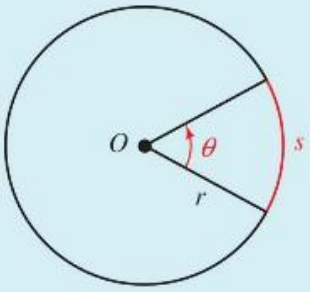
# *Section* 2.2 – Arc Length and Sector Area

## *Arc Length*

### *Definition*

If a central angle $\theta$, in a circle of a radius $r$, cuts off an arc of length $s$, then the measure of $\theta$, in radians is:

$$\theta \text{ (in radians)} = \frac{s}{r}$$




$\theta r = \frac{s}{r} r$

$$s = r\theta \qquad (\theta \text{ in radians})$$

> ***Note***:
> When applying the formula $s = r\theta$, the value of ***must*** be in radian.

***Example***

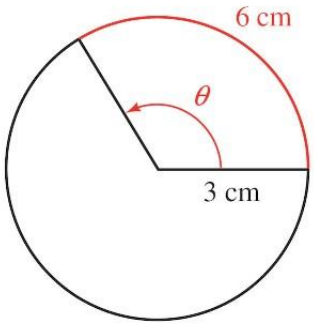
A central angle $\theta$ in a circle of radius 3 *cm* cuts off an arc of length 6 *cm*. What is the radian measure of $\theta$.

*Solution*

$$\theta = \frac{s}{r}$$

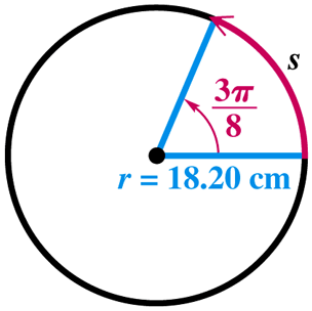
$$= \frac{6\ cm}{3\ cm}$$

$$= 2\ rad$$



***Example***

A circle has radius 18.20 *cm*. Find the length of the arc intercepted by a central angle with measure $\frac{3\pi}{8}$ radians.




*Solution*

***Given***: $\theta = \frac{3\pi}{8}\ rad, \quad r = 18.20\ cm$

$$s = r\theta$$
$$= 18.20\left(\frac{3\pi}{8}\right) cm$$
$$\approx 21.44\ cm$$

***Example***

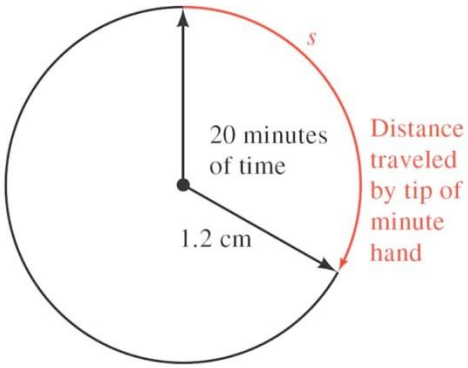
The minute hand of a clock is 1.2 cm long. To two significant digits, how far does the tip of the minute hand move in 20 minutes?

*Solution*

***Given***: $r$ = 1.2 cm

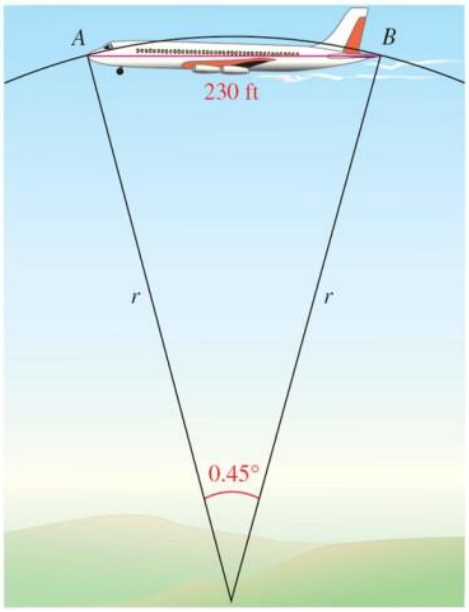
*One complete rotation* = 1 *hour* = 60 *minutes* = $2\pi$

$$\Rightarrow \frac{\theta}{2\pi} = \frac{20}{60}$$
$$\Rightarrow \theta = \frac{20}{60}2\pi$$
$$\Rightarrow \theta = \frac{2\pi}{3}$$
$$s = r\theta$$
$$= 1.2\frac{2\pi}{3}$$
$$\approx 2.5\,cm$$



***Example***

A person standing on the earth notices that a 747 jet flying overhead subtends an angle 0.45°. If the length of the jet is 230 *ft.*, find its altitude to the nearest thousand feet.




