Lecture Four

Section 4.1 – Matrix Transformations from R^n to R^m

Definition

If V and W are vector spaces, and if f is a function with domain V and codomain W, then we say that f is a transformation from V to W or that f maps V to W, which we denote by writing

$$f : V \to W$$

In the special case where $V = W$, the transformation is also called an operator on V.

Matrix Transformation

$$\begin{aligned}
w_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
w_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
\vdots \quad & \quad \vdots \qquad\quad \vdots \qquad\qquad \vdots \\
w_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n
\end{aligned}$$

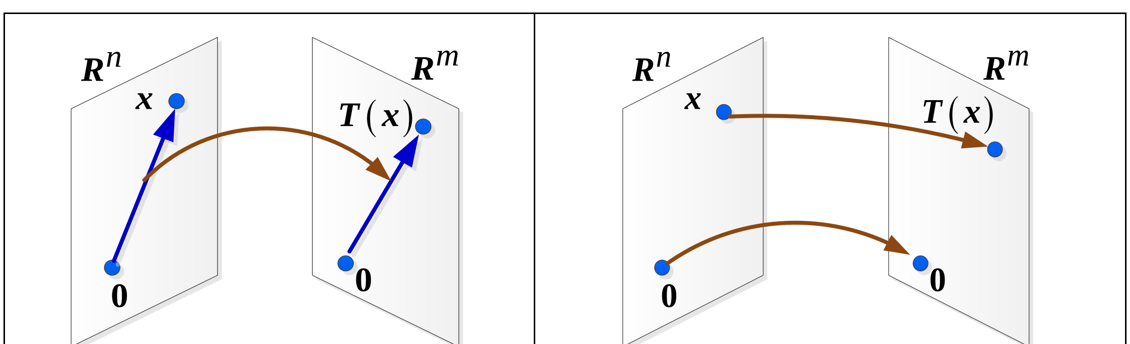
Which we can write in matrix formation

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ & & \\ a_{m1} & a_{m2} & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

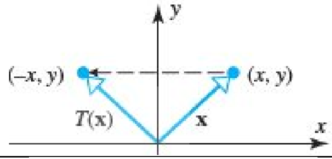
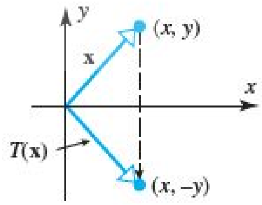
$$w = Ax$$

Although we could view this as a linear system, we will view it instead as a transformation that maps the column vector $\boldsymbol{x}$ in $\boldsymbol{R}^n$ into the column vector w in $\boldsymbol{R}^m$ by multiplying $\boldsymbol{x}$ on the left by A. We call this a ***matrix transformation*** (or ***matrix operator*** if $m = n$) and we denote it by

