Solution Section 8.2 – Sum and Difference Formulas

Exercise

Write the expression as a single trigonometric function $\sin 8x \cos x - \cos 8x \sin x$

Solution

$$\sin 8x \cos x - \cos 8x \sin x = \sin(8x - x)$$
$$= \sin 7x$$

Exercise

Show that $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$

Solution

$$\sin\left(x - \frac{\pi}{2}\right) = \sin x \cos\frac{\pi}{2} - \cos x \sin\frac{\pi}{2}$$
$$= \sin x \cdot (0) - \cos x \cdot (1)$$
$$= -\cos x$$

Exercise

If $\sin A = \frac{4}{5}$ with A in QII, and $\cos B = -\frac{5}{13}$ with B in QIII, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

Solution

$\cos A = -\frac{3}{5}$ $\sin B = -\frac{12}{13}$

a) $\sin(A+B) = \sin A \cos B + \sin B \cos A$

$$= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) + \left(-\frac{12}{13}\right)\left(-\frac{3}{5}\right)$$
$$= -\frac{20}{65} + \frac{36}{65}$$
$$= \frac{16}{65}$$

b) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$= \left(-\frac{3}{5}\right)\left(-\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right)$$
$$= \frac{15}{65} + \frac{48}{65}$$

$$= \boxed{\frac{63}{65}}$$

c) $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)}$

$$= \boxed{\frac{16}{63}}$$

d) $\sin(A-B) = \sin A\cos B - \sin B\cos A$

$$= \left(\tfrac{4}{5}\right)\left(-\tfrac{5}{13}\right) - \left(-\tfrac{12}{13}\right)\left(-\tfrac{3}{5}\right)$$

$$= -\frac{20}{65} - \frac{36}{65}$$

$$= \boxed{-\frac{56}{65}}$$

e) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$$= \left(-\tfrac{3}{5}\right)\left(-\tfrac{5}{13}\right) + \left(\tfrac{4}{5}\right)\left(-\tfrac{12}{13}\right)$$

$$= \frac{15}{65} - \frac{48}{65}$$

$$= \boxed{-\frac{33}{65}}$$

f) $\tan(A-B) = \dfrac{\sin(A-B)}{\cos(A-B)}$

$$= \boxed{\frac{56}{33}}$$

Exercise

If $\sin A = \frac{3}{5}$ $\left(A \in QII\right)$, and $\cos B = -\frac{12}{13}$ $\left(B \in QIII\right)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

Solution

$\cos A = -\frac{4}{5}$ $\sin B = -\frac{5}{13}$

a) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$= \left(\tfrac{3}{5}\right)\left(-\tfrac{12}{13}\right) + \left(-\tfrac{4}{5}\right)\left(-\tfrac{5}{13}\right)$$

$$= \frac{-36+20}{65}$$

$$= \boxed{-\frac{16}{65}}$$

b) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$$= \left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right) - \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right)$$

$$= \frac{48+15}{65}$$

$$= \frac{63}{65}$$

c) $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)}$

$$= -\frac{16}{63}$$

d) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$$= \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) - \left(-\frac{4}{5}\right)\left(-\frac{5}{13}\right)$$

$$= \frac{-36-20}{65}$$

$$= -\frac{56}{65}$$

e) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$$= \left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right) + \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right)$$

$$= \frac{48-15}{65}$$

$$= \frac{33}{65}$$

f) $\tan(A-B) = \dfrac{\sin(A-B)}{\cos(A-B)}$

$$= -\frac{56}{33}$$

Exercise

If $\sin A = \frac{1}{\sqrt{5}}$ $\left(A \in QI\right)$, and $\tan B = \frac{3}{4}$ $\left(B \in QI\right)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

