Solution   Section 2.5 – Subspaces, Span and Null Space

Exercise

Suppose S and T are two subspaces of a vector space **V**.

a) The sum $S+T$ contains all sums $s+t$ of a vector s in S and a vector t in T. Show that $S+T$ satisfies the requirements (addition and scalar multiplication) for a vector space.

b) If S and T are lines in $\boldsymbol{R}^m$, what is the difference between $S+T$ and $S\cup T$? That union contains all vectors from S and T or both. Explain this statement: The span of $S\cup T$ is $S+T$.

Solution

a) Let $s,\ s'$ be vectors in S, Let $t,\ t'$ be vectors in T, and let c be a scalar. Then

$$(s+t)+(s'+t')=(s+s')+(t+t') \text{ and}$$

$$c(s+t)=cs+ct$$

Thus $S+T$ is closed under addition and scalar multiplication, it satisfies the two requirements for a vector space.

b) If S and T are distinct lines, then S and T is a plane, whereas $S\cup T$ is not even closed under addition. The span of $S\cup T$ is the set of all combinations of vectors in this union. In particular, it contains all sums $s+t$ of a vector ***s*** in S and a vector ***t*** in T, and these sums form $S+T$. $S+T$ contains both S and T; so it contains $S\cup T$. $S+T$ is a vector space.

c) So, it contains all combinations of vectors in itself; in particular, it contains the span of $S\cup T$. Thus, the span of $S\cup T$ is $S+T$.

Exercise

Determine which of the following are subspaces of $\boldsymbol{R}^3$?

a) All vectors of the form $(a, 0, 0)$

b) All vectors of the form $(a, 1, 1)$

c) All vectors of the form (a, b, c), where $b = a + c$

d) All vectors of the form (a, b, c), where $b = a + c + 1$

e) All vectors of the form $(a, b, 0)$

Solution

a) $(a_1,\ 0,\ 0)+(a_2,\ 0,\ 0)=(a_1+a_2,\ 0,\ 0)$

$k(a,\ 0,\ 0)=(ka,\ 0,\ 0)$

This is a subspace of $\boldsymbol{R}^3$

b) $(a_1, 1, 1)+(a_2, 1, 1)=(a_1+a_2, 2, 2)$ which is not in the set.

Therefore, this is not a subspace of $\boldsymbol{R}^3$

c) $(a_1, b_1, c_1)+(a_2, b_2, c_2)=(a_1+a_2, b_1+b_2, c_1+c_2)$

$$=(a_1+a_2, a_1+c_1+a_2+c_2, c_1+c_2)$$

$$=(a_1+a_2, (a_1+a_2)+(c_1+c_2), c_1+c_2)$$

$$=(a_1, a_1+c_1, c_1)+(a_2, a_2+c_2, c_2)$$

$$k(a, b, c)=(ka, kb, kc)$$

$$=(ka, k(a+c), kc)$$

$$=k(a, (a+c), c)$$

This is a subspace of $\boldsymbol{R}^3$

d) $k(a+c+1)\neq ka+kc+1$ so $k(a,b,c)$ is not in the set.

Therefore, this is not a subspace of $\boldsymbol{R}^3$

e) $(a_1, b_1, 0)+(a_2, b_2, 0)=(a_1+a_2, b_1+b_2, 0)$

$k(a, b, 0)=(ka, kb, 0)$

This is a subspace of $\boldsymbol{R}^3$

Exercise

Determine which of the following are subspaces of $\boldsymbol{R}^\infty$?

a) All sequences $\boldsymbol{v}$ in $\boldsymbol{R}^\infty$ of the form $v = (v, 0, v, 0, \ldots)=(kv, k, kv, k, \cdots)$

b) All sequences $\boldsymbol{v}$ in $\boldsymbol{R}^\infty$ of the form $v = (v, 1, v, 1, \ldots)$

c) All sequences $\boldsymbol{v}$ in $\boldsymbol{R}^\infty$ of the form $v = (v, 2v, 4v, 8v, 16v, \ldots)$

Solution

a) $(v_1,0,v_1,0,\ldots)+(v_2,0,v_2,0,\ldots)=(v_1+v_2, 0, v_1+v_2, 0, \ldots)$

$kv=k(v,0,v,0,\ldots)=(kv,0,kv,0,\ldots)$

This is a subspace of $\boldsymbol{R}^\infty$

b) $kv=k(v,1,v,1,\ldots)$

kv is not in the set

Since $k \neq 1$, then is not a subspace of $\boldsymbol{R}^{\infty}$

c) $\left(v_1, 2v_1, 4v_1, 8v_1, \ldots\right) + \left(v_2, 2v_2, 4v_2, 8v_2, \ldots\right) = \left(v_1 + v_2, 2v_1 + 2v_2, 4v_1 + 4v_2, 8v_1 + 8v_2, \ldots\right)$

$$= \left(v_1 + v_2, 2\left(v_1 + v_2\right), 4\left(v_1 + v_2\right), 8\left(v_1 + v_2\right), \ldots\right)$$

$k\left(v, 2v, 4v, 8v, \ldots\right) = \left(kv,\ 2kv,\ 4kv,\ 8kv,\ \ldots\right)$

This is a subspace of $\boldsymbol{R}^{\infty}$

Exercise

Which of the following are linear combinations of $\boldsymbol{u} = (0, -2, 2)$ and $\boldsymbol{v} = (1, 3, -1)$?

a) (2, 2, 2) *b)* (3, 1, 5) *c)* (0, 4, 5) *d)* (0, 0 ,0)

Solution

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \\ 2 & -1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

a) $b = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \to \left[\begin{array}{cc|c} 0 & 1 & 2 \\ -2 & 3 & 2 \\ 2 & -1 & 2 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$

(2, 2, 2) = 2$\boldsymbol{u}$ + 2$\boldsymbol{v}$ is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.

b) $b = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \to \left[\begin{array}{cc|c} 0 & 1 & 3 \\ -2 & 3 & 1 \\ 2 & -1 & 5 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$

(3, 1, 5) = 4$\boldsymbol{u}$ + 3$\boldsymbol{v}$ is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.

c) $b = \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} \to \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & 3 & 4 \\ 2 & -1 & 5 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$

(0, 4, 5) is not a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.

d) $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & 3 & 0 \\ 2 & -1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

(0, 0, 0) = 0$\boldsymbol{u}$ + 0$\boldsymbol{v}$ is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.

