Chapter 2.

2.1- **a)** $x''(t)+6x'(t)+9x(t) = \delta(t) \rightarrow sX^2(s)+6sX(s)+9X(s) = 1$
$\rightarrow \quad X(s)[s^2+6s+9] = 1$

$$X(s) = \frac{1}{s^2+6s+9} = \frac{1}{(s+3)^2} \;;$$

Using Laplace transform $x(t) = e^{-3t}u(t)$

b) $x''(t)+3x'(t)+2x(t) = u(t) \rightarrow s^2X(s)+3sX(s)+2X(s) = 1/s$

$$X(s) = \frac{1}{s(s^2+3s+2)} = \frac{1}{2s} + \frac{-1}{s+1} + \frac{1}{2}\frac{1}{(s+2)}$$

$$\rightarrow \mathcal{L}[X(s)] = x(t) = (1/2 - e^{-t} + 1/2\, e^{-2t})u(t)$$

c) $x''(t)+3x'(t)+2x(t) = \cos 4t \rightarrow s^2X(s)+3sX(s)+2X(s) = \dfrac{s}{s^2+16}$

$$\rightarrow X(s)\,(s^2+3s+2) = \frac{s}{s^2+16}$$

$$X(s) = \frac{s}{(s^2+16)(s^2+3s+2)} = \frac{s}{(s+j4)(s-j4)(s+1)(s+2)}$$

$$\rightarrow \phi(ju) = \left.\frac{s}{(s+1)(s+2)}\right|_{s=j4} = .141 - j(.167)$$

$$k_1 = {}^{1}\!/_{10} \;, k_2 = -{}^{1}\!/_{17} \quad \Rightarrow X(s) = \frac{-.141-j.165}{s^2+16} + \frac{-{}^{1}\!/_{17}}{s+1} + \frac{{}^{1}\!/_{10}}{s+2}$$

$$\mathcal{L}[X(s)] = x(t) = \frac{1}{4}e^{-at}(-.165\cos 4t + .141\sin 4t) + \frac{1}{10}\,e^{-2t} - \frac{1}{17}e^{-t}$$

$$= [-.04125\cos 4t + .03525\sin 4t - \frac{1}{17}e^{-2t} + .1\,e^{-2t}]u(t)$$

d) $D^2x(t)+5Dx(t)+4x(t) = 8 \rightarrow s^2X(s)+5sX(s)+4X(s) = \dfrac{8}{s}$

$$\rightarrow X(s)[s^2+5s+4] = \frac{8}{s} \rightarrow X(s) = \frac{8}{s(s+1)(s+4)} = \frac{2}{s} - \frac{8/3}{s+1} + \frac{2/3}{s+4}$$

$$\mathcal{L}[X(s)] = x(t) = (2 - (8/3)e^{-t} + (2/3)e^{-4t})\,u(t)$$

2.2- a) $(s^2+2s+5)\,X(s) - 2s - 4 = \dfrac{10}{s}$

b) $(s^3 + 4s^2 + 8s + 4)X(s) + 4s^2 + 15s + 28 = \dfrac{5}{s^2+25}$

2 Solution.

2.3 - $X(s) = \dfrac{10}{(s+2)\left[(s+2.035)^2+.886\right]\left[(s-1.035)^2+1.88\right]}$

$$= \frac{7.7}{s+2} + \frac{.742(s+1.99)}{(s+2.035)^2+.886} + \frac{.0267(s+.953)}{(s-1.035)^2+1.88}$$

$£\,[X(s)] = x(t) = 7.7e^{-2t} + 0.742e^{-2.035t}\sin(.785t-87.2°)$
$+ .267e^{1.035t}\sin(3.54t-136.6°)$

2.4 - $F(s) = \int f(t)e^{-st}dt = \int_1^3 (t+1)e^{-st}dt + \int_3^4 4e^{-st}dt$

$$= \int_1^3 te^{-st}dt + \int_1^3 e^{-st}dt + 4\int_3^4 e^{-st}dt$$

$$= -e^{-st}\left(\frac{t}{s}+\frac{1}{s^2}\right)\Big|_1^3 - \frac{e^{-st}}{s}\Big|_1^3 - 4\,\frac{e^{-st}}{s}\Big|_3^4$$

$$= -4\,\frac{e^{-4s}}{s} + \frac{e^{-3s}}{s^2} + 2\,\frac{e^{-s}}{s} + \frac{e^{-s}}{s^2}$$

2.5 - **a)** $G(s) = \dfrac{1}{(s+2)(s+3)} = \dfrac{A}{s+2} + \dfrac{B}{s+3}$

$$\left\{\begin{array}{l} A + B = 0 \rightarrow B = -A = -1 \\ 3A + 2B = 1 \rightarrow A = 1 \end{array}\right\} \Rightarrow G(s) = \frac{1}{s+2} - \frac{1}{s+3}$$

$£^{-1}\,[G(s)] = g(t) = e^{-2t} - e^{-3t}$

b) $G(s) = \dfrac{1}{(s+1)^2(s+4)} = \dfrac{A}{s+4} + \dfrac{Cs+d}{(s+1)^2}$

$$\rightarrow \left\{\begin{array}{l} A + C = 0 \\ 2A + 4C + D = 0 \\ A + 4D = 1 \end{array}\right\} \begin{array}{l} A = 1/9 \\ C = -1/9 \\ D = 2/9 \end{array}$$

$$G(s) = \frac{1/9}{s+4} + \frac{1}{9}\,\frac{2-s}{(s+1)^2} = \frac{1}{9}\,\frac{1}{s+4} + \frac{2}{9}\,\frac{1}{(s+1)^2} - \frac{1}{9}\,\frac{s}{(s+1)^2}$$

$\rightarrow\ £^{-1}\,[G(s)] = g(t) = \frac{1}{9}e^{-4t} + \frac{2}{9}te^{-t} - \frac{1}{9}$

c) $G(s) = \frac{A}{s+4} + \frac{B}{s+2} + \frac{Cs+D}{(s+2)^2} + \frac{Es^2+Fs+G}{(s+2)^3} \Rightarrow \begin{cases} A = -\frac{5}{4} \\ B = \frac{5}{4} \\ D = -\frac{5}{2} \\ G = \frac{5}{2} \end{cases}$

$g(t) = \frac{5}{2}\, t^2\, e^{-2t} - \frac{5}{2}\, t\, e^{-2t} + \frac{5}{4}\, e^{-2t} - \frac{5}{4} e^{-4t}$

d) $G(s) = \frac{2(s+1)}{s(s^2+s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+2}$

or this function will be in form of: $\frac{\omega_n^2(1+as)}{s(s^2+2\xi\omega_n+\omega_n^2)}$

where: $a = 1$, $\omega_n^2 = 2$, and $\xi = \frac{1}{2\sqrt{2}} = .353$

$\pounds^{-1}[G(s)] = g(t) = 1 + \frac{1}{\sqrt{1-\xi^2}}\sqrt{1-2a\xi\omega_n + a^2\omega_n^2}\; e^{-\xi\omega_n t}\,\sin(\omega_n\sqrt{1-\xi^2}\, t + \phi)$

$\phi = \tan^{-1}\frac{a\omega_n\sqrt{1-\xi^2}}{1-a\xi\omega_n} - \tan^{-1}\frac{\sqrt{1-\xi^2}}{-\xi} = \tan^{-1}\frac{1.322}{.5} - \tan^{-1}\frac{.9354}{-.353}$

$= 69.3^\circ + 69.3^\circ \cong 138.6^\circ$

$g(t) = 1 + 1.51\, e^{-.5t}\sin(1.323t + 138.6°)$

e) $G(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} \rightarrow A = 1.5$; $B = -3.$; $C = 2.5$

$g(t) = 1.5e^{-t} - 3e^{-2t} + 2.5e^{-3t}$

2.6 - Time delay:

$\pounds^{-1}\left[\frac{2}{s^2-6s+3} = \frac{2}{(s-3)^2+2^2}\right] = e^{3t}\sin 2t$

$\pounds^{-1}\left[\frac{s-1}{(s-1)^2+1}\right] = e^{t}\cos t$

$f(t) = \begin{cases} -\,e^{t}\cos t & 0 < t \le .5 \\ e^{3(t-.5)}\sin 2(t-.5) - e^{t}\cos t & t > .5 \end{cases}$

2.7 - $\pounds\ \{y'''(t) + 2y''(t) + 11y'(t) + 4y(t)\} = 0$

$s^3Y(s)-s^2y(0)-sy(0)-y'(0)+2s^2Y(s)-2sy(0)-2y'(0)+11sY(s)$
$11y(0)+4Y(s) = 0$

$\rightarrow Y(s)\ (s^3+2s^2+11s+4) = \alpha_1 s^2 + (\alpha_2 + 2\alpha_1)s + \alpha_3 + 2\alpha_2 + 11\alpha_1$

$$Y(s) = \frac{\alpha_1 s^2 + (\alpha_2 + 2\alpha_1)s + \alpha_3 + 2\alpha_2 + 11\alpha_1}{s^3 + 2s^2 + 11s + 4}$$

$$= \frac{5s^2 + 6s + 2}{s^3 + 2s^2 + 11s + 4}$$

$$\begin{cases} \alpha_1 = 5 & \\ 2\alpha_1 + \alpha_2 = 6 & \rightarrow \alpha_2 = 4 \\ 11\alpha_1 + 2\alpha_2 + \alpha_3 = 2 & \rightarrow \alpha_3 = -45 \end{cases}$$

2.8 - $\pounds^{-1}\left[\frac{1}{s}\right] = u(t)$; $\pounds^{-1}\left[\frac{1}{s+2}\right]$

the convolution integral:

$$\pounds^{-1}\left\{\frac{1}{s}\,\frac{1}{s+2}\right\} = \int_0^t u(t-\tau)e^{-2\tau}d\tau = -\frac{1}{2}e^{-2\tau}\Big|_0^t = \frac{1}{2}(1 - e^{2t})$$

2.9-

$$f(\infty) = \lim_{s\to 0} s\,F(s) = \frac{0}{0} \Rightarrow f(\infty) = \lim_{s\to 0}\left[\frac{2}{4s^3+39s^2+115s+75}\right] = \frac{2}{75}$$

2.10 - $f(0) = \lim_{s\to\infty} s\,F(s) = \frac{1}{\infty} = 0$

2.11 - For the response to initial conditions, the input r can be taken to be zero, and the transform is :

$[s^2Y - sy(0) - y'(0)] + 7\,[sY - y(0)] + 6Y = 0$

$s^2Y - s - 2 + 7sY - 7 + 6Y = 0$

$$Y(s) = \frac{s+9}{s^2 + 7s + 6} = \frac{s+9}{(s+1)(s+6)} = \frac{A}{s+1} + \frac{B}{s+6} \Rightarrow \begin{cases} A = 1.6 \\ B = -.6 \end{cases}$$

$$Y(s) = \frac{1.6}{s+1} - \frac{.6}{s+6} \Leftrightarrow y(t) = 1.6\,e^{-t} - 0.6\,e^{-6t}$$

2.12 - The forced response is given by :

$$y_b(t) = \int_0^t \omega(t-\tau)\left[3\frac{dx}{dt} + 2x\right]d\tau = 3\int_0^t \omega(t-\tau)\frac{dx}{dt}\,d\tau + 2\int_0^t \omega(t-\tau)x\,d\tau$$

$$\int_0^t \omega(t-\tau)\frac{dx}{d\tau}\,d\tau = \omega(0)x(t) - \omega(t)x(0) - \int_0^t \frac{\partial\omega(t-\tau)}{\partial\tau}x\,d\tau \;;\; \omega(0) = 0.$$

$$y_b(t) = \int_0^t \left[-3\frac{\partial\omega(t-\tau)}{\partial\tau} + 2\omega(t-\tau)\right]x(\tau)d\tau - 3\omega(t)\,x(0)$$

and the forced response is:

$$y_b(t) = 3e^{-2t}\int_0^t e^{2\tau}e^{-3\tau}d\tau - 4te^{-2t}\int_0^t e^{2\tau}e^{-3\tau}d\tau$$

$$+ 4e^{-2t}\int_0^t \tau e^{2\tau}e^{-3\tau}d\tau - 3te^{-2t}$$

$$= 7\left[e^{-2t} - e^{-3t} - te^{-2t}\right]$$

2.13 - $F(s) = -\dfrac{2}{9s} + \dfrac{1}{3s^2} + \dfrac{1}{5(s+1)} + \dfrac{1}{45(s+6)}$

