Solution   Section 1.6 – Minimization Problems ≥ (Duality)

Exercise

Find the transpose of the matrix

$a)\ \begin{bmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 4 & 4 & 4 \end{bmatrix}$ $b)\ \begin{bmatrix} -3 & -8 & 1 \\ 5 & -2 & 5 \\ 9 & 6 & -2 \\ 4 & 5 & 8 \end{bmatrix}$

Solution

$$a)\ \begin{bmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 4 & 4 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 7 & 4 \\ 3 & 8 & 4 \end{bmatrix}$$

$$b)\ \begin{bmatrix} -3 & -8 & 1 \\ 5 & -2 & 5 \\ 9 & 6 & -2 \\ 4 & 5 & 8 \end{bmatrix}^T = \begin{bmatrix} -3 & 5 & 9 & 4 \\ -8 & -2 & 6 & 5 \\ 1 & 5 & -2 & 8 \end{bmatrix}$$

Exercise

Solve the following minimization problem by maximizing the dual:

$Maximize:\ P = 12y_1 + 17y_2$

$$Subject\ to: \begin{cases} 2y_1 + 3y_2 \le 21 \\ 5y_1 + 7y_2 \le 50 \\ y_1, y_2 \ge 0 \end{cases}$$

Solution

Solving the dual problem is a standard maximization problem, we can solve it using the simplex method.

$Maximize:\ P = 12y_1 + 17y_2$

$$Subject\ to: \begin{cases} 2y_1 + 3y_2 \le 21 \\ 5y_1 + 7y_2 \le 50 \\ y_1, y_2 \ge 0 \end{cases}$$

Initial System

$$\begin{cases} 2y_1 + 3y_2 + x_1 = 21 \\ 5y_1 + 7y_2 + x_2 = 50 \\ -12y_1 - 17y_2 + P = 0 \end{cases}$$

Initial Tableau

$$\begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & x_1 & x_2 & P \end{array} \\ \left[\begin{array}{ccccc|c} 2 & (3) & 1 & 0 & 0 & 21 \\ 5 & 7 & 0 & 1 & 0 & 50 \\ \hline -12 & \langle -17 \rangle & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad \begin{array}{l} 21 \div 3 = 7 \\ 50 \div 7 = 7.1 \end{array}$$

$$\begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & x_1 & x_2 & P \end{array} \\ \left[\begin{array}{ccccc|c} \frac{2}{3} & 1 & \frac{1}{3} & 0 & 0 & 7 \\ \frac{1}{3} & 0 & -\frac{7}{3} & 1 & 0 & 1 \\ \hline \left\langle -\frac{2}{3} \right\rangle & 0 & \frac{17}{3} & 0 & 1 & 119 \end{array}\right] \end{array} \quad \begin{array}{l} 7 \div \frac{2}{3} = 10.5 \\ 1 \div \frac{1}{3} = 3 \end{array}$$

<u>Final Tableau</u>

$$\begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & x_1 & x_2 & P \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 1 & 5 & -2 & 0 & 5 \\ (1) & 0 & -7 & 3 & 0 & 3 \\ \hline 0 & 0 & \mathbf{1} & \mathbf{2} & 1 & \mathbf{121} \end{array}\right] \end{array}$$

<u>*Optimal Solution*</u> *(for the original minimization problem):*

Minimum: $C = 121, \quad x_1 = 1, \quad x_2 = 2$

Exercise

Solve the following minimization problem by maximizing the dual:

$$Minimize: \ C = 16x_1 + 8x_2 + 4x_3$$

$$Subject\ to \quad \begin{cases} 3x_1 + 2x_2 + 2x_3 \ge 16 \\ 4x_1 + 3x_2 + x_3 \ge 14 \\ 5x_1 + 3x_2 + x_3 \ge 12 \\ x_1, x_2, x_3 \ge 0 \end{cases}$$

