SOLUTION Section 3.1 – Sequences

Exercise

Find the values of $a_1, a_2, a_3,$ *and* a_4 for $a_n = \frac{1-n}{n^2}$

Solution

$$a_1 = \frac{1-1}{1^2} = 0, \qquad a_2 = \frac{1-2}{2^2} = -\frac{1}{4}$$

$$a_3 = \frac{1-3}{3^2} = -\frac{2}{9}, \quad a_4 = \frac{1-4}{4^2} = -\frac{3}{16}$$

Exercise

Find the values of $a_1, a_2, a_3,$ *and* a_4 for $a_n = \frac{1}{n!}$

Solution

$$a_1 = \frac{1}{1!} = 1, \quad a_2 = \frac{1}{2!} = \frac{1}{4}, \quad a_3 = \frac{1}{3!} = \frac{1}{6}, \quad a_4 = \frac{1}{4!} = \frac{1}{24}$$

Exercise

Find the values of $a_1, a_2, a_3,$ *and* a_4 for $a_n = \frac{(-1)^{n+1}}{2n-1}$

Solution

$$a_1 = \frac{(-1)^{1+1}}{2(1)-1} = 1, \quad a_2 = \frac{(-1)^{2+1}}{2(2)-1} = -\frac{1}{3}, \quad a_3 = \frac{(-1)^{3+1}}{2(3)-1} = \frac{1}{5}, \quad a_4 = \frac{(-1)^{4+1}}{2(4)-1} = -\frac{1}{7}$$

Exercise

Find the values of $a_1, a_2, a_3,$ *and* a_4 for $a_n = 2+(-1)^n$

Solution

$$a_1 = 2+(-1)^1 = 1, \quad a_2 = 2+(-1)^2 = 3, \quad a_3 = 2+(-1)^3 = 1, \quad a_4 = 2+(-1)^4 = 3$$

Exercise

Find the values of $a_1, a_2, a_3,$ *and* a_4 for $a_n = \frac{2^n-1}{2^n}$

Solution

$$a_1 = \frac{2^1-1}{2^1} = \frac{1}{2}, \quad a_2 = \frac{2^2-1}{2^2} = \frac{3}{4}, \quad a_3 = \frac{2^3-1}{2^3} = \frac{7}{8}, \quad a_4 = \frac{2^4-1}{2^5} = \frac{15}{32}$$

Exercise

Write the first ten terms of the sequence $a_1 = 1, \quad a_{n+1} = a_n + \frac{1}{2^n}$

Solution

$$a_2 = a_1 + \frac{1}{2^1} = 1 + \frac{1}{2} = \frac{3}{2}, \quad a_3 = \frac{3}{2} + \frac{1}{2^2} = \frac{7}{4}, \quad a_4 = \frac{7}{4} + \frac{1}{2^3} = \frac{15}{8}, \quad a_5 = \frac{15}{8} + \frac{1}{2^4} = \frac{31}{16},$$

$$a_6 = \frac{31}{16} + \frac{1}{2^5} = \frac{63}{32}, \quad a_7 = \frac{63}{32} + \frac{1}{2^6} = \frac{127}{64}, \quad a_8 = \frac{127}{64} + \frac{1}{2^7} = \frac{255}{128}, \quad a_9 = \frac{255}{128} + \frac{1}{2^8} = \frac{511}{256},$$

$$a_{10} = \frac{511}{256} + \frac{1}{2^9} = \frac{1023}{512}$$

Exercise

Write the first ten terms of the sequence $a_1 = 1, \quad a_{n+1} = \frac{a_n}{n+1}$

Solution

$$a_1 = \boxed{1}, \quad a_2 = \frac{1}{1+1} = \boxed{\frac{1}{2}}, \quad a_3 = \frac{\frac{1}{2}}{2+1} = \boxed{\frac{1}{6}}, \quad a_4 = \frac{\frac{1}{6}}{3+1} = \boxed{\frac{1}{24}},$$

$$a_5 = \frac{\frac{1}{24}}{4+1} = \boxed{\frac{1}{120}}, \quad a_6 = \frac{\frac{1}{120}}{5+1} = \boxed{\frac{1}{720}}, \quad a_7 = \frac{\frac{1}{720}}{6+1} = \boxed{\frac{1}{5040}}, \quad a_8 = \frac{\frac{1}{5040}}{7+1} = \boxed{\frac{1}{40,320}},$$

$$a_9 = \frac{\frac{1}{40,320}}{8+1} = \boxed{\frac{1}{362,880}}, \quad a_{10} = \frac{\frac{1}{362,880}}{9+1} = \boxed{\frac{1}{3,628,800}}$$

