Lecture R – Introduction to Differential Equation

Section R.1 – Derivative

Constant Rule

$$\frac{d}{dx}[c]=0 \quad c \text{ is constant}$$

Example

Find the derivative:

a) $f(x)=-2$ $\qquad f'(x)=0$

b) $y=\pi$ $\qquad y'=0$

Power Rule

$$\frac{d}{dx}[x^n]=nx^{n-1} \qquad n \text{ is any real number}$$

Constant Times a Function

$$\frac{d}{dx}\left[cf(x)\right]=cf'(x)$$

Example

Find the derivative each function

a. $y=\dfrac{9}{4x^2}=\dfrac{9}{4}x^{-2}$

Solution

$$\rightarrow y'=\frac{9}{4}(-2)x^{-3}=-\frac{9}{2x^3}$$

b. $y=\sqrt[3]{x}=x^{1/3}$

Solution

$$\rightarrow y'=\frac{1}{3}x^{(1/3)-1}$$

$$=\frac{1}{3}x^{-2/3}$$

$$=\frac{1}{3\sqrt[3]{x^2}}$$

The Product Rule

The derivative of the product of two differentiable functions is equal to the first function times the derivative of the second plus the second function times the derivative of the first,

$$(f.g)' = f.g' + f'.g$$

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'gh + fg'h + fgh'$$

Example

Find the derivative of $y = \left(4x + 3x^2\right)(6 - 3x)$

Solution

$$y = 24x + 6x^2 - 9x^3$$

$$\begin{aligned} y' &= \left(4x + 3x^2\right)\frac{d}{dx}(6 - 3x) + (6 - 3x)\frac{d}{dx}\left(4x + 3x^2\right) \\ &= \left(4x + 3x^2\right)(-3) + (6 - 3x)(4 + 6x) \\ &= -12x - 9x^2 + 24 + 36x - 12x - 18x^2 \\ &= -27x^2 + 12x + 24 \end{aligned}$$

Example

Find the derivative of $y = \left(\frac{1}{x} + 1\right)(2x + 1)$

Solution

$$\begin{aligned} y' &= \left(x^{-1} + 1\right)\frac{d}{dx}(2x + 1) + (2x + 1)\frac{d}{dx}\left(x^{-1} + 1\right) \\ &= \left(x^{-1} + 1\right)(2) + (2x + 1)\left(-x^{-2}\right) \\ &= \frac{2}{x} + 2 + (2x + 1)\left(-\frac{1}{x^2}\right) \\ &= \frac{2}{x} + 2 - \frac{2x}{x^2} - \frac{1}{x^2} \\ &= \frac{2}{x} + 2 - \frac{2}{x} - \frac{1}{x^2} \\ &= 2 - \frac{1}{x^2} \\ &= \frac{2x^2 - 1}{x^2} \end{aligned}$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}=\frac{gf'-fg'}{g^2}$$

Example

Find the derivative of $y=\dfrac{x+4}{5x-2}$

Solution

$$y'=\frac{(5x-2)\frac{d}{dx}[(x+4)]-(x+4)\frac{d}{dx}[(5x-2)]}{(5x-2)^2}$$

$$=\frac{(5x-2)(1)-(x+4)(5)}{(5x-2)^2}$$

$$=\frac{5x-2-5x-20}{(5x-2)^2}$$

$$=-\frac{22}{(5x-2)^2}$$

Example

Find the derivative of $y=\dfrac{3-\frac{2}{x}}{x+4}$

Solution

$$y=\frac{\frac{3x-2}{x}}{x+4}=\frac{3x-2}{x}\cdot\frac{1}{x+4}=\frac{3x-2}{x^2+4x}$$

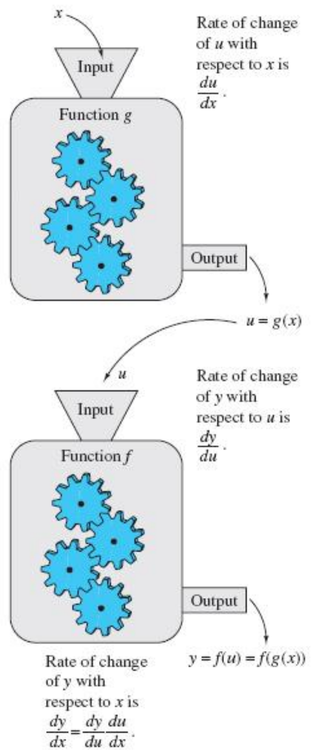
$$y'=\frac{\left(x^2+4x\right)(3)-(3x-2)(2x+4)}{\left[x(x+4)\right]^2}$$

$$\left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$=\frac{3x^2+12x-6x^2-12x+4x+8}{x^2(x+4)^2}$$

$$=\frac{-3x^2+4x+8}{x^2(x+4)^2}$$

The Chain Rule



The Chain Rule

If $y = f(u)$ is a differentiable function of u, and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x, and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

