Section 8.3 – Double-angle and Half-Angle Formulas

$$\begin{aligned}\sin 2A &= \sin(A+A)\\ &= \sin A\cos A + \cos A\sin A\\ &= 2\sin A\cos A\end{aligned}$$

$$\sin 2A \neq 2\sin A$$

$$\begin{aligned}\cos 2A &= \cos(A+A)\\ &= \cos A\cos A - \sin A\sin A\\ &= \cos^2 A - \sin^2 A\end{aligned}$$

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A\\ &= \cos^2 A - \left(1-\cos^2 A\right)\\ &= \cos^2 A - 1 + \cos^2 A\\ &= 2\cos^2 A - 1\end{aligned}$$

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A\\ &= \left(1-\sin^2 A\right) - \sin^2 A\\ &= 1 - \sin^2 A - \sin^2 A\\ &= 1 - 2\sin^2 A\end{aligned}$$

Example

If $\sin A = \frac{3}{5}$ with A in QII, find $\sin 2A$

Solution

$$\cos A = -\frac{4}{5}$$

$$\begin{aligned}\sin 2A &= 2\sin A\cos A\\ &= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)\\ &= -\frac{24}{25}\end{aligned}$$

Example

Prove $\quad (\sin\theta+\cos\theta)^2 = 1+\sin 2\theta$

Solution

$$
\begin{aligned}
(\sin\theta+\cos\theta)^2 &= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta \\
&= \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta \\
&= 1 + 2\sin\theta\cos\theta \\
&= 1+\sin 2\theta \quad \checkmark
\end{aligned}
$$

Example

Prove $\quad \sin 2x = \dfrac{2\cot x}{1+\cot^2 x}$

Solution

$$
\begin{aligned}
\frac{2\cot x}{1+\cot^2 x} &= \frac{2\frac{\cos x}{\sin x}}{1+\frac{\cos^2 x}{\sin^2 x}} \\
&= \frac{2\frac{\cos x}{\sin x}}{\frac{\sin^2 x+\cos^2 x}{\sin^2 x}} \\
&= 2\frac{\cos x}{\sin x}\frac{\sin^2 x}{\sin^2 x+\cos^2 x} \\
&= 2\frac{\cos x}{1}\frac{\sin x}{1} \\
&= 2\cos x\sin x \\
&= \sin 2x \quad \checkmark
\end{aligned}
$$

Example

Prove $\quad \cos 4x = 8\cos^4 x - 8\cos^2 x + 1$

Solution

$$
\begin{aligned}
\cos 4x &= \cos(2.2x) \\
&= 2\cos^2 2x - 1 \\
&= 2(\cos 2x)^2 - 1 \\
&= 2\left(2\cos^2 x - 1\right)^2 - 1
\end{aligned}
$$

$$= 2\left(4\cos^4 x - 4\cos^2 x + 1\right) - 1$$

$$= 8\cos^4 x - 8\cos^2 x + 2 - 1$$

$$\boxed{= 8\cos^4 x - 8\cos^2 x + 1}$$ ✓

$$\tan 2A = \tan(A + A)$$

$$= \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$\boxed{\tan 2A = \frac{2\tan A}{1 - \tan^2 A}}$$

Example

Simplify $\dfrac{2\tan 15^\circ}{1 - \tan^2 15^\circ}$

Solution

$$\frac{2\tan 15^\circ}{1 - \tan^2 15^\circ} = \tan(2 \cdot 15^\circ)$$

$$= \tan(30^\circ)$$

$$\boxed{= \frac{1}{\sqrt{3}}}$$

Example

Prove $\tan\theta = \dfrac{1 - \cos 2\theta}{\sin 2\theta}$

Solution

$$\frac{1 - \cos 2\theta}{\sin 2\theta} = \frac{1 - (1 - 2\sin^2\theta)}{2\sin\theta\cos\theta}$$

$$= \frac{1 - 1 + 2\sin^2\theta}{2\sin\theta\cos\theta}$$

$$= \frac{2\sin^2\theta}{2\sin\theta\cos\theta}$$

$$= \frac{\sin\theta}{\cos\theta}$$

$$\boxed{= \tan\theta}$$ ✓

Example

Given $\cos\theta = \frac{3}{5}$ and $\sin\theta < 0$, find $\sin 2\theta,\ \cos 2\theta,\ and\ \tan 2\theta$

