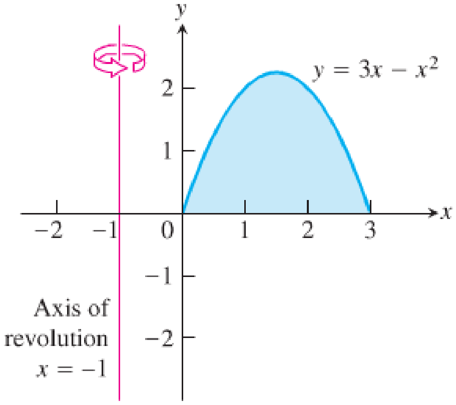
Section 1.4 – Volume by Shells

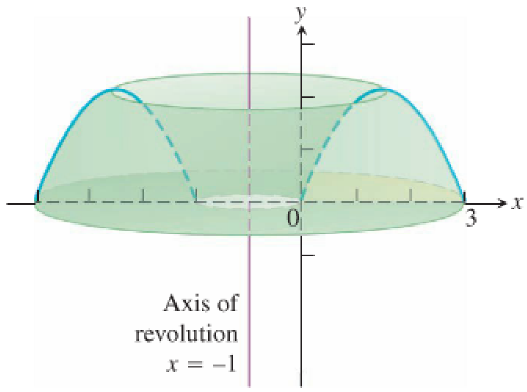
Slicing with Cylinders

Example

The region enclosed by the x-axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the vertical line $x = -1$ to generate a solid. Find the volume of the solid

Solution

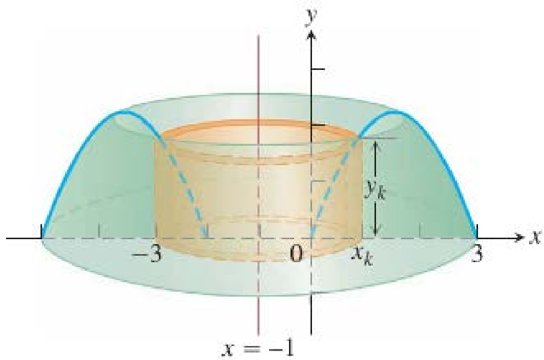


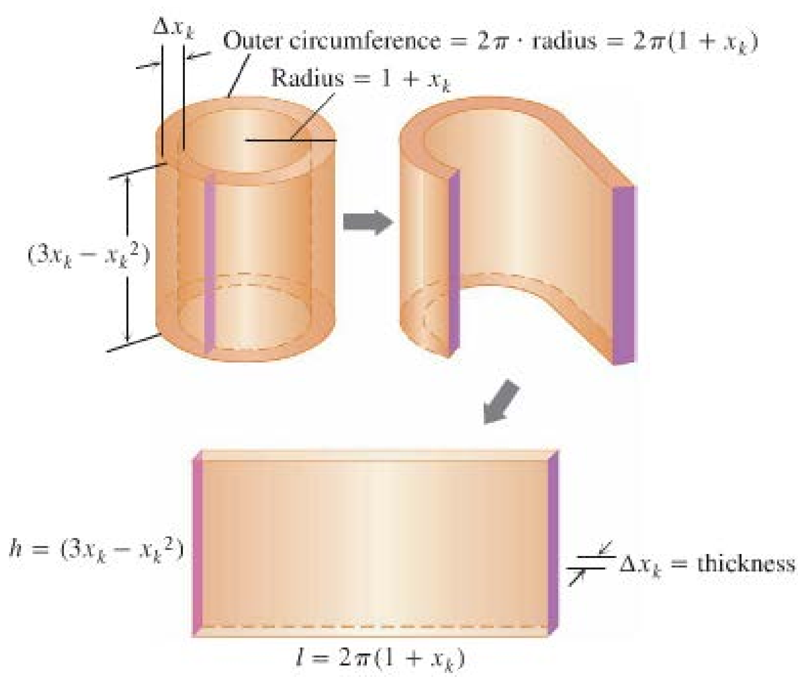


If we rotate a vertical strip of thickness Δx, this rotation produces a cylindrical shell of height y_k above a point x_k within the base of the vertical strip.

$$\Delta V_k = \textit{circumference} \times \textit{height} \times \textit{thickness}$$

$$= 2\pi\left(1 + x_k\right) \cdot \left(3x_k - x_k^2\right) \cdot \Delta x_k$$





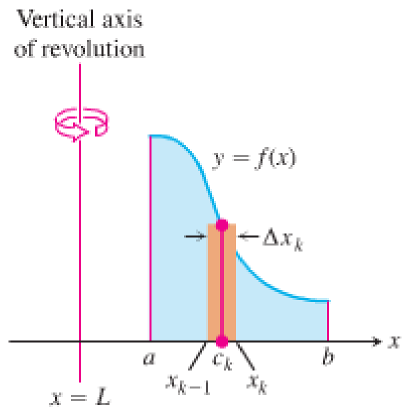
The Riemann sum:

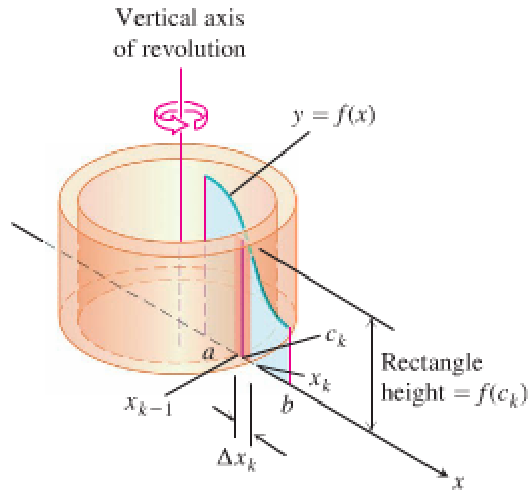
$$\sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} 2\pi\left(1+x_k\right)\cdot\left(3x_k - x_k^2\right)\cdot\Delta x_k$$

Taking the limit as the thickness $\Delta x_k \to 0$ and $n \to \infty$ gives the volume integral

$$\begin{aligned}
V &= \lim_{n\to\infty}\sum_{k=1}^{n} 2\pi\left(1+x_k\right)\cdot\left(3x_k - x_k^2\right)\cdot\Delta x_k \\
&= \int_0^3 2\pi\left(x+1\right)\left(3x-x^2\right)dx \\
&= 2\pi\int_0^3\left(3x^2+3x-x^2-x^3\right)dx \\
&= 2\pi\int_0^3\left(2x^2+3x-x^3\right)dx \\
&= 2\pi\left[\tfrac{2}{3}x^3+\tfrac{3}{2}x^2-\tfrac{1}{4}x^4\right]_0^3 \\
&= 2\pi\left[\tfrac{2}{3}(3)^3+\tfrac{3}{2}(3)^2-\tfrac{1}{4}(3)^4\right] \\
&= \frac{45\pi}{2}\ \ unit^3
\end{aligned}$$

Shell Method



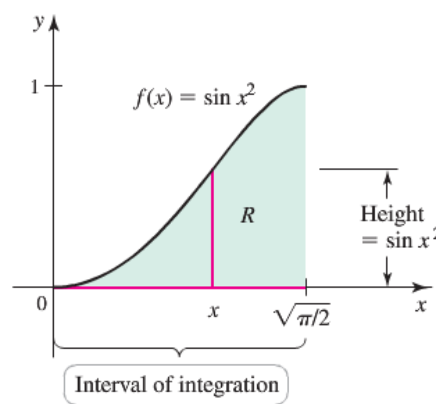


$$V = \int_a^b 2\pi \begin{pmatrix} shell \\ radius \end{pmatrix} \begin{pmatrix} shell \\ height \end{pmatrix} dx$$

Example

Let R be the region bounded by the graph of $f(x) = \sin x^2$, the x-axis, and the vertical line $x = \sqrt{\frac{\pi}{2}}$. Find the volume of the solid generated when R is revolved about the y-axis.

Solution



$$V = \int_a^b 2\pi \begin{pmatrix} shell \\ radius \end{pmatrix} \begin{pmatrix} shell \\ height \end{pmatrix} dx$$

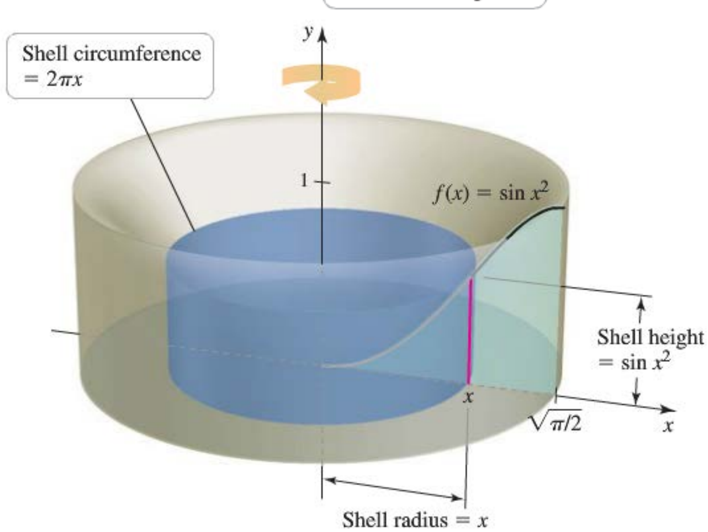
$$= 2\pi \int_0^{\sqrt{\pi/2}} x \sin x^2 \ dx$$

$$= \pi \int_0^{\sqrt{\pi/2}} \sin x^2 \ d\left(x^2\right)$$

$$= -\pi \cos\left(x^2\right) \Big|_0^{\sqrt{\pi/2}}$$

$$= -\pi \left(\cos\left(\frac{\pi}{2}\right) - \cos 0 \right)$$

$$= \pi \ \ unit^3$$



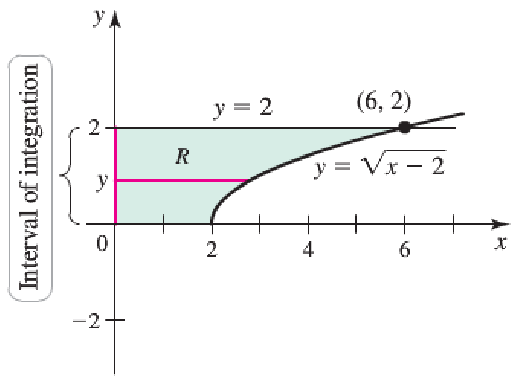
Example

Let R be the region in the first region bounded by the graph $y=\sqrt{x-2}$ and the line $y=2$.

a) Find the volume of the solid generated when R is revolved about the x-axis.

b) Find the volume of the solid generated when R is revolved about the line $y=-2$.

Solution



a) $y=\sqrt{x-2} \;\rightarrow\; y^2=x-2 \;\Rightarrow\; x=y^2+2$

$0\le y\le 2$

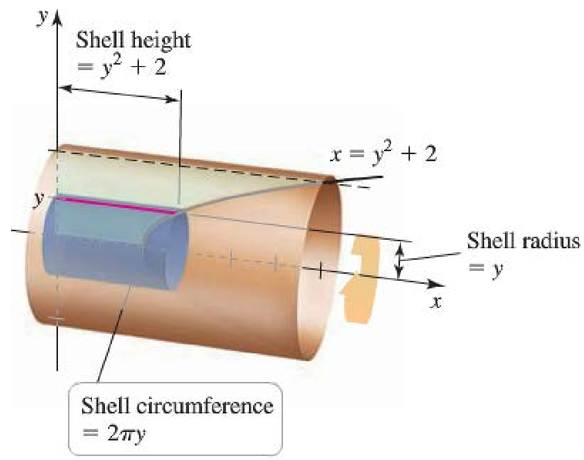
$$V=2\pi\int_c^d \left(\begin{matrix}shell\\ radius\end{matrix}\right)\left(\begin{matrix}shell\\ height\end{matrix}\right)dy$$

$$=2\pi\int_0^2 y\left(y^2+2\right)dy$$

$$=2\pi\int_0^2 \left(y^3+2y\right)dy$$

$$=2\pi\left(\frac{y^4}{4}+y^2\right)\Bigg|_0^2$$

$$\boxed{=16\pi\ \ unit^3}$$



b) Revolved R about the line $y=-2$.

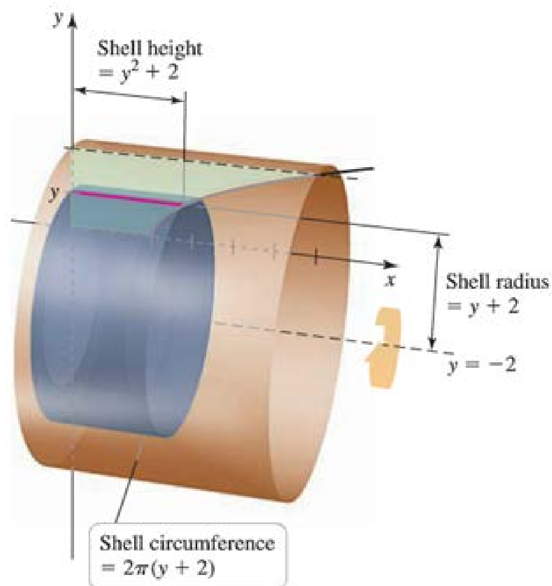
$$V=2\pi\int_c^d \left(\begin{matrix}shell\\ radius\end{matrix}\right)\left(\begin{matrix}shell\\ height\end{matrix}\right)dy$$

$$=2\pi\int_0^2 (y+2)\left(y^2+2\right)dy$$

$$=2\pi\left(\frac{1}{4}y^4+\frac{2}{3}y^3+y^2+4y\right)\Bigg|_0^2$$

$$=2\pi\left(4+\frac{16}{3}+4+8\right)$$

$$\boxed{=\frac{128\pi}{3}\ \ unit^3}$$



Example

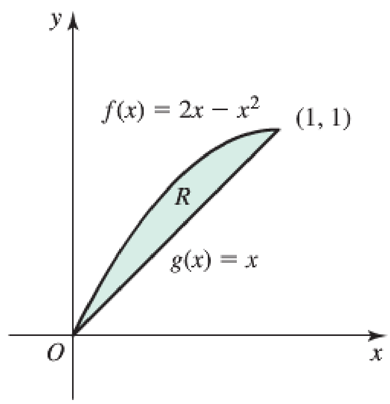
The region R is bounded by the graphs of $f(x)=2x-x^2$ and $g(x)=x$ on the interval $[0,\ 1]$.
Use the washer method and the shell method to find the volume of the solid formed when R is revolved about the x-axis.

Solution

$f(x)=g(x) \quad \rightarrow \quad 2x-x^2=x$

$x^2-x=0 \ \Rightarrow \ \underline{x=0,\ 1}$

Washer Method:



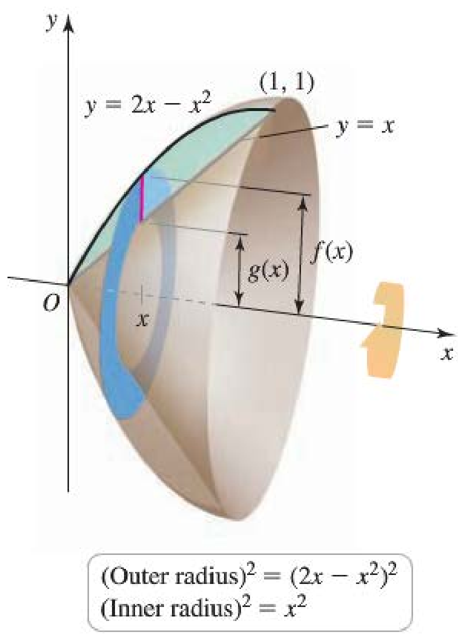
$$V=\pi\int_0^1\left[\left(2x-x^2\right)^2-x^2\right]dx$$

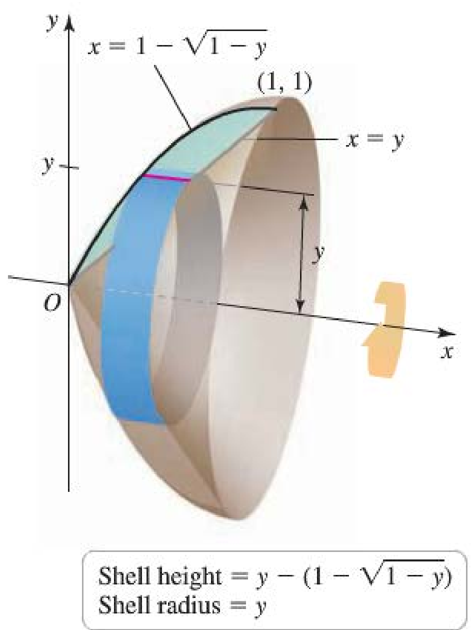
$$=\pi\int_0^1\left(3x^2-4x^3+x^4\right)dx$$

$$=\pi\left(x^3-x^4+\frac{x^5}{5}\right)\Big|_0^1$$

$$=\pi\left(1-1+\frac{1}{5}\right)$$

$$\underline{=\frac{\pi}{5}\ \ unit^3}$$





Shell Method:

$\underline{x=y}\quad y=2x-x^2 \ \rightarrow \ x^2-2x+y=0$

$$x=\frac{2\pm\sqrt{4-4y}}{2}=1-\sqrt{1-y} \qquad \cancel{=1+\sqrt{1-y}}$$

$x = 0 \quad \rightarrow y = 0$

$x = 1 \quad \rightarrow y = 1$

$$V = 2\pi \int_0^1 y\left[y - \left(1 - \sqrt{1-y}\right)\right] dy$$

$$= 2\pi \int_0^1 y\left[y - 1 + \sqrt{1-y}\right] dy$$

$$= 2\pi \int_0^1 \left(y^2 - y + y(1-y)^{1/2}\right) dy$$

$$= 2\pi \left(\frac{1}{3}y^3 - \frac{1}{2}y^2 + \frac{2}{5}(1-y)^{5/2} - \frac{2}{3}(1-y)^{3/2} \right) \Big|_0^1$$

$$= 2\pi \left(\frac{1}{3} - \frac{1}{2} - \left(\frac{2}{5} - \frac{2}{3} \right) \right)$$

$$= 2\pi \left(-\frac{1}{6} + \frac{4}{15} \right)$$

$$= 2\pi \left(\frac{9}{90} \right)$$

$$\boxed{= \frac{\pi}{5} \ \ unit^3}$$

Let $u = 1 - y \ \rightarrow \ y = 1 - u$ & $dy = -du$

$$\int y(1-y)^{1/2}\, dy = -\int (1-u)u^{1/2} du$$

$$= -\int \left(u^{1/2} - u^{3/2} \right) du$$

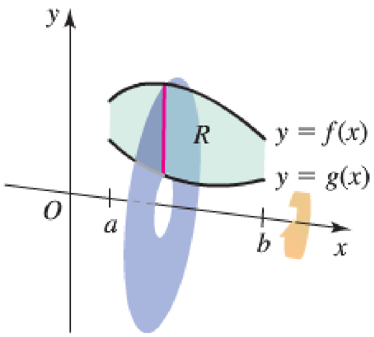
$$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}$$

$$= \frac{2}{5}(1-y)^{5/2} - \frac{2}{3}(1-y)^{3/2}$$

Summary of the Shell Method

1. Draw the region and sketch a line segment across it parallel to the axis of revolution. Label the segment's height or length (*shell height*) and distance from the axis of revolution (*shell radius*)
2. Find the limits of integration for the thickness variable.
3. Integrate the product 2π (*shell radius*) (*shell height*) with respect to the thickness variable (*x* or *y*) to find the volume

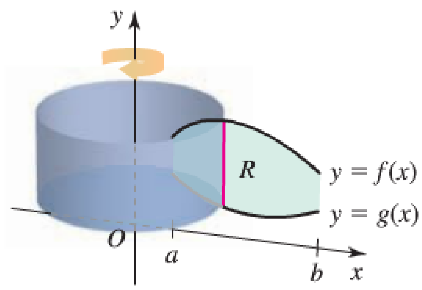
Integration With respect to x



Disk/washer method about the x-axis

Disks/washers are ***perpendicular*** to the *x*-axis

$$V = \pi \int_a^b \left(f(x)^2 - g(x)^2 \right) dx$$

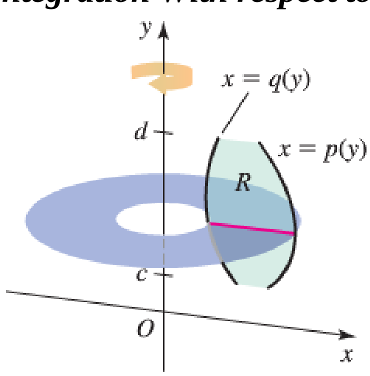


Shell method about the y-axis

Shells are ***parallel*** to the *y*-axis

$$V = 2\pi \int_a^b x\left(f(x) - g(x) \right) dx$$

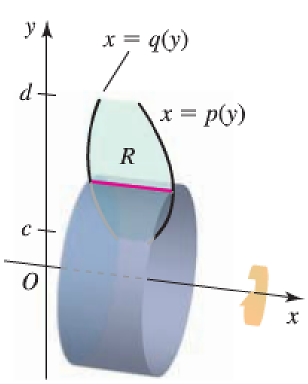
Integration With respect to y



Disk/washer method about the y-axis

Disks/washers are ***perpendicular*** to the *y*-axis

$$V = \pi \int_c^d \left(p(y)^2 - q(y)^2 \right) dy$$



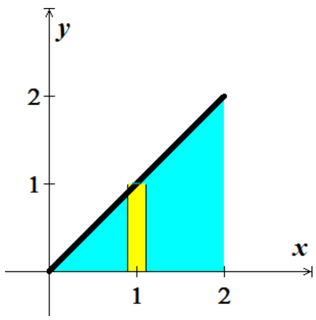
Shells are ***parallel*** to the *x*-axis

$$V = 2\pi \int_c^d y\left(p(y) - q(y) \right) dy$$

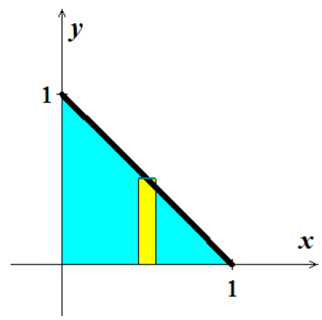
Exercises Section 1.4 – Volume by Shells

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

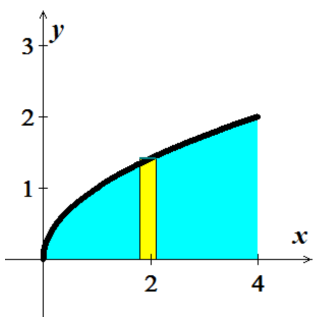
1. $y = x$



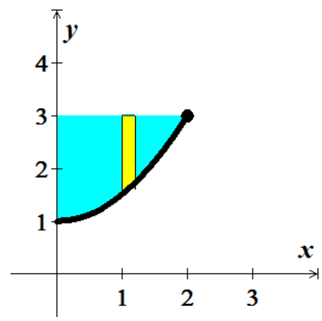
2. $y = 1 - x$



3. $y = \sqrt{x}$



4. $y = \frac{1}{2}x^2 + 1$



5. $y = \frac{1}{4}x^2, \quad y = 0, \quad x = 4$

6. $y = \frac{1}{2}x^3, \quad y = 0, \quad x = 3$

7. $y = x^2, \quad y = 4x - x^2$

8. $y = 9 - x^2, \quad y = 0$

9. $y = 4x - x^2, \quad x = 0, \quad y = 4$

10. $y = x^{3/2}, \quad y = 8, \quad x = 0$

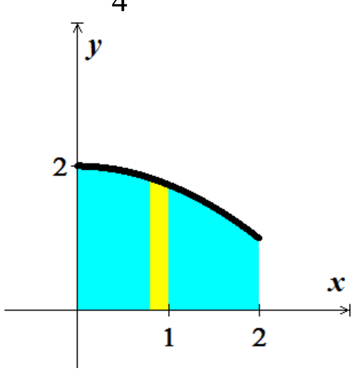
11. $y = \sqrt{x - 2}, \quad y = 0, \quad x = 4$

12. $y = -x^2 + 1, \quad y = 0$

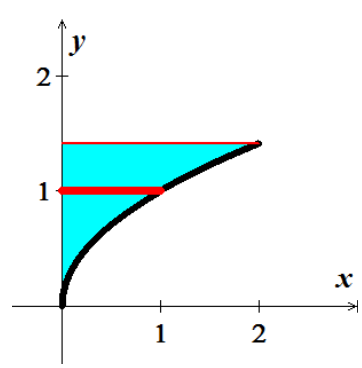
13. $y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad y = 0, \quad x = 0, \quad x = 1$

Use the shell method to find the volume of the solid generated by revolving the shaded region about the indicated axis

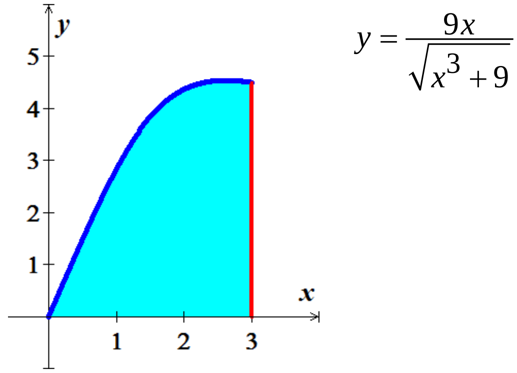
14. $y = 2 - \frac{1}{4}x^2$



15. $x = y^2$



16. Use the shell method to find the volume of the solid generated by revolving the shaded region about the *y*-axis

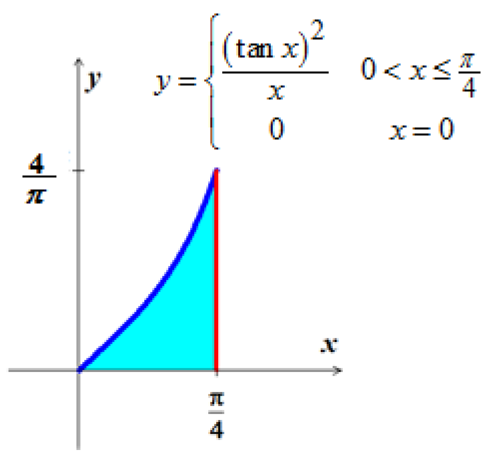


17. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = x^2, \quad y = 2 - x, \quad x = 0, \quad for\ x \ge 0$ about the *y*-axis.

18. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = 2 - x^2, \quad y = x^2, \quad x = 0$ about the *y*-axis.

19. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = \frac{3}{2\sqrt{x}}, \quad y = 0, \quad x = 1, \quad x = 4$ about the *y*-axis.

20. Let $g(x) = \begin{cases} \frac{(\tan x)^2}{x} & 0 < x \le \frac{\pi}{4} \\ 0 & x = 0 \end{cases}$



a) Show that $x \cdot g(x) = (\tan x)^2, \quad 0 \le x \le \frac{\pi}{4}$

b) Find the volume of the solid generated by revolving the shaded region about the *y*-axis.

21. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $x = \sqrt{y}, \quad x = -y, \quad y = 2$ about the *x*-axis.

22. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $x = y^2, \quad x = -y, \quad y = 2, \quad y \ge 0$ about the *x*-axis.