The General Power Rule

$$\frac{dy}{dx} = \frac{d}{dx}\left[u(x)^n\right] = n\,u^{n\text{-}1}\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[u^n\right] = n\,u^{n\text{-}1}u'$$

Example

Find the derivative of $y = \left(x^2 + 3x\right)^4$

Solution

$$u = x^2 + 3x$$

$$y' = n \quad (u)^{n-1} \quad \frac{d}{dx}[u]$$

$$= 4\left(x^2 + 3x\right)^3 \frac{d}{dx}[x^2 + 3x]$$

$$= 4\left(x^2 + 3x\right)^3 (2x + 3)$$

Derivatives of Trigonometric Functions

$(\sin x)' = \cos x$	$(\cos x)' = -\sin x$	$(\tan x)' = \sec^2 x$
$(\csc x)' = -\csc x \cot x$	$(\sec x)' = \sec x \tan x$	$(\cot x)' = -\csc^2 x$

Example

Find the derivatives

a) $y = \sin x \cos x$

$$y' = \sin x (\cos x)' + \cos x (\sin x)'$$

$$= \sin x(-\sin x) + \cos x(\cos x)$$

$$\boxed{= \cos^2 x - \sin^2 x}$$

b) $y = \dfrac{\cos x}{1 - \sin x}$

$$y' = \frac{(1-\sin x)(\cos x)' - \cos x(1-\sin x)'}{(1-\sin x)^2}$$

$$= \frac{(1-\sin x)(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2}$$

$$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} \qquad \sin^2 x + \cos^2 x = 1$$

$$= \frac{1-\sin x}{(1-\sin x)^2}$$

$$\boxed{= \frac{1}{1-\sin x}}$$

Derivative of the Natural Exponential Function

$$\frac{d}{dx}\left[e^{x}\right]=e^{x}$$

$$\frac{d}{dx}\left[e^{U}\right]=e^{U}\,\frac{dU}{dx}$$

Differentiate each function.

a) $f(x)=e^{-2x^3}$

$$f'(x)=e^{-2x^3}\,\tfrac{d}{dx}[-2x^3]$$

$$=e^{-2x^3}\left[-6x^2\right]$$

$$\underline{=-\frac{6x^2}{e^{2x^3}}}\Big|$$

b) $f(x)=4e^{x^2}$

$$f'(x)=4e^{x^2}\,\tfrac{d}{dx}[x^2]$$

$$=4e^{x^2}(2x)$$

$$\underline{=8xe^{x^2}}\Big|$$

c) $y=10e^{3x^2}$

$$y'=10e^{3x^2}\left(3x^2\right)'$$

$$=10e^{3x^2}\left(6x\right)$$

$$\underline{=60x\;e^{3x^2}}\Big|$$

Derivative of ln

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx} = \frac{u'}{u}$$

Note: $\ln U \Rightarrow U > 0$

Derivative of $\log_a x$

$$\frac{d}{dx}\left[\log_a x\right] = \frac{1}{(\ln a)x}$$

Example

Find the Derivatives

a) $f(x) = \ln(x^2 - 4)$

$Let \quad u = x^2 - 4 \quad \Rightarrow \quad \frac{du}{dx} = 2x$

$f'(x) = \frac{1}{u}\frac{du}{dx}$

$= \frac{1}{x^2 - 4}(2x)$

$\boxed{= \frac{2x}{x^2 - 4}}$

b) $f(x) = x^2 \ln x$

$f' = x^2 \frac{d}{dx}[\ln x] + \ln x \frac{d}{dx}\left[x^2\right] \qquad (fg)' = f'g + fg'$