<u>Solution</u>

The Coefficient Matrix ***The Transpose***

$$A = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 3 & 2 & 2 & 16 \\ 4 & 3 & 1 & 14 \\ 5 & 3 & 1 & 12 \\ \hline 16 & 8 & 4 & 1 \end{array}\right] \end{array} \qquad A^T = \begin{array}{c} \begin{matrix} y_1 & y_2 & y_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 3 & 4 & 5 & 16 \\ 2 & 3 & 3 & 8 \\ 2 & 1 & 1 & 4 \\ \hline 16 & 14 & 12 & 1 \end{array}\right] \end{array}$$

The Dual:

$$Maximize: \; P = 16y_1 + 14y_2 + 12y_3$$

$$ST: \begin{cases} 3y_1 + 4y_2 + 5y_3 \le 16 \\ 2y_1 + 3y_2 + 3y_3 \le 8 \\ 2y_1 + y_2 + y_3 \le 4 \\ y_1, y_2, y_3 \ge 0 \end{cases}$$

Initial System

$$\begin{cases} 3y_1 + 4y_2 + 5y_3 + x_1 = 16 \\ 2y_1 + 3y_2 + 3y_3 + x_2 = 8 \\ 2y_1 + y_2 + y_3 + x_3 = 4 \\ -16y_1 - 14y_2 - 12y_3 + P = 0 \end{cases}$$

$$\begin{array}{c} \begin{matrix} y_1 & y_2 & y_3 & x_1 & x_2 & x_3 & P & \end{matrix} \\ \left[\begin{array}{ccccccc|c} 3 & 4 & 5 & 1 & 0 & 0 & 0 & 16 \\ 2 & 3 & 3 & 0 & 1 & 0 & 0 & 8 \\ (2) & 1 & 1 & 0 & 0 & 1 & 0 & 4 \\ \hline \langle -16 \rangle & -14 & -12 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad \begin{array}{l} 16 \div 3 = 5.3 \\ 8 \div 2 = 4 \\ 4 \div 2 = 2 \end{array}$$

$$\begin{array}{c} \begin{matrix} y_1 & y_2 & y_3 & x_1 & x_2 & x_3 & P & \end{matrix} \\ \left[\begin{array}{ccccccc|c} 0 & 5/2 & 7/2 & 1 & 0 & -3/2 & 0 & 10 \\ 0 & (2) & 2 & 0 & 1 & -1 & 0 & 4 \\ 1 & 1/2 & 1/2 & 0 & 0 & 1/2 & 0 & 2 \\ \hline 0 & \langle -6 \rangle & -4 & 0 & 0 & 8 & 1 & 32 \end{array}\right] \end{array} \quad \begin{array}{l} 10 \div (5/2) = 4 \\ 4 \div 2 = 2 \\ 2 \div (1/2) = 4 \end{array}$$

$$\begin{array}{c} \begin{matrix} y_1 & y_2 & y_3 & x_1 & x_2 & x_3 & P & \end{matrix} \\ \left[\begin{array}{ccccccc|c} 0 & 0 & 1 & 1 & -5/2 & -1/4 & 0 & 5 \\ 0 & 1 & 1 & 0 & 1/2 & -1/2 & 0 & 2 \\ 1 & 0 & 0 & 0 & -1/4 & 3/4 & 0 & 1 \\ \hline 0 & 0 & 2 & 0 & 3 & 5 & 1 & 44 \end{array}\right] \end{array}$$

Optimal Solution (for the original minimization problem):

Minimum: $C = 44, \quad x_1 = 0, \quad x_2 = 3 \quad x_4 = 5$

Exercise

Customers buy 14 units of regular beer and 20 units of light beer monthly. The brewery decides to produce extra beer, beyond that needed to satisfy the customers. The cost per unit of regular beer is $33,000 and the cost per unit of light beer is $44,000. Every unit of regular beer brings in $200,000 in revenue, while every unit of light beer brings in $400,000 in revenue. The brewery wants at least $16,000,000 in revenue. At least 18 additional units of beer can be sold. How much of each beer type should be made so as to minimize total production costs? What is the minimum cost?

