Solution Section 4.1 – Parameterizations of Plane Curves

Exercise

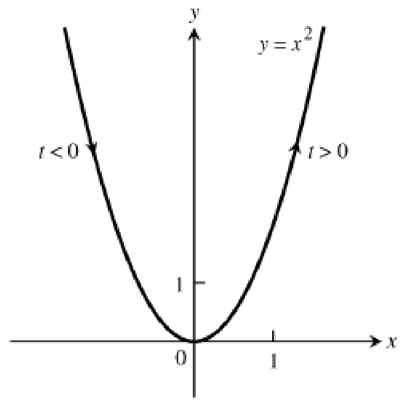
Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$x = 3t, \quad y = 9t^2, \quad -\infty < t < \infty$

Solution

$$x = 3t \quad \Rightarrow \quad t = \frac{x}{3}$$

$$y = 9t^2 = 9\left(\frac{x}{3}\right)^2 = x^2$$



Exercise

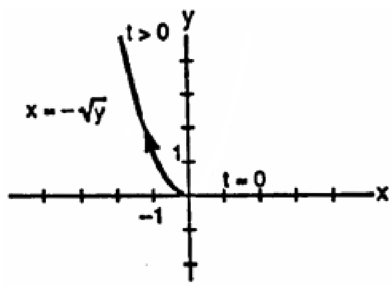
Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$x = -\sqrt{t}, \quad y = t, \quad t \ge 0$

Solution

$$x = -\sqrt{t} = -\sqrt{y}$$

$$y = x^2, \quad x \le 0$$



Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

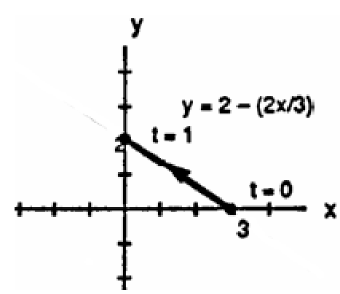
$x = 3 - 3t, \quad y = 2t, \quad 0 \le t \le 1$

Solution

$$x = 3 - 3t \quad \Rightarrow \quad 3t = 3 - x$$

$$t = 1 - \frac{x}{3}$$

$$y = 2\left(1 - \frac{x}{3}\right) = 2 - \frac{2}{3}x$$



Exercise

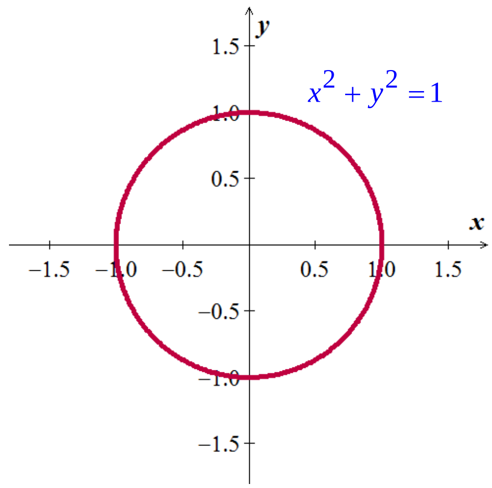
Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$x = \cos 2t, \quad y = \sin 2t, \quad 0 \le t \le \pi$

Solution

$$\cos^2 2t + \sin^2 2t = 1$$

$$\boxed{x^2 + y^2 = 1}$$



Exercise

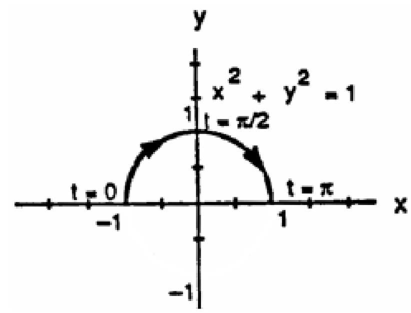
Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$x = \cos(\pi - t), \quad y = \sin(\pi - t), \quad 0 \le t \le \pi$

Solution

$$\cos^2(\pi - t) + \sin^2(\pi - t) = 1$$

$$\boxed{x^2 + y^2 = 1; \quad y \ge 0}$$



Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

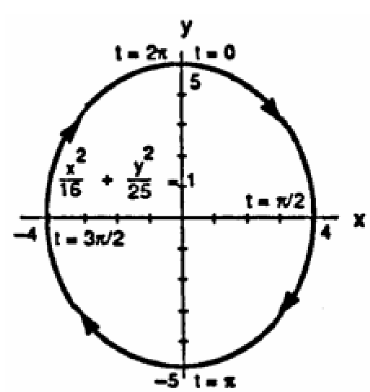
$x = 4\sin t, \quad y = 5\cos t, \quad 0 \le t \le 2\pi$

Solution

$$\sin t = \frac{x}{4}, \quad \cos t = \frac{y}{5}$$

$$\sin^2 t + \cos^2 t = 1$$

$$\boxed{\frac{x^2}{16} + \frac{y^2}{25} = 1}$$



Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

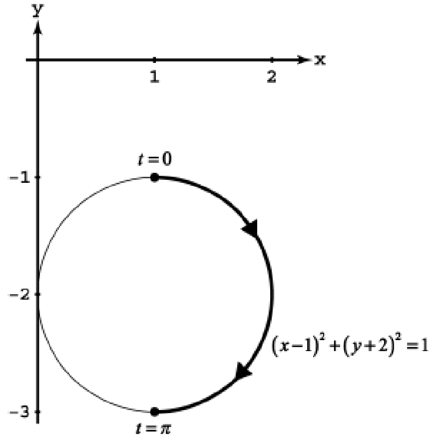
$$x = 1 + \sin t, \quad y = \cos t - 2, \quad 0 \le t \le 2\pi$$

Solution

$\sin t = x - 1, \quad \cos t = y + 2$

$\sin^2 t + \cos^2 t = 1$

$\boxed{(x-1)^2 + (y+2)^2 = 1}$



Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

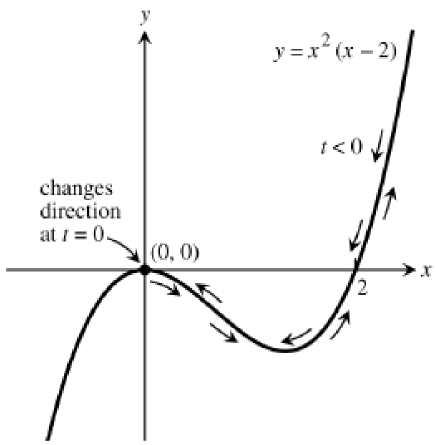
$$x = t^2, \quad y = t^6 - 2t^4, \quad -\infty < t < \infty$$

Solution

$$y = t^6 - 2t^4$$

$$= \left(t^2\right)^3 - 2\left(t^2\right)^2$$

$$\boxed{= x^3 - 2x^2}$$



Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = \frac{t}{t-1}, \quad y = \frac{t-2}{t+1}, \quad -1 < t < 1$$

Solution

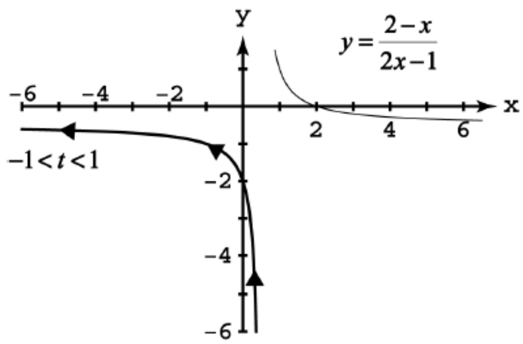
$x = \frac{t}{t-1} \quad \Rightarrow \quad xt - x = t$

$t(x-1) = x \to \quad t = \frac{x}{x-1}$

$$y = \frac{\frac{x}{x-1} - 2}{\frac{x}{x-1} + 1}$$

$$= \frac{x - 2x + 2}{x + x - 1}$$

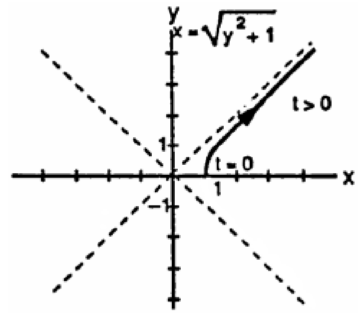
$$= \boxed{\frac{-x+2}{2x-1}}$$



Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x=\sqrt{t+1}, \quad y=\sqrt{t}, \quad t\ge 0$$



Solution

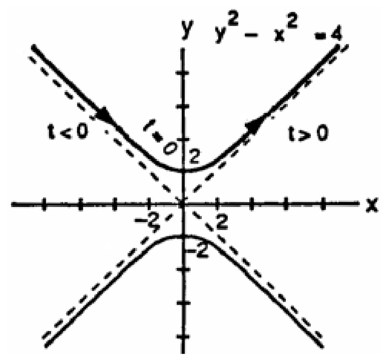
$$y=\sqrt{t} \quad \rightarrow \quad y^2=t$$

$$\boxed{x=\sqrt{y^2+1} \quad y\ge 0}$$

Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x=2\sinh t, \quad y=2\cosh t, \quad -\infty<t<\infty$$



Solution

$$\sinh t=\frac{x}{2}, \quad \cosh t=\frac{y}{2}$$

$$\cosh^2 t-\sinh^2 t=1$$

$$\frac{y^2}{4}-\frac{x^2}{4}=1$$

$$\boxed{y^2-x^2=4}$$

Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

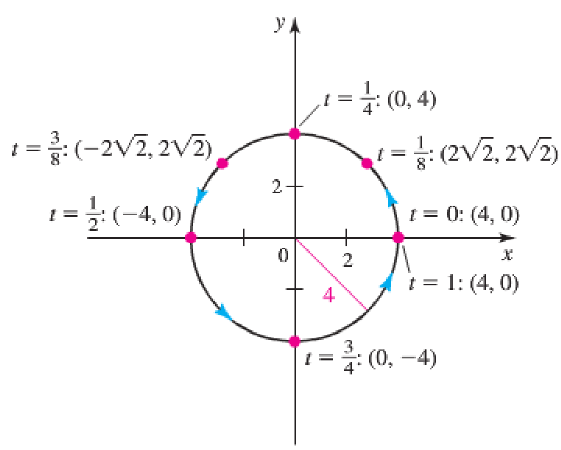
$$x=4\cos 2\pi t, \quad y=4\sin 2\pi t, \quad 0\le t\le 1$$

Solution

$$x^2+y^2=16\cos^2 2\pi t+16\sin^2 2\pi t=16$$

The equation represents a circle with a center at origin of radius 4.

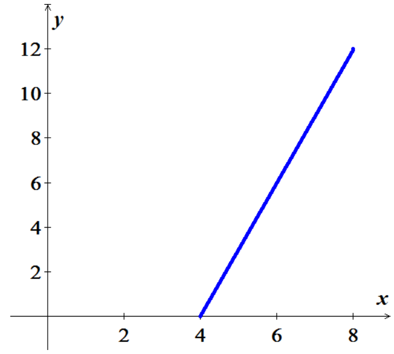
t	$(x,\ y)$
0	$(4,\ 0)$
$\frac{1}{8}$	$(2\sqrt{2},\ 2\sqrt{2})$
0.25	$(0,\ 4)$
0.5	$(-4,\ 0)$
0.75	$(0,\ -4)$
1	$(4,\ 0)$



Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = \sqrt{t} + 4, \quad y = 3\sqrt{t}; \quad 0 \le t \le 16$$



Solution

$\sqrt{t} = \dfrac{y}{3}$

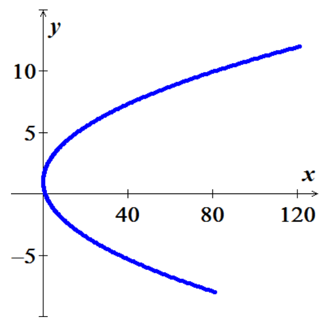
$x = \sqrt{t} + 4 = \dfrac{1}{3}y + 4$

$\boxed{y = 3(x-4)}$ (*Line*)

Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = (t+1)^2, \quad y = t + 2; \quad -10 \le t \le 10$$



Solution

$t = y - 2$

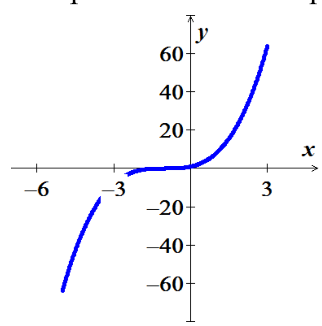
$x = (y-1)^2$

$\quad \boxed{= y^2 - 2y + 1}$

Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = t - 1, \quad y = t^3; \quad -4 \le t \le 4$$



Solution

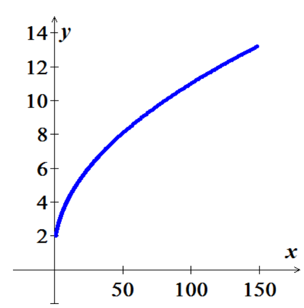
$t = x + 1$

$\boxed{y = (x+1)^3}$

Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = e^{2t}, \quad y = e^t + 1; \quad 0 \le t \le 2.5$$



Solution

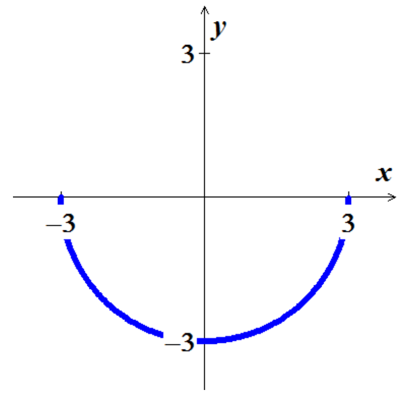
$e^t = \sqrt{x}$

$\boxed{y = \sqrt{x} + 1}$

Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = 3\cos t, \quad y = 3\sin t; \quad \pi \le t \le 2\pi$$



Solution

$$\cos t = \frac{x}{3}, \quad \sin t = \frac{y}{3}$$

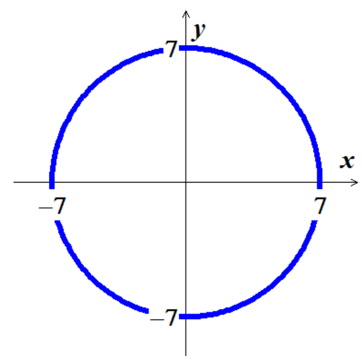
$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \qquad \cos^2 t + \sin^2 t = 1$$

$$\underline{x^2 + y^2 = 9} \qquad -3 \le x \le 3 \quad 0 \le y \le 3$$

Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = -7\cos 2t, \quad y = -7\sin 2t; \quad 0 \le t \le \pi$$



Solution

$$\cos 2t = -\frac{x}{7}, \quad \sin 2t = -\frac{y}{7}$$

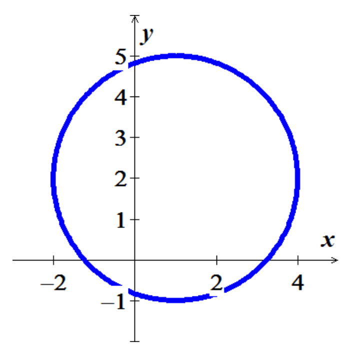
$$\left(-\frac{x}{7}\right)^2 + \left(-\frac{y}{7}\right)^2 = 1 \qquad \cos^2 2t + \sin^2 2t = 1$$

$$\underline{x^2 + y^2 = 49}$$

Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = 1 - 3\sin 4\pi t, \quad y = 2 + 3\cos 4\pi t; \quad 0 \le t \le \frac{1}{2}$$



Solution

$$\sin 4\pi t = \frac{1-x}{3}, \quad \cos 4\pi t = \frac{y-2}{3}$$

$$\left(\frac{1-x}{3}\right)^2 + \left(\frac{y-2}{3}\right)^2 = 1 \qquad \cos^2 2t + \sin^2 2t = 1$$

$$\underline{(x-1)^2 + (y-2)^2 = 49}$$

Exercise

Find parametric equation for the left half of the parabola $y = x^2 + 1$, originating at $(0,\ 1)$

Solution

Let $\quad x = -t \ \to y = t^2 + 1 \qquad$ for $0 \le t \le \infty$

Exercise

Find parametric equation for the line that passes through the points $(1, 1)$ and $(3, 5)$, oriented in the direction of increasing *x*.

Solution

$$y = \frac{5-1}{3-1}(x-1)+1 = 2x-1 \qquad\qquad y = m\left(x-x_1\right)+y_1$$

$$x = 1+(3-1)t = 1+2t$$

$$y = 1+(5-1)t = 1+4t$$

$$\begin{cases} x = 1+2t \\ y = 1+4t \end{cases} \quad -\infty < t < \infty$$

Exercise

Find parametric equation for the lower half of the circle centered at $(-2, 2)$ with radius 6, oriented in the counterclockwise direction.

Solution

$$(x+2)^2 + (y-2)^2 = 36$$

$$\begin{cases} x+2 = -6\cos t \\ y-2 = -6\sin t \end{cases}$$

$(-)$ since it oriented in *ccw* direction and lower half, therefore, *y*-value has to be negative.

$$\begin{cases} x = -2-6\cos t \\ y = 2-6\sin t \end{cases} \quad 0 \le t \le \pi$$

Exercise

Find parametric equation for the upper half of the parabola $x = y^2$, originating at $(0, 0)$

Solution

Let $y = t \quad \Rightarrow x = t^2 \quad 0 \le t < \infty$

Exercise

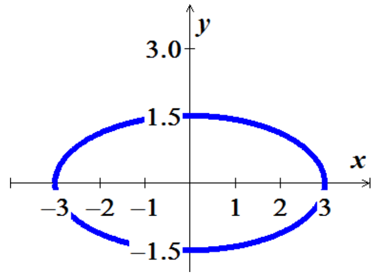
Find parametric equations (not unique) of an ellipse centered at the origin with major axis of length 6 on the *x-axis* and minor axis of length 3 on the *y-axis*, generated counterclockwise. Graph the ellipse and find a description in terms of *x* and *y*.

Solution

$$a = 3 \quad b = \frac{3}{2}$$

$$\frac{x^2}{9}+\frac{4y^2}{9}=1 \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \ \ (Ellipse)$$

$$\begin{cases} x=3\cos t \\ y=\frac{3}{2}\sin t \end{cases} \quad 0\le t\le 2\pi$$



Exercise

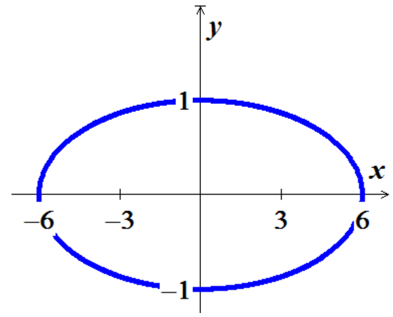
Find parametric equations (not unique) of an ellipse centered at the origin with major axis of length 12 on the x-$axis$ and minor axis of length 2 on the y-$axis$, generated clockwise. Graph the ellipse and find a description in terms of x and y.

Solution

$a=6 \quad b=1$

$$\frac{x^2}{36}+y^2=1 \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \ \ (Ellipse)$$

$$\begin{cases} x=6\cos t \\ y=-\sin t \end{cases} \quad 0\le t\le 2\pi \quad cw$$



Exercise

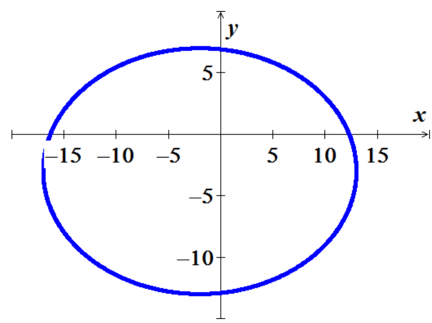
Find parametric equations (not unique) of an ellipse centered at $(-2,\ -3)$ with major and minor axes of lengths 30 and 20, parallel to the x-$axis$ and y-$axis$, respectively. Graph the ellipse and find a description in terms of x and y.

Solution

$a=15 \quad b=10$

$$\frac{(x+2)^2}{15^2}+\frac{(y+3)^2}{100}=1 \qquad \frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$$\begin{cases} x=-2+15\cos t \\ y=-3+10\sin t \end{cases} \quad 0\le t\le 2\pi \quad cw$$



Exercise

Find a parametric equations and a parameter interval for the motion of a particle starting at the point (2, 0) and tracing the top half of the circle $x^2+y^2=4$ four times.

Solution

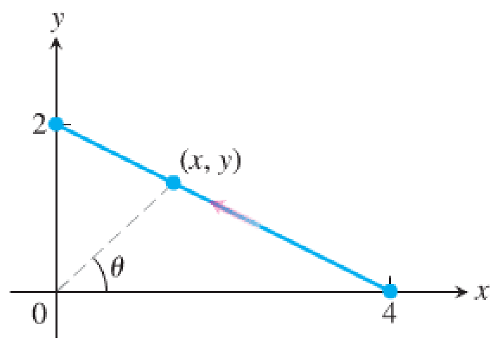
The top half of the circle: $y\ge 0$

$$\boxed{x=2\cos t,\quad y=2|\sin t|,\quad 0\le t\le \frac{\pi}{4}}$$

Exercise

Find a parametrization for the line segment joining points (0,2) and (4, 0) using the angle θ in the accompanying figure as the parameter.

Solution



$\tan\theta = \frac{y}{x} \Rightarrow y = x\tan\theta$

Slope: $m = \frac{0-2}{4-0} = -\frac{1}{2}$

The equation of the line passing thru (0,2) and (4, 0):

$y = -\frac{1}{2}(x-4)$

$y = -\frac{1}{2}x + 2$

$x\tan\theta = -\frac{1}{2}x + 2$

$x\tan\theta + \frac{1}{2}x = 2$

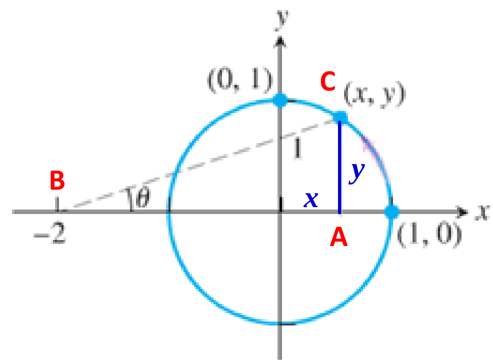
$x(2\tan\theta + 1) = 4 \rightarrow \boxed{x = \frac{4}{1+2\tan\theta}}$

$\underline{\left| y = x\tan\theta = \frac{4\tan\theta}{1+2\tan\theta} \right|} \quad 0 \le \theta < \frac{\pi}{2}$

Exercise

Find a parametrization for the circle $x^2 + y^2 = 1$ starting at (1, 0) and moving counterclockwise to the terminal point (0, 1), using the angle θ in the accompanying figure as the parameter.

Solution



$\tan\theta = \frac{y}{x+2} \quad \Rightarrow y = (x+2)\tan\theta$

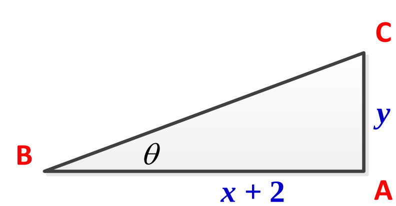
The equation of the circle is given by: $x^2 + y^2 = 1$

$x^2 + (x+2)^2\tan^2\theta = 1$

$x^2 + \left(x^2 + 4x + 4\right)\tan^2\theta = 1$

$\left(1+\tan^2\theta\right)x^2 + 4\tan^2\theta x + 4\tan^2\theta - 1 = 0$

$\sec^2\theta x^2 + 4\tan^2\theta x + 4\tan^2\theta - 1 = 0$



$$x = \frac{-4\tan^2\theta \pm \sqrt{16\tan^4\theta - 4\sec^2\theta\left(4\tan^2\theta - 1\right)}}{2\sec^2\theta}$$

$$= \frac{-4\tan^2\theta \pm \sqrt{16\tan^4\theta - 4\left(1+\tan^2\theta\right)\left(4\tan^2\theta - 1\right)}}{2\sec^2\theta}$$

$$=\frac{-4\tan^2\theta \pm \sqrt{16\tan^4\theta - 16\tan^2\theta + 4 - 16\tan^4\theta + 4\tan^2\theta}}{2\sec^2\theta}$$

$$=\frac{-4\tan^2\theta \pm \sqrt{4-12\tan^2\theta}}{2\sec^2\theta}$$

$$=\frac{-4\tan^2\theta \pm 2\sqrt{1-3\tan^2\theta}}{2\sec^2\theta}$$

$$=\frac{-2\tan^2\theta \pm \sqrt{1-3\tan^2\theta}}{\sec^2\theta}$$

$$=-2\frac{\tan^2\theta}{\sec^2\theta} \pm \cos^2\theta\sqrt{1-3\tan^2\theta}$$

$$=-2\sin^2\theta \pm \cos\theta\sqrt{\cos^2\theta - 3\sin^2\theta}$$

$$=-2\left(1-\cos^2\theta\right) \pm \cos\theta\sqrt{\cos^2\theta - 3\left(1-\cos^2\theta\right)}$$

$$\boxed{=2\cos^2\theta - 2 \pm \cos\theta\sqrt{4\cos^2\theta - 3}}$$

$$y=\left(2\cos^2\theta - 2 \pm \cos\theta\sqrt{4\cos^2\theta-3}+2\right)\tan\theta$$

$$=\left(2\cos^2\theta \pm \cos\theta\sqrt{4\cos^2\theta-3}\right)\tan\theta$$

$$=2\cos^2\theta\frac{\sin\theta}{\cos\theta} \pm \cos\theta\frac{\sin\theta}{\cos\theta}\sqrt{4\cos^2\theta-3}$$

$$\boxed{=2\cos\theta\sin\theta \pm \sin\theta\sqrt{4\cos^2\theta-3}}$$

At the point (0, 1): $y=(x+2)\tan\theta$

$$1=2\tan\theta \quad \Rightarrow \quad \tan\theta=\frac{1}{2} \quad \to \quad \boxed{\theta=\tan^{-1}\frac{1}{2}}$$

Exercise

A common task is to parameterize curves given either by either Cartesian equations or by graphs. Find a parametric representation of the following curves.

a) The segment of the parabola $y=9-x^2, \quad for \quad -1\le x\le 3$

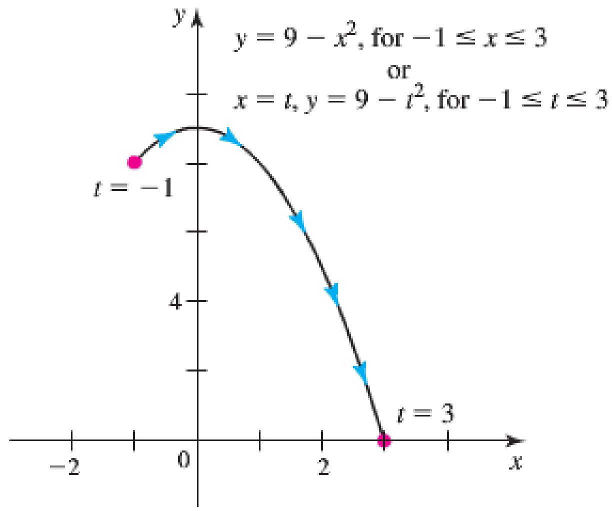
b) The complete curve $x=(y-5)^2+\sqrt{y}$

c) The piecewise linear path that connects $P(-2,\ 0)$ to $Q(0,\ 3)$ to $R(4,\ 0)$ (in that order), where the parameter varies over the interval $0\le t\le 2$

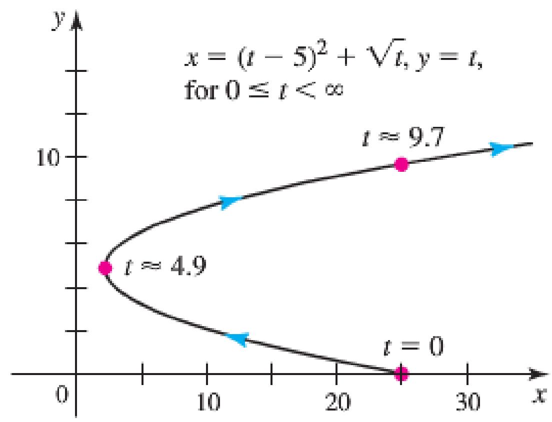
Solution

a) Let $x = t \quad \Rightarrow \quad y = 9 - t^2 \qquad$ for $-1 \le t \le 3$

Which represents a parabola



b) Let $y = t \quad \Rightarrow \quad x = (t-5)^2 + \sqrt{t}$



c) The path consists of 2 line segments that can be parameterized separately. $y = m\left(x - x_0\right) + y_0$

The line segment *PQ*: $P(-2,\ 0) \quad Q(0,\ 3)$

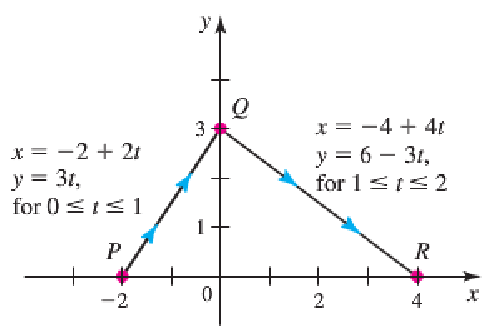
$$y = \frac{3}{2}(x+2) \quad = \frac{3}{2}x + 3 \quad \to \quad 2y - 3x = 6$$

$$x = 2t - 2, \quad y = 3t, \quad \textit{for} \quad 0 \le t \le 1$$

The line segment *QR*: $Q(0,\ 3) \quad R(4,\ 0)$

$$y = \frac{-3}{4}(x-4) \quad = -\frac{3}{4}x + 4 \quad \to \quad 4y + 3x = 16$$

$$x = 4t - 4, \quad y = -3t + 6, \quad \textit{for} \quad 1 \le t \le 2$$



Exercise

A projectile launched from the ground with an initial speed of 20 *m/s* and a launch angle θ follows a trajectory approximated by

$$x = (20\cos\theta)t, \quad y = -4.9t^2 + (20\sin\theta)t$$

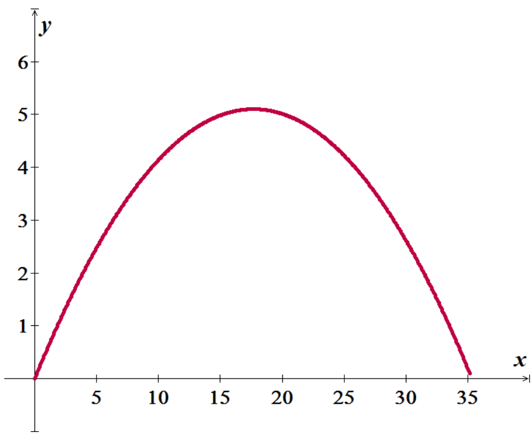
Where *x* and *y* are the horizontal and vertical positions of the projectile relative to the launch point $(0,\ 0)$.

a) Graph the trajectory for various of θ in the range $0 < \theta < \frac{\pi}{2}$.

