Section 3.7 – Implicit Differentiation and Related Rates

Explicit and Implicit Functions

$y = f(x)$ is called explicit form, the variable y is explicitly written as a function of x.

(***Example***: $y = 3x - 5$)

Example

Find dy/dx for the equation $x^2 y = 1$

<u>*Solution*</u>

$$y = \frac{1}{x^2} = x^{-2}$$

$$\frac{dy}{dx} = -2x^{-3}$$

$$\frac{dy}{dx} = -\frac{2}{x^3}$$

Example

Differentiate each expression with respect to x.

a. $x + 5y$

<u>*Solution*</u>

$$\frac{d}{dx}\left[x + 5y\right] = 1 + 5\frac{dy}{dx}$$

b. xy^3

<u>*Solution*</u>

$$\frac{d}{dx}\left[xy^3\right] = x\frac{d}{dx}[y^3] + y^3\frac{d}{dx}[x]$$

$$= x(2y)\frac{dy}{dx} + y^3$$

$$= 2xy\frac{dy}{dx} + y^3$$

Implicit Differentiation

Consider an equation involving x and y in which y is a differentiable function of x. You can use the steps below to find dy/dx.

1. Differentiate both sides of the equation with respect to x.
2. Write the result so that all terms involving dy/dx are on the left side of the equation and all other terms are on the right side of the equation.
3. Factor dy/dx out of terms if necessary.
4. Solve for dy/dx.

Example

Find dy/dx for $x+\sqrt{x}\sqrt{y}=y^2$

<u>*Solution*</u>

$$\frac{d}{dx}\left(x+x^{1/2}y^{1/2}\right)=\frac{d}{dx}y^2$$

$$1+\frac{d}{dx}\left(x^{1/2}\right)y^{1/2}+x^{1/2}\frac{d}{dx}\left(y^{1/2}\right)=2y\frac{dy}{dx}$$

$$1+\frac{1}{2}x^{-1/2}y^{1/2}+x^{1/2}\frac{1}{2}y^{-1/2}\frac{dy}{dx}=2y\frac{dy}{dx}$$

$$1+\frac{y^{1/2}}{2x^{1/2}}+\frac{x^{1/2}}{2y^{1/2}}\frac{dy}{dx}=2y\frac{dy}{dx}$$

$$1+\frac{y^{1/2}}{2x^{1/2}}=2y\frac{dy}{dx}-\frac{x^{1/2}}{2y^{1/2}}\frac{dy}{dx}$$

$$\left(\frac{4y^{3/2}-x^{1/2}}{2y^{1/2}}\right)\frac{dy}{dx}=\frac{2x^{1/2}+y^{1/2}}{2x^{1/2}}$$

$$\frac{dy}{dx}=\frac{2x^{1/2}+y^{1/2}}{2x^{1/2}}\cdot\frac{2y^{1/2}}{4y^{3/2}-x^{1/2}}$$

$$=\frac{4x^{1/2}y^{1/2}+2y}{8x^{1/2}y^{3/2}-2x}$$

Divide every term by 2

$$=\frac{2x^{1/2}y^{1/2}+y}{4x^{1/2}y^{3/2}-x}$$

Example

Find the slope of the tangent line to the circle $x^2 + y^2 = 25$ at the point (3, -4)

Solution

$$\frac{d}{dx}\left[x^2 + y^2\right] = \frac{d}{dx}[25]$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Slope: $\frac{dy}{dx} = -\frac{3}{-4} = \frac{3}{4}$

Example

Find the equation of the tangent line to the circle $x^3 + y^3 = 9xy$ at the point (2, 4)

