Solution ***Section* 3.1 – Inverse Functions**

Exercise

Determine whether the function is one-to-one: $f(x) = 3x - 7$

Solution

$$f(a) = f(b)$$
$$3a - 7 = 3b - 7$$
$$3a = 3b - 7 + 7$$
$$3a = 3b \qquad \textit{Divide both sides by 3}$$
$$a = b$$

$\therefore$ The function is one-to-one

Exercise

Determine whether the function is one-to-one: $f(x) = x^2 - 9$

Solution

$1 \neq -1$ $1^2 - 9 \neq (-1)^2 - 9$ $-8 = -8 \rightarrow$ Contradict the definition	$f(a) = f(b)$ $a^2 - 9 = b^2 - 9$ $a^2 = b^2$ $a = \pm b$

The function is not one-to-one

Exercise

Determine whether the function is one-to-one: $f(x) = \sqrt{x}$

Solution

$$f(a) = f(b)$$
$$\sqrt{a} = \sqrt{b}$$
$$\left(\sqrt{a}\right)^2 = \left(\sqrt{b}\right)^2 \qquad \textit{Square both sides}$$
$$a = b$$

$\therefore$ The function is one-to-one

Exercise

Determine whether the function is one-to-one: $f(x)=\sqrt[3]{x}$

Solution

$$f(a)=f(b)$$

$$\sqrt[3]{a}=\sqrt[3]{b}$$

$$\left(\sqrt[3]{a}\right)^3=\left(\sqrt[3]{b}\right)^3 \qquad \textit{cube both sides}$$

$$a=b$$

$\therefore$ The function is one-to-one

Exercise

Determine whether the function is one-to-one: $f(x)=|x|$

Solution

$$1\neq -1$$

$$|1|\neq |-1|$$

$$1\neq 1 \ (\textit{false})$$

$\therefore$ The function is not one-to-one

Exercise

Given the function f described by $f(x)=\dfrac{2}{x+3}$, prove that f is one-to-one.

Solution

$$f(a)=f(b)$$

$$\frac{2}{a+3}=\frac{2}{b+3}$$

$$(a+3)(b+3)\frac{2}{a+3}=\frac{2}{b+3}(a+3)(b+3)$$

$$2(b+3)=2(a+3)$$

$$b+3=a+3$$

$a=b$ $\qquad$ f is one-to-one

Exercise

Given the function f described by $f(x)=(x-2)^3$, prove that f is one-to-one.

Solution

$$f(a)=f(b)$$

$$(a-2)^3=(b-2)^3$$

$$\left[(a-2)^3\right]^{1/3}=\left[(b-2)^3\right]^{1/3}$$

$a-2=b-2$ *Add 2 on both sides*

$$a=b$$

Exercise

Given the function f described by $y=x^2+2$, prove that f is one-to-one.

Solution

$$f(a)=f(b)$$

$a^2+2=b^2+2$ *Subtract 2*

$$a^2=b^2$$

$a=\pm\sqrt{b^2}$ Function is not a one-to-one

The inverse function doesn't exist.

Exercise

Given the function f described by $f(x)=\frac{x+1}{x-3}$, prove that f is one-to-one.

Solution

$$f(a)=f(b)$$

$\frac{a+1}{a-3}=\frac{b+1}{b-3}$ ***Cross multiplication***

$$(a+1)(b-3)=(b+1)(a-3)$$

$$ab-3a+b-3=ab-3b+a-3$$

$$-3a-a=ab-3b-3-b+3-ab$$

$-4a=-4b$ ***Divide by -4***

$a=b$ Function is one-to-one

Exercise

Find the inverse of $f(x)=(x-2)^3$

Solution

$$y=(x-2)^3$$

$$x=(y-2)^3$$

$$x^{1/3}=\left[(y-2)^3\right]^{1/3}$$

$$x^{1/3}=y-2$$

$$\sqrt[3]{x}+2=y$$

$$\Rightarrow \boxed{f^{-1}(x)=\sqrt[3]{x}+2}$$

Exercise

Find the inverse of $f(x)=\frac{x+1}{x-3}$

Solution

$$y=\frac{x+1}{x-3}$$

$$x=\frac{y+1}{y-3}$$

$$x(y-3)=y+1$$

$$xy-3x=y+1$$

$$xy-y=3x+1$$

$$y(x-1)=3x+1$$

$$y=\boxed{\frac{3x+1}{x-1}=f^{-1}(x)}$$

Exercise

Let $f(x)=x^3-1$ and $g(x)=\sqrt[3]{x+1}$, is g the inverse function of f?

Solution

$$\begin{aligned}(f\circ g)(x)&=f(g(x))\\&=f\left(\sqrt[3]{x+1}\right)\\&=\left(\sqrt[3]{x+1}\right)^3-1\\&=x+1-1\\&=x\end{aligned}$$

$$\begin{aligned}(g\circ f)(x)&=g\left(f(x)\right)\\&=g\left(x^3-1\right)\\&=\sqrt[3]{x^3-1+1}\\&=\sqrt[3]{x^3}\\&=x\end{aligned}$$

g is the inverse function of f

Exercise

Given that $f(x)=5x+8$, use composition of functions to show that $f^{-1}(x)=\frac{x-8}{5}$

Solution

$$\begin{aligned}\left(f^{-1}\circ f\right)(x)&=f^{-1}(f(x))\\&=f^{-1}(5x+8)\\&=\frac{(5x+8)-8}{5}\\&=\frac{5x+8-8}{5}\\&=\frac{5x}{5}=x\end{aligned}$$

$$\begin{aligned}\left(f\circ f^{-1}\right)(x)&=f(f^{-1}(x))=f^{-1}(\tfrac{x-8}{5})\\&=5(\tfrac{x-8}{5})+8=x-8+8=x\end{aligned}$$

Exercise

Given the function $f(x)=(x+8)^3$

a) Find $f^{-1}(x)$

b) Graph f and f^{-1} in the same rectangular coordinate system

c) Find the domain and the range of f and f^{-1}

Solution

a) $y=(x+8)^3$ **Replace f(*x*) with *y***

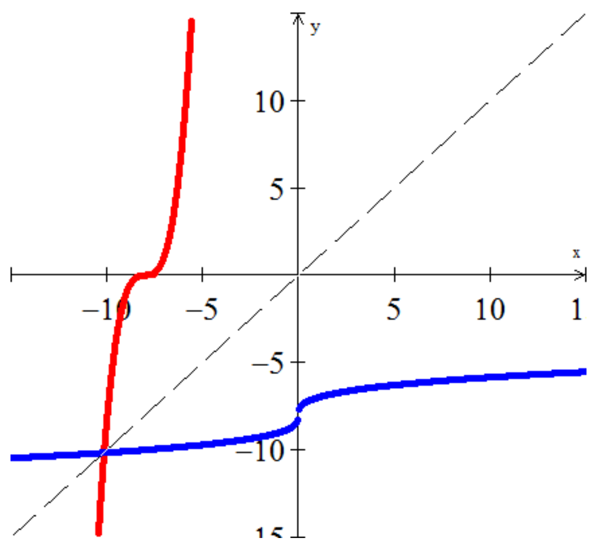
$x=(y+8)^3$ **Interchange *x* and *y***

$(x)^{1/3}=\left((y+8)^3\right)^{1/3}$

$x^{1/3}=y+8$ ***Subtract 8 from both sides.***

$\boxed{x^{1/3}-8}=y=f^{-1}(x)$

b)



c) Domain of f = Range of f^{-1}: $(-\infty,\infty)$

Range of f = Domain of f^{-1}: $(-\infty,\infty)$

Exercise

Find the inverse of $f(x)=\dfrac{2x+1}{x-3}$

Solution

$$y=\frac{2x+1}{x-3}$$

$$x=\frac{2y+1}{y-3}$$

$$x(y-3)=2y+1$$

$$xy-3x=2y+1$$

$$xy-2y=3x+1$$

$$y(x-2)=3x+1$$

$$y=\frac{3x+1}{x-2}=f^{-1}(x)$$

Exercise

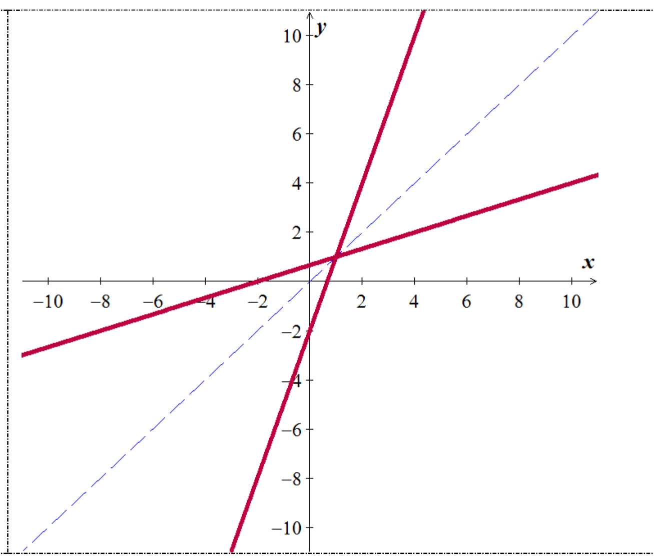
Prove the f *and* g are inverse functions of each other, and sketch the graphs of f *and* g:

$f(x)=3x-2 \quad g(x)=\dfrac{x+2}{3}$

Solution

$$\begin{aligned} f\left(g(x)\right) &= f\left(\frac{x+2}{3}\right) \\ &= 3\frac{x+2}{3}-2 \\ &= x+2-2 \\ &= x \end{aligned}$$

$$\begin{aligned} g\left(f(x)\right) &= g\left(3x-2\right) \\ &= \frac{3x-2+2}{3} \\ &= \frac{3x}{3} \\ &= x \end{aligned}$$



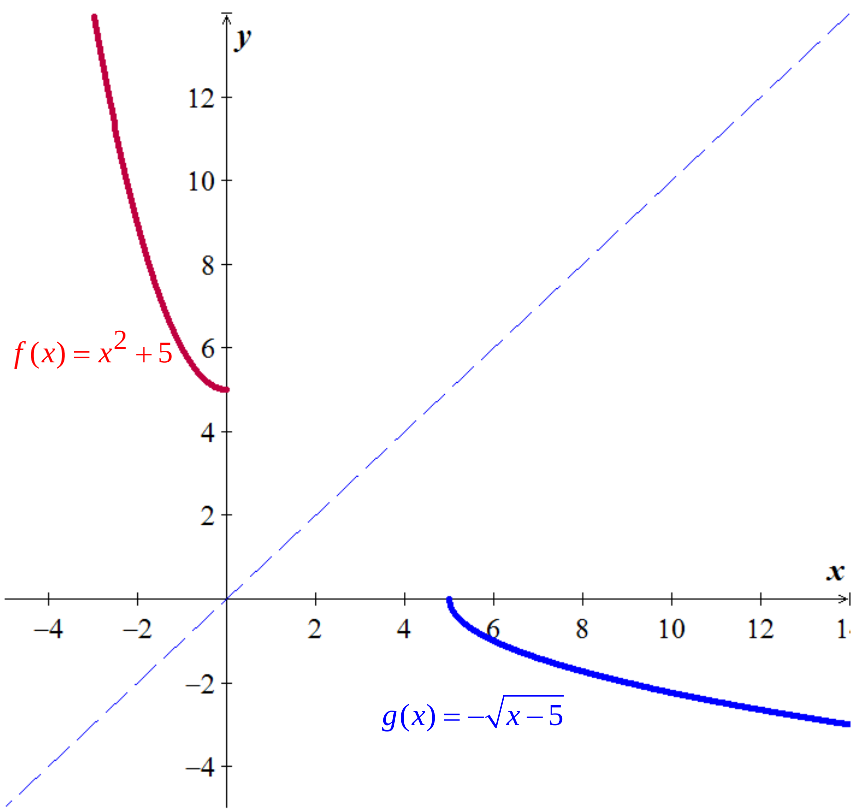
Exercise

Prove the f *and* g are inverse functions of each other, and sketch the graphs of f *and* g:

$f(x) = x^2 + 5,\ x \le 0 \quad g(x) = -\sqrt{x-5}, x \ge 5$

Solution

$f\left(g(x)\right) = f\left(-\sqrt{x-5}\right)$ $= \left(-\sqrt{x-5}\right)^2 + 5$ $= x - 5 + 5$ $= x$	$g\left(f(x)\right) = g\left(x^2 + 5\right)$ $= -\sqrt{x^2 + 5 - 5}$ $= -\sqrt{x^2}$ $= -\lvert x \rvert$ $= -(-x)$ since $x < 0$ $= x$



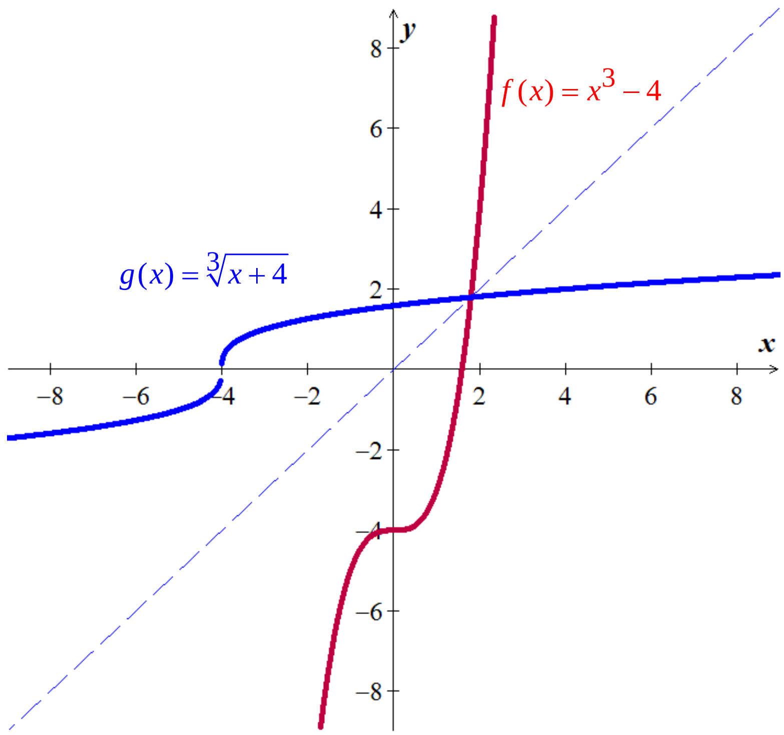
Exercise

Prove the f *and* g are inverse functions of each other, and sketch the graphs of f *and* g:

$f(x)=x^3-4; \quad g(x)=\sqrt[3]{x+4}$

Solution

$f(g(x)) = f\left(\sqrt[3]{x+4}\right)$	$g(f(x)) = g\left(x^3-4\right)$
$= \left(\sqrt[3]{x+4}\right)^3 - 4$	$= \sqrt[3]{x^3-4+4}$
$= x+4-4$	$= \sqrt[3]{x^3}$
$\underline{= x}$	$\underline{= x}$



Exercise

Determine the domain and range of f^{-1}: $f(x)=-\dfrac{2}{x-1}$ (Hint: first find the domain and range of f)

Solution

$x-1 \ne 0 \Rightarrow x \ne 1$

Range of f^{-1} = Domain of f: $\mathbb{R}-\{1\}$ $\qquad (-\infty, 1)\cup(1, \infty)$

Domain of f^{-1} = Range of f: $\mathbb{R}-\{0\}$ $\qquad (-\infty, 0)\cup(0, \infty)$

Exercise

Determine the domain and range of f^{-1}: $f(x)=\dfrac{5}{x+3}$ (Hint: first find the domain and range of *f*)

