Lecture Three - Exponential and Logarithmic Functions

Section 3.1 – Inverse Functions

Inverse Relations

Interchanging the first and second coordinates of each ordered pair in a relation produces the inverse relation.

If a relation is defined by an equation, interchanging the variables produces an equation of the inverse relation

Given the relation: $\{(Zambia, 4.2), (Columbia, 4.5), (Poland, 3.3), (Italy, 3.3), (US, 2.5)\}$

Inverse Relation: $\{(4.2, Zambia), (4.5, Columbia), (3.3, Poland), (3.3, Italy), (2.5, US)\}$

Example

Consider the relation g given by: $G = \{(2, 4), (-1, 3), (-2, 0)\}$

Solution

The inverse relation: $G = \{(4, 2), (3, -1), (0, -2)\}$

Example

Consider the relation given by: $F = \{(-2, 2), (-1, 1), (0, 0), (1, 3), (2, 5)\}$

Solution

The inverse relation: $G = \{(2, -2), (1, -1), (0, 0), (3, 1), (5, 2)\}$

One-to-One Functions

A function *f* is one-to-one (1 – 1) if different inputs have different outputs that is,

$$if \;\; a \neq b, \quad then \quad f(a) \neq f(b)$$

A function *f* is one-to-one (1 – 1) if different outputs the same, the inputs are the same – that is,

$$if \;\; f(a) = f(b), \quad then \;\; a = b$$

Example

Given the function *f* described by $f(x) = 2x - 3$, prove that f is one-to-one.

Solution

$f(a) = f(b)$

$2a - 3 = 2b - 3$ *Add 3 on both sides*

$2a = 2b$ *Divide by 2*

$a = b$ f is one-to-one

Example

Given the function *f* described by $f(x) = -4x + 12$, prove that f is one-to-one.

Solution

$f(a) = f(b)$

$-4a + 12 = -4b + 12$ *Subtract 12 from both sides*

$-4a = -4b$ *Divide by -4*

$a = b$

Example

Given the function *f* described by $f(x) = x^2$, prove that *f* is one-to-one.

Solution

$-1 \neq 1$

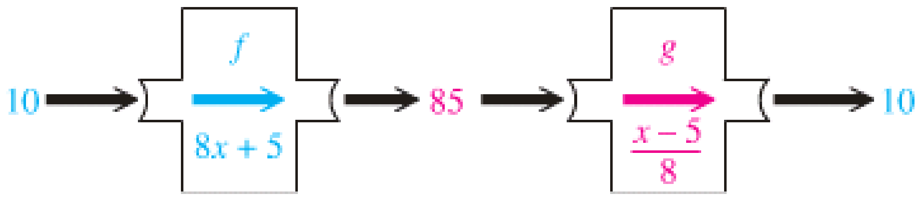
$\begin{cases} f(-1) = 1 \\ f(1) = 1 \end{cases} \Rightarrow f(-1) = f(1)$ f is *not* one-to-one

Definition of the Inverse of a Function

Let f and g be two functions such that

$$f(g(x)) = x \qquad \textit{and} \qquad g(f(x)) = x$$

$$x \underset{g = f^{-1}}{\overset{f}{\rightleftarrows}} f(x) \qquad\qquad g(f(x)) = f^{-1}(f(x)) = x$$



If the inverse of a function f is also a function, it is named f^{-1} read "f - inverse"

The -1 in f^{-1} is not an exponent! And is not equal to $\cancel{\dfrac{1}{f(x)}}$

***Domain* and *Range* of** f *and* f^{-1}

$$\text{domain of } f^{-1} = \text{range of } f$$

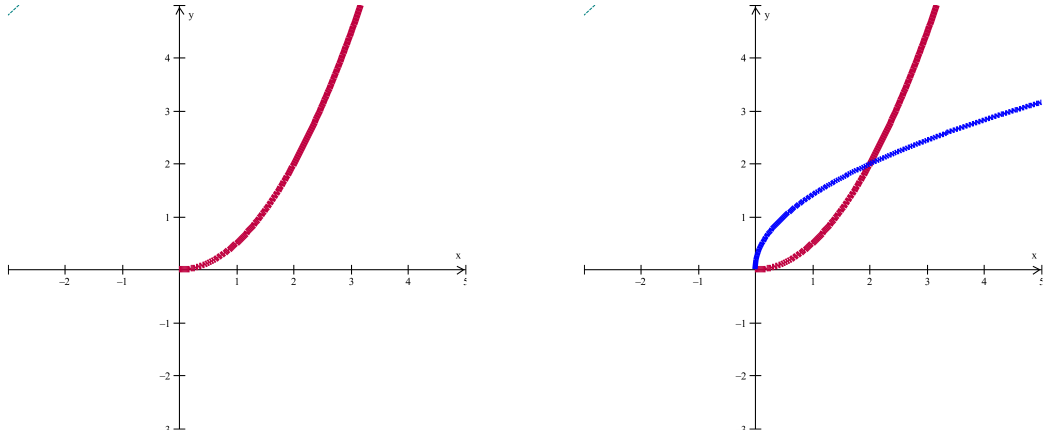
$$\text{range of } f^{-1} = \text{domain of } f$$

If a function f is one-to-one, then f^{-1} is the unique function such that each of the following holds.

$\boxed{\left(f^{-1} \circ f\right)(x) = f^{-1}(f(x)) = x}$ for each x in the domain of f, and

$\boxed{\left(f \circ f^{-1}\right)(x) = f(f^{-1}(x)) = x}$ for each x in the domain of f^{-1}

The condition that f is one-to-one in the definition of inverse function is important; otherwise, g will not define a function



Example

Let $f(x)=x^3-1$ and $g(x)=\sqrt[3]{x+1}$, is g the inverse function of f?

Solution

$$\begin{aligned}(f\circ g)(x)&=f(g(x))\\&=f\left(\sqrt[3]{x+1}\right)\\&=\left(\sqrt[3]{x+1}\right)^3-1\\&=x+1-1\\&=x\end{aligned}$$

$$\begin{aligned}(g\circ f)(x)&=g\left(f(x)\right)\\&=g\left(x^3-1\right)\\&=\sqrt[3]{x^3-1+1}\\&=\sqrt[3]{x^3}\\&=x\end{aligned}$$

g is the inverse function of f

Example

Show that each function is the inverse of the other: $f(x)=4x-7$ *and* $g(x)=\dfrac{x+7}{4}$

Solution

$$\begin{aligned}f(g(x))&=f\left(\frac{x+7}{4}\right)\\&=4\left(\frac{x+7}{4}\right)-7\\&=x+7-7\\&=x\end{aligned}$$

$$\begin{aligned}g(f(x))&=g(4x-7)\\&=\frac{4x-7+7}{4}\\&=\frac{4x}{4}\\&=x\end{aligned}$$

Finding the *Inverse Function*

	Example
Finding an Inverse Function	$f(x) = 2x + 7$
1. Replace $f(x)$ with y	$y = 2x + 7$
2. Interchange x and y	$x = 2y + 7$
3. Solve for y	$x - 7 = 2y$
	$\dfrac{x-7}{2} = y$
4. Replace y with $f^{-1}(x)$	$f^{-1}(x) = \dfrac{x-7}{2}$

Example

Find the inverse of $f(x) = 4x^3 - 1$

Solution

$$y = 4x^3 - 1$$

$$x = 4y^3 - 1$$

$$x + 1 = 4y^3$$

$$\frac{x+1}{4} = y^3$$

$$y = \left(\frac{x+1}{4}\right)^{1/3}$$

$$= \boxed{\sqrt[3]{\frac{x+1}{4}} = f^{-1}(x)}$$

Example

Find a formula for the inverse $f(x)=\frac{5x-3}{2x+1}$

<u>Solution</u>

$$y=\frac{5x-3}{2x+1}$$

$$x=\frac{5y-3}{2y+1}$$

$$x(2y+1)=5y-3$$

$$2xy+x=5y-3$$

$$2xy-5y=-x-3$$

$$y(2x-5)=-x-3$$

$$y=\frac{-x-3}{2x-5}$$

$$\boxed{f^{-1}(x)=-\frac{x+3}{2x-5}}$$

Exercise *Section* 3.1 – Inverse Functions

Determine whether the function is one-to-one

1. $f(x)=3x-7$

2. $f(x)=x^2-9$

3. $f(x)=\sqrt{x}$

4. $f(x)=\sqrt[3]{x}$

5. $f(x)=|x|$

Prove that the given function *f* is one-to-one

6. $f(x)=\dfrac{2}{x+3}$

7. $f(x)=(x-2)^3$

8. $y=x^2+2$

9. $f(x)=\dfrac{x+1}{x-3}$

10. Let $f(x)=x^3-1$ and $g(x)=\sqrt[3]{x+1}$, is *g* the inverse function of *f*?

11. Given that $f(x)=5x+8$, use composition of functions to show that $f^{-1}(x)=\dfrac{x-8}{5}$

12. Given the function $f(x)=(x+8)^3$

a) Find $f^{-1}(x)$

b) Graph f and f^{-1} in the same rectangular coordinate system

c) Find the domain and the range of f and f^{-1}

Prove the f *and* g are inverse functions of each other, and sketch the graphs of f *and* g