Exercise

Write the first ten terms of the sequence $a_1 = 2, \quad a_2 = -1, \quad a_{n+2} = \frac{a_{n+1}}{a_n}$

Solution

$$a_1 = 2, \quad a_2 = -1, \quad a_3 = \frac{-1}{2}, \quad a_4 = \frac{-\frac{1}{2}}{-1} = \frac{1}{2}, \quad a_5 = \frac{\frac{1}{2}}{-\frac{1}{2}} = -1,$$

$$a_6 = \frac{-1}{\frac{1}{2}} = -2, \quad a_7 = \frac{-2}{-1} = 2, \quad a_8 = \frac{2}{-2} = -1, \quad a_9 = \frac{-1}{2} = -\frac{1}{2}, \quad a_{10} = \frac{-\frac{1}{2}}{-1} = \frac{1}{2}$$

Exercise

Find a formula for the nth term of the sequence $-1, 1, -1, 1, -1, \cdots$

Solution

$$\boxed{a_n = (-1)^n} \quad n \in \mathbb{N}$$

Exercise

Find a formula for the nth term of the sequence $1,\ -\frac{1}{4},\ \frac{1}{9},\ -\frac{1}{16},\ \frac{1}{25},\cdots$

Solution

$$a_1 = 1 \quad r = -\frac{1}{4} \quad \to a_n = a_1 r = -\frac{1}{4} = \frac{(-1)^{n+1}}{n^2}$$

$$\boxed{a_n = \frac{(-1)^{n+1}}{n^2}} \quad n \in \mathbb{N}$$

Exercise

Find a formula for the nth term of the sequence $\frac{1}{9},\ \frac{2}{12},\ \frac{2^2}{15},\ \frac{2^3}{18},\ \frac{2^4}{21},\cdots$

Solution

$$\boxed{a_n = \frac{2^{n-1}}{3(n+2)}} \quad n \in \mathbb{N}$$

Exercise

Find a formula for the nth term of the sequence $-3,\ -2,\ -1,\ 0,\ 1,\cdots$

Solution

$$d = -2 - (-3) = 1$$

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= -3 + (n-1)(1) \\ &= -3 + n - 1 \\ &\boxed{= n - 4} \quad n \in \mathbb{N} \end{aligned}$$

Exercise

Find a formula for the nth term of the sequence $\frac{1}{25},\ \frac{8}{125},\ \frac{27}{625},\ \frac{64}{3125},\ \frac{125}{15,625},\cdots$

Solution

$$\frac{1}{5^2},\ \frac{2^3}{5^3},\ \frac{3^3}{5^4},\ \frac{4^3}{5^5},\ \frac{5^3}{5^6},\cdots$$

$$a_n = \frac{n^3}{5^{n+1}} \quad n \in \mathbb{N}$$

Exercise

Find a formula for the *n*th term of the sequence $0,\ 1,\ 1,\ 2,\ 2,\ 3,\ 3,\ 4,\ \cdots$

Solution

$$a_n = \dfrac{n-\frac{1}{2}+(-1)^n\left(\frac{1}{2}\right)}{2} \qquad n \in \mathbb{N}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{n+(-1)^n}{n}$$

Solution

$$\lim_{n\to\infty} \frac{n+(-1)^n}{n} = \lim_{n\to\infty}\left(1+\frac{(-1)^n}{n}\right) \underline{=1|} \quad \Rightarrow \quad \textit{converges}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{1-2n}{1+2n}$$

Solution

$$\lim_{n\to\infty} \frac{1-2n}{1+2n} = \lim_{n\to\infty}\left(\frac{\frac{1}{n}-2}{\frac{1}{n}+2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{-2}{2}\right)$$

$$\underline{=-1|}$$ The limit ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{1-n^3}{70-4n^2}$$

Solution

$$\lim_{n\to\infty} \frac{1-n^3}{70-4n^2} = \lim_{n\to\infty} \frac{\frac{1}{n^2}-n}{\frac{70}{n^2}-4}$$

$$\lim_{n\to\infty} \frac{0-n}{0-4}$$

$$\underline{=\infty|} \quad \Rightarrow \quad \textit{diverges}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \left(2 - \frac{1}{2^n}\right)\left(3 + \frac{1}{2^n}\right)$$

