Solution   *Section* 2.3 – Trigonometric Substitutions

Exercise

Evaluate the integral $\displaystyle\int \frac{3\,dx}{\sqrt{1+9x^2}}$

Solution

$$3x = \tan t \quad \Rightarrow \quad dx = \frac{1}{3}\sec^2 t\ dt$$

$$\sqrt{1+9x^2} = 3\sec^2 t$$

$$\begin{aligned}
\int \frac{3\,dx}{\sqrt{1+9x^2}} &= \frac{1}{3}\int \frac{\sec^2 t}{3\sec t}\,dt \\
&= \int \sec t\ dt \\
&= \int \sec t\,\frac{\sec t+\tan t}{\sec t+\tan t}\,dt \\
&= \int \frac{\sec^2 t+\sec t\tan t}{\sec t+\tan t}\,dt \\
&= \int \frac{1}{\sec t+\tan t}\,d\left(\sec t+\tan t\right) \\
&= \ln\left|\sec t+\tan t\right| + C \\
&= \ln\left|\sqrt{1+u^2}+u\right| + C \\
&= \ln\left|\sqrt{1+9x^2}+3x\right| + C
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^2}{4+x^2}\,dx$

Solution

$$x = 2\tan\theta \quad dx = 2\sec^2\theta d\theta$$

$$4+x^2 = 4\sec^2\theta$$

$$\int \frac{x^2}{4+x^2}\,dx = \int \frac{4\tan^2\theta}{4\sec^2\theta}\ 2\sec^2\theta\,d\theta$$

$$= 2\int \tan^2\theta \, d\theta$$

$$= 2\int \left(\sec^2\theta - 1\right) d\theta \qquad\qquad \int \sec^2\theta d\theta = \tan\theta$$

$$= 2\left(\tan\theta - \theta\right) + C$$

$$= 2\left(\frac{x}{2} - \tan^{-1}\left(\frac{x}{2}\right)\right) + C$$

$$\boxed{= x - 2\tan^{-1}\left(\frac{x}{2}\right) + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+1}}$

Solution

$$x = \tan\theta \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$dx = \sec^2\theta d\theta$$

$$\sqrt{x^2+1} = \sqrt{\tan^2\theta + 1} = \sec\theta$$

$$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec^2\theta \, d\theta}{\tan^2\theta \sec\theta}$$

$$= \int \frac{\sec\theta}{\tan^2\theta} d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta\cos\theta} d\theta$$

$$= \int \frac{\cos\theta}{\sin^2\theta} d\theta$$

$$= \int \sin^{-2}\theta d\left(\sin\theta\right)$$

$$= -\frac{1}{\sin\theta} + C$$

$$= -\frac{\sec\theta}{\tan\theta} + C$$

$$\boxed{= -\frac{\sqrt{x^2+1}}{x} + C}$$

Exercise

Evaluate: $\displaystyle\int \frac{dx}{\sqrt{x^2+4}}$

Solution

Let: $x = 2\tan\theta \quad \to dx = 2\sec^2\theta d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\sqrt{x^2+4} = 2|\sec\theta|$

$$\int \frac{dx}{\sqrt{x^2+4}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}}$$

$$= \int \frac{2\sec^2\theta\, d\theta}{2|\sec\theta|}$$

$$= \int \sec\theta\, d\theta$$

$$= \int \sec\theta\, \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\, d\theta$$

$$= \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\, d\theta$$

$$= \int \frac{1}{\sec\theta+\tan\theta}\, d(\sec\theta+\tan\theta)$$

$$= \ln|\sec\theta+\tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}\right| + C$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\left(1+x^2\right)^2}$

Solution

$x = \tan\theta \qquad 1+x^2 = \left(\sec^2\theta\right)^2$

$dx = \sec^2\theta\, d\theta$

$$\int \frac{dx}{\left(1+x^2\right)^2} = \int \frac{\sec^2\theta}{\sec^4\theta}\, d\theta$$

$$= \int \frac{1}{\sec^2 \theta}\, d\theta$$

$$= \int \cos^2 \theta \, d\theta$$

$$= \frac{1}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{2}\tan^{-1} x + \frac{1}{2}\sin\theta\cos\theta + C$$

$$= \frac{1}{2}\tan^{-1} x + \frac{1}{2}\frac{x}{\sqrt{1+x^2}}\frac{1}{\sqrt{1+x^2}} + C$$

$$= \frac{1}{2}\tan^{-1} x + \frac{x}{2\left(1+x^2\right)} + C$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{4x^2+1}}$

Solution

$2x = \tan\theta \qquad \sqrt{4x^2+1} = \sec\theta$

$dx = \frac{1}{2}\sec^2\theta\, d\theta$

$$\int \frac{dx}{\sqrt{4x^2+1}} = \frac{1}{2}\int \frac{\sec^2\theta}{\sec\theta}\, d\theta$$

$$= \frac{1}{2}\int \sec\theta\, d\theta$$

$$= \int \sec\theta \,\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\, d\theta$$

$$= \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\, d\theta$$

$$= \int \frac{1}{\sec\theta+\tan\theta}\, d\left(\sec\theta+\tan\theta\right)$$

$$= \frac{1}{2}\ln\left|\sec\theta+\tan\theta\right| + C$$

$$= \frac{1}{2}\ln\left|\sqrt{4x^2+1}+2x\right| + C$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\left(x^2+1\right)^{3/2}}$

Solution

$x = \tan\theta \qquad \sqrt{x^2+1} = \sec\theta$

$dx = \sec^2\theta\, d\theta$

$$\int \frac{dx}{\left(x^2+1\right)^{3/2}} = \int \frac{\sec^2\theta}{(\sec\theta)^3}\, d\theta$$

$$= \int \frac{d\theta}{\sec\theta}$$

$$= \int \cos\theta\, d\theta$$

$$= \sin\theta + C \qquad\qquad \sin\theta = \frac{\tan\theta}{\sec\theta} = \frac{x}{\sqrt{x^2+1}}$$

$$\boxed{= \frac{x}{\sqrt{x^2+1}} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{9x^3}{\sqrt{x^2+1}}\, dx$

Solution

$x = \tan\theta \qquad \sqrt{x^2+1} = \sec\theta$

$dx = \sec^2\theta\, d\theta$

$$\int \frac{9x^3}{\sqrt{x^2+1}}\, dx = \int \frac{9\tan^3\theta}{\sec\theta}\left(\sec^2\theta\right) d\theta$$

$$= 9\int \tan^2\theta \tan\theta \sec\theta\, d\theta$$

$$= 9\int \left(\sec^2\theta - 1\right)\, d(\sec\theta)$$

$$= 9\left(\tfrac{1}{3}\sec^3\theta - \sec\theta\right) + C$$

$$= 3\left(x^2+1\right)\sqrt{x^2+1} - 9\sqrt{x^2+1} + C$$

$$= 3\sqrt{x^2+1}\left(x^2+1-3\right)+C$$

$$\boxed{= 3\sqrt{x^2+1}\left(x^2-2\right)+C}$$

Exercise

Evaluate $\displaystyle\int \sqrt{16x^2+9}\, dx$

Solution

$4x = 3\tan\theta \qquad \sqrt{4x^2+9} = 3\sec\theta$

$4dx = 3\sec^2\theta\; d\theta$

$$\int \sqrt{16x^2+9}\, dx = \int 3\sec\theta\left(\frac{3}{4}\sec^2\theta\right)d\theta$$

$$= \frac{9}{4}\int \sec^3\theta\; d\theta$$

$u = \sec x \qquad dv = \sec^2 xdx$

$du = \sec x\tan xdx \qquad v = \tan x$

$$\int \sec^3 xdx = \sec x\tan x - \int \tan x\left(\sec x\tan xdx\right)$$

$$= \sec x\tan x - \int \tan^2 x\sec xdx$$

$$= \sec x\tan x - \int \left(\sec^2 x-1\right)\sec x\, dx$$

$$= \sec x\tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$2\int \sec^3 x\, dx = \sec x\tan x + \ln\left|\sec x+\tan x\right|$$

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln\left|\sec x+\tan x\right|$$

$$= \frac{9}{8}\sec\theta\tan\theta + \frac{9}{8}\ln\left|\sec\theta+\tan\theta\right| + C$$

$$= \frac{9}{8}\frac{\sqrt{4x^2+9}}{3}\frac{4x}{3} + \frac{9}{8}\ln\left|\frac{\sqrt{4x^2+9}}{3}+\frac{4x}{3}\right| + C$$

$$\boxed{= \frac{1}{2}x\sqrt{4x^2+9} + \frac{9}{8}\ln\left|\frac{2x+\sqrt{4x^2+9}}{3}\right| + C}$$

Exercise

Evaluate $\displaystyle\int x\sqrt{x^2+1}\,dx$

Solution

$$\int x\sqrt{x^2+1}\,dx = \frac{1}{2}\int \left(x^2+1\right)^{1/2}\,d\left(x^2+1\right)$$

$$= \frac{1}{3}\left(x^2+1\right)^{3/2} + C$$

OR

$x = \tan\theta \qquad \sqrt{x^2+1} = \sec\theta$

$dx = \sec^2\theta\,d\theta$

$$\int x\sqrt{x^2+1}\,dx = \int \tan\theta\sec^3\theta\,d\theta$$

$$= \int \sec^2\theta\,d(\sec\theta)$$

$$= \frac{1}{3}\sec^3\theta + C$$

$$= \frac{1}{3}\left(x^2+1\right)^{3/2} + C$$

Exercise

Evaluate $\displaystyle\int \frac{\sqrt{25x^2+4}}{x^4}\,dx$

Solution

$5x = 2\tan\theta \qquad \sqrt{25x^2+4} = 2\sec\theta$

$5dx = 2\sec^2\theta\,d\theta$

$$\int \frac{\sqrt{25x^2+4}}{x^4}\,dx = \int \frac{2\sec\theta}{\left(\frac{2}{5}\right)^4\tan^4\theta}\,\frac{2}{5}\sec^2\theta\,d\theta$$

$$= \frac{125}{4}\int \frac{\sec^3\theta}{\tan^4\theta}\,d\theta$$

$$= \frac{125}{4}\int \frac{\cos\theta}{\sin^4\theta}\,d\theta$$

$$= \frac{125}{4} \int \sin^{-4}\theta \, d(\sin\theta)$$

$$= -\frac{125}{12} \frac{1}{\sin^3\theta} + C$$

$$= -\frac{125}{12} \left(\frac{\tan\theta}{\sec\theta}\right)^3 + C$$

$$= -\frac{125}{12} \left(\frac{\sqrt{25x^2+4}}{5x}\right)^3 + C$$

$$\boxed{= -\frac{\left(25x^2+4\right)^{3/2}}{12x^3} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{1}{x\sqrt{4x^2+9}}\, dx$

Solution

$2x = 3\tan\theta \qquad \sqrt{4x^2+9} = 3\sec\theta$

$dx = \frac{3}{2}\sec^2\theta \, d\theta$

$$\int \frac{1}{x\sqrt{4x^2+9}}\, dx = \int \frac{1}{\frac{9}{2}\tan\theta\sec\theta}\left(\frac{3}{2}\sec^2\theta\right) d\theta$$

$$= \int \frac{1}{\frac{9}{2}\tan\theta\sec\theta}\left(\frac{3}{2}\sec^2\theta\right) d\theta$$

$$= \frac{1}{3}\int \frac{\sec\theta}{\tan\theta}\, d\theta$$

$$= \frac{1}{3}\int \frac{1}{\sin\theta}\, d\theta$$

$$= \frac{1}{3}\int \csc\theta \frac{\csc\theta + \cot\theta}{\csc\theta + \cot\theta}\, d\theta$$

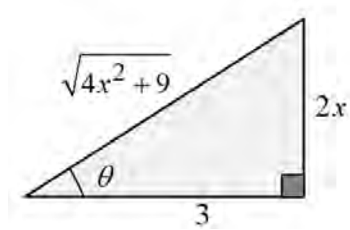
$$= \frac{1}{3}\int \frac{\csc^2\theta + \csc\theta\cot\theta}{\csc\theta + \cot\theta}\, d\theta$$

$$= -\frac{1}{3}\int \frac{1}{\csc\theta + \cot\theta}\, d(\csc\theta + \cot\theta)$$

$$= -\frac{1}{3}\ln\left|\csc\theta + \cot\theta\right| + C$$

$$= -\frac{1}{3}\ln\left|\frac{\sqrt{4x^2+9}}{2x} + \frac{3}{2x}\right| + C$$

$$\boxed{= -\frac{1}{3}\ln\left|\frac{\sqrt{4x^2+9}+3}{2x}\right| + C}$$



Exercise

Evaluate $\displaystyle\int \frac{1}{\left(x^2+5\right)^{3/2}}\,dx$

Solution

$x = \sqrt{5}\tan\theta \qquad \sqrt{x^2+5} = \sqrt{5}\sec\theta$

$dx = \sqrt{5}\sec^2\theta\; d\theta$

$$\int \frac{1}{\left(x^2+5\right)^{3/2}}\,dx = \int \frac{1}{5\sqrt{5}\sec^3\theta}\left(\sqrt{5}\sec^2\theta\right)\,d\theta$$

$$= \frac{1}{5}\int \frac{1}{\sec\theta}\,d\theta$$

$$= \frac{1}{5}\int \cos\theta\,d\theta$$

$$= \frac{1}{5}\sin\theta$$

$$= \frac{1}{5}\frac{\tan\theta}{\sec\theta}$$

$$= \frac{1}{5}\frac{x}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{x^2+5}}$$

$$\boxed{= \frac{1}{5}\frac{x}{\sqrt{x^2+5}} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x\,dx}{\sqrt{x^2+4}}$

Solution

$$x = 2\tan\theta \qquad \sqrt{x^2+4} = 2\sec\theta$$

$$dx = 2\sec^2\theta\, d\theta$$

$$\int \frac{x\,dx}{\sqrt{x^2+4}} = \int \frac{2\tan\theta}{2\sec\theta}\left(2\sec^2\theta\, d\theta\right)$$

$$= 2\int \tan\theta \sec\theta\, d\theta$$

$$= 2\sec\theta$$

$$\boxed{= \sqrt{x^2+4} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^3}{\sqrt{x^2+4}}\,dx$

Solution

$$x = 2\tan\theta \qquad \sqrt{x^2+4} = 2\sec\theta$$

$$dx = 2\sec^2\theta\, d\theta$$

$$\int \frac{x^3\,dx}{\sqrt{x^2+4}} = \int \frac{8\tan^3\theta}{2\sec\theta}\left(2\sec^2\theta\, d\theta\right)$$

$$= 8\int \tan^2\theta \tan\theta \sec\theta\, d\theta$$

$$= 8\int \left(\sec^2\theta - 1\right) d\left(\sec\theta\right)$$

$$= 8\left(\frac{1}{3}\sec^3\theta - \sec\theta\right) + C$$

$$= \frac{8}{3}\frac{\left(x^2+4\right)^{3/2}}{8} - 8\frac{\left(x^2+4\right)^{1/2}}{2} + C$$

$$= \sqrt{x^2+4}\left(\frac{1}{3}\left(x^2+4\right) - 4\right) + C$$

$$= \frac{1}{3}\sqrt{x^2+4}\,\left(x^2+4-12\right) + C$$

$$\boxed{= \frac{1}{3}\sqrt{x^2+4}\,\left(x^2-8\right) + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\left(1+4x^2\right)^{3/2}}$

Solution

$$x = \frac{1}{2}\tan\theta \qquad \sqrt{4x^2+1} = \sec\theta$$

$$dx = \frac{1}{2}\sec^2\theta\, d\theta$$

$$\int \frac{dx}{\left(1+4x^2\right)^{3/2}} = \frac{1}{2}\int \frac{\sec^2\theta}{\left(\sec^2\theta\right)^{3/2}}\, d\theta$$

$$= \frac{1}{2}\int \frac{\sec^2\theta}{\sec^3\theta}\, d\theta$$

$$= \frac{1}{2}\int \frac{1}{\sec\theta}\, d\theta$$

$$= \frac{1}{2}\int \cos\theta\, d\theta$$

$$= \frac{1}{2}\sin\theta + C$$

$$= \frac{1}{2}\frac{\tan\theta}{\sec\theta} + C$$

$$= \frac{1}{2}\frac{2x}{\sqrt{4x^2+1}} + C$$

$$= \frac{x}{\sqrt{4x^2+1}} + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+9}}$

Solution

$$x = 3\tan\theta \qquad \sqrt{x^2+9} = 3\sec\theta$$

$$dx = 3\sec^2\theta\, d\theta$$

$$\int \frac{dx}{x^2\sqrt{x^2+9}} = \int \frac{3\sec^2\theta}{9\tan^2\theta(3\sec\theta)}\, d\theta$$

$$= \frac{1}{9}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

$$= \frac{1}{9}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{9}\int \frac{1}{\sin^2\theta}\, d(\sin\theta)$$

$$= -\frac{1}{9}\frac{1}{\sin\theta} + C$$

$$= -\frac{1}{9}\frac{\sec\theta}{\tan\theta} + C$$

$$\boxed{= -\frac{1}{9}\frac{\sqrt{x^2+9}}{x} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+4}}$

Solution

$$x = 2\tan\theta \qquad \sqrt{x^2+4} = 2\sec\theta$$

$$dx = 2\sec^2\theta\, d\theta$$

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta}{4\tan^2\theta(2\sec\theta)}\, d\theta$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

$$= \frac{1}{4}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{4}\int \frac{1}{\sin^2\theta}\, d(\sin\theta)$$

$$= -\frac{1}{4}\frac{1}{\sin\theta} + C$$

$$= -\frac{1}{4}\frac{\sec\theta}{\tan\theta} + C$$

$$\boxed{= -\frac{1}{4}\frac{\sqrt{x^2+4}}{x} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^2\,dx}{\sqrt{x^2+1}}$

Solution

$$x = \tan\theta \qquad \sqrt{x^2+1} = \sec\theta$$

$$dx = \sec^2\theta\,d\theta$$

$$\int \frac{x^2\,dx}{\sqrt{x^2+1}} = \int \frac{\tan^2\theta}{\sec\theta}\sec^2\theta\,d\theta$$

$$= \int \tan^2\theta\sec\theta\,d\theta$$

$$= \int \left(\sec^2\theta - 1\right)\sec\theta\,d\theta$$

$$= \int \sec^3\theta\,d\theta - \int \sec\theta\,d\theta$$

$$u = \sec\theta \qquad dv = \sec^2\theta d\theta$$

$$du = \sec\theta\tan\theta d\theta \qquad v = \tan\theta$$

$$\int \sec^3\theta d\theta = \sec\theta\tan\theta - \int \tan\theta\left(\sec\theta\tan\theta d\theta\right)$$

$$= \sec\theta\tan\theta - \int \tan^2\theta\sec\theta\,d\theta$$

$$= \sec\theta\tan\theta - \int \left(\sec^2\theta - 1\right)\sec\theta\,d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta\,d\theta + \int \sec\theta\,d\theta$$

$$2\int \sec^3\theta d\theta = \sec\theta\tan\theta + \int \sec\theta\,d\theta$$

$$\int \sec^3\theta d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\,d\theta$$

$$\int \sec\theta\,d\theta = \int \sec\theta\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\,d\theta$$

$$= \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}\,d\theta$$

$$= \int \frac{1}{\sec\theta + \tan\theta}\,d\left(\sec\theta + \tan\theta\right)$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

$$\int \frac{x^2\, dx}{\sqrt{x^2+1}} = \int \sec^3\theta\, d\theta - \int \sec\theta\, d\theta$$

