Solution — *Section* 1.1 – Differential Equations & Solutions

Exercise

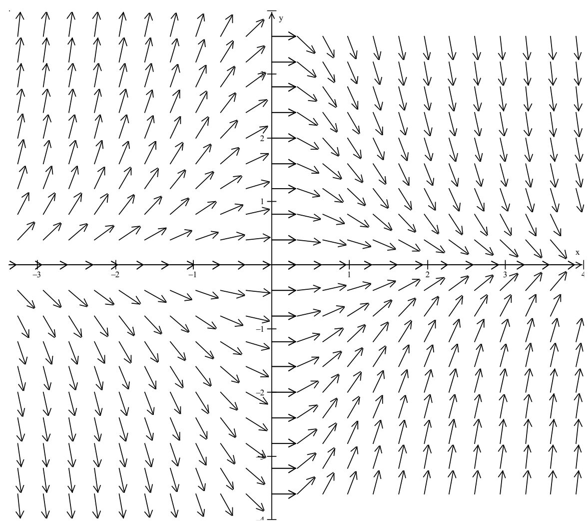
Show that $y(t) = Ce^{-(1/2)t^2}$ is a solution of the 1st order equation $y' = -ty \quad \textit{for} \quad -3 \le C \le 3$

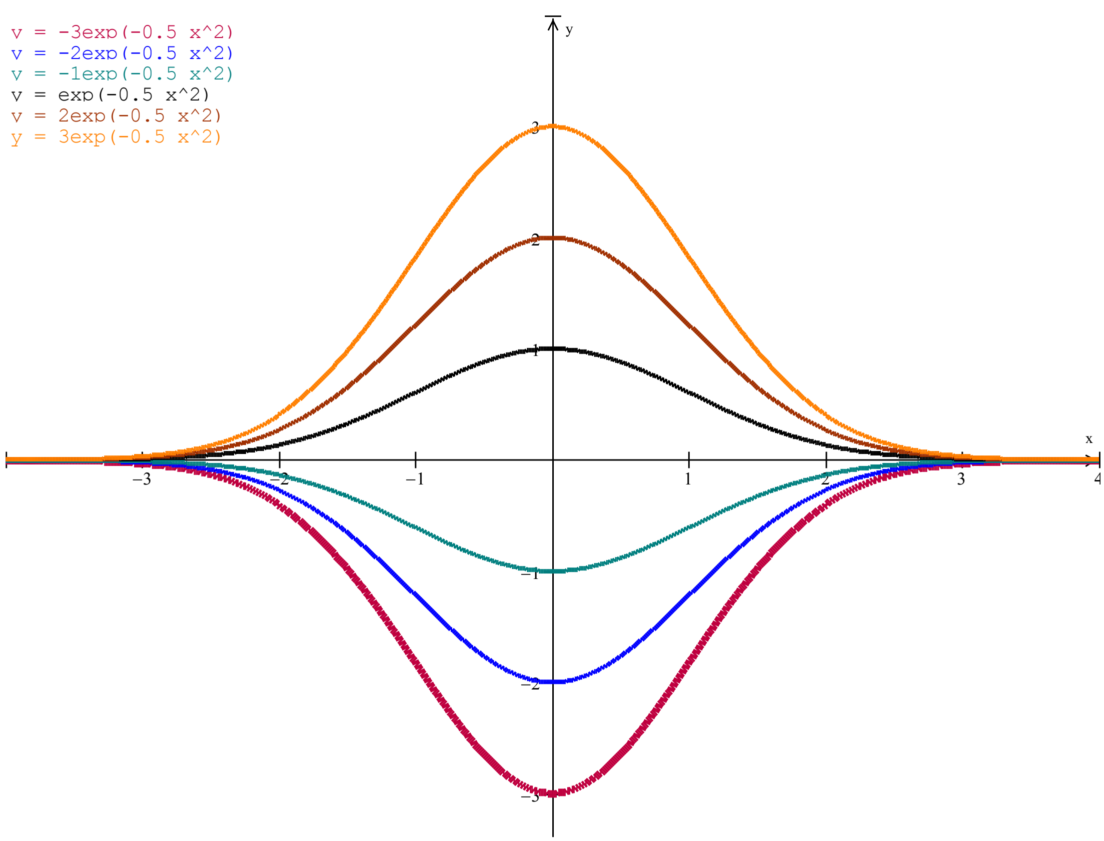
Solution

$$y' = -\frac{1}{2}2tCe^{-(1/2)t^2}$$

$$= -tCe^{-(1/2)t^2}$$

$$\boxed{= -ty}$$





Exercise

Show that $y(t) = \dfrac{4}{1 + Ce^{-4t}}$ is a solution of the 1st order equation $y' = y(4 - y)$

Solution

$$y' = \frac{d}{dt}\left(\frac{4}{1 + Ce^{-4t}}\right)$$

$$= \frac{-4\left(Ce^{-4t}\right)'}{\left(1 + Ce^{-4t}\right)^2}$$

$$= \frac{16Ce^{-4t}}{\left(1 + Ce^{-4t}\right)^2}$$

$$= \frac{A}{1 + Ce^{-4t}} + \frac{B}{\left(1 + Ce^{-4t}\right)^2}$$

$$= \frac{A + ACe^{-4t} + B}{\left(1 + Ce^{-4t}\right)^2}$$

$$\Rightarrow \begin{cases} A = 16 \\ A + B = 0 \rightarrow B = -16 \end{cases}$$

$$= \frac{16}{1 + Ce^{-4t}} - \frac{16}{\left(1 + Ce^{-4t}\right)^2}$$

$$= \frac{4}{1 + Ce^{-4t}}\left(4 - \frac{4}{1 + Ce^{-4t}}\right)$$

$$\underline{= y(4 - y)}$$

$$y(4 - y) =$$

$$= \frac{4}{1 + Ce^{-4t}}\left[4 - \frac{4}{1 + Ce^{-4t}}\right]$$

$$= \frac{4}{1 + Ce^{-4t}}\left[\frac{4 + 4Ce^{-4t} - 4}{1 + Ce^{-4t}}\right]$$

$$= \frac{4}{1 + Ce^{-4t}}\left[\frac{4Ce^{-4t}}{1 + Ce^{-4t}}\right]$$

$$= \frac{16Ce^{-4t}}{\left(1 + Ce^{-4t}\right)^2}$$

Exercise

Show that $y(x) = x^{-3/2}$ is a solution of $4x^2y'' + 12xy' + 3y = 0$ for $x > 0$

Solution

$$y(x) = x^{-3/2}$$

$$y' = -\frac{3}{2}x^{-5/2}$$

$$y'' = \frac{15}{4}x^{-7/2}$$

$$4x^2y'' + 12xy' + 3y \overset{?}{=} 0$$

$$4x^2\left(\frac{15}{4}x^{-7/2}\right)+12x\left(-\frac{3}{2}x^{-5/2}\right)+3x^{-3/2}\overset{?}{=}0$$

$$15x^{-3/2}-18x^{-3/2}+3x^{-3/2}\overset{?}{=}0$$

$$0=0 \quad \checkmark$$

$\therefore \quad y(x)=x^{-3/2}$ is a solution of $4x^2y''+12xy'+3y=0$

Exercise

A general solution may fail to produce all solutions of a differential equation $y(t)=\dfrac{4}{1+Ce^{-4t}}$. Show that $y=0$ is a solution of the differential equation, but no value of C in the given general solution will produce this solution.

Solution

$$y(t)=0 \Rightarrow y'=0$$

$$y(4-y)=0(4-0)=0$$

Exercise

Use the given general solution to find a solution of the differential equation having the given initial condition. $\quad ty'+y=t^2, \quad y(t)=\frac{1}{3}t^2+\frac{C}{t}, \qquad y(1)=2$

Solution

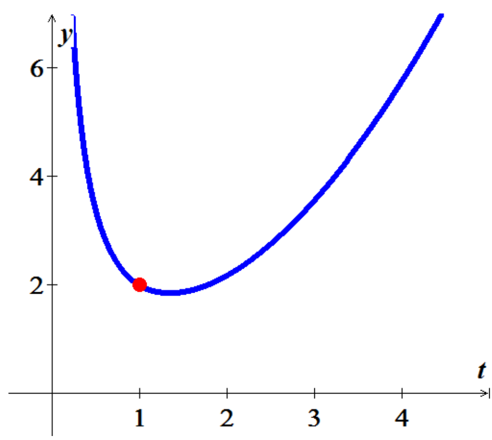
$$y(1)=2$$

$$y(1)=\tfrac{1}{3}(1)^2+\frac{C}{1}$$

$$2=\tfrac{1}{3}+C$$

$$C=2-\tfrac{1}{3}=\tfrac{5}{3}$$

$$\boxed{y(t)=\tfrac{1}{3}t^2+\frac{5}{3t}}$$



The interval of existence is $(0,\ \infty)$

Exercise

Show that $y(t)=2t-2+Ce^{-t}$ is a solution of the 1st order equation $y'+y=2t \ \ for \quad -3\le C\le 3$

Solution

$$y' + y = \left(2t - 2 + Ce^{-t}\right)' + 2t - 2 + Ce^{-t}$$
$$= 2 - Ce^{-t} + 2t - 2 + Ce^{-t}$$
$$= 2t \quad \checkmark$$

Exercise

Use the given general solution to find a solution of the differential equation having the given initial condition. $y' + 4y = \cos t, \quad y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t + Ce^{-4t}, \qquad y(0) = -1$

Solution

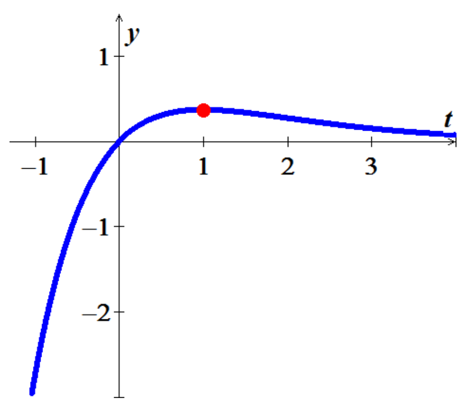
$$y(0) = \frac{4}{17}\cos(0) + \frac{1}{17}\sin(0) + Ce^{-4(0)}$$
$$-1 = \frac{4}{17} + C$$
$$C = -1 - \frac{4}{17}$$
$$= -\frac{21}{17}$$

$$\boxed{y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t - \frac{21}{17}e^{-4t}}$$

Exercise

Use the given general solution to find a solution of the differential equation having the given initial condition. $ty' + (t+1)y = 2te^{-t}, \quad y(t) = e^{-t}\left(t + \frac{C}{t}\right), \quad y(1) = \frac{1}{e}$

Solution



$$y(1) = \frac{1}{e} = e^{-1}$$
$$y(1) = e^{-1}\left(1 + \frac{C}{1}\right)$$
$$e^{-1} = e^{-1}(1 + C)$$
$$1 = 1 + C$$

Hence, $C = 0$

The solution is: $y(t) = te^{-t}$

This function is defined and differentiable on the whole real line. Hence, the interval of existence is the whole real line.

Exercise

Use the given general solution to find a solution of the differential equation having the given initial condition. $y' = y(2+y), \quad y(t) = \dfrac{2}{-1+Ce^{-2t}}, \quad y(0) = -3$

Solution

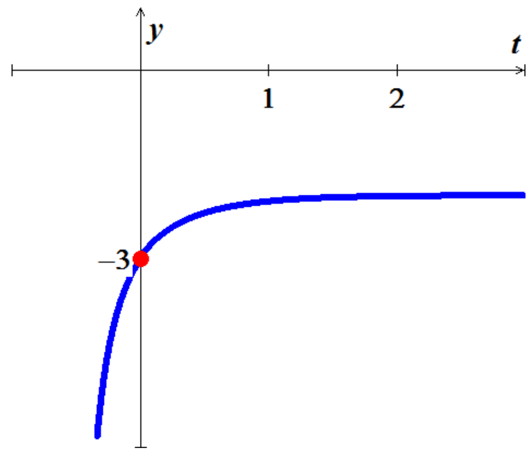
$$y(0) = \frac{2}{-1+Ce^{-2(0)}}$$

$$-3 = \frac{2}{-1+C}$$

$$3 - 3C = 2$$

$$-3C = -1$$

$$\boxed{C = \frac{1}{3}}$$



The solution is:

$$y(t) = \frac{2}{-1+\frac{1}{3}e^{-2t}}$$

$$\boxed{= \frac{6}{-3+e^{-2t}}}$$

Exercise

Find the values of ***m*** so that the function $y = e^{mx}$ is a solution of the given differential equation

a) $y' + 2y = 0$

b) $5y' - 2y = 0$

c) $y'' - 5y' + 6y = 0$

d) $2y'' + 7y' - 4y = 0$

Solution

$$y = e^{mx} \quad \Rightarrow \quad y' = me^{mx} \quad \Rightarrow \quad y'' = m^2e^{mx}$$

a) $y' + 2y = 0$

$$me^{mx} + 2e^{mx} = 0 \quad \Rightarrow \quad (m+2)e^{mx} = 0$$

$$\boxed{m = -2}$$

b) $5y' - 2y = 0$

$$5me^{mx} - 2e^{mx} = 0 \quad \Rightarrow \quad (5m-2)e^{mx} = 0$$

$$\boxed{m = \frac{2}{5}}$$

c) $y'' - 5y' + 6y = 0$

$$m^2e^{mx} - 5me^{mx} + 6e^{mx} = 0 \quad \Rightarrow \quad \left(m^2 - 5m + 6\right)e^{mx} = 0$$

$$\boxed{m = 2, \ \ 3}$$

d) $2y'' + 7y' - 4y = 0$

$$2m^2e^{mx} + 7me^{mx} - 4e^{mx} = 0 \quad \Rightarrow \quad \left(2m^2 + 7m - 4\right)e^{mx} = 0$$

$$\boxed{m = \frac{1}{2}, \quad -4}$$

Exercise

Let $x = c_1 \cos t + c_2 \sin t$ is 2-parameter family solutions of the second order differential equation of $x'' + x = 0$. Find a solution of the second-order consisting of this differential equation and the given initial conditions.

a) $x(0) = -1, \quad x'(0) = 8$

b) $x\left(\frac{\pi}{2}\right) = 0, \quad x'\left(\frac{\pi}{2}\right) = 1$

c) $x\left(\frac{\pi}{6}\right) = \frac{1}{2}, \quad x'\left(\frac{\pi}{6}\right) = 0$

d) $x\left(\frac{\pi}{4}\right) = \sqrt{2}, \quad x'\left(\frac{\pi}{4}\right) = 2\sqrt{2}$

Solution

$$x = c_1 \cos t + c_2 \sin t \quad \Rightarrow \quad x' = -c_1 \sin t + c_2 \cos t$$

a) $x(0) = -1 \quad \Rightarrow \quad \boxed{-1 = c_1}$

$x'(0) = 8 \quad \Rightarrow \quad \boxed{8 = c_2}$

b) $x\left(\frac{\pi}{2}\right) = 0 \quad \Rightarrow \quad \boxed{0 = c_2}$

$x'\left(\frac{\pi}{2}\right) = 1 \quad \Rightarrow \quad \boxed{-1 = c_1}$

c) $x\left(\frac{\pi}{6}\right) = \frac{1}{2} \quad \Rightarrow \quad \frac{\sqrt{3}}{2}c_1 + \frac{1}{2}c_2 = \frac{1}{2} \quad \rightarrow \quad \sqrt{3}\, c_1 + c_2 = 1$

$x'\left(\frac{\pi}{6}\right) = 0 \quad \Rightarrow \quad -\frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2 = 0 \quad \rightarrow \quad -c_1 + \sqrt{3}\, c_2 = 0$

$\boxed{c_1 = \frac{\sqrt{3}}{4}}, \quad \boxed{c_2 = \frac{1}{4}}$

d) $x\left(\frac{\pi}{4}\right) = \sqrt{2} \quad \Rightarrow \quad \frac{\sqrt{2}}{2}c_1 + \frac{\sqrt{2}}{2}c_2 = \sqrt{2} \quad \rightarrow \quad c_1 + c_2 = 2$

$x'\left(\frac{\pi}{4}\right) = 2\sqrt{2} \quad \Rightarrow \quad \frac{\sqrt{2}}{2}c_1 + \frac{\sqrt{2}}{2}c_2 = 2\sqrt{2} \quad \rightarrow \quad -c_1 + c_2 = 4$

$\boxed{c_1 = -1} \quad \boxed{c_2 = 3}$

Exercise

Find values of r such that $y(x)=x^r$ is a solution of $x^2y''-4xy'+6y=0$

Solution

$$y(x)=x^r \quad \Rightarrow \quad y'=rx^{r-1}$$

$$y''=r(r-1)x^{r-2}$$

$$x^2r(r-1)x^{r-2}-4xrx^{r-1}+6x^r=0$$

$$r(r-1)x^r-4rx^r+6x^r=0$$

$$r^2-r-4r+6=0 \quad \text{since} \quad x^r\neq 0$$

$$r^2-5r+6=0 \quad \rightarrow \quad \underline{r=3,\ 2}$$

Exercise

Solve the differential equation $y'=3x^2-2x+4$

Solution

$$y(x)=\int\left(3x^2-2x+4\right)dx$$

$$\underline{=x^3-x^2+4x+C}$$

Exercise

Solve the differential equation $y''=2x+\sin 2x$

Solution

$$y'=\int(2x+\sin 2x)dx$$

$$=x^2-\tfrac{1}{2}\cos 2x+C_1$$

$$y=\int\left(x^2-\tfrac{1}{2}\cos 2x+C_1\right)dx$$

$$\underline{=\tfrac{1}{3}x^3-\tfrac{1}{4}\sin 2x+C_1x+C_2}$$

Exercise

Given the differential equation $x^2y''-2xy'+2y=4x^3$, is the given equation a solution to?

Solution

a) $y = 2x^3 + x^2$

$y' = 6x^2 + 2x$

$y'' = 12x + 2$

$x^2 y'' - 2xy' + 2y \overset{?}{=} 4x^3$

$x^2(12x+2) - 2x\left(6x^2 + 2x\right) + 2\left(2x^3 + x^2\right) = 12x^3 + 2x^2 - 12x^3 - 4x^2 + 4x^3 + 2x^2$

$$= 4x^3 \quad \checkmark$$

$y = 2x^3 + x^2$ is a solution.

b) $y = 2x + x^2$

$y' = 2 + 2x$

$y'' = 2$

$x^2 y'' - 2xy' + 2y \overset{?}{=} 4x^3$

$x^2(2) - 2x(2+2x) + 2\left(2x + x^2\right) = 2x^2 - 4x - 4x^2 + 4x + 2x^2$

$$= 0 \neq 4x^3$$

$y = 2x + x^2$ is ***not*** a solution.

Exercise

Find the general solution of the differential equation $y' = xy$

Solution

$$\frac{dy}{dx} = xy$$

$$\frac{dy}{y} = xdx$$

$$\int \frac{dy}{y} = \int xdx$$

$$\ln|y| = \frac{1}{2}x^2 + C$$

$$|y| = e^{x^2/2+C}$$

$$y(x) = \pm e^{x^2/2}e^C$$

$$= Ae^{x^2/2} \qquad \text{Where } A = \pm e^C$$

Exercise

Find the general solution of the differential equation $xy' = 2y$

Solution

$$x\frac{dy}{dx} = 2y$$

$$\frac{dy}{y} = 2\frac{dx}{x}$$

$$\int \frac{dy}{y} = \int \frac{2}{x}dx$$

$$\ln|y| = 2\ln|x| + C$$

$$= \ln x^2 + C$$

$$y(x) = \pm e^{\ln x^2 + C}$$

$$= \pm e^C x^2$$

$$= Ax^2$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = e^{x-y}$

Solution

$$\frac{dy}{dx} = e^{x}e^{-y}$$

$$\frac{dy}{e^{-y}} = e^{x}dx$$

$$\int e^{y}dy = \int e^{x}dx$$

$$e^{y} = e^{x} + C$$

$$\boxed{y(x) = \ln\left(e^{x} + C\right)}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \left(1 + y^2\right)e^{x}$

Solution

$$\frac{dy}{dx} = \left(1 + y^2\right)e^{x}$$

$$\int \frac{dy}{1 + y^2} = \int e^{x}dx$$

$$\tan^{-1} y = e^{x} + C$$

$$\boxed{y(x) = \tan\left(e^{x} + C\right)}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = xy + y$

Solution

$$\frac{dy}{dx} = (x+1)y$$

$$\int \frac{dy}{y} = \int (x+1)dx$$

$$\ln y = \frac{1}{2}x^2 + x + C$$

$$\boxed{y(x) = e^{x^2/2 + x + C}}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$y' = ye^x - 2e^x + y - 2$

Solution

$$\frac{dy}{dx} = (y-2)e^x + y - 2$$

$$\frac{dy}{dx} = (y-2)\left(e^x + 1\right)$$

$$\frac{dy}{y-2} = \left(e^x + 1\right)dx$$

$$\int \frac{dy}{y-2} = \int \left(e^x + 1\right)dx$$

$$\ln|y-2| = e^x + x + C$$

$$y - 2 = \pm e^{e^x + x + C}$$

$$y - 2 = \pm e^C e^{e^x + x}$$

$$\boxed{y(x) = De^{e^x + x} + 2} \quad D = \pm e^C$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \dfrac{x}{y+2}$

Solution

$$\frac{dy}{dx} = \frac{x}{y+2}$$

$$(y+2)dy = xdx$$

$$\int (y+2)dy = \int xdx$$

$$\tfrac{1}{2}y^2 + 2y = \tfrac{1}{2}x^2 + C$$

$$y^2 + 4y = x^2 + 2C$$

$$y^2 + 4y - x^2 - D = 0, \quad (D = 2C)$$

$$y = \frac{-4 \pm \sqrt{16 - 4\left(-x^2 - D\right)}}{2} = \frac{-4 \pm \sqrt{16 + 4x^2 + 4D}}{2} = \frac{-4 \pm 2\sqrt{x^2 + (4+D)}}{2} \underline{= -2 \pm \sqrt{x^2 + E}} \qquad E = 4 + D$$

$$\boxed{y(x) = -2 \pm \sqrt{x^2 + E}}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \dfrac{xy}{x-1}$

Solution

$$\frac{dy}{dx} = y\left(\frac{x}{x-1}\right)$$

$$\frac{dy}{y} = \left(\frac{x}{x-1}\right)dx$$

$$\int \frac{dy}{y} = \int \left(1+\frac{1}{x-1}\right)dx$$

$$\ln|y| = x + \ln|x-1| + C$$

$$y(x) = \pm e^{x+\ln|x-1|+C}$$

$$= \pm e^{C} e^{x} e^{\ln|x-1|}$$

$$\boxed{= De^{x}|x-1|}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$$y' = \frac{y^2 + ty + t^2}{t^2}$$

Solution

$$y' = \frac{y^2}{t^2} + \frac{y}{t} + 1$$

Let $x = \dfrac{y}{t} \quad \Rightarrow \quad y = xt \ \rightarrow \ y' = x + tx'$

$$y' = \frac{y^2}{t^2} + \frac{y}{t} + 1 = x^2 + x + 1$$

$$x + tx' = x^2 + x + 1$$

$$t\frac{dx}{dt} = x^2 + 1$$

$$\int \frac{dx}{x^2+1} = \int \frac{dt}{t}$$

$$\tan^{-1} x = \ln|t| + C$$

$$\tan^{-1}\frac{y}{t} = \ln|t| + C$$

$$\frac{y}{t} = \tan\left(\ln|t| + C\right)$$

$$\boxed{y(t) = t\tan\left(\ln|t| + C\right)}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}$$

Solution

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}$$

$$\left(4 + y^3\right)dy = \left(4x - x^3\right)dx$$

$$\int \left(4 + y^3\right)dy = \int \left(4x - x^3\right)dx$$

$$4y + \frac{1}{4}y^4 = 2x^2 - \frac{1}{4}x^4 + C_1$$

$$16y + y^4 = 8x^2 - x^4 + C$$

$$\boxed{y^4 + 16y + x^4 - 8x^2 = C}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$$y' = \frac{2xy + 2x}{x^2 - 1}$$

Solution

$$\frac{dy}{dx} = \frac{2x(y+1)}{x^2 - 1}$$

$$\frac{dy}{y+1} = \frac{2x}{x^2 - 1}dx \qquad d\left(x^2 - 1\right) = 2xdx$$

$$\int \frac{d(y+1)}{y+1} = \int \frac{d\left(x^2 - 1\right)}{x^2 - 1}$$

$$\ln|y+1| = \ln\left|x^2 - 1\right| + C$$

$$y + 1 = e^{\ln\left|x^2 - 1\right| + C}$$

$$y = e^C e^{\ln\left|x^2 - 1\right|} - 1$$

$$\boxed{y(x) = Ae^{\ln\left|x^2 - 1\right|} - 1}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} = \sin 5x$

Solution

$$\int dy = \int \sin 5x dx$$

$$\boxed{y(x) = -\frac{1}{5}\cos 5x + C}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} = (x+1)^2$

Solution

$$\int dy = \int \left(x^2 + 2x + 1\right) dx$$

$$\boxed{y(x) = \frac{1}{3}x^3 + x^2 + x + C}$$

Exercise

Find the general solution of the differential equation $dx + e^{3x} dy = 0$

Solution

$$\int dy = -\int e^{-3x} dx$$

$$\boxed{y(x) = \frac{1}{3}e^{-3x} + C}$$

Exercise

Find the general solution of the differential equation $dy - (y-1)^2 dx = 0$

Solution

$$\int \frac{dy}{(y-1)^2} = \int dx$$

$$\int \frac{d(y-1)}{(y-1)^2} = \int dx$$

$$-\frac{1}{y-1} = x + C$$

$$\boxed{y(x) = 1 - \frac{1}{x+C}}$$

Exercise

Find the general solution of the differential equation $x\frac{dy}{dx} = 4y$

Solution

$$\int \frac{dy}{y} = 4\int \frac{dx}{x}$$

$$\ln y = 4\ln x + \ln C$$

$$\ln y = \ln Cx^4$$

$$\boxed{y(x) = Cx^4}$$

Exercise

Find the general solution of the differential equation $\frac{dx}{dy} = y^2 - 1$

Solution

$$\int dx = \int \left(y^2 - 1\right) dy$$

$$\boxed{x = \frac{1}{3}y^3 - y + C}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} = e^{2y}$

Solution

$$\int e^{-2y} dy = \int dx$$

$$-\frac{1}{2}e^{-2y} = x + C$$

$$e^{-2y} = -2x + C_1$$

$$-2y = \ln\left(C_1 - 2x\right)$$

$$\boxed{y(x) = -\frac{1}{2}\ln\left(C_1 - 2x\right)}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} + 2xy^2 = 0$

Solution

$$\frac{dy}{dx} = -2xy^2$$

$$-\int \frac{dy}{y^2} = \int 2x\,dx$$

$$\frac{1}{y} = x^2 + C$$

$$\boxed{y(x) = \frac{1}{x^2 + C}}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} = e^{3x+2y}$

Solution

$$\frac{dy}{dx} = e^{3x}e^{2y}$$

$$\int e^{-2y}dy = \int e^{3x}dx$$

$$-\frac{1}{2}e^{-2y} = \frac{1}{3}e^{3x} + C$$

$$e^{-2y} = C_1 - \frac{2}{3}e^{3x}$$

$$-2y = \ln\left(C_1 - \frac{2}{3}e^{3x}\right)$$

$$\boxed{y(x) = -\frac{1}{2}\ln\left(C_1 - \frac{2}{3}e^{3x}\right)}$$

Exercise

Find the general solution of the differential equation $e^x y\frac{dy}{dx} = e^{-y} + e^{-2x-y}$

Solution

$$e^x y\frac{dy}{dx} = e^{-y}\left(1 + e^{-2x}\right)$$

$$ye^y dy = e^{-x}\left(1 + e^{-2x}\right)dx$$

$$\int ye^y dy = \int \left(e^{-x} + e^{-3x}\right)dx$$

$$\boxed{(y-1)e^y = -e^{-x} - \frac{1}{3}e^{-3x} + C}$$

		$\int e^y$
+	y	e^y
−	1	e^y

Exercise

Find the general solution of the differential equation $y\ln x\dfrac{dx}{dy}=\left(\dfrac{y+1}{x}\right)^2$

Solution

$$x^2\ln x dx=\frac{1}{y}\left(y^2+2y+1\right)dy$$

$$\int x^2\ln x dx=\int\left(y+2+\frac{1}{y}\right)dy$$

$$u=\ln x \qquad dv=x^2dx$$

$$du=\frac{dx}{x} \qquad v=\frac{1}{3}x^3$$

$$\frac{1}{3}x^3\ln x-\frac{1}{3}\int x^2dx=\frac{1}{2}y^2+2y+\ln|y|+C$$

$$\boxed{\frac{1}{3}x^3\ln x-\frac{1}{9}x^3=\frac{1}{2}y^2+2y+\ln|y|+C}$$

Exercise

Find the general solution of the differential equation $\dfrac{dy}{dx}=\left(\dfrac{2y+3}{4x+5}\right)^2$

Solution

$$\int\frac{dy}{(2y+3)^2}=\int\frac{dx}{(4x+5)^2}$$

$$\frac{1}{2}\int\frac{d(2y+3)}{(2y+3)^2}=\frac{1}{4}\int\frac{d(4x+5)}{(4x+5)^2}$$

$$\frac{1}{2}\frac{-1}{2y+3}=\frac{1}{4}\frac{-1}{4x+5}+C$$

$$\boxed{\frac{2}{2y+3}=\frac{1}{4x+5}+C}$$

Exercise

Find the general solution of the differential equation $\csc y dx+\sec^2 x dy=0$

Solution

$$\csc y dx=-\sec^2 x dy$$

$$\frac{dy}{\csc y}=-\frac{dx}{\sec^2 x}$$

$$\sin y dy=-\cos^2 x dx$$

$$\int \sin y\ dy = -\frac{1}{2}\int (1+\cos 2x)\,dx$$

$$-\cos y = -\frac{1}{2}\left(x+\frac{1}{2}\sin 2x\right)+C$$

$$\boxed{\cos y = \frac{1}{2}x+\frac{1}{4}\sin 2x + C}$$

Exercise

Find the general solution of the differential equation $\sin 3x dx + 2y\cos^3 3x dy = 0$