2.14 - $F(s) = E\left[\dfrac{1}{Ts^2} - \dfrac{e^{-Ts}}{s(1-e^{-Ts})}\right]$

2.15 - a) $F(z) = \dfrac{z(z - \cosh 2T)}{z^2 - 2z\cosh 2T + 1}$

b) $F(z) = \dfrac{Tz}{(z-1)^2} - \dfrac{1}{3}\dfrac{z\sin 2T}{z^2 - 2z\cos^2 T + 1}$

c) $F(s) = \dfrac{A}{s} + \dfrac{Bs}{s^2+2} \Rightarrow A = 1; B = -1$

$$\Rightarrow F(z) = \frac{z}{z-1} - \frac{z(z - \cos\sqrt{2}T)}{z^2 - 2z\cos\sqrt{2}\,T + 1}$$

d) $F(z) = \dfrac{e^{-1}z + 1 - 2e^{-1}}{z^2 - (1+e^{-1})z + e^{-1}}$; for T = 1.

e) $\boldsymbol{Z}[ak] = \dfrac{1}{1 - az^{-1}}$ and $\boldsymbol{Z}[x(k-1)] = z^{-1}X(z)$

$$F(z) = z^{-1}\frac{1}{1 - az^{-1}} = \frac{k}{1 - az^{-1}} \qquad ; \; k = 1,2,3...$$

f) By using complex integration $g(k) = \dfrac{x(k)}{k} = k$

$$\mathcal{Z}\left[\frac{x(k)}{k}\right] = G(z) = \sum_{k=0}^{\infty}\frac{x(k)}{k}z^{-1} = \sum_{k=0}^{\infty}kz^{-k}$$

$$\frac{d}{dz}G(z) = -\sum_{k=0}^{\infty}k^2z^{-k-1} = -z^{-1}\sum_{k=0}^{\infty}k^2z^{-k} = -\frac{X(z)}{z}$$

$$G(z) = \mathcal{Z}\left[\frac{x(k)}{k}\right] = \int_z^{\infty}\frac{X(z_1)}{z_1}dz_1 + G(\infty) = \int_z^{\infty}\sum_{k=0}^{\infty}k^2z_1^{-k-1}dz_1 + G(\infty)$$

$$= \sum_{k=0}^{\infty}k^2\frac{z_1^{-k}}{-k}\bigg|_z^{\infty} + G(\infty) = \sum_{k=0}^{\infty}kz^{-k} + \lim_{k\to 0}\frac{x(k)}{k} = \frac{z^{-1}}{\left(1 - z^{-1}\right)^2} + \lim_{k\to 0}k$$

$$= \frac{z^{-1}}{\left(1 - z^{-1}\right)^2}$$

2.16- a)

$$\frac{E(z)}{z} = \frac{1}{(z - 1)(z - 2)} = \frac{-1}{z - 1} + \frac{1}{z - 2}$$

$$\Rightarrow \quad E(z) = \frac{-z}{z - 1} + \frac{z}{z - 2}$$

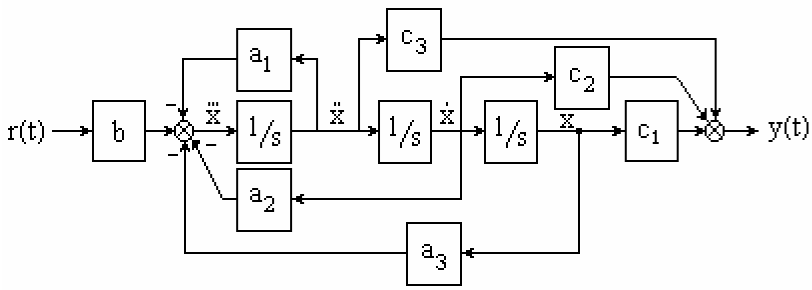
Then: $e(t) = (-1 + 2^t)\,u(t)$

b) $\dfrac{E(z)}{z} = \dfrac{1}{z - 1} + \dfrac{1}{z - e^{-aT}} \quad \Rightarrow \quad E(z) = \dfrac{z}{z - 1} + \dfrac{z}{z - e^{-aT}}$

Then: $e(kT) = 1-e^{-akT}$

Chapter 3.

3.1 -



3.2 - $C(s) = x_1 + x_2$ (1)

$x_1 = (R-C+x_2)(s+10)$ (2)

$x_2 = (R-C-x_1)(s+5)$ (3)

$(2) \to \quad x_1 = (R-x_1-x_2+x_2)(s+10) = (R-x_1)(s+10) \to \quad x_1 = R(s)\,\frac{s+10}{s+11}$

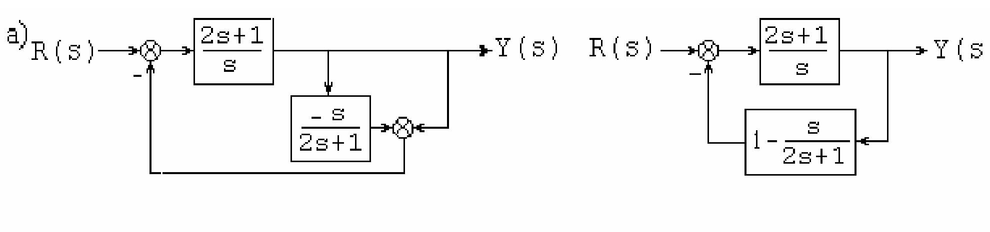
$(3) \to \quad x_2 = (R-2x_1-x_2)(s+5) \quad \to \quad x_2 = -\frac{(s+9)(s+5)}{(s+11)(s+6)}R(s)$

$$C(s) = x_1 + x_2 = \frac{s+10}{s+11}R(s) - \frac{(s+9)(s+5)}{(s+6)(s+11)}R(s)$$

$$= \frac{1}{s+11}\left[s + 10 - \frac{(s+9)(s+5)}{s+6}\right]R(s)$$

$$\frac{C(s)}{R(s)} = \frac{1}{s+11}\left(\frac{2s+15}{s+6}\right) = \frac{2s+15}{s^2+17s+66}$$

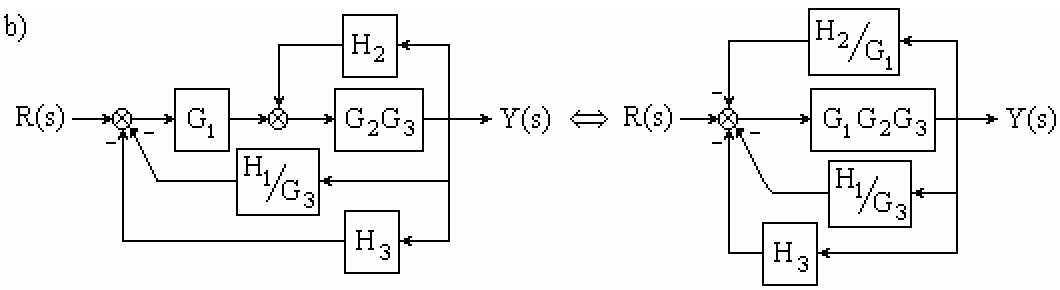
3.3 -

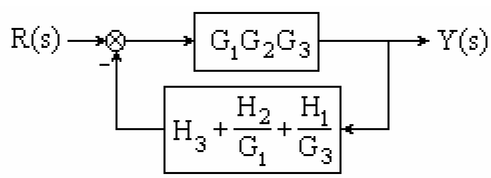


$$\frac{Y(s)}{R(s)} = \frac{\frac{2s+1}{s}}{1 + \left(1 - \frac{s}{2s+1}\right)\left(\frac{2s+1}{s}\right)} = \frac{\frac{2s+1}{s}}{1 + \frac{2s+1}{s} - 1} = 1$$

8 Solution.

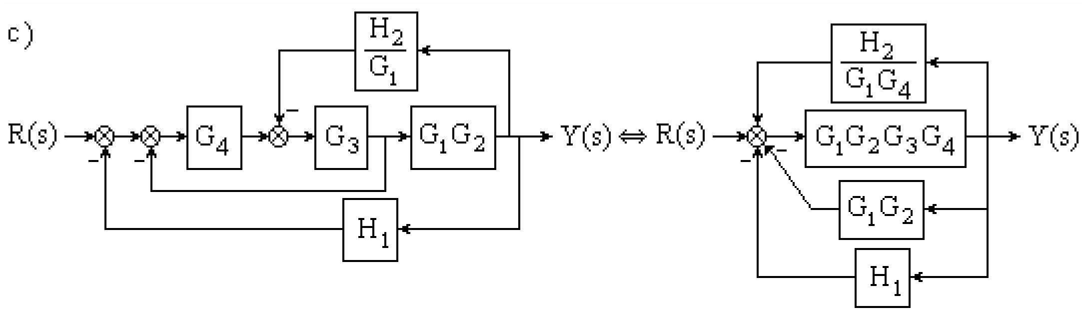
b)

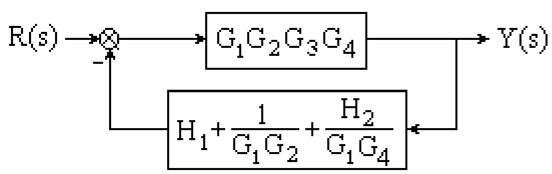




$$\Rightarrow \frac{Y(s)}{R(s)} = \frac{G_1G_2G_3}{1 + G_1G_2G_3\left(H_3 + \frac{H_2}{G_1} + \frac{H_1}{G_3}\right)}$$

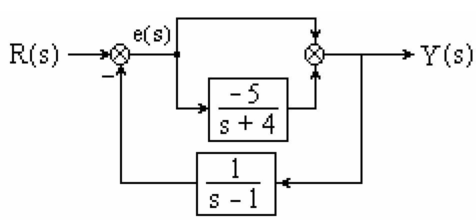
c)





$$\Rightarrow \quad \frac{Y(s)}{R(s)} = \frac{G_1G_2G_3G_4}{1 + 1 + (G_1G_2G_3G_4)\left(H_1 + \frac{1}{G_1G_2} + \frac{H_2}{G_1G_4}\right)}$$

d)



$$\frac{Y(s)}{e(s)} = 1 - \frac{5}{s+4} = \frac{s-1}{s+4} = T$$

$$\Rightarrow \quad \frac{Y(s)}{R(s)} = \frac{T}{1+\frac{T}{s-1}} = \frac{s-1/s+4}{1+\frac{s-1}{s+4}\frac{1}{s-1}} = \frac{s-1}{s+4+1} = \frac{s-1}{s+5}$$

3.4- a)Loop 1: $E_{in}(s) = R_1(I_1(s)-I_2(s)) + (R_2+\frac{1}{sC_1})(I_1(s)-I_2(s))$ [1]

Loop 2: $R_1(I_2(s)-I_1(s)) + (R_3+\frac{1}{sC_2}).I_2(s) = 0$ [2]

Loop 3: $(R_2+\frac{1}{sC_1})(I_3(s)-I_1(s)) + (R_4+sL_1)I_3(s) = 0$ [3]

Loop4: $E_0(s) = R_3\ I_2(s)$ [4]

From [4] $\rightarrow$ $\frac{E_0(s)}{I_2(s)} = R_3 = G_3$

From [3] $\rightarrow$ $\frac{I_3(s)}{I_1(s)} = \frac{R_2+\frac{1}{sC_1}}{R_2+R_4+\frac{1}{sC_1}+sL_1} = G_6$

From [2] $\rightarrow$ $\frac{I_2(s)}{I_1(s)} = \frac{R_1}{R_1+R_3+\frac{1}{sC_2}} = G_2$

From [1] $\rightarrow$ $E_{in}(s) - (-R_1)I_2(s) + (R_2 + \frac{1}{sC_1})I_3(s) = (R_1 + R_2 + \frac{1}{sC_1})I_1(s)$

$\Rightarrow$ $G_4 = R_1$, $G_5 = R_2 + \frac{1}{sC_1}$, $G_1 = \frac{1}{R_1+R_2+\frac{1}{s\ C_1}}$

b) Transfer function: $\frac{E_0(s)}{E_i(s)} = \frac{G_1G_2G_3}{1 - G_5G_6G_1 - G_1G_2G_4}$

3.5- a) [**1**] $- V_1(s) + (R_1+\frac{1}{sC_1})I_1(s) - \frac{1}{sC_1}I_2(s) = 0$

