

Solution

Section 2.1 – Functions and Graphs

Exercise

$$f(x) = \begin{cases} 2+x & \text{if } x < -4 \\ -x & \text{if } -4 \leq x \leq 2 \\ 3x & \text{if } x > 2 \end{cases}$$

Find: $f(-5)$, $f(-1)$, $f(0)$, and $f(3)$

Solution

a) $f(-5) = 2 - 5 = -3$

b) $f(-1) = -(-1) = 1$

c) $f(0) = -0 = 0$

d) $f(3) = 3(3) = 9$

Exercise

$$f(x) = \begin{cases} -2x & \text{if } x < -3 \\ 3x-1 & \text{if } -3 \leq x \leq 2 \\ -4x & \text{if } x > 2 \end{cases}$$

Find: $f(-5)$, $f(-1)$, $f(0)$, and $f(3)$

Solution

a) $f(-5) = -2(-5) = 10$

b) $f(-1) = 3(-1) - 1 = -4$

c) $f(0) = 3(0) - 1 = -1$

d) $f(3) = -4(3) = -12$

Exercise

$$f(x) = \begin{cases} x^3 + 3 & \text{if } -2 \leq x \leq 0 \\ x + 3 & \text{if } 0 < x < 1 \\ 4 + x - x^2 & \text{if } 1 \leq x \leq 3 \end{cases}$$

Find: $f(-5)$, $f(-1)$, $f(0)$, and $f(3)$

Solution

a) $f(-5) = \text{doesn't exist}$

b) $f(-1) = (-1)^3 + 3$
 $\quad \quad \quad = 2$

c) $f(0) = (0)^3 + 3$

$$\underline{= 3}$$

$$d) \quad f(3) = 4 + (3) - (3)^2 \\ \underline{= -2}$$

Exercise

$$h(x) = \begin{cases} \frac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$

Find: $h(5)$, $h(0)$, and $h(3)$

Solution

$$a) \quad h(5) = \frac{5^2 - 9}{5 - 3} \\ \underline{= 8}$$

$$b) \quad h(0) = \frac{0^2 - 9}{0 - 3} \\ \underline{= 3}$$

$$c) \quad \underline{h(3) = 6}$$

Exercise

$$f(x) = \begin{cases} 3x + 5 & \text{if } x < 0 \\ 4x + 7 & \text{if } x \geq 0 \end{cases} \quad \text{Find}$$

$$a) \quad f(0) \quad b) \quad f(-2) \quad c) \quad f(1) \quad d) \quad f(3) + f(-3) \quad e) \quad \text{Graph } f(x)$$

Solution

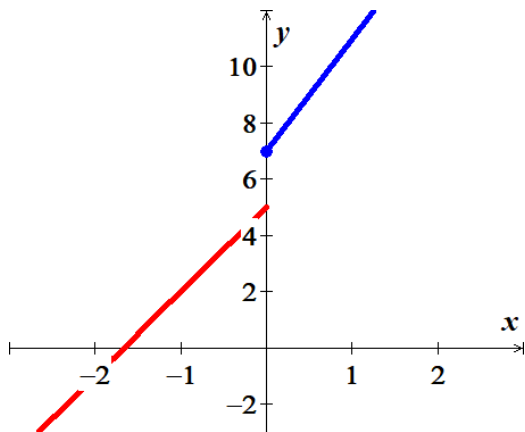
$$a) \quad f(0) = 4(0) + 7 \\ \underline{= 7}$$

$$b) \quad f(-2) = 3(-2) + 5 \\ \underline{= -1}$$

$$c) \quad f(1) = 4(1) + 7 \\ \underline{= 11}$$

$$d) \quad f(3) + f(-3) = 4(3) + 7 + 3(-3) + 5 \\ = 12 + 7 - 9 + 5 \\ \underline{= 15}$$

$$e)$$



Exercise

$$f(x) = \begin{cases} 6x-1 & \text{if } x < 0 \\ 7x+3 & \text{if } x \geq 0 \end{cases} \quad \text{Find}$$

- a) $f(0)$ b) $f(-1)$ c) $f(4)$ d) $f(2) + f(-2)$ e) Graph $f(x)$

Solution

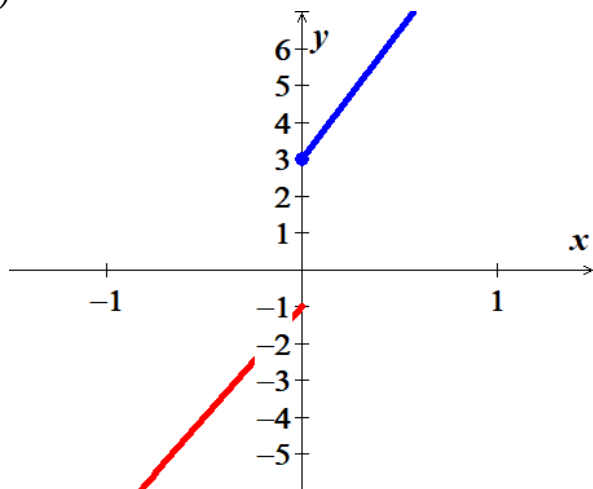
$$\begin{aligned} \text{a) } f(0) &= 7(0) + 3 \\ &= 3 \end{aligned}$$

$$\begin{aligned} \text{b) } f(-2) &= 6(-1) - 1 \\ &= -7 \end{aligned}$$

$$\begin{aligned} \text{c) } f(4) &= 7(4) + 3 \\ &= 31 \end{aligned}$$

$$\begin{aligned} \text{d) } f(2) + f(-2) &= 7(2) + 3 + 6(-2) - 1 \\ &= 14 + 3 - 12 - 1 \\ &= 4 \end{aligned}$$

e)



Exercise

$$f(x) = \begin{cases} 2x+1 & \text{if } x \leq 1 \\ 3x-2 & \text{if } x > 1 \end{cases} \quad \text{Find}$$

- a) $f(0)$ b) $f(2)$ c) $f(-2)$ d) $f(1) + f(-1)$ e) Graph $f(x)$

Solution

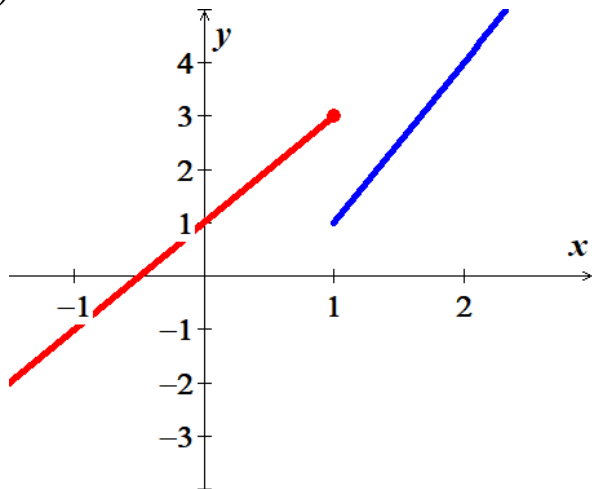
$$\begin{aligned} \text{a) } f(0) &= 2(0) + 1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{b) } f(2) &= 3(2) - 2 \\ &= 4 \end{aligned}$$

$$\begin{aligned} \text{c) } f(-2) &= 2(-2) + 1 \\ &= -3 \end{aligned}$$

$$\begin{aligned} \text{d) } f(1) + f(-1) &= (2(1) + 1) + (2(-1) + 1) \\ &= 2 + 1 - 2 + 1 \\ &= 2 \end{aligned}$$

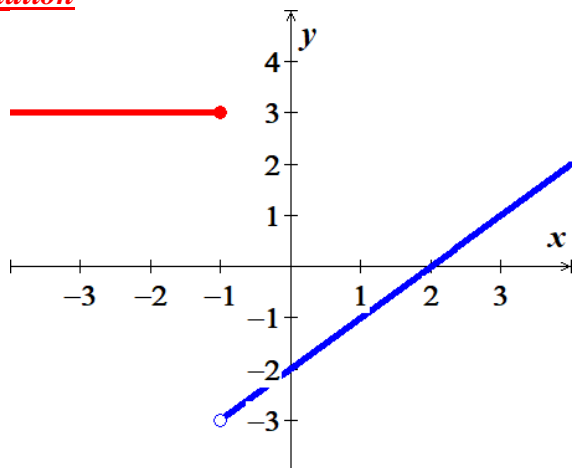
e)



Exercise

Graph the piecewise function defined by $f(x) = \begin{cases} 3 & \text{if } x \leq -1 \\ x-2 & \text{if } x > -1 \end{cases}$

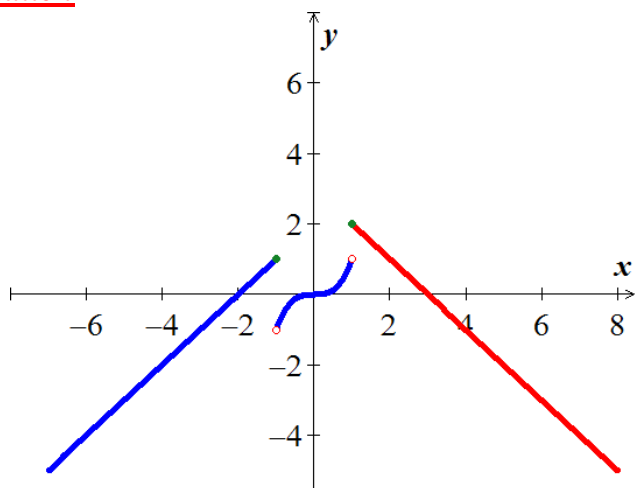
Solution



Exercise

Sketch the graph $f(x) = \begin{cases} x+2 & \text{if } x \leq -1 \\ x^3 & \text{if } -1 < x < 1 \\ -x+3 & \text{if } x \geq 1 \end{cases}$

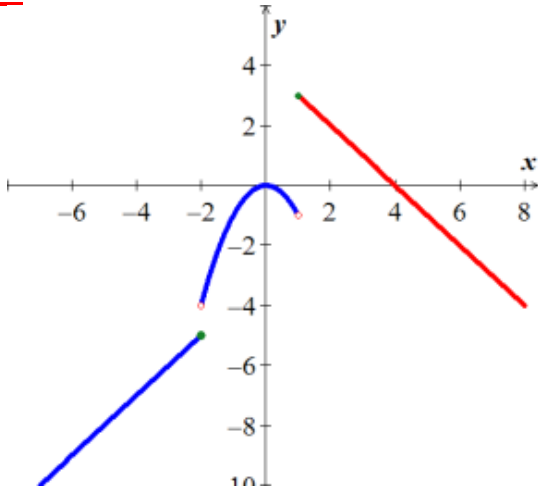
Solution



Exercise

Sketch the graph $f(x) = \begin{cases} x-3 & \text{if } x \leq -2 \\ -x^2 & \text{if } -2 < x < 1 \\ -x+4 & \text{if } x \geq 1 \end{cases}$

Solution



Exercise

Determine any **relative maximum** or **minimum** of the function, determine the intervals on which the function **increasing** or **decreasing**, and then find the **domain** and the **range**.

$$f(x) = x^2 - 2x + 3$$

Solution

Relative Maximum: None

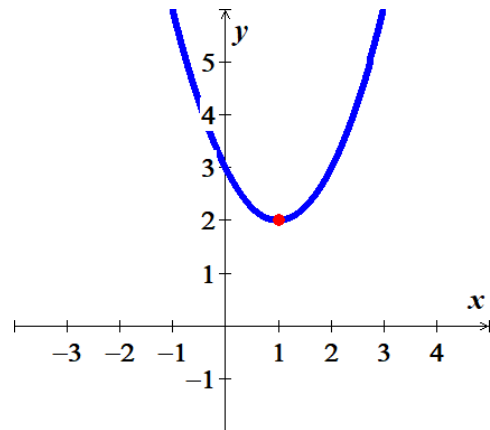
Minimum Point: (1, 2)

Increasing: (1, ∞)

Decreasing: (-∞, 1)

Domain: ℝ

Range: [2, ∞)



Exercise

Determine any **relative maximum** or **minimum** of the function, determine the intervals on which the function **increasing** or **decreasing**, and then find the **domain** and the **range**.

$$f(x) = -x^2 - 2x + 3$$

Solution

Maximum Point: $(-1, 4)$

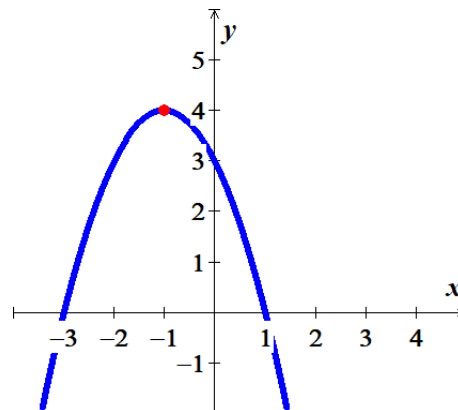
Relative Minimum: *None*

Increasing: $(-\infty, -1)$

Decreasing: $(-1, \infty)$

Domain: \mathbb{R}

Range: $(-\infty, 4]$



Exercise

Determine any **relative maximum** or **minimum** of the function, determine the intervals on which the function **increasing** or **decreasing**, and then find the **domain** and the **range**.

$$f(x) = -x^3 + 3x^2$$

Solution

Relative Maximum: $(2, 4)$

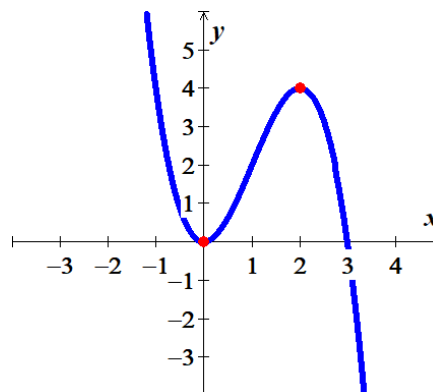
Relative Minimum: $(0, 0)$

Increasing: $(0, 2)$

Decreasing: $(-\infty, 0) \cup (2, \infty)$

Domain: \mathbb{R}

Range: \mathbb{R}



Exercise

Determine any **relative maximum** or **minimum** of the function, determine the intervals on which the function **increasing** or **decreasing**, and then find the **domain** and the **range**.

$$f(x) = x^3 - 3x^2$$

Solution

Relative Maximum: $(0, 0)$

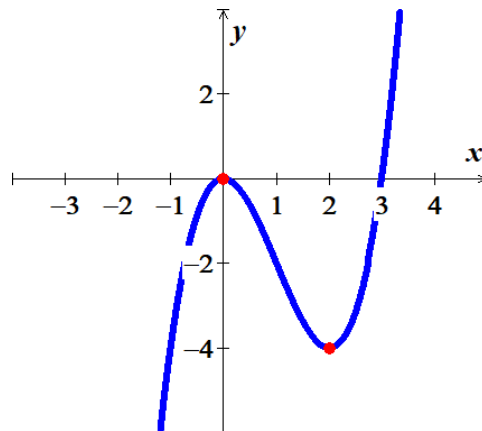
Relative Minimum: $(2, -4)$

Increasing: $(-\infty, 0) \cup (2, \infty)$

Decreasing: $(0, 2)$

Domain: \mathbb{R}

Range: \mathbb{R}



Exercise

Determine any **relative maximum** or **minimum** of the function, determine the intervals on which the function **increasing** or **decreasing**, and then find the **domain** and the **range**.

$$f(x) = \frac{1}{4}x^4 - 2x^2$$

Solution

Relative Maximum: $(0, 0)$

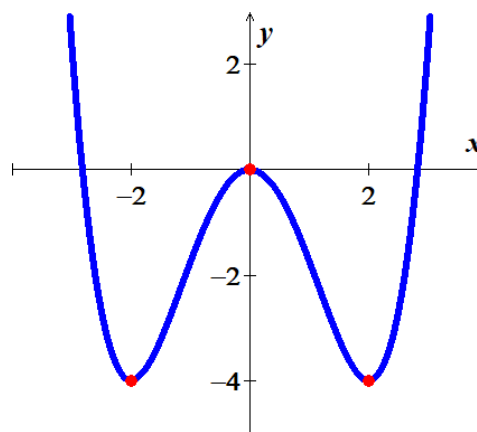
Minimum Points: $(-2, -4)$ & $(2, -4)$

Increasing: $(-2, 0) \cup (2, \infty)$

Decreasing: $(-\infty, -2) \cup (0, 2)$

Domain: \mathbb{R}

Range: $[-4, \infty)$



Exercise

Determine any **relative maximum** or **minimum** of the function, determine the intervals on which the function **increasing** or **decreasing**, and then find the **domain** and the **range**.

$$f(x) = \frac{4}{81}x^4 - \frac{8}{9}x^2 + 4$$

Solution

Relative Maximum: (0, 4)

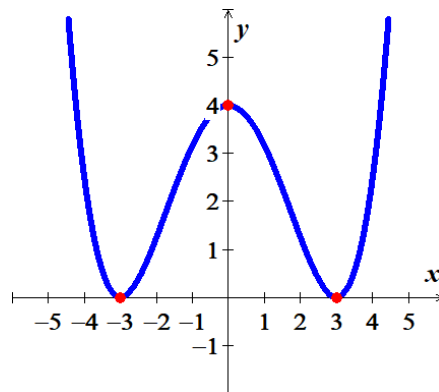
Minimum Points: (-3, 0) & (3, 0)

Increasing: (-3, 0) \cup (3, ∞)

Decreasing: ($-\infty$, -3) \cup (0, 3)

Domain: \mathbb{R}

Range: $[0, \infty)$



Exercise

The elevation H , in *meters*, above sea level at which the boiling point of water is in t degrees Celsius is given by the function

$$H(t) = 1000(100 - t) + 580(100 - t)^2$$

At what elevation is the boiling point 99.5° .

Solution

$$\begin{aligned} H(99.5) &= 1000(100 - 99.5) + 580(100 - 99.5)^2 \\ &= \underline{645 \text{ m}} \end{aligned}$$

Exercise

A hot-air balloon rises straight up from the ground at a rate of 120 ft./min . The balloon is tracked from a rangefinder on the ground at point P, which is 400 ft. from the release point Q of the balloon. Let d = the distance from the balloon to the rangefinder and t – the time, in minutes, since the balloon was released. Express d as a function of t .

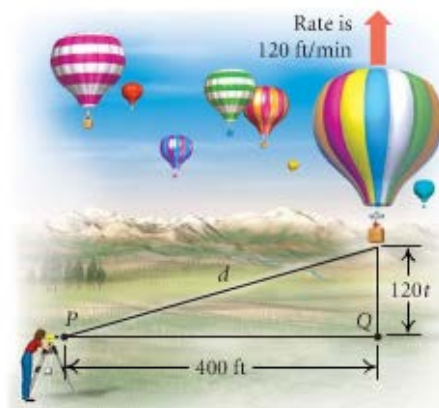
Solution

$$d^2 = (120t)^2 + 400^2$$

$$d = \sqrt{14400t^2 + 160000}$$

$$d = \sqrt{1600(9t^2 + 100)}$$

$$d(t) = \underline{40\sqrt{9t^2 + 100}}$$



Exercise

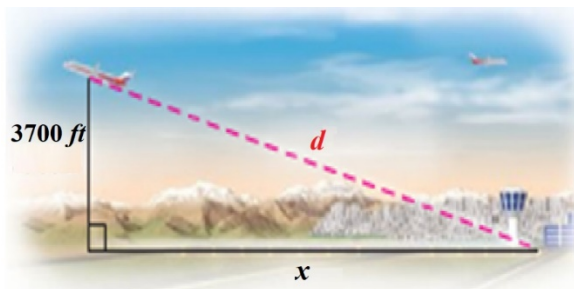
An airplane is flying at an altitude of 3700 *feet*. The slanted distance directly to the airport is d *feet*. Express the horizontal distance x as a function of d .

Solution

$$d^2 = (3,700)^2 + x^2$$

$$h^2 = d^2 - (3700)^2$$

$$h(t) = \sqrt{d^2 - (3,700)^2}$$



Exercise

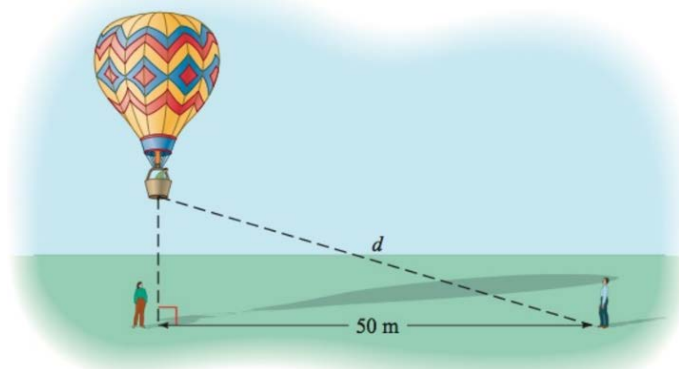
For the first minute of flight, a hot air balloon rises vertically at a rate of 3 *m/sec*. If t is the time in *seconds* that the balloon has been airborne, write the distance d between the balloon and a point on the ground 50 *meters* from the point to lift off as a function of t .

Solution

$$h = 3t \quad v = \frac{h}{t}$$

$$d^2 = h^2 + 50^2$$

$$d(t) = \sqrt{9t^2 + 2,500}$$



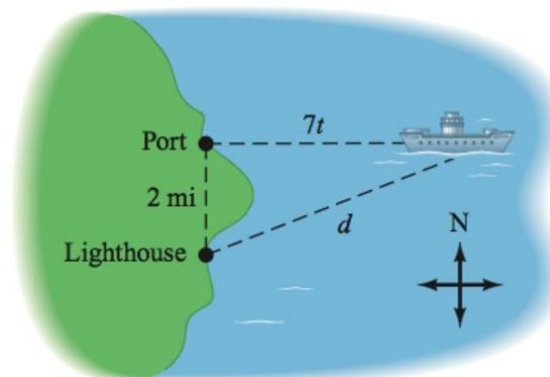
Exercise

A light house is 2 *miles* south of a port. A ship leaves port and sails east at a rate of 7 *miles per hour*. Express the distance d between the ship and the lighthouse as a function of time, given that the ship has been sailing for t hours.

Solution

$$d^2 = 4^2 + (7t)^2$$

$$d(t) = \sqrt{16 + 49t^2}$$



Exercise

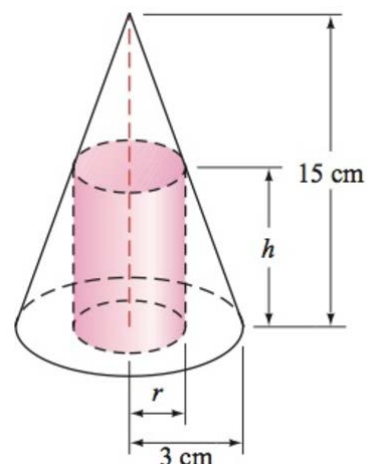
A cone has an altitude of 15 cm and a radius of 3 cm. A right circular cylinder of radius r and height h is inscribed in the cone. Use similar triangles to write h as a function of r .

Solution

$$\frac{15-h}{15} = \frac{r}{3}$$

$$15 - h = 5r$$

$$\underline{h(r) = 15 - 5r}$$



Exercise

Water is flowing into a conical drinking cup with an altitude of 4 inches and a radius of 2 inches.

- Write the radius r of the surface of the water as a function of its depth h .
- Write the volume V of the water as a function of its depth h .

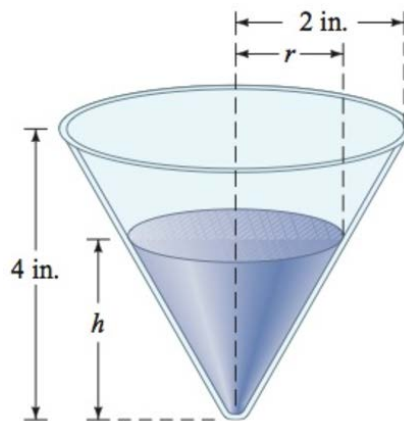
Solution

$$a) \quad \frac{h}{4} = \frac{r}{2}$$

$$\underline{r(h) = \frac{1}{2}h}$$

$$b) \quad \text{Area} = \pi r^2$$

$$\begin{aligned} V &= \frac{1}{3} \pi r^2 h \\ &= \frac{1}{3} \pi \left(\frac{h^2}{4} \right) h \\ &= \frac{1}{12} \pi h^3 \end{aligned}$$



Exercise

A water tank has the shape of a right circular cone with height 16 feet and radius 8 feet. Water is running into the tank so that the radius r (in feet) of the surface of the water is given by $r = 1.5t$, where t is the time (in minutes) that the water has been running.

- The area A of the surface of the water is $A = \pi r^2$. Find $A(t)$ and use it to determine the area of the surface of the water when $t = 2$ minutes.
- The volume V of the water is given by $V = \frac{1}{3} \pi r^2 h$. Find $V(t)$ and use it to determine the volume of the water when $t = 3$ minutes.

Solution

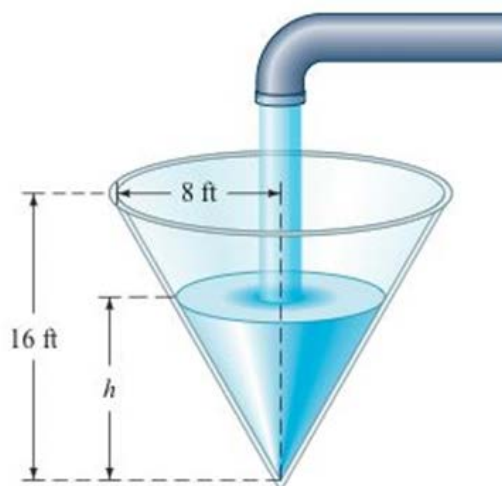
c) $Area = \pi r^2$

$$A(t) = \pi \left(\frac{3}{2}t \right)^2$$
$$= \frac{9\pi}{4}t^2$$

d) $\frac{h}{16} = \frac{r}{8}$

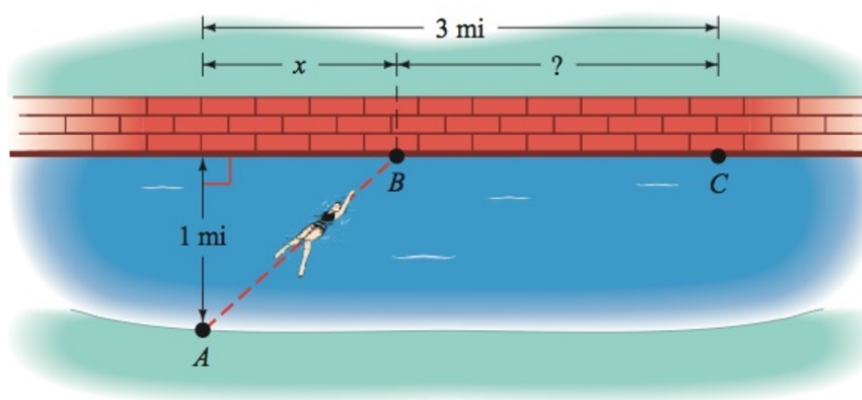
$$h = 2r$$

$$V(t) = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi r^2 (2r)$$
$$= \frac{2}{3}\pi r^3$$
$$= \frac{2}{3}\pi \left(\frac{3}{2}t \right)^3$$
$$= \frac{9}{4}\pi t^3$$



Exercise

An athlete swims from point **A** to point **B** at a rate of 2 miles per hour and runs from point **B** to point **C** at a rate of 8 miles per hour. Use the dimensions in the figure to write the time t required to reach point **C** as a function of x .



Solution

$$\text{Swimming distance} = \sqrt{x^2 + 1}$$

$$t_{\text{swim}} = \frac{\sqrt{x^2 + 1}}{2} \quad t = \frac{d}{v}$$

$$\text{Running distance} = 3 - x$$

$$t_{run} = \frac{3-x}{8} \qquad t = \frac{d}{v}$$

$$t_{total} = \frac{\sqrt{x^2+1}}{2} + \frac{3-x}{8}$$

Exercise

A device used in golf to estimate the distance d , in yards, to a hole measures the size s , in inches, that the 7-foot pin appears to be in a viewfinder. Express the distance d as a function of s .

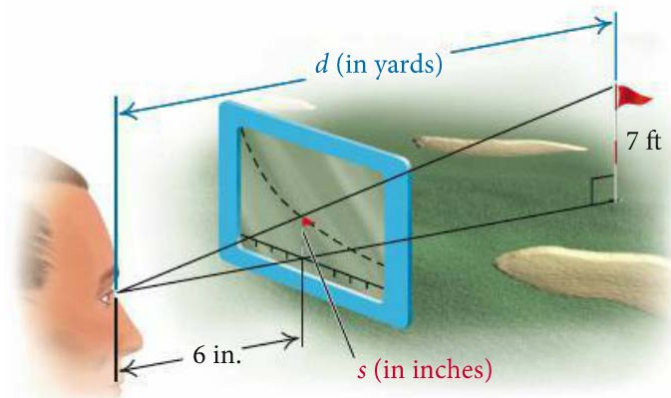
Solution

$$\frac{d}{6} = \frac{7 \text{ ft}}{s \text{ in}}$$

$$d = \frac{7 \text{ ft}}{s \text{ in}} 6 \text{ in}$$

$$d = \frac{42 \text{ ft}}{s} \frac{1 \text{ yd}}{3 \text{ ft}}$$

$$d(s) = \frac{14}{s}$$



Exercise

A rhombus is inscribed in a rectangle that is w meters wide with a perimeter of 40 m. Each vertex of the rhombus is a midpoint of a side of the rectangle. Express the area of the rhombus as a function of the rectangle's width.

Solution

The area of the rhombus = $\frac{1}{2}$ area of the rectangle, since each vertex of the rhombus is a midpoint of a side of the rectangle.

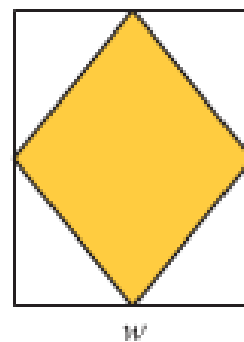
$$\text{Perimeter: } 2l + 2w = 40 \qquad \text{Divide both sides by 2}$$

$$l + w = 20$$

$$l = 20 - w$$

$$\text{Area of the rectangle} = lw = (20 - w)w$$

$$\begin{aligned} \text{Area of the rhombus} &= \frac{1}{2}(20w - w^2) \\ &= -\frac{1}{2}w^2 + 10w \end{aligned}$$



Exercise

The surface area S of a right circular cylinder is given by the formula $S = 2\pi rh + 2\pi r^2$. If the height is twice the radius, find each of the following.

- a) A function $S(r)$ for the surface area as a function of r .
- b) A function $S(h)$ for the surface area as a function of h .

Solution

Given: $h = 2r$

a) $S = 2\pi rh + 2\pi r^2$

$$\begin{aligned} S(r) &= 2\pi r(2r) + 2\pi r^2 \\ &= 4\pi r^2 + 2\pi r^2 \\ &= 6\pi r^2 \end{aligned}$$

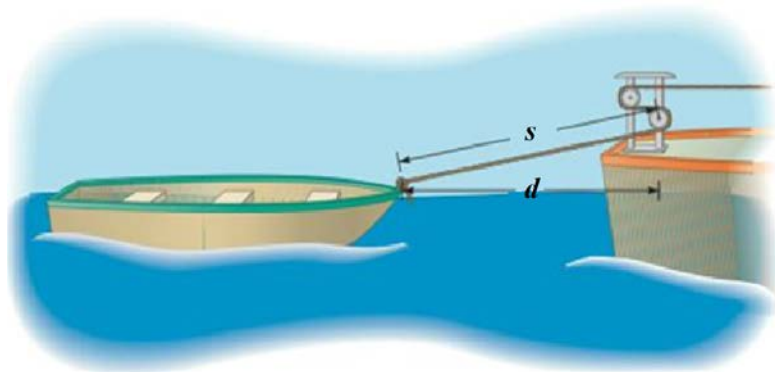
b) $r = \frac{1}{2}h$

$$\begin{aligned} S(h) &= 2\pi\left(\frac{1}{2}h\right)h + 2\pi\left(\frac{1}{2}h\right)^2 \\ &= \pi h^2 + \frac{1}{2}\pi h^2 \\ &= \frac{3}{2}\pi h^2 \end{aligned}$$



Exercise

A boat is towed by a rope that runs through a pulley that is 4 feet above the point where the rope is tied to the boat. The length (in feet) of the rope from the boat to the pulley is given by $s = 48 - t$, where t is the time in seconds that the boat has been in tow. The horizontal distance from the pulley to the boat is d .



- a) Find $d(t)$
- b) Evaluate $s(35)$ and $d(35)$

Solution

$$\begin{aligned}
 a) \quad s^2 &= d^2 + 4^2 \\
 d^2 &= (48 - t)^2 - 16 \\
 d(t) &= \sqrt{2,304 - 96t + t^2 - 16} \\
 &= \sqrt{t^2 - 96t + 2,288}
 \end{aligned}$$

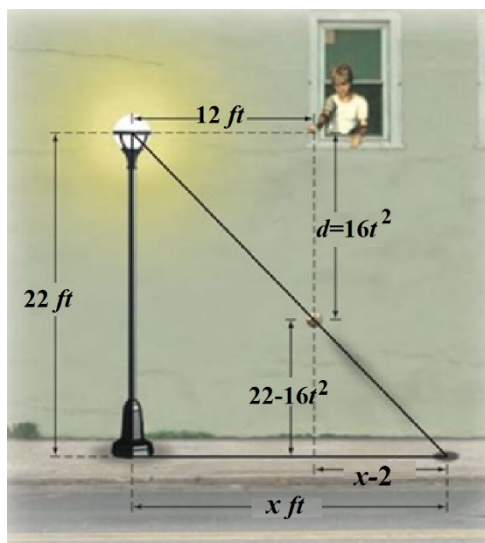
$$\begin{aligned}
 b) \quad s(35) &= 48 - 35 \\
 &= 13 \text{ feet} \\
 d(35) &= \sqrt{(48 - 35)^2 - 16} \\
 &= \sqrt{13^2 - 16} \\
 &= \sqrt{153} \text{ feet}
 \end{aligned}$$

Exercise

The light from a lamppost casts a shadow from a ball that was dropped from a height of 22 feet above the ground. The distance d , in feet, the ball has dropped t seconds after it is released is given by $d(t) = 16t^2$. Find the distance x , in feet, of the shadow from the base of the lamppost as a function of time t .

Solution

$$\begin{aligned}
 \frac{22 - 16t^2}{22} &= \frac{x - 12}{x} \\
 (22 - 16t^2)x &= 22(x - 12) \\
 (22 - 16t^2)x &= 22x - 264 \\
 (22 - 16t^2 - 22)x &= -264 \\
 -16t^2x &= -264 \\
 x(t) &= \frac{33}{2t^2}
 \end{aligned}$$



Exercise

A right circular cylinder of height h and a radius r is inscribed in a right circular cone with a height of 10 feet and a base with radius 6 feet.

- Express the height h of the cylinder as a function of r .
- Express the volume V of the cylinder as a function of r .
- Express the volume V of the cylinder as a function of h .

Solution

$$a) \quad \frac{h}{10} = \frac{6-r}{6}$$

$$\boxed{h(r) = \frac{5}{3}(6-r)}$$

$$b) \quad V = \pi r^2 h$$

$$V(r) = \frac{5}{3} \pi r^2 (6-r)$$

$$\boxed{= \frac{5}{3} \pi (6r^2 - r^3)}$$

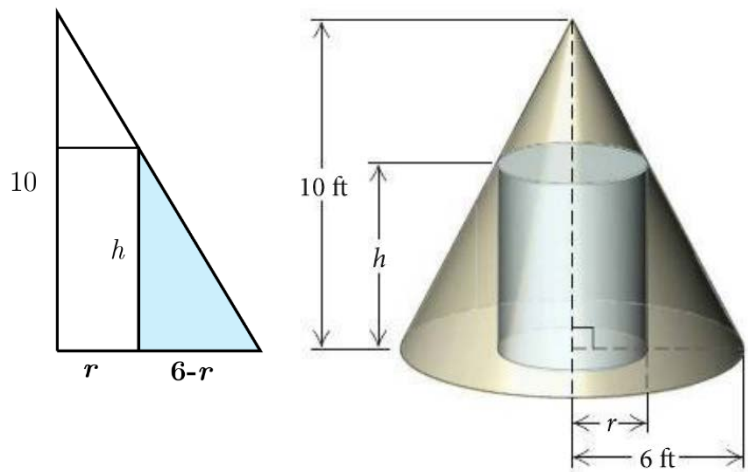
$$c) \quad \frac{3}{5}h = 6-r$$

$$r = 6 - \frac{3}{5}h$$

$$V = \pi r^2 h$$

$$V(h) = \pi \left(\frac{30-3h}{5} \right)^2 h$$

$$\boxed{= \frac{1}{25} \pi h (30-3h)^2}$$



Solution

Section 2.2 – Function Operations

Exercise

Find the domain: $f(x) = 7x + 4$

Solution

Domain: $(-\infty, \infty)$

Exercise

Find the domain: $f(x) = |3x - 2|$

Solution

Domain: \mathbb{R}

Exercise

Find the domain: $f(x) = 3x + \pi$

Solution

Domain: \mathbb{R}

Exercise

Find the domain: $f(x) = \sqrt{7}x + \frac{1}{2}$

Solution

Domain: \mathbb{R}

Exercise

Find the domain: $f(x) = -2x^2 + 3x - 5$

Solution

Domain: \mathbb{R}

Exercise

Find the domain: $f(x) = x^3 - 2x^2 + x - 3$

Solution

Domain: \mathbb{R} |

Exercise

Find the domain: $f(x) = x^2 - 2x - 15$

Solution

Domain: \mathbb{R} |

Exercise

Find the domain $f(x) = 4 - \frac{2}{x}$

Solution

Domain: $x \neq 0$ |

Exercise

Find the domain $f(x) = \frac{1}{x^4}$

Solution

Domain: $x \neq 0$ |

Exercise

Find the domain: $g(x) = \frac{3}{x-4}$

Solution

Domain: $x \neq 4$ |

Exercise

Find the domain $y = \frac{2}{x-3}$

Solution

Domain: $x \neq 3$ |

Exercise

Find the domain $y = \frac{-7}{x-5}$

Solution

Domain: $x \neq 5$ |

Exercise

Find the domain $f(x) = \frac{x+5}{2-x}$

Solution

$$2 - x \neq 0$$

Domain: $x \neq 2$ |

Exercise

Find the domain $f(x) = \frac{8}{x+4}$

Solution

$$x + 4 \neq 0$$

Domain: $x \neq -4$ |

Exercise

Find the domain $f(x) = \frac{1}{x+4}$

Solution

Domain: $x \neq -4$ |

Exercise

Find the domain $f(x) = \frac{1}{x-4}$

Solution

Domain: $x \neq 4$ |

Exercise

Find the domain $f(x) = \frac{3x}{x+2}$

Solution

Domain: $x \neq -2$ |

Exercise

Find the domain $f(x) = x - \frac{2}{x-3}$

Solution

Domain: $x \neq 3$ |

Exercise

Find the domain $f(x) = x + \frac{3}{x-5}$

Solution

Domain: $x \neq 5$ |

Exercise

Find the domain $f(x) = \frac{1}{2}x - \frac{8}{x+7}$

Solution

Domain: $x \neq -7$ |

Exercise

Find the domain $f(x) = \frac{1}{x-3} - \frac{8}{x+7}$

Solution

Domain: $x \neq -7, 3$ |

Exercise

Find the domain $f(x) = \frac{1}{x+4} - \frac{2x}{x-4}$

Solution

Domain: $x \neq \pm 4$ |

Exercise

Find the domain $f(x) = \frac{3x^2}{x+3} - \frac{4x}{x-2}$

Solution

Domain: $x \neq -3, 2$

Exercise

Find the domain $f(x) = \frac{1}{x^2 - 2x + 1}$

Solution

$$x^2 - 2x + 1 \neq 0 \quad a + b + c = 0 \rightarrow x = 1, \frac{c}{a}$$

Domain: $x \neq 1$

Exercise

Find the domain $f(x) = \frac{x}{x^2 + 3x + 2}$

Solution

$$x^2 + 3x + 2 \neq 0 \quad a - b + c = 0 \rightarrow x = -1, -\frac{c}{a}$$

Domain: $x \neq -1, -2$

Exercise

Find the domain $f(x) = \frac{x^2}{x^2 - 5x + 4}$

Solution

$$x^2 - 5x + 4 \neq 0 \quad a + b + c = 0 \rightarrow x = 1, \frac{c}{a}$$

Domain: $x \neq 1, 4$

Exercise

Find the domain $f(x) = \frac{1}{x^2 - 4x - 5}$

Solution

$$x^2 - 4x - 5 \neq 0 \quad a - b + c = 0 \rightarrow x = -1, -\frac{c}{a}$$

Domain: $x \neq -1, 5$

Exercise

Find the domain $g(x) = \frac{2}{x^2 + x - 12}$

Solution

$$x^2 + x - 12 \neq 0$$
$$(x + 4)(x - 3) \neq 0$$

$$\text{Domain: } \underline{x \neq -4, 3} \quad \underline{(-\infty, -4) \cup (-4, 3) \cup (3, \infty)}$$

Exercise

Find the domain $h(x) = \frac{5}{\frac{4}{x} - 1}$

Solution

$$x \neq 0 \quad \frac{4}{x} - 1 \neq 0$$
$$\frac{4 - x}{x} \neq 0$$
$$4 - x \neq 0$$
$$x \neq 4$$

$$\text{Domain: } \underline{x \neq 0, 4} \quad \underline{(-\infty, 0) \cup (0, 4) \cup (4, \infty)}$$

Exercise

Find the domain $y = \sqrt{x}$

Solution

$$x \geq 0$$

$$\text{Domain: } \underline{x \geq 0} \quad \underline{[0, \infty)}$$

Exercise

Find the domain $f(x) = \sqrt{8 - 3x}$

Solution

$$8 - 3x \geq 0$$

$$8 \geq 3x$$

$$\text{Domain: } \underline{x \leq \frac{8}{3}} \quad \underline{\left(-\infty, \frac{8}{3}\right]}$$

Exercise

Find the domain $y = \sqrt{4x+1}$

Solution

$$4x+1 \geq 0 \Rightarrow x \geq -\frac{1}{4}$$

$$\text{Domain: } \underline{x \geq -\frac{1}{4}} \mid \left[-\frac{1}{4}, \infty \right)$$

Exercise

Find the domain $y = \sqrt{7-2x}$

Solution

$$7-2x \geq 0$$

$$-2x \geq -7$$

$$\text{Domain: } \underline{x \leq \frac{7}{2}} \mid \left(-\infty, \frac{7}{2} \right]$$

Exercise

Find the domain $f(x) = \sqrt{8-x}$

Solution

$$8-x \geq 0$$

$$\text{Domain: } \underline{x \leq -8} \mid (-\infty, 8]$$

Exercise

Find the domain $f(x) = \sqrt{3-2x}$

Solution

$$\text{Domain: } \underline{x \leq \frac{3}{2}} \mid$$

Exercise

Find the domain $f(x) = \sqrt{3+2x}$

Solution

$$\text{Domain: } \underline{x \geq -\frac{3}{2}} \mid$$

Exercise

Find the domain $f(x) = \sqrt{5-x}$

Solution

Domain: $x \leq 5$

Exercise

Find the domain $f(x) = \sqrt{x-5}$

Solution

Domain: $x \geq 5$

Exercise

Find the domain $f(x) = \sqrt{6-3x}$

Solution

Domain: $x \leq 2$

Exercise

Find the domain $f(x) = \sqrt{3x-6}$

Solution

Domain: $x \geq 2$

Exercise

Find the domain $f(x) = \sqrt{2x+7}$

Solution

Domain: $x \geq -\frac{7}{2}$

Exercise

Find the domain $f(x) = \sqrt{x^2-16}$

Solution

$$x^2 - 16 = 0$$

$$x^2 = 16$$

$$x = \pm 4$$

$$\text{Domain: } \underline{x \leq -4 \quad x \geq 4}$$

Exercise

Find the domain $f(x) = \sqrt{16 - x^2}$

Solution

$$x = \pm 4$$

$$\text{Domain: } \underline{-4 \leq x \leq 4}$$

Exercise

Find the domain $f(x) = \sqrt{9 - x^2}$

Solution

$$x = \pm 3$$

$$\text{Domain: } \underline{-3 \leq x \leq 3}$$

Exercise

Find the domain $f(x) = \sqrt{x^2 - 25}$

Solution

$$x = \pm 5$$

$$\text{Domain: } \underline{-5 \leq x \leq 5}$$

Exercise

Find the domain $f(x) = \sqrt{x^2 - 5x + 4}$

Solution

$$x^2 - 5x + 4$$

$$a + b + c = 0 \rightarrow x = 1, \frac{c}{a}$$

$$x = 1, 4$$

$$\text{Domain: } \underline{x \leq 1 \quad x \geq 4}$$

Exercise

Find the domain $f(x) = \sqrt{x^2 + 5x + 4}$

Solution

$$x^2 + 5x + 4 \quad a - b + c = 0 \rightarrow x = -1, -\frac{c}{a}$$

$$x = -1, -4$$

$$\text{Domain: } \underline{x \leq -4 \quad x \geq -1}$$

Exercise

Find the domain $f(x) = \sqrt{x^2 + 3x + 2}$

Solution

$$x^2 + 3x + 2 \quad a - b + c = 0 \rightarrow x = -1, -\frac{c}{a}$$

$$x = -1, -2$$

$$\text{Domain: } \underline{x \leq -2 \quad x \geq -1}$$

Exercise

Find the domain $f(x) = \sqrt{x^2 - 3x + 2}$

Solution

$$x^2 - 3x + 2 \quad a + b + c = 0 \rightarrow x = 1, \frac{c}{a}$$

$$x = 1, 2$$

$$\text{Domain: } \underline{x \leq 1 \quad x \geq 2}$$

Exercise

Find the domain $f(x) = \sqrt{x-4} + \sqrt{x+1}$

Solution

$$x \geq 4 \quad x \geq -1$$

$$\text{Domain: } \underline{x \geq 4}$$

Exercise

Find the domain $f(x) = \sqrt{3-x} + \sqrt{x-2}$

Solution

$$x \leq 3 \quad x \geq 2$$

$$\text{Domain: } \underline{2 \leq x \leq 3}$$

Exercise

Find the domain $f(x) = \sqrt{1-x} + \sqrt{4-x}$

Solution

$$x \leq 1 \quad x \leq 4$$

$$\text{Domain: } \underline{x \leq 1}$$

Exercise

Find the domain $f(x) = \sqrt{1-x} - \sqrt{x-3}$

Solution

$$x \leq 1 \quad x \geq 3$$

$$\text{Domain: } \underline{\emptyset}$$

Exercise

Find the domain $f(x) = \sqrt{x+4} - \sqrt{x-1}$

Solution

$$x \geq -4 \quad x \geq 1$$

$$\text{Domain: } \underline{x \geq 1}$$

Exercise

Find the domain $f(x) = \frac{\sqrt{x+1}}{x}$

Solution

$$x+1 \geq 0$$

$$x \neq 0$$

$$x \geq -1$$

$$\text{Domain: } \underline{x \geq -1 \quad x \neq 0} \quad \underline{[-1, 0) \cup (0, \infty)}$$

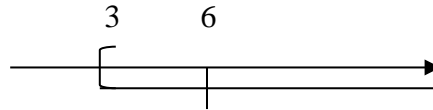
Exercise

Find the domain $g(x) = \frac{\sqrt{x-3}}{x-6}$

Solution

$$\rightarrow \begin{cases} x \geq 3 \\ x \neq 6 \end{cases}$$

Domain: $\underline{x \geq 3 \quad x \neq 6} \mid \underline{[3, 6) \cup (6, \infty)}$



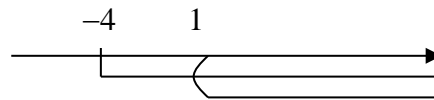
Exercise

Find the domain $f(x) = \frac{\sqrt{x+4}}{\sqrt{x-1}}$

Solution

$$\rightarrow \begin{cases} x \geq -4 \\ x > 1 \end{cases}$$

Domain: $\underline{x > 1} \mid \underline{(1, \infty)}$



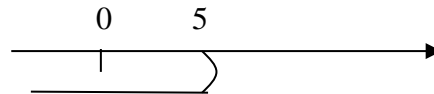
Exercise

Find the domain $f(x) = \frac{\sqrt{5-x}}{x}$

Solution

$$x \leq 5 \quad x \neq 0$$

Domain: $\underline{x \leq 5 \quad x \neq 0} \mid \underline{(-\infty, 0) \cup (0, 5]}$



Exercise

Find the domain $f(x) = \frac{x}{\sqrt{5-x}}$

Solution

Domain: $\underline{x < 5} \mid \underline{(-\infty, 5)}$

Exercise

Find the domain $f(x) = \frac{1}{x\sqrt{5-x}}$

Solution

$$x < 5 \quad x \neq 0$$

$$\text{Domain: } \underline{x < 5 \quad x \neq 0} \mid \underline{(-\infty, 0) \cup (0, 5)}$$

