Calculus II - Review

1/ $\displaystyle\sum_{n=0}^{\infty} 5^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n$

$|r| = \frac{1}{5} < 1$

$S = \dfrac{1}{1-\frac{1}{5}} = \boxed{\dfrac{5}{4}}$

By the Geometric series, the given series converges w/ sum of $\frac{5}{4}$.

2/ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$\displaystyle\int_1^{\infty} \frac{dx}{x^2+1} = \arctan x \Big|_1^{\infty}$

$= \arctan \infty - \arctan 1$

$= \frac{\pi}{2} - \frac{\pi}{4}$

$= \boxed{\frac{\pi}{4}}$

By the Integral Test, the given series converges

3/ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

$$2^n+1 > \frac{1}{2^n}$$

$$\frac{1}{2^n+1} < \left(\frac{1}{2}\right)^n$$

$$\sum \left(\frac{1}{2}\right)^n \qquad r = \frac{1}{2} < 1$$

it converges by geometric series

By the Comparison Test, the given series converges.

4/ $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 (n+2)!}{n!\ 3^{2n}}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2\ (n+3)!}{(n+1)!\ 3^{2n+2}} \cdot \frac{n!\ 3^{2n}}{n^2\ (n+2)!}$$

$$= \left(\frac{n+1}{n}\right)^2 \left(\frac{n+3}{n+1}\right)\left(\frac{1}{9}\right)$$

1

$$\rho = \frac{1}{9} \lim_{n\to\infty} \left(\left(\frac{n+1}{n}\right)^2 \left(\frac{n+3}{n+1}\right) \right)$$

$$= \frac{1}{9} < 1$$

By the Ratio Test, the given series converges

5/ $$\sum_{n=1}^{\infty} \left(\ln\left(e^2+\frac{1}{n}\right)\right)^{n+1}$$

$$\sqrt[n]{\left(\ln\left(e^2+\frac{1}{n}\right)\right)^{n+1}} = \left(\ln\left(e^2+\frac{1}{n}\right)\right)^{\frac{n+1}{n}}$$

$$\rho = \lim_{n\to\infty} \left(\ln\left(e^2+\frac{1}{n}\right)\right)^{\frac{n+1}{n}}$$

$$\frac{n+1}{n} \to 1 \qquad \frac{1}{n} \to 0$$

$$= \ln(e^2)$$

$$= 2 > 1$$

By the Root Test, the given series diverges.

6/ $$\sum_{n=1}^{\infty} (-1)^n \underbrace{\frac{n}{n^2+1}}_{u_n}$$

$u_n > u_{n+1}$

$$u_{n+1} = \frac{n+1}{(n+1)^2+1} \longrightarrow n^2+2n+2$$

$$n < n+1$$

$$2n^2+2n > n^2+n+1$$

$$n^3+2n^2+2n > n^3+n^2+n+1$$

$$n(n^2+2n+2) > n^2(n+1)+(n+1)$$

$$n[(n+1)^2+1] > (n+1)(n^2+1)$$

$$\frac{n}{n^2+1} > \frac{n+1}{(n+1)^2+1} \Rightarrow u_n > u_{n+1}$$

$$\frac{n}{n^2+1} \longrightarrow 0$$

By the Alternating series, the given series converges

$$\sum_{n=1}^{\infty} \frac{5}{n^{2/3}}$$

$$p = \frac{2}{3} < 1$$

By the p-series the given series converges

$$\sum_{n=1}^{\infty} \frac{5}{k^{3/2}}$$

$$p = \frac{3}{2} > 1$$

By the p-series, the given series diverges.

$$\sum_{k=1}^{\infty} k e^{-k}$$

$$\begin{array}{c|l} & \int e^{-x} \\ \hline x & -e^{-x} \\ 1 & e^{-x} \end{array}$$

