Section 2.4 – Nonhomogeneous Equations; Method of undetermined Coefficients

The second order ***nonhomogeneous*** equation is given by: $y'' + p(x)y' + q(x)y = f(x) \quad (N)$

The corresponding ***homogeneous*** equation: $y'' + p(x)y' + q(x)y = 0 \quad (H)$

Theorem

Suppose that y_p is a particular solution to the nonhomogeneous (or inhomogeneous) equation $y'' + py' + qy = f$ and that y_1 and y_2 form a fundamental set of solutions to the homogeneous equation $y'' + py' + qy = 0.$ Then the general solution to the inhomogeneous equation is given by

$$y = y_p + C_1 y_1 + C_2 y_2$$

C_1 *and* C_2 are arbitrary constants.

Theorem

Let $y = y_1(x)$ *and* $y = y_2(x)$ be ***linearly independent*** $\left(W(x) \neq 0\right)$ solutions of the reduced equation (*H*) and let $y_p(x)$ be a ***particular solution*** of (*N*). Then the general solution of (*N*) consists of the general solution of the reduced equation (*H*) ***plus*** a particular solution of (*N*):

$$y(x) = \underbrace{y_p(x)}_{\text{a Particular Solution}} + \underbrace{C_1 y_1(x) + C_2 y_2(x)}_{\text{General Solution}}$$

Forcing Term

If the forcing term f has a form that is replicated under differentiation, then look for a solution with the same general form as the forcing term.

Example

Find a particular solution to the equation $y'' - y' - 2y = 2e^{-2t}$

Solution

The forcing term $f(t) = 2e^{-2t} \Rightarrow$ the particular solution $y = ae^{-2t}$

$$y' = -2ae^{-2t}$$

$$y'' = 4ae^{-2t}$$

$$4ae^{-2t} + 2ae^{-2t} - 2ae^{-2t} = 2e^{-2t}$$

$$4ae^{-2t} = 2e^{-2t}$$

$$4a = 2$$

$$a = \frac{1}{2}$$

$$\boxed{y(t) = \frac{1}{2}e^{-2t}}$$

Trigonometric Forcing Term

$f(t) = A\cos\omega t + B\sin\omega t$

The general solution: $y(t) = a\cos\omega t + b\sin\omega t$

Example

Find a particular solution to the equation $y'' + 2y' - 3y = 5\sin 3t$

Solution

The particular solution: $y(t) = a\cos 3t + b\sin 3t$

$$y' = -3a\sin 3t + 3b\cos 3t$$

$$y'' = -9a\cos 3t - 9b\sin 3t$$

$$\begin{aligned} y'' + 2y' - 3y &= -9a\cos 3t - 9b\sin 3t + 2\left(-3a\sin 3t + 3b\cos 3t\right) - 3\left(a\cos 3t + b\sin 3t\right) \\ &= -9a\cos 3t - 9b\sin 3t - 6a\sin 3t + 6b\cos 3t - 3a\cos 3t - 3b\sin 3t \\ &= \left(-12a + 6b\right)\cos 3t - \left(6a + 12b\right)\sin 3t \\ &= 5\sin 3t \end{aligned}$$

$$\begin{cases} -12a + 6b = 0 \\ -\left(6a + 12b\right) = 5 \end{cases} \Rightarrow a = -\frac{1}{6},\ b = -\frac{1}{3}$$

$$\boxed{y(t) = -\frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t}$$

The Complex Method

Example

Find a particular solution to the equation $y'' + 2y' - 3y = 5\sin 3t$

Solution

$$5e^{3it} = 5\cos 3t + 5i\sin 3t = 5cis3t$$

$$z'' + 2z' - 3z = 5e^{3it}$$

The particular solution: $z(t) = x(t) + i\,y(t)$

$$\begin{aligned} z'' + 2z' - 3z &= (x+iy)'' + 2(x+iy)' - 3(x+iy) \\ &= (x'' + 2x' - 3x) + i(y'' + 2y' - 3y) \\ &= 5\cos 3t + i\ 5\sin 3t \end{aligned}$$

$$x'' + 2x' - 3x = 5\cos 3t$$

$$z(t) = ae^{3it}$$

$$z' = 3iae^{3it}$$

$$z'' = 9i^2ae^{3it} = -9ae^{3it}$$

$$\begin{aligned} z'' + 2z' - 3z &= -9ae^{3it} + 2(3i)ae^{3it} - 3ae^{3it} \\ &= -12ae^{3it} + 6iae^{3it} \\ &= -6(2-i)ae^{3it} \\ &= 5e^{3it} \end{aligned}$$

