Section 1.3 – Solving Linear Programming and Applications

These approaches help management to make a decision

1) Graph the region (feasible region).
2) Identify the corner points.
3) Evaluate the objective function at each corner point.
4) interpret the optimal solution {maximum or minimum and where it occurs}

A solution region of a system of linear inequalities

Bounded: if it can be enclosed within circle

Unbounded: ∞

Corner point is a point in the feasible region where the boundary lines are intersected.

If an optimum value (either maximum or minimum) of the objective function exists, it will occur at one or more of the corner points of the feasible region.

At least = Min. is $\geq$

At the most = Max $\leq$

Maximization

Example

Find the maximum value of the objective function $z = 3x + 4y$ subject to the following constraints

$$\begin{cases} 2x + y \leq 4 & (1) \\ -x + 2y \leq 4 & (2) \\ x, y \geq 0 & (3) \end{cases}$$

$$(1) \cap (2) \quad \rightarrow \quad A = \left(\frac{4}{5},\ \frac{12}{5}\right)$$

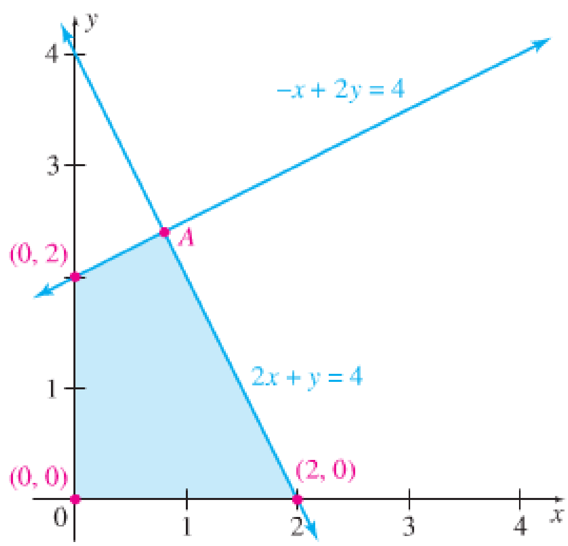
$$(1) \cap x-axis \quad \rightarrow \quad (2,\ 0)$$

$$(2) \cap y-axis \quad \rightarrow \quad (0,\ 2)$$

$$A = \left(\frac{4}{5},\ \frac{12}{5}\right) \Rightarrow z = 3x + 4y$$

$$= 3\frac{4}{5} + 4\frac{12}{5}$$

$$= 12$$



The corner point A leads to the largest value of z.

