Solution *Section* 3.5 – Additional Identities

Exercise

Write $10\cos 5x \sin 3x$ as a sum or difference

Solution

$$10\cos 5x\sin 3x = 10\cdot\frac{1}{2}\left[\sin(5x+3x)-\sin(5x-3x)\right]$$

$$= 5(\sin 8x - \sin 2x)$$

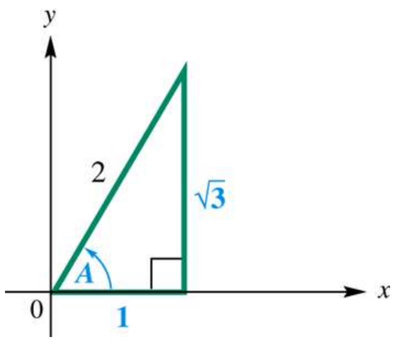
Exercise

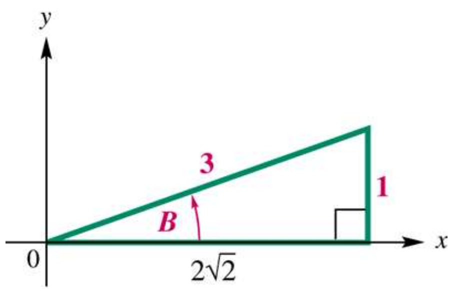
Evaluate without using the calculator $\cos\left(\arctan\sqrt{3}+\arcsin\frac{1}{3}\right)$

Solution

$$\alpha = \arctan\sqrt{3} \Rightarrow \tan\alpha = \sqrt{3}$$

$$\beta = \arcsin\frac{1}{3} \Rightarrow \sin\beta = \frac{1}{3}$$





$\sin\alpha = \frac{\sqrt{3}}{2},\quad \cos\alpha = \frac{1}{2}$ $\qquad$ $\sin\beta = \frac{1}{3},\quad \cos B = \frac{2\sqrt{2}}{3}$

$$\cos\left(\arctan\sqrt{3}+\arcsin\frac{1}{3}\right) = \cos(\alpha+\beta)$$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= \frac{1}{2}\frac{2\sqrt{2}}{3} - \frac{\sqrt{3}}{2}\frac{1}{3}$$

$$= \frac{2\sqrt{2}-\sqrt{3}}{6}$$

Exercise

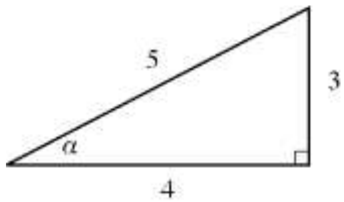
Evaluate without using the calculator $\cos(\arcsin\frac{3}{5} - \arctan 2)$

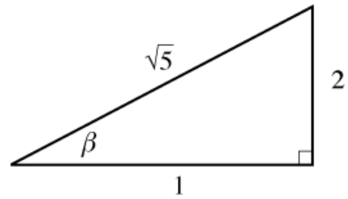
Solution

$$\cos(\arcsin\frac{3}{5} - \arctan 2) = \cos(\alpha - \beta)$$

$$\alpha = \arcsin\frac{3}{5}$$

$$\beta = \arctan 2$$





$$\cos(\arcsin\frac{3}{5} - \arctan 2) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$= \frac{4}{5}\frac{1}{\sqrt{5}} + \frac{3}{5}\frac{2}{\sqrt{5}}$$

$$= \frac{4}{5\sqrt{5}} + \frac{6}{5\sqrt{5}}$$

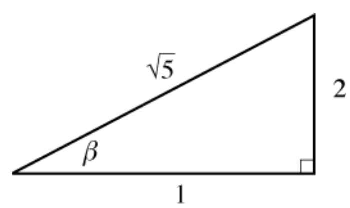
$$= \frac{10}{5\sqrt{5}}$$

$$= \frac{2}{\sqrt{5}}$$

Exercise

Evaluate without using the calculator $\sin\left(2\cos^{-1}\frac{1}{\sqrt{5}}\right)$

Solution



$$\beta = \cos^{-1}\frac{1}{\sqrt{5}}$$

$$\cos\beta = \frac{1}{\sqrt{5}}$$

$$\sin\beta = \frac{2}{\sqrt{5}}$$

$$\sin(2\beta) = 2\sin\beta\cos\beta$$

$$= 2\frac{2}{\sqrt{5}}\frac{1}{\sqrt{5}}$$

$$= \frac{4}{5}$$

Exercise

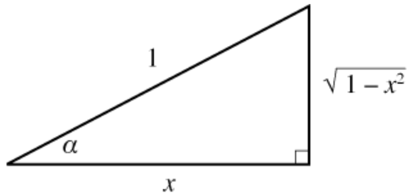
Write $\sin\left(2\cos^{-1} x\right)$ as an equivalent expression involving only x.