Solution

Domain of f^{-1} = Range of f: $\mathbb{R}-\{0\}$ $\quad (-\infty,\ 0)\cup(0,\ \infty)$

Range of f^{-1} = Domain of f: $\mathbb{R}-\{-3\}$ $\quad (-\infty,\ -3)\cup(-3,\ \infty)$

Exercise

Determine the domain and range of f^{-1}: $f(x)=\dfrac{4x+5}{3x-8}$ (Hint: first find the domain and range of *f*)

Solution

Domain of f^{-1} = Range of f: $\mathbb{R}-\left\{\frac{8}{3}\right\}$ $\quad \left(-\infty,\ \frac{8}{3}\right)\cup\left(\frac{8}{3},\ \infty\right)$

Range of f^{-1} = Domain of f: $\mathbb{R}-\left\{\frac{4}{3}\right\}$ $\quad \left(-\infty,\ \frac{4}{3}\right)\cup\left(\frac{4}{3},\ \infty\right)$

Exercise

Find the inverse function of: $f(x)=3x+5$

Solution

$y=3x+5$

$x=3y+5$ *Interchange x and y*

$x-5=3y$ *Solve for y*

$\frac{x-5}{3}=y \quad \rightarrow f^{-1}(x)=\dfrac{x-5}{3}$

Exercise

Find the inverse function of: $f(x)=\dfrac{1}{3x-2}$

Solution

$y=\dfrac{1}{3x-2}$

$x=\dfrac{1}{3y-2}$ ***Interchange x and y***

$x(3y-2)=1$ ***Solve for y***

$3xy-2x=1$

$3xy=1+2x$

$y=\dfrac{1+2x}{3x}=f^{-1}(x)$

Exercise

Find the inverse function of: $f(x)=\dfrac{3x+2}{2x-5}$

Solution

$y=\dfrac{3x+2}{2x-5}$

$x=\dfrac{3y+2}{2y-5}$ ***Interchange x and y***

$x(2y-5)=3y+2$ ***Solve for y***

$2xy-5x=3y+2$

$2xy-3y=5x+2$

$(2x-3)y=5x+2$

$y=\dfrac{5x+2}{2x-3}=f^{-1}(x)$

Exercise

Find the inverse function of: $f(x) = \dfrac{4x}{x-2}$

Solution

$$y = \frac{4x}{x-2}$$

$$x = \frac{4y}{y-2}$$

$$x(y-2) = 4y$$

$$xy - 2x = 4y$$

$$xy - 4y = 2x$$

$$(x-4)y = 2x$$

$$y = \boxed{\frac{2x}{x-4} = f^{-1}(x)}$$

Exercise

Find the inverse function of: $f(x) = 2 - 3x^2; \quad x \le 0$

Solution

$$y = 2 - 3x^2$$

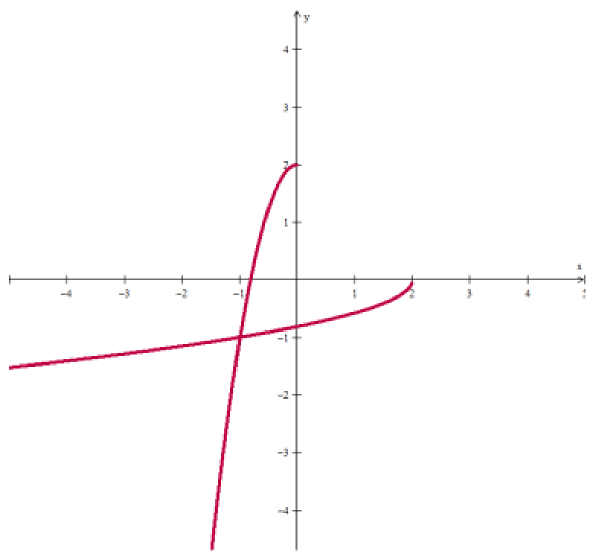
$$x = 2 - 3y^2$$

$$3y^2 = 2 - x$$

$$y^2 = \frac{2-x}{3}$$

$$y = \boxed{-\sqrt{\frac{2-x}{3}} = f^{-1}(x)}$$

Since x < 0



Exercise

Find the inverse function of: $f(x) = 2x^3 - 5$

Solution

$$y = 2x^3 - 5$$

$$y + 5 = 2x^3$$

$$\frac{y+5}{2} = x^3$$

$$x = \sqrt[3]{\frac{y+5}{2}}$$

$$f^{-1}(x) = \sqrt[3]{\frac{x+5}{2}}$$

Exercise

Find the inverse function of: $f(x) = \sqrt{3-x}$

Solution

$$y = \sqrt{3-x} \qquad \boldsymbol{y \ge 0}$$

$$y^2 = 3 - x$$

$$x = 3 - y^2 \qquad \boldsymbol{x \ge 0}$$

$$\boxed{f^{-1}(x) = 3 - x^2}$$

Exercise

Find the inverse function of: $f(x) = \sqrt[3]{x} + 1$

Solution

$$y = \sqrt[3]{x} + 1$$

$$y - 1 = \sqrt[3]{x}$$

$$(y-1)^3 = x$$

$$\boxed{f^{-1}(x) = (x-1)^3}$$

Exercise

Find the inverse function of: $f(x)=\left(x^3+1\right)^5$

Solution

$$y=\left(x^3+1\right)^5$$

$$\sqrt[5]{y}=x^3+1$$

$$\sqrt[5]{y}-1=x^3$$

$$x=\sqrt[3]{\sqrt[5]{y}-1}$$

$$\boxed{f^{-1}(x)=\sqrt[3]{\sqrt[5]{y}-1}}$$

Exercise

Find the inverse function of: $f(x)=x^2-6x; \quad x\ge 3$

Solution

$$y=x^2-6x$$

$$x^2-6x-y=0$$

$$\begin{aligned} x &= \frac{-(-6)\pm\sqrt{(-6)^2-4(1)(-y)}}{2(1)} \\ &= \frac{6\pm\sqrt{36+4y}}{2} \\ &= \frac{6\pm 4\sqrt{9+y}}{2} \\ &= 3\pm\sqrt{9+y} \end{aligned}$$

Since $x\ge 3 \Rightarrow$ we can select $x=3+\sqrt{y+9}$

$$\therefore \boxed{f^{-1}(x)=3+\sqrt{x+9}}$$

Solution Section 3.2 - Exponential Functions

Exercise

Find $2^{3.4}$

Solution

$$2^{3.4} = 10.5561$$

Exercise

Find $5^{\sqrt{3}}$

Solution

$$5^{\sqrt{3}} = 16.2425$$

Exercise

Find $6^{-1.2}$

Solution

$$6^{-1.2} = 0.1165$$

Exercise

Evaluate to four decimal places using a calculator: $e^{-0.75}$

Solution

$$e^{-0.75} = .4724$$

Exercise

Evaluate to four decimal places using a calculator: $e^{2.3}$

Solution

$$e^{2.3} = 9.9742$$

Exercise

Evaluate to four decimal places using a calculator: $e^{-0.95}$

Solution

$$e^{-0.95} = 0.3867$$

Exercise

Sketch the graph: $f(x) = 2^{x} + 3$

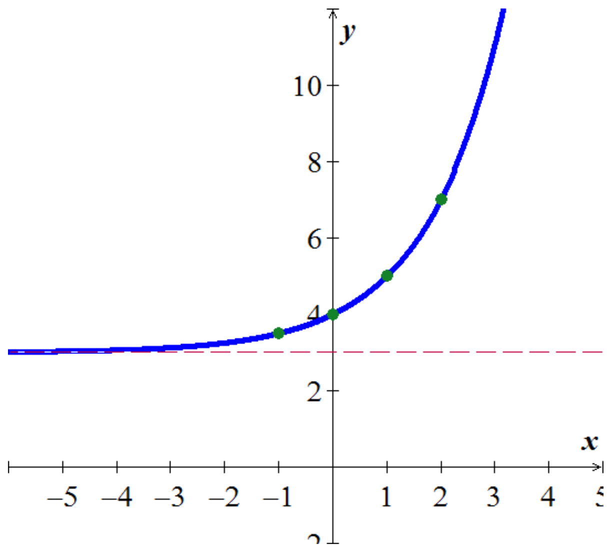
Solution

Asymptote: $y = 3$

Domain: $(-\infty,\ \infty)$

Range: $(3,\ \infty)$

x	$f(x)$
–1	3.5
0	4
1	5
2	7



Exercise

Sketch the graph: $f(x) = 2^{3-x}$

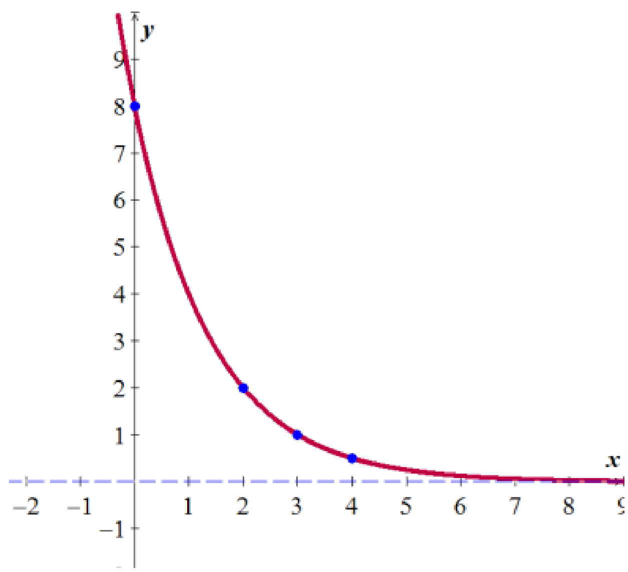
Solution

Asymptote: $y = 0$

Domain: $(-\infty,\ \infty)$

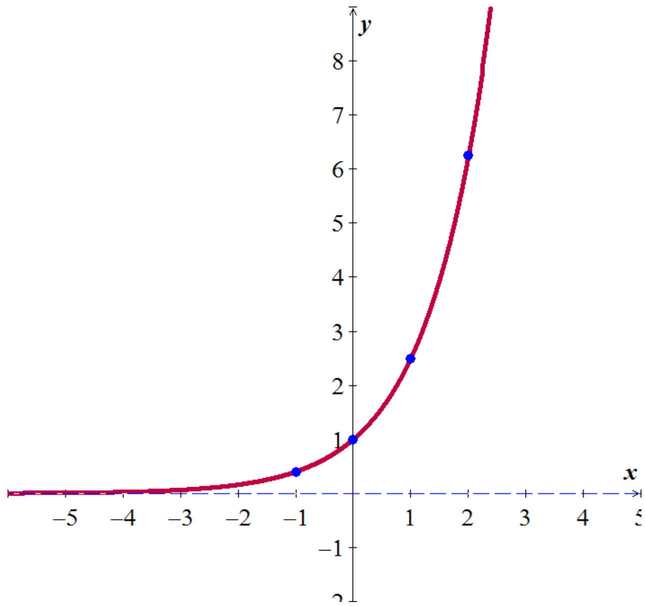
Range: $(0,\ \infty)$

x	$f(x)$
1	4
2	2
3	1
4	.5



Exercise

Sketch the graph: $f(x) = \left(\frac{2}{5}\right)^{-x}$



Solution

Asymptote: $y = 0$

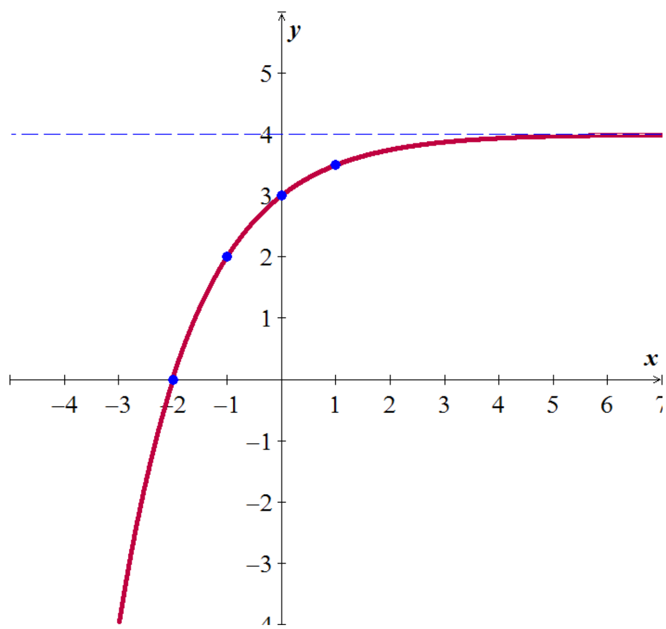
Domain: $(-\infty,\ \infty)$

Range: $(0,\ \infty)$

x	$f(x)$
-1	0.4
0	1
1	2.5
2	6.25

Exercise

Sketch the graph: $f(x) = -\left(\frac{1}{2}\right)^{x} + 4$



Solution

Asymptote: $y = 4$

Domain: $(-\infty,\ \infty)$

Range: $(-\infty,\ 4)$

x	$f(x)$
-2	0
-1	2
0	3
1	3.5

Exercise

Sketch the graph of $f(x) = e^{x+4}$

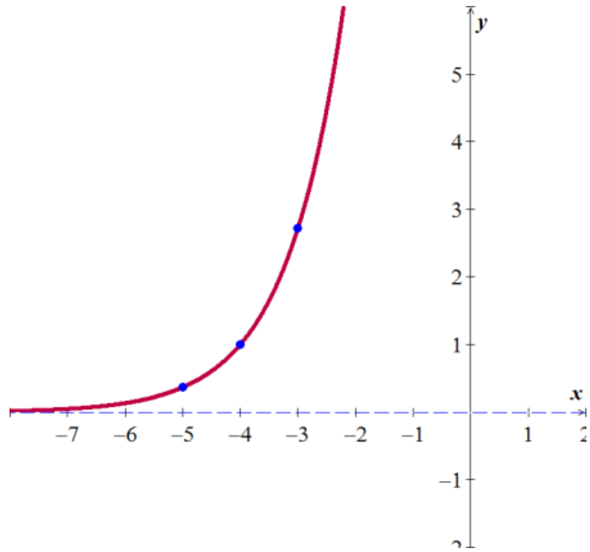
Solution

Asymptote: $y = 0$

Domain: $(-\infty,\ \infty)$

Range: $(0,\ \infty)$

x	$f(x)$
-5	0.4
-4	1
-3	2.7



Exercise

The exponential function $f(x) = 1066e^{0.042x}$ models the gray wolf population of the Western Great Lakes, $f(x)$, in billions, x years after 1978. Project the gray population in the recovery area in 2012.

Solution

$x = 2012 - 1978 = 34$

$$f(x = 34) = 1066e^{0.042(34)} \qquad 1066\ e \wedge (.042 * 34)$$

$$= 4445.6$$

$$\approx 4446$$

Exercise

The function $f(x) = 6.4e^{0.0123x}$ describes world population, $f(x)$, in billions, x years after 2004 subject to a growth rate of 1.23% annually. Use the function to predict world population in 2050.