Solution

Exercise

Acme Micros markets computers with single-sided and double-sided drives. The disk driers are supplied by two other companies, Associated Electronics and Digital Drives. Associated Electronics charges $250 for a single-sided disk drive and $350 for a double-sided disk drive. Digital Drives charges $290 for a single-sided disk drive and $320 for a double-sided disk drive. Associated Electronics can supply at most 1,000 disk drives in any combination of single-sided and double-sided drives. The combined monthly total supplied by Digital Drives cannot exceed 2,000 disk drives. Acme Micros needs at least 1,200 single-sided drives and at least 1,600 double-sided drives each month. How many disk drives of each type should Acme Micros order from each supplier in order to meet its monthly demand and minimize the purchase cost? What is the minimum purchase cost?

Solution

Let x_1 : Number of single-sided - Associated Electronics

x_2 : Number of double-sided - Associated Electronics

x_3 : Number of single-sided - Digital Drives

x_4 : Number of double-sided - Digital Drives

$$\textit{Minimize}: \ C = 250x_1 + 350x_2 + 290x_3 + 320x_4$$

$$\textit{Subject to} \quad \begin{cases} x_1 + x_2 \le 1000 \\ x_3 + x_4 \le 2000 \\ x_1 + x_3 \ge 1200 \\ x_2 + x_4 \ge 1600 \\ x_1, x_2, x_3, x_4 \ge 0 \end{cases}$$

$$Minimize: \; C = 250x_1 + 350x_2 + 290x_3 + 320x_4$$

$$Subject\ to \quad \begin{cases} -x_1 - x_2 \geq -1000 \\ -x_3 - x_4 \geq -2000 \\ x_1 + x_3 \geq 1200 \\ x_2 + x_4 \geq 1600 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

The Coefficient Matrix | ***The Transpose***

$$A = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} -1 & -1 & 0 & 0 & -1000 \\ 0 & 0 & -1 & -1 & -2000 \\ 1 & 0 & 1 & 0 & 1200 \\ 0 & 1 & 0 & 1 & 1600 \\ \hline 250 & 350 & 290 & 320 & 1 \end{array}\right] \end{array}$$

$$A = \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \left[\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 250 \\ -1 & 0 & 0 & 1 & 350 \\ 0 & -1 & 1 & 0 & 290 \\ 0 & -1 & 0 & 1 & 320 \\ \hline -1000 & -2000 & 1200 & 1600 & 1 \end{array}\right] \end{array}$$

The Dual:

$$Maximize: \; P = -1000y_1 - 2000y_2 + 1200y_3 + 1600y_4$$

$$Subject\ \ to: \begin{cases} -y_1 + y_3 \leq 250 \\ -y_1 + y_4 \leq 350 \\ -y_2 + y_3 \leq 290 \\ -y_2 + y_4 \leq 320 \\ y_1, y_2, y_3, y_4 \geq 0 \end{cases}$$

$$\begin{array}{c} \begin{array}{ccccccccc} y_1 & y_2 & y_3 & y_4 & x_1 & x_2 & x_3 & x_4 & P \end{array} \\ \left[\begin{array}{ccccccccc|c} -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 250 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 350 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 290 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 320 \\ \hline 1000 & 2000 & -1200 & -1600 & 100 & 0 & 200 & 1600 & 1 & 1 \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{ccccccccc} y_1 & y_2 & y_3 & y_4 & x_1 & x_2 & x_3 & x_4 & P \end{array} \\ \left[\begin{array}{ccccccccc|c} 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 290 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 70 \\ 1 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 40 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 320 \\ \hline 0 & 200 & 0 & 0 & 100 & 0 & 200 & 1600 & 1 & 820,000 \end{array}\right] \end{array}$$

The minimal purchase cost is \$820,000 for 1000 single-sided and 0 double-sided - Associated Electronics, 200 single-sided and 1600 double-sided - Digital Drives

Exercise

A farmer can buy three types of plant food, mix A, mix B, and mix C. Each cubic yard of mix A contains 20 pounds of phosphoric acid, 10 pounds of nitrogen, and 10 pound of potash. Each cubic yard of mix B contains 10 pounds of phosphoric acid, 10 pounds of nitrogen, and 15 pound of potash. . Each cubic yard of mix C contains 20 pounds of phosphoric acid, 20 pounds of nitrogen, and 5 pound of potash. The minimum monthly requirements are 480 pounds of phosphoric acid, 320 pounds of nitrogen, and 225 pound of potash. If mix A costs \$30 per cubic yard, nix B costs \$36 per cubic yard, and mix C \$39 per cubic yard, how many cubic yards of each mix should the farmer blend to meet the minimum monthly requirements at a minimal cost? What is the minimum cost?

