Solution *Section* 1.1 – Differential Equations & Solutions

Exercise

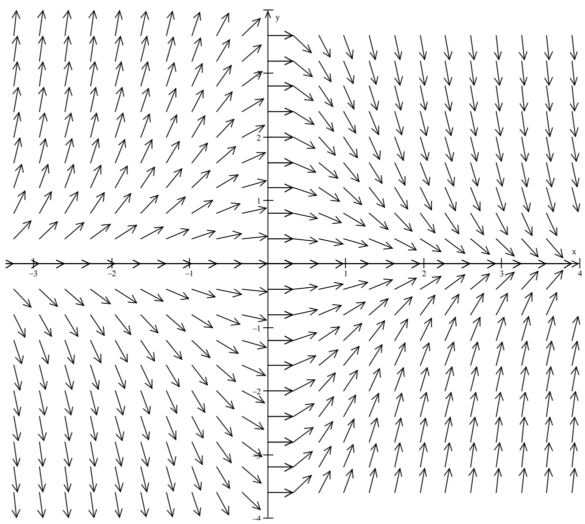
Show that $y(t) = Ce^{-(1/2)t^2}$ is a solution of the 1st order equation $y' = -ty \quad for \quad -3 \le C \le 3$

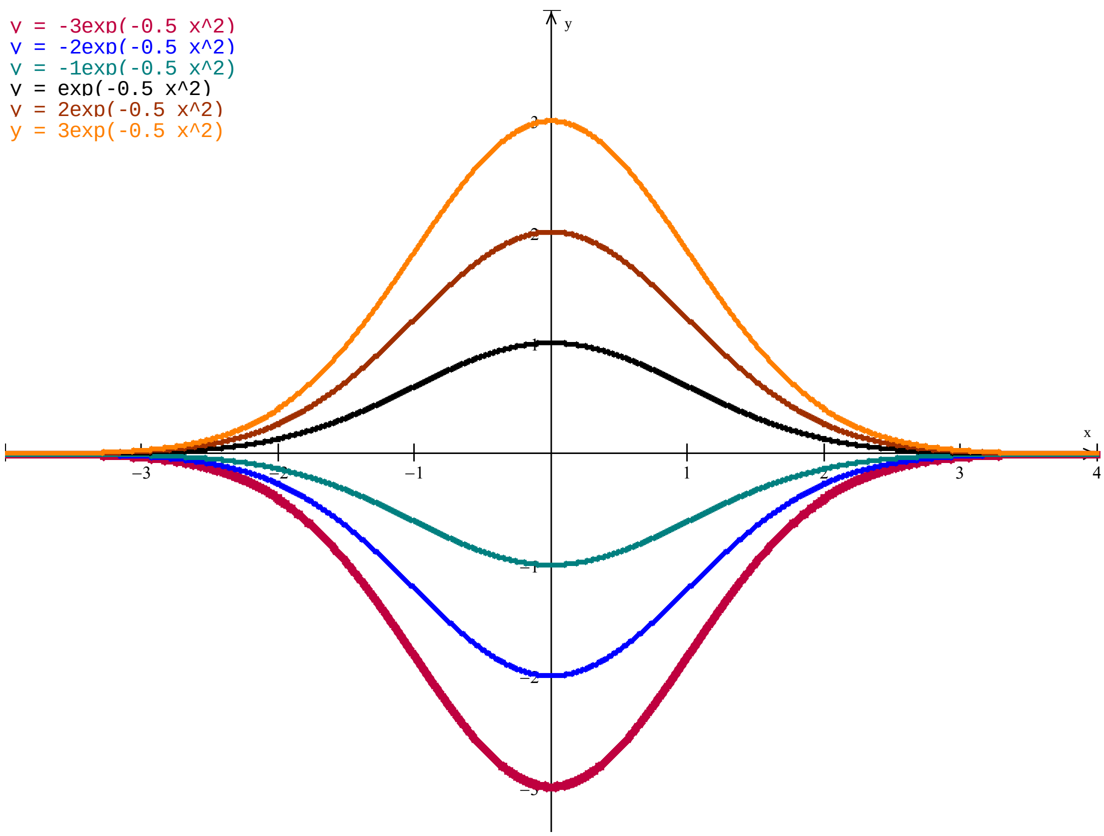
Solution

$$y' = -\frac{1}{2}2tCe^{-(1/2)t^2}$$

$$= -tCe^{-(1/2)t^2}$$

$$\underline{= -ty}$$





Exercise

Show that $y(t)=\dfrac{4}{1+Ce^{-4t}}$ is a solution of the 1st order equation $y'=y(4-y)$

Solution

$$y'=\frac{d}{dt}\left(\frac{4}{1+Ce^{-4t}}\right)$$

$$=\frac{-4\left(Ce^{-4t}\right)'}{\left(1+Ce^{-4t}\right)^2}$$

$$=\frac{16Ce^{-4t}}{\left(1+Ce^{-4t}\right)^2}$$

$$=\frac{A}{1+Ce^{-4t}}+\frac{B}{\left(1+Ce^{-4t}\right)^2}$$

$$=\frac{A+ACe^{-4t}+B}{\left(1+Ce^{-4t}\right)^2}$$

$$\Rightarrow\begin{cases} A=16 \\ A+B=0\to B=-16\end{cases}$$

$$=\frac{16}{1+Ce^{-4t}}-\frac{16}{\left(1+Ce^{-4t}\right)^2}$$

$$=\frac{4}{1+Ce^{-4t}}\left(4-\frac{4}{1+Ce^{-4t}}\right)$$

$$=y(4-y)$$

$$y(4-y)=$$

$$=\frac{4}{1+Ce^{-4t}}\left[4-\frac{4}{1+Ce^{-4t}}\right]$$

$$=\frac{4}{1+Ce^{-4t}}\left[\frac{4+4Ce^{-4t}-4}{1+Ce^{-4t}}\right]$$

$$=\frac{4}{1+Ce^{-4t}}\left[\frac{4Ce^{-4t}}{1+Ce^{-4t}}\right]$$

$$=\frac{16Ce^{-4t}}{\left(1+Ce^{-4t}\right)^2}$$

Exercise

A general solution may fail to produce all solutions of a differential equation $y(t)=\dfrac{4}{1+Ce^{-4t}}$. Show that $y=0$ is a solution of the differential equation, but no value of C in the given general solution will produce this solution.

Solution

$$y(t)=0\Rightarrow y'=0$$

$$y(4-y)=0(4-0)=0$$

Exercise

Use the given general solution to find a solution of the differential equation having the given initial condition. $ty' + y = t^2, \quad y(t) = \frac{1}{3}t^2 + \frac{C}{t}, \qquad y(1) = 2$

Solution

$$y(1) = 2$$

$$y(1) = \frac{1}{3}(1)^2 + \frac{C}{1}$$

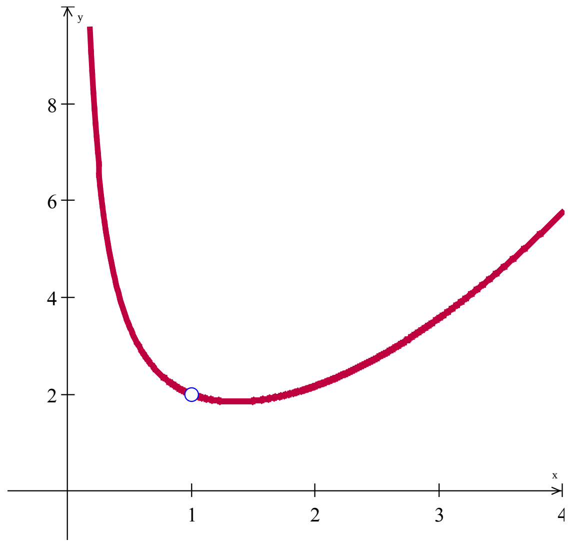
$$2 = \frac{1}{3} + C$$

$$C = 2 - \frac{1}{3}$$

$$= \frac{5}{3}$$

$$\boxed{y(t) = \frac{1}{3}t^2 + \frac{5}{3t}}$$

The interval of existence is $(0, \infty)$



Exercise

Show that $y(t) = 2t - 2 + Ce^{-t}$ is a solution of the 1st order equation $y' + y = 2t \ \ for \quad -3 \le C \le 3$

Solution

$$y' + y = \left(2t - 2 + Ce^{-t}\right)' + 2t - 2 + Ce^{-t}$$

$$= 2 - Ce^{-t} + 2t - 2 + Ce^{-t}$$

$$= 2t \quad \checkmark$$

Exercise

Use the given general solution to find a solution of the differential equation having the given initial condition. $y' + 4y = \cos t, \quad y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t + Ce^{-4t}, \qquad y(0) = -1$

Solution

$$y(0) = \frac{4}{17}\cos(0) + \frac{1}{17}\sin(0) + Ce^{-4(0)}$$

$$-1 = \frac{4}{17} + C$$

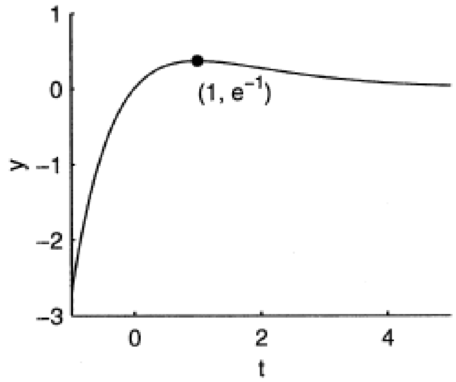
$$C = -1 - \frac{4}{17} = -\frac{21}{17}$$

$$\boxed{y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t - \frac{21}{17}e^{-4t}}$$

Exercise

Use the given general solution to find a solution of the differential equation having the given initial condition. $ty' + (t+1)y = 2te^{-t}, \quad y(t) = e^{-t}\left(t + \frac{C}{t}\right), \quad y(1) = \frac{1}{e}$

Solution



$$y(1) = \frac{1}{e} = e^{-1}$$

$$y(1) = e^{-1}\left(1 + \frac{C}{1}\right)$$

$$e^{-1} = e^{-1}(1+C)$$

$$1 = 1 + C$$

Hence, $C = 0$

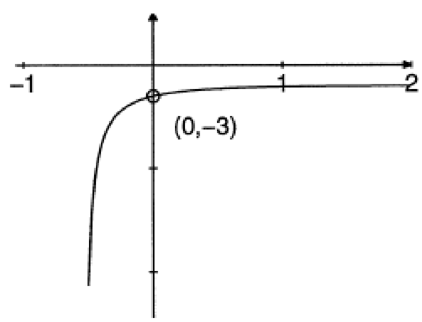
The solution is: $y(t) = te^{-t}$

This function is defined and differentiable on the whole real line. Hence, the interval of existence is the whole real line.

Exercise

Use the given general solution to find a solution of the differential equation having the given initial condition. $y' = y(2+y), \quad y(t) = \frac{2}{-1+Ce^{-2t}}, \quad y(0) = -3$

Solution



$$y(0) = \frac{2}{-1+Ce^{-2(0)}}$$

$$-3 = \frac{2}{-1+C}$$

$$3 - 3C = 2$$

$$-3C = -1$$

$$\boxed{C = \frac{1}{3}}$$

The solution is:

$$y(t) = \frac{2}{-1+\frac{1}{3}e^{-2t}}$$

$$= \frac{6}{-3+e^{-2t}}$$

Exercise

Find the values of $\boldsymbol{m}$ so that the function $y = e^{mx}$ is a solution of the given differential equation

a) $y' + 2y = 0$

b) $5y' - 2y = 0$

c) $y'' - 5y' + 6y = 0$

d) $2y'' + 7y' - 4y = 0$

Solution

$$y = e^{mx} \quad \Rightarrow \quad y' = me^{mx} \quad \Rightarrow \quad y'' = m^2 e^{mx}$$

a) $y' + 2y = 0$

$$me^{mx} + 2e^{mx} = 0 \quad \Rightarrow \quad (m+2)e^{mx} = 0$$

$$\boxed{m = -2}$$

b) $5y' - 2y = 0$

$$5me^{mx} - 2e^{mx} = 0 \quad \Rightarrow \quad (5m-2)e^{mx} = 0$$

$$\boxed{m = \frac{2}{5}}$$

c) $y'' - 5y' + 6y = 0$

$$m^2 e^{mx} - 5me^{mx} + 6e^{mx} = 0 \quad \Rightarrow \quad \left(m^2 - 5m + 6\right)e^{mx} = 0$$

$$\boxed{m = 2, \ \ 3}$$

d) $2y'' + 7y' - 4y = 0$

$$2m^2 e^{mx} + 7me^{mx} - 4e^{mx} = 0 \quad \Rightarrow \quad \left(2m^2 + 7m - 4\right)e^{mx} = 0$$

$$\boxed{m = \frac{1}{2}, \ \ -4}$$

Exercise

Let $x = c_1 \cos t + c_2 \sin t$ is 2-parameter family solutions of the second order differential equation of $x'' + x = 0$. Find a solution of the second-order consisting of this differential equation and the given initial conditions.

a) $x(0) = -1, \quad x'(0) = 8$

b) $x\left(\frac{\pi}{2}\right) = 0, \quad x'\left(\frac{\pi}{2}\right) = 1$

c) $x\left(\frac{\pi}{6}\right) = \frac{1}{2}, \quad x'\left(\frac{\pi}{6}\right) = 0$

d) $x\left(\frac{\pi}{4}\right) = \sqrt{2}, \quad x'\left(\frac{\pi}{4}\right) = 2\sqrt{2}$

Solution

$$x = c_1 \cos t + c_2 \sin t \quad \Rightarrow \quad x' = -c_1 \sin t + c_2 \cos t$$

a) $x(0) = -1 \quad \Rightarrow \quad \underline{-1 = c_1}$

$$x'(0) = 8 \quad \Rightarrow \quad \boxed{8 = c_2}$$

b) $$x\left(\frac{\pi}{2}\right) = 0 \quad \Rightarrow \quad \boxed{0 = c_2}$$

$$x'\left(\frac{\pi}{2}\right) = 1 \quad \Rightarrow \quad \boxed{-1 = c_1}$$

c) $$x\left(\frac{\pi}{6}\right) = \frac{1}{2} \quad \Rightarrow \quad \frac{\sqrt{3}}{2}c_1 + \frac{1}{2}c_2 = \frac{1}{2} \quad \to \quad \sqrt{3}\,c_1 + c_2 = 1$$

$$x'\left(\frac{\pi}{6}\right) = 0 \quad \Rightarrow \quad -\frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2 = 0 \quad \to \quad -c_1 + \sqrt{3}\,c_2 = 0$$

$$\boxed{c_1 = \frac{\sqrt{3}}{4}}, \quad \boxed{c_2 = \frac{1}{4}}$$

d) $$x\left(\frac{\pi}{4}\right) = \sqrt{2} \quad \Rightarrow \quad \frac{\sqrt{2}}{2}c_1 + \frac{\sqrt{2}}{2}c_2 = \sqrt{2} \quad \to \quad c_1 + c_2 = 2$$

$$x'\left(\frac{\pi}{4}\right) = 2\sqrt{2} \quad \Rightarrow \quad \frac{\sqrt{2}}{2}c_1 + \frac{\sqrt{2}}{2}c_2 = 2\sqrt{2} \quad \to \quad -c_1 + c_2 = 4$$

$$\boxed{c_1 = -1} \quad \boxed{c_2 = 3}$$

Solution Section 1.2 – Solutions to Separable Equations

Exercise

Find the general solution of the differential equation $y' = xy$

Solution

$$\frac{dy}{dx} = xy$$

$$\frac{dy}{y} = x\,dx$$

$$\int \frac{dy}{y} = \int x\,dx$$

$$\ln|y| = \frac{1}{2}x^2 + C$$

$$|y| = e^{x^2/2+C}$$

$$y(x) = \pm e^{x^2/2}e^C$$

$$= Ae^{x^2/2} \qquad \text{Where } A = \pm e^C$$

Exercise

Find the general solution of the differential equation $xy' = 2y$

Solution

$$x\frac{dy}{dx} = 2y$$

$$\frac{dy}{y} = 2\frac{dx}{x}$$

$$\int \frac{dy}{y} = \int \frac{2}{x}dx$$

$$\ln|y| = 2\ln|x| + C$$

$$= \ln x^2 + C$$

$$y(x) = \pm e^{\ln x^2 + C}$$

$$= \pm e^C x^2$$

$$= Ax^2$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = e^{x-y}$

Solution

$$\frac{dy}{dx} = e^x e^{-y}$$

$$\frac{dy}{e^{-y}} = e^x dx$$

$$\int e^y dy = \int e^x dx$$

$$e^y = e^x + C$$

$$\boxed{y(x) = \ln\left(e^x + C\right)}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \left(1 + y^2\right)e^x$

Solution

$$\frac{dy}{dx} = \left(1 + y^2\right)e^x$$

$$\frac{dy}{1 + y^2} = e^x dx$$

$$\int \frac{dy}{1 + y^2} = \int e^x dx$$

$$\tan^{-1} y = e^x + C$$

$$\boxed{y(x) = \tan\left(e^x + C\right)}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = xy + y$

Solution

$$\frac{dy}{dx} = (x+1)y$$

$$\frac{dy}{y} = (x+1)dx$$

$$\int \frac{dy}{y} = \int (x+1)dx$$

$$\ln y = \frac{1}{2}x^2 + x + C$$

$$\boxed{y = e^{x^2/2+x+C}}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$y' = ye^x - 2e^x + y - 2$

Solution

$$\frac{dy}{dx} = (y-2)e^x + y - 2$$

$$\frac{dy}{dx} = (y-2)\left(e^x + 1\right)$$

$$\frac{dy}{y-2} = \left(e^x + 1\right)dx$$

$$\int \frac{dy}{y-2} = \int \left(e^x + 1\right)dx$$

$$\ln|y-2| = e^x + x + C$$

$$y - 2 = \pm e^{e^x + x + C}$$

$$y - 2 = \pm e^C e^{e^x + x}$$

$$\boxed{y = De^{e^x + x} + 2} \quad D = \pm e^C$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \dfrac{x}{y+2}$

Solution

$$\frac{dy}{dx} = \frac{x}{y+2}$$

$$(y+2)dy = xdx$$

$$\int (y+2)dy = \int xdx$$

$$\tfrac{1}{2}y^2 + 2y = \tfrac{1}{2}x^2 + C$$

$$y^2 + 4y = x^2 + 2C$$

$$y^2 + 4y - x^2 - D = 0, \quad (D = 2C)$$

$$y = \frac{-4 \pm \sqrt{16 - 4\left(-x^2 - D\right)}}{2}$$

$$= \frac{-4 \pm \sqrt{16 + 4x^2 + 4D}}{2}$$

$$= \frac{-4 \pm 2\sqrt{x^2 + (4+D)}}{2} \qquad E = 4 + D$$

$$= -2 \pm \sqrt{x^2 + E}$$

$$\boxed{y(x) = -2 \pm \sqrt{x^2 + E}}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \dfrac{xy}{x-1}$

Solution

$$\frac{dy}{dx} = y\left(\frac{x}{x-1}\right)$$

$$\frac{dy}{y} = \left(\frac{x}{x-1}\right)dx$$

$$\int \frac{dy}{y} = \int \left(1 + \frac{1}{x-1}\right)dx$$

$$\ln|y| = x + \ln|x-1| + C$$

$$y(x) = \pm e^{x+\ln|x-1|+C}$$

$$= \pm e^{C} e^{x} e^{\ln|x-1|}$$

$$\underline{= De^{x}|x-1|}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$$y' = \frac{y^2 + ty + t^2}{t^2}$$

Solution

$$y' = \frac{y^2}{t^2} + \frac{y}{t} + 1$$

Let $x = \frac{y}{t} \quad \Rightarrow \quad y = xt \ \rightarrow \ y' = x + tx'$

$$y' = \frac{y^2}{t^2} + \frac{y}{t} + 1 = x^2 + x + 1$$

$$x + tx' = x^2 + x + 1$$

$$t\frac{dx}{dt} = x^2 + 1$$

$$\frac{dx}{x^2+1} = \frac{dt}{t}$$

$$\int \frac{dx}{x^2+1} = \int \frac{dt}{t}$$

$$\tan^{-1} x = \ln|t| + C$$

$$\tan^{-1} \frac{y}{t} = \ln|t| + C$$

$$\frac{y}{t} = \tan\left(\ln|t| + C\right)$$

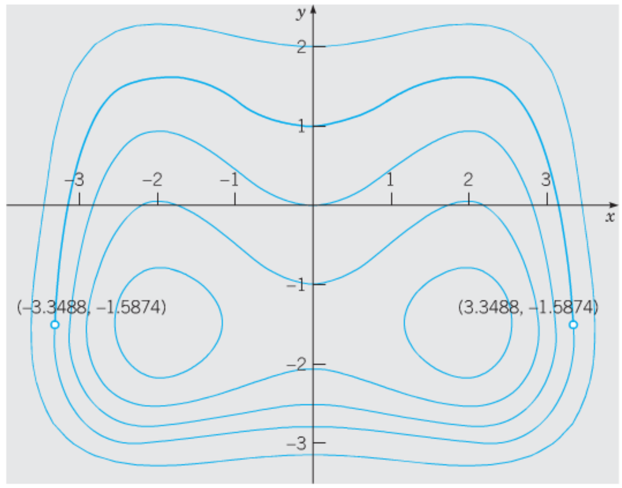
$$\boxed{y = t\tan\left(\ln|t| + C\right)}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}$$

Solution



$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}$$

$$\left(4 + y^3\right)dy = \left(4x - x^3\right)dx$$

$$\int \left(4 + y^3\right)dy = \int \left(4x - x^3\right)dx$$

$$4y + \frac{1}{4}y^4 = 2x^2 - \frac{1}{4}x^4 + C_1$$

$$16y + y^4 = 8x^2 - x^4 + C$$

$$y^4 + 16y + x^4 - 8x^2 = +C$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$$y' = \frac{2xy + 2x}{x^2 - 1}$$

Solution

$$\frac{dy}{dx} = \frac{2x(y+1)}{x^2 - 1}$$

$$\frac{dy}{y+1} = \frac{2x}{x^2 - 1}dx \qquad d\left(x^2 - 1\right) = 2x\,dx$$

$$\int \frac{d(y+1)}{y+1} = \int \frac{d\left(x^2 - 1\right)}{x^2 - 1}$$

$$\ln|y+1| = \ln\left|x^2 - 1\right| + C$$

$$y + 1 = e^{\ln\left|x^2 - 1\right| + C}$$

$$y = e^C e^{\ln\left|x^2 - 1\right|} - 1$$

$$y(x) = Ae^{\ln\left|x^2 - 1\right|} - 1$$

Exercise

Find the exact solution of the initial value problem. Indicate the interval of existence.

$$y' = \frac{y}{x}, \qquad y(1) = -2$$

Solution

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln|y| = \ln|x| + C$$

$$\begin{aligned} y &= \pm e^{\ln|x| + C} \\ &= \pm e^{C} e^{\ln|x|} \\ &= D|x| \\ &= Dx \end{aligned}$$

$$y = Dx \quad \Rightarrow \quad D = \frac{y}{x} = \frac{-2}{1} = -2$$

$$\boxed{y = -2x}$$

Exercise

Find the exact solution of the initial value problem. Indicate the interval of existence.

$$y' = -\frac{2t\left(1 + y^2\right)}{y}, \qquad y(0) = 1$$

Solution

$$\frac{dy}{dt} = -\frac{2t\left(1 + y^2\right)}{y}$$

$$\frac{ydy}{1 + y^2} = -2tdt$$

$$\int \frac{ydy}{1 + y^2} = \int -2tdt$$

$$\frac{1}{2}\int \frac{1}{1 + y^2} d\left(1 + y^2\right) = -2\int tdt$$

$$\frac{1}{2}\ln\left(1+y^2\right)=-t^2+C$$

$$\ln\left(1+y^2\right)=-2t^2+2C$$

$$1+y^2=e^{-2t^2+2C}$$

$$1+y^2=e^{2C}e^{-2t^2}$$

$$1+y^2=De^{-2t^2}$$

$$1+1^2=De^{-2(0)^2}$$

$$\boxed{2=D}$$

$$y^2=2e^{-2t^2}-1$$

$$y^2=2e^{-2t^2}-1$$

$$y=\pm\sqrt{2e^{-2t^2}-1}$$

$$\boxed{y(x)=\sqrt{2e^{-2t^2}-1}}$$

$$2e^{-2t^2}-1>0$$

$$2e^{-2t^2}>1$$

$$e^{-2t^2}>\frac{1}{2}$$

$$-2t^2>\ln\left(\frac{1}{2}\right)$$

$$t^2<-\frac{1}{2}\ln\left(\frac{1}{2}\right)=\frac{1}{2}\ln 2$$

$$t^2<\ln\sqrt{2}$$

$$t<\left|\ln\sqrt{2}\right|$$

The interval of existence: $\left(-\ln\sqrt{2},\ \ln\sqrt{2}\right)$

Exercise

Find the exact solution of the initial value problem. Indicate the interval of existence.

$y' = \dfrac{\sin x}{y}, \quad y\left(\frac{\pi}{2}\right) = 1$

Solution

$$\frac{dy}{dx} = \frac{\sin x}{y}$$

$$ydy = \sin x dx$$

$$\int ydy = \int \sin x dx$$

$$\frac{1}{2}y^2 = -\cos x + C_1$$

$$y^2 = -2\cos x + C \quad \left(C = 2C_1\right)$$

$$y(x) = \pm\sqrt{-2\cos x + C}$$

$$y\left(\frac{\pi}{2}\right) = \sqrt{-2\cos\frac{\pi}{2} + C}$$

$$1 = \sqrt{C} \quad \Rightarrow \quad \boxed{C = 1}$$

$$y(x) = \sqrt{1 - 2\cos x}$$

The interval of existence will be the interval containing $\frac{\pi}{2}$ and $1 - 2\cos x > 0$

$$\cos x < \frac{1}{2} \quad \Rightarrow \quad \boxed{\frac{\pi}{3} < x < \frac{5\pi}{3}}$$

Exercise

Find the exact solution of the initial value problem. Indicate the interval of existence.

$$4tdy = \left(y^2 + ty^2\right)dt, \quad y(1) = 1$$

Solution

$$4tdy = y^2(1+t)dt$$

$$4\frac{dy}{y^2} = \frac{1+t}{t}dt$$

$$4\int \frac{dy}{y^2} = \int\left(\frac{1}{t} + 1\right)dt \qquad\qquad \int \frac{dx}{x^2} = -\frac{1}{x}$$

$$-\frac{4}{y} = \ln|t| + t + C$$

$$y = \frac{-4}{\ln|t| + t + C}$$

$$1 = \frac{-4}{\ln|1| + 1 + C} \quad \Rightarrow \quad 1 = \frac{-4}{1+C} \ \rightarrow \ 1 + C = -4 \ \Rightarrow \ \boxed{C = -5}$$

$$y = \frac{-4}{\ln|t| + t - 5}$$

Exercise

Find the exact solution of the initial value problem. Indicate the interval of existence.

$$y' = \frac{1-2t}{y}, \quad y(1) = -2$$

Solution

$$y\frac{dy}{dt} = 1 - 2t$$

$$\int y\,dy = \int (1-2t)\,dt$$

$$\tfrac{1}{2}y^2 = t - t^2 + C_1$$

$$y^2 = 2t - 2t^2 + C$$

$$(-2)^2 = 2(1) - 2(1)^2 + C \quad \Rightarrow \quad \boxed{C = 4}$$

$$y = -\sqrt{2t - 2t^2 + 4}$$

The negative value is taken to satisfy the initial condition.

Exercise

Find the exact solution of the initial value problem. Indicate the interval of existence.

$$y' = y^2 - 4, \quad y(0) = 0$$

Solution

$$\frac{dy}{dt} = y^2 - 4$$

$$\frac{dy}{y^2 - 4} = dt \qquad\qquad \frac{1}{y^2-4} = \frac{A}{y-2} + \frac{B}{y+2}$$

$$\frac{1}{y^2-4} = \frac{(A+B)y + 2A - 2B}{y-2}$$

$$\to \begin{cases} A + B = 0 \\ 2A - 2B = 1 \end{cases} \quad \Rightarrow A = \frac{1}{4} \quad B = -\frac{1}{4}$$

$$\left(\frac{1}{4(y-2)} - \frac{1}{4(y+2)}\right) dy = dt$$

$$\int \left(\frac{1}{4(y-2)} - \frac{1}{4(y+2)}\right) dy = \int dt$$

$$\frac{1}{4}\left(\ln|y-2| - \ln|y+2|\right) = t + C$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4t + C$$

$$\frac{y-2}{y+2} = \pm e^{4t+C}$$

$$\frac{y-2}{y+2} = \pm e^{C} e^{4t} = ke^{4t}$$

$$y - 2 = ke^{4t} y + 2ke^{4t}$$

$$y - ke^{4t} y = 2 + 2ke^{4t}$$

$$y\left(1 - ke^{4t}\right) = 2 + 2ke^{4t}$$

$$y = \frac{2 + 2ke^{4t}}{1 - ke^{4t}}$$

$$0 = \frac{2 + 2ke^{4(0)}}{1 - ke^{4(0)}}$$

$$0 = 2 + 2k \quad \Rightarrow \quad \boxed{k = -1}$$

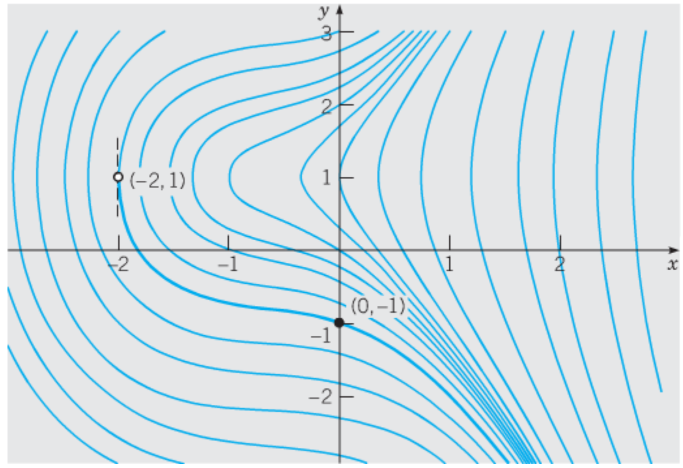
$$y = \frac{2 - 2e^{4t}}{1 + e^{4t}}$$

Exercise

Find the exact solution of the initial value problem. Indicate the interval of existence.

$$\frac{dy}{dx}=\frac{3x^2+4x+2}{2(y-1)}, \quad y(0)=-1$$

Solution



$$\frac{dy}{dx}=\frac{3x^2+4x+2}{2y-2}$$

$$(2y-2)dy=\left(3x^2+4x+2\right)dx$$

$$\int(2y-2)dy=\int\left(3x^2+4x+2\right)dx$$

$$y^2-2y=x^3+2x^2+2x+C$$

$$y(0)=-1$$

$$(-1)^2-2(-1)=(0)^3+2(0)^2+2(0)+C$$

$$\boxed{3=C}$$

$$y^2-2y=x^3+2x^2+2x+3$$

Exercise

Find the exact solution of the initial value problem. Indicate the interval of existence.

$$y'=\frac{x}{1+2y}, \quad y(-1)=0$$

Solution

$$\frac{dy}{dx}=\frac{x}{1+2y}$$

$$(1+2y)dy=xdx$$

$$\int(1+2y)dy=\int xdx$$

$$y+y^2=\frac{1}{2}x^2+C \qquad y(-1)=0$$

$$0=\frac{1}{2}(-1)^2+C \quad \Rightarrow \quad C=-\frac{1}{2}$$

$$y+y^2=\frac{1}{2}x^2-\frac{1}{2}$$

Exercise

A murder victim is discovered at midnight and the temperature of the body is recorded at 31°C. One hour later, the temperature of the body is 29°C. Assume that the surrounding air temperature remains constant at 21°C. Use Newton's law of cooling to calculate the victim's time of death. *Note*: The normal temperature of a living human being is approximately 37°C

Solution

Given:

The initial temperature: $T(0) = 31\ °C$.