*Solution*

$$s = r\theta$$

$$\Rightarrow r = \frac{s}{\theta}$$

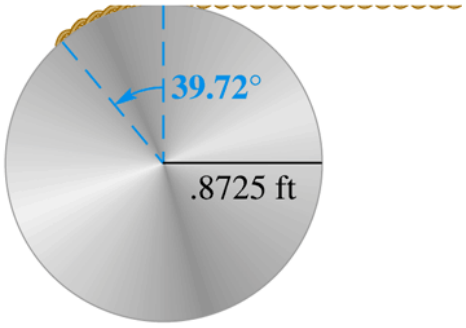
$$= \frac{230}{0.45\left(\frac{\pi}{180}\right)}$$

$$= \frac{230(180)}{0.45\pi}$$

$$= 29{,}000\ ft$$

***Example***

A rope is being wound around a drum with radius 0.8725 ft. How much rope will be wound around the drum if the drum is rotated through an angle of 39.72°?




*Solution*

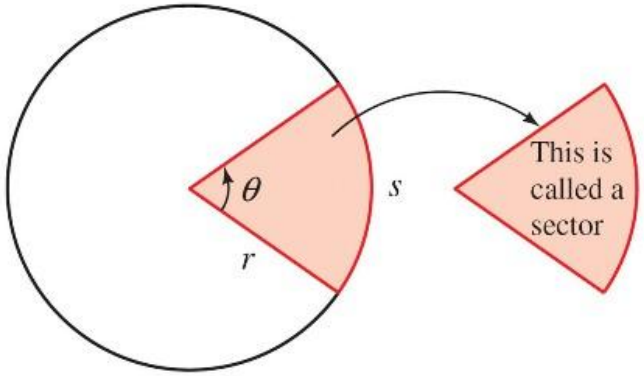
$$s = r\theta$$

$$= 0.8725\left(39.72\frac{\pi}{180}\right)$$

$$\approx 0.6049\ feet$$

## *Area of a Sector*

A sector of a circle is a portion of the interior of a circle intercepted by a central angle.




$$\text{Area of sector} \rightarrow \frac{A}{\pi r^2} = \frac{\theta}{2\pi} \leftarrow \text{Central angle } \theta$$
$$\text{Area of circle} \rightarrow \frac{\quad}{\pi r^2} = \frac{\quad}{2\pi} \leftarrow \text{One full rotation}$$

$$\frac{A}{\pi r^2}\pi r^2 = \frac{\theta}{2\pi}\pi r^2$$

$$A = \frac{1}{2}r^2\theta$$

### *Definition*

If $\theta$ (in radians) is a central angle in a circle with radius $r$, then the area of the sector formed by an angle $\theta$ is given by

$$A = \frac{1}{2}r^2\theta \qquad (\theta \text{ in radians})$$

### *Example*

Find the area of the sector formed by a central angle of 1.4 radians in a circle of radius 2.1 meters

*Solution*

**Given**: $r = 2.1$ m
$\theta = 1.4$

$$A = \frac{1}{2}r^2\theta$$
$$= \frac{1}{2}(2.1)^2(1.4)$$
$$= 3.1m^2$$

***Example***

If the sector formed by a central angle of 15° has an area of $\frac{\pi}{3}\text{cm}^2$, find the radius of a circle.