13. $f(x)=3x-2 \quad g(x)=\dfrac{x+2}{3}$

14. $f(x)=x^2+5, x\le 0 \quad g(x)=-\sqrt{x-5}, \ \ x\ge 5$

15. $f(x)=x^3-4; \quad g(x)=\sqrt[3]{x+4}$

Determine the domain and range of f^{-1}(Hint: first find the domain and range of *f*)

16. $f(x)=-\dfrac{2}{x-1}$

17. $f(x)=\dfrac{5}{x+3}$

18. $f(x)=\dfrac{4x+5}{3x-8}$

Find the inverse function of

19. $f(x)=3x+5$

20. $f(x)=\dfrac{1}{3x-2}$

21. $f(x)=\dfrac{3x+2}{2x-5}$

22. $f(x)=\dfrac{4x}{x-2}$

23. $f(x)=2-3x^2; \quad x\le 0$

24. $f(x)=2x^3-5$

25. $f(x)=\sqrt{3-x}$

26. $f(x)=\sqrt[3]{x}+1$

27. $f(x)=\left(x^3+1\right)^5$

28. $f(x)=x^2-6x; \quad x\ge 3$

29. $f(x)=(x-2)^3$

30. $f(x)=\dfrac{x+1}{x-3}$

31. $f(x)=\dfrac{2x+1}{x-3}$

Section 3.2 - Exponential Functions

Definition

The exponential function f with base $\boldsymbol{b}$ is defined by

$$f(x) = b^x \qquad \text{or} \qquad y = b^x$$

Base

where $b > 0$, $b \neq 1$ and $\boldsymbol{x}$ is any real number.

$$f(x) = 2^x \qquad f(x) = \left(\tfrac{1}{2}\right)^{2x+1} \qquad f(x) = 3^{-x} \qquad \cancel{f(x) = (-2)^x}$$

Example

Given: $f(x) = 13.49\,(0.967)^x - 1$, find $f(60)$

<u>Solution</u>

$$\begin{aligned} f(\mathbf{60}) &= 13.49\,(0.967)^{\mathbf{60}} - 1 \\ &= 0.8014 \end{aligned}$$

Example

If $f(x) = 2^x$, find each of the following. $f(-1)$, $f(3)$, $f\left(\frac{5}{2}\right)$

<u>Solution</u>

a) $f(-1) = 2^{-1} = 0.5$

b) $f(3) = 2^3 = 8$

c) $f\left(\frac{5}{2}\right) = 2^{\frac{5}{2}} = 5.6569$

Graphing Exponential

1. Define the Horizontal Asymptote $f(x) = b^x \pm d$

$y = 0 \pm \text{d}$

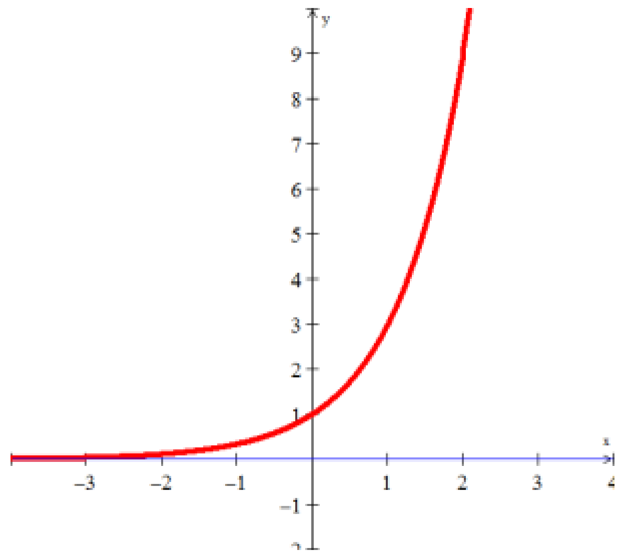
Example

$f(x) = 3^x$

Asymptote: $y = 0$

The exponential function always equals to 0

$x \to \infty \;\; or \;\; x \to -\infty \Rightarrow f(x) \to 0$



2. Define/Make a table

(Force your exponential to = 0, then solve for x)

x	***f(x)***
$x-2$	
$x-1$	
x	
$x+1$	
$x+2$	

x	*f(x)*
–2	1/9
–1	1/3
0	1
1	3
2	9

Domain: $(-\infty,\ \infty)$

Range: (d, ∞)