Solution

$$\sin\theta = -\frac{4}{5}$$

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\left(-\frac{4}{5}\right)\left(\frac{3}{5}\right) \\ &= -\frac{24}{25}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= \left(\frac{3}{5}\right)^2 - \left(-\frac{4}{5}\right)^2 \\ &= \frac{9}{25} - \frac{16}{25} \\ &= -\frac{7}{25}\end{aligned}$$

$$\begin{aligned}\tan 2\theta &= \frac{\sin 2\theta}{\cos 2\theta} \\ &= \frac{24}{7}\end{aligned}$$

Half-Angle Formulas

$\cos 2A = 2\cos^2 A - 1$

$\cos 2x = 2\cos^2 x - 1$

$\cos 2x + 1 = 2\cos^2 x$

$2\cos^2 x = \cos 2x + 1$

$\cos^2 x = \dfrac{\cos 2x + 1}{2}$ *Divide both sides by 2*

$\cos x = \pm\sqrt{\dfrac{\cos 2x + 1}{2}}$ *Replace x with* $\frac{A}{2}$

$$\Rightarrow \boxed{\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}}$$

$\cos 2A = 1 - 2\sin^2 A$

$\cos 2x = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

$\sin^2 x = \dfrac{1 - \cos 2x}{2}$ *Divide both sides by 2*

$\sin x = \pm\sqrt{\dfrac{1 - \cos 2x}{2}}$ *Replace x with* $\frac{A}{2}$

$$\Rightarrow \boxed{\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}}$$

Example

If $\sin A = -\frac{12}{13}$ with $180° < A < 270°$ find the six trigonometric function of $A/2$

Solution

Since $180° < A < 270°$

$\cos A = -\sqrt{1 - \sin^2 A}$

$= -\frac{5}{13}$

$90° < \frac{A}{2} < 135° \quad \Rightarrow \frac{A}{2} \in QII$

$$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}}$$

$$= \sqrt{\frac{1}{2}\left(1 - \frac{-5}{13}\right)}$$

$$= \sqrt{\frac{1}{2}\left(\frac{13+5}{13}\right)}$$

$$= \sqrt{\frac{9}{13}}$$

$$\boxed{= \frac{3}{\sqrt{13}}}$$

$$\cos\frac{A}{2} = -\sqrt{\frac{1+\cos A}{2}}$$

$$= -\sqrt{\frac{1}{2}\left(1 + \frac{-5}{13}\right)}$$

$$= -\sqrt{\frac{1}{2}\frac{8}{13}}$$

$$= -\sqrt{\frac{4}{13}}$$

$$\boxed{= -\frac{2}{\sqrt{13}}}$$

$$\tan\frac{A}{2} = \frac{\frac{3}{\sqrt{13}}}{-\frac{2}{\sqrt{13}}} \qquad \tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}$$

$$\boxed{= -\frac{3}{2}}$$

$$\csc\frac{A}{2} \boxed{= \frac{\sqrt{13}}{3}} \qquad \csc\frac{A}{2} = \frac{1}{\sin\frac{A}{2}}$$

$$\sec\frac{A}{2} \boxed{= -\frac{\sqrt{13}}{2}} \qquad \sec\frac{A}{2} = \frac{1}{\cos\frac{A}{2}}$$

$$\cot\frac{A}{2} \boxed{= -\frac{2}{3}} \qquad \cot\frac{A}{2} = \frac{1}{\tan\frac{A}{2}}$$

$$\boxed{\tan\frac{A}{2} = \frac{1-\cos A}{\sin A}} \qquad \boxed{\tan\frac{A}{2} = \frac{\sin A}{1+\cos A}}$$

Example

Find the exact of $\tan 15°$

Solution

$$\begin{aligned}
\tan 15° &= \tan\frac{30°}{2} \\
&= \frac{1-\cos 30°}{\sin 30°} \\
&= \frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}} \\
&= \frac{\frac{2-\sqrt{3}}{2}}{\frac{1}{2}} \\
&= 2-\sqrt{3}
\end{aligned}$$

Example

Prove $\quad \sin^2\frac{x}{2} = \frac{\tan x - \sin x}{2\tan x}$

Solution

$$\begin{aligned}
\sin^2\frac{x}{2} &= \frac{1-\cos x}{2} \\
&= \frac{\tan x}{\tan x}\frac{1-\cos x}{2} \\
&= \frac{\tan x - \tan x\cos x}{2\tan x} \\
&= \frac{\tan x - \frac{\sin x}{\cos x}\cos x}{2\tan x} \\
&= \frac{\tan x - \sin x}{2\tan x}
\end{aligned}$$

