

# Lecture Two – Techniques of Integration

## Section 2.1 – Integration by Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)dx$$

**Example:**  $\int x \cos x dx$ ,  $\int x^2 e^x dx$ , and  $\int x \ln x dx$

### Integration by Parts Formula

$$\int f(x)g'(x) = f(x)g(x) - \int f'(x)g(x)dx$$

Let  $u$  and  $v$  be differentiable functions of  $x$ .

$$\int u dv = uv - \int v du$$

### Guidelines for integration by Parts

1. Let  $dv$  be the most complicated portion of the integrand that fits a basic integration formula. Let  $u$  be the remaining factor.
2. Let  $u$  be the portion of the integrand whose derivative is a function simpler than  $u$ . Let  $dv$  be the remaining factor.

### Example

Evaluate:  $\int x \cos x dx$

### Solution

$$u = x \quad dv = \cos x dx$$

Let:  $du = dx$   $v = \int dv = \int \cos x dx = \sin x$

$$\begin{aligned} \int x \cos x dx &= x \sin x - \int \sin x dx \\ &= x \sin x + \cos x + C \end{aligned}$$

$$\int u dv = uv - \int v du$$

### Example

Evaluate:  $\int \ln x \, dx$

### Solution

Let:

$$u = \ln x \quad dv = dx$$

$$du = \frac{1}{x} dx \quad v = \int dx = x$$

$$\begin{aligned} \int \ln x \, dx &= x \ln x - \int x \frac{1}{x} dx \\ &= x \ln x - \int dx \\ &= \underline{x \ln x - x + C} \end{aligned}$$

$$\int u \, dv = uv - \int v \, du$$

## Tabular Integration

### Example

Evaluate  $\int x^2 e^x \, dx$

### Solution

$$f(x) = x^2 \quad \text{and} \quad g(x) = e^x$$

$f(x)$ & derivatives			$\int g(x) = \int e^x$
$x^2$	(+)	$\rightarrow$	$e^x$
$2x$	(-)	$\rightarrow$	$e^x$
$2$	(+)	$\rightarrow$	$e^x$

It is called **tabular integration**

$$\int x^2 e^x \, dx = \underline{x^2 e^x - 2x e^x + 2e^x + C}$$

$$u = x^2 \quad dv = e^x dx$$

$$du = 2x dx \quad v = \int e^x dx = e^x$$

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx$$

$$u = x \quad dv = e^x dx$$

Let:  $du = dx \quad v = \int e^x dx = e^x$

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx$$

$$= x^2 e^x - 2 \left[ x e^x - \int e^x dx \right]$$

$$= x^2 e^x - 2(x e^x - e^x) + C$$

$$= \underline{x^2 e^x - 2x e^x + 2e^x + C}$$

### Example

Evaluate  $\int x^3 \sin x \, dx$

### Solution

$$\int x^3 \sin x \, dx = \underline{-x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C}$$

$\int \sin x$		
+	$x^3$	$-\cos x$
-	$3x^2$	$-\sin x$
+	$6x$	$\cos x$
-	$6$	$\sin x$

### Example

Evaluate  $\int e^x \cos x \, dx$

### Solution

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$2 \int e^x \cos x \, dx = e^x (\sin x + \cos x)$$

$$\int e^x \cos x \, dx = \underline{\frac{1}{2} e^x (\sin x + \cos x) + C}$$

		$\int \cos x \, dx$
+	$e^x$	$\sin x$
-	$e^x$	$-\cos x$
+	$e^x$	$-\int \cos x \, dx$



Let:  $u = e^x$   $dv = \cos x \, dx$   
 $du = e^x \, dx$   $v = \int \cos x \, dx = \sin x$

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

$$\begin{aligned} \int e^x \cos x \, dx &= e^x \sin x - \int e^x \sin x \, dx \\ &= e^x \sin x - \left[ -e^x \cos x - \int (-\cos x) e^x \, dx \right] \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx \end{aligned}$$

$$\int e^x \cos x \, dx + \int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx + \int e^x \cos x \, dx$$

Let:  $u = e^x$   $dv = \sin x \, dx$   
 $du = e^x \, dx$   $v = \int \sin x \, dx = -\cos x$

$$2 \int e^x \cos x dx = e^x \sin x + e^x \cos x + C_1$$

$$\int e^x \cos x dx = \underline{\frac{1}{2}e^x \sin x + \frac{1}{2}e^x \cos x + C}$$

### Example

Obtain a formula that expresses the integral  $\int \cos^n x dx$

### Solution

$$u = \cos^{n-1} x \quad dv = \cos x dx$$

$$\begin{aligned} \text{Let: } du &= (n-1) \cos^{n-2} x (-\sin x dx) \\ &= -(n-1) \cos^{n-2} x \sin x dx \end{aligned} \quad v = \int \cos x dx = \sin x$$

$$\int u dv = uv - \int v du$$

$$\int \cos^n x dx = \cos^{n-1} x \sin x - \int \sin x \left( -(n-1) \cos^{n-2} x \sin x dx \right)$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int (\cos^{n-2} x - \cos^n x) dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx - (n-1) \int \cos^n x dx$$

$$\int \cos^n x dx + (n-1) \int \cos^n x dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx$$

$$(1+n-1) \int \cos^n x dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx$$

$$n \int \cos^n x dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx$$

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

**Example:** 
$$\int \cos^3 x \, dx = \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \int \cos x \, dx$$

$$= \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \sin x + C$$

## Evaluating Definite Integrals by Parts

### Example

Find the area of the region bounded by the curve  $y = xe^{-x}$  and the  $x$ -axis from  $x = 0$  to  $x = 4$ .

### Solution

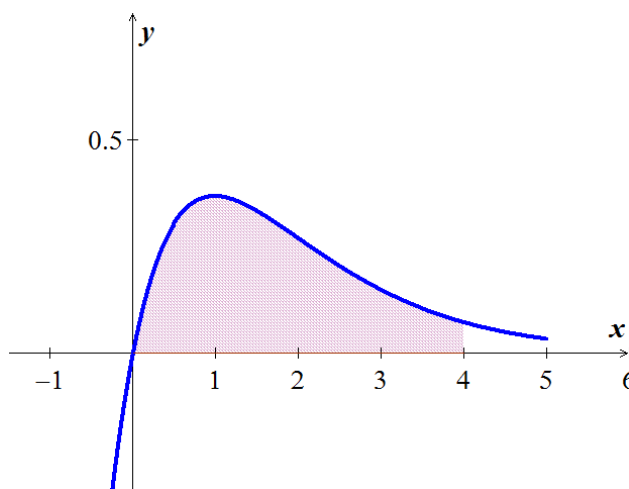
$$A = \int_0^4 x e^{-x} \, dx$$

		$\int e^{-x}$
+	$x$	$-e^{-x}$
-	1	$e^{-x}$

$$A = (-x-1)e^{-x} \Big|_0^4$$

$$= -5e^{-4} + 1$$

$$\approx 0.91 \text{ unit}^2$$



### 2<sup>nd</sup> Method

Let:  $u = x$   $dv = e^{-x} dx$

$$du = dx \quad v = \int e^{-x} dx = -e^{-x}$$

$$\int u dv = uv - \int v du$$

$$\int_0^4 x e^{-x} dx = -x e^{-x} \Big|_0^4 - \int_0^4 (-e^{-x}) dx$$

$$= -(4e^{-4} - 0) + \int_0^4 e^{-x} dx$$

$$\begin{aligned}
&= -4e^{-4} + \left( -e^{-x} \right) \Big|_0^4 \\
&= -4e^{-4} - \left[ e^{-4} - 1 \right] \\
&= -4e^{-4} - e^{-4} + 1 \\
&= 1 - 5e^{-4} \\
&\approx 0.91 \text{ unit}^2
\end{aligned}$$

## Formula

Evaluate  $\int x^n e^{ax} dx$

		$\int e^{ax}$
+	$x^n$	$\frac{1}{a} e^{ax}$
-	$nx^{n-1}$	$\frac{1}{a^2} e^{ax}$
+	$n(n-1)x^{n-2}$	$\frac{1}{a^3} e^{ax}$
-	$n(n-1)(n-2)x^{n-3}$	$\frac{1}{a^4} e^{ax}$
	$\vdots \vdots$	$\vdots \vdots$

$$\begin{aligned}
\int x^n e^{ax} dx &= \frac{1}{a} x^n e^{ax} - \frac{n}{a^2} x^{n-1} e^{ax} + \frac{n(n-1)}{a^3} x^{n-2} e^{ax} - \frac{n(n-1)(n-2)}{a^4} x^{n-3} e^{ax} + \dots \\
&= e^{ax} \sum_{k=0}^n (-1)^k \cdot \frac{n!}{(n-k)!} \cdot \frac{1}{a^{k+1}} \cdot x^{n-k}
\end{aligned}$$

## Exercises      Section 2.1 – Integration by Parts

(1 – 92)    Evaluate the integrals

1.  $\int x \ln x \, dx$

2.  $\int \ln x^2 \, dx$

3.  $\int \ln(3x) \, dx$

4.  $\int \frac{1}{x \ln x} \, dx$

5.  $\int x (\ln x)^2 \, dx$

6.  $\int x^2 (\ln x)^2 \, dx$

7.  $\int \frac{(\ln x)^3}{x} \, dx$

8.  $\int x^2 \ln x^3 \, dx$

9.  $\int \ln(x + x^2) \, dx$

10.  $\int x \ln(x + 1) \, dx$

11.  $\int \frac{(\ln x)^2}{x} \, dx$

12.  $\int x^5 \ln 3x \, dx$

13.  $\int x^5 \ln x \, dx$

14.  $\int \ln(x + 1) \, dx$

15.  $\int \frac{\ln x}{x^{10}} \, dx$

16.  $\int x e^{2x} \, dx$

17.  $\int x^3 e^x \, dx$

18.  $\int \frac{2x}{e^x} \, dx$

19.  $\int \frac{x^3 e^{x^2}}{(x^2 + 1)^2} \, dx$

20.  $\int x^2 e^{-3x} \, dx$

21.  $\int (x^2 - 2x + 1) e^{2x} \, dx$

22.  $\int x^5 e^{x^3} \, dx$

23.  $\int x e^{-4x} \, dx$

24.  $\int \frac{x e^{2x}}{(2x + 1)^2} \, dx$

25.  $\int \frac{5x}{e^{2x}} \, dx$

26.  $\int \frac{e^{1/x}}{x^2} \, dx$

27.  $\int x^2 e^{4x} \, dx$

28.  $\int x^3 e^{-3x} \, dx$

29.  $\int x^4 e^x \, dx$

30.  $\int x^3 e^{4x} \, dx$

31.  $\int (x + 1)^2 e^x \, dx$

32.  $\int 2x e^{3x} \, dx$

33.  $\int x^2 \sin x \, dx$

34.  $\int \theta \cos \pi \theta \, d\theta$

35.  $\int 4x \sec^2 2x \, dx$

36.  $\int x^3 \sin x \, dx$

37.  $\int (x^3 - 2x) \sin 2x \, dx$

38.  $\int x^2 \sin 2x \, dx$

39.  $\int x^2 \sin(1 - x) \, dx$

40.  $\int x \sin x \cos x \, dx$

41.  $\int x \cos x \, dx$

42.  $\int x \csc x \cot x \, dx$

43.  $\int x^2 \cos x \, dx$

44.  $\int x^3 \cos 2x \, dx$

45.  $\int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx$

46.  $\int x \sinh x \, dx$
47.  $\int x^2 \cosh x \, dx$
48.  $\int e^{2x} \cos 3x \, dx$
49.  $\int e^{-3x} \sin 5x \, dx$
50.  $\int e^{-x} \sin 4x \, dx$
51.  $\int e^{-2\theta} \sin 6\theta \, d\theta$
52.  $\int e^{-3x} \sin 4x \, dx$
53.  $\int e^{4x} \cos 2x \, dx$
54.  $\int e^{3x} \cos 3x \, dx$
55.  $\int e^{3x} \cos 2x \, dx$
56.  $\int e^x \sin x \, dx$
57.  $\int e^{-2x} \sin 3x \, dx$
58.  $\int \frac{x}{\sqrt{x-1}} \, dx$
59.  $\int x\sqrt{x-5} \, dx$
60.  $\int \frac{x}{\sqrt{6x+1}} \, dx$
61.  $\int \frac{x}{2\sqrt{x+2}} \, dx$
62.  $\int \frac{2x^2-3x}{(x-1)^3} \, dx$
63.  $\int \frac{x^2+3x+4}{\sqrt[3]{2x+1}} \, dx$
64.  $\int \frac{x}{\sqrt{x+1}} \, dx$
65.  $\int \frac{x^5}{\sqrt{1-2x^3}} \, dx$
66.  $\int x \sqrt{1-3x} \, dx$
67.  $\int \sin(\ln x) \, dx$
68.  $\int \tan^{-1} y \, dy$
69.  $\int \sin^{-1} y \, dy$
70.  $\int x \tan^{-1} x \, dx$
71.  $\int \sinh^{-1} x \, dx$
72.  $\int \tan^{-1} 3x \, dx$
73.  $\int \cos^{-1}\left(\frac{x}{2}\right) \, dx$
74.  $\int \sec^{-1} x \, dx$
75.  $\int_{-1}^0 2x^2 \sqrt{x+1} \, dx$
76.  $\int_0^{1/\sqrt{2}} x \tan^{-1} x^2 \, dx$
77.  $\int_1^e x^2 \ln x \, dx$
78.  $\int_{-1}^{\ln 2} \frac{3t}{e^t} \, dt$
79.  $\int_{\pi}^{2\pi} \cot \frac{x}{3} \, dx$
80.  $\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2) \, dx$
81.  $\int_1^e x^3 \ln x \, dx$
82.  $\int_0^1 x\sqrt{1-x} \, dx$
83.  $\int_0^{\pi/3} x \tan^2 x \, dx$
84.  $\int_0^{\pi} x \sin x \, dx$
85.  $\int_1^e \ln 2x \, dx$
86.  $\int_0^{\pi/2} x \cos 2x \, dx$
87.  $\int_0^{\ln 2} x e^x \, dx$
88.  $\int_1^{e^2} x^2 \ln x \, dx$
89.  $\int_0^3 x e^{x/2} \, dx$
90.  $\int_0^2 x^2 e^{-2x} \, dx$
91.  $\int_0^{\pi/4} x \cos 2x \, dx$
92.  $\int_0^{\pi} x \sin 2x \, dx$
93.  $\int_1^4 e^{\sqrt{x}} \, dx$



(94 – 98) Use integration by parts to establish the reduction formula

$$94. \int x^n \sin x \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx$$

$$95. \int x^n e^{ax} \, dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx, \quad a \neq 0$$

$$96. \int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx$$

$$97. \int_a^b \left( \int_x^b f(t) \, dt \right) dx = \int_a^b (x-a) f(x) \, dx$$

$$98. \int \sqrt{1-x^2} \, dx = \frac{1}{2} x \sqrt{1-x^2} + \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} \, dx$$

$$99. \text{ Find the indefinite integral: } \int 5x^n \ln ax \, dx \quad a \neq 0, n \neq -1$$

100. Find the volume of the solid generated by the region bounded by  $f(x) = x \ln x$ , and the  $x$ -axis on  $[1, e^2]$  is revolved about the  $y$ -axis.

101. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve  $y = e^x$ , and the line  $x = \ln 2$  about the line  $x = \ln 2$

102. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve  $y = e^{-x}$ , and the line  $x = 1$ , about

a) the line  $y$ -axis

b) the line  $x = 1$

103. Find the volume of the solid that is generated by the region bounded by  $f(x) = e^{-x}$ ,  $x = \ln 2$ , and the coordinate axes is revolved about the  $y$ -axis.

104. Find the volume of the solid that is generated by the region bounded by  $f(x) = e^{-x}$ , and the  $x$ -axis on  $[1, \ln 2]$  is revolved about the line  $x = \ln 2$ .

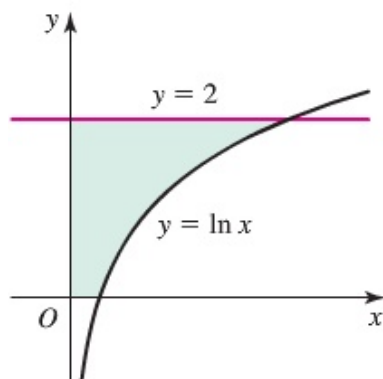
105. Find the volume of the solid that is generated by the region bounded by  $f(x) = \sin x$ , and the  $x$ -axis on  $[0, \pi]$  is revolved about the  $y$ -axis.

106. Find the area of the region generated when the region bounded by  $y = \sin x$  and  $y = \sin^{-1} x$  on the interval  $\left[0, \frac{1}{2}\right]$ .

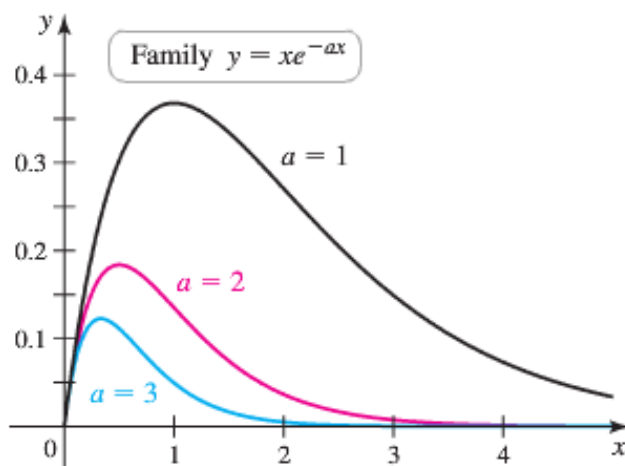
107. Find the area between the curves  $y = \ln x^2$ ,  $y = \ln x$ , and  $x = e^2$

108. Determine the area of the shaded region bounded by

$$y = \ln x, \quad y = 2, \quad y = 0, \quad \text{and} \quad x = 0$$



109. The curves  $y = xe^{-ax}$  are shown in the figure for  $a = 1, 2$ , and  $3$ .



- Find the area of the region bounded by  $y = xe^{-x}$  and the  $x$ -axis on the interval  $[0, 4]$ .
- Find the area of the region bounded by  $y = xe^{-ax}$  and the  $x$ -axis on the interval  $[0, 4]$  where  $a > 0$
- Find the area of the region bounded by  $y = xe^{-ax}$  and the  $x$ -axis on the interval  $[0, b]$ . Because this area depends on  $a$  and  $b$ , we call it  $A(a, b)$  where  $a > 0$  and  $b > 0$ .
- Use part (c) to show that  $A(1, \ln b) = 4A\left(2, \frac{1}{2} \ln b\right)$
- Does this pattern continue? Is it true that  $A(1, \ln b) = a^2 A\left(a, \frac{1}{a} \ln b\right)$

- 110.** Suppose a mass on a spring that is slowed by friction has the position function  $s(t) = e^{-t} \sin t$
- Graph the position function. At what times does the oscillator pass through the position  $s = 0$ ?
  - Find the average value of the position on the interval  $[0, \pi]$ .
  - Generalize part (b) and find the average value of the position on the interval  $[n\pi, (n+1)\pi]$ , for  $n = 0, 1, 2, \dots$
- 111.** Given the region bounded by the graphs of  $y = x \sin x$ ,  $y = 0$ ,  $x = 0$ ,  $x = \pi$ , find
- The area of the region.
  - The volume of the solid generated by revolving the region about the  $x$ -axis
  - The volume of the solid generated by revolving the region about the  $y$ -axis
  - The centroid of the region
- 112.** The region  $R$  is bounded by the curve  $y = \ln x$  and the  $x$ -axis on the interval  $[1, e]$ . Find the volume of the solid that is generated when  $R$  is revolved in the following ways
- About the  $x$ -axis
  - About the  $y$ -axis
  - About the line  $x = 1$
  - About the line  $y = 1$
- 113.** A string stretched between the two points  $(0, 0)$  and  $(2, 0)$  is plucked by displacing the string  $h$  units at its midpoint. The motion of the string is modeled by a **Fourier Sine series** whose coefficients are given by

$$b_n = h \int_0^1 x \sin \frac{n\pi x}{2} dx + h \int_1^2 (-x+2) \sin \frac{n\pi x}{2} dx$$

Find  $b_n$

## Section 2.2 – Trigonometric Integrals

### Products of Powers of *Sines* and *Cosines*

We begin with integrals of the form

$$\int \sin^m x \cos^n x dx$$

#### Example

Evaluate  $\int \sin^3 x \cos^2 x dx$

#### Solution

$$\begin{aligned}\int \sin^3 x \cos^2 x dx &= \int \sin x \sin^2 x \cos^2 x dx \\&= \int (1 - \cos^2 x) \cos^2 x (-d(\cos x)) \quad d(\cos x) = -\sin x dx \Rightarrow \sin x dx = -d(\cos x) \\&= -\int (\cos^2 x - \cos^4 x) d(\cos x) \quad \text{or Assume } u = \cos x \\&= -\left(\frac{1}{3}\cos^3 x - \frac{1}{5}\cos^5 x\right) + C \\&= \underline{\underline{\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C}}\end{aligned}$$

#### Example

Evaluate  $\int \cos^5 x dx$

#### Solution

$$\begin{aligned}\int \cos^5 x dx &= \int \cos^4 x \cos x dx \quad \cos x dx = d(\sin x) \quad \cos^2 x = 1 - \sin^2 x \\&= \int (1 - \sin^2 x)^2 d(\sin x) \\&= \int (1 - 2\sin^2 x + \sin^4 x) d\sin x \\&= \underline{\underline{\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C}}\end{aligned}$$

**Example**

Evaluate  $\int \sin^2 x \cos^4 x \, dx$

**Solution**

$$\begin{aligned}
 \int \sin^2 x \cos^4 x \, dx &= \int \left( \frac{1 - \cos 2x}{2} \right) \left( \frac{1 + \cos 2x}{2} \right)^2 dx & \sin^2 x &= \frac{1 - \cos 2x}{2} & \cos^2 x &= \frac{1 + \cos 2x}{2} \\
 &= \frac{1}{8} \int (1 - \cos 2x) (1 + 2 \cos 2x + \cos^2 2x) dx \\
 &= \frac{1}{8} \int (1 + 2 \cos 2x + \cos^2 2x - \cos 2x - 2 \cos^2 2x - \cos^3 2x) dx \\
 &= \frac{1}{8} \int (1 + \cos 2x - \cos^2 2x - \cos^3 2x) dx \\
 &= \frac{1}{8} \left[ x + \frac{1}{2} \sin 2x - \int (\cos^3 2x + \cos^2 2x) dx \right]
 \end{aligned}$$

$$\begin{aligned}
 \int \cos^3 2x \, dx &= \int (1 - \sin^2 2x) \cos 2x \, dx \\
 &= \frac{1}{2} \int (1 - \sin^2 2x) d(\sin 2x) \\
 &= \frac{1}{2} \left( \sin 2x - \frac{1}{3} \sin^3 2x \right)
 \end{aligned}$$

$$\begin{aligned}
 \int \cos^2 2x \, dx &= \frac{1}{2} \int (1 + \cos 4x) \, dx \\
 &= \frac{1}{2} \left( x + \frac{1}{4} \sin 4x \right)
 \end{aligned}$$

$$\begin{aligned}
 \int \sin^2 x \cos^4 x \, dx &= \frac{1}{8} \left[ x + \frac{1}{2} \sin 2x - \frac{1}{2} \left( \sin 2x - \frac{1}{3} \sin^3 2x \right) - \frac{1}{2} \left( x + \frac{1}{4} \sin 4x \right) \right] + C \\
 &= \frac{1}{8} \left[ x + \frac{1}{2} \sin 2x - \frac{1}{2} \sin 2x + \frac{1}{6} \sin^3 2x - \frac{1}{2} x - \frac{1}{8} \sin 4x \right] + C \\
 &= \frac{1}{8} \left( \frac{1}{2} x + \frac{1}{6} \sin^3 2x - \frac{1}{8} \sin 4x \right) + C \\
 &= \underline{\underline{\frac{1}{16} \left( x + \frac{1}{3} \sin^3 2x - \frac{1}{4} \sin 4x \right) + C}}
 \end{aligned}$$

### Example

Evaluate  $\int_0^{\pi/4} \sqrt{1 + \cos 4x} \, dx$

### Solution

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2} \Rightarrow 1 + \cos 2\theta = 2 \cos^2 \theta$$

$$\theta = 2x \Rightarrow 1 + \cos 4x = 2 \cos^2 2x$$

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x} \, dx = \int_0^{\pi/4} \sqrt{2 \cos^2 2x} \, dx$$

$$= \sqrt{2} \int_0^{\pi/4} |\cos 2x| \, dx$$

$$= \sqrt{2} \int_0^{\pi/4} \cos 2x \, dx$$

$$= \sqrt{2} \left( \frac{\sin 2x}{2} \right) \Big|_0^{\pi/4}$$

$$= \frac{\sqrt{2}}{2} \left[ \sin \frac{\pi}{2} - 0 \right]$$

$$= \frac{\sqrt{2}}{2}$$

$$\sqrt{2 \cos^2 2x} = \sqrt{2} \sqrt{\cos^2 2x} = \sqrt{2} |\cos 2x|$$

$$\cos 2x \geq 0 \quad \text{on} \quad \left[ 0, \frac{\pi}{4} \right]$$

### Example

Evaluate  $\int \sin^3 x \cos^{-2} x \, dx$

### Solution

$$\int \sin^3 x \cos^{-2} x \, dx = \int \sin^2 x \cos^{-2} x \sin x \, dx$$

$$= - \int (1 - \cos^2 x) \cos^{-2} x \, d(\cos x)$$

$$= - \int (\cos^{-2} x - 1) \, d(\cos x)$$

$$= -(-\cos^{-1} x - \cos x) + C$$

$$= \cos x + \sec x + C$$

## Products of Powers of $\tan x$ and $\sec x$

### Example

Evaluate  $\int \tan^4 x \, dx$

### Solution

$$\begin{aligned}\int \tan^4 x \, dx &= \int \tan^2 x \cdot \tan^2 x \, dx \\&= \int \tan^2 x \cdot (\sec^2 x - 1) \, dx \\&= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx \\&= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx \\&= \int \tan^2 x \sec^2 x \, dx - \int \sec^2 x \, dx + \int dx \\&= \int \tan^2 x \, d(\tan x) - \int \sec^2 x \, dx + \int dx \\&= \frac{1}{3} \tan^3 x - \tan x + x + C\end{aligned}$$

$$\tan^2 x = \sec^2 x - 1$$

$$d(\tan x) = \sec^2 x \, dx$$

### Example

Evaluate  $\int \sec^3 x \, dx$

### Solution

Let:  $u = \sec x \quad dv = \sec^2 x \, dx$   
 $du = \sec x \tan x \, dx \quad v = \tan x$

$$\begin{aligned}\int \sec^3 x \, dx &= \sec x \tan x - \int \tan x (\sec x \tan x \, dx) \\&= \sec x \tan x - \int \tan^2 x \sec x \, dx \\&= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx \\&= \sec x \tan x - \int (\sec^3 x - \sec x) \, dx\end{aligned}$$

$$\begin{aligned}
&= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx \\
\int \sec^3 x \, dx + \int \sec^3 x \, dx &= \sec x \tan x - \int \sec^3 x \, dx + \int \sec^3 x \, dx + \int \sec x \, dx \\
2 \int \sec^3 x \, dx &= \sec x \tan x + \int \sec x \, dx \\
&= \sec x \tan x + \ln |\sec x + \tan x| + C_1 \\
\int \sec^3 x \, dx &= \underline{\frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C}
\end{aligned}$$

## Products of Sines and Cosines

Recall the identities

$$\begin{aligned}
\sin mx \sin nx &= \frac{1}{2} [\cos(m-n)x - \cos(m+n)x] \\
\sin mx \cos nx &= \frac{1}{2} [\sin(m-n)x + \sin(m+n)x] \\
\cos mx \cos nx &= \frac{1}{2} [\cos(m-n)x + \cos(m+n)x]
\end{aligned}$$

### Example

Evaluate  $\int \sin 3x \cos 5x \, dx$

### Solution

$$\begin{aligned}
\int \sin 3x \cos 5x \, dx &= \frac{1}{2} \int (\sin(-2x) + \sin 8x) \, dx \\
&= \frac{1}{2} \int [-\sin(2x) + \sin 8x] \, dx \\
&= \frac{1}{2} \left( \frac{1}{2} \cos 2x - \frac{1}{8} \cos 8x \right) + C \\
&= \underline{\frac{1}{4} \cos 2x - \frac{1}{16} \cos 8x + C}
\end{aligned}$$



## Guidelines for Cosine & Sine

**Case 1** If  $m$  is *odd*, we write  $m$  as  $2k + 1$  and use the identity  $\sin^2 x = 1 - \cos^2 x$  to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x$$

Then we combine the single  $\sin x$  with  $dx$  in the integral and set  $\sin x dx = -d(\cos x)$

**Case 2** If  $m$  is *even* and  $n$  is *odd*, in  $\int \sin^m x \cos^n x dx$  we write  $n$  as  $2k + 1$  and use the identity

$\cos^2 x = 1 - \sin^2 x$  to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x$$

Then we combine the single  $\cos x$  with  $dx$  in the integral and set  $\cos x dx = d(\sin x)$