Example

$$f(x) = \left(\frac{1}{3}\right)^x$$

$$= \left(3^{-1}\right)^x$$

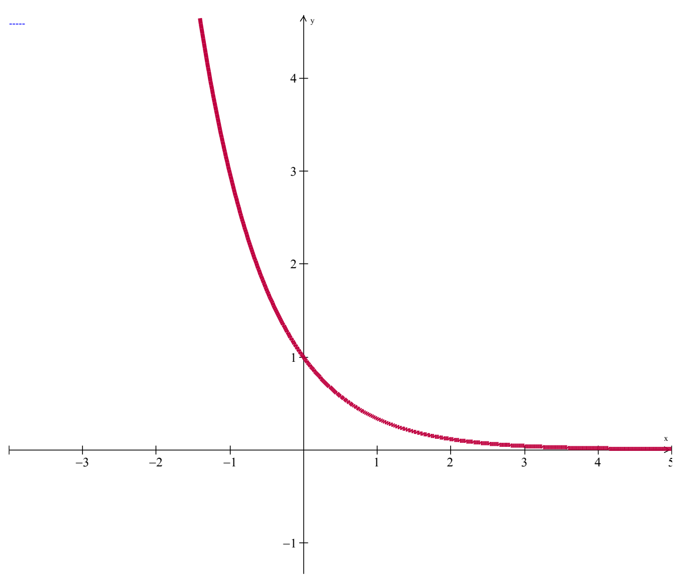
$$= 3^{-x}$$

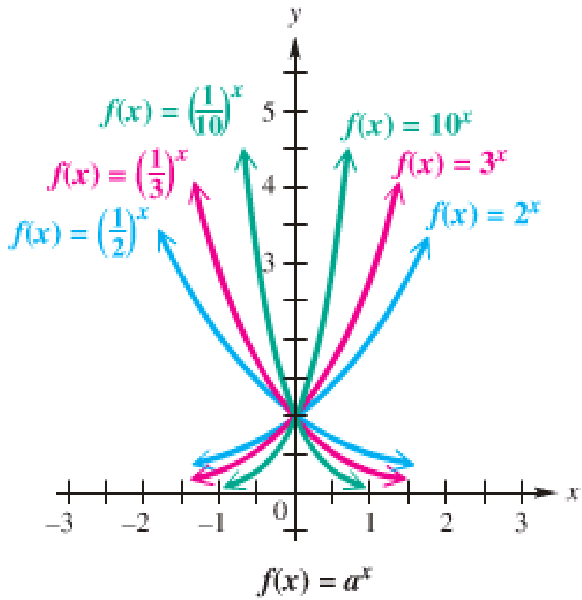
Reflected across y-axis

Asymptote: $y = 0$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$





Example

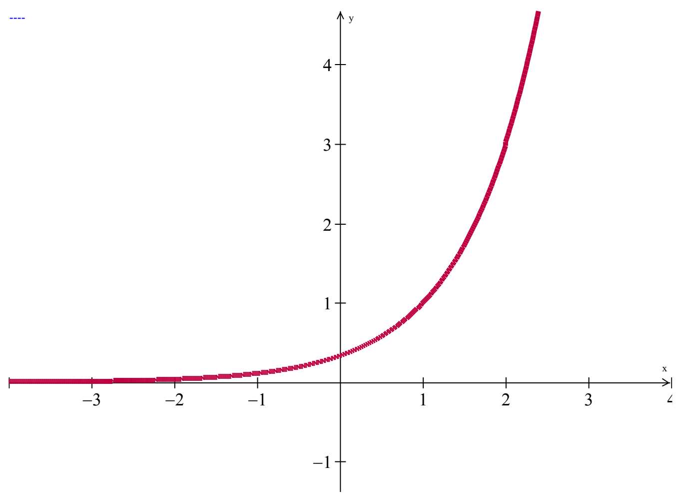
$f(x) = 3^{x-1}$

Shift right 1 unit

Asymptote: $y = 0$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$



Example

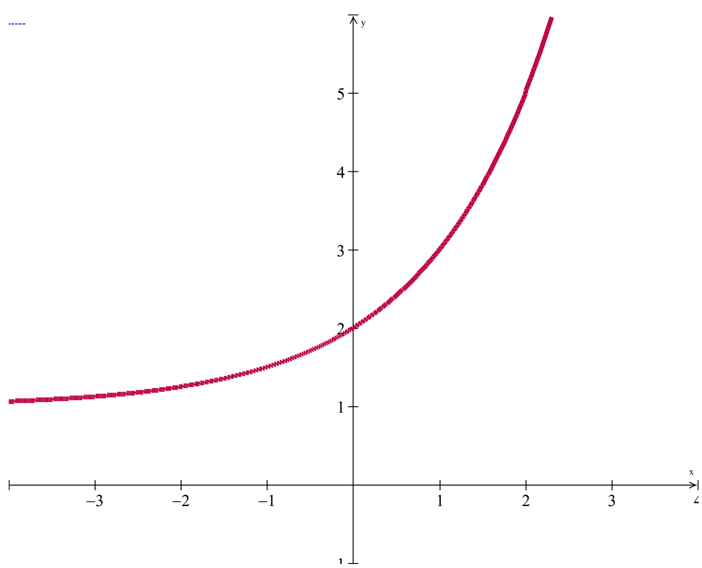
$f(x) = 2^x + 1$

Shift up 1 unit

Asymptote: $y = 1$

Domain: $(-\infty, \infty)$

Range: $(1, \infty)$

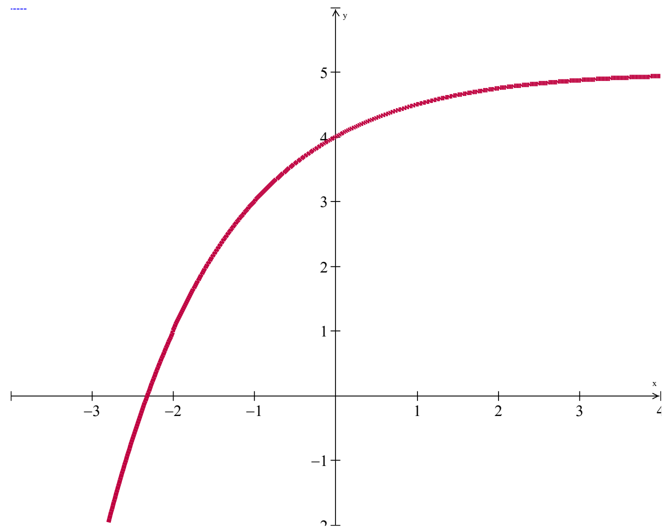


$f(x) = 5 - 2^{-x}$

Shifted up 5 units

Reflected across x-axis and y-axis

Asymptote: $y = 5$



Example

Give the domain and range.

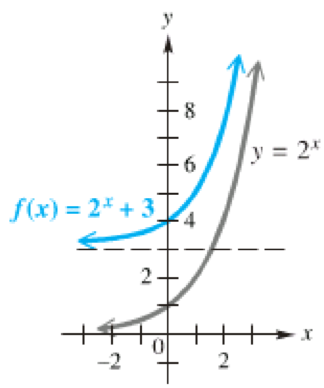
a) $f(x) = -2^{x}$

Reflected across x-axis

Asymptote: $y = 0$

Domain: $(-\infty, \infty)$

Range: $(-\infty, 0)$



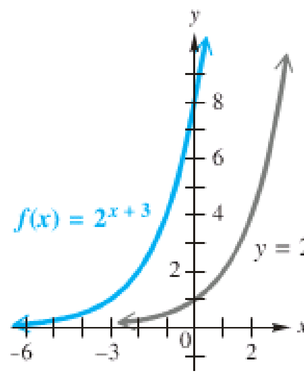
b) $f(x) = 2^{x+3}$

Shifted left 3 units

Asymptote: $y = 0$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$



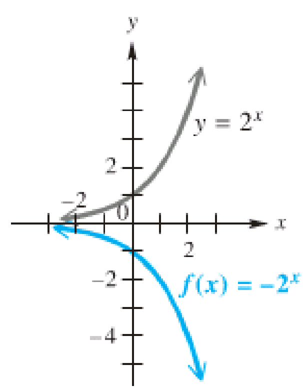
c) $f(x) = 2^{x} + 3$

Shifted up 3 units

Asymptote: $y = 3$

Domain: $(-\infty, \infty)$

Range: $(3, \infty)$



Natural Base *e*

The irrational number e is called natural base

$f(x) = e^x$ is called natural exponential function

$e^0 = 1$ $\qquad$ $e \approx 2.7183$ $\qquad$ $e^2 \approx 7.3891$ $\qquad$ $e^{-1} \approx 0.3679$

Example

The exponential function $f(x) = 1066e^{0.042x}$ models the gray wolf population of the Western Great Lakes, $f(x)$, in billions, x years after 1978. Project the gray population in the recovery area in 2012.

Solution

$x = 2012 - 1978 = 34$

$$f(x = 34) = 1066e^{0.042(34)}$$ $\qquad$ $1066\ e \wedge (.042 * 34)$

$$= 4445.6$$

$$\approx 4446$$

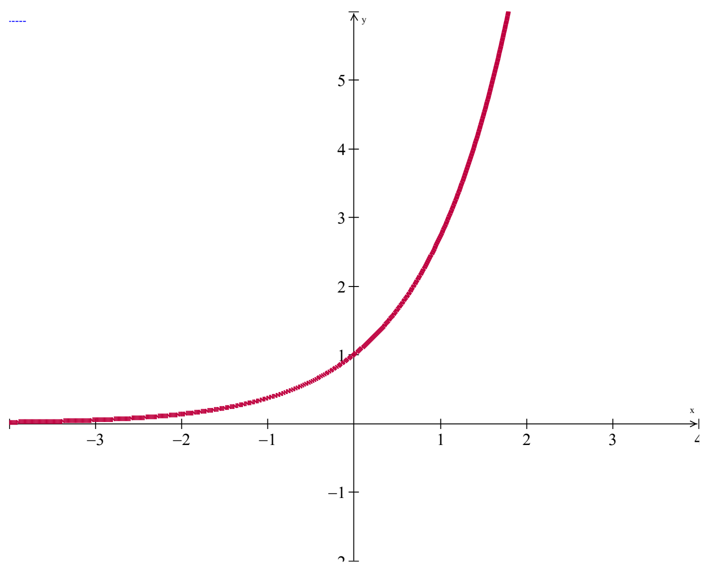
Example

Graph $\qquad f(x) = e^x$

Solution

Asymptote: $y = 0$

x	***f(x)***
-2	.14
-1	.4
0	1
1	2.7
2	7.4



Example

$f(x) = e^{x+3}$

Solution

Shifted left 3 units

Asymptote: $y = 0$

Formulas for Compound Interest

1. For n compounding per year: $A = P\left(1+\frac{r}{n}\right)^{nt}$

2. For Continuous compounding: $A = Pe^{rt}$

P: Principal, initial value
n: number of period per year
t: number of years
r: interest rate

A is also called ***Future value***

P is also called ***Present value***

Example

A sum of \$10,000 is invested at an annual rate of 8%. Find the balance in the account after 5 years subject to quarterly compounding

Solution

Given:
P = \$10,000
r = 8% = 0.08
t = 5

Quarterly n = 4 $\Rightarrow A = P\left(1+\frac{r}{n}\right)^{nt} = 10000\left(1+\frac{0.08}{4}\right)^{4(5)}$ $10000(1+.08/4)\wedge(4*5)$

$\boxed{= \$14,859.47}$

Example

Suppose \$5000 is deposited in an account paying 3% interest compounded continuously for 5 *yrs*. Find the total amount on deposit at the end of 5 *yrs*.