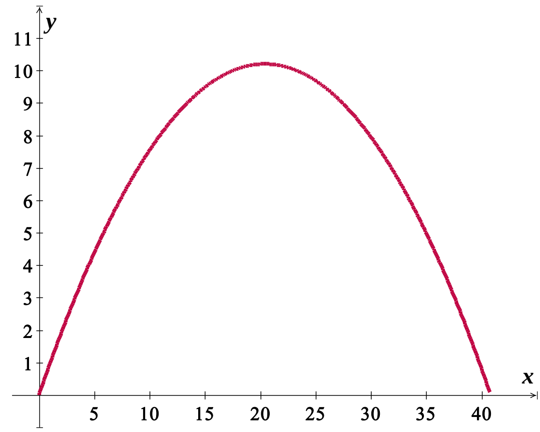
b) Based on your observations, what value of θ gives the greatest range (the horizontal distance between the launch and landing points)?

Solution

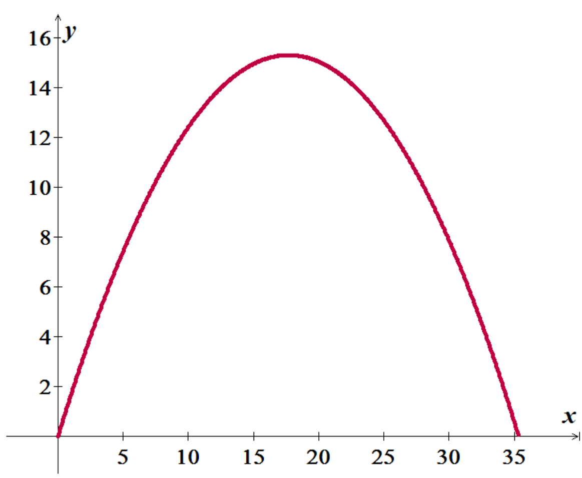
a)



$t = \frac{\pi}{6} \to \quad x = \left(10\sqrt{3}\right)t, \quad y = -4.9t^2 + 10t$



$t = \frac{\pi}{4} \to \quad x = 10\sqrt{2}\ t, \quad y = -4.9t^2 + 10\sqrt{2}\ t$



$t = \frac{\pi}{3} \to \quad x = 10t, \quad y = -4.9t^2 + \left(10\sqrt{3}\right)t$

b) The maximum appears to be reached when $\theta = \frac{\pi}{4}$

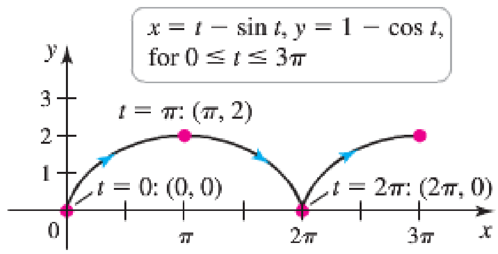
Exercise

Many fascinating curves are generated by points on rolling wheels. The path of a light on the rim of a rolling when is a cycloid, which has the parametric equations

$$x = a(t - \sin t), \quad y = a(1 - \cos t), \quad \textit{for} \;\; t \ge 0$$

Where $a > 0$. Graph the cycloid with $a = 1$. On what interval does the parameter generate one arch of the cycloid?

Solution



The wheel completes one full revolution on the interval $0 \le t \le 3\pi$, which gives one arch of the cycloid.

Exercise

Find parametric equations that describe the circular path of the objects. Assume (x, y) denotes the position of the object relative to the origin at the center of the circle.

A go-cart moves counterclockwise with constant speed around a circular track of radius 400 *m*, completing in 1.5 *min*.

Solution

Let *t* be time in minute, so $0 \le t \le 1.5$

$1.5\theta = \frac{3}{2}\theta = 2\pi t \quad \rightarrow \theta = \frac{4\pi}{3}t$

Since $\begin{cases} x = r\cos\theta & \rightarrow x = 400\cos\left(\frac{4\pi}{3}t\right) \\ y = r\sin\theta & \quad y = 400\sin\left(\frac{4\pi}{3}t\right) \end{cases} \quad \Rightarrow x^2 + y^2 = 400^2$

The path is a circle of radius 400, center at origin and the circle is traversed counterclockwise.

Exercise

Find parametric equations that describe the circular path of the objects. Assume $(x,\ y)$ denotes the position of the object relative to the origin at the center of the circle.
The tip of the 15-*in* second hand of a clock completes one revolution in 60 *sec*.

Solution

Let t be time in second, so $0 \le t \le 60$

$$60\theta = 2\pi t \quad \to \theta = \frac{\pi}{30}t$$

$$\text{Since } \begin{cases} x = r\cos\theta & \to x = 15\cos\left(\frac{\pi}{30}t\right) \\ y = r\sin\theta & \quad y = 15\sin\left(\frac{\pi}{30}t\right) \end{cases} \qquad \Rightarrow x^2 + y^2 = 15^2$$

The path is a circle of radius 15, center at origin and the circle is traversed clockwise.

Exercise

Find parametric equations that describe the circular path of the objects. Assume $(x,\ y)$ denotes the position of the object relative to the origin at the center of the circle.
A Ferris wheel has a radius of 20 *m* and completes a revolution in the clockwise direction at constant speed in 3 *min*. Assume that *x* and *y* measure the horizontal and vertical positions of a seat on the Ferris wheel relative to a coordinate system whose origin is at the low point of the wheel. Assume the seat begins moving at the origin.

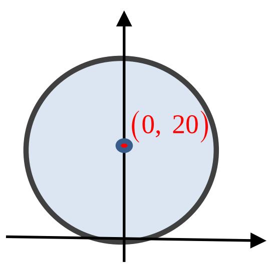
Solution

Let t be time in minute, so $0 \le t \le 3$

Since the low point is the origin, the circle has its center at $(0,\ 20)$ and a radius of 20.

$$3\theta = 2\pi t \quad \to \theta = \frac{2\pi}{3}t$$

$$\text{Since } \begin{cases} x = r\cos\theta & \to x = -20\cos\left(\frac{2\pi}{3}t\right) \\ y = r\sin\theta & \quad y = 20 - 20\sin\left(\frac{2\pi}{3}t\right) \end{cases} \qquad \Rightarrow x^2 + (y-20)^2 = 20^2$$



The path is a circle of radius 20, center at $(0,\ 20)$.

Exercise

A plane traveling horizontally at 80 *m/s* over flat ground at an elevation of 3000 *m* releases an emergency packet. The trajectory of the packet is given by

$$x = 80t, \quad y = -4.9t^2 + 3000, \quad \text{for } \ t \ge 0$$

Where the origin is the point on the ground directly beneath the plane at the moment of the release. Graph the trajectory of the packet and find the coordinates of the point where the packet lands.

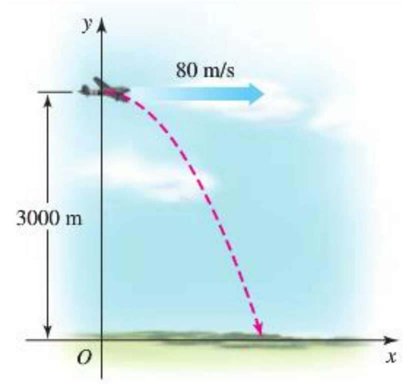
Solution

The packet hits ground when:

$y=0=-4.9t^2+3000$

$$\Rightarrow \quad \underline{t=\sqrt{\frac{3000}{4.9}}} \approx 24.744 \text{ sec}$$

And $\underline{x=80t \approx 80(24.744)} \approx 1979.487\ m$



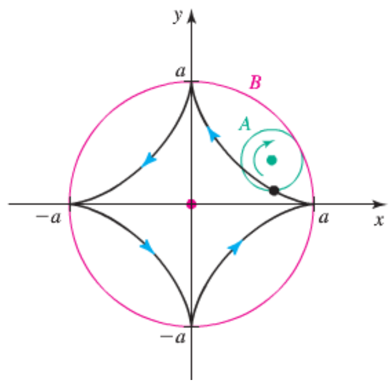
Exercise

The path of a point on circle A with radius $\frac{a}{4}$ that rolls on the inside of circle B with a radius a is an asteroid or hypocycloid. Its parametric equations are

$$x=a\cos^3 t, \quad y=a\sin^3 t, \quad 0 \le t \le 2\pi$$

Graph the asteroid with $a=1$ and find its equation in terms of x and y.

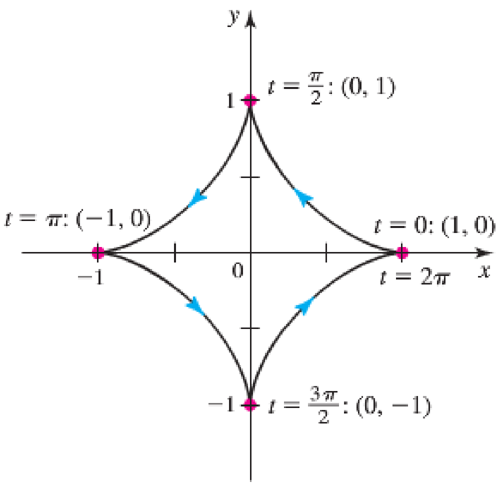
Solution



$$\cos t=\left(\frac{x}{a}\right)^{1/3}, \quad \sin t=\left(\frac{y}{a}\right)^{1/3}$$

$$\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{a}\right)^{2/3}=1 \qquad \cos^2 t+\sin^2 t=1$$

$$x^{2/3}+y^{2/3}=1 \qquad (a=1)$$



Solution Section 4.2 – Calculus with Parametric Curves

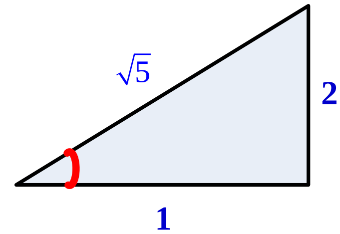
Exercise

Find all the points at which the curve has the given slope. $x = 4\cos t, \quad y = 4\sin t; \quad slope = \frac{1}{2}$

Solution

$$\frac{dy}{dx} = \frac{4\cos t}{-4\sin t} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= -\cot t = \frac{1}{2}$$



$$\cot t = -\frac{1}{2} \quad \Rightarrow \quad t = \cot^{-1}\left(-\frac{1}{2}\right) \quad t \in QII \ \& \ QIV$$

$$x = -\cos\left(\cot^{-1}\frac{1}{2}\right) = -\frac{1}{\sqrt{5}}, \quad y = 4\sin\left(\cot^{-1}\frac{1}{2}\right) = \frac{8}{\sqrt{5}}; \quad \left(-\frac{\sqrt{5}}{5}, \ \frac{8\sqrt{5}}{5}\right)$$

$$x = \cos\left(\cot^{-1}\frac{1}{2}\right) = \frac{1}{\sqrt{5}}, \quad y = -4\sin\left(\cot^{-1}\frac{1}{2}\right) = -\frac{8}{\sqrt{5}}; \quad \left(\frac{\sqrt{5}}{5}, \ -\frac{8\sqrt{5}}{5}\right)$$

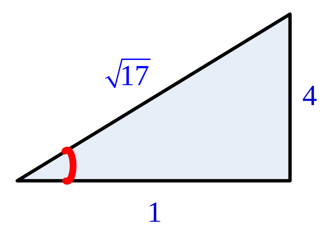
Exercise

Find all the points at which the curve has the given slope. $x = 2\cos t, \quad y = 8\sin t; \quad slope = -1$

Solution

$$\frac{dy}{dx} = \frac{8\cos t}{-2\sin t} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= -4\cot t = -1$$



$$\cot t = \frac{1}{4} \quad \Rightarrow \quad t = \cot^{-1}\left(\frac{1}{4}\right) \quad t \in QI \ \& \ QIII$$

$$x = 2\cos\left(\cot^{-1}\frac{1}{4}\right) = \frac{2}{\sqrt{17}}, \quad y = 8\sin\left(\cot^{-1}\frac{1}{4}\right) = \frac{32}{\sqrt{5}}; \quad \left(\frac{2\sqrt{17}}{17}, \ \frac{32\sqrt{17}}{17}\right)$$

$$x = -2\cos\left(\cot^{-1}\frac{1}{4}\right) = -\frac{2}{\sqrt{17}}, \quad y = -8\sin\left(\cot^{-1}\frac{1}{4}\right) = -\frac{32}{\sqrt{5}}; \quad \left(-\frac{2\sqrt{17}}{17}, \ -\frac{32\sqrt{17}}{17}\right)$$

Exercise

Find all the points at which the curve has the given slope. $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}; \quad slope = 1$

Solution

$$\frac{dy}{dx} = \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

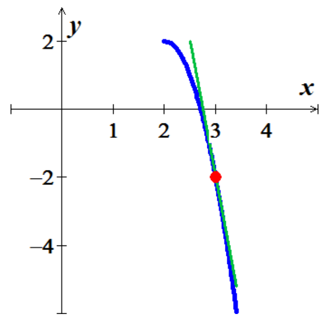
$$= \frac{t^2+1}{t^2-1} = 1$$

$t^2 + 1 \neq 1$ $\therefore$ There are no points on this curve with slope 1.

Exercise

Find all the points at which the curve has the given slope. $\quad x = 2 + \sqrt{t}, \quad y = 2 - 4t; \quad slope = -8$

Solution



$$\frac{dy}{dx} = \frac{-4}{\frac{1}{2\sqrt{t}}} = -8\sqrt{t} = -8 \qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$t = 1 \quad \to \quad (3, -2)$$

Exercise

Find an equation of the line tangent to the curve at the point corresponding to the given value of t.

$$x = \sin t, \quad y = \cos t, \quad t = \frac{\pi}{4}$$

Solution

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{\cos t}$$

$$= -\tan t \Big|_{t=\frac{\pi}{4}}$$

$$= -1$$

At $t = \frac{\pi}{4} \Rightarrow x = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \quad y = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \quad \to \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

The equation of the tangent line is $\quad y = -\left(x - \frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2} = -x + \sqrt{2}$

Exercise

Find an equation of the line tangent to the curve at the point corresponding to the given value of t.

$$x = t^2 - 1, \quad y = t^3 + t, \quad t = 2$$

Solution

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2+1}{2t}\Big|_{t=2}$$

$$= \frac{13}{4}$$

At $t = 2 \Rightarrow \; x = 3, \quad y = 10 \quad \to (3, \; 10)$

The equation of the tangent line is $\quad y = \frac{13}{4}(x-3) + 10 = \frac{13}{4}x + \frac{1}{4}$

Exercise

Find an equation of the line tangent to the curve at the point corresponding to the given value of t.

$$x = e^t, \quad y = \ln(t+1), \quad t = 0$$

Solution

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \left.\frac{\frac{1}{t+1}}{e^t}\right|_{t=0} \quad \boxed{=1}$$

At $t = 0 \Rightarrow \ x = 1, \quad y = 0 \quad \to (1,\ 0)$

The equation of the tangent line is $\boxed{y = x - 1}$

Exercise

Find an equation of the line tangent to the curve at the point corresponding to the given value of t.

$$x = \cos t + t\sin t, \quad y = \sin t - t\cos t, \quad t = \frac{\pi}{4}$$

Solution

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t - \cos t + t\sin t}{-\sin t + \sin t + t\cos t}$$

$$= \tan t\Big|_{t=\frac{\pi}{4}}$$

$$\boxed{=1}$$

At $t = \frac{\pi}{4} \Rightarrow x = \frac{\sqrt{2}}{2} + \frac{\pi}{4}\frac{\sqrt{2}}{2}, \quad y = \frac{\sqrt{2}}{2} - \frac{\pi}{4}\frac{\sqrt{2}}{2} \quad \to \left(\frac{4\sqrt{2}+\pi\sqrt{2}}{8},\ \frac{4\sqrt{2}-\pi\sqrt{2}}{8}\right)$

The equation of the tangent line is $\boxed{y = x - \frac{4\sqrt{2}+\pi\sqrt{2}}{8} + \frac{4\sqrt{2}-\pi\sqrt{2}}{8} = x - \frac{\pi\sqrt{2}}{4}}$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\quad x = \sin 2\pi t, \quad y = \cos 2\pi t, \quad t = -\frac{1}{6}$

Solution

$$x = \sin 2\pi\left(-\frac{1}{6}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$y = \cos 2\pi\left(-\frac{1}{6}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

The point $\left(-\frac{\sqrt{3}}{2},\ \frac{1}{2}\right)$

$$\frac{dx}{dt} = 2\pi\cos 2\pi t, \quad \frac{dy}{dt} = -2\pi\sin 2\pi t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{-2\pi\sin 2\pi t}{2\pi\cos 2\pi t} = -\tan 2\pi t$$

$$\left.\frac{dy}{dx}\right|_{t=-\frac{1}{6}} = -\tan 2\pi\left(-\frac{1}{6}\right) = -\tan\left(-\frac{\pi}{3}\right) = \sqrt{3}$$

The tangent to the curve at the point $\left(-\frac{\sqrt{3}}{2},\ \frac{1}{2}\right)$ is: $\quad y = \sqrt{3}\left(x + \frac{\sqrt{3}}{2}\right) + \frac{1}{2} = \sqrt{3}x + 2$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(-\tan 2\pi t\right) = -2\pi \sec^2 2\pi t$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$= \frac{-2\pi \sec^2 2\pi t}{2\pi \cos 2\pi t}$$

$$= -\frac{1}{\cos^3 2\pi t}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=-\frac{1}{6}} = -\frac{1}{\cos^3\left(-\frac{\pi}{3}\right)} = -8$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\quad x = \cos t, \quad y = \sqrt{3}\cos t, \quad t = \frac{2\pi}{3}$

Solution

$$x = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

$$y = \sqrt{3}\cos\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$ ***The point*** $\left(-\frac{1}{2},\ -\frac{\sqrt{3}}{2}\right)$

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = -\sqrt{3}\sin t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{-\sqrt{3}\sin t}{-\sin t} = \sqrt{3}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{2\pi}{3}} = \sqrt{3}$$

The tangent to the curve at the point $\left(-\frac{1}{2},\ -\frac{\sqrt{3}}{2}\right)$ is: $\quad y = \sqrt{3}\left(x + \frac{1}{2}\right) - \frac{\sqrt{3}}{2} = \sqrt{3}x$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\sqrt{3}\right) = 0$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$= \frac{0}{-\sin t}$$

$$= 0$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=\frac{2\pi}{3}} = 0$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\quad x = t, \quad y = \sqrt{t}, \quad t = \frac{1}{4}$

Solution

$$x = \frac{1}{4}$$

$$y = \sqrt{\frac{1}{4}} = \frac{1}{2} \qquad\qquad \textbf{\textit{The point}}\ \left(\frac{1}{4},\ \frac{1}{2}\right)$$

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = \frac{1}{2\sqrt{t}}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{1}{2\sqrt{t}} \cdot 1 = \frac{1}{2\sqrt{t}}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{1}{4}} = \frac{1}{2\sqrt{\frac{1}{4}}} = 1$$

The tangent is: $\quad y = \left(x - \frac{1}{4}\right) + \frac{1}{2} = x + \frac{1}{4}$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1}{2\sqrt{t}}\right)$$

$$= -\frac{1}{4}t^{-3/2}$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$= \frac{-\frac{1}{4}t^{-3/2}}{1}$$

$$= -\frac{1}{4}t^{-3/2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=\frac{1}{4}} = -\frac{1}{4}\left(\frac{1}{4}\right)^{-3/2} = -2$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\dfrac{d^2y}{dx^2}$ at this point $\quad x=\sec^2 t-1, \quad y=\tan t, \quad t=-\frac{\pi}{4}$

Solution

$$x=\sec^2\left(-\tfrac{\pi}{4}\right)-1=1$$

$$y=\tan\left(-\tfrac{\pi}{4}\right)=-1 \qquad\qquad \textbf{\textit{The point }}(1,\ -1)$$

$$\frac{dx}{dt}=2\sec^2 t\tan t, \quad \frac{dy}{dt}=\sec^2 t$$

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\sec^2 t}{2\sec^2 t\tan t}=\frac{1}{2\tan t}$$

$$\left.\frac{dy}{dx}\right|_{t=-\frac{\pi}{4}}=\frac{1}{2\tan\left(-\frac{\pi}{4}\right)}=\boxed{-\frac{1}{2}}$$

The tangent is: $\quad y=-\frac{1}{2}(x-1)-1=\boxed{-\frac{1}{2}x+\frac{1}{2}}$

$$\frac{dy'}{dt}=\frac{d}{dt}\left(\frac{1}{2}\frac{1}{\tan\theta}\right)$$

$$=\frac{1}{2}\frac{-\sec^2\theta}{\tan^2\theta}=-\frac{1}{2}\frac{\frac{1}{\cos^2\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}$$

$$=-\frac{1}{2}\csc^2\theta$$

$$\frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}$$

$$=\frac{-\frac{1}{2}\csc^2\theta}{2\sec^2 t\tan t} \qquad\qquad \frac{-\frac{1}{2}\csc^2\theta}{2\sec^2 t\tan t}=-\frac{1}{4}\frac{\frac{1}{\sin^2 t}}{\frac{1}{\cos^2 t}\frac{\sin t}{\cos t}}=-\frac{1}{4}\frac{\cos^3 t}{\sin^3 t}=-\frac{1}{4}\cot^3 t$$

$$\boxed{=-\frac{1}{4}\cot^3 t}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=-\frac{\pi}{4}}=-\frac{1}{4}\cot^3\left(-\frac{\pi}{4}\right)=\boxed{\frac{1}{4}}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of d^2y/dx^2 at this point $x=\frac{1}{t+1}$, $y=\frac{t}{t-1}$, $t=2$

Solution

$$x=\frac{1}{2+1}=\frac{1}{3} \qquad y=\frac{2}{2-1}=2 \qquad \textit{The point } \left(\tfrac{1}{3},\ 2\right)$$

$$\frac{dx}{dt}=\frac{-1}{(t+1)^2}, \quad \frac{dy}{dt}=\frac{t-1-t}{(t-1)^2}=\frac{-1}{(t-1)^2}$$

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\frac{-1}{(t-1)^2}}{\frac{-1}{(t+1)^2}}=\frac{(t+1)^2}{(t-1)^2}$$

$$\left.\frac{dy}{dx}\right|_{t=2}=\frac{(2+1)^2}{(2-1)^2}=9$$

The tangent is: $y=9\left(x-\frac{1}{3}\right)+2=9x-1$

$$\frac{dy'}{dt}=\frac{d}{dt}\left(\frac{t+1}{t-1}\right)^2$$

$$=2\left(\frac{t+1}{t-1}\right)\left(\frac{t-1-t-1}{(t-1)^2}\right)$$

$$=-4\frac{t+1}{(t-1)^3}$$

$$\frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}$$

$$=-4\frac{t+1}{(t-1)^3}\frac{(t+1)^2}{-1}$$

$$=4\frac{(t+1)^3}{(t-1)^3}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=2}=4\frac{(2+1)^3}{(2-1)^3}=108$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of d^2y/dx^2 at this point $x = t + e^t, \quad y = 1 - e^t, \quad t = 0$