$$T_A : R^n \to R^m$$



T maps vectors to vectors	*T maps points to points*

<table>
<tr><td>Reflection about the y-axis
$T(x,y)=(-x,y)$</td><td>
</td><td>$T(e_1)=T(1,0)=(-1,0)$
$T(e_2)=T(0,1)=(0,1)$</td><td>$\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$</td></tr>
<tr><td>Reflection about the x-axis
$T(x,y)=(x,-y)$</td><td>
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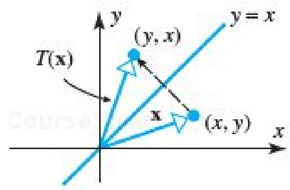
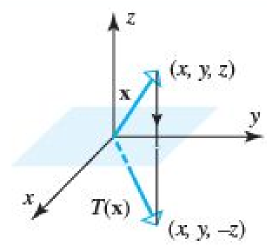
<tr><td>Reflection about the line
y = x
$T(x,y)=(x,-y)$</td><td>
</td><td>$T(e_1)=T(1,0)=(0,1)$
$T(e_2)=T(0,1)=(1,0)$</td><td>$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$</td></tr>
<tr><td>Reflection about the
xy-plane
$T(x,y,z)=(x,y,-z)$</td><td>
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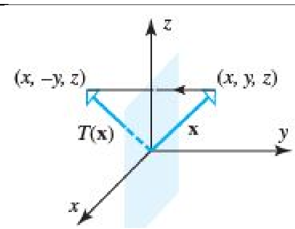
$T(x,y,z)=(x,-y,z)$</td><td>
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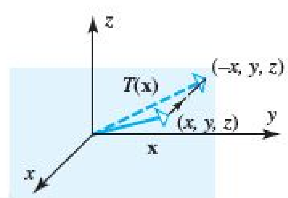
$T(x,y,z)=(-x,y,z)$</td><td>
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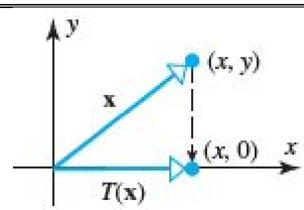
$T(x,y)=(x,0)$</td><td>
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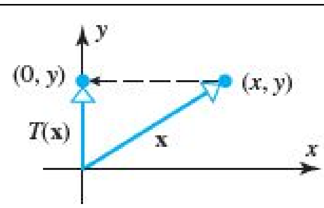
$T(x,y)=(0,y)$</td><td>
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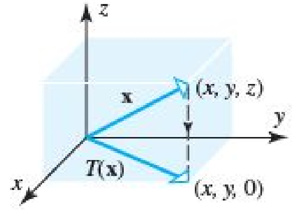
<tr>
<td>Orthogonal projection on the xy-Plane
$T(x,y,z)=(x,y,0)$</td>
<td>
</td>
<td>$T(1,0,0)=(1,0,0)$
$T(0,1,0)=(0,1,0)$
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<td>$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$</td>
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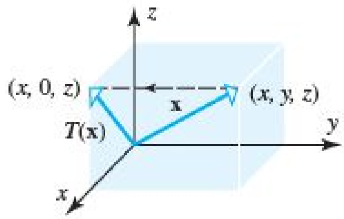
<td>
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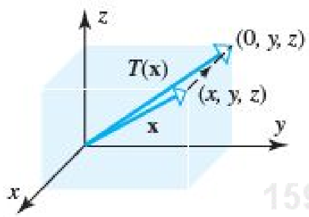
<td>
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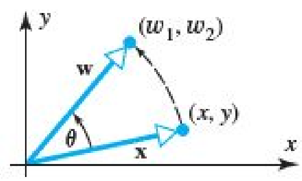