Solution

$$\sin 3x dx = -2y\cos^3 3x dy$$

$$\int \frac{\sin 3x}{\cos^3 3x}dx = -\int 2y dy$$

$$-\frac{1}{3}\int \cos^{-3} 3x\ d(\cos 3x) = -\int 2y dy$$

$$-\frac{1}{6}\cos^{-2} 3x + C = y^2$$

$$\boxed{y^2 = -\frac{1}{6}\sec^2 3x + C}$$

Exercise

Find the general solution of the differential equation $\left(e^y+1\right)^2 e^{-y}dx + \left(e^x+1\right)^3 e^{-x}dy = 0$

Solution

$$\left(e^y+1\right)^2 e^{-y}dx = -\left(e^x+1\right)^3 e^{-x}dy$$

$$\frac{e^x}{\left(e^x+1\right)^3}dx = -\frac{e^y}{\left(e^y+1\right)^2}dy$$

$$\int \left(e^x+1\right)^{-3} d\left(e^x+1\right) = -\int \frac{1}{\left(e^y+1\right)^2}d\left(e^y+1\right)$$

$$\boxed{-\frac{1}{2}\frac{1}{\left(e^x+1\right)^2}+C = \frac{1}{e^y+1}}$$

Exercise

Find the general solution of the differential equation $x\left(1+y^2\right)^{1/2} dx = y\left(1+x^2\right)^{1/2} dy$

Solution

$$\int x\left(1+x^2\right)^{-1/2} dx = \int y\left(1+y^2\right)^{-1/2} dy$$

$$\frac{1}{2}\int \left(1+x^2\right)^{-1/2} d\left(1+x^2\right) = \frac{1}{2}\int \left(1+y^2\right)^{-1/2} d\left(1+y^2\right)$$

$$2\left(1+y^2\right)^{1/2} = 2\left(1+x^2\right)^{1/2} + C$$

$$\left(1+y^2\right)^{1/2} = \left(1+x^2\right)^{1/2} + C$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = y\sin x$

Solution

$$\int \frac{dy}{y} = \int \sin x\ dx$$

$$\ln|y| = -\cos x + C$$

$$y = e^{-\cos x + C}$$

$$= Ae^{-\cos x}$$

Exercise

Find the general solution of the differential equation. $(1+x)\frac{dy}{dx} = 4y$

Solution

$$\int \frac{dy}{y} = \int \frac{4}{1+x} dx$$

$$\ln|y| = 4\ln|1+x| + \ln C$$

$$= \ln(1+x)^4 + \ln C$$

$$= \ln C(1+x)^4$$

$$y(x) = C(1+x)^4$$

Exercise

Find the general solution of the differential equation. $2\sqrt{x}\frac{dy}{dx} = \sqrt{1-y^2}$

Solution

$$\int \frac{dy}{\sqrt{1-y^2}} = \frac{1}{2}\int x^{-1/2}\, dx$$

$$\boxed{\arcsin y = \sqrt{x} + C}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = 3\sqrt{xy}$

Solution

$$\int y^{1/2} dy = 3\int x^{1/2} dx$$

$$\frac{2}{3}y^{3/2} = 2x^{3/2} + C$$

$$\boxed{y^{3/2} = 3x^{3/2} + C_1}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = (64xy)^{1/3}$

Solution

$$\int y^{-1/3} dy = \int 4x^{1/3} dx$$

$$\frac{3}{2}y^{2/3} = 3x^{4/3} + C_1$$

$$\boxed{y^{2/3} = 2x^{4/3} + C}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = 2x\sec y$

Solution

$$\int \cos y\ dy = \int 2x\ dx$$

$$\boxed{\sin y = x^2 + C}$$

Exercise

Find the general solution of the differential equation. $\left(1-x^2\right)\dfrac{dy}{dx}=2y$

Solution

$$\int \frac{dy}{y}=2\int \frac{1}{1-x^2}dx$$

$$\int \frac{dy}{y}=\int \left(\frac{1}{1+x}+\frac{1}{1-x}\right)dx$$

$$\ln|y|=\ln|1+x|-\ln|1-x|+\ln C$$

$$\ln|y|=\ln C\left|\frac{1+x}{1-x}\right|$$

$$\boxed{y(x)=C\frac{1+x}{1-x}}$$

Exercise

Find the general solution of the differential equation. $(1+x)^2\dfrac{dy}{dx}=(1+y)^2$

Solution

$$\int \frac{dy}{(1+y)^2}=\int \frac{1}{(1+x)^2}dx$$

$$-\frac{1}{1+y}=-\frac{1}{1+x}+C$$

$$\frac{1}{1+y}=\frac{1+C+Cx}{1+x}$$

$$y+1=\frac{1+x}{C_1+Cx}$$

$$y=\frac{1+x}{C_1+Cx}-1$$

$$=\frac{1+x-C_1-Cx}{C_1+Cx} \qquad A=1-C_1 \quad B=1-C$$

$$\boxed{=\frac{A+Bx}{C_1+Cx}}$$

Exercise

Find the general solution of the differential equation. $\dfrac{dy}{dx}=xy^3$

Solution

$$\int y^{-3}dy=\int xdx$$

$$-\frac{1}{2y^2} = \frac{1}{2}x^2 + C_1$$

$$\frac{1}{y^2} = -x^2 + C$$

$$\boxed{y^2 = \frac{1}{-x^2 + C}}$$

Exercise

Find the general solution of the differential equation. $y\frac{dy}{dx} = x\left(y^2 + 1\right)$

Solution

$$\int \frac{y}{y^2+1}dy = \int xdx$$

$$\frac{1}{2}\int \frac{1}{y^2+1}d\left(y^2+1\right) = \frac{1}{2}x^2 + C$$

$$\ln\left(y^2+1\right) = x^2 + C$$

$$y^2 + 1 = e^{x^2+C}$$

$$\boxed{y^2 = Ae^{x^2} - 1}$$

Exercise

Find the general solution of the differential equation. $y^3\frac{dy}{dx} = \left(y^4 + 1\right)\cos x$

Solution

$$\int \frac{y^3}{y^4+1}dy = \int \cos x dx$$

$$\frac{1}{4}\ln\left(y^4+1\right) = \sin x + C$$

$$\ln\left(y^4+1\right) = 4\sin x + C$$

$$y^4 + 1 = e^{4\sin x + C}$$

$$\boxed{y^4 = A\,e^{4\sin x} - 1}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = \frac{1+\sqrt{x}}{1+\sqrt{y}}$

Solution

$$\int \left(1 + y^{1/2}\right) dy = \int \left(1 + x^{1/2}\right) dx$$

$$\boxed{y + \frac{2}{3} y^{3/2} = x + \frac{2}{3} x^{3/2} + C}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = \frac{(x-1)y^5}{x^2\left(2y^3 - y\right)}$

Solution

$$\left(\frac{2y^3 - y}{y^5}\right) dy = \left(\frac{x-1}{x^2}\right) dx$$

$$\int \left(2\frac{1}{y^2} - \frac{1}{y^4}\right) dy = \int \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$$

$$-\frac{2}{y} + \frac{1}{3y^3} = \ln|x| + \frac{1}{x} + C$$

$$\boxed{\frac{1 - 6y^2}{3y^3} = \ln|x| + \frac{1}{x} + C}$$

Exercise

Find the general solution of the differential equation. $\left(x^2 + 1\right)(\tan y) y' = x$

Solution

$$\int \tan y \; dy = \int \frac{x}{x^2 + 1} dx$$

$$\ln|\sec y| = \frac{1}{2}\ln\left(x^2 + 1\right) + \ln C$$

$$= \ln C\sqrt{x^2 + 1}$$

$$\boxed{\sec y = C\sqrt{x^2 + 1}}$$

Exercise

Find the general solution of the differential equation. $x^2 y' = 1 - x^2 + y^2 - x^2 y^2$

Solution

$$x^2 y' = 1 - x^2 + \left(1 - x^2\right) y^2$$

$$x^2 y' = \left(1 - x^2\right)\left(1 + y^2\right)$$

$$\int \frac{1}{1 + y^2} dy = \int \frac{1 - x^2}{x^2} dx$$

$$\int \frac{1}{1 + y^2} dy = \int \left(\frac{1}{x^2} - 1\right) dx$$

$$\boxed{\arctan y = -\frac{1}{x} - x + C}$$

Exercise

Find the general solution of the differential equation. $xy' + 4y = 0$

Solution

$$x \frac{dy}{dx} = -4y$$

$$\int \frac{dy}{y} = -4 \int \frac{dx}{x}$$

$$\ln|y| = -4 \ln|x| + C$$

$$\ln|y| = \ln x^{-4} + C$$

$$y(x) = e^{\ln x^{-4} + C}$$

$$= e^C e^{\ln x^{-4}}$$

$$\boxed{= Ax^{-4}}$$

Exercise

Find the general solution of the differential equation. $\left(x^2 + 1\right) y' + 2xy = 0$

Solution

$$\left(x^2 + 1\right) \frac{dy}{dx} = -2xy$$

$$\int \frac{dy}{y} = -\int \frac{2x}{x^2 + 1} dx$$

$$\ln|y| = -\ln\left(x^2+1\right) + \ln C$$

$$\ln|y| = \ln\frac{C}{x^2+1}$$

$$\boxed{y(x) = \frac{C}{x^2+1}}$$

Exercise

Find the general solution of the differential equation. $\dfrac{y'}{\left(x^2+1\right)y} = 3$

Solution

$$\int \frac{1}{y}dy = \int \left(3x^2+3\right)dx$$

$$\ln|y| = x^3 + 3x + C$$

$$\boxed{y(x) = e^{x^3+3x+C}}$$

Exercise

Find the general solution of the differential equation. $y + e^x y' = 0$

Solution

$$e^x \frac{dy}{dx} = -y$$

$$\int \frac{dy}{y} = -\int e^{-x}dx$$

$$\ln|y| = e^{-x} + C$$

$$\boxed{y(x) = e^{e^{-x}+C}}$$

Exercise

Find the general solution of the differential equation. $\dfrac{dx}{dt} = 3xt^2$

Solution

$$\int \frac{dx}{x} = \int 3t^2 dt$$

$$\ln|x| = t^3 + C$$

$$x(t) = e^{t^3+C} \ \boxed{= Ae^{t^3}}$$

Exercise

Find the general solution of the differential equation. $x\frac{dy}{dx} = \frac{1}{y^3}$

Solution

$$\int y^3 dy = \int \frac{1}{x} dx$$

$$\frac{1}{4}y^4 = \ln|x| + C_1$$

$$y^4 = 4\ln|x| + C$$

$$\boxed{y^4 = \ln x^4 + C}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = \frac{x}{y^2\sqrt{x+1}}$

Solution

$$\int y^2 dy = \int \frac{x}{\sqrt{x+1}} dx$$

Let $u = x+1 \quad \to x = u-1 \quad \to du = dx$

$$\frac{1}{3}y^3 = \int \frac{u-1}{u^{1/2}} du$$

$$\frac{1}{3}y^3 = \int \left(u^{1/2} - u^{-1/2}\right) du$$

$$\frac{1}{3}y^3 = \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C_1$$

$$\boxed{y^3 = 2(x+1)^{3/2} - 6(x+1)^{1/2} + C}$$

Exercise

Find the general solution of the differential equation. $\frac{dx}{dt} - x^3 = x$

Solution

$$\frac{dx}{dt} = x^3 + x$$

$$\int \frac{dx}{x\left(x^2+1\right)} = \int dt$$

$$\frac{1}{x\left(x^2+1\right)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$Ax^2 + A + Bx^2 + Cx = 1$$

$$\begin{cases} x^2 & A+B=0 \\ x & \boxed{C=0} \quad \to \boxed{B=-1} \\ x^0 & \boxed{A=1} \end{cases}$$

$$\int \frac{dx}{x} - \int \frac{dx}{x^2+1} = t + K$$

$$\boxed{\ln|x| - \arctan x = t + K}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = \frac{x}{ye^{x+2y}}$

Solution

$$\frac{dy}{dx} = \frac{x}{ye^{2y}e^x}$$

$$\int ye^{2y}dy = \int xe^{-x}dx$$

$$\frac{1}{2}ye^{2y} - \frac{1}{4}e^{2y} = -xe^{-x} - e^{-x} + C_1$$

$$\boxed{(2y-1)e^{2y} = -4(x+1)e^{-x} + C}$$

		$\int e^{2y}$
+	y	$\frac{1}{2}e^{2y}$
–	1	$\frac{1}{4}e^{2y}$

		$\int e^{-x}$
+	x	$-e^{-x}$
–	1	e^{-x}

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = \frac{\sec^2 y}{1+x^2}$

Solution

$$\int \cos^2 y\ dy = \int \frac{dx}{1+x^2}$$

$$\frac{1}{2}\int (1+\cos 2y)dy = \arctan x + C$$

$$\boxed{\frac{1}{2}\left(y + \frac{1}{2}\sin 2y\right) = \arctan x + C}$$

Exercise

Find the general solution of the differential equation. $x\frac{dv}{dx} = \frac{1-4v^2}{3v}$

Solution

$$\int \frac{3v}{1-4v^2}\,dv = \int \frac{dx}{x}$$

$$-\frac{3}{8}\int \frac{1}{1-4v^2}\,d\left(1-4v^2\right) = \int \frac{dx}{x}$$

$$-\frac{3}{8}\ln\left|1-4v^2\right| = \ln|x| + \ln C$$

$$\ln\left(\left|1-4v^2\right|\right)^{-3/8} = \ln|Cx|$$

$$\boxed{\left(1-4v^2\right)^{-3/8} = Cx}$$

Exercise

Find the general solution of the differential equation. $\dfrac{dy}{dx} = 3x^2\left(1+y^2\right)^{3/2}$

Solution

$$\int \left(1+y^2\right)^{-3/2} dy = \int 3x^2 dx$$

$y = \tan\theta \qquad \sqrt{1+y^2} = \sec\theta$

$dy = \sec^2\theta\ d\theta$

$$\int \sec^{-3}\theta\ \sec^2\theta\ d\theta = x^3 + C$$

$$\int \sec\theta\ d\theta = x^3 + C$$

$$\ln|\sec\theta + \tan\theta| = x^3 + C$$

$$\boxed{\frac{1}{\sqrt{1+y^2}} + y = C_1 e^{x^3}}$$

Exercise

Find the general solution of the differential equation. $\dfrac{1}{y}dy + ye^{\cos x}\sin x dx = 0$

Solution

$$\int \frac{1}{y^2}dy = -\int e^{\cos x}\sin x dx$$

$$-\frac{1}{y} = e^{\cos x} + C$$

$$\boxed{y(x) = \frac{-1}{e^{\cos x} + C}}$$

Exercise

Find the general solution of the differential equation. $\left(x+xy^2\right)dx+e^{x^2}ydy=0$

Solution

$$x\left(1+y^2\right)dx=-e^{x^2}ydy$$

$$\int xe^{-x^2}dx=-\int\frac{y}{1+y^2}dy$$

$$-\frac{1}{2}\int e^{-x^2}d\left(e^{-x^2}\right)=-\frac{1}{2}\int\frac{1}{1+y^2}d\left(1+y^2\right)$$

$$\boxed{e^{-x^2}=\ln\left(1+y^2\right)+C}$$

Exercise

Find the exact solution of the initial value problem. $y'=\frac{y}{x},\quad y(1)=-2$

Solution

$$\frac{dy}{dx}=\frac{y}{x}$$

$$\frac{dy}{y}=\frac{dx}{x}$$

$$\int\frac{dy}{y}=\int\frac{dx}{x}$$

$$\ln|y|=\ln|x|+C$$

$$\begin{aligned}y&=\pm e^{\ln|x|+C}\\&=\pm e^{C}e^{\ln|x|}\\&=D|x|\\&=Dx\end{aligned}$$

$$y=Dx\quad\Rightarrow\quad D=\frac{y}{x}=\frac{-2}{1}=-2$$

$$\boxed{y(x)=-2x}$$

Exercise

Find the exact solution of the initial value problem. $y'=-\frac{2t\left(1+y^2\right)}{y},\quad y(0)=1$

Solution

$$\frac{dy}{dt} = -\frac{2t\left(1+y^2\right)}{y}$$

$$\int \frac{ydy}{1+y^2} = \int -2tdt$$

$$\frac{1}{2}\int \frac{1}{1+y^2} d\left(1+y^2\right) = -2\int tdt$$

$$\frac{1}{2}\ln\left(1+y^2\right) = -t^2 + C$$

$$\ln\left(1+y^2\right) = -2t^2 + 2C$$

$$1+y^2 = e^{-2t^2+2C}$$

$$1+y^2 = e^{2C}e^{-2t^2}$$

$$1+y^2 = De^{-2t^2}$$

$$1+1^2 = De^{-2(0)^2} \rightarrow \underline{2 = D}$$

$$y^2 = 2e^{-2t^2} - 1$$

$$y^2 = 2e^{-2t^2} - 1$$

$$y = \pm\sqrt{2e^{-2t^2} - 1}$$

$$\boxed{y(t) = \sqrt{2e^{-2t^2} - 1}}$$

$$2e^{-2t^2} - 1 > 0$$

$$2e^{-2t^2} > 1$$

$$e^{-2t^2} > \frac{1}{2}$$

$$-2t^2 > \ln\left(\frac{1}{2}\right)$$

$$t^2 < -\frac{1}{2}\ln\left(\frac{1}{2}\right) = \frac{1}{2}\ln 2$$

$$t^2 < \ln\sqrt{2}$$

$$t < \left|\ln\sqrt{2}\right|$$

The interval of existence: $\left(-\ln\sqrt{2},\ \ln\sqrt{2}\right)$

Exercise

Find the exact solution of the initial value problem. Indicate the interval of existence.

$$y' = \frac{\sin x}{y}, \qquad y\left(\frac{\pi}{2}\right) = 1$$

Solution

$$\frac{dy}{dx} = \frac{\sin x}{y}$$

$$y\,dy = \sin x\,dx$$

$$\int y\,dy = \int \sin x\,dx$$

$$\frac{1}{2}y^2 = -\cos x + C_1$$

$$y^2 = -2\cos x + C \quad (C = 2C_1)$$

$$y(x) = \pm\sqrt{-2\cos x + C}$$

$$y\left(\frac{\pi}{2}\right) = \sqrt{-2\cos\frac{\pi}{2} + C}$$

$$1 = \sqrt{C} \quad \Rightarrow \quad \boxed{C = 1}$$

$$\boxed{y(x) = \sqrt{1 - 2\cos x}}$$

The interval of existence will be the interval containing $\frac{\pi}{2}$ and $1 - 2\cos x > 0$

$$\cos x < \frac{1}{2} \quad \Rightarrow \quad \boxed{\frac{\pi}{3} < x < \frac{5\pi}{3}}$$

Exercise

Find the exact solution of the initial value problem. $4t\,dy = \left(y^2 + ty^2\right)dt, \quad y(1) = 1$

Solution

$$4t\,dy = y^2(1 + t)\,dt$$

$$4\int \frac{dy}{y^2} = \int\left(\frac{1}{t} + 1\right)dt \qquad\qquad \int \frac{dx}{x^2} = -\frac{1}{x}$$

$$-\frac{4}{y} = \ln|t| + t + C$$

$$y = \frac{-4}{\ln|t| + t + C}$$

$$1 = \frac{-4}{\ln|1| + 1 + C} \quad \Rightarrow \quad 1 = \frac{-4}{1 + C} \quad \to \quad 1 + C = -4 \quad \Rightarrow \quad \boxed{C = -5}$$

$$\boxed{y(t) = \frac{-4}{\ln|t| + t - 5}}$$

Exercise

Find the exact solution of the initial value problem. $y' = \frac{1-2t}{y}, \quad y(1) = -2$

Solution

$$y\frac{dy}{dt} = 1-2t$$

$$\int y\,dy = \int (1-2t)\,dt$$

$$\frac{1}{2}y^2 = t - t^2 + C_1$$

$$y^2 = 2t - 2t^2 + C$$

$$(-2)^2 = 2(1) - 2(1)^2 + C \quad \Rightarrow \quad \boxed{C = 4}$$

$$\boxed{y(t) = -\sqrt{2t - 2t^2 + 4}}$$

The negative value is taken to satisfy the initial condition.

Exercise

Find the exact solution of the initial value problem. $y' = y^2 - 4, \quad y(0) = 0$

Solution

$$\frac{dy}{dt} = y^2 - 4$$

$$\frac{dy}{y^2-4} = dt \qquad \frac{1}{y^2-4} = \frac{A}{y-2} + \frac{B}{y+2}$$

$$\frac{1}{y^2-4} = \frac{(A+B)y + 2A - 2B}{y-2}$$

$$\to \begin{cases} A + B = 0 \\ 2A - 2B = 1 \end{cases} \Rightarrow A = \frac{1}{4} \quad B = -\frac{1}{4}$$

$$\left(\frac{1}{4(y-2)} - \frac{1}{4(y+2)}\right)dy = dt$$

$$\int \left(\frac{1}{4(y-2)} - \frac{1}{4(y+2)}\right)dy = \int dt$$

$$\frac{1}{4}\left(\ln|y-2| - \ln|y+2|\right) = t + C$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4t + C$$

$$\frac{y-2}{y+2} = \pm e^{4t+C}$$

$$\frac{y-2}{y+2} = \pm e^C e^{4t} = ke^{4t}$$

$$y-2 = ke^{4t}y + 2ke^{4t}$$

$$y - ke^{4t}y = 2 + 2ke^{4t}$$

$$y\left(1-ke^{4t}\right) = 2 + 2ke^{4t}$$

$$y = \frac{2+2ke^{4t}}{1-ke^{4t}}$$

$$0 = \frac{2+2ke^{4(0)}}{1-ke^{4(0)}}$$

$$0 = 2 + 2k \quad \Rightarrow \quad \boxed{k=-1}$$

$$y(t) = \frac{2-2e^{4t}}{1+e^{4t}}$$

Exercise

Find the exact solution of the initial value problem. Indicate the interval of existence.

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}, \quad y(0) = -1$$

Solution

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2y-2}$$

$$(2y-2)\,dy = \left(3x^2+4x+2\right)dx$$

$$\int (2y-2)\,dy = \int \left(3x^2+4x+2\right)dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$

$$y(0) = -1$$

$$(-1)^2 - 2(-1) = (0)^3 + 2(0)^2 + 2(0) + C$$

$$\Rightarrow \boxed{C=3}$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3$$

Exercise

Find the exact solution of the initial value problem. $y' = \dfrac{x}{1+2y}, \quad y(-1) = 0$

Solution

$$\frac{dy}{dx} = \frac{x}{1+2y}$$

$$\int (1+2y)\,dy = \int x\,dx$$

$$y + y^2 = \frac{1}{2}x^2 + C \qquad y(-1) = 0$$

$$0 = \frac{1}{2}(-1)^2 + C \quad \Rightarrow \quad C = -\frac{1}{2}$$

$$\boxed{y + y^2 = \frac{1}{2}x^2 - \frac{1}{2}}$$

Exercise

Find the exact solution of the initial value problem $\left(e^{2y} - y\right)\cos x \frac{dy}{dx} = e^{y} \sin 2x, \quad y(0) = 0$

Solution

$$\frac{e^{2y} - y}{e^{y}} dy = \frac{2\sin x \cos x}{\cos x} dx$$

$$\int \left(e^{y} - ye^{-y}\right) dy = \int 2\sin x\,dx$$

		$\int e^{-y} dy$
+	y	$-e^{-y}$
−	1	e^{-y}

$$e^{y} + ye^{-y} + e^{-y} = -2\cos x + C$$

$$y(0) = 0 \quad 1 + 1 = -2 + C$$

$$\rightarrow \boxed{C = 4}$$

$$\boxed{e^{y} + ye^{-y} + e^{-y} = 4 - 2\cos x}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = e^{-x^2}, \quad y(3) = 5$

Solution

$$\int_3^x \frac{dy}{dt} dt = \int_3^x e^{-t^2} dt$$

$$y(x) - y(3) = \int_3^x e^{-t^2} dt$$

$$\boxed{y(x) = 5 + \int_3^x e^{-t^2} dt}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dx}+2y=1, \quad y(0)=\frac{5}{2}$

Solution

$$\frac{dy}{dx}=1-2y$$

$$\frac{dy}{1-2y}=dx$$

$$-\frac{1}{2}\int\frac{d(1-2y)}{1-2y}=\int dx$$

$$-\frac{1}{2}\ln|1-2y|=x+C$$

$$\ln|1-2y|=-2x+C \qquad y(0)=\frac{5}{2}$$

$$\ln|1-5|=C \quad \to \quad \boxed{C=\ln 4}$$

$$1-2y=e^{-2x+\ln 4}$$

$$1-2y=e^{-2x}e^{\ln 4}$$

$$\boxed{y(x)=\frac{1}{2}-2e^{-2x}}$$

Exercise

Find the exact solution of the initial value problem. $\sqrt{1-y^2}\,dx-\sqrt{1-x^2}\,dy=0, \quad y(0)=\frac{\sqrt{3}}{2}$

Solution

$$\int\frac{dx}{\sqrt{1-x^2}}=\int\frac{dy}{\sqrt{1-y^2}} \qquad \int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}$$

$$\sin^{-1}x+C=\sin^{-1}y \qquad y(0)=\frac{\sqrt{3}}{2}$$

$$\sin^{-1}0+C=\sin^{-1}\frac{\sqrt{3}}{2} \quad \Rightarrow \quad \boxed{C=\frac{\pi}{3}}$$

$$\sin^{-1}y=\sin^{-1}x+\frac{\pi}{3}$$

$$y=\sin\left(\sin^{-1}x+\frac{\pi}{3}\right)$$

$$=\sin\left(\sin^{-1}x\right)\cos\frac{\pi}{3}+\cos\left(\sin^{-1}x\right)\sin\frac{\pi}{3} \qquad \alpha=\sin^{-1}x\to\sin\alpha=x \quad \cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-x^2}$$

$$\boxed{y(x)=\frac{x}{2}+\frac{\sqrt{3}}{2}\sqrt{1-x^2}}$$

Exercise

Find the exact solution of the initial value problem. $\left(1+x^4\right)dy + x\left(1+4y^2\right)dx = 0, \quad y(1)=0$

Solution

$$\int \frac{1}{1+(2y)^2}dy = -\int \frac{x}{1+\left(x^2\right)^2}dx$$

$$\int \frac{1}{1+(2y)^2}dy = -\frac{1}{2}\int \frac{1}{1+\left(x^2\right)^2}d\left(x^2\right) \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\frac{1}{2}\tan^{-1}2y = -\frac{1}{2}\tan^{-1}x^2 + C$$

$$\tan^{-1}2y + \tan^{-1}x^2 = C_1 \qquad y(1)=0 \quad \tan^{-1}0 + \tan^{-1}1 = C_1 \quad \Rightarrow \boxed{C_1 = \frac{\pi}{4}}$$

$$\boxed{\tan^{-1}2y + \tan^{-1}x^2 = \frac{\pi}{4}}$$

$$2y = \tan\left(\frac{\pi}{4} - \tan^{-1}x^2\right)$$

$$= \frac{\tan\frac{\pi}{4} - \tan\left(\tan^{-1}x^2\right)}{1+\tan\left(\frac{\pi}{4}\right)\tan\left(\tan^{-1}x^2\right)}$$

$$\boxed{y(x) = \frac{1}{2}\frac{1-x^2}{1+x^2}}$$

Exercise

Find the exact solution of the initial value problem. $e^{-2t}\frac{dy}{dt} = \frac{1+e^{-2t}}{y}, \quad y(0)=0$

Solution

$$ydy = \left(1+e^{-2t}\right)e^{2t}dt$$

$$\int ydy = \int\left(e^{2t}+1\right)dt$$

$$\frac{1}{2}y^2 = \frac{1}{2}e^{2t} + t + C_1$$

$$y^2 = e^{2t} + 2t + C \qquad y(0)=0$$

$$0 = 1 + C \quad \to \boxed{C = -1}$$

$$\boxed{y^2 = e^{2t} + 2t - 1}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt} = \frac{t+2}{y}, \quad y(0) = 2$

Solution

$$\int y\,dy = \int (t+2)\,dt$$

$$\frac{1}{2}y^2 = \frac{1}{2}t^2 + 2t + C_1$$

$$y^2 = t^2 + 4t + C \qquad y(0) = 2 \qquad \underline{4 = C}$$

$$\boxed{y(t) = \sqrt{t^2 + 4t + 4}}$$

Exercise

Find the exact solution of the initial value problem. $\frac{1}{t^2}\frac{dy}{dt} = y, \quad y(0) = 1$

Solution

$$\int \frac{1}{y}\,dy = \int t^2\,dt$$

$$\ln|y| = \frac{1}{3}t^3 + C \qquad y(0) = 1$$

$$\ln|1| = C \to \underline{C = 0}$$

$$\ln|y| = \frac{1}{3}t^3$$

$$\boxed{y(t) = e^{t^3/3}}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt} = -y^2 e^{2t}; \quad y(0) = 1$

Solution

$$-\int \frac{1}{y^2}\,dy = \int e^{2t}\,dt$$

$$\frac{1}{y} = \frac{1}{2}e^{2t} + C \qquad y(0) = 1 \qquad 1 = \frac{1}{2} + C \to \underline{C = \frac{1}{2}}$$

$$\frac{1}{y} = \frac{1}{2}\left(e^{2t} + 1\right)$$

$$\boxed{y(t) = \frac{2}{e^{2t} + 1}}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt}-(2t+1)y=0; \quad y(0)=2$

