Lecture Four - Integration

Section 4.1 – Antiderivatives

Antiderivatives

$$f(x) = x^3 \qquad \Rightarrow \qquad f'(x) = 3x^2$$

Definition of Antiderivatives

A Function F is an ***antiderivative*** of a function f on an interval I if

$$F'(x) = f(x) \qquad \text{for all } x \text{ in } I.$$

Theorem

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

Where C is an arbitrary constant.

Notation for Antiderivatives and indefinite integrals

The notation $\quad \int f(x)dx = F(x) + C$

where C is an arbitrary constant, means that F is an Antiderivative of f.

That is $F'(x) = f(x)$ for all x in the domain of f.

$\int f(x)dx$ Indefinite integral

Integrand

Integral sign

$$\int f(x)dx = \overbrace{F(x) + C}^{\text{Antiderivative}}$$

Differential

Basic Integration Rules

$$\int k\,dx = kx + C$$

$$\int kf(x)\,dx = k\int f(x)\,dx$$

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

Example

Find each indefinite integral.

a) $$\int \frac{1}{x^2}dx = \int x^{-2}dx$$

$$= \frac{x^{-2+1}}{-2+1} + C$$

$$= \frac{x^{-1}}{-1} + C$$

$$= -\frac{1}{x} + C$$

b) $$\int \sqrt[3]{x}\,dx = \int x^{1/3}dx$$

$$= \frac{x^{1/3+1}}{1/3+1} + C$$

$$= \frac{x^{4/3}}{4/3} + C$$

$$= \frac{3}{4}x^{4/3} + C \qquad or \qquad = \frac{3}{4}x\sqrt[3]{x} + C$$

Example

Evaluate $\int \left(4x^3 - 5x + 2\right) dx$

Solution

$$\int (4x^3 - 5x + 2) dx = \int 4x^3 dx - \int 5x dx + \int 2 dx$$

$$= 4\frac{x^4}{4} - 5\frac{x^2}{2} + 2x + C$$

$$\boxed{= x^4 - \frac{5}{2}x^2 + 2x + C}$$

Example

Evaluate $\int \left(x^2 - 2x + 5\right) dx$

Solution

$$\int \left(x^2 - 2x + 5\right) dx = \boxed{\frac{x^3}{3} - x^2 + 5x + C}$$

Example

Evaluate $\int \sin x \, dx$

Solution

$$\int \sin x dx = \boxed{-\cos x + C}$$

Example

Evaluate $\int \cos 3x \, dx$

Solution

$$\int \cos 3x \, dx = \boxed{\frac{1}{3}\sin 3x + C}$$

Definition

The ***natural logarithm*** is the function given by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0$$

Zero width: $\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$

Definition

The ***number e*** is that number in the domain of the natural logarithm satisfying

$$\ln e = 1 \quad \text{and} \quad \int_1^e \frac{1}{t}\,dt = 1$$

Other Indefinite Integrals

$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax} \quad \rightarrow \quad \int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x} \quad \rightarrow \quad \int \frac{dx}{x} = \ln|x| + C$$

$$\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{a^2 - x^2}} \quad \rightarrow \quad \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{a}{x^2 + a^2} \quad \rightarrow \quad \int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\frac{d}{dx}\left(\sec^{-1}\left|\frac{x}{a}\right|\right) = \frac{a}{x\sqrt{x^2 - a^2}} \quad \rightarrow \quad \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$$

Example

Evaluate $\int e^{-10x}\,dx$

Solution

$$\int e^{-10x}\,dx = -\frac{1}{10}e^{-10x} + C$$

Example

Evaluate $\displaystyle\int \frac{5}{x}dx$

Solution

$$\int \frac{5}{x}dx = \boxed{5\ln|x| + C}$$

Example

Evaluate $\displaystyle\int \frac{4}{\sqrt{9-x^2}}dx$

Solution

$$\int \frac{4}{\sqrt{9-x^2}}dx = \boxed{4\sin^{-1}\left(\frac{x}{3}\right) + C} \qquad a^2 = 9 \quad \rightarrow \quad a = 3$$

Example

Evaluate $\displaystyle\int \frac{dx}{16x^2+1}$

Solution

$$\begin{aligned}\int \frac{dx}{16x^2+1} &= \frac{1}{16}\int \frac{dx}{x^2+\frac{1}{16}} \\ &= \frac{1}{16}\int \frac{dx}{x^2+\left(\frac{1}{4}\right)^2} \\ &= \left(\frac{1}{16}\right)4\tan^{-1}(4x) + C \\ &= \boxed{\frac{1}{16}\tan^{-1}(4x) + C}\end{aligned}$$

Exercises *Section* 4.1 – Antiderivatives

Find each indefinite integral.

1. $\int v^2 dv$

2. $\int x^{1/2} dx$

3. $\int 4y^{-3} dy$

4. $\int (x^3 - 4x + 2) dx$

5. $\int \left(3z^2 - 4z + 5\right) dz$

6. $\int \left(x^2 - 1\right)^2 dx$

7. $\int \frac{x^2 + 1}{\sqrt{x}} dx$

8. $\int \left(\sqrt[4]{x^3} + 1\right) dx$

9. $\int \sqrt{x}(x+1) dx$

10. $\int (1+3t)t^2 dt$

11. $\int \frac{x^2 - 5}{x^2} dx$

12. $\int (-40x + 250) dx$

13. $\int \frac{x+2}{\sqrt{x}} dx$

14. $\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx$

15. $\int \left(\sqrt{x} + \sqrt[3]{x}\right) dx$

16. $\int 2x\left(1 - x^{-3}\right) dx$

17. $\int \left(\frac{4 + \sqrt{t}}{t^3}\right) dt$

18. $\int (-2\cos t) dt$

19. $\int 7\sin\frac{\theta}{3} d\theta$

20. $\int \frac{2}{5}\sec\theta\tan\theta d\theta$

21. $\int \left(4\sec x\tan x - 2\sec^2 x\right) dx$

22. $\int (2\cos 2x - 3\sin 3x) dx$

23. $\int \left(1 + \tan^2\theta\right) d\theta$

24. $\int \frac{\csc\theta}{\csc\theta - \sin\theta} d\theta$

25. $\int \left(2e^x - 3e^{-2x}\right) dx$

26. $\int \frac{dx}{\sqrt{9 - x^2}}$

27. $\int \frac{dx}{9 + 3x^2}$

28. $\int \frac{4x^2 - 3x + 2}{x^2} dx$

29. $\int \left(x^8 - 3x^3 + 1\right) dx$

30. $\int (2x+1)^2 dx$

31. $\int \frac{x+1}{x} dx$

32. $\int \left(\frac{1}{x^2} - \frac{2}{x^{5/2}}\right) dx$

33. $\int \frac{x^4 - 2\sqrt{x} + 2}{x^2} dx$

34. $\int (1 + \cos 3\theta) d\theta$

35. $\int 2\sec^2\theta \, d\theta$

36. $\int \sec 2x \tan 2x \, dx$

37. $\int 2e^{2x} dx$

38. $\int \frac{12}{x} dx$

39. $\int \frac{dx}{\sqrt{1 - x^2}}$

40. $\int \frac{dx}{x^2 + 1}$

41. $\int \frac{1 + \tan\theta}{\sec\theta} d\theta$

42. $\int \left(\sqrt[4]{x^3} + \sqrt{x^5}\right) dx$

43. $\int \left(\sqrt[4]{x^3} + 1 \right) \, dx$

44. $\int \left(5x^4 + 3x^2 + 2x + 5 \right) \, dx$

45. $\int \left(5x^{4/3} + 3x^{2/3} + 2x^{1/3} \right) \, dx$

46. $\int \left(5x^{-4/3} + 3x^{-2/3} + 2x^{-1/3} \right) \, dx$

47. $\int \frac{x^4 - 3x^2 + 5}{x^4} \, dx$

48. $\int \left(\frac{3}{x^7} - \frac{5}{x^6} \right) \, dx$

49. $\int \frac{x + 8}{\sqrt{x}} \, dx$

50. $\int \frac{x^2 + 8}{\sqrt[3]{x}} \, dx$

51. $\int \cos\left(\frac{5\pi}{3} x \right) \, dx$

52. $\int \sin\left(\frac{2x}{3} \right) \, dx$

53. $\int \left(5\cos x + 4\sin x + 3\sec^2 x \right) \, dx$

54. $\int \sec\theta \left(\sec\theta + \tan\theta \right) \, d\theta$

55. $\int \left(\tan^2\theta + 1 \right) \, d\theta$

56. $\int \left(\cos^4\theta - \sin^4\theta \right) \, d\theta$

57. $\int \left(\cos^2\theta - \sin^2\theta \right) \, d\theta$

58. $\int \left(\cos^2\theta + \sin^2\theta \right) \, d\theta$

59. $\int \left(\cos 2x \cos 4x - \sin 2x \sin 4x \right) \, dx$

60. $\int \left(\sin 2x \cos 4x - \cos 2x \sin 4x \right) \, dx$

61. $\int \left(\sin 3x \cos 2x + \cos 3x \sin 2x \right) \, dx$

62. $\int \cos 2x \sin 2x \, dx$

63. $\int \left(2\cos^2 x - 1 \right) \, dx$

64. $\int \left(1 - 2\sin^2 x \right) \, dx$

65. $\int e^{-5x} \, dx$

66. $\int 4e^{4x} \, dx$

67. $\int \left(2\sin\theta - 5e^{\theta} \right) \, d\theta$

68. $\int \left(\frac{3}{x} + \sec^2 x \right) \, dx$

69. $\int \left(\sin x + 2^x \right) \, dx$

70. $\int \left(2x - 3^x \right) \, dx$

71. $\int \left(4x - \frac{3}{x} - \csc^2 x \right) \, dx$

72. $\int \left(e^{4x} - \frac{3}{x} + 2\csc x \cot x \right) \, dx$

73. $\int (a + b) e^{(a+b)x} \, dx$

74. $\int \left(a^2 - b^2 \right) e^{(a-b)x} \, dx$

Find the function with the following property

75. $\frac{dy}{dx} = 2x - 7, \quad y(2) = 0$

76. $\frac{dy}{dx} = \frac{1}{x^2} + x, \quad y(2) = 1; \ x > 0$

77. $\frac{ds}{dt} = 1 + \cos t, \quad s(0) = 4$

78. $\frac{ds}{dt} = \cos t + \sin t, \quad s(\pi) = 1$

79. $f'(x) = 3x^2 - 1 \quad \& \ f(0) = 10$

80. $f'(t) = \sin t + 2t \quad \& \ f(0) = 5$

81. $f'(x) = x^2 + x^{-2} \quad \& \ f(1) = 1$

82. $f'(x) = \sin^2 x \quad \& \ f(1) = 1$

83. Find the general solution of $F'(x) = 4x + 2$, and find the particular solution that satisfies the initial condition $F(1) = 8$.

84. Derive the position function if a ball is thrown upward with initial velocity of 32 *feet* per second from an initial height of 48 *feet*. When does the ball hit the ground? With what velocity does the ball hit the ground?

85. Suppose a publishing company has found that the marginal cost at a level of production of x thousand books is given by

$$\frac{dC}{dx} = \frac{50}{\sqrt{x}}$$

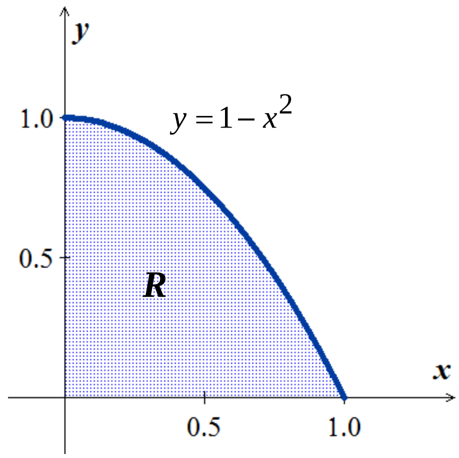
And that the fixed cost (the cost before the first book can be produced) is a $25,000. Find the cost function $C(x)$.

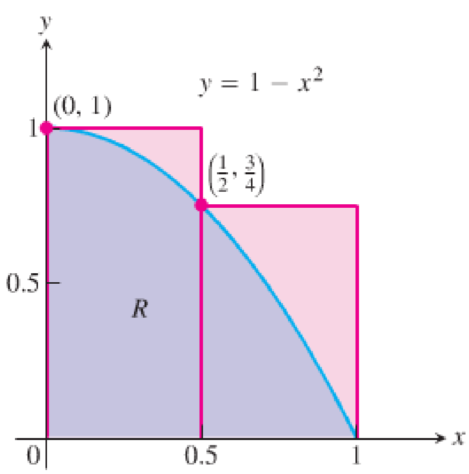
Section 4.2 – Area under Curves

The ***definite integral*** is the key tool in calculus for defining and calculating quantities important to mathematics and science, such as areas, volumes, lengths, and more…

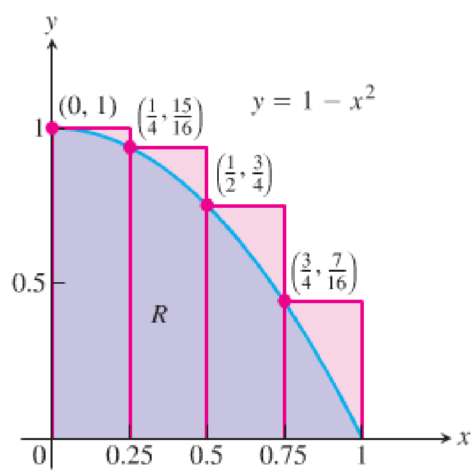
Area

To find the area of the shaded region R that lies above the x-axis, below the graph of $y = 1 - x^2$ and between the vertical lines $x = 0$ and $x = 1$.

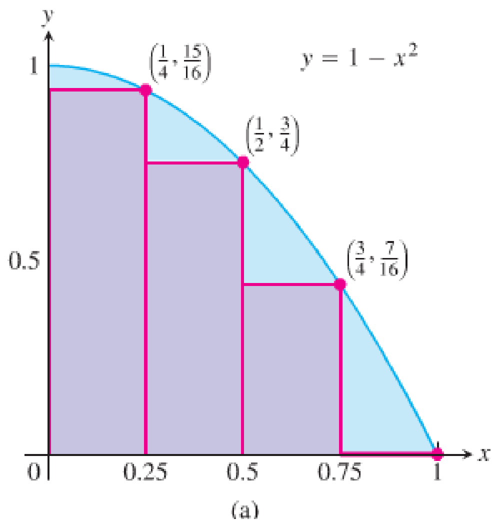


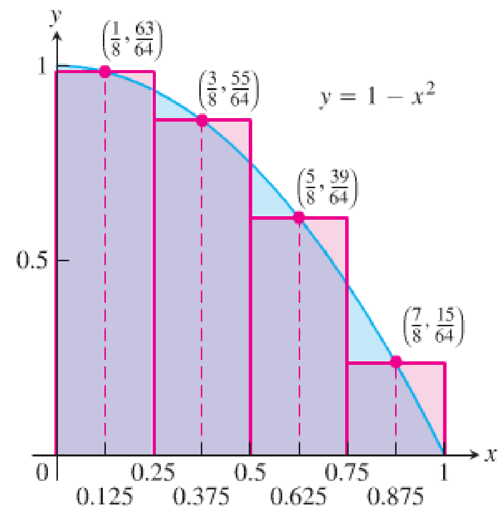


$$Area \approx 1 \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = 0.875$$

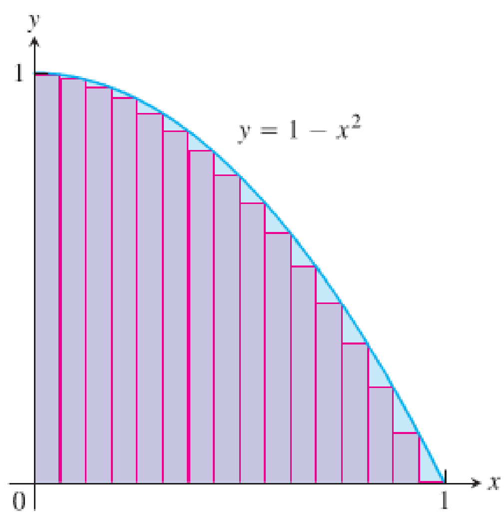


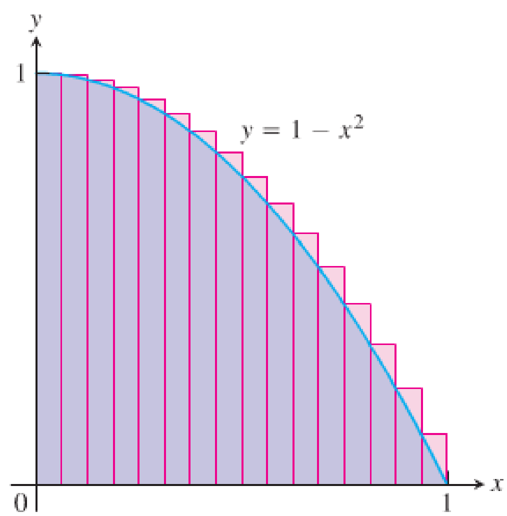
$$Area \approx 1 \cdot \frac{1}{4} + \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} = 0.78125$$





$$Area \approx \frac{15}{16}\cdot\frac{1}{4}+\frac{3}{4}\cdot\frac{1}{4}+\frac{7}{16}\cdot\frac{1}{4}+0\cdot\frac{1}{4}=0.53125 \qquad Area \approx \frac{63}{64}\cdot\frac{1}{4}+\frac{55}{64}\cdot\frac{1}{4}+\frac{39}{64}\cdot\frac{1}{4}+\frac{15}{64}\cdot\frac{1}{4}=0.671875$$





In each case of the computations, the interval $[a, b]$ over which the function f is defined was subdivided into n equal subintervals (also called ***length***) $\Delta x = \frac{b-a}{n}$, and f was evaluated at a point in each subinterval. The finite sums can be given by the form:

$$f\left(c_1\right)\Delta x + f\left(c_2\right)\Delta x + f\left(c_3\right)\Delta x + \cdots + f\left(c_n\right)\Delta x$$

Distance Traveled

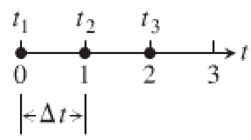
The distance formula is given by: $distance = velocity \times time$

Example

The velocity function of a projectile fired straight up into the air is $f(t) = 160 - 9.8t\ m/\sec$. Use the summation technique to estimate how far the projectile rises during the first 3 *sec*. How close do the sums come to the exact value of 435.9 *m*?

