Chapter 8. An Introduction to Systems

Section 8.1. Definitions and Examples

1. The system

$$x' = v$$
$$v' = -x - 0.02v + 2\cos t,$$

has dependent variables (unknowns) x and v. Therefore, the dimension is 2. The right side depends explicitly on the independent variable t, so the system is nonautonomous.

2. The dimension is 2 since there are two equations and two unknown functions. The system is autonomous since there is no explicit dependence of the right hand sides on the independent variable.

3. The system

$$x' = -ax + ay$$
$$y' = rx - y - xz$$
$$z' = -bz + xy,$$

has dependent variables (unknowns) x, y, and z. Therefore, the dimension is 3. The right side does not depend explicitly on t, so the system is autonomous.

4. The dimension is 3 since there are three equations and three unknown functions. The system is autonomous since there is no explicit dependence of the right hand sides on the independent variable.

5. The system

$$u_1' = u_2$$
$$u_2' = -\frac{1}{2}u_1 + \frac{1}{2}u_3$$
$$u_3' = u_4$$
$$u_4' = \frac{3}{2}u_1 + \frac{1}{2}u_3,$$

has dependent variables (unknowns) u_1, u_2, u_3 and u_4. Therefore, the dimension is 4. The right side does not depend explicitly on t, so the system is autonomous.

6. The system

$$u_1' = u_2$$
$$u_2' = -\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_1}u_3$$
$$u_3' = u_4$$
$$u_4' = \frac{k_2}{m_2}u_1 - \frac{k_2 + k_3}{m_2}u_3 + \cos t$$

has dependent variables u_1, u_2, u_3, and u_4. Therefore, the dimension is 4. The right sides explicitly reference the independent variable t, so the system is not autonomous.

7. If $x(t) = 2e^{2t} - 2e^{-t}$ and $y(t) = -e^{-t} + 2e^{2t}$, then

$$x' = (2e^{2t} - 2e^{-t})' = 4e^{2t} + 2e^{-t},$$

and

$$-4x + 6y = -4(2e^{2t} - 2e^{-t}) + 6(-e^{-t} + 2e^{2t})$$
$$= 4e^{2t} + 2e^{-t},$$

so the first equation is satisfied. Further,

$$y' = (-e^{-t} + 2e^{2t})' = e^{-t} + 4e^{2t},$$

and

$$-3x + 5y = -3(2e^{2t} - 2e^{-t}) + 5(-e^{-t} + 2e^{2t})$$
$$= 4e^{2t} + e^{-t},$$

so the second equation is satisfied. Finally,

$$x(0) = 2e^{2(0)} - 2e^{-0} = 0$$
$$y(0) = -e^{-0} + 2e^{2(0)} = 1,$$

so the initial conditions are also satisfied.

8. If $x(t) = (1+t)e^{-t}$ and $y(t) = -te^{-t}$, then

$$\begin{aligned} x' &= ((1+t)e^{-t})' \\ &= -(1+t)e^{-t} + e^{-t} \\ &= -te^{-t} \\ &= y, \end{aligned}$$

so the first equation is satisfied. Further,

$$y' = (-te^{-t})' = te^{-t} - e^{-t},$$

and

$$\begin{aligned} -x - 2y &= -(1+t)e^{-t} - 2(-te^{-t}) \\ &= te^{-t} - e^{-t}, \end{aligned}$$

and the second equation is satisfied. Finally,

$$\begin{aligned} x(0) &= (1+0)e^{-0} = 1 \\ y(0) &= -0e^{-0} = 0, \end{aligned}$$

and the initial conditions are satisfied.

9. If $x(t) = e^{-t}(-\cos t - \sin t)$ and $v(t) = 2e^{-t}\sin t$, then

$$\begin{aligned} x' &= (e^{-t}(-\cos t - \sin t))' \\ &= e^{-t}(\sin t - \cos t) - e^{-t}(-\cos t - \sin t) \\ &= 2e^{-t}\sin t \\ &= v, \end{aligned}$$

and the first equation is satisfied. Further,

$$v' = (2e^{-t}\sin t)' = 2e^{-t}\cos t - 2e^{-t}\sin t,$$

and

$$\begin{aligned} -2x - 2v &= -2(e^{-t}(-\cos t - \sin t)) \\ &\quad - 2(2e^{-t}\sin t) \\ &= 2e^{-t}\cos t - 2e^{-t}\sin t, \end{aligned}$$

so the second equation is satisfied. Finally,

$$\begin{aligned} x(0) &= e^{-0}(-\cos 0 - \sin 0) = -1 \\ v(0) &= 2e^{-0}\sin 0 = 0, \end{aligned}$$

so the initial conditions are satisfied.

10. If $x(t) = e^t$ and $y(t) = e^{-t}$, then

$$x' = (e^t)' = e^t,$$

and

$$x^2 y = (e^t)^2(e^{-t}) = e^t,$$

and the first equation is satisfied. Further,

$$y' = (e^{-t})' = -e^{-t},$$

and

$$-xy^2 = -e^{-t}(e^{-t})^2 = -e^{-t},$$

so the second equation is satisfied. Finally, $x(0) = e^0 = 1$ and $y(0) = e^{-0} = 1$, so the initial conditions are satisfied.

11. We know that $y'' = -2y' - 4y + 3\cos 2t$, $y(0) = 1$, $y'(0) = 0$. If we let $u_1 = y$ and $u_2 = y'$, then

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= -2u_2 - 4u_1 + 3\cos 2t. \end{aligned}$$

Hence with $\mathbf{u} = (u_1, u_2)^T$,

$$\mathbf{u}' = \begin{pmatrix} u_2 \\ -2u_2 - 4u_1 + 3\cos 2t \end{pmatrix}.$$

Furthermore $\mathbf{u}(0) = (u_1(0), u_2(0))^T = (y(0), y'(0))^T = (1, 0)^T$.

12. We know that $x'' = -(\mu/m)x' - (k/m)x + F_0\cos\omega t$, $x(0) = x_0$, $x'(0) = v_0$. Hence, with $u_1 = x$, and $u_2 = x'$,

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= -\frac{\mu}{m}u_2 - \frac{k}{m}u_1 + F_0\cos\omega t. \end{aligned}$$

Therefore, with $\mathbf{u} = (u_1, u_2)^T$,

$$\mathbf{u}' = \begin{pmatrix} u_2 \\ -(\mu/m)u_2 - (k/m)u_1 + F_0\cos\omega t \end{pmatrix}.$$

Furthermore, $\vec{u}(0) = (u_1(0), u_2(0))^T = (x(0), x'(0))^T = (x_0, v_0)^T$.

13. We know that $x'' = -\delta x' + x - x^3 + \gamma\cos\omega t$, $x(0) = x_0$, $x'(0) = v_0$. If we let $u_1 = x$ and $u_2 = x'$, then

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= -\delta u_2 + u_1 - u_1^3 + \gamma\cos\omega t. \end{aligned}$$

Furthermore, $u_1(0) = x(0) = x_0$ and $u_2(0) = x'(0) = v_0$.

14. Set $u_1 = x$ and $u_2 = x'$. Then

$$u_1' = x' = u_2$$
$$u_2' = x'' = -\mu(x^2 - 1)x' - x$$
$$= -\mu(u_1^2 - 1)u_2 - u_1.$$

Hence with $\mathbf{u} = (u_1, u_2)^T$,

$$\mathbf{u}' = \begin{pmatrix} u_2 \\ -\mu(u_1^2 - 1)u_2 - u_1 \end{pmatrix}.$$

Furthermore $\mathbf{u}(0) = (u_1(0), u_2(0))^T = (x(0), x'(0))^T = (x_0, v_0)^T$.

15. We know that $\omega''' = \omega$, $\omega(0) = \omega_0$, $\omega'(0) = \alpha_0$, and $\omega''(0) = \gamma_0$. If we let $u_1 = \omega$, $u_2 = \omega'$, and $u_3 = \omega''$, then

$$u_1' = u_2$$
$$u_2' = u_3$$
$$u_3' = u_1.$$

Further, $u_1(0) = \omega(0) = \omega_0$, $u_2(0) = \omega'(0) = \alpha_0$, and $u_3(0) = \omega''(0) = \gamma_0$.

16. We know that $y''' = -y'y'' + \sin \omega t$, $y(0) = \alpha$, $y'(0) = \beta$, and $y''(0) = \gamma$. If we let $u_1 = y$, $u_2 = y'$, and $u_3 = y''$, then

$$u_1' = u_2$$
$$u_2' = u_3$$
$$u_3' = -u_2 u_3 + \sin \omega t.$$

With $\mathbf{u} = (u_1, u_2, u_3)^T$,

$$\mathbf{u}' = \begin{pmatrix} u_2 \\ u_3 \\ -u_2 u_3 + \sin \omega t \end{pmatrix}.$$

Furthermore, $\vec{u}(0) = (u_1(0), u_2(0), u_3(0))^T = (y(0), y'(0), y''(0))^T = (\alpha, \beta, \gamma)^T$.

17. The right side of the system

$$u' = v$$
$$v' = -3u - 2v + 5\cos t$$

depends explicitly on the independent variable t. Therefore, the system is nonautonomous.

18. The right side of the system

$$u' = v(u^2 + v^2)$$
$$v' = -v(u^2 + v^2)$$

makes no explicit mention of the independent variable. Therefore, the system is autonomous.

19. The right side of the system

$$u' = v \cos u$$
$$v' = tv$$

depends explicitly on the independent variable t. Therefore, the system is nonautonomous.

20. The right-hand side in the system

$$u' = v \cos u$$
$$v' = u^2 e^u$$

makes no explicit mention of the independent variable. Therefore, the system is autonomous.

21. In the system

$$u' = v + \cos u$$
$$v' = v - t\omega$$
$$\omega' = 5u - 9v + 8\omega,$$

the right side depends explicitly on the independent variable t. Therefore, the system is nonautonomous.

22. The right-hand side of the system

$$u' = w + \cos u - 2v$$
$$v' = u^2 e^v$$
$$w' = u + v + w$$

makes no explicit mention of the independent variable. Therefore, the system is autonomous.

23. Let $x_1 = u$, $x_2 = v$. If we let $\mathbf{x} = (x_1, x_2)'$, then the system

$$x_1' = x_2$$
$$x_2' = -3x_1 - 2x_2 + 5\cos t$$

can be written

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}),$$

where

$$\mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \quad \text{and}$$

$$\mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} x_2 \\ -3x_1 - 2x_2 + 5\cos t \end{pmatrix}.$$

24. Let $x_1 = u$, $x_2 = v$. If we let $\mathbf{x} = (x_1, x_2)'$, then the system

$$x_1' = x_1(x_1^2 + x_2^2)$$
$$x_2' = -x_2(x_1^2 + x_2^2)$$

can be written

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}),$$

where

$$\mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \quad \text{and} \quad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_1(x_1^2 + x_2^2) \\ -x_2(x_1^2 + x_2^2) \end{pmatrix}.$$

25. Let $x_1 = u$, $x_2 = v$. If we let $\mathbf{x} = (x_1, x_2)'$, then the system

$$x_1' = x_2 \cos x_1$$
$$x_2' = t x_2$$

can be written

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}),$$

where

$$\mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \quad \text{and} \quad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} x_2 \cos x_1 \\ t x_2 \end{pmatrix}.$$

26. Let $x_1 = u$ and $x_2 = v$. If we let $\mathbf{x} = (x_1, x_2)^T$, then the system

$$x_1' = x_2 \cos x_1$$
$$x_2' = x_1^2 e^{x_1}$$

can be written

$$\mathbf{x}' = f(\mathbf{x}),$$

where

$$\mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \quad \text{and} \quad f(\mathbf{x}) = \begin{pmatrix} x_2 \cos x_1 \\ x_1^2 e^{x_1} \end{pmatrix}.$$

27. Let $x_1 = u$, $x_2 = v$, and $x_3 = \omega$. If we let $\mathbf{x} = (x_1, x_2, x_3)'$, then the system

$$x_1' = x_2 + \cos x_1$$
$$x_2' = x_2 - t x_3$$
$$x_3' = 5x_1 - 9x_2 + 8x_3$$

can be written

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}),$$

where

$$\mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} \quad \text{and} \quad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} x_2 + \cos x_1 \\ x_2 - t x_3 \\ 5x_1 - 9x_2 + 8x_3 \end{pmatrix}.$$

28. Let $x_1 = u$, $x_2 = v$, and $x_3 = w$. If we let $\vec{x} = (x_1, x_2, x_3)^T$, then the system

$$x_1' = x_3 + \cos x_1 - 2x_2$$
$$x_2' = x_1^2 e^{x_2}$$
$$x_3' = x_1 + x_2 + x_3$$

can be written

$$\mathbf{x}' = f(\mathbf{x}),$$

where

$$\mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} \quad \text{and} \quad f(\mathbf{x}) = \begin{pmatrix} x_3 + \cos x_1 - 2x_2 \\ x_1^2 e^{x_2} \\ x_1 + x_2 + x_3 \end{pmatrix}.$$

29. Let $x_1 = S$, $x_2 = I$, and $x_3 = R$. If we let $\mathbf{x} = (x_1, x_2, x_3)^T$, then the system

$$x_1' = -x_1 x_2$$
$$x_2' = x_1 x_2 - x_2$$
$$x_3' = x_1$$

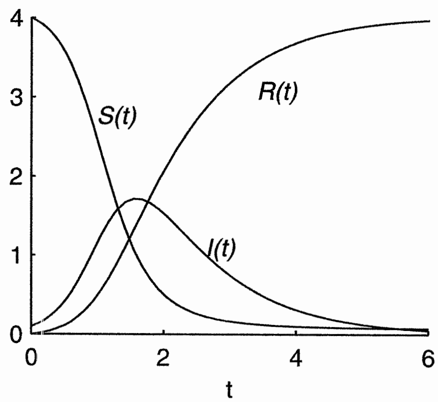
can be written

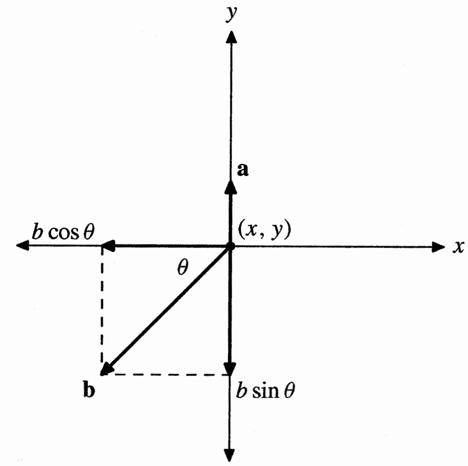
$$\mathbf{x} = f(\mathbf{x}),$$

where

$$\mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} \quad \text{and} \quad f(\mathbf{x}) = \begin{pmatrix} -x_1 x_2 \\ x_1 x_2 - x_2 \\ x_1 \end{pmatrix}.$$

Furthermore, $(x_1(0), x_2(0), x_3(0))^T = (S(0), I(0), R(0))^T = (4, 0, 1, 0)^T$. These equations can be used to produce the following component solutions, $x_1(t) = S(t)$, $x_2(t) = I(t)$, and $x_3(t) = R(t)$.





30. To normalize quantities in the SIR model, let $s = S/N$, $i = I/N$, and $r = R/N$. Starting with

$$\begin{aligned} S' &= -aSI \\ I' &= aSI - bI \\ R' &= bI, \end{aligned}$$

and substituting,

$$\begin{aligned} (sN)' &= -a(sN)(iN) \\ (iN)' &= a(sN)(iN) - b(iN) \\ (rN)' &= b(iN), \end{aligned}$$

or,

$$\begin{aligned} s' &= -asiN \\ i' &= asiN - bi \\ r' &= bi. \end{aligned}$$

31. First, break each velocity vector into vertical and horizontal components, letting b represent the magnitude of the vector **b**.

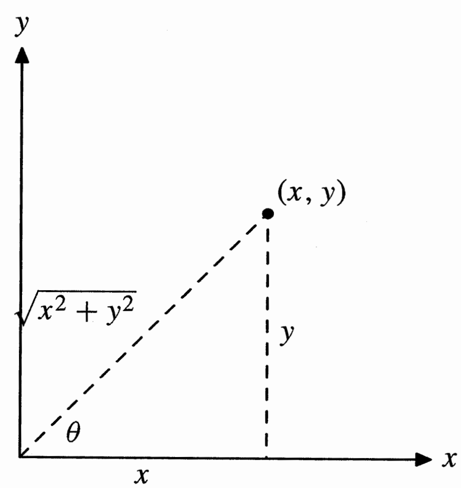
As you can see in the figure, the velocity in the x-direction is $-b\cos\theta$. Thus,

$$\frac{dx}{dt} = -b\cos\theta.$$

In the y-direction, we resolve two components.

$$\frac{dy}{dt} = a - b\sin\theta$$

A second image,



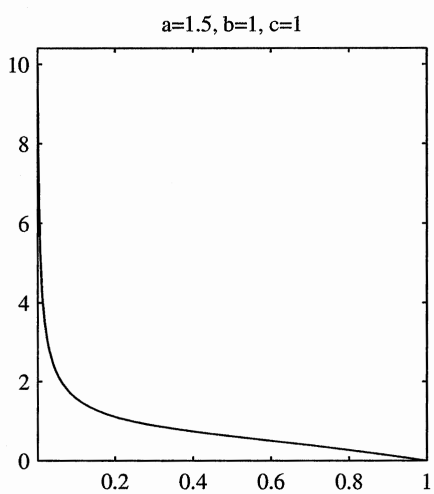
gives us

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$
$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}.$$

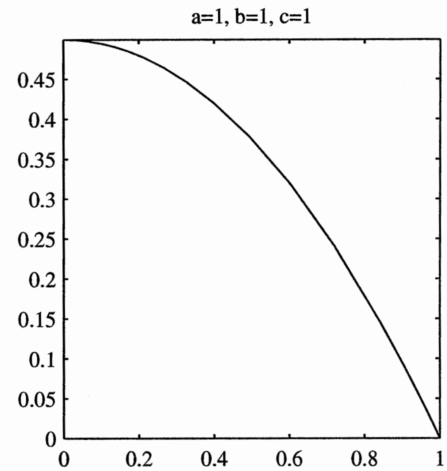
Substituting,

$$\frac{dx}{dt} = \frac{-bx}{\sqrt{x^2+y^2}}$$
$$\frac{dy}{dt} = a - \frac{by}{\sqrt{x^2+y^2}}.$$

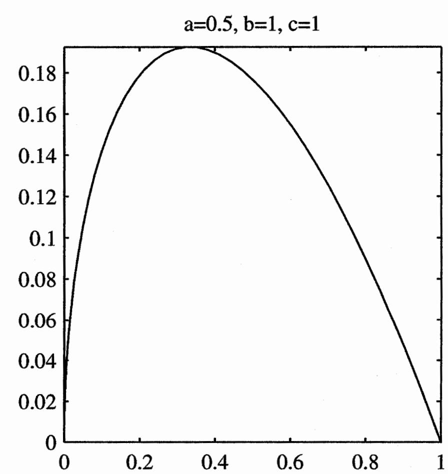
32. Here are the interesting cases. In the first figure, the speed of the current is greater than that of the boat. As a result, the boat drifts downstream.



