CHAPTER 15 MULTIPLE INTEGRALS

15.1 DOUBLE AND ITERATED INTEGRALS OVER RECTANGLES

1. $\int_1^2 \int_0^4 2xy\,dy\,dx = \int_1^2 \left[x\,y^2\right]_0^4 dx = \int_1^2 16x\,dx = \left[8\,x^2\right]_1^2 = 24$

2. $\int_0^2 \int_{-1}^1 (x-y)\,dy\,dx = \int_0^2 \left[xy - \frac{1}{2}y^2\right]_{-1}^1 dx = \int_0^2 2x\,dx = \left[x^2\right]_0^2 = 4$

3. $\int_{-1}^0 \int_{-1}^1 (x+y+1)\,dx\,dy = \int_{-1}^0 \left[\frac{x^2}{2} + yx + x\right]_{-1}^1 dy = \int_{-1}^0 (2y+2)\,dy = \left[y^2+2y\right]_{-1}^0 = 1$

4. $\int_0^1 \int_0^1 \left(1 - \frac{x^2+y^2}{2}\right) dx\,dy = \int_0^1 \left[x - \frac{x^3}{6} - \frac{x\,y^2}{2}\right]_0^1 dy = \int_0^1 \left(\frac{5}{6} - \frac{y^2}{2}\right) dy = \left[\frac{5}{6}y - \frac{y^3}{6}\right]_0^1 = \frac{2}{3}$

5. $\int_0^3 \int_0^2 (4-y^2)\,dy\,dx = \int_0^3 \left[4y - \frac{y^3}{3}\right]_0^2 dx = \int_0^3 \frac{16}{3}\,dx = \left[\frac{16}{3}\,x\right]_0^3 = 16$

6. $\int_0^3 \int_{-2}^0 (x^2y - 2xy)\,dy\,dx = \int_0^3 \left[\frac{x^2y^2}{2} - xy^2\right]_{-2}^0 dx = \int_0^3 (4x - 2x^2)\,dx = \left[2x^2 - \frac{2x^3}{3}\right]_0^3 = 0$

7. $\int_0^1 \int_0^1 \frac{y}{1+x\,y}\,dx\,dy = \int_0^1 \left[\ln|1+x\,y|\right]_0^1 dy = \int_0^1 \ln|1+y|\,dy = \left[y\ln|1+y| - y + \ln|1+y|\right]_0^1 = 2\ln 2 - 1$

8. $\int_1^4 \int_0^4 \left(\frac{x}{2} + \sqrt{y}\right) dx\,dy = \int_1^4 \left[\frac{1}{4}x^2 + x\sqrt{y}\right]_0^4 dy = \int_1^4 \left(4 + 4\,y^{1/2}\right) dy = \left[4y + \frac{8}{3}y^{3/2}\right]_1^4 = \frac{92}{3}$

9. $\int_0^{\ln 2} \int_1^{\ln 5} e^{2x+y}\,dy\,dx = \int_0^{\ln 2} \left[e^{2x+y}\right]_1^{\ln 5} dx = \int_0^{\ln 2} \left(5e^{2x} - e^{2x+1}\right) dx = \left[\frac{5}{2}e^{2x} - \frac{1}{2}e^{2x+1}\right]_0^{\ln 2} = \frac{3}{2}(5-e)$

10. $\int_0^1 \int_1^2 x\,y\,e^x\,dy\,dx = \int_0^1 \left[\frac{1}{2}x\,y^2 e^x\right]_1^2 dx = \int_0^1 \frac{3}{2}x\,e^x\,dx = \left[\frac{3}{2}x\,e^x - \frac{3}{2}e^x\right]_0^1 = \frac{3}{2}$

11. $\int_{-1}^2 \int_0^{\pi/2} y\sin x\,dx\,dy = \int_{-1}^2 \left[-y\cos x\right]_0^{\pi/2} dy = \int_{-1}^2 y\,dy = \left[\frac{1}{2}y^2\right]_{-1}^2 = \frac{3}{2}$

12. $\int_\pi^{2\pi} \int_0^\pi (\sin x + \cos y)\,dx\,dy = \int_\pi^{2\pi} \left[-\cos x + x\cos y\right]_0^\pi dy = \int_\pi^{2\pi} (2 + \pi\cos y)\,dy = \left[2y + \pi\sin y\right]_\pi^{2\pi} = 2\pi$

13. $\iint_R (6\,y^2 - 2\,x)\,dA = \int_0^1 \int_0^2 (6\,y^2 - 2\,x)\,dy\,dx = \int_0^1 \left[2\,y^3 - 2\,x\,y\right]_0^2 dx = \int_0^1 (16 - 4\,x)\,dx = \left[16\,x - 2\,x^2\right]_0^1 = 14$

14. $\iint_R \frac{\sqrt{x}}{y^2}\,dA = \int_0^4 \int_1^2 \frac{\sqrt{x}}{y^2}\,dy\,dx = \int_0^4 \left[-\frac{\sqrt{x}}{y}\right]_1^2 dx = \int_0^4 \frac{1}{2}x^{1/2}\,dx = \left[\frac{1}{3}x^{3/2}\right]_0^4 = \frac{8}{3}$

15. $\iint_R x\,y\cos y\,dA = \int_{-1}^1 \int_0^\pi x\,y\cos y\,dy\,dx = \int_{-1}^1 \left[x\,y\sin y + x\cos y\right]_0^\pi dx = \int_{-1}^1 (-2x)\,dx = \left[-x^2\right]_{-1}^1 = 0$

16. $\iint_R y\sin(x+y)\,dA = \int_{-\pi}^0 \int_0^\pi y\sin(x+y)\,dy\,dx = \int_{-\pi}^0 \left[-y\cos(x+y) + \sin(x+y)\right]_0^\pi dx$

$= \int_{-\pi}^0 (\sin(x+\pi) - \pi\cos(x+\pi) - \sin x)\,dx = \left[-\cos(x+\pi) - \pi\sin(x+\pi) + \cos x\right]_{-\pi}^0 = 4$

17. $\iint_R e^{x-y}\,dA = \int_0^{\ln 2}\int_0^{\ln 2} e^{x-y}\,dy\,dx = \int_0^{\ln 2}\left[-e^{x-y}\right]_0^{\ln 2}\,dx = \int_0^{\ln 2}\left(-e^{x-\ln 2}+e^x\right)dx = \left[-e^{x-\ln 2}+e^x\right]_0^{\ln 2} = \frac{1}{2}$

18. $\iint_R x\,y\,e^{xy^2}\,dA = \int_0^2\int_0^1 x\,y\,e^{xy^2}\,dy\,dx = \int_0^2\left[\frac{1}{2}e^{xy^2}\right]_0^1 dx = \int_0^2\left(\frac{1}{2}e^x-\frac{1}{2}\right)dx = \left[\frac{1}{2}e^x-\frac{1}{2}x\right]_0^2 = \frac{1}{2}(e^2-3)$

19. $\iint_R \frac{xy^3}{x^2+1}\,dA = \int_0^1\int_0^2 \frac{xy^3}{x^2+1}\,dy\,dx = \int_0^1\left[\frac{xy^4}{4(x^2+1)}\right]_0^2 dx = \int_0^1 \frac{4x}{x^2+1}\,dx = \left[2\ln|x^2+1|\right]_0^1 = 2\ln 2$

20. $\iint_R \frac{y}{x^2y^2+1}\,dA = \int_0^1\int_0^1 \frac{y}{(xy)^2+1}\,dx\,dy = \int_0^1\left[\tan^{-1}(xy)\right]_0^1 dy = \int_0^1 \tan^{-1}y\,dy = \left[y\tan^{-1}y - \frac{1}{2}\ln|1+y^2|\right]_0^1 = \frac{\pi}{4}-\frac{1}{2}\ln 2$

21. $\int_1^2\int_1^2 \frac{1}{xy}\,dy\,dx = \int_1^2 \frac{1}{x}(\ln 2 - \ln 1)\,dx = (\ln 2)\int_1^2 \frac{1}{x}\,dx = (\ln 2)^2$

22. $\int_0^1\int_0^{\pi} y\cos xy\,dx\,dy = \int_0^1 [\sin xy]_0^{\pi}\,dy = \int_0^1 \sin \pi y\,dy = \left[-\frac{1}{\pi}\cos \pi y\right]_0^1 = -\frac{1}{\pi}(-1-1) = \frac{2}{\pi}$

23. $V = \iint_R f(x,y)\,dA = \int_{-1}^1\int_{-1}^1 (x^2+y^2)\,dy\,dx = \int_{-1}^1\left[x^2y+\frac{1}{3}y^3\right]_{-1}^1 dx = \int_{-1}^1\left(2x^2+\frac{2}{3}\right)dx = \left[\frac{2}{3}x^3+\frac{2}{3}x\right]_{-1}^1 = \frac{8}{3}$

24. $V = \iint_R f(x,y)\,dA = \int_0^2\int_0^2 (16-x^2-y^2)\,dy\,dx = \int_0^2\left[16y-x^2y-\frac{1}{3}y^3\right]_0^2 dx = \int_0^2\left(\frac{88}{3}-2x^2\right)dx = \left[\frac{88}{3}x-\frac{2}{3}x^3\right]_0^2$
$= \frac{160}{3}$

25. $V = \iint_R f(x,y)\,dA = \int_0^1\int_0^1 (2-x-y)\,dy\,dx = \int_0^1\left[2y-xy-\frac{1}{2}y^2\right]_0^1 dx = \int_0^1\left(\frac{3}{2}-x\right)dx = \left[\frac{3}{2}x-\frac{1}{2}x^2\right]_0^1 = 1$

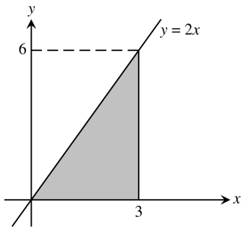
26. $V = \iint_R f(x,y)\,dA = \int_0^4\int_0^2 \frac{y}{2}\,dy\,dx = \int_0^4\left[\frac{y^2}{4}\right]_0^2 dx = \int_0^4 1\,dx = [x]_0^4 = 4$

27. $V = \iint_R f(x,y)\,dA = \int_0^{\pi/2}\int_0^{\pi/4} 2\sin x\cos y\,dy\,dx = \int_0^{\pi/2}[2\sin x\sin y]_0^{\pi/4}dx = \int_0^{\pi/2}\left(\sqrt{2}\sin x\right)dx = \left[-\sqrt{2}\cos x\right]_0^{\pi/2}$
$= \sqrt{2}$

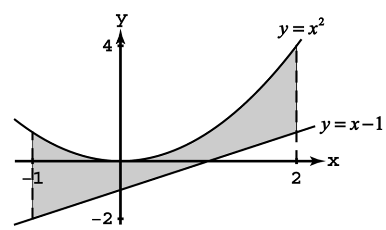
28. $V = \iint_R f(x,y)\,dA = \int_0^1\int_0^2 (4-y^2)\,dy\,dx = \int_0^1\left[4y-\frac{1}{3}y^3\right]_0^2 dx = \int_0^1\left(\frac{16}{3}\right)dx = \left[\frac{16}{3}x\right]_0^1 = \frac{16}{3}$

15.2 DOUBLE INTEGRALS OVER GENERAL REGIONS

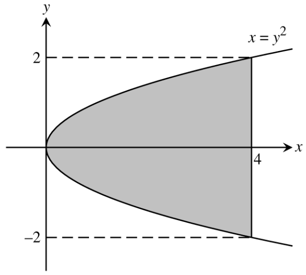
1.



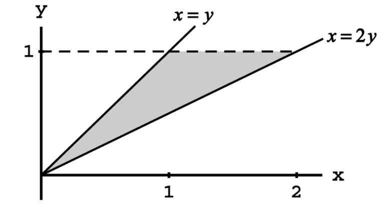
2.



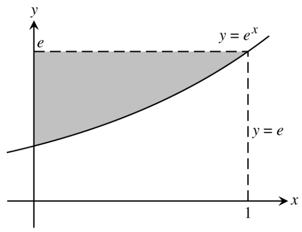
3.



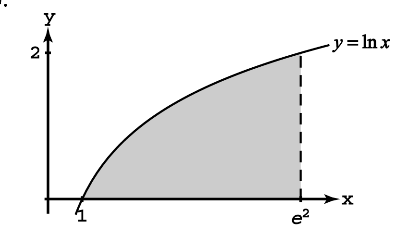
4.



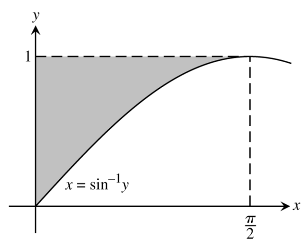
5.



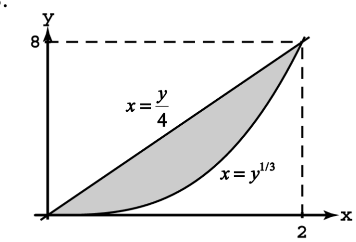
6.



7.



8.



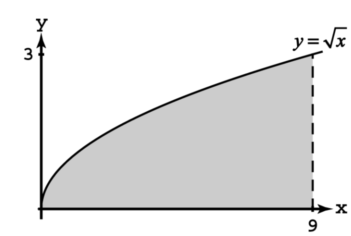
9. (a) $\int_0^2 \int_{x^3}^8 dy\, dx$ (b) $\int_0^8 \int_0^{y^{1/3}} dx\, dy$

10. (a) $\int_0^3 \int_0^{2x} dy\, dx$ (b) $\int_0^6 \int_{y/2}^3 dx\, dy$

11. (a) $\int_0^3 \int_{x^2}^{3x} dy\, dx$ (b) $\int_0^9 \int_{y/3}^{\sqrt{y}} dx\, dy$

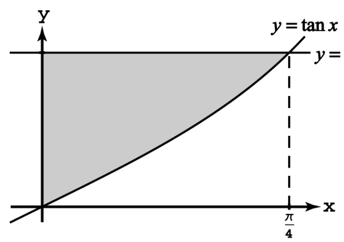
12. (a) $\int_0^2 \int_1^{e^x} dy\, dx$ (b) $\int_1^{e^2} \int_{\ln y}^2 dx\, dy$

13. (a) $\int_0^9 \int_0^{\sqrt{x}} dy\, dx$
(b) $\int_0^3 \int_{y^2}^9 dx\, dy$



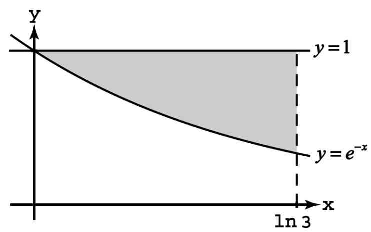
14. (a) $\int_0^{\pi/4}\int_{\tan x}^{1} dy\,dx$

(b) $\int_0^1\int_0^{\tan^{-1} y} dx\,dy$



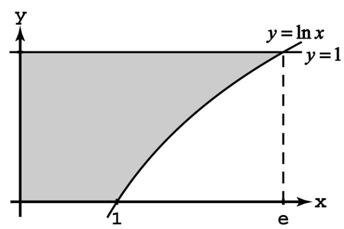
15. (a) $\int_0^{\ln 3}\int_{e^{-x}}^{1} dy\,dx$

(b) $\int_{1/3}^{1}\int_{-\ln y}^{\ln 3} dx\,dy$



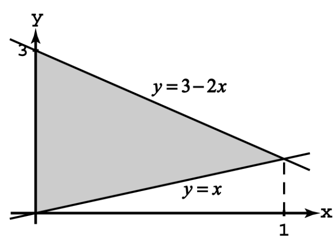
16. (a) $\int_0^1\int_0^1 dy\,dx + \int_1^e\int_{\ln x}^{1} dy\,dx$

(b) $\int_0^1\int_0^{e^y} dx\,dy$



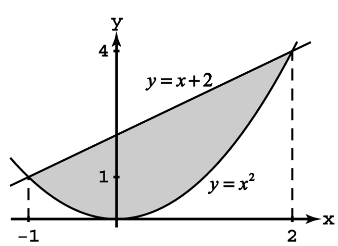
17. (a) $\int_0^1\int_x^{3-2x} dy\,dx$

(b) $\int_0^1\int_0^y dx\,dy + \int_1^3\int_0^{(3-y)/2} dx\,dy$



18. (a) $\int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx$

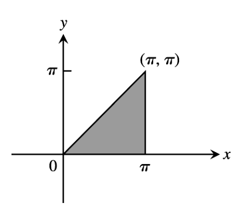
(b) $\int_0^1\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_1^3\int_{y-2}^{\sqrt{y}} dx\,dy$



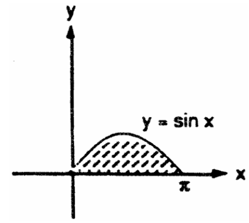
19. $\int_0^{\pi}\int_0^{x}(x\sin y)\,dy\,dx = \int_0^{\pi}\left[-x\cos y\right]_0^x dx$

$= \int_0^{\pi}(x - x\cos x)\,dx = \left[\frac{x^2}{2} - (\cos x + x\sin x)\right]_0^{\pi}$

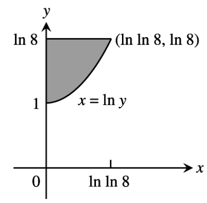
$= \frac{\pi^2}{2} + 2$



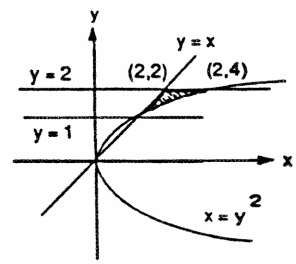
20. $\int_0^{\pi}\int_0^{\sin x} y\,dy\,dx = \int_0^{\pi}\left[\frac{y^2}{2}\right]_0^{\sin x} dx = \int_0^{\pi}\frac{1}{2}\sin^2 x\,dx$
$= \frac{1}{4}\int_0^{\pi}(1-\cos 2x)\,dx = \frac{1}{4}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\pi}{4}$



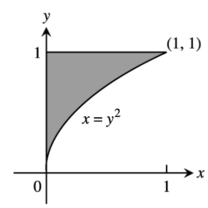
21. $\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\,dx\,dy = \int_1^{\ln 8}\left[e^{x+y}\right]_0^{\ln y} dy = \int_1^{\ln 8}\left(ye^y - e^y\right) dy$
$= \left[(y-1)e^y - e^y\right]_1^{\ln 8} = 8(\ln 8 - 1) - 8 + e$
$= 8\ln 8 - 16 + e$



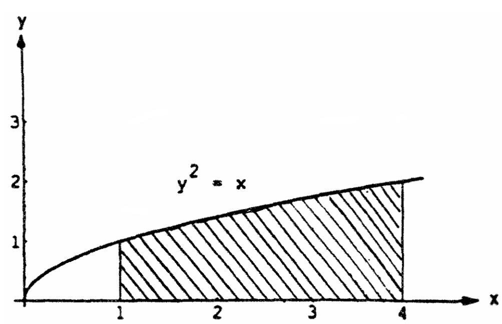
22. $\int_1^2\int_y^{y^2} dx\,dy = \int_1^2 (y^2 - y)\,dy = \left[\frac{y^3}{3} - \frac{y^2}{2}\right]_1^2$
$= \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - \frac{1}{2}\right) = \frac{7}{3} - \frac{3}{2} = \frac{5}{6}$



23. $\int_0^1\int_0^{y^2} 3y^3 e^{xy}\,dx\,dy = \int_0^1 \left[3y^2 e^{xy}\right]_0^{y^2} dy$
$= \int_0^1 \left(3y^2 e^{y^3} - 3y^2\right) dy = \left[e^{y^3} - y^3\right]_0^1 = e - 2$



24. $\int_1^4\int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}}\,dy\,dx = \int_1^4 \left[\frac{3}{2}\sqrt{x}\, e^{y/\sqrt{x}}\right]_0^{\sqrt{x}} dx$
$= \frac{3}{2}(e-1)\int_1^4 \sqrt{x}\,dx = \left[\frac{3}{2}(e-1)\left(\frac{2}{3}\right)x^{3/2}\right]_1^4 = 7(e-1)$



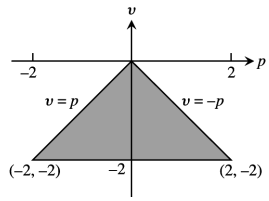
25. $\int_1^2\int_x^{2x} \frac{x}{y}\,dy\,dx = \int_1^2 \left[x \ln y\right]_x^{2x} dx = (\ln 2)\int_1^2 x\,dx = \frac{3}{2}\ln 2$

26. $\int_0^1\int_0^{1-x} (x^2 + y^2)\,dy\,dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx = \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3}\right] dx = \int_0^1 \left[x^2 - x^3 + \frac{(1-x)^3}{3}\right] dx$
$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12}\right]_0^1 = \left(\frac{1}{3} - \frac{1}{4} - 0\right) - \left(0 - 0 - \frac{1}{12}\right) = \frac{1}{6}$

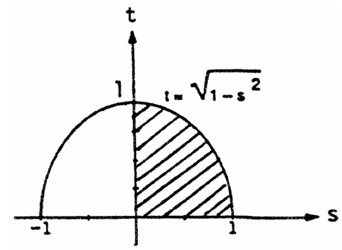
27. $\int_0^1\int_0^{1-u} \left(v - \sqrt{u}\right) dv\,du = \int_0^1 \left[\frac{v^2}{2} - v\sqrt{u}\right]_0^{1-u} du = \int_0^1 \left[\frac{1-2u+u^2}{2} - \sqrt{u}(1-u)\right] du$
$= \int_0^1 \left(\frac{1}{2} - u + \frac{u^2}{2} - u^{1/2} + u^{3/2}\right) du = \left[\frac{u}{2} - \frac{u^2}{2} + \frac{u^3}{6} - \frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2}\right]_0^1 = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} - \frac{2}{3} + \frac{2}{5} = -\frac{1}{2} + \frac{2}{5} = -\frac{1}{10}$

28. $\int_1^2 \int_0^{\ln t} e^s \ln t \, ds \, dt = \int_1^2 \left[e^s \ln t\right]_0^{\ln t} dt = \int_1^2 (t \ln t - \ln t) \, dt = \left[\frac{t^2}{2} \ln t - \frac{t^2}{4} - t \ln t + t\right]_1^2$
$= (2 \ln 2 - 1 - 2 \ln 2 + 2) - \left(-\frac{1}{4} + 1\right) = \frac{1}{4}$

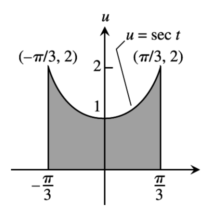
29. $\int_{-2}^0 \int_v^{-v} 2 \, dp \, dv = 2\int_{-2}^0 [p]_v^{-v} \, dv = 2\int_{-2}^0 -2v \, dv$
$= -2\left[v^2\right]_{-2}^0 = 8$



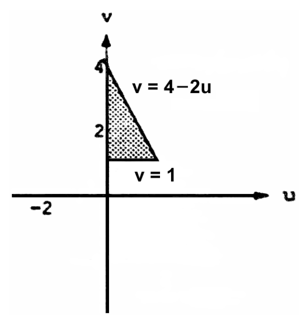
30. $\int_0^1 \int_0^{\sqrt{1-s^2}} 8t \, dt \, ds = \int_0^1 \left[4t^2\right]_0^{\sqrt{1-s^2}} ds$
$= \int_0^1 4\left(1 - s^2\right) ds = 4\left[s - \frac{s^3}{3}\right]_0^1 = \frac{8}{3}$



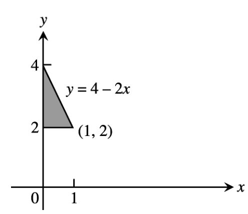
31. $\int_{-\pi/3}^{\pi/3} \int_0^{\sec t} 3 \cos t \, du \, dt = \int_{-\pi/3}^{\pi/3} [(3 \cos t)u]_0^{\sec t}$
$= \int_{-\pi/3}^{\pi/3} 3 \, dt = 2\pi$



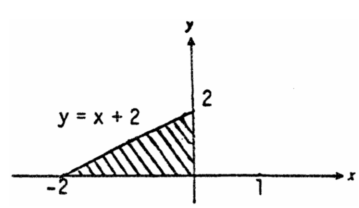
32. $\int_0^{3/2} \int_1^{4-2u} \frac{4-2u}{v^2} \, dv \, du = \int_0^{3/2} \left[\frac{2u-4}{v}\right]_1^{4-2u} du$
$= \int_0^{3/2} (3 - 2u) \, du = \left[3u - u^2\right]_0^{3/2} = \frac{9}{2}$



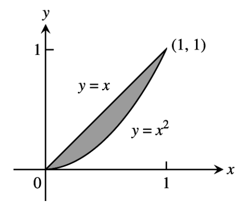
33. $\int_2^4 \int_0^{(4-y)/2} dx \, dy$



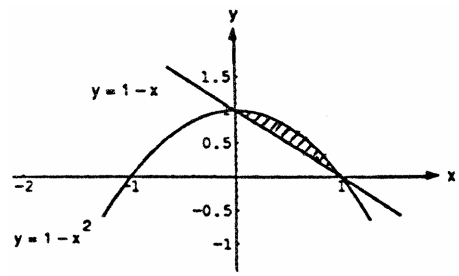
34. $\int_{-2}^0 \int_0^{x+2} dy \, dx$



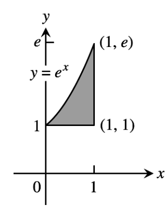
35. $\int_0^1 \int_{x^2}^{x} dy\, dx$



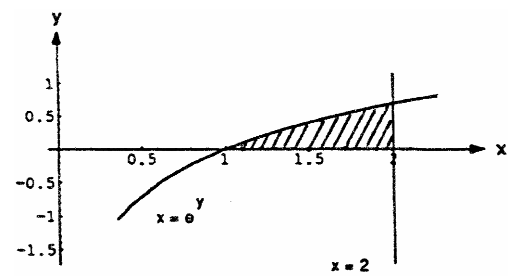
36. $\int_0^1 \int_{1-y}^{\sqrt{1-y}} dx\, dy$



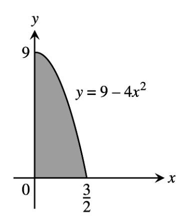
37. $\int_1^e \int_{\ln y}^{1} dx\, dy$



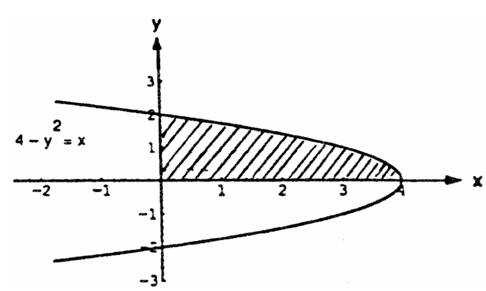
38. $\int_1^2 \int_0^{\ln x} dy\, dx$



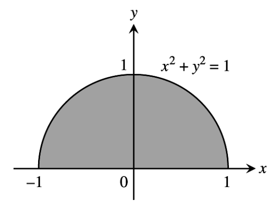
39. $\int_0^9 \int_0^{\frac{1}{2}\sqrt{9-y}} 16x\, dx\, dy$



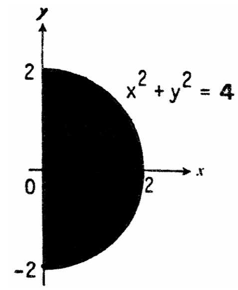
40. $\int_0^4 \int_0^{\sqrt{4-x}} y\, dy\, dx$



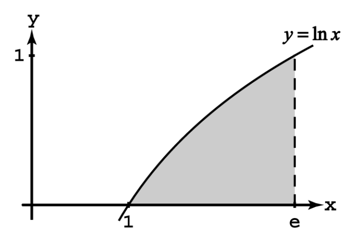
41. $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} 3y\,dy\,dx$



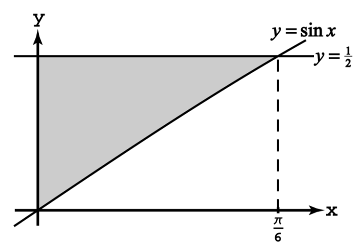
42. $\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} 6x\,dx\,dy$



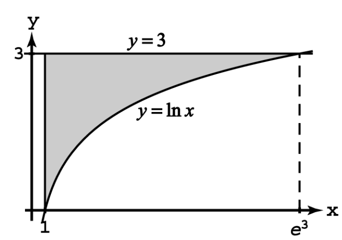
43. $\int_{0}^{1}\int_{e^y}^{e} x\,y\,dx\,dy$



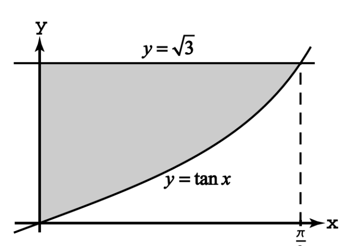
44. $\int_{0}^{1/2}\int_{0}^{\sin^{-1}y} x\,y^2\,dx\,dy$



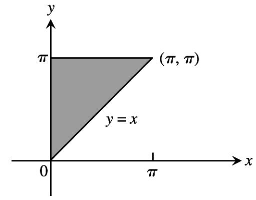
45. $\int_{1}^{e^3}\int_{\ln x}^{3} (x+y)dy\,dx$



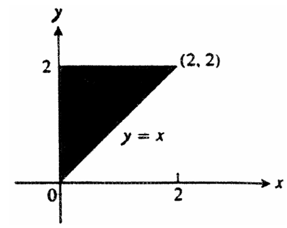
46. $\int_{0}^{\pi/3}\int_{\tan x}^{\sqrt{3}} \sqrt{x\,y}\,dy\,dx$



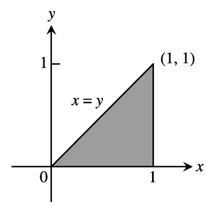
47. $\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\, dy\, dx = \int_0^{\pi}\int_0^{y} \frac{\sin y}{y}\, dx\, dy = \int_0^{\pi} \sin y\, dy = 2$



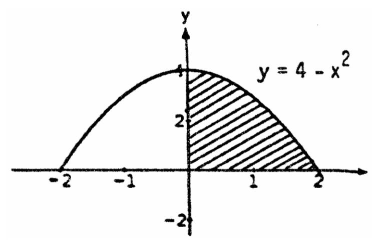
48. $\int_0^2\int_x^2 2y^2 \sin xy\, dy\, dx = \int_0^2\int_0^y 2y^2 \sin xy\, dx\, dy$
$= \int_0^2 [-2y \cos xy]_0^y\, dy = \int_0^2 (-2y \cos y^2 + 2y)\, dy$
$= [-\sin y^2 + y^2]_0^2 = 4 - \sin 4$



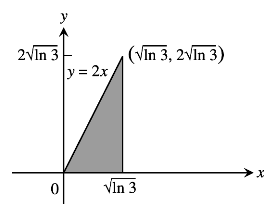
49. $\int_0^1\int_y^1 x^2 e^{xy}\, dx\, dy = \int_0^1\int_0^x x^2 e^{xy}\, dy\, dx = \int_0^1 [xe^{xy}]_0^x\, dx$
$= \int_0^1 (xe^{x^2} - x)\, dx = \left[\frac{1}{2} e^{x^2} - \frac{x^2}{2}\right]_0^1 = \frac{e-2}{2}$



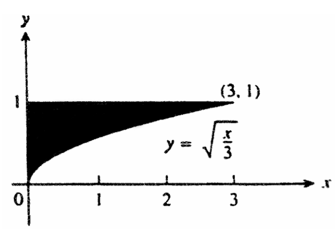
50. $\int_0^2\int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dy\, dx = \int_0^4\int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\, dx\, dy$
$= \int_0^4 \left[\frac{x^2 e^{2y}}{2(4-y)}\right]_0^{\sqrt{4-y}} dy = \int_0^4 \frac{e^{2y}}{2}\, dy = \left[\frac{e^{2y}}{4}\right]_0^4 = \frac{e^8 - 1}{4}$



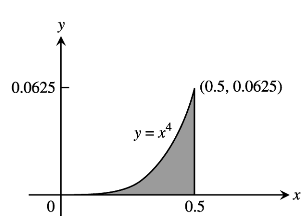
51. $\int_0^{2\sqrt{\ln 3}}\int_{y/2}^{\sqrt{\ln 3}} e^{x^2}\, dx\, dy = \int_0^{\sqrt{\ln 3}}\int_0^{2x} e^{x^2}\, dy\, dx$
$= \int_0^{\sqrt{\ln 3}} 2xe^{x^2}\, dx = [e^{x^2}]_0^{\sqrt{\ln 3}} = e^{\ln 3} - 1 = 2$



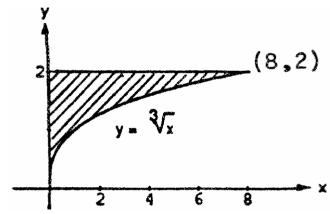
52. $\int_0^3\int_{\sqrt{x/3}}^1 e^{y^3}\, dy\, dx = \int_0^1\int_0^{3y^2} e^{y^3}\, dx\, dy$
$= \int_0^1 3y^2 e^{y^3}\, dy = [e^{y^3}]_0^1 = e - 1$



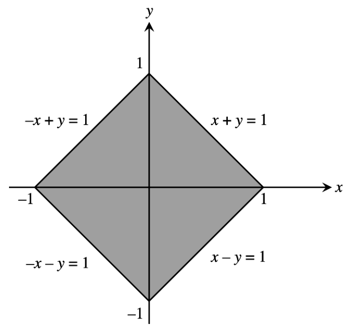
53. $\int_0^{1/16}\int_{y^{1/4}}^{1/2} \cos(16\pi x^5)\, dx\, dy = \int_0^{1/2}\int_0^{x^4} \cos(16\pi x^5)\, dy\, dx$
$= \int_0^{1/2} x^4 \cos(16\pi x^5)\, dx = \left[\frac{\sin(16\pi x^5)}{80\pi}\right]_0^{1/2} = \frac{1}{80\pi}$



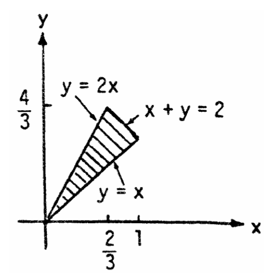
54. $\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{1}{y^4+1}\, dy\, dx = \int_0^2 \int_0^{y^3} \frac{1}{y^4+1}\, dx\, dy$
$= \int_0^2 \frac{y^3}{y^4+1}\, dy = \frac{1}{4}\left[\ln\left(y^4+1\right)\right]_0^2 = \frac{\ln 17}{4}$



55. $\iint_R \left(y - 2x^2\right) dA$
$= \int_{-1}^0 \int_{-x-1}^{x+1} \left(y - 2x^2\right) dy\, dx + \int_0^1 \int_{x-1}^{1-x} \left(y - 2x^2\right) dy\, dx$
$= \int_{-1}^0 \left[\frac{1}{2} y^2 - 2x^2 y\right]_{-x-1}^{x+1} dx + \int_0^1 \left[\frac{1}{2} y^2 - 2x^2 y\right]_{x-1}^{1-x} dx$
$= \int_{-1}^0 \left[\frac{1}{2}(x+1)^2 - 2x^2(x+1) - \frac{1}{2}(-x-1)^2 + 2x^2(-x-1)\right] dx$
$\quad + \int_0^1 \left[\frac{1}{2}(1-x)^2 - 2x^2(1-x) - \frac{1}{2}(x-1)^2 + 2x^2(x-1)\right] dx$
$= -4 \int_{-1}^0 \left(x^3 + x^2\right) dx + 4 \int_0^1 \left(x^3 - x^2\right) dx$
$= -4\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^0 + 4\left[\frac{x^4}{4} - \frac{x^3}{3}\right]_0^1 = 4\left[\frac{(-1)^4}{4} + \frac{(-1)^3}{3}\right] + 4\left(\frac{1}{4} - \frac{1}{3}\right) = 8\left(\frac{3}{12} - \frac{4}{12}\right) = -\frac{8}{12} = -\frac{2}{3}$



56. $\iint_R xy\, dA = \int_0^{2/3} \int_x^{2x} xy\, dy\, dx + \int_{2/3}^1 \int_x^{2-x} xy\, dy\, dx$
$= \int_0^{2/3} \left[\frac{1}{2} xy^2\right]_x^{2x} dx + \int_{2/3}^1 \left[\frac{1}{2} xy^2\right]_x^{2-x} dx$
$= \int_0^{2/3} \left(2x^3 - \frac{1}{2} x^3\right) dx + \int_{2/3}^1 \left[\frac{1}{2} x(2-x)^2 - \frac{1}{2} x^3\right] dx$
$= \int_0^{2/3} \frac{3}{2} x^3\, dx + \int_{2/3}^1 \left(2x - x^2\right) dx$



$= \left[\frac{3}{8} x^4\right]_0^{2/3} + \left[x^2 - \frac{2}{3} x^3\right]_{2/3}^1 = \left(\frac{3}{8}\right)\left(\frac{16}{81}\right) + \left(1 - \frac{2}{3}\right) - \left[\frac{4}{9} - \left(\frac{2}{3}\right)\left(\frac{8}{27}\right)\right] = \frac{6}{81} + \frac{27}{81} - \left(\frac{36}{81} - \frac{16}{81}\right) = \frac{13}{81}$

57. $V = \int_0^1 \int_x^{2-x} \left(x^2 + y^2\right) dy\, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_x^{2-x} dx = \int_0^1 \left[2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3}\right] dx = \left[\frac{2x^3}{3} - \frac{7x^4}{12} - \frac{(2-x)^4}{12}\right]_0^1$
$= \left(\frac{2}{3} - \frac{7}{12} - \frac{1}{12}\right) - \left(0 - 0 - \frac{16}{12}\right) = \frac{4}{3}$

58. $V = \int_{-2}^1 \int_x^{2-x^2} x^2\, dy\, dx = \int_{-2}^1 \left[x^2 y\right]_x^{2-x^2} dx = \int_{-2}^1 \left(2x^2 - x^4 - x^3\right) dx = \left[\frac{2}{3} x^3 - \frac{1}{5} x^5 - \frac{1}{4} x^4\right]_{-2}^1$
$= \left(\frac{2}{3} - \frac{1}{5} - \frac{1}{4}\right) - \left(-\frac{16}{3} + \frac{32}{5} - \frac{16}{4}\right) = \left(\frac{40}{60} - \frac{12}{60} - \frac{15}{60}\right) - \left(-\frac{320}{60} + \frac{384}{60} - \frac{240}{60}\right) = \frac{189}{60} = \frac{63}{20}$

59. $V = \int_{-4}^1 \int_{3x}^{4-x^2} (x+4)\, dy\, dx = \int_{-4}^1 \left[xy + 4y\right]_{3x}^{4-x^2} dx = \int_{-4}^1 \left[x\left(4 - x^2\right) + 4\left(4 - x^2\right) - 3x^2 - 12x\right] dx$
$= \int_{-4}^1 \left(-x^3 - 7x^2 - 8x + 16\right) dx = \left[-\frac{1}{4} x^4 - \frac{7}{3} x^3 - 4x^2 + 16x\right]_{-4}^1 = \left(-\frac{1}{4} - \frac{7}{3} + 12\right) - \left(\frac{64}{3} - 64\right) = \frac{157}{3} - \frac{1}{4} = \frac{625}{12}$

60. $V = \int_0^2 \int_0^{\sqrt{4-x^2}} (3 - y)\, dy\, dx = \int_0^2 \left[3y - \frac{y^2}{2}\right]_0^{\sqrt{4-x^2}} dx = \int_0^2 \left[3\sqrt{4 - x^2} - \left(\frac{4-x^2}{2}\right)\right] dx$
$= \left[\frac{3}{2} x\sqrt{4 - x^2} + 6 \sin^{-1}\left(\frac{x}{2}\right) - 2x + \frac{x^3}{6}\right]_0^2 = 6\left(\frac{\pi}{2}\right) - 4 + \frac{8}{6} = 3\pi - \frac{16}{6} = \frac{9\pi - 8}{3}$

61. $V = \int_0^2 \int_0^3 \left(4 - y^2\right) dx\, dy = \int_0^2 \left[4x - y^2 x\right]_0^3 dy = \int_0^2 \left(12 - 3y^2\right) dy = \left[12y - y^3\right]_0^2 = 24 - 8 = 16$

62. $V=\int_0^2\int_0^{4-x^2}(4-x^2-y)\,dy\,dx=\int_0^2\left[(4-x^2)\,y-\frac{y^2}{2}\right]_0^{4-x^2}dx=\int_0^2\frac{1}{2}(4-x^2)^2\,dx=\int_0^2\left(8-4x^2+\frac{x^4}{2}\right)dx$
$=\left[8x-\frac{4}{3}x^3+\frac{1}{10}x^5\right]_0^2=16-\frac{32}{3}+\frac{32}{10}=\frac{480-320+96}{30}=\frac{128}{15}$

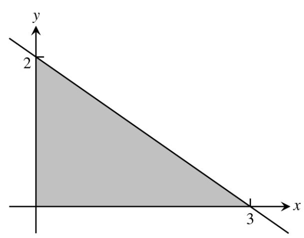
63. $V=\int_0^2\int_0^{2-x}(12-3y^2)\,dy\,dx=\int_0^2\left[12y-y^3\right]_0^{2-x}dx=\int_0^2\left[24-12x-(2-x)^3\right]dx=\left[24x-6x^2+\frac{(2-x)^4}{4}\right]_0^2=20$

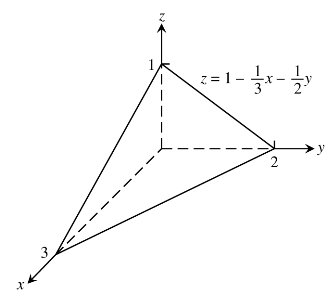
64. $V=\int_{-1}^0\int_{-x-1}^{x+1}(3-3x)\,dy\,dx+\int_0^1\int_{x-1}^{1-x}(3-3x)\,dy\,dx=6\int_{-1}^0(1-x^2)\,dx+6\int_0^1(1-x)^2\,dx=4+2=6$

65. $V=\int_1^2\int_{-1/x}^{1/x}(x+1)\,dy\,dx=\int_1^2\left[xy+y\right]_{-1/x}^{1/x}dx=\int_1^2\left[1+\frac{1}{x}-\left(-1-\frac{1}{x}\right)\right]dx=2\int_1^2\left(1+\frac{1}{x}\right)dx=2\left[x+\ln x\right]_1^2$
$=2(1+\ln 2)$

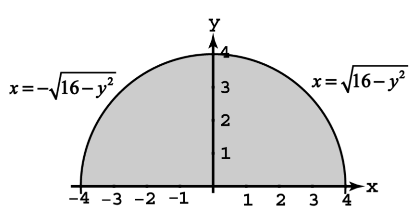
66. $V=4\int_0^{\pi/3}\int_0^{\sec x}(1+y^2)\,dy\,dx=4\int_0^{\pi/3}\left[y+\frac{y^3}{3}\right]_0^{\sec x}dx=4\int_0^{\pi/3}\left(\sec x+\frac{\sec^3 x}{3}\right)dx$
$=\frac{2}{3}\left[7\ln|\sec x+\tan x|+\sec x\tan x\right]_0^{\pi/3}=\frac{2}{3}\left[7\ln\left(2+\sqrt{3}\right)+2\sqrt{3}\right]$

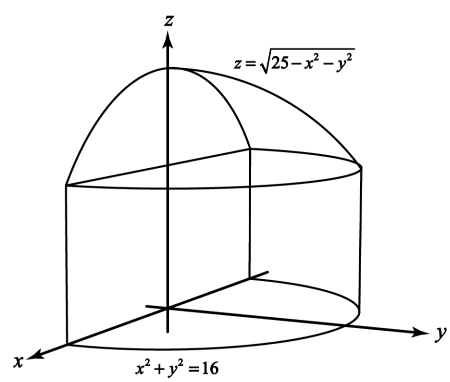
67.