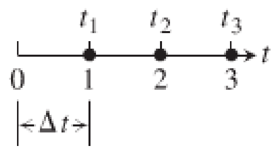
Solution

i. $\Delta t = 1\ \sec \quad \to t = 0, 1, 2$



$$
\begin{aligned}
D &\approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t \\
&= f(0)\Delta t + f(1)\Delta t + f(2)\Delta t \\
&= (160 - 9.8(0))(1) + (160 - 9.8(1))(1) + (160 - 9.8(2))(1) \\
&\underline{= 450.6}
\end{aligned}
$$

ii. $\Delta t = 1\ \sec \quad \to t = 1, 2, 3$



$$
\begin{aligned}
D &\approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t \\
&= (160 - 9.8(1))(1) + (160 - 9.8(2))(1) + (160 - 9.8(3))(1) \\
&\underline{= 421.2}
\end{aligned}
$$

iii. $\Delta t = 0.5\ \sec \quad \to t = 0,\ 0.5,\ 1,\ 1.5,\ 2,\ 2.5$

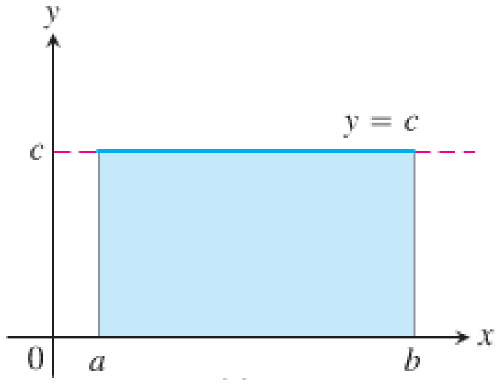
$$
\begin{aligned}
D &\approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t + f(t_4)\Delta t + f(t_5)\Delta t + f(t_6)\Delta t \\
&= (160 - 9.8(0))(1) + (160 - 9.8(0.5))(1) + (160 - 9.8(1))(1) + (160 - 9.8(1.5))(1) \\
&\qquad + (160 - 9.8(2))(1) + (160 - 9.8(2.5))(1) \\
&\underline{\approx 443.25}
\end{aligned}
$$

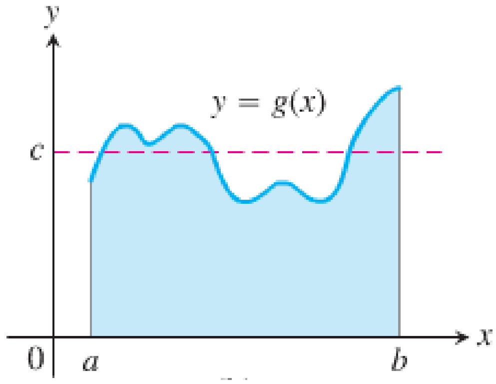
iv. $\Delta t = 0.5\ \sec \quad \to t = 0.5,\ 1,\ 1.5,\ 2,\ 2.5,\ 3$

$$
\begin{aligned}
D &\approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t + f(t_4)\Delta t + f(t_5)\Delta t + f(t_6)\Delta t \\
&= (160 - 9.8(0.5))(1) + (160 - 9.8(1))(1) + (160 - 9.8(1.5))(1) + (160 - 9.8(2))(1) \\
&\qquad + (160 - 9.8(2.5))(1) + (160 - 9.8(3))(1) \\
&\underline{\approx 428.55}
\end{aligned}
$$

The true value is 435.9 if you use more subintervals $\Delta t = 0.25\ \sec$, the interval 436.13 & 435.67
The projectile rose about 436 m during the first 3 sec of flight.

Average Value of a Nonnegative Continuous Function



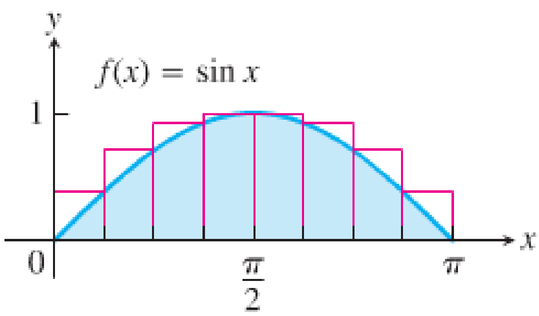


The average value of a collection of *n* numbers $x_1, x_2, \cdots, x_n$ is obtained by adding them together and dividing by *n*.

Example

Estimate the average value of the function $f(x) = \sin x$ on the interval [0, π].

Solution



To get the upper sum approximation with 8 rectangles of equal width $\Delta x = \frac{\pi}{8}$.

$$A \approx \left(\sin\frac{\pi}{8} + \sin\frac{\pi}{4} + \sin\frac{3\pi}{8} + \sin\frac{\pi}{2} + \sin\frac{\pi}{2} + \sin\frac{5\pi}{8} + \sin\frac{3\pi}{4} + \sin\frac{7\pi}{8}\right) \cdot \frac{\pi}{8}$$

≈ 2.365

Finite Sums and Sigma Notation

Sigma notation enables us to write a sum with many terms in the compact form

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

The Greek letter $\sum$ (capital ***sigma***, corresponding to our letter S)

The index k ends at $k = n$

The summation symbol $\longrightarrow \displaystyle\sum_{k=1}^{n} a_k \longrightarrow$ a_k is a formula for the kth term

The index k starts at $k = 1$

Example

Sigma Notation	***Written***	***Value of the Sum***
$\displaystyle\sum_{k=1}^{5} k$	$1+2+3+4+5$	15
$\displaystyle\sum_{k=1}^{4} (-1)^k \cdot k$	$(-1)^1 \cdot 1 + (-1)^2 \cdot 2 + (-1)^3 \cdot 3 + (-1)^4 \cdot 4$	$-1+2-3+4 = 2$
$\displaystyle\sum_{k=1}^{3} \frac{k}{k+1}$	$\dfrac{1}{1+1} + \dfrac{2}{2+1} + \dfrac{3}{3+1}$	$\dfrac{1}{2} + \dfrac{2}{3} + \dfrac{3}{4} = \dfrac{23}{12}$
$\displaystyle\sum_{k=4}^{5} \frac{k^2}{k-1}$	$\dfrac{4^2}{4-1} + \dfrac{5^2}{5-1}$	$\dfrac{16}{3} + \dfrac{25}{4} = \dfrac{139}{12}$

Example

We can write:

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 = \sum_{k=1}^{10} k^2$$

Example

Express the sum 1+ 3 +5 +7 + 9 in sigma notation.

Solution

Starting with $k = 0$: $1+3+5+7+9 = \sum_{k=0}^{4} (2k+1)$

Starting with $k = 1$: $1+3+5+7+9 = \sum_{k=1}^{5} (2k-1)$

Theorem on Sums

If $a_1, a_2, a_3, \ldots, a_n, \ldots$ and $b_1, b_2, b_3, \ldots, b_n, \ldots$ are infinite sequences, then for every positive integer ***n***,

(1) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

(2) $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

(3) $\sum_{k=1}^{n} ca_k = c\left(\sum_{k=1}^{n} a_k\right)$

(4) $\sum_{k=1}^{n} c = n \cdot c$

Proof

$$\sum_{k=1}^{n}\left(a_k+b_k\right)=\left(a_1+b_1\right)+\left(a_2+b_2\right)+\ldots+\left(a_n+b_n\right)$$

$$=\left(a_1+a_2+\ldots+a_n\right)+\left(b_1+b_2+\ldots+b_n\right)$$

$$=\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}b_k$$

Example

(1) $\sum_{k=1}^{n}(k+4)=\sum_{k=1}^{n}k+\sum_{k=1}^{n}4=\sum_{k=1}^{n}k+4\cdot n$

(2) $\sum_{k=1}^{n}\left(3k-k^2\right)=3\sum_{k=1}^{n}k-\sum_{k=1}^{n}k^2$

(3) $\sum_{k=1}^{n}\left(-a_k\right)=\sum_{k=1}^{n}(-1)\cdot\left(a_k\right)=(-1)\cdot\sum_{k=1}^{n}\left(a_k\right)=-\sum_{k=1}^{n}a_k$

(4) $\sum_{k=1}^{n}\frac{1}{n}=n\cdot\frac{1}{n}=1$

Example

Show that the sum of the first n integers is $\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$

Solution

The sum of the first 4 integers is: $\sum_{k=1}^{4}k=\frac{4(5)}{2}=10$

To prove the formula in general:

$$\begin{array}{ccccccccc} 1 & + & 2 & + & 3 & + & \cdots & + & n \\ n & + & (n-1) & + & (n-2) & + & \cdots & + & 1 \\ \hline n+1 & + & n+1 & + & n+1 & + & \cdots & + & n+1 \end{array} \rightarrow n(n+1)$$

Since it is twice the desired quantity, the sum of the first n integers is $\frac{n(n+1)}{2}$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} = (1+2+3+\cdots+n)^2$$

Limits of Finite Sums

Example

Find the limiting value of lower sum approximations to the area of the region R below the graph of $y = 1 - x^2$ and above the interval [0, 1] on the x-axis using equal-width rectangles whose width approach zero and whose number approaches infinity.

Solution

The lower sum approximation using n rectangles of equal width: $\Delta x = \frac{1-0}{n} = \frac{1}{n}$

By subdividing the interval [0, 1] into n equal width subintervals:

$$[0,\ 1] = \left[0,\ \frac{1}{n}\right], \left[\frac{1}{n},\ \frac{2}{n}\right], \left[\frac{2}{n},\ \frac{3}{n}\right],\ \ldots,\ \left[\frac{n-1}{n},\ \frac{n}{n}\right] = \left[\frac{k-1}{n},\ \frac{k}{n}\right]$$

$$f\left(\frac{k}{n}\right) = 1 - \left(\frac{k}{n}\right)^2$$

$$\left[f\left(\frac{1}{n}\right)\right]\cdot\left(\frac{1}{n}\right) + \left[f\left(\frac{2}{n}\right)\right]\cdot\left(\frac{1}{n}\right) + \cdots + \left[f\left(\frac{n}{n}\right)\right]\cdot\left(\frac{1}{n}\right)$$

We can write this in sigma notation:

$$\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot\left(\frac{1}{n}\right) = \sum_{k=1}^{n}\left[1 - \left(\frac{k}{n}\right)^2\right]\cdot\left(\frac{1}{n}\right)$$

$$= \sum_{k=1}^{n}\left(\frac{1}{n} - \frac{k^2}{n^3}\right)$$

$$= \sum_{k=1}^{n}\frac{1}{n} - \sum_{k=1}^{n}\frac{k^2}{n^3}$$

$$= n\cdot\frac{1}{n} - \frac{1}{n^3}\sum_{k=1}^{n} k^2$$

$$= 1 - \frac{1}{n^3}\cdot\frac{n\cdot(n+1)\cdot(2n+1)}{6}$$

$$=1-\frac{2n^3+3n^2+n}{6n^3}$$

$$=\frac{6n^3-2n^3-3n^2-n}{6n^3}$$

$$=\frac{4n^3-3n^2-n}{6n^3}$$

$$\lim_{x\to\infty}\left(\frac{4n^3-3n^2-n}{6n^3}\right)=\frac{4}{6}=\frac{2}{3}$$

The lower sum approximation converge to $\frac{2}{3}$

The upper sum approximation also converge to $\frac{2}{3}$

Review

Definition of Arithmetic Sequence

A sequence $a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots$ is an arithmetic sequence if there is a real number ***d*** such that for every positive integer ***k***,

$$a_{k+1}=a_k+d$$

The number $d=a_{k+1}-a_k$ is called the ***common difference*** of the sequence.

The nth Term of an Arithmetic Sequence: $\boxed{a_n=a_1+(n-1)d}$

Example

Express the sum in terms of summation notation: $4+11+18+25+32$. (Answers are not unique)

Solution

Number of terms: $n = 5$

Difference in terms: $d = 11 - 4 = 7$

$a_n=a_1+(n-1)d$

$a_n=4+(n-1)7=4+7n-7=7n-3$

$$\sum_{n=1}^{5}(7n-3)$$

Theorem

Formulas for S_n

If $a_1, a_2, a_3, \ldots, a_n, \ldots$ is an arithmetic sequence with common difference ***d***, then the *n*th partial sum S_n (that is, the sum of the first ***n*** terms) is given by either

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right] \quad or \quad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

Definition of *Geometric* Sequence

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a geometric sequence if $a_1 \neq 0$ and if there is a real number $r \neq 0$ such that for every positive integer *k*.

$$a_{k+1} = a_k r$$

The number $r = \dfrac{a_{k+1}}{a_k}$ is called the ***common ratio*** of the sequence.

The formula for the n^th^ Term of a Geometric Sequence: $\boxed{a_n = a_1 r^{n-1}}$

The common ratio for: $6,\ -12,\ 24,\ -48,\ \ldots,\ (-2)^{n-1}(6),\ \ldots$ is $= \frac{-12}{6} = -2$

Example

Express the sum in terms of summation notation (Answers are not unique.)

$$\frac{1}{4} - \frac{1}{12} + \frac{1}{36} - \frac{1}{108}$$

Solution

$$\frac{1}{4} - \frac{1}{12} + \frac{1}{36} - \frac{1}{108} = \frac{1}{4} - \frac{1}{4}\frac{1}{3^1} + \frac{1}{4}\frac{1}{3^2} - \frac{1}{4}\frac{1}{3^3}$$

$$= \sum_{n=1}^{4} (-1)^{n+1} \frac{1}{4}\left(\frac{1}{3}\right)^{n-1}$$

Theorem: Formula for S_n

The nth partial sum S_n of a geometric sequence with first term a_1 and common ratio $r \neq 1$ is

$$\boxed{S_n = a_1 \frac{1-r^n}{1-r}}$$

Riemann Sums

The theory of limits of finite approximations was made precise by the German mathematician ***Bernhard Riemann***.

We introduce the notion of a *Riemann sum*, which underlies the theory of the definite integral.

Let a closed interval $[a, b]$ be partitioned by points $a < x_1 < x_2 < \cdots < x_{n-1} < b$

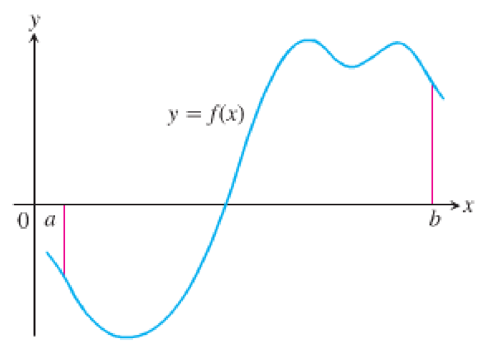
To make the notation consistent, so that

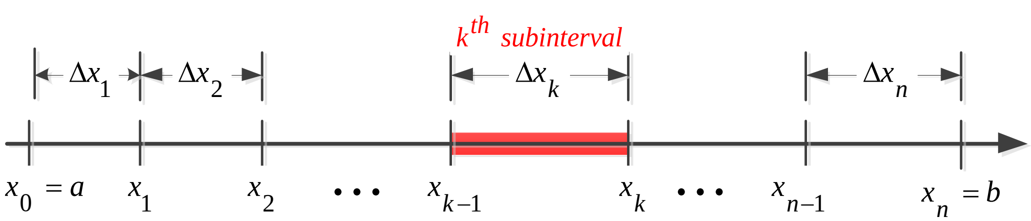
$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

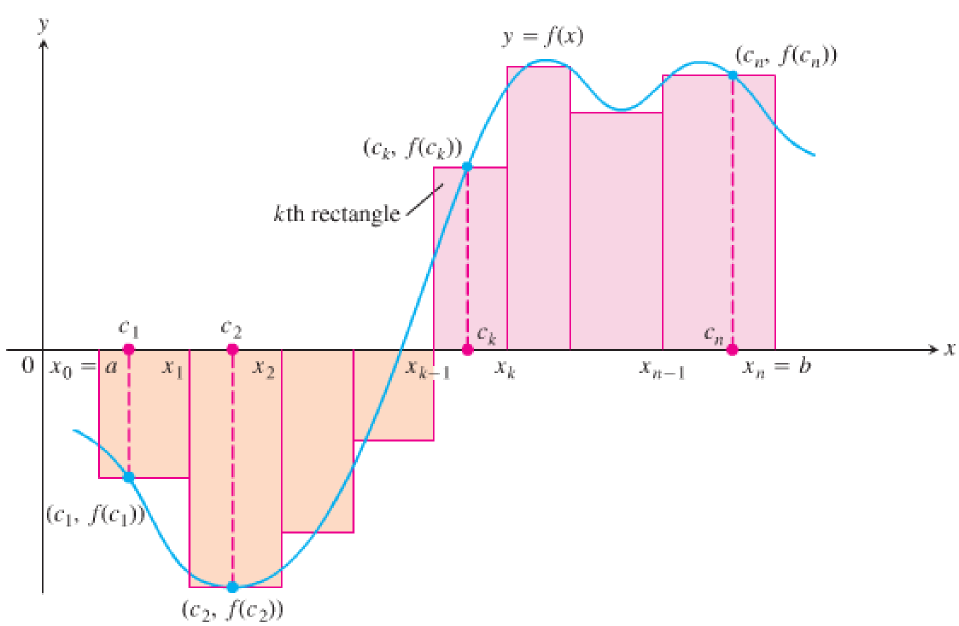
The set: $P = \{x_0, x_1, x_2, \cdots, x_{n-1}, x_n\}$ is called a partition of $[a, b]$.

