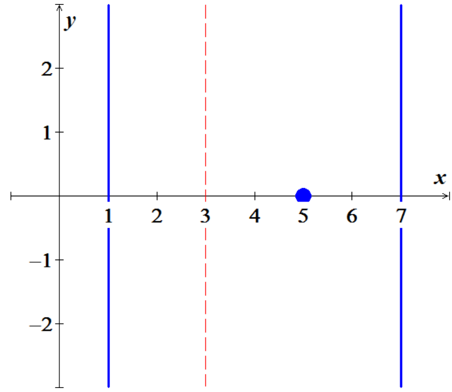
Solution   *Section* 1.6 – Precise Definition of Limits

Exercise

Sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \;\Rightarrow\; a < x < b$ for $a = 1, \quad b = 7, \quad x_0 = 5$



Solution

$$|x - 5| < \delta \;\Rightarrow\; -\delta < x - 5 < \delta$$

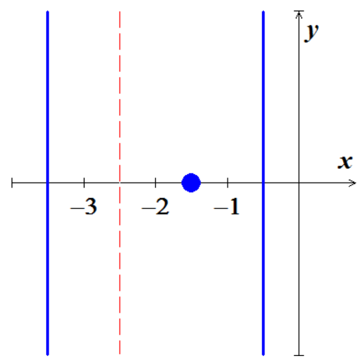
$$-\delta + 5 < x < \delta + 5$$

$$-\delta + 5 = 1 \;\Rightarrow\; \underline{\delta = 4}$$

$$\delta + 5 = 7 \;\Rightarrow\; \underline{\delta = 2}$$

Exercise

Sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \;\Rightarrow\; a < x < b$ for $a = -\frac{7}{2}, \quad b = -\frac{1}{2}, \quad x_0 = -\frac{3}{2}$



Solution

$$\left|x + \frac{3}{2}\right| < \delta \;\Rightarrow\; -\delta < x + \frac{3}{2} < \delta$$

$$-\delta - \frac{3}{2} < x < \delta - \frac{3}{2}$$

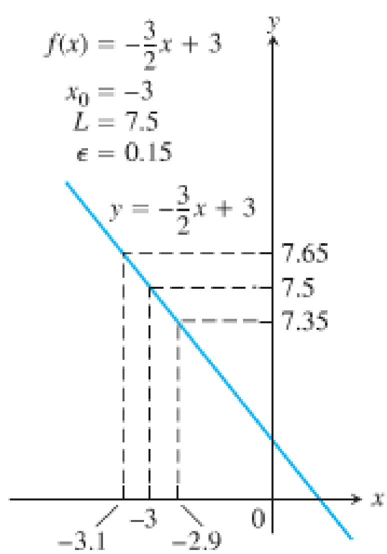
$$-\delta - \frac{3}{2} = -\frac{7}{2} \;\Rightarrow\; \underline{\delta = \frac{7}{2} - \frac{3}{2} = 2}$$

$$\delta - \frac{3}{2} = -\frac{1}{2} \;\Rightarrow\; \underline{\delta = \frac{1}{2} - \frac{3}{2} = -1}$$

Exercise

Use the graph to find a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \;\Rightarrow\; |f(x) - L| < \varepsilon$$



Solution

Given: $a = -3.1, \quad b = -2.9, \quad x_0 = -3$

$$|x + 3| < \delta \;\Rightarrow\; -\delta < x + 3 < \delta$$

$$-\delta - 3 < x < \delta - 3$$

$$-\delta - 3 = -3.1 \;\Rightarrow\; \underline{\delta = 3.1 - 3 = 0.1}$$

$$\delta - 3 = -2.9 \;\Rightarrow\; \underline{\delta = 3 - 2.9 = 0.1}$$

Exercise

Find an open interval about x_0 on which the inequality $\left|f(x)-L\right|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<\left|x-x_0\right|<\delta$ the inequality $\left|f(x)-L\right|<\varepsilon$ holds.