68.





69. $\int_1^\infty\int_{e^{-x}}^1\frac{1}{x^3y}\,dy\,dx=\int_1^\infty\left[\frac{\ln y}{x^3}\right]_{e^{-x}}^1dx=\int_1^\infty-\left(\frac{-x}{x^3}\right)dx=-\lim_{b\to\infty}\left[\frac{1}{x}\right]_1^b=-\lim_{b\to\infty}\left(\frac{1}{b}-1\right)=1$

70. $\int_{-1}^1\int_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}}(2y+1)\,dy\,dx=\int_{-1}^1\left[y^2+y\right]\Big|_{-1/(1-x^2)^{1/2}}^{1/(1-x^2)^{1/2}}dx=\int_{-1}^1\frac{2}{\sqrt{1-x^2}}\,dx=4\lim_{b\to1^-}\left[\sin^{-1}x\right]_0^b$
$=4\lim_{b\to1^-}\left[\sin^{-1}b-0\right]=2\pi$

71. $\int_{-\infty}^\infty\int_{-\infty}^\infty\frac{1}{(x^2+1)(y^2+1)}\,dx\,dy=2\int_0^\infty\left(\frac{2}{y^2+1}\right)\left(\lim_{b\to\infty}\tan^{-1}b-\tan^{-1}0\right)dy=2\pi\lim_{b\to\infty}\int_0^b\frac{1}{y^2+1}\,dy$
$=2\pi\left(\lim_{b\to\infty}\tan^{-1}b-\tan^{-1}0\right)=(2\pi)\left(\frac{\pi}{2}\right)=\pi^2$

72. $\int_0^{\infty}\int_0^{\infty} xe^{-(x+2y)}\,dx\,dy = \int_0^{\infty} e^{-2y} \lim\limits_{b\to\infty} \left[-xe^{-x} - e^{-x}\right]_0^b dy = \int_0^{\infty} e^{-2y} \lim\limits_{b\to\infty} \left(-be^{-b} - e^{-b} + 1\right) dy$
$= \int_0^{\infty} e^{-2y}\,dy = \frac{1}{2} \lim\limits_{b\to\infty} \left(-e^{-2b} + 1\right) = \frac{1}{2}$

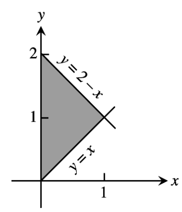
73. $\iint_R f(x,y)\,dA \approx \frac{1}{4} f\left(-\frac{1}{2}, 0\right) + \frac{1}{8} f(0,0) + \frac{1}{8} f\left(\frac{1}{4}, 0\right) = \frac{1}{4}\left(-\frac{1}{2}\right) + \frac{1}{8}\left(0 + \frac{1}{4}\right) = -\frac{3}{32}$

74. $\iint_R f(x,y)\,dA \approx \frac{1}{4}\left[f\left(\frac{7}{4}, \frac{11}{4}\right) + f\left(\frac{9}{4}, \frac{11}{4}\right) + f\left(\frac{7}{4}, \frac{13}{4}\right) + f\left(\frac{9}{4}, \frac{13}{4}\right)\right] = \frac{1}{16}(29 + 31 + 33 + 35) = \frac{128}{16} = 8$

75. The ray $\theta = \frac{\pi}{6}$ meets the circle $x^2 + y^2 = 4$ at the point $\left(\sqrt{3}, 1\right) \Rightarrow$ the ray is represented by the line $y = \frac{x}{\sqrt{3}}$. Thus,
$\iint_R f(x,y)\,dA = \int_0^{\sqrt{3}}\int_{x/\sqrt{3}}^{\sqrt{4-x^2}} \sqrt{4-x^2}\,dy\,dx = \int_0^{\sqrt{3}} \left[(4 - x^2) - \frac{x}{\sqrt{3}}\sqrt{4 - x^2}\right] dx = \left[4x - \frac{x^3}{3} + \frac{(4-x^2)^{3/2}}{3\sqrt{3}}\right]_0^{\sqrt{3}} = \frac{20\sqrt{3}}{9}$

76. $\int_2^{\infty}\int_0^2 \frac{1}{(x^2-x)(y-1)^{2/3}}\,dy\,dx = \int_2^{\infty} \left[\frac{3(y-1)^{1/3}}{(x^2-x)}\right]_0^2 dx = \int_2^{\infty} \left(\frac{3}{x^2-x} + \frac{3}{x^2-x}\right) dx = 6\int_2^{\infty} \frac{dx}{x(x-1)}$
$= 6 \lim\limits_{b\to\infty} \int_2^b \left(\frac{1}{x-1} - \frac{1}{x}\right) dx = 6 \lim\limits_{b\to\infty} \left[\ln(x-1) - \ln x\right]_2^b = 6 \lim\limits_{b\to\infty} \left[\ln(b-1) - \ln b - \ln 1 + \ln 2\right]$
$= 6\left[\lim\limits_{b\to\infty} \ln\left(1 - \frac{1}{b}\right) + \ln 2\right] = 6 \ln 2$

77. $V = \int_0^1\int_x^{2-x} (x^2 + y^2)\,dy\,dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_x^{2-x} dx$
$= \int_0^1 \left[2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3}\right] dx = \left[\frac{2x^3}{3} - \frac{7x^4}{12} - \frac{(2-x)^4}{12}\right]_0^1$
$= \left(\frac{2}{3} - \frac{7}{12} - \frac{1}{12}\right) - \left(0 - 0 - \frac{16}{12}\right) = \frac{4}{3}$



78. $\int_0^2 (\tan^{-1}\pi x - \tan^{-1} x)\,dx = \int_0^2\int_x^{\pi x} \frac{1}{1+y^2}\,dy\,dx = \int_0^2\int_{y/\pi}^{y} \frac{1}{1+y^2}\,dx\,dy + \int_2^{2\pi}\int_{y/\pi}^{2} \frac{1}{1+y^2}\,dx\,dy$
$= \int_0^2 \frac{\left(1-\frac{1}{\pi}\right)y}{1+y^2}\,dy + \int_2^{2\pi} \frac{\left(2-\frac{y}{\pi}\right)}{1+y^2}\,dy = \left(\frac{\pi-1}{2\pi}\right)\left[\ln(1+y^2)\right]_0^2 + \left[2\tan^{-1} y + \frac{1}{2\pi}\ln(1+y^2)\right]_2^{2\pi}$
$= \left(\frac{\pi-1}{2\pi}\right)\ln 5 + 2\tan^{-1} 2\pi - \frac{1}{2\pi}\ln(1+4\pi^2) - 2\tan^{-1} 2 + \frac{1}{2\pi}\ln 5$
$= 2\tan^{-1} 2\pi - 2\tan^{-1} 2 - \frac{1}{2\pi}\ln(1+4\pi^2) + \frac{\ln 5}{2}$

79. To maximize the integral, we want the domain to include all points where the integrand is positive and to exclude all points where the integrand is negative. These criteria are met by the points (x, y) such that $4 - x^2 - 2y^2 \ge 0$ or $x^2 + 2y^2 \le 4$, which is the ellipse $x^2 + 2y^2 = 4$ together with its interior.

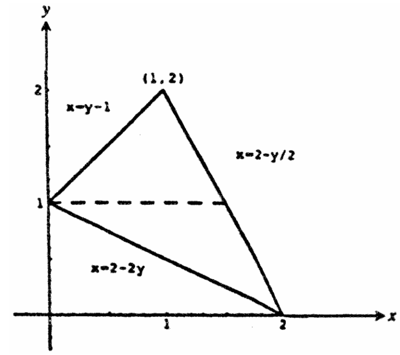
80. To minimize the integral, we want the domain to include all points where the integrand is negative and to exclude all points where the integrand is positive. These criteria are met by the points (x, y) such that $x^2 + y^2 - 9 \le 0$ or $x^2 + y^2 \le 9$, which is the closed disk of radius 3 centered at the origin.

81. No, it is not possible. By Fubini's theorem, the two orders of integration must give the same result.

82. One way would be to partition R into two triangles with the line y = 1. The integral of f over R could then be written as a sum of integrals that could be evaluated by integrating first with respect to x and then with respect to y:

$$\iint_R f(x, y)\, dA$$
$$= \int_0^1 \int_{2-2y}^{2-(y/2)} f(x, y)\, dx\, dy + \int_1^2 \int_{y-1}^{2-(y/2)} f(x, y)\, dx\, dy.$$

Partitioning R with the line x = 1 would let us write the integral of f over R as a sum of iterated integrals with order dy dx.



83. $\int_{-b}^{b}\int_{-b}^{b} e^{-x^2-y^2}\, dx\, dy = \int_{-b}^{b}\int_{-b}^{b} e^{-y^2} e^{-x^2}\, dx\, dy = \int_{-b}^{b} e^{-y^2}\left(\int_{-b}^{b} e^{-x^2}\, dx\right) dy = \left(\int_{-b}^{b} e^{-x^2}\, dx\right)\left(\int_{-b}^{b} e^{-y^2}\, dy\right)$

$= \left(\int_{-b}^{b} e^{-x^2}\, dx\right)^2 = \left(2\int_0^b e^{-x^2}\, dx\right)^2 = 4\left(\int_0^b e^{-x^2}\, dx\right)^2$; taking limits as $b \to \infty$ gives the stated result.

84. $\int_0^1 \int_0^3 \frac{x^2}{(y-1)^{2/3}}\, dy\, dx = \int_0^3 \int_0^1 \frac{x^2}{(y-1)^{2/3}}\, dx\, dy = \int_0^3 \frac{1}{(y-1)^{2/3}}\left[\frac{x^3}{3}\right]_0^1 dy = \frac{1}{3}\int_0^3 \frac{dy}{(y-1)^{2/3}}$

$= \frac{1}{3}\lim_{b \to 1^-}\int_0^b \frac{dy}{(y-1)^{2/3}} + \frac{1}{3}\lim_{b \to 1^+}\int_b^3 \frac{dy}{(y-1)^{2/3}} = \lim_{b \to 1^-}\left[(y-1)^{1/3}\right]_0^b + \lim_{b \to 1^+}\left[(y-1)^{1/3}\right]_b^3$

$= \left[\lim_{b \to 1^-}(b-1)^{1/3} - (-1)^{1/3}\right] - \left[\lim_{b \to 1^+}(b-1)^{1/3} - (2)^{1/3}\right] = (0+1) - \left(0 - \sqrt[3]{2}\right) = 1 + \sqrt[3]{2}$

85-88. Example CAS commands:

Maple:

```
f := (x,y) -> 1/x/y;
q1 := Int( Int( f(x,y), y=1..x ), x=1..3 );
evalf( q1 );
value( q1 );
evalf( value(q1) );
```

89-94. Example CAS commands:

Maple:

```
f := (x,y) -> exp(x^2);
c,d := 0,1;
g1 := y ->2*y;
g2 := y -> 4;
q5 := Int( Int( f(x,y), x=g1(y)..g2(y) ), y=c..d );
value( q5 );
plot3d( 0, x=g1(y)..g2(y), y=c..d, color=pink, style=patchnogrid, axes=boxed, orientation=[-90,0],
        scaling=constrained, title="#89 (Section 15.2)" );
r5 := Int( Int( f(x,y), y=0..x/2 ), x=0..2 ) + Int( Int( f(x,y), y=0..1 ), x=2..4 );
value( r5);
value( q5-r5 );
```

85-94. Example CAS commands:

Mathematica: (functions and bounds will vary)

You can integrate using the built-in integral signs or with the command **Integrate**. In the **Integrate** command, the integration begins with the variable on the right. (In this case, y going from 1 to x).

```
Clear[x, y, f]
f[x_, y_]:= 1 / (x y)
Integrate[f[x, y], {x, 1, 3}, {y, 1, x}]
```

To reverse the order of integration, it is best to first plot the region over which the integration extends. This can be done with ImplicitPlot and all bounds involving both x and y can be plotted. A graphics package must be loaded. Remember to use the double equal sign for the equations of the bounding curves.

```
Clear[x, y, f]
<<Graphics`ImplicitPlot`
ImplicitPlot[{x==2y, x==4, y==0, y==1},{x, 0, 4.1}, {y, 0, 1.1}];
f[x_, y_]:=Exp[x^2]
Integrate[f[x, y], {x, 0, 2}, {y, 0, x/2}] + Integrate[f[x, y], {x, 2, 4}, {y, 0, 1}]
```

To get a numerical value for the result, use the numerical integrator, **NIntegrate**. Verify that this equals the original.

```
Integrate[f[x, y], {x, 0, 2}, {y, 0, x/2}] + NIntegrate[f[x, y], {x, 2, 4}, {y, 0, 1}]
NIntegrate[f[x, y], {y, 0, 1},{x, 2y, 4}]
```

Another way to show a region is with the FilledPlot command. This assumes that functions are given as y = f(x).

```
Clear[x, y, f]
<<Graphics`FilledPlot`
FilledPlot[{x^2, 9},{x, 0,3}, AxesLabels → {x, y}];
f[x_, y_]:= x Cos[y^2]
Integrate[f[x, y], {y, 0, 9}, {x, 0, Sqrt[y]}]
```

85. $\int_1^3 \int_1^x \frac{1}{xy}\, dy\, dx \approx 0.603$

86. $\int_0^1 \int_0^1 e^{-(x^2+y^2)}\, dy\, dx \approx 0.558$

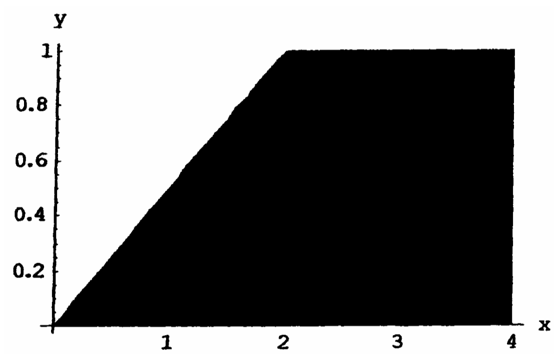
87. $\int_0^1 \int_0^1 \tan^{-1} xy\, dy\, dx \approx 0.233$

88. $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} 3\sqrt{1-x^2-y^2}\, dy\, dx \approx 3.142$

89. Evaluate the integrals:

$$\int_0^1 \int_{2y}^4 e^{x^2}\, dx\, dy$$
$$= \int_0^2 \int_0^{x/2} e^{x^2}\, dy\, dx + \int_2^4 \int_0^1 e^{x^2}\, dy\, dx$$
$$= -\tfrac{1}{4} + \tfrac{1}{4}\left(e^4 - 2\sqrt{\pi}\,\mathrm{erfi}(2) + 2\sqrt{\pi}\,\mathrm{erfi}(4)\right)$$
$$\approx 1.1494 \times 10^6$$

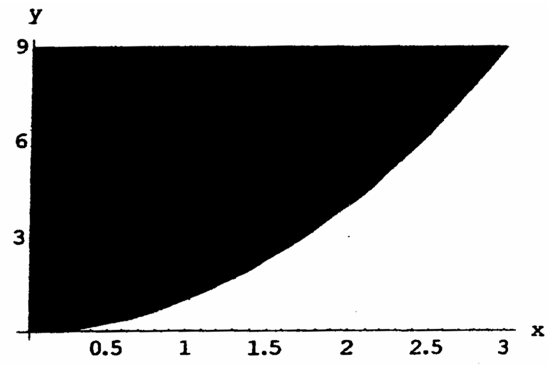
The following graph was generated using Mathematica.



90. Evaluate the integrals:

$$\int_0^3 \int_{x^2}^9 x\cos(y^2)\,dy\,dx = \int_0^9 \int_0^{\sqrt{y}} x\cos(y^2)\,dx\,dy = \frac{\sin(81)}{4} \approx -0.157472$$

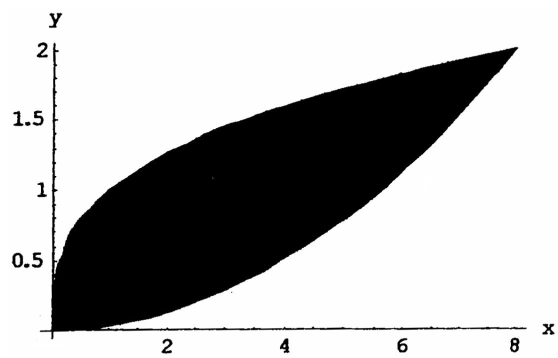
The following graph was generated using Mathematica.



91. Evaluate the integrals:

$$\int_0^2 \int_{y^3}^{4\sqrt{2y}} (x^2y - xy^2)\,dx\,dy = \int_0^8 \int_{x^2/32}^{\sqrt[3]{x}} (x^2y - xy^2)\,dy\,dx = \frac{67{,}520}{693} \approx 97.4315$$

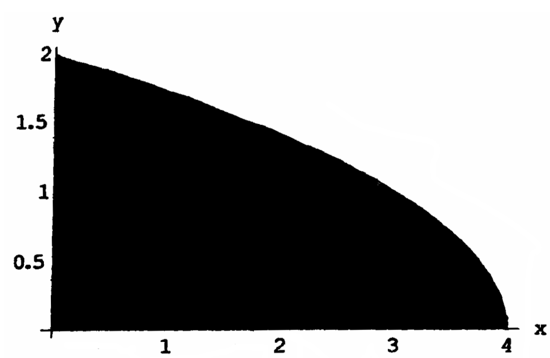
The following graph was generated using Mathematica.



92. Evaluate the integrals:

$$\int_0^2 \int_0^{4-y^2} e^{xy}\,dx\,dy = \int_0^4 \int_0^{\sqrt{4-x}} e^{xy}\,dy\,dx \approx 20.5648$$

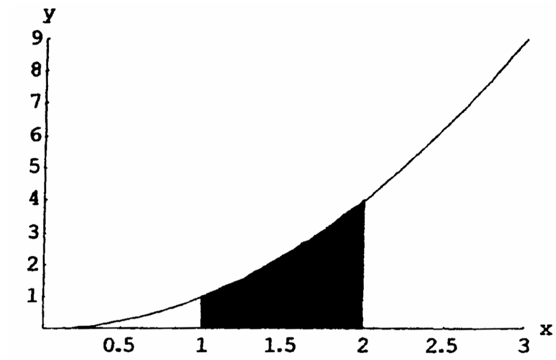
The following graph was generated using Mathematica.



93. Evaluate the integrals:

$$\int_1^2 \int_0^{x^2} \frac{1}{x+y}\, dy\, dx$$
$$= \int_0^1 \int_1^2 \frac{1}{x+y}\, dx\, dy + \int_1^4 \int_{\sqrt{y}}^2 \frac{1}{x+y}\, dx\, dy$$
$$-1 + \ln\left(\tfrac{27}{4}\right) \approx 0.909543$$

The following graph was generated using Mathematica.



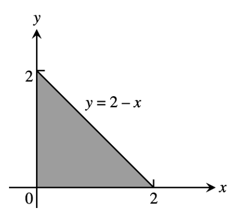
94. Evaluate the integrals:

$$\int_1^2 \int_{y^3}^8 \frac{1}{\sqrt{x^2+y^2}}\, dx\, dy = \int_1^8 \int_1^{\sqrt[3]{x}} \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx$$
$$\approx 0.866649$$

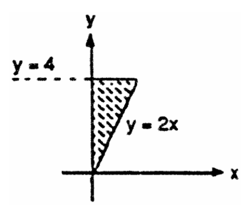
The following graph was generated using Mathematica.

15.3 AREA BY DOUBLE INTEGRATION

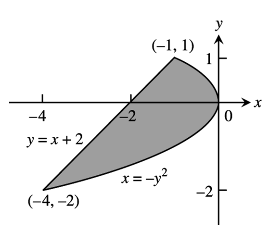
1. $\int_0^2 \int_0^{2-x} dy\, dx = \int_0^2 (2-x)\, dx = \left[2x - \frac{x^2}{2}\right]_0^2 = 2,$
or $\int_0^2 \int_0^{2-y} dx\, dy = \int_0^2 (2-y)\, dy = 2$



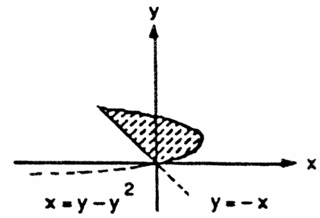
2. $\int_0^2 \int_{2x}^4 dy\, dx = \int_0^2 (4-2x)\, dx = \left[4x - x^2\right]_0^2 = 4,$
or $\int_0^4 \int_0^{y/2} dx\, dy = \int_0^4 \frac{y}{2}\, dy = 4$



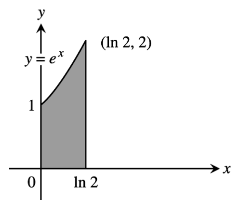
3. $\int_{-2}^1 \int_{y-2}^{-y^2} dx\, dy = \int_{-2}^1 (-y^2 - y + 2)\, dy$
$= \left[-\frac{y^3}{3} - \frac{y^2}{2} + 2y\right]_{-2}^1$
$= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) = \frac{9}{2}$



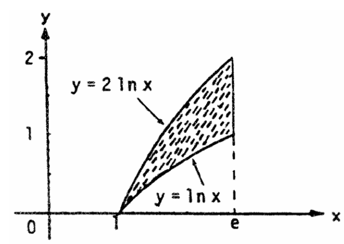
4. $\int_0^2 \int_{-y}^{y-y^2} dx\,dy = \int_0^2 (2y - y^2)\,dy = \left[y^2 - \frac{y^3}{3}\right]_0^2$
$= 4 - \frac{8}{3} = \frac{4}{3}$



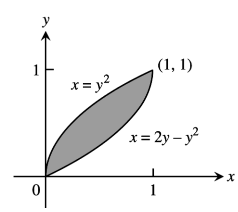
5. $\int_0^{\ln 2} \int_0^{e^x} dy\,dx = \int_0^{\ln 2} e^x\,dx = [e^x]_0^{\ln 2} = 2 - 1 = 1$



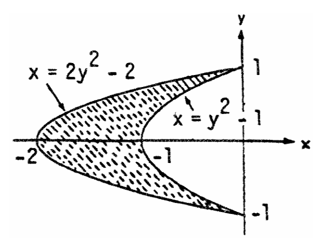
6. $\int_1^e \int_{\ln x}^{2\ln x} dy\,dx = \int_1^e \ln x\,dx = [x \ln x - x]_1^e$
$= (e - e) - (0 - 1) = 1$



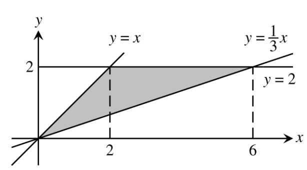
7. $\int_0^1 \int_{y^2}^{2y-y^2} dx\,dy = \int_0^1 (2y - 2y^2)\,dy = \left[y^2 - \frac{2}{3}\,y^3\right]_0^1$
$= \frac{1}{3}$



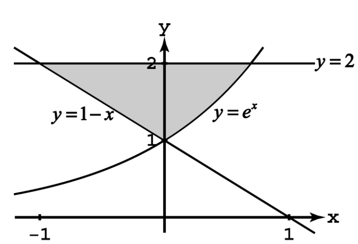
8. $\int_{-1}^1 \int_{2y^2-2}^{y^2-1} dx\,dy = \int_{-1}^1 (y^2 - 1 - 2y^2 + 2)\,dy$
$= \int_{-1}^1 (1 - y^2)\,dy = \left[y - \frac{y^3}{3}\right]_{-1}^1 = \frac{4}{3}$



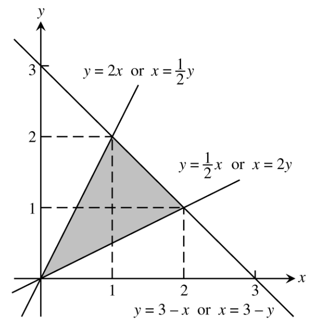
9. $\int_0^2 \int_y^{3y} 1\,dx\,dy = \int_0^2 [x]_y^{3y} dy$
$= \int_0^2 (2y)\,dy = [y^2]_0^2 = 4$



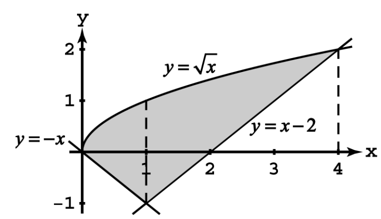
10. $\int_1^2 \int_{1-y}^{\ln y} 1\,dx\,dy = \int_0^2 [x]_{1-y}^{\ln y} dy$
$= \int_1^2 (\ln y - 1 + y)\,dy = \left[y \ln y - 2y + \frac{y^2}{2}\right]_1^2$
$= 2\ln 2 - \frac{1}{2}$



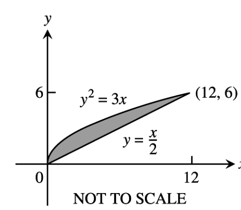
11. $\int_0^1 \int_{x/2}^{2x} 1\, dy\, dx + \int_1^2 \int_{x/2}^{3-x} 1\, dy\, dx$
$= \int_0^1 [y]_{x/2}^{2x} dx + \int_1^2 [y]_{x/2}^{3-x} dx$
$= \int_0^1 \left(\frac{3}{2}x\right) dx + \int_1^2 \left(3 - \frac{3}{2}x\right) dx$
$= \left[\frac{3}{4}x^2\right]_0^1 + \left[3x - \frac{3}{4}x^2\right]_1^2 = \frac{3}{2}$



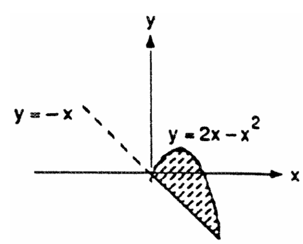
12. $\int_0^1 \int_{-x}^{\sqrt{x}} 1\, dy\, dx + \int_1^4 \int_{x-2}^{\sqrt{x}} 1\, dy\, dx$
$= \int_0^1 [y]_{-x}^{\sqrt{x}} dx + \int_1^4 [y]_{x-2}^{\sqrt{x}} dx$
$= \int_0^1 \left(\sqrt{x} + x\right) dx + \int_1^4 \left(\sqrt{x} - x + 2\right) dx$
$= \left[\frac{2}{3}x^{3/2} + \frac{1}{2}x^2\right]_0^1 + \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right]_1^4 = \frac{13}{3}$



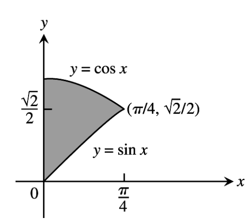
13. $\int_0^6 \int_{y^2/3}^{2y} dx\, dy = \int_0^6 \left(2y - \frac{y^2}{3}\right) dy = \left[y^2 - \frac{y^3}{9}\right]_0^6$
$= 36 - \frac{216}{9} = 12$



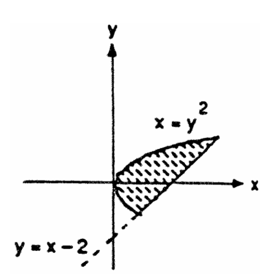
14. $\int_0^3 \int_{-x}^{2x-x^2} dy\, dx = \int_0^3 (3x - x^2)\, dx = \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3$
$= \frac{27}{2} - 9 = \frac{9}{2}$



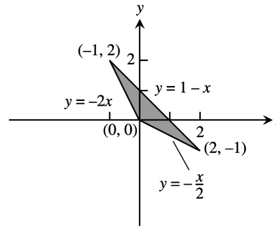
15. $\int_0^{\pi/4} \int_{\sin x}^{\cos x} dy\, dx$
$= \int_0^{\pi/4} (\cos x - \sin x)\, dx = [\sin x + \cos x]_0^{\pi/4}$
$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1) = \sqrt{2} - 1$



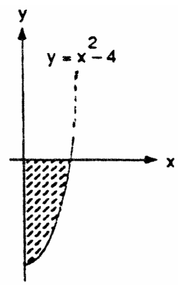
16. $\int_{-1}^2 \int_{y^2}^{y+2} dx\, dy = \int_{-1}^2 (y + 2 - y^2)\, dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^2$
$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = 5 - \frac{1}{2} = \frac{9}{2}$

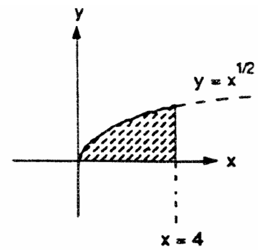


17. $\int_{-1}^{0}\int_{-2x}^{1-x} dy\,dx + \int_{0}^{2}\int_{-x/2}^{1-x} dy\,dx$
$= \int_{-1}^{0}(1+x)\,dx + \int_{0}^{2}\left(1-\frac{x}{2}\right)dx$
$= \left[x+\frac{x^2}{2}\right]_{-1}^{0} + \left[x-\frac{x^2}{4}\right]_{0}^{2} = -\left(-1+\frac{1}{2}\right)+(2-1) = \frac{3}{2}$



18. $\int_{0}^{2}\int_{x^2-4}^{0} dy\,dx + \int_{0}^{4}\int_{0}^{\sqrt{x}} dy\,dx$
$= \int_{0}^{2}(4-x^2)\,dx + \int_{0}^{4} x^{1/2}\,dx$
$= \left[4x-\frac{x^3}{3}\right]_{0}^{2} + \left[\frac{2}{3}x^{3/2}\right]_{0}^{4} = \left(8-\frac{8}{3}\right)+\frac{16}{3} = \frac{32}{3}$





19. (a) average $= \frac{1}{\pi^2}\int_{0}^{\pi}\int_{0}^{\pi}\sin(x+y)\,dy\,dx = \frac{1}{\pi^2}\int_{0}^{\pi}\left[-\cos(x+y)\right]_{0}^{\pi}dx = \frac{1}{\pi^2}\int_{0}^{\pi}\left[-\cos(x+\pi)+\cos x\right]dx$
$= \frac{1}{\pi^2}\left[-\sin(x+\pi)+\sin x\right]_{0}^{\pi} = \frac{1}{\pi^2}\left[(-\sin 2\pi+\sin\pi)-(-\sin\pi+\sin 0)\right] = 0$

(b) average $= \frac{1}{\left(\frac{\pi^2}{2}\right)}\int_{0}^{\pi}\int_{0}^{\pi/2}\sin(x+y)\,dy\,dx = \frac{2}{\pi^2}\int_{0}^{\pi}\left[-\cos(x+y)\right]_{0}^{\pi/2}dx = \frac{2}{\pi^2}\int_{0}^{\pi}\left[-\cos\left(x+\frac{\pi}{2}\right)+\cos x\right]dx$
$= \frac{2}{\pi^2}\left[-\sin\left(x+\frac{\pi}{2}\right)+\sin x\right]_{0}^{\pi} = \frac{2}{\pi^2}\left[\left(-\sin\frac{3\pi}{2}+\sin\pi\right)-\left(-\sin\frac{\pi}{2}+\sin 0\right)\right] = \frac{4}{\pi^2}$

20. average value over the square $= \int_{0}^{1}\int_{0}^{1} xy\,dy\,dx = \int_{0}^{1}\left[\frac{xy^2}{2}\right]_{0}^{1}dx = \int_{0}^{1}\frac{x}{2}\,dx = \frac{1}{4} = 0.25;$

average value over the quarter circle $= \frac{1}{\left(\frac{\pi}{4}\right)}\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} xy\,dy\,dx = \frac{4}{\pi}\int_{0}^{1}\left[\frac{xy^2}{2}\right]_{0}^{\sqrt{1-x^2}}dx$
$= \frac{2}{\pi}\int_{0}^{1}(x-x^3)\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_{0}^{1} = \frac{1}{2\pi} \approx 0.159$. The average value over the square is larger.

21. average height $= \frac{1}{4}\int_{0}^{2}\int_{0}^{2}(x^2+y^2)\,dy\,dx = \frac{1}{4}\int_{0}^{2}\left[x^2y+\frac{y^3}{3}\right]_{0}^{2}dx = \frac{1}{4}\int_{0}^{2}\left(2x^2+\frac{8}{3}\right)dx = \frac{1}{2}\left[\frac{x^3}{3}+\frac{4x}{3}\right]_{0}^{2} = \frac{8}{3}$

22. average $= \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2}\int_{\ln 2}^{2\ln 2}\frac{1}{xy}\,dy\,dx = \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2}\left[\frac{\ln y}{x}\right]_{\ln 2}^{2\ln 2}dx$
$= \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2}\frac{1}{x}(\ln 2+\ln\ln 2-\ln\ln 2)\,dx = \left(\frac{1}{\ln 2}\right)\int_{\ln 2}^{2\ln 2}\frac{dx}{x} = \left(\frac{1}{\ln 2}\right)\left[\ln x\right]_{\ln 2}^{2\ln 2}$
$= \left(\frac{1}{\ln 2}\right)(\ln 2+\ln\ln 2-\ln\ln 2) = 1$

23. $\int_{-5}^{5}\int_{-2}^{0}\frac{10{,}000e^y}{1+\frac{|x|}{2}}\,dy\,dx = 10{,}000\left(1-e^{-2}\right)\int_{-5}^{5}\frac{dx}{1+\frac{|x|}{2}} = 10{,}000\left(1-e^{-2}\right)\left[\int_{-5}^{0}\frac{dx}{1-\frac{x}{2}}+\int_{0}^{5}\frac{dx}{1+\frac{x}{2}}\right]$
$= 10{,}000\left(1-e^{-2}\right)\left[-2\ln\left(1-\frac{x}{2}\right)\right]_{-5}^{0} + 10{,}000\left(1-e^{-2}\right)\left[2\ln\left(1+\frac{x}{2}\right)\right]_{0}^{5}$
$= 10{,}000\left(1-e^{-2}\right)\left[2\ln\left(1+\frac{5}{2}\right)\right] + 10{,}000\left(1-e^{-2}\right)\left[2\ln\left(1+\frac{5}{2}\right)\right] = 40{,}000\left(1-e^{-2}\right)\ln\left(\frac{7}{2}\right) \approx 43{,}329$

24. $\int_{0}^{1}\int_{y^2}^{2y-y^2}100(y+1)\,dx\,dy = \int_{0}^{1}\left[100(y+1)x\right]_{y^2}^{2y-y^2}dy = \int_{0}^{1}100(y+1)\left(2y-2y^2\right)dy = 200\int_{0}^{1}\left(y-y^3\right)dy$
$= 200\left[\frac{y^2}{2}-\frac{y^4}{4}\right]_{0}^{1} = (200)\left(\frac{1}{4}\right) = 50$

25. Let (x_i, y_i) be the location of the weather station in county i for $i = 1, \ldots, 254$. The average temperature in Texas at time t_0 is approximately $\frac{\sum_{i=1}^{254} T(x_i,y_i)\,\Delta_i A}{A}$, where $T(x_i, y_i)$ is the temperature at time t_0 at the weather station in county i, $\Delta_i A$ is the area of county i, and A is the area of Texas.

