

## ***Solution***      ***Section 2.4 – Chain Rule***

### ***Exercise***

Express  $\frac{dw}{dt}$  as a function of  $t$ , then evaluate  $\frac{dw}{dt}$  at the given value of  $t$ .

$$w = x^2 + y^2, \quad x = \cos t, \quad y = \sin t, \quad t = \pi$$

### **Solution**

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \\ &= \frac{\partial}{\partial x} (x^2 + y^2) \frac{d}{dt} (\cos t) + \frac{\partial}{\partial y} (x^2 + y^2) \frac{d}{dt} (\sin t) \\ &= 2x(-\sin t) + 2y \cos t \\ &= -2(\cos t) \sin t + 2(\sin t) \cos t \\ &= \underline{0} \quad | \end{aligned}$$

$$\frac{dw}{dt}(t = \pi) = \underline{0} \quad |$$

$$\begin{aligned}w &= x^2 + y^2 \\ &= \cos^2 t + \sin^2 t \\ &= 1 \\ \frac{dw}{dt} &= 0\end{aligned}$$

### ***Exercise***

Express  $\frac{dw}{dt}$  as a function of  $t$ , then evaluate  $\frac{dw}{dt}$  at the given value of  $t$ .

$$w = x^2 + y^2, \quad x = \cos t + \sin t, \quad y = \cos t - \sin t, \quad t = 0$$

### **Solution**

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \\ &= \frac{\partial}{\partial x} (x^2 + y^2) \frac{d}{dt} (\cos t + \sin t) + \frac{\partial}{\partial y} (x^2 + y^2) \frac{d}{dt} (\cos t - \sin t) \\ &= (2x)(-\sin t + \cos t) + (2y)(-\sin t - \cos t) \\ &= 2(\cos t + \sin t)(\cos t - \sin t) - 2(\cos t - \sin t)(\sin t + \cos t) \\ &= \underline{0} \quad | \end{aligned}$$

$$\frac{dw}{dt}(t = 0) = \underline{0} \quad |$$

### Exercise

Express  $\frac{dw}{dt}$  as a function of  $t$ , then evaluate  $\frac{dw}{dt}$  at the given value of  $t$ .

$$w = \ln(x^2 + y^2 + z^2), \quad x = \cos t, \quad y = \sin t, \quad z = 4\sqrt{t}, \quad t = 3$$

### Solution

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} \\ &= \frac{2x}{x^2 + y^2 + z^2}(-\sin t) + \frac{2y}{x^2 + y^2 + z^2}(\cos t) + \frac{2z}{x^2 + y^2 + z^2} \left( 2 \frac{1}{\sqrt{t}} \right) \\ &= \frac{-2\cos t \sin t + 2\sin t \cos t + 4(4\sqrt{t})(t^{-1/2})}{\cos^2 t + \sin^2 t + 16t} \\ &= \frac{16}{1+16t} \end{aligned}$$

$$\begin{aligned} w &= \ln(x^2 + y^2 + z^2) \\ &= \ln(\cos^2 t + \sin^2 t + 16t) \\ &= \ln(1 + 16t) \end{aligned}$$

$$\frac{dw}{dt} = \frac{16}{1+16t}$$

$$\begin{aligned} \frac{dw}{dt}(3) &= \frac{16}{1+16(3)} \\ &= \frac{16}{49} \end{aligned}$$

### Exercise

Express  $\frac{dw}{dt}$  as a function of  $t$ , then evaluate  $\frac{dw}{dt}$  at the given value of  $t$ .

$$w = z - \sin xy, \quad x = t, \quad y = \ln t, \quad z = e^{t-1}, \quad t = 1$$

### Solution

$$\begin{aligned} \frac{\partial w}{\partial t} &= (-y \cos xy)(1) + (-x \cos xy)\left(\frac{1}{t}\right) + (1)(e^{t-1}) \\ &= -(\ln t) \cos(t \ln t) - \cos(t \ln t) + e^{t-1} \\ &= -(\ln t + 1) \cos(t \ln t) + e^{t-1} \end{aligned}$$

$$w = z - \sin xy$$

$$\begin{aligned}
&= e^{t-1} - \sin(t \ln t) \\
\frac{\partial w}{\partial t} &= e^{t-1} - \cos(t \ln t) \left[ \ln t + t \left( \frac{1}{t} \right) \right] \\
&= e^{t-1} - (\ln t + 1) \cos(t \ln t) \\
\frac{\partial w}{\partial t}(\mathbf{1}) &= -(\ln \mathbf{1} + 1) \cos(\mathbf{1} \ln \mathbf{1}) + e^{\mathbf{1}-1} \\
&= -1 \cos 0 + 1 \\
&= \underline{0}
\end{aligned}$$

### Exercise

Express  $\frac{dw}{dt}$  as a function of  $t$ , then evaluate  $\frac{dw}{dt}$  at the given value of  $t$ .

$$w = \sin(xy + \pi), \quad x = e^t, \quad y = \ln(t+1) \quad t = 0$$

### Solution

$$\begin{aligned}
\frac{\partial w}{\partial t} &= y \cos(xy + \pi) e^t + x \cos(xy + \pi) \frac{1}{t+1} & \frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \\
&= e^t \ln(t+1) \cos(e^t \ln(t+1) + \pi) + e^t \cos(e^t \ln(t+1) + \pi) \frac{1}{t+1} \\
&= \underline{e^t \left( \ln(t+1) + \frac{1}{t+1} \right) \cos(e^t \ln(t+1) + \pi)} \\
\frac{\partial w}{\partial t} \Big|_{t=0} &= \cos \pi \\
&= \underline{-1}
\end{aligned}$$

### Exercise

Express  $\frac{dw}{dt}$  as a function of  $t$ , then evaluate  $\frac{dw}{dt}$  at the given value of  $t$ .

$$w = xe^y + y \sin z - \cos z, \quad x = 2\sqrt{t}, \quad y = t - 1 + \ln t, \quad z = \pi t, \quad t = 1$$