**Case 3** If both  $m$  and  $n$  are *even*, in  $\int \sin^m x \cos^n x dx$ , we substitute

$$\text{To reduce the integrand to one in lower powers of } \cos 2x \quad \int \cos ax dx = \frac{1}{a} \sin ax + C$$

## Guidelines for Tangent & Secant

**Case 1** When the power of the tangent is *odd* and positive.

$$\begin{aligned} \int \sec^m x \tan^{2k+1} x dx &= \int \sec^{m-1} x (\tan^2 x)^k \sec x \tan x dx \\ &= \int \sec^{m-1} x (\sec^2 x - 1)^k d(\sec x) \end{aligned}$$

**Case 2** When the power of the secant is *even* and positive.

$$\int \sec^{2k} x \tan^n x dx = \int (\sec^2 x)^{k-1} \tan^n x \sec^2 x dx = \int (1 + \tan^2 x)^{k-1} \tan^n x d(\tan x)$$

**Case 3** When there are no secant factors

$$\int \tan^n x dx = \int \tan^{n-2} x (\tan^2 x) dx = \int \tan^{n-2} x (\sec^2 x - 1) dx$$

**Case 4** When there are only secant, use integration by parts.

**Case 5** Otherwise, convert to cosines and sines.

## Wallis's Formulas

<b>1.</b> If $n$ is odd ( $n \geq 3$ ), then	$\int_0^{\pi/2} \cos^n x \, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right) \cdots \left(\frac{n-1}{n}\right)$
<b>2.</b> If $n$ is even ( $n \geq 2$ ), then	$\int_0^{\pi/2} \cos^n x \, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right) \cdots \left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$

## Formulas

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

$$\int \sec^n x \, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$$

$$\int \sin^n x \, dx = \frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

$$\int \tan^n x \, dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x \, dx$$

$$\int \tan x \, dx = -\ln |\cos x| + C = \ln |\sec x| + C$$

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C$$

$$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C$$

$$\int \csc x \, dx = -\ln |\csc x + \cot x| + C$$

## Exercises      Section 2.2 – Trigonometric Integrals

(1 – 149) Evaluate the integrals

- |                                           |                                  |                                    |
|-------------------------------------------|----------------------------------|------------------------------------|
| 1. $\int \sin^5 \frac{x}{2} dx$           | 16. $\int \sec^3 \pi x dx$       | 31. $\int \sin^2 x \cos^2 x dx$    |
| 2. $\int \sin^4 6\theta d\theta$          | 17. $\int \sec 4x dx$            | 32. $\int \sin^2 x \cos^3 x dx$    |
| 3. $\int x^2 \sin^2 x dx$                 | 18. $\int \csc^6 x dx$           | 33. $\int \sin^2 x \cos^4 x dx$    |
| 4. $\int \sin^3 3x dx$                    | 19. $\int \tan^5 \frac{x}{2} dx$ | 34. $\int \sin^2 x \cos^5 x dx$    |
| 5. $\int \sin^5 x dx$                     | 20. $\int \tan^5 x dx$           | 35. $\int \sin^3 x \cos^5 x dx$    |
| 6. $\int 8 \cos^4 2\pi x dx$              | 21. $\int \tan^5 3x dx$          | 36. $\int \sin^3 x \cos^4 x dx$    |
| 7. $\int x \cos^3 x dx$                   | 22. $\int \tan^6 3x dx$          | 37. $\int \sin^3 2x \cos^4 x dx$   |
| 8. $\int \cos^4 x dx$                     | 23. $\int 20 \tan^6 x dx$        | 38. $\int \sin^3 2x \cos^3 2x dx$  |
| 9. $\int \cos^4 5x dx$                    | 24. $\int \tan^4 x dx$           | 39. $\int \sin^4 x \cos^2 x dx$    |
| 10. $\int \cos^2 3x dx$                   | 25. $\int \tan^3 \theta d\theta$ | 40. $\int \sin^4 x \cos^3 x dx$    |
| 11. $\int \cos^3 \frac{x}{3} dx$          | 26. $\int \tan^3 4x dx$          | 41. $\int \sin^4 x \cos^4 x dx$    |
| 12. $\int \cos^2 4x dx$                   | 27. $\int \cot^3 2x dx$          | 42. $\int \sin^4 x \cos^5 x dx$    |
| 13. $\int \sqrt{1 + \cos \frac{x}{2}} dx$ | 28. $\int \cot^4 x dx$           | 43. $\int \sin^5 x \cos^5 x dx$    |
| 14. $\int \sec^4 2x dx$                   | 29. $\int \cot^4 3x dx$          | 44. $\int \sin^5 x \cos^{-2} x dx$ |
| 15. $\int 6 \sec^4 x dx$                  | 30. $\int \cot^5 3x dx$          | 45. $\int \sin 3x \cos^6 3x dx$    |

- |                                                             |                                                                |                                                                       |
|-------------------------------------------------------------|----------------------------------------------------------------|-----------------------------------------------------------------------|
| 46. $\int \sin^4 2x \cos 2x \, dx$                          | 61. $\int \sin(-7\theta) \cos 6\theta \, d\theta$              | 76. $\int \frac{dx}{1 - \cos x}$                                      |
| 47. $\int \cos^3 2x \sin^5 2x \, dx$                        | 62. $\int \sin \theta \sin 3\theta \, d\theta$                 | 77. $\int \frac{dx}{1 - \sin x}$                                      |
| 48. $\int 16 \sin^2 x \cos^2 x \, dx$                       | 63. $\int \sin 5\theta \sin 4\theta \, d\theta$                | 78. $\int \frac{\sin \theta \cos \theta}{2 - \cos \theta} \, d\theta$ |
| 49. $\int \sin 2x \cos 3x \, dx$                            | 64. $\int \sin x \cos^5 x \, dx$                               | 79. $\int \tan^3 x \sec^3 x \, dx$                                    |
| 50. $\int \sin^2 \theta \cos 3\theta \, d\theta$            | 65. $\int \sin^7 2x \cos 2x \, dx$                             | 80. $\int \sec x \tan^2 x \, dx$                                      |
| 51. $\int \cos^3 \theta \sin 2\theta \, d\theta$            | 66. $\int \sin^3 2x \sqrt{\cos 2x} \, dx$                      | 81. $\int \sec^2 x \tan^2 x \, dx$                                    |
| 52. $\int \sin^{-3/2} x \cos^3 x \, dx$                     | 67. $\int \sin^3 x \cos^2 x \, dx$                             | 82. $\int \sec^4 x \tan^2 x \, dx$                                    |
| 53. $\int \sin^3 x \cos^{3/2} x \, dx$                      | 68. $\int \frac{\cos^3 \theta}{\sqrt{\sin \theta}} \, d\theta$ | 83. $\int \sec^6 4x \tan 4x \, dx$                                    |
| 54. $\int \sin \theta \sin 2\theta \sin 3\theta \, d\theta$ | 69. $\int \frac{\cos^5 \theta}{\sqrt{\sin \theta}} \, d\theta$ | 84. $\int \sec^2 \frac{x}{2} \tan \frac{x}{2} \, dx$                  |
| 55. $\int \sin 3x \cos 6x \, dx$                            | 70. $\int \frac{\cos^2 x}{\sin^5 x} \, dx$                     | 85. $\int \tan^3 2x \sec^3 2x \, dx$                                  |
| 56. $\int \sin 3x \cos 7x \, dx$                            | 71. $\int \frac{\sin^3 x}{\cos^4 x} \, dx$                     | 86. $\int \tan^5 2x \sec^4 2x \, dx$                                  |
| 57. $\int \sin 5x \cos 4x \, dx$                            | 72. $\int \frac{\sin^4 x}{\cos^6 x} \, dx$                     | 87. $\int \tan^3 x \sec^5 x \, dx$                                    |
| 58. $\int \cos 2\theta \cos 6\theta \, d\theta$             | 73. $\int \frac{2 \cos x + 3 \sin x}{\sin^3 x} \, dx$          | 88. $\int \tan^3 x \sec^4 x \, dx$                                    |
| 59. $\int \cos 5\theta \cos 3\theta \, d\theta$             | 74. $\int \frac{\sin 2x}{1 + \sin^2 x} \, dx$                  | 89. $\int \tan^5 \theta \sec^4 \theta \, d\theta$                     |
| 60. $\int \sin 2\theta \cos 4\theta \, d\theta$             | 75. $\int \frac{2 + \sin x + 2 \cos x}{1 + \cos x} \, dx$      | 90. $\int \tan^5 \theta \sec^7 \theta \, d\theta$                     |

$$91. \int \tan^7 \theta \sec^5 \theta d\theta$$

$$92. \int \sec^4 3x \tan^3 3x dx$$

$$93. \int \tan^3 \frac{\pi x}{2} \sec^2 \frac{\pi x}{2} dx$$

$$94. \int \sec^{-2} x \tan^3 x dx$$

$$95. \int \sqrt{\tan x} \sec^4 x dx$$

$$96. \int \tan^5 \theta \csc^2 \theta d\theta$$

$$97. \int \csc^2 x \cot x dx$$

$$98. \int \csc^{10} x \cot x dx$$

$$99. \int (\cot 2x - \csc 2x)^2 dx$$

$$100. \int \operatorname{sech}^4 x dx$$

$$101. \int \sinh^3 x \cosh^2 x dx$$

$$102. \int \operatorname{sech}^2 x \sinh x dx$$

$$103. \int \frac{\tan^3 x}{\sqrt{\sec x}} dx$$

$$104. \int \frac{\tan^2 x}{\sec x} dx$$

$$105. \int \frac{\sec x}{\tan^2 x} dx$$

$$106. \int \frac{\sec^2 x}{\tan^5 x} dx$$

$$107. \int \frac{\csc^4 x}{\cot^2 x} dx$$

$$108. \int \frac{\sec^4(\ln x)}{x} dx$$

$$109. \int e^x \sec(e^x + 1) dx$$

$$110. \int e^x \sec^3(e^x) dx$$

$$111. \int e^x \sqrt{\tan^2 e^x + 1} dx$$

$$112. \int_0^{\sqrt{\pi/2}} x \sin^3(x^2) dx$$

$$113. \int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}} dx$$

$$114. \int_{\pi/6}^{\pi/2} \frac{dx}{\sin x}$$

$$115. \int_{\pi/6}^{\pi/3} \cot^3 \theta d\theta$$

$$116. \int_0^{\pi/3} \tan^2 x dx$$

$$117. \int_0^{\pi/4} 6 \tan^3 x dx$$

$$118. \int_0^{\pi/4} \tan^4 x dx$$

$$119. \int_0^{\pi} 8 \sin^4 y \cos^2 y dy$$

$$120. \int_0^{\pi/6} 3 \cos^5 3x \, dx$$

$$121. \int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta \, d\theta$$

$$122. \int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}} \, dx$$

$$123. \int_0^{\pi} \sqrt{1-\cos^2 \theta} \, d\theta$$

$$124. \int_0^{\pi/6} \sqrt{1+\sin x} \, dx$$

$$125. \int_{-\pi/3}^{\pi/3} \sqrt{\sec^2 \theta - 1} \, d\theta$$

$$126. \int_0^{\pi} (1-\cos 2x)^{3/2} \, dx$$

$$127. \int_0^{\pi} (1-\cos^2 x)^{3/2} \, dx$$

$$128. \int_{-\pi}^{\pi} (1-\cos^2 x)^{3/2} \, dx$$

$$129. \int_{\pi/4}^{\pi/2} \csc^4 \theta \, d\theta$$

$$130. \int_{-\pi}^{\pi} \sin 3x \sin 3x \, dx$$

$$131. \int_{-\pi/2}^{\pi/2} \cos x \cos 7x \, dx$$

$$132. \int_0^{\pi/4} \cos^5 2x \sin^2 2x \, dx$$

$$133. \int_0^{\pi/6} \sin^5 x \, dx$$

$$134. \int_{-\pi}^{\pi} \sin^2 x \, dx$$

$$135. \int_{-\pi/2}^{\pi/2} (\sin^2 x + 1) \, dx$$

$$136. \int_0^{\pi/3} \sec^{3/2} x \tan x \, dx$$

$$137. \int_0^{\pi/2} \frac{\cos x}{1+\sin x} \, dx$$

$$138. \int_{\pi/6}^{\pi/3} \sin 6x \cos 4x \, dx$$

$$139. \int_{-\pi/2}^{\pi/2} 3 \cos^3 x \, dx$$

$$140. \int_0^{\pi} \sec^2 x \, dx$$

$$141. \int_0^{\ln(\sqrt{3}+2)} \frac{\cosh x}{\sqrt{4-\sinh^2 x}} \, dx$$

$$142. \int_0^{\pi/2} \cos^4 x \, dx$$

$$143. \int_0^{\pi/2} \cos^{10} \theta \, d\theta$$

$$144. \int_0^{\pi/2} \cos^7 x \, dx$$

$$145. \int_0^{\pi/2} \cos^9 x \, dx$$

$$146. \int_0^{\pi/2} \sin^5 x \, dx$$

$$148. \int_0^{\pi/2} \sin^8 x \, dx$$

$$147. \int_0^{\pi/2} \sin^6 x \, dx$$

$$149. \int_0^{\pi/2} \tan^2 \frac{x}{2} \, dx$$

150. Find the area of the region bounded by the graphs of  $y = \tan x$  and  $y = \sec x$  on the interval  $\left[0, \frac{\pi}{4}\right]$

(151 – 154) Find the area of the region bounded by the graphs of the equations

$$151. y = \sin x, \quad y = \sin^3 x, \quad x = 0, \quad x = \frac{\pi}{2}$$

$$152. y = \sin^2 \pi x, \quad y = 0, \quad x = 0, \quad x = 1$$

$$153. y = \cos^2 x, \quad y = \sin^2 x, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$$

$$154. y = \cos^2 x, \quad y = \sin x \cos x, \quad x = -\frac{\pi}{2}, \quad x = \frac{\pi}{4}$$

(155 – 156) Find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the  $x$ -axis

$$155. y = \tan x, \quad y = 0, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$$

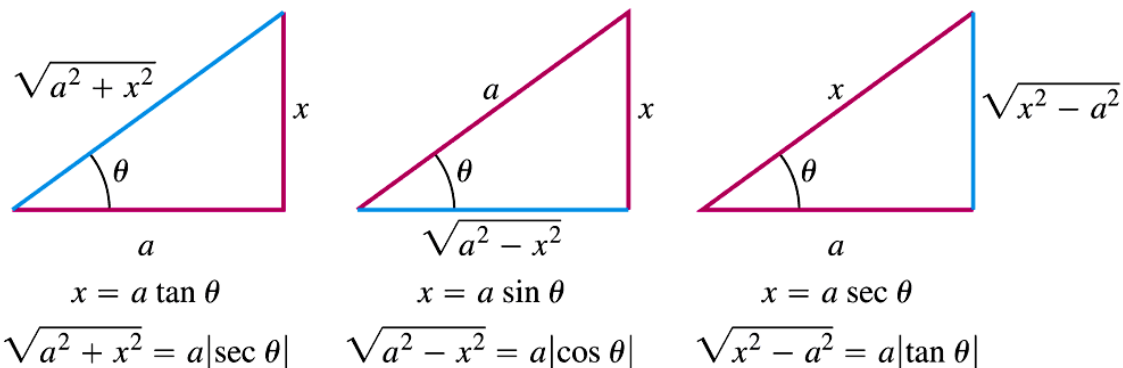
$$156. y = \cos \frac{x}{2}, \quad y = \sin \frac{x}{2}, \quad x = 0, \quad x = \frac{\pi}{2}$$

(157– 158) Find the **volume** of the solid generated by revolving the region bounded by the graphs of the equations about the  $x$ -axis, then find the **centroid** of the region

$$157. y = \sin x, \quad y = 0, \quad x = 0, \quad x = \pi$$

$$158. y = \cos x, \quad y = \sin 0, \quad x = 0, \quad x = \frac{\pi}{2}$$

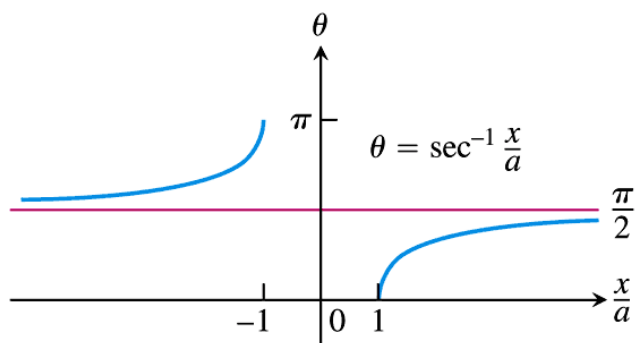
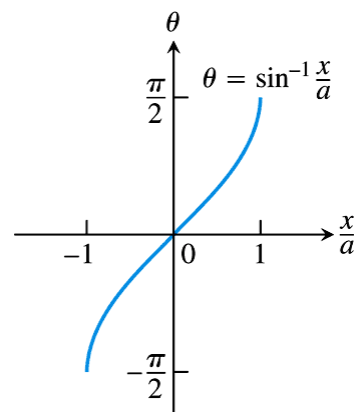
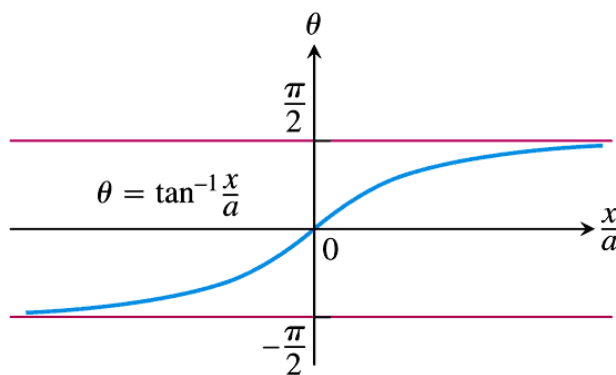
## Section 2.3 – Trigonometric Substitutions



$$x = a \tan \theta \Rightarrow \theta = \tan^{-1} \left( \frac{x}{a} \right) \text{ with } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$x = a \sin \theta \Rightarrow \theta = \sin^{-1} \left( \frac{x}{a} \right) \text{ with } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$x = a \sec \theta \Rightarrow \theta = \sec^{-1} \left( \frac{x}{a} \right) \text{ with } \begin{cases} 0 \leq \theta < \frac{\pi}{2} & \text{if } \frac{x}{a} \geq 1 \\ \frac{\pi}{2} < \theta \leq \pi & \text{if } \frac{x}{a} \leq -1 \end{cases}$$





## Procedure for a Trigonometric Substitution

1. Write down the substitution for  $x$ , calculate the differential  $dx$ , and specify the selected values of  $\theta$  for the substitution.
2. Substitute the trigonometric expression and the calculated differential into the integrand, and then simplify the results algebraically.
3. Integrate the trigonometric integral, keeping in mind the restrictions on the angle  $\theta$  for reversibility.
4. Draw an appropriate reference triangle to reserve the substitution in the integration result and convert it back to the original variable  $x$ .

### Example

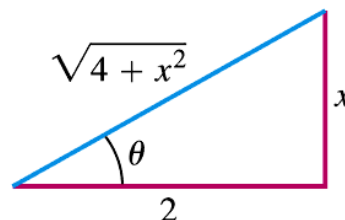
Evaluate  $\int \frac{dx}{\sqrt{4+x^2}}$

### Solution

Let:  $x = 2 \tan \theta \rightarrow dx = 2 \sec^2 \theta d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\begin{aligned} 4 + x^2 &= 4 + 4 \tan^2 \theta \\ &= 4(1 + \tan^2 \theta) \\ &= 4 \sec^2 \theta \end{aligned}$$

$$\begin{aligned} \int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2 \sec^2 \theta d\theta}{\sqrt{4 \sec^2 \theta}} \\ &= \int \frac{2 \sec^2 \theta}{2 |\sec \theta|} d\theta \\ &= \int \sec \theta d\theta \\ &= \ln |\sec \theta + \tan \theta| + C \\ &= \ln \left| \frac{\sqrt{4+x^2}}{2} + \frac{x}{2} \right| + C \end{aligned}$$



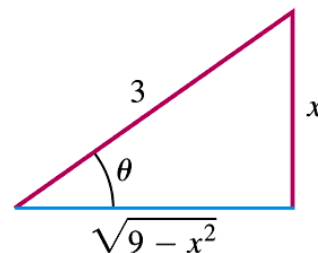
### Example

Evaluate  $\int \frac{x^2 dx}{\sqrt{9-x^2}}$

### Solution

$$x = 3 \sin \theta \rightarrow dx = 3 \cos \theta d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\begin{aligned} 9 - x^2 &= 9 - 9 \sin^2 \theta \\ &= 9(1 - \sin^2 \theta) \\ &= 9 \cos^2 \theta \end{aligned}$$



$$\begin{aligned} \int \frac{x^2 dx}{\sqrt{9-x^2}} &= \int \frac{9 \sin^2 \theta \cdot 3 \cos \theta}{|3 \cos \theta|} d\theta \\ &= 9 \int \sin^2 \theta d\theta \\ &= 9 \int \frac{1 - \cos 2\theta}{2} d\theta \\ &= \frac{9}{2} \left( \theta - \frac{1}{2} \sin 2\theta \right) + C \\ &= \frac{9}{2} \left( \theta - \frac{1}{2} 2 \sin \theta \cos \theta \right) + C \\ &= \frac{9}{2} \left( \sin^{-1} \left( \frac{x}{3} \right) - \frac{x}{3} \cdot \frac{\sqrt{9-x^2}}{3} \right) + C \\ &= \frac{9}{2} \sin^{-1} \left( \frac{x}{3} \right) - \frac{x}{2} \sqrt{9-x^2} + C \end{aligned}$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\sin \theta = \frac{x}{3}, \quad \cos^2 \theta = \frac{9-x^2}{9} \Rightarrow \cos \theta = \frac{\sqrt{9-x^2}}{3}$$

### Example

Evaluate  $\int \frac{dx}{\sqrt{25x^2 - 4}}, \quad x > \frac{2}{5}$

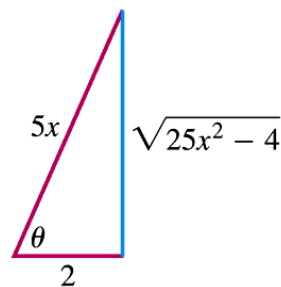
### Solution

$$\begin{aligned}\sqrt{25x^2 - 4} &= \sqrt{25\left(x^2 - \frac{4}{25}\right)} \\ &= 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2}\end{aligned}$$

$$x = \frac{2}{5} \sec \theta \rightarrow dx = \frac{2}{5} \sec \theta \tan \theta d\theta, \quad 0 < \theta < \frac{\pi}{2}$$

$$\begin{aligned}x^2 - \left(\frac{2}{5}\right)^2 &= \frac{4}{25} \sec^2 \theta - \frac{4}{25} \\ &= \frac{4}{25} (\sec^2 \theta - 1) \\ &= \frac{4}{25} \tan^2 \theta\end{aligned}$$

$$\begin{aligned}\int \frac{dx}{\sqrt{25x^2 - 4}} &= \int \frac{dx}{5\sqrt{x^2 - \left(\frac{2}{5}\right)^2}} \\ &= \frac{1}{5} \int \frac{\frac{2}{5} \sec \theta \tan \theta d\theta}{\frac{2}{5} \tan \theta} \\ &= \frac{1}{5} \int \sec \theta d\theta \\ &= \frac{1}{5} \ln |\sec \theta + \tan \theta| + C \\ &= \frac{1}{5} \ln \left| \frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2} \right| + C\end{aligned}$$



**OR**

$$\sqrt{25x^2 - 4}$$

$$5x = 2 \sec \theta \rightarrow dx = \frac{2}{5} \sec \theta \tan \theta d\theta$$

$$\sqrt{25x^2 - 4} = 2 \tan \theta$$

$$\int \frac{dx}{\sqrt{25x^2 - 4}} = \frac{1}{5} \int \sec \theta d\theta$$

## Summary

$\sqrt{a^2 + b^2 x^2}$	$\sqrt{a^2 - b^2 x^2}$	$\sqrt{b^2 x^2 - a^2}$
$bx = a \tan \theta$ $\sqrt{a^2 + b^2 x^2} = a \sec \theta$	$bx = a \sin \theta$ $\sqrt{a^2 - b^2 x^2} = a \cos \theta$	$bx = a \sec \theta$ $\sqrt{b^2 x^2 - a^2} = a \tan \theta$

## Exercises      Section 2.3 – Trigonometric Substitutions

(1 – 100) Evaluate the integrals

1.  $\int \frac{3dx}{\sqrt{1+9x^2}}$

2.  $\int \frac{x^2}{4+x^2} dx$

3.  $\int \frac{dx}{x^2\sqrt{x^2+1}}$

4.  $\int \frac{dx}{\sqrt{x^2+4}}$

5.  $\int \frac{dx}{(1+x^2)^2}$

6.  $\int \frac{dx}{\sqrt{4x^2+1}}$

7.  $\int \frac{dx}{(x^2+1)^{3/2}}$

8.  $\int \frac{9x^3}{\sqrt{x^2+1}} dx$

9.  $\int \sqrt{16x^2+9} dx$

10.  $\int x\sqrt{x^2+1} dx$

11.  $\int \frac{\sqrt{25x^2+4}}{x^4} dx$

12.  $\int \frac{1}{x\sqrt{4x^2+9}} dx$

13.  $\int \frac{1}{(x^2+5)^{3/2}} dx$

14.  $\int \frac{x dx}{\sqrt{x^2+4}}$

15.  $\int \frac{x^3}{\sqrt{x^2+4}} dx$

16.  $\int \frac{dx}{(1+4x^2)^{3/2}}$

17.  $\int \frac{dx}{x^2\sqrt{x^2+9}}$

18.  $\int \frac{dx}{x^2\sqrt{x^2+4}}$

19.  $\int \frac{x^2 dx}{\sqrt{x^2+1}}$

20.  $\int \frac{x^3 dx}{(x^2+a^2)^{3/2}}$

21.  $\int \frac{dx}{(x^2+4)^2}$

22.  $\int \frac{dx}{\sqrt{4x^2+16}}$

23.  $\int \frac{x^4}{x^2+1} dx$

24.  $\int \frac{e^{2x}}{(1+e^{4x})^{3/2}} dx$

25.  $\int \frac{dx}{1+\cos x}$

26.  $\int \frac{dx}{\sqrt{x^2-a^2}}$

27.  $\int \frac{\sqrt{x^2-1}}{x} dx$

28.  $\int \frac{\sqrt{y^2-49}}{y} dy, \quad y > 7$

29.  $\int \frac{5 dx}{\sqrt{25x^2-9}}, \quad x > \frac{3}{5}$

30.  $\int \frac{2dx}{x^3\sqrt{x^2-1}}, \quad x > 1$

31.  $\int \frac{dx}{x^3\sqrt{x^2-100}}$

32.  $\int \frac{x^3 dx}{x^2-1}$

33.  $\int \frac{(1-x^2)^{1/2}}{x^4} dx$

34.  $\int \frac{\sqrt{1-(\ln x)^2}}{x \ln x} dx$

35.  $\int \sqrt{x} \sqrt{1-x} dx$

36.  $\int \frac{\sqrt{x-2}}{\sqrt{x-1}} dx$

37.  $\int \frac{dx}{\sqrt{4x^2-49}}$

38.  $\int \frac{dx}{\sqrt{x^2 - 25}}$
39.  $\int \frac{\sqrt{x^2 - 25}}{x} dx$
40.  $\int \frac{x^3}{\sqrt{x^2 - a^2}} dx$
41.  $\int \frac{x^3}{\sqrt{x^2 - 25}} dx$
42.  $\int x^3 \sqrt{x^2 - 25} dx$
43.  $\int \sqrt{5x^2 - 1} dx$
44.  $\int \frac{dx}{\sqrt{9x^2 - 25}}, \quad x > \frac{5}{3}$
45.  $\int \frac{x dx}{\sqrt{4x^2 - 1}}$
46.  $\int \frac{dx}{\sqrt{x^2 - 81}}$
47.  $\int \frac{\sqrt{x^2 - 9}}{x} dx$
48.  $\int \frac{1}{x^2 \sqrt{9x^2 - 1}} dx$
49.  $\int \frac{dx}{(x^2 - 36)^{3/2}}$
50.  $\int \frac{dx}{\sqrt{36 - x^2}}$
51.  $\int \sqrt{a^2 - x^2} dx$
52.  $\int \frac{dx}{2 - \sqrt{3x}}$
53.  $\int \frac{x dx}{1 - \sqrt{x}}$
54.  $\int \frac{dx}{\sqrt{1 - 2x^2}}$
55.  $\int \frac{x^2}{(100 - x^2)^{3/2}} dx$
56.  $\int \frac{2dx}{\sqrt{1 - 4x^2}}$
57.  $\int \sqrt{\frac{x}{1 - x}} dx$
58.  $\int x \sqrt{2x - x^2} dx$
59.  $\int x^2 \sqrt{a^2 - x^2} dx$
60.  $\int \frac{x}{\sqrt{4x - x^2}} dx$
61.  $\int \frac{x}{\sqrt{ax - x^2}} dx, \quad a > 0$
62.  $\int \frac{x^2}{\sqrt{a^2 - x^2}} dx$
63.  $\int \frac{x^2}{\sqrt{16 - x^2}} dx$
64.  $\int \frac{dx}{(16 - x^2)^{3/2}}$
65.  $\int \frac{dx}{x^2 \sqrt{9 - x^2}}$
66.  $\int \frac{dx}{x^2 \sqrt{4 - x^2}}$
67.  $\int \frac{4}{x^2 \sqrt{16 - x^2}} dx$
68.  $\int \frac{x^3}{\sqrt{9 - x^2}} dx$
69.  $\int \sqrt{25 - 4x^2} dx$
70.  $\int e^x \sqrt{1 - e^{2x}} dx$
71.  $\int \frac{\sqrt{1 - x}}{\sqrt{x}} dx$
72.  $\int \frac{y dy}{\sqrt{16 - y^2}}$
73.  $\int \frac{x^3}{\sqrt{4 - x^2}} dx$
74.  $\int \frac{dx}{\sqrt{4 - x^2}}$
75.  $\int \frac{1}{x^4 + 4x^2 + 4} dx$
76.  $\int \frac{dx}{(1 - x^2)^{3/2}}$
77.  $\int \frac{(1 - x^2)^{5/2}}{x^8} dx$
78.  $\int \frac{dx}{\sqrt{3 - 2x - x^2}}$
79.  $\int \frac{x^3 + x + 1}{x^4 + 2x^2 + 1} dx$

$$80. \int \operatorname{arcsec} 2x \, dx \quad x > \frac{1}{2}$$

$$81. \int x \arcsin x \, dx$$

$$82. \int_0^2 \sqrt{1+4x^2} \, dx$$

$$83. \int_0^3 \frac{x^3}{\sqrt{x^2+9}} \, dx$$

$$84. \int_{\ln(3/4)}^{\ln(4/3)} \frac{e^x \, dx}{(1+e^{2x})^{3/2}}$$

$$85. \int_1^e \frac{e^x \, dx}{(1+e^{2x})^{3/2}}$$

$$86. \int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}}$$

$$87. \int_{1/2}^{1/4} \frac{dy}{y\sqrt{1+(\ln y)^2}}$$

$$88. \int_0^{\sqrt{3}/2} \frac{4}{9+4x^2} \, dx$$

$$89. \int_{1/12}^{1/4} \frac{dx}{\sqrt{x}(1+4x)}$$

$$90. \int_{8\sqrt{2}}^{16} \frac{dx}{\sqrt{x^2-64}}$$

$$91. \int_{\sqrt{2}}^2 \frac{\sqrt{x^2-1}}{x} \, dx$$

$$92. \int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2-1}}$$

$$93. \int_4^6 \frac{x^2}{\sqrt{x^2-9}} \, dx$$

$$94. \int_{\sqrt{3}}^2 \frac{\sqrt{x^2-3}}{x} \, dx$$

$$95. \int_0^{\sqrt{3}/2} \frac{x^2}{(1-x^2)^{3/2}} \, dx$$

$$96. \int_0^{\sqrt{3}/2} \frac{1}{(1-x^2)^{5/2}} \, dx$$

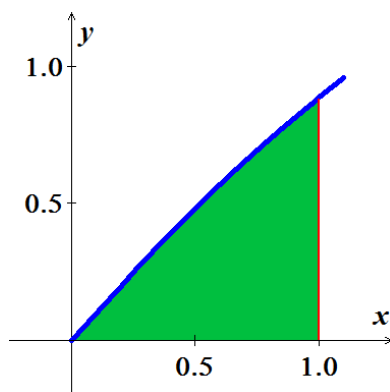
$$97. \int_0^{3/5} \sqrt{9-25x^2} \, dx$$

$$98. \int_0^{3/2} \frac{dx}{\sqrt{9-x^2}}$$

$$99. \int_0^3 \frac{dx}{\sqrt{9-x^2}}$$

$$100. \int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2} \, dx$$

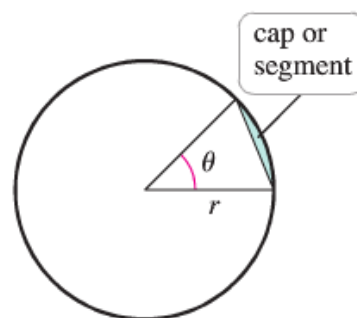
101. Consider the region bounded by the graph  $y = \sqrt{x \tan^{-1} x}$  and  $y = 0$  for  $0 \leq x \leq 1$ . Find the volume of the solid formed by revolving this region about the  $x$ -axis.



