

Solution **Section 3.3 – Gram-Schmidt Process**

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of \mathbb{R}^m .

$$\vec{u}_1 = (1, -3), \quad \vec{u}_2 = (2, 2)$$

Solution

$$\begin{aligned}\vec{v}_1 &= \frac{\vec{u}_1}{\|\vec{u}_1\|} \\ &= \frac{(1, -3)}{\sqrt{1^2 + (-3)^2}} \\ &= \frac{(1, -3)}{\sqrt{10}} \\ &= \left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}} \right)\end{aligned}$$

$$\begin{aligned}\vec{w}_2 &= \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1) \vec{u}_1 \\ &= (2, 2) - \left[(2, 2) \cdot \left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}} \right) \right] \left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}} \right) \\ &= (2, 2) - \left[\frac{2}{\sqrt{10}} - \frac{6}{\sqrt{10}} \right] \left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}} \right) \\ &= (2, 2) - \left[-\frac{4}{\sqrt{10}} \right] \left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}} \right) \\ &= (2, 2) - \left(-\frac{4}{10}, \frac{12}{10} \right) \\ &= (2, 2) - \left(-\frac{2}{5}, \frac{6}{5} \right) \\ &= \left(\frac{12}{5}, \frac{4}{5} \right)\end{aligned}$$

$$\begin{aligned}\|\vec{w}_2\| &= \sqrt{\left(\frac{12}{5} \right)^2 + \left(\frac{4}{5} \right)^2} \\ &= \sqrt{\frac{144}{25} + \frac{16}{25}} \\ &= \sqrt{\frac{160}{25}} \\ &= \frac{4\sqrt{10}}{5}\end{aligned}$$

$$\begin{aligned}
 \vec{v}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\
 &= \frac{4\sqrt{10}}{5} \left(\frac{12}{5}, \frac{4}{5} \right) \\
 &= \left(\frac{48\sqrt{10}}{25}, \frac{16\sqrt{10}}{25} \right)
 \end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of \mathbb{R}^m .

$$\vec{u}_1 = (1, 0), \quad \vec{u}_2 = (3, -5)$$

Solution

$$\begin{aligned}
 \vec{v}_1 &= \frac{\vec{u}_1}{\|\vec{u}_1\|} \\
 &= \frac{(1, 0)}{\sqrt{1^2 + 0^2}} \\
 &= (1, 0)
 \end{aligned}$$

$$\begin{aligned}
 \vec{w}_2 &= \vec{u}_2 - (\vec{u}_2 \cdot \vec{v}_1) \vec{v}_1 \\
 &= (0, -5) \\
 &= (3, -5) - [(3, -5) \cdot (1, 0)](1, 0) \\
 &= (3, -5) - [3](1, 0) \\
 &= (3, -5) - (3, 0) \\
 &= (0, -5)
 \end{aligned}$$

$$\begin{aligned}
 \|\vec{w}_2\| &= \sqrt{0^2 + (-5)^2} \\
 &= 5
 \end{aligned}$$

$$\begin{aligned}
 \vec{v}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\
 &= \frac{1}{5}(0, -5) \\
 &= (0, -1)
 \end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of \mathbb{R}^m .

$$\{(1, 1, 1), (-1, 1, 0), (1, 2, 1)\}$$

Solution

$$\begin{aligned}\vec{u}_1 &= \frac{(1, 1, 1)}{\sqrt{1^2+1^2+1^2}} \\ &= \frac{(1, 1, 1)}{\sqrt{3}}\end{aligned}$$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

$$\begin{aligned}\vec{w}_2 &= \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1) \vec{u}_1 \\ &= (-1, 1, 0) - \left[(-1, 1, 0) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \right] \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\ &= (-1, 1, 0) - \left[-\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + 0 \right] \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\ &= (-1, 1, 0) - 0 \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\ &= (-1, 1, 0)\end{aligned}$$

$$\begin{aligned}\|\vec{w}_2\| &= \sqrt{(-1)^2 + 1^2} \\ &= \sqrt{2}\end{aligned}$$

$$\vec{u}_2 = \frac{(-1, 1, 0)}{\sqrt{2}}$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)$$

$$\begin{aligned}\vec{v}_3 \cdot \vec{u}_1 &= (1, 2, 1) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\ &= \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} \\ &= \frac{3}{\sqrt{3}} \frac{\sqrt{3}}{\sqrt{3}} \\ &= \sqrt{3}\end{aligned}$$

$$\begin{aligned}
 \vec{v}_3 \cdot \vec{u}_2 &= (1, 2, 1) \cdot \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\
 &= -\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} + 0 \\
 &= \frac{1}{\sqrt{2}}
 \end{aligned}$$

$$\begin{aligned}
 \vec{w}_3 &= \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1) \vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2) \vec{u}_2 \\
 &= (1, 2, 1) - \sqrt{3} \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) - \sqrt{2} \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\
 &= (1, 2, 1) - (1, 1, 1) - (-1, 1, 0) \\
 &= (1, 0, 0)
 \end{aligned}$$

$$\begin{aligned}
 \vec{u}_3 &= \frac{(1, 0, 0)}{\sqrt{1}} \\
 &= (1, 0, 0)
 \end{aligned}
 \qquad
 \vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of \mathbf{R}^m .

$$\{(1, 1, 1), (0, 1, 1), (0, 0, 1)\}$$

Solution

$$\begin{aligned}
 \vec{u}_1 &= \frac{(1, 1, 1)}{\sqrt{1+1+1}} \\
 &= \frac{(1, 1, 1)}{\sqrt{3}} \\
 &= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)
 \end{aligned}
 \qquad
 \vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\begin{aligned}
 \vec{w}_2 &= (0, 1, 1) - \left[(0, 1, 1) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \right] \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\
 &= (0, 1, 1) - \left[\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right] \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\
 &= (0, 1, 1) - \left[\frac{2}{\sqrt{3}} \right] \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\
 &= (0, 1, 1) - \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3} \right)
 \end{aligned}
 \qquad
 \vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1) \vec{u}_1$$

$$= \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right) \Big|$$

$$\begin{aligned} \|\vec{w}_2\| &= \sqrt{\left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} \\ &= \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{1}{9}} \\ &= \sqrt{\frac{6}{9}} \\ &= \frac{\sqrt{6}}{3} \Big| \end{aligned}$$

$$\vec{u}_2 = \frac{\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)}{\frac{\sqrt{6}}{3}}$$

$$= \frac{3}{\sqrt{6}} \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right)$$

$$= \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right) \Big|$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$\begin{aligned} \vec{v}_3 \cdot \vec{u}_1 &= (0, 0, 1) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\ &= \frac{1}{\sqrt{3}} \Big| \end{aligned}$$

$$\begin{aligned} \vec{v}_3 \cdot \vec{u}_2 &= (0, 0, 1) \cdot \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right) \\ &= \frac{1}{\sqrt{6}} \Big| \end{aligned}$$

$$\begin{aligned} \vec{w}_3 &= (0, 0, 1) - \frac{1}{\sqrt{3}} \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) - \frac{1}{\sqrt{6}} \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right) & \vec{w}_3 &= \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1) \vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2) \vec{u}_2 \\ &= (0, 0, 1) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) - \left(-\frac{1}{3}, \frac{1}{6}, \frac{1}{6} \right) \\ &= \left(0, -\frac{1}{2}, \frac{1}{2} \right) \Big| \end{aligned}$$

$$\begin{aligned} \vec{u}_3 &= \frac{\left(0, -\frac{1}{2}, \frac{1}{2} \right)}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}} \\ &= \frac{\left(0, -\frac{1}{2}, \frac{1}{2} \right)}{\sqrt{\frac{1}{2}}} \end{aligned}$$

$$\vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$\begin{aligned}
&= \frac{\left(0, -\frac{1}{2}, \frac{1}{2}\right)}{\frac{1}{\sqrt{2}}} \\
&= \sqrt{2} \left(0, -\frac{1}{2}, \frac{1}{2}\right) \\
&= \left(0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \Big|
\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of \mathbb{R}^m .

$$\{(1, 1, 1), (0, 2, 1), (1, 0, 3)\}$$

Solution

$$\begin{aligned}
\vec{u}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
&= \frac{(1, 1, 1)}{\sqrt{1+1+1}} \\
&= \frac{(1, 1, 1)}{\sqrt{3}} \\
&= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{w}_2 &= \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1) \vec{u}_1 \\
&= (0, 2, 1) - \left[(0, 2, 1) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \right] \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\
&= (0, 2, 1) - \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\
&= (0, 2, 1) - \frac{3}{\sqrt{3}} \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\
&= (0, 2, 1) - \sqrt{3} \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\
&= (0, 2, 1) - (1, 1, 1) \\
&= (-1, 1, 0) \Big|
\end{aligned}$$

$$\begin{aligned}
\|\vec{w}_2\| &= \sqrt{(-1)^2 + (1)^2 + (0)^2} \\
&= \sqrt{2} \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{u}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\
&= \frac{(-1, 1, 0)}{\sqrt{2}} \\
&= \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 \cdot \vec{u}_1 &= (1, 0, 3) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\
&= \frac{1}{\sqrt{3}} + \frac{3}{\sqrt{3}} \\
&= \frac{4}{\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 \cdot \vec{u}_2 &= (1, 0, 3) \cdot \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\
&= -\frac{1}{\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
\vec{w}_3 &= \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1) \vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2) \vec{u}_2 \\
&= (1, 0, 3) - \frac{4}{\sqrt{3}} \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) + \frac{1}{\sqrt{2}} \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\
&= (1, 0, 3) - \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3} \right) + \left(-\frac{1}{2}, \frac{1}{2}, 0 \right) \\
&= \left(-\frac{5}{6}, -\frac{5}{6}, \frac{5}{3} \right)
\end{aligned}$$

$$\begin{aligned}
\vec{u}_3 &= \frac{\vec{w}_3}{\|\vec{w}_3\|} \\
&= \frac{1}{\sqrt{\left(-\frac{5}{6}\right)^2 + \left(-\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2}} \left(-\frac{5}{6}, -\frac{5}{6}, \frac{5}{3} \right) \\
&= \frac{1}{\sqrt{\frac{25}{36} + \frac{25}{36} + \frac{25}{9}}} \left(-\frac{5}{6}, -\frac{5}{6}, \frac{5}{3} \right) \\
&= \frac{1}{\frac{5}{6}\sqrt{6}} \left(-\frac{5}{6}, -\frac{5}{6}, \frac{5}{3} \right) \\
&= \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right)
\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of \mathbb{R}^m .

$$\{(2, 2, 2), (1, 0, -1), (0, 3, 1)\}$$

Solution

$$\begin{aligned}\vec{u}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{(2, 2, 2)}{\sqrt{2^2+2^2+2^2}} \\ &= \frac{(2, 2, 2)}{\sqrt{12}} \\ &= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \end{aligned}$$

$$\begin{aligned}\vec{w}_3 &= \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1) \vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2) \vec{u}_2 \\ &= (1, 0, -1) - \left[(1, 0, -1) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \right] \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\ &= (1, 0, -1) - \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} \right) \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\ &= (0, 2, 1) - (0) \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\ &= (1, 0, -1) \end{aligned}$$

$$\begin{aligned}\vec{u}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\ &= \frac{(1, 0, -1)}{\sqrt{2}} \\ &= \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right) \end{aligned}$$

$$\begin{aligned}\vec{v}_3 \cdot \vec{u}_1 &= (0, 3, 1) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\ &= \frac{3}{\sqrt{3}} + \frac{1}{\sqrt{3}} \\ &= \frac{4}{\sqrt{3}} \end{aligned}$$

$$\vec{v}_3 \cdot \vec{u}_2 = (0, 3, 1) \cdot \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right)$$

$$\left| -\frac{1}{\sqrt{2}} \right|$$

$$\begin{aligned}\vec{w}_3 &= \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1) \vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2) \vec{u}_2 \\ &= (0, 3, 1) - \frac{4}{\sqrt{3}} \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) + \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right) \\ &= (0, 3, 1) - \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3} \right) + \left(\frac{1}{2}, 0, -\frac{1}{2} \right) \\ &= \left(-\frac{5}{6}, \frac{5}{3}, -\frac{5}{6} \right) \end{aligned}$$

$$\begin{aligned}\vec{u}_3 &= \frac{\vec{w}_3}{\|\vec{w}_3\|} \\ &= \frac{1}{\sqrt{\left(-\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(-\frac{5}{6}\right)^2}} \left(-\frac{5}{6}, \frac{5}{3}, -\frac{5}{6} \right) \\ &= \frac{1}{\sqrt{\frac{25}{36} + \frac{25}{9} + \frac{25}{36}}} \left(-\frac{5}{6}, \frac{5}{3}, -\frac{5}{6} \right) \\ &= \frac{1}{\frac{5}{6}\sqrt{6}} \left(-\frac{5}{6}, \frac{5}{3}, -\frac{5}{6} \right) \\ &= \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right) \end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of \mathbb{R}^m .