✓

Exercises Section 8.3 – Double-angle Half-Angle Formulas

1. Let $\sin A = -\frac{3}{5}$ with A in Q*III* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin\frac{A}{2}$ *e)* $\cos\frac{A}{2}$ *f)* $\tan\frac{A}{2}$

2. Let $\sin A = \frac{3}{5}$ with A in Q*II* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin\frac{A}{2}$ *e)* $\cos\frac{A}{2}$ *f)* $\tan\frac{A}{2}$

3. Let $\cos A = \frac{3}{5}$ with A in Q*IV* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin\frac{A}{2}$ *e)* $\cos\frac{A}{2}$ *f)* $\tan\frac{A}{2}$

4. Let $\cos A = \frac{5}{13}$ with A in Q*I* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin\frac{A}{2}$ *e)* $\cos\frac{A}{2}$ *f)* $\tan\frac{A}{2}$

5. Let $\cos A = -\frac{12}{13}$ with A in Q*II* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin\frac{A}{2}$ *e)* $\cos\frac{A}{2}$ *f)* $\tan\frac{A}{2}$

6. Let $\sin A = -\frac{7}{25}$ with A in Q*III* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin\frac{A}{2}$ *e)* $\cos\frac{A}{2}$ *f)* $\tan\frac{A}{2}$

7. Let $\sin A = -\frac{24}{25}$ with A in Q*IV* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin\frac{A}{2}$ *e)* $\cos\frac{A}{2}$ *f)* $\tan\frac{A}{2}$

8. Let $\cos A = \frac{15}{17}$ with A in Q*I* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin\frac{A}{2}$ *e)* $\cos\frac{A}{2}$ *f)* $\tan\frac{A}{2}$

9. Let $\cos x = \frac{1}{\sqrt{10}}$ with x in Q*IV* and find $\cot 2x$

10. Verify: $(\cos x - \sin x)(\cos x + \sin x) = \cos 2x$

11. Verify: $\cot x \sin 2x = 1 + \cos 2x$

12. Simplify $\cos^2 7x - \sin^2 7x$

13. Write $\sin 3x$ in terms of $\sin x$

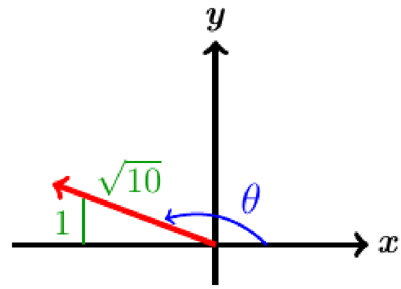
14. Find the values of the six trigonometric functions of θ if $\cos 2\theta = \frac{4}{5}$ *and* $90° < \theta < 180°$

15. Use half-angle formulas to find the exact value of $\sin 105°$

16. Find the exact of $\tan 22.5°$

17. Given: $\cos x = \frac{2}{3}$, $\frac{3\pi}{2} < x < 2\pi$, find $\cos\frac{x}{2}$, $\sin\frac{x}{2}$, and $\tan\frac{x}{2}$