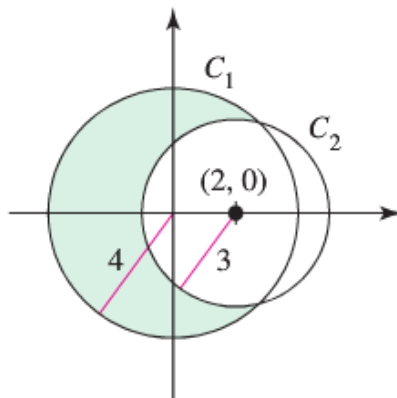
102. Use two approach to show that the area of a cap (or segment) of a circle of radius  $r$  subtended by an angle  $\theta$  is given by

$$A_{seg} = \frac{1}{2} r^2 (\theta - \sin \theta)$$

- a) Find the area using geometry (no calculus).  
b) Find the area using calculus

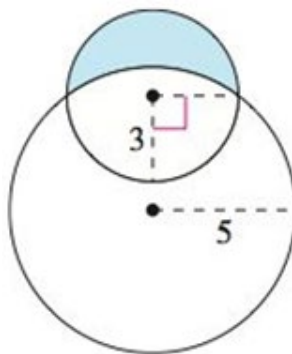


103. A lune is a crescent-shaped region bounded by the arcs of two circles. Let  $C_1$  be a circle of radius 4 centered at the origin. Let  $C_2$  be a circle of radius 3 centered at the point  $(2, 0)$ . Find the area of the lune that lies inside  $C_1$  and outside  $C_2$ .



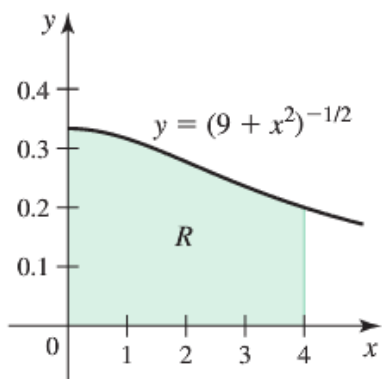


104. The crescent-shaped region bounded by two circles forms a lune. Find the area of the lune given that the radius of the smaller circle is 3 and the radius of the larger circle is 5.



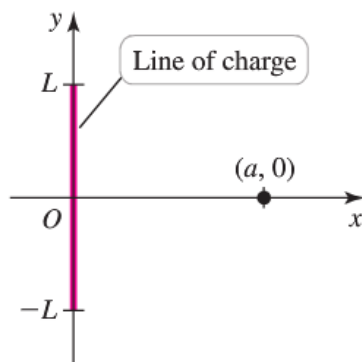
105. Consider the function  $f(x) = (9 + x^2)^{-1/2}$  and the region  $R$  on the interval  $[0, 4]$ .

- Find the area of  $R$ .
- Find the volume of the solid generated when  $R$  is revolved about the  $x$ -axis.
- Find the volume of the solid generated when  $R$  is revolved about the  $y$ -axis.



106. A total of  $Q$  is distributed uniformly on a line segment of length  $2L$  along the  $y$ -axis. The  $x$ -component of the electric field at a point  $(a, 0)$  is given by

$$E_x = \frac{kQa}{2L} \int_{-L}^L \frac{dy}{(a^2 + y^2)^{3/2}}$$



Where  $k$  is a physical constant and  $a > 0$

a) Confirm that  $E_x(a) = \frac{kQ}{a\sqrt{a^2 + L^2}}$

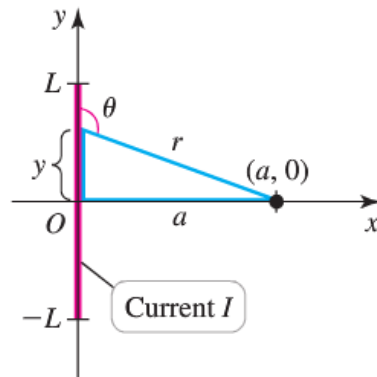
b) Letting  $\rho = \frac{Q}{2L}$  be the charge density on the line segment, show that if

$$L \rightarrow \infty, \text{ then } E_x = \frac{2k\rho}{a}$$

**107.** A long, straight wire of length  $2L$  on the  $y$ -axis carries a current  $I$ . according to the Biot-Savart Law, the magnitude of the field due to the current at a point  $(a, 0)$  is given by

$$B(a) = \frac{\mu_0 I}{4\pi} \int_{-L}^L \frac{\sin \theta}{r^2} dy$$

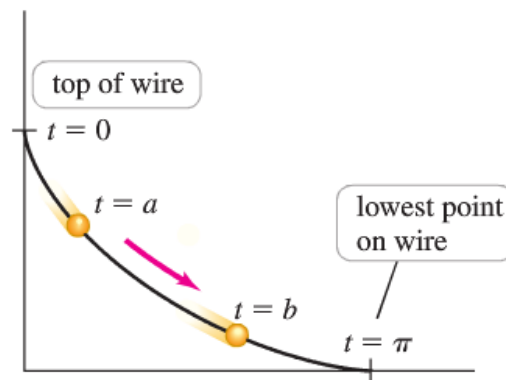
Where  $\mu_0$  is a physical constant,  $a > 0$ , and  $\theta$ ,  $r$ , and  $y$  are related to the figure



a) Show that the magnitude of the magnetic field at  $(a, 0)$  is  $B(a) = \frac{\mu_0 IL}{2\pi a \sqrt{a^2 + L^2}}$

b) What is the magnitude of the magnetic field at  $(a, 0)$  due to an infinitely long wire ( $L \rightarrow \infty$ )?

**108.** The cycloid is the curve traced by a point on the rim of a rolling wheel. Imagine a wire shaped like an inverted cycloid.



A bead sliding down this wire without friction has some remarkable properties. Among all wire shapes, the cycloid is the shape that produces the fastest descent time. It can be shown that the descent time between any two points  $0 \leq a < b \leq \pi$  on the curve is

$$\text{descent time} = \int_a^b \sqrt{\frac{1 - \cos t}{g(\cos a - \cos t)}} dt$$

Where  $g$  is the acceleration due to gravity,  $t = 0$  corresponds to the top of the wire, and  $t = \pi$  corresponds to the lowest point on the wire.

- a) Find the descent time on the interval  $[a, b]$ .
- b) Show that when  $b = \pi$ , the descent time is the same for all values of  $a$ ; that is, the descent time to the bottom of the wire is the same for all starting points.

**109.** Find the area of the region bounded by the curve  $f(x) = (16 + x^2)^{-3/2}$  and the  $x$ -axis on the interval  $[0, 3]$

**110.** Find the length of the curve  $y = ax^2$  from  $x = 0$  to  $x = 10$ , where  $a > 0$  is a real number.

**111.** Find the arc length of the graph of  $f(x) = \frac{1}{2}x^2$  from  $x = 0$  to  $x = 1$

**112.** A projectile is launched from the ground with an initial speed  $V$  at an angle  $\theta$  from the horizontal. Assume that the  $x$ -axis is the horizontal ground and  $y$  is the height above the ground. Neglecting air resistance and letting  $g$  be the acceleration due to gravity, it can be shown that the trajectory of the projectile is given by

$$y = -\frac{1}{2}kx^2 + y_{\max} \quad \text{where} \quad k = \frac{g}{(V \cos \theta)^2}$$

$$\text{and} \quad y_{\max} = \frac{(V \sin \theta)^2}{2g}$$

a) Note that the high point of the trajectory occurs at  $(0, y_{\max})$ . If the projectile is on the ground at  $(-a, 0)$  and  $(a, 0)$ , what is  $a$ ?

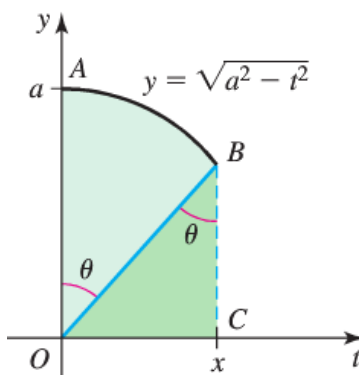
b) Show that the length of the trajectory (arc length) is  $2 \int_0^a \sqrt{1 + k^2 x^2} dx$

c) Evaluate the arc length integral and express your result in the terms of  $V$ ,  $g$ , and  $\theta$ .

d) For fixed value of  $V$  and  $g$ , show that the launch angle  $\theta$  that maximizes the length of the trajectory satisfies  $(\sin \theta) \ln(\sec \theta + \tan \theta) = 1$

113. Let  $F(x) = \int_0^x \sqrt{a^2 - t^2} dt$ . The figure shows that

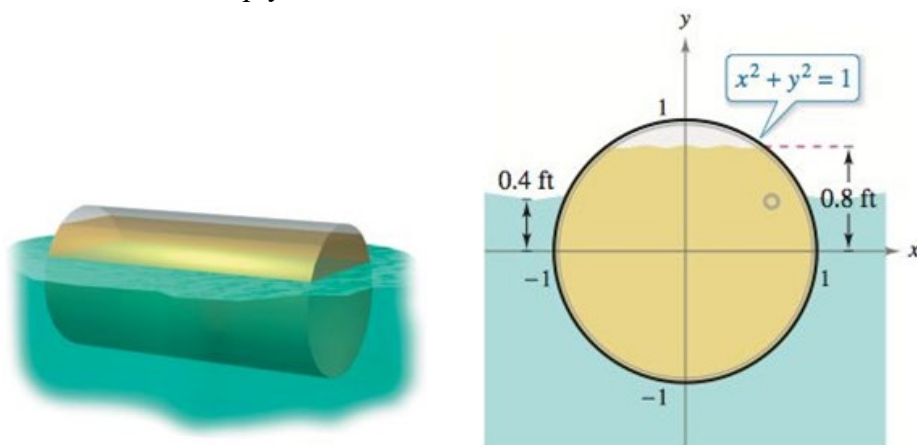
$$F(x) = \text{area of sector } OAB + \text{area of triangle } OBC$$



a) Use the figure to prove that  $F(x) = \frac{a^2 \sin^{-1}\left(\frac{x}{a}\right)}{2} + \frac{x\sqrt{a^2 - x^2}}{2}$

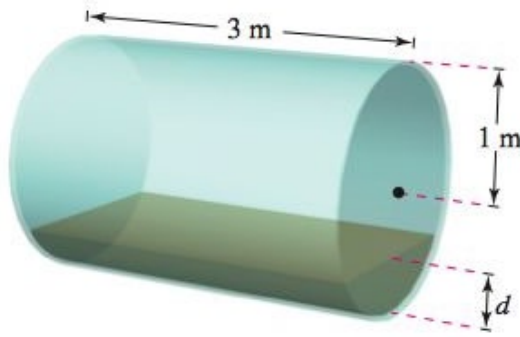
b) Conclude that  $\int \sqrt{a^2 - x^2} dx = \frac{a^2 \sin^{-1}\left(\frac{x}{a}\right)}{2} + \frac{x\sqrt{a^2 - x^2}}{2} + C$

114. A sealed barrel of oil (weighing 48 pounds per cubic foot) is floating in seawater (weighing 64 pounds per cubic foot). The barrel is not completely full of oil. With the barrel lying on its side, the top 0.2 foot of the barrel is empty.



Compare the fluid forces against one end of the barrel from the inside and from the outside.

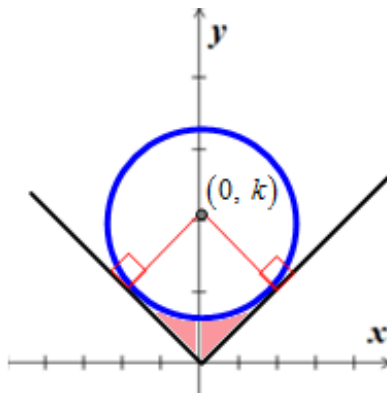
- 115.** The axis of a storage tank in the form of a right circular cylinder is horizontal. The radius and length of the tank are 1 meter and 3 meters, respectively.



- Determine the volume of fluid in the tank as a function of its depth  $d$ .
- Graph the function in part (a).
- Design a dip stick for the tank with markings of  $\frac{1}{4}$ ,  $\frac{1}{2}$ , and  $\frac{3}{4}$
- Fluid is entering the tank at a rate of  $\frac{1}{4} \text{ m}^3/\text{min}$ . Determine the rate of change of the depth of the fluid as a function of its depth  $d$ .
- Graph the function in part (d). When will the rate of change of the depth be minimum?

- 116.** The surface of a machine part is the region between the graphs of

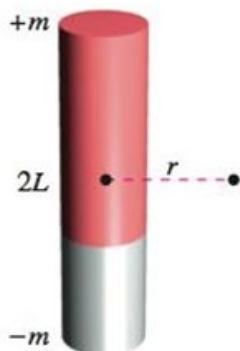
$$y = |x| \quad \text{and} \quad x^2 + (y - k)^2 = 25$$



- Find  $k$  when the circle is tangent to the graph of  $y = |x|$
- Find the area of the surface of the machine part.
- Find the area of the surface of the machine part as a function of the radius  $r$  of the circle.

117. The field strength  $H$  of a magnet of length  $2L$  on a particle  $r$  units from the center of the magnet is

$$H = \frac{2mL}{(r^2 + L^2)^{3/2}}$$



Where  $\pm m$  are the poles of the magnet

Find the average field strength as the particle moves from 0 to  $R$  **units** from the center by evaluating the integral

$$\frac{1}{R} \int_0^R \frac{2mL}{(r^2 + L^2)^{3/2}} dr$$

## Section 2.4 – Partial Fractions

This section shows how to express a rational; function as a sum of simpler functions, called *partial fractions*.

### Example

Evaluate  $\int \frac{5x-3}{x^2-2x-3} dx$

### Solution

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}$$

$$5x-3 = (A+B)x - 3A + B$$

$$x \quad A+B=5$$

$$x^0 \quad -3A+B=-3$$

$$A+B=5$$

$$\begin{array}{r} - \\ 3A-B=3 \end{array}$$
$$\hline 4A=8$$

$$\underline{A=2, \quad B=3}$$

$$\int \frac{5x-3}{x^2-2x-3} dx = \int \left( \frac{2}{x+1} + \frac{3}{x-3} \right) dx$$
$$\underline{= 2\ln|x+1| + 3\ln|x-3| + C}$$

### Example

Use partial fractions to evaluate  $\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)} dx$

### Solution

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

$$x^2+4x+1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)$$

$$= Ax^2 + 4Ax + 3A + Bx^2 + 2Bx - 3B + Cx^2 - C$$

$$x^2 \quad A + B + C = 1$$

$$x \quad 4A + 2B = 4$$

$$x^0 \quad 3A - 3B - C = 1$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 3 & -3 & -1 \end{vmatrix} = -16$$

$$\Delta_A = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 1 & -3 & -1 \end{vmatrix} = -12$$

$$\Delta_B = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 4 & 0 \\ 3 & 1 & -1 \end{vmatrix} = -8$$

$$\Delta_C = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 4 \\ 3 & -3 & 1 \end{vmatrix} = 4$$

$$A = \frac{-12}{-16} = \frac{3}{4}$$

$$B = \frac{-8}{-16} = \frac{1}{2}$$

$$C = \frac{4}{-16} = -\frac{1}{4}$$

$$\int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)} dx = \int \left( \frac{3}{4} \frac{1}{x-1} + \frac{1}{2} \frac{1}{x+1} - \frac{1}{4} \frac{1}{x+3} \right) dx$$

$$= \frac{\frac{3}{4} \ln|x-1| + \frac{1}{2} \ln|x+1| - \frac{1}{4} \ln|x+3| + K}{}$$



## Method of Partial Fractions ( $f(x)/g(x)$ *Proper*)

1. Let  $(x-r)$  be a linear factor of  $g(x)$ . Suppose that  $(x-r)^m$  is the highest power of  $(x-r)$  that divides  $g(x)$ . Then,

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \dots + \frac{A_m}{(x-r)^m}$$

2. Let  $x^2 + px + q$  be an irreducible quadratic function of  $g(x)$  has no real roots. Suppose that  $(x^2 + px + q)^n$  is the highest power. Then

$$\frac{B_1x + C_1}{(x^2 + px + q)} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \dots + \frac{B_nx + C_n}{(x^2 + px + q)^n}$$

3. Set the original fraction  $\frac{f(x)}{g(x)}$  equal to the sum of these partial fractions.
4. Equate the coefficients of corresponding powers of  $x$  and solve the resulting equations for the undetermined coefficients.

### Example

Use partial fractions to evaluate  $\int \frac{6x+7}{(x+2)^2} dx$

### Solution

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$\begin{aligned} 6x+7 &= A(x+2) + B \\ &= Ax + 2A + B \end{aligned}$$

$$\rightarrow \begin{cases} \boxed{A=6} \\ 2A+B=7 \rightarrow \boxed{B=7-12=-5} \end{cases}$$

$$\int \frac{6x+7}{(x+2)^2} dx = \int \left( \frac{6}{x+2} - \frac{5}{(x+2)^2} \right) dx$$

$$d(x+2) = dx$$

$$= \int \frac{6}{x+2} dx - 5 \int (x+2)^{-2} d(x+2)$$

$$= 6 \ln|x+2| + 5(x+2)^{-1} + C$$

$$= \boxed{6 \ln|x+2| + \frac{5}{x+2} + C}$$

### Example

Use partial fractions to evaluate  $\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx$

### Solution

$$\begin{array}{r} 2x \\ x^2 - 2x - 3 \overline{) 2x^3 - 4x^2 - x - 3} \\ \underline{2x^3 - 4x^2 - 6x} \phantom{- 3} \\ 5x - 3 \end{array}$$

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}$$

$$5x - 3 = (A + B)x - 3A + B$$

$$x \quad A + B = 5$$

$$x^0 \quad -3A + B = -3$$

$$A + B = 5$$

$$\begin{array}{r} - \quad 3A - B = 3 \\ \hline 4A = 8 \end{array}$$

$$\boxed{A = 2, \quad B = 3}$$

$$\begin{aligned} \int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx &= \int 2x \, dx + \int \frac{5x - 3}{x^2 - 2x - 3} \, dx \\ &= \int 2x \, dx + \int \frac{2}{x + 1} \, dx + \int \frac{3}{x - 3} \, dx \\ &= \underline{x^2 + 2 \ln|x + 1| + 3 \ln|x - 3| + C} \end{aligned}$$

### Example

Use partial fractions to evaluate  $\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} dx$

### Solution

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}$$

$$-2x + 4 = (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1)$$

$$\begin{aligned}
&= (Ax + B)(x^2 - 2x + 1) + C(x^3 - x^2 + x - 1) + Dx^2 + D \\
&= (A + C)x^3 + (-2A + B - C + D)x^2 + (A - 2B + C)x + B - C + D
\end{aligned}$$

$$x^3 \quad A + C = 0 \quad \rightarrow C = -A$$

$$x^2 \quad -2A + B - C + D = 0$$

$$x \quad A - 2B + C = -2 \quad \rightarrow -2B = -2 \quad \underline{B = 1}$$

$$x^0 \quad B - C + D = 4$$

$$-A + D = -1$$

$$\frac{A + D = 3}{2D = 2} \rightarrow \underline{D = 1 \quad A = 2}$$

$$\rightarrow \underline{C = -2}$$

$$\begin{aligned}
\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} dx &= \int \left( \frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2} \right) dx \\
&= \int \left( \frac{2x}{x^2 + 1} + \frac{1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2} \right) dx \\
&= \ln(x^2 + 1) + \tan^{-1} x - 2 \ln|x - 1| - \frac{1}{x - 1} + K
\end{aligned}$$

### Example

Use partial fractions to evaluate  $\int \frac{dx}{x(x^2 + 1)^2}$

### Solution

$$\frac{1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}$$

$$1 = A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + x(Dx + E)$$

$$1 = (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A$$

$$\begin{cases}
x^4 & A + B = 0 & \rightarrow \underline{B = -1} \\
x^3 & \underline{C = 0} \\
x^2 & 2A + B + D = 0 & \rightarrow \underline{D = -1} \\
x & C + E = 0 & \rightarrow \underline{E = 0} \\
x^0 & \underline{A = 1}
\end{cases}$$

$$\begin{aligned}
\int \frac{dx}{x(x^2+1)^2} &= \int \frac{dx}{x} - \int \frac{x \, dx}{x^2+1} - \int \frac{x \, dx}{(x^2+1)^2} \\
&= \int \frac{dx}{x} - \frac{1}{2} \int \frac{1}{x^2+1} \, d(x^2+1) - \frac{1}{2} \int \frac{1}{(x^2+1)^2} \, d(x^2+1) \\
&= \ln|x| - \frac{1}{2} \ln(x^2+1) + \frac{1}{2} \frac{1}{x^2+1} + K \\
&= \ln|x| - \ln\sqrt{x^2+1} + \frac{1}{2} \frac{1}{x^2+1} + K \\
&= \ln \frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + K
\end{aligned}$$

## Exercises      Section 2.4 – Partial Fractions

(1 – 102) Express the integrand as a sum of partial fractions and evaluate the integrals

1.  $\int \frac{dx}{x^2 + 2x}$

2.  $\int \frac{2x+1}{x^2 - 7x + 12} dx$

3.  $\int \frac{x+3}{2x^3 - 8x} dx$

4.  $\int \frac{x^2}{(x-1)(x^2 + 2x + 1)} dx$

5.  $\int \frac{8x^2 + 8x + 2}{(4x^2 + 1)^2} dx$

6.  $\int \frac{x^2 + x}{x^4 - 3x^2 - 4} dx$

7.  $\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{(\theta^2 + 1)^3} d\theta$

8.  $\int \frac{x^4}{x^2 - 1} dx$

9.  $\int \frac{16x^3}{4x^2 - 4x + 1} dx$

10.  $\int \frac{e^{4x} + 2e^{2x} - e^x}{e^{2x} + 1} dx$

11.  $\int \frac{\sin \theta d\theta}{\cos^2 \theta + \cos \theta - 2}$

12.  $\int \frac{(x-2)^2 \tan^{-1}(2x) - 12x^3 - 3x}{(4x^2 + 1)(x-2)^2} dx$

13.  $\int \frac{\sqrt{x+1}}{x} dx$

14.  $\int \frac{x^3 - 2x^2 + 3x - 4}{x^2 + 1} dx$

15.  $\int \frac{4x^2 + 2x + 4}{x+1} dx$

16.  $\int \frac{3x^2 + 7x - 2}{x^3 - x^2 - 2x} dx$

17.  $\int \frac{3x^2 + 2x + 5}{(x-1)(x^2 - x - 20)} dx$

18.  $\int \frac{5x^2 - 3x + 2}{x^3 - 2x^2} dx$

19.  $\int \frac{7x^2 - 13x + 13}{(x-2)(x^2 - 2x + 3)} dx$

20.  $\int \frac{1}{x^2 - 5x + 6} dx$

21.  $\int \frac{1}{x^2 - 5x + 5} dx$

22.  $\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} dx$

23.  $\int \frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)} dx$

24.  $\int \frac{8x^3 + 13x}{(x^2 + 2)^2} dx$

25.  $\int \frac{\sin x}{\cos x + \cos^2 x} dx$

26.  $\int \frac{5 \cos x}{\sin^2 x + 3 \sin x - 4} dx$

$$27. \int \frac{e^x}{(e^x - 1)(e^x + 4)} dx$$

$$28. \int \frac{e^x}{(e^{2x} + 1)(e^x - 1)} dx$$

$$29. \int \frac{\sqrt{x}}{x-4} dx$$

$$30. \int \frac{1}{\sqrt{x} - \sqrt[3]{x}} dx$$

$$31. \int \frac{dx}{1 + \sin x}$$

$$32. \int \frac{dx}{2 + \cos x}$$

$$33. \int \frac{dx}{1 - \cos x}$$

$$34. \int \frac{dx}{1 + \sin x + \cos x}$$

$$35. \int \frac{1}{x^2 - 9} dx$$

$$36. \int \frac{5}{x^2 + 3x - 4} dx$$

$$37. \int \frac{2}{9x^2 - 1} dx$$

$$38. \int \frac{3-x}{3x^2 - 2x - 1} dx$$

$$39. \int \frac{x^2 + 12x + 12}{x^3 - 4x} dx$$

$$40. \int \frac{x^3 - x + 3}{x^2 + x - 2} dx$$

$$41. \int \frac{5x-2}{(x-2)^2} dx$$

$$42. \int \frac{2x^3 - 4x^2 - 15x + 4}{x^2 - 2x - 8} dx$$

$$43. \int \frac{x+2}{x^2 + 5x} dx$$

$$44. \int \frac{\sec^2 x}{\tan^2 x + 5 \tan x + 6} dx$$

$$45. \int \frac{\sec^2 x}{\tan x (\tan x + 1)} dx$$

$$46. \int \frac{x dx}{x^2 + 4x + 3}$$

$$47. \int \frac{x+1}{x^2 (x-1)} dx$$

$$48. \int \frac{2x^3 + x^2 - 21x + 24}{x^2 + 2x - 8} dx$$

$$49. \int \frac{8x+5}{2x^2 + 3x + 1} dx$$

$$50. \int \frac{2x^2 + 7x + 4}{x^3 + 2x^2 + 2x} dx$$

$$51. \int \frac{3x^3 + 4x^2 + 6x}{(x+1)^2 (x^2 + 4)} dx$$

$$52. \int \frac{x^2 - 4}{x^2 + 4} dx$$

$$53. \int \frac{dx}{x^2 - 2x - 15}$$

$$54. \int \frac{3x^2 + x - 3}{x^2 - 1} dx$$

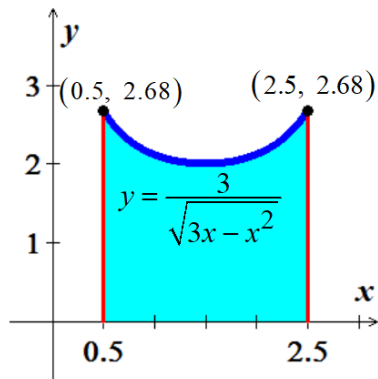
$$55. \int \frac{2x^2 - 4x}{x^2 - 4} dx$$