The partition P divides $[a, b]$ into n closed subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$$







These products are:

$$S_P = \sum_{k=1}^{n} f\left(c_k\right) \Delta x_k$$

The sum S_P is called a ***Riemann sum*** *for f on the interval* [*a*, *b*], and c_k in the subintervals.

If we choose *n* subintervals all having equal width $\Delta x = \frac{b-a}{n}$ to partition [*a*, *b*], then choose the point c_k to be the right-hand endpoints of each subintervals when forming the Riemann sum. This choice leads to the Riemann sum formula

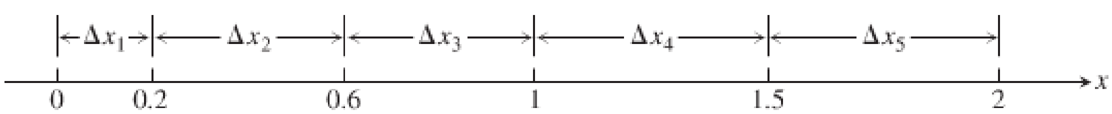
$$S_n = \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right) \cdot \left(\frac{b-a}{n}\right)$$

Example

The set $P = \{0,\ 0.2,\ 0.6,\ 1,\ 1.5,\ 2\}$ is a partition of [0, 2]

Solution

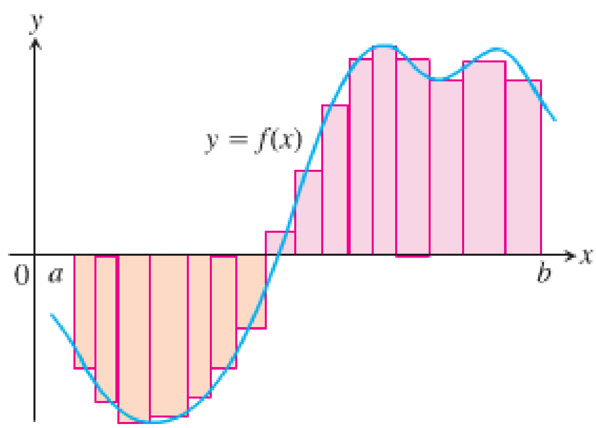
There are five subintervals of *P*: [0, 0.2], [0.2, 0.6], [0.6, 1], [1, 1.5], and [1.5, 2]

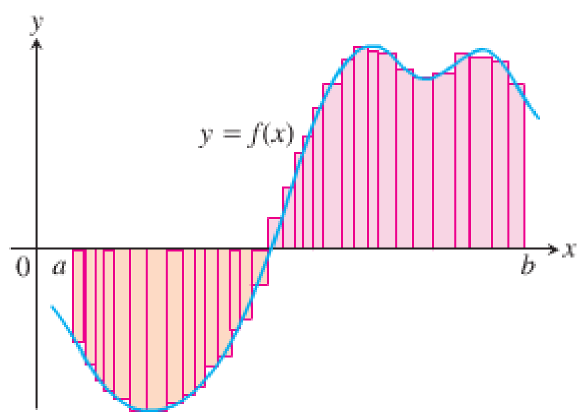


The lengths of the subintervals are:

$$\Delta x_1 = 0.2 \quad \Delta x_2 = 0.4 \quad \Delta x_3 = 0.4 \quad \Delta x_4 = 0.5 \quad \Delta x_5 = 0.5$$

The longest subinterval length is 0.5, so the norm of the partition is $\|P\| = 0.5$





Exercises *Section* 4.2 – Area under Curves

Use finite approximations to estimate the area under the graph of the function using

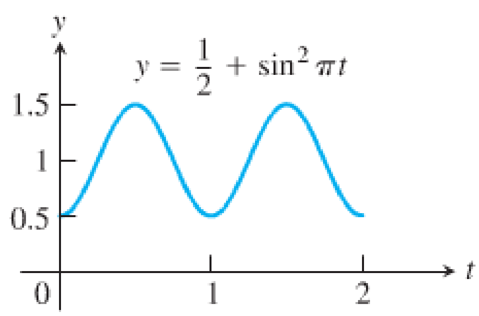
a) A lower sum with two rectangles of equal width
b) A lower sum with four rectangles of equal width
c) A upper sum with two rectangles of equal width
d) A upper sum with four rectangles of equal width

1. $f(x) = \frac{1}{x}$ *between* $x = 1$ *and* $x = 5$

2. $f(x) = 4 - x^2$ *between* $x = -2$ *and* $x = 2$

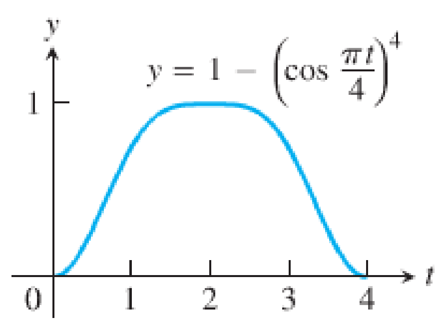
3. Use finite approximations to estimate the average value of *f* on the given interval by partitioning the interval into four subintervals of equal length and evaluating *f* at the subinterval midpoints.

$$f(t) = \frac{1}{2} + \sin^2 \pi t \quad \text{on} \quad [0,\ 2]$$



4. Use finite approximations to estimate the average value of *f* on the given interval by partitioning the interval into four subintervals of equal length and evaluating *f* at the subinterval midpoints.

$$f(t) = 1 - \left(\cos \frac{\pi t}{4}\right)^4 \quad \text{on} \quad [0,\ 4]$$



Write the sums without sigma notation. Then evaluate them:

5. $\sum_{k=1}^{2} \frac{6k}{k+1}$ **6.** $\sum_{k=1}^{3} \frac{k-1}{k}$ **7.** $\sum_{k=1}^{5} \sin k\pi$ **8.** $\sum_{k=1}^{4} (-1)^k \cos k\pi$

9. Write the following expression 1 + 2 + 4 + 8 + 16 + 32 in sigma notation

10. Write the following expression 1 – 2 + 4 – 8 + 16 – 32 in sigma notation

11. Write the following expression $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$ in sigma notation

12. Write the following expression $-\frac{1}{5}+\frac{2}{5}-\frac{3}{5}+\frac{4}{5}-\frac{5}{5}$ in sigma notation

13. Suppose that $\sum_{k=1}^{n} a_k = -5$ *and* $\sum_{k=1}^{n} b_k = 6$. Find the value of $\sum_{k=1}^{n}\left(b_k - 2a_k\right)$

Evaluate the sums

14. $\sum_{k=1}^{10} k^3$

15. $\sum_{k=1}^{7} (-2k)$

16. $\sum_{k=1}^{5} \frac{\pi k}{15}$

17. $\sum_{k=1}^{5} k(3k+5)$

18. $\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3$

19. $\sum_{k=1}^{500} 7$

20. $\sum_{k=18}^{71} k(k-1)$

21. $\sum_{k=1}^{n} \left(\frac{1}{n}+2n\right)$

22. Graph the function $f(x)=x^2-1$ over the given interval [0, 2]. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\sum_{k=1}^{4} f(c_k)\Delta x_k$, given c_k is the

a) Left-hand endpoint

b) Right-hand endpoint

c) Midpoint of kth subinterval.

(Make a separate sketch for each set of rectangles.)

Section 4.3 – Definite Integral

Definition

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the ***definite integral* of f over $[a, b]$** and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\Delta x_k$ if the following condition is satisfied:

> Given any number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \cdots, x_n\}$ of [a, b] with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have
>
> $$\left| \sum_{k=1}^{n} f(c_k)\Delta x_k - J \right| < \varepsilon$$

Leibniz introduced a notation for the definite integral that captures its construction as a limit of Riemann sums.

Upper limit of integration

Function is integrand

Integral sign

$$\int_a^b f(x)\,dx$$

x is the variable of integration

Lower limit of integration

Integral of *f* from *a* to *b*.

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = J = \int_a^b f(x)\,dx$$

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k)\Delta x_k = J = \int_a^b f(x)\,dx$$

Theorem – Integrability of Continuous Functions

If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)\,dx$ exists and f in integrable over $[a, b]$

Properties of Definite Integrals

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \qquad\qquad \int_a^a f(x)\,dx = 0$$

Theorem

When f and g are integrable over the interval $[a, b]$, the definite integral satisfies the rules:

Order of Integration: $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

Zero Width Interval: $\int_a^a f(x)\,dx = 0$

Constant Multiple: $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

Sum and Difference: $\int_a^b \left(f(x) \pm g(x)\right)dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Additivity: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

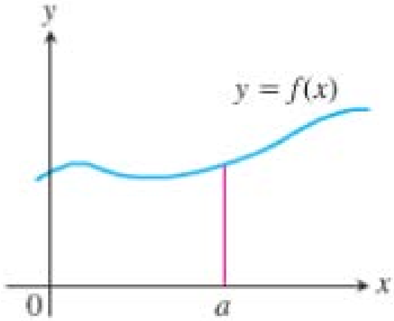
Max-Min Inequality: If f has ***maximum*** value *max* f and ***minimum*** value *min* f on $[a, b]$, then

$$(\min f)\cdot(b-a) \le \int_a^b f(x)\,dx \le (\max f)\cdot(b-a)$$

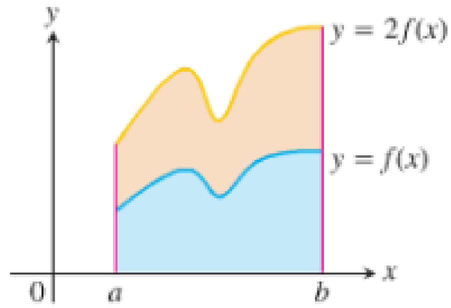
Domination:

$$f(x) \ge g(x) \text{ on } [a,\ b] \quad \Rightarrow \quad \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

$$f(x) \ge 0 \text{ on } [a,\ b] \quad \Rightarrow \quad \int_a^b f(x)\,dx \ge 0$$

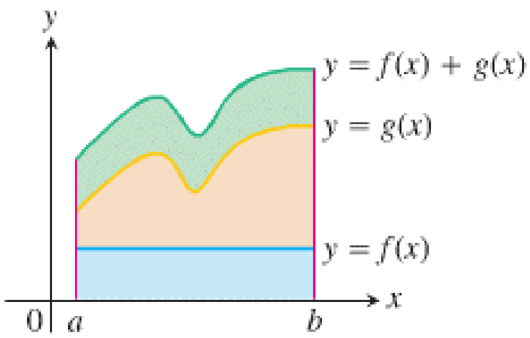


Zero Width Interval: $\int_a^a f(x)dx = 0$



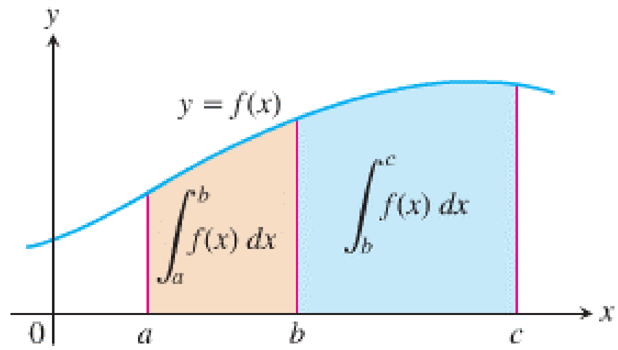
Constant Multiple: (k = 2)

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$



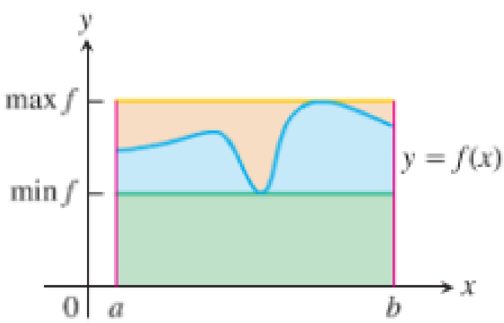
Sum: (*areas add*)

$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$



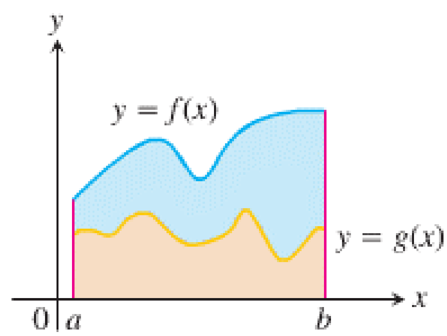
Additive for definite integrals:

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$



Max-Min Inequality:

$$\min f \cdot (b-a) \le \int_a^b f(x)dx \le \max f \cdot (b-a)$$



Domination

$$f(x) \ge g(x) \text{ on } [a,b] \Rightarrow \int_a^b f(x)dx \ge \int_a^b g(x)dx$$

Example

Suppose that $\int_{-1}^{1} f(x)dx = 5,\quad \int_{1}^{4} f(x)dx = -2,\quad \int_{-1}^{1} h(x)dx = 7$. Find:

a) $\int_{4}^{1} f(x)dx$

b) $\int_{-1}^{1} \left[2f(x)+3h(x)\right]dx$

Solution

a) $\int_{4}^{1} f(x)dx = -\int_{1}^{4} f(x)dx = -(-2) = \underline{2}$

b) $\int_{-1}^{1} \left[2f(x)+3h(x)\right]dx = 2\int_{-1}^{1} f(x)dx + 3\int_{-1}^{1} h(x)dx$

$$= 2(5) + 3(7)$$

$$\underline{= 31}$$

Example

Show that the value of $\int_{0}^{1} \sqrt{1+\cos x}\,dx$ is less than or equal to $\sqrt{2}$

Solution

min $f \bullet (b-a)$: is the lower bound

max $f \bullet (b-a)$: is the upper bound

The maximum value of $\sqrt{1+\cos x}$ on [0, 1] is $\sqrt{1+1} = \sqrt{2}$

So, $\int_{0}^{1} \sqrt{1+\cos x}\,dx \le \sqrt{2}\cdot(1-0) = \sqrt{2}$

Area Under the Graph of a Nonnegative Function

Definition

If $y = f(x)$ is nonnegative and integrable over a closed interval [*a*, *b*], then the area under the curve $y = f(x)$ over [*a*, *b*] is the integral of *f* from *a* to *b*,

$$A = \int_a^b f(x)\,dx$$

Example

Compute $\int_0^b x\,dx$ and find the area *A* under *y* = *x* over the interval [0, *b*], *b* > 0.

Solution

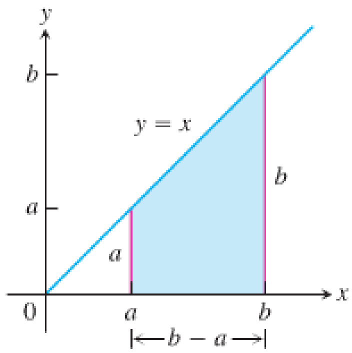
To Compute the definite integral, we consider the partition *P* subdivides the interval [0, *b*] into *n* subintervals of equal width $\Delta x = \frac{b-0}{n} = \frac{b}{n}$.