$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\, d\theta - \int \sec\theta\, d\theta$$

$$= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\int \sec\theta\, d\theta$$

$$= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \frac{1}{2}x\sqrt{x^2+1} - \frac{1}{2}\ln\left|\sqrt{x^2+1} + x\right| + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^3\, dx}{\left(x^2+a^2\right)^{3/2}}$

Solution

$$x = a\tan\theta \qquad \sqrt{x^2+a^2} = a\sec\theta$$

$$dx = a\sec^2\theta\, d\theta$$

$$\int \frac{x^3\, dx}{\left(x^2+a^2\right)^{3/2}} = \int \frac{a^3\tan^3\theta}{a^3\sec^3\theta}\left(a\sec^2\theta\right) d\theta$$

$$= a\int \frac{\tan^3\theta}{\sec\theta}\, d\theta$$

$$= a\int \frac{\sin^3\theta}{\cos^3\theta}\cos\theta\, d\theta$$

$$= a\int \frac{\sin^3\theta}{\cos^2\theta}\, d\theta$$

$$= a\int \sin^2\theta\cos^{-2}\theta\, \left(\sin\theta\, d\theta\right)$$

$$= -a\int \left(1-\cos^2\theta\right)\cos^{-2}\theta\, d\left(\cos\theta\right)$$

$$= -a\int \left(\cos^{-2}\theta - 1\right)\, d\left(\cos\theta\right)$$

$$= -a\left(-\frac{1}{\cos\theta} - \cos\theta\right) + C$$

$$= a\left(\sec\theta + \frac{1}{\sec\theta}\right) + C$$

$$= a\left(\frac{\sqrt{x^2+a^2}}{a} + \frac{a}{\sqrt{x^2+a^2}}\right) + C$$

$$\boxed{= \sqrt{x^2+a^2} + \frac{a^2}{\sqrt{x^2+a^2}} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\left(x^2+4\right)^2}$

Solution

$$x = 2\tan\theta \qquad x^2+4 = 4\sec^2\theta$$

$$dx = 2\sec^2\theta\, d\theta$$

$$\int \frac{dx}{\left(x^2+4\right)^2} = \int \frac{2\sec^2\theta}{\left(4\sec^2\theta\right)^2}\, d\theta$$

$$= \frac{1}{8}\int \frac{\sec^2\theta}{\sec^4\theta}\, d\theta$$

$$= \frac{1}{8}\int \frac{1}{\sec^2\theta}\, d\theta$$

$$= \frac{1}{8}\int \cos^2\theta\, d\theta$$

$$= \frac{1}{16}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{16}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{16}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{1}{16}\left(\theta + \frac{\tan\theta}{\sec\theta}\frac{1}{\sec\theta}\right) + C$$

$$= \frac{1}{16}\left(\arctan\frac{x}{2} + \frac{x}{\sqrt{x^2+4}}\frac{2}{\sqrt{x^2+4}}\right) + C$$

$$\boxed{= \frac{1}{16}\arctan\frac{x}{2} + \frac{1}{8}\frac{x}{x^2+4} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\sqrt{4x^2+16}}$

Solution

$$\int \frac{dx}{\sqrt{4x^2+16}} = \frac{1}{2}\int \frac{dx}{\sqrt{x^2+4}}$$

$$x = 2\tan\theta \qquad \sqrt{x^2+4} = 2\sec\theta$$

$$dx = 2\sec^2\theta\, d\theta$$

$$\begin{aligned}
\int \frac{dx}{\sqrt{4x^2+16}} &= \frac{1}{2}\int \frac{2\sec^2\theta}{2\sec\theta}\, d\theta \\
&= \frac{1}{2}\int \sec\theta\, d\theta \\
&= \frac{1}{2}\int \sec\theta \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\, d\theta \\
&= \frac{1}{2}\int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\, d\theta \\
&= \frac{1}{2}\int \frac{1}{\sec\theta+\tan\theta}\, d\left(\sec\theta+\tan\theta\right) \\
&= \frac{1}{2}\ln\left|\sec\theta+\tan\theta\right| + C \\
&= \frac{1}{2}\ln\left|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}\right| + C
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^4}{x^2+1}\, dx$

Solution

$$x = \tan\theta \qquad x^2+1 = \sec^2\theta$$

$$dx = \sec^2\theta\, d\theta$$

$$\int \frac{x^4}{x^2+1}\, dx = \int \frac{\tan^4\theta}{\sec^2\theta}\, \sec^2\theta\, d\theta$$

$$= \int \tan^4 \theta \, d\theta$$

$$= \int \tan^2 \theta \left(\sec^2 \theta - 1\right) \, d\theta$$

$$= \int \tan^2 \theta \sec^2 \theta \, d\theta - \int \tan^2 \theta \, d\theta$$

$$= \int \tan^2 \theta \, d(\tan \theta) - \int \left(\sec^2 \theta - 1\right) d\theta$$

$$= \frac{1}{3} \tan^3 \theta - \tan \theta + \theta + C$$

$$\boxed{= \frac{1}{3} x^3 - x + \tan^{-1} x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{e^{2x}}{\left(1+e^{4x}\right)^{3/2}} dx$

Solution

$$\int \frac{e^{2x}}{\left(1+e^{4x}\right)^{3/2}} dx = \frac{1}{2} \int \frac{1}{\left(1+\left(e^{2x}\right)^2\right)^{3/2}} d\left(e^{2x}\right) \qquad \textbf{\textit{Let }} y = e^{2x}$$

$$= \frac{1}{2} \int \frac{1}{\left(1+y^2\right)^{3/2}} dy$$

$$y = \tan \theta \qquad y^2 + 1 = \sec^2 \theta$$

$$dy = \sec^2 \theta \, d\theta$$

$$\int \frac{e^{2x}}{\left(1+e^{4x}\right)^{3/2}} dx = \frac{1}{2} \int \frac{1}{\left(\sec^2 \theta\right)^{3/2}} \sec^2 \theta \, d\theta$$

$$= \frac{1}{2} \int \frac{1}{\sec^3 \theta} \sec^2 \theta \, d\theta$$

$$= \frac{1}{2} \int \frac{1}{\sec \theta} \, d\theta$$

$$= \frac{1}{2} \int \cos\theta \, d\theta$$

$$= \frac{1}{2} \sin\theta + C$$

$$= \frac{1}{2} \frac{\tan\theta}{\sec\theta} + C$$

$$= \frac{1}{2} \frac{y}{\sqrt{1+y^2}} + C$$

$$\boxed{= \frac{1}{2} \frac{e^{2x}}{\sqrt{1+e^{4x}}} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{1+\cos x}$

Solution

$$\int \frac{dx}{1+\cos x} = \int \frac{1}{1+\cos x} \frac{1-\cos x}{1-\cos x} \, dx$$

$$= \int \frac{1-\cos x}{1-\cos^2 x} \, dx$$

$$= \int \frac{1-\cos x}{\sin^2 x} \, dx$$

$$= \int \left(\frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x} \right) dx$$

$$= \int \csc^2 x \, dx - \int \frac{1}{\sin^2 x} \, d(\sin x)$$

$$= -\cot x + \frac{1}{\sin x} + C$$

$$\boxed{= -\cot x + \csc x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}}$

Solution

$$x = a\sec\theta \qquad \sqrt{x^2 - a^2} = a\tan\theta$$

$$dx = a\sec\theta \tan\theta \, d\theta$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta}{a\tan\theta}\,d\theta$$

$$= \int \sec\theta\,\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\,d\theta$$

$$= \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\,d\theta$$

$$= \int \frac{1}{\sec\theta+\tan\theta}\,d\left(\sec\theta+\tan\theta\right)$$

$$= \ln\left|\sec\theta+\tan\theta\right| + C$$

$$\boxed{= \ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right| + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sqrt{x^2-1}}{x}\,dx$

Solution

$$x = \sec\theta \qquad \sqrt{x^2-1} = \tan\theta$$

$$dx = \sec\theta\tan\theta\,d\theta$$

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \frac{\tan\theta}{\sec\theta}\left(\sec\theta\tan\theta\right)\,d\theta$$

$$= \int \tan^2\theta\,d\theta$$

$$= \int \left(\sec^2\theta - 1\right)\,d\theta$$

$$= \tan\theta - \theta + C$$

$$\boxed{= \sqrt{x^2-1} - \operatorname{arcsec}\theta + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sqrt{y^2-49}}{y}\,dy, \quad y > 7$

Solution

$$y = 7\sec\theta \quad \to \quad dy = 7\sec\theta\tan\theta\,d\theta$$

$$\sqrt{y^2-49}=7\tan\theta$$

$$\begin{aligned}\int\frac{\sqrt{y^2-49}}{y}\,dy &= \int\frac{(7\tan\theta)}{7\sec\theta}(7\sec\theta\tan\theta)\,d\theta\\ &= 7\int\tan^2\theta\,d\theta\\ &= 7\int\left(\sec^2\theta-1\right)d\theta\\ &= 7(\tan\theta-\theta)+C\\ &= 7\left[\frac{\sqrt{y^2-49}}{7}-\sec^{-1}\left(\frac{y}{7}\right)\right]+C\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int\frac{5\,dx}{\sqrt{25x^2-9}},\quad x>\frac{3}{5}=\sin^{-1}\frac{1}{2}-\sin^{-1}0$

Solution

$$5x=3\sec\theta \quad\to\quad dx=\frac{3}{5}\sec\theta\tan\theta d\theta$$

$$\sqrt{25x^2-9}=3\tan\theta$$

$$\begin{aligned}\int\frac{5dx}{\sqrt{25x^2-9}} &= \int\frac{5\left(\frac{3}{5}\sec\theta\tan\theta d\theta\right)}{3\tan\theta}\\ &= \int\sec\theta d\theta\\ &= \int\sec\theta\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}d\theta\\ &= \int\frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\,d\theta\\ &= \int\frac{1}{\sec\theta+\tan\theta}\,d(\sec\theta+\tan\theta)\\ &= \ln\left|\sec\theta+\tan\theta\right|+C\\ &= \ln\left|\frac{5}{3}x+\frac{1}{3}\frac{\sqrt{25x^2-9}}{3}\right|+C\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{2dx}{x^3\sqrt{x^2-1}}, \quad x>1$

Solution

$x = \sec\theta \quad dx = \sec\theta\tan\theta\, d\theta$

$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$

$$\int \frac{2\,dx}{x^3\sqrt{x^2-1}} = \int \frac{2\sec\theta\tan\theta\, d\theta}{\sec^3\theta\tan\theta}$$

$$= 2\int \cos^2\theta d\theta \qquad\qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= 2\int \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \int (1+\cos 2\theta)\, d\theta$$

$$= \theta + \frac{1}{2}\sin 2\theta + C$$

$$= \theta + \sin\theta\cos\theta + C \qquad\qquad x = \sec\theta = \frac{1}{\cos\theta} \Rightarrow \cos\theta = \frac{1}{x}$$

$$\sin\theta = \tan\theta\cos\theta = \sqrt{x^2-1}\left(\frac{1}{x}\right)$$

$$\boxed{= \sec^{-1} x + \frac{\sqrt{x^2-1}}{x^2} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{x^3\sqrt{x^2-100}}$

Solution

$x = 10\sec\theta \qquad \sqrt{x^2-100} = 10\tan\theta$

$dx = 10\sec\theta\tan\theta d\theta$

$$\int \frac{dx}{x^3\sqrt{x^2-100}} = \int \frac{10\sec\theta\tan\theta\, d\theta}{10^3\sec^3\theta(10\tan\theta)}$$

$$= \frac{1}{10^3}\int \frac{1}{\sec^2\theta}\, d\theta$$

$$= \frac{1}{10^3}\int \cos^2\theta\, d\theta$$

$$= \frac{1}{2\times 10^3} \int (1+\cos 2\theta)\ d\theta$$

$$= \frac{1}{2\times 10^3} \left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{2\times 10^3} (\theta + \sin\theta \cos\theta) + C$$

$$= \frac{1}{2\times 10^3} \left(\tan^{-1} \frac{\sqrt{x^2-100}}{10} + \frac{\tan\theta}{\sec\theta} \frac{1}{\sec\theta} \right) + C$$

$$= \frac{1}{2\times 10^3} \left(\tan^{-1} \frac{\sqrt{x^2-100}}{10} + \frac{\sqrt{x^2-100}}{x} \frac{10}{x} \right) + C$$

$$= \frac{1}{2\times 10^3} \left(\tan^{-1} \frac{\sqrt{x^2-100}}{10} + \frac{10\ \sqrt{x^2-100}}{x^2} \right) + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^3\, dx}{x^2-1}$

Solution

$$\begin{array}{r|l} & x \\ \hline x^2-1 & x^3 \\ & \underline{x^3 - x} \\ & \quad x \end{array}$$

$$d\left(x^2-1\right) = 2xdx \ \Rightarrow \ \frac{1}{2} d\left(x^2-1\right) = xdx$$

$$\int \frac{x^3\, dx}{x^2-1} = \int \left(x + \frac{x}{x^2-1} \right) dx$$

$$= \int xdx + \int \frac{x}{x^2-1} dx$$

$$= \int xdx + \frac{1}{2} \int \frac{d\left(x^2-1\right)}{x^2-1}$$

$$= \frac{1}{2}x^2 + \frac{1}{2}\ln\left|x^2-1\right| + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\left(1-x^2\right)^{1/2}}{x^4}\,dx$

Solution

$$x = \sin\theta \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$dx = \cos\theta\, d\theta$$

$$\left(1-x^2\right)^{1/2} = \left(1-\sin^2 x\right)^{1/2} = \cos\theta$$

$$\begin{aligned}\int \frac{\left(1-x^2\right)^{1/2}}{x^4}\,dx &= \int \frac{\cos\theta}{\sin^4\theta}\cos\theta\, d\theta \\ &= \int \frac{\cos^2\theta}{\sin^2\theta}\frac{1}{\sin^2\theta}\,d\theta \\ &= \int \cot^2\theta\csc^2\theta\, d\theta \\ &= -\frac{1}{3}\cot^3\theta + C \\ &= -\frac{1}{3}\left(\frac{\cos\theta}{\sin\theta}\right)^3 + C \\ &= -\frac{1}{3}\left(\frac{\sqrt{1-x^2}}{x}\right)^3 + C\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}\,dx$

Solution

$$\ln x = \sin\theta \qquad 0 < \theta \le \frac{\pi}{2}$$

$$\frac{1}{x}dx = \cos\theta\, d\theta$$

$$\sqrt{1-(\ln x)^2} = \sqrt{1-\sin^2\theta} = \cos\theta$$

$$\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}\,dx = \int \frac{\cos\theta}{\sin\theta}\cos\theta\, d\theta$$

$$= \int \frac{\cos^2\theta}{\sin\theta}\, d\theta$$

$$= \int \frac{1-\sin^2\theta}{\sin\theta}\, d\theta$$

$$= \int \frac{1}{\sin\theta}\, d\theta - \int \sin\theta\, d\theta$$

$$= \int \csc\theta\, d\theta - \int \sin\theta\, d\theta$$

$$= -\ln\left|\csc\theta + \cot\theta\right| + \cos\theta + C$$

$$\boxed{= -\ln\left|\frac{1}{\ln x} + \frac{\sqrt{1-(\ln x)^2}}{\ln x}\right| + \sqrt{1-(\ln x)^2} + C}$$

Exercise

Evaluate the integral $\int \sqrt{x}\,\sqrt{1-x}\, dx$

Solution

$u = \sqrt{x} \ \rightarrow u^2 = x \quad \Rightarrow dx = 2udu$

$u = \sin\theta \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$du = \cos\theta\, d\theta$

$$\int \sqrt{x}\,\sqrt{1-x}\, dx = \int u\sqrt{1-u^2}\,(2udu)$$

$$= 2\int u^2\sqrt{1-u^2}\, du \qquad\qquad \sqrt{1-u^2} = \sqrt{1-\sin^2\theta} = \cos\theta$$

$$= 2\int \sin^2\theta\cos\theta\cos\theta\, d\theta$$

$$= 2\int \sin^2\theta\cos^2\theta\, d\theta \qquad\qquad \sin 2\theta = 2\sin\theta\cos\theta \quad \rightarrow \sin^2 2\theta = 4\sin^2\theta\cos^2\theta$$

$$= \frac{1}{2}\int \sin^2 2\theta\, d\theta \qquad\qquad \sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$= \frac{1}{2}\int \frac{1-\cos 4\theta}{2}\, d\theta$$

$$= \frac{1}{4}\int d\theta - \frac{1}{4}\int \cos 4\theta d\theta$$

$$= \frac{1}{4}\theta - \frac{1}{16}\sin 4\theta + C$$

$$= \frac{1}{4}\theta - \frac{1}{16} 2\sin 2\theta \cos 2\theta + C$$

$$= \frac{1}{4}\theta - \frac{1}{8} 2\sin\theta\cos\theta\left(2\cos^2\theta - 1\right) + C$$

$$= \frac{1}{4}\theta - \frac{1}{2}\sin\theta\cos^3\theta + \frac{1}{4}\sin\theta\cos\theta + C$$

$$= \frac{1}{4}\sin^{-1} u - \frac{1}{2}u\left(1-u^2\right)^{3/2} + \frac{1}{4}u\sqrt{1-u^2} + C$$

$$\boxed{= \frac{1}{4}\sin^{-1}\sqrt{x} - \frac{1}{2}\sqrt{x}\,(1-x)^{3/2} + \frac{1}{4}\sqrt{x}\sqrt{1-x} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sqrt{x-2}}{\sqrt{x-1}}\,dx$