[**2**] $V_2(s) = I_2(s)R_2 + V_3(s) = I_2(s)R_2+(\frac{1}{sC_2})I_2(s) = (R_2+\frac{1}{sC_2})I_2(s)$

[**3**] $I_2(R_2+\frac{1}{sC_1}+\frac{1}{sC_2}) + I_1(s)(-\frac{1}{sC_1}) = 0$

[**4**] $V_3(s) = \frac{1}{sC_2}I_2(s)$

[**5**] $V_2(s) = \frac{1}{sC_1}(I_1(s) - I_2(s))$

From block diagram:

$I_1(s)= G_1(V_1(s)-G_5I_2(s))$ $\rightarrow$ [1] $\rightarrow$ $I_1(s) = \frac{1}{R_1+\frac{1}{C_1s}}(V_1(s) + \frac{1}{C_1s}I_2(s))$

$V_2(s) = G_2(I_1(s)-I_2(s))$ $\rightarrow$ [5]

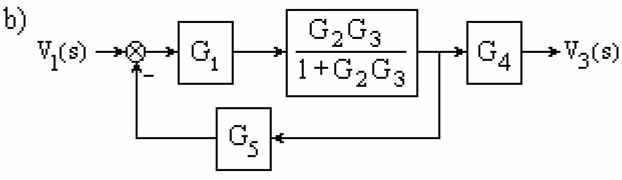
$I_2(s) = G_3V_2(s)$ $\rightarrow$ [2] $\rightarrow$ $I_2(s) = \frac{1}{R_2+\frac{1}{C_2s}}V_2(s)$

$V_3(s) = G_4I_2(s)$ $\rightarrow$ [4] $\rightarrow$ $V_3(s) = \frac{1}{sC_2}I_2 = G_4I_2(s)$

$$\Rightarrow \quad G_4 = \frac{1}{sC_2} \; ; \quad G_1 = \frac{1}{R_1 + \frac{1}{C_1 s}} \; ; \quad G_5 = -\frac{1}{sC_1} \; ; \; G_2 = \frac{1}{sC_1} \; ;$$

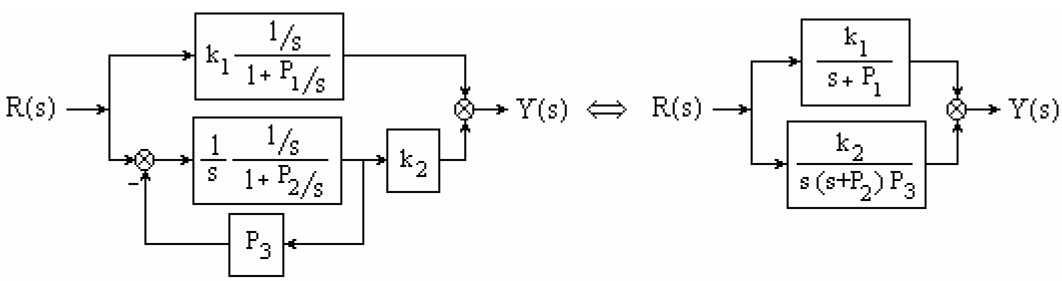
$$G_3 = \frac{1}{R_2 + \frac{1}{C_2 s}}$$

b)



$$\frac{V_3(s)}{V_4(s)} = G_4 \times \frac{\frac{G_1G_2G_3}{1+G_2G_3}}{1+\frac{G_1G_2G_3G_5}{1+G_2G_3}} = \frac{G_1G_2G_3G_4}{1+G_2G_3+G_1G_2G_3G_5}$$

3.6-

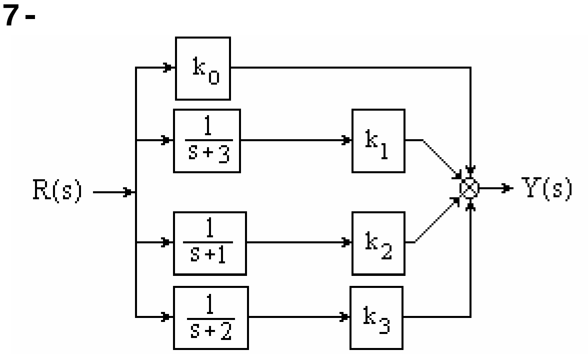


$$\frac{Y(s)}{R(s)} = \frac{k_1}{s+P_1} + \frac{k_2}{s(s+P_2)+P_3} = \frac{k_1 s^2 + (k_1P_2 + k_2)s + k_1P_3 + k_2P_1}{s^3 + (P_1+P_2)s^2 + (P_1P_2+P_3)s + P_1P_3} = \frac{s^2+3s+5}{(s+2)\,(s^2+s+5)}$$

$$\Rightarrow \quad P_1 = 2 \, , \quad P_2 = 1 \, , \quad P_3 = 5 \, , \quad k_1 = 1$$

$$\left.\begin{array}{lll} k_1P_3 + k_2 = 3 & \rightarrow & k_2 = \\ k_1P_3 + k_2P_1 = 5 & \rightarrow & k_2 = \end{array}\right\} \rightarrow \text{Impossible} \qquad \begin{array}{l} 3\text{-}1 = 2 \\ (5\text{-}5)/2 = 0 \end{array}$$

3.7-



$$\frac{Y(s)}{R(s)} = k_o + \frac{k_1}{s+3} + \frac{k_2}{s+1} + \frac{k_3}{s+2}$$

$$= \frac{k_o s^3 + (6k_o + k_1 + k_2 + k_3)s^2 + (11k_o + 3k_1 + 5k_2 + 4k_3)s + 6k_o + 2k_1 + 6k_2 + 3k_3}{(s+1)(s+2)(s+3)}$$

$k_0 = 2$

$6k_0+k_1+k_2+k_3 = 1 \quad \rightarrow \quad k_1+k_2+k_3 = -11$

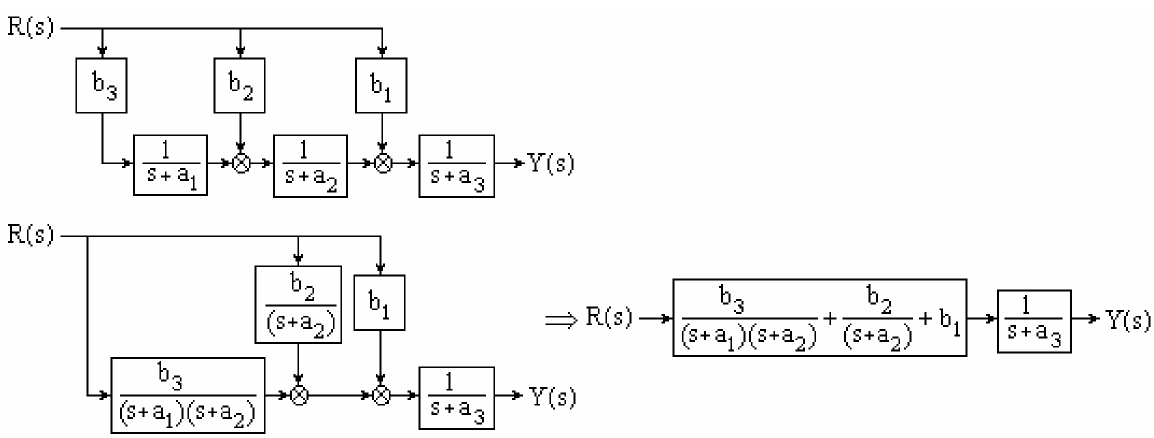
$11k_0+3k_1+5k_2+4k_3 = 2 \quad \rightarrow \quad 3k_1+5k_2+4k_3 = -20$

$6k_0+2k_1+6k_2+3k_3 = 1 \quad \rightarrow \quad 2k_1+6k_2+3k_3 = -2$

$\rightarrow k_1 = -(41/2)\ , \quad k_2 = 7/2\ , \quad k_3 = 6$

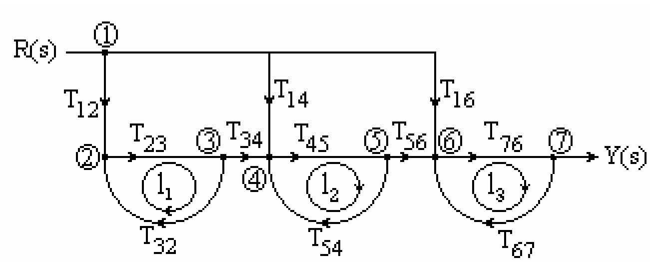
3.8-

a) **i-** Direct block diagram simplification:



Then: $\frac{Y(s)}{R(s)} = \frac{b_3 + b_2(s+a_1) + b_1(s+a_1)(s+a_2)}{(s+a_1)(s+a_2)(s+a_3)}$

ii- Mason Rule:



$l_1 = T_{23}T_{32} = -a_1/s \qquad P_1 = T_{12}T_{23}T_{34}T_{45}T_{56}T_{67} = b_3 / s^3$

$l_2 = T_{45}T_{54} = -a_2/s \qquad P_2 = T_{14}T_{45}T_{56}T_{67} = b_2 / s^2$

$l_3 = T_{67}T_{76} = -a_3/s \qquad P_3 = T_{16}T_{67} = b_1 / s$

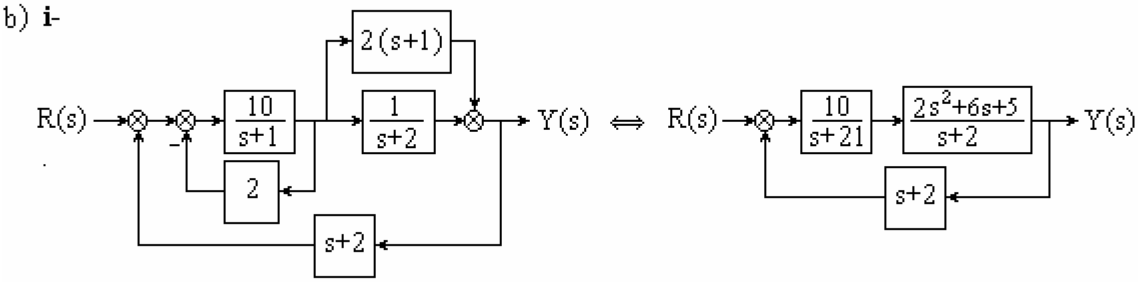
 Solution.

$\Delta = 1-(l_1+l_2+l_3)+(l_1l_2+l_1l_3+l_2l_3) - l_1l_2l_3$

$\Delta_1 = 1$; $\Delta_2 = 1 - l_1$; $\Delta_3 = 1 - (l_1+l_2) + l_1l_2$

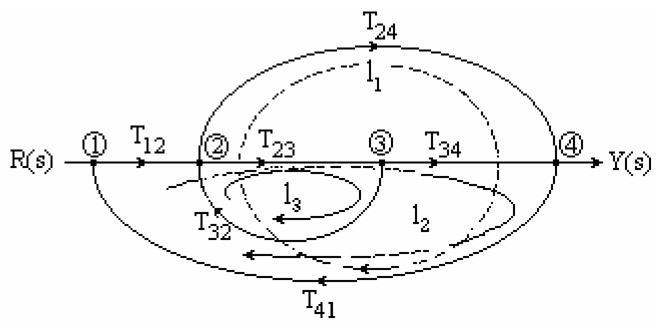
$$\frac{Y(s)}{R(s)} = \frac{P_1\Delta_1+P_2\Delta_2+P_3\Delta_3}{\Delta} = \frac{b_1(s+a_1)(s+a_2)+b_2(s+a_1)+b_3}{(s+a_1)(s+a_2)(s+a_3)}$$

b) **i-**



$$\frac{Y(s)}{R(s)} = \frac{\frac{10(2S^2+6S+5)}{(S+2)(S+21)}}{1+\frac{20S^2+60S+50}{S+21}} = \frac{20s^2+60s+50}{-(s+2)(20s^2+61s+71)}$$

ii- Mason rule:



$T_{12} +1$; $T_{23} = \frac{10}{s+1}$; $T_{34} = \frac{1}{s+2}$; $T_{24} = 20$; $T_{32} = -2$; $T_{41} = s+2$

$P_1 = T_{12}T_{23}T_{34} = \frac{10}{(S+1)(S+2)}$; $l_1 = T_{12}T_{24}T_{41} = 20(s+2)$