Solution

$\cos A = \sqrt{1-\frac{1}{5}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$ $\sin B = \frac{3}{5}$; $\cos B = \frac{4}{5}$

a) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$= \left(\frac{1}{\sqrt{5}}\right)\left(\frac{4}{5}\right) + \left(\frac{3}{5}\right)\left(\frac{2}{\sqrt{5}}\right)$$

$$= \frac{4+6}{5\sqrt{5}}$$

$$= \frac{10}{5\sqrt{5}}$$

$$\boxed{= \frac{2}{\sqrt{5}}}$$

b) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$= \left(\frac{2}{\sqrt{5}}\right)\left(\frac{4}{5}\right) - \left(\frac{1}{\sqrt{5}}\right)\left(\frac{3}{5}\right)$$

$$= \frac{8-3}{5\sqrt{5}}$$

$$\boxed{= \frac{1}{\sqrt{5}}}$$

c) $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)}$

$$\boxed{= 2}$$

d) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

$$= \left(\frac{1}{\sqrt{5}}\right)\left(\frac{4}{5}\right) - \left(\frac{3}{5}\right)\left(\frac{2}{\sqrt{5}}\right)$$

$$= \frac{4-6}{5\sqrt{5}}$$

$$\boxed{= -\frac{2}{5\sqrt{5}}}$$

e) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

$$= \left(\frac{2}{\sqrt{5}}\right)\left(\frac{4}{5}\right) + \left(\frac{1}{\sqrt{5}}\right)\left(\frac{3}{5}\right)$$

$$= \frac{8+3}{5\sqrt{5}}$$

$$\boxed{= \frac{11}{5\sqrt{5}}}$$

f) $\tan(A-B) = \dfrac{\sin(A-B)}{\cos(A-B)}$

$$\boxed{= -\frac{2}{11}}$$

Exercise

If $\sin A = \frac{3}{5}\ (A \in QII)$, and $\cos B = \frac{12}{13}\ (B \in QIV)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

Solution

$\cos A = -\frac{4}{5} \qquad \sin B = -\frac{5}{13}$

a) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$= \left(\frac{3}{5}\right)\left(\frac{12}{13}\right) + \left(-\frac{4}{5}\right)\left(-\frac{5}{13}\right)$$

$$= \frac{36+20}{65}$$

$$= \frac{56}{65}$$

b) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$$= \left(-\frac{4}{5}\right)\left(\frac{12}{13}\right) - \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right)$$

$$= \frac{-48-15}{65}$$

$$= -\frac{63}{65}$$

c) $\tan(A+B) = -\frac{56}{63}$ $\qquad \tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$

d) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$$= \left(\frac{3}{5}\right)\left(\frac{12}{13}\right) - \left(-\frac{4}{5}\right)\left(-\frac{5}{13}\right)$$

$$= \frac{36-20}{65}$$

$$= \frac{16}{65}$$

e) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$$= \left(-\frac{4}{5}\right)\left(\frac{12}{13}\right) + \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right)$$

$$= \frac{-48-15}{65}$$

$$= -\frac{63}{65}$$

f) $\tan(A-B) = -\frac{16}{63}$ $\qquad \tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)}$

Exercise

If $\sin A = \frac{7}{25}\ (A \in QII)$, and $\cos B = -\frac{8}{17}\ (B \in QIII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

Solution

$\cos A = -\frac{24}{25} \qquad \sin B = -\frac{15}{17}$

a)
$$\begin{aligned}\sin(A+B) &= \sin A \cos B + \cos A \sin B \\ &= \left(\frac{7}{25}\right)\left(-\frac{8}{17}\right) + \left(-\frac{24}{25}\right)\left(-\frac{15}{17}\right) \\ &= \frac{-56+360}{425} \\ &= \frac{304}{425}\end{aligned}$$

b)
$$\begin{aligned}\cos(A+B) &= \cos A \cos B - \sin A \sin B \\ &= \left(-\frac{24}{25}\right)\left(-\frac{8}{17}\right) - \left(\frac{7}{25}\right)\left(-\frac{15}{17}\right) \\ &= \frac{192+105}{425} \\ &= \frac{297}{425}\end{aligned}$$

c) $\tan(A+B) = \frac{304}{297}$ $\qquad \tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$

d)
$$\begin{aligned}\sin(A-B) &= \sin A \cos B - \cos A \sin B \\ &= \left(\frac{7}{25}\right)\left(-\frac{8}{17}\right) - \left(-\frac{24}{25}\right)\left(-\frac{15}{17}\right) \\ &= \frac{-56-360}{425} \\ &= -\frac{416}{425}\end{aligned}$$

e)
$$\begin{aligned}\cos(A-B) &= \cos A \cos B + \sin A \sin B \\ &= \left(-\frac{24}{25}\right)\left(-\frac{8}{17}\right) + \left(\frac{7}{25}\right)\left(-\frac{15}{17}\right) \\ &= \frac{192-105}{425} \\ &= -\frac{87}{425}\end{aligned}$$

f) $\tan(A-B) = -\frac{416}{87}$ $\qquad \tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)}$