Solution

$x = 0 + e^0 = 1 \qquad y = 1 - e^0 = 0 \qquad$ ***The point*** $(1,\ 0)$

$$\frac{dx}{dt} = 1 + e^t, \quad \frac{dy}{dt} = -e^t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-e^t}{1+e^t}$$

$$\left.\frac{dy}{dx}\right|_{t=0} = -\frac{e^0}{1+e^0} = -\frac{1}{2}$$

The tangent is: $\quad y = -\frac{1}{2}(x-1) = -\frac{1}{2}x + \frac{1}{2}$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{-e^t}{1+e^t}\right)$$

$$= \frac{-e^t\left(1+e^t\right) - e^t\left(-e^t\right)}{\left(1+e^t\right)^2}$$

$$= \frac{-e^t - e^{2t} + e^{2t}}{\left(1+e^t\right)^2}$$

$$= \frac{-e^t}{\left(1+e^t\right)^2}$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$= \frac{-e^t}{\left(1+e^t\right)^2}\frac{1}{1+e^t}$$

$$= \frac{-e^t}{\left(1+e^t\right)^3}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=0} = \frac{-e^0}{\left(1+e^0\right)^3} = -\frac{1}{8}$$

Exercise

Find the slope of the curve $x = f(t),\ y = g(t)$ at the given value of *t*. Define *x* and *y* as differentiable functions. $x^3 + 2t^2 = 9, \quad 2y^3 - 3t^2 = 4, \quad t = 2$

Solution

$$x^3 + 2(2)^2 = 9 \Rightarrow x^3 = 9 - 8 = 1 \rightarrow \boxed{x = 1}$$

$$2y^3 - 3(2)^2 = 4 \quad \Rightarrow \quad 2y^3 = 4 + 12 = 16 \quad \rightarrow y^3 = 8 \ \Rightarrow \boxed{y = 2}$$

$$x^3 + 2t^2 = 9 \quad \Rightarrow \quad 3x^2\frac{dx}{dt} + 4t = 0$$

$$3x^2\frac{dx}{dt} = -4t$$

$$\frac{dx}{dt} = -\frac{4t}{3x^2}$$

$$2y^3 - 3t^2 = 4 \quad \Rightarrow \quad 6y^2\frac{dy}{dt} - 6t = 0$$

$$y^2\frac{dy}{dt} = t$$

$$\frac{dy}{dt} = \frac{t}{y^2}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{t}{y^2}}{-\frac{4t}{3x^2}} = -\frac{3x^2}{4y^2}$$

$$\left.\frac{dy}{dx}\right|_{t=2} = -\frac{3(1)^2}{4(2)^2} = \underline{-\frac{3}{16}}$$

Exercise

Find the slope of the curve $x = f(t),\ y = g(t)$ at the given value of *t*. Define *x* and *y* as differentiable functions. $x + 2x^{3/2} = t^2 + t, \quad y\sqrt{t+1} + 2t\sqrt{y} = 4, \quad t = 0$

Solution

$$x + 2x^{3/2} = 0^2 + 0 \quad \Rightarrow \quad x\left(1 + 2x^{1/2}\right) = 0$$

$$\rightarrow \boxed{x = 0} \quad \cancel{x^{1/2} = -\frac{1}{2}} \quad (False)$$

$$y\sqrt{0+1} + 2(0)\sqrt{y} = 4 \quad \Rightarrow \quad \boxed{y = 4}$$

$$x + 2x^{3/2} = t^2 + t \quad \Rightarrow \quad \frac{dx}{dt} + 3x^{1/2}\frac{dx}{dt} = 2t + 1$$

$$\frac{dx}{dt}\left(1 + 3x^{1/2}\right) = 2t + 1$$

$$\frac{dx}{dt}=\frac{2t+1}{1+3x^{1/2}}$$

$$y\sqrt{t+1}+2t\sqrt{y}=4 \quad\Rightarrow\quad \frac{dy}{dt}\sqrt{t+1}+\frac{1}{2}y(t+1)^{-1/2}+2\sqrt{y}+2t\left(\frac{1}{2}y^{-1/2}\right)\frac{dy}{dt}=0$$

$$\frac{dy}{dt}\left(\sqrt{t+1}+\frac{t}{\sqrt{y}}\right)=-\frac{y}{2\sqrt{t+1}}-2\sqrt{y}$$

$$\frac{dy}{dt}\left(\frac{\sqrt{t+1}\sqrt{y}+t}{\sqrt{y}}\right)=\frac{-y-4\sqrt{t+1}\sqrt{y}}{2\sqrt{t+1}}$$

$$\frac{dy}{dt}=\frac{-y-4\sqrt{t+1}\sqrt{y}}{2\sqrt{t+1}}\cdot\frac{\sqrt{y}}{\sqrt{t+1}\sqrt{y}+t}$$

$$\frac{dy}{dt}=\frac{-y\sqrt{y}-4y\sqrt{t+1}}{2(t+1)\sqrt{y}+2t\sqrt{t+1}}$$

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-y\sqrt{y}-4y\sqrt{t+1}}{2(t+1)\sqrt{y}+2t\sqrt{t+1}}\cdot\frac{1+3\sqrt{x}}{2t+1}$$

$$\left.\frac{dy}{dx}\right|_{t=0}=\frac{-4\sqrt{4}-4(4)\sqrt{0+1}}{2(0+1)\sqrt{4}+2(0)\sqrt{0+1}}\cdot\frac{1+3\sqrt{0}}{2(0)+1}=\underline{-6}$$

Exercise

Find the slope of the curve $x=f(t),\ y=g(t)$ at the given value of t. Define x and y as differentiable functions. $\quad t=\ln(x-t),\quad y=te^{t},\quad t=0$

Solution

$$0=\ln(x-0)\quad\Rightarrow\quad \ln x=0\ \to\ \boxed{x=1}$$

$$y=(0)e^{0}\quad\Rightarrow\quad \boxed{y=0}$$

$$t=\ln(x-t)\quad\Rightarrow\quad 1=\frac{\frac{dx}{dt}-1}{x-t}$$

$$\frac{dx}{dt}-1=x-t$$

$$\frac{dx}{dt}=x-t+1$$

$$y=te^{t}\quad\Rightarrow\quad \frac{dy}{dt}=e^{t}+te^{t}=e^{t}(1+t)$$

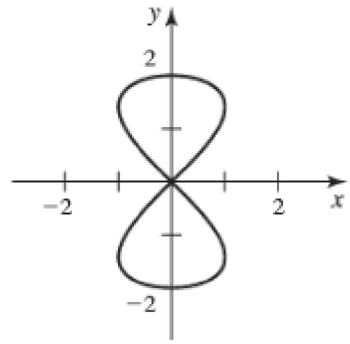
$$\frac{dy}{dx}=\frac{e^{t}(1+t)}{x-t+1} \qquad\qquad \frac{dy}{dx}=\frac{dy/dt}{dx/dt}$$

$$\left.\frac{dy}{dx}\right|_{t=0}=\frac{e^{0}(1+0)}{1-0+1}=\underline{\frac{1}{2}}$$

Exercise

Consider Lissajous curve, estimate the coordinates of the points on the curve at which there is

$$x = \sin 2t, \quad y = 2\sin t; \quad 0 \le t \le 2\pi$$



a) A horizontal tangent line

b) A vertical tangent line.

Solution

a) $\dfrac{dy}{dx} = \dfrac{2\cos t}{2\cos 2t} = 0 \qquad\qquad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

$\cos t = 0 \quad \rightarrow \quad t = \frac{\pi}{2}, \ \frac{3\pi}{2}$

$t = \frac{\pi}{2} \rightarrow \quad x = \sin \pi = 0 \quad y = 2\sin\frac{\pi}{2} = 2 \quad \boxed{(0,\ 2)}$

$t = \frac{3\pi}{2} \rightarrow \quad x = \sin 3\pi = 0 \quad y = 2\sin\frac{3\pi}{2} = -2 \quad \boxed{(0,\ -2)}$

b) Vertical tangent line: $\cos 2t = 0 \quad \cos t \ne 0$

$\cos 2t = 0 \quad \rightarrow \quad 2t = \frac{\pi}{2}, \ \frac{3\pi}{2}, \ \frac{5\pi}{2}. \ \frac{7\pi}{2} \quad \Rightarrow \boxed{t = \frac{\pi}{4}, \ \frac{3\pi}{4}, \ \frac{5\pi}{4}. \ \frac{7\pi}{4}}$

$t = \frac{\pi}{4} \rightarrow \quad x = 1 \quad y = \sqrt{2} \quad \boxed{(1,\ \sqrt{2})}$

$t = \frac{3\pi}{4} \rightarrow \quad x = -1 \quad y = \sqrt{2} \quad \boxed{(-1,\ \sqrt{2})}$

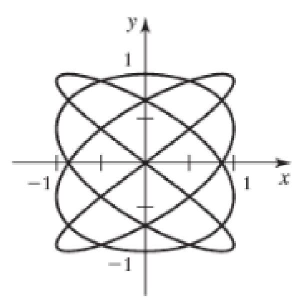
$t = \frac{5\pi}{4} \rightarrow \quad x = -1 \quad y = -\sqrt{2} \quad \boxed{(-1,\ -\sqrt{2})}$

$t = \frac{7\pi}{4} \rightarrow \quad x = 1 \quad y = -\sqrt{2} \quad \boxed{(1,\ -\sqrt{2})}$

Exercise

Consider Lissajous curve, estimate the coordinates of the points on the curve at which there is

$$x = \sin 4t, \quad y = \sin 3t; \quad 0 \le t \le 2\pi$$



a) A horizontal tangent line

b) A vertical tangent line.

Solution

a) $\dfrac{dy}{dx} = \dfrac{3\cos 3t}{4\cos 4t} = 0 \qquad\qquad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

$\cos 3t = 0 \quad \rightarrow \quad 3t = \frac{\pi}{2}, \ \frac{3\pi}{2}, \frac{5\pi}{2}, \ \frac{7\pi}{2}, \frac{9\pi}{2}, \ \frac{11\pi}{2} \quad \boxed{t = \frac{\pi}{6}, \ \frac{\pi}{2}, \frac{5\pi}{6}, \ \frac{7\pi}{6}, \frac{3\pi}{2}, \ \frac{11\pi}{6}}$

$t = \frac{\pi}{6} \rightarrow \quad x = -\frac{\sqrt{3}}{2} \quad y = 1 \quad \boxed{\left(-\frac{\sqrt{3}}{2},\ 1\right)}$

$t = \frac{\pi}{2} \rightarrow \quad x = 0 \quad y = -1 \quad \boxed{(0,\ -1)}$

$t = \frac{5\pi}{6} \to \quad x = \frac{\sqrt{3}}{2} \quad y = -1 \quad \boxed{\left(\frac{\sqrt{3}}{2},\ -1\right)}$

$t = \frac{7\pi}{6} \to \quad x = -\frac{\sqrt{3}}{2} \quad y = 1 \quad \boxed{\left(-\frac{\sqrt{3}}{2},\ -1\right)}$

$t = \frac{3\pi}{2} \to \quad x = 0 \quad y = 1 \quad \boxed{(0,\ 1)}$

$t = \frac{11\pi}{6} \to \quad x = \frac{\sqrt{3}}{2} \quad y = 1 \quad \boxed{\left(\frac{\sqrt{3}}{2},\ 1\right)}$

b) Vertical tangent line: $\cos 4t = 0 \quad \cos 3t \neq 0$

$\cos 4t = 0 \quad \to \quad 4t = \frac{(n+1)\pi}{2} \quad \Rightarrow \quad \boxed{t = \frac{(n+1)\pi}{8}}$

$t = \frac{(n+1)\pi}{8} \to \quad x = \pm 1 \quad y = \pm\sin\frac{3\pi}{8} \quad \boxed{\left(\pm 1,\ \pm\sin\frac{3\pi}{8}\right)}$

Exercise

Find the area under one arch of the cycloid $x = a(t - \sin t), \quad y = a(1 - \cos t)$

Solution

$$A = \int_0^{2\pi} y\,dx$$

$$= \int_0^{2\pi} a(1-\cos t)\,d\left[a(t-\sin t)\right] \qquad\qquad d\left[a(t-\sin t)\right] = a(1-\cos t)\,dt$$

$$= \int_0^{2\pi} a^2(1-\cos t)^2\,dt$$

$$= a^2\int_0^{2\pi}\left(1 - 2\cos t + \cos^2 t\right)dt$$

$$= a^2\int_0^{2\pi}\left(1 - 2\cos t + \frac{1+\cos 2t}{2}\right)dt$$

$$= a^2\int_0^{2\pi}\left(\frac{3}{2} - 2\cos t + \frac{1}{2}\cos 2t\right)dt$$

$$= a^2\left[\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t\right]_0^{2\pi}$$

$$= a^2\left[\frac{3}{2}(2\pi) - 2\sin(2\pi) + \frac{1}{4}\sin 2(2\pi) - 0\right]$$

$$\boxed{= 3\pi a^2}$$

Exercise

Find the area enclosed by the y–axis and the curve $\quad x=t-t^2, \quad y=1+e^{-t}$

Solution

$$x=t-t^2=0 \quad \Rightarrow \quad \underline{t=0,\ 1}$$

$$A=\int_0^1 xdy$$

$$=\int_0^1\left(t-t^2\right)d\left(1+e^{-t}\right)$$

$$=\int_0^1\left(t-t^2\right)\left(-e^{-t}\right)dt$$

$$=-\int_0^1\left(t-t^2\right)e^{-t}dt$$

		$\int e^{-t}$
$t-t^2$	$(+)$	$-e^{-t}$
$1-2t$	$(-)$	e^{-t}
-2	$(+)$	$-e^{-t}$

$$=-\left[\left(t-t^2\right)\left(-e^{-t}\right)-(1-2t)\left(e^{-t}\right)+(-2)\left(-e^{-t}\right)\right]_0^1$$

$$=-\left[e^{-t}\left(t^2-t\right)-e^{-t}(1-2t)+2e^{-t}\right]_0^1$$

$$=-\left[e^{-1}\left(1^2-1\right)-e^{-1}(1-2(1))+2e^{-1}-\left(e^{-0}\left(0^2-0\right)-e^{-0}(1-2(0))+2e^{-0}\right)\right]$$

$$=-\left[e^{-1}+2e^{-1}-(-1+2)\right]$$

$$=-\left(3e^{-1}-1\right)$$

$$=1-3e^{-1}$$

$$\underline{=1-\frac{3}{e}}$$

Exercise

Find the area enclosed by the ellipse $\quad x=a\cos t, \quad y=b\sin t, \quad 0\le t\le 2\pi$

Solution

$$A=\int_0^{2\pi} ydx=2\left|\int_0^{\pi} ydx\right|$$

$$=2\int_0^{\pi} b\sin t\ d\left[a\cos t\right]$$

$$= 2\int_0^{\pi} b\sin t\left(-a\sin t\right)\, dt$$

$$= -2ab\int_0^{\pi} \sin^2 t\, dt$$

$$= -2ab\int_0^{\pi} \left(\frac{1-\cos 2t}{2}\right)\, dt$$

$$= -ab\left[t - \frac{1}{2}\sin 2t\right]_0^{\pi}$$

$$= -ab\left(\pi - \frac{1}{2}\sin 2\pi - 0\right)$$

$$= \left|-\pi ab\right|$$

$$\underline{= \pi ab}$$

Exercise

Find the lengths of the curves $x = \cos t, \quad y = t + \sin t, \quad 0 \le t \le \pi$

Solution

$$x = \cos t \quad \Rightarrow \quad \frac{dx}{dt} = -\sin t$$

$$y = t + \sin t \quad \Rightarrow \quad \frac{dy}{dt} = 1 + \cos t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\sin^2 t + (1+\cos t)^2}$$

$$= \sqrt{\sin^2 t + 1 + 2\cos t + \cos^2 t}$$

$$= \sqrt{2 + 2\cos t}$$

$$L = \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi} \sqrt{2 + 2\cos t}\, dt$$

$$= \sqrt{2}\int_0^{\pi} \sqrt{(1+\cos t)\frac{1-\cos t}{1-\cos t}}\, dt$$

$$= \sqrt{2}\int_0^{\pi} \sqrt{\frac{1-\cos^2 t}{1-\cos t}}\, dt$$

$$L = \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi} \sqrt{4\sin^2 \frac{t}{2}}\, dt \qquad 2\sin^2 \frac{t}{2} = 1 + \cos t$$

$$= 2\int_0^{\pi} \sin \frac{t}{2}\, dt$$

$$= -4\cos \frac{t}{2}\Big|_0^{\pi}$$

$$= -4(0 - 1)$$

$$\underline{= 4}$$

$$= \sqrt{2}\int_0^{\pi} \sqrt{\frac{\sin^2 t}{1-\cos t}}\, dt$$

$$= \sqrt{2}\int_0^{\pi} \frac{\sin t}{\sqrt{1-\cos t}}\, dt \qquad d(1-\cos t) = \sin t dt$$

$$= \sqrt{2}\int_0^{\pi} \frac{d(1-\cos t)}{\sqrt{1-\cos t}}$$

$$= \sqrt{2}\left[2\sqrt{1-\cos t}\right]_0^{\pi}$$

$$= 2\sqrt{2}\left(\sqrt{1-\cos\pi} - \sqrt{1-\cos 0}\right)$$

$$= 2\sqrt{2}\left(\sqrt{2} - 0\right)$$

$$= 4$$

Exercise

Find the lengths of the curves $x = t^3, \quad y = \frac{3}{2}t^2, \quad 0 \le t \le \sqrt{3}$

Solution

$$x = t^3 \quad \Rightarrow \quad \frac{dx}{dt} = 3t^2$$

$$y = \frac{3}{2}t^2 \quad \Rightarrow \quad \frac{dy}{dt} = 3t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9t^4 + 9t^2}$$

$$= 3t\sqrt{t^2+1}$$

$$L = \int_0^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\sqrt{3}} 3t\sqrt{t^2+1}\, dt \qquad u = t^2+1 \quad \Rightarrow \quad du = 2tdt \quad \rightarrow \begin{cases} t = \sqrt{3} \Rightarrow u = 4 \\ t = 0 \Rightarrow u = 1 \end{cases}$$

$$= \frac{3}{2}\int_1^4 \sqrt{u}\, du$$

$$= \frac{3}{2}\frac{2}{3}\left[u^{3/2}\right]_1^4$$

$$= 4^{3/2} - 1$$

$$= 7$$

Exercise

Find the lengths of the curves $x = 8\cos t + 8t\sin t, \quad y = 8\sin t - 8t\cos t, \quad 0 \le t \le \frac{\pi}{2}$

Solution

$$x = 8\cos t + 8t\sin t \quad \Rightarrow \quad \frac{dx}{dt} = -8\sin t + 8\sin t + 8t\cos t = \underline{8t\cos t}$$

$$y = 8\sin t - 8t\cos t \quad \Rightarrow \quad \frac{dy}{dt} = 8\cos t - 8\cos t + 8t\sin t = \underline{8t\sin t}$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(8t\cos t)^2 + (8t\sin t)^2}$$

$$= \sqrt{(8t)^2\cos^2 t + (8t)^2\sin^2 t}$$

$$= 8t\sqrt{\cos^2 t + \sin^2 t} \qquad \cos^2 t + \sin^2 t = 1$$

$$= 8t$$

$$L = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi/2} 8t\, dt$$

$$= 4t^2 \Big|_0^{\pi/2}$$

$$= 4\left(\frac{\pi^2}{4} - 0\right)$$

$$= \underline{\pi^2}$$

Exercise

Find the lengths of the curves $x = \ln(\sec t + \tan t) - \sin t, \quad y = \cos t, \quad 0 \le t \le \frac{\pi}{3}$

Solution

$$x = \ln(\sec t + \tan t) - \sin t \quad \Rightarrow \quad \frac{dx}{dt} = \frac{\sec t\tan t + \sec^2 t}{\sec t + \tan t} - \cos t$$

$$= \frac{\sec t(\tan t + \sec t)}{\sec t + \tan t} - \cos t$$

$$= \underline{\sec t - \cos t}$$

$$y = \cos t \quad \Rightarrow \quad \frac{dy}{dt} = \underline{-\sin t}$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(\sec t - \cos t)^2 + (-\sin t)^2}$$

$$= \sqrt{\sec^2 t - 2\sec t\cos t + \cos^2 t + \sin^2 t}$$

$$= \sqrt{\sec^2 t - 2\frac{1}{\cos t}\cos t + 1}$$
$$= \sqrt{\sec^2 t - 2 + 1}$$
$$= \sqrt{\sec^2 t - 1}$$
$$= \sqrt{\tan^2 t}$$
$$= \tan t$$

$$L = \int_0^{\pi/3} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= \int_0^{\pi/3} \tan t\ dt \qquad\qquad \int_0^{\pi/3} \frac{\sin t}{\cos t}\ dt = \int_0^{\pi/3} -\frac{d(\cos t)}{\cos t}$$

$$= -\ln|\cos t|\Big|_0^{\pi/3}$$
$$= -\ln\cos\frac{\pi}{3} + \ln\cos 0$$
$$= -\ln\frac{1}{2} + \ln 1$$
$$\underline{= \ln 2|}$$

Exercise

Find the areas of the surfaces generated by revolving the curves

$$x = \frac{2}{3}t^{3/2}, \quad y = 2\sqrt{t}, \quad 0 \le t \le \sqrt{3}; \quad x - axis$$

Solution

$$x = \frac{2}{3}t^{3/2} \quad \Rightarrow \quad \frac{dx}{dt} = \underline{t^{1/2}|}$$

$$y = 2\sqrt{t} \quad \Rightarrow \quad \frac{dy}{dt} = \underline{t^{-1/2}|}$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left(t^{1/2}\right)^2 + \left(t^{-1/2}\right)^2}$$
$$= \sqrt{t + t^{-1}}$$
$$= \sqrt{\frac{t^2 + 1}{t}}$$

$$A = 2\pi \int_0^{\sqrt{3}} x ds$$

$$= 2\pi \int_0^{\sqrt{3}} \frac{2}{3}t^{3/2}\sqrt{\frac{t^2 + 1}{t}}dt$$

$$= \frac{4\pi}{3}\int_0^{\sqrt{3}} t\sqrt{t^2+1}dt \qquad u = t^2+1 \quad \Rightarrow \quad du = 2tdt \quad \to \begin{cases} t=\sqrt{3} \Rightarrow u=4 \\ t=0 \Rightarrow u=1 \end{cases}$$

$$= \frac{2\pi}{3}\int_1^4 \sqrt{u}\ du$$

$$= \frac{2\pi}{3}\left[\frac{2}{3}u^{3/2}\right]_1^4 \qquad = \frac{4\pi}{9}\left(4^{3/2}-1\right)$$

$$= \frac{28\pi}{9}$$

Exercise

Find the areas of the surfaces generated by revolving the curves

$$x = t+\sqrt{2}, \quad y = \frac{t^2}{2}+\sqrt{2}t, \quad -\sqrt{2} \le t \le \sqrt{2}; \quad y-axis$$

Solution

$$x = t+\sqrt{2} \quad \Rightarrow \quad \frac{dx}{dt} = 1$$

$$y = \frac{t^2}{2}+\sqrt{2}t \quad \Rightarrow \quad \frac{dy}{dt} = t+\sqrt{2}$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \sqrt{1^2+\left(t+\sqrt{2}\right)^2}$$

$$= \sqrt{1+t^2+2\sqrt{2}t+2}$$

$$= \sqrt{t^2+2\sqrt{2}\ t+3}$$

$$A = 2\pi\int_{-\sqrt{2}}^{\sqrt{2}} xds$$

$$= 2\pi\int_{-\sqrt{2}}^{\sqrt{2}} \left(t+\sqrt{2}\right)\sqrt{t^2+2\sqrt{2}\ t+3}\ dt \qquad u = t^2+2\sqrt{2}\ t+3 \quad \Rightarrow \quad du = \left(2t+2\sqrt{2}\right)d = 2\left(t+\sqrt{2}\right)dt$$

$$\to \begin{cases} t=\sqrt{2} \Rightarrow u=9 \\ t=-\sqrt{2} \Rightarrow u=1 \end{cases}$$

$$= \pi\int_1^9 \sqrt{u}\ du$$

$$= \pi\left[\frac{2}{3}u^{3/2}\right]_1^9$$

$$= \frac{2\pi}{3}\left(9^{3/2}-1\right)$$

$$= \frac{52\pi}{9}$$

Exercise

Find the areas of the surfaces generated by revolving the curves $x = 2t, \quad y = 3t; \quad 0 \le t \le 3 \quad x\text{-axis}$

Solution

$$x = 2t \quad \rightarrow \quad \frac{dx}{dt} = 2$$

$$y = 3t \quad \rightarrow \quad \frac{dy}{dt} = 3$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{4+9} = \sqrt{13}$$

$$S = 2\pi \int_0^3 (3t)\sqrt{13}\, dt \qquad\qquad S = 2\pi \int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 3\pi\sqrt{13}\left[t^2\right]_0^3$$

$$= 27\pi\sqrt{13}\ \ unit^2$$

Exercise

Find the areas of the surfaces generated by revolving the curves $x = 2t, \quad y = 3t; \quad 0 \le t \le 3 \quad y\text{-axis}$

Solution

$$x = 2t \quad \rightarrow \quad \frac{dx}{dt} = 2$$

$$y = 3t \quad \rightarrow \quad \frac{dy}{dt} = 3$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{13}$$

$$S = 2\pi \int_0^3 (2t)\sqrt{13}\, dt \qquad\qquad S = 2\pi \int_a^b x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 2\pi\sqrt{13}\left[t^2\right]_0^3$$

$$= 18\pi\sqrt{13}\ \ unit^2$$

Exercise

Find the areas of the surfaces generated by revolving the curves $x = t, \quad y = 4 - 2t; \quad 0 \le t \le 2 \quad x\text{-axis}$

Solution

$$x = t \quad \to \quad \frac{dx}{dt} = 1$$

$$y = 4 - 2t \quad \to \quad \frac{dy}{dt} = -2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{5}$$

$$S = 2\pi \int_0^2 (4 - 2t)\sqrt{5}\ dt \qquad\qquad S = 2\pi \int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= 2\pi\sqrt{5}\left[4t - t^2\right]_0^2$$

$$= 8\pi\sqrt{5}\ \ unit^2$$

Exercise

Find the areas of the surfaces generated by revolving the curves $x = t, \quad y = 4 - 2t; \quad 0 \le t \le 2 \quad y\text{-}axis$

Solution

$$x = t \quad \to \quad \frac{dx}{dt} = 1$$

$$y = 4 - 2t \quad \to \quad \frac{dy}{dt} = -2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{5}$$

$$S = 2\pi \int_0^2 (t)\sqrt{5}\ dt \qquad\qquad S = 2\pi \int_a^b x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= \pi\sqrt{5}\left[t^2\right]_0^2$$

$$= 4\pi\sqrt{5}\ \ unit^2$$

Exercise

Use the parametric equations $x = t^2\sqrt{3} \quad and \quad y = 3t - \frac{1}{3}t^3$ to

a) Graph the curve on the interval $-3 \le t \le 3$.

b) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

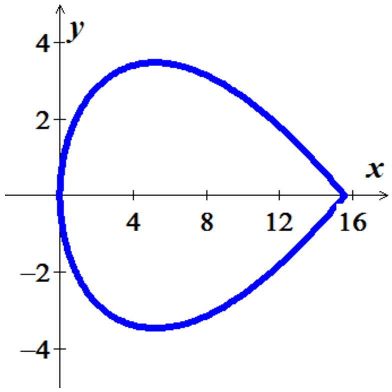
c) Find the equation of the tangent line at the point $\left(\sqrt{3},\ \frac{8}{3}\right)$

d) Find the length of the curve

e) Find the surface area generated by revolving the curve about the $x\text{-}axis$

Solution

a)



b) $\boxed{\dfrac{dy}{dx}=\dfrac{3-t^2}{2t\sqrt{3}}}$ $\qquad\qquad \dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}$

$$\frac{dy'}{dt}=\frac{1}{2\sqrt{3}}\frac{-2t^2-3+t^2}{t^2}=-\frac{t^2+3}{2\sqrt{3}t^2} \qquad\qquad \frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}$$

$$\frac{d^2y}{dx^2}=-\frac{t^2+3}{2\sqrt{3}t^2}\cdot\frac{1}{2t\sqrt{3}}$$

$$\boxed{=-\frac{t^2+3}{12t^3}}$$

c) $\left(\sqrt{3},\ \frac{8}{3}\right)\to x=t^2\sqrt{3}=\sqrt{3}\quad\Rightarrow \boxed{t=1}$

$$m=\left.\frac{dy}{dx}\right|_{t=1}=\left.\frac{3-t^2}{2t\sqrt{3}}\right|_{t=1}=\boxed{\frac{1}{\sqrt{3}}}$$

$$y=\frac{\sqrt{3}}{3}\left(x-\sqrt{3}\right)+\frac{8}{3}$$

$$\boxed{=\frac{\sqrt{3}}{3}x+\frac{5}{3}}$$

d) $\dfrac{dx}{dt}=2t\sqrt{3}\qquad \dfrac{dy}{dt}=3-t^2$

$$L=\int_{-3}^{3}\sqrt{12t^2+9-6t^2+t^4}\ dt \qquad\qquad L=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

$$=\int_{-3}^{3}\sqrt{\left(t^2+3\right)^2}\ dt$$

$$=\int_{-3}^{3}\left(t^2+3\right)\ dt$$

$$=\frac{1}{3}t^3+3t\Big|_{-3}^{3}$$

$$=9+9+9+9$$

$$\boxed{=36\ \ unit}$$

e) $S = 2\pi \int_0^3 \left(3t - \frac{1}{3}t^3\right)\left(t^2 + 3\right) dt$

$S = 2\pi \int_a^b y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$

$= 2\pi \int_0^3 \left(2t^3 - \frac{1}{3}t^5 + 9t\right) dt$

$= 2\pi \left[\frac{1}{2}t^4 - \frac{1}{18}t^6 + \frac{9}{2}t^2 \right]_0^3$

$= 2\pi \left(\frac{81}{2} - \frac{81}{2} + \frac{81}{2} \right)$

$= 81\pi \ \ unit^2$

Solution   *Section* 4.3 – Polar Coordinates

Exercise

Find the Cartesian coordinates of the following points (given in polar coordinates)

$$a)\ \left(\sqrt{2},\ \frac{\pi}{4}\right)\quad b)\ (1,\ 0)\quad c)\ \left(0,\ \frac{\pi}{2}\right)\quad d)\ \left(-\sqrt{2},\ \frac{\pi}{4}\right)$$

Solution

a) $\begin{cases} x = r\cos\theta = \sqrt{2}\cos\frac{\pi}{4} = 1 \\ x = r\sin\theta = \sqrt{2}\sin\frac{\pi}{4} = 1 \end{cases}$ ***Cartesian coordinates*** $(1,\ 1)$

b) $\begin{cases} x = r\cos\theta = 1\cos 0 = 1 \\ x = r\sin\theta = 1\sin 0 = 0 \end{cases}$ ***Cartesian coordinates*** $(1,\ 0)$

c) $\begin{cases} x = r\cos\theta = 0\cos\frac{\pi}{2} = 0 \\ x = r\sin\theta = 0\sin\frac{\pi}{2} = 0 \end{cases}$ ***Cartesian coordinates*** $(0,\ 0)$

d) $\begin{cases} x = r\cos\theta = -\sqrt{2}\cos\frac{\pi}{4} = -1 \\ x = r\sin\theta = -\sqrt{2}\sin\frac{\pi}{4} = -1 \end{cases}$ ***Cartesian coordinates*** $(-1,\ -1)$

Exercise

Find the polar coordinates, $0 \le \theta < 2\pi$ *and* $r \ge 0$, of the following points given in Cartesian coordinates

$$a)\ (1,\ 1)\quad b)\ (-3,\ 0)\quad c)\ \left(\sqrt{3},\ -1\right)\quad d)\ (-3,\ 4)$$

Solution

a) $\begin{cases} r = \sqrt{1^2 + 1^2} = \sqrt{2} \\ \theta = \tan^{-1}\frac{1}{1} = \frac{\pi}{4} \end{cases}$ ***Polar coordinates*** $\left(\sqrt{2},\ \frac{\pi}{4}\right)$

b) $\begin{cases} r = \sqrt{(-3)^2 + 0^2} = 3 \\ \theta = \tan^{-1}\frac{0}{-3} = \pi \end{cases}$ ***Polar coordinates*** $(3,\ \pi)$

c) $\begin{cases} r = \sqrt{\sqrt{3}^2 + (-1)^2} = 2 \\ \theta = \tan^{-1}\frac{-1}{\sqrt{3}} = \frac{11\pi}{6} \end{cases}$ ***Polar coordinates*** $\left(2,\ \frac{11\pi}{6}\right)$

d) $\begin{cases} r = \sqrt{(-3)^2 + 4^2} = 5 \\ \theta = \tan^{-1}\frac{4}{-3} = \pi - \arctan\left(\frac{4}{3}\right) \end{cases}$ ***Polar coordinates*** $\left(5,\ \pi - \arctan\left(\frac{4}{3}\right)\right)$

Exercise

Find the polar coordinates, $-\pi \le \theta < \pi$ *and* $r \ge 0$, of the following points given in Cartesian coordinates

$$a)\ (-2,\ -2) \quad b)\ (0,\ 3) \quad c)\ \left(-\sqrt{3},\ 1\right) \quad d)\ (5,\ -12)$$

Solution

a) $\begin{cases} r = \sqrt{(-2)^2 + (-2)^2} = 2\sqrt{2} \\ \theta = \tan^{-1}\frac{-2}{-2} = -\frac{3\pi}{4} \end{cases}$ ***Polar coordinates*** $\left(2\sqrt{2}, -\frac{3\pi}{4}\right)$

b) $\begin{cases} r = \sqrt{0^2 + 3^2} = 3 \\ \theta = \tan^{-1}\frac{3}{0} = \frac{\pi}{2} \end{cases}$ ***Polar coordinates*** $\left(3,\ \frac{\pi}{2}\right)$

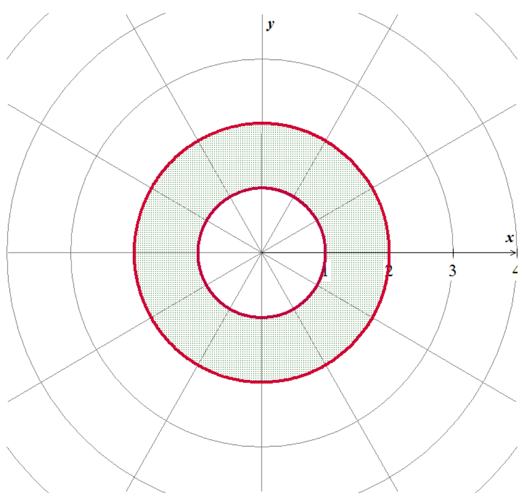
c) $\begin{cases} r = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} = 2 \\ \theta = \tan^{-1}\frac{1}{-\sqrt{3}} = \frac{5\pi}{6} \end{cases}$ ***Polar coordinates*** $\left(2,\ \frac{5\pi}{6}\right)$

d) $\begin{cases} r = \sqrt{5^2 + (-12)^2} = 13 \\ \theta = \tan^{-1}\frac{-12}{5} = -\arctan\left(\frac{12}{5}\right) \end{cases}$ ***Polar coordinates*** $\left(13,\ -\arctan\left(\frac{12}{5}\right)\right)$

Exercise

Graph $1 \le r \le 2$

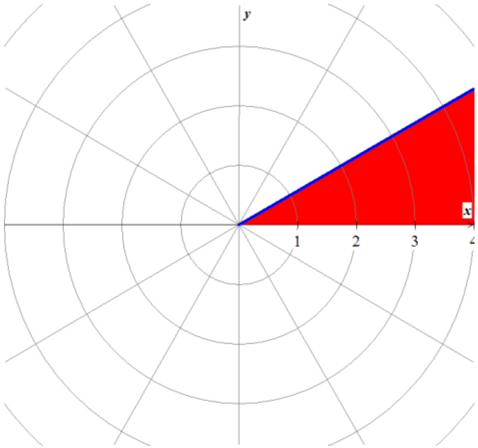
Solution



Exercise

Graph $\quad 0 \le \theta \le \frac{\pi}{6}, \quad r \ge 0$

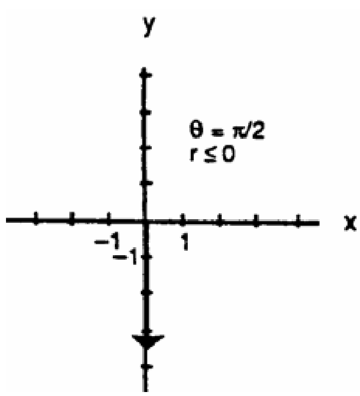
Solution



Exercise

Graph $\quad \theta = \frac{\pi}{2}, \quad r \le 0$

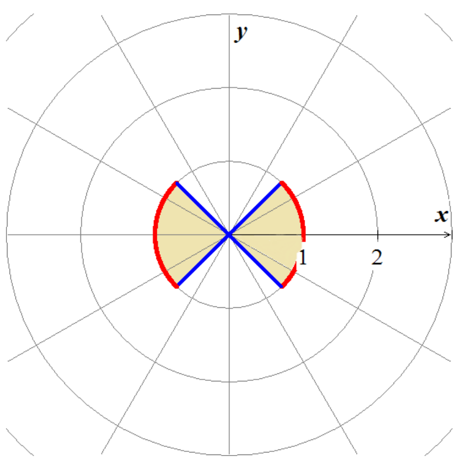
Solution



Exercise

Graph $\quad -\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}, \quad 0 \le r \le 1$

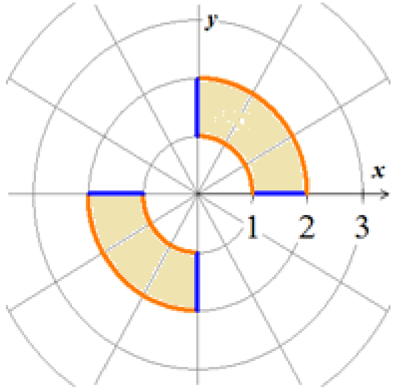
Solution



Exercise

Graph $\quad 0 \le \theta \le \frac{\pi}{2}, \quad 1 \le |r| \le 2$

Solution



Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r\cos\theta = 2$

Solution

$r\cos\theta = 2 \quad \Rightarrow \quad x = 2$, vertical line

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r\sin\theta = -1$

Solution

$r\sin\theta = -1 \quad \Rightarrow \quad y = -1$, horizontal line

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r = -3\sec\theta$

Solution

$$r = -3\sec\theta = -\frac{3}{\cos\theta} \quad \Rightarrow \quad r\cos\theta = -3$$

$x = -3$, vertical line through $(-3,\ 0)$

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r\cos\theta + r\sin\theta = 1$

Solution

$r\cos\theta + r\sin\theta = 1 \quad \Rightarrow \quad x + y = 1$, line with slope -1

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r^2 = 4r\sin\theta$

Solution

$$r^2 = 4r\sin\theta \quad \Rightarrow \quad x^2 + y^2 = 4y$$

$$x^2 + y^2 - 4y = 0$$

$$x^2 + y^2 - 4y + 4 = 4$$

$$x^2 + (y-2)^2 = 4$$

It is a circle with a center $C = (0,\ 2)$ and radius $r = 2$.

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r = \dfrac{5}{\sin\theta - 2\cos\theta}$

Solution

$$r = \frac{5}{\sin\theta - 2\cos\theta} \quad \Rightarrow \quad r\sin\theta - 2r\cos\theta = 5$$

$$y - 2x = 5 \quad \to y = 2x + 5$$

It is a line with slope $m = 2$ and intercept $b = 5$

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r = 4\tan\theta\sec\theta$

Solution

$$r = 4\tan\theta\sec\theta = 4\frac{\sin\theta}{\cos\theta}\frac{1}{\cos\theta} = 4\frac{\sin\theta}{\cos^2\theta} \quad \Rightarrow \quad r\cos^2\theta = 4\sin\theta$$

$$r^2\cos^2\theta = 4r\sin\theta$$

$x^2 = 4y \quad \to \quad y = \frac{1}{4}x^2$ It is a parabola with vertex (0, 0).

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph
$r\sin\theta = \ln r + \ln\cos\theta$

Solution

$$r\sin\theta = \ln r + \ln\cos\theta \qquad \textit{Power Rule}$$
$$= \ln r\cos\theta$$

$y = \ln x$ Graph of the natural logarithm function

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $\cos^2\theta = \sin^2\theta$

Solution

$$\cos^2\theta = \sin^2\theta \quad \rightarrow \quad r^2\cos^2\theta = r^2\sin^2\theta$$
$$x^2 = y^2$$
$$y = \pm x$$

The graph is 2 perpendicular lines through the origin with slopes -1 and 1,

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r = 2\cos\theta + 2\sin\theta$

Solution

$$r = 2\cos\theta + 2\sin\theta \quad \rightarrow \quad r^2 = 2r\cos\theta + 2r\sin\theta$$
$$x^2 + y^2 = 2x + 2y$$
$$x^2 - 2x + y^2 - 2y = 0$$
$$x^2 - 2x + 1 + y^2 - 2y + 1 = 1 + 1$$
$$(x-1)^2 + (y-1)^2 = 2$$

It is a circle with a center $C = (1,\ 1)$ and radius $r = \sqrt{2}$.

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r\sin\left(\frac{2\pi}{3} - \theta\right) = 5$

Solution

$$r\sin\left(\frac{2\pi}{3} - \theta\right) = 5 \quad \rightarrow \quad r\left(\sin\frac{2\pi}{3}\cos\theta - \cos\frac{2\pi}{3}\sin\theta\right) = 5$$

$$\frac{\sqrt{3}}{2}r\cos\theta+\frac{1}{2}r\sin\theta=5$$

$$\frac{\sqrt{3}}{2}x+\frac{1}{2}y=5$$

$$\sqrt{3}x+y=10$$

It is a line with slope $m=-\sqrt{3}$ and intercept $b = 10$

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r=\dfrac{4}{2\cos\theta-\sin\theta}$

Solution

$$2r\cos\theta-r\sin\theta=4$$

$$2x-y=4$$

The graph: Line $2x-y=4$ with slope $m = 2$.

Exercise

Replace the Cartesian equation with equivalent polar equation $x = y$

Solution

$$x=y \quad\Rightarrow\quad r\cos\theta=r\sin\theta$$

$$\cos\theta=\sin\theta$$

$$\boxed{\theta=\frac{\pi}{4}}$$

Exercise

Replace the Cartesian equation with equivalent polar equation $x^2-y^2=1$

Solution

$$x^2-y^2=1 \quad\Rightarrow\quad r^2\cos^2\theta-r^2\sin^2\theta=1$$

$$r^2\left(\cos^2\theta-\sin^2\theta\right)=1$$

$$\boxed{r^2\cos 2\theta=1}$$

Exercise

Replace the Cartesian equation with equivalent polar equation $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$

Solution

$$\frac{x^2}{9}+\frac{y^2}{4}=1 \quad \Rightarrow \quad 4x^2+9y^2=36$$

$$\underline{4r^2\cos^2\theta+9r^2\sin^2\theta=36}$$

Exercise

Replace the Cartesian equation with equivalent polar equation $\quad xy=1$

Solution

$$xy=1 \quad \Rightarrow \quad r^2\cos\theta\sin\theta=1 \qquad\qquad \sin 2\theta=2\cos\theta\sin\theta$$

$$r^2\frac{1}{2}\sin 2\theta=1$$

$$\underline{r^2\sin 2\theta=2}$$

Exercise

Replace the Cartesian equation with equivalent polar equation $\quad x^2+xy+y^2=1$

Solution

$$x^2+xy+y^2=1 \quad \Rightarrow \quad r^2+r^2\cos\theta\sin\theta=1$$

$$\underline{r^2(1+\cos\theta\sin\theta)=1}$$

Exercise

Replace the Cartesian equation with equivalent polar equation $\quad x^2+(y-2)^2=4$

Solution

$$x^2+(y-2)^2=4 \quad \Rightarrow \quad x^2+y^2-4y+4=4$$

$$x^2+y^2-4y=0$$

$$r^2-4r\sin\theta=0$$

$$r^2=4r\sin\theta$$

$$\underline{r=4\sin\theta}$$

Exercise

Replace the Cartesian equation with equivalent polar equation $\quad (x+2)^2+(y-5)^2=16$

Solution

$$(x+2)^2+(y-5)^2=16 \quad \Rightarrow \quad x^2+4x+4+y^2-10y+25=16$$

$$x^2 + 4x + y^2 - 10y = -13$$
$$r^2 + 4r\cos\theta - 10r\sin\theta = -13$$
$$\underline{r^2 = -4r\cos\theta + 10r\sin\theta - 13}$$

Exercise

a) Show that every vertical line in the *xy*–plane has a polar equation of the form $r = a\sec\theta$

b) Find the analogous polar equation for horizontal lines in the *xy*–plane.

Solution

a) $x = a \quad \Rightarrow \quad r\cos\theta = a \qquad \to r = \dfrac{a}{\cos\theta} \underline{= a\sec\theta}$

b) $y = b \quad \Rightarrow \quad r\sin\theta = b \qquad \to r = \dfrac{b}{\sin\theta} \underline{= b\csc\theta}$

Exercise

Identify the symmetries of the curve. Then sketch the curve. $\qquad r = 2 - 2\cos\theta$

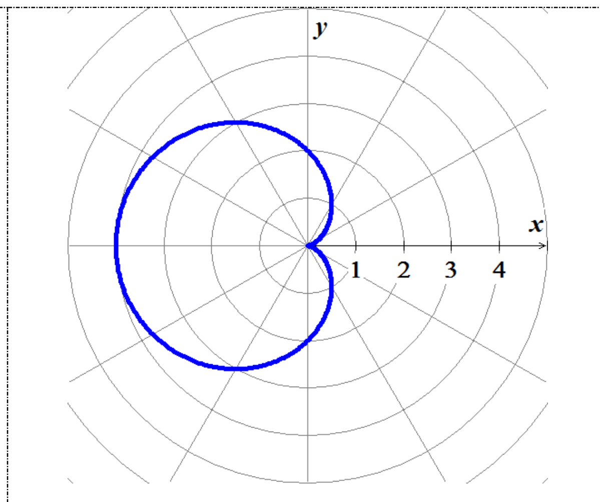
Solution

$2 - 2\cos(-\theta) = 2 - 2\cos\theta = r \quad \Rightarrow$ Symmetric about the *x*-axis

$\begin{cases} 2 - 2\cos(-\theta) \neq -r \\ 2 - 2\cos(\pi - \theta) = 2 + 2\cos\theta \neq r \end{cases} \quad \Rightarrow$ It is not symmetric about the *y*-axis

Therefore, it is *not* symmetric about the origin.

θ	$r = 2 - 2\cos\theta$
0	0
$\frac{\pi}{3}$	1
$\frac{\pi}{2}$	2
$\frac{2\pi}{3}$	1
π	4



Exercise

Identify the symmetries of the curve. Then sketch the curve. $r = 1 + \sin\theta$

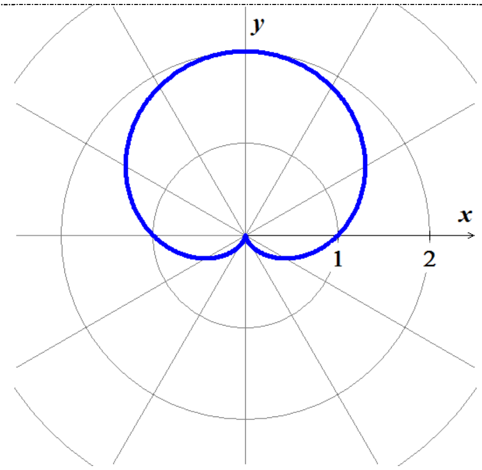
Solution

$$\begin{cases} 1+\sin(-\theta) = 1 - \sin\theta \neq r \\ 1+\sin(\pi-\theta) = 1+\sin\theta \neq -r \end{cases} \Rightarrow \text{ It is not symmetric about the } x\text{-axis}$$

$1+\sin(\pi-\theta) = 1+\sin\theta = r \quad \Rightarrow$ It is symmetric about the y-axis

Therefore, it is not symmetric about the origin.

θ	$r = 1+\sin\theta$
$-\frac{\pi}{2}$	0
$-\frac{\pi}{4}$	.293
0	1
$\frac{\pi}{4}$	1.707
$\frac{\pi}{2}$	2



Exercise

Identify the symmetries of the curve. Then sketch the curve. $r = 2 + \sin\theta$

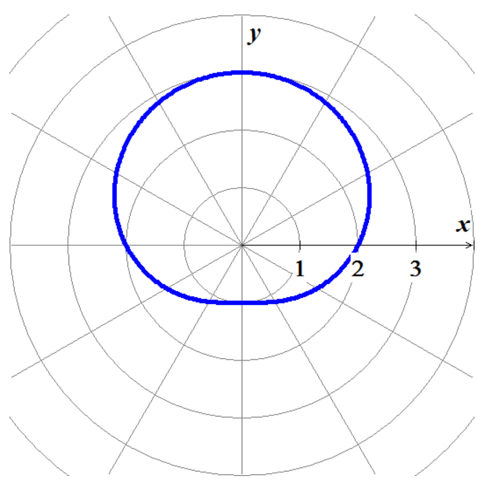
Solution

$$\begin{cases} 2+\sin(-\theta) = 2 - \sin\theta \neq r \\ 2+\sin(\pi-\theta) = 2+\sin\theta \neq -r \end{cases} \Rightarrow \text{ It is not symmetric about the } x\text{-axis}$$

$2+\sin(\pi-\theta) = 2+\sin\theta = r \quad \Rightarrow$ It is symmetric about the y-axis

Therefore, it is not symmetric about the origin.

θ	$r = 2+\sin\theta$
$-\frac{\pi}{2}$	1
$-\frac{\pi}{4}$	1.293
0	2
$\frac{\pi}{4}$	1.707
$\frac{\pi}{2}$	2.707



Exercise

Identify the symmetries of the curve. Then sketch the curve. $r^2 = \sin\theta$

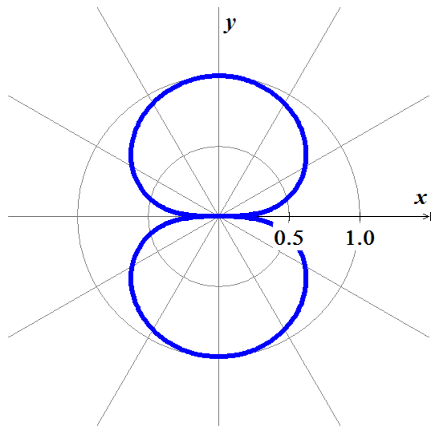
Solution

$\sin(\pi-\theta) = \sin\theta = r^2 \quad \Rightarrow$ It is symmetric about the *x*-axis

$\sin(\pi-\theta) = \sin\theta = r^2 \quad \Rightarrow$ It is symmetric about the *y*-axis

Therefore, it is symmetric about the origin.

θ	$r=\sqrt{\sin\theta}$
0	0
$\frac{\pi}{6}$	0.707
$\frac{\pi}{4}$	0.84
$\frac{\pi}{3}$	0.93
$\frac{\pi}{2}$	1



Exercise

Identify the symmetries of the curve. Then sketch the curve. $r^2 = -\sin\theta$

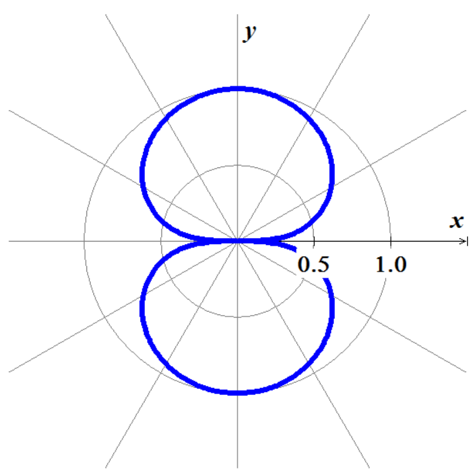
Solution

$-\sin(\pi-\theta) = -\sin\theta = r^2 \quad \Rightarrow$ It is symmetric about the *x*-axis

$-\sin(\pi-\theta) = -\sin\theta = r^2 \quad \Rightarrow$ It is symmetric about the *y*-axis

Therefore, it is symmetric about the origin

θ	$r^2=-\sin\theta$
0	0
$\frac{\pi}{6}$	0.707
$\frac{\pi}{4}$	0.84
$\frac{\pi}{3}$	0.93
$\frac{\pi}{2}$	1



Exercise

Identify the symmetries of the curve. Then sketch the curve. $\qquad r^2 = -\cos\theta$

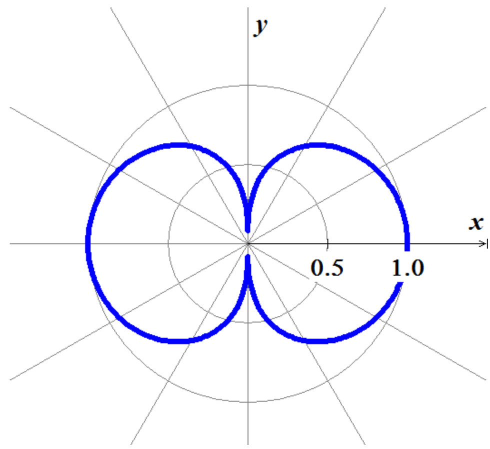
Solution

$-\cos(-\theta) = -\cos\theta = r^2 \quad \Rightarrow$ It is symmetric about the *x*-axis

$\begin{cases} -\cos(-\theta) = -\cos\theta = r^2 \\ (-r)^2 = r^2 = -\cos\theta \end{cases} \quad \Rightarrow$ It is symmetric about the *y*-axis

Therefore, it is symmetric about the origin

θ	$r = \sqrt{-\cos\theta}$
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	0.7
$\frac{3\pi}{4}$	0.84
$\frac{5\pi}{6}$	0.93
	1

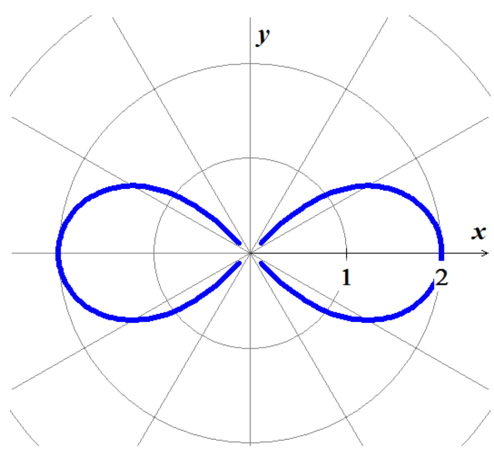


Exercise

Graph the lemniscate. What symmetries do these curves have? $\qquad r^2 = 4\cos 2\theta$

Solution

θ	$r^2 = 4\cos 2\theta$
0	2
$\frac{\pi}{12}$	1.8
$\frac{\pi}{6}$	1.4
$\frac{\pi}{4}$	0



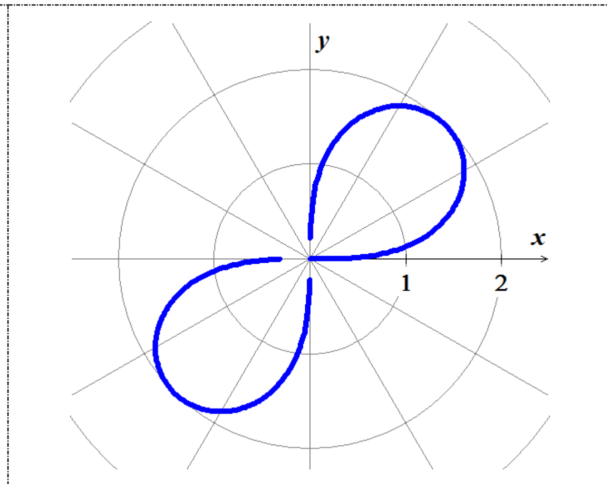
$(\pm r)^2 = 4\cos 2(-\theta) \;\Rightarrow\; r^2 = 4\cos 2\theta$ The graph is symmetric about the *x*-axis and the *y*-axis $\Rightarrow$ The graph is symmetric about the origin.

