SOLUTION *Section* 4.5 – Multiple Eigenvalues Solutions

Exercise

Find the general solution $\quad x' = \begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix} x$

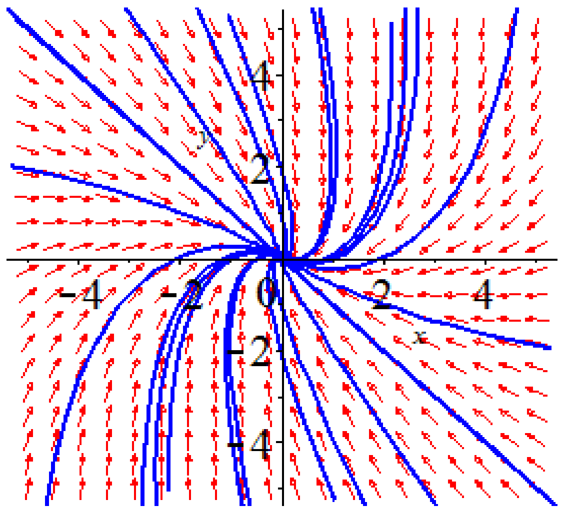
Solution

$$\left|A - \lambda I\right| = \begin{vmatrix} -2-\lambda & 1 \\ -1 & -4-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 9 = 0 \qquad \text{The eigenvalues are: } \lambda_{1,2} = -3 \quad (\textit{multiplicity } 2)$$

$$(A+3I)^2 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$(A-\lambda I)^2 \vec{v}_2 = \vec{0} \;\Rightarrow\; \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\vec{v}_2 = \vec{0} \text{ and } \vec{v}_2 \text{ is a nonzero vector, we can let } \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$(A+3I)\vec{v}_2 = \vec{v}_1 \;\Rightarrow\; \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{v}_1$$



$$\vec{x}_1(t) = \vec{v}_1 e^{\lambda t} \quad \textit{and} \quad \vec{x}_2(t) = \left(\vec{v}_1 t + \vec{v}_2\right)e^{\lambda t}$$

$$\to \begin{cases} \vec{x}_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} \\ \vec{x}_2(t) = \begin{pmatrix} t+1 \\ -t \end{pmatrix} e^{-3t} \end{cases}$$

The general solution: $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$

$$\begin{cases} x_1(t) = \left(c_1 + c_2 + c_2 t\right)e^{-3t} \\ x_2(t) = \left(-c_1 - c_2 t\right)e^{-3t} \end{cases}$$

Exercise

Find the general solution $\quad x' = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} x$

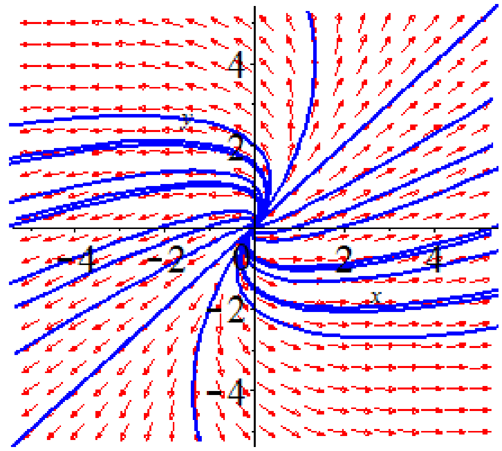
Solution

$$\left|A - \lambda I\right| = \begin{vmatrix} 3-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 2 = 0 \qquad \text{The eigenvalues are: } \lambda_{1,2} = 2 \quad (\textit{multiplicity } 2)$$

$$(A-2I)^2 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$(A-\lambda I)^2 \vec{v}_2 = \vec{0} \;\Rightarrow\; \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\vec{v}_2 = \vec{0} \text{ and } \vec{v}_2 \text{ is a nonzero vector, we can let } \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$(A-2I)\vec{v}_2 = \vec{v}_1 \quad \Rightarrow \quad \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \vec{v}_1$$



$$\vec{x}_1(t) = \vec{v}_1 e^{\lambda t} \quad \textit{and} \quad \vec{x}_2(t) = \left(\vec{v}_1 t + \vec{v}_2\right)e^{\lambda t}$$

$$\begin{cases} \vec{x}_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} \\ \vec{x}_2(t) = \begin{pmatrix} t+1 \\ t \end{pmatrix} e^{2t} \end{cases}$$

The general solution: $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$

$$\begin{cases} x_1(t) = \left(c_1 + c_2 + c_2 t\right)e^{2t} \\ x_2(t) = \left(c_1 + c_2 t\right)e^{2t} \end{cases}$$

