

Solution **Section 1.3 – Matrices and Matrix operations**

Exercise

For the matrices: $A = \begin{bmatrix} p & 0 \\ q & r \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, when does $AB = BA$

Solution

$$\begin{aligned} AB &= \begin{pmatrix} p & 0 \\ q & r \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} p & p \\ q & q+r \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & 0 \\ q & r \end{pmatrix} \\ &= \begin{pmatrix} p+q & r \\ q & r \end{pmatrix} \end{aligned}$$

$$AB = BA$$

$$\begin{pmatrix} p & p \\ q & q+r \end{pmatrix} = \begin{pmatrix} p+q & r \\ q & r \end{pmatrix}$$

$$\begin{cases} p = p+q \\ p = r \\ q+r = r \end{cases} \Rightarrow \begin{cases} q = 0 \\ p = r \\ q = 0 \end{cases}$$

Exercise

Find a combination $x_1 w_1 + x_2 w_2 + x_3 w_3$ that gives the zero vector:

$$w_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}; \quad w_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}; \quad w_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Those vectors are independent or dependent?

The vectors lie in a _____.

The matrix W with those columns is not invertible.

Solution

$$w_1 - 2w_2 + w_3 = 0; \text{ Therefore those vectors are dependent}$$

The vectors lie in a plane

Exercise

The very last words say that the 5 by 5 centered difference matrix is not invertible, Write down the 5 equations $Cx = b$. Find a combination of left sides that gives zero. What combination of b_1, b_2, b_3, b_4, b_5 must be zero?

Solution

The 5 by 5 centered difference matrix is

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

The five equations $Cx = b$ are:

$$x_2 = b_1, \quad -x_1 + x_3 = b_2, \quad -x_2 + x_4 = b_3, \quad -x_3 + x_5 = b_4, \quad -x_4 = b_5.$$

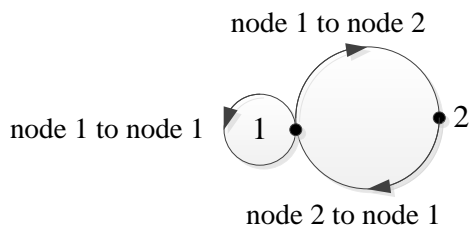
Observe that the sum of the first

$$x_2 - x_2 + x_4 - x_4 = b_1 + b_2 + b_5$$

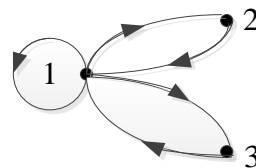
$$0 = b_1 + b_2 + b_5$$

Exercise

A direct graph starts with n nodes. There are n^2 possible edges, each edge leaves one of the n nodes and enters one of the n nodes (possibly itself). The n by n adjacency matrix has $a_{ij} = 1$ when edge leaves node i and enters node j ; if no edge then $a_{ij} = 0$. Here are directed graphs and their adjacency matrices:



$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The i, j entry of A^2 is $a_{i1}a_{1j} + \dots + a_{in}a_{nj}$.

Why does that sum count the two-step paths from i to any node to j ?

The i, j entry of A^k counts k -steps paths:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{array}{l} \text{counts the paths} \\ \text{with two edges} \end{array} \quad \begin{bmatrix} 1 \text{ to } 2 \text{ to } 1, 1 \text{ to } 1 \text{ to } 1 & 1 \text{ to } 1 \text{ to } 2 \\ 2 \text{ to } 1 \text{ to } 1 & 2 \text{ to } 1 \text{ to } 2 \end{bmatrix}$$

List all 3-step paths between each pair of nodes and compare with A^3 . When A^k has **no zeros**, that number k is the diameter of the graph – the number of edges needed to connect the most pair of nodes. What is the diameter of the second graph?

Solution

The number $a_{ik}a_{kj}$ will be “1” if there is an edge from node i to k and an edge from k to j .

This is a 2-step path. The number $a_{ik}a_{kj}$ will be “0” if either of those edge (from node i to k and from k to j) is missing.

The sum of $a_{ik}a_{kj}$ is the number of 2-step paths leaving i and entering j .

Matrix multiplication is right for this count.

The 3-step paths are counted by A^3 ; we look at paths to node 2:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \quad \begin{array}{l} \text{counts the paths} \\ \text{with three steps} \end{array} \quad \begin{bmatrix} \dots & 1 \text{ to } 1 \text{ to } 1 \text{ to } 2, 1 \text{ to } 2 \text{ to } 1 \text{ to } 2 \\ \dots & 2 \text{ to } 1 \text{ to } 1 \text{ to } 2 \end{bmatrix}$$

The A^k contain Fibonacci numbers 0, 1, 1, 2, 3, 5, 8, 13,

Fibonacci's rule $F_{k+2} = F_{k+1} + F_k$ show up in $(A)(A^k) = A^{k+1}$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix} = \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix} = A^{k+1}$$

There are **13 six-step** paths from node one to node 1.