In the second case, the boat speed and current speed are equal. The boat eventually makes it to shore, but a significant distance downstream from its destination.

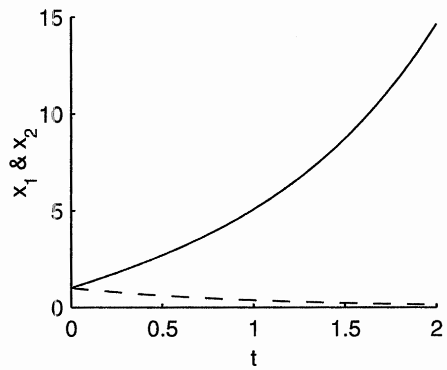


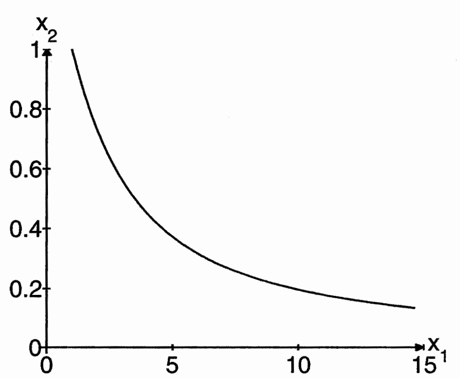
The last case, with the boat speed greater than the current speed, was tricky. We managed to get our variable step solver to stop by solving on the time interval [0, 1.3]. Answers will vary on different systems.

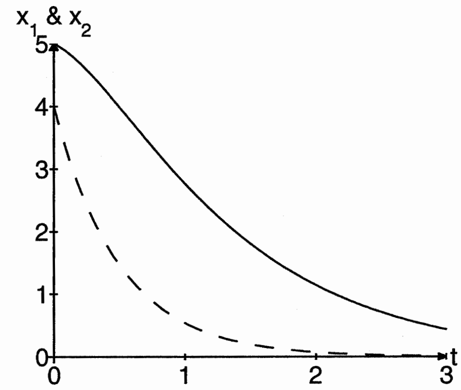


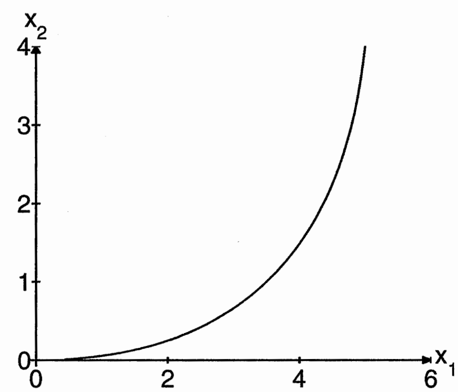
Section 8.2. Geometric Interpretation of Solutions

1. The first figure is a plot of $x_1(t) = 2e^t - e^{-t}$ (the solid curve) and $x_2(t) = e^{-t}$ (the dashed curve) versus t on the time interval $[0, 2]$. The second figure uses the same time interval, but it is a plot of x_2 versus x_1 in the phase plane.



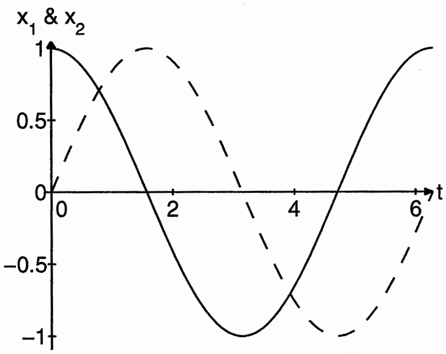


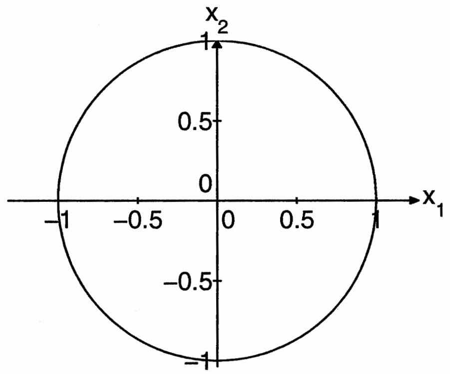




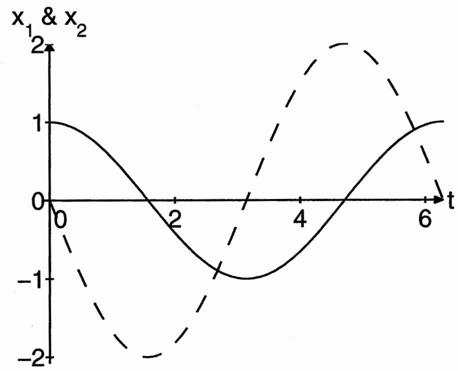
2. The first figure is a plot of $x_1(t) = 9e^{-t} - 4e^{-2t}$ (the solid curve) and $x_2(t) = 4e^{-2t}$ (the dashed curve) versus t on the time interval $[0, 3]$. The second figure uses the same time interval, but it is a plot of x_2 versus x_1 in the phase plane.

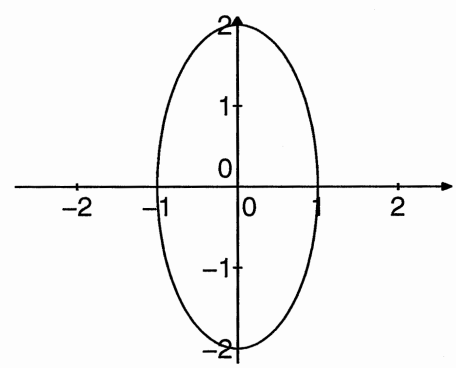
3. The first figure is a plot of $x_1(t) = \cos t$ (the solid curve) and $x_2(t) = \sin t$ (the dashed curve) versus t on the time interval $[0, 2\pi]$. The second figure uses the same time interval, but it is a plot of x_2 versus x_1 in the phase plane.



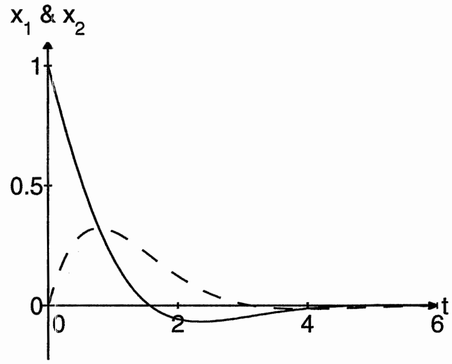


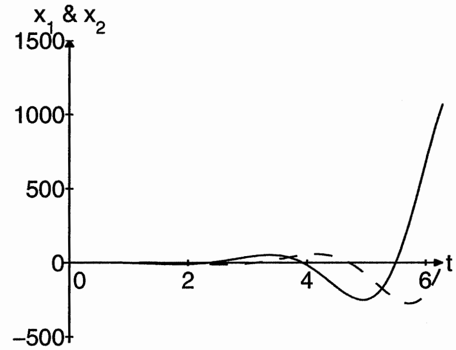
4. The first figure is a plot of $x_1(t) = \cos t$ (the solid curve) and $x_2(t) = -2\sin t$ (the dashed curve) versus t on the time interval $[0, 2\pi]$. The second figure uses the same time interval, but it is a plot of x_2 versus x_1 in the phase plane.

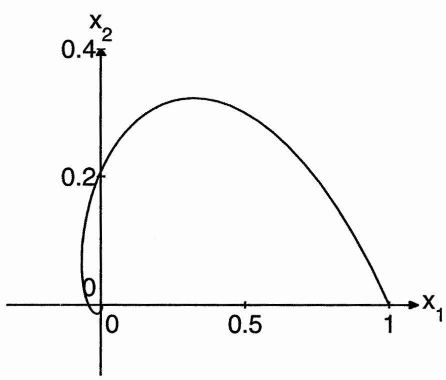


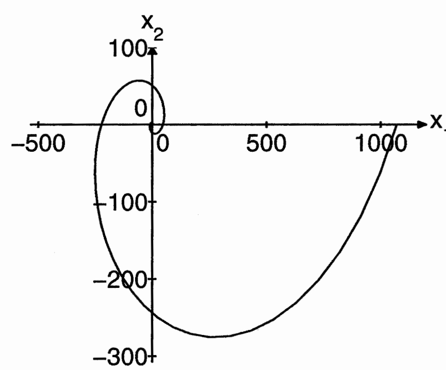


5. The first figure is a plot of $x_1(t) = e^{-t}\cos t$ (the solid curve) and $x_2(t) = e^{-t}\sin t$ (the dashed curve) versus t on the time interval $[0, 2\pi]$. The second figure uses the same time interval, but it is a plot of x_2 versus x_1 in the phase plane.



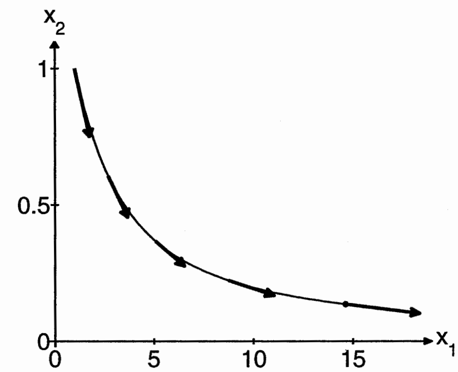




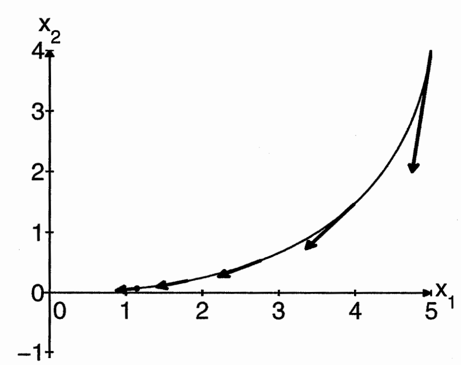


6. The first figure is a plot of $x_1(t) = 2e^t \cos 2t$ (the solid curve) and $x_2(t) = e^t \sin 2t$ (the dashed curve) versus t on the time interval $[0, 2\pi]$. The second figure right uses the same time interval, but it is a plot of x_2 versus x_1 in the phase plane.

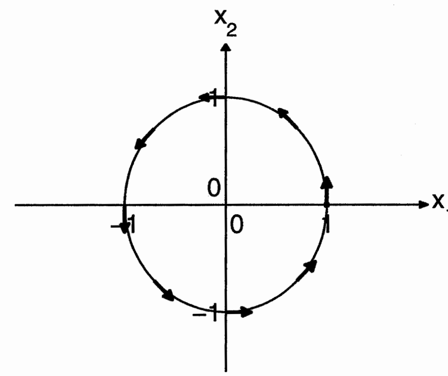
7. If $\mathbf{x}(t) = (2e^t - e^{-t}, e^{-t})$, then $\mathbf{x}'(t) = (2e^t + e^{-t}, -e^{-t})$. In the figure that follows, the derivative was used to plot vectors tangent to the curve at selected points. Tangent vectors are plotted at 25% of their actual length.



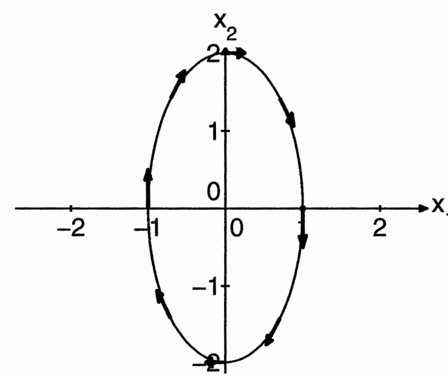
8. If $\mathbf{x}(t) = (9e^{-t} - 4e^{-2t}, 4e^{-2t})$, then $\mathbf{x}'(t) = (-9e^{-t} + 8e^{-2t}, -8e^{-2t})$. In the figure that follows, the derivative was used to plot vectors tangent to the curve at selected points. Tangent vectors are plotted at 25% of their actual length.



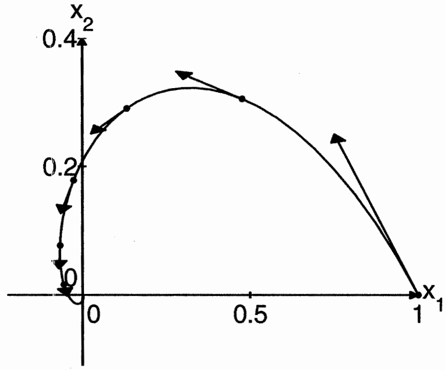
9. If $\mathbf{x}(t) = (\cos t, \sin t)$, then $\mathbf{x}'(t) = (-\sin t, \cos t)$. In the figure that follows, the derivative was used to plot vectors tangent to the curve at selected points. Tangent vectors are plotted at 25% of their actual length.



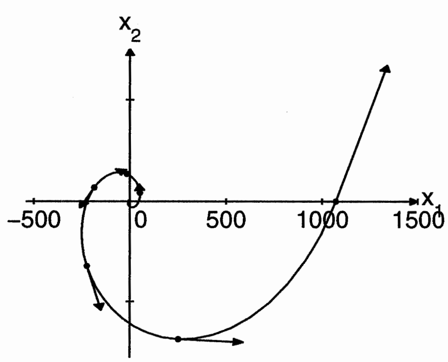
10. If $\mathbf{x}(t) = (\cos t, -2\sin t)$, then $\mathbf{x}'(t) = (-\sin t, -2\cos t)$. In the figure that follows, the derivative was used to plot vectors tangent to the curve at selected points. Tangent vectors are plotted at 25% of their actual length.



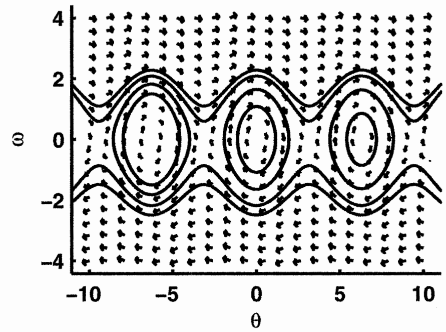
11. If $\mathbf{x}(t) = (e^{-t}\cos t, e^{-t}\sin t)$, then $\mathbf{x}'(t) = (e^{-t}(-\sin t - \cos t), e^{-t}(\cos t - \sin t))$. In the figure that follows, the derivative was used to plot vectors tangent to the curve at selected points. Tangent vectors are plotted at 25% of their actual length.



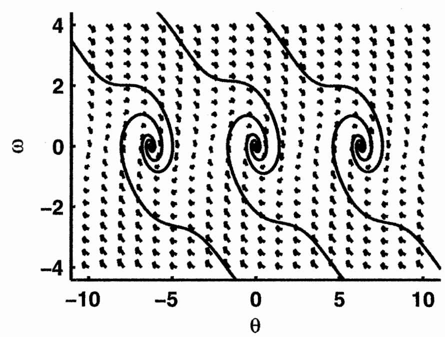
12. If $\mathbf{x}(t) = (2e^t \cos 2t, e^t \sin 2t)$, then $\mathbf{x}'(t) = (e^t(-4\sin 2t + 2\cos 2t), e^t(2\cos 2t + \sin 2t))$. In the figure that follows, the derivative was used to plot vectors tangent to the curve at selected points. Tangent vectors are plotted at 25% of their actual length.



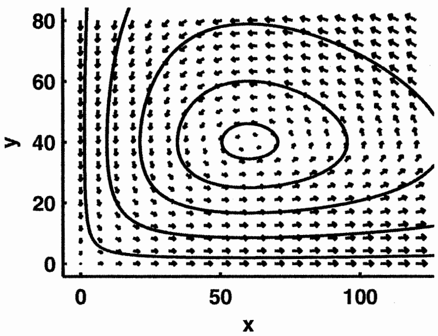
13. The directions and several solution trajectories for $\theta' = \omega$ and $\omega' = -\sin\theta$ follow.



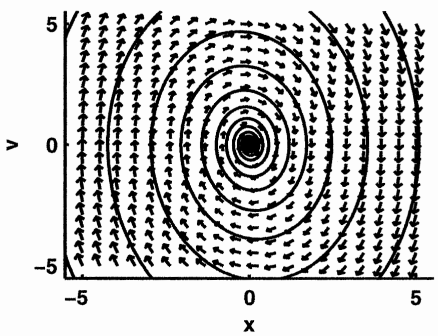
14. The directions and several solution trajectories for $\theta' = \omega$ and $\omega' = -\sin\theta - 0.5\omega$ follow.



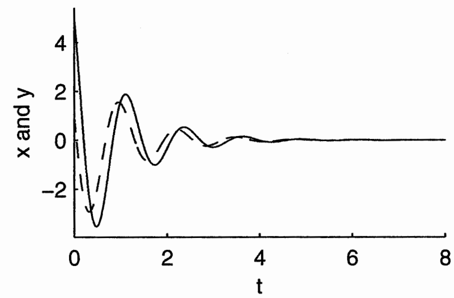
15. The directions and several solution trajectories for $x' = (0.4 - 0.01y)x$ and $y' = (0.005x - 0.3)y$ follow.

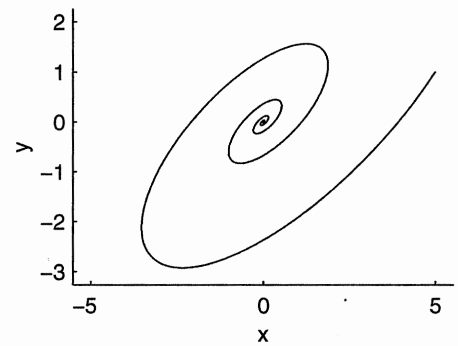


16. The directions and several solution trajectories $x' = v$ and $v' = -3x - 0.2v$ follow.

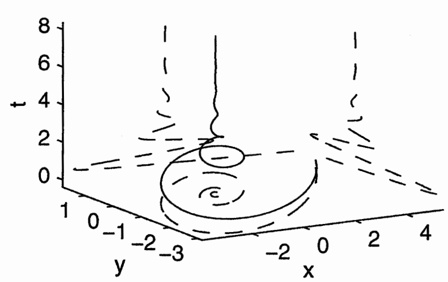


17. In the first figure, component plots, with x the solid curve and y the dashed curve. In the second figure, the solution in the phase plane.

