Lecture 1 – Functions, Exponential & Logarithms

Section 1.1 – Functions

A ***set*** is a collection of objects of some type, and the objects are called ***elements*** of the set.

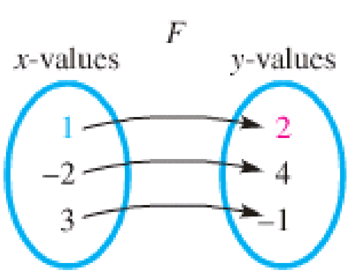
Notation or Terminology	***Meaning***	***Example***
$a \in S$	***a*** is an element of ***S***	$3 \in \mathbb{Z}$
$a \notin S$	***a*** is not an element of ***S***	$\frac{3}{2} \notin \mathbb{Z}$
$S \subset T$	S is a ***subset*** of T Every element of S is an element of T	$\mathbb{Z} \subset \mathbb{R}$
Constant	A letter or symbol that represents a specific element of a set.	$5,\ \sqrt{2},\ \pi$
Variable	A letter or symbol that represents any element of a set.	Let ***x*** denote any real number

Definition of a *Function*

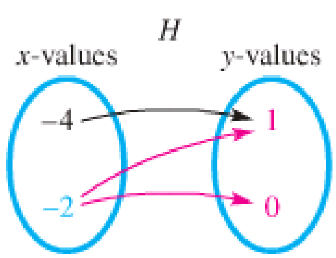
A ***function*** is a relation between two variables such that to matches each element of a first set (called ***domain***) to an element of a second set (called ***range***) in such way that no element in the first set is assigned to two different elements in the second set.

The ***domain*** of the function is the set of all values of the independent variable for which the function is defined.

The ***range*** of the function is the set of all values taken on by the dependent variable.



F is a function.



H is not a function.

The *Domain* of a Function

1. *Rational* function: $\dfrac{f(x)}{h(x)}$ $\qquad \Rightarrow$***Domain***: $h(x) \neq 0$

Example: $f(x) = \dfrac{1}{x-3}$ $\qquad$ ***Domain***: $x \neq 3$

2. *Irrational* function: $\sqrt{g(x)}$ $\qquad \Rightarrow$ ***Domain***: $g(x) \geq 0$

Example: $g(x) = \sqrt{3-x} + 5$ $\qquad \Rightarrow 3 - x \geq 0 \Rightarrow -x \geq -3$

Domain: $x \leq 3$

3. Otherwise: *Domain* all real numbers

Example: $f(x) = x^3 + |x|$ $\qquad$ ***Domain***: All real numbers $(-\infty, \infty)$

(1) & (2)→ Find the domain: $f(x) = \dfrac{x+1}{\sqrt{x-3}}$ $\qquad \Rightarrow$ ***Domain:*** $x > 3$

$$\boxed{\begin{array}{l} ax^2 + bx + c \geq 0 \;\rightarrow\; if\;\; a > 0 \;\Rightarrow\; x \leq x_1,\;\; x \geq x_2 \\ ax^2 + bx + c \leq 0 \;\rightarrow\; if\;\; a > 0 \;\Rightarrow\; x_1 \leq x \leq x_2 \end{array}}$$

Example

Let $g(x) = \dfrac{\sqrt{4+x}}{1-x}$. Find the domain of g.

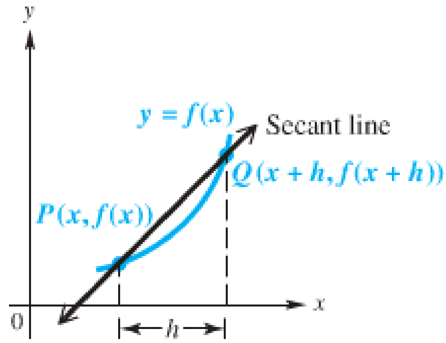
<u>Solution</u>

$$\begin{cases} 4 + x \geq 0 \Rightarrow x \geq -4 \\ 1 - x \neq 0 \Rightarrow x \neq 1 \end{cases} \quad \rightarrow \underline{[-4,\ 1) \cup (1,\ \infty)}$$

Difference Quotients

$$\frac{f(x+h)-f(x)}{(x+h)-x}$$

The difference quotient is given by: $\dfrac{f(x+h)-f(x)}{h}$



Example

For the function *f* given by $f(x)=2x^2-3x$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$
\begin{aligned}
f(x+h) &= 2(\text{——})^2 - 3(\text{——}) \\
&= 2(x+h)^2 - 3(x+h) \qquad\qquad (a+b)^2 = a^2+2ab+b^2 \\
&= 2\left(x^2+2xh+h^2\right) - 3x - 3h \\
&= 2x^2+4xh+2h^2-3x-3h
\end{aligned}
$$

$$
\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\overbrace{2x^2+4xh+2h^2-3x-3h}^{f(x+h)} - \overbrace{(2x^2-3x)}^{f(x)}}{h} \\
&= \frac{2x^2+4xh+2h^2-3x-3h-2x^2+3x}{h} \\
&= \frac{4xh+2h^2-3h}{h} \\
&= \frac{4xh}{h}+\frac{2h^2}{h}-\frac{3h}{h} \\
&= 4x+2h-3
\end{aligned}
$$

Sum	$(f+g)(x)=f(x)+g(x)$
Difference	$(f-g)(x)=f(x)-g(x)$
Product	$(f\cdot g)(x)=f(x)\cdot g(x)$
Quotient	$\left(\dfrac{f}{g}\right)(x)=\dfrac{f(x)}{g(x)}$

Example

Let $f(x)=8x-9$ and $g(x)=\sqrt{2x-1}$. Find $(f+g)(x),\ (f-g)(x),\ (f\cdot g)(x),\ and\ (f/g)(x)$ and give the domain

Solution

Domain of f: $(-\infty,\infty)$

Domain of g: $2x-1\ge 0 \ \to 2x\ge 1 \ \Rightarrow x\ge \frac{1}{2}$

a) $(f+g)(x)$

$$(f+g)(x)=8x-9+\sqrt{2x-1}$$

Domain: $x\ge \frac{1}{2}$ or $\left[\frac{1}{2},\infty\right)$

b) $(f-g)(x)$

$$(f-g)(x)=8x-9-\sqrt{2x-1}$$

Domain: $x\ge \frac{1}{2}$

c) $(fg)(x)$

$$(fg)(x)=(8x-9)\sqrt{2x-1}$$

Domain: $x\ge \frac{1}{2}$

d) $\left(\frac{f}{g}\right)(x)$

$$\left(\frac{f}{g}\right)(x)=\frac{8x-9}{\sqrt{2x-1}} \qquad\qquad 2x-1>0 \ \to 2x>1 \ \Rightarrow x>\frac{1}{2}$$