Exercise

Find the general solution $\quad \boldsymbol{x}' = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}\boldsymbol{x}$

Solution

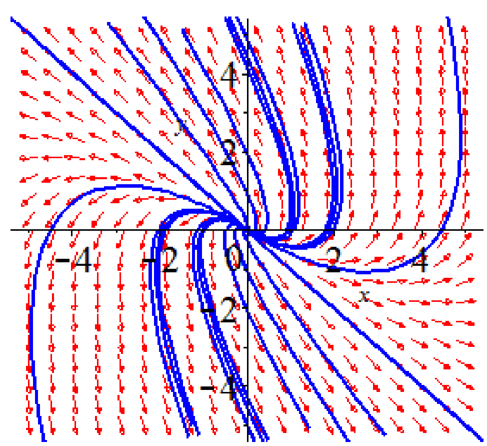
$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & -2 \\ 2 & 5-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = 0$$

The eigenvalues are: $\lambda_{1,2} = 3 \quad (\textit{multiplicity } 2)$

$$(A-3I)^2 = \begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$(A-\lambda I)^2 \vec{v}_2 = \vec{0} \quad \Rightarrow \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\vec{v}_2 = \vec{0}$ and $\vec{v}_2$ is a nonzero vector, we can let $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$(A-3I)\vec{v}_2 = \vec{v}_1 \quad \Rightarrow \quad \begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \vec{v}_1$$



$$\vec{x}_1(t) = \vec{v}_1 e^{\lambda t} \quad \textit{and} \quad \vec{x}_2(t) = \left(\vec{v}_1 t + \vec{v}_2\right)e^{\lambda t}$$

$$\begin{cases} \vec{x}_1(t) = \begin{pmatrix} -2 \\ 2 \end{pmatrix} e^{3t} \\ \vec{x}_2(t) = \begin{pmatrix} -2t+1 \\ 2t \end{pmatrix} e^{3t} \end{cases}$$

The general solution: $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$

$$\begin{cases} x_1(t) = \left(-2c_1 + c_2 - 2c_2 t\right)e^{3t} \\ x_2(t) = \left(2c_1 + 2c_2 t\right)e^{3t} \end{cases}$$

Exercise

Find the general solution $\quad \boldsymbol{x}' = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix} \boldsymbol{x}$

Solution

$$\left|A-\lambda I\right| = \begin{vmatrix} 3-\lambda & -1 \\ 1 & 5-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 16 = 0 \qquad \text{The eigenvalues are: } \lambda_{1,2} = 4 \quad (\textit{multiplicity } 2)$$

$$(A-4I)^2 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$(A-\lambda I)^2 \vec{v}_2 = \vec{0} \quad \Rightarrow \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\vec{v}_2 = \vec{0} \text{ and } \vec{v}_2 \text{ is a nonzero vector, we can let } \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

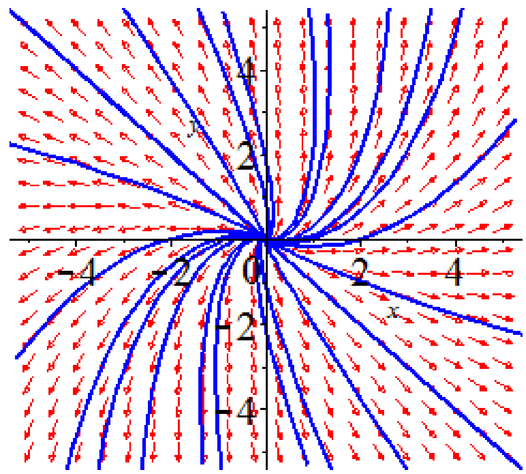
$$(A-2I)\vec{v}_2 = \vec{v}_1 \quad \Rightarrow \quad \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \vec{v}_1}$$

$$\vec{x}_1(t) = \vec{v}_1 e^{\lambda t} \quad \textit{and} \quad \vec{x}_2(t) = \left(\vec{v}_1 t + \vec{v}_2\right)e^{\lambda t}$$

$$\begin{cases} \vec{x}_1(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t} \\ \vec{x}_2(t) = \begin{pmatrix} -t+1 \\ t \end{pmatrix} e^{4t} \end{cases}$$

The general solution: $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$

$$\begin{cases} x_1(t) = \left(-c_1 + c_2 - c_2 t\right)e^{4t} \\ x_2(t) = \left(c_1 + c_2 t\right)e^{4t} \end{cases}$$