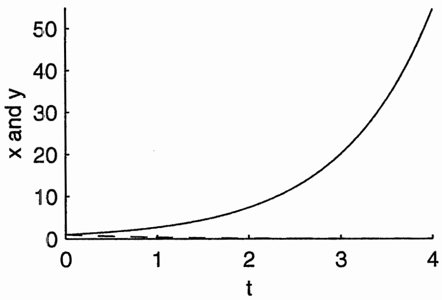


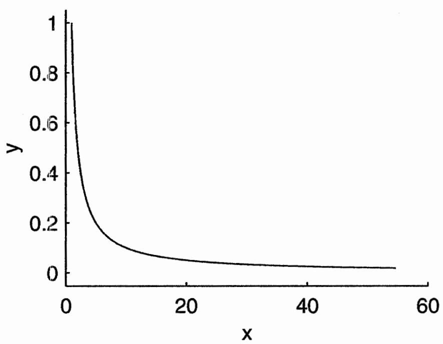


Finally, a composite plot, with the 3D plot solid and the others dashed.

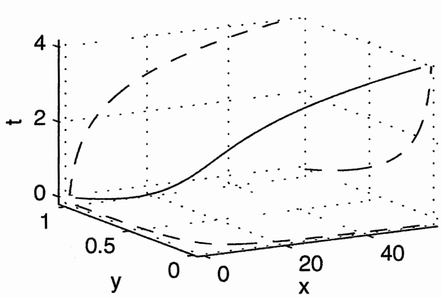


18. In the first figure, component plots, with x the solid curve and y the dashed curve. In the second figure, the solution in the phase plane.

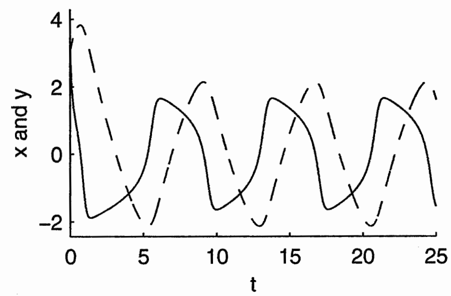


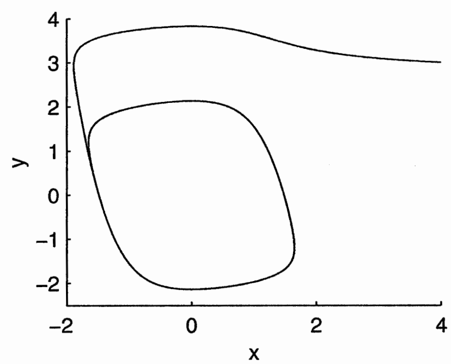


Finally, a composite plot, with the 3D plot solid and the others dashed.

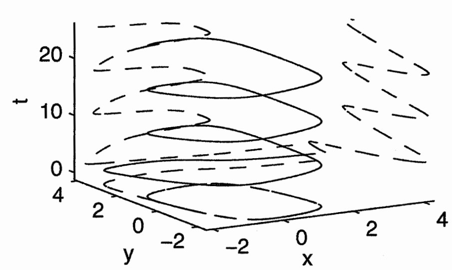


19. In the first figure, component plots, with x the solid curve and y the dashed curve. In the second figure, the solution in the phase plane.

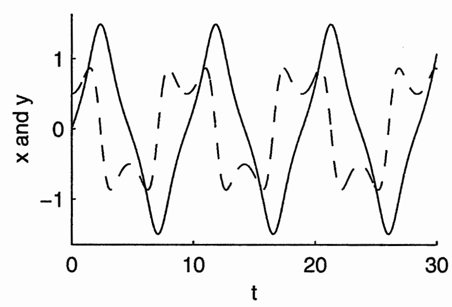


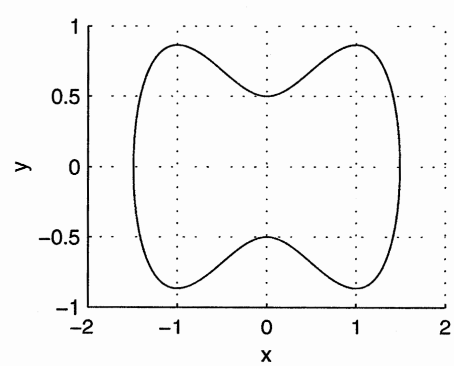


Finally, a composite plot, with the 3D plot solid and the others dashed.

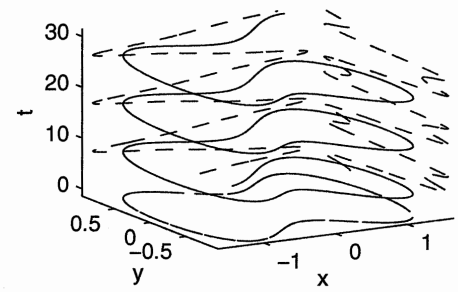


20. In the first figure, component plots, with x the solid curve and y the dashed curve. In the second figure, the solution in the phase plane.



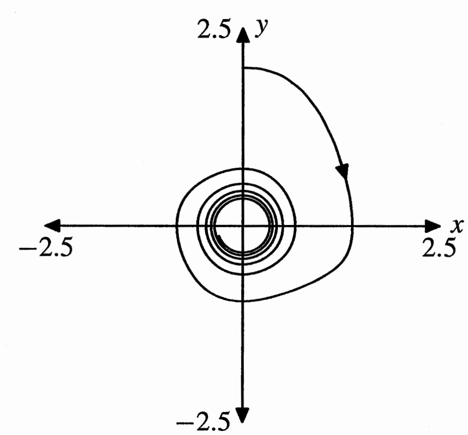


Finally, a composite plot, with the 3D plot solid and the others dashed.

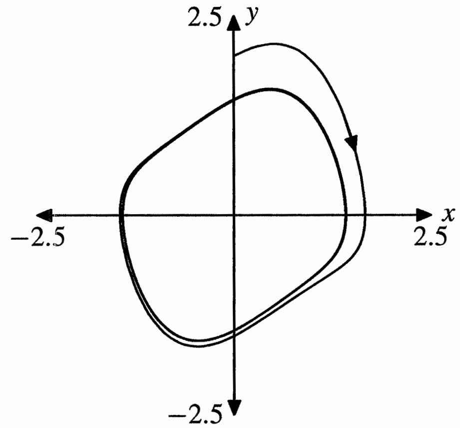


21. In I and IV, x and y exhibit oscillatory behavior. In I, the oscillatory behavior decays, while in IV, the oscillatory behavior eventually becomes periodic. Thus, I $\mapsto$ D and IV $\mapsto$ C. In II and III, one component decays to zero while the other component levels asymptotically to some value. In II, y decays to zero, so II $\mapsto$ A. But, in III, x decays to zero, so III $\mapsto$ B.

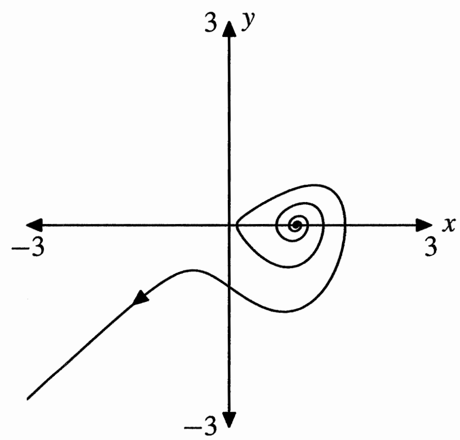
22. Initially, $x(0) = 0$ and $y(0) = 2$, then y decays as x increases, thereafter both x and y oscillate as they decay toward zero.



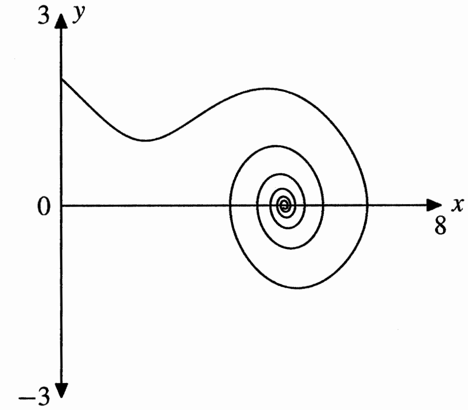
23. Initially, $x(0) = 0$ and $y(0) = 2$. Shortly thereafter, y decays as x increases. Soon, both x and y begin a seemingly periodic motion.



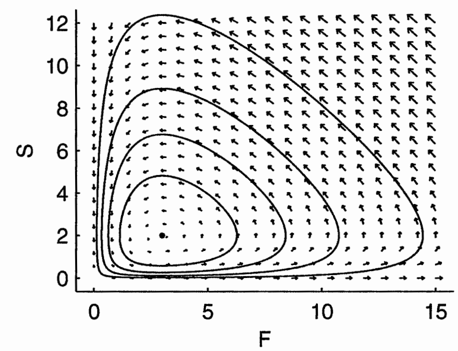
24. At first, x oscillates mildly about 1, while y oscillates mildly about zero. This would indicate a turning about $(1, 0)$ in the phase plane. The oscillations grow larger until both x and y shoot off to $-\infty$. One possible solution follows.



25. Initially, $x(0) = 0$ and $y(0) = 2$. Thereafter, x increases rapidly, then decays asymptotically in an oscillatory manner to about 5 or 6. Meanwhile, y decays, eventually oscillating about zero. One possible solution follows.



26. (a) The solutions appear to be closed, indicating a periodic oscillation of the two species, which eventually return to starting levels.



(b) The solution starting at $(3, 2)$ remains at $(3, 2)$ for all time. This solution, as we shall soon see, is called an equilibrium solution.

27. If the prey population grows according to the logistic model, then

$$F' = r\left(1 - \frac{F}{K}\right)F.$$

Expanding, $F = rF - rF^2/K$, and replacing r with a and r/K with e,

$$F' = aF - eF^2.$$

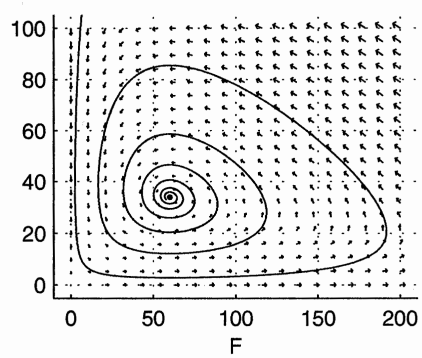
Now, because interaction between species is harmful to the prey,

$$F' = aF - eF^2 - bFS.$$

If predators still follow the Malthusian model, and interactions between species is still considered beneficial to the predators, then

$$S' = -cS + dFS.$$

28. (a) The tricky part on this problem is finding a good viewing window. After some experimentation, we found that $\{(F, S) : 0 \le F \le 200,\ 0 \le S \le 100\}$ provided a nice window into the phase plane. It appears that solutions, rather than being periodic as in the standard predator-prey model, now spiral into a stable point.



(b) The solution starting at $(60, 34)$ remains at $(60, 34)$ for all time. As we shall soon see, this type of solution is called an equilibrium solution.

29. (a) Given that $dF/dt = aF - bFS$ and $dS/dt = -cS + dFS$,

$$\frac{dS}{dF} = \frac{dS/dt}{dF/dt} = \frac{-cS + dFS}{aF - bFS} = \frac{(-c + dF)S}{(a - bS)F}.$$

(b) Using the result in part (a), we separate the variables.

$$\frac{a - bS}{S}\,dS = \frac{-c + dF}{F}\,dF$$

$$\left(\frac{a}{S} - b\right) dS = \left(-\frac{c}{F} + d\right) dF$$

Integrate.

$$a \ln S - bS = -c \ln F + dF + C,$$

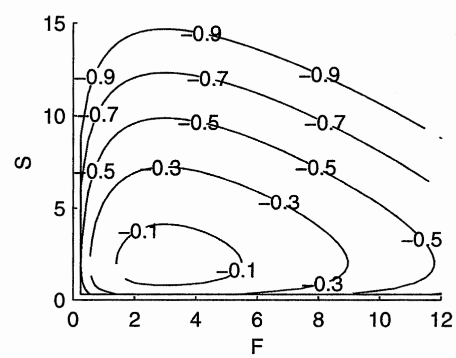
or, equivalently,

$$a \ln S - bS + c \ln F - dF = C.$$

(c) An implicit function plotter was used to sketch

$$0.2 \ln S - 0.1S + 0.3 \ln F - 0.1F = C,$$

for $C = -0.2, -0.4, -0.6, -0.8$, and -1.



Section 8.3. Qualitative Analysis

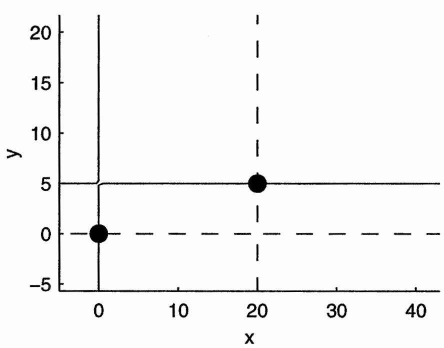
1. Set the right hand side of $x' = 0.2x - 0.04xy$ equal to zero.

$$\begin{aligned} 0.2x - 0.04xy &= 0 \\ 20x - 4xy &= 0 \\ 4x(5 - y) &= 0 \end{aligned}$$

Thus, $x = 0$ and $y = 5$ are the x-nullclines. They appear in a solid line style in the figure. Set the right hand side of $y' = -0.1y + 0.005xy$ equal to zero.

$$\begin{aligned} -0.1y + 0.005xy &= 0 \\ -100y + 5xy &= 0 \\ 5y(-20 + x) &= 0 \end{aligned}$$

Thus, $y = 0$ and $x = 20$ are the y-nullclines. They appear in a dashed line style in the figure.



The equilibrium points appear where the x-nullclines intersect the y-nullclines. These are easily seen to be $(0, 0)$ and $(20, 5)$.

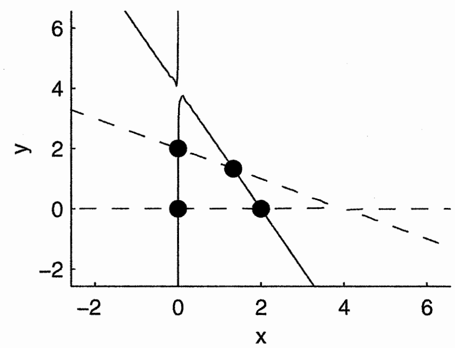
2. Set the right-hand side of $x' = 4x - 2x^2 - xy$ equal to zero.

$$\begin{aligned} 4x - 2x^2 - xy &= 0 \\ x(4 - 2x - y) &= 0 \end{aligned}$$

Thus, $x = 0$ and $4 - 2x - y = 0$ are the x-nullclines. They appear in a solid line style in the figure. Set the right-hand side of $y' = 4y - xy - 2y^2$ equal to zero.

$$\begin{aligned} 4y - xy - 2y^2 &= 0 \\ y(4 - x - 2y) &= 0 \end{aligned}$$

Thus, $y = 0$ and $4 - x - 2y = 0$ are the y-nullclines. They appear in a dashed line style in the figure.



The equilibrium points appear where the x-nullclines intersect the y-nullclines. These are $(0, 0)$, $(2, 0)$, $(0, 2)$, and $(4/3, 4/3)$.

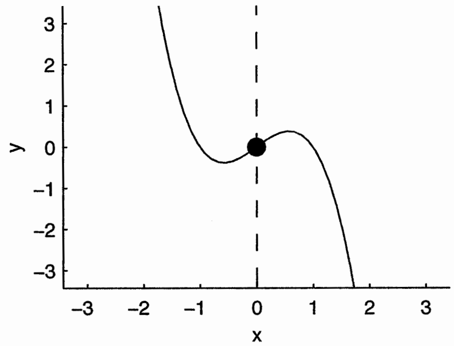
3. Set the right hand side of $x' = x - y - x^3$ equal to zero.

$$\begin{aligned} x - y - x^3 &= 0 \\ y &= x - x^3 \end{aligned}$$

Thus, $y = x - x^3$ is the x-nullcline. It appears in a solid line style in the figure. Set the right hand side of $y' = x$ equal to zero.

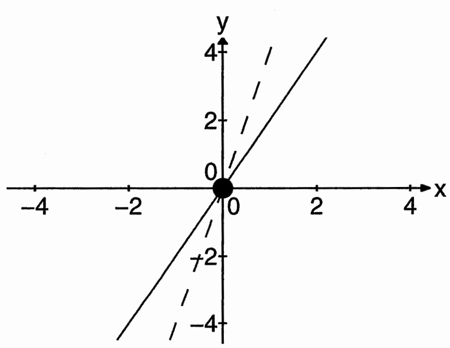
$$x = 0$$

Thus, $x = 0$ is the y-nullcline. It appears in a dashed line style in the figure.



The equilibrium point appears where the x-nullcline intersects the y-nullcline. This is easily seen to be $(0, 0)$.

4. The x-nullcline is the line defined by $2x - y = 0$. It is shown below as the solid line. The y-nullcline is the line defined by $-4x + 2y = 0$. It is the dashed line in the following figure.



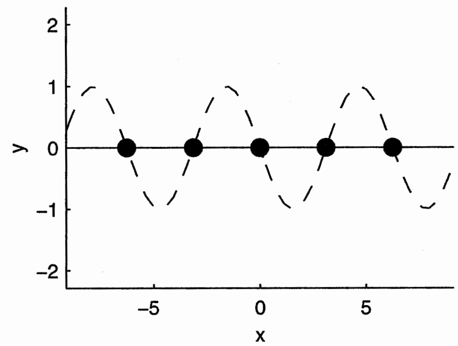
5. Set the right hand side of $x' = y$ equal to zero.

$$y = 0$$

Thus, $y = 0$ is the x-nullcline. It appears in a solid line style in the figure. Set the right hand side of $y' = -\sin x - y$ equal to zero.