$$-6(2-i)a = 5$$

$$\begin{aligned} a &= -\frac{5}{6(2-i)}\frac{2+i}{2+i} \\ &= -\frac{5(2+i)}{6(4+1)} \\ &= -\frac{2+i}{6} \end{aligned}$$

$$\begin{aligned} z(t) &= -\tfrac{1}{6}(2+i)e^{3it} \\ &= -\tfrac{1}{6}(2+i)(\cos 3t + i\sin 3t) \\ &= -\tfrac{1}{6}\left[(2\cos 3t - \sin 3t) + i(\cos 3t + 2\sin 3t)\right] \end{aligned}$$

$$\boxed{y(t) = -\tfrac{1}{6}(\cos 3t + 2\sin 3t)}$$

Polynomial Forcing Term

$$f(t) = a_0 t^n + a_1 t^{n-1} + \dots + a_{n-1} t + a_n$$

Example

Find a particular solution to the equation $y'' + 2y' - 3y = 3t + 4$

Solution

The right-hand side is a polynomial of degree 1.

The particular solution: $y(t) = at + b$

$$y' = a$$
$$y'' = 0$$

$$\begin{aligned} y'' + 2y' - 3y &= 0 + 2a - 3(at + b) \\ &= 2a - 3b - 3at \\ &= 3t + 4 \end{aligned}$$

$$\rightarrow \begin{cases} -3a = 3 \\ 2a - 3b = 4 \end{cases} \quad \Rightarrow a = -1; \quad b = -2$$

$$\underline{y(t) = -t - 2|}$$

Exceptional Cases

Example

Find a particular solution to the equation $y'' - y' - 2y = 3e^{-t}$

Solution

The particular solution $y = ae^{-t}$

$$y'' - y' - 2y = ae^{-t} + ae^{-t} - 2ae^{-t}$$
$$= 0$$

The particular solution $y = ate^{-t}$ or $y = at^2e^{-t}$

$$y' = ae^{-t} - ate^{-t} = ae^{-t}(1-t)$$
$$y'' = -ae^{-t} - ae^{-t} + ate^{-t}$$
$$= ate^{-t} - 2ae^{-t}$$

$$y'' - y' - 2y = ate^{-t} - 2ae^{-t} - ae^{-t} + ate^{-t} - 2ate^{-t}$$
$$= -3ae^{-t}$$

$$-3ae^{-t} = 3e^{-t}$$
$$a = -1$$

The particular solution $\boxed{y = -te^{-t}}$

Summary

$f(t)$	y_p
Any Constant	A
$at + b$	$At + B$
$at^2 + c$	$At^2 + Bt + C$
$at^3 + \cdots + b$	$At^3 + Bt^3 + Ct + E$
$\sin at$ *or* $\cos at$	$A\cos at + B\sin at$
e^{at}	Ae^{at}
$(at+b)e^{at}$	$(At+B)e^{at}$
t^2e^{at}	$\left(At^2 + Bt + C\right)e^{at}$
$e^{at}\sin bt$	$e^{at}(A\cos bt + B\sin bt)$
$t^2\sin bt$	$\left(At^2 + Bt + C\right)\cos bt + \left(Et^2 + Ft + G\right)\sin bt$
$te^{at}\cos bt$	$(At+B)\cos bt + (Ct+E)\sin bt$

Exercises Section 2.4 – Nonhomogeneous Equations; Method of undetermined Coefficients

1. Show that the 3 solutions $y_1 = x, \quad y_2 = x\ln x, \quad y_3 = x^2$ of the 3rd order equation $x^3y''' - x^2y'' + 2xy' - 2y = 0$ are linearly independent on an open interval $x > 0$. Then find a particular solution that satisfies the initial conditions $y(1) = 3, \quad y'(1) = 2, \quad y''(1) = 1$