Exercise

Find the general solution $\quad \boldsymbol{x}' = \begin{bmatrix} 2 & 0 & 0 \\ -7 & 9 & 7 \\ 0 & 0 & 2 \end{bmatrix} \boldsymbol{x}$

Solution

$$\left|A-\lambda I\right| = \begin{vmatrix} 2-\lambda & 0 & 0 \\ -7 & 9-\lambda & 7 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (9-\lambda)(2-\lambda)^2 = 0$$

The eigenvalues are: $\lambda_1 = 9 \quad \lambda_{2,3} = 2$

For $\lambda_1 = 9 \quad \Rightarrow \quad (A-9I)\vec{v}_1 = 0$

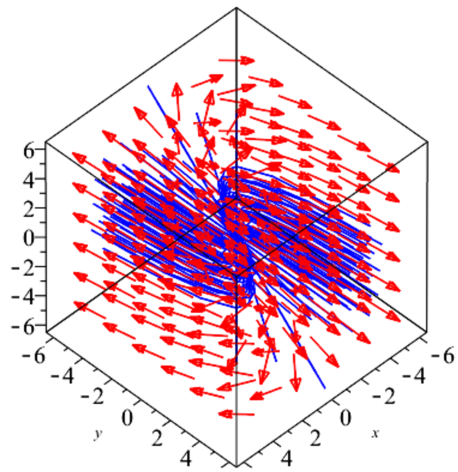
$$\begin{pmatrix} -7 & 0 & 0 \\ -7 & 0 & 7 \\ 0 & 0 & -7 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} a = 0 \\ \\ c = 0 \end{matrix} \quad \rightarrow \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \vec{x}_1(t) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{9t}$$

For $\lambda_{2,3} = 2 \quad \Rightarrow \quad (A - 2I)\vec{v}_2 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ -7 & 7 & 7 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad -a+b+c=0$$

Let $b = 0 \quad \Rightarrow \quad a = c = 1 \; \to \; \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \; \Rightarrow \; \vec{x}_2(t) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}$

Let $c = 0 \quad \Rightarrow \quad a = b = 1 \; \to \; \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \; \Rightarrow \; \vec{x}_3(t) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}$



The general solution: $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$

$$\begin{cases} x_1(t) = (c_2 + c_3)e^{2t} \\ x_2(t) = c_1 e^{9t} + c_3 e^{2t} \\ x_3(t) = c_2 e^{2t} \end{cases}$$

Exercise

Find the general solution $\quad \boldsymbol{x}' = \begin{bmatrix} 25 & 12 & 0 \\ -18 & -5 & 0 \\ 6 & 6 & 13 \end{bmatrix}\boldsymbol{x}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 25-\lambda & 12 & 0 \\ -18 & -5-\lambda & 0 \\ 6 & 6 & 13-\lambda \end{vmatrix} = (25-\lambda)(-5-\lambda)(13-\lambda) + 216(13-\lambda)$$

$$= (13-\lambda)\left(-125 - 20\lambda + \lambda^2 + 216\right)$$

$$= (13-\lambda)\left(\lambda^2 - 20\lambda + 91\right)$$

$$= (13-\lambda)(\lambda - 13)(\lambda - 7) = 0$$

The eigenvalues are: $\lambda_1 = 7 \quad \lambda_{2,3} = 13$

For $\lambda_1 = 7 \quad \Rightarrow \quad (A - 7I)\vec{v}_1 = 0$

$$\begin{pmatrix} 18 & 12 & 0 \\ -18 & -12 & 0 \\ 6 & 6 & 6 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} 3a = -2b \\ \\ a + b + c = 0 \end{matrix} \quad \to \begin{cases} a = 2 \\ b = -3 \\ c = -2 + 3 = 1 \end{cases}$$

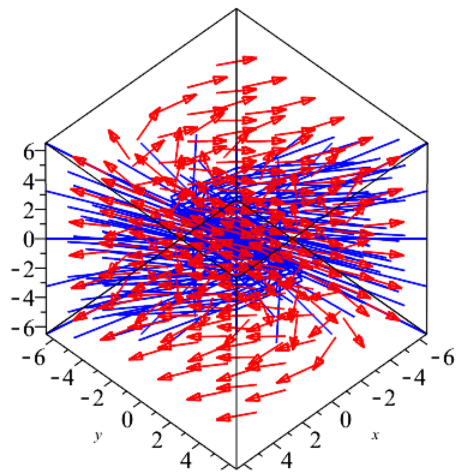
$$\rightarrow \vec{v}_1 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_1(t) = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} e^{7t}$$