</tr>
<tr>
<td colspan="4">Rotation Operators</td>
</tr>
<tr>
<td>Rotation through an angle θ</td>
<td>
</td>
<td>$w_1=x\cos\theta-y\sin\theta$
$w_2=x\sin\theta+y\cos\theta$</td>
<td>$\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}$</td>
</tr>
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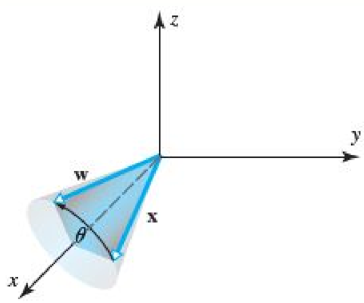
<td>Counterclockwise rotation about the positive x-axis through an angle θ</td>
<td>
</td>
<td>$w_1=x$
$w_2=y\cos\theta-z\sin\theta$
$w_3=x\sin\theta+z\cos\theta$</td>
<td>$\begin{pmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{pmatrix}$</td>
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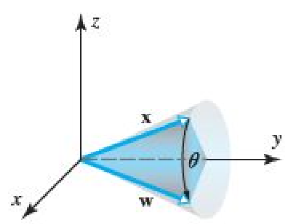
<td>
</td>
<td>$w_1=x\cos\theta+z\sin\theta$
$w_2=y$
$w_3=-x\sin\theta+z\cos\theta$</td>
<td>$\begin{pmatrix}\cos\theta&0&\sin\theta\\0&1&0\\-\sin\theta&0&\cos\theta\end{pmatrix}$</td>
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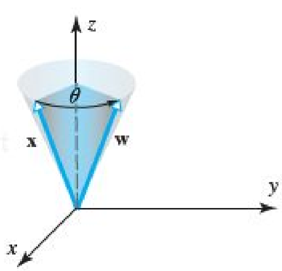
<td>
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<td>$\begin{pmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{pmatrix}$</td>
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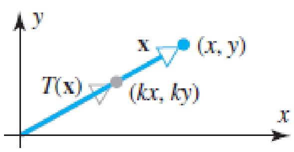
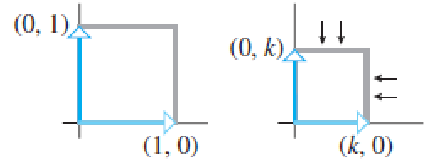
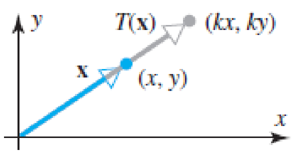
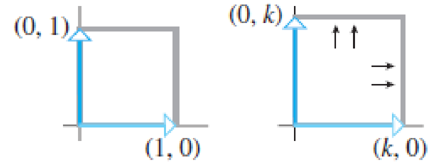
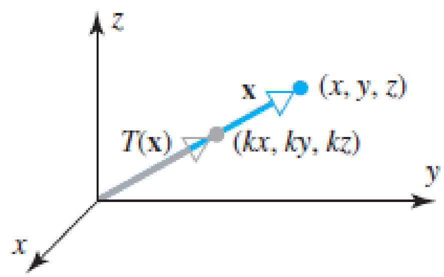
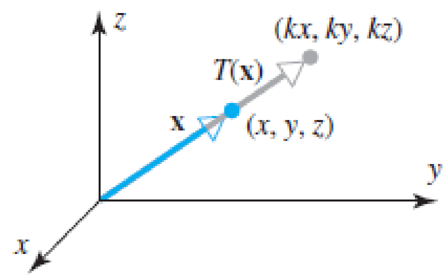
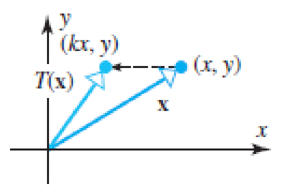
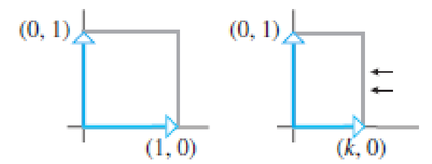
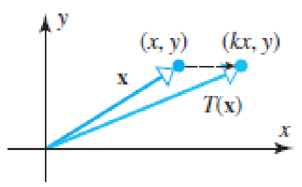
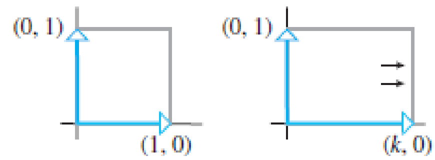
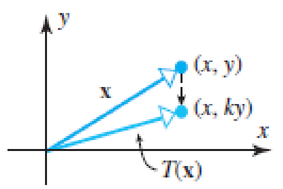
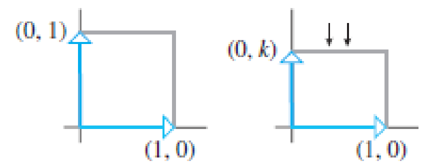
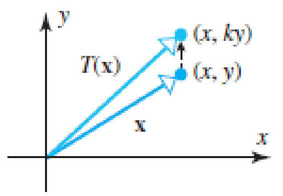
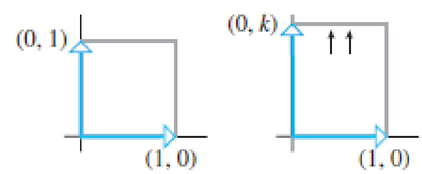
Dilations and Contractions