$$\{(1, -1, 0), (0, 1, 0), (2, 3, 1)\}$$

Solution

$$\begin{aligned}\vec{u}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{(1, -1, 0)}{\sqrt{2}} \\ &= \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right) \end{aligned}$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1) \vec{u}_1$$

$$\begin{aligned}
&= (0, 1, 0) - \left[(0, 1, 0) \cdot \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right) \right] \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right) \\
&= (0, 1, 0) + \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right) \\
&= (0, 1, 0) + \left(\frac{1}{2}, -\frac{1}{2}, 0 \right) \\
&= \left(\frac{1}{2}, \frac{1}{2}, 0 \right) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{u}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\
&= \frac{1}{\sqrt{\frac{1}{4} + \frac{1}{4}}} \left(\frac{1}{2}, \frac{1}{2}, 0 \right) \\
&= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 \cdot \vec{u}_1 &= (2, 3, 1) \cdot \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right) \\
&= \frac{2}{\sqrt{2}} - \frac{3}{\sqrt{2}} \\
&= -\frac{1}{\sqrt{2}} \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 \cdot \vec{u}_2 &= (2, 3, 1) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\
&= \frac{4}{\sqrt{2}} \\
&= 2\sqrt{2} \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{w}_3 &= \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1) \vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2) \vec{u}_2 \\
&= (2, 3, 1) + \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right) - 2\sqrt{2} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\
&= (2, 3, 1) + \left(\frac{1}{2}, -\frac{1}{2}, 0 \right) - (2, 2, 0) \\
&= \left(\frac{1}{2}, \frac{1}{2}, 1 \right) \Big|
\end{aligned}$$

$$\vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{\frac{1}{4} + \frac{1}{4} + 1}} \left(\frac{1}{2}, \frac{1}{2}, 1 \right) \\
&= \frac{2}{\sqrt{6}} \left(\frac{1}{2}, \frac{1}{2}, 1 \right) \\
&= \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right)
\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of \mathbb{R}^m .

$$\{(3, 0, 4), (-1, 0, 7), (2, 9, 11)\}$$

Solution

$$\begin{aligned}
\vec{u}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
&= \frac{(3, 0, 4)}{\sqrt{9+16}} \\
&= \left(\frac{3}{5}, 0, \frac{4}{5} \right)
\end{aligned}$$

$$\begin{aligned}
\vec{w}_2 &= \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1) \vec{u}_1 \\
&= (-1, 0, 7) - \left[(-1, 0, 7) \cdot \left(\frac{3}{5}, 0, \frac{4}{5} \right) \right] \left(\frac{3}{5}, 0, \frac{4}{5} \right) \\
&= (-1, 0, 7) - \left(-\frac{3}{5} + \frac{28}{5} \right) \left(\frac{3}{5}, 0, \frac{4}{5} \right) \\
&= (-1, 0, 7) - 5 \left(\frac{3}{5}, 0, \frac{4}{5} \right) \\
&= (-1, 0, 7) - (3, 0, 4) \\
&= (-4, 0, 3)
\end{aligned}$$

$$\begin{aligned}
\vec{u}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\
&= \frac{1}{\sqrt{16+9}} (-4, 0, 3) \\
&= \left(-\frac{4}{5}, 0, \frac{3}{5} \right)
\end{aligned}$$

$$\vec{v}_3 \cdot \vec{u}_1 = (2, 9, 11) \cdot \left(\frac{3}{5}, 0, \frac{4}{5} \right)$$

$$= \frac{6}{5} + \frac{44}{5}$$

$$= 10 \quad |$$

$$\vec{v}_3 \cdot \vec{u}_2 = (2, 9, 11) \cdot \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$$

$$= -\frac{8}{5} + \frac{33}{5}$$

$$= 5 \quad |$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1) \vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2) \vec{u}_2$$

$$\vec{w}_3 = (2, 9, 11) - 10\left(\frac{3}{5}, 0, \frac{4}{5}\right) - 5\left(-\frac{4}{5}, 0, \frac{3}{5}\right)$$

$$= (0, 9, 0) \quad |$$

$$\vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$= \frac{1}{9}(0, 9, 0)$$

$$= (0, 1, 0) \quad |$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of \mathbb{R}^m .

$$\{(1, 1, 1, 1), (1, 2, 1, 0), (1, 3, 0, 0)\}$$

Solution

$$\vec{v}_1 = \vec{u}_1$$

$$= (1, 1, 1, 1) \quad |$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1, 1, 1, 1)}{\sqrt{4}}$$

$$= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \quad |$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$\begin{aligned}
&= (1, 2, 1, 0) - \frac{(1, 2, 1, 0) \cdot (1, 1, 1, 1)}{\|(1, 1, 1, 1)\|^2} (1, 1, 1, 1) \\
&= (1, 2, 1, 0) - \frac{1+2+1}{4} (1, 1, 1, 1) \\
&= (1, 2, 1, 0) - \frac{4}{4} (1, 1, 1, 1) \\
&= (1, 2, 1, 0) - (1, 1, 1, 1) \\
&= \underline{(0, 1, 0, -1)} \quad |
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \frac{(0, 1, 0, -1)}{\sqrt{1+1}} \\
&= \underline{\left(0, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)} \quad |
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
&= (1, 3, 0, 0) - \frac{(1, 3, 0, 0) \cdot (1, 1, 1, 1)}{\|(1, 1, 1, 1)\|^2} (1, 1, 1, 1) - \frac{(1, 3, 0, 0) \cdot (0, 1, 0, -1)}{\|(0, 1, 0, -1)\|^2} (0, 1, 0, -1) \\
&= (1, 3, 0, 0) - \frac{4}{4} (1, 1, 1, 1) - \frac{3}{2} (0, 1, 0, -1) \\
&= (1, 3, 0, 0) - (1, 1, 1, 1) - \left(0, \frac{3}{2}, 0, -\frac{3}{2}\right) \\
&= \underline{\left(0, \frac{1}{2}, -1, \frac{1}{2}\right)} \quad |
\end{aligned}$$

$$\begin{aligned}
\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
&= \frac{\left(0, \frac{1}{2}, -1, \frac{1}{2}\right)}{\sqrt{\frac{1}{4}+1+\frac{1}{4}}} \\
&= \frac{\left(0, \frac{1}{2}, -1, \frac{1}{2}\right)}{\frac{\sqrt{6}}{2}} \\
&= \frac{2}{\sqrt{6}} \left(0, \frac{1}{2}, -1, \frac{1}{2}\right) \\
&= \underline{\left(0, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)} \quad |
\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of \mathbb{R}^m .

$$\{(0, 2, -1, 1), (0, 0, 1, 1), (-2, 1, 1, -1)\}$$

Solution

$$\begin{aligned}\vec{u}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{(0, 2, -1, 1)}{\sqrt{6}} \\ &= \left(0, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \end{aligned}$$

$$\begin{aligned}\vec{w}_2 &= \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1) \vec{u}_1 \\ &= (0, 0, 1, 1) - \left[(0, 0, 1, 1) \cdot \left(0, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \right] \left(0, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \\ &= (0, 0, 1, 1) - \left[-\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}} \right] \left(0, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \\ &= (0, 0, 1, 1) - [0] \left(0, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \\ &= (0, 0, 1, 1) \end{aligned}$$

$$\begin{aligned}\vec{u}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\ &= \frac{(0, 0, 1, 1)}{\sqrt{2}} \\ &= \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \end{aligned}$$

$$\begin{aligned}\vec{v}_3 \cdot \vec{u}_1 &= (-2, 1, 1, -1) \cdot \left(0, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \\ &= \frac{2}{\sqrt{6}} - \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} \\ &= 0 \end{aligned}$$

$$\begin{aligned}\vec{v}_3 \cdot \vec{u}_2 &= (-2, 1, 1, -1) \cdot \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \\ &= 0 \end{aligned}$$

$$\begin{aligned}
 \vec{w}_3 &= \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1) \vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2) \vec{u}_2 \\
 &= (-2, 1, 1, -1) - 0 - 0 \\
 &= (-2, 1, 1, -1) \quad |
 \end{aligned}$$

$$\begin{aligned}
 \vec{u}_3 &= \frac{\vec{w}_3}{\|\vec{w}_3\|} \\
 &= \frac{(-2, 1, 1, -1)}{\sqrt{(-2)^2 + 1^2 + 1^2 + (-1)^2}} \\
 &= \frac{(-2, 1, 1, -1)}{\sqrt{7}} \\
 &= \left(-\frac{2}{\sqrt{7}}, \frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}, -\frac{1}{\sqrt{7}} \right) \quad |
 \end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of \mathbb{R}^m .

$$\vec{u}_1 = (1, 0, 0), \quad \vec{u}_2 = (3, 7, -2), \quad \vec{u}_3 = (0, 4, 1)$$

Solution

$$\begin{aligned}
 \vec{v}_1 &= \frac{\vec{u}_1}{\|\vec{u}_1\|} \\
 &= \frac{(1, 0, 0)}{\sqrt{1^2 + 0^2 + 0^2}} \\
 &= (1, 0, 0) \quad |
 \end{aligned}$$

$$\begin{aligned}
 \vec{w}_2 &= \vec{u}_2 - (\vec{u}_2 \cdot \vec{v}_1) \vec{v}_1 \\
 &= (3, 7, -2) - [(3, 7, -2) \cdot (1, 0, 0)](1, 0, 0) \\
 &= (3, 7, -2) - 3(1, 0, 0) \\
 &= (0, 7, -2) \quad |
 \end{aligned}$$

$$\begin{aligned}
 \vec{v}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\
 &= \frac{1}{\sqrt{53}}(0, 7, -2) \\
 &= \left(0, \frac{7}{\sqrt{53}}, -\frac{2}{\sqrt{53}} \right) \quad |
 \end{aligned}$$

$$\begin{aligned}\vec{u}_3 \cdot \vec{v}_1 &= (0, 4, 1) \cdot (1, 0, 0) \\ &= 0\end{aligned}$$

$$\begin{aligned}\vec{u}_3 \cdot \vec{v}_2 &= (0, 4, 1) \cdot \left(0, \frac{7}{\sqrt{53}}, -\frac{2}{\sqrt{53}}\right) \\ &= \frac{26}{\sqrt{53}}\end{aligned}$$

$$\begin{aligned}\vec{w}_3 &= \vec{u}_3 - (\vec{u}_3 \cdot \vec{v}_1)\vec{v}_1 - (\vec{u}_3 \cdot \vec{v}_2)\vec{v}_2 \\ &= (0, 4, 1) - 0 - \frac{26}{\sqrt{53}}\left(0, \frac{7}{\sqrt{53}}, -\frac{2}{\sqrt{53}}\right) \\ &= (0, 4, 1) - 0 - \frac{26}{53}\left(0, 7, -2\right) \\ &= (0, 4, 1) - \left(0, \frac{182}{53}, -\frac{52}{53}\right) \\ &= \left(0, \frac{30}{53}, \frac{105}{53}\right)\end{aligned}$$

$$\begin{aligned}\vec{v}_3 &= \frac{\vec{w}_3}{\|\vec{w}_3\|} \\ &= \frac{\left(0, \frac{30}{53}, \frac{105}{53}\right)}{\sqrt{\left(\frac{30}{53}\right)^2 + \left(\frac{105}{53}\right)^2}} \\ &= \frac{53}{\sqrt{11925}}\left(0, \frac{30}{53}, \frac{105}{53}\right) \\ &= \frac{53}{15\sqrt{53}}\left(0, \frac{30}{53}, \frac{105}{53}\right) \\ &= \left(0, \frac{2}{\sqrt{15}}, \frac{7}{\sqrt{15}}\right)\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an *orthonormal* basis for the subspaces of \mathbb{R}^m .

$$\vec{u}_1 = (1, 1, 1, 1), \quad \vec{u}_2 = (1, 2, 4, 5), \quad \vec{u}_3 = (1, -3, -4, -2)$$