Solution

$$\lim_{n\to\infty}\left(2 - \frac{1}{2^n}\right)\left(3 + \frac{1}{2^n}\right) = (2)(3) \underline{= 6|} \quad \Rightarrow \quad \textit{converges}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = n\pi \cos(n\pi)$$

Solution

$$\lim_{n\to\infty} n\pi \cos(n\pi) = \lim_{n\to\infty} n\pi(-1)^n \underline{= \infty|} \quad \Rightarrow \quad \textit{diverges}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = n - \sqrt{n^2 - n}$$

Solution

$$\begin{aligned}
\lim_{n\to\infty} n - \sqrt{n^2 - n} &= \lim_{n\to\infty}\left(n - \sqrt{n^2 - n}\right)\frac{n+\sqrt{n^2-n}}{n+\sqrt{n^2-n}} \\
&= \lim_{n\to\infty} \frac{n^2-\left(n^2-n\right)}{n+\sqrt{n^2-n}} \\
&= \lim_{n\to\infty} \frac{n}{n+\sqrt{n^2-n}} \\
&= \lim_{n\to\infty} \frac{1}{1+\sqrt{1-\frac{1}{n}}} \\
&\underline{= \frac{1}{2}|}
\end{aligned}$$

The given series *converges*.

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \sqrt{\frac{2n}{n+1}}$$

Solution

$$\lim_{n\to\infty}\sqrt{\frac{2n}{n+1}} = \sqrt{\lim_{n\to\infty}\frac{2}{1+\frac{1}{n}}} \underline{= \sqrt{2}|} \quad \Rightarrow \quad \textit{The given series converges}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\sin^2 n}{2^n}$$

Solution

$$0 \le \frac{\sin^2 n}{2^n} \le \frac{1}{2^n} \quad \text{By the Sandwich Theorem for sequences}$$

$$\lim_{n\to\infty} \frac{\sin^2 n}{2^n} \underline{= 0|} \quad \Rightarrow \quad \textit{The given series converges}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\ln n}{\ln 2n}$$

Solution

$$\lim_{n\to\infty} \frac{\ln n}{\ln 2n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{2}{2n}} \underline{= 1|} \quad \Rightarrow \quad \textit{The given series converges}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{3^n \cdot 6^n}{2^{-n} \cdot n!}$$

Solution

$$\begin{aligned}
\lim_{n\to\infty} \frac{3^n \cdot 6^n}{2^{-n} \cdot n!} &= \lim_{n\to\infty} \frac{2^n \cdot 3^n \cdot 6^n}{n!} \\
&= \lim_{n\to\infty} \frac{36^n}{n!} \\
&\underline{= 0|} \quad \Rightarrow \quad \textit{The given series converges}
\end{aligned}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \sqrt{n}\ \sin \frac{1}{\sqrt{n}}$$

Solution

$$\lim_{n\to\infty} \frac{\sin \frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\left(-\frac{1}{2n^{3/2}}\right)\cos\frac{1}{\sqrt{n}}}{-\frac{1}{2n^{3/2}}}$$

$$= \lim_{n\to\infty} \cos\frac{1}{\sqrt{n}}$$

$$= \cos 0$$

$$\underline{= 1|} \quad \Rightarrow \quad \textit{The given series converges}$$

or $\displaystyle \lim_{n\to\infty} \frac{\sin\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{u\to 0} \frac{\sin u}{u} = 1$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{n^2}{2^n - 1}$$

Solution

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{2^n - 1} \qquad \text{L'Hôpital Rule}$$

$$= \lim_{x\to\infty} \frac{2x}{(\ln 2)\cdot 2^x}$$

$$= \lim_{x\to\infty} \frac{2}{(\ln 2)^2 \cdot 2^x}$$

$$\underline{= 0|} \qquad \text{The sequence } \textbf{\textit{converges}}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$\left\{c_n\right\} = \left\{(-1)^n \frac{1}{n!}\right\}$$