Domain: $x>\frac{1}{2}$

Even and Odd Functions

Given the function $f(x)$ then find $f(-x)$ and simplify:

- If $f(-x) = f(x) \Rightarrow f$ is ***even***, or
- If $f(-x) = -f(x) \Rightarrow f$ is ***odd***
- ***Neither***

Example

Decide whether each function is even, odd, or neither

a) $f(x) = 8x^4 - 3x^2$

$$\begin{aligned} f(-x) &= 8(-x)^4 - 3(-x)^2 \\ &= 8x^4 - 3x^2 \\ &= f(x) \end{aligned}$$

Function is *Even*

b) $f(x) = 6x^3 - 9x$

$$\begin{aligned} f(-x) &= 6(-x)^3 - 9(-x) \\ &= -6x^3 + 9x \\ &= -\left(6x^3 - 9x\right) \\ &= -f(x) \end{aligned}$$

Function is *Odd*

c) $f(x) = 3x^2 + 5x$

$$\begin{aligned} f(-x) &= 3(-x)^2 + 5(-x) \\ &= 3x^2 - 5x \end{aligned}$$

Function is *Neither*

Piecewise-Defined Functions

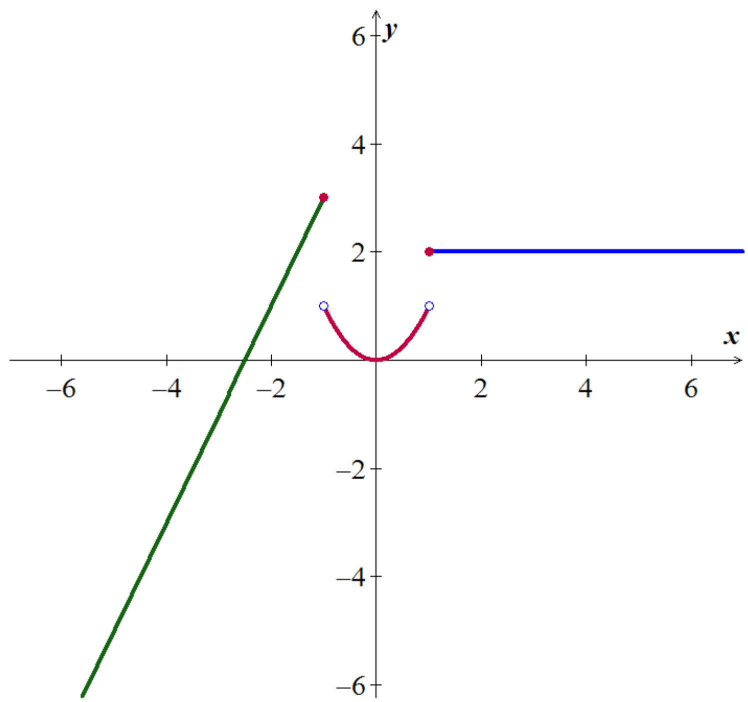
Function are sometimes described by more than one expression, we call such functions ***piecewise-defined functions***.

Example

Graph each function

$$f(x)=\begin{cases} 2x+5 & \text{if} \quad x\le -1 \\ x^2 & \text{if} \quad |x|<1 \\ 2 & \text{if} \quad x\ge 1 \end{cases}$$

Solution

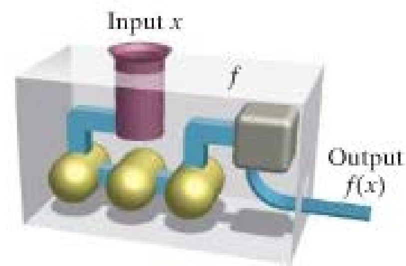


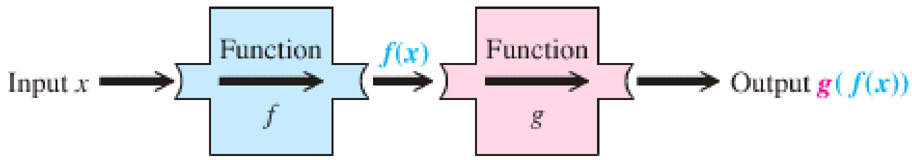
Composition of Functions

The composite function $f \circ g$, the composite of f and g, is defined as

$$\left(f \circ g\right)(x) = f\left(g\left(x\right)\right)$$

Where x is in the domain of g
and $g\left(x\right)$ is in the domain of f





Example

Let $f(x) = x^2 - 1$ *and* $g(x) = 3x + 5$

a) Find $\left(f \circ g\right)(x)$ and the domain of $f \circ g$

b) Find $\left(g \circ f\right)(x)$ and the domain of $g \circ f$

c) Find $\left(f\left(g\right)\right)(2)$ in two different ways: first using the functions f and g separately and second using the composite function $f \circ g$.

Solution

a) $\left(f \circ g\right)(x) = f\left(g(x)\right)$

$= f\left(3x+5\right)$ ***Domain***: $\left(3x+5\right) \to \mathbb{R}$

$= \left(\underline{\quad}\right)^2 - 1$

$= \left(3x+5\right)^2 - 1$

$= 9x^2 + 30x + 25 - 1$

$= 9x^2 + 30x + 24$ ***Domain***: $\left(9x^2 + 30x + 24\right) \to \mathbb{R}$

Domain of $f \circ g$: $\mathbb{R}$

b) $\left(g \circ f\right)(x) = g\left(f(x)\right)$

$= g\left(x^2 - 1\right)$ ***Domain***: $\left(x^2 - 1\right) \to \mathbb{R}$

$= 3\left(x^2 - 1\right) + 5$

$= 3x^2 - 3 + 5$

$= 3x^2 + 2$ ***Domain***: $\left(3x^2 + 2\right) \to \mathbb{R}$

Domain of $g \circ f$: $\mathbb{R}$

c) $g(2)=3(2)+5=11$

$$\begin{aligned}(f\circ g)(2) &= f\big(g(2)\big)\\ &= f(11)\\ &= 11^2-1\\ &= 120\end{aligned}$$

$(f\circ g)(x)=9x^2+30x+24$

$(f\circ g)(2)=9(2)^2+30(2)+24 \ \underline{=120}$

Example

Let $f(x)=x^2-16$ *and* $g(x)=\sqrt{x}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $(f\circ g)(x)=f\big(g(x)\big)$