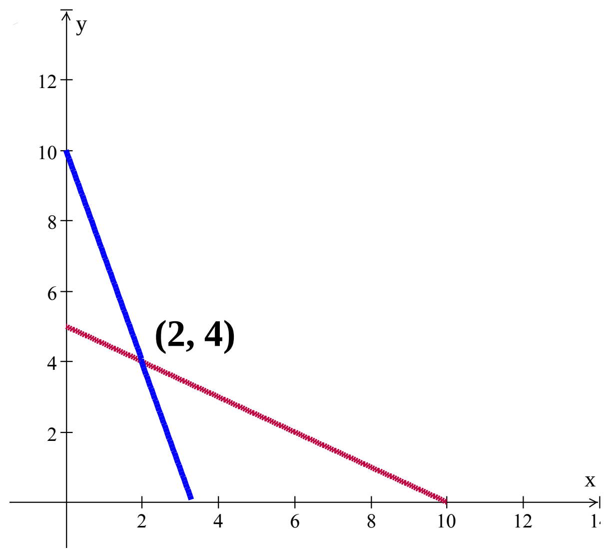
The feasible region is bounded

Minimization

Example

Solve the following linear programming problem

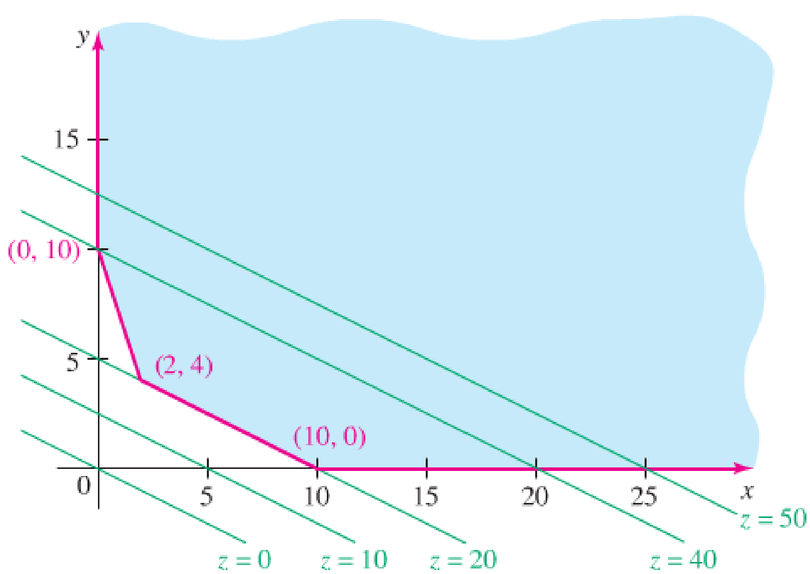
$$\begin{aligned} &\textit{Minimize} && z = 2x + 4y \\ &\textit{Subject to} && x + 2y \ge 10 \\ & && 3x + y \ge 10 \\ & && x, y \ge 0 \end{aligned}$$



The minimum value: $z = 2x + 4y$

$$= 2(2) + 4(4)$$

$$= 20$$



The feasible region is unbounded

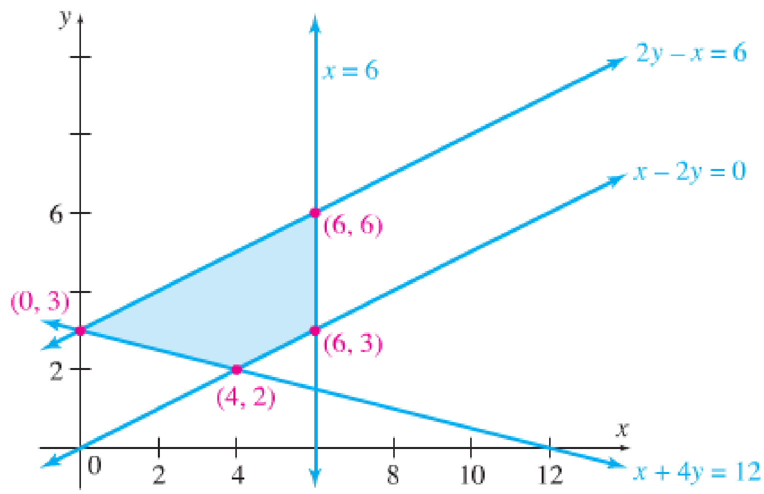
Maximization and Minimization

Example

Solve the following linear programming problem

$$z = x + 10y$$

$$\text{Subject to} \quad \begin{aligned} x + 4y &\ge 12 \\ x - 2y &\le 0 \\ -x + 2y &\le 6 \\ x &\le 6 \end{aligned}$$



At $(0,\ 3) \Rightarrow z = 0 + 10(3) = 30$

At $(4,\ 2) \Rightarrow z = 4 + 10(2) = 24$

At $(6,\ 3) \Rightarrow z = 6 + 10(3) = 36$

At $(6,\ 6) \Rightarrow z = 6 + 10(6) = 66$

The minimum value of ***z*** is 24 at the corner point (4, 2).

The maximum value of ***z*** is 66 at the corner point (6, 6).