Solution

$$\frac{dy}{dt}=(2t+1)y$$

$$\int\frac{dy}{y}=\int(2t+1)dt$$

$$\ln|y|=t^2+t+C \qquad y(0)=2$$

$$\underline{\ln 2=C}$$

$$\ln|y|=t^2+t+\ln 2$$

$$y(t)=e^{\ln 2}e^{t^2+t}$$

$$\boxed{=2e^{t^2+t}}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt}+4ty^2=0; \quad y(0)=1$

Solution

$$-\int\frac{dy}{y^2}=\int 4tdt$$

$$\frac{1}{y}=2t^2+C \qquad y(0)=1 \qquad \underline{1=C}$$

$$\frac{1}{y}=2t^2+1$$

$$\boxed{y(t)=\frac{1}{2t^2+1}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx}=ye^x; \quad y(0)=2e$

Solution

$$\int\frac{dy}{y}=\int e^x dx$$

$$\ln|y|=e^x+\ln C$$

$$y(0)=2e \ \rightarrow \ \ln|2e|=1+\ln C$$

$$\ln 2+1=1+\ln C \ \Rightarrow \ \underline{C=2}$$

$$\ln|y| = e^x + \ln 2$$

$$y(x) = e^{e^x + \ln 2}$$

$$= e^{e^x} e^{\ln 2}$$

$$\boxed{= 2e^{e^x}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 3x^2\left(y^2+1\right)$; $y(0)=1$

Solution

$$\int \frac{1}{y^2+1}dy = \int 3x^2 dx$$

$$\arctan y = x^3 + C$$

$$y(0)=1 \quad \rightarrow \quad \arctan 1 = C \quad \Rightarrow \boxed{C = \frac{\pi}{4}}$$

$$\boxed{y(x) = \tan\left(x^3 + \frac{\pi}{4}\right)}$$

Exercise

Find the exact solution of the initial value problem $2y\frac{dy}{dx} = \frac{x}{\sqrt{x^2-16}}$; $y(5)=2$

Solution

$$\int 2y dy = \int \frac{x}{\sqrt{x^2-16}}dx$$

$$y^2 = \frac{1}{2}\int \left(x^2-16\right)^{-1/2} d\left(x^2-16\right)$$

$$y^2 = \left(x^2-16\right)^{1/2} + C$$

$$y(5)=2 \quad \rightarrow 4 = (9)^{1/2} + C \quad \Rightarrow C = 4-3 \boxed{=1}$$

$$\boxed{y^2 = 1 + \sqrt{x^2-16}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 4x^3y - y$; $y(1)=-3$

Solution

$$\frac{dy}{dx} = \left(4x^3-1\right)y$$

$$\int \frac{dy}{y} = \int \left(4x^3 - 1\right) dx$$

$$\ln|y| = x^4 - x + C$$

$$y = Ce^{x^4 - x}$$

$$y(1) = -3 \quad \boxed{-3 = C}$$

$$\boxed{y(x) = -3e^{x^4 - x}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} + 1 = 2y; \quad y(1) = 1$

Solution

$$\int \frac{dy}{2y - 1} = \int dx$$

$$\frac{1}{2}\ln(2y - 1) = x + C$$

$$\ln(2y - 1) = 2x + C$$

$$2y - 1 = e^{2x + C}$$

$$y(x) = Ae^{2x} + \frac{1}{2}$$

$$y(1) = 1 \quad 1 = Ae^2 + 1 \quad \rightarrow \boxed{A = e^{-2}}$$

$$\boxed{y(x) = e^{2x - 2} + \frac{1}{2}}$$

Exercise

Find the exact solution of the initial value problem $(\tan x)\frac{dy}{dx} = y; \quad y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

Solution

$$\int \frac{dy}{y} = \int \frac{dx}{\tan x} = \int \frac{\cos x dx}{\sin x}$$

$$\ln y = \ln(\sin x) + \ln C$$

$$y(x) = C \sin x$$

$$y\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \quad \rightarrow \boxed{\frac{\pi}{2} = C}$$

$$\boxed{y(x) = \frac{\pi}{2}\sin x}$$

Exercise

Find the exact solution of the initial value problem $x\dfrac{dy}{dx} - y = 2x^2 y; \quad y(1) = 1$

Solution

$$x\frac{dy}{dx} = 2x^2 y + y$$

$$x\frac{dy}{dx} = \left(2x^2 + 1\right)y$$

$$\int \frac{dy}{y} = \int \left(2x + \frac{1}{x}\right)dx$$

$$\ln y = x^2 + \ln x + \ln C$$

$$y(x) = e^{x^2 + \ln x + \ln C}$$

$$= Cxe^{x^2}$$

$$y(1) = 1 \quad \to \quad 1 = Ce \quad \Rightarrow \quad \underline{C = e^{-1}}$$

$$\boxed{y(x) = xe^{x^2 - 1}}$$

Exercise

Find the exact solution of the initial value problem $\dfrac{dy}{dx} = 2xy^2 + 3x^2 y^2; \quad y(1) = -1$

Solution

$$\frac{dy}{dx} = \left(2x + 3x^2\right)y^2$$

$$\int \frac{dy}{y^2} = \int \left(2x + 3x^2\right)dx$$

$$-\frac{1}{y} = x^2 + x^3 + C$$

$$y(x) = \frac{-1}{x^2 + x^3 + C}$$

$$y(1) = -1 \quad \to \quad -1 = \frac{-1}{C} \quad \Rightarrow \quad \underline{C = 1}$$

$$\boxed{y(x) = \frac{-1}{x^2 + x^3 + 1}}$$

Exercise

Find the exact solution of the initial value problem $\dfrac{dy}{dx} = 6e^{2x - y}; \quad y(0) = 0$

Solution

$$\int e^y dy = \int 6e^{2x} dx$$

$$e^y = 3e^{2x} + C$$

$$y(x) = \ln\left(3e^{2x} + C\right)$$

$$y(0) = 0 \quad \rightarrow \quad 0 = \ln(3 + C)$$

$$\Rightarrow 3 + C = 1 \rightarrow \underline{C = -2}$$

$$\boxed{y(x) = \ln\left(3e^{2x} - 2\right)}$$

Exercise

Find the exact solution of the initial value problem $2\sqrt{x}\frac{dy}{dx} = \cos^2 y; \quad y(4) = \frac{\pi}{4}$

Solution

$$\frac{dy}{\cos^2 y} = \frac{1}{2}x^{-1/2} dx$$

$$\int \sec^2 y\, dy = \int \frac{1}{2}x^{-1/2}\, dx$$

$$\tan y = \sqrt{x} + C$$

$$y(x) = \tan^{-1}\left(\sqrt{x} + C\right)$$

$$y(4) = \frac{\pi}{4} \quad \rightarrow \quad \frac{\pi}{4} = \arctan(2 + C)$$

$$\rightarrow 2 + C = 1 \Rightarrow \underline{C = -1}$$

$$\boxed{y(x) = \tan^{-1}\left(\sqrt{x} - 1\right)}$$

Exercise

Find the exact solution of the initial value problem $y' + 3y = 0; \quad y(0) = -3$

Solution

$$\frac{dy}{dx} = -3y$$

$$\int \frac{dy}{y} = -3\int dx$$

$$\ln|y| = -3x + C$$

$$y(x) = e^{-3x+C}$$

$$= Ae^{-3x}$$

$$y(0) = -3 \quad \rightarrow \quad \underline{-3 = A}$$

$$\boxed{y(x) = -3e^{-3x}}$$

Exercise

Find the exact solution of the initial value problem $\quad 2y' - y = 0; \quad y(-1) = 2$

Solution

$$2\frac{dy}{dx} = y$$

$$\int \frac{dy}{y} = \frac{1}{2}\int dx$$

$$\ln|y| = \frac{1}{2}x + C$$

$$y(x) = e^{x/2+C}$$

$$= Ae^{x/2}$$

$$y(-1) = 2 \quad \rightarrow \quad 2 = Ae^{-1/2} \quad \Rightarrow \quad \underline{A = 2e^{1/2}}$$

$$y(x) = 2e^{1/2}e^{x/2}$$

$$\boxed{= 2e^{(x+1)/2}}$$

Exercise

Find the exact solution of the initial value problem $\quad 2xy - y' = 0; \quad y(1) = 3$

Solution

$$\frac{dy}{dx} = 2xy$$

$$\int \frac{dy}{y} = \int 2x\, dx$$

$$\ln|y| = x^2 + C$$

$$y(x) = e^{x^2+C}$$

$$= Ae^{x^2}$$

$$y(1) = 3 \quad \rightarrow \quad 3 = Ae \quad \Rightarrow \quad \underline{A = 3e^{-1}}$$

$$\boxed{y(x) = \frac{3}{e}e^{x^2}}$$

Exercise

Find the exact solution of the initial value problem $y\frac{dy}{dx} - \sin x = 0; \quad y\left(\frac{\pi}{2}\right) = -2$

Solution

$$\int y\, dy = \int \sin x\, dx$$

$$\frac{1}{2}y^2 = -\cos x + C$$

$$y\left(\frac{\pi}{2}\right) = -2 \quad \to \quad \underline{2 = C}$$

$$\frac{1}{2}y^2 = -\cos x + 2$$

$$y^2 = 4 - 2\cos x$$

$$\boxed{y(x) = -\sqrt{4 - 2\cos x}} \quad \textit{Initial value is negative}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dt} = \frac{1}{y^2}; \quad y(1) = 2$

Solution

$$\int y^2 dy = \int dt$$

$$\frac{1}{3}y^3 = t + C$$

$$y(1) = 2 \quad \to \quad \frac{8}{3} = 1 + C \quad \Rightarrow \quad \boxed{C = \frac{5}{3}}$$

$$\frac{1}{3}y^3 = t + \frac{5}{3}$$

$$y^3 = 3t + 5$$

$$\boxed{y(t) = (3t + 5)^{1/3}}$$

Exercise

Find the exact solution of the initial value problem $y' + \frac{1}{y+1} = 0; \quad y(1) = 0$

Solution

$$\frac{dy}{dx} = -\frac{1}{y+1}$$

$$\int (y+1)dy = -\int dx$$

$$\frac{1}{2}y^2 + y = -x + C$$

$$y(1)=0 \quad \to \quad \underline{C=1}$$

$$\frac{1}{2}y^2+y=-x+1$$

$$y^2+2y+2(x-1)=0 \quad \to \quad y=\frac{-2\pm 2\sqrt{1-2x+2}}{2}$$

$$\boxed{y(x)=-1+\sqrt{3-2x}} \quad (\textit{initial condition + sign})$$

Exercise

Find the exact solution of the initial value problem $y'+e^y t=e^y\sin t;\quad y(0)=0$

Solution

$$\frac{dy}{dt}=(-t+\sin t)e^y$$

$$\int e^{-y}dy=\int(-t+\sin t)dt$$

$$-e^{-y}=-\frac{1}{2}t^2-\cos t+C$$

$$e^{-y}=\frac{1}{2}t^2+\cos t+C$$

$$y(0)=0 \quad \to 1=1+C \Rightarrow \underline{C=0}$$

$$e^{-y}=\frac{1}{2}t^2+\cos t$$

$$-y=\ln\left(\frac{1}{2}t^2+\cos t\right)$$

$$\boxed{y(t)=-\ln\left(\frac{1}{2}t^2+\cos t\right)}$$

Exercise

Find the exact solution of the initial value problem $y'-2ty^2=0;\quad y(0)=-1$

Solution

$$\frac{dy}{dt}=2ty^2$$

$$\int\frac{dy}{y^2}=\int 2t\ dt$$

$$-\frac{1}{y}=t^2+C$$

$$y(0)=-1 \quad \Rightarrow \underline{C=1}$$

$$-\frac{1}{y}=t^2+1$$

$$\boxed{y(t)=\frac{-1}{t+1}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 1 + y^2; \quad y\left(\frac{\pi}{4}\right) = -1$

Solution

$$\int \frac{dy}{1+y^2} = \int dx$$

$$\tan^{-1} y = x + C$$

$$y\left(\frac{\pi}{4}\right) = -1 \quad \rightarrow -\frac{\pi}{4} = \frac{\pi}{4} + C \;\Rightarrow\; \boxed{C = -\frac{\pi}{2}}$$

$$\tan^{-1} y = x - \frac{\pi}{2}$$

$$\boxed{y(x) = \tan\left(x - \frac{\pi}{2}\right)}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dt} = t - ty^2; \quad y(0) = \frac{1}{2}$

Solution

$$\frac{dy}{dt} = t\left(1 - y^2\right)$$

$$\int \frac{dy}{1-y^2} = \int t\ dt$$

$$\frac{1}{1-y^2} = \frac{A}{1-y} + \frac{B}{1+y}$$

$$1 = A + Ay + B - By$$

$$\begin{cases} A - B = 0 \\ A + B = 1 \end{cases} \quad \rightarrow \quad \boxed{A = \frac{1}{2} = B}$$

$$\int \left(\frac{1}{2}\frac{1}{1-y} + \frac{1}{2}\frac{1}{1+y}\right) dy = \int t\ dt$$

$$-\frac{1}{2}\ln|1-y| + \frac{1}{2}\ln|1+y| = \frac{1}{2}t + C$$

$$\ln|1+y| - \ln|1-y| = t + C$$

$$\ln\left|\frac{1+y}{1-y}\right| = t + C$$

$$\frac{1+y}{1-y} = Ae^t$$

$$y(0)=\frac{1}{2} \quad \rightarrow \frac{\frac{3}{2}}{\frac{1}{2}}=A \Rightarrow \underline{A=3}|$$

$$\frac{1+y}{1-y}=3e^t$$

$$1+y=3e^t-3ye^t$$

$$y\left(1+3e^t\right)=3e^t-1$$

$$\boxed{y(t)=\frac{3e^t-1}{1+3e^t}}$$

Exercise

Find the exact solution of the initial value problem $\quad 3y^2\frac{dy}{dt}+2t=1; \quad y(-1)=-1$

Solution

$$\int 3y^2 dy=\int(1-2t)\,dt$$

$$y^3=t-t^2+C$$

$$y(-1)=-1 \quad \rightarrow -1=-1-1+C \Rightarrow \underline{C=1}|$$

$$y^3=t-t^2+1$$

$$\boxed{y(t)=\left(t-t^2+1\right)^{1/3}}$$

Exercise

Find the exact solution of the initial value problem $\quad e^x y'+(\cos y)^2=0; \quad y(0)=\frac{\pi}{4}$

Solution

$$e^x\frac{dy}{dx}=-(\cos y)^2$$

$$\int \sec^2 y\,dy=-\int e^{-x}dx$$

$$\tan y=e^{-x}+C$$

$$y(0)=\frac{\pi}{4} \quad \rightarrow 1=1+C \Rightarrow \underline{C=0}|$$

$$\tan y=e^{-x}$$

$$\boxed{y(x)=\arctan\left(e^{-x}\right)}$$

Exercise

Find the exact solution of the initial value problem $\quad (2y-\sin y)y' + x = \sin x; \quad y(0)=0$

Solution

$$(2y-\sin y)\frac{dy}{dx} = -x+\sin x$$

$$\int (2y-\sin y)\,dy = \int (-x+\sin x)\,dx$$

$$y^2+\cos y = -\frac{1}{2}x^2 - \cos x + C$$

$$y(0)=0 \quad \to 1 = -1 + C \;\Rightarrow\; \underline{C=2}$$

$$\boxed{y^2+\cos y = -\frac{1}{2}x^2 - \cos x + 2}$$

Exercise

Find the exact solution of the initial value problem $\quad e^y y' + \frac{x}{y+1} = \frac{2}{y+1}; \quad y(1)=2$

Solution

$$e^y\frac{dy}{dx} = \frac{2-2x}{y+1}$$

$$\int (y+1)e^y\,dy = \int (2-2x)\,dx$$

$$ye^y = 2x - x^2 + C$$

$$y(1)=2 \quad \to 2e^2 = 2-1+C$$

$$\Rightarrow \underline{C = 2e^2-1}$$

$$\boxed{ye^y = 2x - x^2 + 2e^2 - 1}$$

Exercise

Find the exact solution of the initial value problem $\quad (\ln y)y' + x = 1; \quad y(3)=e$

Solution

$$(\ln y)\frac{dy}{dx} = 1-x$$

$$\int (\ln y)\,dy = \int (1-x)\,dx \qquad \begin{array}{ll} u = \ln y & dv = dy \\ du = \frac{1}{y}dy & v = y \end{array}$$

$$y\ln y - \int dy = x - \frac{1}{2}x^2 + C$$

$$y\ln y - y = x - \frac{1}{2}x^2 + C$$

$$y(3) = e \quad \rightarrow \; e \ln e = 3 - \frac{9}{2} + C$$

$$\Rightarrow \; \underline{C = e + \frac{3}{2}}$$

$$\underline{y \ln y - y = x - \frac{1}{2}x^2 + e + \frac{3}{2}}$$

Exercise

Find the exact solution of the initial value problem $\quad y' = x^3(1-y); \quad y(0) = 3$

Solution

$$\int \frac{dy}{1-y} = \int x^3 dx$$

$$-\ln|1-y| = \frac{1}{4}x^4 + C_1$$

$$\ln|1-y| = -\frac{1}{4}x^4 + C$$

$$y(0) = 3 \quad \rightarrow \; \underline{\ln 2 = C}$$

$$1 - y = e^{-\frac{1}{4}x^4 + C}$$

$$y = 1 - e^{\ln 2} e^{-\frac{1}{4}x^4}$$

$$\underline{y(x) = 1 - 2e^{-x^4/4}}$$

Exercise

Find the exact solution of the initial value problem $\quad y' = \left(1 + y^2\right)\tan x; \quad y(0) = \sqrt{3}$

Solution

$$\int \frac{1}{1+y^2} dy = \int \tan x \; dx$$

$$\tan^{-1} y = \ln|\sec x| + C$$

$$y(0) = \sqrt{3} \quad \rightarrow \; \frac{\pi}{3} = \ln 1 + C$$

$$\Rightarrow \; \underline{C = \frac{\pi}{3}}$$

$$\underline{y(x) = \tan\left(\ln|\sec x| + \frac{\pi}{3}\right)}$$

Exercise

Find the exact solution of the initial value problem $\frac{1}{2}\frac{dy}{dx}=\sqrt{1+y}\ \cos x;\quad y(\pi)=0$

Solution

$$\frac{1}{2}\int(1+y)^{-1/2}\,dy=\int\cos x\,dx$$

$$\sqrt{1+y}=\sin x+C$$

$$y(\pi)=0 \quad \underline{1=C|}$$

$$\sqrt{1+y}=\sin x+1$$

$$\boxed{y(x)=(\sin x+1)^2-1}$$

Exercise

Find the exact solution of the initial value problem $x^2\frac{dy}{dx}=\frac{4x^2-x-2}{(x+1)(y+1)};\quad y(1)=1$

Solution

$$(y+1)dy=\frac{4x^2-x-2}{x^2(x+1)}dx$$

$$\frac{4x^2-x-2}{x^2(x+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$$

$$4x^2-x-2=Ax^2+Ax+Bx+B+Cx^2$$

$$\begin{cases} x^2 & A+C=4 & \underline{C=3|} \\ x & A+B=-1 & \underline{A=1|} \\ x^0 & \underline{B=-2|} & \end{cases}$$

$$\int(y+1)dy=\int\frac{dx}{x}-\int\frac{2}{x^2}dx+3\int\frac{dx}{x+1}$$

$$\frac{1}{2}y^2+y=\ln|x|+\frac{2}{x}+3\ln|x+1|+C$$

$$y(1)=1 \quad \to \frac{1}{2}+1=2+3\ln 2+C$$

$$\Rightarrow \boxed{C=-\frac{1}{2}-\ln 8}$$

$$\boxed{\frac{1}{2}y^2+y=\ln|x|+\frac{2}{x}+3\ln|x+1|-\frac{1}{2}-\ln 8}$$

Exercise

Find the exact solution of the initial value problem $\frac{1}{\theta}\frac{dy}{d\theta}=\frac{y\sin\theta}{y^2+1} \quad y(\pi)=1$

Solution

$$\int \frac{y^2+1}{y}dy=\int \theta\sin\theta\, d\theta$$

$$\int\left(y+\frac{1}{y}\right)dy=-\theta\cos\theta+\sin\theta+C$$

$$\frac{1}{2}y^2+\ln|y|=-\theta\cos\theta+\sin\theta+C$$

$$y(\pi)=1 \quad \frac{1}{2}=\pi+C$$

$$\Rightarrow \boxed{C=\frac{1}{2}-\pi}$$

$$\boxed{\frac{1}{2}y^2+\ln|y|=-\theta\cos\theta+\sin\theta+\frac{1}{2}-\pi}$$

Exercise

Find the exact solution of the initial value problem $x^2dx+2ydy=0; \quad y(0)=2$

Solution

$$\int 2ydy=-\int x^2dx$$

$$y^2=-\frac{1}{3}x^3+C$$

$$y(0)=2 \quad \rightarrow \boxed{4=C}$$

$$\boxed{y^2=-\frac{1}{3}x^3+4}$$

Exercise

Find the exact solution of the initial value problem $\frac{1}{t}\frac{dy}{dt}=2\cos^2 y; \quad y(0)=\frac{\pi}{4}$

Solution

$$\int \sec^2 y\, dy=\int 2t\, dt$$

$$\tan y=t^2+C$$

$$y(0)=\frac{\pi}{4} \quad \rightarrow \boxed{1=C}$$

$$\boxed{y(t)=\tan^{-1}\left(t^2+1\right)}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 8x^3 e^{-2y}; \quad y(1) = 0$

Solution

$$\int e^{2y}dy = \int 8x^3 dx$$

$$\frac{1}{2}e^{2y} = 2x^4 + C$$

$$y(1) = 0 \quad \rightarrow \boxed{\frac{1}{2} = C}$$

$$\frac{1}{2}e^{2y} = 2x^4 + \frac{1}{2}$$

$$e^{2y} = 4x^4 + 1$$

$$\boxed{y(x) = \frac{1}{2}\ln\left(4x^4 + 1\right)}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = x^2(1+y); \quad y(0) = 3$

Solution

$$\int \frac{1}{1+y}dy = \int x^2 dx$$

$$\ln|1+y| = \frac{1}{3}x^3 + C$$

$$y(0) = 3 \quad \rightarrow \boxed{\ln 4 = C}$$

$$1 + y = e^{\frac{1}{3}x^3 + \ln 4}$$

$$\boxed{y(x) = 4e^{x^3/3} - 1}$$

Exercise

Find the exact solution of the initial value problem $\sqrt{y}\,dx + (1+x)dy = 0; \quad y(0) = 1$

Solution

$$\int y^{-1/2}dy = -\int \frac{1}{x+1}dx$$

$$2\sqrt{y} = -\ln|x+1| + C$$

$$y(0) = 1 \quad \rightarrow \boxed{2 = C}$$

$$\boxed{2\sqrt{y} = -\ln|x+1| + 2}$$

Exercise

Find the exact solution of the initial value problem $\dfrac{dy}{dx} = 6y^2x, \quad y(1) = \dfrac{1}{25}$

Solution

$$\int \frac{dy}{y^2} = \int 6x\,dx$$

$$-\frac{1}{y} = 3x^2 + C \qquad y(1) = \frac{1}{25}$$

$$-25 = 3 + C \quad \to \quad \underline{C = -28}$$

$$-\frac{1}{y} = 3x^2 - 28$$

$$\boxed{y(x) = \frac{1}{28 - 3x^2}}$$

Exercise

Find the exact solution of the initial value problem $\dfrac{dy}{dx} = \dfrac{3x^2 + 4x - 4}{2y - 4}, \quad y(1) = 3$

Solution

$$\int (2y - 4)\,dy = \int \left(3x^2 + 4x - 4\right)dx$$

$$y^2 - 4y = x^3 + 2x^2 - 4x + C \qquad y(1) = 3$$

$$9 - 12 = 1 + 2 - 4 + C \quad \to \quad \underline{C = -2}$$

$$\boxed{y^2 - 4y = x^3 + 2x^2 - 4x - 2}$$

Exercise

Find the exact solution of the initial value problem $y' = e^{-y}(2x - 4) \quad y(5) = 0$

Solution

$$\int e^y\,dy = \int (2x - 4)\,dx$$

$$e^y = x^2 - 4x + C \qquad y(5) = 0$$

$$e^0 = 25 - 20 + C \quad \to \quad \underline{C = -4}$$

$$e^y = x^2 - 4x - 4$$

$$\boxed{y(x) = \ln\left|x^2 - 4x - 4\right|}$$

Exercise

Find the exact solution of the initial value problem $\frac{dr}{d\theta}=\frac{r^2}{\theta}, \quad r(1)=2$

Solution

$$\int \frac{dr}{r^2}=\int \frac{d\theta}{\theta}$$

$$-\frac{1}{r}=\ln|\theta|+C \qquad r(1)=2$$

$$\boxed{-\frac{1}{2}=C}$$

$$-\frac{1}{r}=\ln|\theta|-\frac{1}{2}$$

$$\frac{1}{r}=\frac{1-2\ln|\theta|}{2}$$

$$\boxed{r(\theta)=\frac{2}{1-2\ln|\theta|}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dt}=e^{y-t}\left(1+t^2\right)\sec y, \quad y(0)=0$

Solution

$$\frac{dy}{dt}=e^{-t}\left(1+t^2\right)e^{y}\sec y$$

$$\int\left(e^{-y}\cos y\right)dy=\int\left(1+t^2\right)e^{-t}dt$$

$$\int\left(e^{-y}\cos y\right)dy=e^{-y}(\sin y-\cos y)-\int\left(e^{-y}\cos y\right)dy$$

$$2\int\left(e^{-y}\cos y\right)dy=e^{-y}(\sin y-\cos y)$$

$$\int\left(e^{-y}\cos y\right)dy=\frac{1}{2}e^{-y}(\sin y-\cos y)$$

$$\int\left(1+t^2\right)e^{-t}dt=e^{-t}\left(-1-t^2-2t-2\right)$$

		$\int \cos y$		$\int e^{-t}$
+	e^{-y}	$\sin y$	$1+t^2$	$-e^{-t}$
−	$-e^{-y}$	$-\cos y$	$2t$	e^{-t}
+	e^{-y}		2	$-e^{-t}$

$$\frac{1}{2}e^{-y}(\sin y-\cos y)=-e^{-t}\left(t^2+2t+3\right)+C \qquad y(0)=0$$

$$-\frac{1}{2}=-3+C \quad\rightarrow\quad \boxed{C=\frac{5}{2}}$$

$$\boxed{\frac{1}{2}e^{-y}(\sin y-\cos y)=-e^{-t}\left(t^2+2t+3\right)+\frac{5}{2}}$$

Exercise

A thermometer reading 100°*F* is placed in a medium having a constant temperature of 70°*F*. After 6 *min*, the thermometer reads 80°*F*. What is the reading after 20 *min*?

Solution

Given: $T_0 = 100°F \quad \& \ A = 70°F$ $\qquad \frac{dT}{dt} = -k(T - A) \ \rightarrow \ T = A + (T_0 - A)e^{-kt}$

$$T(t) = 70 + (100 - 70)e^{-kt}$$

$$= \boxed{70 + 30e^{-kt}}$$

$$T(6) = 70 + 30e^{-6k} = 80$$

$$e^{-6k} = \frac{1}{3} \quad \rightarrow \quad k = -\frac{1}{6}\ln\frac{1}{3} = \boxed{\frac{\ln 3}{6}}$$

$$T(t) = 70 + 30e^{-\frac{\ln 3}{6}t}$$

$$T(20) = 70 + 30e^{-\frac{\ln 3}{6}20}$$

$$\boxed{\approx 70.77\ °F}$$

Exercise

Blood plasma is stored at 40°*F*. Before the plasma can be used, it must be at 90°*F*. When the plasma is placed in an oven at 120°*F*, it takes 45 *min* for the plasma to warm to 90°*F*. How long will it take for the plasma to warm to 90°*F* if the oven temperature is set at:

a) 100°*F*.
b) 140°*F*.
c) 80°*F*.

Solution

Given: $T_0 = 40°F \quad \& \ A = 120°F$ $\qquad \frac{dT}{dt} = -k(T - A) \ \rightarrow \ T = A + (T_0 - A)e^{-kt}$

$$T(t) = 120 - 80e^{-kt}$$

$$T(45) = 120 - 80e^{-45k} = 90$$

$$e^{-45k} = \frac{3}{8} \quad \rightarrow \quad \boxed{k = -\frac{1}{45}\ln\frac{3}{8} \approx .021796}$$

$$\boxed{T(t) = 120 - 80e^{-.021796t}}$$

a) $T_0 = 40°F \quad \& \ A = 100°F$

$$T(t) = 100 - 60e^{-.021796t} = 90$$

$$e^{-.021796t} = \frac{1}{6}$$

$t = -\frac{1}{.021796}\ln\frac{1}{6}$

$\approx 82\ min$

b) $T_0 = 40°F \quad \& \ A = 140°F$

$T(t) = 140 - 100e^{-.021796t} = 90$

$e^{-.021796t} = \frac{1}{2}$

$t = -\frac{1}{.021796}\ln\frac{1}{2}$

$\approx 31.8\ min$

c) $T_0 = 40°F \quad \& \ A = 80°F$

$T(t) = 80 - 40e^{-.021796t} = 90$

$e^{-.021796t} = \frac{1}{4}$

$t = -\frac{1}{.021796}\ln\frac{1}{4}$

$\approx 63.6\ min$

Exercise

A pot of boiling water at 100°*C* is removed from a stove at time $t = 0$ and left to cool in the kitchen. After 5 *min*, the water temperature has decreased to 80°*C*, and another 5 *min* later it has dropped to 65°*C*. Assuming Newton's law for cooling, determine the (constant) temperature of the kitchen.