Exercise

A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. All entries are 1. Which of these matrix operations are allowed, and what are the results?

a) AB

c) ABD

e) ABC

g) $A(B+C)$

b) BA

d) DBA

f) $ABCD$

Solution

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

a) $AB: (3 \times 5)(5 \times 3) = (3 \times 3)$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{pmatrix}$$

b) $BA : (5 \times 3)(3 \times 5) = (5 \times 5)$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix}$$

c) $ABD : (3 \times 5)(5 \times 3)(3 \times 1) = (3 \times 1)$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 15 \\ 15 \\ 15 \end{pmatrix}$$

d) $DBA : (3 \times 1)(5 \times 3)(3 \times 5) = NA$

e) $ABC : (3 \times 5)(5 \times 3)(5 \times 1) = NA$

f) $ABCD : (3 \times 5)(5 \times 3)(5 \times 1)(3 \times 1) = NA$

g) $A(B + C) : (3 \times 5)((5 \times 3) + (5 \times 1)) = NA$

Matrices B and C are not the same size.

Exercise

What rows or columns or matrices do you multiply to find.

- The third column of AB ?
- The second column of AB ?
- The first row of AB ?
- The second row of AB ?
- The entry in row 3, column 4 of AB ?
- The entry in row 2, column 3 of AB ?

Solution

- A (column 3 of B)
- A (column 2 of B)

c) (Row 1 of A) B

d) (Row 2 of A) B

e) (Row 3 of A) (Column 4 of B)

f) (Row 2 of A) (Column 3 of B)

Exercise

Add AB to AC and compare with $A(B + C)$:

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

Solution

$$AB = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 0 & 7 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$$

$$\begin{aligned} A(B + C) &= \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \left(\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} AB + AC &= \begin{bmatrix} 0 & 7 \\ 0 & 7 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix} \end{aligned}$$

$$\boxed{A(B + C) = AB + AC}$$

Exercise

True or False

- a) If A^2 is defined then A is necessarily square.
- b) If AB and BA are defined then A and B are square.
- c) If AB and BA are defined then AB and BA are square.
- d) If $AB = B$, then $A = I$

Solution

- a) True
- b) False, if A has an order m by n and B n by m : $AB: m \times m$ $BA: n \times n$
- c) True; $AB: m \times m$ $BA: n \times n$
- d) False, if B is the matrix of all zeros.

Exercise

- a) Find a nonzero matrix A such that $A^2 = 0$
- b) Find a matrix that has $A^2 \neq 0$ but $A^3 = 0$

Solution

- a) A nonzero matrix A such that $A^2 = 0$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

- b) A matrix that has $A^2 \neq 0$ but $A^3 = 0$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^3 = A^2 A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Exercise

Suppose you solve $Ax = b$ for three special right sides b :

$$Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

If the three solutions x_1, x_2, x_3 are the columns of a matrix X , what is A times X ?

Solution

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Therefore, $Ax = I$

Exercise

Show that $(A + B)^2$ is different from $A^2 + 2AB + B^2$, when

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

Write down the correct rule for $(A + B)(A + B) = A^2 + \underline{\hspace{2cm}} + B^2$

Solution

$$A + B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix}$$

$$(A + B)^2 = \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix}$$

$$2AB = 2 \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 14 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix} \neq \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$$

$$\boxed{(A+B)^2 \neq A^2 + 2AB + B^2}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

$$\begin{aligned} A^2 + AB + BA + B^2 &= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 4 \\ 5 & 6 \end{bmatrix} \end{aligned}$$

$$\boxed{(A+B)(A+B) = A^2 + \textcolor{red}{-}AB + \textcolor{red}{BA}\textcolor{red}{-} + B^2}$$

Exercise

Find the product of the 2 matrices by rows or by columns: $\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

Solution

$$\text{By rows: } \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{pmatrix} (2 \quad 3) \begin{pmatrix} 4 \\ 2 \end{pmatrix} \\ (5 \quad 1) \begin{pmatrix} 4 \\ 2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 14 \\ 22 \end{pmatrix}$$

$$\text{By columns: } \begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 4 \begin{pmatrix} 2 \\ 5 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 14 \\ 22 \end{pmatrix}$$

Exercise

Find the product of the 2 matrices by rows or by columns: $\begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Solution

$$\begin{aligned} \text{By rows: } \begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} &= \begin{pmatrix} (3 \ 6)(2 \ -1) \\ (6 \ 12)(2 \ -1) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{By columns: } \begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} &= 2 \begin{pmatrix} 3 \\ 6 \end{pmatrix} - 1 \begin{pmatrix} 6 \\ 12 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Exercise