$P_2 = T_{12}T_{24} = 20$; $l_2 = T_{12}T_{23}T_{34}T_{41} = \frac{10}{s+1}$

; $l_3 = T_{23}T_{32} = \frac{-20}{S+1}$

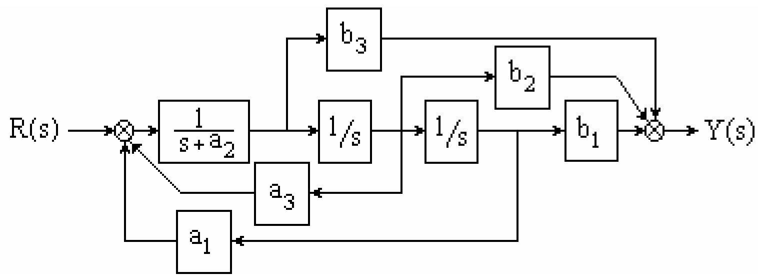
$\Delta = 1-(l_1+l_2+l_3)+(l_1l_2+l_1l_3+l_2l_3)-(l_1l_2l_3) = 1 - (l_1+l_2+l_3)$

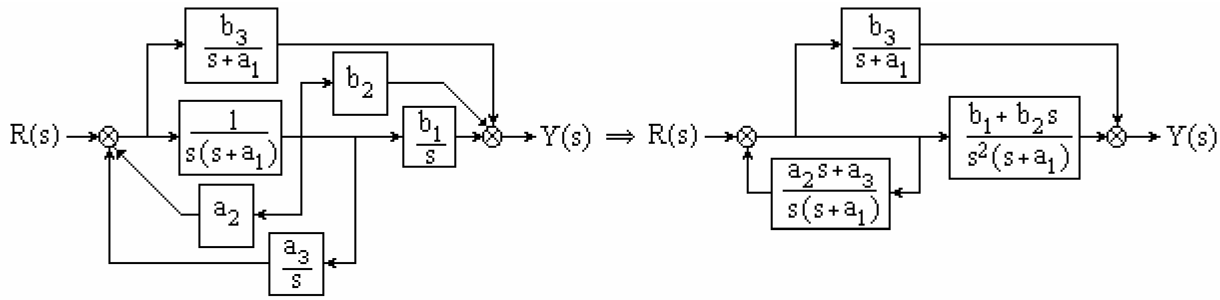
$\Delta_1 = 1- l_1+l_2+l_3)^{**} = 1$

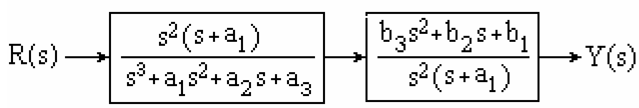
$\Delta_2 = 1$

$$\frac{Y(s)}{R(s)} = \frac{P_1\Delta_1+P_2\Delta_2}{\Delta} = \frac{P_1+P_2}{1-l_1l_2-l_3} = \frac{\frac{10}{(s+1)(s+2)}+20}{1-20s-40-\frac{10}{s+1}+\frac{20}{s+1}} = \frac{20s^2+60s+50}{-(20s^3+99s^2+147s+58)}$$

c)

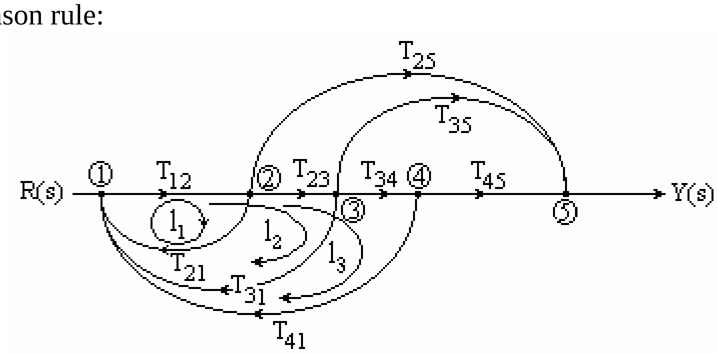






$$\Rightarrow \quad \frac{Y(s)}{R(s)} = \frac{b_3 s^2 + b_2 s + b_1}{s^3 + a_1 s^2 + a_2 s + a_3}$$

ii- Mason rule:



$T_{12} = T_{23} = T_{34} = 1/s$, $T_{21} = -a_1$, $T_{31} = -a_2$, $T_{41} = -a_3$,
$T_{25} = b_3$
$T_{35} = b_2$, $T_{45} = b_1$.

$P_1 = T_{12}T_{23}T_{34}T_{45} = b_1 / s^3$; $l_1 = T_{12}T_{21} = -a_1 / s$
$P_2 = T_{12}T_{23}T_{35} = b_2 / s^2$; $l_2 = T_{12}T_{23}T_{31} = -a_2 / s^2$
$P_3 = T_{12}T_{25} = b_3 / s$; $l_3 = T_{12}T_{23}T_{34}T_{41} = -a_3 / s^3$

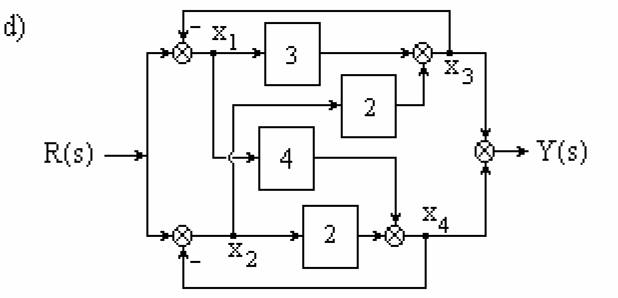
$\Delta = 1-(l_1+l_2+l_3)+(l_1l_2+l_1l_3+l_2l_3)-(l_1l_2l_3) = 1 - (l_1+l_2+l_3)$

$\Delta_1 = 1 - (l_1+l_2+l_3)^{**} = 1$; $\Delta_2 = 1$; $\Delta_3 = 1$

 Solution.

$$\frac{Y(s)}{R(s)} = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3}{\Delta} = \frac{P_1 + P_2 + P_3}{1 - l_1 - l_2 - l_3} = \frac{\frac{b_1}{s^3} + \frac{b_2}{s^2} + \frac{b_3}{s}}{1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}} = \frac{b_3 s^2 + b_2 s + b_1}{s^3 + a_1 s^2 + a_2 s + a_3}$$

d)



$x_1 = R(s) - x_3$; $x_3 = 3x_1 + 2x_2$
$x_2 = R(s) - x_4$; $x_4 = 2x_2 + 4x_1$
$Y = x_3 + x_4$

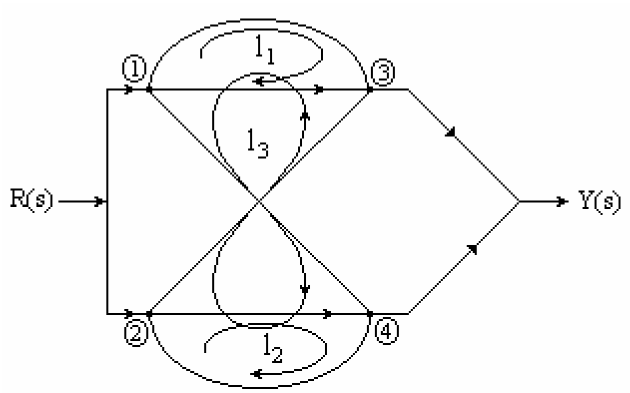
$x_3 = 3R - 3x_3 + 2R - 2x_4$
$\rightarrow \quad x_3 = (5/4)R - (1/2)x_4$

$x_4 = 2R - 2x_4 + 4R - 4x_3$
$\rightarrow \quad x_4 = 2R - (4/3)x_3$

$\Rightarrow x_3 = 5/4\ R - R + (2/3)x_3 \quad \rightarrow \quad x_3 = 3/4\ R \quad \rightarrow \quad x_4 = R$

$Y(s) = 3/4\ R + R = 7/4\ R \quad \Rightarrow \quad \frac{Y(s)}{R(s)} = \frac{7}{4}$

ii- Mason's rule:



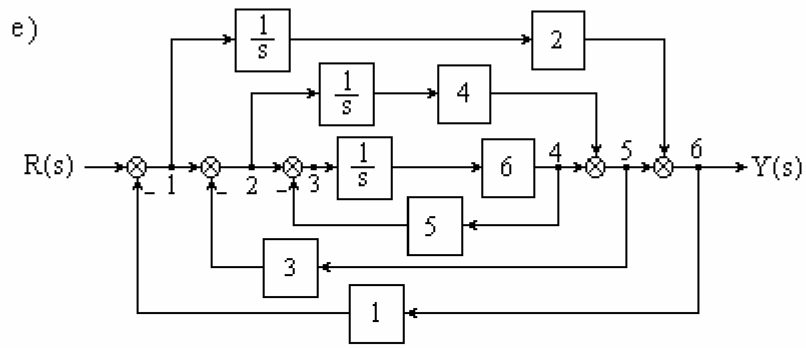
$P_1 = T_{13}T_{35} = 3$; $l_1 = T_{13}T_{31} = -3$
$P_2 = T_{14}T_{45} = 4$; $l_2 = T_{24}T_{42} = -2$
$P_3 = T_{24}T_{45} = 2$; $l_3 = T_{14}T_{42}T_{23}T_{31} = 8$
$P_4 = T_{23}T_{35} = 2$

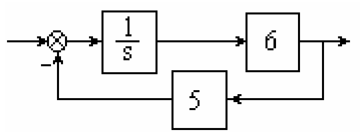
$T_{13} = 3$, $T_{14} = 4$, $T_{23} = T_{24} = 2$, $T_{31} = -1 = T_{42}$

$\Delta = 1 - (l_1 + l_2 + l_3) + (l_1 l_2 + l_1 l_3 + l_2 l_3) - (l_1 l_2 l_3)$
$\quad = 1 - (-3 - 2 + 8) + 6 = 4$
$\Delta_1 = 1 - (l_1 + l_2 + l_3)^*$

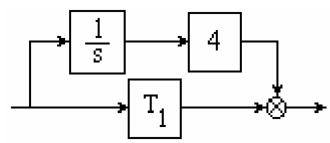
$$\frac{Y(s)}{R(s)} = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4}{\Delta} = \frac{P_1 + P_2 + P_3 + P_4}{\Delta} = \frac{7}{4}$$

e)

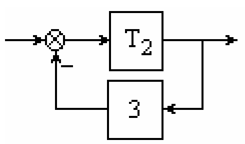




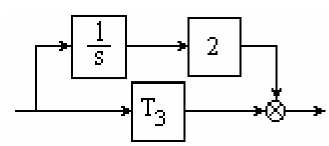
$$T_1 = \frac{6/s}{1+\frac{30}{s}} = \frac{6}{s+30}$$



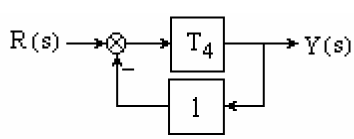
$$T_2 = T_1 + \frac{4}{s} = \frac{10s+120}{s(s+30)}$$



$$T_3 = \frac{T_2}{1+3T_2} = \frac{10s+120}{s^2+60s+360}$$



$$T_4 = T_3 + \frac{2}{s} = \frac{12s^2+240s+720}{s^3+60s^2+360s}$$



$$\Rightarrow \quad \frac{Y(s)}{R(s)} = \frac{T_4}{1+T_4} = \frac{12s^2+240s+720}{s^3+72s^2+600s+720}$$

b) signal flow graph:

$T_{12} = 1$, $T_{23} = 1$, $T_{34} = 6/s$, $T_{45} = 1$, $T_{56} = 1$, $T_{43} = -5$

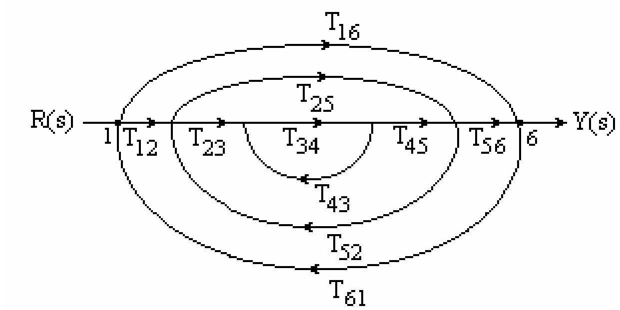
$T_{25} = 4/s$, $T_{16} = 2/s$, $T_{52} = -3$, $T_{61} = -1$

$P_1 = T_{12}T_{23}T_{34}T_{45}T_{56} = 6/s$;

