

$$r = 1 + 2r \cos \theta$$

$$(x^2 + y^2)^2 = (1 + 2x)^2$$

$$x^2 + y^2 = 1 + 4x + 4x^2$$

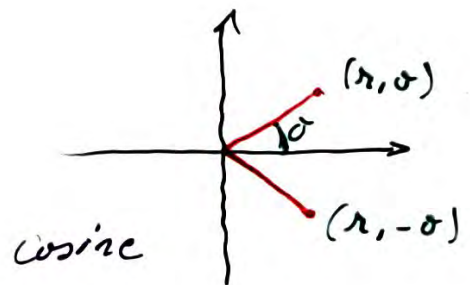
$$y^2 - 3x^2 - 4x - 1 = 0$$

Symmetry

1- About x-axis

$$(r, \theta) \rightarrow (r, -\theta)$$

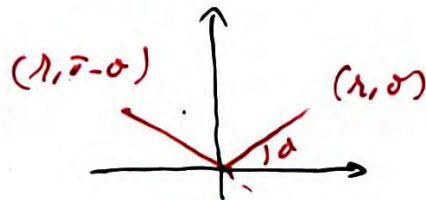
$$(-r, \pi + \theta)$$



2- y-axis

$$(r, \theta) \rightarrow (r, \pi - \theta)$$

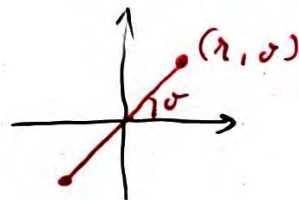
$$(-r, -\theta)$$



3 about the Origin

$$(r, \theta) \rightarrow (r, \pi + \theta)$$

$$(-r, \theta)$$

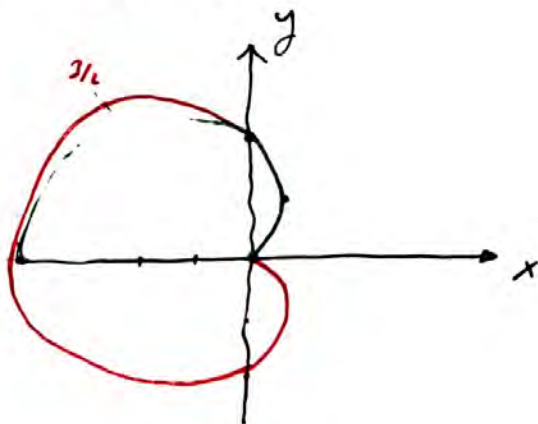


cosine & secant are even fns

IR are even fns

Ex Graph $r = 1 - \cos \theta$

Soln



θ	r
0	0
$\pi/2$	$1/2$
π	1
$3\pi/2$	$3/2$
2π	2

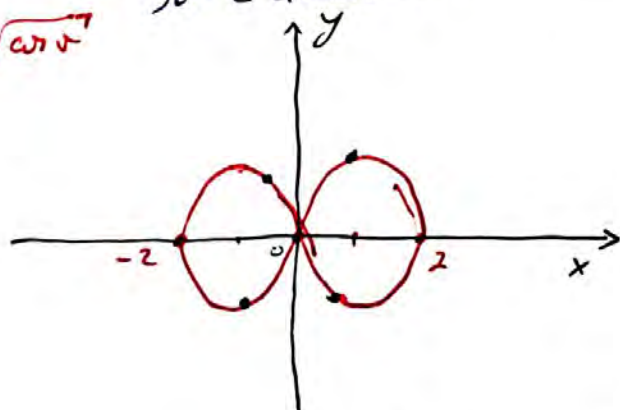
Ex Graph $r^2 = 4 \cos \theta > 0$ $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$

$\theta = 0 \rightarrow 4 \cos(0) = 4 \cos \theta = r^2 \rightarrow x\text{-axis}$

$(-1, 0) \rightarrow (-1)^2 = 4 \cos(\pi) \rightarrow \text{origin}$

$$r^2 = 4 \cos \theta$$

$$r = \pm 2 \sqrt{\cos \theta}$$



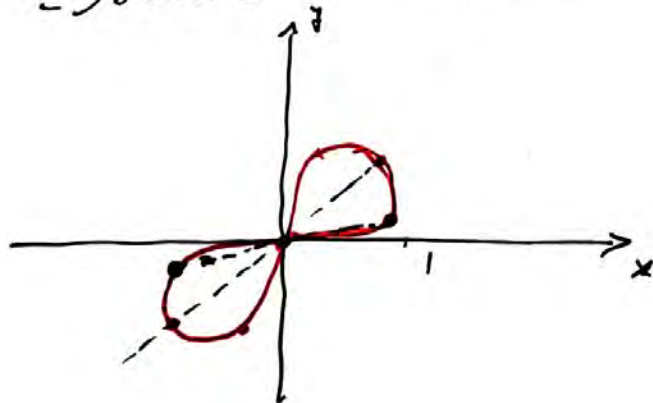
θ	r
0	± 2
$\pi/2$	0
$3\pi/2$	± 2

Ex

$$r^2 = \sin 2\theta$$

lemniscate

$$\sin 2(\pi/4) = \frac{1}{2}$$



4.4 Calculus.

Slope:

$$r = f(\theta)$$

$$\begin{cases} x = r \cos \theta = f(\theta) \cos \theta \\ y = r \sin \theta = f(\theta) \sin \theta \end{cases}$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}$$

$$\frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}$$

Ex $-\pi \leq \theta \leq \pi$

cardioid: $f(\theta) = 1 - \cos \theta = r$

$$\frac{dy}{dx} = \frac{\sin^2 \theta + (1 - \cos \theta) \cos \theta}{\sin \theta \cos \theta - (1 - \cos \theta) \sin \theta}$$

$$= \frac{\sin^2 \theta + \cos \theta - \cos^2 \theta}{\sin \theta \cos \theta - \sin \theta + \cos \theta \sin \theta}$$

$$= \frac{1 - \cos^2 \theta + \cos \theta - \cos^2 \theta}{2 \cos \theta \sin \theta - \sin \theta}$$

$$= - \frac{2 \cos^2 \theta - \cos \theta - 1}{\sin \theta (2 \cos \theta - 1)}$$

$$= - \frac{(\cos \theta - 1)(2 \cos \theta + 1)}{\sin \theta (2 \cos \theta - 1)} = 0$$

$$\cos \theta = 1$$

$$\theta = 0 \neq$$

$$\cos \theta = -\frac{1}{2}$$

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

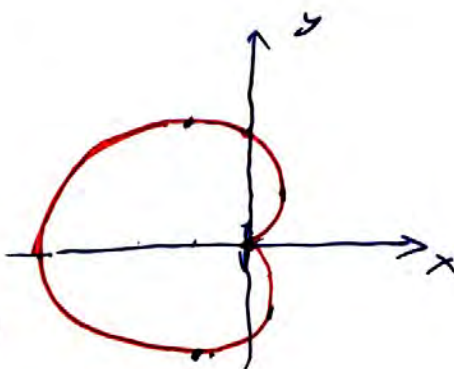
$$\begin{matrix} \sin \theta \neq 0 & \cos \theta \neq \frac{1}{2} \\ \theta \neq 0, \pi & \theta \neq \pm \frac{\pi}{3} \end{matrix}$$

$$\frac{dy}{dx} \bigg|_{\theta=0} = \frac{0}{0}$$

L'hopital Rule

$$= - \frac{-2 \cos \theta \sin \theta + \sin \theta}{2 \cos 2\theta - \cos \theta} \bigg|_{\theta=0}$$

$$= \frac{0}{1} = 0 \checkmark$$



Area

$$Area = \frac{1}{2} r^2 \theta$$

$$A = \frac{1}{2} \int_{\alpha}^{\beta} (f(\theta))^2 d\theta = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$$

Ex A? $r = 2(1 + \cos \theta)$

$$A = \frac{1}{2} \int_0^{2\pi} [2(1 + \cos \theta)]^2 d\theta$$

$$= 2 \int_0^{2\pi} (1 + 2\cos \theta + \cos^2 \theta) d\theta$$

$$= 2 \int_0^{2\pi} \left(\underbrace{1}_{\rightarrow} + 2\cos \theta + \underbrace{\frac{1}{2} + \frac{1}{2}\cos 2\theta}_{\rightarrow} \right) d\theta$$

$$= \int_0^{2\pi} (3 + 4\cos \theta + \cos 2\theta) d\theta$$

$$= 3\theta + 4\sin \theta + \frac{1}{2}\sin 2\theta \Big|_0^{2\pi}$$

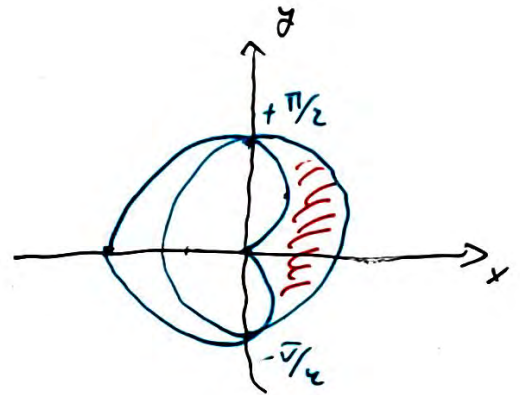
$$= 6\pi \text{ units}^2$$

Ex A? $r=1$ & $r=1-\cos\theta$
inside *outside*

Soln $r=1=1-\cos\theta \Rightarrow \cos\theta=0$
 $\theta = \pm \frac{\pi}{2}$

$\theta = 0$

$$\begin{aligned}
 \text{Area} &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \left(\overset{r_2}{1^2} - \overset{r_1}{(1-\cos\theta)^2} \right) d\theta \\
 &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \left(\underbrace{1-1}_{0} + 2\cos\theta - \underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}} \right) d\theta \\
 &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \left(2\cos\theta - \frac{1}{2} - \frac{1}{2}\cos 2\theta \right) d\theta \\
 &= \frac{1}{2} \left(2\sin\theta - \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta \right) \Big|_{-\pi/2}^{\pi/2} \\
 &= \frac{1}{2} \left(2 - \frac{\pi}{4} + 2 - \frac{\pi}{4} \right) \\
 &= \frac{1}{2} \left(4 - \frac{\pi}{2} \right) \\
 &= \underline{2 - \frac{\pi}{4} \text{ unit}^2}
 \end{aligned}$$