Solution

$$\begin{aligned}\vec{v}_1 &= \frac{\vec{u}_1}{\|\vec{u}_1\|} \\ &= \frac{(1, 1, 1, 1)}{\sqrt{4}} \\ &= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \end{aligned}$$

$$\begin{aligned}\vec{w}_2 &= \vec{u}_2 - (\vec{u}_2 \cdot \vec{v}_1) \vec{v}_1 \\ &= (1, 2, 4, 5) - \left[(1, 2, 4, 5) \cdot \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \right] \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \\ &= (1, 2, 4, 5) - 6 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \\ &= (1, 2, 4, 5) - (3, 3, 3, 3) \\ &= (-2, -1, 1, 2) \end{aligned}$$

$$\begin{aligned}\vec{v}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\ &= \frac{1}{\sqrt{10}} (-2, -1, 1, 2) \\ &= \left(-\frac{2}{\sqrt{10}}, -\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}} \right) \end{aligned}$$

$$\begin{aligned}\vec{u}_3 \cdot \vec{v}_1 &= (1, -3, -4, -2) \cdot \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \\ &= \frac{1-3-4-2}{2} \\ &= -4 \end{aligned}$$

$$\begin{aligned}\vec{u}_3 \cdot \vec{v}_2 &= (1, -3, -4, -2) \cdot \left(-\frac{2}{\sqrt{10}}, -\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}} \right) \\ &= \frac{-2+3-4-4}{\sqrt{10}} \\ &= -\frac{7}{\sqrt{10}} \end{aligned}$$

$$\vec{w}_3 = \vec{u}_3 - (\vec{u}_3 \cdot \vec{v}_1) \vec{v}_1 - (\vec{u}_3 \cdot \vec{v}_2) \vec{v}_2$$

$$\begin{aligned}
&= (1, -3, -4, -2) + 4\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + \frac{7}{\sqrt{10}}\left(-\frac{2}{\sqrt{10}}, -\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right) \\
&= (1, -3, -4, -2) + (2, 2, 2, 2) + \left(-\frac{7}{5}, -\frac{7}{10}, \frac{7}{10}, \frac{7}{5}\right) \\
&= \left(\frac{8}{5}, -\frac{17}{10}, -\frac{27}{10}, \frac{7}{5}\right) \Big| \\
\vec{v}_3 &= \frac{\vec{w}_3}{\|\vec{w}_3\|} \\
&= \frac{1}{\sqrt{\frac{64}{25} + \frac{289}{100} + \frac{289}{100} + \frac{49}{25}}}\left(\frac{8}{5}, -\frac{17}{10}, -\frac{27}{10}, \frac{7}{5}\right) \\
&= \frac{1}{\sqrt{\frac{1030}{100}}}\left(\frac{8}{5}, -\frac{17}{10}, -\frac{27}{10}, \frac{7}{5}\right) \\
&= \left(\frac{16}{\sqrt{1030}}, -\frac{17}{\sqrt{1030}}, -\frac{27}{\sqrt{1030}}, \frac{14}{\sqrt{1030}}\right) \Big|
\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an *orthonormal* basis for the subspaces of \mathbb{R}^m .

$$\vec{u}_1 = (1, 1, 1, 1), \quad \vec{u}_2 = (1, 1, 2, 4), \quad \vec{u}_3 = (1, 2, -4, -3)$$

Solution

$$\begin{aligned}
\vec{v}_1 &= \frac{\vec{u}_1}{\|\vec{u}_1\|} \\
&= \frac{(1, 1, 1, 1)}{\sqrt{4}} \\
&= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \Big| \\
\vec{w}_2 &= \vec{u}_2 - (\vec{u}_2 \cdot \vec{v}_1) \vec{v}_1 \\
&= (1, 1, 2, 4) - \left[(1, 1, 2, 4) \cdot \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right] \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \\
&= (1, 1, 2, 4) - 4\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \\
&= (1, 1, 2, 4) - (2, 2, 2, 2) \\
&= (-1, -1, 0, 2) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{v}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\
&= \frac{1}{\sqrt{1+1+4}}(-1, -1, 0, 2) \\
&= \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}} \right)
\end{aligned}$$

$$\begin{aligned}
\vec{u}_3 \cdot \vec{v}_1 &= (1, 2, -4, -3) \cdot \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \\
&= \frac{1+2-4-3}{2} \\
&= -2
\end{aligned}$$

$$\begin{aligned}
\vec{u}_3 \cdot \vec{v}_2 &= (1, 2, -4, -3) \cdot \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}} \right) \\
&= \frac{-1-2-6}{\sqrt{6}} \\
&= -\frac{9}{\sqrt{6}}
\end{aligned}$$

$$\begin{aligned}
\vec{w}_3 &= \vec{u}_3 - (\vec{u}_3 \cdot \vec{v}_1) \vec{v}_1 - (\vec{u}_3 \cdot \vec{v}_2) \vec{v}_2 \\
&= (1, 2, -4, -3) + 2 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) + \frac{9}{\sqrt{6}} \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}} \right) \\
&= (1, 2, -4, -3) + (1, 1, 1, 1) + \left(-\frac{3}{2}, -\frac{3}{2}, 0, 3 \right) \\
&= \left(\frac{1}{2}, \frac{3}{2}, -3, 1 \right)
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \frac{\vec{w}_3}{\|\vec{w}_3\|} \\
&= \frac{1}{\sqrt{\frac{1}{4} + \frac{9}{4} + 9 + 1}} \left(\frac{1}{2}, \frac{3}{2}, -3, 1 \right) \\
&= \frac{2}{5\sqrt{2}} \left(\frac{1}{2}, \frac{3}{2}, -3, 1 \right) \\
&= \left(\frac{1}{5\sqrt{2}}, \frac{3}{5\sqrt{2}}, -\frac{6}{5\sqrt{2}}, \frac{2}{5\sqrt{2}} \right)
\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an *orthonormal* basis for the subspaces of \mathbb{R}^m .

$$\vec{u}_1 = (0, 2, 1, 0); \quad \vec{u}_2 = (1, -1, 0, 0); \quad \vec{u}_3 = (1, 2, 0, -1); \quad \vec{u}_4 = (1, 0, 0, 1)$$

Solution

$$\begin{aligned}\vec{v}_1 &= \frac{\vec{u}_1}{\|\vec{u}_1\|} \\ &= \frac{(0, 2, 1, 0)}{\sqrt{5}} \\ &= \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right) \end{aligned}$$

$$\begin{aligned}\vec{w}_2 &= \vec{u}_2 - (\vec{u}_2 \cdot \vec{v}_1) \vec{v}_1 \\ &= (1, -1, 0, 0) - \left[(1, -1, 0, 0) \cdot \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)\right] \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right) \\ &= (1, -1, 0, 0) - \left(-\frac{2}{\sqrt{5}}\right) \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right) \\ &= \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right) \end{aligned}$$

$$\begin{aligned}\vec{v}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\ &= \frac{1}{\sqrt{1 + \frac{1}{25} + \frac{4}{25} + 0}} \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right) \\ &= \frac{5}{\sqrt{30}} \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right) \\ &= \left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0\right) \end{aligned}$$

$$\begin{aligned}u_3 \cdot v_1 &= (1, 2, 0, -1) \cdot \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right) \\ &= \frac{4}{\sqrt{5}} \end{aligned}$$

$$\begin{aligned}u_3 \cdot v_2 &= (1, 2, 0, -1) \cdot \left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0\right) \\ &= \frac{5}{\sqrt{30}} - \frac{2}{\sqrt{30}} \end{aligned}$$

$$= \frac{3}{\sqrt{30}} \Big|$$

$$\begin{aligned}\vec{w}_3 &= \vec{u}_3 - (\vec{u}_3 \cdot \vec{v}_1) \vec{v}_1 - (\vec{u}_3 \cdot \vec{v}_2) \vec{v}_2 \\ &= (1, 2, 0, -1) - \left(\frac{4}{\sqrt{5}}\right) \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right) - \left(\frac{3}{\sqrt{30}}\right) \left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0\right) \\ &= (1, 2, 0, -1) - \left(0, \frac{8}{5}, \frac{4}{5}, 0\right) - \left(\frac{1}{2}, -\frac{1}{10}, \frac{1}{5}, 0\right) \\ &= \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right) \Big| \end{aligned}$$

$$\begin{aligned}\vec{v}_3 &= \frac{\vec{w}_3}{\|\vec{w}_3\|} \\ &= \frac{\left(\frac{1}{2}, \frac{1}{2}, -1, -1\right)}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + (-1)^2 + (-1)^2}} \\ &= \frac{1}{\sqrt{5}} \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right) \\ &= \frac{\sqrt{2}}{\sqrt{5}} \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right) = \frac{2}{\sqrt{10}} \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right) \\ &= \left(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, -\frac{2}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right) \Big| \end{aligned}$$

$$\begin{aligned}u_4 \cdot v_1 &= (1, 0, 0, 1) \cdot \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right) \\ &= 0 \Big| \end{aligned}$$

$$\begin{aligned}u_4 \cdot v_2 &= (1, 0, 0, 1) \cdot \left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0\right) \\ &= \frac{5}{\sqrt{30}} \Big| \end{aligned}$$

$$\begin{aligned}u_4 \cdot v_3 &= (1, 0, 0, 1) \cdot \left(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, -\frac{2}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right) \\ &= -\frac{1}{\sqrt{10}} \Big| \end{aligned}$$

$$\begin{aligned}\vec{w}_4 &= \vec{u}_4 - (\vec{u}_4 \cdot \vec{v}_1) \vec{v}_1 - (\vec{u}_4 \cdot \vec{v}_2) \vec{v}_2 - (\vec{u}_4 \cdot \vec{v}_3) \vec{v}_3 \\ &= (1, 2, 0, -1) - (0) - \left(\frac{5}{\sqrt{30}}\right) \left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0\right) + \left(\frac{1}{\sqrt{10}}\right) \left(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, -\frac{2}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right) \end{aligned}$$

$$\begin{aligned}
&= (1, 2, 0, -1) - \left(\frac{5}{6}, -\frac{1}{6}, \frac{1}{3}, 0\right) + \left(\frac{1}{10}, \frac{1}{10}, -\frac{1}{5}, -\frac{1}{5}\right) \\
&= \left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right) \Big| \\
\vec{v}_4 &= \frac{\vec{w}_4}{\|\vec{w}_4\|} \\
&= \frac{\left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right)}{\sqrt{\left(\frac{4}{15}\right)^2 + \left(\frac{4}{15}\right)^2 + \left(-\frac{8}{15}\right)^2 + \left(\frac{4}{5}\right)^2}} \\
&= \frac{1}{\sqrt{\frac{240}{225}}} \left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right) \\
&= \frac{1}{\frac{4}{\sqrt{15}}} \left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right) \\
&= \frac{15}{4\sqrt{15}} \left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right) \\
&= \left(\frac{1}{\sqrt{15}}, \frac{1}{\sqrt{15}}, -\frac{2}{\sqrt{15}}, \frac{3}{\sqrt{15}}\right) \Big|
\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis for the subspaces of \mathbb{R}^m .

$$\{(1, 1, 0), (1, 2, 0), (0, 1, 2)\}$$

Solution

$$\vec{v}_1 = (1, 1, 0) \Big|$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
&= (0, 2, 1) - \frac{(1, 2, 0) \cdot (1, 1, 0)}{1+1+0} (1, 1, 0) \\
&= (0, 2, 1) - \frac{3}{2} (1, 1, 0) \\
&= \left(-\frac{3}{2}, \frac{1}{2}, 1\right) \Big|
\end{aligned}$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(0, 1, 2) \cdot (1, 1, 0)}{2} (1, 1, 0)$$

$$=\left(\frac{1}{2}, \frac{1}{2}, 0\right) \Big|$$

$$\begin{aligned} \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{1}{\frac{9}{4} + \frac{1}{4} + 1} (0, 1, 2) \cdot \left(-\frac{3}{2}, \frac{1}{2}, 1\right) \left(-\frac{3}{2}, \frac{1}{2}, 1\right) \\ &= \frac{2}{7} \frac{5}{2} \left(-\frac{3}{2}, \frac{1}{2}, 1\right) \\ &= \left(-\frac{15}{14}, \frac{5}{14}, \frac{5}{7}\right) \Big| \end{aligned}$$

$$\begin{aligned} \vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= (0, 1, 2) - \left(\frac{1}{2}, \frac{1}{2}, 0\right) - \left(-\frac{15}{14}, \frac{5}{14}, \frac{5}{7}\right) \\ &= \left(\frac{4}{7}, \frac{1}{7}, \frac{9}{7}\right) \Big| \end{aligned}$$

$$\begin{aligned} \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{1}{\sqrt{2}} (1, 1, 0) \\ &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \Big| \end{aligned}$$

$$\begin{aligned} \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{1}{\sqrt{\frac{9}{4} + \frac{1}{4} + 1}} \left(-\frac{3}{2}, \frac{1}{2}, 1\right) \\ &= \frac{2}{\sqrt{14}} \left(-\frac{3}{2}, \frac{1}{2}, 1\right) \\ &= \left(-\frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right) \Big| \end{aligned}$$

$$\begin{aligned} \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= \frac{1}{\sqrt{\frac{16}{49} + \frac{1}{49} + \frac{81}{49}}} \left(\frac{4}{7}, \frac{1}{7}, \frac{9}{7}\right) \\ &= \frac{7}{\sqrt{98}} \left(\frac{4}{7}, \frac{1}{7}, \frac{9}{7}\right) \end{aligned}$$

$$= \frac{7}{7\sqrt{2}} \left(\frac{4}{7}, \frac{1}{7}, \frac{9}{7} \right)$$

$$= \left(\frac{4}{7\sqrt{2}}, \frac{1}{7\sqrt{2}}, \frac{9}{7\sqrt{2}} \right) \Big|$$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis for the subspaces of \mathbb{R}^m .

$$\{(1, -2, 2), (2, 2, 1), (2, -1, -2)\}$$

Solution

$$\underline{\vec{v}_1 = \vec{u}_1 = (1, -2, 2) \Big|}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (2, 2, 1) - \frac{(2, 2, 1) \cdot (1, -2, 2)}{9} (1, -2, 2)$$

$$= (2, 2, 1) - \frac{0}{9} (1, -2, 2)$$

$$= (2, 2, 1) \Big|$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(2, -1, -2) \cdot (1, -2, 2)}{9} (1, -2, 2)$$

$$= \frac{0}{9} (1, -2, 2)$$

$$= (0, 0, 0) \Big|$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{1}{9} [(2, -1, -2) \cdot (2, 2, 1)] (2, 2, 1)$$

$$= (0, 0, 0) \Big|$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (2, -1, -2) - (0, 0, 0) - (0, 0, 0)$$

$$= (2, -1, -2) \Big|$$

$$\begin{aligned}\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{1}{3}(1, -2, 2) \\ &= \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \right) \mid\end{aligned}$$

$$\begin{aligned}\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{1}{3}(2, 2, 1) \\ &= \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right) \mid\end{aligned}$$

$$\begin{aligned}\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= \frac{1}{3}(2, -1, -2) \\ &= \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3} \right) \mid\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis for the subspaces of \mathbb{R}^m .

