Solution Section 2.4 – Cross Product

Exercise

Prove when the cross product $\vec{u}\times\vec{v}$ is perpendicular to $\vec{u}$, then $\vec{u}\bullet(\vec{u}\times\vec{v})=0$

Solution

Let $\vec{u}=(u_1,\ u_2,\ u_3)$ and $\vec{v}=(v_1,\ v_2,\ v_3)$

$$\begin{aligned}
\vec{u}\bullet(\vec{u}\times\vec{v}) &= (u_1,\ u_2,\ u_3)\bullet(u_2v_3-u_3v_2,\ u_3v_1-u_1v_3,\ u_1v_2-u_2v_1)\\
&= u_1(u_2v_3-u_3v_2)+u_2(u_3v_1-u_1v_3)+u_3(u_1v_2-u_2v_1)\\
&= u_1u_2v_3-u_1u_3v_2+u_2u_3v_1-u_2u_1v_3+u_3u_1v_2-u_3u_2v_1\\
&= 0
\end{aligned}$$

Exercise

Find $\vec{u}\times\vec{v}$, where $\vec{u}=(1,\ 2,\ -2)$ *and* $\vec{v}=(3,\ 0,\ 1)$ and show that $\vec{u}\times\vec{v}$ is perpendicular to $\vec{u}$ and to $\vec{v}$.

Solution

$$\vec{u}\times\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -2\\ 3 & 0 & 1\end{vmatrix}$$

$$=\left(\begin{vmatrix}2 & -2\\ 0 & 1\end{vmatrix},\ -\begin{vmatrix}1 & -2\\ 3 & 1\end{vmatrix},\ \begin{vmatrix}1 & 2\\ 3 & 0\end{vmatrix}\right)$$

$$=(2,\ -7,\ -6)$$

$$\begin{aligned}
\vec{u}\bullet(\vec{u}\times\vec{v}) &= (1,\ 2,\ -2)\bullet(2,\ -7,\ -6)\\
&= 2-14+12\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\vec{v}\bullet(\vec{u}\times\vec{v}) &= (3,\ 0,\ 1)\bullet(2,\ -7,\ -6)\\
&= 6-0-6\\
&= 0
\end{aligned}$$

$\vec{u}\times\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

Exercise

Given $\vec{u} = (3,\ 2,\ -1)$, $\vec{v} = (0,\ 2,\ -3)$, *and* $\vec{w} = (2,\ 6,\ 7)$ Compute the vectors

a) $\vec{u} \times \vec{v}$

b) $\vec{v} \times \vec{w}$

c) $\vec{u} \times (\vec{v} \times \vec{w})$

d) $(\vec{u} \times \vec{v}) \times \vec{w}$

e) $\vec{u} \times (\vec{v} - 2\vec{w})$

Solution

a) $$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -1 \\ 0 & 2 & -3 \end{vmatrix}$$

$$= \left(\begin{vmatrix} 2 & -1 \\ 2 & -3 \end{vmatrix},\ -\begin{vmatrix} 3 & -1 \\ 0 & -3 \end{vmatrix},\ \begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} \right)$$

$$\underline{= (-4,\ 9,\ 6)}$$

b) $$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & -3 \\ 2 & 6 & 7 \end{vmatrix}$$

$$= \left(\begin{vmatrix} 2 & -3 \\ 6 & 7 \end{vmatrix},\ -\begin{vmatrix} 0 & -3 \\ 2 & 7 \end{vmatrix},\ \begin{vmatrix} 0 & 2 \\ 2 & 6 \end{vmatrix} \right)$$

$$\underline{= (32,\ -6,\ -4)}$$

c) $$u \times (v \times w) = (3,\ 2,\ -1) \times (32,\ -6,\ -4)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -1 \\ 32 & -6 & -4 \end{vmatrix}$$

