10. Rotation of a rigid object about a fixed axis

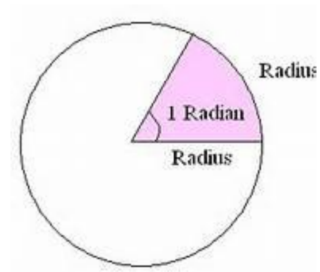
10.1 Angles

An angle is a measure of the inclination between two lines. There are two units of measurement for angles. They are the degree and the radian.

A degree: is defines to be $\left(\frac{1}{360}\right)^{th}$ of a complete circle. Therefore

$$\boxed{\text{one revolution} = 360\ \text{deg.}}$$

A degree: is defined to be a central angle that subtends an arc-length equal to the radius. Generally the radian measure of a central angle is defined to be the ration between the arc-length it subtends and the radius.



$$\boxed{\theta = \frac{s}{r}}$$

$\theta \rightarrow$ angle measure in radians
$s \rightarrow$ arc length
r→ radius

A radian is unit less.

Example: Calculate the radian measure of a central angle that subtends an arc length of 2 cm in a circle of radius 4 cm

$$s = 2cm\,; r = 4\ cm\,; \theta = ??$$

$$\theta = \frac{s}{r} = \frac{2}{4} = 0.5\ radians$$

For one revolution the arc length subtended is the circumference of the circle ($c = 2\pi r$)

$$\text{one revolution} \quad = \frac{2\pi r}{r} = 2\pi$$

$$\boxed{1\ \text{rev.} = 2\pi\ \text{radians}}$$

Since one revolution is equal to 360 deg. & 2π rad.

$$\boxed{\begin{aligned} deg &= \frac{\pi}{180}\, rad \\ rad &= \frac{180}{\pi}\, deg \end{aligned}}$$

Example: Convert the following

a) 120° to radians

$$120° = 120\left(\frac{\pi}{180} rad\right) = \frac{2\pi}{3} rad$$

b) $\frac{3\pi}{2}$ rad to deg

$$\frac{3\pi}{2}\ \text{rad} = \frac{3\pi}{2}\left(\frac{180}{\pi}\right) = 270°$$

c) 5 revolutions to degrees

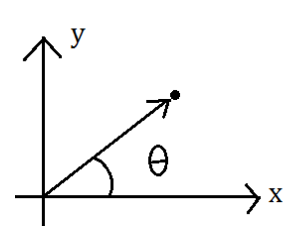
$$5\ \text{rev}\ 5(360°) = 1800°$$

d) 2 revolutions to radians

$$2\ \text{rev} \ = 2(2\pi) = 4\pi\ \text{radians}$$

10.2 Angular Motion Variables

Angular Position (θ): of a particle is defined to be the angle formed between its position vector and the positive x-axis. Unit of measurement for angular position is radian.



Angular Displacement ($\Delta\theta$) : is defined to be the change in the angular position of a particle.

$$\Delta\theta = \theta_f - \theta_i$$

Average Angular Velocity($\bar{\omega}$): is defined to be angular displacement per a unit time. Unit of measurement is radians/second.

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t} = \frac{\theta_f - \theta_i}{\Delta t}$$

Instantaneous Angular Velocity(ω): is angular velocity at a given instant of time.

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Average Angular Acceleration($\bar{\alpha}$): is change in angular velocity per a unit time. Its unit of measurement is radians/second^2.

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_i}{\Delta t}$$

Instantaneous Angular Acceleration(α): is angular acceleration at a given instant of time.

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

10.3 Relationship between angular and linear variables

$$\theta = \frac{s}{r} \Rightarrow \ \Delta\theta = \frac{\Delta s}{\mathrm{r}}$$

$$\boxed{\Delta s = r\Delta\theta}$$

$\Delta s \to$ linear displacement
$\Delta\theta \to$ angular displacement
$r \to$ radius