56.  $\int \frac{dx}{x^3 - 2x^2}$
57.  $\int \frac{dx}{x^2 - x - 2}$
58.  $\int \frac{4x^2 + 13x - 9}{x^3 + 2x^2 - 3x} dx$
59.  $\int \frac{3x^3 - 18x^2 + 29x - 4}{(x+1)(x-2)^3} dx$
60.  $\int \frac{x^2 - x - 21}{2x^3 - x^2 + 8x - 4} dx$
61.  $\int \frac{5x^3 - 3x^2 + 7x - 3}{(x^2 + 1)^2} dx$
62.  $\int \frac{2x^4 - 2x^3 + 6x^2 - 5x + 1}{x^3 - x^2 + x - 1} dx$
63.  $\int \frac{81}{x^3 - 9x^2} dx$
64.  $\int \frac{10x}{x^2 - 2x - 24} dx$
65.  $\int \frac{x+1}{x^2(x^2 + 4)} dx$
66.  $\int \frac{1+x^2}{(x+1)^3} dx$
67.  $\int \frac{6}{x^2 - 1} dx$
68.  $\int \frac{21x^2}{x^3 - x^2 - 12x} dx$
69.  $\int \frac{x+1}{x^3 + 3x^2 - 18x} dx$
70.  $\int \frac{x^2 + 12x - 4}{x^3 - 4x} dx$
71.  $\int \frac{6x^2}{x^4 - 5x^2 + 4} dx$
72.  $\int \frac{4x - 2}{x^3 - x} dx$
73.  $\int \frac{16x^2}{(x-6)(x+2)^2} dx$
74.  $\int \frac{8(x^2 + 4)}{x(x^2 + 8)} dx$
75.  $\int \frac{x^2 + x + 2}{(x+1)(x^2 + 1)} dx$
76.  $\int \frac{2}{x(x^2 + 1)^2} dx$
77.  $\int \frac{1}{(x+1)(x^2 + 2x + 2)^2} dx$
78.  $\int \frac{2-x}{x^2 + x} dx$
79.  $\int \frac{3x+11}{(x+2)(x+3)} dx$
80.  $\int \frac{1}{x^2 - a^2} dx$
81.  $\int \frac{1}{x^2 + 5x + 6} dx$
82.  $\int \frac{x^3 + 6x^2 + 3x + 6}{x^3 + 2x^2} dx$
83.  $\int \frac{2x^2 + 5x - 1}{x^3 + x^2 - 2x} dx$

$$\begin{aligned}
84. & \int \frac{3x+6}{x^3+2x^2-3x} dx \\
85. & \int \frac{3x^2+2x-2}{x^3-1} dx \\
86. & \int \frac{x^3+5x^2+2x-4}{x^4-1} dx \\
87. & \int \frac{x^2+4x}{(x^2+4)(x-2)^2} dx \\
88. & \int \frac{x^2+2x+3}{(x-1)(x+1)^2} dx \\
89. & \int \frac{x^4-x^3+3x^2-x+2}{(x-1)(x^2+2)^2} dx \\
90. & \int \frac{-x^2+11x+18}{(x-1)(x+1)(x^2+3x+3)} dx \\
91. & \int \frac{x^3+5x^2+2x-4}{x(x^2+4)^2} dx \\
92. & \int_{-1}^2 \frac{5x}{x^2-x-6} dx \\
93. & \int_0^5 \frac{2}{x^2-4x-32} dx
\end{aligned}$$

$$\begin{aligned}
94. & \int_0^1 \frac{dx}{(x+1)(x^2+1)} \\
95. & \int_{-1/2}^{1/2} \frac{x^2+1}{x^2-1} dx \\
96. & \int_0^2 \frac{3}{4x^2+5x+1} dx \\
97. & \int_1^5 \frac{x-1}{x^2(x+1)} dx \\
98. & \int_0^1 \frac{x^2-x}{x^2+x+1} dx \\
99. & \int_4^8 \frac{y dy}{y^2-2y-3} \\
100. & \int_1^{\sqrt{3}} \frac{3x^2+x+4}{x^3+x} dx \\
101. & \int_0^{\pi/2} \frac{dx}{\sin x + \cos x} \\
102. & \int_0^{\pi/3} \frac{\sin \theta}{1-\sin \theta} d\theta
\end{aligned}$$

103. Find the volume of the solid generated by the revolving the shaded region about x-axis





(104 – 105) Find the area of the region bounded by the graphs of

104.  $y = \frac{12}{x^2 + 5x + 6}$ ,  $y = 0$ ,  $x = 0$ , and  $x = 1$       105.  $y = \frac{7}{16 - x^2}$  and  $y = 1$

106. Find the length of the graph of the function  $y = \ln(1 - x^2)$ ,  $0 \leq x \leq \frac{1}{2}$

107. The region in the first quadrant that is enclosed by the  $x$ -axis, the curve  $y = \frac{5}{x\sqrt{5-x}}$ , and the lines  $x = 1$  and  $x = 4$  is revolved about the  $x$ -axis to generate a solid. Find the volume of the solid.

108. Consider the region bounded by the graphs  $y = \frac{2x}{x^2 + 1}$ ,  $y = 0$ ,  $x = 0$ , and  $x = 3$ .

- a) Find the volume of the solid generated by revolving the region about the  $x$ -axis
- b) Find the centroid of the region.

109. Consider the region bounded by the graph  $y^2 = \frac{(2-x)^2}{(1+x)^2}$ ,  $0 \leq x \leq 1$ .

Find the volume of the solid generated by revolving this region about the  $x$ -axis.

110. A single infected individual enters a community of  $n$  susceptible individuals. Let  $x$  be the number of newly infected individuals at time  $t$ . The common epidemic model assumes that the disease spreads at a rate proportional to the product of the total number infected and the number not yet infected. So,

$$\frac{dx}{dt} = k(x+1)(n-x) \text{ and you obtain}$$

$$\int \frac{1}{(x+1)(n-x)} dx = \int k dt$$

Solve for  $x$  as a function of  $t$ .

111. Evaluate  $\int_0^1 \frac{x}{1+x^4} dx$  in **two** different ways.

## Section 2.5 – Numerical Integration

### Absolute and Relative Error

#### **Definition**

Suppose  $c$  is a computed numerical solution to a problem having an exact solution  $x$ .

There are two common measures of the error in  $c$  as an approximation to  $x$ :

$$\text{absolute error} = |c - x| \quad \& \quad \text{relative error} = \frac{|c - x|}{|x|} \quad (\text{if } x \neq 0)$$

#### **Example**

The ancient Greeks used  $\frac{22}{7}$  to approximate the value of  $\pi$ . Determine the absolute and relative error in this approximation to  $\pi$ .

#### **Solution**

$$\begin{aligned} \text{absolute error} &= \left| \frac{22}{7} - \pi \right| \\ &\approx 0.00126 \end{aligned}$$

$$\begin{aligned} \text{relative error} &= \frac{\left| \frac{22}{7} - \pi \right|}{\pi} \\ &\approx .000402 \\ &\approx .04\% \end{aligned}$$

## Midpoint Rule

### Definition

Suppose  $f$  is defined and integrable on  $[a, b]$ . The **midpoint Rule Approximation** to  $\int_a^b f(x)dx$  using  $n$  equally spaced subintervals on  $[a, b]$  is

$$\begin{aligned} M(n) &= f(m_1)\Delta x + f(m_2)\Delta x + \cdots + f(m_n)\Delta x \\ &= \sum_{k=1}^n f\left(\frac{x_{k-1} + x_k}{2}\right)\Delta x \end{aligned}$$

Where  $\Delta x = \frac{b-a}{n}$ ,

$$x_0 = a, \quad x_k = a + k\Delta x$$

$$m_k = \frac{x_{k-1} + x_k}{2} \text{ is the midpoint of } [x_{k-1}, x_k], \text{ for } k = 1, 2, \dots, n.$$

### Example

Approximate  $\int_2^4 x^2 dx$  using the Midpoint Rule with  $n = 4$  subinterval

### Solution

$$\text{With } a = 2, b = 4 \rightarrow \Delta x = \frac{4-2}{4} = 0.5$$

The grid points are:

$$x_0 = 2$$

$$x_1 = 2 + 0.5 = \underline{2.5}$$

$$x_2 = 2 + 2(.5) = \underline{3}$$

$$x_3 = 2 + 3(0.5) = \underline{3.5}$$

$$x_4 = 2 + 4(0.5) = \underline{4}$$

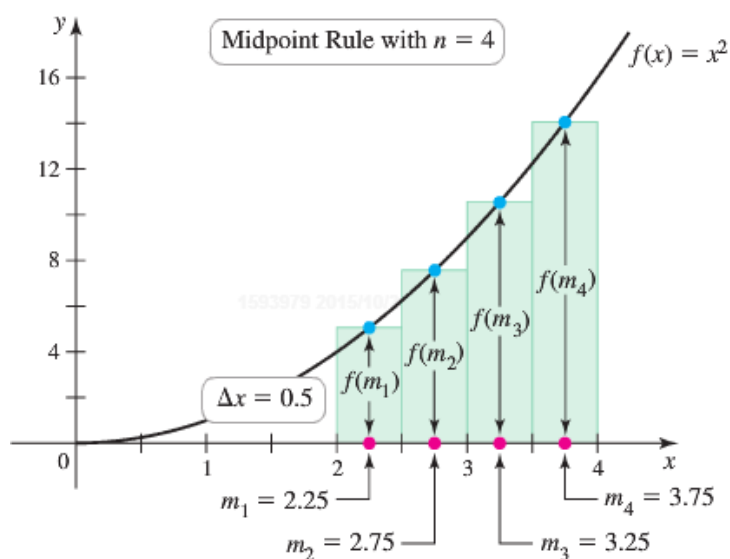
$$m_1 = \frac{2.5+2}{2} = \underline{2.25}$$

$$m_2 = \frac{3+2.5}{2} = \underline{2.75}$$

$$m_3 = \frac{2+3.5}{2} = \underline{3.25}$$

$$m_4 = \frac{3.5+4}{2} = \underline{3.75}$$

$$\begin{aligned}
 M(4) &= f(m_1)\Delta x + f(m_2)\Delta x + f(m_3)\Delta x + f(m_4)\Delta x \\
 &= (2.25^2 + 2.75^2 + 3.25^2 + 3.75^2)(0.5) \\
 &= 18.625
 \end{aligned}$$



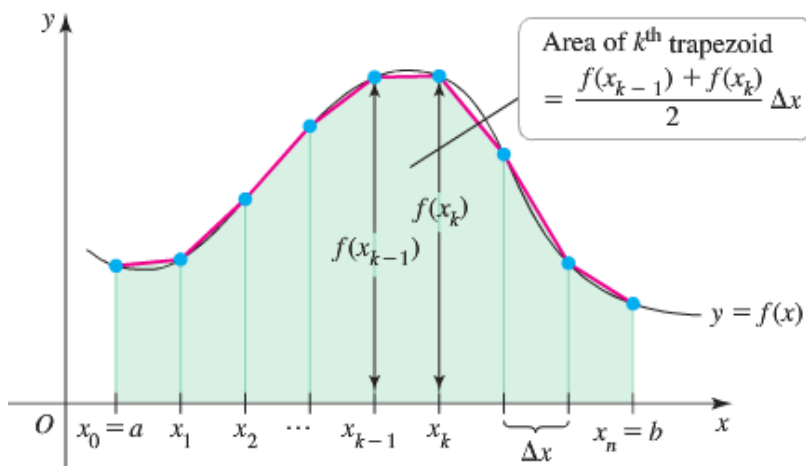
$$\begin{aligned}
 \text{Exact} &= \int_2^4 x^2 \, dx \\
 &= \left. \frac{1}{3}x^3 \right|_2^4 \\
 &= \frac{56}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{absolute error} &= \left| 18.625 - \frac{56}{3} \right| \\
 &\approx 0.0417
 \end{aligned}$$

$$\begin{aligned}
 \text{relative error} &= \frac{\left| 18.625 - \frac{56}{3} \right|}{\frac{56}{3}} \\
 &\approx .00223 = .223\%
 \end{aligned}$$

## Trapezoid Approximations

The **Trapezoid Rule** for the value of a definite integral is based on approximating the region between a curve and the  $x$ -axis with trapezoids instead of rectangles.



The length of each subinterval is  $\Delta x = \frac{b-a}{n}$  is called the **step size** or **mesh size**.

The area of a trapezoid:  $\Delta x \cdot \left( \frac{y_{i-1} + y_i}{2} \right)$

The area is the approximation by adding the areas of all trapezoids:

$$\begin{aligned} T &= \frac{1}{2}(y_0 + y_1)\Delta x + \frac{1}{2}(y_1 + y_2)\Delta x + \cdots + \frac{1}{2}(y_{n-2} + y_{n-1})\Delta x + \frac{1}{2}(y_{n-1} + y_n)\Delta x \\ &= \frac{1}{2}\Delta x (y_0 + y_1 + y_1 + y_2 + \cdots + y_{n-2} + y_{n-1} + y_{n-1} + y_n) \\ &= \frac{1}{2}\Delta x (y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-2} + 2y_{n-1} + y_n) \end{aligned}$$

### The Trapezoid Rule

If  $f$  is continuous on  $[a, b]$  and if a regular partition of  $[a, b]$  is determined by the numbers  $a = x_0, x_1, \dots, x_n = b$ , then

$$\begin{aligned} \int_a^b f(x) dx &\approx \frac{b-a}{2n} [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)] \\ T(n) &= \left( \frac{1}{2}f(x_0) + \sum_{k=1}^{n-1} f(x_k) + \frac{1}{2}f(x_n) \right) \Delta x \end{aligned}$$

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \dots, \quad x_{n-1} = a + (n-1)\Delta x, \quad x_n = b$$

Where  $\Delta x = \frac{b-a}{n}$  and  $x_0 = a, \quad x_k = a + k\Delta x$

## Error Estimate for the Trapezoidal Rule

If  $M$  is a positive real number such that  $|f''(x)| \leq M$  for all  $x$  in  $[a, b]$ , then the error involved in using the

Trapezoidal Rule is not greater than  $\frac{M(b-a)^3}{12n^2}$

### Example

Use the Trapezoid Rule with  $n = 4$  to estimate  $\int_1^2 x^2 dx$ . Compare the estimate with the exact value.

### Solution

$$|\Delta x| = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$$

$$x_0 = 1$$

$$x_1 = 1 + \frac{1}{4} = \frac{5}{4}$$

$$x_2 = 1 + 2\left(\frac{1}{4}\right) = \frac{6}{4}$$

$$x_3 = 1 + 3\left(\frac{1}{4}\right) = \frac{7}{4}$$

$$x_4 = 2$$

$$T = \frac{1}{2} \Delta x (y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$$

$$= \frac{1}{2} \cdot \frac{1}{4} \left( 1^2 + 2\left(\frac{5}{4}\right)^2 + 2\left(\frac{6}{4}\right)^2 + 2\left(\frac{7}{4}\right)^2 + 2^2 \right)$$

$$= \frac{1}{8} \left( 1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4 \right)$$

$$= \frac{75}{32}$$

$$\approx 2.34375$$

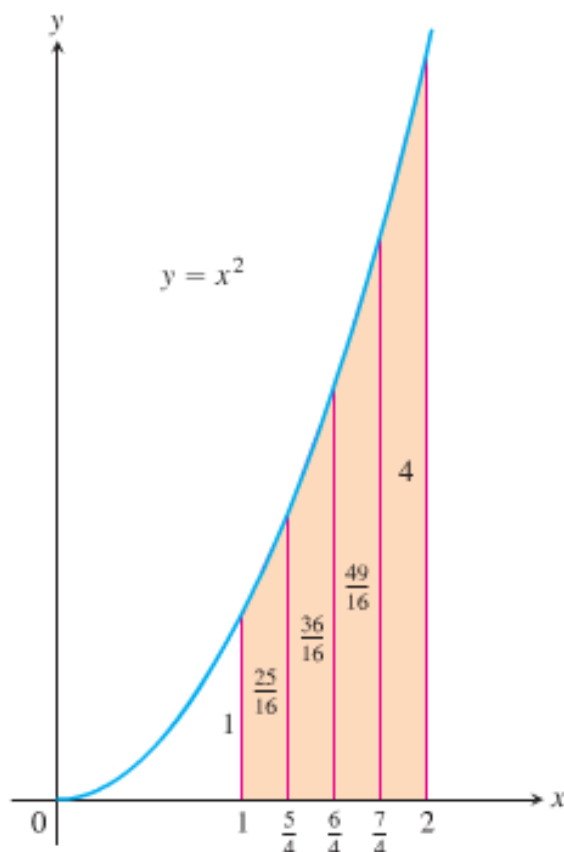
$$\int_1^2 x^2 dx = \frac{1}{3} x^3 \Big|_1^2$$

$$= \frac{1}{3} (2^3 - 1^3)$$

$$= \frac{7}{3} \approx 2.3333$$

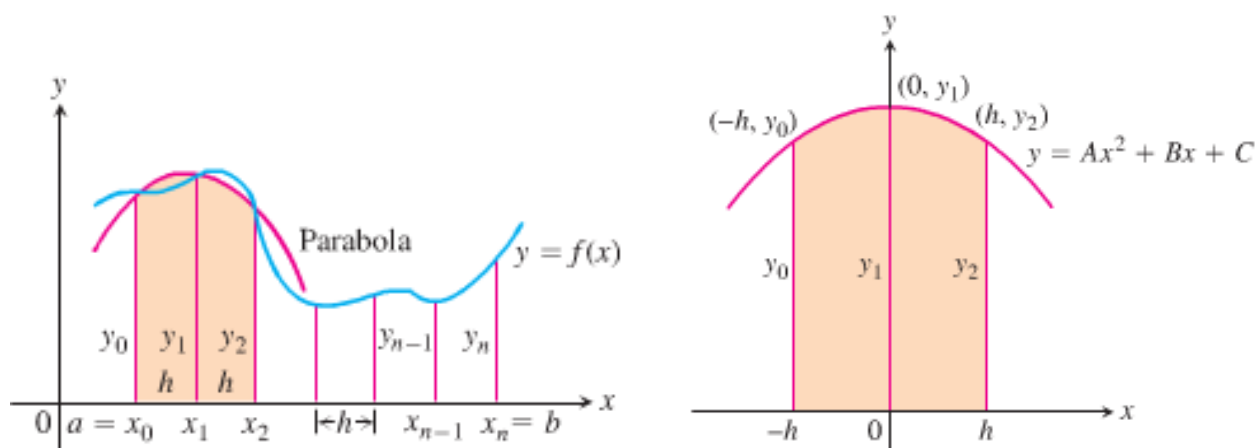
The difference:  $2.34375 - 2.3333 \approx 0.01042$

The percentage error:  $\frac{2.34375 - 2.3333}{2.3333} \approx 0.004466 \approx .446\%$



## Simpson's Rule: Approximations Using Parabolas

We partition the interval  $[a, b]$  into  $n$  subintervals of equal length  $h = \Delta x = \frac{b-a}{n}$   $n$  : even number



The parabola has an equation of the form:  $y = Ax^2 + Bx + C$

So the area under it from  $x = -h$  to  $x = h$  is

$$\begin{aligned}
 A_p &= \int_{-h}^h (Ax^2 + Bx + C) dx \\
 &= \left[ \frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx \right]_{-h}^h \\
 &= \frac{A}{3}h^3 + \frac{B}{2}h^2 + Ch - \left( \frac{A}{3}(-h)^3 + \frac{B}{2}(-h)^2 + C(-h) \right) \\
 &= \frac{A}{3}h^3 + \frac{B}{2}h^2 + Ch + \frac{A}{3}h^3 - \frac{B}{2}h^2 + Ch \\
 &= \frac{2}{3}Ah^3 + 2Ch \\
 &= \frac{h}{3}(2Ah^2 + 6C)
 \end{aligned}$$

Since the curve passes through the three points  $(-h, y_0)$ ,  $(0, y_1)$ , and  $(h, y_2)$

$$y_0 = Ah^2 - Bh + C \quad y_1 = C \quad y_2 = Ah^2 + Bh + C$$

$$C = y_1, \quad Ah^2 - Bh = y_0 - y_1$$

$$\frac{Ah^2 + Bh = y_2 - y_1}{2Ah^2 = y_0 - 2y_1 + y_2}$$

$$2Ah^2 = y_0 - 2y_1 + y_2$$

$$\begin{aligned}
 A_p &= \frac{h}{3}(2Ah^2 + 6C) \\
 &= \frac{h}{3}(y_0 - 2y_1 + y_2 + 6y_1) \\
 &= \frac{h}{3}(y_0 + 4y_1 + y_2)
 \end{aligned}$$

Computing the areas under all the parabolas and adding the results gives the approximation

$$\begin{aligned}\int_a^b f(x)dx &\approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n) \\ &= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)\end{aligned}$$

### ***Simpson's Rule***

To approximate  $\int_a^b f(x)dx$ , use  $S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \dots, \quad x_{n-1} = a + (n-1)\Delta x, \quad x_n = b$$

$$\text{Where } \Delta x = \frac{b-a}{n}$$

$$\int_a^b f(x)dx \approx \frac{b-a}{3n} \left[ f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right]$$

### ***Error Estimate for the Trapezoidal Rule***

If  $M$  is a positive real number such that  $|f^{(4)}(x)| \leq M$  for all  $x$  in  $[a, b]$ , then the error involved in using

the Simpson's Rule is not greater than  $\frac{M(b-a)^5}{180n^4}$

### ***Example***

Use Simpson's Rule with  $n = 4$  to approximate  $\int_0^2 5x^4 dx$

### **Solution**

$$\begin{aligned}\Delta x &= \frac{b-a}{n} \\ &= \frac{2-0}{4} \\ &= \frac{1}{2}\end{aligned}$$

$$x_0 = 0, \quad x_1 = \frac{1}{2}, \quad x_2 = 0 + 2\frac{1}{2} = 1, \quad x_3 = 0 + 3\frac{1}{2} = \frac{3}{2}, \quad x_4 = 2$$



$$\begin{aligned}
S &= \frac{\Delta x}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) \\
&= \frac{1}{3} \frac{1}{2} \left( 5(0)^4 + 4(5)\left(\frac{1}{2}\right)^4 + 2(5)(1)^4 + 4(5)\left(\frac{3}{2}\right)^4 + 5(2)^4 \right) \\
&= \frac{1}{6} \left( 0 + \frac{5}{4} + 10 + \frac{405}{4} + 80 \right) \\
&= \frac{1}{6} \left( \frac{385}{2} \right) \\
&= \frac{385}{12}
\end{aligned}$$

$\approx 32.08333$  |

The exact value is 32.

### Example

The table lists rates of change  $s'(t)$  in global sea level  $s(t)$  in various years from 1995 ( $t = 0$ ) to 2011 ( $t = 16$ ), with rates of change reported in  $mm/yr$ .

Years	1995	1998	2000	2002	2003	2007	2009	2011
$t$	0	3	5	7	8	12	14	16
$s'(t)$ ( $mm/yr$ )	0.51	5.19	4.39	2.21	5.24	0.63	4.19	2.38

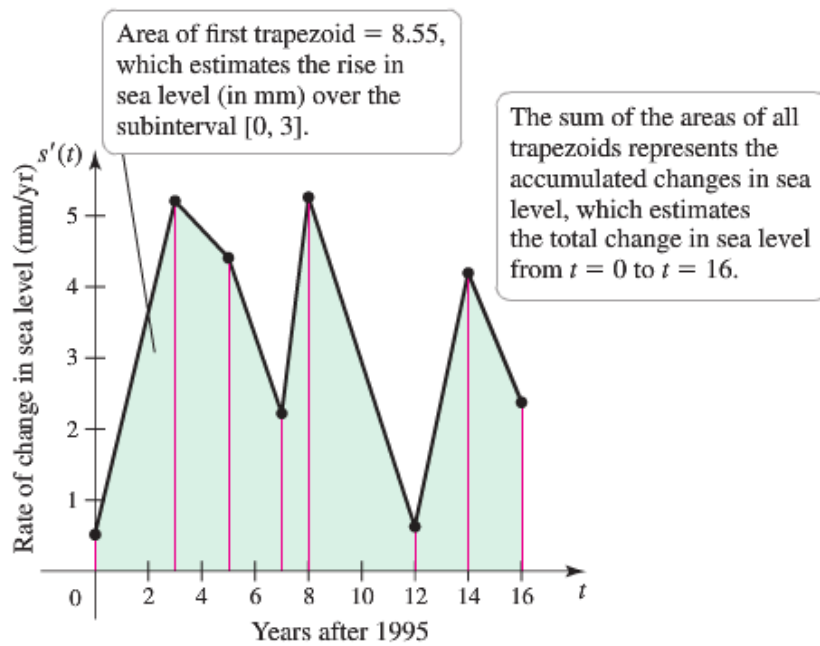
- Assuming  $s'(t)$  is continuous on  $[0, 16]$ , explain how a definite integral can be used to find the net change in sea level from 1995 to 2011; then write the definite integral.
- Use the data in the table and generalize the trapezoid Rule to estimate the value of the integral from part (a).

### Solution

- The net change in any quantity  $Q$  over the interval  $[a, b]$  is  $Q(b) - Q(a)$

Net change in  $s(t) = S(b) - S(a)$

$$= \int_0^{16} s'(t) \, dt$$



b) From the figure the values accompanied by 7 trapezoids whose area approximates  $\int_0^{16} s'(t) dt$

**Area** of the **first** trapezoid:

$$\begin{aligned} T_1 &= \frac{1}{2}(s'(0) + s'(3)) \cdot 3 \\ &= \frac{1}{2}(0.51 + 5.19) \cdot 3 \\ &= 8.55 \end{aligned}$$

$$\begin{aligned} T_2 &= \frac{1}{2}(s'(3) + s'(5)) \cdot 2 \\ &= \frac{1}{2}(5.19 + 4.39) \cdot 2 \\ &= 9.58 \end{aligned}$$

$$\begin{aligned} T_3 &= \frac{1}{2}(s'(5) + s'(7)) \cdot 2 \\ &= \frac{1}{2}(4.39 + 2.21) \cdot 2 \\ &= 6.6 \end{aligned}$$

$$\begin{aligned} T_4 &= \frac{1}{2}(s'(7) + s'(8)) \cdot 1 \\ &= \frac{1}{2}(2.21 + 5.24) \\ &= 3.725 \end{aligned}$$

$$\begin{aligned} T_5 &= \frac{1}{2}(s'(8) + s'(12)) \cdot 4 \\ &= \frac{1}{2}(5.24 + 0.63) \cdot 4 \\ &= 11.74 \end{aligned}$$

$$\begin{aligned}
 T_6 &= \frac{1}{2}(s'(12) + s'(14)) \cdot 2 \\
 &= \frac{1}{2}(0.63 + 4.19) \cdot 2 \\
 &= \underline{4.82} \text{ |}
 \end{aligned}$$

$$\begin{aligned}
 T_7 &= \frac{1}{2}(s'(14) + s'(16)) \cdot 2 \\
 &= \frac{1}{2}(4.19 + 2.38) \cdot 2 \\
 &= \underline{6.57} \text{ |}
 \end{aligned}$$

$$\begin{aligned}
 T(7) &= T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 \\
 &\approx \underline{51.585 \text{ mm}} \text{ |}
 \end{aligned}$$

## Exercises      Section 2.5 – Numerical Integration

(1 – 4) Find the *Midpoint Rule* approximations to

1.  $\int_0^1 \sin \pi x \, dx \quad n = 6 \text{ subintervals}$

3.  $\int_0^1 e^{-\sqrt{x}} \, dx \quad n = 6 \text{ subintervals}$

2.  $\int_0^\pi x^2 \sin x \, dx \quad n = 8 \text{ subintervals}$

4.  $\int_0^1 e^{-x} \, dx \quad n = 8 \text{ subintervals}$

(5 – 7) Estimate the minimum number of subintervals to approximate the integrals with an error of magnitude of  $10^{-4}$  by (a) the *Trapezoid Rule* and (b) *Simpson's Rule*.