$= f\left(\sqrt{x}\right)$ **Domain** : $\left(\sqrt{x}\right)\to x\ge 0$

$=\left(\sqrt{x}\right)^2-16$

$=x-16$ **Domain** : $(x-16)\to\mathbb{R}$

Domain of $f\circ g$: $x\ge 0$

b) $(g\circ f)(x)=g\big(f(x)\big)$

$=g\left(x^2-16\right)$ **Domain** : $\left(x^2-1\right)\to\mathbb{R}$

$=\sqrt{x^2-16}$ **Domain** : $\left(\sqrt{x^2-16}\right)\to|x|\ge 4$

Domain of $g\circ f$: $|x|\ge 4$ *or* $(-\infty,\ -4]\cup[4,\ \infty)$

Exercises *Section* 1.1 –Functions

Find the Domain

1. $f(x) = 7x + 4$
2. $f(x) = |3x - 2|$
3. $f(x) = x^2 - 2x - 15$
4. $g(x) = \frac{3}{x-4}$
5. $y = \frac{2}{x-3}$
6. $y = \frac{-7}{x-5}$
7. $f(x) = 4 - \frac{2}{x}$
8. $f(x) = \frac{1}{x^4}$
9. $f(x) = \frac{x+5}{2-x}$
10. $f(x) = \frac{8}{x+4}$
11. $f(x) = \frac{1}{x^2-4x-5}$
12. $f(x) = \sqrt{8-3x}$
13. $g(x) = \frac{2}{x^2+x-12}$
14. $h(x) = \frac{5}{\frac{4}{x}-1}$
15. $y = \sqrt{x}$
16. $y = \sqrt{4x+1}$
17. $y = \sqrt{7-2x}$
18. $f(x) = \sqrt{8-x}$
19. $f(x) = \frac{\sqrt{x+1}}{x}$
20. $g(x) = \frac{\sqrt{x-3}}{x-6}$
21. $f(x) = \frac{\sqrt{x+4}}{\sqrt{x-1}}$
22. $f(x) = \sqrt{x+4} - \sqrt{x-1}$
23. $f(x) = \sqrt{2x+7}$
24. $f(x) = \sqrt{9-x^2}$
25. $f(x) = \sqrt{x^2-25}$
26. $f(x) = \frac{x+1}{x^3-4x}$
27. $f(x) = \frac{4x}{6x^2+13x-5}$
28. $f(x) = \frac{\sqrt{2x-3}}{x^2-5x+4}$
29. $f(x) = \frac{\sqrt{4x-3}}{x^2-4}$
30. $f(x) = \frac{x-4}{\sqrt{x-2}}$
31. $f(x) = \frac{1}{(x-3)\sqrt{x+3}}$
32. $f(x) = \sqrt{x+2} + \sqrt{2-x}$
33. $f(x) = \sqrt{(x-2)(x-6)}$

Find the difference quotient $\frac{f(x)-f(a)}{x-a}$, for the given function

34. $f(x) = \sqrt{x-3}$,
35. $f(x) = 2x^2$
36. $f(x) = 9x + 5$
37. $f(x) = 6x + 2$
38. $f(x) = 4x + 11$
39. $f(x) = 2x^2 - x - 3$

40. Find $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, *and* $(f/g)(x)$ and the domain of

$f(x) = \sqrt{3-2x}, \quad g(x) = \sqrt{x+4}$

41. Find $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, *and* $(f/g)(x)$ and the domain of

$f(x) = \frac{2x}{x-4}, \quad g(x) = \frac{x}{x+5}$

42. Let $f(x)=\sqrt{4x-1}$ and $g(x)=\frac{1}{x}$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\frac{f}{g}\right)(x)$

43. Given that $f(x)=x+1$ and $g(x)=\sqrt{x+3}$

a) Find $(f+g)(x)$

b) Find the domain of $(f+g)(x)$

c) Find: $(f+g)(6)$

44. Given that $f(x)=x^2-4$ and $g(x)=x+2$

a) Find $(f+g)(x)$ and its domain

b) Find $(f/g)(x)$ and its domain

45. Find $(f\circ g)(x)$, $(g\circ f)(x)$, $f(g(-2))$ *and* $g(f(3))$: $f(x)=2x^2+3x-4$, $g(x)=2x-1$

46. Find $(f\circ g)(x)$, $(g\circ f)(x)$, $f(g(-2))$ *and* $g(f(3))$: $f(x)=x^3+2x^2$, $g(x)=3x$

47. Find $(f\circ g)(x)$, $(g\circ f)(x)$, $f(g(-2))$ *and* $g(f(3))$: $f(x)=|x|$, $g(x)=-7$

48. Let $f(x)=x^2-3x$ *and* $g(x)=\sqrt{x+2}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

49. Let $f(x)=\sqrt{x-2}$ *and* $g(x)=\sqrt{x+5}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

50. Let $f(x)=\frac{3x+5}{2}$ *and* $g(x)=\frac{2x-5}{3}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

51. Let $f(x)=\frac{x-1}{x-2}$ *and* $g(x)=\frac{x-3}{x-4}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

52. Given $f(x)=\sqrt{x}$ and $g(x)=x+3$, find $(f\circ g)(x)$, $(g\circ f)(x)$ and their domain.

53. Given that $f(x)=\sqrt{x}$ and $g(x)=2-3x$, find $(f\circ g)(x)$, $(g\circ f)(x)$ and their domain.

54. Given that $f(x)=\dfrac{1}{x-2}$ and $g(x)=\dfrac{x+2}{x}$, find $(f\circ g)(x)$, $(g\circ f)(x)$ and their domain.

55. Given that $f(x)=2x-5$ and $g(x)=x^2-3x+8$, find $(f\circ g)(x)$, $(g\circ f)(x)$ and their domain then find $(f\circ g)(7)$

56. Given that $f(x)=\sqrt{x}$ and $g(x)=x-1$, find

a) $(f\circ g)(x)=f(g(x))$

b) $(g\circ f)(x)=g(f(x))$

c) $(f\circ g)(2)=f(g(2))$

57. Given that $f(x)=\dfrac{x}{x+5}$ and $g(x)=\dfrac{6}{x}$, find

a) $(f\circ g)(x)=f(g(x))$

b) $(g\circ f)(x)=g(f(x))$

c) $(f\circ g)(2)=f(g(2))$

Determine whether f is even, odd, or neither

58. $f(x)=3x^4+2x^2-5$

59. $f(x)=8x^3-3x^2$

60. $f(x)=\sqrt{x^2+4}$

61. $f(x)=3x^2-5x+1$

62. $f(x)=\sqrt[3]{x^3-x}$

63. $f(x)=|x|-3$

64. $f(x)=x^3-\dfrac{1}{x}$

65. $f(x)=-x^3+2x$

66. $f(x)=x^5-2x^3$

67. $f(x)=.5x^4-2x^2+6$

68. $f(x)=.75x^2+|x|+4$

69. $f(x)=x^3-x+9$

70. $f(x)=x^4-5x+8$

71. $f(x)=x^3+x$

72. $g(x)=x^2-x$

73. $h(x)=2x^2+x^4$

74. $f(x)=2x^2+x^4+1$

75. $f(x)=\frac{1}{5}x^6-3x^2$

76. $f(x)=x\sqrt{1-x^2}$

77. $f(x)=x^2\sqrt{1-x^2}$

78. $f(x)=5x^7-6x^3-2x$

79. $f(x)=5x^6-3x^2-7$

80. $f(x)=x^2+6$

81. $f(x)=7x^3-x$

82. $h(x)=x^5+1$

83. $f(x)=\begin{cases} 2+x & \text{if } x<-4 \\ -x & \text{if } -4\le x\le 2 \\ 3x & \text{if } x>2 \end{cases}$ Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