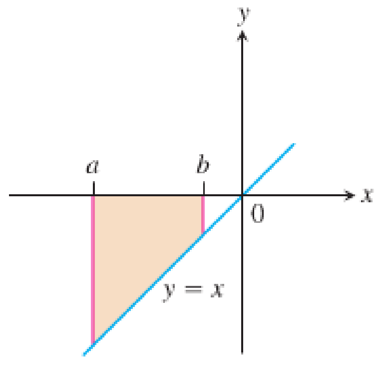
$$P = \left\{0,\ \frac{b}{n},\ \frac{2b}{n},\ \frac{3b}{n},\ \ldots,\ \frac{nb}{n}\right\} \quad \text{and} \quad c_k = \frac{kb}{n}$$

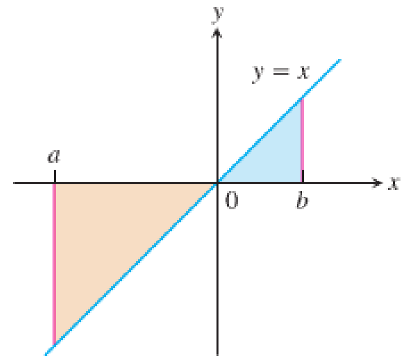
$$\begin{aligned}\sum_{k=1}^{n} f(c_k)\Delta x &= \sum_{k=1}^{n} \frac{kb}{n}\cdot\frac{b}{n}\\ &= \sum_{k=1}^{n}\frac{kb^2}{n^2}\\ &= \frac{b^2}{n^2}\sum_{k=1}^{n} k\\ &= \frac{b^2}{n^2}\cdot\frac{n(n+1)}{2}\\ &= \frac{b^2}{2}\left(1+\frac{1}{n}\right)\end{aligned}$$

$$\lim_{n\to\infty}\left[\frac{b^2}{2}\left(1+\frac{1}{n}\right)\right] = \frac{b^2}{2}$$

$$\int_0^b x\,dx = \boxed{\frac{b^2}{2}}$$







$$A = \int_0^b x dx = \frac{b^2}{2}$$

$$\int_a^b x dx = \int_a^0 x dx + \int_0^b x dx$$

$$= -\int_0^a x dx + \int_0^b x dx$$

$$= -\frac{a^2}{2} + \frac{b^2}{2}$$

$$\boxed{\int_a^b x dx = \frac{b^2}{2} - \frac{a^2}{2} \qquad a < b}$$

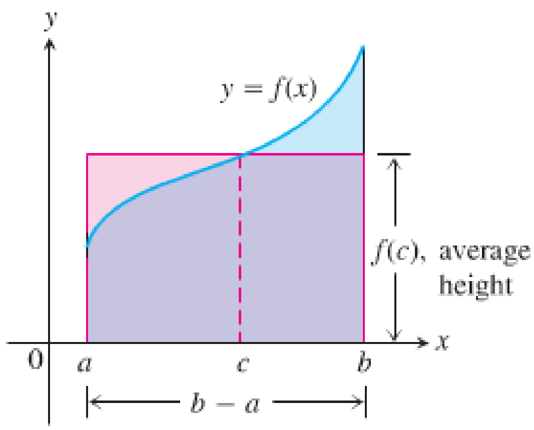
$$\boxed{\int_a^b x^2 dx = \frac{b^3}{3} - \frac{a^3}{3} \qquad a < b}$$

Section 4.4 – Fundamental Theorem of Calculus

Mean Value Theorem for Definite Integrals

If f is continuous on [a, b], then some point c in [a, b],

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$



Theorem – The Fundamental Theorem of Calculus, P-1

If f is continuous on [a, b], then $F(x) = \int_a^x f(t)\,dt$ is continuous on [a, b], and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Theorem – The Fundamental Theorem of Calculus, P-2

If f is continuous at every point in [a, b], then F is any antiderivative of f on [a, b], then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

Example

a) $$\int_0^{\pi} \cos x \, dx = \sin x \Big|_0^{\pi}$$

$$= \sin \pi - \sin 0$$

$$\underline{= 0}$$

b) $$\int_{-\frac{\pi}{4}}^{0} \sec x \tan x \, dx = \sec x \Big|_{-\frac{\pi}{4}}^{0}$$

$$= \sec 0 - \sec\left(-\frac{\pi}{4}\right)$$

$$= \underline{1 - \sqrt{2}}$$

c) $$\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx = \left[x^{3/2} + \frac{4}{x}\right]_1^4$$

$$= \left((4)^{3/2} + \frac{4}{4}\right) - \left((1)^{3/2} + \frac{4}{1}\right)$$

$$= (9) - (5)$$

$$= \underline{4}$$

Theorem – The Net Change Theorem

The net change in a function $F(x)$ over an interval $a \le x \le b$ is the integral of tis rate of change:

$$F(b) - F(a) = \int_a^b F'(x)\,dx$$

Example

Consider the analysis of a heavy rock blown straight up from the ground by a dynamite blast. The velocity of the rock at any time *t* during its motion was given as $v(t) = 160 - 32t \ \ ft/\sec$

a) Find the displacement of the rock during the time period $0 \le t \le 8$

b) Find the total distance traveled during this time period.

Solution

a) $displacement: \quad s(t) = \int_0^8 v(t)\,dt$

$$\begin{aligned} &= \int_0^8 (160 - 32t)\,dt \\ &= \left[160t - 16t^2\right]_0^8 \\ &= \left(160(8) - 16(8)^2\right) - \left(160(0) - 16(0)^2\right) \\ &= 256 \end{aligned}$$

The height if the rock is 256 *ft* above the ground 8 *sec* after the explosion.

b) $v(t) = 160 - 32t = 0 \quad \to \quad \boxed{t = 5 \ \sec}$

The velocity is positive over the time [0, 5] and negative over [5, 8]

$$\begin{aligned} \int_0^8 |v(t)|\,dt &= \int_0^5 |v(t)|\,dt + \int_5^8 |v(t)|\,dt \\ &= \int_0^5 (160 - 32t)\,dt - \int_5^8 (160 - 32t)\,dt \\ &= \left[160t - 16t^2\right]_0^5 - \left[160t - 16t^2\right]_5^8 \\ &= \left[\left(160(5) - 16(5)^2\right) - \left(160(0) - 16(0)^2\right)\right] \\ &\qquad - \left[\left(160(8) - 16(8)^2\right) - \left(160(5) - 16(5)^2\right)\right] \\ &= 400 - (-144) \\ &= 544 \end{aligned}$$

Example

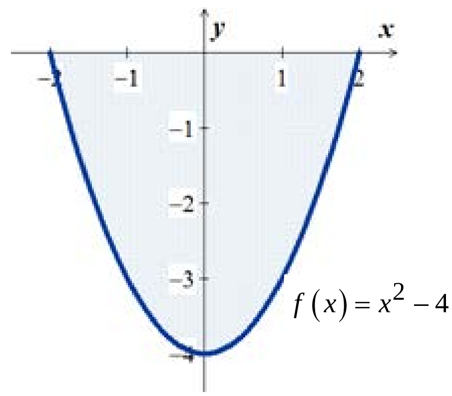
Shows the graph of $f(x)=x^2-4$ and its mirror image $g(x)=4-x^2$ are reflected across the *x*-axis. For each function, compute

a) The definite integral over the interval [–2, 2]

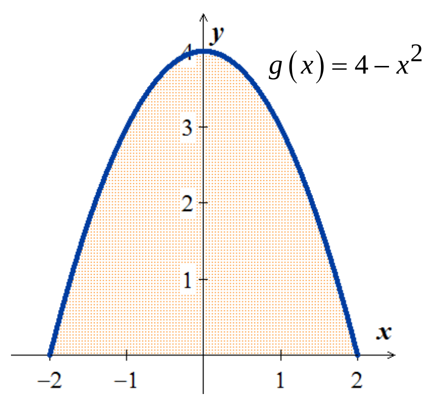
b) The area between the graph and the *x*-axis over [–2, 2]

Solution

a)
$$\int_{-2}^{2} f(x)\,dx=\int_{-2}^{2}\left(x^2-4\right)dx$$
$$=\left[\frac{x^3}{3}-4x\right]_{-2}^{2}$$
$$=\left[\frac{(2)^3}{3}-4(2)\right]-\left[\frac{(-2)^3}{3}-4(-2)\right]$$
$$=\left(\frac{8}{3}-8\right)-\left(-\frac{8}{3}+8\right)$$
$$=-\frac{32}{3}$$



$$\int_{-2}^{2} g(x)\,dx=\int_{-2}^{2}\left(4-x^2\right)dx$$
$$=\left[4x-\frac{x^3}{3}\right]_{-2}^{2}$$
$$=\left[4(2)-\frac{(2)^3}{3}\right]-\left[4(-2)-\frac{(-2)^3}{3}\right]$$
$$=\frac{32}{3}$$



b) In both cases, the area between the curve and the *x*-axis over [–2, 2] is $\frac{32}{3}$ units.

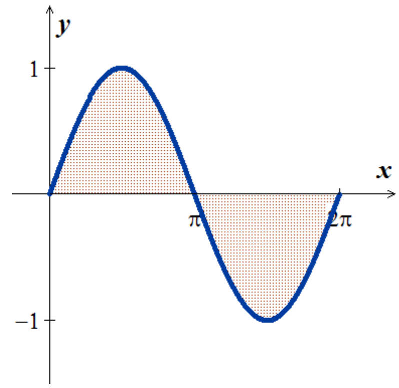
Example

Shows the graph of $f(x)=\sin x$ between $x=0$ and $x=2\pi$. Compute

a) The definite integral of $f(x)$ over [0, 2π]

b) The area between the graph and the *x*-axis over [0, 2π]

Solution



a) $$\int_0^{2\pi}\sin x\,dx=-\cos x\Big|_0^{2\pi}$$

$$=-(\cos 2\pi-\cos 0)$$

$$=-(1-1)$$

$$\underline{=0}$$

b) The area between the graph and the axis is obtained by adding the absolute values

$$Area=\left|\int_0^{\pi}\sin x\,dx\right|+\left|\int_{\pi}^{2\pi}\sin x\,dx\right|$$

$$=\left|-\cos x\right|\Big|_0^{\pi}+\left|-\cos x\right|\Big|_{\pi}^{2\pi}$$

$$=\left|-(\cos\pi-\cos 0)\right|+\left|-(\cos 2\pi-\cos\pi)\right|$$

$$=\left|-(-1-1)\right|+\left|-(1-(-1))\right|$$

$$=|2|+|-2|$$

$$\underline{=4}$$

Summary

To find the area between the graph of $y=f(x)$ and the *x*-axis over the interval [*a*, *b*]:

1. Subdivide [*a*, *b*] at the zeros of *f*.

2. Integrate f over each subinterval.

3. Add the absolute values of the integrals.

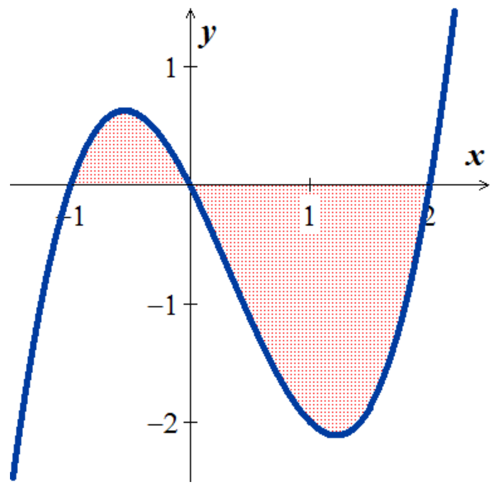
Example

Find the area of the region between the x-axis and the graph of $f(x)=x^3-x^2-2x, \quad -1\le x\le 2$

Solution

The zeros of: $f(x)=x^3-x^2-2x=0$

$$x\left(x^2-x-2\right)=0 \quad \Rightarrow \quad x=0,\ -1,\ 2$$



$$\int_{-1}^{0}\left(x^3-x^2-2x\right)dx=\left[\frac{x^4}{4}-\frac{x^3}{3}-x^2\right]_{-1}^{0}$$

$$=\left[0-\left(\frac{(-1)^4}{4}-\frac{(-1)^3}{3}-(-1)^2\right)\right]$$

$$=-\left(\frac{1}{4}+\frac{1}{3}-1\right)$$

$$=\frac{5}{12}$$

$$\int_{0}^{2}\left(x^3-x^2-2x\right)dx=\left[\frac{x^4}{4}-\frac{x^3}{3}-x^2\right]_{0}^{2}$$

$$=\left[\left(\frac{(2)^4}{4}-\frac{(2)^3}{3}-(2)^2\right)-0\right]$$

$$=\left(4-\frac{8}{3}-4\right)$$

$$=-\frac{8}{3}$$

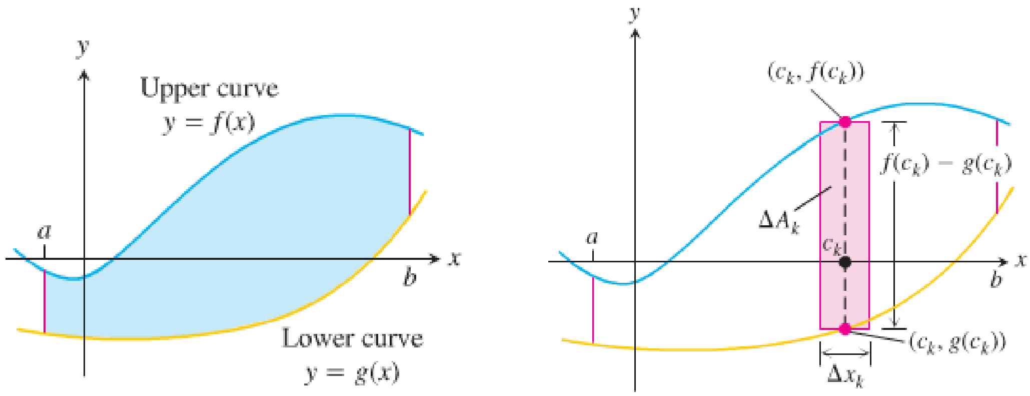
$$Area=\left|\int_{-1}^{0}\left(x^3-x^2-2x\right)dx\right|+\left|\int_{0}^{2}\left(x^3-x^2-2x\right)dx\right|$$

$$=\frac{5}{12}+\left|-\frac{8}{3}\right|$$

$$=\frac{5}{12}+\frac{8}{3}$$

$$=\frac{37}{12}$$

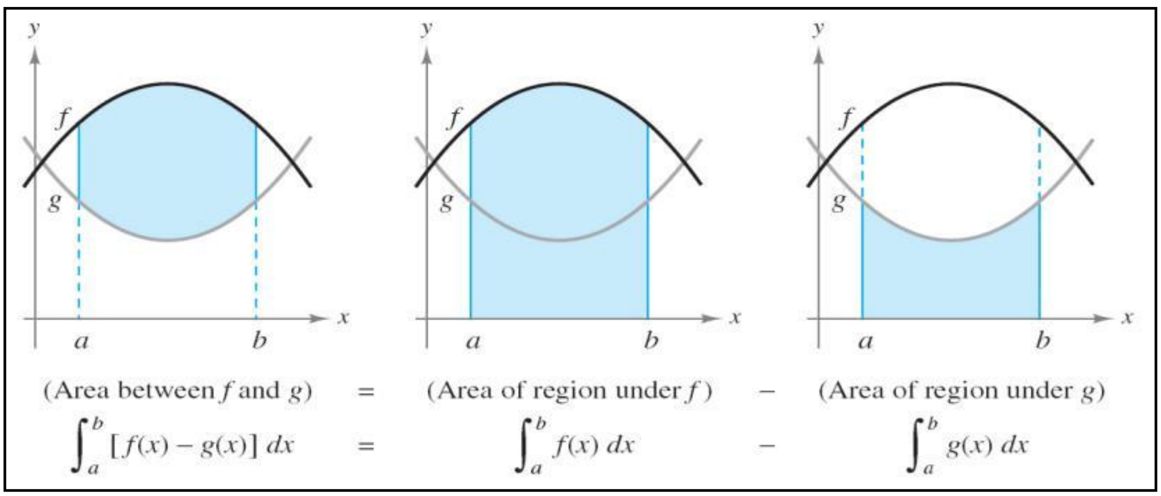
Areas between Curves



Definition

If f and g are continuous with $f(x) \geq g(x)$ throughout [*a*, *b*], then the ***area of the region between the curves*** $y = f(x)$ *and* $y = g(x)$ **from *a* to *b*** is:

$$A = \int_a^b \left[f(x) - g(x) \right] dx$$



Example

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution

The limits of integrations are found by letting:

$$2 - x^2 = -x \qquad \Rightarrow x^2 - x - 2 = 0 \quad \to \quad \underline{x = -1,\ 2}$$

$$A = \int_{-1}^{2} \left[f(x) - g(x) \right] dx$$

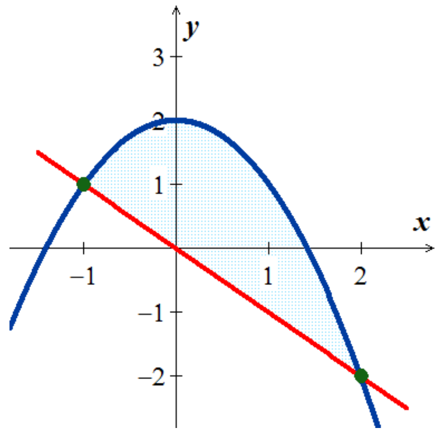
$$= \int_{-1}^{2} \left[2 - x^2 - (-x)\right] dx$$

$$= \int_{-1}^{2} \left(2 - x^2 + x\right) dx$$

$$= \left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{2}$$

$$= \left(4 - \frac{8}{3} + \frac{4}{2}\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right)$$

$$= \frac{9}{2}$$



Example

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below the x-axis and the line $y = x - 2$

Solution

$$\left(y = \sqrt{x}\right) \cap (y = 0) \quad \rightarrow \quad (0,\ 0)$$

$$\left(y = \sqrt{x}\right) \cap (y = x - 2) \quad \rightarrow \quad \sqrt{x} = x - 2$$

