3.2 Power Series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

about $x = 0$

about $x = a$

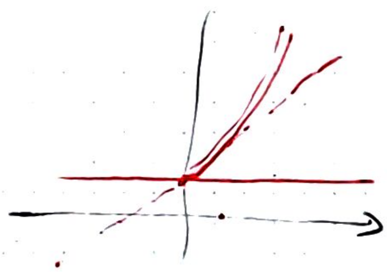
$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots$$

c_i's coeff. constants

center a

Ex $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$

$= \frac{1}{1-x}$ $\quad |x| < 1 \quad -1 < x < 1$



Ex $\quad 1-\frac{1}{2}(x-2)+\frac{1}{4}(x-2)^2-\dots+\left(\frac{-1}{2}\right)^n(x-2)^n+\dots$

$$\left(-\frac{1}{2}\right)^n(x-2)^n=\left(-\frac{x-2}{2}\right)^n$$

$$\lambda=-\frac{x-2}{2}$$

$$|\lambda|=\frac{x-2}{2}<1$$

$$x-2<2$$

$$0<x<4 \quad \text{convergence interval}$$

$$P_0(x)=1$$

$$P_1(x)=1-\frac{1}{2}(x-2)=-\frac{1}{2}x+2$$

$$P_2(x)=1-\frac{1}{2}(x-2)+\frac{1}{4}(x-2)^2$$

$$=3-\frac{3}{2}x+\frac{x^4}{4}$$

Ex $\quad x$? converges $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$

soln $\quad \left|\frac{u_{n+1}}{u_n}\right|=\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right|$

$$=\frac{n}{n+1}|x| \longrightarrow |x|$$

$$\lim_{n\to\infty}\frac{n}{n+1}|x|=|x|$$

If $|x|<1$, the series converges absolutely

$|x|>1$ " diverges

At $x=1 \Rightarrow \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ $\Big\}$ $u_n>u_{n+1}$

$n<n+1$

$\frac{1}{n}>\frac{1}{n+1}$ ✓

$\frac{1}{n}\to 0$ ✓

converges by Alternating series

At $x=-1 \Rightarrow \displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-1)^n}{n}=\sum_{n=1}^{\infty}\frac{(-1)^{2n-1}}{n}$

$$=-\sum_{n=1}^{\infty}\frac{1}{n} \quad p=1 \Rightarrow \text{diverges}$$

∴ series converges: $-1<x\leq 1$

Ex. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1}$

$\left|\frac{u_{n+1}}{u_n}\right| = \frac{2n-1}{2n+1}\left|\frac{x^{2n+1}}{x^{2n-1}}\right|$ $\qquad \frac{x^{2n+1}}{2n+1}$

$= \frac{2n-1}{2n+1} \cdot x^2 \longrightarrow x^2$

$x^2 < 1$ converge, $\quad x^2 > 1$ diverges

at: $x = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ $\qquad \cdot\, 2n+1$

$$\left\{\begin{array}{l} n < n+1 \\ 2n < 2n+1 \\ 2n-1 < 2n+1 \\ \frac{1}{2n-1} > \frac{1}{2n+1} \\ u_n > u_{n+1} \checkmark \\ \frac{1}{2n-1} \to 0 \end{array}\right.$$

converges by Alternating series

at $x = -1 \Rightarrow \sum (-1)^{n-1} \frac{(-1)^{2n-1}}{2n-1} = \sum \frac{(-1)^{3n-2}}{2n-1}$

converges by Alternating, series

series converges $-1 \leq x \leq 1$, and diverges elsewhere

Ex $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{n!}{(n+1)!}\left|\frac{x^{n+1}}{x^n}\right|$$

$$= \frac{1}{n+1}|x| \longrightarrow 0 \quad (\forall x)$$

The series converges absolutely for all x

Ex $\sum_{n=0}^{\infty} n!\, x^n$

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{(n+1)!}{n!}\left|\frac{x^{n+1}}{x^n}\right|$$

$$= (n+1)|x| \longrightarrow \infty$$

The series diverges for all x except $x=0$

$$\sum c_n (x-a)^n$$

1- R (>0) (Radius) diverges $|x-a| > R$

" $|x-a| < R$

$\begin{cases} x-a = -R \\ x-a = +R \end{cases}$ to find

2- $R = \infty \Rightarrow$ series converges

3- $R = 0$ " diverges

$R = \frac{1}{L}$ $\quad L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$

$R = \lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| \Rightarrow \begin{cases} \infty & \text{converges} \\ 0 & \text{diverges.} \end{cases}$

$-R < x-a < R$

$a-R < x < a+R$ (interval convergence)

Ex Center, radius, and interval of convergence

$$\sum_{n=0}^{\infty} \frac{(2x+5)^n}{(n^2+1)\,3^n} \qquad 2^n\left(x+\frac{5}{2}\right)^n = \left[2\left(x+\frac{5}{2}\right)\right]^n$$

