# Section 1.8 – Applications

Network Analysis

Networks composed of branches and junctions are used as models in such fields as economics, traffic analysis, and electrical engineering.

In a network model, you assume that the total flow into a junction is equal to the total flow out of the junction

Example

Set up a system of linear equations to represent the network shown below. Then solve the system for x_i, $i = 1, 2, 3, 4, 5$.
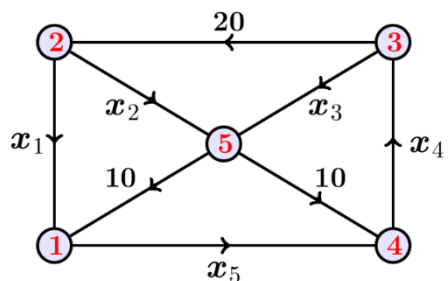
Solution

$$1 \rightarrow \quad x_1 + 10 = x_5 \quad \Rightarrow \quad x_1 - x_5 = -10$$

$$2 \rightarrow \quad x_1 + x_2 = 20$$

$$3 \rightarrow \quad x_4 = x_3 + 20 \quad \Rightarrow \quad -x_3 + x_4 = 20$$

$$4 \rightarrow \quad x_4 = x_5 + 10 \quad \Rightarrow \quad x_4 - x_5 = 10$$

$$5 \rightarrow \quad x_2 + x_3 = 10 + 10 = 20$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & | & -10 \\ 1 & 1 & 0 & 0 & 0 & | & 20 \\ 0 & 0 & -1 & 1 & 0 & | & 20 \\ 0 & 0 & 0 & 1 & -1 & | & 10 \\ 0 & 1 & 1 & 0 & 0 & | & 20 \end{pmatrix} \quad R_2 - R_1$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & | & -10 \\ 0 & 1 & 0 & 0 & 1 & | & 30 \\ 0 & 0 & -1 & 1 & 0 & | & 20 \\ 0 & 0 & 0 & 1 & -1 & | & 10 \\ 0 & 1 & 1 & 0 & 0 & | & 20 \end{pmatrix} \quad R_5 - R_2$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 1 & 0 & -1 & -10 \end{pmatrix} \quad R_5 + R_3$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 1 & -1 & 10 \end{pmatrix} \quad R_5 - R_4$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\rightarrow x_1 - x_5 = -10 \quad \Rightarrow x_1 = x_5 - 10$

$\rightarrow x_2 + x_5 = 30 \quad \Rightarrow x_2 = 30 - x_5$

$\rightarrow -x_3 + x_4 = 20 \quad \Rightarrow x_3 = x_5 - 10$

$\rightarrow x_4 - x_5 = 10 \quad \Rightarrow x_4 = x_5 + 10$

Solution: $\left(x_5 - 10, \quad 30 - x_5, \quad x_5 - 10, \quad 10 + x_5, \quad x_5 \right)$

2nd Method

$$\begin{vmatrix} 1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{vmatrix} - 1 \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{vmatrix} - 1 \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= -1 + 1$$

$$= 0$$

Infinite solution:

$1 \rightarrow \quad x_1 = x_5 - 10$

$2 \rightarrow \quad x_2 = 20 - x_1 = 30 - x_5$

$4 \rightarrow \quad x_4 = x_5 + 10$

$3 \rightarrow \quad x_3 = x_4 - 20 = x_5 - 10$

Electrical network

An electrical network is another type of network where analysis is commonly applied. An analysis of such a system uses two properties of electrical networks known as **Kirchhoff's Laws.**

➢ All the current flowing into a junction must flow out of it.
➢ The sum of the products IR (I is current and R is resistance) around a closed path is equal to the total voltage in the path.