$$= \left(\begin{vmatrix} 2 & -1 \\ -6 & -4 \end{vmatrix},\ -\begin{vmatrix} 3 & -1 \\ 32 & -4 \end{vmatrix},\ \begin{vmatrix} 3 & 2 \\ 32 & -6 \end{vmatrix} \right)$$

$$\underline{= (-14,\ -20,\ -82)}$$

d) $$(u \times v) \times w = (-4,\ 9,\ 6) \times (2,\ 6,\ 7)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & 9 & 6 \\ 2 & 6 & 7 \end{vmatrix}$$

$$=\left(\begin{vmatrix}9 & 6\\ 6 & 7\end{vmatrix},\ -\begin{vmatrix}-4 & 6\\ 2 & 7\end{vmatrix},\ \begin{vmatrix}-4 & 9\\ 2 & 6\end{vmatrix}\right)$$

$$\underline{=\left(27,\ 40,\ -42\right)}$$

e) $u\times(v-2w)=(3,\ 2,\ -1)\times\left[(0,\ 2,\ -3)-2(2,\ 6,\ 7)\right]$

$$=(3,\ 2,\ -1)\times(-4,\ -10,\ -17)$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 2 & -1\\ -4 & -1 & -17\end{vmatrix}$$

$$=\left(\begin{vmatrix}2 & -1\\ -10 & -17\end{vmatrix},\ -\begin{vmatrix}3 & -1\\ -4 & -17\end{vmatrix},\ \begin{vmatrix}3 & 2\\ -4 & -10\end{vmatrix}\right)$$

$$\underline{=\left(-44,\ 47,\ -22\right)}$$

Exercise

Use the cross product to find a vector that is orthogonal to both

a) $\vec{u}=(-6,\ 4,\ 2),\quad \vec{v}=(3,\ 1,\ 5)$

b) $\vec{u}=(1,\ 1,\ -2),\quad \vec{v}=(2,\ -1,\ 2)$

c) $\vec{u}=(-2,\ 1,\ 5),\quad \vec{v}=(3,\ 0,\ -3)$

Solution

a) $\vec{u}\times\vec{v}=(-6,\ 4,\ 2)\times(3,\ 1,\ 5)$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -6 & 4 & 2\\ 3 & 1 & 5\end{vmatrix}$$

$$=\left(\begin{vmatrix}4 & 2\\ 1 & 5\end{vmatrix},\ -\begin{vmatrix}-6 & 2\\ 3 & 5\end{vmatrix},\ \begin{vmatrix}-6 & 4\\ 3 & 1\end{vmatrix}\right)$$

$$\underline{=\left(18,\ 36,\ -18\right)}$$

b) $\vec{u}\times\vec{v}=(1,\ 1,\ -2)\times(2,\ -1,\ 2)$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & -2\\ 2 & -1 & 2\end{vmatrix}$$

$$=\left(\begin{vmatrix}1 & -2\\ -1 & 2\end{vmatrix},\ -\begin{vmatrix}1 & -2\\ 2 & 2\end{vmatrix},\ \begin{vmatrix}1 & 1\\ 2 & -1\end{vmatrix}\right)$$

$$\underline{= (0,\ -6,\ -3)}$$

c) $\vec{u}\times\vec{v} = (-2,\ 1,\ 5)\times(3,\ 0,\ -3)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 5 \\ 3 & 0 & -3 \end{vmatrix}$$

$$= \left(\begin{vmatrix} 1 & 5 \\ 0 & -3 \end{vmatrix},\ -\begin{vmatrix} -2 & 5 \\ 3 & -3 \end{vmatrix},\ \begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix} \right)$$

$$\underline{= (-3,\ 9,\ -3)}$$

Exercise

Find the area of the parallelogram determined by the given vectors

a) $\vec{u} = (1,\ -1,\ 2)$ *and* $\vec{v} = (0,\ 3,\ 1)$

b) $\vec{u} = (3,\ -1,\ 4)$ *and* $\vec{v} = (6,\ -2,\ 8)$

c) $\vec{u} = (2,\ 3,\ 0)$ *and* $\vec{v} = (-1,\ 2,\ -2)$

Solution

a) $Area = \|\vec{u}\times\vec{v}\|$

$$= \left\| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 0 & 3 & 1 \end{matrix} \right\|$$