Solution

$A = Pe^{rt} = 5000e^{.03(5)} \boxed{\approx \$5,809.17}$ $5000\ e(.03*5)$

Exercises *Section* 3.2 - Exponential Functions

Find

1. $2^{3.4}$ **2.** $5^{\sqrt{3}}$ **3.** $6^{-1.2}$

Evaluate to four decimal places using a calculator

4. $e^{-0.75}$ **5.** $e^{2.3}$ **6.** $e^{-0.95}$

Sketch the graph

7. $f(x)=2^x+3$

8. $f(x)=2^{3-x}$

9. $f(x)=\left(\frac{2}{5}\right)^{-x}$

10. $f(x)=-\left(\frac{1}{2}\right)^x+4$

11. $f(x)=e^{x+4}$

12. The exponential function $f(x)=1066e^{0.042x}$ models the gray wolf population of the Western Great Lakes, $f(x)$, in billions, x years after 1978. Project the gray population in the recovery area in 2012.

13. The function $f(x)=6.4e^{0.0123x}$ describes world population, $f(x)$, in billions, x years after 2004 subject to a growth rate of 1.23% annually. Use the function to predict world population in 2050.

14. Find the accumulated value of an investment of $10,000 for 5 years at an interest rate of 5.5% if the money is

a) Compounded semiannually

b) Compounded quarterly

c) Compounded monthly

15. Suppose $1000 is deposited in an account paying 4% interest per year compounded quarterly.

a) Find the amount in the account after 10 years with no withdraws.

b) How much interest is earned over the 10 years period?

16. An investment of 1,000 increased to $13,464 in 20 years. If the interest was compounded continuously, find the interest rate.

17. Becky must pay a lump sum of $6000 in 5 yrs.

a) What amount deposited today at 3.1% compounded annually will grow to $6000 in 5 yrs.?

b) If only $5000 is available to deposit now, what annual interest rate is necessary for the money to increase to $6000 in 5 yrs.?

18. Find the present value of $4,000 if the annual interest rate is 3.5% compounded quarterly for 6 years.

19. How much money will there be in an account at the end of 8 years if \$18,000 is deposited at 3% interest compounded semi-annually?

20. The function defined by $P(x) = 908e^{-0.0001348x}$ approximates the atmospheric pressure (in millibars) at an altitude of x meters. Use P to predict the pressure:

a) At 0 meters

b) At 12,000 meters

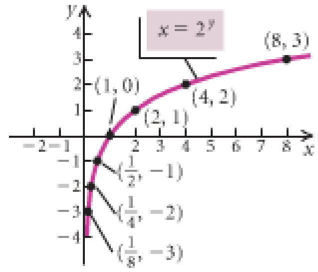
Section 3.3 - Logarithmic Functions

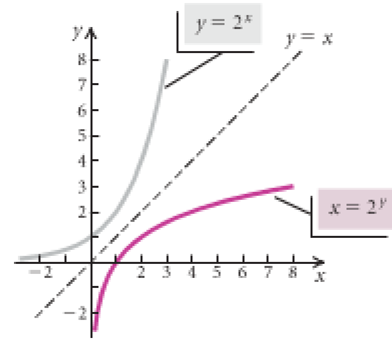
Graph: $x = 2^y$

Find the inverse function of $f(x) = 2^x$

$y = 2^x$

$x = 2^y$ *Solve for y?*





Logarithmic Function (*Definition*)

For $\mathbf{x} > \mathbf{0}$ and $b > 0$, $b \neq 1$

$y = \log_b x$ is equivalent to $x = b^y$

$$y = \log_b x \iff x = b^y$$

Base

The function $f(x) = \log_b x$ is the logarithmic function with base b.

$\log_b x$: <u>*read*</u> log base b of x *log x* ***means*** *log* $_{10}$ *x*

Example

Write each equation in its equivalent exponential form:

a) $3 = \log_7 x \quad \Rightarrow x = 7^3$

b) $2 = \log_b 25 \quad \Rightarrow 25 = b^2$

c) $\log_4 26 = y \quad \Rightarrow 26 = 4^y$

Write each equation in its equivalent logarithmic form:

a) $2^5 = x \quad \Rightarrow 5 = \log_2 x$

b) $27 = b^3 \quad \Rightarrow 3 = \log_b 27$

Basic Logarithmic Properties

$\log_b b = 1 \quad \to b = b^1$

$\log_b 1 = 0 \quad \to 1 = b^0$

Inverse Properties

$\log_b b^x = x$ $\qquad\qquad$ $\log_7 7^8 = 8$

$b^{\log_b x} = x$ $\qquad\qquad$ $3^{\log_3 17} = 17$

Example

Evaluate each expression without using a calculator:

a. $\log_5 \frac{1}{125}$

$\Rightarrow \log_5 \frac{1}{5^3} = x$ $\qquad$ *converts to exponential*

$5^{-3} = 5^x$

$-3 = x$

$\Rightarrow \log_5 \frac{1}{125} = -3$

$\log_5 \frac{1}{125} = \log_5 \frac{1}{5^3} = \log_5 5^{-3} = -3$ $\quad$ (*Inverse Property*)

b. $\log_3 \sqrt[7]{3}$

$\Rightarrow \log_3 3^{1/7} = \frac{1}{7}$

Natural Logarithms

Definition

$$f(x) = \log_e x = lnx$$

The logarithmic function with base ***e*** is called natural logarithmic function.

$\ln x$ *read* "el en of x"

log(–1) = *doesn't exist*	ln(–1) = *doesn't exist*
log0 = *doesn't exist*	ln0 = *doesn't exist*
$\log 0.5 \approx -0.3010$	$\ln 0.5 \approx -0.6931$
log1 = 0	ln1 = 0
$\log 2 \approx 0.3010$	$\ln 2 \approx 0.6931$
log10 = 1	ln$e$ = 1

Change-of-Base Logarithmic

$$\log_b M = \frac{\log_a M}{\log_a b} \qquad \log_b M = \frac{\log M}{\log b} \quad \textbf{\textit{or}} \quad \log_b M = \frac{\ln M}{\ln b}$$

Evaluate

$$\log_7 2506 = \frac{\log 2506}{\log 7}$$ log(2506) / log(7)

$$\approx 4.02$$

Or

$$\log_7 2506 = \frac{\ln 2506}{\ln 7} \approx 4.02$$ ln(2506) / ln(7)

$$\log_5 17 = \frac{\ln 17}{\ln 5} \approx 1.7604$$

$$\log_2 0.1 = \frac{\ln 0.1}{\ln 2} \approx -3.3219$$

Domain

The domain of a logarithmic function of the form $f(x) = \log_b x$ is the set of all positive real numbers. (*Inside* the log has to be > 0)

Range: $(-\infty, \infty)$

Example

Find the domain of

a) $f(x) = \log_4(x-5)$

$$x-5>0 \;\Rightarrow\; x>5$$

Domain: $\boxed{(5,\ \infty)}$

b) $f(x) = \ln(4-x)$

$$4-x>0$$
$$\Rightarrow -x>-4$$
$$\Rightarrow x<4$$

Domain: $\boxed{(-\infty,\ 4)}$

c) $h(x) = \ln(x^2)$

$x^2 > 0 \Rightarrow$ all real numbers except 0.