Solution

Let x_1 : Number of cubic yards of mix A.

x_2 : Number of cubic yards of mix B.

x_3 : Number of cubic yards of mix C.

$$\textit{Minimize}: \ C = 30x_1 + 36x_2 + 39x_3$$

$$\textit{Subject to} \quad \begin{cases} 20x_1 + 10x_2 + 20x_3 \ge 480 \\ 10x_1 + 10x_2 + 20x_3 \ge 320 \\ 10x_1 + 15x_2 + 5x_3 \ge 225 \\ x_1,\ x_2,\ x_3 \ge 0 \end{cases}$$

The Coefficient Matrix

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ A = \left[\begin{array}{ccc|c} 20 & 10 & 20 & 480 \\ 10 & 10 & 20 & 320 \\ 10 & 15 & 5 & 225 \\ \hline 30 & 36 & 39 & 1 \end{array}\right] \end{array}$$

The Transpose

$$\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ A^T = \left[\begin{array}{ccc|c} 20 & 10 & 10 & 30 \\ 10 & 10 & 15 & 36 \\ 20 & 20 & 5 & 39 \\ \hdashline 480 & 320 & 225 & 1 \end{array}\right] \end{array}$$

The Dual:

$$\textit{Maximize}: \ P = 480y_1 + 320y_2 + 225y_3$$

$$ST: \begin{cases} 20y_1 + 10y_2 + 10y_3 \le 30 \\ 10y_1 + 10y_2 + 15y_3 \le 36 \\ 20y_1 + 20y_2 + 5y_3 \le 39 \\ y_1,\ y_2,\ y_3 \ge 0 \end{cases}$$

$$\begin{array}{c} \begin{array}{ccccccc} y_1 & y_2 & y_3 & x_1 & x_2 & x_3 & P \end{array} \\ \left[\begin{array}{ccccccc|c} 20 & 10 & 10 & 1 & 0 & 0 & 0 & 30 \\ 10 & 10 & 15 & 0 & 1 & 0 & 0 & 36 \\ 20 & 20 & 5 & 0 & 0 & 1 & 0 & 39 \\ \hdashline -480 & -320 & -225 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{ccccccccc} y_1 & y_2 & y_3 & x_1 & x_2 & x_3 & P & & \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 0 & 0 & 1 & -.6 & -.7 & 0 & 1.6 \\ 0 & 0 & 1 & 0 & .4 & -.2 & 0 & 6.6 \\ 0 & 1 & 0 & -1 & .4 & .8 & 0 & 15.6 \\ \hdashline 0 & 0 & 0 & 16 & 2 & 7 & 1 & 825 \end{array}\right] \end{array}$$

For the farmer, to meet the minimum monthly requirements at a minimal cost, should blend 16 yd^3 of A, 2 yd^3 of B, and 7 yd^3 of C; and the minimum cost is \$825.00**.**

Exercise

Mark, who is ill, takes vitamin pills. Each day he must have at least 16 units of vitamin A, 5 units of vitamin B, and 20 units of vitamin C. he can choose between pill #1, which contains 8 units of A, 1 of B, and 2 of C; and pill #2, which contains 2 units of A, 1 of B, and 7 of C. Pill 1 costs 15¢, and pill 2 costs 30¢.

a) How many of each pill should be buy in order to minimize his cost?
b) What is the minimum cost?
c) For the solution in part a, Mark is receiving more than he needs of at least one vitamin. Identify that vitamin, and tell how much surplus he is receiving. Is there any ways he can avoid receiving that surplus while still meeting the other constraints and minimizing the cost?