At $t = 1\ hr \quad \Rightarrow \quad T(1) = 29\ °C$

The surrounding temperature: $A = 21\ °C$

The temperature is given by the formula: $T = A + \left(T_0 - A\right)e^{-kt}$

$$T = 21 + (31 - 21)e^{-kt} = 21 + 10e^{-kt}$$

$$29 = 21 + 10e^{-k(1)}$$

$$8 = 10e^{-k}$$

$$e^{-k} = \frac{8}{10}$$

$$-k = \ln(0.8)$$

$$\boxed{k \approx 0.2231}$$

$$T = 21 + 10e^{-0.2231t}$$

$$37 = 21 + 10e^{-0.2231t}$$

$$10e^{-0.2231t} = 16$$

$$e^{-0.2231t} = 1.6$$

$$-0.2231t = \ln 1.6$$

$$t = \frac{\ln 1.6}{-0.2231} \approx -2.1\ hrs \qquad .1 * 60 = 6\ \text{ min}$$

The murder occurred 2 *hours* and 6 *minutes* earlier.

Exercise

Suppose a cold beer at 40°F is placed into a warn room at 70°F. suppose 10 minutes later, the temperature of the beer is 48°F. Use Newton's law of cooling to find the temperature 25 minutes after the beer was placed into the room.

Solution

Given:

The initial temperature: $T(0) = 40\ °F$.

At $t = 10$ min $\Rightarrow$ $T(10) = 48\ °F$

The surrounding temperature: $A = 70\ °F$

Let $T(t)$ be the temperature of the beer at time t minutes after being placed into the room.

From Newton's law of cooling,

$$T'(t) = k(A - T)$$

$$\frac{dT}{dt} = k(70 - T)$$

$$\frac{dT}{70 - T} = kdt$$

$$\int \frac{dT}{70 - T} = \int kdt \qquad\qquad d(70 - T) = -dT$$

$$-\ln(70 - T) = kt + C \qquad 70 > T(t)$$

$$\ln(70 - T) = -kt - C$$

$$70 - T = e^{-kt - C} = e^{-C}e^{-kt} = ce^{-kt}$$

$$T(t) = 70 - ce^{-kt}$$

From the initial condition:

$$T(0) = 70 - ce^{-k(0)}$$

$$40 = 70 - c \quad \Rightarrow \quad c = 30$$

$$\Rightarrow T(t) = 70 - 30e^{-kt}$$

$$T(t = 10) = 70 - 30e^{-k(10)}$$

$$48 = 70 - 30e^{-10k}$$

$$30e^{-10k} = 70 - 48 = 22$$

$$e^{-10k} = \frac{22}{30}$$

$$-10k = \ln\left(\frac{22}{30}\right) \quad \Rightarrow \quad k = -\frac{\ln(11/15)}{10} \approx 0.031$$

$$T(t) = 70 - 30e^{-.031t}$$

$$T(t = 25) = 70 - 30e^{-.031(25)}$$

$$\approx 56.18°\ F$$

Solution Section 1.3 – Models of Motions

Exercise

A stone is released from rest and dropped into a deep well. Eight seconds later, the sound of the stone splashing into the water at the bottom of the well returns to the ear of the person who released the stone. How long does it take the stone to drop to the bottom of the well? How deep is the well? Ignore air resistance. The speed of sound is 340 *m/s*.

Solution

$$\begin{aligned} d &= \frac{1}{2}gt^2 \\ &= \frac{1}{2}9.8t^2 \\ &= 4.9t^2 \end{aligned}$$

$$\begin{aligned} d &= 340s \\ &= 340(8-t) \end{aligned}$$

$$4.9t^2 = 2720 - 340t$$

$$4.9t^2 + 340t - 2720 = 0$$

$$t = 7.2438\ sec$$

$$\begin{aligned} d &= 340(8-7.2438) \\ &= 257.1\ m \end{aligned}$$

Exercise

A rocket is fired vertically and ascends with constant acceleration $a = 100\ m/s^2$ for 1.0 *min*. At that point, the rocket motor shuts off and the rocket continues upward under the influence of gravity. Find the maximum altitude acquired by the rocket and the total time elapsed from the take-off until the rocket returns to the earth. *Ignore air resistance*.

Solution

$$\begin{aligned} d &= \frac{1}{2}(a-g)t^2 \\ &= \frac{1}{2}(100-9.8)t^2 \end{aligned}$$

$$\begin{aligned} d(1hr = 60min) &= \frac{1}{2}(100-9.8)(60)^2 \\ &= 162,360\ m \end{aligned}$$

$$\begin{aligned} v &= d' \\ &= \left(\frac{1}{2}(100-9.8)t^2\right)' \\ &= (100-9.8)t \end{aligned}$$

$$v(60) = (100 - 9.8)(60)$$
$$= 5412 \ m/s$$

The velocity will be reduced: $5412 - 9.8t = 0$

$$\boxed{t = 552.2 \ s}$$

The altitude: $d(t) = -\frac{9.8}{2}t^2 + 5412\,t + 162,360$

$$d(552.2) = -\frac{9.8}{2}(552.2)^2 + 5412(552.2) + 162,360$$
$$= 1.657 \times 10^6 m$$

Back to the ground: $4.9t^2 = 1.657 \times 10^6$

$$\boxed{t_b = 581.5 \ s}$$

Total time: $t = 552.2 + 581.5 = 1193.7 \ s$

Exercise

A ball having mass $m = 0.1\,kg$ falls from rest under the influence of gravity in a medium that provides a resistance that is proportional to its velocity. For a velocity of $0.2\ m/s$ the force due to the resistance of the medium is -1 N. Find the terminal velocity of the ball.

1 N is the force required to accelerate a 1 kg mass at a rate of $1\ m/s^2$: $1N = 1\text{ kg} \cdot m/s^2$

Solution

The resistance force: $R = -rv$

$$-1 = -0.2r$$
$$\Rightarrow \boxed{r = 5}$$

The terminal velocity: $v_{term} = -\frac{mg}{r}$

$$= \frac{0.1(9.8)}{5}$$
$$= -0.196\ m/s$$

Exercise

A ball is projected vertically upward with initial velocity v_0 from ground level. Ignore air resistance.

a) What is the maximum height acquired by the ball?

b) How long does it take the ball to reach its maximum height? How long does it take the ball to return to the ground? Are these times identical?

c) What is the speed of the ball when it impacts with the ground on its return?

Solution

The position: $x(t) = -\frac{1}{2}gt^2 + v_0 t$

a) The maximum height when the velocity is zero

$$v = x' = -gt + v_0 = 0$$

$$t = \frac{v_0}{g}$$

$$\text{Maximum height } = -\frac{1}{2}g\left(\frac{v_0}{g}\right)^2 + v_0\frac{v_0}{g}$$

$$= -\frac{1}{2}\frac{v_0^2}{g} + \frac{v_0^2}{g}$$

$$= \frac{v_0^2}{2g}$$

b) The ball will take to reach the maximum height $t = \frac{v_0}{g}$ and the same to return to the ground, both are equal to $t = \frac{v_0}{g}$

c) When the ball hits the ground the time is equal to zero.

$$v = -g(0) + v_0 = v_0$$

Exercise

An object having mass 70 kg falls from rest under the influence of gravity. The terminal velocity of the object is $-20\ m/s$. Assume that the air resistance is proportional to the velocity.

a) Find the velocity and distance traveled at the end of 2 seconds.

b) How long does it take the object to reach 80% of its terminal velocity?

Solution

a) The terminal velocity: $v=-\frac{mg}{r}$

$$-20=-\frac{70(9.8)}{r}$$

$$\underline{r=\frac{70(9.8)}{20}=34.3}$$

$$v(t)=Ce^{-rt/m}-\frac{mg}{r}$$

$$v(t=0)=Ce^{-r(0)/m}-\frac{mg}{r}$$

$$0=C-\frac{mg}{r}\quad\Rightarrow\quad\boxed{C=\frac{mg}{r}}$$

$$v(t)=\frac{mg}{r}\left(e^{-rt/m}-1\right)$$

$$\underline{v(t=2)=\frac{70(9.8)}{34.3}\left(e^{-34.3(2)/70}-1\right)\approx-12.4938}$$

$$x=\int_0^t v(t)\,dt$$

$$=\frac{mg}{r}\int_0^t\left(e^{-rs/m}-1\right)ds$$

$$=\frac{mg}{r}\left[-\frac{m}{r}e^{-rs/m}-s\right]_0^t$$

$$=\frac{mg}{r}\left[-\frac{m}{r}e^{-rt/m}-t-\left(-\frac{m}{r}-0\right)\right]$$

$$=\frac{mg}{r}\left[\frac{m}{r}\left(1-e^{-rt/m}\right)-t\right]$$

$$\underline{x(2)=\frac{70(9.8)}{34.3}\left[\frac{70}{34.3}\left(1-e^{-34.3(2)/70}\right)-2\right]\approx-14.5025}$$

b) The velocity is 80% of its terminal velocity when $.8=1-e^{-rt/m}$

$$e^{-rt/m}=.2$$

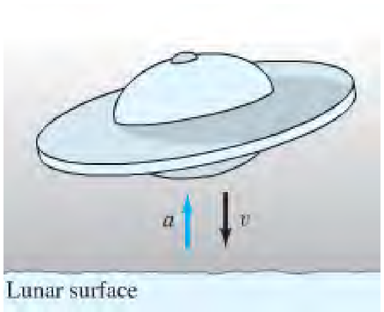
$$-\frac{rt}{m}=\ln(.2)\quad\Rightarrow\quad\underline{t=\frac{m}{r}\ln(.2)\approx3.285\ \text{sec}}$$

Exercise

A lunar lander is falling freely toward the surface of the moon at a speed of 450 m/s. Its retrorockets, when fired, provide a constant deceleration of 2.5 m/s^2 (the gravitational acceleration produced by the moon is assumed to be included in the given acceleration). At What height above the lunar surface should the retrorockets be activated to ensure a "soft touchdown? ($v = 0$ at impact)?

Solution

Given: $t = 0 \quad \to \quad v_0 = -450\ m/s \quad a = +2.5$



Because an upward thrust increases the velocity v (although decreases the speed $|v|$), then

$$v(t) = 2.5t - 450$$

$$x(t) = \int v(t)\,dt$$

$$= \int (2.5t - 450)\,dt$$

$$= 1.25t^2 - 450t + x_0$$

x_0 : is the height of the lander above the lunar surface at the time $t = 0$ when the retrorockets should be activated.

$$v(t) = 2.5t - 450 = 0$$

$$t = \frac{450}{2.5} = \underline{180\ s|}$$

When $x = 0 \quad \to \quad t = 180$

$$0 = 1.25(180)^2 - 450(180) + x_0$$

$$x_0 = 450(180) - 1.25(180)^2 \ \underline{= 40,500|}$$

Thus the retrorockets should be activated when the lunar lander is 40.500 m (40,5 km) above the surface of the moon, and it will touch down softly on the lunar surface after 3 minutes of decelerating descent.

Exercise

A body falling in a relatively dense fluid, oil for example, is acted on by three forces: a resistance force *R,* a buoyant force *B,* and its weight *w* due to gravity. The buoyant force is equal to the weight of the fluid displaced by the object. For a slowly moving spherical body of radius *a*, the resistive force is given by Stokes's law $R = 6\pi\frac{\mu a}{v}$, where *v* is the velocity of the body, and μ is the coefficient of viscosity of the surrounding fluid?

a) Find the limiting velocity of a solid sphere of radius *a* and density ρ falling freely in a medium of density ρ' and coefficient of viscosity μ.

b) In 1910 R. A. Millikan studied the motion of tiny droplets of oil falling in an electric field. A field of strength *E* exerts a force E_e on a droplet with charge *e*. Assume that *E* has been adjusted so the droplet is held stationary $(v = 0)$ and that *w* and *B* are as given. Find an expression for *e*.

Solution

a) The equation of motion is $m\frac{dv}{dt} = w - R - B$

$$\frac{4}{3}\pi a^3\rho\left(\frac{dv}{dt}\right) = \frac{4}{3}\pi a^3\rho g - 6\pi\mu a v - \frac{4}{3}\pi a^3\rho' g$$

The limiting velocity occurs when $\frac{dv}{dt} = 0$

b) Since the droplet is motionless, $v = \frac{dv}{dt} = 0$, we have the equation of motion

$$0 = \frac{4}{3}\pi a^3\rho g - Ee - \frac{4}{3}\pi a^3\rho' g$$

Where ρ is the density of the oil and ρ' is the density of air.

$$Ee = \frac{4}{3}\pi a^3\rho g - \frac{4}{3}\pi a^3\rho' g$$

$$Ee = \frac{4}{3}\pi a^3(\rho - \rho')$$

$$e = \frac{4}{3}\frac{\pi a^3}{E}(\rho - \rho')$$

Exercise

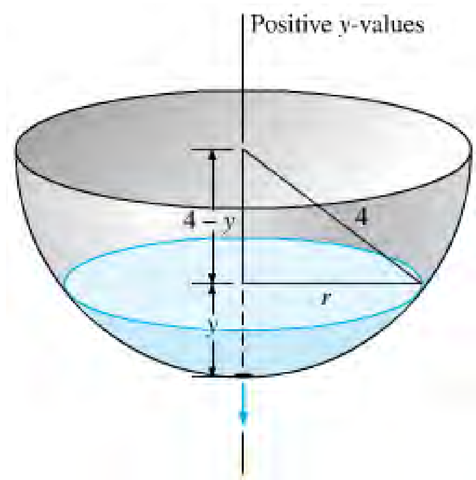
A hemispherical bowl has top radius of 4ft. and at time $t = 0$ is full of water. At that moment a circular hole with diameter 1 in. is opened in the bottom of the tank. How long will it take for all the water to drain from the tank?

Solution

$$r^2 + (4-y)^2 = 16 \quad \rightarrow \quad r^2 = 16 - (4-y)^2 \qquad \text{(\textbf{\textit{Right Triangle}})}$$

$$A(y) = \pi r^2 = \pi\left[16 - (4-y)^2\right] = \pi\left(8y - y^2\right)$$

$$a = \pi r^2 = \pi\left(\frac{1}{2}\,in.\frac{1\,ft.}{12\,in.}\right)^2 = \pi\left(\frac{1}{24}\right)^2$$



$$\frac{dV}{dt} = -a\sqrt{2gy}$$

$$\pi\left(8y - y^2\right)\frac{dy}{dt} = -\pi\left(\frac{1}{24}\right)^2\sqrt{2\cdot 32y}$$

$$\left(8y - y^2\right)\frac{dy}{dt} = -\frac{1}{576}(8)\sqrt{y}$$

$$\frac{8y - y^2}{y^{1/2}}dy = -\frac{1}{72}dt$$

$$\left(8y^{1/2} - y^{3/2}\right)dy = -\frac{1}{72}dt$$

$$\int\left(8y^{1/2} - y^{3/2}\right)dy = -\frac{1}{72}\int dt$$

$$\frac{16}{3}y^{3/2} - \frac{2}{5}y^{5/2} = -\frac{1}{72}t + C$$

$$y(0) = 4$$

$$\frac{16}{3}(4)^{3/2} - \frac{2}{5}(4)^{5/2} = -\frac{1}{72}(0) + C$$

$$\boxed{C = \frac{448}{15}}$$

$$\frac{16}{3}y^{3/2} - \frac{2}{5}y^{5/2} = -\frac{1}{72}t + \frac{448}{15}$$

The tank is empty when y = 0, thus when

$$0 = -\frac{1}{72}t + \frac{448}{15}$$

$$\frac{1}{72}t = \frac{448}{15}$$

$$t = \frac{448}{15}(72) \;\; \underline{\approx 2150 \;\; \text{sec}}$$

That is about 35 min. 50 s. So it takes slightly less than 36 minutes for the tank to drain.

Exercise

Suppose that the tank has a radius of 3 ft. and that its bottom hole is circular with radius 1 in. How long will it take the water (initially 9 ft. deep) to drain completely?

Solution

$$A = \pi r^2 = \pi(3)^2 = 9\pi$$

$$a = \pi r^2 = \pi\left(1 in. \frac{1\,ft.}{12\ in.}\right)^2 = \pi\left(\frac{1}{12}\right)^2$$

$$A(y)\frac{dy}{dt} = -a\sqrt{2gy}$$

$$9\pi\frac{dy}{dt} = -\frac{\pi}{144}\sqrt{2(32)y}$$

$$9\frac{dy}{dt} = -\frac{1}{144}\left(8\sqrt{y}\right)$$

$$9\frac{dy}{dt} = -\frac{1}{18}\sqrt{y}$$

$$162\frac{dy}{\sqrt{y}} = -dt$$

$$162\int y^{-1/2}dy = -\int dt$$

$$324y^{1/2} = -t + C$$

$$y(t=0) = 9$$

$$324\sqrt{9} = -0 + C$$

$$\underline{C = 972}$$

$$324\sqrt{y} = -t + 972$$

Hence $y = 0$ when $\underline{t = 972\sec = 16\min\ \ 12\sec}$

Exercise

At time $t = 0$ the bottom plug (at the vertex) of a full conical water tank 16 ft. high is removed. After 1 hr the water in the tank is 9 ft. deep. When will the tank be empty?

Solution

The radius of the cross-section of the cone at height y is proportional to y, so $A(y)$ is proportional to y^2. Therefore,

$$y^2 y' = -k\sqrt{y} \qquad\qquad A(y)y' = -k\sqrt{y}$$

$$y^{3/2}dy = -kdt$$

$$\int y^{3/2}dy = -\int kdt$$

$$\frac{2}{5}y^{5/2} = -kt + C_1$$

$$2y^{5/2} = -5kt + C$$

With initial condition: $y(0) = 16$

$$2(16)^{5/2} = -5k(0) + C$$

$$\underline{C = 2048}$$

$$2y^{5/2} = -5kt + 2048$$

$$y(1) = 9$$

$$2(9)^{5/2} = -5k(1) + 2048$$

$$486 - 2048 = -5k$$

$$\underline{k = \frac{1562}{5}}$$

$$2y^{5/2} = -1562t + 2048$$

Hence $y = 0$ when $t = \frac{2048}{1562}$ $\underline{\approx 1.31\ hr}$

Exercise

Suppose that a cylindrical tank initially containing V_0 gallons of water drains (through a bottom hole) in T minutes. Use Torricelli's law to show that the volume of water in the tank after $t \le T$ minutes is $V = V_0\left(1 - \frac{t}{T}\right)^2$

Solution

$$\frac{dy}{dt} = -k\sqrt{y} \qquad\qquad A(y)y' = -k\sqrt{y}$$

$$\int y^{-1/2}dy = -\int kdt$$

$$2y^{1/2} = -kt + C$$

With initial condition: $y(0) = h$

$$2\sqrt{h} = -k(0) + C$$

$$\boxed{C = 2\sqrt{h}}$$

$$2\sqrt{y} = -kt + 2\sqrt{h}$$

$$y(t = T) = 0$$

$$0 = -kT + 2\sqrt{h}$$

$$\boxed{k = \frac{2\sqrt{h}}{T}}$$

$$2\sqrt{y} = -\frac{2\sqrt{h}}{T}t + 2\sqrt{h}$$

$$\sqrt{y} = \sqrt{h}\left(1 - \frac{t}{T}\right)$$

$$y = h\left(1 - \frac{t}{T}\right)^2$$

If r denotes the radius of the cylinder, then

$$V(y) = \pi r^2 y$$

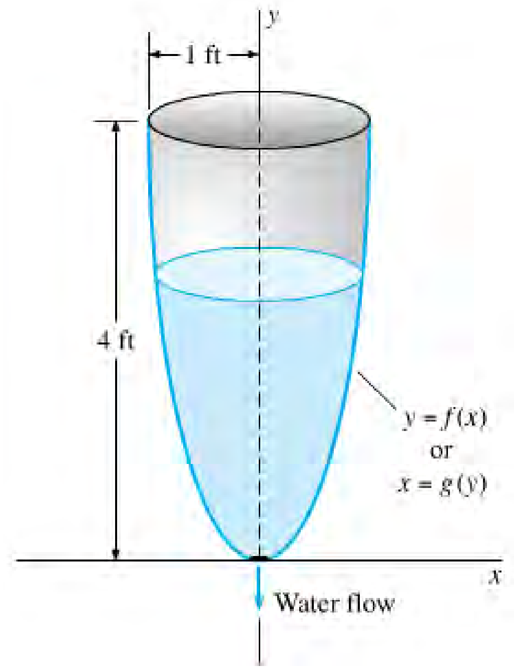
$$= \pi r^2 h\left(1 - \frac{t}{T}\right)^2$$

$$\boxed{= V_0\left(1 - \frac{t}{T}\right)^2} \qquad V_0 = \pi r^2 h$$

Exercise

The clepsydra, or water clock – A 12-*hr* water clock is to be designed with the dimensions, shaped like the surface obtained by revolving the curve $y = f(x)$ around the *y*-axis. What should be this curve, and what should be the radius of the circular bottom hole, in order that the water level will fall at the constant rate of 4 inches per hour?

Solution



The rate of fall of the water level is

$$\frac{dy}{dt} = -4\ in./hr = -\frac{1}{10800}\ ft/\sec$$

$$= -4\ \frac{in}{hr} \cdot \frac{1\,ft}{12\ in} \cdot \frac{1\ hr}{3600\ \sec}$$

$$= -\frac{1}{10800}\ ft/\sec$$

$$A = \pi x^2 \quad and \quad a = \pi r^2$$

$$A\frac{dy}{dt} = -a\sqrt{2gy}$$

$$\pi x^2 \frac{-1}{10800} = -\pi r^2 \sqrt{2(32)y}$$

$$\frac{x^2}{10800} = 8r^2\sqrt{y}$$

The curve is of the form $y = kx^4$

$$4 = k\left(1^4\right) \quad \to \quad k = 4$$

$$y = 4x^4 \quad \to \quad \sqrt{y} = 2x^2$$

$$\frac{x^2}{10800} = 8r^2\left(2x^2\right)$$

$$\frac{1}{10800} = 16r^2$$

$$r = \frac{1}{\sqrt{172800}}$$

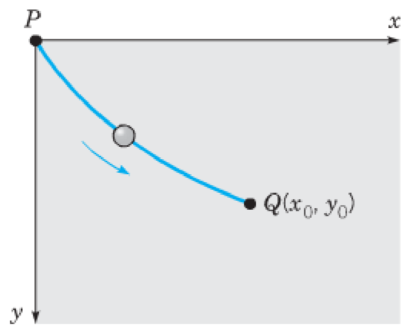
$$\approx 0.0024\,ft \quad or \quad 0.028\ in.$$

Exercise

One of the famous problems in the history of mathematics is the brachistochrone problem: to find the curve along which a particle will slide without friction in the minimum time from one given point *P* to another point *Q*, the second point being lower than the first but not directly beneath it.
This problem was posed by Johann Bernoulli in 1696 as a challenge problem to the mathematicians of his day. Correct solutions were found by Johann Bernoulli and his brother Jakob Bernoulli and by Isaac Newton, Gottfried Leibniz, and the Marquis de L'Hospital. The brachistochrone problem is important in the development of mathematics as one of the forerunners of the calculus of variations.
In solving this problem, it is convenient to take the origin as the upper point *P* and to orient the axes as shown. The lower point Q has coordinates $\left(x_0, y_0\right)$. It is then possible to show that the curve of minimum time is given by a function $y=\phi(x)$ that satisfies the differential equation

$$\left(1+y'^{\,2}\right)y=k^2 \qquad (eq.\ i)$$

Where k^2 is a certain positive constant to be determined later



a) Solve the equation $(eq.\ i)$ for y'. Why is it necessary to choose the positive square root?

b) Introduce the new variable t by the relation

$$y=k^2\sin^2 t \qquad (eq.\ ii)$$

Show that the equation found in part (a) then takes the form

$$k^2\sin^2 t\ dt=dx \qquad (eq.\ iii)$$

c) Letting $\theta=2t$, show that the solution of $(eq.\ iii)$ for which $x=0$ when $y=0$ is given by

$$x=k^2\frac{\theta-\sin\theta}{2}, \quad y=k^2\frac{1-\cos\theta}{2} \qquad (eq.\ iv)$$

Equations (*iv*) are parametric equations of the solution of (eq. *i*) that passes through (0, 0). The graph of Eqs. (*iv*) is called a cycloid.

d) If we make a proper choice of the constant k, then the cycloid also passes through the point $\left(x_0, y_0\right)$ and is the solution of the brachistochrone problem. Find *k* if $x_0=1$ and $y_0=2$

Solution

a) $\left(1+y'^{\,2}\right)y=k^2$

$$1+y'^{\,2}=\frac{k^2}{y}$$

$$y'^2 = \frac{k^2}{y} - 1$$

$$y' = \pm\sqrt{\frac{k^2 - y}{y}}$$

$$\boxed{y' = \sqrt{\frac{k^2 - y}{y}}}$$

The positive answer is chosen, since y is an increasing function of x.

b) Let $y = k^2 \sin^2 t$

$$dy = 2k^2 \sin t \cos t \; dt \qquad\qquad \left(\sin^2 t\right)' = 2\sin t \left(\sin t\right)' \, dt$$

$$y' = \frac{dy}{dx} = \sqrt{\frac{k^2 - y}{y}}$$

$$= \sqrt{\frac{k^2 - k^2\sin^2 t}{k^2 \sin^2 t}}$$

$$= \frac{k\sqrt{1 - \sin^2 t}}{k \sin t}$$

$$= \frac{\sqrt{\cos^2 t}}{\sin t}$$

$$= \boxed{\frac{\cos t}{\sin t}}$$

$$\frac{dy}{dx} = \frac{2k^2 \sin t \cos t \; dt}{dx} = \frac{\cos t}{\sin t}$$

$$\boxed{2k^2 \sin^2 t \; dt = dx}$$

c) Setting $\theta = 2t$, $2k^2 \sin^2\left(\frac{\theta}{2}\right) \, d\left(\frac{\theta}{2}\right) = dx$

$$k^2 \sin^2\left(\frac{\theta}{2}\right) \, d\theta = dx$$

$$\int k^2 \sin^2\left(\frac{\theta}{2}\right) \, d\theta = \int dx$$

$$\int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$k^2\left(\frac{\theta}{2} - \frac{\sin\theta}{2}\right) = x + C$$ $t = \theta = 0$ at the origin.

$$\boxed{C = 0}$$

$$\boxed{x(\theta) = k^2 \cdot \frac{\theta - \sin\theta}{2}}$$

$$y = k^2 \sin^2\left(\frac{\theta}{2}\right)$$

$$= k^2 \cdot \frac{1-\cos\theta}{2}$$

$$\sin^2\left(\frac{x}{2}\right) = \frac{1-\cos x}{2}$$

d) $$\frac{y}{x} = \frac{k^2 \cdot \frac{1-\cos\theta}{2}}{k^2 \cdot \frac{\theta - \sin\theta}{2}}$$

$$= \frac{1-\cos\theta}{\theta - \sin\theta}$$

Letting $x = 1, \quad y = 2$

$$\frac{1-\cos\theta}{\theta-\sin\theta} = 2$$

$$1-\cos\theta = 2\theta - 2\sin\theta$$

The solution is: $\theta \approx 1.401$

$$y = k^2 \cdot \frac{1-\cos\theta}{2}$$

$$2y = (1-\cos\theta)k^2$$

$$k^2 = \frac{2y}{1-\cos\theta} = \frac{2(2)}{1-\cos(1.401)} = 4.183$$

```
4/(1-cos(1.401)
            4.813
√(Ans
            2.194
```