$$\{(1, 0, 0), (1, 1, 1), (1, 1, -1)\}$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (1, 0, 0) \mid$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (1, 1, 1) - \frac{(1, 1, 1) \cdot (1, 0, 0)}{1} (1, 0, 0) \\ &= (1, 1, 1) - (1, 0, 0) \\ &= (0, 1, 1) \mid\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(1, 1, -1) \cdot (1, 0, 0)}{1} (1, 0, 0) \\ &= (1, 0, 0) \mid\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{1}{1} [(1, 1, -1) \cdot (0, 1, 1)] (0, 1, 1) \\ &= 0(0, 1, 1) \\ &= \underline{(0, 0, 0)}\end{aligned}$$

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= (1, 1, -1) - (1, 0, 0) - (0, 0, 0) \\ &= \underline{(0, 1, -1)}\end{aligned}$$

$$\begin{aligned}\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{1}{1} (1, 0, 0) \\ &= \underline{(1, 0, 0)}\end{aligned}$$

$$\begin{aligned}\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{1}{\sqrt{2}} (0, 1, 1) \\ &= \underline{\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}\end{aligned}$$

$$\begin{aligned}\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= \frac{1}{\sqrt{2}} (0, 1, -1) \\ &= \underline{\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)}\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of \mathbb{R}^m .

$$\{(4, -3, 0), (1, 2, 0), (0, 0, 4)\}$$

Solution

$$\underline{\vec{v}_1 = \vec{u}_1 = (4, -3, 0)}$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
&= (1, 2, 0) - \frac{(1, 2, 0) \cdot (4, -3, 0)}{25} (4, -3, 0) \\
&= (1, 2, 0) + \frac{2}{25} (4, -3, 0) \\
&= \left(\frac{33}{25}, \frac{44}{25}, 0 \right) \Big|
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(0, 0, 4) \cdot (4, -3, 0)}{25} (4, -3, 0) \\
&= (0, 0, 0) \Big|
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{225}{3,025} \left[(0, 0, 4) \cdot \left(\frac{33}{25}, \frac{44}{25}, 0 \right) \right] \left(\frac{33}{25}, \frac{44}{25}, 0 \right) \\
&= (0, 0, 0) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
&= (0, 0, 4) - (0, 0, 0) - (0, 0, 0) \\
&= (0, 0, 4) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
&= \frac{1}{\sqrt{16+9}} (4, -3, 0) \\
&= \left(\frac{4}{5}, -\frac{3}{5}, 0 \right) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \frac{25}{\sqrt{3,025}} \left(\frac{33}{25}, \frac{44}{25}, 0 \right) \\
&= \frac{25}{55} \left(\frac{33}{25}, \frac{44}{25}, 0 \right) \\
&= \left(\frac{3}{5}, \frac{4}{5}, 0 \right) \Big|
\end{aligned}$$

$$\begin{aligned}
 \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
 &= \frac{1}{4}(0, 0, 4) \\
 &= \underline{(0, 0, 1)}
 \end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis for the subspaces of \mathbb{R}^m .

$$\{(0, 1, 2), (2, 0, 0), (1, 1, 1)\}$$

Solution

$$\underline{\vec{v}_1 = \vec{u}_1 = (0, 1, 2)}$$

$$\begin{aligned}
 \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
 &= (2, 0, 0) - \frac{(2, 0, 0) \cdot (0, 1, 2)}{5} (0, 1, 2) \\
 &= (2, 0, 0) + \frac{0}{5} (0, 1, 2) \\
 &= \underline{(2, 0, 0)}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(1, 1, 1) \cdot (0, 1, 2)}{5} (0, 1, 2) \\
 &= \frac{3}{5} (0, 1, 2) \\
 &= \underline{\left(0, \frac{3}{5}, \frac{6}{5}\right)}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{1}{4} [(1, 1, 1) \cdot (2, 0, 0)] (2, 0, 0) \\
 &= \frac{1}{2} (2, 0, 0) \\
 &= \underline{(1, 0, 0)}
 \end{aligned}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (1, 1, 1) - (1, 0, 0) - \left(0, \frac{3}{5}, \frac{6}{5}\right)$$

$$= \left(0, \frac{2}{5}, -\frac{1}{5}\right) \Big|$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{5}}(0, 1, 2)$$

$$= \left(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \Big|$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{2}(2, 0, 0)$$

$$= (1, 0, 0) \Big|$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{5}{\sqrt{5}}\left(0, \frac{2}{5}, -\frac{1}{5}\right)$$

$$= \left(0, \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right) \Big|$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of \mathbb{R}^m .

$$\{(0, 1, 1), (1, 1, 0), (1, 0, 1)\}$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (0, 1, 1) \Big|$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (1, 1, 0) - \frac{(1, 1, 0) \cdot (0, 1, 1)}{2} (0, 1, 1)$$

$$= (1, 1, 0) - \frac{1}{2} (0, 1, 1)$$

$$= \left(1, \frac{1}{2}, -\frac{1}{2}\right) \Big|$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(1, 0, 1) \cdot (0, 1, 1)}{2} (0, 1, 1)$$

$$= \left(0, \frac{1}{2}, \frac{1}{2} \right) \Big|$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{4}{6} \left[(1, 0, 1) \cdot \left(1, \frac{1}{2}, -\frac{1}{2} \right) \right] \left(1, \frac{1}{2}, -\frac{1}{2} \right)$$

$$= \frac{1}{3} \left(1, \frac{1}{2}, -\frac{1}{2} \right)$$

$$= \left(\frac{1}{3}, \frac{1}{6}, -\frac{1}{6} \right) \Big|$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (1, 0, 1) - \left(0, \frac{1}{2}, \frac{1}{2} \right) - \left(\frac{1}{3}, \frac{1}{6}, -\frac{1}{6} \right)$$

$$= \left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3} \right) \Big|$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{2}} (0, 1, 1)$$

$$= \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \Big|$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{2}{\sqrt{6}} \left(1, \frac{1}{2}, -\frac{1}{2} \right)$$

$$= \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right) \Big|$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{3}{\sqrt{12}} \left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3} \right)$$

$$= \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \Big|$$

Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis for the subspaces of \mathbb{R}^m .

$$\{(1, 2, -2), (1, 0, -4), (5, 2, 0), (1, 1, -1)\}$$

Solution

$$\underline{\vec{v}_1 = \vec{u}_1 = (1, 2, -2) \mid}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (1, 0, -4) - \frac{(1, 0, -4) \cdot (1, 2, -2)}{9} (1, 2, -2) \\ &= (1, 0, -4) - (1, 2, -2) \\ &= (0, -2, -2) \mid\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(5, 2, 0) \cdot (1, 2, -2)}{9} (1, 2, -2) \\ &= (1, 2, -2) \mid\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{1}{8} [(5, 2, 0) \cdot (0, -2, -2)] (0, -2, -2) \\ &= -\frac{1}{2} (0, -2, -2) \\ &= (0, 1, 1) \mid\end{aligned}$$

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= (5, 2, 0) - (1, 2, -2) - (0, 1, 1) \\ &= (4, -1, 1) \mid\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(1, 1, -1) \cdot (1, 2, -2)}{9} (1, 2, -2) \\ &= \frac{5}{9} (1, 2, -2) \\ &= \left(\frac{5}{9}, \frac{10}{9}, -\frac{10}{9} \right) \mid\end{aligned}$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{1}{8}[(1, 1, -1) \cdot (0, -2, -2)](0, -2, -2)$$

$$= \underline{(0, 0, 0) \mid}$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 = \frac{1}{18}[(1, 1, -1) \cdot (4, -1, 1)](4, -1, 1)$$

$$= \frac{1}{9}(4, -1, 1)$$

$$= \underline{\left(\frac{4}{9}, -\frac{1}{9}, \frac{1}{9}\right) \mid}$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3$$

$$= (1, 1, -1) - \left(\frac{5}{9}, \frac{10}{9}, -\frac{10}{9}\right) - \left(\frac{4}{9}, -\frac{1}{9}, \frac{1}{9}\right)$$

$$= \underline{(0, 0, 0) \mid}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{3}(1, 2, -2)$$

$$= \underline{\left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right) \mid}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{2\sqrt{2}}(0, -2, -2)$$

$$= \underline{\left(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \mid}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{3\sqrt{2}}(4, -1, 1)$$

$$= \underline{\left(\frac{4}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}\right) \mid}$$

$$\vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

$$= \underline{(0, 0, 0) \mid}$$

Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis for the subspaces of \mathbb{R}^m .

$$\{(-3, 1, 2), (1, 1, 1), (2, 0, -1), (1, -3, 2)\}$$

Solution

$$\underline{\vec{v}_1 = \vec{u}_1 = (-3, 1, 2)}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (1, 1, 1) - \frac{(1, 1, 1) \cdot (-3, 1, 2)}{14} (-3, 1, 2) \\ &= (1, 1, 1) - \frac{0}{14} (1, 2, -2) \\ &= (1, 1, 1)\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(2, 0, -1) \cdot (-3, 1, 2)}{14} (-3, 1, 2) \\ &= -\frac{4}{7} (-3, 1, 2) \\ &= \left(\frac{12}{7}, -\frac{4}{7}, -\frac{8}{7} \right)\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{1}{3} [(2, 0, -1) \cdot (1, 1, 1)] (1, 1, 1) \\ &= \frac{1}{3} (1, 1, 1) \\ &= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)\end{aligned}$$

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= (2, 0, -1) - \left(\frac{12}{7}, -\frac{4}{7}, -\frac{8}{7} \right) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \\ &= \left(-\frac{1}{21}, \frac{5}{21}, -\frac{4}{21} \right)\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(1, -3, 2) \cdot (-3, 1, 2)}{14} (-3, 1, 2) \\ &= -\frac{1}{7} (-3, 1, 2)\end{aligned}$$

$$=\left(\frac{3}{7}, -\frac{1}{7}, -\frac{2}{7}\right) \Big|$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{(1, -3, 2) \cdot (1, 1, 1)}{3} (1, 1, 1)$$

$$=(0, 0, 0) \Big|$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 = \frac{441}{42} \left[(1, -3, 2) \cdot \left(-\frac{1}{21}, \frac{5}{21}, -\frac{4}{21}\right) \right] \left(-\frac{1}{21}, \frac{5}{21}, -\frac{4}{21}\right)$$

$$= \frac{441}{42} \left(-\frac{24}{21}\right) \left(-\frac{1}{21}, \frac{5}{21}, -\frac{4}{21}\right)$$

$$=\left(\frac{4}{7}, -\frac{20}{7}, \frac{16}{7}\right) \Big|$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3$$

$$=(1, -3, 2) - \left(\frac{3}{7}, -\frac{1}{7}, -\frac{2}{7}\right) - (0, 0, 0) - \left(\frac{4}{7}, -\frac{20}{7}, \frac{16}{7}\right)$$

$$=(0, 0, 0) \Big|$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{14}} (-3, 1, 2)$$

$$=\left(-\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right) \Big|$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{\sqrt{3}} (1, 1, 1)$$

$$=\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \Big|$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{21}{\sqrt{42}} \left(-\frac{1}{21}, \frac{5}{21}, -\frac{4}{21}\right)$$

$$=\left(-\frac{1}{\sqrt{42}}, \frac{5}{\sqrt{42}}, -\frac{4}{\sqrt{42}}\right) \Big|$$

$$\begin{aligned}\vec{q}_4 &= \frac{\vec{v}_4}{\|\vec{v}_4\|} \\ &= \underline{(0, 0, 0)}\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis for the subspaces of \mathbb{R}^m .

$$\{(2, 1, 1), (0, 3, -1), (3, -4, -2), (-1, -1, 3)\}$$

Solution

$$\underline{\vec{v}_1 = \vec{u}_1 = (2, 1, 1)}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (0, 3, -1) - \frac{(0, 3, -1) \cdot (2, 1, 1)}{6} (2, 1, 1) \\ &= (0, 3, -1) - \frac{1}{3} (2, 1, 1) \\ &= \underline{\left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3}\right)}\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(3, -4, -2) \cdot (2, 1, 1)}{6} (2, 1, 1) \\ &= \underline{(0, 0, 0)}\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{9}{84} \left[(3, -4, -2) \cdot \left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3}\right) \right] \left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3}\right) \\ &= \frac{3}{28} (-10) \left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3}\right) \\ &= \underline{\left(\frac{5}{7}, -\frac{20}{7}, \frac{10}{7}\right)}\end{aligned}$$

