Solution   *Section* 2.3 – Orthogonality

Exercise

Determine whether $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

a) $\boldsymbol{u}=(-6,\ -2),\quad \boldsymbol{v}=(5,\ -7)$

b) $\boldsymbol{u}=(6,\ 1,\ 4),\quad \boldsymbol{v}=(2,\ 0,\ -3)$

c) $\boldsymbol{u}=(1,\ -5,\ 4),\quad \boldsymbol{v}=(3,\ 3,\ 3)$

d) $\boldsymbol{u}=(-2,\ 2,\ 3),\quad \boldsymbol{v}=(1,\ 7,\ -4)$

Solution

a) $\boldsymbol{u}\cdot\boldsymbol{v}=(-6)(5)+(-2)(-7)$

$$=-30+14$$

$$=-16\neq 0$$

$\therefore$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are not orthogonal

b) $\boldsymbol{u}\cdot\boldsymbol{v}=6(2)+1(0)+4(-3)$

$$=0$$

$\therefore$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

c) $\boldsymbol{u}\cdot\boldsymbol{v}=1(3)-5(3)+4(3)$

$$=0$$

$\therefore$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

d) $\boldsymbol{u}\cdot\boldsymbol{v}=-2(1)+2(7)+3(-4)$

$$=0$$

$\therefore$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

Exercise

Determine whether the vectors form an orthogonal set

a) $\vec{v}_1=(2,\ 3),\quad \vec{v}_2=(3,\ 2)$

b) $\vec{v}_1=(1,\ -2),\quad \vec{v}_2=(-2,\ 1)$

c) $\vec{u}=(-4,\ 6,\ -10,\ 1)\quad \vec{v}=(2,\ 1,\ -2,\ 9)$

d) $\vec{u}=(a,\ b)\quad \vec{v}=(-b,\ a)$

e) $\vec{v}_1=(-2,\ 1,\ 1),\quad \vec{v}_2=(1,\ 0,\ 2),\quad \vec{v}_3=(-2,\ -5,\ 1)$

f) $\boldsymbol{v}_1=(1,\ 0,\ 1),\quad \boldsymbol{v}_2=(1,\ 1,\ 1),\quad \boldsymbol{v}_3=(-1,\ 0,\ 1)$

g) $\boldsymbol{v}_1=(2,-2,1),\quad \boldsymbol{v}_2=(2,1,-2),\quad \boldsymbol{v}_3=(1,2,2)$

Solution

a) $\vec{v}_1 \bullet \vec{v}_2 = 2(3) + 3(2) \underline{= 12 \neq 0|}$

$\therefore$ Vectors don't form an orthogonal set

b) $\vec{v}_1 \bullet \vec{v}_2 = 1(-2) - 2(1) \underline{= -4 \neq 0|}$

$\therefore$ Vectors don't form an orthogonal set

c) $\boldsymbol{u} \bullet \boldsymbol{v} = -8 + 6 + 20 + 9 \underline{= 27 \neq 0|}$; These vectors are not orthogonal

d) $\boldsymbol{u} \bullet \boldsymbol{v} = -ab + ab \underline{= 0|}$; These vectors are orthogonal

e) $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = -2(1) + 1(0) + 1(2) \underline{= 0|}$

$\boldsymbol{v}_1 \cdot \boldsymbol{v}_3 = -2(-2) + 1(-5) + 1(1) \underline{= 0|}$

$\boldsymbol{v}_2 \cdot \boldsymbol{v}_3 = 1(-2) + 0(-5) + 2(1) \underline{= 0|}$

$\therefore$ Vectors form an orthogonal set

f) $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 1(1) + 0(1) + 1(1) \underline{= 2 \neq 0|}$

$\therefore$ Vectors don't form an orthogonal set

g) $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 2(2) - 2(1) + 1(-2) \underline{= 0|}$

$\boldsymbol{v}_1 \cdot \boldsymbol{v}_3 = 2(1) - 2(2) + 1(2) \underline{= 0|}$

$\boldsymbol{v}_2 \cdot \boldsymbol{v}_3 = 2(1) + 1(2) - 2(2) \underline{= 0|}$

$\therefore$ Vectors form an orthogonal set

Exercise

Find a unit vector that is orthogonal to both $\boldsymbol{u} = (1, 0, 1)$ and $\boldsymbol{v} = (0, 1, 1)$

Solution

Let $\boldsymbol{w} = (w_1, w_2, w_3)$ be the unit vector that is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$.

$$\boldsymbol{u} \cdot \boldsymbol{w} = 1(w_1) + 0(w_2) + 1(w_3) = \underline{w_1 + w_3 = 0|}$$

$$\boxed{w_3 = -w_1}$$

$$\boldsymbol{v} \cdot \boldsymbol{w} = 0(w_1) + 1(w_2) + 1(w_3) = \underline{w_2 + w_3 = 0|}$$

$$\boxed{w_3 = -w_2}$$

$$w_1 = w_2 = -w_3$$

The orthogonal vector to both $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\boldsymbol{w} = (1,\ 1,\ -1)$, therefore the unit vector is

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \frac{1}{\sqrt{1^2+1^2+(-1)^2}}(1,\ 1,\ -1)$$

$$= \frac{1}{\sqrt{3}}(1,\ 1,\ -1)$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)$$

The possible vectors are: $\pm\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)$

Exercise

a) Show that $\boldsymbol{v} = (a, b)$ and $\boldsymbol{w} = (-b, a)$ are orthogonal vectors.

b) Use the result to find two vectors that are orthogonal to $\boldsymbol{v} = (2, -3)$.

c) Find two unit vectors that are orthogonal to (–3, 4)

Solution

a) $\boldsymbol{v}\cdot\boldsymbol{w} = a(-b)+b(a) = -ab+ab = 0$ are orthogonal vectors.

b) (2, 3) and (–2, 3).

c) $\vec{u}_1 = \frac{1}{\sqrt{4^2+3^2}}(4,\ 3) = \left(\frac{4}{5},\ \frac{3}{5}\right)$

$\vec{u}_2 = -\frac{1}{\sqrt{4^2+3^2}}(4,\ 3) = \left(-\frac{4}{5},\ -\frac{3}{5}\right)$

Exercise

Find the vector component of $\boldsymbol{u}$ along $\boldsymbol{a}$ and the vector component of $\boldsymbol{u}$ orthogonal to $\boldsymbol{a}$.