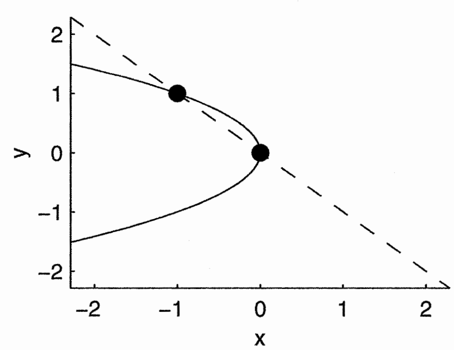
$$-\sin x - y = 0$$
$$y = -\sin x$$

Thus, $y = -\sin x$ is the y-nullcline. It appears in a dashed line style in the figure.



The equilibrium points appear where the x-nullcline intersects the y-nullcline. These occur at the points $(k\pi, 0)$, where k is an integer. A few are shown in the figure.

6. Set the right-hand side of $x' = x + y^2$ equal to zero. Thus, $x + y^2 = 0$ is the x-nullcline, set in a solid line style in the figure. Next, set the right-hand side of $y' = x + y$ equal to zero. Thus, $x + y = 0$ is the y-nullcline, set in a dashed line style in the figure.



The equilibrium points appear where the x-nullclines intersect the y-nullclines. These are at $(0, 0)$ and $(-1, 1)$.

7. (a) If $x(t) = t$ and $y(t) = \sin t$, then

$$x' = (t)' = 1,$$

and

$$1-(y-\sin x)\cos x = 1-(\sin t-\sin t)\cos t = 1,$$

so the first equation is satisfied. Further,

$$y' = (\sin t)' = \cos t,$$

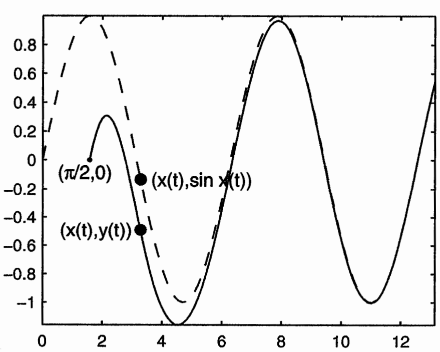
and

$$\cos x - y + \sin x = \cos t - \sin t + \sin t = \cos t,$$

so the second equation is satisfied.

(b) See the figure in part (c).

(c) Because of uniqueness, the solution with initial condition $x(0) = \pi/2$, $y(0) = 0$, cannot cross the solution $x = t$, $y = \sin t$ found in part (a). Thus, it must remain below the solution in part (a) for all time. Therefore, if $(x(t), y(t))$ denotes the second solution, we must have $y(t) < \sin x(t)$ for all time, as shown in the figure.



8. (a) If $x = t$ and $y = e^t$, then

$$x' = (t)' = 1$$

and

$$1 - e^x(y - e^x) = 1 - e^t(e^t - e^t) = 1,$$

so the first equation is satisfied. Further,

$$y' = (e^t)' = e^t$$

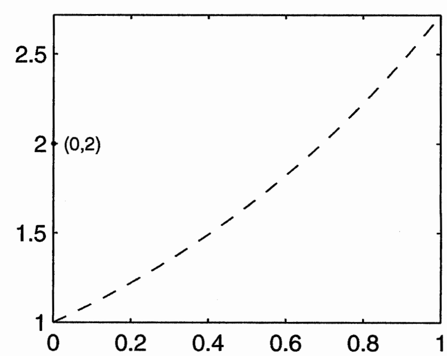
and

$$2e^x - y = 2e^t - e^t = e^t,$$

so the second equation is satisfied.

(b) See the figure in part (c).

(c) Note that the solution in part (a) has $x(0) = 0$, $y(0) = e^0 = 1$. Because of uniqueness, the solution with initial condition $x(0) = 0$, $y(0) = 2$ must not intersect the solution in part (a). Thus, this solution must remain above the solution in part (a) for all time. Actually, things are a little tricky here, as $(0, 2)$ is an equilibrium solution of the system. Thus, the solution with initial condition $(0, 2)$ remains at $(0, 2)$ for all time.



9. (a) If $x = e^t$ and $y = -e^t$, then

$$x' = (e^t)' = e^t,$$

and

$$\begin{aligned} &-y - x(x^2 - y^2) \\ &\quad = -(-e^t) - e^t((e^t)^2 - (-e^t)^2) \\ &\quad = e^t - e^t(e^{2t} - e^{2t}) \\ &\quad = e^t, \end{aligned}$$

so the first equation is satisfied. Further,

$$y' = (-e^t)' = -e^t,$$

and

$$\begin{aligned}
&-x - y(x^2 - y^2)\\
&\quad = -e^t - (-e^t)((e^t)^2 - (-e^t)^2)\\
&\quad = -e^t + e^t(e^{2t} - e^{2t})\\
&\quad = -e^t,
\end{aligned}$$

so the second equation is satisfied.

(b) If $x(t) = e^{-t}$ and $y(t) = e^{-t}$, then

$$x' = (e^{-t})' = -e^{-t},$$

and

$$\begin{aligned}
&-y - x(x^2 - y^2)\\
&\quad = -e^{-t} - e^{-t}((e^{-t})^2 - (e^{-t})^2)\\
&\quad = -e^{-t} - e^{-t}(e^{-2t} - e^{-2t})\\
&\quad = -e^{-t},
\end{aligned}$$

so the first equation is satisfied. Further,

$$y' = (e^{-t})' = -e^{-t},$$

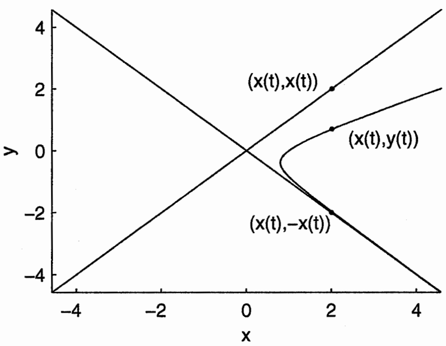
and

$$\begin{aligned}
&-x - y(x^2 - y^2)\\
&\quad = -e^{-t} - e^{-t}((e^{-t})^2 - (e^{-t})^2)\\
&\quad = -e^{-t} - e^{-t}(e^{-2t} - e^{-2t})\\
&\quad = -e^{-t},
\end{aligned}$$

and the second equation is satisfied.

(c) See the figure in part (d).

(d) It is important to recognize that solution (a), $x(t) = e^t$, $y(t) = -e^t$, is a parametric description of the line $y = -x$. Also, solution (b) lies on the line $y = -x$. Uniqueness tells us that this third solution must remain trapped between the solutions of parts (a) and (b) (solutions may not cross). Thus, if $(x(t), y(t))$ is a point on this third solution, the figure shows why $-x(t) < y(t) < x(t)$ for all t.



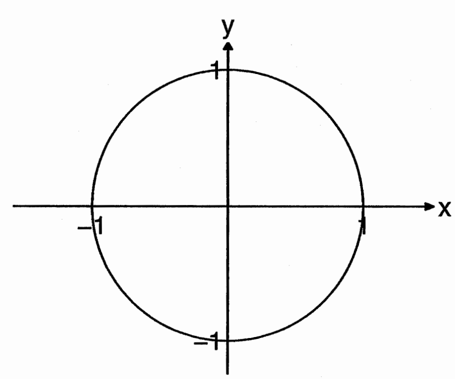
10. (a) Notice that if $x(t) = \sin t$ and $y(t) = \cos t$, then $x^2 + y^2 = 1$. Therefore we have

$$x' = \cos t \quad \text{and}$$
$$y - x(x^2 + y^2 - 1) = y = \cos t.$$

Next

$$y' = -\sin t \quad \text{and}$$
$$-x - y(x^2 + y^2 - 1) = -x = -\sin t.$$

(b) The solution curve in part (a) is the unit circle and is plotted below.



(c) The point $(x(0), y(0)) = (0.5, 0)$ is inside the unit circle, which is the solution curve from part (a). By the uniqueness theorem the solution curve starting at $(0.5, 0)$ cannot cross the unit circle. It must therefore stay inside the unit circle for all time. Hence $x^2(t) + y^2(t) < 1$ for all t.

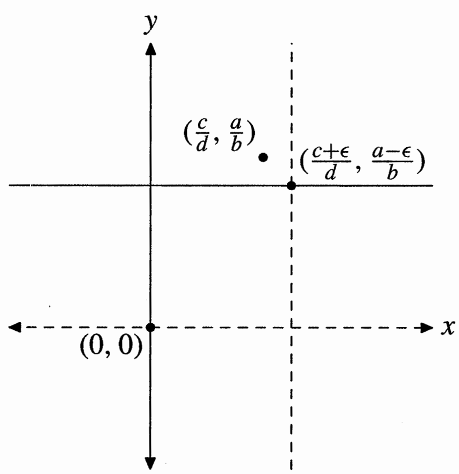
11. (a) Set the right side of $x' = ax - bxy - \epsilon x$ equal to zero.

$$ax - bxy - \epsilon x = 0$$
$$x((a - \epsilon) - by) = 0$$

Thus, $x = 0$ and $y = (a - \epsilon)/b$ are the x-nullclines. Set the right side of $y' = -cy + dxy - \epsilon y$ equal to zero.

$$-cy + dxy - \epsilon y = 0$$
$$y((-c - \epsilon) + dx) = 0$$

Thus, $y = 0$ and $x = (c + \epsilon)/d$ are the y-nullclines. The equilibrium points are intersections of the x- and y-nullclines, easily seen to be $(0, 0)$ and $((c + \epsilon)/d, (a - \epsilon)/b)$.

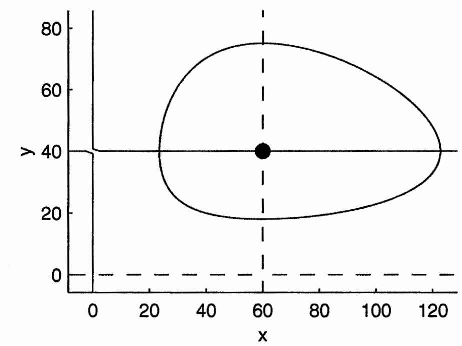


(b) As you can see in the figure in part (a), the harvesting strategy has shifted the equilibrium point. It has moved from

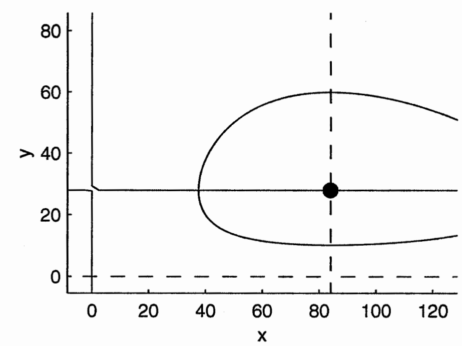
$$\left(\frac{c}{d}, \frac{a}{b}\right) \quad \text{to} \quad \left(\frac{c+\epsilon}{d}, \frac{a-\epsilon}{b}\right).$$

Therefore, the predator population decreases somewhat, but the prey population actually increases. So, this harvesting strategy is a poor one if the intent was to decrease the level of prey (such as spraying with insecticides).

(c) First, the original system, $x' = (0.4 - 0.01y)x$, $y' = (-0.3 + 0.005x)y$, without harvesting, provides this image.



Note that the equilibrium point is $(60, 40)$ for this system. Now, if we add harvesting, as in system 11c, then we get this result. Note that both pictures were crafted with the same initial condition; namely, $x(0) = 40$ and $y(0) = 20$.



Clearly, the predator population has decreased on average, while the prey population has increased on average. The equilibrium point is now at $(84, 28)$.

12. (a) Set the right-hand side of $x' = ax(1 - x/K) -$

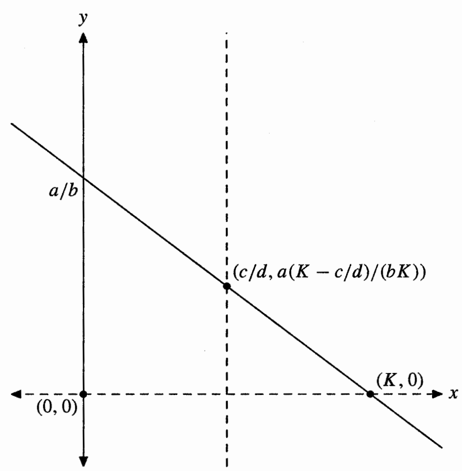
bxy equal to zero.

$$ax\left(1-\frac{x}{K}\right)-bxy=0$$
$$x\left[a\left(1-\frac{x}{K}\right)-by\right]=0$$

Thus, $x=0$ and $a(1-x/K)-by=0$ are the x-nullclines (solid). Set the right-hand side of $y'=-cy+dxy$ equal to zero.

$$-cy+dxy=0$$
$$y(-c+dx)=0$$

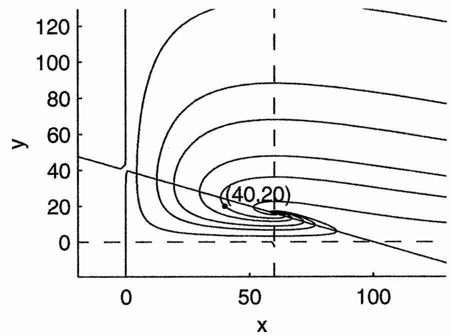
Thus, $y=0$ and $-c+dx=0$ are the y-nullclines (dashed). If $K>c/d$, then the equilibrium point $(c/d, a(K-c/d)/bK)$ falls in the first quadrant.



There are also equilibrium points at $(0,0)$ and $(K,0)$.

(b) This particular trajectory is labelled with initial condition (40,20) in the phase portrait in part (c).

(c) The phase portrait follows.



In this adjustment of the predator-prey model, note that solutions are no longer periodic. Rather, solutions "spiral" and "sink" into the equilibrium point at $(c/d, a(K-c/d)/(bK))$, in this case, the point $(60,16)$.

13. If x grows according to the logistic model, then
$$x'=r\left(1-\frac{x}{K}\right)x=rx-\frac{rx^2}{K}.$$
Of course, this becomes
$$x'=ax-bx^2,$$
if we replace r with a and r/K with b. Similarly, if y grows according to the logistic model, we can eventually write
$$y'=Ay-By^2.$$
Now, in light of the competition for resources, interaction between the two species is harmful to each. Thus, a certain percentage of contacts between species will decrease the rates dx/dt and dy/dt (perhaps differently for each rate), so we write
$$x'=ax-bx^2-cxy$$
$$y'=Ay-By^2-Cxy.$$

14. To name a few:
    - We've seen systems where the solutions are closed periodic solutions that spiral about a "central" equilibrium point.
    - We've seen systems where solutions seemingly "sink" into an equilibrium point.
    - There are others.

15. Setting the right side of $x' = \sigma(y-x)$ equal to zero,

$$\begin{aligned}\sigma(y-x) &= 0\\ y &= x.\end{aligned}$$

Setting the right side of $z' = -\beta z + xy$ equal to zero,

$$\begin{aligned}-\beta z + xy &= 0\\ xy &= \beta z.\end{aligned}$$

However, $y = x$, so we can write

$$x^2 = \beta z.$$

Setting the right side of $y' = \rho x - y - xz$ equal to zero,

$$\rho x - y - xz = 0.$$

However, $y = x$, so

$$\begin{aligned}\rho x - x - xz &= 0\\ x(\rho - 1 - z) &= 0.\end{aligned}$$

Thus, either $x = 0$ or $z = \rho - 1$. If $x = 0$, then y and z are easily shown to also equal zero, giving $(0, 0, 0)$ as one equilibrium point. If $z = \rho - 1$, then

$$\begin{aligned}x^2 &= \beta z\\ x^2 &= \beta(\rho - 1)\\ x &= \pm\sqrt{\beta(\rho - 1)}.\end{aligned}$$

Of course, this last expression is real only if $\rho > 1$, which it is. Thus, the second equilibrium point is $(\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho - 1)$ and the third equilibrium point is $(-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho-1)$.

———————×———————

Section 8.4. Linear Systems

1. Note that the system

$$\begin{aligned}x_1' &= -x_2\\ x_2' &= -x_1 - 2x_2 + 5\sin t\end{aligned}$$

has the form

$$\begin{aligned}x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + f_1(t)\\ x_2' &= a_{21}(t)x_2 + a_{22}(t)x_2 + f_2(t).\end{aligned}$$

Hence, the system is linear. Although $f_1(t) = 0$, because $f_2(t) = 5\sin t$ is not zero, the system is inhomogeneous.

2. Note that the system

$$\begin{aligned}x_1' &= -2x_1 + x_1x_2\\ x_2' &= -3x_1 - x_2,\end{aligned}$$

because of the nonlinear term x_1x_2, does not have the form

$$\begin{aligned}x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + f_1(t)\\ x_2' &= a_{21}(t)x_2 + a_{22}(t)x_2 + f_2(t).\end{aligned}$$

Hence, the system is nonlinear.

3. Note that the system

$$\begin{aligned}x_1' &= -x_2\\ x_2' &= \sin x_1,\end{aligned}$$

because of the nonlinear term $\sin x_1$, does not have the form

$$\begin{aligned}x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + f_1(t)\\ x_2' &= a_{21}(t)x_2 + a_{22}(t)x_2 + f_2(t).\end{aligned}$$

Hence, the system is nonlinear.

4. Note that the system

$$\begin{aligned}x_1' &= x_1 + (\sin t)x_2\\ x_2' &= 2tx_1 - x_2\end{aligned}$$

has the form

$$\begin{aligned}x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + f_1(t)\\ x_2' &= a_{21}(t)x_2 + a_{22}(t)x_2 + f_2(t).\end{aligned}$$

Hence, the system is linear. Because $f_1(t) = 0$ and $f_2(t) = 0$, the system is homogeneous.

5. Note that the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -x_1 - x_2 + \sin t \end{aligned}$$

has the form

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + a_{13}(t)x_3 + f_1(t) \\ x_2' &= a_{21}(t)x_2 + a_{22}(t)x_2 + a_{23}(t)x_3 + f_2(t) \\ x_3' &= a_{31}(t)x_1 + a_{32}(t)x_2 + a_{33}(t)x_3 + f_3(t). \end{aligned}$$

Hence, the system is linear. Although $f_1(t) = 0$ and $f_2(t) = 0$, because $f_3(t) = \sin t$ is nonzero, the system is inhomogeneous.

