Math 1325 - Calculus with Application ***Exam* 3** ***Review***

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1. Find the critical numbers and the open intervals on which the function is increasing or decreasing:

   a. $f(x) = x^3 - 3x + 2$ **b.** $f(x) = \sqrt{9 - x^2}$ ***c.*** $f(x) = \dfrac{x^2}{x^2 + 4}$

   d. $f(x) = x\sqrt{2 - x^2}$ ***e.*** $f(x) = \left(4 - x^2\right)^{2/3}$ ***f.*** $f(x) = \dfrac{x^2 - 3x - 4}{x - 2}$

2. Find all relative extrema of:

   a. $\dfrac{6x}{x^4 + 3}$

   b. $f(x) = 2x^3 - 12x^2 + 2$

   c. $f(x) = x^4 - 8x^3 + 16x^2 + 9$

   d. $f(x) = x^4 - 2x^3 + 1$

3. Find the absolute extrema of

   a. $f(x) = x^3 - 12x$ on the closed interval [0, 3]

   b. $f(x) = (x - 5)^{2/3}$ on the closed interval [–6, 8]

   c. $f(x) = 3x^2 - 6x + 1$ on the closed interval [–2, 2]

4. Find the point(s) of inflection and determine the concavities of

   a. $f(x) = x^4 - 18x^2 + 5$

   b. $f(x) = 6x^3 - 8x^2 - 6x - 2$

   c. $f(x) = x(6 - x)^2$

Applications

5. Suppose the resident population P(in millions) of the United States can be modeled by $P = 0.00000583t^3 + 0.005003t^2 + 0.13776t + 4.658;\ -4 \le t \le 197$, where $t = 0$ corresponds to 1800. Analytically find the minimum and maximum populations in the U.S. for $-4 \le t \le 197$

6. The number of milligrams x of a medication in the bloodstream t hours after a dose is taken can be modeled by $x(t) = \dfrac{2000t}{t^2+3}$; t > 0. Find the maximum value of x. Round your answer to two decimal places

7. The concentration (in milligrams/cubic centimeter) of a certain drug in a patient's body t hr after injection is given by
$$C(t) = \frac{t^2}{2t^3+1} \quad (0 \le t \le 11)$$
At what time is the concentration the maximum?

8. The number of people who denoted to a certain organization between 1990 and 2007 can be modeled by the equation: $D(t) = -10.73\,t^3 + 208.81\,t^2 - 169.8\,t + 9775.23$ donors, where t is the number of year after 1990. Find the point of inflection $0 \le t \le 17$

9. Suppose that the total number of units produced by a worker in t hours of an 8-hour shift can be modeled by the production function $P(t)$: $P(t) = 54t + 24t^2 - 2t^3$. Find the number of hours before the rate of production is maximized. That is, find the point of diminishing returns.

10. If the cost function for a product is $C(x) = 500 + 3x + 0.08x^2$ dollars. Determine how many units $\boldsymbol{x}$ should be produced to minimize the average cost per unit.

11. A travel agency will plan a tour for groups of size 28 or larger. If the group contains exactly 28 people, the price is \$500 per person. However, each person's price is reduced by \$10 for each additional person above the 28 . If the travel agency incurs a price of \$100 per person for the tour, what size group will give the agency the maximum profit?

12. A rectangular box with a square base is to be formed from a square piece of paper with 42" sides. If a square piece with side a is cut from each corner of the paper and the sides are folded up to from an open box the volume of the box is $V = (42 - 2x)^2\,x$. What value of $\boldsymbol{x}$ will maximize the volume of the box?

13. Determine the dimensions of a rectangular solid (with a square base) with maximum volume if its surface area is 400 square feet.

Solution

1. a. CN: $x = \pm 1$; $incr. (-\infty, -1)$ *and* $(1, \infty)$, $decr. (-1, 1)$

   b. CN: $x = 0, \pm 3$; $incr. (-3, 0)$, $decr. (0, 3)$

   c. CN: $x = 0$; $decr. (-\infty, 0)$, $incr. (0, \infty)$

   d. CN: $x = \pm 1, \pm\sqrt{2}$; $incr. (-1, -1)$, $decr. \left(-\sqrt{2}, -1\right)$ *and* $\left(1, \sqrt{2}\right)$

   e. CN: $x = 0, \pm 2$; $incr. (-2, 0) \,\&\, (2, \infty)$, $decr. (-\infty, -2) \,\&\, (0, 2)$

   f. CN: $x = 2$; $incr. (-\infty, 2)$ *and* $(2, \infty)$

2. a. RMAX: $(1, \frac{3}{2})$; RMIN: $(-1, -\frac{3}{2})$

   b. RMAX: (0, 2) ; RMIN: (4, –62)

   c. RMAX: (2, 25) ; RMIN: (0, 9) & (4, 9)

   d. RMIN: (3/2, –0.69)