Exercise

If $\cos A = -\frac{4}{5}$ $(A \in QII)$, and $\sin B = \frac{24}{25}$ $(B \in QII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

Solution

$\sin A = \frac{3}{5}$ $\cos B = -\frac{7}{25}$

a)
$$\begin{aligned}\sin(A+B) &= \sin A\cos B + \cos A\sin B \\ &= \left(\frac{3}{5}\right)\left(-\frac{7}{25}\right)+\left(-\frac{4}{5}\right)\left(\frac{24}{25}\right) \\ &= \frac{-21-96}{125} \\ &= -\frac{117}{125}\end{aligned}$$

b)
$$\begin{aligned}\cos(A+B) &= \cos A\cos B - \sin A\sin B \\ &= \left(-\frac{4}{5}\right)\left(-\frac{7}{25}\right)-\left(\frac{3}{5}\right)\left(\frac{24}{25}\right) \\ &= \frac{28-72}{125} \\ &= -\frac{44}{125}\end{aligned}$$

c) $\tan(A+B) = \frac{117}{44}$ $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$

d)
$$\begin{aligned}\sin(A-B) &= \sin A\cos B - \cos A\sin B \\ &= \left(\frac{3}{5}\right)\left(-\frac{7}{25}\right)-\left(-\frac{4}{5}\right)\left(\frac{24}{25}\right) \\ &= \frac{-21+96}{125} \\ &= \frac{75}{125}\end{aligned}$$

e)
$$\begin{aligned}\cos(A-B) &= \cos A\cos B + \sin A\sin B \\ &= \left(-\frac{4}{5}\right)\left(-\frac{7}{25}\right)+\left(\frac{3}{5}\right)\left(\frac{24}{25}\right) \\ &= \frac{28+72}{125} \\ &= \frac{100}{125}\end{aligned}$$

f) $\tan(A-B) = \frac{75}{100}$ $\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)}$

Exercise

If $\cos A = \frac{15}{17}$ $(A \in QI)$, and $\cos B = -\frac{12}{13}$ $(B \in QII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

Solution

$\sin A = \frac{8}{17}$ $\sin B = \frac{5}{13}$

a)
$$\begin{aligned}\sin(A+B) &= \sin A\cos B + \cos A\sin B \\ &= \left(\frac{8}{17}\right)\left(-\frac{12}{13}\right)+\left(\frac{15}{17}\right)\left(\frac{5}{13}\right) \\ &= \frac{-96+75}{221} \\ &= -\frac{21}{221}\end{aligned}$$

b)
$$\begin{aligned}\cos(A+B) &= \cos A\cos B - \sin A\sin B \\ &= \left(\frac{15}{17}\right)\left(-\frac{12}{13}\right)-\left(\frac{8}{17}\right)\left(\frac{5}{13}\right) \\ &= \frac{-180-40}{221} \\ &= -\frac{220}{221}\end{aligned}$$

c) $\tan(A+B) = \frac{21}{220}$ $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$

d)
$$\begin{aligned}\sin(A-B) &= \sin A\cos B - \cos A\sin B \\ &= \left(\frac{8}{17}\right)\left(-\frac{12}{13}\right)-\left(\frac{15}{17}\right)\left(\frac{5}{13}\right) \\ &= \frac{-96-75}{221} \\ &= -\frac{171}{221}\end{aligned}$$

e)
$$\begin{aligned}\cos(A-B) &= \cos A\cos B + \sin A\sin B \\ &= \left(\frac{15}{17}\right)\left(-\frac{12}{13}\right)+\left(\frac{8}{17}\right)\left(\frac{5}{13}\right) \\ &= \frac{-180+40}{221} \\ &= -\frac{140}{221}\end{aligned}$$

f) $\tan(A-B) = \frac{171}{140}$ $\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)}$