### Solution

$$\begin{aligned}
\frac{\partial w}{\partial t} &= e^y \frac{1}{\sqrt{t}} + (xe^y + \sin z) \left( 1 + \frac{1}{t} \right) + (y \cos z + \sin z)(\pi) & \frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} \\
&= \underline{\frac{1}{\sqrt{t}} e^{t-1+\ln t} + (2\sqrt{t} e^{t-1+\ln t} + \sin(\pi t)) \left( 1 + \frac{1}{t} \right) + \pi (e(t-1+\ln t) \cos(\pi t) + \sin(\pi t))} \\
\frac{\partial w}{\partial t} \Big|_{t=1} &= 1 + (2 + \sin \pi)(2) + \pi \sin(\pi) & &= 1 + 4 + 0 \\
&= \underline{5}
\end{aligned}$$

### Exercise

Express  $\frac{\partial z}{\partial u}$  and  $\frac{\partial z}{\partial v}$  as functions of  $u$  and  $v$  if  $z = 4e^x \ln y$ ,  $x = \ln(u \cos v)$ ,  $y = u \sin v$ , then evaluate  $\frac{\partial z}{\partial u}$  and  $\frac{\partial z}{\partial v}$  at the point  $(u, v) = \left(2, \frac{\pi}{4}\right)$ .

### Solution

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{dx}{du} + \frac{\partial z}{\partial y} \frac{dy}{du} \\&= \left(4e^x \ln y\right) \left(\frac{\cos v}{u \cos v}\right) + \left(4 \frac{e^x}{y}\right) (\sin v) \\&= 4e^x \left(\frac{\ln y}{u} + \frac{\sin v}{y}\right) \\&= 4e^{\ln(u \cos v)} \left(\frac{\ln(u \sin v)}{u} + \frac{\sin v}{u \sin v}\right) \\&= 4(u \cos v) \left(\frac{\ln(u \sin v)}{u} + \frac{1}{u}\right) \\&= \underline{4 \cos v \ln(u \sin v) + 4 \cos v} \quad | \end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \frac{dx}{dv} + \frac{\partial z}{\partial y} \frac{dy}{dv} \\&= \left(4e^x \ln y\right) \left(\frac{-u \sin v}{u \cos v}\right) + \left(4 \frac{e^x}{y}\right) (u \cos v) \\&= 4e^{\ln(u \cos v)} \left[\frac{-\ln(u \sin v)(u \sin v)}{u \cos v} + \frac{u \cos v}{u \sin v}\right] \\&= 4u \cos v \left(\frac{-u \sin^2 v \cdot \ln(u \sin v) + u \cos^2 v}{u \cos v \sin v}\right) \\&= 4 \left(\frac{-u \sin^2 v \cdot \ln(u \sin v) + u \cos^2 v}{\sin v}\right) \\&= \underline{-4u \sin v \cdot \ln(u \sin v) + 4u \frac{\cos^2 v}{\sin v}} \quad | \end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial u} \left(2, \frac{\pi}{4}\right) &= 4 \cos \frac{\pi}{4} \ln \left(2 \sin \frac{\pi}{4}\right) + 4 \cos \frac{\pi}{4} \\&= 2\sqrt{2} \ln \sqrt{2} + 2\sqrt{2} \\&= 2\sqrt{2} \left(\frac{1}{2} \ln 2 + 1\right) \\&= \underline{\sqrt{2} (\ln 2 + 2)} \quad | \end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial v} \left( 2, \frac{\pi}{4} \right) &= -8 \sin \left( \frac{\pi}{4} \right) \cdot \ln \left( 2 \sin \left( \frac{\pi}{4} \right) \right) + 8 \frac{\cos^2 \left( \frac{\pi}{4} \right)}{\sin \left( \frac{\pi}{4} \right)} \\
&= -4\sqrt{2} \ln(\sqrt{2}) + 8 \cdot \frac{1}{2} \cdot \sqrt{2} \\
&= \underline{-2\sqrt{2} \ln 2 + 4\sqrt{2}} \quad |
\end{aligned}$$

### Exercise

Express  $\frac{\partial w}{\partial u}$  and  $\frac{\partial w}{\partial v}$  as functions of  $u$  and  $v$  if  $w = xy + yz + xz$ ,  $x = u + v$ ,  $y = u - v$ ,  $z = uv$ , then

evaluate  $\frac{\partial w}{\partial u}$  and  $\frac{\partial w}{\partial v}$  at the point  $(u, v) = \left( \frac{1}{2}, 1 \right)$ .

### Solution

$$\begin{aligned}
\frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x} \frac{dx}{du} + \frac{\partial w}{\partial y} \frac{dy}{du} + \frac{\partial w}{\partial z} \frac{dz}{du} \\
&= (y + z)(1) + (x + z)(1) + (y + x)(v) \\
&= y + z + x + z + (y + x)(v) \\
&= y + x + 2z + yv + xv \\
&= u - v + u + v + 2uv + uv - v^2 + uv + v^2 \\
&= \underline{2u + 4uv} \quad |
\end{aligned}$$

$$\begin{aligned}
\frac{\partial w}{\partial u} \left( \frac{1}{2}, 1 \right) &= 2 \left( \frac{1}{2} \right) + 4 \left( \frac{1}{2} \right) (1) \\
&= \underline{3} \quad |
\end{aligned}$$

$$\begin{aligned}
\frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x} \frac{dx}{dv} + \frac{\partial w}{\partial y} \frac{dy}{dv} + \frac{\partial w}{\partial z} \frac{dz}{dv} \\
&= (y + z)(1) + (x + z)(-1) + (y + x)(u) \\
&= y + z - x - z + yu + xu \\
&= y - x + yu + xu \\
&= u - v - u - v + u^2 - uv + u^2 + uv \\
&= \underline{-2v + 2u^2} \quad |
\end{aligned}$$

$$\begin{aligned}
\frac{\partial w}{\partial v} \left( \frac{1}{2}, 1 \right) &= -2(1) + 2 \left( \frac{1}{2} \right)^2 \\
&= \underline{-\frac{3}{2}} \quad |
\end{aligned}$$

### Exercise

Express  $\frac{\partial u}{\partial x}$ ,  $\frac{\partial u}{\partial y}$  and  $\frac{\partial u}{\partial z}$  as functions of  $x$ ,  $y$  and  $z$  if  $u = e^{qr} \sin^{-1} p$ ,  $p = \sin x$ ,  $q = z^2 \ln y$ ,  $r = \frac{1}{z}$ , then evaluate  $\frac{\partial u}{\partial x}$ ,  $\frac{\partial u}{\partial y}$  and  $\frac{\partial u}{\partial z}$  at the point  $(x, y, z) = \left(\frac{\pi}{4}, \frac{1}{2}, -\frac{1}{2}\right)$ .