Find the product of the 2 matrices by rows or by columns: $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

Solution

$$\begin{aligned} \text{By rows: } \begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} (1 \ 2 \ 4)(3 \ 1 \ 1) \\ (2 \ 0 \ 1)(3 \ 1 \ 1) \end{pmatrix} \\ &= \begin{pmatrix} 1(3) + 2(1) + 4(1) \\ 2(3) + 0(1) + 1(1) \end{pmatrix} \\ &= \begin{pmatrix} 9 \\ 7 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{By columns: } \begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} &= 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 9 \\ 7 \end{pmatrix} \end{aligned}$$

Exercise

Find the product of the 2 matrices by rows or by columns: $\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

Solution

$$\begin{aligned} \text{By rows: } \begin{pmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} &= \begin{pmatrix} (1 \ 2 \ 4)(2 \ 2 \ 3) \\ (-2 \ 3 \ 1)(2 \ 2 \ 3) \\ (-4 \ 1 \ 2)(2 \ 2 \ 3) \end{pmatrix} \\ &= \begin{pmatrix} 18 \\ 5 \\ 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{By columns: } \begin{pmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} &= 2 \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 18 \\ 5 \\ 0 \end{pmatrix} \end{aligned}$$

Exercise

Given $A = \begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix}$ $B = \begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix}$ Find $A + B$, $2A$, and $-B$

Solution

$$A + B = \begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix} + \begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & -2 \\ 8 & -2 & 0 \end{bmatrix}$$

$$2A = 2 \begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 6 \\ 6 & -2 & -4 \\ 0 & 0 & 8 \end{bmatrix}$$

$$-B = - \begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 3 \\ 1 & 0 & 0 \\ -8 & 2 & 4 \end{bmatrix}$$

Exercise

Given $A = \begin{bmatrix} 3 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Find AB and BA if possible

Solution

$$\begin{aligned} AB &= \begin{bmatrix} 3 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & -10 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3(3) + 2(0) - 3(1) & 3(-4) + 2(1) - 3(0) \\ 0(3) + 1(0) + 0(1) & 0(-4) + 1(1) + 0(0) \end{bmatrix} \\ &= \begin{bmatrix} 6 & -10 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{bmatrix} 3 & -4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 2 & -9 \\ 0 & 1 & 0 \\ 3 & 2 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 3(3) - 4(0) & 3(2) - 4(1) & 3(-3) - 4(0) \\ 0(3) + 1(0) & 0(2) + 1(1) & 0(-3) + 1(0) \\ 1(3) + 0(0) & 1(2) + 0(1) & 1(-3) + 0(0) \end{bmatrix} \\ &= \begin{bmatrix} 9 & 2 & -9 \\ 0 & 1 & 0 \\ 3 & 2 & -3 \end{bmatrix} \end{aligned}$$

Exercise

Given $A = \begin{bmatrix} 5 & 3 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 4 & -2 \\ -2 & 0 \\ 9 & 1 \end{bmatrix}$

Find AB and BA if possible

Solution

$AB = \text{Undefined}$

$$BA = \begin{bmatrix} 4 & -2 \\ -2 & 0 \\ 9 & 1 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 22 & 12 \\ -10 & -6 \\ 44 & 27 \end{bmatrix}$$

Exercise

Given $A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$ Find AB and BA if possible

Solution

$$a) AB = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix}$$

$$b) BA = \text{Undefined}$$

Exercise

Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

Compute the following (where possible):

$$a) D + E \quad b) D - E \quad c) 5A \quad d) -7C \quad e) 2B - C \quad g) -3(D + 2E)$$

Solution

$$a) D + E = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 6 & 5 \\ -2 & 1 & 3 \\ 7 & 3 & 7 \end{bmatrix}$$

$$b) D - E = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 4 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$c) 5A = 5 \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 0 \\ -5 & 10 \\ 5 & 5 \end{bmatrix}$$

$$d) -7C = -7 \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} -7 & -28 & -14 \\ -21 & -7 & -35 \end{bmatrix}$$

$$e) 2B - C = \text{can't be calculated}$$

$$g) -3(D + 2E) = -3 \left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 12 & 2 & 6 \\ -2 & 2 & 4 \\ 8 & 2 & 6 \end{bmatrix} \right) = -3 \begin{bmatrix} 13 & 7 & 8 \\ -3 & 2 & 5 \\ 11 & 4 & 10 \end{bmatrix}$$

$$= -3 \begin{bmatrix} 13 & 7 & 8 \\ -3 & 2 & 5 \\ 11 & 4 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} -39 & -21 & -24 \\ 9 & -6 & -15 \\ -33 & -12 & -30 \end{bmatrix}$$