5.  $\int_1^3 (2x-1) \, dx$

6.  $\int_{-1}^1 (x^2+1) \, dx$

7.  $\int_2^4 \frac{1}{(s-1)^2} \, ds$

(8 – 27) Find the *Trapezoid & Simpson's Rule* approximations and error to

8.  $\int_0^1 \sin \pi x \, dx \quad n = 6 \text{ subintervals}$

15.  $\int_{\pi/2}^\pi \frac{\sin x}{x} \, dx \quad n = 6 \text{ subintervals}$

9.  $\int_0^1 e^{-x} \, dx \quad n = 8 \text{ subintervals}$

16.  $\int_0^{\pi/4} x \tan x \, dx \quad n = 6 \text{ subintervals}$

10.  $\int_1^5 (3x^2 - 2x) \, dx \quad n = 8 \text{ subintervals}$

17.  $\int_0^1 e^{-x^2} \, dx \quad n = 10 \text{ subintervals}$

11.  $\int_0^{\pi/4} 3 \sin 2x \, dx \quad n = 8 \text{ subintervals}$

18.  $\int_0^2 \frac{1}{\sqrt{1+x^2}} \, dx \quad n = 10 \text{ subintervals}$

12.  $\int_0^8 e^{-2x} \, dx \quad n = 8 \text{ subintervals}$

19.  $\int_0^{1/2} \sin(e^{x/2}) \, dx \quad n = 8 \text{ subintervals}$

13.  $\int_{-1}^1 \sqrt{x^2+1} \, dx \quad n = 8 \text{ subintervals}$

20.  $\int_2^3 \frac{1}{\ln x} \, dx \quad n = 10 \text{ subintervals}$

14.  $\int_0^{1/2} \sin(x^2) \, dx \quad n = 4 \text{ subintervals}$

21.  $\int_1^2 e^{1/x} \, dx \quad n = 4 \text{ subintervals}$

22.  $\int_0^1 \ln(1 + e^x) dx \quad n = 8 \text{ subintervals}$

25.  $\int_0^3 \frac{1}{1 + x^5} dx \quad n = 6 \text{ subintervals}$

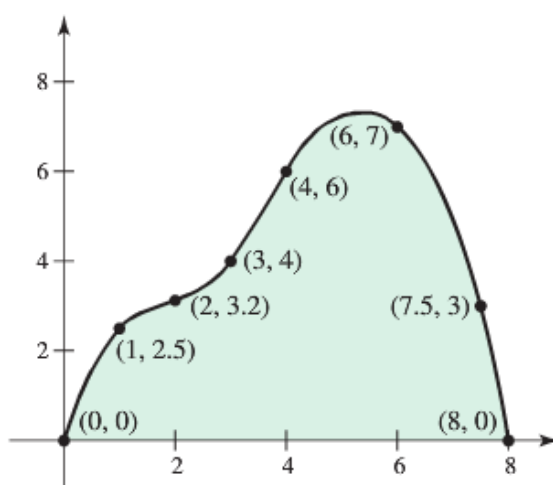
23.  $\int_0^1 x^5 e^x dx \quad n = 10 \text{ subintervals}$

26.  $\int_1^4 \frac{e^x}{x} dx \quad n = 10 \text{ subintervals}$

24.  $\int_0^4 \sqrt{x} \sin x dx \quad n = 8 \text{ subintervals}$

27.  $\int_1^2 \frac{dx}{x} \quad n = 10 \text{ subintervals}$

28. A piece of wood paneling must be cut in the shape shown below. The coordinates of several point on its curved surface are also shown (with units of inches).



- Estimate the surface area of the paneling using the Trapezoid Rule
  - Estimate the surface area of the paneling using a left Riemann sum.
  - Could two identical pieces be cut from a 9-in by 9-in piece of wood?
29. The region bounded by the curves  $y = \frac{1}{1 + e^{-x}}$ ,  $x = 0$  and  $x = 10$  is rotated about  $x$ -axis. Use Simpson's Rule with  $n = 10$  to estimate the volume of the resulting solid.
30. A pendulum with length  $L$  that makes a maximum angle  $\theta_0$  with the vertical. Using Newton's Second Law it can be shown that the period  $T$  (the time for one complete swing) is given by

$$T = 4 \sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

Where  $k = \sin\left(\frac{1}{2}\theta_0\right)$  and  $g$  is the acceleration due to gravity. If  $L = 1 \text{ m}$  and  $\theta_0 = 42^\circ$ , use Simpson's Rule with  $n = 10$  to find the period.

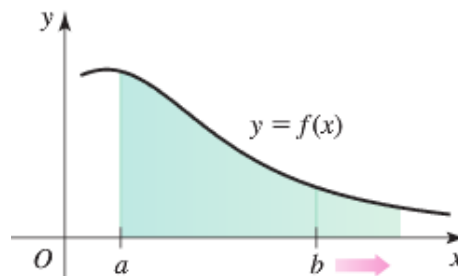
## Section 2.6 – Improper Integrals

### Definition

Integrals with infinite limits of integration are *improper integrals*.

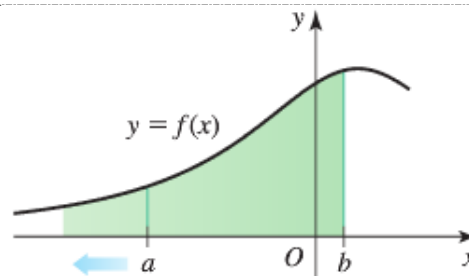
1. If  $f(x)$  is continuous on  $[a, \infty)$ , then

$$\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$$



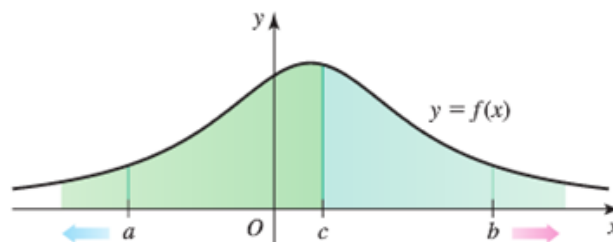
2. If  $f(x)$  is continuous on  $(-\infty, b]$ , then

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$$



3. If  $f(x)$  is continuous on  $(-\infty, \infty)$ , then

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx$$



In each case, if the limit is finite we say that the improper integral *converges* and that the limit is the *value* of the improper integral. If the limit fails to exist, the improper integral *diverges*.

### Example

Is the area under the curve  $y = \frac{\ln x}{x^2}$  from  $x = 1$  to  $x = \infty$  finite? If so, what is its value?

### Solution

$$\begin{aligned} \int_1^b \frac{\ln x}{x^2} dx &= -\frac{1}{x} \ln x \Big|_1^b - \int_1^b \left(-\frac{1}{x}\right) \left(\frac{1}{x}\right) dx \\ &= -\left(\frac{1}{b} \ln b - \ln 1\right) + \int_1^b \frac{1}{x^2} dx \\ &= -\frac{1}{b} \ln b + \left(-\frac{1}{x}\right) \Big|_1^b \end{aligned}$$

$$\begin{aligned} u &= \ln x & dv &= \frac{dx}{x^2} \\ du &= \frac{1}{x} dx & v &= -\frac{1}{x} \end{aligned}$$

$$= -\frac{1}{b} \ln b - \left(\frac{1}{b} - 1\right)$$

$$= -\frac{1}{b} \ln b - \frac{1}{b} + 1$$

$$\int_1^{\infty} \frac{\ln x}{x^2} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x^2} dx$$

$$= \lim_{b \rightarrow \infty} \left( -\frac{1}{b} \ln b - \frac{1}{b} + 1 \right)$$

$$= -\lim_{b \rightarrow \infty} \left( \frac{1/b}{1} \right) - 0 + 1$$

$$= -0 + 1$$

$$= 1$$

*L'Hôpital Rule*

### Example

Evaluate  $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

### Solution

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^{\infty} \frac{dx}{1+x^2}$$

$$\int_{-\infty}^0 \frac{dx}{1+x^2} = \lim_{a \rightarrow -\infty} \int_a^0 \frac{dx}{1+x^2}$$

$$= \lim_{a \rightarrow -\infty} \tan^{-1} x \Big|_a^0$$

$$= \lim_{a \rightarrow -\infty} \left( \tan^{-1} 0 - \tan^{-1} a \right)$$

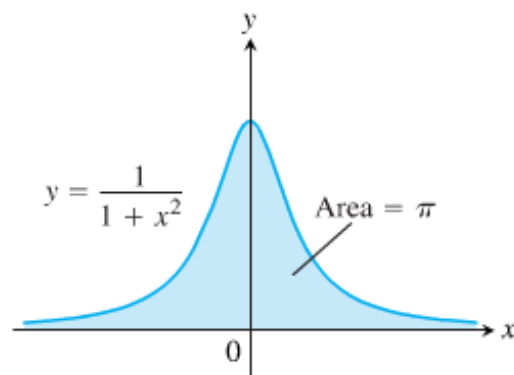
$$= 0 - \left( -\frac{\pi}{2} \right)$$

$$= \frac{\pi}{2}$$

$$\int_0^{\infty} \frac{dx}{1+x^2} = \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{1+x^2}$$

$$= \lim_{b \rightarrow \infty} \tan^{-1} x \Big|_0^b$$

$$= \lim_{b \rightarrow \infty} \left( \tan^{-1} b - \tan^{-1} 0 \right)$$



$$= \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2} \Big|$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^{\infty} \frac{dx}{1+x^2}$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi \Big|$$

### Example

For what value of  $p$  does the integral  $\int_1^{\infty} \frac{dx}{x^p}$  converge? When the integral does converge, what is its value?

### Solution

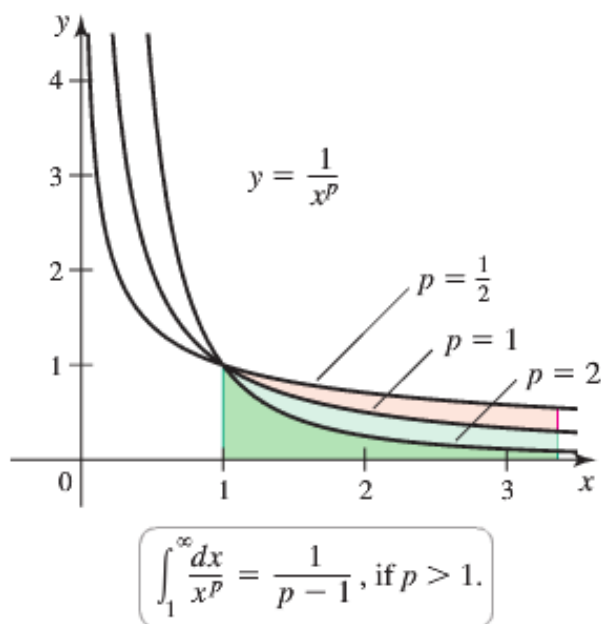
If  $p \neq 1$

$$\begin{aligned} \int_1^b \frac{dx}{x^p} &= \frac{x^{-p+1}}{-p+1} \Big|_1^b \\ &= \frac{1}{1-p} (b^{1-p} - 1) \end{aligned}$$

$$\begin{aligned} \int_1^{\infty} \frac{dx}{x^p} &= \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x^p} \\ &= \lim_{b \rightarrow \infty} \left[ \frac{1}{1-p} (b^{1-p} - 1) \right] \\ &= \begin{cases} \frac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{cases} \end{aligned}$$

If  $p = 1$

$$\begin{aligned} \int_1^{\infty} \frac{dx}{x^p} &= \int_1^{\infty} \frac{dx}{x} \\ &= \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x} \end{aligned}$$





$$\begin{aligned}
&= \lim_{b \rightarrow \infty} (\ln x) \Big|_1^b \\
&= \lim_{b \rightarrow \infty} (\ln b - \ln 1) \\
&= \underline{\infty}
\end{aligned}$$

## Integrands with Vertical Asymptotes

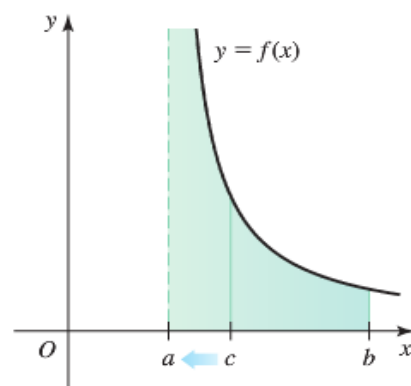
### Definition

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals**.

If the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

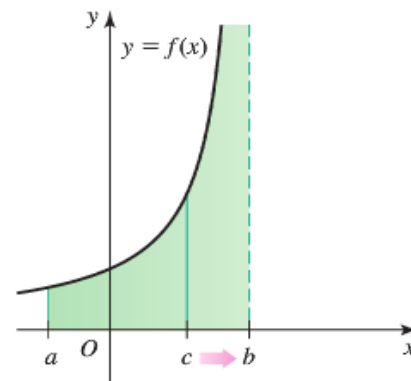
1. If  $f(x)$  is continuous on  $(a, b]$ , then

$$\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx$$



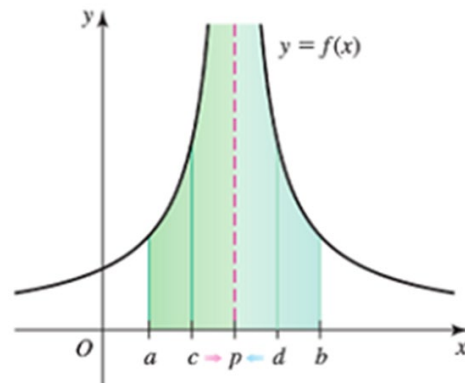
2. If  $f(x)$  is continuous on  $[a, b)$ , then

$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx$$



3. If  $f(x)$  is continuous on  $[a, b]$ , then

$$\int_a^b f(x) dx = \lim_{c \rightarrow p^-} \int_a^c f(x) dx + \lim_{d \rightarrow p^+} \int_d^b f(x) dx$$



### Example

Investigate the convergence of  $\int_0^1 \frac{1}{1-x} dx$

### Solution

$$\begin{aligned}\int_0^1 \frac{1}{1-x} dx &= \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{1-x} dx \\&= \lim_{b \rightarrow 1^-} \left( -\ln|1-x| \right) \Big|_0^b \\&= \lim_{b \rightarrow 1^-} \left( -\ln|1-b| + 0 \right) \\&= \infty\end{aligned}$$

The limit is infinite, so the integral diverges.

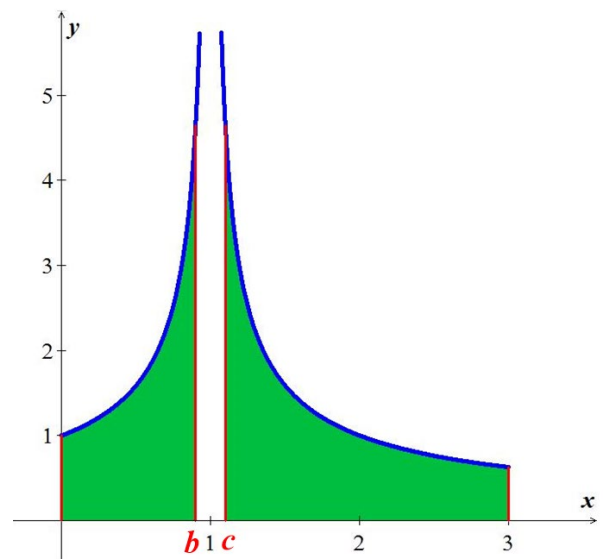
### Example

Evaluate  $\int_0^3 \frac{dx}{(x-1)^{2/3}}$

### Solution

The integrand has a vertical asymptote at  $x = 1$  and is continuous on  $[0, 1)$  and  $(1, 3]$ .

$$\begin{aligned}\int \frac{dx}{(x-1)^{2/3}} &= \int (x-1)^{-2/3} d(x-1) = 3(x-1)^{1/3} \\ \int_0^3 \frac{dx}{(x-1)^{2/3}} &= \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}} \\&= 3(x-1)^{1/3} \Big|_0^{1^-} + 3(x-1)^{1/3} \Big|_{1^+}^3 \\&= 3(0+1) + 3\left(\sqrt[3]{2} - 0\right) \\&= 3 + 3\sqrt[3]{2}\end{aligned}$$



### Example

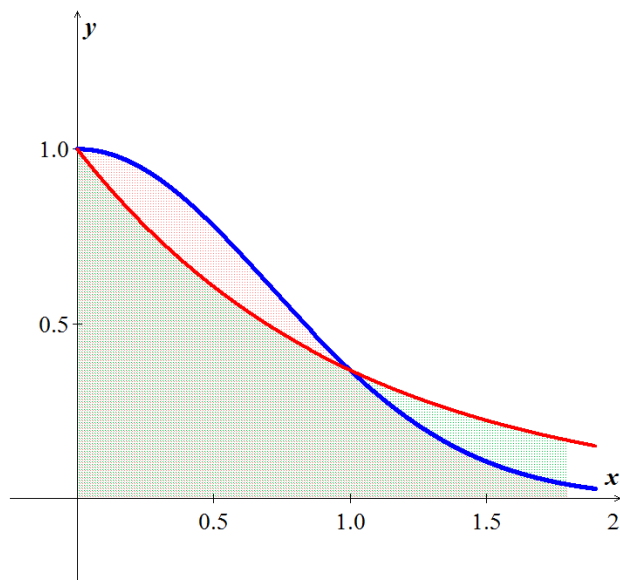
Does the integral  $\int_1^{\infty} e^{-x^2} dx$  converge?

### Solution

$$\int_1^{\infty} e^{-x^2} dx = \lim_{b \rightarrow \infty} \int_1^b e^{-x^2} dx$$

$$\begin{aligned} \int_1^b e^{-x^2} dx &\leq \int_1^b e^{-x} dx \\ &= -e^{-b} + e^{-1} \\ &< e^{-1} \approx 0.36788 \end{aligned}$$

The integral converges



### Theorem – Direct Comparison Test

Let  $f$  and  $g$  be continuous on  $[a, \infty)$  with  $0 \leq f(x) \leq g(x)$  for all  $x \geq a$ . Then

1.  $\int_a^{\infty} f(x) dx$  *converges* if  $\int_a^{\infty} g(x) dx$  *converges*
2.  $\int_a^{\infty} g(x) dx$  *diverges* if  $\int_a^{\infty} f(x) dx$  *diverges*

### Theorem – Limit Comparison Test

If the positive functions  $f$  and  $g$  are continuous on  $[a, \infty)$ , and if

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty$$

Then

$$\int_a^{\infty} f(x) dx \text{ and } \int_a^{\infty} g(x) dx$$

Both converge or both diverge

### Example

Show that  $\int_1^{\infty} \frac{dx}{1+x^2}$  converges by comparison with  $\int_1^{\infty} \frac{dx}{x^2}$ . Find and compare the two integral values.

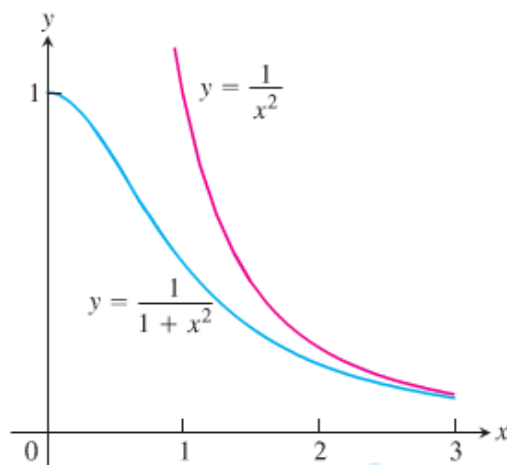
### Solution

The functions  $f(x) = \frac{1}{x^2}$  and  $g(x) = \frac{1}{1+x^2}$  are positive and continuous on  $[1, \infty)$ . Also,

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow \infty} \frac{\frac{1}{x^2}}{\frac{1}{1+x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{1+x^2}{x^2} \\ &= \underline{1}\end{aligned}$$

Therefore,  $\int_1^{\infty} \frac{dx}{1+x^2}$  converges because  $\int_1^{\infty} \frac{dx}{x^2}$  converges.

$$\begin{aligned}\int_1^{\infty} \frac{dx}{1+x^2} &= \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{1+x^2} \\ &= \lim_{b \rightarrow \infty} \left( \tan^{-1} b - \tan^{-1} 1 \right) \\ &= \frac{\pi}{2} - \frac{\pi}{4} \\ &= \underline{\frac{\pi}{4}}\end{aligned}$$



### Example

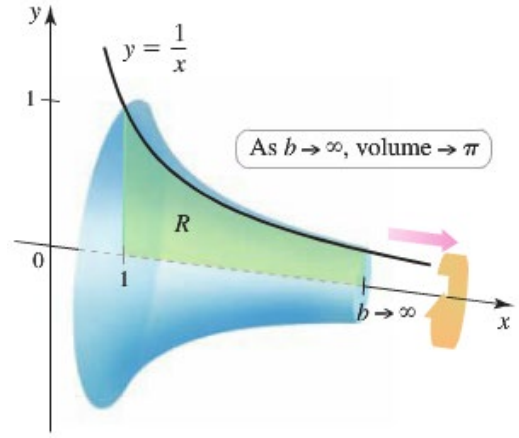
Let  $R$  be the region bounded by the graph of  $y = x^{-1}$  and the  $x$ -axis, for  $x \geq 1$ .

- What is the volume of the solid generated when  $R$  is revolved about the  $x$ -axis?
- What is the surface area of the solid generated when  $R$  is revolved about the  $x$ -axis?
- What is the volume of the solid generated when  $R$  is revolved about the  $y$ -axis?

### Solution

$$\begin{aligned} a) \quad V &= \pi \int_1^{\infty} \frac{1}{x^2} dx \\ &= -\pi \frac{1}{x} \Big|_1^{\infty} \\ &= -\pi(0 - 1) \\ &= \pi \text{ unit}^3 \end{aligned}$$

$$V = \pi \int_a^b (f(x))^2 dx$$



$$\begin{aligned} b) \quad S &= 2\pi \int_1^{\infty} \frac{1}{x} \sqrt{1 + \left(-\frac{1}{x^2}\right)^2} dx & S &= 2\pi \int_a^b f(x) \sqrt{1 + f'(x)^2} dx \\ &= 2\pi \int_1^{\infty} \frac{1}{x^3} \sqrt{x^4 + 1} dx \\ &> 2\pi \int_1^{\infty} \frac{x^2}{x^3} dx & \sqrt{x^4 + 1} &> \sqrt{x^4} = x^2 \\ &= 2\pi \int_1^{\infty} \frac{1}{x} dx \\ &= 2\pi (\ln x) \Big|_1^{\infty} \\ &= \infty \text{ unit}^2 \end{aligned}$$

$$\begin{aligned} c) \quad V &= 2\pi \int_1^{\infty} x \cdot \frac{1}{x} dx & V &= 2\pi \int_a^b x \cdot f(x) dx \quad (\text{Shell method}) \\ &= 2\pi x \Big|_1^{\infty} \\ &= \infty \text{ unit}^3 \end{aligned}$$

## Exercises      Section 2.6 – Improper Integrals

(1 – 81)      Evaluate the integrals

1.  $\int_0^{\infty} \frac{dx}{x^2 + 1}$

2.  $\int_0^4 \frac{dx}{\sqrt{4-x}}$

3.  $\int_{-\infty}^2 \frac{2dx}{x^2 + 4}$

4.  $\int_{-\infty}^{\infty} \frac{xdx}{(x^2 + 4)^{3/2}}$

5.  $\int_1^{\infty} \frac{dx}{x\sqrt{x^2 - 1}}$

6.  $\int_{-\infty}^{\infty} 2xe^{-x^2} dx$

7.  $\int_0^1 (-\ln x) dx$

8.  $\int_{-1}^4 \frac{dx}{\sqrt{|x|}}$

9.  $\int_0^{\infty} e^{-3x} dx$

10.  $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

11.  $\int_1^{10} \frac{dx}{(x-2)^{1/3}}$

12.  $\int_1^{\infty} \frac{dx}{x^2}$

13.  $\int_0^{\infty} \frac{dx}{(x+1)^3}$

14.  $\int_{-\infty}^0 e^x dx$

15.  $\int_1^{\infty} 2^{-x} dx$

16.  $\int_{-\infty}^0 \frac{dx}{\sqrt[3]{2-x}}$

17.  $\int_{4/\pi}^{\infty} \frac{1}{x^2} \sec^2\left(\frac{1}{x}\right) dx$

18.  $\int_{e^2}^{\infty} \frac{dx}{x \ln^p x} \quad p > 1$

19.  $\int_0^{\infty} \frac{p}{\sqrt[5]{p^2 + 1}} dp$

20.  $\int_{-1}^1 \ln y^2 dy$

21.  $\int_{-2}^6 \frac{dx}{\sqrt{|x-2|}}$

22.  $\int_0^{\infty} xe^{-x} dx$

23.  $\int_0^1 x \ln x dx$

24.  $\int_0^1 x \ln x dx$

25.  $\int_1^{\infty} \frac{\ln x}{x^2} dx$

26.  $\int_1^{\infty} (1-x)e^x dx$

27.  $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx$

28.  $\int_0^1 \frac{dx}{\sqrt[3]{x}}$

29.  $\int_1^{\infty} \frac{3}{\sqrt[3]{x}} dx$

30.  $\int_1^{\infty} \frac{4}{\sqrt[4]{x}} dx$

31.  $\int_0^2 \frac{dx}{x^3}$

32.  $\int_1^{\infty} \frac{dx}{x^3}$

33.  $\int_1^{\infty} \frac{6}{x^4} dx$

34.  $\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)}$

35.  $\int_{-\infty}^0 xe^{-4x} dx$

36.  $\int_0^{\infty} x e^{-x/3} dx$
37.  $\int_0^{\infty} x^2 e^{-x} dx$
38.  $\int_0^{\infty} e^{-x} \cos x dx$
39.  $\int_4^{\infty} \frac{1}{x(\ln x)^3} dx$
40.  $\int_1^{\infty} \frac{\ln x}{x} dx$
41.  $\int_{-\infty}^{\infty} \frac{4}{16+x^2} dx$
42.  $\int_0^{\infty} \frac{x^3}{(x^2+1)^2} dx$
43.  $\int_0^{\infty} \frac{1}{e^x + e^{-x}} dx$
44.  $\int_0^{\infty} \frac{e^x}{1+e^x} dx$
45.  $\int_0^{\infty} \cos \pi x dx$
46.  $\int_0^{\infty} \sin \frac{x}{2} dx$
47.  $\int_1^{\infty} \frac{dx}{(x+1)^9}$
48.  $\int_1^{\infty} \frac{3x-1}{4x^3-x^2} dx$
49.  $\int_{-\infty}^{\infty} \frac{4}{x^2+16} dx$
50.  $\int_{-\infty}^{-1} \frac{dx}{(x-1)^4}$
51.  $\int_0^{\infty} x e^{-x} dx$
52.  $\int_0^{\infty} \frac{6x}{1+x^6} dx$
53.  $\int_0^2 \frac{dx}{\sqrt[3]{|x-1|}}$
54.  $\int_{-1}^1 \frac{dx}{x^2+2x+5}$
55.  $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+5}$
56.  $\int_0^{\infty} \cos x dx$
57.  $\int_2^{\infty} \frac{\cos\left(\frac{\pi}{x}\right)}{x^2} dx$
58.  $\int_{-\infty}^a \sqrt{e^x} dx$
59.  $\int_0^{\infty} \frac{e^x}{e^{2x}+1} dx$
60.  $\int_1^{\infty} \frac{dx}{x(x+1)}$
61.  $\int_1^{\infty} \frac{dx}{x^2(x+1)}$
62.  $\int_1^{\infty} \frac{3x^2+1}{x^3+x} dx$
63.  $\int_1^{\infty} \frac{1}{x^2} \sin \frac{\pi}{x} dx$
64.  $\int_2^{\infty} \frac{dx}{(x+2)^2}$
65.  $\int_1^{\infty} \frac{\tan^{-1} x}{x^2+1} dx$
66.  $\int_{-3}^1 \frac{dx}{(2x+6)^{2/3}}$
67.  $\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$
68.  $\int_0^{\ln 3} \frac{e^x}{(e^x-1)^{2/3}} dx$
69.  $\int_1^2 \frac{dx}{\sqrt{x-1}}$
70.  $\int_{-1}^1 \frac{dx}{x^2}$
71.  $\int_0^2 \frac{dx}{(x-1)^2}$
72.  $\int_{-1}^2 \frac{dx}{(x-1)^2}$
73.  $\int_1^{\infty} \frac{dx}{x \sqrt{x^2-1}}$
74.  $\int_0^{\infty} x e^{-x^2} dx$

75.  $\int_{-\infty}^{\infty} x e^{-x^2} dx$

76.  $\int_{-\infty}^{\infty} \frac{x}{x^2 + 1} dx$

77.  $\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2}$

78.  $\int_{-\infty}^{\infty} \frac{dx}{x^2 + 6x + 12}$

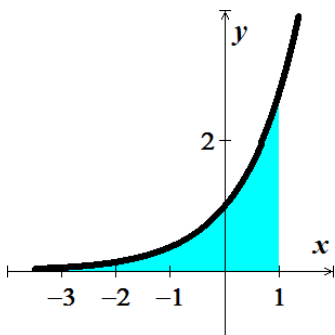
79.  $\int \frac{dx}{2 - \sqrt{3}x}$

80.  $\int \theta \cos(2\theta + 1) d\theta$

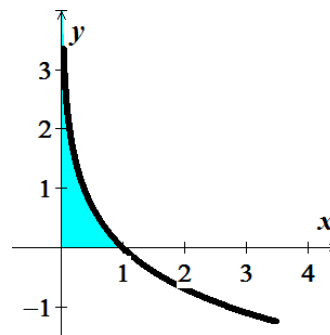
81.  $\int \sqrt{x} \sqrt{1 + \sqrt{x}} dx$

(82 – 85) Find the area of the unbounded shaded region

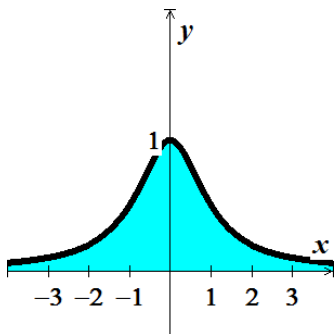
82.  $y = e^x, -\infty < x \leq 1$



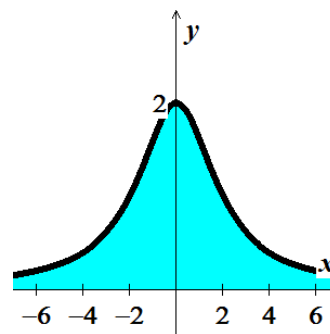
83.  $y = -\ln x$



84.  $y = \frac{1}{x^2 + 1}$



85.  $y = \frac{8}{x^2 + 4}$



86. Find the area of the region  $R$  between the graph of  $f(x) = \frac{1}{\sqrt{9 - x^2}}$  and the  $x$ -axis on the interval  $(-3, 3)$  (if it exists)

87. Find the volume of the region bounded by  $f(x) = (x^2 + 1)^{-1/2}$  and the  $x$ -axis on the interval  $[2, \infty)$  is revolved about the  $x$ -axis.