*Solution*

Given: $\theta = 15° \frac{\pi}{180} = \frac{\pi}{12}$

$A = \frac{\pi}{3}$

$A = \frac{1}{2} r^2 \theta$

$\frac{\pi}{3} = \frac{1}{2} r^2 \frac{\pi}{12}$

$\frac{24}{\pi} \frac{\pi}{3} = \frac{1}{2} r^2 \frac{\pi}{12} \frac{24}{\pi}$

$8 = r^2$

$r = \sqrt{8}$

$r = 2\sqrt{2}\ cm$

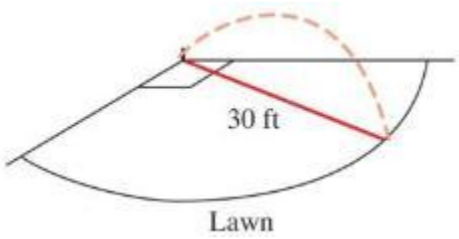
***Example***

A lawn sprinkler located at the corner of a yard is set to rotate 90° and project water out 30.0 ft. To three significant digits, what area of lawn is watered by the sprinkler?

*Solution*

Given: $\theta = 90° = \frac{\pi}{2}$

$r = 30\ ft$




$A = \frac{1}{2} r^2 \theta$

$= \frac{1}{2} (30)^2 \frac{\pi}{2}$

$= 707\ ft^2$

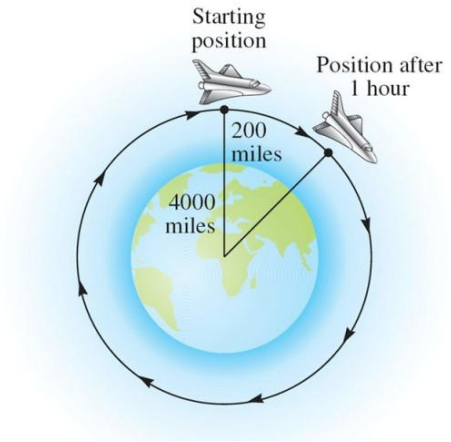
# *Exercises* *Section* 2.2 – Arc Length and Sector Area

**1.** The minute hand of a clock is 1.2 cm long. How far does the tip of the minute hand travel in 40 minutes?

**2.** Find the radian measure if angle $\theta$, if $\theta$ is a central angle in a circle of radius $r = 4$ inches, and $\theta$ cuts off an arc of length $s = 12\pi$ inches.

**3.** Give the length of the arc cut off by a central angle of 2 radians in a circle of radius 4.3 inches

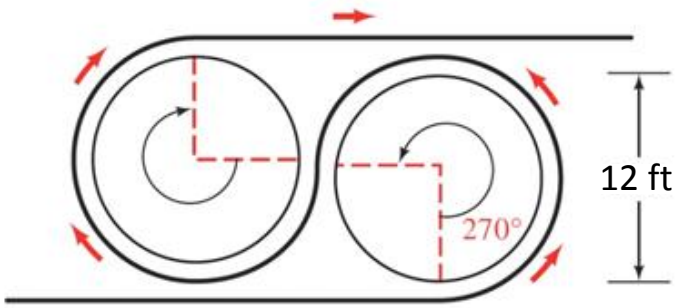
**4.** A space shuttle 200 miles above the earth is orbiting the earth once every 6 hours. How long, in hours, does it take the space shuttle to travel 8,400 miles? (Assume the radius of the earth is 4,000 miles.) Give both the exact value and an approximate value for your answer.




**5.** The pendulum on a grandfather clock swings from side to side once every second. If the length of the pendulum is 4 feet and the angle through which it swings is 20°. Find the total distance traveled in 1 minute by the tip of the pendulum on the grandfather clock.

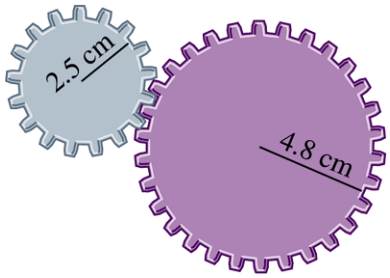
**6.** Reno, Nevada is due north of Los Angeles. The latitude of Reno is 40°, while that of Los Angeles is 34° N. The radius of Earth is about 4000 mi. Find the north-south distance between the two cities.