Solution

$x = 2050 - 2004 = 46$

$$f(x = 46) = 6.4e^{0.0123(46)} \qquad 6.4\ e \wedge (0.0123 * 46)$$

$$\approx 11.27 \text{ billion}$$

Exercise

Find the accumulated value of an investment of \$10,000 for 5 years at an interest rate of 5.5% if the money is

Solution

Given: P = 10000

$t = 5$

$r = 0.055$

a. Semiannually: $n = 2$

$$\Rightarrow A = 10000\left(1+\frac{0.055}{2}\right)^{2(5)} \qquad 10000\ (1+.055/2)\wedge(2*5)$$

$$= \$13116.51$$

b. Quarterly: $n = 4$

$$\Rightarrow A = 10000\left(1+\frac{0.055}{4}\right)^{4(5)} \qquad 10000\ (1+.055/4)\wedge(4*5)$$

$$= \$13140.67$$

c. Monthly: $n = 12$

$$\Rightarrow A = 10000\left(1+\frac{0.055}{12}\right)^{12(5)} \qquad 10000\ (1+.055\ /\ 12)\wedge(12*5)$$

$$= \$13157.04$$

d. $A = 10000e^{(0.055)(5)} = \$13165.31$ $\qquad 10000\ e\wedge(0.055*5)$

$$= \$13165.31$$

Exercise

Suppose \$1000 is deposited in an account paying 4% interest per year compounded quarterly.

a) Find the amount in the account after 10 years with no withdraws.

b) How much interest is earned over the 10 years period?

Solution

Given:

$P = 1000$

$r = .04$

$n = 4$

a) $t = 10$

$$A = P\left(1+\frac{r}{n}\right)^{tn}$$

$$A = 1000\left(1+\frac{.04}{4}\right)^{10(4)} \qquad 1000(1+.04/4)\wedge(4*10)$$

$= \$1,488.86$

$b)$ The interest earned: $\$1488.86 - \$1000 = \$488.86$

Exercise

Becky must pay a lump sum of $6000 in 5 yrs.

a) What amount deposited today at 3.1% compounded annually will grow to $6000 in 5 yrs.?

b) If only $5000 is available to deposit now, what annual interest rate is necessary for the money to increase to $6000 in 5 yrs.?

Solution

$a)\quad A = P\left(1+\frac{r}{n}\right)^{tn}$

$$6000 = P\left(1+\frac{.031}{1}\right)^{5(1)}$$

$$6000 = P(1.031)^5$$

$$\frac{6000}{(1.031)^5} = P \qquad\qquad 6000/1.031\wedge 5$$

$$P \approx \$5{,}150.60$$

$a)\quad A = P\left(1+\frac{r}{n}\right)^{tn}$

$$6000 = 5000\left(1+\frac{r}{1}\right)^{5(1)}$$

$$\frac{6000}{5000} = (1+r)^5$$

$$\frac{6}{5} = (1+r)^5$$

$$\left(\frac{6}{5}\right)^{1/5} = 1+r$$

$$r = \left(\frac{6}{5}\right)^{1/5} - 1 \qquad\qquad (6/5)\wedge(1/5)-1$$

$$\approx .0371$$

The interest rate of 3.71% will produce enough to increase the $5,000 to $6,000 by the end of 5 yr.

Exercise

An investment of 1,000 increased to $13,464 in 20 years. If the interest was compounded continuously, find the interest rate.

Solution

$$A = Pe^{rt}$$

$$13464 = 1000e^{20r}$$

$$13.464 = e^{20r}$$

$$\ln(13.464) = \ln e^{20r}$$

$$20r = \ln 13.464$$

$$r = \frac{\ln 13.464}{20} \approx 0.13$$

The interest rate is 13%.

Exercise

Find the present value of $4,000 if the annual interest rate is 3.5% compounded quarterly for 6 years.

Solution

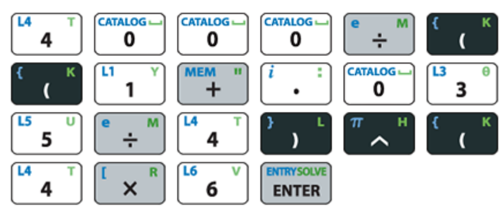
Given: $A = 4000.00, \quad r = 0.035, \quad t = 6, \quad n = 4$

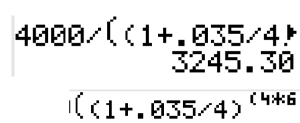
$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$4000 = P\left(1+\frac{0.035}{4}\right)^{4(6)}$$

$$P = \frac{4000}{\left(1+\frac{0.035}{4}\right)^{4(6)}}$$

$$\underline{= \$3245.30}$$





Exercise

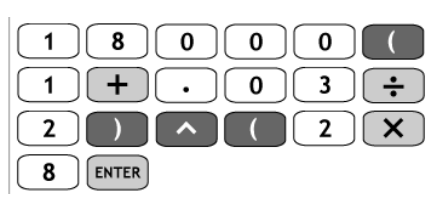
How much money will there be in an account at the end of 8 years if $18,000 is deposited at 3% interest compounded semi-annually?

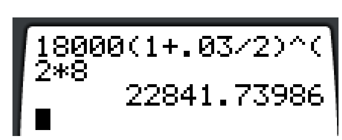
Solution

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$= 18000\left(1+\frac{0.03}{2}\right)^{2(8)}$$

$$\underline{= \$22,841.74}$$





Exercise

The function defined by $P(x) = 908e^{-0.0001348x}$ approximates the atmospheric pressure (in millibars) at an altitude of x meters. Use P to predict the pressure:

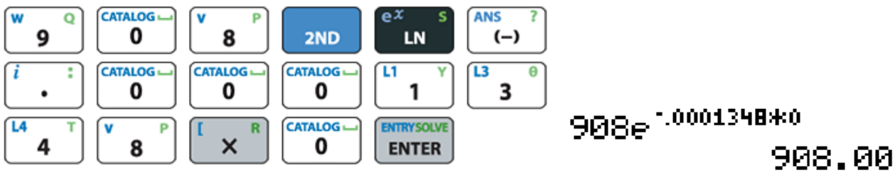
a) At 0 meters

b) At 12,000 meters

Solution

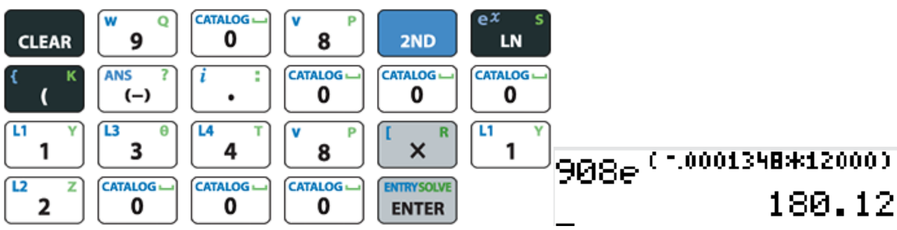
a) At 0 meters

$$P(x = 0) = 908e^{-0.0001348(0)} \ \underline{= 908 \ \ millibars}$$



b) At 12,000 meters

$$P(x = 12{,}000) = 908e^{-0.0001348(12{,}000)} \ \underline{= 180 \ \ millibars}$$



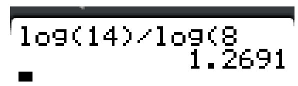
Solution   *Section* 3.3 - Logarithmic Functions

Exercise

Find $\log_8 14$

Solution

$$\log_8 14 = \frac{\log 14}{\log 8}$$
$$\approx 1.2691$$



Exercise

Write the equation in its equivalent logarithmic form $2^6 = 64$

Solution

$$6 = \log_2 64$$

Exercise

Write the equation in its equivalent exponential form $2 = \log_9 x$

Solution

$$\Rightarrow 9^2 = x$$

Exercise

Write the equation in its equivalent logarithmic form $5^4 = 625$

Solution

$$\Rightarrow 4 = \log_5 625$$

Exercise

Write the equation in its equivalent logarithmic form $5^{-3} = \frac{1}{125}$

Solution

$$-3 = \log_5 \frac{1}{125}$$

Exercise

Write the equation in its equivalent logarithmic form $\sqrt[3]{64}=4$

Solution

$$64^{1/3}=4$$
$$\Rightarrow \log_{64}=\frac{1}{3}$$

Exercise

Write the equation in its equivalent logarithmic form $b^3=343$

Solution

$$\Rightarrow \log_b 343=3$$

Exercise

Write the equation in its equivalent logarithmic form $8^y=300$

Solution

$$\Rightarrow \log_8 300=y$$

Exercise

Write the equation in its equivalent logarithmic form: $\left(\frac{2}{3}\right)^{-3}=\frac{27}{8}$

Solution

$$\log_{\frac{2}{3}}\left(\frac{27}{8}\right)=-3$$

Exercise

Write the equation in its equivalent exponential form $\log_5 125=y$

Solution

$$\Rightarrow 5^y=125$$

Exercise

Write the equation in its equivalent exponential form $\log_4 16=x$

Solution

$$16=4^x$$

Exercise

Write the equation in its equivalent exponential form $\log_5 \frac{1}{5} = x$

Solution

$$\frac{1}{5} = 5^x$$

Exercise

Write the equation in its equivalent exponential form $\log_2 \frac{1}{8} = x$

Solution

$$\frac{1}{2^3} = 2^x$$

$$2^{-3} = 2^x$$

Exercise

Write the equation in its equivalent exponential form $\log_6 \sqrt{6} = x$

Solution

$$6^{1/2} = 6^x$$

Exercise

Write the equation in its equivalent exponential form $\log_3 \frac{1}{\sqrt{3}} = x$

Solution

$$3^{-1/2} = 3^x$$

Exercise

Write the equation in its equivalent exponential form: $6 = \log_2 64$

Solution

$$6 = \log_2 64 \Leftrightarrow 2^6 = 64$$

Exercise

Write the equation in its equivalent exponential form: $2 = \log_9 x$

Solution

$$2 = \log_9 x \Leftrightarrow x = 2^9$$

Exercise

Write the equation in its equivalent exponential form: $\log_{\sqrt{3}} 81 = 8$

Solution

$$\log_{\sqrt{3}} 81 = 8 \quad \Leftrightarrow \quad 81 = \left(\sqrt{3}\right)^8$$

Exercise

Write the equation in its equivalent exponential form: $\log_4 \frac{1}{64} = -3$

Solution

$$\log_4 \frac{1}{64} = -3 \quad \Leftrightarrow \quad \frac{1}{64} = x^{-3}$$

Exercise

Evaluate the expression without using a calculator: $\log_4 16$

Solution

$$\log_4 16 = \log_4 4^2 \ \underline{= 2} \qquad\qquad \log_b b^x = x$$

Exercise

Evaluate the expression without using a calculator: $\log_2 \frac{1}{8}$

Solution

$$\log_2 \frac{1}{8} = \log_2 \frac{1}{2^3}$$

$$= \log_2 2^{-3} \qquad\qquad \log_b b^x = x$$

$$\underline{= -3}$$

Exercise

Evaluate the expression without using a calculator: $\log_6 \sqrt{6}$

Solution

$$\log_6 \sqrt{6} = \log_6 6^{1/2} \ \underline{= \frac{1}{2}}$$

Exercise

Evaluate the expression without using a calculator: $\log_3 \frac{1}{\sqrt{3}}$

Solution

$$\log_3 \frac{1}{\sqrt{3}} = \log_3 \frac{1}{3^{1/2}}$$

$$= \log_3 3^{-1/2} \qquad\qquad \log_b b^x = x$$

$$\boxed{= -\frac{1}{2}}$$

Exercise

Evaluate the expression without using a calculator: $\log_3 \sqrt[7]{3}$

Solution

$$\Rightarrow \log_3 3^{1/7} = x \qquad \textit{Converts to exponential}$$

$$3^{1/7} = 3^x$$

$$x = \frac{1}{7}$$

$$\Rightarrow \log_3 \sqrt[7]{3} = \frac{1}{7}$$

Exercise

Find $\log_5 8$ using common logarithms

Solution

$$\log_{\mathbf{5}} \mathbf{8} = \frac{\ln \mathbf{8}}{\ln \mathbf{5}} \approx 1.292$$

Exercise

Find the number $\log_5 1$

Solution

$$\log_5 1 = 0$$

Exercise

Find the number $\log_7 7^2$

Solution

$$\log_7 7^2 = 2$$

Exercise

Find the number $3^{\log_3 8}$

Solution

$$3^{\log_3 8} = 8$$

Exercise

Find the number $10^{\log 3}$

Solution

$$10^{\log 3} = 3$$

Exercise

Find the number $e^{2+\ln 3}$

Solution

$$e^{2+\ln 3} = 22.1672$$

Exercise

Find the number $\ln e^{-3}$

Solution

$$\ln e^{-3} = -3$$

Exercise

Find the domain of $\log_5 (x+4)$

Solution

$$x > -4 \quad \rightarrow \quad (-4, \infty)$$

Exercise

Find the domain of $\log_5(x+6)$

Solution

$$x > -6 \quad \to \ (-6, \infty)$$

Exercise

Find the domain of $\log(2-x)$

Solution

$$2-x>0$$
$$-x>-2$$
$$x<2 \quad \to \ (-\infty, 2)$$

Exercise

Find the domain of $\log(7-x)$

Solution

$$7-x>0$$

$$-x>-7$$

$$x<7 \quad \to \ (-\infty, 7)$$

Exercise

Find the domain of $\ln(x-2)^2$

Solution

$$x-2 \neq 0 \ \Rightarrow \ x \neq 2$$

$$(-\infty, 2) \cup (2, \infty)$$

Exercise

Find the domain of $\ln(x-7)^2$

Solution

$$x-7 \neq 0 \ \Rightarrow \ x \neq 7$$

$$(-\infty, 7) \cup (7, \infty)$$

Exercise

Find the domain of $\log\left(x^2-4x-12\right)$

Solution

$$x^2-4x-12\neq 0 \Rightarrow x\neq -2,6$$

$$(-\infty,-2)\cup(-2,6)\cup(6,\infty)$$

Exercise

Find the domain of $\log\left(\frac{x-2}{x+5}\right)$

Solution

$$\begin{cases} x\neq 2 \\ x\neq -5 \end{cases}$$

<table>
<tr><td colspan="3">-5 0 2</td></tr>
<tr><td>+</td><td>-</td><td>+</td></tr>
</table>