Solution

Let x_1 : Number of #1 pills

x_2 : Number of #2 pills

	Vitamin A	Vitamin B_1	Vitamin C	Cost
#1	8	1	2	\$0.10
#2	2	1	7	\$0.20
Total Needed	16	5	20	

$$Minimize: \ C = 0.1x_1 + 0.2x_2$$

$$Subject\ to \ \begin{cases} 8x_1 + 2x_2 \ge 16 \\ x_1 + x_2 \ge 5 \\ 2x_1 + 7x_2 \ge 20 \\ x_1, x_2 \ge 0 \end{cases}$$

The ***dual***:

$$Maximize: \ P = 16y_1 + 5y_2 + 20y_3$$

$$Subject\ to \ \begin{cases} 8y_1 + y_2 + 2y_3 \ge 0.1 \\ 2y_1 + y_2 + 7y_3 \ge 0.2 \\ y_1, y_2, y_2 \ge 0 \end{cases}$$

$$\begin{array}{cccccc|c} y_1 & y_2 & y_3 & x_1 & x_2 & P & \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 8 & 1 & 2 & 1 & 0 & 0 & 0.1 \\ 2 & 1 & 7 & 0 & 1 & 0 & 0.2 \\ \hline -16 & -5 & -20 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{array}{cccccc|c} y_1 & y_2 & y_3 & x_1 & x_2 & P & \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 10.4 & 1 & 0 & 1.4 & -.4 & 0 & 0.06 \\ -1.2 & 0 & 1 & -.2 & .2 & 0 & 0.02 \\ \hline 12 & 0 & 0 & 3 & 2 & 1 & 0.7 \end{array}\right]$$

The minimum value is 0.7 when $y_1 = 3 \quad y_2 = 2$.

Mark should buy 3 of pills #1 for a minimum cost of 60 cents, and 2 of pills #2 for a minimum cost of 70 cents.

Exercise

One gram of soybean meal provides at least 2.5 units of vitamins and 5 calories. One gram of meat byproducts provides at least 4.5 units of vitamins and 3 calories. One gram of grain provides at least 5 units of vitamins and 10 calories. If a gram of soybean meal costs 6 cents, a gram of meat byproducts 8 cents, and a gram of grain 9 cents, what mixture of these three ingredients will provide at least 54 units of vitamins and 60 calories per serving at minimum cost? What will be the minimum cost?

Solution:

$$\text{objective}: \quad C = 7x_1 + 8x_2 + 9x_3$$

$$\text{Subject to} \quad \begin{cases} 2.5x_1 + 4.5x_2 + 5x_3 \ge 54 \\ 5x_1 + 3x_2 + 10x_3 \ge 60 \\ x_1, x_2, x_3 \ge 0 \end{cases}$$

$$A = \left(\begin{array}{ccc|c} 2.5 & 4.5 & 5 & 54 \\ 5 & 3 & 10 & 60 \\ \hline 7 & 8 & 10 & 1 \end{array}\right) \quad A^T = \left(\begin{array}{cc|c} 2.5 & 5 & 7 \\ 4.5 & 3 & 8 \\ 5 & 10 & 10 \\ \hline 54 & 60 & 1 \end{array}\right)$$

$$\text{Maximize}: \quad P = 54y_1 + 60y_2$$

$$\text{Subject to} \quad \begin{cases} 2.5y_1 + 5y_2 \le 7 \\ 4.5y_1 + 3y_2 \le 8 \\ 5y_1 + 10y_2 \le 10 \\ y_1, y_2 \ge 0 \end{cases}$$

$$\begin{array}{cccccc}
y_1 & y_2 & x_1 & x_2 & x_3 & P
\end{array}$$
$$\left[\begin{array}{cccccc|c}
2.5 & 5 & 1 & 0 & 0 & 0 & 7 \\
4.5 & 3 & 0 & 1 & 0 & 0 & 8 \\
5 & \boxed{10} & 0 & 0 & 1 & 0 & 10 \\
\hline
-54 & \langle -60 \rangle & 0 & 0 & 0 & 1 & 0
\end{array}\right]$$