Exercise

Which of the following are linear combinations of $\boldsymbol{u}$ = (2, 1, 4), $\boldsymbol{v}$ = (1, –1, 3) and $\boldsymbol{w}$ = (3, 2, 5)?

a) (–9, –7, –15)

b) (6, 11, 6)

c) (0, 0 ,0)

Solution

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 3 & 5 \end{pmatrix}$$

a) $$\left[\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Therefore, $(-9,\ -7,\ -15) = -2\boldsymbol{u} + 1\boldsymbol{v} - 2\boldsymbol{w}$

b) $$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 6 \\ 1 & -1 & 2 & 11 \\ 4 & 3 & 5 & 6 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Therefore, $(6,\ 11,\ 6) = 4\boldsymbol{u} - 5\boldsymbol{v} + 1\boldsymbol{w}$

c) $$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore, $(0,\ 0,\ 0) = 0\boldsymbol{u} + 0\boldsymbol{v} + 0\boldsymbol{w}$

Exercise

Determine whether the given vectors span $\boldsymbol{R}^3$

a) $\vec{v}_1 = (2,\ 2,\ 2),\quad \vec{v}_2 = (0,\ 0,\ 3),\quad \vec{v}_3 = (0,\ 1,\ 1)$

b) $\vec{v}_1 = (2,\ -1,\ 3),\quad \vec{v}_2 = (4,\ 1,\ 2),\quad \vec{v}_3 = (8,\ -1,\ 8)$

c) $\vec{v}_1 = (3,\ 1,\ 4),\quad \vec{v}_2 = (2,\ -3,\ 5),\quad \vec{v}_3 = (5,\ -2,\ 9),\quad \vec{v}_4 = (1,\ 4,\ -1)$

Solution

a) $$det\begin{pmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{pmatrix} = -6 \neq 0$$

The system is consistent for all values so the given vectors span $\boldsymbol{R}^3$.

b) $det\begin{pmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{pmatrix} = 0$

The system is not consistent for all values so the given vectors do not span $\boldsymbol{R}^3$.

c) $\left[\begin{array}{cccc|c} 3 & 2 & 5 & 1 & b_1 \\ 1 & -3 & -2 & 4 & b_2 \\ 4 & 5 & 9 & -1 & b_3 \end{array}\right] \xrightarrow{\text{leads to}} \left[\begin{array}{cccc|c} 1 & -3 & -2 & 4 & b_2 \\ 0 & 1 & 1 & -1 & \frac{1}{11}b_1 - \frac{3}{11}b_2 \\ 0 & 0 & 0 & 0 & -\frac{17}{11}b_1 + \frac{7}{11}b_2 + b_3 \end{array}\right]$

The system has a solution only if $-\frac{17}{11}b_1 + \frac{7}{11}b_2 + b_3 = 0$. But since this is a restriction that the given vectors don't span on all of $\boldsymbol{R}^3$. So the given vectors do not span $\boldsymbol{R}^3$.

Exercise

Which of the following are linear combinations of $A = \begin{pmatrix} 4 & 0 \\ -2 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}$

a) $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ *b)* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *c)* $\begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix}$

Solution

$$\begin{pmatrix} 4 & 1 & 0 \\ 0 & -1 & 2 \\ -2 & 2 & 1 \\ -2 & 3 & 4 \end{pmatrix}$$

a) $\left[\begin{array}{ccc|c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & -5 \\ -2 & 2 & 1 & -1 \\ -2 & 3 & 4 & -8 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = 1A + 2B - 3C$ is a linear combinations of *A*, *B*, and *C*.

b) $\left[\begin{array}{ccc|c} 4 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ -2 & 2 & 1 & 0 \\ -2 & 3 & 4 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0A + 0B + 0C$$ is a linear combinations of *A*, *B*, and *C*.

c) $$\left[\begin{array}{ccc|c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & 0 \\ -2 & 2 & 1 & 3 \\ -2 & 3 & 4 & 8 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix} = 1A + 2B + 1C$$ is a linear combination of *A*, *B*, and *C*.

Exercise

Suppose that $\vec{v}_1 = (2,\ 1,\ 0,\ 3)$, $\vec{v}_2 = (3,\ -1,\ 5,\ 2)$, $\vec{v}_3 = (-1,\ 0,\ 2,\ 1)$. Which of the following vectors are in span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

a) (2, 3, –7, 3) *b)* (0, 0, 0, 0) *c)* (1, 1, 1, 1) *d)* (–4, 6, –13, 4)

Solution

In order to be span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, there must exists scalars *a*, *b*, *c* that $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \boldsymbol{w}$

$$A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

a) $$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 1 & -1 & 0 & 3 \\ 0 & 5 & 2 & -7 \\ 3 & 2 & 1 & 3 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This system is consistent, it has only solution is *a* = 2, *b* = –1, *c* = –1

$2\vec{v}_1 - 1\vec{v}_2 - 1\vec{v}_3 = (2,\ 3,\ -7,\ 3)$

Therefore, (2, 3, –7, 3) is in span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

b) The vector (0, 0 , 0, 0) is obviously in span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

Since $0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 = (0,\ 0,\ 0,\ 0)$

c) $\left[\begin{array}{ccc|c} 2 & 3 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 5 & 2 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