Exercise

Graph the lemniscate. What symmetries do these curves have? $r^2 = 4\sin 2\theta$

Solution

θ	$r^2 = 4\sin 2\theta$
0	0
$\frac{\pi}{12}$	1.4
$\frac{\pi}{6}$	1.8
$\frac{\pi}{4}$	2



$(\pm r)^2 = 4\sin 2\theta \Rightarrow r^2 = 4\sin 2\theta$ The graph is symmetric about the origin.

$4\sin 2(-\theta) = -4\sin 2\theta \neq r^2 \Rightarrow$ The graph is *not* symmetric about the *x*-axis

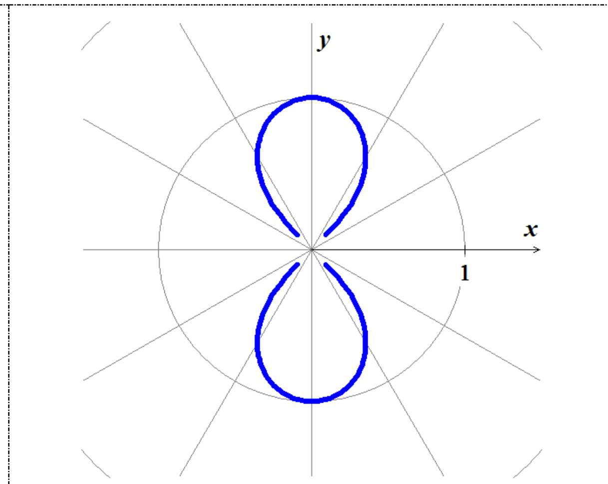
$4\sin 2(\pi - \theta) = 4\sin(2\pi - 2\theta) = 4\sin(-2\theta) = -4\sin 2\theta \neq r^2 \Rightarrow$ The graph is *not* symmetric about the *y*-axis.

Exercise

Graph the lemniscate. What symmetries do these curves have? $r^2 = -\cos 2\theta$

Solution

θ	$r^2 = -\cos 2\theta$
$\frac{\pi}{4}$	0
$\frac{\pi}{3}$	.7
$\frac{\pi}{2}$	1

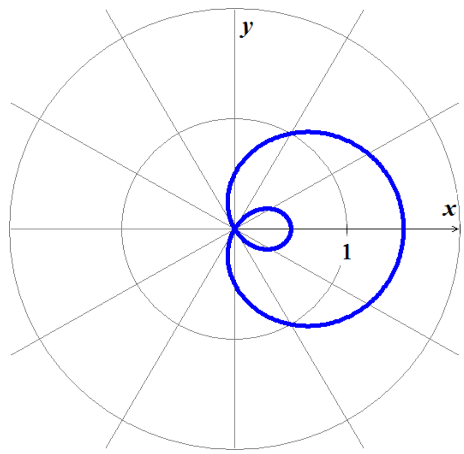


Exercise

Graph the limaçons is Old French for "snail". Equations for limaçons have the form $r = \frac{1}{2} + \cos\theta$

Solution

θ	$r = \frac{1}{2} + \cos\theta$
0	1.5
$\frac{\pi}{6}$	1.36
$\frac{\pi}{4}$	1.2
$\frac{\pi}{3}$	1
$\frac{\pi}{2}$	0.5
$\frac{3\pi}{4}$	−0.2
π	−0.5

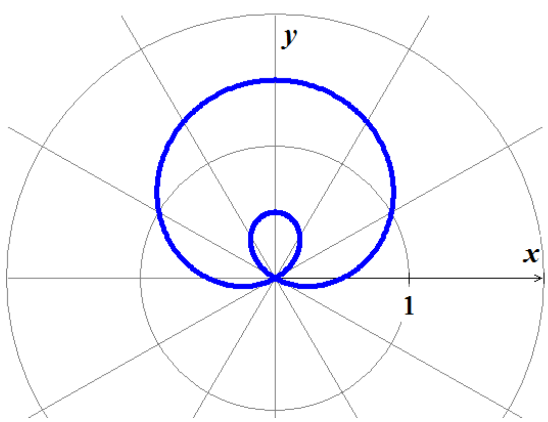


Exercise

Graph the limaçons is Old French for "snail". Equations for limaçons have the form $r = \frac{1}{2} + \sin\theta$

Solution

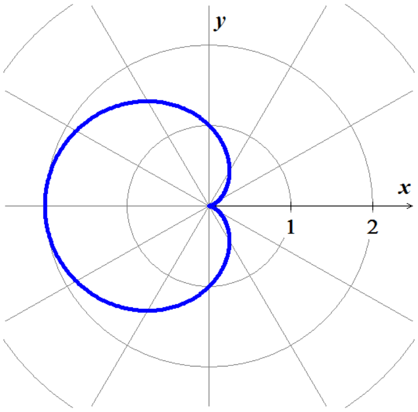
θ	$r = \frac{1}{2} + \sin\theta$
0	0.5
$\frac{\pi}{6}$	1
$\frac{\pi}{4}$	1.2
$\frac{\pi}{3}$	1.36
$\frac{\pi}{2}$	1.5
π	0.5
$\frac{5\pi}{4}$	−0.2
$\frac{3\pi}{2}$	−0.5



Exercise

Graph the limaçons is Old French for "snail". Equations for limaçons have the form $r = 1 - \cos\theta$

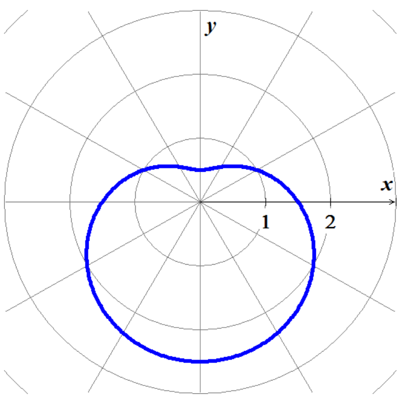
Solution



Exercise

Graph the limaçons is Old French for "snail". Equations for limaçons have the form $r = \frac{3}{2} - \sin\theta$

Solution

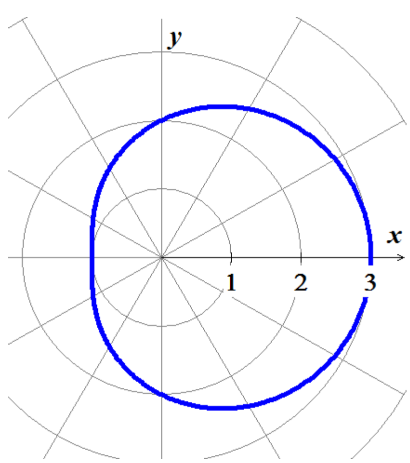


Exercise

Graph the limaçons is Old French for "snail". Equations for limaçons have the form $r = 2 + \cos\theta$

Solution

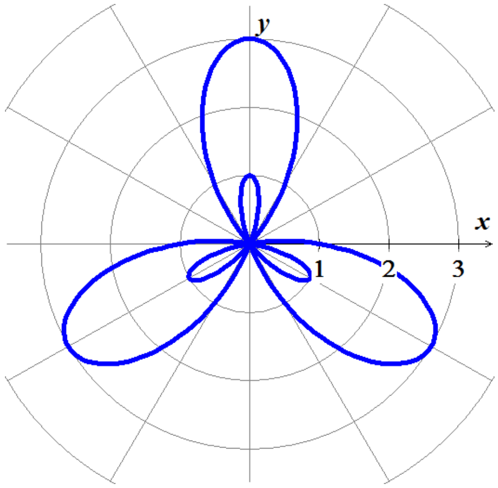
θ	$r = 2 + \cos\theta$
0	3
$\frac{\pi}{6}$	≈ 1.866
$\frac{\pi}{4}$	≈ 1.7
$\frac{\pi}{2}$	2
$\frac{3\pi}{4}$	≈ 1.29
π	1



Exercise

Graph the equation $r = 1 - 2\sin 3\theta$

Solution



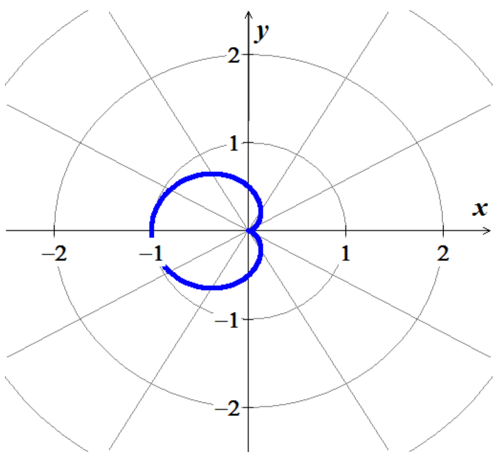
Exercise

Graph the equation $r = \sin^2 \frac{\theta}{2}$

Solution

$$\sin^2\left(-\frac{\theta}{2}\right) = \sin^2\left(\frac{\theta}{2}\right) = r \quad \Rightarrow \text{ It is symmetric about the } x\text{-axis}$$

θ	$r = \sin^2 \frac{\theta}{2}$
0	0
$\frac{\pi}{3}$	0.25
$\frac{\pi}{2}$	0.5
$\frac{2\pi}{3}$	0.75
π	1
$\frac{4\pi}{3}$	0.75
2π	0

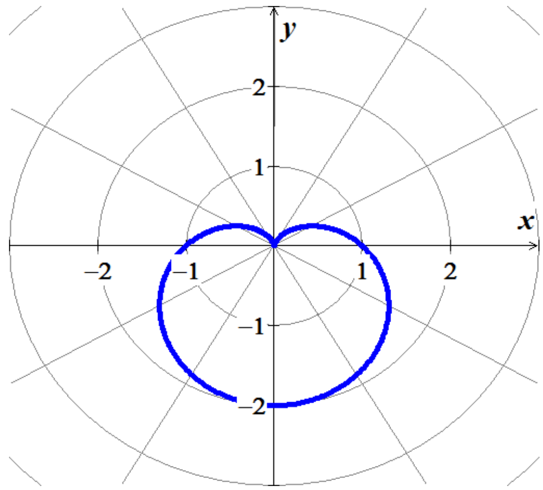


Exercise

Graph the equation $r = 1 - \sin\theta$

Solution

θ	$r = 1 - \sin\theta$
0	1
$\frac{\pi}{6}$	0.5
$\frac{\pi}{4}$	≈ 0.3
$\frac{\pi}{2}$	0
$\frac{3\pi}{4}$	≈ 0.3
π	1
$\frac{7\pi}{6}$	1.5
$\frac{3\pi}{2}$	2



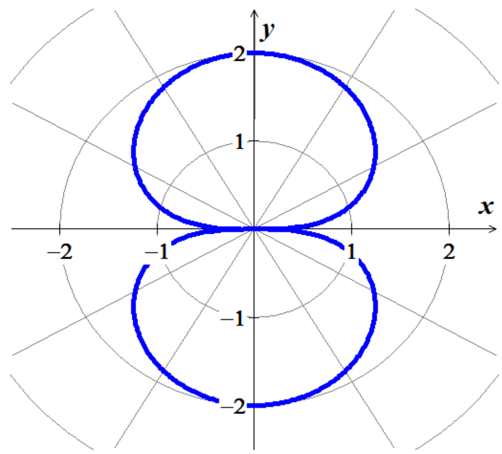
Exercise

Graph the equation $r^2 = 4\sin\theta$

Solution

$4\sin(\pi - \theta) = 4\sin\theta = r \quad \Rightarrow$ It is symmetric about the *y*-axis

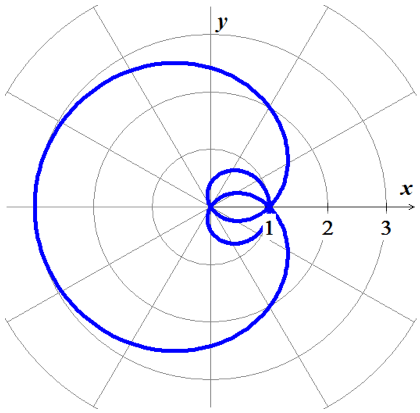
θ	$r = \pm 2\sqrt{\sin\theta}$
0	0
$\frac{\pi}{6}$	$\pm\sqrt{2} \approx \pm 1.4$
$\frac{\pi}{4}$	$\approx \pm 1.7$
$\frac{\pi}{3}$	$\approx \pm 1.9$
$\frac{\pi}{2}$	± 2



Exercise

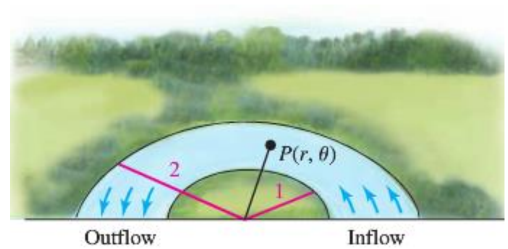
Graph the nephroid of Freeth equation $r = 1 + 2\sin\frac{\theta}{2}$

Solution



Exercise

Water flows in a shallow semicircular channel with inner and outer radii of 1 *m* and 2 *m*. At a point $P(r,\ \theta)$ in the channel, the flow is in the tangential direction (counterclockwise along circles), and it depends only on *r*, the distance from the center of the semicircles.



a) Express the region formed by the channel as a set in polar coordinates.

b) Express the inflow and outflow regions of the channel as sets in polar coordinates.

c) Suppose the tangential velocity of the water in *m/s* is given by $v(r) = 10r$, for $1 \le r \le 2$. Is the velocity greater at $\left(1.5,\ \frac{\pi}{4}\right)$ or $\left(1.2,\ \frac{3\pi}{4}\right)$? Explain.

d) Suppose the tangential velocity of the water is given by $v(r) = \frac{20}{r}$, for . Is the velocity greater $\left(1.8,\ \frac{\pi}{6}\right)$ or $\left(1.3,\ \frac{2\pi}{3}\right)$? Explain.

e) The total amount of water that flows through the channel (across a cross section of the channel $\theta = \theta_0$) is proportional to $\int_1^2 v(r)\,dr$. Is the total flow through the channel greater for the flow in part (*c*) or (*d*)?

Solution

a) The region is given by $\{(r,\ \theta):\ 1 \le r \le 2,\ \ 0 \le \theta \le \pi\}$

b) The inflow is given by $\{(r,\ \theta):\ 1 \le r \le 2,\ \ \theta = 0\}$

The outflow is given by $\{(r,\ \theta):\ 1 \le r \le 2,\ \ \theta = \pi\}$

c) The tangential velocity at $\left(1.5,\ \frac{\pi}{4}\right)$ is $v(1.5) = 10(1.5)\ \underline{= 15\ \ m/s}$

At $\left(1.2,\ \frac{3\pi}{4}\right)$ is $v(1.2) = 10(1.2)\ \underline{= 12\ \ m/s}$ So it is greater at 1.5.

d) The tangential velocity at $\left(1.8,\ \frac{\pi}{6}\right)$ is $v(1.8)=\frac{20}{1.8}$ $\underline{\approx 11.11\ \ m/s}$

At $\left(1.3,\ \frac{2\pi}{3}\right)$ is $v(1.3)=\frac{20}{1.3}$ $\underline{\approx 15.38\ \ m/s}$

So it is greater at 1.3.

e) $\int_1^2 v(r)\,dr = \int_1^2 10r\ dr$

$= 5r^2\Big|_1^2$

$\underline{=15}$

$\int_1^2 v(r)\,dr = \int_1^2 \frac{20}{r}\ dr$

$= 20\ln r\Big|_1^2$

$\underline{=20\ln 2}$ $\underline{\approx 13.86}$ So the flow in part (c) is greater.

Exercise

A simplified model assumes that the orbits of Earth and Mars are circular with radii of 2 and 3, respectively, and that Earth completes one orbit in one year while Mars takes two years. When $t=0$. Earth is at $(2,\ 0)$ and Mars is at $(3,\ 0)$; both orbit the Sum (at $(0,\ 0)$) in the counterclockwise direction. The position of Mars relative to Earth is given by the parametric equations

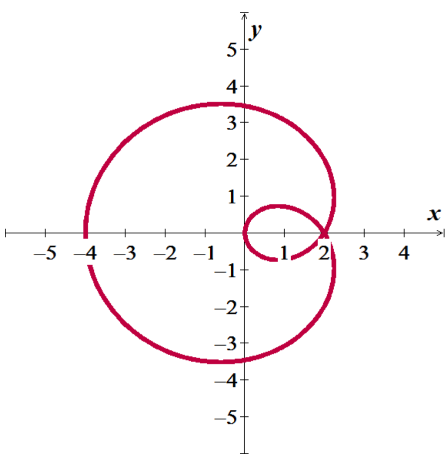
$$x=(3-4\cos\pi t)\cos\pi t+2,\quad y=(3-4\cos\pi t)\sin\pi t$$

a) Graph the parametric equations, for $0\le t\le 2$

b) Letting $r=3-4\cos\pi t$, explain why the path of Mars relative to Earth is a limaçon.

Solution

a)



b) $r=3-4\cos\pi t$ is a limaçon, and $x-2=r\cos\pi t$ and $y=r\,\text{sins}\,\pi t$ is a circle, and the composition of a limaçon and a circle is a limaçon.

Exercise

Find the slopes of the curves at the given points. Sketch the curves along with their tangents at these points. *Cardioid* $r=-1+\cos\theta;\quad \theta=\pm\frac{\pi}{2}$

Solution

$$\theta=\frac{\pi}{2}\quad\Rightarrow\quad r=-1+\cos\frac{\pi}{2}=-1\quad\rightarrow\quad\left(-1,\ \frac{\pi}{2}\right)$$

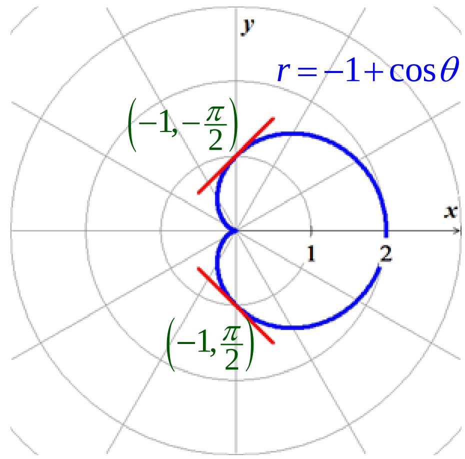
$$\theta=-\frac{\pi}{2}\quad\Rightarrow\quad r=-1+\cos\left(-\frac{\pi}{2}\right)=-1\quad\rightarrow\quad\left(-1,\ -\frac{\pi}{2}\right)$$

$$r'=\frac{dr}{d\theta}=-\sin\theta$$

$$Slope=\frac{r'\sin\theta+r\cos\theta}{r'\cos\theta-r\sin\theta}=\frac{-\sin^2\theta+r\cos\theta}{-\sin\theta\cos\theta-r\sin\theta}$$

$$Slope\Big|_{\left(-1,\frac{\pi}{2}\right)}=\frac{-\sin^2\frac{\pi}{2}+(-1)\cos\frac{\pi}{2}}{-\sin\frac{\pi}{2}\cos\frac{\pi}{2}+\sin\frac{\pi}{2}}=-1$$

$$Slope\Big|_{\left(-1,-\frac{\pi}{2}\right)}=\frac{-\sin^2\left(-\frac{\pi}{2}\right)+(-1)\cos\left(-\frac{\pi}{2}\right)}{-\sin\left(-\frac{\pi}{2}\right)co\left(-\frac{\pi}{2}\right)+\sin\left(-\frac{\pi}{2}\right)}=1$$



Exercise

Find the slopes of the curves at the given points. Sketch the curves along with their tangents at these points. *Cardioid* $r=-1+\sin\theta;\quad \theta=0,\ \pi$

Solution

$$\theta=0\quad\Rightarrow\quad r=-1+\sin 0=-1\quad\rightarrow\quad(-1,\ 0)$$

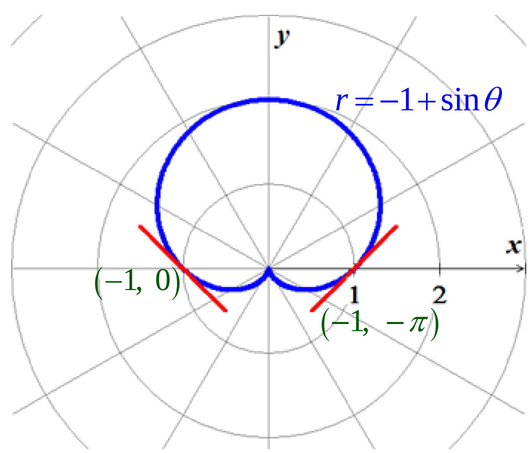
$$\theta=\pi\quad\Rightarrow\quad r=-1+\sin\pi=-1\quad\rightarrow\quad(-1,\ \pi)$$

$$r'=\frac{dr}{d\theta}=\cos\theta$$

$$Slope=\frac{r'\sin\theta+r\cos\theta}{r'\cos\theta-r\sin\theta}=\frac{\cos\theta\sin\theta+r\cos\theta}{\cos^2\theta-r\sin\theta}$$

$$Slope\Big|_{(-1,0)}=\frac{\cos(0)\sin(0)+(-1)\cos(0)}{\cos^2(0)-(-1)\sin(0)}=-1$$

$$Slope\Big|_{(-1,\pi)}=\frac{\cos(\pi)\sin(\pi)+(-1)\cos(\pi)}{\cos^2(\pi)-(-1)\sin(\pi)}=1$$



Exercise

Find the slopes of the curves at the given points. Sketch the curves along with their tangents at these points. $\quad$ *Four – leaved rose* $\quad r = \sin 2\theta; \quad \theta = \pm\frac{\pi}{4}, \ \pm\frac{3\pi}{4}$

Solution

$$\theta = -\frac{\pi}{4} \quad \Rightarrow \quad r = \sin\left(-\frac{\pi}{2}\right) = -1 \quad \to \quad \left(-1, \ -\frac{\pi}{4}\right)$$

$$\theta = \frac{\pi}{4} \quad \Rightarrow \quad r = \sin\left(\frac{\pi}{2}\right) = 1 \quad \to \quad \left(1, \ \frac{\pi}{4}\right)$$

$$\theta = -\frac{3\pi}{4} \quad \Rightarrow \quad r = \sin\left(-\frac{3\pi}{2}\right) = 1 \quad \to \quad \left(1, \ -\frac{3\pi}{4}\right)$$

$$\theta = \frac{3\pi}{4} \quad \Rightarrow \quad r = \sin\left(\frac{3\pi}{2}\right) = -1 \quad \to \quad \left(-1, \ \frac{3\pi}{4}\right)$$

$$r' = \frac{dr}{d\theta} = 2\cos 2\theta$$

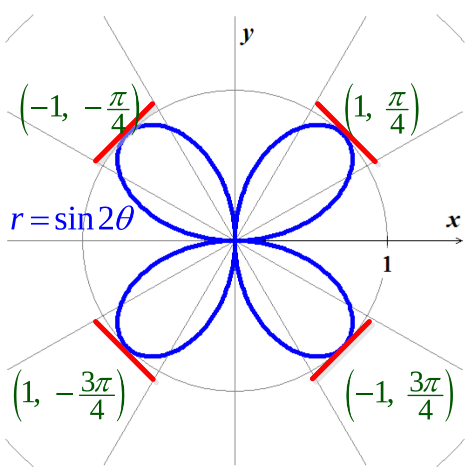
$$Slope = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{2\cos 2\theta \sin\theta + r\cos\theta}{2\cos 2\theta\cos\theta - r\sin\theta}$$

$$Slope\Big|_{\left(-1,-\frac{\pi}{4}\right)} = \frac{2\cos\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{4}\right) + (-1)\cos\left(-\frac{\pi}{4}\right)}{2\cos\left(-\frac{\pi}{2}\right)\cos\left(-\frac{\pi}{4}\right) - (-1)\sin\left(-\frac{\pi}{4}\right)} = \underline{1}$$

$$Slope\Big|_{\left(1,\frac{\pi}{4}\right)} = \frac{2\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right) + (1)\cos\left(\frac{\pi}{4}\right)}{2\cos\left(\frac{\pi}{2}\right)\cos\left(-\frac{\pi}{4}\right) - (1)\sin\left(\frac{\pi}{4}\right)} = -1$$

$$Slope\Big|_{\left(-1,\frac{3\pi}{4}\right)} = \frac{2\cos\left(\frac{3\pi}{2}\right)\sin\left(\frac{3\pi}{4}\right) + (-1)\cos\left(-\frac{3\pi}{4}\right)}{2\cos\left(\frac{3\pi}{2}\right)\cos\left(\frac{3\pi}{4}\right) - (-1)\sin\left(\frac{3\pi}{4}\right)} = 1$$

$$Slope\Big|_{\left(1,-\frac{3\pi}{4}\right)} = \frac{2\cos\left(-\frac{3\pi}{2}\right)\sin\left(-\frac{3\pi}{4}\right) + (1)\cos\left(-\frac{3\pi}{4}\right)}{2\cos\left(-\frac{3\pi}{2}\right)\cos\left(-\frac{3\pi}{4}\right) - (1)\sin\left(-\frac{3\pi}{4}\right)} = -1$$