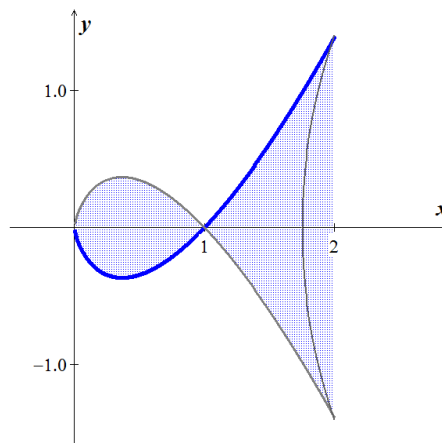


88. Find the volume of the region bounded by  $f(x) = \sqrt{\frac{x+1}{x^3}}$  and the  $x$ -axis on the interval  $[1, \infty)$  is revolved about the  $x$ -axis.
89. Find the volume of the region bounded by  $f(x) = (x+1)^{-3}$  and the  $x$ -axis on the interval  $[0, \infty)$  is revolved about the  $y$ -axis.
90. Find the volume of the region bounded by  $f(x) = \frac{1}{\sqrt{x \ln x}}$  and the  $x$ -axis on the interval  $[2, \infty)$  is revolved about the  $x$ -axis.
91. Find the volume of the region bounded by  $f(x) = \frac{\sqrt{x}}{\sqrt[3]{x^2 + 1}}$  and the  $x$ -axis on the interval  $[0, \infty)$  is revolved about the  $x$ -axis.
92. Find the volume of the region bounded by  $f(x) = (x^2 - 1)^{-1/4}$  and the  $x$ -axis on the interval  $(1, 2]$  is revolved about the  $y$ -axis.
93. Find the volume of the region bounded by  $f(x) = \tan x$  and the  $x$ -axis on the interval  $\left[0, \frac{\pi}{2}\right)$  is revolved about the  $x$ -axis.
94. Find the volume of the region bounded by  $f(x) = -\ln x$  and the  $x$ -axis on the interval  $(0, 1]$  is revolved about the  $x$ -axis.
95. Find the volume of the solid generated by revolving the region bounded by the graphs of  $y = xe^{-x}$ ,  $y = 0$ , and  $x = 0$  about the  $x$ -axis.

96. The region between the  $x$ -axis and the curve

$$f(x) = \begin{cases} 0, & x = 0 \\ x \ln x, & 0 < x \leq 2 \end{cases}$$

is revolved about the  $x$ -axis to generate the solid.  
Find the volume of the solid.



(97 – 98) Consider the region satisfying the inequalities

- a) Find the area of the region
- b) Find the volume of the solid generated by revolving the region about the  $x$ -axis .
- c) Find the volume of the solid generated by revolving the region about the  $y$ -axis .

97.  $y \leq e^{-x}$ ,  $y \geq 0$ ,  $x \geq 0$

98.  $y \leq \frac{1}{x^2}$ ,  $y \geq 0$ ,  $x \geq 1$

99. Find the perimeter of the hypocycloid of four cusps  $x^{2/3} + y^{2/3} = 4$

100. Find the arc length of the graph  $y = \sqrt{16 - x^2}$  over the interval  $[0, 4]$

101. The region bounded by  $(x - 2)^2 + y^2 = 1$  is revolved about the  $y$ -axis to form a torus. Find the surface area of the torus.

102. Find the surface area formed by revolving the graph  $y = 2e^{-x}$  on the interval  $[0, \infty)$  about the  $x$ -axis

103. The magnetic potential  $P$  at a point on the axis of a circular coil is given by

$$P = \frac{2\pi NI r}{k} \int_c^\infty \frac{1}{(r^2 + x^2)^{3/2}} dx$$

Where  $N$ ,  $I$ ,  $r$ ,  $k$ , and  $c$  are constants. Find  $P$ .

104. A “semi-infinite” uniform rod occupies the nonnegative  $x$ -axis . The rod has a linear density  $\delta$ , which means that a segment of length  $dx$  has a mass of  $\delta dx$  . A particle of mass  $M$  is located at the point  $(-a, 0)$  . The gravitational force  $F$  that the rod exerts on the mass is given by

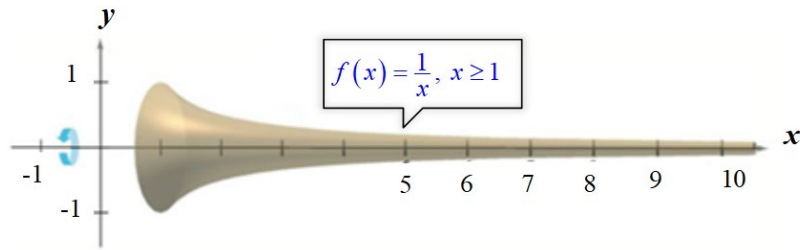
$$F = \int_0^\infty \frac{GM\delta}{(a+x)^2} dx$$

Where  $G$  is the gravitational constant. Find  $F$ .

105. Let  $R$  be the region bounded by the graph of  $f(x) = x^{-p}$  and the  $x$ -axis

- a) Let  $S$  be the solid generated when  $R$  is revolved about the  $x$ -axis . For what values of  $p$  is the volume of  $S$  finite for  $0 < x \leq 1$  ?
- b) Let  $S$  be the solid generated when  $R$  is revolved about the  $y$ -axis . For what values of  $p$  is the volume of  $S$  finite for  $0 < x \leq 1$  ?
- c) Let  $S$  be the solid generated when  $R$  is revolved about the  $x$ -axis . For what values of  $p$  is the volume of  $S$  finite for  $x \geq 1$  ?
- d) Let  $S$  be the solid generated when  $R$  is revolved about the  $y$ -axis . For what values of  $p$  is the volume of  $S$  finite for  $x \geq 1$  ?

106. The solid formed by revolving (about the  $x$ -axis) the unbounded region lying between the graph of  $f(x) = \frac{1}{x}$  and the  $x$ -axis ( $x \geq 1$ ) is called **Gabriel's Horn**.



Show that this solid has a finite volume and an infinite surface area.

107. Water is drained from a 3000-gal tank at a rate that starts at 100 gal/hr. and decreases continuously by 5% /hr. If the drain left open indefinitely, how much water drains from the tank? Can a full tank be emptied at this rate?

108. Let  $I(a) = \int_0^{\infty} \frac{dx}{(1+x^a)(1+x^2)}$ , where  $a$  is a real number.

- a) Evaluate  $I(a)$  and show that its value is independent of  $a$ .

(**Hint:** split the integral into two integrals over  $[0, 1]$  and  $[1, \infty)$ ; then use a change of variables to convert the second integral into an integral over  $[0, 1]$ .)

- b) Let  $f$  be any positive continuous function on  $\left[0, \frac{\pi}{2}\right]$

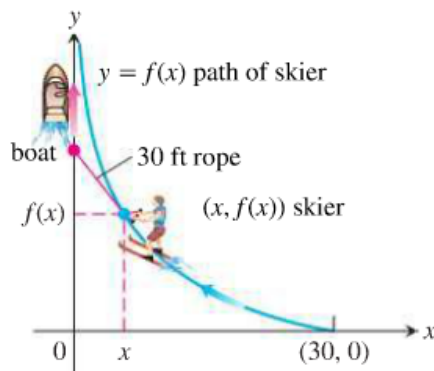
Evaluate  $\int_0^{\pi/2} \frac{f(\cos x)}{f(\cos x) + f(\sin x)} dx$

(**Hint:** Use the identity  $\cos\left(\frac{\pi}{2} - x\right) = \sin x$ )

109. Let  $R$  be the region bounded by  $y = \ln x$ , the  $x$ -axis, and the line  $x = a$ , where  $a > 1$ .

- a) Find the volume  $V_1(a)$  of the solid generated when  $R$  is revolved about the  $x$ -axis (as a function of  $a$ ).
- b) Find the volume  $V_2(a)$  of the solid generated when  $R$  is revolved about the  $y$ -axis (as a function of  $a$ ).
- c) Graph  $V_1$  and  $V_2$ . For what values of  $a > 1$  is  $V_1(a) > V_2(a)$ ?

- 110.** Let  $R$  be the region bounded by the graph of  $f(x) = x^{-p}$  and the  $x$ -axis, for  $x \geq 1$ . Let  $V_1$  and  $V_2$  be the volumes of the solids generated when  $R$  is revolved about the  $x$ -axis and the  $y$ -axis, respectively, if they exist.
- For what values of  $p$  (if any) is  $V_1 = V_2$ ?
  - Repeat part (a) on the interval  $(0, 1]$ .
- 111.** Let  $R_1$  be the region bounded by the graph of  $y = e^{-ax}$  and the  $x$ -axis on the interval  $[0, b]$  where  $a > 0$  and  $b > 0$ . Let  $R_2$  be the region bounded by the graph of  $y = e^{-ax}$  and the  $x$ -axis on the interval  $[b, \infty)$ . Let  $V_1$  and  $V_2$  be the volumes of the solids generated when  $R_1$  and  $R_2$  are revolved about the  $x$ -axis. Find and graph the relationship between  $a$  and  $b$  for which  $V_1 = V_2$ .
- 112.** Suppose that a boat is positioned at the origin with a water skier tethered to the boat at the point  $(30, 0)$  on a rope 30 feet long. As the boat travels along the positive  $y$ -axis, the skier is pulled behind the boat along an unknown path  $y = f(x)$ , as shown



a) Show that  $f'(x) = \frac{-\sqrt{900 - x^2}}{x}$

(Hint: Assume that the skier is always pointed directly at the boat and the rope is on line is on a line tangent to the path  $y = f(x)$ .)

b) Solve the equation in part (a) for  $f(x)$ , using  $f(30) = 0$

- 113.** Many chemical reactions are the result of the interaction of 2 molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If  $a$  is the amount of substance  $A$  and  $b$  is the substance  $B$  at time  $t = 0$ , and if  $x$  is the amount of product at time  $t$ , then the rate of formation of  $x$  may be given by the differential equation

$$\frac{dx}{dt} = k(a-x)(b-x) \quad \text{or} \quad \frac{1}{(a-x)(b-x)} \frac{dx}{dt} = k$$

Where  $k$  is a constant for the reaction. Integrate both sides of this equation to obtain a relation between  $x$  and  $t$ .

**a)** If  $a = b$

**b)** If  $a \neq b$

Assume in each case that  $x = 0$  when  $t = 0$

# First Order Differential Equations

## Section 2.7 – First-Order Linear Equations

### General First-Order Differential Equations and Solutions

A *first-order differential equation* is an equation

$$\frac{dy}{dx} = f(x, y)$$

In which  $f(x, y)$  is a function of two variables defined on a region in the  $xy$ -plane.

#### Example

Show that every member of the family of functions  $y = \frac{C}{x} + 2$  is a solution of the first-order differential equation  $\frac{dy}{dx} = \frac{1}{x}(2 - y)$  on the interval  $(0, \infty)$ , where  $C$  is any constant.

#### Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left( \frac{C}{x} + 2 \right) \\ &= -\frac{C}{x^2}\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{x}(2 - y) \\ -\frac{C}{x^2} &= \frac{1}{x} \left( 2 - \left( \frac{C}{x} + 2 \right) \right) \\ -\frac{C}{x^2} &= \frac{1}{x} \left( 2 - \frac{C}{x} - 2 \right) \\ -\frac{C}{x^2} &= \frac{1}{x} \left( -\frac{C}{x} \right) \\ -\frac{C}{x^2} &= -\frac{C}{x^2} \quad \checkmark\end{aligned}$$

Therefore, for every value of  $C$ , the function  $y = \frac{C}{x} + 2$  is a solution of the first-order differential equation  $\frac{dy}{dx} = \frac{1}{x}(2 - y)$ .

### Example

Show that the function  $y = (x + 1) - \frac{1}{3}e^x$  is a solution of the first-order initial value problem

$$\frac{dy}{dx} = y - x \quad y(0) = \frac{2}{3}.$$

### Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left( x + 1 - \frac{1}{3}e^x \right) \\ &= 1 - \frac{1}{3}e^x\end{aligned}$$

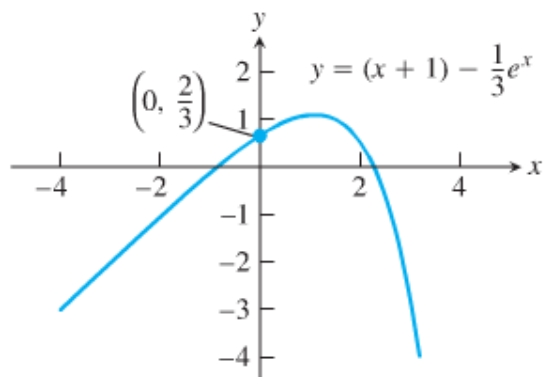
$$y - x = 1 - \frac{1}{3}e^x$$

$$y = x + 1 - \frac{1}{3}e^x$$

$$y(0) = (0 + 1) - \frac{1}{3}e^0$$

$$= 1 - \frac{1}{3}$$

$$= \frac{2}{3}$$

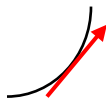


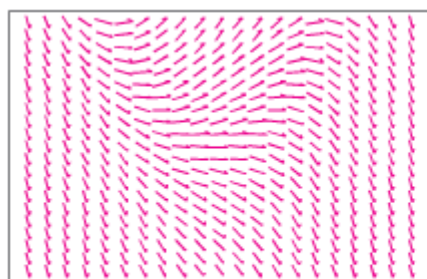
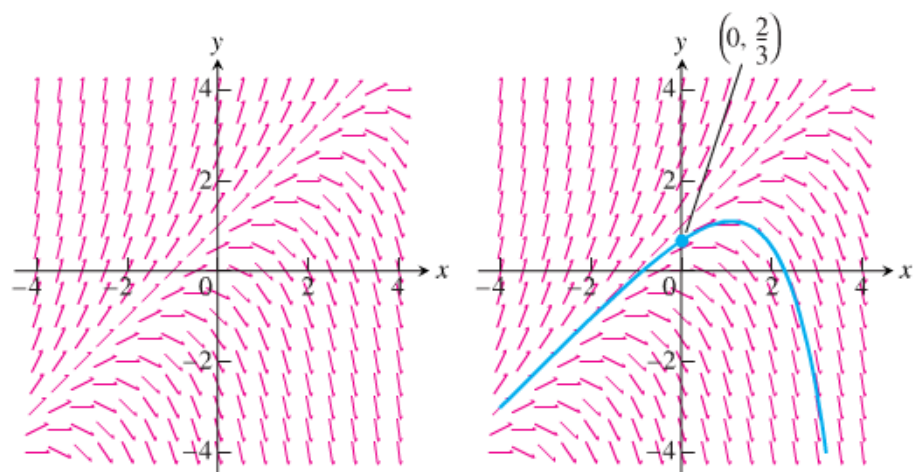
### Slope Fields: Viewing Solution Curves

Each time we specify an initial condition  $y(x_0) = y_0$  for the solution of a differential equation

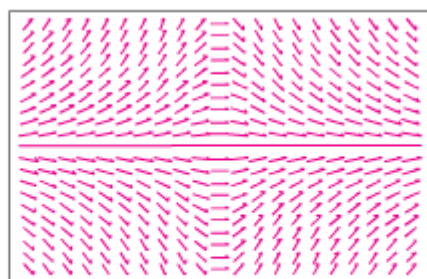
$y' = f(x, y)$ , the solution curve is required to pass through the point  $(x_0, y_0)$  and to have a slope  $f(x_0, y_0)$  there.

What we draw a lineal element at each point  $(x, y)$  with slope  $f(x, y)$  then the collection of these lineal elements is called a **direction field** or a **slope field** of the differential equation  $\frac{dy}{dx} = f(x, y)$ .

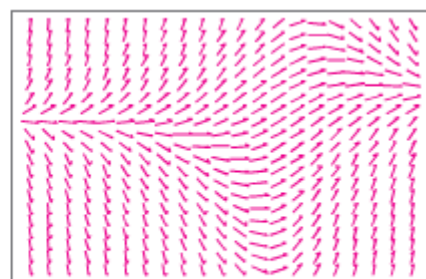




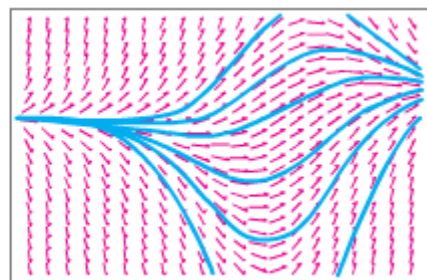
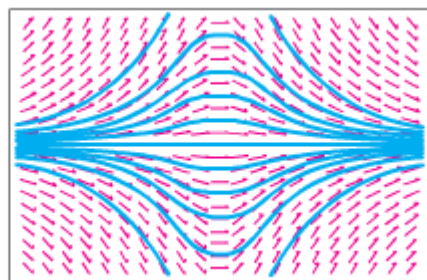
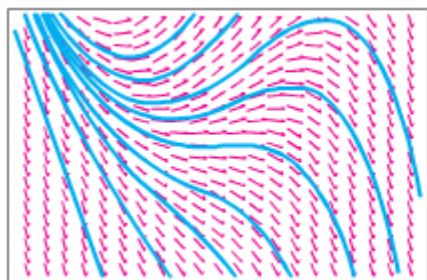
(a)  $y' = y - x^2$



(b)  $y' = -\frac{2xy}{1+x^2}$



(c)  $y' = (1-x)y + \frac{x}{2}$

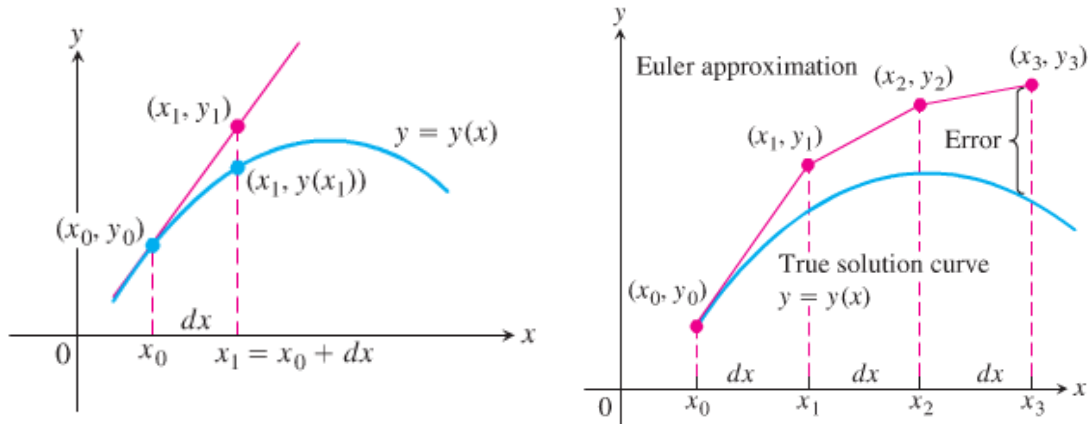




## Euler's Method

**Euler's method** named after *Leonhard Euler* is an example of a **fixed-step** solver.

Euler's method is a first-order numerical procedure for solving ordinary differential equations (ODEs) with a given initial value. It is the most basic kind of explicit method for numerical integration of ordinary differential equations.



$$y' = f(x, y) \quad y(x_0) = y_0$$

The setting size:  $h = \frac{b-a}{k} > 0$  ;  $k = 1, 2, 3, \dots$

$$\text{Then,} \quad x_0 = a$$

$$x_1 = x_0 + h = a + h$$

$$x_k = x_{k-1} + h = a + kh$$

$$\text{Last point} \quad x_k = a + kh = b$$

By the definition of the derivative:

$$y'(x_k) \approx \frac{y(x_{k+1}) - y(x_k)}{h}$$

$$y'(x_k) \approx \frac{y_{k+1} - y_k}{h} = f(x_k, y_k) : \text{slope}$$

The tangent line at the point  $(x_0, y(x_0))$  is:

$$y_{k+1} = y_k + h \cdot f(x_k, y_k)$$

$$y_{k+1} = y_k + \Delta x_{\text{step}} \cdot f(x_k, y_k)$$

$$y_{k+1} = y_k + f(x_k, y_k) dx$$

This method is known as *Euler's Method* with step size  $h$ .

### ***Example***

Find the first three approximations  $y_1, y_2, y_3$  using Euler's method for the initial value problem

$$y' = 1 + y, \quad y(0) = 1$$

Starting at  $x_0 = 0$  with  $dx = 0.1$ .

### **Solution**

$$\begin{aligned} y_1 &= y_0 + f(x_0, y_0)dx \\ &= y_0 + (1 + y_0)dx \\ &= 1 + (1 + 1)(0.1) \\ &= \underline{1.2} \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + f(x_1, y_1)dx \\ &= y_1 + (1 + y_1)dx \\ &= 1.2 + (1 + 1.2)(0.1) \\ &= \underline{1.42} \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + f(x_2, y_2)dx \\ &= y_2 + (1 + y_2)dx \\ &= 1.42 + (1 + 1.42)(0.1) \\ &= \underline{1.662} \end{aligned}$$

### Example

Use Euler's method to solve

$$y' = 1 + y, \quad y(0) = 1$$

On the interval  $0 \leq x \leq 1$ , starting at  $x_0 = 0$  and taking

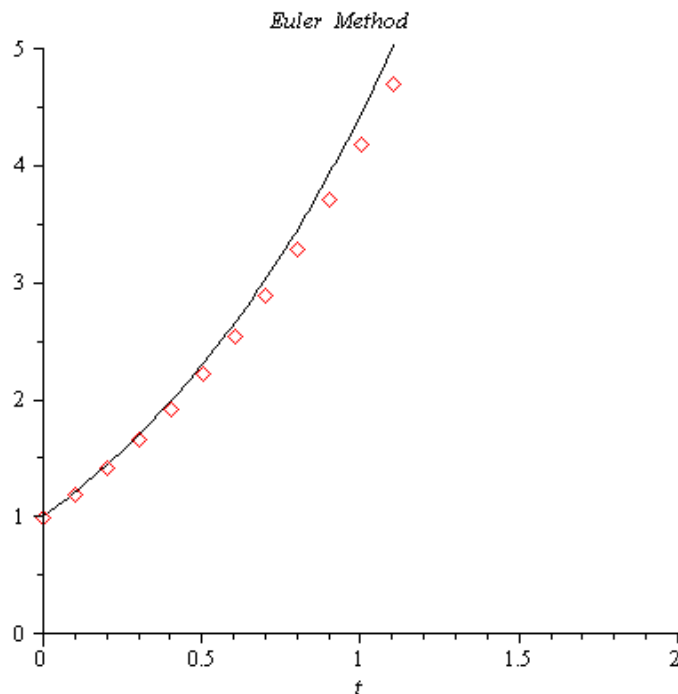
- a)  $dx = 0.1$ .
- b)  $dx = 0.05$ .

Compare the approximations with the values of the exact solution  $y = 2e^x - 1$

### Solution

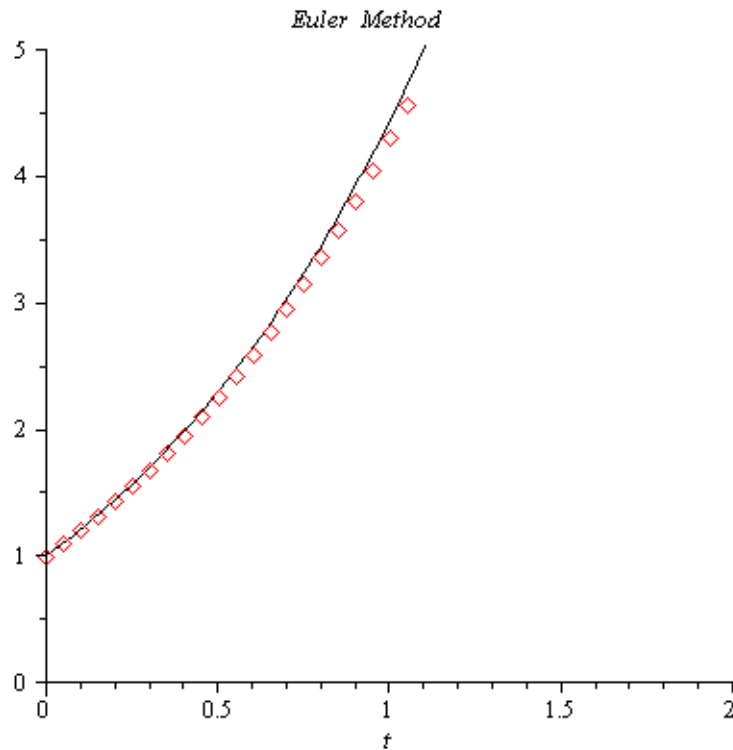
a) Euler Method  $dx = 0.1$

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	1.00000000	1.00000000	0.00000000
0.10	1.20000000	1.21034184	0.01034184
0.20	1.42000000	1.44280552	0.02280552
0.30	1.66200000	1.69971762	0.03771762
0.40	1.92820000	1.98364940	0.05544940
0.50	2.22102000	2.29744254	0.07642254
0.60	2.54312200	2.64423760	0.10111560
0.70	2.89743420	3.02750541	0.13007121
0.80	3.28717762	3.45108186	0.16390424
0.90	3.71589538	3.91920622	0.20331084
1.00	4.18748492	4.43656366	0.24907874



b) Euler Method  $dx = 0.05$

$t$	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	1.00000000	1.00000000	0.00000000
0.05	1.10000000	1.10254219	0.00254219
0.10	1.20500000	1.21034184	0.00534184
0.15	1.31525000	1.32366849	0.00841849
0.20	1.43101250	1.44280552	0.01179302
0.25	1.55256313	1.56805083	0.01548771
0.30	1.68019128	1.69971762	0.01952633
0.35	1.81420085	1.83813510	0.02393425
0.40	1.95491089	1.98364940	0.02873851
0.45	2.10265643	2.13662437	0.03396794
0.50	2.25778925	2.29744254	0.03965329
0.55	2.42067872	2.46650604	0.04582732
0.60	2.59171265	2.64423760	0.05252495
0.65	2.77129828	2.83108166	0.05978337
0.70	2.95986320	3.02750541	0.06764222
0.75	3.15785636	3.23400003	0.07614367
0.80	3.36574918	3.45108186	0.08533268
0.85	3.58403664	3.67929370	0.09525707
0.90	3.81323847	3.91920622	0.10596775
0.95	4.05390039	4.17141932	0.11751893
1.00	4.30659541	4.43656366	0.12996825



A **first-order linear** differential equation is one that can be written in the **standard form**

$$\frac{dy}{dx} + P(x) \cdot y = Q(x)$$

Where  $P$  and  $Q$  are continuous functions of  $x$

## Solving Linear Equations

We solve the equation  $\frac{dy}{dx} + P(x) \cdot y = Q(x)$

## Separable Equation

**Solution of the homogenous equation**

$$\frac{dy}{dx} + P(x)y = 0$$

$$\frac{dy}{dx} = -P(x)y$$

$$\int \frac{dy}{y} = - \int P(x) dx$$

*Integrate both sides*

$$\ln|y| = - \int P(x) dx + C$$

**Convert to exponential form**

$$y(x) = e^{\int P(x) dx + C}$$

$$= e^{\int P(x) dx} e^C$$

$$\underline{y(x) = A e^{\int P(x) dx}}$$

### Example

Solve the differential equation  $y' = ty^2$

#### Solution

$$\frac{dy}{dt} = ty^2$$

$$\frac{dy}{y^2} = t dt$$

$$\int y^{-2} dy = \int t dt$$

$$-y^{-1} = \frac{1}{2}t^2 + C$$

$$-\frac{1}{y} = \frac{t^2 + 2C}{2} \quad \text{Cross multiplication}$$

$$\underline{y(t) = -\frac{2}{t^2 + 2C}}$$

### General Method

1. Separate the variables
2. Integrate both sides
3. Solve for the solution  $y(t)$ , if possible

### Example

Find the general solution of the differential equation.  $y' = \frac{2xy + 2x}{x^2 - 1}$