Example

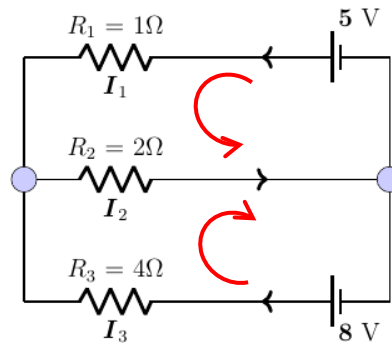Determine the currents I_1, I_2, and I_3 for the electrical network
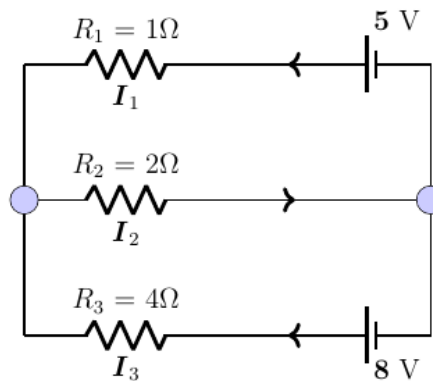
Solution

$I_2 = I_1 + I_3$

$I_1 + 2I_2 = 5$

$2I_2 + 4I_3 = 8$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ I_1 + 2I_2 \quad\quad = 5 \\ \quad\quad I_2 + 2I_3 = 4 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 7 \qquad D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 5 & 2 & 0 \\ 4 & 1 & 2 \end{vmatrix} = 7 \qquad D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 5 & 0 \\ 0 & 4 & 2 \end{vmatrix} = 14 \qquad D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 2 & 5 \\ 0 & 1 & 4 \end{vmatrix} = 7$$

$\underline{I_1 = 1\ A} \qquad \underline{I_2 = 2\ A} \qquad \underline{I_3 = 1\ A}$

Cryptography

A **cryptogram** is a message written according to a secret code (the Greek word **kryptos** means "*hidden*"). One method of using matrix multiplication to **encode** and **decode** messages.

Let assign a number to each letter in the alphabet (with 0 assigned to a blank space), as shown

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

Example

Consider the invertible matrix: $A = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix}$

The message: ***MEET ME MONDAY***

   a) Write the uncoded row matrices 1×3 for the message.
   b) Use the matrix A to encode the message.
   c) Decode a message from part *b*) given the matrix A.

Solution

  a)

$$\begin{matrix} M & E & E \\ [13 & 5 & 5] \end{matrix} \quad \begin{matrix} T & \_ & M \\ [20 & 0 & 13] \end{matrix} \quad \begin{matrix} E & \_ & M \\ [5 & 0 & 13] \end{matrix} \quad \begin{matrix} O & N & D \\ [15 & 14 & 4] \end{matrix} \quad \begin{matrix} A & Y & \_ \\ [1 & 25 & 0] \end{matrix}$$

  b) Let encode the message ***MEET ME MONDAY***

$$[13 \quad 5 \quad 5] \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [13 \quad -26 \quad 21]$$

$$[20 \quad 0 \quad 13] \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [33 \quad -53 \quad -12]$$

$$[5 \quad 0 \quad 13] \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [18 \quad -23 \quad -42]$$

$$[15 \quad 14 \quad 4]\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [5 \quad -20 \quad 56]$$

$$[1 \quad 25 \quad 0]\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [-24 \quad 23 \quad 77]$$

The sequence of coded row matrices is

$$[13 \quad -26 \quad -21] \quad [33 \quad -53 \quad -12] \quad [18 \quad -23 \quad -42] \quad [5 \quad -20 \quad 56] \quad [-24 \quad 23 \quad 77]$$

The cryptogram:

$$13 \quad -26 \quad -21 \quad 33 \quad -53 \quad -12 \quad 18 \quad -23 \quad -42 \quad 5 \quad -20 \quad 56 \quad -24 \quad 23 \quad 77$$

c) To decode a message given the matrix A.

$$|A| = \begin{vmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{vmatrix} = 1$$

$$A^{-1} = \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$$

With the cryptogram:

$$[13 \quad -26 \quad -21] \quad [33 \quad -53 \quad -12] \quad [18 \quad -23 \quad -42] \quad [5 \quad -20 \quad 56] \quad [-24 \quad 23 \quad 77]$$

$$[13 \quad -26 \quad 21]\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = [13 \quad 5 \quad 5]$$

$$[33 \quad -53 \quad -1]\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = [20 \quad 0 \quad 13]$$

$$[18 \quad -23 \quad -42]\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = [5 \quad 0 \quad 13]$$