a) $\boldsymbol{u} = (6,\ 2),\quad \boldsymbol{a} = (3,\ -9)$

b) $\boldsymbol{u} = (3,\ 1,\ -7),\quad \boldsymbol{a} = (1,\ 0,\ 5)$

c) $\boldsymbol{u} = (1,\ 0,\ 0),\quad \boldsymbol{a} = (4,\ 3,\ 8)$

d) $\boldsymbol{u} = (1,\ 1,\ 1),\quad \boldsymbol{a} = (0,\ 2,\ -1)$

e) $\boldsymbol{u} = (2,\ 1,\ 1,\ 2),\quad \boldsymbol{a} = (4,\ -4,\ 2,\ -2)$

f) $\boldsymbol{u} = (5,0,-3,7),\quad \boldsymbol{a} = (2,1,-1,-1)$

Solution

a) $proj_{\boldsymbol{a}}\boldsymbol{u} = \frac{\boldsymbol{u}\bullet\boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$

$$= \frac{6(3)+2(-9)}{3^2+(-9)^2}(3,\ -9)$$

$$= \frac{0}{90}(3,\ -9)$$

$$\boxed{= (0,\ 0)}$$

$$\boldsymbol{u} - proj_{\boldsymbol{a}}\boldsymbol{u} = (6,\ 2) - (0,\ 0)$$

$$\boxed{= (6,\ 2)}$$

b) $$proj_{\boldsymbol{a}}\boldsymbol{u} = \frac{\boldsymbol{u} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} = \frac{3(1)+0-7(5)}{1^2+0+5^2}(1,\ 0,\ 5)$$

$$= \frac{-32}{26}(1,\ 0,\ 5)$$

$$\boxed{= \left(-\frac{16}{13},\ 0,\ -\frac{80}{13}\right)}$$

$$\boldsymbol{u} - proj_{\boldsymbol{a}}\boldsymbol{u} = (1,0,5) - \left(-\frac{16}{13},\ 0,\ -\frac{80}{13}\right)$$

$$\boxed{= \left(\frac{55}{13},\ 1,\ -\frac{11}{13}\right)}$$

c) $$proj_{\boldsymbol{a}}\boldsymbol{u} = \frac{\boldsymbol{u} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

$$= \frac{1(4)+0+0}{4^2+3^2+8^2}(4,\ 3,\ 8)$$

$$= \frac{4}{89}(4,\ 3,\ 8)$$

$$\boxed{= \left(\frac{16}{89},\ \frac{12}{89},\ \frac{32}{89}\right)}$$

$$\boldsymbol{u} - proj_{\boldsymbol{a}}\boldsymbol{u} = (1,\ 0,\ 0) - \left(\frac{16}{89},\ \frac{12}{89},\ \frac{32}{89}\right)$$

$$\boxed{= \left(\frac{73}{89},\ -\frac{12}{89},\ -\frac{32}{89}\right)}$$

d) $$proj_{\boldsymbol{a}}\boldsymbol{u} = \frac{\boldsymbol{u} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

$$= \frac{1(0)+1(2)+1(-1)}{0^2+2^2+(-1)^2}(0,\ 2,\ -1)$$

$$= \frac{1}{5}(0,\ 2,\ -1)$$

$$\boxed{= \left(0,\ \frac{2}{5},\ -\frac{1}{5}\right)}$$

$$\boldsymbol{u} - proj_{\boldsymbol{a}}\boldsymbol{u} = (1,1,1) - \left(0,\ \frac{2}{5},\ -\frac{2}{5}\right)$$

$$= \left(1,\ \frac{3}{5},\ \frac{6}{5}\right)$$

e) $$proj_{\boldsymbol{a}}\boldsymbol{u} = \frac{\boldsymbol{u} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

$$= \frac{2(4)+1(-4)+1(2)+2(-2)}{4^2+(-4)^2+2^2+(-2)^2}(4,\ -4,\ 2,\ -2)$$

$$= \frac{2}{40}(4,\ -4,\ 2,\ -2)$$

$$= \left(\frac{1}{5},\ -\frac{1}{5},\ \frac{1}{10},\ -\frac{1}{10}\right)$$

$$\boldsymbol{u} - proj_{\boldsymbol{a}}\boldsymbol{u} = (2,\ 1,\ 1,\ 2) - \left(\frac{1}{5},\ -\frac{1}{5},\ \frac{1}{10},\ -\frac{1}{10}\right)$$

$$= \left(\frac{9}{5},\ \frac{6}{5},\ \frac{9}{10},\ \frac{21}{10}\right)$$

f) $$proj_{\boldsymbol{a}}\boldsymbol{u} = \frac{\boldsymbol{u} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

$$= \frac{5(2)+0(1)-3(-1)+7(-1)}{2^2+1^2+(-1)^2+(-1)^2}(2,\ 1,\ -1,\ -1)$$

$$= \frac{6}{7}(2,\ 1,\ -1,\ -1)$$

$$= \left(\frac{12}{7},\ \frac{6}{7},\ -\frac{6}{7},\ -\frac{6}{7}\right)$$

$$\boldsymbol{u} - proj_{\boldsymbol{a}}\boldsymbol{u} = (5,\ 0,\ -3,\ 7) - \left(\frac{12}{7},\ \frac{6}{7},\ -\frac{6}{7},\ -\frac{6}{7}\right)$$

$$= \left(\frac{23}{7},\ -\frac{6}{7},\ -\frac{15}{7},\ \frac{55}{7}\right)$$

Exercise

Project the vector $\boldsymbol{v}$ onto the line through $\boldsymbol{a}$, Check that $\boldsymbol{e}$ is perpendicular to $\boldsymbol{a}$:

a) $$v = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad and \quad a = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

b) $$v = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \quad and \quad a = \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix}$$

$c)\quad v=\begin{pmatrix}1\\1\\1\end{pmatrix} \quad and \quad a=\begin{pmatrix}1\\2\\2\end{pmatrix}$