#### Solution

$$\frac{dy}{dx} = \frac{2x(y+1)}{x^2 - 1}$$

$$\frac{dy}{y+1} = \frac{2x}{x^2 - 1} dx$$

$$d(x^2 - 1) = 2x dx$$

$$\int \frac{1}{y+1} d(y+1) = \int \frac{1}{x^2 - 1} d(x^2 - 1)$$

$$\ln|y+1| = \ln|x^2 - 1| + \ln C$$

$$= \ln\left(C|x^2 - 1|\right)$$

$$y+1=e^{C\ln\left|x^2-1\right|}$$

$$y=e^Ce^{\ln\left|x^2-1\right|}-1$$

$$\underline{y(x)=Ae^{\ln\left|x^2-1\right|}-1}$$

## Solution of the Nonhomogeneous Equation

$$y' + p(x)y = f(x)$$

Let assume:  $y = y_h + y_p$  where  $\begin{cases} y_h & \text{Homogeneous Solution} \\ y_p & \text{Particular Solution} \end{cases}$

The homogeneous equation is given by  $y'_h + p(x)y_h = 0$

$$y'_h = -p(x)y_h \Rightarrow y_h = e^{-\int p dx}$$

$$y_p = u(x)y_h = u.e^{-\int p dx}$$

$$y'_p + p(x)y_p = f(x)$$

$$(uy_h)' + puy_h = f$$

$$u'y_h + uy'_h + puy_h = f$$

$$u'y_h + u(y'_h + py_h) = f$$

Since  $y'_h + py_h = 0$  homogeneous

$$u'y_h = f$$

$$\frac{du}{dx} = \frac{f}{y_h}$$

$$du = \frac{f}{e^{-\int p dx}} dx$$

$$= f.e^{\int p dx} dx$$

$$u = \int f.e^{\int p dx} dx$$

$$y_p = u.e^{-\int p dx} = \left( \int f.e^{\int p dx} dx \right) e^{-\int p dx} = e^{-\int p dx} \int f.e^{\int p dx} dx$$

$$y = y_h + y_p$$

$$= Ce^{-\int p dx} + e^{-\int p dx} \int f.e^{\int p dx} dx$$

$$= e^{-\int p dx} \left( C + \int f.e^{\int p dx} dx \right)$$

$$y' + p(x)y = f(x) \Rightarrow y = \frac{1}{e^{\int p dx}} \left( \int f.e^{\int p dx} dx + C \right)$$



### Example

Solve the equation  $x \frac{dy}{dx} = x^2 + 3y, \quad x > 0$

### Solution

$$y' - \frac{3}{x}y = x$$

$$\begin{aligned} e^{\int p dx} &= e^{-3 \int \frac{dx}{x}} \\ &= e^{-3 \ln|x|} \\ &= e^{\ln x^{-3}} \\ &= \underline{x^{-3}} \end{aligned}$$

$$\begin{aligned} \int x \cdot x^{-3} dx &= \int x^{-2} dx \\ &= \underline{-\frac{1}{x}} \end{aligned}$$

$$\begin{aligned} y(x) &= \frac{1}{x^{-3}} \left( -\frac{1}{x} + C \right) \\ &= x^3 \left( -\frac{1}{x} + C \right) \\ &= \underline{-x^2 + Cx^3} \quad x > 0 \end{aligned}$$

### Example

Solve the equation  $3xy' - y = \ln x + 1, \quad x > 0$ , satisfying  $y(1) = -2$

### Solution

$$y' - \frac{1}{3x}y = \frac{\ln x + 1}{3x}$$

$$\begin{aligned} e^{\int p dx} &= e^{\int \left(-\frac{1}{3x}\right) dx} \\ &= e^{-\frac{1}{3} \ln x} \\ &= e^{\ln x^{-1/3}} \\ &= \underline{x^{-1/3}} \end{aligned}$$

$$\begin{aligned} \int \left(x^{-1/3}\right) \frac{\ln x + 1}{3x} dx &= \frac{1}{3} \int (\ln x + 1) x^{-4/3} dx \\ &= \frac{1}{3} \left( -3x^{-1/3} (\ln x + 1) + 3 \int x^{-4/3} dx \right) \end{aligned}$$

$$u = \ln x + 1 \quad dv = \int x^{-4/3} dx$$

$$du = \frac{1}{x} dx \quad v = -3x^{-1/3}$$

$$= \frac{1}{3} \left( -3x^{-1/3} (\ln x + 1) - 9x^{-1/3} \right)$$

$$= \underline{-x^{-1/3} (\ln x + 1) - 3x^{-1/3} }$$

$$y(x) = x^{1/3} \left( -x^{-1/3} (\ln x + 1) - 3x^{-1/3} + C \right)$$

$$= \underline{-\ln x - 4 + Cx^{1/3} }$$

$$y(1) = -\ln(1) - 4 + C(1)^{1/3}$$

$$-2 = -0 - 4 + C$$

$$\Rightarrow \underline{C = 2}$$

$$\underline{y = 2x^{1/3} - \ln x - 4}$$

## Exercises      Section 2.7 – First-Order Linear Equations

(1 – 2) Write an equivalent first-order differential equation and initial condition for  $y$ .

1.  $y = \int_1^x \frac{1}{t} dt$

2.  $y = 2 - \int_0^x (1 + y(t)) \sin t dt$

(3 – 6) Use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round the results to four decimals

3.  $y' = 1 - \frac{y}{x}, \quad y(2) = -1, \quad dx = 0.5$

5.  $y' = y^2(1 + 2x), \quad y(-1) = 1, \quad dx = 0.5$

4.  $y' = x(1 - y), \quad y(1) = 0, \quad dx = 0.2$

6.  $y' = ye^x, \quad y(0) = 2, \quad dx = 0.5$

7. Use the Euler method with  $dx = 0.2$  to estimate  $y(2)$  if  $y' = \frac{y}{x}$  and  $y(1) = 2$ . What is the exact value of  $y(2)$ ?

8. Use Euler's Method to solve  $y' = 1 + y, \quad y(0) = 1$  on the interval  $0 \leq x \leq 1$  and taking  $dx = 0.05$ . Compare the approximation to the values of the exact solution.

9. Use Euler's Method to solve  $y' = 2xy + 2y, \quad y(0) = 3$  on the interval  $0 \leq x \leq 1$  and taking  $dx = 0.1$ . Compare the approximation to the values of the exact solution.

(10 – 16) Verify that the given function  $y$  is a solution of the differential equation that follows it. Assume that  $C, C_1$ , and  $C_2$  are arbitrary constants.

10.  $y = Ce^{-5t}; \quad y'(t) + 5y = 0$

11.  $y = Ct^{-3}; \quad ty'(t) + 3y = 0$

12.  $y = C_1 \sin 4t + C_2 \cos 4t; \quad y''(t) + 16y = 0$

13.  $y = C_1 e^{-x} + C_2 e^x; \quad y''(x) - y = 0$

14.  $y' + 4y = \cos t, \quad y(t) = \frac{4}{17} \cos t + \frac{1}{17} \sin t + Ce^{-4t}, \quad y(0) = -1$

15.  $ty' + (t+1)y = 2te^{-t}, \quad y(t) = e^{-t} \left( t + \frac{C}{t} \right), \quad y(1) = \frac{1}{e}$

16.  $y' = y(2 + y), \quad y(t) = \frac{2}{-1 + Ce^{-2t}}, \quad y(0) = -3$

(17 – 20) Verify that the given function  $y$  is a solution of the initial value problem that follows it.

17.  $y = 16e^{2t} - 10$ ;  $y' - 2y = 20$ ,  $y(0) = 6$

18.  $y = 8t^6 - 3$ ;  $ty' - 6y = 18$ ,  $y(1) = 5$

19.  $y = -3 \cos 3t$ ;  $y'' + 9y = 0$ ,  $y(0) = -3$ ,  $y'(0) = 0$

20.  $y = \frac{1}{4}(e^{2x} - e^{-2x})$ ;  $y'' - 4y = 0$ ,  $y(0) = 0$ ,  $y'(0) = 1$

(21 – 104) Solve the differential equations

21.  $y' = xy$

22.  $xy' = 2y$

23.  $y' = e^{x-y}$

24.  $y' = (1 + y^2)e^x$

25.  $y' = xy + y$

26.  $y' = ye^x - 2e^x + y - 2$

27.  $y' = \frac{x}{y+2}$

28.  $y' = \frac{xy}{x-1}$

29.  $x \frac{dy}{dx} + y = e^x$ ,  $x > 0$

30.  $y' + (\tan x)y = \cos^2 x$ ,  $-\frac{\pi}{2} < x < \frac{\pi}{2}$

31.  $(1+x)y' + y = \sqrt{x}$

32.  $e^{2x}y' + 2e^{2x}y = 2x$

33.  $x \frac{dy}{dx} + 2y = 1 - \frac{1}{x}$ ,  $x > 0$

34.  $(t+1) \frac{ds}{dt} + 2s = 3(t+1) + \frac{1}{(t+1)^2}$ ,  $t > -1$

35.  $\tan \theta \frac{dr}{d\theta} + r = \sin^2 \theta$ ,  $0 < \theta < \frac{\pi}{2}$

36.  $y' = \cos x - y \sec x$

37.  $(1+x^3)y' = 3x^2y + x^2 + x^5$

38.  $\frac{dy}{dt} - 2y = 4 - t$

39.  $y' + y = \frac{1}{1+e^t}$

40.  $y' = 3y - 4$

41.  $y' = -2y - 4$

42.  $y' = -y + 2$

43.  $y' = 2y + 6$

44.  $x(x-1)dy - ydx = 0$

45.  $xy' + 2y = 1 - x^{-1}$

46.  $xy' - y = 2x \ln x$

47.  $(1+e^x)dy + (ye^x + e^{-x})dx = 0$

48.  $(x+3y^2)dy + ydx = 0$

49.  $y' = \frac{y^2 + ty + t^2}{t^2}$

50.  $\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}$

51.  $y' = \frac{2xy + 2x}{x^2 - 1}$

52.  $\frac{dy}{dx} = \sin 5x$

53.  $\frac{dy}{dx} = (x+1)^2$

54.  $dx + e^{3x}dy = 0$

55.  $dy - (y-1)^2 dx = 0$

56.  $x \frac{dy}{dx} = 4y$

57.  $\frac{dx}{dy} = y^2 - 1$

58.  $\frac{dy}{dx} = e^{2y}$
59.  $\frac{dy}{dx} + 2xy^2 = 0$
60.  $\frac{dy}{dx} = e^{3x+2y}$
61.  $e^x y \frac{dy}{dx} = e^{-y} + e^{-2x-y}$
62.  $y \ln x \frac{dx}{dy} = \left( \frac{y+1}{x} \right)^2$
63.  $\frac{dy}{dx} = \left( \frac{2y+3}{4x+5} \right)^2$
64.  $\csc y dx + \sec^2 x dy = 0$
65.  $\sin 3x dx + 2y \cos^3 3x dy = 0$
66.  $\frac{dy}{dx} = (64xy)^{1/3}$
67.  $\frac{dy}{dx} = 2x \sec y$
68.  $\frac{dy}{dx} = \frac{x}{ye^{x+2y}}$
69.  $y' - y = 3e^t$
70.  $y' - y = e^{2t} - 1$
71.  $y' + y = te^{-t} + 1$
72.  $y' + y = 1 + e^{-x} \cos 2x$
73.  $y' + y \cot x = \cos x$
74.  $y' + y \sin t = \sin t$
75.  $y' + (\cot t)y = 2t \csc t$
76.  $y' + (1 + \sin t)y = 0$
77.  $y' + \left( \frac{1}{2} \cos x \right) y = -\frac{3}{2} \cos x$
78.  $\frac{dy}{dx} + y = e^{3x}$
79.  $y' - ty = t$
80.  $y' = 2y + x^2 + 5$
81.  $xy' + 2y = 3$
82.  $y' + 2y = 1$
83.  $y' + 2y = e^{-t}$
84.  $y' + 2y = e^{-2t}$
85.  $y' - 2y = e^{3t}$
86.  $y' + 2y = e^{-x} + x + 1$
87.  $y' + 2xy = x$
88.  $y' - 2ty = t$
89.  $y' + 2ty = 5t$
90.  $y' - 2xy = e^{x^2}$
91.  $y' + 2xy = x^3$
92.  $y' - 2y = t^2 e^{2t}$
93.  $x' - 2 \frac{x}{t+1} = (t+1)^2$
94.  $y' + \frac{2}{t} y = \frac{\cos t}{t^2}$
95.  $y' - 2(\cos 2t)y = 0$
96.  $y' + 2y = \cos 3t$
97.  $y' - 3y = 5$
98.  $y' + 3y = 2xe^{-3x}$
99.  $y' + 3x^2 y = x^2$
100.  $y' + \frac{3}{t} y = \frac{\sin t}{t^3}, \quad (t \neq 0)$
101.  $y' + \frac{3}{x} y = 1 + \frac{1}{x}$
102.  $y' + \frac{3}{2} y = \frac{1}{2} e^x$
103.  $y' + 5y = t + 1$
104.  $xy' - y = x^2 \sin x$

(105 – 202) Solve the initial value problem

105.  $y' = \frac{1-2t}{y}$ ,  $y(1) = -2$

106.  $y' = y^2 - 4$ ,  $y(0) = 0$

107.  $\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}$ ,  $y(0) = -1$

108.  $y' = \frac{x}{1+2y}$ ,  $y(-1) = 0$

109.  $(e^{2y} - y) \cos x \frac{dy}{dx} = e^y \sin 2x$ ,  $y(0) = 0$

110.  $\frac{dy}{dx} = e^{-x^2}$ ,  $y(3) = 5$

111.  $\frac{dy}{dx} + 2y = 1$ ,  $y(0) = \frac{5}{2}$

112.  $\sqrt{1-y^2} dx - \sqrt{1-x^2} dy = 0$ ,  $y(0) = \frac{\sqrt{3}}{2}$

113.  $(1+x^4)dy + x(1+4y^2)dx = 0$ ,  $y(1) = 0$

114.  $e^{-2t} \frac{dy}{dt} = \frac{1+e^{-2t}}{y}$ ,  $y(0) = 0$

115.  $\frac{dy}{dt} = \frac{t+2}{y}$ ,  $y(0) = 2$

116.  $\frac{1}{t^2} \frac{dy}{dt} = y$ ,  $y(0) = 1$

117.  $\frac{dy}{dt} = -y^2 e^{2t}$ ,  $y(0) = 1$

118.  $\frac{dy}{dt} - (2t+1)y = 0$ ,  $y(0) = 2$

119.  $\frac{dy}{dt} + 4ty^2 = 0$ ,  $y(0) = 1$

120.  $\frac{dy}{dx} = ye^x$ ,  $y(0) = 2e$

121.  $\frac{dy}{dx} = 3x^2(y^2 + 1)$ ,  $y(0) = 1$

122.  $2y \frac{dy}{dx} = \frac{x}{\sqrt{x^2 - 16}}$ ,  $y(5) = 2$

123.  $\frac{dy}{dx} = 4x^3 y - y$ ,  $y(1) = -3$

124.  $\frac{dy}{dx} + 1 = 2y$ ,  $y(1) = 1$

125.  $(\tan x) \frac{dy}{dx} = y$ ,  $y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

126.  $x \frac{dy}{dx} - y = 2x^2 y$ ,  $y(1) = 1$

127.  $\frac{dy}{dx} = 2xy^2 + 3x^2 y^2$ ,  $y(1) = -1$

128.  $\frac{dy}{dx} = 6e^{2x-y}$ ,  $y(0) = 0$

129.  $2\sqrt{x} \frac{dy}{dx} = \cos^2 y$ ,  $y(4) = \frac{\pi}{4}$

130.  $y' + 3y = 0$ ,  $y(0) = -3$

131.  $2y' - y = 0$ ,  $y(-1) = 2$

132.  $\sqrt{y} dx + (1+x) dy = 0$ ,  $y(0) = 1$

133.  $\frac{dy}{dx} = 6y^2 x$ ,  $y(1) = \frac{1}{25}$

134.  $\frac{dy}{dx} = \frac{3x^2 + 4x - 4}{2y - 4}$ ,  $y(1) = 3$

135.  $y' - 3y = 4$ ,  $y(0) = 2$

136.  $y' = y + 2xe^{2x}$ ,  $y(0) = 3$

137.  $(1+t^2)y' + 4ty = (1+t^2)^{-2}$ ,  $y(1) = 0$

138.  $y' + y = e^t$ ,  $y(0) = 1$

139.  $y' + \frac{1}{2}y = t$ ,  $y(0) = 1$

140.  $y' = x + 5y$ ,  $y(0) = 3$

141.  $y' = 2x - 3y$ ,  $y(0) = \frac{1}{3}$

142.  $xy' + y = e^x$ ,  $y(1) = 2$

143.  $y \frac{dx}{dy} - x = 2y^2$ ,  $y(1) = 5$

144.  $xy' + y = 4x + 1$ ,  $y(1) = 8$
145.  $y' + 4xy = x^3 e^{x^2}$ ,  $y(0) = -1$
146.  $(x+1)y' + y = \ln x$ ,  $y(1) = 10$
147.  $y' - (\sin x)y = 2 \sin x$ ,  $y\left(\frac{\pi}{2}\right) = 1$
148.  $y' + y = 2$ ,  $y(0) = 0$
149.  $y' - 2y = 3e^{2x}$ ,  $y(0) = 0$
150.  $xy' + 2y = 3x$ ,  $y(1) = 5$
151.  $xy' + 5y = 7x^2$ ,  $y(2) = 5$
152.  $xy' - y = x$ ,  $y(1) = 7$
153.  $xy' + y = 3xy$ ,  $y(1) = 0$
154.  $xy' + 3y = 2x^5$ ,  $y(2) = 1$
155.  $y' + y = e^x$ ,  $y(0) = 1$
156.  $xy' - 3y = x^3$ ,  $y(1) = 10$
157.  $y' + 2xy = x$ ,  $y(0) = -2$
158.  $y' = (1-y)\cos x$ ,  $y(\pi) = 2$
159.  $(1+x)y' + y = \cos x$ ,  $y(0) = 1$
160.  $y' = 1 + x + y + xy$ ,  $y(0) = 0$
161.  $xy' = 3y + x^4 \cos x$ ,  $y(2\pi) = 0$
162.  $y' = 2xy + 3x^2 e^{x^2}$ ,  $y(0) = 5$
163.  $(x^2 + 4)y' + 3xy = x$ ,  $y(0) = 1$
164.  $(x^2 + 1)y' + 3x^3 y = 6xe^{-3x^2/2}$ ,  $y(0) = 1$
165.  $y' - 2y = e^{3x}$ ,  $y(0) = 3$
166.  $y' - 3y = 6$ ,  $y(0) = 1$
167.  $2y' + 3y = e^x$ ,  $y(0) = 0$
168.  $y' + y = 1 + e^{-x} \cos 2x$ ,  $y\left(\frac{\pi}{2}\right) = 0$
169.  $2y' + (\cos x)y = -3 \cos x$ ,  $y(0) = -4$
170.  $y' + 2y = e^{-x} + x + 1$ ,  $y(-1) = e$
171.  $y' + \frac{y}{x} = xe^{-x}$ ,  $y(1) = e - 1$
172.  $y' + 4y = e^{-x}$ ,  $y(1) = \frac{4}{3}$
173.  $x^2 y' + 3xy = x^4 \ln x + 1$ ,  $y(1) = 0$
174.  $y' + \frac{3}{x}y = 3x - 2$ ,  $y(1) = 1$
175.  $y' - (\sin x)y = 2 \sin x$ ,  $y\left(\frac{\pi}{2}\right) = 1$
176.  $y' + (\tan x)y = \cos^2 x$ ,  $y(0) = -1$
177.  $t y' + 2y = t^2 - t + 1$ ,  $y(1) = \frac{1}{2}$
178.  $y' + (\cos t)y = \cos t$ ,  $y(\pi) = 2$
179.  $y' + 2ty = 2t$ ,  $y(0) = 1$
180.  $y' + y = \frac{e^{-t}}{t^2}$ ,  $y(1) = 0$
181.  $ty' + 2y = \sin t$ ,  $y(\pi) = \frac{1}{\pi}$
182.  $t \frac{dy}{dt} + 2y = t^3$ ,  $t > 0$ ,  $y(2) = 1$
183.  $\theta \frac{dy}{d\theta} + y = \sin \theta$ ,  $\theta > 0$ ,  $y\left(\frac{\pi}{2}\right) = 1$
184.  $\frac{dy}{dx} + xy = x$ ,  $y(0) = -6$
185.  $y' = \frac{y}{x}$ ,  $y(1) = -2$
186.  $y' = \frac{\sin x}{y}$ ,  $y\left(\frac{\pi}{2}\right) = 1$
187.  $y' = y + 2xe^{2x}$ ,  $y(0) = 3$
188.  $(x^2 + 1)y' + 3xy = 6x$ ,  $y(0) = -1$

$$189. \quad y' = (4t^3 + 1)y, \quad y(0) = 4$$

$$190. \quad y' = \frac{e^t}{2y}, \quad y(\ln 2) = 1$$

$$191. \quad (\sec x)y' = y^3, \quad y(0) = 3$$

$$192. \quad \frac{dy}{dx} = e^{x-y}, \quad y(0) = \ln 3$$

$$193. \quad y' = 2e^{3y-t}, \quad y(0) = 0$$

$$194. \quad y' = 3y - 6, \quad y(0) = 9$$

$$195. \quad y' = -y + 2, \quad y(0) = -2$$

$$196. \quad y' = -2y - 4, \quad y(0) = 0$$

$$197. \quad \frac{dy}{dx} + 3x^2y = x^2, \quad y(0) = -1$$

$$198. \quad xdy + (y - \cos x)dx = 0, \quad y\left(\frac{\pi}{2}\right) = 0$$

$$199. \quad \frac{dy}{dt} = \frac{t+1}{2ty}, \quad y(1) = 4$$

$$200. \quad \frac{dy}{dt} = \sqrt{y} \sin t, \quad y(0) = 4$$

$$201. \quad y'(t) + 3y = 0, \quad y(0) = 6$$

$$202. \quad y'(t) = 2y + 4, \quad y(0) = 8$$



## Section 2.8 – Applications

### Motion with Resistance Proportional to Velocity

An object with a mass  $m$  is moving along a coordinate line with position function  $s$  and velocity  $v$  at time  $t$ .

From *Newton's second law* of motion, the resistance force opposing the motion is

$$\begin{aligned}\text{Force} &= \text{mass} \times \text{acceleration} \\ &= m \frac{dv}{dt}\end{aligned}$$

If the resisting force is proportional to velocity, we have

$$m \frac{dv}{dt} = -kv \quad \text{or} \quad \frac{dv}{dt} = -\frac{k}{m}v \quad (k > 0)$$

This is a separable differential equation, the solution with initial condition  $v = v_0$  at  $t = 0$  is

$$v = v_0 e^{-(k/m)t}$$

In the English system, where the weight is measured in pounds, mass is measured in slugs. Thus,

$$\text{Pounds} = \text{slugs} \times 32$$

### Example

For a 192-lb ice skater, the  $k$  is about  $\frac{1}{3}$  slug/sec and  $m = \frac{192}{32} = 6$  slugs. How long will it take the skater to coast from 11 ft/sec (7.5 mph) to 1 ft/sec? How far will the skater coast before coming to a complete stop?

### Solution

$$v = v_0 e^{-(k/m)t} \qquad \frac{k}{m} = \frac{\frac{1}{3}}{6} = \frac{1}{18}$$

$$1 = 11e^{-(1/18)t}$$

$$e^{-(1/18)t} = \frac{1}{11}$$

$$-\frac{1}{18}t = \ln\left(\frac{1}{11}\right) = -\ln 11$$

$$t = 18 \ln 11 \approx 43 \text{ sec}$$

$$\text{Distance} = \frac{v_0 m}{k} = \frac{11(6)}{1/3} = 198 \text{ ft}$$

## Mixtures Problems

The physical representation of the rate of change:

$$\frac{dy}{dt} = \text{rate of change} = \text{rate in} - \text{rate out}$$

This is referred to as a **balance law**.

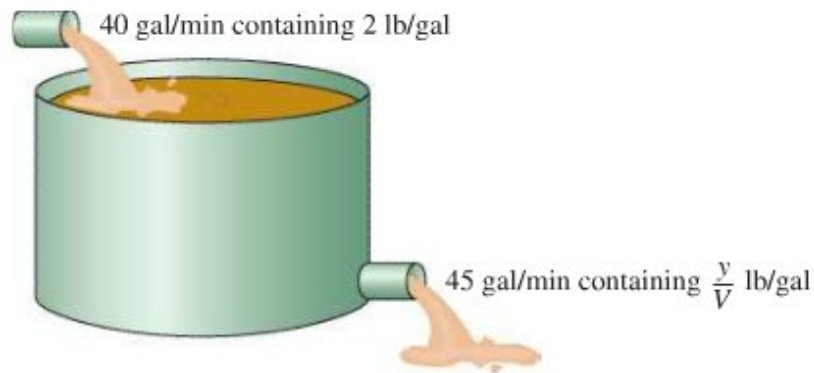
$$\text{Rate} = \text{Volume Rate (gal/min)} \times \text{Concentration (lb/gal)}$$

If  $y(t)$  is the amount of chemical in the container at time  $t$  and  $V(t)$  is the total volume of liquid in the container at time  $t$ , then the departure rate of the chemical at time  $t$  is

$$\begin{aligned} \text{Departure Rate} &= \frac{y(t)}{V(t)} \bullet (\text{outflow rate}) \\ &= \left( \frac{\text{concentration in}}{\text{container at } t} \right) \bullet (\text{outflow rate}) \end{aligned}$$

### Example

In an oil refinery, a storage tank contains 2,000 gal of gasoline that initially has 100 lb. of an additive dissolved in it. In preparation for winter weather, gasoline containing 2 lb. of additive per gallon is pumped into the tank at a rate of 40 gal/min. The well-mixed solution is pumped out at a rate of 45 gal/min. How much of the additive is in the tank 20 min after the pumping process begins?



### Solution

Let  $y$  be the amount (in lb.) of additive in the tank at time  $t$  and  $y(0) = 100$

$$\begin{aligned} V(t) &= 2000 + \left( 40 \frac{\text{gal}}{\text{min}} - 45 \frac{\text{gal}}{\text{min}} \right) (t \text{ min}) \\ &= 2000 - 5t \end{aligned}$$

$$\begin{aligned} \text{Rate out} &= \frac{y(t)}{V(t)} \bullet \text{outflow rate} \\ &= \left( \frac{y}{2000 - 5t} \right) (45) \end{aligned}$$

$$= \frac{45y}{2000-5t} \frac{lb}{min} \Bigg|$$

$$Rate\ in = \left(2 \frac{lb}{gal}\right) \left(40 \frac{gal}{min}\right)$$

$$= 80 \frac{lb}{min} \Bigg|$$

$$\frac{dy}{dt} = 80 - \frac{45y}{2000-5t}$$

$$\frac{dy}{dt} + \frac{45}{2000-5t} y = 80$$

$$\begin{aligned} v(t) &= e^{\int p dt} = e^{\int \frac{45}{2000-5t} dt} & d(2000-5t) &= -5dt \Rightarrow dt = -\frac{1}{5}d(2000-5t) \\ &= e^{\int \frac{45}{2000-5t} \left(-\frac{1}{5}d(2000-5t)\right)} \\ &= e^{-\int \frac{9}{2000-5t} d(2000-5t)} \\ &= e^{-9 \ln(2000-5t)} \\ &= (2000-5t)^{-9} \Bigg| \end{aligned}$$

$$\begin{aligned} y &= \frac{1}{(2000-5t)^{-9}} \left( \int 80 \cdot (2000-5t)^{-9} dt + C \right) \\ &= (2000-5t)^9 \left( 80 \int (2000-5t)^{-9} \left(-\frac{1}{5}d(2000-5t)\right) + C \right) \\ &= (2000-5t)^9 \left( -16 \frac{(2000-5t)^{-8}}{-8} + C \right) \\ &= (2000-5t)^9 \left( 2(2000-5t)^{-8} + C \right) \\ &= 2(2000-5t) + C(2000-5t)^9 \end{aligned}$$

$$y(0) = 2(2000-5(0)) + C(2000-5(0))^9$$

$$100 = 4000 + C(2000)^9$$

$$C(2000)^9 = -3900$$

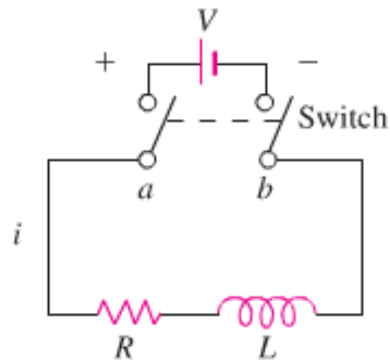
$$C = -\frac{3900}{(2000)^9}$$

$$y = 2(2000 - 5t) - \frac{3900}{(2000)^9}(2000 - 5t)^9$$

$$y(20) = 2(2000 - 5(\textcolor{red}{20})) - \frac{3900}{(2000)^9}(2000 - 5(\textcolor{red}{20}))^9$$

$$\approx \textcolor{blue}{1342} \text{ lb.}$$

## RL Circuits



The diagram represents an electrical circuit whose total resistance is a constant  $R$  ohms and whose self-inductance is coil, is  $L$  henries.

