Lecture Four – Vector Calculus

Section 4.1 – Vector Fields

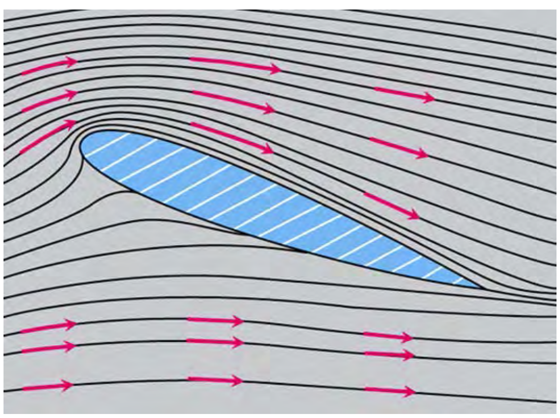
Gravitational and electric forces have both a direction and a magnitude. They are represented by a vector, in a subset of Euclidean space, at each point in their domain, producing a ***vector field***.

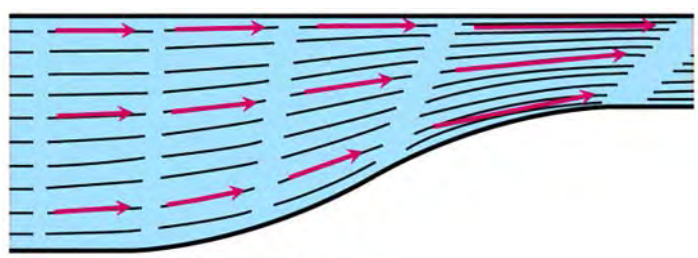
Vector fields are often used to model, for example, the speed and direction of a moving fluid throughout space, or the strength and direction of some force, such as the magnetic or gravitational force, as it changes from point to point. A line integral can be used to find the rate at which the fluid flows along or across a curve within the domain.

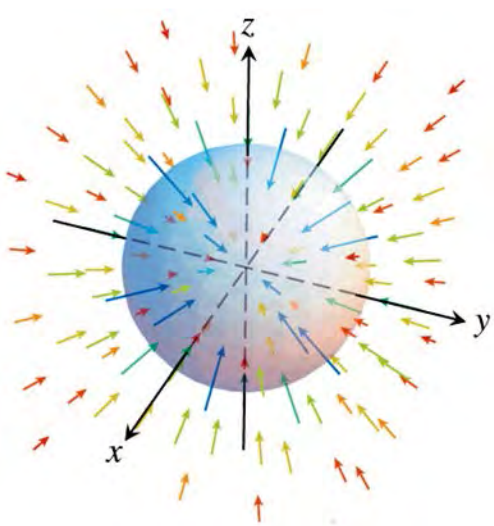
Vector Fields

Suppose a region in the plane or in space occupied by a moving fluid, such air or water. The fluid is made up of a large number of particles, where a particle has a velocity which can vary. Such a fluid flow is an example of a ***vector field***.

Vectors fields are associated with forces such as gravitational attraction, and to magnetic fields, electric fields, and also purely mathematical fields.





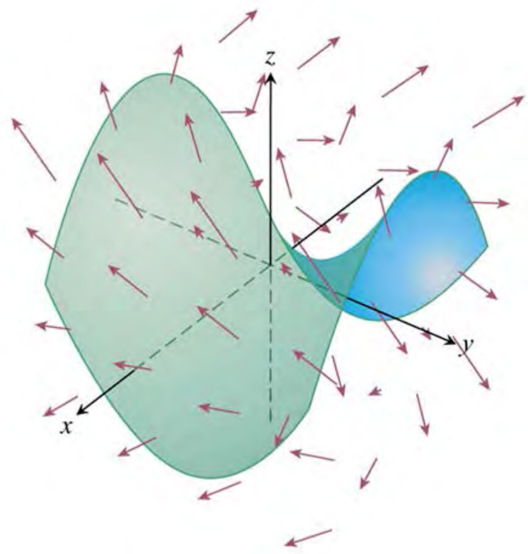


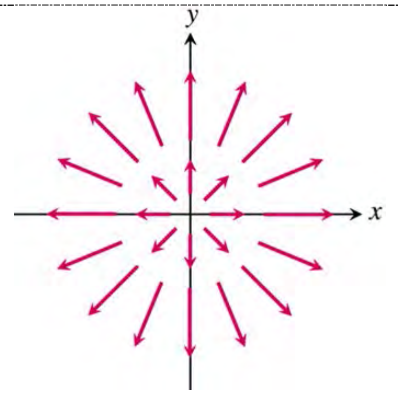
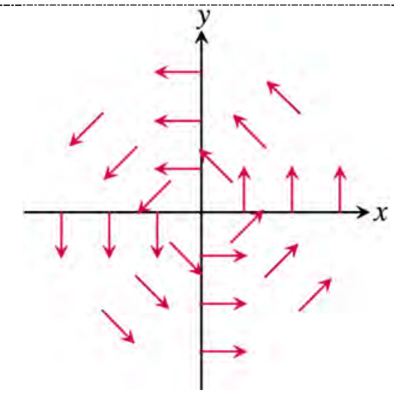
A vector filed is a function that assigns a vector to each point in its domain.

A vector field on a three-dimensional domain in space is given by

$$\boldsymbol{F}(x,y,z) = M(x,y,z)\boldsymbol{i} + N(x,y,z)\boldsymbol{j} + P(x,y,z)\boldsymbol{k}$$

The velocity field expression: $\quad \boldsymbol{v}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$



	
The radial field $\boldsymbol{F} = x\boldsymbol{i} + y\boldsymbol{j}$ of position vectors of points in the plane.	A spin field of rotating unit vectors $\boldsymbol{F} = \dfrac{-y\boldsymbol{i} + x\boldsymbol{j}}{\sqrt{x^2+y^2}}$ In the plane

Definition

Let f and g be defined a region R of $\mathbb{R}^2$. A vector field in $\mathbb{R}^2$ is a function $\boldsymbol{F}$ assigns to each point in R a vector $(f(x, y), g(x, y))$. The vector field is written as

$$\boldsymbol{F}(x, y) = (f(x, y), g(x, y)) \qquad \textit{or}$$

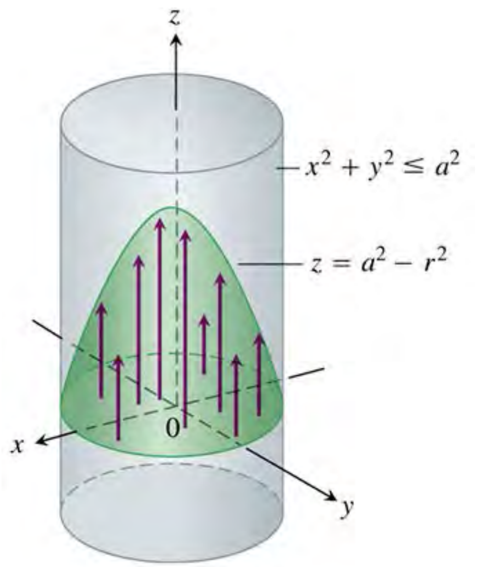
$$F(x, y) = f(x, y)\hat{\boldsymbol{i}} + g(x, y)\hat{\boldsymbol{j}}$$

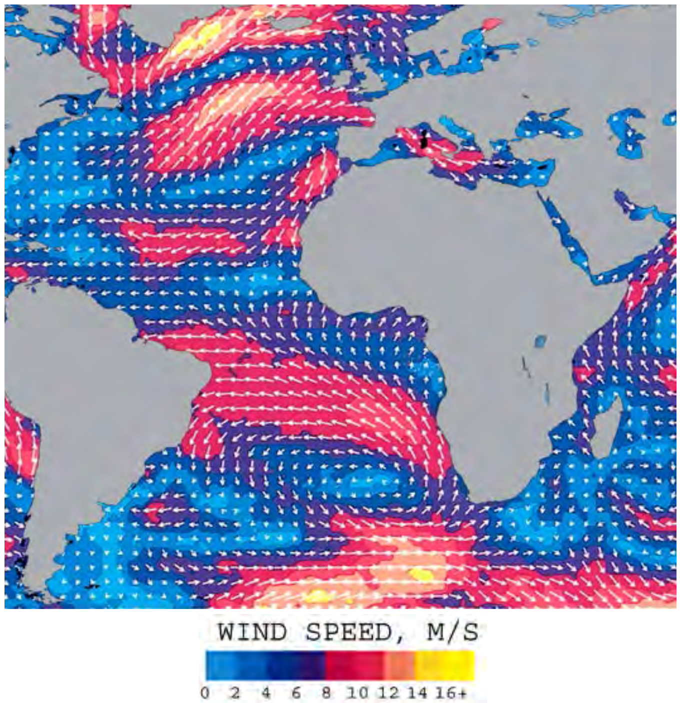
A vector field $\boldsymbol{F} = (f, g)$ is continuous or differentiable on a region R of $\mathbb{R}^2$ if f and g are continuous or differentiable on R, respectively.

Gradient Fields

The gradient vector of a differentiable scalar-valued function at a point gives the direction of greatest increase of the function. An important type of vector field is formed by all the gradient vectors of the function. We define the gradient field of a differentiable function $f(x, y, z)$ to be the field gradient vectors

$$\nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}$$





Example

Suppose that the temperature T at each point (x, y, z) in a region of space is given by

$$T = 100 - x^2 - y^2 - z^2$$

And that $\boldsymbol{F}(x, y, z)$ is defined to be the gradient of T. Find the vector field $\boldsymbol{F}$.

Solution

The gradient field $\boldsymbol{F}$ is the field $\boldsymbol{F} = \nabla T = -2x\boldsymbol{i} - 2y\boldsymbol{j} - 2z\boldsymbol{k}$.

At each point in space, the field F gives the direction for which the increase in temperature is greatest.

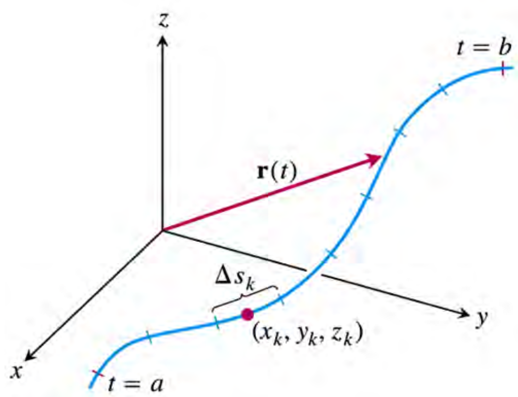
Section 4.2 – Line Integrals

Definition

If f is defined on a curve C given parametrically by $\boldsymbol{r}(t) = g(t)\boldsymbol{i} + h(t)\boldsymbol{j} + k(t)\boldsymbol{k}$, $a \le t \le b$, then the line integral of f over C is

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(x_k, y_k, z_k\right)\Delta s_k$$

Provided this limit exists.



How to Evaluate a Line Integral

1. Find a smooth parametrization of C, $\boldsymbol{r}(t) = g(t)\boldsymbol{i} + h(t)\boldsymbol{j} + k(t)\boldsymbol{k}, \quad a \le t \le b$
2. Evaluate the integral as

$$\int_C f(x,y,z)\,ds = \int_a^b f\left(g(t), h(t), k(t)\right)\,|\boldsymbol{v}(t)|\,dt$$

Example

Integrate $f(x,y,z) = x - 3y^2 + z$ over the line segment C joining the origin to the point (1, 1, 1).

Solution

Assume that: $\boldsymbol{r}(t) = t\boldsymbol{i} + t\boldsymbol{j} + t\boldsymbol{k}, \quad 0 \le t \le 1$

$|\boldsymbol{v}(t)| = |\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3} \ne 0$ (The parameterization is smooth)

$$\int_C f(x,y,z)\,ds = \int_0^1 f(t,t,t)\left(\sqrt{3}\right)dt$$

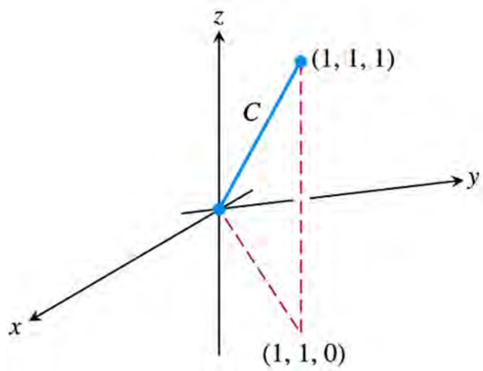
$$= \sqrt{3}\int_0^1 \left(t - 3t^2 + t\right)dt$$

$$= \sqrt{3}\int_0^1 \left(2t - 3t^2\right)dt$$

$$= \sqrt{3}\left[t^2 - t^3\right]_0^1$$

$$= \sqrt{3}(1-1)$$

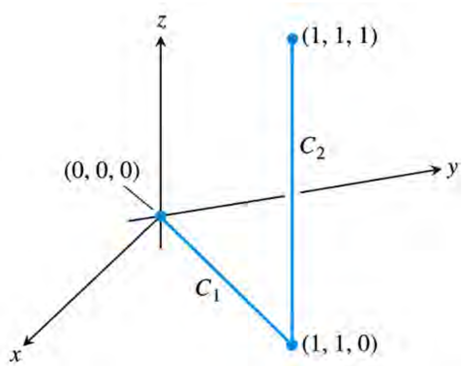
$$\underline{= 0}$$



Example

Integrate $f(x,y,z) = x - 3y^2 + z$ over $C_1 \cup C_2$ using the path the origin to the point (1, 1, 1).

Solution



$$C_1:\quad \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} \quad 0 \le t \le 1 \qquad |\mathbf{v}| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$C_2:\quad \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k} \quad 0 \le t \le 1 \quad |\mathbf{v}| = \sqrt{0^2 + 0^2 + 1^2} = 1$$

$$\int_{C_1 \cup C_2} f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds$$

$$= \int_0^1 f(t,t,0)\sqrt{2}\,dt + \int_0^1 f(1,1,t)(1)\,dt$$

$$= \sqrt{2}\int_0^1 \left(t - 3t^2 + 0\right)dt + \int_0^1 (1 - 3 + t)\,dt$$

$$= \sqrt{2}\left[\tfrac{1}{2}t^2 - t^3\right]_0^1 + \left[-2t + \tfrac{1}{2}t^2\right]_0^1$$

$$= \sqrt{2}\left(\tfrac{1}{2} - 1\right) + \left(-2 + \tfrac{1}{2}\right)$$

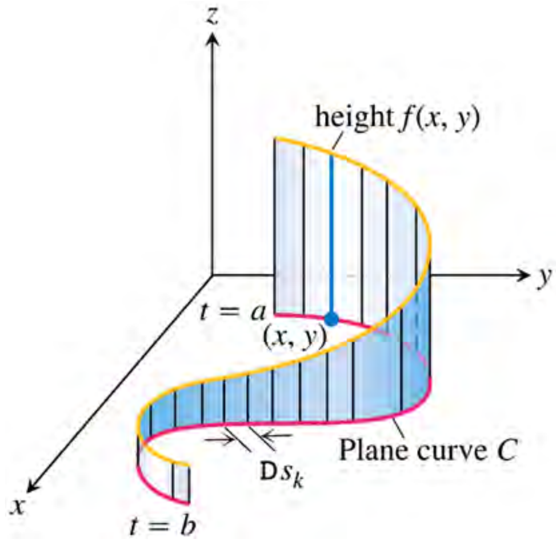
$$\underline{= -\frac{\sqrt{2}}{2} - \frac{3}{2}}$$

- The value of the line integral along a path joining two points can change if you change the path between them.

Line Integrals in the Plane

There is an interesting geometric interpretation for line integrals in the plane. If C is a smooth curve in the xy-plane parametrized by

$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \le t \le b$, we generate a cylindrical surface by moving a straight line along C orthogonal to the plane, holding the line parallel to the z-axis.



The cylinder cuts through the surface, forming a curve on it. The part of the cylindrical surface that lies beneath the surface curve and above the xy-plane is like a ***winding wall*** or ***fence*** standing on the curve C and orthogonal to the plane.

$$\int_C f\ ds = \lim_{n\to\infty}\sum_{k=1}^{n} f\left(x_k, y_k\right)\Delta s_k$$

Where $\Delta s_k \to 0 \ \ as \ \ n\to\infty$, we see that he line integral $\int_C f\ ds$ is the area of the wall.

Line Integrals with Respect to the *xyz* Coordinates

$$\int_C M(x,y,z)dx = \int_a^b M\left(g(t),h(t),k(t)\right)\ g'(t)dt$$

$$\int_C N(x,y,z)dy = \int_a^b N\left(g(t),h(t),k(t)\right)\ h'(t)dt$$

$$\int_C P(x,y,z)dz = \int_a^b P\left(g(t),h(t),k(t)\right)\ k'(t)dt$$

Example

Evaluate the line integral $\int_C -ydx + zdy + 2xdz$, where C is the helix

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k} \quad 0 \le t \le 2\pi$$

Solution

$x = \cos t, \quad y = \sin t, \quad z = t$

$dx = (-\sin t)dt, \quad dy = (\cos t)dt, \quad dz = dt$

$$\int_C -y\,dx + z\,dy + 2x\,dz = \int_0^{2\pi} \left[(-\sin t)(-\sin t) + t\cos t + 2\cos t\right] dt$$

$$= \int_0^{2\pi} \left(\sin^2 t + t\cos t + 2\cos t\right) dt$$

$$= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2t + t\cos t + 2\cos t\right) dt$$

$$= \left[\frac{1}{2}t - \frac{1}{4}\sin 2t + \left(t\sin t + \cos t\right) + 2\sin t\right]_0^{2\pi}$$

$$= \left(\frac{1}{2}(2\pi) + 1\right) - (1)$$

$$= \pi + 1 - 1$$

$$\underline{= \pi}$$

		$\cos t$
+	t	$\sin t$
−	1	$-\cos t$

Exercises Section 4.2 – Line Integrals

1. Evaluate $\int_C (x+y)\,ds$ where C is the straight-line segment $x=t,\ y=(1-t),\ z=0$ from (0, 1, 0) to (1, 0, 0).

2. Evaluate $\int_C (x-y+z-2)\,ds$ where C is the straight-line segment $x=t,\ y=(1-t),\ z=1$ from $(0,\ 1,\ 1)$ to $(1,\ 0,\ 1)$.

3. Evaluate $\int_C (xy+y+z)\,ds$ along the curve $\boldsymbol{r}(t)=2t\boldsymbol{i}+t\boldsymbol{j}+(2-2t)\boldsymbol{k},\quad 0\le t\le 1$

4. Find the integral of $f(x,y,z)=x+y+z$ over the straight line segment from $(1,\ 2,\ 3)$ to $(0,\ -1,\ 1)$

5. Find the integral of $f(x,y,z)=\dfrac{\sqrt{3}}{x^2+y^2+z^2}$ over the curve $\boldsymbol{r}(t)=t\boldsymbol{i}+t\boldsymbol{j}+t\boldsymbol{k},\quad 1\le t\le \infty$

6. Evaluate $\int_C x\,ds$ where C is

a) The straight-line segment $x=t,\ \ y=\frac{t}{2}$, from (0, 0) to (4, 2).

b) The parabolic curve $x=t,\ \ y=t^2$, from (0, 0) to (2, 4).

7. Evaluate $\int_C \sqrt{x+2y}\,ds$ where C is

a) The straight-line segment $x=t,\ \ y=4t$, from (0, 0) to (1, 4).

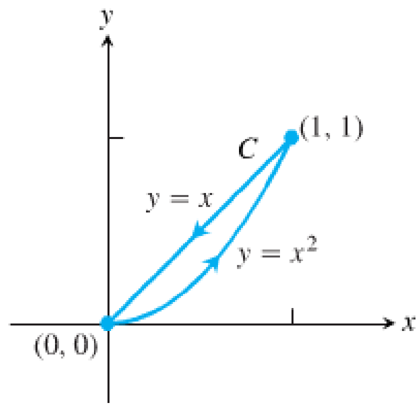
b) $C_1\cup C_2$: C_1 is the line segment (0, 0) to (1, 0) and C_2 is the line segment (1, 0) to (1, 2).

8. Find the line integral of $f(x,y)=\dfrac{\sqrt{y}}{x}$ along the curve $\boldsymbol{r}(t)=t^3\boldsymbol{i}+t^4\boldsymbol{j},\quad \frac{1}{2}\le t\le 1$

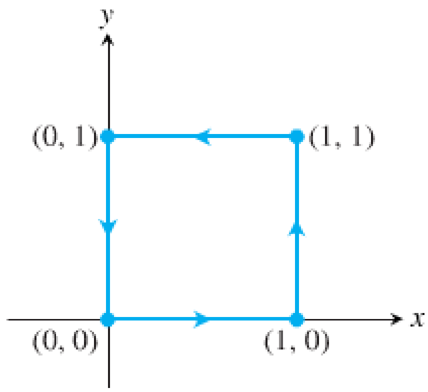
9. Find the line integral of $f(x,y)=\dfrac{x^3}{y}$ over the curve $C:\quad y=\dfrac{x^2}{2},\quad 0\le x\le 2$

10. Find the line integral of $f(x,y)=x^2-y$ over the curve $C:\quad x^2+y^2=4$ in the first quadrant from (0, 2) to $\left(\sqrt{2},\ \sqrt{2}\right)$

11. Evaluate $\int_C \left(x+\sqrt{y}\right)\, ds$ where C is



12. Evaluate $\int_C \frac{1}{x^2+y^2+1}\, ds$ where C is



13. Find the line integral of $f(x,y)=\frac{x^3}{y}$ over the curve $C: \quad y=\frac{x^2}{2}, \quad 0\le x\le 2$

14. Find the line integral of $f(x,y)=x^2-y$ over the curve $C: \quad x^2+y^2=4$ in the first quadrant from (0, 2) to $\left(\sqrt{2},\ \sqrt{2}\right)$

Section 4.3 – Conservative Vector Fields

Line Integrals of Vector Fields

Assume the vector field $\boldsymbol{F}(x,y,z)=M(x,y,z)\boldsymbol{i}+N(x,y,z)\boldsymbol{j}+P(x,y,z)\boldsymbol{k}$ has a continuous components, and the curve C has a smooth parametrization $\boldsymbol{r}(t)=f(t)\boldsymbol{i}+g(t)\boldsymbol{j}+h(t)\boldsymbol{k}$, $a\le t\le b$. $\boldsymbol{r}(t)$ defines along the path C which we call the ***forward direction***. At each point along the path C, the tangent vector $T=\frac{d\boldsymbol{r}}{ds}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}$ is a unit vector tangent to the path and pointing is this forward direction. The tangential component is given by the dot product

$$\boldsymbol{F}\cdot\boldsymbol{T}=\boldsymbol{F}\cdot\frac{d\boldsymbol{r}}{ds}$$

Definition

Let $\boldsymbol{F}$ be a vector field with continuous components defined along a smooth curve C parametrized by $\boldsymbol{r}(t)$, $a\le t\le b$. Then the line integral of $\boldsymbol{F}$ along C is

$$\int_C \boldsymbol{F}\cdot\boldsymbol{T}\,ds=\int_C\left(\boldsymbol{F}\cdot\frac{d\boldsymbol{r}}{ds}\right)ds=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$$

Evaluating the Line Integral of $F=M\boldsymbol{i}+N\boldsymbol{j}+P\boldsymbol{k}$ along *C*: $\boldsymbol{r}(t)=g(t)\boldsymbol{i}+h(t)\boldsymbol{j}+k(t)\boldsymbol{k}$

1. Express the vector field $\boldsymbol{F}$ in terms of the parametrized curve C as $\boldsymbol{F}(\boldsymbol{r}(t))$ by substituting the components $x=g(t),\ y=h(t),\ z=k(t)$ of $\boldsymbol{r}$ into the scalar components $M(x,y,z),\ N(x,y,z),\ P(x,y,z)$ of $\boldsymbol{F}$.
2. Find the derivative (velocity) vector $\frac{d\boldsymbol{r}}{dt}$.
3. Evaluate the line integral with respect to the parameter t, $a\le t\le b$, to obtain

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_a^b \boldsymbol{F}(\boldsymbol{r}(t))\cdot\frac{d\boldsymbol{r}}{dt}dt$$

Example

Evaluate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where $\boldsymbol{F} = z\boldsymbol{i} + xy\boldsymbol{j} - y^2\boldsymbol{k}$ along the curve C given by $\boldsymbol{r}(t) = t^2\boldsymbol{i} + t\boldsymbol{j} + \sqrt{t}\boldsymbol{k} \quad 0 \le t \le 1$.

