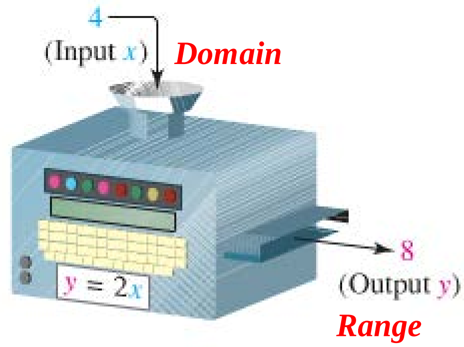
Lecture Two

Section 2.1 – Functions and Graphs

Relations

A ***relation*** is any set of ordered pairs. The set of all first components of ordered pairs is called the domain of the relation and the set of second components is called the range of the relation.

Definition of a Function



Function machine

A ***function*** is a relation between two variables such that to matches each element of a first set (called ***domain***) to an element of a second set (called ***range***) in such way that no element in the first set is assigned to two different elements in the second set.

The ***domain*** of the function is the set of all values of the independent variable for which the function is defined.

The ***range*** of the function is the set of all values taken on by the dependent variable.

Example

Determine whether each relation is a function and *find the domain and the range*.

a) $F = \{(1,2),(-2,4),(3,-1)\}$

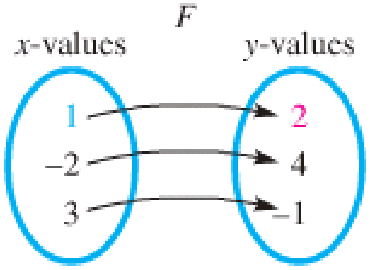
Function: Yes
Domain: {-2, 1, 3}
Range: {-1, 2, 4}

b) $G = \{(1,1),(1,2),(1,3),(2,3)\}$

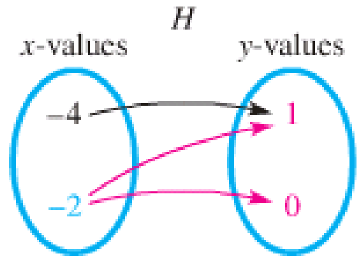
Function: No
Domain: {1, 2}
Range: {1, 2, 3}

c) $H = \{(-4,1),(-2,1),(-2,0)\}$

Function: No
Domain: {-4, -2}
Range: {0, 1}



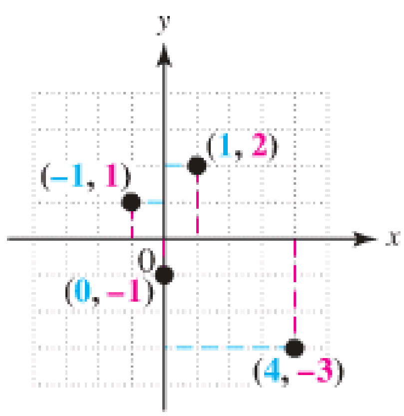
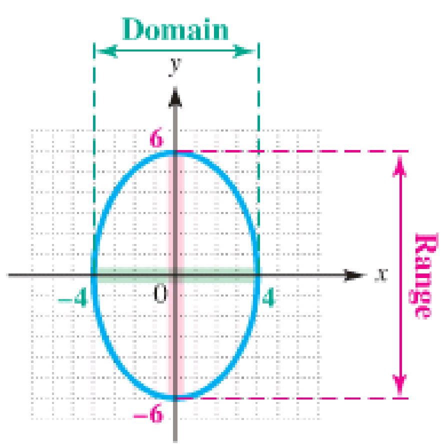
F is a function.



H is not a function.

Example

Give the domain and range of each relation

	***Domain***: {-1, 0, 1, 4} ***Range***: {-3, -1, 1, 2}
	***Domain***: $[-4, 4]$ ***Range***: $[-6, 6]$

Functions as Equations $y = -0.016x^2 + 0.93x + 8.5$

x: independent

y: depend on x

Notation for Functions

$f(x)$ read " f of x " or " f at x " represents the value of the function at the number x.

Example

Let $f(x) = -x^2 + 5x - 3$

a) $f(2)$

$$f(x) = -x^2 + 5x - 3$$

$$f(\text{—}) = -(\text{—})^2 + 5(\text{—}) - 3$$

$$f(2) = -(2)^2 + 5(2) - 3$$ $(-)2 \wedge 2 + 5 * 2 - 3$

$$\underline{= 3}$$

b) $f(q)$

$$f(q) = -(q)^2 + 5(q) - 3$$

$$\underline{= -q^2 + 5q - 3}$$

Example

If $f(x) = x^2 - 2x + 7$, evaluate each of the following:

a) $f(-5)$

b) $f(x+4)$

Solution

a) $f(-5) = ?$

$$f(\text{—}) = (\text{—})^2 - 2(\text{—}) + 7$$

$$f(-5) = (-5)^2 - 2(-5) + 7$$

$= 25 + 10 + 7$

$= 42$

b) $f(x+4)=?$

$$f(\text{—})=(\text{—})^2-2(\text{—})+7$$

$$f(x+4)=(x+4)^2-2(x+4)+7 \qquad (a+b)^2=a^2+2ab+b^2$$

$$=x^2+2(4)x+4^2-2x-8+7$$

$$=x^2+8x+16-2x-1$$

$$\underline{=x^2+6x+15}$$

Example

Let $g(x)=2x+3$, find $g(a+1)$

Solution

$$g(x)=2x+3$$

$$g(a+1)=2(a+1)+3$$

$$=2a+2+3$$

$$\underline{=2a+5}$$

Example

Given: $f(x)=2x^2-x+3$, find the following.

a) $f(0)$

b) $f(-7)$

c) $f(5a)$

Solution

a) $f(x=0)=2(0)^2-(0)+3$

b) $f(-7)=2(-7)^2-(-7)+3$

$$=108$$

c) $f(5a)=2(5a)^2-(5a)+3$

$$=50a^2-5a+3$$

Increasing and Decreasing Functions

- A function *rises from left to right (x-coordinate)*, the function *f* is said to be ***increasing*** on an open interval I (*a*, *b*) (*x*-coordinate)

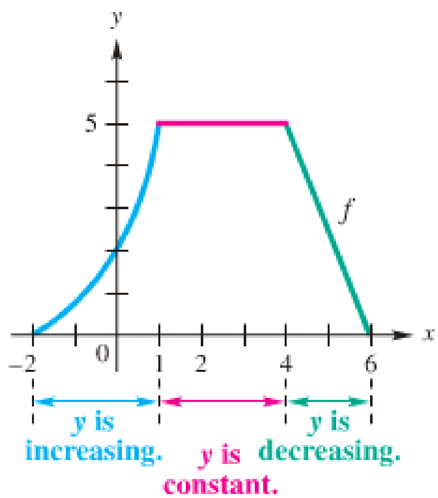
$$a < b \quad \Rightarrow \quad f(a) < f(b)$$

- A function *f* is said to be ***decreasing*** on an open interval I

$$a < b \quad \Rightarrow \quad f(a) > f(b)$$

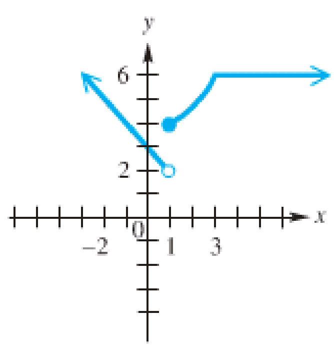
- A function *f* is said to be ***constant*** on an open interval I

$$a < b \quad \Rightarrow \quad f(a) = f(b)$$



Example

Determine the intervals over which the function is increasing, decreasing, or constant



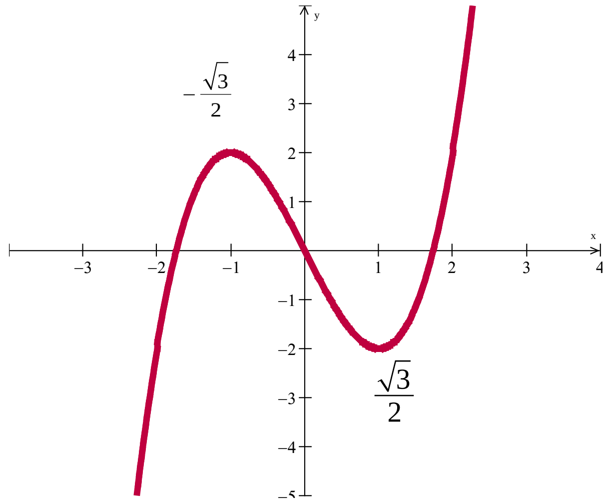
Increasing: [1, 3]

Decreasing: $(-\infty, 1)$

Constant: $[3, \infty)$

Example

State the intervals on which the given function $f(x) = x^3 - 3x$ is increasing, decreasing, or constant

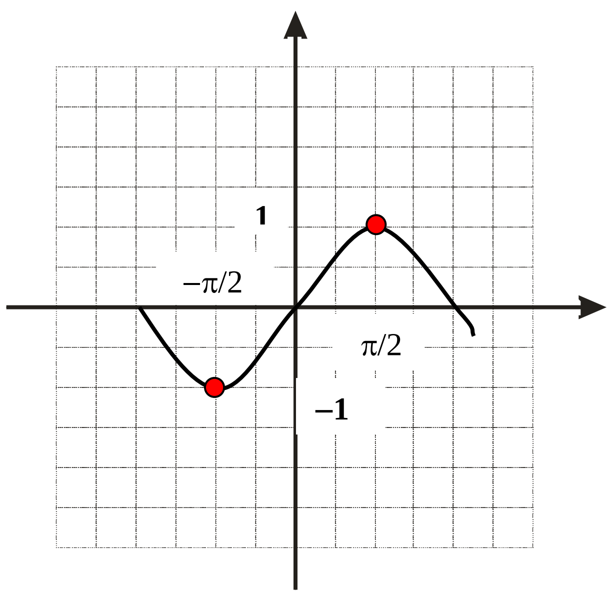


Increasing $\left(-\infty, -\frac{\sqrt{3}}{2}\right), \left(\frac{\sqrt{3}}{2}, \infty\right)$ *Decreasing* $\left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$

Relative *Maxima (um)* and *Minima (um)*

f(a) is a relative maximum if there exists an open interval I about *a* such that *f(a) > f(x),* for all *x* in I.

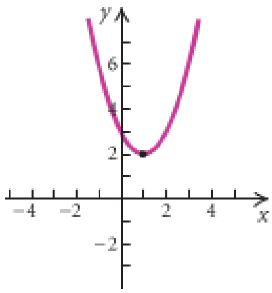
f(a) is a relative minimum if there exists an open interval I about *a* such that *f(a) < f(x),* for all *x* in I.



The relative minimum value of the function is –1 @ $x = -\pi/2$

The relative maximum value of the function is 1 @ $x = \pi/2$

Determine any relative maximum or minimum of the function, determine the intervals on which the function increasing or decreasing, and then find the domain and the range.



Relative Maximum: ____________________

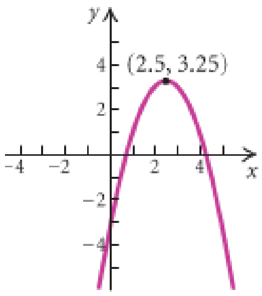
Relative Minimum: ____________________

Increasing: ____________________

Decreasing: ____________________

Domain: ____________________

Range: ____________________



Relative Maximum: ____________________

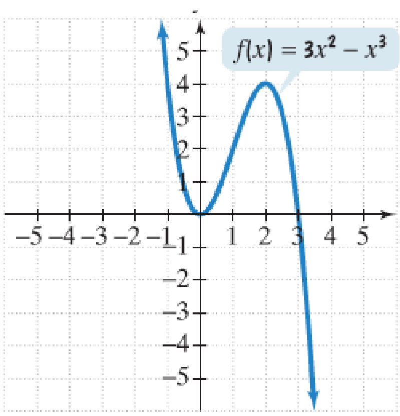
Relative Minimum: ____________________

Increasing: ____________________

Decreasing: ____________________

Domain: ____________________

Range: ____________________



Relative Maximum: ____________________

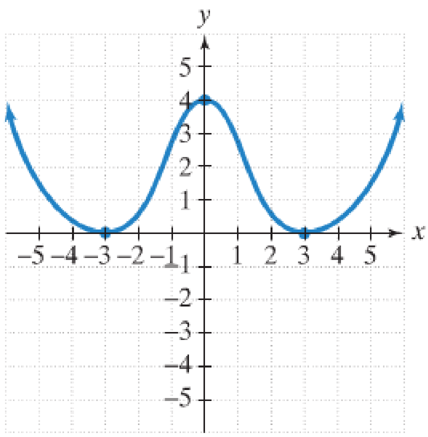
Relative Minimum: ____________________

Increasing: ____________________

Decreasing: ____________________

Domain: ____________________

Range: ____________________



Relative Maximum: ____________________

Relative Minimum: ____________________

Increasing: ____________________

Decreasing: ____________________

Domain: ____________________

Range: ____________________

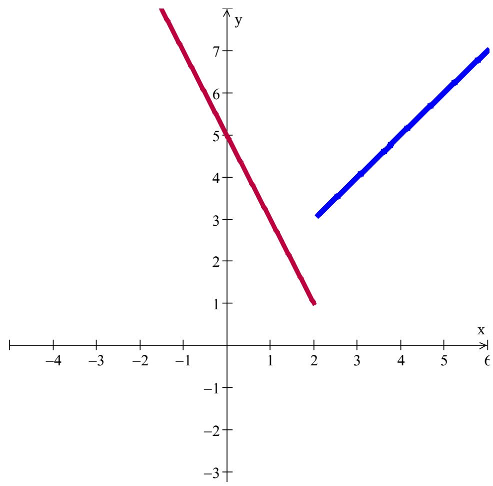
Piecewise-Defined Functions

Function are sometimes described by more than one expression, we call such functions ***piecewise-defined functions***.