Example

Mr. Trenga plans to start a new business called River Explorers, which will rent canoes and kayaks to people to travel 10 miles down the Clarion River in Cook Forest State Park. He has \$45,000 to purchase new boats. He can buy the canoes for \$600 each and the kayaks for \$750 each. His facility can hold up to 65 boats. The canoes will rent for \$25 a day and the kayaks will rent for \$30 a day. How many canoes and how many kayaks should he buy to earn the most revenue?

<u>Solution</u>

x: represents the number of canoes

y : represents the number of kayaks

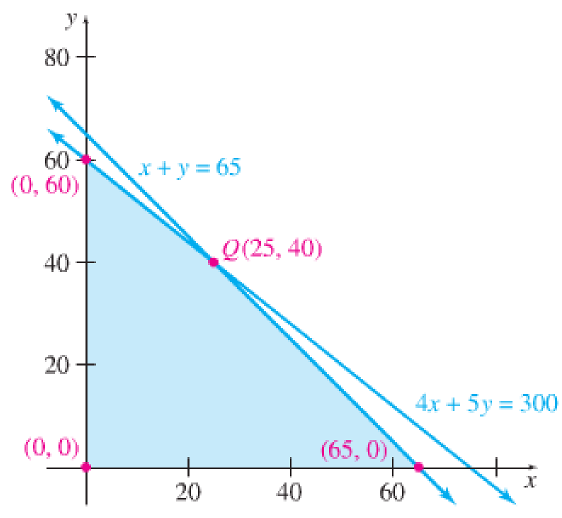
	Canoes	***Kayaks***		***Total***
	x	***y***	$\le$	65
Cost of Each	600	750	$\le$	45,000
Revenue	25	30		

$$\begin{cases} x+y \le 65 \\ 600x+750y \le 45{,}000 \quad \textit{divide both sides by } 150 \end{cases}$$

$$\begin{cases} x+y \le 65 \\ 4x+5y \le 300 \end{cases}$$

The mathematical model for this problem for the given linear programming problem is as follows

$$\begin{array}{lll} \textit{Maximize} & z = 25x+30y & (1) \\ \textit{Subject to} & x+y \le 65 & (2) \\ & 4x+5y \le 300 & (3) \\ & x, y \ge 0 & \end{array}$$



The corner points are

$$(0,\ 0)$$

$(2)\cap(3) \quad \to \ (25,\ 40)$

$(3)\cap x-axis \ \to \ (65,\ 0)$

$(2)\cap y-axis \ \to \ (0,\ 60)$

Corner points	***z = 25x +30y***
(0, 0)	z = 25(0) +30(0) **= 0**
(25, 40)	z = 25(25) +30(40) = **1825**
(65, 0)	z = 25(65) +30(0) = **1625**
(0, 60)	z = 25(0) +30(60) = **1800**

The objective function, which represents revenue, is maximized when x = 25 and y = 40. He should buy 25 canoes and 40 kayaks.

Example

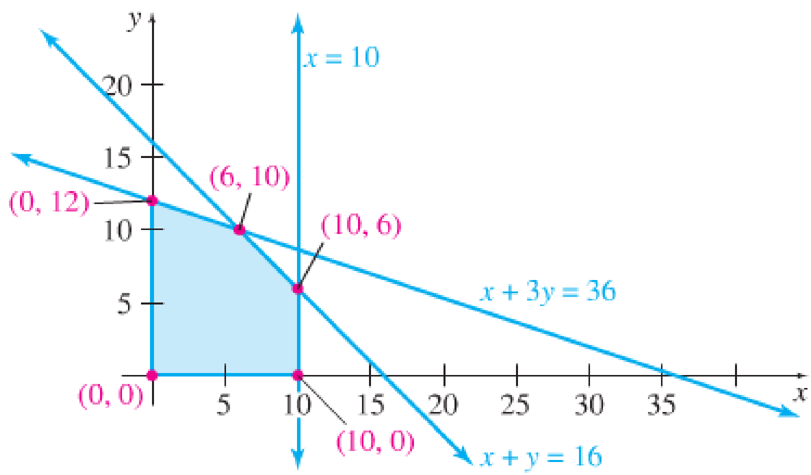
A 4-H member raises only goats and pigs. She wants to raise no more than 16 animals, including no more than 10 goats. She spends $25 to raise a goat and $75 to raise a pig, and she has $900 available for the project. The 4-H member wishes to maximize her profits. Each goat produces $12 in profit and each pig $40 in profit