For $\lambda_{2,3} = 13 \quad \Rightarrow \quad (A - 13I)V = 0$

$$\begin{pmatrix} 12 & 12 & 0 \\ -18 & -18 & 0 \\ 6 & 6 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad a = -b$$

Let $c = 0 \quad \& \quad a = 1,\ b = -1 \ \rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \Rightarrow \vec{x}_2(t) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{13t}$

Let $c = 1 \quad \Rightarrow \quad a = b = 0 \ \rightarrow \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_3(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{13t}$



The general solution: $\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$

$$\begin{cases} x_1(t) = 2c_1 e^{7t} + c_2 e^{13t} \\ x_2(t) = -3c_1 e^{7t} - c_2 e^{13t} \\ x_3(t) = c_1 e^{7t} \qquad\quad + c_3 e^{13t} \end{cases}$$

Exercise

Find the general solution $\quad \boldsymbol{x}' = \begin{bmatrix} -3 & 0 & -4 \\ -1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \boldsymbol{x}$

Solution

$$|A - \lambda I| = \begin{vmatrix} -3-\lambda & 0 & -4 \\ -1 & -1-\lambda & -1 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (-3-\lambda)(-1-\lambda)(1-\lambda) + 4(-1-\lambda)$$

$$= (-1-\lambda)\left(\lambda^2 + 2\lambda + 1\right)$$

$$= -(\lambda + 1)^3 = 0$$

The eigenvalues are: $\lambda_{1,2,3} = -1 \quad (\textit{multiplicity } 3)$

For $\lambda = -1 \quad \Rightarrow \quad (A + I)V = 0$

$$\begin{pmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad a = -2c \quad \rightarrow V = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

The defect of $\lambda = -1$ is 2.

$$(A+I)^2 = \begin{pmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$(A+I)^2 \vec{v}_3 = 0$, therefore any nonzero vector $\vec{v}_3 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ will be a solution

$$\underline{|\vec{v}_2} = (A+I)\vec{v}_3 = \begin{pmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

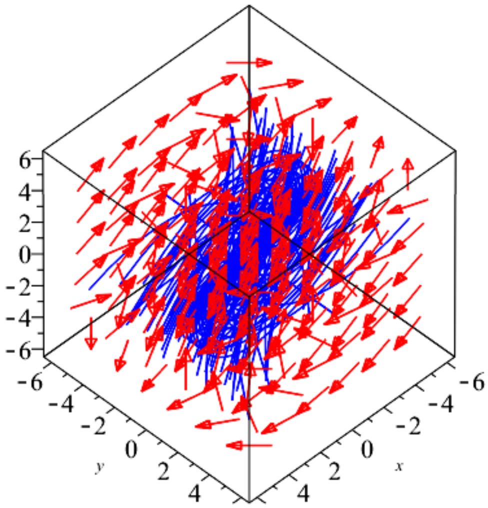
$$\underline{|\vec{v}_1} = (A+I)\vec{v}_2 = \begin{pmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = \vec{v}_1 e^{-t} \\ \vec{x}_2(t) = (\vec{v}_1 t + \vec{v}_2) e^{-t} \\ \vec{x}_3(t) = \left(\frac{1}{2}\vec{v}_1 t^2 + \vec{v}_2 t + \vec{v}_3\right) e^{-t} \end{cases} \rightarrow \begin{cases} \vec{x}_1(t) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-t} \\ \vec{x}_2(t) = \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}\right) e^{-t} \\ \vec{x}_3(t) = \left(\frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t^2 + \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) e^{-t} \end{cases}$$

The general solution:

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

$$\begin{cases} x_1(t) = (-2c_2 + c_3 - 2c_3 t) e^{-t} \\ x_2(t) = \left(\frac{1}{2}c_3 t^2 - (c_3 + c_2) t - c_2 - c_1\right) e^{-t} \\ x_3(t) = (c_2 + c_3 t) e^{-t} \end{cases}$$



Exercise

Find the general solution $\quad \boldsymbol{x}' = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix} \boldsymbol{x}$

Solution

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 0 & 1 \\ 0 & 1-\lambda & -4 \\ 0 & 1 & -3-\lambda \end{vmatrix} = (-3-\lambda)(-1-\lambda)(1-\lambda) + 4(-1-\lambda)$$

$$= (-1-\lambda)\left(\lambda^2 + 2\lambda + 1\right)$$

$$= -(\lambda+1)^3 = 0$$

The eigenvalues are: $\lambda_{1,2,3} = -1 \ \ (multiplicity\ 3)$. The defect of $\lambda = -1$ is 2.