$k \approx 2.194$

Solution Section 1.4 – Linear Equations

Exercise

Find the general solution of $y' - 3y = 5$

Solution

$$u(t) = e^{-\int 3dt}$$
$$= e^{-3t}$$

$$e^{-3t}y' - 3e^{-3t}y = 5e^{-3t}$$

$$\left(e^{-3t}y\right)' = 5e^{-3t}$$

$$e^{-3t}y = \int 5e^{-3t}dt$$

$$e^{-3t}y = -\tfrac{5}{3}e^{-3t} + C$$

$$\boxed{y(t) = -\tfrac{5}{3} + Ce^{3t}}$$

Exercise

Find the general solution of $y' + 2ty = 5t$

Solution

$$u(t) = e^{\int 2tdt}$$
$$= e^{t^2}$$

$$e^{t^2}y' + 2te^{t^2}y = 5te^{t^2}$$

$$\left(e^{t^2}y\right)' = 5te^{t^2}$$

$$e^{t^2}y = \int 5te^{t^2}dt \qquad de^{t^2} = 2te^{t^2}dt$$

$$= 5\int \tfrac{1}{2}de^{t^2}$$

$$= \tfrac{5}{2}e^{t^2} + C$$

$$\boxed{y(t) = \tfrac{5}{2} + Ce^{-t^2}}$$

Exercise

Find the general solution of $x' - 2\frac{x}{t+1} = (t+1)^2$

Solution

$$P(t) = \frac{-2}{t+1}, \quad Q(t) = (t+1)^2$$

$$x_h = e^{\int P(t)dt} = e^{\int -\frac{2}{t+1}dt} = e^{-2\ln(t+1)} = e^{\ln(t+1)^{-2}} = (t+1)^{-2}$$

$$\int Q(t)x_h\,dt = \int (t+1)^2(t+1)^{-2}\,dt = \int dt = t$$

$$x = \frac{1}{e^{\int Pdt}}\left(\int Q.e^{\int Pdt}dt + C\right) = \frac{1}{x_h}\left(\int Q.x_h\,dt + C\right)$$

$$x(t) = \frac{1}{(t+1)^{-2}}(t + C)$$

$$= (t+1)^2(t+C)$$

$$= t(t+1)^2 + C(t+1)^2$$

Exercise

Find the general solution of $(1+x)y' + y = \cos x$

Solution

$$y' + \frac{1}{1+x}y = \frac{\cos x}{1+x} \qquad P(x) = \frac{1}{1+x}, \quad Q(x) = \frac{\cos x}{1+x}$$

$$y_h = e^{\int \frac{1}{1+x}dx} = e^{\ln(1+x)} = 1+x$$

$$\int Q(x)y_h\,dx = \int \frac{\cos x}{1+x}(1+x)\,dx = \int \cos x dx = \sin x$$

$$y = \frac{1}{e^{\int Pdx}}\left(\int Q.e^{\int Pdx}dx + C\right)$$

$$y(x) = \frac{1}{x+1}(\sin x + C)$$

$$\boxed{y(x) = \frac{\sin x + C}{x+1}}$$

Exercise

Find the general solution of $y' - y = 3e^{t}$

Solution

$P(t) = -1, \quad Q(t) = 3e^{t}$

$$y_h = e^{\int -dt} = e^{-t}$$

$$\int Q(t)\, y_h\, dt = \int 3e^{t}e^{-t}dt = \int 3dt = 3t$$

$y(t) = \frac{1}{e^{-t}}(3t + C)$

$\boxed{y(t) = 3te^{t} + Ce^{t}}$

Exercise

Find the general solution of $y' + y = \sin t$

Solution

$P(t) = 1, \quad Q(t) = \sin t$

$$e^{\int dt} = e^{t}$$

$$\int e^{t}\sin t\ dt = e^{t}\sin t - \int e^{t}\cos t\ dt$$

$u = \sin t \qquad dv = e^{t}dt$

$du = \cos t\ dt \qquad v = e^{t}$

$$= e^{t}\sin t - e^{t}\cos t - \int e^{t}\sin t\ dt$$

$u = \cos t \qquad dv = e^{t}dt$

$du = -\sin t\ dt \qquad v = e^{t}$

$$2\int e^{t}\sin t\ dt = e^{t}\sin t - e^{t}\cos t$$

$$\int e^{t}\sin t\ dt = \frac{1}{2}e^{t}(\sin t - \cos t)$$

$y(t) = \frac{1}{e^{t}}\left(\frac{1}{2}e^{t}(\sin t - \cos t) + C\right)$

$\boxed{y(t) = \frac{1}{2}\sin t - \frac{1}{2}\cos t + Ce^{-t}}$

Exercise

Find the general solution of $y' + y = \dfrac{1}{1+e^t}$

Solution

$$P(t) = 1, \quad Q(t) = \frac{1}{1+e^t}$$

$$e^{\int dt} = e^t$$

$$\int \frac{e^t}{1+e^t}\, dt = \int \frac{1}{1+e^t} d\left(1+e^t\right) = \ln\left(1+e^t\right)$$

$$y = \frac{1}{e^{\int Pdx}}\left(\int Q.e^{\int Pdx} dx + C\right)$$

$$y(t) = \frac{1}{e^t}\left(\ln\left(1+e^t\right) + C\right)$$

$$\boxed{y(t) = e^{-t}\ln\left(1+e^t\right) + Ce^{-t}}$$

Exercise

Find the general solution of $y' + 3t^2 y = t^2$

Solution

$$P(t) = 3t^2, \quad Q(t) = t^2$$

$$e^{\int 3t^2 dt} = e^{t^3}$$

$$\int t^2 e^{t^3}\, dt = \frac{1}{3}\int e^{t^3}\, d\left(t^3\right) = \frac{1}{3}e^{t^3}$$

$$y = \frac{1}{e^{\int Pdx}}\left(\int Q.e^{\int Pdx} dx + C\right)$$

$$y(t) = \frac{1}{e^{t^3}}\left(\frac{1}{3}e^{t^3} + C\right)$$

$$\boxed{y(t) = \frac{1}{3} + Ce^{-t^3}}$$

Exercise

Find the general solution of $(\cos t)y' + (\sin t)y = 1$

Solution

$$y' + (\tan t)y = \frac{1}{\cos t} \qquad P(t) = \tan t, \quad Q(t) = \frac{1}{\cos t} = \sec t$$

$$e^{\int \tan t dt} = e^{-\ln|\cos t|} = e^{\ln\frac{1}{|\cos t|}} = \frac{1}{|\cos t|} = \sec t$$

$$\int \sec^2 t \; dt = \tan t$$

$$y = \frac{1}{e^{\int Pdx}}\left(\int Q.e^{\int Pdx}dx + C\right)$$

$$y(t) = \frac{1}{\sec t}(\tan t + C)$$

$$= \cos t\left(\frac{\sin t}{\cos t} + C\right)$$

$$\underline{= \sin t + C\cos t}$$

Exercise

Find the general solution of $y^2 + (y')^2 = 1$

Solution

$$\left(\frac{dy}{dx}\right)^2 = 1 - y^2$$

$$\frac{dy}{dx} = \pm\sqrt{1 - y^2}$$

$$\frac{dy}{\sqrt{1 - y^2}} = \pm dx$$

$$\int \frac{dy}{\sqrt{1 - y^2}} = \pm\int dx$$

$$\sin^{-1} y = \pm(x + c)$$

$$y = \sin(\pm(x + c))$$

$$\boxed{y = \pm\sin(x + c)}$$

Exercise

Find the general solution of $y' + \frac{3}{t}y = \frac{\sin t}{t^3}, \quad (t \neq 0)$

Solution

$$P(t) = \frac{3}{t}, \quad Q(t) = \frac{\sin t}{t^3}$$

$$e^{\int \frac{3}{t}dt} = e^{3\ln t} = e^{\ln t^3} = t^3$$

$$\int \frac{\sin t}{t^3} t^3 \, dt = \int \sin t \, dt = -\cos t$$

$$y(t) = \frac{1}{t^3}(-\cos t + C) \qquad\qquad y = \frac{1}{e^{\int P dx}}\left(\int Q.e^{\int P dx} dx + C\right)$$

$$\boxed{y(t) = \frac{C}{t^3} - \frac{\cos t}{t^3}}$$

Exercise

Find the general solution of $\left(1+e^t\right)y' + e^t y = 0$

Solution

$$y' + \frac{e^t}{1+e^t}y = 0$$

$$P(t) = \frac{e^t}{1+e^t}, \quad Q(t) = 0$$

$$e^{\int \frac{e^t}{1+e^t}dt} = e^{\int \frac{1}{1+e^t}d\left(1+e^t\right)} = e^{\ln\left(1+e^t\right)} = 1+e^t$$

$$y(t) = \frac{1}{1+e^t}(0+c) = \frac{c}{1+e^t}$$

$$\frac{dy}{dt} = -\frac{e^t}{1+e^t}y$$

$$\frac{dy}{y} = -\frac{e^t}{1+e^t}dt$$

$$\int \frac{dy}{y} = -\int \frac{1}{1+e^t}d\left(1+e^t\right)$$

$$\ln y = -\ln\left(1+e^t\right) + C$$

$$\ln y = \ln\left(\frac{1}{1+e^t}\right) + \ln c$$

$$\ln y = \ln\left(\frac{c}{1+e^t}\right)$$

$$\boxed{y = \frac{c}{1+e^t}}$$

Exercise

Find the general solution of $\left(t^2+9\right)y'+ty=0$

Solution

$$y'+\frac{t}{t^2+9}y=0$$

$$P(t)=\frac{t}{t^2+9}, \quad Q(t)=0$$

$$e^{\int\frac{t}{t^2+9}dt}=e^{\frac{1}{2}\int\frac{1}{t^2+9}d\left(t^2+9\right)} \qquad d\left(t^2+9\right)=2tdt \Rightarrow \frac{1}{2}d\left(t^2+9\right)=tdt$$

$$=e^{\frac{1}{2}\ln\left(t^2+9\right)}=e^{\ln\sqrt{t^2+9}}=\sqrt{t^2+9}$$

$$y(t)=\frac{1}{\sqrt{t^2+9}}(0+c)$$

$$=\boxed{\frac{c}{\sqrt{t^2+9}}}$$

Exercise

Find the general solution of $y'+y=\frac{1}{1+e^t}$

Solution

$$e^{\int dt}=e^t$$

$$\int\frac{1}{1+e^t}e^t\ dt=\int\frac{d\left(1+e^t\right)}{1+e^t}=\ln\left(1+e^t\right)$$

$$y(t)=\frac{1}{e^t}\left(\ln\left(1+e^t\right)+C\right)$$

$$\boxed{y(t)=e^{-t}\ln\left(1+e^t\right)+Ce^{-t}}$$

Exercise

Find the general solution of $\frac{dy}{dt} - 2y = 4 - t$

Solution

$$e^{\int -2dt} = e^{-2t}$$

$$\int (4-t)e^{-2t}\, dt = \int \left(4e^{-2t} - te^{-2t}\right)\, dt$$

$$= -2e^{-2t} + \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t}$$

$$= -\frac{7}{4}e^{-2t} + \frac{1}{2}te^{-2t}$$

		$\int e^{-2t}$
+	t	$-\frac{1}{2}e^{-2t}$
−	1	$\frac{1}{4}e^{-2t}$

$$y(t) = \frac{1}{e^{-2t}}\left(-\frac{7}{4}e^{-2t} + \frac{1}{2}te^{-2t} + C\right)$$

$$\boxed{y(t) = \frac{1}{2}t - \frac{7}{4} + Ce^{2t}}$$

Exercise

Find the general solution of $\left(1+x^3\right)y' = 3x^2y + x^2 + x^5$

Solution

$$y' - \frac{3x^2}{1+x^3}y = \frac{x^2\left(1+x^3\right)}{1+x^3} = x^2$$

$$e^{\int -\frac{3x^2}{1+x^3}dx} = e^{-\int \frac{d\left(1+x^3\right)}{1+x^3}} = e^{-\ln\left(1+x^3\right)} = e^{\ln\left(1+x^3\right)^{-1}} = \frac{1}{1+x^3}$$

$$\int \frac{1}{1+x^3}\cdot x^2 dx = \frac{1}{3}\int \frac{d\left(1+x^3\right)}{1+x^3}$$

$$= \frac{1}{3}\ln\left(1+x^3\right)$$

$$y(x) = \left(1+x^3\right)\left(\frac{1}{3}\ln\left(1+x^3\right) + C\right)$$

$$\boxed{= \frac{1}{3}\left(1+x^3\right)\ln\left(1+x^3\right) + C\left(1+x^3\right)}$$

Exercise

Find the general solution of $y' = \cos x - y \sec x$

Solution

$$y' + (\sec x)\, y = \cos x$$

$$e^{\int \sec x dx} = e^{\ln|\sec x + \tan x|} = \sec x + \tan x$$

$$\int \cos x(\sec x + \tan x)\, dx = \int \cos x\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right) dx$$

$$= \int (1 + \sin x)\, dx$$

$$= x - \cos x$$

$$\boxed{y(x) = \frac{1}{\sec x + \tan x}(x - \cos x + C)}$$

Exercise

Solve the differential equation: $\qquad x\dfrac{dy}{dx} + y = e^x, \quad x > 0$

Solution

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{e^x}{x} \rightarrow \quad P(x) = \frac{1}{x} \quad Q(x) = \frac{e^x}{x}$$

$$e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\int \frac{e^x}{x} e^{\int \frac{1}{x}dx} dx = \int x\frac{e^x}{x} dx = \int e^x dx = e^x$$

$$\boxed{y(x) = \frac{1}{x}\left(e^x + C\right)}, \quad x > 0$$

$$y = \frac{1}{e^{\int P dx}}\left(\int Q.e^{\int P dx} dx + C\right)$$

Exercise

Solve the differential equation: $y' + (\tan x)y = \cos^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

Solution

$$y' + (\tan x)y = \cos^2 x, \quad P(x) = \tan x, \quad Q(x) = \cos^2 x$$

$$y_h = e^{\int \tan x dx} = e^{\ln(\cos x)^{-1}} = (\cos x)^{-1} \qquad \int \tan x dx = -\ln|\cos x| = \ln(\cos x)^{-1}$$

$$\int \cos^2 x (\cos x)^{-1} dx = \int \cos x dx = \sin x$$

$$y = \frac{1}{e^{\int P dx}}\left(\int Q.e^{\int P dx} dx + C\right)$$

$$y(x) = \frac{1}{(\cos x)^{-1}}(\sin x + C)$$

$$y(x) = \cos x(\sin x + C)$$

$$= \cos x \sin x + C\cos x$$

Exercise

Solve the differential equation: $x\frac{dy}{dx} + 2y = 1 - \frac{1}{x}, \quad x > 0$

Solution

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{1}{x} - \frac{1}{x^2}, \quad P(x) = \frac{2}{x} \quad Q(x) = \frac{1}{x} - \frac{1}{x^2}$$

$$y_h = e^{\int \frac{2}{x} dx} = e^{2\ln x} = e^{\ln x^2} = x^2$$

$$\int \left(\frac{1}{x} - \frac{1}{x^2}\right)x^2 dx = \int (x-1)dx = \frac{1}{2}x^2 - x$$

$$y = \frac{1}{e^{\int P dx}}\left(\int Q.e^{\int P dx} dx + C\right)$$

$$y(x) = \frac{1}{x^2}\left(\frac{1}{2}x^2 - x + C\right)$$

$$y(x) = \frac{1}{2} - \frac{1}{x} + \frac{C}{x^2}, \quad x > 0$$

Exercise

Solve the differential equation: $(1+x)y' + y = \sqrt{x}$

Solution

$$\frac{dy}{dx} + \frac{1}{1+x}y = \frac{\sqrt{x}}{1+x}, \quad P(x) = \frac{1}{1+x} \quad Q(x) = \frac{\sqrt{x}}{1+x}$$

$$e^{\int \frac{1}{1+x}dx} = e^{\ln(1+x)} = 1+x$$

$$\int \frac{\sqrt{x}}{1+x}(1+x)\,dx = \int x^{1/2}dx = \frac{2}{3}x^{3/2}$$

$$y = \frac{1}{e^{\int Pdx}}\left(\int Q.e^{\int Pdx}dx + C\right)$$

$$y(x) = \frac{1}{1+x}\left(\frac{2}{3}x^{3/2} + C\right)$$

$$= \frac{2x^{3/2}}{3(1+x)} + \frac{C}{1+x}$$

Exercise

Solve the differential equation: $e^{2x}y' + 2e^{2x}y = 2x$

Solution

$$y' + 2y = 2xe^{-2x}, \quad P(x) = 2 \quad Q(x) = 2xe^{-2x}$$

$$e^{\int 2dx} = e^{2x}$$

$$\int 2xe^{-2x}\left(e^{2x}\right)dx = 2\int xdx = x^2$$

$$y(x) = \frac{1}{e^{2x}}\left(x^2 + C\right)$$

$$= x^2e^{-2x} + Ce^{-2x}$$

Exercise

Solve the differential equation: $(t+1)\dfrac{ds}{dt}+2s=3(t+1)+\dfrac{1}{(t+1)^2}, \quad t>-1$

Solution

$$\frac{ds}{dt}+\frac{2}{t+1}s=3+\frac{1}{(t+1)^3}, \quad P(t)=\frac{2}{t+1} \quad Q(t)=3+\frac{1}{(t+1)^3}$$

$$e^{\int \frac{2}{t+1}dt}=e^{2\ln(t+1)}=e^{\ln(t+1)^2}=(t+1)^2$$

$$\int\left(3+\frac{1}{(t+1)^3}\right)(t+1)^2\,dt=\int\left(3(t+1)^2+\frac{1}{t+1}\right)dt \qquad d(t+1)=dt$$

$$=3\int(t+1)^2\,d(t+1)+\int\frac{1}{t+1}d(t+1)$$

$$=(t+1)^3+\ln(t+1)$$

$$s(t)=\frac{1}{(t+1)^2}\left((t+1)^3+\ln(t+1)+C\right)$$

$$\boxed{=t+1+\frac{\ln(t+1)}{(t+1)^2}+\frac{C}{(t+1)^2}, \quad t>-1}$$

Exercise

Solve the differential equation: $\tan\theta\dfrac{dr}{d\theta}+r=\sin^2\theta, \quad 0<\theta<\dfrac{\pi}{2}$

Solution

$$\frac{dr}{d\theta}+\frac{1}{\tan\theta}r=\frac{\sin^2\theta}{\tan\theta}$$

$$\frac{dr}{d\theta}+\frac{1}{\tan\theta}r=\sin^2\theta\frac{\cos\theta}{\sin\theta}$$

$$\frac{dr}{d\theta}+\frac{1}{\tan\theta}r=\sin\theta\cos\theta, \quad P(\theta)=\frac{1}{\tan\theta}=\cot\theta \quad Q(\theta)=\sin\theta\cos\theta$$

$$e^{\int\cot\theta d\theta}=e^{\ln|\sin\theta|}=\sin\theta, \quad 0<\theta<\frac{\pi}{2}$$

$$\int(\sin\theta\cos\theta)(\sin\theta)d\theta=\int\left(\sin^2\theta\cos\theta\right)d\theta \qquad d(\sin\theta)=\cos\theta d\theta$$

$$=\int\sin^2\theta d(\sin\theta)$$

$$=\frac{1}{3}\sin^3\theta$$

$$\left\lfloor r(\theta)=\frac{1}{\sin\theta}\left(\frac{1}{3}\sin^3\theta+C\right)\right.$$

$$\boxed{=\frac{1}{3}\sin^2\theta+\frac{C}{\sin\theta}}$$

Exercise

Find the general solution of $y'-3y=4$; $y(0)=2$

Solution

$$e^{-\int 3dt}=e^{-3t}$$

$$\int 4e^{-3t}dt=-\frac{4}{3}e^{-3t}$$

$$y(t)=e^{3t}\left(-\frac{4}{3}e^{-3t}+C\right)$$

$$\boxed{=-\frac{4}{3}+Ce^{3t}}$$

$$y(0)=-\frac{4}{3}+Ce^{3(0)}$$

$$2=-\frac{4}{3}+C$$

$$C=\frac{4}{3}+2\ =\frac{10}{3}$$

$$\boxed{y(t)=-\frac{4}{3}+\frac{10}{3}e^{3t}}$$

Exercise

Find the general solution of $y'=y+2xe^{2x}$; $y(0)=3$

Solution

$$y'-y=2xe^{2x} \qquad P(x)=-1,\quad Q(x)=2xe^{2x}$$

$$e^{\int -1dx}=e^{-x}$$

$$\int 2xe^{2x}\left(e^{-x}\right)dx=2\int xe^{x}dx=2\left(xe^{x}-e^{x}\right)$$

$$y(x)=\frac{1}{e^{-x}}\left(2xe^{x}-2e^{x}+C\right) \qquad y=\frac{1}{e^{\int Pdx}}\left(\int Q.e^{\int Pdx}dx+C\right)$$

$$= e^{x}\left(2xe^{x} - 2e^{x} + C\right)$$

$$\underline{= 2xe^{2x} - 2e^{2x} + Ce^{x}|}$$

$$y(x=0) = 2(0)e^{2(0)} - 2e^{2(0)} + Ce^{(0)}$$

$$3 = -2 + C$$

$$\boxed{C = 5}$$

$$\boxed{y(x) = 2xe^{2x} - 2e^{2x} + 5e^{x}}$$

Exercise

Find the general solution of $\left(x^2+1\right)y' + 3xy = 6x; \quad y(0) = -1$

Solution

$$y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}$$

$$e^{\int \frac{3x}{x^2+1}dx} = e^{\frac{3}{2}\ln\left(x^2+1\right)} = e^{\ln\left(x^2+1\right)^{\frac{3}{2}}} \underline{= \left(x^2+1\right)^{\frac{3}{2}}|}$$

$$\int \left(x^2+1\right)^{\frac{3}{2}} \frac{6x}{x^2+1}dx = 3\int \left(x^2+1\right)^{\frac{1}{2}} d\left(x^2+1\right)$$

$$= 2\left(x^2+1\right)^{\frac{3}{2}}$$

$$y(x) = 2 + C\left(x^2+1\right)^{-\frac{3}{2}}$$

$$y(0) = 2 + C\left((0)^2+1\right)^{-\frac{3}{2}}$$

$$-1 = 2 + C(1)^{-\frac{3}{2}}$$

$$C = -3$$

$$\boxed{y(x) = 2 - 3\left(x^2+1\right)^{-\frac{3}{2}}}$$

Exercise

Solve the initial value problem: $\quad t\frac{dy}{dt} + 2y = t^3, \quad t > 0, \quad y(2) = 1$

Solution

$$\frac{dy}{dt}+\frac{2}{t}y=t^2, \quad P(t)=\frac{2}{t}, \quad Q(t)=t^2$$

$$e^{\int \frac{2}{t}dt}=e^{2\ln t}=e^{\ln t^2}=t^2$$

$$\int t^2 t^2 dt=\int t^4 dt=\frac{1}{5}t^5$$

$$y(t)=\frac{1}{t^2}\left(\frac{1}{5}t^5+C\right)=\frac{1}{5}t^3+\frac{C}{t^2}$$

$$y(2)=\frac{1}{5}2^3+\frac{C}{2^2}$$

$$1=\frac{8}{5}+\frac{C}{4}$$

$$\frac{C}{4}=1-\frac{8}{5}=-\frac{3}{5}\Rightarrow \boxed{C=-\frac{12}{5}}$$

$$\boxed{y(t)=\frac{1}{5}t^3-\frac{12}{5t^2}}$$

Exercise

Solve the initial value problem: $\theta\dfrac{dy}{d\theta}+y=\sin\theta, \quad \theta>0, \quad y\left(\frac{\pi}{2}\right)=1$

Solution

$$\frac{dy}{d\theta}+\frac{1}{\theta}y=\frac{\sin\theta}{\theta}, \quad P(\theta)=\frac{1}{\theta}, \quad Q(\theta)=\frac{\sin\theta}{\theta}$$

$$e^{\int \frac{1}{\theta}d\theta}=e^{\ln|\theta|}=\theta \quad (>0)$$

$$\int \frac{\sin\theta}{\theta}\theta d\theta=\int \sin\theta d\theta=-\cos\theta$$

$$y(\theta)=\frac{1}{\theta}(-\cos\theta+C)$$

$$y\left(\frac{\pi}{2}\right)=\frac{2}{\pi}\left(-\cos\frac{\pi}{2}+C\right)$$

$$1=\frac{2}{\pi}(0+C)$$

$$1=\frac{2}{\pi}C$$

$$C=\frac{\pi}{2}$$

$$\boxed{y(\theta) = -\frac{\cos\theta}{\theta} + \frac{\pi}{2\theta}}$$

Exercise

Solve the initial value problem: $\frac{dy}{dx} + xy = x, \quad y(0) = -6$

Solution

$$\frac{dy}{dx} + xy = x, \quad P(x) = x, \quad Q(x) = x$$

$$e^{\int x\,dx} = e^{x^2/2}$$

$$\int xe^{x^2/2}dx = \int e^{x^2/2}d\left(\frac{x^2}{2}\right) = e^{x^2/2} \qquad d\left(\frac{x^2}{2}\right) = xdx$$

$$y(x) = \frac{1}{e^{x^2/2}}\left(e^{x^2/2} + C\right)$$

$$y(0) = \frac{1}{e^{0^2/2}}\left(e^{0^2/2} + C\right)$$

$$-6 = 1(1 + C)$$

$$-6 = 1 + C$$

$$C = -7$$

$$y(x) = \frac{1}{e^{x^2/2}}\left(e^{x^2/2} - 7\right)$$

$$\boxed{= 1 - \frac{7}{e^{x^2/2}}}$$

Exercise

Solve the initial value problem: $ty' + 2y = 4t^2, \quad y(1) = 2$

Solution

$y' + \frac{2}{t}y = 4t \qquad P(t) = \frac{2}{t}, \quad Q(t) = 4t$

$e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = e^{\ln t^2} = t^2$

1st method

$\int t^2(4t)\,dt = 4\int t^3 dt = t^4$

$y(t) = \frac{1}{t^2}\left(t^4 + C\right) \qquad y = \frac{1}{e^{\int P dt}}\left(\int Q.e^{\int P dt}dt + C\right)$

2nd method

$t^2 y' + t^2 \frac{2}{t}y = 4t\left(t^2\right)$

$t^2 y' + 2ty = 4t^3$

$\left(t^2 y\right)' = 4t^3$

$t^2 y = t^4 + C$

$y(t) = \frac{1}{t^2}\left(t^4 + C\right)$

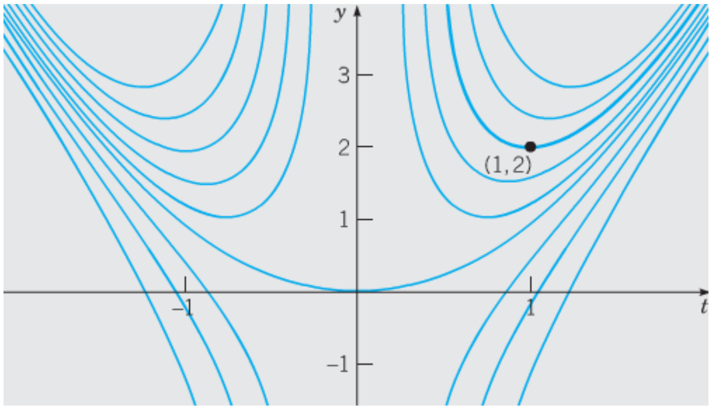
$y(1) = \frac{1}{1^2}\left(1^4 + C\right)$

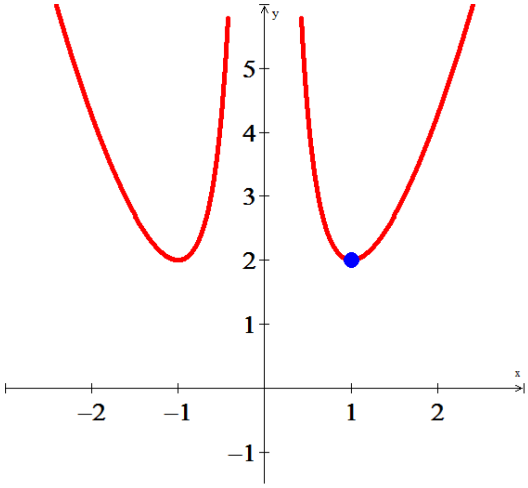
$2 = 1 + C$

$\boxed{C = 1}$

$y(t) = \frac{1}{t^2}\left(t^4 + 1\right)$

$\boxed{y(t) = t^2 + \frac{1}{t^2}}$





Exercise

Find the solution of the initial value problem $\left(1+t^2\right)y' + 4ty = \left(1+t^2\right)^{-2}, \quad y(1) = 0$

Solution

$$y' + \frac{4t}{1+t^2}y = \frac{\left(1+t^2\right)^{-2}}{1+t^2}$$

$$y' + \frac{4t}{1+t^2}y = \left(1+t^2\right)^{-3} \qquad P(t) = \frac{4t}{1+t^2}, \quad Q(t) = \left(1+t^2\right)^{-3}$$

$$e^{\int \frac{4t}{1+t^2}dt} = e^{2\int \frac{d\left(1+t^2\right)}{1+t^2}} = e^{2\ln\left(1+t^2\right)} = e^{\ln\left(1+t^2\right)^2} = \left(1+t^2\right)^2$$

$$\int \left(1+t^2\right)^2\left(1+t^2\right)^{-3}dt = \int \frac{dt}{1+t^2} = \tan^{-1}|t|$$

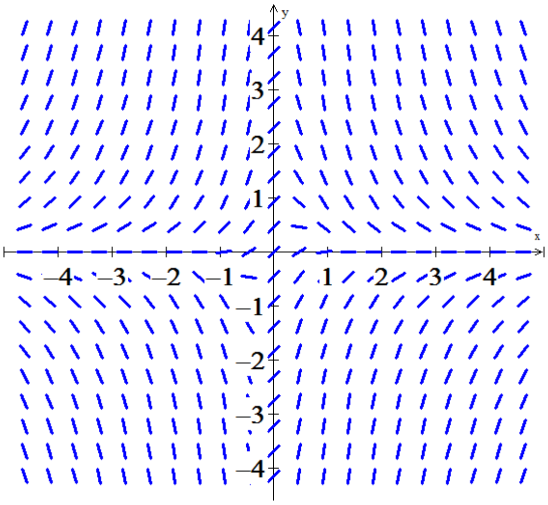
$$y(t) = \frac{1}{\left(1+t^2\right)^2}\left(\tan^{-1}|t| + C\right) \qquad y = \frac{1}{e^{\int Pdx}}\left(\int Q.e^{\int Pdx}dx + C\right)$$

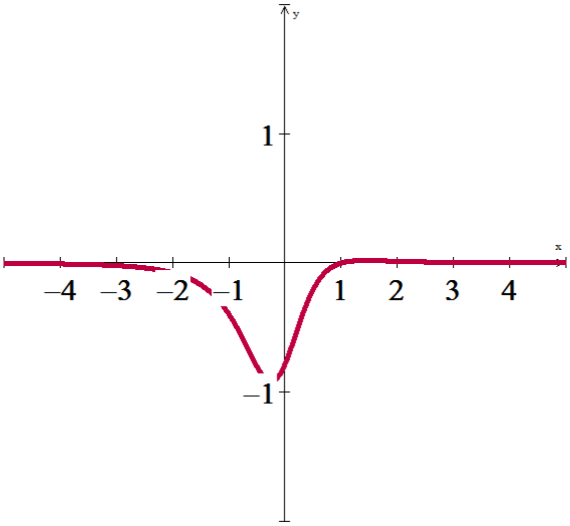
Given $y(1) = 0$, then

$$0 = \frac{1}{\left(1+1^2\right)^2}\left(\tan^{-1}|1| + C\right)$$

$$0 = \frac{\pi}{4} + C \quad \Rightarrow \quad C = -\frac{\pi}{4}$$

$$y(t) = \frac{1}{\left(1+t^2\right)^2}\left(\tan^{-1}|t| - \frac{\pi}{4}\right)$$