Solution

$$u = \sqrt{x-1} \quad \rightarrow u^2 = x-1 \quad \Rightarrow 2udu = dx$$

$$u = \sec\theta \qquad 0 < \theta < \frac{\pi}{2}$$

$$du = \sec\theta\tan\theta\, d\theta$$

$$\int \frac{\sqrt{x-2}}{\sqrt{x-1}}\,dx = \int \frac{\sqrt{u^2-1}}{u}2udu$$

$$= 2\int \sqrt{u^2-1}\, du$$

$$\sqrt{u^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$$

$$= 2\int \tan\theta\sec\theta\tan\theta\; d\theta$$

$$w = \tan\theta \qquad dv = \sec\theta\tan\theta d\theta$$

$$dw = \sec^2\theta d\theta \qquad v = \sec\theta$$

$$2\int \tan\theta\sec\theta\tan\theta\, d\theta = 2\sec\theta\tan\theta - 2\int \sec^3\theta\, d\theta$$

$$= 2\sec\theta\tan\theta - 2\int \sec^2\theta\sec\theta\, d\theta$$

$$= 2\sec\theta\tan\theta - 2\int\left(\tan^2\theta + 1\right)\sec\theta\, d\theta$$

$$= 2\sec\theta\tan\theta - 2\int\tan^2\theta\sec\theta\, d\theta - 2\int\sec\theta\, d\theta$$

$$\int\sec\theta\, d\theta = \int\sec\theta\,\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\, d\theta$$

$$= \int\frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}\, d\theta$$

$$= \int\frac{1}{\sec\theta + \tan\theta}\, d\left(\sec\theta + \tan\theta\right)$$

$$= \ln\left|\sec\theta + \tan\theta\right|$$

$$2\int\tan^2\theta\sec\theta d\theta = 2\sec\theta\tan\theta - 2\int\tan^2\theta\sec\theta\, d\theta - 2\ln\left|\sec\theta + \tan\theta\right|$$

$$4\int\tan^2\theta\sec\theta\, d\theta = 2\sec\theta\tan\theta - 2\ln\left|\sec\theta + \tan\theta\right|$$

$$2\int\tan^2\theta\sec\theta\, d\theta = \sec\theta\tan\theta - \ln\left|\sec\theta + \tan\theta\right|$$

$$\int\frac{\sqrt{x-2}}{\sqrt{x-1}}\, dx = 2\int\tan\theta\sec\theta\tan\theta\, d\theta$$

$$= \sec\theta\tan\theta - \ln\left|\sec\theta + \tan\theta\right| + C$$

$$= u\sqrt{u^2 - 1} - \ln\left|u + \sqrt{u^2 - 1}\right| + C$$

$$\underline{= \sqrt{x-1}\sqrt{x-2} - \ln\left|\sqrt{x-1} + \sqrt{x-2}\right| + C}$$

Exercise

Evaluate: $\displaystyle\int\frac{dx}{\sqrt{4x^2 - 49}}$

Solution

$$2x = 7\sec\theta \quad \rightarrow dx = \frac{7}{2}\sec\theta\tan\theta\, d\theta$$

$$\sqrt{4x^2 - 49} = \frac{7}{2}\tan\theta$$

$$\int\frac{dx}{\sqrt{4x^2 - 49}} = \frac{1}{2}\int\frac{dx}{\sqrt{x^2 - \left(\frac{7}{2}\right)^2}}$$

$$= \frac{1}{2} \int \frac{\frac{7}{2} \sec\theta \tan\theta \, d\theta}{\frac{7}{2} \tan\theta}$$

$$= \frac{1}{2} \int \sec\theta \, d\theta$$

$$= \frac{1}{2} \int \sec\theta \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta} \, d\theta$$

$$= \frac{1}{2} \int \frac{\sec^2\theta + \sec\theta \tan\theta}{\sec\theta + \tan\theta} \, d\theta$$

$$= \frac{1}{2} \int \frac{1}{\sec\theta + \tan\theta} \, d(\sec\theta + \tan\theta)$$

$$= \frac{1}{2} \ln \left| \sec\theta + \tan\theta \right| + C$$

$$= \frac{1}{2} \ln \left| \frac{2x}{7} + \frac{\sqrt{4x^2 - 49}}{7} \right| + C$$

Exercise

Evaluate $\displaystyle \int \frac{dx}{\sqrt{x^2 - 25}}$

Solution

$$\int \frac{dx}{\sqrt{x^2 - 25}} = \int \frac{5 \sec\theta \tan\theta}{5 \tan\theta} \, d\theta$$

$x = 5 \sec\theta$ $\quad \sqrt{x^2 - 25} = 5 \tan\theta$

$dx = 5 \sec\theta \tan\theta \, d\theta$

$$= \int \sec\theta \, d\theta$$

$$= \int \sec\theta \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta} \, d\theta$$

$$= \int \frac{\sec^2\theta + \sec\theta \tan\theta}{\sec\theta + \tan\theta} \, d\theta$$

$$= \int \frac{1}{\sec\theta + \tan\theta} \, d(\sec\theta + \tan\theta)$$

$$= \ln \left| \sec\theta + \tan\theta \right| + C$$

$$= \ln \left| \frac{x}{5} + \frac{1}{5}\sqrt{x^2 - 25} \right| + C$$

Exercise

Evaluate $\displaystyle\int \frac{\sqrt{x^2-25}}{x}\,dx$

Solution

$x = 5\sec\theta \qquad \sqrt{x^2-25} = 5\tan\theta$

$dx = 5\sec\theta\tan\theta\,d\theta$

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \int \frac{5\tan\theta}{5\sec\theta}\left(5\sec\theta\tan\theta\right)d\theta$$

$$= 5\int \tan^2\theta\,d\theta$$

$$= 5\int \left(\sec^2\theta - 1\right)d\theta$$

$$= 5\left(\tan\theta - \theta\right) + C$$

$$\underline{= \sqrt{x^2-25} - 5\operatorname{arcsec}\frac{x}{5} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x^3}{\sqrt{x^2-a^2}}\,dx$

Solution

$x = a\sec\theta \qquad \sqrt{x^2-a^2} = a\tan\theta$

$dx = a\sec\theta\tan\theta\,d\theta$

$$\int \frac{x^3}{\sqrt{x^2-a^2}}\,dx = \int \frac{a^3\sec^3\theta}{a\tan\theta}\left(a\sec\theta\tan\theta\right)d\theta$$

$$= a^3\int \sec^4\theta\,d\theta$$

$$= a^3\int \left(1+\tan^2\theta\right)\sec^2\theta\,d\theta$$

$$= a^3\int \left(1+\tan^2\theta\right)d\left(\tan\theta\right)$$

$$= a^3\left(\tan\theta + \frac{1}{3}\tan^3\theta\right) + C$$

$$= a^3\left(\frac{\sqrt{x^2-a^2}}{a}+\frac{1}{3}\frac{\left(x^2-a^2\right)^{3/2}}{a^3}\right)+C$$

$$= \sqrt{x^2-a^2}\left(a^2+\frac{1}{3}\left(x^2-a^2\right)\right)+C$$

$$\underline{= \frac{1}{3}\sqrt{x^2-a^2}\left(x^2+2a^2\right)+C}$$

Exercise

Evaluate $\displaystyle\int \frac{x^3}{\sqrt{x^2-25}}\,dx$

Solution

$x = 5\sec\theta \qquad \sqrt{x^2-25} = 5\tan\theta$

$dx = 5\sec\theta\tan\theta\,d\theta$

$$\int \frac{x^3}{\sqrt{x^2-25}}\,dx = \int \frac{5^3\sec^3\theta}{5\tan\theta}\left(5\sec\theta\tan\theta\right)\,d\theta$$

$$= 125\int \sec^4\theta\,d\theta$$

$$= 125\int \left(1+\tan^2\theta\right)\sec^2\theta\,d\theta$$

$$= 125\int \left(1+\tan^2\theta\right)\,d\left(\tan\theta\right)$$

$$= 125\left(\tan\theta+\frac{1}{3}\tan^3\theta\right)+C$$

$$= 125\left(\frac{\sqrt{x^2-25}}{5}+\frac{1}{3}\frac{\left(x^2-25\right)^{3/2}}{125}\right)+C$$

$$= \sqrt{x^2-25}\left(25+\frac{x^2-25}{3}\right)+C$$

$$\underline{= \frac{1}{3}\sqrt{x^2-25}\left(x^2+50\right)+C}$$

Exercise

Evaluate $\displaystyle\int x^3\sqrt{x^2-25}\,dx$

Solution

$$x = 5\sec\theta \qquad \sqrt{x^2-25} = 5\tan\theta$$

$$dx = 5\sec\theta\tan\theta\, d\theta$$

$$\begin{aligned}
\int x^3\sqrt{x^2-25}\,dx &= \int 5^3\sec^3\theta\,(5\tan\theta)(5\sec\theta\tan\theta)\,d\theta \\
&= 5^5\int \sec^4\theta\tan^2\theta\,d\theta \\
&= 5^5\int \sec^2\theta\left(1+\tan^2\theta\right)\tan^2\theta\,d\theta \\
&= 5^5\int \left(\tan^2\theta+\tan^4\theta\right)\,d(\tan\theta) \\
&= 5^5\left(\tfrac{1}{3}\tan^3\theta+\tfrac{1}{5}\tan^5\theta\right)+C \\
&= 5^5\left(\frac{1}{3}\frac{1}{5^3}\left(x^2-25\right)^{3/2}+\frac{1}{5^6}\left(x^2-25\right)^{5/2}\right)+C \\
&= \left(x^2-25\right)^{3/2}\left(\frac{25}{3}+\frac{1}{5}\left(x^2-25\right)\right)+C \\
&= \frac{1}{15}\left(x^2-25\right)^{3/2}\left(125+3x^2-75\right)+C \\
&= \boxed{\frac{1}{15}\left(x^2-25\right)^{3/2}\left(3x^2+50\right)+C}
\end{aligned}$$

Exercise

Evaluate $\displaystyle\int \sqrt{5x^2-1}\,dx$

Solution

$$\sqrt{5}x = \sec\theta \qquad \sqrt{5x^2-1} = \tan\theta$$

$$dx = \frac{1}{\sqrt{5}}\sec\theta\tan\theta\,d\theta$$

$$\int\sqrt{5x^2-1}\,dx = \frac{1}{\sqrt{5}}\int \sec\theta\,d\theta$$

$$= \frac{1}{\sqrt{5}} \int \sec\theta \, \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta} \, d\theta$$

$$= \frac{1}{\sqrt{5}} \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta} \, d\theta$$

$$= \frac{1}{\sqrt{5}} \int \frac{1}{\sec\theta + \tan\theta} \, d(\sec\theta + \tan\theta)$$

$$= \frac{1}{\sqrt{5}} \ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \frac{1}{\sqrt{5}} \ln\left|\frac{x}{\sqrt{5}} + \sqrt{5x^2 - 1}\right| + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\sqrt{9x^2 - 25}}, \quad x > \frac{5}{3}$

Solution

$$x = \frac{5}{3}\sec\theta \qquad \sqrt{9x^2 - 25} = 5\tan\theta$$

$$dx = \frac{5}{3}\sec\theta\tan\theta \, d\theta$$

$$\int \frac{dx}{\sqrt{9x^2 - 25}} = \int \frac{1}{5\tan\theta}\left(\frac{5}{3}\sec\theta\tan\theta\right) d\theta$$

$$= \frac{1}{3} \int \sec\theta \, d\theta$$

$$= \frac{1}{3} \int \sec\theta \, \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta} \, d\theta$$

$$= \frac{1}{3} \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta} \, d\theta$$

$$= \frac{1}{3} \int \frac{1}{\sec\theta + \tan\theta} \, d(\sec\theta + \tan\theta)$$

$$= \frac{1}{3} \ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \frac{1}{3} \ln\left|\frac{3x}{5} + \frac{\sqrt{9x^2 - 25}}{5}\right| + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x\,dx}{\sqrt{4x^2-1}}$

Solution

$$x = \frac{1}{2}\sec\theta \qquad \sqrt{4x^2-1} = \tan\theta$$

$$dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta$$

$$\begin{aligned}\int \frac{x\,dx}{\sqrt{4x^2-1}} &= \frac{1}{2}\int \frac{\sec\theta}{\tan\theta}\left(\frac{1}{2}\sec\theta\tan\theta\right)d\theta \\ &= \frac{1}{4}\int \sec^2\theta\,d\theta \\ &= \frac{1}{4}\tan\theta + C \\ &= \frac{1}{4}\sqrt{4x^2-1} + C\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\sqrt{x^2-81}}$

Solution

$$x = 9\sec\theta \qquad \sqrt{x^2-81} = 9\tan\theta$$

$$dx = 9\sec\theta\tan\theta\,d\theta$$

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2-81}} &= \int \frac{9\sec\theta\tan\theta}{9\tan\theta}\,d\theta \\ &= \int \sec\theta\,d\theta \\ &= \int \sec\theta\,\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\,d\theta \\ &= \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\,d\theta \\ &= \int \frac{1}{\sec\theta+\tan\theta}\,d(\sec\theta+\tan\theta) \\ &= \ln\left|\sec\theta+\tan\theta\right| + C \\ &= \ln\left|\frac{x}{9}+\frac{\sqrt{x^2-81}}{9}\right| + C\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sqrt{x^2-9}}{x}\,dx$

Solution

$$x = 3\sec\theta \qquad \sqrt{x^2-9} = 3\tan\theta$$

$$dx = 3\sec\theta\tan\theta\,d\theta$$

$$\begin{aligned}\int \frac{\sqrt{x^2-9}}{x}\,dx &= \int \frac{3\tan\theta}{3\sec\theta}(3\sec\theta\tan\theta)\,d\theta \\ &= 3\int \tan^2\theta\,d\theta \\ &= 3\int \left(\sec^2\theta - 1\right)\,d\theta \\ &= 3(\tan\theta - \theta) + C \\ &= \sqrt{x^2-9} - 3\sec^{-1}\frac{x}{3} + C\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{1}{x^2\sqrt{9x^2-1}}\,dx$

Solution

$$3x = \sec\theta \qquad \sqrt{9x^2-1} = \tan\theta$$

$$dx = \frac{1}{3}\sec\theta\tan\theta\,d\theta$$

$$\begin{aligned}\int \frac{1}{x^2\sqrt{9x^2-1}}\,dx &= \int \frac{1}{\frac{1}{9}\sec^2\theta(\tan\theta)}\cdot\frac{1}{3}\sec\theta\tan\theta\,d\theta \\ &= 3\int \frac{1}{\sec\theta}\,d\theta \\ &= 3\int \cos\theta\,d\theta \\ &= 3\sin\theta + C \qquad\qquad \sin\theta = \frac{\tan\theta}{\sec\theta} \\ &= 3\frac{\sqrt{9x^2-1}}{3x} + C \\ &= \frac{\sqrt{9x^2-1}}{x} + C\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\left(x^2-36\right)^{3/2}}$

Solution

$$x = 6\sec\theta \qquad x^2 - 36 = 36\tan^2\theta$$

$$dx = 6\sec\theta\tan\theta\,d\theta$$

$$\begin{aligned}
\int \frac{dx}{\left(x^2-36\right)^{3/2}} &= \int \frac{6\sec\theta\tan\theta\,d\theta}{\left(36\tan^2\theta\right)^{3/2}} \\
&= \int \frac{6\sec\theta\tan\theta\,d\theta}{6^3\tan^3\theta} \\
&= \frac{1}{36}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta \\
&= \frac{1}{36}\int \frac{1}{\cos\theta}\frac{\cos^2\theta}{\sin^2\theta}\,d\theta \\
&= \frac{1}{36}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta \\
&= \frac{1}{36}\int \frac{1}{\sin^2\theta}\,d(\sin\theta) \\
&= -\frac{1}{36}\frac{1}{\sin\theta} + C \qquad\qquad \sin\theta = \frac{\tan\theta}{\sec\theta} \\
&= -\frac{1}{36}\cdot\frac{x}{6}\cdot\frac{6}{\sqrt{x^2-36}} + C \\
&= -\frac{1}{36}\frac{x}{\sqrt{x^2-36}} + C
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\sqrt{36-x^2}}$

Solution

$$x = 6\sin\theta \qquad \sqrt{36-x^2} = 6\cos\theta$$

$$dx = 6\cos\theta\,d\theta$$

$$\int \frac{dx}{\sqrt{36-x^2}} = \int \frac{6\cos\theta\,d\theta}{6\cos\theta}$$

$$= \int d\theta$$

$$= \theta + C$$

$$\boxed{= \sin^{-1}\frac{x}{6} + C}$$

Exercise

Evaluate the integral $\int \sqrt{a^2 - x^2}\, dx$

Solution

$$x = a\sin\theta \qquad \sqrt{a^2 - x^2} = a\cos\theta$$

$$dx = a\cos\theta\, d\theta$$

$$\int \sqrt{a^2 - x^2}\, dx = \int a\cos\theta\left(a\cos\theta\, d\theta\right)$$

$$= a^2 \int \cos^2\theta\, d\theta$$

$$= \frac{a^2}{2}\int (1 + \cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{a^2}{2}\left(\arcsin\frac{x}{a} + \sin\theta\cos\theta\right) + C$$

$$= \frac{a^2}{2}\left(\arcsin\frac{x}{a} + \frac{x}{a}\frac{\sqrt{a^2 - x^2}}{a}\right) + C$$

$$\boxed{= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{1}{2}x\sqrt{a^2 - x^2} + C}$$

Exercise

Evaluate the integral $\int \frac{dx}{2 - \sqrt{3x}}$

Solution

Let $u^2 = \sqrt{3x} \quad \rightarrow \quad u^4 = 3x$

$$4u^3 du = 3dx$$

$$\int \frac{dx}{2-\sqrt{3x}} = \frac{4}{3}\int \frac{u^3}{2-u^2}du$$

$$u = \sqrt{2}\sin\theta \qquad 2-u^2 = 2\cos^2\theta$$

$$du = \sqrt{2}\cos\theta\, d\theta$$

$$\int \frac{dx}{2-\sqrt{3x}} = \frac{4}{3}\int \frac{2\sqrt{2}\sin^3\theta}{2\cos^2\theta}\left(\sqrt{2}\cos\theta\, d\theta\right)$$

$$= \frac{8}{3}\int \frac{\sin^3\theta}{\cos\theta}\, d\theta$$

$$= \frac{8}{3}\int \frac{\sin^2\theta}{\cos\theta}\, \left(\sin\theta\, d\theta\right)$$

$$= -\frac{8}{3}\int \frac{\left(1-\cos^2\theta\right)}{\cos\theta}\, d\left(\cos\theta\right)$$

$$= -\frac{8}{3}\int \left(\frac{1}{\cos\theta} - \cos\theta\right)\, d\left(\cos\theta\right)$$

$$= -\frac{8}{3}\left(\ln\left|\cos\theta\right| - \frac{1}{2}\cos^2\theta\right) + C$$

$$= -\frac{8}{3}\left(\ln\left|\frac{\sqrt{2-u^2}}{\sqrt{2}}\right| - \frac{1}{2}\frac{2-u^2}{2}\right) + C$$

$$= -\frac{8}{3}\left(\ln\left|\frac{\sqrt{2-\sqrt{3x}}}{\sqrt{2}}\right| - \frac{2-\sqrt{3x}}{4}\right) + C$$

$$= -\frac{8}{3}\ln\left|\frac{\sqrt{2-\sqrt{3x}}}{\sqrt{2}}\right| + \frac{2}{3}\left(2-\sqrt{3x}\right) + C$$

$$= -\frac{8}{3}\ln\left(\frac{2-\sqrt{3x}}{2}\right)^{1/2} + \frac{4}{3} - \frac{2}{3}\sqrt{3x} + C$$

$$\boxed{= -\frac{4}{3}\ln\left(\frac{2-\sqrt{3x}}{2}\right) - \frac{2}{3}\sqrt{3x} + C_1}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x\, dx}{1-\sqrt{x}}$