$$= \left| \left(\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix},\ -\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix},\ \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} \right) \right|$$

$$= |(-7,\ -1,\ 3)|$$

$$= \sqrt{7^2 + 1^2 + 3^2}$$

$$\underline{= \sqrt{59}} \quad \left(unit^2 \right)$$

b) $Area = \|\vec{u}\times\vec{v}\|$

$$= \left\| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 4 \\ 6 & -2 & 8 \end{matrix} \right\|$$

$$= \left| \left(\begin{vmatrix} -1 & 4 \\ -2 & 8 \end{vmatrix},\ -\begin{vmatrix} 3 & 4 \\ 6 & 8 \end{vmatrix},\ \begin{vmatrix} 3 & -1 \\ 6 & -2 \end{vmatrix} \right) \right|$$

$$= \left| (0,\ 0,\ 0) \right|$$

$$\underline{= 0}$$

c) $Area = \left\| \vec{u} \times \vec{v} \right\|$

$$= \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 0 \\ -1 & 2 & -2 \end{vmatrix} \right\|$$

$$= \left| \left(\begin{vmatrix} 3 & 0 \\ 2 & -2 \end{vmatrix},\ -\begin{vmatrix} 2 & 0 \\ -1 & -2 \end{vmatrix},\ \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} \right) \right|$$

$$= \left| (-6,\ 4,\ 7) \right|$$

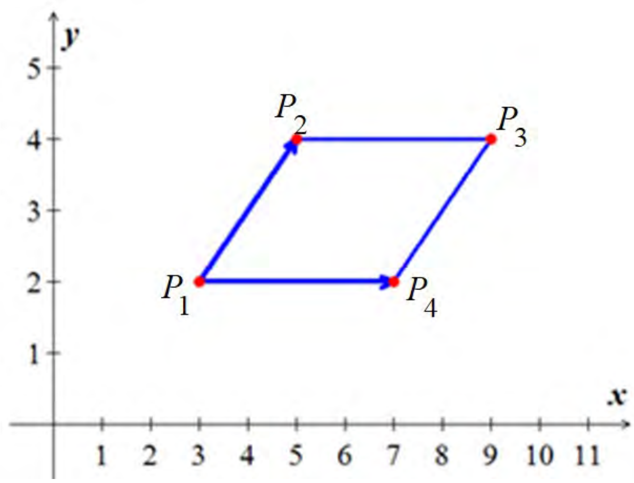
$$= \sqrt{(-6)^2 + 4^2 + 7^2}$$

$$\underline{= \sqrt{101}} \quad \left(unit^2 \right)$$

Exercise

Find the area of the parallelogram with the given vertices $P_1(3,2),\quad P_2(5,4),\quad P_3(9,4),\quad P_4(7,2)$

Solution



$$\overrightarrow{P_1P_2} = (5-3, 4-2) = (2,\ 2)$$

$$\overrightarrow{P_4P_3} = (9-7, 4-2) = (2,\ 2)$$

$$\overrightarrow{P_1P_4} = (7-3, 2-2) = (4,\ 0)$$

$$\overrightarrow{P_2P_3} = (9-5, 4-4) = (4,\ 0)$$

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_2} = (2,2) \times (4,0)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 0 \\ 4 & 0 & 0 \end{vmatrix}$$

$$= \left(\begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix},\ -\begin{vmatrix} 2 & 0 \\ 4 & 0 \end{vmatrix},\ \begin{vmatrix} 2 & 2 \\ 4 & 0 \end{vmatrix} \right)$$

$$= (0,\ 0,\ -8)$$

The area of the parallelogram is

$$\left\| \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_2} \right\| = \sqrt{0 + 0 + (-8)^2}$$

$$\underline{= 8}$$

Exercise

Find the area of the triangle with the given vertices:

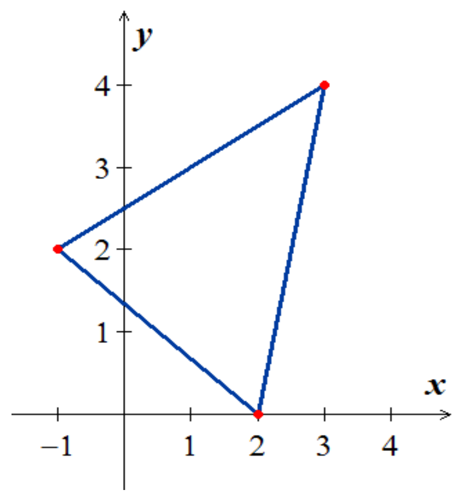
a) $A(2,\ 0)\quad B(3,\ 4)\quad C(-1,\ 2)$

b) $A(1,\ 1)\quad B(2,\ 2)\quad C(3,\ -3)$

c) $P(2,\ 6,\ -1)\quad Q(1,\ 1,\ 1)\quad R=(4,\ 6,\ 2)$

Solution

a) $\overrightarrow{AB}=(1,\ 4)\qquad \overrightarrow{AC}=(-3,\ 2)$



$$\overrightarrow{AB}\times\overrightarrow{AC}=(1,\ 4,\ 0)\times(-3,\ 2,\ 0)$$

$$=\left(\begin{vmatrix}4 & 0\\ 2 & 0\end{vmatrix},\ -\begin{vmatrix}1 & 0\\ -3 & 0\end{vmatrix},\ \begin{vmatrix}1 & 4\\ -3 & 2\end{vmatrix}\right)$$

$$=(0,\ 0,\ 14)$$

$$\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|=\sqrt{0+0+14^2}$$

$$=14$$

The area of the triangle is

$$\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|=\frac{1}{2}14$$

$$=7$$

b) $\overrightarrow{AB}=(1,\ 1)\qquad \overrightarrow{AC}=(2,\ -4)$

$$\overrightarrow{AB}\times\overrightarrow{AC}=(1,\ 1,\ 0)\times(2,\ -4,\ 0)$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 0\\ 2 & -4 & 0\end{vmatrix}$$

$$=\left(\begin{vmatrix}1 & 0\\ -4 & 0\end{vmatrix},\ -\begin{vmatrix}1 & 0\\ 2 & 0\end{vmatrix},\ \begin{vmatrix}1 & 1\\ 2 & -4\end{vmatrix}\right)$$

$$=(0,\ 0,\ -6)$$

$$\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|=\sqrt{0+0+(-6)^2}$$

$$=6$$

The area of the triangle is

$$\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|=\frac{1}{2}(6)$$

$$=3$$

c) $\overrightarrow{PQ}=(-1,\ -5,\ 2)\qquad \overrightarrow{PR}=(2,\ 0,\ 3)$

$$\overrightarrow{PQ}\times\overrightarrow{PR}=(-1,\ -5,\ 2)\times(2,\ 0,\ 3)$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & -5 & 2\\ 2 & 0 & 3\end{vmatrix}\qquad\qquad \begin{matrix}-1 & -5 & 2 & -1 & -5\\ 2 & 0 & 3 & 2 & 0\end{matrix}$$

$$=(-15,\ 7,\ 10)$$

$$\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\|=\sqrt{(-15)^2+7^2+10^2}$$

$$=\sqrt{374}$$

The area of the triangle is

$$\frac{1}{2}\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\|\ \underline{=\frac{1}{2}\sqrt{374}}\quad unit^2$$

Exercise

a) Find the area of the parallelogram with edges $v=(3,\ 2)$ and $w=(1,\ 4)$

b) Find the area of the triangle with sides $\vec{v}$, $\vec{w}$, and $\vec{v}+\vec{w}$. Draw it.

c) Find the area of the triangle with sides $\vec{v}$, $\vec{w}$, and $\vec{v}-\vec{w}$. Draw it.