$$(-\infty,-5)\cup(2,\infty)$$

Exercise

Sketch the graph of $f(x)=\log_4(x-2)$

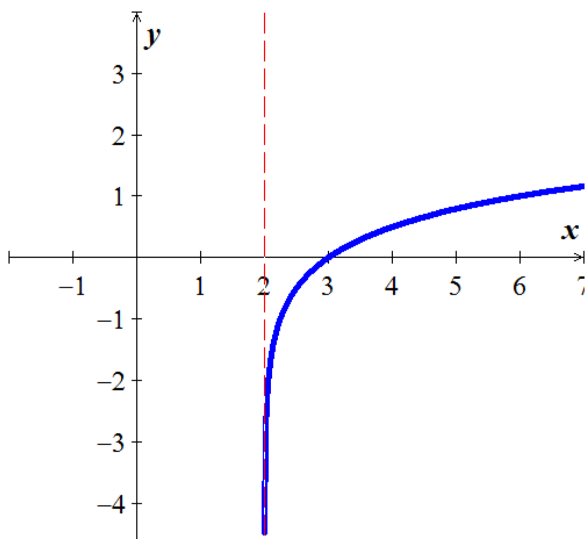
Solution

Asymptote: $x=2$

Domain: $(2,\ \infty)$

Range: $(-\infty,\infty)$

x	***f(x)***
2	
2.5	–.5
3	0
4	.5



Exercise

Sketch the graph of $f(x) = \log_4 |x|$

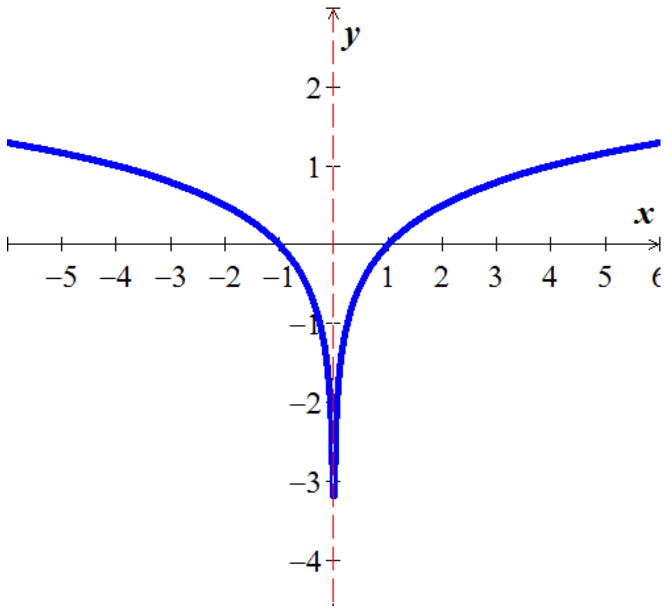
Solution

Asymptote: $x = 0$

Domain: $(-\infty,\ 0) \cup (0,\ \infty)$

Range: $(-\infty, \infty)$

x	$f(x)$
0	
$\pm .5$	$-.5$
± 1	0
± 2	.5



Exercise

Sketch the graph of $f(x) = \left(\log_4 x\right) - 2$

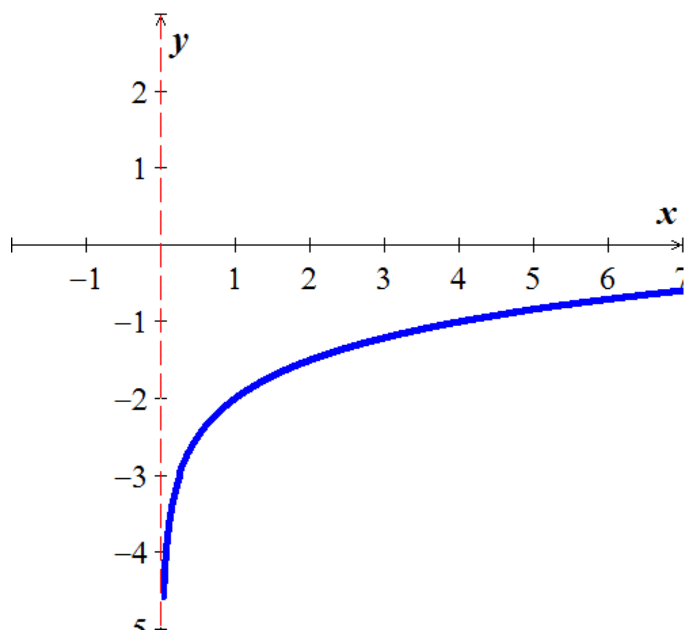
Solution

Asymptote: $x = 0$

Domain: $(0,\ \infty)$

Range: $(-\infty, \infty)$

x	$f(x)$
0	
0.5	-2.5
1	0
2	1.5



Exercise

On a study by psychologists Bornstein and Bornstein, it was found that the average walking speed w, in feet per second, of a person living in a city of population P, in ***thousands***, is given by the function

$$w(P) = 0.37 \ln P + 0.05$$

a) The population is 124,848. Find the average walking speed of people living in Hartford.
b) The population is 1,236,249. Find the average walking speed of people living in San Antonio.

Solution

124,848 = 124.848 ***thousand***

a) $w(P=124.848) = 0.37 \ln(124.848) + 0.05 \approx 1.8$ ft/sec

b) $w(P=1,236.249) = 0.37 \ln(1,236.249) + 0.05 \approx 2.7$ ft/sec

Exercise

The loudness of sounds is measured in a unit called a decibel. To measure with this unit, we first assign an intensity of I_0 to a very faint sound, called the threshold sound. If a particular sound has intensity I, then the decibel rating of this louder sound is

$$d = 10\log\frac{I}{I_0}$$

Find the exact decibel rating of a sound with intensity $10,000I_0$

Solution

$$\begin{aligned} d &= 10\log\frac{10000I_0}{I_0} \\ &= 10\log 10000 \\ &= 40 \end{aligned}$$

Exercise

A model for advertising response is given by the function

$$N(a) = 1000 + 200 \ln a, \qquad a \ge 1$$

Where $N(a)$ is the number of units sold when a is the amount spent on advertising, in thousands of dollars.

a) $N(a = 1)$
b) $N(a = 5)$

Solution

a) $N(a=1) = 1000 + 200\ ln1 = 1000$ *units*

b) $N(a=5) = 1000 + 200\ ln5 = 1322$ *units*

Exercise

Students in an accounting class took a final exam and then took equivalent forms of the exam at monthly intervals thereafter. The average score *S(t)*, as a percent, after *t* months was found to be given by the function

$$S(t) = 78 - 15 \log(t + 1), \quad t \geq 0$$

a) What was the average score when the students initially took the test, $t = 0$?

b) What was the average score after 4 months? 24 months?

Solution

a) What was the average score when the students initially took the test, $t = 0$?

$t = 0 \rightarrow S(t) = 78 - 15 \log(0 + 1) = 78\%$

b) What was the average score after 4 months? 24 months?

After 4 months $\rightarrow S(t = 4) = 78 - 15 \log(4 + 1) = 67.5\%$

24 months $\rightarrow S(t = 24) = 78 - 15 \log(24 + 1) = 57\%$

Exercise

$5^{-3} = \frac{1}{125} \Leftrightarrow \log_5\left(\frac{1}{125}\right) = -3$

$8^{1/3} = 2 \Leftrightarrow \log_8 2 = \frac{1}{3}$

$10^{0.3010} = 2 \Leftrightarrow \log 2 = 0.3010$

$Q^t = x \Leftrightarrow \log_Q x = t$

$e^{-1} = 0.3679 \Leftrightarrow ln0.3679 = -1$

$p^k = 3 \Leftrightarrow \log_p 3 = k$

$t = \log_4 7 \Leftrightarrow 4^t = 7$

$\log 7 = 0.845 \Leftrightarrow 10^{0.845} = 7$

$ln0.38 = -0.9676 \Leftrightarrow \mathrm{e}^{-0.9676} = 0.38$

$\ln W^5 = t \Leftrightarrow \mathrm{e}^t = W^5$

$\log 8 = 0.9031$

ln(-4) *doesn't exist*

ln(0.00037) = -7.9020

Solution   *Section* 3.4 – Properties of Logarithms

Exercise

Express as a sum of logarithms: $\log_3(ab)$

Solution

$$\log_3(ab) = \log_3 a + \log_3 b$$

Exercise

Express as a sum of logarithms: $\log_7(7x)$

Solution

$$\log_7(7x) = \log_7 7 + \log_7 x$$

$$= 1 + \log_7 x$$

Exercise

Express the following in terms of sums and differences of logarithms $\log \frac{x}{1000}$

Solution

$$\log \frac{x}{1000} = \log x - \log 1000$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_5\left(\frac{125}{y}\right)$

Solution

$$\log_5\left(\frac{125}{y}\right) = \log_5 125 - \log_5 y$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b x^7$

Solution

$$\log_b x^7 = 7\log_b x$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln \sqrt[7]{x}$

Solution

$$\begin{aligned}\ln \sqrt[7]{x} &= \ln x^{1/7} \\ &= \frac{1}{7}\ln x\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_a \dfrac{x^2 y}{z^4}$

Solution

$$\begin{aligned}\log_a \frac{x^2 y}{z^4} &= \log_a x^2 y - \log_a z^4 && \textbf{\textit{Quotient Rule}} \\ &= \log_a x^2 + \log_a y - \log_a z^4 && \textbf{\textit{Product Rule}} \\ &= 2\log_a x + \log_a y - 4\log_a z && \textbf{\textit{Power Rule}}\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b \dfrac{x^2 y}{b^3}$

Solution

$$\begin{aligned}\log_b \frac{x^2 y}{b^3} &= \log_b x^2 y - \log_b b^3 \\ &= \log_b x^2 + \log_b y - \log_b b^3 \\ &= 2\log_b x + \log_b y - 3\log_b b \\ &= 2\log_b x + \log_b y - 3\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b \left(\frac{x^3 y}{z^2} \right)$

Solution

$$\log_b \left(\frac{x^3 y}{z^2} \right) = \log_b \left(x^3 y \right) - \log_b z^2$$

$$= \log_b x^3 + \log_b y - \log_b z^2$$

$$= 3\log_b x + \log_b y - 2\log_b z$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b \left(\frac{\sqrt[3]{x}\, y^4}{z^5} \right)$

Solution

$$\log_b \left(\frac{\sqrt[3]{x}\, y^4}{z^5} \right) = \log_b \left(\sqrt[3]{x}\, y^4 \right) - \log_b \left(z^5 \right)$$

$$= \log_b \left(x^{1/3} \right) + \log_b \left(y^4 \right) - \log_b \left(z^5 \right)$$

$$= \frac{1}{3} \log_b (x) + 4\log_b (y) - 5\log_b (z)$$

Exercise

Express the following in terms of sums and differences of logarithms $\log \left(\frac{100x^3 \sqrt[3]{5-x}}{3(x+7)^2} \right)$

Solution

$$\log \left(\frac{100x^3 \sqrt[3]{5-x}}{3(x+7)^2} \right) = \log \left(100x^3 \sqrt[3]{5-x} \right) - \log \left(3(x+7)^2 \right)$$

$$= \log 10^2 + \log x^3 + \log (5-x)^{1/3} - \left[\log 3 + \log \left((x+7)^2 \right) \right]$$

$$= 2\log 10 + 3\log x + \tfrac{1}{3}\log (5-x) - \log 3 - 2\log(x+7)$$

$$= 2 + 3\log x + \tfrac{1}{3}\log (5-x) - \log 3 - 2\log(x+7)$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_a \sqrt[4]{\dfrac{m^8\, n^{12}}{a^3\, b^5}}$

Solution

$$\log_a \sqrt[4]{\frac{m^8\, n^{12}}{a^3\, b^5}} = \log_a \left(\frac{m^8 n^{12}}{a^3 b^5}\right)^{1/4} \qquad \textit{Power Rule}$$

$$= \frac{1}{4}\log_a \left(\frac{m^8 n^{12}}{a^3 b^5}\right) \qquad \textit{Quotient Rule}$$

$$= \frac{1}{4}\left[\log_a m^8 n^{12} - \log_a a^3 b^5\right] \qquad \textit{Product Rule}$$

$$= \frac{1}{4}\left[\log_a m^8 + \log_a n^{12} - \left(\log_a a^3 + \log_a b^5\right)\right] \qquad \textit{Power Rule}$$

$$= \frac{1}{4}\left[8\log_a m + 12\log_a n - 3 - 5\log_a b\right]$$

$$= 2\log_a m + 3\log_a n - \frac{3}{4} - \frac{5}{4}\log_a b$$

Exercise

Use the properties of logarithms to rewrite: $\log_p \sqrt[3]{\dfrac{m^5 n^4}{t^2}}$

Solution

$$\log_p \sqrt[3]{\frac{m^5 n^4}{t^2}} = \log_p \left(\frac{m^5 n^4}{t^2}\right)^{1/3} \qquad \textit{Power Rule}$$

$$= \frac{1}{3}\log_p \left(\frac{m^5 n^4}{t^2}\right) \qquad \textit{Quotient Rule}$$

$$= \frac{1}{3}\left(\log_p m^5 n^4 - \log_p t^2\right) \qquad \textit{Product Rule}$$

$$= \frac{1}{3}\left(\log_p m^5 + \log_p n^4 - \log_p t^2\right) \qquad \textit{Power Rule}$$

$$= \frac{1}{3}\left(5\log_p m + 4\log_p n - 2\log_p t\right)$$

$$= \frac{5}{3}\log_p m + \frac{4}{3}\log_p n - \frac{2}{3}\log_p t$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b \sqrt[n]{\dfrac{x^3y^5}{z^m}}$

Solution

$$\log_b \sqrt[n]{\frac{x^3y^5}{z^m}} = \log_b \left(\frac{x^3y^5}{z^m}\right)^{1/n}$$

$$= \frac{1}{n}\log_b \left(\frac{x^3y^5}{z^m}\right) \qquad \textit{Power Rule}$$

$$= \frac{1}{n}\left(\log_b x^3y^5 - \log_b z^m\right) \qquad \textit{Quotient Rule}$$

$$= \frac{1}{n}\left(\log_b x^3 + \log_b y^5 - \log_b z^m\right) \qquad \textit{Product Rule}$$

$$= \frac{1}{n}\left(3\log_b x + 5\log_b y - m\log_b z\right) \qquad \textit{Power Rule}$$

$$= \frac{3}{n}\log_b x + \frac{5}{n}\log_b y - \frac{m}{n}\log_b z$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_a \sqrt[3]{\dfrac{a^2\, b}{c^5}}$

Solution

$$\log_a \sqrt[3]{\frac{a^2\, b}{c^5}} = \log_a \left(\frac{a^2\, b}{c^5}\right)^{1/3} \qquad \textit{Convert the radical to power}$$

$$= \frac{1}{3}\log_a \left(\frac{a^2\, b}{c^5}\right) \qquad \textit{Power Rule}$$

$$= \frac{1}{3}\left[\log_a a^2\, b - \log_a c^5\right] \qquad \textit{Quotient Rule}$$

$$= \frac{1}{3}\left[\log_a a^2 + \log_a b - \log_a c^5\right] \qquad \textit{Product Rule}$$

$$= \frac{1}{3}\left[2\log_a a + \log_a b - 5\log_a c\right] \qquad \textit{Power Rule}$$

$$= \frac{2}{3}\log_a a + \frac{1}{3}\log_a b - \frac{5}{3}\log_a c$$

$$= \frac{2}{3} + \frac{1}{3}\log_a b - \frac{5}{3}\log_a c$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b\left(x^4\sqrt[3]{y}\right)$

Solution

$$\log_b\left(x^4\sqrt[3]{y}\right) = \log_b\left(x^4\right)+\log_b\left(\sqrt[3]{y}\right)$$
$$= \log_b\left(x^4\right) + \log_b\left(y^{1/3}\right)$$
$$= 4\log_b(x) + \frac{1}{3}\log_b(y)$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_5\left(\frac{\sqrt{x}}{25y^3}\right)$

Solution

$$log_5\left(\frac{\sqrt{x}}{25y^3}\right) = log_5\left(x^{1/2}\right) - log_5\left(25y^3\right)$$
$$= \log_5\left(x^{1/2}\right) - \left[\log_5\left(5^2\right)+\log_5\left(y^3\right)\right]$$
$$= \log_5\left(x^{1/2}\right) - \log_5\left(5^2\right) - \log_5\left(y^3\right)$$
$$= \frac{1}{2}\log_5(x) - 2\log_5(5) - 3\log_5(y)$$
$$= \frac{1}{2}\log_5(x) - 2 - 3\log_5(y)$$

Exercise

Express $\log_a \frac{x^3 w}{y^2 z^4}$ in terms of logarithms of *x*, *y*, *z*, and *w*.

Solution

$$\log_a \frac{x^3 w}{y^2 z^4} = \log_a x^3 w - \log_a y^2 z^4 \qquad \textit{Quotient rule}$$
$$= \log_a x^3 + \log_a w - \left(\log_a y^2 + \log_a z^4\right) \qquad \textit{Product rule}$$

$$= \log_a x^3 + \log_a w - \log_a y^2 - \log_a z^4 \qquad \textit{Distribute minus}$$

$$= 3\log_a x + \log_a w - 2\log_a y - 4\log_a z \qquad \textit{Power rule}$$

Exercise

Express $\log_a \dfrac{\sqrt{y}}{x^4 \sqrt[3]{z}}$ in terms of logarithms of *x*, *y*, and *z*.