Exercise

Find the domain $f(x) = \frac{x+1}{x^3-4x}$

Solution

$$x^3 - 4x \neq 0$$

$$x(x^2 - 4) \neq 0$$

$$\text{Domain: } \underline{x \neq 0, \pm 2} \mid (-\infty, -2) \cup (-2, 0) \cup (0, 2) \cup (2, \infty)$$

Exercise

Find the domain $f(x) = \frac{\sqrt{x+5}}{x}$

Solution

$$x \geq -5 \quad x \neq 0$$

$$\text{Domain: } \underline{x \geq -5 \quad x \neq 0}$$

Exercise

Find the domain $f(x) = \frac{x}{\sqrt{x+5}}$

Solution

$$x > -5$$

$$\text{Domain: } \underline{x > -5}$$

Exercise

Find the domain $f(x) = \frac{1}{x\sqrt{x+5}}$

Solution

$$x > -5 \quad x \neq 0$$

$$\text{Domain: } \underline{x > -5 \quad x \neq 0}$$

Exercise

Find the domain $f(x) = \frac{x+3}{\sqrt{x-3}}$

Solution

$$\text{Domain: } \underline{x > 3}$$

Exercise

Find the domain $f(x) = \frac{\sqrt{x+3}}{\sqrt{x-3}}$

Solution

$$x \geq -3 \quad x > 3$$

$$\text{Domain: } \underline{x > 3}$$

Exercise

Find the domain $f(x) = \frac{\sqrt{x-2}}{\sqrt{x+2}}$

Solution

$$x \geq 2 \quad x > -2$$

$$\text{Domain: } \underline{x \geq 2}$$

Exercise

Find the domain $f(x) = \frac{\sqrt{2-x}}{\sqrt{x+2}}$

Solution

$$x \leq 2 \quad x > -2$$

$$\text{Domain: } \underline{-2 < x \leq 2}$$

Exercise

Find the domain $f(x) = \frac{x-4}{\sqrt{x-2}}$

Solution

Domain: $x > 2$

Exercise

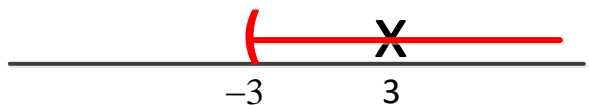
Find the domain of $f(x) = \frac{1}{(x-3)\sqrt{x+3}}$

Solution

$$x-3 \neq 0 \quad x+3 > 0$$

$$x \neq 3 \quad x > -3$$

Domain: $\{x \mid x > -3 \text{ and } x \neq 3\}$
 $(-3, 3) \cup (3, \infty)$



Exercise

Find the domain of $f(x) = \sqrt{x+2} + \sqrt{2-x}$

Solution

$$x+2 \geq 0 \quad 2-x \geq 0$$

$$x \geq -2 \quad -x \geq -2 \rightarrow x \leq 2$$

Domain: $\{x \mid -2 \leq x \leq 2\}$



Exercise

Find the domain of $f(x) = \sqrt{(x-2)(x-6)}$

Solution

$$x-2 \geq 0 \quad x-6 \geq 0$$

$$x \geq 2 \quad x \geq 6$$

Domain: $\{x \mid x \leq 2, x \geq 6\}$

	2	6
-	+	+
-	-	+
+	-	+

Exercise

Find the domain of $f(x) = \sqrt{x+3} - \sqrt{4-x}$

Solution

$$x \geq -3 \quad x \leq 4$$

$$\text{Domain: } \underline{-3 \leq x \leq 4}$$

Exercise

Find the domain of $f(x) = \frac{\sqrt{4x-3}}{x^2-4}$

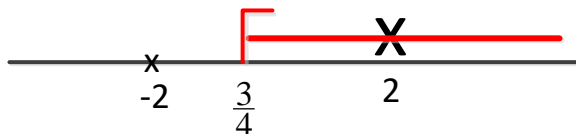
Solution

$$4x-3 \geq 0 \quad x^2-4 \neq 0$$

$$4x \geq 3 \quad x \neq \pm 2$$

$$x \geq \frac{3}{4}$$

$$\text{Domain: } \left[\frac{3}{4}, 2 \right) \cup (2, \infty)$$



Exercise

Find the domain of $f(x) = \frac{4x}{6x^2+13x-5}$

Solution

$$6x^2+13x-5 \neq 0$$

$$x = \frac{-13 \pm \sqrt{169+120}}{12}$$

$$= \begin{cases} \frac{-13-17}{12} = -\frac{5}{2} \\ \frac{-13+17}{12} = \frac{1}{3} \end{cases}$$

$$\text{Domain: } \underline{x \neq -\frac{5}{2}, \frac{1}{3}}$$

Exercise

Find the domain of $f(x) = \frac{\sqrt{2x-3}}{x^2-5x+4}$

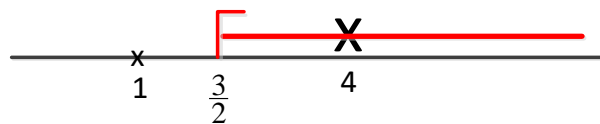
Solution

$$2x-3 \geq 0 \quad x^2-5x+4 \neq 0$$

$$2x \geq 3 \quad x \neq 1, 4$$

$$x \geq \frac{3}{2}$$

$$\text{Domain: } \underline{x \geq \frac{3}{2}, x \neq 4} \quad \left[\frac{3}{2}, 4 \right) \cup (4, \infty)$$



Exercise

Find the domain of $f(x) = \frac{x^2}{\sqrt{x^2 - 5x + 4}}$

Solution

$$x^2 - 5x + 4 \quad a + b + c = 0 \rightarrow x = 1, \frac{c}{a}$$

$$x = 1, 4$$

$$\text{Domain: } \underline{x < 1 \quad x > 4}$$

Exercise

Find the domain of $f(x) = \frac{x+2}{\sqrt{x^2 + 5x + 4}}$

Solution

$$x^2 + 5x + 4 \quad a - b + c = 0 \rightarrow x = -1, -\frac{c}{a}$$

$$x = -1, -4$$

$$\text{Domain: } \underline{x < -4 \quad x > -1}$$

Exercise

Find the domain of $f(x) = \frac{\sqrt{x+2}}{\sqrt{x^2 + 3x + 2}}$

Solution

$$x^2 + 3x + 2 \quad a - b + c = 0 \rightarrow x = -1, -\frac{c}{a}$$

$$x < -2 \quad x > -1$$

$$\sqrt{x+2} \rightarrow x \geq -2$$

$$\text{Domain: } \underline{x > -1}$$

Exercise

Find the domain of $f(x) = \frac{\sqrt{2x+3}}{x^2 - 6x + 5}$

Solution

$$x^2 - 6x + 5 \qquad a + b + c = 0 \rightarrow x = 1, \frac{c}{a}$$

$$x \neq 1, 5$$

$$\sqrt{2x+3} \rightarrow x \geq -\frac{3}{2}$$

$$\text{Domain: } \underline{x \geq -\frac{3}{2} \quad x \neq 1, 5}$$

Exercise

Let $f(x) = 4x - 3$ and $g(x) = 5x + 7$. Find each of the following and give the domain

$$a) (f+g)(x) \qquad b) (f-g)(x) \qquad c) (fg)(x) \qquad d) \left(\frac{f}{g}\right)(x)$$

Solution

$$a) (f+g)(x) = 4x - 3 + 5x + 7 \\ \underline{= 9x + 4}$$

$$\text{Domain: } \underline{\mathbb{R}}$$

$$b) (f-g)(x) = 4x - 3 - (5x + 7) \\ = 4x - 3 - 5x - 7 \\ \underline{= -x - 10}$$

$$\text{Domain: } \underline{\mathbb{R}}$$

$$c) (fg)(x) = (4x - 3)(5x + 7) \\ \underline{= 20x^2 + 13x - 21}$$

$$\text{Domain: } \underline{\mathbb{R}}$$

$$d) \left(\frac{f}{g}\right)(x) = \underline{\frac{4x-3}{5x+7}}$$

$$\text{Domain: } \underline{x \neq -\frac{7}{5}}$$

Exercise

Let $f(x) = 2x^2 + 3$ and $g(x) = 3x - 4$. Find each of the following and give the domain

a) $(f + g)(x)$ b) $(f - g)(x)$ c) $(fg)(x)$ d) $\left(\frac{f}{g}\right)(x)$

Solution

$$\begin{aligned} \text{a) } (f + g)(x) &= 2x^2 + 3 + 3x - 4 \\ &= \underline{2x^2 + 3x - 1} \end{aligned}$$

Domain: \mathbb{R}

$$\begin{aligned} \text{b) } (f - g)(x) &= 2x^2 + 3 - (3x - 4) \\ &= 2x^2 + 3 - 3x + 4 \\ &= \underline{2x^2 - 3x + 7} \end{aligned}$$

Domain: \mathbb{R}

$$\begin{aligned} \text{c) } (fg)(x) &= (2x^2 + 3)(3x - 4) \\ &= \underline{6x^2 + x - 12} \end{aligned}$$

Domain: \mathbb{R}

$$\text{d) } \left(\frac{f}{g}\right)(x) = \underline{\frac{2x^2 + 3}{3x - 4}}$$

Domain: $x \neq -\frac{4}{3}$

Exercise

Let $f(x) = x^2 - 2x - 3$ and $g(x) = x^2 + 3x - 2$. Find each of the following and give the domain

a) $(f + g)(x)$ b) $(f - g)(x)$ c) $(fg)(x)$ d) $\left(\frac{f}{g}\right)(x)$

Solution

$$\begin{aligned} \text{a) } (f + g)(x) &= x^2 - 2x - 3 + x^2 + 3x - 2 \\ &= \underline{2x^2 + x - 5} \end{aligned}$$

Domain: \mathbb{R}

$$\begin{aligned} \text{b) } (f - g)(x) &= x^2 - 2x - 3 - x^2 - 3x + 2 \\ &= \underline{-5x - 1} \end{aligned}$$

Domain: \mathbb{R}

$$\begin{aligned} c) \quad (fg)(x) &= (x^2 - 2x - 3)(x^2 + 3x - 2) \\ &= x^4 + 3x^3 - 2x^2 - 2x^3 - 6x^2 + 4x - 3x^2 - 9x + 6 \\ &= x^4 + x^3 - 11x^2 - 5x + 6 \end{aligned}$$

Domain: \mathbb{R}

$$d) \quad \left(\frac{f}{g}\right)(x) = \frac{x^2 - 2x - 3}{x^2 + 3x - 2}$$

$$\text{Domain: } x \neq \frac{-3 \pm \sqrt{17}}{2}$$

Exercise

Let $f(x) = \sqrt{4x-1}$ and $g(x) = \frac{1}{x}$. Find each of the following and give the domain

a) $(f+g)(x)$

b) $(f-g)(x)$

c) $(fg)(x)$

d) $\left(\frac{f}{g}\right)(x)$

Solution

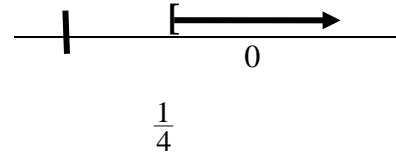
a) $(f+g)(x)$

$$(f+g)(x) = \sqrt{4x-1} + \frac{1}{x}$$

$$4x-1 \geq 0 \quad x \neq 0$$

$$x \geq \frac{1}{4}$$

$$\text{Domain: } \left[\frac{1}{4}, \infty\right)$$



b) $(f-g)(x)$

$$(f-g)(x) = \sqrt{4x-1} - \frac{1}{x}$$

$$4x-1 \geq 0 \quad x \neq 0$$

$$x \geq \frac{1}{4}$$

$$\text{Domain: } \left[\frac{1}{4}, \infty\right)$$

$$\begin{aligned} c) \quad (fg)(x) &= \sqrt{4x-1} \left(\frac{1}{x}\right) \\ &= \frac{\sqrt{4x-1}}{x} \end{aligned}$$

$$4x - 1 \geq 0 \quad x \neq 0$$

$$x \geq \frac{1}{4}$$

$$\text{Domain: } \left[\frac{1}{4}, \infty \right)$$

$$d) \left(\frac{f}{g} \right)(x) = \frac{\sqrt{4x-1}}{\frac{1}{x}} \quad \text{Domain: } x \neq 0$$

$$= x\sqrt{4x-1}$$

$$4x - 1 \geq 0$$

$$x \geq \frac{1}{4}$$

$$\text{Domain: } \left[\frac{1}{4}, \infty \right)$$

Exercise

Given that $f(x) = x + 1$ and $g(x) = \sqrt{x + 3}$

a) Find $(f + g)(x)$

b) Find the domain of $(f + g)(x)$

c) Find: $(f + g)(6)$

Solution

$$a) (f + g)(x) = f(x) + g(x) \\ = x + 1 + \sqrt{x + 3} \quad |$$

$$b) x + 3 \geq 0 \rightarrow x \geq -3$$

$$\text{Domain} = [-3, \infty)$$

$$c) (f + g)(6) = 6 + 1 + \sqrt{6 + 3} \\ = 10 \quad |$$

Exercise

Given that $f(x) = x^2 - 4$ and $g(x) = x + 2$

a) Find $(f + g)(x)$ and its domain

b) Find $(f / g)(x)$ and its domain

Solution

$$\begin{aligned} a) \quad (f + g)(x) &= x^2 - 4 + x + 2 \\ &= x^2 + x - 2 \end{aligned}$$

Domain: \mathbb{R} |

$$b) \quad \frac{f(x)}{g(x)} = \frac{x^2 - 4}{x + 2}$$

$$x \neq -2$$

Domain: $\underline{(-\infty, -2) \cup (-2, \infty)}$ |

Exercise

Let $f(x) = x^2 + 1$ and $g(x) = 3x + 5$. Find $(f + g)(1)$, $(f - g)(-3)$, $(fg)(5)$, and $\left(\frac{f}{g}\right)(0)$

Solution

$$\begin{aligned} a) \quad (f + g)(1) &= f(1) + g(1) \\ &= 1^2 + 1 + 3(1) + 5 \\ &= 10 \end{aligned}$$

$$\begin{aligned} b) \quad (f - g)(-3) &= f(-3) - g(-3) \\ &= (-3)^2 + 1 - (3(-3) + 5) \\ &= 10 \end{aligned}$$

$$\begin{aligned} c) \quad (fg)(5) &= f(5) \cdot g(5) \\ &= (5^2 + 1) \cdot (3(5) + 5) \\ &= (26) \cdot (20) \\ &= 520 \end{aligned}$$

$$\begin{aligned} d) \quad \left(\frac{f}{g}\right)(0) &= \frac{f(0)}{g(0)} \\ &= \frac{0^2 + 1}{3(0) + 5} \\ &= \frac{1}{5} \end{aligned}$$

Exercise

Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $(f / g)(x)$ and the domain of

$$f(x) = \sqrt{3 - 2x}, \quad g(x) = \sqrt{x + 4}$$

Solution

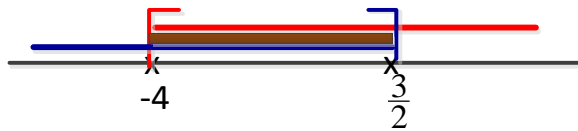
$$f(x) + g(x) = \sqrt{3 - 2x} + \sqrt{x + 4}$$

$$3 - 2x \geq 0 \quad x + 4 \geq 0$$

$$-2x \geq -3 \quad x \geq -4$$

$$x \leq \frac{3}{2}$$

$$\text{Domain: } \left\{ x \mid -4 \leq x \leq \frac{3}{2} \right\}$$



$$f(x) - g(x) = \sqrt{3 - 2x} - \sqrt{x + 4}$$

$$3 - 2x \geq 0 \quad x + 4 \geq 0$$

$$-2x \geq -3 \quad x \geq -4$$

$$x \leq \frac{3}{2}$$

$$\text{Domain: } \left\{ x \mid -4 \leq x \leq \frac{3}{2} \right\}$$

$$(f \cdot g)(x) = (\sqrt{3 - 2x})(\sqrt{x + 4})$$

$$= \sqrt{(3 - 2x)(x + 4)}$$

$$= \sqrt{-2x^2 - 5x + 12}$$

$$3 - 2x \geq 0 \quad x + 4 \geq 0$$

$$-2x \geq -3 \quad x \geq -4$$

$$x \leq \frac{3}{2}$$

$$\text{Domain: } \left\{ x \mid -4 \leq x \leq \frac{3}{2} \right\}$$

$$(f / g)(x) = \frac{\sqrt{3 - 2x}}{\sqrt{x + 4}} \cdot \frac{\sqrt{x + 4}}{\sqrt{x + 4}}$$

$$= \frac{\sqrt{-2x^2 - 5x + 12}}{x + 4}$$

$$3 - 2x \geq 0 \quad x + 4 > 0$$

$$-2x \geq -3 \quad x > -4$$

$$x \leq \frac{3}{2}$$

$$\text{Domain: } \left\{ x \mid -4 < x \leq \frac{3}{2} \right\}$$

$$\left(-4, \frac{3}{2} \right]$$

Exercise

Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $(f / g)(x)$ and the domain of

$$f(x) = \frac{2x}{x-4}, \quad g(x) = \frac{x}{x+5}$$

Solution

$$\begin{aligned}(f + g)(x) &= \frac{2x}{x-4} + \frac{x}{x+5} \\&= \frac{2x(x+5) + x(x-4)}{(x-4)(x+5)} \\&= \frac{2x^2 + 10x + x^2 - 4x}{(x-4)(x+5)} \\&= \frac{3x^2 + 6x}{(x-4)(x+5)}\end{aligned}$$

$$x-4 \neq 0 \quad x+5 \neq 0$$

$$x \neq 4 \quad x \neq -5$$

$$\text{Domain: } \{x \mid x \neq -5, 4\} \quad (-\infty, -5) \cup (-5, 4) \cup (4, \infty)$$

$$\begin{aligned}(f - g)(x) &= \frac{2x}{x-4} - \frac{x}{x+5} \\&= \frac{2x(x+5) - x(x-4)}{(x-4)(x+5)} \\&= \frac{2x^2 + 10x - x^2 + 4x}{(x-4)(x+5)} \\&= \frac{x^2 + 14x}{(x-4)(x+5)}\end{aligned}$$

$$x \neq 4 \quad x \neq -5$$

$$\text{Domain: } \{x \mid x \neq -5, 4\}$$

$$\begin{aligned}(f \cdot g)(x) &= \frac{2x}{x-4} \cdot \frac{x}{x+5} \\&= \frac{2x^2}{(x-4)(x+5)}\end{aligned}$$

$$x \neq 4 \quad x \neq -5$$

$$\text{Domain: } \{x \mid x \neq -5, 4\}$$

$$\begin{aligned}(f / g)(x) &= \frac{2x}{x-4} \div \frac{x}{x+5} \\&= \frac{2x}{x-4} \cdot \frac{x+5}{x}\end{aligned}$$

$$= 2 \frac{x+5}{x-4}$$

$$x \neq 4 \quad x \neq -5$$

Domain: $\{x \mid x \neq -5, 4\}$

Exercise

Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $(f / g)(x)$ of $f(x) = x - 5$ and $g(x) = x^2 - 1$

Solution

$$\begin{aligned} a) \quad (f + g)(x) &= f(x) + g(x) \\ &= x - 5 + x^2 - 1 \\ &= x^2 + x - 6 \end{aligned}$$

$$\begin{aligned} b) \quad (f - g)(x) &= f(x) - g(x) \\ &= x - 5 - (x^2 - 1) \\ &= x - 5 - x^2 + 1 \\ &= -x^2 + x - 4 \end{aligned}$$

$$\begin{aligned} c) \quad (fg)(x) &= f(x)g(x) \\ &= (x - 5)(x^2 - 1) \\ &= x^3 - x - 5x^2 + 5 \\ &= x^3 - 5x^2 - x + 5 \end{aligned}$$

$$\begin{aligned} d) \quad \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} \\ &= \frac{x-5}{x^2-1} \end{aligned}$$

Exercise

For the function f given by $f(x) = 9x + 5$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$f(x+h) = 9(x+h) + 5 = 9x + 9h + 5$$

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\overbrace{9x+9h+5}^{f(x+h)} - \overbrace{(9x+5)}^{f(x)}}{h} \\ &= \frac{9x+9h+5 - 9x-5}{h} \end{aligned}$$

$$= \frac{9h}{h}$$

$$= 9$$

Exercise

For the function f given by $f(x) = 6x + 2$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\frac{f(x+h) - f(x)}{h} = \frac{6(x+h) + 2 - (6x + 2)}{h}$$

$$= \frac{6x + 6h + 2 - 6x - 2}{h}$$

$$= \frac{6h}{h}$$

$$= 6$$

Exercise

For the function f given by $f(x) = 4x + 11$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\frac{f(x+h) - f(x)}{h} = \frac{4(x+h) + 11 - (4x + 11)}{h}$$

$$= \frac{4x + 4h + 11 - 4x - 11}{h}$$

$$= \frac{4h}{h}$$

$$= 4$$

Exercise

For the function f given by $f(x) = 3x - 5$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\frac{f(x+h) - f(x)}{h} = \frac{3(x+h) - 5 - (3x - 5)}{h}$$

$$= \frac{3x + 3h - 5 - 3x + 5}{h}$$

$$= \frac{3h}{h}$$

$$= 3$$

Exercise

For the function f given by $f(x) = -2x - 3$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{-2(x+h) - 3 + 2x + 3}{h} \\ &= \frac{-2x - 2h - 3 + 2x + 3}{h} \\ &= \frac{-2h}{h} \\ &= -2\end{aligned}$$

Exercise

For the function f given by $f(x) = -4x + 3$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{-4(x+h) + 3 + 4x - 3}{h} \\ &= \frac{-4x - 4h + 3 + 4x - 3}{h} \\ &= \frac{-4h}{h} \\ &= -4\end{aligned}$$

Exercise

For the function f given by $f(x) = 3x - 6$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{3(x+h) - 6 - 3x + 6}{h} \\ &= \frac{3x + 3h - 6 - 3x + 6}{h} \\ &= \frac{3h}{h} \\ &= 3\end{aligned}$$

Exercise

For the function f given by $f(x) = -5x - 7$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{-5(x+h) - 7 + 5x + 7}{h} \\ &= \frac{-5x - 5h - 7 + 5x + 7}{h} \\ &= \frac{-5h}{h} \\ &= -5\end{aligned}$$

Exercise

Given the function: $f(x) = 2x^2$. Find and simplify the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}f(x+h) &= 2(x+h)^2 \\ &= 2(x^2 + 2hx + h^2) \\ &= 2x^2 + 4hx + 2h^2 \\ \frac{f(x+h) - f(x)}{h} &= \frac{2x^2 + 4hx + 2h^2 - 2x^2}{h} \\ &= \frac{4hx + 2h^2}{h} \\ &= \frac{4hx}{h} + \frac{2h^2}{h} \\ &= 4x + 2h\end{aligned}$$

Exercise

For the function f given by $f(x) = 5x^2$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{5(x+h)^2 - 5x^2}{h} \\ &= \frac{5(x^2 + 2hx + h^2) - 5x^2}{h}\end{aligned}$$

$$\begin{aligned}
 &= \frac{5x^2 + 10hx + 5h^2 - 5x^2}{h} \\
 &= \frac{10hx + 5h^2}{h} \\
 &= \underline{10x + 5h}
 \end{aligned}$$

Exercise

For the function f given by $f(x) = 3x^2 - 4x$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}
 \frac{f(x+h) - f(x)}{h} &= \frac{3(x+h)^2 - 4x - 3x^2 + 4x}{h} \\
 &= \frac{3(x^2 + 2hx + h^2) - 4(x+h) - 3x^2 + 4x}{h} \\
 &= \frac{3x^2 + 6hx + 3h^2 - 4x - 4h - 3x^2 + 4x}{h} \\
 &= \frac{6hx + 3h^2 - 4h}{h} \\
 &= \underline{6x + 3h - 4}
 \end{aligned}$$

Exercise

For the function f given by $f(x) = 2x^2 - 3x$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}
 f(x+h) &= 2(\text{---})^2 - 3(\text{---}) \\
 &= 2(x+h)^2 - 3(x+h) & (a+b)^2 &= a^2 + 2ab + b^2 \\
 &= 2(x^2 + 2xh + h^2) - 3x - 3h \\
 &= 2x^2 + 4xh + 2h^2 - 3x - 3h \\
 \frac{f(x+h) - f(x)}{h} &= \frac{\overbrace{2x^2 + 4xh + 2h^2 - 3x - 3h}^{f(x+h)} - \overbrace{(2x^2 - 3x)}^{f(x)}}{h} \\
 &= \frac{2x^2 + 4xh + 2h^2 - 3x - 3h - 2x^2 + 3x}{h} \\
 &= \frac{4xh + 2h^2 - 3h}{h}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{4xh}{h} + \frac{2h^2}{h} - \frac{3h}{h} \\
&= \underline{4x + 2h - 3}
\end{aligned}$$

Exercise

For the function f given by $f(x) = 2x^2 - x - 3$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}
f(x+h) &= 2(x+h)^2 - (x+h) - 3 \\
&= 2(x^2 + 2hx + h^2) - x - h - 3 \\
&= 2x^2 + 4hx + 2h^2 - x - h - 3
\end{aligned}$$

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{2x^2 + 2h^2 + 4hx - x - h - 3 - (2x^2 - x - 3)}{h} \\
&= \frac{2x^2 + 2h^2 + 4hx - x - h - 3 - 2x^2 + x + 3}{h} \\
&= \frac{2h^2 + 4hx - h}{h} \\
&= \frac{2h^2}{h} + \frac{4hx}{h} - \frac{h}{h} \\
&= \underline{2h + 4x - 1}
\end{aligned}$$

Exercise

For the given function $f(x) = 2x^2 - x - 3$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{2(x+h)^2 - (x+h) - 3 - 2x^2 + x + 3}{h} \\
&= \frac{2(x^2 + 2hx + h^2) - x - h - 3 - 2x^2 + x + 3}{h} \\
&= \frac{2x^2 + 4hx + 2h^2 - h - 2x^2}{h} \\
&= \frac{4hx + 2h^2 - h}{h} \\
&= \underline{4x + 2h - 1}
\end{aligned}$$

Exercise

For the given function $f(x) = x^2 - 2x + 5$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^2 - 2(x+h) + 5 - x^2 + 2x - 5}{h} \\&= \frac{x^2 + 2hx + h^2 - 2x - 2h - x^2 + 2x}{h} \\&= \frac{2hx + h^2 - 2h}{h} \\&= \underline{2x + h - 2} \quad | \end{aligned}$$

Exercise

For the given function $f(x) = 3x^2 - 2x + 5$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{3(x+h)^2 - 2(x+h) + 5 - 3x^2 + 2x - 5}{h} \\&= \frac{3(x^2 + 2hx + h^2) - 2x - 2h - 3x^2 + 2x}{h} \\&= \frac{3x^2 + 6hx + 3h^2 - 2h - 3x^2}{h} \\&= \frac{6hx + 3h^2 - 2h}{h} \\&= \underline{6x + 3h - 2} \quad | \end{aligned}$$

Exercise

For the given function $f(x) = -2x^2 - 3x + 7$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{-2(x+h)^2 - 3(x+h) + 7 + 2x^2 + 3x - 7}{h} \\&= \frac{-2(x^2 + 2hx + h^2) - 3x - 3h + 2x^2 + 3x}{h} \\&= \frac{-2x^2 - 4hx - 2h^2 - 3h + 2x^2}{h} \end{aligned}$$

$$= \frac{-4hx - 2h^2 - 3h}{h}$$

$$= -4x - 2h - 3$$

Exercise

An open box is to be made from a square piece of cardboard that measures 40 inches on each side, to construct the box, squares that measure x inches on each side are cut from each corner of the cardboard.

- Express the volume V of the box as a function of x .
- Determine the domain of V .

Solution

$$a) \quad V(x) = x(40 - 2x)^2$$

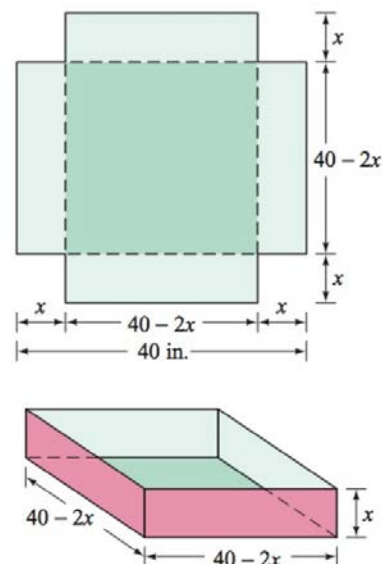
$$= x(1600 - 160x + 4x^2)$$

$$= 4x^3 - 160x^2 + 1600x$$

$$b) \quad 40 - 2x = 0$$

$$x = 20$$

$$\text{Domain: } \{x \mid 0 < x < 20\}$$



Exercise

A child 4 feet tall is standing near a street lamp that is 12 feet high. The light from the lamp casts a shadow.

- Find the length l of the shadow as a function of the distance d of the child from the lamppost.
- What is the domain of the function?
- What is the length of the shadow when the child is 8 feet from the base of the lamppost?

Solution

$$a) \quad \frac{l + d}{12} = \frac{l}{4}$$

$$l + d = 3l$$

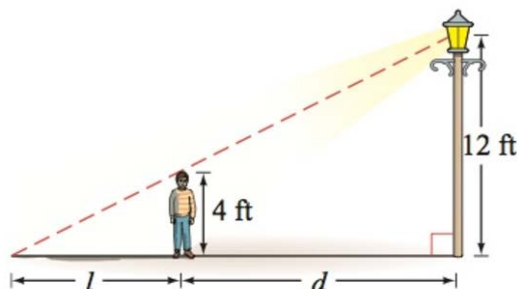
$$2l = d$$

$$l(d) = \frac{1}{2}d$$

$$b) \quad \text{Domain: } \{x \mid 0 \leq d < \infty\}$$

$$c) \quad \text{Given: } d = 8$$

$$l = 4 \text{ feet}$$



Exercise

An open box is to be made from a square piece of cardboard with the dimensions 30 inches by 30 inches by cutting out squares of area x^2 from each corner.

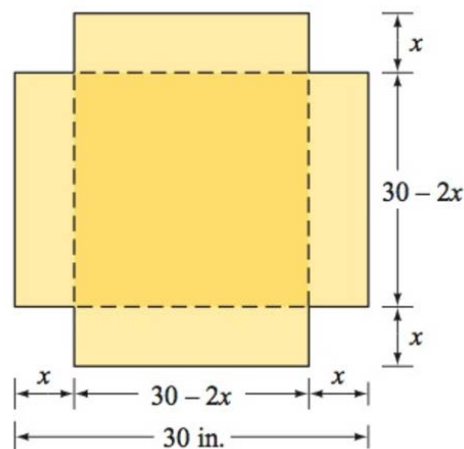
- Express the volume V of the box as a function of x .
- Determine the domain of V .

Solution

$$\begin{aligned} a) \quad V(x) &= x(30 - 2x)^2 \\ &= x(900 - 120x + 4x^2) \\ &= 4x^3 - 120x^2 + 900x \end{aligned}$$

$$\begin{aligned} b) \quad 30 - 2x &= 0 \\ x &= 15 \end{aligned}$$

$$\text{Domain: } \{x \mid 0 < x < 15\}$$



Exercise

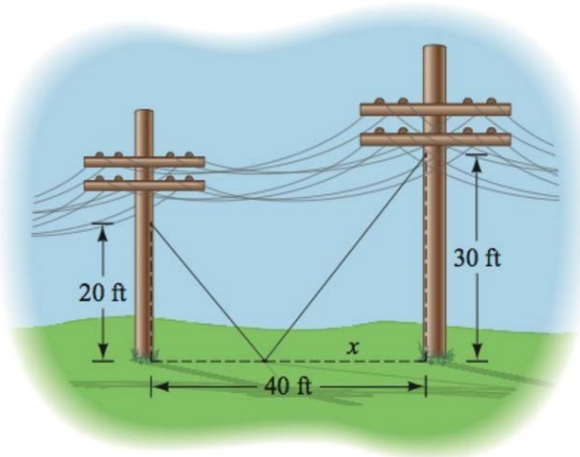
Two guy wires are attached to utility poles that are 40 feet apart.

- Find the total length of the two guy wires as a function of x .
- What is the domain of this function?

Solution

$$\begin{aligned} a) \quad \ell_1 &= \sqrt{(40 - x)^2 + 20^2} \\ &= \sqrt{1,600 - 80x + x^2 + 400} \\ &= \sqrt{2,000 - 80x + x^2} \\ \ell_2 &= \sqrt{x^2 + 30^2} \\ &= \sqrt{x^2 + 900} \\ \ell(x) &= \sqrt{2,000 - 80x + x^2} + \sqrt{x^2 + 900} \end{aligned}$$

$$b) \quad \text{Domain: } [0, 40]$$



Exercise

A rancher has 360 yards of fencing with which to enclose two adjacent rectangular corrals, one for sheep and one for cattle. A river forms one side of the corrals. Suppose the width of each corral is x yards.

- Express the total area of the two corrals as a function of x .
- Find the domain of the function.

Solution

a) $P = 3x + l = 360$

$$l = 360 - 3x$$

$$A = xl$$

$$= x(360 - 3x)$$

$$\underline{A(x) = 360x - 3x^2}$$

b) $x(360 - 3x) = 0$

$$\underline{x = 0}$$

$$360 - 3x = 0$$

$$3x = 360$$

$$\Rightarrow \underline{x = 120}$$

$$\text{Domain: } \underline{0 < x < 120}$$



Exercise

A rectangle is bounded by the x - and y -axis of $y = -\frac{1}{2}x + 4$

- Find the area of the rectangle as a function of x .
- What is the domain of this function.

Solution

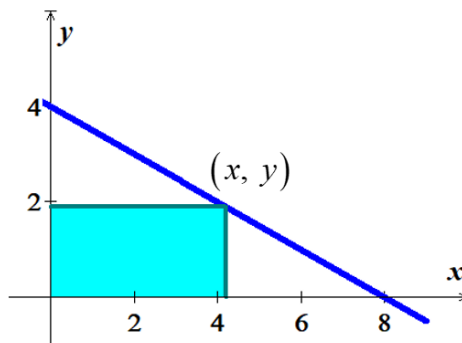
a) $\text{Area} = xy$

$$\begin{aligned} A(x) &= x\left(-\frac{1}{2}x + 4\right) \\ &= \underline{-\frac{1}{2}x^2 + 4x} \end{aligned}$$

b) $x\left(-\frac{1}{2}x + 4\right) = 0$

$$x = 0 \quad x = 8$$

$$\text{Domain: } \underline{0 < x < 8}$$



Solution

Section 2.3 – Composition Functions

Exercise

Given that $f(x) = 2x - 5$ and $g(x) = x^2 - 3x + 8$, find $(f \circ g)(x)$, $(g \circ f)(x)$ and their domain then find $(f \circ g)(7)$

Solution

$$\begin{aligned} f(g(x)) &= f(x^2 - 3x + 8) & \text{Domain: } (-\infty, \infty) \\ &= 2(\text{-----}) - 5 \\ &= 2(2x^2 - 3x + 8) - 5 \\ &= 2x^2 - 6x + 16 - 5 \\ &= 2x^2 - 6x + 11 & \text{Domain: } (-\infty, \infty) \end{aligned}$$

Domain: \mathbb{R}

$$\begin{aligned} g(f(x)) &= g(2x - 5) & \text{Domain: } (-\infty, \infty) \\ &= (\text{---})^2 - 3(\text{---}) + 8 \\ &= (2x - 5)^2 - 3(2x - 5) + 8 \\ &= 4x^2 - 20x + 25 - 6x + 15 + 8 \\ &= 4x^2 - 26x + 48 & \text{Domain: } (-\infty, \infty) \end{aligned}$$

Domain: \mathbb{R}

$$f(g(7)) = 2(7)^2 - 6(7) + 11 = 67$$

Exercise

Given that $f(x) = \sqrt{x}$ and $g(x) = x - 1$, find

- a) $(f \circ g)(x) = f(g(x))$
- b) $(g \circ f)(x) = g(f(x))$
- c) $(f \circ g)(2) = f(g(2))$

Solution

$$\begin{aligned} \text{a) } (f \circ g)(x) &= f(g(x)) \\ &= f(x - 1) \\ &= \sqrt{x - 1} \end{aligned}$$

$$\begin{aligned}
 b) \quad (g \circ f)(x) &= g(f(x)) \\
 &= g(\sqrt{x}) \\
 &= \sqrt{x-1}
 \end{aligned}$$

$$\begin{aligned}
 c) \quad (f \circ g)(2) &= f(g(2)) \\
 &= \sqrt{2-1} \\
 &= 1
 \end{aligned}$$

Exercise

Given that $f(x) = \frac{x}{x+5}$ and $g(x) = \frac{6}{x}$, find

$$a) \quad (f \circ g)(x) = f(g(x))$$

$$b) \quad (g \circ f)(x) = g(f(x))$$

$$c) \quad (f \circ g)(2) = f(g(2))$$

Solution

$$a) \quad (f \circ g)(x) = f(g(x))$$

$$\begin{aligned}
 &= f\left(\frac{6}{x}\right) \\
 &= \frac{\frac{6}{x}}{\frac{6}{x}+5} \\
 &= \frac{\frac{6}{x}}{\frac{6+5x}{x}} \\
 &= \frac{6}{6+5x}
 \end{aligned}$$

$$b) \quad (g \circ f)(x) = g(f(x))$$

$$\begin{aligned}
 &= g\left(\frac{x}{x+5}\right) \\
 &= \frac{6}{\frac{x}{x+5}} \\
 &= \frac{6(x+5)}{x} \\
 &= \frac{6x+30}{x}
 \end{aligned}$$

$$c) \quad (f \circ g)(2) = f(g(2))$$

$$\begin{aligned}
 &= \frac{6}{6+5(2)} \\
 &= \frac{6}{16} \\
 &= \frac{3}{8}
 \end{aligned}$$

Exercise

Find $(f \circ g)(x)$, $(g \circ f)(x)$, $f(g(-2))$ and $g(f(3))$: $f(x) = 2x^2 + 3x - 4$, $g(x) = 2x - 1$

Solution

$$\begin{aligned}
 f(g(x)) &= f(2x - 1) \\
 &= 2(2x - 1)^2 + 3(2x - 1) - 4 \\
 &= 2(4x^2 - 4x + 1) + 6x - 3 - 4 \\
 &= 8x^2 - 8x + 2 + 6x - 7 \\
 &= 8x^2 - 2x - 5
 \end{aligned}$$

$$\begin{aligned}
 g(f(x)) &= g(2x^2 + 3x - 4) \\
 &= 2(2x^2 + 3x - 4) - 1 \\
 &= 4x^2 + 6x - 8 - 1 \\
 &= 4x^2 + 6x - 9
 \end{aligned}$$

$$\begin{aligned}
 f(g(-2)) &= 8(-2)^2 - 2(-2) - 5 \\
 &= 31
 \end{aligned}$$

$$\begin{aligned}
 g(f(3)) &= 4(3)^2 + 6(3) - 9 \\
 &= 45
 \end{aligned}$$

Exercise

Find $(f \circ g)(x)$, $(g \circ f)(x)$, $f(g(-2))$ and $g(f(3))$: $f(x) = x^3 + 2x^2$, $g(x) = 3x$

Solution

$$\begin{aligned}
 f(g(x)) &= f(3x) \\
 &= (3x)^3 + 2(3x)^2
 \end{aligned}$$

$$\begin{aligned}
 & \underline{= 27x^3 + 18x^2} \\
 g(f(x)) &= g(x^3 + 2x^2) \\
 &= 3(x^3 + 2x^2) \\
 &= \underline{3x^3 + 6x^2}
 \end{aligned}$$

$$\begin{aligned}
 f(g(-2)) &= 27(-2)^3 + 18(-2)^2 \\
 &= \underline{288}
 \end{aligned}$$

$$\begin{aligned}
 g(f(3)) &= 3(3)^3 + 6(3)^2 \\
 &= \underline{135}
 \end{aligned}$$

Exercise

Find $(f \circ g)(x)$, $(g \circ f)(x)$, $f(g(-2))$ and $g(f(3))$: $f(x) = |x|$, $g(x) = -7$

Solution

$$\begin{aligned}
 f(g(x)) &= f(-7) \\
 &= |-7| \\
 &= 7
 \end{aligned}$$

$$\begin{aligned}
 g(f(x)) &= g(|x|) \\
 &= -7
 \end{aligned}$$

$$f(g(-2)) = \underline{7}$$

$$g(f(3)) = \underline{-7}$$

Exercise

Given $f(x) = x - 3$ and $g(x) = x + 3$

- Find $(f \circ g)(x)$ and the domain of $f \circ g$
- Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

$$\begin{aligned}
 a) \quad f(g(x)) &= f(x + 3) & \text{Domain: } \mathbb{R} \\
 &= (x + 3) - 3 \\
 &= \underline{x} & \text{Domain: } \mathbb{R}
 \end{aligned}$$

Domain: \mathbb{R}

$$b) \quad g(f(x)) = g(x-3) \quad \text{Domain: } \mathbb{R}$$

$$= (x+3) - 3$$

$$= x \quad \text{Domain: } \mathbb{R}$$

Domain: \mathbb{R}

Exercise

Given $f(x) = \frac{2}{3}x$ and $g(x) = \frac{3}{2}x$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

$$a) \quad f(g(x)) = f\left(\frac{3}{2}x\right) \quad \text{Domain: } \mathbb{R}$$

$$= \frac{2}{3}\left(\frac{3}{2}x\right)$$

$$= x \quad \text{Domain: } \mathbb{R}$$

Domain: \mathbb{R}

$$b) \quad g(f(x)) = g\left(\frac{2}{3}x\right) \quad \text{Domain: } \mathbb{R}$$

$$= \frac{3}{2}\left(\frac{2}{3}x\right)$$

$$= x \quad \text{Domain: } \mathbb{R}$$

Domain: \mathbb{R}

Exercise

Given $f(x) = x-1$ and $g(x) = 3x^2 - 2x - 1$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

$$a) \quad f(g(x)) = f(3x^2 - 2x - 1) \quad \text{Domain: } \mathbb{R}$$

$$= 3(x-1)^2 - 2(x-1) - 1$$

$$\begin{aligned}
&= 3(x^2 - 2x + 1) - 2x + 2 - 1 \\
&= 3x^2 - 6x + 3 - 2x + 1 \\
&= \underline{3x^2 - 8x + 4} \quad \text{Domain: } \mathbb{R}
\end{aligned}$$

Domain: \mathbb{R}

$$\begin{aligned}
b) \quad g(f(x)) &= g(x - 1) \quad \text{Domain: } \mathbb{R} \\
&= 3x^2 - 2x - 1 - 1 \\
&= \underline{3x^2 - 2x - 2} \quad \text{Domain: } \mathbb{R}
\end{aligned}$$

Domain: \mathbb{R}

Exercise

Given $f(x) = 3x - 2$ and $g(x) = x^2 - 5$

- a) Find $(f \circ g)(x)$ and the domain of $f \circ g$
- b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

$$\begin{aligned}
a) \quad f(g(x)) &= f(x^2 - 5) \quad \text{Domain: } \mathbb{R} \\
&= 3(x^2 - 5) - 2 \\
&= 3x^2 - 15 - 2 \\
&= \underline{3x^2 - 17} \quad \text{Domain: } \mathbb{R}
\end{aligned}$$

Domain: \mathbb{R}

$$\begin{aligned}
b) \quad g(f(x)) &= g(3x - 2) \quad \text{Domain: } \mathbb{R} \\
&= (3x - 2)^2 - 5 \\
&= 9x^2 - 12x + 4 - 5 \\
&= \underline{9x^2 - 12x - 1} \quad \text{Domain: } \mathbb{R}
\end{aligned}$$

Domain: \mathbb{R}

Exercise

Given $f(x) = x^2 - 2$ and $g(x) = 4x - 3$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

$$a) \quad f(g(x)) = f(4x - 3) \quad \text{Domain: } \mathbb{R}$$

$$= (4x - 3)^2 - 2$$

$$= 16x^2 - 24x + 9 - 2$$

$$= \underline{16x^2 - 24x + 7} \quad \text{Domain: } \mathbb{R}$$

$$\text{Domain: } \underline{\mathbb{R}}$$

$$b) \quad g(f(x)) = g(x^2 - 2) \quad \text{Domain: } \mathbb{R}$$

$$= 4(x^2 - 2) - 3$$

$$= 4x^2 - 8 - 3$$

$$= \underline{4x^2 - 11} \quad \text{Domain: } \mathbb{R}$$

$$\text{Domain: } \underline{\mathbb{R}}$$

Exercise

Given $f(x) = 4x^2 - x + 10$ and $g(x) = 2x - 7$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

$$a) \quad f(g(x)) = f(2x - 7) \quad \text{Domain: } \mathbb{R}$$

$$= 4(2x - 7)^2 - (2x - 7) + 10$$

$$= 4(4x^2 - 28x + 49) - 2x + 7 + 10$$

$$= 16x^2 - 112x + 196 - 2x + 17$$

$$= \underline{16x^2 - 114x + 213} \quad \text{Domain: } \mathbb{R}$$

$$\text{Domain: } \underline{\mathbb{R}}$$

$$b) \quad g(f(x)) = g(4x^2 - x + 10) \quad \text{Domain: } \mathbb{R}$$

$$\begin{aligned}
&= 2(4x^2 - x + 10) - 7 \\
&= 8x^2 - 2x + 20 - 7 \\
&= 8x^2 - 2x + 13
\end{aligned}$$

Domain: \mathbb{R}

Domain: \mathbb{R}

Exercise

Given $f(x) = \sqrt{x}$ and $g(x) = x + 3$

- Find $(f \circ g)(x)$ and the domain of $f \circ g$
- Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

$$\begin{aligned}
a) \quad f(g(x)) &= f(x+3) \\
&= \sqrt{x+3}
\end{aligned}$$

Domain: \mathbb{R}

Domain: $x \geq -3$

Domain: $x \geq -3$

$$\begin{aligned}
b) \quad g(f(x)) &= g(\sqrt{x}) \\
&= \sqrt{x} + 3
\end{aligned}$$

Domain: $x \geq 0$

Domain: $x \geq 0$

Domain: $x \geq 0$

Exercise

Given $f(x) = \sqrt{x}$ and $g(x) = 2 - 3x$

- Find $(f \circ g)(x)$ and the domain of $f \circ g$
- Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

$$\begin{aligned}
a) \quad f(g(x)) &= f(2-3x) \\
&= \sqrt{2-3x}
\end{aligned}$$

Domain: \mathbb{R}

Domain: $x \leq \frac{2}{3}$

Domain: $x \leq \frac{2}{3}$

$$\begin{aligned}
b) \quad g(f(x)) &= g(\sqrt{x}) \\
&= 2 - 3\sqrt{x}
\end{aligned}$$

