Lecture Four – Vector Calculus

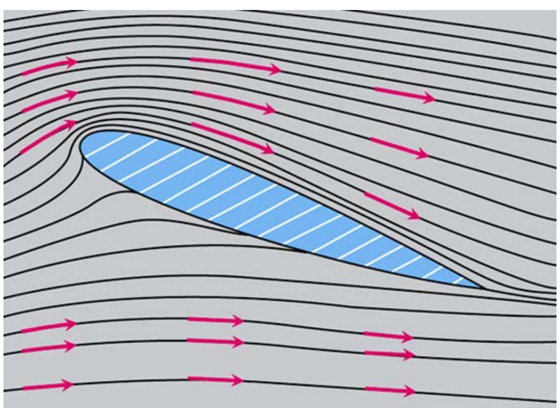
Section 4.1 – Vector Fields

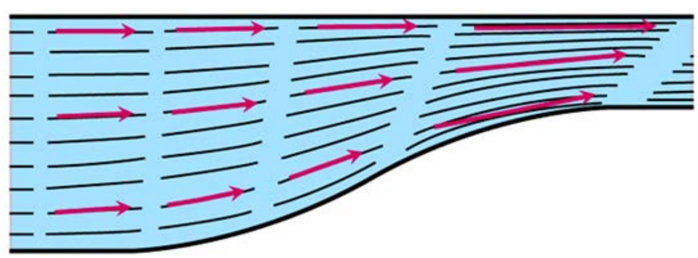
Gravitational and electric forces have both a direction and a magnitude. They are represented by a vector, in a subset of Euclidean space, at each point in their domain, producing a ***vector field***.

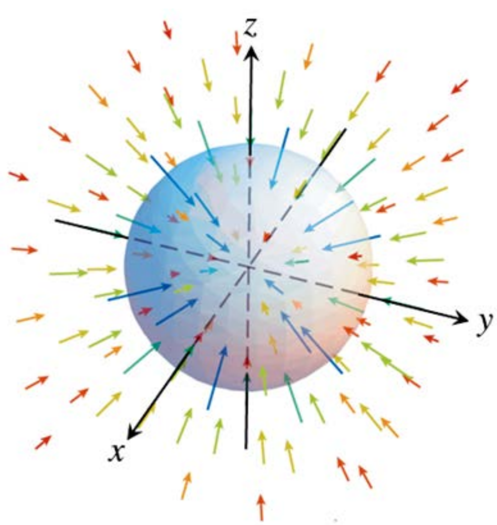
Vector fields are often used to model, for example, the speed and direction of a moving fluid throughout space, or the strength and direction of some force, such as the magnetic or gravitational force, as it changes from point to point. A line integral can be used to find the rate at which the fluid flows along or across a curve within the domain.

Vector Fields

Suppose a region in the plane or in space occupied by a moving fluid, such air or water. The fluid is made up of a large number of particles, where a particle has a velocity which can vary. Such a fluid flow is an example of a ***vector field***.
Vectors fields are associated with forces such as gravitational attraction, and to magnetic fields, electric fields, and also purely mathematical fields.





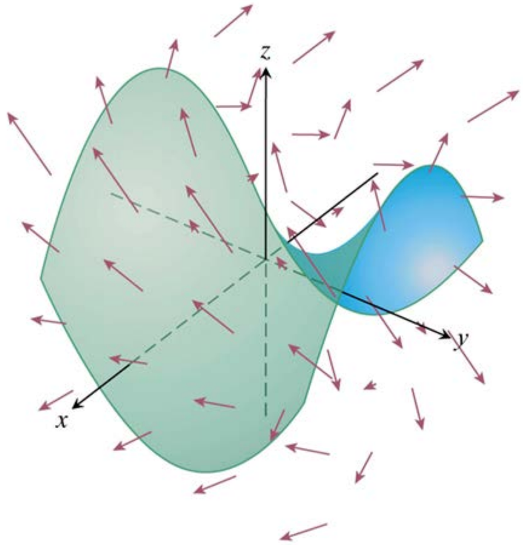


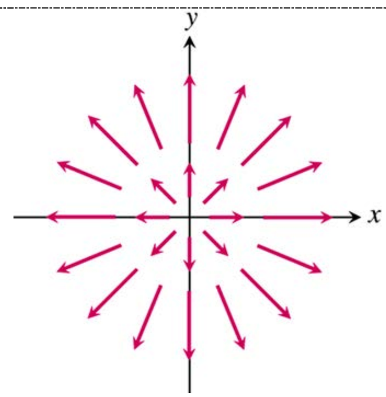
A vector filed is a function that assigns a vector to each point in its domain.

A vector field on a three-dimensional domain in space is given by

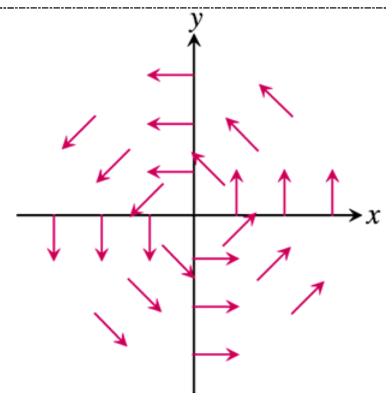
$$\vec{F}(x, y, z) = M(x, y, z)\hat{i} + N(x, y, z)\hat{j} + P(x, y, z)\hat{k}$$

The velocity field expression: $\vec{v}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$





The radial field $\vec{F} = x\hat{i} + y\hat{j}$ of position vectors of points in the plane.



A spin field of rotating unit vectors

$$\vec{F} = \frac{-y\hat{i} + x\hat{j}}{\sqrt{x^2 + y^2}}$$

In the plane

Definition

Let f and g be defined a region R of $\mathbb{R}^2$. A vector field in $\mathbb{R}^2$ is a function $\vec{F}$ assigns to each point in R a vector $\langle f(x, y), \ g(x, y)\rangle$. The vector field is written as

$$\vec{F}(x, y) = \langle f(x, y), \ g(x, y)\rangle \qquad \text{or}$$

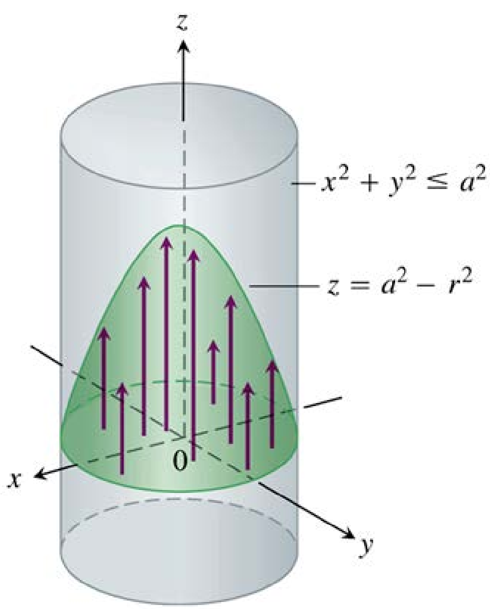
$$\vec{F}(x, y) = f(x, y)\hat{i} + g(x, y)\hat{j}$$

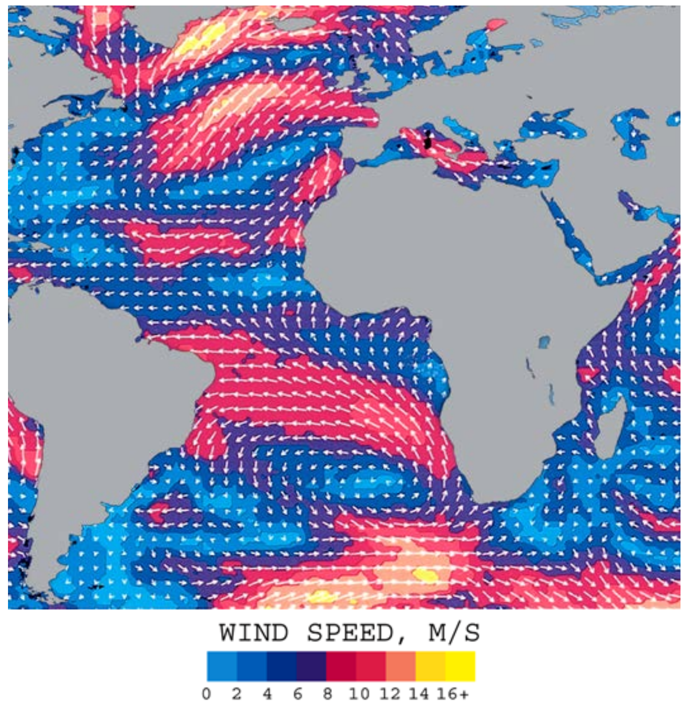
A vector field $\vec{F} = \langle f, \ g\rangle$ is continuous or differentiable on a region R of $\mathbb{R}^2$ if f and g are continuous or differentiable on R, respectively.

Gradient Fields

The gradient vector of a differentiable scalar-valued function at a point gives the direction of greatest increase of the function. An important type of vector field is formed by all the gradient vectors of the function. We define the gradient field of a differentiable function $f(x, y, z)$ to be the field gradient vectors

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$





Example

Suppose that the temperature T at each point (x, y, z) in a region of space is given by

$$T = 100 - x^2 - y^2 - z^2$$

And that $\vec{F}(x, y, z)$ is defined to be the gradient of T. Find the vector field $\vec{F}$.

Solution

The gradient field $\vec{F}$ is the field

$$\begin{aligned}\vec{F} &= \nabla T \\ &= -2x\hat{i} - 2y\hat{j} - 2z\hat{k}\end{aligned}$$

At each point in space, the field $\vec{F}$ gives the direction for which the increase in temperature is greatest.