Exercise

If $\sin A = -\frac{3}{5}\ (A \in QIV)$, and $\sin B = \frac{7}{25}\ (B \in QII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

Solution

$\cos A = -\frac{4}{5}$ $\cos B = -\frac{24}{25}$

a) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$= \left(-\frac{3}{5}\right)\left(-\frac{24}{25}\right) + \left(-\frac{4}{5}\right)\left(\frac{7}{25}\right)$$

$$= \frac{72-28}{125}$$

$$= \frac{44}{125}$$

b) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$$= \left(-\frac{4}{5}\right)\left(-\frac{24}{25}\right) - \left(-\frac{3}{5}\right)\left(\frac{7}{25}\right)$$

$$= \frac{96+21}{125}$$

$$= \frac{117}{125}$$

c) $\tan(A+B) = \frac{44}{117}$ $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$

d) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$$= \left(-\frac{3}{5}\right)\left(-\frac{24}{25}\right) - \left(-\frac{4}{5}\right)\left(\frac{7}{25}\right)$$

$$= \frac{72+28}{125}$$

$$= \frac{100}{125}$$

e) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$$= \left(-\frac{4}{5}\right)\left(-\frac{24}{25}\right) + \left(-\frac{3}{5}\right)\left(\frac{7}{25}\right)$$

$$= \frac{96-21}{125}$$

$$= \frac{75}{125}$$

f) $\tan(A-B) = \frac{100}{75}$ $\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)}$

Exercise

If $\sec A = \sqrt{5}$ with A in QI, and $\sec B = \sqrt{10}$ with B in QI, find $\sec(A+B)$

Solution

$$\sec(A+B) = \frac{1}{\cos(A+B)}$$

$$\sec A = \sqrt{5}$$

$$\cos A = \frac{1}{\sqrt{5}} \quad \sin A = \frac{2}{\sqrt{5}}$$

$$\sec B = \sqrt{10}$$

$$\cos B = \frac{1}{\sqrt{10}} \quad \sin B = \sqrt{1 - \frac{1}{10}} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{5}}$$

$$\begin{aligned}
\cos(A+B) &= \cos A \cos B - \sin A \sin B \\
&= \frac{1}{\sqrt{5}}\frac{1}{\sqrt{10}} - \frac{2}{\sqrt{5}}\frac{3}{\sqrt{10}} \\
&= \frac{1-6}{\sqrt{50}} \\
&= \frac{5}{\sqrt{50}} \\
&= \frac{5}{5\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
\sec(A+B) &= \frac{1}{\frac{1}{\sqrt{2}}} \\
&\underline{= \sqrt{2}\ |}
\end{aligned}$$

Exercise

Prove the identity $\dfrac{\sin(A-B)}{\cos A \cos B} = \tan A - \tan B$

Solution

$$\begin{aligned}
\frac{\sin(A-B)}{\cos A \cos B} &= \frac{\sin A \cos B - \sin B \cos A}{\cos A \cos B} \\
&= \frac{\sin A \cos B}{\cos A \cos B} - \frac{\sin B \cos A}{\cos A \cos B} \\
&= \frac{\sin A}{\cos A} - \frac{\sin B}{\cos B} \\
&\underline{= \tan A - \tan B\ |} \quad \checkmark
\end{aligned}$$

Exercise

Prove the identity $\sec(A+B)=\dfrac{\cos(A-B)}{\cos^2 A-\sin^2 B}$

Solution

$$\sec(A+B)=\frac{1}{\cos(A+B)}$$

$$=\frac{1}{\cos A\cos B-\sin A\sin B}$$

$$=\frac{1}{\cos A\cos B-\sin A\sin B}\frac{\cos(A-B)}{\cos(A-B)}$$

$$=\frac{1}{\cos A\cos B-\sin A\sin B}\frac{\cos(A-B)}{\cos A\cos B+\sin A\sin B}$$