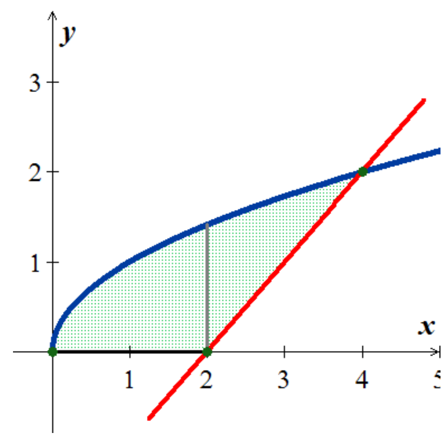
$$\left(\sqrt{x}\right)^2 = (x - 2)^2$$

$$x = x^2 - 4x + 4$$

$$x^2 - 5x + 4 = 0$$

$$\rightarrow \quad x = \cancel{1},\ 4$$

$$(y = 0) \cap (y = x - 2) \quad \rightarrow \quad x = 2$$



$$\text{Total Area } = \int_0^2 \left[\sqrt{x} - 0\right] dx + \int_2^4 \left[\sqrt{x} - (-x + 2)\right] dx$$

$$= \left[\frac{2}{3} x^{3/2}\right]_0^2 + \left[\frac{2}{3} x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

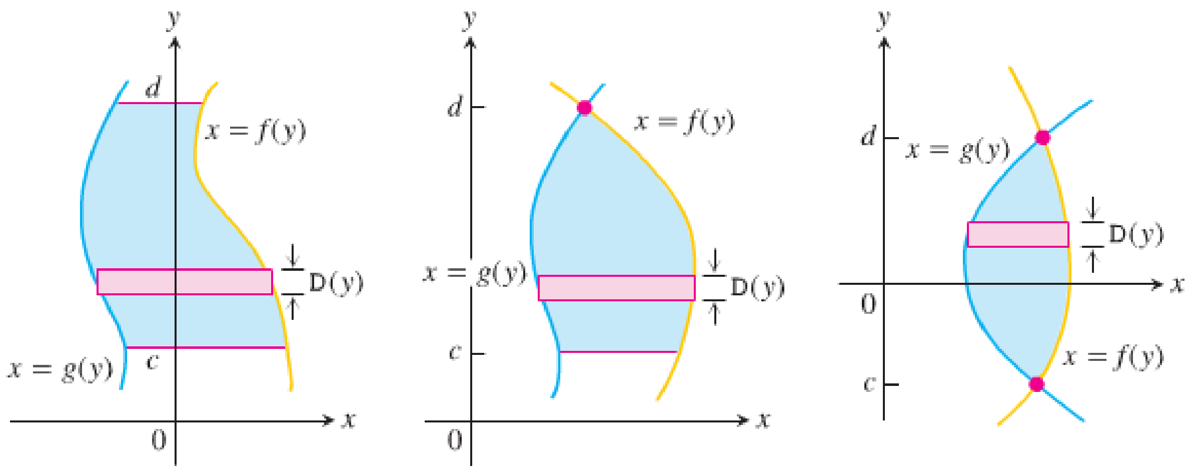
$$= \left[\frac{2}{3}\left(2^{3/2}\right) - 0\right] + \left(\frac{2}{3} 4^{3/2} - \frac{4^2}{2} + 2(4)\right) - \left(\frac{2}{3} 2^{3/2} - \frac{2^2}{2} + 2(2)\right)$$

$$= \frac{2}{3}\left(2^{3/2}\right) + \frac{2}{3} 4^{3/2} - \frac{16}{2} + 8 - \frac{2}{3} 2^{3/2} + \frac{4}{2} - 4$$

$$= \frac{2}{3}(8) - 2$$

$$= \frac{10}{3}$$

Integration with Respect to *y*



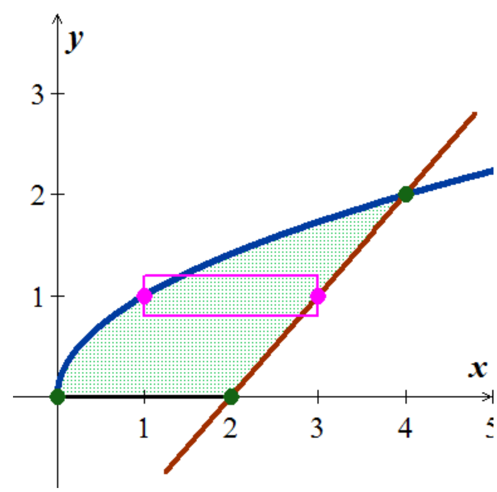
$$A = \int_c^d \left[f(y) - g(y) \right] dy$$

(*From right hand to left* hand)

Example

Find the area of the region by integrating with respect to y, in the first quadrant that is bounded above by $y = \sqrt{x}$ and below the *x*-axis and the line $y = x - 2$.

Solution



$y = \sqrt{x} \quad \rightarrow \quad x = y^2$

$y = x - 2 \quad \rightarrow \quad x = y + 2$

$\left(x = y^2\right) \cap (y = 0) \quad \rightarrow \quad (0,\ 0)$

$\left(x = y^2\right) \cap (x = y + 2) \quad \rightarrow \quad y^2 = y + 2$

$y^2 - y - 2 = 0 \quad \rightarrow \quad y = -1,\ 2$

$(y = 0) \cap (x = y + 2) \quad \rightarrow \quad y = 0$

$$A = \int_0^2 \left[y + 2 - y^2 \right] dy$$

$$= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_0^2$$

$$= \frac{2^2}{2} + 2(2) - \frac{2^3}{3} - 0$$

$$= \frac{10}{3}$$

Exercises Section 4.4 – Fundamental Theorem of Calculus

Evaluate the integrals

1. $\int_0^3 (2x+1)dx$

2. $\int_0^2 x(x-3)dx$

3. $\int_0^4 \left(3x - \frac{x^3}{4}\right)dx$

4. $\int_{-2}^2 \left(x^3 - 2x + 3\right)dx$

5. $\int_0^1 \left(x^2 + \sqrt{x}\right)dx$

6. $\int_0^{\pi/3} 4\sec u \tan u \, du$

7. $\int_{\pi/4}^{3\pi/4} \csc\theta \cot\theta d\theta$

8. $\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right)dt$

9. $\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} dy$

10. $\int_1^8 \frac{\left(x^{1/3}+1\right)\left(2-x^{2/3}\right)}{x^{1/3}}dx$

11. $\int_{\pi/2}^{\pi} \frac{\sin 2x}{2\sin x}dx$

12. $\int_0^{\pi/3} (\cos x + \sec x)^2 dx$

13. $\int_0^{\pi} \frac{1}{2}\left(\cos x + |\cos x|\right)dx$

14. $\int_0^1 2x\left(4 - x^2\right)dx$

15. $\int_0^4 (8-2x)dx$

16. $\int_0^4 \frac{1}{\sqrt{16-x^2}}dx$

17. $\int_{-4}^2 (2x+4)dx$

18. $\int_0^2 (1-x)dx$

19. $\int_0^2 \left(x^2 - 2\right)dx$

20. $\int_0^{\pi/2} \cos x \, dx$

21. $\int_1^7 \frac{dx}{x}$

22. $\int_4^9 3\sqrt{x} \, dx$

23. $\int_{-2}^3 \left(x^2 - x - 6\right)dx$

24. $\int_0^1 \left(1 - \sqrt{x}\right)dx$

25. $\int_0^{\pi/4} 2\cos x \, dx$

26. $\int_{-\pi/4}^{7\pi/4} (\sin x + \cos x)dx$

27. $\int_0^{\ln 8} e^x dx$

28. $\int_1^4 \left(\frac{x-1}{x}\right)dx$

29. $\int_{-2}^{-1} \left(3e^{3x} + \frac{2}{x}\right)dx$

30. $\int_0^2 \frac{dx}{x^2+4}$

Find the total area between the region between the given graph and the *x*-axis

31. $y = -x^2 - 2x, \quad -3 \le x \le 2$

32. $y = x^3 - 3x^2 + 2x, \quad 0 \le x \le 2$

33. $y = x^{1/3} - x, \quad -1 \le x \le 8$

34. $f(x) = x^2 + 1, \quad 2 \le x \le 3$

35. Find the area of the region between the graph of $y = 4x - 8$ and the *x-axis*, for $-4 \le x \le 8$

36. Find the area of the region between the graph of $y = -3x$ and the *x-axis*, for $-2 \le x \le 2$

37. Find the area of the region between the graph of $y = 3x + 6$ and the $x\text{-}axis$, for $0 \le x \le 6$

38. Find the area of the region between the graph of $y = 1 - |x|$ and the $x\text{-}axis$, for $-2 \le x \le 2$

39. Find the area of the region above the $x\text{-}axis$ bounded by $y = 4 - x^2$

40. Find the area of the region above the $x\text{-}axis$ bounded by $y = x^4 - 16$

41. Find the area of the region between the graph of $y = 6\cos x$ and the $x\text{-}axis$, for $-\frac{\pi}{2} \le x \le \pi$

42. Find the area of the region between the graph of $f(x) = \frac{1}{x}$ and the $x\text{-}axis$, for $-2 \le x \le -1$

43. Find the area of the region bounded by the graph of $f(x) = x^2 - 4x + 3 \quad x\text{-}axis \ on \ 0 \le x \le 3$

44. Find the area of the region bounded by the graph of $f(x) = x^2 + 4x + 3 \quad x\text{-}axis \ on \ -3 \le x \le 0$

45. Find the area of the region bounded by the graph of $f(x) = x^2 - 3x + 2 \quad x\text{-}axis \ on \ 0 \le x \le 2$

46. Find the area of the region bounded by the graph of $f(x) = x^2 + 3x + 2 \quad x\text{-}axis \ on \ -2 \le x \le 0$

47. Find the area of the region bounded by the graph of $f(x) = 2x^2 - 4x + 2 \quad x\text{-}axis \ on \ 0 \le x \le 2$

48. Find the area of the region bounded by the graph of $f(x) = 2x^2 + 4x + 2 \quad x\text{-}axis \ on \ -1 \le x \le 1$

49. Find the area of the region bounded by the graphs of $x = y^2 - y$ and $x = 2y^2 - 2y - 6$

50. Find the area of the region bounded by the graphs of $y = x^2 - 4$ & $y = -x^2 - 2x$

Compute the area of the region bounded by the graph of f and the $x\text{-}axis$ on the given interval.

51. $f(x) = \dfrac{1}{x^2 + 1} \quad on \quad \left[-1, \sqrt{3}\right]$

52. $f(x) = 2\sin\frac{x}{4} \quad on \quad [0, 2\pi]$

53. Archimedes, inventor, military engineer, physicist, and the greatest mathematician of classical times in the Western world, discovered that the area under a parabolic arch is two-thirds the base times the height. Sketch the parabolic arch $y = h - \left(\dfrac{4h}{b^2}\right)x^2 \quad -\frac{b}{2} \le x \le \frac{b}{2}$, assuming that h and b are positive.

Then use calculus to find the area of the region enclosed between the arch and the x-axis

54. Suppose that a company's marginal revenue from the manufacture and sale of eggbeaters is

$$\frac{dr}{dx} = 2 - \frac{2}{(x+1)^2}$$

Where r is measured in thousands of dollars and x in thousands of units. How much money should the company expect from a production run of $x = 3$ thousand eggbeaters? To find out, integrate the marginal revenue from $x = 0$ to $x = 3$.

55. The height H (ft) of a palm tree after growing for t years is given by

$$H = \sqrt{t+1} + 5t^{1/3} \quad \textit{for} \quad 0 \le t \le 8$$

a) Find the tree's height when $t = 0$, $t = 4$, and $t = 8$.

b) Find the tree's average height for $0 \le t \le 8$

Section 4.5 – Working with Integrals

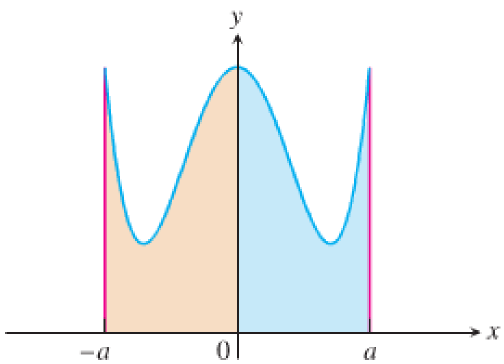
Definite Integrals of Symmetric Functions

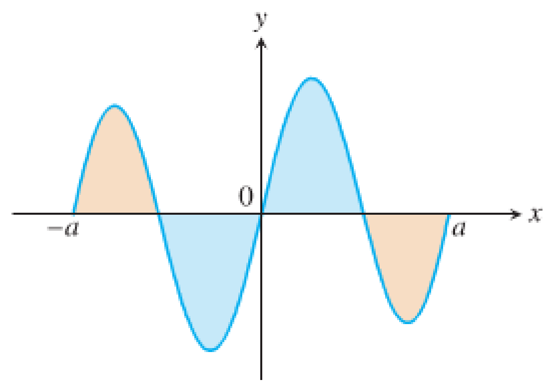
Theorem

Let f be continuous on the symmetric interval $[-a, a]$

- ✓ If f is even, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

- ✓ If f is odd, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$





Example

Evaluate $\displaystyle\int_{-2}^{2}\left(x^4 - 4x^2 + 6\right)dx$

Solution

Since $f(-x) = f(x) \quad\Rightarrow\quad f(x)$ *is even*

$$\int_{-2}^{2}\left(x^4 - 4x^2 + 6\right)dx = 2\int_{0}^{2}\left(x^4 - 4x^2 + 6\right)dx$$

$$= 2\left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\right]_0^2$$

$$= 2\left[\left(\frac{2^5}{5} - \frac{4}{3}2^3 + 6(2)\right) - 0\right]$$

$$= \frac{232}{15}$$

Average Value of a Continuous Function Revisited

The average value of a nonnegative continuous function *f* over an interval [*a*, *b*], leading us to define this average as the area under the graph of $y = f(x)$ divided by *b* – *a*.

$$Average = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Definition

If *f* is integrable on [*a, b*], then its *average value* on [*a*, *b*], also called its ***mean***, is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example

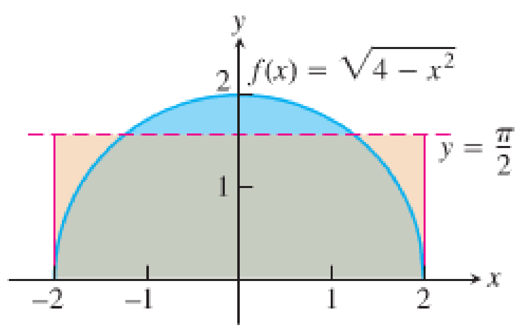
Find the average value of $f(x) = \sqrt{4-x^2}$ *on* $[-2,\ 2]$

Solution

$f(x) = \sqrt{4-x^2}$ is a function of an upper semicircle with a radius 2 and centered at the origin.

The area between the semicircle and the x-axis from –2 to 2 can be computed using the geometry formula:

$$Area = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(2)^2 = 2\pi \text{ -}$$



$$Area = \int_{-2}^{2}\sqrt{4-x^2}\,dx = 2\pi$$

$$av(f) = \frac{1}{2-(-2)}\int_{-2}^{2}\sqrt{4-x^2}\,dx$$

$$= \frac{1}{4}(2\pi)$$

$$= \boxed{\frac{\pi}{2}}$$

Example

We can model the voltage in the electrical wiring of a typical home with the sine function

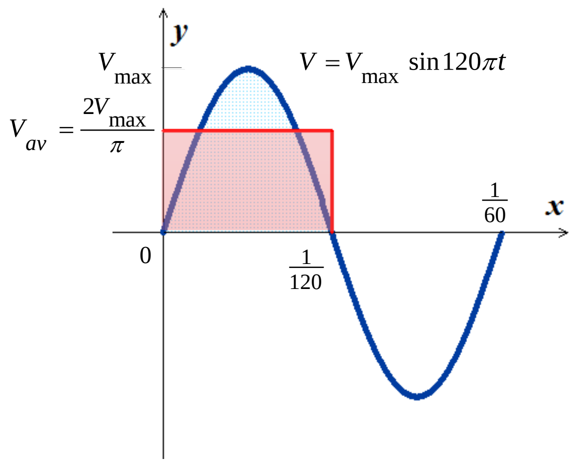
$$V = V_{\max} \sin 120\pi t$$

Which express the voltage V in volts as a function of time t in seconds. The function runs through 60 cycles each second (its frequency is 60 Hz (*hertz*)). The positive constant $V_{\max}$ is the ***peak voltage***.

Solution

The average value of V over the half-cycle from 0 to $\frac{1}{120}$ sec is

$$\begin{aligned}
V_{av} &= \frac{1}{\frac{1}{120} - 0} \int_0^{1/120} V_{\max} \sin 120\pi t \ dt \\
&= 120 V_{\max} \int_0^{1/120} \sin 120\pi t \ dt \\
&= 120 V_{\max} \left[-\frac{1}{120\pi} \cos 120\pi t \right]_0^{1/120} \\
&= -\frac{1}{\pi} V_{\max} \left(\cos\left(120\pi \frac{1}{120}\right) - \cos(120\pi \cdot 0) \right) \\
&= -\frac{1}{\pi} V_{\max} (\cos \pi - \cos 0) \\
&= -\frac{1}{\pi} V_{\max} (-1 - 1) \\
&= \frac{2}{\pi} V_{\max}
\end{aligned}$$



The average value of the voltage over a full cycle is zero.