26. Let $y = f(x)$ be a nonnegative, continuous function on [a, b], then $A = \iint_R dA = \int_a^b \int_0^{f(x)} dy\,dx = \int_a^b [y]_0^{f(x)}\,dx = \int_a^b f(x)\,dx$

15.4 DOUBLE INTEGRALS IN POLAR FORM

1. $x^2 + y^2 = 9^2 \Rightarrow r = 9 \Rightarrow \frac{\pi}{2} \le \theta \le 2\pi,\ 0 \le r \le 9$

2. $x^2 + y^2 = 1^2 \Rightarrow r = 1,\ x^2 + y^2 = 4^2 \Rightarrow r = 4 \Rightarrow -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 1 \le r \le 4$

3. $y = x \Rightarrow \theta = \frac{\pi}{4},\ y = -x \Rightarrow \theta = \frac{3\pi}{4},\ y = 1 \Rightarrow r = \csc\theta \Rightarrow \frac{\pi}{4} \le \theta \le \frac{3\pi}{4},\ 0 \le r \le \csc\theta$

4. $x = 1 \Rightarrow r = \sec\theta,\ y = \sqrt{3}x \Rightarrow \theta = \frac{\pi}{3} \Rightarrow 0 \le \theta \le \frac{\pi}{3},\ 0 \le r \le \sec\theta$

5. $x^2 + y^2 = 1^2 \Rightarrow r = 1,\ x = 2\sqrt{3} \Rightarrow r = 2\sqrt{3}\sec\theta,\ y = 2 \Rightarrow r = 2\csc\theta;\ 2\sqrt{3}\sec\theta = 2\csc\theta \Rightarrow \theta = \frac{\pi}{6}$
$\Rightarrow 0 \le \theta \le \frac{\pi}{6},\ 1 \le r \le 2\sqrt{3}\sec\theta;\ \frac{\pi}{6} \le \theta \le \frac{\pi}{2},\ 1 \le r \le 2\sqrt{3}\csc\theta$

6. $x^2 + y^2 = 2^2 \Rightarrow r = 2,\ x = 1 \Rightarrow r = \sec\theta;\ 2 = \sec\theta \Rightarrow \theta = \frac{\pi}{3}$ or $\theta = -\frac{\pi}{3} \Rightarrow -\frac{\pi}{3} \le \theta \le \frac{\pi}{3},\ \sec\theta \le r \le 2$

7. $x^2 + y^2 = 2x \Rightarrow r = 2\cos\theta \Rightarrow -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2\cos\theta$

8. $x^2 + y^2 = 2y \Rightarrow r = 2\sin\theta \Rightarrow 0 \le \theta \le \pi,\ 0 \le r \le 2\sin\theta$

9. $\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} dy\,dx = \int_0^{\pi}\int_0^{1} r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi} d\theta = \frac{\pi}{2}$

10. $\int_0^1\int_0^{\sqrt{1-y^2}} (x^2 + y^2)\,dx\,dy = \int_0^{\pi/2}\int_0^1 r^3\,dr\,d\theta = \frac{1}{4}\int_0^{\pi/2} d\theta = \frac{\pi}{8}$

11. $\int_0^2\int_0^{\sqrt{4-y^2}} (x^2 + y^2)\,dx\,dy = \int_0^{\pi/2}\int_0^2 r^3\,dr\,d\theta = 4\int_0^{\pi/2} d\theta = 2\pi$

12. $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx = \int_0^{2\pi}\int_0^{a} r\,dr\,d\theta = \frac{a^2}{2}\int_0^{2\pi} d\theta = \pi a^2$

13. $\int_0^6\int_0^y x\,dx\,dy = \int_{\pi/4}^{\pi/2}\int_0^{6\csc\theta} r^2\cos\theta\,dr\,d\theta = 72\int_{\pi/4}^{\pi/2}\cot\theta\csc^2\theta\,d\theta = -36\left[\cot^2\theta\right]_{\pi/4}^{\pi/2} = 36$

14. $\int_0^2\int_0^x y\,dy\,dx = \int_0^{\pi/4}\int_0^{2\sec\theta} r^2\sin\theta\,dr\,d\theta = \frac{8}{3}\int_0^{\pi/4}\tan\theta\sec^2\theta\,d\theta = \frac{4}{3}$

15. $\int_1^{\sqrt{3}}\int_1^x dy\,dx = \int_{\pi/6}^{\pi/4}\int_{\csc\theta}^{\sqrt{3}\sec\theta} r\,dr\,d\theta = \int_{\pi/6}^{\pi/4}\left(\frac{3}{2}\sec^2\theta - \frac{1}{2}\csc^2\theta\right)d\theta = \left[\frac{3}{2}\tan\theta + \frac{1}{2}\cot\theta\right]_{\pi/6}^{\pi/4} = 2 - \sqrt{3}$

16. $\int_{\sqrt{2}}^{2}\int_{\sqrt{4-y^2}}^{y} dy\,dx = \int_{\pi/4}^{\pi/2}\int_2^{2\csc\theta} r\,dr\,d\theta = \int_{\pi/6}^{\pi/4}(2\csc^2\theta - 2)\,d\theta = \left[-2\cot\theta - \frac{1}{2}\theta\right]_{\pi/4}^{\pi/2} = 2 - \frac{\pi}{2}$

17. $\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\,dy\,dx = \int_{\pi}^{3\pi/2}\int_0^1 \frac{2r}{1+r}\,dr\,d\theta = 2\int_{\pi}^{3\pi/2}\int_0^1 \left(1-\frac{1}{1+r}\right) dr\,d\theta = 2\int_{\pi}^{3\pi/2}(1-\ln 2)\,d\theta$
$= (1-\ln 2)\pi$

18. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}\,dy\,dx = 4\int_0^{\pi/2}\int_0^1 \frac{2r}{(1+r^2)^2}\,dr\,d\theta = 4\int_0^{\pi/2}\left[-\frac{1}{1+r^2}\right]_0^1 d\theta = 2\int_0^{\pi/2} d\theta = \pi$

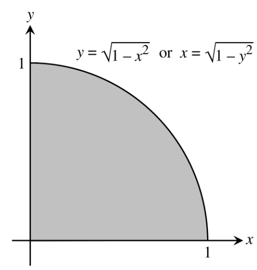
19. $\int_0^{\ln 2}\int_0^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}\,dx\,dy = \int_0^{\pi/2}\int_0^{\ln 2} re^r\,dr\,d\theta = \int_0^{\pi/2}(2\ln 2-1)\,d\theta = \frac{\pi}{2}(2\ln 2-1)$

20. $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1)\,dx\,dy = 4\int_0^{\pi/2}\int_0^1 \ln(r^2+1)\,r\,dr\,d\theta = 2\int_0^{\pi/2}(\ln 4-1)\,d\theta = \pi(\ln 4-1)$

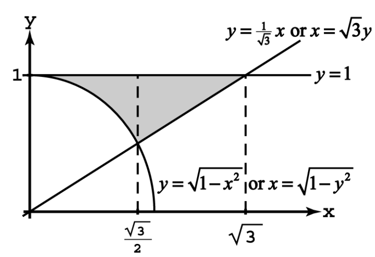
21. $\int_0^1\int_x^{\sqrt{2-x^2}} (x+2y)\,dy\,dx = \int_{\pi/4}^{\pi/2}\int_0^{\sqrt{2}} (r\cos\theta + 2r\sin\theta)\,r\,dr\,d\theta = \int_{\pi/4}^{\pi/2}\left[\frac{r^3}{3}\cos\theta + \frac{2r^3}{3}\sin\theta\right]_0^{\sqrt{2}} d\theta$
$= \int_{\pi/4}^{\pi/2}\left(\frac{2\sqrt{2}}{3}\cos\theta + \frac{4\sqrt{2}}{3}\sin\theta\right) d\theta = \left[\frac{2\sqrt{2}}{3}\sin\theta - \frac{4\sqrt{2}}{3}\cos\theta\right]_{\pi/4}^{\pi/2} = \frac{2(1+\sqrt{2})}{3}$

22. $\int_1^2\int_0^{\sqrt{2x-x^2}} \frac{1}{(x^2+y^2)^2}\,dy\,dx = \int_0^{\pi/4}\int_{\sec\theta}^{2\cos\theta} \frac{1}{r^4}\,r\,dr\,d\theta = \int_0^{\pi/4}\left[-\frac{1}{2r^2}\right]_{\sec\theta}^{2\cos\theta} d\theta = = \int_0^{\pi/4}\left(\frac{1}{2}\cos^2\theta - \frac{1}{8}\sec^2\theta\right) d\theta$
$= \left[\frac{1}{4}\theta + \frac{1}{8}\sin 2\theta - \frac{1}{8}\tan\theta\right]_0^{\pi/4} = \frac{\pi}{16}$

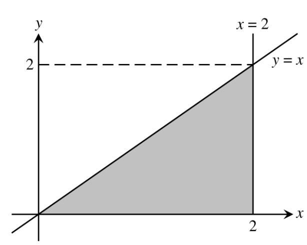
23. $\int_0^1\int_0^{\sqrt{1-x^2}} x\,y\,dy\,dx$ or
$\int_0^1\int_0^{\sqrt{1-y^2}} x\,y\,dx\,dy$



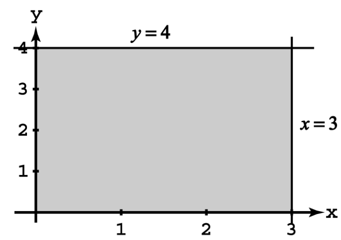
24. $\int_{1/2}^{1}\int_{\sqrt{1-y^2}}^{\sqrt{3}y} x\,dx\,dy$ or
$\int_0^{\sqrt{3}/2}\int_{\sqrt{1-x^2}}^{1} x\,dy\,dx + \int_{\sqrt{3}/2}^{\sqrt{3}}\int_{x/\sqrt{3}}^{1} x\,dy\,dx$



25. $\int_0^2\int_0^x y^2(x^2+y^2)\,dy\,dx$ or
$\int_0^2\int_y^2 y^2(x^2+y^2)\,dx\,dy$



26. $\int_0^3 \int_0^4 (x^2 + y^2)^3\, dy\, dx$ or
$\int_0^4 \int_0^3 (x^2 + y^2)^3\, dx\, dy$



27. $\int_0^{\pi/2} \int_0^{2\sqrt{2 - \sin 2\theta}} r\, dr\, d\theta = 2 \int_0^{\pi/2} (2 - \sin 2\theta)\, d\theta = 2(\pi - 1)$

28. $A = 2 \int_0^{\pi/2} \int_1^{1 + \cos\theta} r\, dr\, d\theta = \int_0^{\pi/2} (2\cos\theta + \cos^2\theta)\, d\theta = \frac{8 + \pi}{4}$

29. $A = 2 \int_0^{\pi/6} \int_0^{12\cos 3\theta} r\, dr\, d\theta = 144 \int_0^{\pi/6} \cos^2 3\theta\, d\theta = 12\pi$

30. $A = \int_0^{2\pi} \int_0^{4\theta/3} r\, dr\, d\theta = \frac{8}{9} \int_0^{2\pi} \theta^2\, d\theta = \frac{64\pi^3}{27}$

31. $A = \int_0^{\pi/2} \int_0^{1 + \sin\theta} r\, dr\, d\theta = \frac{1}{2} \int_0^{\pi/2} \left(\frac{3}{2} + 2\sin\theta - \frac{\cos 2\theta}{2}\right) d\theta = \frac{3\pi}{8} + 1$

32. $A = 4 \int_0^{\pi/2} \int_0^{1 - \cos\theta} r\, dr\, d\theta = 2 \int_0^{\pi/2} \left(\frac{3}{2} - 2\cos\theta + \frac{\cos 2\theta}{2}\right) d\theta = \frac{3\pi}{2} - 4$

33. $\text{average} = \frac{4}{\pi a^2} \int_0^{\pi/2} \int_0^a r\sqrt{a^2 - r^2}\, dr\, d\theta = \frac{4}{3\pi a^2} \int_0^{\pi/2} a^3\, d\theta = \frac{2a}{3}$

34. $\text{average} = \frac{4}{\pi a^2} \int_0^{\pi/2} \int_0^a r^2\, dr\, d\theta = \frac{4}{3\pi a^2} \int_0^{\pi/2} a^3\, d\theta = \frac{2a}{3}$

35. $\text{average} = \frac{1}{\pi a^2} \int_{-a}^{a} \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} \sqrt{x^2 + y^2}\, dy\, dx = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a r^2\, dr\, d\theta = \frac{a}{3\pi} \int_0^{2\pi} d\theta = \frac{2a}{3}$

36. $\text{average} = \frac{1}{\pi} \iint_R [(1 - x)^2 + y^2]\, dy\, dx = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 [(1 - r\cos\theta)^2 + r^2\sin^2\theta]\, r\, dr\, d\theta$
$= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 (r^3 - 2r^2\cos\theta + r)\, dr\, d\theta = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{3}{4} - \frac{2\cos\theta}{3}\right) d\theta = \frac{1}{\pi} \left[\frac{3}{4}\theta - \frac{2\sin\theta}{3}\right]_0^{2\pi} = \frac{3}{2}$

37. $\int_0^{2\pi} \int_1^{\sqrt{e}} \left(\frac{\ln r^2}{r}\right) r\, dr\, d\theta = \int_0^{2\pi} \int_1^{\sqrt{e}} 2\ln r\, dr\, d\theta = 2\int_0^{2\pi} [r\ln r - r]_1^{e^{1/2}}\, d\theta = 2\int_0^{2\pi} \sqrt{e}\left[\left(\frac{1}{2} - 1\right) + 1\right] d\theta = 2\pi\left(2 - \sqrt{e}\right)$

38. $\int_0^{2\pi} \int_1^{e} \left(\frac{\ln r^2}{r}\right) dr\, d\theta = \int_0^{2\pi} \int_1^{e} \left(\frac{2\ln r}{r}\right) dr\, d\theta = \int_0^{2\pi} [(\ln r)^2]_1^{e}\, d\theta = \int_0^{2\pi} d\theta = 2\pi$

39. $V = 2 \int_0^{\pi/2} \int_1^{1 + \cos\theta} r^2\cos\theta\, dr\, d\theta = \frac{2}{3} \int_0^{\pi/2} (3\cos^2\theta + 3\cos^3\theta + \cos^4\theta)\, d\theta$
$= \frac{2}{3}\left[\frac{15\theta}{8} + \sin 2\theta + 3\sin\theta - \sin^3\theta + \frac{\sin 4\theta}{32}\right]_0^{\pi/2} = \frac{4}{3} + \frac{5\pi}{8}$

40. $V = 4 \int_0^{\pi/4} \int_0^{\sqrt{2\cos 2\theta}} r\sqrt{2 - r^2}\, dr\, d\theta = -\frac{4}{3} \int_0^{\pi/4} \left[(2 - 2\cos 2\theta)^{3/2} - 2^{3/2}\right] d\theta$
$= \frac{2\pi\sqrt{2}}{3} - \frac{32}{3} \int_0^{\pi/4} (1 - \cos^2\theta)\sin\theta\, d\theta = \frac{2\pi\sqrt{2}}{3} - \frac{32}{3}\left[\frac{\cos^3\theta}{3} - \cos\theta\right]_0^{\pi/4} = \frac{6\pi\sqrt{2} + 40\sqrt{2} - 64}{9}$

41. (a) $I^2 = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^{\infty}\left(e^{-r^2}\right) r\,dr\,d\theta = \int_0^{\pi/2}\left[\lim_{b\to\infty}\int_0^b re^{-r^2}\,dr\right]d\theta$

$= -\frac{1}{2}\int_0^{\pi/2}\lim_{b\to\infty}\left(e^{-b^2}-1\right)d\theta = \frac{1}{2}\int_0^{\pi/2}d\theta = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2}$

(b) $\lim_{x\to\infty}\int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}}\,dt = \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-t^2}\,dt = \left(\frac{2}{\sqrt{\pi}}\right)\left(\frac{\sqrt{\pi}}{2}\right) = 1$, from part (a)

42. $\int_0^{\infty}\int_0^{\infty}\frac{1}{(1+x^2+y^2)^2}\,dx\,dy = \int_0^{\pi/2}\int_0^{\infty}\frac{r}{(1+r^2)^2}\,dr\,d\theta = \frac{\pi}{2}\lim_{b\to\infty}\int_0^b\frac{r}{(1+r^2)^2}\,dr = \frac{\pi}{4}\lim_{b\to\infty}\left[-\frac{1}{1+r^2}\right]_0^b$

$= \frac{\pi}{4}\lim_{b\to\infty}\left(1-\frac{1}{1+b^2}\right) = \frac{\pi}{4}$

43. Over the disk $x^2+y^2\le\frac{3}{4}$: $\iint_R \frac{1}{1-x^2-y^2}\,dA = \int_0^{2\pi}\int_0^{\sqrt{3}/2}\frac{r}{1-r^2}\,dr\,d\theta = \int_0^{2\pi}\left[-\frac{1}{2}\ln(1-r^2)\right]_0^{\sqrt{3}/2}d\theta$

$= \int_0^{2\pi}\left(-\frac{1}{2}\ln\frac{1}{4}\right)d\theta = (\ln 2)\int_0^{2\pi}d\theta = \pi\ln 4$

Over the disk $x^2+y^2\le 1$: $\iint_R \frac{1}{1-x^2-y^2}\,dA = \int_0^{2\pi}\int_0^1\frac{r}{1-r^2}\,dr\,d\theta = \int_0^{2\pi}\left[\lim_{a\to 1^-}\int_0^a\frac{r}{1-r^2}\,dr\right]d\theta$

$= \int_0^{2\pi}\lim_{a\to 1^-}\left[-\frac{1}{2}\ln(1-a^2)\right]d\theta = 2\pi\cdot\lim_{a\to 1^-}\left[-\frac{1}{2}\ln(1-a^2)\right] = 2\pi\cdot\infty$, so the integral does not exist over $x^2+y^2\le 1$

44. The area in polar coordinates is given by $A = \int_{\alpha}^{\beta}\int_0^{f(\theta)} r\,dr\,d\theta = \int_{\alpha}^{\beta}\left[\frac{r^2}{2}\right]_0^{f(\theta)}d\theta = \frac{1}{2}\int_{\alpha}^{\beta}f^2(\theta)\,d\theta = \int_{\alpha}^{\beta}\frac{1}{2}r^2\,d\theta$, where $r = f(\theta)$

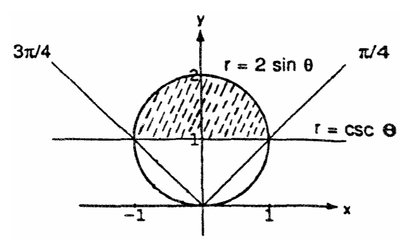
45. average $= \frac{1}{\pi a^2}\int_0^{2\pi}\int_0^a\left[(r\cos\theta-h)^2+r^2\sin^2\theta\right]r\,dr\,d\theta = \frac{1}{\pi a^2}\int_0^{2\pi}\int_0^a\left(r^3-2r^2h\cos\theta+rh^2\right)dr\,d\theta$

$= \frac{1}{\pi a^2}\int_0^{2\pi}\left(\frac{a^4}{4}-\frac{2a^3h\cos\theta}{3}+\frac{a^2h^2}{2}\right)d\theta = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{a^2}{4}-\frac{2ah\cos\theta}{3}+\frac{h^2}{2}\right)d\theta = \frac{1}{\pi}\left[\frac{a^2\theta}{4}-\frac{2ah\sin\theta}{3}+\frac{h^2\theta}{2}\right]_0^{2\pi}$

$= \frac{1}{2}\left(a^2+2h^2\right)$

46. $A = \int_{\pi/4}^{3\pi/4}\int_{\csc\theta}^{2\sin\theta} r\,dr\,d\theta = \frac{1}{2}\int_{\pi/4}^{3\pi/4}\left(4\sin^2\theta-\csc^2\theta\right)d\theta$

$= \frac{1}{2}\left[2\theta-\sin 2\theta+\cot\theta\right]_{\pi/4}^{3\pi/4} = \frac{\pi}{2}$



47-50. Example CAS commands:

Maple:

```
f := (x,y) -> y/(x^2+y^2);
    a,b := 0,1;
    f1 := x -> x;
    f2 := x -> 1;
    plot3d( f(x,y), y=f1(x)..f2(x), x=a..b, axes=boxed, style=patchnogrid, shading=zhue, orientation=[0,180], title="#47(a)
        (Section 15.4)" );                                        # (a)
    q1 := eval( x=a, [x=r*cos(theta),y=r*sin(theta)] );            # (b)
    q2 := eval( x=b, [x=r*cos(theta),y=r*sin(theta)] );
    q3 := eval( y=f1(x), [x=r*cos(theta),y=r*sin(theta)] );
    q4 := eval( y=f2(x), [x=r*cos(theta),y=r*sin(theta)] );
    theta1 := solve( q3, theta );
```

```
theta2 := solve( q1, theta );
r1 := 0;
r2 := solve( q4, r );
plot3d(0,r=r1..r2, theta=theta1..theta2, axes=boxed, style=patchnogrid, shading=zhue, orientation=[-90,0],
        title="#47(c) (Section 15.4)" );
fP := simplify(eval( f(x,y), [x=r*cos(theta),y=r*sin(theta)] ));          # (d)
q5 := Int( Int( fP*r, r=r1..r2 ), theta=theta1..theta2 );
value( q5 );
```

Mathematica: (functions and bounds will vary)

For 47 and 48, begin by drawing the region of integration with the **FilledPlot** command.

```
Clear[x, y, r, t]
<<Graphics`FilledPlot`
FilledPlot[{x, 1}, {x, 0, 1}, AspectRatio → 1, AxesLabel → {x,y}];
```

The picture demonstrates that r goes from 0 to the line y=1 or r = 1/ Sin[t], while t goes from π/4 to π/2.

```
f:= y / (x^2 + y^2)
topolar={x → r Cos[t], y → r Sin[t]};
fp= f/.topolar //Simplify
Integrate[r fp, {t, π/4, π/2}, {r, 0, 1/Sin[t]}]
```

For 49 and 50, drawing the region of integration with the ImplicitPlot command.

```
Clear[x, y]
<<Graphics`ImplicitPlot`
ImplicitPlot[{x==y, x==2 − y, y==0, y==1}, {x, 0, 2.1}, {y, 0, 1.1}];
```

The picture shows that as t goes from 0 to π/4, r goes from 0 to the line x=2 − y. **Solve** will find the bound for r.

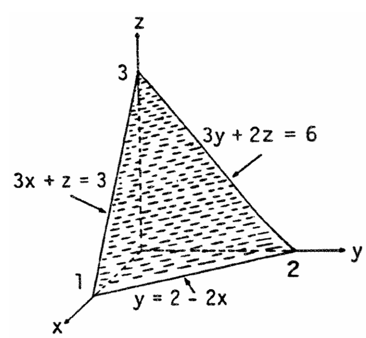
```
bdr=Solve[r Cos[t]==2 − r Sin[t], r]//Simplify
f:=Sqrt[x + y]
topolar={x → r Cos[t], y → r Sin[t]};
fp= f/.topolar //Simplify
Integrate[r fp, {t, 0, π/4}, {r, 0, bdr[[1, 1, 2]]}]
```

15.5 TRIPLE INTEGRALS IN RECTANGULAR COORDINATES

1. $\int_0^1 \int_0^{1-x} \int_{x+z}^1 F(x, y, z)\, dy\, dz\, dx = \int_0^1 \int_0^{1-x} \int_{x+z}^1 dy\, dz\, dx = \int_0^1 \int_0^{1-x} (1 - x - z)\, dz\, dx$
$= \int_0^1 \left[(1-x) - x(1-x) - \frac{(1-x)^2}{2}\right] dx = \int_0^1 \frac{(1-x)^2}{2} dx = \left[-\frac{(1-x)^3}{6}\right]_0^1 = \frac{1}{6}$

2. $\int_0^1 \int_0^2 \int_0^3 dz\, dy\, dx = \int_0^1 \int_0^2 3\, dy\, dx = \int_0^1 6\, dx = 6,\ \int_0^2 \int_0^1 \int_0^3 dz\, dx\, dy,\ \int_0^3 \int_0^2 \int_0^1 dx\, dy\, dz,\ \int_0^2 \int_0^3 \int_0^1 dx\, dz\, dy,$
$\int_0^3 \int_0^1 \int_0^2 dy\, dx\, dz,\ \int_0^1 \int_0^3 \int_0^2 dy\, dz\, dx$

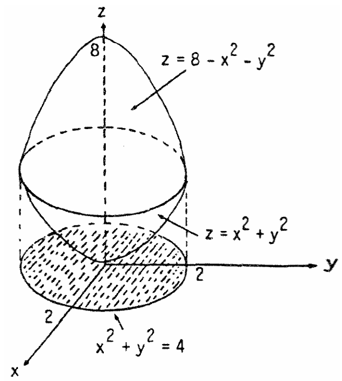
3. $\int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} dz\, dy\, dx$
$= \int_0^1 \int_0^{2-2x} \left(3 - 3x - \frac{3}{2}y\right) dy\, dx$
$= \int_0^1 \left[3(1-x)\cdot 2(1-x) - \frac{3}{4}\cdot 4(1-x)^2\right] dx$
$= 3\int_0^1 (1-x)^2\, dx = \left[-(1-x)^3\right]_0^1 = 1,$
$\int_0^2 \int_0^{1-y/2} \int_0^{3-3x-3y/2} dz\, dx\, dy,\ \int_0^1 \int_0^{3-3x} \int_0^{2-2x-2z/3} dy\, dz\, dx,$



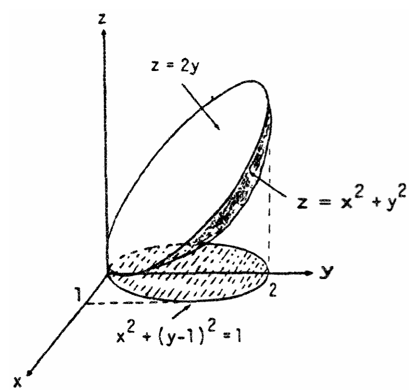
$\int_0^3\int_0^{1-z/3}\int_0^{2-2x-2z/3} dy\,dx\,dz,\ \int_0^2\int_0^{3-3y/2}\int_0^{1-y/2-z/3} dx\,dz\,dy,\ \int_0^3\int_0^{2-2z/3}\int_0^{1-y/2-z/3} dx\,dy\,dz$

4. $\int_0^2\int_0^3\int_0^{\sqrt{4-x^2}} dz\,dy\,dx = \int_0^2\int_0^3 \sqrt{4-x^2}\,dy\,dx = \int_0^2 3\sqrt{4-x^2}\,dx = \frac{3}{2}\left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_0^2 = 6\sin^{-1}1 = 3\pi,$
$\int_0^3\int_0^2\int_0^{\sqrt{4-x^2}} dz\,dx\,dy,\ \int_0^2\int_0^{\sqrt{4-x^2}}\int_0^3 dy\,dz\,dx,\ \int_0^2\int_0^{\sqrt{4-z^2}}\int_0^3 dy\,dx\,dz,\ \int_0^2\int_0^3\int_0^{\sqrt{4-z^2}} dx\,dy\,dz,\ \int_0^3\int_0^2\int_0^{\sqrt{4-z^2}} dx\,dz\,dy$

5. $\int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{8-x^2-y^2} dz\,dy\,dx = 4\int_0^2\int_0^{\sqrt{4-x^2}}\int_{x^2+y^2}^{8-x^2-y^2} dz\,dy\,dx$
$= 4\int_0^2\int_0^{\sqrt{4-x^2}} [8-2(x^2+y^2)]\,dy\,dx$
$= 8\int_0^2\int_0^{\sqrt{4-x^2}} (4-x^2-y^2)\,dy\,dx$
$= 8\int_0^{\pi/2}\int_0^2 (4-r^2)\,r\,dr\,d\theta = 8\int_0^{\pi/2}\left[2r^2 - \frac{r^4}{4}\right]_0^2 d\theta$
$= 32\int_0^{\pi/2} d\theta = 32\left(\frac{\pi}{2}\right) = 16\pi,$
$\int_{-2}^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{x^2+y^2}^{8-x^2-y^2} dz\,dx\,dy,$
$\int_{-2}^2\int_{y^2}^4\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\,dz\,dy + \int_{-2}^2\int_4^{8-y^2}\int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} dx\,dz\,dy,$
$\int_0^4\int_{-\sqrt{z}}^{\sqrt{z}}\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\,dy\,dz + \int_4^8\int_{-\sqrt{8-z}}^{\sqrt{8-z}}\int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} dx\,dy\,dz,\ \int_{-2}^2\int_{x^2}^4\int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dy\,dz\,dx + \int_{-2}^2\int_4^{8-x^2}\int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} dy\,dz\,dx,$
$\int_0^4\int_{-\sqrt{z}}^{\sqrt{z}}\int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dy\,dx\,dz + \int_4^8\int_{-\sqrt{8-z}}^{\sqrt{8-z}}\int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} dy\,dx\,dz$



6. The projection of D onto the xy-plane has the boundary $x^2+y^2 = 2y \Rightarrow x^2+(y-1)^2 = 1$, which is a circle. Therefore the two integrals are:
$\int_0^2\int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}}\int_{x^2+y^2}^{2y} dz\,dx\,dy$ and $\int_{-1}^1\int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}}\int_{x^2+y^2}^{2y} dz\,dy\,dx$



7. $\int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\,dz\,dy\,dx = \int_0^1\int_0^1 \left(x^2+y^2+\frac{1}{3}\right) dy\,dx = \int_0^1 \left(x^2+\frac{2}{3}\right) dx = 1$

8. $\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy = \int_0^{\sqrt{2}}\int_0^{3y} (8-2x^2-4y^2)\,dx\,dy = \int_0^{\sqrt{2}} \left[8x - \frac{2}{3}x^3 - 4xy^2\right]_0^{3y} dy$
$= \int_0^{\sqrt{2}} (24y - 18y^3 - 12y^3)\,dy = \left[12y^2 - \frac{15}{2}y^4\right]_0^{\sqrt{2}} = 24 - 30 = -6$

9. $\int_1^e\int_1^{e^2}\int_1^{e^3} \frac{1}{xyz}\,dx\,dy\,dz = \int_1^e\int_1^{e^2}\left[\frac{\ln x}{yz}\right]_1^{e^3} dy\,dz = \int_1^e\int_1^{e^2} \frac{3}{yz}\,dy\,dz = \int_1^e \left[\frac{\ln y}{z}\right]_1^{e^2} dz = \int_1^e \frac{6}{z}\,dz = 6$

10. $\int_0^1\int_0^{3-3x}\int_0^{3-3x-y} dz\,dy\,dx = \int_0^1\int_0^{3-3x} (3-3x-y)\,dy\,dx = \int_0^1 \left[(3-3x)^2 - \frac{1}{2}(3-3x)^2\right] dx = \frac{9}{2}\int_0^1 (1-x)^2\,dx$
$= -\frac{3}{2}\left[(1-x)^3\right]_0^1 = \frac{3}{2}$

11. $\int_0^{\pi/6}\int_0^1\int_{-2}^3 y\sin z\,dx\,dy\,dz = \int_0^{\pi/6}\int_0^1 5y\sin z\,dy\,dz = \frac{5}{2}\int_0^{\pi/6}\sin z\,dz = \frac{5\left(2-\sqrt{3}\right)}{4}$

12. $\int_{-1}^1\int_0^1\int_0^2 (x+y+z)\,dy\,dx\,dz = \int_{-1}^1\int_0^1 = \left[xy+\frac{1}{2}y^2+zy\right]_0^2 dx\,dz = \int_{-1}^1\int_0^1 (2x+2+2z)\,dx\,dz$
$= \int_{-1}^1 \left[x^2+2x+2zx\right]_0^1 dz = \int_{-1}^1 (3+2z)\,dz = \left[3z+z^2\right]_{-1}^1 = 6$

13. $\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2}} dz\,dy\,dx = \int_0^3\int_0^{\sqrt{9-x^2}} \sqrt{9-x^2}\,dy\,dx = \int_0^3 (9-x^2)\,dx = \left[9x-\frac{x^3}{3}\right]_0^3 = 18$

14. $\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_0^{2x+y} dz\,dx\,dy = \int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (2x+y)\,dx\,dy = \int_0^2 \left[x^2+xy\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy = \int_0^2 (4-y^2)^{1/2}(2y)\,dy$
$= \left[-\frac{2}{3}(4-y^2)^{3/2}\right]_0^2 = \frac{2}{3}(4)^{3/2} = \frac{16}{3}$

15. $\int_0^1\int_0^{2-x}\int_0^{2-x-y} dz\,dy\,dx = \int_0^1\int_0^{2-x} (2-x-y)\,dy\,dx = \int_0^1 \left[(2-x)^2-\frac{1}{2}(2-x)^2\right] dx = \frac{1}{2}\int_0^1 (2-x)^2\,dx$
$= \left[-\frac{1}{6}(2-x)^3\right]_0^1 = -\frac{1}{6}+\frac{8}{6} = \frac{7}{6}$

16. $\int_0^1\int_0^{1-x^2}\int_3^{4-x^2-y} x\,dz\,dy\,dx = \int_0^1\int_0^{1-x^2} x(1-x^2-y)\,dy\,dx = \int_0^1 x\left[(1-x^2)^2-\frac{1}{2}(1-x^2)\right] dx = \int_0^1 \frac{1}{2}x(1-x^2)^2\,dx$
$= \left[-\frac{1}{12}(1-x^2)^3\right]_0^1 = \frac{1}{12}$

17. $\int_0^\pi\int_0^\pi\int_0^\pi \cos(u+v+w)\,du\,dv\,dw = \int_0^\pi\int_0^\pi [\sin(w+v+\pi)-\sin(w+v)]\,dv\,dw$
$= \int_0^\pi [(-\cos(w+2\pi)+\cos(w+\pi))+(\cos(w+\pi)-\cos w)]\,dw$
$= [-\sin(w+2\pi)+\sin(w+\pi)-\sin w+\sin(w+\pi)]_0^\pi = 0$

18. $\int_0^1\int_1^{\sqrt{e}}\int_1^e s\,e^s \ln r\,\frac{(\ln t)^2}{t}\,dt\,dr\,ds = \int_0^1\int_1^{\sqrt{e}} (s\,e^s\ln r)\left[\frac{1}{3}(\ln t)^3\right]_1^e dr\,ds = \int_0^1\int_1^{\sqrt{e}} \frac{s\,e^s}{3}\ln r\,dr\,ds = \int_0^1 \frac{s\,e^s}{3}\left[r\ln r - r\right]_1^{\sqrt{e}} ds$
$= \frac{2-\sqrt{e}}{6}\int_0^1 s\,e^s ds = \frac{2-\sqrt{e}}{6}\left[s\,e^s - e^s\right]_0^1 = \frac{2-\sqrt{e}}{6}$

19. $\int_0^{\pi/4}\int_0^{\ln\sec v}\int_{-\infty}^{2t} e^x\,dx\,dt\,dv = \int_0^{\pi/4}\int_0^{\ln\sec v} \lim_{b\to-\infty}\left(e^{2t}-e^b\right)dt\,dv = \int_0^{\pi/4}\int_0^{\ln\sec v} e^{2t}\,dt\,dv = \int_0^{\pi/4}\left(\frac{1}{2}e^{2\ln\sec v}-\frac{1}{2}\right)dv$
$= \int_0^{\pi/4}\left(\frac{\sec^2 v}{2}-\frac{1}{2}\right)dv = \left[\frac{\tan v}{2}-\frac{v}{2}\right]_0^{\pi/4} = \frac{1}{2}-\frac{\pi}{8}$

20. $\int_0^7\int_0^2\int_0^{\sqrt{4-q^2}} \frac{q}{r+1}\,dp\,dq\,dr = \int_0^7\int_0^2 \frac{q\sqrt{4-q^2}}{r+1}\,dq\,dr = \int_0^7 \frac{1}{3(r+1)}\left[-(4-q^2)^{3/2}\right]_0^2 dr = \frac{8}{3}\int_0^7 \frac{1}{r+1}\,dr = \frac{8\ln 8}{3} = 8\ln 2$

21. (a) $\int_{-1}^1\int_0^{1-x^2}\int_{x^2}^{1-z} dy\,dz\,dx$ (b) $\int_0^1\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{x^2}^{1-z} dy\,dx\,dz$ (c) $\int_0^1\int_0^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy\,dz$
(d) $\int_0^1\int_0^{1-y}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dz\,dy$ (e) $\int_0^1\int_{-\sqrt{y}}^{\sqrt{y}}\int_0^{1-y} dz\,dx\,dy$

22. (a) $\int_0^1\int_0^1\int_{-1}^{-\sqrt{z}} dy\,dz\,dx$ (b) $\int_0^1\int_0^1\int_{-1}^{-\sqrt{z}} dy\,dx\,dz$ (c) $\int_0^1\int_{-1}^{-\sqrt{z}}\int_0^1 dx\,dy\,dz$
(d) $\int_{-1}^0\int_0^{y^2}\int_0^1 dx\,dz\,dy$ (e) $\int_{-1}^0\int_0^1\int_0^{y^2} dz\,dx\,dy$

23. $V = \int_0^1\int_{-1}^1\int_0^{y^2} dz\,dy\,dx = \int_0^1\int_{-1}^1 y^2\,dy\,dx = \frac{2}{3}\int_0^1 dx = \frac{2}{3}$

24. $V = \int_0^1 \int_0^{1-x} \int_0^{2-2z} dy\,dz\,dx = \int_0^1 \int_0^{1-x} (2-2z)\,dz\,dx = \int_0^1 \left[2z - z^2\right]_0^{1-x} dx = \int_0^1 (1-x^2)\,dx = \left[x - \frac{x^3}{3}\right]_0^1 = \frac{2}{3}$

25. $V = \int_0^4 \int_0^{\sqrt{4-x}} \int_0^{2-y} dz\,dy\,dx = \int_0^4 \int_0^{\sqrt{4-x}} (2-y)\,dy\,dx = \int_0^4 \left[2\sqrt{4-x} - \left(\frac{4-x}{2}\right)\right] dx$
$= \left[-\frac{4}{3}(4-x)^{3/2} + \frac{1}{4}(4-x)^2\right]_0^4 = \frac{4}{3}(4)^{3/2} - \frac{1}{4}(16) = \frac{32}{3} - 4 = \frac{20}{3}$

26. $V = 2\int_0^1 \int_{-\sqrt{1-x^2}}^0 \int_0^{-y} dz\,dy\,dx = -2\int_0^1 \int_{-\sqrt{1-x^2}}^0 y\,dy\,dx = \int_0^1 (1-x^2)\,dx = \frac{2}{3}$