Example

Graph each function

$$f(x)=\begin{cases}-2x+5 & \text{if } x\le 2\\ x+1 & \text{if } x>2\end{cases}$$

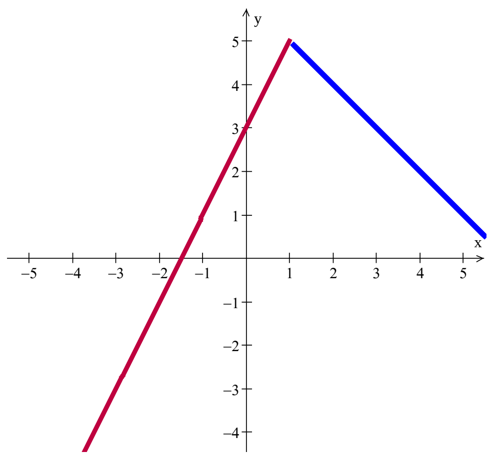


Find: $f(2)=-2(2)+5=1$

$f(0)=-2(0)+5=5$

$f(4)=4+1=5$

$$f(x)=\begin{cases}2x+3 & \text{if } x\le 1\\ -x+6 & \text{if } x>1\end{cases}$$



Example

$$C(t)=\begin{cases}20 & \text{if } 0\le t\le 60\\ 20+0.40(t-60) & \text{if } t>60\end{cases}$$

Find C(40), C(80), and C(60)

Solution

a) $C(40) = 20$

b) $C(80) = 20 + 0.40(80 - 60) = 28$

c) $C(60) = 20$

Exercise *Section* 2.1 – Functions and Graphs

1. Determine whether each relation is a function and *find the domain and the range*.

a) {(1, 2), (3, 4), (5, 6), (5, 8)}

b) {(1, 2), (3, 4), (6, 5), (8, 5)}

c) {(9, -5), (9, 5), (2, 40)}

d) {(-2, 5), (5, 7), (0, 1), (4, -2)}

e) {(-5,3), (0, 3), (6, 3)}

2. Identify the domain and the range: {(5, 12.8), (10, 16.2), (15, 18.9), (20, 20.7), (25, 21.81)}

3. Let $f(x) = -3x + 4$, find $f(0)$

4. Let $g(x) = -x^2 + 4x - 1$, find $g(-x)$

5. Let $f(x) = -3x + 4$, find $f(a+4)$

6. Given: $f(x) = 2|x| + 3x$, find $f(2-h)$.

7. Given: $g(x) = \frac{x-4}{x+3}$, find $g(x+h)$

8. Given: $g(x) = \frac{x}{\sqrt{1-x^2}}$, find $g(0)$ *and* $g(-1)$

9. Given that $g(x) = 2x^2 + 2x + 3$. Find $g(p+3)$

10. If $f(x) = x^2 - 2x + 7$, evaluate each of the following: $f(-5),\ f(x+4),\ f(-x)$

11. Find $g(0),\ g(-4),\ g(7),\ and\ g\left(\frac{3}{2}\right)$ *for* $g(x) = \frac{x}{\sqrt{16-x^2}}$

12. $f(x) = \begin{cases} 2+x & if\ \ x < -4 \\ -x & if\ \ -4 \le x \le 2 \\ 3x & if\ \ x > 2 \end{cases}$ Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

13. $f(x) = \begin{cases} -2x & if\ \ x < -3 \\ 3x-1 & if\ \ -3 \le x \le 2 \\ -4x & if\ \ x > 2 \end{cases}$ Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

14. $f(x) = \begin{cases} x^3+3 & if\ \ -2 \le x \le 0 \\ x+3 & if\ \ 0 < x < 1 \\ 4+x-x^2 & if\ \ 1 \le x \le 3 \end{cases}$ Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

15. $h(x)=\begin{cases}\dfrac{x^2-9}{x-3} & if \quad x\neq 3\\ 6 & if \quad x=3\end{cases}$ Find: $h(5),\ h(0),\ and\ h(3)$

16. Graph the piecewise function defined by $f(x)=\begin{cases}3 & if \quad x\le -1\\ x-2 & if \quad x>-1\end{cases}$

17. Sketch the graph $f(x)=\begin{cases}x+2 & if \quad x\le -1\\ x^3 & if \quad -1<x<1\\ -x+3 & if \quad x\ge 1\end{cases}$

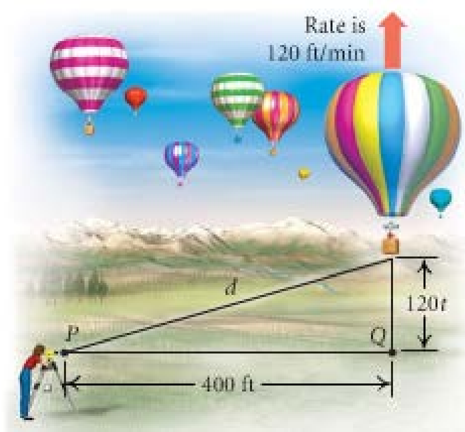
18. Sketch the graph $f(x)=\begin{cases}x-3 & if \quad x\le -2\\ -x^2 & if \quad -2<x<1\\ -x+4 & if \quad x\ge 1\end{cases}$

19. The elevation H, in meters, above sea level at which the boiling point of water is in ***t*** degrees Celsius is given by the function

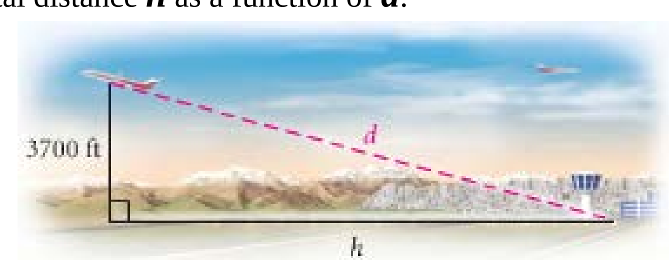
$$H(t)=1000(100-t)+580(100-t)^2$$

At what elevation is the boiling point 99.5°.

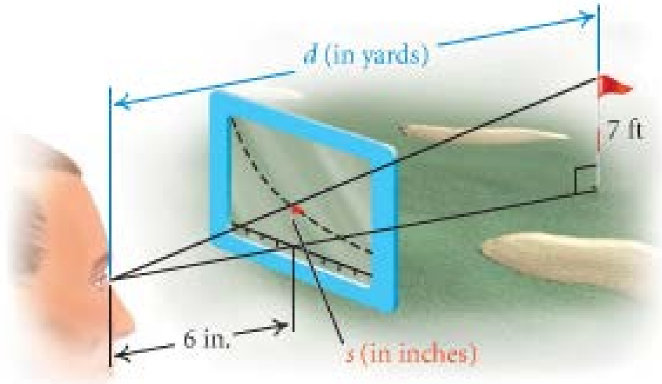
20. A hot-air balloon rises straight up from the ground at a rate of 120 ft./min. The balloon is tracked from a rangefinder on the ground at point P, which is 400 ft. from the release point Q of the balloon. Let ***d*** = the distance from the balloon to the rangefinder and ***t*** – the time, in minutes, since the balloon was released. Express ***d*** as a function of ***t***.



21. An airplane is flying at an altitude of 3700 *ft*. The slanted distance directly to the airport is ***d*** feet. Express the horizontal distance ***h*** as a function of ***d***.



22. A device used in golf to estimate the distance $\boldsymbol{d}$, in yards, to a hole measures the size $\boldsymbol{s}$, in inches, that the 7-ft pin appears to be in a viewfinder. Express the distance $\boldsymbol{d}$ as a function of $\boldsymbol{s}$.

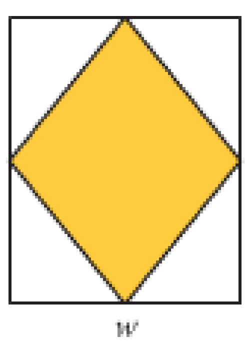


23. A rancher has 360 yd. of fencing with which to enclose two adjacent rectangular corrals, one for sheep and one for cattle. A river forms one side of the corrals. Suppose the width of each corral is $\boldsymbol{x}$ yards.

a) Express the total area of the two corrals as a function of $\boldsymbol{x}$.

b) Find the domain of the function.

24. A rhombus is inscribed in a rectangle that is $\boldsymbol{w}$ meters wide with a perimeter of 40 m. Each vertex of the rhombus is a midpoint of a side of the rectangle. Express the area of the rhombus as a function of the rectangle's width.



Section 2.2 – Transformation of Functions

Vertical Translation

For $d > 0$,

$y = f(x) + d \Rightarrow$ The graph shifted up $\boldsymbol{d}$ units

Example: Describe how the graph $g(x) = x^2 + 3$ can be obtained from $y = x^2$

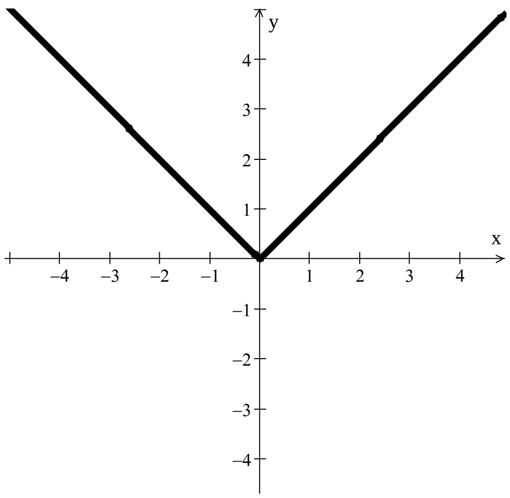
g shifted up 3 units

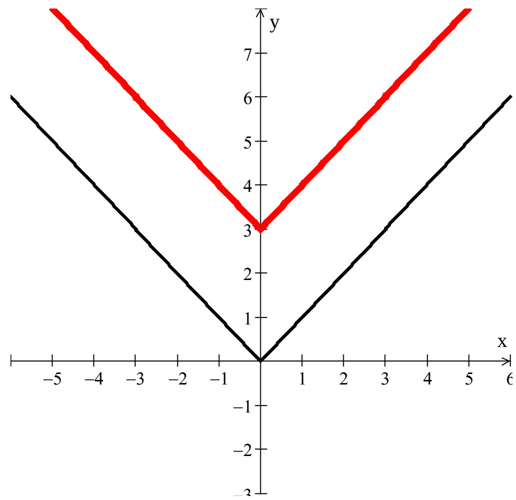
$y = f(x) - d \Rightarrow$ The graph shifted down $\boldsymbol{d}$ units

Example: Describe how the graph $g(x) = x^2 - 3$ can be obtained from $y = x^2$

g shifted down 3 units

Use the graph of $f(x) = |x|$ to obtain the graph of $g(x) = |x| + 3$





Horizontal Translation

For $b > 0$,

$y = f(x-b)$ $\Rightarrow$ The graph shifted right ***b*** units

Example: Describe how the graph $g(x) = |x-3|$ can be obtained from $y = |x|$

g shifted right 3 units

$y = f(x+b)$ $\Rightarrow$ The graph shifted left ***b*** units

Example: Describe how the graph $g(x) = \sqrt{x+2}$ can be obtained from $y = \sqrt{x}$

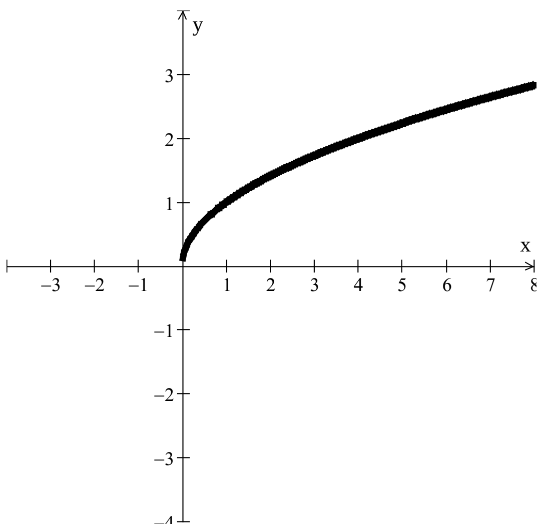
g shifted left 2 units

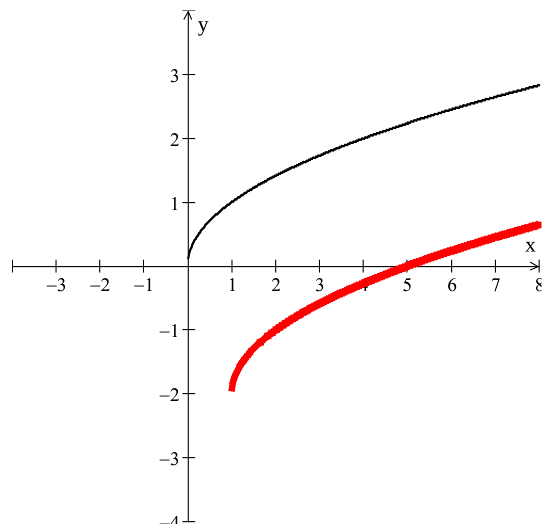
Example: Describe how the graph $h(x) = \sqrt{x+2} - 3$ can be obtained from $y = \sqrt{x}$

h shifted left 2 units and down 3 units

Describe how the graph $f(x) = \sqrt{x-1} - 2$ can be obtained from $h(x) = \sqrt{x}$

f shifted right 1 unit and down 2 units



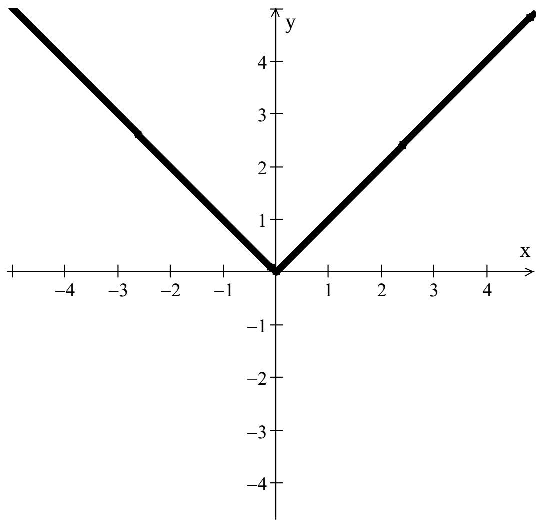


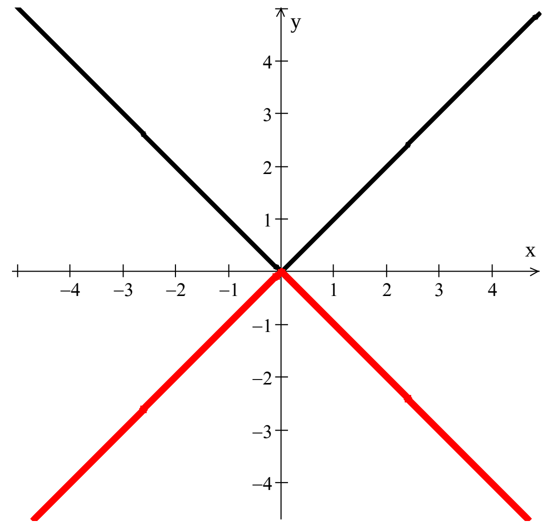
Reflections

The graph of $y = -f(x)$ is the reflection of the graph across the x-axis (upside down)

The graph of $y = f(-x)$ is the reflection of the graph across the y-axis

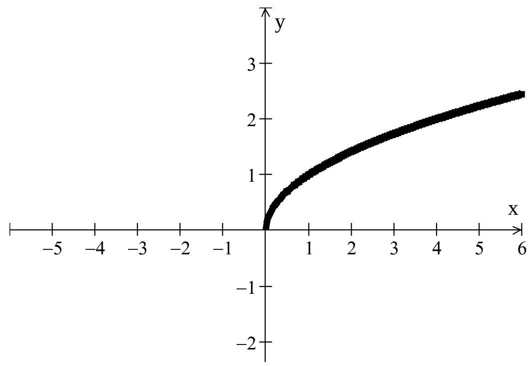
Describe how the graph $g(x) = -|x|$ can be obtained from $h(x) = |x|$