This system is inconsistent, therefore (1, 1, 1, 1) is *not* in span $\left\{\vec{v}_1, \vec{v}_2, \vec{v}_3\right\}$

d) $\left[\begin{array}{ccc|c} 2 & 3 & -1 & -4 \\ 1 & -1 & 0 & 6 \\ 0 & 5 & 2 & -13 \\ 3 & 2 & 1 & 4 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

This system is consistent, it has only solution is $a = 3$, $b = -3$, $c = 1$

$3\vec{v}_1 - 3\vec{v}_2 + 1\vec{v}_3 = (-4,\ 6,\ -13,\ 4)$

Therefore, (–4, 6, –13, 4) is in span $\left\{\vec{v}_1, \vec{v}_2, \vec{v}_3\right\}$

Exercise

Let $\boldsymbol{f} = \cos^2 x$ and $\boldsymbol{g} = \sin^2 x$. Which of the following lie in the space spanned by $\boldsymbol{f}$ and $\boldsymbol{g}$

a) $\cos 2x$ *b)* $3 + x^2$ *c)* $\sin x$ *d)* 0

Solution

a) $\cos 2x = \cos^2 x - \sin^2 x$, therefore $\cos 2x$ is in span $\{\boldsymbol{f}, \boldsymbol{g}\}$

b) In order for $3 + x^2$ to be in span $\{\boldsymbol{f}, \boldsymbol{g}\}$, there must exist scalars *a* and *b* such that

$a\cos^2 x + b\sin^2 x = 3 + x^2$

When $\left.\begin{array}{lll} x = 0 & \Rightarrow & a = 3 \\ x = \pi & \Rightarrow & a = 3 + \pi^2 \end{array}\right\} \rightarrow \textit{contradiction}$

Therefore $3 + x^2$ is *not* in span $\{\boldsymbol{f}, \boldsymbol{g}\}$

c) In order for $\sin x$ to be in span $\{\boldsymbol{f}, \boldsymbol{g}\}$, there must exist scalars *a* and *b* such that

$a\cos^2 x + b\sin^2 x = \sin x$

When $\left.\begin{array}{lll} x = \frac{\pi}{2} & \Rightarrow & b = 1 \\ x = -\frac{\pi}{2} & \Rightarrow & b = -1 \end{array}\right\} \rightarrow \textit{contradiction}$

Therefore $\sin x$ is *not* in span $\{\boldsymbol{f}, \boldsymbol{g}\}$

d) In order for 0 to be in span $\{\boldsymbol{f}, \boldsymbol{g}\}$, there must exist scalars *a* and *b* such that

$$0\cos^2 x + 0\sin^2 x = 0$$

Therefore $\mathbf{0}$ is in span $\{\boldsymbol{f}, \boldsymbol{g}\}$

Exercise

$V = \mathbb{R}^3, \quad S = \{(0,\ s,\ t) \mid s,\ t \text{ are real numbers}\}$ where V is a vector space and S is subset of V

a) Is S closed under addition?

b) Is S closed under scalar multiplication?

c) Is S a subspace of V?

Solution

a) Let $\boldsymbol{u} = (0,\ s_1,\ t_1)$ *and* $\boldsymbol{v} = (0,\ s_2,\ t_2)$

$$\boldsymbol{u} + \boldsymbol{v} = (0,\ s_1 + s_2,\ t_1 + t_2)$$
$$= (0,\ s,\ t)$$

Yes, S is closed under addition

b) $k\boldsymbol{u} = (0,\ ks_1,\ kt_1) = (0,\ s,\ t)$

Yes, S is closed under scalar multiplication

c) Since S is closed under addition and scalar multiplication, then S is a subspace of V.

Exercise

$V = \mathbb{R}^3, \quad S = \{(x,\ y,\ z) \mid x,\ y\ ,\ z\ \geq 0\}$ where V is a vector space and S is subset of V

a) Is S closed under addition?

b) Is S closed under scalar multiplication?

c) Is S a subspace of V?

Solution

a) Let $\boldsymbol{u} = (x_1, y_1, z_1)$ *and* $\boldsymbol{v} = (x_2, y_2, z_2)$

$$\boldsymbol{u} + \boldsymbol{v} = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \qquad \textit{where } x = x_1 + x_2,\ y = y_1 + y_2,\ z = z_1 + z_2$$
$$= (x,\ y,\ z)$$

Yes, S is closed under addition

b) $(-1)\boldsymbol{u} = (-x_1,\ -y_1,\ -z_1)$

S is ***not*** closed under scalar multiplication since $x_1 \geq 0 \Rightarrow -x_1 \leq 0$

c) S is ***not*** a subspace of V.

Exercise

$V = \mathbb{R}^3, \quad S = \{(x,\ y,\ z) \mid z = x + y + 1\}$ where V is a vector space and S is subset of V

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V?

Solution

a) Let $\boldsymbol{u} = (0,\ 1,\ 2)$ *and* $\boldsymbol{v} = (1,\ 2,\ 4)$

$$\boldsymbol{u} + \boldsymbol{v} = (1,\ 3,\ 6)$$
$$\neq (1,\ 3,\ 1+3+1)$$

No, S is *not* closed under addition

b)
$$\begin{aligned} 2\boldsymbol{u} &= (2x_1,\ 2y_1,\ 2z_1) \\ &= (2x_1,\ 2y_1,\ 2(x_1 + y_1 + 1)) \\ &= (2x_1,\ 2y_1,\ 2x_1 + 2y_1 + 2) \qquad \textit{Where} \quad x = 2x_1, \quad y = 2y_1, \quad 2z = 2(x_1 + y_1 + 1) \\ &= (x,\ y,\ z) \end{aligned}$$

Yes, S is closed under scalar multiplication

c) S is ***not*** a subspace of V.

Exercise

Let $S = \left\{(a_1,\ a_2,\ a_3) \in \mathbb{R}^3 :\ a_1 = 3a_2 \ \textit{and} \ a_3 = -a_2\right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\boldsymbol{R}^3$?