Dividing by time interval, Δt, for the change

$$\frac{\Delta \mathrm{s}}{\Delta \mathrm{t}} = r\frac{\Delta\theta}{\Delta \mathrm{t}} \quad but \quad \frac{\Delta \mathrm{s}}{\Delta \mathrm{t}} = v$$

$$\frac{\Delta\theta}{\Delta \mathrm{t}} = \omega$$

$$\boxed{v = r\omega}$$

$v \to$ linear speed
$r \to$ radius
$\omega \to$ angular speed

$$\Delta v = r\Delta\omega \ \Rightarrow \frac{\Delta v}{t} = r\frac{\Delta\omega}{\Delta t}$$

$$\text{But} \quad \frac{\Delta \mathrm{v}}{\Delta \mathrm{t}} = a_t \ \Rightarrow \frac{\Delta\omega}{\Delta \mathrm{t}} = \alpha$$

$$\therefore \ \boxed{a_t = r\alpha}$$

$a_t \to$ tangential acceleration
$\alpha \to$ angular acceleration

10.4 Uniformly Accelerated Angular Motion

Motion with constant angular acceleration. The equations for a uniformly accelerated motion are obtained in the same way as the equations for a uniformly accelerated motion. Hence, they can easily be obtained from the equations of a uniformly accelerated motions by replacing linear variables with angular variables ($i.e.$ $\Delta x \to \Delta\theta$; $v \to \omega$; $a \to \alpha$)

Equations of a uniformly accelerated angular motion

$$\boxed{\begin{aligned} \omega_f &= \omega_i + \alpha t \\ \Delta\theta &= \omega_i t + \frac{1}{2}\alpha t^2 \\ \omega_f^2 &= w_i^2 + 2\alpha\Delta\theta \\ \Delta\theta &= \left(\frac{\omega_i + \omega_f}{2}\right) t \end{aligned}}$$

Only two of these equations are independent. If any 3 of the 5 variables are known, the other 2 can be obtained by using these equations.

Example: The angular speed of a rigid object rotating about a fixed axis increased from 10 rad/s to 30 rad/s in 10 seconds.

a) Calculate its angular acceleration

$$\omega_i = 10 \text{ rad/s} \;\; ; \;\; \omega_f = 30 \text{ rad/s} \;\; ; \;\; t = 10\, s$$
$$\omega_f = \omega_i + \alpha t$$
$$30 = 10 + \alpha(10)$$
$$\underline{\alpha = 2 \text{ rad/s}^2}$$

b) Calculate its angular displacement

$$\Delta\theta = ??$$
$$\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2$$
$$\Delta\theta = (10)(10) + \frac{1}{2}(2)(10)^2$$
$$= 100 + 100$$
$$\Delta\theta = \underline{200 \text{ radians}}$$

c) For a particle on the rigid object at a perpendicular distance of 10cm from the axis of rotation, calculate:

i.) Its tangential acceleration.

$$a_t = ?? \qquad r = 10cm = 0.1\, m$$
$$\therefore \;\; a_t = r\alpha = (0.1)(2) = (0.2) \text{ m}/s^2$$

ii.) The distance travelled

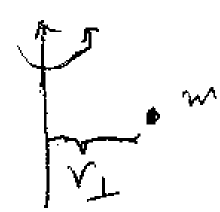
$$\Delta s = ?? \qquad \text{r} = 0.1\text{m}$$
$$\Delta s = \text{r}\Delta\theta = (0.1)(200) = 20\, m$$

iii.) Its final linear speed

$$v_f = ??$$
$$v_f = r\omega_f = (0.1)(30) = 3 \text{ m/s}$$

10.5 Moment of Inertia

The moment of inertia(I) of a particle of mass *m* located at a perpendicular distance $r_\perp$ from the axis of rotation is defined as



$$\boxed{I = mr_\perp^2}$$

The unit of measurement for moment of inertia is $\underline{kg \cdot m^2}$

The moment of inertia of system of particles is obtained by adding the moment of inertia of the particles.

$$\boxed{I = \sum_i m_i r_{\perp i}^2 = m_1 r_{\perp 1}^2 + m_2 r_{\perp 2}^2 + \cdots}$$

10.6 Rotational Kinetic Energy

Suppose a particle of mass *m* is revolving about a fixed axis at a perpendicular distance of $r_\perp$ from the axis with a speed of *v*. Then

$$KE = \frac{1}{2}mv^2 \quad \text{but} \quad v = r_\perp \omega$$

$$KE = \frac{1}{2}m(r_\perp\omega)^2 = \frac{1}{2}(mr_\perp^2)\omega^2$$

$$\text{but} \quad mr_\perp^2 = I \text{ (its moment of inertia)}$$

$$KE = \frac{1}{2}I\omega^2$$

Therefore the rotational kinetic energy (KE_{rot}) of a particle of moment of inertia *I* rotating with an angular speed ω may be defined as

$$\boxed{KE_{rot} = \frac{1}{2}I\omega^2}$$

The rotational kinetic energy of a system of particles rotating about a fixed axis is obtained by adding the kinetic energies of the particles.