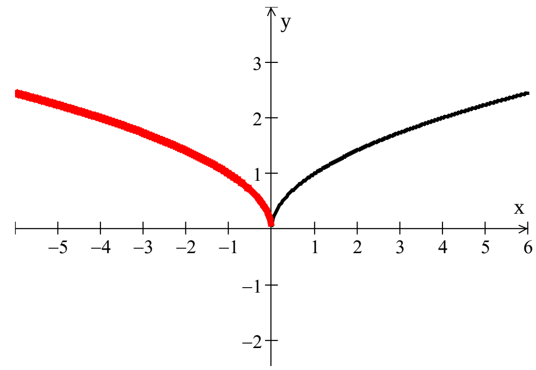




Describe how the graph $g(x) = \sqrt{-x}$ can be obtained from $y = \sqrt{x}$

Is a reflection across y-axis





Describe how the graph $h(x) = \sqrt[3]{-x}$ can be obtained from $f(x) = \sqrt[3]{x}$

Is a reflection across y-axis

Vertical Stretching and Shrinking

The graph of $y = af(x)$ can be obtained from the graph of $y = f(x)$ by

Stretching vertically for $|a| > 1$, or

Shrinking vertically for $0 < |a| < 1$

Horizontal Stretching and Shrinking

The graph of $y = f(cx)$ can be obtained from the graph of $y = f(x)$ by

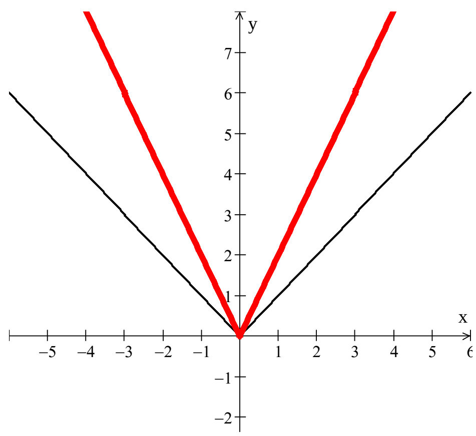
Stretching horizontally for, $0 < |c| < 1$ or

Shrinking horizontally for $|c| > 1$

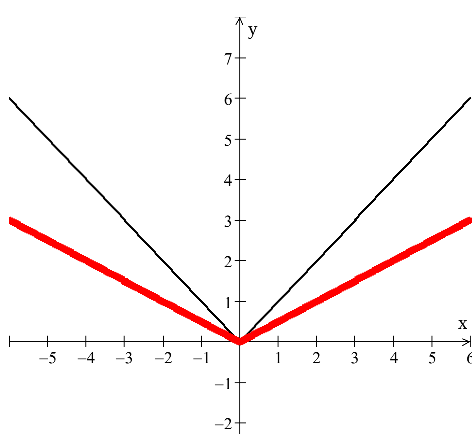
Example

Graph each function

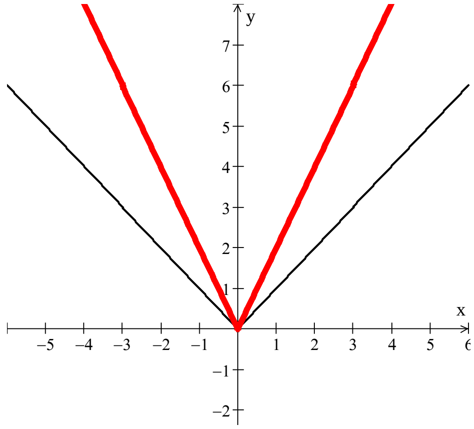
a) $g(x) = 2|x|$



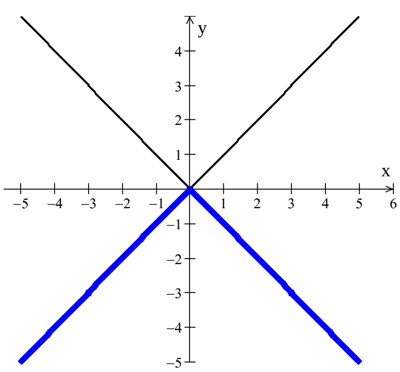
b) $g(x) = \frac{1}{2}|x|$



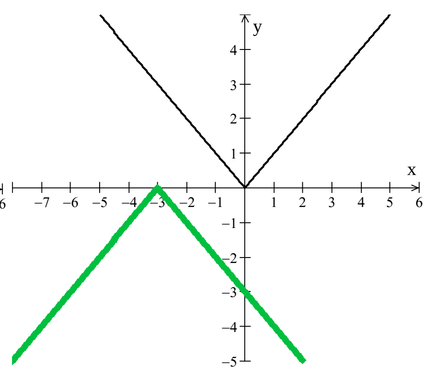
c) $k(x)=|2x|$



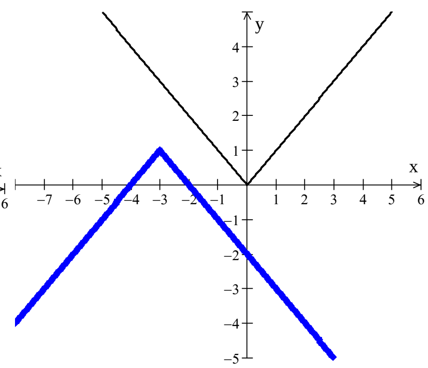
$f(x)=-|x+3|+1$



$f(x)=-|x|$

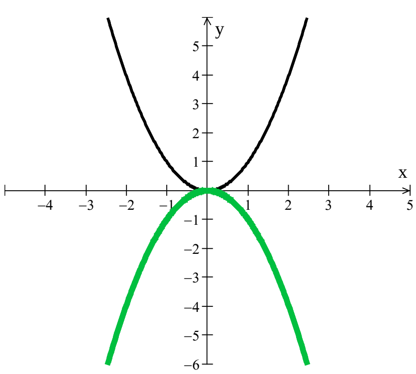


$f(x)=-|x+3|$

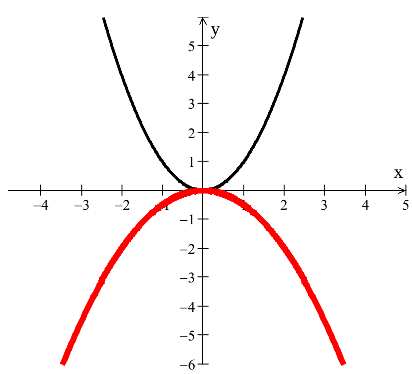


$f(x)=-|x+3|+1$

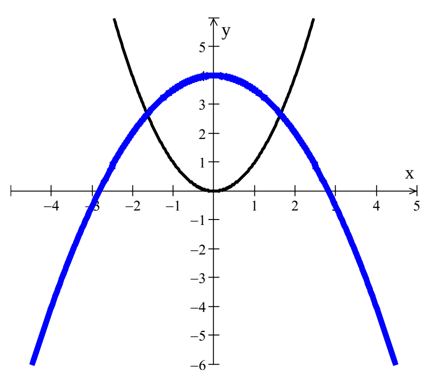
$g(x)=-\frac{1}{2}x^2+4$



$g(x)=-x^2$



$g(x)=-\frac{1}{2}x^2$



$g(x)=-\frac{1}{2}x^2+4$

$$\begin{cases} |a| > 1 & \Rightarrow \quad \textit{Stretching Vertically} \\ 0 < |a| < 1 & \Rightarrow \quad \textit{Shrinking Vertically} \end{cases}$$

Reflected across x-axis

$$\begin{cases} |c| > 1 & \Rightarrow \quad \textit{Shrinking Horizontally} \\ 0 < |c| < 1 & \Rightarrow \quad \textit{Stretching Horizontally} \end{cases}$$

$$\boldsymbol{y = -af(-c\,(x \pm b)) \pm d}$$

Reflected across y-axis

$$\begin{cases} +b & \textit{Shifted Left} \\ -b & \textit{Shifted Right} \end{cases} \qquad \begin{cases} +d & \textit{Shifted up} \\ -d & \textit{Shifted Down} \end{cases}$$

Algebraic Tests of Symmetry

x*-axis**: If replacing ***y with ***-y*** (negative '*y*') produces an equivalent equation, then the graph is *symmetric with respect* to the *x*-axis

Example: $y = x^2 + 2$

$$-y = x^2 + 2$$

$$y = -x^2 - 2$$

It is NOT equivalent $\Rightarrow$ It is not symmetric with respect to the *x*-axis

y*-axis**: If replacing ***x with ***–x*** produces an equivalent equation, then the graph is *symmetric with respect* to the *y*-axis

Example: $y = x^2 + 2$

$$y = (-x)^2 + 2$$

$$y = x^2 + 2$$

$\Rightarrow$ It is symmetric with respect to the *y*-axis

Origin: If replacing ***x*** with ***–x*** *and* ***y*** with ***-y*** produces an equivalent equation, then the graph is *symmetric with respect* to the origin

Example: $y = x^2 + 2$

$$-y = (-x)^2 + 2$$

$-y = x^2 + 2$ $\Rightarrow$ It is not symmetric with respect to the origin

Exercises 2.2 – Transformation of Functions

1. Write an equation for a function that has the shape of $f(x)=x^2$, but upside-down and shifted right 2 units and down 3 units.

2. Describe how the graph of $f(x)=\sqrt{x-3}+2$ can be obtained from the graph of $y=\sqrt{x}$

3. Describe how the graph of $f(x)=-\frac{1}{2}(x-2)^2+1$ can be obtained from the graph of $f(x)=x^2$

4. Describe how the graph of $f(x)=-(x-8)^2$ can be obtained from the graph of $f(x)=x^2$

5. Describe how the graph of $y=\sqrt{x+6}-5$ can be obtained from the graph of $y=\sqrt{x}$

6. Describe how the graph of $y=\sqrt{-(x+2)}-1$ can be obtained from the graph of $y=\sqrt{x}$

7. Describe how the graph of $y=\left|\frac{1}{2}x\right|-5$ can be obtained from the graph of $y=|x|$

8. Explain how the graph $y=f(x-2)+3$ compares to the graph of $y=f(x)$

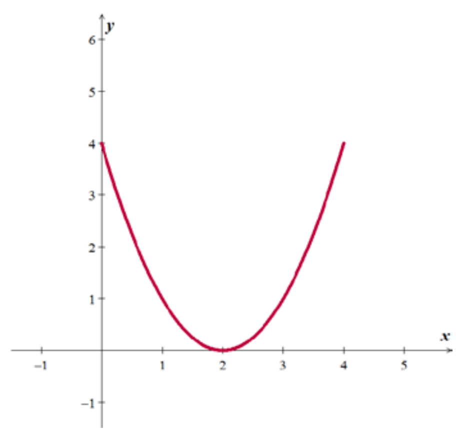
9. Explain how the graph $y=f(-x)-2$ compares to the graph of $y=f(x)$

10. Explain how the graph $y=-\frac{1}{2}f(x)$ compares to the graph of $y=f(x)$

11. Explain how the graph $y=f\left(\frac{1}{2}x\right)-3$ compares to the graph of $y=f(x)$

12. Explain how the graph $y=-2f\left[\frac{1}{2}(x-3)\right]+5$ compares to the graph of $y=f(x)$

13. The graph of a function f with domain [0, 4] is shown:



a) $y=f(x+3)$

b) $y=f(x-2)+3$

c) $y=f\left(-\frac{1}{2}x\right)$

d) $y=|f(x)|$

Section 2.3 – Function Operations and Composition

The *Domain* of a Function

1. ***Rational*** function: $\dfrac{f(x)}{h(x)}$ $\Rightarrow$ ***Domain***: $\boxed{h(x) \neq 0}$

Example: $f(x) = \frac{1}{x-3}$

$$\Rightarrow x - 3 \neq 0$$
$$\Rightarrow x \neq 3$$

Domain: $(-\infty, 3) \cup (3, \infty)$

2. ***Irrational*** function: $\sqrt{g(x)} \Rightarrow$ ***Domain***: $\boxed{g(x) \geq 0}$

Example: $g(x) = \sqrt{3-x} + 5$

$$\Rightarrow 3 - x \geq 0$$
$$-x \geq -3$$
$$\Rightarrow x \leq 3$$

Domain: $(-\infty,\ 3]$

3. ***Otherwise***: Domain all real numbers $(-\infty,\ \infty)$

Example: $f(x) = x^3 + |x|$

Domain: All real numbers $(-\infty,\ \infty)$

(*1*) & (*2*) → Find the domain: $f(x) = \frac{x+1}{\sqrt{x-3}}$

$$\Rightarrow x - 3 > 0$$
$$x > 3$$

Domain: $(3,\ \infty)$

Example

Find the domain

a) $f(x) = x^2 + 3x - 17$

$\Rightarrow$ ***Domain***: All real numbers

b) $g(x) = \frac{5x}{x^2 - 49}$

$\Rightarrow x^2 - 49 \neq 0$

$\rightarrow x^2 = 49$

$\rightarrow x \neq \pm 7$

Domain: $\begin{cases} \{x \mid x \neq \pm 7\} \quad \text{or} \\ (-\infty, -7) \cup (-7, 7) \cup (7, \infty) \end{cases}$

c) $h(x) = \sqrt{9x - 27}$

$\Rightarrow 9x - 27 \geq 0$

$\rightarrow 9x \geq 27$

$\Rightarrow x \geq 3$

Domain: $[3, \infty)$

The *Algebra* of Functions

$$(f+g)(x) = f(x) + g(x)$$

$$(f-g)(x) = f(x) - g(x)$$

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Example

Let $f(x) = x^2 + 1$ and $g(x) = 3x + 5$. Find each of the following

a) $(f+g)(1)$

$$\begin{aligned}(f+g)(1) &= f(1) + g(1)\\ &= 1^2 + 1 + 3(1) + 5\\ &= 1 + 1 + 3 + 5\\ &= 10\end{aligned}$$

b) $(f-g)(-3)$

$$\begin{aligned}(f-g)(-3) &= f(-3) - g(-3)\\ &= (-3)^2 + 1 - \left(3(-3) + 5\right)\\ &= 14\end{aligned}$$

c) $(fg)(5)$

$$\begin{aligned}(fg)(5) &= f(5) \cdot g(5)\\ &= \left(5^2 + 1\right) \cdot \left(3(5) + 5\right)\\ &= (26) \cdot (20)\\ &= 520\end{aligned}$$

d) $\left(\frac{f}{g}\right)(0)$

$$\begin{aligned}\left(\frac{f}{g}\right)(0) &= \frac{f(0)}{g(0)}\\ &= \frac{0^2 + 1}{3(0) + 5}\\ &= \frac{1}{5}\end{aligned}$$

Example

Let $f(x)=8x-9$ and $g(x)=\sqrt{2x-1}$. Find each of the following and give the domain