Solution

$\alpha = \cos^{-1} x$

$\cos\alpha = x$

$\sin\alpha = \dfrac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$



$$\begin{aligned}\sin\left(2\cos^{-1} x\right) &= \sin(2\alpha)\\ &= 2\sin\alpha\cos\alpha\\ &= 2\sqrt{1-x^2}\cdot x\\ &= 2x\sqrt{1-x^2}\end{aligned}$$

Exercise

Write $\sec\left(\tan^{-1}\frac{x-2}{2}\right)$ as an equivalent expression involving only x.

Solution

$\alpha = \tan^{-1}\dfrac{x-2}{2}$

$\tan\alpha = \dfrac{x-2}{2}$

$$\begin{aligned}c &= \sqrt{(x-2)^2 + 2^2}\\ c &= \sqrt{x^2 - 4x + 4 + 4}\\ c &= \sqrt{x^2 - 4x + 8}\end{aligned}$$

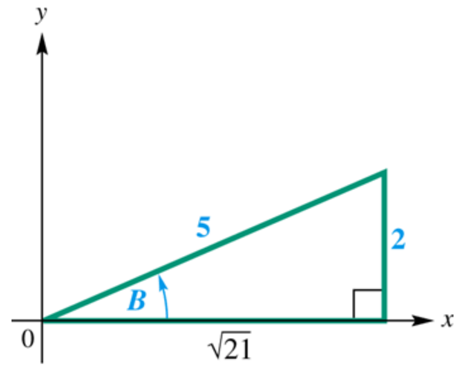
$\cos\alpha = \dfrac{2}{\sqrt{x^2 - 4x + 8}}$

$$\begin{aligned}\sec\alpha &= \frac{1}{\cos\alpha}\\ &= \frac{1}{\frac{\sqrt{x^2-4x+8}}{2}}\\ &= \frac{\sqrt{x^2-4x+8}}{2}\end{aligned}$$

Exercise

Evaluate without using the calculator $\tan\left(2\arcsin\frac{2}{5}\right)$

Solution



$$\alpha = \arcsin\frac{2}{5} \Rightarrow \sin\alpha = \frac{2}{5}$$

$$x = \sqrt{5^2 - 2^2} = \sqrt{21}$$

$$\tan(\alpha) = \frac{2}{\sqrt{21}}$$

$$\tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$= \frac{2\frac{2}{\sqrt{21}}}{1-\left(\frac{2}{\sqrt{21}}\right)^2}$$

$$= \frac{\frac{4}{\sqrt{21}}}{1-\frac{4}{21}}$$

$$= \frac{\frac{4}{\sqrt{21}}}{\frac{21-4}{21}}$$

$$= \frac{\frac{4}{\sqrt{21}}}{\frac{17}{21}}$$

$$= \frac{4}{\sqrt{21}}\frac{21}{17}\frac{\sqrt{21}}{\sqrt{21}}$$

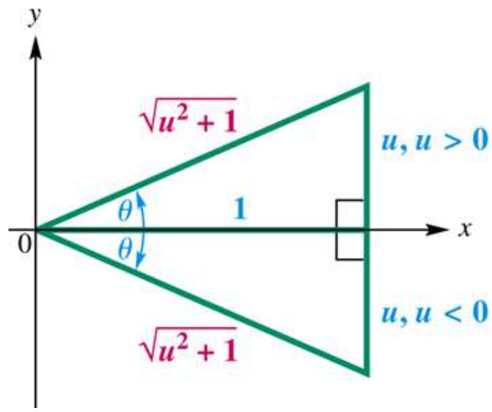
$$= \frac{4(21)\sqrt{21}}{21(17)}$$

$$= \frac{4\sqrt{21}}{17}$$

Exercise

Evaluate without using the calculator $\sin\left(\tan^{-1}u\right)$

Solution



$$\theta = \tan^{-1}u \quad \Rightarrow \quad \tan\theta = u = \frac{u}{1}$$

$$r = \sqrt{x^2+y^2} = \sqrt{1+u^2}$$

$$\sin\theta = \frac{u}{\sqrt{u^2+1}}\frac{\sqrt{u^2+1}}{\sqrt{u^2+1}}$$

$$= \frac{u\sqrt{u^2+1}}{u^2+1}$$

Exercise

Write $\cos\left(2\sin^{-1}u\right)$ as an equivalent expression involving only x.