Find the particular solution for the given differential equation

2. $y'' + 3y' + 2y = 4e^{-3t}$

3. $y'' + 6y' + 8y = -3e^{-t}$

4. $y'' + 2y' + 5y = 12e^{-t}$

5. $y'' + 3y' - 18y = 18e^{2t}$

6. $y'' + 4y = \cos 3t$

7. $y'' + 7y' + 6y = 3\sin 2t$

8. $y'' + 5y' + 4y = 2 + 3t$

9. $y'' + 6y' + 8y = 2t - 3$

10. $y'' + 3y' + 4y = t^3$

11. $y'' + 2y' + 2y = 2 + \cos 2t$

12. $y'' - y = t - e^{-t}$

13. $y'' - 2y' + y = 10e^{-2t}\cos t$

14. $y''' - 4y'' + 4y' = 5t^2 - 6t + 4t^2e^t + 3e^{5t}$

Use the ***complex method*** to find the particular solution for

15. $y'' + 4y' + 3y = \cos 2t + 3\sin 2t$

16. $y'' + 4y = \cos 3t$

Find the general solution for the given differential equation

17. $y'' + y = 2\cos x$

18. $y'' + y = \cos 3x$

19. $y'' + y = 2x\sin x$

20. $y'' - y = x^2e^x + 5$

21. $y'' - y' = -3$

22. $y'' - y' = 2\sin x$

23. $y'' - y' = \sin x$

24. $y'' - y' = -8x + 3$

25. $y'' + y = 2x + 3e^x$

26. $y'' - y = x^2 + e^x$

27. $y'' + y' = 10x^4 + 2$

28. $y'' - y' = 5e^x - \sin 2x$

29. $y'' + y = x\cos x - \cos x$

30. $y'' + y = e^x\sin x$

31. $y'' - y' - 2y = 20\cos x$

32. $y'' - y' + \frac{1}{4}y = 3 + e^{x/2}$

33. $y'' + y' + \frac{1}{4}y = e^x(\sin 3x - \cos 3x)$

34. $y'' - y' - 2y = e^{3x}$

35. $y'' - y' - 6y = 20e^{-2x}$

36. $y'' + y' - 6y = 2x$

37. $y'' - y' - 6y = e^{-x} - 7\cos x$

38. $y'' + y' + 8y = x\cos 3x + (10x^2 + 21x + 9)\sin 3x$

39. $y'' - y' - 12y = e^{4x}$

40. $y'' + 2y' = 2x + 5 - e^{-2x}$

41. $y'' - 2y' = 12x - 10$

42. $y'' + 2y' + y = \sin x + 3\cos 2x$

43. $y'' - 2y' + y = 6e^x$

44. $y'' + 2y' + y = x^2$

45. $y'' + 2y' + y = x^2 e^{-x}$

46. $y'' - 2y' + y = x^3 + 4x$

47. $y'' + 2y' + y = 6\sin 2x$

48. $y'' - 2y' + y = \left(x^2 - 1\right)e^{2x} + (3x+4)e^x$

49. $y'' + 2y' + 2y = 5e^{6x}$

50. $y'' + 2y' + 2y = x^3$

51. $y'' + 2y' + 2y = \cos x + e^{-x}$

52. $y'' - 2y' + 2y = e^x \sin x$

53. $y'' - 2y' + 2y = e^{2x}(\cos x - 3\sin x)$

54. $y'' - 2y' - 3y = 1 - x^2$

55. $y'' - 2y' - 3y = 4e^x - 9$

56. $y'' - 2y' - 3y = 2e^{-x}\cos x + x^2 + xe^{3x}$

57. $y'' - 2y' + 5y = 25x^2 + 12$

58. $y'' - 2y' + 5y = e^x \cos 2x$

59. $y'' - 2y' + 5y = e^x \sin x$

60. $y'' + 2y' - 24y = 16 - (x+2)e^{4x}$

61. $y'' + 3y = -48x^2 e^{3x}$

62. $y'' - 3y' = e^{3x} - 12x$

63. $y'' + 3y' = 4x - 5$

64. $y'' - 3y' = 8e^{3x} + 4\sin x$

65. $y'' + 3y' + 2y = 6$