Solution

Given: $T(0) = 100°C, \quad T(5) = 80°C, \quad T(10) = 65°C$

Let: $B = T_0 - A$

$\frac{dT}{dt} = -k(T - A) \quad \rightarrow \quad T = A + \left(T_0 - A\right)e^{-kt}$

$T(t) = A + Be^{-kt}$

$T(0) = 100°C \quad \rightarrow \ A + B = 100 \quad \Rightarrow \ A = 100 - B$

$T(5) = 80°C \quad \rightarrow A + Be^{-5k} = 80$

$T(10) = 65°C \quad \rightarrow A + Be^{-10k} = 65$

$$\rightarrow \begin{cases} 100 - B + Be^{-5k} = 80 \\ 100 - B + Be^{-10k} = 65 \end{cases} \qquad \begin{cases} B\left(1 - e^{-5k}\right) = 20 & (1) \\ B\left(1 - e^{-10k}\right) = 35 & (2) \end{cases}$$

$(1) \rightarrow \quad B = \frac{20}{1 - e^{-5k}}$

$(2) \rightarrow \quad \frac{20}{1 - e^{-5k}}\left(1 - e^{-10k}\right) = 35$

$$4-4e^{-10k}=7-7e^{-5k}$$

$$4\left(e^{-5k}\right)^2-7e^{-5k}+3=0 \quad\Rightarrow\quad \begin{cases} e^{-5k}=1 \\ e^{-5k}=\frac{3}{4} \end{cases}$$

$$e^{-5k}=1 \rightarrow \cancel{k=0} \qquad\qquad \underline{e^{-5k}=\frac{3}{4}}$$

$$B=\frac{20}{1-e^{-5k}}=\frac{20}{1-\frac{3}{4}}\ \underline{=80}$$

$$A=100-B=100-80$$

$$\underline{=20\ °C}$$

Exercise

A murder victim is discovered at midnight and the temperature of the body is recorded at 31°*C*. One hour later, the temperature of the body is 29°*C*. Assume that the surrounding air temperature remains constant at 21°*C*. Use Newton's law of cooling to calculate the victim's time of death. *Note*: The normal temperature of a living human being is approximately 37°*C*

Solution

Given: The initial temperature: $T(0)=31\ °C$

At $t=1\ hr \quad\Rightarrow\quad T(1)=29\ °C$

The surrounding temperature: $A=21\ °C$

The temperature is given by the formula: $T=A+\left(T_0-A\right)e^{-kt}$

$$T=21+(31-21)e^{-kt}=21+10e^{-kt}$$

$$29=21+10e^{-k(1)}$$

$$8=10e^{-k}$$

$$e^{-k}=\frac{8}{10}$$

$$-k=\ln(0.8)$$

$$\boxed{k\approx 0.2231}$$

$$T=21+10e^{-0.2231t}$$

$$37=21+10e^{-0.2231t}$$

$$10e^{-0.2231t}=16$$

$$e^{-0.2231t}=1.6$$

$$-0.2231t=\ln 1.6$$

$$t=\frac{\ln 1.6}{-0.2231}$$

$\approx \boxed{-2.1\ hrs}$ $\quad .1 * 60 = 6\ \text{min}$

The murder occurred 2 *hours* and 6 *minutes* earlier.

Exercise

Suppose a cold beer at 40°*F* is placed into a warn room at 70°*F*. suppose 10 minutes later, the temperature of the beer is 48°*F*. Use Newton's law of cooling to find the temperature 25 *minutes* after the beer was placed into the room.

Solution

Given: The initial temperature: $T(0) = 40\ °F$.

At $t = 10\ \text{min} \Rightarrow T(10) = 48\ °F$

The surrounding temperature: $A = 70\ °F$

Let $T(t)$ be the temperature of the beer at time t minutes after being placed into the room.

From Newton's law of cooling: $T'(t) = k(A - T)$

$$\frac{dT}{dt} = k(70 - T)$$

$$\frac{dT}{70 - T} = kdt$$

$$\int \frac{dT}{70 - T} = \int kdt \qquad d(70 - T) = -dT$$

$$-\ln(70 - T) = kt + C \qquad 70 > T(t)$$

$$\ln(70 - T) = -kt - C$$

$$70 - T = e^{-kt - C} = e^{-C}e^{-kt} = ce^{-kt}$$

$$T(t) = 70 - ce^{-kt}$$

From the initial condition:

$$T(0) = 70 - ce^{-k(0)}$$

$$40 = 70 - c \Rightarrow c = 30$$

$$\Rightarrow T(t) = 70 - 30e^{-kt}$$

$$T(t = 10) = 70 - 30e^{-k(10)}$$

$$48 = 70 - 30e^{-10k}$$

$$30e^{-10k} = 70 - 48 = 22$$

$$e^{-10k} = \frac{22}{30}$$

$$-10k = \ln\left(\frac{22}{30}\right) \Rightarrow k = -\frac{\ln(11/15)}{10} \approx 0.031$$

$$T(t) = 70 - 30e^{-.031t}$$

$$T(t=25) = 70 - 30e^{-.031(25)}$$

$$\approx 56.18^\circ\ F$$

Exercise

A thermometer is removed from a room where the temperature is $70^\circ\ F$ and is taken outside, where the air temperature is $10^\circ\ F$. After one-half minute the thermometer reads $50^\circ\ F$.

a) What is the reading of the thermometer at $t = 1\ min$?

b) How long will it take for the thermometer to reach $15^\circ\ F$?

Solution

$$\frac{dT}{dt} = -k(T - A) \quad \rightarrow \quad T = A + \left(T_0 - A\right)e^{-kt}$$

a) ***Given***: $T_0 = 70^\circ F \quad \& \ A = 10^\circ F$

$$T(t) = 10 + (70 - 10)e^{-kt}$$

$$= 10 + 60e^{-kt}$$

$$T\left(t = \frac{1}{2}\right) = 10 + 60e^{-k/2} = 50$$

$$60e^{-k/2} = 40$$

$$e^{-k/2} = \frac{2}{3}$$

$$k = -2\ln\frac{2}{3} \approx 0.811$$

$$T(t) = 10 + 60e^{-.811t}$$

$$T(1) = 10 + 60e^{-.811}$$

$$\approx 36.67^\circ\ F$$

b) $T(t) = 10 + 60e^{-.811t} = 15^\circ\ F$

$$60e^{-.811t} = 5$$

$$e^{-.811t} = \frac{1}{12}$$

$$t = -\frac{1}{.811}\ln\frac{1}{12}$$

$$\approx 3.06\ min$$

Exercise

A thermometer is taken from an inside room to the outside, where the air temperature is $5°\ F$. After 1 *minute* the thermometer reads $55°\ F$, and after 5 *minutes* the thermometer reads $30°\ F$. What is the initial temperature of the inside room?

Solution

Given: $A = 5°F$ $\qquad \dfrac{dT}{dt} = -k(T - A) \quad \rightarrow \quad T = A + \left(T_0 - A\right)e^{-kt}$

$$T(t) = 5 + \left(T_0 - 5\right)e^{-kt}$$

$$T(t=1) = 5 + \left(T_0 - 5\right)e^{-k} = 55 \quad \rightarrow \quad \left(T_0 - 5\right)e^{-k} = 50$$

$$T(t=5) = 5 + \left(T_0 - 5\right)e^{-5k} = 30 \quad \rightarrow \quad \left(T_0 - 5\right)e^{-5k} = 25$$

$$\left(T_0 - 5\right) = \frac{50}{e^{-k}} = \frac{25}{e^{-5k}}$$

$$2e^{-5k} = e^{-k}$$

$$e^{-k}\left(2e^{-4k} - 1\right) = 0 \Rightarrow e^{-4k} = \frac{1}{2}$$

$$k = -\frac{1}{4}\ln\frac{1}{2} \approx 0.1733$$

$$T_0 = \frac{50}{e^{-0.1733}} + 5$$

$$\approx 64.461°\ F$$

Exercise

The temperature inside a house is $70°\ F$. A thermometer is taken outside after being inside the house for enough time for it to read $70°\ F$. The outside air temperature is $10°\ F$. After three *minutes* the thermometer reading is found to be $25°\ F$. Find the reading on the thermometer as a function of time.

Solution

Given: $T_0 = 70°F, \quad A = 10°F$

$$T(t=3) = 25°$$

$$T(t) = 10 + (70 - 10)e^{-kt} \qquad \frac{dT}{dt} = -k(T - A) \quad \rightarrow \quad T = A + \left(T_0 - A\right)e^{-kt}$$

$$T(t) = 10 + 60e^{-kt}$$

$$T(3) = 25 \quad \rightarrow \quad 25 = 10 + 60e^{-3k}$$

$$60e^{-3k} = 15$$

$$e^{-3k} = \frac{1}{4} \quad \rightarrow \quad k = -\frac{1}{3}\ln\frac{1}{4} \approx 0.462$$

$$T(t) = 10 + 60e^{-0.462t}$$

Exercise

A metal bar at a temperature of $100° \ F$ is placed in a room at a constant temperature of $0° \ F$. If after 20 *minutes* the temperature of the bar is $50° \ F$.

a) Find the time it will take the bar to reach a temperature of $25° \ F$

b) Find the temperature of the bar after 10 *minutes*.

Solution

Given: $T_0 = 100°F, \quad A = 0°F$

$T(t = 20) = 50°$

$$\frac{dT}{dt} = -k(T - A) \quad \to \quad T = A + \left(T_0 - A\right)e^{-kt}$$

a) $T(t) = 100e^{-kt}$

$T(20) = 50 \quad \to \ 50 = 100e^{-20k}$

$-20k = \ln\frac{1}{2} \quad \to \quad \underline{k \approx 0.035}$

$\boxed{T(t) = 100e^{-0.035t}}$

b) $T(t = 10) = 100e^{-0.035(10)}$

$\approx 70.5°$

Exercise

A small metal bar, whose initial temperature was $20° \ C$, is dropped into a large container of boiling water.

a) How long will it take the bar to reach $90° \ C$ if it is known that its temperature increases $2°$ in 1 *second*?

b) How long will it take the bar to reach $98° \ C$

Solution

a) ***Given***: $T_0 = 20° \ C \quad \& \ A = 100° \ C$

$T(t) = 100 + (20 - 100)e^{-kt} \ \underline{= 100 - 80e^{-kt}}$

$$T = A + \left(T_0 - A\right)e^{-kt}$$

$T(1) = 100 - 80e^{-k} = 22$

$e^{-k} = \frac{78}{80}$

$k = -\ln\frac{39}{40} \ \underline{\approx 0.0253}$

$T(t) = \underline{100 - 80e^{-0.0253t}}$

$100 - 80e^{-0.0253t} = 90$

$e^{-0.0253t} = \frac{1}{8}$

$\underline{t} = -\frac{1}{0.0253}\ln\frac{1}{8}$

$\approx 82.1\ sec$

b) $T = 100 - 80e^{-0.0253t} = 98 \quad \Rightarrow \quad e^{-0.0253t} = \frac{1}{40}$

$t = -\frac{1}{0.0253}\ln\frac{1}{40}$

$\approx 145.8\ sec$

Exercise

Two large containers ***A*** and ***B*** of the same size are filled with different fluids. The fluids in containers ***A*** and ***B*** are maintained at $0°\ C$ and $100°\ C$, respectively. A small metal bar, whose initial temperature is $100°\ C$, is lowered into container ***A***. After 1 *minute* the temperature of the bar is $90°\ C$. After 2 *minutes* the bar is removed and instantly transferred to the other container. After 1 *minute* in container ***B*** the temperature of the bar rises $10°\ C$. How long, measured from the start of the entire process, will it take the bar to reach $99.9°\ C$?

Solution

Given: *Tank* ***A***: $A = 0°\ C,\ \ T_1(0) = 100°\ C\ \ \&\ \ T_1(1) = 90°\ C$

Tank ***B***: $A = 100°\ C\ \ \&\ \ T_2(1) = 110°\ C$

$T_1(t) = (100 - 0)e^{-kt} = 100e^{-kt}$ $\qquad T = A + (T_0 - A)e^{-kt}$

$T_1(1) = 100e^{-k} = 90 \quad \to \ k = -\ln\left(\frac{9}{10}\right) \approx 0.10536$

$T_1(t) = 100e^{-0.10536t}$

$T_1(2) = 100e^{-0.10536(2)} \approx 81°\ C$

Tank B: $\qquad T_1(2) \approx 81°\ C = T_2(0)$

$T_2(t) = 100 + (81 - 100)e^{-kt} = 100 - 19e^{-kt}$ $\qquad T = A + (T_0 - A)e^{-kt}$

$T_2(1) = 100 - 19e^{-k} = 81 + 10 = 91 \quad \to \ k = -\ln\left(\frac{9}{19}\right) \approx 0.7472$

$T_2(t) = 100 - 19e^{-0.7472t}$

$T_2(t) = 100 - 19e^{-0.7472t} = 99.9$

$-0.7472t = \ln\frac{0.1}{19}$

$t = -\frac{1}{0.7472}\ln\frac{0.1}{19}$

$\approx 7.02\ \ min$

$\therefore$ The entire process will take the bar to reach $99.9°\ C$ is approximately 7.02 *minutes*.

Exercise

A thermometer reading $70° \ F$ is placed in an oven preheated to a constant temperature. Through a glass window in the oven door, an observer records that the thermometer reads $110° \ F$ after $\frac{1}{2}$ *minute* and $145° \ F$ after 1 *minute*. How hot is the oven?

Solution

Given: $T_0 = 70° \ F$

$$T(t) = A + (70 - A)e^{-kt} \qquad\qquad T = A + \left(T_0 - A\right)e^{-kt}$$

$$T\left(t = \tfrac{1}{2}\right) = A + (70 - A)e^{-k/2} = 110 \quad \rightarrow \quad e^{-k/2} = \frac{110 - A}{70 - A}$$

$$T(t = 1) = A + (70 - A)e^{-k} = 145 \quad \rightarrow \quad e^{-k} = \frac{145 - A}{70 - A}$$

$$e^{-k} = \left(e^{-k/2}\right)^2 = \left(\frac{110 - A}{70 - A}\right)^2 = \frac{145 - A}{70 - A}$$

$$\frac{(110 - A)^2}{70 - A} = 145 - A$$

$$12{,}100 - 220A + A^2 = 10{,}150 - 215A + A^2$$

$$5A = 1950 \quad \rightarrow \quad \underline{A = 390}$$

∴ The temperature in the oven is $390° \ F$.

Exercise

At $t = 0$ a sealed test tube containing a chemical is immersed in a liquid bath. The initial temperature of the chemical in the test tube is $80° \ F$. the liquid bath has a controlled temperature given by

$T_m(t) = 100 - 40e^{-0.1t}$, $t \ge 0$, where t is measured in *minutes*.

a) Assume that $k = -0.1$, describe in words what you expect the temperature $T(t)$ of the chemical to be like in the short term. In the long term.

b) Solve the initial-value problem.

c) Graph $T(t)$.

Solution

a) **Given**: $T_0 = 80° \ F$

$T_m(0) = 100 - 40 = 60° \ F$

The temperature decreases (or cool off), in the short time.

Over time, the temperature will increase towards 100° since $e^{-0.1t}$ decrease from 1 to 0 as t approaches infinity. Thus, in the long term, the temperature of the chemical should increase or warm toward 100°.

b) $\dfrac{dT}{dt} = -0.1\left(T - 100 + 40e^{-0.1t}\right)$

$\dfrac{dT}{dt} + 0.1T = 10 - 4e^{-0.1t}$

$$e^{\int 0.1dt} = \underline{e^{0.1t}}$$

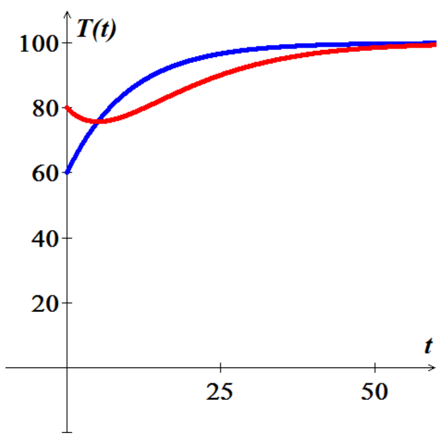
$$\int\left(10 - 4e^{-0.1t}\right)e^{0.1t}dt = \int\left(10e^{0.1t} - 4\right)dt = \underline{100e^{0.1t} - 4t}$$

$T(t) = e^{-0.1t}\left(100e^{0.1t} - 4t + C\right) \qquad T_0 = 80$

$$100 + C = 80 \ \rightarrow\ \underline{C = -20}$$

$$\underline{T(t) = 100 - 4(t+5)e^{-0.1t}}$$

c)



Exercise

The mathematical model for the shape of a flexible cable strung between two vertical supports is given by

$$\frac{dy}{dx} = \frac{W}{T_1}$$

Where *W* denotes the portion of the total vertical load between the points P_1 and P_2

The model is separable under the following conditions that describe a suspension bridge.

Let assume that the *x*-axis runs along the horizontal roadbed, and the *y*-axis passes through $(0,\ a)$, which is the lowest point on one cable over the span of the bridge, coinciding with the interval $\left[-\frac{L}{2},\ \frac{L}{2}\right]$.

In the case of a suspension bridge, the usual assumption is that the vertical load in the given equation is only a uniform roadbed, distributed along the horizontal axis. In other words, it is assumed that the weight of all cables is negligible in comparison to the weight of the roadbed and that the weight per unit length of the roadbed (lb/ft) is a constant ρ. Use this information to set up and solve an appropriate initial-value problem from which the shape (a curve with equation $y = \varphi(x)$) of each of the two cables in a suspension bridge is determined. Express the solution of the IVP in terms of the sag h and span L.

Solution

Since the tension T_1 (or magnitude T_1) acts at the lowest point of the cable, using the symmetry to solve the problem on the interval $\left[0, \frac{L}{2}\right]$.

The assumption that the roadbed is uniform (that is, weighs a constant ρ (lb/ft) implies

$W = \rho x$, where $0 \le x \le \frac{L}{2}$

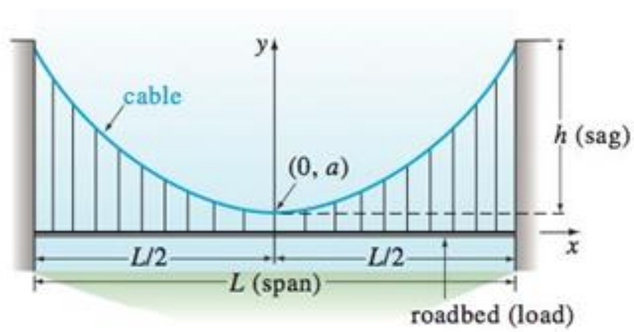
$$\frac{dy}{dx} = \frac{\rho}{T_1}x$$

$$\int dy = \frac{\rho}{T_1}\int x dx$$

$$y(x) = \frac{1}{2}\frac{\rho}{T_1}x^2 + C \qquad y(0) = a$$

$$y(0) = \underline{C = a}$$

$$\boxed{y(x) = \frac{\rho}{2T_1}x^2 + a}$$



$$y\left(\frac{L}{2}\right) = h + a$$

$$\frac{\rho}{2T_1}\frac{L^2}{4} = h \Rightarrow \boxed{\frac{\rho}{2T_1} = \frac{4h}{L^2}}$$

$$\boxed{y(x) = \frac{4h}{L^2}x^2 + a} \quad -\frac{L}{2} \le x \le \frac{L}{2}$$

Exercise

The Brentano-Stevens Law in psychology models the way that a subject reacts to a stimulus. It states that if R represents the reaction to an amount S of stimulus, then the relative rates of increase are proportional:

$$\frac{1}{R}\frac{dR}{dt} = \frac{k}{S}\frac{dS}{dt}$$

Where k is a positive constant. Find R as a function of S.

Solution

$$\frac{1}{R}\frac{dR}{dt} = \frac{k}{S}\frac{dS}{dt}$$

$$\frac{d}{dt}(\ln R) = \frac{d}{dt}(k\ln S)$$

$$\ln R = k\ln S + C$$

$$R(S) = e^{\ln S^k + C}$$

$$= e^{C} e^{\ln S^k} \qquad A = e^{C}$$

$$\underline{= AS^k}$$

Exercise

Barbara weighs 60 *kg* and is on a diet of 1600 *calories* per day, of which 850 are used automatically by basal metabolism. She spends about 15 *cal/kg/day* times her weight doing exercises. If 1 *kg* of fat contains 10,000 *cal*. and we assume that the storage of calories in the form of fat is 100% efficient, formulate a differential equation and solve it to find her weight as a function of time. Does her weight ultimately approach an equilibrium weight?

Solution

$m(t) = m$: mass at time *t*.

The net intake of calories per day at time *t* is

$$c(t) = 1600 - 850 - 15m \ = 750 - 15m$$

$$m = \frac{1}{15}\left(750 - c(t)\right)$$

$$\frac{dm}{dt} = \frac{c(t)}{10,000}$$

$$= \frac{750 - 15m}{10,000}$$

$$= -3\frac{m-50}{2000}$$

$$\int \frac{dm}{m-50} = -\frac{3}{2000}\int dt$$

$$\ln|m-50| = -\frac{3}{2000}t + C$$

$$m - 50 = Ae^{-\frac{3}{2000}t}$$

$$m(0) = 60 \quad \rightarrow \quad \underline{10 = A}$$

$$\underline{m(t) = 10e^{-\frac{3}{2000}t} + 50}$$

$$\lim_{t\to\infty} m(t) = \lim_{t\to\infty}\left(10e^{-\frac{3}{2000}t} + 50\right)$$

$$\underline{= 50}$$

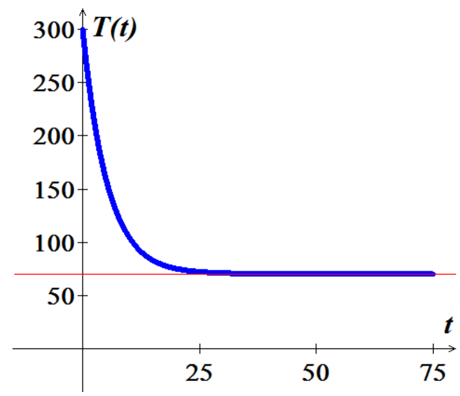
Thus, Barbara's mass gradually settles down to 50 *kg*.

Exercise

When a chicken is removed from an oven, its temperature is measured at 300° *F*. Three minutes later its temperature is 200° *F*. How long will it take for the chicken to cool off to a room temperature of 70° *F*.

Solution

Given: $T_0 = 300°F, \quad A = 70°F, \quad T(3) = 200°F$



$$T(t) = 70 + (300-70)e^{-kt} \qquad T = A + (T_0 - A)e^{-kt}$$

$$T(3) = 70 + 230e^{-3k} = 200$$

$$e^{-3k} = \frac{130}{230}$$

$$k = -\frac{1}{3}\ln\left(\frac{13}{23}\right) \approx 0.19018$$

$$\boxed{T(t) = 70 + 230e^{-0.19018t}}$$

$$70 + 230e^{-0.19018t} = 70$$

$$e^{-0.19018t} = 0 \quad \Rightarrow \quad t \to \infty$$

Exercise

Suppose that a corpse was discovered in a motel room at midnight and its temperature was $80°\ F$. The temperature of the room is kept constant at $60°\ F$. Two hours later the temperature of the corpse dropped to $75°\ F$. Find the time of death.

Solution

First we use the observed temperatures of the corpse to find the constant *k*. We have

$$k = -\frac{1}{2}\ln\left(\frac{75-60}{80-60}\right) = 0.1438$$

In order to find the time of death we need to remember that the temperature of a corpse at time of death is $98.6°\ F$ (assuming the dead person was not sick!). Then we have

$$\begin{aligned} t_d &= -\frac{1}{k}\ln\left(\frac{98.6-60}{80-60}\right) \\ &= -\frac{1}{0.1438}\ln\left(\frac{38.6}{20}\right) \\ &\approx -4.57\ \ hrs \end{aligned}$$

$$4.57\ hrs = \ 4 \quad .57 \times 60 = 34'$$

$$12 - (4\ hrs \quad 34\ \min)$$

Which means that the death happened around 7:26 P.M.

Exercise

Suppose that a corpse was discovered at 10 PM and its temperature was $85^\circ\ F$. Two hours later, its temperature is $74^\circ\ F$. If the ambient temperature is $68^\circ\ F$. Estimate the time of death.

Solution

Given: $A = 68, \quad T_0 = 85, \quad T_2 = 74$

$k = -\frac{1}{2}\ln\left(\frac{74-68}{85-68}\right) = 0.521$

$T(t) = 68 + (85-68)e^{-.521t}$ $\qquad\qquad T(t) = A + \left(T_0 - A\right)e^{-kt}$

$= 68 + 17e^{-.521t}$

$t_d = -\frac{1}{.521}\ln\left(\frac{98.6-68}{85-68}\right)$

≈ -1.13

$1.13\ hrs = \ 1 \quad .13\times 60 \approx 8'$

$10 - (1\ hr \quad 8\ \text{min})$

The death happened around 8:52 P.M

Solution Section 1.3 – Models of Motions

Exercise

A body of mass *m* falls from rest subject to gravity in a medium offering resistance proportional to the square of the velocity. Determine the velocity and position of the body at *t* seconds.

Solution

The resistance force *R*, according the given, is proportional to the square of the velocity. Since this force is an upward direction. $R = -r.v^2$

Let $r = \frac{g}{a^2} \quad \to \quad R = -\frac{gv^2}{a^2}$

$$F = mg + F_r$$

$$m\frac{dv}{dt} = mg - mR$$

$$m\frac{dv}{dt} = mg - m\frac{gv^2}{a^2}$$

$$\frac{dv}{dt} = g\left(1 - \frac{v^2}{a^2}\right)$$

$$\int \frac{dv}{1-\left(\frac{v}{a}\right)^2} = g\int dt$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| \quad or \quad \frac{1}{a}\tanh^{-1}\frac{x}{a}$$

$$a\tanh^{-1}\frac{v}{a} = gt + C_1$$

$$\tanh^{-1}\frac{v}{a} = \frac{g}{a}t + C_2$$

$$\frac{v}{a} = \tanh\left(\frac{g}{a}t + C_2\right)$$

$$v(t) = a\tanh\left(\frac{g}{a}t + C_2\right)$$

$$v(0) = 0 \quad \to 0 = a\tanh\left(C_2\right) \quad \Rightarrow \boxed{C_2 = 0}$$

$$\boxed{v(t) = a\tanh\left(\frac{g}{a}t\right)}$$

$$v(t) = \frac{dy}{dt} = a\tanh\left(\frac{g}{a}t\right)$$

$$\int dy = a\int \tanh\left(\frac{g}{a}t\right)dt$$

$$\int \tanh axdx = \frac{1}{a}\ln\left(\cosh ax\right) + C$$

$$y(t) = \frac{a^2}{g}\ln\left(\cosh\frac{g}{a}t\right) + C_3$$

$$y(0)=0 \quad \to 0=\frac{a^2}{g}\ln(\cosh 0)+C_3$$

$$0=\frac{a^2}{g}\ln(1)+C_3 \quad \to \quad \boxed{C_3=0}$$

$$\boxed{y(t)=\frac{a^2}{g}\ln\left(\cosh\frac{g}{a}t\right)}$$

Exercise

A body of mass *m*, with initial velocity v_0, falls vertically. If the initial position is denoted s_0. Determine the velocity and position of the body at *t* seconds.

Assume the body acted upon by gravity alone and the air resistance proportional to the square of the velocity.

Solution

The resistance force *R*, according the given, is proportional to the square of the velocity. $R=-r.^2v^2$

$$F=mg+F_r$$

$$m\frac{dv}{dt}=mg-mr^2v^2$$

$$\frac{dv}{dt}=g-r^2v^2$$

$$\int\frac{dv}{g-(rv)^2}=\int dt$$

$$\frac{1}{g}\int\frac{dv}{1-\left(\frac{r}{\sqrt{g}}v\right)^2}=\int dt \qquad\qquad \int\frac{dx}{a^2-x^2}=\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| \quad or \quad \frac{1}{a}\tanh^{-1}\frac{x}{a}$$

$$\frac{1}{g}\frac{\sqrt{g}}{r}\tanh^{-1}\left(\frac{r}{\sqrt{g}}v\right)=t+C_1$$

$$\tanh^{-1}\left(\frac{r}{\sqrt{g}}v\right)=r\sqrt{g}t+C_2$$

$$\frac{r}{\sqrt{g}}v=\tanh\left(r\sqrt{g}t+C_2\right)$$

$$v(t)=\frac{\sqrt{g}}{r}\tanh\left(r\sqrt{g}t+C_2\right)$$

$$v(0)=v_0 \quad \to v_0=\frac{\sqrt{g}}{r}\tanh\left(C_2\right) \quad \Rightarrow \boxed{C_2=\tanh^{-1}\left(\frac{rv_0}{\sqrt{g}}\right)}$$

$$\boxed{v(t)=\frac{\sqrt{g}}{r}\tanh\left(r\sqrt{g}t+\tanh^{-1}\left(\frac{rv_0}{\sqrt{g}}\right)\right)}$$

$$v(t)=\frac{ds}{dt}=\frac{\sqrt{g}}{r}\tanh\left(r\sqrt{g}t+C_2\right)$$

$$\int ds=\frac{\sqrt{g}}{r}\int\tanh\left(r\sqrt{g}t+C_2\right)dt \qquad \int\tanh axdx=\frac{1}{a}\ln(\cosh ax)+C$$

$$s(t)=\frac{\sqrt{g}}{r}\frac{1}{r\sqrt{g}}\ln\left(\cosh\left(r\sqrt{g}t+C_2\right)\right)+C_3$$

$$=\frac{1}{r^2}\ln\left(\cosh\left(r\sqrt{g}t+C_2\right)\right)+C_3$$

$$s(0)=s_0 \quad \to s_0=\frac{1}{r^2}\ln\left(\cosh\left(C_2\right)\right)+C_3$$

$$\to \boxed{C_3=s_0-\frac{1}{r^2}\ln\left(\cosh\left(C_2\right)\right)}$$

$$\boxed{s(t)=\frac{1}{r^2}\ln\left(\cosh\left(r\sqrt{g}t+\tanh^{-1}\left(\frac{rv_0}{\sqrt{g}}\right)\right)\right)+s_0-\frac{1}{r^2}\ln\left(\cosh\left(\tanh^{-1}\left(\frac{rv_0}{\sqrt{g}}\right)\right)\right)}$$

Exercise

A body falls from a height of 300 *ft*. What distance has it traveled after 4 *sec*. if subject to *g*, the earth's acceleration?