Solution

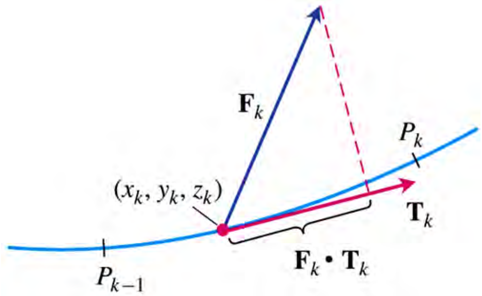
$$\boldsymbol{F}(\boldsymbol{r}(t)) = \sqrt{t}\boldsymbol{i} + t^3\boldsymbol{j} - t^2\boldsymbol{k}$$

$$\frac{d\boldsymbol{r}}{dt} = 2t\boldsymbol{i} + \boldsymbol{j} + \frac{1}{2\sqrt{t}}\boldsymbol{k}$$

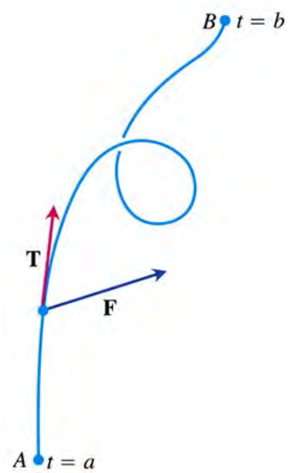
$$\begin{aligned}
\boldsymbol{F}(\boldsymbol{r}(t)) \cdot \frac{d\boldsymbol{r}}{dt} &= \left(\sqrt{t}\boldsymbol{i} + t^3\boldsymbol{j} - t^2\boldsymbol{k}\right) \cdot \left(2t\boldsymbol{i} + \boldsymbol{j} + \frac{1}{2\sqrt{t}}\boldsymbol{k}\right) \\
&= 2t\sqrt{t} + t^3 - \frac{t^2}{2\sqrt{t}} \\
&= 2t^{3/2} + t^3 - \frac{1}{2}t^{3/2} \\
&= \frac{3}{2}t^{3/2} + t^3
\end{aligned}$$

$$\begin{aligned}
\int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_0^1 \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \frac{d\boldsymbol{r}}{dt}\,dt \\
&= \int_0^1 \left(\frac{3}{2}t^{3/2} + t^3\right) dt \\
&= \left[\frac{3}{2}\frac{2}{5}t^{5/2} + \frac{1}{4}t^4\right]_0^1 \\
&= \frac{3}{5} + \frac{1}{4} \\
&= \boxed{\frac{17}{20}}
\end{aligned}$$

Work Done by a Force over a Curve in Space



Definition



Let C be a smooth curve parametrized by $\boldsymbol{r}(t), \ a \le t \le b$, and $\boldsymbol{F}$ be a continuous force field over a region containing C. Then the ***work*** done in moving an object from point $A = \boldsymbol{r}(a)$ to the point $B = \boldsymbol{r}(b)$ along C is

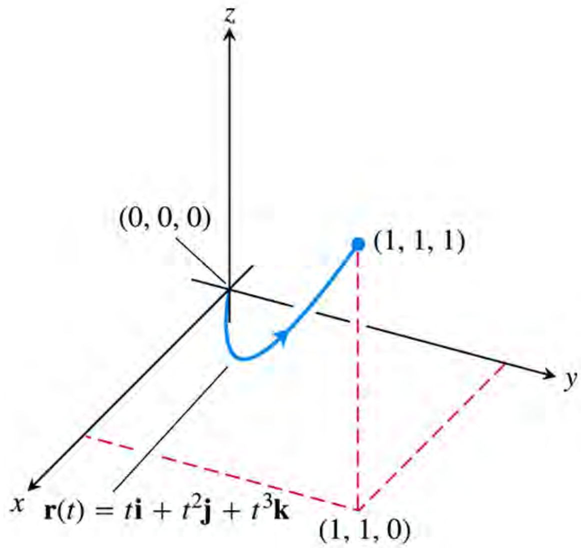
$$W = \int_C \boldsymbol{F} \cdot \boldsymbol{T}\, ds = \int_a^b \boldsymbol{F}\left(\boldsymbol{r}(t)\right) \cdot \frac{d\boldsymbol{r}}{dt} dt$$

Different ways to write the work integral for $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j} + P\boldsymbol{k}$ over the curve C: $\boldsymbol{r}(t) = g(t)\boldsymbol{i} + h(t)\boldsymbol{j} + k(t)\boldsymbol{k}$	
$\boldsymbol{W} = \int_C \boldsymbol{F} \cdot \boldsymbol{T}\, ds$	***The definition***
$= \int_C \boldsymbol{F} \cdot d\boldsymbol{r}$	***Vector differential form***
$= \int_a^b \boldsymbol{F} \cdot \frac{d\boldsymbol{r}}{dt} dt$	***Parametric vector evaluation***
$= \int_a^b \left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt} \right) dt$	***Parametric scalar evaluation***
$= \int_C M dx + N dy + P dz$	***Scalar differential form***

Example

Find the work done by the force field $\boldsymbol{F} = \left(y - x^2\right)\boldsymbol{i} + \left(z - y^2\right)\boldsymbol{j} + \left(x - z^2\right)\boldsymbol{k}$ along the curve $\boldsymbol{r}(t) = t\boldsymbol{i} + t^2\boldsymbol{j} + t^3\boldsymbol{k} \quad 0 \le t \le 1$, form (0, 0, 0) to (1, 1, 1).

Solution



$$\begin{aligned}\boldsymbol{F} &= \left(y - x^2\right)\boldsymbol{i} + \left(z - y^2\right)\boldsymbol{j} + \left(x - z^2\right)\boldsymbol{k} \\ &= \left(t^2 - t^2\right)\boldsymbol{i} + \left(t^3 - t^4\right)\boldsymbol{j} + \left(t - t^6\right)\boldsymbol{k} \\ &= \left(t^3 - t^4\right)\boldsymbol{j} + \left(t - t^6\right)\boldsymbol{k}\end{aligned}$$

$$\begin{aligned}\frac{d\boldsymbol{r}}{dt} &= \frac{d}{dt}\left(t\boldsymbol{i} + t^2\boldsymbol{j} + t^3\boldsymbol{k}\right) \\ &= \boldsymbol{i} + 2t\boldsymbol{j} + 3t^2\boldsymbol{k}\end{aligned}$$

$$\begin{aligned}\boldsymbol{F} \bullet \frac{d\boldsymbol{r}}{dt} &= \left[\left(t^3 - t^4\right)\boldsymbol{j} + \left(t - t^6\right)\boldsymbol{k}\right] \bullet \left(\boldsymbol{i} + 2t\boldsymbol{j} + 3t^2\boldsymbol{k}\right) \\ &= 2t\left(t^3 - t^4\right) + 3t^2\left(t - t^6\right) \\ &= 2t^4 - 2t^5 + 3t^3 - 3t^8\end{aligned}$$

$$\begin{aligned}W &= \int_0^1 \boldsymbol{F} \cdot \frac{d\boldsymbol{r}}{dt}\,dt \\ &= \int_0^1 \left(2t^4 - 2t^5 + 3t^3 - 3t^8\right)dt \\ &= \left[\frac{2}{5}t^5 - \frac{1}{3}t^6 + \frac{3}{4}t^4 - \frac{1}{3}t^9\right]_0^1 \\ &= \frac{2}{5} - \frac{1}{3} + \frac{3}{4} - \frac{1}{3} \\ &= \boxed{\frac{29}{60}}\end{aligned}$$

Example

Find the work done by the force field $\boldsymbol{F} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ in moving an object along the curve C parametrized by $\boldsymbol{r}(t) = \cos(\pi t)\boldsymbol{i} + t^2\boldsymbol{j} + \sin(\pi t)\boldsymbol{k} \quad 0 \le t \le 1$.

Solution

$$\boldsymbol{F}(\boldsymbol{r}(t)) = \cos(\pi t)\boldsymbol{i} + t^2\boldsymbol{j} + \sin(\pi t)\boldsymbol{k}$$

$$\frac{d\boldsymbol{r}}{dt} = -\pi\sin(\pi t)\boldsymbol{i} + 2t\boldsymbol{j} + \pi\cos(\pi t)\boldsymbol{k}$$

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{r}(t)) \cdot \frac{d\boldsymbol{r}}{dt} &= \left(\cos(\pi t)\boldsymbol{i} + t^2\boldsymbol{j} + \sin(\pi t)\boldsymbol{k}\right) \cdot \left(-\pi\sin(\pi t)\boldsymbol{i} + 2t\boldsymbol{j} + \pi\cos(\pi t)\boldsymbol{k}\right) \\ &= -\pi\cos(\pi t)\sin(\pi t) + 2t^3 + \pi\cos(\pi t)\sin(\pi t) \\ &= 2t^3\end{aligned}$$

The work done is the line integral

$$\begin{aligned}W &= \int_0^1 2t^3 dt \\ &= \frac{1}{2}t^4 \Big|_0^1 \\ &= \frac{1}{2}\end{aligned}$$

Flow integrals and Circulation for Velocity Fields

Definitions

If $\boldsymbol{r}(t)$ parametrizes a smooth curve C in the domain of a continuous velocity field $\boldsymbol{F}$, the ***flow*** along the curve point $A = \boldsymbol{r}(a)$ to $B = \boldsymbol{r}(b)$ is

$$Flow = \int_C \boldsymbol{F} \bullet \boldsymbol{T}\ ds$$

The integral in this case is called a ***flow integral***. If the curve starts and ends at the same point, so that $A = B$, the flow is called the ***circulation*** around the curve.

Example

A fluid's velocity field is $\boldsymbol{F} = x\boldsymbol{i} + z\boldsymbol{j} + y\boldsymbol{k}$. Find the flow along the helix

$$\boldsymbol{r}(t) = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} + t\boldsymbol{k}, \quad 0 \le t \le \frac{\pi}{2}$$

Solution

$$\boldsymbol{F} = x\boldsymbol{i} + z\boldsymbol{j} + y\boldsymbol{k}$$

$$= (\cos t)\boldsymbol{i} + t\boldsymbol{j} + (\sin t)\boldsymbol{k}$$

$$\frac{d\boldsymbol{r}}{dt} = (-\sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j} + \boldsymbol{k}$$

$$\boldsymbol{F} \cdot \frac{d\boldsymbol{r}}{dt} = \left((\cos t)\boldsymbol{i} + t\boldsymbol{j} + (\sin t)\boldsymbol{k}\right) \cdot \left((-\sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j} + \boldsymbol{k}\right)$$

$$= -\cos t \sin t + t\cos t + \sin t$$

$$Flow = \int_0^{\pi/2} (-\cos t \sin t + t\cos t + \sin t)\,dt$$

		$\cos t$
+	t	$\sin t$
−	1	$-\cos t$

$$\int -\cos t \sin t\,dt = \int \cos t \; d(\cos t) = \tfrac{1}{2}\cos^2 t$$

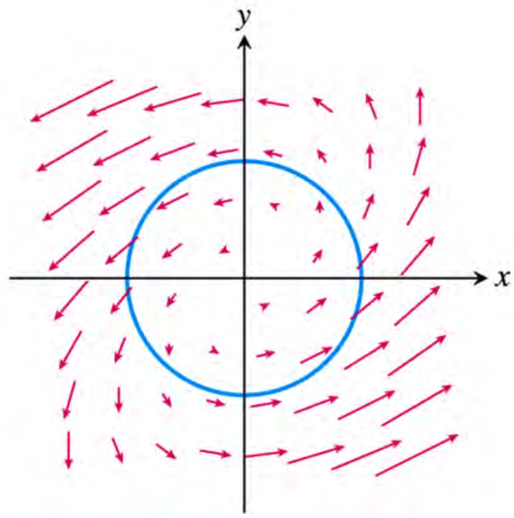
$$= \left[\frac{1}{2}\cos^2 t + t\sin t + \cos t - \cos t\right]_0^{\pi/2}$$

$$= \left[\frac{1}{2}\cos^2 t + t\sin t\right]_0^{\pi/2}$$

$$= \frac{\pi}{2} - \frac{1}{2}$$

Example

Find the circulation of the field $\boldsymbol{F} = (x-y)\boldsymbol{i} + x\boldsymbol{j}$ around the circle $\boldsymbol{r}(t) = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j},\quad 0 \le t \le 2\pi$



Solution

$$\boldsymbol{F} = (x-y)\boldsymbol{i} + x\boldsymbol{j}$$
$$= (\cos t - \sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j}$$

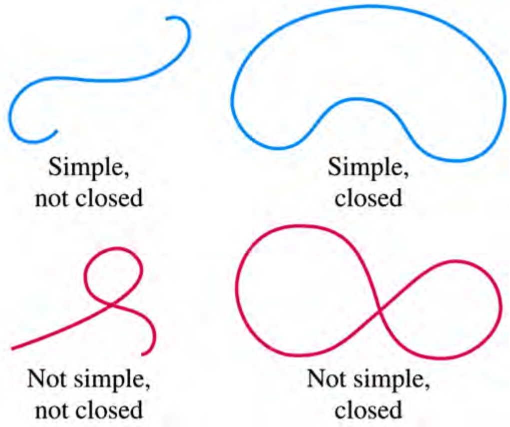
$$\frac{d\boldsymbol{r}}{dt} = (-\sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j}$$

$$\boldsymbol{F} \cdot \frac{d\boldsymbol{r}}{dt} = \left((\cos t - \sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j}\right) \cdot \left((-\sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j}\right)$$
$$= -\cos t \sin t + \sin^2 t + \cos^2 t$$
$$= 1 - \cos t \sin t$$

$$Circulation = \int_0^{2\pi} (1 - \cos t \sin t)\,dt$$
$$= \left[t + \frac{1}{2}\cos^2 t\right]_0^{2\pi}$$
$$= 2\pi + \frac{1}{2} - \frac{1}{2}$$
$$= 2\pi$$

Flux across a Simple Plane Curve

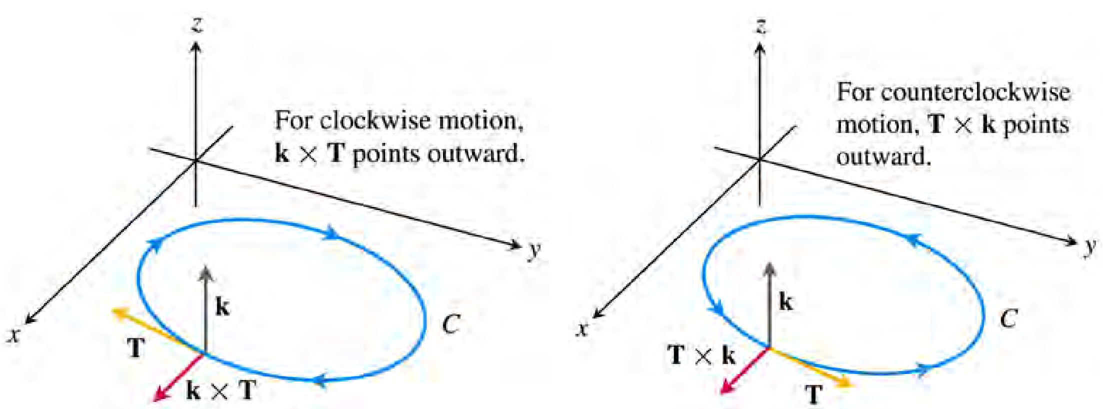
A curve in the *xy*-plane is simple if it does not cross itself. When a curve starts and ends at the same point, it is a ***closed curve*** or ***loop***.



Definition

If C is a smooth simple closed curve in the domain of a continuous velocity field in $\boldsymbol{F} = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$ in the plane, and if $\boldsymbol{n}$ is the outward-pointing unit normal vector on C, the flux of $\boldsymbol{F}$ across C is

$$\textit{Flux of } \boldsymbol{F} \textit{ across } C = \int_C \boldsymbol{F} \bullet \boldsymbol{n}\, ds$$



$$\begin{aligned}\boldsymbol{n} &= \boldsymbol{T} \times \boldsymbol{k} \\ &= \left(\frac{dx}{ds}\boldsymbol{i} + \frac{dy}{ds}\boldsymbol{j}\right) \times \boldsymbol{k} \\ &= \frac{dy}{ds}\boldsymbol{i} - \frac{dx}{ds}\boldsymbol{j}\end{aligned}$$

$$\boldsymbol{F} \bullet \boldsymbol{n} = M(x,y)\frac{dy}{ds} - N(x,y)\frac{dx}{ds}$$

Calculating Flux Across a Smooth Closed Plane Curve

$$\left(Flux\ of\ \ \boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j}\ \ across\ C\right) = \oint_C M dy - N dx$$

The integral can be evaluated from any smooth parametrization $x = g(t),\ y = h(t),\ a \le t \le b$, that traces C counterclockwise exactly once.

Example

Find the flux of $\boldsymbol{F} = (x - y)\boldsymbol{i} + x\boldsymbol{j}$ across the circle $x^2 + y^2 = 1$ in the *xy*-plane. (The vector field and curve)

Solution

The parametrization $\boldsymbol{r}(t) = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j}, \quad 0 \le t \le 2\pi$ traces the circle counterclockwise exactly once.

$$M = x - y = \cos t - \sin t, \qquad dy = d(\sin t) = \cos t\ dt$$
$$N = x = \cos t, \qquad dx = d(\cos t) = -\sin t\ dt$$

$$Flux = \int_C M dy - N dx$$

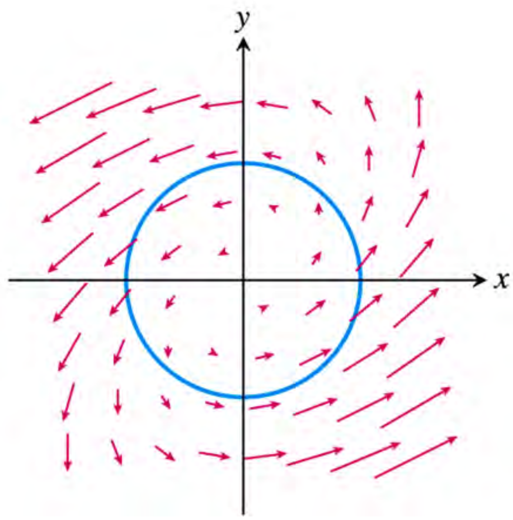
$$= \int_0^{2\pi} \left(\cos^2 t - \sin t \cos t + \cos t \sin t\right) dt$$

$$= \int_0^{2\pi} \cos^2 t\ dt$$

$$= \int_0^{2\pi} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2t\right)\ dt$$

$$= \left[\tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right]_0^{2\pi}$$

$$\underline{= \pi}$$



Exercises *Section* 4.3 – Conservative Vector Fields

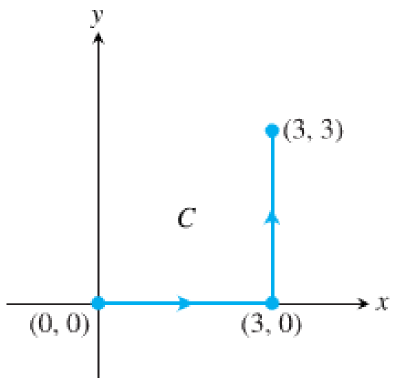
1. Find the gradient field of the function $f(x,y,z)=\left(x^2+y^2+z^2\right)^{-1/2}$

2. Find the gradient field of the function $f(x,y,z)=\ln\sqrt{x^2+y^2+z^2}$

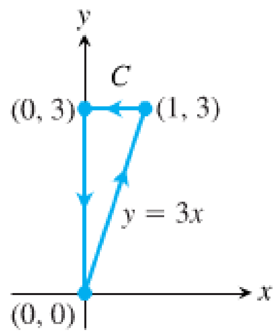
3. Find the gradient field of the function $f(x,y,z)=e^z-\ln\left(x^2+y^2\right)$

4. Find the line integral of $\int_C (x-y)\,dx$ where $C:\ x=t,\quad y=2t+1,\quad for\ 0\le t\le 3$

5. Find the line integral of $\int_C \left(x^2+y^2\right)dy$ where C is



6. Find the line integral of $\int_C \sqrt{x+y}\ dx$ where C is



7. Find the work done by the force field $\boldsymbol{F}=xy\boldsymbol{i}+y\boldsymbol{j}-yz\boldsymbol{k}$ over the curve $\boldsymbol{r}(t)=t\boldsymbol{i}+t^2\boldsymbol{j}+t\boldsymbol{k},\quad 0\le t\le 1$.

8. Find the work done by the force field $\boldsymbol{F}=2y\boldsymbol{i}+3x\boldsymbol{j}+(x+y)\boldsymbol{k}$ over the curve $\boldsymbol{r}(t)=(\cos t)\boldsymbol{i}+(\sin t)\boldsymbol{j}+\frac{t}{6}\boldsymbol{k},\quad 0\le t\le 2\pi$.

9. Find the work done by the force field $\boldsymbol{F} = z\boldsymbol{i} + x\boldsymbol{j} + y\boldsymbol{k}$ over the curve $\boldsymbol{r}(t) = (\sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j} + t\boldsymbol{k}, \quad 0 \le t \le 2\pi$.