18. Use a right triangle in Q*II* to find the value of $\cos\theta$ *and* $\tan\theta$



(19 – 46) Prove the following equation is an identity

19. $\cot\theta = \dfrac{\sin 2\theta}{1-\cos 2\theta}$

20. $\sin 3x = \sin x\left(3\cos^2 x - \sin^2 x\right)$

21. $\cos 3x = \cos^3 x - 3\cos x\sin^2 x$

22. $\cos^4 x - \sin^4 x = \cos 2x$

23. $\sin 2x = -2\sin x\sin\left(x-\frac{\pi}{2}\right)$

24. $\dfrac{\sin 4t}{4} = \cos^3 t\sin t - \sin^3 t\cos t$

25. $\dfrac{\cos 2x}{\sin^2 x} = \csc^2 x - 2$

26. $\dfrac{\cos 2x + \cos 2y}{\sin x + \cos y} = 2\cos y - 2\sin x$

27. $\dfrac{\cos 2x}{\cos^2 x} = \sec^2 x - 2\tan^2 x$

28. $\sin 4x = (4\sin x\cos x)\left(2\cos^2 x - 1\right)$

29. $\cos 2y = \dfrac{1-\tan^2 y}{1+\tan^2 y}$

30. $\cos 4x = \cos^4 x - 6\sin^2 x\cos^2 x + \sin^4 x$

31. $\tan^2 x(1+\cos 2x) = 1-\cos 2x$

32. $\dfrac{\cos 2x}{\sin^2 x} = 2\cot^2 x - \csc^2 x$

33. $\tan x + \cot x = 2\csc 2x$

34. $\tan 2x = \dfrac{2}{\cot x - \tan x}$

35. $\dfrac{1-\tan x}{1+\tan x} = \dfrac{1-\sin 2x}{\cos 2x}$

36. $\sin 2\alpha\sin 2\beta = \sin^2(\alpha+\beta) - \sin^2(\alpha-\beta)$

37. $\cos^2(A-B) - \cos^2(A+B) = \sin 2A\ \sin 2B$

38. $2\csc x\cos^2\frac{x}{2} = \dfrac{\sin x}{1-\cos x}$

39. $\tan\frac{\alpha}{2} = \sin\alpha + \cos\alpha\cot\alpha - \cot\alpha$

40. $\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right) = \dfrac{\sin^2 x}{4}$

41. $\tan\frac{x}{2} + \cot\frac{x}{2} = 2\csc x$

42. $2\sin^2\left(\frac{x}{2}\right) = \dfrac{\sin^2 x}{1+\cos x}$

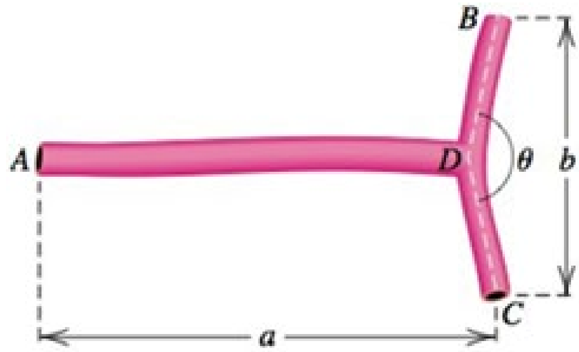
43. $\tan^2\left(\frac{x}{2}\right) = \dfrac{\sec x + \cos x - 2}{\sec x - \cos x}$

44. $\sec^2\left(\frac{x}{2}\right) = \dfrac{2\sec x + 2}{\sec x + 2 + \cos x}$

45. $\dfrac{1-\sin^2\left(\frac{x}{2}\right)}{1+\sin^2\left(\frac{x}{2}\right)} = \dfrac{1+\cos x}{3-\cos x}$

46. $\dfrac{1-\cos^2\left(\frac{x}{2}\right)}{1-\sin^2\left(\frac{x}{2}\right)} = \dfrac{1-\cos x}{1+\cos x}$

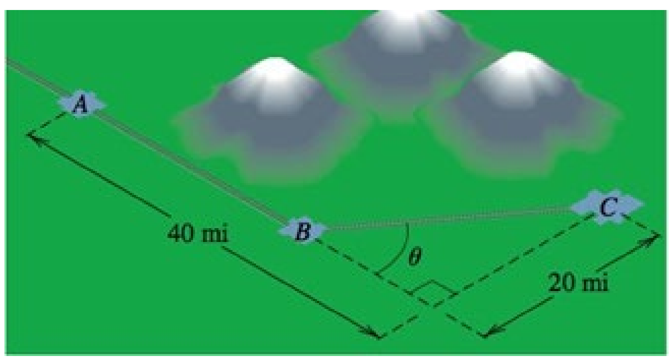
47. A common form of cardiovascular branching is bifurcation, in which an artery splits into two smaller blood vessels. The bifurcation angle θ is the angle formed by the two smaller arteries. The line through *A* and *D* bisects θ and is perpendicular to the line through *B* and *C*.



a) Show that the length l of the artery from *A* to *B* is given by $l = a + \frac{b}{2}\tan\frac{\theta}{4}$.

b) Estimate the length l from the three measurements $a = 10\ mm$, $b = 6\ mm$, and $\theta = 156°$.

48. A proposed rail road route through three towns located at points *A, B*, and *C*. At *B*, the track will turn toward *C* at an angle θ.