Solution

$$\log_a \frac{\sqrt{y}}{x^4 \sqrt[3]{z}} = \log_a y^{1/2} - \log_a x^4 z^{1/3} \qquad \textit{Quotient rule}$$

$$= \log_a y^{1/2} - \left(\log_a x^4 + \log_a z^{1/3}\right) \qquad \textit{Product rule}$$

$$= \log_a y^{1/2} - \log_a x^4 - \log_a z^{1/3} \qquad \textit{Distribute minus}$$

$$= \frac{1}{2}\log_a y - 4\log_a x - \frac{1}{3}\log_a z \qquad \textit{Power rule}$$

Exercise

Express $\ln \sqrt[4]{\dfrac{x^7}{y^5 z}}$ in terms of logarithms of *x*, *y*, and *z*.

Solution

$$\ln \sqrt[4]{\frac{x^7}{y^5 z}} = \ln \left(\frac{x^7}{y^5 z}\right)^{1/4}$$

$$= \frac{1}{4}\ln\left(\frac{x^7}{y^5 z}\right) \qquad \textit{Power rule}$$

$$= \frac{1}{4}\left(\ln x^7 - \ln y^5 z\right) \qquad \textit{Quotient rule}$$

$$= \frac{1}{4}\left(\ln x^7 - \left(\ln y^5 + \ln z\right)\right) \qquad \textit{Product rule}$$

$$= \frac{1}{4}\left(\ln x^7 - \ln y^5 - \ln z\right)$$

$$= \frac{1}{4}\left(7\ln x - 5\ln y - \ln z\right) \qquad \textit{Power rule}$$

$$= \frac{7}{4}\ln x - \frac{5}{4}\ln y - \ln z$$

Exercise

Express $\ln x\sqrt[3]{\dfrac{y^4}{z^5}}$ in terms of logarithms of *x*, *y*, and *z*.

Solution

$$\ln x\sqrt[3]{\frac{y^4}{z^5}} = \ln x + \ln\left(\frac{y^4}{z^5}\right)^{1/3} \qquad \textit{Product rule}$$

$$= \ln x + \ln\left(\frac{y^{4/3}}{z^{5/3}}\right)$$

$$= \ln x + \ln y^{4/3} - \ln z^{5/3} \qquad \textit{Quotient rule}$$

$$= \ln x + \frac{4}{3}\ln y - \frac{5}{3}\ln z \qquad \textit{Power rule}$$

Exercise

Express as one logarithm: $2\log_a x + \frac{1}{3}\log_a(x-2) - 5\log_a(2x+3)$

Solution

$$2\log_a x + \tfrac{1}{3}\log_a(x-2) - 5\log_a(2x+3) = \log_a x^2 + \log_a(x-2)^{1/3} - \log_a(2x+3)^5$$

$$= \log_a x^2(x-2)^{1/3} - \log_a(2x+3)^5$$

$$= \log_a \frac{x^2(x-2)^{1/3}}{(2x+3)^5}$$

Exercise

Express as one logarithm: $5\log_a x - \frac{1}{2}\log_a(3x-4) - 3\log_a(5x+1)$

Solution

$$5\log_a x - \tfrac{1}{2}\log_a(3x-4) - 3\log_a(5x+1) = \log_a x^5 - \log_a(3x-4)^{1/2} - \log_a(5x+1)^3$$

$$= \log_a x^5 - \left[\log_a(3x-4)^{1/2} + \log_a(5x+1)^3\right]$$

$$= \log_a x^5 - \left[\log_a(3x-4)^{1/2}(5x+1)^3\right]$$

$$= \log_a \frac{x^5}{(3x-4)^{1/2}(5x+1)^3}$$

Exercise

Express as one logarithm: $\log\left(x^3y^2\right)-2\log\left(x\sqrt[3]{y}\right)-3\log\left(\frac{x}{y}\right)$

Solution

$$
\begin{aligned}
\log\left(x^3y^2\right)-2\log\left(x\sqrt[3]{y}\right)-3\log\left(\frac{x}{y}\right) &= \log\left(x^3y^2\right)-\log\left(xy^{1/3}\right)^2-\log\left(xy^{-1}\right)^3 \\
&= \log\left(x^3y^2\right)-\left[\log\left(x^2y^{2/3}\right)+\log\left(x^3y^{-3}\right)\right] \\
&= \log\left(x^3y^2\right)-\log\left(x^2y^{2/3}x^3y^{-3}\right) \\
&= \log\left(x^3y^2\right)-\log\left(x^5y^{-7/3}\right) \\
&= \log\left(\frac{x^3y^2}{x^5y^{-7/3}}\right) \\
&= \log\left(\frac{y^2y^{7/3}}{x^2}\right) \\
&= \log\left(\frac{y^{13/3}}{x^2}\right) \\
&= \log\left(\frac{\sqrt[3]{y^{13}}}{x^2}\right) \\
&= \log\left(\frac{y^4\sqrt[3]{y}}{x^2}\right)
\end{aligned}
$$

Exercise

Express as one logarithm: $\ln y^3+\frac{1}{3}\ln\left(x^3y^6\right)-5\ln y$

Solution

$$
\begin{aligned}
\ln y^3+\tfrac{1}{3}\ln\left(x^3y^6\right)-5\ln y &= \ln y^3+\ln\left(x^3y^6\right)^{1/3}-\ln y^5 \\
&= \ln y^3+\ln\left(x^{3/3}y^{6/3}\right)-\ln y^5 \\
&= \ln y^3+\ln\left(xy^2\right)-\ln y^5 \\
&= \ln\left(y^3xy^2\right)-\ln y^5 \\
&= \ln\left(\frac{y^5x}{y^5}\right) \\
&= \ln x
\end{aligned}
$$

Exercise

Express as one logarithm: $2\ln x - 4\ln\left(\frac{1}{y}\right) - 3\ln(xy)$

Solution

$$
\begin{aligned}
2\ln x - 4\ln\left(\frac{1}{y}\right) - 3\ln(xy) &= \ln x^2 - \ln\left(\frac{1}{y}\right)^4 - \ln(xy)^3 \\
&= \ln x^2 - \left[\ln\left(y^{-4}\right) + \ln\left(x^3y^3\right)\right] \\
&= \ln x^2 - \ln\left(y^{-4}x^3y^3\right) \\
&= \ln x^2 - \ln\left(y^{-1}x^3\right) \\
&= \ln \frac{x^2}{y^{-1}x^3} \\
&= \ln \frac{y}{x}
\end{aligned}
$$

Exercise

Write as a single logarithmic $4\ln x + 7\ln y - 3\ln z$

Solution

$$
\begin{aligned}
4\ln x + 7\ln y - 3\ln z &= \ln x^4 + \ln y^7 - \ln z^3 \\
&= \ln\left(x^4y^7\right) - \ln z^3 \\
&= \ln\left(\frac{x^4y^7}{z^3}\right)
\end{aligned}
$$

Exercise

Write as a single logarithmic $\frac{1}{3}\left[5\ln(x+6) - \ln x - \ln(x^2-25)\right]$

Solution

$$
\begin{aligned}
\frac{1}{3}\left[5\ln(x+6) - \ln x - \ln(x^2-25)\right] &= \frac{1}{3}\left[5\ln(x+6) - \left(\ln x + \ln(x^2-25)\right)\right] \\
&= \frac{1}{3}\left[\ln(x+6)^5 - \ln x(x^2-25)\right] \\
&= \frac{1}{3}\left[\ln \frac{(x+6)^5}{x(x^2-25)}\right] \\
&= \ln\left(\frac{(x+6)^5}{x(x^2-25)}\right)^{1/3}
\end{aligned}
$$

Exercise

Write as a single logarithmic $\frac{2}{3}\left[\ln\left(x^2-4\right)-\ln\left(x+2\right)\right]+\ln(x+y)$

Solution

$$\frac{2}{3}\left[\ln\left(x^2-4\right)-\ln\left(x+2\right)\right]+\ln(x+y)=\frac{2}{3}\left[\ln\frac{x^2-4}{x+2}\right]+\ln(x+y)$$

$$=\frac{2}{3}\left[\ln\frac{(x+2)(x-2)}{x+2}\right]+\ln(x+y)$$

$$=\frac{2}{3}\ln(x-2)+\ln(x+y)$$

$$=\ln(x-2)^{2/3}+\ln(x+y)$$

$$=\ln(x-2)^{2/3}(x+y)$$

$$=\ln(x+y)\ \sqrt[3]{(x-2)^2}$$

Exercise

Write as a single logarithmic $\frac{1}{2}\log_b m+\frac{3}{2}\log_b 2n-\log_b m^2n$

Solution

$$\frac{1}{2}\log_b m+\frac{3}{2}\log_b 2n-\log_b m^2n=\log_b m^{1/2}+\log_b (2n)^{3/2}-\log_b m^2n$$

$$=\log_b\left(m^{1/2}(2n)^{3/2}\right)-\log_b m^2n$$

$$=\log_b\frac{m^{1/2}2^{3/2}n^{3/2}}{m^2n}$$

$$=\log_b\frac{2^{3/2}n^{1/2}}{m^{3/2}}$$

$$=\log_b\left(\frac{2^3n}{m^3}\right)^{1/2}$$

$$=\log_b\sqrt{\frac{8n}{m^3}}$$

Exercise

Write the expression as a single logarithm. $\frac{1}{2}\log_y p^3q^4 - \frac{2}{3}\log_y p^4q^3$

Solution

$$\frac{1}{2}\log_y p^3q^4 - \frac{2}{3}\log_y p^4q^3 = \log_y \left(p^3q^4\right)^{1/2} - \log_y \left(p^4q^3\right)^{2/3}$$

$$= \log_y \frac{\left(p^3q^4\right)^{1/2}}{\left(p^4q^3\right)^{2/3}}$$

$$= \log_y \frac{\left(p^3\right)^{1/2}\left(q^4\right)^{1/2}}{\left(p^4\right)^{2/3}\left(q^3\right)^{2/3}}$$

$$= \log_y \frac{p^{3/2}q^2}{p^{8/3}q^2}$$

$$= \log_y \frac{p^{3/2}}{p^{8/3}}$$

$$= \log_y \frac{1}{p^{8/3-3/2}}$$

$$= \log_y \frac{1}{p^{7/6}}$$

Exercise

Write the expression as a single logarithm. $\frac{1}{2}\log_a x + 4\log_a y - 3\log_a x$

Solution

$$\frac{1}{2}\log_a x + 4\log_a y - 3\log_a x = 4\log_a y - \frac{5}{2}\log_a x$$

$$= \log_a y^4 - \log_a x^{5/2}$$

$$= \log_a y^4 - \log_a \sqrt{x^5}$$

Exercise

Write the expression as a single logarithm. $\frac{2}{3}\left[\ln\left(x^2-9\right)-\ln(x+3)\right]+\ln(x+y)$

Solution

$$\frac{2}{3}\left[\ln\left(x^2-9\right)-\ln(x+3)\right]+\ln(x+y)=\frac{2}{3}\ln\frac{x^2-9}{x+3}+\ln(x+y)$$

$$=\frac{2}{3}\ln\frac{(x+3)(x-3)}{x+3}+\ln(x+y)$$

$$=\frac{2}{3}\ln(x-3)+\ln(x+y)$$

$$=\ln(x-3)^{2/3}+\ln(x+y)$$

$$=\ln\left((x-3)^{2/3}(x+y)\right)$$

$$=\ln\left((x+y)\sqrt[3]{(x-3)^2}\right)$$

Exercise

Write the expression as a single logarithm. $\frac{1}{4}\log_b x-2\log_b 5-10\log_b y$

Solution

$$\frac{1}{4}\log_b x-2\log_b 5-10\log_b y=\log_b x^{1/4}-\log_b 5^2-\log_b y^{10}$$

$$=\log_b x^{1/4}-\left[\log_b 5^2+\log_b y^{10}\right]$$

$$=\log_b x^{1/4}-\left[\log_b\left(5^2y^{10}\right)\right]$$

$$=\log_b\frac{\sqrt[4]{x}}{5^2y^{10}}$$

Exercise

Assume that $\log_{10}2=.3010$. Find each logarithm $\log_{10}4$, $\log_{10}5$

Solution

a) $\log_{10}4=\log_{10}2^2$

$$=2\log_{10}2$$

$$=2(.301)$$

$$=.6020$$

b) $\log_{10} 5 = \log_{10} \frac{10}{2}$

$$= \log_{10} 10 - \log_{10} 2$$

$$= 1 - .03010$$

$$= .6990$$

Exercise

Given that: $\log_a 2 \approx 0.301, \log_a 7 \approx 0.845$, and $\log_a 11 \approx 1.041$ find each of the following:

$\log_a \frac{2}{11}$, $\log_a 14$, $\log_a 98$, $\log_a \frac{1}{7}$, $\log_a 9$

Solution

$$\log_a \frac{2}{11} = \log_a 2 - \log_a 11$$

$$= 0.301 - 1.041 \boxed{\approx -0.74}$$

$$\log_a 14 = \log_a 2(7) = \log_a 2 + \log_a 7 = 0.301 + 0.845 \approx 1.146$$

$$\log_a 98 = \log_a 2(7^2) = \log_a 2 + \log_a 7^2 = \log_a 2 + 2\log_a 7 = 0.301 + 2(0.845) \approx 1.991$$