6. Note that the system

$$\begin{aligned} x_1' &= -x_1 + tx_2 - t^2x_3 \\ x_2' &= -2x_1 + 2tx_2 \\ x_3' &= -t^2x_1 + (\sin t)x_2 - \sin x_3, \end{aligned}$$

because of the nonlinear term $\sin x_3$, does not have the form

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + a_{13}(t)x_3 + f_1(t) \\ x_2' &= a_{21}(t)x_2 + a_{22}(t)x_2 + a_{23}(t)x_3 + f_2(t) \\ x_3' &= a_{31}(t)x_1 + a_{32}(t)x_2 + a_{33}(t)x_3 + f_3(t). \end{aligned}$$

Hence, the system is nonlinear. Note that all the other terms are linear.

7. If $x_1(t) = e^t$ and $x_2(t) = -e^t$, then the left side of $x_1' = 3x_1 + 2x_2$ is

$$x_1' = e^t.$$

Substituting in the right hand side,

$$3x_1 + 2x_2 = 3(e^t) + 2(-e^t) = e^t.$$

Hence, the first equation is satisfied. In a similar fashion, the left side of $x_2' = -4x_1 - 3x_2$ is $x_2' = -e^t$ and the right side is $-4x_1 - 3x_2 = -4(e^t) - 3(-e^t) = -e^t$. Hence, the second equation is satisfied.

8. If $x_1(t) = e^{-2t}$ and $x_2(t) = -e^{-2t}$, then the left side of $x_1' = -5x_1 - 3x_2$ is

$$x_1' = -2e^{-2t}.$$

Substituting in the right hand side,

$$-5x_1 - 3x_2 = -5(e^{-2t}) - 3(-e^{-2t}) = -2e^{-2t}.$$

Hence, the first equation is satisfied. In a similar fashion, the left side of $x_2' = 6x_1 + 4x_2$ is $x_2' = 2e^{-2t}$ and the right side is $6x_1 + 4x_2 = 6(e^{-2t}) + 4(-e^{-2t}) = 2e^{-2t}$. Hence, the second equation is satisfied.

9. If $x_1(t) = 2e^{-t} - e^{-2t}$ and $x_2(t) = -2e^{-t}$, then the left side of $x_1' = -2x_1 - x_2$ is

$$x_1' = -2e^{-t} + 2e^{-2t}.$$

Substituting in the right hand side,

$$\begin{aligned} -2x_1 - x_2 &= -2(2e^{-t} - e^{-2t}) - (-2e^{-t}) \\ &= -2e^{-t} + 2e^{-2t}. \end{aligned}$$

Hence, the first equation is satisfied. In a similar fashion, the left side of $x_2' = -x_2$ is $x_2' = 2e^{-t}$ and the right side is $-x_2 = 2e^{-t}$. Hence, the second equation is satisfied.

10. If $x_1(t) = -2e^{-2t} + 3e^{3t}$ and $x_2(t) = -2e^{-2t}$, then the left side of $x_1' = 3x_1 - 5x_2$ is

$$x_1' = 4e^{-2t} + 9e^{3t}.$$

Substituting in the right hand side,

$$\begin{aligned} 3x_1 - 5x_2 &= 3(-2e^{-2t} + 3e^{3t}) - 5(-2e^{-2t}) \\ &= 4e^{-2t} + 9e^{3t}. \end{aligned}$$

Hence, the first equation is satisfied. In a similar fashion, the left side of $x_2' = -2x_2$ is $x_2' = 4e^{-2t}$ and the right side is $-2x_2 = -2(-2e^{-2t}) = 4e^{-2t}$. Hence, the second equation is satisfied.

11. We can write

$$\begin{aligned} x_1' &= -2x_1 + 3x_2 \\ x_2' &= x_1 - 4x_2 \end{aligned}$$

as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -2 & 3 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Therefore, the system is linear.

12. We can write

$$x_1' = tx_1 + (t\cos t)x_2$$
$$x_2' = (\sin t)x_1 + 3x_2$$

as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} t & t\cos t \\ \sin t & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Therefore, the system is linear.

13. The system

$$x_1' = -2x_1 + x_2^2$$
$$x_2' = 3x_1 - x_2$$

is nonlinear. Note the term x_2^2. It cannot be written in the form $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$.

14. We can write

$$x_1' = -2x_1 + 3tx_2 + \cos t$$
$$x_2' = \frac{1}{t}x_1 - 4x_2 + \frac{\sin t}{t}.$$

as

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -2 & 3t \\ 1/t & -4 \end{pmatrix} + \begin{pmatrix} \cos t \\ (\sin t)/t \end{pmatrix}$$

Therefore, the system is linear.

15. The system

$$t^2 x_1' = -2tx_1x_2 + 3x_2$$
$$(1/t)x_2' = tx_1 - (4/t)x_2$$

is nonlinear. Note the nonlinear term $-2tx_1x_2$. Therefore, the system cannot be written in the form $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$.

16. We can write

$$x_1' = -2tx_1 + 3x_2 - (\cos t)x_3$$
$$x_2' = x_1 - 4t^2 x_3$$
$$x_3' = -\frac{3}{t}x_2 - \cos t$$

as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -2t & 3 & -\cos t \\ 1 & 0 & -4t^2 \\ 0 & -3/t & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\cos t \end{pmatrix}.$$

Therefore, the system is linear.

17. Exercises 11, 12, 14, and 16 are linear; 13 and 15 are nonlinear. Of the linear systems, only 11 and 12 are homogeneous.

18. Write the system $x_1' = -3x_1$, $x_2' = -2x_1 - x_2$ in matrix-vector form.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -3 & 0 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Note that this system is in the form $\mathbf{x}' = A\mathbf{x}$. Next, if $\mathbf{x} = (e^{-3t}, e^{-3t})^T$, then $\mathbf{x}' = (-3e^{-3t}, -3e^{-3t})^T$ and

$$A\mathbf{x} = \begin{pmatrix} -3 & 0 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix} = \begin{pmatrix} -3e^{-3t} \\ -3e^{-3t} \end{pmatrix}.$$

Thus, $\mathbf{x}' = A\mathbf{x}$ and $\mathbf{x} = (e^{-3t}, e^{-3t})^T$ is a solution.

19. Write the system $x_1' = 8x_1 - 10x_2$, $x_2' = 5x_1 - 7x_2$ in matrix-vector form.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 8 & -10 \\ 5 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Note that this system is in the form $\mathbf{x}' = A\mathbf{x}$. Next, if $\mathbf{x} = (2e^{3t}, e^{3t})^T$, then $\mathbf{x}' = (6e^{3t}, 3e^{3t})^T$ and

$$A\mathbf{x} = \begin{pmatrix} 8 & -10 \\ 5 & -7 \end{pmatrix} \begin{pmatrix} 2e^{3t} \\ e^{3t} \end{pmatrix} = \begin{pmatrix} 6e^{3t} \\ 3e^{3t} \end{pmatrix}.$$

Thus, $\mathbf{x}' = A\mathbf{x}$ and $\mathbf{x} = (2e^{3t}, e^{3t})^T$ is a solution.

20. Write the system $x_1' = -3x_1 + x_2$, $x_2' = -2x_1$ in matrix-vector form.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Note that this system is in the form $\mathbf{x}' = A\mathbf{x}$. Next, if $\mathbf{x} = (-e^{-2t} + e^{-t}, -e^{-2t} + 2e^{-t})^T$, then $\mathbf{x}' = (2e^{-2t} - e^{-t}, 2e^{-2t} - 2e^{-t})^T$ and

$$A\mathbf{x} = \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} -e^{-2t} + e^{-t} \\ -e^{-2t} + 2e^{-t} \end{pmatrix}$$
$$= \begin{pmatrix} 2e^{-2t} - e^{-t} \\ 2e^{-2t} - 2e^{-t} \end{pmatrix}.$$

Thus, $\mathbf{x}' = A\mathbf{x}$ and $\mathbf{x} = (-e^{-2t} + e^{-t}, -e^{-2t} + 2e^{-t})^T$ is a solution.

21. Write the system$x_1' = -x_1 + 4x_2$, $x_2' = 3x_2$ in matrix-vector form.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -1 & 4 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Note that this system is in the form $\mathbf{x}' = A\mathbf{x}$. Next, if $\mathbf{x} = (e^{3t} - e^{-t}, e^{3t})^T$, then $\mathbf{x}' = (3e^{3t} + e^{-t}, 3e^{3t})^T$ and

$$A\mathbf{x} = \begin{pmatrix} & 4 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} e^{3t} - e^{-t} \\ e^{3t} \end{pmatrix} = \begin{pmatrix} 3e^{3t} + e^{-t} \\ 3e^{3t} \end{pmatrix}.$$

Thus, $\mathbf{x}' = A\mathbf{x}$ and $\mathbf{x} = (e^{3t} - e^{-t}, e^{3t})^T$ is a solution.

22. Let's begin by setting a node at a (see Figure **??**) and labeling current coming out of node a with I_1, I_2, and I_3. Further, let the voltage drops across the capacitor, resistor, and inductor be represented by V_1, V_2, and V_3, respectively. Kirchoff's second law requires that the sum of the currents at node a equal zero. Thus,

$$I_1 + I_2 + I_3 = 0. \tag{4.1}$$

Next, the sum of the voltage drops around the left-most loop must equal zero. Thus,

$$V_1 - V_2 = 0. \tag{4.2}$$

Also, the sum of the voltage drops around the right-most loop must equal zero. Thus,

$$V_2 - V_3 = 0. \tag{4.3}$$

Furthermore, the voltage drops across each circuit element are given by the equations

$$CV_1' = I_1, \qquad V_2 = RI_2, \qquad \text{and} \qquad LI_3' = V_3. \tag{4.4}$$

Note that the independent variables in equations (4.18) and (4.19) are V and I, represented by V_1 and I_3, respectively. Therefore, the goal is to eliminate V_2, V_3, I_1 and I_2 from equations (4.1), (4.2), (4.3), and (4.4), keeping only V_1 and I_3.

First, note that equations (4.2) and (4.3) imply that $V_1 = V_2 = V_3$, an anticipated consequence of the fact that the circuit elements are in parallel. Taking $LI_3' = V_3$ from equation (4.4), then replacing V_3 with V_1 yields the equation

$$LI_3' = V_1,$$

or, alternatively,

$$I_3' = \frac{V_1}{L}. \tag{4.5}$$

Secondly, start with $CV_1' = I_1$ from equation (4.4), then use equation (4.1) to write

$$CV_1' = -I_2 - I_3.$$

Now, from equation (4.4), substitute $I_2 = V_2/R$ to get

$$CV_1' = -\frac{V_2}{R} - I_3.$$

But, $V_1 = V_2$, so

$$CV_1' = -\frac{V_1}{R} - I_3,$$

or, alternatively,

$$V_1' = -\frac{V_1}{RC} - \frac{I_3}{C}. \tag{4.6}$$

Because, $V_1 = V$ and $I_3 = I$, equations (4.5) and (4.6) provide us with

$$I' = \frac{V}{L} \tag{4.7}$$

$$V' = -\frac{V}{RC} - \frac{I}{C}. \tag{4.8}$$

23. Label the currents coming out of node "a", as shown in Figure 8. The sum of the current flowing out of node "a" must equal zero.

$$I + I_2 + I_3 = 0 \tag{4.9}$$

Proceeding in a clockwise direction around the top loop, the sum of the voltage drops across each element must equal zero. Thus,

$$R_1 I + LI' - R_2 I_2 = 0. \tag{4.10}$$

Because we are moving in the direction of the current I, the voltage drops across the resistor R_1 and the inductor L are positive. However, as we continue in a clockwise direction, we move **against** the current I_2 and thus experience a voltage **gain** across the resistor R_2. Hence, the negative sign in equation (4.10).

Secondly, we proceed in a clockwise direction around the bottom loop. Again, the sum of the voltage drops must equal zero.

$$R_2I_2 - V = 0 \tag{4.11}$$

Again, we move opposite the current when moving clockwise through the capacitor. This, coupled with the fact that V represents the voltage drop across the capacitor, explains $-V$ in equation (4.11). Solve equation (4.10) for LI'.

$$LI' = -R_1I + R_2I_2 \tag{4.12}$$

>From equation (4.11), $R_2I_2 = V$, which, when substituted in (4.12), produces the following result.

$$LI' = -R_1I + V \tag{4.13}$$

Note that this is our first equation (4.20).

Next, solve equation (4.11) for I_2.

$$I_2 = \frac{V}{R_2} \tag{4.14}$$

>From equation (4.9), $I_2 = -I - I_3$. With this substitution, equation (4.14) becomes

$$\frac{V}{R_2} = -I - I_3. \tag{4.15}$$

However, the voltage drop across the capacitor is given by

$$V = \frac{q}{C},$$

where q is the charge on the capacitor. Differentiating,

$$CV' = q',$$

and using the fact that $I_3 = q'$,

$$CV' = I_3.$$

Substituting this result in equation (4.15), we obtain

$$\frac{V}{R_2} = -I - CV',$$

or equivalently,

$$CV' = -I - \frac{V}{R_2}, \tag{4.16}$$

which is identical to equation (4.21).

24. **Solution**. We'll represent the current across E and R_1 with the variable I. This leads to the adjustment in the schematic shown in Figure **??**. Kirchoff's second law says that the current into the node at D equals the current out at the same node. Thus,

$$I = I_1 + I_2. \tag{4.17}$$

Traversing the loop $ABCDA$ in a clockwise direction, the sum of the voltage drops at each element must equal zero. Thus,

$$-E + R_1I + L_1I_1' = 0,$$

or equivalently,

$$L_1I_1' = -R_1I + E. \tag{4.18}$$

Use equation (4.17) to replace I with $I_1 + I_2$ in equation (4.18). Thus,

$$L_1I_1' = -R_1(I_1 + I_2) + E,$$

or equivalently,

$$L_1I_1' = -R_1I_1 - R_2I_2 + E.$$

This last equation is the requested equation (4.22).

Next, proceed in a clockwise direction about the loop $ABCJKDA$.[1] Thus,

$$-E + R_1I + R_2I_2 + L_2I_2' = 0,$$

or equivalently,

$$L_2I_2' = -R_1I - R_2I_2 + E. \tag{4.19}$$

We again use equation (4.17) to replace I with $I_1 + I_2$ in equation (4.19).

$$L_2I_2' = -R_1(I_1 + I_2) - R_2I_2 + E$$

A little algebra provides

$$L_2I_2' = -R_1I_1 - (R_1 + R_2)I_2 + E,$$

which is the needed equation (4.23).

[1] We could traverse the loop $CJKDC$, but this would involve both I_1' and I_2'. Although this loop would eventually provide us with the needed result, the loop $ABCJKDA$ only involves I_2', a better strategy for this problem.

25. Let $x_1(t)$ and $x_2(t)$ represent the salt content (in pounds) of the upper and lower tanks, respectively. Because pure water flows into the upper tank, no salt is coming into the tank. However, the rate at which salt is leaving the tank is

$$\begin{aligned}\text{Rate Out} &= 4\,\text{gal/min} \times \frac{x_1(t)}{100}\,\text{lb/gal}\\ &= \frac{x_1(t)}{25}\,\text{lb/min}.\end{aligned}$$

Because $dx_1/dt = \text{Rate In} - \text{Rate Out}$, we write

$$\frac{dx_1}{dt} = -\frac{x_1}{25}. \tag{4.20}$$

Salt enters the lower tank at the same rate as it is leaving the upper tank. Further, the rate at which salt leaves the lower tank is

$$\begin{aligned}\text{Rate Out} &= 4\,\text{gal/min} \times \frac{x_2(t)}{100}\,\text{lb/gal}\\ &= \frac{x_2(t)}{25}\,\text{lb/min}.\end{aligned}$$

Again, because $dx_2/dt = \text{Rate In} - \text{Rate Out}$, we write

$$\frac{dx_2}{dt} = \frac{x_1}{25} - \frac{x_2}{25}. \tag{4.21}$$

In matrix form, the system defined by equations (4.20) and (4.21) is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -1/25 & 0 \\ 1/25 & -1/25 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{4.22}$$

Recall that the initial salt content in the upper and lower tanks is 10 and 20 pounds, respectively. Thus, the initial condition is $\mathbf{x}(0) = (x_1(0), x_2(0)) = (10, 20)$.

Finally, when the system (4.22) is compared with $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, we see that $\mathbf{f} = \mathbf{0}$ and the system is homogeneous.

26. Let $x_1(t)$ and $x_2(t)$ represents the amount of salt (in pounds) in the upper and lower tanks, respectively. The rate at which salt flows into the upper tank is found by multiplying the solution rate by the concentration of the salt solution entering the upper tank.

$$\text{Rate In} = 5\,\text{gal/min} \times 0.2\,\text{lb/gal} = 1\,\text{lb/min}$$

Similarly, salt leaves the upper tank at the following rate.

$$\text{Rate Out} = 5\,\text{gal/min} \times \frac{x_1}{200}\,\text{lb/gal} = \frac{x_1}{40}\,\text{lb/min}$$

Thus, the rate at which the salt content is changing in the upper tank is $dx_1/dt = \text{Rate In} - \text{Rate Out}$, or

$$\frac{dx_1}{dt} = 1 - \frac{x_1}{40}. \tag{4.23}$$

A similar argument shows that the salt content in the lower tank is changing according to the equation

$$\frac{dx_2}{dt} = \frac{x_1}{40} - \frac{x_2}{40}. \tag{4.24}$$

In matrix-vector form, the system becomes

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -1/40 & 0 \\ 1/40 & -1/40 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Because each tank contains pure water at time zero, the initial condition is $\mathbf{x}(0) = (0, 0)^T$. When we compare the system 4.24 with $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, we see that $\mathbf{f} = (1, 0)^T$, so the system is inhomogeneous.