Domain: $x \geq 0$

Domain: $x \geq 0$

Domain: $x \geq 0$

Exercise

Given $f(x) = 3x + 2$ and $g(x) = \sqrt{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

$$a) \quad f(g(x)) = f(\sqrt{x}) \quad \text{Domain: } x \geq 0$$

$$= 3\sqrt{x} + 2 \quad \text{Domain: } x \geq 0$$

$$\text{Domain: } x \geq 0$$

$$b) \quad g(f(x)) = g(3x + 2) \quad \text{Domain: } \mathbb{R}$$

$$= \sqrt{3x + 2} \quad \text{Domain: } x \geq -\frac{2}{3}$$

$$\text{Domain: } x \geq -\frac{2}{3}$$

Exercise

Given $f(x) = x^4$ and $g(x) = \sqrt[4]{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

$$a) \quad f(g(x)) = f(\sqrt[4]{x}) \quad \text{Domain: } x \geq 0$$

$$= (\sqrt[4]{x})^4$$

$$= x$$

$$\text{Domain: } \mathbb{R}$$

$$\text{Domain: } x \geq 0$$

$$b) \quad g(f(x)) = g(x^4) \quad \text{Domain: } \mathbb{R}$$

$$= \sqrt[4]{x^4}$$

$$= x$$

$$\text{Domain: } \mathbb{R}$$

$$\text{Domain: } \mathbb{R}$$

Exercise

Given $f(x) = x^n$ and $g(x) = \sqrt[n]{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

$$a) \quad f(g(x)) = f(\sqrt[n]{x})$$

$$\text{Domain: } \begin{cases} \text{If } n \text{ is even} & x \geq 0 \\ \text{If } n \text{ is odd} & \mathbb{R} \end{cases}$$

$$= (\sqrt[n]{x})^n$$

$$= x$$

$$\text{Domain: } \mathbb{R}$$

$$\text{Domain: } \begin{cases} \text{If } n \text{ is even} & x \geq 0 \\ \text{If } n \text{ is odd} & \mathbb{R} \end{cases}$$

$$b) \quad g(f(x)) = g(x^n)$$

$$\text{Domain: } \mathbb{R}$$

$$= \sqrt[n]{x^n}$$

$$= x$$

$$\text{Domain: } \mathbb{R}$$

$$\text{Domain: } \mathbb{R}$$

Exercise

Given $f(x) = x^2 - 3x$ and $g(x) = \sqrt{x+2}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

$$a) \quad f(g(x)) = f(\sqrt{x+2})$$

$$x+2 \geq 0 \Rightarrow x \geq -2$$

$$= (\sqrt{x+2})^2 - 3\sqrt{x+2}$$

$$= x+2 - 3\sqrt{x+2}$$

$$x+2 \geq 0 \Rightarrow x \geq -2$$

$$\text{Domain: } \{x \mid x \geq -2\}$$

$$b) \quad g(f(x)) = g(x^2 - 3x)$$

$$\mathbb{R}$$

$$= \sqrt{x^2 - 3x + 2}$$

$$x^2 - 3x + 2 \geq 0 \Rightarrow (x-1)(x-2) \geq 0 \Leftrightarrow x \leq 1, x \geq 2$$

$$\text{Domain: } \{x \mid x \leq 1, x \geq 2\}$$

Exercise

Given $f(x) = \sqrt{x-2}$ and $g(x) = \sqrt{x+5}$

- a) Find $(f \circ g)(x)$ and the domain of $f \circ g$
- b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

$$\begin{aligned} \text{a) } f(g(x)) &= f(\sqrt{x+5}) \\ &= \sqrt{\sqrt{x+5}-2} \end{aligned}$$

$$x+5 \geq 0 \Rightarrow x \geq -5$$

$$\sqrt{x+5}-2 \geq 0 \Rightarrow \sqrt{x+5} \geq 2$$

$$x+5 \geq 4$$

$$x \geq -1$$

$$\text{Domain: } \{x \mid x \geq -1\}$$

$$\begin{aligned} \text{b) } g(f(x)) &= g(\sqrt{x-2}) \\ &= \sqrt{\sqrt{x-2}+5} \end{aligned}$$

$$x-2 \geq 0 \Rightarrow x \geq 2$$

$$\sqrt{x-2}+5 \geq 0 \Rightarrow \sqrt{x-2} \geq -5 \quad \text{Always true when } x \geq 2$$

$$\text{Domain: } \{x \mid x \geq 2\}$$

Exercise

Given $f(x) = x^2 + 2$ and $g(x) = \sqrt{3-x}$

- a) Find $(f \circ g)(x)$ and the domain of $f \circ g$
- b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

$$\begin{aligned} \text{a) } f(g(x)) &= f(\sqrt{3-x}) \\ &= (\sqrt{3-x})^2 + 2 \\ &= 3 - x + 2 \\ &= 5 - x \end{aligned}$$

$$\text{Domain: } x \leq 3$$

$$\text{Domain: } x \leq 3$$

$$\begin{aligned} \text{b) } g(f(x)) &= g(x^2 + 2) \\ &= \sqrt{3 - x^2 - 2} \\ &= \sqrt{1 - x^2} \end{aligned}$$

$$\text{Domain: } \mathbb{R}$$

$$\text{Domain: } -1 \leq x \leq 1$$

$$\text{Domain: } -1 \leq x \leq 1$$

Exercise

Given $f(x) = x^5 - 2$ and $g(x) = \sqrt[5]{x+2}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

$$a) \quad f(g(x)) = f\left(\sqrt[5]{x+2}\right) \quad \text{Domain: } \mathbb{R}$$

$$= \left(\sqrt[5]{x+2}\right)^5 - 2$$

$$= x + 2 - 2$$

$$= x$$

Domain: \mathbb{R}

Domain: \mathbb{R}

$$b) \quad g(f(x)) = g(x^5 - 2) \quad \text{Domain: } \mathbb{R}$$

$$= \sqrt[5]{x^5 - 2 + 2}$$

$$= \sqrt[5]{x^5}$$

$$= x$$

Domain: \mathbb{R}

Domain: \mathbb{R}

Exercise

Given $f(x) = 1 - x^2$ and $g(x) = \sqrt{x^2 - 25}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

$$a) \quad f(g(x)) = f\left(\sqrt{x^2 - 25}\right) \quad \text{Domain: } x \leq -5 \quad x \geq 5$$

$$= 1 - \left(\sqrt{x^2 - 25}\right)^2$$

$$= 1 - (x^2 - 25)$$

$$= 1 - x^2 + 25$$

$$= 26 - x^2$$

Domain: \mathbb{R}

Domain: $x \leq -5 \quad x \geq 5$

$$b) \quad g(f(x)) = g(1-x^2) \quad \text{Domain: } \mathbb{R}$$

$$= \sqrt{(1-x^2)^2 - 25}$$

$$= \sqrt{1-2x^2+x^4-25}$$

$$= \sqrt{x^4-2x^2-24}$$

$$x^2 = \frac{2 \pm \sqrt{4+96}}{2}$$

$$= \begin{cases} \frac{2-10}{2} = -4 \text{ } \times \\ \frac{2+10}{2} = 6 \end{cases}$$

$$x^2 = 6 \rightarrow x = \pm\sqrt{6}$$

$$\text{Domain: } x \leq -\sqrt{6} \quad x \geq \sqrt{6}$$

$$\text{Domain: } \underline{x \leq -\sqrt{6} \quad x \geq \sqrt{6}}$$

Exercise

$$\text{Given } f(x) = 2x+3 \quad \text{and} \quad g(x) = \frac{x-3}{2}$$

$$a) \quad \text{Find } (f \circ g)(x) \text{ and the domain of } f \circ g$$

$$b) \quad \text{Find } (g \circ f)(x) \text{ and the domain of } g \circ f$$

Solution

$$a) \quad f(g(x)) = f\left(\frac{x-3}{2}\right) \quad \text{Domain: } \mathbb{R}$$

$$= 2\left(\frac{x-3}{2}\right) + 3$$

$$= x-3+3$$

$$= x$$

$$\text{Domain: } \mathbb{R}$$

$$\text{Domain: } \underline{\mathbb{R}}$$

$$b) \quad g(f(x)) = g(2x+3) \quad \text{Domain: } \mathbb{R}$$

$$= \frac{1}{2}(2x+3-3)$$

$$= x$$

$$\text{Domain: } \mathbb{R}$$

$$\text{Domain: } \underline{\mathbb{R}}$$

Exercise

Given $f(x) = 4x - 5$ and $g(x) = \frac{x+5}{4}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

$$a) \quad f(g(x)) = f\left(\frac{x+5}{4}\right) \quad \text{Domain: } \mathbb{R}$$

$$= 4\left(\frac{x+5}{4}\right) - 5$$

$$= x + 5 - 5$$

$$= x$$

Domain: \mathbb{R}

Domain: \mathbb{R}

$$b) \quad g(f(x)) = g(4x - 5) \quad \text{Domain: } \mathbb{R}$$

$$= \frac{1}{4}(4x - 5 + 5)$$

$$= x$$

Domain: \mathbb{R}

Domain: \mathbb{R}

Exercise

Given $f(x) = \frac{4}{1-5x}$ and $g(x) = \frac{1}{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

$$a) \quad f(g(x)) = f\left(\frac{1}{x}\right) \quad \text{Domain: } x \neq 0$$

$$= \frac{4}{1-5\frac{1}{x}}$$

$$= \frac{4x}{x-5}$$

Domain: $x \neq 5$

Domain: $x \neq 0, 5$

$$b) \quad g(f(x)) = g\left(\frac{4}{1-5x}\right) \quad \text{Domain: } x \neq \frac{1}{5}$$

$$= \frac{1-5x}{4}$$

Domain: \mathbb{R}

Domain: $x \neq \frac{1}{5}$

Exercise

Given $f(x) = \frac{1}{x-2}$ and $g(x) = \frac{x+2}{x}$

- a) Find $(f \circ g)(x)$ and the domain of $f \circ g$
b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\frac{x+2}{x}\right)$ **Domain:** $x \neq 0$

$$= \frac{1}{\frac{x+2}{x} - 2}$$

$$= \frac{1}{x+2-2x}$$

$$= \frac{x}{2-x}$$

Domain: $x \neq 2$

Domain: $x \neq 0, 2$

$(-\infty, 0) \cup (0, 2) \cup (2, \infty)$

b) $g(f(x)) = g\left(\frac{1}{x-2}\right)$ **Domain:** $x \neq 2$

$$= \frac{\frac{1}{x-2} + 2}{\frac{1}{x-2}}$$

$$= \frac{1+2x-4}{x-2}$$

$$= \frac{1}{x-2}$$

$$= 2x-3$$

Domain: \mathbb{R}

Domain: $x \neq 2$

$(-\infty, 2) \cup (2, \infty)$

Exercise

Given $f(x) = \frac{3x+5}{2}$ and $g(x) = \frac{2x-5}{3}$

- a) Find $(f \circ g)(x)$ and the domain of $f \circ g$
b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\frac{2x-5}{3}\right)$ **Domain:** \mathbb{R}

$$= \frac{3 \frac{2x-5}{3} + 5}{2}$$

$$\begin{aligned}
 &= \frac{2x-5+5}{2} \\
 &= \frac{2x}{2} \\
 &= x
 \end{aligned}$$

Domain: \mathbb{R}

$$\begin{aligned}
 b) \quad g(f(x)) &= g\left(\frac{3x+5}{2}\right) \\
 &= \frac{2 \cdot \frac{3x+5}{2} - 5}{3} \\
 &= \frac{3x+5-5}{3} \\
 &= \frac{3x}{3} \\
 &= x
 \end{aligned}$$

Domain: \mathbb{R}

Domain: \mathbb{R}

Exercise

Given $f(x) = \frac{1}{1+x}$ and $g(x) = \frac{1-x}{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

$$\begin{aligned}
 a) \quad f(g(x)) &= f\left(\frac{1-x}{x}\right) \\
 &= \frac{1}{1 + \frac{1-x}{x}} \\
 &= \frac{x}{x+1-x} \\
 &= x
 \end{aligned}$$

Domain: $x \neq 0$

Domain: $x \neq 0$

$$\begin{aligned}
 b) \quad g(f(x)) &= g\left(\frac{1}{x+1}\right) \\
 &= \frac{1 - \frac{1}{x+1}}{\frac{1}{x+1}} \\
 &= x+1-1 \\
 &= x
 \end{aligned}$$

Domain: \mathbb{R}

Domain: $x \neq -1$

Domain: \mathbb{R}

Domain: \mathbb{R}

Exercise

Given $f(x) = \frac{x-1}{x-2}$ and $g(x) = \frac{x-3}{x-4}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

$$a) \quad f(g(x)) = f\left(\frac{x-3}{x-4}\right) \quad \text{Domain: } x \neq 4$$

$$\begin{aligned} &= \frac{\frac{x-3}{x-4} - 1}{\frac{x-3}{x-4} - 2} \\ &= \frac{\frac{x-3-(x-4)}{x-4}}{\frac{x-3-2(x-4)}{x-4}} \\ &= \frac{x-3+x+4}{x-3-2x+8} \end{aligned}$$

$$= \frac{2x+1}{-x+5} \quad \text{Domain: } x \neq 5$$

$$\text{Domain: } \{x \mid x \neq 4, 5\}$$

$$b) \quad g(f(x)) = g\left(\frac{x-1}{x-2}\right) \quad \text{Domain: } x \neq 2$$

$$\begin{aligned} &= \frac{\frac{x-1}{x-2} - 3}{\frac{x-1}{x-2} - 4} \\ &= \frac{\frac{x-1-3(x-2)}{x-2}}{\frac{x-1-4(x-2)}{x-2}} \\ &= \frac{x-1-3x+6}{x-1-4x+8} \end{aligned}$$

$$= \frac{-2x+5}{-3x+7} \quad \text{Domain: } x \neq \frac{7}{3}$$

$$\text{Domain: } \left\{x \mid x \neq 2, \frac{7}{3}\right\}$$

Exercise

Given $f(x) = \frac{6}{x-3}$ and $g(x) = \frac{1}{x}$

- a) Find $(f \circ g)(x)$ and the domain of $f \circ g$
b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $(f \circ g)(x)$

$$f(g(x)) = f\left(\frac{1}{x}\right) \quad \text{Domain: } x \neq 0$$

$$= \frac{6}{\frac{1}{x} - 3}$$

$$= \frac{6}{\frac{1-3x}{x}}$$

$$= \frac{6x}{1-3x} \quad \text{Domain: } x \neq \frac{1}{3}$$

$$\text{Domain: } \underline{x \neq 0, \frac{1}{3}} \quad \underline{(-\infty, 0) \cup \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)}$$

b) $(g \circ f)(x)$

$$g(f(x)) = g\left(\frac{6}{x-3}\right) \quad \text{Domain: } x \neq 3$$

$$= \frac{1}{\frac{6}{x-3}}$$

$$= \frac{x-3}{6} \quad \text{Domain: } (-\infty, \infty)$$

$$\text{Domain: } \underline{x \neq 3} \quad \underline{(-\infty, 3) \cup (3, \infty)}$$

Exercise

Given $f(x) = \frac{6}{x}$ and $g(x) = \frac{1}{2x+1}$

- a) Find $(f \circ g)(x)$ and the domain of $f \circ g$
b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\frac{1}{2x+1}\right) \quad \text{Domain: } x \neq -\frac{1}{2}$

$$= \frac{6}{\frac{1}{2x+1}}$$

$$= \underline{12x+6} \quad \text{Domain: } \mathbb{R}$$

$$\text{Domain: } \underline{x \neq -\frac{1}{2}} \mid$$

$$b) \quad g(f(x)) = g\left(\frac{6}{x}\right) \quad \text{Domain: } x \neq 0$$

$$= \frac{1}{2\frac{6}{x} + 1}$$

$$= \underline{\frac{x}{12+x}} \mid$$

$$\text{Domain: } x \neq -12$$

$$\text{Domain: } \underline{x \neq -12, 0} \mid$$

Exercise

$$\text{Given } f(x) = 3x - 7 \quad \text{and} \quad g(x) = \frac{x+7}{3}$$

$$a) \quad \text{Find } (f \circ g)(x) \text{ and the domain of } f \circ g$$

$$b) \quad \text{Find } (g \circ f)(x) \text{ and the domain of } g \circ f$$

Solution

$$a) \quad f(g(x)) = f\left(\frac{x+7}{3}\right) \quad \text{Domain: } \mathbb{R}$$

$$= 3\frac{x+7}{3} - 7$$

$$= x + 7 - 7$$

$$= \underline{x} \mid$$

$$\text{Domain: } \mathbb{R}$$

$$\text{Domain: } \underline{\mathbb{R}} \mid$$

$$b) \quad g(f(x)) = g(3x - 7) \quad \text{Domain: } \mathbb{R}$$

$$= \frac{3x - 7 + 7}{3}$$

$$= \underline{x} \mid$$

$$\text{Domain: } \mathbb{R}$$

$$\text{Domain: } \underline{\mathbb{R}} \mid$$

Exercise

$$\text{Given } f(x) = \frac{2x+3}{x-4} \quad \text{and} \quad g(x) = \frac{4x+3}{x-2}$$

$$a) \quad \text{Find } (f \circ g)(x) \text{ and the domain of } f \circ g$$

$$b) \quad \text{Find } (g \circ f)(x) \text{ and the domain of } g \circ f$$

Solution

$$a) \quad f(g(x)) = f\left(\frac{4x+3}{x-2}\right) \quad \text{Domain: } x \neq 2$$

$$\begin{aligned} &= \frac{2\frac{4x+3}{x-2} + 3}{\frac{4x+3}{x-2} - 4} \\ &= \frac{8x+6+3x-6}{4x+3-4x+8} \\ &= \frac{11x}{11} \\ &= x \end{aligned}$$

Domain: \mathbb{R}

Domain: $x \neq 2$

$$b) \quad g(f(x)) = g\left(\frac{2x+3}{x-4}\right) \quad \text{Domain: } x \neq 4$$

$$\begin{aligned} &= \frac{4\frac{2x+3}{x-4} + 3}{\frac{2x+3}{x-4} - 2} \\ &= \frac{8x+12+3x-4}{2x+3-2x+8} \\ &= \frac{11x}{11} \\ &= x \end{aligned}$$

Domain: \mathbb{R}

Domain: $x \neq 4$

Exercise

Given $f(x) = \frac{2x+3}{x+4}$ and $g(x) = \frac{-4x+3}{x-2}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

$$a) \quad f(g(x)) = f\left(\frac{-4x+3}{x-2}\right) \quad \text{Domain: } x \neq 2$$

$$\begin{aligned} &= \frac{2\frac{-4x+3}{x-2} + 3}{\frac{-4x+3}{x-2} + 4} \\ &= \frac{-8x+6+3x-6}{-4x+3+4x-8} \\ &= \frac{-5x}{-5} \\ &= x \end{aligned}$$

Domain: \mathbb{R}

Domain: $x \neq 2$ |

b) $g(f(x)) = g\left(\frac{2x+3}{x+4}\right)$ **Domain:** $x \neq -4$

$$\begin{aligned} &= \frac{-4\frac{2x+3}{x+4} + 3}{\frac{2x+3}{x+4} - 2} \\ &= \frac{-8x-12+3x+12}{2x+3-2x-8} \\ &= \frac{-5x}{-5} \\ &= x \end{aligned}$$

Domain: \mathbb{R}

Domain: $x \neq -4$ |

Exercise

Given $f(x) = x+1$ and $g(x) = x^3 - 5x^2 + 3x + 7$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f(x^3 - 5x^2 + 3x + 7)$ **Domain:** \mathbb{R}

$$= x^3 - 5x^2 + 3x + 7 + 1$$

$$= x^3 - 5x^2 + 3x + 8$$
 Domain: \mathbb{R}

Domain: \mathbb{R} |

b) $g(f(x)) = g(x+1)$ **Domain:** \mathbb{R}

$$= (x+1)^3 - 5(x+1)^2 + 3(x+1) + 7$$

$$= x^3 + 3x^2 + 3x + 1 - 5(x^2 + 2x + 1) + 3x + 3 + 7$$

$$= x^3 + 3x^2 + 6x + 11 - 5x^2 - 10x - 5$$

$$= x^3 - 2x^2 - 4x + 6$$
 Domain: \mathbb{R}

Domain: \mathbb{R} |

Exercise

Given $f(x) = x - 1$ and $g(x) = x^3 + 2x^2 - 3x - 9$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

$$a) \quad f(g(x)) = f(x^3 + 2x^2 - 3x - 9) \quad \text{Domain: } \mathbb{R}$$

$$= x^3 + 2x^2 - 3x - 9 - 1$$

$$= \underline{x^3 + 2x^2 - 3x - 10} \quad \text{Domain: } \mathbb{R}$$

$$\text{Domain: } \mathbb{R}$$

$$b) \quad g(f(x)) = g(x - 1) \quad \text{Domain: } \mathbb{R}$$

$$= (x - 1)^3 + 2(x - 1)^2 - (x - 1) - 9$$

$$= x^3 - 3x^2 + 3x - 1 + 2(x^2 - 2x + 1) - 3x + 3 - 9$$

$$= x^3 - 3x^2 - 7 + 2x^2 - 4x + 2$$

$$= \underline{x^3 - x^2 - 4x - 5} \quad \text{Domain: } \mathbb{R}$$

$$\text{Domain: } \mathbb{R}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ and $g(x) = x^2 - 5x$: $(f \circ g)(4)$

Solution

$$(f \circ g)(4) = f(g(4))$$

$$= f(16 - 20)$$

$$= f(-4)$$

$$= -8 - 3$$

$$= \underline{-11}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ and $g(x) = x^2 - 5x$: $(g \circ f)(4)$

Solution

$$(g \circ f)(4) = g(f(4))$$

$$\begin{aligned}
 &= g(8-3) \\
 &= g(5) \\
 &= 25-25 \\
 &= 0
 \end{aligned}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ and $g(x) = x^2 - 5x$: $(f \circ g)(-2)$

Solution

$$\begin{aligned}
 (f \circ g)(-2) &= f(g(-2)) \\
 &= f(4+10) \\
 &= f(14) \\
 &= 28-3 \\
 &= 25
 \end{aligned}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ and $g(x) = x^2 - 5x$: $(g \circ f)(-2)$

Solution

$$\begin{aligned}
 (g \circ f)(-2) &= g(f(-2)) \\
 &= g(-4-3) \\
 &= g(-7) \\
 &= 49+35 \\
 &= 84
 \end{aligned}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ and $g(x) = x^2 - 5x$: $(f \circ f)(-3)$

Solution

$$\begin{aligned}
 (f \circ f)(-3) &= f(f(-3)) \\
 &= f(-6-3) \\
 &= f(-9) \\
 &= -18-3 \\
 &= -21
 \end{aligned}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ and $g(x) = x^2 - 5x$: $(g \circ g)(7)$

Solution

$$\begin{aligned}(g \circ g)(7) &= g(g(7)) \\ &= g(49 - 35) \\ &= g(14) \\ &= 196 - 70 \\ &= 126\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ and $g(x) = x^2 - 5x$: $(f \circ g)(\sqrt{2})$

Solution

$$\begin{aligned}(f \circ g)(\sqrt{2}) &= f(g(\sqrt{2})) \\ &= f(2 - 5\sqrt{2}) \\ &= 2(2 - 5\sqrt{2}) - 3 \\ &= 4 - 10\sqrt{2} - 3 \\ &= 1 - 10\sqrt{2}\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ and $g(x) = x^2 - 5x$: $(g \circ f)(\sqrt{3})$

Solution

$$\begin{aligned}(g \circ f)(\sqrt{3}) &= g(f(\sqrt{3})) \\ &= g(2\sqrt{3} - 3) \\ &= (2\sqrt{3} - 3)^2 - 5(2\sqrt{3} - 3) \\ &= 12 - 12\sqrt{3} + 9 - 10\sqrt{3} + 15 \\ &= 36 - 22\sqrt{3}\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ and $g(x) = x^2 - 5x$: $(f \circ g)(2a)$

Solution

$$\begin{aligned}(f \circ g)(2a) &= f(g(2a)) \\&= f(4a^2 - 10a) \\&= 2(4a^2 - 10a) - 3 \\&= \underline{8a^2 - 20a - 3}\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ and $g(x) = x^2 - 5x$: $(g \circ f)(3b)$

Solution

$$\begin{aligned}(g \circ f)(3b) &= g(f(3b)) \\&= g(6b - 3) \\&= (6b - 3)^2 - 5(6b - 3) \\&= 36b^2 - 36b + 9 - 30b + 15 \\&= \underline{36b^2 - 66b + 24}\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ and $g(x) = x^2 - 5x$: $(f \circ g)(k + 1)$

Solution

$$\begin{aligned}(f \circ g)(k + 1) &= f(g(k + 1)) \\&= f((k + 1)^2 - 5k - 5) \\&= 2((k + 1)^2 - 5k - 5) - 3 \\&= 2(k^2 + 2k + 1) - 10k - 10 - 3 \\&= 2k^2 + 4k + 2 - 10k - 13 \\&= \underline{2k^2 - 6k - 11}\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ and $g(x) = x^2 - 5x$: $(g \circ f)(k - 1)$

Solution

$$\begin{aligned}(g \circ f)(k - 1) &= g(f(k - 1)) \\&= g(2k - 2 - 3) \\&= g(2k - 5) \\&= (2k - 5)^2 - 5(2k - 5) \\&= 4k^2 - 20k + 25 - 10k + 25 \\&= \underline{4k^2 - 30k + 50}\end{aligned}$$

Solution ***Section 2.4 – Properties of Division***

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x) = 2x^4 - x^3 + 7x - 12$; $p(x) = x^2 - 3$

Solution

$$\begin{array}{r} \overline{2x^4-x^3+0x^2+7x-12} \\ \underline{2x^4-6x^2} \\ +6x^2+7x \\ \underline{-x^3+3x} \\ 6x^2+4x-12 \\ \underline{6x^2-18} \\ 4x+6 \end{array}$$

$$\underline{Q(x) = 2x^2 - x + 6; \quad R(x) = 4x + 6}$$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x) = 3x^3 + 2x - 4$; $p(x) = 2x^2 + 1$

Solution

$$\begin{array}{r} \overline{3x^3+0x^2+2x-4} \\ \underline{3x^3+\frac{3}{2}x} \\ \phantom{+\frac{3}{2}x}\frac{1}{2}x-4 \end{array}$$

$$\underline{Q(x) = \frac{3}{2}x; \quad R(x) = \frac{1}{2}x - 4}$$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x) = 7x + 2$; $p(x) = 2x^2 - x - 4$

Solution

$$\begin{array}{r} \frac{2}{7}x - \frac{11}{49} \\ 7x + 2 \overline{) 2x^2 - x - 4} \\ \underline{2x^2 + \frac{4}{7}x} \\ -\frac{11}{7}x - 4 \\ \underline{-\frac{11}{7}x - \frac{22}{49}} \\ -\frac{174}{49} \end{array}$$

$$\underline{Q(x) = \frac{2}{7}x - \frac{11}{49}} \quad \left| \quad \underline{R(x) = -\frac{174}{49}} \right|$$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x) = 9x + 4$; $p(x) = 2x - 5$

Solution

$$\begin{array}{r} \frac{9}{2} \\ 2x - 5 \overline{) 9x + 4} \\ \underline{9x - \frac{45}{2}} \\ -\frac{37}{2} \end{array}$$

$$\underline{Q(x) = \frac{9}{2}}; \quad \left| \quad \underline{R(x) = -\frac{37}{2}} \right|$$

Exercise

Use the remainder theorem to find $f(c)$: $f(x) = x^4 - 6x^2 + 4x - 8$; $c = -3$

Solution

$$\begin{aligned} f(-3) &= (-3)^4 - 6(-3)^2 + 4(-3) - 8 \\ &= 7 \end{aligned}$$

Exercise

Use the remainder theorem to find $f(c)$: $f(x) = x^4 + 3x^2 - 12$; $c = -2$

Solution

$$\begin{aligned} f(-2) &= (-2)^4 + 3(-2)^2 - 12 \\ &= 16 \end{aligned}$$

Exercise

Use the factor theorem to show that $x - c$ is a factor of $f(x)$: $f(x) = x^3 + x^2 - 2x + 12$; $c = -3$

Solution

$$\begin{aligned} f(-3) &= (-3)^3 + (-3)^2 - 2(-3) + 12 \\ &= 0 \end{aligned}$$

From the factor theorem; $x + 3$ is a factor of $f(x)$.

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $2x^3 - 3x^2 + 4x - 5$; $x - 2$

Solution

$$\begin{array}{r|rrrr} 2 & 2 & -3 & 4 & -5 \\ & & 4 & 2 & 12 \\ \hline & 2 & 1 & 6 & \boxed{7} \end{array}$$

$$\underline{Q(x) = 2x^2 + x + 6 \quad R(x) = 7}$$

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $5x^3 - 6x^2 + 15$; $x - 4$

Solution

$$\begin{array}{r|rrrr} 4 & 5 & -6 & 0 & 15 \\ & & 20 & 56 & 224 \\ \hline & 5 & 14 & 56 & \boxed{239} \end{array}$$

$$\underline{Q(x) = 5x^2 + 14x + 56 \quad R(x) = 239}$$

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $9x^3 - 6x^2 + 3x - 4$; $x - \frac{1}{3}$

Solution

$$\begin{array}{r|rrrr} \frac{1}{3} & 9 & -6 & 3 & -4 \\ & & 3 & -1 & \frac{2}{3} \\ \hline & 9 & -3 & 2 & \boxed{-\frac{10}{3}} \end{array}$$

$$\underline{Q(x) = 9x^2 - 3x + 2 \quad R(x) = -\frac{10}{3} \quad |}$$

Exercise

Use the synthetic division to find $f(c)$: $f(x) = 2x^3 + 3x^2 - 4x + 4$; $c = 3$

Solution

$$\begin{array}{r|rrrr} 3 & 2 & 3 & -4 & 4 \\ & & 9 & 36 & 93 \\ \hline & 3 & 12 & 32 & \boxed{97} \end{array}$$

$$\underline{f(3) = 97 \quad |}$$

Exercise

Use the synthetic division to find $f(c)$: $f(x) = 8x^5 - 3x^2 + 7$; $c = \frac{1}{2}$

Solution

$$\begin{array}{r|rrrrrr} \frac{1}{2} & 8 & 0 & 0 & -3 & 0 & 7 \\ & & 4 & 2 & 1 & -1 & -\frac{1}{2} \\ \hline & 8 & 4 & 2 & -2 & -1 & \boxed{\frac{13}{2}} \end{array}$$

$$\underline{f\left(\frac{1}{2}\right) = \frac{13}{2} \quad |}$$

Exercise

Use the synthetic division to find $f(c)$: $f(x) = x^3 - 3x^2 - 8$; $c = 1 + \sqrt{2}$

Solution

$$\begin{array}{r|rrrr}
 1 + \sqrt{2} & 3 & -3 & 0 & -8 \\
 & & 3 + 3\sqrt{2} & 6 + 3\sqrt{2} & 12 + 9\sqrt{2} \\
 \hline
 & 3 & 3\sqrt{2} & 6 + 3\sqrt{2} & \boxed{4 + 9\sqrt{2}}
 \end{array}$$

$$\underline{f(1 + \sqrt{2}) = 4 + 9\sqrt{2}}$$

Exercise

Use the synthetic division to show that c is a zero of $f(x)$: $f(x) = 3x^4 + 8x^3 - 2x^2 - 10x + 4$; $c = -2$

Solution

$$\begin{array}{r|rrrrr}
 -2 & 3 & 8 & -2 & -10 & 4 \\
 & & -6 & -4 & 12 & -4 \\
 \hline
 & 3 & 2 & -6 & 2 & \boxed{0}
 \end{array}$$

$$\underline{f(-2) = 0}$$

Exercise

Use the synthetic division to show that c is a zero of $f(x)$: $f(x) = 27x^4 - 9x^3 + 3x^2 + 6x + 1$; $c = -\frac{1}{3}$

Solution

$$\begin{array}{r|rrrrr}
 -\frac{1}{3} & 27 & -9 & 3 & 6 & 1 \\
 & & -9 & 6 & -3 & -1 \\
 \hline
 & 27 & -18 & 9 & 3 & \boxed{0}
 \end{array}$$

$$\underline{f\left(-\frac{1}{3}\right) = 0}$$

Exercise

Find all values of k such that $f(x)$ is divisible by the given linear polynomial:

$$f(x) = kx^3 + x^2 + k^2x + 3k^2 + 11; \quad x + 2$$

Solution

$$\begin{array}{r|rrrr}
 -2 & k & 1 & k^2 & 3k^2 + 11 \\
 & & -2k & 4k - 2 & -2k^2 - 8k + 4 \\
 \hline
 & k & 1 - 2k & k^2 + 4k - 2 & k^2 - 8k + 15
 \end{array}$$

$$k^2 - 8k + 15 = 0 \Rightarrow \boxed{k = 3, 5}$$

Exercise

Find all solutions of the equation: $x^3 - x^2 - 10x - 8 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{8}{1} \right\} = \pm \{1, 2, 4, 8\}$

$$\begin{array}{r|rrrr}
 -1 & 1 & -1 & -10 & -8 \\
 & & -1 & 2 & 8 \\
 \hline
 & 1 & -2 & -8 & 0 \rightarrow x^2 - 2x - 8 = 0
 \end{array}$$

The solutions are: $\boxed{x = -1, -2, 4}$

Exercise

Find all solutions of the equation: $x^3 + x^2 - 14x - 24 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{24}{1} \right\} = \pm \{1, 2, 3, 4, 6, 8, 12, 24\}$

$$\begin{array}{r|rrrr}
 -2 & 1 & 1 & -14 & -24 \\
 & & -2 & 2 & 24 \\
 \hline
 & 1 & -1 & -12 & 0 \rightarrow x^2 - x - 12 = 0
 \end{array}$$

The solutions are: $\boxed{x = -2, -3, 4}$

Exercise

Find all solutions of the equation: $2x^3 - 3x^2 - 17x + 30 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{30}{2} \right\} = \pm \left\{ 1, 2, 3, 5, 6, 10, 15, 30, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{15}{2} \right\}$

$$\begin{array}{r|rrrr} 2 & 2 & -3 & -17 & 30 \\ & & 4 & 2 & -30 \\ \hline & 2 & 1 & -15 & \boxed{0} \end{array} \rightarrow 2x^2 + x - 15 = 0$$

The solutions are: $x = 2, -3, \frac{5}{2}$

Exercise

Find all solutions of the equation: $12x^3 + 8x^2 - 3x - 2 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{2}{12} \right\} = \pm \left\{ 1, 2, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{2}, \frac{1}{6}, \frac{1}{12} \right\}$

$$\begin{array}{r|rrrr} \frac{1}{2} & 12 & 8 & -3 & -2 \\ & & 6 & 7 & 2 \\ \hline & 12 & 14 & 4 & \boxed{0} \end{array} \rightarrow 12x^2 + 14x + 4 = 0$$

The solutions are: $x = \frac{1}{2}, -\frac{1}{2}, -\frac{2}{3}$

Exercise

Find all solutions of the equation: $x^3 + x^2 - 6x - 8 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{8}{1} \right\} = \pm \{1, 2, 4, 8\}$

$$\begin{array}{r|rrrr} -2 & 1 & 1 & -6 & -8 \\ & & -2 & 2 & 8 \\ \hline & 1 & -1 & -4 & \boxed{0} \end{array} \rightarrow x^2 - x - 4 = 0$$

$$x = \frac{1 \pm \sqrt{1+16}}{2}$$

The solutions are: $x = -2, \frac{1 \pm \sqrt{17}}{2}$

Exercise

Find all solutions of the equation: $x^3 - 19x - 30 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{30}{1} \right\} = \pm \{1, 2, 3, 5, 6, 15, 30\}$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -19 & -30 \\ & & -2 & 4 & 30 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array} \rightarrow x^2 - 2x - 15$$

$$x = \frac{2 \pm \sqrt{4 + 60}}{2}$$

$$= \begin{cases} \frac{2-8}{2} = -3 \\ \frac{2+8}{2} = 5 \end{cases}$$

The solutions are: $x = -2, -3, 5$

Exercise

Find all solutions of the equation: $2x^3 + x^2 - 25x + 12 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{12}{2} \right\} = \pm \left\{ 1, 2, 3, 4, 6, 12, \frac{1}{2}, \frac{3}{2} \right\}$

$$\begin{array}{r|rrrr} 3 & 2 & 1 & -25 & 12 \\ & & 6 & 21 & -12 \\ \hline & 2 & 7 & -4 & \boxed{0} \end{array} \rightarrow 2x^2 + 7x - 4$$

$$x = \frac{-7 \pm \sqrt{49 + 32}}{4}$$

$$= \begin{cases} \frac{-7-9}{4} = -4 \\ \frac{-7+9}{4} = \frac{1}{2} \end{cases}$$

The solutions are: $x = -4, \frac{1}{2}, 3$

Exercise

Find all solutions of the equation: $3x^3 + 11x^2 - 6x - 8 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{8}{3} \right\} = \pm \left\{ 1, 2, 4, 8, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3} \right\}$

$$\begin{array}{r|rrrr} 1 & 3 & 11 & -6 & -8 \\ & & 3 & 14 & 8 \\ \hline & 3 & 14 & 8 & \boxed{0} \end{array} \rightarrow 3x^2 + 14x + 8$$

$$x = \frac{-14 \pm \sqrt{196 - 96}}{6}$$

$$= \begin{cases} \frac{-14-10}{6} = -4 \\ \frac{-14+10}{6} = -\frac{2}{3} \end{cases}$$

The solutions are: $x = -4, -\frac{2}{3}, 1$

Exercise

Find all solutions of the equation: $2x^3 + 9x^2 - 2x - 9 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{9}{2} \right\} = \pm \left\{ 1, 3, 9, \frac{1}{2}, \frac{3}{2}, \frac{9}{2} \right\}$

$$\begin{array}{r|rrrr} 1 & 2 & 9 & -2 & -9 \\ & & 2 & 11 & 9 \\ \hline & 2 & 11 & 9 & \boxed{0} \end{array} \rightarrow 2x^2 + 11x + 9$$

$$x = -1, -\frac{9}{2} \quad a - b + c = 0 \rightarrow x = -1, -\frac{c}{a}$$

The solutions are: $x = -\frac{9}{2}, -1, 1$

Exercise

Find all solutions of the equation: $x^3 + 3x^2 - 6x - 8 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{8}{1} \right\} = \pm \{1, 2, 4, 8\}$

$$\begin{array}{r|rrrr} -1 & 1 & 3 & -6 & -8 \\ & & -1 & -2 & 8 \\ \hline & 1 & 2 & -8 & \boxed{0} \end{array} \rightarrow x^2 + 2x - 8 = 0$$

$$x = \frac{-2 \pm \sqrt{4 + 32}}{2}$$

$$= \begin{cases} \frac{-2-6}{2} = -4 \\ \frac{-2+6}{2} = 2 \end{cases}$$

The solutions are: $x = -4, -1, 2$

Exercise

Find all solutions of the equation: $3x^3 - x^2 - 6x + 2 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{2}{3} \right\} = \pm \left\{ 1, 2, \frac{1}{3}, \frac{2}{3} \right\}$

$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & -1 & -6 & 2 \\ & & 1 & 0 & -2 \\ \hline & 3 & 0 & -6 & \boxed{0} \end{array} \rightarrow 3x^2 - 6 = 0$$

$$x^2 = 2$$

The solutions are: $x = \frac{1}{3}, \pm\sqrt{2}$

Exercise

Find all solutions of the equation: $x^3 - 8x^2 + 8x + 24 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{24}{1} \right\} = \pm \{1, 2, 3, 4, 6, 8, 12, 24\}$

$$\begin{array}{r|rrrr} 6 & 1 & -8 & 8 & 24 \\ & & 6 & -12 & -24 \\ \hline & 1 & -2 & -4 & \boxed{0} \end{array} \rightarrow x^2 - 2x - 4 = 0$$

$$\begin{aligned} x &= \frac{2 \pm \sqrt{4 + 16}}{2} \\ &= \frac{2 \pm 2\sqrt{5}}{2} \end{aligned}$$

The solutions are: $x = 6, 1 \pm \sqrt{5}$

Exercise

Find all solutions of the equation: $x^3 - 7x^2 - 7x + 69 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{69}{1} \right\} = \pm \{1, 3, 23, 69\}$

$$\begin{array}{r|rrrr} -3 & 1 & -7 & -7 & 69 \\ & & -3 & 30 & -69 \\ \hline & 1 & -10 & 23 & \boxed{0} \end{array} \rightarrow x^2 - 10x + 23 = 0$$

$$x = \frac{10 \pm \sqrt{100 - 92}}{2}$$

$$= \frac{10 \pm 2\sqrt{2}}{2}$$

The solutions are: $x = -3, 5 \pm \sqrt{2}$

Exercise

Find all solutions of the equation: $x^3 - 3x - 2 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{2}{1} \right\} = \pm \{1, 2\}$

$$\begin{array}{c|cccc} -1 & 1 & 0 & -3 & -2 \\ & & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array} \rightarrow x^2 - x - 2 = 0$$

$$x = -1, 2 \quad a - b + c = 0 \rightarrow x = -1, -\frac{c}{a}$$

The solutions are: $x = -1, -1, 2$

Exercise

Find all solutions of the equation: $x^3 - 2x + 1 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \{1\}$

$$\begin{array}{c|cccc} 1 & 1 & 0 & -2 & 1 \\ & & 1 & 1 & 1 \\ \hline & 1 & 1 & -1 & \boxed{0} \end{array} \rightarrow x^2 + x - 1 = 0$$

$$x = \frac{-1 \pm \sqrt{5}}{2}$$

The solutions are: $x = 1, \frac{-1 \pm \sqrt{5}}{2}$

Exercise

Find all solutions of the equation: $x^3 - 2x^2 - 11x + 12 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{12}{1} \right\} = \pm \{1, 2, 3, 4, 6, 12\}$

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -11 & 12 \\ & & 1 & -1 & 12 \\ \hline & 1 & -1 & -12 & \boxed{0} \end{array} \rightarrow x^2 - x - 12 = 0$$

$$x = \frac{1 \pm \sqrt{1+48}}{2}$$

$$= \begin{cases} \frac{1-7}{2} = -3 \\ \frac{1+7}{2} = 4 \end{cases}$$

The solutions are: $x = -3, 1, 4$

Exercise

Find all solutions of the equation: $x^3 - 2x^2 - 7x - 4 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{4}{1} \right\} = \pm \{1, 2, 4\}$

$$\begin{array}{r|rrrr} -1 & 1 & -2 & -7 & -4 \\ & & -1 & 3 & 4 \\ \hline & 1 & -3 & -4 & \boxed{0} \end{array} \rightarrow x^2 - 3x - 4 = 0$$

$$x = -1, 4 \quad a - b + c = 0 \rightarrow x = -1, -\frac{c}{a}$$

The solutions are: $x = -1, -1, 4$

Exercise

Find all solutions of the equation: $x^3 - 10x - 12 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{12}{1} \right\} = \pm \{1, 2, 3, 4, 6, 12\}$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -10 & -12 \\ & & -2 & 4 & 12 \\ \hline & 1 & -2 & -6 & \boxed{0} \end{array} \rightarrow x^2 - 2x - 6 = 0$$

$$\begin{aligned} x &= \frac{2 \pm \sqrt{4 + 24}}{2} \\ &= \frac{2 \pm 2\sqrt{7}}{2} \end{aligned}$$

The solutions are: $x = -2, 1 \pm \sqrt{7}$

Exercise

Find all solutions of the equation: $x^3 - 5x^2 + 17x - 13 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{13}{1} \right\} = \pm \{1, 13\}$

$$\begin{array}{r|rrrr} 1 & 1 & -5 & 17 & -13 \\ & & 1 & -4 & 13 \\ \hline & 1 & -4 & 13 & \boxed{0} \end{array} \rightarrow x^2 - 4x + 13 = 0$$

$$\begin{aligned} x &= \frac{4 \pm \sqrt{16 - 52}}{2} \\ &= \frac{4 \pm 6i}{2} \end{aligned}$$

The solutions are: $x = 1, 2 \pm 3i$

Exercise

Find all solutions of the equation: $6x^3 + 25x^2 - 24x + 5 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{5}{6} \right\} = \pm \left\{ 1, 5, \frac{1}{2}, \frac{5}{2}, \frac{1}{3}, \frac{5}{3}, \frac{1}{6}, \frac{5}{6} \right\}$

$$\begin{array}{r|rrrr} -5 & 6 & 25 & -24 & 5 \\ & & -30 & 25 & -5 \\ \hline & 6 & -5 & 1 & \boxed{0} \end{array} \rightarrow 6x^2 - 5x + 1 = 0$$

$$x = \frac{5 \pm \sqrt{25 - 24}}{12}$$

$$= \begin{cases} \frac{5-1}{12} = \frac{1}{3} \\ \frac{5+1}{12} = \frac{1}{2} \end{cases}$$

The solutions are: $x = -5, \frac{1}{3}, \frac{1}{2}$

Exercise

Find all solutions of the equation: $8x^3 + 18x^2 + 45x + 27 = 0$

Solution

$$\begin{aligned} \text{possibilities: } \pm \left\{ \frac{27}{8} \right\} &= \pm \left\{ \frac{1, 3, 9, 27}{1, 2, 4, 8} \right\} \\ &= \pm \left\{ 1, 3, 9, 27, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{9}{2}, \frac{9}{4}, \frac{9}{8}, \frac{27}{2}, \frac{27}{4}, \frac{27}{8} \right\} \end{aligned}$$

$$\begin{array}{r|rrrr} -\frac{3}{4} & 8 & 18 & 45 & 27 \\ & & -6 & -9 & -27 \\ \hline & 8 & 12 & 36 & \boxed{0} \end{array} \rightarrow 8x^2 + 12x + 36 = 0$$

The solutions are: $x = -\frac{3}{4}, -\frac{3}{4} \pm i \frac{3\sqrt{7}}{4}$

Exercise

Find all solutions of the equation: $3x^3 - x^2 + 11x - 20 = 0$

Solution

$$\begin{aligned} \text{possibilities: } \pm \left\{ \frac{20}{3} \right\} &= \pm \left\{ \frac{1, 2, 4, 5, 10, 20}{1, 3} \right\} \\ &= \pm \left\{ 1, 2, 4, 5, 10, 20, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{10}{3}, \frac{20}{3} \right\} \end{aligned}$$

$$\begin{array}{r|rrrr} \frac{4}{3} & 3 & -1 & 11 & -20 \\ & & 4 & 4 & 20 \\ \hline & 3 & 3 & 15 & \boxed{0} \end{array} \rightarrow 3x^2 + 3x + 15 = 0$$

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{9 - 180}}{6} \\ &= \frac{-3 \pm 3\sqrt{19}}{6} \end{aligned}$$

The solutions are: $x = \frac{4}{3}, -\frac{1}{2} \pm i \frac{\sqrt{19}}{2}$

Exercise

Find all solutions of the equation: $x^4 - x^3 - 9x^2 + 3x + 18 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{18}{1} \right\} = \pm \{1, 2, 3, 6, 9, 18\}$

$$\begin{array}{r|rrrrr} -2 & 1 & -1 & -9 & 3 & 18 \\ & & -2 & 6 & 6 & -18 \\ \hline 3 & 1 & -3 & -3 & 9 & 0 \\ & & 3 & 0 & -9 & \\ \hline & 1 & 0 & -3 & 0 & \end{array} \rightarrow x^3 - 3x^2 - 3x + 9 = 0 \rightarrow \pm \left\{ \frac{9}{1} \right\} = \pm \{1, 3, 9\}$$
$$\rightarrow x^2 - 3 = 0 \Rightarrow x = \pm\sqrt{3}$$

The solutions are: $x = -2, 3, \pm\sqrt{3}$

Exercise

Find all solutions of the equation: $2x^4 - 9x^3 + 9x^2 + x - 3 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{3}{2} \right\} = \pm \left\{ 1, 3, \frac{1}{2}, \frac{3}{2} \right\}$

$$\begin{array}{r|rrrrr} 1 & 2 & -9 & 9 & 1 & -3 \\ & & 2 & -7 & 2 & 3 \\ \hline 1 & 2 & -7 & 2 & 3 & 0 \\ & & 2 & -5 & -3 & \\ \hline & 2 & -5 & -3 & 0 & \end{array} \rightarrow 2x^3 - 7x^2 + 2x + 3 = 0 \rightarrow \pm \left\{ \frac{3}{2} \right\} = \pm \left\{ 1, 3, \frac{1}{2}, \frac{3}{2} \right\}$$
$$\rightarrow 2x^2 - 5x - 3 = 0$$

$$x = \frac{5 \pm \sqrt{25 + 24}}{4}$$

$$= \begin{cases} \frac{5-7}{4} = -\frac{1}{2} \\ \frac{5+7}{4} = 3 \end{cases}$$