Solution

a) $$proj_{\boldsymbol{a}}\boldsymbol{v}=\frac{\boldsymbol{v}\bullet\boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

$$=\frac{1(1)+2(1)+2(1)}{1^2+1^2+1^2}(1,\ 1,\ 1)$$

$$=\frac{5}{3}(1,\ 1,\ 1)$$

$$=\left(\frac{5}{3},\ \frac{5}{3},\ \frac{5}{3}\right)$$

$$\boldsymbol{e}=\boldsymbol{v}-proj_{\boldsymbol{a}}\boldsymbol{v}=(1,\ 2,\ 2)-\left(\frac{5}{3},\ \frac{5}{3},\ \frac{5}{3}\right)$$

$$=\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$\boldsymbol{e}\cdot\boldsymbol{a}=\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)\bullet(1,\ 1,\ 1)$$

$$=-\frac{2}{3}+\frac{1}{3}+\frac{1}{3}$$

$$=0$$

$\boldsymbol{e}$ is perpendicular to $\boldsymbol{a}$

b) $$proj_{\boldsymbol{a}}\boldsymbol{v}=\frac{\boldsymbol{v}\bullet\boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

$$=\frac{1(-1)+3(-3)+1(-1)}{(-1)^2+(-3)^2+(-1)^2}(-1,\ -3,\ -1)$$

$$=\frac{-11}{11}(-1,\ -3,\ -1)$$

$$=(1,\ 3,\ 1)$$

$$\boldsymbol{e}=\boldsymbol{v}-proj_{\boldsymbol{a}}\boldsymbol{v}=(1,\ 3,\ 1)-(1,\ 3,\ 1)$$

$$=(0,\ 0,\ 0)$$

$$\boldsymbol{e}\cdot\boldsymbol{a}=(0,\ 0,\ 0)\bullet(-1,\ -3,\ -1)$$

$$=0$$

$\boldsymbol{e}$ is perpendicular to $\boldsymbol{a}$

c) $proj_{\boldsymbol{a}}\boldsymbol{v} = \dfrac{\boldsymbol{v} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$

$$= \frac{1(1)+1(2)+1(2)}{(1)^2+(2)^2+(2)^2}(1,\ 2,\ 2)$$

$$= \frac{5}{9}(1,\ 2,\ 2)$$

$$= \left(\frac{5}{9},\ \frac{10}{9},\ \frac{10}{9}\right)$$

$$\boldsymbol{e} = \boldsymbol{v} - proj_{\boldsymbol{a}}\boldsymbol{v} = (1,\ 1,\ 1) - \left(\frac{5}{9},\ \frac{10}{9},\ \frac{10}{9}\right)$$

$$= \left(\frac{4}{9},\ -\frac{1}{9},\ -\frac{1}{9}\right)$$

$$\boldsymbol{e} \cdot \boldsymbol{a} = \left(\frac{4}{9},\ -\frac{1}{9},\ -\frac{1}{9}\right) \bullet (1,\ 2,\ 2)$$

$$= \frac{4}{9} - \frac{2}{9} - \frac{2}{9}$$

$$= 0$$

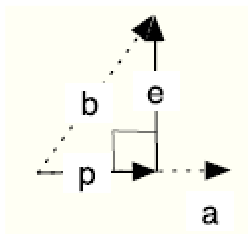
$\boldsymbol{e}$ is perpendicular to $\boldsymbol{a}$

Exercise

Draw the projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ and also compute it $\quad b = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ *and* $a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Solution

$$proj_{\boldsymbol{a}}\boldsymbol{b} = \frac{\boldsymbol{b} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$



$$= \frac{\cos\theta(1)+\sin\theta(0)}{(1)^2+0}(1,0)$$

$$= \cos\theta(1,\ 0)$$

$$= (\cos\theta,\ 0)$$

$$\boldsymbol{e} = \boldsymbol{b} - proj_{\boldsymbol{a}}\boldsymbol{b} = (\cos\theta, \sin\theta) - (\cos\theta, 0)$$

$$= (0,\ \sin\theta)$$

Exercise

Draw the projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ and also compute it $\quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and* $a = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

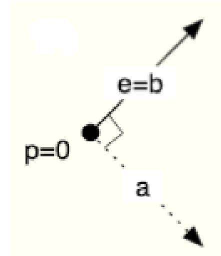
Solution

$$proj_{\boldsymbol{a}}\boldsymbol{b} = \frac{\boldsymbol{b} \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

$$= \frac{1(1)+1(-1)}{1^2+(-1)^2}(1,\ -1)$$

$$= \frac{0}{2}(1,-1)$$

$$\underline{= (0,\ 0)}$$



$$\boldsymbol{e} = \boldsymbol{b} - proj_{\boldsymbol{a}}\boldsymbol{b}$$

$$= (1,\ 1) - (0,\ 0)$$

$$\underline{= (1,\ 1)}$$

Exercise

Find the projection matrix $proj_{\boldsymbol{a}}\boldsymbol{u} = \frac{\boldsymbol{u} \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2}$ onto the line through $a = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Solution

$$a^T a = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 9$$

$$P = \frac{1}{a^T a} a.a^T$$

$$= \frac{1}{9}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix}$$

Exercise

Show that if $\boldsymbol{v}$ is orthogonal to both $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$, then $\boldsymbol{v}$ is orthogonal to $k_1\boldsymbol{w}_1 + k_2\boldsymbol{w}_2$ for all scalars k_1 and k_2.

Solution

$$\begin{aligned}\boldsymbol{v}\cdot\left(k_1\boldsymbol{w}_1+k_2\boldsymbol{w}_2\right)&=\boldsymbol{v}\cdot\left(k_1\boldsymbol{w}_1\right)+\boldsymbol{v}\cdot\left(k_2\boldsymbol{w}_2\right)\\&=k_1\left(\boldsymbol{v}\cdot\boldsymbol{w}_1\right)+k_2\left(\boldsymbol{v}\cdot\boldsymbol{w}_2\right)\\&=k_1(0)+k_2(0)\\&=0\end{aligned}$$

If v is orthogonal to $\boldsymbol{w}_1$ & $\boldsymbol{w}_2$

$\rightarrow \boldsymbol{v}\cdot\boldsymbol{w}_1=\boldsymbol{v}\cdot\boldsymbol{w}_2=0$

Exercise

a) Project the vector $\boldsymbol{v}$ = (3, 4, 4) onto the line through $\boldsymbol{a}$ = (2, 2, 1) and then onto the plane that also contains $\boldsymbol{a}^*=(1,\ 0,\ 0)$.

b) Check that the first error vector $\boldsymbol{v}-\boldsymbol{p}$ is perpendicular to $\boldsymbol{a}$, and the second error vector $\boldsymbol{v}-\boldsymbol{p}^*$ is also perpendicular to $\boldsymbol{a}^*$.