Solution

a) Let $\vec{u} = (a_1, a_2, a_3)$ *and* $\vec{v} = (b_1, b_2, b_3)$

$$\begin{aligned} \vec{u} + \vec{v} &= (a_1, a_2, a_3) + (b_1, b_2, b_3) \\ &= (3a_2, a_2, -a_2) + (3b_2, b_2, -b_2) \\ &= (3a_2 + 3b_2, a_2 + b_2, -a_2 - b_2) \\ &= (3(a_2 + b_2), a_2 + b_2, -(a_2 + b_2)) \qquad c_2 = a_2 + b_2 \\ &= (3c_2, c_2, -c_2) \end{aligned}$$

$$=\left(c_1, c_2, c_3\right): \quad c_1 = 3c_2 \quad c_3 = -c_2$$

S is closed under addition

b) $k\vec{u} = k\left(a_1, a_2, a_3\right)$

$$= k\left(3a_2, a_2, -a_2\right)$$

$$= \left(3ka_2, ka_2, -ka_2\right) \qquad c_2 = ka_2$$

$$= \left(3c_2, c_2, -c_2\right)$$

$$= \left(c_1, c_2, c_3\right): \quad c_1 = 3c_2 \quad c_3 = -c_2$$

S is closed under scalar multiplication.

c) S is a subspace of V.

Exercise

Let $S = \left\{\left(a_1, a_2, a_3\right) \in \mathbb{R}^3 : a_1 = a_3 + 2\right\}$, Determine:

a) Is S closed under addition?

b) Is S closed under scalar multiplication?

c) Is S a subspace of $\boldsymbol{R}^3$?

Solution

a) Let $\vec{u} = (2, 1, 0)$ *and* $\vec{v} = (3, 0, 1)$ $\qquad a_1 = a_3 + 2$

$$\vec{u} + \vec{v} = (2, 1, 0) + (3, 0, 1)$$

$$= (5, 1, 1) \qquad 5 \overset{?}{=} 1 + 2$$

$$\neq (3, 1, 1)$$

S is *not* closed under addition

b) $k\vec{u} = k\left(a_1, a_2, a_3\right)$

$$= k\left(a_3 + 2, a_2, a_3\right)$$

$$= \left(ka_3 + 2k, ka_2, ka_3\right)$$

$$a_1 = a_3 + 2 \quad \rightarrow \quad ka_3 + 2k = a_3 + 2$$

$$2k \neq 2 \quad (\forall k)$$

S is *not* closed under scalar multiplication.

c) S is ***not*** a subspace of V.

Exercise

Let $S = \left\{ \left(a_1,\ a_2,\ a_3 \right) \in \mathbb{R}^3 :\ 2a_1 - 7a_2 + a_3 = 0 \right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\boldsymbol{R}^3$?

Solution

a) Let $\vec{u} = \left(a_1, a_2, a_3 \right)$ *and* $\vec{v} = \left(b_1, b_2, b_3 \right)$

$$2a_1 - 7a_2 + a_3 = 0 \ \rightarrow \ a_3 = 7a_2 - 2a_1$$

$$\begin{aligned}
\vec{u} + \vec{v} &= \left(a_1, a_2, a_3 \right) + \left(b_1, b_2, b_3 \right) \\
&= \left(a_1, a_2, 7a_2 - 2a_1 \right) + \left(b_1, b_2, 7b_2 - 2b_1 \right) \\
&= \left(a_1 + b_1, a_2 + b_2, 7a_2 - 2a_1 + 7b_2 - 2b_1 \right) \\
&= \left(a_1 + b_1, a_2 + b_2, 7\left(a_2 + b_2 \right) - 2\left(a_1 + b_1 \right) \right) && \text{Let } c_1 = a_1 + b_1 \quad c_2 = a_2 + b_2 \\
&= \left(c_1, c_2, 7c_2 - 2c_1 \right) && c_3 = 7c_2 - 2c_1 \ \rightarrow \ 2c_1 - 7c_2 + c_3 = 0 \\
&= \left(c_1, c_2, c_3 \right)
\end{aligned}$$

S is closed under addition

b)
$$\begin{aligned}
k\vec{u} &= k\left(a_1, a_2, a_3 \right) \\
&= k\left(a_1, a_2, 7a_2 - 2a_1 \right) \\
&= \left(ka_1, ka_2, 7ka_2 - 2ka_1 \right) && \text{Let } c_1 = ka_1 \quad c_2 = ka_2 \\
&= \left(c_1, c_2, 7c_2 - 2c_1 \right) && c_3 = 7c_2 - 2c_1 \ \rightarrow \ 2c_1 - 7c_2 + c_3 = 0 \\
&= \left(c_1, c_2, c_3 \right)
\end{aligned}$$

S is closed under scalar multiplication.

c) *S* is a subspace of *V*.

Exercise

Let $S = \left\{ \left(a_1,\ a_2,\ a_3 \right) \in \mathbb{R}^3 :\ a_1 - 4a_2 - a_3 = 0 \right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?

c) Is S a subspace of $\boldsymbol{R}^3$?