Solution

$$a(t)=-g$$

$$v(t)=-\int gdt$$

$$=-gt+C_1$$

$$v(0)=0 \quad \to \quad \boxed{C_1=0}$$

$$\boxed{v(t)=-32.2t}$$

$$v(t)=\frac{dh}{dt}=-32.2t$$

$$\int dh=-32.2\int t\ dt$$

$$h(t)=-16.1t^2+C_2$$

$$h(0)=300 \quad \to \quad \boxed{C_2=300}$$

$$\boxed{h(t)=-16.1t^2+300}$$

$$h(t=4) = -16.1(16) + 300$$
$$\boxed{= 42.4\ ft}$$

Exercise

A body falls from an initial velocity of $1,000\ ft/s$. What distance has it traveled after 3 *sec*. if subject to $g = 32\ ft/s^2$, the earth's acceleration?

Solution

$$a(t) = g$$
$$v(t) = \int g dt$$
$$= gt + C_1$$
$$v(0) = 1000 \quad \to \quad \boxed{C_1 = 1,000}$$
$$v(t) = 32t + 1,000$$
$$v(t) = \frac{dh}{dt} = 32t + 1,000$$
$$\int dh = \int (32t + 1,000)\ dt$$
$$h(t) = 16t^2 + 1,000t + C_2$$
$$h(0) = 0 \quad \to \quad \boxed{C_2 = 0}$$
$$\boxed{h(t) = 16t^2 + 1,000t}$$
$$h(t=3) = 16(9) + 3,000$$
$$\boxed{= 3,144\ ft}$$

Exercise

A projectile is fired straight upwards with an initial velocity of $1,600\ ft/s$. What is its velocity at $40,000\ ft$. $\left(g = 32\ ft/s^2\right)$

Solution

$$a(t) = -g$$
$$v(t) = -\int 32 dt$$
$$= -32t + C_1$$

$$v(0)=1600 \quad \to \quad \boxed{C_1 = 1,600}$$

$$\boxed{v(t) = -32t + 1,600}$$

$$v(t) = \frac{dh}{dt} = -32t + 1,600$$

$$\int dh = \int (-32t + 1,600)\ dt$$

$$h(t) = -16t^2 + 1,600t + C_2$$

$$h(0) = 0 \quad \to \quad \boxed{C_2 = 0}$$

$$h(t) = -16t^2 + 1,600t$$

$$-16t^2 + 1,600t = 40,000$$

$$t^2 - 100t + 2,500 = 0$$

$$(t-50)^2 = 0 \quad \to \quad \boxed{t = 50}$$

$$v(50) = -32(50) + 1,600$$

$$\boxed{= 0 \ \ ft/sec}$$

Exercise

A projectile is fired straight upwards with an initial velocity of $1,000$ *ft/s*. What is its velocity at $8,000$ *ft*. $\left(g = 32\ ft/s^2\right)$

Solution

$$v(t) = -\int 32 dt$$

$$= -32t + C_1$$

$$v(0)=1000 \quad \to \quad \boxed{C_1 = 1,000}$$

$$\boxed{v(t) = -32t + 1,000}$$

$$v(t) = \frac{dh}{dt} = -32t + 1,000$$

$$\int dh = \int (-32t + 1,000)\ dt$$

$$h(t) = -16t^2 + 1,000t + C_2$$

$$h(0) = 0 \quad \to \quad \boxed{C_2 = 0}$$

$$h(t) = -16t^2 + 1,000t$$

$$-16t^2 + 1,000t = 8,000$$

$$2t^2 - 125t + 1,000 = 0$$

$$t = \frac{125 \pm \sqrt{15625 - 8000}}{4} = \frac{125 \pm \sqrt{7,625}}{4} \approx \begin{cases} 9.42 \\ 53.08 \end{cases}$$

$$v(9.42) = -32(9.42) + 1,000$$

$$\approx 698.56 \;\; ft/sec$$

$$v(53.08) = -32(53.08) + 1,000$$

$$\approx -698.56 \;\; ft/sec$$

Exercise

An 8 *lb*. weight falls from rest toward earth. Assuming that the weight is acted upon by air resistance, numerically equal to $2v$, but measured in pounds, find the velocity and distance fallen after t seconds. (The variable v represents the velocity measured in *ft/sec*.)

Solution

The resistance force $F = 2v$

$$F = mg + F_r$$

$$m\frac{dv}{dt} = 8 - 2v$$

$$\frac{8}{32}\frac{dv}{dt} = 8 - 2v$$

$$\frac{dv}{dt} = 8(4 - v)$$

$$\int \frac{dv}{4-v} = \int 8dt$$

$$-\ln|4 - v| = 8t + C_1$$

$$\ln|4 - v| = C_2 - 8t$$

$$4 - v = e^{C_2 - 8t}$$

$$v = 4 - e^{C_2} e^{-8t}$$

$$v(t) = 4 - C_3 e^{-8t}$$

$$v(0) = 0 \quad \rightarrow 0 = 4 - C_3 \quad \Rightarrow C_3 = 4$$

$$v(t) = 4\left(1 - e^{-8t}\right)$$

$$v(t) = \frac{dy}{dt} = 4\left(1 - e^{-8t}\right)$$

$$\int dy = \int 4\left(1-e^{-8t}\right)dt$$

$$y(t) = 4\left(t+\frac{1}{8}e^{-8t}\right)+C_4$$

$$y(0)=0 \quad \to 0 = 4\left(\frac{1}{8}\right)+C_4 \quad \Rightarrow \boxed{C_4 = -\frac{1}{2}}$$

$$\boxed{y(t) = 4\left(t+\frac{1}{8}e^{-8t}\right)-\frac{1}{2}}$$

Exercise

A stone is released from rest and dropped into a deep well. Eight seconds later, the sound of the stone splashing into the water at the bottom of the well returns to the ear of the person who released the stone. How long does it take the stone to drop to the bottom of the well? How deep is the well? Ignore air resistance. The speed of sound is 340 *m/s*.

Solution

$$d = \frac{1}{2}gt^2$$
$$= \frac{1}{2}9.8t^2$$
$$= 4.9t^2$$

$$d = 340s$$
$$= 340(8-t)$$

$$4.9t^2 = 2720 - 340t$$

$$4.9t^2 + 340t - 2720 = 0$$

$$\boxed{t = 7.2438\ sec}$$

$$d = 340(8-7.2438)$$
$$\boxed{= 257.1\ m}$$

Exercise

A rocket is fired vertically and ascends with constant acceleration $a = 100\ m/s^2$ for 1.0 *min*. At that point, the rocket motor shuts off and the rocket continues upward under the influence of gravity. Find the maximum altitude acquired by the rocket and the total time elapsed from the take-off until the rocket returns to the earth. *Ignore air resistance*.

Solution

$$d = \frac{1}{2}(a-g)t^2$$
$$= \frac{1}{2}(100-9.8)t^2$$

$$d(1hr = 60min) = \frac{1}{2}(100 - 9.8)(60)^2$$
$$= 162,360\ m$$

$$v = d'$$
$$= \left(\frac{1}{2}(100 - 9.8)t^2\right)'$$
$$= (100 - 9.8)t$$

$$v(60) = (100 - 9.8)(60)$$
$$= 5412\ m/s$$

The velocity will be reduced: $5412\ - 9.8t = 0$

$$\boxed{t = 552.2\ \ s}$$

The altitude: $d(t) = -\frac{9.8}{2}t^2 + 5412\ t + 162,360$

$$d(552.2) = -\frac{9.8}{2}(552.2)^2 + 5412(552.2) + 162,360$$
$$= 1.657 \times 10^6 m$$

Back to the ground: $4.9t^2 = 1.657 \times 10^6$

$$\boxed{t_b\ = 581.5\ \ s}$$

Total time: $t = 552.2 + 581.5 = 1193.7\ \ \text{sec}$

Exercise

A ball having mass $m = 0.1\ kg$ falls from rest under the influence of gravity in a medium that provides a resistance that is proportional to its velocity. For a velocity of $0.2\ m/s$ the force due to the resistance of the medium is $-1\ N$. Find the terminal velocity of the ball.

1 N is the force required to accelerate a 1 kg mass at a rate of $1\ m/s^2$: $1N = 1\ \text{kg} \cdot m/s^2$

Solution

The resistance force: $R = -rv$

$$-1 = -0.2r\ \Rightarrow \boxed{r = 5}$$

The terminal velocity: $v_{term}\ = -\frac{mg}{r}$

$$= \frac{0.1(9.8)}{5}$$
$$= -0.196\ \ m/s$$

Exercise

A ball is projected vertically upward with initial velocity v_0 from ground level. Ignore air resistance.

a) What is the maximum height acquired by the ball?

b) How long does it take the ball to reach its maximum height? How long does it take the ball to return to the ground? Are these times identical?

c) What is the speed of the ball when it impacts with the ground on its return?

Solution

The position: $x(t) = -\frac{1}{2}gt^2 + v_0 t$

a) The maximum height when the velocity is zero

$$v = x' = -gt + v_0 = 0$$

$$t = \frac{v_0}{g}$$

$$\text{Maximum height } = -\frac{1}{2}g\left(\frac{v_0}{g}\right)^2 + v_0\frac{v_0}{g}$$

$$= -\frac{1}{2}\frac{v_0^2}{g} + \frac{v_0^2}{g}$$

$$= \frac{v_0^2}{2g}$$

b) The ball will take to reach the maximum height $t = \frac{v_0}{g}$ and the same to return to the ground, both are equal to $t = \frac{v_0}{g}$

c) When the ball hits the ground the time is equal to zero.

$$v = -g(0) + v_0$$

$$= v_0$$

Exercise

An object having mass 70 *kg* falls from rest under the influence of gravity. The terminal velocity of the object is $-20\ m/s$. Assume that the air resistance is proportional to the velocity.

a) Find the velocity and distance traveled at the end of 2 *seconds*.

b) How long does it take the object to reach 80% of its terminal velocity?

Solution

a) The terminal velocity: $v = -\frac{mg}{r}$

$$-20 = -\frac{70(9.8)}{r}$$

$$\boxed{r = \frac{70(9.8)}{20} = 34.3}$$

$$v(t) = Ce^{-rt/m} - \frac{mg}{r}$$

$$v(t=0) = Ce^{-r(0)/m} - \frac{mg}{r}$$

$$0 = C - \frac{mg}{r} \quad \Rightarrow \quad \boxed{C = \frac{mg}{r}}$$

$$v(t) = \frac{mg}{r}\left(e^{-rt/m} - 1\right)$$

$$\boxed{v(t=2) = \frac{70(9.8)}{34.3}\left(e^{-34.3(2)/70} - 1\right) \approx -12.4938}$$

$$x = \int_0^t v(t)\,dt$$

$$= \frac{mg}{r}\int_0^t \left(e^{-rs/m} - 1\right)ds$$

$$= \frac{mg}{r}\left[-\frac{m}{r}e^{-rs/m} - s\right]_0^t$$

$$= \frac{mg}{r}\left[-\frac{m}{r}e^{-rt/m} - t - \left(-\frac{m}{r} - 0\right)\right]$$

$$= \frac{mg}{r}\left[\frac{m}{r}\left(1 - e^{-rt/m}\right) - t\right]$$

$$\boxed{x(2) = \frac{70(9.8)}{34.3}\left[\frac{70}{34.3}\left(1 - e^{-34.3(2)/70}\right) - 2\right] \approx -14.5025}$$

b) The velocity is 80% of its terminal velocity when $.8 = 1 - e^{-rt/m}$

$$e^{-rt/m} = .2$$

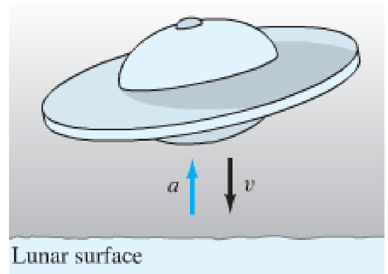
$$-\frac{rt}{m} = \ln(.2) \quad \Rightarrow \quad \boxed{t = \frac{m}{r}\ln(.2) \approx 3.285 \text{ sec}}$$

Exercise

A lunar lander is falling freely toward the surface of the moon at a speed of 450 *m/s*. Its retrorockets, when fired, provide a constant deceleration of 2.5 m/s^2 (the gravitational acceleration produced by the moon is assumed to be included in the given acceleration). At What height above the lunar surface should the retrorockets be activated to ensure a “soft touchdown? (v = 0 at impact)?

Solution

Given: $t=0 \quad \to \quad v_0 = -450\ m/s \quad a = +2.5$



Because an upward thrust increases the velocity v (although decreases the speed $|v|$), then

$$v(t) = 2.5t - 450$$

$$x(t) = \int v(t)\,dt$$

$$= \int (2.5t - 450)\,dt$$

$$= 1.25t^2 - 450t + x_0$$

x_0 : is the height of the lander above the lunar surface at the time $t = 0$ when the retrorockets should be activated.

$$v(t) = 2.5t - 450 = 0$$

$$t = \frac{450}{2.5} = \underline{180\ s}|$$

When $x = 0 \quad \to \quad t = 180$

$$0 = 1.25(180)^2 - 450(180) + x_0$$

$$x_0 = 450(180) - 1.25(180)^2$$

$$\underline{= 40{,}500}|$$

Thus the retrorockets should be activated when the lunar lander is 40.500 m (40,5 km) above the surface of the moon, and it will touch down softly on the lunar surface after 3 minutes of decelerating descent.

Exercise

A body falling in a relatively dense fluid, oil for example, is acted on by three forces: a resistance force R, a buoyant force B, and its weight w due to gravity. The buoyant force is equal to the weight of the fluid displaced by the object. For a slowly moving spherical body of radius a, the resistive force is given by Stokes's law $R = 6\pi \frac{\mu a}{v}$, where v is the velocity of the body, and μ is the coefficient of viscosity of the surrounding fluid?

a) Find the limiting velocity of a solid sphere of radius a and density ρ falling freely in a medium of density ρ' and coefficient of viscosity μ.

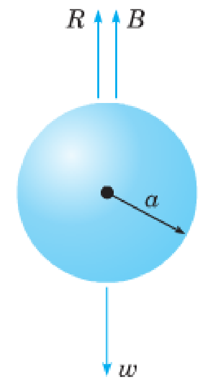
b) In 1910 R. A. Millikan studied the motion of tiny droplets of oil falling in an electric field. A field of strength E exerts a force E_e on a droplet with charge e. Assume that E has been adjusted so the droplet is held stationary $(v = 0)$ and that w and B are as given. Find an expression for e.

Solution

a) The equation of motion is $m\frac{dv}{dt} = w - R - B$

$$\frac{4}{3}\pi a^3 \rho\left(\frac{dv}{dt}\right) = \frac{4}{3}\pi a^3 \rho g - 6\pi\mu a v - \frac{4}{3}\pi a^3 \rho' g$$

The limiting velocity occurs when $\frac{dv}{dt} = 0$



b) Since the droplet is motionless, $v = \frac{dv}{dt} = 0$, we have the equation of motion

$$0 = \frac{4}{3}\pi a^3 \rho g - Ee - \frac{4}{3}\pi a^3 \rho' g$$

Where ρ is the density of the oil and ρ' is the density of air.

$$Ee = \frac{4}{3}\pi a^3 \rho g - \frac{4}{3}\pi a^3 \rho' g$$

$$Ee = \frac{4}{3}\pi a^3 (\rho - \rho')$$

$$\boxed{e = \frac{4}{3}\frac{\pi a^3}{E}(\rho - \rho')}$$

Exercise

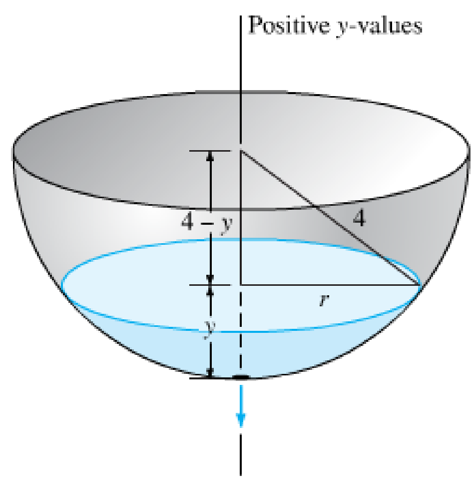
A hemispherical bowl has top radius of 4 *ft*. and at time $t = 0$ is full of water. At that moment a circular hole with diameter 1 *in*. is opened in the bottom of the tank. How long will it take for all the water to drain from the tank?

Solution

$$r^2 + (4-y)^2 = 16 \quad \rightarrow \quad r^2 = 16 - (4-y)^2 \qquad (\textit{Right Triangle})$$

$$A(y) = \pi r^2 = \pi\left[16 - (4-y)^2\right] = \pi\left(8y - y^2\right)$$

$$a = \pi r^2 = \pi\left(\frac{1}{2} in. \frac{1\ ft.}{12\ in.}\right)^2 = \pi\left(\frac{1}{24}\right)^2$$



$$\frac{dV}{dt} = -a\sqrt{2gy}$$

$$\pi\left(8y - y^2\right)\frac{dy}{dt} = -\pi\left(\frac{1}{24}\right)^2\sqrt{2\cdot 32y}$$

$$\left(8y - y^2\right)\frac{dy}{dt} = -\frac{1}{576}(8)\sqrt{y}$$

$$\frac{8y - y^2}{y^{1/2}}dy = -\frac{1}{72}dt$$

$$\left(8y^{1/2} - y^{3/2}\right)dy = -\frac{1}{72}dt$$

$$\int\left(8y^{1/2} - y^{3/2}\right)dy = -\frac{1}{72}\int dt$$

$$\frac{16}{3}y^{3/2}-\frac{2}{5}y^{5/2}=-\frac{1}{72}t+C$$

$$y(0)=4$$

$$\frac{16}{3}(4)^{3/2}-\frac{2}{5}(4)^{5/2}=-\frac{1}{72}(0)+C$$

$$C=\frac{448}{15}$$

$$\frac{16}{3}y^{3/2}-\frac{2}{5}y^{5/2}=-\frac{1}{72}t+\frac{448}{15}$$

The tank is empty when y = 0, thus when

$$0=-\frac{1}{72}t+\frac{448}{15}$$

$$\frac{1}{72}t=\frac{448}{15}$$

$$t=\frac{448}{15}(72)$$

$$\approx 2150 \ \text{sec}$$

That is about 35 *min*. 50 *s*. So it takes slightly less than 36 *minutes* for the tank to drain.

Exercise

Suppose that the tank has a radius of 3 *feet*. and that its bottom hole is circular with radius 1 *in*. How long will it take the water (initially 9 *ft*. deep) to drain completely?

Solution

$$A=\pi r^2=\pi(3)^2=9\pi$$

$$a=\pi r^2=\pi\left(1in.\frac{1\ ft.}{12\ in.}\right)^2=\pi\left(\frac{1}{12}\right)^2$$

$$A(y)\frac{dy}{dt}=-a\sqrt{2gy}$$

$$9\pi\frac{dy}{dt}=-\frac{\pi}{144}\sqrt{2(32)y}$$

$$9\frac{dy}{dt}=-\frac{1}{144}\left(8\sqrt{y}\right)$$

$$9\frac{dy}{dt}=-\frac{1}{18}\sqrt{y}$$

$$162\frac{dy}{\sqrt{y}}=-dt$$

$$162\int y^{-1/2}dy=-\int dt$$

$$324y^{1/2}=-t+C$$

$$y(t=0)=9$$

$$324\sqrt{9}=-0+C \quad \Rightarrow \quad \underline{C=972}$$

$$324\sqrt{y}=-t+972$$

Hence $y=0$ when $\underline{t=972\sec=16\min\ \ 12\sec}$

Exercise

At time *t* = 0 the bottom plug (at the vertex) of a full conical water tank 16 *ft*. high is removed. After 1 *hr* the water in the tank is 9 *ft*. deep. When will the tank be empty?

Solution

The radius of the cross-section of the cone at height *y* is proportional to *y*, so $A(y)$ is proportional to y^2. Therefore,

$$y^2y'=-k\sqrt{y} \qquad\qquad A(y)y'=-k\sqrt{y}$$

$$y^{3/2}dy=-kdt$$

$$\int y^{3/2}dy=-\int kdt$$

$$\frac{2}{5}y^{5/2}=-kt+C_1$$

$$2y^{5/2}=-5kt+C$$

With initial condition: $y(0)=16$

$$2(16)^{5/2}=-5k(0)+C \quad \Rightarrow \quad \underline{C=2048}$$

$$2y^{5/2}=-5kt+2048$$

$$y(1)=9$$

$$2(9)^{5/2}=-5k(1)+2048$$

$$486-2048=-5k \quad \Rightarrow \quad \underline{k=\frac{1562}{5}}$$

$$2y^{5/2}=-1562t+2048$$

Hence $y=0$ when $t=\frac{2048}{1562}\ \underline{\approx 1.31\ hr}$

Exercise

Suppose that a cylindrical tank initially containing V_0 gallons of water drains (through a bottom hole) in *T minutes*. Use Torricelli's law to show that the volume of water in the tank after $t\le T$ minutes is

$$V=V_0\left(1-\frac{t}{T}\right)^2$$

Solution

$$\frac{dy}{dt} = -k\sqrt{y} \qquad\qquad A(y)y' = -k\sqrt{y}$$

$$\int y^{-1/2}dy = -\int kdt$$

$$2y^{1/2} = -kt + C$$

With initial condition: $y(0) = h \quad 2\sqrt{h} = -k(0) + C \rightarrow \boxed{C = 2\sqrt{h}}$

$$2\sqrt{y} = -kt + 2\sqrt{h}$$

$$y(t = T) = 0$$

$$0 = -kT + 2\sqrt{h} \Rightarrow \boxed{k = \frac{2\sqrt{h}}{T}}$$

$$2\sqrt{y} = -\frac{2\sqrt{h}}{T}t + 2\sqrt{h}$$

$$\sqrt{y} = \sqrt{h}\left(1 - \frac{t}{T}\right)$$

$$y = h\left(1 - \frac{t}{T}\right)^2$$

If r denotes the radius of the cylinder, then

$$V(y) = \pi r^2 y$$

$$= \pi r^2 h\left(1 - \frac{t}{T}\right)^2$$

$$\boxed{= V_0\left(1 - \frac{t}{T}\right)^2} \qquad V_0 = \pi r^2 h$$

Exercise

The clepsydra, or water clock – A 12-*hr* water clock is to be designed with the dimensions, shaped like the surface obtained by revolving the curve $y = f(x)$ around the *y*-axis. What should be this curve, and what should be the radius of the circular bottom hole, in order that the water level will fall at the constant rate of 4 *inches* per *hour*?

Solution

The rate of fall of the water level is

$$\frac{dy}{dt} = -4\ in./hr = -\frac{1}{10800}\ ft/\sec$$

$$= -4\ \frac{in}{hr} \cdot \frac{1\,ft}{12\ in} \cdot \frac{1\ hr}{3600\ \sec}$$

$$= -\frac{1}{10800}\ ft/\sec$$

$$A = \pi x^2 \quad and \quad a = \pi r^2$$

$$A\frac{dy}{dt} = -a\sqrt{2gy}$$

$$\pi x^2 \frac{-1}{10800} = -\pi r^2 \sqrt{2(32)y}$$

$$\frac{x^2}{10800} = 8r^2\sqrt{y}$$

The curve is of the form $y = kx^4$

$$4 = k\left(1^4\right) \quad \to \quad k = 4$$

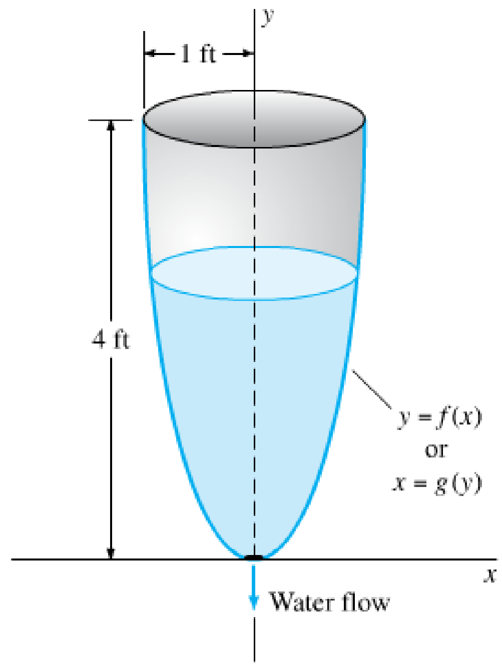
$$y = 4x^4 \quad \to \quad \sqrt{y} = 2x^2$$

$$\frac{x^2}{10800} = 8r^2\left(2x^2\right)$$

$$\frac{1}{10800} = 16r^2$$

$$r = \frac{1}{\sqrt{172800}}$$

$$\approx 0.0024 \ ft \quad or \quad 0.028 \ in.$$



Exercise

One of the famous problems in the history of mathematics is the brachistochrone problem: to find the curve along which a particle will slide without friction in the minimum time from one given point *P* to another point *Q*, the second point being lower than the first but not directly beneath it.

This problem was posed by Johann Bernoulli in 1696 as a challenge problem to the mathematicians of his day. Correct solutions were found by Johann Bernoulli and his brother Jakob Bernoulli and by Isaac Newton, Gottfried Leibniz, and the Marquis de L'Hospital. The brachistochrone problem is important in the development of mathematics as one of the forerunners of the calculus of variations.

In solving this problem, it is convenient to take the origin as the upper point *P* and to orient the axes as shown. The lower point Q has coordinates $\left(x_0, y_0\right)$. It is then possible to show that the curve of minimum time is given by a function $y = \phi(x)$ that satisfies the differential equation

$$\left(1 + y'^{\,2}\right)y = k^2 \qquad (eq.\ i)$$

Where k^2 is a certain positive constant to be determined later

a) Solve the equation $(eq.\ i)$ for y'. Why is it necessary to choose the positive square root?

b) Introduce the new variable t by the relation

$$y = k^2 \sin^2 t \qquad (eq.\ ii)$$

Show that the equation found in part (a) then takes the form

$$k^2 \sin^2 t \ dt = dx \qquad (eq.\ iii)$$

c) Letting $\theta = 2t$, show that the solution of $(eq.\ iii)$ for which $x = 0$ when $y = 0$ is given by

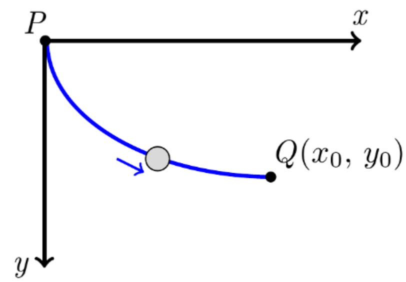
$$x = k^2\,\frac{\theta - \sin\theta}{2}, \quad y = k^2\,\frac{1-\cos\theta}{2} \qquad (eq.\ iv)$$

Equations (*iv*) are parametric equations of the solution of (eq. *i*) that passes through (0, 0). The graph of Eqs. (*iv*) is called a cycloid.

d) If we make a proper choice of the constant k, then the cycloid also passes through the point (x_0, y_0) and is the solution of the brachistochrone problem. Find *k* if $x_0 = 1$ and $y_0 = 2$

Solution

a) $\left(1 + y'^{\,2}\right) y = k^2$



$$1 + y'^{\,2} = \frac{k^2}{y}$$

$$y'^{\,2} = \frac{k^2}{y} - 1$$

$$y' = \pm\sqrt{\frac{k^2 - y}{y}}$$

$$\boxed{y' = \sqrt{\frac{k^2 - y}{y}}}$$

The positive answer is chosen, since *y* is an increasing function of *x*.

b) Let $y = k^2 \sin^2 t$

$$dy = 2k^2 \sin t \cos t\ dt \qquad \left(\sin^2 t\right)' = 2\sin t\left(\sin t\right)'\ dt$$

$$y' = \frac{dy}{dx} = \sqrt{\frac{k^2 - y}{y}}$$

$$= \sqrt{\frac{k^2 - k^2\sin^2 t}{k^2 \sin^2 t}}$$

$$= \frac{k\sqrt{1 - \sin^2 t}}{k \sin t}$$

$$= \frac{\sqrt{\cos^2 t}}{\sin t}$$

$$= \boxed{\frac{\cos t}{\sin t}}$$

$$\frac{dy}{dx} = \frac{2k^2 \sin t \cos t\ dt}{dx} = \frac{\cos t}{\sin t}$$

$$\boxed{2k^2 \sin^2 t\ dt = dx}$$

c) Setting $\theta = 2t$, $2k^2 \sin^2\left(\frac{\theta}{2}\right) d\left(\frac{\theta}{2}\right) = dx$

$$k^2 \sin^2\left(\frac{\theta}{2}\right) d\theta = dx$$

$$\int k^2 \sin^2\left(\frac{\theta}{2}\right) d\theta = \int dx$$

$$\int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$k^2\left(\frac{\theta}{2} - \frac{\sin\theta}{2}\right) = x + C$$

$t = \theta = 0$ at the origin.

$$\underline{C = 0}$$

$$\boxed{x(\theta) = k^2 \cdot \frac{\theta - \sin\theta}{2}}$$

$$y = k^2 \sin^2\left(\frac{\theta}{2}\right)$$

$$\boxed{= k^2 \cdot \frac{1 - \cos\theta}{2}}$$

$$\sin^2\left(\frac{x}{2}\right) = \frac{1 - \cos x}{2}$$

d) $$\frac{y}{x} = \frac{k^2 \cdot \frac{1-\cos\theta}{2}}{k^2 \cdot \frac{\theta - \sin\theta}{2}}$$

$$= \frac{1 - \cos\theta}{\theta - \sin\theta}$$

Letting $x = 1, \quad y = 2 \quad \rightarrow \quad \frac{1 - \cos\theta}{\theta - \sin\theta} = 2$

$$1 - \cos\theta = 2\theta - 2\sin\theta$$

The solution is: $\underline{\theta \approx 1.401}$

$$y = k^2 \cdot \frac{1 - \cos\theta}{2}$$

$$2y = (1 - \cos\theta)k^2$$

$$k^2 = \frac{2y}{1 - \cos\theta} = \frac{2(2)}{1 - \cos(1.401)} = 4.183$$

$$\underline{k \approx 2.194}$$

Exercise

Many chemical reactions are the result of the interaction of 2 molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If a is the amount of substance A and b is the substance B at time $t = 0$, and if x is the amount of product at time t, then the rate of formation of x may be given by the differential equation

$$\frac{dx}{dt} = k(a - x)(b - x) \quad \text{or} \quad \frac{1}{(a - x)(b - x)}\frac{dx}{dt} = k$$

Where k is a constant for the reaction. Integrate both sides of this equation to obtain a relation between x and t.

a) If $a = b$

b) If $a \neq b$

Assume in each case that $x = 0$ when $t = 0$

Solution

$$\frac{1}{(a-x)(b-x)}dx = kdt$$

a) $a = b \quad \Rightarrow \quad \frac{1}{(a-x)^2}dx = kdt$

$$\int \frac{1}{(a-x)^2}dx = \int kdt$$

$$\frac{1}{a-x} = kt + C$$

$$x(t=0) = 0 \quad \Rightarrow \quad \boxed{\frac{1}{a} = C}$$

$$\frac{1}{a-x} = kt + \frac{1}{a} = \frac{kat+1}{a}$$

$$a - x = \frac{a}{kat+1}$$

$$x = a - \frac{a}{kat+1}$$

$$\boxed{= \frac{a^2kt}{kat+1}}$$

b) $a \neq b \quad \Rightarrow \quad \frac{1}{(a-x)(b-x)}dx = kdt$

$$\int \frac{1}{(a-x)(b-x)}dx = \int kdt$$

$$\frac{1}{(a-x)(b-x)} = \frac{A}{a-x} + \frac{B}{b-x}$$

$$\begin{cases} -A - B = 0 \\ bA + aB = 1 \end{cases} \rightarrow \begin{cases} B = \frac{1}{a-b} \\ A = -\frac{1}{a-b} \end{cases}$$

$$\frac{-1}{a-b}\int \frac{1}{a-x}dx + \frac{1}{a-b}\int \frac{1}{b-x}dx = \int kdt$$

$$\frac{1}{a-b}\ln|a-x| - \frac{1}{a-b}\ln|b-x| = kt + C$$

$$\frac{1}{a-b}\ln\left|\frac{a-x}{b-x}\right| = kt + C$$

$$x(0) = 0 \quad \Rightarrow \quad \boxed{\frac{1}{a-b}\ln\left(\frac{a}{b}\right) = C}$$

$$\frac{1}{a-b}\ln\left|\frac{a-x}{b-x}\right| = kt + \frac{1}{a-b}\ln\left(\frac{a}{b}\right)$$

$$\ln\left|\frac{a-x}{b-x}\right| = (a-b)kt + \ln\left(\frac{a}{b}\right)$$

$$\frac{a-x}{b-x} = e^{(a-b)kt + \ln\left(\frac{a}{b}\right)}$$

$$\frac{a-x}{b-x} = \frac{a}{b}e^{(a-b)kt}$$

$$a - x = b\frac{a}{b}e^{(a-b)kt} - x\frac{a}{b}e^{(a-b)kt}$$

$$x\left(\frac{a}{b}e^{(a-b)kt} - 1\right) = ae^{(a-b)kt} - a$$

$$\boxed{x = \frac{abe^{(a-b)kt} - ab}{ae^{(a-b)kt} - b}}$$

Exercise

An open cylindrical tank initially filled with water drains through a hole in the bottom of the tank according to Torricelli's Law. If $h(t)$ is the depth of water in the tank for $t \geq 0$, then Torricelli's Law implies $h'(t) = -2k\sqrt{h}$, where k is a constant that includes the acceleration due to gravity, the radius of the tank, and the radius of the drain. Assume that the initial depth of the water is $h(0) = H$.

a) Find the solution of the initial value problem.

b) Find the solution in the case that $k = 0.1$ and $H = 0.5\ m$.

c) In general, how long does it take the tank to drain in terms of k and H?