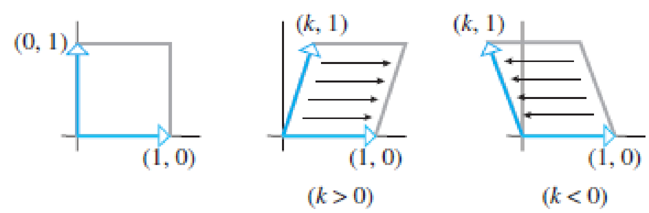
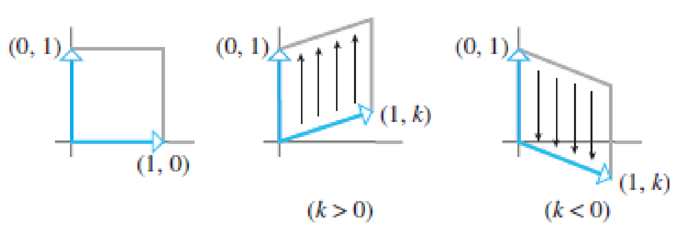
Contraction with factor k on R^2 $(0 \le k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$
Dilation with factor k on R^2 $(k > 1)$			

Contraction with factor k on R^3 $(0 \le k \le 1)$		$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$
Dilation with factor k on R^3 $(k \ge 1)$		

Expansion or Compression

Compression of R^2 in the x-direction with factor k $(0 \le k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Expansion of R^2 in the x-direction with factor k $(k > 1)$			
Compression of R^2 in the y-direction with factor k $(0 \le k < 1)$			$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
Expansion of R^2 in the y-direction with factor k $(k > 1)$			

Shear

Shear of R^2 in the x-direction with factor k $T(x, y) = (x + ky, y)$		$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Shear of R^2 in the y-direction with factor k $T(x, y) = (x, y + kx)$		$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Orthogonal Projections on Lines through the Origin

$$T(e_1) = \begin{bmatrix} \cos^2\theta \\ \sin\theta\cos\theta \end{bmatrix} \quad T(e_2) = \begin{bmatrix} \sin\theta\cos\theta \\ \sin^2\theta \end{bmatrix}$$

$$P_0 = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$$

Example

Find the orthogonal projection of the vector $\boldsymbol{x} = (1, 5)$ on the line through the origin that makes an angle of $\frac{\pi}{6}$ $(=30°)$ with the x-axis

Solution

$$\begin{aligned}
P_0 &= \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos^2\left(\frac{\pi}{6}\right) & \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right) & \sin^2\left(\frac{\pi}{6}\right) \end{pmatrix} \\
&= \begin{pmatrix} \left(\frac{\sqrt{3}}{2}\right)^2 & \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\ \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) & \left(\frac{1}{2}\right)^2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}
\end{aligned}$$

$$P_0\boldsymbol{x} = \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{3+5\sqrt{3}}{4} \\ \frac{\sqrt{3}+5}{4} \end{pmatrix} \approx \boxed{\begin{pmatrix} 2.91 \\ 1.68 \end{pmatrix}}$$

Four Fundamental Subspaces

1. The ***row space*** is $C\left(A^T\right)$, a subspace of $\boldsymbol{R}^n$.
2. The ***column space*** is $C(A)$, a subspace of $\boldsymbol{R}^m$.
3. The ***nullspace*** is $N(A)$, a subspace of $\boldsymbol{R}^n$.
4. The ***left nullspace*** is $N\left(A^T\right)$, a subspace of $\boldsymbol{R}^m$.

The Four Subspaces for R

Consider the matrix 3 by 5:

$$\begin{bmatrix} 1 & 3 & 5 & 0 & 9 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} m=3 & \textit{pivot rows } 1 \textit{ and } 2 \\ n=5 & \\ r=2 & \textit{pivot columns } 1 \textit{ and } 4 \end{matrix}$$

1. Rows 1 and 2 are a basis. The row space contains combination of all 3 rows.
The ***row space*** of ***R*** has dimension 2 (= ***rank***).
The dimension of the row space is r. The nonzero rows of ***R*** form a basis.