Solution

a) $proj_{\boldsymbol{a}}\boldsymbol{v}=\dfrac{\boldsymbol{v}\bullet\boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$

$$\begin{aligned}&=\frac{3(2)+4(2)+4(1)}{(2)^2+(2)^2+(1)^2}(2,\ 2,\ 1)\\&=\frac{18}{9}(2,\ 2,\ 1)\\&=(4,\ 4,\ 2)\end{aligned}$$

The plane contains the vectors $\boldsymbol{a}$ and $\boldsymbol{a}^*$ and is the column space of $\boldsymbol{A}$.

$$A=\begin{bmatrix}2&1\\2&0\\1&0\end{bmatrix}$$

$$A^TA=\begin{bmatrix}2&2&1\\1&0&0\end{bmatrix}\begin{bmatrix}2&1\\2&0\\1&0\end{bmatrix}=\begin{bmatrix}9&2\\2&1\end{bmatrix}\qquad\left(A^TA\right)^{-1}=\begin{bmatrix}9&2\\2&1\end{bmatrix}^{-1}=\frac{1}{5}\begin{bmatrix}1&-2\\-2&9\end{bmatrix}$$

$$\begin{aligned}P&=A\left(A^TA\right)^{-1}A^T\\&=\frac{1}{5}\begin{bmatrix}2&1\\2&0\\1&0\end{bmatrix}\begin{bmatrix}1&-2\\-2&9\end{bmatrix}\begin{bmatrix}2&2&1\\1&0&0\end{bmatrix}\\&=\begin{bmatrix}1&0&0\\0&.8&.4\\0&.4&.2\end{bmatrix}\end{aligned}$$

b) The error vector: $e = v - p = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

$$ae = \begin{pmatrix} 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = 2(-1) + 2(0) + 1(2) = 0.$$

Therefore, ***e*** is perpendicular to ***a***.

$$p^* = Pv = \begin{pmatrix} 1 & 0 & 0 \\ 0 & .8 & .4 \\ 0 & .4 & .2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4.8 \\ 2.4 \end{pmatrix}$$

The error vector: $e^* = v - p^* = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 4.8 \\ 2.4 \end{pmatrix} = \begin{pmatrix} 0 \\ -.8 \\ 1.6 \end{pmatrix}$

$$a^* e^* = \begin{pmatrix} 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & -.8 & 1.6 \end{pmatrix} = 2(0) + 2(-.8) + 1(1.6) = 0.$$

Therefore, ***e**** is perpendicular to ***a****.

Exercise

Compute the projection matrices $\boldsymbol{aa}^T / \boldsymbol{a}^T\boldsymbol{a}$ onto the lines through $\boldsymbol{a}_1 = (-1,\ 2,\ 2)$ and $\boldsymbol{a}_2 = (2,\ 2,\ -1)$. Multiply those projection matrices and explain why their product P_1P_2 is what it is. Project $\boldsymbol{v} = (1,\ 0,\ 0)$ onto the lines $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and also onto $\boldsymbol{a}_3 = (2,\ -1,\ 2)$. Add up the three projections $p_1 + p_2 + p_3$.

Solution

For $\boldsymbol{a}_1 = (-1,\ 2,\ 2)$

$$a_1 a_1^T = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \begin{pmatrix} -1 & 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}$$

$$a_1^T a_1 = \begin{pmatrix} -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

$$\underline{= 9|}$$

$$P_1 = \frac{aa^T}{a^T a}$$

$$= \frac{1}{9}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}$$

For $\boldsymbol{a}_2 = (2,\ 2,\ -1)$

$$a_2 a_2^T = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}\begin{pmatrix} 2 & 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

$$a_2^T a_2 = \begin{pmatrix} 2 & 2 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$\underline{=9}$$

$$P_2 = \frac{aa^T}{a^T a}$$

$$= \frac{1}{9}\begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

$$P_1 P_2 = \frac{1}{9}\left(\frac{1}{9}\right)\begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}\begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

$$= \frac{1}{81}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{=0}$$

This because a_1 *and* a_2 are perpendicular.

For $\boldsymbol{a}_3 = (2,\ -1,\ 2)$

$$a_3 a_3^T = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}$$

$$a_3{}^T a_3 = \begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

$$\underline{= 9}$$

$$P_3 = \frac{a_3 a_3{}^T}{a_3{}^T a_3}$$

$$= \frac{1}{9} \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}$$

$$p_3 = P_3 v$$

$$= \frac{1}{9} \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ \frac{4}{9} \end{pmatrix}$$

$$p_1 = P_1 v$$

$$= \frac{1}{9} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{9} \\ -\frac{2}{9} \\ -\frac{2}{9} \end{pmatrix}$$

$$p_2 = P_2 v$$

$$= \frac{1}{9}\begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{9} \\ \frac{4}{9} \\ -\frac{2}{9} \end{pmatrix}$$

$$p_1 + p_2 + p_3 = \begin{pmatrix} \frac{1}{9} \\ -\frac{2}{9} \\ -\frac{2}{9} \end{pmatrix} + \begin{pmatrix} \frac{4}{9} \\ \frac{4}{9} \\ -\frac{2}{9} \end{pmatrix} + \begin{pmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ \frac{4}{9} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{= \boldsymbol{v}}$$

The reason is that a_3 is perpendicular to a_1 *and* a_2.

Hence, when you compute the three projections of a vector and add them up you get back to the vector you start with.

Exercise

If $P^2 = P$ show that $(I - P)^2 = I - P$. When P projects onto the column space of A, $I - P$ projects onto the ____.

Solution

$$(I - P)^2 v = (I - P)(I - P)v$$

$$= (I - P)(Iv - Pv)$$
$$= I^2 v - IPv - PIv + P^2 v$$
$$= v - Pv - Pv + P^2 v \qquad P^2 v = Pv \quad \textbf{\textit{By definition}}$$
$$= v - Pv - Pv + Pv$$
$$= v - Pv$$

$$(I - P)^2 \vec{v} = (I - P)\vec{v} \;\Rightarrow\; \boxed{(I - P)^2 = (I - P)}$$

When P projects onto the column space of A, then $I - P$ projects onto the left nullspace.

Because $(I - P)^2 v = (I - P)v$; if $P\boldsymbol{v}$ is in the column space of A, then $v - Pv$ is a vector perpendicular to $C(A)$.

Exercise

What linear combination of $(1,\ 2,\ -1)$ and $(1,\ 0,\ 1)$ is closest to $\vec{v} = (2,\ 1,\ 1)$?