66. $y'' + 3y' + 2y = 4x^2$

67. $y'' - 3y' + 2y = 5e^x$

68. $y'' - 3y' + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$

69. $y'' - 3y' + 2y = 14\sin 2x - 18\cos 2x$

70. $y'' + 3y' + 2y = e^{-x} + e^{-2x} - x$

71. $y''' - 3y'' + 3y' - y = 3e^x$

72. $y'' - 3y' - 10y = -3$

73. $y'' - 3y' - 10y = 2x - 3$

74. $y'' + 3y' - 10y = 6e^{4x}$

75. $y'' + 3y' - 10y = x\left(e^x + 1\right)$

76. $y'' - 4y = 4x^2$

77. $y'' + 4y = 3x^3$

78. $y'' + 4y = 3\sin x$

79. $y'' + 4y = 3\sin 2x$

80. $y'' + 4y = 4\cos x + 3\sin x - 8$

81. $y'' - 4y = \left(x^2 - 3\right)\sin 2x$

82. $y'' + 4y' + 4y = 2x + 6$

83. $y'' + 4y' + 5y = 5x + e^{-x}$

84. $y'' + 4y' + 5y = 2e^{-2x} + \cos x$

85. $y'' + 5y' = 15x^2$

86. $y'' - 5y' = 2x^3 - 4x^2 - x + 6$

87. $y'' + 6y' + 8y = 3e^{-2x} + 2x$

88. $y'' - 6y' + 9y = e^{3x}$

89. $y'' + 6y' + 9y = -xe^{4x}$

90. $y'' + 6y' + 13y = e^{-3x}\cos 2x$

91. $y'' - 7y' = -3$

92. $y'' + 7y' = 42x^2 + 5x + 1$

93. $y'' + 8y = 5x + 2e^{-x}$

94. $y'' - 8y' + 20y = 100x^2 - 26xe^x$

95. $y'' - 9y = 54$

96. $y'' + 9y = x^2\cos 3x + 4\sin x$

97. $y'' + 10y' + 25y = 14e^{-5x}$

98. $y'' - 10y' + 25y = 30x + 3$

99. $y'' - 16y = 2e^{4x}$

100. $y'' + 25y = 6\sin x$

101. $y'' + 25y = 20\sin 5x$

102. $\frac{1}{4}y'' + y' + y = x^2 - 2x$

103. $2y'' - 5y' + 2y = -6e^{x/2}$

104. $2y'' - 7y' + 5y = -29$

105. $4y'' + 9y = 15$

106. $4y'' - 4y' - 3y = \cos 2x$

107. $9y'' - 6y' + y = 9xe^{x/3}$

108. $y^{(3)}+y''=8x^2$

109. $y^{(3)}-y''-4y'+4y=5-e^x+e^{2x}$

110. $y^{(3)}+y''=3e^x+4x^2$

111. $y^{(3)}+2y''+y'=10$

112. $y^{(3)}-2y''-4y'+8y=6xe^{2x}$

113. $y^{(3)}-3y''+3y'-y=x-4e^x$

114. $y^{(3)}-4y''+y'+6y=4\sin 2x$

115. $y^{(3)}-3y''+3y'-y=e^x-x+16$

116. $y^{(3)}-6y''=3-\cos x$

117. $y^{(3)}-6y''+11y'-6y=2xe^{-x}$

118. $y^{(3)}+8y''=-6x^2+9x+2$

119. $y^{(4)}+y''=3x^2+4\sin x-2\cos x$

120. $y^{(4)}+2y''+y=(x-2)^2$

121. $y^{(4)}-y''=4x+2xe^{-x}$

122. $\left(D^2+D-2\right)y=2x-40\cos 2x$

123. $\left(D^2-3D+2\right)y=2\sin x$

124. $(D-2)^3\left(D^2+9\right)y=x^2e^{2x}+x\sin 3x$

Find the general solution that satisfy the given initial conditions

125. $y''+y=\cos x;\quad y(0)=1,\quad y'(0)=-1$

126. $y''+y'=x\ ;\quad y(1)=0,\quad y'(1)=1$

127. $y''+y'=-x\ ;\quad y(0)=1,\quad y'(0)=0$

128. $y''+y=8\cos 2t-4\sin t\quad y\left(\frac{\pi}{2}\right)=-1,\quad y'\left(\frac{\pi}{2}\right)=0$