Solution

$$\theta = \sin^{-1}u \quad \Rightarrow \quad \sin\theta = u$$

$$\cos\left(2\sin^{-1}u\right) = \cos 2\theta$$

$$= 1 - 2\sin^2\theta$$

$$= 1 - 2u^2$$

Exercise

Prove the identity: $\quad \dfrac{\sin 3x - \sin x}{\cos 3x - \cos x} = -\cot 2x$

Solution

$$\frac{\sin 3x - \sin x}{\cos 3x - \cos x} = \frac{2\cos\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)}{-2\sin\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)}$$

$$= -\frac{\cos 2x \sin x}{\sin 2x \sin x}$$

$$= -\frac{\cos 2x}{\sin 2x}$$

$$= -\cot 2x$$

Exercise

Prove the following equation is an identity: $\sin(x+y)\cos(x-y)=\sin x\cos x+\cos y\sin y$

Solution

$$\sin(x+y)\cos(x-y)=\frac{1}{2}\left[\sin(x+y+x-y)+\sin(x+y-x+y)\right]$$
$$=\frac{1}{2}\left[\sin(2x)+\sin(2y)\right]$$
$$=\frac{1}{2}\left[2\sin x\cos x+2\sin y\cos y\right]$$
$$=\sin x\cos x+\sin y\cos y$$

Exercise

Prove the following equation is an identity: $2\sin(x+y)\cos(x-y)=\sin 2x+\sin 2y$

Solution

$$2\sin(x+y)\cos(x-y)=\sin(x+y+x-y)+\sin(x+y-(x-y))$$
$$=\sin(2x)+\sin(x+y-x+y)$$
$$=\sin(2x)+\sin(2y)$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin(26k)+\sin(8k)}{\cos(26k)-\cos(8k)}=-\cot(9k)$

Solution

$$\frac{\sin(26k)+\sin(8k)}{\cos(26k)-\cos(8k)}=\frac{2\sin\left(\frac{26k+8k}{2}\right)\cos\left(\frac{26k-8k}{2}\right)}{-2\sin\left(\frac{26k+8k}{2}\right)\sin\left(\frac{26k-8k}{2}\right)}$$
$$=-\frac{\sin(17k)\cos(9k)}{\sin(17k)\sin(9k)}$$
$$=-\cot 9k$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin(26k)-\sin(12k)}{\sin(26k)+\sin(12k)}=\cot(19k)\tan(7k)$

Solution

$$\frac{\sin(26k)-\sin(12k)}{\sin(26k)+\sin(12k)}=\frac{2\cos\left(\frac{26k+12k}{2}\right)\sin\left(\frac{26k-12k}{2}\right)}{2\sin\left(\frac{26k+12k}{2}\right)\cos\left(\frac{26k-12k}{2}\right)}$$

$$=\frac{\cos(19k)\sin(7k)}{\sin(19k)\cos(7k)}$$

$$=\cot(19k)\tan(7k)$$

Exercise

Prove the following equation is an identity: $\sin(x+y)\cos(x-y)=\sin x\cos x+\cos y\sin y$

Solution

$$\sin(x+y)\cos(x-y)=\tfrac{1}{2}\left[\sin(x+y+x-y)+\sin(x+y-(x-y))\right]$$

$$=\tfrac{1}{2}\left[\sin(2x)+\sin(x+y-x+y)\right]$$

$$=\tfrac{1}{2}\left[\sin(2x)+\sin(2y)\right]$$

$$=\tfrac{1}{2}\left[2\sin x\cos x+2\sin y\cos y\right]$$

$$=\tfrac{1}{2}2(\sin x\cos x+\sin y\cos y)$$

$$=\sin x\cos x+\sin y\cos y$$

Exercise

Prove the following equation is an identity: $(\sin\alpha+\cos\alpha)(\sin\beta+\cos\beta)=\sin(\alpha+\beta)+\cos(\alpha-\beta)$