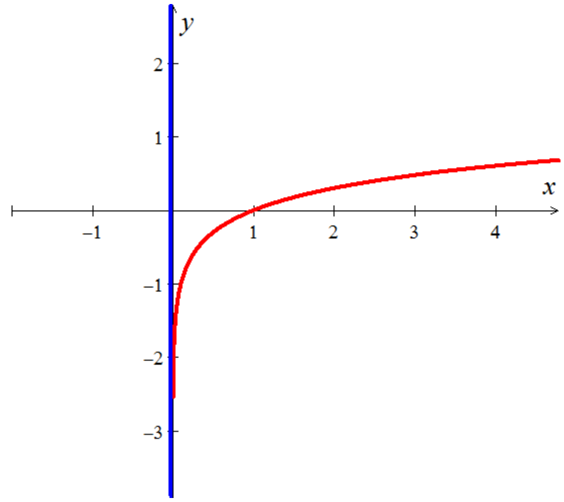
Domain: $\{x | x \neq 0\}$ or $\boxed{(-\infty,\ 0)\cup(0,\ \infty)}$

Graphs of *Logarithmic* Functions

Graph $g(x) = \log x$

Asymptote: x = 0 (*Force inside log to be equal to zero, then solve for x*)

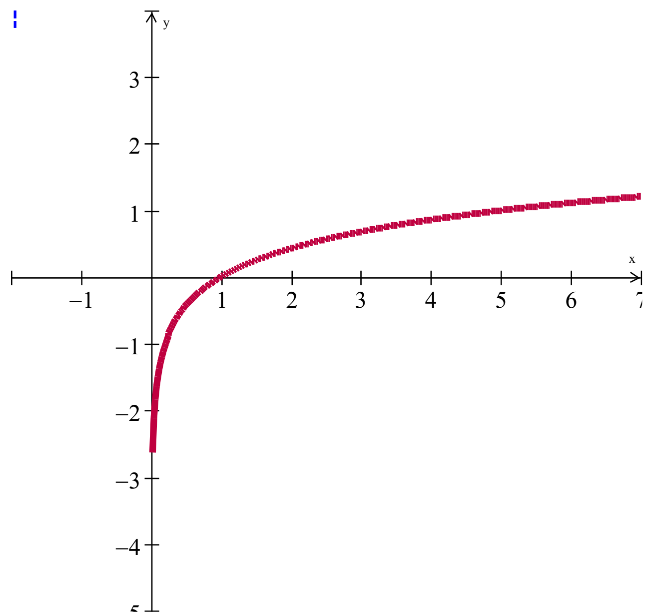
x	$g(x)$
~~0~~	
0.5	–.3
1	0
2	.3
3	.5



$f(x) = \log_5 x$

Asymptote: x = 0

$f(x) = \log_5 x = \dfrac{\log x}{\log 5}$

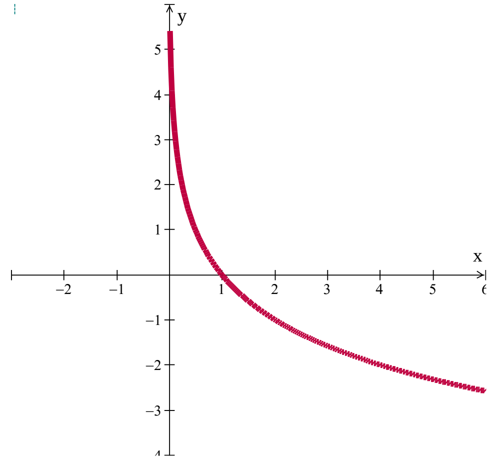


Graph: $f(x) = \log_{1/2} x$

Asymptote: $x = 0$

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$



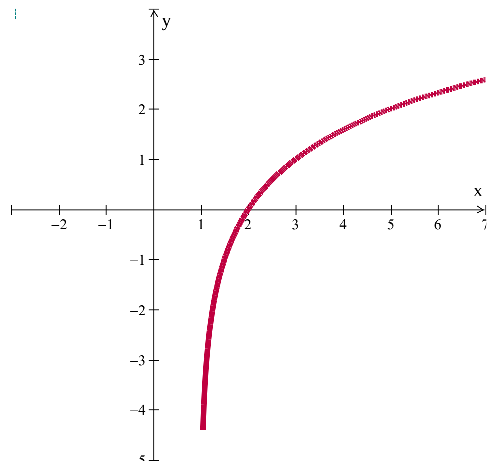
Graph: $f(x) = \log_2 (x-1)$

Asymptote: $x = 1$

Domain: $(1, \infty)$

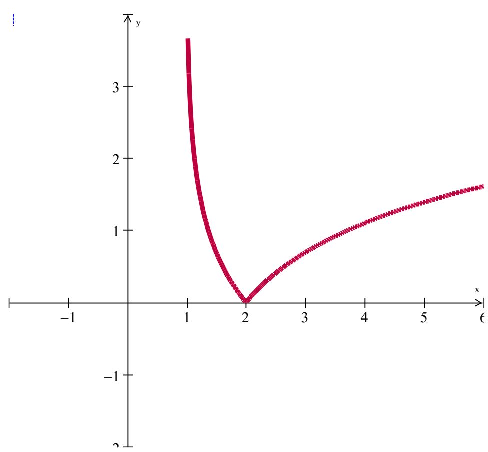
Range: $(-\infty, \infty)$

Shifted 1 unit right.



$f(x) = |\ln(x-1)|$

Asymptote: $x = 1$



Exercises *Section* 3.3 - Logarithmic Functions

1. Find $\log_8 14$

Write the equation in its equivalent logarithmic form

2. $2^6 = 64$

3. $2 = \log_9 x$

4. $5^4 = 625$

5. $5^{-3} = \frac{1}{125}$

6. $\sqrt[3]{64} = 4$

7. $b^3 = 343$

8. $8^y = 300$

9. $\left(\frac{2}{3}\right)^{-3} = \frac{27}{8}$

Write the equation in its equivalent exponential form

10. $\log_5 125 = y$

11. $\log_4 16 = x$

12. $\log_5 \frac{1}{5} = x$

13. $\log_2 \frac{1}{8} = x$

14. $\log_6 \sqrt{6} = x$

15. $\log_3 \frac{1}{\sqrt{3}} = x$

16. $6 = \log_2 64$

17. $2 = \log_9 x$

18. $\log_{\sqrt{3}} 81 = 8$

19. $\log_4 \frac{1}{64} = -3$

Evaluate the expression without using a calculator

20. $\log_4 16$ **21.** $\log_2 \frac{1}{8}$ **22.** $\log_6 \sqrt{6}$ **23.** $\log_3 \frac{1}{\sqrt{3}}$ **24.** $\log_3 \sqrt[7]{3}$

25. Find $\log_5 8$ using common logarithms

Find the number

26. $\log_5 1$

27. $\log_7 7^2$

28. $3^{\log_3 8}$

29. $10^{\log 3}$

30. $e^{2+\ln 3}$

31. $\ln e^{-3}$

Find the domain of

32. $\log_5(x+4)$ **33.** $\log_5(x+6)$ **34.** $\log(2-x)$ **35.** $\log(7-x)$

36. $\ln(x-2)^2$ **37.** $\ln(x-7)^2$ **38.** $\log\left(x^2-4x-12\right)$ **39.** $\log\left(\frac{x-2}{x+5}\right)$

Sketch the graph of

40. $f(x) = \log_4 (x-2)$

41. $f(x) = \log_4 |x|$

42. $f(x) = \left(\log_4 x\right) - 2$

43. On a study by psychologists Bornstein and Bornstein, it was found that the average walking speed w, in feet per second, of a person living in a city of population P, in ***thousands***, is given by the function:

$$w(P) = 0.37 \ln P + 0.05$$

a) The population is 124,848. Find the average walking speed of people living in Hartford.

b) The population is 1,236,249. Find the average walking speed of people living in San Antonio.

44. The loudness of sounds is measured in a unit called a decibel. To measure with this unit, we first assign an intensity of I_0 to a very faint sound, called the threshold sound. If a particular sound has intensity *I*, then the decibel rating of this louder sound is

$$d = 10\log\frac{I}{I_0}$$

Find the exact decibel rating of a sound with intensity $10{,}000I_0$

45. Students in an accounting class took a final exam and then took equivalent forms of the exam at monthly intervals thereafter. The average score *S(t)*, as a percent, after *t* months was found to be given by the function

$$S(t) = 78 - 15\log(t + 1), \quad t \geq 0$$

a) What was the average score when the students initially took the test, t = 0?

b) What was the average score after 4 months? 24 months?

