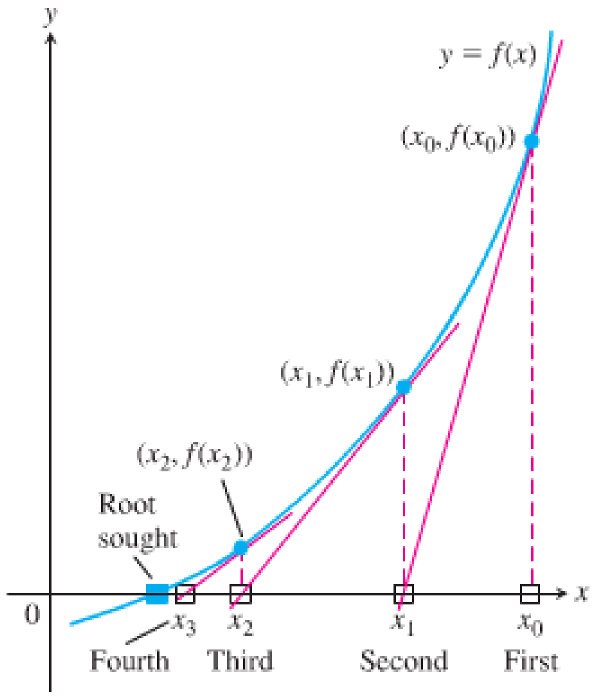
Section 3.6 – Newton's Method

Procedure for *Newton's Method*

The goal of Newton's method, also called the ***Newton-Raphson*** method, for estimating a solution of an equation $f(x)=0$ is to produce a sequence of approximations that approach the solution.



We begin with the first number x_0 of the sequence. Then the function is approximated by its tangent line, and one computes the *x*-intercept of this tangent line. At each step the method approximates a zero of *f* with a zero of one of its linearization.

> Initial estimates, x_0, the method than uses the tangent curve $y=f(x)$ @ $\left(x_0, f\left(x_0\right)\right)$ to approximate the curve, calling the point x_1 where the tangent meets the *x*-axis. The number x_1 usually a better approximation to the solution that is x_0. The point x_2 where the tangent to the curve at $\left(x_1, f\left(x_1\right)\right)$ crosses the *x*-axis is the next approximation in the sequence. We continue on using each approximation to generate the next, until we are close enough to the root to stop.

The point-slope equation for the tangent to the curve at $\left(x_n, f\left(x_n\right)\right)$ is

$$y=f\left(x_n\right)+f'\left(x_n\right)\cdot\left(x-x_n\right)$$

We can find where it crosses the *x*-axis by setting $y=0$.

$$0 = f(x_n) + f'(x_n)\cdot(x - x_n) \quad \Rightarrow \quad x - x_n = -\frac{f(x_n)}{f'(x_n)}$$

$$\boxed{x = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{if } f'(x_n) \neq 0$$

Example

Find the positive root of the equation $f(x) = x^2 - 2 = 0$

Solution

$$f'(x) = 2x$$

$$\begin{aligned} x_{n+1} &= x_n - \frac{x_n^2 - 2}{2x_n} \\ &= x_n - \frac{x_n}{2} + \frac{1}{x_n} \\ &= \frac{x_n}{2} + \frac{1}{x_n} \end{aligned}$$

Example

Find the x-coordinate of the point where the curve $y = x^3 - x$ crosses the horizontal line $y = 1$.

Solution

-- $x^3 - x = 1$

$x^3 - x - 1 = 0$

$f(x) = x^3 - x - 1$

$$\begin{cases} f(1) = -1 \\ f(2) = 5 \end{cases}$$

$f'(x) = 3x^2 - 1$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$
0	1	−1	2	1.5
1	1.5	0.875	5.75	1.347826087
2	1.347826087	0.100682173	4.449905482	1.325200399
3	1.325200399	0.002058362	4.268468292	1.324718174
4	1.324718174	0.000000924	4.264634722	1.324717957
5	1.324717957	−1.8672 E−13	4.264632999	1.324717957

The result: $\boxed{x = 1.324717957}$

Exercises *Section* **3.6 – Newton's Method**

1. Use Newton's method to estimate the on real solution of $x^3+3x+1=0$. Start with $x_0=0$ and then find x_2

2. Use Newton's method to estimate the on real solution of $x^4+x-3=0$. Start with $x_0=-1$ for the left-hand zero and with $x_0=1$ for the zero on the right. Then, in each case, find x_2

3. Use Newton's method to estimate the on real solution of $2x-x^2+1=0$. Start with $x_0=0$ for the left-hand zero and with $x_0=2$ for the zero on the right. Then, in each case, find x_2

4. Use Newton's method to estimate the on real solution of $x^4-2=0$. Start with $x_0=1$ and then find x_2

Use the Newton's method to approximate the roots to ten digits of

5. $f(x)=3x^3-4x^2+1$

6. $f(x)=e^{-2x}+2e^x-6$

7. $f(x)=2x^5-6x^3-4x+2$