$$\log_a \frac{1}{7} = \log_a 1 - \log_a 7 \approx 0 - 0.845 = -0.845$$

$\log_a 9$ Can’t be found from the given information

Solution Section 3.5 – Exponential and logarithmic Equations

Exercise

Solve: $2^{3x-7} = 32$

Solution

$2^{3x-7} = 32 = 2^5$

$3x - 7 = 5$ *add 7 on both sides*

$3x = 12$ *Divide by 3*

$x = 4$

Exercise

Solve $4^{2x-1} = 64$

Solution

$4^{2x-1} = 4^3$

$2x-1 = 3$

$2x = 4$

$x = 2$

Exercise

Solve $3^{1-x} = \frac{1}{27}$

Solution

$3^{1-x} = \frac{1}{3^3}$

$3^{1-x} = 3^{-3}$

$1 - x = -3$

$-x = -4$

$x = 4$

Exercise

Solve $\left(\frac{1}{3}\right)^x = 81$

Solution

$$\left(\frac{1}{3}\right)^x = 81$$

$$\left(3^{-1}\right)^x = 3^4$$

$$3^{-x} = 3^4$$

$$-x = 4$$

$$x = -4$$

Exercise

Solve: $5^x = 134$

Solution

$$\ln 5^x = \ln(134)$$

$$x \ln 5 = \ln(134)$$

$$x = \frac{\ln(134)}{\ln 5}$$

Exercise

Solve: $7^x = 12$

Solution

$$7^x = 12$$

$$\ln 7^x = \ln 12 \qquad \textit{Property of logarithm}$$

$$x \ln 7 = \ln 12 \qquad \textit{Power Rule}$$

$$x = \frac{\ln 12}{\ln 7}$$

$$\approx 1.277$$

Exercise

Solve $9^x = \frac{1}{\sqrt[3]{3}}$

Solution

$$\left(3^3\right)^x = \frac{1}{3^{1/3}}$$

$$3^{3x} = 3^{-1/3}$$

$$3x = -\frac{1}{3}$$

$$x = -\frac{1}{9}$$

Exercise

Solve $9e^x = 107$

Solution

$$e^x = \frac{107}{9}$$

$$\ln e^x = \ln\left(\frac{107}{9}\right)$$

$$x\ln e = \ln\left(\frac{107}{9}\right)$$

$$x = \ln\left(\frac{107}{9}\right)$$

Exercise

Solve $7^{2x+1} = 3^{x+2}$

Solution

$$\ln 7^{2x+1} = \ln 3^{x+2}$$

$$(2x+1)\ln 7 = (x+2)\ln 3$$

$$2x\ln 7 + \ln 7 = x\ln 3 + 2\ln 3$$

$$2x\ln 7 - x\ln 3 = 2\ln 3 - \ln 7$$

$$x(2\ln 7 - \ln 3) = 2\ln 3 - \ln 7$$

$$x = \frac{2\ln 3 - \ln 7}{2\ln 7 - \ln 3}$$

Exercise

Solve: $4^{x+3} = 3^{-x}$

Solution

$$\ln 4^{x+3} = \ln 3^{-x}$$

$$(x+3)\ln 4 = -x\ln 3$$

$$x\ln 4 + 3\ln 4 = -x\ln 3$$

$$x\ln 4 + x\ln 3 = -3\ln 4$$

$$x(\ln 4 + \ln 3) = -3\ln 4$$

$$x = \frac{-3\ln 4}{(\ln 4 + \ln 3)}$$

$$x \approx -1.6737$$

Exercise

Solve $2^{x+4} = 8^{x-6}$

Solution

$$2^{x+4} = \left(2^3\right)^{x-6}$$

$$2^{x+4} = 2^{3x-18}$$

$$x + 4 = 3x - 18$$

$$x + 4 - 3x - 4 = 3x - 18 - 3x - 4$$

$$-2x = -22$$

$$\boxed{x = 11}$$

Exercise

Solve $8^{x+2} = 4^{x-3}$

Solution

$$\left(2^3\right)^{x+2} = \left(2^2\right)^{x-3}$$

$$2^{3(x+2)} = 2^{2(x-3)}$$

$$3(x+2) = 2(x-3)$$

$$3x + 6 = 2x - 6$$

$$3x - 2x = -6 - 6$$

$$x = -12$$

Exercise

Solve $7^x = 12$

Solution

$\ln 7^x = \ln 12$ ***Property of logarithm***

$x \ln 7 = \ln 12$ ***Power Rule***

$x = \dfrac{\ln 12}{\ln 7} \approx 1.277$

Exercise

Solve: $5^{x+4} = 4^{x+5}$

Solution

$\ln 5^{x+4} = \ln 4^{x+5}$

$(x+4)\ln 5 = (x+5)\ln 4$

$x\ln 5 + 4\ln 5 = x\ln 4 + 5\ln 4$

$x\ln 5 - x\ln 4 = 5\ln 4 - 4\ln 5$

$x(\ln 5 - \ln 4) = 5\ln 4 - 4\ln 5$

$\boxed{x = \dfrac{5\ln 4 - 4\ln 5}{\ln 5 - \ln 4}}$

Exercise

Solve: $5^{x+2} = 4^{1-x}$

Solution

$\ln 5^{x+2} = \ln 4^{1-x}$

$(x+2)\ln 5 = (1-x)\ln 4$

$x\ln 5 + 2\ln 5 = \ln 4 - x\ln 4$

$x\ln 5 + x\ln 4 = \ln 4 - 2\ln 5$

$x(\ln 5 + \ln 4) = \ln 4 - 2\ln 5$

$\boxed{x = \dfrac{\ln 4 - 2\ln 5}{\ln 5 + \ln 4}} \approx -0.612$

Exercise

Solve: $27 = 3^{5x}9^{x^2}$

Solution

$$27 = 3^{5x}(3^2)^{x^2}$$

$$3^3 = 3^{5x}3^{2x^2}$$

$$3^3 = 3^{5x+2x^2}$$

$$3 = 5x + 2x^2$$

$$0 = 5x + 2x^2 - 3$$

$$2x^2 + 5x - 3 = 0$$

$$x = \frac{-(5) \pm \sqrt{5^2 - 4(2)(-3)}}{2(2)} = \frac{-5 \pm \sqrt{49}}{4} = \frac{-5 \pm 7}{4}$$

$$x = \begin{cases} \frac{-5+7}{4} = \frac{1}{2} \\ \frac{-5-7}{4} = -3 \end{cases}$$

Exercise

Solve: $3^{2x-1} = 0.4^{x+2}$

Solution

$$\ln 3^{2x-1} = \ln 0.4^{x+2}$$

$$(2x-1)\ln 3 = (x+2)\ln 0.4$$ ***Power Property***

$$2x\ln 3 - \ln 3 = x\ln 0.4 + 2\ln 0.4$$ ***Distributive property***

$$2x\ln 3 - x\ln 0.4 = 2\ln 0.4 + \ln 3$$

$$x(2\ln 3 - \ln 0.4) = 2\ln 0.4 + \ln 3$$

$$x = \frac{2\ln 0.4 + \ln 3}{2\ln 3 - \ln 0.4}$$ $(2\ln(0.4)+\ln(3))/(2\ln(3)-\ln(0.4))$

$$\approx -.236$$

Exercise

Solve: $4^{3x-5} = 16$

Solution

$$4^{3x-5} = 4^2$$

$$3x - 5 = 2$$

$$3x = 7$$

$$x = \frac{7}{3}$$

Exercise

Solve: $4^{x+3} = 3^{-x}$

Solution

$$\ln 4^{x+3} = \ln 3^{-x}$$

$$(x+3)\ln 4 = -x\ln 3$$

$$x\ln 4 + 3\ln 4 = -x\ln 3$$

$$x\ln 4 + x\ln 3 = -3\ln 4$$

$$x(\ln 4 + \ln 3) = -3\ln 4$$

$$\boxed{x = \frac{-3\ln 4}{(\ln 4 + \ln 3)}} \approx -1.6737$$

Exercise

Solve: $3^{x-1} = 7^{2x+5}$

Solution

$$\ln 3^{x-1} = \ln 7^{2x+5}$$

$$(x-1)\ln 3 = (2x+5)\ln 7$$

$$x\ln 3 - \ln 3 = 2x\ln 7 + 5\ln 7$$

$$x\ln 3 - 2x\ln 7 = \ln 3 + 5\ln 7$$

$$x(\ln 3 - 2\ln 7) = \ln 3 + 5\ln 7$$

$$x = \frac{\ln 3 + 5\ln 7}{\ln 3 - 2\ln 7}$$ $(\ln(3)+5\ln(7))/(\ln(3)-2\ln(7))$

$$\approx -3.8766$$

Exercise

Solve: $4^{x-2} = 2^{3x+3}$

Solution

$$\left(2^2\right)^{x-2} = 2^{3x+3}$$

$$2^{2x-4} = 2^{3x+3}$$

$$2x - 4 = 3x + 3$$

$$2x - 3x = 4 + 3$$

$$-x = 7$$

$$x = -7$$

Exercise

Solve: $2^{3x-7} = 32$

Solution

$$2^{3x-7} = 32 = 2^5$$

$$3x - 7 = 5$$

$$3x = 5+7 = 12$$

$$x = \frac{12}{3} = 4$$

Exercise

Solve: $3^{2x-1} = 0.4^{x+2}$

Solution

$$\ln 3^{2x-1} = \ln 0.4^{x+2}$$

$(2x-1)\ln 3 = (x+2)\ln 0.4$ *Power Property*

$2x\ln 3 - \ln 3 = x\ln 0.4 + 2\ln 0.4$ *Distributive property*

$$2x\ln 3 - x\ln 0.4 = 2\ln 0.4 + \ln 3$$

$$x(2\ln 3 - \ln 0.4) = 2\ln 0.4 + \ln 3$$

$x = \dfrac{2\ln 0.4 + \ln 3}{2\ln 3 - \ln 0.4}$ $(2\ln(0.4)+\ln(3))/(2\ln(3)-\ln(0.4))$

$$\approx -.236$$

Exercise

Solve $e^{2x} - 2e^{x} - 3 = 0$

Solution

$$U^2 - 2U - 3 = 0 \quad \Rightarrow \; U = -1, 3$$

$$\begin{cases} U = e^{\,x} = -1 \rightarrow Impossible \\ U = e^{\,x} = 3 \rightarrow ln e^{\,x} = ln3 \rightarrow \boxed{x = ln3} \end{cases}$$

Exercise

Solve: $e^{0.08t} = 2500$

Solution

$$\ln\left(e^{0.08t}\right) = \ln 2500$$

$$0.08t = \ln 2500$$

$$t = \frac{\ln 2500}{0.08} \approx 97.8$$

Exercise

Solve $e^{x^2} = 200$

Solution

$$\ln e^{x^2} = \ln 200 \qquad \textit{Natural Log both sides}$$

$$x^2 = \ln 200 \qquad \ln e = 1$$

$$x = \pm\sqrt{\ln 200}$$

$$\approx \pm 2.302$$

Exercise

Solve $e^{2x+1} \cdot e^{-4x} = 3e$

Solution

$$e^{2x+1-4x} = 3e$$

$$e^{-2x+1} = 3e$$

$$e^{-2x}e = 3e \qquad \textit{Divide by } e$$

$$e^{-2x} = 3$$

$$\ln e^{-2x} = \ln 3$$

$$-2x = \ln 3$$

$$x = -\frac{1}{2}\ln 3$$

$$\approx -.549$$

Exercise

Solve: $e^{2x} - 8e^{x} + 7 = 0$

Solution

$$\textit{Let } \ u = e^{x} \qquad\qquad \left(e^{x}-1\right)\left(e^{x}-7\right) = 0$$

$$u^2 - 8u + 7 = 0$$

$$\Rightarrow u = e^{x} = 1 \qquad\qquad \Rightarrow u = e^{x} = 7$$

$$\textit{ln}e^{x} = \textit{ln}1 \qquad\qquad \ln e^{x} = \ln 7$$

$$\Rightarrow x = 0 \qquad\qquad \Rightarrow x = \textit{ln}7$$

Exercise

Solve: $e^{x} + e^{-x} - 6 = 0$

Solution

$$e^{x}e^{x} + e^{x}e^{-x} - e^{x}6 = e^{x}0$$

$$e^{2x} + 1 - 6e^{x} = 0$$

$$e^{2x} - 6e^{x} + 1 = 0$$

$$u^2 - 6u + 1 = 0$$

$$u = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(1)}}{2(1)} = \frac{6 \pm \sqrt{36-4}}{2} = \frac{6 \pm \sqrt{32}}{2} = \frac{6 \pm 4\sqrt{2}}{2}$$

$$u = 3 \pm 2\sqrt{2}$$

$$e^{x} = 3 \pm 2\sqrt{2} \Rightarrow \ln(e^{x}) = \ln\left(3 \pm 2\sqrt{2}\right)$$

$$\Rightarrow x\ln(e) = \ln\left(3 \pm 2\sqrt{2}\right)$$

$$\Rightarrow \boxed{x = \ln\left(3 \pm 2\sqrt{2}\right)} \qquad \text{or} \pm 1.76$$

Exercise

Solve: $e^{1-3x} \cdot e^{5x} = 2e$

Solution

$$e^{1-3x+5x} = 2e$$

$$e^{1+2x} = 2e$$

$$e^{1}e^{2x} = 2e$$ *Divide by e*

$$e^{2x} = 2$$ *Natural Log both sides*

$$\ln e^{2x} = \ln 2$$

$$2x = \ln 2$$

$$\underline{\left| x = \frac{1}{2}\ln 2 \approx 0.3456 \right|}$$

Exercise

Solve $6\ln(2x) = 30$

Solution

$$\ln(2x) = \frac{30}{6}$$

$$\ln(2x) = 5$$

$$2x = e^{5}$$

$$x = \frac{1}{2}e^{5}$$

Exercise

Solve $\log_5(x-7) = 2$

Solution

$$x - 7 = 5^{2}$$

$$x = 25 + 7$$

$$x = 32$$

Exercise

Solve $\log_5 x + \log_5 (4x-1) = 1$

Solution

$$\log_5 x(4x-1) = 1$$

$$x(4x-1) = 5^1$$

$$4x^2 - x = 5$$

$$4x^2 - x - 5 = 0$$

$$\rightarrow \begin{cases} x = -1 \\ x = \frac{5}{4} \end{cases} \rightarrow \textit{Check} \quad x = \frac{5}{4} \quad \textit{only solution}$$

Exercise

Solve: $\log x + \log(x-3) = 1$

Solution

$$\log[x(x-3)] = 1$$

$$x(x-3) = 10^1 = 10$$

$$x^2 - 3x = 10$$

$$x^2 - 3x - 10 = 0$$

$$(x+2)(x-5) = 0$$

$$\Rightarrow x = -2, 5$$

Check: $x = -2 \Rightarrow \cancel{\log(-2) + \log(x-3) = 1}$

$x = 5 \Rightarrow \log(5) + \log(5-3) = 1$

Exercise

Solve: $\log x - \log(x+3) = 1$

Solution

$$\log \frac{x}{x+3} = 1$$

$$\frac{x}{x+3} = 10^1 = 10$$

$$x = 10x + 30$$

$$9x = -30$$

$x = -\frac{10}{3}$ ***No Solution***

Exercise

Solve: $\log_3 x = -2$

Solution

$x = 3^{-2}$ *Convert to exponential*

$x = \frac{1}{3^2}$

$x = \frac{1}{9}$

Exercise

Solve: $\log(3x+2) + \log(x-1) = 1$

Solution

$\log(3x+2) + \log(x-1) = 1$ *Product Rule*

$\log\left[(3x+2)(x-1)\right] = 1$ *Convert to exponential form*

$(3x+2)(x-1) = 10^1$

$3x^2 - x - 2 = 10$

$3x^2 - x - 12 = 0$ *Solve for x*

$x = \frac{1-\sqrt{145}}{6} < 0$ $x = \frac{1+\sqrt{145}}{6} > 1$

Solution: $\boxed{x = \frac{1+\sqrt{145}}{6}}$

Exercise

Solve: $\log_5(x+2) + \log_5(x-2) = 1$

Solution

$\log_5\left[(x+2)(x-2)\right] = 1$

$(x+2)(x-2) = 5^1$

$x^2 - 4 = 5$

$x^2 = 5 + 4$

$x^2 = 9$

$x = \pm 3$ $\log_5\left[(-3)+2\right] + \log_5\left[(-3)-2\right] = 1$

$\log_5\left[(3)+2\right] + \log_5\left[(3)-2\right] = 1$

Solution: $\boxed{x = 3}$

Exercise

Solve: $\log x + \log(x-9) = 1$

Solution

$\log x(x-9) = 1$

$x(x-9) = 10^1$

$x^2 - 9x - 10 = 0$

$\Rightarrow x$ = -1 (Check; it is not a solution)

$\Rightarrow x$ = 10 (only solution)

Exercise

Solve: $\log_2(x+1) + \log_2(x-1) = 3$

Solution

$\log_2(x+1)(x-1) = 3$

$x^2 - 1 = 2^3$

$x^2 = 8 + 1 = 9 \Rightarrow x = \pm 3$

Check: $x = -3 \ \rightarrow \log_2(-3+1) + \log_2(-3-1) = 3 \quad \Rightarrow$ *It is* not a *Solution*

