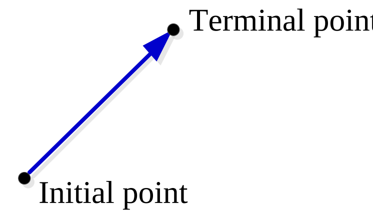
Lecture Two

Section 2.1 – Vectors in 2-Space, 3-Space, and *n*-Space

Vectors in two dimensions are also called **2–*space***

Vectors in three dimensions are also called **3–*space*** by arrow

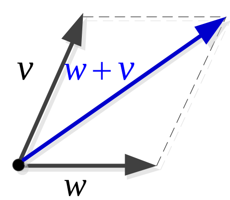
The direction of the arrowhead specifies the ***direction*** of the vector and the ***length*** of the arrow specifies the *magnitude*.



The tail of the arrow is called the ***initial point*** of the vector and the tip the ***terminal point***.

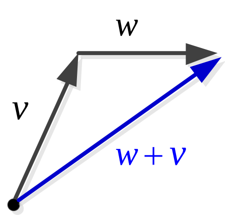
Parallelogram Rule for Vector Addition

If $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors in 2-space or 3-space that are positioned so their initial points coincides, then the vectors form adjacent sides of a parallelogram, and then the sum $\boldsymbol{v} + \boldsymbol{w}$ is the vector represented by the arrow from the common initial point of $\boldsymbol{v}$ and $\boldsymbol{w}$ to the opposite vertex of the parallelogram.

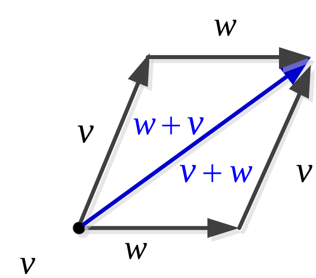


Triangle Rule for Vector Addition

If $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors in 2-space or 3-space that are positioned so the initial point of w is at the terminal point of $\boldsymbol{v}$, then the sum $\boldsymbol{v} + \boldsymbol{w}$ is represented by the arrow from the initial point of $\boldsymbol{v}$ to the terminal point of $\boldsymbol{w}$.



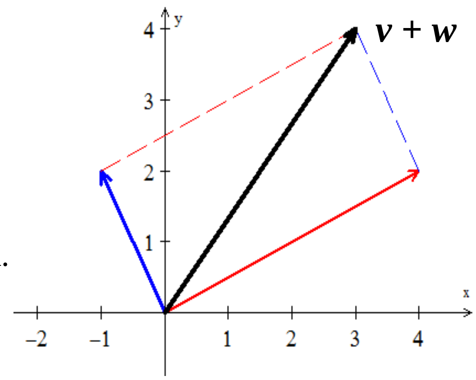
$$\boxed{\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}}$$



Example of Sum and Difference of vectors

Consider the vector $\vec{v}$ is given by the component $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and represented by an arrow. The arrow goes from 4 units to the right and 2 units up.

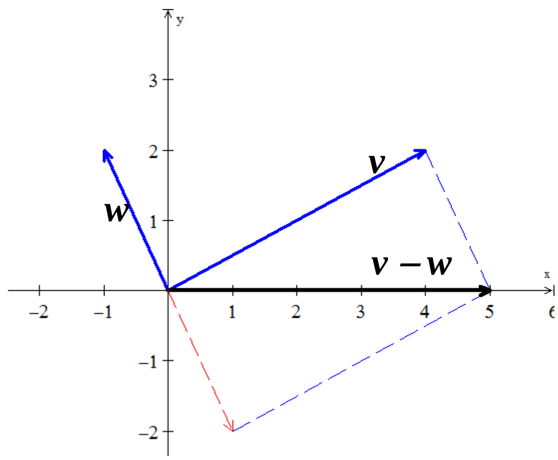
Consider anther vector $\vec{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$



Vector addition (head to tail) at the end of $\vec{v}$, place the start of $\vec{w}$.

The vector addition and w produces the diagonal of a parallelogram.

$$\vec{v} + \vec{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



$$\vec{v} - \vec{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

In 3-dimensional space, the arrow starts at the origin $(0,\ 0,\ 0)$, where the *xyz* axis meet.