### Solution

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial p} \frac{dp}{dx} + \frac{\partial u}{\partial q} \frac{dq}{dx} + \frac{\partial u}{\partial r} \frac{dr}{dx} \\ &= \left( \frac{e^{qr}}{\sqrt{1-p^2}} \right) (\cos x) + \left( re^{qr} \sin^{-1} p \right) (0) + \left( qe^{qr} \sin^{-1} p \right) (0) \\ &= \frac{e^{z \ln y} \cos x}{\sqrt{1-\sin^2 x}} \\ &= e^{\ln y^z} \frac{\cos x}{|\cos x|} \\ &= \underline{y^z} \quad \text{if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial x} \left( \frac{\pi}{4}, \frac{1}{2}, -\frac{1}{2} \right) &= \left( \frac{1}{2} \right)^{-1/2} \\ &= 2^{1/2} \\ &= \underline{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial p} \frac{dp}{dy} + \frac{\partial u}{\partial q} \frac{dq}{dy} + \frac{\partial u}{\partial r} \frac{dr}{dy} \\ &= \left( \frac{e^{qr}}{\sqrt{1-p^2}} \right) (0) + \left( re^{qr} \sin^{-1} p \right) \left( \frac{z^2}{y} \right) + \left( qe^{qr} \sin^{-1} p \right) (0) \\ &= \frac{z^2}{y} \frac{1}{z} e^{z \ln y} \sin^{-1}(\sin x) \\ &= \frac{z}{y} e^{\ln y^z} (x) \\ &= \frac{xz}{y} y^z \\ &= \underline{xzy^{z-1}}\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial y} \left( \frac{\pi}{4}, \frac{1}{2}, -\frac{1}{2} \right) &= \left( \frac{\pi}{4} \right) \left( -\frac{1}{2} \right) \left( \frac{1}{2} \right)^{-1/2-1} \\ &= -\left( \frac{\pi}{8} \right) 2^{3/2}\end{aligned}$$

$$\underline{= -\frac{\pi\sqrt{2}}{4}} \quad \Big|$$

$$\begin{aligned} \frac{\partial u}{\partial z} &= \frac{\partial u}{\partial p} \frac{dp}{dz} + \frac{\partial u}{\partial q} \frac{dq}{dz} + \frac{\partial u}{\partial r} \frac{dr}{dz} \\ &= \left( \frac{e^{qr}}{\sqrt{1-p^2}} \right) (0) + \left( r e^{qr} \sin^{-1} p \right) (2z \ln y) + \left( q e^{qr} \sin^{-1} p \right) \left( -\frac{1}{z^2} \right) \\ &= 2z \ln y \left( \frac{1}{z} y^z \sin^{-1}(\sin x) \right) - \frac{1}{z^2} \left( z^2 (\ln y) y^z \sin^{-1}(\sin x) \right) \\ &= 2xy^z \ln y - xy^z \ln y \\ &\underline{= xy^z \ln y} \quad \Big| \end{aligned}$$

$$\begin{aligned} \frac{\partial u}{\partial z} \left( \frac{\pi}{4}, \frac{1}{2}, -\frac{1}{2} \right) &= \left( \frac{\pi}{4} \right) \left( \frac{1}{2} \right)^{-1/2} \ln \left( \frac{1}{2} \right) \\ &= \left( \frac{\pi}{4} \right) (\sqrt{2}) (-\ln 2) \\ &\underline{= -\frac{\pi\sqrt{2}}{4} \ln 2} \quad \Big| \end{aligned}$$

### Exercise

Find the values of  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  if  $z^3 - xy + yz + y^3 - 2 = 0$  at the point  $(1, 1, 1)$

### Solution

$$F(x, y, z) = z^3 - xy + yz + y^3 - 2$$

$$F_x = -y, \quad F_y = -x + z + 3y^2, \quad F_z = 3z^2 + y$$

$$\begin{aligned} \frac{\partial z}{\partial x} &= -\frac{-y}{3z^2 + y} & \frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} \\ &\underline{= \frac{y}{3z^2 + y}} \quad \Big| \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial x} (1, 1, 1) &= \frac{1}{3(1)^2 + 1} \\ &\underline{= \frac{1}{4}} \quad \Big| \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial y} &= -\frac{-x + z + 3y^2}{3z^2 + y} & \frac{\partial z}{\partial y} &= -\frac{F_y}{F_z} \end{aligned}$$

$$= \frac{x - z - 3y^2}{3z^2 + y} \Big|$$

$$\begin{aligned} \frac{\partial z}{\partial y}(1, 1, 1) &= \frac{1 - 1 - 3(1)^2}{3(1)^2 + 1} \\ &= -\frac{3}{4} \Big| \end{aligned}$$

### ***Exercise***

Find the values of  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  if  $\sin(x + y) + \sin(y + z) + \sin(x + z) = 0$  at the point  $(\pi, \pi, \pi)$

### ***Solution***

$$F(x, y, z) = \sin(x + y) + \sin(y + z) + \sin(x + z)$$

$$F_x = \cos(x + y) + \cos(x + z)$$

$$F_y = \cos(x + y) + \cos(y + z)$$

$$F_z = \cos(y + z) + \cos(x + z)$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\cos(x + y) + \cos(x + z)}{\cos(y + z) + \cos(x + z)}$$

$$\frac{\partial z}{\partial x}(\pi, \pi, \pi) = -\frac{\cos(2\pi) + \cos(2\pi)}{\cos(2\pi) + \cos(2\pi)}$$

$$= -1 \Big|$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\cos(x + y) + \cos(y + z)}{\cos(y + z) + \cos(x + z)}$$

$$\frac{\partial z}{\partial y}(\pi, \pi, \pi) = -\frac{\cos(2\pi) + \cos(2\pi)}{\cos(2\pi) + \cos(2\pi)}$$

$$= -1 \Big|$$



### Exercise

Find the values of  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  if  $xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0$  at the point  $(1, \ln 2, \ln 3)$