$$f(x)=x+1, \quad L=5, \quad x_0=4, \quad \varepsilon=0.01$$

Solution

$$\left|(x+1)-5\right|<.01 \quad \Rightarrow \quad \left|x-4\right|<.01$$

$$-.01<x-4<.01$$

$$-.01+4<x-4+4<.01+4$$

$$3.99<x<4.01$$

$$\left|x-4\right|<\delta \quad \Rightarrow \quad -\delta<x-4<\delta$$

$$-\delta+4<x<\delta+4$$

$$-\delta+4=3.99 \ \Rightarrow \ \delta=4-3.99=0.01$$

$$\delta+4=4.01 \ \Rightarrow \ \delta=4.01-4=0.01$$

$$\Rightarrow \delta=.01$$

Exercise

Find an open interval about x_0 on which the inequality $\left|f(x)-L\right|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<\left|x-x_0\right|<\delta$ the inequality $\left|f(x)-L\right|<\varepsilon$ holds.

$$f(x)=\sqrt{x+1}, \quad L=1, \quad x_0=0, \quad \varepsilon=0.1$$

Solution

$$\left|\sqrt{x+1}-1\right|<0.1 \quad \Rightarrow \quad -0.1<\sqrt{x+1}-1<0.1$$

$$-0.1+1<\sqrt{x+1}-1+1<0.1+1$$

$$.9<\sqrt{x+1}<1.1$$

$$(.9)^2<\left(\sqrt{x+1}\right)^2<(1.1)^2$$

$$.81<x+1<1.21$$

$$.81-1<x+1-1<1.21-1$$

$$-0.19<x<0.21$$

$$\left|x-0\right|<\delta \quad \Rightarrow \quad -\delta<x<\delta$$

$$-\delta=-0.19 \ \Rightarrow \ \delta=0.19$$

$$\delta=0.21$$

$$\rightarrow \boxed{\delta=0.19}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

$$f(x)=\sqrt{x-7}, \quad L=4, \quad x_0=23, \quad \varepsilon=1$$

Solution

$$\left|\sqrt{x-7}-4\right|<1 \quad \Rightarrow \quad -1<\sqrt{x-7}-4<1$$

$$3<\sqrt{x-7}<5$$

$$(3)^2<\left(\sqrt{x-7}\right)^2<(5)^2$$

$$9<x-7<25$$

$$9+7<x-7+7<25+7$$

$$16<x<32$$

$$|x-23|<\delta \quad \Rightarrow \quad -\delta<x-23<\delta$$

$$-\delta+23<x<\delta+23$$

$$\left.\begin{aligned} -\delta+23=16 \ &\Rightarrow \ \delta=23-16=7 \\ \delta+23=32 \ &\Rightarrow \ \delta=32-23=9 \end{aligned}\right\} \rightarrow \boxed{\delta=7}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

$$f(x)=x^2, \quad L=3, \quad x_0=\sqrt{3}, \quad \varepsilon=0.1$$

Solution

$$\left|x^2-3\right|<0.1 \quad \Rightarrow \quad -0.1<x^2-3<0.1$$

$$2.9<x^2<3.1$$

$$\sqrt{2.9}<x<\sqrt{3.1}$$

$$\left|x-\sqrt{3}\right|<\delta \quad \Rightarrow \quad -\delta<x-\sqrt{3}<\delta$$

$$-\delta+\sqrt{3}<x<\delta+\sqrt{3}$$

$$\left.\begin{aligned} -\delta+\sqrt{3}=\sqrt{2.9} \ &\Rightarrow \ \delta=\sqrt{3}-\sqrt{2.9}=.029 \\ \delta+\sqrt{3}=\sqrt{3.1} \ &\Rightarrow \ \delta=\sqrt{3.1}-\sqrt{3}=.029 \end{aligned}\right\} \rightarrow \boxed{\delta=.029}$$

Exercise

Find an open interval about x_0 on which the inequality $\left|f(x)-L\right|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<\left|x-x_0\right|<\delta$ the inequality $\left|f(x)-L\right|<\varepsilon$ holds.