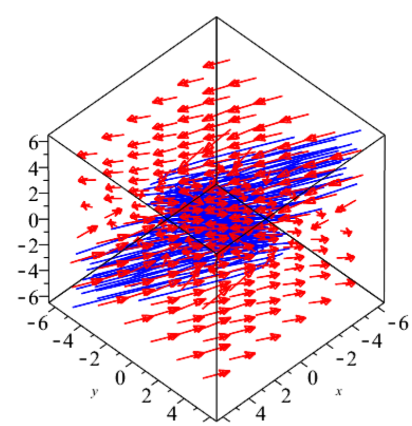
$$(A+I)^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(A+I)^3 = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \to \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\vec{v}_2 = (A+I)\vec{v}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ Contradict the rule $\vec{v}_2 \neq 0$. Then, let assume $\to \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$\vec{v}_2 = (A+I)\vec{v}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \qquad \vec{v}_1 = (A+I)\vec{v}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = \vec{v}_1 e^{-t} \\ \vec{x}_2(t) = \left(\vec{v}_1 t + \vec{v}_2\right) e^{-t} \\ \vec{x}_3(t) = \left(\frac{1}{2}\vec{v}_1 t^2 + \vec{v}_2 t + \vec{v}_3\right) e^{-t} \end{cases} \to \begin{cases} \vec{x}_1(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-t} \\ \vec{x}_2(t) = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}\right) e^{-t} \\ \vec{x}_3(t) = \left(\frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t^2 + \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) e^{-t} \end{cases}$$



The general solution: $\begin{cases} x_1(t) = \left(c_1 + c_2 t + \frac{1}{2} c_3 t^2\right) e^{-t} \\ x_2(t) = \left(2c_2 + c_3 + 2c_3 t\right) e^{-t} \\ x_3(t) = \left(c_2 + c_3 t\right) e^{-t} \end{cases} \qquad \vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$

Exercise

Find the general solution $\quad \boldsymbol{x}' = \begin{bmatrix} 0 & 0 & 1 \\ -5 & -1 & -5 \\ 4 & 1 & -2 \end{bmatrix} \boldsymbol{x}$

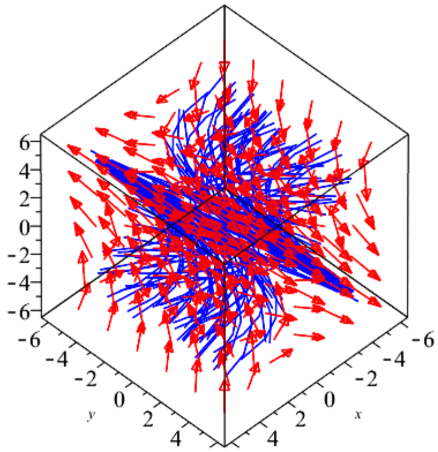
Solution

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 0 & 1 \\ -5 & -1-\lambda & -5 \\ 4 & 1 & -2-\lambda \end{vmatrix} = -\lambda(-1-\lambda)(-2-\lambda)-5-4(-1-\lambda)-5\lambda$$

$$= -\lambda(-1-\lambda)(-2-\lambda)-1-\lambda$$

$$= (-1-\lambda)\left(\lambda^2+2\lambda+1\right)$$

$$= -(\lambda+1)^3 = 0$$



The eigenvalues are: $\lambda_{1,2,3} = -1 \quad (\mathit{multiplicity}\ 3)$

The defect of $\lambda = -1$ is 2.

$$(A+I)^2 = \begin{pmatrix} 1 & 0 & 1 \\ -5 & 0 & -5 \\ 4 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ -5 & 0 & -5 \\ 4 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 0 \\ -25 & -5 & 0 \\ -5 & -1 & 0 \end{pmatrix}$$

$$(A+I)^3 = \begin{pmatrix} 5 & 1 & 0 \\ -25 & -5 & 0 \\ -5 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ -5 & 0 & -5 \\ 4 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rightarrow \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{v}_2 = (A+I)\vec{v}_3 = \begin{pmatrix} 1 & 0 & 1 \\ -5 & 0 & -5 \\ 4 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} \qquad \vec{v}_1 = (A+I)\vec{v}_2 = \begin{pmatrix} 1 & 0 & 1 \\ -5 & 0 & -5 \\ 4 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -25 \\ -5 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = \vec{v}_1 e^{-t} \\ \vec{x}_2(t) = \left(\vec{v}_1 t + \vec{v}_2\right)e^{-t} \\ \vec{x}_3(t) = \left(\frac{1}{2}\vec{v}_1 t^2 + \vec{v}_2 t + \vec{v}_3\right)e^{-t} \end{cases} \rightarrow \begin{cases} \vec{x}_1(t) = \begin{pmatrix} 5 \\ -25 \\ -5 \end{pmatrix} e^{-t} \\ \vec{x}_2(t) = \left(\begin{pmatrix} 5 \\ -25 \\ -5 \end{pmatrix} t + \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}\right) e^{-t} \\ \vec{x}_3(t) = \left(\frac{1}{2}\begin{pmatrix} 5 \\ -25 \\ -5 \end{pmatrix} t^2 + \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) e^{-t} \end{cases}$$