<u>*Solution*</u>

	Goats	***Pigs***		***Total***
Raised	***x***	***y***	$\le$	16
Goat Limit	1		$\le$	10
Cost	$25	$75	$\le$	$900
Profit	$12	$40		

$$25x+75y\le 900 \quad \Rightarrow \quad x+3y\le 36$$

$$\begin{array}{lll} \textit{Maximize} & z=12x+40y & (1) \\ \textit{Subject to} & x+y\le 16 & (2) \\ & x\le 10 & (3) \\ & x+3y\le 36 & (4) \\ & x,y\ge 0 & \end{array}$$



The corner points are

$$(0,\ 0)$$

$$(2)\cap(3) \quad \to \quad (10,\ 6)$$

$$(2)\cap(4) \quad \to \quad (6,\ 10)$$

$$(2)\cap x-axis \quad \to \quad (10,\ 0)$$

$$(4)\cap y-axis \quad \to \quad (0,\ 12)$$

Corner points	***z = 12x + 40y***
(0, 0)	z = 12(0) + 40(0) = **0**
(10, 6)	z = 12(10) + 40(6) = **360**
(6, 10)	z = 12(6) + 40(10) = **472**
(10, 0)	z = 12(10) + 40(0) = **120**
(0, 12)	z = 12(0) + 40(12) = **480**

Therefore, 12 pigs and no goats will produce a maximum profit of $480.

Example

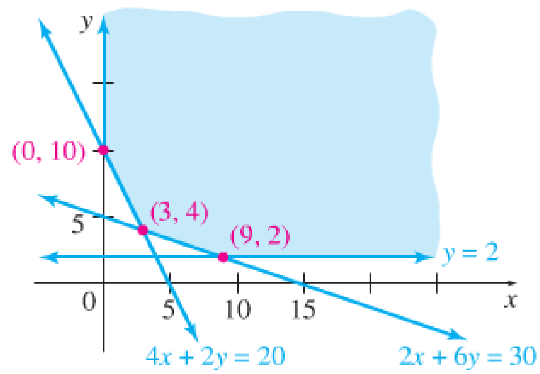
Certain animals in a rescue shelter must have at least 30 *g* of protein and at least 20 *g* of fat per feeding period. These nutrients come from food *A*, which costs 18 cents per unit and supplies 2 *g* of protein and 4 *g* of fat; and food *B*, which costs 12 cents per unit and supplies 6 *g* of protein and 2 *g* of fat. Food *B* is bought under a long-term contract requiring that at least 2 units of *B* be used per serving.

How much of each food must be bought to produce the minimum cost per serving?

<u>Solution</u>

	Food A	***Food B***		***Total***
	x	***y***		
Proteins	2	6	≥	30
Fat	4	2	≥	20
Long-Term Contract		1	≥	2
Cost	18	12		

$$
\begin{array}{lrl}
\textit{Minimize} & z = 0.18x + 0.12y & (1) \\
\textit{Subject to} & 2x + 6y \ge 30 & (2) \\
 & 4x + 2y \ge 20 & (3) \\
 & y \ge 2 & (4) \\
 & x \ge 0 & (5)
\end{array}
$$



The corner points are

$(2)\cap(3) \quad \rightarrow \quad (3,\ 4)$

$(2)\cap(4) \quad \rightarrow \quad (9,\ 2)$

$(3)\cap y-axis \quad \rightarrow \quad (0,\ 10)$

Corner points	***z = 0.18x + 0.12y***
(3, 4)	z = 0.18(3) + 0.12(4) = **1.02**
(9, 2)	z = 0.18(9) + 0.12(2) = **1.86**
(0, 10)	z = 0.18(0) + 0.12(10) = **1.20**

Therefore, 3 units of food *A* and 4 units of food *B* will produce the minimum cost of $1.02 per serving.