Solution

Let $u^2 = \sqrt{x} \rightarrow u^4 = x$

$$4u^3 du = dx$$

$$\int \frac{x\,dx}{1-\sqrt{x}} = 4\int \frac{u^4}{1-u^2}u^3du$$

$$= 4\int \frac{u^7}{1-u^2}\ du$$

$$u = \sin\theta \qquad 1-u^2 = \cos^2\theta$$

$$du = \cos\theta\ d\theta$$

$$\int \frac{x\,dx}{1-\sqrt{x}} = 4\int \frac{\sin^7\theta}{\cos^2\theta}\cos\theta\ d\theta$$

$$= 4\int \frac{\sin^6\theta}{\cos\theta}(\sin\theta\ d\theta)$$

$$= -4\int \frac{\left(1-\cos^2\theta\right)^3}{\cos\theta}d\left(\cos\theta\right)$$

$$= -4\int \frac{1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta}{\cos\theta}\ d\left(\cos\theta\right)$$

$$= -4\int \left(\frac{1}{\cos\theta}-3\cos\theta+3\cos^3\theta-\cos^5\theta\right)\ d\left(\cos\theta\right)$$

$$= -4\left(\ln\left|\cos\theta\right|-\frac{3}{2}\cos^2\theta+\frac{3}{4}\cos^4\theta-\frac{1}{6}\cos^6\theta\right)+C$$

$$= -4\ln\left|\cos\theta\right|+6\cos^2\theta-3\cos^4\theta+\frac{2}{3}\cos^6\theta+C$$

$$= -4\ln\left|\sqrt{1-u^2}\right|+6\left(1-u^2\right)-3\left(1-u^2\right)^2+\frac{2}{3}\left(1-u^2\right)^3+C$$

$$= -4\ln\left(1-\sqrt{x}\right)^{1/2}+6-6\sqrt{x}-3\left(1-\sqrt{x}\right)^2+\frac{2}{3}\left(1-\sqrt{x}\right)^3+C$$

$$= -2\ln\left(1-\sqrt{x}\right)-6\sqrt{x}-3\left(1-2\sqrt{x}+x\right)+\frac{2}{3}\left(1-3\sqrt{x}+3x-x\sqrt{x}\right)$$

$$= -2\ln\left(1-\sqrt{x}\right)-6\sqrt{x}+6\sqrt{x}-3x-2\sqrt{x}+2x-\frac{2}{3}x\sqrt{x}$$

$$\boxed{= -2\ln\left(1-\sqrt{x}\right)-x-2\sqrt{x}-\frac{2}{3}x\sqrt{x}+C_1}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\sqrt{1-2x^2}}$

Solution

$$\sqrt{2}\,x = \sin\theta \qquad \sqrt{1-2x^2} = \cos\theta$$

$$dx = \frac{1}{\sqrt{2}}\cos\theta\, d\theta$$

$$\int \frac{dx}{\sqrt{1-2x^2}} = \frac{1}{\sqrt{2}}\int \frac{\cos\theta\, d\theta}{\cos\theta}$$

$$= \frac{1}{\sqrt{2}}\theta + C$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\sqrt{2}\,x\right) + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^2}{\left(100-x^2\right)^{3/2}}\,dx$

Solution

$$x = 10\sin\theta \qquad \sqrt{100-x^2} = 10\cos\theta$$

$$dx = 10\cos\theta\, d\theta$$

$$\int \frac{x^2}{\left(100-x^2\right)^{3/2}}\,dx = \int \frac{100\sin^2\theta}{(10\cos\theta)^3}(10\cos\theta)\, d\theta$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta}\, d\theta$$

$$= \int \tan^2\theta\, d\theta$$

$$= \int \left(\sec^2\theta - 1\right)\, d\theta$$

$$= \tan\theta - \theta + C$$

$$= \frac{x}{\sqrt{100-x^2}} - \sin^{-1}\frac{x}{10} + C$$

Exercise

Evaluate: $\displaystyle\int \frac{2dx}{\sqrt{1-4x^2}}$

Solution

$$\int \frac{2dx}{\sqrt{1-4x^2}} = \int \frac{d(2x)}{\sqrt{1-(2x)^2}}$$

$$= \sin^{-1} 2x + C$$

Exercise

Evaluate $\displaystyle\int \sqrt{\frac{x}{1-x}}\, dx$

Solution

$$x = \sin^2\theta \qquad \sqrt{1-x} = \cos\theta$$

$$dx = 2\sin x \cos\theta\, d\theta$$

$$\int \sqrt{\frac{x}{1-x}}\, dx = \int \frac{\sin\theta}{\cos\theta} 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int \sin^2\theta\, d\theta$$

$$= \int (1-\cos 2\theta)\, d\theta$$

$$= \theta - \frac{1}{2}\sin 2\theta + C$$

$$= \theta - \sin\theta\cos\theta + C$$

$$= \arcsin\sqrt{x} - \sqrt{x}\sqrt{1-x} + C$$

Exercise

Evaluate $\displaystyle\int x\sqrt{2x-x^2}\, dx$

Solution

$$2x - x^2 = 1 - 1 + 2x - x^2$$

$$= 1 - \left(1 - 2x + x^2\right)$$

$$= 1 - (1-x)^2$$

$$\int x\sqrt{2x-x^2}\,dx = \int x\sqrt{1-(1-x)^2}\,dx$$

$$1-x=\sin\theta \qquad \sqrt{1-(1-x)^2}=\cos\theta$$

$$dx=-\cos\theta\,d\theta$$

$$\begin{aligned}
\int x\sqrt{2x-x^2}\,dx &= \int (1-\sin\theta)(\cos\theta)(-\cos\theta\,d\theta) \\
&= -\int (1-\sin\theta)\left(\cos^2\theta\right)\,d\theta \\
&= -\int \cos^2\theta\,d\theta + \int \left(\sin\theta\cos^2\theta\right)\,d\theta \\
&= -\frac{1}{2}\int (1+\cos 2\theta)\,d\theta - \int \cos^2\theta\,d(\cos\theta) \\
&= -\frac{1}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)-\frac{1}{3}\cos^3\theta + C \\
&= -\frac{1}{2}\theta-\frac{1}{2}\sin\theta\cos\theta-\frac{1}{3}\cos^3\theta + C \\
&= -\frac{1}{2}\arcsin(1-x)-\frac{1}{2}(1-x)\sqrt{2x-x^2}-\frac{1}{3}\left(\sqrt{2x-x^2}\right)^3 + C
\end{aligned}$$

Exercise

Evaluate $\displaystyle\int x^2\sqrt{a^2-x^2}\,dx$

Solution

$$x=a\sin\theta \qquad \sqrt{a^2-x^2}=a\cos\theta$$

$$dx=a\cos\theta\,d\theta$$

$$\begin{aligned}
\int x^2\sqrt{a^2-x^2}\,dx &= \int a^2\sin^2\theta(a\cos\theta)(a\cos\theta\,d\theta) \\
&= a^4\int \sin^2\theta\cos^2\theta\,d\theta \\
&= \frac{a^4}{4}\int (1-\cos 2\theta)(1+\cos 2\theta)\,d\theta \\
&= \frac{1}{4}a^4\int \left(1-\cos^2 2\theta\right)\,d\theta
\end{aligned}$$

$$=\frac{1}{4}a^4\int\left(1-\frac{1}{2}-\frac{1}{2}\cos 4\theta\right)\,d\theta$$

$$=\frac{1}{4}a^4\int\left(\frac{1}{2}-\frac{1}{2}\cos 4\theta\right)\,d\theta$$

$$=\frac{1}{8}a^4\left(\theta-\frac{1}{4}\sin 4\theta\right)+C$$

$$=\frac{1}{8}a^4\left(\theta-\frac{1}{2}\sin 2\theta\cos 2\theta\right)+C$$

$$=\frac{1}{8}a^4\left(\theta-\sin\theta\cos\theta\left(1-2\sin^2\theta\right)\right)+C$$

$$=\frac{1}{8}a^4\left(\arcsin\frac{x}{a}-\frac{x}{a}\frac{\sqrt{a^2-x^2}}{a}\left(1-2\frac{x^2}{a^2}\right)\right)+C$$

$$=\frac{1}{8}a^4\left(\arcsin\frac{x}{a}-\frac{x\sqrt{a^2-x^2}}{a^2}+2\frac{x^3\sqrt{a^2-x^2}}{a^4}\right)+C$$

$$\boxed{=\frac{1}{8}a^4\arcsin\frac{x}{a}-\frac{1}{8}a^2x\sqrt{a^2-x^2}+\frac{1}{4}x^3\sqrt{a^2-x^2}+C}$$

Exercise

Evaluate $\displaystyle\int\frac{x}{\sqrt{4x-x^2}}\,dx$

Solution

$$4x-x^2=4-4+4x-x^2$$

$$=4-\left(4-4x+x^2\right)$$

$$\boxed{=4-(2-x)^2}$$

$$\int\frac{x}{\sqrt{4x-x^2}}\,dx=\int\frac{x}{\sqrt{4-(2-x)^2}}\,dx$$

$$2-x=2\sin\theta \qquad \sqrt{4-(2-x)^2}=2\cos\theta$$

$$dx=-2\cos\theta\,d\theta$$

$$\int\frac{x}{\sqrt{4x-x^2}}\,dx=\int\frac{2-2\sin\theta}{2\cos\theta}(-2\cos\theta\,d\theta)$$

$$=-2\int(1-\sin\theta)\,d\theta$$

$$= -2\left(\theta + \cos\theta\right) + C$$

$$= -2\left(\arcsin\left(1 - \frac{1}{2}x\right) + \frac{\sqrt{4 - (2 - x)^2}}{2}\right) + C$$

$$\boxed{= -2\arcsin\left(2 - x\right) - \sqrt{4x - x^2} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x}{\sqrt{ax - x^2}}\,dx, \quad a > 0$

Solution

$$ax - x^2 = \frac{a^2}{4} - \frac{a^2}{4} + ax - x^2$$

$$= \frac{a^2}{4} - \left(\frac{a^2}{4} - ax + x^2\right)$$

$$\boxed{= \frac{a^2}{4} - \left(\frac{a}{2} - x\right)^2}$$

$$\int \frac{x}{\sqrt{ax - x^2}}\,dx = \int \frac{x}{\sqrt{\frac{a^2}{4} - \left(\frac{a}{2} - x\right)^2}}\,dx$$

$$\frac{a}{2} - x = \frac{a}{2}\sin\theta \qquad \sqrt{\frac{a^2}{4} - \left(\frac{a}{2} - x\right)^2} = \frac{a}{2}\cos\theta$$

$$dx = -\frac{a}{2}\cos\theta\ d\theta$$

$$\int \frac{x}{\sqrt{ax - x^2}}\,dx = \int \frac{\frac{a}{2} - \frac{a}{2}\sin\theta}{\frac{a}{2}\cos\theta}\left(-\frac{a}{2}\cos\theta\ d\theta\right)$$

$$= -\frac{a}{2}\int \left(1 - \sin\theta\right)\ d\theta$$

$$= -\frac{a}{2}\left(\theta + \cos\theta\right) + C$$

$$= -\frac{a}{2}\arcsin\left(1 - \frac{2}{a}x\right) - \sqrt{ax - x^2} + C$$

$$\boxed{= \frac{a}{2}\left(\arcsin\left(\frac{2}{a}x - 1\right) + \frac{2}{a}\sqrt{ax - x^2}\right) + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx$

Solution

$x = a\sin\theta \qquad \sqrt{a^2-x^2} = a\cos\theta$

$dx = a\cos\theta\, d\theta$

$$\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx = \int \frac{a^2\sin^2\theta}{a\cos\theta}(a\cos\theta\, d\theta)$$

$$= a^2 \int \sin^2\theta\, d\theta$$

$$= \frac{1}{2}a^2 \int (1-\cos 2\theta\,)\,d\theta$$

$$= \frac{1}{2}a^2\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{2}a^2(\theta - \sin\theta\cos\theta) + C$$

$$= \frac{1}{2}a^2\left(\arcsin\frac{x}{a} - \frac{x}{a}\frac{\sqrt{a^2-x^2}}{a}\right) + C$$

$$= \frac{1}{2}a^2\arcsin\left(\frac{x}{a}\right) - \frac{1}{2}x\sqrt{a^2-x^2} + C$$

Exercise

Evaluate $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$

Solution

$x = 4\sin\theta \qquad \sqrt{16-x^2} = 4\cos\theta$

$dx = 4\cos\theta\, d\theta$

$$\int \frac{x^2}{\sqrt{16-x^2}}\,dx = \int \frac{16\sin^2\theta}{4\cos\theta}(4\cos\theta\, d\theta)$$

$$= 16\int \sin^2\theta\, d\theta$$

$$= 8\int (1-\cos 2\theta)\, d\theta$$

$$= 8\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$

$$= 8\left(\sin^{-1}\frac{x}{4} - \sin\theta\cos\theta\right) + C$$

$$= 8\left(\sin^{-1}\frac{x}{4} - \frac{x}{4}\frac{\sqrt{16-x^2}}{4}\right) + C$$

$$\boxed{= 8\sin^{-1}\frac{x}{4} - \frac{1}{2}x\sqrt{16-x^2} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\left(16-x^2\right)^{3/2}}$

Solution

$$x = 4\sin\theta \qquad \sqrt{16-x^2} = 4\cos\theta$$

$$dx = 4\cos\theta d\theta$$

$$\int \frac{dx}{\left(16-x^2\right)^{3/2}} = \int \frac{4\cos\theta}{(4\cos\theta)^3} d\theta$$

$$= \frac{1}{16}\int \frac{1}{\cos^2\theta} d\theta$$

$$= \frac{1}{16}\int \sec^2\theta \, d\theta$$

$$\boxed{= \frac{1}{16}\tan\theta + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}}$

Solution

$$x = 3\sin\theta \qquad \sqrt{9-x^2} = 3\cos\theta$$

$$dx = 3\cos\theta d\theta$$

$$\int \frac{dx}{x^2\sqrt{9-x^2}} = \int \frac{3\cos\theta}{9\sin^2\theta(3\cos\theta)} d\theta$$

$$= \frac{1}{9}\int \csc^2\theta \, d\theta$$

$$= -\frac{1}{9}\cot\theta + C$$

$$= -\frac{1}{9}\frac{\cos\theta}{\sin\theta} + C$$

$$= -\frac{1}{9}\frac{\sqrt{9-x^2}}{3}\cdot\frac{3}{x} + C$$

$$\boxed{= -\frac{1}{9}\frac{\sqrt{9-x^2}}{x} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$

Solution

$x = 2\sin\theta \qquad \sqrt{4-x^2} = 2\cos\theta$

$dx = 2\cos\theta \, d\theta$

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta}{4\sin^2\theta(2\cos\theta)}d\theta$$

$$= \frac{1}{4}\int \csc^2\theta \, d\theta$$

$$= -\frac{1}{4}\cot\theta + C$$

$$= -\frac{1}{4}\frac{\cos\theta}{\sin\theta} + C$$

$$= -\frac{1}{4}\frac{\sqrt{4-x^2}}{2}\cdot\frac{2}{x} + C$$

$$\boxed{= -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{4}{x^2\sqrt{16-x^2}}\,dx$

Solution

$x = 4\sin\theta \qquad \sqrt{16-x^2} = 4\cos\theta$

$dx = 4\cos\theta \, d\theta$

$$\int \frac{4}{x^2\sqrt{16-x^2}}\,dx = \int \frac{16\cos\theta}{16\sin^2\theta(4\cos\theta)}\,d\theta$$

$$=\frac{1}{4}\int \csc^2\theta\, d\theta$$

$$=-\frac{1}{4}\cot\theta + C$$

$$=-\frac{1}{4}\frac{\cos\theta}{\sin\theta} + C$$

$$\boxed{=-\frac{1}{4}\frac{\sqrt{16-x^2}}{x} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x^3}{\sqrt{9-x^2}}\, dx$

Solution

$x = 3\sin\theta \qquad \sqrt{9-x^2} = 3\cos\theta$

$dx = 3\cos\theta\, d\theta$

$$\int \frac{x^3}{\sqrt{9-x^2}}\, dx = \int \frac{27\sin^3\theta}{3\cos\theta}\,(3\cos\theta)\, d\theta$$

$$= 27\int \sin^3\theta\, d\theta$$

$$= 27\int \left(1-\cos^2\theta\right)\, d(\cos\theta)$$

$$= 27\left(\cos\theta - \frac{1}{3}\cos^3\theta\right) + C$$

$$= 27\cos\theta - 9\cos^3\theta + C$$

$$= 27\frac{\sqrt{9-x^2}}{3} - 9\left(\frac{\sqrt{9-x^2}}{3}\right)^3 + C$$

$$= 9\sqrt{9-x^2} - \frac{1}{3}\left(9-x^2\right)\sqrt{9-x^2} + C$$

$$= \frac{1}{3}\sqrt{9-x^2}\left(27-9+x^2\right) + C$$

$$\boxed{= \frac{1}{3}\sqrt{9-x^2}\left(18+x^2\right) + C}$$

Exercise

Evaluate $\displaystyle\int \sqrt{25-4x^2}\ dx$

Solution

$2x = 5\sin\theta \qquad \sqrt{25-4x^2} = 5\cos\theta$

$dx = \frac{5}{2}\cos\theta\, d\theta$

$$\int \sqrt{25-4x^2}\ dx = \frac{25}{2}\int \cos^2\theta\ d\theta$$

$$= \frac{25}{4}\int (1+\cos 2\theta)\ d\theta$$

$$= \frac{25}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{25}{4}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{25}{4}\left(\sin^{-1}\frac{2x}{5} + \frac{2x}{5}\frac{\sqrt{25-4x^2}}{5}\right) + C$$

$$\underline{= \frac{25}{4}\sin^{-1}\frac{2x}{5} + \frac{1}{2}x\sqrt{25-4x^2} + C}$$

Exercise

Evaluate $\displaystyle\int e^x\sqrt{1-e^{2x}}\ dx$

Solution

$e^x = \sin\theta \qquad \sqrt{1-e^{2x}} = \cos\theta$

$e^x dx = \cos\theta\ d\theta$

$$\int e^x\sqrt{1-e^{2x}}\ dx = \int \cos^2\theta\ d\theta$$

$$= \frac{1}{2}\int (1+\cos 2\theta)\ d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{2}(\theta + \sin\theta\cos\theta) + C$$

$$\underline{= \frac{1}{2}\left(\arcsin e^x + e^x\sqrt{1-e^{2x}}\right) + C}$$

Exercise

Evaluate $\displaystyle\int \frac{\sqrt{1-x}}{\sqrt{x}}\,dx$

Solution

$$\sqrt{x} = \sin\theta \rightarrow x = \sin^2\theta \quad \sqrt{1-x} = \cos\theta$$

$$dx = 2\sin\theta\cos\theta\,d\theta$$

$$\begin{aligned}\int \frac{\sqrt{1-x}}{\sqrt{x}}\,dx &= \int \frac{\cos\theta}{\sin\theta}(2\sin\theta\cos\theta)\,d\theta \\ &= 2\int \cos^2\theta\,d\theta \\ &= \int (1+\cos 2\theta)\,d\theta \\ &= \theta + \frac{1}{2}\sin 2\theta + C \\ &= \theta + 2\sin\theta\cos\theta + C \\ &= \arcsin\sqrt{x} + 2\sqrt{x}\sqrt{1-x} + C\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{y\,dy}{\sqrt{16-y^2}}$

Solution

$$y = 4\sin\theta \qquad \sqrt{16-y^2} = 4\cos\theta$$

$$dy = 4\cos\theta\,d\theta$$

$$\begin{aligned}\int \frac{y\,dy}{\sqrt{16-y^2}} &= \int \frac{4\sin\theta}{4\cos\theta}(4\cos\theta\,d\theta) \\ &= 4\int \sin\theta\,d\theta \\ &= -4\cos\theta + C \\ &= -\sqrt{16-y^2} + C\end{aligned}$$