Exercise

Solve $xy' + 2y = \sin x$ *for* y' $\qquad y\left(\frac{\pi}{2}\right) = 0$

Solution

$$xy' + 2y = \sin x$$

$$y' + \frac{2}{x}y = \frac{\sin x}{x} \qquad x \neq 0$$

$$u(x) = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = e^{\ln x^2} = x^2$$

$$x^2 y' + x^2 \frac{2}{x} y = x^2 \frac{\sin x}{x}$$

$$x^2 y' + 2xy = x \sin x$$

$$\left(x^2 y\right)' = x \sin x$$

$$x^2 y = \int x \sin x dx$$

Integration by part

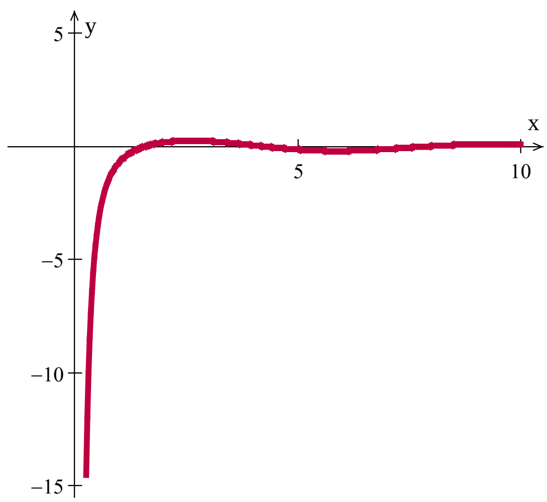
$$u = x \qquad dv = \sin x dx$$

$$du = dx \qquad v = \int \sin x dx = -\cos x$$

$$\int x \sin x dx = -x\cos x + \int \cos x dx = -x\cos x + \sin x + C$$

$$x^2 y = \int x \sin x dx$$

$$= -x\cos x + \sin x + C$$



$$y(x) = -\frac{1}{x}\cos x + \frac{1}{x^2}\sin x + \frac{C}{x^2}$$

$$y\left(\frac{\pi}{2}\right) = -\frac{1}{\left(\frac{\pi}{2}\right)}\cos\left(\frac{\pi}{2}\right) + \frac{1}{\left(\frac{\pi}{2}\right)^2}\sin\left(\frac{\pi}{2}\right) + \frac{C}{\left(\frac{\pi}{2}\right)^2}$$

$$0 = \frac{4}{\pi^2} + \frac{4}{\pi^2}C$$

$$\frac{4}{\pi^2}C = -\frac{4}{\pi^2}$$

$$\boxed{C = -1}$$

$$\boxed{y(x) = -\frac{1}{x}\cos x + \frac{1}{x^2}\sin x - \frac{1}{x^2}} \qquad x \neq 0$$

Exercise

Find the solution of the initial value problem. Discuss the interval of existence and provide a sketch of your solution $(2x+3)y' = y + (2x+3)^{1/2}; \quad y(-1) = 0$

Solution

$$(2x+3)y' - y = (2x+3)^{1/2}$$

$$y' - \frac{1}{2x+3}y = (2x+3)^{-1/2} \qquad P(x) = \frac{-1}{2x+3}, \quad Q(x) = (2x+3)^{-1/2}$$

$$e^{\int \frac{-1}{2x+3}dx} = e^{-\frac{1}{2}\int \frac{1}{2x+3}d(2x+3)} = e^{-\frac{1}{2}\ln(2x+3)} = e^{\ln(2x+3)^{-1/2}} = |2x+3|^{-1/2}$$

$$\int (2x+3)^{-1/2}(2x+3)^{-1/2}dx = \int (2x+3)^{-1}dx$$

$$= \frac{1}{2}\int \frac{d(2x+3)}{2x+3} \qquad d(2x+3) = 2dx$$

$$= \frac{1}{2}\ln|2x+3|$$

$$y = \frac{1}{e^{\int Pdx}}\left(\int Q.e^{\int Pdx}dx + C\right)$$

$$y(x) = \frac{1}{(2x+3)^{-1/2}}\left(\frac{1}{2}\ln(2x+3) + C\right)$$

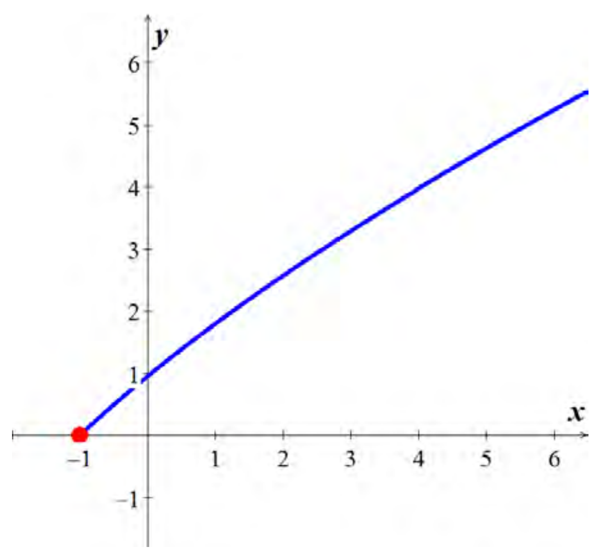
$$= \frac{1}{2}(2x+3)^{1/2}\ln(2x+3) + C(2x+3)^{1/2}$$

$$0 = \frac{1}{2}(2(-1)+3)^{1/2}\ln(2(-1)+3) + C(2(-1)+3)^{1/2}$$

$$0 = \frac{1}{2}(1)^{1/2}\ln(1) + C(1)^{1/2}$$

$$\boxed{C = 0}$$

$$\boxed{y(x) = \frac{1}{2}(2x+3)^{1/2}\ln(2x+3)}$$



Exercise

Find the general solution of the given differential equation. Then find the particular solution satisfying the given initial condition of $y' - 3y = 4, \quad y(0) = 2$

Solution

$$P(x) = -3, \quad Q(x) = 4$$

$$e^{\int -3dx} = e^{-3x}$$

$$\int 4e^{-3x}dx = -\frac{4}{3}e^{-3x}$$

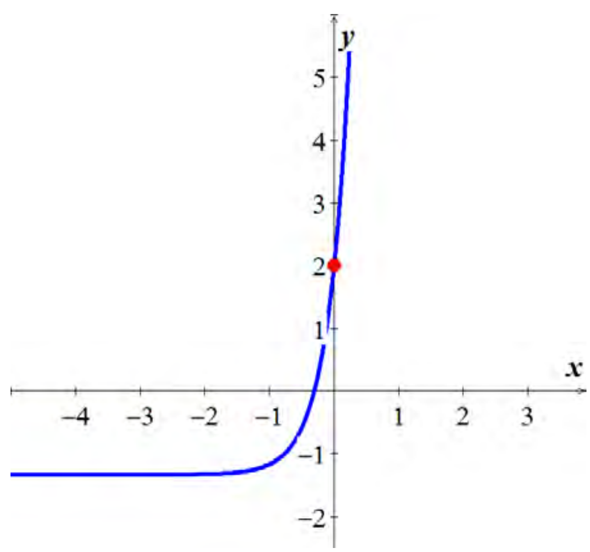
$$y = \frac{1}{e^{\int Pdx}}\left(\int Q.e^{\int Pdx}dx + C\right)$$

$$y(x) = \frac{1}{e^{-3x}}\left(-\frac{4}{3}e^{-3x} + C\right)$$

$$= -\frac{4}{3} + Ce^{3x}$$

$$2 = -\frac{4}{3} + Ce^{0} \quad \Rightarrow \quad C = \frac{10}{3}$$

$$y(x) = -\frac{4}{3} + \frac{10}{3}e^{3x} = \boxed{\frac{10e^{3x} - 4}{3}}$$



Exercise

Find the general solution of the given differential equation. Then find the particular solution satisfying the given initial condition of

$$y' + \frac{1}{2}y = t, \quad y(0) = 1$$

Solution

$$u(t) = e^{\int \frac{1}{2} dt}$$
$$= e^{t/2}$$

$$\left(e^{t/2}y\right)' = te^{t/2}$$

$$e^{t/2}y = \int te^{t/2}dt \qquad\qquad \int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1)$$

$$= \frac{e^{t/2}}{\left(\frac{1}{2}\right)^2}\left(\frac{t}{2} - 1\right) + C$$

$$= 4e^{t/2}\left(\frac{t}{2} - 1\right) + C$$

$$= (2t - 4)e^{t/2} + C$$

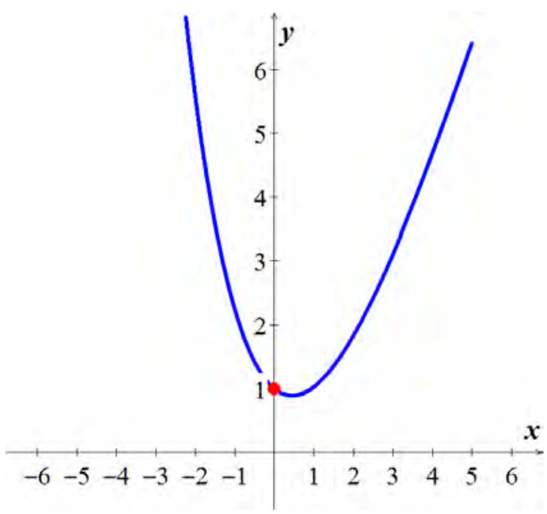
$$y(t) = (2t - 4) + Ce^{-t/2}$$

$$y(t=0) = (2(0) - 4) + Ce^{-0/2}$$

$$1 = -4 + C$$

$$C = 5$$

$$\boxed{y(t) = 2t - 4 + 5e^{-t/2}}$$



Exercise

Find the general solution of the given differential equation. Then find the particular solution satisfying the given initial condition of

$$y' + y = e^t, \quad y(0) = 1$$

Solution

$$P(t) = 1, \quad Q(t) = e^t$$

$$e^{\int dt} = e^t$$

$$\int e^t e^t \, dt = \int e^{2t} dt = \frac{1}{2} e^{2t}$$

$$y = \frac{1}{e^{\int P dt}} \left(\int Q . e^{\int P dt} dt + C \right)$$

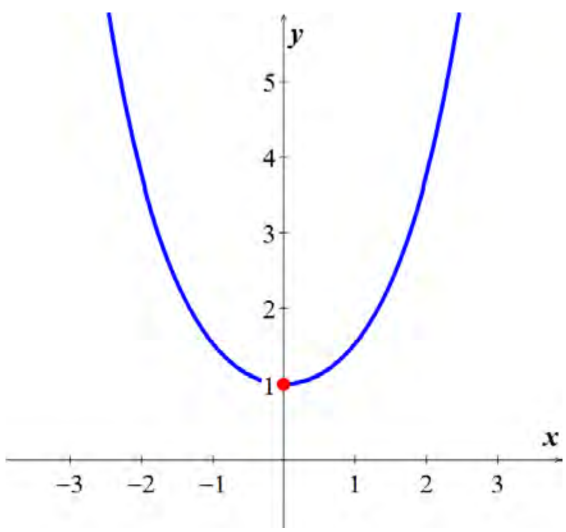
$$y(t) = \frac{1}{e^t} \left(\frac{1}{2} e^{2t} + C \right)$$

$$= \frac{1}{2} e^t + C e^{-t}$$

$$1 = \frac{1}{2} e^0 + C e^{-0}$$

$$1 = \frac{1}{2} + C \quad \Rightarrow \quad C = \frac{1}{2}$$

$$\boxed{y(t) = \frac{1}{2}\left(e^t + e^{-t}\right)}$$



Solution   Section 1.5 – Mixing Problems

Exercise

Consider two tanks, label tank A and tank B for reference. Tank A contains 100 *gal* of solution in which is dissolved 20 *lb* of salt. Tank B contains 200 *gal* of solution which is dissolved 40 *lb* of salt. Pure water flows into the tank A at rate of 5 *gal/s*. There is a drain at the bottom of tank A. The solution leaves tank A via the drain at a rate of 5 *gal/s* and flows immediately into tank B at the same rate. A drain at the bottom of tank B allows the solution to leave tank B at a rate of 2.5 *gal/s*. What is the salt content in tank B at the precise moment that tank B contains 250 *gal* of solution?

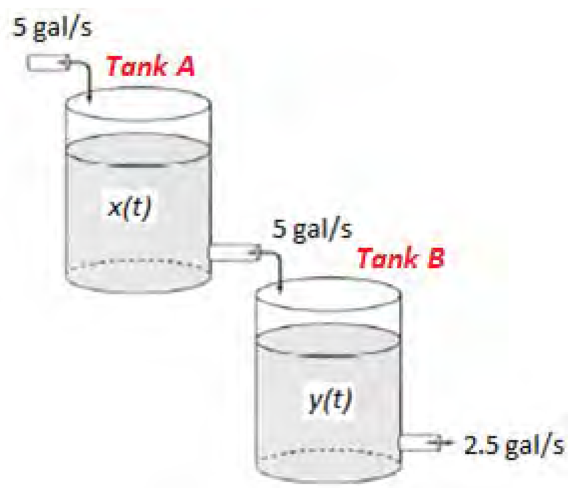
Solution

Tank ***A*** contains 100 *gal* of solution in which is dissolved 20 *lb* of salt

$Volume:\ V(t) = 100 + (5-5)t = 100$

Concentration at time t: $c(t) = \dfrac{x(t)}{V(t)} = \dfrac{x(t)}{100}\ \ lb/gal$

Rate out = Volume Rate x Concentration



$$Rate\ out = 5\frac{gal}{s} \times \frac{x(t)}{100}\frac{lb}{gal}$$

$$= \frac{x}{20}\frac{lb}{s}$$

$$\frac{dx}{dt} = rate\ in - rate\ out$$

$$x' = -\frac{x(t)}{20}\ \ \frac{lb}{s}$$

$$\frac{dx}{x} = -\frac{1}{20}dt$$

$$\int \frac{dx}{x} = -\frac{1}{20}\int dt$$

$$\ln|x| = -\frac{1}{20}t + C$$

$$|x| = e^{-\frac{t}{20}+C}$$

$$x(t) = C_1 e^{-\frac{t}{20}}$$

$$x(0) = C_1 e^{-\frac{0}{20}}$$

$$C_1 = 20$$

$$x(t) = 20e^{-\frac{t}{20}}$$

Tank ***B*** contains 200 *gal* of solution which is dissolved 40 *lb* of salt.

$Volume:\ V(t) = 200 + (5-2.5)t = 200 + 2.5t$

$Rate\ in = Rate\ out\ \ (Tank\ A)$

$$= \frac{x}{20} \frac{lb}{s}$$

$$Rate\ out = 2.5 \frac{gal}{s} \times \frac{y}{200 + 2.5t} \frac{lb}{gal}$$

$$= \frac{y}{80+t} \ \frac{lb}{s}$$

$$\frac{dy}{dt} = rate\ in - rate\ out$$

$$\frac{dy}{dt} = \frac{x}{20} - \frac{y}{80+t}$$

$$\frac{dy}{dt} = \frac{20e^{-\frac{t}{20}}}{20} - \frac{y}{80+t}$$

$$= e^{-\frac{t}{20}} - \frac{1}{80+t} y$$

$$y' + \frac{1}{80+t} y = e^{-\frac{t}{20}}$$

$$u(t) = e^{\int \frac{1}{80+t} dt}$$

$$= e^{\ln(80+t)}$$

$$= 80 + t$$

$$\left[(80+t)y\right]' = (80+t)e^{-\frac{t}{20}}$$

$$(80+t)y = \int \left(80e^{-\frac{t}{20}} + te^{-\frac{t}{20}}\right) dt$$

$$\int 80e^{-\frac{t}{20}} dt = 80\int e^{u}(-20)du \qquad u = -\frac{t}{20} \Rightarrow du = -\frac{1}{20}dt \to -20du = dt$$

$$= -1600e^{-\frac{t}{20}}$$

$$\int te^{-\frac{t}{20}} dt = \frac{e^{-\frac{t}{20}}}{\left(-\frac{1}{20}\right)^2}\left(-\frac{1}{20}t - 1\right) \qquad \int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1)$$

$$= 400e^{-\frac{t}{20}}\left(-\frac{1}{20}t - 1\right)$$

$$= -20te^{-\frac{t}{20}} - 400e^{-\frac{t}{20}}$$

$$(80+t)y = \int \left(80e^{-\frac{t}{20}} + te^{-\frac{t}{20}}\right) dt$$

$$= -1600e^{-\frac{t}{20}} - 20te^{-\frac{t}{20}} - 400e^{-\frac{t}{20}} + C$$
$$= -2000e^{-\frac{t}{20}} - 20te^{-\frac{t}{20}} + C$$
$$= -20(100+t)e^{-\frac{t}{20}} + C$$

$$y(t) = -20\left(\frac{100+t}{80+t}\right)e^{-\frac{t}{20}} + \frac{C}{80+t}$$

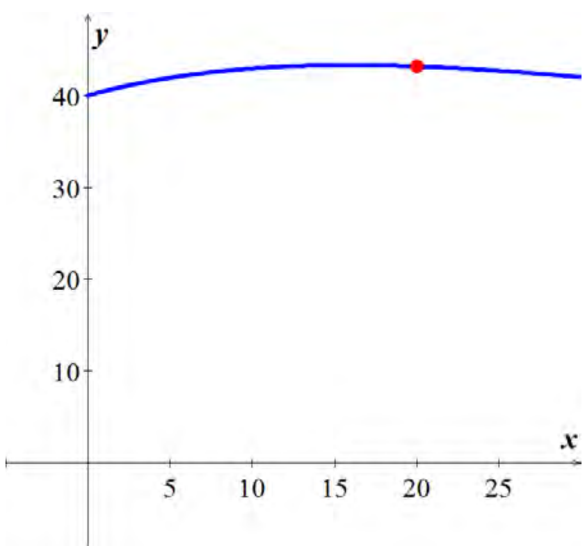
$$y(t=0) = -20\left(\frac{100+0}{80+0}\right)e^{-\frac{0}{20}} + \frac{C}{80+0}$$

$$40 = -20\left(\frac{100}{80}\right) + \frac{C}{80}$$
$$40 = -25 + \frac{C}{80}$$
$$\frac{C}{80} = 65$$
$$C = 5200$$

$$y(t) = -20\left(\frac{100+t}{80+t}\right)e^{-\frac{t}{20}} + \frac{5200}{80+t}$$

$$V(t) = 200 + 2.5t = 250$$
$$t = \frac{50}{2.5} = 20$$

$$y(t=20) = -20\left(\frac{100+20}{80+20}\right)e^{-\frac{20}{20}} + \frac{5200}{80+20}$$
$$= 43.1709\ lb$$



Exercise

A tank contains 100 *gal* of pure water. At time zero, a sugar-water solution containing 0.2 *lb* of sugar per gal enters the tank at a rate of 3 *gal/min*. Simultaneously, a drain is opened at the bottom of the tank allowing the sugar solution to leave the tank at 3 *gal/min*. Assume that the solution in the tank is kept perfectly mixed at all times.

a) What will be the sugar content in the tank after 20 minutes?
b) How long will it take the sugar content in the tank to reach 15 lb?
c) What will be the eventual sugar content in the tank?

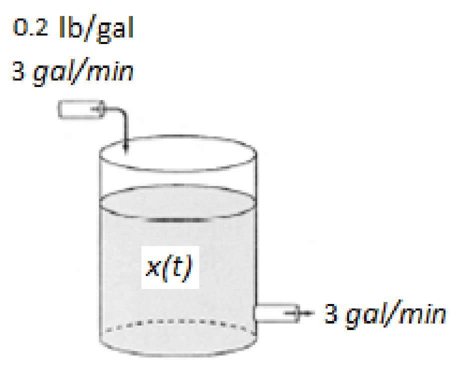
Solution

a) What will be the sugar content in the tank after 20 minutes?

$$Rate\ in = 3\frac{gal}{min}\times 0.2\frac{lb}{gal} = 0.6\frac{lb}{min}$$

$$Rate\ out = 3\frac{gal}{min}\times\frac{x(t)}{100}\frac{lb}{gal}$$

$$= \frac{3x(t)}{100}\frac{lb}{min}$$



$$\frac{dx}{dt} = 0.6 - \frac{3x}{100}$$

$$x' + \frac{3}{100}x = 0.6$$

$$u(t) = e^{\int \frac{3}{100}dt} = e^{0.03t}$$

$$\int 0.6e^{0.03t}dt = \frac{0.6}{0.03}e^{.03t} = 20e^{.03t}$$

$$x(t) = \frac{1}{e^{.03t}}\left(20e^{.03t} + C\right)$$

$$x(t) = 20 + Ce^{-.03t} \qquad x(t=0) = 20 + Ce^{-.03(0)}$$

$$0 = 20 + C \quad \rightarrow \quad \boxed{C = -20}$$

$$x(t) = 20 - 20e^{-.03t}$$

$$x(20) = 20 - 20e^{-.03(20)}$$

$$= 9.038\ \ lb$$

b) How long will it take the sugar content in the tank to reach 15 lb?

$$15 = 20 - 20e^{-.03t}$$

$$-5 = -20e^{-.03t}$$

$$e^{-.03t} = \frac{5}{20}$$

$$-.03t = \ln\frac{1}{4} \qquad \Rightarrow \boxed{t = \frac{\ln\frac{1}{4}}{-.03} \approx 46\ \ min}$$

c) What will be the eventual sugar content in the tank?

$$t\to\infty \Rightarrow e^{-.03t} \rightarrow 0$$

$$\boxed{x(t)\rightarrow 20}$$

Exercise

A tank initially contains 50 gal of sugar water having a concentration of 2 lb. of sugar for each gal of water. At time zero, pure water begins pouring into the tank at a rate of 2 gal per minute. Simultaneously, a drain is opened at the bottom of the tank so that the volume of sugar-water solution in the tank remains constant.

a) How much sugar is in the tank after 10 minutes?

b) How long will it take the sugar content in the tank to dip below 20 lb.?

c) What will be the eventual sugar content in the tank?

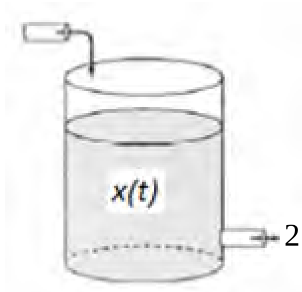
Solution

$x(t)$ represents the number of pounds of sugar.

a) $Rate\ in = 0$

$$Rate\ out = 2\frac{gal}{min} \times \frac{x(t)}{50}\frac{lb}{gal}$$

$$= \frac{x(t)}{25}\ \frac{lb}{min}$$



$$\frac{dx}{dt} = 0 - \frac{x}{25}$$

$$x(t) = Ae^{-t/25}$$

The initial condition: $x(0) = 50gal \times 2\frac{lb}{gal} = 100\ lb$

$$A = 100$$

$$x(t) = 100e^{-.04t}$$

$$x(t = 10) = 100e^{-.04(10)}$$

$$\underline{= 67.032\ \ lb|}$$

b) $x(t) = 100e^{-.04t} = 20$

$$e^{-.04t} = .2$$

$$-.04t = \ln(.2)$$

$$t = \frac{\ln(.2)}{-.04}$$

$$\underline{\approx 40.236\ \ \min|}$$

c) $x(t) = \lim\limits_{t\to\infty} 100e^{-.04t}\ \underline{= 0|}$

Exercise

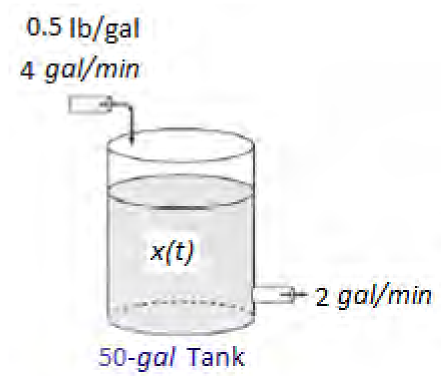
A 50-gal tank initially contains 20 *gal* of pure water. Salt-water solution containing 0.5 *lb* of salt for each gallon of water begins entering the tank at a rate of 4 *gal/min*. Simultaneously, a drain is opened at the bottom of the tank, allowing the salt-water solution to leave the tank at a rate of 2 *gal/min*. What is the salt content (*lb*) in the tank at the precise moment that the tank is full of salt-water solution?

Solution

$$V(t) = 20 + (4-2)t$$
$$= 20 + 2t$$

$$Rate\ in = 4\frac{gal}{min} \times 0.5\frac{lb}{gal} = 2\frac{lb}{min}$$

$$Rate\ out = 2\frac{gal}{min} \times \frac{x(t)}{20+2t}\ \frac{lb}{gal}$$
$$= \frac{x(t)}{10+t}\ \frac{lb}{min}$$



$$x' = 2 - \frac{x}{10+t}$$

$$x' + \frac{1}{10+t}x = 2$$

$$u(t) = e^{\int \frac{1}{t+10}dt}$$
$$= e^{\ln|t+10|}$$
$$= t+10$$

$$\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b|$$

$$\left[(t+10)x\right]' = 2(t+10)$$

$$(t+10)x = 2\int (t+10)\,dt$$

$$(t+10)x = (t+10)^2 + C$$

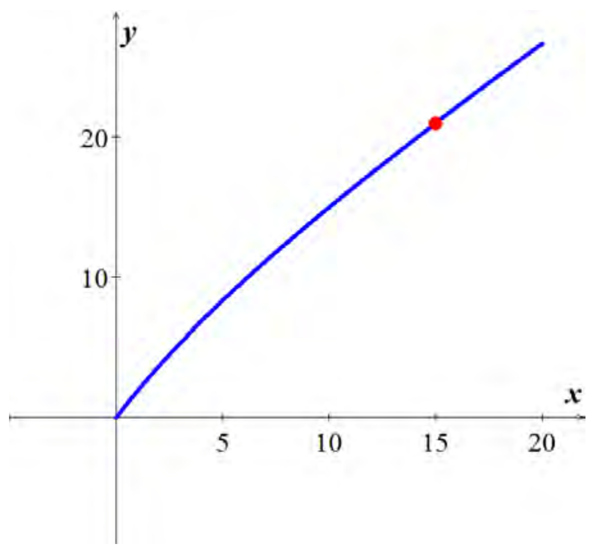
$$x(t) = t + 10 + \frac{C}{t+10}$$

$$x(t=0) = 0 + 10 + \frac{C}{0+10}$$

$$0 = 10 + \frac{C}{10}$$

$$C = -100$$

$$x(t) = t + 10 - \frac{100}{t+10}$$



Full tank: $V(t) = 20 + 2t = 40$

$2t = 30 \rightarrow t = 15\ min$

$$x(t=15) = 15 + 10 - \frac{100}{15+10}$$
$$= 25 - \frac{100}{25}$$
$$= 21\ lb$$

Exercise

A tank contains 500 *gal* of a salt-water solution containing 0.05 *lb* of salt per gallon of water. Pure water is poured into the tank and a drain at the bottom of the tank is adjusted so as to keep the volume of solution in the tank constant. At what rate (*gal/min*) should the water be poured into the tank to lower the salt concentration to 0.01 *lb/gal* of water in less than one hour?

Solution

$$V(t) = 500$$

$$Rate\ in = 0\,\frac{lb}{min}$$

$$Rate\ out = r\,\frac{gal}{min} \times \frac{x(t)}{500}\ \ \frac{lb}{gal} = \frac{r}{500}x(t)\ \ \frac{lb}{min}$$

$$\frac{dx}{dt} = -\frac{r}{500}x$$

$$\frac{dx}{x} = -\frac{r}{500}dt$$

$$\int \frac{dx}{x} = -\int \frac{r}{500}dt$$

$$\ln x = -\frac{r}{500}t + C$$

$$x = Ae^{-\frac{r}{500}t}$$

$$.05 = Ae^{-\frac{r}{500}0} \quad \Rightarrow \quad A = .05$$

The concentration at time t is given by: $c(t) = .05e^{-\frac{r}{500}t}$

The concentration reaches 1% in one hour or 60 minutes.

$$.01 = .05e^{-\frac{r}{500}60}$$

$$.2 = e^{-\frac{6}{50}r}$$

$$-\frac{6}{50}r = \ln .2$$

$$\boxed{r = -\frac{25}{3}\ln .2 \approx 13.4\ \ gal\,/\,min}$$

Exercise

A tank contains 100 gal of fresh water. A solution containing 1 lb./gal of soluble lawn fertilizer runs into the tank at the rate of 1 gal/min, and the mixture is pumped out of the tank at a rate of 3 gal/min. Find the maximum amount of fertilizer in the tank and the time required to reach the maximum.

Solution

Volume of the tank at time t is:

$$V(t) = 100\ gal + \left(1\frac{gal}{\text{min}} - 3\frac{gal}{\text{min}}\right)(t\ \text{ min}) = 100 - 2t$$

$$\frac{dy}{dt} = Rate\ in - \ Rate\ out$$

$$= \left(1\ \frac{lb}{gal}\right)\left(1\ \frac{gal}{\text{min}}\right) - \left(\frac{y}{100-2t}\ \frac{lb}{gal}\right)\left(3\ \frac{gal}{\text{min}}\right)$$

$$= 1 - \frac{3y}{100-2t}$$

$$\frac{dy}{dt} + \frac{3}{100-2t}y = 1 \quad \rightarrow \quad P(t) = \frac{3}{100-2t} \quad Q(t) = 1$$

$$e^{\int \frac{3dt}{100-2t}} = e^{\frac{3}{2}\int \frac{-dt}{100-2t}} = e^{-\frac{3}{2}\ln(100-2t)} = e^{\ln(100-2t)^{-3/2}} = (100-2t)^{-3/2}$$

$$\int 1(100-2t)^{-3/2}dt = -\frac{1}{2}\int (100-2t)^{-3/2}d(100-2t) = (100-2t)^{-1/2}$$

$$y(t) = \frac{1}{(100-2t)^{-3/2}}\left[(100-2t)^{-1/2} + C\right]$$

$$y(t) = 100 - 2t + C(100-2t)^{3/2}$$

$$y(0) = 100 - 2(0) + C(100 - 2(0))^{3/2}$$

$$0 = 100 + C(100)^{3/2}$$

$$\boxed{C = -100^{-1/2} = -\frac{1}{10}}$$

$$y(t) = 100 - 2t - 0.1(100-2t)^{3/2}$$

$$\frac{dy}{dx} = -2 - 0.1\frac{3}{2}(100-2t)^{1/2}(-2) = \boxed{-2 + 0.3(100-2t)^{1/2} = 0}$$

$$(100-2t)^{1/2} = \frac{2}{0.3} \quad \Rightarrow \quad 100 - 2t = \left(\frac{2}{0.3}\right)^2 = \frac{4}{0.09} = \frac{400}{9}$$

$$2t = 100 - \frac{400}{9} = \frac{500}{9}$$

$$\boxed{t = \frac{500}{18} \approx 12.78\ \text{ min}}$$

The maximum amount is: $\quad y(t = 12.78) = 100 - 2(12.78) - 0.1(100 - 2(12.78))^{3/2}$

$$\boxed{y \approx 14.8\ lb}$$

Exercise

A 200-gal tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 lb./gal of concentrate enters the tank at the rate of 5 gal/min, and the well-stirred mixture is withdrawn at the rate of 3 gal/min.

a) At what time will the tank be full?

b) At the time the tank is full, how many pounds of concentrate will it contain?