Solution

a) $Area=\begin{vmatrix}3 & 1\\ 2 & 4\end{vmatrix}$

$$\underline{=10}$$

which is the parallelogram *OABC*

b) The area of the triangle with sides $\vec{v}$, $\vec{w}$, and $\vec{v}+\vec{w}$ is the triangle *OCB* or *OAB* which it is half the parallelogram (by definition).

$$Area=5$$

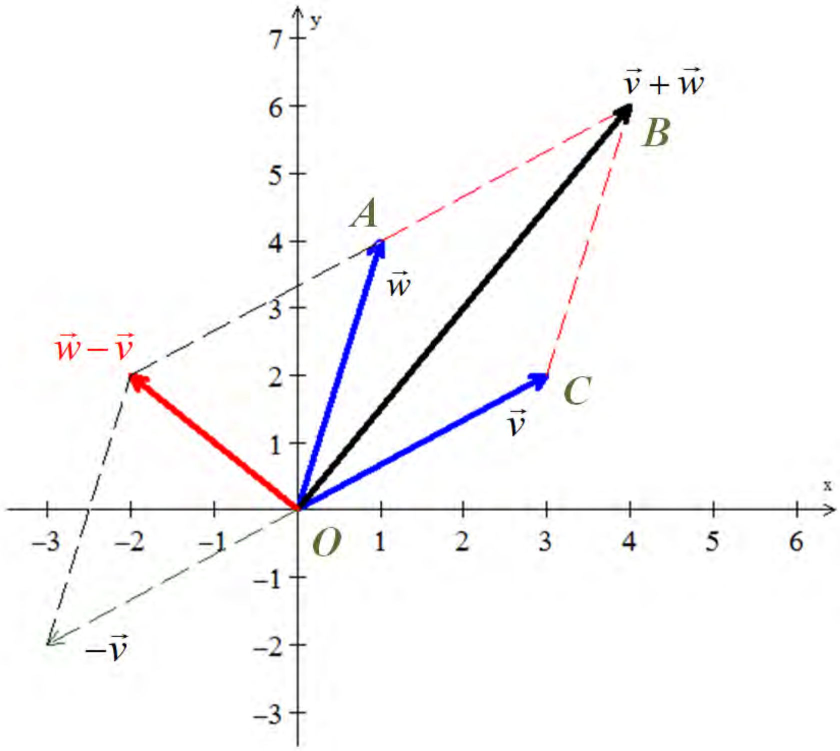
$$\vec{v}+\vec{w}=(3,\ 2)+(1,\ 4)$$

$$=(4,\ 6)$$

$$Area=\frac{1}{2}\begin{vmatrix}3 & 4\\ 2 & 6\end{vmatrix}$$

$$=\frac{1}{2}(10)$$

$$\underline{=5}$$



c) The area of the triangle with sides $\vec{v}$, $\vec{w}$, and $\vec{v}-\vec{w}$ is equivalent to the triangle *OAC* which it is half the parallelogram (by definition).

$$Area = 5$$

$$Area = \frac{1}{2}\begin{vmatrix} 2 & -2 \\ -3 & -2 \end{vmatrix}$$

$$= \frac{1}{2}|-10|$$

$$= 5$$

Exercise

Find the volume of the parallelepiped with sides $\vec{u}$, $\vec{v}$, and $\vec{w}$.

a) $\vec{u} = (2,\ -6,\ 2),\quad \vec{v} = (0,\ 4,\ -2),\quad \vec{w} = (2,\ 2,\ -4)$

b) $\vec{u} = (3,\ 1,\ 2),\quad \vec{v} = (4,\ 5,\ 1),\quad \vec{w} = (1,\ 2,\ 4)$