27. $V = \int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} dz\,dy\,dx = \int_0^1 \int_0^{2-2x} \left(3 - 3x - \frac{3}{2}y\right) dy\,dx = \int_0^1 \left[6(1-x)^2 - \frac{3}{4} \cdot 4(1-x)^2\right] dx$
$= \int_0^1 3(1-x)^2\,dx = \left[-(1-x)^3\right]_0^1 = 1$

28. $V = \int_0^1 \int_0^{1-x} \int_0^{\cos(\pi x/2)} dz\,dy\,dx = \int_0^1 \int_0^{1-x} \cos\left(\frac{\pi x}{2}\right) dy\,dx = \int_0^1 \left(\cos\frac{\pi x}{2}\right)(1-x)\,dx$
$= \int_0^1 \cos\left(\frac{\pi x}{2}\right) dx - \int_0^1 x\cos\left(\frac{\pi x}{2}\right) dx = \left[\frac{2}{\pi}\sin\frac{\pi x}{2}\right]_0^1 - \frac{4}{\pi^2}\int_0^{\pi/2} u\cos u\,du = \frac{2}{\pi} - \frac{4}{\pi^2}\left[\cos u + u\sin u\right]_0^{\pi/2}$
$= \frac{2}{\pi} - \frac{4}{\pi^2}\left(\frac{\pi}{2} - 1\right) = \frac{4}{\pi^2}$

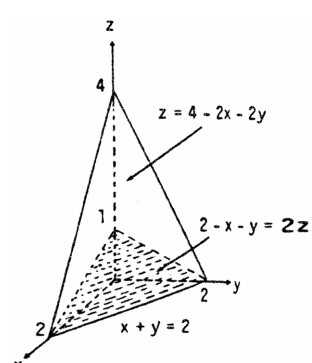
29. $V = 8\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz\,dy\,dx = 8\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\,dy\,dx = 8\int_0^1 (1-x^2)\,dx = \frac{16}{3}$

30. $V = \int_0^2 \int_0^{4-x^2} \int_0^{4-x^2-y} dz\,dy\,dx = \int_0^2 \int_0^{4-x^2} (4-x^2-y)\,dy\,dx = \int_0^2 \left[\left(4-x^2\right)^2 - \frac{1}{2}\left(4-x^2\right)^2\right] dx$
$= \frac{1}{2}\int_0^2 \left(4-x^2\right)^2 dx = \int_0^2 \left(8 - 4x^2 + \frac{x^4}{2}\right) dx = \frac{128}{15}$

31. $V = \int_0^4 \int_0^{\left(\sqrt{16-y^2}\right)/2} \int_0^{4-y} dx\,dz\,dy = \int_0^4 \int_0^{\left(\sqrt{16-y^2}\right)/2} (4-y)\,dz\,dy = \int_0^4 \frac{\sqrt{16-y^2}}{2}(4-y)\,dy$
$= \int_0^4 2\sqrt{16-y^2}\,dy - \frac{1}{2}\int_0^4 y\sqrt{16-y^2}\,dy = \left[y\sqrt{16-y^2} + 16\sin^{-1}\frac{y}{4}\right]_0^4 + \left[\frac{1}{6}\left(16-y^2\right)^{3/2}\right]_0^4$
$= 16\left(\frac{\pi}{2}\right) - \frac{1}{6}(16)^{3/2} = 8\pi - \frac{32}{3}$

32. $V = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{3-x} dz\,dy\,dx = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (3-x)\,dy\,dx = 2\int_{-2}^2 (3-x)\sqrt{4-x^2}\,dx$
$= 3\int_{-2}^2 2\sqrt{4-x^2}\,dx - 2\int_{-2}^2 x\sqrt{4-x^2}\,dx = 3\left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_{-2}^2 + \left[\frac{2}{3}\left(4-x^2\right)^{3/2}\right]_{-2}^2$
$= 12\sin^{-1}1 - 12\sin^{-1}(-1) = 12\left(\frac{\pi}{2}\right) - 12\left(-\frac{\pi}{2}\right) = 12\pi$

33. $\int_0^2 \int_0^{2-x} \int_{(2-x-y)/2}^{4-2x-2y} dz\,dy\,dx = \int_0^2 \int_0^{2-x} \left(3 - \frac{3x}{2} - \frac{3y}{2}\right) dy\,dx$
$= \int_0^2 \left[3\left(1 - \frac{x}{2}\right)(2-x) - \frac{3}{4}(2-x)^2\right] dx$
$= \int_0^2 \left[6 - 6x + \frac{3x^2}{2} - \frac{3(2-x)^2}{4}\right] dx$
$= \left[6x - 3x^2 + \frac{x^3}{2} + \frac{(2-x)^3}{4}\right]_0^2 = (12 - 12 + 4 + 0) - \frac{2^3}{4} = 2$



34. $V = \int_0^4 \int_z^8 \int_z^{8-z} dx\,dy\,dz = \int_0^4 \int_z^8 (8-2z)\,dy\,dz = \int_0^4 (8-2z)(8-z)\,dz = \int_0^4 (64-24z+2z^2)\,dz$
$= \left[64z - 12z^2 + \frac{2}{3}z^3\right]_0^4 = \frac{320}{3}$

35. $V = 2\int_{-2}^2 \int_0^{\sqrt{4-x^2}/2} \int_0^{x+2} dz\,dy\,dx = 2\int_{-2}^2 \int_0^{\sqrt{4-x^2}/2} (x+2)\,dy\,dx = \int_{-2}^2 (x+2)\sqrt{4-x^2}\,dx$
$= \int_{-2}^2 2\sqrt{4-x^2}\,dx + \int_{-2}^2 x\sqrt{4-x^2}\,dx = \left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_{-2}^2 + \left[-\frac{1}{3}(4-x^2)^{3/2}\right]_{-2}^2$
$= 4\left(\frac{\pi}{2}\right) - 4\left(-\frac{\pi}{2}\right) = 4\pi$

36. $V = 2\int_0^1 \int_0^{1-y^2} \int_0^{x^2+y^2} dz\,dx\,dy = 2\int_0^1 \int_0^{1-y^2} (x^2+y^2)\,dx\,dy = 2\int_0^1 \left[\frac{x^3}{3} + xy^2\right]_0^{1-y^2} dy$
$= 2\int_0^1 (1-y^2)\left[\frac{1}{3}(1-y^2)^2 + y^2\right] dy = 2\int_0^1 (1-y^2)\left(\frac{1}{3} + \frac{1}{3}y^2 + \frac{1}{3}y^4\right) dy = \frac{2}{3}\int_0^1 (1-y^6)\,dy$
$= \frac{2}{3}\left[y - \frac{y^7}{7}\right]_0^1 = \left(\frac{2}{3}\right)\left(\frac{6}{7}\right) = \frac{4}{7}$

37. average $= \frac{1}{8}\int_0^2 \int_0^2 \int_0^2 (x^2+9)\,dz\,dy\,dx = \frac{1}{8}\int_0^2 \int_0^2 (2x^2+18)\,dy\,dx = \frac{1}{8}\int_0^2 (4x^2+36)\,dx = \frac{31}{3}$

38. average $= \frac{1}{2}\int_0^1 \int_0^1 \int_0^2 (x+y-z)\,dz\,dy\,dx = \frac{1}{2}\int_0^1 \int_0^1 (2x+2y-2)\,dy\,dx = \frac{1}{2}\int_0^1 (2x-1)\,dx = 0$

39. average $= \int_0^1 \int_0^1 \int_0^1 (x^2+y^2+z^2)\,dz\,dy\,dx = \int_0^1 \int_0^1 \left(x^2+y^2+\frac{1}{3}\right) dy\,dx = \int_0^1 \left(x^2+\frac{2}{3}\right) dx = 1$

40. average $= \frac{1}{8}\int_0^2 \int_0^2 \int_0^2 xyz\,dz\,dy\,dx = \frac{1}{4}\int_0^2 \int_0^2 xy\,dy\,dx = \frac{1}{2}\int_0^2 x\,dx = 1$

41. $\int_0^4 \int_0^1 \int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}}\,dx\,dy\,dz = \int_0^4 \int_0^2 \int_0^{x/2} \frac{4\cos(x^2)}{2\sqrt{z}}\,dy\,dx\,dz = \int_0^4 \int_0^2 \frac{x\cos(x^2)}{\sqrt{z}}\,dx\,dz = \int_0^4 \left(\frac{\sin 4}{2}\right) z^{-1/2}\,dz$
$= \left[(\sin 4)z^{1/2}\right]_0^4 = 2\sin 4$

42. $\int_0^1 \int_0^1 \int_{x^2}^1 12xz\,e^{zy^2}\,dy\,dx\,dz = \int_0^1 \int_0^1 \int_0^{\sqrt{y}} 12xz\,e^{zy^2}\,dx\,dy\,dz = \int_0^1 \int_0^1 6yz\,e^{zy^2}\,dy\,dz = \int_0^1 \left[3e^{zy^2}\right]_0^1 dz$
$= 3\int_0^1 (e^z - z)\,dz = 3\left[e^z - 1\right]_0^1 = 3e - 6$

43. $\int_0^1 \int_{\sqrt[3]{z}}^1 \int_0^{\ln 3} \frac{\pi e^{2x}\sin(\pi y^2)}{y^2}\,dx\,dy\,dz = \int_0^1 \int_{\sqrt[3]{z}}^1 \frac{4\pi\sin(\pi y^2)}{y^2}\,dy\,dz = \int_0^1 \int_0^{y^3} \frac{4\pi\sin(\pi y^2)}{y^2}\,dz\,dy$
$= \int_0^1 4\pi y\sin(\pi y^2)\,dy = \left[-2\cos(\pi y^2)\right]_0^1 = -2(-1) + 2(1) = 4$

44. $\int_0^2 \int_0^{4-x^2} \int_0^x \frac{\sin 2z}{4-z}\,dy\,dz\,dx = \int_0^2 \int_0^{4-x^2} \frac{x\sin 2z}{4-z}\,dz\,dx = \int_0^4 \int_0^{\sqrt{4-z}} \left(\frac{\sin 2z}{4-z}\right) x\,dx\,dz = \int_0^4 \left(\frac{\sin 2z}{4-z}\right)\frac{1}{2}(4-z)\,dz$
$= \left[-\frac{1}{4}\cos 2z\right]_0^4 = \left[-\frac{1}{4} + \frac{1}{2}\sin^2 z\right]_0^4 = \frac{\sin^2 4}{2}$

45. $\int_0^1 \int_0^{4-a-x^2} \int_a^{4-x^2-y} dz\,dy\,dx = \frac{4}{15} \Rightarrow \int_0^1 \int_0^{4-a-x^2} (4-x^2-y-a)\,dy\,dx = \frac{4}{15}$
$\Rightarrow \int_0^1 \left[(4-a-x^2)^2 - \frac{1}{2}(4-a-x^2)^2\right] dx = \frac{4}{15} \Rightarrow \frac{1}{2}\int_0^1 (4-a-x^2)^2\,dx = \frac{4}{15} \Rightarrow \int_0^1 [(4-a)^2 - 2x^2(4-a) + x^4]\,dx$
$= \frac{8}{15} \Rightarrow \left[(4-a)^2 x - \frac{2}{3}x^3(4-a) + \frac{x^5}{5}\right]_0^1 = \frac{8}{15} \Rightarrow (4-a)^2 - \frac{2}{3}(4-a) + \frac{1}{5} = \frac{8}{15} \Rightarrow 15(4-a)^2 - 10(4-a) - 5 = 0$
$\Rightarrow 3(4-a)^2 - 2(4-a) - 1 = 0 \Rightarrow [3(4-a)+1][(4-a)-1] = 0 \Rightarrow 4-a = -\frac{1}{3}$ or $4-a = 1 \Rightarrow a = \frac{13}{3}$ or $a = 3$

46. The volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{4abc\pi}{3}$ so that $\frac{4(1)(2)(c)\pi}{3} = 8\pi \Rightarrow c = 3$.

47. To minimize the integral, we want the domain to include all points where the integrand is negative and to exclude all points where it is positive. These criteria are met by the points (x, y, z) such that $4x^2 + 4y^2 + z^2 - 4 \le 0$ or $4x^2 + 4y^2 + z^2 \le 4$, which is a solid ellipsoid centered at the origin.

48. To maximize the integral, we want the domain to include all points where the integrand is positive and to exclude all points where it is negative. These criteria are met by the points (x, y, z) such that $1 - x^2 - y^2 - z^2 \ge 0$ or $x^2 + y^2 + z^2 \le 1$, which is a solid sphere of radius 1 centered at the origin.

49-52. Example CAS commands:

Maple:

```
F := (x,y,z) -> x^2*y^2*z;
q1 := Int( Int( Int( F(x,y,z), y=-sqrt(1-x^2)..sqrt(1-x^2) ), x=-1..1 ), z=0..1 );
value( q1 );
```

Mathematica: (functions and bounds will vary)

```
Clear[f, x, y, z];
f:= x^2 y^2 z
Integrate[f, {x,−1,1}, {y,−Sqrt[1 − x^2], Sqrt[1 − x^2]}, {z, 0, 1}]
N[%]
topolar={x → r Cos[t], y → r Sin[t]};
fp= f/.topolar //Simplify
Integrate[r fp, {t, 0, 2π}, {r, 0, 1},{z, 0, 1}]
N[%]
```

15.6 MOMENTS AND CENTERS OF MASS

1. $M = \int_0^1 \int_x^{2-x^2} 3\,dy\,dx = 3\int_0^1 (2 - x^2 - x)\,dx = \frac{7}{2}$; $M_y = \int_0^1 \int_x^{2-x^2} 3x\,dy\,dx = 3\int_0^1 [xy]_x^{2-x^2}\,dx$
$= 3\int_0^1 (2x - x^3 - x^2)\,dx = \frac{5}{4}$; $M_x = \int_0^1 \int_x^{2-x^2} 3y\,dy\,dx = \frac{3}{2}\int_0^1 [y^2]_x^{2-x^2}\,dx = \frac{3}{2}\int_0^1 (4 - 5x^2 + x^4)\,dx = \frac{19}{5}$
$\Rightarrow \bar{x} = \frac{5}{14}$ and $\bar{y} = \frac{38}{35}$

2. $M = \delta \int_0^3 \int_0^3 dy\,dx = \delta \int_0^3 3\,dx = 9\delta$; $I_x = \delta \int_0^3 \int_0^3 y^2\,dy\,dx = \delta \int_0^3 \left[\frac{y^3}{3}\right]_0^3 dx = 27\delta$;
$I_y = \delta \int_0^3 \int_0^3 x^2\,dy\,dx = \delta \int_0^3 [x^2 y]_0^3\,dx = \delta \int_0^3 3x^2\,dx = 27\delta$

3. $M = \int_0^2 \int_{y^2/2}^{4-y} dx\,dy = \int_0^2 \left(4 - y - \frac{y^2}{2}\right) dy = \frac{14}{3}$; $M_y = \int_0^2 \int_{y^2/2}^{4-y} x\,dx\,dy = \frac{1}{2}\int_0^2 [x^2]_{y^2/2}^{4-y}\,dy$
$= \frac{1}{2}\int_0^2 \left(16 - 8y + y^2 - \frac{y^4}{4}\right) dy = \frac{128}{15}$; $M_x = \int_0^2 \int_{y^2/2}^{4-y} y\,dx\,dy = \int_0^2 \left(4y - y^2 - \frac{y^3}{2}\right) dy = \frac{10}{3}$
$\Rightarrow \bar{x} = \frac{64}{35}$ and $\bar{y} = \frac{5}{7}$

4. $M = \int_0^3 \int_0^{3-x} dy\,dx = \int_0^3 (3 - x)\,dx = \frac{9}{2}$; $M_y = \int_0^3 \int_0^{3-x} x\,dy\,dx = \int_0^3 [xy]_0^{3-x}\,dx = \int_0^3 (3x - x^2)\,dx = \frac{9}{2}$
$\Rightarrow \bar{x} = 1$ and $\bar{y} = 1$, by symmetry

5. $M = \int_0^a \int_0^{\sqrt{a^2-x^2}} dy\,dx = \frac{\pi a^2}{4}$; $M_y = \int_0^a \int_0^{\sqrt{a^2-x^2}} x\,dy\,dx = \int_0^a [xy]_0^{\sqrt{a^2-x^2}}\,dx = \int_0^a x\sqrt{a^2 - x^2}\,dx = \frac{a^3}{3}$
$\Rightarrow \bar{x} = \bar{y} = \frac{4a}{3\pi}$, by symmetry

6. $M = \int_0^{\pi}\int_0^{\sin x} dy\,dx = \int_0^{\pi} \sin x\,dx = 2$; $M_x = \int_0^{\pi}\int_0^{\sin x} y\,dy\,dx = \frac{1}{2}\int_0^{\pi} \left[y^2\right]_0^{\sin x} dx = \frac{1}{2}\int_0^{\pi} \sin^2 x\,dx$
$= \frac{1}{4}\int_0^{\pi}(1-\cos 2x)\,dx = \frac{\pi}{4} \Rightarrow \overline{x} = \frac{\pi}{2}$ and $\overline{y} = \frac{\pi}{8}$

7. $I_x = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y^2\,dy\,dx = \int_{-2}^{2}\left[\frac{y^3}{3}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx = \frac{2}{3}\int_{-2}^{2}(4-x^2)^{3/2}\,dx = 4\pi$; $I_y = 4\pi$, by symmetry;
$I_o = I_x + I_y = 8\pi$

8. $I_y = \int_{\pi}^{2\pi}\int_0^{(\sin^2 x)/x^2} x^2\,dy\,dx = \int_{\pi}^{2\pi}(\sin^2 x - 0)\,dx = \frac{1}{2}\int_{\pi}^{2\pi}(1-\cos 2x)\,dx = \frac{\pi}{2}$

9. $M = \int_{-\infty}^{0}\int_0^{e^x} dy\,dx = \int_{-\infty}^{0} e^x\,dx = \lim_{b\to-\infty}\int_b^0 e^x\,dx = 1 - \lim_{b\to-\infty} e^b = 1$; $M_y = \int_{-\infty}^{0}\int_0^{e^x} x\,dy\,dx = \int_{-\infty}^{0} xe^x\,dx$
$= \lim_{b\to-\infty}\int_b^0 xe^x\,dx = \lim_{b\to-\infty}\left[xe^x - e^x\right]_b^0 = -1 - \lim_{b\to-\infty}(be^b - e^b) = -1$; $M_x = \int_{-\infty}^{0}\int_0^{e^x} y\,dy\,dx$
$= \frac{1}{2}\int_{-\infty}^{0} e^{2x}\,dx = \frac{1}{2}\lim_{b\to-\infty}\int_b^0 e^{2x}\,dx = \frac{1}{4} \Rightarrow \overline{x} = -1$ and $\overline{y} = \frac{1}{4}$

10. $M_y = \int_0^{\infty}\int_0^{e^{-x^2/2}} x\,dy\,dx = \lim_{b\to\infty}\int_0^b xe^{-x^2/2}\,dx = -\lim_{b\to\infty}\left[\frac{1}{e^{x^2/2}} - 1\right]_0^b = 1$

11. $M = \int_0^2\int_{-y}^{y-y^2}(x+y)\,dx\,dy = \int_0^2\left[\frac{x^2}{2} + xy\right]_{-y}^{y-y^2} dy = \int_0^2\left(\frac{y^4}{2} - 2y^3 + 2y^2\right) dy = \left[\frac{y^5}{10} - \frac{y^4}{2} + \frac{2y^3}{3}\right]_0^2 = \frac{8}{15}$;
$I_x = \int_0^2\int_{-y}^{y-y^2} y^2(x+y)\,dx\,dy = \int_0^2\left[\frac{x^2y^2}{2} + xy^3\right]_{-y}^{y-y^2} dy = \int_0^2\left(\frac{y^6}{2} - 2y^5 + 2y^4\right) dy = \frac{64}{105}$;

12. $M = \int_{-\sqrt{3}/2}^{\sqrt{3}/2}\int_{4y^2}^{\sqrt{12-4y^2}} 5x\,dx\,dy = 5\int_{-\sqrt{3}/2}^{\sqrt{3}/2}\left[\frac{x^2}{2}\right]_{4y^2}^{\sqrt{12-4y^2}} dy = \frac{5}{2}\int_{-\sqrt{3}/2}^{\sqrt{3}/2}(12 - 4y^2 - 16y^4)\,dy = 23\sqrt{3}$

13. $M = \int_0^1\int_x^{2-x}(6x+3y+3)\,dy\,dx = \int_0^1\left[6xy + \frac{3}{2}y^2 + 3y\right]_x^{2-x} dx = \int_0^1(12 - 12x^2)\,dx = 8$;
$M_y = \int_0^1\int_x^{2-x} x(6x+3y+3)\,dy\,dx = \int_0^1(12x - 12x^3)\,dx = 3$; $M_x = \int_0^1\int_x^{2-x} y(6x+3y+3)\,dy\,dx$
$= \int_0^1(14 - 6x - 6x^2 - 2x^3)\,dx = \frac{17}{2} \Rightarrow \overline{x} = \frac{3}{8}$ and $\overline{y} = \frac{17}{16}$

14. $M = \int_0^1\int_{y^2}^{2y-y^2}(y+1)\,dx\,dy = \int_0^1(2y - 2y^3)\,dy = \frac{1}{2}$; $M_x = \int_0^1\int_{y^2}^{2y-y^2} y(y+1)\,dx\,dy = \int_0^1(2y^2 - 2y^4)\,dy = \frac{4}{15}$;
$M_y = \int_0^1\int_{y^2}^{2y-y^2} x(y+1)\,dx\,dy = \int_0^1(2y^2 - 2y^4)\,dy = \frac{4}{15} \Rightarrow \overline{x} = \frac{8}{15}$ and $\overline{y} = \frac{8}{15}$; $I_x = \int_0^1\int_{y^2}^{2y-y^2} y^2(y+1)\,dx\,dy$
$= 2\int_0^1(y^3 - y^5)\,dy = \frac{1}{6}$

15. $M = \int_0^1\int_0^6(x+y+1)\,dx\,dy = \int_0^1(6y+24)\,dy = 27$; $M_x = \int_0^1\int_0^6 y(x+y+1)\,dx\,dy = \int_0^1 y(6y+24)\,dy = 14$;
$M_y = \int_0^1\int_0^6 x(x+y+1)\,dx\,dy = \int_0^1(18y+90)\,dy = 99 \Rightarrow \overline{x} = \frac{11}{3}$ and $\overline{y} = \frac{14}{27}$; $I_y = \int_0^1\int_0^6 x^2(x+y+1)\,dx\,dy$
$= 216\int_0^1\left(\frac{y}{3} + \frac{11}{6}\right) dy = 432$

16. $M = \int_{-1}^{1}\int_{x^2}^{1}(y+1)\,dy\,dx = -\int_{-1}^{1}\left(\frac{x^4}{2} + x^2 - \frac{3}{2}\right) dx = \frac{32}{15}$; $M_x = \int_{-1}^{1}\int_{x^2}^{1} y(y+1)\,dy\,dx = \int_{-1}^{1}\left(\frac{5}{6} - \frac{x^6}{3} - \frac{x^4}{2}\right) dx$
$= \frac{48}{35}$; $M_y = \int_{-1}^{1}\int_{x^2}^{1} x(y+1)\,dy\,dx = \int_{-1}^{1}\left(\frac{3x}{2} - \frac{x^5}{2} - x^3\right) dx = 0 \Rightarrow \overline{x} = 0$ and $\overline{y} = \frac{9}{14}$; $I_y = \int_{-1}^{1}\int_{x^2}^{1} x^2(y+1)\,dy\,dx$
$= \int_{-1}^{1}\left(\frac{3x^2}{2} - \frac{x^6}{2} - x^4\right) dx = \frac{16}{35}$

17. $M = \int_{-1}^{1}\int_{0}^{x^2} (7y+1)\,dy\,dx = \int_{-1}^{1}\left(\frac{7x^4}{2} + x^2\right) dx = \frac{31}{15}$; $M_x = \int_{-1}^{1}\int_{0}^{x^2} y(7y+1)\,dy\,dx = \int_{-1}^{1}\left(\frac{7x^6}{3} + \frac{x^4}{2}\right) dx = \frac{13}{15}$;
$M_y = \int_{-1}^{1}\int_{0}^{x^2} x(7y+1)\,dy\,dx = \int_{-1}^{1}\left(\frac{7x^5}{2} + x^3\right) dx = 0 \Rightarrow \overline{x} = 0$ and $\overline{y} = \frac{13}{31}$; $I_y = \int_{-1}^{1}\int_{0}^{x^2} x^2(7y+1)\,dy\,dx$
$= \int_{-1}^{1}\left(\frac{7x^6}{2} + x^4\right) dx = \frac{7}{5}$

18. $M = \int_{0}^{20}\int_{-1}^{1}\left(1 + \frac{x}{20}\right) dy\,dx = \int_{0}^{20}\left(2 + \frac{x}{10}\right) dx = 60$; $M_x = \int_{0}^{20}\int_{-1}^{1} y\left(1 + \frac{x}{20}\right) dy\,dx = \int_{0}^{20}\left[\left(1 + \frac{x}{20}\right)\left(\frac{y^2}{2}\right)\right]_{-1}^{1} dx = 0$;
$M_y = \int_{0}^{20}\int_{-1}^{1} x\left(1 + \frac{x}{20}\right) dy\,dx = \int_{0}^{20}\left(2x + \frac{x^2}{10}\right) dx = \frac{2000}{3} \Rightarrow \overline{x} = \frac{100}{9}$ and $\overline{y} = 0$; $I_x = \int_{0}^{20}\int_{-1}^{1} y^2\left(1 + \frac{x}{20}\right) dy\,dx$
$= \frac{2}{3}\int_{0}^{20}\left(1 + \frac{x}{20}\right) dx = 20$

19. $M = \int_{0}^{1}\int_{-y}^{y} (y+1)\,dx\,dy = \int_{0}^{1}(2y^2 + 2y)\,dy = \frac{5}{3}$; $M_x = \int_{0}^{1}\int_{-y}^{y} y(y+1)\,dx\,dy = 2\int_{0}^{1}(y^3 + y^2)\,dy = \frac{7}{6}$;
$M_y = \int_{0}^{1}\int_{-y}^{y} x(y+1)\,dx\,dy = \int_{0}^{1} 0\,dy = 0 \Rightarrow \overline{x} = 0$ and $\overline{y} = \frac{7}{10}$; $I_x = \int_{0}^{1}\int_{-y}^{y} y^2(y+1)\,dx\,dy = \int_{0}^{1}(2y^4 + 2y^3)\,dy$
$= \frac{9}{10}$; $I_y = \int_{0}^{1}\int_{-y}^{y} x^2(y+1)\,dx\,dy = \frac{1}{3}\int_{0}^{1}(2y^4 + 2y^3)\,dy = \frac{3}{10} \Rightarrow I_o = I_x + I_y = \frac{6}{5}$

20. $M = \int_{0}^{1}\int_{-y}^{y} (3x^2+1)\,dx\,dy = \int_{0}^{1}(2y^3 + 2y)\,dy = \frac{3}{2}$; $M_x = \int_{0}^{1}\int_{-y}^{y} y(3x^2+1)\,dx\,dy = \int_{0}^{1}(2y^4 + 2y^2)\,dy = \frac{16}{15}$;
$M_y = \int_{0}^{1}\int_{-y}^{y} x(3x^2+1)\,dx\,dy = 0 \Rightarrow \overline{x} = 0$ and $\overline{y} = \frac{32}{45}$; $I_x = \int_{0}^{1}\int_{-y}^{y} y^2(3x^2+1)\,dx\,dy = \int_{0}^{1}(2y^5 + 2y^3)\,dy = \frac{5}{6}$;
$I_y = \int_{0}^{1}\int_{-y}^{y} x^2(3x^2+1)\,dx\,dy = 2\int_{0}^{1}\left(\frac{3}{5}y^5 + \frac{1}{3}y^3\right) dy = \frac{11}{30} \Rightarrow I_o = I_x + I_y = \frac{6}{5}$

21. $I_x = \int_{0}^{a}\int_{0}^{b}\int_{0}^{c}(y^2 + z^2)\,dz\,dy\,dx = \int_{0}^{a}\int_{0}^{b}\left(cy^2 + \frac{c^3}{3}\right) dy\,dx = \int_{0}^{a}\left(\frac{cb^3}{3} + \frac{c^3b}{3}\right) dx = \frac{abc\,(b^2+c^2)}{3}$
$= \frac{M}{3}(b^2 + c^2)$ where $M = abc$; $I_y = \frac{M}{3}(a^2 + c^2)$ and $I_z = \frac{M}{3}(a^2 + b^2)$, by symmetry

22. The plane $z = \frac{4-2y}{3}$ is the top of the wedge $\Rightarrow I_x = \int_{-3}^{3}\int_{-2}^{4}\int_{-4/3}^{(4-2y)/3}(y^2 + z^2)\,dz\,dy\,dx$
$= \int_{-3}^{3}\int_{-2}^{4}\left[\frac{8y^2}{3} - \frac{2y^3}{3} + \frac{8(2-y)^3}{81} + \frac{64}{81}\right] dy\,dx = \int_{-3}^{3}\frac{104}{3}\,dx = 208$; $I_y = \int_{-3}^{3}\int_{-2}^{4}\int_{-4/3}^{(4-2y)/3}(x^2 + z^2)\,dz\,dy\,dx$
$= \int_{-3}^{3}\int_{-2}^{4}\left[\frac{(4-2y)^3}{81} + \frac{x^2(4-2y)}{3} + \frac{4x^2}{3} + \frac{64}{81}\right] dy\,dx = \int_{-3}^{3}\left(12x^2 + \frac{32}{3}\right) dx = 280$;
$I_z = \int_{-3}^{3}\int_{-2}^{4}\int_{-4/3}^{(4-2y)/3}(x^2 + y^2)\,dz\,dy\,dx = \int_{-3}^{3}\int_{-2}^{4}(x^2 + y^2)\left(\frac{8}{3} - \frac{2y}{3}\right) dy\,dx = 12\int_{-3}^{3}(x^2 + 2)\,dx = 360$

23. $M = 4\int_{0}^{1}\int_{0}^{1}\int_{4y^2}^{4} dz\,dy\,dx = 4\int_{0}^{1}\int_{0}^{1}(4 - 4y^2)\,dy\,dx = 16\int_{0}^{1}\frac{2}{3}\,dx = \frac{32}{3}$; $M_{xy} = 4\int_{0}^{1}\int_{0}^{1}\int_{4y^2}^{4} z\,dz\,dy\,dx$
$= 2\int_{0}^{1}\int_{0}^{1}(16 - 16y^4)\,dy\,dx = \frac{128}{5}\int_{0}^{1}dx = \frac{128}{5} \Rightarrow \overline{z} = \frac{12}{5}$, and $\overline{x} = \overline{y} = 0$, by symmetry;
$I_x = 4\int_{0}^{1}\int_{0}^{1}\int_{4y^2}^{4}(y^2 + z^2)\,dz\,dy\,dx = 4\int_{0}^{1}\int_{0}^{1}\left[\left(4y^2 + \frac{64}{3}\right) - \left(4y^4 + \frac{64y^6}{3}\right)\right] dy\,dx = 4\int_{0}^{1}\frac{1976}{105}\,dx = \frac{7904}{105}$;
$I_y = 4\int_{0}^{1}\int_{0}^{1}\int_{4y^2}^{4}(x^2 + z^2)\,dz\,dy\,dx = 4\int_{0}^{1}\int_{0}^{1}\left[\left(4x^2 + \frac{64}{3}\right) - \left(4x^2y^2 + \frac{64y^6}{3}\right)\right] dy\,dx = 4\int_{0}^{1}\left(\frac{8}{3}x^2 + \frac{128}{7}\right) dx$
$= \frac{4832}{63}$; $I_z = 4\int_{0}^{1}\int_{0}^{1}\int_{4y^2}^{4}(x^2 + y^2)\,dz\,dy\,dx = 16\int_{0}^{1}\int_{0}^{1}(x^2 - x^2y^2 + y^2 - y^4)\,dy\,dx$
$= 16\int_{0}^{1}\left(\frac{2x^2}{3} + \frac{2}{15}\right) dx = \frac{256}{45}$

24. (a) $M = \int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2}\int_{0}^{2-x} dz\,dy\,dx = \int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2}(2-x)\,dy\,dx = \int_{-2}^{2}(2-x)\left(\sqrt{4-x^2}\right) dx = 4\pi$;
$M_{yz} = \int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2}\int_{0}^{2-x} x\,dz\,dy\,dx = \int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2} x(2-x)\,dy\,dx = \int_{-2}^{2} x(2-x)\left(\sqrt{4-x^2}\right) dx = -2\pi$;

$M_{xz} = \int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2}\int_{0}^{2-x} y\,dz\,dy\,dx = \int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2} y(2-x)\,dy\,dx$
$= \frac{1}{2}\int_{-2}^{2}(2-x)\left[\frac{4-x^2}{4} - \frac{4-x^2}{4}\right] dx = 0 \Rightarrow \overline{x} = -\frac{1}{2}$ and $\overline{y} = 0$

(b) $M_{xy} = \int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2}\int_{0}^{2-x} z\,dz\,dy\,dx = \frac{1}{2}\int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2}(2-x)^2\,dy\,dx = \frac{1}{2}\int_{-2}^{2}(2-x)^2\left(\sqrt{4-x^2}\right)dx$
$= 5\pi \Rightarrow \overline{z} = \frac{5}{4}$

25. (a) $M = 4\int_{0}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{4} dz\,dy\,dx = 4\int_{0}^{\pi/2}\int_{0}^{2}\int_{r^2}^{4} r\,dz\,dr\,d\theta = 4\int_{0}^{\pi/2}\int_{0}^{2}(4r - r^3)\,dr\,d\theta = 4\int_{0}^{\pi/2} 4\,d\theta = 8\pi;$
$M_{xy} = \int_{0}^{2\pi}\int_{0}^{2}\int_{r^2}^{4} zr\,dz\,dr\,d\theta = \int_{0}^{2\pi}\int_{0}^{2}\frac{r}{2}(16 - r^4)\,dr\,d\theta = \frac{32}{3}\int_{0}^{2\pi} d\theta = \frac{64\pi}{3} \Rightarrow \overline{z} = \frac{8}{3}$, and $\overline{x} = \overline{y} = 0$, by symmetry

(b) $M = 8\pi \Rightarrow 4\pi = \int_{0}^{2\pi}\int_{0}^{\sqrt{c}}\int_{r^2}^{c} r\,dz\,dr\,d\theta = \int_{0}^{2\pi}\int_{0}^{\sqrt{c}}(cr - r^3)\,dr\,d\theta = \int_{0}^{2\pi}\frac{c^2}{4}\,d\theta = \frac{c^2\pi}{2} \Rightarrow c^2 = 8 \Rightarrow c = 2\sqrt{2},$
since $c > 0$

26. $M = 8$; $M_{xy} = \int_{-1}^{1}\int_{3}^{5}\int_{-1}^{1} z\,dz\,dy\,dx = \int_{-1}^{1}\int_{3}^{5}\left[\frac{z^2}{2}\right]_{-1}^{1} dy\,dx = 0$; $M_{yz} = \int_{-1}^{1}\int_{3}^{5}\int_{-1}^{1} x\,dz\,dy\,dx$
$= 2\int_{-1}^{1}\int_{3}^{5} x\,dy\,dx = 4\int_{-1}^{1} x\,dx = 0$; $M_{xz} = \int_{-1}^{1}\int_{3}^{5}\int_{-1}^{1} y\,dz\,dy\,dx = 2\int_{-1}^{1}\int_{3}^{5} y\,dy\,dx = 16\int_{-1}^{1} dx = 32$
$\Rightarrow \overline{x} = 0, \overline{y} = 4, \overline{z} = 0$; $I_x = \int_{-1}^{1}\int_{3}^{5}\int_{-1}^{1}(y^2 + z^2)\,dz\,dy\,dx = \int_{-1}^{1}\int_{3}^{5}\left(2y^2 + \frac{2}{3}\right)dy\,dx = \frac{2}{3}\int_{-1}^{1} 100\,dx = \frac{400}{3}$;
$I_y = \int_{-1}^{1}\int_{3}^{5}\int_{-1}^{1}(x^2 + z^2)\,dz\,dy\,dx = \int_{-1}^{1}\int_{3}^{5}\left(2x^2 + \frac{2}{3}\right)dy\,dx = \frac{4}{3}\int_{-1}^{1}(3x^2 + 1)\,dx = \frac{16}{3}$;
$I_z = \int_{-1}^{1}\int_{3}^{5}\int_{-1}^{1}(x^2 + y^2)\,dz\,dy\,dx = 2\int_{-1}^{1}\int_{3}^{5}(x^2 + y^2)\,dy\,dx = 2\int_{-1}^{1}\left(2x^2 + \frac{98}{3}\right)dx = \frac{400}{3}$

27. The plane $y + 2z = 2$ is the top of the wedge $\Rightarrow I_L = \int_{-2}^{2}\int_{-2}^{4}\int_{-1}^{(2-y)/2}\left[(y-6)^2 + z^2\right] dz\,dy\,dx$
$= \int_{-2}^{2}\int_{-2}^{4}\left[\frac{(y-6)^2(4-y)}{2} + \frac{(2-y)^3}{24} + \frac{1}{3}\right] dy\,dx$; let $t = 2 - y \Rightarrow I_L = 4\int_{-2}^{4}\left(\frac{13t^3}{24} + 5t^2 + 16t + \frac{49}{3}\right) dt = 1386;$
$M = \frac{1}{2}(3)(6)(4) = 36$

28. The plane $y + 2z = 2$ is the top of the wedge $\Rightarrow I_L = \int_{-2}^{2}\int_{-2}^{4}\int_{-1}^{(2-y)/2}\left[(x-4)^2 + y^2\right] dz\,dy\,dx$
$= \frac{1}{2}\int_{-2}^{2}\int_{-2}^{4}(x^2 - 8x + 16 + y^2)(4 - y)\,dy\,dx = \int_{-2}^{2}(9x^2 - 72x + 162)\,dx = 696$; $M = \frac{1}{2}(3)(6)(4) = 36$

29. (a) $M = \int_{0}^{2}\int_{0}^{2-x}\int_{0}^{2-x-y} 2x\,dz\,dy\,dx = \int_{0}^{2}\int_{0}^{2-x}(4x - 2x^2 - 2xy)\,dy\,dx = \int_{0}^{2}(x^3 - 4x^2 + 4x)\,dx = \frac{4}{3}$

(b) $M_{xy} = \int_{0}^{2}\int_{0}^{2-x}\int_{0}^{2-x-y} 2xz\,dz\,dy\,dx = \int_{0}^{2}\int_{0}^{2-x} x(2-x-y)^2\,dy\,dx = \int_{0}^{2}\frac{x(2-x)^3}{3}\,dx = \frac{8}{15}$; $M_{xz} = \frac{8}{15}$ by symmetry; $M_{yz} = \int_{0}^{2}\int_{0}^{2-x}\int_{0}^{2-x-y} 2x^2\,dz\,dy\,dx = \int_{0}^{2}\int_{0}^{2-x} 2x^2(2-x-y)\,dy\,dx = \int_{0}^{2}(2x - x^2)^2\,dx = \frac{16}{15}$
$\Rightarrow \overline{x} = \frac{4}{5}$, and $\overline{y} = \overline{z} = \frac{2}{5}$