$P_2 = T_{12}T_{23}T_{56} = 4/s$

$P_3 = T_{16} = 2/s$

16 Solution.



$l_1 = T_{34}T_{43} = -30/s$

$l_2 = T_{23}T_{34}T_{45}T_{52} = -18/s$

$l_3 = T_{12}T_{23}T_{34}T_{45}T_{56}T_{61} = -6/s$

$l_4 = T_{25}T_{52} = -12/s$

$l_5 = T_{16}T_{61} = -2/s$

$l_6 = T_{12}T_{25}T_{56}T_{61} = -\ 4/s$

$\Delta = 1-(l_1+l_2+l_3+l_4+l_5+l_6)+(l_1l_4+l_1l_5+l_1l_6+l_2l_5+l_4l_5)-(l_1l_4l_5)$

$$= 1 - \frac{-72}{s} + \frac{360}{s^2} + \frac{60}{s^2} + \frac{120}{s^2} + \frac{36}{s^2} + \frac{24}{s^2} - \frac{-720}{s^3}$$

$$= 1 + \frac{72}{s} + \frac{600}{s^2} + \frac{720}{s^3}$$

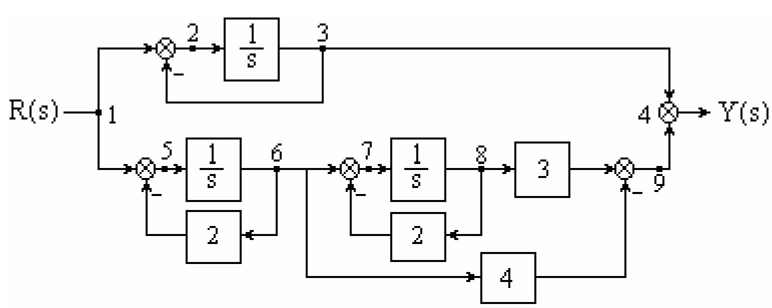
$\Delta_1 = 1$, $\Delta_2 = 1 - l_1 = 1$,

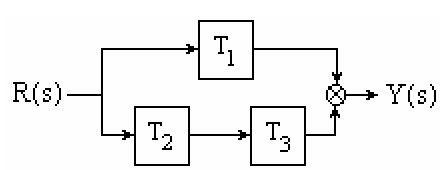
$\Delta_3 = 1-(l_1+l_2+l_4)+ l_1l_4 = 1+60/s + 360/s^2$

$$\frac{Y(s)}{R(s)} = \frac{P_1\Delta_1+P_2\Delta_2+P_3\Delta_3}{\Delta} = \frac{6/S(1)+(4/S)(1+30/S)+(2/S)(1+60/S+360/S^2)}{1+72/S+600/S^2+720/S^3}$$

$$= \frac{12s^2+240s+720}{s^3+72s^2+600s+720}$$

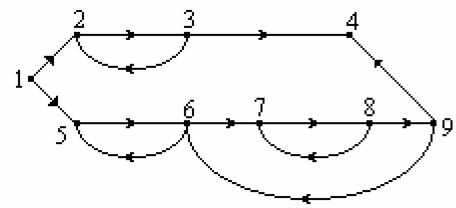
f) a) Block diagram:





$$\frac{Y(s)}{R(s)} = T_1 + T_2T_3 = \frac{1}{s+1} + \frac{1}{s+2}(4 + \frac{3}{s+2}) = \frac{1}{s+1} + \frac{4s+11}{(s+2)^2} = \frac{5s^2+19s+15}{s^3+5s^2+8s+4}$$

b) Signal flow graph:



$T_{12} = 1,\ T_{23} = 1/s,\ T_{34} = 1,$
$T_{15} = 1\ ,\ T_{56} = 1/s\ ,\ T_{67} = 1,$
$T_{78} = 1/s,\ T_{89} = 3,\ T_{32} = -1,$
$T_{69} = 4,\ T_{87} = -2,\ T_{94} = 1$

$P_1 = T_{12}T_{23}T_{34} = 1/s$
$P_2 = T_{15}T_{56}\ T_{67}T_{78}\ T_{89} = 3/s^2$
$P_3 = T_{15}T_{56}\ T_{69}T_{94} = 4/s$

$l_1 = T_{23}T_{32} = -1/s\ ,\ l_2 = T_{56}T_{65} = -2/s\ ,\ l_3 = T_{78}T_{87} = -2/s$

$\Delta = 1 - (l_1+l_2+l_3) + (l_1l_2+l_1l_3+l_2l_3) - (l_1l_2l_3)$
$= 1 - (-5/s) + (2/s^2+2/s^2+4/s^2) - (-4/s^3)$
$= 1 + 5/s + 8/s^2 + 4/s^3$

$\Delta_1 = 1 - (l_2+l_3) + l_2l_3 = 1 + 4/s + 4/s^2$

$\Delta_2 = 1 - l_1 = 1 + 1/s$

$\Delta_3 = 1 - (l_1+l_3) + l_1l_3 = 1 + 3/s + 2/s^2$

$$\frac{Y(s)}{R(s)} = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3}{\Delta} = \frac{(1/s)(1+4/s+4/s^2)+(3/s^2)(1+1/s)+4/s(1+3/s+2/s^2)}{1+5/s+8/s^2+4/s^3}$$
$$= \frac{5s^2+19s+15}{s^3+5s^2+8s+4}$$

3.9- Open Loop: $C(s) = G(s)R(s)$; where $R(s)=\frac{1}{s}$

$$C(s) = \frac{s+2}{s(s^2+4s+3)}\frac{1}{s} = \frac{s+2}{s^2(s+3)(s+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+3} + \frac{D}{s+1}$$

$\Rightarrow A=2/3;\ B=-5/9;\ C=1/18;\ D=1/2$

$$c(t) = \left(-\frac{5}{9} + \frac{2}{3}t + \frac{1}{8}e^{-3t} + \frac{1}{2}e^{-t}\right)u(t)$$

Closed Loop: $$G_{CL}(s)=\frac{G(s)}{1+G(s)} = \frac{\frac{s+2}{s^3+4s^2+3s}}{\frac{s^3+4s^2+3s}{s^3+4s^2+3s} + \frac{s+2}{s^3+4s^2+3s}} = \frac{s+2}{s^3+4s^2+4s+2}$$

$C(s)= G_{CL}(s)R(s)$; where $R(s)=\frac{1}{s}$

$$C(s)= \frac{s+2}{s(s+2.8)(s^2+1.2s+.7)} = \frac{1}{s} + \frac{.05}{s+2.8} - \frac{1.05s+1.07}{s^2+1.2s+.7}$$
$$= \frac{1}{s} + \frac{.05}{s+2.8} - 1.05\left(\frac{s+.6}{(s+.6)^2+(.58)^2} + .72\frac{.58}{(s+.6)^2+(.58)^2}\right)$$

$$c(t) = \left(1+.05e^{-2.8t} - 1.05e^{-.6t}(\cos .58t + .72\sin .58t)\right)u(t)$$

3.10- $$G_{CL}(s) = \frac{G(s)}{1+G(s)} = \frac{20}{s^2+8s+40}$$

C(s) = G(s)R(s) ; where R(s)=1/s

$$C(s) = \frac{20}{s^3+8s^2+40s} = \frac{1/2}{s} + \frac{.32\ \angle+39.23^\circ}{s+4-4.9j} + \frac{.32\ \angle-39.23^\circ}{s+4-4.9j}$$

$$c(t) = \left(1/2-.64e^{-4t}\cos(4.9t-39.23^\circ)\right)u(t)$$

3.11 - $1+G(s)H(s)= \begin{bmatrix} 1+\frac{1}{s+1} & \frac{2}{s(s+2)} \\ \frac{5}{s} & 1+10 \end{bmatrix} = \begin{bmatrix} \frac{s+2}{s+1} & \frac{2}{s(s+2)} \\ \frac{5}{s} & 11 \end{bmatrix}$

The closed-loop transfer matrix:

$$M(s) = [1+G(s)H(s)]^{-1}G(s)=\frac{1}{\Delta}\begin{bmatrix} 11 & \frac{-2}{s(s+2)} \\ -\frac{5}{s} & \frac{s+2}{s+1} \end{bmatrix}\begin{bmatrix} \frac{1}{s+1} & \frac{2}{s(s+2)} \\ \frac{5}{s} & 10 \end{bmatrix}$$

where: $\Delta = 11\frac{s+2}{s+1} - \frac{5}{s}\frac{2}{s(s+2)} = \frac{11s^4+44s^3+44s^2-10s-10}{s^2(s+1)(s+2)}$

Thus

$$M(s) = \frac{s^2(s+1)(s+2)}{11s^4+44s^3+44s^2-10s-10}\begin{bmatrix} \frac{11s^3+22s^2-10s-10}{s^2(s+1)(s+2)} & \frac{2}{s(s+2)} \\ \frac{5}{s} & -10\frac{s^2(s+2)^2-(s+1)}{s^2(s+1)(s+2)} \end{bmatrix}$$

3.11 - $G(s)=\frac{1}{s(s+2)} \Rightarrow G(z)= \frac{1}{2}\frac{(1-e^{-2T})z}{(z-1)(z-e^{-2T})} = \frac{1}{2}\frac{.865z}{(z-1)(z-.135)}$ (T=1)

$$\frac{C(z)}{R(z)} = \frac{.83\ z}{(z-1)(z-.135)}$$

3.12 - z.o.h = $\frac{1-e^{-Ts}}{s}$; $G(s) = \frac{1-e^{-Ts}}{s}\frac{5}{s(s+1)} = 5\frac{1-e^{-Ts}}{s^2(s+1)}$; $e^{-1}=0.3679$ (T=1)

$$G(z) = \frac{.3679z+0.2642}{(z-0.3679)(z-1)} = \frac{C(z)}{R(z)}$$

Chapter 4.

4.1- $m\ddot{x} = -kx - c\dot{x} + f(t) \Rightarrow m\ddot{x} + c\dot{x} + kx = f(t)$

For a transfer function model:

$(ms^2 + cs + k)\, X(s) = F(s)$

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}$$

4.2 -

$$\begin{cases} m\ddot{x} = -kx - k_1(x - x_1) - c_1(\dot{x} - x_1) + f \\ m_1\ddot{x}_1 = k_1(x - x_1) + c_1(\dot{x} - \dot{x}_1) \end{cases}$$

$$\begin{cases} m\ddot{x} + c_1\dot{x} + (k + k_1)x = c_1\dot{x}_1 + k_1x_1 + f \\ m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 = c_1\dot{x} + k_1x \end{cases}$$

$$\Rightarrow \begin{cases} (ms^2 + c_1s + k + k_1)X(s) = (c_1s + k_1)X_1(s) + F(s) \\ (m_1s^2 + c_1s + k_1)X_1(s) = (c_1s + k_1)X(s) \end{cases}$$

$$\Rightarrow X_1 = \frac{c_1s + k_1}{m_1s^2 + c_1s + k_1} X$$

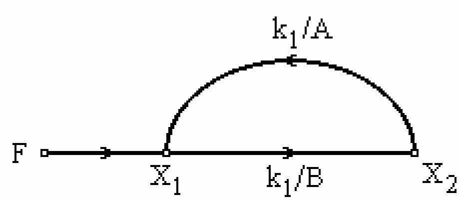
then :

$$\frac{X(s)}{F(s)} = \frac{m_1s^2 + c_1s + k_1}{mm_1s^4 + (mc_1 + c_1)s^3 + (mk_1 + m_1k + m_1k_1)s^2 + c_1ks + kk_1}$$

4.3 - $\begin{cases} F + k_1X_2 = (M_1s^2 + f_1s + k_1)X_2 \\ k_1X_1 = (M_2s^2 + f_2s + k_1 + k_2)X_2 \end{cases}$

$A = M_1s^2 + f_1s + k_1 \quad \& \quad B = M_2s^2 + f_2s + k_1 + k_2$

$$\left.\begin{aligned} \frac{1}{A}F + \frac{k_1}{A}X_2 &= X_1 \\ \frac{k_1}{B}X_1 &= X_2 \end{aligned}\right\} \Rightarrow \text{Signal flow graph is}$$