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= (3, -4, -2) - (0, 0, 0) - \left(\frac{5}{7}, -\frac{20}{7}, \frac{10}{7}\right) \\ &= \underline{\left(\frac{16}{7}, -\frac{8}{7}, -\frac{24}{7}\right)}\end{aligned}$$

$$\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(-1, -1, 3) \cdot (2, 1, 1)}{6} (2, 1, 1)$$

$$= (0, 0, 0) \mid$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{9}{84} \left[(-1, -1, 3) \cdot \left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right) \right] \left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right)$$

$$= \frac{3}{28} \left(-\frac{18}{3} \right) \left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right)$$

$$= \left(\frac{3}{7}, -\frac{12}{7}, \frac{6}{7} \right) \mid$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 = \frac{49}{896} \left[(-1, -1, 3) \cdot \left(\frac{16}{7}, -\frac{8}{7}, -\frac{24}{7} \right) \right] \left(\frac{16}{7}, -\frac{8}{7}, -\frac{24}{7} \right)$$

$$= \frac{7}{128} \left(-\frac{80}{7} \right) \left(\frac{16}{7}, -\frac{8}{7}, -\frac{24}{7} \right)$$

$$= \left(-\frac{10}{7}, \frac{5}{7}, \frac{15}{7} \right) \mid$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3$$

$$= (-1, -1, 3) - (0, 0, 0) - \left(\frac{3}{7}, -\frac{12}{7}, \frac{6}{7} \right) - \left(-\frac{10}{7}, \frac{5}{7}, \frac{15}{7} \right)$$

$$= (0, 0, 0) \mid$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{6}} (2, 1, 1)$$

$$= \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right) \mid$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{3}{2\sqrt{21}} \left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right)$$

$$= \left(-\frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, -\frac{2}{\sqrt{21}} \right) \mid$$

$$\begin{aligned}
 \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
 &= \frac{7}{8\sqrt{14}} \left(\frac{16}{7}, -\frac{8}{7}, -\frac{24}{7} \right) \\
 &= \left(\frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, -\frac{3}{\sqrt{14}} \right) \Big|
 \end{aligned}$$

$$\begin{aligned}
 \vec{q}_4 &= \frac{\vec{v}_4}{\|\vec{v}_4\|} \\
 &= (0, 0, 0) \Big|
 \end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis for the subspaces of \mathbb{R}^m .

$$\vec{u}_1 = (1, 1, 0, -1), \quad \vec{u}_2 = (1, 3, 0, 1), \quad \vec{u}_3 = (4, 2, 2, 0)$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (1, 1, 0, -1) \Big|$$

$$\begin{aligned}
 \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
 &= (1, 3, 0, 1) - \frac{(1, 3, 0, 1) \cdot (1, 1, 0, -1)}{3} (1, 1, 0, -1) \\
 &= (1, 3, 0, 1) - (1, 1, 0, -1) \\
 &= (0, 2, 0, 2) \Big|
 \end{aligned}$$

$$\begin{aligned}
 \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(4, 2, 2, 0) \cdot (1, 1, 0, -1)}{3} (1, 1, 0, -1) \\
 &= (2, 2, 0, -2) \Big|
 \end{aligned}$$

$$\begin{aligned}
 \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{(4, 2, 2, 0) \cdot (0, 2, 0, 2)}{8} (0, 2, 0, 2) \\
 &= \frac{1}{2} (0, 2, 0, 2) \\
 &= (0, 1, 0, 1) \Big|
 \end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
&= (4, 2, 2, 0) - (2, 2, 0, -2) - (0, 1, 0, 1) \\
&= (2, -1, 2, 1) \quad |
\end{aligned}$$

$$\begin{aligned}
\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
&= \frac{1}{\sqrt{3}}(1, 1, 0, -1) \\
&= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}} \right) \quad |
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \frac{1}{2\sqrt{2}}(0, 2, 0, 2) \\
&= \left(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \quad |
\end{aligned}$$

$$\begin{aligned}
\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
&= \frac{1}{\sqrt{10}}(2, -1, 2, 1) \\
&= \left(\frac{2}{\sqrt{10}}, -\frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right) \quad |
\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of \mathbb{R}^m .

$$\vec{u}_1 = (1, 1, 1, 1), \quad \vec{u}_2 = (1, 1, 2, 4), \quad \vec{u}_3 = (1, 2, -4, -3)$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (1, 1, 1, 1) \quad |$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
&= (1, 1, 2, 4) - \frac{(1, 1, 2, 4) \cdot (1, 1, 1, 1)}{4} (1, 1, 1, 1)
\end{aligned}$$

$$= (1, 1, 2, 4) - 2(1, 1, 1, 1)$$

$$= \underline{(-1, -1, 0, 2)}$$

$$\begin{aligned} \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(1, 2, -4, -3) \cdot (1, 1, 1, 1)}{4} (1, 1, 1, 1) \\ &= \underline{(-1, -1, -1, -1)} \end{aligned}$$

$$\begin{aligned} \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{(1, 2, -4, -3) \cdot (-1, -1, 0, 2)}{6} (-1, -1, 0, 2) \\ &= -\frac{3}{2}(-1, -1, 0, 2) \\ &= \underline{\left(\frac{3}{2}, \frac{3}{2}, 0, -3\right)} \end{aligned}$$

$$\begin{aligned} \vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= (1, 2, -4, -3) - (-1, -1, -1, -1) - \left(\frac{3}{2}, \frac{3}{2}, 0, -3\right) \\ &= \underline{\left(\frac{1}{2}, \frac{3}{2}, -3, 1\right)} \end{aligned}$$

$$\begin{aligned} \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{1}{2}(1, 1, 1, 1) \\ &= \underline{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)} \end{aligned}$$

$$\begin{aligned} \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{1}{\sqrt{6}}(-1, -1, 0, 2) \\ &= \underline{\left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}}\right)} \end{aligned}$$

$$\begin{aligned} \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= \frac{2}{\sqrt{50}}\left(\frac{1}{2}, \frac{3}{2}, -3, 1\right) \end{aligned}$$

$$= \frac{2}{5\sqrt{2}} \left(\frac{1}{2}, \frac{3}{2}, -3, 1 \right)$$

$$= \left(\frac{1}{5\sqrt{2}}, \frac{3}{5\sqrt{2}}, -\frac{6}{5\sqrt{2}}, \frac{2}{5\sqrt{2}} \right) \Big|$$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis for the subspaces of \mathbb{R}^m .

$$\{(3, 4, 0, 0), (-1, 1, 0, 0), (2, 1, 0, -1), (0, 1, 1, 0)\}$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (3, 4, 0, 0) \Big|$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (-1, 1, 0, 0) - \frac{(-1, 1, 0, 0) \cdot (3, 4, 0, 0)}{25} (3, 4, 0, 0)$$

$$= (-1, 1, 0, 0) - \frac{1}{25} (3, 4, 0, 0)$$

$$= \left(-\frac{28}{25}, \frac{21}{25}, 0, 0 \right) \Big|$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(2, 1, 0, -1) \cdot (3, 4, 0, 0)}{25} (3, 4, 0, 0)$$

$$= \frac{10}{25} (3, 4, 0, 0)$$

$$= \left(\frac{6}{5}, \frac{8}{5}, 0, 0 \right) \Big|$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{625}{1,225} \left[(2, 1, 0, -1) \cdot \left(-\frac{28}{25}, \frac{21}{25}, 0, 0 \right) \right] \left(-\frac{28}{25}, \frac{21}{25}, 0, 0 \right)$$

$$= \frac{25}{49} \left(-\frac{35}{25} \right) \left(-\frac{28}{25}, \frac{21}{25}, 0, 0 \right)$$

$$= \left(\frac{4}{5}, -\frac{3}{5}, 0, 0 \right) \Big|$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (2, 1, 0, -1) - \left(\frac{6}{5}, \frac{8}{5}, 0, 0 \right) - \left(\frac{4}{5}, -\frac{3}{5}, 0, 0 \right)$$

$$= \underline{(0, 0, 0, -1)}$$

$$\begin{aligned} \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(0, 1, 1, 0) \cdot (3, 4, 0, 0)}{25} (3, 4, 0, 0) \\ &= \frac{4}{25} (3, 4, 0, 0) \\ &= \underline{\left(\frac{12}{25}, \frac{16}{25}, 0, 0 \right)} \end{aligned}$$

$$\begin{aligned} \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{25}{49} \left[(0, 1, 1, 0) \cdot \left(-\frac{28}{25}, \frac{21}{25}, 0, 0 \right) \right] \left(-\frac{28}{25}, \frac{21}{25}, 0, 0 \right) \\ &= \frac{25}{49} \left(\frac{21}{25} \right) \left(-\frac{28}{25}, \frac{21}{25}, 0, 0 \right) \\ &= \underline{\left(-\frac{12}{25}, \frac{9}{25}, 0, 0 \right)} \end{aligned}$$

$$\begin{aligned} \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 &= [(0, 1, 1, 0) \cdot (0, 0, 0, -1)] (0, 0, 0, -1) \\ &= \underline{(0, 0, 0, 0)} \end{aligned}$$

$$\begin{aligned} \vec{v}_4 &= \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 \\ &= (0, 1, 1, 0) - \left(\frac{12}{25}, \frac{16}{25}, 0, 0 \right) - \left(-\frac{12}{25}, \frac{9}{25}, 0, 0 \right) - (0, 0, 0, 0) \\ &= \underline{(0, 0, 1, 0)} \end{aligned}$$

$$\begin{aligned} \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{1}{5} (3, 4, 0, 0) \\ &= \underline{\left(\frac{3}{5}, \frac{4}{5}, 0, 0 \right)} \end{aligned}$$

$$\begin{aligned} \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{25}{35} \left(-\frac{28}{25}, \frac{21}{25}, 0, 0 \right) \\ &= \underline{\left(-\frac{4}{5}, \frac{3}{5}, 0, 0 \right)} \end{aligned}$$

$$\begin{aligned}\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= (0, 0, 0, -1) \end{aligned}$$

$$\begin{aligned}\vec{q}_4 &= \frac{\vec{v}_4}{\|\vec{v}_4\|} \\ &= (0, 0, 1, 0) \end{aligned}$$

Exercise

Find the QR -decomposition of

$$a) \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

$$c) \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$$

$$e) \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$b) \begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix}$$

$$d) \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$$

Solution

$$a) \text{ Since } \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5 \neq 0, \text{ The matrix is invertible}$$

$$\vec{u}_1 (1, 2), \quad \vec{u}_2 = (-1, 3)$$

$$\vec{v}_1 = \vec{u}_1 = (1, 2)$$

$$\begin{aligned}\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{(1, 2)}{\sqrt{1^2 + 2^2}} \\ &= \frac{(1, 2)}{\sqrt{5}} \\ &= \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) \end{aligned}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - (\vec{u}_2 \cdot \vec{v}_1) \vec{v}_1 \\ &= (-1, 3) - \left[(-1, 3) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) \right] \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)\end{aligned}$$

$$\begin{aligned}
&= (-1, 3) - \left(\frac{5}{\sqrt{5}}\right) \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \\
&= (-1, 3) - (1, 2) \\
&= \underline{(-2, 1)}
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \frac{(-2, 1)}{\sqrt{(-2)^2 + 1^2}} \\
&= \underline{\left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)}
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_1, \vec{q}_1 \rangle &= (1, 2) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \\
&= \frac{5}{\sqrt{5}} \\
&= \underline{\sqrt{5}}
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_2, \vec{q}_1 \rangle &= (-1, 3) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \\
&= \frac{5}{\sqrt{5}} \\
&= \underline{\sqrt{5}}
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_2, \vec{q}_2 \rangle &= (-1, 3) \cdot \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \\
&= \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{5}} \\
&= \underline{\sqrt{5}}
\end{aligned}$$

$$\begin{aligned}
R &= \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}
\end{aligned}$$

The QR -decomposition of the matrix is

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}$$

b) The column vectors of are: $\vec{u}_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$$\vec{v}_1 = \vec{u}_1 = (3, -4)$$

$$\begin{aligned} \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{(3, -4)}{\sqrt{9+16}} \\ &= \left(\frac{3}{5}, -\frac{4}{5}\right) \end{aligned}$$

$$\begin{aligned} \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (5, 0) - \frac{(5, 0) \cdot (3, -4)}{25} (3, -4) \\ &= (5, 0) - \frac{15}{25} (3, -4) \\ &= (5, 0) - \frac{3}{5} (3, -4) \\ &= (5, 0) - \left(\frac{9}{5}, -\frac{12}{5}\right) \\ &= \left(\frac{16}{5}, \frac{12}{5}\right) \end{aligned}$$

$$\begin{aligned} \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{\left(\frac{16}{5}, \frac{12}{5}\right)}{\sqrt{\frac{256}{25} + \frac{144}{25}}} \\ &= \frac{1}{\sqrt{400}} \left(\frac{16}{5}, \frac{12}{5}\right) \\ &= \frac{1}{\sqrt{16}} \left(\frac{16}{5}, \frac{12}{5}\right) \\ &= \frac{1}{4} \left(\frac{16}{5}, \frac{12}{5}\right) \\ &= \left(\frac{4}{5}, \frac{3}{5}\right) \end{aligned}$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} 3\left(\frac{3}{5}\right) - 4\left(-\frac{4}{5}\right) & 5\left(\frac{3}{5}\right) + 0\left(-\frac{4}{5}\right) \\ 0 & 5\left(\frac{4}{5}\right) - 0\left(\frac{3}{5}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 \\ 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \cdot \begin{bmatrix} 5 & 3 \\ 0 & 4 \end{bmatrix}$$

$$\mathbf{A} = \mathbf{Q} \mathbf{R}$$

c) Since the column vectors $\vec{u}_1 (1, 0, 1)$, $\vec{u}_2 = (2, 1, 4)$ are linearly independent, so has a QR -decomposition.