$(f+g)(x),\quad (f-g)(x),\quad (fg)(x),\ \left(\frac{f}{g}\right)(x)$

Solution

Domain of f: $(-\infty,\infty)$

Domain of g: $\left[\frac{1}{2},\infty\right)$ $\quad \sqrt{2x-1\geq 0}\ \rightarrow 2x\geq 1\ \Rightarrow x\geq \frac{1}{2}$

a) $(f+g)(x)$

$(f+g)(x)=8x-9+\sqrt{2x-1}$ ***Domain***: $\left[\frac{1}{2},\infty\right)$

b) $(f-g)(x)$

$(f-g)(x)=8x-9-\sqrt{2x-1}$ ***Domain***: $\left[\frac{1}{2},\infty\right)$

c) $(fg)(x)$

$(fg)(x)=(8x-9)\sqrt{2x-1}$ ***Domain***: $\left[\frac{1}{2},\infty\right)$

d) $\left(\frac{f}{g}\right)(x)$

$\left(\frac{f}{g}\right)(x)=\frac{8x-9}{\sqrt{2x-1}}$ $\quad 2x-1>0\ \rightarrow 2x>1\ \Rightarrow x>\frac{1}{2}$

Domain: $\left(\frac{1}{2},\infty\right)$

Example

Let $f(x)=\sqrt{x-3}$ and $g(x)=\sqrt{x+1}$

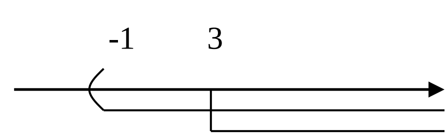
Domain $f(x): x-3\geq 0\Rightarrow x\geq 3$ *and* ***Domain*** $g(x): x+1\geq 0\Rightarrow x\geq -1$

a. $(f+g)(x)=\sqrt{x-3}+\sqrt{x+1}$

b. Domain: $x\geq 3$ *and* $x\geq -1\ \Rightarrow$ ***Domain***: $x\geq 3$

c. Domain: $\left(\frac{f}{g}\right)(x)$

$$\frac{f}{g}=\frac{\sqrt{x-3}}{\sqrt{x+1}}$$



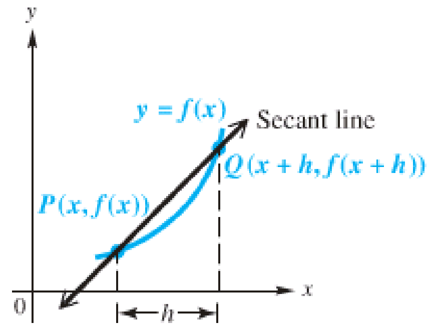
$$\rightarrow\begin{cases} x-3\geq 0\Rightarrow \boxed{x\geq 3} \\ x+1>0\Rightarrow \boxed{x>-1}\end{cases}$$

Domain: $x\geq 3\quad [3,\infty)$

Difference Quotients

$$\frac{f(x+h)-f(x)}{(x+h)-x}$$

The difference quotient is given by: $\dfrac{f(x+h)-f(x)}{h}$



Example

For the function *f* given by $f(x)=2x^2-3x$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$f(x+h)=2(\text{—})^2-3(\text{—})$$

$$=2(x+h)^2-3(x+h) \qquad\qquad (a+b)^2=a^2+2ab+b^2$$

$$=2\left(x^2+2xh+h^2\right)-3x-3h$$

$$=2x^2+4xh+2h^2-3x-3h$$

$$\frac{f(x+h)-f(x)}{h}=\frac{\overbrace{2x^2+4xh+2h^2-3x-3h}^{f(x+h)}-\overbrace{(2x^2-3x)}^{f(x)}}{h}$$

$$=\frac{2x^2+4xh+2h^2-3x-3h-2x^2+3x}{h}$$

$$=\frac{4xh+2h^2-3h}{h}$$

$$=\frac{4xh}{h}+\frac{2h^2}{h}-\frac{3h}{h}$$

$$=4x+2h-3$$

Example

For the function *f* given by $f(x)=-2x^2+x+5$, find the difference quotient $\frac{f(x+h)-f(x)}{h}$

Solution

$$f(x+h)=-2(--)^2+(--)+5$$

$$f(x+h)=-2(\boldsymbol{x+h})^2+(\boldsymbol{x+h})+5 \qquad (a+b)^2=a^2+2ab+b^2$$

$$f(x+h)=-2\left(x^2+2hx+h^2\right)+x+h+5$$

$$f(x+h)=-2x^2-4hx-2h^2+x+h+5$$

$$\frac{f(x+h)\ -f(x)}{h}=\frac{-2x^2-4hx-2h^2+x+h+5-(-2x^2+x+5)}{h}$$

$$=\frac{-2x^2-4hx-2h^2+x+h+5+2x^2-x-5}{h}$$

$$=\frac{-4hx-2h^2+h}{h}$$

$$=\frac{-4hx}{h}-\frac{2h^2}{h}+\frac{h}{h}$$

$$\underline{=-4x-2h+1}$$

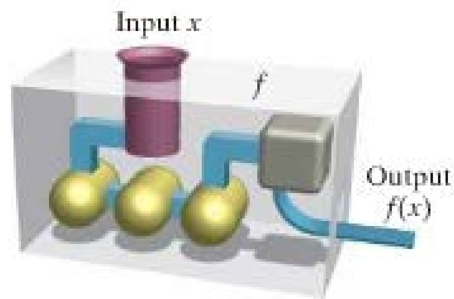
Composition of Functions

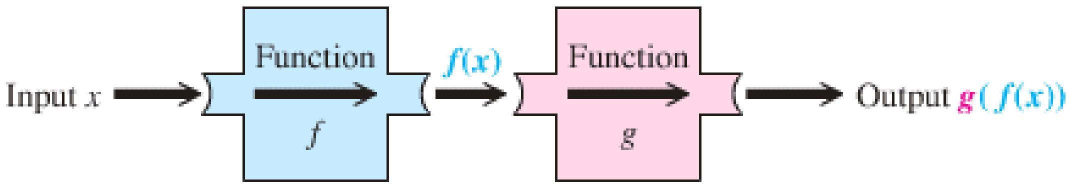
The composite function $f \circ g$, the composite of f and g, is defined as

$$(f \circ g)(x) = f(g(x))$$

Where x is in the domain of g

and $g(x)$ is in the domain of f





Example

Given that $f(x) = 5x + 6$ and $g(x) = 2x^2 - x - 1$, find $(f \circ g)(x)$ and $(g \circ f)(x)$

Solution

$(f \circ g)(x) = f(g(x))$ $\quad = f(2x^2 - x - 1)$ ***Domain:*** All real numbers

$= 5(\text{---------}) + 6$

$= 5(2x^2 - x - 1) + 6$

$= 10x^2 - 5x - 5 + 6$

$\underline{= 10x^2 - 5x + 1}$ ***Domain:*** All real numbers

$(g \circ f)(x) = g(f(x))$

$= g(5x + 6)$ ***Domain:*** All real numbers

$= 2(\qquad)^2 - (\qquad) - 1$

$= 2(5x + 6)^2 - (5x + 6) - 1$

$= 2(25x^2 + 60x + 36) - 5x - 6 - 1$

$= 50x^2 + 120x + 72 - 5x - 7$

$\underline{= 50x^2 + 115x + 65}$ ***Domain:*** All real numbers

Example

Let $f(x)=\sqrt{x}$ and $g(x)=4x+2$, find each of the following and its domain.

a) $(f\circ g)(x)$

$$(f\circ g)(x)=f(g(x))$$
$$=f(4x+2) \qquad (-\infty,\infty)$$
$$=\sqrt{4x+2}$$

$$4x+2\ge 0$$
$$4x\ge -2$$
$$x\ge -\frac{2}{4}$$
$$x\ge -\frac{1}{2}$$

Domain: $\left[-\frac{1}{2},\infty\right)$

b) $(g\circ f)(x)$

$$(g\circ f)(x)=g(f(x))$$
$$=g(\sqrt{x}) \qquad x\ge 0$$
$$=4\sqrt{x}+2 \qquad x\ge 0$$

Domain: $[0,\infty)$

Example

Let $f(x)=2x-1$ and $g(x)=\dfrac{4}{x-1}$ Find:

a) $(f\circ g)(2)$

b) $(g\circ f)(-3)$

Solution

a) $(f\circ g)(2)=f(g(2))$

$$=f\left(\frac{4}{2-1}\right)$$
$$=f(4)$$
$$=2(4)-1$$
$$\underline{=7}$$

b) $(g\circ f)(-3)=g(f(-3))$

$$=g(2(-3)-1)$$
$$=g(-7)$$
$$=\frac{4}{-7-1}$$
$$=\frac{4}{-8}$$
$$\underline{=-\frac{1}{2}}$$

Example

Given that $f(x)=\dfrac{4}{x+2}$ and $g(x)=\dfrac{1}{x}$, find

a) $(f\circ g)(x)$

b) Domain of $(f\circ g)(x)$

Solution

a) $(f\circ g)(x)=f\left(g(x)\right)$

$$=f\left(\frac{1}{x}\right) \qquad \textit{Domain:}: x\neq 0$$

$$=\frac{4}{\frac{1}{x}+2}$$

$$=\frac{4}{\frac{1+2x}{x}}$$

$$=4\div\frac{1+2x}{x}$$

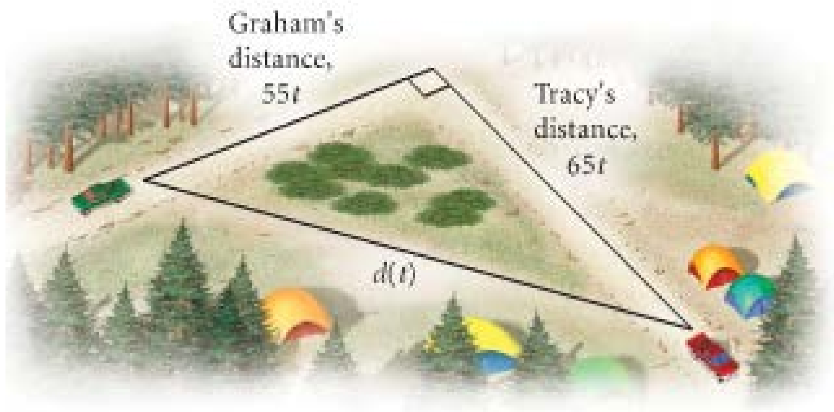
$$=4\frac{x}{1+2x}$$

$$\boxed{=\frac{4x}{1+2x}} \qquad \textit{Domain:}: x\neq -\frac{1}{2}$$

b) Domain: $\left(-\infty,-\frac{1}{2}\right)\cup\left(-\frac{1}{2},0\right)\cup(0,\infty)$

Example

Tracy and Graham drive away from a camp-ground at right angles to each other. Tracy's speed is 65 mph and Graham's is 55 mph.



a) Express the distance between the cars as a function of time.

b) Find the domain of the function.

Solution

a) *Distance = velocity * time*

Use Pythagorean Theorem:

$$d^2(t) = (65t)^2 + (55t)^2$$

$$d^2 = 4225t^2 + 3025t^2$$

$$= 7250t^2$$

$$d(t) = \sqrt{7250t^2}$$

$$= \sqrt{7250}\sqrt{t^2}$$

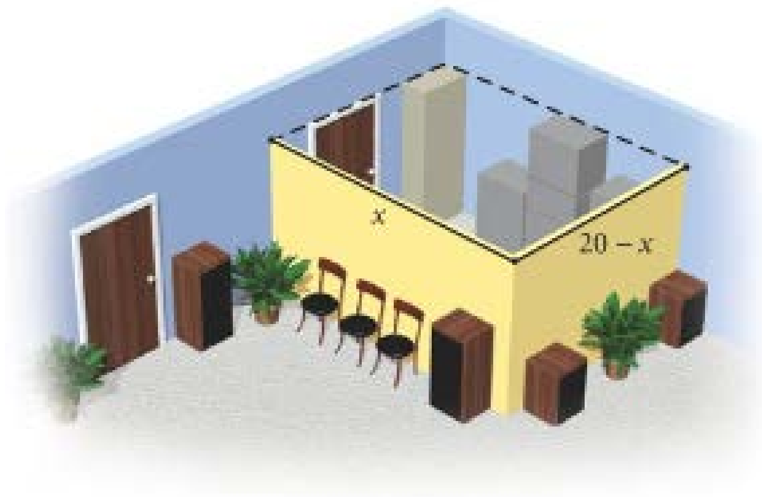
$$\approx 85.15|t|$$

$$\underline{= 85.15\ t|}$$

b) Domain: $t \geq 0$

Example: (storage area)

The sound Shop has 20 *ft*. of dividers with which to set off a rectangular area for the storage of overstock. If a corner of the store is used for the storage area, the partition need only form two sides of a rectangle.



a) Express the floor area of the storage space as a function of the length of the partition.

b) Find the domain of the function.

<u>Solution</u>

Let x = the length

$\Rightarrow$ *width* + *length* = 20

$\Rightarrow$ *width* = 20 - *length*

a) Area = *length* * *width*

$= x(20 - x)$

$= 20x - x^2$

b) Domain: x value varies from 0 to 20 $\Rightarrow$ $(0, 20)$

Exercises *Section* 2.3 – Function Operations and Composition

Find the Domain

1. $f(x)=7x+4$

2. $f(x)=|3x-2|$

3. $f(x)=x^2-2x-15$

4. $f(x)=4-\frac{2}{x}$

5. $f(x)=\frac{1}{x^4}$

6. $g(x)=\frac{3}{x-4}$

7. $y=\frac{2}{x-3}$

8. $y=\frac{-7}{x-5}$

9. $f(x)=\frac{x+5}{2-x}$

10. $f(x)=\frac{8}{x+4}$

11. $f(x)=\frac{1}{x^2-4x-5}$

12. $g(x)=\frac{2}{x^2+x-12}$

13. $h(x)=\frac{5}{\frac{4}{x}-1}$

14. $y=\sqrt{x}$

15. $y=\sqrt{4x+1}$

16. $y=\sqrt{7-2x}$

17. $f(x)=\sqrt{8-x}$

18. $f(x)=\frac{\sqrt{x+1}}{x}$

19. $g(x)=\frac{\sqrt{x-3}}{x-6}$

20. $f(x)=\frac{\sqrt{x+4}}{\sqrt{x-1}}$

21. $f(x)=\sqrt{x+4}-\sqrt{x-1}$

22. $f(x)=\sqrt{2x+7}$

23. $f(x)=\sqrt{8-3x}$

24. $f(x)=\sqrt{9-x^2}$

25. $f(x)=\sqrt{x^2-25}$

26. $f(x)=\frac{x+1}{x^3-4x}$

27. $f(x)=\frac{4x}{6x^2+13x-5}$

28. $f(x)=\frac{\sqrt{2x-3}}{x^2-5x+4}$

29. $f(x)=\frac{\sqrt{4x-3}}{x^2-4}$

30. $f(x)=\frac{x-4}{\sqrt{x-2}}$

31. $f(x)=\frac{1}{(x-3)\sqrt{x+3}}$

32. $f(x)=\sqrt{x+2}+\sqrt{2-x}$

33. $f(x)=\sqrt{(x-2)(x-6)}$

34. Let $f(x)=\sqrt{4x-1}$ and $g(x)=\frac{1}{x}$. Find each of the following and give the domain

a) $(f+g)(x)$

b) $(f-g)(x)$

c) $(fg)(x)$

d) $\left(\frac{f}{g}\right)(x)$

35. Given that $f(x)=x+1$ and $g(x)=\sqrt{x+3}$

a) Find $(f+g)(x)$

b) Find the domain of $(f+g)(x)$

c) Find: $(f+g)(6)$

36. Given that $f(x)=x^2-4$ and $g(x)=x+2$

a) Find $(f+g)(x)$ and its domain

b) Find $(f/g)(x)$ and its domain

37. Let $f(x)=x^2+1$ and $g(x)=3x+5$. Find $(f+g)(1)$, $(f-g)(-3)$, $(fg)(5)$, and $\left(\frac{f}{g}\right)(0)$

38. Find $(f+g)(x)$, $(f-g)(x)$, $(f\cdot g)(x)$, *and* $(f/g)(x)$ and the domain of $f(x)=\sqrt{3-2x}$, $g(x)=\sqrt{x+4}$