To measure the voltage effectively, we can use an instrument the square root of the average value of the square of the voltage, namely:

$$V_{rms} = \sqrt{\left(V^2\right)_{av}}$$

"rms" : root mean square.

$$V_{av}^2 = \frac{1}{\frac{1}{60} - 0} \int_0^{1/60} \left(V_{\max}\right)^2 \sin^2 120\pi t \ dt = \frac{\left(V_{\max}\right)^2}{2}$$

The rms voltage is: $V_{rms} = \sqrt{\left(V^2\right)_{av}} = \dfrac{V_{\max}}{\sqrt{2}}$

The "115 volts ac" means that the rms voltage is 115. The peak voltage is:

$$V_{\max} = \sqrt{2}\ V_{rms} = \sqrt{2}\ (115) \approx 163 \ \ volts$$

Exercises Section 4.5 – Working with Integrals

1. If *f* is an odd function, why is $\int_{-a}^{a} f(x)\,dx = 0$?

2. If *f* is an even function, why is $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

3. Is x^{12} an even or odd function? Is $\sin\left(x^2\right)$ an even or odd function?

Use symmetry to evaluate the following integrals

4. $\int_{-2}^{2} x^9\,dx$

5. $\int_{-200}^{200} 2x^5\,dx$

6. $\int_{-\pi/4}^{\pi/4} \cos x\ dx$

7. $\int_{-2}^{2} \left(x^9 - 3x^5 + 2x^2 - 10\right)dx$

8. $\int_{-\pi/2}^{\pi/2} \left(\cos 2x + \cos x \sin x - 3\sin x^5\right)\ dx$

Find the average value of the following functions on the given interval.

9. $f(x) = x^3 \quad on\ [-1,\ 1]$

10. $f(x) = \frac{1}{x^2+1} \quad on\ [-1,\ 1]$

11. $f(x) = \frac{1}{x} \quad on\ [1,\ e]$

12. $f(x) = e^{2x} \quad on\ [0,\ \ln 2]$

Suppose that $\int_{0}^{4} f(x)\,dx = 10$ and $\int_{0}^{4} g(x)\,dx = 20$. Furthermore, suppose that f is an even function and g is an odd function. Evaluate the integrals

13. $\int_{-4}^{4} f(x)\,dx$

14. $\int_{-4}^{4} 3g(x)\,dx$

15. $\int_{0}^{1} 8xf\left(4x^2\right)dx$

16. $\int_{-2}^{2} 3xf(x)\,dx$

17. $\int_{-4}^{4} \left(4f(x) - 3g(x)\right)dx$

Zuppose that f is an even function with $\int_0^8 f(x)\,dx = 9$. Evaluate the integral

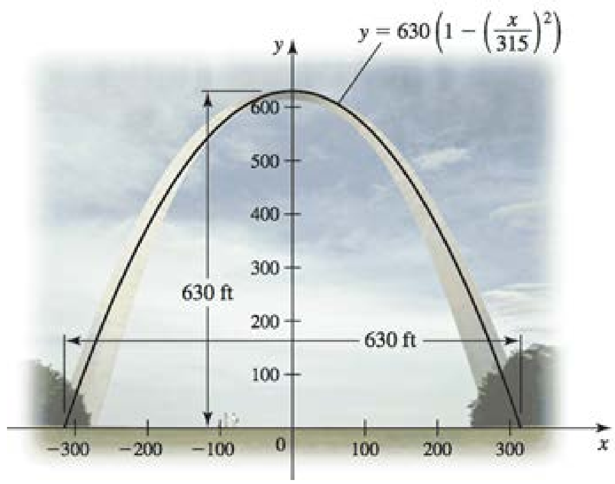
18. $\int_{-1}^{1} x\, f\left(x^2\right) dx$ **19.** $\int_{-2}^{2} x^2\, f\left(x^3\right) dx$

Suppose that p is a nonzero real number and f is an odd integrable function with $\int_0^1 f(x)\,dx = \pi$.

Evaluate the integral

20. $\int_0^{\frac{\pi}{2p}} (\cos px)\, f(\sin px)\,dx$ **21.** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)\, f(\sin x)\,dx$

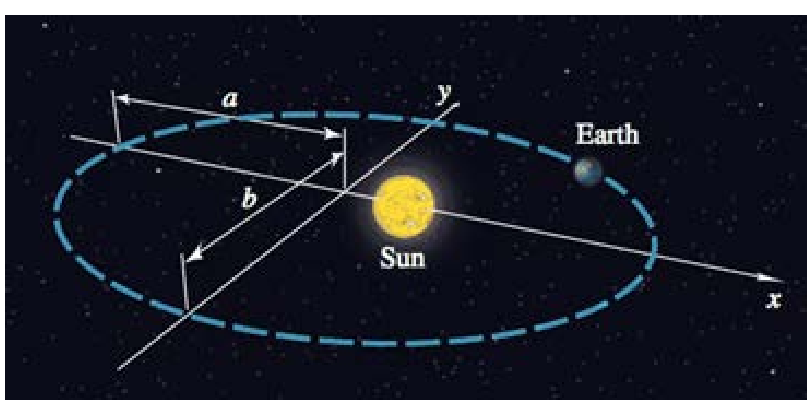
22. The Gateway Arch in St. Louis is 630 *ft* high and has a 630-*ft* base. Its shape can be modeled by the parabola



$$y = 630\left(1 - \left(\frac{x}{315}\right)^2\right)$$

Find the average height of the arch above the ground.

23. The planets orbit the Sun in elliptical orbits with the Sun at one focus. The equation of an ellipse whose dimensions are 2 a in the x-direction and 2 b in the y-direction is



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ W}$$

a) Let d^2 denote the square of the distance from a planet to the center of the ellipse at (0, 0). Integrate over the interval $[-a,\ a]$ to show that the average value of d^2 is $\frac{a^2+2b^2}{3}$

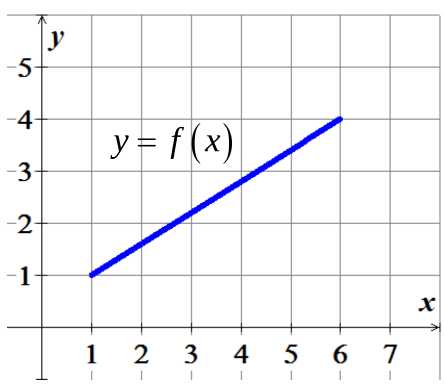
b) Show that in the case of a circle ($a = b = R$), the average value in part (a) is R^2.

c) Assuming $0 < b < a$, the coordinates of the Sun are $\left(\sqrt{a^2-b^2},\ 0\right)$. Let D^2 denote the square of the distance from the planet to the Sun. Integrate over the interval $[-a,\ a]$ to show that the average value of D^2 is $\frac{4a^2-b^2}{3}$.

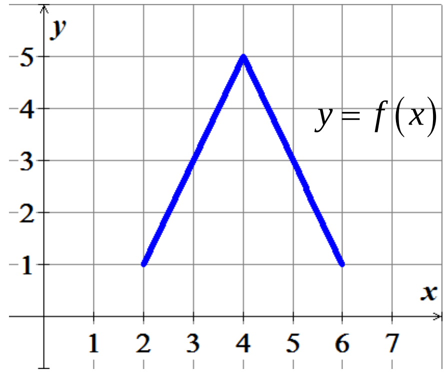
24. A particle moves along a line with a velocity given by $v(t)=5\sin\pi t$ starting with an initial position $s(0)=0$. Find the displacement of the particle between $t=0$ and $t=2$, which is given by $s(t)=\int_0^2 v(t)\,dt$. Find the distance traveled by the particle during this interval, which is $\int_0^2 |v(t)|\,dt$.

25. A baseball is launched into the outfield on a parabolic trajectory given by $y=0.01x(200-x)$. Find the average height of the baseball over the horizontal extent of its flight.

26. Find the average value of f shown in the figure on the interval $[1,\ 6]$ and then find the point(s) c in $(1,\ 6)$ guaranteed to exist by the Mean Value Theorem for Integrals



27. Find the average value of f shown in the figure on the interval $[2,\ 6]$ and then find the point(s) c in $(2,\ 6)$ guaranteed to exist by the Mean Value Theorem for Integrals



Section 4.6 – Substitution Rule

Substitution: Running the Chain Rule Backwards

The Chain rule formula is:

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right)=u^n\frac{du}{dx}$$

We can see that $\frac{u^{n+1}}{n+1}$ is an antiderivative of the function $u^n\frac{du}{dx}$. Therefore, if we integrate both sides

$$\int u^n\frac{du}{dx}dx=\int\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right)dx$$

$$\boxed{\int u^n du=\frac{u^{n+1}}{n+1}+C}$$

Example

Find the integral $\int\left(x^3+x\right)^5\left(3x^2+1\right)dx$

Solution

Let: $u=x^3+x$

$$du=\frac{du}{dx}dx$$

$$=\left(3x^2+1\right)dx$$

$$\int\left(x^3+x\right)^5\left(3x^2+1\right)dx=\int u^5du$$

$$=\frac{u^6}{6}+C$$

$$=\frac{1}{6}\left(x^3+x\right)^6+C$$

$$d\left(x^3+x\right)=\left(3x^2+1\right)dx$$

$$\int\left(x^3+x\right)^5\left(3x^2+1\right)dx=\int\left(x^3+x\right)^5d\left(x^3+x\right)$$

$$=\frac{1}{6}\left(x^3+x\right)^6+C$$

Example

Find the integral $\int \sqrt{2x+1}dx$

Solution

Let: $u = 2x+1 \quad \Rightarrow \quad du = \frac{du}{dx}dx = 2dx$

$$dx = \frac{1}{2}du$$

$$\int \sqrt{2x+1}dx = \frac{1}{2}\int u^{1/2}du$$

$$= \frac{1}{2}\frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{3}(2x+1)^{3/2} + C$$

$d(2x+1) = 2dx$

$$\int \sqrt{2x+1}\,dx = \frac{1}{2}\int (2x+1)^{1/2}d(2x+1)$$

$$= \frac{1}{3}(2x+1)^{3/2} + C$$

Theorem – The Substitution Rule

If $u = g(x)$ is differentiable function whose range is an interval *I*, and *f* is continuous on *I*, then

$$\int f((g(x)))g'(x)dx = \int f(u)du$$

Proof

By the Chain Rule, $F(g(x))$ is an antiderivative of $f(g(x))\cdot g'(x)$ whenever *F* is an antiderivative of *f*:

$$\frac{d}{dx}F(g(x)) = F'(g(x))\cdot g'(x)$$

$$= f(g(x))\cdot g'(x)$$

If we make the substitution $u = g(x)$, then

$$\int f(g(x))g'(x)dx = \int \frac{d}{dx}F(g(x))dx$$

$$= F(g(x)) + C$$

$$= F(u) + C$$

$$= \int F'(u)du$$

$$= \int f(u)du$$

Integral of $\int \frac{1}{u} du$

If u is a differentiable function that is never zero $\qquad \int \frac{1}{u} du = \ln|u| + C$

Example

Evaluate the integral $\int \tan x dx$

Solution

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx \qquad u = \cos x > 0 \quad \to du = -\sin x dx \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\begin{aligned} \int \tan x dx &= -\int \frac{d(\cos x)}{\cos x} \\ &= -\ln|\cos x| + C \\ &= \ln \frac{1}{|\cos x|} + C \\ &= \ln|\sec x| + C \end{aligned}$$

Example

Evaluate the integral $\int_0^2 \frac{2x}{x^2 - 5} dx$

Solution

$$u = x^2 - 5 \quad \to du = 2x dx \quad \begin{cases} x = 0 & \to u = -5 \\ x = 2 & \to u = -1 \end{cases}$$

$$\begin{aligned} \int_0^2 \frac{2x}{x^2-5} dx &= \int_{-5}^{-1} \frac{du}{u} \\ &= \ln|u| \Big|_{-5}^{-1} \\ &= \ln|-1| - \ln|-5| \\ &= \ln 1 - \ln 5 \\ &= -\ln 5 \end{aligned}$$

$$d\left(x^2 - 5\right) = 2x dx$$

$$\begin{aligned} \int_0^2 \frac{2x}{x^2-5} dx &= \int_0^2 \frac{d\left(x^2-5\right)}{x^2-5} \\ &= \ln\left|x^2 - 5\right| \Big|_0^2 \\ &= \ln|-1| - \ln|-5| \\ &= -\ln 5 \end{aligned}$$

Example

Find the integral $\int \sec^2(5t+1)\cdot 5dt$

Solution

$$\int \sec^2(5t+1)\cdot 5dt = \int \sec^2(5t+1)\,d(5t+1) \qquad d(5t+1)=5dt$$

$$\underline{= \tan(5t+1)+C} \qquad \frac{d}{du}\tan u = \sec^2 u$$

The General Antiderivative of the Exponential Function

$$\int e^u du = e^u + C$$

Example

Evaluate the integral $\int_0^{\ln 2} e^{3x}dx$

Solution

$$u=3x \quad du=3dx \to dx=\frac{1}{3}dx \quad \begin{cases} x=0 & \to u=0 \\ x=\ln 2 & \to u=3\ln 2=\ln 2^3=\ln 8 \end{cases}$$

$$\int_0^{\ln 2} e^{3x}dx = \int_0^{\ln 8} e^u \cdot \frac{1}{3}du$$

$$= \frac{1}{3}\int_0^{\ln 8} e^u du$$

$$= \frac{1}{3}e^u \Big|_0^{\ln 8}$$

$$= \frac{1}{3}\left(e^{\ln 8} - e^0\right)$$

$$= \frac{1}{3}(8-1)$$

$$\underline{= \frac{7}{3}}$$

Example

Find the integral $\int \cos(7\theta+3)\,d\theta$

Solution

$$\int \cos(7\theta+3)\,d\theta = \frac{1}{7}\int \cos(7\theta+3)\ d(7\theta+3) \qquad\qquad d(7\theta+3) = 7d\theta$$

$$\boxed{= \frac{1}{7}\sin(7\theta+3)+C}$$

Example

Find the integral $\int x^2 \sin\left(x^3\right)dx$

Solution

$$\int x^2 \sin\left(x^3\right)dx = \frac{1}{3}\int \sin\left(x^3\right)d\left(x^3\right)$$

$$\boxed{= -\frac{1}{3}\cos\left(x^3\right)+C}$$

Example

Find the integral $\int x\sqrt{2x+1}\ dx$

Solution

Let: $u = 2x+1$

$du = 2dx$

$dx = \frac{1}{2}du$

$u = 2x+1$

$2x = u-1$

$\Rightarrow \quad x = \frac{u-1}{2}$

$$\int x\sqrt{2x+1}\ dx = \int \frac{1}{2}(u-1)\sqrt{u}\ \ \frac{1}{2}du$$

$$= \frac{1}{4}\int (u-1)u^{1/2}\ du$$

$$= \frac{1}{4}\int \left(u^{3/2} - u^{1/2}\right) du$$

$$= \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C$$

$$= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C$$

Example

Find the integral $\displaystyle\int \frac{2z\, dz}{\sqrt[3]{z^2+1}}$

Solution

Let: $u = z^2 + 1 \quad \Rightarrow \quad du = 2zdz$

$$\int \frac{2z\, dz}{\sqrt[3]{z^2+1}} = \int \frac{du}{u^{1/3}}$$

$$= \int u^{-1/3} du$$

$$= \frac{3}{2}u^{2/3} + C$$

$$= \frac{3}{2}\left(z^2+1\right)^{2/3} + C$$

Or let: $u = \sqrt[3]{z^2+1} \quad \rightarrow \quad u^3 = z^2 + 1$

$$3u^2 du = 2zdz$$

$$\int \frac{2z\, dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2 du}{u}$$

$$= 3\int u\, du$$

$$= 3 \cdot \frac{u^2}{2} + C$$

$$= \frac{3}{2}\left(z^2+1\right)^{2/3} + C$$

Definition

If $a > 0$ and u is a differentiable of x, then a^u is a differentiable function of x and

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx} \quad \rightarrow \quad \int a^u du = \frac{a^u}{\ln a} + C$$

Example

- $\displaystyle\int 2^x dx = \frac{2^x}{\ln 2} + C$

- $\displaystyle\int 2^{\sin x} \cos x dx = \int 2^{\sin x} d(\sin x)$

$$= \frac{2^{\sin x}}{\ln 2} + C$$

➢ $$\int \frac{\log_2 x}{x}dx = \int \frac{1}{x}\frac{\ln x}{\ln 2}dx$$

$$= \frac{1}{\ln 2}\int \frac{\ln x}{x}dx$$

$$= \frac{1}{\ln 2}\int \ln x \, d(\ln x)$$

$$= \frac{1}{\ln 2}\cdot\frac{1}{2}(\ln x)^2 + C$$

$$= \boxed{\frac{(\ln x)^2}{2\ln 2} + C}$$

Substitution Formula

Theorem

If g' is continuous on the interval [a, b] and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x))\cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Proof