**7.** The first cable railway to make use of the figure-eight drive system was a Sutter Street Railway. Each drive sheave was 12 feet in diameter. Find the length of cable riding on one of the drive sheaves.

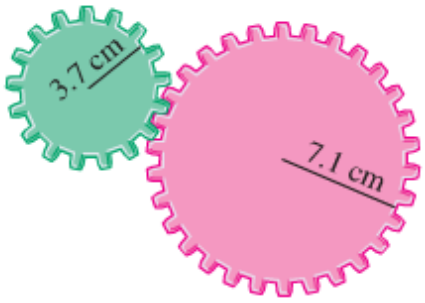


**8.** The diameter of a model of George Ferris's Ferris wheel is 250 feet, and $\theta$ is the central angle formed as a rider travels from his or her initial position $P_0$ to position $P_1$. Find the distance traveled by the rider if $\theta = 45°$ and if $\theta = 105°$.

**9.** Two gears are adjusted so that the smaller gear drives the larger one. If the smaller gear rotates through an angle of 225°, through how many degrees will the larger gear rotate?

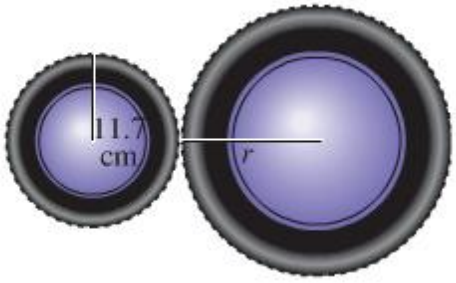



**10.** Two gears are adjusted so that the smaller gear drives the larger one. If the smaller gear rotates through an angle of 300°, through how many degrees will the larger rotate?




**11.** The rotation of the smaller wheel causes the larger wheel to rotate. Through how many degrees will the larger wheel rotate if the smaller one rotates through 60.0°?

**12.** Find the radius of the larger wheel if the smaller wheel rotates 80° when the larger wheel rotates 50°.

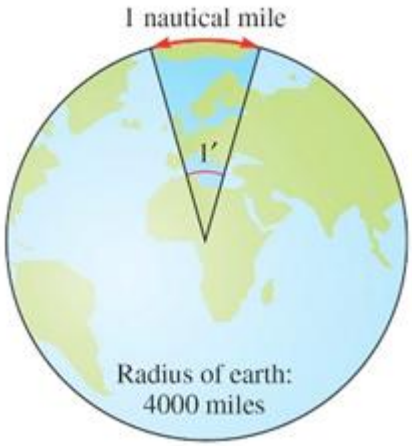



**13.** Los Angeles and New York City are approximately 2,500 miles apart on the surface of the earth. Assuming that the radius of the earth is 4,000 miles, find the radian measure of the central angle with its vertex at the center of the earth that has Los Angeles on one side and New York City in the other side.

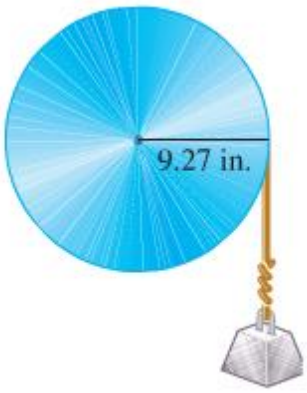
**14.** Find the number of regular (statute) miles in 1 nautical mile to the nearest hundredth of a mile. (Use 4,000 miles for the radius of the earth).

**15.** If two ships are 20 nautical miles apart on the ocean, how many statute miles apart are they?

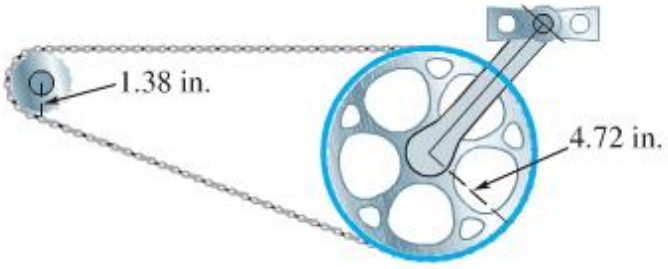
**16.** If a central angle with its vertex at the center of the earth has a measure of 1′, then the arc on the surface of the earth that is cut off by this angle (knows as the great circle distance) has a measure of 1 nautical mile.