### Solution

$$F(x, y, z) = xe^y + ye^z + 2\ln x - 2 - 3\ln 2$$

$$F_x = e^y + \frac{2}{x}$$

$$F_y = xe^y + e^z$$

$$F_z = ye^z$$

$$\frac{\partial z}{\partial x} = -\frac{e^y + \frac{2}{x}}{ye^z}$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

$$= -\frac{xe^y + 2}{xye^z}$$

$$\frac{\partial z}{\partial x}(1, \ln 2, \ln 3) = -\frac{(1)e^{\ln 2} + 2}{\ln 2 e^{\ln 3}}$$

$$= -\frac{2+2}{3\ln 2}$$

$$= -\frac{4}{3\ln 2}$$

$$\frac{\partial z}{\partial y} = -\frac{xe^y + e^z}{ye^z}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

$$\frac{\partial z}{\partial y}(1, \ln 2, \ln 3) = -\frac{e^{\ln 2} + e^{\ln 3}}{\ln 2 e^{\ln 3}}$$

$$= -\frac{2+3}{3\ln 2}$$

$$= -\frac{5}{3\ln 2}$$

### Exercise

Find  $\frac{\partial w}{\partial r}$  when  $r = 1, s = -1$  if  $w = (x + y + z)^2$ ,  $x = r - s$ ,  $y = \cos(r + s)$ ,  $z = \sin(r + s)$

### Solution

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{dx}{dr} + \frac{\partial w}{\partial y} \frac{dy}{dr} + \frac{\partial w}{\partial z} \frac{dz}{dr}$$

$$= 2(x + y + z)(1) + 2(x + y + z)(-\sin(r + s)) + 2(x + y + z)(\cos(r + s))$$

$$= 2(x + y + z)[1 - \sin(r + s) + \cos(r + s)]$$

$$= 2(r - s + \cos(r + s) + \sin(r + s))(1 - \sin(r + s) + \cos(r + s))$$

$$\frac{\partial w}{\partial r}(1, -1) = 2(1 - (-1) + \cos(1 - 1) + \sin(1 - 1))(1 - \sin(1 - 1) + \cos(1 - 1))$$

$$\begin{aligned}
&= 2(1+1+1+0)(1-0+1) \\
&= 2(3)(2) \\
&= 12
\end{aligned}$$

### Exercise

Find  $\frac{\partial z}{\partial u}$  when  $u = 0$ ,  $v = 1$  if  $z = \sin xy + x \sin y$ ,  $x = u^2 + v^2$ ,  $y = uv$

### Solution

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{dx}{du} + \frac{\partial z}{\partial y} \frac{dy}{du} \\
&= (y \cos x + \sin y)(2u) + (x \cos xy + x \cos y)(v) \\
&= 2u(uv \cos(u^2 + v^2) + \sin uv) + v((u^2 + v^2) \cos(u^3v + uv^3) + (u^2 + v^2) \cos uv) \\
&= 2u(uv \cos(u^2 + v^2) + \sin uv) + v(u^2 + v^2)(\cos(u^3v + uv^3) + \cos uv) \\
\left. \frac{\partial z}{\partial u} \right|_{u=0, v=1} &= 2(0)(0 \cos(1) + \sin 0) + 1(1)(\cos(0) + \cos 0) \\
&= 0 + 1(1+1) \\
&= 2
\end{aligned}$$

### Exercise

Find  $\frac{\partial z}{\partial u}$  and  $\frac{\partial z}{\partial v}$  when  $u = \ln 2$ ,  $v = 1$  if  $z = 5 \tan^{-1} x$ ,  $x = e^u + \ln v$

### Solution

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{dz}{dx} \frac{\partial x}{\partial u} = \left( \frac{5}{1+x^2} \right) e^u = \left( \frac{5}{1+(e^u + \ln v)^2} \right) e^u \\
\left. \frac{\partial z}{\partial u} \right|_{u=\ln 2, v=1} &= \left( \frac{5}{1+(e^{\ln 2} + \ln 1)^2} \right) e^{\ln 2} \\
&= \left( \frac{5}{1+(2+0)^2} \right) (2) \\
&= 2\left(\frac{5}{5}\right) \\
&= 2
\end{aligned}$$

### Exercise

Find  $\frac{\partial z}{\partial u}$  and  $\frac{\partial z}{\partial v}$  when  $u=1$ ,  $v=-2$  if  $z = \ln q$ ,  $q = \sqrt{v+3} \tan^{-1} u$

### Solution

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{dz}{dq} \frac{\partial q}{\partial u} \\&= \left(\frac{1}{q}\right) \left(\sqrt{v+3} \frac{1}{1+u^2}\right) \\&= \frac{1}{\sqrt{v+3} \tan^{-1} u} \cdot \frac{\sqrt{v+3}}{1+u^2} \\&= \frac{1}{(1+u^2) \tan^{-1} u} \Big| \\ \frac{\partial z}{\partial u} \Big|_{u=1, v=-2} &= \frac{1}{(1+1^2) \tan^{-1} 1} \\&= \frac{1}{2 \cdot \frac{\pi}{4}} \\&= \frac{2}{\pi} \Big|\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{dz}{dq} \frac{\partial q}{\partial v} \\&= \left(\frac{1}{q}\right) \left(\frac{1}{2\sqrt{v+3}} \tan^{-1} u\right) \\&= \left(\frac{1}{\sqrt{v+3} \tan^{-1} u}\right) \left(\frac{\tan^{-1} u}{2\sqrt{v+3}}\right) \\&= \frac{1}{2(v+3)} \Big|\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial v} \Big|_{u=1, v=-2} &= \frac{1}{2(-2+3)} \\&= \frac{1}{2} \Big|\end{aligned}$$

### Exercise

Find  $\frac{\partial w}{\partial r}$  and  $\frac{\partial w}{\partial s}$  when  $r = \pi$  and  $s = 0$  if  $w = \sin(2x - y)$ ,  $x = r + \sin s$ ,  $y = rs$