The forward path gain is $P_1 = \dfrac{k_1}{AB}$

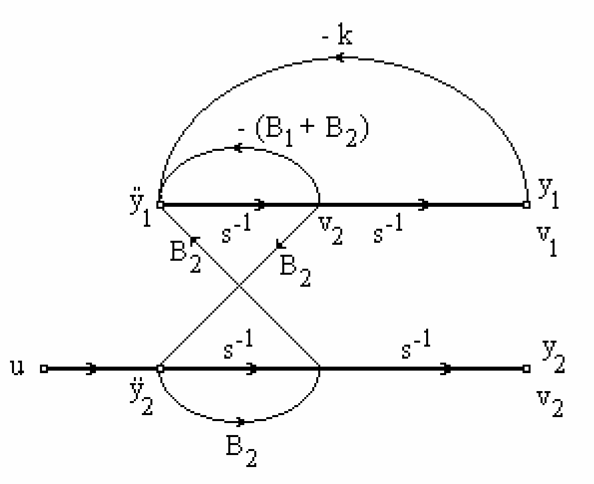
feedback loop gain $P_{11} = \dfrac{k_1^2}{AB}$

then: $\Delta = 1 - P_{11} = \dfrac{AB - k_1^2}{AB} + \Delta_1 = 1$

$$\frac{X_2}{F} = \frac{P_1\Delta_1}{\Delta} = \frac{k_1}{AB-k_1^2} = \frac{k_1}{(M_1s^2+f_1s+k_1)(M_2s^2+f_2s+k_1+k_2)-k_1^2}$$

4.5 -

$$\begin{cases} \ddot{y}_1 + B_1\dot{y}_1 + ky_1 + B_2(\dot{y}_1 - \dot{y}_2) = 0 \\ y_2 + B_2(\dot{y}_2 - \dot{y}_1) = u(t) \end{cases}$$



$$\dot{V} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k & -(B_1+B_2) & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & B_2 & 0 & -B_2 \end{bmatrix} V + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \; ; \quad y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} V$$

$$\left.\begin{aligned} \dot{v}(t) &= A_c v(t) + b_c u(t) \\ y(t) &= C_c v(t) + d_c u(t) \end{aligned}\right\} \rightarrow \begin{cases} sV(s) - V(0) = A_c V(s) + b_c U(s) \\ V(s) = (I_s - A_c)^{-1} V(0) + [I_s - A_c]^{-1} b_c U(s) \end{cases}$$

$$\phi_c(t) = \pounds^{-1} \; [I_s - A_c]^{-1} \rightarrow \; \phi_c(t) = A_o + A_1 t + A_2 t^2 + \ldots$$

$$\phi_c(t) = I + A_c t + A_c^2 \frac{t^2}{2!} + A_c^3 \frac{t^3}{3!} + \ldots$$

4.6 - $k(\theta_1 - \theta_2) = J_1\ddot{\theta}_2 + C(\dot{\theta}_2 - \dot{\theta}_3)$

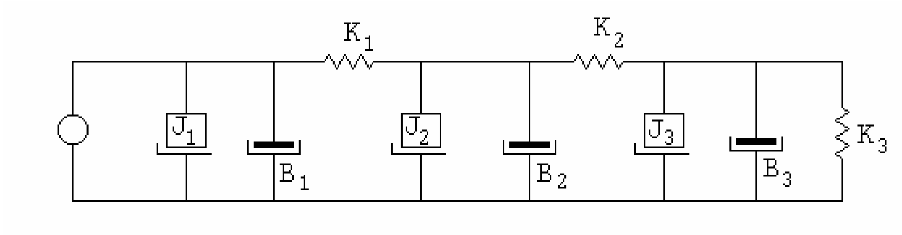
It's the damping torque which in turn accelerates inertia T_2

$$C(\dot{\theta}_2 - \dot{\theta}_3) = J_2\ddot{\theta}_3$$

$$\left.\begin{aligned} J_1\ddot{\theta}_2 + C\dot{\theta}_2 + k\theta_2 &= C\dot{\theta}_3 + k\theta_1 \\ J_2\ddot{\theta}_3 + C\dot{\theta}_3 &= C\dot{\theta}_2 \end{aligned}\right\} \rightarrow \begin{aligned} (J_1s^2 + Cs + k)\theta_2 &= Cs\theta_3 + k\theta_1 \\ (J_2s^2 + Cs)\theta_3 &= Cs\theta_2 \end{aligned}$$

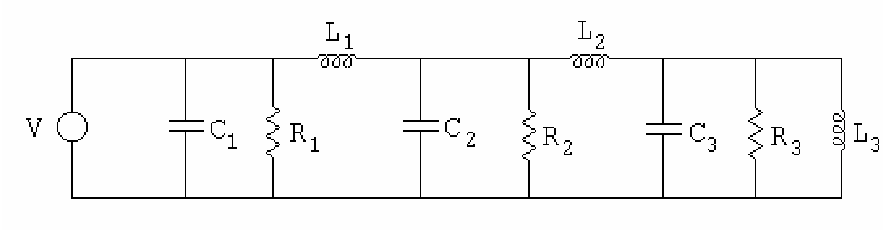
$$\theta_2 = \left(\frac{J_2}{C}s + 1\right)\theta_3 \rightarrow \frac{\theta_3}{\theta_1} = \frac{k}{(J_1J_2/C)s^3 + (J_1+J_2)s^2 + (J_2k/C)s + k}$$

4.7 - a) The mechanical network:



b) (1) $T = (J_1D^2+B_1D+K_1)\theta_1 - K_1\theta_2$
(2) $0 = -K_1\theta_1 +[J_2D^2+B_2D+(K_1+K_2)]\theta_2 - K_2\theta_2$
(3) $0 = -K_2\theta_2 +[J_2D^2+B_3D+(K_2+K_3)]\theta_3$

c)



d) T =
$\theta_3(t)=-16.4 + 2t + 21.7e^{-0.22t} - 4.81e^{-0.48t} + 0.852e^{-0.389t}\sin(.744t-.03)$
$+ 0.113e^{-0.411t}\sin(1.19t-4.83)$

4.8 - a) $f(t) = \frac{l_2}{l_1}(MD^2+BD+K)y$

Chapter 5.

5.1 -

a) $D(s) = (s^2+1)(s+2)(s+3) = (s-j)(s+j)(s+2)(s+3) = 0$
$\rightarrow$ Conditionally stable, 2 roots on the jw axis.

b) $D(s)= s^2(s+1)^2(s+2)(s+3) = 0$ $\rightarrow$ Unstable, repeated roots at origin.

c) $D(s)= s(s+1)(s+2)(s+3) = 0$ $\rightarrow$ Conditionally stable, simple roots at origin.

d) $D(S)=(s+1)^2(s+2)(s+3)=0$ $\rightarrow$ Asymptotically stable, all roots in L.H.P.

e) $D(s) = s^2(s+1)(s+2) = 0$ $\rightarrow$ Unstable repeated poles at origin.

f) $D(s) = s(s+1)(s+2) = 0$ $\rightarrow$ Conditionally stable, pole at origin.

g) $D(s) = (s+1)(s+2) = 0$ $\rightarrow$ Asymptotically stable, poles in the L.H.P.

h) $D(s) = (s+1)^2(s+2) = 0$ $\rightarrow$ Asymptotically stable, poles in the L.H.P.

i) $D(s) = (s+1)^2(s-2) = 0$ $\rightarrow$ Unstable, pole in the R.H.P.

j) $D(s) = (s+1)(s^2+s+1) = 0$ $\rightarrow$ Roots are: -1, -0.5 ± 0.866j.
All roots in L.H.P. Asymptotically stable.

k) $D(s)= s^2(s+1)(s^2+s+1) = 0$ $\rightarrow$ Unstable, repeated pair of roots at s=0.

l) $D(s)=s(s+1)(s^2+s+1)=0$ $\rightarrow$ Conditionally stable, simple root at s=0.

m) $D(s)=(s^2+1)(s^2+s+1)=0$ $\rightarrow$ Conditionally stable, simple roots at $s = \pm j$

n) $D(s)=(s+1)^2(s^2+s+1)=0$ $\rightarrow$ Roots are: -1, -1, -0.5 ± 0.566j .
All roots in L.H.P. Asymptotically stable.

o) $D(s)=s(s+1)(s-2)(s^2+s+1)=0$ $\rightarrow$ s=2 ; root in R.H.P. Unstable.

p) $D(s) = s^7+s^6-s^5+2s^4+3s^3+4s+4 = 0$

Routh array:

s^7		1	-1	3	4
s^6	1	1	2	0	4
s^5	-1/3	-3	3	0	
s^4					
s^3					
s^2					
s^1					
s^0					

The negative sign (-3) shows that the system is unstable.

q) $D(s) = s^7+s^6+s^5+s^3+s^2+1 = 0$
$\rightarrow$ Unstable, missing coefficient, it is neither odd or even

polynomial ; $a_n = 0$

r) $D(s) = s^6+2s^4+s^2+1 = 0 \rightarrow D'(s) = 6s^5+8s^3+2s$

$$\begin{array}{c|c|cccc}
s^6 & & 1 & 2 & 3 & 4 \\
 & 1/6 & & & & \\
s^5 & & 6 & 8 & 2 & 0 \\
 & 9 & & & & \\
s^4 & & 2/3 & 2/3 & 1 & \\
 & 1/3 & & & & \\
s^3 & & 2 & -7 & 0 & \\
 & 2/3 & & & & \\
s^2 & & 3 & 1 & & \\
s^1 & & -23/3 & 0 & & \\
s^0 & & 1 & & &
\end{array}$$

One sign change in the first column $\rightarrow$ the system is unstable.

s) $D(s) = s^5+6s^4+11s^3+6s^2 = s^2(s^3+6s^2+11s+6) = 0$
$\rightarrow$ Unstable, repeated roots at origin.

t) $D(s) = s^5+2s^3+s = s(s^4+2s^2+1)=s(s^2+1)^2=s(s+j)(s-j)(s+j)(s-j) = 0$
$\rightarrow$ Unstable, repeated roots on the jw axis.

u) $D(s) = s^4+10s^3+35s^2+50s+24$

$$\begin{array}{c|c|ccc}
s^4 & & 1 & 35 & 24 \\
 & 1/10 & & & \\
s^3 & & 10 & 50 & 0 \\
 & 1/3 & & & \\
s^2 & & 30 & 24 & \\
 & 30/42 & & & \\
s^1 & & 42 & & \\
s^0 & & 24 & &
\end{array}$$

No sign changes in the first column Asymptotically stable.

v) $D(s)=s^4+2s^3-5s^2+4s+2=0 \rightarrow$ Unstable, negative coefficient on s^2 term.

w) $D(s)=s^4+2s^3+5s^2+4s+2 = 0$

$$\begin{array}{c|c|ccc}
s^4 & & 1 & 5 & 2 \\
 & 1/2 & & & \\
s^3 & & 2 & 4 & 0 \\
 & 2/3 & & & \\
s^2 & & 3 & 2 & \\
 & 9/8 & & & \\
s^1 & & 8/3 & & \\
 & 4/3 & & & \\
s^0 & & 2 & &
\end{array}$$

This system is asymptotically stable.

x) $D(s) = s^4+5s^2+4 = (s^2+1)(s^2+4) = 0$
$\rightarrow$ Conditionally stable, roots on the jw-axis.

y) $D(s) = s^4-3s^3-5s^2+4s+3 = 0 \rightarrow$ Unstable, sign change.

z) $D(s) = s^3+6s^2+11s+6 = 0 \rightarrow$ Roots are: $s = -1, -2, -3$.
All roots in L.H.P Asymptotically stable.

5.2- 1) (a) $d(s) = s^4+8s^3+36s^2+80s+k = 0$

i. The system is asymptotically stable when : $k > 0$
$80 - (4/13)k > 0 \rightarrow k < [(80)(13)/4] = 260 \rightarrow 0<k<260$

ii. The system is conditionally stable: $k = 0$ or $k = 260$

iii. The system is unstable: $k > 260$, $k < 0$.

s^4		1	36	k
	1/8			
s^3		8	80	0
	4/13			
s^2		26	k	
s^1		$\frac{1040-4k}{13}$		
s^0		k		

(b) i. When k = 260 → auxiliary equation: $26s^2+260 = 0$
→ $s^2+10 = 0$; then by long division: $d(s)=(s^2+10)(s^2+8s+26)=0$
⇒ **Roots**: $s_{1,2} = \pm j\sqrt{10}$, $s_{3,4} = -4 \pm j\sqrt{10}$.

ii. When k = 0 → $d(s)=s(s^3+8s^2+36s+80)=s(s+4)(s^2+4s+20)=0$
⇒ **Roots**: s = 0 ,-4 , $-2 \pm j4$.