Solution

$$(\sin\alpha+\cos\alpha)(\sin\beta+\cos\beta)=\sin\alpha\sin\beta+\sin\alpha\cos\beta+\cos\alpha\sin\beta+\cos\alpha\cos\beta$$

$$=\tfrac{1}{2}\cos(\alpha-\beta)-\tfrac{1}{2}\cos(\alpha+\beta)+\tfrac{1}{2}\sin(\alpha+\beta)+\tfrac{1}{2}\sin(\alpha-\beta)$$
$$+\tfrac{1}{2}\sin(\alpha+\beta)-\tfrac{1}{2}\sin(\alpha-\beta)+\tfrac{1}{2}\cos(\alpha+\beta)+\tfrac{1}{2}\cos(\alpha-\beta)$$

$$=\cos(\alpha-\beta)+\sin(\alpha+\beta)$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos x - \cos 3x}{\cos x + \cos 3x} = \tan 2x \tan x$

Solution

$$\frac{\cos x - \cos 3x}{\cos x + \cos 3x} = \frac{-2\sin\left(\frac{x+3x}{2}\right)\sin\left(\frac{x-3x}{2}\right)}{2\cos\left(\frac{x+3x}{2}\right)\cos\left(\frac{x-3x}{2}\right)}$$

$$= -\frac{\sin(2x)\sin(-x)}{\cos(2x)\cos(-x)}$$

$$= -\tan(2x)\frac{-\sin(x)}{\cos(x)}$$

$$= \tan(2x)\tan(x)$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos 5x + \cos 3x}{\cos 5x - \cos 3x} = -\cot 4x \cot x$

Solution

$$\frac{\cos 5x + \cos 3x}{\cos 5x - \cos 3x} = \frac{2\cos\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)}{-2\sin\left(\frac{5x+3x}{2}\right)\sin\left(\frac{5x-3x}{2}\right)}$$

$$= -\frac{\cos(4x)\cos(x)}{\sin(4x)\sin(x)}$$

$$= -\cot(4x)\cot(x)$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin 3t - \sin t}{\cos 3t + \cos t} = \tan t$

Solution

$$\frac{\sin 3t - \sin t}{\cos 3t + \cos t} = \frac{2\cos\left(\frac{3t+t}{2}\right)\sin\left(\frac{3t-t}{2}\right)}{2\cos\left(\frac{3t+t}{2}\right)\cos\left(\frac{3t-t}{2}\right)}$$

$$= \frac{\cos(2t)\sin(t)}{\cos(2t)\cos(t)}$$

$$= \frac{\sin t}{\cos t}$$

$$= \tan t$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin 3x + \sin 5x}{\sin 3x - \sin 5x} = -\dfrac{\tan 4x}{\tan x}$

Solution

$$\begin{aligned}
\frac{\sin 3x + \sin 5x}{\sin 3x - \sin 5x} &= \frac{2\sin\left(\frac{3x+5x}{2}\right)\cos\left(\frac{3x-5x}{2}\right)}{2\cos\left(\frac{3x+5x}{2}\right)\sin\left(\frac{3x-5x}{2}\right)} \\
&= \frac{\sin(4x)\cos(-x)}{\cos(4x)\sin(-x)} \\
&= \tan(4x)\frac{\cos(x)}{-\sin(x)} \\
&= -\tan(4x)\cot x \\
&= -\tan(4x)\frac{1}{\tan x} \\
&= -\frac{\tan 4x}{\tan x}
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cos^2 x - \cos^2 y = -\sin(x+y)\sin(x-y)$

Solution

$$\begin{aligned}
\cos^2 x - \cos^2 y &= (\cos x - \cos y)(\cos x + \cos y) \\
&= \left(-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\right)\left(2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\right) \\
&= -2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x+y}{2}\right)2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x-y}{2}\right) \\
&= -\sin 2\left(\frac{x+y}{2}\right)\sin 2\left(\frac{x-y}{2}\right) \\
&= -\sin(x+y)\sin(x-y)
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos 8x-\cos 2x}{2\sin 5x}=-\sin 3x$

Solution

$$\begin{aligned}\frac{\cos 8x-\cos 2x}{2\sin 5x}&=\frac{-2\sin\left(\frac{8x+2x}{2}\right)\sin\left(\frac{8x-2x}{2}\right)}{2\sin 5x}\\ &=\frac{-\sin(5x)\sin(3x)}{\sin 5x}\\ &=-\sin 3x\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin 9x+\sin 3x}{\cos 9x+\cos 3x}=\tan 6x$