Exercises Section 1.3 – Solving Linear Programming and Applications

1. Maximize and minimize $z = 4x + 2y$ subject to the constraints

$$\begin{cases} 2x + y \le 20 & (1) \\ 10x + y \ge 36 & (2) \\ 2x + 5y \ge 36 & (3) \\ x, y \ge 0 \end{cases}$$

2. A manufacturing plant makes two types of inflatable boats, a two-person boat and a four-person boat. Each two-person boat requires 0.9 labor-hour in the cutting department and 0.8 labor-hour in the assembly department. Each four-person boat requires 1.8 labor-hours in the cutting department and 1.2 labor-hours in the assembly department. The maximum labor-hours available each month in the cutting and assembly departments are 864 and 672, respectively. The company makes a profit of $25 on each two-person boat and $40 on each four-person boat

a) Identify the decision variables
b) Summarize the relevant material in a table
c) Write the objective function P.
d) Write the problem constraints and the nonnegative constraints
e) Determine how many boats should be manufactured each month to maximize the profit. What is the maximum profit?

3. A chicken farmer can buy a special food mix A at 20¢ per pound and a special food mix B at 40¢ per pound. Each pound of mix A contains 3,000 units of nutrient N_1 and 1,000 units of nutrient N_2, and Each pound of mix B contains 4,000 units of nutrient N_1 and 4,000 units of nutrient N_2. If the minimum daily requirements for the chickens collectively are 36,000 units of nutrient N_1 and 20,000 units of nutrient N_2, how many pounds of each food mix should be used each day to minimize daily food costs while meeting (or exceeding) the minimum daily nutrient requirements? What is the minimum daily cost? Construct a mathematical model and solve using the geometric method.

4. A company produces small engines for several manufacturers. The company receives orders from two assembly plants for their Top-flight engine. Plant I needs at least 45 engines, and plant II needs at least 32 engines. The company can send at most 90 engines to these two assembly plants. It costs $30 per engine to ship to plant I and $40 per engine to ship to plant II. Plant I gives the company $20 in rebates toward its products for each engine they buy, while plant II gives similar $15 rebates. The company estimates that they need at least $1200 in rebates to cover products they plan to buy from the two plants. How many engines should be shipped to each plant to minimize shipping costs? What is the minimum cost?

5. The Muro Manufacturing Company makes two kinds of plasma screen TV sets. It produces the Flexscan set that sells for $350 profit and the Panoramic I that sells for $500 profit. On the assembly line, the Flexscan requires 5 hours, and the Panoramic I takes 7 hours. The cabinet shop spends 1 hour on the cabinet for the Flexscan and 2 hours on the cabinet for the Panoramic I. Both sets require 4 hours for testing and packing. On a particular production run, the Muro Company has available 3600 work-hours on the assembly line, 900 work-hours in the cabinet shop, and 2600 work-hours in the testing and packing department. How many sets of each type should it produce to make a maximum profit? What is the maximum profit?

6. The manufacturing process requires that oil refineries must manufacture at least 2 gal of gasoline for every gallon of fuel oil. To meet the winter demand for fuel oil, at least 3 million gal a day must be produced. The demand for gasoline is no more than 6.4 million gal per day. It takes 0.25 hour to ship each million gal of gasoline and 1 hour to ship each million gal of fuel oil out of the warehouse. No more than 4.65 hours are available for shipping. If the refinery sells gasoline for $2.50 per gal and fuel oil for $2 per gal, how many of each should be produced to maximize revenue? Find the maximum revenue.