39. Find $(f+g)(x)$, $(f-g)(x)$, $(f\cdot g)(x)$, *and* $(f/g)(x)$ and the domain of $f(x)=\frac{2x}{x-4}$, $g(x)=\frac{x}{x+5}$

40. Find $(f+g)(x)$, $(f-g)(x)$, $(f\cdot g)(x)$, *and* $(f/g)(x)$ of $f(x)=x-5$ and $g(x)=x^2-1$

41. Given the function: $f(x)=2x^2$. Find and simplify the difference quotient $\frac{f(x+h)-f(x)}{h}$

42. For the function *f* given by $f(x)=9x+5$, find the difference quotient $\frac{f(x+h)-f(x)}{h}$

43. For the function *f* given by $f(x)=6x+2$, find the difference quotient $\frac{f(x+h)-f(x)}{h}$

44. For the function *f* given by $f(x)=4x+11$, find the difference quotient $\frac{f(x+h)-f(x)}{h}$

45. For the function *f* given by $f(x)=2x^2-x-3$, find the difference quotient $\frac{f(x+h)-f(x)}{h}$

46. Given $f(x)=\sqrt{x}$ and $g(x)=x+3$, find $(f\circ g)(x)$, $(g\circ f)(x)$ and their domain.

47. Given that $f(x)=\sqrt{x}$ and $g(x)=2-3x$, find $(f\circ g)(x)$, $(g\circ f)(x)$ and their domain.

48. Given that $f(x)=\frac{1}{x-2}$ and $g(x)=\frac{x+2}{x}$, find $(f\circ g)(x)$, $(g\circ f)(x)$ and their domain.

49. Given that $f(x)=2x-5$ and $g(x)=x^2-3x+8$, find $(f\circ g)(x)$, $(g\circ f)(x)$ and their domain then find $(f\circ g)(7)$

50. Given that $f(x)=\sqrt{x}$ and $g(x)=x-1$, find

a) $(f\circ g)(x)=f(g(x))$

b) $(g\circ f)(x)=g(f(x))$

c) $(f\circ g)(2)=f(g(2))$

51. Given that $f(x)=\dfrac{x}{x+5}$ and $g(x)=\dfrac{6}{x}$, find

a) $(f\circ g)(x)=f(g(x))$

b) $(g\circ f)(x)=g(f(x))$

c) $(f\circ g)(2)=f(g(2))$

52. Find $\left(f\circ g\right)(x)$, $\left(g\circ f\right)(x)$, $f\left(g(-2)\right)$ *and* $g\left(f(3)\right)$: $f(x)=2x^2+3x-4$, $g(x)=2x-1$

53. Find $\left(f\circ g\right)(x)$, $\left(g\circ f\right)(x)$, $f\left(g(-2)\right)$ *and* $g\left(f(3)\right)$: $f(x)=x^3+2x^2$, $g(x)=3x$

54. Find $\left(f\circ g\right)(x)$, $\left(g\circ f\right)(x)$, $f\left(g(-2)\right)$ *and* $g\left(f(3)\right)$: $f(x)=|x|$, $g(x)=-7$

55. Let $f(x)=x^2-3x$ *and* $g(x)=\sqrt{x+2}$

a) Find $\left(f\circ g\right)(x)$ and the domain of $f\circ g$

b) Find $\left(g\circ f\right)(x)$ and the domain of $g\circ f$

56. Let $f(x)=\sqrt{x-2}$ *and* $g(x)=\sqrt{x+5}$

a) Find $\left(f\circ g\right)(x)$ and the domain of $f\circ g$

b) Find $\left(g\circ f\right)(x)$ and the domain of $g\circ f$

57. Let $f(x)=\dfrac{3x+5}{2}$ *and* $g(x)=\dfrac{2x-5}{3}$

a) Find $\left(f\circ g\right)(x)$ and the domain of $f\circ g$

b) Find $\left(g\circ f\right)(x)$ and the domain of $g\circ f$

58. Let $f(x)=\dfrac{x-1}{x-2}$ *and* $g(x)=\dfrac{x-3}{x-4}$

a) Find $\left(f\circ g\right)(x)$ and the domain of $f\circ g$

b) Find $\left(g\circ f\right)(x)$ and the domain of $g\circ f$

59. Let $f(x)=\dfrac{6}{x-3}$ *and* $g(x)=\dfrac{1}{x}$

c) Find $\left(f\circ g\right)(x)$ and the domain of $f\circ g$

d) Find $\left(g\circ f\right)(x)$ and the domain of $g\circ f$

Section 2.4 – Quadratic Functions and Models

Quadratic Function

A function f is a ***quadratic function*** if $f(x) = ax^2 + bx + c$

Vertex of a Parabola

The ***vertex*** of the graph of $f(x)$ is

$$V_x \text{ or } x_v = -\frac{b}{2a}$$

$$V_y \text{ or } y_v = f\left(-\frac{b}{2a}\right)$$

$$\textit{Vertex}\text{ Point } \left(-\frac{b}{2a},\ f\left(-\frac{b}{2a}\right)\right)$$

$$f(x) = x^2 - 4x - 2$$

$$x = -\frac{b}{2a} = -\frac{-4}{2(1)} = 2$$

$$y = f\left(-\frac{b}{2a}\right) = f(2)$$

$$= (2)^2 - 4(2) - 2$$

$$= -6$$

Vertex point: $(2, -6)$

Axis of Symmetry: $x = V_x = -\frac{b}{2a}$

Axis of Symmetry: $x = 2$

Minimum or Maximum Point

If $a > 0 \Rightarrow f(x)$ has a ***minimum*** point

If $a < 0 \Rightarrow f(x)$ has a ***maximum*** point

@ vertex point (V_x, V_y)

Minimum point @ $(2, -6)$

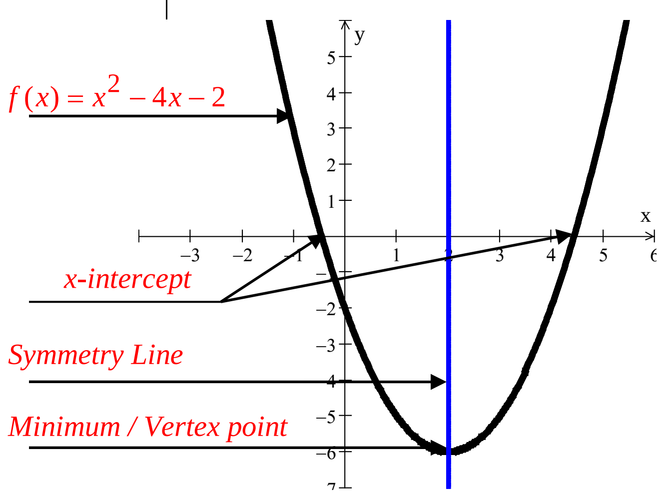
Range

If $a > 0 \Rightarrow [V_y,\ \infty)$

If $a < 0 \Rightarrow (-\infty,\ V_y]$

$[-6, \infty)$

Domain: $(-\infty, \infty)$



Example

For the graph of the function $f(x) = -x^2 - 2x + 8$

a. **Find the vertex point**

$$x = -\frac{-2}{2(-1)} = -1$$

$$y = f(-1) = -(-1)^2 - 2(-1) + 8 = 9$$

Vertex point (–1, 9)

b. **Find the line of symmetry:** $x = -1$

c. **State whether there is a maximum or minimum value *and* find that value**

Minimum point, value (–1, 9)

d. **Find the *x*-intercept**

$x = -4,\ 2$

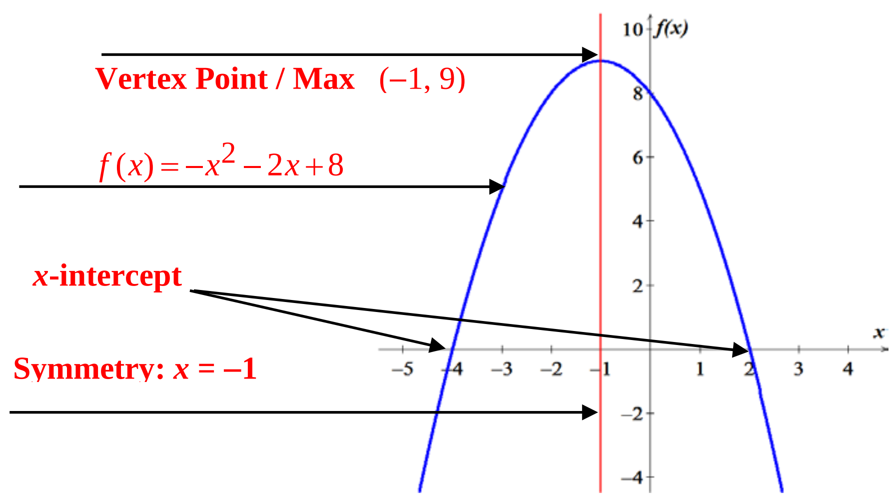
e. **Find the y-intercept**

$y = 8$

f. **Find the range and the domain of the function.**

Range: (–∞, 9] *Domain*: (–∞, ∞)

g. **Graph the function and label, show part *a thru d* on the plot below**



h. **On what intervals is the function increasing? Decreasing?**

Increasing: (–∞,–1) *Decreasing*: (–1, ∞)

Example

Find the axis and vertex of the parabola having equation $f(x) = 2x^2 + 4x + 5$

Solution

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{4}{2(2)} \\ &= -1 \end{aligned}$$

Axis of the parabola: $x = -1$

$$\begin{aligned} y &= f(-1) \\ &= 2(-1)^2 + 4(-1) + 5 \\ &= 3 \end{aligned}$$

Vertex point: $\boxed{(-1, 3)}$

Maximizing Area

You have 120 ft of fencing to enclose a rectangular region. Find the dimensions of the rectangle that maximize the enclosed area. What is the maximum area?

Solution

$$P = 2l + 2w$$

$$120 = 2l + 2w$$

$$60 = l + w \qquad \rightarrow \boxed{l = 60 - w}$$

$$\begin{aligned} A &= lw \\ &= (60 - w)w \\ &= 60w - w^2 \\ &= -w^2 + 60w \end{aligned}$$

Vertex: $w = -\frac{60}{2(-1)} = 30$

$\rightarrow l = 60 - w = 30$

$A = lw = (30)(30) \boxed{= 900\ ft^2}$

Example

A stone mason has enough stones to enclose a rectangular patio with 60 *ft* of stone wall. If the house forms one side of the rectangle, what is the maximum area that the mason can enclose? What should the dimensions of the patio be in order to yield this area?

Solution

$$P = l + 2w = 60 \quad \Rightarrow \boxed{l = 60 - 2w}$$

$$\begin{aligned} A &= lw \\ &= (60 - 2w)w \\ &= 60w - 2w^2 \\ &= -2w^2 + 60w \end{aligned}$$

$$\begin{aligned} w &= -\frac{b}{2a} \\ &= -\frac{60}{2(-2)} \\ &= 15\ ft \end{aligned}$$

$$\Rightarrow l = 60 - 2w = 60 - 2(15) = 30\ ft$$

$$Area = (15)(30) = 450\ ft^2$$

Position Function (*Projectile Motion*)

Example

A model rocket is launched with an initial velocity of 100 ft/sec from the top of a hill that is 20 ft high. Its height *t* seconds after it has been launched is given by the function $s(t) = -16t^2 + 100t + 20$. Determine the time at which the rocket reaches its maximum height and find the maximum height.

Solution

$$\begin{aligned} t &= -\frac{b}{2a} \\ &= -\frac{100}{2(-16)} \\ &= 3.125 \ \text{sec} \end{aligned}$$

$$s(t = 3.125) = -16(3.125)^2 + 100(3.125) + 20 \underline{= 176.25\ ft}$$

Exercises *Section* 2.4 – Quadratic Functions and Models

1. Give the vertex, axis, domain, and range. Then, graph the function $f(x)=x^2+6x+5$

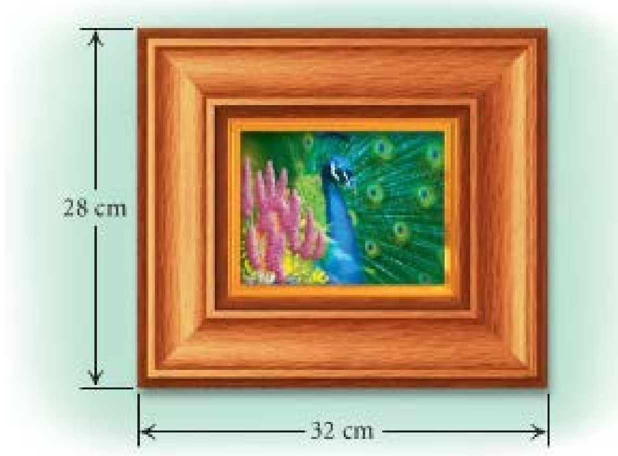
2. Give the vertex, axis, domain, and range. Then, graph the function $f(x)=-x^2-6x-5$

3. Give the vertex, axis, domain, and range. Then, graph the function $f(x)=x^2-4x+2$

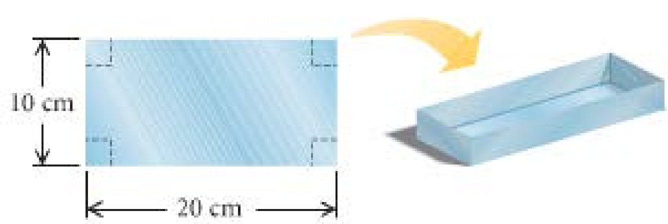
4. Give the vertex, axis, domain, and range. Then, graph the function $f(x)=-2x^2+16x-26$

5. You have 600 *ft.* of fencing to enclose a rectangular plot that borders on a river. If you do not fence the side along the river, find the length and width of the plot that will maximize the area. What is the largest area that can be enclosed?

6. A picture frame measures 28 cm by 32 cm and is of uniform width. What is the width of the frame if 192 cm^2 of the picture shows?