2. The ***column space*** of ***R*** has dimension *r* = 2.
The pivot columns 1 and 4 form a basis. They are independent because they start with the *r* by *r* identity matrix.
There are 3 special solutions:

$$C_2 = 3C_1 \qquad \textit{The special solution is } (-3,\ 1,\ 0,\ 0,\ 0)$$

$$C_3 = 5C_1 \qquad \textit{The special solution is } (-5,\ 0,\ 1,\ 0,\ 0)$$

$$C_5 = 9C_1 + 8C_2 \qquad \textit{The special solution is } (-9,\ 0,\ 0,\ -8,\ 1)$$

The dimension of the column space is r. The pivot columns form a basis.

3. The ***nullspace*** has dimension *n* – *r* = 5 – 2 = 3 (free variables). Here x_2, x_3, x_5 are free (no pivots in those columns). They yield the three special solutions to $Rx = 0$. Set a free variable to 1, and solve for x_1 and x_4.

$$s_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad s_3 = \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad s_5 = \begin{bmatrix} -9 \\ 0 \\ 0 \\ -8 \\ 1 \end{bmatrix}$$

$Rx = 0$ has the complete solution: $x = x_2 s_2 + x_3 s_3 + x_5 s_5$
The nullspace has dimension *n* – *r*. The special solutions form a basis.

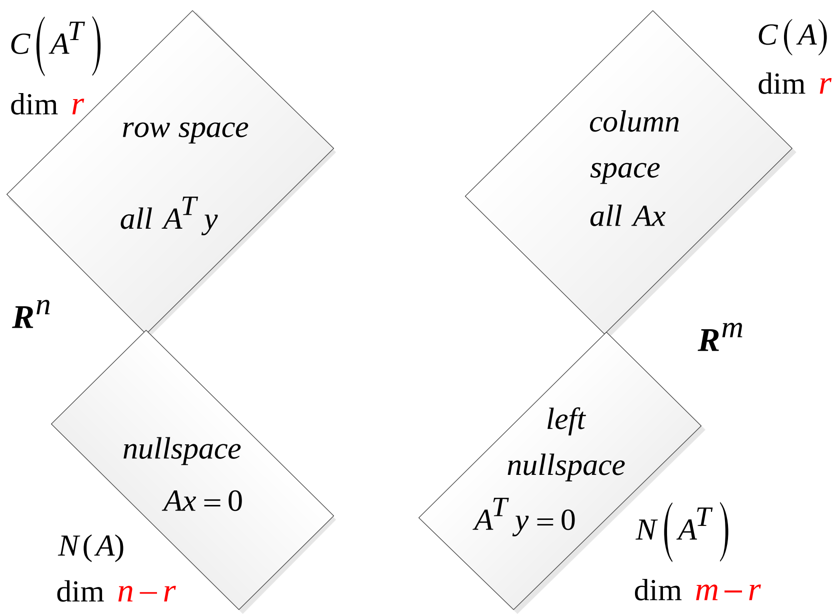
4. The ***nullspace*** of R^T has dimension $m - r = 3 - 2 = 1$

The equation $R^T y = 0$: $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 0 & 0 \\ 0 & 1 & 0 \\ 9 & 8 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} y_1 = 0 \\ y_2 = 0 \\ y_3 \text{ anything} \end{cases}$

The nullspace of R^T contains all vectors $y = (0,\ 0,\ y_3)$ and it is the line of the basis vector $(0,\ 0,\ 1)$.

The left nullspace has dimension m – r. The solutions are $y = (0, \ldots, y_{r+1}, \ldots, y_m)$

- In $\boldsymbol{R}^n$ the row space and nullspace have dimensions r and $n - r$ (adding to n)
- In $\boldsymbol{R}^m$ the column space and left nullspace have dimensions r and $m - r$ (total m)



The Four Subspaces for A

The subspace dimensions for A are the same as for R.

These matrices are connected by an invertible matrix E. $EA = R \quad and \quad A = E^{-1}R$

1. ***A** has the same row space as **R***. Same dimension r and same basis
   Every row of ***A*** is a combination of the rows of ***R***. Also every row of ***R*** is a combination of the rows of ***A***.

