I. 7/2

2.7. O.D.E.

First Order D.E.

$$y' = \frac{dy}{dx} = f(x,y)$$

Ex $y = \frac{c}{x} + 2 \Rightarrow y' = \frac{1}{x}(2-y)$ $(0,\infty)$

$$y' = -\frac{c}{x^2}$$

$$y' \stackrel{?}{=} \frac{1}{x}(2-y)$$

$$-\frac{c}{x^2} \stackrel{?}{=} \frac{1}{x}\left(2-\frac{c}{x}+2\right)$$

$$= \frac{1}{x}\left(-\frac{c}{x}\right)$$

$$= -\frac{c}{x^2} \checkmark.$$

Ex $y = (x+1) - \frac{1}{3}e^x$ is soln! $y' = y - x$ $y(0) = \frac{2}{3}$

$$y' = 1 - \frac{1}{3}e^x$$

$$y' = y - x$$

$$1 - \frac{1}{3}e^x = x + 1 - \frac{1}{3}e^x - x$$

$$= 1 - \frac{1}{3}e^x \checkmark.$$

$$y(0) = \frac{2}{3} \stackrel{?}{=} 0 + 1 - \frac{1}{3}e^0$$

$$= 1 - \frac{1}{3}$$

$$= \frac{2}{3} \checkmark.$$

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y' : slope

$$y' + Py = G$$

Method: Separable Eqn.

$$\frac{dy}{dx} + P(x)\,y = 0 \qquad \text{Homogeneous.}$$

$$\frac{dy}{dx} = -Py$$

$$\int \frac{dy}{y} = -\int P(x)\,dx$$

$$\ln|y| = -\int P(x)\,dx + C$$

$$y = e^{-\int P\,dx + C}$$

$$= e^{-\int P\,dx}\, e^{C} \qquad e^{C} \text{ is constant}$$

$$= A\, e^{-\int P\,dx}$$

Ex $y' = ty^2$

$$\frac{dy}{dt} = ty^2$$

$$\int \frac{dy}{y^2} = \int t\,dt$$

$$-\frac{1}{y} = \frac{1}{2}t^2 + C$$

$$= \frac{t^2 + 2C}{2}$$

$$y = -\frac{2}{t^2 + 2C}$$

Ex: $y' = \frac{2xy + 2x}{x^2 - 1}$

$$\frac{dy}{dx} = \frac{2x(y+1)}{x^2-1}$$

$$\frac{dy}{y+1} = \frac{2x}{x^2-1}\,dx$$

$$\int \frac{d(y+1)}{y+1} = \int \frac{d(x^2-1)}{x^2-1}$$

$$\ln|y+1| = \ln|x^2-1| + \ln C$$

$$= \ln C|x^2-1|$$

$$y+1 = C|x^2-1|$$

$$y = C|x^2-1| - 1$$

$$y' + py = f$$

Ex $x\frac{dy}{dx} = x^2 + 3y$

$$y' - \frac{3}{x}y = x$$

$$e^{-\int \frac{3}{x}dx} = e^{-3\ln x} = e^{\ln x^{-3}}$$

$$= x^{-3}$$

$$\int x\,x^{-3}dx = \int x^{-2}dx$$

$$= -\frac{1}{x}$$

$$y(x) = \frac{1}{x^{-3}}\left(-\frac{1}{x} + C\right) \quad x^3$$

$$= -x^2 + Cx^3 \qquad x > 0$$

$$y(x) = \frac{1}{e^{\int P(x)dx}}\left(\int f(x)\,e^{\int P(x)dx} + C\right)$$

$3xy' - y = \ln x + 1 \qquad y(1) = -2$

$$y' - \frac{1}{3x} y = \frac{\ln x + 1}{3x}$$

$$e^{-\int \frac{1}{3x} dx} = e^{-\frac{1}{3}\ln x} = e^{\ln x^{-1/3}} = x^{-1/3}$$

$$\int \frac{\ln x + 1}{3x} (x^{-1/3}) dx = \frac{1}{3} \int x^{-4/3} (\ln x + 1) dx$$

$u = \ln x + 1 \qquad v = \int x^{-4/3} dx$

$du = \frac{1}{x} dx \qquad = -3x^{-1/3}$

$$= \frac{1}{3}\left[-3x^{-1/3}(\ln x + 1) + 3\int \frac{x^{-1/3}}{x} dx\right]$$

$$= -x^{-1/3}(\ln x + 1) + \int x^{-4/3} dx$$

$$= -x^{-1/3}(\ln x + 1) - 3x^{-1/3}$$

$$y(x) = x^{1/3}\left(-\frac{\ln x + 1}{x^{1/3}} - \frac{3}{x^{1/3}} + C\right)$$

$$= -\ln x - 1 - 3 + C x^{1/3}$$

$$= -\ln x - 4 + C x^{1/3}$$

$y(1) = -2$

$-2 = -4 + C$

$C = 2$

$$y(x) = -\ln x - 4 + 2\sqrt[3]{x}$$

2.7

#11.21. $y' = xy = \frac{dy}{dx}$

$$\int \frac{dy}{y} = \int x\,dx$$

$$\ln|y| = \frac{1}{2}x^2 + C$$

$$y = e^{\frac{1}{2}x^2 + C}$$

$$= e^{1/2 x^2} e^C$$

$$= C e^{x^2/2}$$

#23 $y' = e^{x-y}$

$$\frac{dy}{dx} = e^x e^{-y}$$

$$\int e^y\,dy = \int e^x\,dx$$

$$e^y = e^x + C$$

$$y = \ln(e^x + C) \qquad C \geq 0$$

#26 $y' = ye^x - 2e^x + y - 2$

$$= y(e^x+1) - 2(e^x+1)$$

$$\frac{dy}{dx} = (y-2)(e^x+1)$$

$$\int \frac{d(y-2)}{y-2} = \int (e^x+1)\,dx \qquad d(y-2) = dy$$

$$\ln|y-2| = e^x + x + C$$

#30. $y' + (\tan x)\, y = \cos^2 x$ $\quad (\tan x \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2})$