27. Let $x_1(t)$, $x_2(t)$, and $x_3(t)$ represent the salt content (in pounds) of the first, second, and third tanks, respectively. Salt enters the first tank at a rate

$$5\,\text{gal/min} \times 2\,\text{lb/gal} = 10\,\text{lb/min}.$$

Salt leaves the first tank at a rate

$$5\,\text{gal/min} \times \frac{x_1}{100}\,\text{lb/gal} = \frac{x_1}{20}\,\text{lb/gal}.$$

Because $dx_1/dt = \text{Rate In} - \text{Rate Out}$, the differential equation governing the salt content in the first tank is

$$\frac{dx_1}{dt} = 10 - \frac{x_1}{20}. \tag{4.25}$$

Similarly, the equations governing the salt content in the second and third tanks are

$$\frac{dx_2}{dt} = \frac{x_1}{20} - \frac{x_2}{16}, \tag{4.26}$$

and

$$\frac{dx_3}{dt} = \frac{x_2}{16} - \frac{x_3}{12}, \tag{4.27}$$

respectively. In matrix-vector form, the system created by equations (4.25), (4.26), and (4.27), becomes

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{bmatrix} -1/20 & 0 & 0 \\ 1/20 & -1/16 & 0 \\ 0 & 1/16 & -1/12 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}.$$

Comparing this system with $\mathbf{x}' = A\mathbf{x}+\mathbf{f}$, we see that $\mathbf{f}(t) = (10, 0, 0)^T$, so the system is inhomogeneous. Because each tank contains pure water at time $t = 0$, the initial condition is $\mathbf{x}(0) = (0, 0, 0)^T$.

28. As in Example 4.8, the forces on mass m_1 are $-k_1 x$ and $k_2(y - x)$. Thus, by Newton's law,

$$m_1 x'' = -k_1 x + k_2(y - x),$$

or equivalently,

$$x'' = -\frac{k_1}{m_1} x + \frac{k_2}{m_1}(y - x). \tag{4.28}$$

On the second mass m_2, the spring must exert an equal but opposite force to that which it exerts on the first mass m_1; i.e., $-k_2(y - x)$. There is also the driving force $F = A\cos\omega t$ exerted on this second mass. Thus, by Newton's law,

$$m_2 y'' = -k_2(y - x) + A\cos\omega t,$$

or equivalently,

$$y'' = -\frac{k_2}{m_2}(y - x) + \frac{A}{m_2}\cos\omega t. \tag{4.29}$$

To change to a system of first order equations, let $x_1 = x$, $x_2 = x'$, $x_3 = y$, and $x_4 = y'$. Then,

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{k_1}{m_1} x_1 + \frac{k_2}{m_1}(x_3 - x_1) \\ x_3' &= x_4 \\ x_4' &= -\frac{k_2}{m_2}(x_3 - x_1) + \frac{A}{m_2}\cos\omega t. \end{aligned} \tag{4.30}$$

In matrix form, this can be written

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -(k_1 + k_2)/m_1 & 0 & k_2/m_1 & 0 \\ 0 & 0 & 0 & 1 \\ k_2/m_2 & 0 & -k_2/m_2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ (A/m_2)\cos\omega t \end{pmatrix}.$$

Note that this has the form $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, with $\mathbf{f}(t) = (0, 0, 0, (A/m_2)\cos\omega t)^T$. Hence, the system is inhomogeneous. Because the masses are released from rest from their equilibrium positions,

$$\begin{aligned} \mathbf{x}(0) &= (x_1(0), x_2(0), x_3(0), x_4(0))^T \\ &= (x(0), x'(0), y(0), y'(0))^T \\ &= (0, 0, 0, 0)^T. \end{aligned}$$

29. As in Example 4.8, the forces on mass the leftmost mass are $-kx$ and $k(y-x)$. Thus, by Newton's law,

$$mx'' = -kx + k(y - x) = -2kx + ky,$$

or equivalently,

$$x'' = -\frac{2k}{m} x + \frac{k}{m} y. \tag{4.31}$$

On the middle mass m, the spring connecting it to the left most mass must exert an equal but opposite force to that which it exerts on the leftmost mass; i.e., $-k(y-x)$. On the other side, the spring connecting the middle mass to the rightmost mass exerts a force $k(z - y)$. Thus, by Newton's law,

$$my'' = -k(y - x) + k(z - y) = kx - 2ky + kz,$$

or equivalently,

$$y'' = \frac{k}{m} x - \frac{2k}{m} y + \frac{k}{m} z. \tag{4.32}$$

Finally, the spring connecting the rightmost mass to the middle mass must exert an equal but opposite force to that which it exerts on the middle mass; i.e.,

$-k(z-y)$. On the other side, the spring connecting the right most mass to the vertical support on the right exerts a force of $-kz$. Hence, by Newton's law,

$$mz'' = -k(z-y) - kz = ky - 2kz,$$

or equivalently,

$$z'' = \frac{k}{m}y - \frac{2k}{m}z. \tag{4.33}$$

To change to a system of first order equations, let $x_1 = x$, $x_2 = x'$, $x_3 = y$, $x_4 = y'$, $x_5 = z$, and $x_6 = z'$. Then,

$$\begin{aligned}
x_1' &= x_2\\
x_2' &= -\frac{2k}{m}x_1 + \frac{k}{m}x_3\\
x_3' &= x_4\\
x_4' &= \frac{k}{m}x_1 - \frac{2k}{m}x_3 + \frac{k}{m}x_5\\
x_5' &= x_6\\
x_6' &= \frac{k}{m}x_3 - \frac{2k}{m}x_5.
\end{aligned} \tag{4.34}$$

In matrix form, this can be written $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6)^T$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ -2k/m & 0 & k/m & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ k/m & 0 & -2k/m & 0 & k/m & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & k/m & 0 & -2k/m & 0 \end{pmatrix}.$$

Note that this has the form $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, with $\mathbf{f}(t) = \mathbf{0}$. Hence, the system is homogeneous. Because the masses are released from rest, each displaced 10 cm to the right of their equilibrium positions,

$$\begin{aligned}
\mathbf{x}(0) &= (x_1(0), x_2(0), x_3(0), x_4(0), x_5(0), x_6(0))^T\\
&= (x(0), x'(0), y(0), y'(0), z(0), z'(0))^T\\
&= (10, 0, 10, 0, 10, 0)^T.
\end{aligned}$$

Section 8.5. Properties of Linear Systems

1. The system

$$\begin{aligned} x_1' &= -x_1 + 3x_2\\ x_2' &= 2x_2 \end{aligned}$$

can be written

$$\begin{pmatrix} x_1\\ x_2 \end{pmatrix}' = \begin{pmatrix} -1 & 3\\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1\\ x_2 \end{pmatrix}.$$

2. The system

$$\begin{aligned} x_1' &= 6x_1 + 4x_2\\ x_2' &= -8x_1 - 6x_2 \end{aligned}$$

can be written

$$\begin{pmatrix} x_1\\ x_2 \end{pmatrix}' = \begin{pmatrix} 6 & 4\\ -8 & -6 \end{pmatrix}\begin{pmatrix} x_1\\ x_2 \end{pmatrix}.$$

3. The system

$$\begin{aligned} x_1' &= x_1 + x_2\\ x_2' &= -x_1 + x_2 \end{aligned}$$

can be written

$$\begin{pmatrix} x_1\\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 1\\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1\\ x_2 \end{pmatrix}.$$

4. The system

$$\begin{aligned} x_1' &= -x_2\\ x_2' &= x_1 \end{aligned}$$

can be written

$$\begin{pmatrix} x_1\\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & -1\\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1\\ x_2 \end{pmatrix}.$$

5. The system

$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= -x_1 + x_2 + e^t \end{aligned}$$

can be written

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

6. The system

$$\begin{aligned} x_1' &= -x_2 + \sin t \\ x_2' &= x_1 \end{aligned}$$

can be written

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \sin t \\ 0 \end{pmatrix}.$$

7. We saw that the system in Exercise 1 can be written as $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}.$$

If $\mathbf{x}(t) = (e^{-t}, 0)^T$, then

$$\mathbf{x}'(t) = \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix}' = \begin{pmatrix} -e^{-t} \\ 0 \end{pmatrix},$$

and

$$A\mathbf{x}(t) = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ 0 \end{pmatrix}.$$

Thus, $\mathbf{x}$ satisfies the system. If $\vec{y}(t) = (e^{2t}, e^{2t})^T$, then

$$\mathbf{y}(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}' = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix},$$

and

$$A\mathbf{y}(t) = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix}.$$

Thus, $\mathbf{y}$ satisfies the system. Now,

$$C_1\mathbf{x}(t) + C_2\mathbf{y}(t) = \begin{pmatrix} C_1 e^{-t} + C_2 e^{2t} \\ C_2 e^{2t} \end{pmatrix}.$$

Thus,

$$(C_1\mathbf{x}(t) + C_2\mathbf{y}(t))' = \begin{pmatrix} -C_1 e^{-t} + 2C_2 e^{2t} \\ 2C_2 e^{2t} \end{pmatrix}.$$

But,

$$\begin{aligned} &A(C_1\mathbf{x}(t) + C_2\mathbf{y}(t)) \\ &= \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} C_1 e^{-t} + C_2 e^{2t} \\ C_2 e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} -C_1 e^{-t} + 2C_2 e^{2t} \\ 2C_2 e^{2t} \end{pmatrix}, \end{aligned}$$

So $C_1\mathbf{x}(t) + C_2\mathbf{y}(t)$ is a solution of the system.

8. We saw that the system in Exercise 2 can be written as $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 6 & 4 \\ -8 & -6 \end{pmatrix}.$$

If $\mathbf{x}(t) = (-e^{2t}, e^{2t})^T$, then

$$\mathbf{x}'(t) = \begin{pmatrix} -e^{2t} \\ e^{2t} \end{pmatrix}' = \begin{pmatrix} -2e^{2t} \\ 2e^{2t} \end{pmatrix},$$

and

$$A\mathbf{x}(t) = \begin{pmatrix} 6 & 4 \\ -8 & -6 \end{pmatrix} \begin{pmatrix} -e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} -2e^{2t} \\ 2e^{2t} \end{pmatrix}.$$

Thus, $\mathbf{x}$ satisfies the system. If $\mathbf{y}(t) = (-e^{-2t}, 2e^{-2t})^T$, then

$$\mathbf{y}'(t) = \begin{pmatrix} -e^{-2t} \\ 2e^{-2t} \end{pmatrix}' = \begin{pmatrix} 2e^{-2t} \\ -4e^{-2t} \end{pmatrix},$$

and

$$A\mathbf{y}(t) = \begin{pmatrix} 6 & 4 \\ -8 & -6 \end{pmatrix} \begin{pmatrix} -e^{-2t} \\ 2e^{-2t} \end{pmatrix} = \begin{pmatrix} 2e^{-2t} \\ -4e^{-2t} \end{pmatrix}.$$

Thus, $\mathbf{y}$ satisfies the system. Now,

$$C_1\mathbf{x}(t) + C_2\mathbf{y}(t) = \begin{pmatrix} -C_1 e^{2t} - C_2 e^{-2t} \\ C_1 e^{2t} + 2C_2 e^{-2t} \end{pmatrix}.$$

Thus,

$$(C_1\mathbf{x}(t) + C_2\mathbf{y}(t))' = \begin{pmatrix} -2C_1 e^{2t} + 2C_2 e^{-2t} \\ 2C_1 e^{2t} - 4C_2 e^{-2t} \end{pmatrix}.$$

But,

$$\begin{aligned}A(C_1\mathbf{x}(t)+C_2\mathbf{y}(t)) &= \begin{pmatrix}6 & 4\\ -8 & -6\end{pmatrix}\begin{pmatrix}-C_1e^{2t}-C_2e^{-2t}\\ C_1e^{2t}+2C_2e^{-2t}\end{pmatrix}\\ &= \begin{pmatrix}-2C_1e^{2t}+2C_2e^{-2t}\\ 2C_1e^{2t}-4C_2e^{-2t}\end{pmatrix},\end{aligned}$$

so $C_1\mathbf{x}(t)+C_2\mathbf{y}(t)$ is a solution of the system.

9. We saw that the system in Exercise 3 can be written as $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}.$$

If $\mathbf{x}(t) = (e^t\cos t, -e^t\sin t)^T$, then

$$\mathbf{x}(t) = \begin{pmatrix}e^t\cos t\\ -e^t\sin t\end{pmatrix}' = \begin{pmatrix}-e^t\sin t+e^t\cos t\\ -e^t\cos t-e^t\sin t\end{pmatrix}.$$

But,

$$\begin{aligned}A\mathbf{x}(t) &= \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}e^t\cos t\\ -e^t\sin t\end{pmatrix}\\ &= \begin{pmatrix}e^t\cos t-e^t\sin t\\ -e^t\cos t-e^t\sin t\end{pmatrix},\end{aligned}$$

so $\mathbf{x}(t)$ is a solutions of the system. Next, if $\mathbf{y}(t) = (e^t\sin t, e^t\cos t)^T$, then

$$\mathbf{y}'(t) = \begin{pmatrix}e^t\sin t\\ e^t\cos t\end{pmatrix}' = \begin{pmatrix}e^t\cos t+e^t\sin t\\ -e^t\sin t+e^t\cos t\end{pmatrix}.$$

But,

$$\begin{aligned}A\mathbf{y}(t) &= \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}e^t\sin t\\ e^t\cos t\end{pmatrix}\\ &= \begin{pmatrix}e^t\sin t+e^t\cos t\\ -e^t\sin t+e^t\cos t\end{pmatrix},\end{aligned}$$

so $\mathbf{y}(t)$ is a solution of the system. Now,

$$C_1\mathbf{x}(t)+C_2\mathbf{y}(t) = \begin{pmatrix}C_1e^t\cos t+C_2e^t\sin t\\ -C_1e^t\sin t+C_2e^t\cos t\end{pmatrix}.$$

Thus,

$$\begin{aligned}(C_1\mathbf{x}(t)+C_2\mathbf{y}(t))' &= \begin{pmatrix}-C_1e^t\sin t+C_1e^t\cos t\\ \quad +C_2e^t\cos t+C_2e^t\sin t\\ -C_1e^t\cos t-C_1e^t\sin t\\ \quad -C_2e^t\sin t+C_2e^t\cos t\end{pmatrix}\\ &= \begin{pmatrix}(C_1+C_2)e^t\cos t\\ \quad +(-C_1+C_2)e^t\sin t\\ (-C_1+C_2)e^t\cos t\\ \quad +(-C_1-C_2)e^t\sin t\end{pmatrix}.\end{aligned}$$

But,

$$\begin{aligned}A(C_1\mathbf{x}(t)+C_2\mathbf{y}(t)) &= \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}C_1e^t\cos t+C_2e^t\sin t\\ -C_1e^t\sin t+C_2e^t\cos t\end{pmatrix}\\ &= \begin{pmatrix}(C_1+C_2)e^t\cos t\\ \quad +(-C_1+C_2)e^t\sin t\\ (-C_1+C_2)e^t\cos t\\ \quad +(-C_1-C_2)e^t\sin t\end{pmatrix},\end{aligned}$$

so $C_1\mathbf{x}(t)+C_2\mathbf{y}(t)$ is a solution of the system.

10. We saw that the system in Exercise 4 can be written as $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

If $\mathbf{x}(t) = (\cos t, \sin t)^T$, then

$$\mathbf{x}'(t) = \begin{pmatrix}\cos t\\ \sin t\end{pmatrix}' = \begin{pmatrix}-\sin t\\ \cos t\end{pmatrix},$$

and

$$A\mathbf{x}(t) = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos t\\ \sin t\end{pmatrix} = \begin{pmatrix}-\sin t\\ \cos t\end{pmatrix}.$$

Thus, $\mathbf{x}$ is a solution of the system in Exercise 4. Next, if $\mathbf{y}(t) = (\sin t, -\cos t)^T$, then

$$\mathbf{y}'(t) = \begin{pmatrix}\sin t\\ -\cos t\end{pmatrix}' = \begin{pmatrix}\cos t\\ \sin t\end{pmatrix},$$

and

$$A\mathbf{y}(t) = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sin t\\ -\cos t\end{pmatrix} = \begin{pmatrix}\cos t\\ \sin t\end{pmatrix}.$$

Thus, $\mathbf{y}$ is a solution of the system in Exercise 4. Now,

$$C_1\mathbf{x}(t) + C_2\mathbf{y}(t) = \begin{pmatrix} C_1\cos t + C_2\sin t \\ C_1\sin t - C_2\cos t \end{pmatrix},$$

and

$$(C_1\mathbf{x}(t) + C_2\mathbf{y}(t))' = \begin{pmatrix} -C_1\sin t + C_2\cos t \\ C_1\cos t + C_2\sin t \end{pmatrix}.$$

But,

$$\begin{aligned} &A(C_1\mathbf{x}(t) + C_2\mathbf{y}(t)) \\ &\quad = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} C_1\cos t + C_2\sin t \\ C_1\sin t - C_2\cos t \end{pmatrix} \\ &\quad = \begin{pmatrix} -C_1\sin t + C_2\cos t \\ C_1\cos t + C_2\sin t \end{pmatrix} \end{aligned}$$

so $C_1\mathbf{x}(t) + C_2\mathbf{y}(t)$ is a solution of the system in Exercise 4.

11. We saw in Exercise 5 that the system can be written as $\mathbf{x}' = A\mathbf{x} + \mathbf{w}$, where

$$A' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

If $\mathbf{x}_p(t) = (e^t, 0)^T$, then

$$\mathbf{x}_p'(t) = \begin{pmatrix} e^t \\ 0 \end{pmatrix}' = \begin{pmatrix} e^t \\ 0 \end{pmatrix}.$$

But,

$$\begin{aligned} A\mathbf{x}_p + \mathbf{w} &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ e^t \end{pmatrix} \\ &= \begin{pmatrix} e^t \\ -e^t \end{pmatrix} + \begin{pmatrix} 0 \\ e^t \end{pmatrix} \\ &= \begin{pmatrix} e^t \\ 0 \end{pmatrix}. \end{aligned}$$

Therefore, $\mathbf{x}_p$ is a solution of the system. Next, if $\mathbf{z}(t) = \mathbf{x}_p + C_1\mathbf{x}(t) + C_2\mathbf{y}(t)$, then

$$\begin{aligned} \mathbf{z}' &= \mathbf{x}_p' + C_1\mathbf{x}'(t) + C_2\mathbf{y}'(t) \\ &= A\mathbf{x}_p + \mathbf{w} + C_1A\mathbf{x} + C_2A\mathbf{y} \\ &= A[\mathbf{x}_p + C_1\mathbf{x} + C_2\mathbf{y}] + \mathbf{w} \\ &= A\mathbf{z} + \mathbf{w}. \end{aligned}$$

Thus, $\mathbf{z}$ is a solution of the system.