$$[5 \quad -20 \quad 56]\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = [15 \quad 14 \quad 4]$$
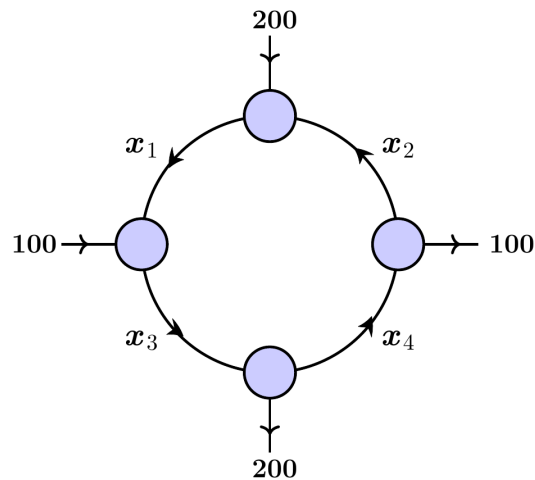
$$\begin{bmatrix} -24 & 23 & 77 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 25 & 0 \end{bmatrix}$$

The message is:

13	5	5	20	0	13	5	0	13	15	14	4	1	25	0
M	*E*	*E*	*T*	*_*	*M*	*E*	*_*	*M*	*O*	*N*	*D*	*A*	*Y*	*_*

# Exercises   *Section* 1.8 – Applications

1. The flow of traffic, in vehicles per hour, through a network of streets as is shown below
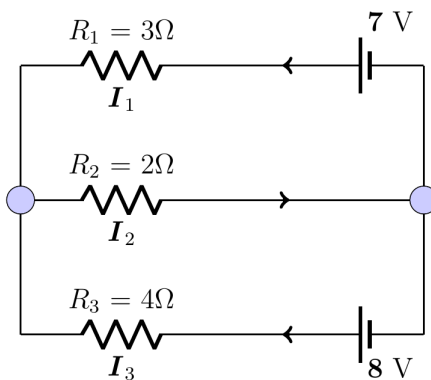

a) Solve this system for x_i, $i = 1, 2, 3, 4$.

b) Find the traffic flow when $x_4 = 0$.

c) Find the traffic flow when $x_4 = 100$.

d) Find the traffic flow when $x_1 = 2x_2$.

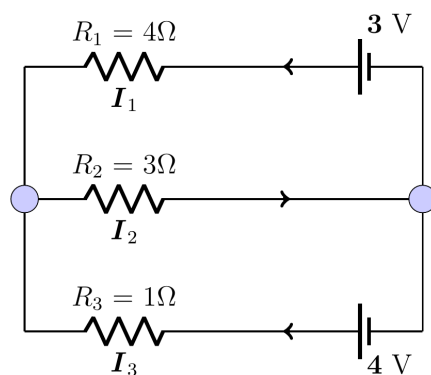**2.** Through a network, Express x_n's in terms of the parameters s and t.
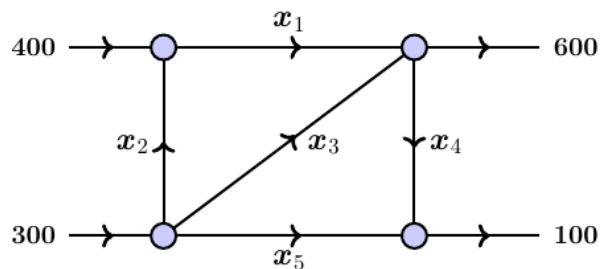
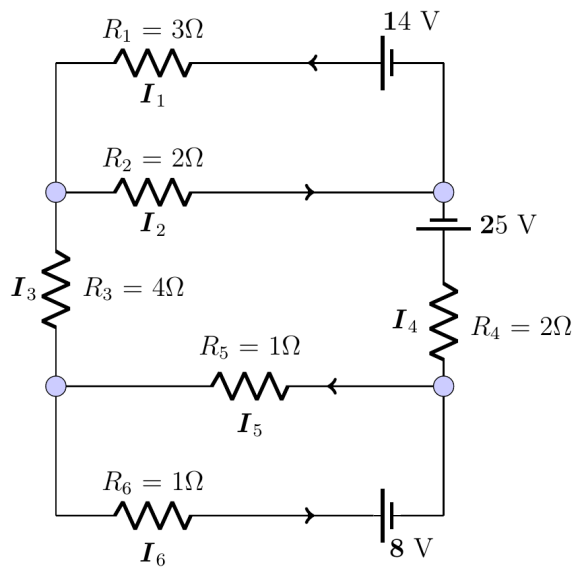**3.** Water is flowing through a network of pipes. Express x_n's in terms of the parameters s and t.
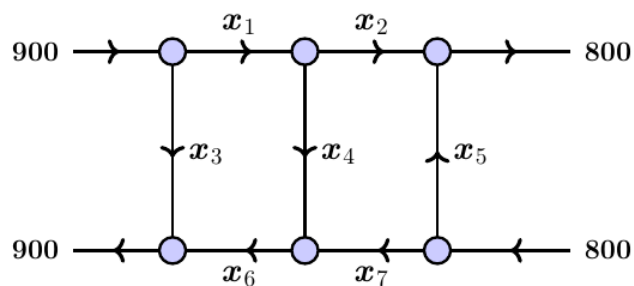
**4.** Determine the currents I_1, I_2, *and* I_3 for the electrical network shown below