Solution

a) $\frac{dh}{dt} = -2k\sqrt{h}$

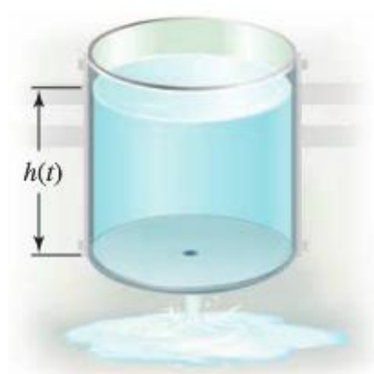
$$\int \frac{dh}{\sqrt{h}} = -2\int kdt$$

$$2\sqrt{h} = 2kt + C_1$$

$$h(t) = (kt + C)^2$$

$$h(0) = H \quad \rightarrow H = C^2 \Rightarrow C = \sqrt{H}$$

$$\boxed{h(t) = \left(kt + \sqrt{H}\right)^2}$$



b) Given: $k = 0.1 \quad H = 0.5\ m$

$$h(t) = \left(0.1t + \sqrt{0.5}\right)^2$$

$$\boxed{= (0.1t + 0.707)^2}$$

c) The tank is drained when $h(t) = 0$

$$\left(kt + \sqrt{H}\right)^2 = 0$$

$$kt + \sqrt{H} = 0$$

$$\boxed{t = -\frac{\sqrt{H}}{k}}$$

Exercise

An object in free fall may be modeled by assuming that the only forces at work are the gravitational force and resistance (friction due to the medium in which the objects falls). By Newton's second law (mass × acceleration = the sum of the external forces), the velocity of the object satisfies the differential equation

$$\underset{mass}{m} \cdot \underset{acceleration}{v'(t)} = \underbrace{mg + f(v)}_{external\ force}$$

Where f is a function that models the resistance and the positive direction is downward. One common assumption (often used for motion in air) is that $f(v) = -kv^2$, where $k > 0$ is a drag coefficient.

a) Show that the equation can be written in the form $v'(t) = g - av^2$ where $a = \frac{k}{m}$

b) For what (positive) value of v is $v'(t) = 0$? (This equilibrium solution is called the ***terminal velocity***.)

c) Find the solution of this separable equation assuming $v(0) = 0$ *and* $0 < v(t)^2 < \frac{g}{a}$ for $t \ge 0$

d) Graph the solution found in part (***c***) with $g = 9.8\ m/s^2,\ \ m = 1\ kg,\ \ and\ \ k = 0.1\ kg/m$, and verify the terminal velocity agrees with the value found in part (***b***).

Solution

a) **Given**: $f(v) = -kv^2$

$$mv'(t) = mg + f(v)$$

$$mv'(t) = mg - kv^2$$

$$v'(t) = g - \frac{k}{m}v^2$$

$$\boxed{v'(t) = g - av^2} \qquad \text{where } a = \frac{k}{m}$$

b) $v'(t) = g - av^2 = 0 \quad \Rightarrow \quad v^2 = \frac{g}{a}$

$$\boxed{v = \sqrt{\frac{g}{a}}}$$

c) $\dfrac{dv}{dt} = g - av^2$

$$\int \frac{dv}{g - av^2} = \int dt$$

$$-\frac{1}{a}\int \frac{dv}{v^2 - \frac{g}{a}} = \int dt$$

$$-\frac{1}{2a}\sqrt{\frac{a}{g}}\int \frac{dv}{v - \sqrt{\frac{g}{a}}} + \frac{1}{2a}\sqrt{\frac{a}{g}}\int \frac{dv}{v + \sqrt{\frac{g}{a}}} = \int dt$$

$$\frac{1}{2}\sqrt{\frac{1}{ag}}\left(-\ln\left|\sqrt{\frac{g}{a}} - v\right| + \ln\left|\sqrt{\frac{g}{a}} + v\right|\right) = t + C_1$$

$$\frac{1}{v^2 - \frac{g}{a}} = \frac{A}{v - \sqrt{\frac{g}{a}}} + \frac{B}{v + \sqrt{\frac{g}{a}}}$$

$$1 = A\sqrt{\frac{g}{a}} + Av + Bv - B\sqrt{\frac{g}{a}}$$

$$\begin{cases} A + B = 0 \ \rightarrow \boxed{A = -B} \\ A\sqrt{\frac{g}{a}} - B\sqrt{\frac{g}{a}} = 1 \end{cases}$$

$$\boxed{A = -B = \frac{1}{2}\sqrt{\frac{a}{g}}}$$

$$\ln\frac{\sqrt{\frac{g}{a}}+v}{\sqrt{\frac{g}{a}}-v} = 2\sqrt{ag}\,t + C_2$$

$$\frac{\sqrt{\frac{g}{a}}+v}{\sqrt{\frac{g}{a}}-v} = e^{2\sqrt{ag}\,t+C_2}$$

$$\sqrt{\frac{g}{a}}+v = Ce^{2\sqrt{ag}\,t}\left(\sqrt{\frac{g}{a}}-v\right) \qquad v(0)=0 \;\Rightarrow\; \sqrt{\frac{g}{a}}=\sqrt{\frac{g}{a}}C \to \underline{C=1}$$

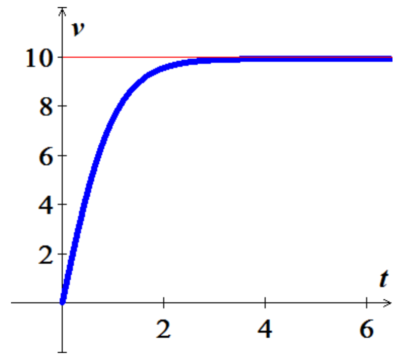
$$v\left(1+e^{2\sqrt{ag}\,t}\right) = \sqrt{\frac{g}{a}}e^{2\sqrt{ag}\,t} - \sqrt{\frac{g}{a}}$$

$$\boxed{v(t) = \frac{e^{2\sqrt{ag}\,t}-1}{1+e^{2\sqrt{ag}\,t}}\sqrt{\frac{g}{a}}}$$

d) $g = 9.8\ m/s^2, \quad m = 1\ kg, \quad and \quad k = 0.1\ kg/m$

$$\to a = \frac{k}{m} \underline{= 0.1}$$

$$\boxed{v(t) = \sqrt{98}\,\frac{e^{2\sqrt{.98}t}-1}{1+e^{2\sqrt{.98}t}}}$$



Exercise

Suppose a small cannonball weighing 16 *pounds* is shot vertically upward, with an initial velocity $v_0 = 300$ *ft/s*

The answer to the question "How high does the cannonball go?" depends on whether we take air resistance into account.

a) Suppose air resistance is ignored. If the positive direction is upward, then a model for the state of the cannonball is given by $\frac{d^2s}{dt^2} = -g$. Since $\frac{ds}{dt} = v(t)$ the last differential equation is the same as $\frac{dv}{dt} = -g$, where we take $g = 32$ *ft/s*^{2}. Find the velocity $v(t)$ of the cannonball at time *t*.

b) Use the result in part (*a*) to determine the height $s(t)$ of the cannonball measured from ground level. Find the maximum height attained by the cannonball.

Solution

a) $$\int \frac{d^2s}{dt} = \int -g\,dt$$

$$v(t) = \frac{ds}{dt} = -gt + C$$

$$v_0 = 300 \quad \underline{300 = C}$$

$\boxed{v(t) = -32t + 300}$

b) $\int ds = \int (-32t + 300)\,dt$

$s(t) = -16t^2 + 300t + C$

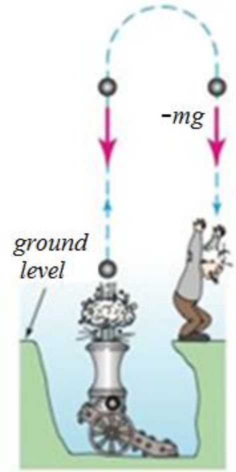
$s_0 = 0 \quad \boxed{0 = C}$

$\boxed{s(t) = -16t^2 + 300t}$

$v(t) = -32t + 300 = 0$

$t = \frac{300}{32} = \boxed{\frac{75}{8} \approx 9.375}$

The maximum height: $\boxed{s(t = 9.375) = 1406.25 \ ft}$



Exercise

Two chemicals *A* and *B* are combined to form a chemical *C*. The resulting reaction between the two chemicals is such that for each *gram* of *A*, 4 *grams* of *B* is used. It is observed that 30 *grams* of the compound *C* is formed in 10 *minutes*.

a) Determine the amount of *C* at time *t* if the rate of the reaction is proportional to the amounts of *A* and *B* remaining and if initially there are 50 *grams* of *A* and 32 *grams* of *B*.

b) How much of the compound *C* is present at 15 *minutes*.

c) Interpret the solution as $t \to \infty$

Solution

a) Let assume that we used *a grams* of *A* and *b grams* of *B*.

Then there *x grams* of compound *C*.

$a + b = x$ and $b = 4a$

$a + 4a = 5a = x \ \to \ \boxed{a = \frac{1}{5}x} \quad \Rightarrow \ \boxed{b = \frac{4}{5}x}$

The rate at which compound *C* is formed satisfies:

$$\frac{dx}{dt} = k\left(50 - \frac{1}{5}x\right)\left(32 - \frac{4}{5}x\right)$$

$$= \frac{4k}{5}(250 - x)(40 - x) \qquad k = \frac{4k}{5} \quad (constant)$$

$$= k(250 - x)(40 - x)$$

$$\frac{dx}{(250 - x)(40 - x)} = k\ dt$$

$$\frac{1}{(250 - x)(40 - x)} = \frac{A}{250 - x} + \frac{B}{40 - x}$$

$$1 = 40A - Ax + 250B - Bx$$

$$\begin{cases} -A-B=0 \\ 40A+250B=1 \end{cases} \rightarrow A=-\frac{1}{210} \quad B=\frac{1}{210}$$

$$\int\left(-\frac{1}{210}\frac{1}{250-x}+\frac{1}{210}\frac{1}{40-x}\right)dx=\int k\ dt$$

$$\frac{1}{210}\ln(250-x)-\frac{1}{210}\ln(40-x)=kt+C$$

$$\ln(250-x)-\ln(40-x)=210(kt+C)$$

$$\ln\left(\frac{250-x}{40-x}\right)=210kt+C_1$$

$$\frac{250-x}{40-x}=C_2e^{210kt}$$

$$x(0)=0 \quad \rightarrow \quad \boxed{\frac{25}{4}=C_2}$$

$$\frac{250-x}{40-x}=\frac{25}{4}e^{210kt}$$

$$x(10)=30 \quad \rightarrow \quad 22=\frac{25}{4}e^{2100k}$$

$$e^{2100k}=\frac{88}{25}$$

$$\boxed{k=\frac{1}{2100}\ln\frac{88}{25}}$$

$$\frac{250-x}{40-x}=\frac{25}{4}e^{\frac{1}{10}t\ln\frac{88}{25}}$$

$$1000-4x=(1000-25x)e^{0.126t}$$

$$\left(25e^{0.126t}-4\right)x=1000e^{0.126t}-1000$$

$$\boxed{x(t)=1000\frac{e^{0.126t}-1}{25e^{0.126t}-4}}$$

b) $x(15)=1000\dfrac{e^{0.126(15)}-1}{25e^{0.126(15)}-4}$

$$\boxed{\approx 34.78\ g}$$

c) $x(t)=1000\dfrac{1-e^{-0.126t}}{25-4e^{-0.126t}}$

$$\lim_{t\to\infty} x(t)=\lim_{t\to\infty} 1000\frac{1-e^{-0.126t}}{25-4e^{-0.126t}}$$

$$=\frac{1000}{25}$$

$$\boxed{=40}$$

$$a=\frac{1}{5}(40)=8\ g$$

For chemical A: $50-8 \boxed{=42\ g}$

$b=\frac{4}{5}(40)=32\ g$

For chemical B: $32-32 \boxed{=0\ g}$

Exercise

Two chemicals A and B are combined to form a chemical C. The rate, or velocity, of the reaction is proportional to the product of the instantaneous amounts of A and B not converted to chemical C. Initially, there are 40 *grams* of A and 50 *grams* of B, and each gram of B, 2 *grams* of A is used. It is observed that 10 *grams* of C is formed in 5 *minutes*.

a) How much is formed in 20 *minutes*?

b) What is the limiting amount of C after a long time?

c) How much of chemicals A and B remains after a long time?

d) If 100 *grams* of chemical A is present initially, at what time is chemical C half-formed?

Solution

a) Let assume that we used a *grams* of A and b *grams* of B.

Then there x *grams* of compound C.

$a+b=x$ and $a=2b$

$2b+b=3b=x \rightarrow \boxed{b=\frac{1}{3}x} \Rightarrow \boxed{a=\frac{2}{3}x}$

The rate at which compound C is formed satisfies:

$$\frac{dx}{dt}=k\left(40-\frac{2}{3}x\right)\left(50-\frac{1}{3}x\right)$$

$$=\frac{k}{3}(120-2x)(150-x) \quad k=\frac{2k}{3} \quad (\textit{constant})$$

$$=k(60-x)(150-x)$$

$$\frac{dx}{(60-x)(150-x)}=k\ dt$$

$$\frac{1}{(60-x)(150-x)}=\frac{A}{60-x}+\frac{B}{150-x}$$

$$1=150A-Ax+60B-Bx$$

$$\begin{cases} -A-B=0 \\ 150A+60B=1 \end{cases} \rightarrow A=\frac{1}{90} \quad B=-\frac{1}{90}$$

$$\int\left(\frac{1}{90}\frac{1}{60-x}-\frac{1}{90}\frac{1}{150-x}\right)dx=\int k\ dt$$

$$-\frac{1}{90}\ln(60-x)+\frac{1}{90}\ln(150-x)=kt+C$$

$$\ln(150-x)-\ln(60-x)=90(kt+C)$$

$$\ln\left(\frac{150-x}{60-x}\right)=90kt+C_1$$

$$\frac{150-x}{60-x}=C_2e^{90kt}$$

$$x(0)=0 \quad \to \quad \boxed{\frac{5}{2}=C_2}$$

$$\frac{150-x}{60-x}=\frac{5}{2}e^{90kt}$$

$$x(5)=10 \quad \to \quad \frac{14}{5}=\frac{5}{2}e^{450k}$$

$$e^{450k}=\frac{28}{25}$$

$$k=\frac{1}{450}\ln\frac{28}{25}\ \boxed{\approx 2.518\times 10^{-4}}$$

$$\frac{150-x}{60-x}=\frac{5}{2}e^{0.02267t}$$

$$250-2x=300e^{0.02267t}-5xe^{0.02267t}$$

$$\left(5e^{0.02267t}-2\right)x=300e^{0.02267t}-250$$

$$\boxed{x(t)=\frac{300e^{0.02267t}-250}{5e^{0.02267t}-2}}$$

$$x(20)=\frac{300e^{0.02267(20)}-250}{5e^{0.02267(20)}-2}$$

$$\boxed{\approx 29.3\ g}$$

b) $x(t)=\dfrac{300-250e^{-0.02267t}}{5-2e^{-0.02267t}}$

$$\lim_{t\to\infty}x(t)=\lim_{t\to\infty}\frac{300-250e^{-0.02267t}}{5-2e^{-0.02267t}}$$

$$=\frac{300}{5}$$

$$\boxed{=60\ g}$$

c) $a=\frac{2}{3}x=\frac{2}{3}60\ \boxed{=40}$

For chemical A: $40-40\ \boxed{=0\ g}$

$b=\frac{1}{3}60\ \boxed{=20}$

For chemical B: $50-20\ \boxed{=30\ g}$

$$2b+b=3b=x \ \to\ \boxed{b=\frac{1}{3}x} \quad\Rightarrow\quad \boxed{a=\frac{2}{3}x}$$

$$\frac{dx}{dt}=k\left(100-\frac{2}{3}x\right)\left(50-\frac{1}{3}x\right)$$

$$= \frac{k}{3}(300-2x)(150-x) \qquad k = \frac{2k}{3} \quad (constant)$$

$$= k(150-x)^2$$

$$\int \frac{dx}{(150-x)^2} = \int k\ dt$$

$$\frac{1}{150-x} = kt + C_3$$

$$x(0) = 0 \quad \rightarrow \quad \boxed{\frac{1}{150} = C_3}$$

$$\frac{1}{150-x} = kt + \frac{1}{150}$$

$$x(5) = 10 \quad \rightarrow \quad \frac{1}{140} = 5k + \frac{1}{150}$$

$$5k = \frac{1}{2100} \rightarrow \boxed{k = \frac{1}{10500}}$$

$$\frac{1}{150-x} = \frac{1}{10500}t + \frac{1}{150} = \frac{t+70}{10500}$$

$$150 - x = \frac{10500}{t+70}$$

$$\boxed{x(t) = 150 - \frac{10500}{t+70}}$$

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty}\left(150 - \frac{10500}{t+70}\right)$$

$$\boxed{=150}$$

When the chemical C is half-formed, that implies to $x(t) = 75$

$$x(t) = 150 - \frac{10500}{t+70} = 75$$

$$\frac{10500}{t+70} = 75$$

$$t + 70 = \frac{10500}{75} = 140$$

$$t = 140 - 70$$

$$\boxed{= 70\ \ min}$$

Exercise

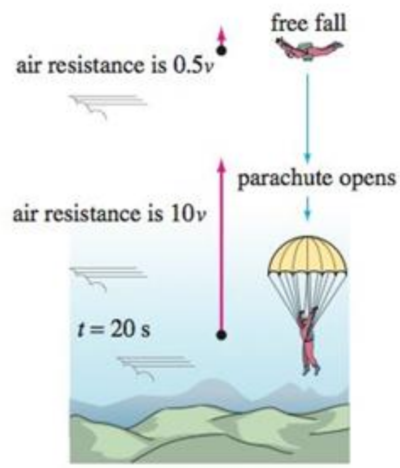
A skydiver weighs 125 *pounds*, and her parachute and equipment combined weigh another 35 *pounds*. After exiting from a plane at an altitude of 15,000 *feet*, she waits 15 *seconds* and opens her parachute. Assume that the constant of proportionality has the value $k=0.5$ during free fall and $k=10$ after the parachute is opened.
Assume that her initial velocity on leaving the plane is *zero*.

a) What is her velocity and how far has she traveled 20 *seconds* after leaving the plane?
b) How does her velocity at 20 *seconds* compare with her terminal velocity?
c) How long does it take her to reach the ground?

Solution

a) Assume that the air resistance is proportional to velocity and the positive direction is downward.

$$m\frac{dv}{dt}=mg-kv \qquad s(0)=0$$



$$\int \frac{m}{mg-kv}dv=\int dt$$

$$-\frac{m}{k}\ln(mg-kv)=t+C_1$$

$$\ln(mg-kv)=-\frac{k}{m}t+C_2$$

$$mg-kv=e^{-\frac{k}{m}t+C_2}$$

$$kv=mg-C_3e^{-\frac{k}{m}t}$$

$$v(t)=\frac{mg}{k}+Ce^{-\frac{k}{m}t}$$

$$v(0)=0 \quad \Rightarrow 0=\frac{mg}{k}+C \quad \rightarrow \quad \underline{C=-\frac{mg}{k}}$$

$$\underline{v(t)=\frac{mg}{k}\left(1-e^{-\frac{k}{m}t}\right)}$$

Given: $k=0.5$, $g=32$, and $m=\frac{w}{g}=\frac{125+35}{32}=\frac{160}{32}=5$

$$v(t)=\frac{160}{0.5}\left(1-e^{-\frac{0.5}{5}t}\right)$$

$$\underline{=320\left(1-e^{-0.1t}\right)}$$

$$s(t)=\int 320\left(1-e^{-0.1t}\right)dt$$

$$\underline{=320\left(t+10e^{-0.1t}\right)+C}$$

$$s(0)=0 \quad \Rightarrow 0=3200+C \quad \rightarrow \quad \underline{C=-3,200}$$

$$\underline{s(t)=320t+3200e^{-0.1t}-3200}$$

When the parachute opens at $t = 15$ sec

$$v(15) = 320\left(1 - e^{-1.5}\right)$$

$$\boxed{\approx 248.598 \ \ ft/s^2}$$

$$s(15) = 320(15) + 3200e^{-1.5} - 3200$$

$$\boxed{\approx 2314.02 \ ft}$$

With the parachute opens at $t = 15$ sec then $t = 0$ (reset) with the given information $k = 10$, $g = 32$, and $m = 5$

$$v(t) = \frac{mg}{k} + Ce^{-\frac{k}{m}t} = 16 + Ce^{-2t}$$

$$v(0) = 248.598 \quad \Rightarrow 248.598 = 16 + C$$

$$\rightarrow \ \boxed{C = 232.598}$$

$$\boxed{v_P(t) = 16 + 232.598e^{-2t}}$$

$$s(t) = \int \left(16 + 232.598e^{-2t}\right) dt$$

$$= 16t - 116.299e^{-2t} + C$$

$$s(0) = 0 \quad \Rightarrow 0 = -116.299 + C$$

$$\rightarrow \ \boxed{C = 116.299}$$

$$\boxed{s_P(t) = 16t - 116.299e^{-2t} + 116.299}$$

After the parachute opens at $t = 5$ sec

$$v_P(5) = 16 + 232.598e^{-10}$$

$$\boxed{\approx 16.011 \ \ ft/s^2}$$

$$s_P(5) = 16(5) - 116.299e^{-10} + 116.299$$

$$\boxed{\approx 196.29 \ ft}$$

After leaving the plane, at $t = 20$ sec

$$v(20) \ \boxed{\approx 16.011 \ \ ft/s^2}$$

$$s(20) = 2314.02 + 196.29$$

$$\boxed{\approx 2510.31 \ ft}$$

b) $$\lim_{t \to \infty} v(t) = \lim_{t \to \infty} \left(16 + 232.598e^{-2t}\right)$$

$$\boxed{= 16 \ \ ft/s^2}$$

She has very nearly reached her terminal velocity in 5 *seconds* after the parachute opens.

c) She exited the plane at an altitude of 15,000 *feet.*
When she open the parachute the distance to the ground is

$$15,000 - s(15) = 15,000 - 2314.02 \ \boxed{\approx 12,686 \ \ ft}$$

$$s_P(t) = 16t - 116.299e^{-2t} + 116.299 = 12,686$$

$$16t - 116.299e^{-2t} - 12,569.701 = 0$$

$$\boxed{t \approx 785.606 \ \ \text{sec}}$$

The total time:

$$t = 15 + 785.61 \ \approx 800.61 \, sec$$

$$\boxed{\approx 13.34 \ \ min}$$

Exercise

A tank in the form of a right-circular cylinder standing on end is leaking water through a circular hole in its bottom. When friction and contraction of water at the hole are ignored, the height *h* of water in the tank is described by

$$\frac{dh}{dt} = -\frac{A_h}{A_w}\sqrt{2gh}$$

Where A_w and A_h are the cross-sectional areas of the water and the hole, respectively.

a) Find $h(t)$ if the initial height of the water is *H*.

b) Sketch the graph $h(t)$ and give the interval *I* of definition in terms of the symbols A_w, A_h, and *H*.

$\left(g = 32 \ \ ft/s^2\right)$

c) Suppose the tank is 10 *feet* high and has radius 2 *feet* and the circular hole has radius $\frac{1}{2}$ *inch*. If the tank is initially full, how long will it take to empty?

Solution

a) $\dfrac{dh}{dt} = -\dfrac{A_h}{A_w}\sqrt{2gh} = -8\dfrac{A_h}{A_w}\sqrt{h}$

$$\int h^{-1/2}dh = -8\frac{A_h}{A_w}\int dt$$

$$2\sqrt{h} = -8\frac{A_h}{A_w}t + C$$

$$h(0) = A \ \ \rightarrow \ \boxed{2\sqrt{H} = C}$$

$$2\sqrt{h} = -8\frac{A_h}{A_w}t + 2\sqrt{H}$$

$$\boxed{h(t) = \left(\sqrt{H} - 4\frac{A_h}{A_w}t\right)^2}$$

b) $\sqrt{h} = \sqrt{H} - 4\frac{A_h}{A_w}t \ \geq 0$

$$\sqrt{H} \geq 4\frac{A_h}{A_w}t$$

$$\boxed{0 \leq t \leq \frac{A_w\sqrt{H}}{4A_h}}$$

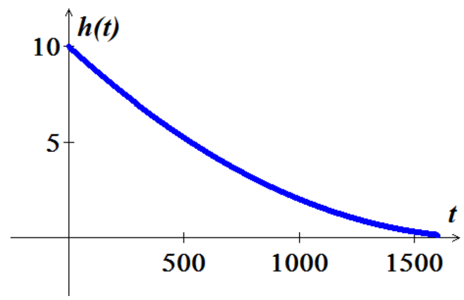
c) ***Given***: $H = 10\ ft,\quad R = 2\ ft,\quad r_h = \frac{1}{2}\frac{1}{12} = \frac{1}{24}\ ft$

$$A_w = \pi R^2 = 4\pi$$

$$A_h = \pi r_h^2 = \frac{\pi}{576}$$

$$h(t) = \left(\sqrt{10} - 4\frac{\pi}{576}\left(\frac{1}{4\pi}\right)t\right)^2$$

$$\boxed{= \left(\sqrt{10} - \frac{1}{576}t\right)^2}$$



$$h(t) = \left(\sqrt{10} - \frac{1}{576}t\right)^2 = 0$$

$$\sqrt{10} - \frac{1}{576}t = 0$$

$$\boxed{t = 576\sqrt{10}\ sec} \quad \approx 1821.47\ sec \approx 30.36\ min$$

Exercise

A tank in the form of a right-circular cylinder cone standing on end, vertex down, is leaking water through a circular hole in its bottom.

a) Suppose the tank is 20 *feet* high and has radius 8 *inches*. Show that the differential equation governing the height h of water leaking from a tank is

$$\frac{dh}{dt} = -\frac{5}{6h^{3/2}}$$

In this model, friction and contraction of the water at the hole were taken into account with $c = 0.6$ and $g = 32\ ft/s^2$. If the tank is initially full, how long will it take the tank to empty?

b) Suppose the tank has a vertex angle of 60° and the circular hole has radius 2 *inches*. Determine the differential equation governing the height h of water. Use $c = 0.6$ and $g = 32\ ft/s^2$.

c) If the height of the water is initially 9 *feet*, how long will it take the tank to empty?