$$\underline{\vec{v}_1 = \vec{u}_1 = (1, 0, 1) \mid}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1, 0, 1)}{\sqrt{1^2 + 0 + 1^2}}$$

$$= \frac{(1, 0, 1)}{\sqrt{2}}$$

$$= \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \mid$$

$$\vec{v}_2 = \vec{u}_2 - \langle \vec{u}_2, \vec{v}_1 \rangle \vec{v}_1$$

$$= (2, 1, 4) - \left[(2, 1, 4) \cdot \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \right] \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)$$

$$= (2, 1, 4) - \left(\frac{6}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)$$

$$= (2, 1, 4) - (3, 0, 3)$$

$$= (-1, 1, 1) \mid$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{(-1, 1, 1)}{\sqrt{(-1)^2 + 1^2 + 1^2}}$$

$$= \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \Big|$$

$$\begin{aligned} \langle \vec{u}_1, \vec{q}_1 \rangle &= (1, 0, 1) \cdot \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \\ &= \frac{2}{\sqrt{2}} \\ &= \sqrt{2} \Big| \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{q}_1 \rangle &= (2, 1, 4) \cdot \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \\ &= \frac{2}{\sqrt{2}} + \frac{4}{\sqrt{2}} \\ &= 3\sqrt{2} \Big| \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{q}_2 \rangle &= (2, 1, 4) \cdot \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\ &= -\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{4}{\sqrt{3}} \\ &= \frac{3}{\sqrt{3}} \\ &= \sqrt{3} \Big| \end{aligned}$$

$$\begin{aligned} R &= \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix} \end{aligned}$$

The QR -decomposition of the matrix is

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$$

$\mathbf{A} = \mathbf{Q} \mathbf{R}$

d) Since $\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{vmatrix} = -4 \neq 0,$

The matrix is invertible, so it has a QR -decomposition.

$$\vec{u}_1 = (1, 1, 0), \quad \vec{u}_2 = (2, 1, 3), \quad \vec{u}_3 = (1, 1, 1)$$

$$\vec{v}_1 = \vec{u}_1 = (1, 1, 0)$$

$$\begin{aligned} \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{(1, 1, 0)}{\sqrt{1^2 + 1^2 + 0}} \\ &= \frac{(1, 1, 0)}{\sqrt{2}} \\ &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \end{aligned}$$

$$\begin{aligned} \vec{v}_2 &= \vec{u}_2 - (\vec{u}_2 \cdot \vec{v}_1) \vec{v}_1 \\ &= (2, 1, 3) - \left[(2, 1, 3) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \right] \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\ &= (2, 1, 3) - \frac{3}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\ &= (2, 1, 3) - \left(\frac{3}{2}, \frac{3}{2}, 0 \right) \\ &= \left(\frac{1}{2}, -\frac{1}{2}, 3 \right) \end{aligned}$$

$$\begin{aligned} \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{\left(\frac{1}{2}, -\frac{1}{2}, 3 \right)}{\sqrt{\left(\frac{1}{2} \right)^2 + \left(-\frac{1}{2} \right)^2 + 3^2}} \\ &= \frac{\left(\frac{1}{2}, -\frac{1}{2}, 3 \right)}{\sqrt{\frac{19}{2}}} \\ &= \frac{\sqrt{2}}{\sqrt{19}} \left(\frac{1}{2}, -\frac{1}{2}, 3 \right) \\ &= \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}} \right) \end{aligned}$$

$$\vec{v}_3 = \vec{u}_3 - (\vec{u}_3 \cdot \vec{v}_1) \vec{v}_1 - (\vec{u}_3 \cdot \vec{v}_2) \vec{v}_2$$

$$\begin{aligned}
&= (1,1,1) - \left[(1,1,1) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \right] \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\
&\quad - \left[(1,1,1) \cdot \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}} \right) \right] \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}} \right) \\
&= (1,1,1) - \frac{2}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) - \frac{6}{\sqrt{38}} \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}} \right) \\
&= (1,1,1) - (1,1,0) - \left(\frac{3}{19}, -\frac{3}{19}, \frac{18}{19} \right) \\
&= \left(-\frac{3}{19}, \frac{3}{19}, \frac{1}{19} \right) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
&= \frac{\left(-\frac{3}{19}, \frac{3}{19}, \frac{1}{19} \right)}{\sqrt{\left(-\frac{3}{19} \right)^2 + \left(\frac{3}{19} \right)^2 + \left(\frac{1}{19} \right)^2}} \\
&= \frac{19}{\sqrt{19}} \left(-\frac{3}{19}, \frac{3}{19}, \frac{1}{19} \right) \\
&= \left(-\frac{3}{\sqrt{19}}, \frac{3}{\sqrt{19}}, \frac{1}{\sqrt{19}} \right) \Big|
\end{aligned}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix}$$

$$\begin{aligned}
\langle \vec{u}_1, \vec{q}_1 \rangle &= (1, 1, 0) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\
&= \frac{2}{\sqrt{2}} \\
&= \sqrt{2} \Big|
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_2, \vec{q}_1 \rangle &= (2, 1, 3) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\
&= \frac{3}{\sqrt{2}} \Big|
\end{aligned}$$

$$\langle \vec{u}_2, \vec{q}_2 \rangle = (2, 1, 3) \cdot \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}} \right)$$

$$\begin{aligned}
&= \frac{2-1+18}{\sqrt{38}} \\
&= \frac{19}{\sqrt{38}} \\
&= \frac{19}{\sqrt{2}\sqrt{19}} \\
&= \frac{\sqrt{19}}{\sqrt{2}} \Big|
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{q}_1 \rangle &= (1, 1, 1) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\
&= \frac{2}{\sqrt{2}} \\
&= \sqrt{2} \Big|
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{q}_2 \rangle &= (1, 1, 1) \cdot \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}} \right) \\
&= \frac{1-1+6}{\sqrt{38}} \\
&= \frac{6}{\sqrt{2}\sqrt{19}} \frac{\sqrt{2}}{\sqrt{2}} \\
&= \frac{3\sqrt{2}}{\sqrt{19}} \Big|
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{q}_3 \rangle &= (1, 1, 1) \cdot \left(-\frac{3}{\sqrt{19}}, \frac{3}{\sqrt{19}}, \frac{1}{\sqrt{19}} \right) \\
&= \frac{-3+3+1}{\sqrt{19}} \\
&= \frac{1}{\sqrt{19}} \Big|
\end{aligned}$$

$$\begin{aligned}
R &= \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle & \langle \vec{u}_3, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle & \langle \vec{u}_3, \vec{q}_2 \rangle \\ 0 & 0 & \langle \vec{u}_3, \vec{q}_3 \rangle \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix}
\end{aligned}$$

The QR -decomposition of the matrix is

$$\begin{matrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix} \\ \mathbf{A} \end{matrix} = \begin{matrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix} \\ \mathbf{Q} \end{matrix} \begin{matrix} \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix} \\ \mathbf{R} \end{matrix}$$

$$e) \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{matrix} R_2 + R_1 \\ R_3 - R_1 \\ R_4 + R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{matrix} \\ R_4 - R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix is linearly dependent, so *doesn't* have a QR -decomposition.

Exercise

Verify that the Cauchy-Schwarz inequality holds for the given vectors using the Euclidean inner product

$$\vec{u} = (0, -2, 2, 1), \quad \vec{v} = (-1, -1, 1, 1)$$

Solution

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= 0 - 2(-1) + 2(1) + 1(1) \\ &= 5 \end{aligned}$$

$$\|\langle \vec{u}, \vec{v} \rangle\| = \sqrt{5}$$

$$\begin{aligned} \|\vec{u}\| \|\vec{v}\| &= \sqrt{0+4+4+1} \sqrt{1+1+1+1} \\ &= \sqrt{9} \sqrt{4} \\ &= 6 \end{aligned}$$

$$\sqrt{5} < 6 \Rightarrow \|\langle \vec{u}, \vec{v} \rangle\| \leq \|\vec{u}\| \|\vec{v}\|$$

Exercise

Apply the Gram-Schmidt *orthonormalization* process in $C^0[-1, 1]$ spanned by the functions, using the inner product $f_1(x) = x + 2$, $f_2(x) = x^2 - 3x + 4$

Solution

$$\text{Let } \vec{u}_1 = f_1 = x + 2, \quad \vec{u}_2 = f_2 = x^2 - 3x + 4$$

$$\underline{\vec{v}_1 = \vec{u}_1 = x + 2}$$

$$\begin{aligned} \langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^1 (x+2)^2 dx \\ &= \frac{1}{3}(x+2)^3 \Big|_{-1}^1 \\ &= \frac{1}{3}(27-1) \\ &= \frac{26}{3} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^1 (x^2 - 3x + 4)(x+2) dx \\ &= \int_{-1}^1 (x^3 - x^2 - 2x + 8) dx \\ &= \left(\frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 + 8x \right) \Big|_{-1}^1 \\ &= \frac{1}{4} - \frac{1}{3} - 1 + 8 - \frac{1}{4} - \frac{1}{3} + 1 + 8 \\ &= \frac{46}{3} \end{aligned}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$\vec{v}_2 = x^2 - 3x + 4 - \frac{46}{3} \left(\frac{3}{26} \right) (x+2)$$

$$= x^2 - 3x + 4 - \frac{23}{13}x - \frac{46}{13}$$

$$\underline{= x^2 - \frac{62}{13}x + \frac{6}{13}}$$

The orthogonal basis is $\left\{ x+2, \quad x^2 - \frac{62}{13}x + \frac{6}{13} \right\}$

$$\begin{aligned}
\langle \vec{v}_2, \vec{v}_2 \rangle &= \int_{-1}^1 \left(x^2 - \frac{62}{13}x + \frac{6}{13} \right)^2 dx \\
&= \frac{1}{169} \int_{-1}^1 \left(13x^2 - 62x + 6 \right)^2 dx \\
&= \frac{1}{169} \int_{-1}^1 \left(169x^4 + 3,844x^2 + 36 - 1,612x^3 + 156x^2 - 744x \right) dx \\
&= \frac{1}{169} \left(\frac{169}{5}x^5 + \frac{4,000}{3}x^3 + 36x - 403x^4 - 372x^2 \right) \Big|_{-1}^1 \\
&= \frac{1}{169} \left(\frac{169}{5} + \frac{4,000}{3} + 36 - 403 - 372 + \frac{169}{5} + \frac{4,000}{3} + 36 + 403 + 372 \right) \\
&= \frac{1}{169} \left(\frac{338}{5} + \frac{8,000}{3} + 72 \right) \\
&= \frac{3,238}{195}
\end{aligned}$$

$$\begin{aligned}
\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
&= \frac{\sqrt{3}}{\sqrt{26}}(x+2)
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \frac{\sqrt{195}}{\sqrt{3238}} \left(x^2 - \frac{62}{13}x + \frac{6}{13} \right)
\end{aligned}$$

The orthonormal basis is $\left\{ \frac{\sqrt{3}}{\sqrt{26}}(x+2), \frac{\sqrt{195}}{\sqrt{3238}} \left(x^2 - \frac{62}{13}x + \frac{6}{13} \right) \right\}$

Exercise

Apply the Gram-Schmidt *orthonormalization* process in $C^0[-1, 1]$ spanned by the functions, using the inner product $f_1(x) = x$, $f_2(x) = x^3$, $f_3(x) = x^5$