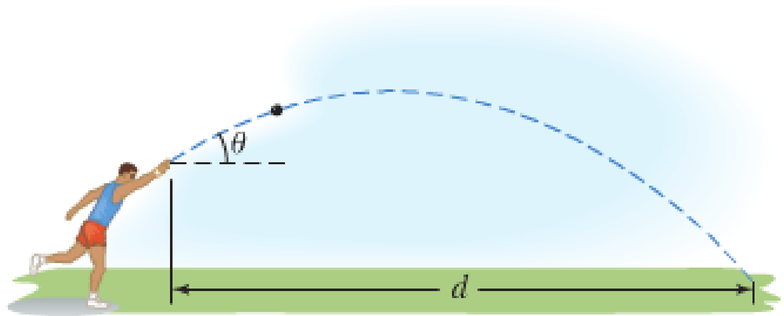
a) Show that the total distance *d* from *A* to *C* is given by $d = 20\tan\frac{1}{2}\theta + 40$

b) Because of mountains between *A* and *C*, the turning point *B* must be at least 20 *miles* from *A*.Is there a route that avoids the mountains and measures exactly 50 *miles*?

49. Throwing events in track and field include the shot put, the discus throw, the hammer throw, and the javelin throw. The distance that the athlete can achieve depends on the initial speed of the object thrown and the angle above the horizontal at which the object leaves the hand. This angle is represented by θ. The distance, *d*, in *feet*, that the athlete throws is modeled by the formula

$$d = \frac{v_0^2}{16}\sin\theta\cos\theta$$

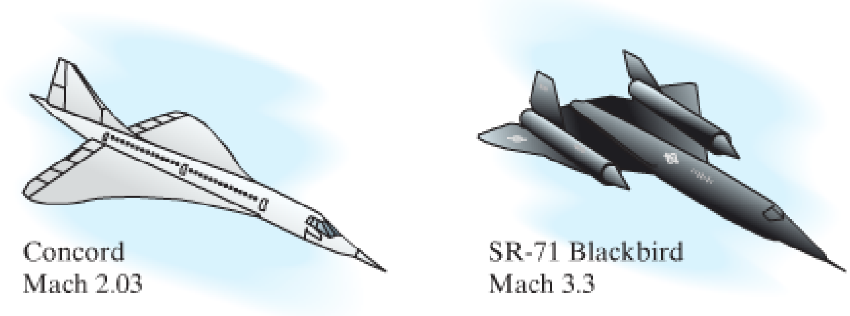
In which v_0 is the initial speed of the object thrown, in *feet* per *second*, and θ is the angle, in *degrees*, at which the object leaves the hand.



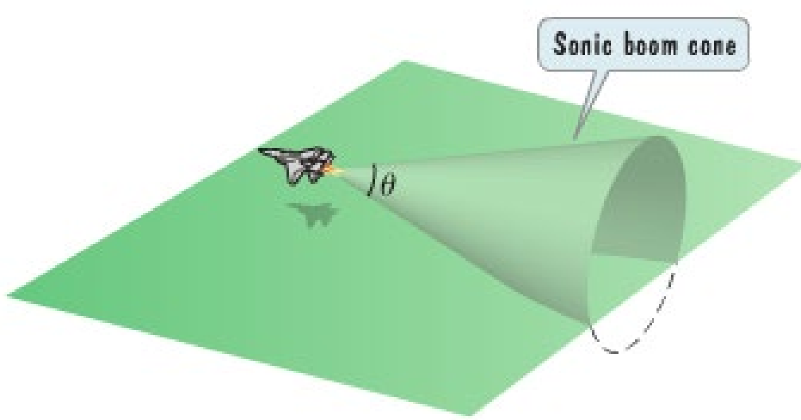
a) Use the identity to express the formula so that it contains the since function only.

b) Use the formula from part (*a*) to find the angle, θ, that produces the maximum distance, d, for a given initial speed, v_0 .

50. The speed of a supersonic aircraft is usually represented by a Mach number. A Mach number is the speed of the aircraft, in *miles* per *hour*, divided by the speed of sound, approximately 740 *mph*. Thus, a plane flying at twice the speed of sound has a speed, M, of Mach 2.



If an aircraft has a speed greater than Mach 1, a sonic boom is heard, created by sound waves that form a cone with a vertex angle θ.



The relationship between the cone's vertex angle θ, and the Mach speed, M, of an aircraft that is flying faster than the speed of sound is given by

$$\sin\frac{\theta}{2} = \frac{1}{M}$$

a) If $\theta = \frac{\pi}{6}$, determine the Mach speed, M, of the aircraft. Express the speed as an exact value and as decimal to the nearest tenth.

b) If $\theta = \frac{\pi}{4}$, determine the Mach speed, M, of the aircraft. Express the speed as an exact value and as decimal to the nearest tenth.