Solution

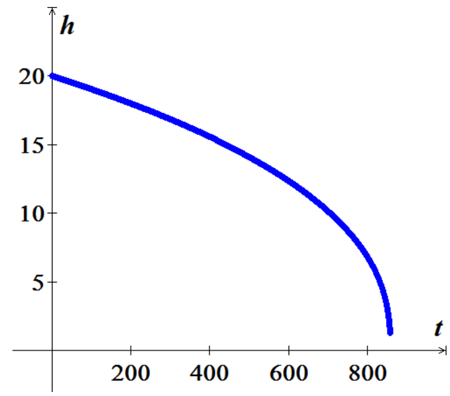
a) $\int 6h^{3/2}dh = -\int 5dt$

$\frac{12}{5}h^{5/2} = -5t + C$

$h^{5/2} = -\frac{25}{12}t + C$

$h(0) = 20 \quad \rightarrow 20^{5/2} = C \Rightarrow \underline{C = 800\sqrt{5}}$

$\underline{h(t) = \left(-\frac{25}{12}t + 800\sqrt{5}\right)^{2/5}}$



$h(t) = \left(-\frac{25}{12}t + 800\sqrt{5}\right)^{2/5} = 0$

$-\frac{25}{12}t + 800\sqrt{5} = 0$

$t = \frac{9600\sqrt{5}}{25} \underline{= 384\sqrt{5} \ \ sec} \quad \underline{\approx 858.65 \ \ sec}$

$\underline{t \approx 14.61 \ \ min}$

b) $\tan 30° = \frac{r}{h} \quad \rightarrow \ r = h\tan 30°$

$r = \frac{h}{\sqrt{3}}$

$A_w = \pi r^2 = \frac{1}{3}\pi h^2$

$\frac{dh}{dt} = -c\frac{A_h}{A_w}\sqrt{2gh}$

$= -(0.6)\pi\left(\frac{2}{12}\right)^2 \frac{3}{\pi h^2}\sqrt{64h}$

$= -\frac{2}{5h^{3/2}}$

$\int h^{3/2}dh = -\frac{2}{5}\int dt$

$\frac{2}{5}h^{5/2} = -\frac{2}{5}t + C_2$

$h^{5/2} = -t + C_2$

$h(0) = 9 \quad \rightarrow 9^{5/2} = C \Rightarrow \underline{C = 243}$

$\underline{h(t) = (-t + 243)^{2/5}}$

$h(t) = (-t + 243)^{2/5} = 0$

$\underline{t = 243 \ \ sec = 4.05 \ min}$

Exercise

Suppose that the conical tank is inverted and that water leaks out a circular hole of radius 2 *inches* in the center of its circular base. Is the time it takes to empty a full tank the same as for the tank with vertex down? Take the friction/.contraction coefficient to be $c = 0.6$ and $g = 32\ ft/s^2$

Solution

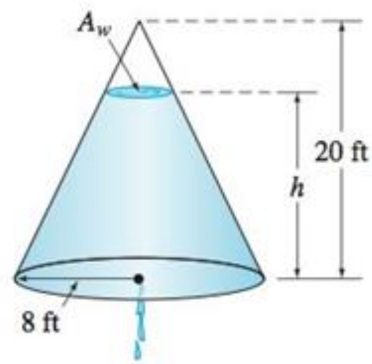
$$\frac{r}{8} = \frac{20-h}{20}$$

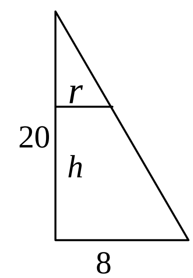
$$r = \frac{2}{5}(20-h)$$

$$A_w = \pi r^2 = \frac{4\pi}{25}(20-h)^2$$

$$\frac{dh}{dt} = -(0.6)(\pi)\left(\frac{2}{12}\right)^2 \frac{25}{4\pi(20-h)^2}\sqrt{64h} \qquad \frac{dh}{dt} = -c\frac{A_h}{A_w}\sqrt{2gh}$$

$$= -\frac{5}{6}\frac{\sqrt{h}}{(20-h)^2}$$





$$\int \frac{400-40h+h^2}{h^{1/2}}dh = -\frac{5}{6}\int dt$$

$$\int\left(400h^{-1/2} - 40h^{1/2} + h^{3/2}\right)dh = -\frac{5}{6}\int dt$$

$$800\sqrt{h} - \frac{80}{3}h^{3/2} + \frac{2}{5}h^{5/2} = -\frac{5}{6}t + C$$

$$h(0) = 20 \quad \rightarrow\ 1600\sqrt{5} - \frac{3200}{3}\sqrt{5} + \frac{1600}{5}\sqrt{5} = C$$

$$\boxed{C = \frac{2560\sqrt{5}}{3}}$$

$$800\sqrt{h} - \frac{80}{3}h^{3/2} + \frac{2}{5}h^{5/2} = -\frac{5}{6}t + \frac{2560\sqrt{5}}{3}$$

$$0 = -\frac{5}{6}t + \frac{2560\sqrt{5}}{3}$$

$$\boxed{t = 1024\sqrt{5}\ \ sec \quad \approx 38.16\ min}$$

The tank empties more slowly when the base of the cone is on the bottom.

Exercise

A differential equation for the velocity v of a falling mass m subjected to air resistance proportional to the square of the instantaneous velocity is

$$m\frac{dv}{dt} = mg - kv^2$$

Where $k > 0$ is a constant of proportionality. The positive direction is downward.

a) Solve the equation subject to the initial condition $v(0) = v_0$.

b) Use the solution in part (*a*) to determine the limiting, or terminal, velocity of the mass.

c) If the distance *s*, measured from the point where the mass was released above the ground, is related to velocity *v* by $\frac{ds}{dt}=v(t)$, find an explicit expression for $s(t)$ if $s(0)=0$

Solution

a) $\int \frac{m}{mg-kv^2}dv=\int dt \qquad \int \frac{dx}{a^2-x^2}=\frac{1}{a}\tanh^{-1}\frac{x}{a}$

$$\frac{1}{\sqrt{k}}\frac{m}{\sqrt{mg}}\tanh^{-1}\left(\frac{\sqrt{k}}{\sqrt{mg}}v\right)=t+C_1$$

$$\tanh^{-1}\left(\sqrt{\frac{k}{mg}}v\right)=\sqrt{\frac{kg}{m}}t+C$$

$$v(0)=v_0 \qquad \tanh^{-1}\left(\sqrt{\frac{k}{mg}}\,v_0\right)=C$$

$$\sqrt{\frac{k}{mg}}v=\tanh\left(\sqrt{\frac{kg}{m}}t+\tanh^{-1}\left(\sqrt{\frac{k}{mg}}\,v_0\right)\right)$$

$$\boxed{v(t)=\sqrt{\frac{mg}{k}}\,\tanh\left(\sqrt{\frac{kg}{m}}\,t+\tanh^{-1}\left(\sqrt{\frac{k}{mg}}\,v_0\right)\right)}$$

b) $\tanh t\xrightarrow[t\to\infty]{}1$

$$\lim_{t\to\infty}v(t)=\lim_{t\to\infty}\left(\sqrt{\frac{mg}{k}}\,\tanh\left(\sqrt{\frac{kg}{m}}\,t+\tanh^{-1}\left(\sqrt{\frac{k}{mg}}\,v_0\right)\right)\right)$$

$$\boxed{=\sqrt{\frac{mg}{k}}}$$

c) $s(t)=\sqrt{\frac{mg}{k}}\int\tanh\left(\sqrt{\frac{kg}{m}}\,t+\tanh^{-1}\left(\sqrt{\frac{k}{mg}}\,v_0\right)\right)dt$

$$=\sqrt{\frac{mg}{k}}\sqrt{\frac{m}{kg}}\,\ln\left[\cosh\left(t+\tanh^{-1}\left(\sqrt{\frac{k}{mg}}\,v_0\right)\right)\right]+C_2$$

$$\boxed{=\frac{m}{k}\,\ln\left[\cosh\left(t+\tanh^{-1}\left(\sqrt{\frac{k}{mg}}\,v_0\right)\right)\right]+C_2}$$

Exercise

An object is dropped from altitude y_0

a) Determine the impact velocity if the drag force is proportional to the square of velocity, with drag coefficient κ.

b) If the terminal velocity is known to –120 *mph* and the impact velocity was –90 *mph*, what was the initial altitude y_0?

Solution

a) $m\frac{dv}{dt}=-mg+\kappa v^2 \quad v\left(y_0\right)=0$

$$\frac{dv}{dt}=v\frac{dv}{dy}$$

$$mv\frac{dv}{dy}=-mg+\kappa v^2$$

$$\int\frac{mv}{\kappa v^2-mg}dv=\int dy$$

$$\frac{m}{2\kappa}\int\frac{1}{\kappa v^2-mg}d\left(\kappa v^2-mg\right)=\int dy$$

$$\frac{m}{2\kappa}\ln\left|\kappa v^2-mg\right|=y+C$$

$$v\left(y_0\right)=0 \quad\rightarrow\quad \frac{m}{2\kappa}\ln mg=y_0+C \quad\Rightarrow\quad \boxed{C=\frac{m}{2\kappa}\ln mg-y_0}$$

$$\frac{m}{2\kappa}\ln\left|\kappa v^2-mg\right|=y+\frac{m}{2\kappa}\ln mg-y_0$$

$$\ln\left|\kappa v^2-mg\right|=\frac{2\kappa}{m}y+\ln\left(mg\right)-\frac{2\kappa}{m}y_0$$

Since the object is falling: $\quad \kappa v^2-mg<0$

$$-\left(\kappa v^2-mg\right)=e^{\frac{2\kappa}{m}\left(y-y_0\right)+\ln\left(mg\right)}$$

$$\kappa v^2=-e^{\frac{2\kappa}{m}\left(y-y_0\right)}e^{\ln\left(mg\right)}+mg$$

$$v^2=\frac{mg}{\kappa}-\frac{mg}{\kappa}e^{\frac{2\kappa}{m}\left(y-y_0\right)}$$

$$\boxed{v=\sqrt{\frac{mg}{\kappa}}\sqrt{1-e^{\frac{2\kappa}{m}\left(y-y_0\right)}}}$$

b) $v_{terminal}=120\frac{mi}{hr}\frac{5280\ ft}{mi}\frac{1\ hr}{3600\ sec}=\frac{528}{3}=\boxed{176\ \ ft/s}$

$$v_{impact}=90\frac{mi}{hr}\frac{5280\ ft}{mi}\frac{1\ hr}{3600\ sec}=\frac{528}{4}=\boxed{132\ \ ft/s}$$

$$v_{terminal}=\sqrt{\frac{mg}{\kappa}}=176 \quad\rightarrow\quad \frac{32m}{\kappa}=176^2\Rightarrow\boxed{\frac{m}{\kappa}=\frac{176^2}{32}=968}$$

$$v(0)=\sqrt{\frac{mg}{\kappa}}\sqrt{1-e^{\frac{2\kappa}{m}\left(y-y_0\right)}}=132$$

$$176\sqrt{1-e^{-1936y_0}}=132$$

$$\sqrt{1-e^{-1936y_0}}=\frac{132}{176}=\frac{3}{4}$$

$$1-e^{-1936y_0}=\frac{9}{16}$$

$$e^{-1936 y_0} = \frac{7}{16}$$

$$-1936 y_0 = \ln\frac{7}{16}$$

$$y_0 = -\frac{1}{1936}\ln\frac{7}{16}$$

$$\boxed{\approx 400.11 \ ft}$$

Exercise

An object is dropped from altitude y_0

a) Assume that the drag force is proportional to the velocity, with drag coefficient κ. Obtain an implicit solution relating velocity and altitude.

b) If the terminal velocity is known to -120 *mph* and the impact velocity was -90 *mph*, what was the initial altitude y_0?

Solution

a) $m\frac{dv}{dt} = -mg - \kappa v$

$$mv\frac{dv}{dy} = -mg - \kappa v \qquad \frac{dv}{dt} = v\frac{dv}{dy}$$

$$m\frac{v}{\kappa v + mg}dv = -dy$$

$$\frac{m}{\kappa}\frac{v}{v + \frac{mg}{\kappa}}dv = -dy$$

$$\frac{m}{\kappa}\int\left(1 - \frac{\frac{mg}{\kappa}}{v + \frac{mg}{\kappa}}\right)dv = -\int dy$$

$$\frac{m}{\kappa}\left(v - \frac{mg}{\kappa}\ln\left|v + \frac{mg}{\kappa}\right|\right) = -y + C$$

$$v\left(y_0\right) = 0 \quad \rightarrow \quad -g\left(\frac{m}{\kappa}\right)^2\ln\left(\frac{mg}{\kappa}\right) = -y_0 + C$$

$$\Rightarrow \boxed{C = y_0 - g\left(\frac{m}{\kappa}\right)^2\ln\left(\frac{mg}{\kappa}\right)}$$

$$\frac{m}{\kappa}\left(v - \frac{mg}{\kappa}\ln\left|v + \frac{mg}{\kappa}\right|\right) = -y + y_0 - g\left(\frac{m}{\kappa}\right)^2\ln\left(\frac{mg}{\kappa}\right)$$

$$y_0 - y = \frac{m}{\kappa}v - \frac{1}{g}\left(\frac{mg}{\kappa}\right)^2\ln\left|v + \frac{mg}{\kappa}\right| + \frac{1}{g}\left(\frac{mg}{\kappa}\right)^2\ln\left(\frac{mg}{\kappa}\right)$$

$$= \frac{m}{\kappa}v - \frac{1}{g}\left(\frac{mg}{\kappa}\right)^2\left(\ln\left|v + \frac{mg}{\kappa}\right| - \ln\left(\frac{mg}{\kappa}\right)\right)$$

$$\boxed{= \frac{m}{\kappa}v - \frac{1}{g}\left(\frac{mg}{\kappa}\right)^2 \ln\left|\frac{\kappa}{mg}v + 1\right|}$$

b) $v_{terminal} = -120\frac{mi}{hr}\frac{5280\ ft}{mi}\frac{1\ hr}{3600\ sec} = -\frac{528}{3}\ \underline{= -176\ \ ft/s}$

$v_{impact} = -90\frac{mi}{hr}\frac{5280\ ft}{mi}\frac{1\ hr}{3600\ sec} = -\frac{528}{4}\ \underline{= -132\ \ ft/s}$

$v_{terminal} = -\frac{mg}{\kappa} = -176 \quad \rightarrow \frac{32m}{\kappa} = 176 \Rightarrow \boxed{\frac{m}{\kappa}} = \frac{176}{32}\ \underline{= 5.5}$

$$y_0 = 5.5v_{imp} - \frac{1}{32}(176)^2 \ln\left|\frac{1}{176}v_{imp} + 1\right|$$

$$= 5.5(-132) - \frac{1}{32}(176)^2 \ln\left|-\frac{132}{176} + 1\right|$$

$$\underline{\approx 615.93\ \ ft}$$

Exercise

An object of mass 3 *kg* is released from rest 500 *m* above the ground and allowed to fall under the influence of gravity. Assume the gravitational force constant, with $g = 9.81\ m/s^2$, and the force due to air resistance is proportional to the velocity of the object with proportionality constant $\kappa = 3\ N\text{-}sec/m$. Determine when the object will hit the ground.

Solution

By Newton second law: $\quad m\frac{dv}{dt} = mg - \kappa v$

$$m\frac{dv}{\kappa v - mg} = -dt$$

$$\frac{m}{\kappa}\int \frac{d(\kappa v - mg)}{\kappa v - mg} = -\int dt$$

$$\frac{m}{\kappa}\ln|\kappa v - mg| = -t + C_1$$

$$\ln(\kappa v - mg) = -\frac{\kappa}{m}t + C_2$$

$$\kappa v - mg = Ce^{-\kappa t/m}$$

$$v(t) = \frac{C}{\kappa}e^{-\kappa t/m} + \frac{mg}{\kappa}$$

$$v(0) = 0 \quad \rightarrow \frac{C}{\kappa} + \frac{mg}{\kappa} = 0$$

$$\Rightarrow\ \underline{C = -mg}$$

Given: $\quad m = 3, \quad g = 9.81, \quad \kappa = 3$

$$v(t) = -\frac{mg}{\kappa}e^{-\kappa t/m} + \frac{mg}{\kappa}$$

$$= \frac{3(9.81)}{3}\left(1-e^{-t}\right)$$

$$\boxed{= 9.81\left(1-e^{-t}\right)}$$

$$x(t) = 9.81\int\left(1-e^{-t}\right)dt$$

$$= 9.81t + 9.81e^{-t} + C_3$$

$$x(0) = 0 \quad \rightarrow \quad \boxed{C_3 = -9.81}$$

$$x(t) = 9.81e^{-t} + 9.81t - 9.81 = 500$$

$$t + e^{-t} = \frac{500+9.81}{9.81} = 51.97$$

$$\boxed{t = 51.97\ sec} \qquad \cancel{t = -4.02}$$

Exercise

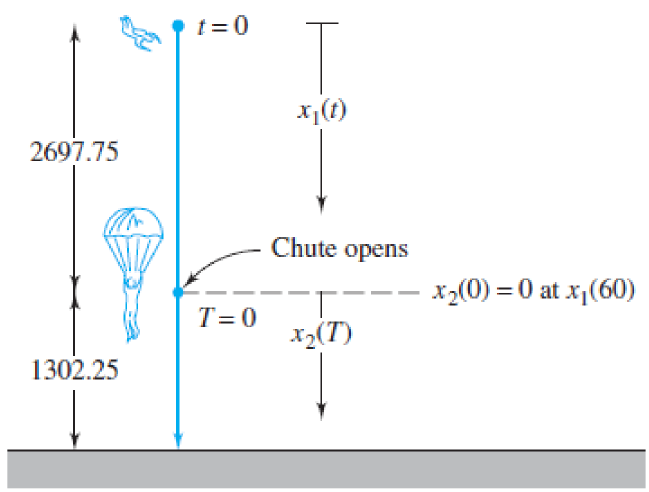
A parachutist whose mass is 75 *kg* drops from helicopter hovering 4000 *m* above the ground and falls toward the earth under the influence of gravity. Assume the gravitational force is constant. Assume also that the force due to air resistance is proportional to the velocity of the parachutist, with the proportionality constant $\kappa_1 = 15$ *N-sec/m* when the chute is closed and with constant $\kappa_2 = 105$ *N-sec/m* when the chute is open. If the chute does not open until 1 *min* after the parachutist leaves the helicopter, after how many *seconds* will he hit the ground?

Solution

Given: $m = 75, \quad x(t) = 4000, \quad g = 9.81, \quad \kappa_1 = 15, \quad v(0) = 0$

Before the chute is open

By Newton second law: $m\frac{dv}{dt} = mg - \kappa v$



$$m\frac{dv}{\kappa v - mg} = -dt$$

$$\frac{m}{\kappa}\int\frac{d(\kappa v - mg)}{\kappa v - mg} = -\int dt$$

$$\frac{m}{\kappa}\ln|\kappa v - mg| = -t + C_1$$

$$\kappa v - mg = Ce^{-\kappa t/m}$$

$$v_1(t) = \frac{C}{\kappa_1}e^{-\kappa_1 t/m} + \frac{mg}{\kappa_1}$$

$$v(0) = 0 \quad \rightarrow \frac{C}{\kappa} + \frac{mg}{\kappa} = 0 \;\Rightarrow\; \boxed{C = -mg}$$

$$v_1(t) = \frac{75(9.81)}{15}\left(1 - e^{-t/5}\right)$$

$$= \boxed{49.05\left(1 - e^{-t/5}\right)}$$

$$v_1(t = 1min = 60s) = 49.05\left(1 - e^{-60/5}\right)$$

$$\boxed{\approx 49.05\ m/s}$$

$$x_1(t) = 49.05\int\left(1 - e^{-t/5}\right)dt$$

$$= 49.05t + 245.25e^{-t/5} + C_2$$

$$x(0) = 0 \quad \to \boxed{C_2 = -245.25}$$

$$x_1(t = 60) = 49.05(60) + 245.25e^{-12} - 245.25$$

$$\boxed{\approx 2697.75\ \ m}$$

When the chute is opened, he is $4000 - 2697.5 \approx 1302.25\ m$ above the ground.

Given: $m = 75$, $x_2(T) = 1302.25$, $g = 9.81$, $\kappa_2 = 105$ $v(0) = 49.05$

$$v_2(t) = \frac{C}{\kappa_2}e^{-\kappa_2 t/m} + \frac{mg}{\kappa_2}$$

$$= \frac{C}{105}e^{-1.4t} + \frac{75(9.81)}{105}$$

$$v(0) = 49.05 \quad \to \frac{C}{105} + 7.007 = 49.05$$

$$\Rightarrow \boxed{C = 4,414.5}$$

$$v_2(t) = 42.043e^{-1.4t} + 7.007$$

$$x_2(t) = \int\left(42.043e^{-1.4t} + 7.01\right)dt$$

$$= -30.03e^{-1.4t} + 7.01t + C_4$$

$$x(0) = 0 \quad \to \boxed{C_4 = 30.03}$$

$$x_2(t) = -30.03e^{-1.4t} + 7.01t + 30.03 = 1302.25$$

$$-30.03e^{-1.4t} + 7.01t - 1272.22 = 0$$

$$\boxed{t = 181.49\ sec}$$

Exercise

A parachutist whose mass is 75 *kg* drops from helicopter hovering 2000 *m* above the ground and falls toward the earth under the influence of gravity. Assume the gravitational force is constant. Assume also that the force due to air resistance is proportional to the velocity of the parachutist, with the proportionality constant $\kappa_1 = 30$ *N-sec/m* when the chute is closed and with constant $\kappa_2 = 90$ *N-sec/m* when the chute is open. If the chute does not open until the velocity of the parachutist reaches 20 *m/sec*, after how many seconds will he reach the ground?

Solution

Given: $m = 75, \quad x(t) = 2000, \quad g = 9.8, \quad \kappa_1 = 30, \quad v(0) = 0, \quad v(t_1) = 20$

Before the chute is open

By Newton's second law: $m\dfrac{dv}{dt} = mg - \kappa_1 v$

$$\frac{dv}{dt} = 9.8 - 0.4v$$

$$\int \frac{dv}{0.4v - 9.8} = -\int dt$$

$$\frac{1}{0.4}\ln|0.4v - 9.8| = -t + C_1$$

$$0.4v - 9.8 = C_2 e^{-0.4t}$$

$$v_1(t) = 2.5C_2 e^{-0.4t} + 24.5$$

$$v(0) = 0 \quad \to 2.5C_2 + 24.5 = 0$$

$$\Rightarrow \ \underline{C_2 = -9.8|}$$

$$v_1(t) = -24.5e^{-0.4t} + 24.5 = 20$$

$$e^{-0.4t} = \frac{4.5}{24.5} \quad \to \quad \underline{\left|t = -\frac{1}{0.4}\ln\frac{4.5}{24.5} \approx 4.24\right|}$$

$$x_1(t) = \int\left(-24.5e^{-0.4t} + 24.5\right)dt$$

$$= 61.25e^{-0.4t} + 24.5t + C_3$$

$$x(0) = 0 \quad \to \quad \underline{C_3 = -61.25|}$$

$$x_1(t) = 61.25e^{-0.4t} + 24.5t - 61.25$$

$$x_1(4.24) = 61.25e^{-0.4(4.24)} + 24.5(4.24) - 61.25$$

$$\underline{\approx 53.86\ m\ |}$$

When the chute is opened, he is $2000 - 53.86 \approx 1946.14$ *m* above the ground.

Given: $m = 75, \quad x_2(T) = 1946.14, \quad g = 9.81, \quad \kappa_2 = 90 \quad v(0) = 20$

$$v_2(t) = \frac{1}{\kappa_2} C_4 e^{-\kappa_2 t/m} + \frac{mg}{\kappa_2}$$

$$= \frac{1}{90} C_4 e^{-1.2t} + 8.1667$$

$$v(0) = 20 \quad \to \frac{1}{90} C_4 + 8.1667 = 20$$

$$\Rightarrow \boxed{C_4 = 1064.997}$$

$$v_2(t) = 11.833 e^{-1.2t} + 8.1667$$

$$x_2(t) = \int \left(11.833 e^{-1.2t} + 8.1667\right) dt$$

$$= -9.86 e^{-1.2t} + 8.1667t + C_5$$

$$x(0) = 0 \quad \to \boxed{C_4 = 9.86}$$

$$x_2(t) = -9.86 e^{-1.2t} + 8.1667t + 9.86 = 1946.14$$

$$-9.86 e^{-1.2t} + 8.1667t - 1936.28 = 0$$

$$\boxed{t = 237 \ sec}$$

Exercise

An object of mass 5 *kg* is released from rest 1000 *m* above the ground and allowed to fall under the influence of gravity. Assume the gravitational force constant, with $g = 9.8 \ m/s^2$, and the force due to air resistance is proportional to the velocity of the object with proportionality constant $\kappa = 50 \ N\text{-}sec/m$. Determine when the object will hit the ground.

Solution

By Newton's second law: $m\frac{dv}{dt} = mg - \kappa v$

$$m \frac{dv}{\kappa v - mg} = -dt$$

$$\frac{m}{\kappa} \int \frac{d(\kappa v - mg)}{\kappa v - mg} = -\int dt$$

$$\frac{m}{\kappa} \ln|\kappa v - mg| = -t + C_1$$

$$\kappa v - mg = Ce^{-\kappa t/m}$$

Given: $m = 5, \quad g = 9.8, \quad \kappa = 50, \quad v(0) = 0, \quad x(0) = 0$

$$v(t) = \frac{C}{50} e^{-\kappa t/m} + \frac{5(9.8)}{50}$$

$$= \frac{C}{50} e^{-10t} + 0.98$$

$$v(0) = 0 \quad \to \frac{C}{50} + 0.98 = 0$$

$$\Rightarrow \boxed{C = -49}$$

$$\boxed{v(t) = -0.98e^{-10t} + 0.98}$$

$$x(t) = 0.98\int\left(1 - e^{-10t}\right)dt$$

$$= 0.98t + 0.098e^{-10t} + C_2$$

$$x(0) = 0 \quad \rightarrow \boxed{C_2 = -0.098}$$

$$x(t) = 0.98t + 0.098e^{-t} - 0.098 = 1000$$

$$0.98t + 0.098e^{-t} - 1000.098 = 0$$

$$\boxed{t = 1020\ sec}$$

Exercise

An object of mass 500 *kg* is released from rest 1000 *m* above the ground and allowed to fall under the influence of gravity. Assume the gravitational force constant, with $g = 9.8\ m/s^2$, and the force due to air resistance is proportional to the velocity of the object with proportionality constant $\kappa = 50\ N\text{-}sec/m$. Determine when the object will hit the ground.

Solution

Given: $m = 500, \quad g = 9.8, \quad \kappa = 50, \quad v(0) = 0, \quad x(0) = 0$

By Newton's second law: $m\dfrac{dv}{dt} = mg - \kappa v$

$$500\frac{dv}{dt} = 4{,}900 - 50v$$

$$\frac{dv}{dt} = 9.8 - 0.1v$$

$$-\int\frac{dv}{0.1v - 9.8} = \int dt$$

$$10\ln|0.1v - 9.8| = -t + C_1$$

$$0.1v - 9.8 = C_2e^{-0.1t}$$

$$v(t) = 10C_2e^{-0.1t} + 98$$

$$v(0) = 0 \quad \rightarrow 10C_2 + 98 = 0 \Rightarrow \boxed{C_2 = -9.8}$$

$$\boxed{v(t) = 98\left(1 - e^{-0.1t}\right)}$$

$$x(t) = 98\int\left(1 - e^{-0.1t}\right)dt$$

$$= 98t + 980e^{-0.1t} + C_3$$

$$x(0) = 0 \quad \rightarrow \quad \boxed{C_3 = -980}$$

$$x(t) = 98t + 980e^{-0.1t} - 980 = 1000$$

$$98t + 980e^{-0.1t} - 1980 = 0$$

$$\boxed{t = 18.7\ sec}$$

Exercise

A 400-*lb* object is released from rest 500 *ft* above the ground and allowed to fall under the influence of gravity. Assuming that the force in pounds due to air resistance is $-10v$, where v is the velocity of the object in *ft/s*, determine the equation of motion of the object. When will the object hit the ground?

Solution

Given: $g = 32, \quad m = \frac{400}{32} = 12.5, \quad -10v \quad v(0) = 0, \quad x(0) = 0$

By Newton second law: $m\frac{dv}{dt} = mg - 10v$

$$12.5\frac{dv}{dt} = 400 - 10v$$

$$\frac{dv}{dt} = 32 - 0.8v$$

$$-\int \frac{dv}{0.8v - 32} = \int dt$$

$$\frac{1}{0.8}\ln|0.8v - 32| = -t + C_1$$

$$\ln|0.8v - 32| = -0.8t + C_1$$

$$0.8v - 32 = C_1 e^{-0.8t}$$

$$v(t) = 40 - \frac{C_2}{0.8}e^{-0.8t}$$

$$v(0) = 0 \quad \rightarrow \quad -\frac{1}{0.8}C_2 + 40 = 0$$

$$\Rightarrow \quad \boxed{C_2 = 32}$$

$$\boxed{v(t) = 40\left(1 - e^{-0.8t}\right)}$$

$$x(t) = 40\int \left(1 - e^{-0.8t}\right)dt$$

$$= 40t + 50e^{-0.8t} + C_3$$

$$x(0) = 0 \quad \rightarrow \quad 50 + C_3 = 0$$

$$\Rightarrow \boxed{C_3 = -50}$$

$$x(t) = 40t + 50e^{-0.8t} - 50 = 500$$

$$4t + 5e^{-0.8t} - 55 = 0 \quad \rightarrow \quad 4t - 55 = 0$$

$$\boxed{t = 13.75 \ sec}$$

Exercise

An object of mass 8 *kg* is given an upward initial velocity of 20 *m/sec* and then allowed to fall under the influence of gravity. Assume that the force in Newton due to air resistance is $-16v$, where v is the velocity of the object in *m/sec*.

a) Determine the equation of motion of the object.

b) If the object is initially 100 m above the ground, determine when the object will hit the ground.