Solution

$$\text{Let } \vec{u}_1 = f_1 = x, \quad \vec{u}_2 = f_2 = x^3, \quad \vec{u}_3 = f_3 = x^5$$

$$\underline{\vec{v}_1 = \vec{u}_1 = x}$$

$$\begin{aligned} \langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^1 x^2 dx \\ &= \frac{1}{3} x^3 \Big|_{-1}^1 \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^1 x^4 dx \\ &= \frac{1}{5} x^5 \Big|_{-1}^1 \\ &= \frac{2}{5} \end{aligned}$$

$$\begin{aligned} \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= x^3 - \frac{2}{5} \left(\frac{3}{2} \right) (x) \\ &= x^3 - \frac{3}{5} x \end{aligned}$$

$$\begin{aligned} \langle \vec{v}_2, \vec{v}_2 \rangle &= \int_{-1}^1 \left(x^3 - \frac{3}{5} x \right)^2 dx \\ &= \int_{-1}^1 \left(x^6 - \frac{6}{5} x^4 + \frac{9}{25} x^2 \right) dx \\ &= \left(\frac{1}{7} x^7 - \frac{6}{25} x^5 + \frac{3}{25} x^3 \right) \Big|_{-1}^1 \\ &= 2 \left(\frac{1}{7} - \frac{6}{25} + \frac{3}{25} \right) \end{aligned}$$

$$= \frac{8}{175} \Big|$$

$$\begin{aligned} \langle \vec{u}_3, \vec{v}_1 \rangle &= \int_{-1}^1 x^6 \, dx \\ &= \frac{1}{7} x^7 \Big|_{-1}^1 \\ &= \frac{2}{7} \Big| \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_3, \vec{v}_2 \rangle &= \int_{-1}^1 x^5 \left(x^3 - \frac{3}{5}x \right) \, dx \\ &= \int_{-1}^1 \left(x^8 - \frac{3}{5}x^6 \right) \, dx \\ &= \left(\frac{1}{9}x^9 - \frac{3}{35}x^7 \right) \Big|_{-1}^1 \\ &= 2 \left(\frac{1}{9} - \frac{3}{35} \right) \\ &= \frac{16}{315} \Big| \end{aligned}$$

$$\begin{aligned} \vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= x^5 - \frac{16}{315} \left(\frac{175}{8} \right) \left(x^3 - \frac{3}{5}x \right) - \frac{2}{7} \left(\frac{3}{2} \right) x \\ &= x^5 - \frac{70}{63} \left(x^3 - \frac{3}{5}x \right) - \frac{3}{7}x \\ &= x^5 - \frac{70}{63}x^3 + \frac{14}{21}x - \frac{3}{7}x \\ &= x^5 - \frac{70}{63}x^3 + \frac{5}{21}x \Big| \end{aligned}$$

The orthogonal basis is $\left\{ x, \, x^3 - \frac{3}{5}x, \, x^5 - \frac{70}{63}x^3 + \frac{5}{21}x \right\}$

$$\begin{aligned} \langle \vec{v}_3, \vec{v}_3 \rangle &= \int_{-1}^1 \left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x \right)^2 \, dx \\ &= \int_{-1}^1 \frac{1}{3,969} \left(63x^5 - 70x^3 + 15x \right)^2 \, dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3,969} \int_{-1}^1 \left(3,969x^{10} - 8,820x^8 + 1,890x^6 - 2,100x^4 + 4,900x^6 + 225x^2 \right) dx \\
&= \frac{1}{3,969} \left(\frac{3,969}{11} x^{11} - 980x^9 + 970x^7 - 420x^5 + 75x^3 \right) \Big|_{-1}^1 \\
&= \frac{2}{3,969} \left(\frac{3,969}{11} - 980 + 970 - 420 + 75 \right) \\
&= \frac{2}{3,969} \left(\frac{3,969}{11} - 355 \right) \\
&= \frac{2}{3,969} \left(\frac{64}{11} \right) \\
&= \frac{128}{43,659}
\end{aligned}$$

$$\begin{aligned}
\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
&= \frac{x}{\sqrt{2/3}} \\
&= \frac{\sqrt{3}}{\sqrt{2}} x
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \sqrt{\frac{175}{8}} \left(x^3 - \frac{3}{5}x \right) \\
&= \frac{5\sqrt{7}}{2\sqrt{2}} \left(x^3 - \frac{3}{5}x \right)
\end{aligned}$$

$$\begin{aligned}
\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
&= \sqrt{\frac{43,659}{128}} \left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x \right) \\
&= \frac{63\sqrt{11}}{8\sqrt{2}} \left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x \right)
\end{aligned}$$

The orthonormal basis is $\left\{ \frac{\sqrt{3}}{\sqrt{2}}x, \frac{5}{2}\sqrt{\frac{7}{2}}\left(x^3 - \frac{3}{5}x\right), \frac{63}{8}\sqrt{\frac{11}{2}}\left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x\right) \right\}$

Exercise

Apply the Gram-Schmidt **orthonormalization** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $f_1(x) = 1, f_2(x) = x, f_3(x) = \frac{1}{2}(3x^2 - 1)$

Solution

$$\text{Let } \vec{u}_1 = f_1 = 1, \quad \vec{u}_2 = f_2 = x, \quad \vec{u}_3 = f_3 = \frac{3}{2}x^2 - \frac{1}{2}$$

$$\underline{\vec{v}_1 = \vec{u}_1 = 1}$$

$$\begin{aligned} \langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^1 1 \, dx \\ &= x \Big|_{-1}^1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^1 x \, dx \\ &= \frac{1}{2}x^2 \Big|_{-1}^1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= x - \frac{0}{2}(1) \\ &= x \end{aligned}$$

$$\begin{aligned} \langle \vec{v}_2, \vec{v}_2 \rangle &= \int_{-1}^1 x^2 \, dx \\ &= \frac{1}{3}x^3 \Big|_{-1}^1 \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_3, \vec{v}_1 \rangle &= \frac{1}{2} \int_{-1}^1 (3x^2 - 1) \, dx \\ &= \frac{1}{2} (x^3 - x) \Big|_{-1}^1 \end{aligned}$$

$$\underline{=0}$$

$$\begin{aligned}\langle \vec{u}_3, \vec{v}_2 \rangle &= \frac{1}{2} \int_{-1}^1 x(3x^2 - 1) \, dx \\ &= \frac{1}{2} \int_{-1}^1 (3x^3 - x) \, dx \\ &= \frac{1}{2} \left(\frac{3}{4}x^4 - \frac{1}{2}x^2 \right) \Big|_{-1}^1 \\ &\underline{=0}\end{aligned}$$

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= \frac{3}{2}x^2 - \frac{1}{2} - \frac{0}{1}(1) - \frac{0}{\frac{2}{3}}(x) \\ &\underline{= \frac{1}{2}(3x^2 - 1)}\end{aligned}$$

The orthogonal basis is $\left\{ 1, \, x, \, \frac{1}{2}(3x^2 - 1) \right\}$

$$\begin{aligned}\langle \vec{v}_3, \vec{v}_3 \rangle &= \frac{1}{4} \int_{-1}^1 (3x^2 - 1)^2 \, dx \\ &= \frac{1}{4} \int_{-1}^1 (9x^4 - 6x^2 + 1) \, dx \\ &= \frac{1}{4} \left(\frac{9}{5}x^5 - 2x^3 + x \right) \Big|_{-1}^1 \\ &= \frac{1}{2} \left(\frac{9}{5} - 2 + 1 \right) \\ &\underline{= \frac{2}{5}}\end{aligned}$$

$$\begin{aligned}\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &\underline{= \frac{1}{\sqrt{2}}}\end{aligned}$$

$$\begin{aligned}\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &\underline{= \sqrt{\frac{3}{2}} \, x}\end{aligned}$$

$$\begin{aligned}
 \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
 &= \frac{1}{\sqrt{\frac{2}{5}}} \frac{1}{2} (3x^2 - 1) \\
 &= \frac{1}{2} \sqrt{\frac{5}{2}} (3x^2 - 1)
 \end{aligned}$$

The orthonormal basis is $\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \frac{1}{2}\sqrt{\frac{5}{2}}(3x^2 - 1) \right\}$

Exercise

Apply the Gram-Schmidt **orthonormalization** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $f_1(x) = 1, f_2(x) = \sin \pi x, f_3(x) = \cos \pi x$

Solution

Let $\vec{u}_1 = f_1 = 1, \vec{u}_2 = f_2 = \sin \pi x, \vec{u}_3 = f_3 = \cos \pi x$

$$\vec{v}_1 = \vec{u}_1 = 1$$

$$\begin{aligned}
 \langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^1 1 dx \\
 &= x \Big|_{-1}^1 \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^1 \sin \pi x \, dx \\
 &= -\frac{1}{\pi} \cos \pi x \Big|_{-1}^1 \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
 &= \sin \pi x
 \end{aligned}$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^1 \sin^2 \pi x \, dx$$

$$\begin{aligned}
&= \frac{1}{2} \int_{-1}^1 (1 - \cos 2\pi x) \, dx \\
&= \frac{1}{2} \left(x - \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^1 \\
&= \underline{1}
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{v}_1 \rangle &= \int_{-1}^1 \cos \pi x \, dx \\
&= \frac{1}{\pi} \sin \pi x \Big|_{-1}^1 \\
&= \underline{0}
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{v}_2 \rangle &= \int_{-1}^1 \cos \pi x \sin \pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 \sin 2\pi x \, dx \\
&= -\frac{1}{4\pi} \cos 2\pi x \Big|_{-1}^1 \\
&= \underline{0}
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
&= \cos \pi x - \underline{0} - \underline{0} \\
&= \underline{\cos \pi x}
\end{aligned}$$

The orthogonal basis is $\left\{ \underline{1}, \underline{\sin \pi x - \frac{1}{\pi}}, \underline{\cos \pi x} \right\}$

$$\begin{aligned}
\langle \vec{v}_3, \vec{v}_3 \rangle &= \int_{-1}^1 \cos^2 \pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 (1 + \cos 2\pi x) \, dx \\
&= \frac{1}{2} \left(x + \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^1 \\
&= \underline{1}
\end{aligned}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{2}}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \sin \pi x$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \cos \pi x$$

The orthonormal basis is $\left\{ \frac{1}{\sqrt{2}}, \sin \pi x, \cos \pi x \right\}$

Exercise

Apply the Gram-Schmidt **orthonormalization** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $f_1(x) = \sin \pi x$, $f_2(x) = \sin 2\pi x$, $f_3(x) = \sin 3\pi x$

Solution

Let $\vec{u}_1 = f_1 = \sin \pi x$, $\vec{u}_2 = f_2 = \sin 2\pi x$, $\vec{u}_3 = f_3 = \sin 3\pi x$

$$\vec{v}_1 = \vec{u}_1 = \sin \pi x$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^1 \sin^2 \pi x \, dx$$

$$= \frac{1}{2} \int_{-1}^1 (1 - \cos 2\pi x) \, dx$$

$$= \frac{1}{2} \left(x - \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^1$$

$$= 1$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_{-1}^1 \sin \pi x \sin 2\pi x \, dx$$

$$= \frac{1}{2} \int_{-1}^1 (\cos 3\pi x - \cos(-\pi x)) \, dx$$

$$\sin a \sin b = \frac{1}{2} [\cos(a+b) - \cos(a-b)]$$

$$\begin{aligned}
&= \frac{1}{2} \int_{-1}^1 (\cos 3\pi x - \cos \pi x) \, dx \\
&= \frac{1}{2} \left(\frac{1}{3\pi} \sin 3\pi x - \frac{1}{\pi} \sin \pi x \right) \Big|_{-1}^1 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
&= \sin 2\pi x
\end{aligned}$$

$$\begin{aligned}
\langle \vec{v}_2, \vec{v}_2 \rangle &= \int_{-1}^1 \sin^2 2\pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 (1 - \cos 4\pi x) \, dx \\
&= \frac{1}{2} \left(x - \frac{1}{4\pi} \sin 4\pi x \right) \Big|_{-1}^1 \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{v}_1 \rangle &= \int_{-1}^1 \sin \pi x \sin 3\pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 (\cos 4\pi x - \cos(-2\pi x)) \, dx \\
&= \frac{1}{2} \int_{-1}^1 (\cos 4\pi x - \cos 2\pi x) \, dx \\
&= \frac{1}{2} \left(\frac{1}{4\pi} \sin 4\pi x - \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^1 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{v}_2 \rangle &= \int_{-1}^1 \sin 3\pi x \sin 2\pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 (\cos 5\pi x - \cos \pi x) \, dx \\
&= \frac{1}{2} \left(\frac{1}{5\pi} \sin 5\pi x - \frac{1}{\pi} \sin \pi x \right) \Big|_{-1}^1 \\
&= 0
\end{aligned}$$

$$\sin a \sin b = \frac{1}{2} [\cos(a+b) - \cos(a-b)]$$

$$\sin a \sin b = \frac{1}{2} [\cos(a+b) - \cos(a-b)]$$

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= \sin 3\pi x \quad | \end{aligned}$$

The orthogonal basis is $\{\sin \pi x, \sin 2\pi x, \sin 3\pi x\}$

$$\begin{aligned}\langle \vec{v}_3, \vec{v}_3 \rangle &= \int_{-1}^1 \sin^2 3\pi x \, dx \\ &= \frac{1}{2} \int_{-1}^1 (1 - \cos 6\pi x) \, dx \\ &= \frac{1}{2} \left(x - \frac{1}{6\pi} \sin 6\pi x \right) \Big|_{-1}^1 \\ &= 1 \quad | \end{aligned}$$

$$\begin{aligned}\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \sin \pi x \quad | \end{aligned}$$

$$\begin{aligned}\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \sin 2\pi x \quad | \end{aligned}$$

$$\begin{aligned}\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= \sin 3\pi x \quad | \end{aligned}$$

The orthonormal basis is $\{\sin \pi x, \sin 2\pi x, \sin 3\pi x\}$

Exercise

Apply the Gram-Schmidt **orthonormalization** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $f_1(x) = \cos \pi x$, $f_2(x) = \cos 2\pi x$, $f_3(x) = \cos 3\pi x$