30. (a) $M = \int_{0}^{2}\int_{0}^{\sqrt{x}}\int_{0}^{4-x^2} kxy\,dz\,dy\,dx = k\int_{0}^{2}\int_{0}^{\sqrt{x}} xy(4 - x^2)\,dy\,dx = \frac{k}{2}\int_{0}^{2}(4x^2 - x^4)\,dx = \frac{32k}{15}$

(b) $M_{yz} = \int_{0}^{2}\int_{0}^{\sqrt{x}}\int_{0}^{4-x^2} kx^2y\,dz\,dy\,dx = k\int_{0}^{2}\int_{0}^{\sqrt{x}} x^2y(4 - x^2)\,dy\,dx = \frac{k}{2}\int_{0}^{2}(4x^3 - x^5)\,dx = \frac{8k}{3}$
$\Rightarrow \overline{x} = \frac{5}{4}$; $M_{xz} = \int_{0}^{2}\int_{0}^{\sqrt{x}}\int_{0}^{4-x^2} kxy^2\,dz\,dy\,dx = k\int_{0}^{2}\int_{0}^{\sqrt{x}} xy^2(4 - x^2)\,dy\,dx = \frac{k}{3}\int_{0}^{2}\left(4x^{5/2} - x^{9/2}\right)dx$
$= \frac{256\sqrt{2}k}{231} \Rightarrow \overline{y} = \frac{40\sqrt{2}}{77}$; $M_{xy} = \int_{0}^{2}\int_{0}^{\sqrt{x}}\int_{0}^{4-x^2} kxyz\,dz\,dy\,dx = \int_{0}^{2}\int_{0}^{\sqrt{x}} xy(4 - x^2)^2\,dy\,dx$
$= \frac{k}{4}\int_{0}^{2}(16x^2 - 8x^4 + x^6)\,dx = \frac{256k}{105} \Rightarrow \overline{z} = \frac{8}{7}$

31. (a) $M = \int_{0}^{1}\int_{0}^{1}\int_{0}^{1}(x + y + z + 1)\,dz\,dy\,dx = \int_{0}^{1}\int_{0}^{1}\left(x + y + \frac{3}{2}\right)dy\,dx = \int_{0}^{1}(x + 2)\,dx = \frac{5}{2}$

(b) $M_{xy} = \int_0^1\int_0^1\int_0^1 z(x+y+z+1)\,dz\,dy\,dx = \frac{1}{2}\int_0^1\int_0^1 \left(x+y+\frac{5}{3}\right) dy\,dx = \frac{1}{2}\int_0^1 \left(x+\frac{13}{6}\right) dx = \frac{4}{3}$
$\Rightarrow M_{xy} = M_{yz} = M_{xz} = \frac{4}{3}$, by symmetry $\Rightarrow \overline{x} = \overline{y} = \overline{z} = \frac{8}{15}$

(c) $I_z = \int_0^1\int_0^1\int_0^1 (x^2+y^2)(x+y+z+1)\,dz\,dy\,dx = \int_0^1\int_0^1 (x^2+y^2)\left(x+y+\frac{3}{2}\right) dy\,dx$
$= \int_0^1 \left(x^3+2x^2+\frac{1}{3}x+\frac{3}{4}\right) dx = \frac{11}{6} \Rightarrow I_x = I_y = I_z = \frac{11}{6}$, by symmetry

32. The plane $y+2z=2$ is the top of the wedge.

(a) $M = \int_{-1}^1\int_{-2}^4\int_{-1}^{(2-y)/2} (x+1)\,dz\,dy\,dx = \int_{-1}^1\int_{-2}^4 (x+1)\left(2-\frac{y}{2}\right) dy\,dx = 18$

(b) $M_{yz} = \int_{-1}^1\int_{-2}^4\int_{-1}^{(2-y)/2} x(x+1)\,dz\,dy\,dx = \int_{-1}^1\int_{-2}^4 x(x+1)\left(2-\frac{y}{2}\right) dy\,dx = 6;$

$M_{xz} = \int_{-1}^1\int_{-2}^4\int_{-1}^{(2-y)/2} y(x+1)\,dz\,dy\,dx = \int_{-1}^1\int_{-2}^4 y(x+1)\left(2-\frac{y}{2}\right) dy\,dx = 0;$

$M_{xy} = \int_{-1}^1\int_{-2}^4\int_{-1}^{(2-y)/2} z(x+1)\,dz\,dy\,dx = \frac{1}{2}\int_{-1}^1\int_{-2}^4 (x+1)\left(\frac{y^2}{4}-y\right) dy\,dx = 0 \Rightarrow \overline{x} = \frac{1}{3}$, and $\overline{y} = \overline{z} = 0$

(c) $I_x = \int_{-1}^1\int_{-2}^4\int_{-1}^{(2-y)/2} (x+1)(y^2+z^2)\,dz\,dy\,dx = \int_{-1}^1\int_{-2}^4 (x+1)\left[2y^2+\frac{1}{3}-\frac{y^3}{2}+\frac{1}{3}\left(1-\frac{y}{2}\right)^3\right] dy\,dx = 45;$

$I_y = \int_{-1}^1\int_{-2}^4\int_{-1}^{(2-y)/2} (x+1)(x^2+z^2)\,dz\,dy\,dx = \int_{-1}^1\int_{-2}^4 (x+1)\left[2x^2+\frac{1}{3}-\frac{x^2y}{2}+\frac{1}{3}\left(1-\frac{y}{2}\right)^3\right] dy\,dx = 15;$

$I_z = \int_{-1}^1\int_{-2}^4\int_{-1}^{(2-y)/2} (x+1)(x^2+y^2)\,dz\,dy\,dx = \int_{-1}^1\int_{-2}^4 (x+1)\left(2-\frac{y}{2}\right)(x^2+y^2)\,dy\,dx = 42$

33. $M = \int_0^1\int_{z-1}^{1-z}\int_0^{\sqrt{z}} (2y+5)\,dy\,dx\,dz = \int_0^1\int_{z-1}^{1-z} \left(z+5\sqrt{z}\right) dx\,dz = \int_0^1 2\left(z+5\sqrt{z}\right)(1-z)\,dz$
$= 2\int_0^1 \left(5z^{1/2}+z-5z^{3/2}-z^2\right) dz = 2\left[\frac{10}{3}z^{3/2}+\frac{1}{2}z^2-2z^{5/2}-\frac{1}{3}z^3\right]_0^1 = 2\left(\frac{9}{3}-\frac{3}{2}\right) = 3$

34. $M = \int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2(x^2+y^2)}^{16-2(x^2+y^2)} \sqrt{x^2+y^2}\,dz\,dy\,dx = \int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,[16-4(x^2+y^2)]\,dy\,dx$
$= 4\int_0^{2\pi}\int_0^2 r(4-r^2)\,r\,dr\,d\theta = 4\int_0^{2\pi}\left[\frac{4r^3}{3}-\frac{r^5}{5}\right]_0^2 d\theta = 4\int_0^{2\pi}\frac{64}{15}\,d\theta = \frac{512\pi}{15}$

35. (a) $\overline{x} = \frac{M_{yz}}{M} = 0 \Rightarrow \iiint\limits_R x\delta(x,y,z)\,dx\,dy\,dz = 0 \Rightarrow M_{yz} = 0$

(b) $I_L = \iiint\limits_D |\mathbf{v}-h\mathbf{i}|^2\,dm = \iiint\limits_D |(x-h)\,\mathbf{i}+y\mathbf{j}|^2\,dm = \iiint\limits_D (x^2-2xh+h^2+y^2)\,dm$
$= \iiint\limits_D (x^2+y^2)\,dm - 2h\iiint\limits_D x\,dm + h^2\iiint\limits_D dm = I_x - 0 + h^2m = I_{c.m.} + h^2m$

36. $I_L = I_{c.m.} + mh^2 = \frac{2}{5}ma^2 + ma^2 = \frac{7}{5}ma^2$

37. (a) $(\overline{x},\overline{y},\overline{z}) = \left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right) \Rightarrow I_z = I_{c.m.} + abc\left(\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}\right)^2 \Rightarrow I_{c.m.} = I_z - \frac{abc(a^2+b^2)}{4}$
$= \frac{abc(a^2+b^2)}{3} - \frac{abc(a^2+b^2)}{4} = \frac{abc(a^2+b^2)}{12}$; $R_{c.m.} = \sqrt{\frac{I_{c.m.}}{M}} = \sqrt{\frac{a^2+b^2}{12}}$

(b) $I_L = I_{c.m.} + abc\left(\sqrt{\frac{a^2}{4}+\left(\frac{b}{2}-2b\right)^2}\right)^2 = \frac{abc(a^2+b^2)}{12} + \frac{abc(a^2+9b^2)}{4} = \frac{abc(4a^2+28b^2)}{12}$
$= \frac{abc(a^2+7b^2)}{3}$; $R_L = \sqrt{\frac{I_L}{M}} = \sqrt{\frac{a^2+7b^2}{3}}$

38. $M = \int_{-3}^3\int_{-2}^4\int_{-4/3}^{(4-2y)/3} dz\,dy\,dx = \int_{-3}^3\int_{-2}^4 \frac{2}{3}(4-y)\,dy\,dx = \int_{-3}^3 \frac{2}{3}\left[4y-\frac{y^2}{2}\right]_{-2}^4 dx = 12\int_{-3}^3 dx = 72;$

$\overline{x} = \overline{y} = \overline{z} = 0$ from Exercise 22 $\Rightarrow I_x = I_{c.m.} + 72\left(\sqrt{0^2+0^2}\right)^2 = I_{c.m.} \Rightarrow I_L = I_{c.m.} + 72\left(\sqrt{16+\frac{16}{9}}\right)^2$
$= 208 + 72\left(\frac{160}{9}\right) = 1488$

15.7 TRIPLE INTEGRALS IN CYLINDRICAL AND SPHERICAL COORDINATES

1. $\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 \left[r\left(2-r^2\right)^{1/2} - r^2\right] dr\, d\theta = \int_0^{2\pi}\left[-\frac{1}{3}\left(2-r^2\right)^{3/2} - \frac{r^3}{3}\right]_0^1 d\theta$
$= \int_0^{2\pi}\left(\frac{2^{3/2}}{3} - \frac{2}{3}\right) d\theta = \frac{4\pi\left(\sqrt{2}-1\right)}{3}$

2. $\int_0^{2\pi}\int_0^3\int_{r^2/3}^{\sqrt{18-r^2}} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^3 \left[r\left(18-r^2\right)^{1/2} - \frac{r^3}{3}\right] dr\, d\theta = \int_0^{2\pi}\left[-\frac{1}{3}\left(18-r^2\right)^{3/2} - \frac{r^4}{12}\right]_0^3 d\theta$
$= \frac{9\pi\left(8\sqrt{2}-7\right)}{2}$

3. $\int_0^{2\pi}\int_0^{\theta/2\pi}\int_0^{3+24r^2} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^{\theta/2\pi}\left(3r+24r^3\right) dr\, d\theta = \int_0^{2\pi}\left[\frac{3}{2}r^2 + 6r^4\right]_0^{\theta/2\pi} d\theta = \frac{3}{2}\int_0^{2\pi}\left(\frac{\theta^2}{4\pi^2} + \frac{4\theta^4}{16\pi^4}\right) d\theta$
$= \frac{3}{2}\left[\frac{\theta^3}{12\pi^2} + \frac{\theta^5}{20\pi^4}\right]_0^{2\pi} = \frac{17\pi}{5}$

4. $\int_0^{\pi}\int_0^{\theta/\pi}\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} z\, dz\, r\, dr\, d\theta = \int_0^{\pi}\int_0^{\theta/\pi}\frac{1}{2}\left[9\left(4-r^2\right)-\left(4-r^2\right)\right] r\, dr\, d\theta = 4\int_0^{\pi}\int_0^{\theta/\pi}\left(4r-r^3\right) dr\, d\theta$
$= 4\int_0^{\pi}\left[2r^2 - \frac{r^4}{4}\right]_0^{\theta/\pi} = 4\int_0^{\pi}\left(\frac{2\theta^2}{\pi^2} - \frac{\theta^4}{4\pi^4}\right) d\theta = \frac{37\pi}{15}$

5. $\int_0^{2\pi}\int_0^1\int_r^{\left(2-r^2\right)^{-1/2}} 3\, dz\, r\, dr\, d\theta = 3\int_0^{2\pi}\int_0^1\left[r\left(2-r^2\right)^{-1/2} - r^2\right] dr\, d\theta = 3\int_0^{2\pi}\left[-\left(2-r^2\right)^{1/2} - \frac{r^3}{3}\right]_0^1 d\theta$
$= 3\int_0^{2\pi}\left(\sqrt{2} - \frac{4}{3}\right) d\theta = \pi\left(6\sqrt{2}-8\right)$

6. $\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2}\left(r^2\sin^2\theta + z^2\right) dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1\left(r^3\sin^2\theta + \frac{r}{12}\right) dr\, d\theta = \int_0^{2\pi}\left(\frac{\sin^2\theta}{4} + \frac{1}{24}\right) d\theta = \frac{\pi}{3}$

7. $\int_0^{2\pi}\int_0^3\int_0^{z/3} r^3\, dr\, dz\, d\theta = \int_0^{2\pi}\int_0^3 \frac{z^4}{324}\, dz\, d\theta = \int_0^{2\pi}\frac{3}{20}\, d\theta = \frac{3\pi}{10}$

8. $\int_{-1}^1\int_0^{2\pi}\int_0^{1+\cos\theta} 4r\, dr\, d\theta\, dz = \int_{-1}^1\int_0^{2\pi} 2(1+\cos\theta)^2\, d\theta\, dz = \int_{-1}^1 6\pi\, d\theta = 12\pi$

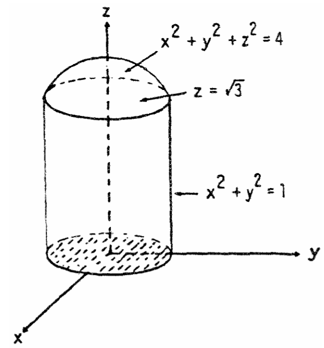
9. $\int_0^1\int_0^{\sqrt{z}}\int_0^{2\pi}\left(r^2\cos^2\theta + z^2\right) r\, d\theta\, dr\, dz = \int_0^1\int_0^{\sqrt{z}}\left[\frac{r^2\theta}{2} + \frac{r^2\sin 2\theta}{4} + z^2\theta\right]_0^{2\pi} r\, dr\, dz = \int_0^1\int_0^{\sqrt{z}}\left(\pi r^3 + 2\pi r z^2\right) dr\, dz$
$= \int_0^1\left[\frac{\pi r^4}{4} + \pi r^2 z^2\right]_0^{\sqrt{z}} dz = \int_0^1\left(\frac{\pi z^2}{4} + \pi z^3\right) dz = \left[\frac{\pi z^3}{12} + \frac{\pi z^4}{4}\right]_0^1 = \frac{\pi}{3}$

10. $\int_0^2\int_{r-2}^{\sqrt{4-r^2}}\int_0^{2\pi}(r\sin\theta + 1)\, r\, d\theta\, dz\, dr = \int_0^2\int_{r-2}^{\sqrt{4-r^2}} 2\pi r\, dz\, dr = 2\pi\int_0^2\left[r\left(4-r^2\right)^{1/2} - r^2 + 2r\right] dr$
$= 2\pi\left[-\frac{1}{3}\left(4-r^2\right)^{3/2} - \frac{r^3}{3} + r^2\right]_0^2 = 2\pi\left[-\frac{8}{3} + 4 + \frac{1}{3}(4)^{3/2}\right] = 8\pi$

11. (a) $\int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} dz\, r\, dr\, d\theta$

(b) $\int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^1 r\, dr\, dz\, d\theta + \int_0^{2\pi}\int_{\sqrt{3}}^2\int_0^{\sqrt{4-z^2}} r\, dr\, dz\, d\theta$

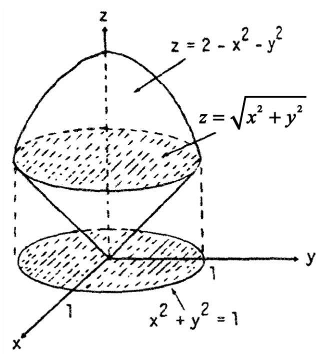
(c) $\int_0^1\int_0^{\sqrt{4-r^2}}\int_0^{2\pi} r\, d\theta\, dz\, dr$



12. (a) $\int_0^{2\pi}\int_0^1\int_r^{2-r^2} dz\, r\, dr\, d\theta$

(b) $\int_0^{2\pi}\int_0^1\int_0^z r\, dr\, dz\, d\theta + \int_0^{2\pi}\int_1^2\int_0^{\sqrt{2-z}} r\, dr\, dz\, d\theta$

(c) $\int_0^1\int_r^{2-r^2}\int_0^{2\pi} r\, d\theta\, dz\, dr$



13. $\int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta}\int_0^{3r^2} f(r, \theta, z)\, dz\, r\, dr\, d\theta$

14. $\int_{-\pi/2}^{\pi/2}\int_0^1\int_0^{r\cos\theta} r^3\, dz\, dr\, d\theta = \int_{-\pi/2}^{\pi/2}\int_0^1 r^4 \cos\theta\, dr\, d\theta = \frac{1}{5}\int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta = \frac{2}{5}$

15. $\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{4-r\sin\theta} f(r, \theta, z)\, dz\, r\, dr\, d\theta$

16. $\int_{-\pi/2}^{\pi/2}\int_0^{3\cos\theta}\int_0^{5-r\cos\theta} f(r, \theta, z)\, dz\, r\, dr\, d\theta$

17. $\int_{-\pi/2}^{\pi/2}\int_1^{1+\cos\theta}\int_0^4 f(r, \theta, z)\, dz\, r\, dr\, d\theta$

18. $\int_{-\pi/2}^{\pi/2}\int_{\cos\theta}^{2\cos\theta}\int_0^{3-r\sin\theta} f(r, \theta, z)\, dz\, r\, dr\, d\theta$

19. $\int_0^{\pi/4}\int_0^{\sec\theta}\int_0^{2-r\sin\theta} f(r, \theta, z)\, dz\, r\, dr\, d\theta$

20. $\int_{\pi/4}^{\pi/2}\int_0^{\csc\theta}\int_0^{2-r\sin\theta} f(r, \theta, z)\, dz\, r\, dr\, d\theta$

21. $\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{8}{3}\int_0^{\pi}\int_0^{\pi} \sin^4\phi\, d\phi\, d\theta = \frac{8}{3}\int_0^{\pi}\left(\left[-\frac{\sin^3\phi\cos\phi}{4}\right]_0^{\pi} + \frac{3}{4}\int_0^{\pi}\sin^2\phi\, d\phi\right) d\theta$
$= 2\int_0^{\pi}\int_0^{\pi} \sin^2\phi\, d\phi\, d\theta = \int_0^{\pi}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} d\theta = \int_0^{\pi} \pi\, d\theta = \pi^2$

22. $\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 (\rho\cos\phi)\,\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/4} 4\cos\phi\sin\phi\, d\phi\, d\theta = \int_0^{2\pi}\left[2\sin^2\phi\right]_0^{\pi/4} d\theta = \int_0^{2\pi} d\theta = 2\pi$

23. $\int_0^{2\pi}\int_0^{\pi}\int_0^{(1-\cos\phi)/2} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{1}{24}\int_0^{2\pi}\int_0^{\pi}(1-\cos\phi)^3 \sin\phi\, d\phi\, d\theta = \frac{1}{96}\int_0^{2\pi}\left[(1-\cos\phi)^4\right]_0^{\pi} d\theta$
$= \frac{1}{96}\int_0^{2\pi}(2^4 - 0)\, d\theta = \frac{16}{96}\int_0^{2\pi} d\theta = \frac{1}{6}(2\pi) = \frac{\pi}{3}$

24. $\int_0^{3\pi/2}\int_0^{\pi}\int_0^1 5\rho^3 \sin^3\phi\, d\rho\, d\phi\, d\theta = \frac{5}{4}\int_0^{3\pi/2}\int_0^{\pi} \sin^3\phi\, d\phi\, d\theta = \frac{5}{4}\int_0^{3\pi/2}\left(\left[-\frac{\sin^2\phi\cos\phi}{3}\right]_0^{\pi} + \frac{2}{3}\int_0^{\pi}\sin\phi\, d\phi\right) d\theta$
$= \frac{5}{6}\int_0^{3\pi/2}\left[-\cos\phi\right]_0^{\pi} d\theta = \frac{5}{3}\int_0^{3\pi/2} d\theta = \frac{5\pi}{2}$

25. $\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^2 3\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/3}(8-\sec^3\phi)\sin\phi\, d\phi\, d\theta = \int_0^{2\pi}\left[-8\cos\phi - \frac12\sec^2\phi\right]_0^{\pi/3} d\theta$
$= \int_0^{2\pi}\left[(-4-2)-\left(-8-\frac12\right)\right]d\theta = \frac52\int_0^{2\pi}d\theta = 5\pi$

26. $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi}\rho^3\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta = \frac14\int_0^{2\pi}\int_0^{\pi/4}\tan\phi\sec^2\phi\,d\phi\,d\theta = \frac14\int_0^{2\pi}\left[\frac12\tan^2\phi\right]_0^{\pi/4}d\theta = \frac18\int_0^{2\pi}d\theta = \frac{\pi}{4}$

27. $\int_0^2\int_{-\pi}^0\int_{\pi/4}^{\pi/2}\rho^3\sin 2\phi\,d\phi\,d\theta\,d\rho = \int_0^2\int_{-\pi}^0\rho^3\left[-\frac{\cos 2\phi}{2}\right]_{\pi/4}^{\pi/2}d\theta\,d\rho = \int_0^2\int_{-\pi}^0\frac{\rho^3}{2}\,d\theta\,d\rho = \int_0^2\frac{\rho^3\pi}{2}\,d\rho = \left[\frac{\pi\rho^4}{8}\right]_0^2 = 2\pi$

28. $\int_{\pi/6}^{\pi/3}\int_{\csc\phi}^{2\csc\phi}\int_0^{2\pi}\rho^2\sin\phi\,d\theta\,d\rho\,d\phi = 2\pi\int_{\pi/6}^{\pi/3}\int_{\csc\phi}^{2\csc\phi}\rho^2\sin\phi\,d\rho\,d\phi = \frac{2\pi}{3}\int_{\pi/6}^{\pi/3}\left[\rho^3\sin\phi\right]_{\csc\phi}^{2\csc\phi}d\phi = \frac{14\pi}{3}\int_{\pi/6}^{\pi/3}\csc^2\phi\,d\phi = \frac{28\pi}{3\sqrt3}$

29. $\int_0^1\int_0^\pi\int_0^{\pi/4}12\rho\sin^3\phi\,d\phi\,d\theta\,d\rho = \int_0^1\int_0^\pi\left(12\rho\left[\frac{-\sin^2\phi\cos\phi}{3}\right]_0^{\pi/4} + 8\rho\int_0^{\pi/4}\sin\phi\,d\phi\right)d\theta\,d\rho$
$= \int_0^1\int_0^\pi\left(-\frac{2\rho}{\sqrt2} - 8\rho\left[\cos\phi\right]_0^{\pi/4}\right)d\theta\,d\rho = \int_0^1\int_0^\pi\left(8\rho-\frac{10\rho}{\sqrt2}\right)d\theta\,d\rho = \pi\int_0^1\left(8\rho-\frac{10\rho}{\sqrt2}\right)d\rho = \pi\left[4\rho^2-\frac{5\rho^2}{\sqrt2}\right]_0^1$
$= \frac{\left(4\sqrt2-5\right)\pi}{\sqrt2}$

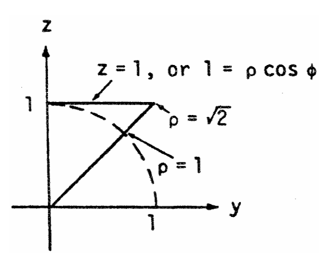
30. $\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\csc\phi}^2 5\rho^4\sin^3\phi\,d\rho\,d\theta\,d\phi = \int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}(32-\csc^5\phi)\sin^3\phi\,d\theta\,d\phi = \int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}(32\sin^3\phi-\csc^2\phi)\,d\theta\,d\phi$
$= \pi\int_{\pi/6}^{\pi/2}(32\sin^3\phi-\csc^2\phi)\,d\phi = \pi\left[-\frac{32\sin^2\phi\cos\phi}{3}\right]_{\pi/6}^{\pi/2} + \frac{64\pi}{3}\int_{\pi/6}^{\pi/2}\sin\phi\,d\phi + \pi\left[\cot\phi\right]_{\pi/6}^{\pi/2}$
$= \pi\left(\frac{32\sqrt3}{24}\right) - \frac{64\pi}{3}\left[\cos\phi\right]_{\pi/6}^{\pi/2} - \pi\left(\sqrt3\right) = \frac{\sqrt3}{3}\pi + \left(\frac{64\pi}{3}\right)\left(\frac{\sqrt3}{2}\right) = \frac{33\pi\sqrt3}{3} = 11\pi\sqrt3$

31. (a) $x^2+y^2=1 \Rightarrow \rho^2\sin^2\phi = 1$, and $\rho\sin\phi=1 \Rightarrow \rho=\csc\phi$; thus
$\int_0^{2\pi}\int_0^{\pi/6}\int_0^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta + \int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_0^{\csc\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

(b) $\int_0^{2\pi}\int_1^2\int_{\pi/6}^{\sin^{-1}(1/\rho)}\rho^2\sin\phi\,d\phi\,d\rho\,d\theta + \int_0^{2\pi}\int_0^2\int_0^{\pi/6}\rho^2\sin\phi\,d\phi\,d\rho\,d\theta$

32. (a) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

(b) $\int_0^{2\pi}\int_0^1\int_0^{\pi/4}\rho^2\sin\phi\,d\phi\,d\rho\,d\theta$
$+\int_0^{2\pi}\int_1^{\sqrt2}\int_{\cos^{-1}(1/\rho)}^{\pi/4}\rho^2\sin\phi\,d\phi\,d\rho\,d\theta$



33. $V = \int_0^{2\pi}\int_0^{\pi/2}\int_{\cos\phi}^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac13\int_0^{2\pi}\int_0^{\pi/2}(8-\cos^3\phi)\sin\phi\,d\phi\,d\theta$
$= \frac13\int_0^{2\pi}\left[-8\cos\phi+\frac{\cos^4\phi}{4}\right]_0^{\pi/2}d\theta = \frac13\int_0^{2\pi}\left(8-\frac14\right)d\theta = \left(\frac{31}{12}\right)(2\pi) = \frac{31\pi}{6}$

34. $V = \int_0^{2\pi}\int_0^{\pi/2}\int_1^{1+\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac13\int_0^{2\pi}\int_0^{\pi/2}(3\cos\phi+3\cos^2\phi+\cos^3\phi)\sin\phi\,d\phi\,d\theta$
$= \frac13\int_0^{2\pi}\left[-\frac32\cos^2\phi-\cos^3\phi-\frac14\cos^4\phi\right]_0^{\pi/2}d\theta = \frac13\int_0^{2\pi}\left(\frac32+1+\frac14\right)d\theta = \frac{11}{12}\int_0^{2\pi}d\theta = \left(\frac{11}{12}\right)(2\pi) = \frac{11\pi}{6}$

35. $V = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac13\int_0^{2\pi}\int_0^{\pi}(1-\cos\phi)^3\sin\phi\,d\phi\,d\theta = \frac13\int_0^{2\pi}\left[\frac{(1-\cos\phi)^4}{4}\right]_0^{\pi}d\theta$
$= \frac{1}{12}(2)^4\int_0^{2\pi}d\theta = \frac43(2\pi) = \frac{8\pi}{3}$

36. $V = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{1-\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} (1-\cos\phi)^3 \sin\phi \, d\phi \, d\theta = \frac{1}{3}\int_0^{2\pi}\left[\frac{(1-\cos\phi)^4}{4}\right]_0^{\pi/2} d\theta$
$= \frac{1}{12}\int_0^{2\pi} d\theta = \frac{1}{12}(2\pi) = \frac{\pi}{6}$

37. $V = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{8}{3}\int_0^{2\pi}\int_{\pi/4}^{\pi/2} \cos^3\phi \sin\phi \, d\phi \, d\theta = \frac{8}{3}\int_0^{2\pi}\left[-\frac{\cos^4\phi}{4}\right]_{\pi/4}^{\pi/2} d\theta$
$= \left(\frac{8}{3}\right)\left(\frac{1}{16}\right)\int_0^{2\pi} d\theta = \frac{1}{6}(2\pi) = \frac{\pi}{3}$

38. $V = \int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{8}{3}\int_0^{2\pi}\int_{\pi/3}^{\pi/2} \sin\phi \, d\phi \, d\theta = \frac{8}{3}\int_0^{2\pi}[-\cos\phi]_{\pi/3}^{\pi/2} d\theta = \frac{4}{3}\int_0^{2\pi} d\theta = \frac{8\pi}{3}$

39. (a) $8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ (b) $8\int_0^{\pi/2}\int_0^{2}\int_0^{\sqrt{4-r^2}} dz \, r \, dr \, d\theta$
(c) $8\int_0^{2}\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} dz \, dy \, dx$

40. (a) $\int_0^{\pi/2}\int_0^{3/\sqrt{2}}\int_r^{\sqrt{9-r^2}} dz \, r \, dr \, d\theta$ (b) $\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{3} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$
(c) $\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{3} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 9\int_0^{\pi/2}\int_0^{\pi/4} \sin\phi \, d\phi \, d\theta = -9\int_0^{\pi/2}\left(\frac{1}{\sqrt{2}} - 1\right) d\theta = \frac{9\pi\left(2-\sqrt{2}\right)}{4}$

41. (a) $V = \int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ (b) $V = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} dz \, r \, dr \, d\theta$
(c) $V = \int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}}\int_1^{\sqrt{4-x^2-y^2}} dz \, dy \, dx$
(d) $V = \int_0^{2\pi}\int_0^{\sqrt{3}}\left[r\left(4-r^2\right)^{1/2} - r\right] dr \, d\theta = \int_0^{2\pi}\left[-\frac{\left(4-r^2\right)^{3/2}}{3} - \frac{r^2}{2}\right]_0^{\sqrt{3}} d\theta = \int_0^{2\pi}\left(-\frac{1}{3} - \frac{3}{2} + \frac{4^{3/2}}{3}\right) d\theta$
$= \frac{5}{6}\int_0^{2\pi} d\theta = \frac{5\pi}{3}$

42. (a) $I_z = \int_0^{2\pi}\int_0^{1}\int_0^{\sqrt{1-r^2}} r^2 \, dz \, r \, dr \, d\theta$
(b) $I_z = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{1} \left(\rho^2 \sin^2\phi\right)\left(\rho^2 \sin\phi\right) d\rho \, d\phi \, d\theta$, since $r^2 = x^2 + y^2 = \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta = \rho^2 \sin^2\phi$
(c) $I_z = \int_0^{2\pi}\int_0^{\pi/2} \frac{1}{5}\sin^3\phi \, d\phi \, d\theta = \frac{1}{5}\int_0^{2\pi}\left(\left[-\frac{\sin^2\phi\cos\phi}{3}\right]_0^{\pi/2} + \frac{2}{3}\int_0^{\pi/2}\sin\phi \, d\phi\right) d\theta = \frac{2}{15}\int_0^{2\pi}[-\cos\phi]_0^{\pi/2} d\theta$
$= \frac{2}{15}(2\pi) = \frac{4\pi}{15}$

43. $V = 4\int_0^{\pi/2}\int_0^{1}\int_{r^4-1}^{4-4r^2} dz \, r \, dr \, d\theta = 4\int_0^{\pi/2}\int_0^{1}\left(5r - 4r^3 - r^5\right) dr \, d\theta = 4\int_0^{\pi/2}\left(\frac{5}{2} - 1 - \frac{1}{6}\right) d\theta = 4\int_0^{\pi/2} d\theta = \frac{8\pi}{3}$

44. $V = 4\int_0^{\pi/2}\int_0^{1}\int_{-\sqrt{1-r^2}}^{1-r} dz \, r \, dr \, d\theta = 4\int_0^{\pi/2}\int_0^{1}\left(r - r^2 + r\sqrt{1-r^2}\right) dr \, d\theta = 4\int_0^{\pi/2}\left[\frac{r^2}{2} - \frac{r^3}{3} - \frac{1}{3}\left(1-r^2\right)^{3/2}\right]_0^1 d\theta$
$= 4\int_0^{\pi/2}\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{3}\right) d\theta = 2\int_0^{\pi/2} d\theta = 2\left(\frac{\pi}{2}\right) = \pi$

45. $V = \int_{3\pi/2}^{2\pi}\int_0^{3\cos\theta}\int_0^{-r\sin\theta} dz \, r \, dr \, d\theta = \int_{3\pi/2}^{2\pi}\int_0^{3\cos\theta} -r^2 \sin\theta \, dr \, d\theta = \int_{3\pi/2}^{2\pi}\left(-9\cos^3\theta\right)(\sin\theta) \, d\theta = \left[\frac{9}{4}\cos^4\theta\right]_{3\pi/2}^{2\pi}$
$= \frac{9}{4} - 0 = \frac{9}{4}$

46. $V = 2\int_{\pi/2}^{\pi}\int_0^{-3\cos\theta}\int_0^{r} dz \, r \, dr \, d\theta = 2\int_{\pi/2}^{\pi}\int_0^{-3\cos\theta} r^2 \, dr \, d\theta = \frac{2}{3}\int_{\pi/2}^{\pi} -27\cos^3\theta \, d\theta$
$= -18\left(\left[\frac{\cos^2\theta\sin\theta}{3}\right]_{\pi/2}^{\pi} + \frac{2}{3}\int_{\pi/2}^{\pi}\cos\theta \, d\theta\right) = -12\,[\sin\theta]_{\pi/2}^{\pi} = 12$

47. $V = \int_0^{\pi/2} \int_0^{\sin\theta} \int_0^{\sqrt{1-r^2}} dz\, r\, dr\, d\theta = \int_0^{\pi/2} \int_0^{\sin\theta} r\sqrt{1-r^2}\, dr\, d\theta = \int_0^{\pi/2} \left[-\frac{1}{3}\left(1-r^2\right)^{3/2} \right]_0^{\sin\theta} d\theta$

$= -\frac{1}{3} \int_0^{\pi/2} \left[\left(1-\sin^2\theta\right)^{3/2} - 1 \right] d\theta = -\frac{1}{3} \int_0^{\pi/2} (\cos^3\theta - 1)\, d\theta = -\frac{1}{3} \left(\left[\frac{\cos^2\theta \sin\theta}{3} \right]_0^{\pi/2} + \frac{2}{3} \int_0^{\pi/2} \cos\theta\, d\theta \right) + \left[\frac{\theta}{3} \right]_0^{\pi/2}$

$= -\frac{2}{9} [\sin\theta]_0^{\pi/2} + \frac{\pi}{6} = \frac{-4+3\pi}{18}$

48. $V = \int_0^{\pi/2} \int_0^{\cos\theta} \int_0^{3\sqrt{1-r^2}} dz\, r\, dr\, d\theta = \int_0^{\pi/2} \int_0^{\cos\theta} 3r\sqrt{1-r^2}\, dr\, d\theta = \int_0^{\pi/2} \left[-\left(1-r^2\right)^{3/2} \right]_0^{\cos\theta} d\theta$

$= \int_0^{\pi/2} \left[-\left(1-\cos^2\theta\right)^{3/2} + 1 \right] d\theta = \int_0^{\pi/2} (1 - \sin^3\theta)\, d\theta = \left[\theta + \frac{\sin^2\theta\cos\theta}{3} \right]_0^{\pi/2} - \frac{2}{3} \int_0^{\pi/2} \sin\theta\, d\theta$

$= \frac{\pi}{2} + \frac{2}{3} [\cos\theta]_0^{\pi/2} = \frac{\pi}{2} - \frac{2}{3} = \frac{3\pi-4}{6}$

49. $V = \int_0^{2\pi} \int_{\pi/3}^{2\pi/3} \int_0^{a} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} \int_{\pi/3}^{2\pi/3} \frac{a^3}{3} \sin\phi\, d\phi\, d\theta = \frac{a^3}{3} \int_0^{2\pi} [-\cos\phi]_{\pi/3}^{2\pi/3} d\theta = \frac{a^3}{3} \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2} \right) d\theta = \frac{2\pi a^3}{3}$

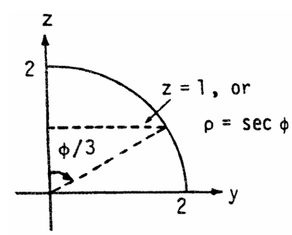
50. $V = \int_0^{\pi/6} \int_0^{\pi/2} \int_0^{a} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{a^3}{3} \int_0^{\pi/6} \int_0^{\pi/2} \sin\phi\, d\phi\, d\theta = \frac{a^3}{3} \int_0^{\pi/6} d\theta = \frac{a^3\pi}{18}$

51. $V = \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec\phi}^{2} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

$= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/3} (8\sin\phi - \tan\phi\sec^2\phi)\, d\phi\, d\theta$

$= \frac{1}{3} \int_0^{2\pi} \left[-8\cos\phi - \frac{1}{2}\tan^2\phi \right]_0^{\pi/3} d\theta$

$= \frac{1}{3} \int_0^{2\pi} \left[-4 - \frac{1}{2}(3) + 8 \right] d\theta = \frac{1}{3} \int_0^{2\pi} \frac{5}{2}\, d\theta = \frac{5}{6}(2\pi) = \frac{5\pi}{3}$



52. $V = 4 \int_0^{\pi/2} \int_0^{\pi/4} \int_{\sec\phi}^{2\sec\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{4}{3} \int_0^{\pi/2} \int_0^{\pi/4} (8\sec^3\phi - \sec^3\phi) \sin\phi\, d\phi\, d\theta$

$= \frac{28}{3} \int_0^{\pi/2} \int_0^{\pi/4} \sec^3\phi \sin\phi\, d\phi\, d\theta = \frac{28}{3} \int_0^{\pi/2} \int_0^{\pi/4} \tan\phi \sec^2\phi\, d\phi\, d\theta = \frac{28}{3} \int_0^{\pi/2} \left[\frac{1}{2}\tan^2\phi \right]_0^{\pi/4} d\theta = \frac{14}{3} \int_0^{\pi/2} d\theta = \frac{7\pi}{3}$

53. $V = 4 \int_0^{\pi/2} \int_0^{1} \int_0^{r^2} dz\, r\, dr\, d\theta = 4 \int_0^{\pi/2} \int_0^{1} r^3\, dr\, d\theta = \int_0^{\pi/2} d\theta = \frac{\pi}{2}$