The solutions are: $x = 1, 1, -\frac{1}{2}, 3$

Exercise

Find all solutions of the equation: $6x^4 + 5x^3 - 17x^2 - 6x = 0$

Solution

$$x(6x^3 + 5x^2 - 17x - 6) = 0 \rightarrow \underline{x=0}$$

$$\text{possibilities: } \pm \left\{ \frac{6}{6} \right\} = \pm \left\{ 1, 2, 3, 6, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{2}{3} \right\}$$

$$\begin{array}{r|rrrr} -2 & 6 & 5 & -17 & -6 \\ & & -12 & 14 & 6 \\ \hline & 6 & -7 & -3 & \boxed{0} \end{array} \rightarrow 6x^2 - 7x - 3 = 0$$

$$x = \frac{7 \pm \sqrt{49 + 72}}{12}$$

$$= \begin{cases} \frac{7-11}{12} = -\frac{1}{3} \\ \frac{7+11}{12} = \frac{3}{2} \end{cases}$$

$$\text{The solutions are: } \underline{x=0, -2, -\frac{1}{3}, \frac{3}{2}}$$

Exercise

Find all solutions of the equation: $x^4 - 2x^2 - 16x - 15 = 0$

Solution

$$\text{possibilities: } \pm \left\{ \frac{15}{1} \right\} = \pm \{1, 3, 5, 15\}$$

$$\begin{array}{r|rrrrr} -1 & 1 & 0 & -2 & -16 & -15 \\ & & -1 & 1 & 1 & 15 \\ \hline 3 & 1 & -1 & -1 & -15 & \boxed{0} \\ & & 3 & 6 & 15 & \\ \hline & 1 & 2 & 5 & \boxed{0} & \end{array} \rightarrow x^3 - x^2 - x - 15 = 0 \rightarrow \pm \left\{ \frac{15}{1} \right\} = \pm \{1, 3, 5, 15\}$$
$$\rightarrow x^2 + 2x + 5 = 0$$

$$x = \frac{-2 \pm \sqrt{-16}}{2}$$

$$= -1 \pm 2i$$

$$\text{The solutions are: } \underline{x=-1, 3, -1 \pm 2i}$$

Exercise

Find all solutions of the equation: $x^4 - 2x^3 - 5x^2 + 8x + 4 = 0$

Solution

possibilities: $\pm \left\{ \frac{4}{1} \right\} = \pm \{1, 2, 4\}$

$$\begin{array}{r|rrrrr} 2 & 1 & -2 & -5 & 8 & 4 \\ & & 2 & 0 & -10 & -4 \\ \hline -2 & 1 & 0 & -5 & -2 & 0 \\ & & -2 & 4 & 2 & \\ \hline & 1 & -2 & -1 & 0 & \end{array} \rightarrow x^3 - 5x - 2 = 0 \rightarrow \pm \left\{ \frac{2}{1} \right\} = \pm \{1, 2\}$$
$$\rightarrow x^2 - 2x - 1 = 0$$

$$x = \frac{2 \pm \sqrt{8}}{2}$$
$$= \frac{2 \pm 2\sqrt{2}}{2}$$

The solutions are: $x = -2, 2, 1 \pm \sqrt{2}$

Exercise

Find all solutions of the equation: $2x^4 - 17x^3 + 4x^2 + 35x - 24 = 0$

Solution

possibilities: $\pm \left\{ \frac{24}{2} \right\} = \pm \left\{ 1, 2, 3, 4, 6, 8, 12, 24, \frac{1}{2}, \frac{3}{2} \right\}$

$$\begin{array}{r|rrrrr} 1 & 2 & -17 & 4 & 35 & -24 \\ & & 2 & -15 & -11 & 24 \\ \hline 1 & 2 & -15 & -11 & 24 & 0 \\ & & 2 & -13 & 24 & \\ \hline & 2 & -13 & -24 & 0 & \end{array} \rightarrow 2x^3 - 15x^2 - 11x + 24 = 0 \rightarrow \pm \left\{ \frac{2}{1} \right\} = \pm \{1, 2\}$$
$$\rightarrow 2x^2 - 13x - 24 = 0$$

$$x = \frac{13 \pm \sqrt{169 + 192}}{4}$$

$$= \begin{cases} \frac{13-19}{4} = -\frac{3}{2} \\ \frac{13+19}{4} = 8 \end{cases}$$

The solutions are: $x = -\frac{3}{2}, 1, 1, 8$

Exercise

Find all solutions of the equation: $x^4 + x^3 - 3x^2 - 5x - 2 = 0$

Solution

possibilities: $\pm \left\{ \frac{2}{1} \right\} = \pm \{1, 2\}$

$$\begin{array}{r|rrrrr} -1 & 1 & 1 & -3 & -5 & -2 \\ & & -1 & 0 & 3 & 2 \\ \hline -1 & 1 & 0 & -3 & -2 & 0 \\ & & -1 & 1 & 2 & \\ \hline & 1 & -1 & -2 & 0 & \end{array} \rightarrow x^3 - 3x - 2 = 0 \rightarrow \pm \left\{ \frac{2}{1} \right\} = \pm \{1, 2\}$$
$$\rightarrow x^2 - x - 2 = 0$$

$$x = \frac{1 \pm \sqrt{9}}{2}$$
$$= \begin{cases} \frac{1-3}{2} = -1 \\ \frac{1+3}{2} = 2 \end{cases}$$

The solutions are: $x = -1, -1, -1, 2$

Exercise

Find all solutions of the equation: $6x^4 - 17x^3 - 11x^2 + 42x = 0$

Solution

$$x(6x^3 - 17x^2 - 11x + 42) = 0$$

$$x = 0 \quad 6x^3 - 17x^2 - 11x + 42 = 0$$

possibilities: $\pm \left\{ \frac{42}{6} \right\} = \pm \left\{ 1, 2, 3, 6, 7, 14, 21, 42, \frac{1}{2}, \frac{3}{2}, \frac{7}{2}, \frac{21}{2}, \frac{1}{3}, \frac{2}{3}, \frac{7}{3}, \frac{14}{3}, \frac{1}{6}, \frac{7}{6}, \frac{21}{6} \right\}$

$$\begin{array}{r|rrrr} 2 & 6 & -17 & -11 & 42 \\ & & 12 & -10 & -42 \\ \hline & 6 & -5 & -21 & 0 \end{array} \rightarrow 6x^2 - 5x - 21 = 0$$

$$x = \frac{5 \pm \sqrt{25 + 504}}{12}$$
$$= \begin{cases} \frac{5-23}{12} = -\frac{3}{2} \\ \frac{5+23}{12} = \frac{7}{3} \end{cases}$$

The solutions are: $x = -\frac{3}{2}, 0, 2, \frac{7}{3}$

Exercise

Find all solutions of the equation: $x^4 - 5x^2 - 2x = 0$

Solution

$$x(x^3 - 5x - 2) = 0$$

$$x = 0 \quad x^3 - 5x - 2 = 0$$

$$\text{possibilities: } \pm \left\{ \frac{2}{1} \right\} = \pm \{1, 2\}$$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -5 & -2 \\ & & -2 & 4 & 2 \\ \hline & 1 & -2 & -1 & \boxed{0} \end{array} \rightarrow x^2 - 2x - 1 = 0$$

$$x = \frac{2 \pm \sqrt{4 + 4}}{2}$$

$$= \frac{2 \pm 2\sqrt{2}}{2}$$

The solutions are: $x = -2, 0, 1 \pm \sqrt{2}$

Exercise

Find all solutions of the equation: $3x^4 - 4x^3 - 11x^2 + 16x - 4 = 0$

Solution

$$\text{possibilities: } \pm \left\{ \frac{4}{3} \right\} = \pm \left\{ 1, 2, 4, \frac{1}{3}, \frac{2}{3}, \frac{4}{3} \right\}$$

$$\begin{array}{r|rrrrr} 1 & 3 & -4 & -11 & 16 & -4 \\ & & 3 & -1 & -12 & 4 \\ \hline 2 & 3 & -1 & -12 & 4 & \boxed{0} \\ & & 6 & 10 & -4 & \\ \hline & 3 & 5 & -2 & \boxed{0} & \end{array} \rightarrow 3x^3 - x^2 - 12x + 4 = 0 \rightarrow \pm \left\{ \frac{4}{3} \right\} = \pm \left\{ 1, 2, 4, \frac{1}{3}, \frac{2}{3}, \frac{4}{3} \right\}$$
$$\rightarrow 3x^2 + 5x - 2 = 0$$

$$x = \frac{-5 \pm \sqrt{25 + 24}}{6}$$

$$= \begin{cases} \frac{-5-7}{6} = -2 \\ \frac{-5+7}{6} = \frac{1}{3} \end{cases}$$

The solutions are: $x = -2, \frac{1}{3}, 1, 2$

Exercise

Find all solutions of the equation: $6x^4 + 23x^3 + 19x^2 - 8x - 4 = 0$

Solution

possibilities: $\pm \left\{ \frac{4}{6} \right\} = \pm \left\{ 1, 2, 4, \frac{1}{6}, \frac{2}{3}, \frac{2}{3} \right\}$

$$\begin{array}{r|rrrrr}
 -2 & 6 & 23 & 19 & -8 & -4 \\
 & & -12 & -22 & 6 & 4 \\
 \hline
 -2 & 6 & 11 & -3 & -2 & 0 \\
 & & -12 & 2 & 2 & \\
 \hline
 & 6 & -1 & -1 & 0 &
 \end{array} \rightarrow 6x^3 + 11x^2 - 3x - 2 = 0 \rightarrow \pm \left\{ \frac{2}{6} \right\} = \pm \left\{ 1, 2, \frac{1}{6}, \frac{1}{3} \right\}$$

$$\rightarrow 6x^2 - x - 1 = 0$$

$$\begin{aligned}
 x &= \frac{1 \pm \sqrt{25}}{12} \\
 &= \begin{cases} \frac{1-5}{12} = -\frac{1}{3} \\ \frac{1+5}{12} = \frac{1}{2} \end{cases}
 \end{aligned}$$

The solutions are: $x = -2, -2, -\frac{1}{3}, \frac{1}{2}$

Exercise

Find all solutions of the equation: $4x^4 - 12x^3 + 3x^2 + 12x - 7 = 0$

Solution

possibilities: $\pm \left\{ \frac{7}{4} \right\} = \pm \left\{ 1, 7, \frac{1}{2}, \frac{7}{2}, \frac{1}{4}, \frac{7}{4} \right\}$

$$\begin{array}{r|rrrrr}
 1 & 4 & -12 & 3 & 12 & -7 \\
 & & 4 & -8 & -5 & 7 \\
 \hline
 -1 & 4 & -8 & -5 & 7 & 0 \\
 & & -4 & 12 & -7 & \\
 \hline
 & 4 & -12 & 7 & 0 &
 \end{array} \rightarrow 4x^3 - 8x^2 - 5x + 7 = 0 \rightarrow \pm \left\{ \frac{7}{4} \right\} = \pm \left\{ 1, 7, \frac{1}{2}, \frac{7}{2}, \frac{1}{4}, \frac{7}{4} \right\}$$

$$\rightarrow 4x^2 - 12x + 7 = 0$$

$$\begin{aligned}
 x &= \frac{12 \pm \sqrt{144 - 112}}{8} \\
 &= \frac{12 \pm 4\sqrt{2}}{8}
 \end{aligned}$$

The solutions are: $x = -1, 1, \frac{3 \pm \sqrt{2}}{2}$

Exercise

Find all solutions of the equation: $2x^4 - 9x^3 - 2x^2 + 27x - 12 = 0$

Solution

$$\text{possibilities: } \pm \left\{ \frac{12}{2} \right\} = \pm \left\{ 1, 2, 3, 4, 6, 12, \frac{1}{2}, \frac{3}{2} \right\}$$

$$\begin{array}{r|rrrrr} 4 & 2 & -9 & -2 & 27 & -12 \\ & & 8 & -4 & -24 & 12 \\ \hline \frac{1}{2} & 2 & -1 & -6 & 3 & 0 \\ & & 1 & 0 & -3 & \\ \hline & 2 & 0 & -6 & 0 & \end{array} \rightarrow 2x^3 - x^2 - 6x + 3 = 0 \rightarrow \pm \left\{ \frac{3}{2} \right\} = \pm \left\{ 1, 3, \frac{1}{2}, \frac{3}{2} \right\}$$
$$\rightarrow 2x^2 - 6 = 0$$

$$x^2 = 3$$

$$\text{The solutions are: } x = \frac{1}{2}, 4, \pm\sqrt{3}$$

Exercise

Find all solutions of the equation: $2x^4 - 19x^3 + 51x^2 - 31x + 5 = 0$

Solution

$$\text{possibilities: } \pm \left\{ \frac{5}{2} \right\} = \pm \left\{ 1, 5, \frac{1}{2}, \frac{5}{2} \right\}$$

$$\begin{array}{r|rrrrr} 5 & 2 & -19 & 51 & -31 & 5 \\ & & 10 & -45 & 30 & -5 \\ \hline \frac{1}{2} & 2 & -9 & 6 & -1 & 0 \\ & & 1 & -4 & 1 & \\ \hline & 2 & -8 & 2 & 0 & \end{array} \rightarrow 2x^3 - 9x^2 + 6x - 1 = 0 \rightarrow \pm \left\{ \frac{1}{2} \right\}$$
$$\rightarrow 2x^2 - 8x + 2 = 0$$

$$x = \frac{8 \pm \sqrt{64 - 16}}{4}$$
$$= \frac{8 \pm 4\sqrt{3}}{4}$$

$$\text{The solutions are: } x = \frac{1}{2}, 5, 2 \pm \sqrt{3}$$

Exercise

Find all solutions of the equation: $4x^4 - 35x^3 + 71x^2 - 4x - 6 = 0$

Solution

$$\text{possibilities: } \pm \left\{ \frac{6}{4} \right\} = \pm \left\{ 1, 2, 3, 6, \frac{1}{2}, \frac{3}{2}, \frac{1}{4}, \frac{3}{4} \right\}$$

$$\begin{array}{r|rrrrr}
 3 & 4 & -35 & 71 & -4 & -6 \\
 & & 12 & -69 & 6 & 6 \\
 \hline
 -\frac{1}{4} & 4 & -23 & 2 & 2 & 0 \\
 & & -1 & 6 & -2 & \\
 \hline
 & 4 & -24 & 8 & 0 &
 \end{array} \rightarrow 4x^3 - 23x^2 + 2x + 2 = 0 \rightarrow \pm\left\{\frac{2}{4}\right\} = \pm\left\{1, 2, \frac{1}{2}, \frac{1}{4}\right\}$$

$$\rightarrow 4x^2 - 24x + 8 = 0$$

$$x^2 - 6x + 2 = 0$$

$$x = \frac{6 \pm \sqrt{36 - 8}}{2}$$

$$= \frac{6 \pm 2\sqrt{7}}{4}$$

The solutions are: $x = -\frac{1}{4}, 3, 3 \pm \sqrt{7}$

Exercise

Find all solutions of the equation: $2x^4 + 3x^3 - 4x^2 - 3x + 2 = 0$

Solution

$$\text{possibilities: } \pm\left\{\frac{2}{2}\right\} = \pm\left\{1, 2, \frac{1}{2}\right\}$$

$$\begin{array}{r|rrrrr}
 1 & 2 & 3 & -4 & -3 & 2 \\
 & & 2 & 5 & 1 & -2 \\
 \hline
 -1 & 2 & 5 & 1 & -2 & 0 \\
 & & -2 & -3 & 2 & \\
 \hline
 & 2 & 3 & -2 & 0 &
 \end{array} \rightarrow 2x^3 - 23x^2 + 2x - 2 = 0 \rightarrow \pm\left\{\frac{2}{2}\right\} = \pm\left\{1, 2, \frac{1}{2}\right\}$$

$$\rightarrow 2x^2 + 3x - 2 = 0$$

$$x = \frac{-3 \pm \sqrt{9 + 16}}{4}$$

$$= \begin{cases} \frac{-3 - 5}{4} = -2 \\ \frac{-3 + 5}{4} = \frac{1}{2} \end{cases}$$

The solutions are: $x = -2, -1, \frac{1}{2}, 1$

Exercise

Find all solutions of the equation: $x^4 + 3x^3 - 30x^2 - 6x + 56 = 0$

Solution

$$\text{possibilities for } \frac{c}{d} : \pm\left\{\frac{56}{1}\right\} = \pm\{1, 2, 4, 7, 8, 14, 28, 56\}$$

$$\begin{array}{r|rrrrr}
 4 & 1 & 3 & -3 & -6 & 56 \\
 & & 4 & 28 & -8 & -56 \\
 \hline
 -7 & 1 & 7 & -2 & -14 & 0 \\
 & & -7 & 0 & 14 & \\
 \hline
 & 1 & 0 & -2 & &
 \end{array} \rightarrow x^3 + 7x^2 - 2x - 14 = 0 \Rightarrow \frac{c}{d} = \pm \left\{ \frac{14}{1} \right\} = \pm \{1, 2, 7, 14\}$$

$$\rightarrow x^2 - 2 = 0$$

$$\Rightarrow x = \pm\sqrt{2}$$

The solutions are: $x = 4, -7, \pm\sqrt{2}$

Exercise

Find all solutions of the equation: $3x^5 - 10x^4 - 6x^3 + 24x^2 + 11x - 6 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{6}{3} \right\} = \pm \left\{ 1, 2, 3, 6, \frac{1}{3}, \frac{2}{3} \right\}$

$$\begin{array}{r|rrrrrr}
 -1 & 3 & -10 & -6 & 24 & 11 & -6 \\
 & & -3 & 13 & -7 & -17 & 6 \\
 \hline
 -1 & 3 & -13 & 7 & 17 & -6 & 0 \\
 & & -3 & 16 & -23 & 6 & \\
 \hline
 2 & 3 & -16 & 23 & -6 & 0 & \\
 & & 6 & 20 & 6 & & \\
 \hline
 & 3 & -10 & 3 & 0 & &
 \end{array} \quad \begin{array}{l} x^4 - 13x^3 + 7x^2 + 17x - 6 = 0 \rightarrow \pm \left\{ \frac{6}{3} \right\} = \pm \left\{ 1, 2, 3, 6, \frac{1}{3}, \frac{2}{3} \right\} \\ \\ 3x^3 - 16x^2 + 26x - 6 = 0 \rightarrow \pm \left\{ 1, 2, 3, 6, \frac{1}{3}, \frac{2}{3} \right\} \\ \\ 3x^2 - 10x + 3 = 0 \end{array}$$

$$x = \frac{10 \pm \sqrt{100 - 36}}{6}$$

$$= \begin{cases} \frac{10-8}{6} = \frac{1}{3} \\ \frac{10+8}{6} = 3 \end{cases}$$

The solutions are: $x = -1, -1, \frac{1}{3}, 2, 3$

Exercise

Find all solutions of the equation: $6x^5 + 19x^4 + x^3 - 6x^2 = 0$

Solution

$$x^2 (6x^3 + 19x^2 + x - 6) = 0 \rightarrow x = 0, 0$$

$$6x^3 + 19x^2 + x - 6 = 0$$

possibilities for $\frac{c}{d} : \pm \left\{ \frac{6}{6} \right\} = \pm \left\{ 1, 2, 3, 6, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{2}{3} \right\}$

$$\begin{array}{r|rrrr} -3 & 6 & 19 & 1 & -6 \\ & & -18 & -3 & 6 \\ \hline & 6 & 1 & -2 & \boxed{0} \end{array} \quad 6x^2 + x - 2 = 0$$

$$x = \frac{-1 \pm \sqrt{1+48}}{12}$$

$$= \begin{cases} \frac{-1-7}{12} = -\frac{2}{3} \\ \frac{-1+7}{12} = \frac{1}{2} \end{cases}$$

The solutions are: $x = 0, 0, -\frac{2}{3}, -3, \frac{1}{2}$

Exercise

Find all solutions of the equation: $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 = 0$

Solution

$$x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 = (x+1)^5 = 0$$

possibilities for $\frac{c}{d} : \pm \{1\}$

$$\begin{array}{r|rrrrrr} -1 & 1 & 5 & 10 & 10 & 5 & 1 \\ & & -1 & -4 & -6 & -4 & -1 \\ \hline -1 & 1 & 4 & 6 & 4 & 1 & \boxed{0} \\ & & -1 & -3 & -3 & -1 \\ \hline -1 & 1 & 3 & 3 & 1 & \boxed{0} \\ & & -1 & -2 & -1 \\ \hline & 1 & 2 & 1 & \boxed{0} \end{array} \quad \begin{aligned} & \rightarrow x^4 + 4x^3 + 6x^2 + 4x + 1 = 0 \rightarrow \pm \{1\} \\ & \rightarrow x^3 + 3x^2 + 3x + 1 = 0 \rightarrow \pm \{1\} \\ & \rightarrow x^2 + 2x + 1 = 0 \end{aligned}$$

$$x^2 + 2x + 1 = (x+1)^2$$

The solutions are: $x = -1, -1, -1, -1, -1$

Exercise

Find all solutions of the equation: $x^5 - x^4 - 7x^3 + 7x^2 + 12x - 12 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\{1, 2, 3, 4, 6, 12\}$

1	1	-1	-7	7	12	-12	
		1	0	-7	0	12	
2	1	0	-7	0	12	0	$\rightarrow x^4 - 7x^2 - 12 = 0 \rightarrow \pm\{1, 2, 3, 4, 6, 12\}$
		2	4	-6	-12		
-2	1	2	-3	-6	0		$\rightarrow x^3 + 2x^2 - 3x - 6 = 0 \rightarrow \pm\{1, 2, 3, 6\}$
		-2	0	6			
	1	0	-3	0			$\rightarrow x^2 - 3 = 0$

$$x^2 = 3$$

The solutions are: $x = -2, 1, 2, \pm\sqrt{3}$

Exercise

Find all solutions of the equation: $x^5 - 2x^3 - 8x = 0$

Solution

$$x(x^4 - 2x^2 - 8) = 0$$

$$x = 0$$

$$x^4 - 2x^2 - 8 = 0$$

$$x^2 = \frac{2 \pm \sqrt{4 + 32}}{2}$$

$$= \begin{cases} \frac{2-6}{2} = -2 \\ \frac{2+6}{2} = 4 \end{cases}$$

$$\begin{cases} x^2 = -2 \rightarrow x = \pm i\sqrt{2} \\ x^2 = 4 \rightarrow x = \pm 2 \end{cases}$$

The solutions are: $x = 0, \pm 2, \pm i\sqrt{2}$

Exercise

Find all solutions of the equation: $x^5 - 32 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\{1, 2, 4, 8, 16, 32\}$

$$\begin{array}{r|rrrrrr} 2 & 1 & 0 & 0 & 0 & 0 & -32 \\ & & 2 & 4 & 8 & 16 & 32 \\ \hline & 1 & 2 & 4 & 8 & 16 & 0 \end{array} \rightarrow x^4 + 2x^3 + 4x^2 + 8x + 16 = 0 \rightarrow \pm\{1, 2, 4, 8, 16\}$$

$$x^4 + 2x^3 + 4x^2 + 8x + 16 = 0$$

Cannot be solved using rational zero theorem.

Therefore; using program

$$\text{The solutions are: } x = 2, \frac{-1 - \sqrt{5} \pm i\sqrt{2}\sqrt{5 - \sqrt{5}}}{2}, \frac{-1 + \sqrt{5} \pm i\sqrt{2}\sqrt{5 - \sqrt{5}}}{2}$$

Exercise

Find all solutions of the equation: $3x^6 - 10x^5 - 29x^4 + 34x^3 + 50x^2 - 24x - 24 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{24}{3}\right\} = \pm\left\{1, 2, 3, 4, 6, 8, 12, 24, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}\right\}$

$$\begin{array}{r|rrrrrrrr} 1 & 3 & -10 & -29 & 34 & 50 & -24 & -24 \\ & & 3 & -7 & -36 & -2 & 48 & 24 \\ \hline -1 & 3 & -7 & -36 & -2 & 48 & 24 & 0 \\ & & -3 & 10 & 26 & -24 & -24 & \\ \hline -2 & 3 & -10 & -26 & 24 & 24 & 0 \\ & & -6 & 32 & -12 & -24 & \\ \hline -\frac{2}{3} & 3 & -16 & 6 & 12 & 0 \\ & & -2 & 12 & -12 & \\ \hline & 3 & -18 & 18 & 0 \end{array} \rightarrow \begin{array}{l} 3x^5 - 7x^4 - 36x^3 - 2x^2 + 48x + 24 = 0 \\ 3x^4 - 10x^3 - 26x^2 + 24x + 24 = 0 \\ 3x^3 - 16x^2 + 12x - 12 = 0 \\ 3x^2 - 18x + 18 = 0 \end{array}$$

$$x^2 - 6x + 6 = 0$$

$$\begin{aligned} x &= \frac{6 \pm \sqrt{36 - 24}}{2} \\ &= \frac{6 \pm 2\sqrt{3}}{2} \end{aligned}$$

$$\text{The solutions are: } x = -2, -1, 1, -\frac{2}{3}, 3 \pm \sqrt{3}$$

Exercise

Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T(k) = \frac{1}{6}(k^3 + 3k^2 + 2k)$$

Where k is the number of levels in the pyramid. If 220 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

Solution

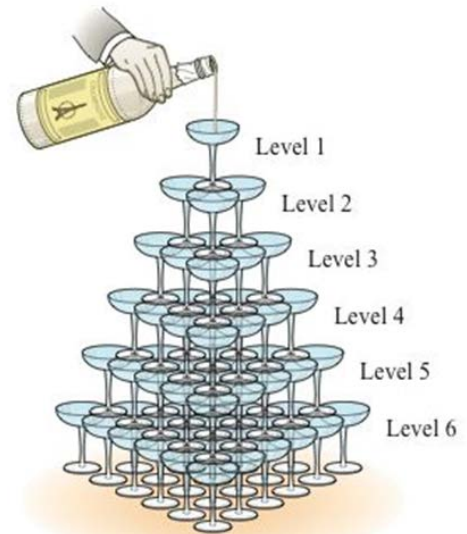
$$\frac{1}{6}(k^3 + 3k^2 + 2k) = 220$$

$$k^3 + 3k^2 + 2k - 1,320 = 0$$

$$\begin{array}{r|rrrr} 10 & 1 & 3 & 2 & -1320 \\ & & 10 & 130 & 1320 \\ \hline & 1 & 13 & 132 & 0 \end{array} \rightarrow k^2 + 13k + 132 = 0$$

$$k = \frac{-13 \pm \sqrt{-359}}{2} \quad \text{C}$$

The are 10 levels in the pyramid.



Exercise

Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T(k) = \frac{1}{6}(2k^3 + 3k^2 + k)$$

Where k is the number of levels in the pyramid. If 140 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

Solution

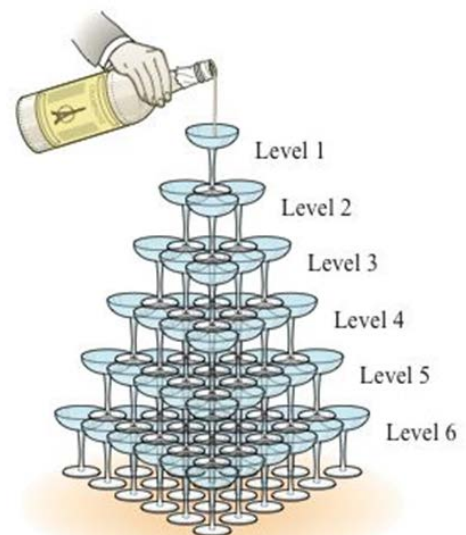
$$\frac{1}{6}(2k^3 + 3k^2 + k) = 150$$

$$2k^3 + 3k^2 + k - 840 = 0$$

$$\begin{array}{r|rrrr} 7 & 2 & 3 & 1 & -840 \\ & & 14 & 119 & 840 \\ \hline & 2 & 17 & 120 & 0 \end{array} \rightarrow 2k^2 + 17k + 120 = 0$$

$$k = \frac{-17 \pm \sqrt{-671}}{4} \quad \text{C}$$

The are 7 levels in the pyramid.



Exercise

A carbon dioxide cartridge for a paintball rifle has the shape of a right circular cylinder with a hemisphere at each end. The cylinder is 4 inches long, and the volume of the cartridge is $2\pi \text{ in}^3$.

The common interior radius of the cylinder and the hemispheres is denoted by x . Estimate the length of the radius x .

Solution

$$\text{Volume of the Cartridge} = 2 \times (\text{Volume of Hemisphere}) + \text{Volume of Cylinder}$$

$$\text{Volume of Sphere} = \frac{4}{3}\pi x^3$$

$$\text{Volume of Cylinder} = 4\pi x^2$$

$$\text{Volume of Cartridge} = \frac{4}{3}\pi x^3 + 4\pi x^2$$

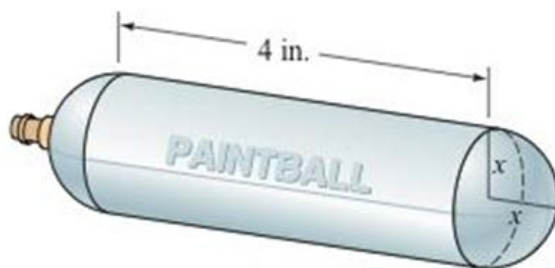
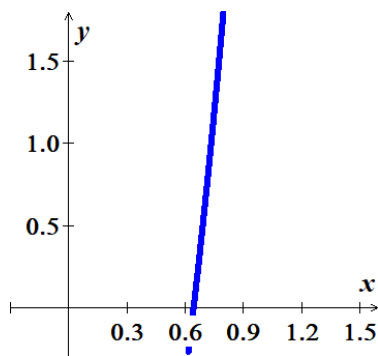
$$2\pi = \frac{4}{3}\pi x^3 + 4\pi x^2$$

$$2x^3 + 6x^2 = 3$$

$$2x^3 + 6x^2 - 3 = 0$$

Using Graph:

$$x \approx 0.64 \text{ in.}$$



Exercise

A propane tank has the shape of a circular cylinder with a hemisphere at each end. The cylinder is 6 feet long and the volume of the tank is $9\pi \text{ ft}^3$. Find the length of the radius x .

Solution

$$\text{Volume of the Cartridge} = 2 \times (\text{Volume of Hemisphere}) + \text{Volume of Cylinder}$$

$$\text{Volume of Sphere} = \frac{4}{3}\pi x^3$$

$$\text{Volume of Cylinder} = 6\pi x^2$$

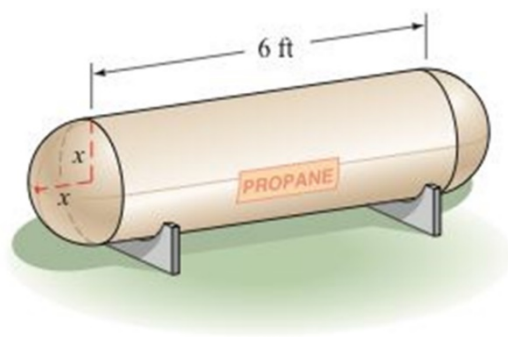
$$\text{Volume of Cartridge} = \frac{4}{3}\pi x^3 + 6\pi x^2$$

$$9\pi = \frac{4}{3}\pi x^3 + 6\pi x^2$$

$$27 = 4x^3 + 18x^2$$

$$4x^3 + 18x^2 - 27 = 0$$

$$\text{possibilities for } \frac{c}{d} : \pm \left\{ \frac{27}{4} \right\} = \pm \left\{ 1, 3, 9, 27, \frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \frac{27}{2}, \frac{1}{4}, \frac{3}{4}, \frac{9}{4}, \frac{27}{4} \right\}$$



$$\begin{array}{c|cccc} -\frac{3}{2} & 4 & 18 & 0 & -27 \\ & & -6 & -18 & 27 \\ \hline & 4 & 12 & -18 & 0 \end{array} \rightarrow 4x^2 + 12x - 18 = 0$$

$$2x^2 + 6x - 9 = 0$$

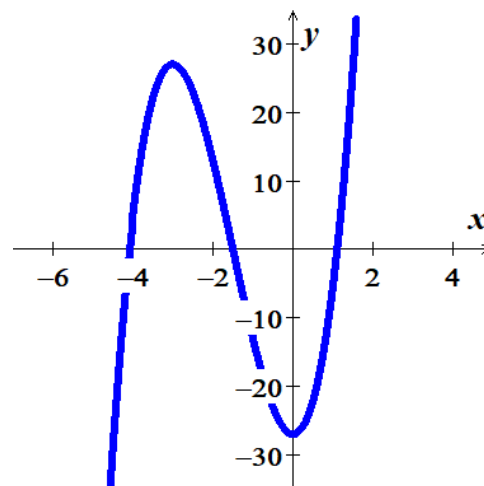
$$x = \frac{-6 \pm \sqrt{36 + 72}}{4}$$

$$= \frac{-6 \pm 6\sqrt{3}}{4}$$

$$= \frac{-3 \pm 3\sqrt{3}}{2}$$

$$x = -\frac{3}{2}, \frac{-3-3\sqrt{3}}{2}, \frac{-3+3\sqrt{3}}{2}$$

\therefore the length of the radius x is $\frac{-3+3\sqrt{3}}{2} \approx 1.1$ foot



Exercise

A cube measures n inches on each edge. If a slice 2 inches thick is cut from one face of the cube, the resulting solid has a volume of 567 in^3 . Find n .

Solution

$$\text{Volume} = n^2(n-2)$$

$$n^3 - 2n^2 = 567$$

$$n^3 - 2n^2 - 567 = 0$$

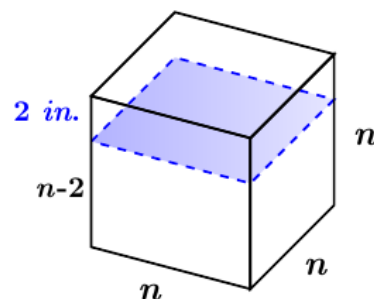
possibilities for $\frac{c}{d} := \pm\{1, 3, 7, 9, 21, 27, 63, 81, 189, 567\}$

$$\begin{array}{c|cccc} 9 & 1 & -2 & 0 & -567 \\ & & 9 & 63 & 567 \\ \hline & 1 & 7 & 63 & 0 \end{array} \rightarrow n^2 + 7n + 63 = 0$$

$$n = \frac{-7 \pm \sqrt{49 - 252}}{2}$$

$$= \frac{-7 \pm i\sqrt{203}}{2} \quad \times$$

$$\therefore n = 9$$



Exercise

A cube measures n inches on each edge. If a slice 1 inch thick is cut from one face of the cube and then a slice 3 inches thick is cut from another face of the cube, the resulting solid has a volume of 1560 in^3 . Find the dimensions of the original cube.

Solution

$$\text{Volume} = n(n-1)(n-3)$$

$$n^3 - 4n^2 + 3n = 1560$$

$$n^3 - 4n^2 + 3n - 1560 = 0$$

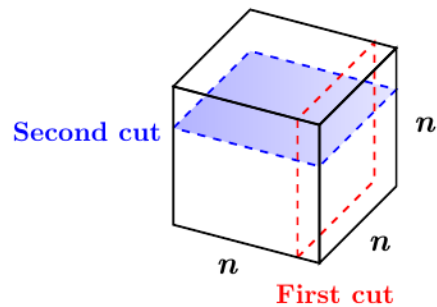
$$\text{possibilities for } \frac{c}{d} := \pm \left\{ \begin{array}{l} 1, 2, 4, 5, 6, 8, 10, 12, 13, 15, 20, 24, 26, 30, 39, \\ 40, 52, 60, 65, 78, 104, 120, 130, 156, 195, 260, 312, 390, 780, 1560 \end{array} \right\}$$

$$\begin{array}{r|rrrr} 13 & 1 & -4 & 3 & -1560 \\ & & 13 & 117 & 1560 \\ \hline & 1 & 9 & 120 & 0 \end{array} \rightarrow n^2 + 9n + 120 = 0$$

$$n = \frac{-9 \pm \sqrt{81 - 480}}{2}$$

$$= \frac{-9 \pm i\sqrt{399}}{2} \quad \times$$

$$\therefore n = 13$$



Exercise

For what value of x will the volume of the following solid be 112 in^3

Solution

$$\text{Volume of the bottom portion} = x^2(x+1)$$

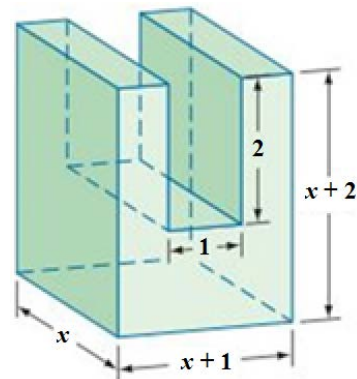
$$\begin{aligned} \text{Volume of one side portion} &= 2x\left(\frac{1}{2}x\right) \\ &= x^2 \end{aligned}$$

$$\text{Total Volume} = x^2(x+1) + 2x^2$$

$$112 = x^3 + 3x^2$$

$$x^3 + 3x^2 - 112 = 0$$

$$\text{possibilities for } \frac{c}{d} := \pm \{1, 2, 4, 8, 14, 28, 56, 112\}$$



$$\begin{array}{c|cccc} 4 & 1 & 3 & 0 & -112 \\ & & 4 & 28 & 112 \\ \hline & 1 & 7 & 28 & 0 \end{array} \rightarrow x^2 + 7x + 28 = 0$$

$$x = \frac{-7 \pm \sqrt{49 - 112}}{2}$$

$$= \frac{-7 \pm 3i\sqrt{7}}{2} \quad \times$$

$$\therefore \underline{x = 4}$$

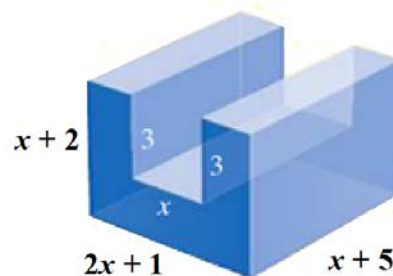
Exercise

For what value of x will the volume of the following solid be 208 in^3

Solution

$$\begin{aligned} \text{Volume of the bottom portion} &= (2x+1)(x+5)(x+2-3) \\ &= (2x^2 + 11x + 5)(x-1) \\ &= 2x^3 + 11x^2 + 5x - 2x^2 - 11x - 5 \\ &= 2x^3 + 9x^2 - 6x - 5 \end{aligned}$$

$$\begin{aligned} \text{Volume of one side portion} &= (3)\frac{1}{2}(2x+1-x)(x+5) \\ &= \frac{3}{2}(x+1)(x+5) \\ &= \frac{3}{2}(x^2 + 6x + 5) \end{aligned}$$



$$\text{Total Volume} = 2x^3 + 9x^2 - 6x - 5 + 2\left(\frac{3}{2}\right)(x^2 + 6x + 5)$$

$$208 = 2x^3 + 9x^2 - 6x - 5 + 3x^2 + 18x + 15$$

$$2x^3 + 12x^2 + 12x - 198 = 0$$

$$x^3 + 6x^2 + 6x - 99 = 0$$

possibilities for $\frac{c}{d} := \pm\{1, 3, 9, 11, 33, 99\}$

$$\begin{array}{c|cccc} 3 & 1 & 6 & 6 & -99 \\ & & 3 & 27 & 99 \\ \hline & 1 & 9 & 33 & 0 \end{array} \rightarrow x^2 + 9x + 33 = 0$$

$$x = \frac{-9 \pm \sqrt{81 - 132}}{2}$$

$$= \frac{-9 \pm i\sqrt{51}}{2} \quad \times$$

$$\therefore \underline{x = 3}$$

Exercise

The length of rectangular box is 1 *inch* more than twice the height of the box, and the width is 3 *inches* more than the height. If the volume of the box is 126 in^3 , find the dimensions of the box.

Solution

$$\text{Volume} = x(2x+1)(x+3)$$

$$2x^3 + 7x^2 + 3x = 126$$

$$2x^3 + 7x^2 + 3x - 126 = 0$$

$$\text{possibilities for } \frac{c}{d} := \pm \left\{ \frac{126}{2} \right\}$$

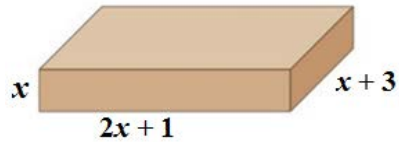
$$= \pm \left\{ 1, 2, 3, 6, 9, 14, 21, 42, 63, 126, \frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \frac{21}{2}, \frac{63}{2} \right\}$$

$$\begin{array}{r|rrrr} 3 & 2 & 7 & 3 & -126 \\ & & 6 & 39 & 126 \\ \hline & 2 & 13 & 42 & 0 \end{array} \rightarrow 2x^2 + 13x + 42 = 0$$

$$x = \frac{-13 \pm \sqrt{169 - 336}}{4}$$

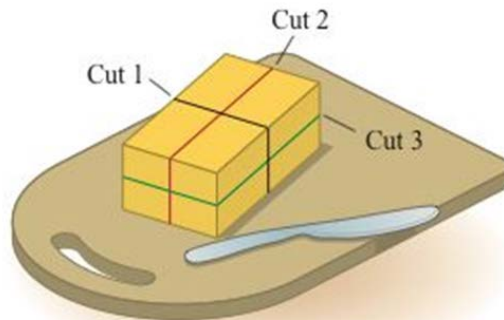
$$= \frac{-13 \pm i\sqrt{167}}{4} \quad \times$$

$$\therefore \underline{x=3}$$



Exercise

One straight cut through a thick piece of cheese produces two pieces. Two straight cuts can produce a maximum of four pieces. Two straight cuts can produce a maximum of four pieces. Three straight cuts can produce a maximum of eight pieces.



You might be inclined to think that every additional cut double number of pieces. However, for four straight cuts, you get a maximum of 15 pieces. The maximum number of pieces P that can be produced by n straight cuts is given by

$$P(n) = \frac{n^3 + 5n + 6}{6}$$

- a) Determine number of pieces that can be produced by five straight cuts.
 b) What is the fewest number of straight cuts that are needed to produce 64 pieces?

Solution

$$a) \quad P(5) = \frac{5^3 + 25 + 6}{6} \\ = 26$$

$$b) \quad \frac{n^3 + 5n + 6}{6} = 64$$

$$n^3 + 5n + 6 = 384$$

$$n^3 + 5n - 378 = 0$$

$$\text{possibilities for } \frac{c}{d} := \pm \{378\}$$

$$= \pm \{1, 2, 3, 6, 7, 9, 14, 18, 21, 27, 42, 54, 63, 126, 189, 378\}$$

$$\begin{array}{r|rrrr} 7 & 1 & 0 & 5 & -378 \\ & & 7 & 49 & 378 \\ \hline & 1 & 7 & 54 & 0 \end{array} \rightarrow n^2 + 7n + 54 = 0$$

$$n = \frac{-7 \pm \sqrt{49 - 216}}{2}$$

$$= \frac{-7 \pm i\sqrt{167}}{2} \quad \times$$

$$\therefore n = 7$$

Exercise

The number of ways one can select three cards from a group of n cards (the order of the selection matters), where $n \geq 3$, is given by $P(n) = n^3 - 3n^2 + 2n$. For a certain card trick, a magician has determined that there are exactly 504 ways to choose three cards from a given group. How many cards are in the group?

Solution

$$P(n) = n^3 - 3n^2 + 2n = 504$$

$$n^3 - 3n^2 + 2n - 504 = 0$$

$$\text{possibilities for } \frac{c}{d} := \pm \{504\}$$

$$= \pm \left\{ \begin{array}{l} 1, 2, 3, 4, 6, 7, 8, 9, 12, 14, 18, 21, \\ 24, 28, 36, 42, 56, 63, 72, 84, 126, 168, 252, 504 \end{array} \right\}$$

$$\begin{array}{r|rrrr} 9 & 1 & -3 & 2 & -504 \\ & & 9 & 54 & 504 \\ \hline & 1 & 6 & 56 & 0 \end{array} \rightarrow n^2 + 6n + 56 = 0$$

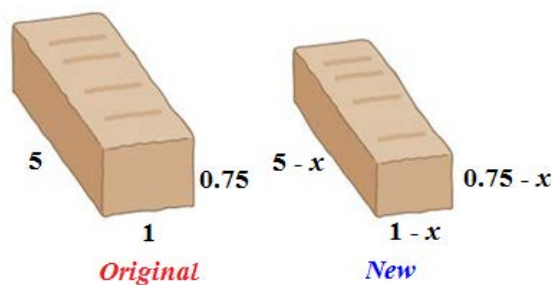
$$n = \frac{-6 \pm \sqrt{36 - 224}}{2}$$

$$= -3 \pm i\sqrt{47} \quad \times$$

$$\therefore n = 9$$

Exercise

A nutrition bar in the shape of a rectangular solid measure 0.75 in. by 1 in. by 5 inches.



To reduce costs, the manufacturer has decided to decrease each of the dimensions of the nutrition bar by x inches, what value of x will produce a new bar with a volume that is 0.75 in^3 less than the present bar's volume.

Solution

$$\begin{aligned} V_{\text{original}} &= (5)(1)\left(\frac{3}{4}\right) \\ &= \frac{15}{4} \end{aligned}$$

$$V_{\text{new}} = (5-x)(1-x)\left(\frac{3}{4}-x\right) \quad \left(x < \frac{3}{4}\right)$$

$$\left(5-6x+x^2\right)\left(\frac{3-4x}{4}\right) = \frac{15}{4} - \frac{3}{4}$$

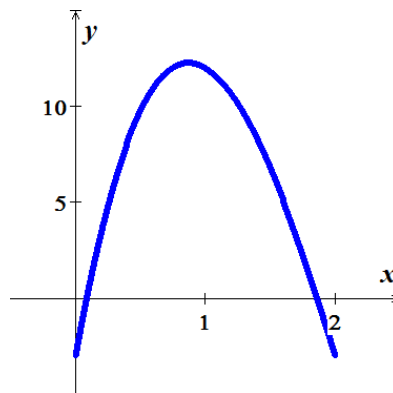
$$15 - 20x - 18x + 24x^2 + 3x^2 - 4x^3 = 4(3)$$

$$4x^3 - 27x^2 + 38x - 3 = 0$$

From graph table:

$$\begin{array}{rr} 0.08200 & -0.06334 \\ 0.08400 & 0.00386 \end{array}$$

$$x \approx 0.083 \text{ in.}$$



Exercise

A rectangular box is square on two ends and has length plus girth of 81 *inches*. (Girth: distance *around* the box). Determine the possible lengths l ($l > w$) of the box if its volume is 4900 in^3 .

Solution

$$81 = l + 4w$$

$$l = 81 - 4w$$

$$V = lw^2$$

$$= (81 - 4w)w^2$$

$$-4w^3 + 81w^2 = 4900$$

$$4w^3 - 81w^2 + 4900 = 0$$

$$\text{possibilities for } \frac{c}{d} := \pm \left\{ \frac{4900}{4} \right\} = \pm \left\{ \begin{array}{l} 1, 2, 4, 7, 10, 14, 20, 28, 49, \\ 100, 175, 245, 350, 490, 700, 1225, 2450, 4900, \dots \end{array} \right\}$$

$$\begin{array}{r|rrrr} 14 & 4 & -81 & 0 & 4900 \\ & & 56 & -350 & -4900 \\ \hline & 4 & -25 & -350 & 0 \end{array} \rightarrow 4w^2 - 25w - 350 = 0$$

$$w = \frac{25 \pm \sqrt{625 + 5600}}{8}$$

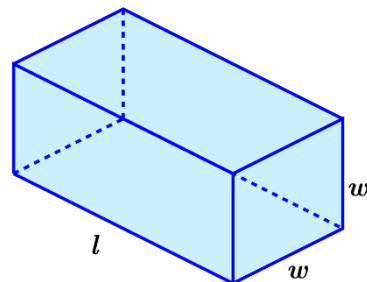
$$= \frac{25 \pm 5\sqrt{249}}{8}$$

$$= \left\{ \begin{array}{l} \frac{25 - 5\sqrt{249}}{8} < 0 \\ \frac{25 + 5\sqrt{249}}{8} \approx 13 \end{array} \right.$$

$$l = 81 - 4(14) = 25$$

$$l = 81 - 4(13) = 29$$

\therefore the possible lengths l are around **25 in.** *or* **29 in.**



Solution **Section 2.5 – Graphing Polynomial Functions**

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = 5x^3 + 7x^2 - x + 9$

Solution

Leading term: $5x^3$ with 3^{rd} degree (n is **odd**)

$$x \rightarrow -\infty \Rightarrow f(x) \rightarrow -\infty \quad f(x) \text{ falls left}$$

$$x \rightarrow \infty \Rightarrow f(x) \rightarrow \infty \quad f(x) \text{ rises right}$$

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = 11x^3 - 6x^2 + x + 3$

Solution

Leading term: $11x^3$ with 3^{rd} degree (n is **odd**)

$$x \rightarrow -\infty \Rightarrow f(x) \rightarrow -\infty \quad f(x) \text{ falls left}$$

$$x \rightarrow \infty \Rightarrow f(x) \rightarrow \infty \quad \text{rises right}$$

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = -11x^3 - 6x^2 + x + 3$

Solution

Leading term: $-11x^3$ with 3^{rd} degree (n is **odd**)

$$x \rightarrow -\infty \Rightarrow f(x) \rightarrow \infty \quad f(x) \text{ rises left}$$

$$x \rightarrow \infty \Rightarrow f(x) \rightarrow -\infty \quad f(x) \text{ falls right}$$

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = 2x^3 + 3x^2 - 23x - 42$

Solution

Leading term: $2x^3$ with 3^{rd} degree (n is **odd**)

$$x \rightarrow -\infty \Rightarrow f(x) \rightarrow -\infty \quad f(x) \text{ falls left}$$

$$x \rightarrow \infty \Rightarrow f(x) \rightarrow \infty \quad f(x) \text{ rises right}$$

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = 5x^4 + 7x^2 - x + 9$

Solution

Leading term: $5x^4$ with 4^{rd} degree (n is *even*)

$x \rightarrow -\infty \Rightarrow f(x) \rightarrow \infty$ $f(x)$ rises left

$x \rightarrow \infty \Rightarrow f(x) \rightarrow \infty$ $f(x)$ rises right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = 11x^4 - 6x^2 + x + 3$

Solution

Leading term: $11x^4$ with 4^{rd} degree (n is *even*)

$x \rightarrow -\infty \Rightarrow f(x) \rightarrow \infty$ $f(x)$ rises left

$x \rightarrow \infty \Rightarrow f(x) \rightarrow \infty$ $f(x)$ rises right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = -5x^4 + 7x^2 - x + 9$

Solution

Leading term: $-5x^4$ with 4^{rd} degree (n is *even*)

$x \rightarrow -\infty \Rightarrow f(x) \rightarrow -\infty$ $f(x)$ falls left

$x \rightarrow \infty \Rightarrow f(x) \rightarrow -\infty$ $f(x)$ falls right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = -11x^4 - 6x^2 + x + 3$

Solution

Leading term: $-11x^4$ with 4^{rd} degree (n is *even*)

$x \rightarrow -\infty \Rightarrow f(x) \rightarrow -\infty$ $f(x)$ falls left

$x \rightarrow \infty \Rightarrow f(x) \rightarrow -\infty$ $f(x)$ falls right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = 5x^5 - 16x^2 - 20x + 64$

Solution

Leading term: $5x^5$ with 5^{th} degree (n is **odd**)

$$x \rightarrow -\infty \Rightarrow f(x) \rightarrow -\infty \quad f(x) \text{ falls left}$$

$$x \rightarrow \infty \Rightarrow f(x) \rightarrow \infty \quad f(x) \text{ rises right}$$

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = -5x^5 - 16x^2 - 20x + 64$

Solution

Leading term: $-5x^5$ with 5^{th} degree (n is **odd**)

$$x \rightarrow -\infty \Rightarrow f(x) \rightarrow \infty \quad f(x) \text{ rises left}$$

$$x \rightarrow \infty \Rightarrow f(x) \rightarrow -\infty \quad f(x) \text{ falls right}$$

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = -3x^6 - 16x^3 + 64$

Solution

Leading term: $-3x^6$ with 6^{th} degree (n is **even**)

$$x \rightarrow -\infty \Rightarrow f(x) \rightarrow -\infty \quad f(x) \text{ falls left}$$

$$x \rightarrow \infty \Rightarrow f(x) \rightarrow -\infty \quad f(x) \text{ falls right}$$

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = 3x^6 - 16x^3 + 4$

Solution

Leading term: $3x^6$ with 6^{th} degree (n is **even**)

$$x \rightarrow -\infty \Rightarrow f(x) \rightarrow \infty \quad f(x) \text{ rises left}$$

$$x \rightarrow \infty \Rightarrow f(x) \rightarrow \infty \quad f(x) \text{ rises right}$$

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = x^3 - x - 1$; between 1 and 2

Solution

$$f(1) = (1)^3 - (1) - 1 \\ = -1$$

$$f(2) = (2)^3 - (2) - 1 \\ = 5$$

Since $f(1)$ and $f(2)$ have opposite signs.