12. We saw in Exercise 6 that the system can be written $\mathbf{x}' = A\mathbf{x} + \mathbf{w}$, where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} \sin t \\ 0 \end{pmatrix}.$$

If $\mathbf{x}_p(t) = (1/2)(t\sin t - \cos t, -t\cos t)^T$, then

$$\begin{aligned} \mathbf{x}_p'(t) &= \frac{1}{2}\begin{pmatrix} t\sin t - \cos t \\ -t\cos t \end{pmatrix}' \\ &= \frac{1}{2}\begin{pmatrix} t\cos t + 2\sin t \\ t\sin t - \cos t \end{pmatrix}. \end{aligned}$$

But,

$$\begin{aligned} &A\mathbf{x}_p + \mathbf{w} \\ &\quad = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} t\sin t - \cos t \\ -t\cos t \end{pmatrix} + \begin{pmatrix} \sin t \\ 0 \end{pmatrix} \\ &\quad = \frac{1}{2}\begin{pmatrix} t\cos t + 2\sin t \\ t\sin t - \cos t \end{pmatrix}. \end{aligned}$$

Therefore, $\mathbf{x}_p$ is a solution of the system in Exercise 10. Next, if $\mathbf{z}(t) = \mathbf{x}_p(t) + C_1\mathbf{x}(t) + C_2\mathbf{y}(t)$, then

$$\begin{aligned} \mathbf{z}' &= \mathbf{x}_p' + C_1\mathbf{x}'(t) + C_2\mathbf{y}'(t) \\ &= A\mathbf{x}_p + \mathbf{w} + C_1A\mathbf{x} + C_2A\mathbf{y} \\ &= A[\mathbf{x}_p + C_1\mathbf{x} + C_2\mathbf{y}] + \mathbf{w} \\ &= A\mathbf{z} + \mathbf{w}. \end{aligned}$$

Thus, $\mathbf{z}$ is a solution of the system in Exercise 10.

13. In Exercise 7 we saw that $\mathbf{x}(t) = (e^{-t}, 0)^T$ and $\mathbf{y}(t) = (e^{2t}, e^{2t})^T$ were solutions of

$$\mathbf{x}' = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}\mathbf{x}.$$

If we evaluate these solutions at $t = 0$, then

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Because,

$$\det([\mathbf{x}(0), \mathbf{y}(0)]) = \det\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1 \neq 0,$$

these vectors are independent and form a fundamental solution set for the system. Hence, the general solution of the system is

$$\mathbf{z}(t) = C_1 \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}.$$

To find the solution with initial condition $\mathbf{z}(0) = (0, 1)^T$, substitute the initial condition in the general solution to get

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

This system has solutions $C_1 = -1$ and $C_2 = 1$. Thus, the solution is

$$\mathbf{z}(t) = -1 \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} + 1 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} -e^{-t} + e^{2t} \\ e^{2t} \end{pmatrix}.$$

14. In Exercise 8, we saw that $\mathbf{x}(t) = (-e^{2t}, e^{2t})^T$ and $\mathbf{y}(t) = (-e^{-2t}, 2e^{-2t})^T$ were solutions of

$$\mathbf{x}' = \begin{pmatrix} 6 & 4 \\ -8 & -6 \end{pmatrix} \mathbf{x}.$$

If we evaluate these solutions at $t = 0$, then

$$\mathbf{x}(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{y}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Because

$$\det([\mathbf{x}(0), \mathbf{y}(0)]) = \det \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} = -1 \neq 0,$$

these vectors are independent and form a fundamental solution set for the system. Hence, the general solution of the system is

$$\mathbf{z}(t) = C_1 \begin{pmatrix} -e^{2t} \\ e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} -e^{-2t} \\ 2e^{-2t} \end{pmatrix}.$$

To find the solution with initial condition $\mathbf{z}(0) = (1, -4)^T$, substitute $t = 0$ in the general solution to get

$$\begin{pmatrix} 1 \\ -4 \end{pmatrix} = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix},$$

This system has solution $C_1 = 2$ and $c_2 = -3$. Thus, the solution is

$$\begin{aligned} \mathbf{z}(t) &= 2 \begin{pmatrix} -e^{2t} \\ e^{2t} \end{pmatrix} - 3 \begin{pmatrix} -e^{-2t} \\ 2e^{-2t} \end{pmatrix} \\ &= \begin{pmatrix} -2e^{2t} + 3e^{-2t} \\ 2e^{2t} - 6e^{-2t} \end{pmatrix}. \end{aligned}$$

15. In Exercise 9, we saw that $\mathbf{x}(t) = (e^t \cos t, -e^t \sin t)^T$ and $\mathbf{y}(t) = (e^t \sin t, e^t \cos t)^T$ were solutions of

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{x}.$$

If we evaluate the solutions at $t = 0$, then

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{y}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Because $\det([\mathbf{x}(0), \mathbf{y}(0)]) = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \neq 0$, the vectors are independent and form a fundamental set of solutions. Hence, then general solution of the system is

$$\mathbf{z}(t) = C_1 \begin{pmatrix} e^t \cos t \\ -e^t \sin t \end{pmatrix} + C_2 \begin{pmatrix} e^t \sin t \\ e^t \cos t \end{pmatrix}.$$

To find the particular solution, substitute the initial condition $\mathbf{z}(0) = (-2, 3)^T$.

$$\begin{pmatrix} -2 \\ 3 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, $C_1 = -2$ and $C_2 = 3$ and the solution is

$$\begin{aligned} \mathbf{z}(t) &= -2 \begin{pmatrix} e^t \cos t \\ -e^t \sin t \end{pmatrix} + 3 \begin{pmatrix} e^t \sin t \\ e^t \cos t \end{pmatrix} \\ &= \begin{pmatrix} -2e^t \cos t + 3e^t \sin t \\ 2e^t \sin t + 3e^t \cos t \end{pmatrix}. \end{aligned}$$

16. In Exercise 10, we saw that $\mathbf{x}(t) = (\cos t, \sin t)^T$ and $\mathbf{y}(t) = (\sin t, -\cos t)^T$ were solutions of

$$\mathbf{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x}.$$

If we evaluate the solutions at $t = 0$, then

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{y}(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Because

$$\det([\mathbf{x}(0), \mathbf{y}(0)]) = \det \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -1 \neq 0,$$

the vectors are independent and form a fundamental solution set for the system. Hence, the general solution of the system is

$$\mathbf{z}(t) = C_1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + C_2 \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}.$$

To find the solution with initial condition $\mathbf{z}(0) = (3, 2)^T$, substitute $t = 0$ in the general solution to get

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$
$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

This system has solutions $C_1 = 3$ and $C_2 = -2$. Thus, the solution is

$$\mathbf{z}(t) = 3 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} - 2 \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$$
$$\mathbf{z}(t) = \begin{pmatrix} 3\cos t - 2\sin t \\ 3\sin t + 2\cos t \end{pmatrix}.$$

17. (a) If $\mathbf{x} = (x_1, x_2)^T = (0, e^t)^T$, then

$$x_1' = 0 = 0 \cdot e^t = x_1 x_2, \text{ and}$$
$$x_2' = e^t = x_2.$$

Hence, $\mathbf{x}$ is a solution of the system

$$x_1' = x_1 x_2$$
$$x_2' = x_2.$$

If $\mathbf{y} = (y_1, y_2)^T = (1, 0)^T$, then

$$x_1' = 0 = 1 \cdot 0 = x_1 x_2, \text{ and}$$
$$x_2' = 0 = x_2,$$

so $\mathbf{y}$ is also a solution of the system.

(b) If $\mathbf{z}(t) = \mathbf{x}(t) + \mathbf{y}(t)$, then $\vec{z} = (z_1, z_2)^T = (1, e^t)^T$. But,

$$\mathbf{z}_1' = 0 \neq 1 \cdot e^t = z_1 z_2,$$

so $\mathbf{z}$ is *not* a solution of the system. There is no contradiction of Theorem 5.1 because the system in *not* linear.

18. Let $\mathbf{x}(t)$ be a solution of

$$\mathbf{y}' = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix} \mathbf{y}$$

with initial condition $\mathbf{x}(0) = (1, -1)^T$. We know that

$$\mathbf{y}(t) = C_1 \begin{pmatrix} 2e^{-t} \\ e^{-t} \end{pmatrix} + C_2 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$$

is a solution of the system. Set $\mathbf{y}(0) = (1, -1)^T$.

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

This system has solution $C_1 = 2$ and $C_2 = -3$. Thus,

$$\mathbf{y}(t) = 2 \begin{pmatrix} 2e^{-t} \\ e^{-t} \end{pmatrix} - 3 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$$

is a solution of the system with $\mathbf{y}(0) = (1, -1)^T$. By the uniqueness theorem,

$$x(t) = y(t) = 2 \begin{pmatrix} 2e^{-t} \\ e^{-t} \end{pmatrix} - 3 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$$

for all t.

19. Evaluate each solution $\mathbf{y}_1(t) = (-e^{-t}, -e^{-t}, e^{-t})^T$, $\mathbf{y}_2(t) = (0, -e^t, 2e^t)^T$, and $\mathbf{y}_3(t) = (e^{2t}, 0, 2e^{2t})^T$ at $t = 0$.

$$\mathbf{y}_1(0) = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{y}_2(0) = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \quad \text{and}$$
$$\mathbf{y}_3(0) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

Because

$$\det\begin{pmatrix}-1 & 0 & 1\\ -1 & -1 & 0\\ 1 & 2 & 2\end{pmatrix} = 1 \neq 0,$$

these vectors are independent. Therefore, by Proposition 5.12, the solutions $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are independent for all t.

20. Evaluate each solution $\mathbf{y}_1(t) = (e^{-t}, e^{-t}, 2e^{-t})^T$, $\mathbf{y}_2(t) = (-e^t, 3e^t, 0)^T$, and $\mathbf{y}_3(t) = (2e^{-t} - 3e^t, 2e^{-t} + 9e^t, 4e^{-t})^T$ at $t = 0$.

$$\mathbf{y}_1(0) = \begin{pmatrix}1\\1\\2\end{pmatrix}, \quad \mathbf{y}_2(0) = \begin{pmatrix}-1\\3\\0\end{pmatrix}, \quad \text{and}$$

$$\mathbf{y}_3(0) = \begin{pmatrix}-1\\11\\4\end{pmatrix}$$

Because

$$\det\begin{pmatrix}1 & -1 & -1\\ 1 & 3 & 11\\ 2 & 0 & 4\end{pmatrix} = 0,$$

these vectors are dependent. Therefore, by Proposition 5.12, the solutions $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are dependent for all t.

21. Evaluate each solution $\mathbf{y}_1(t) = (-e^{-t} + e^{2t}, -e^{-t} + e^{2t}, e^{2t})^T$, $\mathbf{y}_2(t) = (-e^t + e^{-t}, e^{-t}, 0)^T$, and $\mathbf{y}_3(t) = (e^t, 0, 0)^T$ at $t = 0$.

$$\mathbf{y}_1(0) = \begin{pmatrix}0\\0\\1\end{pmatrix}, \quad \mathbf{y}_2(0) = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \text{and}$$

$$\mathbf{y}_3(0) = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

Because

$$\det\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix} = -1 \neq 0,$$

then these vectors are independent. Therefore, by Proposition 5.12, the solutions $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are independent for all t.

22. Evaluate each solution $\mathbf{y}_1(t) = (e^{-t}, 2e^{-t}, 2e^{-t})^T$, $\mathbf{y}_2(t) = (e^t, e^t, 0)^T$, $\mathbf{y}_3(t) = (0, 0, e^t)^T$ at $t = 0$.

$$\mathbf{y}_1(0) = \begin{pmatrix}1\\2\\2\end{pmatrix}, \quad \mathbf{y}_2(0) = \begin{pmatrix}1\\1\\0\end{pmatrix}, \quad \text{and}$$

$$\mathbf{y}_3(0) = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Because

$$\det\begin{pmatrix}1 & 1 & 0\\ 2 & 1 & 0\\ 2 & 0 & 1\end{pmatrix} = -1,$$

these vectors are independent. Therefore, by Proposition 5.12, the solutions $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are independent for all t.

23. If $\mathbf{y}_1(t) = (-e^{2t}, 2e^{2t})^T$, then

$$\mathbf{y}_1' = \begin{pmatrix}-e^{2t}\\2e^{2t}\end{pmatrix}' = \begin{pmatrix}-2e^{2t}\\4e^{2t}\end{pmatrix},$$

and

$$\begin{pmatrix}-6 & -4\\ 8 & 6\end{pmatrix}\mathbf{y}_1 = \begin{pmatrix}-6 & -4\\ 8 & 6\end{pmatrix}\begin{pmatrix}-e^{2t}\\2e^{2t}\end{pmatrix} = \begin{pmatrix}-2e^{2t}\\4e^{2t}\end{pmatrix}.$$

Therefore, $\mathbf{y}_1$ is a solution. In a similar manner, you can also show that $\mathbf{y}_2(t) = (-e^{-2t}, e^{-2t})^T$ is a solution. Evaluating each solution at $t = 0$,

$$\mathbf{y}_1(0) = \begin{pmatrix}-1\\2\end{pmatrix} \quad \text{and} \quad \mathbf{y}_2(0) = \begin{pmatrix}-1\\1\end{pmatrix}.$$

These vectors are independent ($\mathbf{y}_1$ is not a multiple of $\mathbf{y}_2$), so the solutions are independent for all t and form a fundamental set of solutions. Thus, the general solution is

$$\mathbf{y}(t) = C_1\begin{pmatrix}-e^{2t}\\2e^{2t}\end{pmatrix} + C_2\begin{pmatrix}-e^{-2t}\\e^{-2t}\end{pmatrix}.$$

Substitute the initial condition $\mathbf{y}(0) = (-5, 8)^T$.

$$\begin{pmatrix}-5\\8\end{pmatrix} = C_1\begin{pmatrix}-1\\2\end{pmatrix} + C_2\begin{pmatrix}-1\\1\end{pmatrix}$$

$$\begin{pmatrix}-5\\8\end{pmatrix} = \begin{pmatrix}-1 & -1\\ 2 & 1\end{pmatrix}\begin{pmatrix}C_1\\C_2\end{pmatrix}$$

This system has solution $C_1 = 3$ and $C_2 = 2$, so the final solution is

$$\mathbf{y}(t) = 3\begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix} + 2\begin{pmatrix} -e^{-2t} \\ e^{-2t} \end{pmatrix} = \begin{pmatrix} -3e^{2t} - 2e^{-2t} \\ 6e^{2t} + 2e^{-2t} \end{pmatrix}.$$

24. If $\mathbf{y}_1(t) = (e^{-2t}, 0)^T$, then

$$\mathbf{y}_1' = \begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix}' = \begin{pmatrix} -2e^{-2t} \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} -2 & -2 \\ 0 & -4 \end{pmatrix}\mathbf{y}_1 = \begin{pmatrix} -2 & -2 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix} = \begin{pmatrix} -2e^{-2t} \\ 0 \end{pmatrix}.$$

Therefore, $\mathbf{y}_1$ is a solution. If $\vec{y}_2(t) = (e^{-4t}, e^{-4t})^T$, then

$$\mathbf{y}_2' = \begin{pmatrix} e^{-4t} \\ e^{-4t} \end{pmatrix}' = \begin{pmatrix} -4e^{-4t} \\ -4e^{-4t} \end{pmatrix},$$

and

$$\begin{pmatrix} -2 & -2 \\ 0 & -4 \end{pmatrix}\mathbf{y}_2 = \begin{pmatrix} -2 & -2 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} e^{-4t} \\ e^{-4t} \end{pmatrix} = \begin{pmatrix} -4e^{-4t} \\ -4e^{-4t} \end{pmatrix}$$

Therefore, $\mathbf{y}_2$ is a solution. Further, evaluating each solution at $t = 0$,

$$\mathbf{y}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{y}_2(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

These vectors are clearly independent ($\mathbf{y}_1$ is not a multiple of $\mathbf{y}_2$), so the solutions $\mathbf{y}_1$ and $\vec{y}_2$ are independent for all t and form a fundamental set of solutions. Thus, the general solution is

$$\mathbf{y} = C_1\begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix} + C_2\begin{pmatrix} e^{-4t} \\ e^{-4t} \end{pmatrix}.$$

Substitute the initial condition $\mathbf{y}(0) = (3, 1)^T$.