54. $V = 4 \int_0^{\pi/2} \int_0^{1} \int_{r^2}^{r^2+1} dz\, r\, dr\, d\theta = 4 \int_0^{\pi/2} \int_0^{1} r\, dr\, d\theta = 2 \int_0^{\pi/2} d\theta = \pi$

55. $V = 8 \int_0^{\pi/2} \int_1^{\sqrt{2}} \int_0^{r} dz\, r\, dr\, d\theta = 8 \int_0^{\pi/2} \int_1^{\sqrt{2}} r^2\, dr\, d\theta = 8 \left(\frac{2\sqrt{2}-1}{3} \right) \int_0^{\pi/2} d\theta = \frac{4\pi\left(2\sqrt{2}-1\right)}{3}$

56. $V = 8 \int_0^{\pi/2} \int_1^{\sqrt{2}} \int_0^{\sqrt{2-r^2}} dz\, r\, dr\, d\theta = 8 \int_0^{\pi/2} \int_1^{\sqrt{2}} r\sqrt{2-r^2}\, dr\, d\theta = 8 \int_0^{\pi/2} \left[-\frac{1}{3}\left(2-r^2\right)^{3/2} \right]_1^{\sqrt{2}} d\theta = \frac{8}{3} \int_0^{\pi/2} d\theta = \frac{4\pi}{3}$

57. $V = \int_0^{2\pi} \int_0^{2} \int_0^{4-r\sin\theta} dz\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^{2} (4r - r^2\sin\theta)\, dr\, d\theta = 8 \int_0^{2\pi} \left(1 - \frac{\sin\theta}{3}\right) d\theta = 16\pi$

58. $V = \int_0^{2\pi} \int_0^{2} \int_0^{4-r\cos\theta-r\sin\theta} dz\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^{2} [4r - r^2(\cos\theta + \sin\theta)]\, dr\, d\theta = \frac{8}{3} \int_0^{2\pi} (3 - \cos\theta - \sin\theta)\, d\theta = 16\pi$

59. The paraboloids intersect when $4x^2 + 4y^2 = 5 - x^2 - y^2 \Rightarrow x^2 + y^2 = 1$ and $z = 4$

$\Rightarrow V = 4 \int_0^{\pi/2} \int_0^{1} \int_{4r^2}^{5-r^2} dz\, r\, dr\, d\theta = 4 \int_0^{\pi/2} \int_0^{1} (5r - 5r^3)\, dr\, d\theta = 20 \int_0^{\pi/2} \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^{1} d\theta = 5 \int_0^{\pi/2} d\theta = \frac{5\pi}{2}$

60. The paraboloid intersects the xy-plane when $9 - x^2 - y^2 = 0 \Rightarrow x^2 + y^2 = 9 \Rightarrow V = 4 \int_0^{\pi/2} \int_1^3 \int_0^{9-r^2} dz\, r\, dr\, d\theta$
$= 4 \int_0^{\pi/2} \int_1^3 (9r - r^3)\, dr\, d\theta = 4 \int_0^{\pi/2} \left[\frac{9r^2}{2} - \frac{r^4}{4}\right]_1^3 d\theta = 4 \int_0^{\pi/2} \left(\frac{81}{4} - \frac{17}{4}\right) d\theta = 64 \int_0^{\pi/2} d\theta = 32\pi$

61. $V = 8 \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} dz\, r\, dr\, d\theta = 8 \int_0^{2\pi} \int_0^1 r\,(4 - r^2)^{1/2}\, dr\, d\theta = 8 \int_0^{2\pi} \left[-\frac{1}{3}(4 - r^2)^{3/2}\right]_0^1 d\theta$
$= -\frac{8}{3} \int_0^{2\pi} \left(3^{3/2} - 8\right) d\theta = \frac{4\pi\left(8 - 3\sqrt{3}\right)}{3}$

62. The sphere and paraboloid intersect when $x^2 + y^2 + z^2 = 2$ and $z = x^2 + y^2 \Rightarrow z^2 + z - 2 = 0$
$\Rightarrow (z + 2)(z - 1) = 0 \Rightarrow z = 1$ or $z = -2 \Rightarrow z = 1$ since $z \geq 0$. Thus, $x^2 + y^2 = 1$ and the volume is given by the triple integral $V = 4 \int_0^{\pi/2} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} dz\, r\, dr\, d\theta = 4 \int_0^{\pi/2} \int_0^1 \left[r\,(2 - r^2)^{1/2} - r^3\right] dr\, d\theta$
$= 4 \int_0^{\pi/2} \left[-\frac{1}{3}(2 - r^2)^{3/2} - \frac{r^4}{4}\right]_0^1 d\theta = 4 \int_0^{\pi/2} \left(\frac{2\sqrt{2}}{3} - \frac{7}{12}\right) d\theta = \frac{\pi\left(8\sqrt{2} - 7\right)}{6}$

63. average $= \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \int_{-1}^1 r^2\, dz\, dr\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 2r^2\, dr\, d\theta = \frac{1}{3\pi} \int_0^{2\pi} d\theta = \frac{2}{3}$

64. average $= \frac{1}{\left(\frac{4\pi}{3}\right)} \int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^2\, dz\, dr\, d\theta = \frac{3}{4\pi} \int_0^{2\pi} \int_0^1 2r^2 \sqrt{1 - r^2}\, dr\, d\theta$
$= \frac{3}{2\pi} \int_0^{2\pi} \left[\frac{1}{8} \sin^{-1} r - \frac{1}{8} r \sqrt{1 - r^2}\,(1 - 2r^2)\right]_0^1 d\theta = \frac{3}{16\pi} \int_0^{2\pi} \left(\frac{\pi}{2} + 0\right) d\theta = \frac{3}{32} \int_0^{2\pi} d\theta = \left(\frac{3}{32}\right)(2\pi) = \frac{3\pi}{16}$

65. average $= \frac{1}{\left(\frac{4\pi}{3}\right)} \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^3 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{3}{16\pi} \int_0^{2\pi} \int_0^{\pi} \sin\phi\, d\phi\, d\theta = \frac{3}{8\pi} \int_0^{2\pi} d\theta = \frac{3}{4}$

66. average $= \frac{1}{\left(\frac{2\pi}{3}\right)} \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^3 \cos\phi \sin\phi\, d\rho\, d\phi\, d\theta = \frac{3}{8\pi} \int_0^{2\pi} \int_0^{\pi/2} \cos\phi \sin\phi\, d\phi\, d\theta = \frac{3}{8\pi} \int_0^{2\pi} \left[\frac{\sin^2\phi}{2}\right]_0^{\pi/2} d\theta$
$= \frac{3}{16\pi} \int_0^{2\pi} d\theta = \left(\frac{3}{16\pi}\right)(2\pi) = \frac{3}{8}$

67. $M = 4 \int_0^{\pi/2} \int_0^1 \int_0^r dz\, r\, dr\, d\theta = 4 \int_0^{\pi/2} \int_0^1 r^2\, dr\, d\theta = \frac{4}{3} \int_0^{\pi/2} d\theta = \frac{2\pi}{3}$; $M_{xy} = \int_0^{2\pi} \int_0^1 \int_0^r z\, dz\, r\, dr\, d\theta$
$= \frac{1}{2} \int_0^{2\pi} \int_0^1 r^3\, dr\, d\theta = \frac{1}{8} \int_0^{2\pi} d\theta = \frac{\pi}{4} \Rightarrow \overline{z} = \frac{M_{xy}}{M} = \left(\frac{\pi}{4}\right)\left(\frac{3}{2\pi}\right) = \frac{3}{8}$, and $\overline{x} = \overline{y} = 0$, by symmetry

68. $M = \int_0^{\pi/2} \int_0^2 \int_0^r dz\, r\, dr\, d\theta = \int_0^{\pi/2} \int_0^2 r^2\, dr\, d\theta = \frac{8}{3} \int_0^{\pi/2} d\theta = \frac{4\pi}{3}$; $M_{yz} = \int_0^{\pi/2} \int_0^2 \int_0^r x\, dz\, r\, dr\, d\theta$
$= \int_0^{\pi/2} \int_0^2 r^3 \cos\theta\, dr\, d\theta = 4 \int_0^{\pi/2} \cos\theta\, d\theta = 4$; $M_{xz} = \int_0^{\pi/2} \int_0^2 \int_0^r y\, dz\, r\, dr\, d\theta = \int_0^{\pi/2} \int_0^2 r^3 \sin\theta\, dr\, d\theta$
$= 4 \int_0^{\pi/2} \sin\theta\, d\theta = 4$; $M_{xy} = \int_0^{\pi/2} \int_0^2 \int_0^r z\, dz\, r\, dr\, d\theta = \frac{1}{2} \int_0^{\pi/2} \int_0^2 r^3\, dr\, d\theta = 2 \int_0^{\pi/2} d\theta = \pi \Rightarrow \overline{x} = \frac{M_{yz}}{M} = \frac{3}{\pi}$,
$\overline{y} = \frac{M_{xz}}{M} = \frac{3}{\pi}$, and $\overline{z} = \frac{M_{xy}}{M} = \frac{3}{4}$

69. $M = \frac{8\pi}{3}$; $M_{xy} = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^2 z\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^2 \rho^3 \cos\phi \sin\phi\, d\rho\, d\phi\, d\theta = 4 \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \cos\phi \sin\phi\, d\phi\, d\theta$
$= 4 \int_0^{2\pi} \left[\frac{\sin^2\phi}{2}\right]_{\pi/3}^{\pi/2} d\theta = 4 \int_0^{2\pi} \left(\frac{1}{2} - \frac{3}{8}\right) d\theta = \frac{1}{2} \int_0^{2\pi} d\theta = \pi \Rightarrow \overline{z} = \frac{M_{xy}}{M} = (\pi)\left(\frac{3}{8\pi}\right) = \frac{3}{8}$, and $\overline{x} = \overline{y} = 0$, by symmetry

70. $M = \int_0^{2\pi} \int_0^{\pi/4} \int_0^a \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{a^3}{3} \int_0^{2\pi} \int_0^{\pi/4} \sin\phi\, d\phi\, d\theta = \frac{a^3}{3} \int_0^{2\pi} \frac{2 - \sqrt{2}}{2}\, d\theta = \frac{\pi a^3 \left(2 - \sqrt{2}\right)}{3}$;
$M_{xy} = \int_0^{2\pi} \int_0^{\pi/4} \int_0^a \rho^3 \sin\phi \cos\phi\, d\rho\, d\phi\, d\theta = \frac{a^4}{4} \int_0^{2\pi} \int_0^{\pi/4} \sin\phi \cos\phi\, d\phi\, d\theta = \frac{a^4}{16} \int_0^{2\pi} d\theta = \frac{\pi a^4}{8}$

$\Rightarrow\ \bar{z} = \frac{M_{xy}}{M} = \left(\frac{\pi a^4}{8}\right)\left[\frac{3}{\pi a^3\left(2-\sqrt{2}\right)}\right] = \left(\frac{3a}{8}\right)\left(\frac{2+\sqrt{2}}{2}\right) = \frac{3\left(2+\sqrt{2}\right)a}{16}$, and $\bar{x} = \bar{y} = 0$, by symmetry

71. $M = \int_0^{2\pi}\int_0^4\int_0^{\sqrt{r}} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^4 r^{3/2}\, dr\, d\theta = \frac{64}{5}\int_0^{2\pi} d\theta = \frac{128\pi}{5}$; $M_{xy} = \int_0^{2\pi}\int_0^4\int_0^{\sqrt{r}} z\, dz\, r\, dr\, d\theta$
$= \frac{1}{2}\int_0^{2\pi}\int_0^4 r^2\, dr\, d\theta = \frac{32}{3}\int_0^{2\pi} d\theta = \frac{64\pi}{3}\ \Rightarrow\ \bar{z} = \frac{M_{xy}}{M} = \frac{5}{6}$, and $\bar{x} = \bar{y} = 0$, by symmetry

72. $M = \int_{-\pi/3}^{\pi/3}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} dz\, r\, dr\, d\theta = \int_{-\pi/3}^{\pi/3}\int_0^1 2r\sqrt{1-r^2}\, dr\, d\theta = \int_{-\pi/3}^{\pi/3}\left[-\frac{2}{3}\left(1-r^2\right)^{3/2}\right]_0^1 d\theta$
$= \frac{2}{3}\int_{-\pi/3}^{\pi/3} d\theta = \left(\frac{2}{3}\right)\left(\frac{2\pi}{3}\right) = \frac{4\pi}{9}$; $M_{yz} = \int_{-\pi/3}^{\pi/3}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^2\cos\theta\, dz\, dr\, d\theta = 2\int_{-\pi/3}^{\pi/3}\int_0^1 r^2\sqrt{1-r^2}\cos\theta\, dr\, d\theta$
$= 2\int_{-\pi/3}^{\pi/3}\left[\frac{1}{8}\sin^{-1} r - \frac{1}{8} r\sqrt{1-r^2}\left(1-2r^2\right)\right]_0^1 \cos\theta\, d\theta = \frac{\pi}{8}\int_{-\pi/3}^{\pi/3}\cos\theta\, d\theta = \frac{\pi}{8}\left[\sin\theta\right]_{-\pi/3}^{\pi/3} = \left(\frac{\pi}{8}\right)\left(2\cdot\frac{\sqrt{3}}{2}\right) = \frac{\pi\sqrt{3}}{8}$
$\Rightarrow\ \bar{x} = \frac{M_{yz}}{M} = \frac{9\sqrt{3}}{32}$, and $\bar{y} = \bar{z} = 0$, by symmetry

73. We orient the cone with its vertex at the origin and axis along the z-axis $\Rightarrow\ \phi = \frac{\pi}{4}$. We use the the x-axis which is through the vertex and parallel to the base of the cone $\Rightarrow\ I_x = \int_0^{2\pi}\int_0^1\int_r^1 \left(r^2\sin^2\theta + z^2\right) dz\, r\, dr\, d\theta$
$= \int_0^{2\pi}\int_0^1 \left(r^3\sin^2\theta - r^4\sin^2\theta + \frac{r}{3} - \frac{r^4}{3}\right) dr\, d\theta = \int_0^{2\pi}\left(\frac{\sin^2\theta}{20} + \frac{1}{10}\right) d\theta = \left[\frac{\theta}{40} - \frac{\sin 2\theta}{80} + \frac{\theta}{10}\right]_0^{2\pi} = \frac{\pi}{20} + \frac{\pi}{5} = \frac{\pi}{4}$

74. $I_z = \int_0^{2\pi}\int_0^a\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r^3\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^a 2r^3\sqrt{a^2-r^2}\, dr\, d\theta = 2\int_0^{2\pi}\left[\left(-\frac{r^2}{5} - \frac{2a^2}{15}\right)\left(a^2-r^2\right)^{3/2}\right]_0^a d\theta = 2\int_0^{2\pi}\frac{2}{15}a^5\, d\theta$
$= \frac{8\pi a^5}{15}$

75. $I_z = \int_0^{2\pi}\int_0^a\int_{\left(\frac{h}{a}\right)r}^h \left(x^2+y^2\right) dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^a\int_{\frac{hr}{a}}^h r^3\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^a \left(hr^3 - \frac{hr^4}{a}\right) dr\, d\theta = \int_0^{2\pi} h\left[\frac{r^4}{4} - \frac{r^5}{5a}\right]_0^a d\theta$
$= \int_0^{2\pi} h\left(\frac{a^4}{4} - \frac{a^5}{5a}\right) d\theta = \frac{ha^4}{20}\int_0^{2\pi} d\theta = \frac{\pi ha^4}{10}$

76. (a) $M = \int_0^{2\pi}\int_0^1\int_0^{r^2} z\, dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{2}r^5\, dr\, d\theta = \frac{1}{12}\int_0^{2\pi} d\theta = \frac{\pi}{6}$; $M_{xy} = \int_0^{2\pi}\int_0^1\int_0^{r^2} z^2\, dz\, r\, dr\, d\theta$
$= \frac{1}{3}\int_0^{2\pi}\int_0^1 r^7\, dr\, d\theta = \frac{1}{24}\int_0^{2\pi} d\theta = \frac{\pi}{12}\ \Rightarrow\ \bar{z} = \frac{1}{2}$, and $\bar{x} = \bar{y} = 0$, by symmetry; $I_z = \int_0^{2\pi}\int_0^1\int_0^{r^2} zr^3\, dz\, dr\, d\theta$
$= \frac{1}{2}\int_0^{2\pi}\int_0^1 r^7\, dr\, d\theta = \frac{1}{16}\int_0^{2\pi} d\theta = \frac{\pi}{8}$

(b) $M = \int_0^{2\pi}\int_0^1\int_0^{r^2} r^2\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^1 r^4\, dr\, d\theta = \frac{1}{5}\int_0^{2\pi} d\theta = \frac{2\pi}{5}$; $M_{xy} = \int_0^{2\pi}\int_0^1\int_0^{r^2} zr^2\, dz\, dr\, d\theta$
$= \frac{1}{2}\int_0^{2\pi}\int_0^1 r^6\, dr\, d\theta = \frac{1}{14}\int_0^{2\pi} d\theta = \frac{\pi}{7}\ \Rightarrow\ \bar{z} = \frac{5}{14}$, and $\bar{x} = \bar{y} = 0$, by symmetry; $I_z = \int_0^{2\pi}\int_0^1\int_0^{r^2} r^4\, dz\, dr\, d\theta$
$= \int_0^{2\pi}\int_0^1 r^6\, dr\, d\theta = \frac{1}{7}\int_0^{2\pi} d\theta = \frac{2\pi}{7}$

77. (a) $M = \int_0^{2\pi}\int_0^1\int_r^1 z\, dz\, r\, dr\, d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^1 \left(r - r^3\right) dr\, d\theta = \frac{1}{8}\int_0^{2\pi} d\theta = \frac{\pi}{4}$; $M_{xy} = \int_0^{2\pi}\int_0^1\int_r^1 z^2\, dz\, r\, dr\, d\theta$
$= \frac{1}{3}\int_0^{2\pi}\int_0^1 \left(r - r^4\right) dr\, d\theta = \frac{1}{10}\int_0^{2\pi} d\theta = \frac{\pi}{5}\ \Rightarrow\ \bar{z} = \frac{4}{5}$, and $\bar{x} = \bar{y} = 0$, by symmetry; $I_z = \int_0^{2\pi}\int_0^1\int_r^1 zr^3\, dz\, dr\, d\theta$
$= \frac{1}{2}\int_0^{2\pi}\int_0^1 \left(r^3 - r^5\right) dr\, d\theta = \frac{1}{24}\int_0^{2\pi} d\theta = \frac{\pi}{12}$

(b) $M = \int_0^{2\pi}\int_0^1\int_r^1 z^2\, dz\, r\, dr\, d\theta = \frac{\pi}{5}$ from part (a); $M_{xy} = \int_0^{2\pi}\int_0^1\int_r^1 z^3\, dz\, r\, dr\, d\theta = \frac{1}{4}\int_0^{2\pi}\int_0^1 \left(r - r^5\right) dr\, d\theta$
$= \frac{1}{12}\int_0^{2\pi} d\theta = \frac{\pi}{6}\ \Rightarrow\ \bar{z} = \frac{5}{6}$, and $\bar{x} = \bar{y} = 0$, by symmetry; $I_z = \int_0^{2\pi}\int_0^1\int_r^1 z^2 r^3\, dz\, dr\, d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^1 \left(r^3 - r^6\right) dr\, d\theta$
$= \frac{1}{28}\int_0^{2\pi} d\theta = \frac{\pi}{14}$

78. (a) $M = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^4 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{a^5}{5}\int_0^{2\pi}\int_0^{\pi} \sin\phi \, d\phi \, d\theta = \frac{2a^5}{5}\int_0^{2\pi} d\theta = \frac{4\pi a^5}{5}$; $I_z = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^6 \sin^3\phi \, d\rho \, d\phi \, d\theta$
$= \frac{a^7}{7}\int_0^{2\pi}\int_0^{\pi} (1-\cos^2\phi)\sin\phi \, d\phi \, d\theta = \frac{a^7}{7}\int_0^{2\pi}\left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi} d\theta = \frac{4a^7}{21}\int_0^{2\pi} d\theta = \frac{8a^7\pi}{21}$

(b) $M = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^3 \sin^2\phi \, d\rho \, d\phi \, d\theta = \frac{a^4}{4}\int_0^{2\pi}\int_0^{\pi} \frac{(1-\cos 2\phi)}{2} \, d\phi \, d\theta = \frac{\pi a^4}{8}\int_0^{2\pi} d\theta = \frac{\pi^2 a^4}{4}$;
$I_z = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^5 \sin^4\phi \, d\rho \, d\phi \, d\theta = \frac{a^6}{6}\int_0^{2\pi}\int_0^{\pi} \sin^4\phi \, d\phi \, d\theta$
$= \frac{a^6}{6}\int_0^{2\pi}\left(\left[\frac{-\sin^3\phi\cos\phi}{4}\right]_0^{\pi} + \frac{3}{4}\int_0^{\pi}\sin^2\phi \, d\phi\right) d\theta = \frac{a^6}{8}\int_0^{2\pi}\left[\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right]_0^{\pi} d\theta = \frac{\pi a^6}{16}\int_0^{2\pi} d\theta = \frac{a^6\pi^2}{8}$

79. $M = \int_0^{2\pi}\int_0^{a}\int_0^{\frac{h}{a}\sqrt{a^2-r^2}} dz \, r \, dr \, d\theta = \int_0^{2\pi}\int_0^{a} \frac{h}{a} r\sqrt{a^2-r^2} \, dr \, d\theta = \frac{h}{a}\int_0^{2\pi}\left[-\frac{1}{3}(a^2-r^2)^{3/2}\right]_0^{a} d\theta$
$= \frac{h}{a}\int_0^{2\pi} \frac{a^3}{3} \, d\theta = \frac{2ha^2\pi}{3}$; $M_{xy} = \int_0^{2\pi}\int_0^{a}\int_0^{\frac{h}{a}\sqrt{a^2-r^2}} z \, dz \, r \, dr \, d\theta = \frac{h^2}{2a^2}\int_0^{2\pi}\int_0^{a} (a^2 r - r^3) \, dr \, d\theta$
$= \frac{h^2}{2a^2}\int_0^{2\pi}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) d\theta = \frac{a^2h^2\pi}{4} \Rightarrow \bar{z} = \left(\frac{\pi a^2 h^2}{4}\right)\left(\frac{3}{2ha^2\pi}\right) = \frac{3}{8}h$, and $\bar{x} = \bar{y} = 0$, by symmetry

80. Let the base radius of the cone be a and the height h, and place the cone's axis of symmetry along the z-axis with the vertex at the origin. Then $M = \frac{\pi a^2 h}{3}$ and $M_{xy} = \int_0^{2\pi}\int_0^{a}\int_{\left(\frac{h}{a}\right) r}^{h} z \, dz \, r \, dr \, d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^{a}\left(h^2 r - \frac{h^2}{a^2} r^3\right) dr \, d\theta$
$= \frac{h^2}{2}\int_0^{2\pi}\left[\frac{r^2}{2} - \frac{r^4}{4a^2}\right]_0^{a} d\theta = \frac{h^2}{2}\int_0^{2\pi}\left(\frac{a^2}{2} - \frac{a^2}{4}\right) d\theta = \frac{h^2 a^2}{8}\int_0^{2\pi} d\theta = \frac{h^2 a^2 \pi}{4} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{h^2 a^2 \pi}{4}\right)\left(\frac{3}{\pi a^2 h}\right) = \frac{3}{4}h$, and
$\bar{x} = \bar{y} = 0$, by symmetry $\Rightarrow$ the centroid is one fourth of the way from the base to the vertex

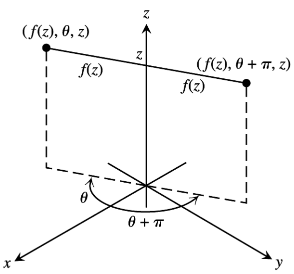
81. The density distribution function is linear so it has the form $\delta(\rho) = k\rho + C$, where ρ is the distance from the center of the planet. Now, $\delta(R) = 0 \Rightarrow kR + C = 0$, and $\delta(\rho) = k\rho - kR$. It remains to determine the constant k: $M = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} (k\rho - kR)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi}\left[k\frac{\rho^4}{4} - kR\frac{\rho^3}{3}\right]_0^{R} \sin\phi \, d\phi \, d\theta$
$= \int_0^{2\pi}\int_0^{\pi} k\left(\frac{R^4}{4} - \frac{R^4}{3}\right)\sin\phi \, d\phi \, d\theta = \int_0^{2\pi} -\frac{k}{12}R^4\left[-\cos\phi\right]_0^{\pi} d\theta = \int_0^{2\pi} -\frac{k}{6}R^4 \, d\theta = -\frac{k\pi R^4}{3} \Rightarrow k = -\frac{3M}{\pi R^4}$
$\Rightarrow \delta(\rho) = -\frac{3M}{\pi R^4}\rho + \frac{3M}{\pi R^4}R$. At the center of the planet $\rho = 0 \Rightarrow \delta(0) = \left(\frac{3M}{\pi R^4}\right)R = \frac{3M}{\pi R^3}$.

82. The mass of the plant's atmosphere to an altitude h above the surface of the planet is the triple integral
$M(h) = \int_0^{2\pi}\int_0^{\pi}\int_R^{h} \mu_0 e^{-c(\rho-R)}\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_R^{h}\int_0^{2\pi}\int_0^{\pi} \mu_0 e^{-c(\rho-R)}\rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$
$= \int_R^{h}\int_0^{2\pi}\left[\mu_0 e^{-c(\rho-R)}\rho^2(-\cos\phi)\right]_0^{\pi} d\theta \, d\rho = 2\int_R^{h}\int_0^{2\pi} \mu_0 e^{cR} e^{-c\rho}\rho^2 \, d\theta \, d\rho = 4\pi\mu_0 e^{cR}\int_R^{h} e^{-c\rho}\rho^2 \, d\rho$
$= 4\pi\mu_0 e^{cR}\left[-\frac{\rho^2 e^{-c\rho}}{c} - \frac{2\rho e^{-c\rho}}{c^2} - \frac{2e^{-c\rho}}{c^3}\right]_R^{h}$ (by parts)
$= 4\pi\mu_0 e^{cR}\left(-\frac{h^2 e^{-ch}}{c} - \frac{2he^{-ch}}{c^2} - \frac{2e^{-ch}}{c^3} + \frac{R^2 e^{-cR}}{c} + \frac{2Re^{-cR}}{c^2} + \frac{2e^{-cR}}{c^3}\right)$.
The mass of the planet's atmosphere is therefore $M = \lim_{h\to\infty} M(h) = 4\pi\mu_0\left(\frac{R^2}{c} + \frac{2R}{c^2} + \frac{2}{c^3}\right)$.

83. (a) A plane perpendicular to the x-axis has the form $x = a$ in rectangular coordinates $\Rightarrow r\cos\theta = a \Rightarrow r = \frac{a}{\cos\theta}$ $\Rightarrow r = a\sec\theta$, in cylindrical coordinates.
(b) A plane perpendicular to the y-axis has the form $y = b$ in rectangular coordinates $\Rightarrow r\sin\theta = b \Rightarrow r = \frac{b}{\sin\theta}$ $\Rightarrow r = b\csc\theta$, in cylindrical coordinates.

84. $ax + by = c \Rightarrow a(r\cos\theta) + b(r\sin\theta) = c \Rightarrow r(a\cos\theta + b\sin\theta) = c \Rightarrow r = \frac{c}{a\cos\theta + b\sin\theta}$.

85. The equation $r = f(z)$ implies that the point $(r, \theta, z) = (f(z), \theta, z)$ will lie on the surface for all θ. In particular $(f(z), \theta + \pi, z)$ lies on the surface whenever $(f(z), \theta, z)$ does $\Rightarrow$ the surface is symmetric with respect to the z-axis.

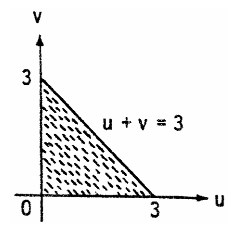


86. The equation $\rho = f(\phi)$ implies that the point $(\rho, \phi, \theta) = (f(\phi), \phi, \theta)$ lies on the surface for all θ. In particular, if $(f(\phi), \phi, \theta)$ lies on the surface, then $(f(\phi), \phi, \theta + \pi)$ lies on the surface, so the surface is symmetric wiith respect to the z-axis.

15.8 SUBSTITUTIONS IN MULTIPLE INTEGRALS

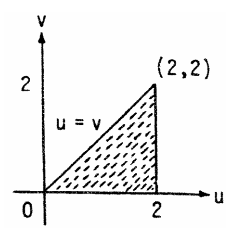
1. (a) $x - y = u$ and $2x + y = v \Rightarrow 3x = u + v$ and $y = x - u \Rightarrow x = \frac{1}{3}(u + v)$ and $y = \frac{1}{3}(-2u + v)$;
$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$

(b) The line segment $y = x$ from $(0, 0)$ to $(1, 1)$ is $x - y = 0 \Rightarrow u = 0$; the line segment $y = -2x$ from $(0, 0)$ to $(1, -2)$ is $2x + y = 0 \Rightarrow v = 0$; the line segment $x = 1$ from $(1, 1)$ to $(1, -2)$ is $(x - y) + (2x + y) = 3 \Rightarrow u + v = 3$. The transformed region is sketched at the right.



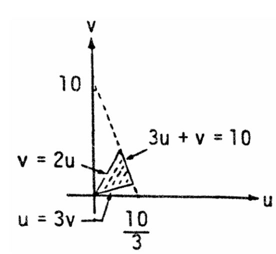
2. (a) $x + 2y = u$ and $x - y = v \Rightarrow 3y = u - v$ and $x = v + y \Rightarrow y = \frac{1}{3}(u - v)$ and $x = \frac{1}{3}(u + 2v)$;
$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}$

(b) The triangular region in the xy-plane has vertices $(0, 0)$, $(2, 0)$, and $\left(\frac{2}{3}, \frac{2}{3}\right)$. The line segment $y = x$ from $(0, 0)$ to $\left(\frac{2}{3}, \frac{2}{3}\right)$ is $x - y = 0 \Rightarrow v = 0$; the line segment $y = 0$ from $(0, 0)$ to $(2, 0) \Rightarrow u = v$; the line segment $x + 2y = 2$ from $\left(\frac{2}{3}, \frac{2}{3}\right)$ to $(2, 0) \Rightarrow u = 2$. The transformed region is sketched at the right.



3. (a) $3x + 2y = u$ and $x + 4y = v \Rightarrow -5x = -2u + v$ and $y = \frac{1}{2}(u - 3x) \Rightarrow x = \frac{1}{5}(2u - v)$ and $y = \frac{1}{10}(3v - u)$;
$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{vmatrix} = \frac{6}{50} - \frac{1}{50} = \frac{1}{10}$

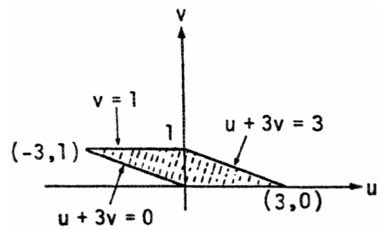
(b) The x-axis $y = 0 \Rightarrow u = 3v$; the y-axis $x = 0 \Rightarrow v = 2u$; the line $x + y = 1 \Rightarrow \frac{1}{5}(2u - v) + \frac{1}{10}(3v - u) = 1 \Rightarrow 2(2u - v) + (3v - u) = 10 \Rightarrow 3u + v = 10$. The transformed region is sketched at the right.



4. (a) $2x-3y=u$ and $-x+y=v \Rightarrow -x=u+3v$ and $y=v+x \Rightarrow x=-u-3v$ and $y=-u-2v$;

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1 & -3 \\ -1 & -2 \end{vmatrix} = 2-3=-1$$

(b) The line $x=-3 \Rightarrow -u-3v=-3$ or $u+3v=3$; $x=0 \Rightarrow u+3v=0$; $y=x \Rightarrow v=0$; $y=x+1 \Rightarrow v=1$. The transformed region is the parallelogram sketched at the right.



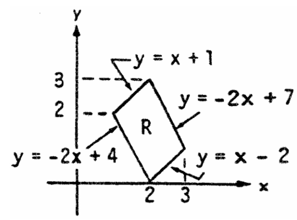
5. $\int_0^4\int_{y/2}^{(y/2)+1}\left(x-\frac{y}{2}\right)dx\,dy = \int_0^4\left[\frac{x^2}{2}-\frac{xy}{2}\right]_{\frac{y}{2}}^{\frac{y}{2}+1}dy = \frac{1}{2}\int_0^4\left[\left(\frac{y}{2}+1\right)^2-\left(\frac{y}{2}\right)^2-\left(\frac{y}{2}+1\right)y+\left(\frac{y}{2}\right)y\right]dy$

$= \frac{1}{2}\int_0^4 (y+1-y)\,dy = \frac{1}{2}\int_0^4 dy = \frac{1}{2}(4)=2$

6. $\iint_R (2x^2-xy-y^2)\,dx\,dy = \iint_R (x-y)(2x+y)\,dx\,dy$

$= \iint_G uv\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv = \frac{1}{3}\iint_G uv\,du\,dv;$

We find the boundaries of G from the boundaries of R, shown in the accompanying figure:



xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$y=-2x+4$	$\frac{1}{3}(-2u+v)=-\frac{2}{3}(u+v)+4$	$v=4$
$y=-2x+7$	$\frac{1}{3}(-2u+v)=-\frac{2}{3}(u+v)+7$	$v=7$
$y=x-2$	$\frac{1}{3}(-2u+v)=\frac{1}{3}(u+v)-2$	$u=2$
$y=x+1$	$\frac{1}{3}(-2u+v)=\frac{1}{3}(u+v)+1$	$u=-1$

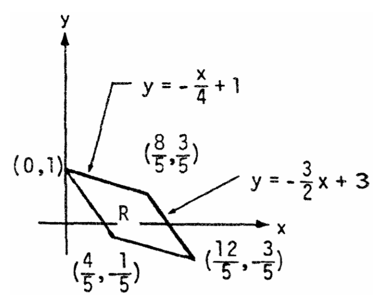
$\Rightarrow \frac{1}{3}\iint_G uv\,du\,dv = \frac{1}{3}\int_{-1}^{2}\int_4^7 uv\,dv\,du = \frac{1}{3}\int_{-1}^{2} u\left[\frac{v^2}{2}\right]_4^7 du = \frac{11}{2}\int_{-1}^{2} u\,du = \left(\frac{11}{2}\right)\left[\frac{u^2}{2}\right]_{-1}^{2} = \left(\frac{11}{4}\right)(4-1) = \frac{33}{4}$

7. $\iint_R (3x^2+14xy+8y^2)\,dx\,dy$

$= \iint_R (3x+2y)(x+4y)\,dx\,dy$

$= \iint_G uv\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv = \frac{1}{10}\iint_G uv\,du\,dv;$

We find the boundaries of G from the boundaries of R, shown in the accompanying figure:



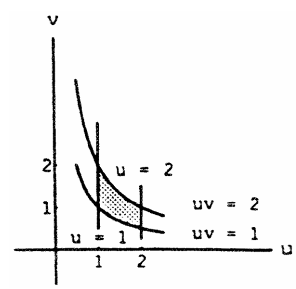
xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$y=-\frac{3}{2}x+1$	$\frac{1}{10}(3v-u)=-\frac{3}{10}(2u-v)+1$	$u=2$
$y=-\frac{3}{2}x+3$	$\frac{1}{10}(3v-u)=-\frac{3}{10}(2u-v)+3$	$u=6$
$y=-\frac{1}{4}x$	$\frac{1}{10}(3v-u)=-\frac{1}{20}(2u-v)$	$v=0$
$y=-\frac{1}{4}x+1$	$\frac{1}{10}(3v-u)=-\frac{1}{20}(2u-v)+1$	$v=4$

$\Rightarrow \frac{1}{10} \iint\limits_G uv\,du\,dv = \frac{1}{10} \int_2^6 \int_0^4 uv\,dv\,du = \frac{1}{10} \int_2^6 u \left[\frac{v^2}{2}\right]_0^4 du = \frac{4}{5} \int_2^6 u\,du = \left(\frac{4}{5}\right) \left[\frac{u^2}{2}\right]_2^6 = \left(\frac{4}{5}\right)(18-2) = \frac{64}{5}$

8. $\iint\limits_R 2(x-y)\,dx\,dy = \iint\limits_G -2v \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint\limits_G -2v\,du\,dv$; the region G is sketched in Exercise 4
$\Rightarrow \iint\limits_G -2v\,du\,dv = \int_0^1 \int_{-3v}^{3-3v} -2v\,du\,dv = \int_0^1 -2v(3-3v+3v)\,dv = \int_0^1 -6v\,dv = \left[-3v^2\right]_0^1 = -3$

9. $x = \frac{u}{v}$ and $y = uv \Rightarrow \frac{y}{x} = v^2$ and $xy = u^2$; $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} v^{-1} & -uv^{-2} \\ v & u \end{vmatrix} = v^{-1}u + v^{-1}u = \frac{2u}{v}$;
$y = x \Rightarrow uv = \frac{u}{v} \Rightarrow v = 1$, and $y = 4x \Rightarrow v = 2$; $xy = 1 \Rightarrow u = 1$, and $xy = 9 \Rightarrow u = 3$; thus
$\iint\limits_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\,dy = \int_1^3 \int_1^2 (v+u)\left(\frac{2u}{v}\right) dv\,du = \int_1^3 \int_1^2 \left(2u + \frac{2u^2}{v}\right) dv\,du = \int_1^3 \left[2uv + 2u^2 \ln v\right]_1^2 du$
$= \int_1^3 (2u + 2u^2 \ln 2)\,du = \left[u^2 + \frac{2}{3}u^2 \ln 2\right]_1^3 = 8 + \frac{2}{3}(26)(\ln 2) = 8 + \frac{52}{3}(\ln 2)$

10. (a) $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix} = u$, and
the region G is sketched at the right

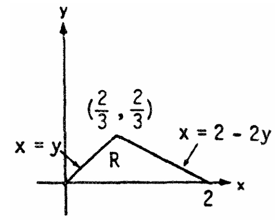


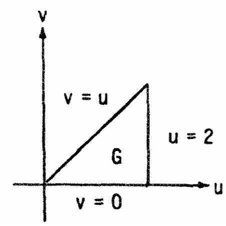
(b) $x = 1 \Rightarrow u = 1$, and $x = 2 \Rightarrow u = 2$; $y = 1 \Rightarrow uv = 1 \Rightarrow v = \frac{1}{u}$, and $y = 2 \Rightarrow uv = 2 \Rightarrow v = \frac{2}{u}$; thus,
$\int_1^2 \int_1^2 \frac{y}{x}\,dy\,dx = \int_1^2 \int_{1/u}^{2/u} \left(\frac{uv}{u}\right) u\,dv\,du = \int_1^2 \int_{1/u}^{2/u} uv\,dv\,du = \int_1^2 u \left[\frac{v^2}{2}\right]_{1/u}^{2/u} du = \int_1^2 u\left(\frac{2}{u^2} - \frac{1}{2u^2}\right) du$
$= \frac{3}{2} \int_1^2 u\left(\frac{1}{u^2}\right) du = \frac{3}{2}\left[\ln u\right]_1^2 = \frac{3}{2} \ln 2$; $\int_1^2 \int_1^2 \frac{y}{x}\,dy\,dx = \int_1^2 \left[\frac{1}{x} \cdot \frac{y^2}{2}\right]_1^2 dx = \frac{3}{2} \int_1^2 \frac{dx}{x} = \frac{3}{2}\left[\ln x\right]_1^2 = \frac{3}{2} \ln 2$