The general solution:

$$\begin{cases} x_1(t) = \left(5c_1 + c_2 + c_3 + 5c_2 t + c_3 t + \frac{5}{2}c_3 t^2\right)e^{-t} \\ x_2(t) = \left(-25c_1 - 5c_2 - 25c_2 t - 5c_3 t - \frac{25}{2}c_3 t^2\right)e^{-t} \\ x_3(t) = \left(-5c_1 + 4c_2 - 5c_2 t + 4c_3 t - \frac{5}{2}c_3 t^2\right)e^{-t} \end{cases} \qquad \vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Exercise

Find the general solution $\quad \boldsymbol{x}' = \begin{bmatrix} 39 & 8 & -16 \\ -36 & -5 & 16 \\ 72 & 16 & -29 \end{bmatrix} \boldsymbol{x}$

Solution

$$\left|A-\lambda I\right| = \begin{vmatrix} 39-\lambda & 8 & -16 \\ -36 & -5-\lambda & 16 \\ 72 & 16 & -29-\lambda \end{vmatrix} = (39-\lambda)(-5-\lambda)(-29-\lambda) + 13032 - 1080\lambda - 9984 + 256\lambda - 8352 - 288\lambda$$

The eigenvalues are: $\lambda_1 = -1, \ \lambda_{2,3} = 3 \ (multiplicity \ 2)$

For $\lambda_1 = -1 \ \Rightarrow \ (A+I)\vec{v}_1 = 0$

$$\begin{pmatrix} 40 & 8 & -16 \\ -36 & -4 & 16 \\ 72 & 16 & -28 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} 5a+b-2c=0 \\ -9a-b+4c=0 \\ 18a+4b-7c=0 \end{matrix} \quad \to \begin{cases} 2a = c \\ 2b = -c \end{cases}$$

$$\to \vec{v}_1 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \quad \Rightarrow \vec{x}_1(t) = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} e^{-t}$$

For $\lambda_{2,3} = 3 \ \Rightarrow \ (A-3I)V = 0$

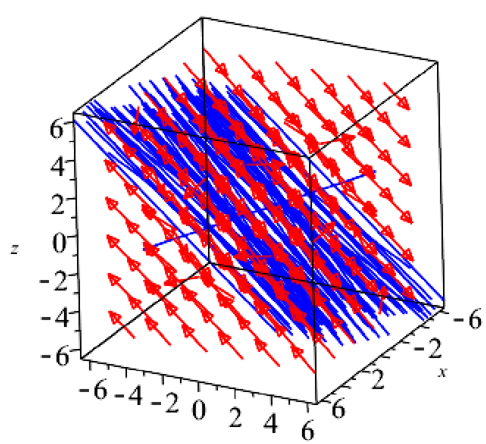
$$\begin{pmatrix} 36 & 8 & -16 \\ -36 & -8 & 16 \\ 72 & 16 & -32 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} 9a+2b-4c=0 \\ -9a-2b+4c=0 \\ 9a+2b-4c=0 \end{matrix} \quad \to \begin{cases} 9a+2b-4c=0 \end{cases}$$

Let $b = 0 \quad \to 9a = 4c \quad a = 4, \ c = 9 \ \to \vec{v}_2 = \begin{pmatrix} 4 \\ 0 \\ 9 \end{pmatrix} \ \Rightarrow \vec{x}_2(t) = \begin{pmatrix} 4 \\ 0 \\ 9 \end{pmatrix} e^{3t}$

Let $c = 0 \quad \to 9a = -2b \quad a = -2, \ b = 9 \ \to \vec{v}_3 = \begin{pmatrix} -2 \\ 9 \\ 0 \end{pmatrix} \ \Rightarrow \vec{x}_3(t) = \begin{pmatrix} -2 \\ 9 \\ 0 \end{pmatrix} e^{3t}$