Solution

a) $V(t) = 100 + \left(5\frac{gal}{\text{min}} - 3\frac{gal}{\text{min}}\right)(t\ \text{ min}) = 100 + 2t$

$$200 = 100 + 2t$$

$$100 = 2t \quad \Rightarrow \quad \boxed{t = 50\ \text{ min}}$$

b) Let $y(t)$ be the amount of concentrate in the tank at time t.

$$\frac{dy}{dt} = \textit{Rate in} - \textit{Rate out}$$

$$\frac{dy}{dt} = \left(0.5\ \frac{lb}{gal}\right)\left(5\ \frac{gal}{\text{min}}\right) - \left(\frac{y}{100+2t}\ \frac{lb}{gal}\right)\left(3\ \frac{gal}{\text{min}}\right)$$

$$= \frac{5}{2} - \frac{3y}{100+2t}$$

$$\frac{dy}{dt} + \frac{3}{100+2t}y = \frac{5}{2} \quad \rightarrow \quad P(t) = \frac{3}{100+2t} \quad Q(t) = \frac{5}{2}$$

$$e^{\int \frac{3dt}{100+2t}} = e^{\frac{3}{2}\int \frac{dt}{50+t}} = e^{\frac{3}{2}\ln(50+t)} = e^{\ln(50+t)^{3/2}} = (50+t)^{3/2}$$

$$\int \frac{5}{2}(50+t)^{3/2}dt = (t+50)^{5/2}$$

$$y(t) = \frac{1}{(t+50)^{3/2}}\left[(t+50)^{5/2} + C\right]$$

$$= t + 50 + \frac{C}{(t+50)^{3/2}}$$

$$y(0) = 0 + 50 + \frac{C}{(0+50)^{3/2}}$$

$$0 = 50 + \frac{C}{50^{3/2}} \qquad \rightarrow \frac{C}{50^{3/2}} = -50 \quad \Rightarrow \quad \boxed{C = -50^{5/2}}$$

$$y(t) = t + 50 - \frac{50^{5/2}}{(t+50)^{3/2}}$$

$$y(t=50) = 50 + 50 - \frac{50^{5/2}}{(50+50)^{3/2}} \approx 83.22\ \ \underline{lb\ of\ concentrate}$$

Exercise

Suppose that an Iowa class battleship has mass 51,000 metric tons (51,000,000 kg) and $k \approx 59{,}000\ kg/\sec$. Assume that the ship loses power when it is moving at a speed of 9 m/sec.

a) About how far will the ship coast before it is dead in the water?

b) About how long will it take the ship's speed to drop to 1 m/sec?

Solution

$$v = v_0 e^{-(k/m)t} = 9e^{-(59{,}000/51{,}000{,}000)t} = 9e^{-(59/51{,}000)t}$$

a) $s(t) = \int v(t)\,dt = \int 9e^{-(59/51{,}000)t}\,dt$

$$= 9\left(-\frac{51000}{59}\right)e^{-(59/51{,}000)t} + C$$

$$= -\frac{459{,}000}{59}e^{-(59/51{,}000)t} + C$$

$$s(0) = -\frac{51{,}000}{59}e^{-(59/51{,}000)(0)} + C$$

$$0 = -\frac{51{,}000}{59} + C$$

$$\boxed{C = \frac{51{,}000}{59}}$$

$$s(t) = -\frac{459{,}000}{59}e^{-(59/51{,}000)t} + \frac{459{,}000}{59}$$

$$= \frac{459{,}000}{59}\left(1 - e^{-(59/51{,}000)t}\right)$$

$$\lim_{t\to\infty} s(t) = \frac{459{,}000}{59}\lim_{t\to\infty}\left(1 - e^{-(59/51{,}000)t}\right)$$

$$= \frac{51{,}000}{59}(1-0)$$

$$\approx 7780\ m$$

The ship will coast about 7780 meters or 7.78 km.

b) $1 = 9e^{-(59/51{,}000)t}$

$$e^{-(59/51{,}000)t} = \frac{1}{9}$$

$$-\frac{59}{51000}t = \ln\frac{1}{9}$$

$$t = -\frac{51000}{59}\ln\frac{1}{9} \approx 1899.3\ \sec$$

It will take about $\frac{1899.3}{60} \approx 61.65$ *minutes*

Exercise

A 66-kg cyclist on a 7-kg bicycle starts coasting on level ground at 9 m/sec. The $k \approx 3.9\ kg/\sec$

a) About how far will the cyclist coast before reaching a complete stop?

b) How long will it take the cyclist's speed to drop to 1 m/sec?

Solution

Mass: $m = 66 + 7 = 73\ kg$

$$v = v_0 e^{-(k/m)t} = 9e^{-(3.9/73)t}$$

a) $s(t) = \int v(t)\,dt = \int 9e^{-(3.9/73)t}\,dt$

$$= 9\left(-\frac{73}{3.9}\right)e^{-(3.9/73)t} + C$$

$$= -\frac{219}{1.3}e^{-(3.9/73)t} + C$$

$$= -\frac{2190}{13}e^{-(3.9/73)t} + C$$

$$s(0) = -\frac{2190}{13}e^{-(3.9/73)(0)} + C$$

$$0 = -\frac{2190}{13} + C$$

$$\boxed{C = \frac{2190}{13}}$$

$$s(t) = -\frac{2190}{13}e^{-(3.9/73)t} + \frac{2190}{13} = \frac{2190}{13}\left(1 - e^{-(3.9/73)t}\right)$$

$$\lim_{t\to\infty} s(t) = \frac{2190}{13}\lim_{t\to\infty}\left(1 - e^{-(3.9/73)t}\right)$$

$$= \frac{2190}{13}(1-0)$$

$$\approx 168.5$$

The cyclist coast about 168.5 meters.

b) $1 = 9e^{-(3.9/76)t}$

$$\frac{1}{9} = e^{-(3.9/73)t} \quad \Rightarrow \quad -\frac{3.9}{73}t = \ln\frac{1}{9}$$

$$t = -\frac{73}{3.9}\ln\frac{1}{9} \approx 41.13\ \sec$$

It will take about 41.13 seconds.

Exercise

An Executive conference room of a corporation contains 4500 ft^3 of air initially free of carbon monoxide. Starting at time *t* = 0, cigarette smoke containing 4% carbon monoxide is blown into the room at the rate of 0.3 $ft^3 / \min$. A ceiling fan keeps the air in the room well circulated and the air leaves the room at the same rate of 0.3 $ft^3 / \min$. Find the time when the concentration of carbon monoxide in the room reaches 0.01%.

Solution

Let $y(t)$ be the amount of carbon monoxide (CO) in the room at time *t*.

$$\frac{dy}{dt} = Rate\ in - \ Rate\ out$$

$$\frac{dy}{dt} = (0.04)(0.3) - \left(\frac{y}{4500}\right)(0.3)$$

$$\frac{dy}{dt} = \frac{12}{1000} - \frac{y}{15,000}$$

$$\frac{dy}{dt} + \frac{1}{15,000}y = \frac{12}{1000} \quad \rightarrow \quad P(t) = \frac{1}{15,000} \quad Q(t) = \frac{12}{1000}$$

$$e^{\int \frac{dt}{15000}} = e^{\frac{1}{15000}t}$$

$$\int \frac{12}{1000}e^{\frac{1}{15000}t}dt = \frac{12}{1000}15000e^{\frac{1}{15000}t} = 180e^{\frac{1}{15000}t}$$

$$y(t) = \frac{1}{e^{\frac{1}{15000}t}}\left[180e^{\frac{1}{15000}t} + C\right]$$

$$y(t) = 180 + Ce^{\frac{-1}{15000}t}$$

$$y(0) = 180 + Ce^{\frac{-1}{15000}0}$$

$$0 = 180 + C \quad \Rightarrow \quad \boxed{C = -180}$$

$$y(t) = 180 - 180e^{\frac{-1}{15000}t}$$

When the concentration of CO is 0.01% in the room, the amount of CO satisfies

$$\frac{y}{4500} = \frac{.01}{100} \quad \Rightarrow \quad y = 0.45\,ft^3$$

When the room contains the amount $y = 0.45\,ft^3$

$$0.45 = 180 - 180e^{\frac{-1}{15000}t}$$

$$180e^{\frac{-1}{15000}t} = 179.55$$

$$e^{\frac{-1}{15000}t} = \frac{179.55}{180}$$

$$\frac{-1}{15000}t = \ln\left(\frac{179.55}{180}\right)$$

$$t = -15000\ln\left(\frac{179.55}{180}\right)$$

$$\boxed{t \approx 37.55\ \text{min}}$$

Exercise

Consider the cascade of 2 tanks with $V_1 = 100\ gal$ and $V_2 = 200\ gal$ the volumes of brine in the 2 tanks. Each tank also initially contains 50 lb. of salt. The three flow rates indicated in the figure are each 5 gal/min, with pure water flowing into tank.

a) Find the amount $x(t)$ of salt in tank 1 at time *t*.

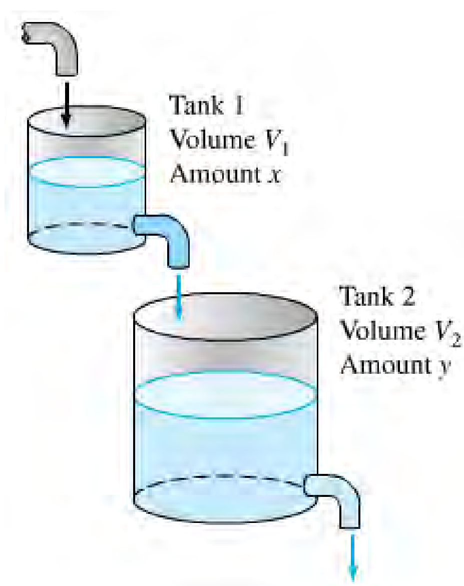
b) Suppose that $y(t)$ is the amount of salt in tank 2 at time *t*. Show first that

$$\frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200}$$

And then solve for $y(t)$, using the function $x(t)$ found in part (*a*).

c) Finally, find the maximum amount of salt ever in tank 2.

Solution



a) $\frac{dx}{dt} = -\frac{x}{20}$ *and* $x(0) = 50$

$$\frac{dx}{x} = -\frac{1}{20}dt$$

$$\int \frac{dx}{x} = -\frac{1}{20}\int dt$$

$$\ln|x| = -\frac{t}{20} + C$$

$$x = e^{-\frac{t}{20}+C} = Ae^{-\frac{t}{20}}$$

$$50 = Ae^{-\frac{0}{20}}$$

$$\boxed{50 = A}$$

$$x(t) = 50e^{-t/20}$$

b) $\frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200}$

$$= \frac{1}{20}\left(50e^{-t/20}\right) - \frac{y}{40}$$

$$= \frac{5}{2}e^{-t/20} - \frac{1}{40}y$$

$$y' + \frac{1}{40}y = \frac{5}{2}e^{-t/20}$$

$$e^{\int \frac{1}{40}dt} = e^{t/40}$$

$$\int \frac{5}{2}e^{-t/20}e^{t/40}dt = \frac{5}{2}\int e^{-t/40}dt = -100e^{-t/40}$$

$$y(t) = \frac{1}{e^{t/40}}\left(-100e^{-t/40} + C\right)$$

$$y(t) = -100e^{-t/20} + Ce^{-t/40}$$

With $y(0) = 50$

$$50 = -100e^{-0/20} + Ce^{-0/40}$$

$$50 = -100 + C$$

$$\boxed{C = 150}$$

$$y(t) = 150e^{-t/40} - 100e^{-t/20}$$

c) The maximum value of y occurs when

$$y'(t) = -\frac{15}{4}e^{-t/40} + 5e^{-t/20}$$

$$= 5e^{-t/40}\left(-\frac{3}{4} + e^{-t/40}\right) = 0$$

$$-\frac{3}{4} + e^{-t/40} = 0$$

$$e^{-t/40} = \frac{3}{4}$$

$$-\frac{t}{40} = \ln\left(\frac{3}{4}\right)$$

$$t = -40\ln\left(\frac{3}{4}\right) \underline{\approx 11.51 \text{ min.}}$$

Exercise

Suppose that in the cascade tank 1 initially 100 gal of pure ethanol and tank 2 initially contains 100 gal of pure water. Pure water flows into tank 1 at 10 gal/min, and the other two flow rates are also 10 gal/min.

a) Find the amounts $x(t)$ and $y(t)$ of ethanol in the two tanks at time $t \geq 0$.

b) Find the maximum amount of ethanol ever in tank 2.

Solution

a) The initial value problem $\frac{dx}{dt} = -\frac{x}{10}, \quad x(0) = 100$

For ***Tank*** 1:

$$\frac{1}{x}dx = -\frac{1}{10}dt$$

$$\int \frac{1}{x}dx = -\frac{1}{10}\int dt$$

$$\ln|x| = -\frac{1}{10}t + C$$

$$x = e^{-t/10+C}$$

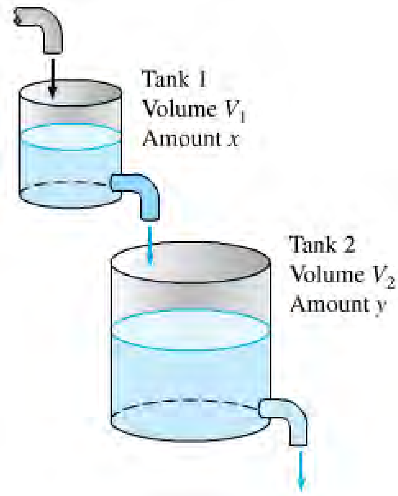
$$x = e^{C}e^{-t/10}$$

$$x = Ae^{-t/10}$$

$$100 = Ae^{-0/10}$$

$$\boxed{100 = A}$$

$$x(t) = 100e^{-t/10}$$



The initial value problem $\frac{dy}{dt} = \frac{x}{10} - \frac{y}{10}, \quad y(0) = 0$

For ***Tank*** 2:

$$\frac{dy}{dt} = \frac{100e^{-t/10}}{10} - \frac{y}{10} = 10e^{-t/10} - \frac{y}{10}$$

$$\frac{dy}{dt} + \frac{y}{10} = 10e^{-t/10} \qquad P(t) = \frac{1}{10}, \quad Q(t) = 10e^{-t/10}$$

$$e^{\int \frac{1}{10}dt} = e^{t/10}$$

$$\int 10e^{-t/10}e^{t/10}\,dt = 10\int dt = 10t$$

$$y(t) = \frac{1}{e^{t/10}}(10t + C) \qquad y = \frac{1}{e^{\int Pdt}}\left(\int Q.e^{\int Pdt}dt + C\right)$$

$$y(t=0) = \frac{1}{e^{0/10}}(10(0) + C)$$

$$\boxed{0 = C}$$

$$y(t) = \frac{1}{e^{t/10}}(10t)$$

$$y(t) = 10te^{-t/10}$$

b) The maximum value of y occurs when

$$y'(t) = 10e^{-t/10} - te^{-t/10} = 0$$

$$(10-t)e^{-t/10} = 0$$

$$10 - t = 0 \quad \rightarrow \quad \boxed{t = 10}$$

Thus when $t = 10$,

$$y_{max} = 10(10)e^{-10/10}$$

$$= 100e^{-1}$$

$$\approx 36.79 \ \ gal$$

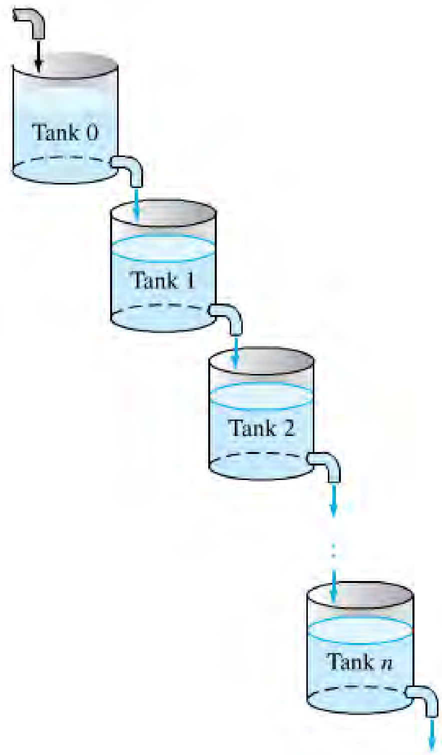
Exercise

A multiple cascade is shown in the figure. At time $t = 0$, tank 0 contains 1 gal of ethanol and 1 gal of water; all the remaining tanks contain 2 gal of pure water each. Pure water is pumped into tank 0 at 1 gal./min, and the varying mixture in each tank is pumped into the one below it at the same rate. Assume, as usual, that the mixtures are kept perfectly uniform by stirring. Let $x_n(t)$ denote the amount of ethanol in tank n at time t.

a) Show that $x_0(t) = e^{-t/2}$

b) Show that the maximum value of $x_n(t)$ for $n > 0$ is $M_n = x_n(2n) = \dfrac{n^n e^{-n}}{n!}$

Solution



a) For ***Tank 0***: $\dfrac{dx}{dt} = -\dfrac{x}{1+1}, \quad x(0) = 1$

$$\frac{1}{x}dx = -\frac{1}{2}dt$$

$$\int \frac{1}{x}dx = -\frac{1}{2}\int dt$$

$$\ln|x| = -\frac{1}{2}t + C$$

$$x_0 = e^{-t/2+C}$$

$$x_0 = Ae^{-t/2}$$

$$1 = Ae^{-0/2}$$

$$\boxed{1 = A}$$

$$\boxed{x_0(t) = e^{-t/2}}$$

b) Using the induction

For $\boldsymbol{n} = \mathbf{0}$, then $x_0(t) = \dfrac{t^0 e^{-t/2}}{0!\ 2^0} = e^{-t/2}$ √ True

Assume it is true for n: $x_n(t) = \dfrac{t^n e^{-t/2}}{n!\ 2^n}$

We need to prove that the equation for

$x_{n+1} = \dfrac{t^{n+1} e^{-t/2}}{(n+1)!\ 2^{n+1}}$ is also true.

$$\frac{dx_{n+1}}{dt} = \frac{1}{2}x_n - \frac{1}{2}x_{n+1}$$

$$= \frac{1}{2}\frac{t^n e^{-t/2}}{n!\ 2^n} - \frac{1}{2}x_{n+1}$$

$$= \frac{t^n e^{-t/2}}{n!\ 2^{n+1}} - \frac{1}{2}x_{n+1}$$

$$\frac{dx_{n+1}}{dt}+\frac{1}{2}x_{n+1}=\frac{1}{n!\,2^{n+1}}t^n e^{-t/2}$$

$$P(t)=\frac{1}{2},\quad Q(t)=\frac{1}{n!\,2^{n+1}}t^n e^{-t/2}$$

$$e^{\int \frac{1}{2}dt}=e^{t/2}$$

$$\int \frac{1}{n!\,2^{n+1}}t^n e^{-t/2}e^{t/2}\,dt=\frac{1}{n!\,2^{n+1}}\int t^n dt$$

$$=\frac{1}{n!\,2^{n+1}}\cdot\frac{t^{n+1}}{n+1}$$

$$=\frac{t^{n+1}}{(n+1)!\,2^{n+1}}$$

$$x_{n+1}=\frac{1}{e^{t/2}}\left(\frac{t^{n+1}}{(n+1)!\,2^{n+1}}+C\right)$$

$$x=\frac{1}{e^{\int Pdt}}\left(\int Q.e^{\int Pdt}dt+C\right)$$

$$x_{n+1}(0)=0$$

$$0=\frac{1}{e^{0/2}}\left(\frac{0^{n+1}}{(n+1)!\,2^{n+1}}+C\right)$$

$$0=\frac{1}{1}(0+C)$$

$$\boxed{0=C}$$

$$x_{n+1}=\frac{t^{n+1}\,e^{-t/2}}{(n+1)!\,2^{n+1}}$$

√ It is also true

c) $$(x_n)'=\frac{1}{n!\,2^n}\left(t^n e^{-t/2}\right)'$$

$$=\frac{1}{n!\,2^n}\left(nt^{n-1}e^{-t/2}-\frac{1}{2}t^n e^{-t/2}\right)$$

$$=\frac{e^{-t/2}t^{n-1}}{n!\,2^n}\left(n-\frac{1}{2}t\right)=0$$

$$\begin{cases} t=0 \\ n-\frac{1}{2}t=0 \end{cases}\to\ t=2n$$

$$M_n=x_n(t=2n)=\frac{(2n)^n\,e^{-2n/2}}{n!\,2^n}$$

$$=\frac{2^n n^n e^{-n}}{n!\,2^n}$$

$$=\frac{n^n e^{-n}}{n!}$$

Exercise

Assume that Lake Erie has a volume of 480 km^3 and that its rate of inflow (from Lake Huron) and outflow (to Lake Ontario) are both 350 km^3 per year. Suppose that at the time $t = 0$ (years), the pollutant concentration of Lake Erie – caused by past industrial pollution that has now been ordered to cease – is 5 times that of Lake Huron. If the outflow henceforth is perfectly mixed lake water, how long will it take to reduce the pollution concentration in Lake Erie to twice that of Lake Huron?

Solution

Given: $V = 480\ km^3$

$r_i = r_o = r = 350\ km^3 / yr$

$c_i = c$ (The pollutant concentration of Lake Huron)

$x_0 = x(0) = 5cV$

$$\frac{dx}{dt} = rc - \frac{r}{V}x$$

$$\frac{dx}{dt} + \frac{r}{V}x = rc \qquad P(t) = \frac{r}{V}, \quad Q(t) = rc$$

$$e^{\int \frac{r}{V}dt} = e^{rt/V}$$

$$\int rce^{rt/V}\, dt = rc\frac{V}{r}e^{rt/V} = cVe^{rt/V}$$

$$x(t) = \frac{1}{e^{rt/V}}\left(cVe^{rt/V} + C\right) \qquad x = \frac{1}{e^{\int Pdt}}\left(\int Q.e^{\int Pdt}dt + C\right)$$

$$x(t) = cV + Ce^{-rt/V}$$

$$x(0) = 5cV$$

$$5cV = cV + Ce^{0}$$

$$\boxed{C = 4cV}$$

$$x(t) = cV + 4cVe^{-rt/V}$$

When is $x(t) = 2cV$

$$2cV = cV + 4cVe^{-rt/V}$$

$$4cVe^{-rt/V} = cV$$

$$e^{-rt/V} = \frac{1}{4}$$

$$e^{rt/V} = 4$$

$$\frac{r}{V}t = \ln 4$$

$$t = \frac{V}{r}\ln 4 = \frac{480}{350}\ln 4 \approx 1.901\ years$$

Exercise

A 120 gal tank initially contains 90 lb. of salt dissolved in 90 gal of water. Brine containing 2 lb./gal of salt flows into the tank at rate of 4 gal/min, and the well-stirred mixture flows out the tank at the rate of 3 gal/min. How much salt does the tank contain when it is full?

Solution

The volume of brine in the tank increase steadily with: $V(t) = 90 + t$

The change Δx in the amount x of salt in the tank from time t to time $t + \Delta t$ is given by:

$$\Delta x = 4(2)\Delta t - 3\left(\frac{x}{90+t}\right)\Delta t$$

$$dx = \left(8 - \frac{3x}{90+t}\right)dt$$

$$\frac{dx}{dt} = 8 - \frac{3x}{90+t}$$

$$\frac{dx}{dt} + \frac{3x}{90+t} = 8 \qquad P(t) = \frac{3}{90+t}, \quad Q(t) = 8$$

$$e^{\int \frac{3}{90+t}dt} = e^{3\ln(90+t)} = e^{\ln(90+t)^3} = (90+t)^3$$

$$\int 8(90+t)^3\,dt = 8\int (90+t)^3\,d(90+t) = 2(90+t)^4$$

$$x(t) = \frac{1}{(90+t)^3}\left(2(90+t)^4 + C\right) \qquad x = \frac{1}{e^{\int Pdt}}\left(\int Q.e^{\int Pdt}dt + C\right)$$

$$x(t) = 180 + 2t + C(90+t)^{-3}$$

$$x(0) = 90$$

$$90 = 180 + 2(0) + C(90+0)^{-3}$$

$$-90 = C(90)^{-3}$$

$$C = -(90)^4$$

$$x(t) = 180 + 2t - (90)^4(90+t)^{-3}$$

$$x(t) = 180 + 2t - \frac{90^4}{(90+t)^3}$$

The tank is full after 30 min, Therefore when $t = 30$,

$$x(t=30) = 180 + 2(30) - \frac{90^4}{(90+30)^3}$$

$$x(30) = 240 - \frac{90^4}{120^3} \underline{\approx 202\ lb}$$

The tank contains 202 lb. of salt.

Solution   *Section* 1.6 – Exact Differential Equations

Exercise

Solve the differential equation $(2x+y)dx+(x-6y)dy=0$

Solution

$$\frac{\partial\psi}{\partial x}=M=2x+y \quad\Rightarrow\quad M_y=1$$

$$\frac{\partial\psi}{\partial y}=N=x-6y \quad\Rightarrow\quad N_x=1$$

$$\Rightarrow M_y=N_x$$

$$\frac{\partial\psi}{\partial x}=2x+y \quad\Rightarrow\quad \psi=\int(2x+y)dx=x^2+xy+h(y)$$

$$\psi_y=x+h'(y)=x-6y \quad\Rightarrow\quad h'(y)=-6y$$

$$h(y)=\int -6y\,dy=-3y^2$$

$$\psi(x,y)=\underline{x^2+xy-3y^2=C}$$

Exercise

Solve the differential equation $(2x+3)dx+(2y-2)dy=0$

Solution

$$\frac{\partial\psi}{\partial x}=M=2x+3 \quad\Rightarrow\quad M_y=2$$

$$\frac{\partial\psi}{\partial y}=N=2y-2 \quad\Rightarrow\quad N_x=2$$

$$\Rightarrow M_y=N_x$$

$$\frac{\partial\psi}{\partial x}=2x+3 \quad\Rightarrow\quad \psi=\int(2x+3)dx=x^2+3x+h(y)$$

$$\psi_y=h'(y)=2y-2 \quad\Rightarrow\quad h(y)=\int(2y-2)dy$$

$$=y^2-2y+C$$

$$\psi(x,y)=\underline{x^2+3x+y^2-2y=C}$$

Exercise

Solve the differential equation $\left(1-y\sin x\right)+\left(\cos x\right)y'=0$

Solution

$$\frac{\partial\psi}{\partial x}=M=1-y\sin x \quad\Rightarrow\quad M_y=-\sin x$$

$$\frac{\partial\psi}{\partial y}=N=\cos x \quad\Rightarrow\quad N_x=-\sin x$$

$$\Rightarrow M_y=N_x$$

$$\frac{\partial\psi}{\partial x}=1-y\sin x \quad\Rightarrow\quad \psi=\int\left(1-y\sin x\right)dx=x+y\cos x+h\left(y\right)$$

$$\psi_y=\cos x+h'\left(y\right)=\cos x \quad\Rightarrow\quad h'\left(y\right)=0$$

$$h\left(y\right)=C$$

$$\psi\left(x,\ y\right)=\underline{x+y\cos x=C}$$

Exercise

Solve the differential equation $\frac{dy}{dx}=-\frac{ax+by}{bx+cy}$

Solution

$$\left(bx+cy\right)dy=-\left(ax+by\right)dx$$

$$\left(ax+by\right)dx+\left(bx+cy\right)dy=0$$

$$\frac{\partial\psi}{\partial x}=M=ax+by \quad\Rightarrow\quad M_y=b$$

$$\frac{\partial\psi}{\partial y}=N=bx+cy \quad\Rightarrow\quad N_x=b$$

$$\Rightarrow M_y=N_x$$

$$\frac{\partial\psi}{\partial x}=ax+by \quad\Rightarrow\quad \psi=\int\left(ax+by\right)dx=\frac{1}{2}ax^2+bxy+h\left(y\right)$$

$$\psi_y=bx+h'\left(y\right)=bx+cy \quad\Rightarrow\quad h'\left(y\right)=cy$$

$$h\left(y\right)=\int cydy=\frac{1}{2}cy^2$$

$$\psi\left(x,\ y\right)=\frac{1}{2}ax^2+bxy+\frac{1}{2}cy^2=D$$

$$\boxed{ax^2+2bxy+cy^2=E} \qquad \left(E=2D\right)$$

Exercise

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{3x^2 + y}{3y^2 - x}$

Solution

$$\left(3x^2 + y\right)dx - \left(3y^2 - x\right)dy = 0$$

$$\frac{\partial \psi}{\partial x} = M = 3x^2 + y \quad \Rightarrow \quad M_y = 1$$

$$\frac{\partial \psi}{\partial y} = N = -3y^2 + x \quad \Rightarrow \quad N_x = 1$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = 3x^2 + y \quad \Rightarrow \quad \psi = \int \left(3x^2 + y\right)dx = x^3 + xy + h(y)$$

$$\psi_y = x + h'(y) = -3y^2 + x \quad \Rightarrow \quad h'(y) = -3y^2$$

$$h(y) = \int -3y^2 dy = -y^3$$

$$\psi(x,\, y) = \underline{x^3 + xy - y^2 = C}$$

Exercise

Solve the differential equation $\left(3x^2 - 2xy + 2\right) + \left(6y^2 - x^2 + 3\right)y' = 0$

Solution

$$\frac{\partial \psi}{\partial x} = M = 3x^2 - 2xy + 2 \quad \Rightarrow \quad M_y = -2x$$

$$\frac{\partial \psi}{\partial y} = N = 6y^2 - x^2 + 3 \quad \Rightarrow \quad N_x = -2x$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = 3x^2 - 2xy + 2 \quad \Rightarrow \quad \psi = \int \left(3x^2 - 2xy + 2\right)dx = x^3 - x^2 y + 2x + h(y)$$

$$\psi_y = -x^2 + h'(y) = 6y^2 - x^2 + 3 \quad \Rightarrow \quad h'(y) = 6y^2 + 3$$

$$h(y) = \int \left(6y^2 + 3\right)dy = 2y^3 + 3y$$

$$\psi(x,\, y) = \underline{x^3 - x^2 y + 2x + 2y^3 + 3y = C}$$

Exercise

Solve the differential equation $\left(e^x \sin y - 2y\sin x\right)dx + \left(e^x \cos y + 2\cos x\right)dy = 0$

Solution

$$\frac{\partial \psi}{\partial x} = M = e^x \sin y - 2y\sin x \quad \Rightarrow \quad M_y = e^x \cos y - 2\sin x$$

$$\frac{\partial \psi}{\partial y} = N = e^x \cos y + 2\cos x \quad \Rightarrow \quad N_x = e^x \cos y - 2\text{sinx}$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = e^x \sin y - 2y\sin x \quad \Rightarrow \quad \psi = \int \left(e^x \sin y - 2y\sin x\right)dx = e^x \sin y + 2y\cos x + h(y)$$

$$\psi_y = e^x \cos y + 2\cos x + h'(y) = e^x \cos y + 2\cos x \quad \Rightarrow \quad h'(y) = 0$$

$$h(y) = C$$

$$\underline{e^x \sin y + 2y\cos x = C}$$

Exercise

Solve the differential equation $\left(\frac{y}{x} + 6x\right)dx + (\ln x - 2)dy = 0, \qquad x > 0$