11. $x = ar\cos\theta$ and $y = ar\sin\theta \Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} = J(r,\theta) = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr\cos^2\theta + abr\sin^2\theta = abr$;
$I_0 = \iint\limits_R (x^2+y^2)\,dA = \int_0^{2\pi} \int_0^1 r^2 (a^2\cos^2\theta + b^2\sin^2\theta)\,|J(r,\theta)|\,dr\,d\theta = \int_0^{2\pi} \int_0^1 abr^3 (a^2\cos^2\theta + b^2\sin^2\theta)\,dr\,d\theta$
$= \frac{ab}{4} \int_0^{2\pi} (a^2\cos^2\theta + b^2\sin^2\theta)\,d\theta = \frac{ab}{4}\left[\frac{a^2\theta}{2} + \frac{a^2\sin 2\theta}{4} + \frac{b^2\theta}{2} - \frac{b^2\sin 2\theta}{4}\right]_0^{2\pi} = \frac{ab\pi(a^2+b^2)}{4}$

12. $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$; $A = \iint\limits_R dy\,dx = \iint\limits_G ab\,du\,dv = \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} ab\,dv\,du$
$= 2ab \int_{-1}^1 \sqrt{1-u^2}\,du = 2ab\left[\frac{u}{2}\sqrt{1-u^2} + \frac{1}{2}\sin^{-1} u\right]_{-1}^1 = ab\left[\sin^{-1} 1 - \sin^{-1}(-1)\right] = ab\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = ab\pi$

13. The region of integration R in the xy-plane is sketched in the figure at the right. The boundaries of the image G are obtained as follows, with G sketched at the right:

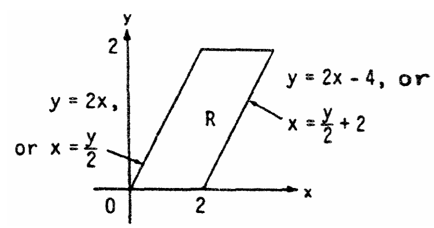




xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$x = y$	$\frac{1}{3}(u + 2v) = \frac{1}{3}(u - v)$	$v = 0$
$x = 2 - 2y$	$\frac{1}{3}(u + 2v) = 2 - \frac{2}{3}(u - v)$	$u = 2$
$y = 0$	$0 = \frac{1}{3}(u - v)$	$v = u$

Also, from Exercise 2, $\frac{\partial(x,y)}{\partial(u,v)} = J(u, v) = -\frac{1}{3} \Rightarrow \int_0^{2/3} \int_y^{2-2y} (x + 2y)\, e^{(y-x)}\, dx\, dy = \int_0^2 \int_0^u ue^{-v} \left|-\frac{1}{3}\right| dv\, du$
$= \frac{1}{3} \int_0^2 u \left[-e^{-v}\right]_0^u du = \frac{1}{3} \int_0^2 u\,(1 - e^{-u})\, du = \frac{1}{3} \left[u\,(u + e^{-u}) - \frac{u^2}{2} + e^{-u}\right]_0^2 = \frac{1}{3}\left[2\,(2 + e^{-2}) - 2 + e^{-2} - 1\right]$
$= \frac{1}{3}(3e^{-2} + 1) \approx 0.4687$

14. $x = u + \frac{v}{2}$ and $y = v \Rightarrow 2x - y = (2u + v) - v = 2u$ and $\frac{\partial(x,y)}{\partial(u,v)} = J(u, v) = \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = 1$; next, $u = x - \frac{v}{2}$ $= x - \frac{y}{2}$ and $v = y$, so the boundaries of the region of integration R in the xy-plane are transformed to the boundaries of G:



xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$x = \frac{y}{2}$	$u + \frac{v}{2} = \frac{v}{2}$	$u = 0$
$x = \frac{y}{2} + 2$	$u + \frac{v}{2} = \frac{v}{2} + 2$	$u = 2$
$y = 0$	$v = 0$	$v = 0$
$y = 2$	$v = 2$	$v = 2$

$\Rightarrow \int_0^2 \int_{y/2}^{(y/2)+2} y^3(2x - y)\, e^{(2x-y)^2}\, dx\, dy = \int_0^2 \int_0^2 v^3(2u)\, e^{4u^2}\, du\, dv = \int_0^2 v^3 \left[\frac{1}{4} e^{4u^2}\right]_0^2 dv = \frac{1}{4} \int_0^2 v^3\,(e^{16} - 1)\, dv$
$= \frac{1}{4}(e^{16} - 1)\left[\frac{v^4}{4}\right]_0^2 = e^{16} - 1$

15. $x = \frac{u}{v}$ and $y = uv \Rightarrow \frac{y}{x} = v^2$ and $xy = u^2$; $\frac{\partial(x,y)}{\partial(u,v)} = J(u, v) = \begin{vmatrix} v^{-1} & -uv^{-2} \\ v & u \end{vmatrix} = v^{-1}u + v^{-1}u = \frac{2u}{v}$;
$y = x \Rightarrow uv = \frac{u}{v} \Rightarrow v = 1$, and $y = 4x \Rightarrow v = 2$; $xy = 1 \Rightarrow u = 1$, and $xy = 4 \Rightarrow u = 2$; thus
$\int_1^2 \int_{1/y}^{y} (x^2 + y^2)\, dx\, dy + \int_2^4 \int_{y/4}^{4/y} (x^2 + y^2)\, dx\, dy = \int_1^2 \int_1^2 \left(\frac{u^2}{v^2} + u^2v^2\right)\left(\frac{2u}{v}\right) du\, dv = \int_1^2 \int_1^2 \left(\frac{2u^3}{v^3} + 2u^3v\right) du\, dv$

$= \int_1^2 \left[\frac{u^4}{2v^3} + \frac{1}{2}u^4 v\right]_1^2 dv = \int_1^2 \left(\frac{15}{2v^3} + \frac{15v}{2}\right) dv = \left[-\frac{15}{4v^2} + \frac{15v^2}{4}\right]_1^2 = \frac{225}{16}$

16. $x = u^2 - v^2$ and $y = 2uv$; $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 = 4(u^2+v^2)$;

$y = 2\sqrt{1-x} \Rightarrow y^2 = 4(1-x) \Rightarrow (2uv)^2 = 4(1-(u^2-v^2)) \Rightarrow u = \pm 1$; $y = 0 \Rightarrow 2uv = 0 \Rightarrow u = 0$ or $v = 0$; $x = 0 \Rightarrow u^2 - v^2 = 0 \Rightarrow u = v$ or $u = -v$; This gives us four triangular regions, but only the one in the quadrant where both u, v are positive maps into the region R in the xy-plane.

$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2+y^2}\, dx\, dy = \int_0^1 \int_0^u \sqrt{(u^2-v^2)^2 + (2uv)^2} \cdot 4(u^2+v^2)\, dv\, du = 4\int_0^1 \int_0^u (u^2+v^2)^2\, dv\, du$

$= 4\int_1^2 \left[u^4 v + \frac{2}{3}u^2 v^3 + \frac{1}{5}v^5\right]_0^u du = \frac{112}{15}\int_1^2 u^5\, du = \frac{112}{15}\left[\frac{1}{6}u^6\right]_1^2 = \frac{56}{45}$

17. (a) $x = u\cos v$ and $y = u\sin v \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix} = u\cos^2 v + u\sin^2 v = u$

(b) $x = u\sin v$ and $y = u\cos v \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sin v & u\cos v \\ \cos v & -u\sin v \end{vmatrix} = -u\sin^2 v - u\cos^2 v = -u$

18. (a) $x = u\cos v,\ y = u\sin v,\ z = w \Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \cos v & -u\sin v & 0 \\ \sin v & u\cos v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u\cos^2 v + u\sin^2 v = u$

(b) $x = 2u - 1,\ y = 3v - 4,\ z = \frac{1}{2}(w-4) \Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{vmatrix} = (2)(3)\left(\frac{1}{2}\right) = 3$

19. $\begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$

$= (\cos\phi)\begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} + (\rho\sin\phi)\begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$

$= (\rho^2\cos\phi)(\sin\phi\cos\phi\cos^2\theta + \sin\phi\cos\phi\sin^2\theta) + (\rho^2\sin\phi)(\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta)$

$= \rho^2\sin\phi\cos^2\phi + \rho^2\sin^3\phi = (\rho^2\sin\phi)(\cos^2\phi + \sin^2\phi) = \rho^2\sin\phi$

20. Let $u = g(x) \Rightarrow J(x) = \frac{du}{dx} = g'(x) \Rightarrow \int_a^b f(u)\, du = \int_{g(a)}^{g(b)} f(g(x))g'(x)\, dx$ in accordance with Theorem 7 in Section 5.6. Note that $g'(x)$ represents the Jacobian of the transformation $u = g(x)$ or $x = g^{-1}(u)$.

21. $\int_0^3 \int_0^4 \int_{y/2}^{1+(y/2)} \left(\frac{2x-y}{2} + \frac{z}{3}\right) dx\, dy\, dz = \int_0^3 \int_0^4 \left[\frac{x^2}{2} - \frac{xy}{2} + \frac{xz}{3}\right]_{y/2}^{1+(y/2)} dy\, dz = \int_0^3 \int_0^4 \left[\frac{1}{2}(y+1) - \frac{y}{2} + \frac{z}{3}\right] dy\, dz$

$= \int_0^3 \left[\frac{(y+1)^2}{4} - \frac{y^2}{4} + \frac{yz}{3}\right]_0^4 dz = \int_0^3 \left(\frac{9}{4} + \frac{4z}{3} - \frac{1}{4}\right) dz = \int_0^3 \left(2 + \frac{4z}{3}\right) dz = \left[2z + \frac{2z^2}{3}\right]_0^3 = 12$

22. $J(u,v,w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$; the transformation takes the ellipsoid region $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$ in xyz-space into the spherical region $u^2 + v^2 + w^2 \le 1$ in uvw-space $\left(\text{which has volume } V = \frac{4}{3}\pi\right)$

$\Rightarrow V = \iiint_R dx\, dy\, dz = \iiint_G abc\, du\, dv\, dw = \frac{4\pi abc}{3}$

23. $J(u,v,w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$; for R and G as in Exercise 22, $\iiint\limits_R |xyz|\, dx\, dy\, dz$

$= \iiint\limits_G a^2b^2c^2uvw\, dw\, dv\, du = 8a^2b^2c^2 \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (\rho \sin\phi \cos\theta)(\rho \sin\phi \sin\theta)(\rho\cos\phi)\,(\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$

$= \frac{4a^2b^2c^2}{3}\int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\cos\theta\sin^3\phi\cos\phi\, d\phi\, d\theta = \frac{a^2b^2c^2}{3}\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta = \frac{a^2b^2c^2}{6}$

24. $u = x$, $v = xy$, and $w = 3z \Rightarrow x = u$, $y = \frac{v}{u}$, and $z = \frac{1}{3}w \Rightarrow J(u,v,w) = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{3u}$;

$\iiint\limits_D (x^2y + 3xyz)\, dx\, dy\, dz = \iiint\limits_G \left[u^2\left(\frac{v}{u}\right) + 3u\left(\frac{v}{u}\right)\left(\frac{w}{3}\right)\right] |J(u,v,w)|\, du\, dv\, dw = \frac{1}{3}\int_0^3\int_0^2\int_1^2 \left(v + \frac{vw}{u}\right) du\, dv\, dw$

$= \frac{1}{3}\int_0^3\int_0^2 (v + vw\ln 2)\, dv\, dw = \frac{1}{3}\int_0^3 (1 + w\ln 2)\left[\frac{v^2}{2}\right]_0^2 dw = \frac{2}{3}\int_0^3 (1 + w\ln 2)\, dw = \frac{2}{3}\left[w + \frac{w^2}{2}\ln 2\right]_0^3$

$= \frac{2}{3}\left(3 + \frac{9}{2}\ln 2\right) = 2 + 3\ln 2 = 2 + \ln 8$

25. The first moment about the xy-coordinate plane for the semi-ellipsoid, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ using the transformation in Exercise 23 is, $M_{xy} = \iiint\limits_D z\, dz\, dy\, dx = \iiint\limits_G cw\, |J(u,v,w)|\, du\, dv\, dw$

$= abc^2 \iiint\limits_G w\, du\, dv\, dw = (abc^2)\cdot(M_{xy}$ of the hemisphere $x^2 + y^2 + z^2 = 1$, $z \ge 0) = \frac{abc^2\pi}{4}$;

the mass of the semi-ellipsoid is $\frac{2abc\pi}{3} \Rightarrow \bar{z} = \left(\frac{abc^2\pi}{4}\right)\left(\frac{3}{2abc\pi}\right) = \frac{3}{8}c$

26. A solid of revolution is symmetric about the axis of revolution, therefore, the height of the solid is solely a function of r. That is, $y = f(x) = f(r)$. Using cylindrical coordinates with $x = r\cos\theta$, $y = y$ and $z = r\sin\theta$, we have

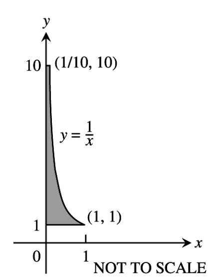
$V = \iiint\limits_G r\, dy\, d\theta\, dr = \int_a^b\int_0^{2\pi}\int_0^{f(r)} r\, dy\, d\theta\, dr = \int_a^b\int_0^{2\pi}[r\, y]_0^{f(r)}\, d\theta\, dr = \int_a^b\int_0^{2\pi} r\, f(r)\, d\theta\, dr = \int_a^b [r\theta f(r)]_0^{2\pi}\, dr$

$\int_a^b 2\pi r f(r) dr$. In the last integral, r is a dummy or stand-in variable and as such it can be replaced by any variable name. Choosing x instead of r we have $V = \int_a^b 2\pi x f(x) dx$, which is the same result obtained using the shell method.

CHAPTER 15 PRACTICE EXERCISES

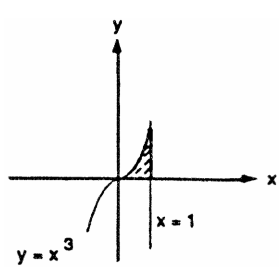
1. $\int_1^{10}\int_0^{1/y} ye^{xy}\, dx\, dy = \int_1^{10} [e^{xy}]_0^{1/y}\, dy$

$= \int_1^{10} (e - 1)\, dy = 9e - 9$

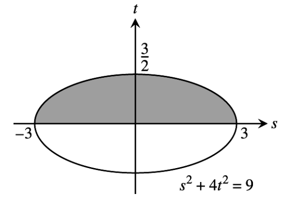


2. $\int_0^1\int_0^{x^3} e^{y/x}\, dy\, dx = \int_0^1 x\left[e^{y/x}\right]_0^{x^3} dx$

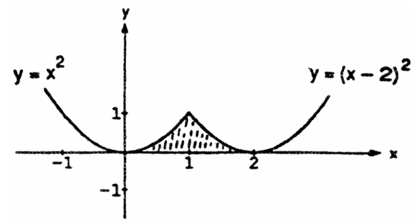
$= \int_0^1 \left(xe^{x^2} - x\right) dx = \left[\frac{1}{2}e^{x^2} - \frac{x^2}{2}\right]_0^1 = \frac{e-2}{2}$



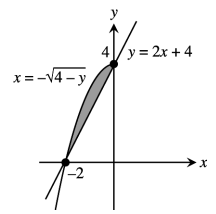
3. $\int_0^{3/2}\int_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}} t\,ds\,dt = \int_0^{3/2} [ts]_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}}\,dt$
$= \int_0^{3/2} 2t\sqrt{9-4t^2}\,dt = \left[-\frac{1}{6}\left(9-4t^2\right)^{3/2}\right]_0^{3/2}$
$= -\frac{1}{6}\left(0^{3/2}-9^{3/2}\right) = \frac{27}{6} = \frac{9}{2}$



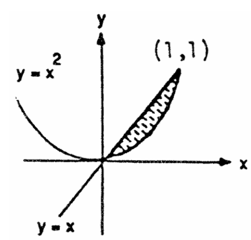
4. $\int_0^1\int_{\sqrt{y}}^{2-\sqrt{y}} xy\,dx\,dy = \int_0^1 y\left[\frac{x^2}{2}\right]_{\sqrt{y}}^{2-\sqrt{y}}\,dy$
$= \frac{1}{2}\int_0^1 y\left(4-4\sqrt{y}+y-y\right)dy$
$= \int_0^1 \left(2y-2y^{3/2}\right)dy = \left[y^2-\frac{4y^{5/2}}{5}\right]_0^1 = \frac{1}{5}$



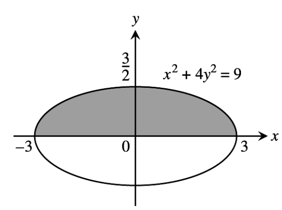
5. $\int_{-2}^{0}\int_{2x+4}^{4-x^2} dy\,dx = \int_{-2}^{0}\left(-x^2-2x\right)dx$
$= \left[-\frac{x^3}{3}-x^2\right]_{-2}^{0} = -\left(\frac{8}{3}-4\right) = \frac{4}{3}$
$\int_0^4\int_{-\sqrt{4-y}}^{(y-4)/2} dx\,dy = \int_0^4\left(\frac{y-4}{2}+\sqrt{4-y}\right)dy$
$= \left[\frac{y^2}{2}-2y-\frac{2}{3}(4-y)^{3/2}\right]_0^4 = 4-8+\frac{2}{3}\cdot 4^{3/2}$
$= -4+\frac{16}{3} = \frac{4}{3}$



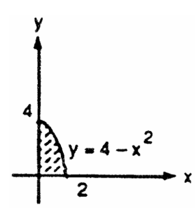
6. $\int_0^1\int_y^{\sqrt{y}} \sqrt{x}\,dx\,dy = \int_0^1\left[\frac{2}{3}x^{3/2}\right]_y^{\sqrt{y}}\,dy$
$= \frac{2}{3}\int_0^1\left(y^{3/4}-y^{3/2}\right)dy = \frac{2}{3}\left[\frac{4}{7}y^{7/4}-\frac{2}{5}y^{5/2}\right]_0^1$
$= \frac{2}{3}\left(\frac{4}{7}-\frac{2}{5}\right) = \frac{4}{35}$
$\int_0^1\int_{x^2}^{x}\sqrt{x}\,dy\,dx = \int_0^1 x^{1/2}\left(x-x^2\right)dx = \int_0^1\left(x^{3/2}-x^{5/2}\right)dx$
$= \left[\frac{2}{5}x^{5/2}-\frac{2}{7}x^{7/2}\right]_0^1 = \frac{2}{5}-\frac{2}{7} = \frac{4}{35}$



7. $\int_{-3}^{3}\int_0^{(1/2)\sqrt{9-x^2}} y\,dy\,dx = \int_{-3}^{3}\left[\frac{y^2}{2}\right]_0^{(1/2)\sqrt{9-x^2}}\,dx$
$= \int_{-3}^{3}\frac{1}{8}\left(9-x^2\right)dx = \left[\frac{9x}{8}-\frac{x^3}{24}\right]_{-3}^{3}$
$= \left(\frac{27}{8}-\frac{27}{24}\right)-\left(-\frac{27}{8}+\frac{27}{24}\right) = \frac{27}{6} = \frac{9}{2}$
$\int_0^{3/2}\int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y\,dx\,dy = \int_0^{3/2} 2y\sqrt{9-4y^2}\,dy$
$= -\frac{1}{4}\cdot\frac{2}{3}\left(9-4y^2\right)^{3/2}\Big|_0^{3/2} = \frac{1}{6}\cdot 9^{3/2} = \frac{27}{6} = \frac{9}{2}$



8. $\int_0^2\int_0^{4-x^2} 2x\,dy\,dx = \int_0^2 [2xy]_0^{4-x^2}\,dx$
$= \int_0^2\left(2x\left(4-x^2\right)\right)dx = \int_0^2\left(8x-2x^3\right)dx$
$= \left[4x^2-\frac{x^4}{2}\right]_0^2 = 16-\frac{16}{2} = 8$
$\int_0^4\int_0^{\sqrt{4-y}} 2x\,dx\,dy = \int_0^4 \left[x^2\right]_0^{\sqrt{4-y}}\,dy$
$= \int_0^4 (4-y)\,dy = \left[4y-\frac{y^2}{2}\right]_0^4 = 16-\frac{16}{2} = 8$



9. $\int_0^1 \int_{2y}^2 4\cos(x^2)\, dx\, dy = \int_0^2 \int_0^{x/2} 4\cos(x^2)\, dy\, dx = \int_0^2 2x\cos(x^2)\, dx = \left[\sin(x^2)\right]_0^2 = \sin 4$

10. $\int_0^2 \int_{y/2}^1 e^{x^2}\, dx\, dy = \int_0^1 \int_0^{2x} e^{x^2}\, dy\, dx = \int_0^1 2xe^{x^2}\, dx = \left[e^{x^2}\right]_0^1 = e - 1$

11. $\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{1}{y^4+1}\, dy\, dx = \int_0^2 \int_0^{y^3} \frac{1}{y^4+1}\, dx\, dy = \frac{1}{4}\int_0^2 \frac{4y^3}{y^4+1}\, dy = \frac{\ln 17}{4}$

12. $\int_0^1 \int_{\sqrt[3]{y}}^1 \frac{2\pi \sin(\pi x^2)}{x^2}\, dx\, dy = \int_0^1 \int_0^{x^3} \frac{2\pi\sin(\pi x^2)}{x^2}\, dy\, dx = \int_0^1 2\pi x \sin(\pi x^2)\, dx = \left[-\cos(\pi x^2)\right]_0^1 = -(-1) - (-1) = 2$

13. $A = \int_{-2}^0 \int_{2x+4}^{4-x^2} dy\, dx = \int_{-2}^0 (-x^2 - 2x)\, dx = \frac{4}{3}$

14. $A = \int_1^4 \int_{2-y}^{\sqrt{y}} dx\, dy = \int_1^4 \left(\sqrt{y} - 2 + y\right) dy = \frac{37}{6}$

15. $V = \int_0^1 \int_x^{2-x} (x^2+y^2)\, dy\, dx = \int_0^1 \left[x^2y + \frac{y^3}{3}\right]_x^{2-x} dx = \int_0^1 \left[2x^2 + \frac{(2-x)^3}{3} - \frac{7x^3}{3}\right] dx = \left[\frac{2x^3}{3} - \frac{(2-x)^4}{12} - \frac{7x^4}{12}\right]_0^1$
$= \left(\frac{2}{3} - \frac{1}{12} - \frac{7}{12}\right) + \frac{2^4}{12} = \frac{4}{3}$

16. $V = \int_{-3}^2 \int_x^{6-x^2} x^2\, dy\, dx = \int_{-3}^2 \left[x^2 y\right]_x^{6-x^2} dx = \int_{-3}^2 (6x^2 - x^4 - x^3)\, dx = \frac{125}{4}$

17. average value $= \int_0^1 \int_0^1 xy\, dy\, dx = \int_0^1 \left[\frac{xy^2}{2}\right]_0^1 dx = \int_0^1 \frac{x}{2}\, dx = \frac{1}{4}$

18. average value $= \frac{1}{\left(\frac{\pi}{4}\right)} \int_0^1 \int_0^{\sqrt{1-x^2}} xy\, dy\, dx = \frac{4}{\pi}\int_0^1 \left[\frac{xy^2}{2}\right]_0^{\sqrt{1-x^2}} dx = \frac{2}{\pi}\int_0^1 (x - x^3)\, dx = \frac{1}{2\pi}$

19. $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}\, dy\, dx = \int_0^{2\pi} \int_0^1 \frac{2r}{(1+r^2)^2}\, dr\, d\theta = \int_0^{2\pi} \left[-\frac{1}{1+r^2}\right]_0^1 d\theta = \frac{1}{2}\int_0^{2\pi} d\theta = \pi$

20. $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1)\, dx\, dy = \int_0^{2\pi} \int_0^1 r\ln(r^2+1)\, dr\, d\theta = \int_0^{2\pi} \int_1^2 \frac{1}{2}\ln u\, du\, d\theta = \frac{1}{2}\int_0^{2\pi} \left[u\ln u - u\right]_1^2 d\theta$
$= \frac{1}{2}\int_0^{2\pi} (2\ln 2 - 1)\, d\theta = [\ln(4) - 1]\,\pi$

21. $(x^2+y^2)^2 - (x^2-y^2) = 0 \Rightarrow r^4 - r^2\cos 2\theta = 0 \Rightarrow r^2 = \cos 2\theta$ so the integral is $\int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\, dr\, d\theta$
$= \int_{-\pi/4}^{\pi/4} \left[-\frac{1}{2(1+r^2)}\right]_0^{\sqrt{\cos 2\theta}} d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \left(1 - \frac{1}{1+\cos 2\theta}\right) d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \left(1 - \frac{1}{2\cos^2\theta}\right) d\theta$
$= \frac{1}{2}\int_{-\pi/4}^{\pi/4} \left(1 - \frac{\sec^2\theta}{2}\right) d\theta = \frac{1}{2}\left[\theta - \frac{\tan\theta}{2}\right]_{-\pi/4}^{\pi/4} = \frac{\pi-2}{4}$

22. (a) $\iint_R \frac{1}{(1+x^2+y^2)^2}\, dx\, dy = \int_0^{\pi/3} \int_0^{\sec\theta} \frac{r}{(1+r^2)^2}\, dr\, d\theta = \int_0^{\pi/3} \left[-\frac{1}{2(1+r^2)}\right]_0^{\sec\theta} d\theta$
$= \int_0^{\pi/3} \left[\frac{1}{2} - \frac{1}{2(1+\sec^2\theta)}\right] d\theta = \frac{1}{2}\int_0^{\pi/3} \frac{\sec^2\theta}{1+\sec^2\theta}\, d\theta;\ \begin{bmatrix} u = \tan\theta \\ du = \sec^2\theta\, d\theta \end{bmatrix} \to \frac{1}{2}\int_0^{\sqrt{3}} \frac{du}{2+u^2}$
$= \frac{1}{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}}\right]_0^{\sqrt{3}} = \frac{\sqrt{2}}{4}\tan^{-1}\sqrt{\frac{3}{2}}$

(b) $\iint_R \frac{1}{(1+x^2+y^2)^2}\, dx\, dy = \int_0^{\pi/2} \int_0^{\infty} \frac{r}{(1+r^2)^2}\, dr\, d\theta = \int_0^{\pi/2} \lim_{b\to\infty} \left[-\frac{1}{2(1+r^2)}\right]_0^b d\theta$
$= \int_0^{\pi/2} \lim_{b\to\infty} \left[\frac{1}{2} - \frac{1}{2(1+b^2)}\right] d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}$

23. $\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \cos(x+y+z)\,dx\,dy\,dz = \int_0^{\pi}\int_0^{\pi} [\sin(z+y+\pi) - \sin(z+y)]\,dy\,dz$
$= \int_0^{\pi} [-\cos(z+2\pi) + \cos(z+\pi) - \cos z + \cos(z+\pi)]\,dz = 0$

24. $\int_{\ln 6}^{\ln 7}\int_0^{\ln 2}\int_{\ln 4}^{\ln 5} e^{(x+y+z)}\,dz\,dy\,dx = \int_{\ln 6}^{\ln 7}\int_0^{\ln 2} e^{(x+y)}\,dy\,dx = \int_{\ln 6}^{\ln 7} e^x\,dx = 1$

25. $\int_0^1\int_0^{x^2}\int_0^{x+y} (2x-y-z)\,dz\,dy\,dx = \int_0^1\int_0^{x^2}\left(\frac{3x^2}{2} - \frac{3y^2}{2}\right)dy\,dx = \int_0^1\left(\frac{3x^4}{2} - \frac{x^6}{2}\right)dx = \frac{8}{35}$

26. $\int_1^e\int_1^x\int_0^z \frac{2y}{z^3}\,dy\,dz\,dx = \int_1^e\int_1^x \frac{1}{z}\,dz\,dx = \int_1^e \ln x\,dx = [x\ln x - x]_1^e = 1$

27. $V = 2\int_0^{\pi/2}\int_{-\cos y}^{0}\int_0^{-2x} dz\,dx\,dy = 2\int_0^{\pi/2}\int_{-\cos y}^{0} -2x\,dx\,dy = 2\int_0^{\pi/2}\cos^2 y\,dy = 2\left[\frac{y}{2} + \frac{\sin 2y}{4}\right]_0^{\pi/2} = \frac{\pi}{2}$

28. $V = 4\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{4-x^2} dz\,dy\,dx = 4\int_0^2\int_0^{\sqrt{4-x^2}}(4-x^2)\,dy\,dx = 4\int_0^2 (4-x^2)^{3/2}\,dx$
$= \left[x(4-x^2)^{3/2} + 6x\sqrt{4-x^2} + 24\sin^{-1}\frac{x}{2}\right]_0^2 = 24\sin^{-1}1 = 12\pi$

29. average $= \frac{1}{3}\int_0^1\int_0^3\int_0^1 30xz\sqrt{x^2+y}\,dz\,dy\,dx = \frac{1}{3}\int_0^1\int_0^3 15x\sqrt{x^2+y}\,dy\,dx = \frac{1}{3}\int_0^3\int_0^1 15x\sqrt{x^2+y}\,dx\,dy$
$= \frac{1}{3}\int_0^3\left[5(x^2+y)^{3/2}\right]_0^1 dy = \frac{1}{3}\int_0^3\left[5(1+y)^{3/2} - 5y^{3/2}\right]dy = \frac{1}{3}\left[2(1+y)^{5/2} - 2y^{5/2}\right]_0^3 = \frac{1}{3}\left[2(4)^{5/2} - 2(3)^{5/2} - 2\right]$
$= \frac{1}{3}\left[2\left(31 - 3^{5/2}\right)\right]$

30. average $= \frac{3}{4\pi a^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^3\sin\phi\,d\rho\,d\phi\,d\theta = \frac{3a}{16\pi}\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta = \frac{3a}{8\pi}\int_0^{2\pi}d\theta = \frac{3a}{4}$

31. (a) $\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} 3\,dz\,dx\,dy$

(b) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 3\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

(c) $\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} 3\,dz\,r\,dr\,d\theta = 3\int_0^{2\pi}\int_0^{\sqrt{2}}\left[r(4-r^2)^{1/2} - r^2\right]dr\,d\theta = 3\int_0^{2\pi}\left[-\frac{1}{3}(4-r^2)^{3/2} - \frac{r^3}{3}\right]_0^{\sqrt{2}} d\theta$
$= \int_0^{2\pi}\left(-2^{3/2} - 2^{3/2} + 4^{3/2}\right)d\theta = \left(8 - 4\sqrt{2}\right)\int_0^{2\pi}d\theta = 2\pi\left(8 - 4\sqrt{2}\right)$

32. (a) $\int_{-\pi/2}^{\pi/2}\int_0^1\int_{-r^2}^{r^2} 21(r\cos\theta)(r\sin\theta)^2\,dz\,r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_0^1\int_{-r^2}^{r^2} 21r^3\cos\theta\sin^2\theta\,dz\,r\,dr\,d\theta$

(b) $\int_{-\pi/2}^{\pi/2}\int_0^1\int_{-r^2}^{r^2} 21r^3\cos\theta\sin^2\theta\,dz\,r\,dr\,d\theta = 84\int_0^{\pi/2}\int_0^1 r^6\sin^2\theta\cos\theta\,dr\,d\theta = 12\int_0^{\pi/2}\sin^2\theta\cos\theta\,d\theta = 4$

33. (a) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

(b) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4}(\sec\phi)(\sec\phi\tan\phi)\,d\phi\,d\theta = \frac{1}{3}\int_0^{2\pi}\left[\frac{1}{2}\tan^2\phi\right]_0^{\pi/4} d\theta = \frac{1}{6}\int_0^{2\pi}d\theta = \frac{\pi}{3}$

34. (a) $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{x^2+y^2}} (6+4y)\,dz\,dy\,dx$

(b) $\int_0^{\pi/2}\int_0^1\int_0^r (6 + 4r\sin\theta)\,dz\,r\,dr\,d\theta$

(c) $\int_0^{\pi/2}\int_{\pi/4}^{\pi/2}\int_0^{\csc\phi} (6 + 4\rho\sin\phi\sin\theta)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$

(d) $\int_0^{\pi/2}\int_0^1\int_0^r (6 + 4r\sin\theta)\,dz\,r\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 (6r^2 + 4r^3\sin\theta)\,dr\,d\theta = \int_0^{\pi/2}[2r^3 + r^4\sin\theta]_0^1\,d\theta$
$= \int_0^{\pi/2}(2 + \sin\theta)\,d\theta = [2\theta - \cos\theta]_0^{\pi/2} = \pi + 1$

35. $\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} z^2yx\, dz\, dy\, dx + \int_1^{\sqrt{3}} \int_0^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} z^2yx\, dz\, dy\, dx$

36. (a) Bounded on the top and bottom by the sphere $x^2 + y^2 + z^2 = 4$, on the right by the right circular cylinder $(x-1)^2 + y^2 = 1$, on the left by the plane $y = 0$

(b) $\int_0^{\pi/2} \int_0^{2\cos\theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dz\, r\, dr\, d\theta$

37. (a) $V = \int_0^{2\pi} \int_0^2 \int_2^{\sqrt{8-r^2}} dz\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^2 \left(r\sqrt{8-r^2} - 2r\right) dr\, d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(8-r^2)^{3/2} - r^2\right]_0^2 d\theta$

$= \int_0^{2\pi} \left[-\frac{1}{3}(4)^{3/2} - 4 + \frac{1}{3}(8)^{3/2}\right] d\theta = \int_0^{2\pi} \frac{4}{3}\left(-2 - 3 + 2\sqrt{8}\right) d\theta = \frac{4}{3}\left(4\sqrt{2} - 5\right) \int_0^{2\pi} d\theta = \frac{8\pi\left(4\sqrt{2}-5\right)}{3}$

(b) $V = \int_0^{2\pi} \int_0^{\pi/4} \int_{2\sec\phi}^{\sqrt{8}} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{8}{3} \int_0^{2\pi} \int_0^{\pi/4} \left(2\sqrt{2}\sin\phi - \sec^3\phi \sin\phi\right) d\phi\, d\theta$

$= \frac{8}{3} \int_0^{2\pi} \int_0^{\pi/4} \left(2\sqrt{2}\sin\phi - \tan\phi \sec^2\phi\right) d\phi\, d\theta = \frac{8}{3} \int_0^{2\pi} \left[-2\sqrt{2}\cos\phi - \frac{1}{2}\tan^2\phi\right]_0^{\pi/4} d\theta$

$= \frac{8}{3} \int_0^{2\pi} \left(-2 - \frac{1}{2} + 2\sqrt{2}\right) d\theta = \frac{8}{3} \int_0^{2\pi} \left(\frac{-5+4\sqrt{2}}{2}\right) d\theta = \frac{8\pi\left(4\sqrt{2}-5\right)}{3}$

38. $I_z = \int_0^{2\pi} \int_0^{\pi/3} \int_0^2 (\rho\sin\phi)^2 (\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \int_0^2 \rho^4 \sin^3\phi\, d\rho\, d\phi\, d\theta$

$= \frac{32}{5} \int_0^{2\pi} \int_0^{\pi/3} (\sin\phi - \cos^2\phi \sin\phi)\, d\phi\, d\theta = \frac{32}{5} \int_0^{2\pi} \left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi/3} d\theta = \frac{8\pi}{3}$

39. With the centers of the spheres at the origin, $I_z = \int_0^{2\pi} \int_0^{\pi} \int_a^b \delta(\rho\sin\phi)^2 (\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$

$= \frac{\delta(b^5 - a^5)}{5} \int_0^{2\pi} \int_0^{\pi} \sin^3\phi\, d\phi\, d\theta = \frac{\delta(b^5 - a^5)}{5} \int_0^{2\pi} \int_0^{\pi} (\sin\phi - \cos^2\phi\sin\phi)\, d\phi\, d\theta$

$= \frac{\delta(b^5 - a^5)}{5} \int_0^{2\pi} \left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi} d\theta = \frac{4\delta(b^5 - a^5)}{15} \int_0^{2\pi} d\theta = \frac{8\pi\delta(b^5 - a^5)}{15}$

40. $I_z = \int_0^{2\pi} \int_0^{\pi} \int_0^{1-\cos\theta} (\rho\sin\phi)^2 (\rho^2\sin\phi)\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi} \int_0^{1-\cos\theta} \rho^4 \sin^3\phi\, d\rho\, d\phi\, d\theta$

$= \frac{1}{5} \int_0^{2\pi} \int_0^{\pi} (1-\cos\phi)^5 \sin^3\phi\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi} (1-\cos\phi)^6 (1+\cos\phi) \sin\phi\, d\phi\, d\theta;$

$\begin{bmatrix} u = 1 - \cos\phi \\ du = \sin\phi\, d\phi \end{bmatrix} \to \frac{1}{5} \int_0^{2\pi} \int_0^2 u^6(2-u)\, du\, d\theta = \frac{1}{5} \int_0^{2\pi} \left[\frac{2u^7}{7} - \frac{u^8}{8}\right]_0^2 d\theta = \frac{1}{5} \int_0^{2\pi} \left(\frac{1}{7} - \frac{1}{8}\right) 2^8\, d\theta$

$= \frac{1}{5} \int_0^{2\pi} \frac{2^3 \cdot 2^5}{56}\, d\theta = \frac{32}{35} \int_0^{2\pi} d\theta = \frac{64\pi}{35}$

41. $M = \int_1^2 \int_{2/x}^2 dy\, dx = \int_1^2 \left(2 - \frac{2}{x}\right) dx = 2 - \ln 4$; $M_y = \int_1^2 \int_{2/x}^2 x\, dy\, dx = \int_1^2 x\left(2 - \frac{2}{x}\right) dx = 1$;

$M_x = \int_1^2 \int_{2/x}^2 y\, dy\, dx = \int_1^2 \left(2 - \frac{2}{x^2}\right) dx = 1 \Rightarrow \bar{x} = \bar{y} = \frac{1}{2 - \ln 4}$

42. $M = \int_0^4 \int_{-2y}^{2y-y^2} dx\, dy = \int_0^4 (4y - y^2)\, dy = \frac{32}{3}$; $M_x = \int_0^4 \int_{-2y}^{2y-y^2} y\, dx\, dy = \int_0^4 (4y^2 - y^3)\, dy = \left[\frac{4y^3}{3} - \frac{y^4}{4}\right]_0^4 = \frac{64}{3}$;

$M_y = \int_0^4 \int_{-2y}^{2y-y^2} x\, dx\, dy = \int_0^4 \left[\frac{(2y-y^2)^2}{2} - 2y^2\right] dy = \left[\frac{y^5}{10} - \frac{y^4}{2}\right]_0^4 = -\frac{128}{5} \Rightarrow \bar{x} = \frac{M_y}{M} = -\frac{12}{5}$ and $\bar{y} = \frac{M_x}{M} = 2$