$= x^2\left(\frac{1}{x}\right) + 2x \ln x$

$= x + 2x \ln x$

$\boxed{= x(1 + 2\ln x)}$

c) $f(x) = -\frac{\ln x}{x^2}$

$f' = -\frac{x^2 \frac{d}{dx}[\ln x] - \ln x \frac{d}{dx}[x^2]}{(x^2)^2}$

$= -\frac{x^2 \frac{1}{x} - 2x \ln x}{x^4}$

$$= -\frac{x - 2x\ln x}{x^4}$$

$$= -\frac{x(1 - 2\ln x)}{x^4}$$

$$\boxed{= -\frac{1 - 2\ln x}{x^3}}$$

Other Bases and Differentiation

$$\frac{d}{dx}\left[a^x\right] = a^x \ln a$$

$$\frac{d}{dx}\left[a^u\right] = a^u (\ln a)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\log_a x\right] = \left(\frac{1}{\ln a}\right)\frac{1}{x} \quad \frac{d}{dx}\left[\log_a u\right] = \left(\frac{1}{\ln a}\right)\left(\frac{1}{u}\right)\frac{du}{dx}$$

Formula $\left(U^m V^n W^p\right)' = U^{m-1}V^{n-1}W^{p-1}\left(mU'VW + nUV'W + pUVW'\right)$

Proof

$$\left(U^m V^n W^p\right)' = \left(U^m\right)' V^n W^p + U^m\left(V^n\right)' W^p + U^m V^n \left(W^p\right)'$$

$$= mU^{m-1}U'V^nW^p + nU^mV^{n-1}V'W^p + pU^mV^nW^{p-1}W' \qquad \textit{factor } \; U^{m-1}V^{n-1}W^{p-1}$$

$$= U^{m-1}V^{n-1}W^{p-1}\left(mU'VW + nUV'W + pUVW'\right)$$

$$\left(U^m V^n\right)' = U^{m-1}V^{n-1}\left(mU'V + nUV'\right)$$

Exercises   *Section* R.1 – Derivative

Find the derivative to the following functions

1. $f(t) = -3t^2 + 2t - 4$
2. $g(x) = 4\sqrt[3]{x} + 2$
3. $f(x) = x\left(x^2 + 1\right)$
4. $f(x) = \frac{2x^2 - 3x + 1}{x}$
5. $f(x) = \frac{4x^3 - 3x^2 + 2x + 5}{x^2}$
6. $f(x) = \frac{-6x^3 + 3x^2 - 2x + 1}{x}$
7. $f(x) = x\left(1 - \frac{2}{x+1}\right)$
8. $g(s) = \frac{s^2 - 2s + 5}{\sqrt{s}}$
9. $f(x) = \frac{x+1}{\sqrt{x}}$
10. $f(x) = 3x\left(2x^2 + 5x\right)$
11. $y = 3\left(2x^2 + 5x\right)$
12. $y = \frac{x^2 + 4x}{5}$
13. $y = \frac{3x^4}{5}$
14. $y = \frac{x^2 - 4}{2x + 5}$
15. $y = \frac{(1+x)(2x-1)}{x-1}$
16. $y = \frac{4}{2x+1}$
17. $y = \frac{2}{(x-1)^3}$

Use the General Power Rule to find the derivative of the function

18. $f(x) = \sqrt{2t^2 + 5t + 2}$
19. $f(x) = \frac{1}{\left(x^2 - 3x\right)^2}$
20. $y = t^2\sqrt{t-2}$
21. $y = \left(\frac{6 - 5x}{x^2 - 1}\right)^2$
22. $y = x^2\sqrt{x^2 + 1}$
23. $y = \left(\frac{x+1}{x-5}\right)^2$
24. $y = \sqrt[3]{(x+4)^2}$

Find the derivative of the trigonometric function

25. $y = x^2 \sin x$
26. $y = \frac{\sin x}{x}$
27. $y = \frac{\cot x}{1 + \cot x}$
28. $y = x^2 \sin x + 2x\cos x - 2\sin x$
29. $y = x^3 \sin x \cos x$
30. $y = \frac{4}{\cos x} + \frac{1}{\tan x}$

Differentiate each function.

31. $f(x) = x^2 e^x$
32. $f(x) = \frac{e^x + e^{-x}}{2}$
33. $f(x) = (1 + 2x)e^{4x}$
34. $y = x^2 e^{5x}$
35. $y = e^{x^2+1}\sqrt{5x+2}$
36. $f(x) = \frac{e^x}{x^2}$
37. $f(x) = x^2 e^x - e^x$

38. $f(x) = \ln \sqrt[3]{x+1}$

39. $f(x) = \ln\left[x^2\sqrt{x^2+1}\right]$

40. $y = \ln \dfrac{x^2}{x^2+1}$

41. $y = \ln \dfrac{1+e^x}{1-e^x}$

42. $y = x\ 3^{x+1}$

43. $f(t) = \dfrac{\log_8\left(t^{3/2}+1\right)}{t}$