Therefore, the polynomial *has a real zero* between 1 and 2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = x^3 - 4x^2 + 2$; between 0 and 1

Solution

$$f(0) = (0)^3 - 4(0)^2 + 2 \\ = 2$$

$$f(1) = (1)^3 - 4(1)^2 + 2 \\ = -1$$

Since $f(0)$ and $f(1)$ have opposite signs.

Therefore, the polynomial *has a real zero* between 0 and 1.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = 2x^4 - 4x^2 + 1$; between -1 and 0

Solution

$$f(-1) = 2(-1)^4 - 4(-1)^2 + 1 \\ = -1$$

$$f(0) = 2(0)^4 - 4(0)^2 + 1 \\ = 1$$

Since $f(0)$ and $f(-1)$ have opposite signs.

Therefore, the polynomial *has a real zero* between -1 and 0 .

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = x^4 + 6x^3 - 18x^2$; between 2 and 3

Solution

$$f(2) = (2)^4 + 6(2)^3 - 18(2)^2 \\ = -8$$

$$f(3) = (3)^4 + 6(3)^3 - 18(3)^2 \\ = 81$$

Since $f(2)$ and $f(3)$ have opposite signs.

Therefore, the polynomial *has a real zero* between 2 and 3.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = x^3 + x^2 - 2x + 1$; between -3 and -2

Solution

$$f(-3) = (-3)^3 + (-3)^2 - 2(-3) + 1 \\ = -11$$

$$f(-2) = (-2)^3 + (-2)^2 - 2(-2) + 1 \\ = 1$$

Since $f(-3)$ and $f(-2)$ have opposite signs.

Therefore, the polynomial *has a real zero* between -2 and -3 .

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = x^5 - x^3 - 1$; between 1 and 2

Solution

$$f(1) = (1)^5 - (1)^3 - 1$$

$$\underline{= -1}$$

$$f(2) = (2)^5 - (2)^3 - 1$$

$$\underline{= 23}$$

Since $f(1)$ and $f(2)$ have opposite signs.

Therefore, the polynomial *has a real zero* between 1 and 2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = 3x^3 - 10x + 9$; between -3 and -2

Solution

$$f(-3) = 3(-3)^3 - 10(-3) + 9$$

$$\underline{= -42}$$

$$f(-2) = 3(-2)^3 - 10(-2) + 9$$

$$\underline{= 5}$$

Since $f(-3)$ and $f(-2)$ have opposite signs.

Therefore, the polynomial *has a real zero* between -3 and -2 .

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = 3x^3 - 8x^2 + x + 2$; between 2 and 3

Solution

$$f(2) = 3(2)^3 - 8(2)^2 + (2) + 2$$

$$\underline{= -4}$$

$$f(3) = 3(3)^3 - 8(3)^2 + (3) + 2$$

$$\underline{= 14}$$

Since $f(2)$ and $f(3)$ have opposite signs.

Therefore, the polynomial *has a real zero* between 2 and 3.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = 3x^3 - 8x^2 + x + 2$; between 1 and 2

Solution

$$\begin{aligned} f(1) &= 3(1)^3 - 8(1)^2 + (1) + 2 \\ &= -2 \end{aligned}$$

$$\begin{aligned} f(2) &= 3(2)^3 - 8(2)^2 + (2) + 2 \\ &= -4 \end{aligned}$$

Since $f(1)$ and $f(2)$ have same signs.

Therefore, *cannot be determined*.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3$; between 0 and 1

Solution

$$\begin{aligned} f(0) &= (0)^5 + 2(0)^4 - 6(0)^3 + 2(0) - 3 \\ &= -3 \end{aligned}$$

$$\begin{aligned} f(1) &= (1)^5 + 2(1)^4 - 6(1)^3 + 2(1) - 3 \\ &= -4 \end{aligned}$$

Since $f(0)$ and $f(1)$ have same signs.

Therefore, *cannot be determined*.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 2x^3 + 3x^2 - 23x - 42$, $a = 3$, $b = 4$

Solution

$$\begin{aligned} P(3) &= 54 + 27 - 69 - 42 \\ &= -30 \end{aligned}$$

$$\begin{aligned} P(4) &= 128 + 48 - 92 - 42 \\ &= 90 \end{aligned}$$

Since $P(3)$ and $P(4)$ have opposite signs.

Therefore, the polynomial *has a real zero* between 3 and 4.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 4x^3 - x^2 - 6x + 1$, $a = 0$, $b = 1$

Solution

$$P(0) = \underline{1}$$

$$P(1) = 4 - 1 - 6 + 1 \\ = \underline{-2}$$

Since $P(0)$ and $P(1)$ have opposite signs.

Therefore, the polynomial *has a real zero* between 0 and 1.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 3x^3 + 7x^2 + 3x + 7$, $a = -3$, $b = -2$

Solution

$$P(-3) = -81 + 63 - 9 + 7 \\ = \underline{-20}$$

$$P(-2) = -24 + 28 - 6 + 7 \\ = \underline{5}$$

Since $P(-3)$ and $P(-2)$ have opposite signs.

Therefore, the polynomial *has a real zero* between -3 and -2 .

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 2x^3 - 21x^2 - 2x + 25$, $a = 1$, $b = 2$

Solution

$$P(1) = 2 - 21 - 2 + 25 \\ = \underline{4}$$

$$P(2) = 16 - 84 - 4 + 25$$

$$= -47$$

Since $P(1)$ and $P(2)$ have opposite signs.

Therefore, the polynomial *has a real zero* between 1 and 2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 4x^4 + 7x^3 - 11x^2 + 7x - 15$, $a = 1$, $b = \frac{3}{2}$

Solution

$$P(1) = 4 + 7 - 11 + 7 - 15$$

$$= -8$$

$$P\left(\frac{3}{2}\right) = 81 + \frac{189}{8} - \frac{99}{4} + \frac{21}{2} - 15$$

$$= 66 + \frac{189 - 198 + 84}{8}$$

$$= 66 + \frac{75}{8}$$

$$= \frac{603}{8}$$

Since $P(1)$ and $P\left(\frac{3}{2}\right)$ have opposite signs.

Therefore, the polynomial *has a real zero* between 1 and $\frac{3}{2}$.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 5x^3 - 16x^2 - 20x + 64$, $a = 3$, $b = \frac{7}{2}$

Solution

$$P(3) = 135 - 144 - 60 + 64$$

$$= -5$$

$$P\left(\frac{7}{2}\right) = \frac{1715}{8} - 196 - 70 + 64$$

$$= \frac{1715}{8} - 202$$

$$= \frac{99}{8}$$

Since $P(3)$ and $P\left(\frac{7}{2}\right)$ have opposite signs.

Therefore, the polynomial *has a real zero* between 3 and $\frac{7}{2}$.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = x^4 - x^2 - x - 4$, $a = 1$, $b = 2$

Solution

$$\begin{aligned} P(1) &= 1 - 1 - 1 - 4 \\ &= -5 \end{aligned}$$

$$\begin{aligned} P(2) &= 16 - 4 - 2 - 4 \\ &= 6 \end{aligned}$$

Since $P(1)$ and $P(2)$ have opposite signs.

Therefore, the polynomial *has a real zero* between 1 and 2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = x^3 - x - 8$, $a = 2$, $b = 3$

Solution

$$\begin{aligned} P(2) &= 8 - 2 - 8 \\ &= -2 \end{aligned}$$

$$\begin{aligned} P(3) &= 27 - 3 - 8 \\ &= 16 \end{aligned}$$

Since $P(2)$ and $P(3)$ have opposite signs.

Therefore, the polynomial *has a real zero* between 2 and 3.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = x^3 - x - 8$, $a = 0$, $b = 1$

Solution

$$\underline{P(0) = -8}$$

$$P(1) = 1 - 1 - 8$$

$$= -8$$

Since $P(0)$ and $P(1)$ have same sign.

Therefore, *cannot be determined*.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = x^3 - x - 8$, $a = 2.1$, $b = 2.2$

Solution

$$P(2.1) = P\left(\frac{21}{10}\right)$$

$$= \frac{9261}{1000} - \frac{21}{10} - 8$$

$$= \frac{9261 - 2100 - 8000}{1000}$$

$$= -\frac{839}{1,000}$$

$$P(2.2) = P\left(\frac{22}{10}\right)$$

$$= \frac{10,648}{1000} - \frac{22}{10} - 8$$

$$= \frac{10,648 - 2,200 - 8,000}{1000}$$

$$= \frac{448}{1,000}$$

$$= 0.448$$

Since $P(2.1)$ and $P(2.2)$ have opposite signs.

Therefore, the polynomial *has a real zero* between 2.1 and 2.2.

Exercise

Let $f(x) = x^4 - 4x^2$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f .

Solution

$$f(x) = x^2(x^2 - 4)$$

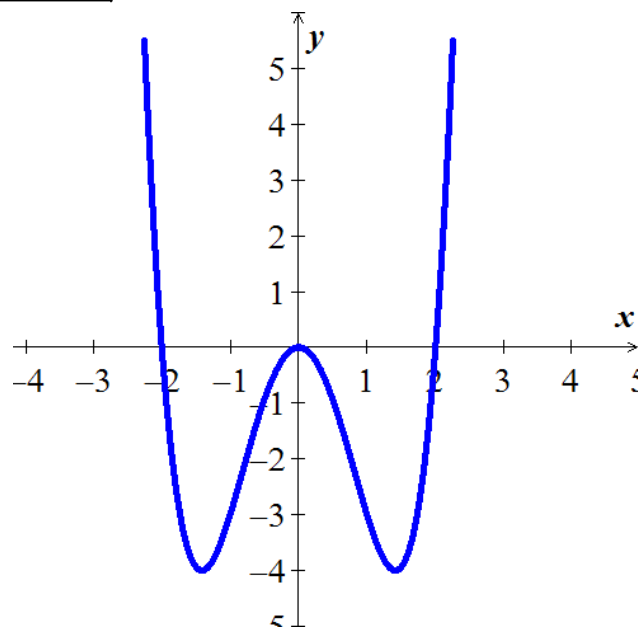
$$= x^2(x - 2)(x + 2)$$

The zeros are: 0, 0, 2, -2.

$-\infty$	-2	0,0	2	∞
+		-		+

$$f(x) < 0 \quad \underline{(-2, 0) \cup (0, 2)} \quad |$$

$$f(x) > 0 \quad \underline{(-\infty, -2) \cup (2, \infty)} \quad |$$



Exercise

Let $f(x) = x^4 + 3x^3 - 4x^2$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f .

Solution

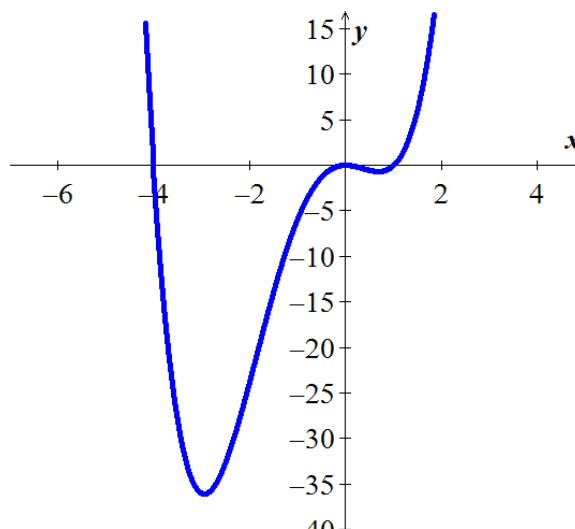
$$f(x) = x^2(x^2 + 3x - 4)$$

The zeros are: 0, 0, 1, -4.

$-\infty$	-4	0,0	1	∞
+		-		+

$$f(x) > 0 \quad \underline{(-\infty, -4) \cup (1, \infty)} \quad |$$

$$f(x) < 0 \quad \underline{(-4, 0) \cup (0, 1)} \quad |$$



Exercise

Let $f(x) = x^3 + 2x^2 - 4x - 8$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f .

Solution

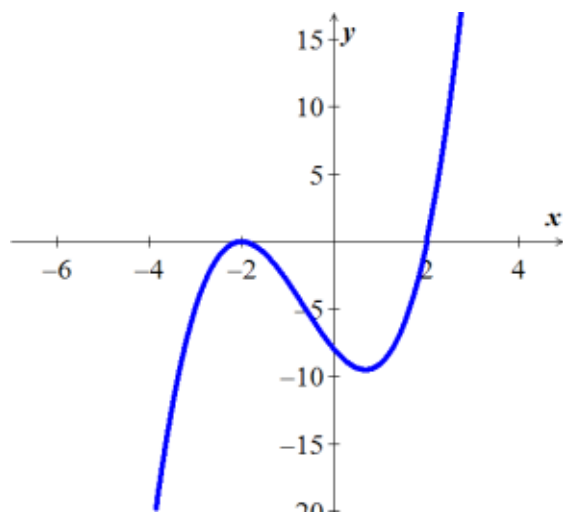
$$\begin{aligned} f(x) &= x^2(x+2) - 4(x+2) \\ &= (x+2)(x^2 - 4) \\ &= (x+2)(x+2)(x-2) = 0 \end{aligned}$$

The zeros are: 2, -2, -2

$-\infty$	-2	0	2	∞
$-$		$-$		$+$

$$f(x) > 0 \quad (2, \infty)$$

$$f(x) < 0 \quad (-\infty, -2) \cup (-2, 2)$$



Exercise

Let $f(x) = x^3 - 3x^2 - 9x + 27$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f .

Solution

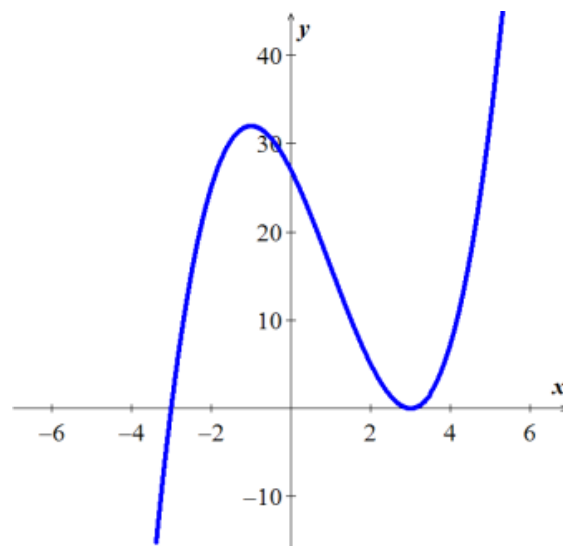
$$\begin{aligned} f(x) &= x^2(x-3) - 9(x-3) \\ &= (x-3)(x^2 - 9) \\ &= (x-3)(x-3)(x+3) \end{aligned}$$

The zeros are: -3, 3 (multiplicity)

$-\infty$	-3	0	3	∞
$-$		$+$		$+$

$$f(x) > 0 \quad (-3, 3) \cup (3, \infty)$$

$$f(x) < 0 \quad (-\infty, -3)$$



Exercise

Let $f(x) = -x^4 + 12x^2 - 27$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f .

Solution

$$x^2 = \frac{-12 \pm \sqrt{36}}{-2}$$

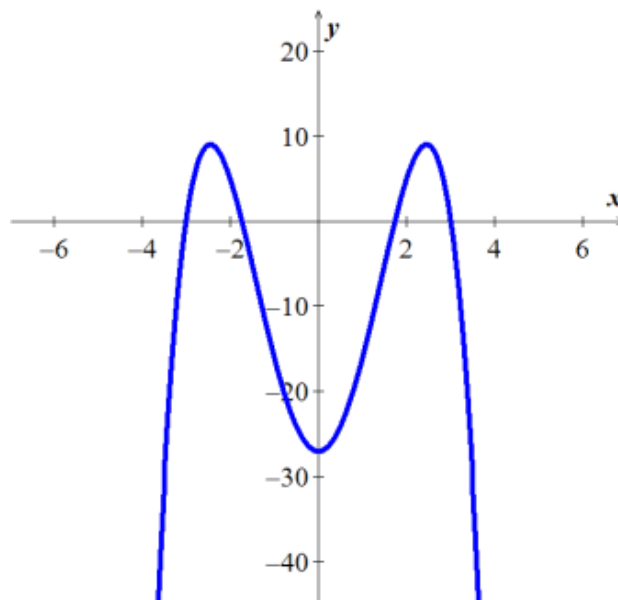
$$= \begin{cases} \frac{-12-6}{-2} = 9 \\ \frac{-12+6}{-2} = 3 \end{cases}$$

$$\rightarrow \begin{cases} x^2 = 9 \\ x^2 = 3 \end{cases} \Rightarrow \begin{cases} x = \pm 3 \\ x = \pm \sqrt{3} \end{cases}$$

-3	$-\sqrt{3}$	$\sqrt{3}$	3	
-	+	-	+	-

$$f(x) > 0 \quad \underline{(-3, -\sqrt{3}) \cup (\sqrt{3}, 3)}$$

$$f(x) < 0 \quad \underline{(-\infty, -3) \cup (-\sqrt{3}, \sqrt{3}) \cup (3, \infty)}$$



Exercise

Let $f(x) = x^2(x+2)(x-1)^2(x-2)$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f .

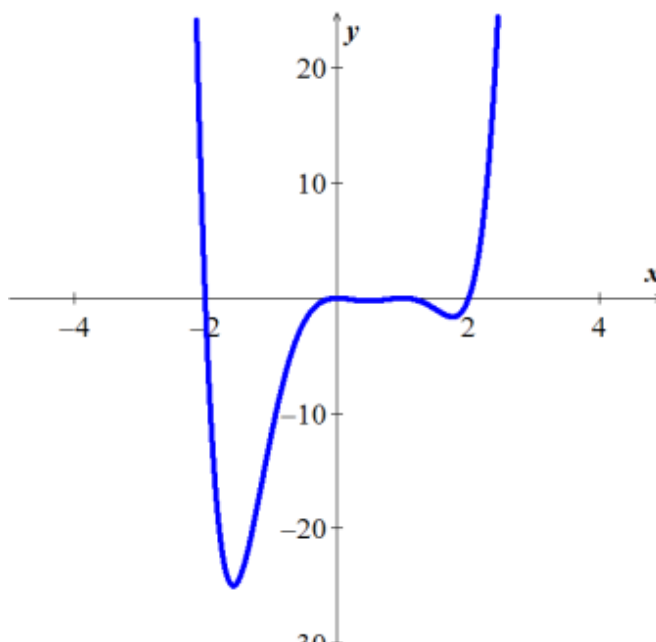
Solution

The zeros are: -2, 2, 0, 0, 1, 1

-2	0,0	1,1	2	
+	-	-	+	

$$f(x) > 0 \quad \underline{(-\infty, -2) \cup (2, \infty)}$$

$$f(x) < 0 \quad \underline{(-2, 0) \cup (0, 1) \cup (1, 2)}$$



Exercise

Let $f(x) = 2x^3 + 11x^2 - 7x - 6$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f .

Solution

$$\begin{aligned} \text{possibilities : } \pm \left\{ \frac{6}{2} \right\} &= \pm \left\{ \frac{1, 2, 3, 6}{1, 2} \right\} \\ &= \pm \left\{ 1, 2, 3, 5, \frac{1}{2}, \frac{3}{2} \right\} \end{aligned}$$

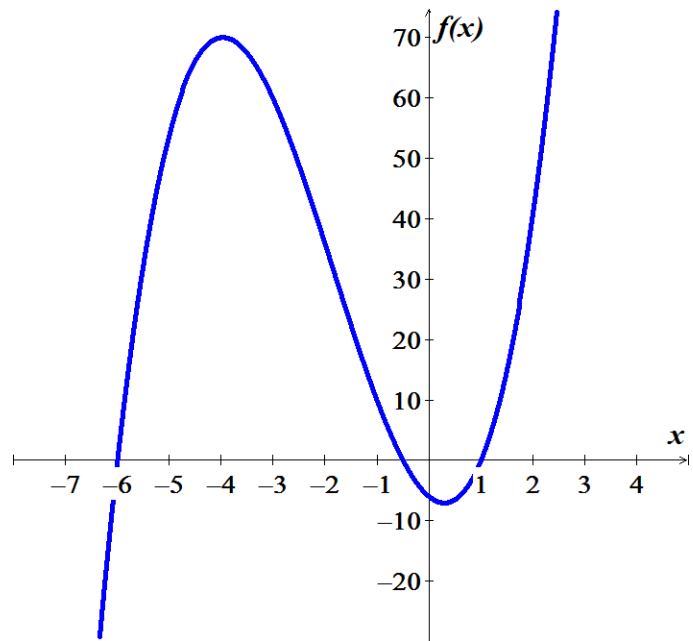
$$\begin{array}{r|rrrr} 1 & 2 & 11 & -7 & -6 \\ & & 2 & 13 & 6 \\ \hline & 2 & 13 & 6 & \boxed{0} \end{array} \rightarrow 2x^2 + 13x + 6 = 0$$

The zeros are: $x = 1, -\frac{1}{2}, -6$

-6	$-\frac{1}{2}$	1	
-	+	-	+

$$f(x) > 0 \quad \left(-6, -\frac{1}{2} \right) \cup (1, \infty)$$

$$f(x) < 0 \quad (-\infty, -6) \cup \left(-\frac{1}{2}, 1 \right)$$



Exercise

Let $f(x) = x^3 + 2x^2 - 5x - 6$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f .

Solution

$$\begin{aligned} \text{possibilities : } \pm \left\{ \frac{6}{1} \right\} \\ &= \pm \{1, 2, 3, 6\} \end{aligned}$$

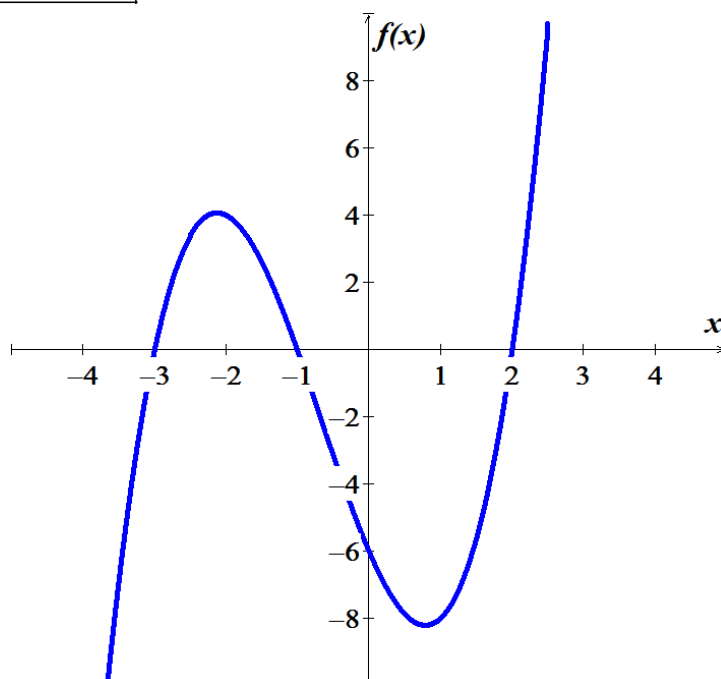
$$\begin{array}{r|rrrr} -1 & 1 & 2 & -5 & -6 \\ & & -1 & -1 & 6 \\ \hline & 1 & 1 & -6 & \boxed{0} \end{array} \rightarrow x^2 + x - 6 = 0$$

The zeros are: $x = -1, -3, 2$

-3	-1	2	
-	+	-	+

$$f(x) > 0 \quad \underline{(-3, -1) \cup (2, \infty)}$$

$$f(x) < 0 \quad \underline{(-\infty, -3) \cup (-1, 2)}$$



Exercise

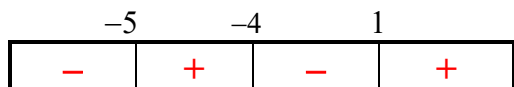
Let $f(x) = x^3 + 8x^2 + 11x - 20$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f .

Solution

$$\text{possibilities : } \pm \left\{ \frac{20}{1} \right\} = \pm \{1, 2, 4, 5, 20, 20\}$$

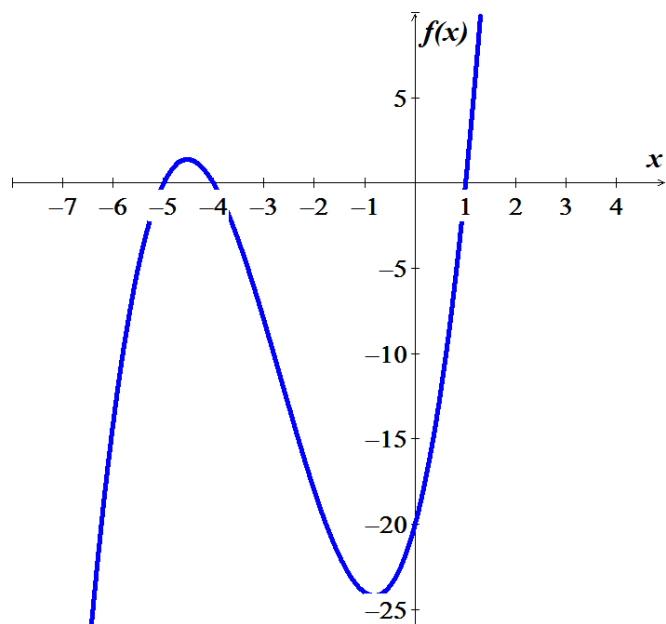
$$\begin{array}{r|rrrr} 1 & 1 & 8 & 11 & -20 \\ & & 1 & 9 & 20 \\ \hline & 1 & 9 & 20 & \boxed{0} \end{array} \rightarrow x^2 + 9x + 20 = 0$$

The zeros are: $\underline{x = -5, -4, 1}$



$$f(x) > 0 \quad \underline{(-5, -4) \cup (1, \infty)}$$

$$f(x) < 0 \quad \underline{(-\infty, -5) \cup (-4, 1)}$$



Exercise

Let $f(x) = x^4 + x^2 - 2$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f .

Solution

possibilities: $\pm\{1, 2\}$

$$\begin{array}{r|rrrrr} 1 & 1 & 0 & 1 & 0 & -2 \\ & & 1 & 1 & 2 & 1 \\ \hline -1 & 1 & 1 & 2 & 2 & 0 \\ & & -1 & 0 & -2 & \\ \hline & 1 & 0 & 2 & 0 & \end{array} \rightarrow x^3 + x^2 + 2x + 1 = 0 \rightarrow \pm\{1, 2\}$$

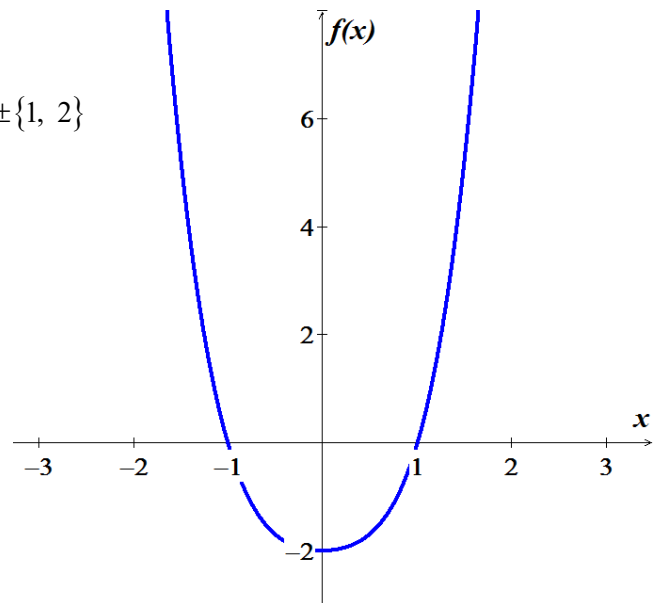
$$\rightarrow x^2 + 2 = 0 \Rightarrow x = \pm i\sqrt{2}$$

The zeros are: $x = \pm 1$

	-1		1	
	+		-	
				+

$$f(x) > 0 \quad (-\infty, -1) \cup (1, \infty)$$

$$f(x) < 0 \quad (-1, 1)$$



Exercise

Let $f(x) = x^4 - x^3 - 6x^2 + 4x + 8$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f .

Solution

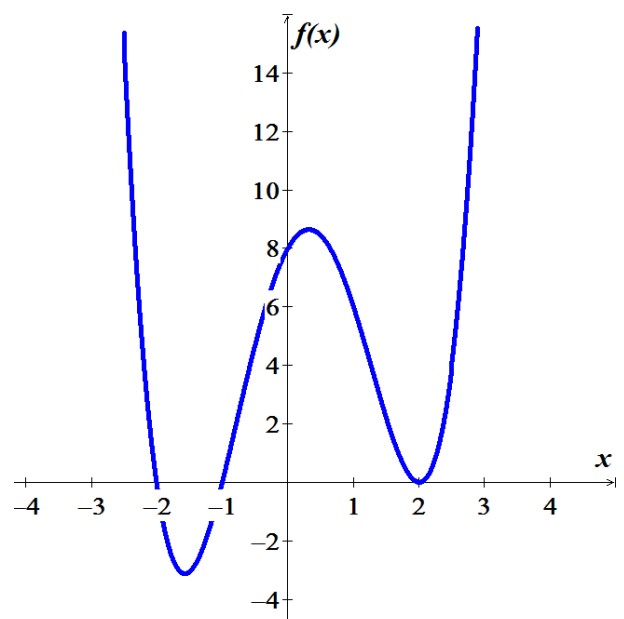
possibilities: $\pm\{1, 2, 4, 8\}$

$$\begin{array}{r|rrrrr} -1 & 1 & -1 & -6 & 4 & 8 \\ & & -1 & 2 & 4 & -8 \\ \hline -2 & 1 & -2 & -4 & 8 & 0 \\ & & -2 & 8 & -8 & \\ \hline & 1 & -4 & 4 & 0 & \end{array} \rightarrow x^3 - 2x^2 - 4x + 8 = 0 \rightarrow \pm\{1, 2, 4, 8\}$$

$$\rightarrow x^2 - 4x + 4 = 0 \Rightarrow x = 2, 2$$

The zeros are: $x = -2, -1, 2, 2$

	-2		-1		2	
	+		-		+	
					+	



$$f(x) > 0 \quad \underline{(-\infty, -1) \cup (1, \infty)}$$

$$f(x) < 0 \quad \underline{(-1, 1)}$$

Exercise

Let $f(x) = 2x^4 - x^3 - 5x^2 + 2x + 2$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f .

Solution

$$\text{possibilities: } \pm \left\{ \frac{2}{2} \right\} = \pm \left\{ 1, 2, \frac{1}{2} \right\}$$

$$\begin{array}{r|rrrrr} 1 & 2 & -1 & -5 & 2 & 2 \\ & & 2 & 1 & -4 & -2 \\ \hline -\frac{1}{2} & 2 & 1 & -4 & -2 & 0 \\ & & -1 & 0 & 2 & \\ \hline & 2 & 0 & -4 & 0 & \end{array} \rightarrow 2x^3 + x^2 - 4x - 2 = 0$$

$$\rightarrow \pm \left\{ 1, 2, \frac{1}{2} \right\}$$

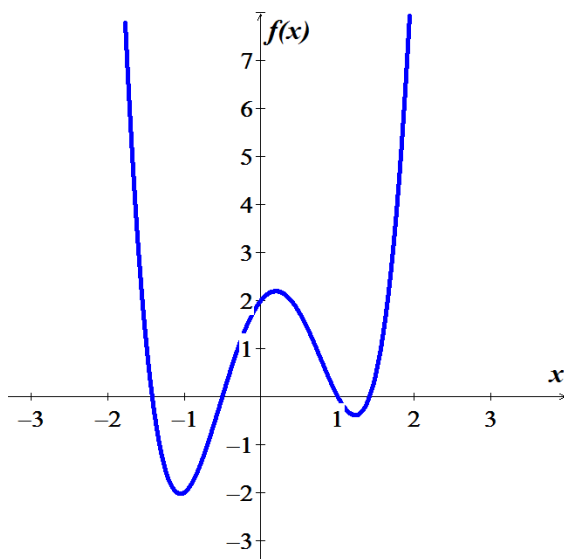
$$\rightarrow 2x^2 - 4 = 0 \Rightarrow x = \pm\sqrt{2}$$

$$\text{The zeros are: } \underline{x = -\frac{1}{2}, 1, -\sqrt{2}, \sqrt{2}}$$

$-\sqrt{2}$	$-\frac{1}{2}$	1	$\sqrt{2}$	
+	-	+	-	+

$$f(x) > 0 \quad \underline{(-\infty, -\sqrt{2}) \cup (-\frac{1}{2}, 1) \cup (\sqrt{2}, \infty)}$$

$$f(x) < 0 \quad \underline{(-\sqrt{2}, -\frac{1}{2}) \cup (1, \sqrt{2})}$$



Exercise

Let $f(x) = 4x^5 - 8x^4 - x + 2$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f .

Solution

$$\begin{aligned} f(x) &= 4x^4(x-2) - (x-2) \\ &= (x-2)(4x^4 - 1) = 0 \end{aligned}$$

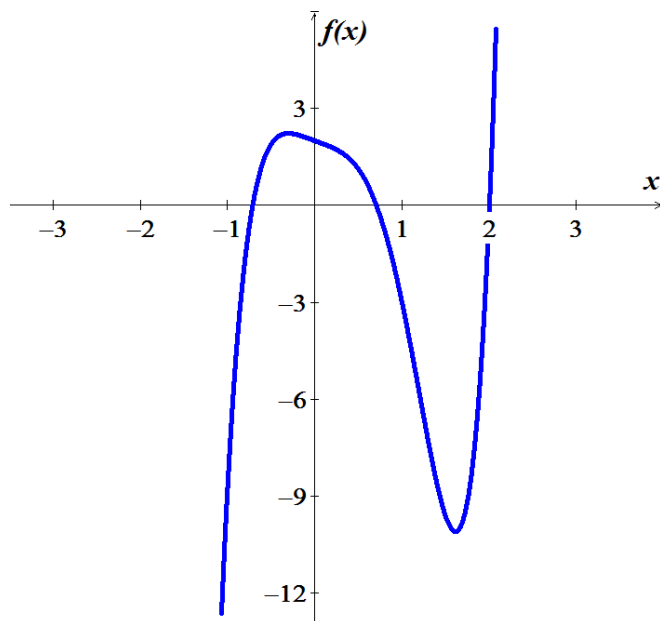
$$4x^4 - 1 = 0 \Rightarrow \begin{cases} x^2 = -\frac{1}{2} \\ x^2 = \frac{1}{2} \end{cases} \quad x = \pm \frac{\sqrt{2}}{2} \quad \text{C}$$

The zeros are: $x = 2, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$

	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	2	
	-	+	-	+

$$f(x) > 0 \quad \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \cup (2, \infty)$$

$$f(x) < 0 \quad \left(-\infty, -\frac{\sqrt{2}}{2} \right) \cup \left(\frac{\sqrt{2}}{2}, 2 \right)$$



Exercise

Let $f(x) = x^5 - 7x^4 + 19x^3 - 37x^2 + 60x - 36$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f .

Solution

possibilities: $\pm \left\{ \frac{36}{1} \right\} = \pm \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

1	1	-7	19	-37	60	-36
		1	-6	13	-24	36
3	1	-6	13	-24	36	0
		3	-9	12	-36	
3	1	-3	4	-12	0	
		3	0	12		
	1	0	4	0		

$$x^4 - 6x^3 + 13x^2 - 24x + 36 = 0 \rightarrow \pm \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$$

$$x^3 - 3x^2 + 4x - 12 = 0 \rightarrow \pm \{1, 2, 3, 4, 6, 12\}$$

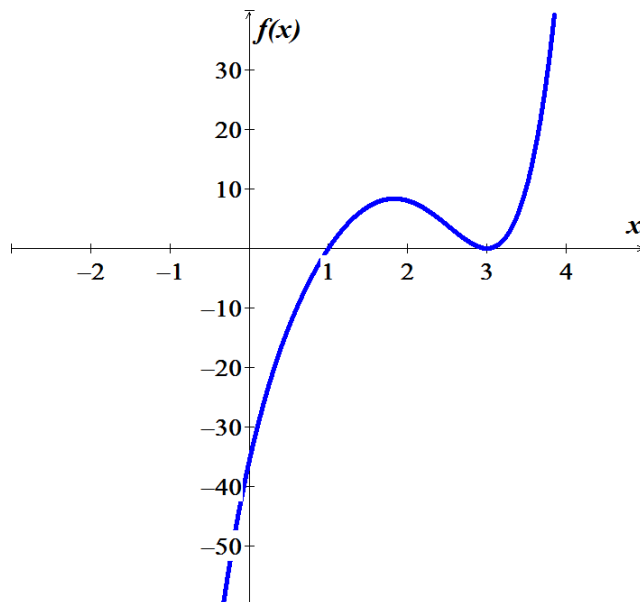
$$x^2 + 4 = 0 \Rightarrow x = \pm 2i$$

The zeros are: $x = 1, 3, 3$

	1	3	
	-	+	+

$$f(x) > 0 \quad (1, 3) \cup (3, \infty)$$

$$f(x) < 0 \quad (-\infty, 1)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

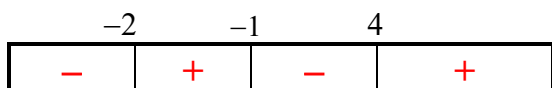
$$f(x) = x^3 - x^2 - 10x - 8$$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{8}{1} \right\} = \pm \{1, 2, 4, 8\}$

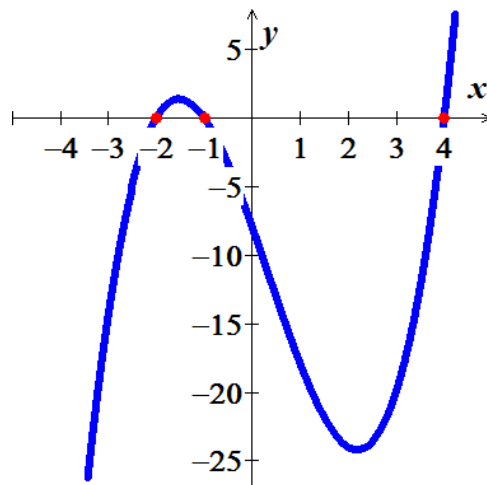
$$\begin{array}{r|rrrr} -1 & 1 & -1 & -10 & -8 \\ & & -1 & 2 & 8 \\ \hline & 1 & -2 & -8 & 0 \end{array} \rightarrow x^2 - 2x - 8 = 0$$

$$x = -1, -2, 4$$



$$f(x) > 0 \quad (-2, -1) \cup (4, \infty)$$

$$f(x) < 0 \quad (-\infty, -2) \cup (-1, 4)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

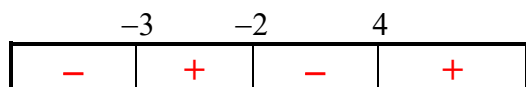
$$f(x) = x^3 + x^2 - 14x - 24$$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{24}{1} \right\} = \pm \{1, 2, 3, 4, 6, 8, 12, 24\}$

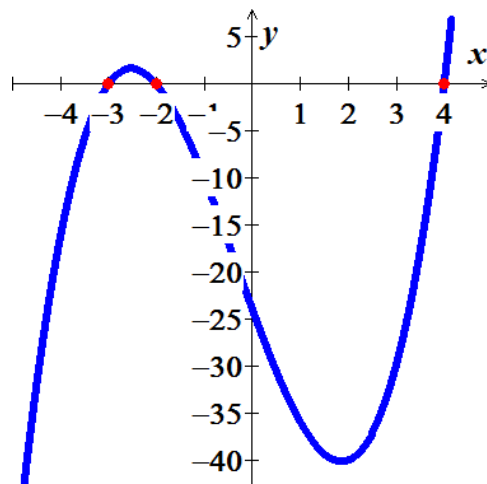
$$\begin{array}{r|rrrr} -2 & 1 & 1 & -14 & -24 \\ & & -2 & 2 & 24 \\ \hline & 1 & -1 & -12 & 0 \end{array} \rightarrow x^2 - x - 12 = 0$$

$$x = -2, -3, 4$$



$$f(x) > 0 \quad (-3, -2) \cup (4, \infty)$$

$$f(x) < 0 \quad (-\infty, -3) \cup (-2, 4)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

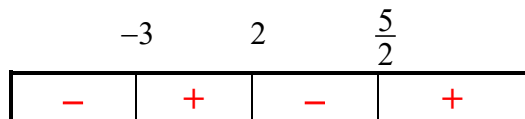
$$f(x) = 2x^3 - 3x^2 - 17x + 30$$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{30}{2} \right\} = \pm \left\{ 1, 2, 3, 5, 6, 10, 15, 30, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{15}{2} \right\}$

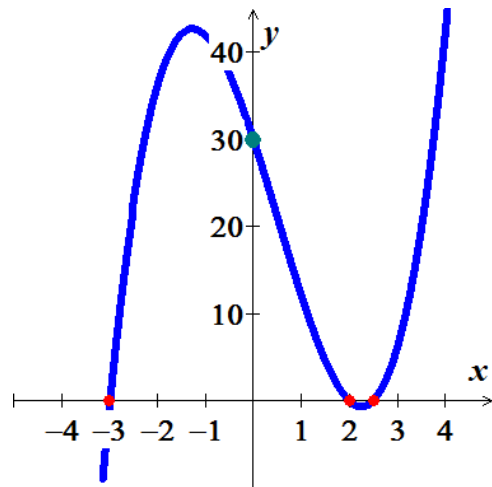
$$\begin{array}{r|rrrr} 2 & 2 & -3 & -17 & 30 \\ & & 4 & 2 & -30 \\ \hline & 2 & 1 & -15 & \boxed{0} \end{array} \rightarrow 2x^2 + x - 15 = 0$$

$$x = 2, -3, \frac{5}{2}$$



$$f(x) > 0 \quad \underline{(-3, 2) \cup \left(\frac{5}{2}, \infty \right)}$$

$$f(x) < 0 \quad \underline{(-\infty, -3) \cup \left(2, \frac{5}{2} \right)}$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

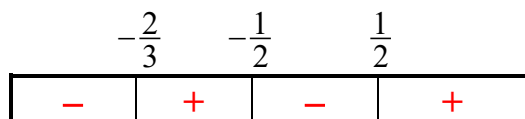
$$f(x) = 12x^3 + 8x^2 - 3x - 2$$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{2}{12} \right\} = \pm \left\{ 1, 2, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{2}, \frac{1}{6}, \frac{1}{12} \right\}$

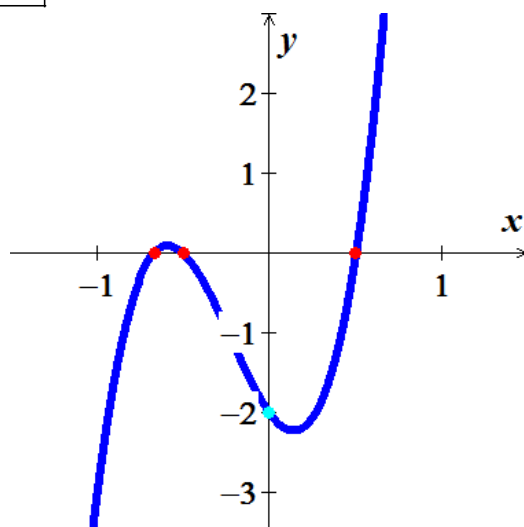
$$\begin{array}{r|rrrr} \frac{1}{2} & 12 & 8 & -3 & -2 \\ & & 6 & 7 & 2 \\ \hline & 12 & 14 & 4 & \boxed{0} \end{array} \rightarrow 12x^2 + 14x + 4 = 0$$

$$x = \frac{1}{2}, -\frac{1}{2}, -\frac{2}{3}$$



$$f(x) > 0 \quad \underline{\left(-\frac{2}{3}, -\frac{1}{2} \right) \cup \left(\frac{1}{2}, \infty \right)}$$

$$f(x) < 0 \quad \left(-\infty, -\frac{2}{3} \right) \cup \left(-\frac{1}{2}, \frac{1}{2} \right)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 + x^2 - 6x - 8$$

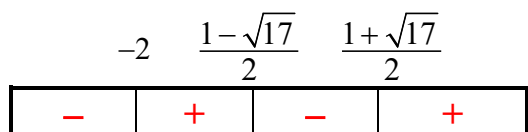
Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{8}{1} \right\} = \pm \{1, 2, 4, 8\}$

$$\begin{array}{r|rrrr} -2 & 1 & 1 & -6 & -8 \\ & & -2 & 2 & 8 \\ \hline & 1 & -1 & -4 & \boxed{0} \end{array} \rightarrow x^2 - x - 4 = 0$$

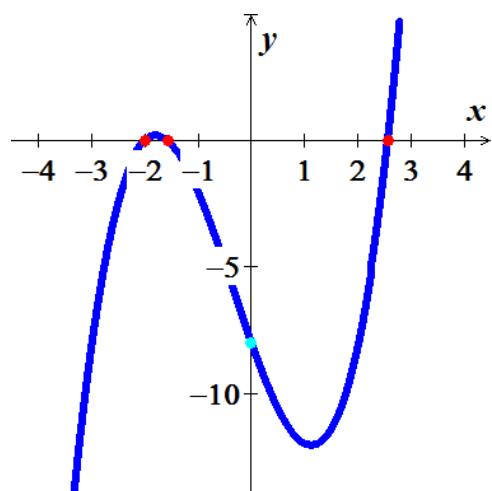
$$x = \frac{1 \pm \sqrt{1+16}}{2}$$

$$x = -2, \frac{1 \pm \sqrt{17}}{2}$$



$$f(x) > 0 \quad \left(-2, \frac{1-\sqrt{17}}{2} \right) \cup \left(\frac{1+\sqrt{17}}{2}, \infty \right)$$

$$f(x) < 0 \quad \left(-\infty, -2 \right) \cup \left(\frac{1-\sqrt{17}}{2}, \frac{1+\sqrt{17}}{2} \right)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 19x - 30$$

Solution

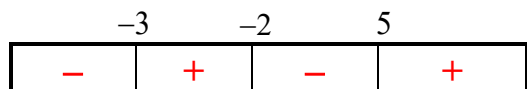
possibilities for $\frac{c}{d} : \pm \left\{ \frac{30}{1} \right\} = \pm \{1, 2, 3, 5, 6, 15, 30\}$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -19 & -30 \\ & & -2 & 4 & 30 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array} \rightarrow x^2 - 2x - 15$$

$$x = \frac{2 \pm \sqrt{4 + 60}}{2}$$

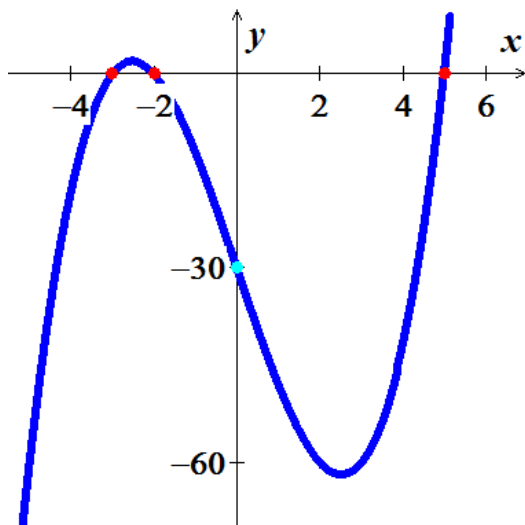
$$= \begin{cases} \frac{2-8}{2} = -3 \\ \frac{2+8}{2} = 5 \end{cases}$$

$$x = -2, -3, 5$$



$$f(x) > 0 \quad (-3, -2) \cup (5, \infty)$$

$$f(x) < 0 \quad (-\infty, -3) \cup (-2, 5)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 2x^3 + x^2 - 25x + 12$$