Length

$$L = \int_a^b \sqrt{r^2 + (r')^2} d\theta$$

Ex $L?$ $r = 1 - \cos \theta$

Soln $0 \leq \theta \leq 2\pi$

$$\begin{aligned}\sqrt{r^2 + (r')^2} &= \sqrt{(1 - \cos \theta)^2 + (\sin \theta)^2} \\&= \sqrt{1 - 2\cos \theta + \cos^2 \theta + \sin^2 \theta} \\&= \sqrt{2 - 2\cos \theta} \\&= \sqrt{2(1 - \cos \theta)} \\&= \sqrt{2(2\sin^2 \theta/2)} \\&= 2\sin \theta/2\end{aligned}$$

$$\sin^2 \theta/2 = \frac{1 - \cos 2\theta/2}{2}$$

$$2\sin^2 \theta/2 = 1 - \cos \theta$$

$$L = 2 \int_0^{2\pi} \sin \frac{\theta}{2} d\theta$$

$$= -4 \cos \frac{\theta}{2} \Big|_0^{2\pi}$$

$$= -4(-1 - 1)$$

$$= \underline{8 \text{ unit}}$$

Surface (revolution

About

$$S = 2\pi \int_{\alpha}^{\beta} f(\theta) \sin \theta \sqrt{(f(\theta))^2 + (f'(\theta))^2} d\theta \quad \text{polar axis}$$

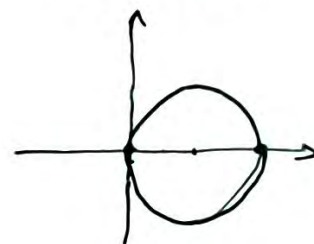
$$= 2\pi \int_{\alpha}^{\beta} f(\theta) \cos \theta \sqrt{(f(\theta))^2 + (f'(\theta))^2} d\theta \quad \text{line: } \theta = \frac{\pi}{2}$$

Ex Area S ? $f(\theta) = \cos \theta$ about line $\theta = \frac{\pi}{2}$

$$0 \leq \theta \leq \pi$$

$$r = \cos \theta$$

$$\begin{aligned} \sqrt{r^2 + r'^2} &= \sqrt{\cos^2 \theta + \sin^2 \theta} \\ &= 1 \end{aligned}$$



$$S = 2\pi \int_0^{\pi} \cos^2 \theta d\theta$$

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

$$= \pi \int_0^{\pi} (1 + \cos 2\theta) d\theta$$

$$= \pi \left(\theta + \frac{1}{2} \sin 2\theta \right) \Big|_0^{\pi}$$

$$= \pi (\pi)$$

$$= \pi^2 \text{ unit}^2$$

#8 Inside oval Limaçon: $R = 4 + 2 \sin \theta$

soln

$$\begin{aligned} A &= \frac{1}{2} \int_0^{2\pi} (4 + 2 \sin \theta)^2 d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (16 + 16 \sin \theta + \cancel{4} \sin^2 \theta) d\theta \\ &= \left(\frac{1}{2} \right) \int_0^{2\pi} \underline{16} + 16 \sin \theta + \underline{2} - 2 \cos 2\theta d\theta \quad \left[\sin^2 \theta = \frac{1 - \cos 2\theta}{2} \right] \\ &= \int_0^{2\pi} (9 + 8 \sin \theta - \cos 2\theta) d\theta \\ &= 9\theta - 8 \cos \theta - \frac{1}{2} \sin 2\theta \Big|_0^{2\pi} \\ &= 18\pi - 8 + 8 \\ &= \underline{18\pi \text{ unit}^2} \end{aligned}$$

#17 Inside 1 leave of $r = \cos 3\theta$

$$\text{Area} = \frac{1}{2} \int_0^{2\pi} \frac{1}{3} \cos^2 3\theta \, d\theta$$

$$\cos^2 3\theta = \frac{1 + \cos 6\theta}{2}$$

$$= \frac{1}{12} \int_0^{2\pi} (1 + \cos 6\theta) \, d\theta$$

$$= \frac{1}{12} \left(\theta + \frac{1}{6} \sin 6\theta \right) \Big|_0^{2\pi}$$

$$= \frac{1}{12} (2\pi)$$

$$= \frac{\pi}{6}$$

Graphing

$$A = \frac{1}{2} \int_{-\pi/6}^{\pi/6} (\cos^2 3\theta) \, d\theta$$

$$= \frac{1}{4} \int_{-\pi/6}^{\pi/6} (1 + \cos 6\theta) \, d\theta$$

$$= \frac{1}{4} \left(\theta + \frac{1}{6} \sin 6\theta \right) \Big|_{-\pi/6}^{\pi/6}$$

$$= \frac{1}{4} \left(\frac{\pi}{6} + \frac{\pi}{6} \right)$$

$$= \frac{\pi}{12} \text{ unit}^2$$