$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ is also written as $(1,\ 2,\ 2)$

Notes:

1. The picture of the combinations $c\vec{u}$ fills a line
2. The picture of the combinations $c\vec{u} + d\vec{v}$ fills a plane
3. The picture of the combinations $c\vec{u} + d\vec{v} + e\vec{w}$ fills a 3-dimensional space.

Linear Combination

Definition

The sum of $c\vec{v}$ and $d\vec{w}$ is a linear combination of vectors $\vec{v}$ and $\vec{w}$; c, d are constants.

4-Special Linear Combinations:

$$\begin{aligned} 1\vec{v} + 1\vec{w} &= \textit{sum of vectors} \\ 1\vec{v} - 1\vec{w} &= \textit{difference of vectors} \\ 0\vec{v} + 0\vec{w} &= \textit{zero vectors} \\ c\vec{v} + 0\vec{w} &= \textit{vector } c\vec{v} \textit{ in the direction of } \vec{v} \end{aligned}$$

Vectors in Coordinate Systems

It is sometimes necessary to consider vectors whose initial are not at the origin. If $\overrightarrow{P_1P_2}$ denotes the vector with initial point $P_1(x_1, y_1)$ and terminal point $P_2(x_2, y_2)$, then the components of this vector are given by the formula

$$\boxed{\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)}$$

If $P_1(x_1, y_1, z_1)$ *and* $P_2(x_2, y_2, z_2)$

$$\boxed{\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)}$$

Example

The components of the vector $v = \overrightarrow{P_1P_2}$ with initial point $P_1(2, -1, 4)$ and terminal point $P_2(7, 5, -8)$, find $\boldsymbol{v}$?

Solution

$$\begin{aligned} \vec{v} &= (7-2, 5-(-1), -8-4) \\ &= (5, 6, -12) \end{aligned}$$

$n-$ Space

The vector spaces are denoted by $\boldsymbol{R}^1, \boldsymbol{R}^2, \boldsymbol{R}^3, \boldsymbol{R}^4 \cdots$. Each space $\boldsymbol{R}^n$ consists of a whole collection of vectors.

Definition

The space $\boldsymbol{R}^n$ consists of all column vectors *v* with *n* components.

Example

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad (1,\ 2,\ 3,\ 0,\ 1) \qquad \begin{bmatrix} 1+i \\ 1-i \end{bmatrix}$$

$$\boldsymbol{R}^3 \qquad\qquad \boldsymbol{R}^5 \qquad\qquad \boldsymbol{C}^2$$

The one-dimensional space $\boldsymbol{R}^1$ is a line (like the *x*-axis)

The two essential vector operations go on inside the vector space that we can add any vectors in $\boldsymbol{R}^n$, and we can multiply any vector by any scalar. The ***result*** stays in the space.

A real vector space is a set of "***vectors***" together with rules for vector addition and for multiplication by real numbers. The addition and the multiplication must produce vectors that are in the space.

Here are three other spaces other than $\boldsymbol{R}^n$:

M The vector space of ***all real 2 by 2 matrices***.

F The vector space of ***all real functions*** $f(x)$.

Z The vector space that consists only of a ***zero vector***.

The zero vector in $\boldsymbol{R}^3$ is the vector (0, 0, 0).

Operation on Vectors in R^n

Definition

If n is a positive integer, then an ordered ***n-tuple*** is a sequence of real numbers $\left(v_1,\ v_2,\ \dots,\ v_n\right)$. The set of all ordered n-tuples is called ***n-space*** and is denoted by $\boldsymbol{R}^n$

Definition

Vectors $\boldsymbol{v}=\left(v_1,\ v_2,\ \dots,\ v_n\right)$ and $\boldsymbol{w}=\left(w_1,\ w_2,\ \dots,\ w_n\right)$ in $\boldsymbol{R}^n$ are said to be ***equivalent*** (also called ***equal***) if