Solution

$$n! = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdots n = 24\cdot \underbrace{5\cdot 6\cdots n}_{n-4}$$

$$2^2 = 2\cdot 2\cdot 2\cdot 2\cdot 2\cdots 2 = 16\cdot \underbrace{2\cdot 2\cdots 2}_{n-4}$$

$$\frac{-1}{2^n} \le (-1)^n \frac{1}{n!} \le \frac{1}{2^n} \qquad n \ge 4$$

By the Squeeze Theorem

$$\lim_{n\to\infty} (-1)^n \frac{1}{n!} \underline{= 0|} \qquad \text{The sequence } \textbf{\textit{converges}}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{5}{n+2}$$

Solution

$$\lim_{n\to\infty} \frac{5}{n+2} \, \underline{=0|}$$ The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = 8 + \frac{5}{n}$$

Solution

$$\lim_{n\to\infty} \left(8 + \frac{5}{n}\right) \underline{=8|}$$ The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = (-1)^n \left(\frac{n}{n+1}\right)$$

Solution

$$\lim_{n\to\infty} (-1)^n \left(\frac{n}{n+1}\right)$$ does not exist (oscillates between -1 and 1)

The sequence ***diverges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{1+(-1)^n}{n^2}$$

Solution

$$\lim_{n\to\infty} \frac{1+(-1)^n}{n^2} \, \underline{=0|}$$ The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{10n^2 + 3n + 7}{2n^2 - 6}$$

Solution

$$\lim_{n\to\infty} \frac{10n^2 + 3n + 7}{2n^2 - 6} = \lim_{n\to\infty} \frac{10n^2}{2n^2}$$

$$\underline{= 5|} \qquad \text{The sequence } \textbf{\textit{converges}}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\sqrt[3]{n}}{\sqrt[3]{n} + 1}$$

Solution

$$\lim_{n\to\infty} \frac{\sqrt[3]{n}}{\sqrt[3]{n} + 1} \underline{= 1|} \qquad \text{The sequence } \textbf{\textit{converges}}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\ln\left(n^3\right)}{2n}$$

Solution

$$\lim_{n\to\infty} \frac{\ln\left(n^3\right)}{2n} = \lim_{n\to\infty} \frac{3\ln(n)}{2n}$$

$$= \lim_{n\to\infty} \frac{3}{2} \frac{\frac{1}{n}}{1}$$

$$\underline{= 0|} \qquad \text{The sequence } \textbf{\textit{converges}}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{5^n}{3^n}$$

Solution

$$\lim_{n\to\infty} \frac{5^n}{3^n} = \lim_{n\to\infty} \left(\frac{5}{3}\right)^n \underline{= \infty|} \qquad \text{The sequence } \textbf{\textit{diverges}}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{(n+1)!}{n!}$$

Solution

$$\lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) \underline{\underline{= \infty}}$$ The sequence ***diverges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{(n-2)!}{n!}$$

Solution

$$\lim_{n\to\infty} \frac{(n-2)!}{n!} = \lim_{n\to\infty} \frac{1}{n(n-1)} \underline{\underline{= 0}}$$ The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{n^p}{e^n}, \quad p > 0$$

Solution

$$\lim_{n\to\infty} \frac{n^p}{e^n} \underline{\underline{= 0}}$$ The sequence ***converges*** $(p > 0, \quad n \ge 2)$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = n \sin \frac{1}{n}$$

Solution

$$\lim_{n\to\infty} n \sin \frac{1}{n} = \lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}}$$ Let $x = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$

$$= \lim_{x\to 0} \frac{\sin x}{x}$$ Since $\lim_{x\to 0} \frac{\sin x}{x} = 1 \quad \lim_{x\to 0} \frac{\cos x}{1} = 1$

$$\underline{\underline{= 1}}$$ The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = 2^{1/n}$$

Solution

$$\lim_{n\to\infty} 2^{1/n} = 2^0 \underline{\underline{= 1}}$$ The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = -3^{-n}$$

Solution

$$\lim_{n\to\infty} -3^{-n} = \lim_{n\to\infty}\left(-\frac{1}{3^n}\right) \underline{= 0|}$$ The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\sin n}{n}$$

Solution

$$\lim_{n\to\infty} \frac{\sin n}{n} = \lim_{n\to\infty} \frac{1}{n}(\sin n) \underline{= 0|} \text{ since } \frac{1}{n} \to 0$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\cos \pi n}{n^2}$$

Solution

$$\lim_{n\to\infty} \frac{\cos \pi n}{n^2} = \lim_{n\to\infty} \frac{1}{n^2}(\cos \pi n) \underline{= 0|}$$ The sequence ***converges***