2. *The column space of* ***A*** *has dimension r*. The number of independent columns equals the number of independent rows.

3. ***A** has the same nullspace as **R***. Dimension $n - r$ and same basis.

   (*dimension of column space*) + (*dimension of nullspace*) = *dimension of* $\boldsymbol{R}^n$

4. *The left nullspace* ***A*** (the nullspace of A^T) *has dimension* $m - r$.

Fundamental Theorem of Linear Algebra, (Part 1)

The column space and row space both have dimension r.

The nullspaces have dimensions $n - r$ and $m - r$.

Example

Consider $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

A has $m = 1$, $n = 3$, and rank: $r = 1$.

The row space is a line in $\boldsymbol{R}^3$.

The nullspace is the plane $Ax = x_1 + 2x_2 + 3x_3 = 0$. This plane has dimension 2 (which is 3 – 1).

The columns of this is 1 by 3 matrix are in $\boldsymbol{R}^1$. The column space is all of $\boldsymbol{R}^1$.

The left nullspace contains only the zero vector.

The only solution to $A^T y = 0$ *is* $y = 0$, the only combination of the row that gives the zero row. Thus $N\left(A^T\right)$ *is* $\boldsymbol{Z}$, the zero space with dimension 0 ($m - r$). In $\boldsymbol{R}^m$ the dimensions (1 + 0) = 1.

Example

Consider $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

A has m = 2, n = 3, and rank: r = 1.

The row space is a line in $\boldsymbol{R}^3$.

The nullspace is the plane $x_1 + 2x_2 + 3x_3 = 0$. This plane has dimension 3 (1 + 2).

The columns are multiples of the first column (1, 1).

The left nullspace contains more than one zero vector. The solution to $A^T y = 0$ has the solution $y = (1, \ -1)$.

The column space and nullspace are perpendicular lines in $\boldsymbol{R}^2$. Their dimensions are 1 and 1 = 2.

Column space = line through $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Left nullspace=line through $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Exercises *Section* 4.1 – Matrix Transformations from R^n to R^m

1. Find the standard matrix for the transformation defined by the equations

a) $\begin{cases} w_1 = 2x_1 - 3x_2 + 4x_4 \\ w_2 = 3x_1 + 5x_2 - x_4 \end{cases}$

b) $\begin{cases} w_1 = 7x_1 + 2x_2 - 8x_3 \\ w_2 = \quad\quad -x_2 + 5x_3 \\ w_3 = 4x_1 + 7x_2 - x_3 \end{cases}$

c) $\begin{cases} w_1 = x_1 \\ w_2 = x_1 + x_2 \\ w_3 = x_1 + x_2 + x_3 \\ w_4 = x_1 + x_2 + x_3 + x_4 \end{cases}$

2. Find the standard matrix for the operator T defined by the formula

a) $T\left(x_1, x_2\right) = \left(2x_1 - x_2,\ x_1 + x_2\right)$

b) $T\left(x_1, x_2, x_3\right) = \left(x_1 + 2x_2 + x_3,\ x_1 + 5x_2,\ x_3\right)$

c) $T\left(x_1, x_2, x_3\right) = \left(4x_1,\ 7x_2,\ -8x_3\right)$

3. Find the standard matrix for the transformation T defined by the formula

a) $T\left(x_1, x_2\right) = \left(x_2,\ -x_1,\ x_1 + 3x_2,\ x_1 - x_2\right)$

b) $T\left(x_1, x_2, x_3\right) = (0,\ 0,\ 0,\ 0,\ 0)$

c) $T\left(x_1, x_2, x_3, x_4\right) = \left(x_4,\ x_1,\ x_3,\ x_2,\ x_1 - x_3\right)$

d) $T\left(x_1, x_2, x_3, x_4\right) = \left(7x_1 + 2x_2 - x_3 + x_4,\ x_2 + x_3,\ -x_1\right)$