84. $f(x)=\begin{cases} -2x & \text{if } x<-3 \\ 3x-1 & \text{if } -3\le x\le 2 \\ -4x & \text{if } x>2 \end{cases}$ Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

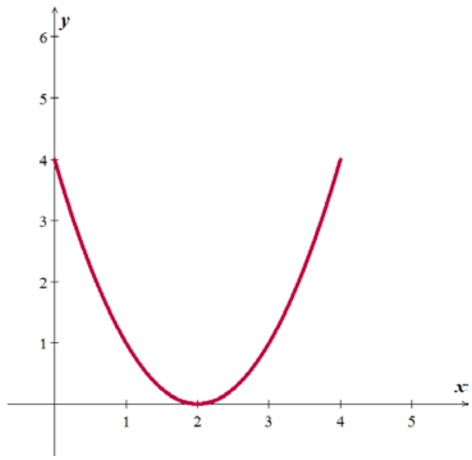
85. The graph of a function f with domain [0, 4] is shown:

a) $y=f(x+3)$

b) $y=f(x-2)+3$

c) $y=f\left(-\frac{1}{2}x\right)$

d) $y=\left|f(x)\right|$



86. $f(x)=\begin{cases} x^3+3 & \text{if } -2\le x\le 0 \\ x+3 & \text{if } 0<x<1 \\ 4+x-x^2 & \text{if } 1\le x\le 3 \end{cases}$ Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

87. $h(x)=\begin{cases} \dfrac{x^2-9}{x-3} & \text{if } x\ne 3 \\ 6 & \text{if } x=3 \end{cases}$ Find: $h(5),\ h(0),\ and\ h(3)$

88. Graph the piecewise function defined by $f(x)=\begin{cases} 3 & \text{if } x\le -1 \\ x-2 & \text{if } x>-1 \end{cases}$

89. Sketch the graph $f(x)=\begin{cases} x+2 & \text{if } x\le -1 \\ x^3 & \text{if } -1<x<1 \\ -x+3 & \text{if } x\ge 1 \end{cases}$

90. Sketch the graph $f(x)=\begin{cases} x-3 & \text{if } x\le -2 \\ -x^2 & \text{if } -2<x<1 \\ -x+4 & \text{if } x\ge 1 \end{cases}$

Section 1.2 – Polynomial Functions & Graphs

Polynomial Function

A *Polynomial function* $P(x)$ in x is a sum of the form is given by:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$$

Where the coefficients $a_n,\ a_{n-1},\ \ldots, a_2,\ a_1, a_0$ are real numbers and the exponents are whole numbers.

$$a_n x^n$$

Degree

Leading Term

Leading Coefficient

Non-polynomial Functions: $\frac{1}{x} + 2x;\quad \sqrt{x^2 - 3} + x;\quad \frac{x-5}{x^2+2}$

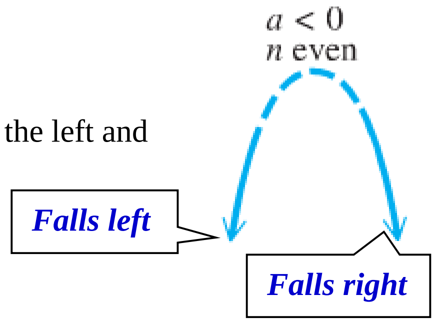
Degree of f	***Form of f(x)***	***Graph of f(x)***
0	$f(x) = a_0$	A horizontal line
1	$f(x) = a_1 x + a_0$	A line with slope a_1
2	$f(x) = a_2 x^2 + a_1 x + a_0$	A parabola with a vertical axis

All polynomial functions are ***continuous functions***.

End Behavior $\left(a_n x^n\right)$

If n (degree) is even:

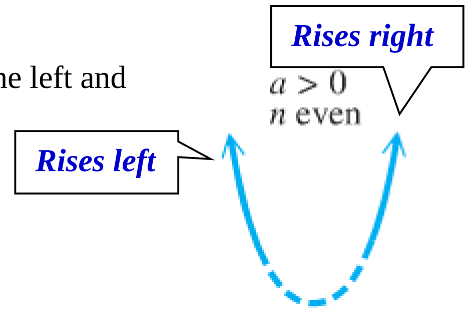
If $a_n < 0$ (in front x^n is negative), then the function falls from the left and right side



$$x \to -\infty \quad \Rightarrow \quad f(x) \to -\infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to -\infty$$

If $a_n > 0$ (in front x^n is positive), then the function rises from the left and right side

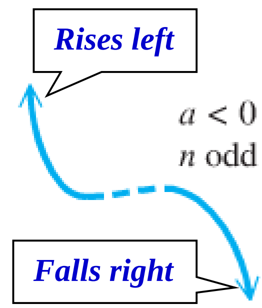


$$x \to -\infty \quad \Rightarrow \quad f(x) \to \infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to \infty$$

If n (degree) is odd:

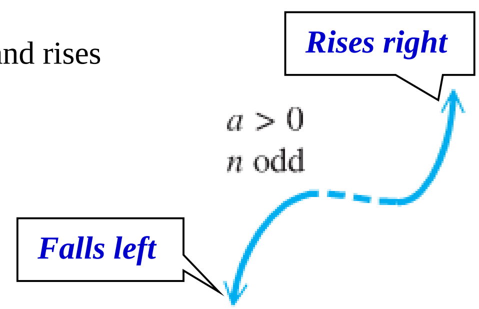
If $a_n < 0$ (negative), then the function rises from the left side and falls from the right side



$$x \to -\infty \quad \Rightarrow \quad f(x) \to \infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to -\infty$$

If $a_n > 0$ (positive), then the function falls from the left side and rises from the right side



$$x \to -\infty \quad \Rightarrow \quad f(x) \to -\infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to \infty$$

Example

Determine the end behavior of the graph of the polynomial function $f(x) = -4x^5 + 7x^2 - x + 9$

Solution

Leading term: $-4x^5$ with 5th degree (n is odd)

$x \to -\infty \quad \Rightarrow \quad f(x) = -(-)^5 = (-)(-) = + \to \infty \qquad f(x)$ rises left

$x \to \infty \quad \Rightarrow \quad f(x) \to -\infty \qquad f(x)$ falls right

The intermediate value *Theorem*

For any polynomial function $f(x)$ with real coefficients and $f(a) \neq f(b)$ for $a < b$, then f takes on every value between $f(a)$ and $f(b)$ in the interval $[a, b]$.