Solution

a) $$\vec{u} \bullet (\vec{v}\times\vec{w}) = \begin{vmatrix} 2 & -6 & 2 \\ 0 & 4 & -2 \\ 2 & 2 & -4 \end{vmatrix}$$

$$= -16$$

The volume of the parallelepiped is $|-16| = 16$ $unit^3$

b) $\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 3 & 1 & 2 \\ 4 & 5 & 1 \\ 1 & 2 & 4 \end{vmatrix}$

$= 45$

The volume of the parallelepiped is $45 \quad unit^3$

Exercise

Compute the scalar triple product $\vec{u} \cdot (\vec{v} \times \vec{w})$

a) $\vec{u} = (-2,\ 0,\ 6), \quad \vec{v} = (1,\ -3,\ 1), \quad \vec{w} = (-5,\ -1,\ 1)$

b) $\vec{u} = (-1,\ 2,\ 4), \quad \vec{v} = (3,\ 4,\ -2), \quad \vec{w} = (-1,\ 2,\ 5)$

c) $\vec{u} = (a,\ 0,\ 0), \quad \vec{v} = (0,\ b,\ 0), \quad \vec{w} = (0,\ 0,\ c)$

d) $\vec{u} = 3\hat{i} - 2\hat{j} - 5\hat{k}, \quad \vec{v} = \hat{i} + 4\hat{j} - 4\hat{k}, \quad \vec{w} = 3\hat{j} + 2\hat{k}$

e) $\vec{u} = (3,\ -1,\ 6) \quad \vec{v} = (2,\ 4,\ 3) \quad \vec{w} = (5,\ -1,\ 2)$

Solution

a) $\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} -2 & 0 & 6 \\ 1 & -3 & 1 \\ -5 & -1 & 1 \end{vmatrix}$

$= -92$

b) $\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} -1 & 2 & 4 \\ 3 & 4 & -2 \\ -1 & 2 & 5 \end{vmatrix}$

$= -10$

c) $\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$

$= abc$

d) $\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix}$

$= 49$

e) $\vec{u} \bullet (\vec{v} \times \vec{w}) = \begin{vmatrix} 3 & -1 & 6 \\ 2 & 4 & 3 \\ 5 & -1 & 2 \end{vmatrix}$

$$\underline{= -110}$$

Exercise

Use the cross product to find the sine of the angle between the vectors $\vec{u} = (2,\ 3,\ -6)$, $\vec{v} = (2,\ 3,\ 6)$

Solution

$$\vec{u} \times \vec{v} = (2,\ 3,\ -6) \times (2,\ 3,\ 6)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -6 \\ 2 & 3 & 6 \end{vmatrix}$$

$$= \left(\begin{vmatrix} 3 & -6 \\ 3 & 6 \end{vmatrix},\ -\begin{vmatrix} 2 & -6 \\ 2 & 6 \end{vmatrix},\ \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} \right)$$

$$\underline{= (36,\ -24,\ 0)}$$

$$\|\vec{u} \times \vec{v}\| = \sqrt{36^2 + (-24)^2 + 0}$$

$$= \sqrt{1872}$$

$$\underline{= 12\sqrt{13}}$$

$$\sin\theta = \left(\frac{\|\vec{u} \times \vec{v}\|}{\|\vec{u}\|\|\vec{v}\|} \right)$$