Solution

$$\begin{aligned}\frac{\sin 9x+\sin 3x}{\cos 9x+\cos 3x}&=\frac{2\sin\left(\frac{9x+3x}{2}\right)\cos\left(\frac{9x-3x}{2}\right)}{2\cos\left(\frac{9x+3x}{2}\right)\cos\left(\frac{9x-3x}{2}\right)}\\ &=\frac{2\sin(6x)\cos(3x)}{2\cos(6x)\cos(3x)}\\ &=\frac{\sin(6x)}{\cos(6x)}\\ &=\tan 6x\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos 2x-\cos 6x}{\sin 2x+\sin 6x}=\tan 2x$

Solution

$$\begin{aligned}\frac{\cos 2x-\cos 6x}{\sin 2x+\sin 6x}&=\frac{-2\sin\left(\frac{2x+6x}{2}\right)\sin\left(\frac{2x-6x}{2}\right)}{2\sin\left(\frac{2x+6x}{2}\right)\cos\left(\frac{2x-6x}{2}\right)}\\ &=-\frac{\sin(4x)\sin(-2x)}{\sin(4x)\cos(-2x)}\\ &=-\frac{-\sin 2x}{\cos 2x}\\ &=\tan 2x\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\sin 8x+\sin 2x}{\sin 8x-\sin 2x}=\frac{\tan 5x}{\tan 3x}$

Solution

$$\frac{\sin 8x+\sin 2x}{\sin 8x-\sin 2x}=\frac{2\sin\left(\frac{8x+2x}{2}\right)\cos\left(\frac{8x-2x}{2}\right)}{2\cos\left(\frac{8x+2x}{2}\right)\sin\left(\frac{8x-2x}{2}\right)}$$

$$=\frac{\sin(5x)\cos(3x)}{\cos(5x)\sin(3x)}$$

$$=\tan 5x\cot 3x$$

$$=\tan 5x\frac{1}{\tan 3x}$$

$$=\frac{\tan 5x}{\tan 3x}$$

Exercise

Prove the following equation is an identity: $\frac{\cos 6x-\cos 2x}{\cos 6x+\cos 2x}=-\tan 4x\tan 2x$

Solution

$$\frac{\cos 6x-\cos 2x}{\cos 6x+\cos 2x}=\frac{-2\sin\left(\frac{6x+2x}{2}\right)\sin\left(\frac{6x-2x}{2}\right)}{2\cos\left(\frac{6x+2x}{2}\right)\cos\left(\frac{6x-2x}{2}\right)}$$

$$=-\frac{\sin(4x)\sin(2x)}{\cos(4x)\cos(2x)}$$

$$=-\frac{\sin 4x}{\cos 4x}\frac{\sin 2x}{\cos 2x}$$

$$=-\tan 4x\tan 2x$$

Exercise

Prove the following equation is an identity: $\sin x(\sin x+\sin 5x)=\cos 2x(\cos 2x-\cos 4x)$

Solution

$$\begin{aligned}
\sin x(\sin x+\sin 5x) &= \sin x\left(2\sin\left(\tfrac{x+5x}{2}\right)\cos\left(\tfrac{x-5x}{2}\right)\right)\\
&= \sin x\left(2\sin 3x\cos(-2x)\right)\\
&= 2\sin x\sin 3x\cos 2x\\
&= 2\cos 2x(\sin x\sin 3x)\\
&= 2\cos 2x\left(\tfrac{1}{2}\left[\cos(x-3x)-\cos(x+3x)\right]\right)\\
&= \cos 2x\left(\cos(-2x)-\cos 4x\right)\\
&= \cos 2x(\cos 2x-\cos 4x)
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos x+\cos y}{\sin x-\sin y}=\cot\dfrac{x-y}{2}$

Solution

$$\begin{aligned}
\frac{\cos x+\cos y}{\sin x-\sin y} &= \frac{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}{2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)}\\
&= \frac{\cos\left(\frac{x-y}{2}\right)}{\sin\left(\frac{x-y}{2}\right)}\\
&= \cot\left(\frac{x-y}{2}\right)
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin 6x+\sin 2x}{2\sin 4x}=\cos 2x$

Solution

$$\begin{aligned}
\frac{\sin 6x+\sin 2x}{2\sin 4x} &= \frac{2\sin\left(\frac{6x+2x}{2}\right)\cos\left(\frac{6x-2x}{2}\right)}{2\sin 4x}\\
&= \frac{\sin(4x)\cos(2x)}{\sin 4x}
\end{aligned}$$