7. A small country can grow only two crops for export, coffee and cocoa. The country has 500,000 hectares of land available for the crops. Long-term contracts require that at least 100,000 hectares be devoted to coffee and at least 200,000 hectares to cocoa. Cocoa must be processed locally, and production bottlenecks limit cocoa to 270,000 hectares. Coffee requires two workers per hectare, with cocoa requiring five. No more than 1,750,000 people are available for working with these corps. Coffee produces a profit of $220 per hectares and cocoa a profit of 4550 per hectare. How many hectares should the country devote to each crop in order to maximize profit? Find the maximum profit.

8. A pension fund manager decides to invest a total of at most $39 million in U.S. treasury bonds paying 4% annual interest and in mutual funds paying 8% annual interest. He plans to invest at least $5 million in bonds and at least $10 million in mutual funds. Bonds have an initial fee of $100 per million dollars, while the fee for mutual funds is $200 per million. The fund manager is allowed to spend no more than $5000 on fees. How much should be invested in each to maximize annual interest? What is the maximum annual interest?

9. Mark, who is ill, takes vitamin pills. Each day he must have at least 16 units of vitamin A, 5 units of vitamin B, and 20 units of vitamin C. he can choose between pill 1, which contains 8 units of A, 1 of B, and 2 of C; and pill 2, which contains 2 units of A, 1 of B, and 7 of C. Pill 1 costs 15¢, and pill 2 costs 30¢.

a) How many of each pill should be buy in order to minimize his cost?

b) What is the minimum cost?

c) For the solution in part a, Mark is receiving more than he needs of at least one vitamin. Identify that vitamin, and tell how much surplus he is receiving. Is there any ways he can avoid receiving that surplus while still meeting the other constraints and minimizing the cost?

10. A certain predator requires at least 10 units of protein and 8 units of fat per day. One prey of species I provides 5 units of protein and 2 units of fat; one prey of species II provides 3 units of protein and 4 units of fat. Capturing and digesting each species-II prey requires 3 units of energy, and capturing and digesting each species-I prey requires 2 units of energy. How many of each prey would meet the predator's daily food requirements with the least expenditure of energy?

11. A dietician is planning a snack package of fruit and nuts. Each ounce of fruit will supply zero units of protein, 2 units of carbohydrates, and 1 unit of fat, and will contain 20 calories. Each ounce of nuts will supply 3 units of protein, 1 unit of carbohydrates, and 2 units of fat, and will contain 30 calories. Every package must provide at least 6 units of protein, at least 10 units of carbohydrates, and no more than 9 units of fat. Find the number of ounces of fruit and number of ounces of nuts that will meet the requirement with the least number of calories. What is the least number of calories?

12. An anthropology article presents a hypothetical situation that could be described by a linear programming model. Suppose a population gathers plants and animals for survival. They need at least 360 units of energy, 300 units of protein, and 8 hides during some time period. One unit of plants provides 30 units of energy, 10 units of protein, and no hides. One animal provides 20 units of energy, 25 units of protein, and 1 hide.

13. In a small town in South Carolina, zoning rules require that the window space (in square feet) in a house be at least one-sixth of the space used up by solid walls. The cost to build windows is \$10 per ft^2, while the cost to build solid walls is \$20 per ft^2. The total amount available for building walls and windows is no more than \$12,000. The estimated monthly cost to heat the house is \$0,32 for each square foot of windows and \$0.20 for each square foot of solid walls. Find the maximum total area (windows plus walls) if no more than \$160 per month is available to pay for heat.

14. A manufacturing company makes two types of water skis, a trick ski and a slalom ski. The trick ski requires 9 labor-hours for fabricating and 1 labor-hour for finishing. The slalom ski requires 5 labor-hours for fabricating and 1 labor-hour for finishing. The maximum labor-hours available per day for fabricating and finishing are 135 and 20 respectively. If x is the number of trick skis and y is the number of slalom skis produced per day, write a system of linear inequalities that indicates appropriate restraints on x and y. Find the set of feasible solutions graphically for the number of each type of ski that can be produced.