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = C_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

This system has solution $C_1 = 2$ and $C_2 = 1$, so the final solution is

$$\mathbf{y}(t) = 2\begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix} + \begin{pmatrix} e^{-4t} \\ e^{-4t} \end{pmatrix} = \begin{pmatrix} 2e^{-2t} + e^{-4t} \\ e^{-4t} \end{pmatrix}.$$

25. If $\mathbf{y}_1(t) = (e^{2t}, e^{2t})^T$, then

$$\mathbf{y}_1' = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}' = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix},$$

and

$$\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}\mathbf{y}_1 = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix}.$$

In a similar manner, you can also show that $\vec{y}_2(t) = (e^{2t}(t+2), e^{2t}(t+1))^T$ is a solution of the system. Further, evaluating each solution at $t = 0$,

$$\mathbf{y}_1(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{y}_2(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The vectors are independent ($\mathbf{y}_1$ is not a multiple of $\mathbf{y}_2$), so the solutions $\mathbf{y}_1$ and $\mathbf{y}_2$ are independent for all t and form a fundamental set of solutions. Thus, the general solution is

$$\mathbf{y}(t) = C_1\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} + C_2\begin{pmatrix} e^{2t}(t+2) \\ e^{2t}(t+1) \end{pmatrix}.$$

Substitute the initial condition $\mathbf{y}(0) = (0, 1)^T$.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

This system has solution $C_1 = 2$ and $C_2 = -1$. Thus, the final solution is

$$\mathbf{y}(t) = 2\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} - \begin{pmatrix} e^{2t}(t+2) \\ e^{2t}(t+1) \end{pmatrix} = \begin{pmatrix} -te^{2t} \\ (1-t)e^{2t} \end{pmatrix}.$$

26. If $\mathbf{y}_1(t) = ((1/2)\cos t - (1/2)\sin t, \cos t)^T$, then

$$\mathbf{y}_1' = \begin{pmatrix} (1/2)\cos t - (1/2)\sin t \\ \cos t \end{pmatrix}' = \begin{pmatrix} -(1/2)\sin t - (1/2)\cos t \\ -\sin t \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}\mathbf{y}_1 = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} (1/2)\cos t - (1/2)\sin t \\ \cos t \end{pmatrix} = \begin{pmatrix} -(1/2)\sin t - (1/2)\cos t \\ -\sin t \end{pmatrix}.$$

Therefore, $\mathbf{y}_1$ is a solution. If $\mathbf{y}_2((1/2)\sin t + (1/2)\cos t, \sin t)^T$, then

$$\mathbf{y}_2' = \begin{pmatrix} (1/2)\sin t + (1/2)\cos t \\ \sin t \end{pmatrix}' = \begin{pmatrix} (1/2)\cos t - (1/2)\sin t \\ \cos t \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}\mathbf{y}_2 = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} (1/2)\sin t + (1/2)\cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} (1/2)\cos t - (1/2)\sin t \\ \cos t \end{pmatrix}.$$

Therefore, $\mathbf{y}_2$ is a solution. Further, evaluating each solution at $t = 0$,

$$\mathbf{y}_1(0) = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{y}_2(0) = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}.$$

These vectors are clearly independent ($\mathbf{y}_1$ is not a multiple of $\mathbf{y}_2$), so the solutions $\mathbf{y}_1$ and $\vec{y}_2$ are independent for all t and form a fundamental set of solutions. Thus, the general solution is

$$\mathbf{y}(t) = C_1\begin{pmatrix} (1/2)\cos t - (1/2)\sin t \\ \cos t \end{pmatrix} + C_2\begin{pmatrix} (1/2)\sin t + (1/2)\cos t \\ \sin t \end{pmatrix}.$$

Substitute the initial condition $\mathbf{y}(0) = (1, 0)^T$.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = C_1\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

This system has solution $C_1 = 0$ and $C_2 = 2$, so the final solution is

$$\mathbf{y}(t) = 2\begin{pmatrix} ((1/2)\sin t + (1/2)\cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} \sin t + \cos t \\ 2\sin t \end{pmatrix}.$$

27. A careful examination of the flow rates in and out of each tank will indicate that the volume of solution in each tank remains unchanged (3L) over time. Let $x_A(t)$ and $x_B(t)$ represent the salt content (kg) of tanks A and B, respectively. Let's begin with an analysis of tank A. We use the so-called "balance law,"

$$\frac{dx_A}{dt} = \text{Rate in} - \text{Rate out}.$$

Pure water enters tank A at 5 L/min, so there is no salt entering there. However, solution from tank B enters tank A at a rate of 4 L/min. The salt concentration in tank B is $x_B(t)/3$ kg/L, so the rate at which salt in entering tank A from tank B is

$$4\,\text{L/min} \times x_B(t)/3\,\text{kg/L} = 4x_B(t)/3\,\text{kg/min}.$$

Solution is leaving tank A for tank B at a rate of 9 L/min. The salt concentration of tank A is $x_A(t)/3$ kg/L. Thus, salt is leaving tank A at a rate of

$$9\,\text{L/min} \times x_A(t)/3\,\text{kg/L} = 3x_A(t)\,\text{kg/min}.$$

Thus, using the balance law above,

$$\frac{dx_A}{dt} = \frac{4x_B}{3} - 3x_A.$$

On tank B,

$$\frac{dx_B}{dt} = \text{Rate in} - \text{Rate out}.$$

We already know that salt enters tank B from tank A at a rate of $3x_A(t)$ kg/min. We also know that salt is leaving tank B for tank A at a rate of $4x_B(t)/3$ kg/min. Let's examine the drain, where solution leaves tank B at a rate of 5 L/min. Because the concentration of salt in tank B is $x_B(t)/3$ kg/L, salt leaves tank B through the drain at

$$5\,\text{L/min} \times x_B/3 = 5x_B/3\,\text{kg/min}.$$

Using the balance law,

$$\frac{dx_B}{dt} = 3x_A - \frac{4x_B}{3} - \frac{5x_B}{3} = 3x_A - 3x_B.$$

28. A careful examination of the flow rates in and out of each tank will indicate that the volume of solution in each tank remains unchanged (200 gal) over time. We let $x_A(t)$, $x_B(t)$, and $x_C(t)$ represent the salt content (in lb) of Tank A, Tank B, and Tank C, respectively. Let's begin with an analysis of Tank A. We use the so-called "balance law."

$$\frac{dx_A}{dt} = \text{Rate in} - \text{Rate out}.$$

Pure water is poured into Tank A at 5 gal/min, so there is no salt entering there. However, solution from Tank B enters Tank A at a rate of 4 gal/min. The salt concentration in Tank B is $x_B(t)/200$ lb/gal, so the rate at which salt is entering Tank A from Tank B is

$$4\,\text{gal/min} \times x_B(t)/200\,\text{lb/gal} = x_B(t)/50\,\text{lb/min}.$$

Solution is leaving Tank A for Tank B at a rate of 9 gal/min. The salt concentration of Tank A is $x_A(t)/200$ lb/gal. Thus, salt is leaving Tank A at a rate of

$$9\,\text{gal/min} \times x_A(t)/200\,\text{lb/gal} = 9x_A(t)/200\,\text{lb/min}.$$

Thus, using the balance law above,

$$\frac{dx_A}{dt} = \frac{x_B}{50} - \frac{9x_A}{200}.$$

On to Tank B.

$$\frac{dx_B}{dt} = \text{Rate in} - \text{Rate out}$$

We already have salt exchanges from Tank A to Tank B and vice versa, so let's analyze the salt exchange between Tank B and Tank C. Solution enters Tank B from Tank C at a rate of 4 gal/min. But the concentration of salt in Tank C is $x_C(t)/200$ lb/gal. Thus, salt enters Tank B from Tank C at a rate

$$4\,\text{gal/min} \times x_C(t)/200\,\text{lb/gal} = x_C(t)/50\,\text{lb/min}.$$

Solution leaves Tank B and enters Tank C at a rate of 9 gal/min. However, the concentration of salt in Tank B is $x_B(t)/200$ lb/gal. Thus, salt leaves Tank B and enters Tank C at a rate

$$9\,\text{gal/min} \times x_B(t)/200\,\text{lb/gal} = 9x_B(t)/200\,\text{lb/min}.$$

Substituting salt exchanges in the balance law above,

$$\begin{aligned}\frac{dx_B}{dt} &= \left(\frac{9x_A}{200} + \frac{x_C}{50}\right) - \left(\frac{x_B}{50} + \frac{9x_B}{200}\right)\\ &= \frac{9x_A}{200} - \frac{13x_B}{200} + \frac{x_C}{50}.\end{aligned}$$

One tank to go.

$$\frac{dx_C}{dt} = \text{Rate in} - \text{Rate out}$$

We already know the salt exchanges between Tank B and Tank C, so we only need to compute the rate at which salt is leaving from Tank C through the remaining spigot. Solution leaves Tank C through this spigot at a rate of 5 gal/min. As the concentration of salt in Tank C is $x_C(t)/200$ lb/gal, the rate at which salt leaves Tank C through this spigot is

$$5\,\text{gal/min} \times x_C(t)/200\,\text{lb/gal} = x_C(t)/40\,\text{lb/min}.$$

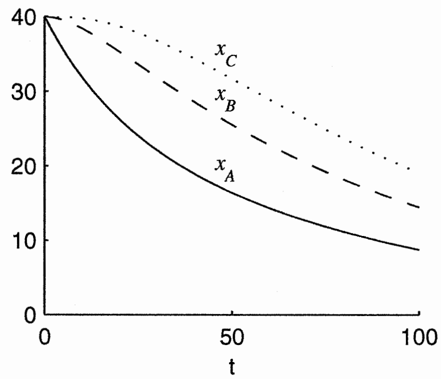
Thus, using the balance law above,

$$\begin{aligned}\frac{dx_C}{dt} &= \frac{9x_B}{200} - \left(\frac{x_C}{50} + \frac{x_C}{40}\right)\\ &= \frac{9x_B}{200} - \frac{9x_C}{200}.\end{aligned}$$

29. In Exercise 28, the salt content in tanks A, B, and C was modeled by the system

$$\begin{aligned}\frac{dx_A}{dt} &= \frac{x_B}{50} - \frac{9x_A}{200}\\ \frac{dx_B}{dt} &= \frac{9x_A}{200} - \frac{13x_B}{200} + \frac{x_C}{50}\\ \frac{dx_C}{dt} &= \frac{9x_B}{200} - \frac{9x_C}{200}.\end{aligned}$$

Setting $x_A(0) = x_B(0) = x_C(0) = 40$, the salt content in each tank is pictured below.



30. Because solution is entering and leaving each tank at the same rate (10 gal/min), each tank maintains a constant volume of solution (200 gal). Let $x(t)$, $y(t)$, and $z(t)$ represent the amount of salt in the first, second, and third tanks in the cascade, respectively. We invoke the balance law for the first tank.

$$\frac{dx}{dt} = \text{Rate in} - \text{Rate out}$$

No salt enters the first tank, but salt solution leaves the first tank at a rate of 10 gal/min. The concentration of salt in the first tank is $x(t)/200$, so the rate at which salt leaves the first tank is

$$10\,\text{gal/min} \times x(t)/200\,\text{lb/gal} = x(t)/20\,\text{lb/min}.$$

Using these salt rates in the above balance law,

$$\frac{dx}{dt} = -\frac{x}{20}.$$

The balance law for the second tank is

$$\frac{dy}{dt} = \text{Rate in} - \text{Rate out}.$$

The rate at which salt enters the second tank is the same as the rate at which salt leaves the first tank, $x/20$ lb/min. Salt solution leaves the second tank at a rate of 10 gal/min. The concentration of salt in the second tank is $y(t)/200$ lb/gal, so the rate at which salt leaves the second tank is

$$10\,\text{gal/min} \times y(t)/200\,\text{lb/gal} = y(t)/20\,\text{lb/min}.$$

Using these salt rates in the above balance law,

$$\frac{dy}{dt} = \frac{x}{20} - \frac{y}{20}.$$

The balance law for the third tank is

$$\frac{dz}{dt} = \text{Rate in} - \text{Rate out}.$$

The rate at which salt enters the third tank is the same as the rate at which salt leaves the second tank, $y(t)/20$ lb/min. Salt solution leaves the third tank at a rate of 10 gal/min. The concentration of salt in the third tank is $z(t)/200$ lb/gal, so the rate at which salt leaves the third tank is

$$10\,\text{gal/min} \times z(t)/200\,\text{lb/gal} = z(t)/20\,\text{lb/min}.$$

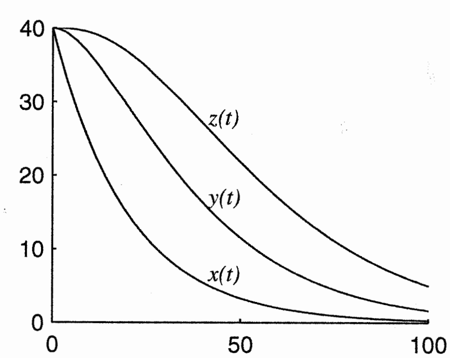
Using these salt rates in the above balance law,

$$\frac{dz}{dt} = \frac{y}{20} - \frac{z}{20}.$$

31. In Exercise 30, we saw that

$$\begin{aligned}\frac{dx}{dt} &= -\frac{x}{20}\\ \frac{dy}{dt} &= \frac{x}{20} - \frac{y}{20}\\ \frac{dz}{dt} &= \frac{y}{20} - \frac{z}{20}\end{aligned}$$

modeled the salt content in three cascading tanks. If $x(0) = y(0) = z(0) = 40$, then the following figure shows the salt content in each tank over time.



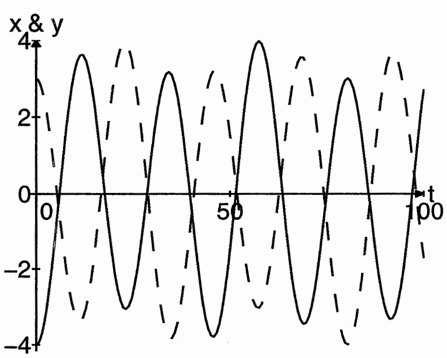
32. Given $k_1 = k_2 = k_3 = 1$ dyne/cm, and $m1 = m2 = 40$ g, the equations of motion for the coupled oscillator become

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= -\frac{k_1+k_2}{m_1}u_1 + \frac{k_2}{m_1}u_3 = -\frac{1}{20}u_1 + \frac{1}{40}u_3 \\ u_3' &= u_4 \\ u_4' &= \frac{k_2}{m_2}u_1 - \frac{k_2+k_3}{m_2}u_3 = \frac{1}{40}u_1 - \frac{1}{20}u_3. \end{aligned}$$

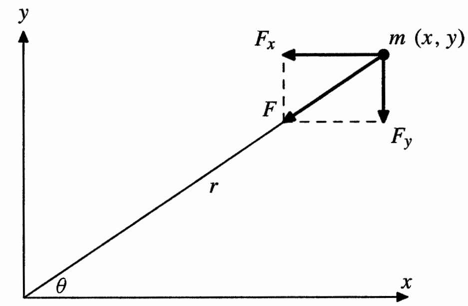
Initial displacements are $x(0) = -4$ cm and $y(0) = 3$ cm. Both masses are released from rest, so $x'(0) = y'(0) = 0$. These lead to initial conditions

$$\begin{aligned} u_1(0) &= x(0) = -4 \\ u_2(0) &= x'(0) = 0 \\ u_3(0) &= y(0) = 3 \\ u_4(0) &= y'(0) = 0 \end{aligned}$$

This system and initial conditions lead to the solution that follows. The solid curve is x and the dashed curve is y.



33. In the figure,



F represents the magnitude of the radially inward force. It is easy to see that

$$\begin{aligned} F_x &= F\cos\theta \\ F_y &= F\sin\theta. \end{aligned}$$

Furthermore,

$$\begin{aligned} \cos\theta &= \frac{x}{r} \\ \sin\theta &= \frac{y}{r}, \end{aligned}$$

and with these substitutions and the fact that $F = km/r^2$, the former equations become

$$\begin{aligned} F_x &= -\frac{km}{r^2}\cdot\frac{x}{r} = -\frac{kmx}{r^3}, \\ F_y &= -\frac{km}{r^2}\cdot\frac{y}{r} = -\frac{kmy}{r^3}, \end{aligned}$$

the minus signs being present because the force is directed opposite the displacement. Next, Newton gives us $F_x = ma_x = mx''$ and $F_y = ma_y = my''$, so

$$\begin{aligned} mx'' &= -\frac{kmx}{r^3} \\ my'' &= -\frac{kmy}{r^3}, \end{aligned}$$

or,

$$\begin{aligned} x'' &= -\frac{kx}{r^3} \\ y'' &= -\frac{ky}{r^3}. \end{aligned}$$

If we now let $u_1 = x$, $u_2 = x'$, $u_3 = y$, and $u_4 = y'$, then

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= -\frac{ku_1}{r^3} \\ u_3' &= u_4 \\ u_4' &= -\frac{ku_3}{r^3}. \end{aligned}$$

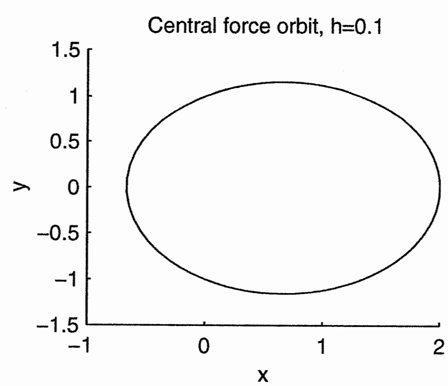
Note that $r = (x^2 + y^2)^{1/2} = (u_1^2 + u_3^2)^{1/2}$, so we actually have

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= -\frac{ku_1}{(u_1^2 + u_3^2)^{3/2}} \\ u_3' &= u_4 \\ u_4' &= -\frac{ku_3}{(u_1^2 + u_3^2)^{3/2}}. \end{aligned}$$

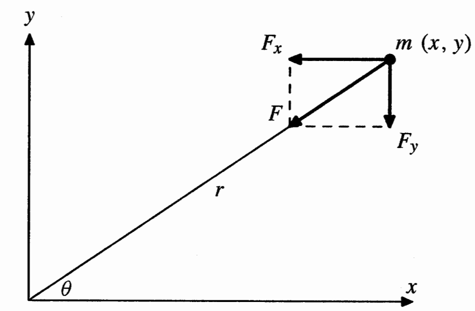
34. In Exercise 33, we saw that (with $k = 1$)

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= -\frac{u_1}{(u_1^2 + u_3^2)^{3/2}} \\ u_3' &= u_4 \\ u_4' &= -\frac{u_3}{(u_1^2 + u_3^2)^{3/2}} \end{aligned}$$

modeled the motion of the particle in the central force field. With $u_1(0) = x(0) = 2$, $u_2(0) = x'(0) = 0$, $u_3(0) = y(0) = 0$, and $u_4(0) = y'(0) = -0.5$, the motion of the particle in the xy plane follows. We used RK4 with step size $h = 0.1$ to produce the orbit.



35. In the figure,



F represents the magnitude of the radially inward force. It is easy to see that

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Furthermore,

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and with these substitutions and the fact that $F = km/r^3$, the former equations become

$$\begin{aligned} F_x &= -\frac{km}{r^3} \cdot \frac{x}{r} = -\frac{kmx}{r^4}, \\ F_y &= -\frac{km}{r^3} \cdot \frac{y}{r} = -\frac{kmy}{r^4}, \end{aligned}$$

the minus signs being present because the force is directed opposite the displacement. Next, Newton gives us $F_x = ma_x = mx''$ and $F_y = ma_y = my''$, so

$$\begin{aligned} mx'' &= -\frac{kmx}{r^4} \\ my'' &= -\frac{kmy}{r^4}, \end{aligned}$$

or,

$$\begin{aligned} x'' &= -\frac{kx}{r^4} \\ y'' &= -\frac{ky}{r^4}. \end{aligned}$$

If we now let $u_1 = x$, $u_2 = x'$, $u_3 = y$, and $u_4 = y'$, then

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= -\frac{ku_1}{r^4} \\ u_3' &= u_4 \\ u_4' &= -\frac{ku_3}{r^4}. \end{aligned}$$

Note that $r = (x^2 + y^2)^{1/2} = (u_1^2 + u_3^2)^{1/2}$, so we actually have

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= -\frac{ku_1}{(u_1^2 + u_3^2)^2} \\ u_3' &= u_4 \\ u_4' &= -\frac{ku_3}{(u_1^2 + u_3^2)^2}. \end{aligned}$$