$R_1 = 3\Omega$
I_1

7 V

$R_2 = 2\Omega$
I_2

$R_3 = 4\Omega$
I_3

8 V

5. Determine the currents I_1, I_2, *and* I_3 for the electrical network shown below

$R_1 = 4\Omega$
I_1

3 V

$R_2 = 3\Omega$
I_2

$R_3 = 1\Omega$
I_3

4 V

6. Determine the currents I_1, I_2, I_3, I_4, I_5, *and* I_6 for the electrical network shown below

$R_1 = 3\Omega$
I_1

14 V

$R_2 = 2\Omega$
I_2

25 V

I_3 $R_3 = 4\Omega$

I_4 $R_4 = 2\Omega$

$R_5 = 1\Omega$
I_5

$R_6 = 1\Omega$
I_6

8 V

7. Determine the currents $I_1, I_2, I_3, I_4, I_5,$ *and* I_6 for the electrical network shown below
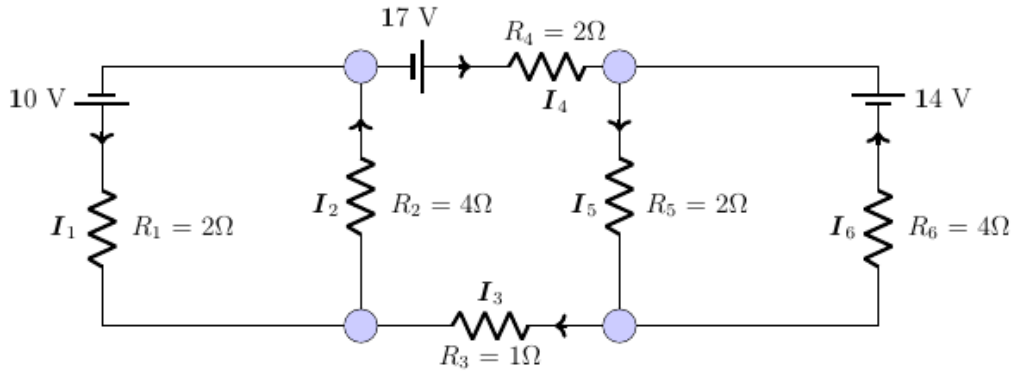
8. Consider the invertible matrix: $A = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -2 & -2 & 7 \end{pmatrix}$

The message: ***ICEBERG DEAD AHEAD***

a) Write the uncoded row matrices 1×3 for the message.
b) Use the matrix A to encode the message.
c) Decode a message from part b) given the matrix A.

9. You want to send the message: ***LINEAR ALGEBRA***   with a key word ***MATH***

a) Write the matrix A.
b) Write the uncoded row matrices 1×2 for the message.
c) Use the matrix A to encode the message.
d) Decode a message from part b) given the matrix A.

10. Consider the invertible matrix: $A = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix}$

Decode the cryptogram

$$1 \quad -5 \quad 11 \quad 19 \quad -25 \quad -45 \quad 11 \quad -16 \quad -28 \quad 20 \quad -29 \quad -27$$
$$12 \quad -12 \quad -53 \quad 40 \quad -61 \quad -35 \quad 8 \quad -17 \quad 7$$