129. $y''-y'-2y=4x^2;\quad y(0)=1,\quad y'(0)=4$

130. $y''-y'-2y=e^{3x};\quad y(1)=2,\quad y'(1)=1$

131. $y''-y'-2y=e^{3x};\quad y(0)=1,\quad y'(0)=2$

132. $y''-y'-2y=e^{3x};\quad y(0)=2,\quad y'(0)=1$

133. $y''+2y'+y=2\cos t\ ;\quad y(0)=3,\quad y'(0)=0$

134. $y''-2y'+y=t^3;\quad y(0)=1,\quad y'(0)=0$

135. $y''-2y'+y=-3-x+x^2\ ;\quad y(0)=-2,\quad y'(0)=1$

136. $y''-2y'+2y=x+1;\quad y(0)=3,\quad y'(0)=0$

137. $y''+2y'+2y=\sin 3x;\quad y(0)=2,\quad y'(0)=0$

138. $y''+2y'+2y=2\cos 2t;\quad y(0)=-2,\quad y'(0)=0$

139. $y''-2y'-3y=2e^x-10\sin x\ ;\quad y(0)=2,\quad y'(0)=4$

140. $y''+2y'+10y=4+26x+6x^2+10x^3\ ;\quad y(0)=2,\quad y'(0)=9$

141. $y''-2y'+10y=6\cos 3t-\sin 3t\ ;\quad y(0)=2,\quad y'(0)=-8$

142. $y''+3y'+2y=e^x;\quad y(0)=0,\quad y'(0)=3$

143. $y''-3y'+2y=3e^{-x}-10\cos 3x\quad y(0)=1\quad y'(0)=2$

144. $y'' + 4y = -2$; $y\left(\frac{\pi}{8}\right) = \frac{1}{2}$, $y'\left(\frac{\pi}{8}\right) = 2$

145. $y'' + 4y = 2x$; $y(0) = 1$, $y'(0) = 2$

146. $y'' + 4y = \sin^2 2t$; $x\left(\frac{\pi}{8}\right) = 0$ $x'\left(\frac{\pi}{8}\right) = 0$

147. $y'' - 4y' + 8y = x^3$; $y(0) = 2$, $y'(0) = 4$

148. $y'' + 4y' + 4y = (3 + x)e^{-2x}$; $y(0) = 2$, $y'(0) = 5$

149. $y'' + 4y' + 4y = 4 - t$; $y(0) = -1$, $y'(0) = 0$

150. $y'' - 4y' + 4y = e^x$; $y(0) = 2$, $y'(0) = 0$

151. $y'' - 4y' - 5y = 4e^{-2t}$; $y(0) = 0$, $y'(0) = -1$

152. $y'' + 4y' + 5y = 35e^{-4x}$; $y(0) = -3$, $y'(0) = 1$

153. $y'' + 4y' + 8y = \sin t$; $y(0) = 1$, $y'(0) = 0$

154. $y'' - 4y' - 12y = 3e^{5t}$; $y(0) = \frac{18}{7}$, $y'(0) = -\frac{1}{7}$

155. $y'' - 4y' - 12y = \sin 2t$; $y(0) = 0$, $y'(0) = 0$

156. $y'' - 5y' = t - 2$; $y(0) = 0$, $y'(0) = 2$

157. $y'' + 5y' - 6y = 10e^{2x}$; $y(0) = 1$, $y'(0) = 1$

158. $y'' + 6y' + 10y = 22 + 20x$; $y(0) = 2$, $y'(0) = -2$

159. $y'' + 7y' + 12y = -2\cos 2x + 36\sin 2x$; $y(0) = -3$, $y'(0) = 3$

160. $y'' + 8y' + 7y = 10e^{-2x}$; $y(0) = -2$, $y'(0) = 10$

161. $y'' + 9y = \sin 2x$; $y(0) = 1$, $y'(0) = 0$

162. $y'' - 64y = 16$; $y(0) = 1$, $y'(0) = 0$

163. $2y'' + 3y' - 2y = 14x^2 - 4x + 11$; $y(0) = 0$, $y'(0) = 0$

164. $5y'' + y' = -6x$; $y(0) = 0$, $y'(0) = -10$

165. $x'' + 9x = 10\cos 2t$; $x(0) = x'(0) = 0$

166. $x'' + 4x = 5\sin 3t$; $x(0) = x'(0) = 0$

167. $x'' + 100x = 225\cos 5t + 300\sin 5t$; $x(0) = 375$, $x'(0) = 0$

168. $x'' + 25x = 90\cos 4t$; $x(0) = 0$, $x'(0) = 90$

169. $y^{(3)} - y' = 4e^{-x} + 3e^{2x}$; $y(0) = 0$, $y'(0) = -1$, $y''(0) = 2$

170. $y^{(3)} + y'' = x + e^{-x}$; $y(0) = 1$, $y'(0) = 0$, $y''(0) = 1$

171. $y^{(3)} - 2y'' + y' = 1 + xe^x$; $y(0) = y'(0) = 0$, $y''(0) = 1$

172. $y^{(4)} - 4y'' = x^2$; $y(0) = y'(0) = 1$, $y''(0) = y^{(3)}(0) = -1$

173. $y^{(4)} - y = 5$; $y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0$

174. $y^{(4)} - y''' = x + e^{x}$; $\quad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0$

175. If k and b are positive constants, then find the general solution of $\quad y'' + k^{2}y = \sin bx$