Exercise

Find the slopes of the curves at the given points. Sketch the curves along with their tangents at these points. $\quad$ *Four – leaved rose* $\quad r = \cos 2\theta;\quad \theta = 0,\ \pm\frac{\pi}{2},\ \pi$

Solution

$$\theta = 0 \ \Rightarrow\ r = \cos(0) = 1 \ \rightarrow\ (1,\ 0)$$

$$\theta = \frac{\pi}{2} \ \Rightarrow\ r = \cos(\pi) = -1 \ \rightarrow\ \left(-1,\ \frac{\pi}{2}\right)$$

$$\theta = -\frac{\pi}{2} \ \Rightarrow\ r = \cos(-\pi) = -1 \ \rightarrow\ \left(-1,\ -\frac{\pi}{2}\right)$$

$$\theta = \pi \ \Rightarrow\ r = \cos(2\pi) = 1 \ \rightarrow\ (1,\ \pi)$$

$$r' = \frac{dr}{d\theta} = -2\sin 2\theta$$

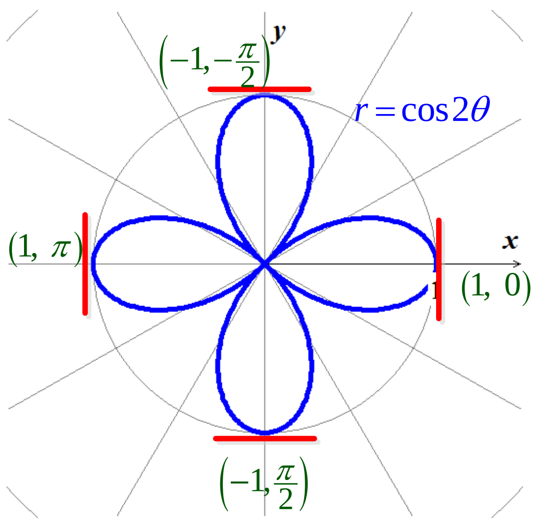
$$Slope = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{-2\sin 2\theta\sin\theta + r\cos\theta}{-2\sin 2\theta\cos\theta - r\sin\theta}$$

$$Slope\Big|_{(1,0)} = \frac{-2\sin(0)\sin(0) + (1)\cos(0)}{-2\sin(0)\cos(0) - (1)\sin(0)} = undefined$$

$$Slope\Big|_{\left(-1,\frac{\pi}{2}\right)} = \frac{-2\sin(\pi)\sin\left(\frac{\pi}{2}\right) + (1)\cos\left(\frac{\pi}{2}\right)}{-2\sin(\pi)\cos\left(\frac{\pi}{2}\right) - (1)\sin\left(\frac{\pi}{2}\right)} = \underline{0|}$$

$$Slope\Big|_{\left(-1,-\frac{\pi}{2}\right)} = \frac{-2\sin(-\pi)\sin\left(-\frac{\pi}{2}\right) + (1)\cos\left(-\frac{\pi}{2}\right)}{-2\sin(-\pi)\cos\left(-\frac{\pi}{2}\right) - (1)\sin\left(-\frac{\pi}{2}\right)} = \underline{0|}$$

$$Slope\Big|_{(1,\pi)} = \frac{-2\sin(2\pi)\sin(\pi) + (1)\cos(\pi)}{-2\sin(2\pi)\cos(\pi) - (1)\sin(\pi)} = undefined$$



Exercise

Find the area of the region bounded by the spiral $r=\theta$ *for* $0\le\theta\le\pi$

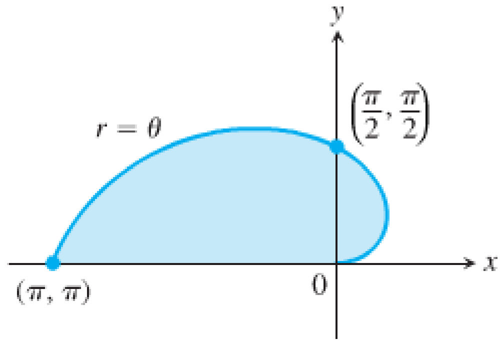
Solution

$$A=\frac{1}{2}\int_0^{\pi} r^2 d\theta$$

$$=\frac{1}{2}\int_0^{\pi} \theta^2 d\theta$$

$$=\frac{1}{6}\theta^3\Big|_0^{\pi}$$

$$=\frac{\pi^3}{6}$$



Exercise

Find the area of the region bounded by the circle $r=2\sin\theta$ *for* $\frac{\pi}{4}\le\theta\le\frac{\pi}{2}$

Solution

$$A=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(2\sin\theta)^2\, d\theta$$

$$=2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sin^2\theta d\theta$$

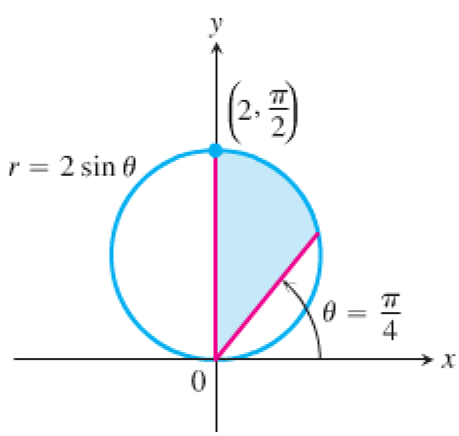
$$=2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{1-\cos 2\theta}{2}d\theta$$

$$=\left[\theta-\frac{1}{2}\sin 2\theta\right]\Big|_{\pi/4}^{\pi/2}$$

$$=\left(\frac{\pi}{2}-\frac{1}{2}\sin\pi\right)-\left(\frac{\pi}{4}-\frac{1}{2}\sin\frac{\pi}{2}\right)$$

$$=\frac{\pi}{2}-\frac{\pi}{4}+\frac{1}{2}\left(-\frac{1}{2}\sin\pi\right)-\left(-\sin\frac{\pi}{2}\right)$$

$$=\frac{\pi}{4}+\frac{1}{2}$$



Exercise

Find the area of the region inside the oval limaçon $r = 4 + 2\sin\theta$

Solution

$$A = \frac{1}{2}\int_0^{2\pi} (4 + 2\sin\theta)^2\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\left(16 + 16\sin\theta + 4\sin^2\theta\right) d\theta$$

$$= \int_0^{2\pi} \left(8 + 8\sin\theta + 2\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \int_0^{2\pi} (8 + 8\sin\theta + 1 - \cos 2\theta)\, d\theta$$

$$= \int_0^{2\pi} (9 + 8\sin\theta - \cos 2\theta)\, d\theta$$

$$= \left[9\theta - 8\cos\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi}$$

$$= 18\pi - 8\cos 2\pi - \frac{1}{2}\sin 4\pi - \left(0 - 8\cos 0 - \frac{1}{2}\sin 0\right)$$

$$= 18\pi - 8 + 8$$

$$= 18\pi$$

Exercise

Find the area of the region inside the cardioid $r = a(1 + \cos\theta), \quad a > 0$

Solution

$$A = \frac{1}{2}\int_0^{2\pi} a^2 (1 + \cos\theta)^2\, d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{a^2}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$$

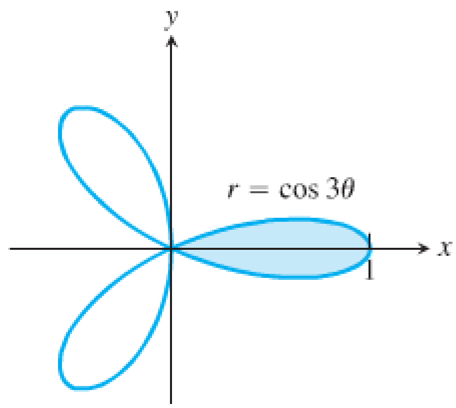
$$= \frac{a^2}{2}(3\pi)$$

$$= \frac{3}{2}\pi a^2$$

Exercise

Find the area of the region inside one leaf of the three-leaved rose $r = \cos 3\theta$

Solution



$$A = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (\cos 3\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2 3\theta\, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1+\cos 6\theta}{2}\, d\theta$$

$$= \frac{1}{4}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (1+\cos 6\theta)\, d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{1}{6}\sin 6\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$= \frac{1}{4}\left[\left(\frac{\pi}{6} + \frac{1}{6}\sin\pi\right) - \left(-\frac{\pi}{6} + \frac{1}{6}\sin(-\pi)\right)\right]$$

$$= \frac{1}{4}\left(\frac{\pi}{6} + \frac{\pi}{6}\right)$$

$$= \frac{\pi}{12}$$

Exercise

Find the area of the region inside the six-leaved rose $r^2 = 2\sin 3\theta$

Solution

$$A = \int_0^{2\pi} \frac{1}{2} r^2 d\theta$$

$$= (6)(2)\int_0^{\pi/2} \frac{1}{2} 2\sin 3\theta \, d\theta$$

$$= 12\int_0^{\pi/2} \sin 3\theta \, d\theta$$

$$= 12\left[-\frac{1}{3}\cos 3\theta\right]_0^{\pi/2}$$

$$= -4\cos\frac{3\pi}{2}$$

$$\underline{= 4}$$

Exercise

Find the area of the region inside the curve $r = \sqrt{\cos\theta}$

Solution

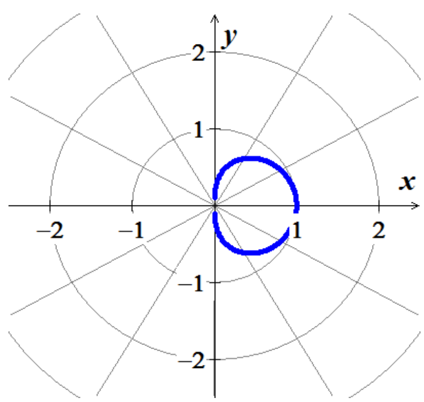
$$r = \sqrt{\cos\theta} \geq 0 \quad \Rightarrow \quad \cos\theta \geq 0 \quad \Rightarrow -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$A = \frac{1}{2}\int_0^{\pi/2} 2\left(\sqrt{\cos\theta}\right)^2 d\theta \qquad\qquad A = \int_\alpha^\beta \frac{1}{2} r^2 d\theta$$

$$= \int_0^{\pi/2} \cos\theta d\theta$$

$$= \sin\theta \Big|_0^{\pi/2}$$

$$\underline{= 1}$$



Exercise

Find the area of the region inside the right lobe of $r = \sqrt{\cos 2\theta}$

Solution

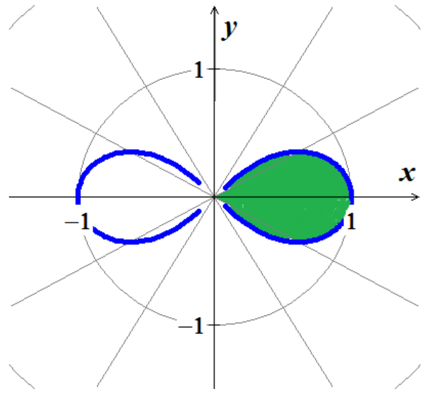
$$r = \sqrt{\cos 2\theta} \geq 0 \quad \Rightarrow \quad \cos 2\theta \geq 0 \quad \rightarrow -\frac{\pi}{2} \leq 2\theta \leq \frac{\pi}{2} \Rightarrow -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$$

$$A = \frac{1}{2}\int_0^{\pi/4} 2\left(\sqrt{\cos 2\theta}\right)^2 d\theta \qquad A = \int_\alpha^\beta \frac{1}{2} r^2 d\theta$$

$$= \int_0^{\pi/2} \cos 2\theta d\theta$$

$$= \frac{1}{2}\sin 2\theta \Big|_0^{\pi/4}$$

$$= \frac{1}{2}$$



Exercise

Find the area of the region inside the cardioid $r = 4 + 4\sin\theta$

Solution

$$A = \frac{1}{2}\int_0^{2\pi} (4 + 4\sin\theta)^2 d\theta \qquad A = \frac{1}{2}\int_\alpha^\beta r^2 d\theta$$

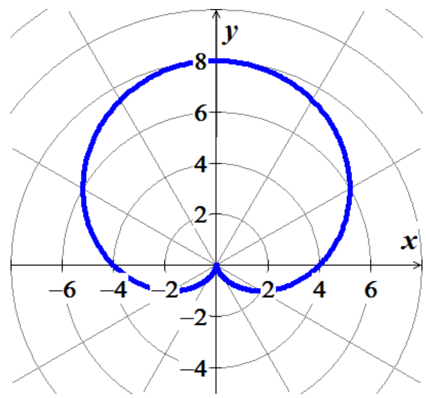
$$= 8\int_0^{2\pi} \left(1 + 2\sin\theta + \sin^2\theta\right) d\theta$$

$$= 8\int_0^{2\pi} \left(1 + 2\sin\theta + \frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= 8\left(\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi}$$

$$= 8(3\pi - 2 + 2)$$

$$= 24\pi$$



Exercise

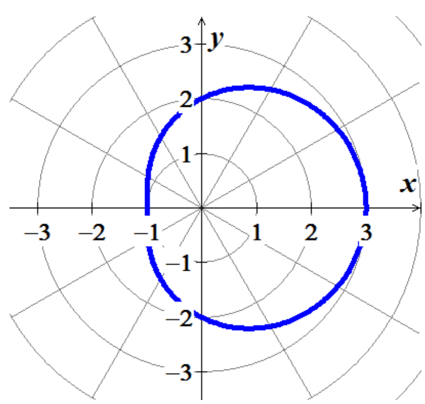
Find the area of the region inside the limaçon $r = 2 + \cos\theta$

Solution

$$A = \frac{1}{2}\int_0^{\pi} 2(2 + \cos\theta)^2 d\theta \qquad A = \frac{1}{2}\int_\alpha^\beta r^2 d\theta$$

$$= \int_0^{\pi} \left(4 + 4\cos\theta + \cos^2\theta\right) d\theta$$

$$= \int_0^{\pi} \left(4 + 4\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$



$$= \left(\frac{9}{2}\theta + 4\sin\theta + \frac{1}{4}\sin 2\theta \right) \Big|_0^{\pi}$$

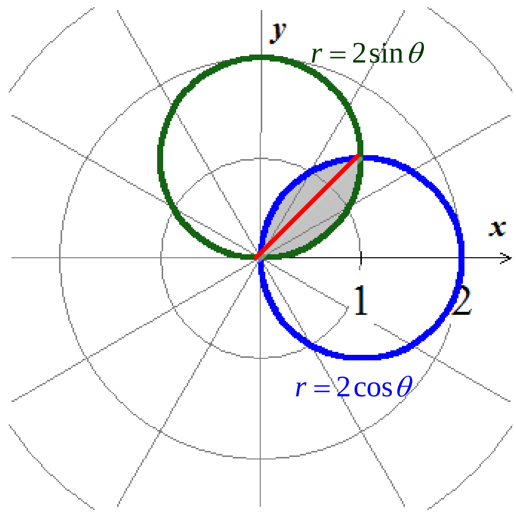
$$\boxed{= \frac{9\pi}{2}}$$

Exercise

Find the area of the region shared by the circles $r = 2\cos\theta$ *and* $r = 2\sin\theta$

Solution

$$r = 2\cos\theta = 2\sin\theta \quad \Rightarrow \quad \cos\theta = \sin\theta \to \boxed{\theta = \frac{\pi}{4}}$$



$$A = 2\int_0^{\pi/4} \frac{1}{2}(2\sin\theta)^2\, d\theta$$

$$= \int_0^{\pi/4} 4\sin^2\theta d\theta$$

$$= \int_0^{\pi/4} 2(1 - \cos 2\theta)\, d\theta$$

$$= \left[2\theta - \sin 2\theta\right]_0^{\pi/4}$$

$$= 2\frac{\pi}{4} - \sin\frac{\pi}{2} - 0$$

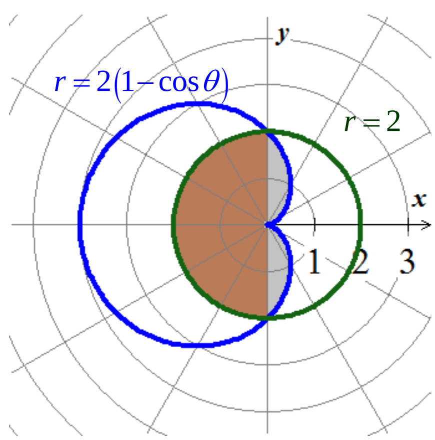
$$\boxed{= \frac{\pi}{2} - 1}$$

Exercise

Find the area of the region shared by the circle $r = 2$ and the cardioid $r = 2(1 - \cos\theta)$

Solution

$$r = 2 - 2\cos\theta = 2 \quad \to \quad \cos\theta = 0 \Rightarrow \boxed{\theta = \pm\frac{\pi}{2}}$$



$$A = \frac{1}{2}\, \textit{Area of the circle} + 2\int_0^{\pi/2} \frac{1}{2}\left[2(1 - \cos\theta)\right]^2 d\theta$$

$$= \frac{1}{2}\pi\left(2^2\right) + 4\int_0^{\pi/2} \left(1 - 2\cos\theta + \cos^2\theta\right) d\theta$$

$$= 2\pi + \int_0^{\pi/2} \left(4 - 8\cos\theta + 2 + 2\cos 2\theta\right) d\theta$$

$$= 2\pi + \int_0^{\pi/2} (6 - 8\cos\theta + 2\cos 2\theta)\, d\theta$$

$$= 2\pi + \left[6\theta - 8\sin\theta + \sin 2\theta\right]_0^{\pi/2}$$

$$= 2\pi + 3\pi - 8$$

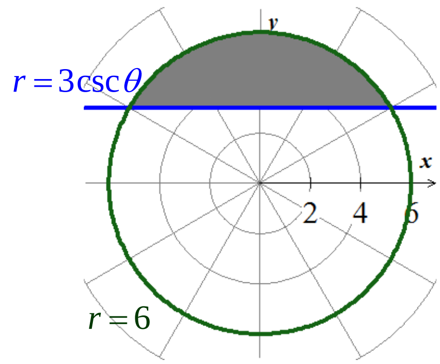
$$\underline{= 5\pi - 8}$$

Exercise

Find the area of the region inside the circle $r = 6$ above the line $r = 3\csc\theta$

Solution

$$r = 3\csc\theta = \frac{3}{\sin\theta} = 6 \to \sin\theta = \frac{3}{6} = \frac{1}{2} \quad \Rightarrow \boxed{\theta = \frac{\pi}{6}, \frac{5\pi}{6}}$$



$$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}\left[6^2 - (3\csc\theta)^2\right] d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left[36 - 9\csc^2\theta\right] d\theta$$

$$= \frac{9}{2}\left[4\theta + \cot\theta\right]_{\pi/6}^{5\pi/6}$$

$$= \frac{9}{2}\left[\left(\frac{10\pi}{3} - \sqrt{3}\right) - \left(\frac{2\pi}{3} + \sqrt{3}\right)\right]$$

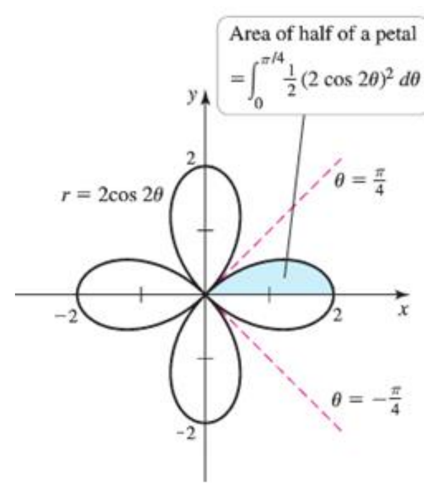
$$= \frac{9}{2}\left(\frac{8\pi}{3} - 2\sqrt{3}\right)$$

$$\underline{= 12\pi - 9\sqrt{3}}$$

Exercise

Find the area of the region in the plane enclosed by the four-leaf rose $r = f(\theta) = 2\cos 2\theta$

Solution



The curve is symmetric about the x-axis:

$$r = 2\cos(-2\theta) = 2\cos 2\theta$$

$$(r, \theta) = (r, -\theta)$$

The curve is symmetric about the y-axis:

$$-r = 2\cos 2(\pi - \theta) = -2\cos 2\theta$$

$$r = 2\cos 2\theta$$

$$(r, \theta) = (-r, \pi - \theta)$$

The graph of the rose appears to be symmetric about x- and y-axes.

$$A = 8\int_0^{\pi/4} \frac{1}{2}r^2 d\theta = 4\int_0^{2\pi} (2\cos 2\theta)^2\, d\theta$$

$$= 16\int_0^{\pi/4} \cos^2 2\theta\ d\theta$$

$$= 8\int_0^{\pi/4} (1+\cos 4\theta)\ d\theta$$

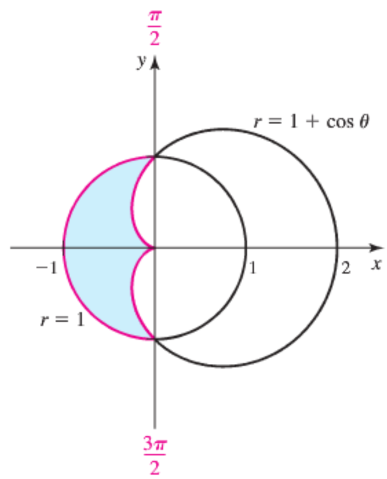
$$= 8\left(\theta + \frac{1}{4}\sin 4\theta\right)\Big|_0^{\pi/4}$$

$$= 8\left[\frac{\pi}{4} + 0 - (0+0)\right]$$

$$\underline{= 2\pi|}$$

Exercise

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 + \cos\theta$



Solution

$$A = \int_{\pi/2}^{3\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta$$

$$= 2\int_{\pi/2}^{\pi} \frac{1}{2}\left(1^2 - (1+\cos\theta)^2\right) d\theta$$

$$= \int_{\pi/2}^{\pi} \left(1 - \left(1 + 2\cos\theta + \cos^2\theta\right)\right) d\theta$$

$$= \int_{\pi/2}^{\pi} \left(-2\cos\theta - \frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \left[-2\sin\theta - \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_{\pi/2}^{\pi}$$

$$= -\frac{\pi}{2} + 2 + \frac{\pi}{4}$$

$$\underline{= 2 - \frac{\pi}{4}|}$$

Exercise

Find the area of the region inside the inner loop $r = \cos\theta - \frac{1}{2}$

Solution

$$A = \frac{1}{2}\int_0^{\pi/3} 2\left(\cos\theta - \frac{1}{2}\right)^2 d\theta \qquad\qquad A = \frac{1}{2}\int_\alpha^\beta r^2 d\theta$$

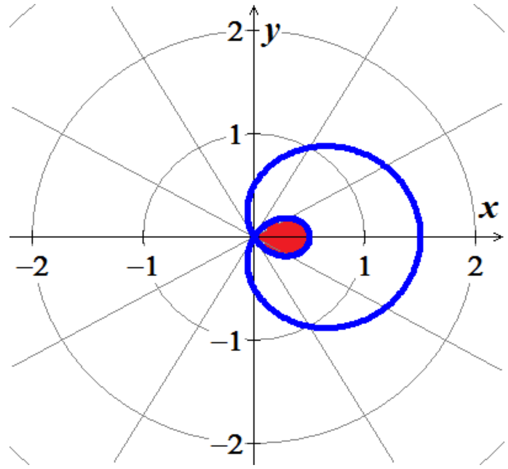
$$= \int_0^{\pi/3} \left(\cos^2\theta - \cos\theta + \frac{1}{4}\right) d\theta$$

$$= \int_0^{\pi/3} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta - \cos\theta + \frac{1}{4}\right) d\theta$$

$$= \left(\frac{3}{4}\theta + \frac{1}{4}\sin 2\theta - \sin\theta\right)\Big|_0^{\pi/3}$$

$$= \frac{\pi}{4} + \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{4} - \frac{3\sqrt{3}}{8}$$



Exercise

Find the area of the region outside the circle $r = \frac{1}{2}$ and inside the circle $r = \cos\theta$

Solution

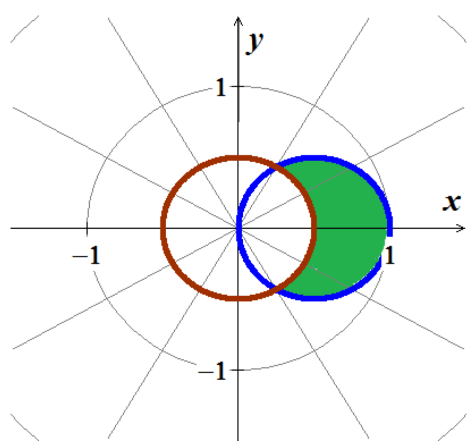
$$r = \cos\theta = \frac{1}{2} \quad \to \quad \theta = \pm\frac{\pi}{3}$$

$$A = \frac{1}{2}\int_0^{\pi/3} 2\left(\cos^2\theta - \frac{1}{4}\right) d\theta \qquad\qquad A = \frac{1}{2}\int_\alpha^\beta \left(r_2^2 - r_1^2\right) d\theta$$

$$= \int_0^{\pi/3} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta - \frac{1}{4}\right) d\theta$$

$$= \left(\frac{1}{4}\theta + \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/3}$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{8}$$

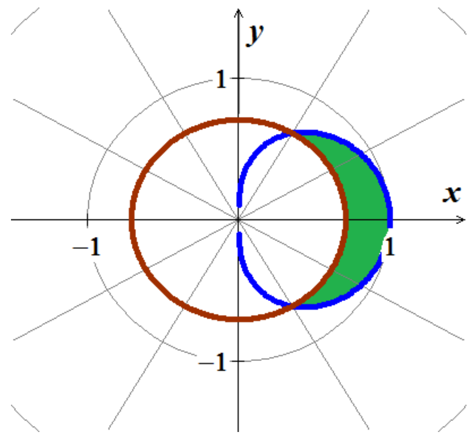


Exercise

Find the area of the region outside the circle $r = \frac{1}{\sqrt{2}}$ and inside the curve $r = \sqrt{\cos\theta}$