Solution

$$a_1 - 4a_2 - a_3 = 0 \quad \to \quad a_1 = 4a_2 + a_3$$

a) Let $\vec{u} = (a_1, a_2, a_3)$ *and* $\vec{v} = (b_1, b_2, b_3)$

$$\vec{u} + \vec{v} = (a_1, a_2, a_3) + (b_1, b_2, b_3)$$

$$= (4a_2 + a_3, a_2, a_3) + (4b_2 + b_3, b_2, b_3)$$

$$= (4a_2 + a_3 + 4b_2 + b_3, a_2 + b_2, a_3 + b_3)$$

$$= \big(4(a_2 + b_2) + (a_3 + b_3), a_2 + b_2, a_3 + b_3\big) \qquad \text{Let } c_2 = a_2 + b_2 \quad c_3 = a_3 + b_3$$

$$= (4c_2 + c_3, c_2, c_3) \qquad c_1 - 4c_2 - c_3 = 0 \to c_1 = 4c_2 + c_3$$

$$= (c_1, c_2, c_3)$$

S is closed under addition

b) $k\vec{u} = k(a_1, a_2, a_3)$

$$= k(4a_2 + a_3, a_2, a_3)$$

$$= (4ka_2 + ka_3, ka_2, ka_3) \qquad \text{Let } c_2 = ka_2 \quad c_3 = ka_3$$

$$= (4c_2 + c_3, c_2, c_3) \qquad c_1 = 4c_2 + c_3 \to c_1 - 4c_2 - c_3 = 0$$

$$= (c_1, c_2, c_3)$$

S is closed under scalar multiplication.

c) S is a subspace of V.

Exercise

Let $S = \left\{(a_1, a_2, a_3) \in \mathbb{R}^3 :\ a_1 + 2a_2 - 3a_3 = 0\right\}$, Determine:

a) Is S closed under addition?

b) Is S closed under scalar multiplication?

c) Is S a subspace of $\boldsymbol{R}^3$?

Solution

$$a_1 + 2a_2 - 3a_3 = 0 \quad \to \quad a_1 = -2a_2 + 3a_3$$

a) Let $\vec{u}=\left(a_1, a_2, a_3\right)$ *and* $\vec{v}=\left(b_1, b_2, b_3\right)$

$$\vec{u}+\vec{v}=\left(a_1, a_2, a_3\right)+\left(b_1, b_2, b_3\right)$$

$$=\left(-2a_2+3a_3, a_2, a_3\right)+\left(-2b_2+3b_3, b_2, b_3\right)$$

$$=\left(-2a_2+3a_3-2b_2+3b_3, a_2+b_2, a_3+b_3\right)$$

$$=\left(-2\left(a_2+b_2\right)+3\left(a_3+b_3\right), a_2+b_2, a_3+b_3\right) \qquad \text{Let } c_2=a_2+b_2 \quad c_3=a_3+b_3$$

$$=\left(-2c_2+3c_3, c_2, c_3\right) \qquad c_1+2c_2-3c_3=0 \to c_1=-2c_2+3c_3$$

$$=\left(c_1, c_2, c_3\right)$$

S is closed under addition

b) $k\vec{u}=k\left(a_1, a_2, a_3\right)$

$$=k\left(4a_2+a_3, a_2, a_3\right)$$

$$=\left(-2ka_2+3ka_3, ka_2, ka_3\right) \qquad \text{Let } c_2=ka_2 \quad c_3=ka_3$$

$$=\left(-2c_2+3c_3, c_2, c_3\right) \qquad c_1=-2c_2+3c_3 \to c_1-2c_2+3c_3=0$$

$$=\left(c_1, c_2, c_3\right)$$

S is closed under scalar multiplication.

c) S is a subspace of V.

Exercise

Let $S=\left\{\left(a_1, a_2, a_3\right)\in\mathbb{R}^3: a_1+2a_2-3a_3=1\right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\boldsymbol{R}^3$?

Solution

$$a_1+2a_2-3a_3=1 \quad\to\quad a_1=1-2a_2+3a_3$$

a) Let $\vec{u}=\left(a_1, a_2, a_3\right)$ *and* $\vec{v}=\left(b_1, b_2, b_3\right)$

$$\vec{u}+\vec{v}=\left(a_1, a_2, a_3\right)+\left(b_1, b_2, b_3\right)$$

$$=\left(1-2a_2+3a_3, a_2, a_3\right)+\left(1-2b_2+3b_3, b_2, b_3\right)$$

$$=\left(1-2a_2+3a_3+1-2b_2+3b_3,\ a_2+b_2,\ a_3+b_3\right)$$

$$=\left(2-2\left(a_2+b_2\right)+3\left(a_3+b_3\right),\ a_2+b_2,\ a_3+b_3\right) \qquad \text{Let } c_2=a_2+b_2 \quad c_3=a_3+b_3$$

$$=\left(2-2c_2+3c_3,\ c_2,\ c_3\right) \qquad c_1+2c_2-3c_3=1 \ \rightarrow\ c_1=1-2c_2+3c_3$$

$$\neq\left(1-2c_2+3c_3,\ c_2,\ c_3\right)$$

S is *not* closed under addition

b) $\vec{u}=(2,\ 1,\ 1)$

$k\vec{u}=k(2,\ 1,\ 1)$

$$=(2k,\ k,\ k) \qquad a_1+2a_2-3a_3=1 \ \rightarrow\ 2k+2k-3k\overset{?}{=}1$$

$$k\neq 1 \ \ (\forall k)$$

S is *not* closed under scalar multiplication.

c) *S* is ***not*** a subspace of *V*.

Exercise

Let $S=\left\{\left(a_1,\ a_2,\ a_3\right)\in\mathbb{R}^3:\ 5a_1^2-3a_2^2+6a_3^2=0\right\}$, Determine:

a) Is *S* closed under addition?

b) Is *S* closed under scalar multiplication?

c) Is *S* a subspace of $\boldsymbol{R}^3$?