Solution

$$\tfrac{1}{2}(1,\ 2,\ -1) + \tfrac{3}{2}(1,\ 0,\ 1) = (2,\ 1,\ 1)$$

So, this $\boldsymbol{v}$ is actually in the span of the two given vectors.

Exercise

Show that $\vec{u} - \vec{v}$ is orthogonal to $\vec{u} + \vec{v}$ if and only if $\|\vec{u}\| = \|\vec{v}\|$

Solution

Suppose that $\vec{u} - \vec{v}$ is orthogonal to $\vec{u} + \vec{v}$. Then

$$0 = \langle \vec{u} - \vec{v},\ \vec{u} + \vec{v} \rangle$$
$$= (\vec{u} - \vec{v})^T (\vec{u} + \vec{v})$$
$$= \left(\vec{u}^T - \vec{v}^T\right)(\vec{u} + \vec{v})$$
$$= \vec{u}^T \vec{u} + \vec{u}^T \vec{v} - \vec{v}^T \vec{u} - \vec{v}^T \vec{v}$$
$$= \langle \vec{u},\ \vec{u} \rangle + \langle \vec{u},\ \vec{v} \rangle - \langle \vec{v},\ \vec{u} \rangle - \langle \vec{v},\ \vec{v} \rangle \qquad \langle \vec{u},\ \vec{v} \rangle = \langle \vec{v},\ \vec{u} \rangle$$
$$= \langle \vec{u},\ \vec{u} \rangle - \langle \vec{v},\ \vec{v} \rangle$$

So $\langle \vec{u},\ \vec{u} \rangle = \langle \vec{v},\ \vec{v} \rangle$. Therefore, $\|\vec{u}\|^2 = \|\vec{v}\|^2 \;\Rightarrow\; \|\vec{u}\| = \|\vec{v}\|$.

Suppose $\|\vec{u}\| = \|\vec{v}\|$. Then

$$\langle \vec{u} - \vec{v},\ \vec{u} + \vec{v} \rangle = (\vec{u} - \vec{v})^T (\vec{u} + \vec{v})$$
$$= \left(\vec{u}^T - \vec{v}^T\right)(\vec{u} + \vec{v})$$

$$= \vec{u}^T \vec{u} + \vec{u}^T \vec{v} - \vec{v}^T \vec{u} - \vec{v}^T \vec{v}$$

$$= \langle \vec{u}, \vec{u} \rangle + \langle \vec{u}, \vec{v} \rangle - \langle \vec{v}, \vec{u} \rangle - \langle \vec{v}, \vec{v} \rangle \qquad \langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$$

$$= \langle \vec{u}, \vec{u} \rangle - \langle \vec{v}, \vec{v} \rangle$$

$$= \|\vec{u}\|^2 - \|\vec{v}\|^2$$

$$= 0$$

So we can see that $\vec{u} - \vec{v}$ is orthogonal to $\vec{u} + \vec{v}$

We conclude that $\vec{u} - \vec{v}$ is orthogonal to $\vec{u} + \vec{v}$ if and only if $\|\vec{u}\| = \|\vec{v}\|$, as desired.

Exercise

Given $\boldsymbol{u} = (3, \ -1, \ 2)$ $\quad \boldsymbol{v} = (4, \ -1, \ 5)$ $\quad$ *and* $\quad \boldsymbol{w} = (8, \ -7, \ -6)$

a) Find $3\boldsymbol{v} - 4(5\boldsymbol{u} - 6\boldsymbol{w})$

b) Find $\boldsymbol{u} \bullet \boldsymbol{v}$ and then the angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$.

Solution

a)
$$3\boldsymbol{v} - 4(5\boldsymbol{u} - 6\boldsymbol{w}) = 3(4, \ -1, \ 5) - 4(5(3, \ -1, \ 2) - 6(8, \ -7, \ -6))$$
$$= (12, \ -3, \ 15) - 4((15, \ -5, \ 10) - (48, \ -42, \ -36))$$
$$= (12, \ -3, \ 15) - 4(-33, \ 37, \ 46)$$
$$= (12, \ -3, \ 15) - (-132, \ 148, \ 184)$$
$$= (144, \ -151, \ -169)$$

b)
$$\boldsymbol{u} \bullet \boldsymbol{v} = (3, \ -1, \ 2) \bullet (1, \ 1, \ -1)$$
$$= 3 - 1 - 2$$
$$= 0$$

$$\theta = 90°$$

Exercise

Given: $\boldsymbol{u} = (3, \ 1, \ 3)$ $\quad \boldsymbol{v} = (4, \ 1, \ -2)$

a) Compute the projection $\boldsymbol{w}$ of $\boldsymbol{u}$ on $\boldsymbol{v}$

b) Find $\boldsymbol{p} = \boldsymbol{u} - \boldsymbol{w}$ and show that $\boldsymbol{p}$ is perpendicular to $\boldsymbol{v}$.

Solution

a)
$$\boldsymbol{w} = proj_{\boldsymbol{v}} \boldsymbol{u} = \frac{\boldsymbol{u} \bullet \boldsymbol{v}}{\|\boldsymbol{v}\|^2} \boldsymbol{v}$$
$$= \frac{(3, \ 1, \ 3) \bullet (4, \ 1, \ -2)}{4^2 + 1^2 + (-2)^2} (4, \ 1, \ -2)$$

$$=\frac{12+1-6}{21}(4,\ 1,\ -2)$$

$$=\frac{7}{21}(4,\ 1,\ -2)$$

$$=\frac{1}{3}(4,\ 1,\ -2)$$

$$=\left(\frac{4}{3},\ \frac{1}{3},\ -\frac{2}{3}\right)$$

b) $\boldsymbol{p}=(3,\ 1,\ 3)-\left(\frac{4}{3},\ \frac{1}{3},\ -\frac{2}{3}\right)$

$$=\left(\frac{5}{3},\ \frac{2}{3},\ \frac{11}{3}\right)$$

$$\boldsymbol{p}\bullet\boldsymbol{u}=\left(\frac{5}{3},\ \frac{2}{3},\ \frac{11}{3}\right)\bullet(4,\ 1,\ -2)$$

$$=\frac{20}{3}+\frac{2}{3}-\frac{22}{3}$$

$$=0$$

p is perpendicular to ***v***.