$$\begin{array}{cccccc}
y_1 & y_2 & x_1 & x_2 & x_3 & P
\end{array}$$
$$\left[\begin{array}{cccccc|c}
0 & 0 & 1 & 0 & 0 & 0 & 2 \\
\boxed{3} & 0 & 0 & 1 & 0 & 0 & 5 \\
.5 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
-24 & 0 & 0 & 0 & 0 & 1 & 60
\end{array}\right]$$

$$\begin{array}{cccccc}
y_1 & y_2 & x_1 & x_2 & x_3 & P
\end{array}$$
$$\left[\begin{array}{cccccc|c}
0 & 0 & 1 & 0 & -\frac{1}{2} & 0 & 2 \\
1 & 0 & 0 & \frac{1}{3} & -\frac{1}{10} & 0 & \frac{5}{3} \\
0 & 1 & 0 & -\frac{1}{6} & \frac{3}{20} & 0 & \frac{1}{6} \\
\hline
0 & 0 & \boxed{0} & \boxed{8} & \boxed{\frac{18}{5}} & 1 & \boxed{100}
\end{array}\right]$$

$$\boxed{x_1 = 0, \quad x_2 = 8, \quad x_3 = \frac{18}{5} = 3.6, \quad C = 100}$$

$Soybean = 0, \quad Meat = 8, \quad Grain = 3.6, \quad Cost = 100$

Exercise

A metropolitan school district has two high-schools that are overcrowded and two that are underenrolled. In order to balance the enrollment, the school board has decided to bus students from the crowded schools to the underenrolled schools. North Division High School has 300 more students than it should have, and South Division High School has 500 more students than it should have. Central High School can accommodate 400 additional students and Washington High School can accommodate 500 additional students. The weekly cost of busing a student from North Division to Central is \$5, from North Division to Washington is \$2, from South Division to Central is \$3, and from South Division to Washington is \$4. Determine the number of students that should be bused from each of the overcrowded schools to each of the underenrolled schools in order to balance the enrollment and minimize the cost of busing the students. What is the minimum cost?

Solution

Let x_1 : Number of students from N.Div. to Central

x_2 : Number of students from N.D. to Washington

x_3 : Number of students from S.D. to Central

x_4 : Number of students from S.D. to Washington

$$Minimize: \ C = 5x_1 + 2x_2 + 3x_3 + 4x_4$$

$$\text{Subject to} \quad \begin{cases} x_1 + x_2 \geq 300 \\ x_3 + x_4 \geq 500 \\ x_1 + x_3 \leq 500 \\ x_2 + x_4 \leq 500 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases} \qquad \text{Subject to} \quad \begin{cases} x_1 + x_2 \geq 300 \\ x_3 + x_4 \geq 500 \\ -x_1 - x_3 \geq -500 \\ -x_2 - x_4 \geq -500 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

<u>The Coefficient Matrix</u>

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ A = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 300 \\ 0 & 0 & 1 & 1 & 500 \\ -1 & 0 & -1 & 0 & -500 \\ 0 & -1 & 0 & -1 & -500 \\ \hdashline 5 & 2 & 3 & 4 & 1 \end{array}\right] \end{array}$$

<u>The Transpose</u>

$$\begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ A = \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 5 \\ 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & -1 & 4 \\ \hdashline 300 & 500 & -500 & -500 & 1 \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{ccccccccc} y_1 & y_2 & y_3 & y_4 & x_1 & x_2 & x_3 & x_4 & P \end{array} \\ \left[\begin{array}{ccccccccc|c} 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 5 \\ 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 4 \\ \hdashline 300 & 500 & -500 & -500 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{ccccccccc} y_1 & y_2 & y_3 & y_4 & x_1 & x_2 & x_3 & x_4 & P \end{array} \\ \left[\begin{array}{ccccccccc|c} 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 4 \\ 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 1 \\ \hdashline 0 & 0 & 0 & 100 & 0 & 300 & 400 & 100 & 1 & 2,200 \end{array}\right] \end{array}$$

The minimal cost is $2,200 when 300 students are bused from North Division to Washington, 400 students are bused from South Division to Central, and 100 students are bused from South Division to Washington. No students bused from North Division to Central.