### Solution

$$\begin{aligned}\frac{\partial w}{\partial x} &= 2 \cos(2x - y) \\ &= 2 \cos(2r + 2 \sin s - rs) \Big|_{r=\pi \quad s=0} \\ &= 2 \cos(2\pi) \\ &= 2\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial y} &= -\cos(2x - y) \\ &= -\cos(2r + 2 \sin s - rs) \Big|_{r=\pi \quad s=0} \\ &= -\cos(2\pi) \\ &= -1\end{aligned}$$

$$\frac{dx}{dr} = 1$$

$$\frac{dy}{dr} = s \Big|_{r=\pi \quad s=0} = 0$$

$$\begin{aligned}\frac{\partial w}{\partial r} &= 2(1) + (-1)(0) \\ &= 2\end{aligned} \qquad \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r}$$

$$\frac{dx}{ds} = \cos s \Big|_{s=0} = 1$$

$$\frac{dy}{ds} = r \Big|_{r=\pi} = \pi$$

$$\begin{aligned}\frac{\partial w}{\partial s} &= 2(1) + (-1)(\pi) \\ &= 2 - \pi\end{aligned} \qquad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}$$

### Exercise

Assume that  $w = f(s^3 + t^2)$  and  $f'(x) = e^x$ . Find  $\frac{\partial w}{\partial t}$  and  $\frac{\partial w}{\partial s}$

### Solution

$$w = f(s^3 + t^2) = f(x) \rightarrow x = s^3 + t^2$$

$$\frac{\partial w}{\partial t} = f'(x) \cdot 2t$$

$$= 2te^x$$

$$= 2te^{s^3+t^2}$$

$$\frac{\partial w}{\partial t} = \frac{dw}{dx} \frac{\partial x}{\partial t}$$

$$\frac{\partial w}{\partial s} = \left( e^x \right) \left( 3s^2 \right)$$

$$= 3s^2 e^{s^3+t^2}$$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx} \frac{\partial x}{\partial s}$$

### Exercise

Evaluate the derivatives  $w'(t)$ , where  $w = xy \sin z$ ,  $x = t^2$ ,  $y = 4t^3$ , and  $z = t + 1$

### Solution

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$$

$$= y \sin z (2t) + x \sin z (12t^2) + xy \cos z$$

$$= 8t^4 \sin(t+1) + 12t^4 \sin(t+1) + 4t^5 \cos(t+1)$$

$$= 20t^4 \sin(t+1) + 4t^5 \cos(t+1)$$

**Or**

$$w(t) = 4t^5 \sin(t+1)$$

$$w' = 20t^4 \sin(t+1) + 4t^5 \cos(t+1)$$

### Exercise

Evaluate the derivatives  $w'(t)$ , where  $w = \sqrt{x^2 + y^2 + z^2}$ ,  $x = \sin t$ ,  $y = \cos t$ , and  $z = \cos t$

### Solution

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$$

$$= \frac{x}{\sqrt{x^2 + y^2 + z^2}} \cos t - \frac{y}{\sqrt{x^2 + y^2 + z^2}} \sin t - \frac{z}{\sqrt{x^2 + y^2 + z^2}} \sin t$$

$$= \frac{\sin t \cos t - \cos t \sin t - \cos t \sin t}{\sqrt{\sin^2 t + \cos^2 t + \sin^2 t}}$$

$$= -\frac{\cos t \sin t}{\sqrt{1 + \sin^2 t}}$$

### Exercise

Evaluate the derivatives  $w_s$  and  $w_t$ , where  $w = xyz$ ,  $x = 2st$ ,  $y = st^2$ , and  $z = s^2t$

### Solution

$$\begin{aligned}w_s &= w_x x_s + w_y y_s + w_z z_s \\&= yz(2t) + xzt^2 + xy(2st) \\&= 2s^3t^4 + 2s^3t^4 + 4s^3t^4 \\&= 8s^3t^4\end{aligned}$$

$$\begin{aligned}w_t &= w_x x_t + w_y y_t + w_z z_t \\&= yz(2s) + xz(2st) + xy(s^2) \\&= 2s^4t^3 + 4s^4t^3 + 2s^4t^3 \\&= 8s^4t^3\end{aligned}$$

**Or**

$$\begin{aligned}w &= (2st)(st^2)(s^2t) \\&= 2s^4t^4\end{aligned}$$

$$w_s = 8s^3t^4$$

$$w_t = 8s^4t^3$$

### Exercise

Evaluate the derivatives  $w_r$ ,  $w_s$ , and  $w_t$ , where  $w = \ln(xy^2)$ ,  $x = rst$ , and  $y = r + s$

### Solution

$$\begin{aligned}w_r &= w_x x_r + w_y y_r \\&= \frac{1}{y^2}(st) + \frac{2}{xy} \\&= \frac{st}{(r+s)^2} + \frac{2}{rst(r+s)} \\&= \frac{rs^2t^2 + 2r + 2s}{rst(r+s)^2}\end{aligned}$$

$$w_s = w_x x_s + w_y y_s$$

$$\begin{aligned}
&= \frac{rt}{y^2} + \frac{2}{xy} \\
&= \frac{rt}{(r+s)^2} + \frac{2}{rst(r+s)} \\
&= \frac{r^2st^2 + 2r + 2s}{rst(r+s)^2}
\end{aligned}$$

$$\begin{aligned}
w_t &= w_x x_t + w_y y_t \\
&= \frac{rs}{y^2} + \frac{2}{xy}(0) \\
&= \frac{rs}{(r+s)^2}
\end{aligned}$$

**Or**  $w = \ln(rst(r+s)^2)$

### Exercise

The voltage  $V$  in a circuit that satisfies the law  $V = IR$  is slowly dropping as the battery wears out. At the same time, the resistance  $R$  is increasing as the resistor heats up. Use the equation

$$\frac{dV}{dt} = \frac{\partial V}{\partial I} \frac{dI}{dt} + \frac{\partial V}{\partial R} \frac{dR}{dt}$$

To find how the current is changing at the instant when  $R = 600 \Omega$ ,  $I = 0.04 A$ ,  $\frac{dR}{dt} = 0.5 \text{ ohm} / \text{sec}$ ,

and  $\frac{dV}{dt} = -0.01 \text{ volt} / \text{sec}$

### Solution

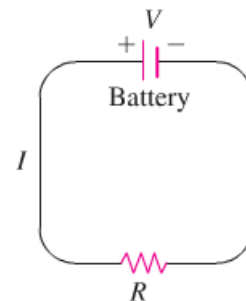
$$V = IR \rightarrow \frac{\partial V}{\partial I} = R, \quad \frac{\partial V}{\partial R} = I$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial I} \frac{dI}{dt} + \frac{\partial V}{\partial R} \frac{dR}{dt}$$

$$-0.01 = (600) \frac{dI}{dt} + (0.04)(0.5)$$

$$-0.02 - 0.01 = 600 \frac{dI}{dt}$$

$$\frac{dI}{dt} = \underline{-0.00005 \text{ amps} / \text{sec}}$$



### Exercise

The lengths  $a$ ,  $b$ , and  $c$  of the edges of a rectangular box are changing with time. At the instant in question,  $a = 1$  m,  $b = 2$  m,  $c = 3$  m,  $\frac{da}{dt} = \frac{db}{dt} = 1$  m / sec, and  $\frac{dc}{dt} = -3$  m / sec. At what rates the box's volume  $V$  and surface area  $S$  changing at that instant? Are the box's interior diagonals increasing in length or decreasing?