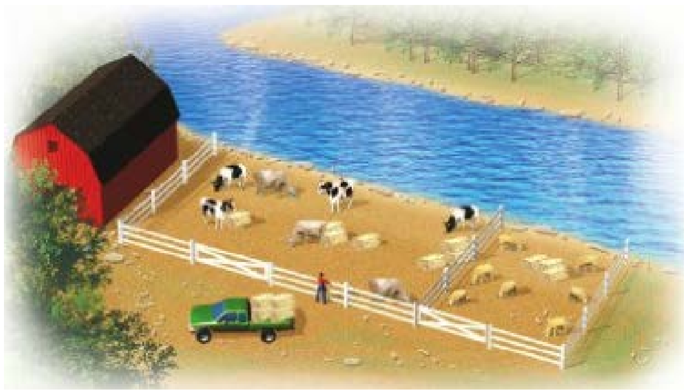


7. An open box is made from a 10-cm by 20-cm of tin by cutting a square from each corner and folding up the edges. The area of the resulting base is 96 cm^2. What is the length of the sides of the squares?



8. A fourth-grade class decides to enclose a rectangular garden, using the side of the school as one side of the rectangle. What is the maximum area that the class can enclose with 32 ft. of fence? What should the dimensions of the garden be in order to yield this area?

9. A rancher needs to enclose two adjacent rectangular corrals, one for cattle and one for sheep. If a river forms one side of the corrals and 240 yd of fencing is available, what is the largest total area that can be enclosed?



10. A Norman window is a rectangle with a semicircle on top. Sky Blue Windows is designing a Norman window that will require 24 ft of trim on the outer edges. What dimensions will allow the maximum amount of light to enter a house?

A Norman window

11. A frog leaps from a stump 3.5 ft. high and lands 3.5 ft. from the base of the stump.

It is determined that the height of the frog as a function of its distance, *x*, from the base of the stump is given by the function $h(x) = -0.5x^2 + 0.75x + 3.5$ where *h* is in feet.

a) How high is the frog when its horizontal distance from the base of the stump is 2 ft.?

b) At what two distances from the base of the stump after is jumped was the frog 3.6 ft. above the ground?

c) At what distance from the base did the frog reach its highest point?

d) What was the maximum height reached by the frog?

Section 2.5 – Polynomial Functions

Polynomial Function

A *Polynomial function* $P(x)$ in x is a sum of the form is given by:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$$

Where the coefficients $a_n,\ a_{n-1},\ \ldots, a_2,\ a_1, a_0$ are real numbers and the exponents are whole numbers.

$$\underline{a_n}\,\underline{x^n}$$

Degree

Leading Term

Leading Coefficient

Non-polynomial Functions: $\frac{1}{x} + 2x;\quad \sqrt{x^2 - 3} + x;\quad \frac{x-5}{x^2+2}$

Degree of f	***Form of f(x)***	***Graph of f(x)***
0	$f(x) = a_0$	A horizontal line
1	$f(x) = a_1 x + a_0$	A line with slope a_1
2	$f(x) = a_2 x^2 + a_1 x + a_0$	A parabola with a vertical axis

All polynomial functions are ***continuous functions***.

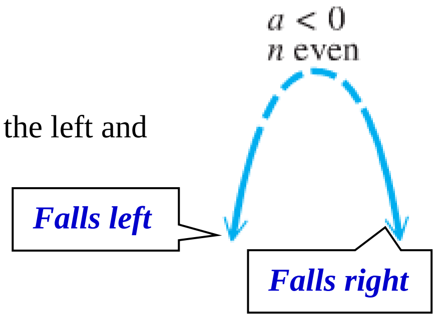
End Behavior $\left(a_n x^n\right)$

If n (degree) is even:

If $a_n < 0$ (in front x^n is negative), then the function falls from the left and right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to -\infty$$

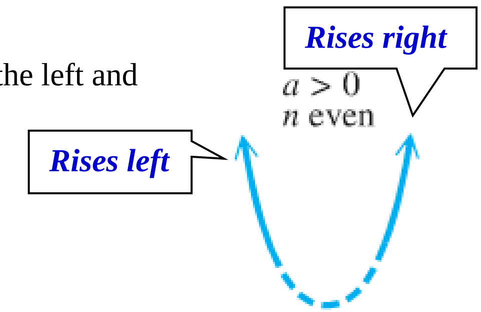
$$x \to \infty \quad \Rightarrow \quad f(x) \to -\infty$$



If $a_n > 0$ (in front x^n is positive), then the function rises from the left and right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to \infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to \infty$$

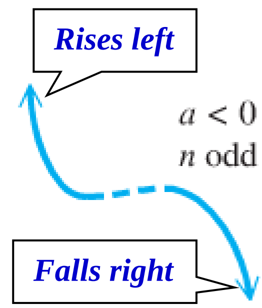


If n (degree) is odd:

If $a_n < 0$ (negative), then the function rises from the left side and falls from the right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to \infty$$

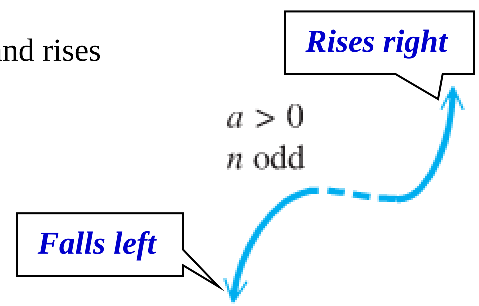
$$x \to \infty \quad \Rightarrow \quad f(x) \to -\infty$$



If $a_n > 0$ (positive), then the function falls from the left side and rises from the right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to -\infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to \infty$$



Example

Determine the end behavior of the graph of the polynomial function $f(x) = -4x^5 + 7x^2 - x + 9$

Solution

Leading term: $-4x^5$ with 5th degree (n is odd)

$x \to -\infty \quad \Rightarrow \quad f(x) = -(-)^5 = (-)(-) = + \to \infty \qquad f(x)$ rises left

$x \to \infty \quad \Rightarrow \quad f(x) \to -\infty \qquad f(x)$ falls right

The Intermediate Value *Theorem*

For any polynomial function $f(x)$ with real coefficients and $f(a) \neq f(b)$ for $a < b$, then f takes on every value between $f(a)$ and $f(b)$ in the interval $[a,\ b]$.

$\therefore$ $f(a)$ and $f(b)$ are the opposite signs. Then the function has a real zero between a and b.

Example

Using the intermediate value theorem, determine, if possible, whether the function has a real zero between a and b.

a) $f(x) = x^3 + x^2 - 6x; \quad a = -4,\ b = -2$

b) $f(x) = x^3 + x^2 - 6x; \quad a = -1,\ b = 3$

Solution

a) $f(x) = x^3 + x^2 - 6x; \quad a = -4,\ b = -2$

$f(-4) = (-4)^3 + (-4)^2 - 6(-4) = -24$

$f(-2) = (-2)^3 + (-2)^2 - 6(-2) = 8$ $\qquad$ $f(x)$ has a zero between -4 and -2

b) $f(x) = x^3 + x^2 - 6x; \quad a = -1,\ b = 3$

$f(-1) = (-1)^3 + (-1)^2 - 6(-1) = 6$

$f(3) = (3)^3 + (3)^2 - 6(3) = 18$ $\qquad$ *Can't be determined*

Example

Show that $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3$ has a zero between 1 and 2.

Solution

$f(1) = (1)^5 + 2(1)^4 - 6(1)^3 + 2(1) - 3 = -4$

$f(2) = (2)^5 + 2(2)^4 - 6(2)^3 + 2(2) - 3 = 17$

Since $f(1)$ *and* $f(2)$ have opposite signs; therefore, $f(c) = 0$ for at least one real number c between 1 and 2.

Exercises Section 2.5 – Polynomial Functions

Determine the end behavior of the graph of the polynomial function

1. $f(x) = 5x^3 + 7x^2 - x + 9$

2. $f(x) = 11x^3 - 6x^2 + x + 3$

3. $f(x) = -11x^3 - 6x^2 + x + 3$

4. $f(x) = 5x^4 + 7x^2 - x + 9$

5. $f(x) = 11x^4 - 6x^2 + x + 3$

6. $f(x) = -5x^4 + 7x^2 - x + 9$

7. $f(x) = -11x^4 - 6x^2 + x + 3$

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers.

8. $f(x) = x^3 - x - 1;\quad between\ \ 1\ and\ \ 2$

9. $f(x) = x^3 - 4x^2 + 2;\quad between\ \ 0\ and\ \ 1$

10. $f(x) = 2x^4 - 4x^2 + 1;\quad between\ \ -1\ and\ \ 0$

11. $f(x) = x^4 + 6x^3 - 18x^2;\quad between\ \ 2\ and\ \ 3$

12. $f(x) = x^3 + x^2 - 2x + 1;\quad between\ \ -3\ and\ \ -2$

13. $f(x) = x^5 - x^3 - 1;\quad between\ \ 1\ and\ \ 2$

14. $f(x) = 3x^3 - 10x + 9;\quad between\ \ -3\ and\ \ -2$

15. $f(x) = 3x^3 - 8x^2 + x + 2;\quad between\ \ 2\ and\ \ 3$

16. $f(x) = 3x^3 - 8x^2 + x + 2;\quad between\ \ 1\ and\ \ 2$

17. $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3;\quad between\ \ 0\ and\ \ 1$

Section 2.6 – Properties of Division

Long Division

Divide $\left(x^3+2x^2-5x-6\right)\ \div\ (x+1)$

$$
\begin{array}{rl}
 & \overbrace{x^2+x-6}^{\text{Quotient}} \\
\underset{\text{Divisor}}{x+1} & \overline{\big)\, x^3+2x^2-5x-6} \longleftarrow \text{Dividend} \\
 & \underline{x^3+x^2} \\
 & \quad x^2-5x \\
 & \quad \underline{x^2-x} \\
 & \qquad -6x-6 \\
 & \qquad \underline{-6x-6} \\
 & \qquad\qquad 0 \longleftarrow \text{Remainder}
\end{array}
$$

$\boxed{Q(x)=x^2+x-6}$

$\boxed{R(x)=0}$

Example

Use the long division to find the quotient and the remainder: $\left(x^4-16\right)\div\left(x^2+3x+1\right)$

Solution

$$
\begin{array}{rl}
 & \qquad\qquad x^2-3x+8 \\
x^2+3x+1 & \overline{\big)\, x^4+0x^3+0x^2+0x-16} \\
 & \underline{x^4+3x^3+x^2} \\
 & \quad -3x^3-x^2 \\
 & \quad \underline{-3x^3-9x^2-3x} \\
 & \qquad\qquad 8x^2+\ \ 3x-16 \\
 & \qquad\qquad \underline{8x^2+24x+\ 8} \\
 & \qquad\qquad\quad -21x-24
\end{array}
$$

$$\frac{x^4-16}{x^2+3x+1}=x^2-3x+8+\frac{-21x-24}{x^2+3x+1}$$

$$x^4-16=\boxed{\left(x^2+3x+1\right)\left(x^2-3x+8\right)+\left(-21x-24\right)}$$

Remainder *Theorem*

If a number c is substituted for x in the polynomial $f(x)$, then the result $f(c)$ is the remainder that would be obtained by dividing $f(x)$ by $x-c$.

That is, if $f(x)=(x-c)Q(x)+R(x)$ *then* $f(c)=R$

Example

If $f(x)=x^3-3x^2+x+5$, use the remainder theorem to find $f(2)$

Solution

$$
\begin{array}{r|l}
 & \quad x^2-x-1 \\
\hline
x-2 & x^3-3x^2+x+5 \\
 & \underline{x^3-2x^2} \\
 & \quad -x^2+x \\
 & \quad \underline{-x^2+2x} \\
 & \qquad\quad -x+5 \\
 & \qquad\quad \underline{-x+2} \\
 & \qquad\qquad\quad \boxed{3}
\end{array}
$$

$f(2)=3$

Factor *Theorem*

A polynomial $f(x)$ has a factor $x-c$ if and only if $f(c)=0$

Example

Show that $x-2$ is a factor of $f(x)=x^3-4x^2+3x+2$.

Solution

Since $f(2)=(2)^3-4(2)^2+3(2)=0$

From the factor theorem; $x-2$ is a factor of $f(x)$.

Synthetic Division

Use synthetic division to find the quotient and the remainder of $\left(4x^3 - 3x^2 + x + 7\right) \div (x-2)$

$$\begin{array}{c|cccc} & x^3 & x^2 & x^1 & x^0 \\ 2 & 4 & -3 & 1 & 7 \\ & \downarrow & 8 & 10 & 22 \\ \hline & 4 & 5 & 11 & \boxed{29} \\ & x^2 & x^1 & x^0 & \end{array}$$

Quotient : $Q(x) = 4x^2 + 5x + 11$

Remainder : $R(x) = 29$

Example

If $f(x) = 3x^5 - 38x^3 + 5x^2 - 1$, use the synthetic division to find $f(4)$.

Solution

$$\begin{array}{c|cccccc} 4 & 3 & 0 & -38 & 5 & 0 & -1 \\ & & 12 & 48 & 40 & 180 & 720 \\ \hline & 3 & 12 & 10 & 45 & 180 & \boxed{719} \end{array}$$

$f(4) = 719$

Example

Show that -11 is a zero of the polynomial $f(x) = x^3 + 8x^2 - 29x + 44$

Solution

$$\begin{array}{c|cccc} -11 & 1 & 8 & -29 & 44 \\ & & -11 & 33 & -44 \\ \hline & 1 & -3 & 4 & \boxed{0} \end{array}$$

Thus, $f(-11) = 0$, and -11 is a zero of f.

The Rational Zeros *Theorem*

If the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ *has integer* coefficients and if $\frac{c}{d}$ is a rational zero of $f(x)$ such that ***c*** and ***d*** have no common prime factor, then

1. The numerator c of the zero is a factor of the constant term a_0
2. The denominator d of the zero is a factor of the leading coefficient a_n

$$\text{possible rational zeros} = \frac{\text{factors of the constant term } a_0}{\text{factors of the leading coefficient } a_n} = \frac{\text{possibilities for } a_0}{\text{possibilities for } a_n}$$

Example

Find all rational solutions of the equation: $3x^4 + 14x^3 + 14x^2 - 8x - 8 = 0$

Solution

possibilities for a_0	$\pm1,\ \pm2,\ \pm4,\ \pm8$
possibilities for a_n	$\pm1,\ \pm3$
possibilities for c/d	$\pm1,\ \pm2,\ \pm4,\ \pm8,\ \pm\frac{1}{3},\ \pm\frac{2}{3},\ \pm\frac{4}{3},\ \pm\frac{8}{3}$

Using the calculator, the result will show that –2 is a zero.

$$\begin{array}{r|rrrrr} -2 & 3 & 14 & 14 & -8 & -8 \\ & & -6 & -16 & 4 & 8 \\ \hline & 3 & 8 & -2 & -4 & \boxed{0} \end{array}$$

We have the factorization of: $(x+2)\left(3x^3 + 8x^2 - 2x - 4\right) = 0$

For $3x^3 + 8x^2 - 2x - 4 \quad \Rightarrow \quad \frac{c}{d} = \frac{\pm1,\ \pm2,\ \pm4}{\pm1,\ \pm3}$

$x = -\frac{2}{3}$ is another solution.

$$\begin{array}{r|rrrr} -\frac{2}{3} & 3 & 8 & -2 & -4 \\ & & -2 & -4 & 4 \\ \hline & 3 & 6 & -6 & \boxed{0} \end{array}$$

We have the factorization of: $(x+2)\left(x + \frac{2}{3}\right)\left(3x^2 + 6x - 6\right) = 0$

By applying quadratic formula to solve: $3x^2 + 6x - 6 = 0 \Rightarrow x = -1 \pm \sqrt{3}$

Hence, the polynomial has two rational roots $x = -2$ *and* $-\frac{2}{3}$ and two irrational roots $x = -1 \pm \sqrt{3}$.