Solution

$$5a_1^2-3a_2^2+6a_3^2=0 \ \rightarrow\ a_2^2=\frac{5}{3}a_1^2+2a_3^2$$

a) Let $\vec{u}=\left(0,\ \sqrt{2},\ 1\right)$ *and* $\vec{v}=\left(3,\ \sqrt{17},\ 1\right)$

$$\vec{u}+\vec{v}=\left(0,\ \sqrt{2},\ 1\right)+\left(3,\ \sqrt{17},\ 1\right)$$

$$=\left(3,\ \sqrt{2}+\sqrt{17},\ 2\right)$$

$$a_2^2=\frac{5}{3}a_1^2+2a_3^2 \ \rightarrow\ \left(\sqrt{2}+\sqrt{17}\right)^2\neq 15+8$$

S is *not* closed under addition

b) $k\vec{u}=k\left(a_1,\ a_2,\ a_3\right)$

$$=\left(ka_1,\ ka_2,\ ka_3\right)$$

$$5\left(ka_1\right)^2-3\left(ka_2\right)^2+6\left(ka_3\right)^2=0$$

$$5k^2a_1^2-3k^2a_2^2+6k^2a_3^2=0$$

$$5a_1^2-3a_2^2+6a_3^2=0$$

S is closed under scalar multiplication.

c) *S* is ***not*** a subspace of *V*.

Exercise

Let $S=\left\{\left(a_1,\ a_2,\ a_3\right)\in\mathbb{R}^3:\ a_3=a_1+a_2\right\}$, Determine:

a) Is *S* closed under addition?

b) Is *S* closed under scalar multiplication?

c) Is *S* a subspace of $\boldsymbol{R}^3$?

Solution

a) Let $\vec{u}=\left(a_1,a_2,a_3\right)$ *and* $\vec{v}=\left(b_1,b_2,b_3\right)$

$$\vec{u}+\vec{v}=\left(a_1,a_2,a_3\right)+\left(b_1,b_2,b_3\right)$$

$$=\left(a_1,a_2,a_1+a_2\right)+\left(b_1,b_2,b_1+b_2\right)$$

$$=\left(a_1+b_1,a_2+b_2,a_1+a_2+b_1+b_2\right)$$ Let $c_1=a_1+b_1 \quad c_2=a_2+b_2$

$$=\left(c_1,c_2,c_1+c_2\right)$$ Then, $c_3=c_1+c_2$

$$=\left(c_1,c_2,c_3\right)$$

S is closed under addition

b) $k\vec{u}=k\left(a_1,a_2,a_3\right)$

$$=k\left(a_1,a_2,a_1+a_2\right)$$

$$=\left(ka_1,ka_2,k\left(a_1+a_2\right)\right)$$

$$=\left(ka_1,ka_2,ka_1+ka_2\right)$$ *Where* $c_1=ka_1,\ c_2=ka_2,\ c_3=ka_1+ka_2$

$$=\left(c_1,c_2,c_3\right)$$ $c_3=ka_1+ka_2=c_1+c_2$

S is closed under scalar multiplication.

c) *S* is a subspace of *V*.

Exercise

Let $S = \left\{ \left(a_1, \ a_2, \ a_3 \right) \in \mathbb{R}^3 : \ a_1 + a_2 + a_3 = 0 \right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\boldsymbol{R}^3$?

Solution

a) Let $\vec{u} = \left(a_1, a_2, a_3 \right) \ni \ a_1 + a_2 + a_3 = 0$

$\vec{v} = \left(b_1, b_2, b_3 \right) \ni \ b_1 + b_2 + b_3 = 0$

$$\vec{u} + \vec{v} = \left(a_1, a_2, a_3 \right) + \left(b_1, b_2, b_3 \right)$$
$$= \left(a_1 + b_1, a_2 + b_2, a_3 + b_3 \right)$$

Since $a_1 + a_2 + a_3 = 0 \quad \& \quad b_1 + b_2 + b_3 = 0$

Then, $\rightarrow \ \left(a_1 + b_1 \right) + \left(a_2 + b_2 \right) + \left(a_3 + b_3 \right) = 0$

S is closed under addition

b) $k\vec{u} = k\left(a_1, a_2, a_3 \right)$
$$= \left(ka_1, ka_2, ka_3 \right)$$
$$ka_1 + ka_2 + ka_3 = k\left(a_1 + a_2 + a_3 \right) = k(0) = 0$$

S is closed under scalar multiplication.

c) *S* is a subspace of *V*.

Exercise

Let $S = \left\{ \left(x_1, \ x_2, \ 1 \right) : \ x_1 \ and \ x_2 \ are \ real \ numbers \right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\boldsymbol{R}^3$?

Solution

a) Let $\vec{u} = \left(x_1, x_2, 1 \right) \quad \& \quad \vec{v} = \left(y_1, y_2, 1 \right)$

$$\vec{u} + \vec{v} = \left(x_1, x_2, 1 \right) + \left(y_1, y_2, 1 \right)$$

$$= \left(x_1 + y_1,\, x_2 + y_2,\, 2\right) \qquad \text{If we let } z_1 = x_1 + y_1 \quad z_2 = x_2 + y_2$$

$$= \left(z_1,\, z_2,\, 2\right)$$

$$\neq \left(z_1,\, z_2,\, 1\right)$$

S is ***not*** closed under addition

b) $k\vec{u} = k\left(x_1,\, x_2,\, 1\right)$

$$= \left(kx_1,\, kx_2,\, k\right) \qquad \text{If we let } z_1 = kx_1 \quad z_2 = kx_2$$

$$\neq \left(z_1,\, z_2,\, 1\right) \qquad k \neq 1 \quad (\forall k)$$

S is ***not*** closed under scalar multiplication.

c) S is ***not*** a subspace of V.

Exercise

Let $S = \left\{\left(x_1,\, x_2,\, x_3\right) \in \mathbb{R}^3 :\ x_2 = x_1 + 2x_3\right\}$, Determine:

a) Is S closed under addition?

b) Is S closed under scalar multiplication?

c) Is S a subspace of $\boldsymbol{R}^3$?