$$f(x)=\frac{120}{x}, \quad L=5, \quad x_0=24, \quad \varepsilon=1$$

Solution

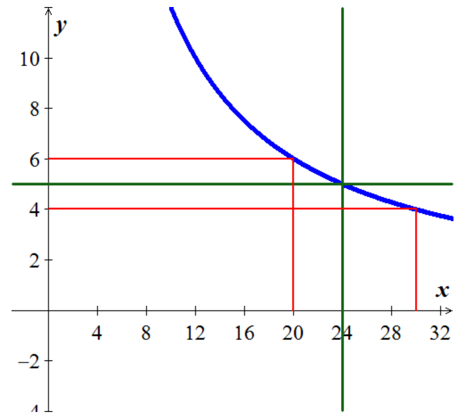
$$\left|\frac{120}{x}-5\right|<0.1 \quad\Rightarrow\quad -1<\frac{120}{x}-5<1$$

$$4<\frac{120}{x}<6$$

$$\frac{1}{6}<\frac{x}{120}<\frac{1}{4}$$

$$\frac{1}{6}(120)<x<\frac{1}{4}(120)$$

$$20<x<30$$



$$|x-24|<\delta \quad\Rightarrow\quad -\delta<x-24<\delta$$

$$-\delta+24<x<\delta+24$$

$$-\delta+24=20 \;\Rightarrow\; \lfloor\delta=24-20=4\rfloor$$

$$\delta+24=30 \;\Rightarrow\; \lfloor\delta=30-24=6\rfloor \quad\to\quad \boxed{\delta=4}$$

Exercise

Prove that $\quad \lim\limits_{x\to 4}(9-x)=5$

Solution

$$\left|(9-x)-5\right|<\varepsilon \quad\Rightarrow\quad -\varepsilon<4-x<\varepsilon$$

$$-\varepsilon-4<-x<\varepsilon-4 \qquad \textit{divide by } (-).$$

$$\varepsilon+4>x>4-\varepsilon$$

$$4-\varepsilon<x<\varepsilon+4$$

$$|x-4|<\delta \quad\Rightarrow\quad -\delta<x-4<\delta$$

$$-\delta+4<x<\delta+4$$

$$-\delta+4=4-\varepsilon \;\Rightarrow\; -\delta=-\varepsilon\Rightarrow\delta=\varepsilon$$

$$\delta+4=\varepsilon+4 \;\Rightarrow\; \delta=\varepsilon \quad\to\quad \boxed{\delta=\varepsilon}$$

Exercise

Prove that $\lim\limits_{x\to 1}\frac{1}{x}=1$

Solution

$$\left|\frac{1}{x}-1\right|<\varepsilon \quad\Rightarrow\quad -\varepsilon<\frac{1}{x}-1<\varepsilon$$

$$-\varepsilon+1<\frac{1}{x}<\varepsilon+1$$

$$\frac{1}{\varepsilon+1}>x>\frac{1}{-\varepsilon+1}$$

$$\frac{1}{1+\varepsilon}<x<\frac{1}{1-\varepsilon}$$

$$|x-1|<\delta \quad\Rightarrow\quad -\delta<x-1<\delta$$

$$1-\delta<x<1+\delta$$

$$\left.\begin{array}{l} 1-\delta=\frac{1}{1+\varepsilon} \quad\Rightarrow\quad \delta=1+\frac{1}{1+\varepsilon}=\frac{2+\varepsilon}{1+\varepsilon} \\ 1+\delta=\frac{1}{1-\varepsilon} \quad\Rightarrow\quad \delta=\frac{1}{1-\varepsilon}-1=\frac{\varepsilon}{1-\varepsilon} \end{array}\right. \to \text{ the smallest}: \boxed{\delta=\frac{\varepsilon}{1-\varepsilon}}$$

Exercise

Prove that $\lim\limits_{x\to 5}\frac{x^2-25}{x-5}=10$

Solution

$$\left|\frac{x^2-25}{x-5}-10\right|<\varepsilon \quad\Rightarrow\quad -\varepsilon<\frac{(x-5)(x+5)}{x-5}-10<\varepsilon$$

$$-\varepsilon+10<x+5<\varepsilon+10$$

$$-\varepsilon+5<x<\varepsilon+15$$

$$|x-10|<\delta \quad\Rightarrow\quad -\delta<x-10<\delta$$

$$10-\delta<x<10+\delta$$

$$\left.\begin{array}{l} 10-\delta=5-\varepsilon \quad\Rightarrow\quad \delta=5+\varepsilon \\ 10+\delta=\varepsilon+15 \quad\Rightarrow\quad \delta=\varepsilon+5 \end{array}\right. \to \text{ the smallest}: \underline{\delta=\varepsilon+5}$$