$$KE = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + \cdots$$

$$but\ v_i = r_{\perp i}\omega$$

$$KE = \frac{1}{2}m_1(r_{\perp 1}\omega)^2 + \frac{1}{2}m_2(r_{\perp 2}\omega)^2 + \cdots$$

$$= \frac{1}{2}\omega^2[m_1r_{\perp 1}^2 + m_2r_{\perp 2}^2 + \cdots]$$

$$\text{But } m_1r_{\perp 1}^2 + m_2r_{\perp 2}^2 + \cdots = I$$

$$\therefore \boxed{KE_{rot} = \frac{1}{2}Iw^2}$$

Where *I* is the total moment of inertia of the system of particles

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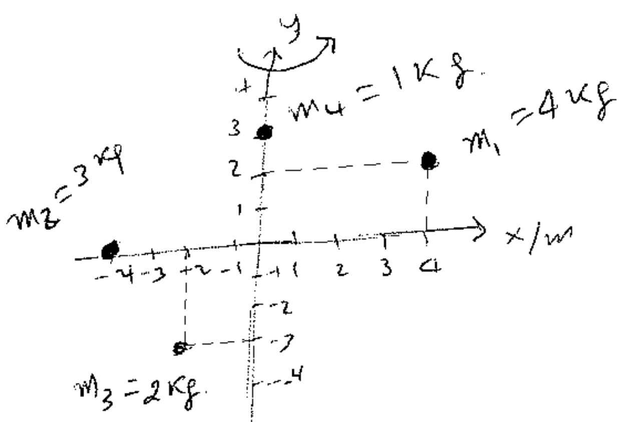
Example: The system of particles shown is revolving around the y-axis with an angular speed of 2 rad/s.



a) Calculate the moment of inertia of the system

For rotation about the y-axis the perpendicular distance of a particle is equal to the absolute value of the x-coordinate of the particle.

$$r_\perp = x$$

$m_1 = 4kg$	$m_2 = 3kg$	$m_3 = 2kg$	$m_4 = 1kg$
$r_{\perp 1} = \lvert x_1\rvert = 4$	$r_{\perp 2} = \lvert x_2\rvert = 4$	$r_{\perp 3} = \lvert x_3\rvert = 2$	$r_{\perp 4} = \lvert x_4\rvert = 0$

$$I = \frac{1}{2}m_1r_{\perp 1}^2 + \frac{1}{2}m_2r_{\perp 2}^2 + \frac{1}{2}m_3r_{\perp 3}^2 + \frac{1}{2}m_4r_{\perp 4}^2$$

$$= \frac{1}{2}(4)(4)^2 + \frac{1}{2}(3)(4)^2 + \frac{1}{2}(2)(2)^2 + \frac{1}{2}(1)(0)^2$$

$$= 32 + 24 + 4 + 0$$

$$\underline{I = 60\ kg \cdot m^2}$$

b) Calculate the rotational kinetic energy of the system.

$I = 60\ kg \cdot m^2$ $\omega = 2$ rad/s $KE_{rot} = ??$

$$KE_{rot} = \frac{1}{2} I \omega^2$$

$$= \frac{1}{2}(60)(2)^2$$

$$KE_{rot} = 120\ J$$

10.7 Moment of Inertia of Solid Objects (Lecture 2)

The moment of inertia of a solid object may be obtained by treating the solid object as made up of small mass elements, $\Delta m_i's$, and then taking the limiting values as the $\Delta m_i's$ approach which of course makes it an integral.

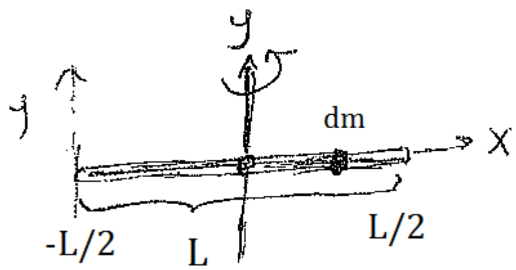
$$I = \lim_{\Delta m_i \to 0} \sum_i \Delta m_i \, r_{\perp i}^2 = \int r_\perp^2 \, dm$$

$$\boxed{I = \int r_\perp^2 \, dm}$$

Where $r_\perp$ is the perpendicular distance between the

small mass element dm and the axis of rotation.