Solution

Given: $m = 8, \quad g = 9.8, \quad \kappa = 16, \quad v(0) = -20, \quad x(0) = 0$

a) By Newton's second law: $m\frac{dv}{dt} = mg - 16v$

$$\frac{dv}{dt} = 9.8 - 2v$$

$$-\int \frac{dv}{2v - 9.8} = \int dt$$

$$\frac{1}{2}\ln|2v - 9.8| = -t + C_1$$

$$2v - 9.8 = C_2 e^{-2t}$$

$$v(t) = \frac{1}{2}C_2 e^{-2t} + 4.9$$

$$v(0) = -20 \quad \rightarrow \frac{1}{2}C_2 + 4.9 = -20$$

$$\Rightarrow \boxed{C_2 = -49.8}$$

$$\boxed{v(t) = -24.9e^{-2t} + 4.9}$$

$$x(t) = \int \left(-24.9e^{-2t} + 4.9\right)dt$$

$$= 4.9t + 12.45e^{-2t} + C_3$$

$$x(0) = 0 \quad \rightarrow \ 12.45 + C_3 = 0$$

$$\Rightarrow \boxed{C_3 = -12.45}$$

$$\boxed{x(t) = 4.9t + 12.45e^{-2t} - 12.45}$$

b) $x(t) = 4.9t + 12.45e^{-2t} - 12.45 = 100$

$4.9t + 12.45e^{-2t} - 112.45 = 0$

$\underline{t \approx 22.9 \ \ sec}$

Exercise

An object of mass 5 *kg* is given an downward initial velocity of 50 *m/sec* and then allowed to fall under the influence of gravity. Assume that the force in Newton due to air resistance is $-10v$, where v is the velocity of the object in *m/sec*.

a) Determine the equation of motion of the object.

b) If the object is initially 100 *m* above the ground, determine when the object will hit the ground.

Solution

Given: $m = 5, \quad g = 9.8, \quad \kappa = 10, \quad v(0) = 50, \quad x(0) = 0$

a) By Newton's second law: $m\dfrac{dv}{dt} = mg - 10v$

$$\frac{dv}{dt} = 9.8 - 2v$$

$$-\int \frac{dv}{2v - 9.8} = \int dt$$

$$\frac{1}{2}\ln|2v - 9.8| = -t + C_1$$

$$2v - 9.8 = C_2 e^{-2t}$$

$$v(t) = \frac{1}{2}C_2 e^{-2t} + 4.9$$

$$v(0) = 50 \quad \to \frac{1}{2}C_2 + 4.9 = 50$$

$$\Rightarrow \underline{C_2 = 90.2}$$

$$\underline{v(t) = 45.1e^{-2t} + 4.9}$$

$$x(t) = \int \left(45.1e^{-2t} + 4.9\right)dt$$

$$= 4.9t - 22.55e^{-2t} + C_3$$

$$x(0) = 0 \quad \to \quad \underline{C_3 = 22.55}$$

$$\underline{x(t) = 4.9t - 22.55e^{-2t} + 22.55}$$

b) $x(t) = 4.9t - 22.55e^{-2t} + 22.55 = 100$

$4.9t - 22.55e^{-2t} - 77.45 = 0$

$\underline{t \approx 15.81 \ \ sec}$

Exercise

A shell of mass 2 *kg* is shot upward with an initial velocity of $200\ m/sec$. The magnitude of the force on the shell due to air resistance is $\frac{|v|}{20}$.

a) When will the shell reach its maximum height above the ground?

b) What is the maximum height?

Solution

Given: $m = 2, \quad g = 9.81, \quad \frac{|v|}{20}, \quad v(0) = 200, \quad x(0) = 0$

a) $m\frac{dv}{dt} = -mg - \frac{v}{20}$

$$\frac{dv}{dt} = -9.8 - \frac{v}{40}$$

$$\int \frac{dv}{v + 392.4} = -\frac{1}{40}\int dt$$

$$\ln|v + 392.4| = -\frac{1}{40}t + C_1$$

$$v(t) = C_2 e^{-t/40} - 392.4$$

$$v(0) = 200 \quad \rightarrow \quad 200 = C_2 - 392.4$$

$$\Rightarrow \boxed{C_2 = 592.4}$$

$$v(t) = 592.4e^{-t/40} - 392.4$$

At maximum height, $v(t) = 0$

$$v(t) = 592.4e^{-t/40} - 392.4 = 0$$

$$t = -40\ln\frac{392.4}{592.4}$$

$$\boxed{\approx 16.476\ \ sec}$$

b) $x(t) = \int \left(592.4e^{-t/40} - 392.4\right)dt$

$$= -23,696e^{-t/40} - 392.4t + C_3$$

$$x(0) = 0 \quad \rightarrow \quad 23,696 = C_3$$

$$x(t) = -23,696e^{-t/40} - 392.4t + 23,696$$

$$x(t = 16.476) = -23,696e^{-16.476/40} - 392.4(16.476) + 23,696$$

$$\boxed{\approx 1534.81\ \ m}$$

Exercise

We need to design a ballistics chamber to decelerate test projectiles fired into it. Assume the resistive force encountered by the projectile is proportional to the square of its velocity and neglect gravity.

The chamber is to be constructed so that the coefficient κ associated with this resistive force is not constant but is, in fact, a linearly increasing function of distance into the chamber:

Let $\kappa(x) = \kappa_0 x$, where κ_0 is a constant; the resistive force then has the form $\kappa(x)v^2 = \kappa_0 x v^2$.

If we use time *t* as the independent variable, Newton's law of motion leads us to the differential equation

$$m\frac{dv}{dt} + \kappa_0 x v^2 = 0 \quad with \quad v = \frac{dx}{dt}$$

a) Adopt distance *x* into the chamber as the new independent variable and rewrite the given differential equation as a first order equation in terms of the new independent variable.

b) Determine the value κ_0 needed if the chamber is to reduce projectile velocity to 1% of its incoming value within *d* units of distance.

Solution

a) $m\frac{dv}{dt} + \kappa_0 x v^2 = 0 \quad with \quad v = \frac{dx}{dt}$

$$m\frac{dv}{dx}\frac{dx}{dt} + \kappa_0 x v^2 = 0 \qquad v = \frac{dx}{dt}$$

$$\boxed{mv\frac{dv}{dx} + \kappa_0 x v^2 = 0} \quad \text{when } x = 0$$

b) $\frac{dv}{dx} + \frac{\kappa_0}{m} x v = 0$

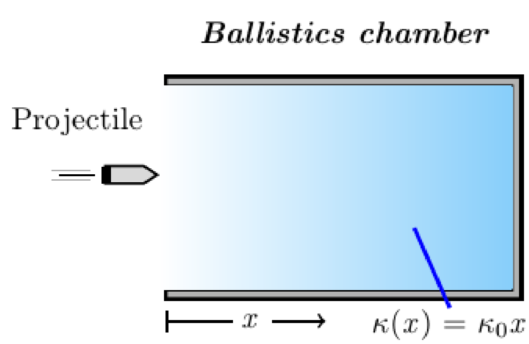
$$\frac{dv}{dx} + \frac{\kappa_0}{m} x v = 0$$

$$e^{\int \frac{\kappa_0}{m} x dx} = e^{\frac{\kappa_0}{2m}x^2}$$

$$v(x) = Ce^{-\frac{\kappa_0}{2m}x^2}$$

$$v(0) = v_0 \quad \to \quad v_0 = C$$

$$\boxed{v(x) = v_0 e^{-\frac{\kappa_0}{2m}x^2}}$$



Let $v = 0.01v_0 \quad and \quad x = d$

$$0.01v_0 = v_0 e^{-\frac{\kappa_0}{2m}d^2}$$

$$e^{-\frac{\kappa_0}{2m}d^2} = 0.01$$

$$-\frac{\kappa_0}{2m}d^2 = \ln(0.01)$$

$$\boxed{\kappa_0 = \frac{2m}{d^2}\ln(100)}$$

Exercise

When the velocity v of an object is very large, the magnitude of the force due to air resistance is proportional to v^2 with the force acting in opposition to the motion of the object. A shell of mass 3 *kg* is hot upward from the ground with an initial velocity of 500 *m/sec*. If the magnitude of the force due to air resistance is $(0.1)v^2$.

a) When will the shell reach its maximum height above the ground?

b) What is the maximum height?

Solution

Given: $m=3, \quad g=9.81, \quad 0.1v^2, \quad v(0)=500, \quad x(0)=0$

a) $m\frac{dv}{dt} = -mg - 0.1v^2$

$$\frac{dv}{dt} = -9.81 - \frac{1}{30}v^2$$

$$\frac{dv}{dt} = -\frac{1}{30}\left(294.3 + v^2\right)$$

$$\int \frac{dv}{294.3+v^2} = -\frac{1}{30}\int dt$$

$$\frac{1}{\sqrt{294.3}}\tan^{-1}\frac{v}{\sqrt{294.3}} = -\frac{1}{30}t + C$$

$$v(0)=500 \quad \to \quad 0.058\tan^{-1}(29.07) = C \;\Rightarrow\; \boxed{C = 0.089}$$

$$\frac{1}{\sqrt{294.3}}\tan^{-1}\frac{v}{\sqrt{294.3}} = -\frac{1}{30}t + 0.08956$$

$$\boxed{v(t) = \sqrt{294.3}\tan(-0.572t + 1.54)}$$

$$v(t) = 17.155\tan(-0.572t+1.54) = 0$$

$$-0.572t + 1.54 = 0$$

$$\boxed{t = 2.69 \ \ sec}$$

b) $x(t) = 17.155\int \tan(-0.572t+1.54)\ dt$

$$= -30\ln\left|\sec(-0.572t+1.54)\right| + C$$

$$x(0)=0 \quad \to C = 30\ln\left|\sec(1.54)\right| \;\boxed{\approx 104.4}$$

$$x(t) = -30\ln\left|\sec(-0.572t+1.54)\right| + 104.4$$

$$\boxed{x(2.69) \approx 104.4 \ \ m}$$

Exercise

A sailboat has been running (on a straight course) under a light wind at 1 *m/sec*. Suddenly the wind picks up, blowing hard enough to apply a constant force of 600 *N* to the sailboat. The only other force acting on the boat is water resistance that is proportional to the velocity of the boat. If the proportionality constant for water resistance is $\kappa = 100$ *N-sec/m* and the mass of the sailboat is 50 *kg*.

a) Find the equation of motion of the sailboat.

b) What is the limiting velocity of the sailboat under this wind?

c) When the velocity of the sailboat reaches 5 *m/sec*, the boat begins to rise out of the water and plane. When this happens, the proportionality constant for the water resistance drop to $\kappa = 60$ *N-sec/m*. Find the equation of motion of the sailboat.

d) What is the limiting velocity of the sailboat under this wind as it is planning?

Solution

Given: $m = 50 \quad F_w = 600 \quad \kappa = 100 \quad v(0) = 1$

a) $m\frac{dv}{dt} = F_w - \kappa v$

$$50\frac{dv}{dt} = 600 - 100v$$

$$\frac{dv}{dt} = 12 - 2v$$

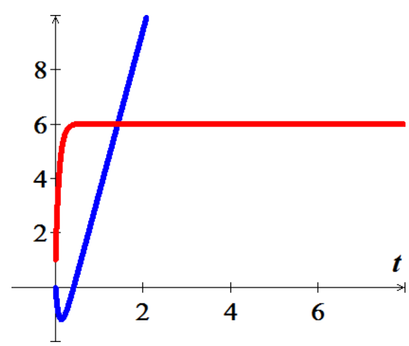
$$\int \frac{dv}{v-6} = -2\int dt$$

$$\ln|v-6| = -2t + C_1$$

$$v = C_2 e^{-2t} + 6$$

$$v(0) = 1 \quad \rightarrow \quad \boxed{C_2 = -5}$$

$$\boxed{v(t) = -5e^{-2t} + 6}$$



$$x(t) = \int \left(-5e^{-2t} + 6\right) dt$$

$$= \frac{5}{2}e^{-12t} + 6t + C_3$$

$$x(0) = 0 \quad \rightarrow \quad \boxed{C_3 = -\frac{5}{2}}$$

$$\boxed{x(t) = \frac{5}{2}e^{-12t} + 6t - \frac{5}{2}}$$

b) $\lim\limits_{t\to\infty} v(t) = \lim\limits_{t\to\infty}\left(-5e^{-2t} + 6\right) \boxed{= 6 \ \ m/s}$

c) $50\frac{dv}{dt} = 600 - 60v$

$$\frac{dv}{dt} = 12 - 1.2v$$

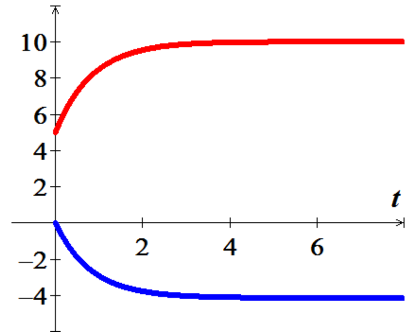
$$\int \frac{dv}{v-10} = -1.2\int dt$$

$$\ln|v-10| = -1.2t + C_4$$

$$v(t) = C_5 e^{-6t/5} + 10$$

$$v(0) = 5 \quad \rightarrow \quad \boxed{C_5 = -5}$$

$$\boxed{v(t) = -5e^{-6t/5} + 10}$$



$$x(t) = \int \left(-5e^{-6t/5} + 10\right) dt$$

$$= \frac{25}{6} e^{-6t/5} + 10 + C_6$$

$$x(0) = 0 \quad \rightarrow \quad \boxed{C_3 = -\frac{85}{6}}$$

$$x(t) = \frac{25}{6} e^{-6t/5} + 10 - \frac{85}{6}$$

$$\boxed{x(t) = \frac{25}{6} e^{-6t/5} - \frac{25}{6}}$$

d) $\lim\limits_{t\to\infty} v(t) = \lim\limits_{t\to\infty} \left(-5e^{-6t/5} + 10\right)$

$$\boxed{= 10 \ \ m/s}$$

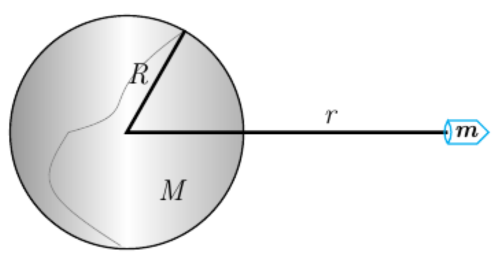
Exercise

According to Newton's law of gravitation, the attractive force between two objects varies inversely as the square of the distances between them. That is, $F_g = \dfrac{GM_1 M_2}{r^2}$

Where M_1 and M_2 are the masses of the objects, r is the distance between them (center to center), F_g is the attractive force, and G is the constant of proportionality.

Consider ta projectile of constant mass m being fired vertically from Earth.

Let t represent time and v the velocity of the projectile.



a) Show that the motion of the projectile, under Earth's gravitational force, is governed by the equation

$$\frac{dv}{dt} = -\frac{gR^2}{r^2},$$

Where r is the distance between the projectile and the center of Earth, R is the radius of Earth, M is the mass of Earth, and $g = \frac{GM}{R^2}$.

b) Use the fact the $\frac{dr}{dt} = v$ to obtain $v\frac{dv}{dr} = -\frac{gR^2}{r^2}$

c) If the projectile leaves Earth's surface with velocity v_0, show that

$$v^2 = \frac{2gR^2}{r} + v_0^2 - 2gR$$

d) Use the result of part (*c*) to how that the velocity of the projectile remains positive if and only if $v_0^2 - 2gR > 0$. The velocity $v_e = \sqrt{2gR}$ is called the escape velocity?

e) If $g = 9.81\ m/sec^2$ and $R = 6370\ km$ for Earth, what is Earth's escape velocity?

f) If the acceleration due to gravity for the Moon is $g_m = \frac{g}{6}$ and the radius of the Moon is $R_m = 1738\ km$, what is the escape velocity of the Moon?

Solution

a) $F_g = \frac{GmM}{r^2}$ $\qquad g = \frac{GM}{R^2} \rightarrow gR^2 = GM$

$$= \frac{mgR^2}{r^2}$$

$$m\frac{dv}{dt} = -F_g$$

$$m\frac{dv}{dt} = -\frac{mgR^2}{r^2}$$

$$\boxed{\frac{dv}{dt} = -\frac{gR^2}{r^2}}$$

b) ***Given***: $\frac{dr}{dt} = v$

$$\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = -\frac{gR^2}{r^2}$$

$$\boxed{v\frac{dv}{dr} = -\frac{gR^2}{r^2}}$$

c) $\int v\,dv = \int -\frac{gR^2}{r^2}\,dr$

$$\frac{1}{2}v^2 = \frac{gR^2}{r} + C_1$$

$$v^2 = \frac{2gR^2}{r} + C_2$$

$$v(R) = v_0 \quad \to \boxed{v_0 - 2gR = C_1}$$

$$\boxed{v^2 = \frac{2gR^2}{r} + v_0^2 - 2gR}$$

d) $v^2 = \frac{2gR^2}{r} + v_0^2 - 2gR > 0$

Since $\frac{2gR^2}{r} > 0 \quad \Rightarrow \quad v_0^2 - 2gR > 0$

$v_e = \sqrt{2gR}$

e) ***Given***: $g = 9.81 \quad R = 6370$

$$v_e = \sqrt{2\left(9.81 \frac{1\ km}{1000\ m}\right)(6370)} \qquad v_e = \sqrt{2gR}$$

$$\boxed{\approx 11.18\ \ km/sec}$$

f) ***Given***: $g_m = \frac{g}{6} = \frac{9.81}{6} \quad R_m = 1738$

$$v_e = \sqrt{2\left(\frac{9.81}{6} \frac{1\ km}{1000\ m}\right)(1738)} \qquad v_e = \sqrt{2gR}$$

$$\boxed{\approx 2.38\ \ km/sec}$$

Exercise

A 180-*lb* skydiver drops from a hot-air balloon. After 10 *sec* of free fall, a parachute is opened. The parachute immediately introduces a drag force proportional to velocity. After an additional 4 *sec*, the parachutist reaches the ground. Assume that air resistance is negligible during free fall and that the parachute is designed so that a 200-*lb* person will reach a terminal velocity of -10 *mph*.

a) What is the speed of the skydiver immediately before the parachute is opened?
b) What is the parachutist's impact velocity?
c) At what altitude was the parachute opened?
d) What is the balloon's altitude?

Solution

Given: $mg = 180 \quad g = 32$

a) $0 \le t \le 10 \quad v(0) = 0$

$v' = -g \quad \to \quad v(t) = -gt$

$v(10) = -(32)(10) \ \boxed{= -320\ \ ft/sec}$

b) $10 \le t \le 14 \quad mg = 180 \to m = \frac{180}{32} = \frac{45}{8}, \quad y(14) = 0, \quad v(0) = -320$

$$mv' + kv = -mg$$

$$\frac{mg}{k} = \frac{200}{k} = 10\frac{mi}{hr}\frac{5280\ ft}{1\ mi}\frac{1\ hr}{3600\ sec}$$

$$k = \frac{(200)(3600)}{52800} = \boxed{\frac{450}{33}}$$

$$v' + \frac{k}{m}v = -g$$

$$v' + \frac{450}{33}\left(\frac{8}{45}\right)v = -32$$

$$\frac{dv}{dt} = -\left(\frac{80}{33}v + 32\right)$$

$$= -\frac{80}{33}\left(v + \frac{66}{5}\right)$$

$$\int \frac{dv}{v + \frac{66}{5}} = -\frac{80}{33}\int dt$$

$$\ln\left|v + \frac{66}{5}\right| = -\frac{80}{33}t + C_1$$

$$v(t) = C_2 e^{-80t/33} - \frac{66}{5}$$

$$v(0) = -320 \quad \to -320 = C_2 - \frac{66}{5}$$

$$\Rightarrow \boxed{C_2 = -\frac{1534}{5}}$$

$$\boxed{v(t) = -\frac{1534}{5}e^{-80t/33} - \frac{66}{5}}$$

$$v(t = 4) = -\frac{1534}{5}e^{-320/33} - \frac{66}{5}$$

$$\boxed{\approx 13.219 \ \ ft/sec}$$

c) $y(t) = -\int\left(-\frac{1534}{5}e^{-80t/33} - \frac{66}{5}\right)\, dt$

$$= -\frac{25,311}{200}e^{-80t/33} + \frac{66}{5}t + C_3$$

$$y(0) = 0 \quad \to C_3 = \frac{25,311}{200}$$

$$\boxed{y(t) = \frac{25,311}{200}e^{-80t/33} - \frac{66}{5}t - \frac{25,311}{200}}$$

$$y(4) = \frac{25,311}{200}e^{-320/33} - \frac{66}{5}(4)\frac{25,311}{200}$$

$$\boxed{\approx 179.347 \ \ ft}$$

d) $h_{balloon} = y(4) + \frac{1}{2}32(10)^2$

$= 179.347 + 1600$

$\approx 1779.347\ \ ft$

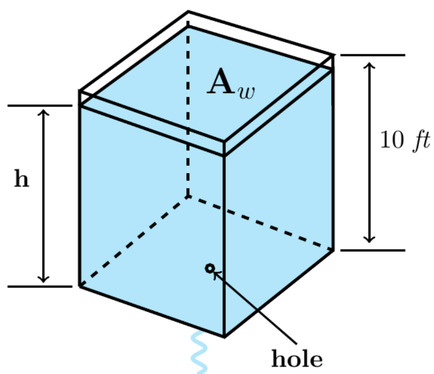
Exercise

Suppose water is leaking from a tank through a circular hole of area A_h at its bottom. When water leaks through a hole, friction and contraction of the stream near the hole reduce the volume of water leaving the tank per second to $cA_h\sqrt{2gh}$, where c $(0 < c < 1)$ is an empirical constant.

Determine a differential equation for the height h of water at time t for the cubical tank. The radius of the hole is 2 *in*., $g = 32\ \ ft/s^2$, and the friction/contraction factor is $c = 0.6$

Solution

Given: $\frac{dV}{dt} = -cA_h\sqrt{2gh}$



$A_h = \pi\left(2in\frac{1\ ft}{12\ in}\right)^2 = \frac{\pi}{36}$

$A_w = 10^2 = 100$

Volume: $V = hA_w$

$h = \frac{1}{A_w}V$

$\frac{dh}{dt} = \frac{1}{A_w}\frac{dV}{dt}$

$= \frac{1}{A_w}\left(-cA_h\sqrt{2gh}\right)$

$= -\frac{A_h}{A_w}\left(c\sqrt{2gh}\right)$

$= -\frac{1}{100}\frac{\pi}{36}\left(c\sqrt{64h}\right)$

$= -\frac{\pi}{750}\sqrt{h}$ The negative sign indicates downward.

$h^{-1/2}dh = \frac{\pi}{750}dt$

$\int h^{-1/2}dh = -\frac{\pi}{750}\int dt$

$2\sqrt{h} = -\frac{\pi}{750}t + C$

$h(t) = \frac{1}{4}\left(C - \frac{\pi}{750}t\right)^2$

Exercise

The right-circular tank loses water out of a circular hole at its bottom.

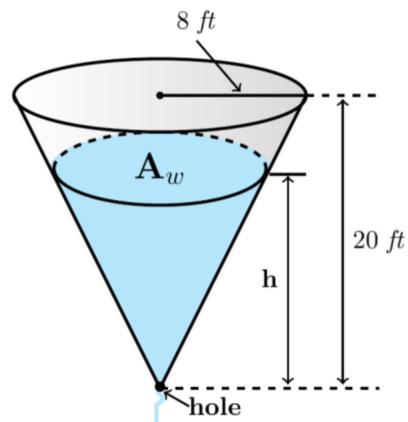
The radius of the hole is 2 *in*., and $g = 32\ ft/s^2$, and the friction/contraction factor is $c = 0.6$.

a) Determine a differential equation for the height *h* of water at time *t* for the cubical tank.

b) Find the height in function of time.

Solution

a) The volume of the tank at time *t* is



$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{r}{h} = \frac{8}{20}$$

$$r = \frac{2}{5}h$$

$$V = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h$$

$$= \frac{4}{75}\pi h^3$$

$$\frac{dV}{dt} = \frac{4}{25}\pi h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{25}{4\pi h^2}\frac{dV}{dt} \qquad \frac{dV}{dt} = -cA_h\sqrt{2gh}$$

$$= -\frac{25c}{4\pi h^2}A_h\sqrt{2gh} \qquad A_h = \pi\left(2in\frac{1\ ft}{12\ in}\right)^2 = \frac{\pi}{36}$$

$$= -\frac{25}{4\pi h^2}\frac{6}{10}\frac{\pi}{36}\sqrt{64h}$$

$$\boxed{= -\frac{5}{6\pi h^{3/2}}}$$

b) $$\frac{dh}{dt} = -\frac{5}{6\pi h^{3/2}}$$

$$\int h^{3/2}dh = -\frac{5}{6\pi}\int dt$$

$$\frac{2}{5}h^{5/2} = -\frac{5}{6\pi}t + C_1$$

$$h^{5/2} = C - \frac{25}{12\pi}t$$

$$\boxed{h(t) = \left(C - \frac{25}{12\pi}t\right)^{2/5}}$$

Exercise

In meteorology, the term virga refers to falling raindrops or ice particles that evaporate before they reach the ground. Assume that a typical raindrop is spherical. Starting at some time, which we can designate as $t = 0$, the raindrop of radius r_0 falls from rest from a cloud and begins to evaporate.

a) If it is assumed that a raindrop evaporates in such a manner that its shape remains spherical, then it also makes sense to assume that the rate at which the raindrop evaporates – that is, the rate at which it loses mass – is proportional to its surface area, Show that this latter assumption implies that the rate at which the radius r of the raindrop decreases is a constant. Find $r(t)$.

b) If the positive direction is downward, construct a mathematical model for the velocity v of the falling raindrop at time $t > 0$. Ignore air resistance.

Solution

a) If ρ is the mass density of the raindrop, then $m = \rho V$

$$\begin{aligned}\frac{dm}{dt} &= \rho\frac{dV}{dt}\\ &= \rho\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)\\ &= 4\rho\pi r^2\frac{dr}{dt}\end{aligned}$$

$$\frac{dm}{dt} = kS = k\left(4\pi r^2\right)$$

$$4\pi r^2 k = 4\rho\pi r^2\frac{dr}{dt}$$

$$k = \rho\frac{dr}{dt}$$

$$\int dr = \frac{k}{\rho}\int dt$$

$$r(t) = \frac{k}{\rho}t + C$$

$$r(0) = r_0 \qquad \boxed{r_0 = C}$$

$$\boxed{r(t) = \frac{k}{\rho}t + r_0}$$

b) From Newton's second law

$$\frac{d}{dt}(mv) = mg$$

$$m\frac{dv}{dt} + v\frac{dm}{dt} = mg$$

$$\rho\left(\frac{4}{3}\pi r^3\right)\frac{dv}{dt} + v\left(k4\pi r^2\right) = \rho\left(\frac{4}{3}\pi r^3\right)g$$

$$\rho r\frac{dv}{dt} + 3kv = \rho rg$$

$$\frac{dv}{dt} + \frac{3k}{r\rho}v = g$$

$$r = \frac{k}{\rho}t + r_0$$

$$= \frac{kt + \rho r_0}{\rho}$$

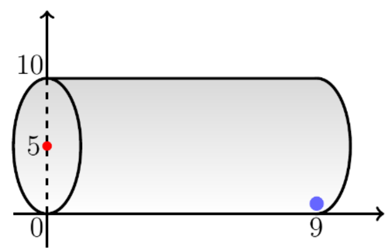
$$\boxed{\frac{dv}{dt} + \frac{3k}{\rho r_0 + kt}v = g}$$

Exercise

A horizontal cylindrical tank of length $9\ ft$, and radius $5\ ft$, is filled with oil. At $t = 0$ a plug at the lowest point of the tank is removed and a flow results.

Find y the depth of the oil in the tank at any time t while the tank is draining. The constriction coefficient is $k = \frac{1}{15}$

Solution



$$x^2 + (y-5)^2 = 25$$

$$x^2 = 25 - \left(y^2 - 10y + 25\right)$$

$$x = \pm\sqrt{10y - y^2}$$

$$dV = Surface \times dy$$

$$Surface = (9)2\sqrt{10y - y^2} = 18\sqrt{10y - y^2}$$

When the liquid of volume V flows out the rate at which it flows through that opening is $k\sqrt{y}$

$$-\frac{dV}{dt} = k\sqrt{y}$$

$$18\sqrt{10y - y^2}\frac{dy}{dt} = -\frac{1}{15}\sqrt{y}$$

$$18\frac{\sqrt{10y - y^2}}{\sqrt{y}}dy = -\frac{1}{15}dt$$

$$18\sqrt{10 - y}dy = -\frac{1}{15}dt$$

$$-\int (10-y)^{1/2} d(10-y) = -\frac{1}{270}\int dt$$

$$\frac{2}{3}(10-y)^{3/2} = \frac{1}{270}t + C$$

$$(10-y)^{3/2} = \frac{1}{180}t + C_1$$

$$y(0) = 10 \quad \rightarrow \quad \boxed{0 = C_1}$$

$$(10-y)^{3/2} = \frac{1}{180}t$$

$$10-y = \left(\frac{1}{180}t\right)^{3/2}$$

$$y(t) = 10 - \frac{1}{1{,}080\sqrt{5}}t^{3/2}$$