2) $d(s) = s^3+16s^2+650s+800k = 0$

(a)

s^3		1	650
	1/16		
s^2		16	800k
s^1		650-50k	
s^0		800k	

i. The system is asymptotically stable if 0 < k < 13

ii. The system is conditionally stable if k = 13 or k = 0.

iii. The system is unstable if k > 13, k < 0

(b) i. when k = 13 → $16s^2 + 800k = 0$ → $s^2 + 650 = 0$
⇒ **Roots**: $s_{1,2} = \pm j\ 25.5$, $s_3 = -16$.
Or $s^3+16s^2+650s+1040 = (s+16)(s^2+650) = 0$

ii. When k = 0 → (650 - 50k)s = 0 ⇒ **Roots**: $s_1 = 0$
$s^3+16s^2+650s+800k = s(s^2 +16s+650) = 0$ → $s^2+16s+650 = 0$
⇒ **Roots**: $s_{2,3} = -8 \pm j\ 24.207$.

5.3 - a) 20 = 17-(1/7)Q → Q = -21
P = Q - (7/20)6 → P = - 23.1

b) The system is unstable because P < 0.

c) 2 Asymptotically stable roots, and 2 Unstable roots because there exist 2 sign changes.

5.4 - $$\frac{C(s)}{R(s)} = \frac{\frac{k}{s(s+5)(s+10)}}{1+\frac{k}{s(s+5)(s+10)}} = \frac{k}{s^3+15s^2+50s+k}$$

1) Characteristic equation is obtained as:
$D(s) = s^3 + 15s^2 + 50s + k$
The Routh table is shown below:
i- For Asymptotic stability : k > 0 and 50-k/15 > 0.
Or k > 0 and 750 > k. **or** 0<k<750 (Asymptotically stable).

$$\begin{array}{l|c|cc} s^3 & 1/15 & 1 & 50 \\ s^2 & & 15 & k \\ s^1 & & 50-\frac{k}{15} & \\ s^0 & & k & \end{array}$$

ii- Unstable: $k<0$ or $k>750$.
iii- Conditionally stable: $k=0$ or $k=750$.

2) Auxiliary equation: when $k = 0$. $(50- 0/15)s = 0 \rightarrow s = 0$.
when $k=750 \Rightarrow 15s^2+750 = 0 \rightarrow s^2+50 = 0$

3) Poles: when $k = 0 \Rightarrow D(s) = s^3+15s^2+50s = s(s+5)(s+10) = 0$
$\Rightarrow s_1 = 0$; $s_2 = -5$; $s_3 = -10$
when $k = 750 \Rightarrow D(s) = s^3+15s^2+50s = (s+15)(s^2+50) = 0$
$\Rightarrow s_1 = -15$; $s_2 = j\sqrt{50}$; $s_3 = -j\sqrt{50}$

5.5 - a) $d(s) = s[s(s+2)(s^2+s+10)+k] = s^5+3s^4+12s^3+20s^2+ks$

$$\begin{array}{llll} s^5 & & 1 & 12 \quad k \\ s^4 & 1/3 & 3 & 20 \\ s^3 & 9/16 & 16/3 & k \\ s^2 & & 20-\frac{9k}{16} & \\ s & & k & \end{array}$$

The system is stable for $20-\frac{9k}{16}>0$; $k>0$

Then: $0 < k > 35.5$

b) $d(s) = 2x10^{-4}s^4 + 0.03s^3 + s^2 + ks$

$$\begin{array}{llll} s^4 & & .0002 & 1 \\ s^3 & .02/3 & 0.03 & k \\ s^2 & \frac{9}{300-2k} & 1-\frac{.02k}{3} > 0 \Rightarrow k < 150 & \\ s & & k > 0 & \end{array}$$

Then: $0 < k > 150$

c) $d(s) = s^5 + 13s^4 + 60s^3 + (100+k)s^2 + 5ks$

$$\begin{array}{lllll} s^5 & & 1 & 60 & 5k \\ s^4 & 1/13 & 13 & 100+k & \\ s^3 & \frac{169}{680-k} & \frac{680-k}{13} & 5k & \\ s^2 & & \frac{-k^2-265k+68000}{680-k}>0 & & \\ s & & 5k > 0 & & \end{array}$$

$\Rightarrow \frac{680-k}{13}>0 \rightarrow k < 680$

$-425<k<160$

Then: $0 < k < 160$

d) $d(s) = s^5 + 8s^4 + 17s^3 + 10s^2 + ks$

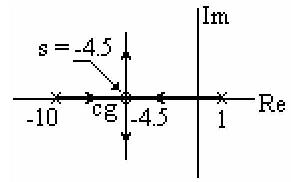
$$\begin{array}{llll} s^5 & & 1 & 17 \quad k \\ s^4 & 1/8 & 8 & 10 \\ s^3 & .5 & 15.75 & k \end{array}$$

26 Solution.

s^2	10-.5k	$\Rightarrow$ k < 20	
s	k	k>0	then: 0 < k < 20

Chapter 6.

6.1- **a**) $GH(s) = \dfrac{k}{(s-1)(s+10)}$



i) starting points: (k = 0) 1, -10
   ending points: (k = ∞) ∞, ∞

ii) real roots branches: [-10,1]

iii) center of gravity: $cg = \dfrac{-10+1}{2-0} = -4.5$

iv) The asymptotic: $\theta_0 = 90°$, $\theta_1 = 270°$

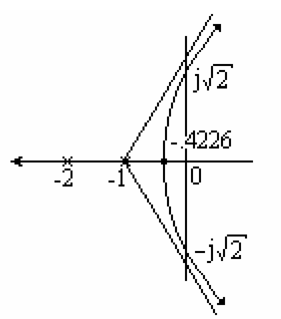
v) The breakaway points: $\dfrac{d[GH(s)]}{ds} = \dfrac{-k(2s+9)}{(s^2+2s-10)^2} = 0 \Rightarrow 2s+9=0 \Rightarrow s=-4.5$

vi) $d_c(s) = s^2 + 9s + (k-10)$

s^2	1	k-10
s^1	9	
s^0	k-10	

$\Rightarrow k = 10 \; ; \; s = 0$

b) $GH(s) = \dfrac{k}{s(s+1)(s+2)}$



i) starting points: 0, -1, -2
   ending points: ∞, ∞, ∞

ii) Branches: [0,-1], [-2,-∞]

iii) $cg = \dfrac{0-1-2}{3-0} = -1$

iv) Asymptotic lines: $\theta_0 = 60°$, $\theta_1 = 180°$, $\theta_2 = 300°$

v) Breakaway points: $\dfrac{d[GH(s)]}{ds} = \dfrac{-k[3s^2+6s+2]}{[s^3+3s^2+2s]^2} = 0$

$\Rightarrow \; 3s^2 + 6s + 2 = (s+0.4226)(s+1.5773) = 0$;
breakaway point is - 0.4226

$GH(s) = \dfrac{k}{s(s+1)(s+2)} = -1$; k = 0.3849

vi) $d_c(s) = s^3+3s^2+2s+k$

s^3	1	2
s^2	3	k
s^1	$2-\frac{1}{3}k$	
s^0	k	

$\Rightarrow 2-\frac{1}{3}k = 0 \rightarrow k = 6$

auxiliary equation: $3s^2+6 = 0$; $s = \pm j\sqrt{2}$

c) $GH(s) = \dfrac{k(s+6)}{s(s+2)(s+4)(s+10)}$

i) starting points: 0, -2, -4, -10
   ending points: -6, ∞, ∞, ∞

ii) Branches:]-∞,-10], [-6,-4], [-2,0]

iii) $cg = \frac{-16-(-6)}{4-1} = -\frac{10}{3}$

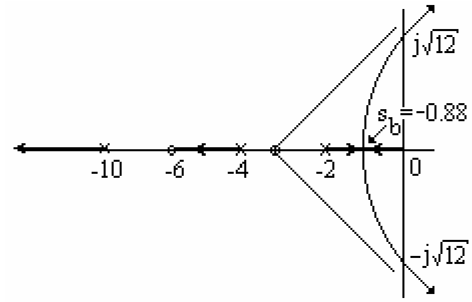
iv) Asymptotic lines: $\theta_o = 60°$, $\theta_1 = 180°$, $\theta_2 = 300°$

v) Breakaway points:

$$\frac{d[GH(s)]}{ds} = \frac{-3s^4-56s^3-356s^2-816s-480}{(s^4+16s^3+68s^2+80s)^2} = 0$$

$s_{1,2} = -7.269 \pm j1.70038$, $s_3 = -3.2475$

$s_4 = -0.884006 \Rightarrow s_b = -0.884$



vi) $d_c(s) = s^4+16s^3+68s^2+(80+k)s+6k$

s^4	1	68	6k
s^3	16	80+k	
s^2	63-k/16	6k	
s^1	$\frac{80640-608k-k^2}{1008-k}$	0	
s^0	6k		

$\Rightarrow k^2+608k-80640 = 0$; $k = 112$.

auxiliary eq. $(63-\frac{112}{16})s^2 + 6(112) = 0 \Rightarrow s = \pm j\sqrt{12}$

d) $\frac{C(s)}{R(s)} = \frac{(s+k)(s+0.27)}{s(s+0.1)+(s+k)(s+0.2)}$; $GH(s) = \frac{k(s+0.2)}{s(2s+0.3)}$

i) Starting points: 0, -0.15

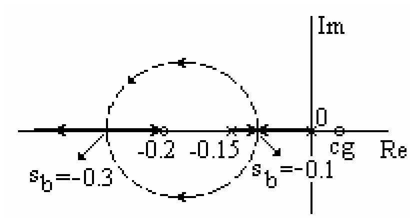
ending points: -0.2, ∞

ii) Branches: $]-\infty,-0.2]$, $[-0.15,0]$

iii) $cg = \frac{-0.15-(-0.2)}{2-1} = 0.05$

iv) Asymptotic lines: $\theta_o = \pi$.

v) Breakaway points:



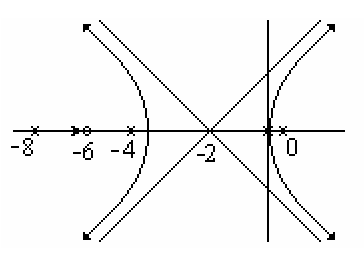
$$\frac{d[GH(s)]}{ds} = \frac{(2s^2+0.3s)-4s^2+1.1s+0.06}{[s(2s+0.3)]^2} = 0 \Rightarrow s_b = -0.3, \; -0.1$$

vi) $d_c(s) = s^4+16s^3+68s^2+(80+k)s+6k$

s^2	2	0.2k
s^1	k+0.3	
s^0	0.2k	

$\Rightarrow k+0.3=0$, $k=-0.3$ but $0 \le k \le \infty \Rightarrow$ no roots on jw axis.

e) $GH(s) = \frac{k(s+6)}{s^2(s+2)(s+4)(s+8)}$



i) Starting points: 0, 0, -2, -4, -8

ending points: -6, ∞, ∞, ∞, ∞

ii) Branches: [-2,-4] , [-6,-8]

iii) $cg = \frac{(-8-4-2)-(-6)}{5-1} = -2$

iv) Asymptotic lines: $\theta_o = 45°$, $\theta_1 = 135°$, $\theta_2 = 225°$, $\theta_3 = 315°$

v) $GH'(s) = \frac{-4ks(s^4+18s^3+112s^2+268s+182)}{[s^2(s+2)(s+4)(s+8)]^2} = 0$;

$s = -1.1, -3.727, -6.58 \pm j.95699 \Rightarrow s_b = -3.727$

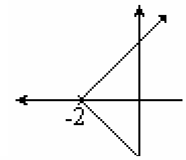
6.2- a) -2,-2,-2 & $-\infty,\infty,\infty$