10. Evaluate $\int_C \boldsymbol{F} \cdot \boldsymbol{T} ds$ for the vector field $\boldsymbol{F} = x^2\boldsymbol{i} - y\boldsymbol{j}$ along the curve $x = y^2$ from $(4, \ 2)$ to $(1, \ -1)$

11. Find the circulation and flux of the fields $\boldsymbol{F}_1 = x\boldsymbol{i} + y\boldsymbol{j}$ *and* $\boldsymbol{F}_2 = -y\boldsymbol{i} + x\boldsymbol{j}$ around and across each of the following curves.

a) The circle $\boldsymbol{r}(t) = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j}, \quad 0 \le t \le 2\pi$

b) The ellipse $\boldsymbol{r}(t) = (\cos t)\boldsymbol{i} + (4\sin t)\boldsymbol{j}, \quad 0 \le t \le 2\pi$

12. Find the circulation and flux of the fields $\boldsymbol{F}_1 = 2x\boldsymbol{i} - 3y\boldsymbol{j}$ *and* $\boldsymbol{F}_2 = 2x\boldsymbol{i} + (x - y)\boldsymbol{j}$ across the circle $\boldsymbol{r}(t) = (a\cos t)\boldsymbol{i} + (a\sin t)\boldsymbol{j}, \quad 0 \le t \le 2\pi$

13. Find a field $\boldsymbol{F} = M(x, y)\boldsymbol{i} + N(x, y)\boldsymbol{j}$ in the *xy*-plane with the property that at each point $(x, y) \ne (0, 0)$, $\boldsymbol{F}$ points toward the origin and $|\boldsymbol{F}|$ is

a) The distance from (x, y) to the origin

b) Inversely proportional to the distance from (x, y) to the origin.
(The field is undefined at (0, 0).)

14. A fluid's velocity field is $\boldsymbol{F} = -4xy\boldsymbol{i} + 8y\boldsymbol{j} + 2\boldsymbol{k}$. Find the flow along the curve $\boldsymbol{r}(t) = t\boldsymbol{i} + t^2\boldsymbol{j} + \boldsymbol{k}, \quad 0 \le t \le 2$

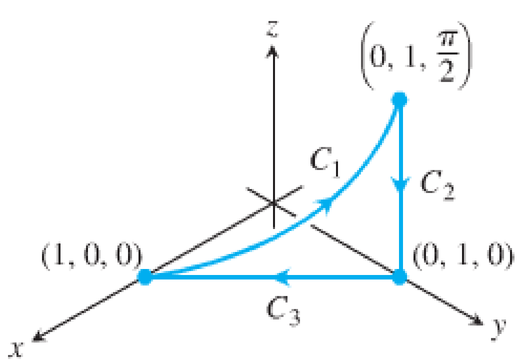
15. A fluid's velocity field is $\boldsymbol{F} = x^2\boldsymbol{i} + yz\boldsymbol{j} + y^2\boldsymbol{k}$. Find the flow along the curve $\boldsymbol{r}(t) = 3t\boldsymbol{j} + 4t\boldsymbol{k}, \quad 0 \le t \le 1$

16. Find the circulation of $\boldsymbol{F} = 2x\boldsymbol{i} + 2z\boldsymbol{j} + 2y\boldsymbol{k}$ around the closed path consisting of the following three curves traversed in the direction of increasing *t*.

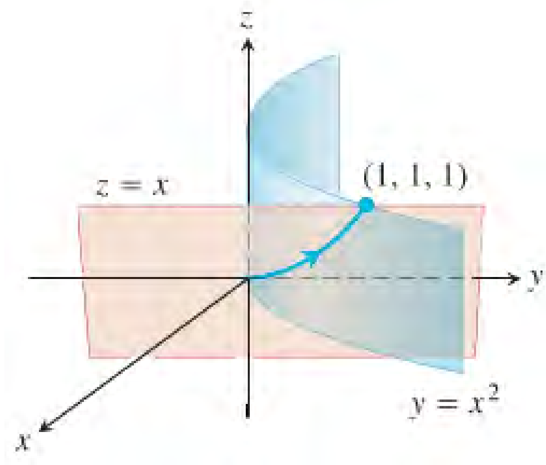
$C_1: \quad \boldsymbol{r}(t) = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} + t\boldsymbol{k}, \quad 0 \le t \le \frac{\pi}{2}$

$C_2: \quad \boldsymbol{r}(t) = \boldsymbol{j} + \frac{\pi}{2}(1 - t)\boldsymbol{k}, \quad 0 \le t \le 1$

$C_3: \quad \boldsymbol{r}(t) = t\boldsymbol{i} + (1 - t)\boldsymbol{j}, \quad 0 \le t \le 1$



17. The field $\boldsymbol{F} = xy\boldsymbol{i} + y\boldsymbol{j} - yz\boldsymbol{k}$ is the velocity field of a flow in space. Find the flow from $(0,\ 0,\ 0)$ to $(1,\ 1,\ 1)$ along the curve of intersection of the cylinder $y = x^2$ and the plane $z = x$.
(***Hint***: Use $t = x$ as the parameter.)



18. Evaluate $\int_C (x-y)\,dx + (x+y)\,dy$ counterclockwise around the triangle with vertices (0, 0), $(1,\ 0)$ and $(0,\ 1)$

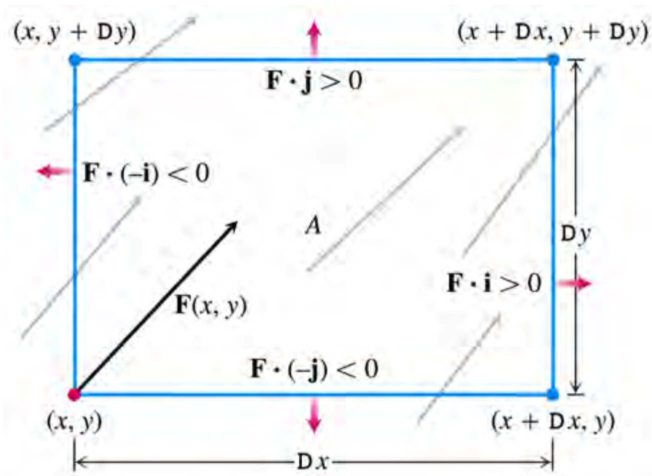
Section 4.4 – Green's Theorem

Green's theorem gives the relationship between a line integral around a simple closed curve *C* and a double integral over the plane region *D* bounded by *C*. It is the two-dimensional special case of the more general ***Stokes' theorem***, and is named after British mathematician *George Green*.

Green’s theorem applies to any vector field, independent of any particular interpretation of the field, provided assumptions of the theorem are satisfied, We introduce two new ideas for Green’s theorem: *divergence* and *circulation density* around an axis perpendicular to the plane.

Divergence

Suppose that $\boldsymbol{F}(x,y)=M(x,y)\boldsymbol{i}+N(x,y)\boldsymbol{j}$ is the velocity field of fluid flowing in the plane and that the first partial derivatives of *M* and *N* are continuous at each point of a region *R*.



Fluid Flow Rates:

Top: $\boldsymbol{F}(x,\ y+\Delta y)\cdot\boldsymbol{j}\ \Delta x=N(x,y+\Delta y)\Delta x$

Bottom: $\boldsymbol{F}(x,\ y)\cdot(-\boldsymbol{j})\ \Delta x=-N(x,y)\Delta x$

Right: $\boldsymbol{F}(x+\ \Delta x,\ y)\cdot\boldsymbol{i}\ \Delta y=M(x+\Delta x,y)\Delta y$

Left: $\boldsymbol{F}(x,\ y)\cdot(-\boldsymbol{i}\ )\Delta y=-M(x,y)\Delta y$

Top and Bottom: $\big(N(x,\ y+\Delta y)-N(x,\ y)\big)\Delta x\approx\left(\dfrac{\partial N}{\partial y}\Delta y\right)\Delta x$

Right and Left: $\big(M(x+\Delta x,\ y)-M(x,\ y)\big)\Delta y\approx\left(\dfrac{\partial M}{\partial x}\Delta x\right)\Delta y$

Adding the last two equations gives the net effect of the flow rates:

$$\textit{Flux across rectangle boundary}\approx\left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right)\Delta x\Delta y$$

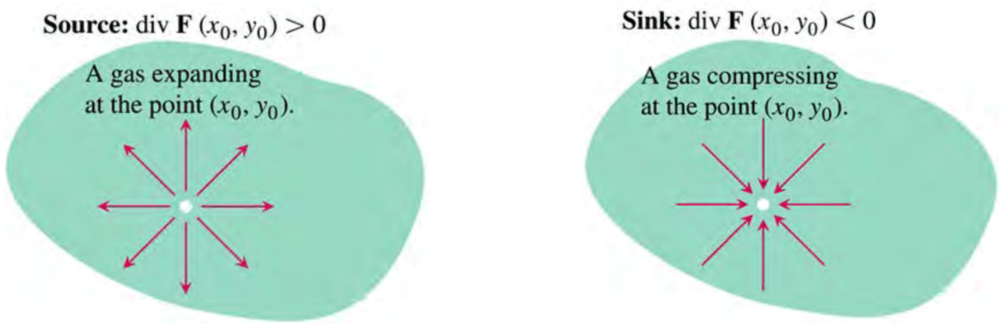
The estimate of the total flux per unit area or flux density for the rectangle:

$$\frac{\textit{Flux across rectangle boundary}}{\textit{rectangle area}} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$$

Definition

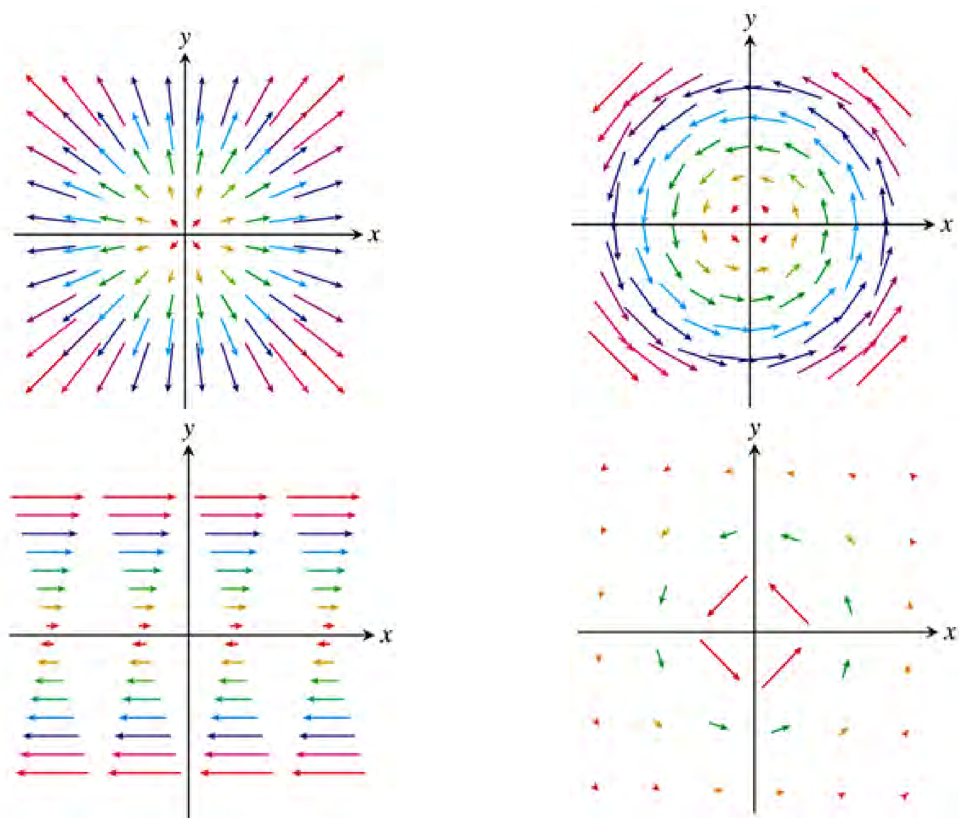
The divergence (flux density) of a vector field $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j}$ at the point (x, y) is

$$div\boldsymbol{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$



Example

The following vector fields represent the velocity of a gas flowing in the *xy*-plane. Find the divergence of each vector field and interpret its physical meaning.



a) Uniform expansion or compression: $\boldsymbol{F}(x,\ y) = cx\boldsymbol{i} + cy\boldsymbol{j}$

b) Uniform rotation: $\boldsymbol{F}(x,\ y) = -cy\boldsymbol{i} + cx\boldsymbol{j}$

c) Shearing flow: $\boldsymbol{F}(x,\ y) = y\boldsymbol{i}$

d) Whirlpool effect: $\boldsymbol{F}(x,\ y) = \dfrac{-y}{x^2+y^2}\boldsymbol{i} + \dfrac{x}{x^2+y^2}\boldsymbol{j}$

Solution

a) $div\boldsymbol{F} = \dfrac{\partial}{\partial x}(cx) + \dfrac{\partial}{\partial y}(cy) = c + c = 2c$

If $c > 0$, the gas is undergoing uniform expansion

If $c < 0$, the gas is undergoing uniform compression

b) $div\boldsymbol{F} = \dfrac{\partial}{\partial x}(-cy) + \dfrac{\partial}{\partial y}(cx) = 0$

The gas is neither expanding nor compressing.

c) $div\boldsymbol{F} = \dfrac{\partial}{\partial x}(y) = 0$

The gas is neither expanding nor compressing.

d) $div\boldsymbol{F} = \dfrac{\partial}{\partial x}\left(\dfrac{-y}{x^2+y^2}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{x}{x^2+y^2}\right)$

$$= \frac{2xy}{\left(x^2+y^2\right)^2} - \frac{2xy}{\left(x^2+y^2\right)^2}$$

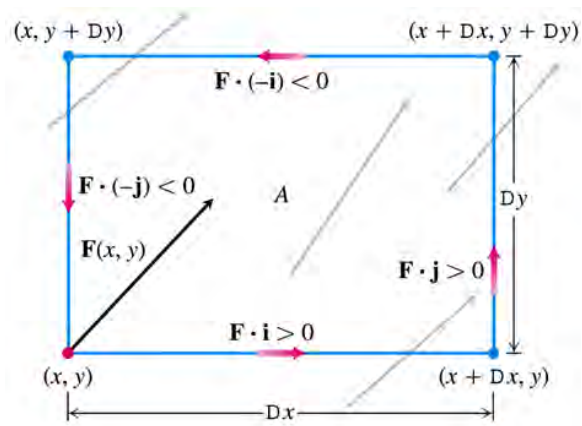
$$= 0$$

The divergence is zero at all points in the domain of the velocity field.

Spin Around an Axis: The *k*-Component of Curl

The Green's Theorem has to do with measuring how a floating paddle wheel, with axis perpendicular to the plane, spins at a point in fluid flowing in a plane region. Sometimes refer to *circulation density* of a vector field $\boldsymbol{F}$ at a point.

$$\boldsymbol{F}(x,\ y) = M(x,\ y)\boldsymbol{i} + N(x,\ y)\boldsymbol{j}$$



The circulation rate of $\boldsymbol{F}$ around the boundary of A is the sum of flow rates along the sides in the tangential direction.

Top: $\boldsymbol{F}(x,\ y+\Delta y)\cdot(-\boldsymbol{i})\ \Delta x = -M(x, y+\Delta y)\Delta x$

Bottom: $\boldsymbol{F}(x,\ y)\cdot\boldsymbol{i}\ \Delta x = M(x,y)\Delta x$

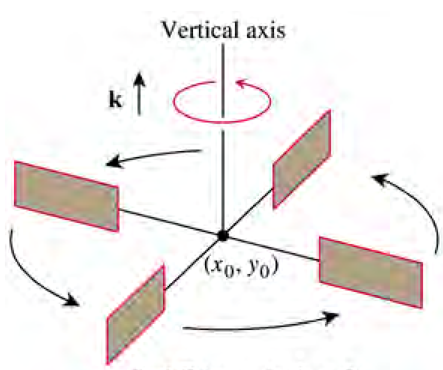
Right: $\boldsymbol{F}(x+\ \Delta x,\ y)\cdot\boldsymbol{j}\ \Delta y = N(x+\Delta x, y)\Delta y$

Left: $\boldsymbol{F}(x,\ y)\cdot(-\boldsymbol{j}\ )\Delta y = -N(x,y)\Delta y$

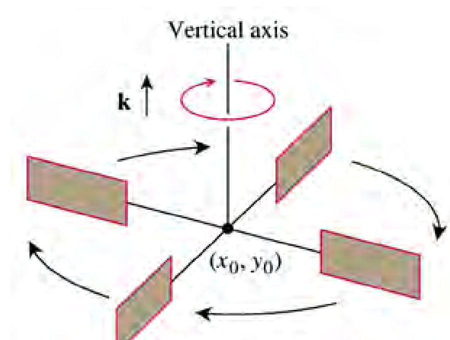
Top and Bottom: $-\left(M(x,\ y+\Delta y) - M(x,\ y)\right)\Delta x \approx -\left(\dfrac{\partial M}{\partial y}\Delta y\right)\Delta x$

Right and Left: $\left(N(x+\Delta x,\ y) - N(x,\ y)\right)\Delta y \approx \left(\dfrac{\partial N}{\partial x}\Delta x\right)\Delta y$

$$\frac{\textit{Circulation around rectangle}}{\textit{rectangle area}} \approx \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$



Curl $\mathbf{F}(x_0, y_0) \cdot \mathbf{k} > 0$
Counterclockwise circulation



Curl $\mathbf{F}(x_0, y_0) \cdot \mathbf{k} < 0$
Clockwise circulation

Definition

The ***circulation density*** of a vector field $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j}$ at the point (x, y) is the scalar expression

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

This expression is also called the ***k-component of the curl***, denoted by $(curl\ \boldsymbol{F}) \bullet \boldsymbol{k}$

Example

Find the circulation density, and interpret what it means, for each vector field

Solution

a) *Uniform expansion*: $(curl\boldsymbol{F}) \bullet k = \frac{\partial}{\partial x}(cy) - \frac{\partial}{\partial y}(cx) = 0$.

The gas is not circulating at very small scales.

b) *Rotation*: $(curl\boldsymbol{F}) \bullet k = \frac{\partial}{\partial x}(cx) - \frac{\partial}{\partial y}(-cy) = 2c$

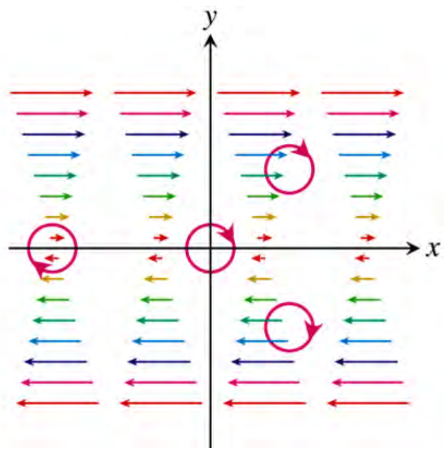
The constant circulation density indicates rotation at every point.

If $c > 0$, the rotation is counterclockwise

If $c < 0$, the rotation is clockwise

c) *Shear*: $(curl\boldsymbol{F}) \bullet k = -\frac{\partial}{\partial y}(y) = -1$

The circulation density is constant and negative, so a paddle wheel floating in water undergoing such a shearing flow spins clockwise. The rate of rotation is the same at each point. The average effect of the fluid flow is to push fluid clockwise around each of the small circles.



d) *Whirlpool*: $(curl\boldsymbol{F}) \bullet k = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)$

$$= \frac{y^2-x^2}{\left(x^2+y^2\right)^2} - \frac{y^2-x^2}{\left(x^2+y^2\right)^2}$$

$$= 0$$

The circulation density is 0 at every point away from the origin (where the vector field is undefined and the whirlpool effect is taking place), and the gas is not circulating at any point for which the vector field is defined.

Theorem – Green's Theorem (Flux-Divergence or Normal Form)

Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the outward flux of $\boldsymbol{F}$ across C equals the double integral of div $\boldsymbol{F}$ over the region R enclosed by C.

$$\underbrace{\oint_C F \cdot N ds}_{\text{Outward flux}} = \oint_C M dy - N dx = \underbrace{\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy}_{\text{Divergence integral}}$$

Theorem – Green's Theorem (Circulation-Curl or Tangential Form)

Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the counterclockwise circulation of $\boldsymbol{F}$ around C equals the double integral of $(curl\ \boldsymbol{F}) \cdot \boldsymbol{k}$ over R.

$$\underbrace{\oint_C F \cdot T ds}_{\text{Counterclockwise circulation}} = \oint_C M dx + N dy = \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy}_{\text{Curl integral}}$$

Example

Verify both forms of Green's Theorem for the vector field $\boldsymbol{F}(x, y) = (x - y)\boldsymbol{i} + x\boldsymbol{j}$

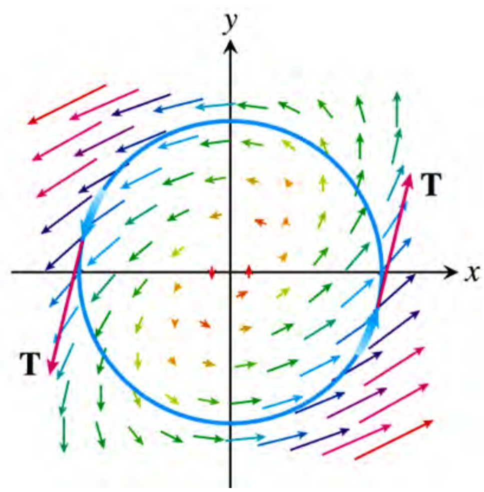
And the region R bounded by the unit circle $C: \quad \boldsymbol{r}(t) = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j}, \quad 0 \le t \le 2\pi$

<u>Solution</u>

$$M = x - y = \cos t - \sin t, \quad dx = d(\cos t) = -\sin t\ dt$$

$$N = x = \cos t, \qquad dy = d(\sin t) = \cos t\ dt$$

$$\frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = 1, \quad \frac{\partial N}{\partial y} = 0$$



1. $$\oint_C M\,dy - N\,dx = \int_0^{2\pi} (\cos t - \sin t)(\cos t\,dt) - (\cos t)(-\sin t\,dt)$$

$$= \int_0^{2\pi} \cos^2 t\ dt$$

$$= \int_0^{2\pi} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2t\right)\ dt$$

$$= \left[\tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right]_0^{2\pi}$$

$$\underline{= \pi}$$

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy = \iint_R (1+0)\,dxdy$$

$$= \iint_R dxdy$$

$$= \textit{area inside the unit circle}$$

$$\underline{= \pi}$$

2. $$\oint_C M\,dx + N\,dy = \int_0^{2\pi} (\cos t - \sin t)(-\sin t\,dt) + \cos t(\cos t\,dt)$$

$$= \int_0^{2\pi} \left(-\cos t \sin t + \sin^2 t + \cos^2 t\right) dt$$

$$= \int_0^{2\pi} \left(-\tfrac{1}{2}\sin 2t + 1\right) dt$$

$$= \tfrac{1}{4}\cos 2t + t \Big|_0^{2\pi}$$

$$\underline{= 2\pi}$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R (1-(-1))\,dxdy$$

$$= 2\iint_R dxdy$$

$$\underline{= 2\pi}$$

Example

Evaluate the line integral $\oint_C xydy - y^2 dx$

Where C is the square cut from the first quadrant by the lines $x=1$ and $y=1$

Solution

With the Normal Form Equation: $M = xy \quad N = y^2$

$$\oint_C xydy - y^2 dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \iint_R (y + 2y) dxdy$$

$$= \int_0^1 \int_0^1 3ydxdy$$

$$= \int_0^1 [3xy]_0^1 \, dy$$

$$= 3\int_0^1 ydy$$

$$= \frac{3}{2}\left[y^2 \right]_0^1$$

$$\underline{= \frac{3}{2}}$$

With the Tangential Form Equation: $M = -y^2 \quad N = xy$

$$\oint_C -y^2 dx + xydy = \iint_R (y - (-2y)) dxdy$$

$$= \int_0^1 \int_0^1 3ydxdy$$

$$\underline{= \frac{3}{2}}$$

Example

Calculate the outward flux of the vector field $\boldsymbol{F}(x, y) = x\boldsymbol{i} + y^2\boldsymbol{j}$ across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$