Soln

$2x+5=0$

Centre of convergence : $x=-\frac{5}{2}$

$$R = \lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right|$$

$$= \lim_{n\to\infty} \frac{((n+1)^2+1)\,3^{n+1}}{(n^2+1)\cdot 3^n}\cdot\frac{2^n}{2^{n+1}}$$

$$= \frac{3}{2}$$

$$-\frac{3}{2} < x+\frac{5}{2} < \frac{3}{2}$$

$$-\frac{5}{2}-\frac{3}{2} < x < \frac{3}{2}-\frac{5}{2}$$

$$-4 < x < -1$$

at $x=-4 \Rightarrow \sum \frac{(-3)^n}{(n^2+1)\,3^n} = \sum \frac{(-1)^n}{n^2+1}$

$$n^2+1 < (n+1)^2+1$$

$$\frac{1}{n^2+1} > \frac{1}{(n+1)^2+1}$$

$$u_n > u_{n+1}$$

$$\frac{1}{n^2+1} \longrightarrow 0$$

converges by Alternating series

at $x=-1 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{n^2+1}$ $\qquad \int_0^{\infty} \frac{dx}{x^2+1} = \tan^{-1}x\Big|_0^{\infty} = \frac{\pi}{2}$

converges by Integral Test.

interval of convergence $-4 \le x \le -1$

11.4.3. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{\sqrt{n+1}}$

centre of convergence: $\underline{x=0}$

$$R = \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n+2}} \qquad \frac{\sqrt{n}}{\sqrt{n}}$$

$= 1$

Radius of convergence is 1. $\quad -1 < x < 1$

at $x = -1 \Rightarrow \displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$

$$\int_0^{\infty} \frac{dx}{(x+1)^{1/2}} = \int_0^{\infty} (x+1)^{-1/2} d(x+1)$$

$$= 2(x+1)^{1/2}\Big|_0^{\infty}$$

$$= \infty$$

diverges by integral test

at $x = 1 \Rightarrow \sum \frac{1}{\sqrt{n+1}}$ diverges by Integ (similar)

Interval of convergence: $-1 < x < 1$

#44 $\sum_{n=0}^{\infty} 3n\,(x+1)^n$

Centre of convergence : $x = -1$

$R = \lim_{n\to\infty} \frac{3n}{3(n+1)}$

$= 1$

Radius of convergence is 1

$-1 < x+1 < 1$

$-2 < x < 0$

$\left.\begin{matrix} u_n > u_{n+1} \\ u_n \to 0 \end{matrix}\right\}$

at $x = -2 \Rightarrow \sum 3n(-1)^n$ $\quad 3n \to \infty$

diverges by Alternanting series

at $x = 0 \Rightarrow \sum_{n=0}^{\infty} 3n \to \infty$ diverges

Interval of convergence $-2 < x < 0$

3.8 Taylor & Maclaurin Series

$$f^{(n)}(x) = n!\, a_n \Rightarrow a_n = \frac{f^{(n)}(x)}{n!}$$

Taylor Series:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{n}(a)}{n!}(x-a)^n + \cdots$$

Maclaurin $a=0$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

Ex. $f(x) = \frac{1}{x}$ at $a = 2$

$f(x) = x^{-1}$ $\qquad f(2) = \frac{1}{2}$

$f'(x) = -x^{-2}$ $\qquad f'(2) = -\frac{1}{2^2}$

$f''(x) = 2x^{-3}$ $\qquad f''(2) = \frac{2}{2^3}$

$f'''(x) = -3!\,x^{-4}$

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots$$

$$\frac{1}{2} - \frac{1}{2^2}(x-2) + \frac{1}{2!}\frac{2}{2^3}(x-2)^2 + \cdots + \frac{n!\,(-1)^n}{2^{n+1}\,n!}(x-2)^n + \cdots$$

$$\frac{1}{2} - \frac{x-2}{4} + \frac{1}{2^3}(x-2)^2 - \cdots + \frac{(-1)^n}{2^{n+1}}(x-2)^n + \cdots$$

Ex - $f(x) = e^x$ @ $x=0$

$f(0)=1$

$f'(x) = e^x$ $\quad f'(0)=1$

$\vdots$

$f^{(n)}(x) = e^x$ $\quad f^{(n)}(0)=1$

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

$$1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{5!}+\frac{1}{6!}$$

$f(x) = \cos x$ @ $x=0$

$f(x) = \cos x \to f(0)=1$ $\qquad f'(x) = -\sin x \to f'(0)=0$

$f''(x) = -\cos x \to f''(0)=-1$ $\qquad f^{(3)}(x) = \sin x$

$f^{(n+1)}(0) = 0.$

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}}{3!}x^3 + \dots + \frac{f^{(n)}}{n!}x^n + \dots$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots + \frac{(-1)^n}{(2n)!}x^{2n} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$$

$2(n)!$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots$$

11) $f(x) = e^{2x} \quad a = 0 \quad 0, 1, 2, 3$

$$f(x) = e^{2x} \longrightarrow f(0) = 1$$
$$f'(x) = 2e^{2x} \longrightarrow f'(0) = 2$$
$$f''(x) = 4e^{2x} \longrightarrow f''(0) = 4$$
$$f'''(x) = 8e^{2x} \longrightarrow f'''(0) = 8$$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$
$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

28) $f(x) = \sin x \quad n = 5$

$$f(x) = \sin x \qquad f(0) = 0$$
$$f'(x) = \cos x \Big|_{x=0} = 1$$
$$f''(x) = -\sin x \Big|_{x=0} = 0$$
$$f'''(x) = -\cos x \Big|_{x=0} = -1$$
$$f^{(4)}(x) = \sin x \Big|_{x=0} = 0$$
$$f^{(5)}(x) = \cos x \Big|_{x=0} = 1$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \qquad 2 \cdot 3 \cdot 4 \cdot 5$$
$$= x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$