Solution

$$\frac{\partial \psi}{\partial x} = M = \frac{y}{x} + 6x \quad \Rightarrow \quad M_y = \frac{1}{x}$$

$$\frac{\partial \psi}{\partial y} = N = \ln x - 2 \quad \Rightarrow \quad N_x = \frac{1}{x}$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = e^x \sin y - 2y\sin x \quad \Rightarrow \quad \psi = \int \left(\frac{y}{x} + 6x\right)dx = y\ln x + 3x^2 + h(y)$$

$$\psi_y = \ln x + h'(y) = \ln x - 2 \quad \Rightarrow \quad h'(y) = -2$$

$$h(y) = \int -2dy = -2y$$

$$\underline{y\ln x + 3x^2 - 2y = C}$$

Exercise

Solve the differential equation $$\frac{xdx}{\left(x^2+y^2\right)^{3/2}}+\frac{ydy}{\left(x^2+y^2\right)^{3/2}}=0$$

Solution

Multiply both side by $\left(x^2+y^2\right)^{3/2}$ since $x^2+y^2\neq 0$

$$xdx+ydy=0$$

$$\frac{\partial\psi}{\partial x}=M=x \quad\Rightarrow\quad M_y=0 \qquad \frac{\partial\psi}{\partial y}=N=y \quad\Rightarrow\quad N_x=0 \qquad \Rightarrow M_y=N_x$$

$$\frac{\partial\psi}{\partial x}=x \quad\Rightarrow\quad \psi=\int xdx=\frac{1}{2}x^2+h(y)$$

$$\psi_y=h'(y)=y \quad\Rightarrow\quad h(y)=\int ydy=\frac{1}{2}y^2$$

$$\frac{1}{2}x^2+\frac{1}{2}y^2=C_1$$

$$\underline{x^2+y^2=C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$x^2y^3+x\left(1+y^2\right)y'=0, \qquad \mu(x,y)=\frac{1}{xy^3}$$

Solution

$$M_y=\frac{\partial}{\partial y}\left(x^2y^3\right)=3x^2y^2 \qquad N_x=\frac{\partial}{\partial x}\left(x+xy^2\right)=1+y^2 \quad\Rightarrow M_y\neq N_x$$

$$x^2y^3\left(\frac{1}{xy^3}\right)+x\left(1+y^2\right)\left(\frac{1}{xy^3}\right)y'=0$$

$$x+\left(\frac{1+y^2}{y^3}\right)\frac{dy}{dx}=0 \quad\Rightarrow\quad \left(\frac{1+y^2}{y^3}\right)dy=-xdx$$

$$\int\left(y^{-3}+\frac{1}{y}\right)dy=-\int xdx$$

$$-\frac{1}{2}y^{-2}+\ln|y|=-\frac{1}{2}x^2+C_0$$

$$\underline{x^2-y^{-2}+\ln|y|=C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$y^2 - xy + \left(x^2\right)y' = 0, \qquad \mu(x, y) = \frac{1}{xy^2}$$

Solution

$$M_y = \frac{\partial}{\partial y}\left(y^2 - xy\right) = 2y - x \qquad N_x = \frac{\partial}{\partial x}\left(x^2\right) = 2x \quad \Rightarrow M_y \neq N_x$$

$$\left(y^2 - xy\right)\left(\frac{1}{xy^2}\right) + \left(x^2\right)\left(\frac{1}{xy^2}\right)y' = 0$$

$$\left(\frac{1}{x} - \frac{1}{y}\right) + \left(\frac{x}{y^2}\right)y' = 0$$

$$M_y = \frac{\partial}{\partial y}\left(\frac{1}{x} - \frac{1}{y}\right) = \frac{1}{y^2} \qquad N_x = \frac{\partial}{\partial x}\left(\frac{x}{y^2}\right) = \frac{1}{y^2} \quad \Rightarrow M_y = N_x$$

$$\frac{\partial\psi}{\partial x} = \frac{1}{x} - \frac{1}{y} \quad \Rightarrow \quad \psi = \int\left(\frac{1}{x} - \frac{1}{y}\right)dx = \ln|x| - \frac{x}{y} + h(y)$$

$$\psi_y = \frac{x}{y^2} + h'(y) = \frac{x}{y^2} \quad \Rightarrow \quad h'(y) = 0 \quad \rightarrow h(y) = C$$

$$\boxed{\ln|x| - \frac{x}{y} = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$x^2y^3 - y + x\left(1 + x^2y^2\right)y' = 0, \qquad \mu(x, y) = \frac{1}{xy}$$

Solution

$$M_y = \frac{\partial}{\partial y}\left(x^2y^3 - y\right) = 3y^2 - 1 \qquad N_x = \frac{\partial}{\partial x}\left(x + x^3y^2\right) = 1 + 3x^2y^2 \quad \Rightarrow M_y \neq N_x$$

$$\left(x^2y^3 - y\right)\left(\frac{1}{xy}\right) + x\left(1 + x^2y^2\right)\left(\frac{1}{xy}\right)y' = 0$$

$$\left(xy^2 - \frac{1}{x}\right) + \left(\frac{1}{y} + x^2y\right)y' = 0$$

$$M_y = \frac{\partial}{\partial y}\left(xy^2 - \frac{1}{x}\right) = 2xy \qquad N_x = \frac{\partial}{\partial x}\left(\frac{1}{y} + x^2y\right) = 2xy \quad \Rightarrow M_y = N_x$$

$$\frac{\partial\psi}{\partial x} = xy^2 - \frac{1}{x} \quad \Rightarrow \quad \psi = \int \left(xy^2 - \frac{1}{x}\right)dx = \frac{1}{2}x^2y^2 - \ln|x| + h(y)$$

$$\psi_y = x^2 y + h'(y) = \frac{1}{y} + x^2 y \quad \Rightarrow \quad h'(y) = \frac{1}{y} \quad \rightarrow h(y) = \ln|y|$$

$$\boxed{\frac{1}{2}x^2y^2 - \ln x + \ln y = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$\left(\frac{\sin y}{y} - 2e^{-x}\sin x\right)dx + \left(\frac{\cos y + 2e^{-x}\cos x}{y}\right)dy = 0, \qquad \mu(x, y) = ye^x$$

Solution

$$M_y = \frac{\partial}{\partial y}\left(\frac{\sin y}{y} - 2e^{-x}\sin x\right) = \frac{y\cos y - \sin y}{y^2}$$

$$N_x = \frac{\partial}{\partial x}\left(\frac{\cos y + 2e^{-x}\cos x}{y}\right) = \frac{1}{y}\left(-2e^{-x}\cos x - 2e^{-x}\sin x\right)$$

$$\Rightarrow M_y \neq N_x$$

$$\left(ye^x\right)\left(\frac{\sin y}{y} - 2e^{-x}\sin x\right)dx + \left(ye^x\right)\left(\frac{\cos y + 2e^{-x}\cos x}{y}\right)dy = 0$$

$$\left(e^x \sin y - 2y\sin x\right)dx + \left(e^x\cos y + 2\cos x\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(e^x\sin y - 2y\sin x\right) = e^x\cos y - 2\sin x \qquad N_x = \frac{\partial}{\partial x}\left(e^x\cos y + 2\cos x\right) = e^x\cos y - 2\sin x$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial\psi}{\partial x} = e^x\sin y - 2y\sin x \quad \Rightarrow \quad \psi = \int\left(e^x\sin y - 2y\sin x\right)dx = e^x\sin y + 2y\cos x + h(y)$$

$$\psi_y = e^x\cos y + 2\cos x + h'(y) = e^x\cos y + 2\cos x \quad \Rightarrow \quad h'(y) = 0 \quad \rightarrow \; h(y) = C$$

$$\boxed{e^x\sin y + 2y\cos x = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$(x+2)\sin y\,dx + x\cos y\,dy = 0, \qquad \mu(x,\ y) = xe^{x}$

Solution

$$M_y = \frac{\partial}{\partial y}\big((x+2)\sin y\big) = (x+2)\cos y \qquad N_x = \frac{\partial}{\partial x}(x\cos y) = -x\sin y \qquad \Rightarrow M_y \neq N_x$$

$$\left(xe^{x}\right)(x+2)\sin y\,dx + \left(xe^{x}\right)x\cos y\,dy = 0$$

$$\left(x^2+2x\right)e^{x}\sin y\,dx + x^2e^{x}\cos y\,dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(\left(x^2+2x\right)e^{x}\sin y\right) = \left(x^2+2x\right)e^{x}\cos y \qquad N_x = \frac{\partial}{\partial x}\left(x^2e^{x}\cos y\right) = \left(2xe^{x}+x^2\right)e^{x}\cos y$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial\psi}{\partial y} = x^2e^{x}\cos y \quad \Rightarrow \quad \psi = \int\left(x^2e^{x}\cos y\right)dy = x^2e^{x}\sin y + h(x)$$

$$\psi_x = \left(x^2+2x\right)e^{x}\sin y + h'(x) = \left(x^2+2x\right)e^{x}\sin y \quad \Rightarrow \quad h'(x) = 0 \ \rightarrow\ h(x) = C$$

$$\boxed{x^2e^{x}\sin y = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$\left(x^2+y^2-x\right)dx - y\,dy = 0, \qquad \mu(x,\ y) = \dfrac{1}{x^2+y^2}$

Solution

$$M_y = \frac{\partial}{\partial y}\left(x^2+y^2-x\right) = 2y \qquad N_x = \frac{\partial}{\partial x}(-y) = 0 \qquad \Rightarrow M_y \neq N_x$$

$$\frac{1}{x^2+y^2}\left(x^2+y^2-x\right)dx - \frac{y}{x^2+y^2}dy = 0$$

$$\left(1-\frac{x}{x^2+y^2}\right)dx - \frac{y}{x^2+y^2}dy = 0$$

$$M_y = \left(1-\frac{x}{x^2+y^2}\right) = \frac{2xy}{\left(x^2+y^2\right)^2} \qquad N_x = \left(\frac{-y}{x^2+y^2}\right) = \frac{2xy}{\left(x^2+y^2\right)^2} \qquad \Rightarrow M_y = N_x$$

$$\frac{d\psi}{dx} = 1 - \frac{x}{x^2+y^2} \quad \Rightarrow \quad \psi = \int\left(1 - \frac{x}{x^2+y^2}\right)dx = \int dx - \frac{1}{2}\int \frac{1}{x^2+y^2}d\left(x^2+y^2\right)$$

$$= \int dx - \frac{1}{2}\int \frac{1}{x^2+y^2}d\left(x^2+y^2\right)$$

$$= x - \frac{1}{2}\ln\left(x^2+y^2\right) + h(y)$$

$$\psi_y = -\frac{y}{x^2+y^2} + h'(y) = -\frac{y}{x^2+y^2} \quad \Rightarrow \quad h'(y) = 0 \quad \to h(y) = C$$

$$\boxed{x - \frac{1}{2}\ln\left(x^2+y^2\right) = C}$$

Exercise

Find the general solution of the homogenous equation $\left(x^2+y^2\right)dx - 2xydy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}\left(x^2+y^2\right) = 2y \qquad N_x = \frac{\partial}{\partial x}(-2xy) = -2y \quad \Rightarrow M_y \ne N_x$$

$$\frac{M_y - N_x}{N} = \frac{2y+2y}{-2xy} = -\frac{4y}{2xy} = -\frac{2}{x}$$

$$\frac{d\mu}{dx} = -\mu\frac{2}{x} \quad \Rightarrow \quad \int\frac{d\mu}{\mu} = -2\int\frac{dx}{x}$$

$$\ln\mu = -2\ln x$$

$$\ln\mu = \ln x^{-2} \quad \Rightarrow \quad \boxed{\mu = \frac{1}{x^2}}$$

$$\frac{1}{x^2}\left(x^2+y^2\right)dx - \frac{1}{x^2}2xydy = 0 \qquad \Rightarrow \quad \left(1+\frac{y^2}{x^2}\right)dx - \frac{2y}{x}dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(1+\frac{y^2}{x^2}\right) = \frac{2y}{x^2} \qquad N_x = \frac{\partial}{\partial x}\left(-\frac{2y}{x}\right) = \frac{2y}{x^2} \quad \Rightarrow M_y = N_x$$

$$\psi = \int\left(1+\frac{y^2}{x^2}\right)dx = x - \frac{y^2}{x} + h(y)$$

$$\psi_y = -\frac{2y}{x} + h'(y) = -\frac{2y}{x} \quad \Rightarrow \quad h'(y) = 0 \qquad \Rightarrow h(y) = C$$

$$x - \frac{y^2}{x} = C \qquad\qquad \textit{multiply by } x$$

$$\boxed{x^2 - y^2 = Cx}$$

Exercise

Find the general solution of the homogenous equation $(x+y)dx+(y-x)dy=0$

Solution

$$(x+y)dx=-(y-x)dy$$

$$\frac{dy}{dx}=\frac{x+y}{x-y}$$

$$=\frac{\frac{x+y}{x}}{\frac{x-y}{x}}$$

$$=\frac{1+\frac{y}{x}}{1-\frac{y}{x}}$$

$$\frac{dy}{dx}=\frac{1+v}{1-v}=x\frac{dv}{dx}+v$$

$$x\frac{dv}{dx}=\frac{1+v}{1-v}-v=\frac{1+v^2}{1-v}$$

$$\frac{1-v}{1+v^2}dv=\frac{dx}{x}$$

$$\int\frac{dx}{x}=\int\frac{1}{1+v^2}dv-\int\frac{v}{1+v^2}dv$$

$$\int\frac{1}{a^2+x^2}dx=\arctan\frac{x}{a}$$

$$\ln x=\arctan v-\frac{1}{2}\int\frac{1}{1+v^2}d\left(1+v^2\right)$$

$$\ln x+C=\arctan v-\frac{1}{2}\ln\left(1+v^2\right)$$

$$\ln x+C=\arctan\frac{y}{x}-\frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right)$$

$$\arctan\frac{y}{x}-\frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right)-\ln x=C$$

Exercise

Find the general solution of the homogenous equation $\dfrac{dy}{dx}=\dfrac{y\left(x^2+y^2\right)}{xy^2-2x^3}$

Solution

$$\frac{dy}{dx}=\frac{y}{x}\frac{\frac{x^2+y^2}{x^2}}{\frac{y^2-2x^2}{x^2}}$$

$$=\frac{y}{x}\frac{1+\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)^2-2}$$

$$=v\frac{1+v^2}{v^2-2}=x\frac{dv}{dx}+v$$

$$\frac{v+v^3}{v^2-2}-v=x\frac{dv}{dx}$$

$$x\frac{dv}{dx}=\frac{v+v^3-v^3+2v}{v^2-2}=\frac{3v}{v^2-2}$$

$$\int\frac{dx}{x}=\int\frac{v^2-2}{3v}dv=\frac{1}{3}\int\left(v-\frac{2}{v}\right)dv$$

$$3\ln x+C=\frac{1}{2}v^2-2\ln v$$

$$3\ln x+C=\frac{1}{2}\frac{y^2}{x^2}-2\ln\frac{y}{x}$$

$$3\ln x+C=\frac{1}{2}\frac{y^2}{x^2}-2(\ln y-\ln x)$$

$$6\ln x+C=\frac{y^2}{x^2}-4\ln y+4\ln x$$

$$\frac{y^2}{x^2}-4\ln y-2\ln x=C$$

Exercise

Find an integrating factor and solve the given equation $\left(3x^2y+2xy+y^3\right)dx+\left(x^2+y^2\right)dy=0$

Solution

$$M_y=\frac{\partial}{\partial y}\left(3x^2y+2xy+y^3\right)=3x^2+2x+3y^2 \quad N_x=\frac{\partial}{\partial x}\left(x^2+y^2\right)=2x \quad M_y\neq N_x$$

$$\frac{M_y-N_x}{N}=\frac{3x^2+2x+3y^2-2x}{x^2+y^2}=3$$

$$\frac{d\mu}{dx}=3\mu \quad\Rightarrow\quad \int\frac{d\mu}{\mu}=3\int dx$$

$$\ln\mu=3x \quad\Rightarrow\quad \mu=e^{3x}$$

$$e^{3x}\left(3x^2y+2xy+y^3\right)dx+e^{3x}\left(x^2+y^2\right)dy=0$$

$$M_y=\frac{\partial}{\partial y}\left[e^{3x}\left(3x^2y+2xy+y^3\right)\right]=e^{3x}\left(3x^2+2x+3y^2\right)$$

$$N_x=\frac{\partial}{\partial x}e^{3x}\left(x^2+y^2\right)=3e^{3x}\left(x^2+y^2\right)+2xe^{3x}=e^{3x}\left(3x^2+3y^2+2x\right)$$

$$\Rightarrow M_y=N_x$$

$$\psi=\int\left(e^{3x}\left(3x^2y+2xy+y^3\right)\right)dx=+h(y)$$

$$=e^{3x}\left(x^2y+\frac{2}{3}xy+\frac{1}{3}y^3-\frac{2}{3}xy-\frac{2}{9}y+\frac{2}{9}y\right)+h(y)$$

$$=e^{3x}\left(x^2y+\frac{1}{3}y^3\right)+h(y)$$

	$\int e^{3x}$
$3x^2y+2xy+y^3$	$\frac{1}{3}e^{3x}$
$6xy+2y$	$\frac{1}{9}e^{3x}$
$6y$	$\frac{1}{27}e^{3x}$

$$\psi_y=e^{3x}\left(x^2+y^2\right)+h'(y)=e^{3x}\left(x^2+y^2\right) \quad\Rightarrow\quad h'(y)=0$$

$$\Rightarrow h(y)=C$$

$$\boxed{e^{3x}\left(x^2y+\frac{1}{3}y^3\right)=C}$$

Exercise

Find an integrating factor and solve the given equation $dx + \left(\frac{x}{y} - \sin y\right)dy = 0$

Solution

$$y\,dx + (x - y\sin y)\,dy = 0 \qquad \textit{Multiply by } \boldsymbol{y} \textit{ both sides}$$

$$M_y = \frac{\partial}{\partial y}(y) = 1; \quad N_x = \frac{\partial}{\partial x}(x - y\sin y) = 1; \quad M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = 2x + y^2 \quad \Rightarrow \quad \psi = \int y\,dx = xy + h(y)$$

$$\psi_y = x + h'(y) = x - y\sin y \quad \Rightarrow \quad h'(y) = -y\sin y$$

	$\int -\sin y$
y	$\cos y$
1	$\sin y$

$$h(y) = -\int y\sin y\,dy = y\cos y - \sin y$$

$$\boxed{xy + y\cos y - \sin y = C}$$

Exercise

Find an integrating factor and solve the given equation $e^x dx + \left(e^x \cot y + 2y\csc y\right)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}\left(e^x\right) = 0 \qquad N_x = \frac{\partial}{\partial x}\left(e^x \cot y + 2y\csc y\right) = e^x \qquad \Rightarrow M_y \neq N_x$$

$$e^x dx + \left(e^x \frac{\cos y}{\sin y} + 2y\frac{1}{\sin y}\right)dy = 0 \qquad \textit{Multiply by } \boldsymbol{\sin y} \textit{ both sides}$$

$$(\sin y)e^x dx + (\sin y)\left(e^x \frac{\cos y}{\sin y} + 2y\frac{1}{\sin y}\right)dy = 0$$

$$e^x \sin y\,dx + \left(e^x \cos y + 2y\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(e^x \sin y\right) = e^x \cos y \qquad N_x = \frac{\partial}{\partial x}\left(e^x \cos y + 2y\right) = e^x \cos y \;\Rightarrow\; \underline{M_y = N_x}$$

$$\psi = \int \left(e^x \sin y\right)dx = e^x \sin y + h(y)$$

$$\psi_y = e^x \cos y + h'(y) = e^x \cos y + 2y \quad \Rightarrow \quad h'(y) = 2y$$

$$\Rightarrow\; h(y) = y^2$$

$$\psi(x, y) = e^x \sin y + y^2 = C$$

$$\boxed{e^x \sin y + y^2 = C}$$

Exercise

Find an integrating factor and solve the given equation $\left(3x+\frac{6}{y}\right)dx+\left(\frac{x^2}{y}+3\frac{y}{x}\right)dy=0$

Solution

$$xy\left(3x+\frac{6}{y}\right)dx+xy\left(\frac{x^2}{y}+3\frac{y}{x}\right)dy=0$$

$$\left(3x^2y+6x\right)dx+\left(x^3+3y^2\right)dy=0$$

$$M_y=\frac{\partial}{\partial y}\left(3x^2y+6x\right)=3x^2 \qquad N_x=\frac{\partial}{\partial x}\left(x^3+3y^2\right)=3x^2 \qquad \Rightarrow \underline{M_y=N_x}$$

$$\psi=\int\left(3x^2y+6x\right)dx=x^3y+3x^2+h(y)$$

$$\psi_y=x^3+h'(y)=x^3+3y^2 \quad\Rightarrow\quad h'(y)=3y^2 \ \Rightarrow\ h(y)=y^3$$

$$\psi(x,y)=x^3y+3x^2+y^3=C$$

$$\boxed{x^3y+3x^2+y^3=C}$$

Exercise

Solve the given initial-value problem $\frac{dy}{dx}=\frac{xy^2-\cos x\sin x}{y\left(1-x^2\right)}, \qquad y(0)=2$

Solution

$$\left(xy^2-\cos x\sin x\right)dx-y\left(1-x^2\right)dy=0$$

$$M_y=\frac{\partial}{\partial y}\left(xy^2-\cos x\sin x\right)=2xy \qquad N_x=\frac{\partial}{\partial x}\left(-y+yx^2\right)=2xy \qquad \Rightarrow \underline{M_y=N_x}$$

$$\psi=\int\left(xy^2-\cos x\sin x\right)dx=\int\left(xy^2-\frac{1}{2}\sin 2x\right)dx=\frac{1}{2}x^2y^2+\frac{1}{4}\cos 2x+h(y)$$

$$\psi_y=x^2y+h'(y)=-y+yx^2 \quad\Rightarrow\quad h'(y)=-y \qquad\Rightarrow\ h(y)=-\frac{1}{2}y^2$$

$$\psi=\frac{1}{2}x^2y^2+\frac{1}{4}\cos 2x-\frac{1}{2}y^2=C$$

$$y(0)=2 \quad\Rightarrow\quad \frac{1}{4}-\frac{1}{2}(4)=C\to\boxed{C=-\frac{7}{4}}$$

$$\frac{1}{2}x^2y^2+\frac{1}{4}\cos 2x-\frac{1}{2}y^2=-\frac{7}{4}$$

$$\boxed{2x^2y^2+\cos 2x-2y^2=-7} \qquad\qquad 2x^2y^2+2\cos^2 x-1-2y^2=-7$$

$$\boxed{x^2y^2+\cos^2 x-y^2=-3}$$

Exercise

Solve the given initial-value problem $\quad (x+y)^2\,dx+\left(2xy+x^2-1\right)dy, \qquad y(1)=1$

Solution

$$M_y=\frac{\partial}{\partial y}(x+y)^2=2(x+y) \qquad N_x=\frac{\partial}{\partial x}\left(2xy+x^2-1\right)=2y+2x \quad \Rightarrow \underline{M_y=N_x|}$$

$$\psi=\int\left(x^2+2xy+y^2\right)dx=\frac{1}{3}x^3+x^2y+xy^2+h(y)$$

$$\psi_y=x^2+2xy+h'(y)=2xy+x^2-1 \quad \Rightarrow \quad h'(y)=-1 \quad \Rightarrow h(y)=-y$$

$$\psi=\frac{1}{3}x^3+x^2y+xy^2-y=C$$

$$y(1)=1 \quad \Rightarrow \quad \frac{1}{3}+1+1-1=C \ \rightarrow \ \underline{C=\frac{4}{3}|}$$

$$\boxed{\frac{1}{3}x^3+x^2y+xy^2-y=\frac{4}{3}}$$

Exercise

Solve the given initial-value problem $\quad (4y+2x-5)dx+(6y+4x-1)dy, \qquad y(-1)=2$

Solution

$$M_y=\frac{\partial}{\partial y}(4y+2x-5)=4, \quad N_x=\frac{\partial}{\partial x}(6y+4x-1)=4, \quad \Rightarrow \underline{M_y=N_x|}$$

$$\psi=\int(4y+2x-5)dx=4xy+x^2-5x+h(y)$$

$$\psi_y=4x+h'(y)=6y+4x-1 \quad \Rightarrow h'(y)=6y-1 \quad \rightarrow h(y)=3y^2-y$$

$$\psi=4xy+x^2-5x+3y^2-y=C$$

$$y(-1)=2 \quad \Rightarrow \quad 4(-1)(2)+1+5+12-2=C \ \rightarrow \ \underline{C=8|}$$

$$\boxed{4xy+x^2-5x+3y^2-y=8}$$

Exercise

Solve the given initial-value problem $\left(e^{x}+y\right)dx+\left(2+x+ye^{y}\right)dy, \qquad y(0)=1$

Solution

$$M_y=\frac{\partial}{\partial y}\left(e^{x}+y\right)=1, \quad N_x=\frac{\partial}{\partial x}\left(2+x+ye^{y}\right)=1, \quad \Rightarrow \underline{M_y=N_x}$$

$$\psi=\int\left(e^{x}+y\right)dx=e^{x}+xy+h(y)$$

$$\psi_y=x+h'(y)=2+x+ye^{y} \quad \Rightarrow h'(y)=2+ye^{y} \quad \to h(y)=2y+e^{y}(y-1)$$

$$e^{x}+xy+2y+e^{y}(y-1)=C$$

$$y(0)=1 \quad \Rightarrow \quad 1+2=C \quad \to \underline{C=3}$$

$$\boxed{e^{x}+xy+2y+e^{y}(y-1)=3}$$

Exercise

Solve the given initial-value problem $(2x-y)dx+(2y-x)dy, \qquad y(1)=3$

Solution

$$M_y=\frac{\partial}{\partial y}(2x-y)=-1, \quad N_x=\frac{\partial}{\partial x}(2y-x)=-1, \quad \Rightarrow \underline{M_y=N_x}$$

$$\psi=\int(2x-y)dx=x^{2}-xy+h(y)$$

$$\psi_y=-x+h'(y)=2y-x \quad \Rightarrow h'(y)=2y \quad \to h(y)=y^{2}$$

$$x^{2}-xy+y^{2}=C$$

$$y(1)=3 \quad \Rightarrow \quad 1-3+9=C \quad \to \underline{C=7}$$

$$\boxed{x^{2}-xy+y^{2}=7}$$

$$y^{2}-xy+x^{2}-7=0 \quad \to \quad y=\frac{x\pm\sqrt{x^{2}-4x^{2}+28}}{2}=\frac{x\pm\sqrt{-3x^{2}+28}}{2}$$

$$since\ y(1)=3 \quad \to \quad y=\frac{1}{2}\left(1\pm\sqrt{-3(1)^{2}+28}\right)=\frac{1}{2}(1\pm5)\begin{cases}\frac{1+5}{2}=3\\ \cancel{\frac{1-5}{2}=-2}\end{cases}$$

$$y=\frac{x+\sqrt{-3x^{2}+28}}{2} \qquad |x|<\sqrt{\frac{28}{3}}$$

Exercise

Solve the given initial-value problem $\left(9x^2+y-1\right)dx-(4y-x)dy, \qquad y(1)=0$

Solution

$$M_y=\frac{\partial}{\partial y}\left(9x^2+y-1\right)=1, \quad N_x=\frac{\partial}{\partial x}(x-4y)=1, \quad \Rightarrow \underline{M_y=N_x}$$

$$\psi=\int\left(9x^2+y-1\right)dx=3x^3+xy-x+h(y)$$

$$\psi_y=x+h'(y)=x-4y \quad \Rightarrow h'(y)=-4y \quad \rightarrow h(y)=-2y^2$$

$$3x^3+xy-x-2y^2=C$$

$$y(1)=0 \quad \Rightarrow \quad 3-1=C \quad \rightarrow \quad \underline{C=2}$$

$$\boxed{-2y^2+xy+3x^3-x=2}$$

$$-2y^2+xy+3x^3-x-2=0 \quad \rightarrow \quad y=\frac{-x\pm\sqrt{x^2+24x^3-8x-16}}{-4}$$

$$since \;\; y(1)=0 \quad \rightarrow \quad y=-\frac{1}{4}\left(-1\pm\sqrt{1+24-8-16}\right)=-\frac{1}{4}(-1\pm1)\begin{cases}-\frac{-1+1}{4}=0\\ \cancel{\frac{-1-1}{-4}=\frac{1}{2}}\end{cases}$$

$$y=\frac{x-\sqrt{x^2+24x^3-8x-16}}{4}$$

Exercise

Find the general solution $\quad y'=\dfrac{x^2+y^2}{2xy}$

Solution

Let $y=xv \quad \Rightarrow \quad y'=v+xv'$

$$v+xv'=\frac{x^2+x^2v^2}{2x^2v}$$

$$xv'=\frac{1+v^2}{2v}-v$$

$$x\frac{dv}{dx}=\frac{1-v^2}{2v}$$

$$\frac{2v}{1-v^2}dv=\frac{dx}{x}$$

$$-\int \frac{d\left(1-v^2\right)}{1-v^2} = \int \frac{dx}{x}$$

$$-\ln\left|1-v^2\right| = \ln|x| + \ln C$$

$$\ln\frac{1}{\left|1-v^2\right|} = \ln|\mathrm{C}x|$$

$$\frac{1}{1-\left(\frac{y}{x}\right)^2} = \mathrm{C}x$$

$$\frac{x^2}{x^2-y^2} = \mathrm{C}x$$

$$\frac{x}{\mathrm{C}} = x^2 - y^2$$

$$\boxed{y^2 = x^2 - C_1 x}$$

Exercise

Find the general solution $\quad 2xyy' = x^2 + 2y^2$

Solution

Let $y = xv \quad \Rightarrow \quad y' = v + xv'$

$$2x(xv)(v+xv') = x^2 + 2x^2v^2 \qquad \textit{Divide both side by } x^2$$

$$2v^2 + 2xvv' = 1 + 2v^2$$

$$2xv\frac{dv}{dx} = 1$$

$$\int 2v\,dv = \int \frac{dx}{x}$$

$$v^2 = \ln x + C \qquad \left(v = \frac{y}{x}\right)$$

$$\frac{y^2}{x^2} = \ln x + C$$

$$\boxed{y^2 = x^2(\ln x + C)}$$

Exercise

Find the general solution $\qquad xy' = y + 2\sqrt{xy}$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$x(v + xv') = vx + 2\sqrt{x^2 v}$