Solution

Let $\vec{u}_1 = f_1 = \cos \pi x$, $\vec{u}_2 = f_2 = \cos 2\pi x$, $\vec{u}_3 = f_3 = \cos 3\pi x$

$$\vec{v}_1 = \vec{u}_1 = \cos \pi x \quad |$$

$$\begin{aligned}
\langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^1 \cos^2 \pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 (1 + \cos 2\pi x) \, dx \\
&= \frac{1}{2} \left(x + \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^1 \\
&= \underline{1}
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^1 \cos 2\pi x \cos \pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 (\cos 3\pi x + \cos \pi x) \, dx \\
&= \frac{1}{2} \left(\frac{1}{3\pi} \sin 3\pi x + \frac{1}{\pi} \sin \pi x \right) \Big|_{-1}^1 \\
&= \underline{0}
\end{aligned}$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
&= \underline{\cos 2\pi x}
\end{aligned}$$

$$\begin{aligned}
\langle \vec{v}_2, \vec{v}_2 \rangle &= \int_{-1}^1 \cos^2 2\pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 (1 + \cos 4\pi x) \, dx \\
&= \frac{1}{2} \left(x + \frac{1}{4\pi} \sin 4\pi x \right) \Big|_{-1}^1 \\
&= \underline{1}
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{v}_1 \rangle &= \int_{-1}^1 \cos 3\pi x \cos \pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 (\cos 4\pi x + \cos 2\pi x) \, dx \\
&= \frac{1}{2} \left(\frac{1}{4\pi} \sin 4\pi x + \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^1
\end{aligned}$$

$$\cos a \cos b = \frac{1}{2} [\cos(a+b) + \cos(a-b)]$$

$$\cos a \cos b = \frac{1}{2} [\cos(a+b) + \cos(a-b)]$$

$$\underline{=0}$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^1 \cos 3\pi x \cos 2\pi x \, dx$$

$$\cos a \cos b = \frac{1}{2} [\cos(a+b) + \cos(a-b)]$$

$$= \frac{1}{2} \int_{-1}^1 (\cos 5\pi x + \cos \pi x) \, dx$$

$$= \frac{1}{2} \left(\frac{1}{5\pi} \sin 5\pi x + \frac{1}{\pi} \sin \pi x \right) \Big|_{-1}^1$$

$$\underline{=0}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$\underline{= \cos 3\pi x}$$

The orthogonal basis is $\{\cos \pi x, \cos 2\pi x, \cos 3\pi x\}$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_{-1}^1 \cos^2 3\pi x \, dx$$

$$= \frac{1}{2} \int_{-1}^1 (1 + \cos 6\pi x) \, dx$$

$$= \frac{1}{2} \left(x + \frac{1}{6\pi} \sin 6\pi x \right) \Big|_{-1}^1$$

$$\underline{=1}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\underline{= \cos \pi x}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$\underline{= \cos 2\pi x}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$\underline{= \cos 3\pi x}$$

The orthonormal basis is $\{\cos \pi x, \cos 2\pi x, \cos 3\pi x\}$

Exercise

For $\mathbb{P}_3[x]$, define the inner product over \mathbb{R} as

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x) dx$$

a) If $f(x) = 1$ is a unit vector in $\mathbb{P}_3[x]$?

b) Find an orthonormal basis for the subspace spanned by x and x^2 .

c) Complete the basis in part (b) to an orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product.

d) Is

$$[f, g] = \int_0^1 f(x)g(x) dx$$

Also, an inner product for $\mathbb{P}_3[x]$

e) Find a pair of vectors \vec{v} and \vec{w} such that

$$\langle \vec{v}, \vec{w} \rangle = 0 \quad \text{but} \quad [\vec{v}, \vec{w}] \neq 0$$

f) Is the basis found in part (c) an orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product in part (d)?

Solution

a) $f(x) = 1$

$$\begin{aligned} \langle f, f \rangle &= \int_{-1}^1 f(x)f(x) dx \\ &= \int_{-1}^1 dx \\ &= x \Big|_{-1}^1 \\ &= 1 + 1 \\ &= 2 \neq 1 \end{aligned}$$

Therefore, when $f(x) = 1$ is **not** a unit vector in $\mathbb{P}_3[x]$

b) Let $\vec{u}_1 = f = x$, $\vec{u}_2 = g = x^2$

$$\underline{\vec{v}_1 = \vec{u}_1 = x}$$

$$\begin{aligned}
 \left\langle \vec{v}_1, \vec{v}_1 \right\rangle &= \int_{-1}^1 x^2 \, dx \\
 &= \frac{1}{3} x^3 \Big|_{-1}^1 \\
 &= \frac{1}{3} (1+1) \\
 &= \frac{2}{3} \Big|
 \end{aligned}$$

$$\begin{aligned}
 \left\langle \vec{u}_2, \vec{v}_1 \right\rangle &= \int_{-1}^1 x^2 (x) \, dx \\
 &= \int_{-1}^1 x^3 \, dx \\
 &= \frac{1}{4} x^4 \Big|_{-1}^1 \\
 &= \frac{1}{4} (1-1) \\
 &= 0 \Big|
 \end{aligned}$$

$$\begin{aligned}
 \vec{v}_2 &= \vec{u}_2 - \frac{\left\langle \vec{u}_2, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1 \\
 &= x^2 - \frac{0}{*} (\,) \\
 &= x^2 \Big|
 \end{aligned}$$

$$\begin{aligned}
 \left\langle \vec{v}_2, \vec{v}_2 \right\rangle &= \int_{-1}^1 x^4 \, dx \\
 &= \frac{1}{5} x^5 \Big|_{-1}^1 \\
 &= \frac{1}{5} (1+1) \\
 &= \frac{2}{5} \Big|
 \end{aligned}$$

$$\begin{aligned}
 \vec{q}_1 &= \frac{\vec{v}_1}{\left\| \vec{v}_1 \right\|} \\
 &= \frac{x}{\sqrt{\frac{2}{3}}}
 \end{aligned}$$

$$= \sqrt{\frac{3}{2}} x \Big|$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{x^2}{\sqrt{\frac{2}{5}}}$$

$$= \sqrt{\frac{5}{2}} x^2 \Big|$$

The orthonormal basis is $\left\{ \sqrt{\frac{3}{2}} x, \sqrt{\frac{5}{2}} x^2 \right\}$

c) Since $\vec{u}_1 = x$, $\vec{u}_2 = x^2$ in $\mathcal{P}_3[x]$

Then, let $\vec{u}_3 = 1$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_{-1}^1 (1)(x) dx$$

$$= \int_{-1}^1 x dx$$

$$= \frac{1}{2} x^2 \Big|_{-1}^1$$

$$= \frac{1}{2} (1-1)$$

$$= 0 \Big|$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^1 (1)(x^2) dx$$

$$= \int_{-1}^1 x^2 dx$$

$$= \frac{1}{3} x^3 \Big|_{-1}^1$$

$$= \frac{1}{3} (1+1)$$

$$= \frac{2}{3} \Big|$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= 1 - \frac{0}{\frac{2}{3}}(x) - \frac{\frac{2}{3}}{\frac{2}{5}}(x^2)$$

$$\underline{= 1 - \frac{5}{3}x^2}$$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_{-1}^1 \left(1 - \frac{5}{3}x^2\right)^2 dx$$

$$= \int_{-1}^1 \left(1 - \frac{10}{3}x^2 + \frac{25}{9}x^4\right) dx$$

$$= \left(x - \frac{10}{9}x^3 + \frac{5}{9}x^5\right) \Big|_{-1}^1$$

$$= 2\left(1 - \frac{10}{9} + \frac{5}{9}\right)$$

$$= 2\left(\frac{9-5}{9}\right)$$

$$\underline{= \frac{8}{9}}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \left(\sqrt{\frac{9}{8}}\right)\left(1 - \frac{5}{3}x^2\right)$$

$$= \frac{3}{2\sqrt{2}}\left(1 - \frac{5}{3}x^2\right)$$

$$\underline{= \frac{3}{2\sqrt{2}} - \frac{5}{2\sqrt{2}}x^2}$$

The orthonormal basis is $\left\{ \sqrt{\frac{3}{2}}x, \sqrt{\frac{5}{2}}x^2, \frac{3}{2\sqrt{2}} - \frac{5}{2\sqrt{2}}x^2 \right\}$

d) $[f, g] = \int_0^1 f(x)g(x) dx$

Let $\vec{u}_1 = 1, \vec{u}_2 = x, \vec{u}_3 = x^2$

$$\underline{\vec{v}_1 = \vec{u}_1 = 1}$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_0^1 1 dx$$

$$=x \bigg|_0^1$$

$$\underline{=1}$$

$$\left\langle \vec{u}_2,\vec{v}_1\right\rangle =\int_0^1x(1)\;dx$$

$$=\int_0^1x\;dx$$

$$=\frac{1}{2}x^2\bigg|_0^1$$

$$\underline{=\frac{1}{2}}$$

$$\vec{v}_2=\vec{u}_2-\frac{\left\langle \vec{u}_2,\vec{v}_1\right\rangle }{\left\| \vec{v}_1\right\| ^2}\vec{v}_1$$

$$=x-\frac{1}{2}(1)$$

$$\underline{=x-\frac{1}{2}}$$

$$\left\langle \vec{v}_2,\vec{v}_2\right\rangle =\int_0^1\left(x-\frac{1}{2}\right)^2\;dx$$

$$=\int_0^1\left(x-\frac{1}{2}\right)^2\;d\left(x-\frac{1}{2}\right)$$

$$=\frac{1}{3}\left(x-\frac{1}{2}\right)^3\bigg|_0^1$$

$$=\frac{1}{3}\left(\left(\frac{1}{2}\right)^3-\left(-\frac{1}{2}\right)^3\right)$$

$$=\frac{1}{3}\left(\frac{1}{8}+\frac{1}{8}\right)$$

$$\underline{=\frac{1}{12}}$$

$$\left\langle \vec{u}_3,\vec{v}_1\right\rangle =\int_0^1\left(x^2\right)(1)\;dx$$

$$=\int_0^1x^2\;dx$$

$$= \frac{1}{3} x^3 \Big|_0^1$$

$$= \frac{1}{3} \Big|$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_0^1 \left(x^2 \right) \left(x - \frac{1}{2} \right) dx$$

$$= \int_0^1 \left(x^3 - \frac{1}{2} x^2 \right) dx$$

$$= \left(\frac{1}{4} x^4 - \frac{1}{6} x^3 \right) \Big|_0^1$$

$$= \frac{1}{4} - \frac{1}{6}$$

$$= \frac{1}{12} \Big|$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= x^2 - \frac{\frac{1}{3}}{1} (1) - \frac{\frac{1}{12}}{\frac{1}{12}} \left(x - \frac{1}{2} \right)$$

$$= x^2 - \frac{1}{3} - x + \frac{1}{2}$$

$$= x^2 - x + \frac{1}{6} \Big|$$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_0^1 \left(x^2 - x + \frac{1}{6} \right)^2 dx$$

$$= \int_0^1 \left(x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36} \right) dx$$

$$= \left(\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{4}{9}x^3 - \frac{1}{6}x^2 + \frac{1}{36}x \right) \Big|_0^1$$

$$= \frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36}$$

$$= \frac{2-5}{10} + \frac{16-6+1}{36}$$

$$= -\frac{3}{10} + \frac{11}{36}$$

$$= \frac{-108+110}{360}$$

$$= \frac{1}{180} \Big|$$

$$\begin{aligned} \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= 1 \Big| \end{aligned}$$

$$\begin{aligned} \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{x - \frac{1}{2}}{\sqrt{\frac{1}{12}}} \\ &= 2\sqrt{3} \left(x - \frac{1}{2} \right) \Big| \end{aligned}$$

$$\begin{aligned} \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= (\sqrt{180}) \left(x^2 - x + \frac{1}{6} \right) \\ &= (6\sqrt{5}) \left(x^2 - x + \frac{1}{6} \right) \\ &= 6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5} \Big| \end{aligned}$$

The orthonormal basis is $\left\{ 1, 2\sqrt{3} \left(x - \frac{1}{2} \right), \sqrt{5} \left(6x^2 - 6x + 1 \right) \right\}$

Therefore, $[f, g] = \int_0^1 f(x)g(x) dx$ is an inner product for $\mathbb{P}_3[x]$

e) Let assume: $\vec{v} = 1$ and $\vec{w} = x$

$$\begin{aligned} \langle \vec{v}, \vec{w} \rangle &= \int_{-1}^1 1(x) dx \\ &= \int_{-1}^1 x dx \\ &= \frac{1}{2}x^2 \Big|_{-1}^1 \\ &= \frac{1}{2}(1-1) \\ &= 0 \Big| \quad \checkmark \end{aligned}$$

$$\begin{aligned}
 [\vec{v}, \vec{w}] &= \int_0^1 1(x) \, dx \\
 &= \frac{1}{2} x^2 \Big|_0^1 \\
 &= \frac{1}{2} \neq 0 \quad \checkmark
 \end{aligned}$$

f) The orthonormal basis in part (c) $\left\{ \sqrt{\frac{3}{2}} x, \sqrt{\frac{5}{2}} x^2, \frac{3}{2\sqrt{2}} - \frac{5}{2\sqrt{2}} x^2 \right\}$ are ***not*** the same as
 the orthonormal basis in part (d) $\left\{ 1, 2\sqrt{3} \left(x - \frac{1}{2} \right), \sqrt{5} (6x^2 - 6x + 1) \right\}$