$$= \frac{12\sqrt{13}}{\sqrt{2^2 + 3^2 + (-6)^2}\sqrt{2^2 + 3^2 + 6^2}}$$

$$= \frac{12\sqrt{13}}{(7)(7)}$$

$$\underline{= \frac{12}{49}\sqrt{13}}$$

Exercise

Simplify $(\vec{u}+\vec{v})\times(\vec{u}-\vec{v})$

Solution

$$
\begin{aligned}
(\vec{u}+\vec{v})\times(\vec{u}-\vec{v}) &= (\vec{u}+\vec{v})\times\vec{u}-(\vec{u}+\vec{v})\times\vec{v} \\
&= (\vec{u}\times\vec{u})+(\vec{v}\times\vec{u})-\left[(\vec{u}\times\vec{v})+(\vec{v}\times\vec{v})\right] \\
&= 0+(\vec{v}\times\vec{u})-\left[(\vec{u}\times\vec{v})+0\right] \\
&= (\vec{v}\times\vec{u})-(\vec{u}\times\vec{v}) \\
&= (\vec{v}\times\vec{u})-\left(-(\vec{v}\times\vec{u})\right) \\
&= (\vec{v}\times\vec{u})+(\vec{v}\times\vec{u}) \\
&= 2(\vec{v}\times\vec{u})
\end{aligned}
$$

Exercise

Prove Lagrange's identity: $\|\vec{u}\times\vec{v}\|^2=\|\vec{u}\|^2\ \|\vec{v}\|^2-(\vec{u}\bullet\vec{v})^2$

Solution

Let $\vec{u}=(u_1,\ u_2,\ u_3)$ and $\vec{v}=(v_1,\ v_2,\ v_3)$

$$\|\vec{u}\|^2=u_1^2+u_2^2+u_3^2$$

$$\|\vec{v}\|^2=v_1^2+v_2^2+v_3^2$$

$$(\vec{u}\bullet\vec{v})^2=(u_1v_1+u_2v_2+u_3v_3)^2$$

$$
\begin{aligned}
\|\vec{u}\times\vec{v}\|^2 &= (u_2v_3-u_3v_2)^2+(u_3v_1-u_1v_3)^2+(u_1v_2-u_2v_1)^2 \\
&= u_2^2v_3^2-2u_2v_3u_3v_2+u_3^2v_2^2+u_3^2v_1^2-2u_3v_1u_1v_3+u_1^2v_3^2+u_1^2v_2^2-2u_2v_1u_2v_1+u_2^2v_1^2
\end{aligned}
$$

$$\|\vec{u}\|^2\ \|\vec{v}\|^2-(\vec{u}\bullet\vec{v})^2=\left(u_1^2+u_2^2+u_3^2\right)\left(v_1^2+v_2^2+v_3^2\right)-(u_1v_1+u_2v_2+u_3v_3)^2$$

$$= u_1^2 v_1^2 + u_1^2 v_2^2 + u_1^2 v_3^2 + u_2^2 v_1^2 + u_2^2 v_2^2 + u_2^2 v_3^2 + u_3^2 v_1^2 + u_3^2 v_2^2 + u_3^2 v_3^2$$

$$-u_1^2 v_1^2 - u_1 v_1 u_2 v_2 - u_1 v_1 u_3 v_3$$

$$-u_2 v_2 u_1 v_1 - u_2^2 v_2^2 - u_2 v_2 u_3 v_3$$

$$-u_1 v_1 u_3 v_3 - u_2 v_2 u_3 v_3 - u_3^2 v_3^2$$

$$= u_2^2 v_3^2 - 2u_2 v_2 u_3 v_3 + u_3^2 v_2^2$$

$$+ u_3^2 v_1^2 - 2u_1 v_1 u_3 v_3 + u_1^2 v_3^2$$

$$+ u_1^2 v_2^2 - 2u_1 v_1 u_2 v_2 + u_2^2 v_1^2$$

$$\Rightarrow \ \|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \ \|\vec{v}\|^2 - (\vec{u} \bullet \vec{v})^2$$

Exercise

Polar coordinates satisfy $x = r\cos\theta \quad and \quad y = \sin\theta$. Polar area $J\ dr\ d\theta$ includes J:

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

The two columns are orthogonal. Their lengths are ______. Thus J = ______.