Solution

$$M = x \quad N = y^2$$

$$Flux = \oint_C F \bullet n\ ds = \oint_C Mdy - Ndx$$

$$= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

Green's Theorem

$$= \int_{-1}^{1} \int_{-1}^{1} (1 + 2y)\, dxdy$$

$$= \int_{-1}^{1} (1 + 2y) x \Big|_{-1}^{1} dy$$

$$= \int_{-1}^{1} (1 + 2y)(1 - (-1))\, dy$$

$$= 2\int_{-1}^{1} (1 + 2y)\, dy$$

$$= 2\left[y + y^2 \right]_{-1}^{1}$$

$$= 2\left[1 + 1 - (-1 + 1) \right]$$

$$\underline{= 4}$$

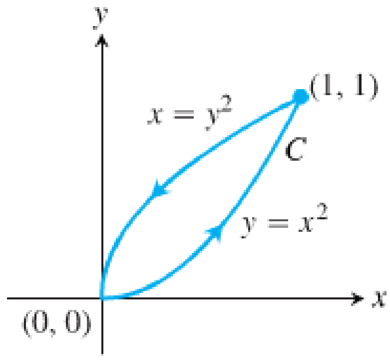
Exercises Section 4.4 – Green's Theorem

1. Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\boldsymbol{F} = (x-y)\boldsymbol{i} + (y-x)\boldsymbol{j}$ and curve C is the square bounded by $x=0,\ x=1,\ y=0,\ y=1$

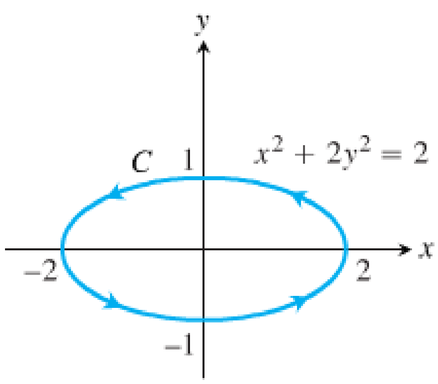
2. Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\boldsymbol{F} = \left(x^2+4y\right)\boldsymbol{i} + \left(x+y^2\right)\boldsymbol{j}$ and curve C is the square bounded by $x=0,\ x=1,\ y=0,\ y=1$

3. Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\boldsymbol{F} = (x+y)\boldsymbol{i} - \left(x^2+y^2\right)\boldsymbol{j}$ and curve C is the triangle bounded by $y=0,\ x=1,\ y=x$

4. Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\boldsymbol{F} = \left(xy+y^2\right)\boldsymbol{i} + (x-y)\boldsymbol{j}$ and curve C



5. Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\boldsymbol{F} = (x+3y)\boldsymbol{i} + (2x-y)\boldsymbol{j}$ and curve C



6. Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\boldsymbol{F} = \left(x+e^x\sin y\right)\boldsymbol{i} + \left(x+e^x\cos y\right)\boldsymbol{j}$ and curve C is the right-hand loop of the lemniscate $r^2 = \cos 2\theta$

7. Find the outward flux for the field $\boldsymbol{F} = \left(3xy - \dfrac{x}{1+y^2}\right)\boldsymbol{i} + \left(e^x + \tan^{-1} y\right)\boldsymbol{j}$ across the cardioid $r = a(1+\cos\theta),\ \ a > 0$

8. Find the work done by $\boldsymbol{F} = 2xy^3\boldsymbol{i} + 4x^2y^2\boldsymbol{j}$ in moving a particle once counterclockwise around the curve *C*: The boundary of the triangular region in the first quadrant enclosed by the *x*-axis, the line $x = 1$ and the curve $y = x^3$

9. Apply Green's Theorem to evaluate the integral $\oint\limits_C \left(y^2dx + x^2dy\right)$

C: The triangle bounded by $x = 0,\ x + y = 1,\ y = 0$

10. Apply Green's Theorem to evaluate the integral $\oint\limits_C (3ydx + 2xdy)$

C: The boundary of $0 \le x \le \pi, \quad 0 \le y \le \sin x$

Section 4.5 – Divergence and Curl

Green's Theorem out of the plane $\left(\mathbb{R}^2\right)$ and into space $\left(\mathbb{R}^3\right)$, it is done as follows:

- The circulation form of Green's Theorem relates a line integral over a simple closed oriented curve in the plane to a double integral over the enclosed region. Stoke's Theorem relates a line integral over a simple closed oriented curve in $\mathbb{R}^3$ to a double integral over a surface whose boundary is the same curve.
- The flux form of Green's Theorem relates a line integral over a simple closed oriented curve in the plane to a double integral over the enclosed region. Similarly, the Divergence Theorem relates an integral over a closed oriented surface in $\mathbb{R}^3$ to a triple integral over the region enclosed by the surface.

Definition

The divergence of a vector field $\boldsymbol{F} = (f,\ g,\ h)$ that is differentiable on a region of $\mathbb{R}^3$ is

$$div\boldsymbol{F} = \nabla\cdot\boldsymbol{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

If $\nabla\cdot\boldsymbol{F} = 0$, the vector field is ***source free***.

Example

Compute the divergence of the following vector fields

a) $\boldsymbol{F} = (x,\ y,\ z)$ $\quad (radial\ field)$

b) $\boldsymbol{F} = (-y,\ x-z,\ y)$ $\quad (rotation\ field)$

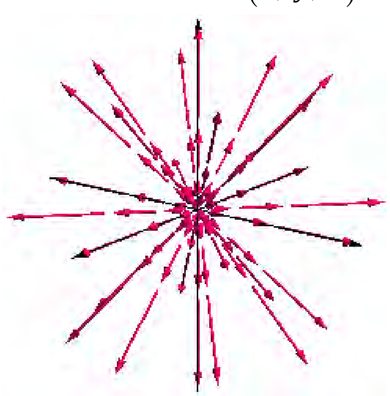
c) $\boldsymbol{F} = (-y,\ x,\ z)$ $\quad (spiral\ flow)$

Solution

a) The divergence is $\nabla\cdot\boldsymbol{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = 1+1+1 = 3$

Because the divergence is positive, the flow expands outward at all points.

Radial field $\boldsymbol{F} = (x,\ y,\ z)$

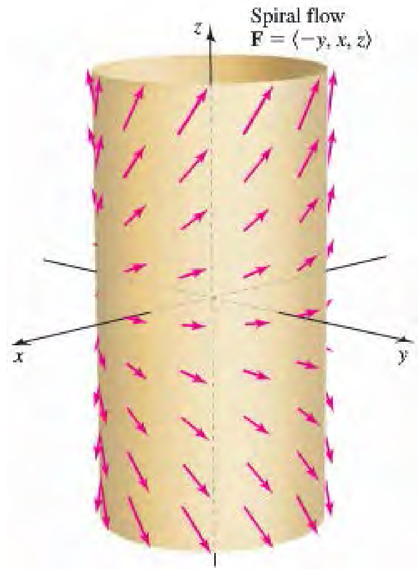


b) The divergence is $\nabla \cdot \boldsymbol{F} = \nabla \cdot (-y,\ x-z,\ y) = \dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} + \dfrac{\partial h}{\partial z}\ \underline{= 0}$

The field is source free.

c) $\nabla \cdot F = \nabla \cdot (-y,\ x,\ z) = \dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} + \dfrac{\partial h}{\partial z}\ \underline{= 1}$

The rotational part of the field in *x* and *y* does not contribute to the divergence. However, the *z*-component of the field produces a nonzero divergence.



Example

Compute the divergence of the radial vector field

$$\boldsymbol{F} = \frac{\boldsymbol{r}}{|\boldsymbol{r}|} = \frac{(x,\ y,\ z)}{\sqrt{x^2+y^2+z^2}}$$

Solution

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\left(x^2+y^2+z^2\right)^{1/2} - x^2\left(x^2+y^2+z^2\right)^{-1/2}}{x^2+y^2+z^2} = \frac{y^2+z^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\left(x^2+y^2+z^2\right)^{1/2} - y^2\left(x^2+y^2+z^2\right)^{-1/2}}{x^2+y^2+z^2} = \frac{x^2+z^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\left(x^2+y^2+z^2\right)^{1/2} - z^2\left(x^2+y^2+z^2\right)^{-1/2}}{x^2+y^2+z^2} = \frac{x^2+y^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\begin{aligned}\nabla \cdot \boldsymbol{F} &= \frac{y^2+z^2+x^2+z^2+x^2+y^2}{\left(x^2+y^2+z^2\right)^{3/2}} \\ &= \frac{2\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{3/2}} \\ &= \frac{2}{\sqrt{x^2+y^2+z^2}}\ \underline{= \frac{2}{|\boldsymbol{r}|}}\end{aligned}$$

Theorem

For a real number p, the divergence of the radial vector field

$$\boldsymbol{F}=\frac{\boldsymbol{r}}{|\boldsymbol{r}|^p}=\frac{(x,\ y,\ z)}{\left(x^2+y^2+z^2\right)^{p/2}} \quad \rightarrow \quad \nabla\cdot\boldsymbol{F}=\frac{3-p}{|\boldsymbol{r}|^p}$$

Example

To gain some intuition about the divergence, consider the two-dimensional vector field $\boldsymbol{F}=(f,\ g)=\left(x^2,\ y\right)$ and a circle C of radius 2 centered at the origin.

a) Without computing it, determine whether the two-dimensional divergence is positive or negative at the point $Q(1,\ 1)$. Why?

b) Confirm tour conjecture in part (*a*) by computing the two-dimensional divergence at Q.

c) Based on part (*b*), over what regions within the circle is the divergence positive and over what regions within the circle is the divergence is negative?

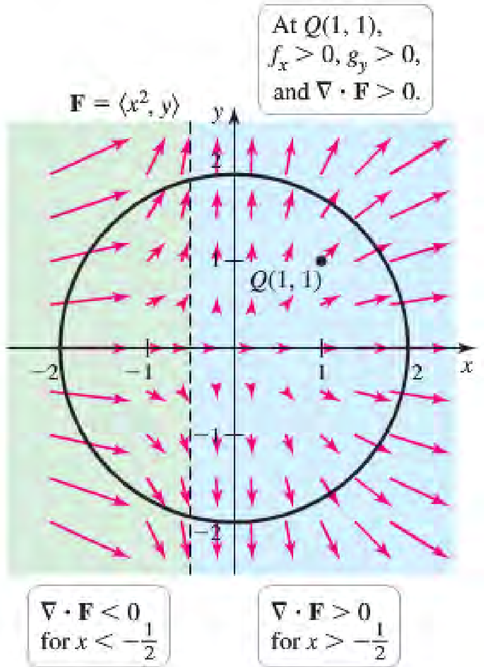
d) By inspection of the figure, on what part of the circle is the flux across the boundary outward? Is the net flux out of the circle positive or negative?

Solution

a) At $Q(1,\ 1)$ the x-component and the y-component of the field are increasing $\left(f_x>0 \text{ and } g_y>0\right)$, so the field is expanding at that point and the two-dimensional divergence is positive.

b) $\nabla\cdot\boldsymbol{F}=\frac{\partial}{\partial x}\left(x^2\right)+\frac{\partial}{\partial y}(y)\ \underline{=2x+1|}$

$\nabla\cdot\boldsymbol{F}\Big|_{Q(1,1)}\ \underline{=3|}$, the divergence is 3.



c) $\nabla\cdot\boldsymbol{F}=2x+1>0 \quad\Rightarrow\quad x>-\frac{1}{2}$

$\nabla\cdot\boldsymbol{F}=2x+1<0 \quad\Rightarrow\quad x<-\frac{1}{2}$

To the left of the line $x=-\frac{1}{2}$ the field is contracting and to the right of the line the field is expanding

d) It appears that the field is tangent to the circle at two points with $x\approx-\frac{1}{2}$.

For points on the circle with $x<-\frac{1}{2}$, the flow is into the circle.

For points on the circle with $x>-\frac{1}{2}$, the flow is out the circle.

It appears that the net outward flux across C is positive. The points where the field changes from inward to outward may be determined exactly.

Curl

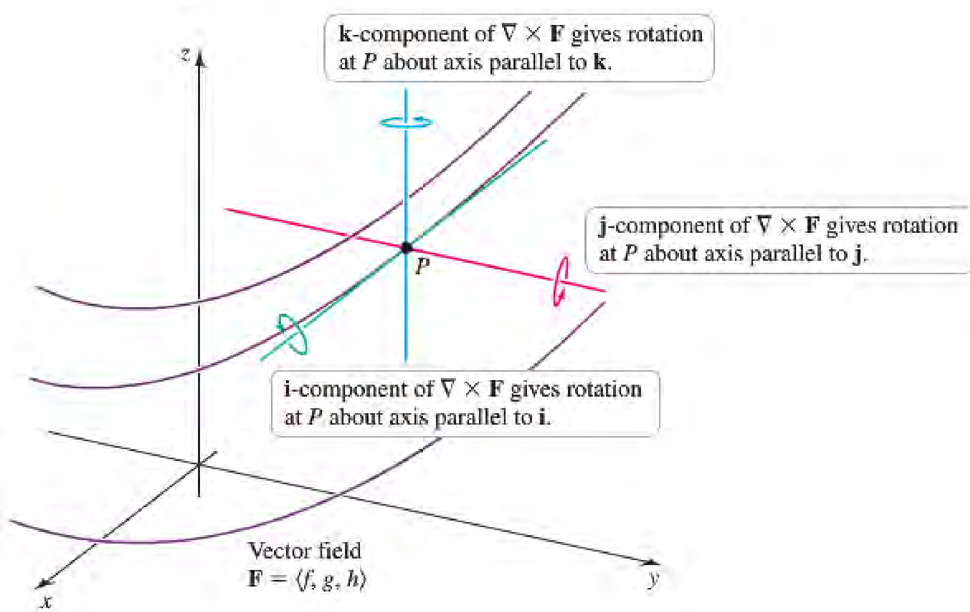
Definition

The curl of a vector field $\boldsymbol{F} = (f,\ g,\ h)$ that is differentiable on a region of $\mathbb{R}^3$ is

$$curl\ \boldsymbol{F} = \nabla\times\boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{\boldsymbol{i}} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\hat{\boldsymbol{j}} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{\boldsymbol{k}}$$

If $\nabla\times\boldsymbol{F} = 0$, the vector field is ***irrotational***.



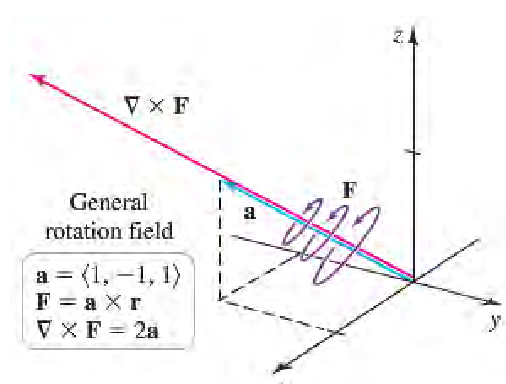
Example

Consider the vector field $\boldsymbol{F} = \boldsymbol{a}\times\boldsymbol{r}$, where $\boldsymbol{a} = \langle a_1,\ a_2,\ a_3\rangle$ is a nonzero vector and $\boldsymbol{r} = \langle x,\ y,\ z\rangle$

$$\boldsymbol{F} = \boldsymbol{a}\times\boldsymbol{r} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= (a_2 z - a_3 y)\hat{\boldsymbol{i}} + (a_3 x - a_1 z)\hat{\boldsymbol{j}} + (a_1 y - a_2 x)\hat{\boldsymbol{k}}$$

This vector field is a general rotation field in 3-dimensions.



Suppose a paddle wheel is placed in the vector field $\boldsymbol{F}$ at a point P with the axis of the wheel in the direction of a unit vector $\boldsymbol{n}$.

$$(\nabla\times\boldsymbol{F})\bullet\vec{\boldsymbol{n}} = (\nabla\times\boldsymbol{F})\cos\theta \qquad (\vec{\boldsymbol{n}}=1)$$

Where θ is the angle between $\nabla\times\boldsymbol{F}$ and $\boldsymbol{n}$.

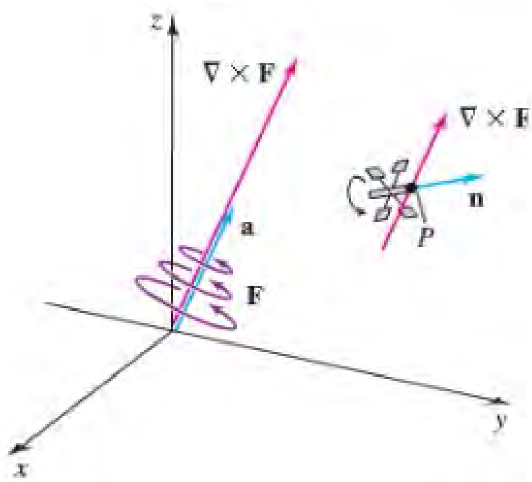
The scalar component is greatest in magnitude and the paddle wheel spins fastest when $\theta = 0$ *or* π; that is when $\nabla\times\boldsymbol{F}$ and $\boldsymbol{n}$ are parallel.

If the axis of the paddle wheel is orthogonal to $\nabla\times\boldsymbol{F}$ $\left(\theta=\pm\frac{\pi}{2}\right)$, the wheel doesn't spin.

General Rotation Vector Field

The general rotation vector field is $\boldsymbol{F} = \boldsymbol{a}\times\boldsymbol{r}$ where the nonzero constant vector $\boldsymbol{a} = (a_1,\ a_2,\ a_3)$ is the axis of rotation and $\boldsymbol{r} = (x,\ y,\ z)$. For all choices of $\boldsymbol{a}$, $\nabla\times\boldsymbol{F} = 2|\boldsymbol{a}|$ and $\nabla\bullet\boldsymbol{F} = 0$. The constant angular speed of the vector field is

$$\omega = |\boldsymbol{a}| = \frac{1}{2}|\nabla\times\boldsymbol{F}|$$



Paddle wheel at P with axis $\boldsymbol{n}$ measures rotation about $\boldsymbol{n}$.
Rotation is a maximum when $\nabla\times\boldsymbol{F}$ is parallel to $\boldsymbol{n}$.

Example

Compute the curl of the rotation field $\boldsymbol{F} = \boldsymbol{a}\times\boldsymbol{r}$ where $\boldsymbol{a} = \langle 1,\ -1,\ 1\rangle$ is the axis of rotation and $\boldsymbol{r} = \langle x,\ y,\ z\rangle$. What is the direction and the magnitude of the curl?

Solution

$$\boldsymbol{F} = \boldsymbol{a}\times\boldsymbol{r} = (-z-y)\hat{\boldsymbol{i}} + (x-z)\hat{\boldsymbol{j}} + (y+x)\hat{\boldsymbol{k}}$$

$$curl\ \boldsymbol{F} = \nabla\times\boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z-y & x-z & y+x \end{vmatrix} = 2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}} = 2\boldsymbol{a}$$

The direction of the curl is the direction of $\boldsymbol{a}$, which is the axis rotation.

The magnitude of $\nabla\times\boldsymbol{F} = |2\boldsymbol{a}| = 2\sqrt{3}$

Working with Divergence and Curl

Theorem

Suppose that $\boldsymbol{F}$ is a conservative vector field on an open region $\boldsymbol{D}$ of $\mathbb{R}^3$ Let $\boldsymbol{F} = \nabla\phi$, where ϕ is a potential function with continuous second partial derivatives on $\boldsymbol{D}$. Then $\nabla\times\boldsymbol{F} = \nabla\times\nabla\phi = 0$; that is, the curl of the gradient is the zero vector and $\boldsymbol{F}$ is irrotational.

$$\begin{aligned}
\nabla\times\nabla\phi &= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi_x & \phi_y & \phi_z \end{vmatrix} \\
&= \left(\phi_{zy} - \phi_{yz}\right)\hat{\boldsymbol{i}} + \left(\phi_{xz} - \phi_{zx}\right)\hat{\boldsymbol{j}} + \left(\phi_{yx} - \phi_{xy}\right)\hat{\boldsymbol{k}} \\
&= 0
\end{aligned}$$

Product Rule for the Divergence

Theorem

Let u be a scalar-valued function that is differentiable on a region D and let $\boldsymbol{F}$ be a vector field that us differentiable on D. Then

$$\nabla\cdot\left(u\boldsymbol{F}\right) = \nabla u\cdot\boldsymbol{F} + u\left(\nabla\cdot\boldsymbol{F}\right)$$

Example

Let $\boldsymbol{r} = \langle x,\ y,\ z \rangle$ and let $\phi = \frac{1}{|\boldsymbol{r}|} = \left(x^2 + y^2 + z^2\right)^{-1/2}$ be a potential function.

a) Find the associated gradient field $\boldsymbol{F} = \nabla\left(\frac{1}{|\boldsymbol{r}|}\right)$

b) Compute $\nabla \cdot \boldsymbol{F}$

Solution

a) $$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right)^{-1/2} = -x\left(x^2 + y^2 + z^2\right)^{-3/2} = -\frac{x}{|\boldsymbol{r}|^3}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + y^2 + z^2\right)^{-1/2} = -y\left(x^2 + y^2 + z^2\right)^{-3/2} = -\frac{y}{|\boldsymbol{r}|^3}$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial}{\partial z}\left(x^2 + y^2 + z^2\right)^{-1/2} = -z\left(x^2 + y^2 + z^2\right)^{-3/2} = -\frac{z}{|\boldsymbol{r}|^3}$$

$$\boldsymbol{F} = \nabla\left(\frac{1}{|\boldsymbol{r}|}\right) = -\frac{x\hat{\boldsymbol{i}} + y\boldsymbol{j} + z\boldsymbol{k}}{|\boldsymbol{r}|^3} = -\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}$$

This result reveals that $\boldsymbol{F}$ is an inverse square vector field and its potential function is $\phi = \frac{1}{|\boldsymbol{r}|}$

b) $$\nabla \cdot \boldsymbol{F} = \nabla \cdot \left(-\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}\right) = -\nabla \cdot \boldsymbol{r}\frac{1}{|\boldsymbol{r}|^3} - \boldsymbol{r} \cdot \nabla\frac{1}{|\boldsymbol{r}|^3}$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right)^{-3/2} = -3x\left(x^2 + y^2 + z^2\right)^{-5/2} = -\frac{3x}{|\boldsymbol{r}|^5}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + y^2 + z^2\right)^{-3/2} = -3y\left(x^2 + y^2 + z^2\right)^{-5/2} = -\frac{3y}{|\boldsymbol{r}|^5}$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial}{\partial z}\left(x^2 + y^2 + z^2\right)^{-3/2} = -3z\left(x^2 + y^2 + z^2\right)^{-5/2} = -\frac{3z}{|\boldsymbol{r}|^5}$$

$$\nabla\frac{1}{|\boldsymbol{r}|^3} = -3\frac{x\hat{\boldsymbol{i}} + y\boldsymbol{j} + z\boldsymbol{k}}{|\boldsymbol{r}|^5} = -3\frac{\boldsymbol{r}}{|\boldsymbol{r}|^5}$$

$$\nabla \cdot \boldsymbol{F} = \nabla \cdot \left(-\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}\right) = -\frac{1}{|\boldsymbol{r}|^3}\nabla \cdot \boldsymbol{r} + 3\boldsymbol{r} \cdot \frac{\boldsymbol{r}}{|\boldsymbol{r}|^5}$$

$$= -\frac{3}{|\boldsymbol{r}|^3} + \frac{3}{|\boldsymbol{r}|^3}$$

$$\underline{= 0}$$

Properties of a Conservative a Vector Field

Let ***F*** be a conservative vector field whose components have continuous second partial derivatives on an open connected region *D* in $\mathbb{R}^3$.

