

## ***Solution***      **Section 3.6 – Newton’s Method**

### ***Exercise***

Use Newton’s method to estimate the on real solution of  $x^3 + 3x + 1 = 0$ .  
Start with  $x_0 = 0$  and then find  $x_2$

### **Solution**

$$y = x^3 + 3x + 1$$

$$y' = 3x^2 + 3$$

$$x_{n+1} = x_n - \frac{x_n^3 + 3x_n + 1}{3x_n^2 + 3}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\underline{x_0 = 0}$$

$$x_1 = x_0 - \frac{x_0^3 + 3x_0 + 1}{3x_0^2 + 3}$$

$$= 0 - \frac{0 + 3(0) + 1}{3(0) + 3}$$

$$\underline{= -\frac{1}{3}}$$

$$x_2 = x_1 - \frac{x_1^3 + 3x_1 + 1}{3x_1^2 + 3}$$
$$= -\frac{1}{3} - \frac{\left(-\frac{1}{3}\right)^3 + 3\left(-\frac{1}{3}\right) + 1}{3\left(-\frac{1}{3}\right) + 3}$$

$$\underline{= -0.3222}$$

### ***Exercise***

Use Newton’s method to estimate the on real solution of  $x^4 + x - 3 = 0$ . Start with  $x_0 = -1$  for the left-hand zero and with  $x_0 = 1$  for the zero on the right. Then, in each case, find  $x_2$

### **Solution**

$$y = x^4 + x - 3$$

$$y' = 4x^3 + 1$$

$$x_{n+1} = x_n - \frac{x_n^4 + x_n - 3}{4x_n^3 + 1}$$

$$\underline{x_0 = -1} \mid$$

$$x_1 = x_0 - \frac{x_0^4 + x_0 - 3}{4x_0^3 + 1}$$

$$= -1 - \frac{(-1)^4 + (-1) - 3}{4(-1)^3 + 1}$$

$$\underline{= -2} \mid$$

$$x_2 = x_1 - \frac{x_1^4 + x_1 - 3}{4x_1^3 + 1}$$

$$= -2 - \frac{(-2)^4 + (-2) - 3}{4(-2)^3 + 1}$$

$$\underline{= -1.64516} \mid$$

$$\underline{x_0 = 1} \mid$$

$$x_1 = x_0 - \frac{x_0^4 + x_0 - 3}{4x_0^3 + 1}$$

$$= 1 - \frac{(1)^4 + (1) - 3}{4(1)^3 + 1}$$

$$\underline{= \frac{6}{5}} \mid$$

$$x_2 = x_1 - \frac{x_1^4 + x_1 - 3}{4x_1^3 + 1}$$

$$= \frac{6}{5} - \frac{\left(\frac{6}{5}\right)^4 + \left(\frac{6}{5}\right) - 3}{4\left(\frac{6}{5}\right)^3 + 1}$$

$$\underline{= 1.16542} \mid$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

### Exercise

Use Newton's method to estimate the on real solution of  $2x - x^2 + 1 = 0$ . Start with  $x_0 = 0$  for the left-hand zero and with  $x_0 = 2$  for the zero on the right. Then, in each case, find  $x_2$

### Solution

$$y = 2x - x^2 + 1$$

$$y' = 2 - 2x$$

$$x_{n+1} = x_n - \frac{2x_n - x_n^2 + 1}{2 - 2x_n}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\underline{x_0 = 0}$$

$$x_1 = x_0 - \frac{2x_0 - x_0^2 + 1}{2 - 2x_0}$$

$$= 0 - \frac{2(0) - (0)^2 + 1}{2 - 2(0)}$$

$$\underline{= -0.5}$$

$$x_2 = x_1 - \frac{2x_1 - x_1^2 + 1}{2 - 2x_1}$$

$$= -0.5 - \frac{2(-0.5) - (-0.5)^2 + 1}{2 - 2(-0.5)}$$

$$\underline{= -0.41667}$$

$$\underline{x_0 = 2}$$

$$x_1 = x_0 - \frac{2x_0 - x_0^2 + 1}{2 - 2x_0}$$

$$= 2 - \frac{2(2) - (2)^2 + 1}{2 - 2(2)}$$

$$\underline{= 2.5}$$

$$x_2 = x_1 - \frac{2x_1 - x_1^2 + 1}{2 - 2x_1}$$

$$= 2.5 - \frac{2(2.5) - (2.5)^2 + 1}{2 - 2(2.5)}$$

$$\underline{= 2.41667}$$

### Exercise

Use Newton's method to estimate the on real solution of  $x^4 - 2 = 0$ . Start with  $x_0 = 1$  and then find  $x_2$

### Solution

$$y = x^4 - 2$$

$$y' = 4x^3$$

$$x_{n+1} = x_n - \frac{x_n^4 - 2}{4x_n^3}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\underline{x_0 = 1}$$

$$x_1 = x_0 - \frac{x_0^4 - 2}{4x_0^3}$$

$$= 1 - \frac{(1)^4 - 2}{4(1)^3}$$

$$= \frac{5}{4}$$

$$\underline{= 1.25}$$

$$x_2 = x_1 - \frac{x_1^4 - 2}{4x_1^3}$$

$$= 1.25 - \frac{(1.25)^4 - 2}{4(1.25)^3}$$

$$\underline{\approx 1.1935}$$

### Exercise

Use the Newton's method to approximate the roots to ten digits of  $f(x) = 3x^3 - 4x^2 + 1$

### Solution

By inspection:  $\underline{x_1 = 1}$  (root)

$$\begin{array}{c|cccc} 1 & 3 & -4 & 0 & 1 \\ & & 3 & -1 & -1 \\ \hline & 3 & -1 & -1 & 0 \end{array} \rightarrow 3x^2 - x - 1$$

$$f(x) = (x-1)(3x^2 - x - 1)$$

We apply Newton's method to  $g(x) = 3x^2 - x - 1$

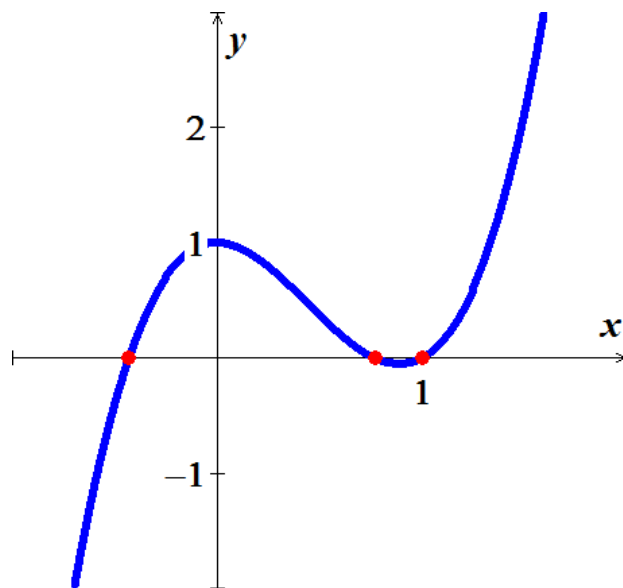
$$g(0) = -1 \quad g(1) = 1 \quad g'(x) = 6x - 1$$

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	0.0000000000	-1	-1	-1
1	-1	3.000000	-7.000000	-0.5714285714
2	-0.5714285714	0.5510204082	-4.4285714286	-0.4470046083
3	-0.4470046083	0.0464439678	-3.6820276497	-0.4343909149
4	-0.4343909149	0.0004773158	-3.6063454894	-0.4342585605
5	-0.4342585605	0.0000000525	-3.6055513629	-0.4342585459
6	-0.4342585459	0.0000000000	-3.6055512755	-0.4342585459

$$x_2 \approx -0.4342585459$$

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	1	1	5	0.8
1	0.8000000000	0.1200000000	3.8000000000	0.7684210526
2	0.7684210526	0.0029916898	3.6105263158	0.7675924505
3	0.7675924505	0.0000020597	3.6055547030	0.7675918792
4	0.7675918792	-0.0000000000	3.6055512755	0.7675918792

$$x_3 \approx 0.7675918792$$



### Exercise

Use the Newton's method to approximate the roots to ten digits of  $f(x) = e^{-2x} + 2e^x - 6$