$$v_1=w_1,\quad v_2=w_2,\quad \dots\quad v_n=w_n$$

We indicate this by $\boldsymbol{v}=\boldsymbol{w}$

Example

$$(a,b,c,d)=(1,-4,2,7)$$

Solution

$$\textit{Iff}\ \ a=1,\quad b=-4,\quad c=2,\quad d=7$$

Vector Space of Infinite Sequences of Real Numbers

If $\boldsymbol{v}=\left(v_1,\ v_2,\ \dots,\ v_n\right)$ and $\boldsymbol{w}=\left(w_1,\ w_2,\ \dots,\ w_n\right)$ are vectors in $\boldsymbol{R}^n$, and if $\boldsymbol{k}$ is any scalar, then we defined

$$\begin{aligned}
\boldsymbol{v}+\boldsymbol{w}&=\left(u_1,\ u_2,\ \dots,\ u_n\right)+\left(w_1,\ w_2,\ \dots,\ w_n\right)\\
&=\left(v_1+w_1,\ v_2+w_2,\ \dots,\ v_n+w_n\right)
\end{aligned}$$

$$\boldsymbol{kv}=\left(kv_1,\ kv_2,\ \dots,\ kv_n\right)$$

$$-\boldsymbol{v}=\left(-v_1,\ -v_2,\ \dots,\ -v_n\right)$$

$$\boldsymbol{w}-\boldsymbol{v}=\boldsymbol{w}+(-\boldsymbol{v})=\left(w_1-v_1,\ w_2-v_2,\ \dots,\ w_n-v_n\right)$$

The *Zero* Vector Space

Let V consist of a single object, which we denote by $\vec{0}$, and define

$$\mathbf{0}+\mathbf{0}=\mathbf{0}\quad \textit{and}\quad k\mathbf{0}=\mathbf{0}$$

Theorem

If $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are vectors in $\boldsymbol{R}^n$, and if $\boldsymbol{k}$ and $\boldsymbol{m}$ are scalars, then

a) $\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$

b) $(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$

c) $\boldsymbol{u}+0=0+\boldsymbol{u}=\boldsymbol{u}$

d) $\boldsymbol{u}+(-\boldsymbol{u})=0$

e) $k(\boldsymbol{u}+\boldsymbol{v})=k\boldsymbol{u}+k\boldsymbol{v}$

f) $(k+m)\boldsymbol{u}=k\boldsymbol{u}+m\boldsymbol{u}$

g) $k(m\boldsymbol{u})=(km)\boldsymbol{u}$

h) $1\boldsymbol{u}=\boldsymbol{u}$

i) $0\boldsymbol{v}=0$

j) $\boldsymbol{k}0=0$

k) $(-1)\boldsymbol{v}=-\boldsymbol{v}$

Proof: $(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$

$$\begin{aligned}
(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w} &= \left(\left(u_1,\ u_2,\ \ldots,\ u_n\right)+\left(v_1,\ v_2,\ \ldots,\ v_n\right)\right)+\left(w_1,\ w_2,\ \ldots,\ w_n\right)\\
&= \left(u_1+v_1,\ u_2+v_2,\ \ldots,\ u_n+v_n\right)+\left(w_1,\ w_2,\ \ldots,\ w_n\right)\\
&= \left(\left(u_1+v_1\right)+w_1,\ \left(u_2+v_2\right)+w_2,\ \ldots,\ \left(u_n+v_n\right)+w_n\right)\\
&= \left(u_1+\left(v_1+w_1\right),\ u_2+\left(v_2+w_2\right),\ \ldots,\ u_n+\left(v_n+w_n\right)\right)\\
&= \left(u_1,\ u_2,\ \ldots,\ u_n\right)+\left(\left(v_1,\ v_2,\ \ldots,\ v_n\right)+\left(w_1,\ w_2,\ \ldots,\ w_n\right)\right)\\
&= \boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})
\end{aligned}$$

Exercises   *Section* 2.1 – Vectors in 2-Space, 3-Space, and n-Space

1. Sketch the following vectors with initial points located at the origin

a) $P_1(4,8)\quad P_2(3,7)$ *b)* $P_1(-1,0,2)\quad P_2(0,-1,0)$ *c)* $P_1(3,-7,2)\quad P_2(-2,5,-4)$

2. Find the components of the vector $\overrightarrow{P_1P_2}$

a) $P_1(3,5)\quad P_2(2,8)$ *b)* $P_1(5,-2,1)\quad P_2(2,4,2)$ *c)* $P_1(0,\ 0,\ 0)\quad P_2(-1,\ 6,\ 1)$