3. a. Absolute Max: (0, 0); Absolute Min: (2, –16)

   b. Absolute Max: 4.95; Absolute Min: 0

   c. Absolute Max: (–2, 25); Absolute Min: (1, –2)

4. a. $(-\sqrt{3}, -40)$ *and* $(\sqrt{3}, -40)$ *Concave up:* $\left(-\infty, -\sqrt{3}\right)$ $\left(\sqrt{3}, \infty\right)$ *Concave down:* $\left(-\sqrt{3}, \sqrt{3}\right)$

   b. $x = \frac{4}{9}$ *Concave up:* $\left(\frac{4}{9}, \infty\right)$ *Concave down:* $\left(-\infty, \frac{4}{9}\right)$

   c. $(4, 16)$ *Concave up:* $(4, \infty)$ *Concave down:* $(-\infty, 4)$

5. The population is minimum at $t = -4$ and maximum at $t = 197$

6. 577.35 mg

7. maximum at $t = 1$ sec

8. There is one inflection point @ $t = 6.49$

9. $t = 4$

10. 79 units

11. 34

12. 7

13. square base side $\frac{10\sqrt{6}}{3}$; height $\frac{10\sqrt{6}}{3}$

1-***b*** $f(x)=\sqrt{9-x^2}$ Domain: $-3 \le x \le 3$

$$f(x)=\left(9-x^2\right)^{1/2}$$

$$f'(x)=\tfrac{1}{2}(-2x)\left(9-x^2\right)^{-1/2} \qquad \left(9-x^2\right)'=-2x$$

$$=-x\left(9-x^2\right)^{-1/2}$$

$$=-\frac{x}{\sqrt{9-x^2}}=0$$

You have to consider what make the denominator is equal to zero as CN.

Solve for x

$$\Rightarrow \boxed{x=0} \qquad 9-x^2=0$$

$$x^2=9 \to \boxed{x=\pm 3}$$

-3 0	3
$f'(-1)>0$	$f'(1)<0$

$$f'(-1)=-\frac{-1}{\sqrt{9-(-1)^2}}=\frac{1}{\sqrt{9-1}}>0$$

$$incr.(-3,0),\quad decr.\ (0,\ 3)$$

6. Suppose the resident population *P* (in millions) of the United States can be modeled by

$P=0.00000583t^3+0.005003t^2+0.13776t+4.658;\ -4 \le t \le 197$**, where *t* = 0 corresponds to 1800.**

Analytically find the minimum and maximum populations in the U.S. for $-4 \le t \le 197$

$$P'=0.00001749t^2+0.010006t+0.13776$$

$$t=\frac{-0.010006 \pm \sqrt{(.010006)^2-4(.00001749)(.13776)}}{2(.00001749)}=\frac{-0.010006 \pm \sqrt{.00009048}}{0.00003498}=\frac{-0.010006 \pm .0095}{0.00003498}$$

$$t=\frac{-0.010006 \pm .0095}{0.00003498}\begin{cases}-14.47\\-557\end{cases}$$ But since t is $-4 \le t \le 197$

That imply the population is minimum @ t = -4 and maximum @ t =197

7. The number of milligrams *x* of a medication in the bloodstream *t* hours after a dose is taken can be modeled by

$x(t)=\dfrac{2000t}{t^2+3}$ **; t > 0. Find the maximum value of *x*. Round your answer to two decimal places**

$$x'(t)=\frac{2000(t^2+3)-2000t(2t)}{(t^2+3)^2}=2000\frac{t^2+3-2t^2}{(t^2+3)^2}=2000\frac{3-t^2}{(t^2+3)^2}$$

$$3-t^2=0 \Rightarrow t^2=3 \Rightarrow t=\pm\sqrt{3} \Rightarrow t=\sqrt{3}$$

$$x(t=\sqrt{3})=\frac{2000\sqrt{3}}{(\sqrt{3})^2+3}=\frac{2000\sqrt{3}}{3+3}=577.35mg$$

8. **The concentration (in milligrams/cubic centimeter) of a certain drug in a patient's body *t* hr after injection is given by** $C(t)=\dfrac{t^2}{2t^3+1}$ $(0\le t\le 11)$ **At what time is the concentration the maximum?**

$$C'(t)=\frac{2t(2t^3+1)-t^2(6t^2)}{(2t^3+1)^2}=\frac{4t^4+2t-6t^4}{(2t^3+1)^2}=\frac{2t-2t^4}{(2t^3+1)^2}=\frac{2t(1-t^3)}{(2t^3+1)^2}=0$$

$$2t(1-t^3)=0\rightarrow\begin{cases}2t=0 \Rightarrow t=0\\ 1-t^3=0\rightarrow t^3=1\Rightarrow t=1\end{cases}\qquad t=1$$