43. $I_o = \int_0^2 \int_{2x}^4 (x^2 + y^2)(3)\, dy\, dx = 3 \int_0^2 \left(4x^2 + \frac{64}{3} - \frac{14x^3}{3}\right) dx = 104$

44. (a) $I_o = \int_{-2}^2 \int_{-1}^1 (x^2 + y^2)\, dy\, dx = \int_{-2}^2 \left(2x^2 + \frac{2}{3}\right) dx = \frac{40}{3}$

(b) $I_x = \int_{-a}^{a}\int_{-b}^{b} y^2 \, dy\, dx = \int_{-a}^{a} \frac{2b^3}{3}\, dx = \frac{4ab^3}{3}$; $I_y = \int_{-b}^{b}\int_{-a}^{a} x^2 \, dx\, dy = \int_{-b}^{b} \frac{2a^3}{3}\, dy = \frac{4a^3b}{3} \Rightarrow I_o = I_x + I_y$
$= \frac{4ab^3}{3} + \frac{4a^3b}{3} = \frac{4ab\,(b^2+a^2)}{3}$

45. $M = \delta \int_0^3\int_0^{2x/3} dy\, dx = \delta \int_0^3 \frac{2x}{3}\, dx = 3\delta$; $I_x = \delta \int_0^3\int_0^{2x/3} y^2\, dy\, dx = \frac{8\delta}{81}\int_0^3 x^3\, dx = \left(\frac{8\delta}{81}\right)\left(\frac{3^4}{4}\right) = 2\delta$

46. $M = \int_0^1\int_{x^2}^{x} (x+1)\, dy\, dx = \int_0^1 (x - x^3)\, dx = \frac{1}{4}$; $M_x = \int_0^1\int_{x^2}^{x} y(x+1)\, dy\, dx = \frac{1}{2}\int_0^1 (x^3 - x^5 + x^2 - x^4)\, dx = \frac{13}{120}$;
$M_y = \int_0^1\int_{x^2}^{x} x(x+1)\, dy\, dx = \int_0^1 (x^2 - x^4)\, dx = \frac{2}{15} \Rightarrow \overline{x} = \frac{8}{15}$ and $\overline{y} = \frac{13}{30}$; $I_x = \int_0^1\int_{x^2}^{x} y^2(x+1)\, dy\, dx$
$= \frac{1}{3}\int_0^1 (x^4 - x^7 + x^3 - x^6)\, dx = \frac{17}{280} \Rightarrow R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{17}{70}}$; $I_y = \int_0^1\int_{x^2}^{x} x^2(x+1)\, dy\, dx = \int_0^1 (x^3 - x^5)\, dx = \frac{1}{12}$

47. $M = \int_{-1}^{1}\int_{-1}^{1} \left(x^2 + y^2 + \frac{1}{3}\right) dy\, dx = \int_{-1}^{1} \left(2x^2 + \frac{4}{3}\right) dx = 4$; $M_x = \int_{-1}^{1}\int_{-1}^{1} y\left(x^2 + y^2 + \frac{1}{3}\right) dy\, dx = \int_{-1}^{1} 0\, dx = 0$;
$M_y = \int_{-1}^{1}\int_{-1}^{1} x\left(x^2 + y^2 + \frac{1}{3}\right) dy\, dx = \int_{-1}^{1} \left(2x^3 + \frac{4}{3}x\right) dx = 0$

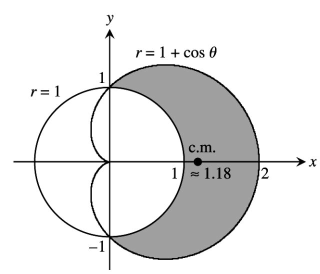
48. Place the ΔABC with its vertices at $A(0,0)$, $B(b,0)$ and $C(a,h)$. The line through the points A and C is $y = \frac{h}{a}x$; the line through the points C and B is $y = \frac{h}{a-b}(x-b)$. Thus, $M = \int_0^h \int_{ay/h}^{(a-b)y/h+b} \delta\, dx\, dy$
$= b\delta \int_0^h \left(1 - \frac{y}{h}\right) dy = \frac{\delta bh}{2}$; $I_x = \int_0^h \int_{ay/h}^{(a-b)y/h+b} y^2\delta\, dx\, dy = b\delta \int_0^h \left(y^2 - \frac{y^3}{h}\right) dy = \frac{\delta bh^3}{12}$

49. $M = \int_{-\pi/3}^{\pi/3}\int_0^3 r\, dr\, d\theta = \frac{9}{2}\int_{-\pi/3}^{\pi/3} d\theta = 3\pi$; $M_y = \int_{-\pi/3}^{\pi/3}\int_0^3 r^2\cos\theta\, dr\, d\theta = 9\int_{-\pi/3}^{\pi/3} \cos\theta\, d\theta = 9\sqrt{3} \Rightarrow \overline{x} = \frac{3\sqrt{3}}{\pi}$,
and $\overline{y} = 0$ by symmetry

50. $M = \int_0^{\pi/2}\int_1^3 r\, dr\, d\theta = 4\int_0^{\pi/2} d\theta = 2\pi$; $M_y = \int_0^{\pi/2}\int_1^3 r^2\cos\theta\, dr\, d\theta = \frac{26}{3}\int_0^{\pi/2}\cos\theta\, d\theta = \frac{26}{3} \Rightarrow \overline{x} = \frac{13}{3\pi}$, and
$\overline{y} = \frac{13}{3\pi}$ by symmetry

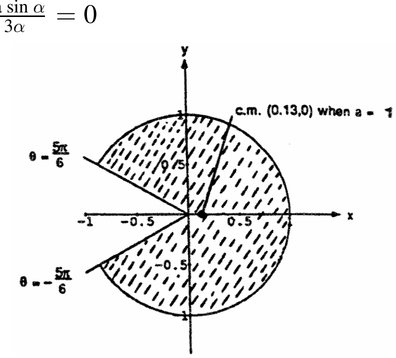
51. (a) $M = 2\int_0^{\pi/2}\int_1^{1+\cos\theta} r\, dr\, d\theta$
$= \int_0^{\pi/2} \left(2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta = \frac{8+\pi}{4}$;
$M_y = \int_{-\pi/2}^{\pi/2}\int_1^{1+\cos\theta} (r\cos\theta)\, r\, dr\, d\theta$
$= \int_{-\pi/2}^{\pi/2} \left(\cos^2\theta + \cos^3\theta + \frac{\cos^4\theta}{3}\right) d\theta$
$= \frac{32+15\pi}{24} \Rightarrow \overline{x} = \frac{15\pi+32}{6\pi+48}$, and
$\overline{y} = 0$ by symmetry

(b)



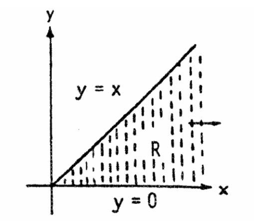
52. (a) $M = \int_{-\alpha}^{\alpha}\int_0^a r\, dr\, d\theta = \int_{-\alpha}^{\alpha} \frac{a^2}{2}\, d\theta = a^2\alpha$; $M_y = \int_{-\alpha}^{\alpha}\int_0^a (r\cos\theta)\, r\, dr\, d\theta = \int_{-\alpha}^{\alpha} \frac{a^3\cos\theta}{3}\, d\theta = \frac{2a^3\sin\alpha}{3}$
$\Rightarrow \overline{x} = \frac{2a\sin\alpha}{3\alpha}$, and $\overline{y} = 0$ by symmetry; $\lim_{\alpha\to\pi^-} \overline{x} = \lim_{\alpha\to\pi^-} \frac{2a\sin\alpha}{3\alpha} = 0$

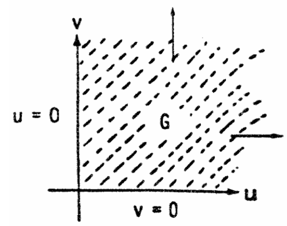
(b) $\overline{x} = \frac{2a}{5\pi}$ and $\overline{y} = 0$



53. $x = u + y$ and $y = v \Rightarrow x = u + v$ and $y = v$

$\Rightarrow J(u, v) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$; the boundary of the image G is obtained from the boundary of R as follows:





xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$y = x$	$v = u + v$	$u = 0$
$y = 0$	$v = 0$	$v = 0$

$$\Rightarrow \int_0^{\infty}\int_0^{x} e^{-sx} f(x - y, y)\, dy\, dx = \int_0^{\infty}\int_0^{\infty} e^{-s(u+v)} f(u, v)\, du\, dv$$

54. If $s = \alpha x + \beta y$ and $t = \gamma x + \delta y$ where $(\alpha\delta - \beta\gamma)^2 = ac - b^2$, then $x = \frac{\delta s - \beta t}{\alpha\delta - \beta\gamma}$, $y = \frac{-\gamma s + \alpha t}{\alpha\delta - \beta\gamma}$,

and $J(s, t) = \frac{1}{(\alpha\delta - \beta\gamma)^2}\begin{vmatrix} \delta & -\beta \\ -\gamma & \alpha \end{vmatrix} = \frac{1}{\alpha\delta - \beta\gamma} \Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(s^2+t^2)} \frac{1}{\sqrt{ac - b^2}}\, ds\, dt$

$= \frac{1}{\sqrt{ac - b^2}}\int_0^{2\pi}\int_0^{\infty} re^{-r^2}\, dr\, d\theta = \frac{1}{2\sqrt{ac - b^2}}\int_0^{2\pi} d\theta = \frac{\pi}{\sqrt{ac - b^2}}$. Therefore, $\frac{\pi}{\sqrt{ac - b^2}} = 1 \Rightarrow ac - b^2 = \pi^2$.

CHAPTER 15 ADDITIONAL AND ADVANCED EXERCISES

1. (a) $V = \int_{-3}^{2}\int_{x}^{6-x^2} x^2\, dy\, dx$ (b) $V = \int_{-3}^{2}\int_{x}^{6-x^2}\int_0^{x^2} dz\, dy\, dx$

(c) $V = \int_{-3}^{2}\int_{x}^{6-x^2} x^2\, dy\, dx = \int_{-3}^{2}\int_{x}^{6-x^2} (6x^2 - x^4 - x^3)\, dx = \left[2x^3 - \frac{x^5}{5} - \frac{x^4}{4}\right]_{-3}^{2} = \frac{125}{4}$

2. Place the sphere's center at the origin with the surface of the water at $z = -3$. Then $9 = 25 - x^2 - y^2 \Rightarrow x^2 + y^2 = 16$ is the projection of the volume of water onto the xy-plane

$\Rightarrow V = \int_0^{2\pi}\int_0^{4}\int_{-\sqrt{25-r^2}}^{-3} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^{4}\left(r\sqrt{25 - r^2} - 3r\right) dr\, d\theta = \int_0^{2\pi}\left[-\frac{1}{3}(25 - r^2)^{3/2} - \frac{3}{2}r^2\right]_0^4 d\theta$

$= \int_0^{2\pi}\left[-\frac{1}{3}(9)^{3/2} - 24 + \frac{1}{3}(25)^{3/2}\right] d\theta = \int_0^{2\pi}\frac{26}{3}\, d\theta = \frac{52\pi}{3}$

3. Using cylindrical coordinates, $V = \int_0^{2\pi}\int_0^{1}\int_0^{2-r(\cos\theta + \sin\theta)} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^{1}(2r - r^2\cos\theta - r^2\sin\theta)\, dr\, d\theta$

$= \int_0^{2\pi}\left(1 - \frac{1}{3}\cos\theta - \frac{1}{3}\sin\theta\right) d\theta = \left[\theta - \frac{1}{3}\sin\theta + \frac{1}{3}\cos\theta\right]_0^{2\pi} = 2\pi$

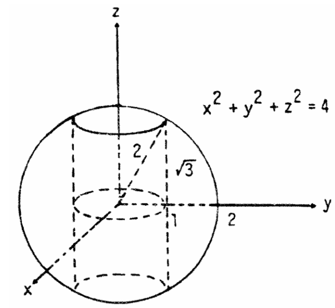
4. $V = 4\int_0^{\pi/2}\int_0^{1}\int_{r^2}^{\sqrt{2-r^2}} dz\, r\, dr\, d\theta = 4\int_0^{\pi/2}\int_0^{1}\left(r\sqrt{2 - r^2} - r^3\right) dr\, d\theta = 4\int_0^{\pi/2}\left[-\frac{1}{3}(2 - r^2)^{3/2} - \frac{r^4}{4}\right]_0^1 d\theta$

$= 4\int_0^{\pi/2}\left(-\frac{1}{3} - \frac{1}{4} + \frac{2\sqrt{2}}{3}\right) d\theta = \left(\frac{8\sqrt{2}-7}{3}\right)\int_0^{\pi/2} d\theta = \frac{\pi\left(8\sqrt{2} - 7\right)}{6}$

5. The surfaces intersect when $3 - x^2 - y^2 = 2x^2 + 2y^2 \Rightarrow x^2 + y^2 = 1$. Thus the volume is
$$V = 4\int_0^1\int_0^{\sqrt{1-x^2}}\int_{2x^2+2y^2}^{3-x^2-y^2} dz\,dy\,dx = 4\int_0^{\pi/2}\int_0^1\int_{2r^2}^{3-r^2} dz\,r\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^1 (3r - 3r^3)\,dr\,d\theta = 3\int_0^{\pi/2} d\theta = \tfrac{3\pi}{2}$$

6. $$V = 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2\sin\phi} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \tfrac{64}{3}\int_0^{\pi/2}\int_0^{\pi/2} \sin^4\phi\,d\phi\,d\theta$$
$$= \tfrac{64}{3}\int_0^{\pi/2}\left[-\frac{\sin^3\phi\cos\phi}{4}\Big|_0^{\pi/2} + \tfrac{3}{4}\int_0^{\pi/2}\sin^2\phi\,d\phi\right]d\theta = 16\int_0^{\pi/2}\left[\tfrac{\phi}{2} - \tfrac{\sin 2\phi}{4}\right]_0^{\pi/2} d\theta = 4\pi\int_0^{\pi/2} d\theta = 2\pi^2$$

7. (a) The radius of the hole is 1, and the radius of the sphere is 2.



(b) $$V = 2\int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-z^2}} r\,dr\,dz\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} (3 - z^2)\,dz\,d\theta = 2\sqrt{3}\int_0^{2\pi} d\theta = 4\sqrt{3}\pi$$

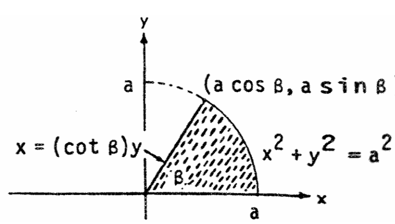
8. $$V = \int_0^{\pi}\int_0^{3\sin\theta}\int_0^{\sqrt{9-r^2}} dz\,r\,dr\,d\theta = \int_0^{\pi}\int_0^{3\sin\theta} r\sqrt{9-r^2}\,dr\,d\theta = \int_0^{\pi}\left[-\tfrac{1}{3}(9-r^2)^{3/2}\right]_0^{3\sin\theta} d\theta$$
$$= \int_0^{\pi}\left[-\tfrac{1}{3}(9 - 9\sin^2\theta)^{3/2} + \tfrac{1}{3}(9)^{3/2}\right]d\theta = 9\int_0^{\pi}\left[1 - (1-\sin^2\theta)^{3/2}\right]d\theta = 9\int_0^{\pi}(1 - \cos^3\theta)\,d\theta$$
$$= \int_0^{\pi}(1 - \cos\theta + \sin^2\theta\cos\theta)\,d\theta = 9\left[\theta - \sin\theta + \tfrac{\sin^3\theta}{3}\right]_0^{\pi} = 9\pi$$

9. The surfaces intersect when $x^2 + y^2 = \frac{x^2+y^2+1}{2} \Rightarrow x^2 + y^2 = 1$. Thus the volume in cylindrical coordinates is
$$V = 4\int_0^{\pi/2}\int_0^1\int_{r^2}^{(r^2+1)/2} dz\,r\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^1\left(\tfrac{r}{2} - \tfrac{r^3}{2}\right)dr\,d\theta = 4\int_0^{\pi/2}\left[\tfrac{r^2}{4} - \tfrac{r^4}{8}\right]_0^1 d\theta = \tfrac{1}{2}\int_0^{\pi/2} d\theta = \tfrac{\pi}{4}$$

10. $$V = \int_0^{\pi/2}\int_1^2\int_0^{r^2\sin\theta\cos\theta} dz\,r\,dr\,d\theta = \int_0^{\pi/2}\int_1^2 r^3\sin\theta\cos\theta\,dr\,d\theta = \int_0^{\pi/2}\left[\tfrac{r^4}{4}\right]_1^2 \sin\theta\cos\theta\,d\theta$$
$$= \tfrac{15}{4}\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = \tfrac{15}{4}\left[\tfrac{\sin^2\theta}{2}\right]_0^{\pi/2} = \tfrac{15}{8}$$

11. $$\int_0^{\infty}\tfrac{e^{-ax}-e^{-bx}}{x}\,dx = \int_0^{\infty}\int_a^b e^{-xy}\,dy\,dx = \int_a^b\int_0^{\infty} e^{-xy}\,dx\,dy = \int_a^b\left(\lim_{t\to\infty}\int_0^t e^{-xy}\,dx\right)dy$$
$$= \int_a^b \lim_{t\to\infty}\left[-\tfrac{e^{-xy}}{y}\right]_0^t dy = \int_a^b \lim_{t\to\infty}\left(\tfrac{1}{y} - \tfrac{e^{-yt}}{y}\right)dy = \int_a^b \tfrac{1}{y}\,dy = [\ln y]_a^b = \ln\left(\tfrac{b}{a}\right)$$

12. (a) The region of integration is sketched at the right



$$\Rightarrow \int_0^{a\sin\beta}\int_{y\cot\beta}^{\sqrt{a^2-y^2}} \ln(x^2+y^2)\,dx\,dy$$
$$= \int_0^{\beta}\int_0^{a} r\ln(r^2)\,dr\,d\theta;$$
$$\begin{bmatrix} u = r^2 \\ du = 2r\,dr \end{bmatrix} \to \tfrac{1}{2}\int_0^{\beta}\int_0^{a^2}\ln u\,du\,d\theta$$
$$= \tfrac{1}{2}\int_0^{\beta}[u\ln u - u]_0^{a^2}\,d\theta$$
$$= \tfrac{1}{2}\int_0^{\beta}\left[2a^2\ln a - a^2 - \lim_{t\to 0} t\ln t\right]d\theta = \tfrac{a^2}{2}\int_0^{\beta}(2\ln a - 1)\,d\theta = a^2\beta\left(\ln a - \tfrac{1}{2}\right)$$

(b) $$\int_0^{a\cos\beta}\int_0^{(\tan\beta)x}\ln(x^2+y^2)\,dy\,dx + \int_{a\cos\beta}^{a}\int_0^{\sqrt{a^2-x^2}}\ln(x^2+y^2)\,dy\,dx$$

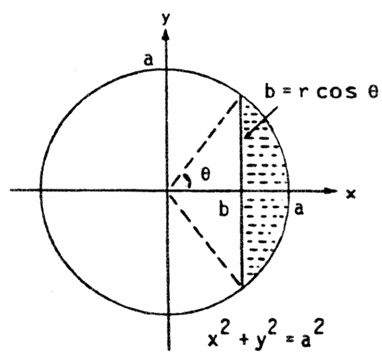
13. $\int_0^x \int_0^u e^{m(x-t)} f(t)\,dt\,du = \int_0^x \int_t^x e^{m(x-t)} f(t)\,du\,dt = \int_0^x (x-t)e^{m(x-t)} f(t)\,dt$; also
$\int_0^x \int_0^v \int_0^u e^{m(x-t)} f(t)\,dt\,du\,dv = \int_0^x \int_t^x \int_t^v e^{m(x-t)} f(t)\,du\,dv\,dt = \int_0^x \int_t^x (v-t)e^{m(x-t)} f(t)\,dv\,dt$
$= \int_0^x \left[\frac{1}{2}(v-t)^2 e^{m(x-t)} f(t)\right]_t^x dt = \int_0^x \frac{(x-t)^2}{2} e^{m(x-t)} f(t)\,dt$

14. $\int_0^1 f(x)\left(\int_0^x g(x-y)f(y)\,dy\right) dx = \int_0^1 \int_0^x g(x-y)f(x)f(y)\,dy\,dx$
$= \int_0^1 \int_y^1 g(x-y)f(x)f(y)\,dx\,dy = \int_0^1 f(y)\left(\int_y^1 g(x-y)f(x)\,dx\right) dy$;
$\int_0^1 \int_0^1 g(|x-y|)\,f(x)f(y)\,dx\,dy = \int_0^1 \int_0^x g(x-y)f(x)f(y)\,dy\,dx + \int_0^1 \int_x^1 g(y-x)f(x)f(y)\,dy\,dx$
$= \int_0^1 \int_y^1 g(x-y)f(x)f(y)\,dx\,dy + \int_0^1 \int_x^1 g(y-x)f(x)f(y)\,dy\,dx$
$= \int_0^1 \int_y^1 g(x-y)f(x)f(y)\,dx\,dy + \underbrace{\int_0^1 \int_y^1 g(x-y)f(y)f(x)\,dx\,dy}_{\text{simply interchange x and y variable names}}$
$= 2\int_0^1 \int_y^1 g(x-y)f(x)f(y)\,dx\,dy$, and the statement now follows.

15. $I_o(a) = \int_0^a \int_0^{x/a^2} (x^2+y^2)\,dy\,dx = \int_0^a \left[x^2y + \frac{y^3}{3}\right]_0^{x/a^2} dx = \int_0^a \left(\frac{x^3}{a^2} + \frac{x^3}{3a^6}\right) dx = \left[\frac{x^4}{4a^2} + \frac{x^4}{12a^6}\right]_0^a$
$= \frac{a^2}{4} + \frac{1}{12}a^{-2}$; $I_o'(a) = \frac{1}{2}a - \frac{1}{6}a^{-3} = 0 \Rightarrow a^4 = \frac{1}{3} \Rightarrow a = \sqrt[4]{\frac{1}{3}} = \frac{1}{\sqrt[4]{3}}$. Since $I_o''(a) = \frac{1}{2} + \frac{1}{2}a^{-4} > 0$, the value of a does provide a <u>minimum</u> for the polar moment of inertia $I_o(a)$.

16. $I_o = \int_0^2 \int_{2x}^4 (x^2+y^2)(3)\,dy\,dx = 3\int_0^2 \left(4x^2 - \frac{14x^3}{3} + \frac{64}{3}\right) dx = 104$

17. $M = \int_{-\theta}^{\theta} \int_{b\sec\theta}^{a} r\,dr\,d\theta = \int_{-\theta}^{\theta} \left(\frac{a^2}{2} - \frac{b^2}{2}\sec^2\theta\right) d\theta$
$= a^2\theta - b^2\tan\theta = a^2\cos^{-1}\left(\frac{b}{a}\right) - b^2\left(\frac{\sqrt{a^2-b^2}}{b}\right)$
$= a^2\cos^{-1}\left(\frac{b}{a}\right) - b\sqrt{a^2-b^2}$; $I_o = \int_{-\theta}^{\theta} \int_{b\sec\theta}^{a} r^3\,dr\,d\theta$
$= \frac{1}{4}\int_{-\theta}^{\theta} (a^4 + b^4\sec^4\theta)\,d\theta$
$= \frac{1}{4}\int_{-\theta}^{\theta} [a^4 + b^4(1+\tan^2\theta)(\sec^2\theta)]\,d\theta$
$= \frac{1}{4}\left[a^4\theta - b^4\tan\theta - \frac{b^4\tan^3\theta}{3}\right]_{-\theta}^{\theta}$
$= \frac{a^4\theta}{2} - \frac{b^4\tan\theta}{2} - \frac{b^4\tan^3\theta}{6}$
$= \frac{1}{2}a^4\cos^{-1}\left(\frac{b}{a}\right) - \frac{1}{2}b^3\sqrt{a^2-b^2} - \frac{1}{6}b^3(a^2-b^2)^{3/2}$



18. $M = \int_{-2}^{2} \int_{1-(y^2/4)}^{2-(y^2/2)} dx\,dy = \int_{-2}^{2} \left(1 - \frac{y^2}{4}\right) dy = \left[y - \frac{y^3}{12}\right]_{-2}^{2} = \frac{8}{3}$; $M_y = \int_{-2}^{2} \int_{1-(y^2/4)}^{2-(y^2/2)} x\,dx\,dy$
$= \int_{-2}^{2} \left[\frac{x^2}{2}\right]_{1-(y^2/4)}^{2-(y^2/2)} dy = \int_{-2}^{2} \frac{3}{32}(4-y^2)\,dy = \frac{3}{32}\int_{-2}^{2} (16 - 8y^2 + y^4)\,dy = \frac{3}{16}\left[16y - \frac{8y^3}{3} + \frac{y^5}{5}\right]_0^2$
$= \frac{3}{16}\left(32 - \frac{64}{3} + \frac{32}{5}\right) = \left(\frac{3}{16}\right)\left(\frac{32\cdot 8}{15}\right) = \frac{48}{15} \Rightarrow \overline{x} = \frac{M_y}{M} = \left(\frac{48}{15}\right)\left(\frac{3}{8}\right) = \frac{6}{5}$, and $\overline{y} = 0$ by symmetry

19. $\int_0^a \int_0^b e^{\max(b^2x^2, a^2y^2)}\,dy\,dx = \int_0^a \int_0^{bx/a} e^{b^2x^2}\,dy\,dx + \int_0^b \int_0^{ay/b} e^{a^2y^2}\,dx\,dy$

$= \int_0^a \left(\frac{b}{a}x\right) e^{b^2x^2}\,dx + \int_0^b \left(\frac{a}{b}y\right) e^{a^2y^2}\,dy = \left[\frac{1}{2ab} e^{b^2x^2}\right]_0^a + \left[\frac{1}{2ba} e^{a^2y^2}\right]_0^b = \frac{1}{2ab}\left(e^{b^2a^2} - 1\right) + \frac{1}{2ab}\left(e^{a^2b^2} - 1\right)$
$= \frac{1}{ab}\left(e^{a^2b^2} - 1\right)$

20. $\int_{y_0}^{y_1}\int_{x_0}^{x_1} \frac{\partial^2 F(x,y)}{\partial x\,\partial y}\,dx\,dy = \int_{y_0}^{y_1}\left[\frac{\partial F(x,y)}{\partial y}\right]_{x_0}^{x_1} dy = \int_{y_0}^{y_1}\left[\frac{\partial F(x_1,y)}{\partial y} - \frac{\partial F(x_0,y)}{\partial y}\right] dx = \left[F(x_1,y) - F(x_0,y)\right]_{y_0}^{y_1}$
$= F(x_1,y_1) - F(x_0,y_1) - F(x_1,y_0) + F(x_0,y_0)$

21. (a) (i) Fubini's Theorem
(ii) Treating G(y) as a constant
(iii) Algebraic rearrangement
(iv) The definite integral is a constant number

(b) $\int_0^{\ln 2}\int_0^{\pi/2} e^x \cos y\,dy\,dx = \left(\int_0^{\ln 2} e^x\,dx\right)\left(\int_0^{\pi/2} \cos y\,dy\right) = \left(e^{\ln 2} - e^0\right)\left(\sin\frac{\pi}{2} - \sin 0\right) = (1)(1) = 1$

(c) $\int_1^2\int_{-1}^1 \frac{x}{y^2}\,dx\,dy = \left(\int_1^2 \frac{1}{y^2}\,dy\right)\left(\int_{-1}^1 x\,dx\right) = \left[-\frac{1}{y}\right]_1^2 \left[\frac{x^2}{2}\right]_{-1}^1 = \left(-\frac{1}{2} + 1\right)\left(\frac{1}{2} - \frac{1}{2}\right) = 0$

22. (a) $\nabla f = x\mathbf{i} + y\mathbf{j} \Rightarrow D_u f = u_1x + u_2y$; the area of the region of integration is $\frac{1}{2}$

$\Rightarrow$ average $= 2\int_0^1\int_0^{1-x}(u_1x + u_2y)\,dy\,dx = 2\int_0^1\left[u_1x(1-x) + \frac{1}{2}u_2(1-x)^2\right] dx$

$= 2\left[u_1\left(\frac{x^2}{2} - \frac{x^3}{3}\right) - \left(\frac{1}{2}u_2\right)\frac{(1-x)^3}{3}\right]_0^1 = 2\left(\frac{1}{6}u_1 + \frac{1}{6}u_2\right) = \frac{1}{3}(u_1 + u_2)$

(b) average $= \frac{1}{\text{area}}\iint_R (u_1x + u_2y)\,dA = \frac{u_1}{\text{area}}\iint_R x\,dA + \frac{u_2}{\text{area}}\iint_R y\,dA = u_1\left(\frac{M_y}{M}\right) + u_2\left(\frac{M_x}{M}\right) = u_1\bar{x} + u_2\bar{y}$

23. (a) $I^2 = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^\infty \left(e^{-r^2}\right) r\,dr\,d\theta = \int_0^{\pi/2}\left[\lim_{b\to\infty}\int_0^b re^{-r^2}\,dr\right] d\theta$

$= -\frac{1}{2}\int_0^{\pi/2}\lim_{b\to\infty}\left(e^{-b^2} - 1\right) d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2}$

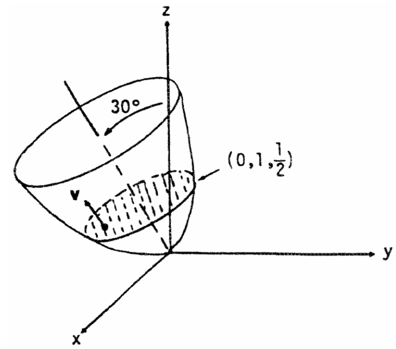
(b) $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2}e^{-t}\,dt = \int_0^\infty \left(y^2\right)^{-1/2} e^{-y^2}(2y)\,dy = 2\int_0^\infty e^{-y^2}\,dy = 2\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}$, where $y = \sqrt{t}$

24. $Q = \int_0^{2\pi}\int_0^R kr^2(1 - \sin\theta)\,dr\,d\theta = \frac{kR^3}{3}\int_0^{2\pi}(1 - \sin\theta)\,d\theta = \frac{kR^3}{3}\left[\theta + \cos\theta\right]_0^{2\pi} = \frac{2\pi kR^3}{3}$

25. For a height h in the bowl the volume of water is $V = \int_{-\sqrt{h}}^{\sqrt{h}}\int_{-\sqrt{h-x^2}}^{\sqrt{h-x^2}}\int_{x^2+y^2}^{h} dz\,dy\,dx$

$= \int_{-\sqrt{h}}^{\sqrt{h}}\int_{-\sqrt{h-x^2}}^{\sqrt{h-x^2}}\left(h - x^2 - y^2\right) dy\,dx = \int_0^{2\pi}\int_0^{\sqrt{h}}\left(h - r^2\right) r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{hr^2}{2} - \frac{r^4}{4}\right]_0^{\sqrt{h}} d\theta = \int_0^{2\pi}\frac{h^2}{4}\,d\theta = \frac{h^2\pi}{2}$.

Since the top of the bowl has area 10π, then we calibrate the bowl by comparing it to a right circular cylinder whose cross sectional area is 10π from $z = 0$ to $z = 10$. If such a cylinder contains $\frac{h^2\pi}{2}$ cubic inches of water to a depth w then we have $10\pi w = \frac{h^2\pi}{2} \Rightarrow w = \frac{h^2}{20}$. So for 1 inch of rain, $w = 1$ and $h = \sqrt{20}$; for 3 inches of rain, $w = 3$ and $h = \sqrt{60}$.



26. (a) An equation for the satellite dish in standard position is $z = \frac{1}{2}x^2 + \frac{1}{2}y^2$. Since the axis is tilted 30°, a unit vector $\mathbf{v} = 0\mathbf{i} + a\mathbf{j} + b\mathbf{k}$ normal to the plane of the water level satisfies $b = \mathbf{v}\cdot\mathbf{k} = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\Rightarrow\ a = -\sqrt{1-b^2} = -\frac{1}{2} \ \Rightarrow\ \mathbf{v} = -\frac{1}{2}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k}$

$\Rightarrow\ -\frac{1}{2}(y-1) + \frac{\sqrt{3}}{2}\left(z - \frac{1}{2}\right) = 0$

$\Rightarrow\ z = \frac{1}{\sqrt{3}}\,y + \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)$

is an equation of the plane of the water level. Therefore the volume of water is $V = \iint\limits_R \int_{\frac{1}{2}x^2+\frac{1}{2}y^2}^{\frac{1}{\sqrt{3}}y+\frac{1}{2}-\frac{1}{\sqrt{3}}} dz\,dy\,dx$, where R is the interior of the ellipse

$x^2 + y^2 - \frac{2}{\sqrt{3}}\,y - 1 + \frac{2}{\sqrt{3}} = 0$. When $x = 0$, then $y = \alpha$ or $y = \beta$, where $\alpha = \frac{\frac{2}{\sqrt{3}} + \sqrt{\frac{4}{3} - 4\left(\frac{2}{\sqrt{3}} - 1\right)}}{2}$

and $\beta = \frac{\frac{2}{\sqrt{3}} - \sqrt{\frac{4}{3} - 4\left(\frac{2}{\sqrt{3}} - 1\right)}}{2}$ $\ \Rightarrow\ V = \int_\alpha^\beta \int_{-\left(\frac{2}{3}y+1-\frac{2}{\sqrt{3}}-y^2\right)^{1/2}}^{\left(\frac{2}{3}y+1-\frac{2}{\sqrt{3}}-y^2\right)^{1/2}} \int_{\frac{1}{2}x^2+\frac{1}{2}y^2}^{\frac{1}{\sqrt{3}}y+\frac{1}{2}-\frac{1}{\sqrt{3}}} 1\,dz\,dx\,dy$

(b) $x = 0 \ \Rightarrow\ z = \frac{1}{2}y^2$ and $\frac{dz}{dy} = y$; $y = 1 \ \Rightarrow\ \frac{dz}{dy} = 1 \ \Rightarrow$ the tangent line has slope 1 or a 45° slant $\Rightarrow$ at 45° and thereafter, the dish will not hold water.

27. The cylinder is given by $x^2 + y^2 = 1$ from $z = 1$ to $\infty \ \Rightarrow\ \iiint\limits_D z\,(r^2+z^2)^{-5/2}\,dV$

$$= \int_0^{2\pi}\int_0^1\int_1^\infty \frac{z}{(r^2+z^2)^{5/2}}\,dz\,r\,dr\,d\theta = \lim_{a\to\infty}\int_0^{2\pi}\int_0^1\int_1^a \frac{rz}{(r^2+z^2)^{5/2}}\,dz\,dr\,d\theta$$

$$= \lim_{a\to\infty}\int_0^{2\pi}\int_0^1 \left[\left(-\frac{1}{3}\right)\frac{r}{(r^2+z^2)^{3/2}}\right]_1^a dr\,d\theta = \lim_{a\to\infty}\int_0^{2\pi}\int_0^1 \left[\left(-\frac{1}{3}\right)\frac{r}{(r^2+a^2)^{3/2}} + \left(\frac{1}{3}\right)\frac{r}{(r^2+1)^{3/2}}\right] dr\,d\theta$$

$$= \lim_{a\to\infty}\int_0^{2\pi}\left[\frac{1}{3}(r^2+a^2)^{-1/2} - \frac{1}{3}(r^2+1)^{-1/2}\right]_0^1 d\theta = \lim_{a\to\infty}\int_0^{2\pi}\left[\frac{1}{3}(1+a^2)^{-1/2} - \frac{1}{3}\left(2^{-1/2}\right) - \frac{1}{3}(a^2)^{-1/2} + \frac{1}{3}\right] d\theta$$

$$= \lim_{a\to\infty} 2\pi\left[\frac{1}{3}(1+a^2)^{-1/2} - \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{3}\left(\frac{1}{a}\right) + \frac{1}{3}\right] = 2\pi\left[\frac{1}{3} - \left(\frac{1}{3}\right)\frac{\sqrt{2}}{2}\right].$$

28. Let's see?

The length of the "unit" line segment is: $L = 2\int_0^1 dx = 2$.

The area of the unit circle is: $A = 4\int_0^1\int_0^{\sqrt{1-x^2}} dy\,dx = \pi$.

The volume of the unit sphere is: $V = 8\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} dz\,dy\,dx = \frac{4}{3}\pi$.

Therefore, the hypervolume of the unit 4-sphere should be:

$$V_{\text{hyper}} = 16\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\int_0^{\sqrt{1-x^2-y^2-z^2}} dw\,dz\,dy\,dx.$$

Mathematica is able to handle this integral, but we'll use the brute force approach.

$$V_{\text{hyper}} = 16\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\int_0^{\sqrt{1-x^2-y^2-z^2}} dw\,dz\,dy\,dx = 16\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \sqrt{1-x^2-y^2-z^2}\,dz\,dy\,dx$$

$$= 16\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \sqrt{1-x^2-y^2}\sqrt{1-\frac{z^2}{1-x^2-y^2}}\,dz\,dy\,dx = \begin{bmatrix} \frac{z}{\sqrt{1-x^2-y^2}} = \cos\theta \\ dz = -\sqrt{1-x^2-y^2}\,\sin\theta\,d\theta \end{bmatrix}$$

$$= 16\int_0^1\int_0^{\sqrt{1-x^2}}(1-x^2-y^2)\int_{\pi/2}^0 -\sqrt{1-\cos^2\theta}\,\sin\theta\,d\theta\,dy\,dx = 16\int_0^1\int_0^{\sqrt{1-x^2}}(1-x^2-y^2)\int_{\pi/2}^0 -\sin^2\theta\,d\theta\,dy\,dx$$

$$= 16\int_0^1\int_0^{\sqrt{1-x^2}}\frac{\pi}{4}(1-x^2-y^2)\,dy\,dx = 4\pi\int_0^1\left(\sqrt{1-x^2} - x^2\sqrt{1-x^2} - \frac{1}{3}(1-x^2)^{3/2}\right)dx$$

$$= 4\pi\int_0^1\sqrt{1-x^2}\left[(1-x^2) - \frac{1-x^3}{3}\right]dx = \frac{8}{3}\pi\int_0^1(1-x^2)^{3/2}\,dx = \begin{bmatrix} x = \cos\theta \\ dx = -\sin\theta\,d\theta \end{bmatrix} = -\frac{8}{3}\pi\int_{\pi/2}^0 \sin^4\theta\,d\theta$$

$$= -\frac{8}{3}\pi\int_{\pi/2}^0\left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta = -\frac{2}{3}\pi\int_{\pi/2}^0(1 - 2\cos 2\theta + \cos^2 2\theta)\,d\theta = -\frac{2}{3}\pi\int_{\pi/2}^0\left(\frac{3}{2} - 2\cos 2\theta + \frac{\cos 4\theta}{2}\right)d\theta = \frac{\pi^2}{2}$$

NOTES: