

Solution **Section 4.6 – Substitution Rule**

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int 2(2x+4)^5 dx, \quad u = 2x+4$$

Solution

$$\text{Let } u = 2x + 4 \Rightarrow du = 2x dx$$

$$\begin{aligned} \int 2(2x+4)^5 dx &= \int u^5 du \\ &= \frac{1}{6}u^6 + C \\ &= \frac{1}{6}(2x+4)^6 + C \end{aligned}$$

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int \frac{4x^3}{(x^4+1)^2} dx, \quad u = x^4+1$$

Solution

$$\text{Let } u = x^4 + 1$$

$$du = 4x^3 dx$$

$$\begin{aligned} \int \frac{4x^3}{(x^4+1)^2} dx &= \int \frac{du}{u^2} \\ &= -\frac{1}{u} + C \\ &= -\frac{1}{x^4+1} + C \end{aligned}$$

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int x \sin(2x^2) dx, \quad u = 2x^2$$

Solution

$$\text{Let } u = 2x^2$$

$$du = 4x dx \rightarrow \frac{1}{4} du = x dx$$

$$\begin{aligned} \int x \sin(2x^2) dx &= \int \frac{1}{4} \sin u \, du \\ &= -\frac{1}{4} \cos u + C \\ &= -\frac{1}{4} \cos(2x^2) + C \end{aligned}$$

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int 12(y^4 + 4y^2 + 1)^2 (y^3 + 2y) dx, \quad u = y^4 + 4y^2 + 1$$

Solution

$$\text{Let } u = y^4 + 4y^2 + 1$$

$$du = (4y^3 + 8y) dx \rightarrow du = 4(y^3 + 2y) dx$$

$$\begin{aligned} \int 12(y^4 + 4y^2 + 1)^2 (y^3 + 2y) dx &= \int 12u^2 \left(\frac{1}{4} du\right) \\ &= 3 \int u^2 \, du \\ &= 3 \frac{u^3}{3} + C \\ &= (y^4 + 4y^2 + 1)^3 + C \end{aligned}$$

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int \csc^2 2\theta \cot 2\theta \, d\theta \rightarrow \begin{cases} \text{a) Using } u = \cot 2\theta \\ \text{b) Using } u = \csc 2\theta \end{cases}$$

Solution

$$\text{Let } u = \cot 2\theta \Rightarrow du = -2 \csc^2 2\theta d\theta \rightarrow -\frac{1}{2} du = \csc^2 2\theta d\theta$$

$$\begin{aligned} \int \csc^2 2\theta \cot 2\theta \, d\theta &= -\int \frac{1}{2} u du \\ &= -\frac{1}{2} \frac{u^2}{2} + C \\ &= -\frac{1}{4} \cot^2 2\theta + C \end{aligned}$$

$$\text{Let } u = \csc 2\theta$$

$$du = -2 \csc 2\theta \cot 2\theta d\theta$$

$$-\frac{1}{2} du = \csc 2\theta \cot 2\theta d\theta$$

$$\begin{aligned} \int \csc^2 2\theta \cot 2\theta \, d\theta &= \int \csc 2\theta (\csc 2\theta \cot 2\theta \, d\theta) \\ &= -\int \frac{1}{2} u \, du \\ &= -\frac{1}{2} \frac{u^2}{2} + C \\ &= -\frac{1}{4} \csc^2 2\theta + C \end{aligned}$$

Exercise

Evaluate the integrals $\int \frac{1}{\sqrt{5s+4}} \, ds$

Solution

$$\text{Let } u = 5s + 4$$

$$du = 5ds$$

$$\frac{1}{5} du = ds$$

$$\int \frac{1}{\sqrt{5s+4}} \, ds = \frac{1}{5} \int u^{-1/2} \, du$$

$$\begin{aligned}
 &= \frac{1}{5} \frac{u^{1/2}}{1/2} + C \\
 &= \frac{2}{5} \sqrt{5s+4} + C
 \end{aligned}$$

Exercise

Evaluate the integrals $\int \theta \sqrt[4]{1-\theta^2} d\theta$

Solution

$$\text{Let } d(1-\theta^2) = -2\theta d\theta$$

$$\begin{aligned}
 \int \theta \sqrt[4]{1-\theta^2} d\theta &= -\frac{1}{2} \int (1-\theta^2)^{1/4} d(1-\theta^2) \\
 &= -\frac{2}{5} (1-\theta^2)^{5/4} + C
 \end{aligned}$$

Exercise

Evaluate the integrals $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2} dx$

Solution

$$d(1+\sqrt{x}) = \frac{1}{2\sqrt{x}} dx$$

$$\begin{aligned}
 \int \frac{1}{\sqrt{x}(1+\sqrt{x})^2} dx &= \int \frac{2}{(1+\sqrt{x})^2} d(1+\sqrt{x}) \\
 &= -\frac{2}{1+\sqrt{x}} + C
 \end{aligned}$$

Exercise

Evaluate the integrals $\int \tan^2 x \sec^2 x dx$

Solution

$$d(\tan x) = \sec^2 x dx$$

$$\int \tan^2 x \sec^2 x \, dx = \int \tan^2 x \, d(\tan x)$$

$$\underline{= \frac{1}{3} \tan^3 x + C}$$

Exercise

Evaluate the integrals $\int \sin^5 \frac{x}{3} \cos \frac{x}{3} \, dx$

Solution

$$\text{Let } d\left(\sin\left(\frac{x}{3}\right)\right) = \frac{1}{3} \cos\left(\frac{x}{3}\right) dx$$

$$\int \sin^5 \frac{x}{3} \cos \frac{x}{3} \, dx = 3 \int \sin^5 \frac{x}{3} \, d\left(\sin \frac{x}{3}\right)$$

$$\underline{= \frac{1}{2} \sin^6\left(\frac{x}{3}\right) + C}$$

Exercise

Evaluate the integrals $\int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2} \, dx$

Solution

$$\text{Let } u = \tan\left(\frac{x}{2}\right)$$

$$du = \frac{1}{2} \sec^2\left(\frac{x}{2}\right) dx \rightarrow 2du = \sec^2\left(\frac{x}{2}\right) dx$$

$$\int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2} \, dx = 2 \int u^7 \, du$$

$$= 2 \frac{1}{8} u^8 + C$$

$$\underline{= \frac{1}{4} \tan^8 \frac{x}{2} + C}$$

$$\int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2} \, dx = 2 \int \tan^7 \frac{x}{2} \, d\left(\tan \frac{x}{2}\right)$$

$$\underline{= \frac{1}{4} \tan^8 \frac{x}{2} + C}$$

Exercise

Evaluate the integrals $\int r^4 \left(7 - \frac{r^5}{10}\right)^3 dr$

Solution

$$\text{Let } d\left(7 - \frac{r^5}{10}\right) = -\frac{1}{2} r^4 \, dr$$

$$\int r^4 \left(7 - \frac{r^5}{10}\right)^3 dr = -2 \int \left(7 - \frac{r^5}{10}\right)^3 d\left(7 - \frac{r^5}{10}\right)$$

$$= -\frac{1}{2} \left(7 - \frac{r^5}{10}\right)^4 + C$$

Exercise

Evaluate the integrals $\int x^{1/2} \sin(x^{3/2} + 1) dx$

Solution

$$d(x^{3/2} + 1) = \frac{3}{2} x^{1/2} dx$$

$$\int x^{1/2} \sin(x^{3/2} + 1) dx = \frac{2}{3} \int \sin(x^{3/2} + 1) d(x^{3/2} + 1)$$

$$= -\frac{2}{3} \cos(x^{3/2} + 1) + C$$

Let $u = x^{3/2} + 1$

$$du = \frac{3}{2} x^{1/2} dx$$

$$\frac{2}{3} du = x^{1/2} dx$$

$$\int x^{1/2} \sin(x^{3/2} + 1) dx = \int \sin u \left(\frac{2}{3} du\right)$$

$$= \frac{2}{3} \int \sin u du$$

$$= \frac{2}{3} (-\cos u) + C$$

$$= -\frac{2}{3} \cos(x^{3/2} + 1) + C$$

Exercise

Evaluate the integrals $\int \csc\left(\frac{v-\pi}{2}\right) \cot\left(\frac{v-\pi}{2}\right) dv$

Solution

$$d\left(\csc\left(\frac{v-\pi}{2}\right)\right) = -\frac{1}{2} \csc\left(\frac{v-\pi}{2}\right) \cot\left(\frac{v-\pi}{2}\right) dv$$

$$\frac{d}{dv}\left(\frac{v-\pi}{2}\right) = \frac{1}{2}$$

$$\int \csc\left(\frac{v-\pi}{2}\right) \cot\left(\frac{v-\pi}{2}\right) dv = -\frac{1}{2} \int d\left(\csc\left(\frac{v-\pi}{2}\right)\right)$$

$$\underline{= -2 \csc\left(\frac{v-\pi}{2}\right) + C}$$

Let $u = \csc\left(\frac{v-\pi}{2}\right)$

$$du = -\frac{1}{2} \csc\left(\frac{v-\pi}{2}\right) \cot\left(\frac{v-\pi}{2}\right) dv$$

$$-2du = \csc\left(\frac{v-\pi}{2}\right) \cot\left(\frac{v-\pi}{2}\right) dv$$

$$\int \csc\left(\frac{v-\pi}{2}\right) \cot\left(\frac{v-\pi}{2}\right) dv = \int -2du$$

$$= -2u + C$$

$$\underline{= -2 \csc\left(\frac{v-\pi}{2}\right) + C}$$

Exercise

Evaluate the integrals $\int \frac{\sin(2t+1)}{\cos^2(2t+1)} dt$

Solution

$$\int \frac{\sin(2t+1)}{\cos^2(2t+1)} dt = -\frac{1}{2} \int \frac{d(\cos(2t+1))}{\cos^2(2t+1)}$$

$$\underline{= \frac{1}{2 \cos(2t+1)} + C}$$

Exercise

Evaluate the integrals $\int \frac{\sec z \tan z}{\sqrt{\sec z}} dz$

Solution

$$d(\sec z) = \sec z \tan z dz$$

$$\int \frac{\sec z \tan z}{\sqrt{\sec z}} dz = \int (\sec z)^{-1/2} d(\sec z)$$

$$\underline{= 2\sqrt{\sec z} + C}$$

Let $u = \sec z \Rightarrow du = \sec z \tan z dz$

$$\begin{aligned}\int \frac{\sec z \tan z}{\sqrt{\sec z}} dz &= \int \frac{du}{u^{1/2}} \\ &= \int u^{-1/2} du \\ &= 2u^{1/2} + C \\ &= \underline{2\sqrt{\sec z} + C}\end{aligned}$$

Exercise

Evaluate the integrals $\int \frac{1}{\sqrt{t}} \cos(\sqrt{t} + 3) dt$

Solution

$$d(\sqrt{t} + 3) = \frac{1}{2\sqrt{t}} dt$$

$$\begin{aligned}\int \frac{1}{\sqrt{t}} \cos(\sqrt{t} + 3) dt &= 2 \int \cos(\sqrt{t} + 3) d(\sqrt{t} + 3) \\ &= \underline{2 \sin(\sqrt{t} + 3) + C}\end{aligned}$$

$$u = \sqrt{t} + 3$$

$$du = \frac{1}{2\sqrt{t}} dt$$

$$2du = \frac{1}{\sqrt{t}} dt$$

$$\begin{aligned}\int \frac{1}{\sqrt{t}} \cos(\sqrt{t} + 3) dt &= \int (\cos u)(2du) \\ &= 2 \int \cos u du \\ &= 2 \sin u + C \\ &= \underline{2 \sin(\sqrt{t} + 3) + C}\end{aligned}$$

Exercise

Evaluate the integrals $\int \frac{1}{\theta^2} \sin \frac{1}{\theta} \cos \frac{1}{\theta} d\theta$

Solution

$$\frac{d}{d\theta}\left(\frac{1}{\theta}\right) = -\frac{1}{\theta^2}$$

$$d\left(\cos\frac{1}{\theta}\right) = \frac{1}{\theta^2} \sin\frac{1}{\theta} d\theta$$

$$\begin{aligned} \int \frac{1}{\theta^2} \sin\frac{1}{\theta} \cos\frac{1}{\theta} d\theta &= \int \cos\frac{1}{\theta} d\left(\cos\frac{1}{\theta}\right) \\ &= \frac{1}{2} \cos^2\frac{1}{\theta} + C \end{aligned}$$

Let $u = \sin\frac{1}{\theta}$

$$\begin{aligned} du &= \left(\cos\frac{1}{\theta}\right)\left(\frac{1}{\theta}\right)' \\ &= \left(\cos\frac{1}{\theta}\right)\left(-\frac{1}{\theta^2}\right) d\theta \end{aligned}$$

$$-du = \frac{1}{\theta^2} \cos\frac{1}{\theta} d\theta$$

$$\begin{aligned} \int \frac{1}{\theta^2} \sin\frac{1}{\theta} \cos\frac{1}{\theta} d\theta &= -\int u du \\ &= -\frac{1}{2} u^2 + C \\ &= -\frac{1}{2} \sin^2\frac{1}{\theta} + C \end{aligned}$$

Exercise

Evaluate the integrals $\int t^3(1+t^4)^3 dt$

Solution

$$d(1+t^4) = 4t^3 dt$$

$$\begin{aligned} \int t^3(1+t^4)^3 dt &= \frac{1}{4} \int (1+t^4)^3 d(1+t^4) \\ &= \frac{1}{16} (1+t^4)^4 + C \end{aligned}$$

$$u = 1+t^4$$

$$du = 4t^3 dt$$

$$\frac{1}{4} du = t^3 dt$$

$$\begin{aligned} \int t^3 (1+t^4)^3 dt &= \frac{1}{4} \int u^3 du \\ &= \frac{1}{4} \left(\frac{u^4}{4} \right) + C \\ &= \frac{1}{16} (1+t^4)^4 + C \end{aligned}$$

Exercise

Evaluate the integrals $\int \frac{1}{x^3} \sqrt{\frac{x^2-1}{x^2}} dx$

Solution

$$\begin{aligned} d\left(\frac{x^2-1}{x^2}\right) &= d\left(1-x^{-2}\right) \\ &= \frac{1}{x^3} dx \end{aligned}$$

$$\begin{aligned} \int \frac{1}{x^3} \sqrt{\frac{x^2-1}{x^2}} dx &= \int \left(1-\frac{1}{x^2}\right)^{1/2} d\left(1-\frac{1}{x^2}\right) \\ &= \frac{1}{3} \left(1-\frac{1}{x^2}\right)^{3/2} + C \end{aligned}$$

$$\begin{aligned} \text{Let } u &= \frac{x^2-1}{x^2} \\ &= 1-\frac{1}{x^2} \\ &= 1-x^{-2} \end{aligned}$$

$$du = 2x^{-3} dx$$

$$\frac{1}{2} du = \frac{1}{x^3} dx$$

$$\begin{aligned} \int \frac{1}{x^3} \sqrt{\frac{x^2-1}{x^2}} dx &= \int u^{1/2} \left(\frac{1}{2} du\right) \\ &= \frac{1}{2} \left(\frac{2}{3} u^{3/2}\right) + C \end{aligned}$$

$$\left. = \frac{1}{3} \left(1 - \frac{1}{x^2} \right)^{3/2} + C \right|$$

Exercise

Evaluate the integrals $\int x^3 \sqrt{x^2 + 1} \, dx$

Solution

$$\text{Let } u = x^2 + 1 \Rightarrow x^2 = u - 1$$

$$du = 2x \, dx$$

$$\frac{1}{2} du = x \, dx$$

$$\begin{aligned} \int x^3 \sqrt{x^2 + 1} \, dx &= \int x^2 \sqrt{x^2 + 1} \, x \, dx \\ &= \int (u - 1) u^{1/2} \left(\frac{1}{2} du \right) \\ &= \frac{1}{2} \int (u^{3/2} - u^{1/2}) \, du \\ &= \frac{1}{2} \left(\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right) + C \\ &= \frac{1}{5} (x^2 + 1)^{5/2} - \frac{1}{3} (x^2 + 1)^{3/2} + C \end{aligned} \left| \right.$$

Exercise

Evaluate the integrals $\int \frac{x}{(x^2 - 4)^3} \, dx$

Solution

$$d(x^2 - 4) = 2x \, dx$$

$$\begin{aligned} \int \frac{x}{(x^2 - 4)^3} \, dx &= \frac{1}{2} \int (x^2 - 4)^{-3} d(x^2 - 4) \\ &= -\frac{1}{4(x^2 - 4)^2} + C \end{aligned} \left| \right.$$

$$u = x^2 - 4$$

$$du = 2x dx$$

$$\frac{1}{2} du = x dx$$

$$\begin{aligned} \int \frac{x}{(x^2 - 4)^3} dx &= \frac{1}{2} \int u^{-3} du \\ &= -\frac{1}{4} u^{-2} + C \\ &= -\frac{1}{4(x^2 - 4)^2} + C \end{aligned}$$

Exercise

Evaluate the integrals $\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}} dr$

Solution

$$d\left(\sqrt{3(2r-1)^2+6}\right) = \frac{1}{2} \frac{6(2)(2r-1)}{\sqrt{3(2r-1)^2+6}} dr$$

$$\begin{aligned} \int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}} dr &= \frac{1}{6} \int \cos\sqrt{3(2r-1)^2+6} d\left(\sqrt{3(2r-1)^2+6}\right) \\ &= \frac{1}{6} \sin\sqrt{3(2r-1)^2+6} + C \end{aligned}$$

Let $u = \sqrt{3(2r-1)^2+6}$

$$\begin{aligned} du &= \frac{1}{2} \left(3(2r-1)^2+6\right)^{-1/2} (6(2r-1)(2)) dr \\ &= \frac{6(2r-1)}{\left(3(2r-1)^2+6\right)^{1/2}} dr \end{aligned}$$

$$\rightarrow \frac{1}{6} du = \frac{2r-1}{\sqrt{3(2r-1)^2+6}} dr$$

$$\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}} dr = \int \cos u \left(\frac{1}{6} du\right)$$

$$= \frac{1}{6} \sin u + C$$

$$= \frac{1}{6} \sin \sqrt{3(2r-1)^2 + 6} + C$$

Exercise

Evaluate the integrals $\int \frac{\sin \sqrt{\theta}}{\sqrt{\theta} \cos^3 \sqrt{\theta}} d\theta$

Solution

$$d(\cos \sqrt{\theta}) = -\frac{1}{2\sqrt{\theta}} \sin \sqrt{\theta} d\theta \quad \frac{d}{du} - u' \sin u$$

$$\int \frac{\sin \sqrt{\theta}}{\sqrt{\theta} \sqrt{\cos^3 \sqrt{\theta}}} d\theta = -2 \int \cos^{3/2} \sqrt{\theta} d(\cos \sqrt{\theta})$$

$$= \frac{4}{\sqrt{\cos \sqrt{\theta}}} + C$$

Let $u = \cos \sqrt{\theta}$

$$du = (-\sin \sqrt{\theta}) \left(\frac{1}{2\sqrt{\theta}} \right) d\theta$$

$$-2du = \frac{1}{\sqrt{\theta}} \sin \sqrt{\theta} d\theta$$

$$\int \frac{\sin \sqrt{\theta}}{\sqrt{\theta} \cos^3 \sqrt{\theta}} d\theta = \int \frac{\sin \sqrt{\theta}}{\sqrt{\theta} \sqrt{\cos^3 \sqrt{\theta}}} d\theta$$

$$= \int \frac{1}{u^{3/2}} (-2du)$$

$$= -2 \int u^{-3/2} du$$

$$= -2 \frac{u^{-1/2}}{-1/2} + C$$

$$= \frac{4}{\sqrt{\cos \sqrt{\theta}}} + C$$

Exercise

Evaluate the integrals. $\int 2x \sqrt{x^2 - 2} \, dx$

Solution

$$d(x^2 - 2) = 2x \, dx$$

$$\begin{aligned} \int 2x \sqrt{x^2 - 2} \, dx &= \int (x^2 - 2)^{1/2} d(x^2 - 2) \\ &= \frac{2}{3} (x^2 - 2)^{3/2} + C \end{aligned}$$

Exercise

Evaluate the integrals $\int x^3 (3x^4 + 1)^2 \, dx$

Solution

$$d(3x^4 + 1) = 12x^3 \, dx$$

$$\begin{aligned} \int x^3 (3x^4 + 1)^2 \, dx &= \int (3x^4 + 1)^2 d(3x^4 + 1) \\ &= \frac{1}{36} (3x^4 + 1)^3 + C \end{aligned}$$

Exercise

Evaluate the integrals $\int 2(3x^4 + 1)^2 \, dx$

Solution

$$\begin{aligned} \int 2(3x^4 + 1)^2 \, dx &= \int 2(9x^8 + 6x^4 + 1) \, dx \\ &= \int (18x^8 + 12x^4 + 2) \, dx \\ &= 2x^9 + \frac{12}{5}x^5 + 2x + C \end{aligned}$$

$$(a + b)^2 = a^2 + 2ab + b^2$$

Exercise

Evaluate the integrals $\int 5x \sqrt{x^2 - 1} \, dx$

Solution

$$d(x^2 - 1) = 2x \, dx$$

$$\begin{aligned} \int 5x \sqrt{x^2 - 1} \, dx &= \frac{5}{2} \int (x^2 - 1)^{1/2} d(x^2 - 1) \\ &= \frac{5}{3} (x^2 - 1)^{3/2} + C \end{aligned}$$

$$u = x^2 - 1$$

$$du = 2x \, dx$$

$$\Rightarrow \frac{1}{2} du = x \, dx$$

$$\begin{aligned} \int 5x (x^2 - 1)^{1/2} \, dx &= 5 \int u^{1/2} \frac{1}{2} du && \text{Substitute for } x \text{ and } dx \\ &= 5 \int u^{1/2} \frac{1}{2} du \\ &= \frac{5}{2} \int u^{1/2} du \\ &= \frac{5}{2} \frac{u^{3/2}}{3/2} + C \\ &= \frac{5}{3} u^{3/2} + C \\ &= \frac{5}{3} (x^2 - 1)^{3/2} + C \end{aligned}$$

Exercise

Find the integral $\int (x^2 - 1)^3 (2x) \, dx$

Solution

$$\begin{aligned} \int (x^2 - 1)^3 (2x) \, dx &= \int (x^2 - 1)^3 d(x^2 - 1) \\ &= \frac{1}{4} (x^2 - 1)^4 + C \end{aligned}$$

Exercise

Find the integral $\int \frac{6x}{(1+x^2)^3} dx$

Solution

$$d(1+x^2) = 2x dx$$

$$\begin{aligned} \int \frac{6x}{(1+x^2)^3} dx &= 3 \int (1+x^2)^{-3} d(1+x^2) \\ &= -\frac{3}{2} (1+x^2)^{-2} + C \\ &= -\frac{3}{2} \frac{1}{(1+x^2)^2} + C \end{aligned}$$

Exercise

Find the integral $\int u^3 \sqrt{u^4+2} du$

Solution

$$d(u^4+2) = 4u^3 du$$

$$\begin{aligned} \int u^3 \sqrt{u^4+2} du &= \frac{1}{4} \int (u^4+2)^{1/2} d(u^4+2) \\ &= \frac{1}{6} (u^4+2)^{3/2} + C \end{aligned}$$

Exercise

Find the integral $\int \frac{t+2t^2}{\sqrt{t}} dt$

Solution

$$\begin{aligned} \int \frac{t+2t^2}{\sqrt{t}} dt &= \int \left(\frac{t}{t^{1/2}} + 2 \frac{t^2}{t^{1/2}} \right) dt \\ &= \int (t^{1/2} + 2t^{3/2}) dt \\ &= \frac{2}{3} t^{3/2} + \frac{4}{5} t^{5/2} + C \end{aligned}$$

Exercise

Find the integral $\int \left(1 + \frac{1}{t}\right)^3 \frac{1}{t^2} dt$

Solution

$$\begin{aligned} \int \left(1 + \frac{1}{t}\right)^3 \frac{1}{t^2} dt &= - \int \left(1 + \frac{1}{t}\right)^3 d\left(1 + \frac{1}{t}\right) \\ &= \underline{-\frac{1}{4} \left(1 + \frac{1}{t}\right)^4 + C} \end{aligned}$$

Exercise

Find the integral $\int (7 - 3x - 3x^2)(2x + 1) dx$

Solution

$$\begin{aligned} d(7 - 3x - 3x^2) &= (-3 - 6x) dx \\ &= -3(1 + 2x^2) dx \\ \int (7 - 3x - 3x^2)(2x + 1) dx &= -\frac{1}{3} \int (7 - 3x - 3x^2)(7 - 3x - 3x^2) dx \\ &= \underline{-\frac{1}{6} (7 - 3x - 3x^2)^2 + C} \end{aligned}$$

Exercise

Find the integral $\int \sqrt{x} (4 - x^{3/2})^2 dx$

Solution

$$\begin{aligned} d(4 - x^{3/2}) &= -\frac{3}{2} x^{1/2} dx \\ \int \sqrt{x} (4 - x^{3/2})^2 dx &= -\frac{2}{3} \int (4 - x^{3/2})^2 d(4 - x^{3/2}) \\ &= \underline{-\frac{2}{9} (4 - x^{3/2})^3 + C} \end{aligned}$$

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$$u = 4 - x^{3/2}$$

$$du = -\frac{3}{2} x^{1/2} dx$$

$$\rightarrow -\frac{2}{3} du = \sqrt{x} dx$$

$$\begin{aligned} \int \sqrt{x} (4 - x^{3/2})^2 dx &= \int u^2 \left(-\frac{2}{3}\right) du \\ &= -\frac{2}{3} \int u^2 du \\ &= -\frac{2}{9} u^3 + C \\ &= -\frac{2}{9} (4 - x^{3/2})^3 + C \end{aligned}$$

Exercise

Find the integral $\int \frac{1}{\sqrt{x} + \sqrt{x+1}} dx$

Solution

$$\begin{aligned} \int \frac{1}{\sqrt{x} + \sqrt{x+1}} dx &= \int \frac{1}{\sqrt{x} + \sqrt{x+1}} \frac{\sqrt{x} - \sqrt{x+1}}{\sqrt{x} - \sqrt{x+1}} dx \\ &= \int \frac{\sqrt{x} - \sqrt{x+1}}{x - (x+1)} dx \\ &= - \int (x^{1/2} - (x+1)^{1/2}) dx \\ &= - \left(\frac{2}{3} x^{3/2} - \frac{2}{3} (x+1)^{3/2} \right) + C \\ &= \frac{2}{3} (x+1)^{3/2} - \frac{2}{3} x^{3/2} + C \end{aligned}$$

Exercise

Find the integral $\int \sqrt{1-x} dx$

Solution

$$\begin{aligned} \int \sqrt{1-x} dx &= - \int (1-x)^{1/2} d(1-x) \\ &= -\frac{2}{3} (1-x)^{3/2} + C \end{aligned} \quad d(1-x) = -dx$$

Exercise

Find the integral $\int x \sqrt{x^2 + 4} \, dx$

Solution

$$\begin{aligned} \int \sqrt{x^2 + 4} \, x \, dx &= \frac{1}{2} \int (x^2 + 4)^{1/2} d(x^2 + 4) & d(x^2 + 4) &= 2x \, dx \\ &= \frac{1}{3} (x^2 + 4)^{3/2} + C \end{aligned}$$

Exercise

Find the integral $\int \sin^2\left(\theta + \frac{\pi}{6}\right) d\theta$

Solution

$$\begin{aligned} \int \sin^2\left(\theta + \frac{\pi}{6}\right) d\theta &= \frac{1}{2} \int \left(1 - \cos\left(2\theta + \frac{\pi}{3}\right)\right) d\theta \\ &= \frac{1}{2} \left(\theta - \frac{1}{2} \sin\left(2\theta + \frac{\pi}{3}\right)\right) + C \\ &= \frac{\theta}{2} - \frac{1}{4} \sin\left(2\theta + \frac{\pi}{3}\right) + C \end{aligned}$$

Exercise

Find the integral $\int \cos^2(8\theta) d\theta$

Solution

$$\begin{aligned} \int \cos^2(8\theta) d\theta &= \frac{1}{2} \int (1 + \cos(16\theta)) d\theta \\ &= \frac{1}{2} \left(\theta + \frac{1}{16} \sin(16\theta)\right) + C \\ &= \frac{\theta}{2} + \frac{1}{32} \sin(16\theta) + C \end{aligned}$$

Exercise

Find the integral $\int \sin^2(2\theta) d\theta$

Solution

$$\begin{aligned}
 \int \sin^2(2\theta) d\theta &= \frac{1}{2} \int (1 - \cos(4\theta)) d\theta \\
 &= \frac{1}{2} \left(\theta - \frac{1}{4} \sin(4\theta) \right) + C \\
 &= \underline{\frac{1}{2} \theta - \frac{1}{8} \sin(4\theta) + C}
 \end{aligned}$$

Exercise

Evaluate the integral $\int 8 \cos^4 2\pi x \, dx$

Solution

$$\begin{aligned}
 \int 8 \cos^4 2\pi x \, dx &= 8 \int (\cos 2\pi x)^4 \, dx & \cos^2 \alpha &= \frac{1 + \cos 2\alpha}{2} \\
 &= 8 \int \left(\frac{1 + \cos 4\pi x}{2} \right)^2 \, dx \\
 &= 2 \int (1 + \cos 4\pi x)^2 \, dx \\
 &= 2 \int (1 + 2 \cos 4\pi x + \cos^2 4\pi x) \, dx \\
 &= 2 \int dx + 4 \int \cos 4\pi x \, dx + 2 \int \cos^2 4\pi x \, dx \\
 &= 2x + 4 \frac{1}{4\pi} \sin 4\pi x + 2 \int \frac{1 + \cos 8\pi x}{2} \, dx \\
 &= 2x + \frac{1}{\pi} \sin 4\pi x + \int (1 + \cos 8\pi x) \, dx \\
 &= 2x + \frac{1}{\pi} \sin 4\pi x + x + \frac{1}{8\pi} \sin 8\pi x + C \\
 &= \underline{3x + \frac{1}{\pi} \sin 4\pi x + \frac{1}{8\pi} \sin 8\pi x + C}
 \end{aligned}$$

Exercise

Evaluate the integral $\int \sec x \, dx$

Solution

$$\int \sec x \, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$$

$$\begin{aligned}
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx & d(\sec x + \tan x) &= (\sec x \tan x + \sec^2 x) dx \\
&= \int \frac{d(\sec x + \tan x)}{\sec x + \tan x} \\
&= \ln |\sec x + \tan x| + C
\end{aligned}$$

Exercise

Evaluate $\int \frac{dx}{\sqrt{1-4x^2}}$

Solution

Let $u = 2x$

$$du = 2dx$$

$$\frac{1}{2} du = dx$$

$$\begin{aligned}
\int \frac{dx}{\sqrt{1-4x^2}} &= \int \frac{dx}{\sqrt{1-(2x)^2}} \\
&= \frac{1}{2} \int \frac{du}{\sqrt{1-u^2}} \\
&= \frac{1}{2} \sin^{-1} u + C \\
&= \frac{1}{2} \sin^{-1}(2x) + C
\end{aligned}$$

Exercise

Evaluate $\int \frac{dx}{\sqrt{3-4x^2}}$

Solution

$$a^2 = 3 \rightarrow a = \sqrt{3} \quad u^2 = 4x^2 = (2x)^2 \rightarrow u = 2x \quad du = 2dx$$

$$\begin{aligned}
\int \frac{dx}{\sqrt{3-4x^2}} &= \frac{1}{2} \int \frac{du}{\sqrt{a^2-u^2}} \\
&= \frac{1}{2} \sin^{-1} \left(\frac{u}{a} \right) + C \\
&= \frac{1}{2} \sin^{-1} \left(\frac{2x}{\sqrt{3}} \right) + C
\end{aligned}$$

Exercise

Evaluate $\int \frac{dx}{\sqrt{e^{2x} - 6}}$

Solution

$$a^2 = 6 \rightarrow a = \sqrt{6}$$

$$u^2 = e^{2x} \rightarrow u = e^x$$

$$du = e^x dx$$

$$dx = \frac{du}{e^x} = \frac{du}{u}$$

$$\begin{aligned} \int \frac{dx}{\sqrt{e^{2x} - 6}} &= \int \frac{du}{u\sqrt{u^2 - a^2}} \\ &= \frac{1}{a} \sec^{-1} \left| \frac{u}{a} \right| + C \\ &= \frac{1}{\sqrt{6}} \sec^{-1} \left| \frac{e^x}{\sqrt{6}} \right| + C \end{aligned}$$

Exercise

Evaluate $\int \frac{dx}{\sqrt{4x - x^2}}$

Solution

$$\begin{aligned} 4x - x^2 &= -(x^2 - 4x) - 4 + 4 \\ &= -(x^2 - 4x + 4) + 4 \\ &= 4 - (x - 2)^2 \end{aligned}$$

$$a = 2$$

$$u = x - 2 \rightarrow du = dx$$

$$\begin{aligned} \int \frac{dx}{\sqrt{4x - x^2}} &= \int \frac{dx}{\sqrt{4 - (x - 2)^2}} \\ &= \sin^{-1} \left(\frac{x - 2}{2} \right) + C \end{aligned}$$

Using Completing the Square

Exercise

Evaluate $\int \frac{dx}{4x^2 + 4x + 2}$

Solution

$$\begin{aligned} 4x^2 + 4x + 2 &= 4\left(x^2 + x\right) + 2 \\ &= 4\left(x^2 + x + \frac{1}{4}\right) + 2 - 4\left(\frac{1}{4}\right) \\ &= 4\left(x + \frac{1}{2}\right)^2 + 1 \\ &= (2x + 1)^2 + 1 \end{aligned}$$

$$a = 1 \quad u = 2x + 1 \rightarrow du = 2dx$$

$$\begin{aligned} \int \frac{dx}{4x^2 + 4x + 2} &= \int \frac{dx}{(2x + 1)^2 + 1} \\ &= \frac{1}{2} \int \frac{du}{u^2 + 1} \\ &= \frac{1}{2} \cdot \frac{1}{1} \tan^{-1}\left(\frac{2x + 1}{1}\right) + C \\ &= \frac{1}{2} \tan^{-1}(2x + 1) + C \end{aligned}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

Exercise

Find the integral $\int \frac{1}{6x - 5} dx$

Solution

$$\begin{aligned} \int \frac{1}{6x - 5} dx &= \frac{1}{6} \int \frac{d(6x - 5)}{6x - 5} \\ &= \frac{1}{6} \ln|6x - 5| + C \end{aligned}$$

Exercise

Find the integral $\int \frac{x^2 + 2x + 3}{x^3 + 3x^2 + 9x + 1} dx$

Solution

$$d(x^3 + 3x^2 + 9x + 1) = (3x^2 + 6x + 9) dx$$

$$\begin{aligned} \int \frac{x^2 + 2x + 3}{x^3 + 3x^2 + 9x + 1} dx &= \frac{1}{3} \int \frac{d(x^3 + 3x^2 + 9x + 1)}{x^3 + 3x^2 + 9x + 1} \\ &= \frac{1}{3} \ln|x^3 + 3x^2 + 9x + 1| + C \end{aligned}$$

Exercise

Find the integral $\int \frac{1}{x(\ln x)^2} dx$

Solution

$$d(\ln x) = \frac{1}{x} dx$$

$$\begin{aligned} \int \frac{1}{x(\ln x)^2} dx &= \int \frac{1}{(\ln x)^2} d(\ln x) \\ &= -\frac{1}{\ln x} + C \end{aligned}$$

Exercise

Find the integral $\int \frac{x-3}{x+3} dx$

Solution

$$\begin{aligned} \int \frac{x-3}{x+3} dx &= \int \left(1 - \frac{6}{x+3}\right) dx \\ &= x - 6 \ln|x+3| + C \end{aligned}$$

Exercise

Find the indefinite integral. $\int \frac{3x}{x^2 + 4} dx$

Solution

$$d(x^2 + 4) = 2x dx$$

$$\int \frac{3x}{x^2 + 4} dx = \frac{3}{2} \int \frac{1}{x^2 + 4} d(x^2 + 4)$$

$$= \frac{3}{2} \ln(x^2 + 4) + C$$

$$u = x^2 + 4$$

$$du = 2x dx$$

$$\frac{1}{2} du = x dx$$

$$\begin{aligned} \int \frac{3x}{x^2 + 4} dx &= \frac{1}{2} \int \frac{3}{u} du \\ &= \frac{3}{2} \ln|u| + C \\ &= \frac{3}{2} \ln(x^2 + 4) + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{2\sqrt{x} + 2x}$

Solution

$$\begin{aligned} \int \frac{dx}{2\sqrt{x} + 2x} &= \int \frac{dx}{2\sqrt{x}(1 + \sqrt{x})} \\ &= \int \frac{du}{u} \\ &= \ln u + C \\ &= \ln(1 + \sqrt{x}) + C \end{aligned}$$

$$u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} dx$$

Exercise

Evaluate the integral $\int \frac{\sec x \, dx}{\sqrt{\ln(\sec x + \tan x)}}$

Solution

Let $u = \sec x + \tan x$

$$du = (\sec x \tan x + \sec^2 x) dx$$

$$= \sec x (\tan x + \sec x) dx$$

$$\sec x \, dx = \frac{du}{\tan x + \sec x} = \frac{du}{u}$$

$$\begin{aligned}
 \int \frac{\sec x dx}{\sqrt{\ln(\sec x + \tan x)}} &= \int \frac{du}{u \sqrt{\ln u}} \\
 &= \int (\ln u)^{-1/2} d(\ln u) & d(\ln u) = \frac{1}{u} du \\
 &= 2(\ln u)^{1/2} + C \\
 &= \underline{2\sqrt{\ln(\sec x + \tan x)} + C}
 \end{aligned}$$

Exercise

Evaluate the integral $\int 8e^{(x+1)} dx$

Solution

$$d(x+1) = dx$$

$$\begin{aligned}
 \int 8e^{(x+1)} dx &= 8 \int e^{(x+1)} d(x+1) \\
 &= \underline{8e^{(x+1)} + C}
 \end{aligned}$$

Exercise

Find the indefinite integral. $\int 4x e^{x^2} dx$

Solution

$$d(x^2) = 2x dx$$

$$\begin{aligned}
 \int 4x e^{x^2} dx &= 2 \int e^{x^2} d(x^2) \\
 &= \underline{2e^{x^2} + C}
 \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{e^{-\sqrt{r}}}{\sqrt{r}} dr$

Solution

$$d(-\sqrt{r}) = -\frac{1}{2\sqrt{r}} dr$$

$$\int \frac{e^{-\sqrt{r}}}{\sqrt{r}} dr = -2 \int e^{-\sqrt{r}} d(-\sqrt{r})$$

$$= -2e^{-\sqrt{r}} + C$$

$$u = -r^{1/2}$$

$$du = -\frac{1}{2} r^{-1/2} dr$$

$$-2du = \frac{1}{r^{1/2}} dr$$

$$\int \frac{e^{-\sqrt{r}}}{\sqrt{r}} dr = \int e^u (-2du)$$

$$= -2e^u + C$$

$$= -2e^{-\sqrt{r}} + C$$

Exercise

Evaluate the integral $\int t^3 e^{t^4} dt$

Solution

$$d(t^4) = 4t^3 dt$$

$$\int t^3 e^{t^4} dt = \frac{1}{4} \int e^{t^4} d(t^4)$$

$$= \frac{1}{4} e^{t^4} + C$$

Exercise

Evaluate the integral $\int e^{\sec \pi t} \sec \pi \tan \pi t dt$

Solution

$$d(\sec \pi t) = \pi \sec \pi t \tan \pi t dt$$

$$\int e^{\sec \pi t} \sec \pi \tan \pi t dt = \frac{1}{\pi} \int e^{\sec \pi t} d(\sec \pi t)$$

$$= \frac{1}{\pi} e^{\sec \pi t} + C \Big|$$

$$u = \sec \pi t$$

$$du = \pi \sec \pi t \tan \pi t \, dt$$

$$\frac{1}{\pi} du = \sec \pi t \tan \pi t \, dt$$

$$\int e^{\sec \pi t} \sec \pi t \tan \pi t \, dt = \frac{1}{\pi} \int e^u du$$

$$= \frac{1}{\pi} e^u + C$$

$$= \frac{1}{\pi} e^{\sec \pi t} + C \Big|$$

Exercise

Find the integral $\int (2x+1) e^{x^2+x} \, dx$

Solution

$$d(x^2 + x) = (2x+1) \, dx$$

$$\int (2x+1) e^{x^2+x} \, dx = \int e^{x^2+x} d(x^2 + x)$$

$$= e^{x^2+x} + C \Big|$$

Exercise

Evaluate the integral $\int \frac{dx}{1+e^x}$

Solution

$$\int \frac{dx}{1+e^x} = \int \frac{e^{-x}}{e^{-x} + 1} \, dx$$

$$= \int \frac{e^{-x} dx}{e^{-x} + 1}$$

$$= - \int \frac{1}{e^{-x} + 1} d(e^{-x} + 1)$$

$$= -\ln(e^{-x} + 1) + C \Big|$$

$$d(e^{-x} + 1) = -e^{-x} dx$$

Exercise

Find the integral $\int \frac{e^x}{1+e^x} dx$

Solution

$$d(e^x + 1) = e^x dx$$

$$\begin{aligned} \int \frac{e^x}{1+e^x} dx &= \int \frac{1}{1+e^x} d(1+e^x) \\ &= \ln(1+e^x) + C \end{aligned}$$

$$u = 1 + e^x$$

$$du = e^x dx$$

$$\int \frac{1}{u} du = \ln|u| + C$$

$$= \ln(1+e^x) + C$$

Exercise

Find the integral $\int \frac{2}{e^{-x} + 1} dx$

Solution

$$\int \frac{2}{e^{-x} + 1} dx = \int \frac{2}{e^{-x} + 1} \frac{e^x}{e^x} dx$$

$$= 2 \int \frac{e^x}{1+e^x} dx$$

$$= 2 \int \frac{d(e^x + 1)}{1+e^x}$$

$$= 2 \ln(e^x + 1) + C$$

Exercise

Find the integral $\int \frac{1}{x^3} e^{1/4x^2} dx$

Solution

$$d\left(\frac{1}{4}x^{-2}\right) = -\frac{1}{2}x^{-3}dx$$

$$\begin{aligned}\int \frac{1}{x^3} e^{1/4x^2} dx &= -2 \int e^{1/4x^2} d\left(\frac{1}{4x^2}\right) \\ &= -2e^{1/4x^2} + C\end{aligned}$$

$$u = \frac{1}{4x^2} = \frac{1}{4}x^{-2}$$

$$du = -\frac{1}{2}x^{-3}dx$$

$$-2du = \frac{1}{x^3}dx$$

$$\begin{aligned}\int e^u (-2)du &= -2 \int e^u du \\ &= -2e^u + C \\ &= -2e^{1/4x^2} + C\end{aligned}$$

Exercise

Find the integral $\int \frac{e^{1/\sqrt{x}}}{x^{3/2}} dx$

Solution

$$d\left(\frac{1}{\sqrt{x}}\right) = -\frac{1}{2x^{3/2}}$$

$$\begin{aligned}\int \frac{e^{1/\sqrt{x}}}{x^{3/2}} dx &= -2 \int e^{1/\sqrt{x}} d\left(\frac{1}{\sqrt{x}}\right) \\ &= -2e^{1/\sqrt{x}} + C\end{aligned}$$

$$u = \frac{1}{\sqrt{x}} = x^{-1/2}$$

$$du = -\frac{1}{2}x^{-3/2}dx$$

$$-2du = \frac{1}{x^{3/2}} dx$$

$$\begin{aligned} \int \frac{e^{1/\sqrt{x}}}{x^{3/2}} dx &= \int e^u (-2du) \\ &= -2 \int e^u du \\ &= -2e^u + C \\ &= \underline{-2e^{1/\sqrt{x}} + C} \end{aligned}$$

Exercise

Find the integral $\int \frac{-e^{3x}}{2-e^{3x}} dx$

Solution

$$\begin{aligned} d(2-e^{3x}) &= -3e^{3x} dx \\ \int \frac{-e^{3x}}{2-e^{3x}} dx &= \frac{1}{3} \int \frac{1}{2-e^{3x}} d(2-e^{3x}) \\ &= \underline{\frac{1}{3} \ln|2-e^{3x}| + C} \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{7e^{7x}}{3+e^{7x}} dx$

Solution

$$\begin{aligned} d(3+e^{7x}) &= 7e^{7x} dx \\ \int \frac{7e^{7x}}{3+e^{7x}} dx &= \int \frac{1}{3+e^{7x}} d(3+e^{7x}) \\ &= \underline{\ln(3+e^{7x}) + C} \end{aligned}$$

$$u = 3 + e^{7x}$$

$$du = 7e^{7x} dx$$

$$\int \frac{7e^{7x}}{3+e^{7x}} dx = \int \frac{du}{u}$$

$$\begin{aligned}
 &= \ln|u| \\
 &= \ln(3 + e^{7x}) + C
 \end{aligned}$$

Exercise

Find the integral $\int \frac{2(e^x - e^{-x})}{(e^x + e^{-x})^2} dx$

Solution

$$d(e^x + e^{-x}) = (e^x - e^{-x}) dx$$

$$\begin{aligned}
 \int \frac{2(e^x - e^{-x})}{(e^x + e^{-x})^2} dx &= \int \frac{2}{(e^x + e^{-x})^2} d(e^x + e^{-x}) \\
 &= -\frac{2}{e^x + e^{-x}} + C
 \end{aligned}$$

$$u = e^x + e^{-x}$$

$$du = (e^x - e^{-x}) dx$$

$$\begin{aligned}
 \int \frac{2(e^x - e^{-x})}{(e^x + e^{-x})^2} dx &= 2 \int \frac{1}{u^2} du \\
 &= 2 \int u^{-2} du \\
 &= 2 \frac{u^{-1}}{-1} + C \\
 &= -2 \frac{1}{u} + C \\
 &= -\frac{2}{e^x + e^{-x}} + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{3^x}{3-3^x} dx$

Solution

$$d(3-3^x) = (-3^x \ln 3) dx$$

$$\begin{aligned} \int \frac{3^x}{3-3^x} dx &= -\frac{1}{\ln 3} \int \frac{1}{3-3^x} d(3-3^x) \\ &= -\frac{1}{\ln 3} \ln|3-3^x| + C \end{aligned}$$

Let $u = 3-3^x$

$$du = (-3^x \ln 3) dx$$

$$-\frac{1}{\ln 3} du = 3^x dx$$

$$\begin{aligned} \int \frac{3^x}{3-3^x} dx &= -\frac{1}{\ln 3} \int \frac{du}{u} \\ &= -\frac{1}{\ln 3} \ln|u| + C \\ &= -\frac{1}{\ln 3} \ln|3-3^x| + C \end{aligned}$$

Exercise

Find the integral $\int (6x+e^x) \sqrt{3x^2+e^x} dx$

Solution

$$d(3x^2+e^x) = (6x+e^x) dx$$

$$\begin{aligned} \int (6x+e^x) \sqrt{3x^2+e^x} dx &= \int (3x^2+e^x)^{1/2} d(3x^2+e^x) \\ &= \frac{2}{3} (3x^2+e^x)^{3/2} + C \end{aligned}$$

$$u = 3x^2 + e^x$$

$$du = (6x+e^x) dx$$

$$\frac{du}{6x+e^x} = dx$$

$$\begin{aligned}
 \int (6x + e^x) \sqrt{3x^2 + e^x} \, dx &= \int (6x + e^x) \sqrt{u} \frac{du}{6x + e^x} \\
 &= \int u^{1/2} du \\
 &= \frac{2}{3} u^{3/2} + C \\
 &= \frac{2}{3} (3x^2 + e^x)^{3/2} + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{x 2^{x^2}}{1 + 2^{x^2}} \, dx$

Solution

$$d(1 + 2^{x^2}) = 2x(\ln 2) 2^{x^2} \, dx$$

$$\begin{aligned}
 \int \frac{x 2^{x^2}}{1 + 2^{x^2}} \, dx &= \frac{1}{2 \ln 2} \int \frac{1}{1 + 2^{x^2}} d(1 + 2^{x^2}) \\
 &= \frac{1}{2 \ln 2} \ln(1 + 2^{x^2}) + C
 \end{aligned}$$

Let $u = 1 + 2^{x^2}$

$$du = 2x 2^{x^2} \ln(2) \, dx$$

$$\frac{du}{2 \ln 2} = x 2^{x^2} \, dx$$

$$\begin{aligned}
 \int \frac{x 2^{x^2}}{1 + 2^{x^2}} \, dx &= \frac{1}{2 \ln 2} \int \frac{du}{u} \\
 &= \frac{1}{2 \ln 2} \ln u + C \\
 &= \frac{1}{2 \ln 2} \ln(1 + 2^{x^2}) + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{x(\log_8 x)^2}$

Solution

$$\begin{aligned}\int \frac{dx}{x(\log_8 x)^2} &= \int \frac{dx}{x\left(\frac{\ln x}{\ln 8}\right)^2} \\ &= (\ln 8)^2 \int \frac{dx}{x(\ln x)^2} \\ &= (\ln 8)^2 \int \frac{d(\ln x)}{(\ln x)^2} \\ &= -(\ln 8)^2 \frac{1}{\ln x} + C\end{aligned}$$

$$d(\ln x) = \frac{1}{x} dx$$

Exercise

Evaluate $\int \frac{dx}{x\sqrt{25x^2 - 2}}$

Solution

Let $u = 5x$

$$du = 5dx$$

$$\frac{1}{5} du = dx$$

$$\begin{aligned}\int \frac{dx}{x\sqrt{25x^2 - 2}} &= \int \frac{du/5}{\frac{u}{5}\sqrt{u^2 - 2}} \\ &= \int \frac{du}{u\sqrt{u^2 - (\sqrt{2})^2}} \\ &= \frac{1}{\sqrt{2}} \sec^{-1} \left| \frac{u}{\sqrt{2}} \right| + C \\ &= \frac{1}{\sqrt{2}} \sec^{-1} \left| \frac{5x}{\sqrt{2}} \right| + C\end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

Exercise

Evaluate $\int \frac{6dr}{\sqrt{4-(r+1)^2}}$

Solution

$$u = r + 1 \Rightarrow du = dr$$

$$a^2 = 4 \rightarrow a = 2$$

$$\begin{aligned} \int \frac{6dr}{\sqrt{4-(r+1)^2}} &= 6 \int \frac{du}{\sqrt{4-u^2}} \\ &= 6 \sin^{-1} \frac{u}{2} + C \\ &= \underline{6 \sin^{-1} \left(\frac{r+1}{2} \right) + C} \end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

Exercise

Evaluate $\int \frac{dx}{2+(x-1)^2}$

Solution

$$u = x - 1 \Rightarrow du = dx$$

$$a^2 = 2 \rightarrow a = \sqrt{2}$$

$$\begin{aligned} \int \frac{dx}{2+(x-1)^2} &= \int \frac{du}{2+u^2} \\ &= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{u}{\sqrt{2}} \right) + C \\ &= \underline{\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x-1}{\sqrt{2}} \right) + C} \end{aligned}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

Exercise

Evaluate $\int \frac{\sec^2 y \, dy}{\sqrt{1 - \tan^2 y}}$

Solution

$$u = \tan y \Rightarrow du = \sec^2 y \, dy$$

$$a^2 = 1 \rightarrow a = 1$$

$$\int \frac{\sec^2 y \, dy}{\sqrt{1 - \tan^2 y}} = \int \frac{du}{\sqrt{1 - u^2}}$$

$$= \sin^{-1}(u) + C$$

$$= \sin^{-1}(\tan y) + C$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

Exercise

Evaluate $\int \frac{dx}{\sqrt{-x^2 + 4x - 3}}$

Solution

$$\int \frac{dx}{\sqrt{-x^2 + 4x - 3}} = \int \frac{dx}{\sqrt{1 - x^2 + 4x - 3 - 1}}$$

$$= \int \frac{dx}{\sqrt{1 - (x^2 - 4x + 4)}}$$

$$= \int \frac{dx}{\sqrt{1 - (x + 2)^2}}$$

$$= \int \frac{du}{\sqrt{1 - u^2}} \int \frac{dx}{\sqrt{2x - x^2}}$$

$$= \sin^{-1} u + C$$

$$= \sin^{-1}(x - 2) + C$$

$$u = x + 2 \Rightarrow du = dx$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

Exercise

Evaluate $\int \frac{dx}{\sqrt{2x - x^2}}$

Solution

$$\int \frac{dx}{\sqrt{2x - x^2}} = \int \frac{dx}{\sqrt{1 + 2x - x^2 - 1}}$$

$$= \int \frac{dx}{\sqrt{1 - (x^2 - 2x + 1)}}$$

$$= \int \frac{dx}{\sqrt{1-(x-1)^2}}$$

$$= \int \frac{du}{\sqrt{1-u^2}}$$

$$= \sin^{-1}(x-1) + C$$

$$u = x-1 \Rightarrow du = dx$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

Exercise

Evaluate $\int \frac{x-2}{x^2-6x+10} dx$

Solution

$$\int \frac{x-2}{x^2-6x+10} dx = \int \frac{x-2}{x^2-6x+9+1} dx$$

$$= \int \frac{x-2-1+1}{(x-3)^2+1} dx$$

$$= \int \frac{x-3+1}{(x-3)^2+1} dx$$

$$= \int \frac{u+1}{u^2+1} du$$

$$= \int \frac{u}{u^2+1} du + \int \frac{1}{u^2+1} du$$

$$= \frac{1}{2} \int \frac{dw}{w} + \int \frac{1}{u^2+1} du$$

$$= \frac{1}{2} \ln w + \tan^{-1} u + C$$

$$= \frac{1}{2} \ln((x-3)^2+1) + \tan^{-1}(x-3) + C$$

$$= \frac{1}{2} \ln(x^2-6x+10) + \tan^{-1}(x-3) + C$$

$$u = x-3 \Rightarrow du = dx$$

$$w = u^2+1 \Rightarrow dw = 2u du \rightarrow \frac{1}{2} dw = u du$$

Exercise

Evaluate $\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$

Solution

$$\begin{aligned}
 \int \frac{dx}{(x+1)\sqrt{x^2+2x}} &= \int \frac{dx}{(x+1)\sqrt{x^2+2x+1-1}} \\
 &= \int \frac{dx}{(x+1)\sqrt{(x+1)^2-1}} \\
 &= \sec^{-1}|x+1| + C
 \end{aligned}$$

$$\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u|$$

Exercise

Evaluate $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$

Solution

$$\begin{aligned}
 \int \frac{dx}{(x-2)\sqrt{x^2-4x+3}} &= \int \frac{dx}{(x-2)\sqrt{x^2-4x+4-1}} \\
 &= \int \frac{dx}{(x-2)\sqrt{(x-2)^2-1}} \\
 &= \int \frac{du}{u\sqrt{u^2-1}} \\
 &= \sec^{-1}u + C \\
 &= \sec^{-1}|x-2| + C
 \end{aligned}$$

$$\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u|$$

Exercise

Evaluate $\int \frac{e^{\cos^{-1}x} dx}{\sqrt{1-x^2}}$

Solution

$$\begin{aligned}
 \int \frac{e^{\cos^{-1}x} dx}{\sqrt{1-x^2}} &= - \int e^{\cos^{-1}x} d(\cos^{-1}x) \\
 &= -e^{\cos^{-1}x} + C
 \end{aligned}$$

$$d(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}} dx$$

Exercise

Evaluate $\int \frac{(\sin^{-1} x)^2 dx}{\sqrt{1-x^2}}$

Solution

$$d(\sin^{-1} x) = \frac{dx}{\sqrt{1-x^2}}$$

$$\begin{aligned} \int \frac{(\sin^{-1} x)^2 dx}{\sqrt{1-x^2}} &= \int (\sin^{-1} x)^2 d(\sin^{-1} x) \\ &= \frac{1}{3} (\sin^{-1} x)^3 + C \end{aligned}$$

Exercise

Evaluate $\int \frac{dy}{(\sin^{-1} y) \sqrt{1+y^2}}$

Solution

$$d(\sin^{-1} y) = \frac{dy}{\sqrt{1-y^2}}$$

$$\begin{aligned} \int \frac{dy}{(\sin^{-1} y) \sqrt{1+y^2}} &= \int \frac{1}{\sin^{-1} y} d(\sin^{-1} y) \\ &= \ln |\sin^{-1} y| + C \end{aligned}$$

Exercise

Evaluate $\int \frac{1}{\sqrt{x}(x+1) \left((\tan^{-1} \sqrt{x})^2 + 9 \right)} dx$

Solution

$$\begin{aligned} d(\tan^{-1} \sqrt{x}) &= \frac{1}{2\sqrt{x}} \frac{1}{1+(\sqrt{x})^2} dx \\ &= \frac{1}{2\sqrt{x}(1+x)} dx \end{aligned}$$

$$\int \frac{1}{\sqrt{x}(x+1)\left(\left(\tan^{-1}\sqrt{x}\right)^2+9\right)} dx = 2 \int \frac{1}{\left(\tan^{-1}\sqrt{x}\right)^2+9} d\left(\tan^{-1}\sqrt{x}\right)$$

$$= \frac{2}{3} \tan^{-1}\left(\frac{\tan^{-1}\sqrt{x}}{3}\right) + C$$

Exercise

Evaluate the integral $\int 2x(x^2+1)^4 dx$

Solution

$$d(x^2+1) = 2x dx$$

$$\int 2x(x^2+1)^4 dx = \int (x^2+1)^4 d(x^2+1)$$

$$= \frac{1}{5}(x^2+1)^5 + C$$

Exercise

Evaluate the integral $\int 8x \cos(4x^2+3) dx$

Solution

$$d(4x^2+3) = 8x dx$$

$$\int 8x \cos(4x^2+3) dx = \int \cos(4x^2+3) d(4x^2+3)$$

$$= \sin(4x^2+3) + C$$

Exercise

Evaluate the integral $\int \sin^3 x \cos x dx$

Solution

$$d(\sin x) = \cos x dx$$

$$\int \sin^3 x \cos x \, dx = \int \sin^3 x \, d(\sin x)$$

$$= \frac{1}{4} \sin^4 x + C$$

Exercise

Evaluate the integral $\int (6x+1)\sqrt{3x^2+x} \, dx$

Solution

$$d(3x^2+x) = (6x+1)dx$$

$$\int (6x+1)\sqrt{3x^2+x} \, dx = \int (3x^2+x)^{1/2} d(3x^2+x)$$

$$= \frac{2}{3} (3x^2+x)^{3/2} + C$$

Exercise

Evaluate the integral $\int 2x(x^2-1)^{99} \, dx$

Solution

$$d(x^2-1) = 2x \, dx$$

$$\int 2x(x^2-1)^{99} \, dx = \int (x^2-1)^{99} d(x^2-1)$$

$$= \frac{1}{100} (x^2-1)^{100} + C$$

Exercise

Evaluate the integral $\int x e^{x^2} \, dx$

Solution

$$\int x e^{x^2} \, dx = \frac{1}{2} \int e^{x^2} d(x^2)$$

$$= \frac{1}{2} e^{x^2} + C$$

$$d(x^2) = 2x \, dx$$

Exercise

Evaluate the integral $\int \frac{2x^2}{\sqrt{1-4x^3}} dx$

Solution

$$d(1-4x^3) = -12x^2 dx$$

$$\begin{aligned} \int \frac{2x^2}{\sqrt{1-4x^3}} dx &= -\frac{1}{6} \int (1-4x^3)^{-1/2} d(1-4x^3) \\ &= \underline{-\frac{1}{3} \sqrt{1-4x^3} + C} \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{(\sqrt{x}+1)^4}{2\sqrt{x}} dx$

Solution

$$d(\sqrt{x}+1) = \frac{1}{2\sqrt{x}} dx$$

$$\begin{aligned} \int \frac{(\sqrt{x}+1)^4}{2\sqrt{x}} dx &= \int (\sqrt{x}+1)^4 d(\sqrt{x}+1) \\ &= \underline{\frac{1}{5}(\sqrt{x}+1)^5 + C} \end{aligned}$$

Exercise

Evaluate the integral $\int (x^2+x)^{10} (2x+1) dx$

Solution

$$d(x^2+x) = (2x+1) dx$$

$$\begin{aligned} \int (x^2+x)^{10} (2x+1) dx &= \int (x^2+x)^{10} d(x^2+x) \\ &= \underline{\frac{1}{11} (x^2+x)^{11} + C} \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{10x-3}$

Solution

$$d(10x-3) = 10dx$$

$$\begin{aligned} \int \frac{dx}{10x-3} &= \frac{1}{10} \int \frac{d(10x-3)}{10x-3} \\ &= \ln|10x-3| + C \end{aligned}$$

Exercise

Evaluate the integral $\int x^3 (x^4 + 16)^6 dx$

Solution

$$d(x^4 + 16) = 4x^3 dx$$

$$\begin{aligned} \int x^3 (x^4 + 16)^6 dx &= \frac{1}{4} \int (x^4 + 16)^6 d(x^4 + 16) \\ &= \frac{1}{28} (x^4 + 16)^7 + C \end{aligned}$$

Exercise

Evaluate the integral $\int \sin^{10} \theta \cos \theta d\theta$

Solution

$$d(\sin \theta) = \cos \theta d\theta$$

$$\begin{aligned} \int \sin^{10} \theta \cos \theta d\theta &= \int \sin^{10} \theta d(\sin \theta) \\ &= \frac{1}{11} \sin^{11} \theta + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{\sqrt{1-9x^2}}$

Solution

$$d(3x) = 3dx$$

$$\int \frac{dx}{\sqrt{1-9x^2}} = \frac{1}{3} \int \frac{d(3x)}{\sqrt{1-(3x)^2}} \qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a}$$
$$= \frac{1}{3} \arcsin 3x + C$$

Exercise

Evaluate the integral $\int x^9 \sin x^{10} dx$

Solution

$$d(x^{10}) = 10x^9 dx$$

$$\int x^9 \sin x^{10} dx = \frac{1}{10} \int \sin x^{10} d(x^{10})$$
$$= -\frac{1}{10} \cos x^{10} + C$$

Exercise

Evaluate the integral $\int (x^6 - 3x^2)^4 (x^5 - x) dx$

Solution

$$d(x^6 - 3x^2) = 6(x^5 - x) dx$$

$$\int (x^6 - 3x^2)^4 (x^5 - x) dx = \frac{1}{6} \int (x^6 - 3x^2)^4 d(x^6 - 3x^2)$$
$$= \frac{1}{30} (x^6 - 3x^2)^5 + C$$

Exercise

Evaluate the integral $\int \frac{x}{x-2} dx$

Solution

$$\int \frac{x}{x-2} dx = \int \left(1 + \frac{2}{x-2}\right) dx$$
$$\underline{= x + 2\ln|x-2| + C}$$

$$\begin{array}{r} x-2 \overline{)x} \\ \underline{-x+2} \\ 2 \end{array}$$

Exercise

Evaluate the integral $\int \frac{dx}{1+4x^2}$

Solution

$$d(2x) = 2dx$$

$$\int \frac{dx}{1+4x^2} = \frac{1}{2} \int \frac{d(2x)}{1+(2x)^2}$$
$$\underline{= \frac{1}{2} \arctan 2x + C}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

Exercise

Evaluate the integral $\int \frac{3}{1+25y^2} dy$

Solution

$$d(5y) = 5dy$$

$$\int \frac{3}{1+25y^2} dy = \frac{3}{5} \int \frac{d(5y)}{1+(5y)^2}$$
$$\underline{= \frac{3}{5} \arctan 5y + C}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

Exercise

Evaluate the integral $\int \frac{2}{x\sqrt{4x^2-1}} dx \quad \left(x > \frac{1}{2}\right)$

Solution

$$\int \frac{2}{x\sqrt{4x^2-1}} dx = \int \frac{d(2x)}{x\sqrt{(2x)^2-1}} \\ = \underline{\underline{\operatorname{arcsec}(2x) + C}}$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \sec^{-1} \left| \frac{x}{a} \right|$$

Exercise

Evaluate the integral $\int \frac{8x+6}{2x^2+3x} dx$

Solution

$$2d(2x^2+3x) = 2(4x+3) dx$$

$$\int \frac{8x+6}{2x^2+3x} dx = 2 \int \frac{1}{2x^2+3x} d(2x^2+3x) \\ = \underline{\underline{2 \ln|2x^2+3x| + C}}$$

Exercise

Evaluate the integral $\int \frac{x}{\sqrt{x-4}} dx$

Solution

$$u = x - 4 \rightarrow x = u + 4$$

$$dx = du$$

$$\int \frac{x}{\sqrt{x-4}} dx = \int \frac{u+4}{u^{1/2}} du \\ = \int (u^{1/2} + 4u^{-1/2}) du \\ = \frac{2}{3} u^{3/2} + 8u^{1/2} + C \\ = \underline{\underline{\frac{2}{3}(x-4)^{3/2} + 8(x-4)^{1/2} + C}}$$

Exercise

Evaluate the integral $\int \frac{x^2}{(x+1)^4} dx$

Solution

$$u = x + 1 \rightarrow x = u - 1$$

$$dx = du$$

$$\begin{aligned} \int \frac{x^2}{(x+1)^4} dx &= \int \frac{(u-1)^2}{u^4} du \\ &= \int \frac{u^2 - 2u + 1}{u^4} du \\ &= \int \left(\frac{1}{u^2} - 2u^{-3} + u^{-4} \right) du \\ &= -\frac{1}{u} + u^{-2} - \frac{1}{3}u^{-3} + C \\ &= -\frac{1}{x+1} + \frac{1}{(x+1)^2} - \frac{1}{3(x+1)^3} + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{x}{\sqrt[3]{x+4}} dx$

Solution

$$u = x + 4 \rightarrow x = u - 4$$

$$dx = du$$

$$\begin{aligned} \int \frac{x}{\sqrt[3]{x+4}} dx &= \int \frac{u-4}{u^{1/3}} du \\ &= \int \left(u^{2/3} - 4u^{-1/3} \right) du \\ &= \frac{3}{5}u^{5/3} - 6u^{2/3} + C \\ &= \frac{3}{5}(x+4)^{5/3} - 6(x+4)^{2/3} + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

Solution

$$\begin{aligned} \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx &= \int \frac{1}{e^x + e^{-x}} d(e^x + e^{-x}) \\ &= \ln(e^x + e^{-x}) + C \end{aligned}$$

$$d(e^x + e^{-x}) = (e^x - e^{-x}) dx$$

Exercise

Evaluate the integral $\int x \sqrt[3]{2x+1} \, dx$

Solution

$$u = 2x + 1 \rightarrow x = \frac{1}{2}(u - 1)$$

$$dx = \frac{1}{2} du$$

$$\begin{aligned} \int x \sqrt[3]{2x+1} \, dx &= \int \frac{1}{2}(u-1)u^{1/3} \left(\frac{1}{2} du\right) \\ &= \frac{1}{4} \int (u^{4/3} - u^{1/3}) du \\ &= \frac{1}{4} \left(\frac{3}{7} u^{7/3} - \frac{3}{4} u^{4/3} \right) + C \\ &= \frac{3}{28} (2x+1)^{7/3} - \frac{3}{16} (2x+1)^{4/3} + C \end{aligned}$$

Exercise

Evaluate the integral $\int (x+1)\sqrt{3x+2} \, dx$

Solution

$$u = 3x + 2 \rightarrow x = \frac{1}{3}(u - 2)$$

$$dx = \frac{1}{3} du$$

$$\begin{aligned} \int (x+1)\sqrt{3x+2} \, dx &= \int \left(\frac{1}{3}u - 2 + 1\right)u^{1/2} \frac{1}{3} du \\ &= \frac{1}{3} \int \left(\frac{1}{3}u - 1\right)u^{1/2} du \\ &= \frac{1}{3} \int \left(\frac{1}{3}u^{3/2} - u^{1/2}\right) du \\ &= \frac{1}{3} \left(\frac{2}{15} u^{5/2} - \frac{2}{3} u^{3/2} \right) + C \\ &= \frac{2}{45} (3x+2)^{5/2} - \frac{2}{9} (3x+2)^{3/2} + C \end{aligned}$$

Exercise

Evaluate the integral $\int \sin^2 x \, dx$

Solution

$$\begin{aligned} \int \sin^2 x \, dx &= \frac{1}{2} \int (1 - \cos 2x) \, dx & \sin^2 \theta &= \frac{1}{2}(1 - \cos 2\theta) \\ &= \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) + C \end{aligned}$$

Exercise

Evaluate the integral $\int \sin^2 \left(\theta + \frac{\pi}{6} \right) d\theta$

Solution

$$\begin{aligned} \int \sin^2 \left(\theta + \frac{\pi}{6} \right) d\theta &= \int \frac{1}{2} \left(1 - \cos 2 \left(\theta + \frac{\pi}{6} \right) \right) d\theta & \sin^2 \theta &= \frac{1}{2}(1 - \cos 2\theta) \\ &= \frac{1}{2} \int d\theta - \frac{1}{4} \int \cos \left(2\theta + \frac{\pi}{3} \right) d \left(2\theta + \frac{\pi}{3} \right) & d \left(2\theta + \frac{\pi}{3} \right) &= 2d\theta \\ &= \frac{1}{2} \theta - \frac{1}{4} \sin \left(2\theta + \frac{\pi}{3} \right) + C \end{aligned}$$

Exercise

Evaluate the integral $\int x \cos^2(x^2) \, dx$

Solution

$$\begin{aligned} d(x^2) &= 2x \, dx \\ \int x \cos^2(x^2) \, dx &= \frac{1}{2} \int \cos^2(x^2) \, d(x^2) \\ &= \frac{1}{4} \int (1 + \cos(2x^2)) \, d(x^2) \\ &= \frac{1}{4} \int d(x^2) + \frac{1}{8} \int \cos(2x^2) \, d(2x^2) & d(x^2) &= 2d(x^2) \\ &= \frac{1}{4} x^2 + \frac{1}{8} \sin(2x^2) + C \end{aligned}$$

$$\begin{aligned}
 \int x \cos^2(x^2) dx &= \frac{1}{2} \int x(1 + \cos(2x^2)) dx \\
 &= \frac{1}{2} \int x dx + \frac{1}{2} \int x \cos(2x^2) dx \\
 &= \frac{1}{4} x^2 + \frac{1}{8} \int \cos(2x^2) d(2x^2) \\
 &= \frac{1}{4} x^2 + \frac{1}{8} \sin(2x^2) + C
 \end{aligned}$$

$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$$

Exercise

Evaluate the integral $\int \sec 4x \tan 4x dx$

Solution

$$d(\sec 4x) = 4 \sec 4x \tan 4x$$

$$\begin{aligned}
 \int \sec 4x \tan 4x dx &= \frac{1}{4} \int d(\sec 4x) \\
 &= \frac{1}{4} \sec 4x + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \sec^2 10x dx$

Solution

$$\begin{aligned}
 \int \sec^2 10x dx &= \frac{1}{10} \int \sec^2 10x d(10x) \\
 &= \frac{1}{10} \tan 10x + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int (\sin^5 x + 3 \sin^3 x - \sin x) \cos x dx$

Solution

$$d(\sin x) = \cos x dx$$

$$\int (\sin^5 x + 3\sin^3 x - \sin x) \cos x \, dx = \int (\sin^5 x + 3\sin^3 x - \sin x) d(\sin x)$$

$$= \frac{1}{6} \sin^6 x + \frac{3}{4} \sin^4 x - \frac{1}{2} \sin^2 x + C$$

Exercise

Evaluate the integral $\int \frac{\csc^2 x}{\cot^3 x} dx$

Solution

$$d(\cot x) = -\csc^2 x \, dx$$

$$\int \frac{\csc^2 x}{\cot^3 x} dx = - \int \cot^{-3} x \, d(\cot x)$$

$$= \frac{1}{2} \cot^{-2} x + C$$

$$= \frac{1}{2 \cot^2 x} + C$$

$$= \frac{1}{2} \tan^2 x + C$$

Exercise

Evaluate the integral $\int (x^{3/2} + 8)^5 \sqrt{x} \, dx$

Solution

$$d(x^{3/2} + 8) = \frac{3}{2} x^{1/2} dx$$

$$\int (x^{3/2} + 8)^5 \sqrt{x} \, dx = \frac{2}{3} \int (x^{3/2} + 8)^5 d(x^{3/2} + 8)$$

$$= \frac{1}{9} (x^{3/2} + 8)^6 + C$$

Exercise

Evaluate the integral $\int \sin x \sec^8 x \, dx$

Solution

$$d(\cos x) = -\sin x \, dx; \quad \sec x = \frac{1}{\cos x}$$

$$\begin{aligned}\int \sin x \sec^8 x \, dx &= -\int \cos^{-8} x \, d(\cos x) \\ &= \frac{1}{7} \cos^{-7} x + C \\ &= \frac{1}{7} \sec^7 x + C\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{e^{2x}}{e^{2x} + 1} dx$

Solution

$$d(e^{2x} + 1) = 2e^{2x} dx$$

$$\begin{aligned}\int \frac{e^{2x}}{e^{2x} + 1} dx &= \frac{1}{2} \int \frac{1}{e^{2x} + 1} d(e^{2x} + 1) \\ &= \frac{1}{2} \ln(e^{2x} + 1) + C\end{aligned}$$

Exercise

Evaluate the integral $\int \sec^3 \theta \tan \theta \, d\theta$

Solution

$$d(\sec \theta) = \tan \theta \sec \theta \, d\theta$$

$$\begin{aligned}\int \sec^3 \theta \tan \theta \, d\theta &= \int \sec^2 \theta \sec \theta \tan \theta \, d\theta \\ &= \int \sec^2 \theta \, d(\sec \theta) \\ &= \frac{1}{3} \sec^3 \theta + C\end{aligned}$$

Exercise

Evaluate the integral $\int x \sin^4 x^2 \cos x^2 \, dx$

Solution

$$d(\sin x^2) = 2x \cos x^2 \, dx$$

$$\int x \sin^4 x^2 \cos x^2 dx = \frac{1}{2} \int \sin^4 x^2 d(\sin x^2)$$

$$= \frac{1}{10} \sin^5(x^2) + C \quad \Big|$$

Exercise

Evaluate the integral $\int \frac{dx}{\sqrt{1+\sqrt{1+x}}}$

Solution

$$u = 1 + \sqrt{1+x} \rightarrow \sqrt{1+x} = u - 1$$

$$du = \frac{1}{2\sqrt{1+x}} dx$$

$$dx = 2(u-1) du$$

$$\int \frac{dx}{\sqrt{1+\sqrt{1+x}}} = 2 \int \frac{(u-1)}{u^{1/2}} du$$

$$= 2 \int (u^{1/2} - u^{-1/2}) du$$

$$= 2 \left(\frac{2}{3} u^{3/2} - 2u^{1/2} \right) + C$$

$$= \frac{4}{3} (1+\sqrt{1+x})^{3/2} - 4(1+\sqrt{1+x})^{1/2} + C \quad \Big|$$

Exercise

Evaluate the integral $\int \tan^{10} 4x \sec^2 4x dx$

Solution

$$d(\tan 4x) = 4 \sec^2(4x) dx$$

$$\int \tan^{10} 4x \sec^2 4x dx = \frac{1}{4} \int \tan^{10} 4x d(\tan 4x)$$

$$= \frac{1}{44} \tan^{11} 4x + C \quad \Big|$$

Exercise

Evaluate the integral $\int \frac{x^2}{x^3 + 27} dx$

Solution

$$d(x^3 + 27) = 3x^2 dx$$

$$\begin{aligned} \int \frac{x^2}{x^3 + 27} dx &= \frac{1}{3} \int \frac{1}{x^3 + 27} d(x^3 + 27) \\ &= \frac{1}{3} \ln|x^3 + 27| + C \end{aligned}$$

Exercise

Evaluate the integral $\int y^2 (3y^3 + 1)^4 dy$

Solution

$$d(3y^3 + 1) = 9y^2 dy$$

$$\begin{aligned} \int y^2 (3y^3 + 1)^4 dy &= \frac{1}{9} \int (3y^3 + 1)^4 d(3y^3 + 1) \\ &= \frac{1}{45} (3y^3 + 1)^5 + C \end{aligned}$$

Exercise

Evaluate the integral $\int x \sin x^2 \cos^8 x^2 dx$

Solution

$$d(\cos x^2) = -2x \sin x^2 dx$$

$$\begin{aligned} \int x \sin x^2 \cos^8 x^2 dx &= -\frac{1}{2} \int \cos^8 x^2 d(\cos x^2) \\ &= -\frac{1}{18} \cos^9(x^2) + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\sin 2x}{1 + \cos^2 x} dx$

Solution

$$\begin{aligned} d(1 + \cos^2 x) &= -2 \cos x \sin x \, dx \\ &= -\sin 2x \, dx \end{aligned}$$

$$\begin{aligned} \int \frac{\sin 2x}{1 + \cos^2 x} dx &= - \int \frac{1}{1 + \cos^2 x} d(1 + \cos^2 x) \\ &= -\ln|1 + \cos^2 x| + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$

Solution

$$d(\sin^{-1} x) = \frac{dx}{\sqrt{1-x^2}}$$

$$\begin{aligned} \int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx &= \int \sin^{-1} x \, d(\sin^{-1} x) \\ &= \frac{1}{2} (\sin^{-1} x)^2 + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{(\tan^{-1} x)(1+x^2)}$

Solution

$$d(\tan^{-1} x) = \frac{dx}{1+x^2}$$

$$\begin{aligned} \int \frac{dx}{(\tan^{-1} x)(1+x^2)} &= \int \frac{1}{\tan^{-1} x} d(\tan^{-1} x) \\ &= \ln|\tan^{-1} x| + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{(\tan^{-1} x)^5}{1+x^2} dx$

Solution

$$d(\tan^{-1} x) = \frac{dx}{1+x^2}$$

$$\begin{aligned} \int \frac{(\tan^{-1} x)^5}{1+x^2} dx &= \int (\tan^{-1} x)^5 d(\tan^{-1} x) \\ &= \frac{1}{6} (\tan^{-1} x)^6 + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{1}{x^2} \sin \frac{1}{x} dx$

Solution

$$d\left(\frac{1}{x}\right) = -\frac{1}{x^2} dx$$

$$\begin{aligned} \int \frac{1}{x^2} \sin \frac{1}{x} dx &= - \int \sin \frac{1}{x} d\left(\frac{1}{x}\right) \\ &= \cos \frac{1}{x} + C \end{aligned}$$

Exercise

Evaluate the integral $\int_{-1}^2 x^2 e^{x^3+1} dx$

Solution

$$d(x^3 + 1) = 3x^2 dx$$

$$\begin{aligned} \int_{-1}^2 x^2 e^{x^3+1} dx &= \frac{1}{3} \int_{-1}^2 e^{x^3+1} d(x^3 + 1) \\ &= \frac{1}{3} e^{x^3+1} \Big|_{-1}^2 \\ &= \frac{1}{3} (e^9 - 1) \end{aligned}$$

Exercise

Evaluate the integral $\int_0^2 x^2 e^{x^3} dx$

Solution

$$d(x^3) = 3x^2 dx$$

$$\begin{aligned}\int_0^2 x^2 e^{x^3} dx &= \frac{1}{3} \int_0^2 e^{x^3} d(x^3) \\ &= \frac{1}{3} e^{x^3} \Big|_0^2 \\ &= \frac{1}{3} (e^8 - e) \end{aligned}$$

Exercise

Evaluate the integral $\int_0^4 \frac{x}{x^2 + 1} dx$

Solution

$$\begin{aligned}\int_0^4 \frac{x}{x^2 + 1} dx &= \frac{1}{2} \int_0^4 \frac{1}{x^2 + 1} d(x^2 + 1) \\ &= \frac{1}{2} \ln(x^2 + 1) \Big|_0^4 \\ &= \frac{1}{2} (\ln 17 - \ln 1) \\ &= \frac{1}{2} \ln 17 \end{aligned}$$

Exercise

Evaluate the integrals $\int \frac{18 \tan^2 x \sec^2 x}{(2 + \tan^3 x)^2} dx$

- a) $u = \tan x$, followed by $v = u^3$ then by $w = 2 + v$
- b) $u = \tan^3 x$, followed by $v = 2 + u$
- c) $u = 2 + \tan^3 x$

Solution

a) Let $u = \tan x \Rightarrow du = \sec^2 x dx$

$$v = u^3 \Rightarrow dv = 3u^2 du$$

$$w = 2 + v \Rightarrow dw = dv$$

$$\begin{aligned} \int \frac{18 \tan^2 x \sec^2 x}{(2 + \tan^3 x)^2} dx &= \int \frac{18u^2 du}{(2 + u^3)^2} \\ &= \int \frac{6 dv}{(2 + v)^2} \\ &= \int \frac{6 dw}{w^2} \\ &= 6 \int w^{-2} dw \\ &= 6 \frac{w^{-1}}{-1} + C \\ &= -\frac{6}{w} + C \\ &= -\frac{6}{2 + v} + C \\ &= -\frac{6}{2 + u^3} + C \\ &= -\frac{6}{2 + \tan^3 x} + C \end{aligned}$$

b) $d(2 + \tan^3 x) = 3 \tan^2 x \sec^2 x dx$

$$\begin{aligned} \int \frac{18 \tan^2 x \sec^2 x}{(2 + \tan^3 x)^2} dx &= \int \frac{6}{(2 + \tan^3 x)^2} d(2 + \tan^3 x) \\ &= -\frac{6}{2 + \tan^3 x} + C \end{aligned}$$

Let $u = \tan^3 x \Rightarrow du = 3 \tan^2 x \sec^2 x dx$

$$v = 2 + u \Rightarrow dv = du$$

$$\begin{aligned} \int \frac{18 \tan^2 x \sec^2 x}{(2 + \tan^3 x)^2} dx &= \int \frac{6 du}{(2 + u)^2} \\ &= \int \frac{6 dv}{v^2} \end{aligned}$$

$$\begin{aligned}
&= \int 6v^{-2} dv \\
&= -6v^{-1} + C \\
&= -\frac{6}{v} + C \\
&= -\frac{6}{2+u} + C \\
&= -\frac{6}{2+\tan^3 x} + C
\end{aligned}$$

c) Let $u = 2 + \tan^3 x$

$$du = 3 \tan^2 x \sec^2 x dx$$

$$\frac{1}{3} du = \tan^2 x \sec^2 x dx$$

$$\begin{aligned}
\int \frac{18 \tan^2 x \sec^2 x}{(2 + \tan^3 x)^2} dx &= \int \frac{18}{u^2} \left(\frac{1}{3} du \right) \\
&= 6 \int u^{-2} du \\
&= -6u^{-1} + C \\
&= -\frac{6}{u} + C \\
&= -\frac{6}{2 + \tan^3 x} + C
\end{aligned}$$

Exercise

Evaluate: $\int_0^1 (2t+3)^3 dt$

Solution

$$d(2t+3) = 2dt \rightarrow \frac{1}{2} d(2t+3) = dt$$

$$\begin{aligned}
\int_0^1 (2t+3)^3 dt &= \frac{1}{2} \int_0^1 (2t+3)^3 d(2t+3) \\
&= \frac{1}{8} (2t+3)^4 \Big|_0^1 \\
&= \frac{1}{8} \left[(2(1)+3)^4 - (2(0)+3)^4 \right] \\
&= \frac{1}{8} (5^4 - 3^4)
\end{aligned}$$

$$= 68 \mid$$

Exercise

Evaluate the integral $\int_0^2 \sqrt{4-x^2} \, dx$

Solution

$$\int_0^2 \sqrt{4-x^2} \, dx = \left(\frac{1}{2} x \sqrt{4-x^2} + \frac{4}{2} \sin^{-1} \frac{x}{2} \right) \bigg|_0^2$$

$\sqrt{4-x^2}$ is a semi-circle with center (0, 0) and radius = 2.

Since x from 0 to 2

$$\text{Area} = \frac{1}{4} (\text{Area of this circle})$$

$$= \frac{1}{4} 2\pi 2^2$$

$$= 2\pi \text{ unit}^2 \mid$$

Exercise

Evaluate the integral $\int_0^3 \sqrt{y+1} \, dy$

Solution

$$d(y+1) = dy$$

$$\int_0^3 \sqrt{y+1} \, dy = \int_0^3 (y+1)^{1/2} d(y+1)$$

$$= \frac{2}{3} (y+1)^{3/2} \bigg|_0^3$$

$$= \frac{2}{3} \left[(3+1)^{3/2} - (0+1)^{3/2} \right]$$

$$= \frac{2}{3} (8-1)$$

$$= \frac{14}{3} \mid$$

Exercise

Evaluate the integral $\int_{-1}^1 r\sqrt{1-r^2} \, dr$

Solution

$$d(1-r^2) = -2rdr$$

$$\begin{aligned} \int_{-1}^1 r\sqrt{1-r^2} \, dr &= -\frac{1}{2} \int_{-1}^1 (1-r^2)^{1/2} d(1-r^2) \\ &= -\frac{1}{3} (1-r^2)^{3/2} \Big|_{-1}^1 \\ &= -\frac{1}{3} \left[(1-(1)^2)^{3/2} - (1-(-1)^2)^{3/2} \right] \\ &= -\frac{1}{3} (0-0) \\ &= \underline{0} \end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/4} \tan x \sec^2 x \, dx$

Solution

$$d(\tan x) = \sec^2 x \, dx$$

$$\begin{aligned} \int_0^{\pi/4} \tan x \sec^2 x \, dx &= \int_0^{\pi/4} \tan x \, d(\tan x) \\ &= \frac{1}{2} \tan^2 x \Big|_0^{\pi/4} \\ &= \frac{1}{2} (1^2 - 0^2) \\ &= \underline{\frac{1}{2}} \end{aligned}$$

Exercise

Evaluate the integral $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x \, dx$

Solution

$$d(\cos x) = -\sin x \, dx$$

$$\begin{aligned} \int_{2\pi}^{3\pi} 3\cos^2 x \sin x \, dx &= - \int_{2\pi}^{3\pi} 3\cos^2 x \, d(\cos x) \\ &= -\cos^3 x \Big|_{2\pi}^{3\pi} \\ &= -\left((-1)^3 - 1^3\right) \\ &= \underline{2} \end{aligned}$$

Exercise

Evaluate the integral $\int_0^1 t^3 (1+t^4)^3 \, dt$

Solution

$$d(1+t^4) = 4t^3 \, dt$$

$$\begin{aligned} \int_0^1 t^3 (1+t^4)^3 \, dt &= \frac{1}{4} \int_0^1 (1+t^4)^3 \, d(1+t^4) \\ &= \frac{1}{16} (1+t^4)^4 \Big|_0^1 \\ &= \frac{1}{16} (2^4 - 1^4) \\ &= \underline{\frac{15}{16}} \end{aligned}$$

Exercise

Evaluate the integral $\int_0^1 \frac{r}{(4+r^2)^2} \, dr$

Solution

$$d(4+r^2) = 2r \, dr$$

$$\int_0^1 \frac{r}{(4+r^2)^2} \, dr = \frac{1}{2} \int_0^1 \frac{d(4+r^2)}{(4+r^2)^2}$$

$$\begin{aligned}
&= -\frac{1}{2} \left(\frac{1}{4+r^2} \right) \Big|_0^1 \\
&= -\frac{1}{2} \left(\frac{1}{5} - \frac{1}{4} \right) \\
&= -\frac{1}{40}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2} dv$

Solution

$$d(1+v^{3/2}) = \frac{3}{2}\sqrt{v} dv$$

$$\begin{aligned}
\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2} dv &= \frac{20}{3} \int_0^1 \frac{1}{(1+v^{3/2})^2} d(1+v^{3/2}) \\
&= -\frac{20}{3} \left(\frac{1}{1+v^{3/2}} \right) \Big|_0^1 \\
&= -\frac{20}{3} \left(\frac{1}{2} - 1 \right) \\
&= \frac{10}{3}
\end{aligned}$$

Exercise

Evaluate the integral $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}} dx$

Solution

$$d(x^2+1) = 2x dx$$

$$\begin{aligned}
\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}} dx &= 2 \int_{-\sqrt{3}}^{\sqrt{3}} (x^2+1)^{-1/2} d(x^2+1) \\
&= 4\sqrt{x^2+1} \Big|_{-\sqrt{3}}^{\sqrt{3}} \\
&= 4(2-2)
\end{aligned}$$

$$= 0 \Big|$$

Let $u = x^2 + 1$

$$du = 2x dx \rightarrow \frac{1}{2} du = x dx$$

$$\rightarrow \begin{cases} x = \sqrt{3} & \rightarrow u = 4 \\ x = -\sqrt{3} & \rightarrow u = 4 \end{cases}$$

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2 + 1}} dx = 4 \int_4^4 \frac{1}{u} \left(\frac{1}{2} du \right)$$

$$= 0 \Big|$$

Exercise

Evaluate the integral $\int_0^1 \frac{x^3}{\sqrt{x^4 + 9}} dx$

Solution

$$d(x^4 + 9) = 4x^3 dx$$

$$\begin{aligned} \int_0^1 \frac{x^3}{\sqrt{x^4 + 9}} dx &= \frac{1}{4} \int_0^1 (x^4 + 9)^{-1/2} d(x^4 + 9) \\ &= \frac{1}{2} (x^4 + 9)^{1/2} \Big|_0^1 \\ &= \frac{1}{2} (10^{1/2} - 9^{1/2}) \\ &= \frac{\sqrt{10} - 3}{2} \Big| \end{aligned}$$

$$u = x^4 + 9$$

$$du = 4x^3 dx$$

$$\frac{1}{4} du = x^3 dx$$

$$\begin{cases} x = 1 \rightarrow u = 10 \\ x = 0 \rightarrow u = 9 \end{cases}$$

$$\int_0^1 \frac{x^3}{\sqrt{x^4 + 9}} dx = \frac{1}{4} \int_9^{10} u^{-1/2} du$$

$$\begin{aligned}
&= \frac{1}{4} \left(2u^{1/2} \right) \Big|_9^{10} \\
&= \frac{1}{2} \left(\textcolor{red}{10}^{1/2} - \textcolor{blue}{9}^{1/2} \right) \\
&= \underline{\underline{\frac{\sqrt{10}-3}{2}}}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/6} (1 - \cos 3t) \sin 3t \, dt$

Solution

$$d(1 - \cos 3t) = 3 \sin 3t \, dt$$

$$\begin{aligned}
\int_0^{\pi/6} (1 - \cos 3t) \sin 3t \, dt &= \frac{1}{3} \int_0^{\pi/6} (1 - \cos 3t) \, d(1 - \cos 3t) \\
&= \frac{1}{6} (1 - \cos 3t)^2 \Big|_0^{\pi/6} \\
&= \frac{1}{6} (1^2 - 0^2) \\
&= \underline{\underline{\frac{1}{6}}}
\end{aligned}$$

Exercise

Evaluate the integral $\int_{-\pi/2}^{\pi/2} \left(2 + \tan \frac{t}{2} \right) \sec^2 \frac{t}{2} \, dt$

Solution

$$d\left(2 + \tan \frac{t}{2} \right) = \frac{1}{2} \sec^2 \frac{t}{2} \, dt$$

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} \left(2 + \tan \frac{t}{2} \right) \sec^2 \frac{t}{2} \, dt &= 2 \int_{-\pi/2}^{\pi/2} \left(2 + \tan \frac{t}{2} \right) d\left(2 + \tan \frac{t}{2} \right) \\
&= \left(2 + \tan \frac{t}{2} \right)^2 \Big|_{-\pi/2}^{\pi/2} \\
&= 3^2 - 1 \\
&= \underline{\underline{8}}
\end{aligned}$$



$$u = 2 + \tan \frac{t}{2}$$

$$du = \frac{1}{2} \sec^2 \frac{t}{2} dt$$

$$2du = \sec^2 \frac{t}{2} dt$$

$$\begin{cases} t = \frac{\pi}{2} & \rightarrow u = 3 \\ t = -\frac{\pi}{2} & \rightarrow u = 1 \end{cases}$$

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} \left(2 + \tan \frac{t}{2}\right) \sec^2 \frac{t}{2} dt &= \int_1^3 u(2du) \\ &= 2 \left(\frac{u^2}{2} \right) \Big|_1^3 \\ &= 3^2 - 1^2 \\ &= 8 \end{aligned}$$

Exercise

Evaluate the integral $\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}} dz$

Solution

$$d(4+3\sin z) = 3\cos z dz$$

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}} dz &= \frac{1}{3} \int_{-\pi}^{\pi} (4+3\sin z)^{-1/2} d(4+3\sin z) \\ &= \frac{2}{3} \sqrt{4+3\sin z} \Big|_{-\pi}^{\pi} \\ &= \frac{2}{3} (2-2) \\ &= 0 \end{aligned}$$

Let $u = 4 + 3\sin z$

$$du = 3\cos z dz$$

$$\frac{1}{3} du = \cos z dz$$

$$\begin{cases} z = \pi & \rightarrow u = 4 \\ z = -\pi & \rightarrow u = 4 \end{cases}$$

$$\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}} dz = \frac{1}{3} \int_4^4 \frac{1}{\sqrt{u}} du$$

$$\underline{=0}$$

Exercise

Evaluate the integral $\int_{-\pi/2}^0 \frac{\sin w}{(3+2\cos w)^2} dw$

Solution

$$d(3+2\cos w) = -2\sin w dw$$

$$\begin{aligned} \int_{-\pi/2}^0 \frac{\sin w}{(3+2\cos w)^2} dw &= -\frac{1}{2} \int_{-\pi/2}^0 \frac{d(3+2\cos w)}{(3+2\cos w)^2} \\ &= \frac{1}{2} \left(\frac{1}{3+2\cos w} \right) \Big|_{-\pi/2}^0 \\ &= \frac{1}{2} \left(\frac{1}{5} - \frac{1}{3} \right) \\ &\underline{= -\frac{1}{15}} \end{aligned}$$

$$d\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

Exercise

Evaluate the integral $\int_0^1 \sqrt{t^5+2t} (5t^4+2) dt$

Solution

$$d(t^5+2t) = (5t^4+2) dt$$

$$\begin{aligned} \int_0^1 \sqrt{t^5+2t} (5t^4+2) dt &= \int_0^1 (t^5+2t)^{1/2} d(t^5+2t) \\ &= \frac{2}{3} (t^5+2t)^{3/2} \Big|_0^1 \\ &= \frac{2}{3} (3^{3/2}) \\ &\underline{= 2\sqrt{3}} \end{aligned}$$

Exercise

Evaluate the integral $\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2}$

Solution

$$d(1+\sqrt{y}) = \frac{1}{2\sqrt{y}} dy$$

$$\begin{aligned}\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2} &= \int_1^4 \frac{1}{(1+\sqrt{y})^2} d(1+\sqrt{y}) \\ &= -\frac{1}{1+\sqrt{y}} \Big|_1^4 \\ &= -\left(\frac{1}{3} - \frac{1}{2}\right) \\ &= \frac{1}{6}\end{aligned}$$

Exercise

Evaluate the integral $\int_0^1 (4y - y^2 + 4y^3 + 1)^{-2/3} (12y^2 - 2y + 4) dy$

Solution

$$d(4y - y^2 + 4y^3 + 1) = (4 - 2y + 12y^2) dy$$

$$\begin{aligned}\int_0^1 (4y - y^2 + 4y^3 + 1)^{-2/3} (12y^2 - 2y + 4) dy &= \int_0^1 (4y - y^2 + 4y^3 + 1)^{-2/3} d(4y - y^2 + 4y^3 + 1) \\ &= 3 (4y - y^2 + 4y^3 + 1)^{1/3} \Big|_0^1 \\ &= 3 (2 - 1) \\ &= 3\end{aligned}$$

Let $u = 4y - y^2 + 4y^3 + 1$

$$du = (4 - 2y + 12y^2) dy$$

$$\rightarrow \begin{cases} y=1 & \rightarrow u=8 \\ y=0 & \rightarrow u=1 \end{cases}$$

$$\begin{aligned}
 \int_0^1 (4y - y^2 + 4y^3 + 1)^{-2/3} (12y^2 - 2y + 4) dy &= \int_1^8 u^{-2/3} du \\
 &= 3u^{1/3} \Big|_1^8 \\
 &= 3(8^{1/3} - 1^{1/3}) \\
 &= 3
 \end{aligned}$$

Exercise

Evaluate the integral $\int_0^5 |x-2| dx$

Solution

$$|x-2| = \begin{cases} x-2 & x > 2 \\ -(x-2) & x < 2 \end{cases}$$

$$\begin{aligned}
 \int_0^5 |x-2| dx &= \int_0^2 -(x-2) dx + \int_2^5 (x-2) dx \\
 &= \left(-\frac{x^2}{2} + 2x \right) \Big|_0^2 + \left(\frac{x^2}{2} - 2x \right) \Big|_2^5 \\
 &= -\frac{4}{2} + 4 - 0 + \left(\frac{25}{2} - 10 - \left(\frac{4}{2} - 4 \right) \right) \\
 &= -2 + 4 + \frac{25}{2} - 10 - 2 + 4 \\
 &= \frac{25}{2} - 6 \\
 &= \frac{13}{2}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/2} e^{\sin x} \cos x dx$

Solution

$$d(\sin x) = \cos x dx$$

$$\int_0^{\pi/2} e^{\sin x} \cos x dx = \int_0^{\pi/2} e^{\sin x} d(\sin x)$$

$$\begin{aligned}
 &= e^{\sin x} \Big|_0^{\pi/2} \\
 &= e^{\sin \frac{\pi}{2}} - e^{\sin 0} \\
 &= e^1 - e^0 \\
 &= \underline{e-1}
 \end{aligned}$$

Exercise

Evaluate $\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$

Solution

$$\begin{aligned}
 \int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} &= \sin^{-1} x \Big|_{\sqrt{2}/2}^{\sqrt{3}/2} \\
 &= \sin^{-1} \left(\frac{\sqrt{3}}{2} \right) - \sin^{-1} \left(\frac{\sqrt{2}}{2} \right) \\
 &= \frac{\pi}{3} - \frac{\pi}{4} \\
 &= \underline{\frac{\pi}{12}}
 \end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

Exercise

Evaluate the integral $\int_0^{\pi/3} \frac{4 \sin \theta}{1-4 \cos \theta} d\theta$

Solution

$$\begin{aligned}
 \int_0^{\pi/3} \frac{4 \sin \theta}{1-4 \cos \theta} d\theta &= \int_0^{\pi/3} \frac{d(1-4 \cos \theta)}{1-4 \cos \theta} \\
 &= \ln |1-4 \cos \theta| \Big|_0^{\pi/3} \\
 &= \ln \left| 1-4 \cos \frac{\pi}{3} \right| - \ln |1-4 \cos 0| \\
 &= \ln |-1| - \ln |-3| \\
 &= \ln 1 - \ln 3 \\
 &= -\ln 3 \\
 &= \underline{\frac{1}{\ln 3}}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_1^2 \frac{2 \ln x}{x} dx$

Solution

$$d(\ln x) = \frac{1}{x} dx$$

$$\begin{aligned} \int_1^2 \frac{2 \ln x}{x} dx &= 2 \int_1^2 \ln x \, d(\ln x) \\ &= (\ln x)^2 \Big|_1^2 \\ &= (\ln 2)^2 \end{aligned}$$

$$u = \ln x$$

$$du = \frac{1}{x} dx$$

$$\begin{cases} x=1 & u = \ln 1 = 0 \\ x=2 & u = \ln 2 \end{cases}$$

$$\begin{aligned} \int_1^2 \frac{2 \ln x}{x} dx &= \int_0^{\ln 2} 2u \, du \\ &= u^2 \Big|_0^{\ln 2} \\ &= (\ln 2)^2 \end{aligned}$$

Exercise

Evaluate the integral $\int_2^{16} \frac{dx}{2x \sqrt{\ln x}}$

Solution

$$d(\ln x) = \frac{1}{x} dx$$

$$\begin{aligned} \int_2^{16} \frac{dx}{2x \sqrt{\ln x}} &= \frac{1}{2} \int_2^{16} (\ln x)^{-1/2} d(\ln x) \\ &= \sqrt{\ln x} \Big|_2^{16} \end{aligned}$$

$$\begin{aligned}
 &= \sqrt{\ln 2^4} - \sqrt{\ln 2} \\
 &= 2\sqrt{\ln 2} - \sqrt{\ln 2} \\
 &= \sqrt{\ln 2}
 \end{aligned}$$

$$u = \ln x$$

$$du = \frac{1}{x} dx$$

$$\begin{cases} x = 2 & u = \ln 2 \\ x = 16 & u = \ln 16 = \ln 2^4 \end{cases}$$

$$\begin{aligned}
 \int_2^{16} \frac{dx}{2x \sqrt{\ln x}} &= \int_{\ln 2}^{4 \ln 2} \frac{1}{2} u^{-1/2} du \\
 &= u^{1/2} \Big|_{\ln 2}^{4 \ln 2} \\
 &= (4 \ln 2)^{1/2} - (\ln 2)^{1/2} \\
 &= 2\sqrt{\ln 2} - \sqrt{\ln 2} \\
 &= \sqrt{\ln 2}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/2} \tan \frac{x}{2} dx$

Solution

$$d \cos \frac{x}{2} = -\frac{1}{2} \sin \frac{x}{2} dx$$

$$\begin{aligned}
 \int_0^{\pi/2} \tan \frac{x}{2} dx &= \int_0^{\pi/2} \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} dx \\
 &= \int_0^{\pi/2} \frac{-2}{\cos \frac{x}{2}} d \cos \frac{x}{2} \\
 &= -2 \ln \left| \cos \frac{x}{2} \right| \Big|_0^{\pi/2} \\
 &= -2 \left(\ln \left| \cos \frac{\pi}{4} \right| - \ln |\cos 0| \right)
 \end{aligned}$$

$$\begin{aligned}
 &= -2 \left(\ln \left| \frac{1}{\sqrt{2}} \right| - \ln |1| \right) \\
 &= -2 \ln \left(2^{-1/2} \right) \\
 &= \ln 2 \quad |
 \end{aligned}$$

Exercise

Evaluate the integral $\int_{\pi/4}^{\pi/2} \cot x \, dx$

Solution

$$\begin{aligned}
 \int_{\pi/4}^{\pi/2} \cot x \, dx &= \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x} \, dx \\
 &= \int_{\pi/4}^{\pi/2} \frac{d(\sin x)}{\sin x} \\
 &= \ln(\sin x) \Big|_{\pi/4}^{\pi/2} \\
 &= \ln 1 - \ln \frac{1}{\sqrt{2}} \\
 &= -\ln \frac{1}{\sqrt{2}} \\
 &= \ln \sqrt{2} \quad |
 \end{aligned}$$

Exercise

Evaluate the integral $\int_{-\ln 2}^0 e^{-x} \, dx$

Solution

$$\begin{aligned}
 \int_{-\ln 2}^0 e^{-x} \, dx &= -e^{-x} \Big|_{-\ln 2}^0 \\
 &= -(e^0 - e^{\ln 2}) \\
 &= -(1 - 2) \\
 &= 1 \quad |
 \end{aligned}$$

Exercise

Evaluate the integral $\int_{\pi/4}^{\pi/2} (1 + e^{\cot \theta}) \csc^2 \theta \, d\theta$

Solution

$$d(\cot \theta) = -\csc^2 \theta \, d\theta$$

$$\begin{aligned} \int_{\pi/4}^{\pi/2} (1 + e^{\cot \theta}) \csc^2 \theta \, d\theta &= - \int_{\pi/4}^{\pi/2} (1 + e^{\cot \theta}) \, d(\cot \theta) \\ &= - \left(\cot \theta + e^{\cot \theta} \right) \Big|_{\pi/4}^{\pi/2} \\ &= - \left(e^0 - 1 - e \right) \\ &= e \end{aligned}$$

Let $u = \cot \theta$

$$du = -\csc^2 \theta \, d\theta$$

$$\begin{cases} \theta = \frac{\pi}{2} & \Rightarrow u = 0 \\ \theta = \frac{\pi}{4} & \Rightarrow u = 1 \end{cases}$$

$$\begin{aligned} \int_{\pi/4}^{\pi/2} (1 + e^{\cot \theta}) \csc^2 \theta \, d\theta &= \int_{\pi/4}^{\pi/2} \csc^2 \theta \, d\theta + \int_{\pi/4}^{\pi/2} e^{\cot \theta} \csc^2 \theta \, d\theta \\ &= -\cot \theta \Big|_{\pi/4}^{\pi/2} + \int_0^1 e^u \, du \\ &= - \left(\cot \frac{\pi}{2} - \cot \frac{\pi}{4} \right) + e^u \Big|_0^1 \\ &= -(0 - 1) + e^1 - 1 \\ &= e \end{aligned}$$

Exercise

Evaluate the integral $\int_1^4 \frac{2\sqrt{x}}{\sqrt{x}} \, dx$

Solution

$$d(\sqrt{x}) = \frac{1}{2\sqrt{x}} \, dx$$

$$\begin{aligned}
 \int_1^4 \frac{2\sqrt{x}}{\sqrt{x}} dx &= 2 \int_1^4 2^{\sqrt{x}} d(\sqrt{x}) \\
 &= \frac{2}{\ln 2} \left(2^{\sqrt{x}} \right) \Big|_1^4 \\
 &= \frac{2}{\ln 2} (4 - 2) \\
 &= \frac{4}{\ln 2}
 \end{aligned}$$

Let $u = \sqrt{x}$

$$du = \frac{1}{2\sqrt{x}} dx$$

$$2du = \frac{1}{\sqrt{x}} dx$$

$$\begin{cases} x=1 & u=1 \\ x=4 & u=\sqrt{4}=2 \end{cases}$$

$$\begin{aligned}
 \int_1^4 \frac{2\sqrt{x}}{\sqrt{x}} dx &= \int_1^2 2^u (2du) \\
 &= 2 \int_1^2 2^u du \\
 &= 2 \left(\frac{2^u}{\ln 2} \right) \Big|_1^2 \\
 &= \frac{2}{\ln 2} (2^2 - 2^1) \\
 &= \frac{2}{\ln 2} (2) \\
 &= \frac{4}{\ln 2}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan t} \sec^2 t \, dt$

Solution

$$d(\tan t) = \sec^2 t \, dt$$

$$\begin{aligned}
 \int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan t} \sec^2 t \, dt &= \int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan t} d(\tan t) \\
 &= \frac{1}{\ln \frac{1}{3}} \left(\frac{1}{3}\right)^{\tan t} \Big|_0^{\pi/4} \\
 &= -\ln 3 \left(\frac{1}{3} - 1\right) \\
 &= \frac{2}{3 \ln 3}
 \end{aligned}$$

$$u = \tan t$$

$$du = \sec^2 t \, dt$$

$$\begin{cases} t = \frac{\pi}{4} & \rightarrow u = 1 \\ t = 0 & \rightarrow u = 0 \end{cases}$$

$$\begin{aligned}
 \int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan t} \sec^2 t \, dt &= \int_0^1 \left(\frac{1}{3}\right)^u du \\
 &= \frac{1}{\ln \frac{1}{3}} \left(\frac{1}{3}\right)^u \Big|_0^1 \\
 &= \frac{1}{-\ln 3} \left(\frac{1}{3} - 1\right) \\
 &= \frac{1}{-\ln 3} \left(\frac{-2}{3}\right) \\
 &= \frac{2}{3 \ln 3}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_1^e x^{(\ln 2)-1} dx$

Solution

$$\begin{aligned}
 \int_1^e x^{(\ln 2)-1} dx &= \frac{1}{\ln 2} x^{\ln 2} \Big|_1^e \\
 &= \frac{1}{\ln 2} (e^{\ln 2} - 1) \\
 &= \frac{1}{\ln 2} (2 - 1) \\
 &= \frac{1}{\ln 2}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_1^e \frac{2 \ln 10 \log_{10} x}{x} dx$

Solution

$$\begin{aligned} \int_1^e \frac{2 \ln 10 \log_{10} x}{x} dx &= 2 \ln 10 \int_1^e \frac{1}{x} \frac{\ln x}{\ln 10} dx &= 2 \int_1^e \frac{\ln x}{x} dx \quad d(\ln x) = \frac{1}{x} dx \\ &= 2 \int_1^e \ln x \, d(\ln x) \\ &= 2 \left(\frac{1}{2} (\ln x)^2 \right) \Big|_1^e \\ &= (\ln e)^2 - (\ln 1)^2 \\ &= 1 \end{aligned}$$

Exercise

Evaluate the integral $\int_0^9 \frac{2 \log_{10} (x+1)}{x+1} dx$

Solution

$$\begin{aligned} \int_0^9 \frac{2 \log_{10} (x+1)}{x+1} dx &= 2 \int_0^9 \frac{1}{x+1} \frac{\ln(x+1)}{\ln 10} dx \\ &= \frac{2}{\ln 10} \int_0^9 \frac{\ln(x+1)}{x+1} dx & d(\ln(x+1)) = \frac{1}{x+1} dx \\ &= \frac{2}{\ln 10} \int_0^9 \ln(x+1) \, d(x+1) \\ &= \frac{2}{\ln 10} \left(\frac{1}{2} (\ln(x+1))^2 \right) \Big|_0^9 \\ &= \frac{1}{\ln 10} \left[(\ln 10)^2 - (\ln 1)^2 \right] \\ &= \frac{1}{\ln 10} \left[(\ln 10)^2 \right] \\ &= \ln 10 \end{aligned}$$

Exercise

Evaluate the integral $\int_1^{e^x} \frac{1}{t} dt$

Solution

$$\begin{aligned}\int_1^{e^x} \frac{1}{t} dt &= \ln|t| \Big|_1^{e^x} \\ &= \ln|e^x| - \ln 1 \\ &= x\end{aligned}$$

Exercise

Evaluate the integral $\frac{1}{\ln a} \int_1^x \frac{1}{t} dt \quad x > 0$

Solution

$$\begin{aligned}\frac{1}{\ln a} \int_1^x \frac{1}{t} dt &= \frac{1}{\ln a} (\ln|t|) \Big|_1^x \\ &= \frac{1}{\ln a} (\ln x - \ln 1) \\ &= \frac{\ln x}{\ln a} \\ &= \log_a x\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\sqrt{\ln \pi}} 2x e^{x^2} \cos(e^{x^2}) dx$

Solution

$$\begin{aligned}\int_0^{\sqrt{\ln \pi}} 2x e^{x^2} \cos(e^{x^2}) dx &= \int_0^{\sqrt{\ln \pi}} \cos(e^{x^2}) d(e^{x^2}) \\ &= \sin(e^{x^2}) \Big|_0^{\sqrt{\ln \pi}} \\ &= \sin \pi - \sin 1 \\ &= -\sin 1 \approx -0.84147\end{aligned}$$

Exercise

Evaluate $\int_0^{3\sqrt{2}/4} \frac{dx}{\sqrt{9-4x^2}}$

Solution

Let: $u = 2x$

$$du = 2dx \rightarrow \frac{du}{2} = dx$$

$$\begin{cases} x = \frac{3\sqrt{2}}{4} & \rightarrow u = \frac{3\sqrt{2}}{2} \\ x = 0 & \rightarrow u = 0 \end{cases}$$

$$\begin{aligned} \int_0^{3\sqrt{2}/4} \frac{dx}{\sqrt{9-4x^2}} &= \frac{1}{2} \int_0^{3\sqrt{2}/2} \frac{du}{\sqrt{9-u^2}} \\ &= \frac{1}{2} \sin^{-1} \frac{u}{3} \Big|_0^{3\sqrt{2}/2} \\ &= \frac{1}{2} \left(\sin^{-1} \frac{\sqrt{2}}{2} - \sin^{-1} 0 \right) \\ &= \frac{1}{2} \left(\frac{\pi}{4} - 0 \right) \\ &= \frac{\pi}{8} \end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

Exercise

Evaluate $\int_{\pi/6}^{\pi/4} \frac{\csc^2 x}{1+(\cot x)^2} dx$

Solution

$$d(\cot x) = -\csc^2 x dx$$

$$\begin{aligned} \int_{\pi/6}^{\pi/4} \frac{\csc^2 x}{1+(\cot x)^2} dx &= - \int_{\pi/6}^{\pi/4} \frac{1}{1+(\cot x)^2} d(\cot x) \\ &= -\arctan(\cot x) \Big|_{\pi/6}^{\pi/4} \\ &= -\left(\arctan(1) - \arctan(\sqrt{3}) \right) \\ &= -\left(\frac{\pi}{4} - \frac{\pi}{3} \right) \end{aligned}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$\left. = \frac{\pi}{12} \right|$$

$$u = \cot x \quad du = -\csc^2 x \, dx$$

$$a^2 = 1 \quad \rightarrow a = 1$$

$$\begin{cases} x = \frac{\pi}{4} & \rightarrow u = \cot \frac{\pi}{4} = 1 \\ x = \frac{\pi}{6} & \rightarrow u = \cot \frac{\pi}{6} = \sqrt{3} \end{cases}$$

$$\int_{\pi/6}^{\pi/4} \frac{\csc^2 x \, dx}{1 + (\cot x)^2} = - \int_{\sqrt{3}}^1 \frac{du}{1 + u^2}$$

$$= -\tan^{-1} u \Big|_{\sqrt{3}}^1$$

$$= -\left(\tan^{-1} 1 - \tan^{-1} \sqrt{3} \right)$$

$$= -\left(\frac{\pi}{4} - \frac{\pi}{3} \right)$$

$$\left. = \frac{\pi}{12} \right|$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

Exercise

Evaluate $\int_1^{e^{\pi/4}} \frac{4dt}{t(1 + \ln^2 t)}$

Solution

$$d(\ln t) = \frac{1}{t} dt$$

$$\int_1^{e^{\pi/4}} \frac{4dt}{t(1 + \ln^2 t)} = 4 \int_1^{e^{\pi/4}} \frac{1}{1 + \ln^2 t} d(\ln t)$$

$$= -\arctan(\ln t) \Big|_1^{e^{\pi/4}}$$

$$= -\left(\arctan(0) - \arctan\left(\frac{\pi}{4}\right) \right)$$

$$\left. = 4 \arctan\left(\frac{\pi}{4}\right) \right|$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$\ln e^u = u$$

$$u = \ln t \Rightarrow du = \frac{dt}{t}$$

$$a^2 = 1 \rightarrow a = 1$$

$$\begin{cases} u = e^{\pi/4} & \rightarrow u = \ln e^{\pi/4} = \frac{\pi}{4} \\ u = 1 & \rightarrow u = \ln 1 = 0 \end{cases}$$

$$\begin{aligned} \int_1^{e^{\pi/4}} \frac{4dt}{t(1+\ln^2 t)} &= 4 \int_0^{\pi/4} \frac{du}{1+u^2} \\ &= \tan^{-1} u \Big|_0^{\pi/4} \\ &= 4 \left(\tan^{-1} \frac{\pi}{4} - \tan^{-1} 0 \right) \\ &= \underline{4 \tan^{-1} \frac{\pi}{4}} \end{aligned}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

Exercise

Evaluate $\int_{1/2}^1 \frac{6}{\sqrt{-4x^2 + 4x + 3}} dx$

Solution

$$\begin{aligned} -4x^2 + 4x + 3 &= -4x^2 + 4x + 3 + 1 - 1 \\ &= 4 - 4x^2 + 4x - 1 \\ &= 4 - (4x^2 - 4x + 1) \\ &= 2^2 - (2x - 1)^2 \end{aligned}$$

$$\int_{1/2}^1 \frac{6}{\sqrt{-4x^2 + 4x + 3}} dx = \int_{1/2}^1 \frac{6}{\sqrt{2^2 - (2x-1)^2}} dx$$

$$u = 2x - 1 \Rightarrow du = 2dx \rightarrow \frac{du}{2} = dx$$

$$\begin{aligned} &= \int_{1/2}^1 \frac{3}{\sqrt{2^2 - u^2}} du \\ &= 3 \sin^{-1} \left(\frac{2x-1}{2} \right) \Big|_{1/2}^1 \\ &= 3 \left(\sin^{-1} \left(\frac{1}{2} \right) - \sin^{-1} (0) \right) \end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

$$= 3 \left(\frac{\pi}{6} - 0 \right)$$

$$= \frac{\pi}{2}$$

Exercise

Evaluate $\int_{2/\sqrt{3}}^2 \frac{\cos(\sec^{-1} x)}{x \sqrt{x^2 - 1}} dx$

Solution

$$d(\sec^{-1} x) = \frac{1}{x \sqrt{x^2 - 1}} dx$$

$$\begin{aligned} \int_{2/\sqrt{3}}^2 \frac{\cos(\sec^{-1} x)}{x \sqrt{x^2 - 1}} dx &= \int_{2/\sqrt{3}}^2 \cos(\sec^{-1} x) d(\sec^{-1} x) \\ &= \sin(\sec^{-1} x) \Big|_{2/\sqrt{3}}^2 \\ &= \sin(\sec^{-1} 2) - \sin\left(\sec^{-1} \frac{2}{\sqrt{3}}\right) \\ &= \sin \frac{\pi}{3} - \sin \frac{\pi}{6} \\ &= \frac{\sqrt{3} - 1}{2} \end{aligned}$$

$$u = \sec^{-1} x \quad du = \frac{dx}{x \sqrt{x^2 - 1}}$$

$$\begin{cases} x = 2 & \rightarrow u = \sec^{-1} 2 = \frac{\pi}{3} \\ x = \frac{2}{\sqrt{3}} & \rightarrow u = \sec^{-1} \frac{2}{\sqrt{3}} = \frac{\pi}{6} \end{cases}$$

$$\begin{aligned} \int_{2/\sqrt{3}}^2 \frac{\cos(\sec^{-1} x)}{x \sqrt{x^2 - 1}} dx &= \int_{\pi/6}^{\pi/3} \cos u \, du \\ &= \sin u \Big|_{\pi/6}^{\pi/3} \\ &= \sin \frac{\pi}{3} - \sin \frac{\pi}{6} \\ &= \frac{\sqrt{3}}{2} - \frac{1}{2} \\ &= \frac{\sqrt{3} - 1}{2} \end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^3 \frac{x}{\sqrt{25-x^2}} dx$

Solution

$$d(25-x^2) = -2x dx$$

$$\begin{aligned}\int_0^3 \frac{x}{\sqrt{25-x^2}} dx &= -\frac{1}{2} \int_0^3 (25-x^2)^{-1/2} d(25-x^2) \\ &= -\sqrt{25-x^2} \Big|_0^3 \\ &= -(4-5) \\ &= 1\end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^\pi \sin^2 5\theta \, d\theta$

Solution

$$\begin{aligned}\int_0^\pi \sin^2 5\theta \, d\theta &= \frac{1}{2} \int_0^\pi (1 - \cos 10\theta) \, d\theta \\ &= \frac{1}{2} \left(\theta - \frac{1}{10} \sin 10\theta \right) \Big|_0^\pi \\ &= \frac{\pi}{2}\end{aligned}$$
$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

Exercise

Evaluate the definite integral $\int_0^\pi (1 - \cos^2 3\theta) \, d\theta$

Solution

$$\begin{aligned}\int_0^\pi (1 - \cos^2 3\theta) \, d\theta &= \int_0^\pi \left(1 - \frac{1}{2} - \cos 6\theta\right) \, d\theta \\ &= \int_0^\pi \left(\frac{1}{2} - \cos 6\theta\right) \, d\theta\end{aligned}$$
$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$= \frac{1}{2}\theta - \frac{1}{6}\sin 6\theta \bigg|_0^{\pi}$$

$$= \frac{\pi}{2}$$

Exercise

Evaluate the definite integral $\int_2^3 \frac{x^2 + 2x - 2}{x^3 + 3x^2 - 6x} dx$

Solution

$$d(x^3 + 3x^2 - 6x) = (3x^2 + 6x - 6)dx$$

$$= 3(x^2 + 2x - 2)dx$$

$$\int_2^3 \frac{x^2 + 2x - 2}{x^3 + 3x^2 - 6x} dx = \frac{1}{3} \int_2^3 \frac{1}{x^3 + 3x^2 - 6x} d(x^3 + 3x^2 - 6x)$$

$$= \frac{1}{3} \ln |x^3 + 3x^2 - 6x| \bigg|_2^3$$

$$= \frac{1}{3} (\ln 36 - \ln 8)$$

$$= \frac{1}{3} (\ln 6^2 - \ln 2^3)$$

$$= \frac{1}{3} (2 \ln 6 - 3 \ln 2)$$

$$= \frac{2}{3} \ln 6 - \ln 2$$

Exercise

Evaluate the definite integral $\int_0^{\ln 2} \frac{e^x}{1 + e^{2x}} dx$

Solution

$$d(e^x) = e^x dx$$

$$\int_0^{\ln 2} \frac{e^x}{1 + e^{2x}} dx = \int_0^{\ln 2} \frac{1}{1 + (e^x)^2} d(e^x)$$

$$\begin{aligned}
 &= \arctan e^x \Big|_0^{\ln 2} \\
 &= \arctan e^{\ln 2} - \arctan 1 \\
 &= \arctan 2 - \frac{\pi}{4}
 \end{aligned}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

Exercise

Evaluate the definite integral $\int_1^3 x \sqrt[3]{x^2 - 1} \, dx$

Solution

$$d(x^2 - 1) = 2x \, dx$$

$$\begin{aligned}
 \int_1^3 x \sqrt[3]{x^2 - 1} \, dx &= \frac{1}{2} \int_1^3 (x^2 - 1)^{1/3} d(x^2 - 1) \\
 &= \frac{3}{8} (x^2 - 1)^{4/3} \Big|_1^3 \\
 &= \frac{3}{8} (8^{4/3} - 0) \\
 &= \frac{3}{8} (2^4) \\
 &= 6
 \end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^2 (x+3)^3 \, dx$

Solution

$$d(x+3) = dx$$

$$\begin{aligned}
 \int_0^2 (x+3)^3 \, dx &= \int_0^2 (x+3)^3 d(x+3) \\
 &= \frac{1}{4} (x+3)^4 \Big|_0^2 \\
 &= \frac{1}{4} (5^4 - 3^4)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{4}(625 - 81) \\
 &= \frac{544}{4} \\
 &= 136 \quad |
 \end{aligned}$$

Exercise

Evaluate the definite integral $\int_{-2}^2 e^{4x+8} dx$

Solution

$$d(4x + 8) = 4dx$$

$$\begin{aligned}
 \int_{-2}^2 e^{4x+8} dx &= \frac{1}{4} \int_{-2}^2 e^{4x+8} d(4x + 8) \\
 &= \frac{1}{4} e^{4x+8} \bigg|_{-2}^2 \\
 &= \frac{1}{4} (e^{16} - 1) \quad |
 \end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^1 \sqrt{x}(\sqrt{x} + 1) dx$

Solution

$$\begin{aligned}
 \int_0^1 \sqrt{x}(\sqrt{x} + 1) dx &= \int_0^1 (x + x^{1/2}) dx \\
 &= \frac{1}{2}x + \frac{2}{3}x^{3/2} \bigg|_0^1 \\
 &= \frac{1}{2} + \frac{2}{3} \\
 &= \frac{7}{6} \quad |
 \end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$

Solution

$$\begin{aligned} \int_0^1 \frac{dx}{\sqrt{4-x^2}} &= \sin^{-1} \frac{x}{2} \Big|_0^1 \\ &= \sin^{-1} \frac{1}{2} - \sin^{-1} 0 \\ &= \frac{\pi}{3} \end{aligned} \qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a}$$

Exercise

Evaluate the definite integral $\int_0^2 \frac{2x}{(x^2+1)^2} dx$

Solution

$$\begin{aligned} \int_0^2 \frac{2x}{(x^2+1)^2} dx &= \int_0^2 \frac{1}{(x^2+1)^2} d(x^2+1) \\ &= -\frac{1}{x^2+1} \Big|_0^2 \\ &= -\left(\frac{1}{5} - 1\right) \\ &= \frac{4}{5} \end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^{\pi/2} \sin^2 \theta \cos \theta d\theta$

Solution

$$d(\sin \theta) = \cos \theta d\theta$$

$$\int_0^{\pi/2} \sin^2 \theta \cos \theta d\theta = \int_0^{\pi/2} \sin^2 \theta d(\sin \theta)$$

$$= \frac{1}{3} \sin^3 \theta \bigg|_0^{\pi/2}$$

$$= \frac{1}{3}$$

Exercise

Evaluate the definite integral $\int_0^{\pi/4} \frac{\sin x}{\cos^2 x} dx$

Solution

$$\int_0^{\pi/4} \frac{\sin x}{\cos^2 x} dx = - \int_0^{\pi/4} \frac{1}{\cos^2 x} d(\cos x)$$

$$= \frac{1}{\cos x} \bigg|_0^{\pi/4}$$

$$= \sqrt{2} - 1$$

Exercise

Evaluate the definite integral $\int_{1/3}^{1/\sqrt{3}} \frac{4}{9x^2 + 1} dx$

Solution

$$d(3x) = 3dx$$

$$\int_{1/3}^{1/\sqrt{3}} \frac{4}{9x^2 + 1} dx = \frac{4}{3} \int_{1/3}^{1/\sqrt{3}} \frac{1}{(3x)^2 + 1} d(3x)$$

$$= \frac{4}{3} \arctan(3x) \bigg|_{1/3}^{1/\sqrt{3}}$$

$$= \frac{4}{3} (\arctan(\sqrt{3}) - \arctan 1)$$

$$= \frac{4}{3} \left(\frac{\pi}{3} - \frac{\pi}{4} \right)$$

$$= \frac{\pi}{9}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

Exercise

Evaluate the definite integral $\int_0^{\ln 4} \frac{e^x}{3+2e^x} dx$

Solution

$$\begin{aligned}\int_0^{\ln 4} \frac{e^x}{3+2e^x} dx &= \frac{1}{2} \int_0^{\ln 4} \frac{1}{3+2e^x} d(3+2e^x) \\&= \frac{1}{2} \ln(3+2e^x) \Big|_0^{\ln 4} \\&= \frac{1}{2} (\ln(3+2e^{\ln 4}) - \ln 5) \\&= \frac{1}{2} (\ln 11 - \ln 5) \\&= \frac{1}{2} \ln \frac{11}{5}\end{aligned}$$

Exercise

Evaluate the definite integral $\int_{-\pi}^{\pi} \cos^2 x \, dx$

Solution

$$\begin{aligned}\int_{-\pi}^{\pi} \cos^2 x \, dx &= \frac{1}{2} \int_{-\pi}^{\pi} (1 + \cos 2x) \, dx \\&= \frac{1}{2} \left(x + \frac{1}{2} \sin 2x \right) \Big|_{-\pi}^{\pi} \\&= \frac{1}{2} (\pi + \pi) \\&= \pi\end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^{\pi/4} \cos^2 8\theta \, d\theta$

Solution

$$\int_0^{\pi/4} \cos^2 8\theta \, d\theta = \frac{1}{2} \int_0^{\pi/4} (1 + \cos 16\theta) \, d\theta$$

$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\begin{aligned}
&= \frac{1}{2} \left(x + \frac{1}{16} \sin 16\theta \right) \bigg|_0^{\pi/4} \\
&= \frac{1}{2} \left(\frac{\pi}{4} \right) \\
&= \frac{\pi}{8}
\end{aligned}$$

Exercise

Evaluate the definite integral $\int_{-\pi/4}^{\pi/4} \sin^2 2\theta \, d\theta$

Solution

$$\begin{aligned}
\int_{-\pi/4}^{\pi/4} \sin^2 2\theta \, d\theta &= \frac{1}{2} \int_{-\pi/4}^{\pi/4} (1 - \cos 4\theta) \, d\theta \\
&= \frac{1}{2} \left(\theta - \frac{1}{4\theta} \sin 4\theta \right) \bigg|_{-\pi/4}^{\pi/4} \\
&= \frac{1}{2} \left(\frac{\pi}{4} + \frac{\pi}{4} \right) \\
&= \frac{\pi}{4}
\end{aligned}$$

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

Exercise

Evaluate the definite integral $\int_0^{\pi/6} \frac{\sin 2x}{\sin^2 x + 2} \, dx$

Solution

$$\begin{aligned}
d(\sin^2 x + 2) &= 2 \sin x \cos x \, dx \\
&= \sin 2x \, dx
\end{aligned}$$

$$\begin{aligned}
\int_0^{\pi/6} \frac{\sin 2x}{\sin^2 x + 2} \, dx &= \int_0^{\pi/6} \frac{1}{\sin^2 x + 2} d(\sin^2 x + 2) \\
&= \ln |\sin^2 x + 2| \bigg|_0^{\pi/6} \\
&= \ln \frac{9}{4} - \ln 2 \\
&= \ln \frac{9}{8}
\end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^{\pi/2} \sin^4 \theta \, d\theta$

Solution

$$\begin{aligned}\int_0^{\pi/2} \sin^4 \theta \, d\theta &= \int_0^{\pi/2} \left(\frac{1 - \cos 2\theta}{2} \right)^2 d\theta & \sin^2 \theta &= \frac{1}{2}(1 - \cos 2\theta) \\&= \frac{1}{4} \int_0^{\pi/2} (1 - 2\cos 2\theta + \cos^2 2\theta) \, d\theta \\&= \frac{1}{4} \int_0^{\pi/2} \left(1 - 2\cos 2\theta + \frac{1}{2} + \frac{1}{2}\cos 4\theta \right) d\theta \\&= \frac{1}{4} \int_0^{\pi/2} \left(\frac{3}{2} - 2\cos 2\theta + \frac{1}{2}\cos 4\theta \right) d\theta \\&= \frac{1}{4} \left(\frac{3}{2}\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta \right) \bigg|_0^{\pi/2} \\&= \frac{1}{4} \left(\frac{3}{2} \frac{\pi}{2} \right) \\&= \frac{3\pi}{16}\end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^1 x \sqrt{1-x^2} \, dx$

Solution

$$\begin{aligned}d(1-x^2) &= -2x \, dx \\ \int_0^1 x \sqrt{1-x^2} \, dx &= -\frac{1}{2} \int_0^1 (1-x^2)^{1/2} d(1-x^2) \\&= -\frac{1}{3} (1-x^2)^{3/2} \bigg|_0^1 \\&= -\frac{1}{3} (0-1) \\&= \frac{1}{3}\end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^{1/4} \frac{x}{\sqrt{1-16x^2}} dx$

Solution

$$d(1-16x^2) = -32x dx$$

$$\begin{aligned} \int_0^{1/4} \frac{x}{\sqrt{1-16x^2}} dx &= -\frac{1}{32} \int_0^{1/4} (1-16x^2)^{-1/2} d(1-16x^2) \\ &= -\frac{1}{16} (1-16x^2)^{1/2} \Big|_0^{1/4} \\ &= -\frac{1}{16} (0-1) \\ &= \frac{1}{16} \end{aligned}$$

Exercise

Evaluate the definite integral $\int_2^3 \frac{x}{\sqrt[3]{x^2-1}} dx$

Solution

$$d(x^2-1) = 2x dx$$

$$\begin{aligned} \int_2^3 \frac{x}{\sqrt[3]{x^2-1}} dx &= \frac{1}{2} \int_2^3 (x^2-1)^{-1/3} d(x^2-1) \\ &= \frac{3}{4} (x^2-1)^{2/3} \Big|_2^3 \\ &= \frac{3}{4} (8^{2/3} - 1) \\ &= \frac{3}{4} (4-1) \\ &= \frac{9}{4} \end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^{6/5} \frac{dx}{25x^2 + 36}$

Solution

$$\begin{aligned}\int_0^{6/5} \frac{dx}{25x^2 + 36} &= \int_0^{6/5} \frac{dx}{25\left(x^2 + \frac{36}{25}\right)} \\&= \int_0^{6/5} \frac{dx}{25\left(x^2 + \left(\frac{6}{5}\right)^2\right)} \\&= \frac{1}{25}\left(\frac{5}{6}\right) \tan^{-1} \frac{5x}{6} \Bigg|_0^{6/5} \\&= \frac{1}{30} \left(\tan^{-1} 1 - \tan^{-1} 0 \right) \\&= \frac{1}{30} \left(\frac{\pi}{4} \right) \\&= \underline{\underline{\frac{\pi}{120}}}\end{aligned}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

Exercise

Evaluate the definite integral $\int_0^2 x^3 \sqrt{16 - x^4} \, dx$

Solution

$$d(16 - x^4) = -4x^3 dx$$

$$\begin{aligned}\int_0^2 x^3 \sqrt{16 - x^4} \, dx &= -\frac{1}{4} \int_0^2 (16 - x^4)^{1/2} d(16 - x^4) \\&= -\frac{1}{6} (16 - x^4)^{3/2} \Bigg|_0^2 \\&= -\frac{1}{6} (0 - 4^3) \\&= \underline{\underline{\frac{32}{3}}}\end{aligned}$$

Exercise

Evaluate the definite integral $\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^2 x} dx$

Solution

$$d(\sin x) = \cos x \, dx$$

$$\begin{aligned} \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^2 x} dx &= \int_{\pi/4}^{\pi/2} \frac{1}{\sin^2 x} d(\sin x) \\ &= -\frac{1}{\sin x} \Big|_{\pi/4}^{\pi/2} \\ &= -(1 - \sqrt{2}) \\ &= \underline{\sqrt{2} - 1} \end{aligned}$$

Exercise

Evaluate the definite integral $\int_{-1}^1 (x-1)(x^2-2x)^7 dx$

Solution

$$\begin{aligned} d(x^2 - 2x) &= (2x - 2) dx \\ &= 2(x - 1) dx \end{aligned}$$

$$\begin{aligned} \int_{-1}^1 (x-1)(x^2-2x)^7 dx &= \frac{1}{2} \int_{-1}^1 (x^2-2x)^7 d(x^2-2x) \\ &= \frac{1}{16} (x^2-2x)^8 \Big|_{-1}^1 \\ &= \frac{1}{16} (1 - 3^8) \\ &= \frac{6560}{16} \\ &= \underline{410} \end{aligned}$$

Exercise

Evaluate the definite integral $\int_{-\pi}^0 \frac{\sin x}{2 + \cos x} dx$

Solution

$$d(2 + \cos x) = -\sin x \, dx$$

$$\begin{aligned} \int_{-\pi}^0 \frac{\sin x}{2 + \cos x} dx &= - \int_{-\pi}^0 \frac{1}{2 + \cos x} d(2 + \cos x) \\ &= -\ln|2 + \cos x| \Big|_{-\pi}^0 \\ &= -(\ln 3 - \ln 1) \\ &= \underline{-\ln 3} \end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^1 \frac{(x+1)(x+2)}{2x^3 + 9x^2 + 12x + 36} dx$

Solution

$$\begin{aligned} d(2x^3 + 9x^2 + 12x + 36) &= (6x^2 + 18x + 12) dx \\ &= 6(x^2 + 3x + 2) dx \end{aligned}$$

$$\begin{aligned} \int_0^1 \frac{(x+1)(x+2)}{2x^3 + 9x^2 + 12x + 36} dx &= \int_0^1 \frac{x^2 + 3x + 2}{2x^3 + 9x^2 + 12x + 36} dx \\ &= \frac{1}{6} \int_0^1 \frac{1}{2x^3 + 9x^2 + 12x + 36} d(2x^3 + 9x^2 + 12x + 36) \\ &= \frac{1}{6} \ln|2x^3 + 9x^2 + 12x + 36| \Big|_0^1 \\ &= \frac{1}{6} (\ln 59 - \ln 36) \\ &= \underline{\frac{1}{6} \ln \frac{59}{36}} \end{aligned}$$

Exercise

Evaluate the definite integral $\int_1^2 \frac{4}{9x^2 + 6x + 1} dx$

Solution

$$\begin{aligned}\int_1^2 \frac{4}{9x^2 + 6x + 1} dx &= \int_1^2 \frac{4}{(3x+1)^2} dx \\&= \frac{4}{3} \int_1^2 \frac{1}{(3x+1)^2} d(3x+1) && d(3x+1) = 3x dx \\&= -\frac{4}{3} \frac{1}{3x+1} \Big|_1^2 \\&= -\frac{4}{3} \left(\frac{1}{7} - \frac{1}{4} \right) \\&= -\frac{4}{3} \left(-\frac{3}{28} \right) \\&= \frac{1}{7}\end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^{\pi/4} e^{\sin^2 x} \sin 2x dx$

Solution

$$\begin{aligned}d(\sin^2 x) &= 2 \sin x \cos x dx \\&= \sin 2x dx\end{aligned}$$

$$\begin{aligned}\int_0^{\pi/4} e^{\sin^2 x} \sin 2x dx &= \int_0^{\pi/4} e^{\sin^2 x} d(\sin^2 x) \\&= e^{\sin^2 x} \Big|_0^{\pi/4} \\&= e^{\sin^2 \frac{\pi}{4}} - e^{\sin^2 0} \\&= e^{\frac{1}{2}} - 1 \\&= \sqrt{e} - 1\end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^1 x \sqrt{x+a} \, dx \quad (a > 0)$

Solution

$$\text{Let } u = x + a \rightarrow x = u - a$$

$$\Rightarrow du = dx$$

$$\begin{aligned} \int_0^1 x \sqrt{x+a} \, dx &= \int_0^1 (u-a)u^{1/2} \, du \\ &= \int_0^1 \left(u^{3/2} - au^{1/2} \right) du \\ &= \left. \frac{2}{5}u^{5/2} - \frac{2}{3}au^{3/2} \right|_0^1 \\ &= \left. \frac{2}{5}(x+a)^{5/2} - \frac{2}{3}a(x+a)^{3/2} \right|_0^1 \\ &= \frac{2}{5}(1+a)^{5/2} - \frac{2}{3}a(1+a)^{3/2} - \frac{2}{5}a^{5/2} + \frac{2}{3}a(a^{3/2}) \\ &= \frac{2}{5}(1+a)^{5/2} - \frac{2}{3}a(1+a)^{3/2} - \frac{2}{5}a^{5/2} + \frac{2}{3}a^{5/2} \\ &= \frac{2}{5}(1+a)^{5/2} - \frac{2}{3}a(1+a)^{3/2} + \frac{4}{15}a^{5/2} \\ &= \frac{2}{5}(1+a)^2\sqrt{1+a} - \frac{2}{3}a(1+a)\sqrt{1+a} + \frac{4}{15}a^2\sqrt{a} \\ &= \left(\frac{2}{5}(1+a)^2 - \frac{2}{3}(a+a^2) \right) \sqrt{1+a} + \frac{4}{15}a^2\sqrt{a} \end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^1 x \sqrt[p]{x+a} \, dx \quad (a > 0)$

Solution

$$\text{Let } u = x + a \rightarrow x = u - a$$

$$du = dx$$

$$\int_0^1 x \sqrt[p]{x+a} \, dx = \int_0^1 (u-a)u^{1/p} \, du$$

$$\begin{aligned}
&= \int_0^1 \left(u^{1+1/p} - au^{1/p} \right) du \\
&= \frac{p}{2p+1} u^{2+1/p} - \frac{p}{p+1} au^{1+1/p} \Bigg|_0^1 \\
&= \frac{p}{2p+1} (x+a)^{2+1/p} - \frac{p}{p+1} a(x+a)^{1+1/p} \Bigg|_0^1 \\
&= \frac{p}{2p+1} (1+a)^{2+1/p} - \frac{p}{p+1} a(1+a)^{1+1/p} - \frac{p}{2p+1} a^{2+1/p} + \frac{p}{p+1} a(a)^{1+1/p} \\
&= \frac{p}{2p+1} (1+a)^2 \sqrt[p]{1+a} - \frac{p}{p+1} a(1+a) \sqrt[p]{1+a} - \frac{p}{2p+1} a^{2+1/p} + \frac{p}{p+1} a^{2+1/p} \\
&= \left(\frac{p}{2p+1} (1+a)^2 - \frac{p}{p+1} a(1+a) \right) \sqrt[p]{1+a} + \left(\frac{p}{p+1} - \frac{p}{2p+1} \right) a^{2+1/p} \\
&= \left(\frac{p}{2p+1} (1+a)^2 - \frac{p}{p+1} (a+a^2) \right) \sqrt[p]{1+a} + \left(\frac{2p^2 + p - p^2 - p}{(p+1)(2p+1)} \right) a^{2+1/p} \\
&= \left(\frac{p}{2p+1} (1+a)^2 - \frac{p}{p+1} (a+a^2) \right) \sqrt[p]{1+a} + \frac{p^2}{(p+1)(2p+1)} a^{2+1/p} \Bigg|
\end{aligned}$$

Or

$$\text{Let } u = \sqrt[p]{x+a} \rightarrow u^p = x+a$$

$$x = u^p - a \rightarrow dx = pu^{p-1} du$$

$$\begin{aligned}
\int_0^1 x \sqrt[p]{x+a} dx &= \int_0^1 (u^p - a) \cdot u \cdot (pu^{p-1}) du \\
&= p \int_0^1 (u^p - a) \cdot u^p du \\
&= p \int_0^1 (u^{2p} - au^p) du \\
&= p \left(\frac{1}{2p+1} \left(\sqrt[p]{x+a} \right)^{2p+1} - \frac{1}{p+1} a \left(\sqrt[p]{x+a} \right)^{p+1} \right) \Bigg|_0^1 \\
&= p \left(\frac{1}{2p+1} \left(\sqrt[p]{1+a} \right)^{2p+1} - \frac{1}{p+1} a \left(\sqrt[p]{1+a} \right)^{p+1} - \frac{1}{2p+1} \left(\sqrt[p]{a} \right)^{2p+1} + \frac{1}{p+1} a \left(\sqrt[p]{a} \right)^{p+1} \right)
\end{aligned}$$

$$\begin{aligned}
&= p \left(\frac{1}{2p+1} (1+a)^{(2p+1)/p} - \frac{1}{p+1} a (1+a)^{(p+1)/p} \right. \\
&\quad \left. - \frac{1}{2p+1} (a)^{(2p+1)/p} + \frac{1}{p+1} a (a)^{(p+1)/p} \right) \\
&= p \left(\frac{1}{2p+1} (1+a)^{(2p+1)/p} - \frac{1}{p+1} a (1+a)^{(p+1)/p} \right. \\
&\quad \left. - \frac{1}{2p+1} (a)^{(2p+1)/p} + \frac{1}{p+1} (a)^{(2p+1)/p} \right)
\end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^1 x \sqrt{1-\sqrt{x}} \, dx$

Solution

$$u = 1 - \sqrt{x} \rightarrow x = (1-u)^2$$

$$dx = -2(1-u)du$$

$$\int_0^1 x \sqrt{1-\sqrt{x}} \, dx = -2 \int_0^1 (1-u)^2 u^{1/2} (1-u) du$$

$$= -2 \int_0^1 (1-u)^3 u^{1/2} du$$

$$= -2 \int_0^1 (1-3u+3u^2-u^3) u^{1/2} du$$

$$= -2 \int_0^1 (u^{1/2} - 3u^{3/2} + 3u^{5/2} - u^{7/2}) du$$

$$= -2 \left(\frac{2}{3} (1-\sqrt{x})^{3/2} - \frac{6}{5} (1-\sqrt{x})^{5/2} + \frac{6}{7} (1-\sqrt{x})^{7/2} - \frac{2}{9} (1-\sqrt{x})^{9/2} \right) \Bigg|_0^1$$

$$= -2 \left(0 - \frac{2}{3} + \frac{6}{5} - \frac{6}{7} + \frac{2}{9} \right)$$

$$= -2 \left(-\frac{32}{315} \right)$$

$$= \frac{34}{315}$$

Exercise

Evaluate the definite integral $\int_0^1 \sqrt{x-x\sqrt{x}} \, dx$

Solution

$$u = 1 - \sqrt{x} \rightarrow x = (1-u)^2$$

$$\Rightarrow dx = -2(1-u)du$$

$$\begin{aligned} \int_0^1 \sqrt{x-x\sqrt{x}} \, dx &= \int_0^1 \sqrt{x(1-\sqrt{x})} \, dx \\ &= -2 \int_0^1 \sqrt{(1-u)^2 u} (1-u) \, du \\ &= -2 \int_0^1 \sqrt{(1-u)^2 u} (1-u) \, du \\ &= -2 \int_0^1 (1-u)^2 \sqrt{u} \, du \\ &= -2 \int_0^1 (1-2u+u^2) u^{1/2} \, du \\ &= -2 \int_0^1 (u^{1/2} - 2u^{3/2} + u^{5/2}) \, du \\ &= -2 \left(\frac{2}{3}(1-\sqrt{x})^{3/2} - \frac{4}{5}(1-\sqrt{x})^{5/2} + \frac{2}{7}(1-\sqrt{x})^{7/2} \right) \Bigg|_0^1 \\ &= -2 \left(0 - \frac{2}{3} + \frac{4}{5} - \frac{2}{7} \right) \\ &= -2 \left(\frac{-16}{105} \right) \\ &= \underline{\underline{\frac{32}{105}}} \end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^{\pi/2} \frac{\cos \theta \sin \theta}{\sqrt{\cos^2 \theta + 16}} \, d\theta$

Solution

$$d(\cos^2 \theta + 16) = -2 \cos \theta \sin \theta \, d\theta$$

$$\begin{aligned}
 \int_0^{\pi/2} \frac{\cos \theta \sin \theta}{\sqrt{\cos^2 \theta + 16}} d\theta &= -\frac{1}{2} \int_0^{\pi/2} \left(\cos^2 \theta + 16 \right)^{-1/2} d\left(\cos^2 \theta + 16 \right) \\
 &= -\sqrt{\cos^2 \theta + 16} \Big|_0^{\pi/2} \\
 &= -\left(4 - \sqrt{17} \right) \\
 &= \underline{\sqrt{17} - 4}
 \end{aligned}$$

Exercise

Evaluate the definite integral $\int_{\frac{2}{5\sqrt{3}}}^{\frac{2}{5}} \frac{dx}{x \sqrt{25x^2 - 1}}$

Solution

$$\begin{aligned}
 \int_{\frac{2}{5\sqrt{3}}}^{\frac{2}{5}} \frac{dx}{x \sqrt{25x^2 - 1}} &= \int_{\frac{2}{5\sqrt{3}}}^{\frac{2}{5}} \frac{d(5x)}{(5x) \sqrt{(5x)^2 - 1}} \\
 &= \sec^{-1}(5x) \Big|_{\frac{2}{5\sqrt{3}}}^{\frac{2}{5}} \\
 &= \sec^{-1}(2) - \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) \\
 &= \frac{\pi}{3} - \frac{\pi}{6} \\
 &= \underline{\frac{\pi}{6}}
 \end{aligned}$$

$$\int \frac{dx}{x \sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left| \frac{x}{a} \right|$$

Exercise

Evaluate the definite integral $\int_0^4 \frac{x}{\sqrt{9 + x^2}} dx$

Solution

$$d(9 + x^2) = 2x dx$$

$$\int_0^4 \frac{x}{\sqrt{9 + x^2}} dx = \frac{1}{2} \int_0^4 \left(9 + x^2 \right)^{-1/2} d\left(9 + x^2 \right)$$

$$\begin{aligned}
 &= \sqrt{9+x^2} \Big|_0^4 \\
 &= 5 - 3 \\
 &= \underline{2}
 \end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^{\pi/4} \frac{\sin \theta}{\cos^3 \theta} d\theta$

Solution

$$d(\cos \theta) = -\sin \theta$$

$$\begin{aligned}
 \int_0^{\pi/4} \frac{\sin \theta}{\cos^3 \theta} d\theta &= - \int_0^{\pi/4} \cos^{-3} \theta d(\cos \theta) \\
 &= \frac{1}{2} \frac{1}{\cos^2 \theta} \Big|_0^{\pi/4} \\
 &= \frac{1}{2} (2 - 1) \\
 &= \underline{\frac{1}{2}}
 \end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^1 2x(4-x^2) dx$

Solution

$$d(4-x^2) = -2x dx$$

$$\begin{aligned}
 \int_0^1 2x(4-x^2) dx &= - \int_0^1 (4-x^2) d(4-x^2) \\
 &= -\frac{1}{2} (4-x^2)^2 \Big|_0^1 \\
 &= -\frac{1}{2} (9-16) \\
 &= \underline{\frac{7}{2}}
 \end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^3 \frac{x^2+1}{\sqrt{x^3+3x+4}} dx$

Solution

$$d(x^3+3x+4) = (3x^2+3)dx$$

$$\begin{aligned}\int_0^3 \frac{x^2+1}{\sqrt{x^3+3x+4}} dx &= \frac{1}{3} \int_0^3 (x^3+3x+4)^{-1/2} d(x^3+3x+4) \\ &= \frac{2}{3} \sqrt{x^3+3x+4} \Big|_0^3 \\ &= \frac{2}{3}(\sqrt{40}-2) \\ &= \frac{4}{3}(\sqrt{10}-1)\end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^4 \frac{x}{x^2+1} dx$

Solution

$$d(x^2+1) = 2x dx$$

$$\begin{aligned}\int_0^4 \frac{x}{x^2+1} dx &= \frac{1}{2} \int_0^4 \frac{1}{x^2+1} d(x^2+1) \\ &= \frac{1}{2} \ln(x^2+1) \Big|_0^4 \\ &= \frac{1}{2}(\ln 17 - \ln 1) \\ &= \frac{1}{2} \ln 17\end{aligned}$$

Exercise

Evaluate the definite integral $\int_1^{e^2} \frac{\ln x}{x} dx$

Solution

$$d(\ln x) = \frac{1}{x} dx$$

$$\begin{aligned} \int_1^{e^2} \frac{\ln x}{x} dx &= \int_1^{e^2} \ln x \, d(\ln x) \\ &= \frac{1}{2} (\ln x)^2 \Big|_1^{e^2} \\ &= \frac{1}{2} \left((\ln e^2)^2 - (\ln 1)^2 \right) \\ &= \frac{1}{2} (2)^2 \\ &= 2 \end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^3 \frac{x^2+1}{\sqrt{x^3+3x+4}} dx$

Solution

$$\begin{aligned} d(x^3+3x+4) &= (3x^2+3) dx \\ &= 3(x^2+1) dx \end{aligned}$$

$$\begin{aligned} \int_0^3 \frac{x^2+1}{\sqrt{x^3+3x+4}} dx &= \frac{1}{3} \int_0^3 (x^3+3x+4)^{-1/2} d(x^3+3x+4) \\ &= \frac{2}{3} \sqrt{x^3+3x+4} \Big|_0^3 \\ &= \frac{2}{3} (\sqrt{40} - 2) \\ &= \frac{2}{3} (\sqrt{10} - 1) \end{aligned}$$

Exercise

Evaluate the definite integral $\int_{-\pi/4}^{\pi/4} \sin^2 2\theta \, d\theta$

Solution

$$\begin{aligned}
 \int_{-\pi/4}^{\pi/4} \sin^2 2\theta \, d\theta &= \frac{1}{2} \int_{-\pi/4}^{\pi/4} (1 - \cos 4\theta) \, d\theta & \sin^2 \theta &= \frac{1}{2}(1 - \cos 2\theta) \\
 &= \frac{1}{2} \left(\theta - \frac{1}{4} \sin 4\theta \right) \bigg|_{-\pi/4}^{\pi/4} \\
 &= \frac{1}{2} \left(\frac{\pi}{4} + \frac{\pi}{4} \right) \\
 &= \frac{\pi}{4}
 \end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^1 (y^3 + 6y^2 - 12y + 9)^{-1/2} (y^2 + 4y - 4) dy$

Solution

$$\begin{aligned}
 d(y^3 + 6y^2 - 12y + 9) &= (3y^2 + 12y - 12) dy \\
 &= 3(y^2 + 4y - 4) dy
 \end{aligned}$$

$$\begin{aligned}
 \int_0^1 (y^3 + 6y^2 - 12y + 9)^{-1/2} (y^2 + 4y - 4) dy \\
 &= \frac{1}{3} \int_0^1 (y^3 + 6y^2 - 12y + 9)^{-1/2} (y^3 + 6y^2 - 12y + 9) dy \\
 &= \frac{2}{3} \sqrt{y^3 + 6y^2 - 12y + 9} \bigg|_0^1 \\
 &= \frac{2}{3} (2 - 3) \\
 &= -\frac{2}{3}
 \end{aligned}$$

Exercise

Solve the initial value problem $\frac{dy}{dt} = e^t \sin(e^t - 2)$, $y(\ln 2) = 0$

Solution

$$\frac{dy}{dt} = e^t \sin(e^t - 2)$$

$$y = \int e^t \sin(e^t - 2) dt$$

$$\text{Let } u = e^t - 2 \rightarrow du = e^t dt$$

$$y = \int \sin u \, du$$

$$= -\cos u + C$$

$$= -\cos(e^t - 2) + C$$

$$y(\ln 2) = -\cos(e^{\ln 2} - 2) + C = 0$$

$$C = \cos(2 - 2)$$

$$= \cos(0)$$

$$= 1$$

$$\underline{y(t) = -\cos(e^t - 2) + 1}$$

Exercise

Solve the initial value problem $\frac{dy}{dt} = e^{-t} \sec^2(\pi e^{-t})$, $y(\ln 4) = \frac{2}{\pi}$

Solution

$$\frac{dy}{dt} = e^{-t} \sec^2(\pi e^{-t})$$

$$y = \int e^{-t} \sec^2(\pi e^{-t}) dt$$

$$d(\pi e^{-t}) = -\pi e^{-t} dt$$

$$y = \int e^{-t} \sec^2(\pi e^{-t}) dt$$

$$= -\frac{1}{\pi} \int \sec^2(\pi e^{-t}) d(\pi e^{-t})$$

$$= -\frac{1}{\pi} \tan(\pi e^{-t}) + C$$

$$y(\ln 4) = -\frac{1}{\pi} \tan(\pi e^{-\ln 4}) + C$$

$$\underline{= \frac{2}{\pi}}$$

$$\begin{aligned}
 C &= \frac{2}{\pi} + \frac{1}{\pi} \tan\left(\pi \cdot \frac{1}{4}\right) \\
 &= \frac{2}{\pi} + \frac{1}{\pi} \\
 &= \frac{3}{\pi}
 \end{aligned}$$

$$y(t) = -\frac{1}{\pi} \tan\left(\pi e^{-t}\right) + \frac{3}{\pi}$$

Exercise

Verify the integration formula: $\int \frac{\tan^{-1} x}{x^2} dx = \ln x - \frac{1}{2} \ln(1+x^2) - \frac{\tan^{-1} x}{x} + C$

Solution

$$\text{If } y = \ln x - \frac{1}{2} \ln(1+x^2) - \frac{\tan^{-1} x}{x} + C$$

$$dy = \left(\frac{1}{x} - \frac{1}{2} \frac{2x}{1+x^2} - \frac{\frac{x}{1+x^2} - \tan^{-1} x}{x^2} \right) dx$$

$$dy = \left(\frac{1}{x} - \frac{x}{1+x^2} - \frac{x - (1+x^2) \tan^{-1} x}{x^2 (1+x^2)} \right) dx$$

$$dy = \left(\frac{x(1+x^2) - x^3 - x + (1+x^2) \tan^{-1} x}{x^2 (1+x^2)} \right) dx$$

$$dy = \left(\frac{x + x^3 - x^3 - x + (1+x^2) \tan^{-1} x}{x^2 (1+x^2)} \right) dx$$

$$dy = \frac{(1+x^2) \tan^{-1} x}{x^2 (1+x^2)} dx$$

$$dy = \frac{\tan^{-1} x}{x^2} dx \quad \checkmark$$

Which verifies the formula

Exercise

Verify the integration formula: $\int \ln(a^2 + x^2) dx = x \ln(a^2 + x^2) - 2x + 2a \tan^{-1} \frac{x}{a} + C$

Solution

$$\text{If } y = x \ln(a^2 + x^2) - 2x + 2a \tan^{-1} \frac{x}{a} + C$$

$$dy = \left(\ln(a^2 + x^2) + x \frac{2x}{a^2 + x^2} - 2 + 2a \frac{\frac{1}{a}}{1 + \frac{x^2}{a^2}} \right) dx$$

$$dy = \left(\ln(a^2 + x^2) + \frac{2x^2}{a^2 + x^2} - 2 + \frac{2}{\frac{a^2 + x^2}{a^2}} \right) dx$$

$$dy = \left(\ln(a^2 + x^2) + \frac{2x^2}{a^2 + x^2} - 2 + \frac{2a^2}{a^2 + x^2} \right) dx$$

$$dy = \left(\ln(a^2 + x^2) + \frac{2x^2 + 2a^2}{a^2 + x^2} - 2 \right) dx$$

$$dy = \left(\ln(a^2 + x^2) + \frac{2(x^2 + a^2)}{a^2 + x^2} - 2 \right) dx$$

$$dy = \left(\ln(a^2 + x^2) + 2 - 2 \right) dx$$

$$dy = \ln(a^2 + x^2) dx \quad \checkmark$$

Which verifies the formula

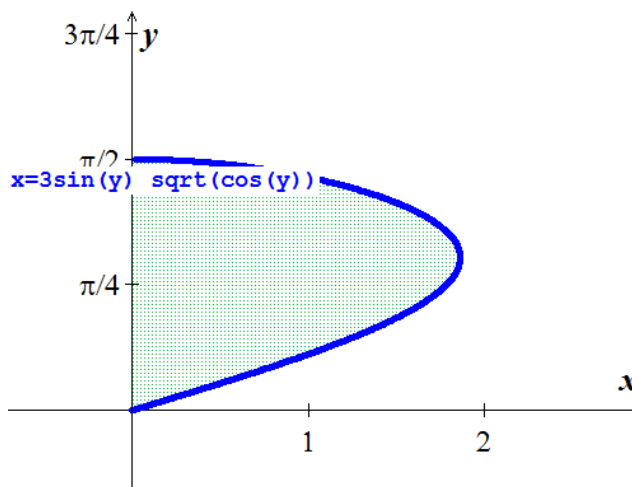
Exercise

Find the area of the region bounded by the graphs of $x = 3 \sin y \sqrt{\cos y}$, and $x = 0$, $0 \leq y \leq \frac{\pi}{2}$

Solution

$$d(\cos y) = -\sin y \, dy$$

$$\begin{aligned} A &= \int_0^{\pi/2} (3 \sin y \sqrt{\cos y} - 0) dx \\ &= -3 \int_0^{\pi/2} \cos^{1/2} y \, d(\cos y) \\ &= -3 \left(\frac{2}{3} \cos^{3/2} y \right) \Big|_0^{\pi/2} \\ &= -2(0 - 1) \\ &= \underline{2 \text{ unit}^2} \end{aligned}$$



Exercise

Find the area of the region bounded by the graph of $f(x) = \frac{x}{\sqrt{x^2 - 9}}$ on $3 \leq x \leq 4$

Solution

$$\begin{aligned} A &= \int_3^4 \frac{x}{\sqrt{x^2 - 9}} dx \\ &= \frac{1}{2} \int_3^4 (x^2 - 9)^{-1/2} d(x^2 - 9) \quad d(x^2 - 9) = 2x dx \\ &= \sqrt{x^2 - 9} \Big|_3^4 \\ &= \sqrt{7} - 0 \\ &= \underline{\sqrt{7} \text{ unit}^2} \end{aligned}$$

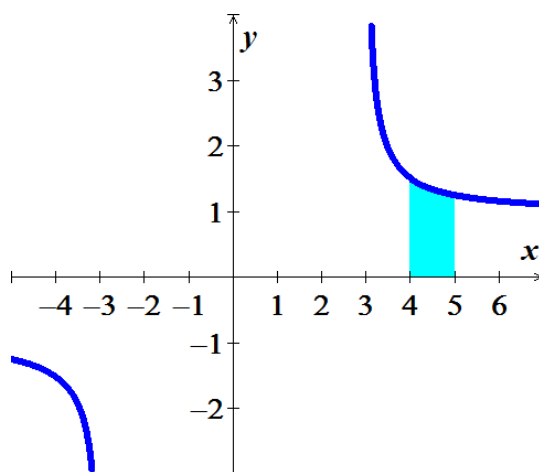
Exercise

Find the area of the region bounded by the graph of $f(x) = \frac{x}{\sqrt{x^2 - 9}}$ and the x -axis between $x = 4$ and $x = 5$.

Solution

$$d(x^2 - 9) = 2x \, dx$$

$$\begin{aligned} A &= \int_4^5 \frac{x}{\sqrt{x^2 - 9}} \, dx \\ &= \frac{1}{2} \int_4^5 (x^2 - 9)^{-1/2} d(x^2 - 9) \\ &= \sqrt{x^2 - 9} \Big|_4^5 \\ &= 4 - \sqrt{7} \text{ unit}^2 \end{aligned}$$

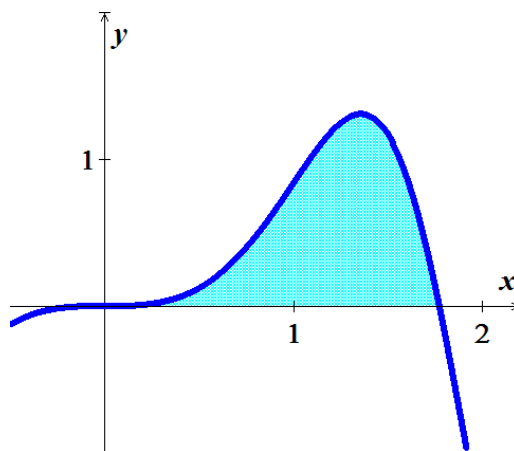
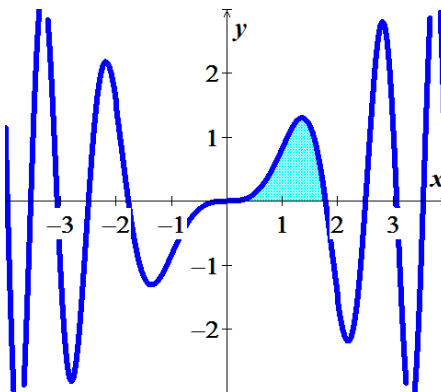


Exercise

Find the area of the region bounded by the graph of $f(x) = x \sin x^2$ and the x -axis between $x = 0$ and $x = \sqrt{\pi}$.

Solution

$$\begin{aligned} A &= \int_0^{\sqrt{\pi}} x \sin x^2 \, dx \\ &= \frac{1}{2} \int_0^{\sqrt{\pi}} \sin x^2 \, d(x^2) \\ &= -\frac{1}{2} \cos(x^2) \Big|_0^{\sqrt{\pi}} \\ &= -\frac{1}{2}(-1 - 1) \\ &= 1 \text{ unit}^2 \end{aligned}$$



Exercise

Find the area of the region bounded by the graph of $f(\theta) = \cos \theta \sin \theta$ and the θ -axis between $\theta = 0$ and $\theta = \frac{\pi}{2}$.

Solution

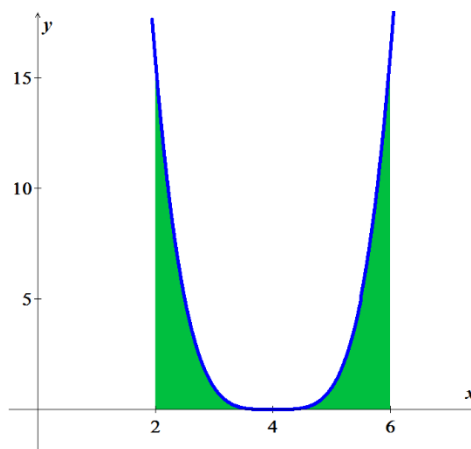
$$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} \cos \theta \sin \theta \, d\theta \\ &= \int_0^{\frac{\pi}{2}} \sin \theta \, d(\sin \theta) \\ &= \frac{1}{2} \sin^2 \theta \Big|_0^{\pi/2} \\ &= \underline{1 \text{ unit}^2} \end{aligned}$$

Exercise

Find the area of the region bounded by the graph of $f(x) = (x-4)^4$ and the x -axis between $x = 2$ and $x = 6$.

Solution

$$\begin{aligned} A &= \int_2^6 (x-4)^4 \, dx \\ &= \int_2^6 (x-4)^4 \, d(x-4) \\ &= 2 \left(\frac{1}{5} \right) (x-4)^5 \Big|_2^4 \\ &= \underline{\frac{64}{5} \text{ unit}^2} \end{aligned}$$



Exercise

Perhaps the simplest change of variables is the shift or translation given by $u = x + c$, where c is a real number.

a) Prove that shifting a function does not change the net area under the curve, in the sense that

$$\int_a^b f(x+c)dx = \int_{a+c}^{b+c} f(u)du$$

b) Draw a picture to illustrate this change of variables in the case that $f(x) = \sin x$, $a = 0$, $b = \pi$, and

$$c = \frac{\pi}{2}$$

Solution

a) Let $u = x + c \rightarrow du = dx$

$$\begin{cases} x = b & \rightarrow u = b + c \\ x = a & \rightarrow u = a + c \end{cases}$$

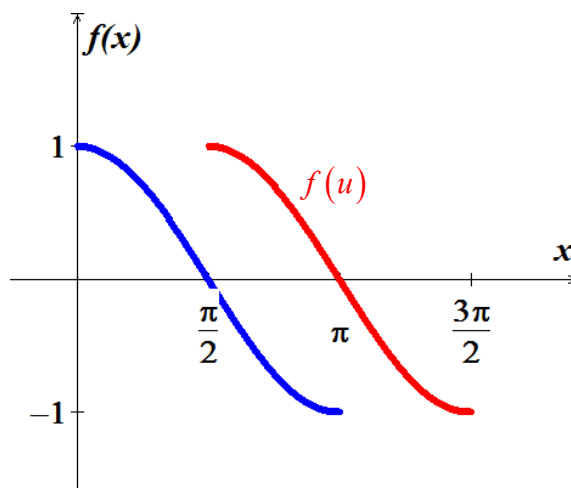
$$\int_a^b f(x+c)dx = \int_{a+c}^{b+c} f(u)du$$

b) Given: $f(x) = \sin x$, $a = 0$, $b = \pi$, & $c = \frac{\pi}{2}$

$$f(x+c) = \sin\left(x + \frac{\pi}{2}\right)$$

$$\begin{cases} b = \pi & \rightarrow f\left(\pi + \frac{\pi}{2}\right) = \sin \frac{3\pi}{2} = -1 \\ a = 0 & \rightarrow f\left(0 + \frac{\pi}{2}\right) = \sin \frac{\pi}{2} = 1 \end{cases}$$

$$f(u) \rightarrow \begin{cases} b+c = \frac{3\pi}{2} \\ a+c = \frac{\pi}{2} \end{cases}$$



Exercise

Another change of variables that can be interpreted geometrically is the scaling $u = cx$, where c is a real number. Prove and interpret the fact that

$$\int_a^b f(cx) dx = \frac{1}{c} \int_{ac}^{bc} f(u) du$$

Draw a picture to illustrate this change of variables in the case that $f(x) = \sin x$, $a = 0$, $b = \pi$, and

$$c = \frac{1}{2}$$

Solution

$$\text{Let } u = cx \rightarrow du = c dx$$

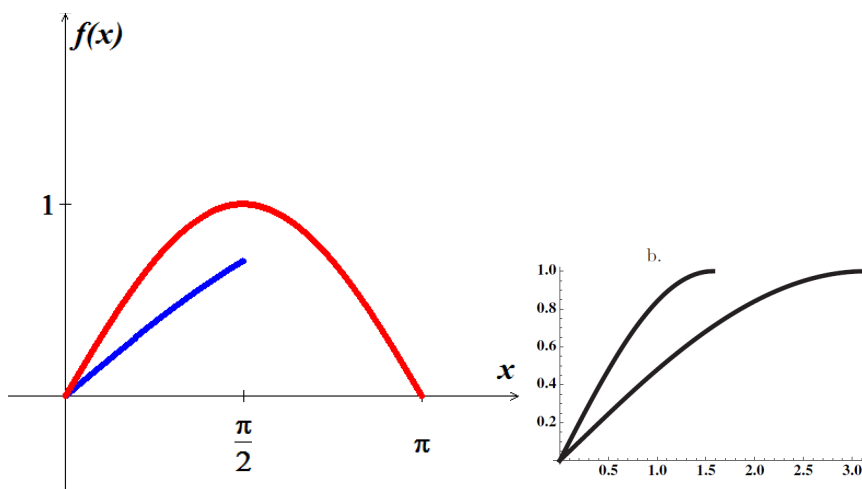
$$\begin{cases} x = b & \rightarrow u = bc \\ x = a & \rightarrow u = ac \end{cases}$$

$$\int_a^b f(cx) dx = \frac{1}{c} \int_{ac}^{bc} f(u) du$$

$$\text{Given: } f(x) = \sin x, \quad a = 0, \quad b = \pi, \quad \& \quad c = \frac{1}{2}$$

$$f(cx) = f\left(\frac{x}{2}\right) = \sin \frac{x}{2}$$

$$\begin{cases} a = 0 & \rightarrow ac = 0 \\ b = \pi & \rightarrow bc = \frac{\pi}{2} \end{cases}$$



Exercise

The function f satisfies the equation $3x^4 - 48 = \int_2^x f(t) dt$. Find f and check your answer by substitution.

Solution

$$\frac{d}{dx}(3x^4 - 48) = \frac{d}{dx} \int_2^x f(t) dt$$

$$12x^3 = f(x)$$

$$\begin{aligned} \int_2^x 12t^3 dt &= 3t^4 \Big|_2^x \\ &= 3x^4 - 3(2)^4 \\ &= 3x^4 - 48 \end{aligned} \quad \checkmark$$

Exercise

Assume f' is continuous on $[2, 4]$, $\int_1^2 f'(2x) dx = 10$, and $f(2) = 4$. Evaluate $f(4)$.

Solution

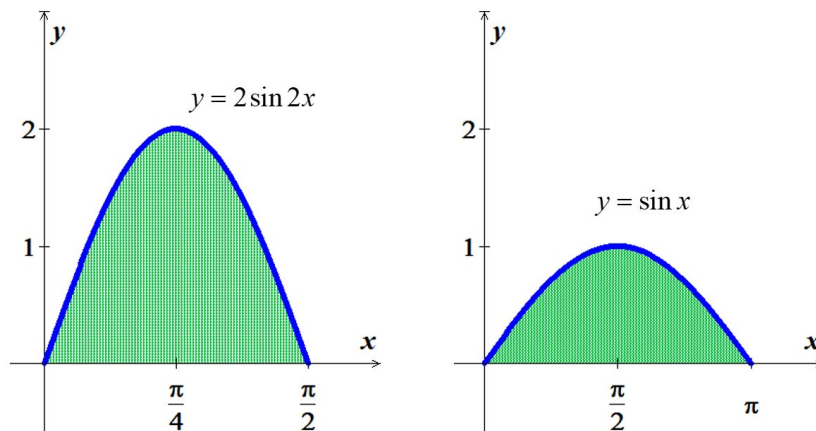
$$\begin{aligned} \int_1^2 f'(2x) dx &= f(2x) \Big|_1^2 \\ &= f(4) - f(2) = 10 \end{aligned}$$

$$f(4) - 4 = 10$$

$$f(4) = 14$$

Exercise

The area of the shaded region under the curve $y = 2 \sin 2x$ in



- Equals the area on the shaded region under the curve $y = \sin x$
- Explain why this is true without computation areas.

Solution

$$\begin{aligned} a) \quad A &= \int_0^{\pi/2} 2 \sin 2x \, dx \\ u &= 2x \rightarrow du = 2dx \\ \begin{cases} x = \frac{\pi}{2} & \rightarrow u = \pi \\ x = 0 & \rightarrow u = 0 \end{cases} \\ &= \int_0^{\pi} \sin u \, du \\ &= \int_0^{\pi} \sin x \, dx \end{aligned}$$

b) Let A_1 = area of $\sin x$ $0 \leq x \leq \pi$

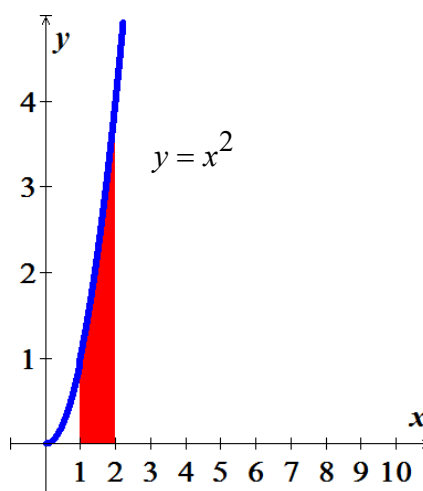
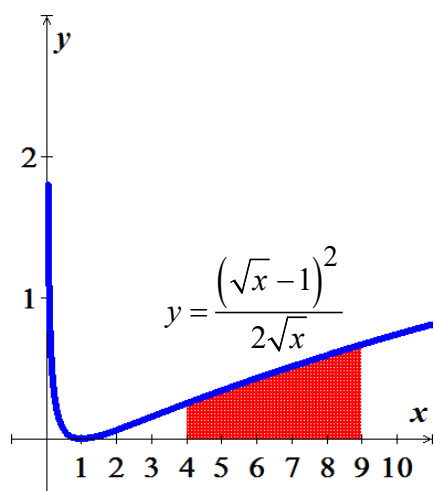
$$A_2 = \text{area of } \sin 2x \quad 0 \leq 2x \leq \pi \rightarrow 0 \leq x \leq \frac{\pi}{2}$$

Area of $0 \leq x \leq \frac{\pi}{2}$ is $\frac{1}{2} A_1$

$$A_2 = 2 \frac{1}{2} A_1 = A_1 \quad \checkmark$$

Exercise

The area of the shaded region under the curve $y = \frac{(\sqrt{x}-1)^2}{2\sqrt{x}}$ on the interval $[4, 9]$



a) Equals the area on the shaded region under the curve $y = x^2$ on the interval $[1, 2]$

b) Explain why this is true without computation areas.

Solution

a) Let $u = \sqrt{x} - 1 \rightarrow x = (u + 1)^2$

$$dx = 2(u + 1) du$$

$$\begin{cases} x = 9 & \rightarrow u = 2 \\ x = 4 & \rightarrow u = 1 \end{cases}$$

$$A_1 = \int_4^9 \frac{(\sqrt{x} - 1)^2}{2\sqrt{x}} dx$$

$$= \int_1^2 \frac{u^2}{2(u + 1)} 2(u + 1) du$$

$$= \int_1^2 u^2 du \quad \checkmark$$

$$= \frac{1}{3}(\sqrt{x} - 1)^3 \Big|_4^9$$

$$= \frac{1}{3}(2^3 - 1)$$

$$= \frac{7}{3}$$

$$A_2 = \int_1^2 x^2 dx$$

$$= \frac{1}{3}x^3 \Big|_1^2$$

$$= \frac{1}{3}(2^3 - 1)$$

$$= \frac{7}{3}$$

b) $\int_4^9 \frac{(\sqrt{x} - 1)^2}{2\sqrt{x}} dx = \int_1^2 u^2 du = \int_1^2 x^2 dx \quad \checkmark$

Exercise

The family of parabolas $y = \frac{1}{a} - \frac{x^2}{a^3}$, where $a > 0$, has the property that for $x \geq 0$, the x -intercept is $(a, 0)$ and the y -intercept is $(0, \frac{1}{a})$. Let $A(a)$ be the area of the region in the first quadrant bounded by the parabola and the x -axis. Find $A(a)$ and determine whether it is increasing, decreasing, or constant function of a .

Solution

Given: $y = \frac{1}{a} - \frac{x^2}{a^3}$ $(a, 0)$ & $(0, \frac{1}{a})$

$$\begin{aligned} A &= \int_0^a \left(\frac{1}{a} - \frac{x^2}{a^3} \right) dx \\ &= \frac{1}{a}x - \frac{1}{3} \frac{x^3}{a^3} \Bigg|_0^a \\ &= 1 - \frac{1}{3} \\ &= \frac{2}{3} \end{aligned}$$

$A(a) = \frac{2}{3}$ is a constant function.

Exercise

Consider the right triangle with vertices $(0, 0)$, $(0, b)$, and $(a, 0)$, where $a > 0$ and $b > 0$. Show that the average vertical distance from points on the x -axis to the hypotenuse is $\frac{b}{2}$, for all $a > 0$.

Solution

$$\begin{aligned} y &= \frac{b-0}{0-a}(x-0) + b & y &= m(x-x_0) + y_0 \\ &= -\frac{b}{a}x + b \end{aligned}$$

Average vertical distance is:

$$\begin{aligned} \frac{1}{a-0} \int_0^a \left(-\frac{b}{a}x + b \right) dx &= \frac{1}{a} \int_0^a \left(b - \frac{b}{a}x \right) dx \\ &= \frac{1}{a} \left(bx - \frac{b}{2a}x^2 \right) \Bigg|_0^a \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{a} \left(ba - \frac{b}{2a} a^2 \right) \\
&= b - \frac{b}{2} \\
&= \frac{b}{2} \quad |
\end{aligned}$$

Exercise

Consider the integral $I = \int \sin^2 x \cos^2 x \, dx$

- Find I using the identity $\sin 2x = 2 \sin x \cos x$
- Find I using the identity $\cos^2 x = 1 - \sin^2 x$
- Confirm that the results in part (a) and (b) are consistent and compare the work involved in each method.

Solution

a) $\sin 2x = 2 \sin x \cos x$

$$\sin^2 2x = 4 \sin^2 x \cos^2 x$$

$$\sin^2 x \cos^2 x = \frac{1}{4} \sin^2 2x$$

$$I = \int \sin^2 x \cos^2 x \, dx$$

$$= \frac{1}{4} \int \sin^2 2x \, dx$$

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$= \frac{1}{8} \int (1 - \cos 4x) \, dx$$

$$= \frac{1}{8} \left(x - \frac{1}{4} \sin 4x \right) + C$$

$$= \frac{1}{8} x - \frac{1}{32} \sin 4x + C \quad |$$

b) $\cos^2 x = 1 - \sin^2 x$

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$= 1 - \frac{1}{2} + \frac{1}{2} \cos 2x$$

$$= \frac{1}{2} + \frac{1}{2} \cos 2x$$

$$I = \int \frac{1}{2}(1 - \cos 2x) \frac{1}{2}(1 + \cos 2x) \, dx$$

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$= \frac{1}{4} \int (1 - \cos^2 2x) \, dx$$

$$\cos^2 x = 1 - \sin^2 x$$

$$= \frac{1}{4} \int \sin^2 2x \, dx \quad \text{From part (a)}$$

$$= \frac{1}{8}x - \frac{1}{32}\sin 4x + C$$

c) The results from part *a* & *b* are consistent.

Exercise

Let $H(x) = \int_0^x \sqrt{4-t^2} \, dt$, for $-2 \leq x \leq 2$.