Exercise

a) Show that ***v*** = (*a*, *b*) and ***w*** = (–*b, a*) are orthogonal vectors

b) Use the result in part (*a*) to find two vectors that are orthogonal to ***v*** = (2, –3)

c) Find two unit vectors that are orthogonal to (–3, 4)

Solution

a) $\boldsymbol{u}\cdot\boldsymbol{v}=-ab+ba=0$; 2 vectors are orthogonal vectors.

b) ***v*** = (2, –3) $\Rightarrow$ ***w*** = (–3, –2) and ***w*** = (3, 2)

c) (–3, 4) $\Rightarrow u=\dfrac{(-3,\ 4)}{\sqrt{9+16}}=\left(-\frac{3}{5},\ \frac{4}{5}\right)$

$$\boldsymbol{u}_1=\left(\frac{4}{5},\ \frac{3}{5}\right)\quad and \quad \boldsymbol{u}_2=\left(-\frac{4}{5},\ -\frac{3}{5}\right)$$

Exercise

Show that *A*(3, 0, 2), .*B*(4, 3, 0), and *C*(8, 1, –1) are vertices of a right triangle. At which vertex is the right angle?

Solution

$$AB=(4-3,\ 3-0,\ 0-2)=(1,\ 3,\ -2)$$

$$AC=(5,\ 1,\ -3)$$

$BC = (4,\ -2,\ -1)$

$AB \bullet AC = 5 + 3 + 6 = 14$
$AB \bullet BC = 4 - 6 + 2 = 0$
$AC \bullet BC = 20 - 2 + 3 = 21$

The right triangle at point B

Exercise

Establish the identity: $\boldsymbol{u}\cdot\boldsymbol{v} = \frac{1}{4}\|\boldsymbol{u}+\boldsymbol{v}\|^2 - \frac{1}{4}\|\boldsymbol{u}-\boldsymbol{v}\|^2$

Solution

Let $\boldsymbol{u}(u_1,\ u_2,\ \ldots,\ u_n)$ *and* $\boldsymbol{v} = (v_1,\ v_2,\ \ldots,\ v_n)$

$\boldsymbol{u}\cdot\boldsymbol{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n$

$\boldsymbol{u}+\boldsymbol{v} = (u_1+v_1,\ u_2+v_2,\ \ldots, u_n+v_n)$

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 = (u_1+v_1)^2 + (u_2+v_2)^2 + \cdots + (u_n+v_n)^2$$
$$= u_1^2 + v_1^2 + 2u_1v_1 + u_2^2 + v_2^2 + 2u_2v_2 + \cdots + u_2^2 + v_n^2 + 2u_nv_n$$

$\boldsymbol{u}-\boldsymbol{v} = (u_1-v_1,\ u_2-v_2,\ \ldots, u_n-v_n)$

$$\|\boldsymbol{u}-\boldsymbol{v}\|^2 = (u_1-v_1)^2 + (u_2-v_2)^2 + \cdots + (u_n-v_n)^2$$
$$= u_1^2 + v_1^2 - 2u_1v_1 + u_2^2 + v_2^2 - 2u_2v_2 + \cdots + u_2^2 + v_n^2 - 2u_nv_n$$

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 - \|\boldsymbol{u}-\boldsymbol{v}\|^2 = u_1^2 + v_1^2 + 2u_1v_1 + u_2^2 + v_2^2 + 2u_2v_2 + \cdots + u_2^2 + v_n^2 + 2u_nv_n$$
$$-\left(u_1^2 + v_1^2 - 2u_1v_1 + u_2^2 + v_2^2 - 2u_2v_2 + \cdots + u_2^2 + v_n^2 - 2u_nv_n\right)$$
$$= u_1^2 + v_1^2 + 2u_1v_1 + u_2^2 + v_2^2 + 2u_2v_2 + \cdots + u_2^2 + v_n^2 + 2u_nv_n$$
$$- u_1^2 - v_1^2 + 2u_1v_1 - u_2^2 - v_2^2 + 2u_2v_2 - \cdots - u_2^2 - v_n^2 + 2u_nv_n$$
$$= 4u_1v_1 + 4u_2v_2 + \cdots + 4u_nv_n$$

$$\frac{1}{4}\left(\|\boldsymbol{u}+\boldsymbol{v}\|^2 - \|\boldsymbol{u}-\boldsymbol{v}\|^2\right) = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

Therefore; $\boldsymbol{u}\cdot\boldsymbol{v} = \frac{1}{4}\|\boldsymbol{u}+\boldsymbol{v}\|^2 - \frac{1}{4}\|\boldsymbol{u}-\boldsymbol{v}\|^2$ is true.

2nd *method*:

$$\frac{1}{4}\|\boldsymbol{u}+\boldsymbol{v}\|^2 - \frac{1}{4}\|\boldsymbol{u}-\boldsymbol{v}\|^2 = \frac{1}{4}\left[(\boldsymbol{u}+\boldsymbol{v})(\boldsymbol{u}+\boldsymbol{v}) - (\boldsymbol{u}-\boldsymbol{v})(\boldsymbol{u}-\boldsymbol{v})\right]$$

$$=\frac{1}{4}\left[\boldsymbol{uu}+2\boldsymbol{uv}+\boldsymbol{vv}-\left(\boldsymbol{uu}-2\boldsymbol{uv}+\boldsymbol{vv}\right)\right]$$
$$=\frac{1}{4}\left[\boldsymbol{uu}+2\boldsymbol{uv}+\boldsymbol{vv}-\boldsymbol{uu}+2\boldsymbol{uv}-\boldsymbol{vv}\right]$$
$$=\frac{1}{4}(4\boldsymbol{uv})$$
$$=\boldsymbol{u}\bullet\boldsymbol{v}$$

Exercise

Find the Euclidean inner product $\boldsymbol{u}\bullet\boldsymbol{v}$: $\boldsymbol{u}=(-1,\ 1,\ 0,\ 4,\ -3)$ $\boldsymbol{v}=(-2,\ -2,\ 0,\ 2,\ -1)$

Solution

$$\boldsymbol{u}\bullet\boldsymbol{v}=2-2+0+8+3\ \underline{=11}$$

Exercise

Find the Euclidean distance between $\boldsymbol{u}$ and $\boldsymbol{v}$: $\boldsymbol{u}=(3,\ -3,\ -2,\ 0,\ -3)$ $\boldsymbol{v}=(-4,\ 1,\ -1,\ 5,\ 0)$

Solution

$$d(\boldsymbol{u},\ \boldsymbol{v})=\|\boldsymbol{u}-\boldsymbol{v}\|=\sqrt{(u_1-v_1)^2+(u_2-v_2)^2+\ldots+(u_n-v_n)^2}$$
$$=\sqrt{(3+4)^2+(-3-1)^2+(-2+1)^2+(0-5)^2+(-3-0)^2}$$
$$=\sqrt{49+16+1+25+9}$$
$$=\sqrt{100}$$
$$\underline{=10}$$