OR

$$\int \frac{y\,dy}{\sqrt{16-y^2}} = -\frac{1}{2}\int \left(16-y^2\right)^{-1/2} d\left(16-y^2\right)$$

$$= -\left(16 - y^2\right)^{1/2} + C$$

$$\boxed{= -\sqrt{16 - y^2} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^3}{\sqrt{4-x^2}}\,dx$

Solution

$x = 2\sin\theta \qquad \sqrt{4-x^2} = 2\cos\theta$

$dx = 2\cos\theta\, d\theta$

$$\int \frac{x^3}{\sqrt{4-x^2}}\,dx = \int \frac{8\sin^3\theta}{2\cos\theta}(2\cos\theta\, d\theta)$$

$$= 8\int \sin^2\theta \sin\theta\, d\theta$$

$$= -8\int \left(1 - \cos^2\theta\right)\, d(\cos\theta)$$

$$= -8\left(\cos\theta - \frac{1}{3}\cos^3\theta\right) + C$$

$$= -8\frac{\sqrt{4-x^2}}{2} + \frac{8}{3}\frac{\left(4-x^2\right)\sqrt{4-x^2}}{8} + C$$

$$= \frac{1}{3}\sqrt{4-x^2}\left(-12 + 4 - x^2\right) + C$$

$$\boxed{= -\frac{1}{3}\sqrt{4-x^2}\left(x^2 + 8\right) + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$

Solution

$x = 2\sin\theta \qquad \sqrt{4-x^2} = 2\cos\theta$

$dx = 2\cos\theta\, d\theta$

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{1}{2\cos\theta}(2\cos\theta\, d\theta)$$

$$= \int d\theta$$

$$= \theta + C$$

$$\boxed{= \sin^{-1}\frac{x}{2} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\left(1-x^2\right)^{3/2}}$

Solution

$x = \sin\theta \qquad \sqrt{1-x^2} = \cos\theta$

$dx = \cos\theta\ d\theta$

$$\int \frac{dx}{\left(1-x^2\right)^{3/2}} = \int \frac{\cos\theta\ d\theta}{\cos^3\theta}$$

$$= \int \frac{d\theta}{\cos^2\theta}$$

$$= \int \sec^2\theta\ d\theta$$

$$= \tan\theta + C$$

$$= \frac{\sin\theta}{\cos\theta} + C$$

$$\boxed{= \frac{x}{\sqrt{1-x^2}} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\left(1-x^2\right)^{5/2}}{x^8}dx$

Solution

$x = \sin\theta \qquad 1-x^2 = \cos^2\theta$

$dx = \cos\theta\ d\theta$

$$\int \frac{\left(1-x^2\right)^{5/2}}{x^8}dx = \int \frac{\left(\cos^2\theta\right)^{5/2}}{\sin^8\theta}\cos\theta\ d\theta$$

$$= \int \frac{\cos^5\theta}{\sin^8\theta} \cos\theta \, d\theta$$

$$= \int \frac{\cos^6\theta}{\sin^6\theta} \cdot \frac{1}{\sin^2\theta} \, d\theta$$

$$= \int \cot^6\theta \csc^2\theta \, d\theta$$

$$= -\int \cot^6\theta \, d(\cot\theta)$$

$$= -\frac{1}{7}\cot^7\theta + C$$

$$= -\frac{1}{7}\left(\frac{\cos\theta}{\sin\theta}\right)^7 + C$$

$$\boxed{= -\frac{1}{7}\left(\frac{\sqrt{1-x^2}}{x}\right)^7 + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{3-2x-x^2}}$

Solution

$$3-2x-x^2 = 3+1-1-2x-x^2$$

$$= 4-\left(1+2x+x^2\right)$$

$$\boxed{= 4-(1+x)^2}$$

$$\int \frac{dx}{\sqrt{3-2x-x^2}} = \int \frac{dx}{\sqrt{4-(1+x)^2}}$$

$x+1 = 2\sin\theta \qquad \sqrt{4-(1+x)^2} = 2\cos\theta$

$dx = 2\cos\theta \, d\theta$

$$\int \frac{dx}{\sqrt{3-2x-x^2}} = \int \frac{2\cos\theta}{2\cos\theta} \, d\theta$$

$$= \int d\theta$$

$$= \theta + C$$

$$\boxed{= \sin^{-1}\left(\frac{x+1}{2}\right) + C}$$

Exercise

Evaluate $\displaystyle\int \frac{1}{x^4+4x^2+4}\,dx$

Solution

$x=\sqrt{2}\tan\theta \qquad x^2+2=2\sec^2\theta$

$dx=\sqrt{2}\sec^2\theta\,d\theta$

$$\int \frac{1}{x^4+4x^2+4}\,dx=\int \frac{dx}{\left(x^2+2\right)^2}$$

$$=\int \frac{\sqrt{2}\sec^2\theta}{4\sec^4\theta}\,d\theta$$

$$=\frac{\sqrt{2}}{4}\int \cos^2\theta\,d\theta$$

$$=\frac{\sqrt{2}}{8}\int (1+\cos 2\theta)\,d\theta$$

$$=\frac{\sqrt{2}}{8}\left(\theta+\frac{1}{2}\sin 2\theta\right)+C$$

$$=\frac{\sqrt{2}}{8}(\theta+\sin\theta\cos\theta)+C$$

$$=\frac{\sqrt{2}}{8}\left(\theta+\frac{\tan\theta}{\sec\theta}\frac{1}{\sec\theta}\right)+C$$

$$=\frac{\sqrt{2}}{8}\left(\arctan\frac{x}{\sqrt{2}}+\frac{x\sqrt{2}}{x^2+2}\right)+C$$

Exercise

Evaluate $\displaystyle\int \frac{x^3+x+1}{x^4+2x^2+1}\,dx$

Solution

$x=\tan\theta \qquad x^2+1=\sec^2\theta$

$dx=\sec^2\theta\,d\theta$

$$\int \frac{x^3+x+1}{x^4+2x^2+1}\,dx=\int \frac{x^3+x}{x^4+2x^2+1}\,dx+\int \frac{1}{\left(x^2+1\right)^2}dx$$

$$= \frac{1}{4}\int \frac{1}{x^4+2x^2+1}\, d\left(x^4+2x^2+1\right) + \int \frac{1}{\sec^2\theta}\, d\theta$$

$$= \frac{1}{4}\ln\left(x^2+1\right)^2 + \int \cos^2\theta\, d\theta$$

$$= \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{2}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{2}\left(\theta + \frac{\tan\theta}{\sec\theta}\frac{1}{\sec\theta}\right) + C$$

$$= \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right) + C$$

Exercise

Evaluate $\int \operatorname{arcsec} 2x\, dx \quad x > \frac{1}{2}$

Solution

$$u = \operatorname{arcsec} 2x \qquad dv = dx$$

$$du = \frac{dx}{x\sqrt{4x^2-1}} \qquad v = x$$

$$\int \operatorname{arcsec} 2x\, dx = x \operatorname{arcsec} 2x - \int \frac{dx}{\sqrt{4x^2-1}}$$

$$2x = \sec\theta \qquad \sqrt{4x^2-1} = \tan\theta$$

$$dx = \frac{1}{2}\sec\theta\tan\theta\, d\theta$$

$$= x\operatorname{arcsec} 2x - \frac{1}{2}\int \frac{\sec\theta\tan\theta}{\tan\theta}\, d\theta$$

$$= x\operatorname{arcsec} 2x - \frac{1}{2}\int \sec\theta\, d\theta$$

$$= x\operatorname{arcsec} 2x - \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C$$

$$= x\operatorname{arcsec} 2x - \frac{1}{2}\ln\left|2x + \sqrt{4x^2-1}\right| + C$$

Exercise

Evaluate $\displaystyle\int x \ \arcsin x \ dx$

Solution

$u = \arcsin x \qquad dv = xdx$

$du = \dfrac{dx}{\sqrt{1-x^2}} \qquad v = \frac{1}{2}x^2$

$$\int x \ \arcsin x \ dx = \frac{1}{2}x^2 \arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}} dx$$

$x = \sin\theta \qquad \sqrt{1-x^2} = \cos\theta$

$dx = \cos\theta \ d\theta$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\cos\theta}\cos\theta \ d\theta$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{2}\int \sin^2\theta \ d\theta$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{4}\int (1-\cos 2\theta) \ d\theta$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{4}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{4}(\theta - \sin\theta\cos\theta) + C$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{4}\left(\arcsin x - \frac{1}{x\sqrt{1-x^2}}\right) + C$$

$$= \frac{1}{4}\left(2x^2-1\right)\arcsin x + \frac{1}{4}\frac{1}{x\sqrt{1-x^2}} + C$$

Exercise

Evaluate $\displaystyle\int_0^2 \sqrt{1+4x^2} \ dx$

Solution

$2x = \tan\theta \qquad \sqrt{1+4x^2} = \sec\theta$

$dx = \frac{1}{2}\sec^2\theta \, d\theta$

$$\int_0^2 \sqrt{1+4x^2} \ dx = \frac{1}{2}\int_0^2 \sec^3\theta \ d\theta$$

$$u = \sec x \qquad dv = \sec^2 x dx$$

$$du = \sec x \tan x \, dx \qquad v = \tan x$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \tan x \left(\sec x \tan x \, dx\right)$$

$$= \sec x \tan x - \int \tan^2 x \sec x \, dx$$

$$= \sec x \tan x - \int \left(\sec^2 x - 1\right) \sec x \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$2\int \sec^3 x \, dx = \sec x \tan x + \ln\left|\sec x + \tan x\right|$$

$$\int \sec^3 x \, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right|$$

$$\int_0^2 \sqrt{1+4x^2} \, dx = \frac{1}{4} \left(\sec\theta \tan\theta + \ln\left|\sec\theta + \tan\theta\right| \right. \Big|_0^2$$

$$= \frac{1}{4} \left(2x\sqrt{1+4x^2} + \ln\left|2x + \sqrt{1+4x^2}\right| \right. \Big|_0^2$$

$$= \frac{1}{4}\left(4\sqrt{17} + \ln\left|4 + \sqrt{17}\right|\right)$$

$$\boxed{= \sqrt{17} + \frac{1}{4}\ln\left(4 + \sqrt{17}\right)}$$

Exercise

Evaluate $\displaystyle \int_0^3 \frac{x^3}{\sqrt{x^2+9}} dx$

Solution

$$x = 3\tan\theta \qquad \sqrt{x^2+9} = 3\sec\theta$$

$$dx = 3\sec^2\theta \, d\theta$$

$$\int_0^3 \frac{x^3}{\sqrt{x^2+9}} dx = \int_0^3 \frac{27\tan^3\theta}{3\sec\theta} 3\sec^2\theta \, d\theta$$

$$= 27\int_0^3 \tan^2\theta \tan\theta \sec\theta \ d\theta$$

$$= 27\int_0^3 \left(\sec^2\theta - 1\right) \ d\left(\sec\theta\right)$$

$$= 27 \ \left(\frac{1}{3}\sec^3\theta - \sec\theta \ \right|_0^3$$

$$= 9\sqrt{x^2+9}\left(\frac{x^2+9}{27} - 1\right) \ \Big|_0^3$$

$$= \frac{1}{3}\sqrt{x^2+9}\left(x^2-18\right) \ \Big|_0^3$$

$$\underline{= -9\sqrt{2} + 18}$$

Exercise

Evaluate the integral $\displaystyle\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^x \, dx}{\left(1+e^{2x}\right)^{3/2}}$

Solution

$e^x = \tan\theta \qquad\qquad \tan^{-1}\left(\frac{3}{4}\right) < \theta < \tan^{-1}\left(\frac{4}{3}\right)$

$x = \ln\left(\tan\theta\right)$

$dx = \dfrac{\sec^2\theta}{\tan\theta} d\theta$

$$\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^x \, dx}{\left(1+e^{2x}\right)^{3/2}} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{\tan\theta}{\left(\sec^2\theta\right)^{3/2}} \ \frac{\sec^2\theta}{\tan\theta} \ d\theta$$

$$= \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{\sec^2\theta}{\sec^3\theta} d\theta$$

$$= \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{1}{\sec\theta} d\theta$$

$$= \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \cos\theta \ d\theta$$

$$= \sin\theta \Big|_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)}$$

$$= \sin\left(\tan^{-1}(3/4)\right) - \sin\left(\tan^{-1}(4/3)\right)$$

$$= \frac{4}{5} - \frac{3}{5}$$

$$\boxed{= \frac{1}{5}}$$

Exercise

Evaluate the integral $\displaystyle\int_1^e \frac{e^x dx}{\left(1+e^{2x}\right)^{3/2}}$

Solution

$$\int_1^e \frac{e^x dx}{\left(1+e^{2x}\right)^{3/2}} = \int_1^e \frac{1}{\left(1+\left(e^x\right)^2\right)^{3/2}}\, d\left(e^x\right) \qquad \text{Let } y = e^x$$

$$= \int_1^e \frac{1}{\left(1+y^2\right)^{3/2}}\, dy$$

$$y = \tan\theta \qquad \sqrt{y^2+1} = \sec\theta$$

$$dy = \sec^2\theta\, d\theta$$

$$\int_1^e \frac{e^x dx}{\left(1+e^{2x}\right)^{3/2}} = \int_1^e \frac{1}{\left(\sec^2\theta\right)^{3/2}} \sec^2\theta\, d\theta$$

$$= \int_1^e \frac{1}{\sec^3\theta} \sec^2\theta\, d\theta$$

$$= \int_1^e \frac{1}{\sec\theta}\, d\theta$$

$$= \int_1^e \cos\theta\, d\theta$$

$$= \sin\theta \Big|_1^e$$

$$= \frac{\tan\theta}{\sec\theta}\Bigg|_1^e$$

$$= \frac{y}{\sqrt{1+y^2}}\Bigg|_1^e$$

$$= \frac{e^x}{\sqrt{1+e^{2x}}}\Bigg|_1^e$$

$$\boxed{= \frac{e^e}{\sqrt{1+e^{2e}}} - \frac{e}{\sqrt{1+e^2}}}$$

Exercise

Evaluate the integral $\displaystyle\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}}$

Solution

$y = e^{\tan\theta} \quad 1 \le y \le e \to \quad 0 \le \theta = \tan^{-1}(\ln y) \le \frac{\pi}{4}$

$dy = e^{\tan\theta}\sec^2\theta\, d\theta$

$$\sqrt{1+(\ln y)^2} = \sqrt{1+\tan^2\theta}$$

$$\boxed{= \sec\theta}$$

$$\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}} = \int_0^{\pi/4} \frac{e^{\tan\theta}\sec^2\theta}{e^{\tan\theta}\sec\theta}\, d\theta$$

$$= \int_0^{\pi/4} \sec\theta\, d\theta$$

$$= \left(\ln|\sec\theta + \tan\theta|\right)\Big|_0^{\pi/4}$$

$$= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 + \tan 0|$$

$$\boxed{= \ln\left(1+\sqrt{2}\right)}$$

Exercise

Evaluate the integral $\displaystyle\int_{1/2}^{1/4} \frac{dy}{y\sqrt{1+(\ln y)^2}}$

Solution

Let $x = \ln y \quad \to \quad y = e^x$

$dy = e^x dx$

$$\int_{1/2}^{1/4} \frac{dy}{y\sqrt{1+(\ln y)^2}} = \int_{1/2}^{1/4} \frac{e^x\, dx}{e^x\sqrt{1+x^2}}$$

$$= \int_{1/2}^{1/4} \frac{dx}{\sqrt{1+x^2}}$$

$x = \tan\theta \qquad \sqrt{x^2+1} = \sec\theta$

$dx = \sec^2\theta\, d\theta$

$$\int_{1/2}^{1/4} \frac{dy}{y\sqrt{1+(\ln y)^2}} = \int_{1/2}^{1/4} \frac{\sec^2\theta\, d\theta}{\sec\theta}$$

$$= \int_{1/2}^{1/4} \sec\theta\, d\theta$$

$$= \int_{1/2}^{1/4} \sec\theta\, \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\, d\theta$$

$$= \int_{1/2}^{1/4} \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\, d\theta$$

$$= \int_{1/2}^{1/4} \frac{1}{\sec\theta+\tan\theta}\, d\left(\sec\theta+\tan\theta\right)$$

$$= \ln\left(\sec\theta+\tan\theta\right) \Big|_{1/2}^{1/4}$$

$$= \ln\left(\sqrt{x^2+1}+x\right) \Big|_{1/2}^{1/4}$$

$$= \ln\left(\sqrt{\ln^2 y+1}+\ln y\right) \Big|_{1/2}^{1/4}$$

$$= \ln\left|\sqrt{\left(\ln\frac{1}{4}\right)^2+1}+\ln\frac{1}{4}\right| - \ln\left|\sqrt{\left(\ln\frac{1}{2}\right)^2+1}+\ln\frac{1}{2}\right|$$

$$= \ln\left|\sqrt{(-\ln 4)^2+1}-\ln 4\right| - \ln\left|\sqrt{(-\ln 2)^2+1}-\ln 2\right|$$

$$\boxed{= \ln\left|\frac{\sqrt{(\ln 4)^2+1}-\ln 4}{\sqrt{(\ln 2)^2+1}-\ln 2}\right|}$$

Exercise

Evaluate $\displaystyle\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2-1}}$

Solution

Let: $u = 3y \quad \Rightarrow \quad du = 3dy \quad \rightarrow \quad \frac{du}{3} = dy$

$$\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2-1}} = \frac{1}{3}\int_{-2/3}^{-\sqrt{2}/3} \frac{1}{\frac{u}{3}\sqrt{u^2-1}}\, du$$

$$= \int_{-2/3}^{-\sqrt{2}/3} \frac{du}{u\sqrt{u^2-1}} \qquad\qquad \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right|$$

$$= \sec^{-1}|3y| \Big|_{-2/3}^{-\sqrt{2}/3}$$

$$= \sec^{-1}\left|-\sqrt{2}\right| - \sec^{-1}|-2|$$

$$= \frac{\pi}{4} - \frac{\pi}{3}$$

$$\boxed{= -\frac{\pi}{12}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\sqrt{3}/2} \frac{4}{9+4x^2}\,dx$