46. A model for advertising response is given by the function

$$N(a) = 1000 + 200\ln a, \qquad a \geq 1$$

Where *N(a)* is the number of units sold when a is the amount spent on advertising, in thousands of dollars.

a) $N(a = 1)$

b) $N(a = 5)$

Section 3.4 – Properties of Logarithms

Product Rule

$$\log_b MN = \log_b M + \log_b N$$

<u>For M > 0 and N > 0</u>

$$\begin{cases} \log_b M = x \quad \Rightarrow \quad M = b^x \\ \log_b N = y \quad \Rightarrow \quad N = b^y \end{cases} \qquad \Rightarrow MN = b^x b^y = b^{x+y}$$

Convert back to logarithmic form: $\log_b MN = x + y$

$$\log_b MN = \log_b M + \log_b N$$

Example

Use the product rule to expand the logarithmic expression

$$\log(100x) = \log 100 + \log x$$

Power Rule

$$\log_b M^{p} = p \log_b M$$

Example

Use the power rule to expand each logarithmic expression

$$\ln \sqrt[3]{x} = \ln(x)^{1/3} = \frac{1}{3}\ln x$$

Quotient Rule

$$\log_b \frac{M}{N} = \log_b M - \log_b N$$

Example

Use the quotient rule to expand the logarithmic expression

$$\ln\left(\frac{e^5}{11}\right) = \ln e^5 - \ln 11 = 5 - \ln 11$$

Express each of the following in terms of sums and differences of logarithms

a) $\log_6(7.9)$

$\log_6(7.9) = \log_6 7 + \log_6 9$ *Product Rule*

b) $\log_9\left(\frac{15}{7}\right)$

$\log_9\left(\frac{15}{7}\right) = \log_9 15 - \log_9 7$ *Quotient Rule*

c) $\log_5 \sqrt{8}$

$\log_5 \sqrt{8} = \log_5 8^{1/2}$

$= \frac{1}{2}\log_5 8$ *Power Rule*

d) $\log_b\left(x^4 \sqrt[3]{y}\right) = \log_b\left(x^4\right) + \log_b\left(\sqrt[3]{y}\right)$ *Product Rule*

$= \log_b\left(x^4\right) + \log_b\left(y^{1/3}\right)$ *Power Rule*

$= 4\log_b(x) + \frac{1}{3}\log_b(y)$

e) $\log_a\left(\frac{mnq}{p^2r^4}\right) = \log_a(mnq) - \log_a\left(p^2r^4\right)$ *Quotient Rule*

$= \log_a m + \log_a n + \log_a q - \left(\log_a p^2 + \log_a r^4\right)$ *Product Rule*

$= \log_a m + \log_a n + \log_a q - \log_a p^2 - \log_a r^4$

$= \log_a m + \log_a n + \log_a q - 2\log_a p - 4\log_a r$ *Power Rule*

f) $\log_5\left(\frac{\sqrt{x}}{25y^3}\right) = \log_5\left(x^{1/2}\right) - \log_5\left(25y^3\right)$ *Quotient Rule*

$= \log_5\left(x^{1/2}\right) - \left[\log_5\left(5^2\right) + \log_5\left(y^3\right)\right]$ *Product Rule*

$= \log_5\left(x^{1/2}\right) - \log_5\left(5^2\right) - \log_5\left(y^3\right)$ $\log_5\left(5^2\right) = 2$

$= \frac{1}{2}\log_5(x) - 2 - 3\log_5(y)$

Example

Write as a single logarithmic

a) $\log 25 + \log 4 = \log\left[(25).(4)\right]$

$$= \log 100$$

$$= \log 10^2$$

$$= 2$$

b) $\log(7x+6) - \log x = \log \dfrac{7x+6}{x}$

c) $\log_3(x+2) + \log_3 x - \log_3 2 = \log_3 x(x+2) - \log_3 2$ *Product Rule*

$$= \log_3 \frac{x(x+2)}{2}$$ *Quotient Rule*

d) $2\ln x + \frac{1}{3}\ln(x+5) = \ln x^2 + \ln(x+5)^{1/3}$ *Power Rule*

$$= \ln x^2\ (x+5)^{1/3}$$ *Product Rule*

$$= \ln x^2\ \sqrt[3]{x+5}$$

e) $2\log(x-3) - \log x = \log(x-3)^2 - \log x$ *Power Rule*

$$= \log \frac{(x-3)^2}{x}$$ *Quotient Rule*

f) $\dfrac{1}{4}\log_b x - 2\log_b 5 - 10\log_b y = \log_b x^{1/4} - \log_b 5^2 - \log_b y^{10}$ *Power Rule*

$$= \log_b x^{1/4} - \left[\log_b 5^2 + \log_b y^{10}\right]$$ *Factor the minus*

$$= \log_b x^{1/4} - \left[\log_b \left(5^2 y^{10}\right)\right]$$ *Product Rule*

$$= \log_b \frac{\sqrt[4]{x}}{5^2 y^{10}}$$ *Quotient Rule*

Exercises *Section* 3.4 – Properties of Logarithms

1. Express as a sum of logarithms: $\log_3(ab)$

2. Express as a sum of logarithms: $\log_7(7x)$

Express the following in terms of sums and differences of logarithms

3. $\log\frac{x}{1000}$

4. $\log_5\left(\frac{125}{y}\right)$

5. $\log_b x^7$

6. $\ln\sqrt[7]{x}$

7. $\log_a \frac{x^2 y}{z^4}$

8. $\log_b \frac{x^2 y}{b^3}$

9. $\log_b\left(\frac{x^3 y}{z^2}\right)$

10. $\log_b\left(\frac{\sqrt[3]{x}y^4}{z^5}\right)$

11. $\log_a \sqrt[4]{\frac{m^8 n^{12}}{a^3 b^5}}$

12. $\log_p \sqrt[3]{\frac{m^5 n^4}{t^2}}$

13. $\log\left(\frac{100x^3\sqrt[3]{5-x}}{3(x+7)^2}\right)$

14. $\log_b \sqrt[n]{\frac{x^3 y^5}{z^m}}$

15. $\log_a \sqrt[3]{\frac{a^2 b}{c^5}}$

16. $\log_b\left(x^4\sqrt[3]{y}\right)$

17. $\log_5\left(\frac{\sqrt{x}}{25y^3}\right)$

18. $\log_a \frac{x^3 w}{y^2 z^4}$

19. $\log_a \frac{\sqrt{y}}{x^4\sqrt[3]{z}}$

20. $\ln\sqrt[4]{\frac{x^7}{y^5 z}}$

21. $\ln x\sqrt[3]{\frac{y^4}{z^5}}$

Write the expression as a single logarithm

22. $2\log_a x+\frac{1}{3}\log_a(x-2)-5\log_a(2x+3)$

23. $5\log_a x-\frac{1}{2}\log_a(3x-4)-3\log_a(5x+1)$

24. $\log\left(x^3y^2\right)-2\log\left(x\sqrt[3]{y}\right)-3\log\left(\frac{x}{y}\right)$

25. $\ln y^3+\frac{1}{3}\ln\left(x^3y^6\right)-5\ln y$

26. $2\ln x-4\ln\left(\frac{1}{y}\right)-3\ln(xy)$

27. $\frac{1}{2}\log_a x+4\log_a y-3\log_a x$

28. $\frac{2}{3}\left[\ln\left(x^2-9\right)-\ln(x+3)\right]+\ln(x+y)$

29. $4\ln x+7\ln y-3\ln z$

30. $\frac{1}{3}\left[5\ln(x+6)-\ln x-\ln(x^2-25)\right]$

31. $\frac{2}{3}\left[\ln\left(x^2-4\right)-\ln(x+2)\right]+\ln(x+y)$

32. $\frac{1}{2}\log_b m+\frac{3}{2}\log_b 2n-\log_b m^2n$

33. $\frac{1}{2}\log_y p^3q^4-\frac{2}{3}\log_y p^4q^3$

34. $\frac{1}{4}\log_b x-2\log_b 5-10\log_b y$

35. Assume that $\log_{10} 2=.3010$. Find each logarithm $\log_{10} 4$, $\log_{10} 5$

36. Given that: $\log_a 2\approx 0.301, \log_a 7\approx 0.845$, and $\log_a 11\approx 1.041$ find each of the following:

$\log_a\frac{2}{11},\ \log_a 14,\ \log_a 98,\ \log_a\frac{1}{7},\ \log_a 9$

Section 3.5 – Exponential and logarithmic Equations

Exponential Equations

$$b^M = b^N \quad \leftrightarrow \quad M = N \quad \text{for any b} > 0, \neq 1$$