The general solution: $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$

$$\begin{cases} x_1(t) = 2c_1e^{-t} + 4c_2e^{3t} - 2c_3e^{3t} \\ x_2(t) = -2c_1e^{-t} \qquad\qquad + 9c_3e^{3t} \\ x_3(t) = c_1e^{-t} + 9c_2e^{3t} \end{cases}$$



Exercise

Find the general solution $\quad \boldsymbol{x}' = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \boldsymbol{x}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 0 & 1 \\ 0 & 2-\lambda & 1 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^4 = 0$$

The eigenvalues are: $\lambda_{1,2,3,4} = 2 \ (multiplicity\ 4)$ and defect 3.

$$(A-2I)^2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$$

$$(A-2I)^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$$

$$(A-2I)^4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \to \quad let\ \ \vec{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\underline{|\vec{v}_3} = (A-2I)\vec{v}_4 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\underline{|\vec{v}_2} = (A-2I)\vec{v}_3 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{|\vec{v}_1} = (A-2I)\vec{v}_2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The general solution:

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + c_4\vec{x}_4(t)$$

$$\vec{x}(t) = \left[c_1\vec{v}_1 + c_2\left(\vec{v}_1 t + \vec{v}_2\right) + c_3\left(\tfrac{1}{2}\vec{v}_1 t^2 + \vec{v}_2 t + \vec{v}_3\right) + c_4\left(\tfrac{1}{3!}\vec{v}_1 t^3 + \tfrac{1}{2}\vec{v}_2 t^2 + \vec{v}_3 t + \vec{v}_4\right)\right]e^{2t}$$

$$\begin{cases} x_1(t) = \left(c_1 + c_3 + c_2 t + c_4 t + \frac{1}{2} c_3 t^2 + \frac{1}{6} c_4 t^3\right) e^{2t} \\ x_2(t) = \left(c_2 + c_3 t + \frac{1}{2} c_4 t^2\right) e^{2t} \\ x_3(t) = \left(c_3 + c_4 t\right) e^{2t} \\ x_4(t) = c_4 e^{2t} \end{cases}$$

Exercise

Find the general solution $\quad \boldsymbol{x}' = \begin{bmatrix} -1 & -4 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \boldsymbol{x}$

Solution

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & -4 & 0 & 0 \\ 1 & 3-\lambda & 0 & 0 \\ 1 & 2 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^4 = 0$$

The eigenvalues are: $\lambda_{1,2,3,4} = 1 \ (\textit{multiplicity } 4)$ and defect 2.

$$(A-I)^2 = \begin{pmatrix} -2 & -4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & -4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix} \neq 0$$

$$(A-I)^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & -4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \rightarrow \quad \textit{let } \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{\vec{v}_2} = (A-I)\vec{v}_3 = \begin{pmatrix} -2 & -4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\underline{\vec{v}_1} = (A-I)\vec{v}_2 = \begin{pmatrix} -2 & -4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \textit{let } \vec{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

The general solution:

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t) + c_4 \vec{x}_4(t)$$

$$\vec{x}(t) = \left[c_1 \vec{v}_1 + c_2 \left(\vec{v}_1 t + \vec{v}_2 \right) + c_3 \left(\frac{1}{2} \vec{v}_1 t^2 + \vec{v}_2 t + \vec{v}_3 \right) + c_4 \vec{v}_4 \right] e^t$$

$$\begin{cases} x_1(t) = \left(-2c_2 + c_3 - 2c_3 t\right)e^t \\ x_2(t) = \left(c_2 + c_3 t\right)e^t \\ x_3(t) = \left(c_2 + c_4 + c_3 t\right)e^t \\ x_4(t) = \left(c_1 + c_2 t + \frac{1}{2}c_3 t^2\right)e^t \end{cases}$$

Exercise

The characteristic equation of the coefficient matrix $\boldsymbol{A}$ of the system

$$\boldsymbol{x}' = \begin{bmatrix} 3 & -4 & 1 & 0 \\ 4 & 3 & 0 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{bmatrix}\boldsymbol{x} \qquad \text{is} \quad p(\lambda) = \left(\lambda^2 - 6\lambda + 25\right)^2 = 0$$

Therefore, $\boldsymbol{A}$ has the repeated complex pair $3 \pm 4i$ of eigenvalues. First show that the complex vectors $\vec{v}_1 = \begin{bmatrix}1 & i & 0 & 0\end{bmatrix}^T$ *and* $\vec{v}_2 = \begin{bmatrix}0 & 0 & 1 & i\end{bmatrix}^T$ form a length 2 chain $\left\{\vec{v}_1, \vec{v}_2\right\}$ associated with the eigenvalue $\lambda = 3 - 4i$. Then calculate the real and imaginary parts of the complex-valued solutions

$$\vec{v}_1 e^{\lambda t} \quad \text{and} \quad \left(\vec{v}_1 t + \vec{v}_2\right)e^{\lambda t}$$