- a) Evaluate $H(0)$
- b) Evaluate $H'(1)$
- c) Evaluate $H'(2)$
- d) Use geometry to evaluate $H(2)$
- e) Find the value of s such that $H(x) = sH(-x)$

Solution

$$a) \quad H(0) = \int_0^0 \sqrt{4-t^2} \, dt$$

$$= 0$$

$$b) \quad H'(x) = \sqrt{4-x^2} \frac{d}{dx}(x)$$

$$= \sqrt{4-x^2}$$

$$H'(1) = \sqrt{3}$$

$$c) \quad H'(2) = \sqrt{4-4}$$

$$= 0$$

$$d) \quad H(2) = \int_0^2 \sqrt{4-t^2} \, dt \quad \text{is the area inside a circle in the first quadrant of radius 2}$$

$$= \frac{1}{4}\pi(2)^2$$

$$= \pi$$

$$e) \quad H(x) = \int_0^{-x} \sqrt{4-t^2} \, dt \quad \sqrt{4-t^2} \text{ is an even function}$$

$$\begin{aligned}
 &= - \int_{-x}^0 \sqrt{4-t^2} \, dt \\
 &= -H(x)
 \end{aligned}$$

$$\therefore \underline{s = -1}$$

$$t = 2 \sin u$$

$$dt = 2 \cos u \, du$$

$$\sqrt{4-t^2} = 2 \cos u$$

$$H(x) = \int_0^x \sqrt{4-t^2} \, dt$$

$$= \int_0^x 2 \cos u \, 2 \cos u \, du$$

$$= \int_0^x 4 \cos^2 u \, du$$

$$= \int_0^x 2(1 + \cos 2u) \, du$$

$$= 2 \left(u + \frac{1}{2} \sin 2u \right) \Big|_0^x$$

$$= 2 \left(\sin^{-1} \frac{t}{2} + \sin u \cos u \right) \Big|_0^x$$

$$= 2 \left(\sin^{-1} \frac{t}{2} + \frac{t}{4} \sqrt{4-t^2} \right) \Big|_0^x$$

$$= 2 \left(\sin^{-1} \frac{x}{2} + \frac{x}{4} \sqrt{4-x^2} \right)$$

$$\underline{= 2 \sin^{-1} \frac{x}{2} + \frac{x}{2} \sqrt{4-x^2}}$$

$$t = 2 \sin u \quad \rightarrow \quad u = \sin^{-1} \frac{t}{2}$$

$$\sqrt{4-t^2} = 2 \cos u \quad \rightarrow \quad \cos u = \frac{1}{2} \sqrt{4-t^2}$$

Exercise

Evaluate the limits $\lim_{x \rightarrow 2} \frac{\int_2^x e^{t^2} dt}{x-2}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{\int_2^x e^{t^2} dt}{x-2} &= \frac{\int_2^2 e^{t^2} dt}{2-2} = \frac{0}{0} \\ &= \lim_{x \rightarrow 2} \frac{e^{x^2} \frac{d}{dx}(x)}{1} \\ &= \lim_{x \rightarrow 2} e^{x^2} \\ &= e^4\end{aligned}$$

Exercise

Evaluate the limits $\lim_{x \rightarrow 1} \frac{\int_1^{x^2} e^{t^3} dt}{x-1}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{\int_1^{x^2} e^{t^3} dt}{x-1} &= \lim_{x \rightarrow 1} \frac{\int_1^1 e^{t^3} dt}{1-1} = \frac{0}{0} \\ &= \lim_{x \rightarrow 1} \frac{2xe^{x^3}}{1} \\ &= 2e\end{aligned}$$

Exercise

Prove that for nonzero constants a and b , $\int \frac{dx}{a^2x^2 + b^2} = \frac{1}{ab} \tan^{-1}\left(\frac{ax}{b}\right) + C$

Solution

$$\int \frac{dx}{a^2x^2 + b^2} = \int \frac{dx}{a^2 \left(x^2 + \left(\frac{b}{a}\right)^2 \right)} \qquad \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$= \frac{1}{a^2} \frac{a}{b} \tan^{-1} \frac{x}{\frac{b}{a}} + C$$

$$= \frac{1}{ab} \tan^{-1} \left(\frac{ax}{b} \right) + C \quad \checkmark$$

Exercise

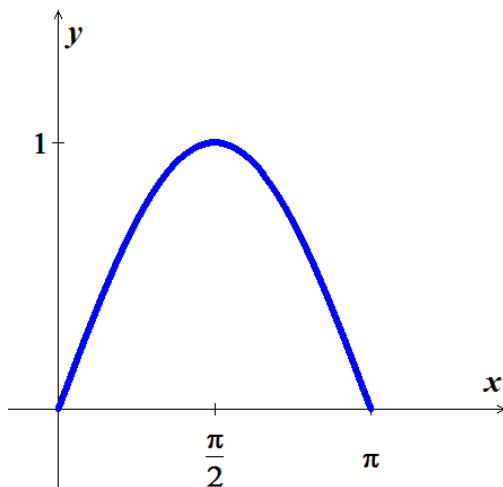
Let $a > 0$ be a real number and consider the family of functions $f(x) = \sin ax$ on the interval $\left[0, \frac{\pi}{a}\right]$.

a) Graph f , for $a = 1, 2, 3$.

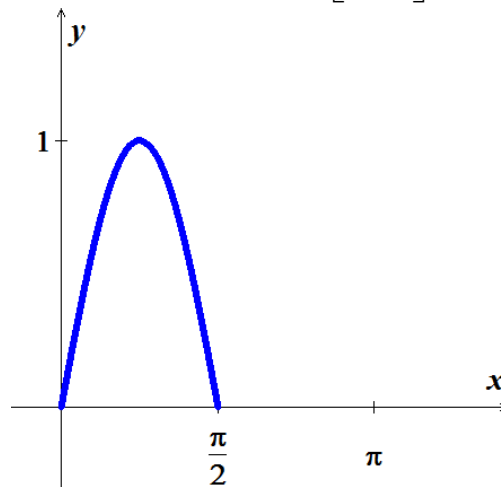
b) Let $g(a)$ be the area of the region bounded by the graph of f and the x -axis on the interval $\left[0, \frac{\pi}{a}\right]$. Graph g for $0 < a < \infty$. Is g an increasing function, a decreasing function, or neither?

Solution

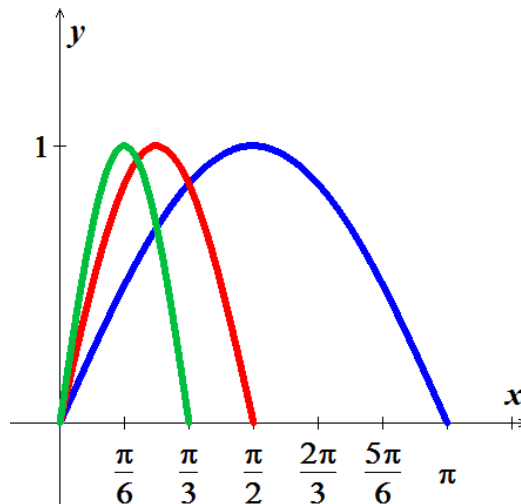
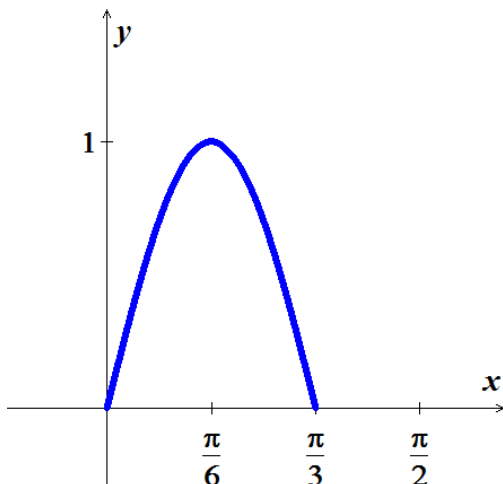
a) $a = 1 \rightarrow f(x) = \sin x \quad x \in [0, \pi]$



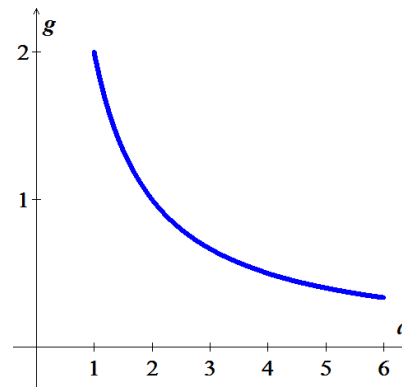
$a = 2 \rightarrow f(x) = \sin 2x \quad x \in \left[0, \frac{\pi}{2}\right]$



$a = 3 \rightarrow f(x) = \sin 3x \quad x \in \left[0, \frac{\pi}{3}\right]$



$$\begin{aligned}
 b) \quad g(x) &= \int_0^{\pi/a} \sin ax \, dx \\
 &= -\frac{1}{a} \cos ax \Big|_0^{\pi/a} \\
 &= -\frac{1}{a} (\cos \pi - \cos 0) \\
 &= -\frac{1}{a} (-1 - 1) \\
 &= \frac{2}{a}
 \end{aligned}$$



The function is decreasing as $a \geq 1$ is increasing.

Exercise

Explain why if a function u satisfies the equation $u(x) + 2 \int_0^x u(t) dt = 10$, then it also satisfies the equation $u'(x) + 2u(x) = 0$. Is it true that if u satisfies the second equation, then it satisfies the first equation?

Solution

$$\frac{d}{dx} u(x) + 2 \frac{d}{dx} \int_0^x u(t) dt = \frac{d}{dx} (10)$$

$$u'(x) + 2 \frac{d}{dx} u(x) \frac{d}{dx} x = 0$$

$$u'(x) + 2u(x) = 0 \quad \checkmark$$

Exercise

$$\text{Let } f(x) = \int_0^x (t-1)^{15} (t-2)^9 dt$$

- Find the interval on which f is increasing and the intervals on which f is decreasing.
- Find the intervals on which f is concave up and the intervals on which f is concave down.
- For what values of x does f have local minima? Local maxima?
- Where are the inflection points of f ?

Solution

$$a) \quad f'(x) = (x-1)^{15} (x-2)^9 = 0$$

$$CN: \underline{x=1, 2}$$

Where $x=1$ is multiplicity of 15

$x=2$ is multiplicity of 9

Therefore, the sign will change.

f is increasing on $(-\infty, 1) \cup (2, \infty)$

f is decreasing on $(1, 2)$

0	1	2
+	-	+

$$b) f''(x) = (x-1)^{14}(x-2)^8(15(x-2)+9(x-1))$$

$$= (x-1)^{14}(x-2)^8(24x-39) = 0$$

$$\underline{x=1, 2, \frac{13}{8}}$$

0	1	$\frac{13}{8}$	2
-	-	+	+

$$(x-1)^{14}(x-2)^8 \geq 0 \text{ (always)}$$

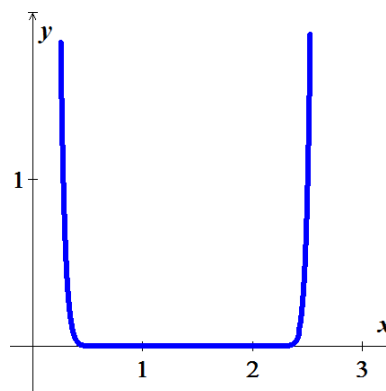
Concave up: $(\frac{13}{8}, \infty)$

Concave down: $(-\infty, \frac{13}{8})$

$$c) \text{ LMIN: } (1, 0)$$

$$\text{ LMAX: } (2, 0)$$

$$d) \text{ point of inflection: } \underline{x = \frac{13}{8}}$$



Exercise

A company is considering a new manufacturing process in one of its plants. The new process provides substantial initial savings, with the savings declining with time t (in years) according to the rate-of-savings function

$$S'(t) = 100 - t^2$$

where $S'(t)$ is in thousands of dollars per year. At the same time, the cost of operating the new process increases with time t (in years), according to the rate-of-cost function (in thousands of dollars per year)

$$C'(t) = t^2 + \frac{14}{3}t$$

- a) For how many years will the company realize savings?
- b) What will be the net total savings during this period?

Solution

- a) For how many years will the company realize savings?

$$C'(t) = S'(t)$$

$$t^2 + \frac{14}{3}t = 100 - t^2$$

$$2t^2 + \frac{14}{3}t - 100 = 0$$

$$\rightarrow t = -\frac{25}{3} \text{ or } 6$$

$$\underline{t = 6}$$

The company should use this type for 6 years.

- b) What will be the net total savings during this period?

$$\text{Total savings} = \int_0^6 \left((100 - t^2) - \left(t^2 + \frac{14}{3}t \right) \right) dt$$

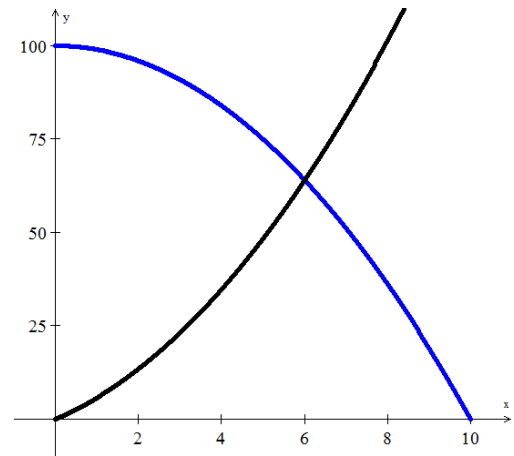
$$= \int_0^6 \left(100 - 2t^2 - \frac{14}{3}t \right) dt$$

$$= 100t - \frac{2}{3}t^3 - \frac{7}{3}t^2 \Big|_0^6$$

$$= 100(6) - \frac{2}{3}(6)^3 - \frac{7}{3}(6)^2 - \left(100(0) - \frac{2}{3}(0)^3 - \frac{7}{3}(0)^2 \right)$$

$$\underline{= 372}$$

The company will save a total of \$372,000. Over the 6-year period



Exercise

Find the producers' surplus if the supply function for pork bellies is given by

$$S(x) = x^{5/2} + 2x^{3/2} + 50$$

Assume supply and demand are in equilibrium at $x = 16$.

Solution

The equilibrium price:

$$\begin{aligned} p_0 &= S(x = 16) = 16^{5/2} + 2(16)^{3/2} + 50 \\ &= 1202 \end{aligned}$$

$$\begin{aligned} \text{Producer's surplus} &= \int_0^{x_0} (p_0 - S(x)) dx \\ &= \int_0^{16} (1202 - (x^{5/2} + 2x^{3/2} + 50)) dx \\ &= \int_0^{16} (1152 - x^{5/2} - 2x^{3/2}) dx \\ &= 1152x - \frac{2}{7}x^{7/2} - \frac{4}{5}x^{5/2} \Big|_0^{16} \\ &= \left(1152(16) - \frac{2}{7}(16)^{7/2} - \frac{4}{5}(16)^{5/2}\right) - \left(1152(0) - \frac{2}{7}(0)^{7/2} - \frac{4}{5}(0)^{5/2}\right) \\ &= \$12,931.66 \end{aligned}$$

The producers' surplus is \$12,931.66

Exercise

An object moves along a line with a velocity in m/s given by $v(t) = 8\cos\left(\frac{\pi t}{6}\right)$. Its initial position is $s(0) = 0$.

a) Graph the velocity function.

b) The position of the object is given by $s(t) = \int_0^t v(y) dy$, for $t \geq 0$. Find the position function, for $t \geq 0$.

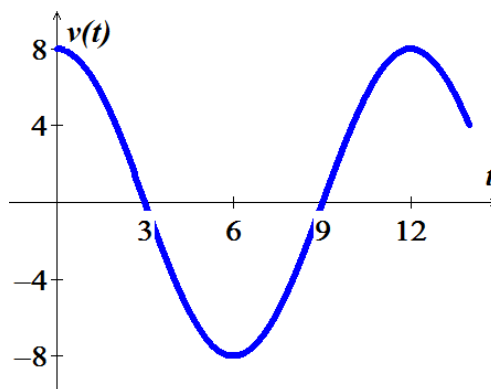
c) What is the period of the motion – that is, starting at any point, how long does it take the object to return to that position?

Solution

$$a) \quad v(t) = 8\cos\left(\frac{\pi t}{6}\right)$$

$$|A| = 8 \quad P = 12$$

t	$v(t)$
0	8
3	0
6	-8
9	0
12	8



$$\begin{aligned}
 b) \quad s(t) &= \int_0^t v(y) dy \\
 &= \int_0^t 8 \cos\left(\frac{\pi}{6} y\right) dy \\
 &= \frac{48}{\pi} \sin\left(\frac{\pi}{6} y\right) \Bigg|_0^t \\
 &= \frac{48}{\pi} \sin \frac{\pi}{6} t
 \end{aligned}$$

$$\begin{aligned}
 c) \quad \text{Period: } P &= \frac{2\pi}{\frac{\pi}{6}} \\
 &= 12
 \end{aligned}$$

Exercise

The population of a culture of bacteria has a growth rate given by $p'(t) = \frac{200}{(t+1)^r}$ bacteria per hour, for

$t \geq 0$, where $r > 1$ is a real number. It is shown that the increase in the population over time interval

$[0, t]$ is given by $\int_0^t p'(s) ds$. (note that the growth rate decreases in time, reflecting competition for space and food.)

- Using the population model with $r = 2$, what is the increase in the population over the time interval $0 \leq t \leq 4$?
- Using the population model with $r = 3$, what is the increase in the population over the time interval $0 \leq t \leq 6$?
- Let ΔP be the increase in the population over a fixed time interval $[0, T]$. For fixed T , does ΔP increase or decrease with the parameter r ? Explain.

- d) A lab technician measures an increase in the population of 350 bacteria over the 10-hr period $[0, 10]$. Estimate the value of r that best fits this data point.
- e) Use the population model in part (b) to find the increase in population over time interval $[0, T]$, for any $T > 0$. If the culture is allowed to grow indefinitely ($T \rightarrow \infty$), does the bacteria population increase without bound? Or does it approach a finite limit?

Solution

a) $r = 2$ & $0 \leq t \leq 4$

$$\begin{aligned} \int_0^4 \frac{200}{(t+1)^2} dt &= \int_0^4 \frac{200}{(t+1)^2} d(t+1) \\ &= -\frac{200}{t+1} \Big|_0^4 \\ &= -(40 - 200) \\ &= \underline{160} \end{aligned}$$

b) $r = 3$ & $0 \leq t \leq 6$

$$\begin{aligned} \int_0^6 \frac{200}{(t+1)^3} dt &= 200 \int_0^6 (t+1)^{-3} d(t+1) \\ &= -100 \frac{1}{(t+1)^2} \Big|_0^6 \\ &= -100 \left(\frac{1}{49} - 1 \right) \\ &= \underline{\frac{4800}{49}} \end{aligned}$$

c) $\Delta P = \int_0^T \frac{200}{(t+1)^r} dt$ decreases as r increases.

Because $\frac{200}{(t+1)^r} > \frac{200}{(t+1)^{r+1}}$

d) $\int_0^{10} \frac{200}{(t+1)^r} dt = 350$

$$200 \int_0^{10} (t+1)^{-r} d(t+1) = 350$$

$$\left. \frac{1}{1-r}(t+1)^{1-r} \right|_0^{10} = \frac{7}{4}$$

$$\frac{1}{1-r}(11^{1-r} - 1) = \frac{7}{4}$$

$$4(11)^{1-r} - 4 = 7 - r$$

$$4(11)^{1-r} + r - 44 = 0 \xrightarrow{\text{using software}} \underline{r \approx 1.278}$$

$$\begin{aligned} e) \int_0^T \frac{200}{(t+1)^3} dt &= -100 \frac{1}{(t+1)^2} \Big|_0^T \\ &= -100 \left(\frac{1}{(T+1)^2} - 1 \right) \\ &= 100 - \frac{100}{(T+1)^2} \end{aligned}$$

$$\lim_{T \rightarrow \infty} \left(100 - \frac{100}{(T+1)^2} \right) = \underline{100}$$

\therefore The bacteria approach a finite limit of 100.

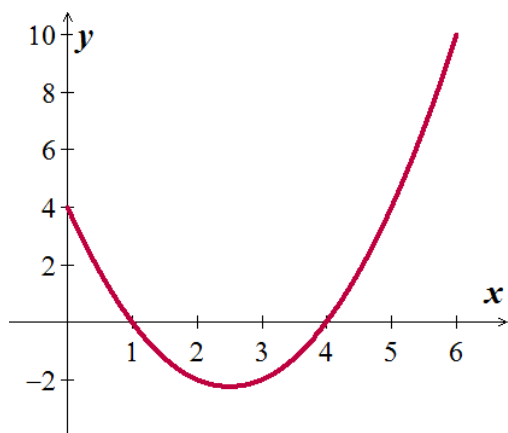
Exercise

Consider the function $f(x) = x^2 - 5x + 4$ and the area function $A(x) = \int_0^x f(t) dt$.

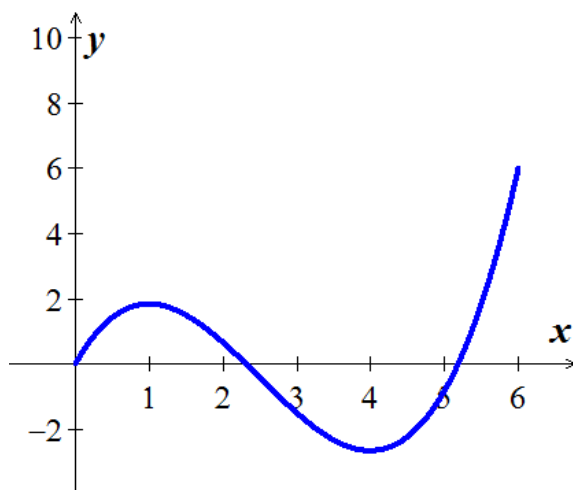
- Graph f on the interval $[0, 6]$.
- Compute and graph A on the interval $[0, 6]$.
- Show that the local extrema of A occur at the zeros of f .
- Give a geometric and analytical explanation for the observation in part (c).
- Find the approximate zeros of A , other than 0, and call them x_1 and x_2 .
- Find b such that the area bounded by the graph of f and the x -axis on the interval $[0, x_1]$ equals the area bounded by the graph of f and the x -axis on the interval $[x_1, b]$.
- If f is an integrable function and $A(x) = \int_0^x f(t) dt$, is it always true that the local extrema of A occur at the zeros of f ? Explain

Solution

a)



$$\begin{aligned}
 b) \quad A(x) &= \int_0^x f(t) dt \\
 &= \int_0^x (t^2 - 5t + 4) dt \\
 &= \left. \frac{1}{3}t^3 - \frac{5}{2}t^2 + 4t \right|_0^x \\
 &= \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x
 \end{aligned}$$



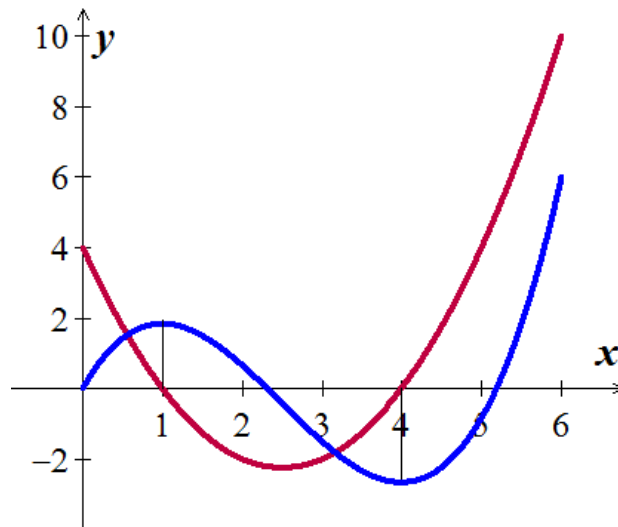
$$c) \quad f(x) = x^2 - 5x + 4 = 0$$

$$\rightarrow \underline{x = 0, 4}$$

$$A(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$$

$$A'(x) = f(x)$$

The zeros of f are at 1 and 4, and A has a local maximum at $x = 1$ and local minimum at $x = 4$.



- d)* Since f is above the axis from 0 to 1, the net area A is increasing and switches to decreasing to the right of 1. When x is between 1 and 4, the function f is below x -axis (negative sign), the Area A is decreasing.

By the fundamental Theorem: $A'(x) = f(x)$, the zeros of f are critical points of A .

$$\begin{aligned} e) \quad A(x) &= \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x \\ &= \frac{1}{6}x(2x^2 - 15x + 24) \end{aligned}$$

$$x = \frac{15 \pm \sqrt{33}}{4}$$

$$\rightarrow \begin{cases} x_1 = \frac{15 - \sqrt{33}}{4} \approx 2.31386 \\ x_2 = \frac{15 + \sqrt{33}}{4} \approx 5.18614 \end{cases}$$

$$\begin{aligned} f) \quad A_1 &= \int_0^{x_1} f(x) dx \\ &= \int_0^1 f(x) dx + \left| \int_1^{x_1} f(x) dx \right| \\ &= \left(\frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x \right) \Big|_0^1 + \left| \left(\frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x \right) \Big|_1^{x_1} \right| \\ &= \left(\frac{1}{3} - \frac{5}{2} + 4 \right) - 0 + \left| 0 - \left(\frac{1}{3} - \frac{5}{2} + 4 \right) \right| \\ &= 2 \left(\frac{1}{3} - \frac{5}{2} + 4 \right) \\ &= 2 \left(\frac{11}{6} \right) \end{aligned}$$

$$= \frac{11}{3} \Big|$$

$$\begin{aligned} A_2 &= \left| \int_{x_1}^{x_2} f(x) dx \right| + \int_{x_2}^b f(x) dx \\ &= \left| \left(\frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x \right) \Big|_{x_1}^{x_2} \right| + \left(\frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x \right) \Big|_{x_2}^b \\ &= 0 + \left[\left(\frac{1}{3}b^3 - \frac{5}{2}b^2 + 4b \right) - 0 \right] \\ &= \frac{1}{3}b^3 - \frac{5}{2}b^2 + 4b \end{aligned}$$

Since $A_1 = A_2$

$$\frac{1}{3}b^3 - \frac{5}{2}b^2 + 4b = \frac{11}{3}$$

$$\frac{1}{3}b^3 - \frac{5}{2}b^2 + 4b - \frac{11}{3} = 0$$

→ $b = 5.744348$ (and 2 complex numbers)

g) No, if the function is a piecewise function.

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ -1 & \text{if } 1 \leq x \leq 2 \end{cases}$$

Then $A(x)$ has a maximum at $x = 1$ even though f is never zero.

This is a case where an extreme point occurs at a singular point rather than a stationary point.