Example

Solve $5^{3x-6} = 125$

Solution

$5^{3x-6} = 5^3$

$3x - 6 = 3$

$3x = 9$

$\Rightarrow \boxed{x = 3}$

Example

Solve $8^{x+2} = 4^{x-3}$

Solution

$\left(2^3\right)^{x+2} = \left(2^2\right)^{x-3}$

$2^{3(x+2)} = 2^{2(x-3)}$

$3(x+2) = 2(x-3)$

$3x + 6 = 2x - 6$

$3x - 2x = -6 - 6$

$\boxed{x = -12}$

Using *Natural Logarithms*

1. Isolate the exponential expression
2. Take the natural logarithm on both sides of the equation
3. Simplify using one of the following properties: $\ln b^x = x \ln b$ *or* $\ln e^x = x$
4. Solve for the variable

Example

Solve: $7e^{2x} - 5 = 58$

Solution

$7e^{2x} - 5 = 58$ ***Isolate the exponential expression***

$7e^{2x} = 63$ ***Divide by 7 both sides***

$e^{2x} = 9$ ***Natural logarithm on both sides***

$\ln e^{2x} = \ln 9$ ***Use inverse Property***

$2x = \ln 9$

$\Rightarrow \boxed{x = \frac{\ln 9}{2} \approx 1.0986}$

Example

Solve: $3^{2x-1} = 7^{x+1}$

Solution

$\ln 3^{2x-1} = \ln 7^{x+1}$ ***Natural logarithm on both sides***

$(2x-1)\ln 3 = (x+1)\ln 7$ ***Power Rule***

$2x\ln 3 - \ln 3 = x\ln 7 + \ln 7$

$2x\ln 3 - x\ln 7 = \ln 3 + \ln 7$

$x(2\ln 3 - \ln 7) = \ln 3 + \ln 7$

$\boxed{x = \frac{\ln 3 + \ln 7}{2\ln 3 - \ln 7}} \approx 12.1143$

Logarithmic Equations

1. Express the equation in the form $\log_b M = c$
2. Use the definition of a logarithm to rewrite the equation in exponential form:

$$\log_b M = c \quad \Rightarrow \quad b^c = M$$

3. Solve for the variable
4. Check proposed solution in the original equation. Include only the set for M > 0

Example

Solve: $\log x + \log(x-3) = 1$

Solution

$\log[x(x-3)] = 1$ ***Product Rule***

$x(x-3) = 10^1$ ***Convert to exponential form***

$x^2 - 3x = 10$

$x^2 - 3x - 10 = 0$ ***Solve for x***

$\Rightarrow x = -2, 5$

Check: $x = -2 \Rightarrow \cancel{\log(-2) + \log(x-3) = 1}$

$x = 5 \Rightarrow \log(5) + \log(5-3) = 1$

Example

Solve: $\log_6(3x+2) + \log_6(x-1) = 1$

Solution

$\log_6[(3x+2)(x-1)] = 1$ ***Product Rule***

$(3x+2)(x-1) = 6^1$ ***Convert to exponential form***

$3x^2 - x - 2 = 6$

$3x^2 - x - 8 = 0$ ***Solve for x***

$x = \frac{1-\sqrt{97}}{6} < 0$ $\quad$ $x = \frac{1+\sqrt{97}}{6} > 1$

Solution: $\boxed{x = \frac{1+\sqrt{97}}{6}}$

Property of Logarithmic Equality

For any $M > 0$, $N > 0$, $b > 0, \neq 1$

$$\log_b M = \log_b N \quad \Rightarrow \quad M = N$$

Example

Solve: $\ln(x-3) = \ln(7x-23) - \ln(x+1)$

Solution

$$\ln(x-3) = \ln\left(\frac{7x-23}{x+1}\right)$$ ***Quotient Rule***

$$x-3 = \frac{7x-23}{x+1}$$

$$(x-3)(x+1) = 7x-23$$

$$x^2 - 2x - 3 = 7x - 23$$

$$x^2 - 9x + 20 = 0 \qquad \Rightarrow x = 4, 5$$

Check: $x = 4 \ \Rightarrow \ \ln(4-3) = \ln(7(4)-23) - \ln(4+1)$

$x = 5 \ \Rightarrow \ \ln(5-3) = \ln(7(5)-23) - \ln(5+1)$

Example

Solve: $\log(x+6) - \log(x+2) = \log x$

Solution

$$\log\frac{x+6}{x+2} = \log x$$ ***Quotient Rule***

$$\frac{x+6}{x+2} = x$$ ***Multiply by x + 2***

$$x+6 = x(x+2)$$

$$x+6 = x^2+2x$$

$$0 = x^2+2x-x-6$$

$$x^2+x-6 = 0$$ ***Solve for x***

$$x = -3, \ 2$$

Check: $x = -3 \rightarrow \log(-3+6) - \log(-3+2) = \log(-3)$ *Or* ***Domain***

$x = 2 \rightarrow \log(2+6) - \log(2+2) = \log(2)$

Solution: $\boxed{x = 2}$

Exercises *Section* 3.5 – Exponential and logarithmic Equations

Solve

1. $2^{3x-7}=32$

2. $4^{2x-1}=64$

3. $3^{1-x}=\frac{1}{27}$

4. $\left(\frac{1}{3}\right)^{x}=81$

5. $5^{x}=134$

6. $7^{x}=12$

7. $9^{x}=\frac{1}{\sqrt[3]{3}}$

8. $9e^{x}=107$

9. $7^{2x+1}=3^{x+2}$

10. $4^{x+3}=3^{-x}$

11. $2^{x+4}=8^{x-6}$

12. $8^{x+2}=4^{x-3}$

13. $7^{x}=12$

14. $5^{x+4}=4^{x+5}$

15. $5^{x+2}=4^{1-x}$

16. $27=3^{5x}9^{x^2}$

17. $3^{2x-1}=0.4^{x+2}$

18. $4^{3x-5}=16$

19. $4^{x+3}=3^{-x}$

20. $3^{x-1}=7^{2x+5}$

21. $4^{x-2}=2^{3x+3}$

22. $2^{3x-7}=32$

23. $3^{2x-1}=0.4^{x+2}$

24. $e^{2x}-2e^{x}-3=0$

25. $e^{0.08t}=2500$

26. $e^{x^2}=200$

27. $e^{2x+1}\cdot e^{-4x}=3e$

28. $e^{2x}-8e^{x}+7=0$

29. $e^{x}+e^{-x}-6=0$

30. $e^{1-3x}\cdot e^{5x}=2e$

31. $6\ln(2x)=30$

32. $\log_5(x-7)=2$

33. $\log_5 x+\log_5(4x-1)=1$

34. $\log x+\log(x-3)=1$

35. $\log x-\log(x+3)=1$

36. $\log_3 x=-2$

37. $\log(3x+2)+\log(x-1)=1$

38. $\log_5(x+2)+\log_5(x-2)=1$

39. $\log x+\log(x-9)=1$

40. $\log_2(x+1)+\log_2(x-1)=3$

41. $\log_8(x+1)-\log_8 x=2$

42. $\log(x+6)-\log(x+2)=\log x$

43. $\ln(x+8)+\ln(x-1)=2\ln x$

44. $\ln(4x+6)-\ln(x+5)=\ln x$

45. $\ln(5+4x)-\ln(x+3)=\ln 3$

46. $\ln(x-5)-\ln(x+4)=\ln(x-1)-\ln(x+2)$

47. $\ln(x-3)=\ln(7x-23)-\ln(x+1)$

48. $27=3^{5x}9^{x^2}$

49. $\ln \sqrt[4]{x} = \sqrt{\ln x}$

50. $\sqrt{\ln x} = \ln \sqrt{x}$

51. $7^{x+6} = 7^{3x-4}$

52. $2^{-100x} = (0.5)^{x-4}$

53. $4^x \left(\frac{1}{2}\right)^{3-2x} = 8.\left(2^x\right)^2$

54. $5^{3x-6} = 125$

55. $e^{x^2} = e^{7x-12}$

56. $f(x) = xe^x + e^x$

57. $f(x) = x^3\left(4e^{4x}\right) + 3x^2e^{4x}$

58. $\log_4 x = \log_4 (8-x)$

59. $\log_7 (x-5) = \log_7 (6x)$

60. $\ln x^2 = \ln(12-x)$

61. $e^{x\ln 3} = 27$

62. $\log_6 (2x-3) = \log_6 12 - \log_6 3$

63. $\ln(-4-x) + \ln 3 = \ln(2-x)$

64. $\log_2 (x+7) + \log_2 x = 3$

65. $\log_3 (x+3) + \log_3 (x+5) = 1$

66. $\ln x = 1 - \ln(x+2)$

67. $\ln x = 1 + \ln(x+1)$

68. $\log_3 (x-2) = \log_3 27 - \log_3 (x-4) - 5^{\log_5 1}$

69. $\log_2 (x+3) = \log_2 (x-3) + \log_3 9 + 4^{\log_4 3}$

70. $\log_3 x - \log_9 (x+42) = 0$

71. Solve for t using logarithms with base a: $2a^{t/3} = 5$

72. Solve for t using logarithms with base a: $K = H - Ca^t$

Find the exact solution (2-decimal place approximation)