23. Compute the volume of the solid generated by revolving the region bounded by the lines $y = x$ *and* $y = x^2$ about each coordinate axis using

a) The *shell* method

b) The *washer* method

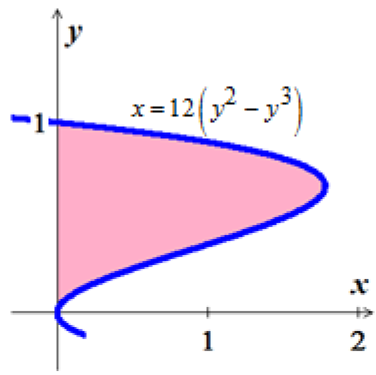
24. Use the shell method to find the volumes of the solids generated by revolving the shaded regions about the *indicated* axes.

a) The x-axis

b) The line $y = 1$

c) The line $y = \frac{8}{5}$

d) The line $y = -\frac{2}{5}$



25. Use the *washer* method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = \sqrt{x}, \quad y = 2, \quad x = 0$ about

a) the x-axis

b) the y-axis

c) 1the line $x = 4$

d) the line $y = 1$

26. The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ to generate a solid. Find the volume of the solid.

a) revolved about the x-axis

b) revolved about the y-axis

27. A cylinder hole with radius r is drilled symmetrically through the center of a sphere with radius R, where $r \le R$. What is the volume of the remaining material?

28. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = \sqrt{x}, \quad y = 2 - x, \quad y = 0$ about the x-axis.

29. Find the volume of the region bounded by $y = \frac{\ln x}{x^2}, \quad y = 0, \quad x = 1, \quad \text{and } x = 3$ revolved about the *y-axis*

30. Find the volume of the region bounded by $y = \frac{e^x}{x}, \quad y = 0, \quad x = 1, \quad \text{and } x = 2$ revolved about the *y-axis*

31. Find the volume of the region bounded by $y^2 = \ln x, \quad y^2 = \ln x^3, \quad \text{and } y = 2$ revolved about the *x-axis*

Find the volume using both the *disk/washer* and *shell* methods of

32. $y=(x-2)^3-2,\quad x=0,\quad y=25;$ revolved about the *y-axis*

33. $y=\sqrt{\ln x},\quad y=\sqrt{\ln x^2},\quad y=1;$ revolved about the *x-axis*

34. $y=\dfrac{6}{x+3},\quad y=2-x;$ revolved about the *x-axis*

Use the shell method to find the volume of the solid generated by the revolving the plane region about the given line

35. $y=2x-x^2,\quad y=0,\quad$ *about the line* $\quad x=4$

36. $y=\sqrt{x},\quad y=0,\quad x=4,\quad$ *about the line* $\quad x=6$

37. $y=x^2,\quad y=4x-x^2,\quad$ *about the line* $\quad x=4$

38. $y=\frac{1}{3}x^3,\quad y=6x-x^2,\quad$ *about the line* $\quad x=3$

Use the disk method or shell method to find the volume of the solid generated vy revolving the region bounded by the graph of the equations about the given lines.

39. $y=x^3,\quad y=0,\quad x=2$

a) *the x-axis* *b)* *the y-axis* *c)* *the line* $x=4$

40. $y=\dfrac{10}{x^2},\quad y=0,\quad x=1,\quad x=5$

a) *the x-axis* *b)* *the y-axis* *c)* *the line* $y=10$

41. Let V_1 and V_2 be the volumes of the solids that result when the plane region bounded by $y=\frac{1}{x}$, $y=0$, $x=\frac{1}{4}$, and $x=c$ $\left(\textit{where } c>\frac{1}{4}\right)$ is revolved about the *x-axis* and the *y-axis*, respectively. Find the value of c for which $V_1=V_2$

42. The region bounded by $y=r^2-x^2$, $y=0$, and $x=0$ is revolved about the *y-axis* to form a paraboloid. A hole, centered along the axis of revolution, is drilled through this solid. The hole has a radius k, $0<k<r$. Find the volume of the resulting ring

a) By integrating with respect to *x*.
b) By integrating with respect to *y*.