$$\int_1^{\infty} x e^{-x}\, dx = (-x-1)e^{-x} \Big|_1^{\infty}$$

$$= -(-2)e^{-1}$$

$$= \frac{2}{e}$$

By the Integral Test, the given series converges

$$\sum_{k=2}^{\infty} \frac{(\ln k)^2}{k}$$

$$\int_2^{\infty} \frac{(\ln x)^2}{x} dx = \int_2^{\infty} (\ln x)^2 d(\ln x)$$

$$= \frac{1}{3} (\ln x)^3 \Big|_2^{\infty}$$

$$= \infty$$

By the Integral Test, the given series diverges

Centre, R, Interval of convergence

$$\sum_{n=1}^{\infty} \frac{e^n}{n^3} (4-x)^n$$

centre of convergence: $x = 4$

$$R = \lim_{n\to\infty} \frac{e^n}{n^3} \cdot \frac{(n+1)^3}{e^{n+1}}$$

$$= \frac{1}{e} \lim \left(\frac{n+1}{n}\right)^3$$

$$= \frac{1}{e}$$

Radius of convergence is $\frac{1}{e}$

$$a - R < x < a + R$$

$$4 - \frac{1}{e} < +x < 4 + \frac{1}{e}$$

@ $x = 4 - \frac{1}{e} \Rightarrow \sum \frac{e^n}{n^3}\left(-\frac{1}{e}\right)^n = \sum \frac{1}{n^3}$

$p = 3 > 1$

By the p-series Test, it converges

@ $x = 4 + \frac{1}{e} \Rightarrow \sum \frac{e^n}{n^3}\left(-\frac{1}{e}\right)^n = \sum \frac{(-1)^n}{n^3}$

$n < n+1$

$n^3 < (n+1)^3$

$\frac{1}{n^3} > \frac{1}{(n+1)^3}$

$u_n > u_{n+1}$ ✓

$\frac{1}{n^3} \to 0$

By the Alternating series, it converges

∴ The interval of convergence:

$$4 - \frac{1}{e} \leq x \leq 4 + \frac{1}{e}$$

$$\sum \frac{1+5^{-n}}{n!} x^n$$

centre of convergence: $x = 0$

$$R = \lim_{n\to\infty} \frac{1+5^{-n}}{n!} \frac{(n+1)!}{1+5^{-n+1}}$$

$$= \lim_{n\to\infty} \frac{1+5^{-n}}{1+5^{-n+1}}(n+1)$$

$$= \infty$$

Interval of convergence: $(-\infty, \infty)$

$$\frac{f^{(n)}}{n!}x^n$$

0, 1, 2, 3 $\quad f(x) = \cos x \quad a = \frac{\pi}{2}$

$f(x) = \cos x \qquad f(\frac{\pi}{2}) = 0$

$f'(x) = -\sin x \qquad f'(\frac{\pi}{2}) = -1$

$f''(x) = -\cos x \qquad f''(\frac{\pi}{2}) = 0$

$f'''(x) = \sin x \qquad f'''(\frac{\pi}{2}) = 1$

$$\boxed{P_3(x) = -(x-\frac{\pi}{2}) + \frac{1}{3!}(x-\frac{\pi}{2})^3}$$

$f(x) = \sec x \quad n=2 \quad$ Maclaurin $a=0$

$f(x) = \sec x \qquad f(0) = 1$

$f'(x) = \sec x \tan x \qquad f'(0) = 0$

$f''(x) = \sec x \tan^2 x + \sec^3 x \qquad f''(0) = 1$

$$\boxed{P_2(x) = 1 + \frac{1}{2}x^2}$$

$f(x) = 5\cos\pi x$

$f(x) = 5\cos\pi x \Big|_{x=0} = 5$ $\quad$ $f'(x) = -5\pi \sin\pi x \Big|_{x=0} = 0$

$f''(x) = -5\pi^2 \cos\pi x \Big|_{x=0} = -5\pi^2$ $\quad$ $f'''(x) = 5\pi^3 \sin\pi x$

$f^{(4)}(x) = 5\pi^4 \cos\pi x \Big|_{x=0} = 5\pi^4$

0

$$5\cos\pi x = 5 - \frac{5\pi^2}{2!}x^2 + \frac{5\pi^4}{4!}x^4 - \cdots$$

$$= 5\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n}}{(2n)!} x^{2n}$$

$$= 5\sum_{n=0}^{\infty}(-1)^n \frac{(\pi x)^{2n}}{(2n)!}$$

$$f(x) = \frac{1}{4+x^2}$$

$a = 0$ ↓ $0, 1, 2, 3$

$(u^n v^m)' = u^{n-1} v^{m-1}(nu'v + muv')$

$$f(x) = \frac{1}{4+x^2} \qquad f(0) = \frac{1}{4}$$

$$f'(x) = \frac{-2x}{(4+x^2)^2} \qquad f'(0) = 0$$

$$f''(x) = -2\,\frac{4+x^2-4x^2}{(4+x^2)^3} = \frac{-8+6x^2}{(4+x^2)^3} \qquad f''(0) = -\frac{1}{8}$$