Exercise

Find for $\vec{v}=2\hat{\boldsymbol{i}}-4\hat{\boldsymbol{j}}+\sqrt{5}\hat{\boldsymbol{k}},\quad \vec{u}=-2\hat{\boldsymbol{i}}+4\hat{\boldsymbol{j}}-\sqrt{5}\hat{\boldsymbol{k}}$

a) $\boldsymbol{v}\bullet\boldsymbol{u},\ |\boldsymbol{v}|,\ |\boldsymbol{u}|$

b) The cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$

c) The scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$

d) The vector $proj_{\boldsymbol{v}}\boldsymbol{u}$

Solution

a) $\boldsymbol{v}\bullet\boldsymbol{u}=\left(2\boldsymbol{i}-4\boldsymbol{j}+\sqrt{5}\boldsymbol{k}\right)\cdot\left(-2\boldsymbol{i}+4\boldsymbol{j}-\sqrt{5}\boldsymbol{k}\right)$

$$=-4-16-5$$
$$\underline{=-25}$$

$$|\boldsymbol{v}| = \sqrt{2^2 + (-4)^2 + \left(\sqrt{5}\right)^2}$$
$$= \sqrt{4+16+5}$$
$$= \sqrt{25}$$
$$\underline{= 5}$$

$$|\boldsymbol{u}| = \sqrt{(-2)^2 + 4^2 + \left(-\sqrt{5}\right)^2}$$
$$= \sqrt{25}$$
$$\underline{= 5}$$

b) $\cos\theta = \dfrac{\boldsymbol{u} \bullet \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$

$$= \frac{-25}{(5)(5)}$$
$$\underline{= -1}$$

c) $|\boldsymbol{u}|\cos\theta = (5)(-1)$

$$\underline{= -5}$$

d) $proj_{\boldsymbol{v}}\boldsymbol{u} = \left(\dfrac{\boldsymbol{u} \bullet \boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\boldsymbol{v}$

$$= \left(\frac{-25}{5^2}\right)\left(2\boldsymbol{i} - 4\boldsymbol{j} + \sqrt{5}\boldsymbol{k}\right)$$
$$= -\left(2\boldsymbol{i} - 4\boldsymbol{j} + \sqrt{5}\boldsymbol{k}\right)$$
$$\underline{= -2\boldsymbol{i} + 4\boldsymbol{j} - \sqrt{5}\boldsymbol{k}}$$

Exercise

Find for $\boldsymbol{v} = \frac{3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{k}, \quad \boldsymbol{u} = 5\boldsymbol{i} + 12\boldsymbol{j}$

a) $\boldsymbol{v} \bullet \boldsymbol{u},\ |\boldsymbol{v}|,\ |\boldsymbol{u}|$

b) The cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$

c) The scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$

d) The vector $proj_{\boldsymbol{v}}\boldsymbol{u}$

Solution

a) $\boldsymbol{v} \bullet \boldsymbol{u} = \left(\frac{3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{k}\right) \cdot (5\boldsymbol{i} + 12\boldsymbol{j})$

$$\underline{= 3}$$

$$|\boldsymbol{v}| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$$

$$= \sqrt{\frac{9}{25} + \frac{16}{25}}$$

$$= \sqrt{\frac{25}{25}}$$

$$\underline{= 1}$$

$$|\boldsymbol{u}| = \sqrt{5^2 + 12^2}$$

$$\underline{= 13}$$

b) $\cos\theta = \dfrac{\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$

$$= \frac{3}{(1)(13)}$$

$$= \underline{\frac{3}{13}}$$

c) $|\boldsymbol{u}|\cos\theta = (13)\left(\frac{3}{13}\right) = \underline{3}$

d) $proj_{\boldsymbol{v}}\,\boldsymbol{u} = \left(\dfrac{\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\boldsymbol{v}$

$$= \left(\frac{3}{1^2}\right)\left(\frac{3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{k}\right)$$

$$\underline{= \frac{9}{5}\hat{i} + \frac{12}{5}\hat{k}}$$

Exercise

Find for $\boldsymbol{v} = 2\boldsymbol{i} + 10\boldsymbol{j} - 11\boldsymbol{k}$, $\quad \boldsymbol{u} = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

a) $\boldsymbol{v}\cdot\boldsymbol{u}$, $|\boldsymbol{v}|$, $|\boldsymbol{u}|$

b) The cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$

c) The scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$

d) The vector $proj_{\boldsymbol{v}}\,\boldsymbol{u}$

Solution

a) $\boldsymbol{v}\cdot\boldsymbol{u} = \left(2\hat{i} + 10\hat{j} - 11\hat{k}\right)\cdot\left(2\hat{i} + 2\hat{j} + \hat{k}\right)$

$$= 4 + 20 - 11$$

$$\underline{= 13}$$

$$|\boldsymbol{v}| = \sqrt{2^2 + 10^2 + (-11)^2}$$
$$= \sqrt{4 + 100 + 121}$$
$$= \sqrt{225}$$
$$\underline{= 15|}$$

$$|\boldsymbol{u}| = \sqrt{2^2 + 2^2 + 1^2}$$
$$\underline{= 3|}$$

b) $\cos\theta = \dfrac{\boldsymbol{u} \bullet \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$

$$= \frac{13}{(3)(15)}$$
$$= \frac{13}{45}$$

c) $|\boldsymbol{u}|\cos\theta = (3)\left(\dfrac{13}{45}\right) = \dfrac{13}{15}$

d) $proj_{\boldsymbol{v}}\boldsymbol{u} = \left(\dfrac{\boldsymbol{u} \bullet \boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\boldsymbol{v}$

$$= \left(\frac{13}{15^2}\right)(2\boldsymbol{i} + 10\boldsymbol{j} - 11\boldsymbol{k})$$
$$= \frac{13}{225}(2\boldsymbol{i} + 10\boldsymbol{j} - 11\boldsymbol{k})$$