Solution

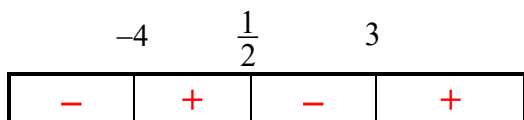
$$\text{possibilities for } \frac{c}{d} : \pm \left\{ \frac{12}{2} \right\} = \pm \left\{ 1, 2, 3, 4, 6, 12, \frac{1}{2}, \frac{3}{2} \right\}$$

$$\begin{array}{r|rrrr} 3 & 2 & 1 & -25 & 12 \\ & & 6 & 21 & -12 \\ \hline & 2 & 7 & -4 & \boxed{0} \end{array} \rightarrow 2x^2 + 7x - 4$$

$$x = \frac{-7 \pm \sqrt{49 + 32}}{4}$$

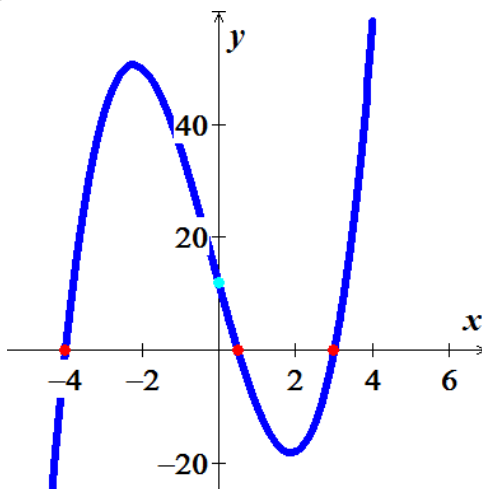
$$= \begin{cases} \frac{-7-9}{4} = -4 \\ \frac{-7+9}{4} = \frac{1}{2} \end{cases}$$

$$x = -4, \frac{1}{2}, 3$$



$$f(x) > 0 \quad \left(-4, \frac{1}{2} \right) \cup (3, \infty)$$

$$f(x) < 0 \quad \left(-\infty, -4 \right) \cup \left(\frac{1}{2}, 3 \right)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 3x^3 + 11x^2 - 6x - 8$$

Solution

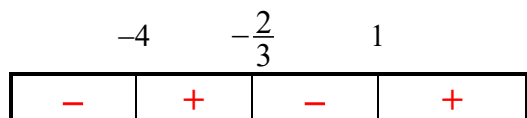
$$\text{possibilities for } \frac{c}{d} : \pm \left\{ \frac{8}{3} \right\} = \pm \left\{ 1, 2, 4, 8, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3} \right\}$$

$$\begin{array}{r|rrrr} 1 & 3 & 11 & -6 & -8 \\ & & 3 & 14 & 8 \\ \hline & 3 & 14 & 8 & \boxed{0} \end{array} \rightarrow 3x^2 + 14x + 8$$

$$x = \frac{-14 \pm \sqrt{196 - 96}}{6}$$

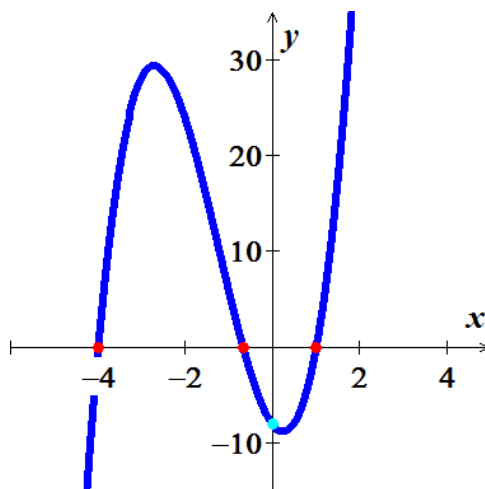
$$= \begin{cases} \frac{-14 - 10}{6} = -4 \\ \frac{-14 + 10}{6} = -\frac{2}{3} \end{cases}$$

$$x = -4, -\frac{2}{3}, 1$$



$$f(x) > 0 \quad \left(-4, -\frac{2}{3} \right) \cup (1, \infty)$$

$$f(x) < 0 \quad \left(-\infty, -4 \right) \cup \left(-\frac{2}{3}, 1 \right)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 2x^3 + 9x^2 - 2x - 9$$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{9}{2} \right\} = \pm \left\{ 1, 3, 9, \frac{1}{2}, \frac{3}{2}, \frac{9}{2} \right\}$

$$\begin{array}{r|rrrr} 1 & 2 & 9 & -2 & -9 \\ & & 2 & 11 & 9 \\ \hline & 2 & 11 & 9 & \boxed{0} \end{array} \rightarrow 2x^2 + 11x + 9$$

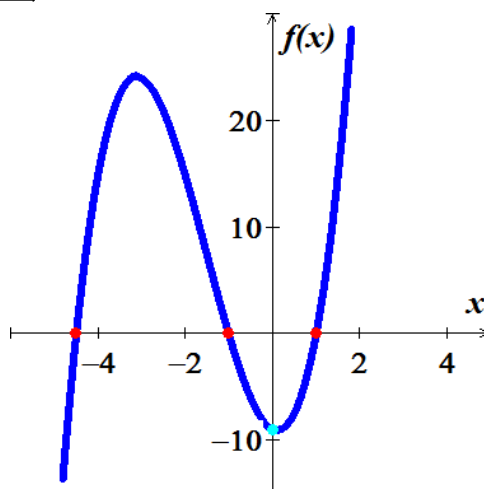
$$x = -1, -\frac{9}{2} \quad a - b + c = 0 \rightarrow x = -1, -\frac{c}{a}$$

$$x = -\frac{9}{2}, -1, 1$$

$-\frac{9}{2}$	-1	1
$-$	$+$	$-$
$+$	$-$	$+$

$$f(x) > 0 \quad \left(-\frac{9}{2}, -1 \right) \cup (1, \infty)$$

$$f(x) < 0 \quad \left(-\infty, -\frac{9}{2} \right) \cup (-1, 1)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 + 3x^2 - 6x - 8$$

Solution

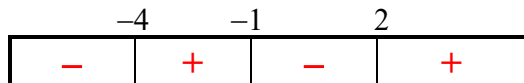
possibilities for $\frac{c}{d} : \pm \left\{ \frac{8}{1} \right\} = \pm \{1, 2, 4, 8\}$

$$\begin{array}{r|rrrr} -1 & 1 & 3 & -6 & -8 \\ & & -1 & -2 & 8 \\ \hline & 1 & 2 & -8 & \boxed{0} \end{array} \rightarrow x^2 + 2x - 8 = 0$$

$$x = \frac{-2 \pm \sqrt{4 + 32}}{2}$$

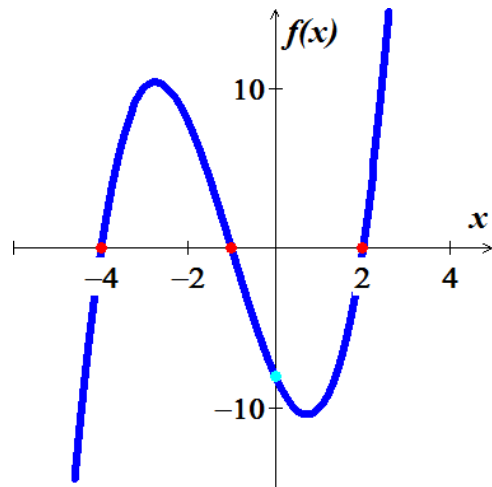
$$= \begin{cases} \frac{-2 - 6}{2} = -4 \\ \frac{-2 + 6}{2} = 2 \end{cases}$$

$$x = -4, -1, 2$$



$$f(x) > 0 \quad (-4, -1) \cup (2, \infty)$$

$$f(x) < 0 \quad (-\infty, -4) \cup (-1, 2)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 3x^3 - x^2 - 6x + 2$$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{2}{3} \right\} = \pm \left\{ 1, 2, \frac{1}{3}, \frac{2}{3} \right\}$

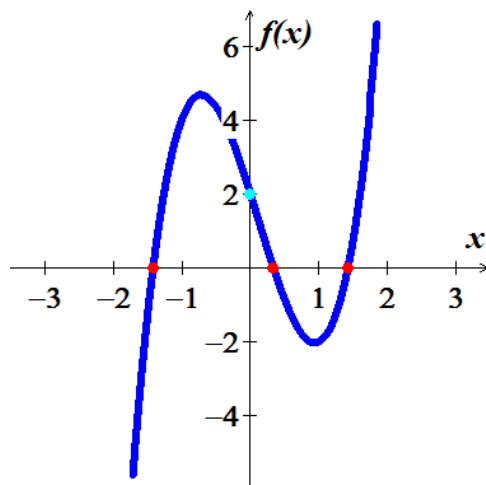
$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & -1 & -6 & 2 \\ & & 1 & 0 & -2 \\ \hline & 3 & 0 & -6 & \boxed{0} \end{array} \rightarrow 3x^2 - 6 = 0$$

$$x = \frac{1}{3}, \pm \sqrt{2}$$

	$-\sqrt{2}$	$\frac{1}{3}$	$\sqrt{2}$	
	-	+	-	+

$$f(x) > 0 \quad \left(-\sqrt{2}, \frac{1}{3} \right) \cup \left(\sqrt{2}, \infty \right)$$

$$f(x) < 0 \quad \left(-\infty, -\sqrt{2} \right) \cup \left(\frac{1}{3}, \sqrt{2} \right)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 8x^2 + 8x + 24$$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{24}{1} \right\} = \pm \{1, 2, 3, 4, 6, 8, 12, 24\}$

$$\begin{array}{r|rrrr} 6 & 1 & -8 & 8 & 24 \\ & & 6 & -12 & -24 \\ \hline & 1 & -2 & -4 & \boxed{0} \end{array} \rightarrow x^2 - 2x - 4 = 0$$

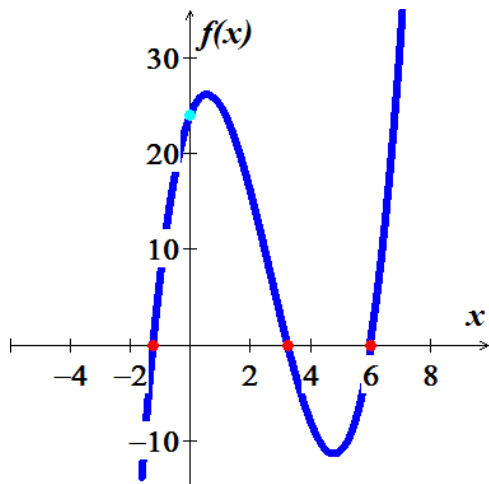
$$x = \frac{2 \pm \sqrt{4 + 16}}{2}$$

$$x = 6, 1 \pm \sqrt{5}$$

	$1 - \sqrt{5}$	$1 + \sqrt{5}$	6	
	-	+	-	+

$$f(x) > 0 \quad \left(1 - \sqrt{5}, 1 + \sqrt{5} \right) \cup \left(6, \infty \right)$$

$$f(x) < 0 \quad \underline{(-\infty, 1-\sqrt{5}) \cup (1+\sqrt{5}, 6)}$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 7x^2 - 7x + 69$$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{69}{1} \right\} = \pm \{1, 3, 23, 69\}$

$$\begin{array}{r|rrrr} -3 & 1 & -7 & -7 & 69 \\ & & -3 & 30 & -69 \\ \hline & 1 & -10 & 23 & \boxed{0} \end{array} \rightarrow x^2 - 10x + 23 = 0$$

$$x = \frac{10 \pm \sqrt{100 - 92}}{2}$$

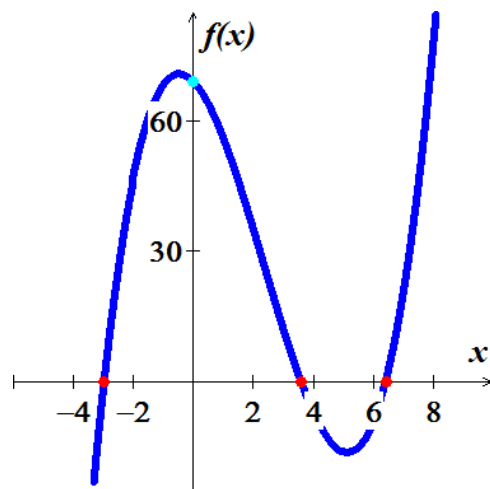
$$= \frac{10 \pm 2\sqrt{2}}{2}$$

$$\underline{x = -3, 5 \pm \sqrt{2}}$$

-3	$5 - \sqrt{2}$	$5 + \sqrt{2}$
-	+	-
-	+	-

$$f(x) > 0 \quad \underline{(-3, 5 - \sqrt{2}) \cup (5 + \sqrt{2}, \infty)}$$

$$f(x) < 0 \quad \underline{(-\infty, -3) \cup (5 - \sqrt{2}, 5 + \sqrt{2})}$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 3x - 2$$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{2}{1} \right\} = \pm \{1, 2\}$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -3 & -2 \\ & & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array} \rightarrow x^2 - x - 2 = 0$$

$$x = -1, 2$$

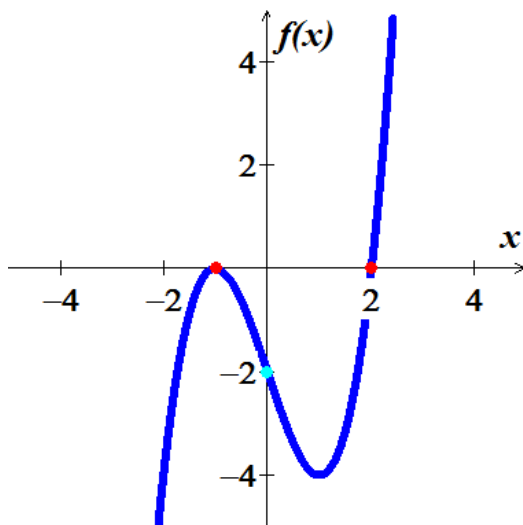
$$a - b + c = 0 \rightarrow x = -1, -\frac{c}{a}$$

$$\underline{x = -1, -1, 2}$$

	-1		2	
	-		-	
	+			

$$f(x) > 0 \quad (2, \infty)$$

$$f(x) < 0 \quad (-\infty, -1) \cup (-1, 2)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 2x + 1$$

Solution

possibilities for $\frac{c}{d} : \pm\{1\}$

$$\begin{array}{c|cccc} 1 & 1 & 0 & -2 & 1 \\ & & 1 & 1 & 1 \\ \hline & 1 & 1 & -1 & \boxed{0} \end{array} \rightarrow x^2 + x - 1 = 0$$

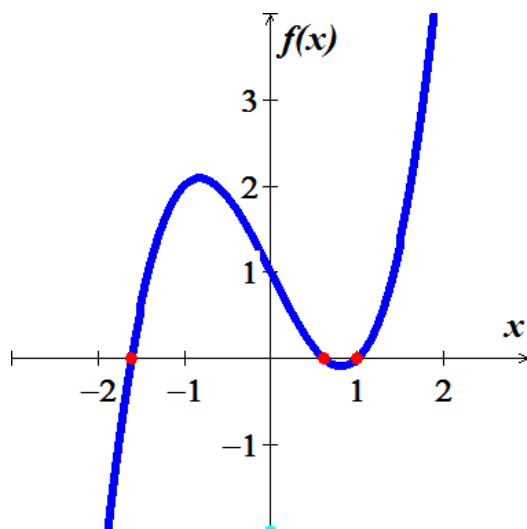
$$x = \frac{-1 \pm \sqrt{5}}{2}$$

$$x = 1, \frac{-1 \pm \sqrt{5}}{2}$$

$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	1
-	+	-
+	-	+

$$f(x) > 0 \quad \left(\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2} \right) \cup (1, \infty)$$

$$f(x) < 0 \quad \left(-\infty, \frac{-1-\sqrt{5}}{2} \right) \cup \left(\frac{-1+\sqrt{5}}{2}, 1 \right)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 2x^2 - 11x + 12$$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{12}{1} \right\} = \pm \{1, 2, 3, 4, 6, 12\}$

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -11 & 12 \\ & & 1 & -1 & 12 \\ \hline & 1 & -1 & -12 & \boxed{0} \end{array} \rightarrow x^2 - x - 12 = 0$$

$$x = \frac{1 \pm \sqrt{1 + 48}}{2}$$

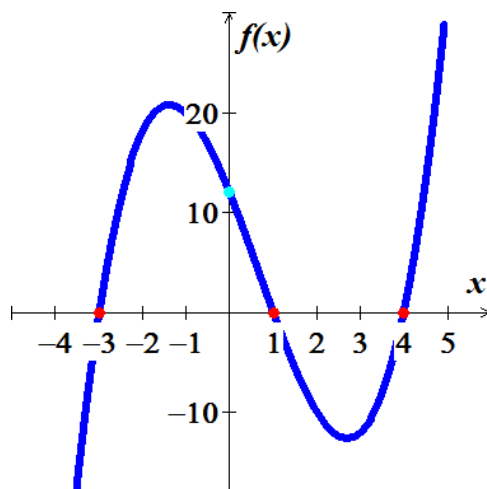
$$= \begin{cases} \frac{1-7}{2} = -3 \\ \frac{1+7}{2} = 4 \end{cases}$$

$$x = -3, 1, 4 \mid$$

-3	1	4	
-	+	-	+

$$f(x) > 0 \quad \underline{(-3, 1) \cup (4, \infty)} \mid$$

$$f(x) < 0 \quad \underline{(-\infty, -3) \cup (1, 4)} \mid$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 2x^2 - 7x - 4$$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{4}{1} \right\} = \pm \{1, 2, 4\}$

$$\begin{array}{r|rrrr} -1 & 1 & -2 & -7 & -4 \\ & & -1 & 3 & 4 \\ \hline & 1 & -3 & -4 & \boxed{0} \end{array} \rightarrow x^2 - 3x - 4 = 0$$

$$x = -1, 4 \mid$$

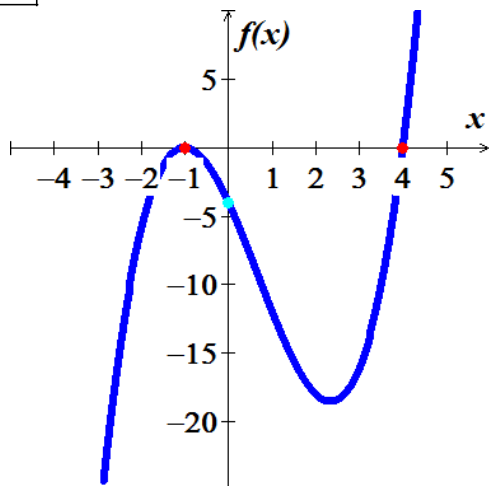
$$a - b + c = 0 \rightarrow x = -1, -\frac{c}{a}$$

$$x = -1, -1, 4 \mid$$

	-1		4	
-		-		+

$$f(x) > 0 \quad (4, \infty)$$

$$f(x) < 0 \quad (-\infty, -1) \cup (-1, 4)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 10x - 12$$

Solution

$$\text{possibilities for } \frac{c}{d} : \pm \left\{ \frac{12}{1} \right\} = \pm \{1, 2, 3, 4, 6, 12\}$$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -10 & -12 \\ & & -2 & 4 & 12 \\ \hline & 1 & -2 & -6 & \boxed{0} \end{array} \rightarrow x^2 - 2x - 6 = 0$$

$$x = \frac{2 \pm \sqrt{4 + 24}}{2}$$

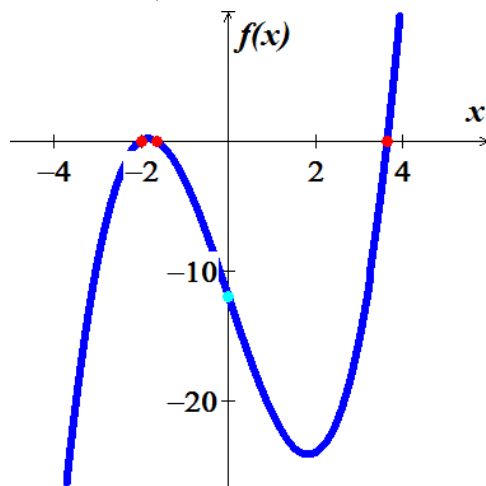
$$= \frac{2 \pm 2\sqrt{7}}{2}$$

$$x = -2, 1 \pm \sqrt{7}$$

	-2		$1 - \sqrt{7}$		$1 + \sqrt{7}$	
-		+		-		+

$$f(x) > 0 \quad (-2, 1 - \sqrt{7}) \cup (1 + \sqrt{7}, \infty)$$

$$f(x) < 0 \quad \underline{(-\infty, -2) \cup (1 - \sqrt{7}, 1 + \sqrt{7})}$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 5x^2 + 17x - 13$$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{13}{1} \right\} = \pm \{1, 13\}$

$$\begin{array}{r|rrrr} 1 & 1 & -5 & 17 & -13 \\ & & 1 & -4 & 13 \\ \hline & 1 & -4 & 13 & \boxed{0} \end{array} \rightarrow x^2 - 4x + 13 = 0$$

$$x = \frac{4 \pm \sqrt{16 - 52}}{2}$$

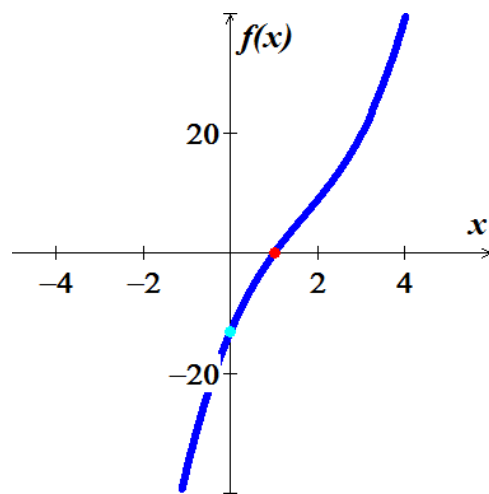
$$= \frac{4 \pm 6i}{2}$$

$$\underline{x = 1, 2 \pm 3i}$$

$$\begin{array}{c|c} 1 & \\ \hline - & + \end{array}$$

$$f(x) > 0 \quad \underline{(1, \infty)}$$

$$f(x) < 0 \quad \underline{(-\infty, 1)}$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 6x^3 + 25x^2 - 24x + 5$$

Solution

$$\text{possibilities for } \frac{c}{d} : \pm \left\{ \frac{5}{6} \right\} = \pm \left\{ 1, 5, \frac{1}{2}, \frac{5}{2}, \frac{1}{3}, \frac{5}{3}, \frac{1}{6}, \frac{5}{6} \right\}$$

$$\begin{array}{r|rrrr} -5 & 6 & 25 & -24 & 5 \\ & & -30 & 25 & -5 \\ \hline & 6 & -5 & 1 & \boxed{0} \end{array} \rightarrow 6x^2 - 5x + 1 = 0$$

$$x = \frac{5 \pm \sqrt{25 - 24}}{12}$$

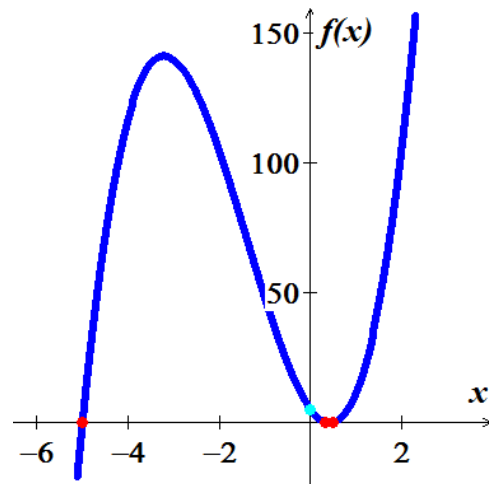
$$= \begin{cases} \frac{5-1}{12} = \frac{1}{3} \\ \frac{5+1}{12} = \frac{1}{2} \end{cases}$$

$$x = -5, \frac{1}{3}, \frac{1}{2}$$

-5	$\frac{1}{3}$	$\frac{1}{2}$
-	+	-
+	-	+

$$f(x) > 0 \quad \left(-5, \frac{1}{3} \right) \cup \left(\frac{1}{2}, \infty \right)$$

$$f(x) < 0 \quad \left(-\infty, -5 \right) \cup \left(\frac{1}{3}, \frac{1}{2} \right)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 8x^3 + 18x^2 + 45x + 27$$

Solution

$$\text{possibilities : } \pm \left\{ \frac{27}{8} \right\} = \pm \left\{ \frac{1}{1}, \frac{3}{2}, \frac{9}{4}, \frac{27}{8} \right\}$$

$$= \pm \left\{ 1, 3, 9, 27, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{9}{2}, \frac{9}{4}, \frac{9}{8}, \frac{27}{2}, \frac{27}{4}, \frac{27}{8} \right\}$$

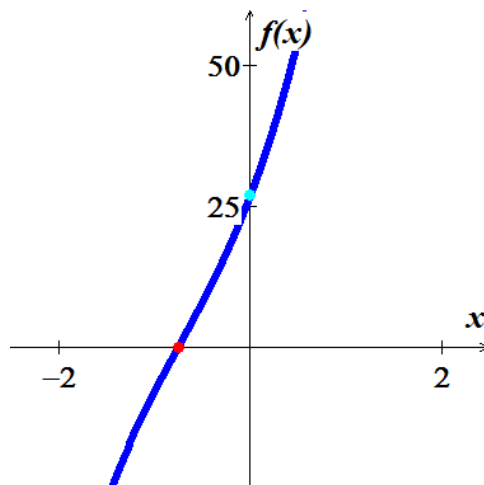
$$\begin{array}{r|rrrr} -\frac{3}{4} & 8 & 18 & 45 & 27 \\ & & -6 & -9 & -27 \\ \hline & 8 & 12 & 36 & \boxed{0} \end{array} \rightarrow 8x^2 + 12x + 36 = 0$$

$$x = -\frac{3}{4}, -\frac{3}{4} \pm i\frac{3\sqrt{7}}{4}$$

$-\frac{3}{4}$
<div style="display: flex; justify-content: space-around;"> - + </div>

$$f(x) > 0 \quad \left(-\frac{3}{4}, \infty\right)$$

$$f(x) < 0 \quad \left(-\infty, -\frac{3}{4}\right)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 3x^3 - x^2 + 11x - 20$$

Solution

$$\text{possibilities: } \pm \left\{ \frac{20}{3} \right\} = \pm \left\{ \frac{1, 2, 4, 5, 10, 20}{1, 3} \right\}$$

$$= \pm \left\{ 1, 2, 4, 5, 10, 20, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{10}{3}, \frac{20}{3} \right\}$$

$$\begin{array}{r|rrrr} \frac{4}{3} & 3 & -1 & 11 & -20 \\ & & 4 & 4 & 20 \\ \hline & 3 & 3 & 15 & \boxed{0} \end{array} \rightarrow 3x^2 + 3x + 15 = 0$$

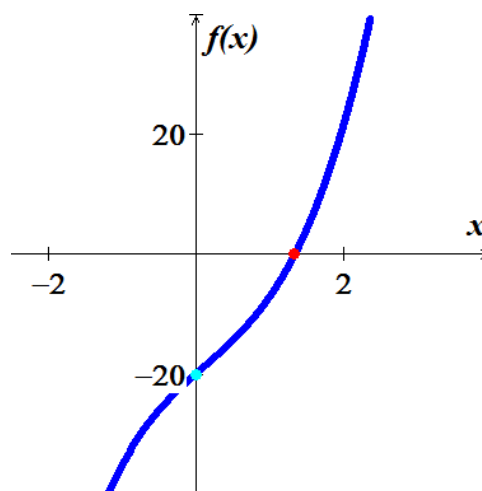
$$x = \frac{-3 \pm \sqrt{9 - 180}}{6}$$

$$x = \frac{4}{3}, -\frac{1}{2} \pm i\frac{\sqrt{19}}{2}$$

$\frac{4}{3}$
<div style="display: flex; justify-content: space-around;"> - + </div>

$$f(x) > 0 \quad \left(\frac{4}{3}, \infty\right)$$

$$f(x) < 0 \quad \left(-\infty, \frac{4}{3}\right)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^4 - x^3 - 9x^2 + 3x + 18$$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{18}{1} \right\} = \pm \{1, 2, 3, 6, 9, 18\}$

$$\begin{array}{r|rrrrr} -2 & 1 & -1 & -9 & 3 & 18 \\ & & -2 & 6 & 6 & -18 \\ \hline 3 & 1 & -3 & -3 & 9 & 0 \\ & & 3 & 0 & -9 & \\ \hline & 1 & 0 & -3 & 0 & \end{array} \rightarrow x^3 - 3x^2 - 3x + 9 = 0 \rightarrow \pm \left\{ \frac{9}{1} \right\} = \pm \{1, 3, 9\}$$

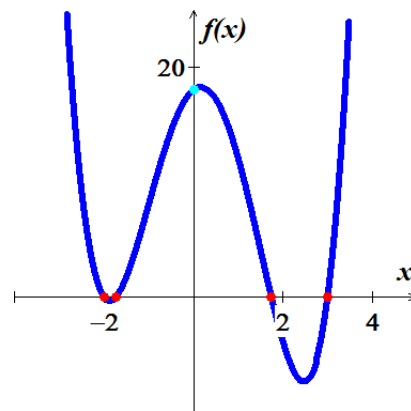
$$\rightarrow x^2 - 3 = 0 \Rightarrow x = \pm \sqrt{3}$$

$$x = -2, 3, \pm \sqrt{3}$$

	-2	$-\sqrt{3}$	$\sqrt{3}$	3
	+	-	+	-

$$f(x) > 0 \quad (-\infty, -2) \cup (-\sqrt{3}, \sqrt{3}) \cup (3, \infty)$$

$$f(x) < 0 \quad (-2, -\sqrt{3}) \cup (\sqrt{3}, 3)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 2x^4 - 9x^3 + 9x^2 + x - 3$$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{3}{2} \right\} = \pm \left\{ 1, 3, \frac{1}{2}, \frac{3}{2} \right\}$

$$\begin{array}{r|rrrrr} 1 & 2 & -9 & 9 & 1 & -3 \\ & & 2 & -7 & 2 & 3 \\ \hline 1 & 2 & -7 & 2 & 3 & 0 \\ & & 2 & -5 & -3 & \\ \hline & 2 & -5 & -3 & 0 & \end{array} \rightarrow 2x^3 - 7x^2 + 2x + 3 = 0 \rightarrow \pm \left\{ \frac{3}{2} \right\} = \pm \left\{ 1, 3, \frac{1}{2}, \frac{3}{2} \right\}$$

$$\rightarrow 2x^2 - 5x - 3 = 0$$

$$x = \frac{5 \pm \sqrt{25 + 24}}{4}$$

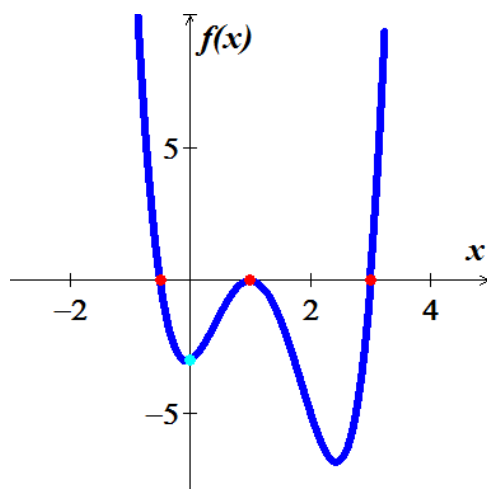
$$= \begin{cases} \frac{5-7}{4} = -\frac{1}{2} \\ \frac{5+7}{4} = 3 \end{cases}$$

$$\underline{x = 1, 1, -\frac{1}{2}, 3}$$

$-\frac{1}{2}$	1	3
$+$	$-$	$-$
$+$	$-$	$+$

$$f(x) > 0 \quad \underline{\left(-\infty, -\frac{1}{2}\right) \cup (3, \infty)}$$

$$f(x) < 0 \quad \underline{\left(-\frac{1}{2}, 1\right) \cup (1, 3)}$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 6x^4 + 5x^3 - 17x^2 - 6x$$

Solution

$$x(6x^3 + 5x^2 - 17x - 6) = 0 \rightarrow \underline{x = 0}$$

$$\text{possibilities: } \pm \left\{ \frac{6}{6} \right\} = \pm \left\{ 1, 2, 3, 6, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{2}{3} \right\}$$

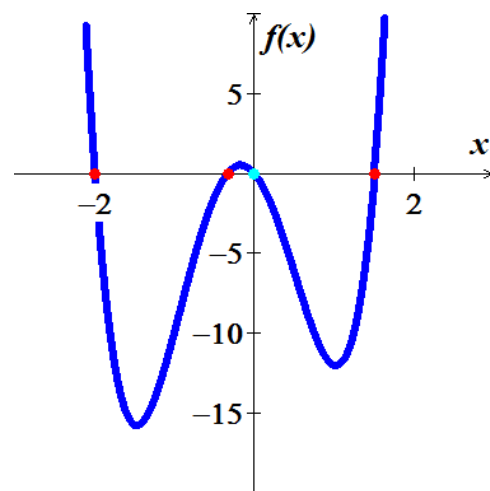
$$\begin{array}{r|rrrr} -2 & 6 & 5 & -17 & -6 \\ & & -12 & 14 & 6 \\ \hline & 6 & -7 & -3 & \boxed{0} \end{array} \rightarrow 6x^2 - 7x - 3 = 0$$

$$\underline{x = 0, -2, -\frac{1}{3}, \frac{3}{2}}$$

-2	$-\frac{1}{3}$	0	$\frac{3}{2}$
$+$	$-$	$+$	$-$
$+$	$-$	$+$	$-$

$$f(x) > 0 \quad \underline{\left(-\infty, -2\right) \cup \left(-\frac{1}{3}, 0\right) \cup \left(\frac{3}{2}, \infty\right)}$$

$$f(x) < 0 \quad \underline{\left(-2, -\frac{1}{3}\right) \cup \left(0, \frac{3}{2}\right)}$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^4 - 2x^2 - 16x - 15$$

Solution

possibilities : $\pm \left\{ \frac{15}{1} \right\} = \pm \{1, 3, 5, 15\}$

$$\begin{array}{r|rrrrr} -1 & 1 & 0 & -2 & -16 & -15 \\ & & -1 & 1 & 1 & 15 \\ \hline 3 & 1 & -1 & -1 & -15 & 0 \\ & & 3 & 6 & 15 & \\ \hline & 1 & 2 & 5 & 0 & \end{array} \rightarrow x^3 - x^2 - x - 15 = 0 \rightarrow \pm \left\{ \frac{15}{1} \right\} = \pm \{1, 3, 5, 15\}$$

$$\rightarrow x^2 + 2x + 5 = 0$$

$$x = \frac{-2 \pm \sqrt{-16}}{2}$$

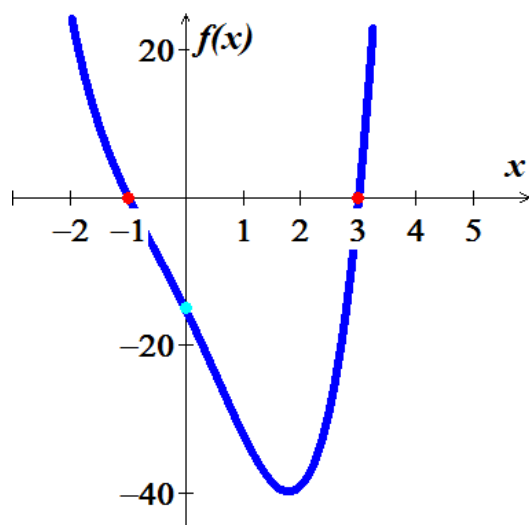
$$= -1 \pm 2i$$

$$\underline{x = -1, 3, -1 \pm 2i}$$

-1	3
+	-

$$f(x) > 0 \quad \underline{(-\infty, -1) \cup (3, \infty)}$$

$$f(x) < 0 \quad \underline{(-1, 3)}$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^4 - 2x^3 - 5x^2 + 8x + 4$$

Solution

possibilities: $\pm \left\{ \frac{4}{1} \right\} = \pm \{1, 2, 4\}$

$$\begin{array}{r|rrrrr} 2 & 1 & -2 & -5 & 8 & 4 \\ & & 2 & 0 & -10 & -4 \\ \hline -2 & 1 & 0 & -5 & -2 & 0 \\ & & -2 & 4 & 2 & \\ \hline & 1 & -2 & -1 & 0 & \end{array} \rightarrow x^3 - 5x - 2 = 0 \rightarrow \pm \left\{ \frac{2}{1} \right\} = \pm \{1, 2\}$$

$$\rightarrow x^2 - 2x - 1 = 0$$

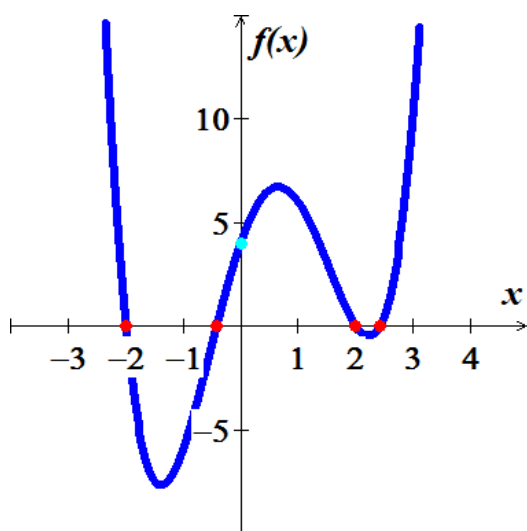
$$x = \frac{2 \pm \sqrt{8}}{2}$$

$$x = -2, 2, 1 \pm \sqrt{2}$$

-2	$1 - \sqrt{2}$	2	$1 + \sqrt{2}$
+	-	+	-

$$f(x) > 0 \quad \underline{(-\infty, -2) \cup (1 - \sqrt{2}, 2) \cup (1 + \sqrt{2}, \infty)}$$

$$f(x) < 0 \quad \underline{(-2, 1 - \sqrt{2}) \cup (2, 1 + \sqrt{2})}$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 2x^4 - 17x^3 + 4x^2 + 35x - 24$$

Solution

possibilities : $\pm \left\{ \frac{24}{2} \right\} = \pm \left\{ 1, 2, 3, 4, 6, 8, 12, 24, \frac{1}{2}, \frac{3}{2} \right\}$

$$\begin{array}{r|rrrrrr} 1 & 2 & -17 & 4 & 35 & -24 \\ & & 2 & -15 & -11 & 24 \\ \hline 1 & 2 & -15 & -11 & 24 & 0 \\ & & 2 & -13 & 24 & \\ \hline & 2 & -13 & -24 & 0 & \end{array} \rightarrow 2x^3 - 15x^2 - 11x + 24 = 0 \rightarrow \pm \left\{ \frac{2}{1} \right\} = \pm \{1, 2\}$$

$$\rightarrow 2x^2 - 13x - 24 = 0$$

$$x = \frac{13 \pm \sqrt{169 + 192}}{4}$$

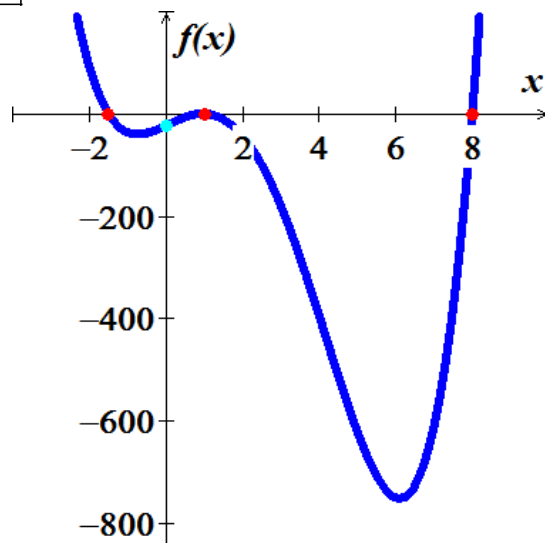
$$= \begin{cases} \frac{13-19}{4} = -\frac{3}{2} \\ \frac{13+19}{4} = 8 \end{cases}$$

$$x = -\frac{3}{2}, 1, 1, 8$$

$-\frac{3}{2}$	1	8
$+$	$-$	$+$

$$f(x) > 0 \quad \left(-\infty, -\frac{3}{2} \right) \cup (8, \infty)$$

$$f(x) < 0 \quad \left(-\frac{3}{2}, 1 \right) \cup (1, 8)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^4 + x^3 - 3x^2 - 5x - 2$$

Solution

possibilities: $\pm \left\{ \frac{2}{1} \right\} = \pm \{1, 2\}$

$$\begin{array}{r|rrrrr} -1 & 1 & 1 & -3 & -5 & -2 \\ & & -1 & 0 & 3 & 2 \\ \hline -1 & 1 & 0 & -3 & -2 & 0 \\ & & -1 & 1 & 2 & \\ \hline & 1 & -1 & -2 & 0 & \end{array} \rightarrow x^3 - 3x - 2 = 0 \rightarrow \pm \left\{ \frac{2}{1} \right\} = \pm \{1, 2\}$$

$$\rightarrow x^2 - x - 2 = 0$$

$$x = \frac{1 \pm \sqrt{1+8}}{2}$$

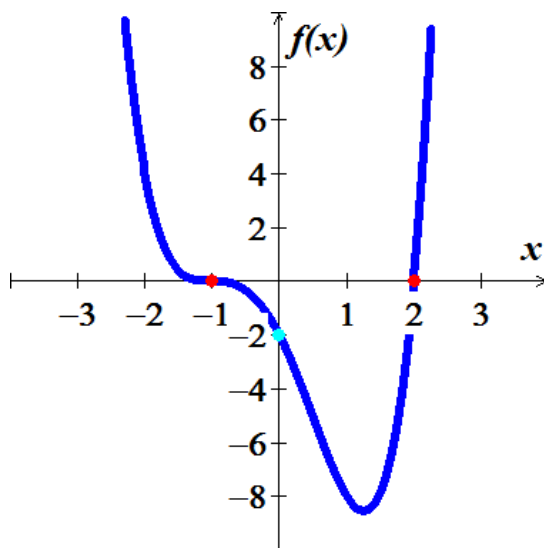
$$= \begin{cases} \frac{1-3}{2} = -1 \\ \frac{1+3}{2} = 2 \end{cases}$$

$$x = -1, -1, -1, 2$$

-1	2
+	-

$$f(x) > 0 \quad (-\infty, -1) \cup (2, \infty)$$

$$f(x) < 0 \quad (-2, 2)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 6x^4 - 17x^3 - 11x^2 + 42x$$

Solution

$$x(6x^3 - 17x^2 - 11x + 42) = 0$$

$$x = 0 \quad 6x^3 - 17x^2 - 11x + 42 = 0$$

$$\text{possibilities: } \pm \left\{ \frac{42}{6} \right\} = \pm \left\{ 1, 2, 3, 6, 7, 14, 21, 42, \frac{1}{2}, \frac{3}{2}, \frac{7}{2}, \frac{21}{2}, \frac{1}{3}, \frac{2}{3}, \frac{7}{3}, \frac{14}{3}, \frac{1}{6}, \frac{7}{6}, \frac{21}{6} \right\}$$

$$\begin{array}{r|rrrr} 2 & 6 & -17 & -11 & 42 \\ & & 12 & -10 & -42 \\ \hline & 6 & -5 & -21 & \boxed{0} \end{array} \rightarrow 6x^2 - 5x - 21 = 0$$

$$x = \frac{5 \pm \sqrt{25 + 504}}{12}$$

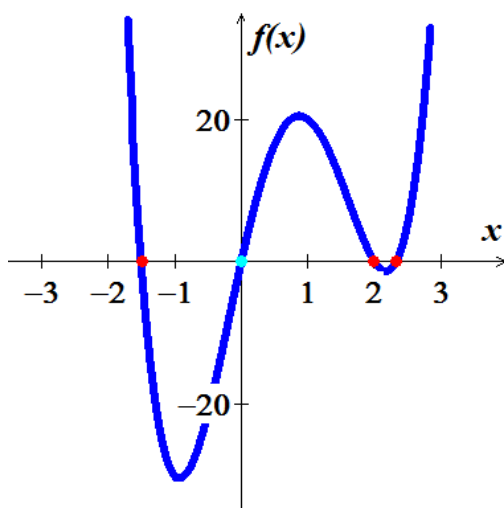
$$= \begin{cases} \frac{5 - 23}{12} = -\frac{3}{2} \\ \frac{5 + 23}{12} = \frac{7}{3} \end{cases}$$

$$x = -\frac{3}{2}, 0, 2, \frac{7}{3}$$

$-\frac{3}{2}$	0	2	$\frac{7}{3}$
$+$	$-$	$+$	$-$

$$f(x) > 0 \quad \left(-\infty, -\frac{3}{2} \right) \cup (0, 2) \cup \left(\frac{7}{3}, \infty \right)$$

$$f(x) < 0 \quad \left(-\frac{3}{2}, 0 \right) \cup \left(2, \frac{7}{3} \right)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^4 - 5x^2 - 2x$$

Solution

$$x(x^3 - 5x - 2) = 0$$

$$x = 0 \quad x^3 - 5x - 2 = 0$$

$$\text{possibilities: } \pm \left\{ \frac{2}{1} \right\} = \pm \{1, 2\}$$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -5 & -2 \\ & & -2 & 4 & 2 \\ \hline & 1 & -2 & -1 & \boxed{0} \end{array} \rightarrow x^2 - 2x - 1 = 0$$

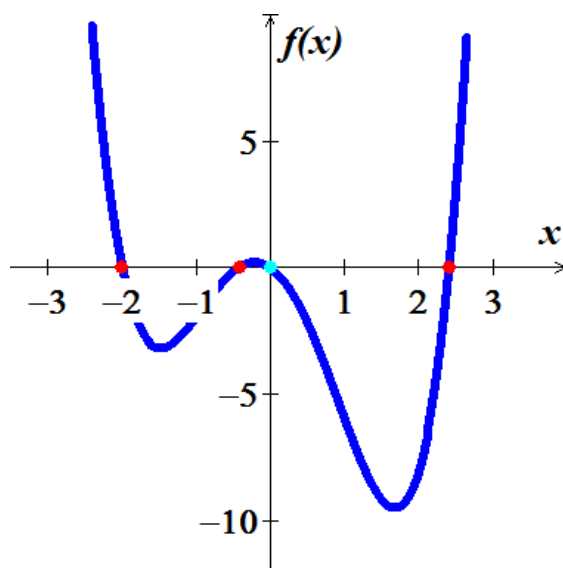
$$x = \frac{2 \pm \sqrt{8}}{2}$$

$$x = -2, 0, 1 \pm \sqrt{2}$$

-2	$1 - \sqrt{2}$	2	$1 + \sqrt{2}$
+	-	+	-

$$f(x) > 0 \quad \underline{(-\infty, -2) \cup (1 - \sqrt{2}, 2) \cup (1 + \sqrt{2}, \infty)}$$

$$f(x) < 0 \quad \underline{(-2, 1 - \sqrt{2}) \cup (2, 1 + \sqrt{2})}$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 3x^4 - 4x^3 - 11x^2 + 16x - 4$$