### Solution

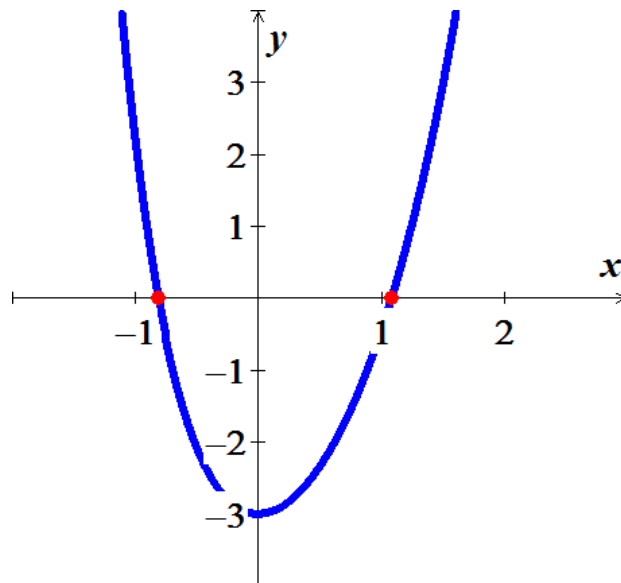
$$f'(x) = -2e^{-2x} + 2e^x$$

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	-1.0000000000	2.1248149813	-14.0423533155	-0.8486852642
1	-0.8486852642	0.3155271886	-10.0631909420	-0.8173306780
2	-0.8173306780	0.0109389885	-9.3722247034	-0.8161635070
3	-0.8161635070	0.0000145618	-9.3472814463	-0.8161619491
4	-0.8161619491	0.0000000000	-9.3472481901	-0.8161619491

$$x \approx -0.8161619491$$

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	1.0000000000	-0.4281010598	5.1658930904	1.0828706774
1	1.0828706774	0.0209547377	5.6769600505	1.0791794875
2	1.0791794875	0.0000433190	5.6534997356	1.0791718252
3	1.0791718252	0.0000000002	5.6534511061	1.0791718251

$$x \approx 1.0791718251$$



### Exercise

Use the Newton's method to approximate the roots to ten digits of  $f(x) = 2x^5 - 6x^3 - 4x + 2$

### Solution

$$f(x) = 10x^4 - 18x^2 - 4$$

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	0.0000000000	2.0000000000	-4.0000000000	0.5000000000
1	0.5000000000	-0.6875000000	-7.8750000000	0.4126984127
2	0.4126984127	-0.0485945125	-6.7756706818	0.4055265009
3	0.4055265009	-0.0003088207	-6.6896876153	0.4054803372
4	0.4054803372	-0.0000000128	-6.6891368363	0.4054803353

$$x \approx 0.4054803353$$

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	-2.0000000000	-6.0000000000	84.0000000000	-1.9285714286
1	-1.9285714286	-0.6062020289	67.3894731362	-1.9195759282
2	-1.9195759282	-0.0087501134	65.4495366742	-1.9194422357
3	-1.9194422357	-0.0000019108	65.4209537375	-1.9194422065
4	-1.9194422065	0.0000000000	65.4209474938	-1.9194422065

$$x \approx -1.9194422065$$

$n$	$x_n$	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	2.0000000000	10.0000000000	84.0000000000	1.8809523810
1	1.8809523810	1.6364873659	57.4894829058	1.8524865245
2	1.8524865245	0.0789319071	51.9953690888	1.8509684681
3	1.8509684681	0.0002159404	51.7110173775	1.8509642921
4	1.8509642921	0.0000000016	51.7102363689	1.8509642921

$$x \approx 1.8509642921$$