$$e^{\int \tan x\, dx} = e^{\ln \sec x} = \sec x$$

$$\int \cos^2 x \frac{1}{\cos x}\, dx = \int \cos x\, dx = \sin x$$

$$y(x) = \cos x\,(\sin x + C)$$

136. $y' = y + 2x e^{2x}$ $\quad y(0) = 3$

$$y' - y = 2x e^{2x}$$

$$e^{\int -dx} = e^{-x}$$

$$\int 2x e^{2x} e^{-x}\, dx = 2 \int x e^{x}\, dx$$
$$= 2e^{x}(x-1)$$

$$y(x) = e^{x}\left(2e^{x}(x-1) + C\right)$$
$$= 2(x-1)e^{2x} + Ce^{x}$$

$$y(0) = 3 = -2 + C$$
$$C = 5$$

$$y(x) = 2(x-1)e^{2x} + 5e^{x}$$

Mixture Problems

Rate of change = rate in − rate out

$\frac{dy}{dt}$

Rate = Volume $\frac{gal}{min}$ × concentration lb/gal

$\text{Rate}_{out} = \frac{y(t)}{V(t)} \cdot \text{outflow rate}$

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gal, lb, min

$V(0) = 3{,}000$ gal $y(0) = 100$ lb

Rate in : 2 lb/gal   40 gal/min

" out : ?   45 gal/min

$V(t) = 3000 + (40 - 45)t$
$= 3000 - 5t$

* Tank to be empty : $3000 - 5t = 0$
$t = 600$ min

$\text{Rate out} = \frac{y(t)}{3000-5t} \cdot 45$

$\text{Rate in} = (2)(40) = 80$

$\frac{dy}{dt} = 80 - \frac{45}{3000-5t}\, y.$

$y' + \frac{45}{3000-5t}\, y = 80$

$e^{\int \frac{45}{3000-5t} dt} = e^{-9\int \frac{d(3000-5t)}{3000-5t}}$

$= e^{-9 \ln|3000-5t|}$

$= (3000-5t)^{-9}$

$$\int 80(3000-5t)^{-9}\,dt = -16\int (3000-5t)^{-9}\,d(3000-5t)$$

$$= 2(3000-5t)^{-8}$$

$$y(t) = (3000-5t)^{9}\left[2(3000-5t)^{-8} + C\right]$$

$$= 3000-5t + C(3000-5t)^{9}$$

$$y(0) = 100$$

$$100 = 3000 + C(3000)^{9}$$

$$C = -\frac{2900}{(3000)^{9}}$$

$$y(t) = 3000-5t - \frac{2900}{(3000)^{9}}(3000-5t)^{9}$$

$$y(20) = 2900 - \frac{2900}{3000^{9}}\,2900^{9}$$

Exam 2 → #19

2.7
202. $y'(t) = 2y + 4 \qquad y(0) = 8$

$$y' - 2y = 4$$

$$e^{\int -2dt} = e^{-2t}$$

$$\int 4e^{-2t}dt = -2e^{-2t}$$

$$y(t) = e^{2t}(-2e^{-2t} + C)$$

$$= -2 + Ce^{2t}$$

$$y(0) = 8 = -2 + C$$

$$C = 10$$

$$y(t) = -2 + 10e^{2t}$$

198. $x\,dy + (y - \cos x)dx = 0 \qquad y\left(\frac{\pi}{2}\right) = 0$

$$x\frac{dy}{dx} + y - \cos x = 0$$

$$y' + \frac{1}{x}y = \frac{\cos x}{x}$$

$$e^{\int \frac{dx}{x}} = e^{\ln x} = x$$

$$\int \frac{\cos x}{x} x\,dx = \int \cos x\,dx = \sin x$$

$$y(x) = \frac{1}{x}(\sin x + C)$$

$$y\left(\frac{\pi}{2}\right) = 0 = \frac{2}{\pi}(1 + C)$$

$$1 + C = 0 \Rightarrow C = -1$$

$$y(x) = \frac{1}{x}(\sin x - 1)$$

#108 $(x^2+1)y' + 3xy = 6x \qquad y(0) = -1$

$$y' + \frac{3x}{x^2+1}\,y = \frac{6x}{x^2+1}$$

$$e^{3\int \frac{x}{x^2+1}dx} = e^{\frac{3}{2}\int \frac{d(x^2+1)}{x^2+1}} = e^{\frac{3}{2}\ln(x^2+1)}$$

$$= (x^2+1)^{3/2}$$

$$\int \frac{6x}{x^2+1}(x^2+1)^{3/2}dx = 6\int x(x^2+1)^{1/2}dx$$

$$= 3\int (x^2+1)^{1/2}d(x^2+1)$$

$$= 2(x^2+1)^{3/2}$$

$$y(x) = \frac{1}{(x^2+1)^{3/2}}\left(2(x^2+1)^{3/2} + C\right)$$

$$\boxed{= 2 + \frac{C}{(x^2+1)^{3/2}}}$$

$$y(0) = -1 = 2 + C$$

$$\boxed{C = -3}$$

$$\boxed{y(x) = 2 - \frac{3}{(x^2+1)^{3/2}}}$$