9. **The number of people who denoted to a certain organization between 1990 and 2007 can be modeled by the equation:** $D(t)\ =\ -10.73\,t^3\ +\ 208.81\,t^2\ -\ 169.8\,t\ +\ 9775.23$ **donors, where t is the number of year after 1990. Find the point of inflection** $0\le t\le 17$

$$D'(t) = -32.19\, t^2 + 417.62\, t - 169.8$$

$$D''(t)\ =\ -64.38\,t\ +\ 417.62\ =0\quad\Rightarrow t=\frac{417.62}{64.38}\approx 6.49$$

10. **Suppose that the total number of units produced by a worker in *t* hours of an 8-hour shift can be modeled by the production function *P*(*t*) :** $P(t) = 54t + 24t^2 - 2t^3$ **. Find the number of hours before the rate of production is maximized. That is, find the point of diminishing returns.**

$$P'=54+48t-6t^2$$
$$P''=48-12t=0\Rightarrow -12t=-48\Rightarrow t=4$$

11. **If the cost function for a product is** $C(x) = 500 + 3x + 0.08x^2$ **dollars. Determine how many units x should be produced to minimize the average cost per unit.**

$$\overline{C}=\frac{C}{x}=\frac{500}{x}+3\frac{x}{x}+0.08\frac{x^2}{x}=\frac{500}{x}+3+0.08x=500x^{-1}+3+0.08x$$

$$\overline{C'}=-500x^{-2}+0.08=0$$

$$x^2\left(-500x^{-2}+0.08=0\right)\Rightarrow -500+0.08x^2=0$$

$$\Rightarrow 0.08x^2=500\Rightarrow x^2=\frac{500}{0.08}\rightarrow x=\sqrt{\frac{500}{0.08}}=79\ units$$

12. **A travel agency will plan a tour for groups of size 28 or larger. If the group contains exactly 28 people, the price is \$500 per person. However, each person's price is reduced by \$10 for each additional person above the 28 . If the travel agency incurs a price of \$100 per person for the tour, what size group will give the agency the maximum profit?**

$$p=500-10(x-28)=500-10x+280$$
$$p=-10x+780$$
$$P=xp-C=x(-10x+780)-100x$$
$$P=-10x^2+780x-100x$$
$$P=-10x^2+680x$$

$$P' = -20x + 680 = 0 \Rightarrow -20x = -680 \Rightarrow x = \frac{-680}{-20} = 34$$

13. **A rectangular box with a square base is to be formed from a square piece of paper with 42” sides. If a square piece with side a is cut from each corner of the paper and the sides are folded up to from an open box the volume of the box is V = (42 – 2x)^2 x. What value of x will maximize the volume of the box?**

$$V' = (42-2x)^2 + 2x(42-2x)(-2)$$

$$V' = (42-2x)\left[42-2x-4x\right]$$

$$V' = (42-2x)(42-6x) = 0 \Rightarrow \begin{cases} 42-2x=0 \to x=21 \\ 42-6x \to x=\frac{42}{6}=7 \end{cases}$$

For x = 21 $\Rightarrow V(21) = (42-2(21))^2(21) = 0(21) = 0$ is not a solution

For x = 7 $V(7) = (42-2(7))^2(7) = 28^2(7) = 5488$

14. **Determine the dimensions of a rectangular solid (with a square base) with maximum volume if its surface area is 400 square feet.**

Area for the base = x^2.

Area of each side = xh

$$S = 2x^2 + 4xh = 400 \Rightarrow x^2 + 2xh = 200$$

$$\Rightarrow 2xh = 200 - x^2 \qquad \Rightarrow h = \frac{200-x^2}{2x}$$

$$V = x^2 h = x^2\,\frac{200-x^2}{2x} = \frac{x(200-x^2)}{2} = \frac{200x-x^3}{2} = 100x - \frac{x^3}{2}$$

$$V' = 100 - \tfrac{3}{2}x^2 = 0$$

$$-\tfrac{3}{2}x^2 = -100 \Rightarrow x^2 = \frac{200}{3}$$

$$\Rightarrow x = \sqrt{\frac{200}{3}} = \frac{\sqrt{200}}{\sqrt{3}} = \frac{\sqrt{100}\sqrt{2}}{\sqrt{3}}\,\frac{\sqrt{3}}{\sqrt{3}} = \frac{10\sqrt{6}}{3}$$

$$\Rightarrow h = \frac{200-x^2}{2x} = \frac{200-\left(\frac{10\sqrt{6}}{3}\right)^2}{2\frac{10\sqrt{6}}{3}} = \frac{200-\frac{600}{9}}{\frac{20\sqrt{6}}{3}} = (200 - \frac{200}{3})\frac{3}{20\sqrt{6}}$$

$$\Rightarrow h = \left(\frac{600-200}{3}\right)\frac{3}{20\sqrt{6}}\,\frac{\sqrt{6}}{\sqrt{6}} = \left(\frac{400}{3}\right)\frac{3\sqrt{6}}{20(6)} = \frac{10\sqrt{6}}{3}$$