$$=\frac{\cos(A-B)}{\cos^2 A\cos^2 B-\sin^2 A\sin^2 B}$$

$$=\frac{\cos(A-B)}{\cos^2 A\left(1-\sin^2 B\right)-\left(1-\cos^2 A\right)\sin^2 B}$$

$$=\frac{\cos(A-B)}{\cos^2 A-\cos^2 A\sin^2 B-\sin^2 B+\cos^2 A\sin^2 B}$$

$$=\frac{\cos(A-B)}{\cos^2 A-\sin^2 B} \quad \checkmark$$

Exercise

Prove the identity $\dfrac{\cos 4\alpha}{\sin\alpha}-\dfrac{\sin 4\alpha}{\cos\alpha}=\dfrac{\cos 5\alpha}{\sin\alpha\cos\alpha}$

Solution

$$\frac{\cos 4\alpha}{\sin\alpha}-\frac{\sin 4\alpha}{\cos\alpha}=\frac{\cos 4\alpha\cos\alpha-\sin 4\alpha\sin\alpha}{\sin\alpha\cos\alpha}$$

$$=\frac{\cos(4\alpha+\alpha)}{\sin\alpha\cos\alpha}$$

$$=\frac{\cos 5\alpha}{\sin\alpha\cos\alpha} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos(x+y)}{\cos(x-y)} = \dfrac{\cot y - \tan x}{\cot y + \tan x}$

Solution

$$\frac{\cos(x+y)}{\cos(x-y)} = \frac{\cos x\cos y - \sin x\sin y}{\cos x\cos y + \sin x\sin y}$$

$$= \frac{\frac{\cos x\cos y}{\cos x\sin y} - \frac{\sin x\sin y}{\cos x\sin y}}{\frac{\cos x\cos y}{\cos x\sin y} + \frac{\sin x\sin y}{\cos x\sin y}}$$

$$= \frac{\cot y - \tan x}{\cot y + \tan x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\cot y + \cot x}{\cot y - \cot x}$

Solution

$$\frac{\sin(x+y)}{\sin(x-y)} = \frac{\sin x\cos y + \sin y\cos x}{\sin x\cos y - \sin y\cos x}$$

$$= \frac{\frac{\sin x\cos y}{\sin x\sin y} + \frac{\sin y\cos x}{\sin x\sin y}}{\frac{\sin x\cos y}{\sin x\sin y} - \frac{\sin y\cos x}{\sin x\sin y}}$$

$$= \frac{\cot y + \cot x}{\cot y - \cot x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos(x+y)}{\cos(x-y)} = \dfrac{\cot y - \tan x}{\cot y + \tan x}$

Solution

$$\frac{\cos(x+y)}{\cos(x-y)} = \frac{\cos x\cos y - \sin x\sin y}{\cos x\cos y + \sin x\sin y}$$

$$= \frac{\frac{\cos x\cos y}{\cos x\sin y} - \frac{\sin x\sin y}{\cos x\sin y}}{\frac{\cos x\cos y}{\cos x\sin y} + \frac{\sin x\sin y}{\cos x\sin y}}$$

$$= \frac{\cot y - \tan x}{\cot y + \tan x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin(x-y)}{\sin x\cos y} = 1 - \cot x \tan y$

Solution

$$\frac{\sin(x-y)}{\sin x\cos y} = \frac{\sin x\cos y - \cos x\sin y}{\sin x\cos y}$$

$$= \frac{\sin x\cos y}{\sin x\cos y} - \frac{\cos x\sin y}{\sin x\cos y}$$

$$= 1 - \cot x\tan y \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin(x-y)}{\sin x\sin y} = \cot y - \cot x$

Solution

$$\frac{\sin(x-y)}{\sin x\sin y} = \frac{\sin x\cos y - \cos x\sin y}{\sin x\sin y}$$

$$= \frac{\sin x\cos y}{\sin x\sin y} - \frac{\cos x\sin y}{\sin x\sin y}$$

$$= \frac{\cos y}{\sin y} - \frac{\cos x}{\sin x}$$

$$= \cot y - \cot x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos(x+y)}{\cos x\sin y} = \cot y - \tan x$