$\therefore$ $f(a)$ and $f(b)$ are the opposite signs. Then the function has a real zero between a and b.

Example

Using the intermediate value theorem, determine, if possible, whether the function has a real zero between *a* and *b*.

a) $f(x) = x^3 + x^2 - 6x; \quad a = -4, \ b = -2$

b) $f(x) = x^3 + x^2 - 6x; \quad a = -1, \ b = 3$

Solution

a) $f(x) = x^3 + x^2 - 6x; \quad a = -4, \ b = -2$

$f(-4) = (-4)^3 + (-4)^2 - 6(-4) = -24$

$f(-2) = (-2)^3 + (-2)^2 - 6(-2) = 8$ *f(x)* has a zero between –4 and –2.

b) $f(x) = x^3 + x^2 - 6x; \quad a = -1, \ b = 3$

$f(-1) = (-1)^3 + (-1)^2 - 6(-1) = 6$

$f(3) = (3)^3 + (3)^2 - 6(3) = 18$ *Can't be determined*.

Example

Show that $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3$ has a zero between 1 and 2.

Solution

$f(1) = 1^5 + 2(1)^4 - 6(1)^3 + 2(1) - 3 = -4$

$f(2) = 2^5 + 2(2)^4 - 6(2)^3 + 2(2) - 3 = 17$

Since $f(1)$ *and* $f(2)$ have opposite signs; therefore, $f(c) = 0$ for at least one real number *c* between 1 and 2.

Properties of Division

Long Division

Divide $\left(x^3+2x^2-5x-6\right) \div (x+1)$

$$
\begin{array}{r l}
 & \overbrace{x^2+x-6}^{\textit{Quotient}} \\
\cline{2-2}
x+1 \big) & x^3+2x^2-5x-6 \;\longleftarrow\; \textit{Dividend} \\
 & \underline{x^3+x^2} \\
 & \quad x^2-5x \\
 & \quad \underline{x^2-x} \\
 & \qquad -6x-6 \\
 & \qquad \underline{-6x-6} \\
 & \qquad\qquad 0 \;\longleftarrow\; \textit{Remainder}
\end{array}
$$

Divisor

$Q(x) = x^2+x-6$

$R(x) = 0$

Remainder *Theorem*

If a number c is substituted for x in the polynomial $f(x)$, then the result $f(c)$ is the remainder that would be obtained by dividing $f(x)$ by $x-c$.

That is, if $f(x) = (x-c)Q(x) + R(x)$ *then* $f(c) = R$

Factor *Theorem*

A polynomial $f(x)$ has a factor $x-c$ if and only if $f(c) = 0$

Synthetic Division

Use synthetic division to find the quotient and the remainder of $\left(4x^3-3x^2+x+7\right) \div (x-2)$

$$
\begin{array}{c|cccc}
 & x^3 & x^2 & x^1 & x^0 \\
 & 4 & -3 & 1 & 7 \\
2 & \downarrow & 8 & 10 & 22 \\
\hline
 & 4 & 5 & 11 & \boxed{29} \\
 & x^2 & x^1 & x^0 &
\end{array}
$$

Quotient: $Q(x) = 4x^2+5x+11$

Remainder: $R(x) = 29$

The Rational Zeros *Theorem*

If the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ *has integer* coefficients and if $\frac{c}{d}$ is a rational zero of $f(x)$ such that $\boldsymbol{c}$ and $\boldsymbol{d}$ have no common prime factor, then

1. The numerator c of the zero is a factor of the constant term a_0
2. The denominator d of the zero is a factor of the leading coefficient a_n

$$\textit{possible rational zeros} = \frac{\textit{factors of the constant term } a_0}{\textit{factors of the leading coefficient } a_n} = \frac{\textit{possibilities for } a_0}{\textit{possibilities for } a_n}$$

Example

Find all rational solutions of the equation: $3x^4 + 14x^3 + 14x^2 - 8x - 8 = 0$

Solution

possibilities for a_0	$\pm 1,\ \pm 2,\ \pm 4,\ \pm 8$
possibilities for a_n	$\pm 1,\ \pm 3$
possibilities for c/d	$\pm 1,\ \pm 2,\ \pm 4,\ \pm 8,\ \pm\frac{1}{3},\ \pm\frac{2}{3},\ \pm\frac{4}{3},\ \pm\frac{8}{3}$

Using the calculator, the result will show that –2 is a zero.

$$\begin{array}{r|rrrrr}
-2 & 3 & 14 & 14 & -8 & -8 \\
 & & -6 & -16 & 4 & 8 \\ \hline
-\frac{2}{3} & 3 & 8 & -2 & -4 & 0 \\
 & & -2 & -4 & 4 & \\ \hline
 & 3 & 6 & -6 & 0 &
\end{array}$$

$\to 3x^3 + 8x^2 - 2x - 4 \to \pm\left\{\frac{4}{3}\right\} = \pm\left\{1,\ 2,\ 4,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3}\right\}$

$\to 3x^2 + 6x - 6 = 0 \Rightarrow \underline{x = -1 \pm \sqrt{3}}$

Hence, the polynomial has roots $\underline{x = -2,\ -\frac{2}{3},\ -1 \pm \sqrt{3}}$

Sketching

Example

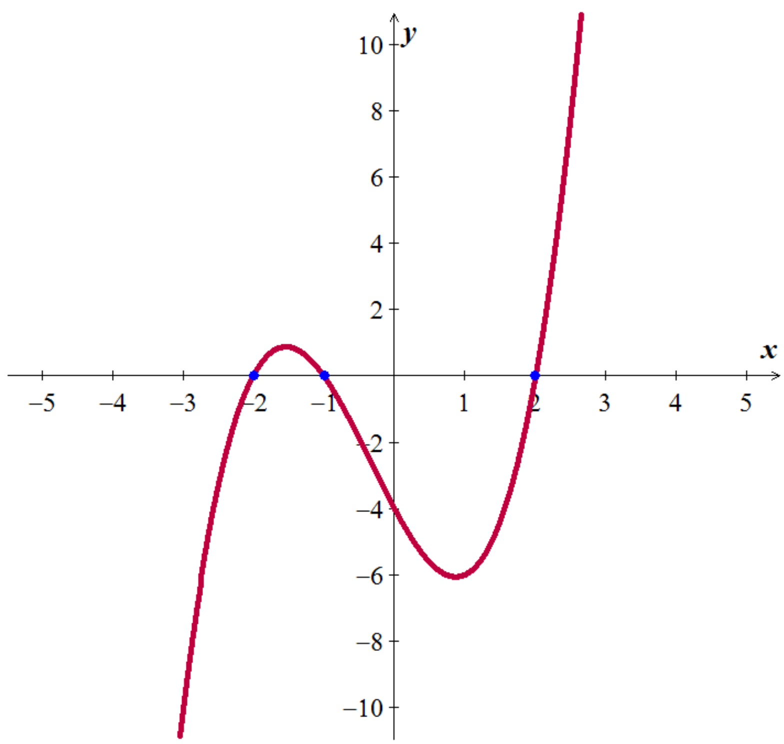
Let $f(x) = x^3 + x^2 - 4x - 4$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$$\begin{aligned} f(x) &= x^3 + x^2 - 4x - 4 \\ &= x^2(x+1) - 4(x+1) \\ &= (x+1)\left(x^2 - 4\right) \\ &= (x+1)(x+2)(x-2) \end{aligned}$$