Solution

$$x = \frac{3}{2}\tan\theta \qquad 9+4x^2 = 9\sec^2\theta$$

$$dx = \frac{3}{2}\sec^2\theta\, d\theta$$

$$\int_0^{\sqrt{3}/2} \frac{4}{9+4x^2}\,dx = \int_0^{\sqrt{3}/2} \frac{4}{9\sec^2\theta}\left(\frac{3}{2}\sec^2\theta\, d\theta\right)$$

$$= \frac{2}{3}\int_0^{\sqrt{3}/2} d\theta$$

$$= \frac{2}{3}\theta \Big|_0^{\sqrt{3}/2}$$

$$= \frac{2}{3}\arctan\left(\frac{2}{3}x\right) \Big|_0^{\sqrt{3}/2}$$

$$= \frac{2}{3}\left(\arctan\left(\frac{\sqrt{3}}{3}\right) - \arctan(0)\right)$$

$$= \frac{2}{3}\arctan\frac{1}{\sqrt{3}}$$

$$= \frac{2}{3}\frac{\pi}{6}$$

$$\boxed{= \frac{\pi}{9}}$$

Exercise

Evaluate the integral $\displaystyle\int_{1/12}^{1/4} \frac{dx}{\sqrt{x}(1+4x)}$

Solution

Let $u = \sqrt{x} \quad \to \quad x = u^2$

$dx = 2u\ du$

$$\int_{1/12}^{1/4} \frac{dx}{\sqrt{x}(1+4x)} = \int_{1/12}^{1/4} \frac{2u\ du}{u\left(1+4u^2\right)}$$

$$= \int_{1/12}^{1/4} \frac{d(2u)}{1+(2u)^2}$$

$$= \arctan 2u \Big|_{1/12}^{1/4}$$

$$= \arctan 2\sqrt{x} \Big|_{1/12}^{1/4}$$

$$= \arctan 1 - \arctan\frac{1}{\sqrt{3}}$$

$$= \frac{\pi}{4} - \frac{\pi}{6}$$

$$\boxed{= \frac{\pi}{12}}$$

Exercise

Evaluate the integral $\displaystyle\int_{8\sqrt{2}}^{16} \frac{dx}{\sqrt{x^2-64}}$

Solution

$$x = 8\sec\theta \qquad \sqrt{x^2-64} = 8\tan\theta$$

$$dx = 8\sec\theta\tan\theta\, d\theta$$

$$\int_{8\sqrt{2}}^{16} \frac{dx}{\sqrt{x^2-64}} = \int_{8\sqrt{2}}^{16} \frac{8\sec\theta\tan\theta\, d\theta}{8\tan\theta}$$

$$= \int_{8\sqrt{2}}^{16} \sec\theta\, d\theta$$

$$= \int_{8\sqrt{2}}^{16} \sec\theta\, \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\, d\theta$$

$$= \int_{8\sqrt{2}}^{16} \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\, d\theta$$

$$= \int_{8\sqrt{2}}^{16} \frac{1}{\sec\theta+\tan\theta}\, d\left(\sec\theta+\tan\theta\right)$$

$$= \ln\left|\sec\theta+\tan\theta\right| \Big|_{8\sqrt{2}}^{16}$$

$$= \ln\left|\frac{x}{8}+\frac{\sqrt{x^2-64}}{8}\right| \Big|_{8\sqrt{2}}^{16}$$

$$= \ln\left|2+\frac{\sqrt{16^2-64}}{8}\right| - \ln\left|\sqrt{2}+\frac{\sqrt{128-64}}{8}\right|$$

$$= \ln\left|2+\frac{8\sqrt{4-1}}{8}\right| - \ln\left(\sqrt{2}+1\right)$$

$$= \ln\left(2+\sqrt{3}\right) - \ln\left(\sqrt{2}+1\right)$$

$$= \ln\left(\frac{2+\sqrt{3}}{1+\sqrt{2}}\right)$$

Exercise

Evaluate the integral $\int_{\sqrt{2}}^{2} \frac{\sqrt{x^2-1}}{x}dx$

Solution

$$x = \sec\theta \qquad \sqrt{x^2-1} = \tan\theta$$

$$dx = \sec\theta\tan\theta\, d\theta$$

$$\int_{\sqrt{2}}^{2} \frac{\sqrt{x^2-1}}{x}dx = \int_{\sqrt{2}}^{2} \frac{\tan\theta}{\sec\theta} \sec\theta\tan\theta\, d\theta$$

$$= \int_{\sqrt{2}}^{2} \tan^2\theta\, d\theta$$

$$= \int_{\sqrt{2}}^{2} \left(\sec^2\theta - 1\right) d\theta$$

$$= \tan\theta - \theta \Big|_{\sqrt{2}}^{2}$$

$$x = 2 = \sec\theta \quad \rightarrow \theta = \arccos\frac{1}{2} = \frac{\pi}{3}$$

$$x = \sqrt{2} = \sec\theta \quad \rightarrow \theta = \arccos\frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

$$= \tan\theta - \theta \Big|_{\pi/4}^{\pi/3}$$

$$= \tan\frac{\pi}{3} - \frac{\pi}{3} - \tan\frac{\pi}{4} + \frac{\pi}{4}$$

$$= \sqrt{3} - 1 + \frac{\pi}{12}$$

Exercise

Evaluate $\int_{4}^{6} \frac{x^2}{\sqrt{x^2-9}}dx$

Solution

$$x = 3\sec\theta \qquad \sqrt{x^2-9} = 3\tan\theta$$

$$dx = 3\sec\theta\tan\theta\, d\theta$$

$$\int_{4}^{6} \frac{x^2}{\sqrt{x^2-9}}dx = \int_{4}^{6} \frac{9\sec^2\theta}{3\tan\theta}(3\sec\theta\tan\theta)d\theta$$

$$= 9\int_4^6 \sec^3\theta\, d\theta$$

$$u = \sec x \qquad dv = \sec^2 x\, dx$$
$$du = \sec x \tan x\, dx \qquad v = \tan x$$

$$\int \sec^3 x\, dx = \sec x \tan x - \int \tan x\left(\sec x \tan x\, dx\right)$$

$$= \sec x \tan x - \int \tan^2 x \sec x\, dx$$

$$= \sec x \tan x - \int \left(\sec^2 x - 1\right)\sec x\, dx$$

$$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$2\int \sec^3 x\, dx = \sec x \tan x + \ln\left|\sec x + \tan x\right|$$

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right|$$

$$= \frac{9}{2}\left(\sec\theta \tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right.\Bigg|_4^6$$

$$= \frac{1}{2}x\sqrt{x^2-9} + \frac{9}{2}\ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| \Bigg|_4^6$$

$$= \frac{9}{2}\left(2\sqrt{3} + \ln\left(2+\sqrt{3}\right) - \frac{4\sqrt{7}}{9} - \ln\left(\frac{4+\sqrt{7}}{3}\right)\right)$$

$$= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\ln\left(\frac{6+3\sqrt{3}}{4+\sqrt{7}}\right)$$

Exercise

Evaluate $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\, dx$

Solution

$$x = \sqrt{3}\sec\theta \qquad \sqrt{x^2-3} = \sqrt{3}\tan\theta$$

$$dx = \sqrt{3}\sec\theta\tan\theta\, d\theta$$

$$\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\, dx = \int_{\sqrt{3}}^{2} \frac{\sqrt{3}\tan\theta}{\sqrt{3}\sec\theta}\left(\sqrt{3}\sec\theta\tan\theta\right) d\theta$$

$$= \sqrt{3} \int_{\sqrt{3}}^{2} \tan^2 \theta \, d\theta$$

$$= \sqrt{3} \int_{\sqrt{3}}^{2} \left(\sec^2 \theta - 1\right) \, d\theta$$

$$= \sqrt{3} \left(\tan \theta - \theta \, \right|_{\sqrt{3}}^{2}$$

$$= \sqrt{3} \left(\frac{\sqrt{x^2 - 3}}{\sqrt{3}} - \operatorname{arcsec} \frac{x}{\sqrt{3}} \, \right|_{\sqrt{3}}^{2}$$

$$= \sqrt{3}\left(\frac{1}{\sqrt{3}} - \frac{\pi}{6} \right)$$

$$= 1 - \frac{\pi\sqrt{3}}{6}$$

Exercise

Evaluate $\displaystyle\int_0^{\sqrt{3}/2} \frac{x^2}{\left(1 - x^2\right)^{3/2}} dx$

Solution

$x = \sin \theta \qquad \sqrt{1 - x^2} = \cos \theta$

$dx = \cos \theta \; d\theta$

$$\int_0^{\sqrt{3}/2} \frac{x^2}{\left(1 - x^2\right)^{3/2}} \, dx = \int_0^{\sqrt{3}/2} \frac{\sin^2 \theta}{\cos^3 \theta} (\cos \theta) d\theta$$

$$= \int_0^{\sqrt{3}/2} \tan^2 \theta \, d\theta$$

$$= \int_0^{\sqrt{3}/2} \left(\sec^2 \theta - 1\right) \, d\theta$$

$$= \tan \theta - \theta \, \Big|_0^{\sqrt{3}/2}$$

$$= \frac{x}{\sqrt{1 - x^2}} - \arcsin x \, \Big|_0^{\sqrt{3}/2}$$

$$=\frac{\sqrt{3}}{2}\frac{1}{\sqrt{1-\frac{3}{4}}}-\frac{\pi}{3}$$

$$=\sqrt{3}-\frac{\pi}{3}$$

Exercise

Evaluate $\displaystyle\int_0^{\sqrt{3}/2}\frac{1}{\left(1-x^2\right)^{5/2}}dx$

Solution

$x=\sin\theta \qquad \sqrt{1-x^2}=\cos\theta$

$dx=\cos\theta\, d\theta$

$$\int_0^{\sqrt{3}/2}\frac{1}{\left(1-x^2\right)^{5/2}}dx=\int_0^{\sqrt{3}/2}\frac{1}{\cos^5\theta}\cos\theta\, d\theta$$

$$=\int_0^{\sqrt{3}/2}\sec^4\theta\, d\theta$$

$$=\int_0^{\sqrt{3}/2}\left(1+\tan^2\theta\right)\sec^2\theta\, d\theta$$

$$=\int_0^{\sqrt{3}/2}\left(1+\tan^2\theta\right)\, d\left(\tan\theta\right)$$

$$=\tan\theta+\frac{1}{3}\tan^3\theta\ \Big|_0^{\sqrt{3}/2}$$

$$=\frac{x}{\sqrt{1-x^2}}+\frac{1}{3}\frac{x^3}{\left(1-x^2\right)^{3/2}}\ \Big|_0^{\sqrt{3}/2}$$

$$=\frac{\sqrt{3}}{2}\frac{1}{\sqrt{1-\frac{3}{4}}}+\frac{\sqrt{3}}{8}\frac{1}{\left(\frac{1}{4}\right)^{3/2}}$$

$$=\sqrt{3}+\sqrt{3}$$

$$=2\sqrt{3}$$

Exercise

Evaluate $\displaystyle\int_0^{3/5} \sqrt{9-25x^2}\ dx$

Solution

$5x = 3\sin\theta \qquad \sqrt{9-25x^2} = 3\cos\theta$

$dx = \frac{3}{5}\cos\theta\, d\theta$

$$\int_0^{3/5} \sqrt{9-25x^2}\ dx = \frac{9}{5}\int_0^{3/5} \cos^2\theta\ d\theta$$

$$= \frac{9}{10}\int_0^{3/5} (1+\cos 2\theta)\ d\theta$$

$$= \frac{9}{10}\left(\theta + \frac{1}{2}\sin 2\theta\right. \Big|_0^{3/5}$$

$$= \frac{9}{10}\left(\theta + \sin\theta\cos\theta\right. \Big|_0^{3/5}$$

$$= \frac{9}{10}\left(\arcsin\frac{5x}{3} + 2\cdot\frac{5x}{3}\cdot\frac{5\sqrt{9-25x^2}}{3}\right. \Bigg|_0^{3/5}$$

$$= \frac{9}{10}\left(\arcsin\frac{5x}{3} + \frac{25}{9}x\sqrt{9-25x^2}\right. \Bigg|_0^{3/5}$$

$$\boxed{= \frac{9\pi}{20}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}}$

Solution

$$\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\frac{x}{3}\ \Big|_0^{3/2}$$

$$\boxed{= \frac{\pi}{6}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

Solution

$$\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\frac{x}{3}\Big|_0^3$$

$$= \sin^{-1}1 - \sin^{-1}0$$

$$\boxed{= \frac{\pi}{2}}$$

Exercise

Evaluate $\displaystyle\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2}dx$

Solution

$$x+2 = 3\sec\theta \qquad \sqrt{(x+2)^2-9} = 3\tan\theta$$

$$dx = 3\sec\theta\tan\theta\ d\theta$$

$$\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2}dx = \int_1^4 \frac{\sqrt{(x+2)^2-9}}{x+2}dx$$

$$= \int_1^4 \frac{3\tan\theta}{3\sec\theta}(3\sec\theta\tan\theta)\ d\theta$$

$$= 3\int_1^4 \tan^2\theta\ d\theta$$

$$= 3\int_1^4 \left(\sec^2\theta - 1\right)\ d\theta$$

$$= 3\left(\tan\theta - \theta\right.\Big|_1^4$$

$$= \sqrt{(x+2)^2-9} - 3\sec^{-1}\left(\frac{x+2}{3}\right)\Big|_1^4$$

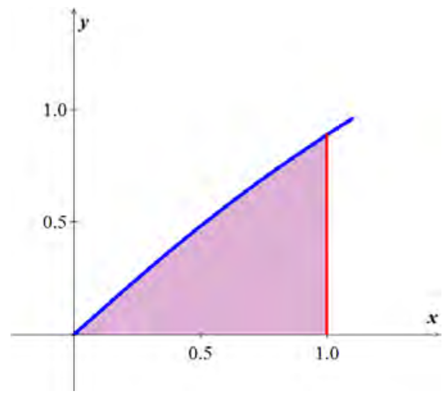
$$= \sqrt{27} - 3\sec^{-1}(2) + 3\sec^{-1}(1)$$

$$\boxed{= 3\sqrt{3} - \pi}$$

Exercise

Consider the region bounded by the graph $y = \sqrt{x \tan^{-1} x}$ and $y = 0$ for $0 \le x \le 1$. Find the volume of the solid formed by revolving this region about the x-axis.

Solution



$$V = \pi \int_0^1 \left(\sqrt{x \tan^{-1} x} \right)^2 dx$$

$$= \pi \int_0^1 x \tan^{-1} x \, dx$$

$$u = \tan^{-1} x \qquad v = \int x \, dx$$

$$du = \frac{1}{x^2 + 1} dx \qquad v = \frac{1}{2} x^2$$

$$V = \pi \left(\frac{1}{2} \left(x^2 \tan^{-1} x \, \Big|_0^1 \right. - \frac{1}{2} \int_0^1 \frac{x^2}{1 + x^2} dx \right)$$

$$= \frac{\pi}{2} \left(\left(\tan^{-1} 1 - 0 \right) - \int_0^1 \left(1 - \frac{1}{1 + x^2} \right) dx \right)$$

$$= \frac{\pi}{2} \left(\frac{\pi}{4} - \int_0^1 dx + \int_0^1 \frac{1}{1 + x^2} dx \right)$$

$$= \frac{\pi}{2} \left(\frac{\pi}{4} - \left(x \, \Big|_0^1 \right. + \left(\tan^{-1} x \, \Big|_0^1 \right. \right)$$

$$= \frac{\pi}{2} \left(\frac{\pi}{4} - 1 + \tan^{-1} 1 \right)$$

$$= \frac{\pi}{2} \left(\frac{\pi}{4} - 1 + \frac{\pi}{4} \right)$$

$$= \frac{\pi}{2} \left(\frac{\pi}{2} - 1 \right)$$

$$= \frac{\pi^2}{4} - \frac{\pi}{2} \quad unit^3$$

Exercise

Use two approach to show that the area of a cap (or segment) of a circle of radius r subtended by an angle θ is given by

$$A_{seg} = \frac{1}{2}r^2(\theta - \sin\theta)$$

a) Find the area using geometry (no calculus).

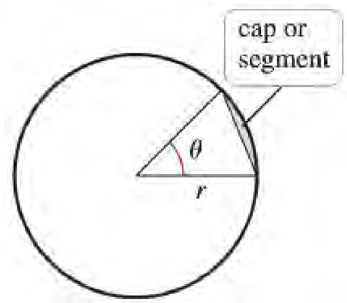
b) Find the area using calculus

Solution

a) Area of a segment (*cap*) = Area of a sector ***minus*** Area of the isosceles triangle

The area of a sector: $A = \frac{1}{2}\theta r^2$

Area of the isosceles triangle: $A = \frac{1}{2}r^2 \sin\theta$



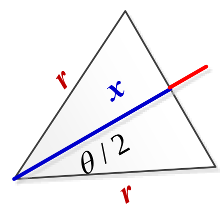
$$A_{seg} = \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin\theta$$

$$= \frac{1}{2}r^2(\theta - \sin\theta)$$

b) $0 \le \theta \le \pi \quad \to \quad 0 \le \frac{\theta}{2} \le \frac{\pi}{2}$

$x = r\cos\frac{\alpha}{2} \quad \to dx = -\frac{1}{2}r\sin\frac{\alpha}{2}\, d\alpha$

$\sqrt{r^2 - x^2} = r\sin\frac{\alpha}{2}$



$$A_{cap} = 2\int_{r\cos\theta/2}^{r} \sqrt{r^2 - x^2}\, dx$$

$$= 2\int_{\theta}^{0} \left(r\sin\frac{\alpha}{2}\right)\left(-\frac{1}{2}r\sin\frac{\alpha}{2}\right) d\alpha$$

$$= r^2 \int_0^{\theta} \left(\sin^2\frac{\alpha}{2}\right) d\alpha$$

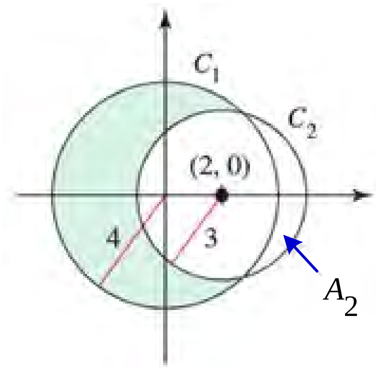
$$= \frac{1}{2}r^2 \int_0^{\theta} (1 - \cos\alpha)\, d\alpha$$

$$= \frac{1}{2}r^2 (\alpha - \sin\alpha)\Big|_0^{\theta}$$

$$= \frac{1}{2}r^2(\theta - \sin\theta) \quad unit^2$$

Exercise

A lune is a crescent-shaped region bounded by the arcs of two circles. Let C_1 be a circle of radius 4 centered at the origin. Let C_2 be a circle of radius 3 centered at the point $(2,\ 0)$. Find the area of the lune that lies inside C_1 and outside C_2.