Example: Obtain the moment of inertia of a uniform thin rod of length, *l*, and mass, *m*, about an axis passing through the midpoint of the rod perpendicular as shown.



Using a coordinate system where the y-axis (which is also the axis of rotation) passes through the center of the rod as shown, the perpendicular distance between *dm* and the axis of rotation is simply the absolute value of the x-coordinate of *dm*.

$$\therefore \;\; r_\perp = |x| \;\; \& \;\; r_\perp^2 = x^2$$

Uniform rod ⇒ its linear density is a constant

$$\therefore \;\; \rho = \frac{dm}{dx} = \frac{M}{L}$$

$$dm = \rho \, dx = \frac{M}{L} dx$$

$$I = \int r_\perp^2 \, dm = \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \left(\frac{M}{L} dx\right)$$

$$= \frac{M}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \, dx = \frac{M}{L} \left[\frac{x^3}{3} \Big|_{-\frac{L}{2}}^{\frac{L}{2}} \right]$$

$$= \frac{M}{L} \left[\frac{L^3}{24} + \frac{L^3}{24} \right] = \frac{M}{L} \cdot \frac{2L^3}{24}$$

$$\boxed{I = \frac{ML^2}{12}}$$

$M \rightarrow$ total mass of rod
$L \rightarrow$ length of rod
$I \rightarrow$ moment of inertia about an axis that passes through the midpoint of rod perpendicularly

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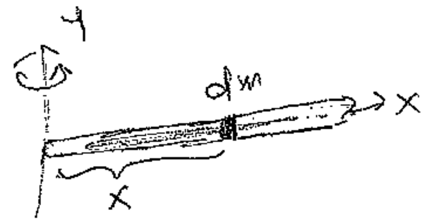
Example: Find the moment of inertia of a uniform thin rod of length, L, and mass, M, about an axis that passes through one of its end perpendicularly.



Using the coordinate system shown, the perpendicular distance between the mass element dm & the axis (y-axis) is simply the x-coordinate of dm.

$$\therefore r_\perp = x$$

Uniform rod $\Rightarrow$ linear density (ρ) is a constant

$$\therefore \quad \rho = \frac{dm}{dx} = \frac{M}{L}$$

$$dm = \rho\, dx = \frac{M}{L} dx$$

$$I = \int r_\perp^2\, dm = \int_0^L x^2 \left(\frac{M}{L} dx\right) = \frac{M}{L} \int_0^L x^2\, dx$$

$$= \frac{M}{L}\left[\frac{x^3}{3}\Big|_0^L\right] = \frac{M}{L} \cdot \frac{L^3}{3} = \frac{ML^2}{3}$$

$$\boxed{I = \frac{ML^2}{3}}$$

$M \rightarrow$ total mass of rod
$L \rightarrow$ length of rod
$I \rightarrow$ moment of inertia of the rod about an axis that passes through one of its ends perpendicularly

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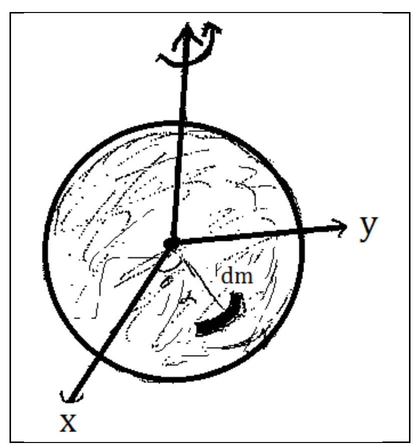
Example: Obtain the moment of inertia of a uniform thin disc of mass, M, and radius, R, about an axis that passes through its center perpendicularly.

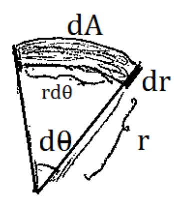


Uniform disk $\Rightarrow$ areal density (ρ) is constant

$$\rho = \frac{dm}{dA} = \frac{M}{\pi R^2}$$

$$\Rightarrow dm = \rho dA = \frac{M}{\pi R^2} dA$$

Using polar coordinates (r, θ) $\quad dA = r dr\, d\theta$



The perpendicular distance between the mass element dm & the axis is simply the r coordinate in polar coordinates

$$\therefore \quad r_\perp = r$$

$$I = \int r_\perp^2 dm = \iint r \cdot \frac{M}{\pi R^2}\, r dr d\theta$$