Exercise

Prove that $\lim\limits_{x\to 0} f(x)=0 \quad if \quad f(x)=\begin{cases} 2x, & x<0 \\ \frac{x}{2}, & x\ge 0 \end{cases}$

Solution

For $x<0$: $|2x-0|<\varepsilon \quad\Rightarrow\quad -\varepsilon<2x<0$

$$-\frac{\varepsilon}{2}<x<0$$

For $x\ge 0$: $\left|\frac{x}{2}-0\right|<\varepsilon \quad\Rightarrow\quad 0\le\frac{x}{2}<\varepsilon$

$$0 \le x < 2\varepsilon$$

$$|x-0| < \delta \quad \Rightarrow \quad -\delta < x < \delta$$

$$-\delta = -\frac{\varepsilon}{2} \;\Rightarrow\; \delta = \frac{\varepsilon}{2} \quad \to \textit{ the smallest}: \boxed{\delta = \frac{\varepsilon}{2}}$$

$$\delta = 2\varepsilon$$

Exercise

Prove that $\lim\limits_{x \to 1} (5x-2) = 3$

Solution

$$|(5x-2)-3| < \varepsilon \quad \Rightarrow \quad -\varepsilon < 5x-5 < \varepsilon$$

$$5-\varepsilon < 5x < \varepsilon + 5$$

$$1-\frac{1}{5}\varepsilon < x < 1+\frac{1}{5}\varepsilon$$

$$|x-3| < \delta \quad \Rightarrow \quad -\delta < x-3 < \delta$$

$$3-\delta < x < 3+\delta$$

$$3-\delta = 1-\frac{1}{5}\varepsilon \quad \Rightarrow \quad \delta = \frac{1}{5}\varepsilon + 2$$

$$3+\delta = 1+\frac{1}{5}\varepsilon \quad \Rightarrow \quad \delta = \frac{1}{5}\varepsilon - 2$$

$$\to \textit{ the smallest}: \boxed{\delta = \frac{1}{5}\varepsilon - 2}$$

Exercise

Prove that $\lim\limits_{x \to 2} \frac{1}{(x-2)^4} = \infty$

Solution

Let $N > 0$ and let $\delta = \frac{1}{\sqrt[4]{N}}$

Suppose that $0 < |x-2| < \delta$

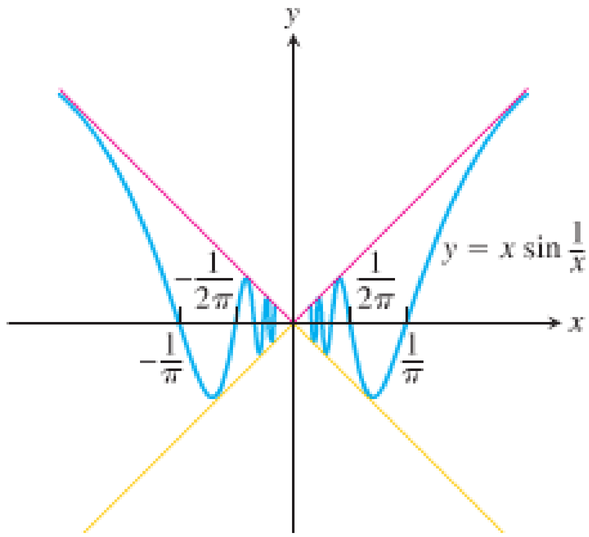
$$|x-2| < \delta = \frac{1}{\sqrt[4]{N}}$$

$$\frac{1}{|x-2|} > \sqrt[4]{N}$$

$$\boxed{\frac{1}{(x-2)^4} > N} \quad \checkmark$$

Exercise

Prove that $\lim\limits_{x\to 0} x\dfrac{1}{\sin x} = 0$



Solution

$$\left.\begin{aligned} -x \le x\sin\frac{1}{x} \le x, \quad \forall x > 0 \\ -x \ge x\sin\frac{1}{x} \ge x, \quad \forall x < 0 \end{aligned}\right\} \to \lim_{x\to 0}(-x) = \lim_{x\to 0}(x) = 0$$

Then by the sandwich theorem, $\boxed{\lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0}$