1. There exists a potential function ϕ such that $\boldsymbol{F} = \nabla\phi$
2. $\int_C F \cdot dr = \phi(B) - \phi(A)$ for all points *A* and *B* in ***D*** and all piecewise-smooth oriented curves *C* from *A* to *B*.
3. $\int_C F \cdot dr = 0$ on all simple piecewise-smooth closed oriented curves *C* in *D*.
4. $\nabla \times \boldsymbol{F} = 0$ at all points of *D*.

Exercises *Section* 4.5 – Divergence and Curl

Find the divergence of the following vector fields

1. $\boldsymbol{F} = \langle 2x,\ 4y,\ -3z \rangle$

2. $\boldsymbol{F} = \langle -2y,\ 3x,\ z \rangle$

3. $\boldsymbol{F} = \left\langle x^2yz,\ -xy^2z,\ -xyz^2 \right\rangle$

4. $\boldsymbol{F} = \left\langle x^2 - y^2,\ y^2 - z^2,\ z^2 - x^2 \right\rangle$

5. $\boldsymbol{F} = \left\langle e^{-x+y},\ e^{-y+z},\ e^{-z+x} \right\rangle$

6. $\boldsymbol{F} = \langle yz\cos x,\ xz\cos y,\ xy\cos z \rangle$

Calculate the divergence of the following radial fields. Express the result in terms of the position vector $\boldsymbol{r}$ and its length $|\boldsymbol{r}|$.

7. $\boldsymbol{F} = \dfrac{\langle x,\ y,\ z \rangle}{x^2+y^2+z^2} = \dfrac{\boldsymbol{r}}{|\boldsymbol{r}|^2}$

8. $\boldsymbol{F} = \langle x,\ y,\ z \rangle \left(x^2+y^2+z^2\right) = \boldsymbol{r}|\boldsymbol{r}|^2$

9. $\boldsymbol{F} = \dfrac{\langle x,\ y,\ z \rangle}{\left(x^2+y^2+z^2\right)^{3/2}} = \dfrac{\boldsymbol{r}}{|\boldsymbol{r}|^3}$

Consider the following vector fields, the circle C, and two points P and Q.

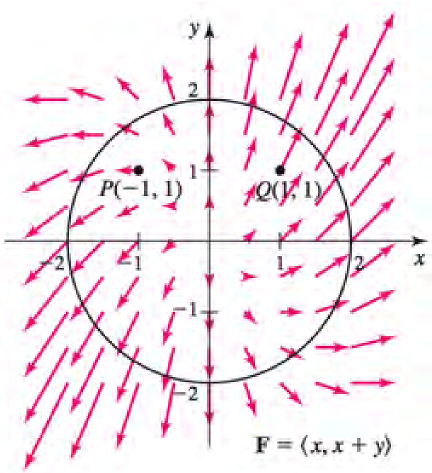
a) Without computing the divergence, does the graph suggest that the divergence is positive or negative at P and Q?

b) Compute the divergence and confirm your conjecture in part (*a*).

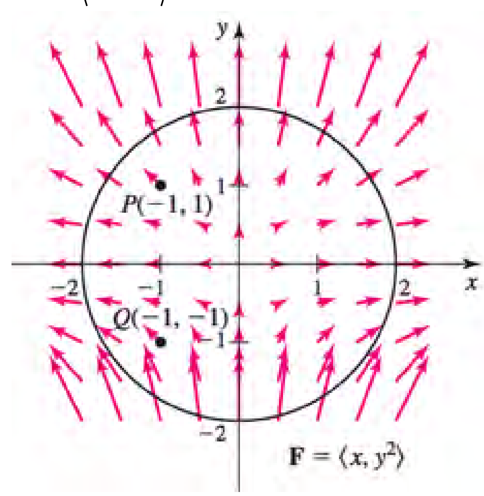
c) On what part of C is the flux outward? Inward?

d) Is the net outward flux across C positive or negative?

10. $\boldsymbol{F} = \langle x,\ x+y \rangle$



11. $\boldsymbol{F} = \left\langle x,\ y^2 \right\rangle$



Consider the following vector fields, where $\boldsymbol{r} = \langle x,\ y,\ z \rangle$

a) Compute the curl field and verify that it has the same direction as the axis of rotation

b) Compute the magnitude of the curl of the field

12. $\boldsymbol{F} = \langle 1,\ 0,\ 0 \rangle \times \boldsymbol{r}$

13. $\boldsymbol{F} = \langle 1,\ -1,\ 0 \rangle \times \boldsymbol{r}$

14. $\boldsymbol{F} = \langle 1,\ -1,\ 1 \rangle \times \boldsymbol{r}$

15. $\boldsymbol{F} = \langle 1,\ -2,\ -3 \rangle \times \boldsymbol{r}$

Compute the curl of the following vector fields

16. $\boldsymbol{F} = \left\langle x^2 - y^2,\ xy,\ z \right\rangle$

17. $\boldsymbol{F} = \left\langle 0,\ z^2 - y^2,\ -yz \right\rangle$

18. $\boldsymbol{F} = \left\langle z^2 \sin y,\ xz^2 \cos y,\ 2xz \sin y \right\rangle$

19. $\boldsymbol{F} = \boldsymbol{r} = \langle x,\ y,\ z \rangle$

20. $\boldsymbol{F} = \dfrac{\langle x,\ y,\ z \rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}} = \dfrac{\boldsymbol{r}}{|\boldsymbol{r}|^3}$

21. $\boldsymbol{F} = \left\langle 3xz^3 e^{y^2},\ 2xz^3 e^{y^2},\ 3xz^2 e^{y^2} \right\rangle$

22. Show that the general rotation field $\boldsymbol{F} = \boldsymbol{a} \times \boldsymbol{r}$, where $\boldsymbol{a}$ is a nonzero constant vector and $\boldsymbol{r} = \langle x,\ y,\ z \rangle$, has zero divergence.

23. Let $\boldsymbol{a} = \langle 0,\ 1,\ 0 \rangle$, $\boldsymbol{r} = \langle x,\ y,\ z \rangle$ and consider the rotation field $\boldsymbol{F} = \boldsymbol{a} \times \boldsymbol{r}$. Use the right-hand rule for cross product to find the direction of $\boldsymbol{F}$ at the points $(0,\ 1,\ 1)$, $(1,\ 1,\ 0)$, $(0,\ 1,\ -1)$, and $(-1,\ 1,\ 0)$

24. Find the exact points on the circle $x^2 + y^2 = 2$ at which the field $\boldsymbol{F} = \langle f,\ g \rangle = \left\langle x^2,\ y \right\rangle$ switches from pointing inward to outward on the circle, or vice versa.

25. Suppose a solid object in $\mathbb{R}^3$ has a temperature distribution given by $T(x,\ y,\ z)$. The heat flow vector field in the object is $\boldsymbol{F} = -k\nabla T$, where the conductivity $k > 0$ is a property of the material. Note that the heat flow vector points in the direction opposite to that of the gradient, which is the direction of greatest temperature decrease. The divergence of the heat flow vector is $\boldsymbol{F} = -k\nabla \cdot \nabla T = -k\nabla^2 T$ (the Laplacian of *T*). Compute the heat flow vector field and its divergence for the following temperature distribution.

a) $T(x,\ y,\ z) = 100\, e^{-\sqrt{x^2+y^2+z^2}}$

b) $T(x,\ y,\ z) = 100\, e^{-x^2+y^2+z^2}$

c) $T(x,\ y,\ z) = 100\left(1 + \sqrt{x^2 + y^2 + z^2}\right)$

Section 4.6 – Surfaces Integrals

We have defined curves in the plane in three different ways:

Explicit form: $y = f(x)$

Implicit form: $F(x, y) = 0$

Parametric vector form: $\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} \quad a \le t \le b$

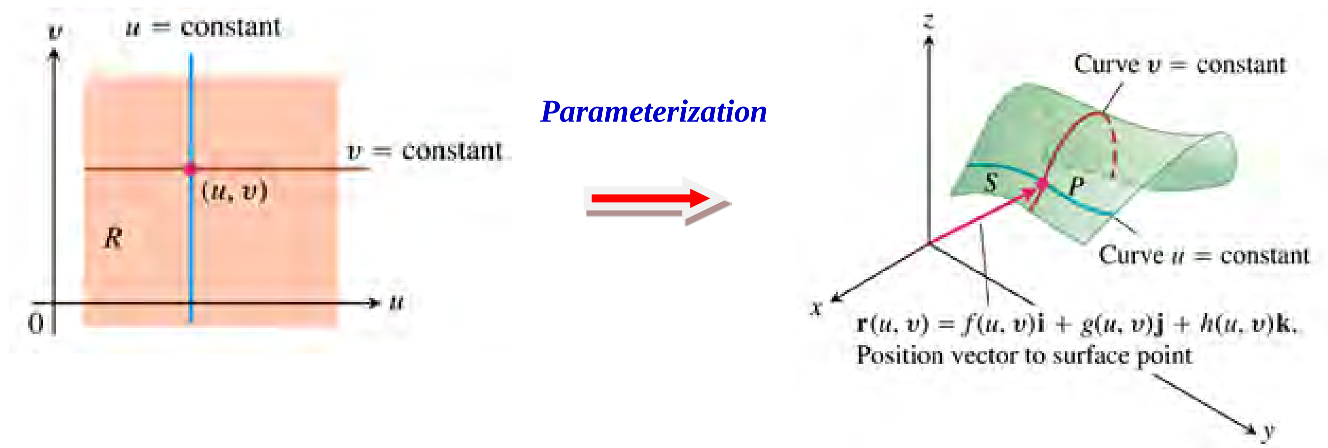
And

Explicit form: $z = f(x, y)$

Implicit form: $F(x, y, z) = 0$

Parameterizations of Surfaces

Suppose:



We call the range of $\boldsymbol{r}$ the ***surface*** S defined or traced by $\boldsymbol{r}$.

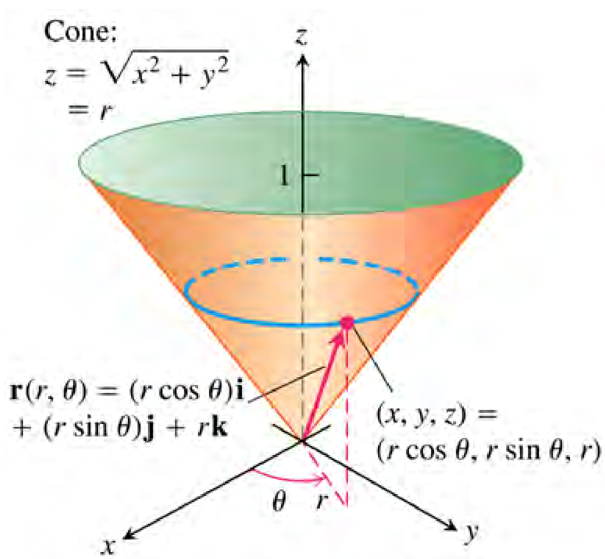
u and v: variable parameters

R: parameter domain

Example

Find a parameterization of the cone

$$z = \sqrt{x^2 + y^2}, \quad 0 \le z \le 1$$



Solution

$x = \cos\theta, \quad y = \sin\theta$

$z = \sqrt{x^2 + y^2} = r$

Assume $u = r \quad and \quad v = \theta$

$\boldsymbol{r}(r, \theta) = (r\cos\theta)\boldsymbol{i} + (r\sin\theta)\boldsymbol{j} + r\boldsymbol{k}$

$0 \le r \le 1, \quad 0 \le \theta \le 2\pi$

Example

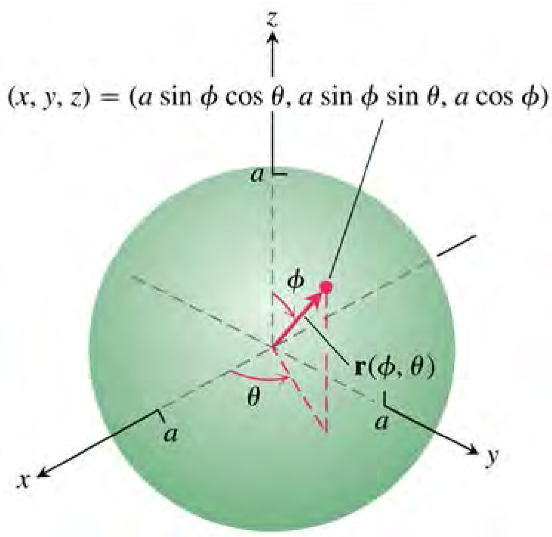
Find a parameterization of the cone $x^2 + y^2 + z^2 = a^2$

Solution

A typical point $(x,\ y,\ z)$ on the sphere has

$$x = a\sin\phi\cos\theta, \quad y = a\sin\phi\sin\theta, \quad z = a\cos\phi$$

$$0 \le \phi \le 2\pi, \quad 0 \le \theta \le 2\pi$$



Taking $u = \phi$ *and* $v = \theta$

$$\boldsymbol{r}(\phi,\ \theta) = (a\sin\phi\cos\theta)\boldsymbol{i} + (a\sin\phi\sin\theta)\boldsymbol{j} + (a\cos\phi)\boldsymbol{k}$$

The parameterization is one-to-one on the interior of the domain *R*, though not on its boundary "*poles*" where $\phi = 0$ *or* $\theta = \pi$

Example

Find a parameterization of the cone $x^2 + (y-3)^2 = 9, \quad 0 \le z \le 5$

Solution

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

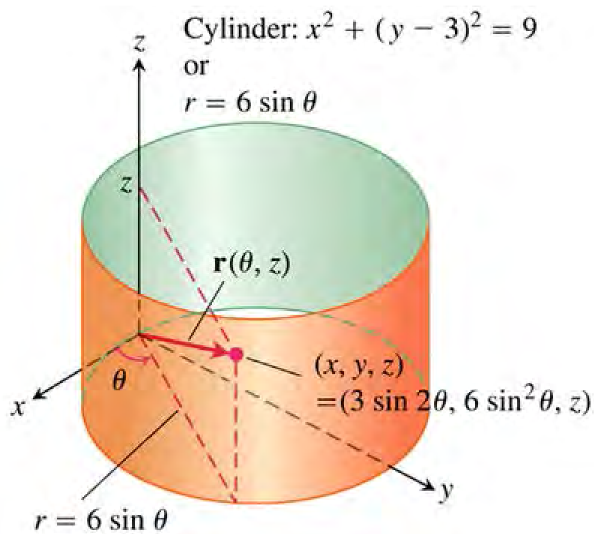
$$x^2 + y^2 - 6y + 9 = 9$$

$$x^2 + y^2 - 6y = 0$$

$$r^2 - 6r\sin\theta = 0$$

$$r(r - 6\sin\theta) = 0$$

$$r = 6\sin\theta, \quad 0 \le \theta \le \pi$$



A typical point on the cylinder has

$$\begin{cases} x = r\cos\theta = 6\sin\theta\cos\theta = 3\sin 2\theta \\ y = r\sin\theta = 6\sin^2\theta \\ z = z \end{cases}$$

Taking $u = \theta$ *and* $v = z$

$$\boldsymbol{r}(\theta,\ z) = (3\sin 2\theta)\boldsymbol{i} + \left(6\sin^2\theta\right)\boldsymbol{j} + z\boldsymbol{k} \quad 0 \le \theta \le \pi, \quad 0 \le z \le 5$$

Surface Area

Calculating the area of a curved surface S based on the parameterization

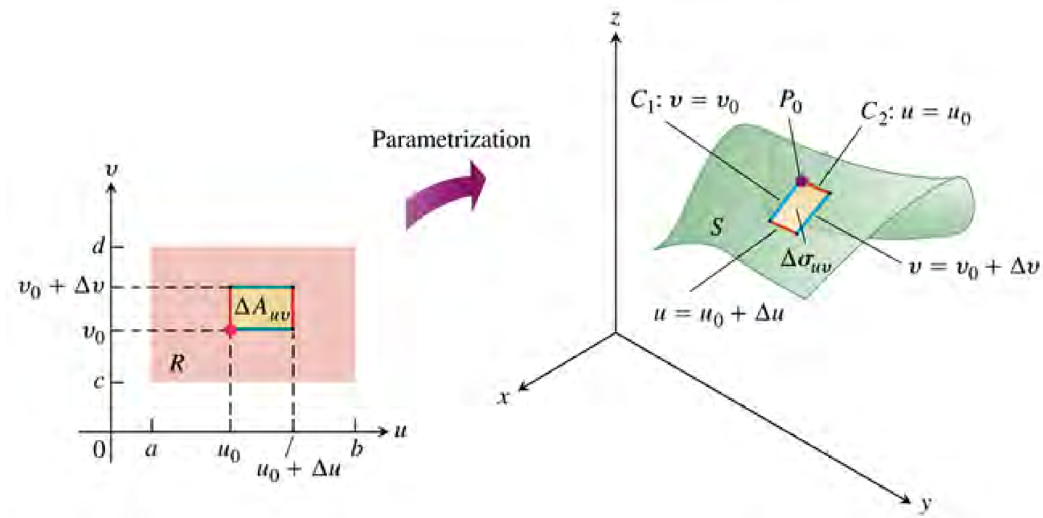
$$\boldsymbol{r}(u,\ v) = f(u,\ v)\boldsymbol{i} + g(u,\ v)\boldsymbol{j} + h(u,\ v)\boldsymbol{k}, \quad a \le u \le b, \quad c \le v \le d$$

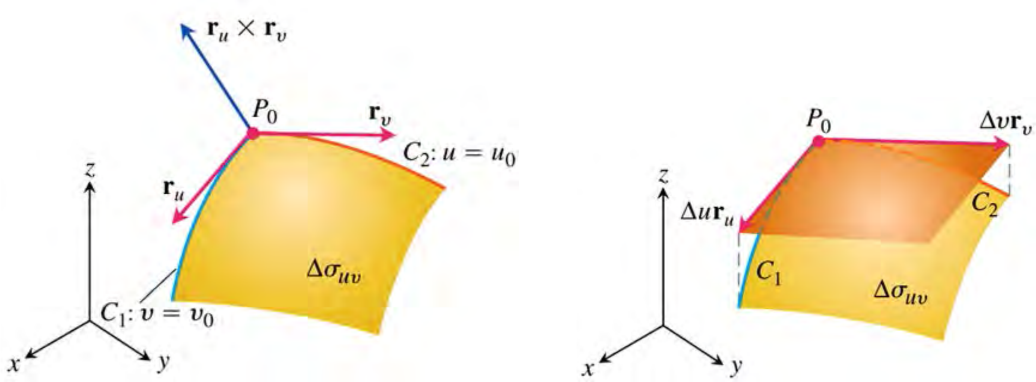
The definition of smoothness involves the partial derivatives of r with respect to u and v:

$$\boldsymbol{r}_u = \frac{\partial \boldsymbol{r}}{\partial u} = \frac{\partial f}{\partial u}\boldsymbol{i} + \frac{\partial g}{\partial u}\boldsymbol{j} + \frac{\partial h}{\partial u}\boldsymbol{k} \qquad \boldsymbol{r}_v = \frac{\partial \boldsymbol{r}}{\partial v} = \frac{\partial f}{\partial v}\boldsymbol{i} + \frac{\partial g}{\partial v}\boldsymbol{j} + \frac{\partial h}{\partial v}\boldsymbol{k}$$

Definition

A ***parameterized*** surface $\boldsymbol{r}(u,v) = f(u,v)\boldsymbol{i} + g(u,v)\boldsymbol{j} + h(u,v)\boldsymbol{k}$ is smooth if $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are continuous and $\boldsymbol{r}_u \times \boldsymbol{r}_v$ is never zero on the interior of the parameter domain.





Definition

The area of the smooth surface

$$\boldsymbol{r}(u,v) = f(u,v)\boldsymbol{i} + g(u,v)\boldsymbol{j} + h(u,v)\boldsymbol{k}, \quad a \le u \le b, \quad c \le v \le d$$

is $$\boldsymbol{A} = \iint_R \left|\boldsymbol{r}_u \times \boldsymbol{r}_v\right| dA = \int_c^d \int_a^b \left|\boldsymbol{r}_u \times \boldsymbol{r}_v\right| du\,dv$$

Surface area Differential for a Parameterized Surface

$$d\sigma = \left|\boldsymbol{r}_u \times \boldsymbol{r}_v\right| dudv \qquad \iint_S d\sigma$$

Surface area differential *Differntial formula for surface area*

Example

Find the surface area of the cone $z = \sqrt{x^2 + y^2}, \quad 0 \le z \le 1$

Solution

$x = r\cos\theta, \quad y = r\sin\theta$, and $z = \sqrt{x^2 + y^2} = r$

$\boldsymbol{r}(r,\ \theta) = (r\cos\theta)\boldsymbol{i} + (r\sin\theta)\boldsymbol{j} + r\boldsymbol{k}$

$0 \le r \le 1, \quad 0 \le \theta \le 2\pi$

$$\boldsymbol{r}_r \times \boldsymbol{r}_\theta = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= -(r\cos\theta)\boldsymbol{i} - (r\sin\theta)\boldsymbol{j} + \left(r\cos^2\theta + r\sin^2\theta\right)\boldsymbol{k} \qquad r\cos^2\theta + r\sin^2\theta = r\left(\cos^2\theta + \sin^2\theta\right) = r$$

$$= -(r\cos\theta)\boldsymbol{i} - (r\sin\theta)\boldsymbol{j} + r\boldsymbol{k}$$

$$\left|\boldsymbol{r}_r \times \boldsymbol{r}_\theta\right| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{r^2 + r^2} = r\sqrt{2}$$

$$A = \int_0^{2\pi}\int_0^1 \left|\boldsymbol{r}_r \times \boldsymbol{r}_\theta\right| dr d\theta$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{2}\ r\ dr d\theta$$

$$= \frac{\sqrt{2}}{2}\int_0^{2\pi} \left[r^2\right]_0^1 d\theta$$

$$= \frac{\sqrt{2}}{2}\int_0^{2\pi} d\theta$$

$$= \frac{\sqrt{2}}{2}[\theta]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{2}(2\pi)$$

$$= \pi\sqrt{2}\ \ units^2$$

Example

Find the surface area of a sphere of radius *a*.