73. $3^{x+4} = 2^{1-3x}$

74. $3^{2-3x} = 4^{2x+1}$

75. $2^{-x^2} = 5$

76. $2^{-x} = 8$

77. $\log\left(x^2+4\right) - \log(x+2) = 2 + \log(x-2)$

78. $5^x + 125\left(5^{-x}\right) = 30$

79. $4^x - 3\left(4^{-x}\right) = 8$

Solve the equation without using the calculator

80. $\log x^2 = (\log x)^2$

81. $\log(\log x) = 2$

82. $\log\sqrt{x^3 - 9} = 2$

83. $e^{2x} + 2e^x - 15 = 0$

84. How long, to the nearest tenth of a year, will it take \$1000 to grow to \$3600 at 8% annual interest compounded quarterly?

Section 3.6 – Exponential Growth and Decay

Exponential Growth and Decay

The mathematical model for exponential growth or decay is given by

$$A(t) = A_0 e^{kt}$$

$A(t)$: Exponential Function (After time t)

A_0 : At time zero (initial value).

t: Time

k: Exponential rate.

- If $\boldsymbol{k > 0}$, the function models of a growing entity
- If $\boldsymbol{k < 0}$, the function models of a decay entity.

Example

In 1990, the population of Africa was 643 million and by 2000 it had grown to 813 million

a. Use the exponential growth function $A(t) = A_0 e^{kt}$, in which t is the number of years after 1990, to find the exponential growth function that models data

b. By which year will Africa's population reach 2000 million, or two billion?

Solution

a. $A(t) = A_0 e^{kt}$

From 1990 to 2000, is 10 years, that implies in 10 years the population grows from 643 to 813

$$813 = 643e^{k(10)}$$

$$\frac{813}{643} = e^{10k}$$

$$\ln \frac{813}{643} = \ln e^{10k}$$

$$\ln \frac{813}{643} = 10k$$

$$\frac{1}{10} \ln \frac{813}{643} = k$$

$$k \approx 0.023$$

$$\Rightarrow A(t) = 643e^{0.023t}$$

b. $2000 = 643e^{0.023t}$

$$\frac{2000}{643} = e^{0.023t}$$

$$\ln\frac{2000}{643} = \ln e^{0.023t}$$

$$\ln\frac{2000}{643} = 0.023t$$

$$\frac{\ln\frac{2000}{643}}{0.023} = t$$

$$\Rightarrow t \approx 49 \quad \to \boxed{Year:\ 2039}$$

Doubling Time

$$P(t) = P_0 e^{kt}$$

$$2P_0 = P_0 e^{kt} \qquad P(t) = 2P_0$$

$$2 = e^{kt}$$

$$\ln 2 = \ln e^{kt}$$

$$\ln 2 = kt \ln e \qquad \ln e = 1$$

$$\ln 2 = kt$$

$$t = \frac{\ln 2}{k}$$

Growth Rate and Doubling Time

$$\boxed{Tk = \ln 2} \qquad or \quad T = \frac{\ln 2}{k} \qquad or \quad k = \frac{\ln 2}{T}$$

Example

A country's population doubled in 45 years. What was the exponential growth rate?

Solution

$$k = \frac{\ln 2}{t}$$

$$= \frac{\ln 2}{45}$$

$$\approx 0.0154$$

Example

According to the U.S. Census Bureau, the world population reached 6 billion people on July 18, 1999, and was growing exponentially. By the end of 2000, the population had grown to 6.079 billion. The projected world population (in billion of people) t years after 2000, is given by the function defined by

$$f(t) = 6.079e^{.0126t}$$

a) Based on this model, what will the world population be in 2010?

b) In what year will the world population reach 7 billion?

Solution

a) In 2010 $\rightarrow t = 10$

$$f(t=10) = 6.079e^{.0126(10)}$$
$$\approx 6.895$$

b) $7 = 6.079e^{.0126t}$

$$\frac{7}{6.079} = e^{.0126t}$$

$$\ln\frac{7}{6.079} = \ln e^{.0126t}$$

$$\ln\left(\frac{7}{6.079}\right) = .0126t \qquad \ln e = 1$$

$$t = \frac{1}{.0126}\ln\left(\frac{7}{6.079}\right) \qquad \ln(7/6.079)/.0126$$

$$\approx 11.2$$

$$\boxed{Year:\ 2011}$$

Example

Strontium-90 is a waste product from nuclear reactors. As a consequence of fallout from atmosphere nuclear tests, we all have a measurable amount of strontium-90 in our bones.

a. The half-life of Strontium-90 is 28 years, meaning that all after 28 years a given amount of the substance will have decayed to half the original amount. Find the exponential decay model for Strontium-90.

b. Suppose the nuclear accident occurs and releases 60 grams of Strontium-90 into the atmosphere. How long will it take for Strontium-90 to decay to a level of 10 grams?

<u>Solution</u>

a. $A = A_0 e^{kt}$

$$\frac{1}{2}A_0 = A_0 e^{k(28)}$$

$$\frac{1}{2} = e^{28k}$$

$$\ln\frac{1}{2} = \ln e^{28k}$$

$$\ln\frac{1}{2} = 28k$$

$$k = \frac{1}{28}\ln\frac{1}{2} \approx -0.0248$$

$$\boxed{A = A_0 e^{-0.0248t}}$$

b. $A = A_0 e^{-0.0248t}$

$$10 = 60e^{-0.0248t}$$

$$\frac{1}{6} = e^{-0.0248t}$$

$$\ln\frac{1}{6} = \ln e^{-0.0248t}$$

$$\ln\frac{1}{6} = -0.0248t$$

$$t = \frac{\ln\frac{1}{6}}{-0.0248} \approx 72.25\ yrs$$

Exercises Section 3.6 – Exponential Growth and Decay

1. Suppose that \$10,000 is invested at interest rate of 5.4% per year, compounded continuously.

a) Find the exponential growth function
b) What will the balance be after, 1 *yr*. 10 *yrs*.?
c) What will the balance be after, 1 *yr*. 10 *yrs*.?

2. In 1990, the population of Africa was 643 million and by 2000 it had grown to 813 million

a) Use the exponential growth function $A(t) = A_0 e^{kt}$, in which t is the number of years after 1990, to find the exponential growth function that models data
b) By which year will Africa's population reach 2000 million, or two billion?

3. The radioactive element carbon-14 has a half-life of 5750 yr. The percentage of carbon-14 present in the remains of organic matter can be used to determine the age of that organic matter. Archaeologists discovered that the linen wrapping from one of the Dead Sea Scrolls had lost 22.3% of its carbon-14 at the time it was found. How old was the linen wrapping?

4. Suppose that \$2000 is invested at interest rate $\boldsymbol{k}$, compounded continuously, and grows to \$2983.65 in 5 yrs.

a) What is the interest rate?
b) Find the exponential growth function
c) What will the balance be after 10 yrs?
d) After how long will the \$2000 have doubled?

5. In 2005, the population of China was about 1.306 billion, and the exponential growth rate was 0.6% per year.

a) Find the exponential growth function
b) Estimate the population in 2008
c) After how long will the population be double what it was in 2005?

6. How long will it take for the money in an account that is compounded continuously at 3% interest to double?

7. If 600 *g* of radioactive substance are present initially and 3 *yrs*. later only 300 *g* remain, how much of the substance will be present after 6 *yrs*.?

8. The population of an endangered species of bird was 4200 in 1990. Thirteen years later, in 2003, the bird population declined to 3000. The population of the birds is decreasing exponentially according to the function $A(t) = 4200e^{kt}$ where A is the bird population t years after 1990. Use this information to find the value of $\boldsymbol{k}$.