$x(v + xv') = vx + 2x\sqrt{v}$ $\qquad$ ***Divide both side by*** x

$v + xv' = v + 2\sqrt{v}$

$x\dfrac{dv}{dx} = 2\sqrt{v}$

$\displaystyle\int \frac{dv}{2\sqrt{v}} = \int \frac{dx}{x}$

$\sqrt{v} = \ln x + C \qquad \left(v = \dfrac{y}{x}\right)$

$\sqrt{\dfrac{y}{x}} = \ln x + C$

$\dfrac{y}{x} = (\ln x + C)^2$

$\underline{y = x(\ln x + C)^2}$

Exercise

Find the general solution $\qquad xy^2 y' = x^3 + y^3$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$x^3 v^2 (v + xv') = x^3 + x^3 v^3$ $\qquad$ ***Divide both side by*** x^3

$v^2 (v + xv') = 1 + v^3$

$v^3 + xv^2 v' = 1 + v^3$

$xv^2 \dfrac{dv}{dx} = 1$

$\displaystyle\int v^2 dv = \int \frac{dx}{x}$

$\dfrac{1}{3}v^3 = \ln x + C \qquad \left(v = \dfrac{y}{x}\right)$

$\dfrac{y^3}{x^3} = 3(\ln x + C)$

$\underline{y^3 = 3x^3(\ln x + C)}$

Exercise

Find the general solution $\quad x^2 y' = xy + x^2 e^{y/x}$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$x^2\left(v + xv'\right) = x^2 v + x^2 e^{vx/x}$ ***Divide both side by*** x^2

$v + xv' = v + e^v$

$x\dfrac{dv}{dx} = e^v$

$\displaystyle\int e^{-v}dv = \int \frac{dx}{x}$

$e^{-v} = \ln x + \ln C \qquad \left(v = \dfrac{y}{x}\right)$

$e^{-\frac{y}{x}} = \ln Cx$

$-\dfrac{y}{x} = \ln\left(\ln Cx\right)$

$\boxed{y = -x\ln\left(\ln Cx\right)}$

Exercise

Find the general solution $\quad x^2 y' = xy + y^2$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$x^2\left(v + xv'\right) = x^2 v + x^2 v^2$ ***Divide both side by*** x^2

$v + xv' = v + v^2$

$x\dfrac{dv}{dx} = v^2$

$\displaystyle\int v^{-2}dv = \int \frac{dx}{x}$

$-v^{-1} = \ln x + \ln C \qquad \left(v = \dfrac{y}{x}\right)$

$\dfrac{x}{y} = -\ln Cx$

$\dfrac{x}{y} = \ln\dfrac{1}{Cx}$

$\boxed{y = \dfrac{x}{\ln\frac{1}{Cx}}}$

Exercise

Find the general solution $\quad xyy' = x^2 + 3y^2$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$vx^2\left(v + xv'\right) = x^2 + 3x^2v^2$ ***Divide both side by x^2***

$v^2 + xvv' = 1 + 3v^2$

$xv\dfrac{dv}{dx} = 1 + 2v^2$

$\dfrac{v}{1+2v^2}dv = \dfrac{dx}{x}$

$\dfrac{1}{4}\displaystyle\int \dfrac{1}{1+2v^2}d\left(1+2v^2\right) = \int \dfrac{dx}{x}$

$\dfrac{1}{4}\ln\left(1+2v^2\right) = \ln x + \ln C$

$\ln\left(1+2v^2\right) = 4\ln Cx$

$\ln\left(1+2\dfrac{y^2}{x^2}\right) = \ln Cx^4$

$\dfrac{x^2+2y^2}{x^2} = Cx^4$

$\underline{x^2 + 2y^2 = Cx^6}$

Exercise

Find the general solution $\quad \left(x^2 - y^2\right)y' = 2xy$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$\left(x^2 - v^2x^2\right)\left(v + xv'\right) = 2x^2v$ ***Divide both side by x^2***

$\left(1 - v^2\right)\left(v + xv'\right) = 2v$

$v + xv' = \dfrac{2v}{1-v^2}$

$x\dfrac{dv}{dx} = \dfrac{2v}{1-v^2} - v$

$x\dfrac{dv}{dx} = \dfrac{v^3+v}{1-v^2}$

$$\int \frac{1-v^2}{v^3+v}dv = \int \frac{dx}{x}$$

$$\frac{1-v^2}{v\left(v^2+1\right)} = \frac{A}{v} + \frac{Bv+C}{v^2+1} = \frac{(A+B)v^2+Cv+A}{v\left(v^2+1\right)}$$

$$\begin{cases} A+B=-1 \\ A=1 \qquad \to B=-2 \end{cases}$$

$$\int \left(\frac{1}{v} - \frac{2v}{v^2+1}\right)dv = \int \frac{dx}{x}$$

$$\int \frac{1}{v}dv - \int \frac{1}{v^2+1}d\left(v^2+1\right) = \int \frac{dx}{x}$$

$$\ln|v| - \ln\left(v^2+1\right) = \ln|x| + \ln C$$

$$\ln \frac{|v|}{v^2+1} = \ln \mathrm{C}x$$

$$\frac{y/x}{\frac{y^2}{x^2}+1} = \mathrm{C}x$$

$$\frac{y}{x}\frac{x^2}{y^2+x^2} = \mathrm{C}x$$

$$\boxed{y = \mathrm{C}\left(y^2+x^2\right)}$$

Exercise

Find the general solution $\quad xyy' = y^2 + x\sqrt{4x^2+y^2}$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$$vx^2\left(v+xv'\right) = x^2v^2 + x\sqrt{4x^2+v^2x^2}$$

$$vx^2\left(v+xv'\right) = x^2v^2 + x^2\sqrt{4+v^2} \qquad \textit{Divide both side by } x^2$$

$$v^2 + xvv' = v^2 + \sqrt{4+v^2}$$

$$xv\frac{dv}{dx} = \sqrt{4+v^2}$$

$$\int \frac{v}{\sqrt{4+v^2}}dv = \int \frac{dx}{x}$$

$$\frac{1}{2}\int \frac{1}{\sqrt{4+v^2}} d\left(4+v^2\right) = \int \frac{dx}{x}$$

$$\sqrt{4+v^2} = \ln x + C$$

$$\sqrt{4+\frac{y^2}{x^2}} = \ln x + C$$

$$\frac{4x^2+y^2}{x^2} = (\ln x + C)^2$$

$$\boxed{4x^2+y^2 = x^2(\ln x + C)^2}$$

Exercise

Find the general solution $\quad xy' = y + \sqrt{x^2+y^2}$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$$x(v+xv') = xv + \sqrt{x^2+v^2x^2}$$

$$x(v+xv') = xv + x\sqrt{1+v^2} \qquad \textit{Divide both side by } x$$

$$v + xv' = v + \sqrt{1+v^2}$$

$$x\frac{dv}{dx} = \sqrt{1+v^2}$$

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$$

$$\ln\left|v+\sqrt{1+v^2}\right| = \ln x + \ln C$$

$$\ln\left|\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}\right| = \ln Cx$$

$$\frac{y}{x}+\frac{1}{x}\sqrt{x^2+y^2} = Cx$$

$$\boxed{y+\sqrt{x^2+y^2} = Cx^2}$$

Exercise

Find the general solution $\quad y^2 y' + 2xy^3 = 6x$

Solution

$$y' + 2xy = 6xy^{-2}$$

Let $\quad u = y^{1+2} = y^3 \quad \Rightarrow \quad y = u^{1/3}$

$$\frac{du}{dx} = 3y^2 \frac{dy}{dx} \quad \Rightarrow \quad y' = \frac{1}{3}y^{-2}u' = \frac{1}{3}u^{-2/3}u'$$

$$\frac{1}{3}u^{-2/3}u' + 2xu^{1/3} = 6xu^{-2/3} \qquad \textit{Multiply both sides by } 3u^{2/3}$$

$$u' + 6xu = 18x$$

$$e^{\int 6x\,dx} = e^{3x^2}$$

$$\int 18xe^{3x^2}dx = 3\int e^{3x^2}d\left(3x^2\right) = 3e^{3x^2}$$

$$u = e^{-3x^2}\left(3e^{3x^2} + C\right)$$

$$\underline{y^3 = 3 + Ce^{-3x^2}}$$

Exercise

Find the general solution $\quad x^2 y' + 2xy = 5y^4$

Solution

$$y' + 2\frac{1}{x}y = \frac{5}{x^2}y^4 \qquad \textit{Divide by } x^2$$

Let $\quad u = y^{1-4} = y^{-3} \quad \Rightarrow \quad y = u^{-1/3}$

$$\frac{du}{dx} = -3y^{-4}\frac{dy}{dx} \quad \Rightarrow \quad y' = -\frac{1}{3}y^4u' = -\frac{1}{3}u^{-4/3}u'$$

$$-\frac{1}{3}u^{-4/3}u' + \frac{2}{x}u^{-1/3} = \frac{5}{x^2}u^{-4/3} \qquad \textit{Multiply both sides by } -3u^{4/3} \quad u = \frac{1}{x^{-6}}\left(\frac{15}{7}x^{-7} + C\right)$$

$$u' - \frac{6}{x}u = -\frac{15}{x^2}$$

$$e^{\int -\frac{6}{x}dx} = e^{-6\ln x} = e^{\ln x^{-6}} = x^{-6}$$

$$\int x^{-6}\left(-\frac{15}{x^2}\right)dx = -15\int x^{-8}dx = \frac{15}{7}x^{-7}$$

$$y^{-3} = \frac{15+7Cx^7}{7x}$$

$$\boxed{y^3 = \frac{7x}{15+7Cx^7}}$$

Exercise

Find the general solution $\quad 2xy' + y^3e^{-2x} = 2xy$

Solution

$$2xy' - 2xy = -e^{-2x}y^3$$

$$y' - y = -\frac{e^{-2x}}{2x}y^3 \qquad \textit{Divide by } 2x$$

Let $\quad u = y^{1-3} = y^{-2} \quad \Rightarrow \quad y = u^{-1/2}$

$$\frac{du}{dx} = -2y^{-3}\frac{dy}{dx} \quad \Rightarrow \quad y' = -\frac{1}{2}y^3u' = -\frac{1}{2}u^{-3/2}u'$$

$$-\frac{1}{2}u^{-3/2}u' - u^{-1/2} = -\frac{e^{-2x}}{2x}u^{-3/2} \qquad \textit{Multiply both sides by } -2u^{3/2}$$

$$u' + 2u = \frac{e^{-2x}}{x}$$

$$e^{\int 2dx} = e^{2x}$$

$$\int \frac{e^{-2x}}{x}e^{2x}dx = \int \frac{dx}{x} = \ln x$$

$$u = \frac{1}{e^{-2x}}(\ln x + C)$$

$$\frac{1}{y^2} = \frac{\ln x + C}{e^{2x}}$$

$$\boxed{y^2 = \frac{e^{2x}}{\ln x + C}}$$

Exercise

Find the general solution $\quad y^2(xy' + y)\left(1+x^4\right)^{1/2} = x$

Solution

$$y^2xy' + y^3 = x\left(1+x^4\right)^{-1/2}$$

$$y' + \frac{1}{x}y = \left(1+x^4\right)^{-1/2}y^{-2} \qquad \textit{Divide both sides by } xy^2$$

Let $u = y^{1+2} = y^3 \quad \Rightarrow \quad y = u^{1/3}$

$$\frac{du}{dx} = 3y^2\frac{dy}{dx} \quad \Rightarrow \quad y' = \frac{1}{3}y^{-2}u' = \frac{1}{3}u^{-2/3}u'$$

$$\frac{1}{3}u^{-2/3}u' + \frac{1}{x}u^{1/3} = \left(1+x^4\right)^{-1/2}u^{-2/3}$$ ***Multiply both sides by*** $3u^{2/3}$

$$u' + \frac{3}{x}u = 3\left(1+x^4\right)^{-1/2}$$

$$e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$$

$$\int 3\left(1+x^4\right)^{-1/2}x^3dx = \frac{3}{4}\int\left(1+x^4\right)^{-1/2}d\left(1+x^4\right) = \frac{3}{2}\sqrt{1+x^4}$$

$$u = \frac{1}{x^3}\left(\frac{3}{2}\sqrt{1+x^4} + C\right)$$

$$\boxed{y^3 = \frac{1}{x^3}\left(\frac{3}{2}\sqrt{1+x^4} + C\right)}$$

Exercise

Find the general solution $\quad 3y^2y' + y^3 = e^{-x}$

Solution

$$3y' + y = e^{-x}y^{-2}$$ ***Divide both sides by*** y^2

Let $u = y^{1+2} = y^3 \quad \Rightarrow \quad y = u^{1/3}$

$$\frac{du}{dx} = 3y^2\frac{dy}{dx} \quad \Rightarrow \quad y' = \frac{1}{3}y^{-2}u' = \frac{1}{3}u^{-2/3}u'$$

$$u^{-2/3}u' + u^{1/3} = e^{-x}u^{-2/3}$$ ***Multiply both sides by*** $u^{2/3}$

$$u' + u = e^{-x}$$

$$e^{\int dx} = e^x$$

$$\int e^{-x}e^xdx = \int dx = x$$

$$u = \frac{1}{e^x}(x+C)$$

$$\boxed{y^3 = e^{-x}(x+C)}$$

Exercise

Find the general solution $\quad 3xy^2y' = 3x^4 + y^3$

Solution

$3xy^2y' - y^3 = 3x^4$ ***Divide both sides by*** xy^2

$3y' - \frac{1}{x}y = 3x^3y^{-2}$

Let $\quad u = y^{1+2} = y^3 \quad \Rightarrow \quad y = u^{1/3} \quad \Rightarrow y' = \frac{1}{3}u^{-2/3}u'$

$u^{-2/3}u' - \frac{1}{x}u^{1/3} = 3x^3u^{-2/3}$ ***Multiply both sides by*** $u^{2/3}$

$u' - \frac{1}{x}u = 3x^3$

$$e^{\int \frac{-1}{x}dx} = e^{-\ln x} = x^{-1}$$

$$\int 3x^3x^{-1}dx = \int 3x^2dx = x^3$$

$u = x\left(x^3 + C\right)$

$y^3 = x^4 + Cx$

$\boxed{y = \sqrt[3]{x^4 + Cx}}$

Exercise

Find the general solution $\quad xe^yy' = 2\left(e^y + x^3e^{2x}\right)$

Solution

Let $u = e^y \quad \Rightarrow \quad y = \ln u \quad \Rightarrow y' = \frac{u'}{u}$

$xu\frac{1}{u}u' = 2u + 2x^3e^{2x}$

$u' - \frac{2}{x}u = 2x^2e^{2x}$

$$e^{\int \frac{-2}{x}dx} = e^{-2\ln x} = x^{-2}$$

$$\int 2x^2e^{2x}x^{-2}dx = 2\int e^{2x}dx = e^{2x}$$

$u = x^2\left(e^{2x} + C\right)$

$e^y = x^2e^{2x} + Cx^2$

$\boxed{y = \ln\left(x^2e^{2x} + Cx^2\right)}$

Exercise

Find the general solution $\left(2x\sin y\cos y\right)y' = 4x^2 + \sin^2 y$

Solution

Let $u = \sin y \quad \Rightarrow \quad u' = (\cos y)y'$

$$2xuu' = 4x^2 + u^2$$

$$u' = 2x\frac{1}{u} + \frac{1}{2x}u$$

$$u' - \frac{1}{2x}u = 2xu^{-1}$$

Let $v = u^{1+1} = u^2 \quad \Rightarrow \quad u = v^{1/2}$

$$v' = 2uu' \quad \Rightarrow \quad u' = \frac{1}{2}u^{-1}v' = \frac{1}{2}v^{-1/2}v'$$

$$\frac{1}{2}v^{-1/2}v' - \frac{1}{2x}v^{1/2} = 2xv^{-1/2}$$ ***Multiply both sides by*** $2v^{1/2}$

$$v' - \frac{1}{x}v = 4x$$

$$e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

$$\int x^{-1}(4x)dx = \int 4dx = 4x$$

$$v = x(4x + C)$$

$$u^2 = 4x^2 + Cx$$

$$\boxed{\sin^2 y = 4x^2 + Cx}$$

Exercise

Find the general solution $\left(x + e^y\right)y' = xe^{-y} - 1$

Solution

Let $u = e^y \quad \Rightarrow \quad y = \ln u \quad \Rightarrow \quad y' = \frac{u'}{u}$

$$(x+u)\frac{u'}{u} = xu^{-1} - 1$$

$$(x+u)u' = x - u$$

Let $u = vx \quad \Rightarrow \quad u' = v + xv'$

$$(x+vx)(v+xv') = x - vx$$

$$x(1+v)(v+xv') = x(1-v)$$

$$(1+v)(v+xv') = 1 - v$$

$$v+v^2+x(1+v)v'=1-v$$

$$x(1+v)\frac{dv}{dx}=1-2v-v^2$$

$$\int\frac{1+v}{1-2v-v^2}dv=\int\frac{dx}{x}$$

$$-\frac{1}{2}\ln\left|1-2v-v^2\right|=\ln x+\ln C$$

$$\ln\left|1-2v-v^2\right|=-2\ln \mathrm{C}x \qquad v=\frac{u}{x}=\frac{e^y}{x}$$

$$\ln\left|1-2\frac{e^y}{x}-\frac{e^{2y}}{x^2}\right|=\ln(\mathrm{C}x)^{-2}$$

$$\frac{x^2-2xe^y-e^{2y}}{x^2}=\frac{1}{(\mathrm{C}x)^2}$$

$$x^2-2xe^y-e^{2y}=C_1$$

Solution *Section* 1.7 - Existence and Uniqueness of Solutions

Exercise

Which of the initial value problems are guaranteed a unique solution. $y' = 4 + y^2, \quad y(0) = 1$

Solution

$f(t, y) = 4 + y^2 \rightarrow f$ is continuous

$\dfrac{\partial f}{\partial y} = 2y$ is also continuous on the whole plane.

Hence the hypotheses are satisfied and guarantee a unique solution.

Exercise

Which of the initial value problems are guaranteed a unique solution? $y' = \sqrt{y}, \quad y(4) = 0$

Solution

$f(t, y) = \sqrt{y} \quad \Rightarrow \; y \ge 0$

$\dfrac{\partial f}{\partial y} = \dfrac{1}{2\sqrt{y}} \quad \rightarrow \; y > 0 \;\; (only)$

Initial condition: $y(4) = 0 \Rightarrow y_0 = 0 \; and \; t_0 = 4$

Both $f \; and \; \dfrac{\partial f}{\partial y}$ are not continuous in the rectangle containing $\left(t_0, y_0\right)$

Hence the hypotheses are not satisfied.

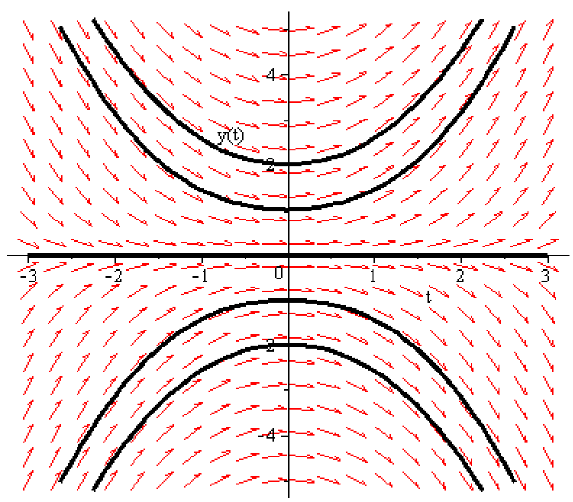
Exercise

Which of the initial value problems are guaranteed a unique solution? $y' = t\tan^{-1} y, \quad y(0) = 2$

Solution

The right hand side of the equation is $f\left(t, y\right) = t\tan^{-1} y$, which is continuous in the whole plane. $\dfrac{\partial f}{\partial y} = \dfrac{t}{t + y^2}$ is also continuous in the whole plane.

Hence the hypotheses are satisfied and the theorem guarantees a unique solution.



Exercise

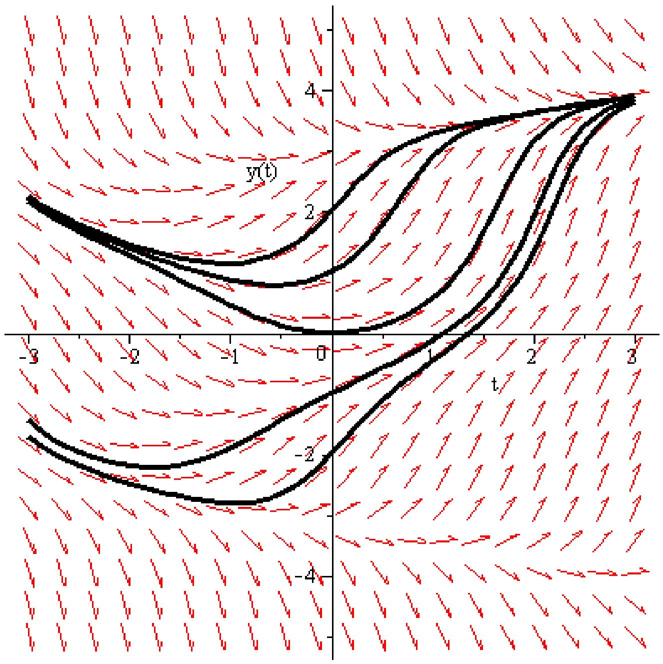
Which of the initial value problems are guaranteed a unique solution? $\omega' = \omega \sin \omega + s, \quad \omega(0) = -1$

Solution

The right hand side of the equation is $f(s, \omega) = \omega \sin \omega + s$, which is continuous in the whole plane.

$\frac{\partial f}{\partial \omega} = \sin \omega + \omega \cos \omega$ is also continuous in the whole plane.

Hence the hypotheses are satisfied and the theorem guarantees a unique solution.



Exercise

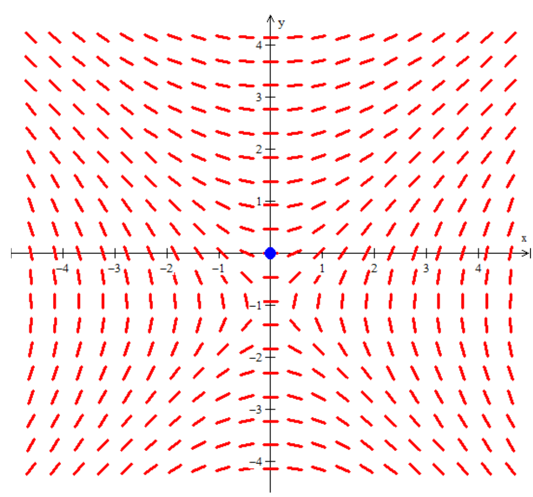
Which of the initial value problems are guaranteed a unique solution? $x' = \frac{t}{x+1}, \quad x(0) = 0$

Solution

The right hand side of the equation is $f(t, x) = \frac{t}{x+1}$, which is continuous in the whole plane, except where $x = -1$.

$\frac{\partial f}{\partial x} = -\frac{t}{(x+1)^2}$ is also continuous in the whole plane, except where $x = -1$.

Hence the hypotheses are satisfied in a rectangle containing the initial point (0, 0), so the theorem guarantees a unique solution.



Exercise

Which of the initial value problems are guaranteed a unique solution? $y' = \frac{1}{x}y + 2, \quad y(0) = 1$

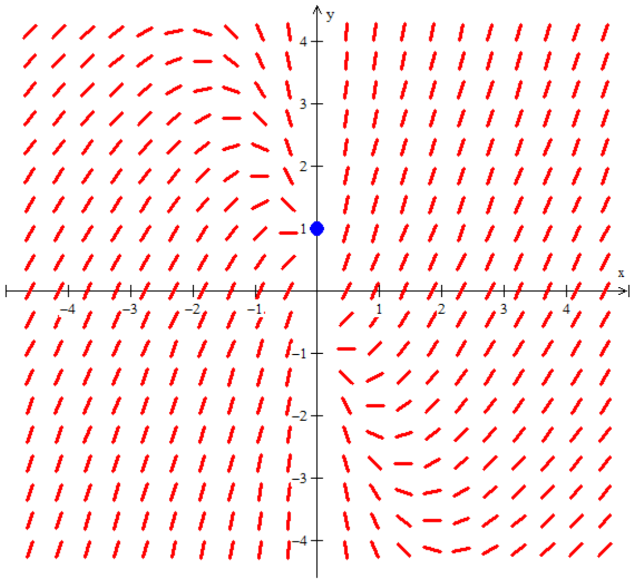
Solution

The right hand side of the equation is $f(x, y) = \frac{1}{x}y + 2$, which is continuous in the whole plane, except where $x = 0$.

Since the initial point is (0, 1), *f* is discontinuous there,

Consequently there is no rectangle containing this point in which *f* is continuous.

The hypotheses are not satisfied, so the theorem doesn't guarantee a unique solution.



Exercise

Show that $y(t) = 0$ and $y(t) = t^3$ are both solutions of the initial value problem $y' = 3y^{2/3}$, where $y(0) = 0$. Explain why this fact doesn't contradict Theorem

Solution

$f(t, y) = 3y^{2/3}$

$f' = 2y^{-1/3}$ which is not continuous at $y = 0$

Exercise

Use a numerical solver to sketch the solution of the given initial value problem

$$\frac{dy}{dt} = \frac{t}{y+1}, \qquad y(2) = 0$$

a) Where does your solver experience difficulty? why? Use the image of your solution to estimate the interval of existence.

b) Find an explicit solution; then use your formula to determine the interval of existence. How does it compare with the approximation found in part (*a*).

Solution

a) $(y+1)dy = tdt$

$$\int (y+1)dy = \int tdt$$

$$\frac{1}{2}y^2 + y = \frac{1}{2}t^2 + C$$

$$y^2 + 2y = t^2 + C$$

$$(0)^2 + 2(0) = 2^2 + C$$

$$0 = 4 + C$$

$$C = -4$$

$$y^2 + 2y = t^2 - 4$$

$$y^2 + 2y - t^2 + 4 = 0$$

Solve for y:

$$y = \frac{-2 \pm \sqrt{2^2 - 4(1)(-t^2+4)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4t^2 - 12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{t^2 - 3}}{2}$$

$$= -1 \pm \sqrt{t^2 - 3}$$

b) The only solution is: $y = -1 + \sqrt{t^2 - 3}$ and $t^2 - 3 > 0 \Rightarrow t > \sqrt{3}$

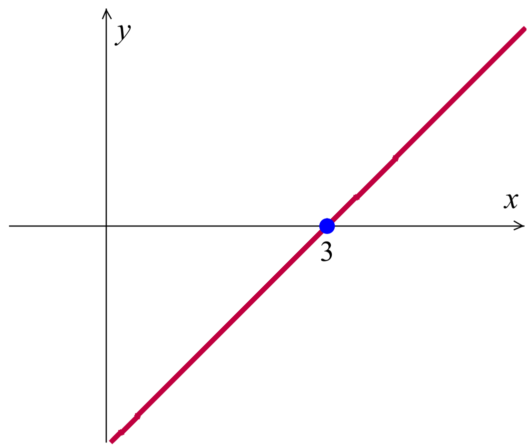
The interval of the solution $\left(\sqrt{3},\ \infty\right)$

Solution   *Section* 1.8 - Autonomous Equations and Stability

Exercise

The graph of the right-hand side $y' = f(y)$ is shown. Identify the equilibrium points and sketch the equilibrium solutions in the ty-plane. Classify each equilibrium point as either unstable or asymptotically stable.

Solution

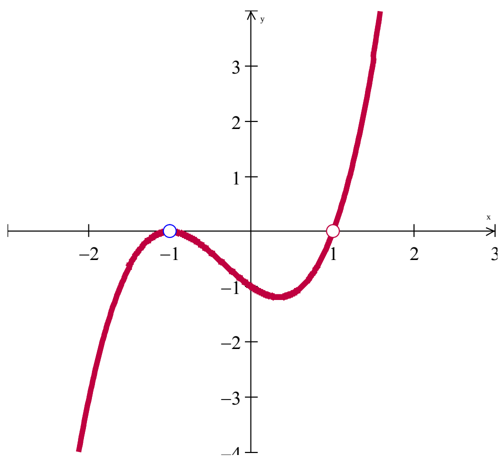


The equilibrium point is: 3 and is stable

Exercise

The graph of the right-hand side $y' = f(y)$ is shown. Identify the equilibrium points and sketch the equilibrium solutions in the ty-plane. Classify each equilibrium point as either unstable or asymptotically stable.

Solution

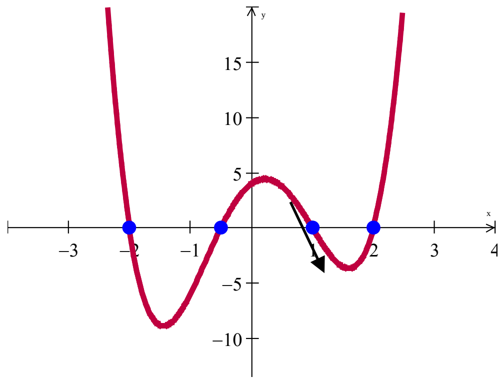


The equilibrium points are: $-1,\ 1$ and both are unstable

Exercise

The graph of the right-hand side $y' = f(y)$ is shown. Identify the equilibrium points and sketch the equilibrium solutions in the *ty*-plane. Classify each equilibrium point as either unstable or asymptotically stable.

Solution



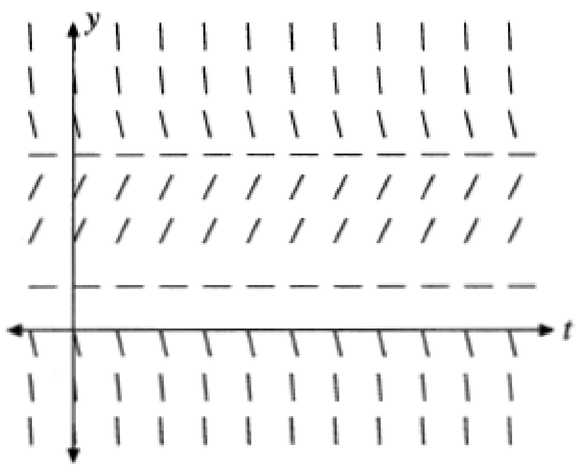
The equilibrium points are: $-2, \ -\frac{1}{2}, \ 1, \ 2$

$-2, \ 1$ are asymptotically stable

$-\frac{1}{2}, \ 2$ are unstable

Exercise

Impose the equilibrium solution(s), classifying each as either unstable or asymptotically stable



Solution

Because the $y' = f(y)$ is autonomous, the slope at any point $(t, \ x)$ in the direction field does not depend on *t*, only on *y*.

There are two equilibrium points. The smaller of them is unstable and the other is asymptotically stable.