Solution

$$\mathbf{r}(\phi,\ \theta)=(a\sin\phi\cos\theta)\,\boldsymbol{i}+(a\sin\phi\sin\theta)\,\boldsymbol{j}+(a\cos\phi)\,\boldsymbol{k} \qquad 0\le\phi\le\pi,\quad 0\le\theta\le 2\pi$$

$$\mathbf{r}_\phi\times\mathbf{r}_\theta=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi\\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0\end{vmatrix}$$

$$=\left(a^2\sin^2\phi\cos\theta\right)\boldsymbol{i}+\left(a^2\sin^2\phi\sin^2\theta\right)\boldsymbol{j}+\left(a^2\cos\phi\sin\phi\cos^2\theta+a^2\cos\phi\sin\phi\sin^2\theta\right)\boldsymbol{k}$$

$$=\left(a^2\sin^2\phi\cos\theta\right)\boldsymbol{i}+\left(a^2\sin^2\phi\sin\theta\right)\boldsymbol{j}+\left(a^2\cos\phi\sin\phi\right)\boldsymbol{k}$$

$$\left|\mathbf{r}_\phi\times\mathbf{r}_\theta\right|=\sqrt{a^4\sin^4\phi\cos^2\theta+a^4\sin^4\phi\sin^2\theta+a^4\cos^2\phi\sin^2\phi}$$

$$=\sqrt{a^4\sin^4\phi\left(\cos^2\theta+\sin^2\theta\right)+a^4\cos^2\phi\sin^2\phi}$$

$$=\sqrt{a^4\sin^4\phi+a^4\cos^2\phi\sin^2\phi}$$

$$=\sqrt{a^4\sin^2\phi\left(\sin^2\phi+\cos^2\phi\right)}$$

$$=a^2\sin\phi$$

$$A=\int_0^{2\pi}\int_0^{\pi}a^2\sin\phi\ d\phi d\theta$$

$$=a^2\int_0^{2\pi}\left[-\cos\phi\right]_0^{\pi}\ d\theta$$

$$=a^2\int_0^{2\pi}\left[-(-1-1)\right]d\theta$$

$$=2a^2\theta\Big|_0^{2\pi}$$

$$\boxed{=4\pi a^2\ \ unit^2}$$

Example

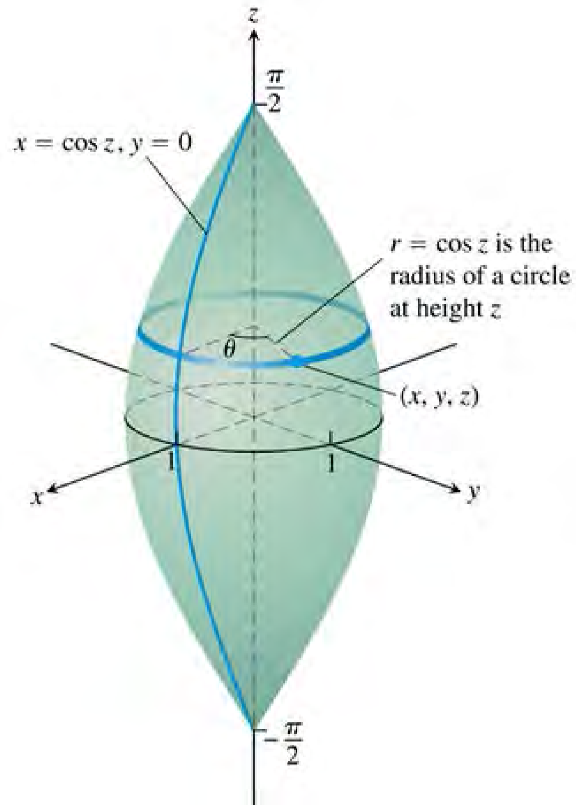
Let S be the "football" surface formed by rotating the curve $x = \cos z, \quad y = 0, \quad -\frac{\pi}{2} \le z \le \frac{\pi}{2}$ around the z-axis. Find the parameterization for S and compute its surface area.

Solution

Let (x, y, z) be an arbitrary point on the circle.

The parameters: $u = z$ and $v = \theta$.

We have:



$$\begin{cases} x = r\cos\theta = \cos u \cos v \\ y = r\sin\theta = \cos u \sin v \\ z = u \end{cases}$$

$$\boldsymbol{r}(u, v) = (\cos u \cos v)\boldsymbol{i} + (\cos u \sin v)\boldsymbol{j} + u\boldsymbol{k}$$

$$-\frac{\pi}{2} \le u \le \frac{\pi}{2}, \quad 0 \le v \le 2\pi$$

$$\boldsymbol{r}_u = (-\sin u \cos v)\boldsymbol{i} - (\sin u \sin v)\boldsymbol{j} + \boldsymbol{k}$$

$$\boldsymbol{r}_v = (-\cos u \sin v)\boldsymbol{i} + (\cos u \cos v)\boldsymbol{j}$$

$$\boldsymbol{r}_u \times \boldsymbol{r}_v = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -\sin u \cos v & -\sin u \sin v & 1 \\ -\cos u \sin v & \cos u \cos v & 0 \end{vmatrix}$$

$$= (-\cos u \cos v)\boldsymbol{i} - (\cos u \sin v)\boldsymbol{j} - \left(\sin u \cos u \cos^2 v + \cos u \sin u \sin^2 v\right)\boldsymbol{k}$$

$$\left|\boldsymbol{r}_u \times \boldsymbol{r}_v\right| = \sqrt{\cos^2 u \cos^2 v + \cos^2 u \sin^2 v + \left(\sin u \cos u\left(\cos^2 v + \sin^2 v\right)\right)^2}$$

$$= \sqrt{\cos^2 u\left(\cos^2 v + \sin^2 v\right) + \sin^2 u \cos^2 u}$$

$$= \sqrt{\cos^2 u\left(1 + \sin^2 u\right)}$$

$$= \cos u\sqrt{1 + \sin^2 u}$$

$$A = \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \cos u\sqrt{1 + \sin^2 u}\ dudv$$

$$w = \sin u \ \Rightarrow\ dw = \cos u du \to \begin{cases} u = -\frac{\pi}{2} & \to w = -1 \\ u = \frac{\pi}{2} & \to w = 1 \end{cases}$$

$$= \int_0^{2\pi}\int_{-1}^{1} \sqrt{1 + w^2}\ dwdv$$

$$= \int_0^{2\pi} \left[\frac{w}{2}\sqrt{1+w^2} + \frac{1}{2}\ln\left(w + \sqrt{1+w^2} \right) \right]_{-1}^{1} dv$$

$$= \int_0^{2\pi} \left[\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln\left(1+\sqrt{2}\right) + \frac{1}{2}\sqrt{2} - \frac{1}{2}\ln\left(-1+\sqrt{2}\right) \right] dv$$

$$\ln\left(-1+\sqrt{2} \cdot \frac{1+\sqrt{2}}{1+\sqrt{2}} \right) = \ln\left(\frac{1}{1+\sqrt{2}} \right) = -\ln\left(1+\sqrt{2}\right)$$

$$= \int_0^{2\pi} \left[\sqrt{2} + \frac{1}{2}\ln\left(1+\sqrt{2}\right) + \frac{1}{2}\ln\left(1+\sqrt{2}\right) \right] dv$$

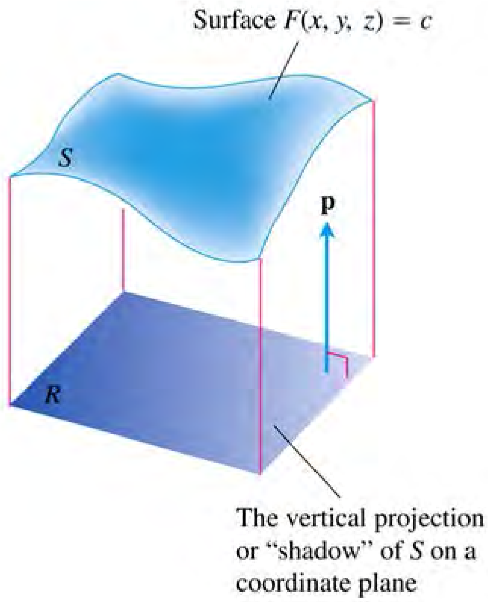
$$= \int_0^{2\pi} \left[\sqrt{2} + \ln\left(1+\sqrt{2}\right) \right] dv$$

$$= \left(\sqrt{2} + \ln\left(1+\sqrt{2}\right) \right) \left[v \right]_0^{2\pi}$$

$$= 2\pi \left[\sqrt{2} + \ln\left(1+\sqrt{2}\right) \right] \quad unit^2$$

Implicit Surfaces

Surfaces are often presented as level sets of a function $F(x, y, z) = c$ for some constant c. Such a level surface does not come with an explicit parameterization, and is called am *implicit defined surface*.



The surface is defined by the equation $F(x, y, z) = c$ and **p** is a unit vector normal to the plane region *R*.

$$\nabla F \cdot \mathbf{p} = \nabla F \cdot \boldsymbol{k} = F_z \neq 0$$

Define the parameters *u* and *v* by *u* = *x* and *v* = *y*. Then *z* = *h*(*u*, *v*) and

$$\boldsymbol{r}(u,v)=u\boldsymbol{i}+v\boldsymbol{j}+h(u,v)\boldsymbol{k}$$

Calculating the partial derivatives of $\boldsymbol{r}$,

$$\boldsymbol{r}_u=\boldsymbol{i}+\frac{\partial h}{\partial u}\boldsymbol{k} \quad and \quad \boldsymbol{r}_v=\boldsymbol{j}+\frac{\partial h}{\partial v}\boldsymbol{k}$$

$$\frac{\partial h}{\partial u}=-\frac{F_x}{F_z} \quad and \quad \frac{\partial h}{\partial v}=-\frac{F_y}{F_z}$$

$$\boldsymbol{r}_u=\boldsymbol{i}-\frac{F_x}{F_z}\boldsymbol{k} \quad and \quad \boldsymbol{r}_v=\boldsymbol{j}-\frac{F_y}{F_z}\boldsymbol{k}$$

$$\begin{aligned}\boldsymbol{r}_u\times\boldsymbol{r}_v &=\frac{F_x}{F_z}\boldsymbol{i}+\frac{F_y}{F_z}\boldsymbol{j}+\boldsymbol{k}\\ &=\frac{1}{F_z}\left(F_x\boldsymbol{i}+F_y\boldsymbol{j}+F_z\boldsymbol{k}\right)\\ &=\frac{\nabla F}{F_z}\\ &=\frac{\nabla F}{\nabla F\cdot\boldsymbol{k}}\\ &=\frac{\nabla F}{\nabla F\cdot\boldsymbol{p}}\end{aligned}$$

Therefore, the surface area differential is given by

$$d\sigma=\left|\boldsymbol{r}_u\times\boldsymbol{r}_v\right|dudv=\frac{|\nabla F|}{|\nabla F\bullet p|}dxdy \qquad u=x \;\; and \;\; v=y$$

Formula for the Surface Area of an Implicit Surface

The area of the surface $F(x,y,z)=c$ over a closed and bounded plane region R is

$$Surface\ area=\iint_R\frac{|\nabla F|}{|\nabla F\cdot\boldsymbol{p}|}dA$$

Where $\boldsymbol{p}=\boldsymbol{i},\boldsymbol{j},\ or\ \boldsymbol{k}$ is normal to R and $\nabla F\cdot\boldsymbol{p}\neq 0$

Example

Find the area of the surface cut from the bottom of the paraboloid $x^2+y^2-z=0$ by the plane $z=4$.

Solution

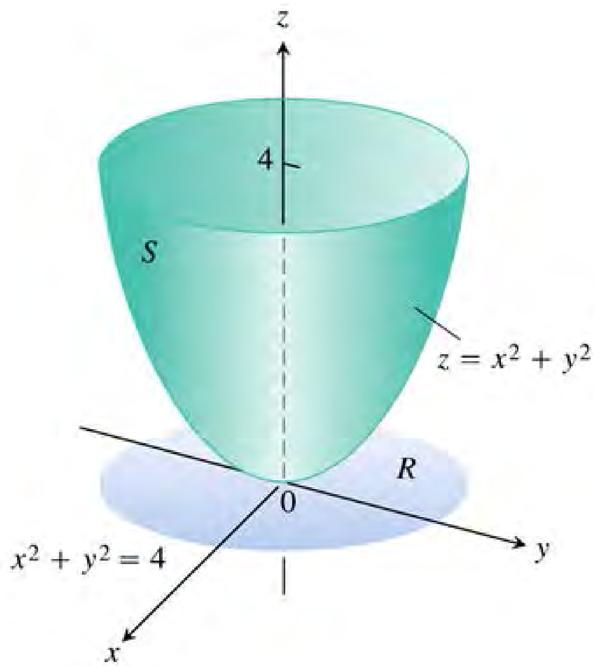
Let $F(x,y,z)=x^2+y^2-z=0$ and R the disk $x^2+y^2\le 4$

$\nabla F = 2x\boldsymbol{i}+2y\boldsymbol{j}-\boldsymbol{k}$

$|\nabla F|=\sqrt{(2x)^2+(2y)^2+(-1)^2} = \sqrt{4x^2+4y^2+1}$

$|\nabla F\cdot \boldsymbol{p}|=|\nabla F\cdot \boldsymbol{k}|=|-1|=1$

In the region R, $dA=dxdy$. Therefore,



$$\begin{aligned}
Surface\ area &= \iint\limits_R \frac{|\nabla F|}{|\nabla F\cdot p|}dA \\
&= \iint\limits_{x^2+y^2\le 4} \sqrt{x^2+y^2+1}\ dxdy \\
&= \int_0^{2\pi}\int_0^2 \sqrt{r^2+1}\ rdrd\theta \\
&= \int_0^{2\pi} d\theta \int_0^2 \sqrt{4r^2+1}\ \tfrac{1}{8}d\left(4r^2+1\right) \\
&= (2\pi)\frac{1}{12}\left[\left(4r^2+1\right)^{3/2}\right]_0^2 \\
&= \frac{\pi}{6}\left(17^{3/2}-1\right) \\
&= \frac{\pi}{6}\left(17\sqrt{17}-1\right)
\end{aligned}$$

Formula for the Surface Area of a Graph $z=f(x,y)$

For a graph $z=f(x,y)$ over the region R in the xy-plane, the surface area formula is

$$A=\iint\limits_R \sqrt{f_x^2+f_y^2+1}\ dxdy$$

Surface	***Equation***	***Explicit Description***	
		Normal Vector $\pm\left\langle -z_x,\ -z_y,\ 0\right\rangle$	***Magnitude*** $\left\lvert\left\langle -z_x,\ -z_y,\ 0\right\rangle\right\rvert$
Cylinder	$x^2+y^2=a^2$ $0\le z\le h$	$\langle x,\ y,\ 0\rangle$	a
Cone	$z^2=x^2+y^2$ $0\le z\le h$	$\left\langle \frac{x}{z},\ \frac{y}{z},\ -1\right\rangle$	$\sqrt{2}$
Sphere	$x^2+y^2+z^2=a^2$	$\left\langle \frac{x}{z},\ \frac{y}{z},\ 1\right\rangle$	$\frac{a}{z}$
Paraboloid	$z=x^2+y^2$ $0\le z\le h$	$\langle 2x,\ 2y,\ -1\rangle$	$\sqrt{1+4\left(x^2+y^2\right)}$

Surface	***Equation***	***Parametric Description***	
		Normal Vector $\boldsymbol{t}_u\times\boldsymbol{t}_v$	***Magnitude*** $\lvert\boldsymbol{t}_u\times\boldsymbol{t}_v\rvert$
Cylinder	$\boldsymbol{r}=\langle a\cos u,\ a\sin u,\ v\rangle$ $0\le u\le 2\pi,\ \ 0\le v\le h$	$\langle a\cos u,\ a\sin u,\ 0\rangle$	a
Cone	$\boldsymbol{r}=\langle v\cos u,\ v\sin u,\ v\rangle$ $0\le u\le 2\pi,\ \ 0\le v\le h$	$\langle v\cos u,\ v\sin u,\ -v\rangle$	$\sqrt{2}\ v$
Sphere	$\boldsymbol{r}=\langle a\sin u\cos v,\ a\sin u\sin v,\ a\cos u\rangle$ $0\le u\le \pi,\ \ 0\le v\le 2\pi$	$\langle a^2\sin^2 u\cos v,\ a^2\sin^2 u\sin v,$ $a^2\sin^2 u\cos u\rangle$	$a^2\sin u$
Paraboloid	$\boldsymbol{r}=\left\langle v\cos u,\ v\sin u,\ v^2\right\rangle$ $0\le u\le 2\pi,\ \ 0\le v\le \sqrt{h}$	$\left\langle 2v^2\cos u,\ 2v^2\sin u,\ -v\right\rangle$	$v\sqrt{1+4v^2}$

Exercises *Section* 4.6 – Surfaces Integrals

1. Find a parametrization of the surface: The paraboloid $z = x^2 + y^2,\ z \le 4$

2. Find a parametrization of the surface: The portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 4$

3. Find a parametrization of the surface cut from the sphere $x^2 + y^2 + z^2 = 8$ by the plane $z = -2$

4. Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral of the portion of the plane $y + 2z = 2$ inside the cylinder $x^2 + y^2 = 1$

5. Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral of the portion of the cone $z = \frac{\sqrt{x^2 + y^2}}{3}$ between the planes $z = 1$ and $z = \frac{4}{3}$

6. Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral of the portion of the cylinder $x^2 + z^2 = 10$ between the planes $y = -1$ and $y = 1$

7. Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral of the portion of the cap cut from the paraboloid $z = x^2 + y^2$ between the planes $z = 1$ and $z = 4$

8. Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 2$.

9. Find the area of the portion of the surface $x^2 - 2z = 0$ that lies above the triangle bounded by the lines $x = \sqrt{3},\ y = 0$, and $y = x$ in the *xy*-plane.

10. Find the area of the cap cut from the sphere $x^2 + y^2 + z^2 = 2$ by the cone $z = \sqrt{x^2 + y^2}$.

11. Find the area of the ellipse cut from the plane $z = cx$ (c a constant) by the cylinder $x^2 + y^2 = 1$.

12. Find the area of the surface cut from the nose of the paraboloid $x = 1 - y^2 - z^2$ by *yz*-plane.

13. Find the area of the surface in the first octant cut from the cylinder $y = \frac{2}{3}z^{3/2}$ by the planes $x = 1$ and $y = \frac{16}{3}$

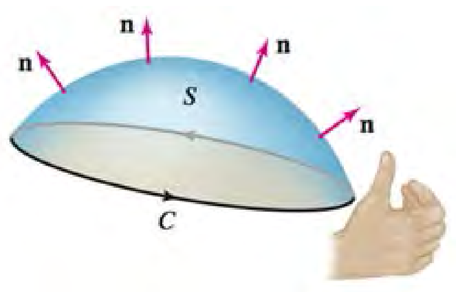
Section 4.7 – Stokes' Theorem

Stokes' Theorem

Stokes' Theorem is the three-dimensional version of the circulation from of Green's Theorem.

If C is a closed simple piecewise-smooth oriented curve in the xy-plane enclosing a region R and $\boldsymbol{F} = \langle f, g \rangle$ is a differentiable vector field on R. Green's Theorem says that

$$\underbrace{\oint \boldsymbol{F} \cdot d\boldsymbol{r}}_{\text{circulation}} = \underbrace{\iint_R \left(g_x - f_y \right) dA}_{\text{curl or rotation}}$$



If the fingers of your right hand curl in the positive direction around C, then your right thumb points in the direction of the vectors normal to S.

Theorem

Let S be an oriented surface in $\mathbb{R}^3$ with a piecewise-smooth closed boundary C whose orientation is consistent with that of S. Assume that $\boldsymbol{F} = \langle f, g, h \rangle$ is a vector field whose components have continuous first partial derivatives on S. Then

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, dS$$

Where $\boldsymbol{n}$ is the unit vector normal to S determined by the orientation of S.

Example

Confirm that Stokes' Theorem holds for the vector field $\boldsymbol{F} = \langle z - y,\ x,\ -x \rangle$ where S is the hemisphere $x^2 + y^2 + z^2 = 4$, for $z \ge 0$, and C is the circle $x^2 + y^2 = 4$ oriented counterclockwise.

Solution

The orientation of C says that the vectors normal to S point in the outward direction. The vector field is a rotation field $\boldsymbol{a} \times \boldsymbol{r}$, where $\boldsymbol{a} = \langle 0, 1, 1 \rangle$ and $\boldsymbol{r} = \langle x, y, z \rangle$, so the axis of rotation points in the direction of the vector $\langle 0, 1, 1 \rangle$.