Solution

The length of the first column is:

$$\ell_1 = \sqrt{\cos^2\theta + \sin^2\theta}$$

$$= 1$$

The length of the second column is:

$$\ell_2 = \sqrt{r^2\sin^2\theta + r^2\cos^2\theta}$$

$$= \sqrt{r^2\left(\sin^2\theta + \cos^2\theta\right)}$$

$$= \sqrt{r^2}$$

$$= r$$

So, J is the product $1.\ r = r$.

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$

$$= r\left(\cos^2\theta + \sin^2\theta\right)$$

$$\underline{= r}$$

Exercise

Prove that $\|\vec{u}+\vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$ if and only if $\vec{u}$ and $\vec{v}$ are parallel vectors.

Solution

If $\vec{u}$ and $\vec{v}$ are parallel vectors, then $\vec{u}\times\vec{v} = 0$

Which the two vectors are collinear, which implies that $\vec{u} = a\vec{v}$

$$\begin{aligned} \|\vec{u}+\vec{v}\| &= \|\vec{u}+a\vec{u}\| \\ &= \|(1+a)\vec{u}\| \\ &= (1+a)\|\vec{u}\| \\ &= \|\vec{u}\| + a\|\vec{u}\| \\ &= \|\vec{u}\| + \|a\vec{u}\| \\ &= \|\vec{u}\| + \|\vec{v}\| \quad \checkmark \end{aligned}$$

Exercise

State the following statements as True or False

a) The cross product of two nonzero vectors $\vec{u}$ and $\vec{v}$ is a nonzero vector if and only if $\vec{u}$ and $\vec{v}$ are not parallel.

b) A normal vector to a plane can be obtained by taking the cross product of two nonzero and noncollinear vectors lying in the plane.

c) The scalar triple product of $\vec{u}$, $\vec{v}$, and $\vec{w}$ determines a vector whose length is equal to the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$.

d) If $\vec{u}$ and $\vec{v}$ are vectors in 3-space, then $\|\vec{u}\times\vec{v}\|$ is equal to the area of the parallelogram determine by $\vec{u}$ and $\vec{v}$.

e) For all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in R^3, the vectors $(\vec{u}\times\vec{v})\times\vec{w}$ and $\vec{u}\times(\vec{v}\times\vec{w})$ are the same.

f) If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in R^3, where $\vec{u}$ is nonzero and $\vec{u}\times\vec{v} = \vec{u}\times\vec{w}$, then $\vec{v} = \vec{w}$

Solution

a) True,

$\vec{u}\times\vec{v} = \|\vec{u}\|\ \|\vec{v}\|\sin\theta = 0$ if $\theta = 0$ which the two vectors are parallel.

b) True;

The cross product of two nonzero and non collinear vectors will be perpendicular to both vectors, hence normal to the plane containing the vectors.

c) False;
The scalar triple product is a scalar, not a vector.

d) True;

e) False;
Let $\vec{u} = \hat{i} \quad \vec{v} = \vec{w} = \hat{j}$

$$\begin{aligned}(\vec{u} \times \vec{v}) \times \vec{w} &= (\hat{i} \times \hat{j}) \times \hat{j} \\ &= \hat{k} \times \hat{j} \\ &\boxed{= -\hat{i}}\end{aligned}$$

$$\begin{aligned}\vec{u} \times (\vec{v} \times \vec{w}) &= \hat{i} \times (\hat{j} \times \hat{j}) \\ &= \hat{i} \times \vec{0} \\ &\boxed{= \vec{0}}\end{aligned}$$

Hence, $(\vec{u} \times \vec{v}) \times \vec{w} \neq (\vec{u} \times \vec{v}) \times \vec{w}$

f) False;
Let $\vec{u} = \hat{i} + \hat{j} \quad \vec{v} = \hat{i} + \hat{j} + \hat{k} \quad \vec{w} = -\hat{i} - \hat{j} + \hat{k}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\boxed{= \hat{i} - \hat{j}}$$

$$\vec{u} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{vmatrix}$$

$$\boxed{= \hat{i} - \hat{j}}$$

$\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$, but $\vec{v} \neq \vec{w}$