Solution

a) Let $\vec{u} = \left(x_1,\, x_2,\, x_3\right)$ & $\vec{v} = \left(y_1,\, y_2,\, y_3\right)$

$$\vec{u} + \vec{v} = \left(x_1,\, x_2,\, x_3\right) + \left(y_1,\, y_2,\, y_3\right)$$

$$= \left(x_1 + y_1,\, x_2 + y_2,\, x_3 + y_3\right) \qquad S = \left\{\left(x_1,\, x_2,\, x_3\right) \in \mathbb{R}^3 :\ x_2 = x_1 + 2x_3\right\}$$

$$x_2 + y_2 = x_1 + 2x_3 + y_1 + 2y_3$$

$$= x_1 + y_1 + 2\left(x_3 + y_3\right)$$

S is closed under addition

b) $k\vec{u} = k\left(x_1,\, x_2,\, x_3\right)$

$$= \left(kx_1,\, kx_2,\, kx_3\right) \qquad \text{If we let } z_1 = kx_1 \quad z_2 = kx_2$$

$$kx_2 = kx_1 + 2kx_3$$

$$kx_2 = k\left(x_1 + 2x_3\right)$$

$$x_2 = x_1 + 2x_3$$

S is closed under scalar multiplication.

c) S is a subspace of V.

Exercise

Let $S=\left\{\begin{pmatrix} a & 1 \\ c & d \end{pmatrix} \in M_{2\times 2} \mid a,\ c,\ d \in \mathbb{R}\right\}$ and $V=M_{2,2}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V ?

Solution

a) Let $A=\begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix}$ & $B=\begin{pmatrix} a_2 & 1 \\ c_2 & d_2 \end{pmatrix}$

$$A+B=\begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix}+\begin{pmatrix} a_2 & 1 \\ c_2 & d_2 \end{pmatrix}$$

$$=\begin{pmatrix} a_1+a_2 & 2 \\ c_1+c_2 & d_1+d_2 \end{pmatrix}$$ If we let $a=a_1+a_2 \quad c=c_1+c_2 \quad d=d_1+d_2$

$$=\begin{pmatrix} a & 2 \\ c & d \end{pmatrix} \neq \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$$

S is ***not*** closed under addition

b) $$kA=k\begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix}$$

$$=\begin{pmatrix} ka_1 & k \\ kc_1 & kd_1 \end{pmatrix}$$ If we let $a=ka_1 \quad c=kc_1 \quad d=kd_1$

$$=\begin{pmatrix} a & k \\ c & d \end{pmatrix} \neq \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$$ $k \neq 1 \quad (\forall k)$

S is ***not*** closed under scalar multiplication.

c) S is ***not*** a subspace of V.

Exercise

Let $S = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in M_{2\times 2} \mid a,\ c,\ d \in \mathbb{R} \right\}$ and $V = M_{2,2}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V ?

Solution

a) Let $A = \begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix}$ & $B = \begin{pmatrix} a_2 & 1 \\ c_2 & d_2 \end{pmatrix}$

$$A + B = \begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & 1 \\ c_2 & d_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 + a_2 & 2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$ If we let $a = a_1 + a_2 \quad c = c_1 + c_2 \quad d = d_1 + d_2$

$$= \begin{pmatrix} a & 2 \\ c & d \end{pmatrix} \neq \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$$

S is ***not*** closed under addition

b) $kA = k\begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix}$

$$= \begin{pmatrix} ka_1 & k \\ kc_1 & kd_1 \end{pmatrix}$$ If we let $a = ka_1 \quad c = kc_1 \quad d = kd_1$

$$= \begin{pmatrix} a & k \\ c & d \end{pmatrix} \neq \begin{pmatrix} a & 1 \\ c & d \end{pmatrix} \qquad k \neq 1 \quad (\forall k)$$

S is ***not*** closed under scalar multiplication.

c) S is ***not*** a subspace of V.

Exercise

Let $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in M_{2\times 2} \mid a,\ d \in \mathbb{R} \ \&\ ad \geq 0 \right\}$ and $V = M_{2,2}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V ?

Solution

a) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \ \rightarrow 1(2) > 0$ & $B = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow (-2)(-1) > 0$

$$A + B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$ad \ge 0 \quad \rightarrow \quad (-1)(1) = -1 < 0$

S is ***not*** closed under addition

b) $kA = k\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$

$$= \begin{pmatrix} ka & 0 \\ 0 & kd \end{pmatrix}$$

$(ka)(kd) = k^2(ad)$

Since, $ad \ge 0$ & $k^2 \ge 0$

$k^2(ad) \ge 0$

S is closed under scalar multiplication.

c) *S* is ***not*** a subspace of *V*.

Exercise

$V = M_{33}$, $S = \{A \mid A \text{ is invertible}\}$ where *V* is a vector space and *S* is subset of *V*

a) Is *S* closed under addition?

b) Is *S* closed under scalar multiplication?

c) Is *S* a subspace of *V*?

Solution

a) Let assume: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ are invertible

But $A + B = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$ is not invertible.

S is *not* closed under addition

b) *S* is *not* closed under scalar multiplication if $k = 0$

c) *S* is ***not*** a subspace of *V*.

Exercise

Let $S = \left\{ p(t) = a + 2at + 3at^3 \ \mid a \in \mathbb{R} \ \ \& \ \ p(t) \in P_2 \right\}$ and $V = P_2$, Determine:

a) Is S closed under addition?

b) Is S closed under scalar multiplication?

c) Is S a subspace of V ?

Solution

a) Let $p_1(t) = a + 2at + 3at^3 \quad \& \quad p_2(t) = b + 2bt + 3bt^3$

$$\begin{aligned} p_1(t) + p_2(t) &= a + 2at + 3at^3 + b + 2bt + 3bt^3 \\ &= (a+b) + 2(a+b)t + 3(a+b)t^3 \qquad \textit{Let } c = a + b \in \mathbb{R} \\ &= c + 2ct + 3ct^3 \end{aligned}$$

S is closed under addition

b) $kp_1(t) = k\left(a + 2at + 3at^3\right)$

$$\begin{aligned} &= ka + 2kat + 3kat^3 \qquad \textit{Let } c = ka \in \mathbb{R} \\ &= c + 2ct + 3ct^3 \end{aligned}$$

S is closed under scalar multiplication.

c) S is a subspace of V.