Compute first the circulation integral in Stokes' Theorem. The curve C with the given orientation is parametrized as $\boldsymbol{r}(t) = \langle 2\cos t,\ 2\sin t,\ 0 \rangle$, for $0 \le t \le 2\pi$

$$\Rightarrow \ \boldsymbol{r}'(t) = \langle -2\sin t,\ 2\cos t,\ 0 \rangle$$

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^{2\pi} \boldsymbol{F} \cdot \boldsymbol{r}'(t)\, dt$$

$$= \int_0^{2\pi} \langle z - y,\ x,\ -x \rangle \cdot \langle -2\sin t,\ 2\cos t,\ 0 \rangle\ dt$$

$$= \int_0^{2\pi} 4\left(\sin^2 t + \cos^2 t\right)\ dt \qquad \sin^2 t + \cos^2 t = 1$$

$$= 4\int_0^{2\pi} dt$$

$$\underline{= 8\pi}$$

$$\nabla \times \boldsymbol{F} = \nabla \times \langle z - y,\ x,\ -x \rangle$$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z - y & x & -x \end{vmatrix}$$

$$= \left(\tfrac{\partial}{\partial y}(-x) - \tfrac{\partial}{\partial z}(x)\right)\hat{\boldsymbol{i}} + \left(\tfrac{\partial}{\partial z}(z-y) - \tfrac{\partial}{\partial x}(-x)\right)\hat{\boldsymbol{j}} + \left(\tfrac{\partial}{\partial x}(x) - \tfrac{\partial}{\partial y}(z-y)\right)\hat{\boldsymbol{k}}$$

$$= 0\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$$

$$\underline{= \langle 0,\ 2,\ 2 \rangle}$$

The region of integration is the base of the hemisphere in the xy-plane, which is

$$R = \left\{(x, y) : x^2 + y^2 \le 4\right\} = \left\{(r, \theta) :\ 0 \le r \le 2,\ 0 \le \theta \le 2\pi\right\}$$

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\ dS = \iint_R \langle 0,\ 2,\ 2 \rangle \cdot \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle dA$$

$$= \iint_R \left(\frac{2y}{\sqrt{4 - x^2 - y^2}} + 2 \right) dA \qquad x^2 + y^2 + z^2 = 4 \ \rightarrow \ z = \sqrt{4 - x^2 - y^2}$$

$$= \int_0^{2\pi} \int_0^2 \left(\frac{2r\sin\theta}{\sqrt{4 - r^2}} + 2 \right) r\,dr\,d\theta$$

$$= \int_0^2 \int_0^{2\pi} \left(\frac{2r^2}{\sqrt{4 - r^2}} \sin\theta + 2r \right) d\theta\ dr$$

$$= \int_0^2 \left(-\frac{2r^2}{\sqrt{4-r^2}}\cos\theta + 2r\theta \right)\Big|_0^{2\pi} dr$$

$$= \int_0^2 \left(-\frac{2r^2}{\sqrt{4-r^2}} + 4\pi r + \frac{2r^2}{\sqrt{4-r^2}} \right) dr$$

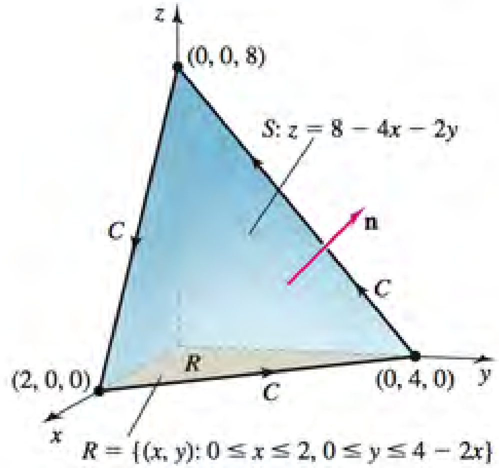
$$= 4\pi \int_0^2 r\, dr$$

$$= 2\pi r^2 \Big|_0^2$$

$$\underline{= 8\pi}$$

Example

Evaluate the line integral $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$ where $\boldsymbol{F} = \langle z,\ -z,\ x^2 - y^2 \rangle$ and C consists of the three line segments that bound the plane $z = 8 - 4x - 2y$ in the first octant, oriented as shown



Solution

$$\nabla \times \boldsymbol{F} = \nabla \times \langle z,\ -z,\ x^2 - y^2 \rangle$$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -z & x^2 - y^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}\left(x^2 - y^2\right) - \frac{\partial}{\partial z}(-z) \right)\hat{\boldsymbol{i}} + \left(\frac{\partial}{\partial z}(z) - \frac{\partial}{\partial x}\left(x^2 - y^2\right) \right)\hat{\boldsymbol{j}} + \left(\frac{\partial}{\partial x}(-z) - \frac{\partial}{\partial y}(z) \right)\hat{\boldsymbol{k}}$$

$$\underline{= \langle 1 - 2y,\ 1 - 2x,\ 0 \rangle}$$

$$z = 8 - 4x - 2y \Rightarrow 4x + 2y + z = 8 \quad \rightarrow \quad \boldsymbol{n} = \langle 4,\ 2,\ 1 \rangle$$

$$\iint_S (\nabla\times \boldsymbol{F})\cdot \boldsymbol{n}\; dS = \iint_R \langle 1-2y,\ 1-2x,\ 0\rangle \cdot \langle 4,\ 2,\ 1\rangle dA$$

$$= \int_0^2 \int_0^{4-2x} (6-4x-8y)\; dydx$$

$$= \int_0^2 \left[6y-4xy-4y^2 \right]_0^{4-2x} dx$$

$$= \int_0^2 \left(24-12x-16x+8x^2-4\left(16-16x+4x^2\right)\right) dx$$

$$= \int_0^2 \left(-8x^2+36x-40 \right) dx$$

$$= \left[-\frac{8}{3}x^3+18x^2-40x \right]_0^2$$

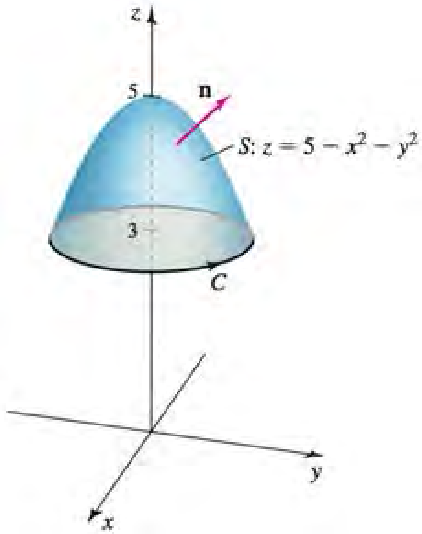
$$= -\frac{64}{3}+72-80$$

$$\boxed{= -\frac{88}{3}}$$

Example

Evaluate the line integral $\iint_S (\nabla\times \boldsymbol{F})\cdot \boldsymbol{n}\; dS$ where $\boldsymbol{F} = \left\langle -xz,\ yz,\ xye^z \right\rangle$ and S is the cap of the paraboloid $z=5-x^2-y^2$ above the plane $z=3$. Assume $\boldsymbol{n}$ points in the upward direction on S.

Solution



$$z=5-x^2-y^2=3 \quad \Rightarrow \quad x^2+y^2=2$$

$$\boldsymbol{r}(t)=\left\langle \sqrt{2}\cos t,\ \sqrt{2}\sin t,\ 0\right\rangle \qquad \boldsymbol{r}(t)=\langle r\cos t,\ r\sin t,\ 0\rangle$$

$$\boldsymbol{r}'(t)=\left\langle -\sqrt{2}\sin t,\ \sqrt{2}\cos t,\ 0\right\rangle$$

$$\left\langle -xz, yz,\ xye^z\right\rangle\cdot\left\langle -\sqrt{2}\sin t, \sqrt{2}\cos t, 0\right\rangle = xz\sqrt{2}\sin t + yz\sqrt{2}\cos t$$

$$= 3\sqrt{2}\cos t\sqrt{2}\sin t + 3\sqrt{2}\sin t\sqrt{2}\cos t$$

$$= 12\sin t\cos t$$

$$\boxed{= 6\sin 2t}$$

$$\iint_S (\nabla\times \boldsymbol{F})\cdot \boldsymbol{n}\; dS = \int_0^{2\pi} \left\langle -xz, yz,\ xye^z\right\rangle\cdot\left\langle -\sqrt{2}\sin t, \sqrt{2}\cos t, 0\right\rangle dt$$

$$= \int_0^{2\pi} 6\sin 2t dt$$

$$= -3\cos 2t \Big|_0^{2\pi}$$

$$\underline{= 0}$$

Interpreting the Curl

Stokes' Theorem leads to another interpretation of the curl at a point in a vector field. We need the idea of the average circulation. If C is the boundary of an oriented surface C, we define the average circulation of $\boldsymbol{F}$ over S as

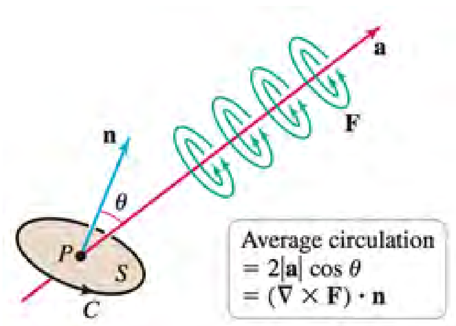
$$\frac{1}{area(S)}\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \frac{1}{area(S)}\iint_S (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\, dS$$

Where Stokes' Theorem is used to convert the circulation integral to a surface integral.

Consider the vector field $\boldsymbol{F} = \boldsymbol{a}\times\boldsymbol{r}$, where $\boldsymbol{a} = (a_1,\ a_2,\ a_3)$ is a nonzero vector and $\boldsymbol{r} = (x,\ y,\ z)$

$$\boldsymbol{F} = \boldsymbol{a}\times\boldsymbol{r} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= (a_2 z - a_3 y)\hat{\boldsymbol{i}} + (a_3 x - a_1 z)\hat{\boldsymbol{j}} + (a_1 y - a_2 x)\hat{\boldsymbol{k}}$$



$$curl\ \boldsymbol{F} = \nabla\times\boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$= (2a_1)\hat{\boldsymbol{i}} + (2a_2)\hat{\boldsymbol{j}} + (2a_3)\hat{\boldsymbol{k}}$$

$$\underline{= 2\boldsymbol{a}}$$

Let S to be a small circular disk centered at a point P, whose normal vector $\boldsymbol{n}$ makes an angle θ with the axis $\boldsymbol{a}$.

Let C be the boundary of S with a counterclockwise orientation.

The average circulation of this vector field on S is

$$\frac{1}{area(S)}\iint_S (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\, dS = \frac{1}{area(S)}(\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\ \cdot area(S)$$

$$= (2\boldsymbol{a})\cdot\boldsymbol{n}$$

$$\underline{= 2|\boldsymbol{a}|\cos\theta}$$

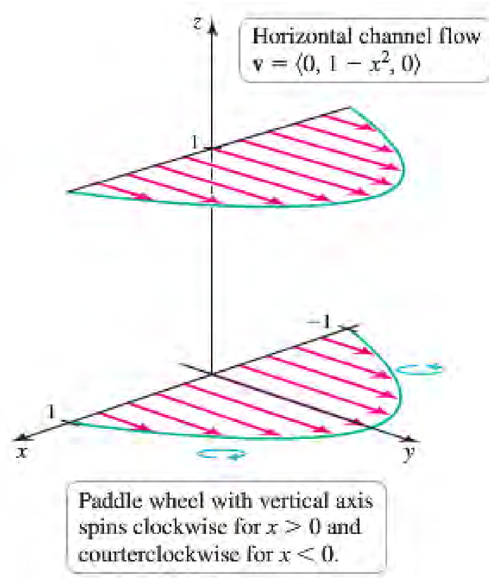
Example

Consider the velocity field $\vec{v} = \langle 0,\ 1-x^2,\ 0 \rangle$, for $|x| \le 1$ and $|z| \le 1$, which represents a horizontal flow in the *y*-direction.

a) Suppose you place a paddle wheel at the point $P\left(\frac{1}{2}, 0, 0\right)$.

Using physical arguments, in which of the coordinate directions should the axis of the wheel point in order for the wheel to spin? In which direction does it spin? What happens if you place the wheel at $Q\left(-\frac{1}{2}, 0, 0\right)$?

b) Compute and graph the curl of $\vec{v}$ and provide an interpretation.



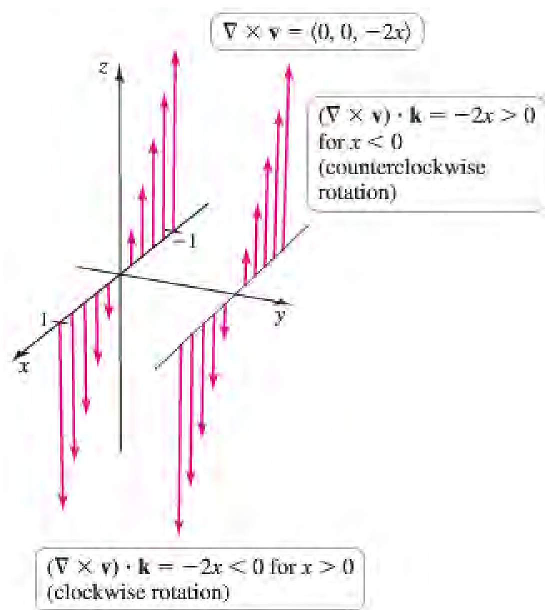
Solution

a) If the axis of the wheel is aligned with the *x*-axis at *P*, the flow strikes the upper and lower halves of the wheel symmetrically and the wheel does not spin. If the axis of the wheel is aligned with the *z*-axis at *P*, the flow in the *y*-direction is greater for $x < \frac{1}{2}$ than it is for $x > \frac{1}{2}$.

Therefore, a wheel located at $\left(\frac{1}{2}, 0, 0\right)$ spins in the clockwise direction, looking from above.

Using the similar argument, we conclude that a vertically oriented paddle wheel placed at $Q\left(-\frac{1}{2}, 0, 0\right)$ spins in the counterclockwise direction (when viewing from above).

b) $$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 1-x^2 & 0 \end{vmatrix} = -2x\hat{k}$$



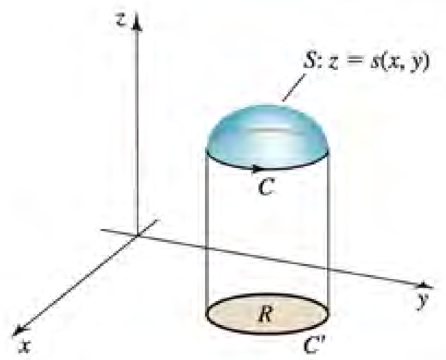
The curl points in the *z*-direction, which is the direction of the paddle wheel axis that gives the maximum angular speed of the wheel. Consider the z-component of the curl, which is $(\nabla \times \vec{v}) \cdot \hat{k} = -2x$

At *x* = 0, this component is zero, meaning the wheel does not spin at any point along the *y*-axis when its axis of the wheel is aligned with the *z*-axis. For $x > 0$, we see that $(\nabla \times \vec{v}) \cdot \hat{k} < 0$, which corresponds to clockwise rotation of the vector field.

For $x < 0$, we see that $(\nabla \times \vec{v}) \cdot \hat{k} > 0$, which corresponds to counterclockwise rotation.

Proof of Stokes' Theorem

Consider the case in which the surface S is the graph of the function $z = s(x, y)$, defined on a region in the xy-plane. Let C be the curve that bounds S with a counterclockwise orientation, let R be the projection of S in the xy-plane, and let C' the projection of C in the xy-plane.



C' is the projection of C in the xy-plane

Let $\boldsymbol{F} = \langle f, g, h\rangle$ the line integral in Stokes' Theorem is

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \oint_C f\ dx + g\ dy + h\ dz$$

$$= \oint_C f\ dx + g\ dy + h\left(z_x dx + z_y dy\right)$$

$$= \oint_C \underbrace{\left(f + hz_x\right)}_{M(x,y)}dx + \underbrace{\left(g + hz_y\right)}_{N(x,y)}dy$$

Where $M(x,y) = f + hz_x \quad N(x,y) = g + hz_y$

Applying Green's Theorem: $\displaystyle\oint_{C'} Mdx + Ndy = \iint_R \left(N_x - M_y\right)dA$

$$M(x,y) = f + hz_x \quad \to M_y = f_y + f_z z_y + hz_{xy} + z_x\left(h_y + h_z z_y\right) \qquad \frac{df}{dy} = f_x x_y + f_y y_y + f_z z_y$$

$$N(x,y) = g + hz_y \quad \to N_x = g_x + g_z z_x + hz_{yx} + z_y\left(h_x + h_z z_x\right)$$

$$N_x - M_y = g_x + g_z z_x + hz_{yx} + h_x z_y + h_z z_x z_y - f_y - f_z z_y - hz_{xy} - h_y z_x - h_z z_y z_x$$

$$= g_x - f_y + z_x\left(g_z - h_y\right) + z_y\left(h_x - f_z\right)$$

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_R \left(g_x - f_y + z_x\left(g_z - h_y\right) + z_y\left(h_x - f_z\right)\right)dA$$

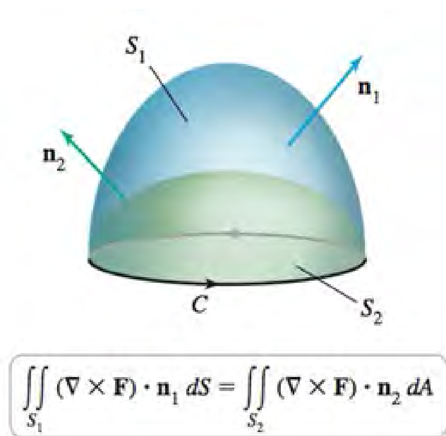
$$\iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, dS = \iint_R \left(\left(h_y - g_z\right)\left(-z_x\right) + \left(f_z - h_x\right)\left(-z_y\right) + \left(g_x - f_y\right) \right) dA$$

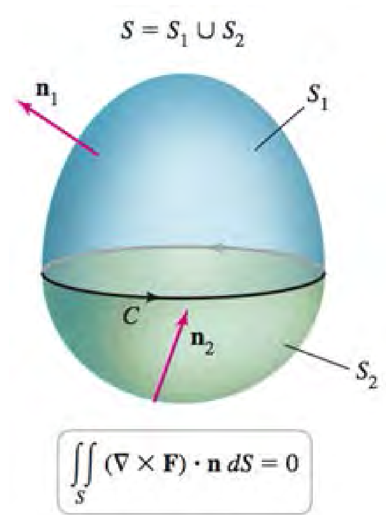
Where the upward vector normal $\boldsymbol{n} = \left\langle -z_x,\ -z_y,\ 1 \right\rangle$

Notes on Stokes' Theorem

1. Stokes' Theorem allows a surface integral $\iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, dS$ to be evaluated using only the values of the vector field in the boundary *C*.

$$\iint_{S_1} (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}_1\ dS = \iint_{S_2} (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}_2\ dS$$





Since $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are equal in magnitude and of opposite sign; therefore

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\ dS = \iint_{S_1} (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}_1\ dS + \iint_{S_2} (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}_2\ dS \ \underline{= 0}$$

2. If $\boldsymbol{F}$ is conservative vector field, then $\nabla \times \boldsymbol{F} = 0$.

Theorem Curl $\boldsymbol{F} = 0$ Implies $\boldsymbol{F}$ is Conservative

Suppose that $\nabla \times \boldsymbol{F} = 0$ throughout an open simply connected region *D* of $\mathbb{R}^3$. Then $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$ on all closed simple smooth curves *C* in *D* and $\boldsymbol{F}$ is a conservative vector field on *D*.

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_S \underbrace{(\nabla \times \boldsymbol{F})}_{0} \cdot \boldsymbol{n}\ dS = 0$$

Exercises Section 4.7 – Stokes' Theorem

Verify that the line integral and the surface integral of Stokes' Theorem are equal for the following vector fields, surfaces *S*, and closed curves *C*. Assume that *C* has counterclockwise orientation and *S* has a consistent orientation.

1. $\boldsymbol{F} = \langle y,\ -x,\ 10 \rangle$; *S* is the upper half of the sphere $x^2+y^2+z^2=1$ and *C* is the circle $x^2+y^2=1$ in the *xy*-plane

2. $\boldsymbol{F} = \langle 0,\ -x,\ y \rangle$; *S* is the upper half of the sphere $x^2+y^2+z^2=4$ and *C* is the circle $x^2+y^2=4$ in the *xy*-plane

3. $\boldsymbol{F} = \langle x,\ y,\ z \rangle$; *S* is the paraboloid $z=8-x^2-y^2$ for $0 \le z \le 8$ and *C* is the circle $x^2+y^2=8$ in the *xy*-plane

4. $\boldsymbol{F} = \langle 2z,\ -4x,\ 3y \rangle$; *S* is the cap of the sphere $x^2+y^2+z^2=169$ above the plane $z=12$ and *C* is the boundary of *S*.

5. $\boldsymbol{F} = \langle y-z,\ z-x,\ x-y \rangle$; *S* is the cap of the sphere $x^2+y^2+z^2=16$ above the plane $z=\sqrt{7}$ and *C* is the boundary of *S*.

Evaluate the line integral $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$ by evaluating the surface integral in Stokes' Theorem with an appropriate choice of *S*. Assume that *C* has a counterclockwise orientation,

6. $\boldsymbol{F} = \langle 2y,\ -z,\ x \rangle$; *C* is the circle $x^2+y^2=12$ in the plane $z=0$.

7. $\boldsymbol{F} = \langle y,\ xz,\ -y \rangle$; *C* is the ellipse $x^2+\frac{y^2}{4}=1$ in the plane $z=1$.

8. $\boldsymbol{F} = \left\langle x^2-z^2,\ y,\ 2xz \right\rangle$; *C* is the boundary of the plane $z=4-x-y$ in the plane first octant.

9. $\boldsymbol{F} = \left\langle y^2,\ -z^2,\ x \right\rangle$; *C* is the circle $\boldsymbol{r}(t) = \langle 3\cos t,\ 4\cos t,\ 5\sin t \rangle$ for $0 \le t \le 2\pi$.

10. $\boldsymbol{F} = \left\langle 2xy\sin z,\ x^2 \operatorname{sinz},\ x^2 y\cos z \right\rangle$; *C* is the boundary of the plane $z=8-2x-4y$ in the first octant.

Evaluate the line integral in Stokes' Theorem to evaluate the surface integral $\iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, dS$. Assume that ***n*** points in an upward direction,

11. $\boldsymbol{F} = \langle x,\ y,\ z \rangle$; *S* is the upper half of the ellipsoid $\frac{x^2}{4}+\frac{y^2}{9}+z^2=1$

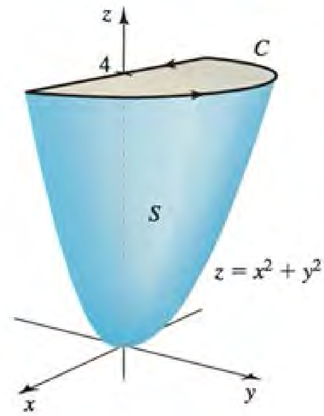
12. $\boldsymbol{F} = \langle 2y,\ -z,\ x-y-z \rangle$; *S* is the cap of the sphere $x^2+y^2+z^2=25$ *for* $3 \le x \le 5$

13. $\boldsymbol{F} = \langle x+y,\ y+z,\ x+z \rangle$; *S* is the tilted disk enclosed $\boldsymbol{r}(t) = \left\langle \cos t,\ 2\sin t,\ \sqrt{3}\cos t \right\rangle$

14. Use Stokes' Theorem and a surface integral to find the circulation on C of the vector field $\boldsymbol{F} = \langle -y,\ x,\ 0 \rangle$ as a function of ϕ. For what value of ϕ is the circulation a maximum?

15. A circle C in the plane $x + y + z = 8$ has a radius of 4 and center (2, 3, 3). Evaluate $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$ for $\boldsymbol{F} = \langle 0,\ -z,\ 2y \rangle$ where C has a counterclockwise orientation when viewed from above. Does the circulation depend on the radius of the circle? Does it depend on the location of the center of the circle?

16. Begin with the paraboloid $z = x^2 + y^2$, for $0 \le z \le 4$, and slice it with the plane $y = 0$. Let S be the surface that remains for $y \ge 0$ (including the planar surface in the *xz*-plane). Let C be the semicircle and line segment that bound the cap of S in the plane $z = 4$ with counterclockwise orientation. Let $\boldsymbol{F} = \langle 2z + y,\ 2x + z,\ 2y + x \rangle$



a) Describe the direction of the vectors normal to the surface that are consistent with the orientation of C.

b) Evaluate $\iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, dS$

c) Evaluate $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$ and check for argument with part (*b*).

17. The French Physicist André-Marie Ampère discovered that an electrical current I in a wire produces a magnetic field B. A special case of Ampère's Law relates the current to the magnetic field through the equation $\oint_C \boldsymbol{B} \cdot d\boldsymbol{r} = \mu I$, where C is any closed curve through which the wire passes and μ is a physical constant. Assume that the current I is given in terms of the current density $\boldsymbol{J}$ as $I = \iint_S \boldsymbol{J} \cdot \boldsymbol{n}\, dS$, where S is an oriented surface with C as a boundary. Use Stokes' Theorem to show that an equivalent form of Ampère's Law is $\nabla \times \boldsymbol{B} = \mu \boldsymbol{J}$.