60°, 180°, 300°

cg = -2

G' = 0 = $-3s^2-12s-10$

s = -1.184, -2.816

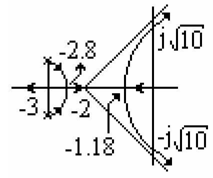


b) 0, -3±j1

cg = -2

60°, 180°, 300°

s = $\pm j\sqrt{10}$ for k = 60



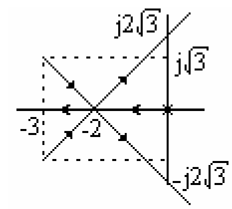
c) 0, $-3\pm j\sqrt{3}$

cg = -2

180°, 360°

$-3s^2-12s-12 = 0$; s = -2

s = $\pm j2\sqrt{3}$ for k = 72



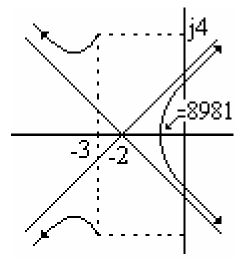
d) 0, -2, -3±j4

cg = -2

45°, 135°, 225°, 315°

$-4s^3-24s^2-74s-50 = 0$; s = -.8981

s = ±j25 for k = 192.1875

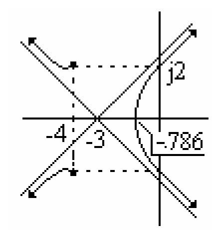


e) s = 0, -2, -4 ±j2

cg = - 2.5

45°, 135°, 225°, 315°

$s^3+7.5s^2+18s+10 = 0$; s = -.786

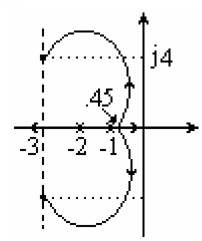


f) s = 0, -1, -2

cg = 3

180°

$s^3+36s^2+182s+150 = 0$; s = -.45

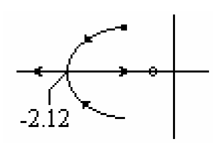


g) s = -1.5 ± j

cg = -2

180°

$s^2+2s-.25 = 0$; s = -2.12

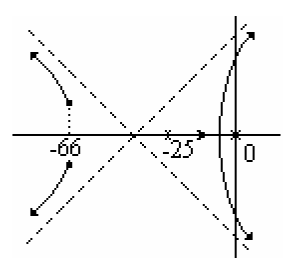


h) s = 0, -25, -50±j10

cg = 125/4

45°, 135°, 225°, 315°

$4s^3+405s^2+10200s+65000 = 0 \rightarrow$ s = -25, -66.



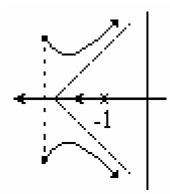
i) $s = -1,\ -2\pm j$

$cg = -5/3$

$60°,\ 180°,\ 300°$

$3s^2+10s+11 = 0$

$s = -1.67\pm j.94$

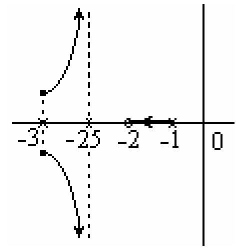


j) $s = -1,\ -3\pm j$

$cg = -2.5$

$90°$

$3s^2+14s+16 = 0 \Rightarrow s = -2.,\ -2.7$



6.3- $D(s) = s^3+6s^2+6s+5 = 0$; first divisor is: $6s^2+6s+5$ or $s^2+s+0.83$

$$\begin{array}{r|l} & s + 5 \\ \hline s^2+s+0.83 & s^3+6s^2+6s+5 \\ & \underline{s^3+s^2+0.83s} \\ & 5s^2+5.17s+5 \rightarrow \\ & \underline{5s^2+5s+4.15} \\ & 0.17s + 0.85 \end{array}$$

$$\begin{array}{r|l} & s + 4.966 \\ \hline s^2+1.034s+1 & s^3+6s^2+6s+5 \\ & \underline{s^3+1.043s^2+s} \\ & 4.966s^2+5s+5 \rightarrow \\ & \underline{4.966s^2+5.1348s+4.966} \\ & -0.1348s+0.034 \end{array}$$

$$\begin{array}{r|l} & s + 4.9932 \\ \hline s^2+1.0068s+1.006 & s^3+6s^2+6s+5 \\ & \underline{s^3+1.0068s^2+1.0068s} \\ & 4.9932s^2+4.9932s+5 \rightarrow \\ & \underline{4.9932s^2+5.027s+5.027} \\ & -0.034s - 0.027 \end{array}$$

$$\begin{array}{r|l} & s + 5 \\ \hline s^2+s+1.0014 & s^3+6s^2+6s+5 \\ & \underline{s^3+s^2+1.0014s} \\ & 5s^2+4.9986s+5 \rightarrow \\ & \underline{5s^2+5s+5.007} \\ & -0.0014s-0.007 \end{array}$$

$|\alpha| = 0.0014 \ll 0.01 \quad |\beta| = 0.007 \ll 0.001$

6.4 - Starting points: 0, -10, -20

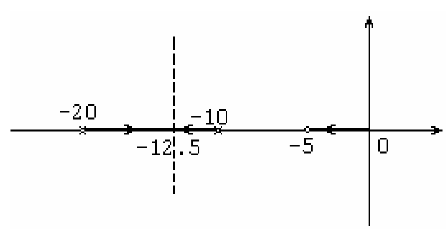
Ending points: $\propto,\ \propto,\ -5$

[-5,0] ; [-20,-10]

$cg = \dfrac{-30-(-5)}{3-1} = -12.5$

angles: $90°,\ 270°$

$s^3+30s^2+(200+k)s+5k=0 \rightarrow k=24$

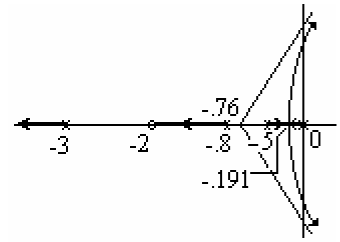


6.5 - a) $D(s)= s(s+0.5)(s+0.8)(s+3)+0.2(s+2) = s^4+4.3s^3+4.3s^2+1.4s+.4$

s^4	1	4.3	.4
s^3	4.3	1.4	
s^2	3.97	.4	
s^1	.97	0	
s^0	.4		

This system is stable.

b) s = 0, -0.5, -0.8, -3 & zeros -2, ∞, ∞, ∞.



$cg = \frac{-.5-.8-3-(-2)}{4-1} = -.766$

θ = 60°, 180°, 300°

GH' = $4s^3+12.9s^2+8.6s+1.2 = 0$

s = -.191, -.66, -2.37

$d(s) = s^4+4.3s^3+4.3s^2+(1.2+k)s+2k$

s^4	1	4.3	2k
s^3		4.3	k+1.2
s^2	4.02-k/4.3	2k	
s^1	$\frac{-k^2-20.9k+20.76}{17.3-k}$		
s^0	2k		

, $-k^2-20.9k+20.76 = 0$; k = -19.84, -1.04

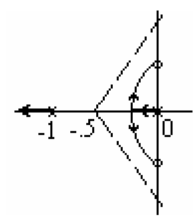
6.6- $T(s) = \dfrac{k(s+1)}{s^3+(2k+1)s^2+ks+k}$

a) $d(s) = s^3+21s^2+10s+10$

s^3	1	10
s^2	21	10
s^1	9.52	
s^0	10	

this system is stable.

b) $1+\frac{k(2s^2+s+1)}{s^2(s+1)} \rightarrow GH = \dfrac{(2s^2+s+1)}{s^2(s+1)}$



poles 0, 0, -1 ; zeros: $\frac{1}{4}(1 \pm j\sqrt{7})$

cg = - 1/2

Angles = 60°, 180°, 300°

$k=\frac{s^2(s+1)}{2s^2+s+1}$; $\frac{d}{ds}(2s^2+s+1)=4s+1 = 0$; s = -1/4

$d(s)= s^3+(2k+1)s^2+ks+k$

s^3	1	k
s^2	2k+1	k
s^1	2k	
s^0	k	

$k = 0 \rightarrow s = 0, -1$

6.7 - $d(s) = s^3 + as^2 + (k+2)s + k+1 = 0$

s^3	1	k+2
s^2 (1/a)	a	k+1
s^1	$k+2-\frac{k+1}{a}$	
s^0	k+1	

Conditionally stable: k+1 =0 $\rightarrow$ k=-1 (imp)

or $\frac{a(k+2)-(k+1)}{a} = 0 \Rightarrow a = \frac{k+1}{k+2}$

Given oscillates frequency: ω_n=2 rd/sec and ζ = 0.

$S_{1,2}= \pm j2 \Rightarrow s^2 = -4 \Rightarrow s^2+4 = 0$

$$
\begin{array}{r l}
 & \qquad\quad s + a \\
s^2 + 4 \big) & s^3 + as^2 + (k+2)s + k + 1 \\
 & \underline{-s^3 \qquad\quad -4s} \\
 & \quad as^2 + (k-2)s + k+1 \\
 & \quad \underline{-as^2 \qquad\qquad - 4a} \\
 & \qquad\quad (k-2)s + k+1-4a \qquad\qquad \Rightarrow k-2=0 \;\Leftrightarrow\; k = 2 \\
 & \qquad\qquad 0 \qquad\quad 0 \qquad\qquad\qquad \Rightarrow k+1-4a = 0 \;\Leftrightarrow\; a = 3/4
\end{array}
$$

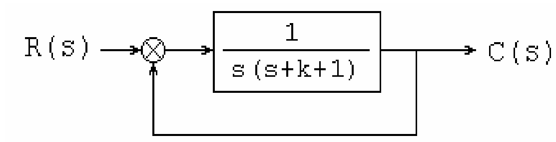
$d(s) = s^3 + \frac{3}{4}s^2 + 4s + 3$

Prove for: $s = j2 \Rightarrow d_c(j2) = -8j-3+8j+3 = 0$

$d_c(-j2) = 8j-3-8j+3 = 0$

$d_c(-\frac{3}{4}) = -\frac{27}{64} + \frac{27}{64} - 3 + 3 = 0$

6.8 -



1) $\dfrac{C(s)}{R(s)} = \dfrac{1}{s^2 + (k+1)s + 1}$

$d_c(s) = s^2 + s + 1 + ks = 1 + k\dfrac{s}{s^2 + s + 1} = 1 + GH(s)$

i) Starting pts: (k=0) $s_{1,2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

Ending pts: ($k=\infty$) $s = 0, \infty$

ii) Real root branches: $[-\infty, 0]$

iii) center of gravity: $cg = \frac{-1}{1} = -1$

iv) Asymptotic: $\theta_0 = \pi$

v) breaking pts: $\frac{d}{ds}[GH(s)] = 0 = \dfrac{s^2+s+1-s(2s+1)}{(s^2+s+1)^2}$

$s^2+s+1-2s^2-s = -s^2+1 = 0 \Rightarrow s_{1,2} = \pm 1 \rightarrow s_{b1} = -1$

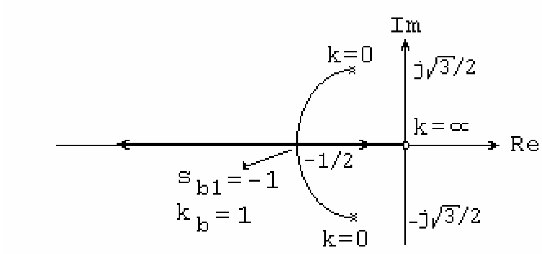
$d_c(s)\Big|_{s=s_{b1}} = 1 + k_{b1}(-1/1) = 0 \rightarrow k_{b1} = 1$

vi) The imaginary gain k_m:

$d_c(s) = s^2 + (k+1)s + 1$

s^2	1	1
s	$k+1$	0
s^0	1	

$k+1 = 0 \rightarrow k = -1$ (but: $0 \le k \le \infty$)



2) The poles of the closed-loop system is: $\frac{G}{1+GH}$

$\frac{C(s)}{R(s)} = \frac{1}{s^2 + (k+1)s + 1}$ when $k = \infty$; $s_1 = 0$ and $s_2 = \infty$

6.9 - $G(s) = \frac{k}{s(1+.01s)(1+0.02s)} = \frac{5000k}{s(s+50)(s+100)}$

1) starting points: 0, -50, -100
Ending points: ∞, ∞, ∞
Branches: [0,-50] , [-100,- ∞[
center gravity: $cg = \frac{0-50-100}{3-0} = -50$
Asymptotic: $\theta = \pm 60°$, 180°
Breakaway point: $\frac{dG}{ds} = 5000K \frac{3s^2+300s+5000}{(s^3+150s^2+5000s)^2}$
$3s^2+300s+5000=0 \rightarrow s = -21.2$ (k_1=9.5)
$s = \pm j70.7$ (k_1=150)

2) $d_c(s) = s^3 + 150s + 5000s + 500k$

s^3	1	5000
s^2	150 (1/150)	5000k
s	$5000-\frac{5000k}{150}$	$\Rightarrow k = 150$

3)