Exercise

Find for $\boldsymbol{v} = 5\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}, \quad \boldsymbol{u} = 2\hat{\boldsymbol{i}} + \sqrt{17}\,\hat{\boldsymbol{j}}$

a) $\boldsymbol{v} \bullet \boldsymbol{u}$, $|\boldsymbol{v}|$, $|\boldsymbol{u}|$

b) The cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$

c) The scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$

d) The vector $proj_{\boldsymbol{v}}\boldsymbol{u}$

Solution

a) $\boldsymbol{v} \bullet \boldsymbol{u} = \left(5\hat{i} + \hat{j}\right) \cdot \left(2\hat{i} + \sqrt{17}\,\hat{j}\right)$

$$\boldsymbol{v} \bullet \boldsymbol{u} = (5\boldsymbol{i} + \boldsymbol{j}) \cdot \left(2\boldsymbol{i} + \sqrt{17}\,\boldsymbol{j}\right) \underline{= 10 + \sqrt{17}}$$

$$|\boldsymbol{v}| = \sqrt{25 + 1} \underline{= \sqrt{26}}$$

$$|\boldsymbol{u}| = \sqrt{4 + 17} \underline{= \sqrt{21}}$$

b) $\cos\theta = \dfrac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$

$$= \frac{10+\sqrt{17}}{\sqrt{21}\sqrt{26}}$$

$$\boxed{= \frac{10+\sqrt{17}}{\sqrt{546}}}$$

c) $|\boldsymbol{u}|\cos\theta = \left(\sqrt{21}\right)\left(\dfrac{10+\sqrt{17}}{\sqrt{546}}\right)$

$$\boxed{= \frac{10+\sqrt{17}}{\sqrt{26}}}$$

d) $proj_{\boldsymbol{v}}\boldsymbol{u} = \left(\dfrac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\boldsymbol{v}$

$$\boxed{= \left(\frac{10+\sqrt{17}}{26}\right)(5\boldsymbol{i}+\boldsymbol{j})}$$

Exercise

Find for $\vec{v} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}\right)$, $\vec{u} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}}\right)$

a) $\boldsymbol{v}\bullet\boldsymbol{u}$, $|\boldsymbol{v}|$, $|\boldsymbol{u}|$

b) The cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$

c) The scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$

d) The vector $proj_{\boldsymbol{v}}\boldsymbol{u}$

Solution

a) $\boldsymbol{v}\bullet\boldsymbol{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}\right)\bullet\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}}\right) = \frac{1}{2}-\frac{1}{3}\ \boxed{=\frac{1}{6}}$

$$|\boldsymbol{v}| = \sqrt{\frac{1}{2}+\frac{1}{3}} = \frac{\sqrt{5}}{\sqrt{6}}\frac{\sqrt{6}}{\sqrt{6}}\ \boxed{=\frac{\sqrt{30}}{6}}$$

$$|\boldsymbol{u}| = \sqrt{\frac{1}{2}+\frac{1}{3}} = \frac{\sqrt{5}}{\sqrt{6}}\frac{\sqrt{6}}{\sqrt{6}}\ \boxed{=\frac{\sqrt{30}}{6}}$$

b) $\cos\theta = \dfrac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$

$$= \frac{\frac{1}{6}}{\frac{\sqrt{30}}{6}\frac{\sqrt{30}}{6}}$$

$$= \frac{1}{6}\left(\frac{36}{30}\right)$$

$$= \frac{1}{5}$$

c) $|\boldsymbol{u}|\cos\theta = \left(\frac{\sqrt{30}}{6}\right)\left(\frac{1}{5}\right)$

$$= \frac{\sqrt{30}}{30}$$

$$= \frac{1}{\sqrt{30}}$$

d) $proj_{\boldsymbol{v}}\boldsymbol{u} = \left(\frac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\boldsymbol{v}$

$$= \frac{1}{6}\left(\frac{36}{30}\right)\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{5}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}}\right)$$

Exercise

Suppose Ted weighs 180 *lb*. and he is sitting on an inclined plane that drops 3 *units* for every 4 horizontal units. The gravitational force vector is $\vec{F}_g = \begin{pmatrix} 0 \\ -180 \end{pmatrix}$.

a) Find the force pushing Ted down the slope.
b) Find the force acting to hold Ted against the slope

Solution

A vector parallel to the slope of the inclined plane is $\vec{v} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$.

a) The vector of the force acting to push Ted down the slope is

$$\vec{F}_s = \frac{\vec{v}\bullet\vec{F}_g}{|\vec{v}|^2}\vec{v}$$

$$= \frac{(4,\ -3)\bullet(0,\ -180)}{16+9}(4,\ -3)$$

$$= \frac{540}{25}(4,\ -3)$$

$$= \left(\frac{432}{5}, \ -\frac{324}{5} \right)$$

The magnitude of the force pushing Ted down the slope is

$$\left\| \vec{F}_s \right\| = \sqrt{\left(\frac{432}{5}\right)^2 + \left(\frac{324}{5}\right)^2}$$

$$= \frac{540}{5}$$

$$= 108 \ \ lb$$

b) The vector of the force acting to hold Ted against the slope is

$$\vec{F}_p = \vec{F}_g - \vec{F}_s$$

$$= \begin{pmatrix} 0 \\ -180 \end{pmatrix} - \begin{pmatrix} \frac{432}{5} \\ -\frac{324}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{432}{5} \\ -\frac{576}{5} \end{pmatrix}$$

$$\left\| \vec{F}_p \right\| = \sqrt{\left(\frac{432}{5}\right)^2 + \left(\frac{576}{5}\right)^2}$$

$$= \frac{720}{5}$$

$$= 144 \ \ lb$$

Exercise

Prove that is two vectors $\vec{u}$ and $\vec{v}$ in R^2 are orthogonal to nonzero vector $\vec{w}$ in R^2, then $\vec{u}$ and $\vec{v}$ are scalar multiples of each other.

Solution

Since $\vec{u}$ is orthogonal to $\vec{w} \ \rightarrow \ \vec{u} \bullet \vec{w} = 0$

$\vec{v}$ is orthogonal to $\vec{w} \ \rightarrow \ \vec{v} \bullet \vec{w} = 0$

$\Rightarrow \ \vec{u} \bullet \vec{w} = \vec{v} \bullet \vec{w} = 0$

There exist $a \in \mathbb{R}$ such that $(a\vec{v}) \bullet \vec{w} = a(\vec{v} \bullet \vec{w}) = 0$

$\vec{u} = a\vec{v} \quad \vec{u} \bullet \vec{w} = \vec{v} \bullet \vec{w} = 0 = (a\vec{v}) \bullet \vec{w}$

Therefore, $\vec{u}$ and $\vec{v}$ are scalar multiples of each other