Exercises Section 2.6 – Properties of Division

1. Find the quotient and remainder if $f(x)$ is divided by $p(x)$:
$f(x) = 2x^4 - x^3 + 7x - 12; \quad p(x) = x^2 - 3$

2. Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x) = 3x^3 + 2x - 4; \quad p(x) = 2x^2 + 1$

3. Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x) = 7x + 2; \quad p(x) = 2x^2 - x - 4$

4. Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x) = 9x + 4; \quad p(x) = 2x - 5$

5. Use the remainder theorem to find $f(c)$: $f(x) = x^4 - 6x^2 + 4x - 8; \quad c = -3$

6. Use the remainder theorem to find $f(c)$: $f(x) = x^4 + 3x^2 - 12; \quad c = -2$

7. Use the factor theorem to show that $x - c$ is a factor of $f(x)$: $f(x) = x^3 + x^2 - 2x + 12; \quad c = -3$

8. Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $2x^3 - 3x^2 + 4x - 5; \quad x - 2$

9. Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $5x^3 - 6x^2 + 15; \quad x - 4$

10. Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $9x^3 - 6x^2 + 3x - 4; \quad x - \frac{1}{3}$

11. Use the synthetic division to find $f(c)$: $f(x) = 2x^3 + 3x^2 - 4x + 4; \quad c = 3$

12. Use the synthetic division to find $f(c)$: $f(x) = 8x^5 - 3x^2 + 7; \quad c = \frac{1}{2}$

13. Use the synthetic division to find $f(c)$: $f(x) = x^3 - 3x^2 - 8; \quad c = 1 + \sqrt{2}$

14. Use the synthetic division to show that c is a zero of $f(x)$:
$f(x) = 3x^4 + 8x^3 - 2x^2 - 10x + 4; \quad c = -2$

15. Use the synthetic division to show that c is a zero of $f(x)$:
$f(x) = 27x^4 - 9x^3 + 3x^2 + 6x + 1; \quad c = -\frac{1}{3}$

16. Find all values of k such that $f(x)$ is divisible by the given linear polynomial:
$f(x) = kx^3 + x^2 + k^2x + 3k^2 + 11; \quad x + 2$

17. Find all solutions of the equation: $x^3 - x^2 - 10x - 8 = 0$

18. Find all solutions of the equation: $x^3 + x^2 - 14x - 24 = 0$

19. Find all solutions of the equation: $2x^3 - 3x^2 - 17x + 30 = 0$

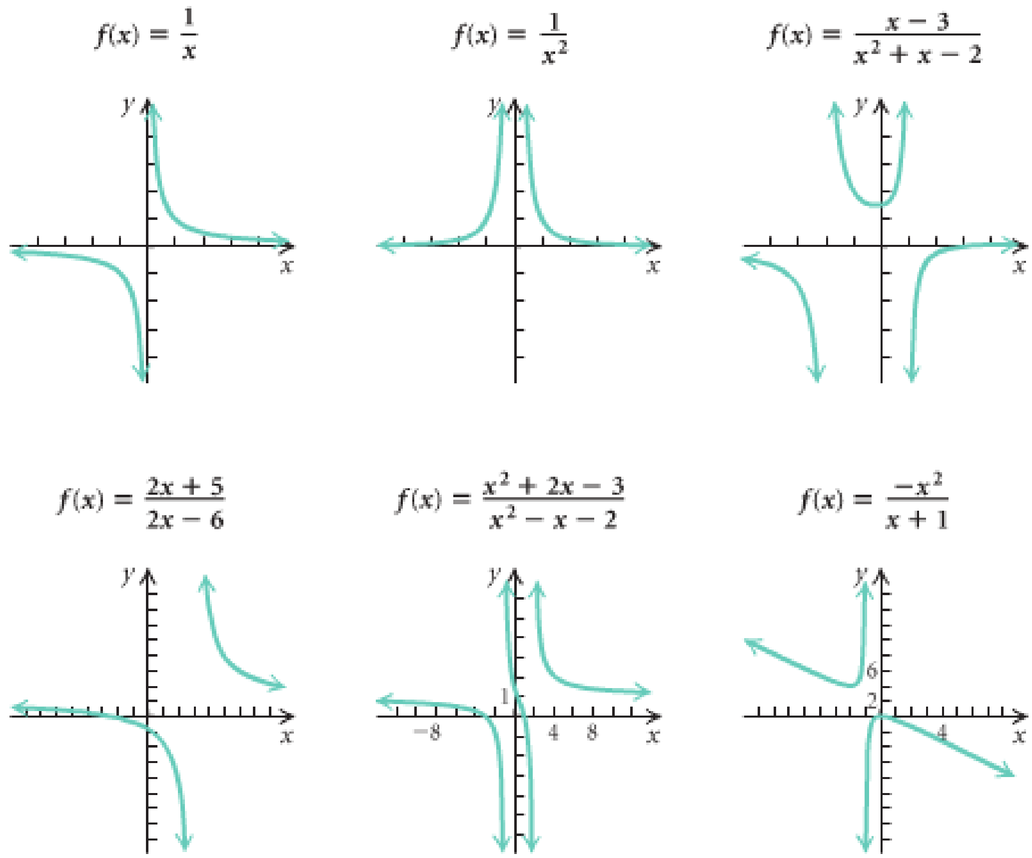
20. Find all solutions of the equation: $12x^3 + 8x^2 - 3x - 2 = 0$

21. Find all solutions of the equation: $x^4 + 3x^3 - 30x^2 - 6x + 56 = 0$

22. Find all solutions of the equation: $3x^5 - 10x^4 - 6x^3 + 24x^2 + 11x - 6 = 0$

23. Find all solutions of the equation: $6x^5 + 19x^4 + x^3 - 6x^2 = 0$

Section 2.7 – Rational Functions



Rational Function

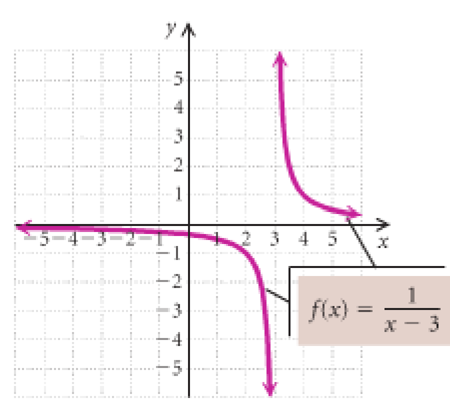
A rational function is a function f that is a quotient of two polynomials, that is,

$$f(x)=\frac{g(x)}{h(x)}$$

Where $g(x)$ and $h(x)$ are polynomials. The domain of f consists of all real numbers ***except*** the zeros of the denominator $h(x)$.

The Domain of a Rational Function

Example



Consider: $f(x)=\dfrac{1}{x-3}$

Find the domain and graph *f*.

Solution

$x-3=0 \quad \Rightarrow \quad \boxed{x=3}$

Thus the domain is: $\{x \mid x \neq 3\}$ *or* $(-\infty,\ 3)\cup(3,\ \infty)$

Function	***Domain***	
$f(x)=\frac{1}{x}$	$\{x \mid x\neq 0\}$	$(-\infty,\ 0)\cup(0,\ \infty)$
$f(x)=\frac{1}{x^2}$	$\{x \mid x\neq 0\}$	$(-\infty,\ 0)\cup(0,\ \infty)$
$f(x)=\frac{x-3}{x^2+x-2}$	$\{x \mid x\neq -2 \text{ and } x\neq 1\}$	$(-\infty,\ -2)\cup(-2,\ 1)\cup(1,\ \infty)$
$f(x)=\frac{2x+5}{2x-6}=\frac{2x+5}{2(x-3)}$	$\{x \mid x\neq 3\}$	$(-\infty,\ 3)\cup(3,\ \infty)$

Asymptotes

Vertical Asymptote (*VA*) - ***Think Domain***

The line $x=a$ is a ***vertical asymptote*** for the graph of a function f if

$$f(x)\to\infty \quad or \quad f(x)\to-\infty$$

As ***x*** approaches ***a*** from either the left or the right

Example

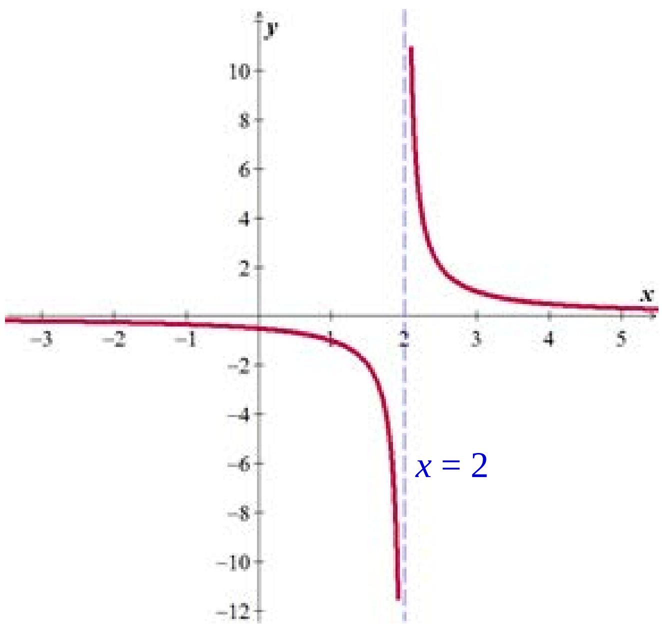
Find the vertical asymptote of $f(x)=\frac{1}{x-2}$, and sketch the graph.

Solution

VA: $x=2$

$f(x)\to\infty \quad as \quad x\to 2^{+}$

$f(x)\to-\infty \quad as \quad x\to 2^{-}$



Horizontal Asymptote (*HA*)

The line $y = c$ is a ***horizontal asymptote*** for the graph of a function f if

$$f(x) \to c \quad as \quad x \to -\infty \quad or \quad x \to -\infty$$

Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0} = \dfrac{a_n x^n}{b_m x^m}$ be a rational function.

1. If the degree of numerator is less than of denominator $(n < m) \Rightarrow$ y = 0

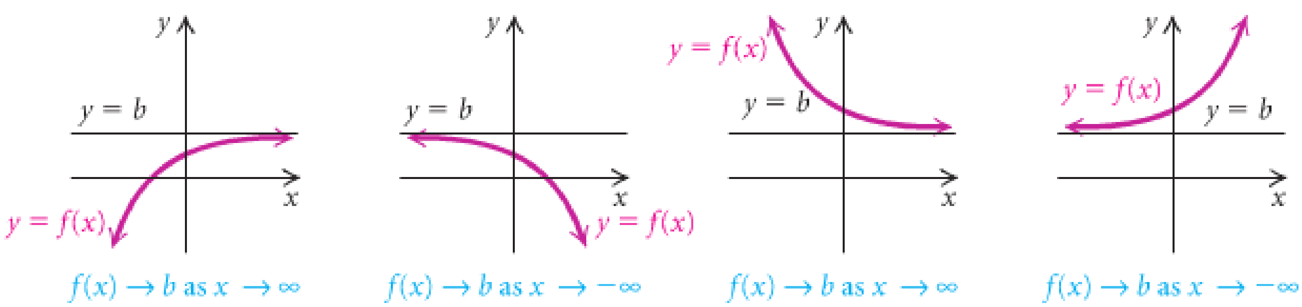
$$y = \frac{2x+1}{4x^2+5} \quad \Rightarrow \boxed{y = 0}$$

2. If the degree of numerator is equal of denominator $(n = m) \Rightarrow y = \dfrac{a_n}{b_m}$

$$y = \frac{2x^2+1}{4x^2+5} \quad \Rightarrow \boxed{y = \frac{2}{4} = \frac{1}{2}}$$

3. If the degree of numerator is greater than of denominator $(n > m) \Rightarrow$ No horizontal asymptote

$$y = \frac{2x^3+1}{4x^2+5} \quad \Rightarrow No \;\; HA$$



Example

Determine the horizontal asymptote of $f(x) = \dfrac{-7x^4 - 10x^2 + 1}{11x^4 + x - 2}$.