Exercise

An autonomous differential equation is given by $y' = 2 - y$

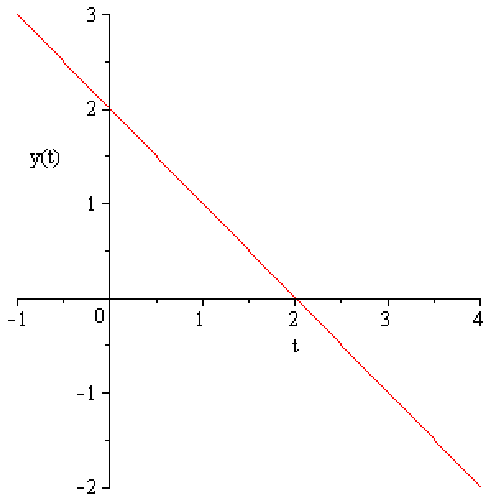
a) Sketch a graph of $f(y)$

b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

Solution

a) $f(y) = 2 - y$

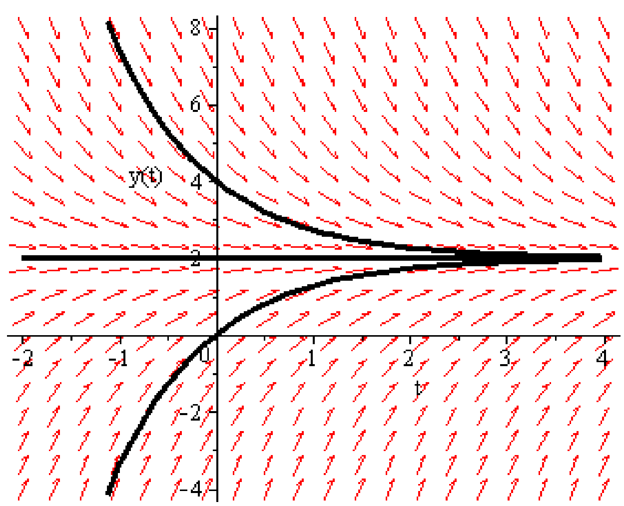


b) The phase line for the autonomous equation is

$y = 2$ is asymptotically stable

c) The phase line indicates that the solutions increase if $y < 2$ and decrease if $y > 2$.

The stable equilibrium solution is $y = 2$



Exercise

An autonomous differential equation is given by $y' = (y+1)(y-4)$

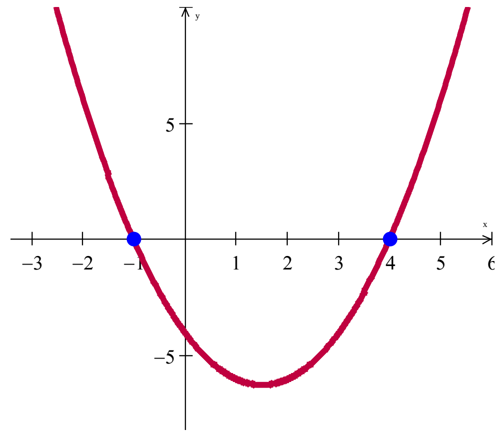
a) Sketch a graph of $f(y)$

b) Use the graph of f to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

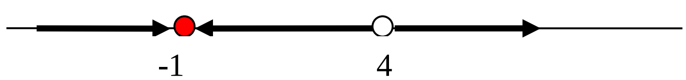
c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

Solution

a) $f(y) = (y+1)(y-4)$

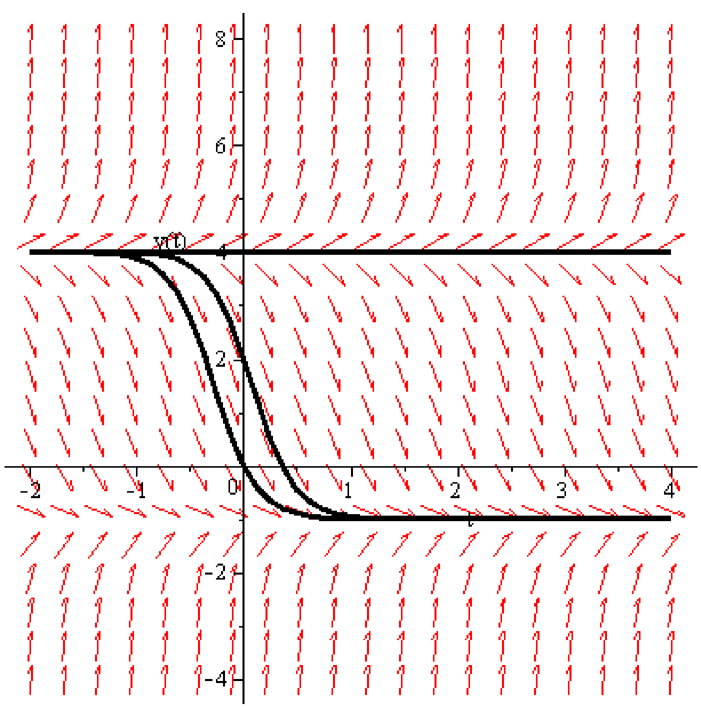


b) The phase line for the autonomous equation is



$y = -1$ is asymptotically stable and $y = 4$ is unstable

c)



Exercise

An autonomous differential equation is given by $y' = 9y - y^3$

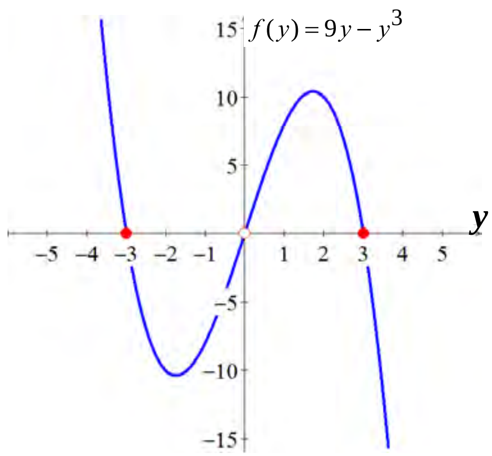
a) Sketch a graph of $f(y)$

b) Use the graph of f to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

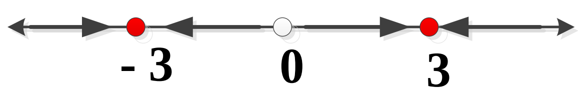
c) Sketch the equilibrium solutions in the ty-plane into regions. Sketch at least one solution trajectory in each of these regions.

Solution

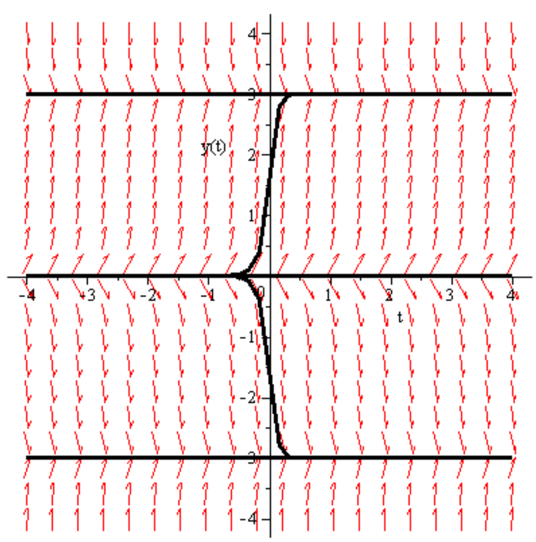
a) $f(y) = 9y - y^3 = y\left(9 - y^2\right)$



b) The phase line for the autonomous equation is



c) The solutions increase if $y < -3$, decrease for $-3 < y < 0$, increase if $0 < y < 3$, and decrease for $y > 3$.



The stable equilibrium solutions at $y(t) = -3, \quad y(t) = 3$ and unstable equilibrium solutions at $y(t) = 0$

Exercise

An autonomous differential equation is given by $y' = \sin y$

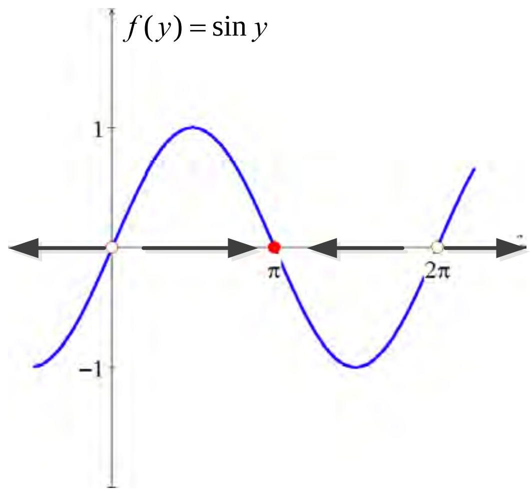
a) Sketch a graph of $f(y)$

b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

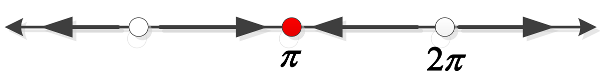
c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

Solution

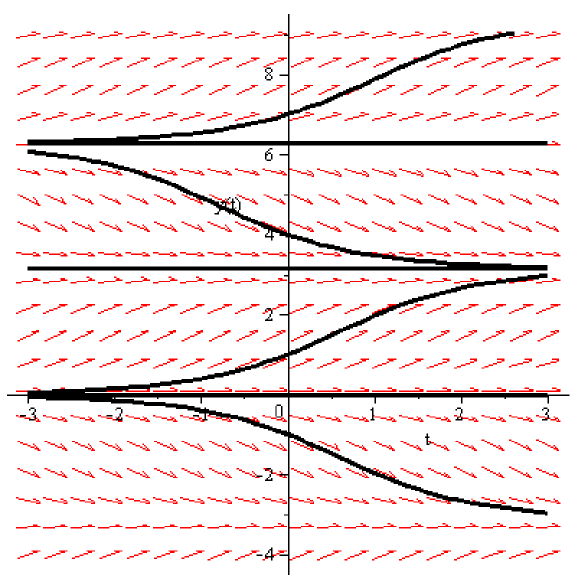
a) $f(y) = \sin y$



b) The phase line for the autonomous equation is



c) The solutions decrease if $-\pi < y < 0$, increase for $0 < y < \pi$, increase if $\pi < y < 2\pi$



The stable equilibrium solutions at $y(t) = \pi$ and unstable equilibrium solutions at $y(t) = 0,\ 2\pi$

Exercise

Determine the stability of the equilibrium solutions $x' = 4 - x^2$

Solution

$$f(x) = x' = 4 - x^2 = 0$$

$$\Rightarrow x^2 = 4$$

The equilibrium points $x = \pm 2$

$$f'(x) = -2x$$

$f'(-2) = -2(-2) > 0$ $\quad$ $x = -2$ is unstable

$f'(2) = -2(2) < 0$ $\quad$ $x = 2$ is asymptotically stable

Exercise

Determine the stability of the equilibrium solutions $x' = x(x-1)(x+2)$

Solution

The equation $f(x) = x(x-1)(x+2)$.

$f(x) = 0 \quad \Rightarrow$ The equilibrium points are $x = 0,\ 1,\ -2$.

$$f(x) = x\left(x^2 + x - 2\right) = x^3 + x^2 - 2x$$

$$f'(x) = 3x^2 + 2x - 2$$

$f'(0) = -2 < 0 \quad \Rightarrow \quad x = 0$ ***Asymptotically stable***

$f'(1) = 3 > 0 \quad \Rightarrow \quad x = 1$ ***Unstable***

$f'(-2) = 2 > 0 \quad \Rightarrow \quad x = -2$ ***Unstable***

Exercise

A tank contains 100 gal of pure water. A salt solution with concentration 3 lb/gal enters the tank at a rate of 2 gal/min. Solution drains from the tank at a rate of 2 gal/min. Use the qualitative analysis to find the eventual concentration of the salt in the tank.

Solution

Let $x(t)$ represents the amount of salt.

$$Rate\ in = 2\frac{gal}{min} \times 3\frac{lb}{gal} = 6\frac{lb}{\min}$$

$$Rate\ out = 2\frac{gal}{min} \times \frac{x(t)}{100}\frac{lb}{gal} = \frac{x(t)}{50}\frac{lb}{gal}$$

$$\frac{dx}{dt} = 6 - \frac{1}{50}x$$

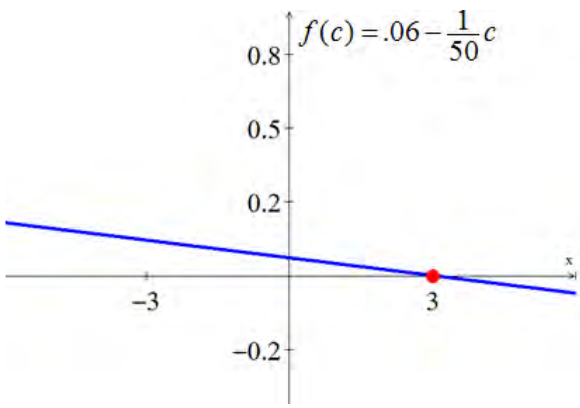
Let $c(t)$ represents the concentration of salt. Thus, $c(t) = \dfrac{x(t)}{100} \quad \to \quad x' = 100c'$

$$100c' = 6 - \frac{1}{50}(100c)$$

$$100c' = .06 - \frac{1}{50}c$$

$$\Rightarrow f(c) = .06 - \frac{1}{50}c = 0$$

$$\frac{1}{50}c = .06 \quad \Rightarrow \quad c = 3$$



$c = 3$ is stable equilibrium point so a trajectory should approach the stable equilibrium solution $c(t) = 3$

Solution Section 1.9 - Modeling Population Growth

Exercise

The rate of growth of bacteria in a petri dish is proportional to the number of bacteria in the dish.

Solution

$$\underline{y' = ky(t)}$$

Exercise

The rate of growth of a population of field mice is inversely proportional to the square root of the population.

Solution

$$\underline{y' = \frac{k}{\sqrt{y(t)}}}$$

Exercise

A biologist starts with 100 cells in a culture. After 24 *hrs,* he counts 300. Assuming a Malthusian model, what the reproductive rate? What will be the number of cells of the end of 5 days?

Solution

$$P = 100e^{rt}$$

$$300 = 100e^{r(1)} \qquad \textbf{24 } \textbf{\textit{hrs}} \textbf{ = 1 } \textbf{\textit{day P}} \textbf{ = 300}$$

$$3 = e^{r}$$

$$r = \ln 3$$

$$\approx 1.0986$$

$$P(t) = 100e^{1.0986t}$$

$$P(5) = 100e^{1.0986(5)}$$

$$\underline{\approx 24300}$$

Exercise

A biologist prepares a culture. After 1 *day* of growth, the biologist counts 1000 cells. After 2*days,* he counts 3000. Assuming a Malthusian model, what the reproductive rate and how many cells were present initially?

Solution

Given: $P(1) = 1000, \quad P(2) = 3000$

The equation of the Malthusian model is $P(t) = Ce^{rt}$

$$P(1) = Ce^{r(1)}$$

$$1000 = Ce^{r}$$

$$e^{r} = \frac{1000}{C} \rightarrow \boxed{r = \ln\left(\frac{1000}{C}\right)}$$

$$P(2) = Ce^{r(2)}$$

$$3000 = Ce^{2r}$$

$$e^{2r} = \frac{3000}{C} \rightarrow \boxed{r = \frac{1}{2}\ln\left(\frac{3000}{C}\right)}$$

$$r = \frac{1}{2}\ln\left(\frac{3000}{C}\right) = \ln\left(\frac{1000}{C}\right)$$

$$\ln\left(\frac{3000}{C}\right) = 2\ln\left(\frac{1000}{C}\right)$$

$$\ln\left(\frac{3000}{C}\right) = \ln\left(\frac{1000}{C}\right)^2$$

$$\frac{3000}{C} = \frac{10^6}{C^2}$$

$$3000C^2 = 10^6 C$$

$$3000C^2 - 10^6 C = 0$$

$$C\left(3000C - 10^6\right) = 0 \quad \Rightarrow \quad \boxed{C = \frac{10^6}{3000} = \frac{1000}{3}}$$

$$r = \ln\left(\frac{1000}{\frac{1000}{3}}\right) = \ln 3$$

$$P(t) = \frac{1000}{3} e^{(\ln 3)t}$$

$$\boxed{P(t = 0) = \frac{1000}{3}}$$

Exercise

A population of bacteria is growing according to the Malthusian model. If the population is triples in 10 *hrs*, what is the reproduction rate? How often does the population double itself?

Solution

$$P(t) = P_0 e^{t}$$

$$rT = \ln(X) \quad \rightarrow \quad \boxed{r = \frac{\ln 3}{10} \approx 0.1099}$$

$$P(t) = P_0 e^{(t\ln 3)/10}$$

$$t = \frac{\ln 2}{r}$$
$$= \frac{\ln 2}{\frac{\ln 3}{10}} = \frac{10\ln 2}{\ln 3}$$
$$\approx 6.3093 \ \ hrs$$

Exercise

Consider a lake that is stocked with walleye pike and that the population of pike is governed by the logistic equation

$$P' = 0.1P\left(1 - \frac{P}{10}\right)$$

where time is measured in days and P in thousands of fish. Suppose that fishing is started in this lake and that 100 fish are removed each day.

Solution

a) Modify the logistic model to account for the fishing.

The modified logistic model

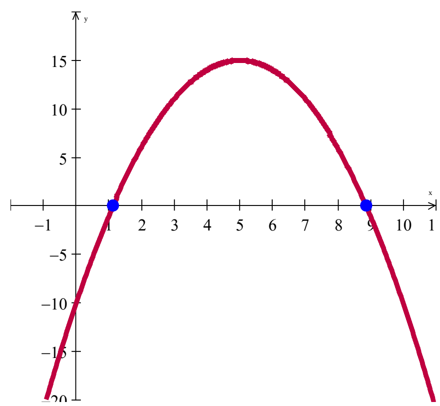
$$P' = P' - \frac{100}{1000} \quad (\text{in } \textit{thousand})$$
$$= 0.1P\left(1 - \frac{P}{10}\right) - 0.1$$

b) Find and classify the equilibrium points for your model.

$$P' = 0.1P - \frac{0.1P^2}{10} - 0.1 = 0 \qquad \textit{Multiply 100 each term}$$
$$10P - P^2 - 10 = 0$$
$$P^2 - 10P + 10 = 0 \qquad \textit{Solve for P}$$
$$P = 5 \pm \sqrt{15}$$

$P = 5 - \sqrt{15}$ Asymptotically stable

$P = 5 + \sqrt{15}$ Unstable



c) Use qualitative analysis to completely discuss the fate of the fish population with this model. In particular, if the initial fish population is 1000, what happens to the fish as time passes? what will happen to an initial population having 2000 fish?

$$P' > 0 \Rightarrow \ 5 - \sqrt{15} < P < 5 + \sqrt{15}$$

For the 1000 (= 1) population, the population decreases until it dies out (doomed);

For the 2000 (= 2) population, the population tend towards the equilibrium $P_2 = 5 + \sqrt{15}$

$$P' = 0.1(1)\left(1 - \frac{1}{10}\right) - .1 = -.01$$

$$P' = 0.1(2)\left(1 - \frac{2}{10}\right) - .1 = .06$$

Exercise

Suppose that in 1885 the population of a certain country was 50 million and was growing at the rate of 750,000 people per year at that time. Suppose also that in 1940 its population was 100 million and was then growing at the rate of 1 million per year. Assume that this population satisfies the logistic equation. Determine both the limiting population M and the predicted population for the year 2000.

Solution

$$P' = kP(M-P)$$

$$.75 = 50k(M-50) \quad (1) \quad (in\ million)$$

$$1 = 100k(M-100) \quad (2) \quad (in\ million)$$

$$\begin{cases} (1) \quad k = \dfrac{.75}{50(M-50)} \\ (2) \quad k = \dfrac{1}{100(M-100)} \end{cases}$$

$$\frac{.75}{50(M-50)} = \frac{1}{100(M-100)}$$

$$2(0.75)(M-100) = M-50$$

$$1.5M - 150 = M - 50$$

$$.5M = 100$$

$$\underline{M = 200}$$

$$k = \frac{1}{100(200-100)} \ \underline{= 0.0001}$$

$$P(60) = \frac{100 \bullet 200}{100 + (200-100)e^{-0.0001(200)(60)}}$$

$$\underline{\approx 153.7 \ \ million\ people}$$

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}$$

Exercise

The time rate of change of a rabbit population P is proportional to the square root of P. At time $t = 0$ (months) the population numbers 100 rabbits and is increasing at the rate of 20 rabbits per month. How many rabbits will there be one year later?

Solution

Given: $P(0) = 100, \quad P'(0) = 20$

$$P' = k\sqrt{P}$$

$$at\ t = 0 \quad \Rightarrow \quad 20 = k\sqrt{100} \ \rightarrow \ \boxed{k = 2}$$

$$\frac{dP}{dt} = 2\sqrt{P}$$

$$\int \frac{1}{2\sqrt{P}}dP = \int dt$$

$$\sqrt{P} = t + C \quad P(0) = 100 \quad \Rightarrow C = 10$$

$$\boxed{P(t) = (t+10)^2}$$

$$P(t=12) = (12+10)^2$$

$$\boxed{= 484 \ \ rabbits}$$

Exercise

Suppose that the fish population $P(t)$ in a lake is attacked by a disease at time $t = 0$, with the result that the fish cease to reproduce (so that the birth rate is $\beta = 0$) and the death rate δ (deaths per week per fish) is thereafter proportional to $\frac{1}{\sqrt{P}}$. If there were initially 900 fish in the lake and 441 were left after 6 weeks, how long did it take all the fish in the lake to die?

Solution

Given: $P(0) = 900, \quad P(6) = 441$

$$P' = -\delta P = -\frac{k}{\sqrt{P}}P = -k\sqrt{P}$$

$$\frac{dP}{dt} = -k\sqrt{P}$$

$$\int \frac{dP}{\sqrt{P}} = -\int kdt$$

$$2\sqrt{P} = -kt + C$$

$$2\sqrt{900} = -k(0) + C \quad \Rightarrow \quad \boxed{C = 60}$$

$$2\sqrt{441} = -k(6) + 60 \quad \Rightarrow \quad \boxed{k = 3}$$

$$2\sqrt{P} = -3t + 60$$

$$0 = -3t + 60 \quad \Rightarrow \quad \boxed{t = 20}$$

It will take 20 weeks for the fish to die in the lake.

Exercise

Suppose that when a certain lake is stocked with fish, the birth and death rates β and δ are both inversely proportional to $\sqrt{P}$

a) Show that $P(t)=\left(\frac{1}{2}kt+\sqrt{P_0}\right)^2$, where k is a constant.

b) If $P_0=100$ and after 6 months there are 169 fish in the lake, how many will there be after 1 year?

Solution

a) $\frac{dP}{dt}=k\sqrt{P}$

$$\int\frac{dP}{\sqrt{P}}=\int kdt$$

$$2\sqrt{P}=kt+C_1$$

$$\sqrt{P}=\frac{1}{2}kt+C$$

$$P(t)=\left(\frac{1}{2}kt+C\right)^2$$

$$P(t=0)=\left(\frac{1}{2}k(0)+C\right)^2=P_0 \quad\Rightarrow\quad C=\sqrt{P_0}$$

$$\boxed{P(t)=\left(\frac{1}{2}kt+\sqrt{P_0}\right)^2}$$

b) ***Given***: $P_0=100, \quad P(6)=169$

$$169=\left(\frac{1}{2}k(6)+\sqrt{100}\right)^2$$

$$13=3k+10 \quad\Rightarrow\quad \boxed{k=1}$$

$$P(t)=\left(\frac{1}{2}t+10\right)^2$$

$$P(t=1yr=12mths)=(6+10)^2=256$$

There are 256 fish after 1 year.

Exercise

The time rate of change of an alligator population P in a swamp is proportional to the square of P. The swamp contained a dozen alligators in 1988, two dozen in 1998.

a) When will there be four dozen alligators in the swamp?

b) What happens thereafter?

Solution

Given: $P_0 = 12, \quad P(10) = 24$

a)

$$\frac{dP}{dt} = kP^2$$

$$\int \frac{dP}{P^2} = \int k\,dt$$

$$-\frac{1}{P} = kt + C$$

$$P(t) = -\frac{1}{kt + C}$$

$$P(0) = -\frac{1}{C} = 12 \quad \Rightarrow \quad C = -\frac{1}{12}$$

$$P(t) = -\frac{1}{kt - \frac{1}{12}}$$

$$P(t) = \frac{12}{1 - 12kt}$$

$$P(10) = \frac{12}{1 - 120k} = 24 \quad \Rightarrow \quad 1 - 120k = \frac{1}{2}$$

$$\boxed{k = \frac{1}{240}}$$

$$P(t) = \frac{12}{1 - \frac{1}{20}t}$$

$$= \frac{240}{20 - t}$$

$$48 = \frac{240}{20 - t}$$

$$20 - t = \frac{240}{48} = 5$$

$t = 15$, that is, in the year 2003

b)

$$P = \frac{240}{20 - t} \xrightarrow[t \to 20]{} \infty$$

The population approaches infinity as t approaches 20 years.

Exercise

Consider a prolific breed of rabbits whose birth and death rates, β and δ, are each proportional to the rabbit population $P = P(t)$, with $\beta > \delta$

a) Show that $P(t) = \dfrac{P_0}{1 - kP_0 t}, \quad k\ constant$

Note that $P(t) \to +\infty \quad as \quad t \to \dfrac{1}{kP_0}$. This is doomsday

b) Suppose that $P_0 = 6$ and that there are nine rabbits after ten months. When does doomsday occur?

c) With $\beta < \delta$, repeat part (a)

d) What now happens to the rabbit population in the long run?

Solution

a) If the birth & death both are proportional to P^2 with $\beta > \delta$

$$\frac{dP}{dt} = kP^2$$

$$\int \frac{dP}{P^2} = \int k\,dt$$

$$-\frac{1}{P} = kt + C$$

$$P(t) = -\frac{1}{kt + C}$$

$$P(0) = -\frac{1}{C} = P_0 \quad \Rightarrow \quad \boxed{C = -\frac{1}{P_0}}$$

$$P(t) = -\frac{1}{kt - \frac{1}{P_0}} \quad \boxed{= \frac{P_0}{1 - P_0 kt}}$$

b) $P_0 = 6 \quad \Rightarrow \quad P(t) = \dfrac{6}{1 - 6kt}$

Given: $P(10) = 9$

$$\frac{6}{1 - 60k} = 9$$

$$1 - 60k = \frac{2}{3}$$

$$k = \frac{1}{180}$$

$$P(t) = \frac{6}{1 - \frac{1}{30}t} \quad \boxed{= \frac{180}{30 - t}}$$

$P = \dfrac{180}{30 - t} \xrightarrow[t \to 30]{} \infty$ (doomsday)

c) If the birth & death both are proportional to P^2 with $\beta < \delta$

$$\frac{dP}{dt} = -kP^2$$

$$\int \frac{dP}{P^2} = -\int k\,dt$$

$$-\frac{1}{P} = -kt - C$$

$$P(t) = \frac{1}{kt + C}$$

$$P(0) = \frac{1}{C} = P_0 \quad \Rightarrow \quad \boxed{C = \frac{1}{P_0}}$$

$$P(t) = \boxed{\frac{P_0}{1 + P_0 kt}}$$

d) $$\frac{P_0}{1 + P_0 kt} \xrightarrow[t \to \infty]{} 0$$

Therefore $P(t) \to 0 \quad as \quad t \to \infty$, so the population die out in the long run.

Exercise

Consider a population $P(t)$ satisfying the logistic equation $\frac{dP}{dt} = aP - bP^2$, where $B = aP$ is the time rate at which births occur and $D = bP^2$ is the rate at which deaths occur.

a) If the initial population is $P(0) = P_0$, and B_0 births per month and D_0 deaths per month are occurring at time $t = 0$, show that the limiting population is $M = \frac{B_0 P_0}{D_0}$.

b) If the initial population is 120 rabbits and there are 8 births per month and 6 deaths per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 95% of the limiting population *M*?

c) If the initial population is 240 rabbits and there are 9 births per month and 12 deaths per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 105% of the limiting population *M*?

Solution

a) $P' = aP - bP^2 = bP\left(\frac{a}{b} - P\right)$ $\qquad$ $P' = kP(M - P)$

$$\Rightarrow M = \frac{a}{b}$$

$$\frac{B_0 P_0}{D_0} = \frac{aP_0 P_0}{bP_0^2} = \frac{a}{b} = M \quad \checkmark$$

b) Given: $P_0 = 120, \quad B_0 = 8, \quad D_0 = 6$

$$a = \frac{B_0}{P_0} = \frac{8}{120} = \frac{1}{15}, \quad b = \frac{D_0}{P_0^2} = \frac{6}{120^2} = \frac{1}{2400}$$

$$M = \frac{B_0 P_0}{D_0} = \frac{(8)(120)}{6} = 160, \quad k = b = \frac{1}{2400}$$

$$P(t) = \frac{(160)(120)}{120 + (160-120)e^{-\frac{160}{2400}t}} \qquad P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}$$

$$= \frac{19200}{120 + 40e^{-\frac{1}{15}t}}$$

$$= \frac{480}{3 + e^{-\frac{1}{15}t}}$$

For $P = .95M$

$$.95(160) = \frac{480}{3 + e^{-\frac{1}{15}t}}$$

$$3 + e^{-\frac{1}{15}t} = \frac{3}{.95}$$

$$e^{-\frac{1}{15}t} = \frac{3}{.95} - 3 = \frac{3}{19}$$

$$-\frac{t}{15} = \ln\frac{3}{19}$$

$$t = -15\ln\frac{3}{19} \underline{\approx 27.69 \ \ months}$$

c) Given: $P_0 = 240, \quad B_0 = 9, \quad D_0 = 12$

$$M = \frac{B_0 P_0}{D_0} = \frac{(9)(240)}{12} = 180, \quad k = b = \frac{D_0}{P_0^2} = \frac{12}{240^2} = \frac{1}{4800}$$

$$P(t) = \frac{(180)(240)}{240 + (180-240)e^{-\frac{180}{4800}t}} \qquad P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}$$

$$= \frac{43200}{240 - 60e^{-\frac{3}{80}t}}$$

$$= \frac{720}{4 - e^{-\frac{3}{80}t}}$$

For $P = 1.05M$

$$1.05(180) = \frac{720}{4 - e^{-\frac{3}{80}t}}$$

$$4 - e^{-\frac{3}{80}t} = \frac{720}{189}$$

$$e^{-\frac{3}{80}t} = 4 - \frac{720}{189} = \frac{36}{189} = \frac{4}{21}$$

$$-\frac{3t}{80} = \ln\frac{4}{21}$$

$$t = -\left(\frac{80}{3}\right)\ln\frac{4}{21} \underline{\approx 44.22 \ \ months}$$