Solution

$$3x^2 + 3y^2y' = 9y + 9xy'$$

$$3y^2y' - 9xy' = 9y - 3x^2$$

$$\left(3y^2 - 9x\right)y' = 9y - 3x^2$$

$$y' = \frac{3\left(3y - x^2\right)}{3\left(y^2 - 3x\right)}$$

$$= \frac{3y - x^2}{y^2 - 3x}$$

$$m\Big|_{(2,4)} = \frac{3(4) - 2^2}{4^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}$$

$$y - 4 = \frac{4}{5}(x - 2)$$

$$y - 4 = \frac{4}{5}x - \frac{8}{5}$$

$$y = \frac{4}{5}x - \frac{8}{5} + 4$$

$$\boxed{y = \frac{4}{5}x + \frac{12}{5}}$$

Example

The demand function for a certain commodity is given by

$$p = \frac{500,000}{2q^3 + 400q + 5000}$$

Where p is the price in dollars and q is the demand in hundreds of units. Find the rate of change (dq/dp) of demand with respect to price when $q = 100$.

<u>*Solution*</u>

The rate of change is $\frac{dq}{dp}$

$$1 = \frac{0 - 500,000\left(6q^2 q' + 400q'\right)}{\left(2q^3 + 400q + 5000\right)^2}$$

$$\left(2q^3 + 400q + 5000\right)^2 = -500,000\left(6q^2 + 400\right)\frac{dq}{dp}$$

$$\frac{dq}{dp} = -\frac{\left(2q^3+400q+5000\right)^2}{500,000\left(6q^2+400\right)}$$

$$\frac{dq}{dp} = -\frac{\left(2(100)^3+400(100)+5000\right)^2}{500,000\left(6(100)^2+400\right)}$$

$$\approx -138$$

This means when the demand is 10,000 (100), demand is decreasing of the rate of 138.

Example

Suppose that x and y are both functions of t, which can be considered to represent time, and that x and y are related by the equation

$$xy^2 + y = x^2 + 17$$

Suppose further that when $x = 2$ and $y = 3$, then $\frac{dx}{dt} = 13$. Find the value of the $\frac{dy}{dt}$ at that moment.

<u>*Solution*</u>

$$y^2 \frac{dx}{dt} + 2xy\frac{dy}{dt} + \frac{dy}{dt} = 2x\frac{dx}{dt}$$

$$3^2(13) + 2(2)(3)\frac{dy}{dt} + \frac{dy}{dt} = 2(2)(13)$$

$$117 + 12\frac{dy}{dt} + \frac{dy}{dt} = 52$$

$$13\frac{dy}{dt} = -65$$

$$\boxed{\frac{dy}{dt} = \frac{-65}{13} = -5}$$

Example

A cone-shaped icicle is dripping from the roof. The radius of the icicle is decreasing at a rate of 0.2 cm per hour, while the length is increasing at a rate of 0.8 cm per hour. If the icicle is currently 4 cm in radius and 20 cm long, is the volume of the icicle increasing or decreasing and at what rate?

<u>*Solution*</u>

The volume of the cone is given by the formula: $V = \frac{1}{3}\pi r^2 h$.

$$\frac{dV}{dt} = \frac{1}{3}\pi\left[2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right]$$

Given the values:

$$\frac{dr}{dt} = -0.2 \qquad \frac{dh}{dt} = 0.8 \qquad r = 4 \qquad h = 20$$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left[2(4)(20)(-0.2) + 4^2(0.8)\right]$$
$$= -20$$

The volume is decreasing at a rate of 20 cm^3 per hour.

Exercises *Section* 3.7 – Implicit Differentiation

1. Find dy/dx for the equation $y^2 + x^2 - 2y - 4x = 4$

2. Find dy/dx: $x^2 y^2 - 2x = 3$

3. Find $\frac{dy}{dx}$, $e^{xy} + x^2 - y^2 = 10$

4. Find dy/dx: $x^2 - xy + y^2 = 4$ and evaluate the derivative at the given point $(0, -2)$

5. Find the slope of the tangent line to the circle $x^2 - 9y^2 = 16$ at the point (5, 1)

6. Find the rate of change of x with respect to p. $p = \sqrt{\frac{200 - x}{2x}}$, $0 < x \le 200$

7. The demand function for a product is given by $P = \frac{2}{0.001x^2 + x + 1}$. Find dx/dp implicitly.