Solution

possibilities : $\pm \left\{ \frac{4}{3} \right\} = \pm \left\{ 1, 2, 4, \frac{1}{3}, \frac{2}{3}, \frac{4}{3} \right\}$

$$\begin{array}{r|rrrrr} 1 & 3 & -4 & -11 & 16 & -4 \\ & & 3 & -1 & -12 & 4 \\ \hline 2 & 3 & -1 & -12 & 4 & 0 \\ & & 6 & 10 & -4 & \\ \hline & 3 & 5 & -2 & 0 & \end{array} \rightarrow 3x^3 - x^2 - 12x + 4 = 0 \rightarrow \pm \left\{ \frac{4}{3} \right\} = \pm \left\{ 1, 2, 4, \frac{1}{3}, \frac{2}{3}, \frac{4}{3} \right\}$$

$$\rightarrow 3x^2 + 5x - 2 = 0$$

$$x = \frac{-5 \pm \sqrt{25 + 24}}{6}$$

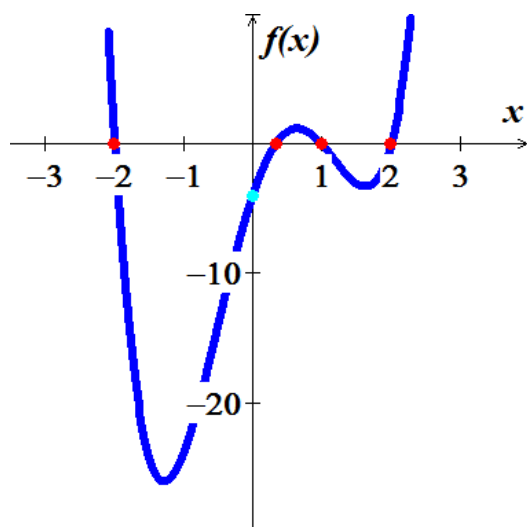
$$= \begin{cases} \frac{-5-7}{6} = -2 \\ \frac{-5+7}{6} = \frac{1}{3} \end{cases}$$

$$x = -2, \frac{1}{3}, 1, 2$$

-2	$\frac{1}{3}$	1	2
+	-	+	-

$$f(x) > 0 \quad \left(-\infty, -2 \right) \cup \left(\frac{1}{3}, 1 \right) \cup (2, \infty)$$

$$f(x) < 0 \quad \left(-2, \frac{1}{3} \right) \cup (1, 2)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 6x^4 + 23x^3 + 19x^2 - 8x - 4$$

Solution

$$\text{possibilities: } \pm \left\{ \frac{4}{6} \right\} = \pm \left\{ 1, 2, 4, \frac{1}{6}, \frac{2}{3}, \frac{2}{3} \right\}$$

$$\begin{array}{r|rrrrr} -2 & 6 & 23 & 19 & -8 & -4 \\ & & -12 & -22 & 6 & 4 \\ \hline -2 & 6 & 11 & -3 & -2 & 0 \\ & & -12 & 2 & 2 & \\ \hline & 6 & -1 & -1 & 0 & \end{array} \rightarrow 6x^3 + 11x^2 - 3x - 2 = 0 \rightarrow \pm \left\{ \frac{2}{6} \right\} = \pm \left\{ 1, 2, \frac{1}{6}, \frac{1}{3} \right\}$$

$$\rightarrow 6x^2 - x - 1 = 0$$

$$x = \frac{1 \pm \sqrt{25}}{12}$$

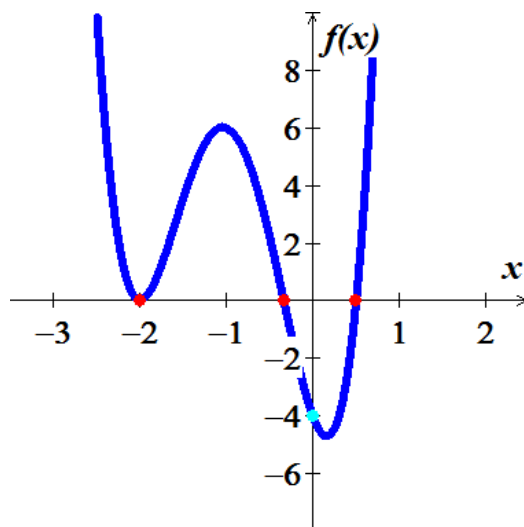
$$= \begin{cases} \frac{1-5}{12} = -\frac{1}{3} \\ \frac{1+5}{12} = \frac{1}{2} \end{cases}$$

$$x = -2, -2, -\frac{1}{3}, \frac{1}{2} \mid$$

$$\begin{array}{c|c|c|c|c} -2 & -\frac{1}{3} & \frac{1}{2} & & \\ \hline + & + & - & + & \end{array}$$

$$f(x) > 0 \quad \underline{(-\infty, -2) \cup (-2, -\frac{1}{3}) \cup (\frac{1}{2}, \infty)}$$

$$f(x) < 0 \quad \underline{(-\frac{1}{3}, \frac{1}{2})}$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 4x^4 - 12x^3 + 3x^2 + 12x - 7$$

Solution

possibilities : $\pm \left\{ \frac{7}{4} \right\} = \pm \left\{ 1, 7, \frac{1}{2}, \frac{7}{2}, \frac{1}{4}, \frac{7}{4} \right\}$

$$\begin{array}{r|rrrrr} 1 & 4 & -12 & 3 & 12 & -7 \\ & & 4 & -8 & -5 & 7 \\ \hline -1 & 4 & -8 & -5 & 7 & 0 \\ & & -4 & 12 & -7 & \\ \hline & 4 & -12 & 7 & 0 & \end{array} \rightarrow 4x^3 - 8x^2 - 5x + 7 = 0 \rightarrow \pm \left\{ \frac{7}{4} \right\} = \pm \left\{ 1, 7, \frac{1}{2}, \frac{7}{2}, \frac{1}{4}, \frac{7}{4} \right\}$$

$$\rightarrow 4x^2 - 12x + 7 = 0$$

$$x = \frac{12 \pm \sqrt{144 - 112}}{8}$$

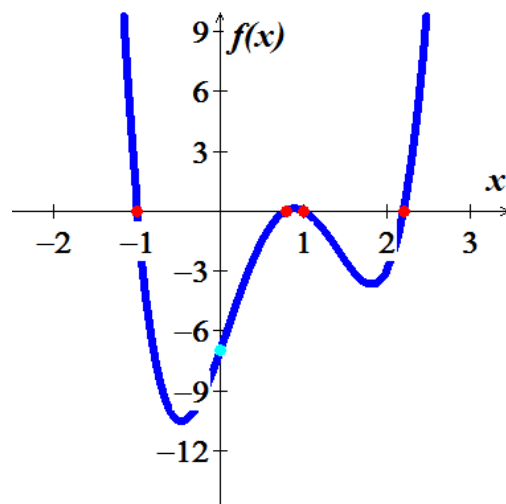
$$= \frac{12 \pm 4\sqrt{2}}{8}$$

$$x = -1, 1, \frac{3 \pm \sqrt{2}}{2}$$

-1	$\frac{3-\sqrt{2}}{2}$	1	$\frac{3+\sqrt{2}}{2}$
+	-	+	-

$$f(x) > 0 \quad \left(-\infty, -1 \right) \cup \left(\frac{3-\sqrt{2}}{2}, 1 \right) \cup \left(\frac{3+\sqrt{2}}{2}, \infty \right)$$

$$f(x) < 0 \quad \left(-1, \frac{3-\sqrt{2}}{2} \right) \cup \left(1, \frac{3+\sqrt{2}}{2} \right)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 2x^4 - 9x^3 - 2x^2 + 27x - 12$$

Solution

$$\text{possibilities: } \pm \left\{ \frac{12}{2} \right\} = \pm \left\{ 1, 2, 3, 4, 6, 12, \frac{1}{2}, \frac{3}{2} \right\}$$

$$\begin{array}{r|rrrrr} 4 & 2 & -9 & -2 & 27 & -12 \\ & & 8 & -4 & -24 & 12 \\ \hline \frac{1}{2} & 2 & -1 & -6 & 3 & 0 \\ & & 1 & 0 & -3 & \\ \hline & 2 & 0 & -6 & 0 & \end{array} \rightarrow 2x^3 - x^2 - 6x + 3 = 0 \rightarrow \pm \left\{ \frac{3}{2} \right\} = \pm \left\{ 1, 3, \frac{1}{2}, \frac{3}{2} \right\}$$

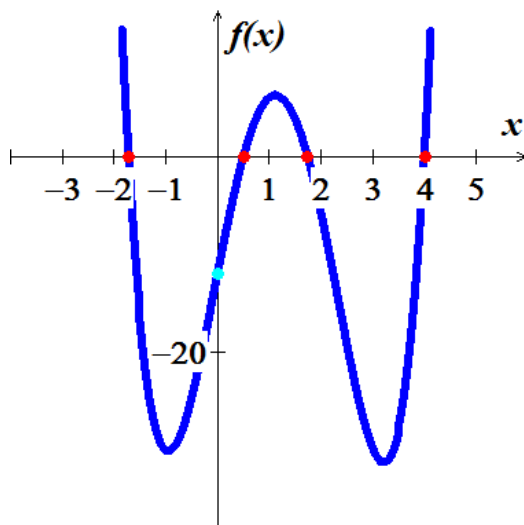
$$\rightarrow 2x^2 - 6 = 0$$

$$x = \frac{1}{2}, 4, \pm\sqrt{3}$$

$-\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	4	
+	-	+	-	+

$$f(x) > 0 \quad \left(-\infty, -\sqrt{3} \right) \cup \left(\frac{1}{2}, \sqrt{3} \right) \cup (4, \infty)$$

$$f(x) < 0 \quad \left(-\sqrt{3}, \frac{1}{2} \right) \cup (\sqrt{3}, 4)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 2x^4 - 19x^3 + 51x^2 - 31x + 5$$

Solution

possibilities : $\pm \left\{ \frac{5}{2} \right\} = \pm \left\{ 1, 5, \frac{1}{2}, \frac{5}{2} \right\}$

$$\begin{array}{r|rrrrr} 5 & 2 & -19 & 51 & -31 & 5 \\ & & 10 & -45 & 30 & -5 \\ \hline \frac{1}{2} & 2 & -9 & 6 & -1 & 0 \\ & & 1 & -4 & 1 & \\ \hline & 2 & -8 & 2 & 0 & \end{array} \rightarrow 2x^3 - 9x^2 + 6x - 1 = 0 \rightarrow \pm \left\{ \frac{1}{2} \right\}$$

$$\rightarrow 2x^2 - 8x + 2 = 0$$

$$x = \frac{8 \pm \sqrt{64 - 16}}{4}$$

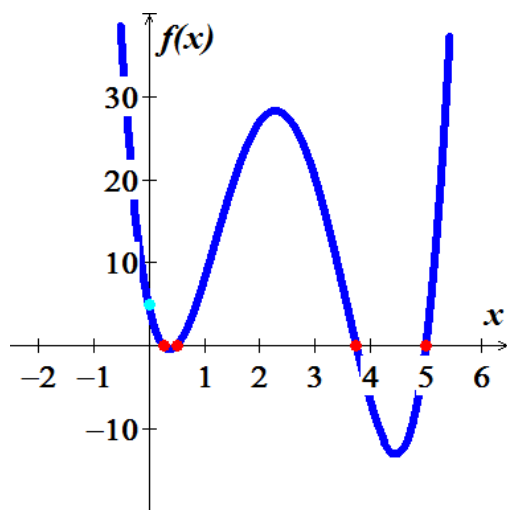
$$= \frac{8 \pm 4\sqrt{3}}{4}$$

$$x = \frac{1}{2}, 5, 2 \pm \sqrt{3}$$

$2 - \sqrt{3}$	$\frac{1}{2}$	$2 + \sqrt{3}$	5
+	-	+	-

$$f(x) > 0 \quad \left(-\infty, 2 - \sqrt{3} \right) \cup \left(\frac{1}{2}, 2 + \sqrt{3} \right) \cup (5, \infty)$$

$$f(x) < 0 \quad \left(2 - \sqrt{3}, \frac{1}{2} \right) \cup (2 + \sqrt{3}, 5)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 4x^4 - 35x^3 + 71x^2 - 4x - 6$$

Solution

$$\text{possibilities: } \pm \left\{ \frac{6}{4} \right\} = \pm \left\{ 1, 2, 3, 6, \frac{1}{2}, \frac{3}{2}, \frac{1}{4}, \frac{3}{4} \right\}$$

$$\begin{array}{r|rrrrr} 3 & 4 & -35 & 71 & -4 & -6 \\ & & 12 & -69 & 6 & 6 \\ \hline -\frac{1}{4} & 4 & -23 & 2 & 2 & 0 \\ & & -1 & 6 & -2 & \\ \hline & 4 & -24 & 8 & 0 & \end{array} \rightarrow 4x^3 - 23x^2 + 2x + 2 = 0 \rightarrow \pm \left\{ \frac{2}{4} \right\} = \pm \left\{ 1, 2, \frac{1}{2}, \frac{1}{4} \right\}$$

$$\rightarrow 4x^2 - 24x + 8 = 0$$

$$x^2 - 6x + 2 = 0$$

$$x = \frac{6 \pm \sqrt{36 - 8}}{2}$$

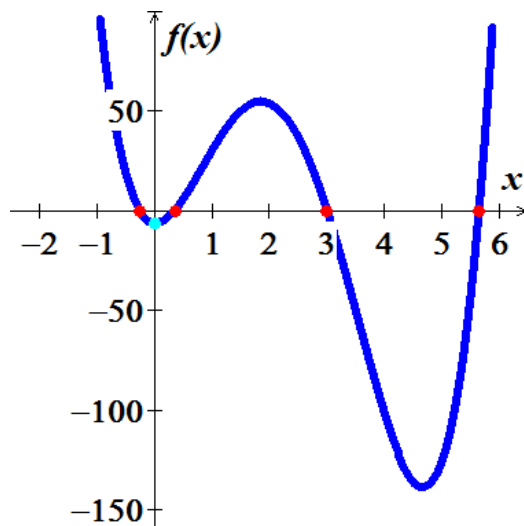
$$= \frac{6 \pm 2\sqrt{7}}{4}$$

$$x = -\frac{1}{4}, 3, 3 \pm \sqrt{7} \quad |$$

$-\frac{1}{4}$	$3 - \sqrt{7}$	3	$3 + \sqrt{7}$
+	-	+	-

$$f(x) > 0 \quad \left(-\infty, -\frac{1}{4} \right) \cup (3 - \sqrt{7}, 3) \cup (3 + \sqrt{7}, \infty) \quad |$$

$$f(x) < 0 \quad \left(-\frac{1}{4}, 3 - \sqrt{7} \right) \cup (3, 3 + \sqrt{7}) \quad |$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 2x^4 + 3x^3 - 4x^2 - 3x + 2$$

Solution

possibilities : $\pm \left\{ \frac{2}{2} \right\} = \pm \left\{ 1, 2, \frac{1}{2} \right\}$

$$\begin{array}{r|rrrrr} 1 & 2 & 3 & -4 & -3 & 2 \\ & & 2 & 5 & 1 & -2 \\ \hline -1 & 2 & 5 & 1 & -2 & 0 \\ & & -2 & -3 & 2 & \\ \hline & 2 & 3 & -2 & 0 & \end{array} \rightarrow 2x^3 - 23x^2 + 2x - 2 = 0 \rightarrow \pm \left\{ \frac{2}{2} \right\} = \pm \left\{ 1, 2, \frac{1}{2} \right\}$$

$$\rightarrow 2x^2 + 3x - 2 = 0$$

$$x = \frac{-3 \pm \sqrt{9 + 16}}{4}$$

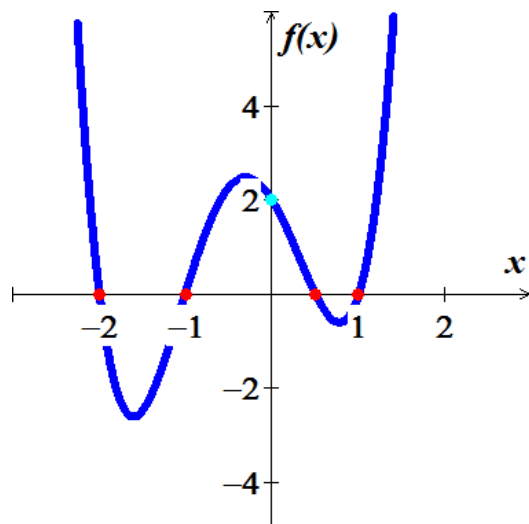
$$= \begin{cases} \frac{-3 - 5}{4} = -2 \\ \frac{-3 + 5}{4} = \frac{1}{2} \end{cases}$$

$$x = -2, -1, \frac{1}{2}, 1$$

-2	-1	$\frac{1}{2}$	1	
+	-	+	-	+

$$f(x) > 0 \quad \left(-\infty, -2 \right) \cup \left(-1, \frac{1}{2} \right) \cup (1, \infty)$$

$$f(x) < 0 \quad \left(-2, -1 \right) \cup \left(\frac{1}{2}, 1 \right)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^4 + 3x^3 - 30x^2 - 6x + 56$$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{56}{1} \right\} = \pm \{1, 2, 4, 7, 8, 14, 28, 56\}$

$$\begin{array}{r|rrrrr} 4 & 1 & 3 & -30 & -6 & 56 \\ & & 4 & 28 & -8 & -56 \\ \hline -7 & 1 & 7 & -2 & -14 & 0 \\ & & -7 & 0 & 14 & \\ \hline & 1 & 0 & -2 & & \end{array} \rightarrow x^3 + 7x^2 - 2x - 14 = 0 \Rightarrow \frac{c}{d} = \pm \left\{ \frac{14}{1} \right\} = \pm \{1, 2, 7, 14\}$$

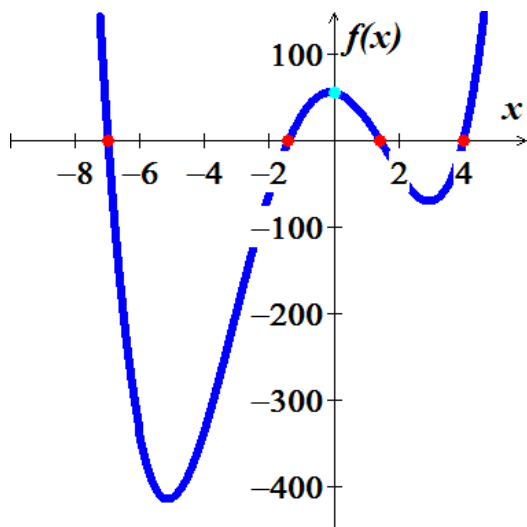
$$\rightarrow x^2 - 2 = 0 \Rightarrow x = \pm\sqrt{2}$$

$$x = 4, -7, \pm\sqrt{2}$$

-7	$-\sqrt{2}$	$\sqrt{2}$	4	
+	-	+	-	+

$$f(x) > 0 \quad (-\infty, -7) \cup (-\sqrt{2}, \sqrt{2}) \cup (4, \infty)$$

$$f(x) < 0 \quad (-7, -\sqrt{2}) \cup (\sqrt{2}, 4)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 3x^5 - 10x^4 - 6x^3 + 24x^2 + 11x - 6$$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{6}{3} \right\} = \pm \left\{ 1, 2, 3, 6, \frac{1}{3}, \frac{2}{3} \right\}$

-1	3	-10	-6	24	11	-6
		-3	13	-7	-17	6
-1	3	-13	7	17	-6	0
		-3	16	-23	6	
2	3	-16	23	-6	0	
		6	20	6		
	3	-10	3	0		

$$x^4 - 13x^3 + 7x^2 + 17x - 6 = 0 \rightarrow \pm \left\{ \frac{6}{3} \right\} = \pm \left\{ 1, 2, 3, 6, \frac{1}{3}, \frac{2}{3} \right\}$$

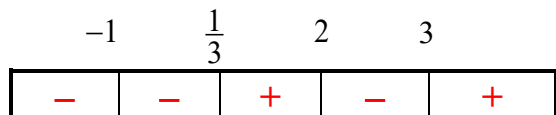
$$3x^3 - 16x^2 + 26x - 6 = 0 \rightarrow \pm \left\{ 1, 2, 3, 6, \frac{1}{3}, \frac{2}{3} \right\}$$

$$3x^2 - 10x + 3 = 0$$

$$x = \frac{10 \pm \sqrt{100 - 36}}{6}$$

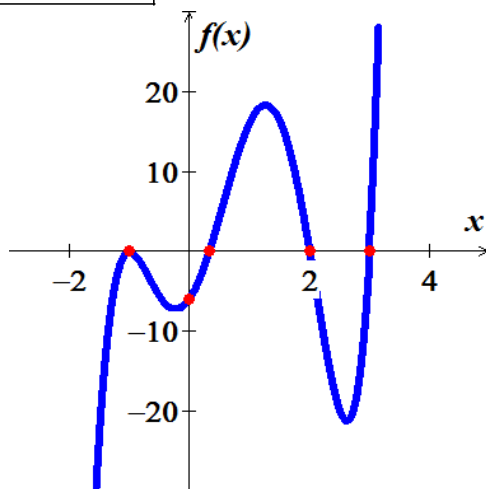
$$= \begin{cases} \frac{10-8}{6} = \frac{1}{3} \\ \frac{10+8}{6} = 3 \end{cases}$$

$$x = -1, -1, \frac{1}{3}, 2, 3$$



$$f(x) > 0 \quad \left(\frac{1}{3}, 2 \right) \cup (3, \infty)$$

$$f(x) < 0 \quad (-\infty, -1) \cup \left(-1, \frac{1}{3}\right) \cup (2, 3)$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 6x^5 + 19x^4 + x^3 - 6x^2$$

Solution

$$x^2(6x^3 + 19x^2 + x - 6) = 0 \rightarrow \underline{x = 0, 0}$$

$$6x^3 + 19x^2 + x - 6 = 0$$

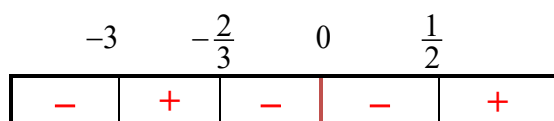
$$\text{possibilities for } \frac{c}{d} : \pm \left\{ \frac{6}{6} \right\} = \pm \left\{ 1, 2, 3, 6, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{2}{3} \right\}$$

$$\begin{array}{r|rrrr} -3 & 6 & 19 & 1 & -6 \\ & & -18 & -3 & 6 \\ \hline & 6 & 1 & -2 & \boxed{0} \end{array} \quad 6x^2 + x - 2 = 0$$

$$x = \frac{-1 \pm \sqrt{1+48}}{12}$$

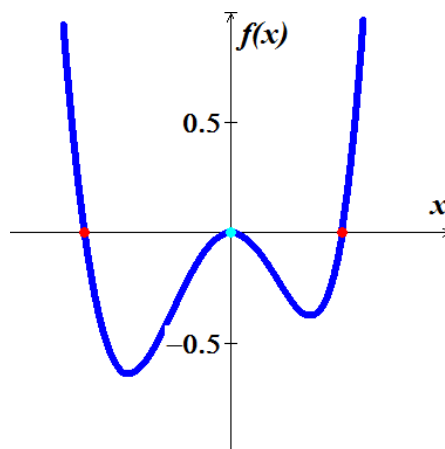
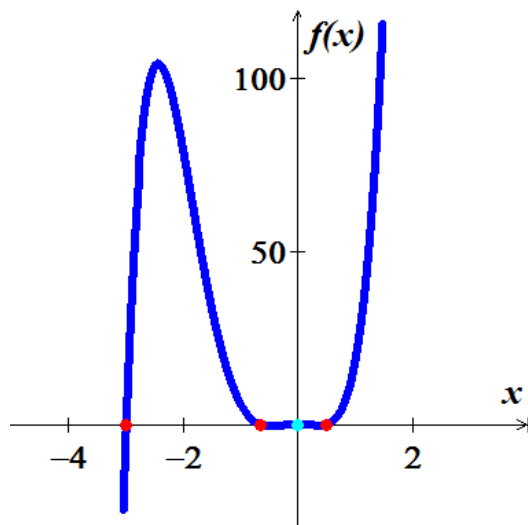
$$= \begin{cases} \frac{-1-7}{12} = -\frac{2}{3} \\ \frac{-1+7}{12} = \frac{1}{2} \end{cases}$$

$$\underline{x = 0, 0, -\frac{2}{3}, -3, \frac{1}{2}}$$



$$f(x) > 0 \quad \underline{\left(-3, -\frac{2}{3}\right) \cup \left(\frac{1}{2}, \infty\right)}$$

$$f(x) < 0 \quad \underline{\left(-\infty, -3\right) \cup \left(-\frac{2}{3}, 0\right) \cup \left(0, \frac{1}{2}\right)}$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$$

Solution

$$x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 = (x+1)^5 = 0$$

possibilities for $\frac{c}{d} : \pm\{1\}$

$$\begin{array}{c|cccccc} -1 & 1 & 5 & 10 & 10 & 5 & 1 \\ & & -1 & -4 & -6 & -4 & -1 \end{array}$$

$$\begin{array}{c|cccccc} -1 & 1 & 4 & 6 & 4 & 1 & 0 \\ & & -1 & -3 & -3 & -1 & \end{array} \rightarrow x^4 + 4x^3 + 6x^2 + 4x + 1 = 0 \rightarrow \pm\{1\}$$

$$\begin{array}{c|cccc} -1 & 1 & 3 & 3 & 1 & 0 \\ & & -1 & -2 & -1 & \end{array} \rightarrow x^3 + 3x^2 + 3x + 1 = 0 \rightarrow \pm\{1\}$$

$$\begin{array}{c|cccc} & 1 & 2 & 1 & 0 \end{array} \rightarrow x^2 + 2x + 1 = 0$$

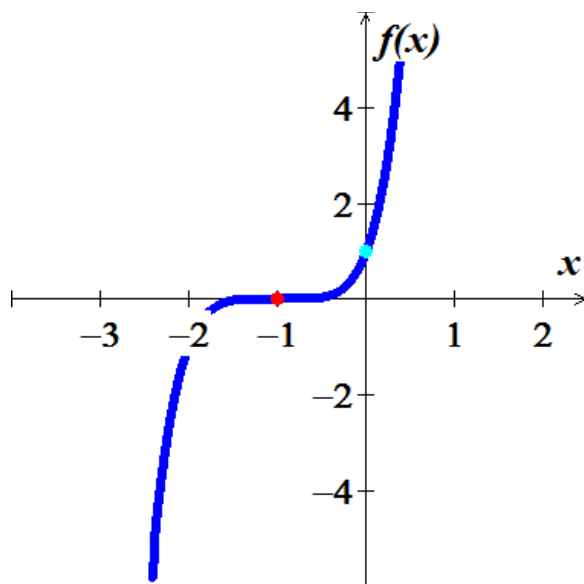
$$x^2 + 2x + 1 = (x+1)^2$$

$x = -1$ | (multiplicity of 5)

$$\begin{array}{c|ccc} -1 & & & \\ \hline - & & & + \end{array}$$

$$f(x) > 0 \quad \underline{(-1, \infty)} \quad |$$

$$f(x) < 0 \quad \underline{(-\infty, -1)} \quad |$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^5 - x^4 - 7x^3 + 7x^2 + 12x - 12$$

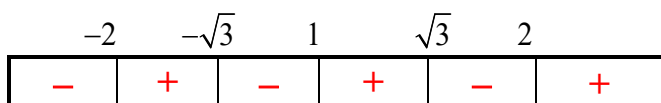
Solution

possibilities for $\frac{c}{d} : \pm\{1, 2, 3, 4, 6, 12\}$

1	1	-1	-7	7	12	-12	
		1	0	-7	0	12	
2	1	0	-7	0	12	0	$\rightarrow x^4 - 7x^2 - 12 = 0 \rightarrow \pm\{1, 2, 3, 4, 6, 12\}$
		2	4	-6	-12		
-2	1	2	-3	-6	0		$\rightarrow x^3 + 2x^2 - 3x - 6 = 0 \rightarrow \pm\{1, 2, 3, 6\}$
		-2	0	6			
	1	0	-3	0			$\rightarrow x^2 - 3 = 0$

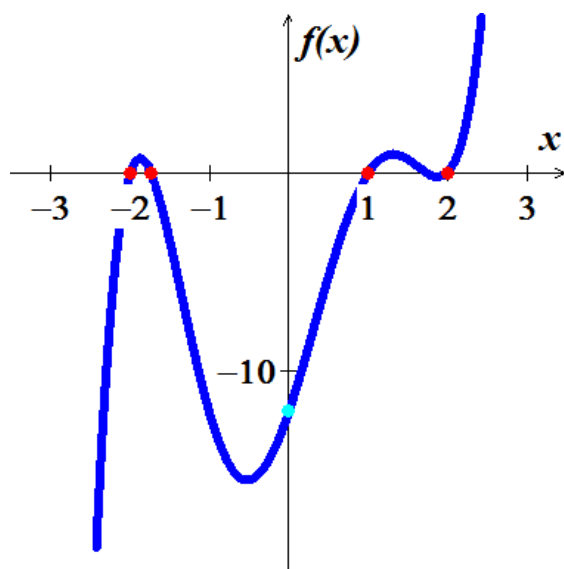
$$x^2 = 3$$

$$x = -2, 1, 2, \pm\sqrt{3}$$



$$f(x) > 0 \quad \underline{(-2, -\sqrt{3}) \cup (1, \sqrt{3}) \cup (2, \infty)}$$

$$f(x) < 0 \quad \underline{(-\infty, -2) \cup (-\sqrt{3}, 1) \cup (\sqrt{3}, 2)}$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^5 - 2x^3 - 8x$$

Solution

$$x(x^4 - 2x^2 - 8) = 0$$

$$\underline{x = 0}$$

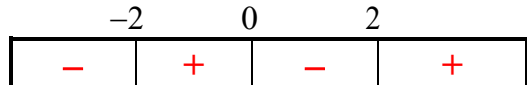
$$x^4 - 2x^2 - 8 = 0.$$

$$x^2 = \frac{2 \pm \sqrt{4+32}}{2}$$

$$= \begin{cases} \frac{2-6}{2} = -2 \\ \frac{2+6}{2} = 4 \end{cases}$$

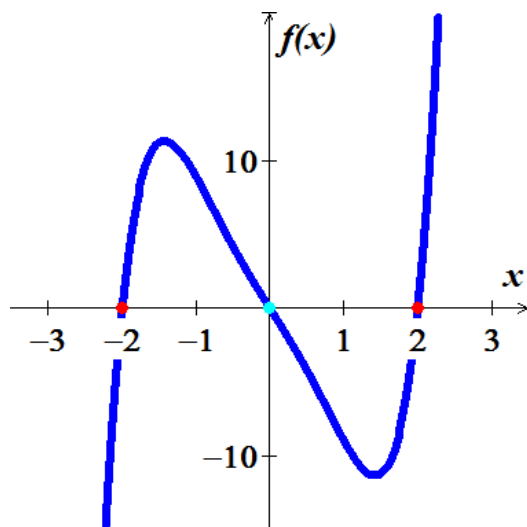
$$\begin{cases} x^2 = -2 \rightarrow x = \pm i\sqrt{2} \\ x^2 = 4 \rightarrow x = \pm 2 \end{cases}$$

$$\underline{x = 0, \pm 2, \pm i\sqrt{2}}$$



$$f(x) > 0 \quad \underline{(-2, 0) \cup (2, \infty)}$$

$$f(x) < 0 \quad \underline{(-\infty, -2) \cup (0, 2)}$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 3x^6 - 10x^5 - 29x^4 + 34x^3 + 50x^2 - 24x - 24$$

Solution

$$\begin{aligned} \text{possibilities for } \frac{c}{d} : & \pm \left\{ \frac{24}{3} \right\} \\ & = \pm \left\{ 1, 2, 3, 4, 6, 8, 12, 24, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3} \right\} \end{aligned}$$

1	3 -10 -29 34 50 -24 -24	
	3 -7 -36 -2 48 24	
-1	3 -7 -36 -2 48 24 0	$\rightarrow 3x^5 - 7x^4 - 36x^3 - 2x^2 + 48x + 24 = 0$
	-3 10 26 -24 -24	
-2	3 -10 -26 24 24 0	$\rightarrow 3x^4 - 10x^3 - 26x^2 + 24x + 24 = 0$
	-6 32 -12 -24	
$-\frac{2}{3}$	3 -16 6 12 0	$\rightarrow 3x^3 - 16x^2 + 12x - 12 = 0$
	-2 12 -12	
	3 -18 18 0	$\rightarrow 3x^2 - 18x + 18 = 0$

$$x^2 - 6x + 6 = 0$$

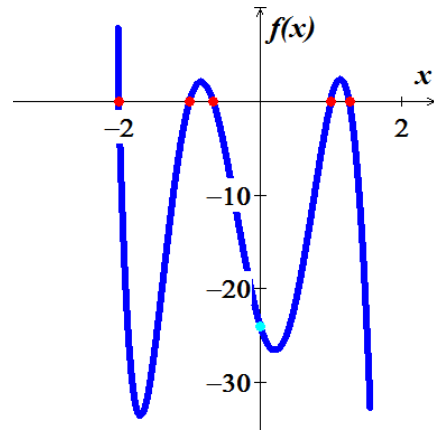
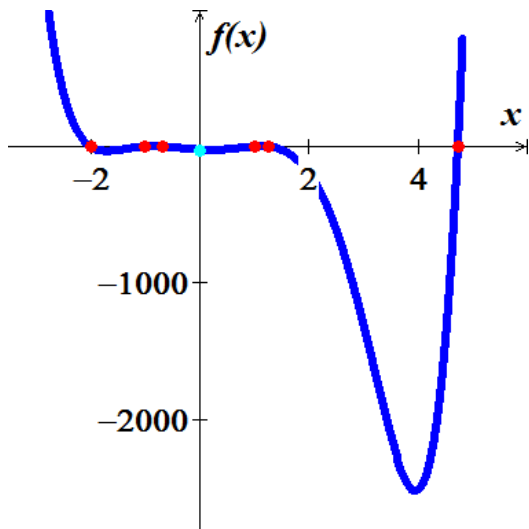
$$\begin{aligned} x &= \frac{6 \pm \sqrt{36 - 24}}{2} \\ &= \frac{6 \pm 2\sqrt{3}}{2} \end{aligned}$$

$$x = -2, -1, 1, -\frac{2}{3}, 3 \pm \sqrt{3} \quad |$$

-2	-1	$-\frac{2}{3}$	1	$3 - \sqrt{3}$	$3 + \sqrt{3}$
+	-	+	-	+	-

$$f(x) > 0 \quad \left(-\infty, -2 \right) \cup \left(-1, -\frac{2}{3} \right) \cup \left(1, 3 - \sqrt{3} \right) \cup \left(3 + \sqrt{3}, \infty \right) \quad |$$

$$f(x) < 0 \quad \left(-2, -1 \right) \cup \left(-\frac{2}{3}, 1 \right) \cup \left(3 - \sqrt{3}, 3 + \sqrt{3} \right) \quad |$$



Solution

Section 2.6 – Graphing Rational Functions

Exercise

Determine all asymptotes of the function: $y = \frac{3x}{1-x}$

Solution

$$VA: x = 1$$

$$HA: y = -3$$

$$Hole: n/a$$

$$Oblique\ asymptote: n/a$$

Exercise

Determine all asymptotes of the function: $y = \frac{x^2}{x^2 + 9}$

Solution

$$VA: n/a \quad x^2 + 9 \neq 0$$

$$HA: y = 1$$

$$Hole: n/a$$

$$Oblique\ asymptote: n/a$$

Exercise

Determine all asymptotes of the function: $y = \frac{x-2}{x^2 - 4x + 3}$

Solution

$$x^2 - 4x + 3 = 0 \Rightarrow x = 1, 3$$

$$y = \frac{x}{x^2} \rightarrow 0$$

$$VA: x = 1, x = 3$$

$$HA: y = 0$$

$$Hole: n/a$$

$$Oblique\ asymptote: n/a$$

Exercise

Determine all asymptotes of the function: $y = \frac{3}{x-5}$

Solution

$$VA: x = 5$$

$$HA: y = 0$$

$$Hole: n/a$$

$$Oblique\ asymptote: n/a$$

Exercise

Determine all asymptotes of the function: $y = \frac{x^3 - 1}{x^2 + 1}$

Solution

VA: none

HA: none

Hole: n/a

Oblique asymptote: $y = x$

$$\begin{array}{r} x \\ x^2 + 1 \overline{) x^3 - 1} \\ \underline{-x^3 - x} \\ -x - 1 \end{array}$$
$$y = x - \frac{x+1}{x^2+1}$$

Exercise

Determine all asymptotes of the function: $y = \frac{x^3 + 3x^2 - 2}{x^2 - 4}$

Solution

VA: $x = \pm 2$

HA: n/a

Hole: n/a

Oblique asymptote: $y = x + 3$

$$\begin{array}{r} x+3 \\ x^2 - 4 \overline{) x^3 + 3x^2 - 2} \\ \underline{-x^3 + 4x} \\ 3x^2 + 4x - 2 \\ \underline{-3x^2 + 12} \\ 4x + 10 \end{array}$$
$$y = x + 3 + \frac{4x+10}{x^2-4}$$

Exercise

Determine all asymptotes of the function: $y = \frac{3x^2 - 27}{(x+3)(2x+1)}$

Solution

$$y = \frac{3x^2 - 27}{(x+3)(2x+1)} = \frac{3(x^2 - 9)}{(x+3)(2x+1)} = \frac{3(x+3)(x-3)}{(x+3)(2x+1)} = \frac{3(x-3)}{(2x+1)}$$

VA: $x = -3, -\frac{1}{2}$

HA: $y = \frac{3}{2}$

Hole: n/a

Oblique asymptote: n/a

Exercise

Determine all asymptotes of the function: $y = \frac{x-3}{x^2-9}$

Solution

$$x^2 - 9 = 0 \rightarrow \boxed{x = \pm 3}$$

$$y = \frac{x-3}{(x-3)(x+3)}$$

$$= \frac{1}{x+3}$$

VA: $x = 3$

HA: $y = 0$

Hole: $x = 3 \rightarrow y = \frac{1}{6}$

Oblique asymptote: n / a

Exercise

Determine all asymptotes of the function: $y = \frac{6}{\sqrt{x^2 - 4x}}$

Solution

$$x^2 - 4x = 0$$

$$\Rightarrow x(x-4) = 0 \rightarrow \boxed{x = 0, 4}$$

VA: $x = 0, x = 4$

HA: $y = 0$

Hole: n / a

Oblique asymptote: n / a

Exercise

Determine all asymptotes of the function: $y = \frac{5x-1}{1-3x}$

Solution

VA: $x = \frac{1}{3}$

HA: $y = -\frac{5}{3}$

Hole: n / a

Oblique asymptote: n / a

Exercise

Determine all asymptotes of the function: $f(x) = \frac{2x-11}{x^2+2x-8}$

Solution

VA: $x = 2, x = -4$

HA: $y = 0$

Hole: n / a

Oblique asymptote: n / a

Exercise

Determine all asymptotes of the function: $f(x) = \frac{x^2 - 4x}{x^3 - x}$

Solution

$$\begin{aligned} f(x) &= \frac{x(x-4)}{x(x^2-1)} \\ &= \frac{x-4}{x^2-1} \end{aligned}$$

VA: $x = -1, x = 1$ **HA:** $y = 0$

Hole: $x = 0 \rightarrow y = 4$ **Oblique asymptote:** n / a

Exercise

Determine all asymptotes of the function: $f(x) = \frac{x-2}{x^3-5x}$

Solution

VA: $x = 0, x = \pm\sqrt{5}$ **HA:** $y = 0$

Hole: n / a **Oblique asymptote:** n / a

Exercise

Determine all asymptotes of the function $f(x) = \frac{4x}{x^2 + 10x}$

Solution

$$x^2 + 10x = 0 \rightarrow x = 0, -10$$

Domain: $(-\infty, -10) \cup (-10, 0) \cup (0, \infty)$

$$\begin{aligned} f(x) &= \frac{4x}{x(x+10)} \\ &= \frac{4}{x+10} \end{aligned}$$

VA: $x = -10$ **HA:** $y = 0$

Hole: $x = 0 \rightarrow y = \frac{4}{10} \Rightarrow$ **hole** $\left(0, \frac{2}{5}\right)$ **Oblique asymptote:** n / a

Exercise

Determine all asymptotes of the function $f(x) = \frac{3-x}{(x-4)(x+6)}$

Solution

$$VA: x = -6 \text{ and } x = 4$$

$$HA: y = 0$$

$$Hole: n/a$$

$$Oblique \text{ asymptote: } n/a$$

Exercise

Determine all asymptotes of the function $f(x) = \frac{x^3}{2x^3 - x^2 - 3x}$

Solution

$$2x^3 - x^2 - 3x = x(2x^2 - x - 3) = 0 \rightarrow x = 0, -1, \frac{3}{2}$$

$$\begin{aligned} f(x) &= \frac{x^3}{2x^3 - x^2 - 3x} \\ &= \frac{x^3}{x(2x^2 - x - 3)} \\ &= \frac{x^2}{2x^2 - x - 3} \end{aligned}$$

$$VA: x = -1 \text{ and } x = \frac{3}{2}$$

$$HA: y = \frac{1}{2}$$

$$Hole: x = 0 \rightarrow y = 0 \Rightarrow \text{hole } (0, 0)$$

$$Oblique \text{ asymptote: } n/a$$

Exercise

Determine all asymptotes of the function $f(x) = \frac{3x^2 + 5}{4x^2 - 3}$

Solution

$$4x^2 - 3 = 0 \rightarrow x = \pm \frac{\sqrt{3}}{2}$$

$$\text{Domain: } \left(-\infty, -\frac{\sqrt{3}}{2}\right) \cup \left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right) \cup \left(\frac{\sqrt{3}}{2}, \infty\right)$$

$$VA: x = -\frac{\sqrt{3}}{2} \text{ and } x = \frac{\sqrt{3}}{2}$$

$$HA: y = \frac{3}{4}$$

$$Hole: n/a$$

$$Oblique \text{ asymptote: } n/a$$

Exercise

Determine all asymptotes of the function $f(x) = \frac{x+6}{x^3 + 2x^2}$

Solution

$$x^3 + 2x^2 = x^2(x+2) = 0 \rightarrow x = 0, -2$$

Domain: $(-\infty, -2) \cup (-2, 0) \cup (0, \infty)$

VA: $x = 0$ and $x = 2$

HA: $y = 0$

Hole: n/a

Oblique asymptote: n/a

Exercise

Determine all asymptotes of the function $f(x) = \frac{x^2 + 4x - 1}{x + 3}$

Solution

VA: $x = -3$

HA: n/a

Hole: n/a

Oblique asymptote: $y = x + 1$

$$\begin{array}{r} x+1 \\ x+3 \overline{) x^2 + 4x - 1} \\ \underline{-x^2 - 3x} \\ x - 1 \\ \underline{-x - 3} \\ -4 \end{array}$$
$$f(x) = \frac{x^2 + 4x - 1}{x + 3} = x + 1 - \frac{4}{x + 3}$$

Exercise

Determine all asymptotes of the function $f(x) = \frac{x^2 - 6x}{x - 5}$

Solution

$$x - 5 = 0 \rightarrow x = 5$$

Domain: $(-\infty, 5) \cup (5, \infty)$

$$\begin{aligned} f(x) &= \frac{x^2 - 6x}{x - 5} \\ &= x - 1 - \frac{5}{x - 5} \end{aligned}$$

VA: $x = 5$

HA: N/A

Hole: N/A

Oblique asymptote: $y = x - 1$

$$\begin{array}{r} x-1 \\ x-5 \overline{) x^2 - 6x} \\ \underline{-x^2 + 5x} \\ -x \end{array}$$
$$\frac{x-5}{-5}$$

Exercise

Determine all asymptotes of the function $f(x) = \frac{x^3 - x^2 + x - 4}{x^2 + 2x - 1}$

Solution

$$x^2 + 2x - 1 = 0 \rightarrow x = -1 \pm \sqrt{2}$$

$$\text{Domain: } (-\infty, -1 - \sqrt{2}) \cup (-1 - \sqrt{2}, -1 + \sqrt{2}) \cup (-1 + \sqrt{2}, \infty)$$

$$\begin{aligned} f(x) &= \frac{x^3 - x^2 + x - 4}{x^2 + 2x - 1} \\ &= x - 3 + \frac{8x - 7}{x^2 + 2x - 1} \end{aligned}$$

$$\text{VA: } x = -1 \pm \sqrt{2}$$

$$\text{HA: } n/a$$

$$\text{Hole: } n/a$$

$$\text{Oblique asymptote: } y = x - 3$$

$$\begin{array}{r} x^2 + 2x - 1 \overline{) \begin{array}{r} x^3 - x^2 + x - 4 \\ -x^3 - 2x^2 + x \\ \hline -3x^2 + 2x - 4 \\ 3x^2 + 6x - 3 \\ \hline 8x - 7 \end{array}} \end{array}$$

Exercise

Determine all asymptotes of the function $f(x) = \frac{4x}{x^2 + 10x}$

Solution

$$x^2 + 10x = 0 \rightarrow x = 0, -10 \quad \text{Domain: } (-\infty, -10) \cup (-10, 0) \cup (0, \infty)$$

$$f(x) = \frac{4x}{x(x+10)} = \frac{4}{x+10}$$

$$\text{VA: } x = -10$$

$$\text{HA: } y = 0$$

$$\text{Hole: } x = 0 \rightarrow y = \frac{4}{10} \Rightarrow \text{hole } \left(0, \frac{2}{5}\right)$$

$$\text{Oblique asymptote: } n/a$$

Exercise

Determine all asymptotes of the function $f(x) = \frac{3-x}{(x-4)(x+6)}$

Solution

$$\text{Domain: } (-\infty, -6) \cup (-6, 4) \cup (4, \infty)$$

$$VA: x = -6 \text{ and } x = 4$$

$$HA: y = 0$$

$$Hole: n/a$$

$$Oblique \text{ asymptote: } n/a$$

Exercise

Determine all asymptotes of the function $f(x) = \frac{x^3}{2x^3 - x^2 - 3x}$

Solution

$$2x^3 - x^2 - 3x = x(2x^2 - x - 3) = 0 \rightarrow x = 0, -1, \frac{3}{2}$$

$$Domain: (-\infty, -1) \cup (-1, 0) \cup (0, \frac{3}{2}) \cup (\frac{3}{2}, \infty)$$

$$f(x) = \frac{x^3}{2x^3 - x^2 - 3x} = \frac{x^3}{x(2x^2 - x - 3)} = \frac{x^2}{2x^2 - x - 3}$$

$$VA: x = -1 \text{ and } x = \frac{3}{2}$$

$$HA: y = \frac{1}{2}$$

$$Hole: x = 0 \rightarrow y = 0 \Rightarrow \text{hole } (0, 0)$$

$$Oblique \text{ asymptote: } n/a$$

Exercise

Determine all asymptotes of the function $f(x) = \frac{3x^2 + 5}{4x^2 - 3}$

Solution

$$4x^2 - 3 = 0 \rightarrow x = \pm \frac{\sqrt{3}}{2}$$

$$Domain: \left(-\infty, -\frac{\sqrt{3}}{2}\right) \cup \left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right) \cup \left(\frac{\sqrt{3}}{2}, \infty\right)$$

$$VA: x = -\frac{\sqrt{3}}{2} \text{ and } x = \frac{\sqrt{3}}{2}$$

$$HA: y = \frac{3}{4}$$

$$Hole: n/a$$

$$Oblique \text{ asymptote: } n/a$$

Exercise

Determine all asymptotes of the function $f(x) = \frac{x+6}{x^3 + 2x^2}$

Solution

$$x^3 + 2x^2 = x^2(x+2) = 0 \rightarrow x = 0, -2 \quad Domain: (-\infty, -2) \cup (-2, 0) \cup (0, \infty)$$

$$VA: x = 0 \text{ and } x = -2$$

$$HA: y = 0$$

$$Hole: n/a$$

$$Oblique \text{ asymptote: } n/a$$

Exercise

Determine all asymptotes of the function $f(x) = \frac{x^2 + 4x - 1}{x + 3}$

Solution

$$x + 3 = 0 \rightarrow x = -3$$

$$\text{Domain: } (-\infty, -3) \cup (-3, \infty)$$

$$\begin{array}{r} x+1 \\ x+3 \overline{) x^2 + 4x - 1} \end{array}$$

$$\underline{-x^2 - 3x}$$

$$x - 1$$

$$\underline{-x - 3}$$

$$-4$$

$$f(x) = \frac{x^2 + 4x - 1}{x + 3} = x + 1 - \frac{4}{x + 3}$$

$$\text{VA: } x = -3$$

$$\text{HA: } n/a$$

$$\text{Hole: } n/a$$

$$\text{Oblique asymptote: } y = x + 1$$

Exercise

Determine all asymptotes of the function $f(x) = \frac{x^2 - 6x}{x - 5}$

Solution

$$x - 5 = 0 \rightarrow x = 5$$

$$\text{Domain: } (-\infty, 5) \cup (5, \infty)$$

$$\begin{array}{r} x-1 \\ x-5 \overline{) x^2 - 6x} \end{array}$$

$$\underline{-x^2 + 5x}$$

$$-x$$

$$\underline{x - 5}$$

$$-5$$

$$f(x) = \frac{x^2 - 6x}{x - 5} = x - 1 - \frac{5}{x - 5}$$

$$\text{VA: } x = 5$$

$$\text{HA: N/A}$$

$$\text{Hole: N/A}$$

$$\text{Oblique asymptote: } y = x - 1$$

Exercise

Determine all asymptotes of the function $f(x) = \frac{x^3 - x^2 + x - 4}{x^2 + 2x - 1}$