Solution

$$r = \sqrt{\cos\theta} = \frac{1}{\sqrt{2}} \quad \to \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \underline{\theta = \pm\frac{\pi}{3}}$$



$$A = \frac{1}{2}\int_0^{\pi/3} 2\left(\cos\theta - \frac{1}{2}\right)d\theta \qquad\qquad A = \frac{1}{2}\int_\alpha^\beta \left(r_2^2 - r_1^2\right)d\theta$$

$$= \left(\sin\theta - \frac{1}{2}\theta\right)\Big|_0^{\pi/3}$$

$$= \frac{\sqrt{3}}{2} - \frac{\pi}{6}\ \ unit^2$$

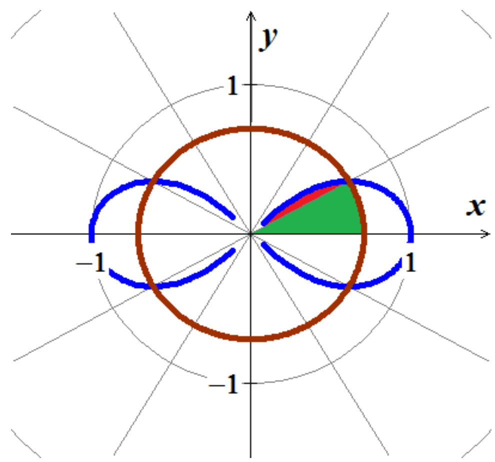
Exercise

Find the area of the region inside the circle $r = \frac{1}{\sqrt{2}}$ in QI and inside the right lobe of $r = \sqrt{\cos 2\theta}$

Solution

$$r = \sqrt{\cos 2\theta} = \frac{1}{\sqrt{2}} \quad \to \quad \cos 2\theta = \frac{1}{2} \quad \Rightarrow \underline{\theta = \frac{\pi}{6}}$$

$$\sqrt{\cos 2\theta} = 0 \quad \to \quad \cos 2\theta = \frac{\pi}{2} \quad \Rightarrow \underline{\theta = \frac{\pi}{4}}$$



$$A = A_1 + A_2$$

$$A = \frac{1}{2}\int_0^{\pi/6}\left(\frac{1}{\sqrt{2}}\right)^2 d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/4}\left(\sqrt{\cos 2\theta}\right)^2 d\theta$$

$$= \frac{1}{2}\int_0^{\pi/6}\frac{1}{2}d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/4}\cos 2\theta d\theta$$

$$= \frac{1}{4}\theta\Big|_0^{\pi/6} + \frac{1}{4}\sin 2\theta\Big|_{\pi/6}^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}}{2} - \frac{\pi}{12}\right) + \frac{1}{4}\left(1 - \frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{24} + \frac{1}{4} - \frac{\sqrt{3}}{8}\ \ unit^2$$

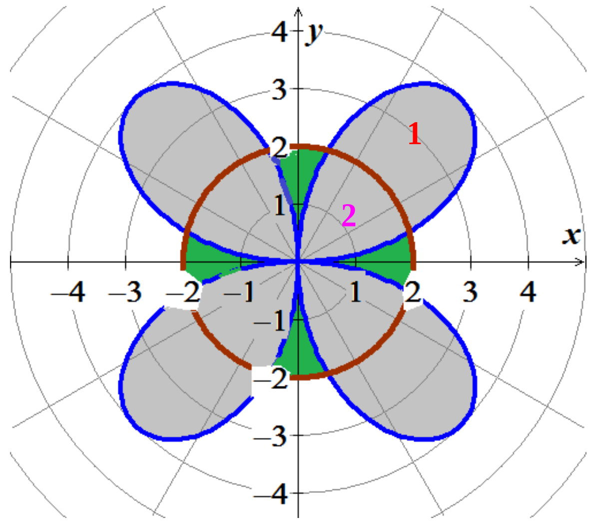
Exercise

Find the area of the region inside the rose $r = 4\sin 2\theta$ and inside the circle $r = 2$

Solution

$$r = 4\sin 2\theta = 2 \quad \to \quad \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6} \quad \Rightarrow \theta = \frac{\pi}{12}, \ \frac{5\pi}{12}$$

The area (1) inside one leaf but outside the circle is:



$$A = \frac{1}{2}\int_{\pi/12}^{5\pi/12} \left(16\sin^2 2\theta - 4\right)d\theta$$

$$= \frac{1}{2}\int_{\pi/12}^{5\pi/12} \left(8 - 8\cos 4\theta - 4\right)d\theta$$

$$= \int_{\pi/12}^{5\pi/12} \left(2 - 4\cos 4\theta\right)d\theta$$

$$= 2\theta - \sin 4\theta \Big|_{\pi/12}^{5\pi/12}$$

$$= \frac{5\pi}{6} + \frac{\sqrt{3}}{2} - \frac{\pi}{6} + \frac{\sqrt{3}}{2}$$

$$= \frac{2\pi}{3} + \sqrt{3}$$

Area inside one leaf (2) is:

$$A = \frac{1}{2}\int_{0}^{\pi/2} 16\sin^2(2\theta)\,d\theta$$

$$= \int_{0}^{\pi/2} \left(4 - 4\cos 4\theta\right)d\theta$$

$$= 4\theta - \sin 4\theta \Big|_{0}^{\pi/2}$$

$$= 2\pi$$

$$\text{Total Area } = 4\left(2\pi - \frac{2\pi}{3} - \sqrt{3}\right) = \frac{16\pi}{3} - 4\sqrt{3} \ \ unit^2$$

Exercise

Find the area of the region inside the lemniscate $r^2 = 2\sin 2\theta$ and outside the circle $r = 1$

Solution

$$r^2 = 2\sin 2\theta = 1 \quad \to \quad \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}, \ \frac{5\pi}{6}$$

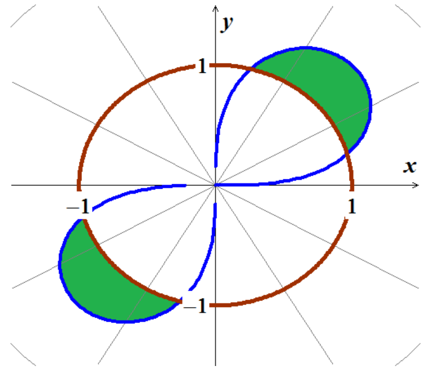
$$\Rightarrow \theta = \frac{\pi}{12}, \ \frac{5\pi}{12}$$

$$A = \frac{1}{2}2\int_{\pi/12}^{5\pi/12} (2\sin 2\theta - 1)\, d\theta$$

$$= -\cos 2\theta - \theta \Big|_{\pi/12}^{5\pi/12}$$

$$= \frac{\sqrt{3}}{2} - \frac{5\pi}{12} + \frac{\sqrt{3}}{2} + \frac{\pi}{12}$$

$$= \sqrt{3} - \frac{\pi}{3} \ \ unit^2$$



Exercise

Find the area of the region inside all the leaves of the rose $r = 3\sin 2\theta$

Solution

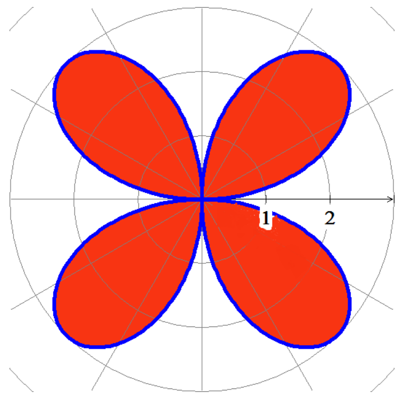
$$A = \frac{1}{2}(8)\int_0^{\pi/4} 9\sin^2 2\theta \, d\theta \qquad \left(\text{Using symmetry } \tfrac{1}{2} - leaf\ \right)$$

$$= 18\int_0^{\pi/4} (1 - \cos 4\theta)\, d\theta$$

$$= 18\left(\theta - \frac{1}{4}\sin 4\theta\right)\Big|_0^{\pi/4}$$

$$= 18\left(\frac{\pi}{4}\right)$$

$$= \frac{9\pi}{2} \ \ unit^2$$



Exercise

Find the area of the region inside one leaf of the rose $r = \cos 5\theta$

Solution

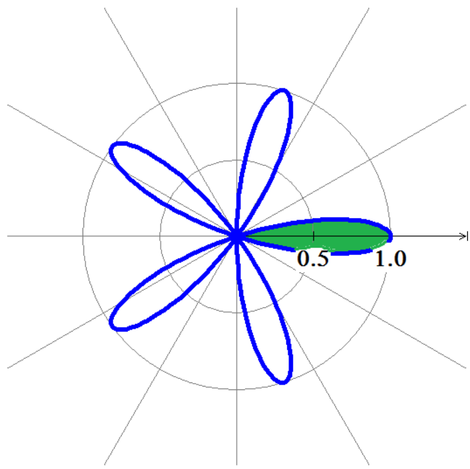
$$0 \le 5\theta \le \frac{\pi}{2} \ \ \rightarrow \ \ 0 \le \theta \le \frac{\pi}{10}$$

$$A = \frac{1}{2}(2)\int_0^{\pi/10} \cos^2 5\theta \, d\theta \quad \left(\textbf{Using symmetry } \tfrac{1}{2} - leaf\ \right)$$

$$= \frac{1}{2}\int_0^{\pi/10} (1 + \cos 10\theta)\, d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{10}\sin 10\theta\right)\Big|_0^{\pi/10}$$

$$= \frac{\pi}{20} \ \ unit^2$$

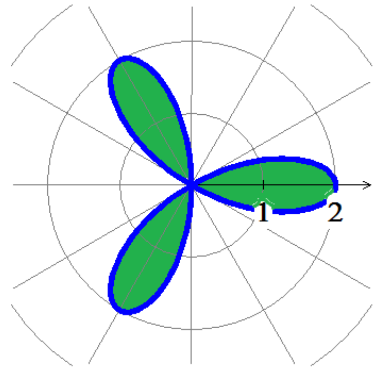


Exercise

Find the area of the region of a complete three-leaf rose $r = 2\cos 3\theta$

Solution

$$0 \le 3\theta \le \frac{\pi}{2} \quad \rightarrow \quad 0 \le \theta \le \frac{\pi}{6}$$



$$A = (6)\frac{1}{2}\int_0^{\pi/6} 4\cos^2 3\theta \, d\theta \qquad \left(\textit{Using symmetry } \tfrac{1}{2} - \textit{leaf}\right)$$

$$= 6\int_0^{\pi/6} (1 + \cos 6\theta) \, d\theta$$

$$= 6\left(\theta + \frac{1}{6}\sin 6\theta\right)\Big|_0^{\pi/6}$$

$$= \pi \ \ unit^2$$

Exercise

Find the area of the region inside the rose $r = 4\cos 2\theta$ and outside the circle $r = 2$

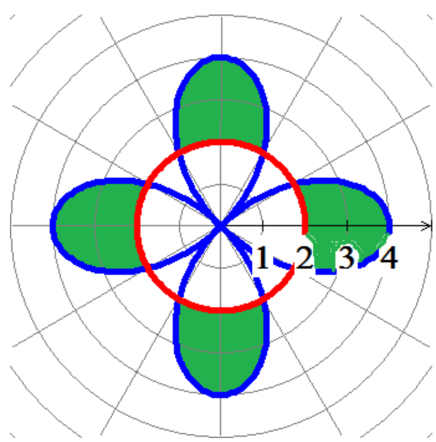
Solution

$$r = 4\cos 2\theta = 2 \quad \rightarrow \quad \cos 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{3}$$

$$0 \le \theta \le \frac{\pi}{6}$$

$$A = (8)\frac{1}{2}\int_0^{\pi/6} \left(16\cos^2 2\theta - 4\right) d\theta \qquad \left(\textit{Using symmetry } \tfrac{1}{2} - \textit{leaf}\right)$$

$$= 4\int_0^{\pi/6} (8 + 8\cos 4\theta - 4) \, d\theta$$



$$= 4\int_0^{\pi/6} (4 + 8\cos 4\theta) \, d\theta$$

$$= 4(4\theta + 2\sin 4\theta)\Big|_0^{\pi/6}$$

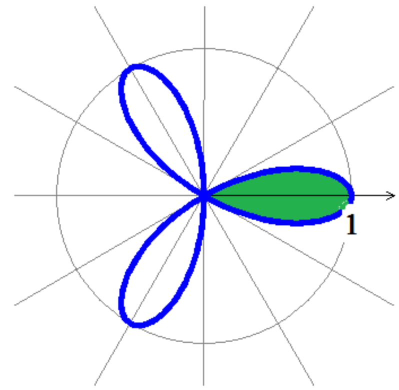
$$= 4\left(\frac{2\pi}{3} + \sqrt{3}\right)$$

$$= \frac{8\pi}{3} + 4\sqrt{3} \ \ unit^2$$

Exercise

Find the area of the region inside one leave of $r = \cos 3\theta$

Solution



$$0 \le 3\theta \le \frac{\pi}{2} \quad \to \quad 0 \le \theta \le \frac{\pi}{6}$$

$$A = (2)\frac{1}{2}\int_0^{\pi/6} \cos^2 3\theta \; d\theta \qquad \left(\textit{Using symmetry } \frac{1}{2} - \textit{leaf}\ \right)$$

$$= \frac{1}{2}\int_0^{\pi/6} (1+\cos 6\theta)\; d\theta$$

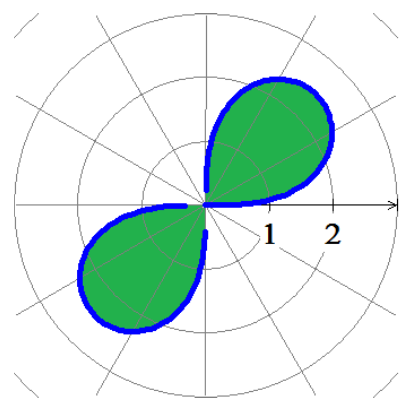
$$= \frac{1}{2}\left(\theta + \frac{1}{6}\sin 6\theta\right)\Big|_0^{\pi/6}$$

$$\boxed{= \frac{\pi}{12}\ \ unit^2}$$

Exercise

Find the area of the region bounded by the lemniscate $r^2 = 6\sin 2\theta$

Solution



$$A = (2)\frac{1}{2}\int_0^{\pi/2} 6\sin 2\theta \; d\theta \qquad (\textit{Using symmetr})$$

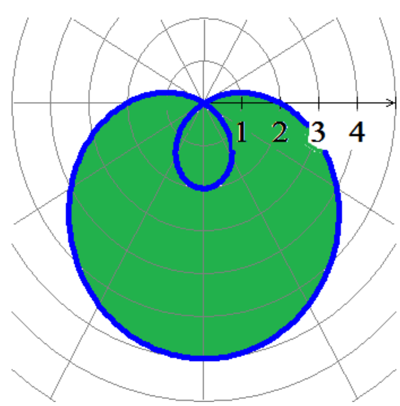
$$= 6\left(-\frac{1}{2}\cos 2\theta\right)\Big|_0^{\pi/2}$$

$$\boxed{= 6\ \ unit^2}$$

Exercise

Find the area of the region bounded by the limaçon $r = 2 - 4\sin\theta$

Solution



$$2 - 4\sin\theta = 0 \ \to \ \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$$

$$A = (2)\frac{1}{2}\int_{-\pi/2}^{\pi/6} (2-4\sin\theta)^2 \; d\theta$$

$$= \int_{-\pi/2}^{\pi/6} \left(4 - 16\sin\theta + 16\sin^2\theta\right)\; d\theta$$

$$= \int_{-\pi/2}^{\pi/6} (4 - 16\sin\theta + 8 - 8\cos 2\theta)\; d\theta$$

$$= \left(12\theta + 16\cos\theta - 4\sin 2\theta\right)\Big|_{-\pi/2}^{\pi/6}$$

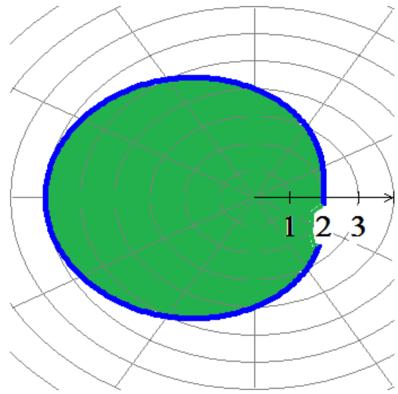
$$= 2\pi + 8\sqrt{3} - 2\sqrt{3} + 6\pi$$

$$\underline{= 8\pi + 6\sqrt{3}\ \ unit^2}$$

Exercise

Find the area of the region bounded by the limaçon $r = 4 - 2\cos\theta$

Solution



$$A = \frac{1}{2}\int_0^{2\pi} \left(4 - 2\cos\theta\right)^2\ d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(16 - 16\cos\theta + 4\cos^2\theta\right)\ d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(16 - 16\cos\theta + 2 + 2\cos 2\theta\right)\ d\theta$$

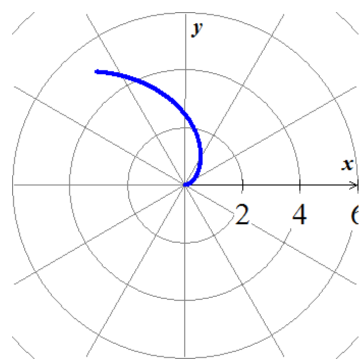
$$= \frac{1}{2}\left(18\theta - 16\sin\theta - \sin 2\theta\right)\Big|_0^{2\pi}$$

$$\underline{= 18\pi\ \ unit^2}$$

Exercise

Find the length of the spiral $r = \theta^2, \quad 0 \le \theta \le \sqrt{5}$

Solution



$$r = \theta^2 \quad \Rightarrow \frac{dr}{d\theta} = 2\theta$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{\theta^4 + 4\theta^2} = |\theta|\sqrt{\theta^2 + 4}$$

$$L = \int_0^{\sqrt{5}} \theta\sqrt{\theta^2 + 4}\ d\theta \qquad\qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$= \frac{1}{2}\int_4^9 u^{1/2}\ du \qquad\qquad u = \theta^2 + 4 \quad\rightarrow\quad du = 2\theta d\theta \;\Rightarrow\; \boxed{\theta d\theta = \frac{1}{2}du} \quad \begin{cases} \theta = \sqrt{5} & u = 9 \\ \theta = 0 & u = 4 \end{cases}$$

$$= \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_4^9$$

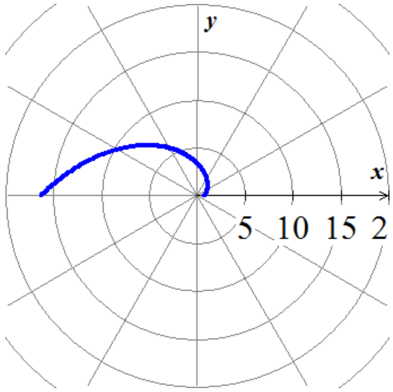
$$= \frac{1}{9}\left(9^{3/2} - 4^{3/2}\right)$$

$$\underline{= \frac{19}{3}\ \ unit}$$

Exercise

Find the length of the spiral $r = \frac{e^{\theta}}{\sqrt{2}}, \quad 0 \le \theta \le \pi$

Solution



$$r = \frac{e^{\theta}}{\sqrt{2}} \quad \Rightarrow \frac{dr}{d\theta} = \frac{1}{\sqrt{2}} e^{\theta}$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{\frac{1}{2}e^{2\theta} + \frac{1}{2}e^{2\theta}} = \sqrt{e^{2\theta}} = e^{\theta}$$

$$L = \int_0^{\pi} e^{\theta}\, d\theta \qquad\qquad L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \left[e^{\theta} \right]_0^{\pi}$$

$$\underline{= e^{\pi} - 1}$$

Exercise

Find the length of the curve $r = a\sin^2\left(\frac{\theta}{2}\right), \quad 0 \le \theta \le \pi, \quad a > 0$

Solution

$$r = a\sin^2\left(\frac{\theta}{2}\right) \quad \Rightarrow \frac{dr}{d\theta} = a\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2\sin^4\left(\frac{\theta}{2}\right) + a^2\sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)}$$

$$= a\sin\left(\frac{\theta}{2}\right)\sqrt{\sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right)} \qquad\qquad \sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right) = 1$$

$$= a\left|\sin\left(\frac{\theta}{2}\right)\right|$$

$$L = \int_0^{\pi} a\sin\left(\frac{\theta}{2}\right)\, d\theta \qquad\qquad L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \left[-2a\cos\left(\frac{\theta}{2}\right) \right]_0^{\pi}$$

$$= -2a\left(\cos\left(\frac{\pi}{2}\right) - \cos 0\right)$$

$$\underline{= 2a}$$

Exercise

Find the length of the parabolic segment $r = \dfrac{6}{1+\cos\theta}, \quad 0 \le \theta \le \dfrac{\pi}{2}$

Solution

$$r = \frac{6}{1+\cos\theta} \quad \Rightarrow \frac{dr}{d\theta} = \frac{6\sin\theta}{(1+\cos\theta)^2}$$

$$\begin{aligned}
\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} &= \sqrt{\frac{36}{(1+\cos\theta)^2} + \frac{36\sin^2\theta}{(1+\cos\theta)^4}} \\
&= \frac{6}{|1+\cos\theta|}\sqrt{1 + \frac{\sin^2\theta}{(1+\cos\theta)^2}} \\
&= \frac{6}{|1+\cos\theta|}\sqrt{\frac{1+2\cos\theta+\cos^2\theta+\sin^2\theta}{(1+\cos\theta)^2}} \\
&= \frac{6}{(1+\cos\theta)^2}\sqrt{2+2\cos\theta} \\
&= \frac{6\sqrt{2}}{(1+\cos\theta)^2}(1+\cos\theta)^{1/2} \\
&= \frac{6\sqrt{2}}{(1+\cos\theta)^{3/2}}
\end{aligned}$$

$$L = \int_0^{\pi/2} \frac{6\sqrt{2}}{(1+\cos\theta)^{3/2}}\, d\theta \qquad\qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 6\sqrt{2}\int_0^{\pi/2} \frac{d\theta}{\left(2\cos^2\frac{\theta}{2}\right)^{3/2}}$$

$$= 6\sqrt{2}\int_0^{\pi/2} \frac{d\theta}{2^{3/2}\cos^3\frac{\theta}{2}}$$

$$= \frac{6}{2}\int_0^{\pi/2} \sec^3\frac{\theta}{2}d\theta$$

$$= 3\int_0^{\pi/2} \sec^3\frac{\theta}{2}d\theta \qquad\qquad u = \frac{\theta}{2} \to du = \frac{1}{2}d\theta \quad \begin{cases} \theta = \frac{\pi}{2} & u = \frac{\pi}{4} \\ \theta = 0 & u = 0 \end{cases}$$

$$= 6\int_0^{\pi/4} \sec^3 u\ du$$

$$= 6\left(\left[\tfrac{1}{2}\sec u \tan u\right]_0^{\pi/4} + \tfrac{1}{2}\int_0^{\pi/4} \sec u \; du\right)$$

$$= 6\left(\tfrac{1}{2}\left(\sqrt{2}(1) - 0\right) + \tfrac{1}{2}\left[\ln\left|\sec u + \tan u\right|\right]_0^{\pi/4}\right)$$

$$= 6\left(\frac{\sqrt{2}}{2} + \tfrac{1}{2}\left(\ln\left|\sqrt{2}+1\right| - \ln 1\right)\right)$$

$$= 3\sqrt{2} + 3\ln\left(\sqrt{2}+1\right) \;\; unit$$

Exercise

Find the length of the curve $r = \cos^3\left(\frac{\theta}{3}\right), \quad 0 \le \theta \le \frac{\pi}{4}$

Solution

$$r = \cos^3\left(\frac{\theta}{3}\right) \;\; \Rightarrow \frac{dr}{d\theta} = -\cos^2\left(\frac{\theta}{3}\right)\sin\left(\frac{\theta}{3}\right)$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{\cos^6\left(\frac{\theta}{3}\right) + \cos^4\left(\frac{\theta}{3}\right)\sin^2\left(\frac{\theta}{3}\right)}$$

$$= \left|\cos^2\left(\frac{\theta}{3}\right)\right|\sqrt{\cos^2\left(\frac{\theta}{3}\right) + \sin^2\left(\frac{\theta}{3}\right)}$$

$$= \cos^2\left(\frac{\theta}{3}\right)$$

$$L = \int_0^{\pi/4} \cos^2\left(\frac{\theta}{3}\right) d\theta \qquad\qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \left(1 + \cos\frac{2\theta}{3}\right) d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{3}{2}\sin\frac{2\theta}{3}\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{3}{2}\sin\frac{\pi}{6} - 0\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{3}{4}\right)$$

$$= \frac{\pi}{8} + \frac{3}{8} \;\; unit$$

Exercise

Find the length of the curve $r = \sqrt{1+\sin 2\theta}, \quad 0 \le \theta \le \pi\sqrt{2}$

Solution

$$r = \sqrt{1+\sin 2\theta} \quad \Rightarrow \frac{dr}{d\theta} = \frac{1}{2}(1+\sin 2\theta)^{-1/2}(2\cos 2\theta) = \cos 2\theta(1+\sin 2\theta)^{-1/2}$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{1+\sin 2\theta + \cos^2 2\theta(1+\sin 2\theta)^{-1}}$$

$$= \sqrt{1+\sin 2\theta + \frac{\cos^2 2\theta}{1+\sin 2\theta}}$$

$$= \sqrt{\frac{1+2\sin 2\theta + \sin^2 2\theta + \cos^2 2\theta}{1+\sin 2\theta}}$$

$$= \sqrt{\frac{2+2\sin 2\theta}{1+\sin 2\theta}}$$

$$= \sqrt{\frac{2(1+\sin 2\theta)}{1+\sin 2\theta}}$$

$$= \sqrt{2}$$

$$L = \int_0^{\pi\sqrt{2}} \sqrt{2}\, d\theta \qquad\qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \sqrt{2}\theta \Big|_0^{\pi\sqrt{2}}$$

$$= \sqrt{2}\left(\pi\sqrt{2} - 0\right)$$

$$\boxed{= 2\pi \ unit}$$

Exercise

Find the length of $\quad r = 8 \quad 0 \le \theta \le 2\pi$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{64+0} = 8$$

$$L = \int_0^{2\pi} 8\, d\theta \qquad\qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$\boxed{= 16\pi \ \ unit}$$

Exercise

Find the length of $r = a \quad 0 \le \theta \le 2\pi$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2 + 0} \; \underline{= a}$$

$$L = \int_0^{2\pi} a \, d\theta \qquad\qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$\underline{= 2\pi a \;\; unit}$$

Exercise

Find the length of $r = 4\sin\theta \quad 0 \le \theta \le \pi$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{16\sin^2\theta + 16\cos^2\theta}$$

$$= 4\sqrt{\sin^2\theta + \cos^2\theta} \qquad \sin^2\theta + \cos^2\theta = 1$$

$$\underline{= 4}$$

$$L = \int_0^{\pi} 4 \, d\theta \qquad\qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$\underline{= 4\pi \;\; unit}$$

Exercise

Find the length of $r = 2a\cos\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{4a^2\cos^2\theta + 4a^2\sin^2\theta} \; \underline{= 2a} \qquad \sin^2\theta + \cos^2\theta = 1$$

$$L = 2a\int_{-\pi/2}^{\pi/2} d\theta \qquad\qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 2a\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$\underline{= 2\pi a \;\; unit}$$