The zeros of $f(x)$ (x-intercepts) are: $-2,\ -1,\ \textit{and}\ 2$

Interval	$-\infty$ $\quad$ -2	-1	**0** $\quad$ 2	∞
Sign of $f(x)$	**–**	**+**	**–**	**+**
Position	**Below *x*-axis**	**Above *x*-axis**	**Below *x*-axis**	**Above *x*-axis**



We can conclude from the chart and the graph that:

$$f(x) > 0 \quad \textit{if } x \textit{ is in } (-2,\ -1) \cup (2,\ \infty)$$

$$f(x) < 0 \quad \textit{if } x \textit{ is in } (-\infty,\ -2) \cup (-1,\ 2)$$

Example

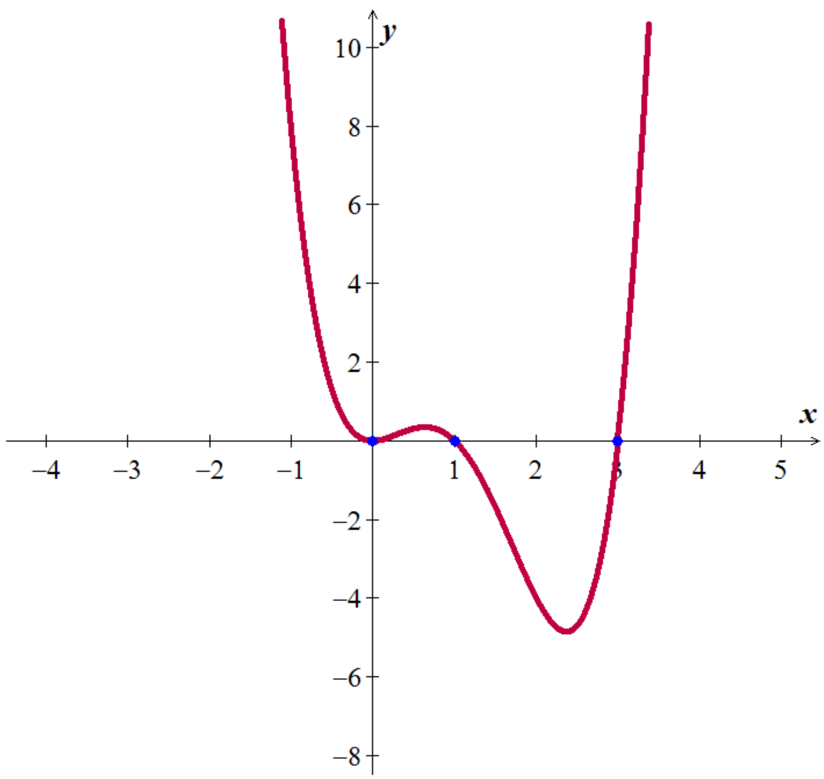
Let $f(x)=x^4-4x^3+3x^2$. Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of f.

Solution

$$f(x)=x^2\left(x^2-4x+3\right)$$
$$=x^2(x-1)(x-3)$$

The zeros are: 0, 1, 3. Since the factor x^2 is always positive, it has no factor

$-\infty$	1	2	3	∞
+		–		+



$$f(x)>0 \quad \text{if } x \text{ is in } (-\infty,\ 0)\cup(0,\ 1)\cup(3,\ \infty)$$

$$f(x)<0 \quad \text{if } x \text{ is in } (1,\ 3)$$

Fundamental Theorem of Algebra

If a polynomial $f(x)$ has positive degree and complex coefficients, then $f(x)$ has at least one complex zero.

Complete Factorization Theorem for Polynomials

If $f(x)$ is a polynomial of degree $n>0$, then there exist n complex numbers $c_1,\ c_2,\ \ldots,\ c_n$ such that:

$$f(x)=a\left(x-c_1\right)\left(x-c_2\right)\ldots\left(x-c_n\right),$$

Where a is the leading coefficient of $f(x)$. Each number c_k is a zero of $f(x)$.

Example

$f(x)$	***Factored From***	***Zeros of*** $f(x)$
$3x^2-(12+6i)x+24i$	$3(x-4)(x-2i)$	$4,\ 2i$
$-6x^3-2x^2-6x-2$	$-6\left(x+\frac{1}{3}\right)(x+i)(x-i)$	$-\frac{1}{3},\ \pm i$

Example

Express $f(x)=x^5-4x^4+13x^3$ as a product of linear factors, and find the five zeros of $f(x)$

Solution

$f(x)=x^3\left(x^2-4x+13\right)$ *factor out* x^3

$$x^2-4x+13=0\to x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(13)}}{2(1)}$$

$$=\frac{4\pm\sqrt{-36}}{2}$$

$$=\frac{4\pm 6i}{2}$$

$$=2\pm 3i$$

$$f(x)=x.x.x(x-2-3i)(x-2+3i)$$

The number 0 is a zero of multiplicity of 3. $\therefore$ $\boxed{0,\ 0,\ 0,\ 2-3i,\ 2+3i}$

Exercises   *Section* 1.2 – Polynomial Functions & Graphs

Find the quotient and remainder if $f(x)$ is divided by $p(x)$

1. $f(x)=2x^4-x^3+7x-12;\quad p(x)=x^2-3$

2. $f(x)=3x^3+2x-4;\quad p(x)=2x^2+1$

3. $f(x)=7x+2;\quad p(x)=2x^2-x-4$

4. $f(x)=9x+4;\quad p(x)=2x-5$

Use the remainder theorem to find $f(c)$

5. $f(x)=x^4-6x^2+4x-8;\quad c=-3$

6. $f(x)=x^4+3x^2-12;\quad c=-2$

7. Use the factor theorem to show that $x-c$ is a factor of $f(x)$: $f(x)=x^3+x^2-2x+12;\quad c=-3$

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second

8. $2x^3-3x^2+4x-5;\quad x-2$

9. $5x^3-6x^2+15;\quad x-4$

10. $9x^3-6x^2+3x-4;\quad x-\frac{1}{3}$

Use the synthetic division to find $f(c)$

11. $f(x)=2x^3+3x^2-4x+4;\quad c=3$

12. $f(x)=8x^5-3x^2+7;\quad c=\frac{1}{2}$

13. $f(x)=x^3-3x^2-8;\quad c=1+\sqrt{2}$

14. Use the synthetic division to show that c is a zero of $f(x)$:

$f(x)=3x^4+8x^3-2x^2-10x+4;\quad c=-2$

15. Use the synthetic division to show that c is a zero of $f(x)$:

$f(x)=27x^4-9x^3+3x^2+6x+1;\quad c=-\frac{1}{3}$

Find all values of k such that $f(x)$ is divisible by the given linear polynomial:

16. $f(x)=kx^3+x^2+k^2x+3k^2+11;\quad x+2$

17. $f(x)=x^3+k^3x^2++2kx-2k^4;\quad x-1.6$

18. $f(x)=k^2x^3-4kx+3;\quad x-1$

Find all solutions of the equation

19. $x^3-x^2-10x-8=0$

20. $x^3+x^2-14x-24=0$

21. $2x^3-3x^2-17x+30=0$

22. $12x^3+8x^2-3x-2=0$

23. $x^4 + 3x^3 - 30x^2 - 6x + 56 = 0$

24. $3x^5 - 10x^4 - 6x^3 + 24x^2 + 11x - 6 = 0$

25. $6x^5 + 19x^4 + x^3 - 6x^2 = 0$

26. $x^4 - x^3 - 9x^2 + 3x + 18 = 0$

27. $2x^4 - 9x^3 + 9x^2 + x - 3 = 0$

28. $8x^3 + 18x^2 + 45x + 27 = 0$

29. $3x^3 - x^2 + 11x - 20 = 0$

30. $6x^4 + 5x^3 - 17x^2 - 6x = 0$

31. If $f(x) = 3x^3 - kx^2 + x - 5k$, find a number k such that the graph of f contains the point (–1, 4).

32. If $f(x) = kx^3 + x^2 - kx + 2$, find a number k such that the graph of f contains the point (2, 12).

33. If one zero of $f(x) = x^3 - 2x^2 - 16x + 16k$ is 2, find two other zeros.

34. If one zero of $f(x) = x^3 - 3x^2 - kx + 12$ is –2, find two other zeros.

35. Find a polynomial $f(x)$ of degree 3 that has the zeros –1, 2, 3; and satisfies the given condition: $f(-2) = 80$

36. Find a polynomial $f(x)$ of degree 3 that has the zeros $-2i$, $2i$, 3; and satisfies the given condition: $f(1) = 20$

37. Find a polynomial $f(x)$ of degree 4 with leading coefficient 1 such that both –4 and 3 are zeros of multiplicity 2, and sketch the graph of f.

Find the zeros of the following functions and state the multiplicity of each zero

38. $f(x) = x^2(3x+2)(2x-5)^3$

39. $f(x) = 4x^5 + 12x^4 + 9x^3$

40. $f(x) = \left(x^2 + x - 12\right)^3 \left(x^2 - 9\right)^2$

41. $f(x) = \left(6x^2 + 7x - 5\right)^4 \left(4x^2 - 1\right)^2$

42. $f(x) = x^4 + 7x^2 - 144$

43. $f(x) = x^4 + 21x^2 - 100$

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f

44. $f(x) = x^4 - 4x^2$

45. $f(x) = x^4 + 3x^3 - 4x^2$

46. $f(x) = x^3 + 2x^2 - 4x - 8$

47. $f(x) = x^3 - 3x^2 - 9x + 27$

48. $f(x) = -x^4 + 12x^2 - 27$

49. $f(x) = x^2(x+2)(x-1)^2(x-2)$

50. $f(x) = 2x^3 + 11x^2 - 7x - 6$

51. $f(x) = x^3 + 2x^2 - 5x - 6$

52. $f(x) = x^3 + 8x^2 + 11x - 20$

53. $f(x) = x^4 + x^2 - 2$

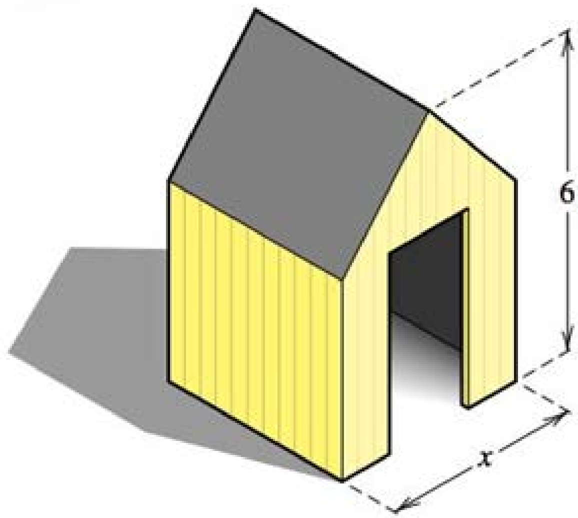
54. $f(x) = x^4 - x^3 - 6x^2 + 4x + 8$

55. $f(x) = 4x^5 - 8x^4 - x + 2$

56. $f(x) = 2x^4 - x^3 - 5x^2 + 2x + 2$

57. $f(x) = x^5 - 7x^4 + 19x^3 - 37x^2 + 60x - 36$

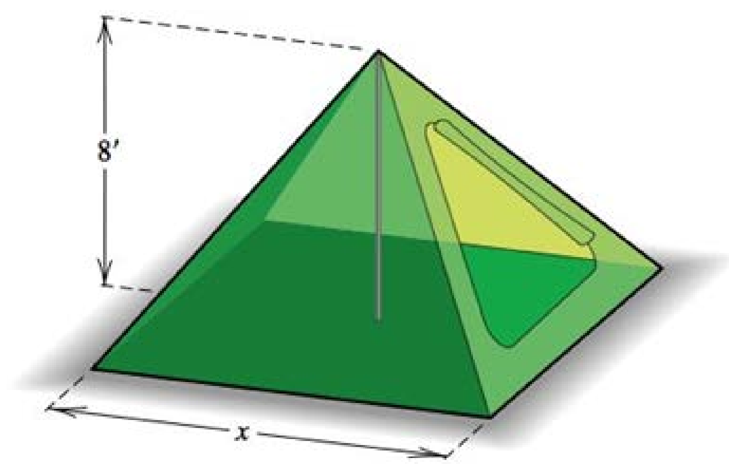
58. A storage shelter is to be constructed in the shape of a cube with a triangular prism forming the roof. The length x of a side of the cube is yet to be determined.



a) If the total height of the structure is 6 *feet*, show that its volume V is given by $V = x^3 + \frac{1}{2}x^2(6-x)$

b) Determine x so that the volume is $80\ ft^3$

59. A canvas camping tent is to be constructed in the shape of a pyramid with a square base. An 8–foot pole will form the center support. Find the length x of a side of the base so that the total amount of canvas needed for the sides and bottom is $384\ ft^2$



Section 1.3 – Rational Functions

A function f is a ***rational function*** if $f(x)=\dfrac{g(x)}{h(x)}$,

Where $g(x)$ and $h(x)$ are polynomials. The domain of f consists of all real numbers ***except*** the zeros of the denominator $h(x)$.