3. Find the terminal point of the vector that is equivalent to $\boldsymbol{u} = (1, 2)$ and whose initial point is $A(1,1)$

4. Find the initial point of the vector that is equivalent to $\boldsymbol{u} = (1, 1, 3)$ and whose terminal point is $B(-1,-1,2)$

5. Find a nonzero vector $\boldsymbol{u}$ with initial point $P(-1, 3, -5)$ such that

a) $\boldsymbol{u}$ has the same direction as $\boldsymbol{v} = (6, 7, -3)$

b) $\boldsymbol{u}$ is oppositely directed as $\boldsymbol{v} = (6, 7, -3)$

6. Let $\boldsymbol{u} = (-3, 1, 2)$, $\boldsymbol{v} = (4, 0, -8)$, and $\boldsymbol{w} = (6, -1, -4)$. Find the components

a) $\vec{v}-\vec{w}$

b) $6\vec{u}+2\vec{v}$

c) $5(\vec{v}-4\vec{u})$

d) $-3(\vec{v}-8\vec{w})$

e) $(2\vec{u}-7\vec{w})-(8\vec{v}+\vec{u})$

f) $-\vec{u}+(\vec{v}-4\vec{w})$

7. Let $\boldsymbol{u} = (2, 1, 0, 1, -1)$ and $\boldsymbol{v} = (-2, 3, 1, 0, 2)$. Find scalars a and b so that $a\boldsymbol{u}+b\boldsymbol{v}=(-8,\ 8,\ 3,\ -1,\ 7)$

8. Find all scalars $c_1,\ c_2,\ and\ c_3$ such that $c_1(1,\ 2,\ 0)+c_2(2,\ 1,\ 1)+c_3(0,\ 3,\ 1)=(0,\ 0,\ 0)$

9. Find the distance between the given points $\begin{bmatrix}5 & 1 & 8 & -1 & 2 & 9\end{bmatrix}$, $\begin{bmatrix}4 & 1 & 4 & 3 & 2 & 8\end{bmatrix}$

10. Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operation on $\boldsymbol{u}=(u_1,\ u_2)\quad \boldsymbol{v}=(v_1,\ v_2)$

$$\boldsymbol{u}+\boldsymbol{v}=(u_1+v_1+1,\ u_2+v_2+1)\qquad k\boldsymbol{u}=(ku_1,\ ku_2)$$

a) Compute $\boldsymbol{u}+\boldsymbol{v}$ and $k\boldsymbol{u}$ for $\boldsymbol{u} = (0, 4)$, $\boldsymbol{v} = (1, -3)$, and $k = 2$.

b) Show that $(0, 0) \neq \mathbf{0}$.

c) Show that $(-1, -1) = 0$.

d) Show that $\boldsymbol{u}+(-\boldsymbol{u})=0$ for $\boldsymbol{u}=(u_1,\ u_2)$

e) Find two vector space axioms that fail to hold.

11. Find $\vec{w}$ given that $10\vec{u}+3\vec{w}=4\vec{v}-2\vec{w}$, $\vec{u}=\begin{pmatrix}1\\-6\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}-20\\5\end{pmatrix}$

12. Find $\vec{w}$ given that $\vec{u}+3\vec{v}-2\vec{w}=5\vec{u}+\vec{v}-4\vec{w}$, $\vec{u}=\begin{pmatrix}1\\-1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}-2\\3\end{pmatrix}$

13. Find $\vec{w}$ given that $2\vec{u}+\vec{v}-3\vec{w}=5\vec{u}+7\vec{v}+3\vec{w}$, $\vec{u}=\begin{pmatrix}1\\-1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}-2\\3\end{pmatrix}$

Draw $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$, and $\vec{u}+2\vec{v}$

14. $\vec{u}=\begin{pmatrix}1\\-1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}-2\\3\end{pmatrix}$

15. $\vec{u}=\begin{pmatrix}-1\\1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}2\\1\end{pmatrix}$

16. $\vec{u}=\begin{pmatrix}1\\0\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}1\\2\end{pmatrix}$

17. $\vec{u}=\begin{pmatrix}-1\\0\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}1\\-1\end{pmatrix}$