Exercise

Find the length of $\quad r = 1 + \sin\theta \quad 0 \le \theta \le 2\pi$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{(1+\sin\theta)^2 + \cos^2\theta}$$

$$= \sqrt{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta} \qquad \sin^2\theta + \cos^2\theta = 1$$

$$= \sqrt{2 + 2\sin\theta}$$

$$L = \sqrt{2}\int_0^{2\pi} \sqrt{1+\sin\theta}\ d\theta \qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$= \sqrt{2}\int_0^{2\pi} \sqrt{1+\sin\theta}\,\frac{\sqrt{1-\sin\theta}}{\sqrt{1-\sin\theta}}\ d\theta \qquad \cos\theta = \pm\sqrt{1-\sin^2\theta}$$

$$= 2\sqrt{2}\int_{\pi/2}^{3\pi/2} \frac{-\cos\theta}{\sqrt{1-\sin\theta}}\ d\theta \qquad \frac{\pi}{2} < \theta < \frac{3\pi}{2} \ \rightarrow \cos\theta < 0$$

$$= 2\sqrt{2}\int_{\pi/2}^{3\pi/2} (1-\sin\theta)^{-1/2}\ d(1-\sin\theta)$$

$$= 4\sqrt{2}\sqrt{1-\sin\theta}\,\Big|_{\pi/2}^{3\pi/2}$$

$$= 4\sqrt{2}\left(\sqrt{2} - 0\right)$$

$$= 8 \ \ unit$$

Exercise

Find the length of $\quad r = 8(1+\cos\theta) \quad 0 \le \theta \le 2\pi$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{64(1+\cos\theta)^2 + 64\sin^2\theta}$$

$$= 8\sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta} \qquad \sin^2\theta + \cos^2\theta = 1$$

$$= 8\sqrt{2 + 2\cos\theta}$$

$$L = 8\sqrt{2}\int_0^{2\pi} \sqrt{1+\cos\theta}\ d\theta \qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$= 8\sqrt{2}\int_0^{2\pi} \sqrt{1+\cos\theta}\,\frac{\sqrt{1-\cos\theta}}{\sqrt{1-\cos\theta}}\ d\theta \qquad \sin\theta = \pm\sqrt{1-\cos^2\theta}$$

$$= 16\sqrt{2}\int_0^{\pi} \frac{\sin\theta}{\sqrt{1-\cos\theta}}\, d\theta \qquad 0 < \theta < \pi \quad \to \sin\theta > 0$$

$$= 16\sqrt{2}\int_0^{\pi} (1-\cos\theta)^{-1/2}\, d(1-\cos\theta)$$

$$= 32\sqrt{2}\sqrt{1-\cos\theta}\Big|_0^{\pi}$$

$$= 32\sqrt{2}\left(\sqrt{2}-0\right)$$

$$= 64 \ \ unit$$

Exercise

Find the length of $\qquad r = 2\theta \quad 0 \le \theta \le \frac{\pi}{2}$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{4\theta^2 + 4} = 2\sqrt{1+\theta^2}$$

$$L = 2\int_0^{\pi/2} \sqrt{1+\theta^2}\ d\theta \qquad L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$= 2\int_0^{\pi/2} \sec^3\alpha\ d\alpha \qquad \theta = \tan\alpha \quad \sqrt{\theta^2+1} = \sec\alpha \quad d\theta = \sec^2\alpha\ d\alpha$$

$$= 2\left[\frac{1}{2}\sec\alpha\tan\alpha + \frac{1}{2}\ln\left|\sec\alpha + \tan\alpha\right|\right]_0^{\pi/2} \qquad \int \sec^3 x\ dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right|$$

$$= 2\left[\frac{1}{2}\theta\sqrt{1+\theta^2} + \frac{1}{2}\ln\left|\sqrt{1+\theta^2} + \theta\right|\right]_0^{\pi/2}$$

$$= \frac{\pi}{2}\sqrt{1+\frac{\pi^2}{4}} + \ln\left(\sqrt{1+\frac{\pi^2}{4}} + \frac{\pi}{2}\right)\ \ unit \qquad \approx 4.158$$

Exercise

Find the length of $\qquad r = \sec\theta \quad 0 \le \theta \le \frac{\pi}{3}$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{\sec^2\theta + \sec^2\theta\tan^2\theta}$$

$$= \sec\theta\sqrt{1+\tan^2\theta}$$

$$= \sec^2 \theta$$

$$L = \int_0^{\pi/3} \sec^2 \theta \, d\theta \qquad\qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

$$= \tan \theta \Big|_0^{\pi/3}$$

$$\boxed{= \sqrt{3} \ \ unit}$$

Exercise

Find the length of $\quad r = \frac{1}{\theta} \quad \pi \le \theta \le 2\pi$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{\frac{1}{\theta^2} + \frac{1}{\theta^4}}$$

$$= \frac{1}{\theta^2}\sqrt{\theta^2 + 1}$$

$$L = \int_\pi^{2\pi} \frac{1}{\theta^2}\sqrt{\theta^2 + 1} \, d\theta \qquad\qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

$$= \sinh^{-1}\theta - \frac{\sqrt{1+\theta^2}}{\theta}\Bigg|_\pi^{2\pi}$$

$$\boxed{= \sinh^{-1} 2\pi - \frac{\sqrt{1+4\pi^2}}{2\pi} - \sinh^{-1}\pi + \frac{\sqrt{1+\pi^2}}{\pi}}$$

$$= 2.5376 - 1.01259 - 1.8623 + 1.04944 \ \ \boxed{\approx 0.71215}$$

Exercise

Find the length of $\quad r = e^\theta \quad 0 \le \theta \le \pi$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{e^{2\theta} + e^{2\theta}} \ \boxed{= \sqrt{2}e^\theta}$$

$$L = \sqrt{2}\int_0^\pi e^\theta \, d\theta \qquad\qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

$$\boxed{= \sqrt{2}\left(e^\pi - 1\right) \ \ unit}$$

Exercise

Find the surface area bounded by $r = 6\cos\theta \quad 0 \le \theta \le \frac{\pi}{2}$ *revolving about Polar axis*

Solution

$$\sqrt{r^2 + (r')^2} = \sqrt{36\cos^2\theta + 36\sin^2\theta} \ \underline{= 6|}$$

$$S = 2\pi\int_0^{\pi/2} 6\cos\theta\sin\theta(6)\ d\theta \qquad\qquad S = 2\pi\int_\alpha^\beta f(\theta)\sin\theta\sqrt{(f(\theta))^2 + (f'(\theta))^2}\ d\theta$$

$$= 36\pi\int_0^{\pi/2} \sin 2\theta\ d\theta \qquad\qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= -18\pi\cos 2\theta\Big|_0^{\pi/2}$$

$$= -18\pi(-1-1)$$

$$\underline{= 36\pi \ \ unit|}$$

Exercise

Find the surface area bounded by $r = a\cos\theta \quad 0 \le \theta \le \frac{\pi}{2}$ *revolving about* $\theta = \frac{\pi}{2}$

Solution

$$\sqrt{r^2 + (r')^2} = \sqrt{a^2\cos^2\theta + a^2\sin^2\theta} \ \underline{= a|}$$

$$S = 2\pi\int_0^{\pi/2} a^2\cos^2\theta\ d\theta \qquad\qquad S = 2\pi\int_\alpha^\beta f(\theta)\cos\theta\sqrt{(f(\theta))^2 + (f'(\theta))^2}\ d\theta$$

$$= a^2\pi\int_0^{\pi/2} (1+\cos 2\theta)\ d\theta \qquad\qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \pi a^2\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/2}$$

$$= \pi a^2\left(\frac{\pi}{2}\right)$$

$$\underline{= \frac{1}{2}\pi^2 a^2 \ \ unit|}$$

Exercise

Find the surface area bounded by $r = e^{a\theta} \quad 0 \le \theta \le \frac{\pi}{2}$ *revolving about* $\theta = \frac{\pi}{2}$

Solution

$$\sqrt{r^2 + (r')^2} = \sqrt{e^{2a\theta} + a^2e^{2a\theta}} \ \underline{= e^{a\theta}\sqrt{1+a^2}|}$$

$$S = 2\pi\sqrt{1+a^2}\int_0^{\pi/2} e^{a\theta}\cos\theta\left(e^{a\theta}\right)\, d\theta \qquad S = 2\pi\int_\alpha^\beta f(\theta)\cos\theta\sqrt{(f(\theta))^2+(f'(\theta))^2}\, d\theta$$

$$= 2\pi\sqrt{1+a^2}\int_0^{\pi/2} e^{2a\theta}\cos\theta\, d\theta$$

$$\int e^{2a\theta}\cos\theta\, d\theta = e^{2a\theta}\sin\theta + 2ae^{2a\theta}\cos\theta - 4a^2\int e^{2a\theta}\cos\theta\, d\theta$$

$$\left(1+4a^2\right)\int e^{2a\theta}\cos\theta\, d\theta = e^{2a\theta}(\sin\theta + 2a\cos\theta)$$

		$\int \cos\theta$
+	$e^{2a\theta}$	$\sin\theta$
−	$2ae^{2a\theta}$	$-\cos\theta$
+	$4a^2e^{2a\theta}$	$-\int\cos\theta$

$$= \frac{2\pi\sqrt{1+a^2}}{1+4a^2}e^{2a\theta}(\sin\theta+2a\cos\theta)\Big|_0^{\pi/2}$$

$$= \boxed{\frac{2\pi\sqrt{1+a^2}}{1+4a^2}\left(e^{a\pi}-2a\right)}$$

Exercise

Find the area surface bounded by $r = a(1+\cos\theta) \quad 0 \le \theta \le \pi \quad$ *revolving about polar axis*

Solution

$$\sqrt{r^2+(r')^2} = \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}$$

$$= a\sqrt{\left(1+2\cos\theta+\cos^2\theta\right)+\sin^2\theta}$$

$$= \boxed{a\sqrt{2+2\cos\theta}}$$

$$S = 2a^2\pi\sqrt{2}\int_0^{\pi/2}(1+\cos\theta)\sin\theta\left(\sqrt{1+\cos\theta}\right)\, d\theta \qquad S = 2\pi\int_\alpha^\beta f(\theta)\sin\theta\sqrt{(f(\theta))^2+(f'(\theta))^2}\, d\theta$$

$$= -2a^2\pi\sqrt{2}\int_0^{\pi/2}(1+\cos\theta)^{3/2}\, d(1+\cos\theta)$$

$$= -\frac{4\sqrt{2}}{5}a^2\pi(1+\cos\theta)^{5/2}\Big|_0^{\pi/2}$$

$$= -\frac{4\sqrt{2}}{5}a^2\pi\left(1-1-2^{5/2}\right)$$

$$= \frac{4\sqrt{2}}{5}a^2\pi\left(4\sqrt{2}\right)$$

$$= \boxed{\frac{32}{5}\pi a^2}$$

Exercise

Find the surface area of the torus generated by revolving the circle given by $r = 2$ about the line $r = 5\sec\theta$

Solution

$$\sqrt{r^2 + (r')^2} = \sqrt{4+0} \; \underline{= 2|}$$

$$S = 4\pi \int_0^{2\pi} \left(\frac{5}{\cos\theta} - 2\right)\cos\theta \, d\theta \qquad S = 2\pi \int_\alpha^\beta \left(r_2 - r\right)\cos\theta\sqrt{r^2 + (r')^2} \, d\theta$$

$$= 4\pi \int_0^{2\pi} (5 - 2\cos\theta) \, d\theta$$

$$= 4\pi (5\theta - 2\sin\theta) \Big|_0^{2\pi}$$

$$= 4\pi (10\pi)$$

$$\underline{= 40\pi^2|}$$

Exercise

Find the surface area of the torus generated by revolving the circle given by $r = a$ about the line $r = b\sec\theta$, where $0 < a < b$

Solution

$$\sqrt{r^2 + (r')^2} = \sqrt{a^2+0} \; \underline{= a|}$$

$$S = 2\pi a \int_0^{2\pi} \left(\frac{b}{\cos\theta} - a\right)\cos\theta \, d\theta \qquad S = 2\pi \int_\alpha^\beta \left(r_2 - r\right)\cos\theta\sqrt{r^2 + (r')^2} \, d\theta$$

$$= 2\pi a \int_0^{2\pi} (b - a\cos\theta) \, d\theta$$

$$= 2\pi a (b\theta - a\sin\theta) \Big|_0^{2\pi}$$

$$= 2\pi a (2b\pi)$$

$$\underline{= 4\pi^2 ab|}$$

Exercise

The curve represented by the equation $r = ae^{b\theta}$, where a and b are constants, is called a logarithmic spiral. The figure shows the graph of $r = e^{\theta/6}$. $-2\pi \le \theta \le 2\pi$. Find the area of the shaded region.

Solution

$r = e^{\theta/6}$

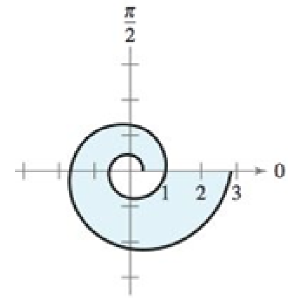
$$A = \frac{1}{2}\int_0^{2\pi} \left(e^{\theta/6}\right)^2 d\theta - \frac{1}{2}\int_{-2\pi}^{0} \left(e^{\theta/6}\right)^2 d\theta$$

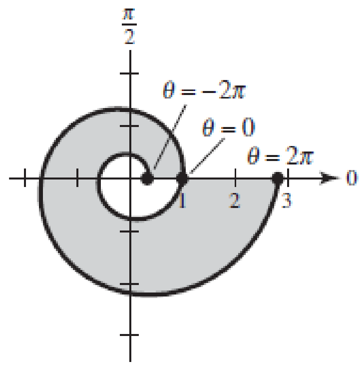
$$= \frac{1}{2}\int_0^{2\pi} e^{\theta/3}\, d\theta - \frac{1}{2}\int_{-2\pi}^{0} e^{\theta/3}\, d\theta$$

$$= \frac{3}{2}e^{\theta/3}\Big|_0^{2\pi} - \frac{3}{2}e^{\theta/3}\Big|_{-2\pi}^{0}$$

$$= \frac{3}{2}\left(e^{2\pi/3} - 1\right) - \frac{3}{2}\left(1 - e^{-2\pi/3}\right)$$

$$= \frac{3}{2}\left(e^{2\pi/3} - 2 + e^{-2\pi/3}\right) \approx 9.3655$$





Exercise

The larger circle in the figure is the graph of $r = 1$.
Find the polar equation of the smaller circle such that the shaded regrions are equal.

Solution

Small circle: $r = a\cos\theta$ with center at $\left(1\cos\frac{\pi}{4},\ 0\right) = \left(\frac{\sqrt{2}}{2},\ 0\right)$

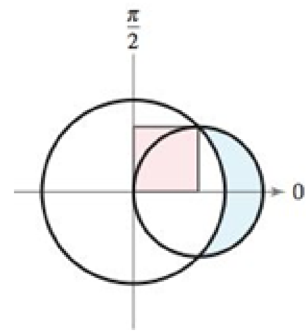
$$A = 2\frac{1}{2}\int_0^{\pi/4} \left[(a\cos\theta)^2 - 1\right] d\theta$$

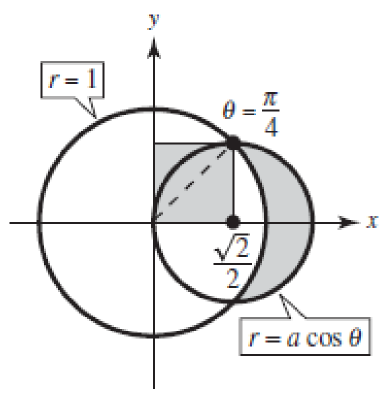
$$= \int_0^{\pi/4} \left(a^2\cos^2\theta - 1\right) d\theta$$

$$= \int_0^{\pi/4} \left(\frac{a^2}{2}(1 + \cos 2\theta) - 1\right) d\theta$$

$$= \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) - \theta\Big|_0^{\pi/4}$$

$$= \frac{a^2}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right) - \frac{\pi}{4}$$



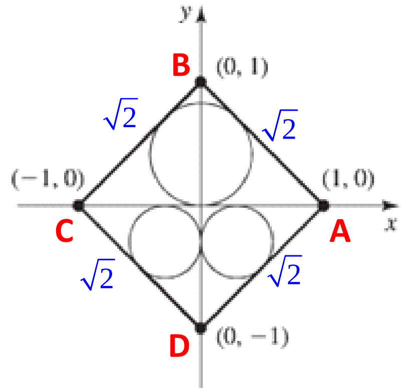


Exercise

Find equations of the circles in the figure.
Determine whether the combined area of the circles is greater than or less than the area of the region inside the square but outside the circles.

Solution

$$Area(\Delta ABC) = \frac{1}{2}bh = \frac{1}{2}(2)(1) = 1$$



The radius of a circle inscribed in the triangle *ABC* is

For the bigger circle, the radius is:

$$R = \frac{Area}{\frac{1}{2}\,perimeter} = \frac{2}{2+\sqrt{2}+\sqrt{2}} = \frac{1}{1+\sqrt{2}}$$

$$Area(\Delta AOD) = Area(\Delta COD) = \frac{1}{2}(1)(1) = \frac{1}{2}$$

The radius of the small circle inscribed in the triangle *COD & AOD* is

$$R_S = \frac{Area}{\frac{1}{2}\left(1+1+\sqrt{2}\right)} = \frac{1}{2+\sqrt{2}}$$

The area inside the 3 circles is:

$$Area = \pi\left(\frac{1}{1+\sqrt{2}}\right)^2 + 2\pi\left(\frac{1}{2+\sqrt{2}}\right)^2$$

$$= \frac{\pi}{\left(1+\sqrt{2}\right)^2} + \frac{2\pi}{\left(2+\sqrt{2}\right)^2}$$

$$\approx 1.078$$

The area of the square is $=\left(\sqrt{2}\right)^2 \ = 2$

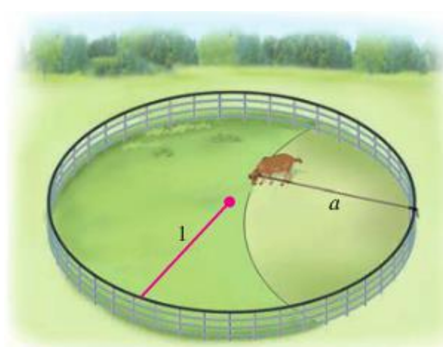
The area outside the circle but inside the square is $\approx 2 - 1.078 \ = 0.922$

Therefore, the area inside the circles is more than outside the circles but inside the square.

Exercise

A circular corral of unit radius is enclosed by a fence. A goat inside the corral is tied to the fence with a rope of length $0 \le a \le 2$.

What is the area of the region (inside the corral) that the goat can graze? Check your answer with the special cases $a = 0$ and $a = 2$



Solution

Suppose that the goal is tethered at the origin, and that the center of the coral is $(1,\ \pi)$.

The circle that the goat can graze is $r = a$, and the corral is given by $r = -2\cos\theta$.

The intersection occurs for $\theta = \cos^{-1}\left(-\frac{a}{2}\right)$

The area grazed by the goat is twice the area of the sector of the circle $r = a$ between $\cos^{-1}\left(-\frac{a}{2}\right)$ and π, plus twice the area of the circle $r = -2\cos\theta$ between $\frac{\pi}{2}$ and $\cos^{-1}\left(-\frac{a}{2}\right)$.

$$A=\int_{\cos^{-1}\left(-\frac{a}{2}\right)}^{\pi} a^2 d\theta+\int_{\pi/2}^{\cos^{-1}\left(-\frac{a}{2}\right)} 4\cos^2\theta\, d\theta$$

$$=a^2\theta\Big|_{\cos^{-1}\left(-\frac{a}{2}\right)}^{\pi}+2\int_{\pi/2}^{\cos^{-1}\left(-\frac{a}{2}\right)}(1+\cos 2\theta)\, d\theta$$

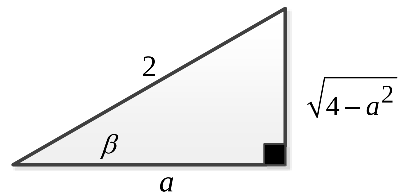
$$=a^2\left(\pi-\cos^{-1}\left(-\tfrac{a}{2}\right)\right)+(2\theta+\sin 2\theta)\Big|_{\pi/2}^{\cos^{-1}\left(-\frac{a}{2}\right)}$$

$\sin 2\beta = 2\sin\beta\cos\beta$

$$=a^2\left(\pi-\cos^{-1}\left(-\tfrac{a}{2}\right)\right)+2\cos^{-1}\left(-\tfrac{a}{2}\right)+\sin\left(2\cos^{-1}\left(-\tfrac{a}{2}\right)\right)-\pi$$

$\sin 2\beta = 2\frac{\sqrt{4-a^2}}{2}\frac{a}{2}$

$$=a^2\left(\pi-\cos^{-1}\left(-\tfrac{a}{2}\right)\right)+2\cos^{-1}\left(-\tfrac{a}{2}\right)-\tfrac{1}{2}a\sqrt{4-a^2}-\pi$$

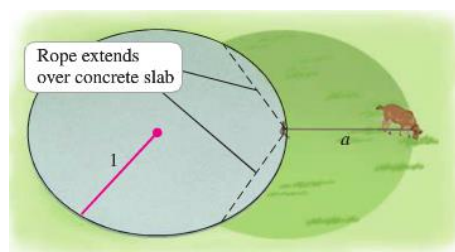


Case $a=0$: $\quad A=\pi-\pi \ \underline{=0}$

Case $a=2$: $\quad A=4(\pi-\pi)+2\pi-\pi \ \underline{=\pi}$

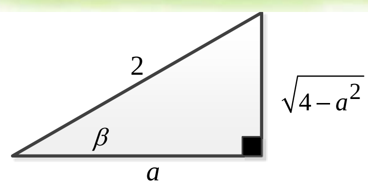
Exercise

A circular concrete slab of unit radius is surrounded by grass. A goat is tied to the edge of the slab with a rope of length $0\le a\le 2$. What is the area of the grassy region that the goat can graze? Note that the rope can extend over the concrete slab. Check your answer with the special cases $a=0$ and $a=2$



Solution

$$A=\int_{\cos^{-1}\left(\frac{a}{2}\right)}^{\pi} a^2 d\theta+\int_{\pi/2}^{\cos^{-1}\left(\frac{a}{2}\right)} 4\cos^2\theta\, d\theta$$



$$=a^2\left(\pi-\cos^{-1}\left(\tfrac{a}{2}\right)\right)+(2\theta+\sin 2\theta)\Big|_{\pi/2}^{\cos^{-1}\left(\frac{a}{2}\right)}$$

$\sin 2\beta = 2\sin\beta\cos\beta$

$$=a^2\pi-a^2\cos^{-1}\left(\tfrac{a}{2}\right)+2\cos^{-1}\left(\tfrac{a}{2}\right)+\sin\left(2\cos^{-1}\left(\tfrac{a}{2}\right)\right)-\pi$$

$\sin 2\beta = 2\frac{\sqrt{4-a^2}}{2}\frac{a}{2}$

$$=\pi\left(a^2-1\right)+\left(2-a^2\right)\cos^{-1}\left(\tfrac{a}{2}\right)+\tfrac{1}{2}a\sqrt{4-a^2}$$

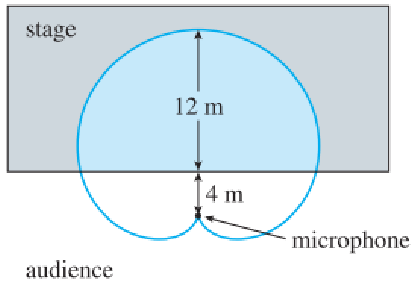
Case $a=0$: $\quad A=-\pi+2\frac{\pi}{2} \ \underline{=0}$

Case $a=2$: $\quad A \ \underline{=3\pi}$

Exercise

When recording live performance, sound engineers often use a microphone with a cardioid pickup pattern because it suppresses noise from the audience. Suppose the microphone is placed 4 *m* from the front of the stage and the boundary of the optimal pickup region is given by the cardioid $r = 8 + 8\sin\theta$, where r if measured in meters and the microphone is at the pole.

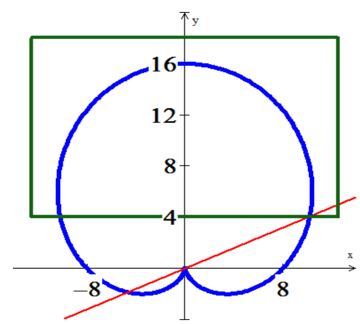
The musicians want to know the area they will have on stage within the optimal pickup range of the microphone, Answer their question.



Solution

At $y = 4 = r\sin\theta$, the line represents the front stage with angle $\theta = \alpha$. $\Leftrightarrow$ $r = \dfrac{4}{\sin\theta}$

The line intersects the curve:



$$r = 8 + 8\sin\theta = \frac{4}{\sin\theta}$$

$$2\sin^2\theta + 2\sin\theta - 1 = 0$$

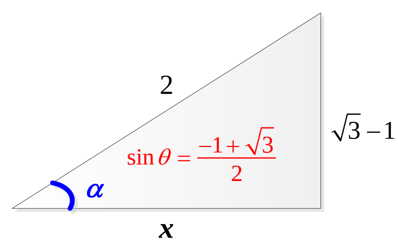
$$\sin\theta = \frac{-2 \pm \sqrt{12}}{4} = \frac{-1 \pm \sqrt{3}}{2} \Rightarrow \sin\theta = \frac{-1+\sqrt{3}}{2}$$

$$\boxed{\theta = \alpha = \sin^{-1}\frac{\sqrt{3}-1}{2}}$$

$$A = 2\int_{\alpha}^{\pi/2} \frac{1}{2}\left[(8+8\sin\theta)^2 - \left(\frac{4}{\sin\theta}\right)^2\right] d\theta$$

$$= \int_{\alpha}^{\pi/2} \left(64 + 128\sin\theta + 64\sin^2\theta - 16\csc^2\theta\right) d\theta$$

$$= 16\int_{\alpha}^{\pi/2} \left(4 + 8\sin\theta + 2 - 2\cos 2\theta - \csc^2\theta\right) d\theta$$



$$= 16\left(6\theta - 8\cos\theta - \sin 2\theta - \cot\theta\right)\Big|_{\alpha}^{\pi/2}$$

$$= 16\left(3\pi - 6\alpha + 8\cos\alpha + \sin 2\alpha + \cot\alpha\right)$$

$$x^2 + \left(\sqrt{3}-1\right)^2 = 4 \quad \to \quad x = \sqrt{4 - 3 + 2\sqrt{3} - 1} = \sqrt{2\sqrt{3}} = \sqrt{\sqrt{4}\sqrt{3}} = \left(\sqrt{12}\right)^{1/2} = \sqrt[4]{12}$$

$$= 16\left(3\pi - 6\sin^{-1}\frac{\sqrt{3}-1}{2} + 4\sqrt[4]{12} + \frac{\sqrt[4]{12}\sqrt{3}-1}{2} + \frac{\sqrt{3}-1}{\sqrt[4]{12}}\right)$$

$$\boxed{\approx 204.16\ m^2}$$