Let F denote any antiderivative of f. Then

$$\int_a^b f(g(x))\cdot g'(x)dx = F(g(x))\Big|_{x=a}^{x=b}$$

$$= F(g(b)) - F(g(a))$$

$$= F(u)\Big|_{u=g(a)}^{u=g(b)}$$

$$= \int_{g(a)}^{g(b)} f(u)du$$

Example

Evaluate $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$

Solution

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \int_{-1}^{1}\left(x^3+1\right)^{1/2} d\left(x^3+1\right) \qquad d\left(x^3+1\right)=3x^2dx$$

$$=\frac{2}{3}\left(x^3+1\right)^{3/2}\Big|_{-1}^{1}$$

$$=\frac{2}{3}\left(2^{3/2}-0^{3/2}\right)$$

$$=\frac{2}{3}2^{3/2} \qquad 2^{5/2}=\sqrt{2^5}=2^2\sqrt{2}$$

$$=\frac{4\sqrt{2}}{3}$$

Example

Evaluate $\displaystyle\int_{\pi/4}^{\pi/2}\cot\theta\csc^2\theta\,d\theta$

Solution

Let $u=\cot\theta \quad\Rightarrow\quad du=-\csc^2\theta d\theta \to -du=\csc^2\theta d\theta$

$$\to\begin{cases}\theta=\frac{\pi}{4} & u=\cot\frac{\pi}{4}=1\\ \theta=\frac{\pi}{2} & u=\cot\frac{\pi}{2}=0\end{cases}$$

$$\int_{\pi/4}^{\pi/2}\cot\theta\csc^2\theta\,d\theta=\int_{1}^{0}u\cdot(-du)$$

$$=-\int_{1}^{0}u\,du$$

$$=-\frac{u^2}{2}\Big|_{1}^{0}$$

$$=-\left(0-\frac{1}{2}\right)$$

$$=\frac{1}{2}$$

Example

Evaluate the integral $\int_0^{\pi/6} \tan 2x dx$

Solution

$$\int_0^{\pi/6} \tan 2x dx = \int_0^{\pi/6} \tan u \cdot \left(\frac{du}{2}\right) \qquad u = 2x \quad \to du = 2dx \quad \Rightarrow dx = \frac{du}{2}$$

$$= \frac{1}{2}\int_0^{\pi/6} \tan u \cdot du$$

$$= \frac{1}{2}\ln\left|\sec 2x\right| \Big|_0^{\pi/6}$$

$$= \frac{1}{2}\left[\ln\left|\sec 2\frac{\pi}{6}\right| - \ln\left|\sec 0\right|\right]$$

$$= \frac{1}{2}(\ln 2 - \ln 1)$$

$$\boxed{= \frac{1}{2}\ln 2}$$

Integrals of $\sin^2 x$ *and* $\cos^2 x$

Example

Find the integral $\int \sin^2 x\, dx$

Solution

$$\int \sin^2 x\, dx = \frac{1}{2}\int (1-\cos 2x)\,dx \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + C$$

$$\boxed{= \frac{1}{2}x - \frac{1}{4}\sin 2x + C}$$

Example

Find the integral $\int \cos^2 x\, dx$

Solution

$$\int \cos^2 x\, dx = \int \frac{1+\cos 2x}{2}\,dx \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right] + C$$

$$\boxed{= \frac{x}{2} + \frac{\sin 2x}{4} + C}$$

Integration Formulas

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C \qquad \left(\textit{Valid for } u^2 < a^2\right)$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \qquad \left(\textit{Valid for all } u\right)$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C \qquad \left(\textit{Valid for } |u| > a > 0\right)$$

Exercises   Section 4.6 – Substitution Rule

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

1. $\int 2(2x+4)^5\,dx, \quad u = 2x+4$

2. $\int \frac{4x^3}{(x4+1)^2}dx, \quad u = x^4+1$

3. $\int x\sin\left(2x^2\right)dx, \quad u = 2x^2$

4. $\int 12\left(y^4+4y^2+1\right)^2\left(y^3+2y\right)dx, \quad u = y^4+4y^2+1$

5. $\int \csc^2 2\theta \cot 2\theta\, d\theta \to \begin{cases} a)\ U\sin g \quad u = \cot 2\theta \\ b)\ U\sin g \quad u = \csc 2\theta \end{cases}$

Evaluate the integrals

6. $\int \frac{1}{\sqrt{5s+4}}ds$

7. $\int \theta\sqrt[4]{1-\theta^2}\,d\theta$

8. $\int \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2}dx$

9. $\int \tan^2 x\sec^2 x\,dx$

10. $\int \sin^5\frac{x}{3}\cos\frac{x}{3}\,dx$

11. $\int \tan^7\frac{x}{2}\sec^2\frac{x}{2}\,dx$

12. $\int r^4\left(7-\frac{r^5}{10}\right)^3 dr$

13. $\int x^{1/2}\sin\left(x^{3/2}+1\right)dx$

14. $\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv$

15. $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}dt$

16. $\int \frac{\sec z\ \tan z}{\sqrt{\sec z}}dz$

17. $\int \frac{1}{\sqrt{t}}\cos\left(\sqrt{t}+3\right)dt$

18. $\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\,d\theta$

19. $\int t^3\left(1+t^4\right)^3 dt$

20. $\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}dx$

21. $\int x^3\sqrt{x^2+1}\,dx$

22. $\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}d\theta$

23. $\int 2x\sqrt{x^2-2}\,dx$

24. $\int \frac{x}{\left(x^2-4\right)^3}dx$

25. $\int x^3\left(3x^4+1\right)^2 dx$

26. $\int 2\left(3x^4+1\right)^2 dx$

27. $\int 5x\sqrt{x^2-1}\,dx$

28. $\int \left(x^2-1\right)^3(2x)dx$

29. $\int \frac{6x}{\left(1+x^2\right)^3}dx$

30. $\displaystyle\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}\,dr$

31. $\displaystyle\int u^3\sqrt{u^4+2}\;du$

32. $\displaystyle\int \frac{t+2t^2}{\sqrt{t}}\;dt$

33. $\displaystyle\int \left(1+\frac{1}{t}\right)^3 \frac{1}{t^2}\;dt$

34. $\displaystyle\int \left(7-3x-3x^2\right)(2x+1)\;dx$

35. $\displaystyle\int \sqrt{x}\left(4-x^{3/2}\right)^2\;dx$

36. $\displaystyle\int \frac{1}{\sqrt{x}+\sqrt{x+1}}\,dx$

37. $\displaystyle\int \sqrt{1-x}\;dx$

38. $\displaystyle\int x\sqrt{x^2+4}\;dx$

39. $\displaystyle\int \sin^2\left(\theta+\frac{\pi}{6}\right)d\theta$

40. $\displaystyle\int \cos^2(8\theta)\,d\theta$

41. $\displaystyle\int \sin^2(2\theta)\,d\theta$

42. $\displaystyle\int 8\cos^4 2\pi x\;dx$

43. $\displaystyle\int \sec x\,dx$

44. $\displaystyle\int \frac{dx}{\sqrt{1-4x^2}}$

45. $\displaystyle\int \frac{dx}{\sqrt{3-4x^2}}$

46. $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-6}}$

47. $\displaystyle\int \frac{dx}{\sqrt{4x-x^2}}$

48. $\displaystyle\int \frac{dx}{4x^2+4x+2}$

49. $\displaystyle\int \frac{1}{6x-5}\,dx$

50. $\displaystyle\int \frac{x^2+2x+3}{x^3+3x^2+9x+1}\,dx$

51. $\displaystyle\int \frac{1}{x(\ln x)^2}\,dx$

52. $\displaystyle\int \frac{x-3}{x+3}\,dx$

53. $\displaystyle\int \frac{3x}{x^2+4}\,dx$

54. $\displaystyle\int \frac{dx}{2\sqrt{x}+2x}$

55. $\displaystyle\int \frac{\sec x\,dx}{\sqrt{\ln(\sec x+\tan x)}}$

56. $\displaystyle\int 8e^{(x+1)}\,dx$

57. $\displaystyle\int 4xe^{x^2}\,dx$

58. $\displaystyle\int (2x+1)e^{x^2+x}\,dx$

59. $\displaystyle\int \frac{e^{-\sqrt{r}}}{\sqrt{r}}\,dr$

60. $\displaystyle\int t^3 e^{t^4}\,dt$

61. $\displaystyle\int e^{\sec\pi t}\sec\pi\ \ \tan\pi t\;dt$

62. $\displaystyle\int (2x+1)e^{x^2+x}\,dx$

63. $\displaystyle\int \frac{dx}{1+e^x}$

64. $\displaystyle\int \frac{e^x}{1+e^x}\,dx$

65. $\displaystyle\int \frac{2}{e^{-x}+1}\,dx$

66. $\displaystyle\int \frac{1}{x^3}e^{1/4x^2}\,dx$

67. $\displaystyle\int \frac{e^{1/\sqrt{x}}}{x^{3/2}}\,dx$

68. $\displaystyle\int \frac{-e^{3x}}{2-e^{3x}}\,dx$

69. $\displaystyle\int \frac{7e^{7x}}{3+e^{7x}}\,dx$

70. $\displaystyle\int \frac{2(e^x-e^{-x})}{(e^x+e^{-x})^2}\,dx$

71. $\displaystyle\int \frac{3^x}{3-3^x}\,dx$

72. $\displaystyle\int \frac{x\ 2^{x^2}}{1+2^{x^2}}\,dx$

73. $\int (6x+e^x)\sqrt{3x^2+e^x}\,dx$

74. $\int \frac{dx}{x\left(\log_8 x\right)^2}$

75. $\int \frac{dx}{x\sqrt{25x^2-2}}$

76. $\int \frac{6dr}{\sqrt{4-(r+1)^2}}$

77. $\int \frac{dx}{2+(x-1)^2}$

78. $\int \frac{\sec^2 y\,dy}{\sqrt{1-\tan^2 y}}$

79. $\int \frac{dx}{\sqrt{-x^2+4x-3}}$

80. $\int \frac{dx}{\sqrt{2x-x^2}}$

81. $\int \frac{x-2}{x^2-6x+10}dx$

82. $\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$

83. $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$

84. $\int \frac{e^{\cos^{-1}x}dx}{\sqrt{1-x^2}}$

85. $\int \frac{\left(\sin^{-1}x\right)^2 dx}{\sqrt{1-x^2}}$

86. $\int \frac{dy}{\left(\sin^{-1}y\right)\sqrt{1+y^2}}$

87. $\int \frac{1}{\sqrt{x}(x+1)\left(\left(\tan^{-1}\sqrt{x}\right)^2+9\right)}dx$

88. $\int 2x\left(x^2+1\right)^4 dx$

89. $\int 8x\cos\left(4x^2+3\right)dx$

90. $\int \sin^3 x\cos x\,dx$

91. $\int (6x+1)\sqrt{3x^2+x}\,dx$

92. $\int 2x\left(x^2-1\right)^{99} dx$

93. $\int xe^{x^2} dx$

94. $\int \frac{2x^2}{\sqrt{1-4x^3}}dx$

95. $\int \frac{\left(\sqrt{x}+1\right)^4}{2\sqrt{x}}dx$

96. $\int \left(x^2+x\right)^{10}(2x+1)dx$

97. $\int \frac{dx}{10x-3}$

98. $\int x^3\left(x^4+16\right)^6 dx$

99. $\int \sin^{10}\theta\,\cos\theta\,d\theta$

100. $\int \frac{dx}{\sqrt{1-9x^2}}$

101. $\int x^9\sin x^{10}dx$

102. $\displaystyle\int \left(x^6-3x^2\right)^4\left(x^5-x\right)dx$

103. $\displaystyle\int \frac{x}{x-2}\,dx$

104. $\displaystyle\int \frac{dx}{1+4x^2}$

105. $\displaystyle\int \frac{3}{1+25y^2}\,dy$

106. $\displaystyle\int \frac{2}{x\sqrt{4x^2-1}}\,dx \quad \left(x>\frac{1}{2}\right)$

107. $\displaystyle\int \frac{8x+6}{2x^2+3x}\,dx$

108. $\displaystyle\int \frac{x}{\sqrt{x-4}}\,dx$

109. $\displaystyle\int \frac{x^2}{(x+1)^4}\,dx$

110. $\displaystyle\int \frac{x}{\sqrt[3]{x+4}}\,dx$

111. $\displaystyle\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

112. $\displaystyle\int x\,\sqrt[3]{2x+1}\,dx$

113. $\displaystyle\int (x+1)\sqrt{3x+2}\,dx$

114. $\displaystyle\int \sin^2 x\,dx$

115. $\displaystyle\int \sin^2\left(\theta+\frac{\pi}{6}\right)d\theta$

116. $\displaystyle\int x\cos^2\left(x^2\right)dx$

117. $\displaystyle\int \sec 4x\,\tan 4x\,dx$

118. $\displaystyle\int \sec^2 10x\,dx$

119. $\displaystyle\int \left(\sin^5 x+3\sin^3 x-\sin x\right)\cos x\,dx$

120. $\displaystyle\int \frac{\csc^2 x}{\cot^3 x}\,dx$

121. $\displaystyle\int \left(x^{3/2}+8\right)^5\sqrt{x}\,dx$

122. $\displaystyle\int \sin x\,\sec^8 x\,dx$

123. $\displaystyle\int \frac{e^{2x}}{e^{2x}+1}\,dx$

124. $\displaystyle\int \sec^3\theta\,\tan\theta\,d\theta$

125. $\displaystyle\int x\,\sin^4 x^2\,\cos x^2\,dx$

126. $\displaystyle\int \frac{dx}{\sqrt{1+\sqrt{1+x}}}$

127. $\displaystyle\int \tan^{10} 4x\,\sec^2 4x\,dx$

128. $\displaystyle\int \frac{x^2}{x^3+27}\,dx$

129. $\displaystyle\int y^2\left(3y^3+1\right)^4 dy$

130. $\displaystyle\int x\sin x^2\cos^8 x^2\,dx$

131. $\displaystyle\int \frac{\sin 2x}{1+\cos^2 x}\,dx$

132. $\displaystyle\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$

133. $\displaystyle\int \frac{dx}{\left(\tan^{-1} x\right)\left(1+x^2\right)}$

134. $\displaystyle\int \frac{\left(\tan^{-1} x\right)^5}{1+x^2}\,dx$

135. $\displaystyle\int \frac{1}{x^2}\sin\frac{1}{x}\,dx$

136. Evaluate the integral $\displaystyle\int \frac{18\tan^2 x\sec^2 x}{\left(2+\tan^3 x\right)^2}\,dx$

a) $u=\tan x$, followed by $v=u^3$ *then by* $w=2+v$

b) $u=\tan^3 x$, followed by $v=2+u$

c) $u=2+\tan^3 x$

Evaluate the integrals

137. $\displaystyle\int_0^1 (2t+3)^3\,dt$

138. $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$

139. $\displaystyle\int_0^3 \sqrt{y+1}\,dy$

140. $\displaystyle\int_{-1}^1 r\sqrt{1-r^2}\,dr$

141. $\displaystyle\int_0^{\pi/4} \tan x\sec^2 x\,dx$

142. $\displaystyle\int_{2\pi}^{3\pi} 3\cos^2 x\sin x\,dx$

143. $\displaystyle\int_0^1 t^3\left(1+t^4\right)^3\,dt$

144. $\displaystyle\int_0^1 \frac{r}{\left(4+r^2\right)^2}\,dr$

145. $\displaystyle\int_0^1 \frac{10\sqrt{v}}{\left(1+v^{3/2}\right)^2}\,dv$

146. $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$

147. $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$

148. $\displaystyle\int_0^{\pi/6} (1-\cos 3t)\sin 3t\,dt$

149. $\displaystyle\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$

150. $\displaystyle\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$

151. $\displaystyle\int_{-\pi/2}^{0} \frac{\sin w}{(3+2\cos w)^2}\,dw$

152. $\int_0^1 \sqrt{t^5+2t}\,\left(5t^4+2\right)dt$

153. $\int_1^4 \frac{dy}{2\sqrt{y}\left(1+\sqrt{y}\right)^2}$

154. $\int_0^1 \left(4y-y^2+4y^3+1\right)^{-2/3}\left(12y^2-2y+4\right)dy$

155. $\int_0^5 |x-2|\,dx$

156. $\int_0^{\pi/2} e^{\sin x}\cos x\,dx$

157. $\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$

158. $\int_0^{\pi/3} \frac{4\sin\theta}{1-4\cos\theta}\,d\theta$

159. $\int_1^2 \frac{2\ln x}{x}\,dx$

160. $\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$

161. $\int_0^{\pi/2} \tan\frac{x}{2}\,dx$

162. $\int_{\pi/4}^{\pi/2} \cot x\,dx$

163. $\int_{-\ln 2}^{0} e^{-x}dx$

164. $\int_{\pi/4}^{\pi/2} \left(1+e^{\cot\theta}\right)\csc^2\theta\,d\theta$

165. $\int_0^{\sqrt{\ln\pi}} 2x\,e^{x^2}\cos\left(e^{x^2}\right)dx$

166. $\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx$

167. $\int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan t}\sec^2 t\,dt$