$x = 3 \ \rightarrow \log_2(3+1) + \log_2(3-1) = 3 \quad \Rightarrow$ *Solution*

Exercise

Solve: $\log_8(x+1) - \log_8 x = 2$

Solution

$\log_8\left(\frac{x+1}{x}\right) = 2$

$\frac{x+1}{x} = 8^2 = 64$

$x + 1 = 64x$

$1 = 63x$

$\boxed{x = \frac{1}{63}}$

Exercise

Solve: $\log(x+6)-\log(x+2)=\log x$

Solution

$\log(x+6)-\log(x+2)=\log x$ ***Quotient Rule***

$\log\frac{x+6}{x+2}=\log x$

$\frac{x+6}{x+2}=x$ ***Multiply by x + 2***

$x+6=x(x+2)$

$x+6=x^2+2x$

$0=x^2+2x-x-6$

$x^2+x-6=0$ ***Solve for x***

$x=-3,\ 2$

Check: $x=-3\to\log(-3+6)-\log(-3+2)=\log(-3)$ *Or* ***Domain***

$x=2\to\log(2+6)-\log(2+2)=\log(2)$

Solution: $x=2$

Exercise

Solve: $\ln(x+8)+\ln(x-1)=2\ln x$

Solution

$\ln[(x+8)(x-1)]=\ln x^2$

$(x+8)(x-1)=x^2$

$x^2-x+8x-8=x^2$

$x^2-x+8x-8-x^2=0$

$7x-8=0$

$7x=8$

$\boxed{x=\frac{8}{7}}$ Check: $\ln(\frac{8}{7}+8)+\ln(\frac{8}{7}-1)=2\ln\frac{8}{7}$

Exercise

Solve: $\ln(4x+6)-\ln(x+5)=\ln x$

Solution

$$\ln\left(\frac{4x+6}{x+5}\right)=\ln x$$

$$\frac{4x+6}{x+5}=x$$

$$4x+6=x(x+5)$$

$$4x+6=x^2+5x$$

$$0=x^2+5x-4x-6$$

$$0=x^2+x-6$$

$$x^2+x-6=0$$

$$(x+3)(x-2)=0$$

$$\Rightarrow x=-3,\ 2$$

Check: x = -3 *no solution* $\ln(4x+6)-\ln(x+5)=\ln(-3)$

x = 2 *(only solution)*

Exercise

Solve: $\ln(5+4x)-\ln(x+3)=\ln 3$

Solution

$$\ln\frac{5+4x}{x+3}=\ln 3$$

$$\frac{5+4x}{x+3}=3$$

$$5+4x=3(x+3)$$

$$5+4x=3x+9$$

$$4x-3x=9-5$$

$$x=4$$

Check: $\ln(5+4(4))-\ln((4)+3)=\ln 3$

Solution: $x=4$

Exercise

Solve $\ln(x-5)-\ln(x+4)=\ln(x-1)-\ln(x+2)$

Solution

$$\ln\frac{x-5}{x+4}=\ln\frac{x-1}{x+2}$$

$$\frac{x-5}{x+4}=\frac{x-1}{x+2}$$

$$(x-5)(x+2)=(x-1)(x+4)$$

$$x^2+2x-5x-10=x^2+4x-x-4$$

$$x^2-3x-10=x^2+3x-4$$

$$x^2-3x-10-x^2-3x+4=0$$

$$-6x-6=0$$

$$-6x=6\Rightarrow x=-1\ \textit{No solution}$$

Exercise

Solve $\ln(x-3)=\ln(7x-23)-\ln(x+1)$

Solution

$$\ln(x-3)=\ln\left(\frac{7x-23}{x+1}\right)$$

$$x-3=\frac{7x-23}{x+1}$$

$$(x-3)(x+1)=7x-23$$

$$x^2-2x-3=7x-23$$

$$x^2-9x+20=0$$

$$\Rightarrow x=4,5$$

Check: $x=4 \quad \Rightarrow \ln(4-3)=\ln(7(4)-23)-\ln(4+1)$

$x=5 \quad \Rightarrow \ln(5-3)=\ln(7(5)-23)-\ln(5+1)$

$\boxed{x=4,\ 5}$

Exercise

How long, to the nearest tenth of a year, will it take \$1000 to grow to \$3600 at 8% annual interest compounded quarterly?

Solution

Given: $A = \$3600$

$P = \$1000$

$r = 8\% = 0.08$

$n = 4$

$$\Rightarrow A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$3600 = 1000\left(1+\frac{0.08}{4}\right)^{4t}$$

$$3.6 = (1.02)^{4t}$$

$$\ln 3.6 = \ln(1.02)^{4t}$$

$$\ln 3.6 = 4t\ln(1.02)$$

$$\frac{\ln 3.6}{4\ln 1.02} = t$$

$$\boxed{t \approx 16.2\, yr}$$

Exercise

Solve: $27 = 3^{5x}9^{x^2}$

Solution

$$27 = 3^{5x}(3^2)^{x^2}$$

$$3^3 = 3^{5x}3^{2x^2}$$

$$3^3 = 3^{5x+2x^2}$$

$$3 = 5x + 2x^2$$

$$0 = 5x + 2x^2 - 3$$

$$2x^2 + 5x - 3 = 0$$

$$x = \frac{-(5) \pm \sqrt{5^2 - 4(2)(-3)}}{2(2)} = \frac{-5 \pm \sqrt{49}}{4} = \frac{-5 \pm 7}{4} = \begin{cases} \frac{-5+7}{4} = \frac{1}{2} \\ \frac{-5-7}{4} = -3 \end{cases}$$

Exercise

Solve: $ln\sqrt[4]{x} = \sqrt{lnx}$

Solution

$$lnx^{1/4} = \sqrt{lnx}$$

$$\frac{1}{4}lnx = \sqrt{lnx}$$

$$\left(\frac{1}{4}lnx\right)^2 = \left(\sqrt{lnx}\right)^2$$

$$\frac{1}{6}ln^2x = lnx$$

$$ln^2x = 6lnx$$

$$ln^2x - 6lnx = 0$$

$$lnx(lnx-6) = 0$$

$$\begin{cases} lnx = 0 \to x = 1 \\ lnx - 6 = 0 \Rightarrow \ln x = 6 \to x = e^6 \end{cases}$$

Exercise

Solve: $\sqrt{lnx} = ln\sqrt{x}$

Solution

$$\sqrt{lnx} = lnx^{1/2}$$

$$\sqrt{lnx} = \frac{1}{2}lnx$$

$$\left(\sqrt{lnx}\right)^2 = \left(\frac{1}{2}lnx\right)^2$$

$$lnx = \frac{1}{4}ln^2x$$

$$4lnx = ln^2x$$

$$ln^2x - 4lnx = 0$$

$$lnx(lnx-4) = 0$$

$$\begin{cases} lnx = 0 \to x = 1 \\ lnx - 4 = 0 \Rightarrow \ln x = 4 \to x = e^4 \end{cases}$$

Exercise

Solve the equation: $7^{x+6} = 7^{3x-4}$

Solution

$$x + 6 = 3x - 4$$
$$4 + 6 = 3x - x$$
$$10 = 2x$$
$$\boxed{x = 5}$$

Exercise

Solve the equation: $2^{-100x} = (0.5)^{x-4}$

Solution

$$2^{-100x} = \left(\frac{1}{2}\right)^{x-4}$$
$$2^{-100x} = \left(2^{-1}\right)^{x-4}$$
$$2^{-100x} = 2^{-x+4}$$
$$-100x = -x + 4$$
$$-100x + x = 4$$
$$-99x = 4$$
$$\boxed{x = -\frac{4}{99}}$$

Exercise

Solve the equation: $4^x\left(\frac{1}{2}\right)^{3-2x} = 8.\left(2^x\right)^2$

Solution

$$\left(2^2\right)^x\left(2^{-1}\right)^{3-2x} = 2^3.2^{2x}$$
$$2^{2x}2^{2x-3} = 2^{3+2x}$$
$$2^{2x+2x-3} = 2^{3+2x}$$
$$2^{4x-3} = 2^{3+2x}$$
$$4x - 3 = 3 + 2x$$
$$4x - 2x = 3 + 3$$
$$2x = 6$$
$$\boxed{x = 3}$$

Exercise

Solve the equation: $5^{3x-6} = 125$

Solution

$$5^{3x-6} = 5^3$$

$$3x - 6 = 3$$

$$3x = 9$$

$$\Rightarrow x = 3$$

Exercise

Solve the equation $e^{x^2} = e^{7x-12}$

Solution

$$e^{x^2} = e^{7x-12}$$

$$x^2 = 7x - 12$$

$$x^2 - 7x + 12 = 0$$

$$\boxed{x = 3,\ 4}$$

Exercise

Solve the equation $f(x) = xe^x + e^x$

Solution

$$xe^x + e^x = 0$$

$$e^x(x+1) = 0$$

$$e^x = 0 \qquad x + 1 = 0$$

$$\boxed{x = -1} \text{ (Only solution)}$$

Exercise

Solve for t using logarithms with base a: $2a^{t/3} = 5$

Solution

$$a^{t/3} = \frac{5}{2}$$

$$\log a^{t/3} = \log \frac{5}{2}$$

$$\frac{t}{3}\log a = \log \frac{5}{2}$$

$$\frac{t}{3} = \frac{\log \frac{5}{2}}{\log a}$$

$$\frac{t}{3} = \log_a \frac{5}{2}$$

$$\boxed{t = 3\log_a \frac{5}{2}}$$

Exercise

Solve for t using logarithms with base a: $K = H - Ca^t$

Solution

$$Ca^t = H - K$$

$$a^t = \frac{H-K}{C}$$

$$\log a^t = \log \frac{H-K}{C}$$

$$t\log a = \log \frac{H-K}{C}$$

$$t = \frac{\log \frac{H-K}{C}}{\log a} = \log_a \frac{H-K}{C}$$

Exercise

Solve the equation: $\log_4 x = \log_4 (8-x)$

Solution

$$x = 8 - x$$

$$x + x = 8$$

$$2x = 8 \to x = 4$$

Check: $\boxed{x = 4}$

Exercise

Solve the equation: $\log_7 (x-5) = \log_7 (6x)$

Solution

$$x-5=6x$$
$$x-6x=5$$
$$-5x=5$$
$$x=-1$$

Check: $\log_7 (-1-5) = \log_7 (6(-1))$

No solution (no negative inside the log)

Exercise

Solve the equation: $\ln x^2 = \ln(12-x)$

Solution

$$\ln x^2 = \ln(12-x)$$
$$x^2 = 12-x$$
$$x^2+x-12=0 \quad \rightarrow \quad x=-4,\ 3$$

Check: $x=-4 \quad \Rightarrow \quad \ln(-4)^2 = \ln(12+4)$

$x=3 \quad \Rightarrow \quad \ln(3)^2 = \ln(12-3)$

The solutions are: $\boxed{x=-4,\ 3}$

Exercise

Solve the equation: $e^{x\ln 3} = 27$

Solution

$$\ln e^{x\ln 3} = \ln 27$$
$$x\ln 3 = \ln 3^3$$
$$x = \frac{3\ln 3}{\ln 3} = 3$$

Exercise

Solve the equation: $\log_4 x = \log_4 (8-x)$

Solution

$$x = 8 - x$$

$$x + x = 8$$

$$2x = 8 \rightarrow x = 4$$

Check: $\boxed{x = 4}$

Exercise

Solve the equation: $\log_7 (x-5) = \log_7 (6x)$

Solution

$$x - 5 = 6x$$
$$x - 6x = 5$$
$$-5x = 5$$
$$x = -1$$

Check: $\log_7 (-1-5) = \log_7 (6(-1))$

No solution (no negative inside the log)

Exercise

Solve the equation: $\ln x^2 = \ln(12 - x)$

Solution

$$\ln x^2 = \ln(12 - x)$$

$$x^2 = 12 - x$$

$$x^2 + x - 12 = 0 \quad \rightarrow \quad x = -4,\ 3$$

Check: $x = -4 \quad \Rightarrow \quad \ln(-4)^2 = \ln(12+4)$

$x = 3 \quad \Rightarrow \quad \ln(3)^2 = \ln(12-3)$

The solutions are: $\boxed{x = -4,\ 3}$

Exercise

Solve the equation: $e^{x\ln 3} = 27$

Solution

$$\ln e^{x\ln 3} = \ln 27$$

$$x\ln 3 = \ln 3^3$$

$$x = \frac{3\ln 3}{\ln 3} \underline{= 3|}$$

Exercise

Solve the equation $\log_6(2x-3) = \log_6 12 - \log_6 3$

Solution

$$\log_6(2x-3) = \log_6 \frac{12}{3}$$

$$\log_6(2x-3) = \log_6 4$$

$$2x - 3 = 4$$

$$2x = 7$$

$$\boxed{x = \frac{7}{2}}$$ ***Check***

Exercise

Solve the equation $\ln(-4-x) + \ln 3 = \ln(2-x)$

Solution

$$\ln 3(-4-x) = \ln(2-x)$$

$$-12 - 3x = 2 - x$$

$$-12 - 2 = 3x - x$$

$$-14 = 2x$$

$$x = -7$$

Check: $\ln(-4-(-7)) + \ln 3 = \ln(2-(-7))$

$$\ln(3) + \ln 3 = \ln(9)$$

$$\ln 3(3) = \ln(9)$$

The solution is $\boxed{x = -7}$

Exercise

Solve the equation $\log_2(x+7)+\log_2 x=3$

Solution

$$\log_2 x(x+7)=3$$

$$x(x+7)=2^3 \qquad \textit{Convert to Exponential Form}$$

$$x^2+7x=8$$

$$x^2+7x-8=0 \quad \Rightarrow \quad x=1, \ -8$$

Check: $x=1 \Rightarrow \quad \log_2(1+7)+\log_2 1=3 \to \log_2 8=3$

$$x=-8 \Rightarrow \quad \log_2(-8+7)+\cancel{\log_2(-8)}=3$$

The solution is $\boxed{x=1}$

Exercise

Solve the equation $\log_3(x+3)+\log_3(x+5)=1$

Solution

$$\log_3(x+3)(x+5)=1$$

$$x^2+3x+5x+15=3^1 \qquad \textit{Convert to Exponential Form}$$

$$x^2+8x+15-3=0$$

$$x^2+8x+12=0$$

$$x=-2, \ -6$$

Check: $x=-2 \Rightarrow \quad \log_3(-2+3)+\log_3(-2+5)=1$

$$\log_3(1)+\log_3(3)=1$$

$$x=-6 \Rightarrow \quad \log_3(-6+3)+\log_3(-6+5)=1$$

$$\cancel{\log_3(-3)}+\log_3(-1)=1$$

The solution is $\boxed{x=-2}$

Exercise

Solve the equation $\ln x = 1 - \ln(x+2)$

Solution

$$\ln x + \ln(x+2) = 1$$

$$\ln x(x+2) = 1$$

$$x^2 + 2x = e^1$$ ***Convert to Exponential Form***

$$x^2 + 2x - e = 0$$

$$x = \frac{-2 \pm \sqrt{4+4e}}{2} = \frac{-2 \pm 2\sqrt{1+e}}{2} = \begin{cases} -1 - \sqrt{1+e} < 0 \\ -1 + \sqrt{1+e} = 0.923 \end{cases}$$

The solution is $\boxed{x = -1 + \sqrt{1+e}}$

Exercise

Solve the equation $\ln x = 1 + \ln(x+1)$

Solution

$$\ln x - \ln(x+1) = 1$$

$$\ln \frac{x}{x+1} = 1$$

$$\frac{x}{x+1} = e^1$$

$$x = (x+1)e$$

$$x = ex + e$$

$$x - ex = e$$

$$x(1-e) = e$$

$$x = \frac{e}{1-e} < 0$$

No solution

Exercise

Solve the equation $\log_3(x-2) = \log_3 27 - \log_3(x-4) - 5^{\log_5 1}$

Solution

$$\log_3(x-2) + \log_3(x-4) = \log_3 3^3 - 1$$

$$\log_3(x-2)(x-4) = 3-1$$

$$\log_3\left(x^2 - 6x + 8\right) = 2$$

$$x^2 - 6x + 8 = 3^2$$

$$x^2 - 6x + 8 = 9$$

$$x^2 - 6x - 1 = 0$$

$$x = 3+\sqrt{10} \qquad x = 3-\sqrt{10}$$

Check: $x = 3+\sqrt{10} \quad \Rightarrow \log_3\left(3+\sqrt{10}-2\right) = \log_3 27 - \log_3\left(3+\sqrt{10}-4\right) - 5^{\log_5 1}$