Solution

$$x^2 + 2x - 1 = 0 \rightarrow x = -1 \pm \sqrt{2}$$

Domain: $(-\infty, -1-\sqrt{2}) \cup (-1-\sqrt{2}, -1+\sqrt{2}) \cup (-1+\sqrt{2}, \infty)$

$$x^2 + 2x - 1 \overline{) x^3 - x^2 + x - 4}$$

$$\begin{array}{r} -x^3 - 2x^2 + x \\ \hline -3x^2 + 2x - 4 \end{array}$$

$$\begin{array}{r} 3x^2 + 6x - 3 \\ \hline 8x - 7 \end{array}$$

$$\begin{aligned} f(x) &= \frac{x^3 - x^2 + x - 4}{x^2 + 2x - 1} \\ &= x - 3 + \frac{8x - 7}{x^2 + 2x - 1} \end{aligned}$$

VA: $x = -1 \pm \sqrt{2}$

HA: N/A

Hole: N/A

Oblique asymptote: $y = x - 3$

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote, Hole, Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{-3x}{x+2}$$

Solution

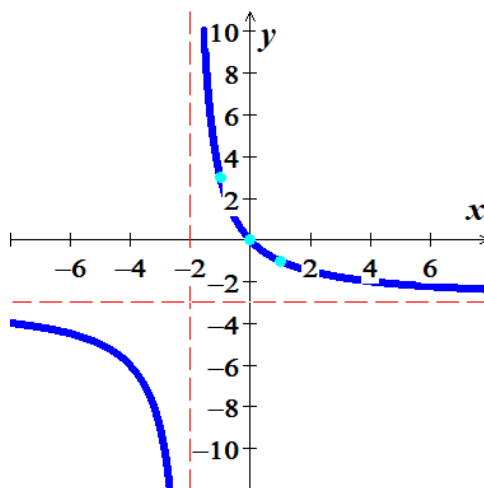
VA: $x = -2$

HA: $y = -3$

Hole: n / a

OA: n / a

x	y
0	0
1	-1
-1	3



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph of

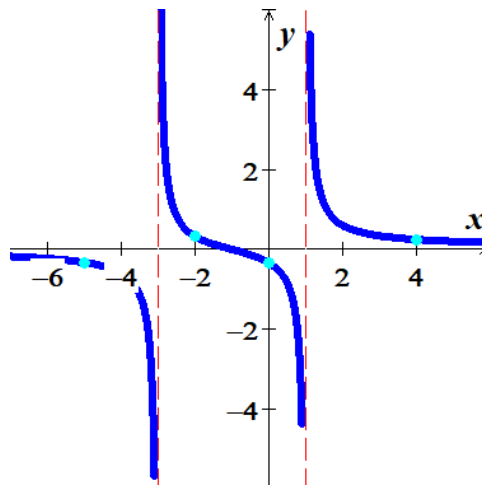
$$f(x) = \frac{x+1}{x^2 + 2x - 3}$$

Solution

VA: $x = 1, x = -3$ **HA:** $y = 0$

Hole: n/a **Oblique asymptote:** n/a

x	y
-5	-0.33
-2	0.33
0	-1/3
4	0.24



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph of

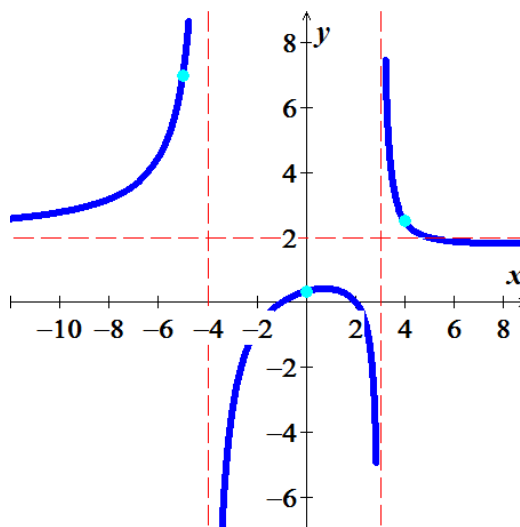
$$f(x) = \frac{2x^2 - 2x - 4}{x^2 + x - 12}$$

Solution

VA: $x = -4, 3$ **HA:** $y = 2$

Hole: n/a **OA:** n/a

x	y
-5	7
-2	-0.8
0	1/3
4	2.5
5	2



Exercise

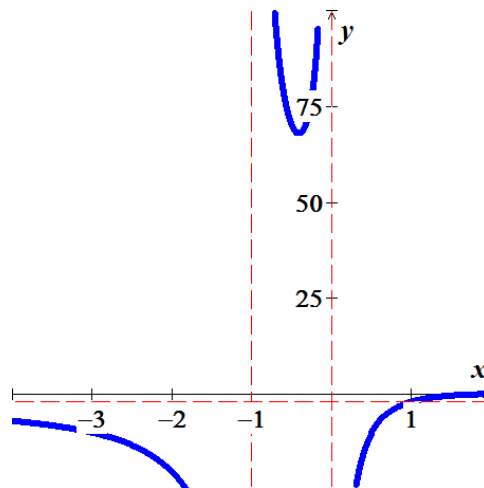
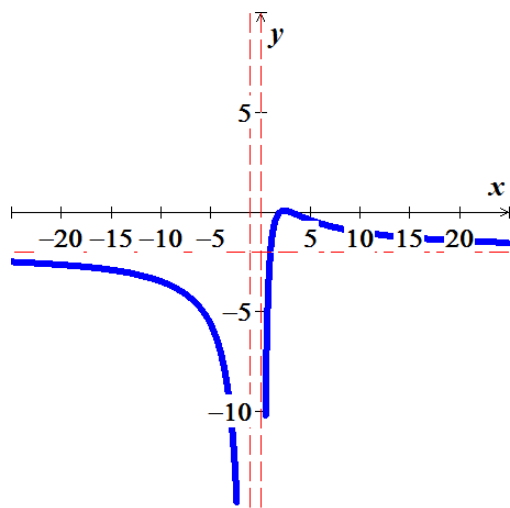
Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph

$$f(x) = \frac{-2x^2 + 10x - 12}{x^2 + x}$$

Solution

VA: $x = -1, 0$ **HA:** $y = -2$

Hole: n/a **OA:** n/a



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph

$$f(x) = \frac{x^2 - x - 6}{x + 1}$$

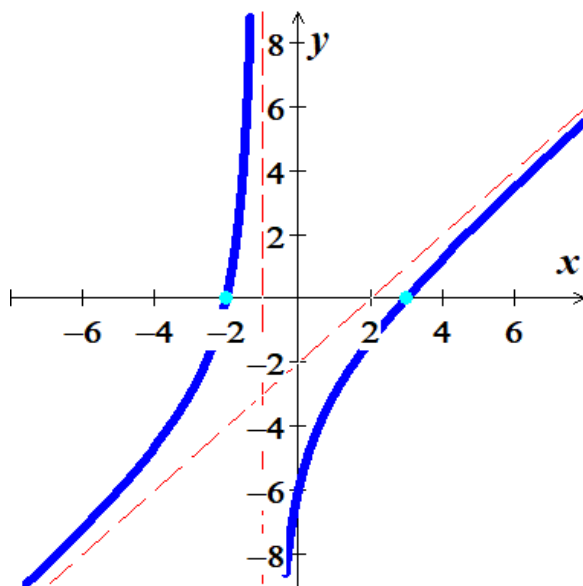
Solution

$$\begin{array}{r} x-2 \\ x+1 \overline{) x^2 - x - 6} \\ \underline{x^2 + x} \\ -2x - 6 \\ \underline{-2x - 2} \\ -4 \end{array}$$

VA: $x = -1$ **HA:** n/a

Hole: n/a **OA:** $y = x - 2$

x	y
2	0
-2	0
0	-6



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph

$$f(x) = \frac{x^3 + 1}{x - 2}$$

Solution

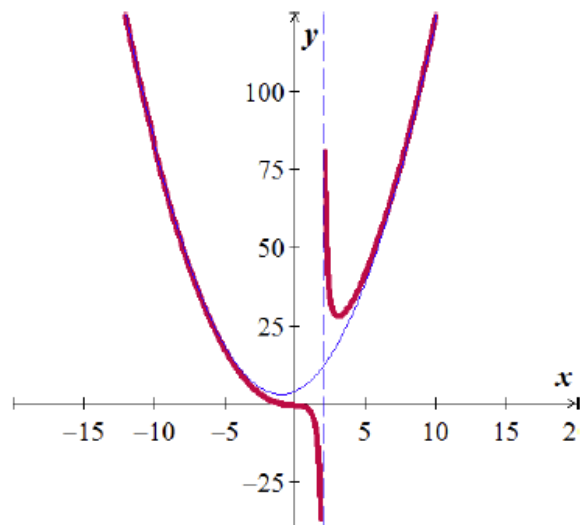
$$\begin{array}{r} x^2 + 2x + 4 \\ x-2 \overline{) x^3 - 1} \\ \underline{x^3 - 2x^2} \\ 2x^2 \\ \underline{2x^2 - 4x} \\ 4x - 1 \\ \underline{4x - 8} \\ 7 \end{array}$$

VA: $x = 2$

HA: n/a

Hole: n/a

OA: $y = x^2 + 2x + 4$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph

$$f(x) = \frac{2x^2 + x - 6}{x^2 + 3x + 2}$$

Solution

$$\begin{aligned} f(x) &= \frac{(2x-3)(x+2)}{(x+1)(x+2)} \\ &= \frac{2x-3}{x+1} \end{aligned}$$

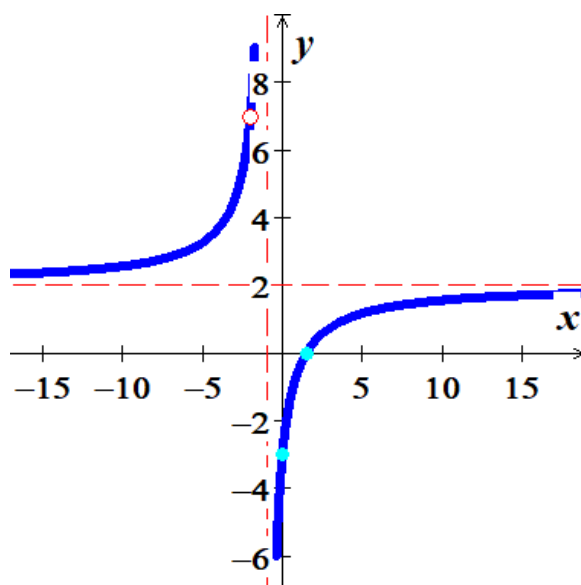
VA: $x = -1$

HA: $y = 2$

Hole: $(-2, 7)$

OA: n/a

x	y
0	-3
$-\frac{3}{2}$	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph

$$f(x) = \frac{x-1}{1-x^2}$$

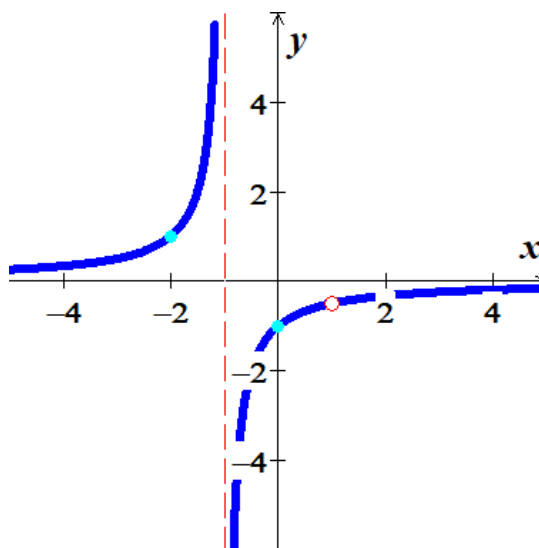
Solution

$$\begin{aligned} f(x) &= \frac{x-1}{(x+1)(1-x)} \\ &= -\frac{1}{x+1} \end{aligned}$$

$$VA: x = -1 \quad HA: y = 0$$

$$Hole: \left(1, -\frac{1}{2}\right) \quad OA: n/a$$

x	y
0	-1
-2	1



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph

$$f(x) = \frac{x^2 + x - 2}{x + 2}$$

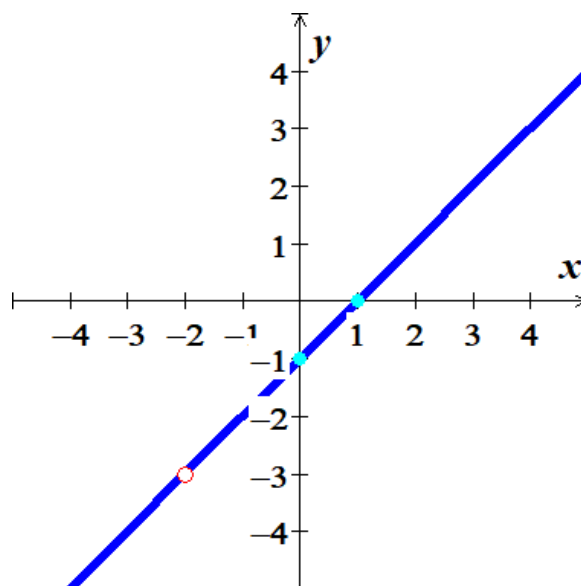
Solution

$$\begin{aligned} f(x) &= \frac{(x+2)(x-1)}{x+2} \\ &= x-1 \end{aligned}$$

$$VA: n/a \quad HA: n/a$$

$$Hole: (-2, -3) \quad OA: n/a$$

x	y
0	-1
1	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph

$$f(x) = \frac{x^3 - 2x^2 - 4x + 8}{x - 2}$$

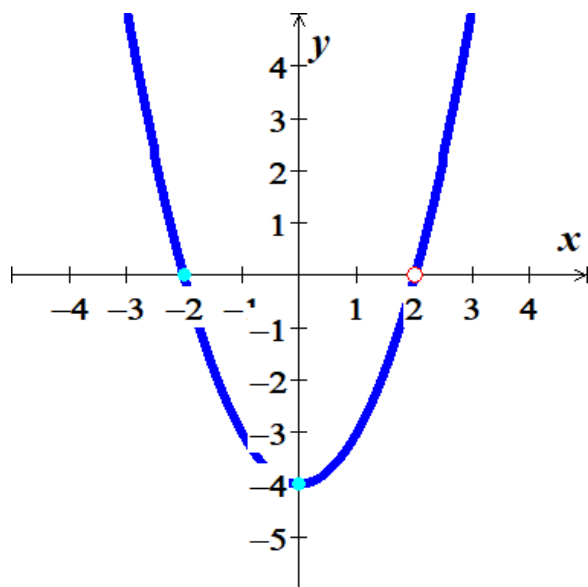
Solution

$$\begin{aligned} f(x) &= \frac{(x^2 - 4)(x - 2)}{x - 2} \\ &= x^2 - 4 \end{aligned}$$

VA: n/a **HA:** n/a

Hole: $(2, 0)$ **OA:** n/a

x	y
0	-4
-2	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph

$$f(x) = \frac{2x^2 - 3x - 1}{x - 2}$$

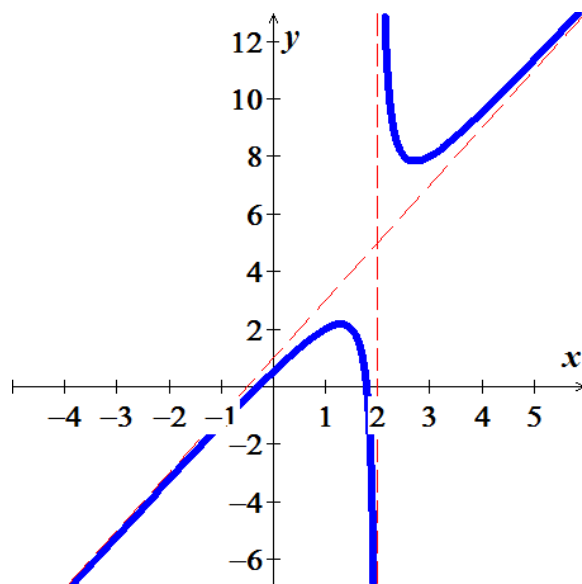
Solution

$$\begin{array}{r} 2x+1 \\ x-2 \overline{) 2x^2 - 3x - 1} \\ \underline{-2x^2 + 4x} \\ x - 1 \\ \underline{-x + 2} \\ 1 \end{array}$$

$$\begin{aligned} f(x) &= \frac{2x^2 - 3x - 1}{x - 2} \\ &= (2x + 1) + \frac{1}{x - 2} \end{aligned}$$

VA: $x = 2$ **HA:** $y = 1$

Hole: n/a **OA:** $y = 2x + 1$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph

$$f(x) = \frac{2x+3}{3x^2+7x-6}$$

Solution

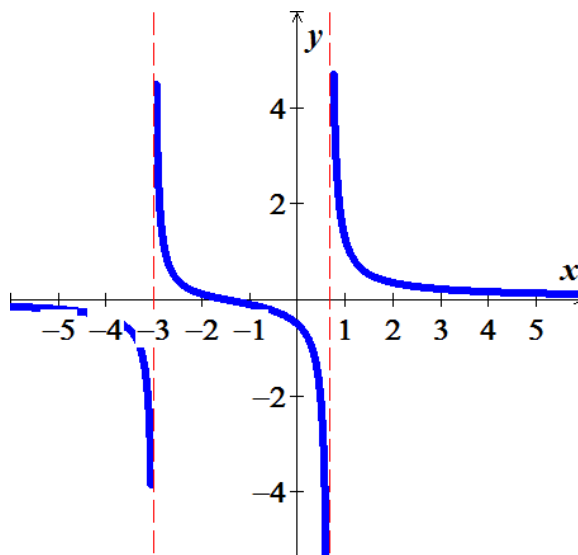
$$3x^2 + 7x - 6 = 0 \Rightarrow x = -3, \frac{2}{3}$$

$$VA: x = -3 \text{ and } x = \frac{2}{3}$$

$$HA: y = 0$$

$$Hole: n/a$$

$$OA: n/a$$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph

$$f(x) = \frac{x^2-1}{x^2+x-6}$$

Solution

$$x^2 + x - 6 = 0 \Rightarrow x = -3, 2$$

$$VA: x = -3 \text{ and } x = 2$$

$$HA: y = 1$$

$$Hole: n/a$$

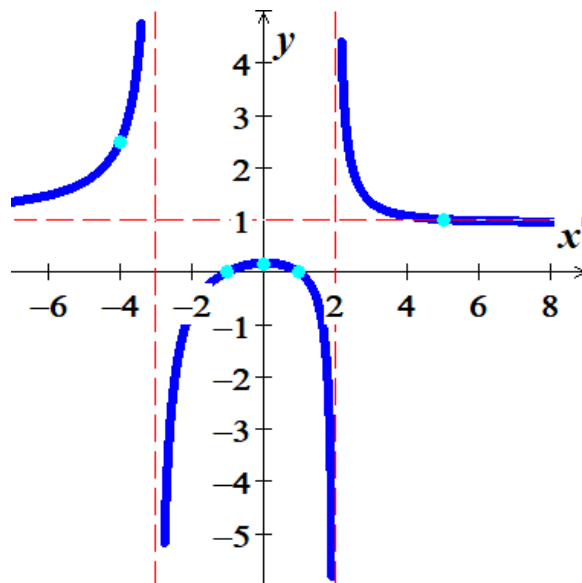
$$OA: n/a$$

$$1 = \frac{x^2-1}{x^2+x-6}$$

$$x^2 + x - 6 = x^2 - 1$$

$$x = 5$$

x	y
0	$\frac{1}{6}$
5	1
± 1	0
-4	$\frac{5}{2}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{-2x^2 - x + 15}{x^2 - x - 12}$$

Solution

$$x^2 - x - 12 = 0 \Rightarrow x = -3, 4$$

$$\text{Domain: } (-\infty, -3) \cup (-3, 4) \cup (4, \infty)$$

$$\begin{aligned} f(x) &= \frac{(-2x+5)(x+3)}{(x-4)(x+3)} \\ &= \frac{-2x+5}{x-4} \end{aligned}$$

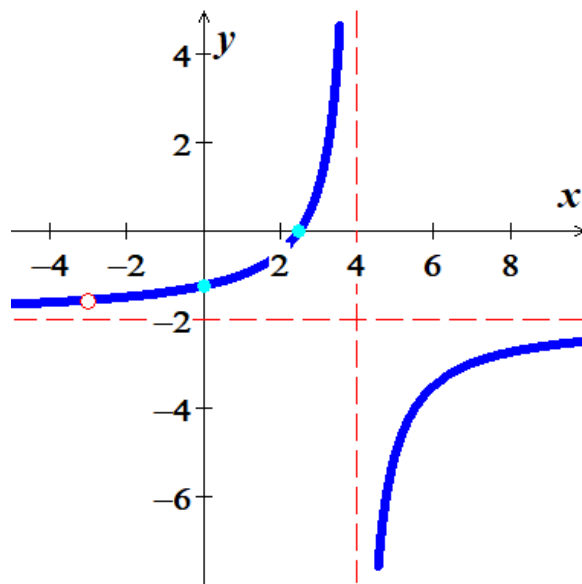
$$\text{VA: } x = 4$$

$$\text{HA: } y = -2$$

$$\text{Hole: } \left(-3, -\frac{11}{7}\right)$$

$$\text{OA: } n/a$$

x	y
0	$-\frac{5}{4}$
$\frac{5}{2}$	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{1}{x-3}$$

Solution

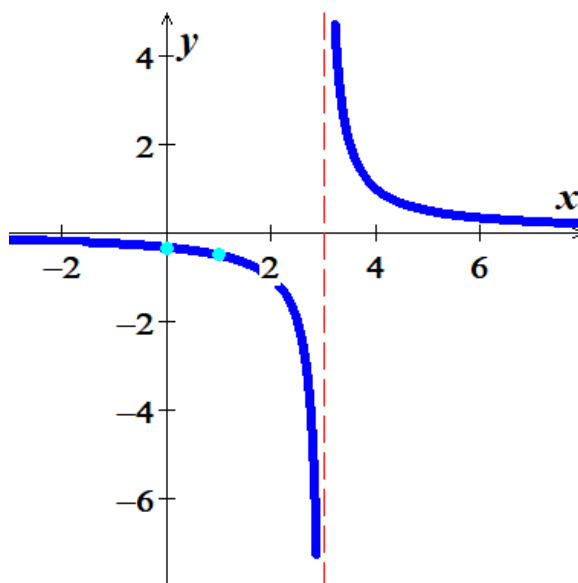
$$\text{VA: } x = 3$$

$$\text{HA: } y = 0$$

$$\text{Hole: } n/a$$

$$\text{OA: } n/a$$

x	y
0	$-\frac{1}{3}$
1	$-\frac{1}{2}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{-2}{x+3}$$

Solution

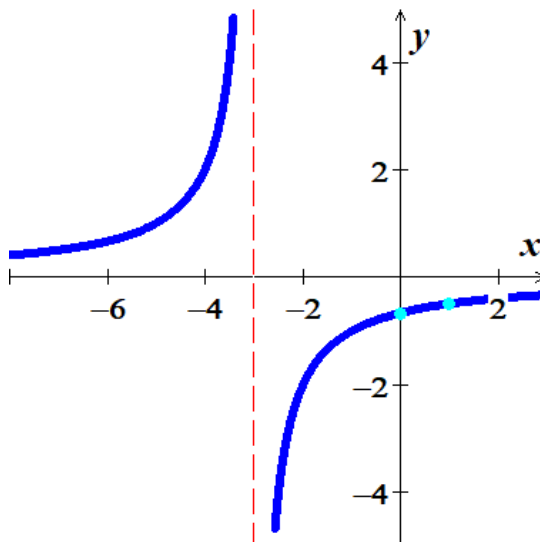
VA: $x = -3$

HA: $y = 0$

Hole: n/a

OA: n/a

x	y
0	$-\frac{2}{3}$
1	$-\frac{1}{2}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x}{x+2}$$

Solution

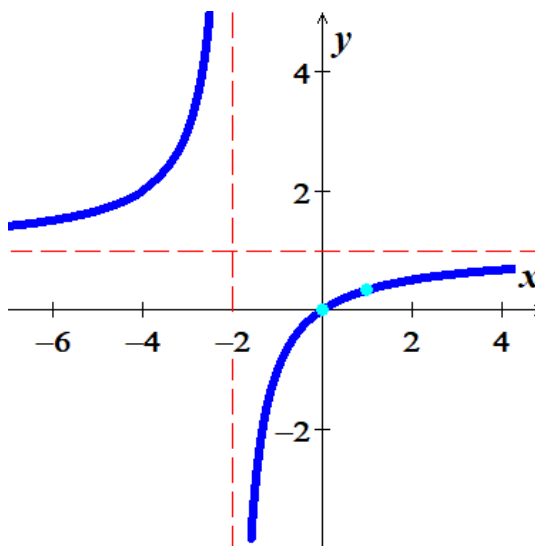
VA: $x = -2$

HA: $y = 1$

Hole: n/a

OA: n/a

x	y
0	0
1	$\frac{1}{3}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph of

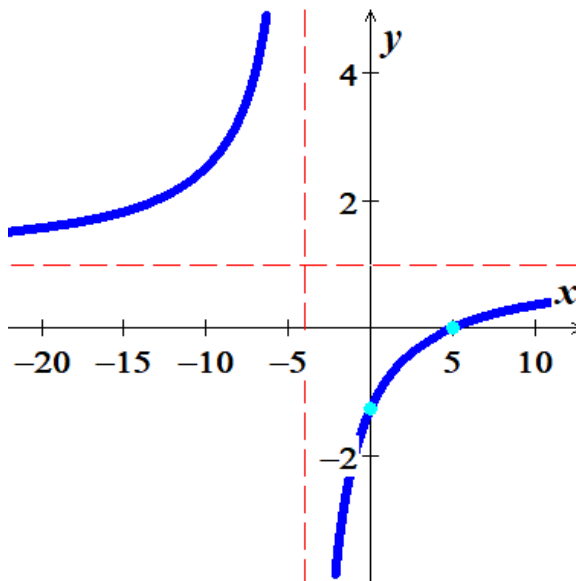
$$f(x) = \frac{x-5}{x+4}$$

Solution

$$VA: x = -4 \quad HA: y = 1$$

$$Hole: n/a \quad OA: n/a$$

x	y
0	$-\frac{5}{4}$
5	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{2x^2 - 2}{x^2 - 9}$$

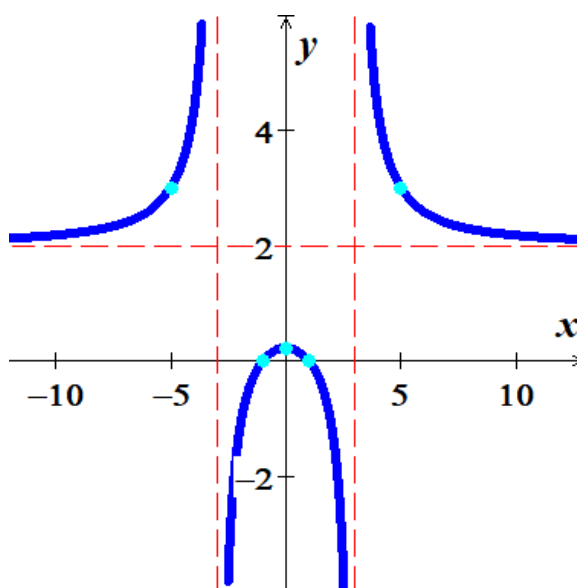
Solution

$$x^2 = 9 \rightarrow x = \pm 3$$

$$VA: x = \pm 3 \quad HA: y = 2$$

$$Hole: n/a \quad OA: n/a$$

x	y
0	$\frac{2}{9}$
± 1	0
± 5	3



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x^2 - 3}{x^2 + 4}$$

Solution

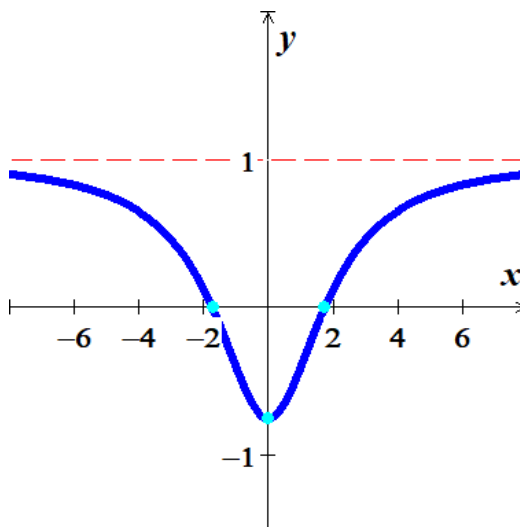
VA: n/a

HA: $y = 1$

Hole: n/a

OA: n/a

x	y
0	$-\frac{3}{4}$
$\pm\sqrt{3}$	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x^2 + 4}{x^2 - 3}$$

Solution

$$x^2 - 3 = 0 \rightarrow x = \pm\sqrt{3}$$

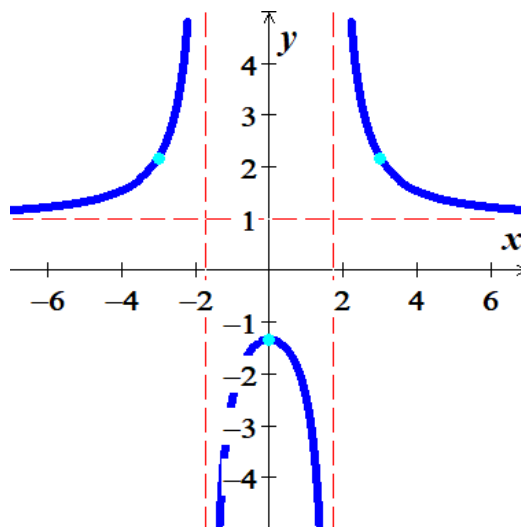
VA: $x = \pm\sqrt{3}$

HA: $y = 1$

Hole: n/a

OA: n/a

x	y
0	$-\frac{4}{3}$
± 3	$\frac{13}{6}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x^2}{x^2 - 6x + 9}$$

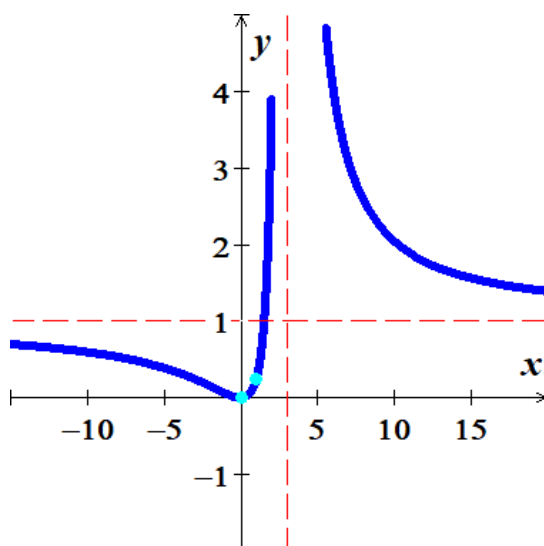
Solution

$$x^2 - 6x + 9 = 0 \rightarrow x = 3$$

$$VA: x = 3 \quad HA: y = 1$$

$$Hole: n/a \quad OA: n/a$$

x	y
0	0
1	$\frac{1}{4}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x^2 + x + 4}{x^2 + 2x - 1}$$

Solution

$$x^2 + 2x - 1 = 0$$

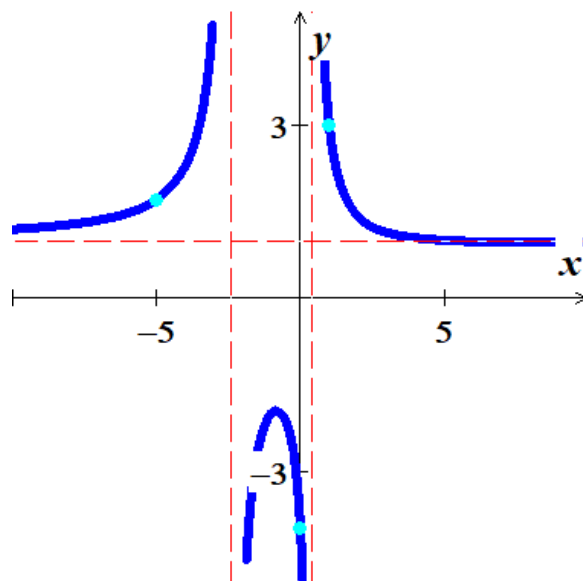
$$x = \frac{-2 \pm \sqrt{8}}{2}$$

$$= -1 \pm \sqrt{2}$$

$$VA: x = -1 \pm \sqrt{2} \quad HA: y = 1$$

$$Hole: n/a \quad OA: n/a$$

x	y
0	-4
1	3
-5	$\frac{12}{7}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{2x^2 + 14}{x^2 - 6x + 5}$$

Solution

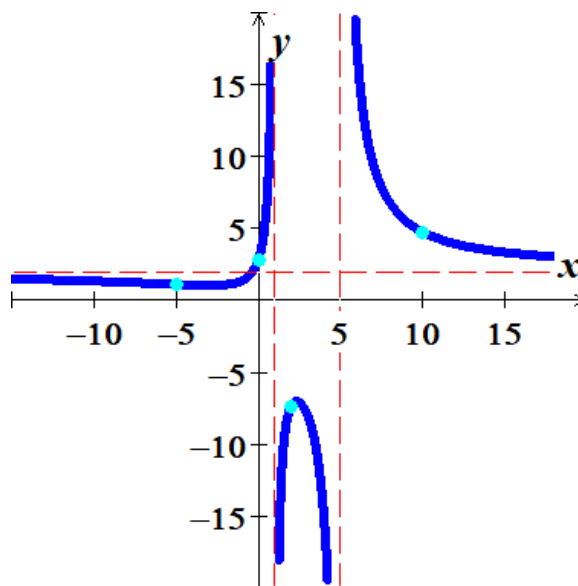
VA: $x = 1, 5$

HA: $y = 2$

Hole: n/a

OA: n/a

x	y
0	$\frac{14}{5}$
2	$-\frac{22}{3}$
-5	$\frac{16}{15}$
10	$\frac{214}{45}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x^2 - 4x - 5}{2x + 5}$$

Solution

$$\begin{array}{r} \frac{1}{2}x - \frac{13}{4} \\ 2x + 5 \overline{) x^2 - 4x - 5} \\ \underline{x^2 + \frac{5}{2}x} \\ -\frac{13}{2}x - 5 \end{array}$$

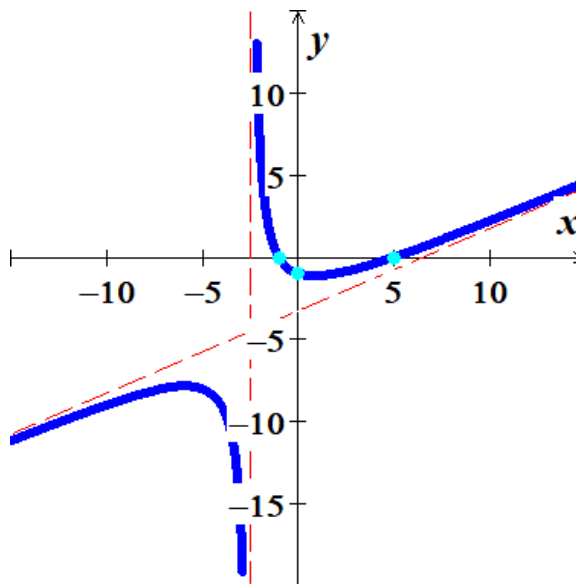
VA: $x = -\frac{5}{2}$

HA: n/a

Hole: n/a

OA: $y = \frac{1}{2}x - \frac{13}{2}$

x	y
0	-1
-1, 5	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x-3}{x^2-3x+2}$$

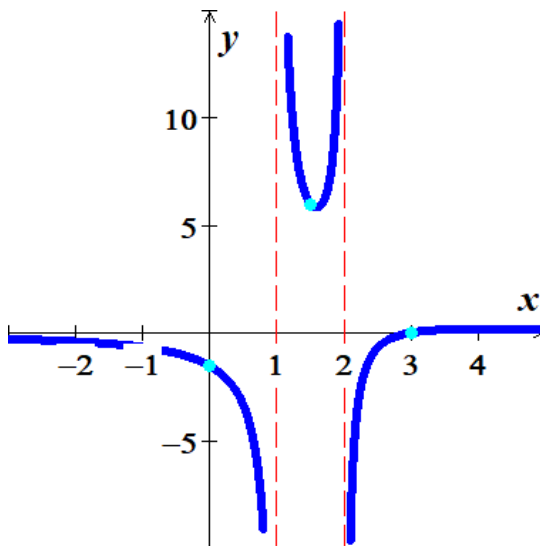
Solution

$$x^2 - 3x + 2 \rightarrow \underline{x=1, 2}$$

$$VA: x=1, 2 \quad HA: y=0$$

$$Hole: n/a \quad OA: n/a$$

x	y
0	$-\frac{3}{2}$
3	0
$\frac{3}{2}$	6



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x^2+2}{x^2+3x+2}$$

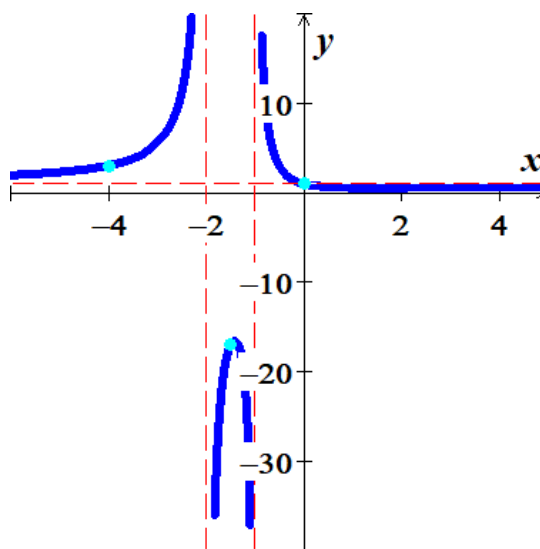
Solution

$$x^2 + 3x + 2 \rightarrow \underline{x=-1, -2}$$

$$VA: x=-1, -2 \quad HA: y=1$$

$$Hole: n/a \quad OA: n/a$$

x	y
0	1
$-\frac{3}{2}$	-17
-4	3



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x-2}{x^2-3x+2}$$

Solution

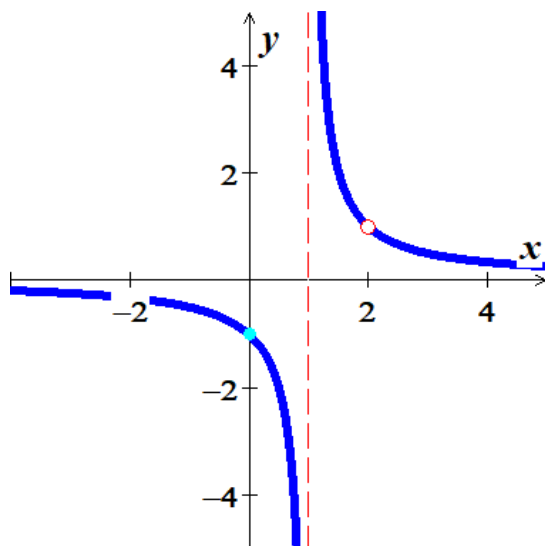
$$x^2 - 3x + 2 \rightarrow x = 1, 2$$

$$f(x) = \frac{x-2}{(x-2)(x-1)} = \frac{1}{x-1}$$

VA: $x = 1$ **HA:** $y = 0$

Hole: $(2, 1)$ **OA:** n/a

x	y
0	-1



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x^2+x}{x+1}$$

Solution

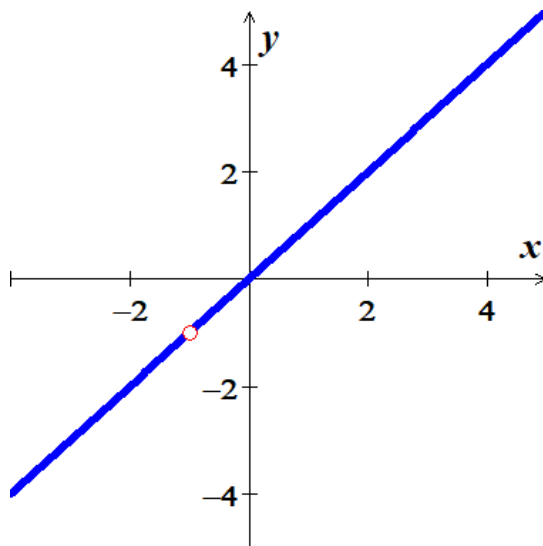
$$f(x) = \frac{x(x+1)}{x+1} = x$$

VA: n/a **HA:** n/a

Hole: $(-1, -1)$ **OA:** n/a

Hole: $(-3, -\frac{11}{7})$ **OA:** n/a

x	y
0	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x^2 - 2x}{x - 2}$$

Solution

$$f(x) = \frac{x(x-2)}{x-2}$$

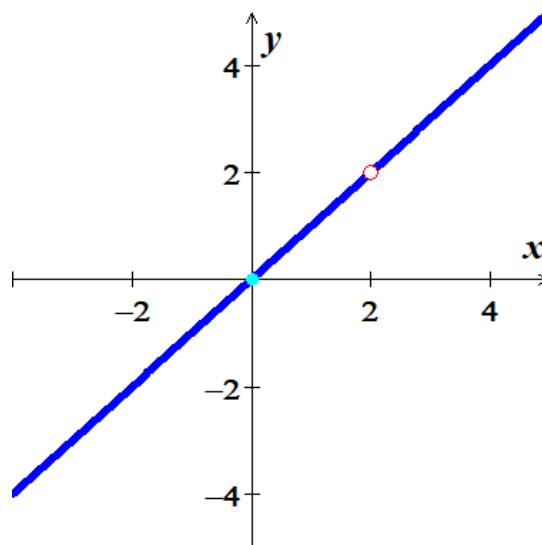
$$= x$$

VA: n/a **HA:** n/a

Hole: $(2, 2)$ **OA:** n/a

Hole: $(-3, -\frac{11}{7})$ **OA:** n/a

x	y
0	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x^2 - 3x}{x + 3}$$

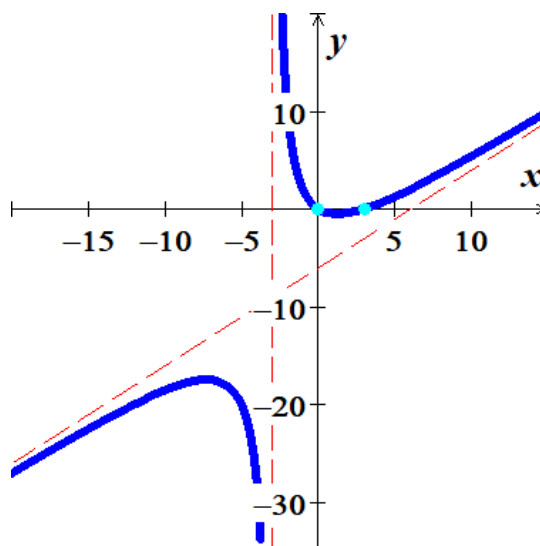
Solution

$$x + 3 \overline{) \begin{array}{r} x - 6 \\ x^2 - 3x \\ \underline{x^2 + 3x} \\ -6x - 5 \end{array}}$$

VA: $x = -3$ **HA:** n/a

Hole: n/a **OA:** $y = x - 6$

x	y
0	0
3	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote; Hole; Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x^3 + 3x^2 - 4x + 6}{x + 2}$$

Solution

$$\begin{array}{r} x^2 + x - 6 \\ x + 2 \overline{) x^3 + 3x^2 - 4x + 6} \\ \underline{x^3 + 2x^2} \\ x^2 - 4x \\ \underline{x^2 + 2x} \\ -6x + 6 \end{array}$$

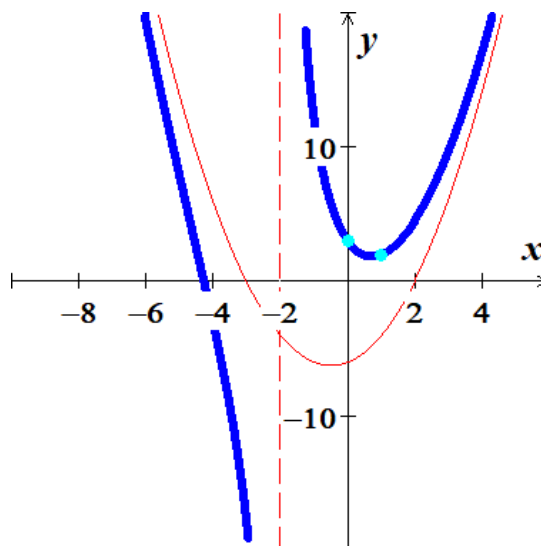
VA: $x = -2$

HA: n/a

Hole: n/a

OA: $y = x^2 + x - 6$

x	y
0	3
1	2



Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} \text{vertical asymptote: } x = 4 \\ \text{horizontal asymptote: } y = -1 \\ x\text{-intercept: } 3 \end{cases}$$

Solution

Vertical Asymptote: $f(x) = \frac{\quad}{x - 4}$

Horizontal Asymptote: $f(x) = \frac{-x + a}{x - 4}$

x -intercept: $f(x = 3) = \frac{-3 + a}{3 - 4} = 0 \Rightarrow a = 3$

$$\underline{f(x) = \frac{-x + 3}{x - 4}}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} \text{vertical asymptote: } x = -4, x = 5 \\ \text{horizontal asymptote: } y = \frac{3}{2} \\ x\text{-intercept: } -2 \end{cases}$$

Solution

Vertical Asymptote: $f(x) = \frac{\quad}{(x+4)(x-5)}$

Horizontal Asymptote: $f(x) = \frac{\textcolor{red}{3}(x+a)(x+b)}{\textcolor{red}{2}(x+4)(x-5)}$

x-intercept: $f(x = \textcolor{red}{-2}) = \frac{3(\textcolor{red}{-2}+a)(\textcolor{red}{-2}+b)}{2}$

$$0 = (\textcolor{red}{-2}+a)(\textcolor{red}{-2}+b)$$

$$\boxed{a = b = 2}$$

$$\begin{aligned} f(x) &= \frac{3}{2} \frac{(x-2)^2}{x^2 - x - 20} \\ &= \frac{\textcolor{blue}{3x^2 - 12x + 12}}{\textcolor{blue}{2x^2 - 2x - 40}} \end{aligned}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} \text{vertical asymptote: } x = 5 \\ \text{horizontal asymptote: } y = -1 \\ x\text{-intercept: } 2 \end{cases}$$

Solution

Vertical Asymptote: $f(x) = \frac{\quad}{x-5}$

x-intercept: $f(x) = \frac{x-2}{x-5}$

Horizontal Asymptote: $f(x) = -\frac{x-2}{x-5}$

$$\underline{f(x) = -\frac{x-2}{x-5}}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} \text{vertical asymptote: } x = -2, x = 0 \\ \text{horizontal asymptote: } y = 0 \\ x\text{-intercept: } 2, \quad f(3) = 1 \end{cases}$$

Solution

Vertical Asymptote: $f(x) = \frac{\quad}{x(x+2)}$

x-intercept : $f(x) = \frac{x-2}{x(x+2)}$

Horizontal Asymptote: $f(x) = \frac{a(x-2)}{x(x+2)}$

$$f(3) = 1 \rightarrow \frac{a(1)}{(3)(5)} = 1 \Rightarrow \underline{a = 15}$$

$$\underline{f(x) = \frac{15x-30}{x^2+2x}}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} \text{vertical asymptote: } x = -3, x = 1 \\ \text{horizontal asymptote: } y = 0 \\ x\text{-intercept: } -1, \quad f(0) = -2 \\ \text{hole: } x = 2 \end{cases}$$

Solution

Vertical Asymptote: $f(x) = \frac{\quad}{(x+3)(x-1)}$

x-intercept : $f(x) = \frac{(x+1)}{(x+3)(x-1)}$

Horizontal Asymptote: $f(x) = \frac{a(x+1)}{(x+3)(x-1)}$

$$f(0) = -2 \rightarrow \frac{a}{-3} = -2 \Rightarrow \underline{a = 6}$$

Hole at $x = 2$: $f(x) = \frac{6(x+1)(x-2)}{(x^2+2x-3)(x-2)}$

$$\underline{f(x) = \frac{6x^2-6x-12}{x^2-7x+6}}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} \text{vertical asymptote: } x = -1, \ x = 3 \\ \text{horizontal asymptote: } y = 2 \\ x\text{-intercept: } -2, \ 1 \\ \text{hole: } x = 0 \end{cases}$$

Solution

Vertical Asymptote: $f(x) = \frac{\quad}{(x+1)(x-3)}$

Horizontal Asymptote: $f(x) = \frac{2}{(x+1)(x-3)}$

x-intercept : $f(x) = \frac{2(x+2)(x-1)}{(x+1)(x-3)}$

Hole at $x = 0$: $f(x) = \frac{2x(x+2)(x-1)}{x(x+1)(x-3)}$

$$\underline{f(x) = \frac{2x^3 + 2x^2 - 4x}{x^3 - 2x^2 - 3x}}$$