**Ohm's Law:**  $V = Ri$

$$L \frac{di}{dt} + Ri = V$$

$i$ : current in amperes

$t$ : time in seconds

### Example

The switch in the  $RL$  circuit is closed at time  $t = 0$ . How will the current flow as a function of time?

### Solution

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}$$

$$e^{\int p dt} = e^{\int \left(\frac{R}{L}\right) dt}$$

$$= e^{\frac{R}{L}t}$$

$$i = \frac{1}{e^{\int p dt}} \left( \int f \cdot e^{\int p dt} dt + C \right)$$

$$= \frac{1}{e^{\frac{R}{L}t}} \left( \int \frac{V}{L} \cdot e^{\frac{R}{L}t} dt + C \right)$$

$$= e^{-\frac{R}{L}t} \left( \frac{V}{L} \frac{L}{R} e^{\frac{R}{L}t} + C \right)$$

$$= e^{-\frac{R}{L}t} \left( \frac{V}{R} e^{\frac{R}{L}t} + C \right)$$

$$t = 0 \Rightarrow i = 0 \quad i = e^{-\frac{R}{L}(0)} \left( \frac{V}{R} e^{\frac{R}{L}(0)} + C \right)$$

$$0 = \frac{V}{R} + C$$

$$C = -\frac{V}{R}$$

$$i(t) = e^{-\frac{R}{L}t} \left( \frac{V}{R} e^{\frac{R}{L}t} - \frac{V}{R} \right)$$

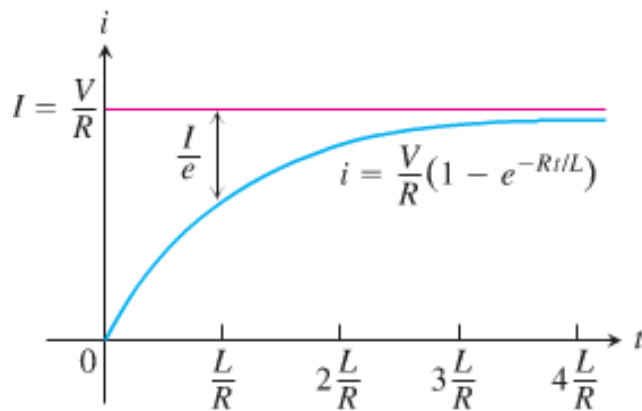
$$= \frac{V}{R} - \frac{V}{R} e^{-\frac{R}{L}t}$$

$$\lim_{t \rightarrow \infty} i = \lim_{t \rightarrow \infty} \left( \frac{V}{R} - \frac{V}{R} e^{-\frac{R}{L}t} \right)$$

$$= \frac{V}{R} - \frac{V}{R} \cdot 0$$

$$= \frac{V}{R}$$

The current approaches the **steady-state value**  $\frac{V}{R}$ .



## Exercises      Section 2.8 – Applications

1. A 66-kg cyclist on a 7-kg bicycle starts coasting on level ground at 9 m/sec. The  $k \approx 3.9 \text{ kg / sec}$ 
  - a) About how far will the cyclist coast before reaching a complete stop?
  - b) How long will it take the cyclist's speed to drop to 1 m/sec?
2. Suppose that an Iowa class battleship has mass 51,000 metric tons (51,000,000 kg) and  $k \approx 59,000 \text{ kg / sec}$ . Assume that the ship loses power when it is moving at a speed of 9 m/sec.
  - a) About how far will the ship coast before it is dead in the water?
  - b) About how long will it take the ship's speed to drop to 1 m/sec?
3. A 200-gal tank is half full of distilled water. At time  $t = 0$ , a solution containing 0.5 lb./gal of concentrate enters the tank at the rate of 5 gal/min, and the well-stirred mixture is withdrawn at the rate of 3 gal/min.
  - a) At what time will the tank be full?
  - b) At the time the tank is full, how many pounds of concentrate will it contain?
4. A tank contains 100 gal of fresh water. A solution containing 1 lb./gal of soluble lawn fertilizer runs into the tank at the rate of 1 gal/min, and the mixture is pumped out of the tank at a rate of 3 gal/min. Find the maximum amount of fertilizer in the tank and the time required to reach the maximum.
5. An Executive conference room of a corporation contains 4500  $\text{ft}^3$  of air initially free of carbon monoxide. Starting at time  $t = 0$ , cigarette smoke containing 4% carbon monoxide is blown into the room at the rate of 0.3  $\text{ft}^3 / \text{min}$ . A ceiling fan keeps the air in the room well circulated and the air leaves the room at the same rate of 0.3  $\text{ft}^3 / \text{min}$ . Find the time when the concentration of carbon monoxide in the room reaches 0.01%.
6. Many chemical reactions are the result of the interaction of 2 molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If  $a$  is the amount of substance  $A$  and  $b$  is the substance  $B$  at time  $t = 0$ , and if  $x$  is the amount of product at time  $t$ , then the rate of formation of  $x$  may be given by the differential equation

$$\frac{dx}{dt} = k(a-x)(b-x) \quad \text{or} \quad \frac{1}{(a-x)(b-x)} \frac{dx}{dt} = k$$

Where  $k$  is a constant for the reaction. Integrate both sides of this equation to obtain a relation between  $x$  and  $t$ .

c) If  $a = b$

d) If  $a \neq b$

Assume in each case that  $x = 0$  when  $t = 0$

7. The tank initially holds 100 *gal* of pure water. At time  $t = 0$ , a solution containing 2 *lb* of salt per gallon begins to enter the tank at the rate of 3 gallons per minute. At the same time a drain is opened at the bottom of the tank so that the volume of solution in the tank remains constant. How much salt is in the tank after 60 *min*? What will be the eventual salt content in the tank?
8. The 600-*gal* tank is filled with 300 *gal* of pure water. A spigot is opened above the tank and a salt solution containing 1.5 *lb.* of salt per gallon of solution begins flowing into the tank at the rate of 3 *gal/min*. Simultaneously, a drain is opened at the bottom of the tank allowing the solution to leave tank at a rate of 1 *gal/min*. What will be the salt content in the tank at the precise moment that the volume of solution in the tank is equal to the tank's capacity (600 *gal*)?
9. A tank with a 2,000 *gal* capacity initially contains 500 *gal* of brine containing 100 *lbs.* of salt starting at time  $t = 0$ , brine containing 0.1 *lb/gal* of salt is added at a rate of 60 *gal/min* and the mixed solution is drained off at a rate of 40 *gal/min*. How much salt is in the tank when it reaches the point of over flowing?
10. The amount of drug in the blood of a patient (in *mg*) due to an intravenous line is governed by the initial value problem

$$y'(t) = -0.02y + 3, \quad y(0) = 0 \quad \text{for } t \geq 0$$

Where  $t$  is measured in hours

- Find and graph the solution of the initial value problem.
  - What is the steady-state level of the drug?
  - When does the drug level reach 90% of the steady-state value?
11. A fish hatchery has 500 *fish* at time  $t = 0$ , when harvesting begins at a rate of  $b$  *fish/yr.* where  $b > 0$ . The fish population is modeled by the initial value problem.

$$y'(t) = 0.1y - b, \quad y(0) = 500 \quad \text{for } t \geq 0$$

Where  $t$  is measured in years.

- Find the fish population for  $t \geq 0$  in terms of the harvesting rate  $b$ .
  - Graph the solution in the case that  $b = 40$  *fish / yr*. Describe the solution.
  - Graph the solution in the case that  $b = 60$  *fish / yr*. Describe the solution.
12. A community of hares on an island has a population of 50 when observations begin at  $t = 0$ . The population for  $t \geq 0$  is modeled by the initial value problem.

$$\frac{dP}{dt} = 0.08P \left( 1 - \frac{P}{200} \right), \quad P(0) = 50$$

- Find the solution of the initial value problem.
- What is the steady-state population?



13. When an infected person is introduced into a closed and otherwise healthy community, the number of people who become infected with the disease (in the absence of any intervention) may be modeled by the logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{A}\right), \quad P(0) = P_0$$

Where  $k$  is a positive infection rate,  $A$  is the number of people in the community, and  $P_0$  is the number of infected people at  $t = 0$ . The model assumes no recovery or intervention.

- a) Find the solution of the initial value problem in terms of  $k$ ,  $A$ , and  $P_0$ .
  - b) Graph the solution in the case that  $k = 0.025$ ,  $A = 300$ , and  $P_0 = 1$ .
  - c) For fixed values of  $k$  and  $A$ , describe the long-term behavior of the solutions for any  $P_0$  with  $0 < P_0 < A$
14. An object of mass  $m$  is released from a balloon. Find the distance it falls in  $t$  seconds, if the force of resistance due to the air is directly proportional to the speed of the object.
15. A body falls from a height of 300 *ft*. What distance has it traveled after 4 *sec*. if subject to  $g$ , the earth's acceleration?
16. A body falls from an initial velocity of 1,000 *ft/s*. What distance has it traveled after 3 *sec*. if subject to  $g = 32 \text{ ft/s}^2$ , the earth's acceleration?
17. A projectile is fired straight upwards with an initial velocity of 1,600 *ft/s*. What is its velocity at 40,000 *ft*. ( $g = 32 \text{ ft/s}^2$ )
18. A projectile is fired straight upwards with an initial velocity of 1,000 *ft/s*. What is its velocity at 8,000 *ft*. ( $g = 32 \text{ ft/s}^2$ )
19. A stone is released from rest and dropped into a deep well. Eight seconds later, the sound of the stone splashing into the water at the bottom of the well returns to the ear of the person who released the stone. How long does it take the stone to drop to the bottom of the well? How deep is the well? Ignore air resistance. The speed of sound is 340 *m/s*.
20. A rocket is fired vertically and ascends with constant acceleration  $a = 100 \text{ m/s}^2$  for 1.0 *min*. At that point, the rocket motor shuts off and the rocket continues upward under the influence of gravity. Find the maximum altitude acquired by the rocket and the total time elapsed from the take-off until the rocket returns to the earth. *Ignore air resistance.*

21. A ball is projected vertically upward with initial velocity  $v_0$  from ground level. Ignore air resistance.
- What is the maximum height acquired by the ball?
  - How long does it take the ball to reach its maximum height? How long does it take the ball to return to the ground? Are these times identical?
  - What is the speed of the ball when it impacts with the ground on its return?

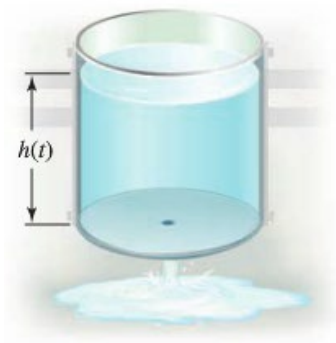
22. An object having mass  $70 \text{ kg}$  falls from rest under the influence of gravity. The terminal velocity of the object is  $-20 \text{ m/s}$ . Assume that the air resistance is proportional to the velocity.
- Find the velocity and distance traveled at the end of  $2 \text{ seconds}$ .
  - How long does it take the object to reach 80% of its terminal velocity?

23. An object in free fall may be modeled by assuming that the only forces at work are the gravitational force and resistance (friction due to the medium in which the objects falls). By Newton's second law (mass  $\times$  acceleration = the sum of the external forces), the velocity of the object satisfies the differential equation

$$\underbrace{m}_{\text{mass}} \cdot \underbrace{v'(t)}_{\text{acceleration}} = \underbrace{mg + f(v)}_{\text{external force}}$$

Where  $f$  is a function that models the resistance and the positive direction is downward. One common assumption (often used for motion in air) is that  $f(v) = -kv^2$ , where  $k > 0$  is a drag coefficient.

- Show that the equation can be written in the form  $v'(t) = g - av^2$  where  $a = \frac{k}{m}$
  - For what (positive) value of  $v$  is  $v'(t) = 0$ ? (This equilibrium solution is called the **terminal velocity**.)
  - Find the solution of this separable equation assuming  $v(0) = 0$  and  $0 < v(t)^2 < \frac{g}{a}$  for  $t \geq 0$
  - Graph the solution found in part (c) with  $g = 9.8 \text{ m/s}^2$ ,  $m = 1 \text{ kg}$ , and  $k = 0.1 \text{ kg/m}$ , and verify the terminal velocity agrees with the value found in part (b).
24. An open cylindrical tank initially filled with water drains through a hole in the bottom of the tank according to Torricelli's Law. If  $h(t)$  is the depth of water in the tank for  $t \geq 0$ , then Torricelli's Law implies  $h'(t) = -2k\sqrt{h}$ , where  $k$  is a constant that includes the acceleration due to gravity, the radius of the tank, and the radius of the drain. Assume that the initial depth of the water is  $h(0) = H$



- a) Find the solution of the initial value problem.
  - b) Find the solution in the case that  $k = 0.1$  and  $H = 0.5$  m.
  - c) In general, how long does it take the tank to drain in terms of  $k$  and  $H$ ?
- 25.** The reaction of chemical compounds can often be modeled by differential equations. Let  $y(t)$  be the concentration of a substance in reaction for  $t \geq 0$  (typical units of  $y$  are *moles/L*). The change in the concentration of a substance, under appropriate conditions, is  $\frac{dy}{dt} = -ky^n$ , where  $k > 0$  is a rate constant and the positive integer  $n$  is the order of the reaction.
- a) Show that for a first-order reaction ( $n = 1$ ), the concentration obeys an exponential decay law.
  - b) Solve the initial value problem for a second-order reaction ( $n = 2$ ) assuming  $y(0) = y_0$
  - c) Graph and compare the concentration for a first-order and second-order reaction with  $k = 0.1$  and  $y_0 = 1$
- 26.** The consumption of a substrate in a reaction involving an enzyme is often modeled using Michaelis-Menton Kinetics, which involves the initial value problem  $\frac{ds}{dt} = \frac{Qs}{K + s}$ ,  $s(0) = s_0$ , where  $s(t)$  is the amount of substrate present at time  $t \geq 0$ , and  $Q$  and  $K$  are positive constants. Solve the initial value problem with  $Q = 10$ ,  $K = 5$ , and  $s_0 = 50$ . Notice that the solution can be expressed explicitly only with  $t$  as a function of  $s$ . Describe how  $s$  behaves as  $t \rightarrow \infty$ .
- 27.** An investment account, which earns interest and has regular deposits, can be modeled by the initial value problem  $B'(t) = aB + m$  for  $t \geq 0$ , with  $B(0) = B_0$ . The constant  $a$  reflects the monthly interest rate,  $m$  is the rate of monthly deposits, and  $B_0$  is the initial balance in the account. Solve the initial value problem with  $a = 0.005$ ,  $m = \$100 / \text{month}$ , and  $B_0 = \$100$ . After how many months does the account have a balance of \$7,500?
- 28.** The growth of cancer tumors may be modeled by the Gomperts growth equation. Let  $M(t)$  be the mass of the tumor for  $t \geq 0$ . The relevant initial value problem is

$$\frac{dM}{dt} = -aM \ln \frac{M}{K}, \quad M(0) = M_0$$

Where  $a$  and  $K$  are positive constants and  $0 < M_0 < K$

- a) Graph the growth rate function  $R(M) = -aM \ln \frac{M}{K}$  assuming  $a = 1$  and  $K = 4$ . For what values of  $M$  is the growth rate positive? For what values of  $M$  is maximum?
- b) Solve the initial value problem and graph the solution for  $a = 1$ ,  $K = 4$ , and  $M_0 = 1$ . Describe the growth pattern of the tumor. Is the growth unbounded? If not, what is the limiting size of the tumor?
- c) In the general equation, what is the meaning of  $K$ ?

29. An endowment is an investment account in which the balance ideally remains constant and withdrawals are made on the interest earned by the account. Such an account may be modeled by the initial value problem  $B'(t) = aB - m$  for  $t \geq 0$ , with  $B(0) = B_0$ . The constant  $a$  reflects the annual interest rate,  $m$  is the annual rate of withdrawal, and  $B_0$  is the initial balance in the account.

- a) Solve the initial value problem with  $a = 0.05$ ,  $m = \$1000 / \text{yr.}$  and  $B_0 = \$15,000$ . Does the balance in the account increase or decrease?
- b) If  $a = 0.05$  and  $B_0 = \$50,000$ , what is the annual withdrawal rate  $m$  that ensures a constant balance in the account? What is the constant balance?

30. The halibut fishery has been modeled by the differential equation  $\frac{dy}{dt} = ky \left( 1 - \frac{y}{M} \right)$

Where  $y(t)$  is the biomass (the total mass of the members of the population) in kilograms at time  $t$  (measured in years), the carrying capacity is estimated to be  $M = 8 \times 10^7 \text{ kg}$  and  $k = 0.71 \text{ per year}$ .

- a) If  $y(0) = 2 \times 10^7 \text{ kg}$ , find the biomass a year later.
- b) How long will it take for the biomass to reach  $4 \times 10^7 \text{ kg}$ .

31. Suppose a population  $P(t)$  satisfies  $\frac{dP}{dt} = 0.4P - 0.001P^2$ ,  $P(0) = 50$  where  $t$  is measured in years.

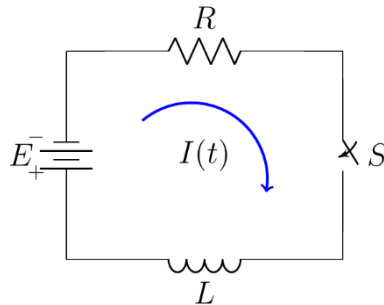
- a) What is the carrying capacity?
- b) What is  $P'(0)$ ?
- c) When will the population reach 50% of the carrying capacity?

32. Let  $P(t)$  be the performance level of someone learning a skill as a function of the training time  $t$ . The graph of  $P$  is called a **learning curve**. We proposed the differential equation

$$\frac{dP}{dt} = k(M - P(t))$$

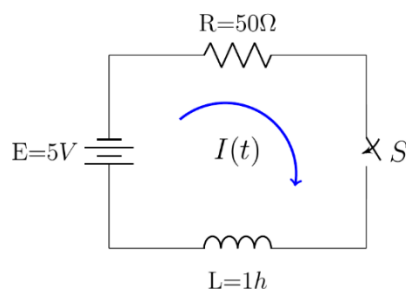
As a reasonable model for learning, where  $k$  is a positive constant. Solve it as a linear differential equation and use your solution to graph the learning curve.

33. An inductor ( $L = 1 \text{ H}$ ) and a resistor ( $R = 0.1 \Omega$ ) are joined in series with an electronic force (emf)  $E = E(t)$  and no charge on the capacitor at  $t = 0$ . Find the ensuing charge current in the current at time  $t$  for the given  $E(t) = 10 - 2t$
34. An inductor ( $L = 1 \text{ H}$ ) and a resistor ( $R = 0.1 \Omega$ ) are joined in series with an electronic force (emf)  $E = E(t)$  and no charge on the capacitor at  $t = 0$ . Find the ensuing current in the current at time  $t$  for the given  $E(t) = 4\cos 3t$
35. An inductor ( $L = 1 \text{ H}$ ) and a resistor ( $R = 0.1 \Omega$ ) are joined in series with an electronic force (emf)  $E = E(t)$  and no charge on the capacitor at  $t = 0$ . Find the ensuing current in the current at time  $t$  for the given  $E(t) = 4\sin 2\pi t$
36. An  $RL$  circuit with a  $1 - \Omega$  resistor and a  $0.01\text{-H}$  inductor is driven by a voltage  $E(t) = \sin 100t \text{ V}$ . If the initial inductor current is zero, determine the subsequence resistor and inductor current and the voltages.
37. An  $RL$  circuit with a  $5 - \Omega$  resistor and a  $0.05\text{-H}$  inductor is driven by a voltage  $E(t) = 5\cos 120t \text{ V}$ . If the initial inductor current is  $1 \text{ A}$ , determine the subsequence resistor and inductor current and the voltages.
38. For the given  $RL$ -circuit



Which has a constant impressed voltage  $E$ , a resistor of resistance  $R$ , and a coil of impedance  $L$ . Find the current  $I(t)$  flowing in the circuit.

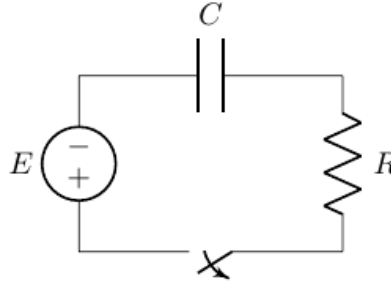
39. For the given  $RL$ -circuit



Which has a constant impressed voltage  $E$ , a resistor of resistance  $R$ , and a coil of impedance  $L$ . Find the current  $I(t)$  flowing in the circuit.

40. A circuit containing an electromotive force, a capacitor with a capacitance of  $C$  farads ( $F$ ), and a resistor with a resistance of  $R$  ohms ( $\Omega$ ). The voltage drop across the capacitor is  $\frac{Q}{C}$ , where  $Q$  is the charge (in coulombs), so in this case **Kirchhoff's Law** gives

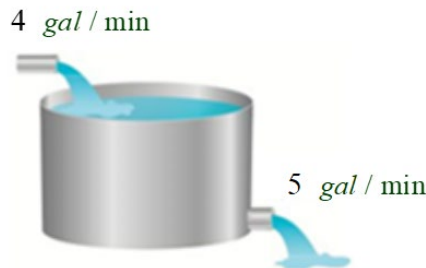
$$RI + \frac{Q}{C} = E(t)$$



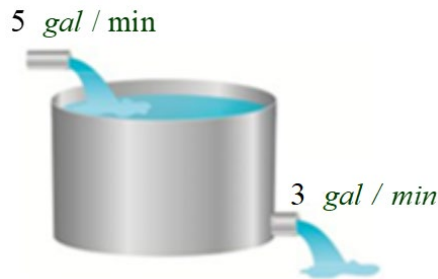
But  $I = \frac{dQ}{dt}$ , so we have  $R \frac{dQ}{dt} + \frac{1}{C}Q = E(t)$

Find the charge and the current at time  $t$

- Suppose the resistance is  $5 \Omega$ , the capacitance is  $0.05 F$ , a battery gives voltage of  $60 V$  and initial charge is  $Q(0) = 0 C$
  - Suppose the resistance is  $2 \Omega$ , the capacitance is  $0.01 F$ ,  $E(t) = 10 \sin 60t$  and initial charge is  $Q(0) = 0 C$
41. A 30-volt electromotive force is applied to an  $LR$ -series circuit in which the inductance is  $0.1$  henry and the resistance is  $50$  ohms.
- Find the current  $i(t)$  if  $i(0) = 0$
  - Determine the current as  $t \rightarrow \infty$
  - Solve the equation when  $E(t) = E_0 \sin \omega t$  and  $i(0) = i_0$
42. A tank contains  $50$  gallons of a solution composed of  $90\%$  water and  $10\%$  alcohol. A second solution containing  $50\%$  water and  $50\%$  alcohol is added to the tank at the rate of  $4 \text{ gal / min}$ . As the second solution is being added, the tank is being drained at a rate of  $5 \text{ gal / min}$ . The solution in the tank is stirred constantly. How much alcohol is in the tank after  $10$  minutes?



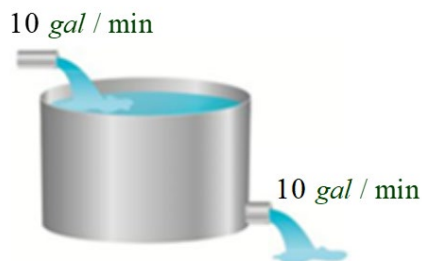
43. A 200-gallon tank is half full of distilled water. At time  $t = 0$ , a concentrate solution containing  $0.5 \text{ lb/gal}$  enters the tank at the rate of  $5 \text{ gal/min}$ , and well-stirred mixture is withdrawn at the rate of  $3 \text{ gal/min}$ .



- a) At what time will the tank be full?  
b) At the time the tank is full, how many pounds of concentrate will it contain?
44. A 200-gallon tank is half full of distilled water. At time  $t = 0$ , a concentrate solution containing  $1 \text{ lb/gal}$  enters the tank at the rate of  $5 \text{ gal/min}$ , and well-stirred mixture is withdrawn at the rate of  $3 \text{ gal/min}$ .



- a) At what time will the tank be full?  
b) At the time the tank is full, how many pounds of concentrate will it contain?
45. A 200-gallon tank is full of a concentrate solution containing  $25 \text{ lb}$ . Starting at time  $t = 0$ , distilled water is admitted to the tank at the rate of  $10 \text{ gal/min}$ , and well-stirred mixture is withdrawn at the same rate.

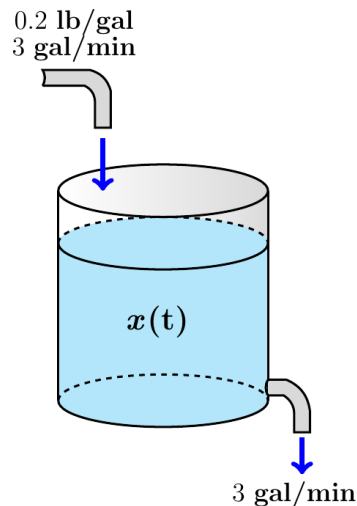


- a) Find the amount of concentrate in the solution as a function of  $t$ .  
b) Find the time at which the amount of concentrate in the tank reaches 15 pounds.  
c) Find the quantity of the concentrate in the solution as  $t \rightarrow \infty$ .

46. A tank contains 300 *litres* of fluid in which 20 *grams* of salt is dissolved. Brine containing 1 g of salt per *litre* is then pumped into the tank at a rate of 4 *L/min*; the well-mixed solution is pumped out at the same rate. Find the number  $x(t)$  of grams of salt in the tank at time  $t$ .

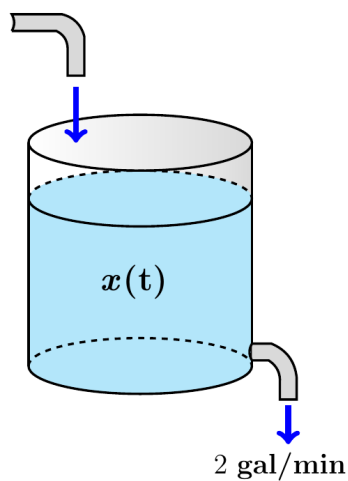


47. A 1500 *gallon* tank initially contains 600 *gallon* of water with 5 *lbs.* of salt dissolved in it. Water enters the tank at a rate of 9 *gal/hr.* and the water entering the tank at a rate has a salt concentration of  $\frac{1}{5}(1 + \cos t)$  *lbs./gal.* If a well-mixed solution leaves the tank at a rate of 6 *gal/hr.*, how much salt is in the tank when it overflows?
48. A tank contains 100 *gal* of pure water. At time zero, a sugar-water solution containing 0.2 *lb* of sugar per *gal* enters the tank at a rate of 3 *gal/min*. Simultaneously, a drain is opened at the bottom of the tank allowing the sugar solution to leave the tank at 3 *gal/min*. Assume that the solution in the tank is kept perfectly mixed at all times.



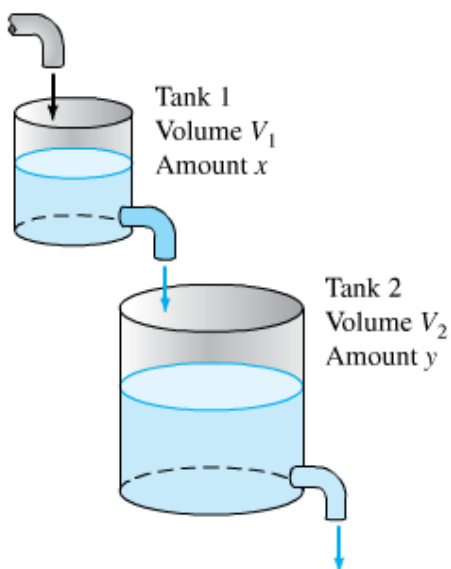
- What will be the sugar content in the tank after 20 *minutes*?
  - How long will it take the sugar content in the tank to reach 15 *lbs*?
  - What will be the eventual sugar content in the tank?
49. A tank initially contains 50 *gal* of sugar water having a concentration of 2 *lbs.* of sugar for each *gal* of water. At time zero, pure water begins pouring into the tank at a rate of 2 *gal per minute*. Simultaneously, a drain is opened at the bottom of the tank so that the volume of sugar-water solution in the tank remains constant.





- a) How much sugar is in the tank after 10 minutes?
- b) How long will it take the sugar content in the tank to dip below 20 lb.?
- c) What will be the eventual sugar content in the tank?

50. Suppose that in the cascade tank 1 initially 100 gal of pure ethanol and tank 2 initially contains 100 gal of pure water. Pure water flows into tank 1 at 10 gal/min, and the other two flow rates are also 10 gal/min.



- a) Find the amounts  $x(t)$  and  $y(t)$  of ethanol in the two tanks at time  $t \geq 0$ .
- b) Find the maximum amount of ethanol ever in tank 2.