To find four independent real-valued solutions of $\boldsymbol{x}' = A\boldsymbol{x}$

Solution

For $\lambda = 3 - 4i \quad \Rightarrow \quad \left(A - (3-4i)I\right)\vec{v}_1 = 0$

$$A - \lambda I = \begin{pmatrix} 4i & -4 & 1 & 0 \\ 4 & 4i & 0 & 1 \\ 0 & 0 & 4i & -4 \\ 0 & 0 & 4 & 4i \end{pmatrix}$$

$$(A - \lambda I)^2 = \begin{pmatrix} 4i & -4 & 1 & 0 \\ 4 & 4i & 0 & 1 \\ 0 & 0 & 4i & -4 \\ 0 & 0 & 4 & 4i \end{pmatrix}\begin{pmatrix} 4i & -4 & 1 & 0 \\ 4 & 4i & 0 & 1 \\ 0 & 0 & 4i & -4 \\ 0 & 0 & 4 & 4i \end{pmatrix} = \begin{pmatrix} -32 & -32i & 8i & -8 \\ 32i & -32 & 8 & 8i \\ 0 & 0 & -32 & -32i \\ 0 & 0 & 32i & -32 \end{pmatrix} \to \begin{pmatrix} -4 & -4i & i & -1 \\ 4i & -4 & 1 & i \\ 0 & 0 & -4 & -4i \\ 0 & 0 & 4i & -4 \end{pmatrix}$$

$$\begin{pmatrix} -4 & -4i & i & -1 \\ 4i & -4 & 1 & i \\ 0 & 0 & -4 & -4i \\ 0 & 0 & 4i & -4 \end{pmatrix} \xrightarrow{R_2 + iR_1} \begin{pmatrix} -4 & -4i & i & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & -4i \\ 0 & 0 & 4i & -4 \end{pmatrix} \xrightarrow[\frac{1}{4}R_4]{-\frac{1}{4}R_3} \begin{pmatrix} -4 & -4i & i & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & i & -1 \end{pmatrix}$$

$$\begin{pmatrix} -4 & -4i & i & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & i & -1 \end{pmatrix} \xrightarrow[R_4 - iR_3]{R_1 - iR_3} \begin{pmatrix} -4 & -4i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-\frac{1}{4}R_1} \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \vec{v}_1 = \begin{bmatrix}1 & i & 0 & 0\end{bmatrix}^T$ *and* $\vec{v}_2 = \begin{bmatrix}0 & 0 & 1 & i\end{bmatrix}^T$

$$\vec{x}_1 = \vec{v}_1 e^{(3-4i)t} \quad and \quad \vec{x}_2 = \left(\vec{v}_1 t + \vec{v}_2\right) e^{(3-4i)t} \qquad\qquad e^{\alpha t i} = cis\alpha t$$

$$\vec{x}_1 = \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} e^{-4t} e^{3t} = \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} (\cos 4t - i\sin 4t) e^{3t} = \begin{pmatrix} \cos 4t - i\sin 4t \\ \sin 4t + i\cos 4t \\ 0 \\ 0 \end{pmatrix} e^{3t}$$

$$\vec{x}_2 = \left(\begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 0 \\ 1 \\ i \end{pmatrix} \right) e^{-4t} e^{3t} = \begin{pmatrix} t \\ ti \\ 1 \\ i \end{pmatrix} (\cos 4t - i\sin 4t) e^{3t} = \begin{pmatrix} t\cos 4t - it\sin 4t \\ t\sin 4t + it\cos 4t \\ \cos 4t - i\sin 4t \\ \sin 4t + i\cos 4t \end{pmatrix} e^{3t}$$

The general solution:

$$x_1(t) = \begin{pmatrix} \cos 4t \\ \sin 4t \\ 0 \\ 0 \end{pmatrix} e^{3t} \qquad x_2(t) = \begin{pmatrix} -\sin 4t \\ \cos 4t \\ 0 \\ 0 \end{pmatrix} e^{3t}$$

$$x_3(t) = \begin{pmatrix} t\cos 4t \\ t\sin 4t \\ \cos 4t \\ \sin 4t \end{pmatrix} e^{3t} \qquad x_4(t) = \begin{pmatrix} -t\sin 4t \\ t\cos 4t \\ -\sin 4t \\ \cos 4t \end{pmatrix} e^{3t}$$