### Solution

$$\begin{aligned} V = abc &\Rightarrow \frac{\partial V}{\partial t} = \frac{\partial V}{\partial a} \frac{da}{dt} + \frac{\partial V}{\partial b} \frac{db}{dt} + \frac{\partial V}{\partial c} \frac{dc}{dt} \\ \frac{\partial V}{\partial t} &= (bc) \frac{da}{dt} + (ac) \frac{db}{dt} + (ab) \frac{dc}{dt} \\ &= (2m)(3m)(1 \text{ m / sec}) + (1m)(3m)(1 \text{ m / sec}) + (1m)(2m)(-3 \text{ m / sec}) \\ &= \underline{3 \text{ m}^3 / \text{sec}} \end{aligned}$$

### Exercise

Let  $T = f(x, y)$  be the temperature at the point  $(x, y)$  on the circle  $x = \cos t$ ,  $y = \sin t$ ,  $0 \leq t \leq 2\pi$  and suppose that

$$\frac{\partial T}{\partial x} = 8x - 4y, \quad \frac{\partial T}{\partial y} = 8y - 4x$$

- a) Find where the maximum and minimum temperatures on the circle occur by examining the derivatives  $\frac{dT}{dt}$  and  $\frac{d^2T}{dt^2}$ .
- b) Suppose that  $T = 4x^2 - 4xy + 4y^2$ . Find the maximum and minimum values of  $T$  on the circle.

### Solution

$$\begin{aligned} a) \quad \frac{dT}{dt} &= \frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt} \\ &= (8x - 4y)(-\sin t) + (8y - 4x)(\cos t) \\ &= (8\cos t - 4\sin t)(-\sin t) + (8\sin t - 4\cos t)(\cos t) \\ &= -8\cos t \sin t + 4\sin^2 t + 8\cos t \sin t - 4\cos^2 t \\ &= \underline{4\sin^2 t - 4\cos^2 t} \\ \frac{dT}{dt} = 0 &\Rightarrow 4\sin^2 t - 4\cos^2 t = 0 \\ \sin^2 t &= \cos^2 t \\ \sin t &= \pm \cos t \end{aligned}$$



$$t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \quad \text{on the interval } 0 \leq t \leq 2\pi$$

$$\begin{aligned} \frac{d^2T}{dt^2} &= 8 \sin t \cos t + 8 \cos t \sin t \\ &= 16 \sin t \cos t \end{aligned}$$

$$\left. \frac{d^2T}{dt^2} \right|_{t=\frac{\pi}{4}} = 16 \sin \frac{\pi}{4} \cos \frac{\pi}{4} > 0 \quad \Rightarrow T \text{ has a minimum at } (x, y) = \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$$

$$\left. \frac{d^2T}{dt^2} \right|_{t=\frac{3\pi}{4}} = 16 \sin \frac{3\pi}{4} \cos \frac{3\pi}{4} < 0 \quad \Rightarrow T \text{ has a maximum at } (x, y) = \left( -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$$

$$\left. \frac{d^2T}{dt^2} \right|_{t=\frac{5\pi}{4}} = 16 \sin \frac{5\pi}{4} \cos \frac{5\pi}{4} > 0 \quad \Rightarrow T \text{ has a minimum at } (x, y) = \left( -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right)$$

$$\left. \frac{d^2T}{dt^2} \right|_{t=\frac{7\pi}{4}} = 16 \sin \frac{7\pi}{4} \cos \frac{7\pi}{4} < 0 \quad \Rightarrow T \text{ has a maximum at } (x, y) = \left( \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right)$$

**b)**  $T = 4x^2 - 4xy + 4y^2$

$$T\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = 4\left(\frac{\sqrt{2}}{2}\right)^2 - 4\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + 4\left(\frac{\sqrt{2}}{2}\right)^2 = 2 - 2 + 2 = \underline{2}$$

$$T\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = 4\left(-\frac{\sqrt{2}}{2}\right)^2 - 4\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + 4\left(\frac{\sqrt{2}}{2}\right)^2 = 2 + 2 + 2 = \underline{6}$$

$$T\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 4\left(-\frac{\sqrt{2}}{2}\right)^2 - 4\left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) + 4\left(-\frac{\sqrt{2}}{2}\right)^2 = 2 - 2 + 2 = \underline{2}$$

$$T\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 4\left(\frac{\sqrt{2}}{2}\right)^2 - 4\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) + 4\left(-\frac{\sqrt{2}}{2}\right)^2 = 2 + 2 + 2 = \underline{6}$$

The maximum value is 6 at  $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$  and  $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

The minimum value is 2 at  $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$  and  $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

### Exercise

Evaluate  $\frac{dy}{dx}$ :  $x^2 - 2y^2 - 1 = 0$

#### Solution

$$F(x, y) = x^2 - 2y^2 - 1$$

$$\frac{dy}{dx} = -\frac{2x}{-4y} \qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$
$$\underline{= \frac{x}{2y} \mid}$$

### Exercise

Evaluate  $\frac{dy}{dx}$ :  $x^3 + 3xy^2 - y^5 = 0$

#### Solution

$$F(x, y) = x^3 + 3xy^2 - y^5$$

$$\frac{dy}{dx} = -\frac{3x^2 + 3y^2}{6xy - 5y^4} \qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

### Exercise

Evaluate  $\frac{dy}{dx}$ :  $2 \sin xy = 1$

#### Solution

$$F(x, y) = 2 \sin xy - 1$$

$$\frac{dy}{dx} = -\frac{2y \cos xy}{2x \cos xy} \qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$
$$\underline{= -\frac{y}{x} \mid}$$