Solution

$$f(x) = \frac{-7x^4 - 10x^2 + 1}{11x^4 + x - 2} \to \frac{-7x^4}{11x^4} = -\frac{7}{11}$$

Therefore, the horizontal asymptote (HA) is: $\boxed{y = -\dfrac{7}{11}}$

Example

Find the vertical and the horizontal asymptote for the graph of f, if it exists

a) $f(x) = \dfrac{3x-1}{x^2-x-6}$

b) $f(x) = \dfrac{5x^2+1}{3x^2-4}$

c) $f(x) = \dfrac{2x^4-3x^2+5}{x^2+1}$

Solution

a) $f(x) = \dfrac{3x-1}{x^2-x-6}$

$$x^2 - x - 6 = 0 \;\rightarrow\; x = -2,\ 3$$

VA: $x = -2, \quad x = 3$

HA: $y = 0$

b) $f(x) = \dfrac{5x^2+1}{3x^2-4}$

$$3x^2 - 4 = 0 \;\rightarrow\; 3x^2 = 4 \;\rightarrow\; x^2 = \frac{4}{3} \;\rightarrow\; \boxed{x = \pm\frac{2}{\sqrt{3}}}$$

VA: $x = -\dfrac{2}{\sqrt{3}}, \quad x = \dfrac{2}{\sqrt{3}}$

HA: $y = \dfrac{5}{3}$

c) $f(x) = \dfrac{2x^4-3x^2+5}{x^2+1}$

$$x^2 + 1 = 0 \;\rightarrow\; x^2 = -1$$

VA: *n/a*

HA: *n/a*

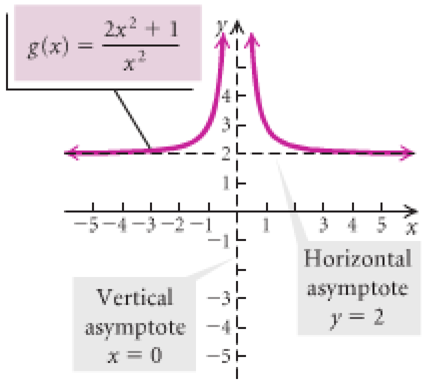
Example

Graph $f(x) = \frac{2x^2+1}{x^2}$. Include and label all asymptotes

Solution

VA: $x = 0$

HA: $y = 2$



x	$g(x)$
−2	2.25
−1.5	$2.\overline{4}$
−1	3
−0.5	6
0.5	6
1	3
1.5	$2.\overline{4}$
2	2.25

Slant or *Oblique* Asymptotes

When the degree of the numerator is one greater than the degree of the numerator, the graph has a slant or oblique asymptote and it is a line $y = ax + b, \quad a \ne 0$. To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

$$y = \frac{3x^2 - 1}{x + 2}$$

$$\begin{array}{r} 3x - 6 \\ x + 2 \overline{\smash{\big)} 3x^2 + 0x - 1} \\ \underline{3x^2 + 6x} \quad\;\; \\ -6x - 1 \\ \underline{-6x - 12} \\ R = 11 \end{array}$$

$$y = \frac{3x^2 - 1}{x + 2} = 3x - 6 + \frac{11}{x + 2}$$

The ***oblique asymptote*** is the line ***y* = 3*x* – 6**

Example

Find all the asymptotes of $f(x) = \dfrac{2x^2 - 3x - 1}{x - 2}$

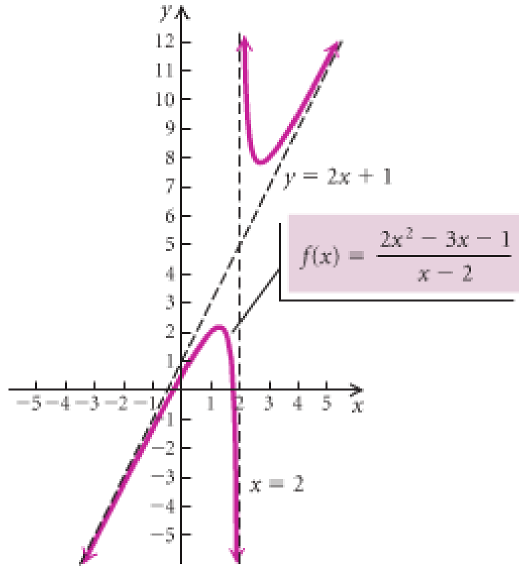
Solution

$$\begin{array}{r|l} & \quad 2x+1 \\ \hline x-2 & 2x^2 - 3x - 1 \\ & \underline{-2x^2 + 4x} \\ & \qquad\quad x - 1 \\ & \qquad \underline{-x + 2} \\ & \qquad\qquad 1 \end{array}$$

$$f(x) = \frac{2x^2 - 3x - 1}{x - 2} = (2x + 1) + \frac{1}{x - 2}$$

The ***oblique asymptote*** is the line $y = 2x + 1$

VA:: $x = 2$



Graph That Has a *Hole*

Example

Sketch the graph of g if $g(x) = \dfrac{(3x+4)(x-1)}{(2x-5)(x-1)}$

Solution

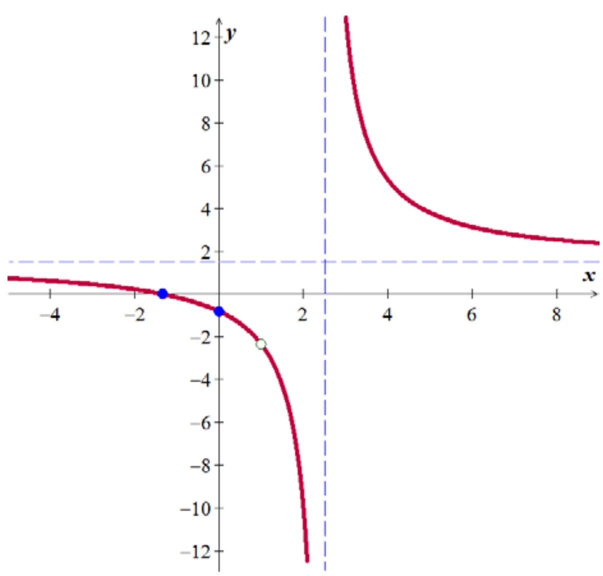
$$g(x) = \frac{(3x+4)(x-1)}{(2x-5)(x-1)} = \frac{3x+4}{2x-5} = f(x)$$

VA: $x = \dfrac{5}{2}$

HA: $y = \dfrac{3}{2}$

The only different between the graphs that g has a ***hole*** at $x = 1 \rightarrow f(1) = -\dfrac{7}{3}$

x	y
–4	.6
1.3	0
0	–.8
4	5.3
6	3.1



Exercises Section 2.7 – Rational Functions

Find the vertical and horizontal asymptotes (if any) of

1. $y=\dfrac{3x}{1-x}$

2. $y=\dfrac{x^2}{x^2+9}$

3. $y=\dfrac{x-2}{x^2-4x+3}$

4. $y=\dfrac{3}{x-5}$

5. $y=\dfrac{x^3-1}{x^2+1}$

6. $y=\dfrac{3x^2-27}{(x+3)(2x+1)}$

7. $y=\dfrac{x^3+3x^2-2}{x^2-4}$

8. $y=\dfrac{x-3}{x^2-9}$

9. $y=\dfrac{6}{\sqrt{x^2-4x}}$

10. $y=\dfrac{5x-1}{1-3x}$

11. $f(x)=\dfrac{2x-11}{x^2+2x-8}$

12. $f(x)=\dfrac{x^2-4x}{x^3-x}$

13. $f(x)=\dfrac{x-2}{x^3-5x}$

Determine all asymptotes of the function

14. $f(x)=\dfrac{4x}{x^2+10x}$

15. $f(x)=\dfrac{3-x}{(x-4)(x+6)}$

16. $f(x)=\dfrac{x^3}{2x^3-x^2-3x}$

17. $f(x)=\dfrac{3x^2+5}{4x^2-3}$

18. $f(x)=\dfrac{x+6}{x^3+2x^2}$

19. $f(x)=\dfrac{x^2+4x-1}{x+3}$

20. $f(x)=\dfrac{x^2-6x}{x-5}$

21. $f(x)=\dfrac{x^3-x^2+x-4}{x^2+2x-1}$

22. $f(x)=\dfrac{-3x}{x+2}$

23. $f(x)=\dfrac{x+1}{x^2+2x-3}$

24. $f(x)=\dfrac{2x^2-2x-4}{x^2+x-12}$

25. $f(x)=\dfrac{-2x^2+10x-12}{x^2+x}$

26. $f(x)=\dfrac{x^2-x-6}{x+1}$

27. $f(x)=\dfrac{x^3+1}{x-2}$

28. $f(x)=\dfrac{2x^2+x-6}{x^2+3x+2}$

29. $f(x)=\dfrac{x-1}{1-x^2}$

30. $f(x)=\dfrac{x^2+x-2}{x+2}$

31. $f(x)=\dfrac{x^3-2x^2-4x+8}{x-2}$

32. Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} \text{vertical asymptote}: x=4 \\ \text{horizontal asymptote}: y=-1 \\ x-\text{intercept}: 3 \end{cases}$$

33. Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} \text{vertical asymptote}: x=-3, x=1 \\ \text{horizontal asymptote}: y=0 \\ x-\text{intercept}: -1,\ f(0)=-2 \\ \text{hole at } x=2 \end{cases}$$

34. Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} \textit{vertical asymptote}: \ x = -4, x = 5 \\ \textit{horizontal asymptote}: \ y = \dfrac{3}{2} \\ x - \textit{intercept}: \ -2 \end{cases}$$

Section 2.8 – Polynomial and Rational Inequalities

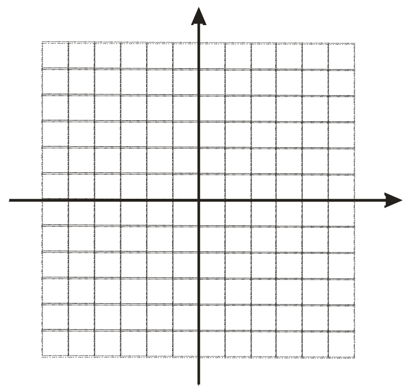
Definition of a Polynomial Inequality

A polynomial inequality is any inequality that can be put into one of the forms

$$f(x) < 0 \qquad f(x) > 0 \qquad f(x) \le 0 \qquad f(x) \ge 0$$

Where f is a polynomial function.

$f(x) = x^2 - 5x + 4 \quad (x = 1, 4)$



Procedure for Solving Polynomial Inequalities

Example

1. Express the inequality in the form $f(x)\ ?\ 0$

$x^2 - x < 12$

$x^2 - x - 12 < 0$

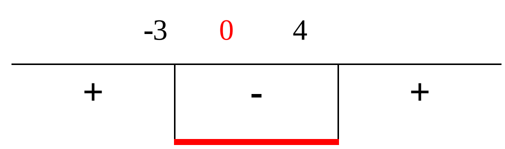
2. Solve $f(x) = 0$

$x^2 - x - 12 = 0$

$(x+3)(x-4) = 0$

$x = -3,\ 4$

3. Locate the boundary
4. Choose one test value
5. Write the solution set



$(-3,\ 4)$

$$\checkmark \quad ax^2 + bx + c \ge 0 \ \rightarrow\ if\ \ a > 0 \ \Rightarrow\ x \le x_1,\ \ x \ge x_2$$

$$\checkmark \quad ax^2 + bx + c \le 0 \ \rightarrow\ if\ \ a > 0 \ \Rightarrow\ x_1 \le x \le x_2$$

Example

Solve $2x^2 + 5x - 12 \ge 0$

Solution

$$2x^2 + 5x - 12 = 0$$
$$(2x-3)(x+4) = 0$$

$2x - 3 = 0 \qquad x + 4 = 0$

$x = \frac{3}{2} \qquad x = -4$

	-4	0	$\frac{3}{2}$	
+		-		+

$$(-\infty,\ -4] \cup \left[\frac{3}{2},\ \infty\right)$$

Example

Solve: $x^3 + 3x^2 \le x + 3$

Solution

$$\Rightarrow x^3 + 3x^2 - x - 3 = 0$$
$$x^2(x+3) - (x+3) = 0$$
$$(x+3)(x^2-1) = 0$$

$x + 3 = 0 \qquad x^2 - 1 = 0$

$x = -3 \qquad x^2 = 1$

$x = -3 \qquad x = \pm 1$

	-3		-1	0	1	
-		+		-		+

$$(-\infty,\ -3] \cup [-1,\ 1]$$

Rational Inequality

Example

Solve: $\dfrac{2x}{x+1} \ge 1$

Solution

$$\frac{2x}{x+1} = 1 \qquad \to \textit{Cond.}: x+1 \ne 0 \Rightarrow \boxed{x \ne -1}$$

$$(x+1)\frac{2x}{x+1} - 1(x+1) = 0$$

$$2x - x - 1 = 0$$

$$x - 1 = 0$$

$$x = 1$$

	-1	0	1	
$+$		$-$		$+$

$$\boxed{(-\infty,\ -1) \cup [1,\ \infty)}$$

Example

Solve $\dfrac{5}{x+4} \ge 1$

Solution

$$\frac{5}{x+4} - 1 = 0 \qquad \textit{Exception}: \ x+4 \ne 0 \ \Rightarrow x \ne -4$$

$$(x+4)\frac{5}{x+4} - 1(x+4) = 0$$

$$5 - x - 4 = 0$$

$$1 - x = 0$$

$$x = 1$$

	-4	0	1	
$-$		$+$		$-$

$$(-4,\ 1]$$

Example

Solve $\dfrac{2x-1}{3x+4} < 5$

Solution

$$\frac{2x-1}{3x+4} - 5 = 0 \qquad \textit{Exception}: \ 3x+4 \ne 0 \ \Rightarrow 3x \ne -4 \ \Rightarrow x \ne -\frac{4}{3}$$

$$(3x+4)\frac{2x-1}{3x+4} - 5(3x+4) = 0$$

$$2x - 1 - 15x - 20 = 0$$

$$-13x - 21 = 0$$

$$-13x = 21$$

$$x = -\frac{21}{13}$$

	$-\frac{21}{13}$		$-\frac{4}{3}$	0
$-$		$+$		$-$

$$\left(-\infty,\ -\frac{21}{13}\right) \cup \left(-\frac{4}{3},\ \infty\right)$$

Position Function

An object that is falling or vertically projected into the air has its height above the ground, s(t), in feet, given by

$$s(t) = -16t^2 + v_o t + s_o$$

v_0 is the original velocity (initial velocity) of the object, in feet per second
t is the time that the object is in motion, in second
s_0 is the original height (initial height) of the object, in feet

Example

An object is propelled straight up from ground level with an initial velocity of 80 ft per second. Its height at time t is modeled by

$$s(t) = -16t^2 + 80t$$

Where the height s(t), is measured in feet and the time, t, is measured in seconds. In which time interval will the object be more than 64 feet above the ground?

Solution

$$-16t^2 + 80t > 64$$

$$-16t^2 + 80t - 64 > 0$$

$$\Rightarrow \quad -16t^2 + 80t - 64 = 0$$

$$-t^2 + 5t - 4 = 0$$

$$t^2 - 5t + 4 = 0$$

$$(t-1)(t-4) = 0$$

$$t - 1 = 0 \qquad t - 4 = 0$$

$$t = 1 \qquad t = 4$$

The time interval [1, 4]

Exercises *Section* 2.8 – Polynomial and Rational Inequalities

Solve the inequality

1. $x^2 - 7x + 10 > 0$

2. $2x^2 - 9x \le 18$

3. $x^2 - 5x + 4 > 0$

4. $x^2 + x - 2 > 0$

5. $x^2 - 4x + 12 < 0$

6. $x^3 - 3x^2 - 9x + 27 < 0$

7. $x^3 - x > 0$

8. $x^3 + 3x^2 \le x + 3$

9. $x^3 + x^2 \ge 48x$

10. $\dfrac{x}{x-3} > 0$

11. $\dfrac{x-2}{x+2} \le 2$

12. $\dfrac{x+2}{3+2x} \le 5$

13. $\dfrac{x-3}{x+4} \ge \dfrac{x+2}{x-5}$

14. $\dfrac{x-4}{x+3} - \dfrac{x+2}{x-1} \le 0$

15. $\dfrac{2x-1}{x+3} \ge \dfrac{x+1}{3x+1}$

16. $\dfrac{x+6}{x-14} \ge 1$

17. A car can be rented from Basic Rental for \$260 per week with no extra charge for mileage. Continental charges \$80 per week plus 25 cents for each mile driven to rent the same car. How many miles must be driven in a week to make the rental cost for Basic Rental a better deal than Continental's?

18. If a projectile is launched from ground level with an initial velocity of 96 *ft.* per *sec*, its height in feet ***t*** seconds after launching is ***s*** feet, where

$$s = -16t^2 + 96t$$

When will the projectile be greater than 80 *ft*. above the ground?

19. A projectile is fired straight up from ground level. After ***t*** seconds, its height above the ground is ***s*** *ft*., where

$$s = -16t^2 + 220t$$

For what time period is the projectile at least 624 *ft*. above the ground?