Solution

$$C_1 \quad \to \quad x^2 + y^2 = 16$$

$$y^2 = 16 - x^2$$

$$C_2 \quad \to \quad (x-2)^2 + y^2 = 9$$

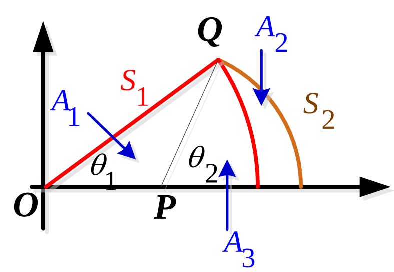
$$y^2 = 9 - (x-2)^2$$

$$16 - x^2 = 9 - x^2 + 4x - 4$$

$$11 = 4x \quad \to \quad \boxed{x = \frac{11}{4}}$$

$$y = \pm\frac{\sqrt{135}}{4}$$

$$= \boxed{\pm\frac{3\sqrt{15}}{4}}$$



For **sector** C_1: $\theta_1 = \tan^{-1}\frac{y}{x}$

$$= \boxed{\tan^{-1}\left(\frac{3\sqrt{15}}{11}\right)}$$

Area: $S_1 = \frac{1}{2}r^2\theta_1$

$$= \boxed{8\tan^{-1}\left(\frac{3\sqrt{15}}{11}\right)}$$

For **sector** C_2: $x_2 = \frac{11}{4} - 2 = \frac{3}{4}$

$$\theta_2 = \tan^{-1}\frac{y}{x_2}$$

$$= \tan^{-1}\sqrt{15}$$

Area: $S_2 = \frac{1}{2}r_2^2\theta_2$

$$= \boxed{\frac{9}{2}\tan^{-1}\left(\sqrt{15}\right)}$$

$OQ = 4, \quad PQ = 3, \quad OP = 2$

$Area\left(\Delta APQ\right) = A_1 = \frac{1}{2}(4)(2)\sin\theta_1$

$$= 4\frac{y}{4}$$

$$\boxed{= \frac{3\sqrt{15}}{4}}$$

$$A_2 = S_2 - S_1 + A_1$$

$$\boxed{= \frac{9}{2}\tan^{-1}\left(\sqrt{15}\right) - 8\tan^{-1}\left(\frac{3\sqrt{15}}{11}\right) + \frac{3\sqrt{15}}{4}}$$

$$A_{lune} = A_{C_1} - A_{C_2} + 2A_2$$

$$= 16\pi - 9\pi + 9\tan^{-1}\left(\sqrt{15}\right) - 16\tan^{-1}\left(\frac{3\sqrt{15}}{11}\right) + \frac{3\sqrt{15}}{2}$$

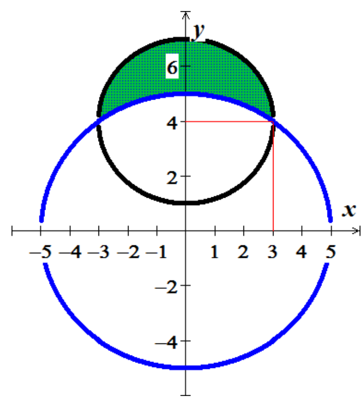
$$\boxed{= 7\pi + 9\tan^{-1}\left(\sqrt{15}\right) - 16\tan^{-1}\left(\frac{3\sqrt{15}}{11}\right) + \frac{3\sqrt{15}}{2} \quad unit^2}$$

$$\boxed{\approx 26.66 \quad unit^2}$$

Exercise

The crescent-shaped region bounded by two circles forms a lune. Find the area of the lune given that the radius of the smaller circle is 3 and the radius of the larger circle is 5.

Solution



Large Circle: $x^2 + y^2 = 25$

$$y = \sqrt{25 - x^2}$$

Small Circle: $r = 3 \;\rightarrow\; y = \sqrt{25 - 9} \;\boxed{= 4}$

$$x^2 + (y-4)^2 = 9$$

$$y = 4 + \sqrt{9 - x^2}$$

$$A = 2\int_0^3 \left(4 + \sqrt{9 - x^2} - \sqrt{25 - x^2}\right) dx$$

$$= 2\left(4x + \frac{1}{2}\left(9\arcsin\left(\frac{x}{3}\right) + x\sqrt{9 - x^2}\right) - \frac{1}{2}\left(25\arcsin\left(\frac{x}{5}\right) + x\sqrt{25 - x^2}\right)\right)\Bigg|_0^3$$

$$= 2\left[12 + \frac{1}{2}\left(9\frac{\pi}{2}\right) - \frac{1}{2}\left(25\arcsin\left(\frac{3}{5}\right) + 12\right)\right]$$

$$\boxed{= 12 + \frac{9\pi}{2} - 25\arcsin\left(\frac{3}{5}\right) \quad unit^2}$$

Exercise

The surface of a machine part is the region between the graphs of $y=|x|$ *and* $x^2+(y-k)^2=25$

a) Find k when the circle is tangent to the graph of $y=|x|$

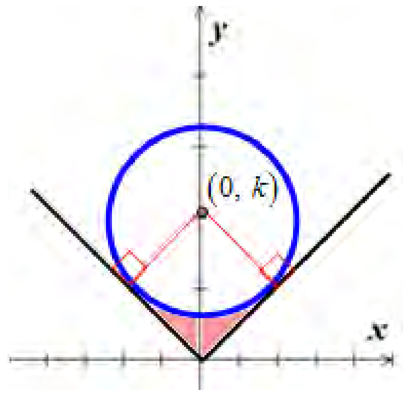
b) Find the area of the surface of the machine part.

c) Find the area of the surface of the machine part as a function of the radius r of the circle.

Solution

a) $x^2+(y-k)^2=25 \quad\to\quad \underline{r=5}$

$k^2=5^2+5^2=50 \quad\to\quad \underline{k=5\sqrt{2}}$



b) $Area = area\ square - \frac{1}{4}(area\ circle)$

$$=5^2-\frac{1}{4}\pi 5^2$$

$$=25\left(1-\frac{\pi}{4}\right)\quad unit^2$$

c) $Area = area\ square - \frac{1}{4}(area\ circle)$

$$=r^2-\frac{1}{4}\pi r^2$$

$$=r^2\left(1-\frac{\pi}{4}\right)\quad unit^2$$

Exercise

Consider the function $f(x)=\left(9+x^2\right)^{-1/2}$ and the region $\boldsymbol{R}$ on the interval [0, 4].

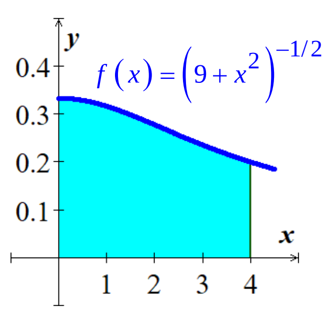
a) Find the area of *R*.

b) Find the volume of the solid generated when *R* is revolved about the *x-axis*.

c) Find the volume of the solid generated when *R* is revolved about the *y-axis*.

Solution

a) $A=\int_0^4 \frac{dx}{\sqrt{9+x^2}}$



$x=3\tan\theta \quad\to\quad dx=3\sec^2\theta\, d\theta$

$\sqrt{9+x^2}=3\sec\theta$

$$=\int_0^4 \frac{3\sec^2\theta}{3\sec\theta}\,d\theta$$

$$=\int_0^4 \sec\theta\, d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| \Big|_0^4$$

$$= \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| \Big|_0^4$$

$$= \ln\left(\frac{5}{3} + \frac{4}{3}\right) - 0$$

$$\underline{= \ln 3 \quad unit^2}$$

b) $x = 3\tan\theta \;\rightarrow\; dx = 3\sec^2\theta\, d\theta$

$9 + x^2 = 9\sec^2\theta$

$$V = \pi\int_0^4 \frac{dx}{9+x^2}$$

$$= \pi\int_0^4 \frac{3\sec^2\theta\, d\theta}{9\sec^2\theta}$$

$$= \frac{\pi}{3}\int_0^4 d\theta$$

$$= \frac{\pi}{3}\theta \Big|_0^4$$

$$= \frac{\pi}{3}\tan^{-1}\frac{x}{3} \Big|_0^4$$

$$\underline{= \frac{\pi}{3}\tan^{-1}\frac{4}{3} \quad unit^3}$$

c) $V = 2\pi\int_0^4 \frac{x}{\sqrt{9+x^2}}dx$ $\qquad\qquad d\left(9+x^2\right) = 2xdx$

$$= \pi\int_0^4 \left(9+x^2\right)^{-1/2} d\left(9+x^2\right)$$

$$= 2\pi\left(9+x^2\right)^{1/2} \Big|_0^4$$

$$= 2\pi(5-3)$$

$$\underline{= 4\pi \quad unit^3}$$

Exercise

A total of $\boldsymbol{Q}$ is distributed uniformly on a line segment of length $2L$ along the y-*axis*. The x-component of the electric field at a point $(a,\ 0)$ is given by

$$E_x = \frac{kQa}{2L}\int_{-L}^{L} \frac{dy}{\left(a^2+y^2\right)^{3/2}}$$

Where k is a physical constant and $a > 0$

a) Confirm that $E_x(a) = \dfrac{kQ}{a\sqrt{a^2+L^2}}$

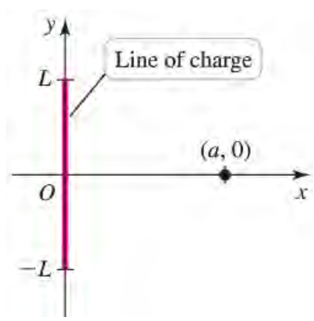
b) Letting $\rho = \dfrac{Q}{2L}$ be the charge density on the line segment, show that if $L \to \infty$, *then* $E_x = \dfrac{2k\rho}{a}$

Solution

a) $E_x = \dfrac{kQa}{2L}\displaystyle\int_{-L}^{L} \frac{dy}{\left(a^2+y^2\right)^{3/2}}$

$$y = a\tan\theta \ \to dy = a\sec^2\theta\, d\theta$$

$$\sqrt{a^2+y^2} = a\sec\theta$$



$$= \frac{kQa}{2L}\int_{-L}^{L} \frac{a\sec^2\theta\, d\theta}{a^3\sec^3\theta}$$

$$= \frac{kQ}{2aL}\int_{-L}^{L} \frac{d\theta}{\sec\theta}$$

$$= \frac{kQ}{2aL}\int_{-L}^{L} \cos\theta d\theta$$

$$= \frac{kQ}{2aL}\sin\theta\ \Big|_{-L}^{L}$$

$$= \frac{kQ}{2aL}\left(\frac{y}{\sqrt{a^2+y^2}}\ \Big|_{-L}^{L}\right.$$

$$= \frac{kQ}{2aL}\left(\frac{2L}{\sqrt{a^2+L^2}}\right)$$

$$\boxed{= \frac{kQ}{a\sqrt{a^2+L^2}}}$$

b) Let $\rho = \dfrac{Q}{2L} \quad \to \quad Q = 2\rho L$

$$E_x(a) = \frac{kQa}{2L} \lim_{L\to\infty} \int_{-L}^{L} \frac{dy}{\left(a^2+y^2\right)^{3/2}}$$

$$= \frac{kQa}{2L} \lim_{L\to\infty} \left(\frac{2L}{a^2\sqrt{a^2+L^2}} \right)$$

$$= k\rho a \frac{2}{a^2}$$

$$\boxed{= \frac{2k\rho}{a}}$$

Exercise

A long, straight wire of length $2L$ on the *y-axis* carries a current I. according to the Biot-Savart Law, the magnitude of the field due to the current at a point $(a,\ 0)$ is given by

$$B(a) = \frac{\mu_0 I}{4\pi} \int_{-L}^{L} \frac{\sin\theta}{r^2} dy$$

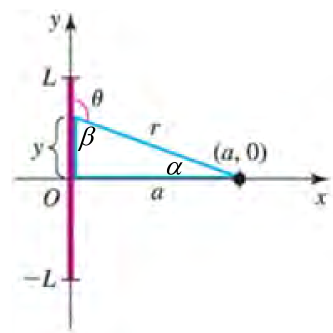
Where μ_0 is a physical constant, $a > 0$, and θ, r, and y are related to the figure

a) Show that the magnitude of the magnetic field at $(a,\ 0)$ is $B(a) = \dfrac{\mu_0 IL}{2\pi a\sqrt{a^2+L^2}}$

b) What is the magnitude of the magnetic field at $(a,\ 0)$ due to an infinitely long wire $(L\to\infty)$?

Solution

a) $\beta = \pi - \theta \quad \& \quad \alpha + \beta = \frac{\pi}{2}$



$$\sin\theta = \sin(\pi - \beta) = \sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha = \frac{a}{r}$$

$$r^2 = y^2 + a^2$$

$$\frac{\sin\theta}{r^2} = \frac{a}{r^3} = \frac{a}{\left(a^2+y^2\right)^{3/2}}$$

$$B(a) = \frac{\mu_0 I}{4\pi} \int_{-L}^{L} \frac{\sin\theta}{r^2} dy$$

$$= \frac{\mu_0 I}{4\pi} \int_{-L}^{L} \frac{a}{\left(a^2+y^2\right)^{3/2}} dy$$

$$y = a\tan u \qquad \sqrt{a^2+y^2} = a\sec u$$

$$dy = a\sec^2 u\ du$$

$$= \frac{\mu_0 I}{2\pi} \int_0^L \frac{a^2 \sec^2 u \, du}{a^3 \sec^3 u}$$

$$= \frac{\mu_0 I}{2a\pi} \int_0^L \frac{1}{\sec u} du$$

$$= \frac{\mu_0 I}{2a\pi} \int_0^L \cos u \, du$$

$$= \frac{\mu_0 I}{2a\pi} \sin u \Big|_0^L$$

$$= \frac{\mu_0 I}{2a\pi} \frac{y}{\sqrt{a^2 + y^2}} \Big|_0^L$$

$$\boxed{= \frac{\mu_0 IL}{2a\pi\sqrt{a^2 + L^2}}}$$

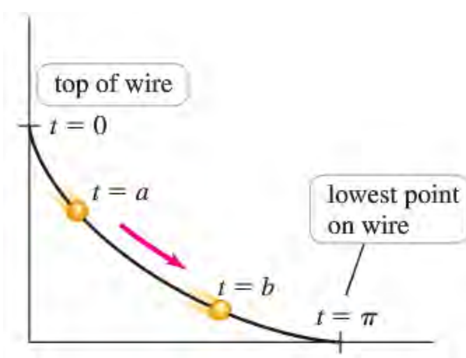
b) $\lim\limits_{L\to\infty} B(a) = \lim\limits_{L\to\infty} \frac{\mu_0 IL}{2a\pi\sqrt{a^2 + L^2}}$

$$= \frac{\mu_0 I}{2a\pi} \lim_{L\to\infty} \frac{L}{\sqrt{a^2 + L^2}} \qquad \lim_{L\to\infty} \frac{L}{\sqrt{a^2 + L^2}} = \lim_{L\to\infty} \frac{L}{\sqrt{L^2}} = 1$$

$$\boxed{= \frac{\mu_0 I}{2a\pi}}$$

Exercise

The cycloid is the curve traced by a point on the rim of a rolling wheel. Imagine a wire shaped like an inverted cycloid.



A bead sliding down this wire without friction has some remarkable properties. Among all wire shapes, the cycloid is the shape that produces the fastest descent time. It can be shown that the descent time between any two points $0 \le a < b \le \pi$ on the curve is

$$descent\ time = \int_a^b \sqrt{\frac{1-\cos t}{g(\cos a - \cos t)}}\, dt$$

Where g is the acceleration due to gravity, $t = 0$ corresponds to the top of the wire, and $t = \pi$ corresponds to the lowest point on the wire.

a) Find the descent time on the interval $[a,\ b]$.

b) Show that when $b = \pi$, the descent time is the same for all values of a; that is, the descent time to the bottom of the wire is the same for all starting points.

Solution

a)
$$\int_a^b \sqrt{\frac{1-\cos t}{g(\cos a - \cos t)}}\, dt = \int_a^b \sqrt{\frac{(1-\cos t)(1+\cos t)}{g(\cos a - \cos t)(1+\cos t)}}\, dt$$

$$= \frac{1}{\sqrt{g}} \int_a^b \sqrt{\frac{\left(1-\cos^2 t\right)}{\cos a + (\cos a - 1)\cos t - \cos^2 t}}\, dt$$

$$= \frac{1}{\sqrt{g}} \int_a^b \frac{\sin t}{\sqrt{\cos a + \left(\frac{\cos a - 1}{2}\right)^2 - \left(\frac{\cos a - 1}{2}\right)^2 + (\cos a - 1)\cos t - \cos^2 t}}\, dt$$

$$= \frac{1}{\sqrt{g}} \int_a^b \frac{\sin t}{\sqrt{\cos a + \left(\frac{\cos a - 1}{2}\right)^2 - \left(\left(\frac{\cos a - 1}{2}\right) - \cos t\right)^2}}\, dt$$

Let: $v = \sqrt{\cos a + \left(\frac{\cos a - 1}{2}\right)^2}$

$$= \frac{1}{2}\sqrt{4\cos a + \cos^2 a - 2\cos a + 1}$$

$$= \frac{1}{2}(\cos a + 1)$$

$$\frac{\cos a - 1}{2} - \cos t = v \sin\theta \quad \rightarrow \quad \sin t\ dt = v\cos\theta\ d\theta$$

$$\sqrt{v - \left(\left(\frac{\cos a - 1}{2}\right) - \cos t\right)^2} = v\cos\theta$$

$$= \frac{1}{\sqrt{g}} \int_a^b \frac{v\cos\theta}{v\cos\theta}\, d\theta$$

$$= \frac{1}{\sqrt{g}}\theta \Big|_a^b$$

$$\theta = \sin^{-1}\left(\frac{\cos a - 1 - 2\cos t}{2}\,\frac{2}{1+\cos a}\right)$$

$$= \frac{1}{\sqrt{g}} \sin^{-1}\left(\frac{\cos a - 1 - 2\cos t}{1+\cos a}\right) \Big|_a^b$$

$$= \frac{1}{\sqrt{g}} \left(\sin^{-1}\left(\frac{\cos a - 1 - 2\cos b}{1+\cos a}\right) - \sin^{-1}(-1)\right)$$

$$= \frac{1}{\sqrt{g}} \left(\sin^{-1}\left(\frac{\cos a - 1 - 2\cos b}{1+\cos a}\right) + \frac{\pi}{2}\right)$$

b) $\frac{1}{\sqrt{g}} \left(\sin^{-1}\left(\frac{\cos a - 1 - 2\cos b}{1+\cos a}\right) + \frac{\pi}{2}\right)\Big|_{b=\pi} = \frac{1}{\sqrt{g}} \left(\sin^{-1}\left(\frac{\cos a - 1 + 2}{1+\cos a}\right) + \frac{\pi}{2}\right)$

$$= \frac{1}{\sqrt{g}} \left(\sin^{-1}(1) + \frac{\pi}{2}\right)$$

$$= \frac{\pi}{\sqrt{g}}$$

Exercise

Find the area of the region bounded by the curve $f(x) = \left(16 + x^2\right)^{-3/2}$ and the x-*axis* on the interval $[0,\ 3]$

Solution

$$A = \int_0^3 \frac{dx}{\left(16 + x^2\right)^{3/2}}$$

$$x = 4\tan\theta \quad \rightarrow dx = 4\sec^2\theta\ d\theta$$

$$16 + x^2 = 16\sec^2\theta$$

$$= \int_0^3 \frac{4\sec^2\theta d\theta}{\left(16\sec^2\theta\right)^{3/2}}$$

$$= \int_0^3 \frac{4\sec^2\theta}{4^3\sec^3\theta} d\theta$$

$$= \frac{1}{16}\int_0^3 \cos\theta\ d\theta$$

$$= \frac{1}{16}\sin\theta\ \Big|_0^3$$

$$= \frac{1}{16}\frac{\tan\theta}{\sec\theta}\ \Big|_0^3$$

$$= \frac{1}{16}\frac{x}{\sqrt{16+x^2}}\ \Big|_0^3$$

$$=\frac{1}{16}\left(\frac{3}{5}-0\right)$$

$$\boxed{=\frac{3}{80}\quad unit^2}$$

Exercise

Find the length of the curve $y=ax^2$ from $x=0$ to $x=10$, where $a>0$ is a real number.