### Exercise

Evaluate  $\frac{dy}{dx}$ :  $ye^{xy} - 2 = 0$

#### Solution

$$F(x, y) = ye^{xy} - 2$$

$$\frac{dy}{dx} = -\frac{y^2 e^{xy}}{e^{xy} + xy e^{xy}} \qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= -\frac{y^2}{1+xy} \Big|$$

### Exercise

Evaluate  $\frac{dy}{dx}$ :  $\sqrt{x^2 + 2xy + y^4} = 3$

### Solution

$$F(x, y) = \sqrt{x^2 + 2xy + y^4} - 3$$

$$\frac{dy}{dx} = -\frac{\frac{1}{2}(2x + 2y)(x^2 + 2xy + y^4)^{-1/2}}{\frac{1}{2}(2x + 4y^3)(x^2 + 2xy + y^4)^{-1/2}}$$

$$= -\frac{x+y}{x+2y^3} \Big|$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

### Exercise

Evaluate  $\frac{dy}{dx}$ :  $y \ln(x^2 + y^2 + 4) = 3$

### Solution

$$F(x, y) = y \ln(x^2 + y^2 + 4) - 3$$

$$\frac{dy}{dx} = -\frac{\frac{2xy}{x^2 + y^2 + 4}}{\ln(x^2 + y^2 + 4) + \frac{2y^2}{x^2 + y^2 + 4}}$$

$$= -\frac{2xy}{2y^2 + (x^2 + y^2 + 4) \ln(x^2 + y^2 + 4)} \Big|$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

### Exercise

Evaluate  $\frac{dy}{dx}$ :  $y \ln(x^2 + y^2) = 4$

### Solution

$$F(x, y) = y \ln(x^2 + y^2) - 4$$

$$\begin{aligned} \frac{dy}{dx} &= - \frac{\frac{2xy}{x^2 + y^2}}{\ln(x^2 + y^2) + \frac{2y^2}{x^2 + y^2}} \\ &= - \frac{2xy}{(x^2 + y^2) \ln(x^2 + y^2) + 2y^2} \end{aligned}$$

$$\frac{dy}{dx} = - \frac{F_x}{F_y}$$

### Exercise

Evaluate  $\frac{dy}{dx}$  :  $2x^2 + 3xy - 3y^4 = 2$

### Solution

$$F(x, y) = 2x^2 + 3xy - 3y^4 - 2$$

$$\frac{dy}{dx} = - \frac{4x + 3y}{3x - 12y^3}$$

$$\frac{dy}{dx} = - \frac{F_x}{F_y}$$

### Exercise

Find  $\frac{dz}{dx}$  and  $\frac{dz}{dy}$  at the given point.  $z^3 - xy + yz + y^3 - 2 = 0$ ;  $(1, 1, 1)$

### Solution

$$F(x, y, z) = z^3 - xy + yz + y^3 - 2$$

$$F_x = -y, \quad F_y = -x + z + 3y^2, \quad \text{and} \quad F_z = 3z^2 + y \Big|_{(1,1,1)} = 4 \neq 0$$

$$\frac{dz}{dx} = - \frac{F_x}{F_z} = - \frac{-y}{3z^2 + y}$$

$$\frac{dz}{dx} \Big|_{(1,1,1)} = - \frac{-1}{4} = \underline{\underline{\frac{1}{4}}}$$

$$\frac{dz}{dy} = - \frac{F_y}{F_z} = - \frac{e^{xz} - z \sin y}{2z + xye^{xz} + \cos y}$$

$$\frac{dz}{dy} \Big|_{(1,1,1)} = \underline{\underline{-\frac{3}{4}}}$$

### Exercise

Find  $\frac{dz}{dx}$  and  $\frac{dz}{dy}$  at the given point.  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0$ ;  $(2, 3, 6)$

### Solution

$$F(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1$$

$$F_x = -\frac{1}{x^2} \Big|_{(2,3,6)} = -\frac{1}{4}$$

$$F_y = -\frac{1}{y^2} \Big|_{(2,3,6)} = -\frac{1}{9}$$

$$F_z = -\frac{1}{z^2} \Big|_{(2,3,6)} = -\frac{1}{36} \neq 0$$

$$\begin{aligned} \frac{dz}{dx} \Big|_{(2,3,6)} &= -\frac{F_x}{F_z} \Big|_{(2,3,6)} \\ &= -\frac{-\frac{1}{4}}{-\frac{1}{36}} \\ &= \underline{\underline{-9}} \end{aligned}$$

$$\begin{aligned} \frac{dz}{dy} \Big|_{(2,3,6)} &= -\frac{F_y}{F_z} \Big|_{(2,3,6)} \\ &= -\frac{-\frac{1}{9}}{-\frac{1}{36}} \\ &= \underline{\underline{-4}} \end{aligned}$$

### Exercise

Find  $\frac{dz}{dx}$  and  $\frac{dz}{dy}$  at the given point.  $\sin(x+y) + \sin(y+z) + \sin(x+z) = 0; \quad (\pi, \pi, \pi)$

### Solution

$$F(x, y, z) = \sin(x+y) + \sin(y+z) + \sin(x+z)$$

$$\begin{aligned} F_x &= \cos(x+y) + \cos(x+z) \Big|_{(\pi, \pi, \pi)} \\ &= \cos 2\pi + \cos 2\pi \\ &= \underline{\underline{2}} \end{aligned}$$

$$\begin{aligned} F_y &= \cos(x+y) + \cos(y+z) \Big|_{(\pi, \pi, \pi)} \\ &= \cos 2\pi + \cos 2\pi \\ &= \underline{\underline{2}} \end{aligned}$$

$$F_z = \cos(y+z) + \cos(x+z) \Big|_{(\pi, \pi, \pi)}$$

$$= \cos 2\pi + \cos 2\pi$$

$$= \underline{2} \quad \neq 0$$

$$\begin{aligned} \left. \frac{dz}{dx} \right|_{(\pi, \pi, \pi)} &= - \frac{F_x}{F_z} \bigg|_{(\pi, \pi, \pi)} \\ &= - \frac{2}{2} \\ &= \underline{-1} \end{aligned}$$

$$\begin{aligned} \left. \frac{dz}{dy} \right|_{(\pi, \pi, \pi)} &= - \frac{F_y}{F_z} \bigg|_{(\pi, \pi, \pi)} \\ &= - \frac{2}{2} \\ &= \underline{-1} \end{aligned}$$