$$f'''(x) = \frac{12x(4+x^2) - 6x(-8+6x^2)}{(4+x^2)^4}$$

$$= \frac{96x - 24x^3}{(4+x^2)^4} \qquad f'''(0) = 0$$

$$T_3(x) = \frac{1}{4} - \frac{1}{8}\frac{x^2}{2!}$$

$$= \frac{1}{4} - \frac{1}{16}x^2$$

Centre, Radius & Interval of convergence

$$\sum_{n=0}^{\infty} 3n(x+1)^n$$

Centre of convergence: $x = -1$

$$R = \lim_{n\to\infty} \frac{3n}{3(n+1)}$$

$$= 1$$

$$-1-1 < x < 1-1$$

$$-2 < x < 0$$

@ $x = -2 \Rightarrow \sum_{n=0}^{\infty} 3n(-1)^n$

$3n \to \infty$, it diverges by Alternating series.

@ $x = 0 \Rightarrow \sum 3n$

$$\int_0^{\infty} 3x\,dx = \frac{3}{2}x^2 \Big|_0^{\infty}$$

$$= \infty$$

diverges

Interval of converges: $-2 < x < 0$

$(-2, 0)$

$$\sum_{k=1}^{\infty} \frac{k^2-1}{k^3+4}$$

$$\frac{k^2-1}{k^3+4} \longrightarrow \frac{1}{k} \qquad \frac{k^2}{k^3}$$

$p=1 \Rightarrow \sum \frac{1}{k}$ diverges by p-series

$$\lim \frac{a_k}{b_n} = \lim_{n\to\infty} \frac{k^2-1}{k^3+4} \cdot k$$

$$= 1 \neq 0$$

$$\sum_{k=0}^{\infty} \frac{k^2}{2^k}$$

$$\sqrt[k]{\frac{k^2}{2^k}} = \frac{k^{2/k}}{2}$$

$$\rho = \lim_{k\to\infty} \frac{k^{2/k}}{2}$$

$$= \frac{1}{2} < 1$$

By Root test, the given series converges

$$\frac{(k+1)^2}{2^{k+1}} \cdot \frac{2^k}{k^2} = \frac{1}{2} \left(\frac{k+1}{k}\right)^2 \to \frac{1}{2} <$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{e^n}{(n+1)!}$$

$$\frac{e^{n+1}}{(n+2)!}$$

$$\begin{cases} (n+1)! < (n+2)! \\ \dfrac{1}{(n+1)!} > \dfrac{1}{(n+2)!} \\ \dfrac{e^n}{(n+1)!} > \dfrac{e^n}{(n+2)!} \end{cases} \quad e$$

$$\left| (-1)^{n+1} \frac{e^n}{(n+1)!} \right| = \frac{e^n}{(n+1)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{e^{n+1}}{(n+2)!} \cdot \frac{(n+1)!}{e^n}$$

$$= \frac{e}{n+2}$$

$$\rho = \lim_{n \to \infty} \frac{e}{n+2}$$

$$= 0$$

converges absolutely

$$\sum_{k=2}^{0} (-1)^k \frac{1}{k \ln k}$$

$$k < k+1 \qquad \ln k < \ln(k+1)$$

$$k(\ln k) < (k+1)\ln(k+1)$$

$$\frac{1}{k \ln k} > \frac{1}{(k+1)\ln(k+1)}$$

$$u_k > u_{k+1} \checkmark$$

$$\frac{1}{k \ln k} \to 0 \quad \checkmark$$

By the Alternating series, the given series Converges

$$\sum_{k=2}^{\infty} \frac{1}{k \ln k}$$

$$\int_2^{\infty} \frac{1}{x \ln x} = \int_2^{\infty} \frac{d(\ln x)}{\ln(x)}$$

$$= \ln(\ln x)\Big|_2^{\infty}$$

$$= \infty$$

$$\sum_{k=0}^{\infty} \frac{2(4^k)}{(2k+1)!}$$

$$\frac{a_{k+1}}{a_k} = \frac{2\ 4^{k+1}}{(2k+3)!} \cdot \frac{(2k+1)!}{2\ 4^k}$$

$$= \frac{4}{(2k+2)(2k+3)}$$

$$\rho = \lim_{k\to\infty} \frac{4}{(2k+2)(2k+3)}$$

$$= 0 < 1$$

By the ratio test, the given series converges.