Solution

$$1+(y')^2=1+(2ax)^2$$

$$L=\int_0^{10}\sqrt{1+4a^2x^2}\ dx$$

$$=\int_0^{10}2a\sqrt{\frac{1}{4a^2}+x^2}\ dx$$

$$x=\frac{1}{2a}\tan\theta \qquad \frac{1}{4a^2}+x^2=\frac{1}{4a^2}\sec^2\theta$$

$$dx=\frac{1}{4a^2}\sec^2\theta\ d\theta$$

$$=\int_0^{10}2a\frac{1}{2a}\sec\theta\frac{1}{4a^2}\sec^2\theta\ d\theta$$

$$=\frac{1}{2a}\int_0^{10}\sec^3\theta\ d\theta$$

$$u=\sec x \qquad dv=\sec^2 xdx$$

$$du=\sec x\tan xdx \qquad v=\tan x$$

$$\int\sec^3 xdx=\sec x\tan x-\int\tan x(\sec x\tan xdx)$$

$$=\sec x\tan x-\int\tan^2 x\sec xdx$$

$$=\sec x\tan x-\int\left(\sec^2 x-1\right)\sec x\ dx$$

$$=\sec x\tan x-\int\sec^3 x\ dx+\int\sec x\ dx$$

$$\int\sec^3 x\ dx+\int\sec^3 x\ dx=\sec x\tan x-\int\sec^3 x\ dx+\int\sec^3 x\ dx+\int\sec x\ dx$$

$$2\int\sec^3 x\ dx=\sec x\tan x+\int\sec x\ dx$$

$$= \sec x \tan x + \ln\left|\sec x + \tan x\right|$$

$$= \frac{1}{4a}\left(\sec\theta \tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right. \Big|_0^{10}$$

$$= \frac{1}{4a}\left(2a\sqrt{\frac{1}{4a^2} + x^2}\,(2ax) + \ln\left|\sqrt{1+4a^2x^2} + 2ax\right|\right) \Big|_0^{10}$$

$$= \frac{1}{4a}\left((2ax)\sqrt{1+4a^2x^2} + \ln\left|\sqrt{1+4a^2x^2} + 2ax\right|\right) \Big|_0^{10}$$

$$= \frac{1}{4a}\left((20a)\sqrt{1+400a^2} + \ln\left|\sqrt{1+400a^2} + 20a\right|\right) \quad \textit{unit}$$

Exercise

Find the arc length of the graph of $f(x) = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$

Solution

$$1 + (f')^2 = 1 + x^2$$

$$L = \int_0^1 \sqrt{1+x^2}\, dx \qquad x = \tan\theta \qquad \sqrt{x^2+1} = \sec\theta$$

$$dx = \sec^2\theta\, d\theta$$

$$= \int_0^1 \sec^3\theta\, d\theta$$

$$= \frac{1}{2}\left(\sec\theta \tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right. \Big|_0^1$$

$$= \frac{1}{2}\left(x\sqrt{x^2+1} + \ln\left|x + \sqrt{x^2+1}\right|\right) \Big|_0^1$$

$$= \frac{1}{2}\left(\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right) \quad \textit{unit}$$

Exercise

A projectile is launched from the ground with an initial speed V at an angle θ from the horizontal. Assume that the x-*axis* is the horizontal ground and y is the height above the ground. Neglecting air resistance and letting g be the acceleration due to gravity, it can be shown that the trajectory of the projectile is given by

$$y = -\frac{1}{2}kx^2 + y_{max} \quad \text{where } k = \frac{g}{(V\cos\theta)^2}$$

$$\text{and} \qquad y_{max} = \frac{(V\sin\theta)^2}{2g}$$

a) Note that the high point of the trajectory occurs at $\left(0,\ y_{max}\right)$. If the projectile is on the ground at $(-a,\ 0)$ and $(a,\ 0)$, what is a?

b) Show that the length of the trajectory (arc length) is $2\int_0^a \sqrt{1+k^2x^2}\ dx$

c) Evaluate the arc length integral and express your result in the terms of V, g, and θ.

d) For fixed value of V and g, show that the launch angle θ that maximizes the length of the trajectory satisfies $(\sin\theta)\ln(\sec\theta + \tan\theta) = 1$

Solution

a) At $(\pm a,\ 0) \to y = 0 = -\frac{1}{2}ka^2 + y_{max}$

$$a^2 = \frac{2}{k}y_{max}$$

$$\boxed{a = \sqrt{\frac{2y_{max}}{k}}}$$

b) $y' = -kx \ \Rightarrow\ 1 + (y')^2 = 1 + k^2x^2$

$$L = \int_{-a}^{a} \sqrt{1+k^2x^2}\ dx \qquad \text{since } y(x) \text{ is an even function}$$

$$= 2\int_0^a \sqrt{1+k^2x^2}\ dx$$

c) $L = 2\int_0^a \sqrt{1+k^2x^2}\ dx$

$$x = \frac{1}{k}\tan\theta \ \Rightarrow\ dx = \frac{1}{k}\sec^2\theta\ d\theta$$

$$1 + k^2x^2 = \sec^2\theta$$

$$= 2\int_0^a \frac{1}{k}\sec\theta\sec^2\theta\ d\theta$$

$$= \frac{2}{k}\int_0^a \sec^3\theta \, d\theta$$

$$= \frac{1}{k}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right|_0^a$$

$$= \frac{1}{k}\left(\sqrt{1+k^2x^2}\,(kx) + \ln\left|\sqrt{1+k^2x^2} + kx\right|\right|_0^a$$

$$\boxed{= \frac{1}{k}\left(ak\sqrt{1+k^2a^2} + \ln\left|\sqrt{1+k^2a^2} + ka\right|\right)}$$

$$a = \sqrt{\frac{2}{k}\frac{(V\sin\theta)^2}{2g}}$$

$$= \frac{V\sin\theta}{\sqrt{g\frac{g}{(V\cos\theta)^2}}}$$

$$= \frac{V^2}{g}\sin\theta\cos\theta$$

$$k = \frac{g}{(V\cos\theta)^2}$$

$$ak = \tan\theta$$

$$L(\theta) = \frac{(V\cos\theta)^2}{g}\left(\tan\theta\sqrt{1+\tan^2\theta} + \ln\left|\sqrt{1+\tan^2\theta} + \tan\theta\right|\right)$$

$$= \frac{V^2\cos^2\theta}{g}\left(\tan\theta\sec\theta + \ln\left|\sec\theta + \tan\theta\right|\right)$$

$$= \frac{V^2}{g}\sin\theta + \frac{V^2}{g}\cos^2\theta\ln\left|\sec\theta + \tan\theta\right|$$

$$\boxed{= \frac{V^2}{g}\left(\sin\theta + \cos^2\theta\sinh^{-1}(\tan\theta)\right) \quad \textit{unit}}$$

d) $$L'(\theta) = \frac{V^2}{g}\left(\cos\theta - 2\cos\theta\sin\theta\sinh^{-1}(\tan\theta) + \cos^2\theta\frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\right)$$

$$= \frac{V^2}{g}\left(\cos\theta - 2\cos\theta\sin\theta\sinh^{-1}(\tan\theta) + \cos^2\theta\sec\theta\right)$$

$$= \frac{2V^2\cos\theta}{g}\left(1 - \sin\theta\sinh^{-1}(\tan\theta)\right) = 0$$

$$\sin\theta\sinh^{-1}(\tan\theta) = 1$$

$$\sin\theta\ln(\sec\theta + \tan\theta) = 1 \quad \checkmark$$

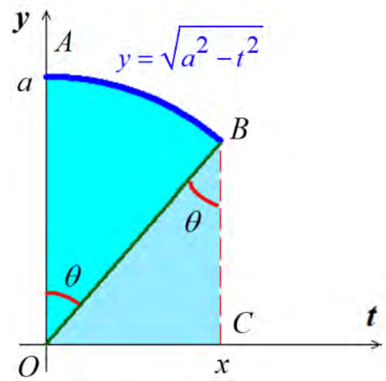
Exercise

Let $F(x)=\int_0^x \sqrt{a^2-t^2}\,dt$. The figure shows that $F(x)=$ *area of sector OAB + area of triangle OBC*

a) Use the figure to prove that $F(x)=\dfrac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\dfrac{x\sqrt{a^2-x^2}}{2}$

b) Conclude that $\displaystyle\int \sqrt{a^2-x^2}\,dx=\dfrac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\dfrac{x\sqrt{a^2-x^2}}{2}+C$

Solution



a) Area of sector *OAB* is $\frac{1}{2}\theta a^2$

From the triangle *OBC*: $\sin\theta=\frac{x}{a}\ \to\ \theta=\sin^{-1}\frac{x}{a}$

$$|BC|=\sqrt{a^2-x^2}$$

Area of sector *OAB* is $\frac{1}{2}a^2\sin^{-1}\frac{x}{a}$

Area of triangle *OBC*: $\frac{1}{2}x\sqrt{a^2-x^2}$

$F(x)=$ *area of sector OAB + area of triangle OBC*

$$=\frac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\frac{x\sqrt{a^2-x^2}}{2}$$

b)
$$\frac{d}{dx}\left(\frac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\frac{x\sqrt{a^2-x^2}}{2}+C\right)=\frac{a^2}{2}\frac{\frac{1}{a}}{\sqrt{1-\left(\frac{x}{a}\right)^2}}+\frac{1}{2}\sqrt{a^2-x^2}-\frac{1}{2}\frac{x^2}{\sqrt{a^2-x^2}}$$

$$=\frac{1}{2}\frac{a^2}{\sqrt{a^2-x^2}}+\frac{1}{2}\frac{a^2-2x^2}{\sqrt{a^2-x^2}}$$

$$=\frac{1}{2}\frac{2a^2-2x^2}{\sqrt{a^2-x^2}}$$

$$=\frac{a^2-x^2}{\sqrt{a^2-x^2}}$$

$$=\sqrt{a^2-x^2}$$

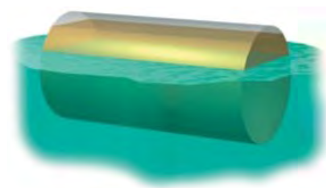
By the antiderivative:

$$\int\sqrt{a^2-x^2}\,dx=\frac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\frac{x\sqrt{a^2-x^2}}{2}+C \quad \checkmark$$

Exercise

A sealed barrel of oil (weighing 48 pounds per cubic foot) is floating in seawater (weighing 64 pounds per cubic foot). The barrel is not completely full of oil. With the barrel lying on its side, the top 0.2 *foot* of the barrel is empty.

Compare the fluid forces against one end of the barrel from the inside and from the outside.



Solution

$$x^2+y^2=1 \quad \rightarrow \quad 2x=2\sqrt{1-y^2}$$

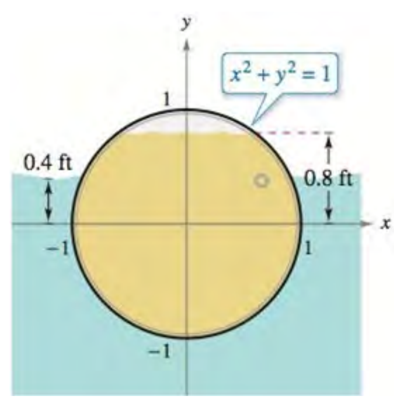
$$F_{inside}=48\int_{-1}^{0.8}(0.8-y)(2)\sqrt{1-y^2}\,dy \qquad\qquad F=w\int_c^d h(y)L(y)dy$$

$$=76.8\int_{-1}^{0.8}\sqrt{1-y^2}\,dy\;-\;96\int_{-1}^{0.8}y\sqrt{1-y^2}\,dy$$

$$=76.8\int_{-1}^{0.8}\sqrt{1-y^2}\,dy\;+\;48\int_{-1}^{0.8}\left(1-y^2\right)^{1/2}\,d\left(1-y^2\right)$$

$$y=\sin\theta \qquad \sqrt{1-y^2}=\cos\theta$$

$$dy=\cos\theta\,d\theta$$



$$=76.8\int_{-1}^{0.8}\cos^2\theta\,d\theta\;+\;32\left(1-y^2\right)^{3/2}\Big|_{-1}^{0.8}$$

$$=38.4\int_{-1}^{0.8}(1+\cos 2\theta)\,d\theta+32(0.16)^{3/2}$$

$$=38.4\left(\theta+\frac{1}{2}\sin 2\theta\right.\Big|_{-1}^{0.8}\;+\;32(0.4)^3$$

$$=38.4\left(\arcsin y+y\sqrt{1-y^2}\right.\Big|_{-1}^{0.8}\;+\;2.048$$

$$=38.4\left(\arcsin 0.8+0.32+\frac{\pi}{2}\right)+2.048$$

$$\approx 121.3 \quad lbs$$

$$F_{outside}=64\int_{-1}^{0.4}(0.4-y)(2)\sqrt{1-y^2}\,dy \qquad\qquad F=w\int_c^d h(y)L(y)dy$$

$$=51.2\int_{-1}^{0.4}\sqrt{1-y^2}\,dy-128\int_{-1}^{0.4}y\sqrt{1-y^2}\,dy$$

$$= 25.6\left(\arcsin y + y\sqrt{1-y^2}\right)\Big|_{-1}^{0.4} + \frac{128}{3}\left(1-y^2\right)^{3/2}\Big|_{-1}^{0.4}$$

$$\approx 93.0\ \ lbs$$

Exercise

The axis of a storage tank in the form of a right circular cylinder is horizontal. The radius and length of the tank are 1 *meter* and 3 *meters*, respectively.

a) Determine the volume of fluid in the tank as a function of its depth *d*.

b) Graph the function in part (a).

c) Design a dip stick for the tank with markings of $\frac{1}{4}$, $\frac{1}{2}$, *and* $\frac{3}{4}$

d) Fluid is entering the tank at a rate of $\frac{1}{4}\ m^3/min$. Determine the rate of change of the depth of the fluid as a function of its depth *d*.

e) Graph the function in part (*d*).\When will the rate of change of the depth be minimum?

Solution

a) Consider the center at $(0,\ 1)$: $x^2 + (y-1)^2 = 1$

$$x = \sqrt{1-(y-1)^2}$$

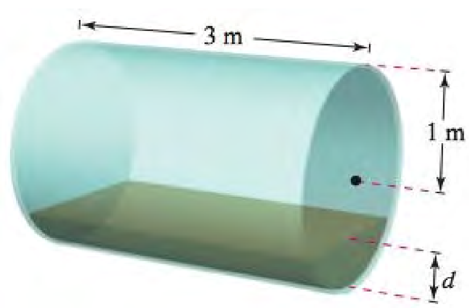
The depth: $0 \le d \le 2$

$$V = \int_0^d (3)\left(2\sqrt{1-(y-1)^2}\right)dy$$

$$= 6\int_0^d \sqrt{1-(y-1)^2}\ d(y-1)$$

$$y-1 = \sin\theta \qquad \sqrt{1-(y-1)^2} = \cos\theta$$

$$d(y-1) = \cos\theta\ d\theta$$



$$= 6\int_0^d \cos^2\theta\ d\theta$$

$$= 3\int_0^d (1+\cos 2\theta)\ d\theta$$

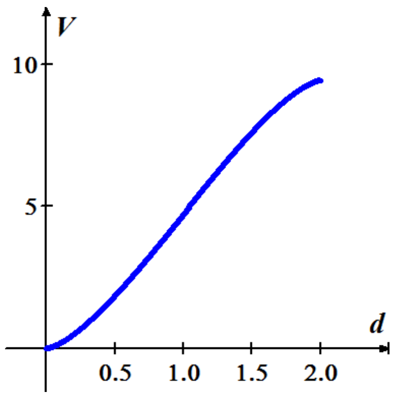
$$= 3\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^d$$

$$= 3\left(\theta + \sin\theta\cos\theta\right)\Big|_0^d$$

$$= 3\left(\arcsin(y-1)+(y-1)\sqrt{1-(y-1)^2}\right)\Big|_0^d$$

$$\boxed{= 3\arcsin(d-1)+3(d-1)\sqrt{2d-d^2}+\frac{3\pi}{2} \quad unit^3}$$

b)



c) The full tank holds $3\pi\ m^3$

A dip stick for the tank with markings of $\frac{1}{4},\ \frac{1}{2},\ and\ \frac{3}{4}$

The horizontal lines are: $y=\frac{3\pi}{4},\quad y=\frac{3\pi}{2},\quad y=\frac{9\pi}{4}$

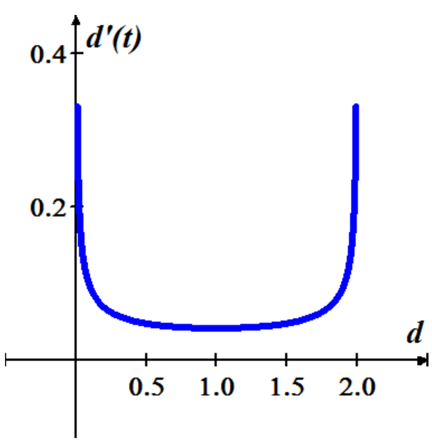
Intersect the curve at $\boxed{d=0.596,\quad d=1.0,\quad d=1.404}$

d) $V=6\int_0^d \sqrt{1-(y-1)^2}\ dy \quad \to \quad \frac{dV}{dt}=\frac{dV}{dd}\frac{dd}{dt}$

$$\frac{dV}{dt}=6\sqrt{1-(d-1)^2}\cdot d'(t)=\frac{1}{4}$$

$$\boxed{d'(t)=\frac{1}{24\sqrt{1-(d-1)^2}}}$$

e)



From the graph, the minimum occurs at $d=1$, which is the widest part of the tank.

Exercise

The field strength H of a magnet of length $2L$ on a particle r units from the center of the magnet is

$$H = \frac{2mL}{\left(r^2+L^2\right)^{3/2}}$$

Where $\pm m$ are the poles of the magnet.

Find the average field strength as the particle moves from 0 to R units from the center by evaluating the integral

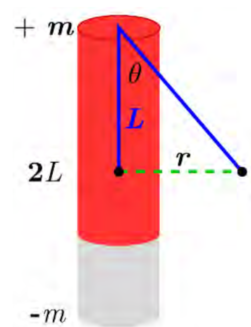
$$\frac{1}{R}\int_0^R \frac{2mL}{\left(r^2+L^2\right)^{3/2}}\,dr$$

<u>***Solution***</u>

$$r = L\tan\theta \quad \to \quad dr = L\sec^2\theta\,d\theta$$

$$r^2+L^2 = L^2\tan^2\theta + L^2$$

$$= L^2\sec^2\theta$$



$$\frac{1}{R}\int_0^R \frac{2mL}{\left(r^2+L^2\right)^{3/2}}\,dr = \frac{1}{R}\int_0^R \frac{2mL}{(L\sec\theta)^3}L\sec^2\theta\;d\theta$$

$$= \frac{2m}{RL}\int_0^R \frac{1}{\sec\theta}\;d\theta$$

$$= \frac{2m}{RL}\int_0^R \cos\theta\;d\theta$$

$$= \frac{2m}{RL}\sin\theta\;\Big|_0^R$$

$$= \frac{2m}{RL}\frac{r}{\sqrt{r^2+L^2}}\;\Big|_0^R$$

$$= \frac{2m}{L\sqrt{R^2+L^2}}$$