168. $\int_1^e x^{(\ln 2)-1}dx$

169. $\int_0^9 \frac{2\log_{10}(x+1)}{x+1}\,dx$

170. $\int_1^e \frac{2\ln 10\log_{10} x}{x}\,dx$

171. $\int_1^{e^x} \frac{1}{t}\,dt$

172. $\frac{1}{\ln a}\int_1^x \frac{1}{t}\,dt \quad x>0$

173. $\int_0^{3\sqrt{2}/4} \frac{dx}{\sqrt{9-4x^2}}$

174. $\int_{\pi/6}^{\pi 4} \frac{\csc^2 x dx}{1+(\cot x)^2}$

175. $\int_1^{e^{\pi/4}} \frac{4dt}{t\left(1+\ln^2 t\right)}$

176. $\int_{1/2}^{1} \frac{6dx}{\sqrt{-4x^2+4x+3}}$

177. $\displaystyle\int_{2/\sqrt{3}}^{2} \frac{\cos\left(\sec^{-1} x\right) dx}{x\sqrt{x^2-1}}$

178. $\displaystyle\int_0^3 \frac{x}{\sqrt{25-x^2}}\,dx$

179. $\displaystyle\int_0^{\pi} \sin^2 5\theta \, d\theta$

180. $\displaystyle\int_0^{\pi} \left(1-\cos^2 3\theta\right) \, d\theta$

181. $\displaystyle\int_2^3 \frac{x^2+2x-2}{x^3+3x^2-6x}\,dx$

182. $\displaystyle\int_0^{\ln 2} \frac{e^x}{1+e^{2x}}\,dx$

183. $\displaystyle\int_1^3 x\sqrt[3]{x^2-1}\,dx$

184. $\displaystyle\int_0^2 (x+3)^3\,dx$

185. $\displaystyle\int_{-2}^{2} e^{4x+8} dx$

186. $\displaystyle\int_0^1 \sqrt{x}\left(\sqrt{x}+1\right) dx$

187. $\displaystyle\int_0^1 \frac{dx}{\sqrt{4-x^2}}$

188. $\displaystyle\int_0^2 \frac{2x}{\left(x^2+1\right)^2}\,dx$

189. $\displaystyle\int_0^{\pi/2} \sin^2\theta \ \cos\theta \, d\theta$

190. $\displaystyle\int_0^{\pi/4} \frac{\sin x}{\cos^2 x}\,dx$

191. $\displaystyle\int_{1/3}^{1/\sqrt{3}} \frac{4}{9x^2+1}\,dx$

192. $\displaystyle\int_0^{\ln 4} \frac{e^x}{3+2e^x}\,dx$

193. $\displaystyle\int_{-\pi}^{\pi} \cos^2 x \, dx$

194. $\displaystyle\int_0^{\pi/4} \cos^2 8\theta \, d\theta$

195. $\displaystyle\int_{-\pi/4}^{\pi/4} \sin^2 2\theta \, d\theta$

196. $\displaystyle\int_0^{\pi/6} \frac{\sin 2x}{\sin^2 x+2}\,dx$

197. $\displaystyle\int_0^{\pi/2} \sin^4\theta \, d\theta$

198. $\displaystyle\int_0^1 x\sqrt{1-x^2}\ dx$

199. $\displaystyle\int_0^{1/4} \frac{x}{\sqrt{1-16x^2}}\,dx$

200. $\displaystyle\int_2^3 \frac{x}{\sqrt[3]{x^2-1}}\,dx$

201. $\displaystyle\int_0^{6/5} \frac{dx}{25x^2+36}$

202. $\displaystyle\int_0^2 x^3\ \sqrt{16-x^4}\ dx$

203. $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^2 x}\,dx$

204. $\displaystyle\int_{-1}^{1} (x-1)\left(x^2-2x\right)^7 dx$

205. $\displaystyle\int_{-\pi}^{0} \frac{\sin x}{2+\cos x}\,dx$

206. $\displaystyle\int_{0}^{1} \frac{(x+1)(x+2)}{2x^3+9x^2+12x+36}\,dx$

207. $\displaystyle\int_{1}^{2} \frac{4}{9x^2+6x+1}\,dx$

208. $\displaystyle\int_{0}^{\pi/4} e^{\sin^2 x}\sin 2x\,dx$

209. $\displaystyle\int_{0}^{1} x\sqrt{x+a}\,dx \quad (a>0)$

210. $\displaystyle\int_{0}^{1} x\sqrt[p]{x+a}\,dx \quad (a>0)$

211. $\displaystyle\int_{0}^{1} x\sqrt{1-\sqrt{x}}\,dx$

212. $\displaystyle\int_{0}^{1} \sqrt{x-x\sqrt{x}}\,dx$

213. $\displaystyle\int_{0}^{\pi/2} \frac{\cos\theta\sin\theta}{\sqrt{\cos^2\theta+16}}\,d\theta$

214. $\displaystyle\int_{\frac{2}{5\sqrt{3}}}^{\frac{2}{5}} \frac{dx}{x\sqrt{25x^2-1}}$

215. $\displaystyle\int_{0}^{4} \frac{x}{\sqrt{9+x^2}}\,dx$

216. $\displaystyle\int_{0}^{\pi/4} \frac{\sin\theta}{\cos^3\theta}\,d\theta$

217. $\displaystyle\int_{0}^{1} 2x\left(4-x^2\right) dx$

218. $\displaystyle\int_{0}^{3} \frac{x^2+1}{\sqrt{x^3+3x+4}}\,dx$

219. $\displaystyle\int_{0}^{4} \frac{x}{x^2+1}\,dx$

220. $\displaystyle\int_{1}^{e^2} \frac{\ln x}{x}\,dx$

221. $\displaystyle\int_{0}^{3} \frac{x^2+1}{\sqrt{x^3+3x+4}}\,dx$

222. $\displaystyle\int_{0}^{1} \left(y^3+6y^2-12y+9\right)^{-1/2}\left(y^2+4y-4\right)dy$

223. $\displaystyle\int_{-1}^{2} x^2 e^{x^3+1}\,dx$

224. $\displaystyle\int_{0}^{2} x^2 e^{x^3}\,dx$

225. $\displaystyle\int_{0}^{4} \frac{x}{x^2+1}\,dx$

Solve the initial value problem

226. $\dfrac{dy}{dt} = e^t \sin\left(e^t - 2\right), \quad y(\ln 2) = 0$

227. $\dfrac{dy}{dt} = e^{-t} \sec^2\left(\pi e^{-t}\right), \quad y(\ln 4) = \dfrac{2}{\pi}$

228. Verify the integration formula: $\displaystyle\int \frac{\tan^{-1} x}{x^2}\,dx = \ln x - \tfrac{1}{2}\ln\left(1+x^2\right) - \frac{\tan^{-1} x}{x} + C$

229. Verify the integration formula: $\displaystyle\int \ln\left(a^2 + x^2\right)dx = x\ln\left(a^2 + x^2\right) - 2x + 2a\tan^{-1}\frac{x}{a} + C$

230. Find the area of the region bounded by the graphs of $x = 3\sin y\sqrt{\cos y}$, *and* $x = 0$, $0 \le y \le \frac{\pi}{2}$

231. Find the area of the region bounded by the graph of $f(x) = \dfrac{x}{\sqrt{x^2 - 9}}$ *on* $3 \le x \le 4$

232. Find the area of the region bounded by the graph of $f(x) = \dfrac{x}{\sqrt{x^2 - 9}}$ and the *x-axis* between $x = 4$ and $x = 5$.

233. Find the area of the region bounded by the graph of $f(x) = x\ \sin x^2$ and the *x-axis* between $x = 0$ and $x = \sqrt{\pi}$.

234. Find the area of the region bounded by the graph of $f(\theta) = \cos\theta\ \sin\theta$ and the *θ-axis* between $\theta = 0$ and $\theta = \frac{\pi}{2}$.

235. Find the area of the region bounded by the graph of $f(x) = (x-4)^4$ and the *x-axis* between $x = 2$ and $x = 6$.

236. Perhaps the simplest change of variables is the shift or translation given by $u = x + c$, where c is a real number.

a) Prove that shifting a function does not change the net area under the curve, in the sense that

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(u)\,du$$

b) Draw a picture to illustrate this change of variables in the case that $f(x) = \sin x$, $a = 0$, $b = \pi$, and $c = \frac{\pi}{2}$

237. Another change of variables that can be interpreted geometrically is the scaling $u = cx$, where c is a real number. Prove and interpret the fact that

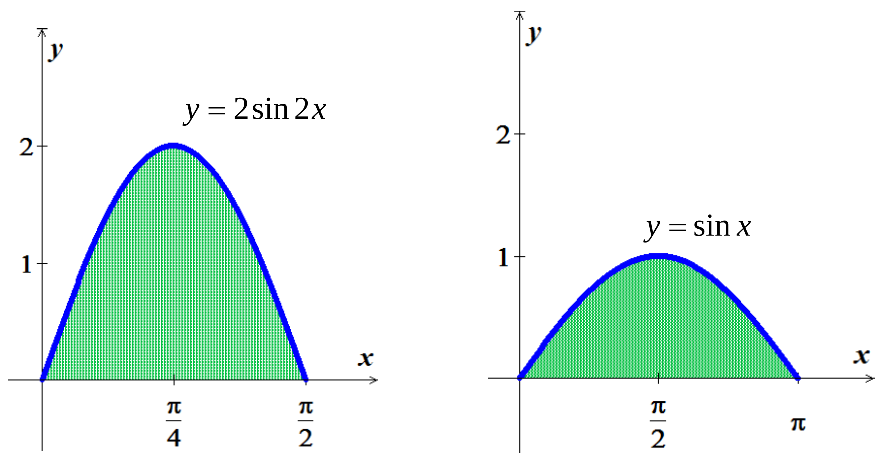
$$\int_a^b f(cx)\,dx = \frac{1}{c}\int_{ac}^{bc} f(u)\,du$$

Draw a picture to illustrate this change of variables in the case that $f(x) = \sin x$, $a = 0$, $b = \pi$, and $c = \frac{1}{2}$

238. The function f satisfies the equation $3x^4 - 48 = \int_2^x f(t)\,dt$. Find f and check your answer by substitution.

239. Assume f' is continuous on $[2,\ 4]$, $\int_1^2 f'(2x)\,dx = 10$, and $f(2) = 4$. Evaluate $f(4)$.

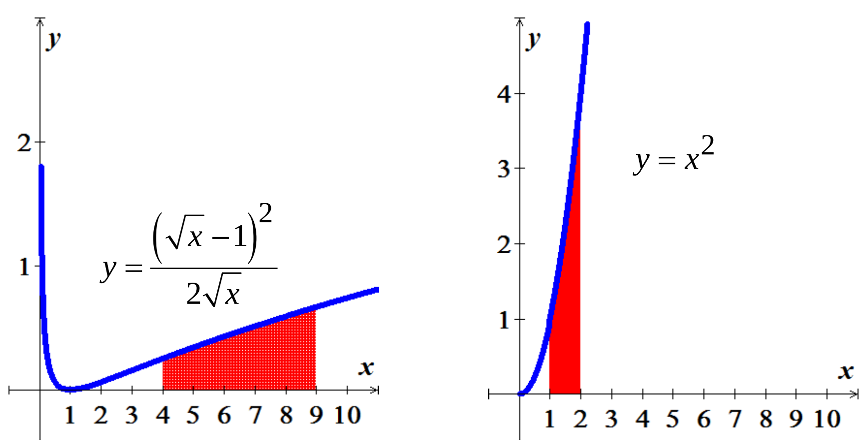
240. The area of the shaded region under the curve $y = 2\sin 2x$ in



a) Equals the area on the shaded region under the curve $y = \sin x$

b) Explain why this is true without computation areas.

241. The area of the shaded region under the curve $y = \dfrac{\left(\sqrt{x}-1\right)^2}{2\sqrt{x}}$ on the interval $[4,\ 9]$



a) Equals the area on the shaded region under the curve $y = x^2$ on the interval $[1,\ 2]$

b) Explain why this is true without computation areas.

242. The family of parabolas $y = \frac{1}{a} - \frac{x^2}{a^3}$, where $a > 0$, has the property that for $x \ge 0$, the *x*-intercept is $(a,\ 0)$ and the *y*-intercept is $\left(0,\ \frac{1}{a}\right)$. Let $A(a)$ be the area of the region in the first quadrant bounded by the parabola and the *x-axis*. Find $A(a)$ and determine whether it is increasing, decreasing, or constant function of a.

243. Consider the right triangle with vertices $(0,\ 0)$, $(0,\ b)$, and $(a,\ 0)$, where $a > 0$ and $b > 0$. Show that the average vertical distance from points on the *x-axis* to the hypotenuse is $\frac{b}{2}$, for all $a > 0$.

244. Consider the integral $I = \int \sin^2 x\ \cos^2 x\ dx$

a) Find I using the identity $\sin 2x = 2\sin x \cos x$

b) Find I using the identity $\cos^2 x = 1 - \sin^2 x$

c) Confirm that the results in part (*a*) and (*b*) are consistent and compare the work involved in each method.

245. Let $H(x) = \int_0^x \sqrt{4 - t^2}\, dt$, for $-2 \le x \le 2$.

a) Evaluate $H(0)$

b) Evaluate $H'(1)$

c) Evaluate $H'(2)$

d) Use geometry to evaluate $H(2)$

e) Find the value of s such that $H(x) = sH(-x)$

Evaluate the limits

246. $\lim\limits_{x \to 2} \dfrac{\int_2^x e^{t^2}\, dt}{x - 2}$

247. $\lim\limits_{x \to 1} \dfrac{\int_1^{x^2} e^{t^3}\, dt}{x - 1}$

248. Prove that for nonzero constants a and b, $\int \frac{dx}{a^2 x^2 + b^2} = \frac{1}{ab} \tan^{-1}\left(\frac{ax}{b}\right) + C$

249. Let $a > 0$ be a real number and consider the family of functions $f(x) = \sin ax$ on the interval $\left[0, \frac{\pi}{a}\right]$.

a) Graph f, for a = 1, 2, 3.

b) Let $g(a)$ be the area of the region bounded by the graph of f and the *x-axis* on the interval $\left[0, \frac{\pi}{a}\right]$. Graph g for $0 < a < \infty$. Is g an increasing function, a decreasing function, or neither?

250. Explain why if a function u satisfies the equation $u(x) + 2\int_0^x u(t)\,dt = 10$, then it also satisfies the equation $u'(x) + 2u(x) = 0$. Is it true that is u satisfies the second equation, then it satisfies the first equation?

251. Let $f(x) = \int_0^x (t-1)^{15}(t-2)^9\,dt$

a) Find the interval on which f is increasing and the intervals on which f is decreasing.

b) Find the intervals on which f is concave up and the intervals on which f is concave down.

c) For what values of x does f have local minima? Local maxima?

d) Where are the inflection points of f?

252. A company is considering a new manufacturing process in one of its plants. The new process provides substantial initial savings, with the savings declining with time t (in years) according to the rate-of-savings function

$$S'(t) = 100 - t^2$$

where $S'(t)$ is in thousands of dollars per year. At the same time, the cost of operating the new process increases with time t (in years), according to the rate-of-cost function (in thousands of dollars per year)

$$C'(t) = t^2 + \tfrac{14}{3}t$$

a) For how many years will the company realize savings?

b) What will be the net total savings during this period?

253. Find the producers' surplus if the supply function for pork bellies is given by

$$S(x) = x^{5/2} + 2x^{3/2} + 50$$

Assume supply and demand are in equilibrium at $x = 16$.

254. An object moves along a line with a velocity in m/s given by $v(t) = 8\cos\left(\frac{\pi t}{6}\right)$. Its initial position is $s(0) = 0$.

a) Graph the velocity function.

b) The position of the object is given by $s(t)=\int_0^t v(y)\,dy$, for $t\geq 0$. Find the position function, for $t\geq 0$.

c) What is the period of the motion – that is, starting at any point, how long does it take the object to return to that position?

255. The population of a culture of bacteria has a growth rate given by $p'(t)=\dfrac{200}{(t+1)^r}$ bacteria per hour, for $t\geq 0$, where $r>1$ is a real number. It is shown that the increase in the population over time interval $[0,\ t]$ is given by $\int_0^t p'(s)\,ds$. (note that the growth rate decreases in time, reflecting competition for space and food.)

a) Using the population model with $r=2$, what is the increase in the population over the time interval $0\leq t\leq 4$?

b) Using the population model with $r=3$, what is the increase in the population over the time interval $0\leq t\leq 6$?

c) Let ΔP be the increase in the population over a fixed time interval $[0,\ T]$. For fixed *T*, does ΔP increase or decrease with the parameter *r*? Explain.

d) A lab technician measures an increase in the population of 350 bacteria over the 10-hr period $[0,\ 10]$. Estimate the value of *r* that best fits this data point.

e) Use the population model in part (*b*) to find the increase in population over time interval $[0,\ T]$, for any $T>0$. If the culture is allowed to grow indefinitely $(T\to\infty)$, does the bacteria population increase without bound? Or does it approach a finite limit?

256. Consider the function $f(x)=x^2-5x+4$ and the area function $A(x)=\int_0^x f(t)\,dt$.

a) Graph f on the interval [0, 6].

b) Compute and graph *A* on the interval [0, 6].

c) Show that the local extrema of *A* occur at the zeros of f.

d) Give a geometric and analytical explanation for the observation in part (c).

e) Find the approximate zeros of *A*, other than 0, and call them x_1 and x_2.

f) Find *b* such that the area bounded by the graph of f and the *x*-axis on the interval $[0,\ x_1]$ equals the area bounded by the graph of f and the *x*-axis on the interval $[x_1,\ b]$.

g) If f is an integrable function and $A(x)=\int_0^x f(t)\,dt$, is it always true that the local extrema of *A* occur at the zeros of f? Explain