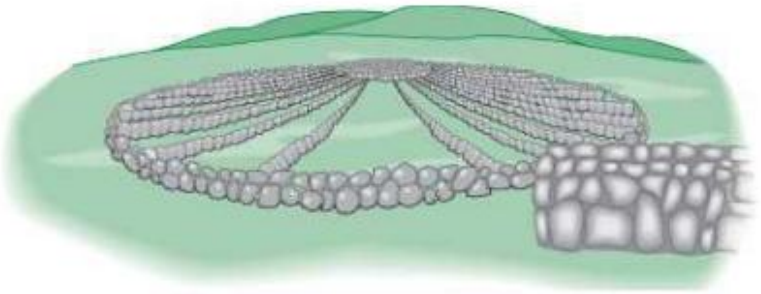

**17.** How many inches will the weight rise if the pulley is rotated through an angle of 71° 50′? Through what angle, to the nearest minute, must the pulley be rotated to raise the weight 6 in?




**18.** The figure shows the chain drive of a bicycle. How far will the bicycle move if the pedals are rotated through 180°? Assume the radius of the bicycle wheel is 13.6 in.



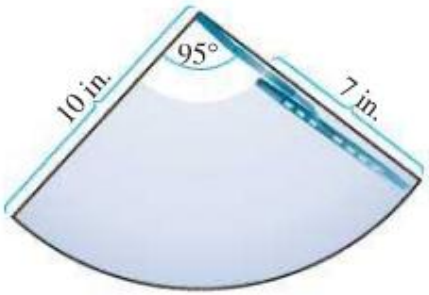
**19.** The circular of a Medicine Wheel is 2500 yrs old. There are 27 aboriginal spokes in the wheel, all equally spaced.



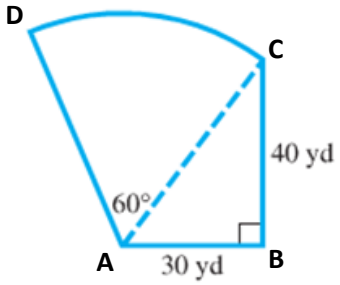
*a)* Find the measure of each central angle in degrees and in radians.
*b)* The radius measure of each of the wheel is 76.0 ft, find the circumference.
*c)* Find the length of each arc intercepted by consecutive pairs of spokes.
*d)* Find the area of each sector formed by consecutive spokes,

**20.** Find the radius of the pulley if a rotation of 51.6° raises the weight 11.4 cm.

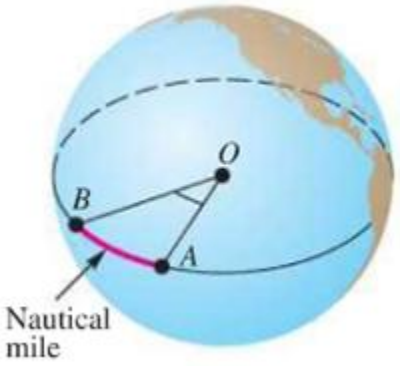

**21.** The total arm and blade of a single windshield wiper was 10 in. long and rotated back and forth through an angle of 95°. The shaded region in the figure is the portion of the windshield cleaned by the 7-in. wiper blade. What is the area of the region cleaned?



**22.** A frequent problem in surveying city lots and rural lands adjacent to curves of highways and railways is that of finding the area when one or more of the boundary lines is the arc of the circle. Find the area of the lot.




**23.** Nautical miles are used by ships and airplanes. They are different from statue miles, which equal 5280 ft. A nautical mile is defined to be the arc length along the equator intercepted by a central angle AOB of 1 min. If the equatorial radius is 3963 mi, use the arc length formula to approximate the number of statute miles in 1 nautical mile.




**24.** The distance to the moon is approximately 238,900 mi. Use the arc length formula to estimate the diameter $d$ of the moon if angle $\theta$ is measured to be 0.5170°.