Solution

$$\frac{\cos(x+y)}{\cos x\sin y} = \frac{\cos x\cos y - \sin x\sin y}{\cos x\sin y}$$

$$= \frac{\cos x\cos y}{\cos x\sin y} - \frac{\sin x\sin y}{\cos x\sin y}$$

$$= \frac{\cos y}{\sin y} - \frac{\sin x}{\cos x}$$

$$= \cot y - \tan x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin(x+y)}{\cos(x-y)} = \dfrac{1+\cot x\tan y}{\cot x+\tan y}$

Solution

$$\frac{\sin(x+y)}{\cos(x-y)} = \frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y+\sin x\sin y}$$

$$= \frac{\frac{\sin x\cos y}{\sin x\cos y}+\frac{\cos x\sin y}{\sin x\cos y}}{\frac{\cos x\cos y}{\sin x\cos y}+\frac{\sin x\sin y}{\sin x\cos y}}$$

$$= \frac{1+\cot x\tan y}{\cot x+\tan y} \quad \checkmark$$

Exercise

Prove the identity $\sin\left(\frac{\pi}{4}+x\right)+\sin\left(\frac{\pi}{4}-x\right)=\sqrt{2}\cos x$

Solution

$$\sin\left(\frac{\pi}{4}+x\right)+\sin\left(\frac{\pi}{4}-x\right) = \sin\frac{\pi}{4}\cos x+\sin x\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\cos x-\sin x\cos\frac{\pi}{4}$$

$$= \sin\frac{\pi}{4}\cos x+\sin\frac{\pi}{4}\cos x$$

$$= 2\sin\frac{\pi}{4}\cos x$$

$$= 2\frac{\sqrt{2}}{2}\cos x$$

$$= \sqrt{2}\cos x \quad \checkmark$$

Exercise

Prove the identity $\cos(A+B)+\cos(A-B)=2\cos A\cos B$

Solution

$$\cos(A+B)+\cos(A-B) = \cos A\cos B-\sin A\sin B+\cos A\cos B+\sin A\sin B$$

$$= \cos A\cos B+\cos A\cos B$$

$$= 2\cos A\cos B \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\sin(x-y)-\sin(y-x)=2\sin x\cos y-2\cos x\sin y$

Solution

$$\begin{aligned}\sin(x-y)-\sin(y-x)&=\sin x\cos y-\sin y\cos x-(\sin y\cos x-\sin x\cos y)\\&=\sin x\cos y-\sin y\cos x-\sin y\cos x+\sin x\cos y\\&=2\sin x\cos y-2\sin y\cos x\ \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cos(x-y)+\cos(y-x)=2\cos x\cos y+2\sin x\sin y$

Solution

$$\begin{aligned}\cos(x-y)+\cos(y-x)&=\cos x\cos y+\sin x\sin y+\cos y\cos x+\sin y\sin x\\&=2\cos x\cos y+2\sin x\sin y\ \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\tan(x+y)\tan(x-y)=\dfrac{\tan^2 x-\tan^2 y}{1-\tan^2 x\tan^2 y}$

Solution

$$\begin{aligned}\tan(x+y)\tan(x-y)&=\frac{\tan x+\tan y}{1-\tan x\tan y}\,\frac{\tan x-\tan y}{1+\tan x\tan y}\\&=\frac{\tan^2 x-\tan^2 y}{1-\tan x^2\tan^2 y}\ \checkmark\end{aligned}$$

$(a+b)(a-b)=a^2-b^2$

Exercise

Prove the following equation is an identity: $\dfrac{\cos(\alpha-\beta)}{\sin(\alpha+\beta)}=\dfrac{1-\tan\alpha\tan\beta}{\tan\alpha-\tan\beta}$

Solution

$$\begin{aligned}\frac{\cos(\alpha-\beta)}{\sin(\alpha+\beta)}&=\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\sin\alpha\cos\beta+\sin\beta\cos\alpha}\\&=\frac{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\sin\beta\cos\alpha}{\cos\alpha\cos\beta}}\\&=\frac{1+\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}\ \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\sec(x+y)=\dfrac{\cos x\cos y+\sin x\sin y}{\cos^2 x-\sin^2 y}$

Solution

$$\sec(x+y)=\frac{1}{\cos(x+y)}\frac{\cos(x-y)}{\cos(x-y)}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{(\cos x\cos y-\sin x\sin y)(\cos x\cos y+\sin x\sin y)}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{\cos^2 x\cos^2 y-\sin^2 x\sin^2 y}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{\cos^2 x\cos^2 y-\sin^2 x\sin^2 y}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{\cos^2 x\left(1-\sin^2 y\right)-\left(1-\cos^2 x\right)\sin^2 y}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{\cos^2 x-\cos^2 x\sin^2 y-\sin^2 y+\cos^2 x\sin^2 y}$$

$$\boxed{=\frac{\cos x\cos y+\sin x\sin y}{\cos^2 x-\sin^2 y}}\quad\checkmark$$

Exercise

Prove the following equation is an identity: $\csc(x-y)=\dfrac{\sin x\cos y+\cos x\sin y}{\sin^2 x-\sin^2 y}$

Solution

$$\csc(x-y)=\frac{1}{\sin(x-y)}\frac{\sin x\cos y+\cos x\sin y}{\sin x\cos y+\cos x\sin y}$$

$$=\frac{\sin x\cos y+\cos x\sin y}{(\sin x\cos y-\cos x\sin y)(\sin x\cos y+\cos x\sin y)}$$

$$=\frac{\sin x\cos y+\cos x\sin y}{\sin^2 x\cos^2 y-\cos^2 x\sin^2 y}$$

$$=\frac{\sin x\cos y+\cos x\sin y}{\sin^2 x\left(1-\sin^2 y\right)-\left(1-\sin^2 x\right)\sin^2 y}$$

$$=\frac{\sin x\cos y+\cos x\sin y}{\sin^2 x-\sin^2 x\sin^2 y-\sin^2 y+\sin^2 x\sin^2 y}$$

$$\boxed{=\frac{\sin x\cos y+\cos x\sin y}{\sin^2 x-\sin^2 y}}\quad\checkmark$$

Exercise

Prove the following equation is an identity: $\tan(x+y)\tan(x-y)=\dfrac{\tan^2 x-\tan^2 y}{1-\tan^2 x\tan^2 y}$

Solution

$$\tan(x+y)\tan(x-y)=\frac{\tan x+\tan y}{1-\tan x\tan y}\cdot\frac{\tan x-\tan y}{1+\tan x\tan y} \qquad (a+b)(a-b)=a^2-b^2$$

$$=\frac{\tan^2 x+\tan^2 y}{1-\tan x^2\tan^2 y} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos(\alpha-\beta)}{\sin(\alpha+\beta)}=\dfrac{1-\tan\alpha\tan\beta}{\tan\alpha-\tan\beta}$

Solution

$$\frac{\cos(\alpha-\beta)}{\sin(\alpha+\beta)}=\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\sin\alpha\cos\beta+\sin\beta\cos\alpha}$$

$$=\frac{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\sin\beta\cos\alpha}{\cos\alpha\cos\beta}}$$

$$=\frac{1+\tan\alpha\tan\beta}{\tan\alpha+\tan\beta} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\sec(x+y)=\dfrac{\cos x\cos y+\sin x\sin y}{\cos^2 x-\sin^2 y}$

Solution

$$\sec(x+y)=\frac{1}{\cos(x+y)}\frac{\cos(x-y)}{\cos(x-y)}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{(\cos x\cos y-\sin x\sin y)(\cos x\cos y+\sin x\sin y)}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{\cos^2 x\cos^2 y-\sin^2 x\sin^2 y}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{\cos^2 x\cos^2 y-\sin^2 x\sin^2 y}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{\cos^2 x\left(1-\sin^2 y\right)-\left(1-\cos^2 x\right)\sin^2 y}$$

$$= \frac{\cos x \cos y + \sin x \sin y}{\cos^2 x - \cos^2 x \sin^2 y - \sin^2 y + \cos^2 x \sin^2 y}$$

$$= \frac{\cos x \cos y + \sin x \sin y}{\cos^2 x - \sin^2 y} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\csc(x - y) = \dfrac{\sin x \cos y + \cos x \sin y}{\sin^2 x - \sin^2 y}$