18. Let S be the paraboloid $z = a\left(1 - x^2 - y^2\right)$, for $z \ge 0$, where $a > 0$ is a real number. Let $\boldsymbol{F} = \langle x - y,\ y + z,\ z - x \rangle$. For what value(s) of a (if any) does $\iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, dS$ have its maximum value?

19. The goal is to evaluate $A = \iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, dS$, where $\boldsymbol{F} = \langle yz,\ -xz,\ xy \rangle$ and S ids the surface of the upper half of the ellipsoid $x^2 + y^2 + 8z^2 = 1\ \ (z \ge 0)$

a) Evaluate a surface integral over a more convenient surface to find the value of A.

b) Evaluate A using a line integral.

Section 4.8 – Divergence Theorem

Divergence Theorem

The Divergence Theorem is the 3-dimensional version of the flux form of Green's Theorem.
If R is a region in the xy-plane, C is the simple closed piecewise-smooth oriented boundary of R, and $\boldsymbol{F} = \langle f, g \rangle$ is a vector field, Green's Theorem says that

$$\underbrace{\oint_C \boldsymbol{F} \cdot \boldsymbol{n}\, dS}_{\text{flux across } C} = \iint_R \underbrace{\left(f_x + g_y\right)}_{\text{divergence}} dA$$

Theorem

Let $\boldsymbol{F}$ be a vector field whose components have continuous first partial derivatives in a connected and simply connected region D enclosed by a smooth oriented surface S. Then

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS = \iiint_D \nabla \cdot \boldsymbol{F}\, dV$$

Where $\boldsymbol{n}$ is the unit outward normal vector on S.

Example

Consider the radial field $\boldsymbol{F} = \langle x, y, z \rangle$ and let S be the sphere $x^2 + y^2 + z^2 = a^2$ that encloses the region D. Assume $\boldsymbol{n}$ is the outward normal vector on the sphere. Evaluate both integrals of the Divergence Theorem.

Solution

The divergence of $\boldsymbol{F}$: $\quad \nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$

$$\begin{aligned}
\iiint_D \nabla \cdot \boldsymbol{F}\, dV &= \iiint_D (3)\, dV \\
&= 3 \cdot volume(D) \\
&= 3 \cdot \frac{4\pi}{3} a^3 \\
&= 4\pi a^3
\end{aligned}$$

$\boldsymbol{r} = \langle x, y, z \rangle = \langle a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi \rangle \qquad R = \{(\phi, \theta):\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\}$

$t_\phi = \langle a\cos\phi\cos\theta,\ a\cos\phi\sin\theta,\ -a\sin\phi \rangle$

$t_\theta = \langle -a\sin\phi\sin\theta,\ a\sin\phi\cos\theta,\ 0 \rangle$

The required vector normal to the surface is

$$t_\phi \times t_\theta = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \left(a^2\sin^2\phi\cos\theta\right)\hat{\imath} + \left(a^2\sin^2\phi\sin\theta\right)\hat{\boldsymbol{j}} + \left(a^2\sin\phi\cos\phi\cos^2\theta + a^2\sin\phi\cos\phi\sin^2\theta\right)\hat{\boldsymbol{k}}$$

$$= \left\langle a^2\sin^2\phi\cos\theta,\ a^2\sin^2\phi\sin\theta,\ a^2\sin^2\phi\cos\phi\right\rangle$$

$$\boldsymbol{F}\cdot\left(t_\phi \times t_\theta\right) = \langle a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi\rangle \cdot \left\langle a^2\sin^2\phi\cos\theta,\ a^2\sin^2\phi\sin\theta,\ a^2\sin^2\phi\cos\phi\right\rangle$$

$$= a^3\sin^3\phi\cos^2\theta + a^3\sin^3\phi\sin^2\theta + a^3\sin\phi\cos^2\phi \qquad \cos^2\theta+\sin^2\theta=1$$

$$= a^3\sin\phi\left(\sin^2\phi+\cos^2\phi\right)$$

$$\underline{= a^3\sin\phi}$$

$$\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\ dS = \iint_R \nabla\boldsymbol{F}\cdot\left(T_\phi \times T_\theta\right)\ dA$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} a^3\sin\phi\ d\phi$$

$$= a^3\ \theta\Big|_0^{2\pi}\ \left[-\cos\theta\right]_0^{\pi}$$

$$\underline{= 4\pi a^3}$$

∴ the two integral of the Divergence Theorem are equal.

Example

Consider the rotation field $\boldsymbol{F} = \boldsymbol{a}\times\boldsymbol{r} = \langle 1,\ 0,\ 1\rangle \times \langle x,\ y,\ z\rangle = \langle -y,\ x-z,\ y\rangle$

Let S be the sphere $x^2+y^2+z^2=a^2$ for $z\ge 0$, together with its base in the *xy*-plane.
Find the net outward flux across S.

Solution

$$\nabla\cdot\boldsymbol{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x-z) + \frac{\partial}{\partial z}(y)\ \underline{=0}$$

∴ The flux across the hemisphere is zero.

However, with the Divergence Theorem, radial fields are interesting and have many physical applications

Example

Find the net outward flux of the field $\boldsymbol{F} = xyz\langle 1,\ 1,\ 1\rangle$ across the boundaries of the cube $D = \{(x,y,z):\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$

Solution

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(xyz)$$

$$= yz + xz + xy$$

$$\iiint_D \nabla \cdot \boldsymbol{F}\ dV = \int_0^1 \int_0^1 \int_0^1 (yz + xz + xy)\,dxdydz$$

$$= \int_0^1 \int_0^1 \left[yzx + \frac{1}{2}x^2 z + \frac{1}{2}x^2 y \right]_0^1 dydz$$

$$= \int_0^1 \int_0^1 \left(yz + \frac{1}{2}z + \frac{1}{2}y \right) dydz$$

$$= \int_0^1 \left[\frac{1}{2}y^2 z + \frac{1}{2}zy + \frac{1}{4}y^2 \right]_0^1 dz$$

$$= \int_0^1 \left(z + \frac{1}{4} \right) dz$$

$$= \left[\frac{1}{2}z^2 + \frac{1}{4}z \right]_0^1$$

$$= \frac{3}{4}$$

Interpretation of the Divergence Using Mass Transport

Suppose that $\boldsymbol{v}$ is the velocity field of a material, such as water or molasses, and ρ is its constant density. The vector field $\boldsymbol{F} = \rho\boldsymbol{v} = \langle f,\ g,\ h\rangle$ describes the ***mass transport*** of the material, with units of

$\frac{mass}{vol.} \times \frac{length}{time} = \frac{mass}{area - time}$ typical units of mass transport are $g/m^2/s$.

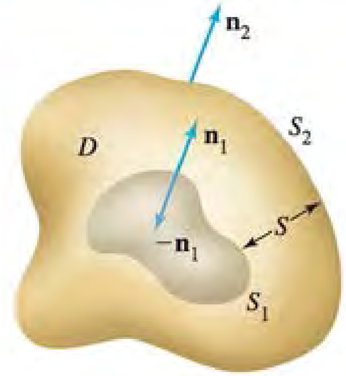
This means that $\boldsymbol{F}$ gives the mass material flowing past a point (in each of the three coordinate direction) per unit of surface area per unit of time.

When $\boldsymbol{F}$ is multiplied by an area, the result in the *flux*, with units of mass/unit time.

Divergence *Theorem* for Hollow Regions

Suppose the vector field $\boldsymbol{F}$ satisfies the conditions of the Divergence Theorem on a region D bounded by two smooth oriented surfaces S_1 and S_2, where S_1 lies within S_2.

Let S be the entire boundary of $D\left(S = S_1 \cup S_2\right)$ and let $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ be the outward unit normal vectors for S_1 and S_2, respectively.



$$\iiint\limits_D \nabla\cdot\boldsymbol{F}\ dV = \iint\limits_S \boldsymbol{F}\cdot\boldsymbol{n}\ dS = \iint\limits_{S_2} \boldsymbol{F}\cdot\boldsymbol{n}_2\ dS - \iint\limits_{S_1} \boldsymbol{F}\cdot\boldsymbol{n}_1\ dS$$

Example

Consider the inverse square vector field $\quad \boldsymbol{F} = \dfrac{\boldsymbol{r}}{|\boldsymbol{r}|^3} = \dfrac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^{3/2}}$

a) Find the net outward flux of $\boldsymbol{F}$ across the surface of the region $D = \left\{(x,y,z):\ a^2 \le x^2+y^2+z^2 \le b^2\right\}$ that lies between concentric spheres with radii a and b.

b) Find the outward flux of $\boldsymbol{F}$ across any sphere that encloses the origin,

Solution

a) $\nabla\cdot\boldsymbol{F} = \nabla\cdot\left(\dfrac{\boldsymbol{r}}{|\boldsymbol{r}|^3}\right) = -\nabla\cdot\boldsymbol{r}\dfrac{1}{|\boldsymbol{r}|^3} - \boldsymbol{r}\cdot\nabla\dfrac{1}{|\boldsymbol{r}|^3}$

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{-3/2} = -3x\left(x^2+y^2+z^2\right)^{-5/2} = -\frac{3x}{|\boldsymbol{r}|^5}$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)^{-3/2} = -3y\left(x^2+y^2+z^2\right)^{-5/2} = -\frac{3y}{|\boldsymbol{r}|^5}$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)^{-3/2} = -3z\left(x^2+y^2+z^2\right)^{-5/2} = -\frac{3z}{|\boldsymbol{r}|^5}$$

$$\nabla\frac{1}{|\boldsymbol{r}|^3} = -3\frac{x\hat{\boldsymbol{i}}+y\boldsymbol{j}+z\boldsymbol{k}}{|\boldsymbol{r}|^5} = -3\frac{\boldsymbol{r}}{|\boldsymbol{r}|^5}$$

$$\begin{aligned}\nabla\cdot\boldsymbol{F} = \nabla\cdot\left(-\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}\right) &= -\frac{1}{|\boldsymbol{r}|^3}\nabla\cdot\boldsymbol{r} + 3\boldsymbol{r}\cdot\frac{\boldsymbol{r}}{|\boldsymbol{r}|^5}\\ &= -\frac{3}{|\boldsymbol{r}|^3} + \frac{3}{|\boldsymbol{r}|^3}\\ &= 0\end{aligned}$$

Let $S = S_2\left(\textit{with radius } b \textit{ larger}\right) \cup S_1\left(\textit{with radius } a\right)$

Because $\iiint\limits_D \nabla \cdot \boldsymbol{F}\ dV = 0$, the divergence Theorem implies that

$$\iint\limits_S \boldsymbol{F} \cdot \boldsymbol{n}\ dS = \iint\limits_{S_2} \boldsymbol{F} \cdot \boldsymbol{n}_2\ dS - \iint\limits_{S_1} \boldsymbol{F} \cdot \boldsymbol{n}_1\ dS = 0$$

Therefore, the net flux across S is zero.

b) $\iint\limits_{S_2} \boldsymbol{F} \cdot \boldsymbol{n}_2\ dS = \iint\limits_{S_1} \boldsymbol{F} \cdot \boldsymbol{n}_1\ dS$

$$\begin{aligned}\iint\limits_{S_1} \boldsymbol{F} \cdot \boldsymbol{n}_1\ dS &= \iint\limits_{S_1} \frac{\boldsymbol{r}}{|\boldsymbol{r}|^3} \cdot \frac{\boldsymbol{r}}{|\boldsymbol{r}|}\ dS \\ &= \iint\limits_{S_1} \frac{\boldsymbol{r}^2}{|\boldsymbol{r}|^4}\ dS \qquad \left(|r| = a\right) \\ &= \iint\limits_{S_1} \frac{1}{a^2}\ dS \qquad\qquad \textit{Surface Area} = 4\pi a^2 \\ &= \frac{4\pi a^2}{a^2} \\ &= 4\pi \end{aligned}$$

$$\iint\limits_{S_2} \boldsymbol{F} \cdot \boldsymbol{n}_2\ dS = \iint\limits_{S_2} \frac{\boldsymbol{r}}{|\boldsymbol{r}|^3} \cdot \frac{\boldsymbol{r}}{|\boldsymbol{r}|}\ dS = \frac{4\pi b^2}{b^2} = 4\pi$$

The flux of the inverse square field across any surface enclosing the origin is 4π.

Gauss' Law

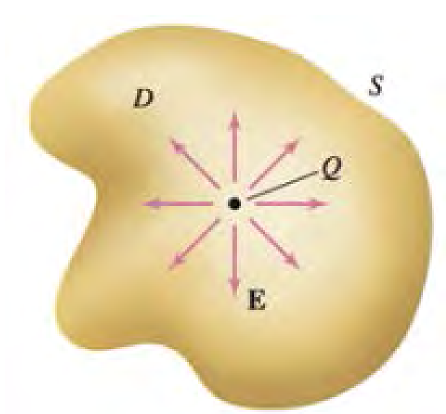
Applying the Divergence Theorem to electric fields leads to one of the fundamental laws of physics. The electric field due to a point charge Q located at the origin is given by the inverse square law.

$$\boldsymbol{E}(x, y, z) = \frac{Q}{4\pi\varepsilon_0}\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}$$

Where $\boldsymbol{r}(x, y, z)$ and ε_0 is a physical constant called the ***permittivity of free square***.

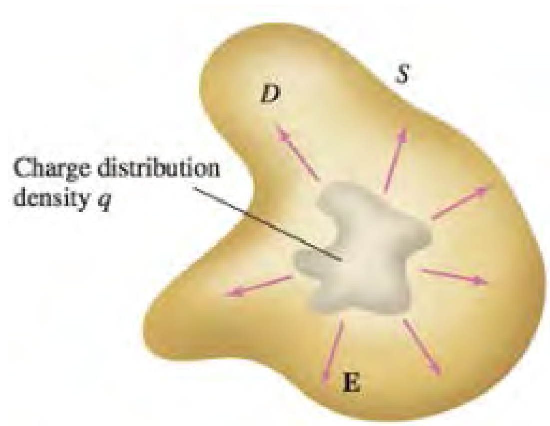
This is one statement of Gauss' Law: If S is a surface that encloses a point charge Q, then the flux of the electric field across S is

$$\iint_S \boldsymbol{E} \cdot \boldsymbol{n}\, dS = \frac{Q}{\varepsilon_0}$$



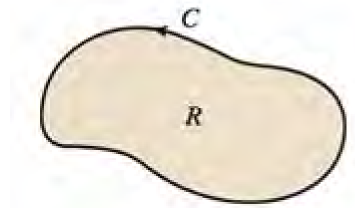
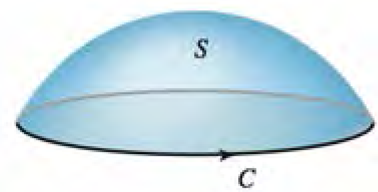
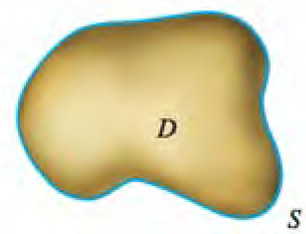
Gauss' Law: Flux of electric field across S due to point charge Q =

$$\iint_S \boldsymbol{E} \cdot \boldsymbol{n}\, dS = \frac{Q}{\varepsilon_0}$$



Gauss' Law: Flux of electric field across S due to charge distribution q =

$$\iint_S \boldsymbol{E} \cdot \boldsymbol{n}\, dS = \frac{1}{\varepsilon_0}\iiint_D q\, dV$$

Fundamental Theorem of Calculus	$\int_a^b f'(x)\,dx = f(b) - f(a)$	
Fundamental Theorem of Line Integrals	$\int_C \nabla f \cdot d\mathbf{x} = f(B) - f(A)$	
Green's Theorem (Circulation Form)	$\iint\limits_R \left(g_x - f_y\right) dA = \oint\limits_C f\,dx + g\,dy$	
Stokes' Theorem	$\iint\limits_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, dS = \oint\limits_C \boldsymbol{F} \cdot d\boldsymbol{r}$	
Divergence Theorem	$\iint\limits_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS = \iiint\limits_D \nabla \cdot \boldsymbol{F}\, dV$	

Exercises *Section* 4.8 – Divergence Theorem

Evaluate both integrals of the Divergence Theorem for the following vector fields and region. Check for agreement.

1. $\boldsymbol{F} = \langle 2x,\ 3y,\ 4z\rangle \quad D = \left\{(x,y,z):\ x^2+y^2+z^2 \le 4\right\}$

2. $\boldsymbol{F} = \langle -x,\ -y,\ -z\rangle \quad D = \left\{(x,y,z):\ |x|\le 1,\ |y|\le 1,\ |z|\le 1\right\}$

3. $\boldsymbol{F} = \langle z-y,\ x,\ -x\rangle \quad D = \left\{(x,y,z):\ \dfrac{x^2}{4}+\dfrac{y^2}{8}+\dfrac{z^2}{12} \le 1\right\}$

4. $\boldsymbol{F} = \left\langle x^2,\ y^2,\ z^2\right\rangle \quad D = \left\{(x,y,z):\ |x|\le 1,\ |y|\le 2,\ |z|\le 3\right\}$

5. Find the net outward flux of the field $\boldsymbol{F} = \langle 2z-y,\ x,\ -2x\rangle$ across the sphere of radius 1 centered at the origin.

6. Find the net outward flux of the field $\boldsymbol{F} = \langle bz-cy,\ cx-az,\ ay-bx\rangle$ across any smooth closed surface $\mathbb{R}^3$, where *a*, *b*, and *c* are constants.

Use the Divergence Theorem to compute the net outward flux of the following fields across the given surface *S or D.*

7. $\boldsymbol{F} = \langle x,\ -2y,\ 3z\rangle$; *S* is the sphere $\left\{(x,y,z):\ x^2+y^2+z^2 = 6\right\}$

8. $\boldsymbol{F} = \left\langle x^2,\ 2xz,\ y^2\right\rangle$; *S* is surface of the cube cut from the first octant by the planes *x* = 1, *y* = 1, and $z=1$

9. $\boldsymbol{F} = \langle x,\ 2y,\ z\rangle$; *S* is boundary of the tetrahedron in the first octant formed by the plane $x+y+z=1$

10. $\boldsymbol{F} = \left\langle x^2,\ y^2,\ z^2\right\rangle$; *S* is the sphere $\left\{(x,y,z):\ x^2+y^2+z^2 = 25\right\}$

11. $\boldsymbol{F} = \langle x,\ y,\ z\rangle$; *S* is the surface of the paraboloid $z = 4-x^2-y^2$, for $z \ge 0$, plus its base in the *xy*-plane

12. $\boldsymbol{F} = \langle x,\ y,\ z\rangle$; *S* is the surface of the cone $z^2 = x^2+y^2$, for $0 \le z \le 4$, plus its top surface in the plane $z=4$

13. $\boldsymbol{F} = \langle z-x,\ x-y,\ 2y-z\rangle$; *D* is the region between the spheres of radius 2 and 4 centered at origin.

14. $\boldsymbol{F} = \boldsymbol{r}|\boldsymbol{r}| = \langle x,\ y,\ z\rangle\sqrt{x^2+y^2+z^2}$; *D* is the region between the spheres of radius 1 and 2 centered at origin.

15. $\boldsymbol{F} = \dfrac{\boldsymbol{r}}{|\boldsymbol{r}|} = \dfrac{\langle x,\ y,\ z\rangle}{\sqrt{x^2+y^2+z^2}}$; *D* is the region between the spheres of radius 1 and 2 centered at origin.

16. $\boldsymbol{F} = \langle z-y,\ x-z,\ 2y-x \rangle$; $D = \{(x,y,z):\ 1 \le |x| \le 3,\ 1 \le |y| \le 3,\ 1 \le |z| \le 3\}$ is the region between two cubes

17. $\boldsymbol{F} = \langle x,\ 2y,\ 3z \rangle$; D is the region between the cylinders $x^2+y^2=1$ and $x^2+y^2=4$ for $0 \le z \le 8$

18. Compute the outward flux of the following vector field across the given surface $\boldsymbol{F} = \left\langle x^2 e^y \cos z,\ -4xe^y \cos z,\ 2xe^y \sin z \right\rangle$; S is the boundary of the ellipsoid $\frac{x^2}{4}+y^2+z^2=1$

19. Compute the outward flux of the following vector field across the given surface $\boldsymbol{F} = \langle -yz,\ xz,\ 1 \rangle$; S is the boundary of the ellipsoid $\frac{x^2}{4}+\frac{y^2}{4}+z^2=1$

20. Compute the outward flux of the following vector field across the given surface $\boldsymbol{F} = \langle x\sin y,\ -\cos y,\ z\sin y \rangle$; S is the boundary of the region bounded by the planes $x=1$, $y=0$, $y=\frac{\pi}{2}$, $z=0$, and $z=x$

21. The electric field due to a point charge Q is $\boldsymbol{E} = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}$, where $\boldsymbol{r} = \langle x,\ y,\ z \rangle$ and ε_0 is a constant

a) Show that the flux of the field across a sphere of radius a centered at the origin is $\iint_S \boldsymbol{E} \cdot \boldsymbol{n}\, dS = \frac{Q}{\varepsilon_0}$

b) Let S be the boundary of the origin between two spheres centered of radius a and b with $a < b$. Use the Divergence Theorem to show that the net outward flux across S is zero.

c) Suppose there is a distribution of charge within a region D. Let $q(x,\ y,\ z)$ be the charge density (charge per unit volume). Interpret the statement that

$$\iint_S \boldsymbol{E} \cdot \boldsymbol{n}\, dS = \frac{1}{\varepsilon_0} \iiint_D q(x,\ y,\ z)\ dV$$

d) Assuming $\boldsymbol{E}$ satisfies the conditions of the Divergence Theorem, conclude from part (*c*) that $\nabla \cdot \boldsymbol{E} = \frac{q}{\varepsilon_0}$

e) Because the electric force is conservative, it has a potential function ϕ. From part (*d*) conclude that $\nabla^2\phi = \nabla \cdot \nabla\phi = \frac{q}{\varepsilon_0}$

Fourier's Law of heat transfer (or heat conduction) states that the heat flow vector $\boldsymbol{F}$ at a point is proportional to the negative gradient of the temperature that is, $\boldsymbol{F} = -k\nabla T$, which means that heat energy flows from hot regions to cold region. The constant $k > 0$ is called the *conductivity,* which has metric units of $J / m\text{-}s\text{-}K$. A temperature function for a region D is given. Find the net outward heat flux $\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS = -k \iint_S \nabla T \cdot \boldsymbol{n}\, dS$ across the boundary S of D. In some cases it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume $k = 1$.

22. $T(x, y, z) = 100 + x + 2y + z; \quad D = \{(x, y, z):\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$

23. $T(x, y, z) = 100 + x^2 + y^2 + z^2; \quad D = \{(x, y, z):\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$

24. $T(x, y, z) = 100 + e^{-z}; \quad D = \{(x, y, z):\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$

25. $T(x, y, z) = 100e^{-x^2 - y^2 - z^2}$; D is the sphere of radius a centered at the origin.