# *Section* 2.8 – Row and Column Spaces

## *Definition*

For an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

The vectors

$$\begin{aligned} \vec{v}_1 &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \\ \vec{v}_2 &= \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \\ \vdots & \qquad\quad \vdots \\ \vec{v}_{\text{m}} &= \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{aligned}$$

In $\mathbb{R}^n$ that are formed from the rows of *A* are called the ***row vectors*** of *A*, and the vectors

$$\vec{v}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \cdots \quad \vec{v}_3 = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

In $\mathbb{R}^m$ that are formed from the rows of *A* are called the ***column vectors*** of *A*.

## *Definition*

If *A* is $m \times n$ matrix, then the subspace of $\mathbb{R}^n$ spanned by the row vectors of *A* is called the ***row space*** of *A* and is denoted by $RS(A)$ *or* $R(A)$, and the subspace $\mathbb{R}^m$ spanned by the row vectors of *A* is called the ***column space*** of *A* and is denoted by $CS(A)$ *or* $C(A)$. The solution space of the homogeneous system of equations $Ax = 0$, which is a subspace of $\mathbb{R}^n$ , is called the null space of *A*.

## The *Column Space* of *A*

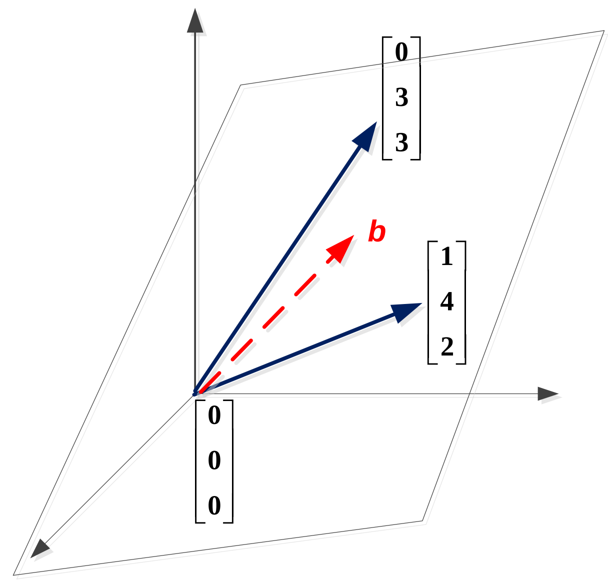
The most important subspaces are tied directly to a matrix $A$, to solve $A\vec{x} = b$.

### *Definition*

The column space consists of all linear combinations of the columns. The combination are all possible vectors $Ax$. They fill the column space $C(A)$.

$$A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$$

$$b = x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$




To solve $A\vec{x} = \vec{b}$ is to express $b$ as a combination of the columns.

The column space $CS(A)$ is a plane that containing the two columns. $A\vec{x} = b$ is solvable when $b$ in on that plane.

### *Theorem*

The system $A\vec{x} = b$ is solvable if and only if $b$ is in the column space of $A$.

### *Example*

Let $A\vec{x} = b$ be the linear system

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Show that ***b*** is in the column space of $A$ by expressing it as a linear combination of the column vectors of $A$.

***Solution***

$$\left[\begin{array}{ccc|c} -1 & 3 & 2 & 1 \\ 1 & 2 & -3 & -9 \\ 2 & 1 & -2 & -3 \end{array}\right] \quad \begin{array}{l} \\ R_2 + R_1 \\ R_3 + 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} -1 & 3 & 2 & 1 \\ 0 & 5 & -1 & -8 \\ 0 & 7 & 2 & -1 \end{array}\right] \quad \begin{array}{l} 5R_1 - 3R_2 \\ \\ 5R_3 - 7R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} -5 & 0 & 13 & 29 \\ 0 & 5 & -1 & -8 \\ 0 & 0 & 17 & 51 \end{array}\right] \quad \begin{array}{l} 17R_1 - 13R_3 \\ 17R_2 + R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} -85 & 0 & 0 & -170 \\ 0 & 85 & 0 & -85 \\ 0 & 0 & 17 & 51 \end{array}\right] \quad \begin{array}{l} -\frac{1}{85}R_1 \\ \frac{1}{85}R_2 \\ \frac{1}{17}R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

That implies to $x_1 = 2, \quad x_2 = -1, \quad x_3 = 3$

It follows that

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

### *Example*

Describe the column spaces (they are subspaces of $\mathbb{R}^2$) for

$$I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 4 \end{bmatrix}$$

***Solution***

The column space of $I$ is the whole space $\mathbb{R}^2$. Every vector is a combination of the columns of $I$. In the space language $CS(I)$ *is* $\mathbb{R}^2$.

The column space of $A$ is only a line, the second column (2, 4) is a multiple of the first column (1, 2) and (2, 4) and all other vectors $(c,\ 2c)$ along that line. The equation $A\vec{x} = \vec{b}$ is only solvable when $\vec{b}$ is on the line.

The column space $C(B)$ is all of $\mathbb{R}^2$. Every $b$ is attainable. The vector $\vec{b} = (3,\ 4)$ is summation of column 1 and 2.

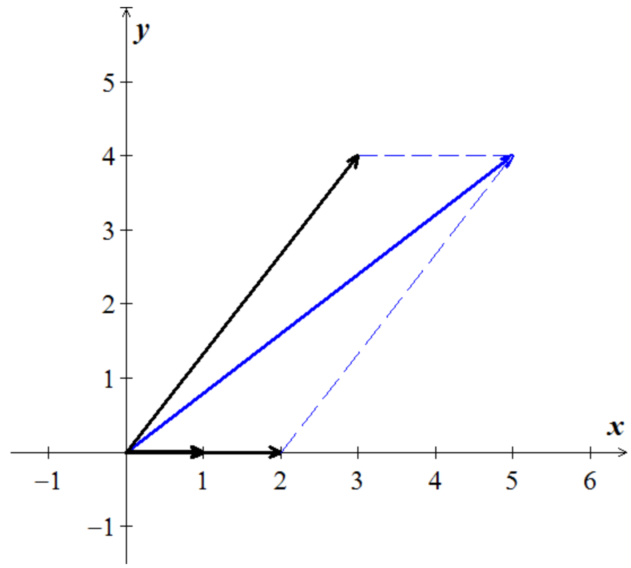
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ \qquad 4x_3 = 4 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 + 2x_2 = 2 \\ \quad x_3 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 1 \end{cases} \quad or \quad \Rightarrow \begin{cases} x_1 = 2 \\ x_2 = 0 \end{cases}$$

$$x = (0,\ 1,\ 1) \quad also \quad x = (2,\ 0,\ 1)$$




This matrix has the same column as $I$ and any $\vec{b}$ is allowed. $\vec{x}$ has an extra component (more solutions).

## *Pivot Columns*

The pivot columns of $R$ have 1's in the pivots and 0's everywhere else.

$$\text{Pivot columns: } A = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix}$$

$$\text{Yields to: } R = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- ***The pivot columns are not combinations of earlier columns. The free columns are combinations of columns which are the special solutions!***

## *Complete* **Solution** *to* $AX = B$

To solve $A\vec{x} = \vec{b}$, we need to put into an ***augmented*** form where $\vec{b}$ is not zero.

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \qquad B = \vec{b} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix}$$

$$X = \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The augmented matrix is just $\begin{bmatrix} A & B \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} R & \boldsymbol{d} \end{bmatrix}$$

### *Special Solutions*

Each special solution to $A\vec{x}=0$ and $R\vec{x}=0$ has one free variable equal to 1.

$$R\vec{x}=\begin{array}{c}\begin{bmatrix}1 & 3 & 0 & 2 & -1\\ 0 & 0 & 1 & 4 & -3\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}\\ \begin{matrix}\phantom{1} & F & \phantom{0} & F & F\end{matrix}\end{array}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

The ***free variables*** are $x_2,\ x_4,\ x_5$

$$\to\begin{cases}x_1+3x_2+2x_4-x_5=0\\ x_3+4x_4-3x_5=0\end{cases}$$

**1.** Set $x_2=1,\ x_4=x_5=0\Rightarrow\begin{cases}x_1=-3\\ x_3=0\end{cases}$ ***(Column 2)***

The special solution: $s_1=(-3,\ 1,\ 0,\ 0,\ 0)$

**2.** Set $x_4=1,\ x_2=x_5=0\Rightarrow\begin{cases}x_1=-2\\ x_3=-4\end{cases}$ ***(Column 4)***

The special solution: $s_2=(-2,\ 0,\ -4,\ 1,\ 0)$

**3.** Set $x_5=1,\ x_2=x_4=0\Rightarrow\begin{cases}x_1=1\\ x_3=3\end{cases}$ ***(Column 5)***

The special solution: $s_3=(1,\ 0,\ 3,\ 0,\ 1)$

The nullspace matrix $N$ contains the 3 special solutions in its columns.

$$N=\begin{bmatrix}-3 & -2 & 1\\ 1 & 0 & 0\\ 0 & -4 & 3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{array}{c}\textit{not free}\\ \textit{free}\\ \textit{not free}\\ \textit{free}\\ \textit{free}\end{array}$$

The linear combinations of these three columns give all vectors in the nullspace.

## One *Particular* Solution

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} R & \boldsymbol{d} \end{bmatrix}$$

The ***free variables*** for $R$ to be $x_2 = x_4$.

Then the equations give the ***pivot variables*** $x_1 = 1 \quad x_3 = 6$

The ***particular solution*** is: $\underline{(1,\ 0,\ 6,\ 0)}|$

The two special (nullspace) solutions to $Rx = 0$:

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{r} x_1 + 3x_2 + x_4 = 0 \\ x_3 + 4x_4 = 0 \end{array}$$

$$\Rightarrow \begin{cases} x_1 = -3x_2 - x_4 \\ x_3 = -4x_4 \end{cases}$$

$x_2 = 1,\ x_4 = 0$

$$\Rightarrow x_1 = -3,\ x_3 = 0 \quad \to \underline{(-3,\ 1,\ 0,\ 0)}|$$

$x_2 = 0,\ x_4 = 1$

$$\Rightarrow x_1 = -2,\ x_3 = -4 \to \underline{(-2,\ 0,\ -4,\ 1)}|$$

The ***complete solution***:

$$x = x_p + x_n$$

$$= \begin{pmatrix} 1 \\ 0 \\ 6 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix}$$

### *Example*

Find the condition on $(b_1, b_2, b_3)$ for $A\vec{x}=\vec{b}$ to be solvable, if

$$A=\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} \quad \vec{b}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

**Solution**

The augmented form:

$$\begin{bmatrix} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{bmatrix} \quad \begin{matrix} \\ R_2 - R_1 \\ R_3 + 2R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & -1 & b_3 + 2b_1 \end{bmatrix} \quad \begin{matrix} R_1 - R_2 \\ \\ R_3 + R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 2b_1 - b_2 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 + b_1 + b_2 \end{bmatrix} \quad \to \quad \underline{b_1 + b_2 + b_3 = 0|}$$

The last equation is 0 = 0 provided $b_1 + b_2 + b_3 = 0$.

There are ***no*** *free variables* and ***no*** *special solutions.*

The nullspace solution: $x_n = 0$

The complete solution:

$$x = x_p + x_n$$

$$= \begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $b_1 + b_2 + b_3 \neq 0$, there is no solution to $A\vec{x}=\vec{b}$ and $\vec{x}_p$ doesn't exist.

### *Example*

*a)* Find a subset of the vectors

$$\vec{v}_1 = (1,-2,0,3) \quad \vec{v}_2 = (2,-5,-3,6), \quad \vec{v}_3 = (0,1,3,0), \quad \vec{v}_4 = (2,-1,4,-7), \quad \vec{v}_5 = (5,-8,1,2)$$

That forms a basis for the space spanned by these vectors

*b)* Express each vector not in the basis as a linear combination of the basis vectors

***Solution***

***a)*** Construct the vectors as its column vectors

$$\begin{array}{cl} \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix} & \begin{array}{l} \\ R_2 + 2R_1 \\ \\ R_4 - 3R_1 \end{array} \end{array}$$

$$\begin{array}{cl} \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & -3 & 3 & 4 & 1 \\ 0 & 0 & 0 & -13 & -13 \end{bmatrix} & \begin{array}{l} R_1 + 2R_2 \\ \\ R_3 - 3R_2 \\ \\ \end{array} \end{array}$$

$$\begin{array}{cl} \begin{bmatrix} 5 & 0 & 10 & 0 & 5 \\ 0 & -5 & 5 & 0 & -5 \\ 0 & 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{l} \frac{1}{5}R_1 \\ -\frac{1}{5}R_2 \\ -\frac{1}{5}R_3 \\ \\ \end{array} \end{array}$$

$$\begin{array}{ccccc} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 & \vec{w}_4 & \vec{w}_5 \end{array} \end{array}$$

The leading 1's occurs in columns 1, 2, and 4.

$\left\{\vec{w}_1, \vec{w}_2, \vec{w}_4\right\}$ is a basis for the column space, and consequently $\left\{\vec{v}_1, \vec{v}_2, \vec{v}_4\right\}$

***b)*** $\vec{w}_1 = (1,\ 0,\ 0,\ 0) \quad \vec{w}_2 = (0,\ 1,\ 0,\ 0), \quad \vec{w}_3 = (2,\ -1,\ 0,\ 0)$

$\vec{w}_4 = (0,\ 0,\ 1,\ 0), \quad \vec{w}_5 = (1,\ 1,\ 1,\ 0)$

$$\vec{w}_3 = 2\vec{w}_1 - \vec{w}_2$$

$$\vec{w}_3 = \vec{w}_1 + \vec{w}_2 + \vec{w}_4$$

We call these ***dependency equations***

The corresponding relationships are:

$$\vec{v}_3 = 2\vec{v}_1 - \vec{v}_2$$

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 + \vec{v}_4$$

## Solving $Ax = 0$ by *elimination*

Matrix $A$ is rectangular and we still use the elimination.

1. Forward elimnation from $A$ to a triangular $U$.
2. Back substitution in $Ax = 0$ to find $x$.

Consider the matrix $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 4 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 4R_2 \end{matrix}$$

***Triangular U***: $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

***P***: The ***pivot*** variables are $x_1$ *and* $x_3$, since columns 1 and 3 contains pivots.

***F***: The ***free*** variables are $x_2$ *and* $x_4$, since columns 2 and 4 have no pivots.

Special solutions to:

$$\begin{cases} x_1 + x_2 + 2x_3 + 3x_4 = 0 \\ \quad 4x_3 + 4x_4 = 0 \end{cases} \rightarrow \begin{cases} x_1 = -x_2 - x_4 \\ x_3 = -x_4 \end{cases}$$

Complete solution: $x = x_2 \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{Special} + x_4 \underbrace{\begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{Special} = \underbrace{\begin{pmatrix} -x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix}}_{Complete}$

The special solution are in the nullspace $NS(A)$, and their combinations fill out the whole Nullspace.

## *Exercises* *Section* 2.8 – Row and Column Spaces

**1.** List the row vectors and column vectors of the matrix

$$\begin{bmatrix} 2 & -1 & 0 & 1 \\ 3 & 5 & 7 & -1 \\ 1 & 4 & 2 & 7 \end{bmatrix}$$

**(2 – 4)** Express the product $A\vec{x}$ as a linear combination of the column vectors of $A$.

**2.** $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

**3.** $\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$

**4.** $\begin{bmatrix} -3 & 6 & 2 \\ 5 & -4 & 0 \\ 2 & 3 & -1 \\ 1 & 8 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$

**(5 – 8)** Determine whether $\vec{b}$ is in the column space of $A$, and if so, express $\vec{b}$ as a linear combination of the column vectors of $A$.

**5.** $A = \begin{bmatrix} 1 & 3 \\ 4 & -6 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$

**6.** $A = \begin{bmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$

**7.** $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

**8.** $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ 3 \\ 5 \\ 7 \end{bmatrix}$

**9.** Suppose that $x_1 = -1, \quad x_2 = 2, \quad x_3 = 4, \quad x_4 = -3$ is a solution of a nonhomogeneous linear system $A\vec{x} = \vec{b}$ and that the solution set of the homogeneous system $A\vec{x} = \vec{0}$ is given by the formulas $\quad x_1 = -3r + 4s, \quad x_2 = r - s, \quad x_3 = r, \quad x_4 = s$

*a)* Find a vector form of the general solution of $A\vec{x} = \vec{0}$

*b)* Find a vector form of the general solution of $A\vec{x} = \vec{b}$

**(10 – 13)** Find the vector form of the general solution of the given linear system $A\vec{x} = \vec{b}$; then use that result to find the vector form of the general solution of $A\vec{x} = \vec{0}$.

**10.** $\begin{cases} x_1 - 3x_2 = 1 \\ 2x_1 - 6x_2 = 2 \end{cases}$

**11.** $\begin{cases} x_1 + x_2 + 2x_3 = 5 \\ x_1 \qquad + x_3 = -2 \\ 2x_1 + x_2 + 3x_3 = 3 \end{cases}$

**12.** $\begin{cases} x_1 + 2x_2 - 3x_3 + x_4 = 4 \\ -2x_1 + x_2 + 2x_3 + x_4 = -1 \\ -x_1 + 3x_2 - x_3 + 2x_4 = 3 \\ 4x_1 - 7x_2 \qquad - 5x_4 = -5 \end{cases}$

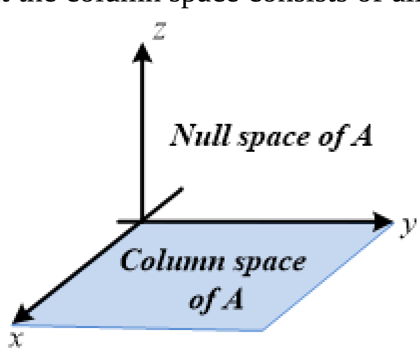
**13.** $\begin{cases} x_1 - 2x_2 + x_3 + 2x_4 = -1 \\ 2x_1 - 4x_2 + 2x_3 + 4x_4 = -2 \\ -x_1 + 2x_2 - x_3 - 2x_4 = 1 \\ 3x_1 - 6x_2 + 3x_3 + 6x_4 = -3 \end{cases}$

**14.** Given the vectors $\vec{v}_1 = (1,\ 2,\ 0)$ and $\vec{v}_2 = (2,\ 3,\ 0)$

*a)* Are they linearly independent?
*b)* Are they a basis for any space?
*c)* What space **V** do they span?
*d)* What is the dimension of that space?
*e)* What matrices $\boldsymbol{A}$ have **V** as their column space?
*f)* Which matrices have **V** as their nullspace?
*g)* Describe all vectors $\vec{v}_3$ that complete a basis $\vec{v}_1, \vec{v}_2, \vec{v}_3$ for $\mathbb{R}^3$.

**15.** *a*) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Show that relative to an *xyz*-coordinate system in 3-space the null space of *A* consists of all points on the *z*-axis and that the column space consists of all points in the *xy*-plane.




*b*) Find a 3 x 3 matrix whose null space is the *x*-axis and whose column space is the *yz*-plane.

**16.** If we add an extra column $\vec{b}$ to a matrix $A$, then the column space gets larger unless _____. Give an example where the column space gets larger and an example where it doesn't. Why is $A\vec{x}=\vec{b}$ solvable exactly when the column space doesn't get larger – it is the same for $A$ and $\begin{bmatrix} A & \vec{b} \end{bmatrix}$?

**17.** For which right sides (find a condition on $b_1$, $b_2$, $b_3$) are these solvable. (Use the column space $C(A)$ and the equation $A\vec{x}=\vec{b}$ )

*a)* $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

*b)* $\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

**18.** Show that the matrices $A$ and $\begin{bmatrix} A & AB \end{bmatrix}$ (with extra columns) have the same column space. But find a square matrix with $C\left(A^2\right)$ smaller than $C\left(A\right)$. Important point: An $n$ by $n$ matrix has $C\left(A\right)=\mathbb{R}^n$ exactly when $A$ is an ______ matrix.

**19.** The column of $AB$ are combinations of the columns of $A$. This means: The column space of $AB$ is contained in (possibly equal to) to the column space of $A$. Give an example where the column spaces $A$ and $AB$ are not equal.

**20.** Find a square matrix $A$ where $C\left(A^2\right)$ (the column space of $A^2$ is smaller than $C\left(A\right)$.

**21.** Suppose $A\vec{x}=\vec{b}$ and $C\vec{x}=\vec{b}$ have the same (complete) solutions for every $\vec{b}$ . Is true that $A=C$?

**22.** Apply Gauss-Jordan elimination to $U\vec{x}=0$ and $U\vec{x}=c$. Reach $R\vec{x}=0$ and $R\vec{x}=d$:

$$\begin{bmatrix} U & 0 \end{bmatrix}=\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \qquad \begin{bmatrix} U & c \end{bmatrix}=\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

Solve $R\vec{x}=0$ to find $x_n$ (its free variable is $x_2=1$).

Solve $R\vec{x}=d$ to find $x_p$ (its free variable is $x_2=0$).

***The subspace requirements:*** $x+y$ ***and*** $cx$ ***(and then all linear combinations*** $cx+dy$***) must stay in the subspace.***

**23.** Which of the following subsets of $\mathbb{R}^3$ are actually subspaces?

*a)* The plane of vectors $(b_1, b_2, b_3)$ with $b_1 = b_2$

*b)* The plane of vectors with $b_1 = 1$.

*c)* The vectors with $b_1 b_2 b_3 = 0$.

*d)* All linear combinations of $v = (1, 4, 0)$ and $w = (2, 2, 2)$.

*e)* All vectors that satisfies $b_1 + b_2 + b_3 = 0$

*f)* All vectors with $b_1 \le b_2 \le b_3$.

**24.** We are given three different vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3$. Construct a matrix so that the equations $A\vec{x} = \vec{b}_1$ and $A\vec{x} = \vec{b}_2$ are solvable, but $A\vec{x} = \vec{b}_3$ is not solvable.

***a)*** How can you decide if this possible?

***b)*** How could you construct $A$?

**25.** For which vectors $(b_1, b_2, b_3)$ do these systems have a solution?

*a)* $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

*c)* $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

*b)* $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

**26.** Find a basis for the null space of $A$. $A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

**27.** Is it true that is $m = n$ then the row space of $A$ equals the column space.

**28.** If the row space equals the column space the $A^T = A$

**29.** If $A^T = -A$, then the row space of $A$ equals the column space.

**30.** Does the matrices $A$ and $-A$ share the same 4 subspaces?

**31.** Is $A$ and $B$ share the same 4 subspaces then $A$ is multiple of $B$.

**32.** Suppose $A\vec{x} = \vec{b}$ & $C\vec{x} = \vec{b}$ have the same (complete) solutions for every $\vec{b}$ . Is it true that $A = C$

**33.** $A$ and $A^T$ have the same left nullspace?