### Exercise

Find  $\frac{dz}{dx}$  and  $\frac{dz}{dy}$  at the given point.  $xe^y + ye^z + 2 \ln x - 2 - 3 \ln 2 = 0$ ;  $(1, \ln 2, \ln 3)$

### Solution

$$F(x, y, z) = xe^y + ye^z + 2 \ln x - 2 - 3 \ln 2$$

$$F_x = e^y + \frac{2}{x} \bigg|_{(1, \ln 2, \ln 3)} = 2 + 2 = \underline{4}$$

$$\begin{aligned} F_y &= xe^y + e^z \bigg|_{(1, \ln 2, \ln 3)} \\ &= e^{\ln 2} + e^{\ln 3} \\ &= 2 + 3 \\ &= \underline{5} \end{aligned}$$

$$F_z = ye^z \bigg|_{(1, \ln 2, \ln 3)} = \ln 2 e^{\ln 3} = \underline{3 \ln 2} \neq 0$$

$$\begin{aligned} \left. \frac{dz}{dx} \right|_{(1, \ln 2, \ln 3)} &= - \frac{F_x}{F_z} \bigg|_{(1, \ln 2, \ln 3)} \\ &= \underline{- \frac{4}{3 \ln 2}} \end{aligned}$$

$$\begin{aligned} \left. \frac{dz}{dy} \right|_{(1, \ln 2, \ln 3)} &= - \frac{F_y}{F_z} \bigg|_{(1, \ln 2, \ln 3)} \\ &= \underline{- \frac{5}{3 \ln 2}} \end{aligned}$$

### Exercise

Consider the surface and parameterized curves  $C$  in the  $xy$ -plane

$$z = 4x^2 + y^2 - 2; \quad C: x = \cos t, \quad y = \sin t, \quad \text{for } 0 \leq t \leq 2\pi$$

- a) Find  $z'(t)$  on  $C$ .
- b) Imagine that you are walking on the surface directly above  $C$  consistent with the positive orientation of  $C$ . Find the values of  $t$  for which you are walking uphill.

### Solution

$$\begin{aligned} a) \quad z'(t) &= z_x x_t + z_y y_t \\ &= 8x(-\sin t) + 2y \cos t \\ &= -8 \cos t \sin t + 2 \sin t \cos t \\ &= -6 \cos t \sin t \\ &= \underline{-3 \sin 2t} \end{aligned}$$

$$\begin{aligned} b) \quad \text{Walking uphill} &\rightarrow z'(t) > 0 \\ -3 \sin 2t > 0 &\rightarrow \sin 2t < 0 \\ \sin 2t = 0 &\rightarrow 2t = 0, \pi, 2\pi, 3\pi, 4\pi \\ t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \\ \underline{\frac{\pi}{2} \leq t \leq \pi \quad \& \quad \frac{3\pi}{2} \leq t \leq 2\pi} \end{aligned}$$

### Exercise

Consider the surface and parameterized curves  $C$  in the  $xy$ -plane

$$z = 4x^2 - 2y^2 + 4; \quad C: x = 2 \cos t, \quad y = 2 \sin t, \quad \text{for } 0 \leq t \leq 2\pi$$

- a) Find  $z'(t)$  on  $C$ .
- b) Imagine that you are walking on the surface directly above  $C$  consistent with the positive orientation of  $C$ . Find the values of  $t$  for which you are walking uphill.

### Solution

$$\begin{aligned} a) \quad z'(t) &= z_x x_t + z_y y_t \\ &= 8x(-2 \sin t) - 4y(2 \cos t) \\ &= -16 \cos t \sin t - 16 \sin t \cos t \\ &= -32 \cos t \sin t \\ &= \underline{-16 \sin 2t} \end{aligned}$$

$$\begin{aligned} b) \quad \text{Walking uphill} &\rightarrow z'(t) > 0 \\ -16 \sin 2t > 0 &\rightarrow \sin 2t < 0 \\ \sin 2t = 0 &\rightarrow 2t = 0, \pi, 2\pi, 3\pi, 4\pi \end{aligned}$$

$$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$$

$$\frac{\pi}{2} \leq t \leq \pi \quad \& \quad \frac{3\pi}{2} \leq t \leq 2\pi$$

### Exercise

Find the value of the derivative of  $f(x, y, z) = xy + yz + xz$  with respect to  $t$  on the curve

$$x = \cos t, \quad y = \sin t, \quad z = \cos 2t \quad \text{at } t = 1$$

### Solution

$$f_x = y + z = \sin t + \cos 2t$$

$$f_y = x + z = \cos t + \cos 2t$$

$$f_z = y + x = \sin t + \cos t$$

$$\frac{dx}{dt} = -\sin t \quad \frac{dy}{dt} = \cos t \quad \frac{dz}{dt} = -2 \sin 2t$$

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$$

$$= (\sin t + \cos 2t)(-\sin t) + (\cos t + \cos 2t)(\cos t) + (\cos t + \sin t)(-2 \sin 2t)$$

$$= -\sin^2 t - \sin t \cos 2t + \cos^2 t + \cos t \cos 2t - 2 \sin 2t \cos t - 2 \sin 2t \sin t$$

$$= \cos^2 t - \sin^2 t + (\cos t - \sin t) \cos 2t - 2 \sin 2t (\cos t + \sin t)$$

$$\left. \frac{df}{dt} \right|_{t=1} = \cos 2 + (\cos 1 - \sin 1) \cos 2 - 2(\cos 1 + \sin 1) \sin 2$$

### Exercise

Define  $y$  as a differentiable function of  $x$  for  $2xy + e^{x+y} - 2 = 0$ , find the values of  $\frac{dy}{dx}$  at point

$$P(0, \ln 2)$$

### Solution

$$F(x, y) = 2xy + e^{x+y} - 2$$

$$F_x = 2y + e^{x+y} \quad F_y = 2x + e^{x+y}$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2y + e^{x+y}}{2x + e^{x+y}}$$

$$\left. \frac{dy}{dx} \right|_{(0, \ln 2)} = -\frac{2 \ln 2 + e^{\ln 2}}{0 + e^{\ln 2}}$$



$$= -\frac{2\ln 2 + 2}{2}$$

$$= \underline{-\ln 2 - 1}$$