Solution

$$\csc(x - y) = \frac{1}{\sin(x - y)} \cdot \frac{\sin x \cos y + \cos x \sin y}{\sin x \cos y + \cos x \sin y}$$

$$= \frac{\sin x \cos y + \cos x \sin y}{(\sin x \cos y - \cos x \sin y)(\sin x \cos y + \cos x \sin y)}$$

$$= \frac{\sin x \cos y + \cos x \sin y}{\sin^2 x \cos^2 y - \cos^2 x \sin^2 y}$$

$$= \frac{\sin x \cos y + \cos x \sin y}{\sin^2 x\left(1 - \sin^2 y\right) - \left(1 - \sin^2 x\right)\sin^2 y}$$

$$= \frac{\sin x \cos y + \cos x \sin y}{\sin^2 x - \sin^2 x \sin^2 y - \sin^2 y + \sin^2 x \sin^2 y}$$

$$= \frac{\sin x \cos y + \cos x \sin y}{\sin^2 x - \sin^2 y} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\tan(x + y) + \tan(x - y) = \dfrac{2 \tan x}{\cos^2 y\left(1 - \tan^2 x \tan^2 y\right)}$

Solution

$$\tan(x + y) + \tan(x - y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} + \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

$$= \frac{(\tan x + \tan y)(1 + \tan x \tan y) + (\tan x - \tan y)(1 - \tan x \tan y)}{(1 - \tan x \tan y)(1 + \tan x \tan y)}$$

$$= \frac{\tan x + \tan^2 x \tan y + \tan y + \tan x \tan^2 y + \tan x - \tan^2 x \tan y - \tan y + \tan x \tan^2 y}{\left(1 - \tan^2 x \tan^2 y\right)}$$

$$= \frac{2 \tan x + 2 \tan x \tan^2 y}{\left(1 - \tan^2 x \tan^2 y\right)}$$

$$= \frac{2\tan x\left(1+\tan^2 y\right)}{\left(1-\tan^2 x\tan^2 y\right)}$$

$$= \frac{2\tan x\sec^2 y}{\left(1-\tan^2 x\tan^2 y\right)}$$

$$= \frac{2\tan x}{\cos^2 y\left(1-\tan^2 x\tan^2 y\right)} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos(x-y)}{\cos(x+y)} = \dfrac{1+\tan x\tan y}{1-\tan x\tan y}$

Solution

$$\frac{\cos(x-y)}{\cos(x+y)} = \frac{\cos x\cos y+\sin x\sin y}{\cos x\cos y-\sin x\sin y}$$

$$= \frac{\frac{\cos x\cos y}{\cos x\cos y}+\frac{\sin x\sin y}{\cos x\cos y}}{\frac{\cos x\cos y}{\cos x\cos y}-\frac{\sin x\sin y}{\cos x\cos y}}$$

$$= \frac{1+\tan x\tan y}{1-\tan x\tan y} \quad \checkmark$$

Exercise

Common household current is called ***alternating current*** because the current alternates direction within the wires. The voltage V in a typical 115-*volt* outlet can be expressed by the function $V(t)=163\sin\omega t$ where ω is the angular speed (in *radians* per *second*) of the rotating generator at the electrical plant, and t is time measured in seconds.

a) It is essential for electric generators to rotate at precisely 60 cycles per second so household appliances and computers will function properly. Determine ω for these electric generators.

b) Determine a value of ϕ so that the graph of $V(t)=163\cos(\omega t-\phi)$ is the same as the graph of $V(t)=163\sin\omega t$

Solution

a) $\omega = 60\frac{cycles}{\sec}\frac{2\pi\ rad}{cycles}$

$$=120\pi\ \frac{rad}{\sec}$$

b) $V(t)=163\cos(\omega t-\phi)=163\sin\omega t$

$$\cos(120\pi t)\cos\phi + \sin(120\pi t)\sin\phi = \sin 120\pi t$$

$$\begin{cases} \cos(120\pi t)\cos\phi = 0 \\ \sin\phi = 1 \end{cases}$$

$$\phi = \frac{\pi}{2}$$