Exercise

Given: $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix}$

a) Find $NS(A)$

b) For which n is $NS(A)$ a subspace of $\mathbb{R}^n$

c) Sketch $NS(A)$ in $\mathbb{R}^2$ or $\mathbb{R}^3$

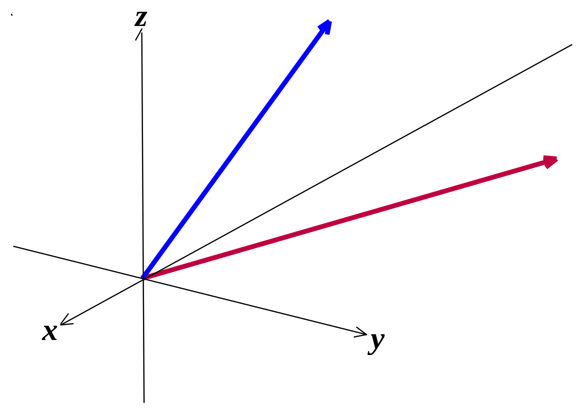
Solution

a) $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix} \xrightarrow[R_2 - 2R_1]{rref} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad x = -3y - 2z$

$$\left\{ y \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \middle| \, y, z \in \mathbb{R} \right\}$$

b) $n = 3$

c)



Exercise

Determine which of the following are subspaces of M_{22}

a) All 2 × 2 matrices with integer entries

b) All matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a+b+c+d=0$

Solution

a) Let $A=\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B=\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ where $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are integers

$A+B=\begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_{3+b_3} & a_4+b_4 \end{bmatrix}$ where $a_1+b_1, a_2+b_2, a_3+b_3, a_4+b_4$ are integers too.

Then, it is closed under addition.

$$\frac{1}{2}A=\frac{1}{2}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}=\begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix}$$

It is not closed under multiplication if the scalar is a real number.

Therefore; it is ***not*** a subspace of M_{22}

b) Let $A=\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ $a_1+a_2+a_3+a_4=0$ and $B=\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ $b_1+b_2+b_3+b_4=0$

$A+B=\begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_{3+b_3} & a_4+b_4 \end{bmatrix}$

$$a_1+a_2+a_3+a_4+b_1+b_2+b_3+b_4=0$$

$$\left(a_1+b_1\right)+\left(a_2+b_2\right)+\left(a_3+b_3\right)+\left(a_4+b_4\right)=0$$

Then, it is closed under addition.

$$kA = \begin{bmatrix} ka_1 & ka_2 \\ ka_3 & ka_4 \end{bmatrix} \quad ka_1 + ka_2 + ka_3 + ka_4 = k\left(a_1 + a_2 + a_3 + a_4\right) = k(0) = 0$$

It is closed under multiplication

Therefore; it is a subspace of M_{22}

Exercise

Let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \ ad - bc = 1 \right\}$. Is V a vector space?

Solution

$$k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

$$\begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} = k^2 ad - k^2 bc$$

$$= k^2 (ad - bc)$$

$$= k^2 \neq k$$

$\therefore$ V is *not* a vector space

Exercise

Let $V = \left\{ (x, 0, y) : \ x \,\&\, y \ \textit{are arbitrary} \ \mathbb{R} \right\}$. Define addition and scalar multiplication as follows:

$$\begin{cases} \left(x_1,\ 0,\ y_1\right) + \left(x_2,\ 0,\ y_2\right) = \left(x_1 + x_2,\ \ y_1 + y_2\right) \\ \quad c(x,\ 0,\ y) = (cx,\ \ cy) \end{cases}$$

Is V a vector space?

Solution

Let $\vec{V}_1\left(x_1,\ 0,\ y_1\right)$ & $\vec{V}_2\left(x_2,\ 0,\ y_2\right)$

$$\vec{V}_1 + \vec{V}_2 = \left(x_1,\ 0,\ y_1\right) + \left(x_2,\ 0,\ y_2\right)$$

$$= \left(x_1 + x_2,\ \ y_1 + y_2\right)$$

$$\neq \left(x_1 + x_2,\ \ 0,\ \ y_1 + y_2\right) = \vec{V}_1 + \vec{V}_2$$

$\therefore$ V is *not* a vector space

Exercise

Construct a matrix whose column space contains $(1, 1, 0)$, $(0, 1, 1)$, and whose nullspace contains $(1, 0, 1)$ and $(0, 0, 1)$

Solution

It is *not* possible.
Since a matrix (A) must be 3×3.
Since the nullspace contains 2 independent vectors, then A can have at most $3-2=1$ pivot.
But the column space contains 2 independent vectors, A must have at least 2 pivots.
These 2 conditions can't both be met.

Exercise

How is the nullspace $N(C)$ related to the spaces $N(A)$ and $N(B)$, is $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

Solution

$$N(C) = N(A) \cap N(B)$$

$$Cx = \begin{bmatrix} Ax \\ Bx \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Iff $Ax = 0 \quad \& \quad Bx = 0$

Exercise

True or False (check addition or give a counterexample)

a) If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V.
b) The empty set is a subspace of every vector space.
c) If V is a vector space other than the zero vector space, then V contains a subspace W such that $W \neq V$.
d) The intersection of any two subsets of V is a subspace of V.
e) Let W be the *xy*-plane in $\boldsymbol{R}^3$; that is, $W = \{(a_1, a_2, 0): a_1, a_2 \in \boldsymbol{R}\}$. Then $W = \boldsymbol{R}^2$

Solution

a) False
W is a subset of V, but not necessary that the scalar of a vector in W is in V.
Therefore, W is *not* a subspace of V

b) False

c) True

d) False

e) False