$x = 3+\sqrt{10} \quad \Rightarrow \cancel{\log_3\left(3-\sqrt{10}-2\right)} = \log_3 27 - \log_3\left(3-\sqrt{10}-4\right) - 5^{\log_5 1}$

The solution is $\boxed{x = 3+\sqrt{10}}$

Exercise

Solve the equation $\log_2(x+3) = \log_2(x-3) + \log_3 9 + 4^{\log_4 3}$

Solution

$$\log_2(x+3) - \log_2(x-3) = 2+3$$

$$\log_2 \frac{x+3}{x-3} = 5$$

$$\frac{x+3}{x-3} = 2^5$$

$$x+3 = 32(x-3)$$

$$x+3 = 32x - 96$$

$$96 + 3 = 32x - x$$

$$31x = 99$$

$x = \frac{99}{31} > 3$ ***Domain***: $x > 3$

The solution is: $\boxed{x = \frac{99}{31}}$

Exercise

Find the exact solution (2-decimal place approximation): $3^{x+4}=2^{1-3x}$

Solution

$\ln 3^{x+4}=\ln 2^{1-3x}$ ***'ln' both sides***

$(x+4)\ln 3=(1-3x)\ln 2$ ***Power Rule***

$x\ln 3+4\ln 3=\ln 2-3x\ln 2$ ***Distribute***

$x\ln 3+3x\ln 2=\ln 2-4\ln 3$

$x(\ln 3+3\ln 2)=\ln 2-4\ln 3$

$$\underline{x=\frac{\ln 2-4\ln 3}{\ln 3+3\ln 2}\approx -1.16}$$

Exercise

Find the exact solution (2-decimal place approximation): $3^{2-3x}=4^{2x+1}$

Solution

$\ln 3^{2-3x}=\ln 4^{2x+1}$ ***'ln' both sides***

$(2-3x)\ln 3=(2x+1)\ln 4$ ***Power Rule***

$2\ln 3-3x\ln 3=2x\ln 4+\ln 4$

$-3x\ln 3-2x\ln 4=\ln 4-2\ln 3$

$-x(3\ln 3+2\ln 4)=\ln 4-2\ln 3$

$$x=-\frac{\ln 4-2\ln 3}{3\ln 3+2\ln 4}$$

$$=-\frac{\ln 4-\ln 3^2}{\ln 3^3+\ln 4^2}$$

$$=\frac{\ln 9-\ln 4}{\ln 27+\ln 16}$$

$$=\frac{\ln\frac{9}{4}}{\ln 432}$$

$$\underline{\approx 0.13}$$

Exercise

Find the exact solution (2-decimal place approximation): $2^{-x^2} = 5$

Solution

$$\ln 2^{-x^2} = \ln 5$$

$$-x^2 \ln 2 = \ln 5$$

$$x^2 = -\frac{\ln 5}{\ln 2} \quad \Rightarrow \quad \textit{No Solution}$$

Exercise

Find the exact solution (2-decimal place approximation): $2^{-x} = 8$

Solution

$$2^{-x} = 2^3$$

$$-x = 3$$

$$\boxed{x = -3}$$

Exercise

Find the exact solution (2-decimal place approximation): $\log\left(x^2+4\right) - \log(x+2) = 2 + \log(x-2)$

Solution

$$\log\left(x^2+4\right) - \log(x+2) - \log(x-2) = 2$$

$$\log\left(x^2+4\right) - \left[\log(x+2) + \log(x-2)\right] = 2$$

$$\log\left(x^2+4\right) - \log(x+2)(x-2) = 2$$

$$\log\left(\frac{x^2+4}{x^2-4}\right) = 2$$

$$\frac{x^2+4}{x^2-4} = 10^2$$

$$x^2 + 4 = 100x^2 - 400$$

$$400 + 4 = 100x^2 - x^2$$

$$99x^2 = 404$$

$$x^2 = \frac{404}{99}$$

$$x = \pm\sqrt{\frac{404}{99}} \approx \pm 2.02 \qquad \boxed{x = 2.02} \text{ is the only solution}$$

Exercise

Find the exact solution (2-decimal place approximation): $5^x + 125\left(5^{-x}\right) = 30$

Solution

$$5^x 5^x + 125\left(5^{-x}\right)5^x = 30\left(5^x\right)$$

$$5^{2x} + 125 = 30\left(5^x\right)$$

$$5^{2x} - 30\left(5^x\right) + 125 = 0 \qquad \textit{Solve for } 5^x$$

$$5^x = 5 \qquad 5^x = 25 = 5^2$$

$$x = 1 \qquad x = 2$$

$$\boxed{x = 1,\ 2}$$

Exercise

Find the exact solution (2-decimal place approximation): $4^x - 3\left(4^{-x}\right) = 8$

Solution

$$4^x 4^x - 3\left(4^{-x}\right)4^x = 8\left(4^x\right)$$

$$4^{2x} - 3 = 8\left(4^x\right)$$

$$4^{2x} - 8\left(4^x\right) - 3 = 0 \qquad \textit{Solve for } 4^x$$

$$4^x = 4 + \sqrt{19} \qquad 4^x = 4 - \sqrt{19} < 0$$

$$x \ln 4 = \ln\left(4 + \sqrt{19}\right)$$

$$x = \frac{\ln\left(4+\sqrt{19}\right)}{\ln 4} \approx 1.53$$

Exercise

Solve the equation without using the calculator: $\log x^2 = (\log x)^2$

Solution

$$2\log x = (\log x)^2$$

$$(\log x)^2 - 2\log x = 0$$

$$\log x(\log x - 2) = 0$$

$$\log x = 0 \qquad\qquad \log x - 2 = 0$$

$$x = 1 \qquad\qquad \log x = 2$$

$$x = 10^2 = 100$$

$$\boxed{x = 1,\ 100}$$

Exercise

Solve the equation without using the calculator: $\log(\log x) = 2$

Solution

$\log x = 10^2$ ***Convert to exponential***

$$\boxed{x = 10^{100}}$$

Exercise

Solve the equation without using the calculator: $\log\sqrt{x^3 - 9} = 2$

Solution

$\sqrt{x^3 - 9} = 10^2$ ***Convert to exponential***

$$\left(\sqrt{x^3 - 9}\right)^2 = (100)^2$$

$$x^3 - 9 = 10000$$

$$x^3 = 10009$$

$$\boxed{x = \sqrt[3]{10,009}}$$

Exercise

Solve the equation without using the calculator: $e^{2x} + 2e^x - 15 = 0$

Solution

$\left(e^x\right)^2 + 2e^x - 15 = 0$ ***Solve for*** e^x

$$e^x = 3 \qquad\qquad e^x \cancel{=} -5 < 0$$

$$\boxed{x = \ln 3}$$

Exercise

Solve the equation: $\log_3 x - \log_9 (x+42) = 0$

Solution

$$\frac{\ln x}{\ln 3} - \frac{\ln(x+42)}{\ln 9} = 0$$

$$\frac{\ln x}{\ln 3} - \frac{\ln(x+42)}{\ln 3^2} = 0$$

$$\frac{\ln x}{\ln 3} - \frac{\ln(x+42)}{2\ln 3} = 0$$

$$\frac{\ln x}{\ln 3} - \frac{1}{2}\frac{\ln(x+42)}{\ln 3} = 0$$

$$\frac{\ln x - \ln(x+42)^{1/2}}{\ln 3} = 0$$

$$\ln x - \ln(x+42)^{1/2} = 0$$

$$\ln x = \ln(x+42)^{1/2}$$

$$x = (x+42)^{1/2}$$

$$(x)^2 = \left((x+42)^{1/2}\right)^2$$

$$x^2 = x+42$$

$$x^2 - x - 42 = 0 \Rightarrow x = -6,\ 7$$

The solution: $\boxed{x=7}$

Exercise

Solve the equation $f(x) = x^3\left(4e^{4x}\right) + 3x^2e^{4x}$

Solution

$$x^3\left(4e^{4x}\right) + 3x^2e^{4x} = 0$$

$$x^2e^{4x}(4x+3) = 0$$

$$x^2 = 0 \qquad 4x+3=0$$

$$x = 0,\ 0 \qquad x = -\frac{3}{4}$$

The solutions are: $\boxed{x = 0,\ 0,\ -\frac{3}{4}}$

Solution Section 3.6 – Exponential Growth and Decay

Exercise

Suppose that \$10,000 is invested at interest rate of 5.4% per year, compounded continuously.

a) Find the exponential growth function

b) What will the balance be after, 1 yr 10 yrs?

c) What will the balance be after, 1 yr 10 yrs?

Solution

a) Find the exponential growth function

$$P(t) = 10000e^{0.054t}$$

b) What will the balance be after, 1 yr 10 yr?

$$P(t=1) = 10000e^{0.054(1)} \approx \$10,555$$

$$P(t=10) = 10000e^{0.054(10)} \approx \$17,160$$

c) What is the doubling time?

$$T = \frac{\ln 2}{k} = \frac{\ln 2}{0.054} \approx 12.8\ yr$$

Exercise

In 1990, the population of Africa was 643 million and by 2000 it had grown to 813 million

a. Use the exponential growth function $A(t) = A_0 e^{kt}$, in which t is the number of years after 1990, to find the exponential growth function that models data

b. By which year will Africa's population reach 2000 million, or two billion?

Solution

a. $A(t) = A_0 e^{kt}$ From 1990 to 2000, is 10 years

$$813 = 643e^{k(10)}$$

$$\frac{813}{643} = e^{10k}$$

$$\ln\frac{813}{643} = \ln e^{10k}$$

$$\ln\frac{813}{643} = 10k$$

$$\frac{1}{10}\ln\frac{813}{643} = k \qquad k \approx 0.023 \qquad \Rightarrow A(t) = 643e^{0.023t}$$

b. $2000 = 643e^{0.023t}$

$$\frac{2000}{643} = e^{0.023t}$$

$$\ln\frac{2000}{643} = \ln e^{0.023t}$$

$$\ln\frac{2000}{643} = 0.023t$$

$$\frac{\ln\frac{2000}{643}}{0.023} = t \qquad \Rightarrow t \approx 49 \quad \rightarrow Year: \; 2039$$

Exercise

The radioactive element carbon-14 has a half-life of 5750 yr. The percentage of carbon-14 present in the remains of organic matter can be used to determine the age of that organic matter. Archaeologists discovered that the linen wrapping from one of the Dead Sea Scrolls had lost 22.3% of its carbon-14 at the time it was found. How old was the linen wrapping?

Solution

When t = 5750 (half-life) $\rightarrow$ P(t) will be half P_0 $\rightarrow P(t) = \frac{1}{2}P_0$

$$P(t) = P_0 e^{-kt}$$

$$\frac{1}{2}P_0 = P_0 e^{-k(5750)}$$

$$\frac{1}{2} = e^{-k(5750)}$$

$$\ln\frac{1}{2} = \ln e^{-k(5750)}$$

$$\ln\frac{1}{2} = -k(5750)$$

$$k = -\frac{\ln\frac{1}{2}}{5750} \approx 0.00012$$

$$P(t) = P_0 e^{-(0.00012)t}$$

Lost 22.3% $\Rightarrow$ 100 – 22.3 = 77.7% left from it is original.

$$0.777P_0 = P_0 e^{-0.00012t}$$

$$0.777 = e^{-0.00012t}$$

$$\ln 0.777 = \ln e^{-0.00012t}$$

$$\ln 0.777 = -0.00012t$$

$$-0.00012t = \ln 0.777$$

$$t = \frac{\ln 0.777}{-0.00012} \approx 2103$$

Exercise

Suppose that \$2000 is invested at interest rate $\boldsymbol{k}$, compounded continuously, and grows to \$2983.65 in 5 yrs.

a) What is the interest rate?

b) Find the exponential growth function

c) What will the balance be after 10 yrs?

d) After how long will the \$2000 have doubled?

Solution

a) $P(t) = P_0 e^{kt}$

$$P(t=5) = P_0 e^{k5} = 2983.65$$

$$2000e^{k5} = 2983.65$$

$$e^{k5} = \frac{2983.65}{2000}$$

$$\ln e^{k5} = \ln\left(\frac{2983.65}{2000}\right)$$

$$5k \ln e = \ln\left(\frac{2983.65}{2000}\right)$$

$$k = \frac{1}{5}\ln\left(\frac{2983.65}{2000}\right) \approx 0.08$$

or k = 8%

b) $P(t) = 2000e^{0.08t}$

c) $P(t=10) = 2000e^{0.08(10)} \approx \$4451.08$

d) $T = \frac{\ln 2}{k} = \frac{\ln 2}{0.08} \approx 8.7\ yr$

Exercise

In 2005, the population of China was about 1.306 billion, and the exponential growth rate was 0.6% per year.

a) Find the exponential growth function

b) Estimate the population in 2008

c) After how long will the population be double what it was in 2005?

Solution

a) In 2005 $\Rightarrow$ t = 0

$$k = \frac{0.6}{100} = 0.006$$

P_0: 1.306

$$P(t) = 1.306e^{0.006t}$$

b) $P(t = 3) = 1.306e^{0.006(3)} \approx 1.33$

c) $2(1.306) = 1.306e^{0.006t}$

$$2 = e^{0.006t}$$

$$e^{0.006t} = 2$$

$$\ln e^{0.006t} = \ln 2$$

$$0.006t = \ln 2$$

$$t = \frac{\ln 2}{0.006} \approx 116$$

Exercise

How long will it take for the money in an account that is compounded continuously at 3% interest to double?

Solution

$$T = \frac{\ln 2}{k}$$

$$= \frac{\ln 2}{0.03}$$

$$\approx 23\ yr$$

Exercise

If 600 g of radioactive substance are present initially and 3 *yr* later only 300 g remain, how much of the substance will be present after 6 *yr*?

Solution

$$y(t) = y_0 e^{kt}$$

$$y(t) = 600e^{kt}$$

When $t = 3 \to y = 300$

$$300 = 600e^{k(3)}$$

$$\frac{300}{600} = e^{3k}$$

$$\ln\frac{300}{600} = \ln e^{3k} \qquad \ln e = 1$$

$$3k = \ln\frac{300}{600}$$

$$k = \frac{1}{3}\ln\frac{300}{600} \qquad \ln(.5)/3$$

$$\approx -.231$$

$$y(t) = 600e^{-.231t}$$

$$y(6) = 600e^{-.231(6)} \qquad 600e\text{^}(-.231*6)$$

$$\approx 150\ g$$

Exercise

The population of an endangered species of bird was 4200 in 1990. Thirteen years later, in 2003, the bird population declined to 3000. The population of the birds is decreasing exponentially according to the function $A(t) = 4200e^{kt}$ where *A* is the bird population t years after 1990. Use this information to find the value of ***k***.

Solution

$$3000 = 4200e^{k(13)}$$

$$\frac{3000}{4200} = e^{13k}$$

$$\ln\frac{3000}{4200} = \ln e^{13k}$$

$$\ln\frac{3000}{4200} = 13k$$

$$k = \frac{\ln\left(\frac{3000}{4200}\right)}{13} \approx -0.26$$