Notation	***Terminology***
$x \to a^-$	x approaches a from the left (through values ***less*** than a)
$x \to a^+$	x approaches a from the right (through values ***greater*** than a)
$f(x) \to \infty$	$f(x)$ increases without bound (can be made as large positive as desired)
$f(x) \to -\infty$	$f(x)$ decreases without bound (can be made as large negative as desired)

The Domain of a Rational Function

Example

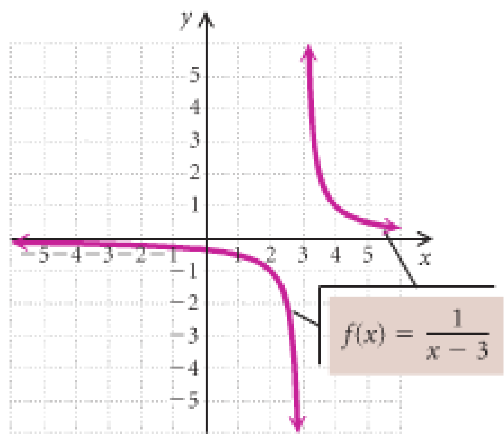
Consider: $\quad f(x)=\dfrac{1}{x-3}$

Find the domain and graph *f*.

Solution

$x-3=0 \quad \Rightarrow \quad \boxed{x=3}$

Thus the domain is: $\{x \mid x \neq 3\}$ *or* $(-\infty,\ 3)\cup(3,\ \infty)$



Function	***Domain***	
$f(x)=\dfrac{1}{x}$	$\{x \mid x \neq 0\}$	$(-\infty,\ 0)\cup(0,\ \infty)$
$f(x)=\dfrac{1}{x^2}$	$\{x \mid x \neq 0\}$	$(-\infty,\ 0)\cup(0,\ \infty)$
$f(x)=\dfrac{x-3}{x^2+x-2}$	$\{x \mid x \neq -2 \text{ and } x \neq 1\}$	$(-\infty,\ -2)\cup(-2,\ 1)\cup(1,\ \infty)$
$f(x)=\dfrac{2x+5}{2x-6}=\dfrac{2x+5}{2(x-3)}$	$\{x \mid x \neq 3\}$	$(-\infty,\ 3)\cup(3,\ \infty)$

Asymptotes

Vertical Asymptote (*VA*) - ***Think Domain***

The line $x = a$ is a ***vertical asymptote*** for the graph of a function f if

$$f(x) \to \infty \quad or \quad f(x) \to -\infty$$

As ***x*** approaches ***a*** from either the left or the right

Example

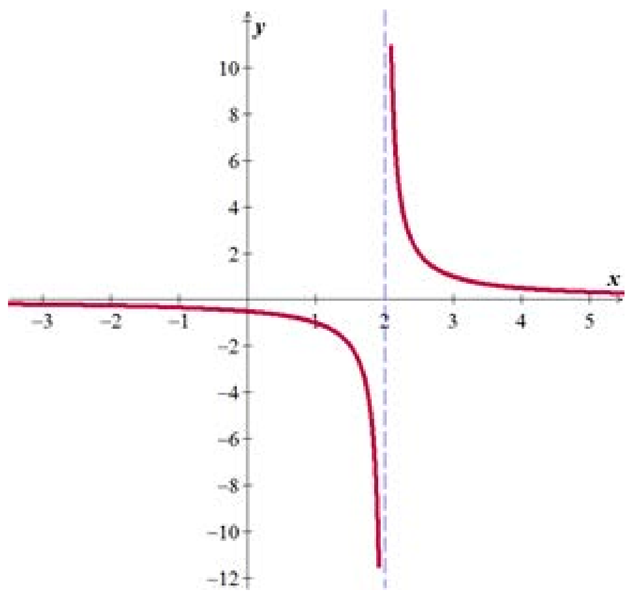
Find the vertical asymptote of $f(x) = \dfrac{1}{x-2}$, and sketch the graph.

Solution

VA: $x = 2$

$$f(x) \to \infty \quad as \quad x \to 2^+$$

$$f(x) \to -\infty \quad as \quad x \to 2^-$$



Hole

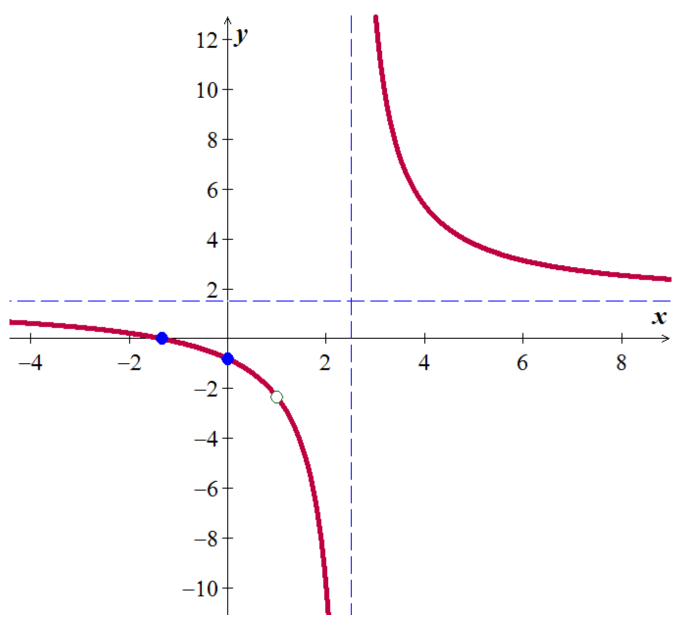
Example

Sketch the graph of g if $g(x) = \dfrac{3x^2 + x - 4}{2x^2 - 7x + 5}$

Solution

$$g(x) = \frac{(3x+4)(x-1)}{(2x-5)(x-1)} = \frac{3x+4}{2x-5} = f(x)$$

g has a hole at $x = 1 \to f(1) = -\dfrac{7}{3}$



Horizontal Asymptote (*HA*)

The line $y = c$ is a ***horizontal asymptote*** for the graph of a function f if

$$f(x) \to c \quad as \quad x \to -\infty \quad or \quad x \to -\infty$$

Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0} = \dfrac{a_n x^n}{b_m x^m}$ be a rational function.

1. If the degree of numerator is less than of denominator $(n < m) \Rightarrow$ y = 0

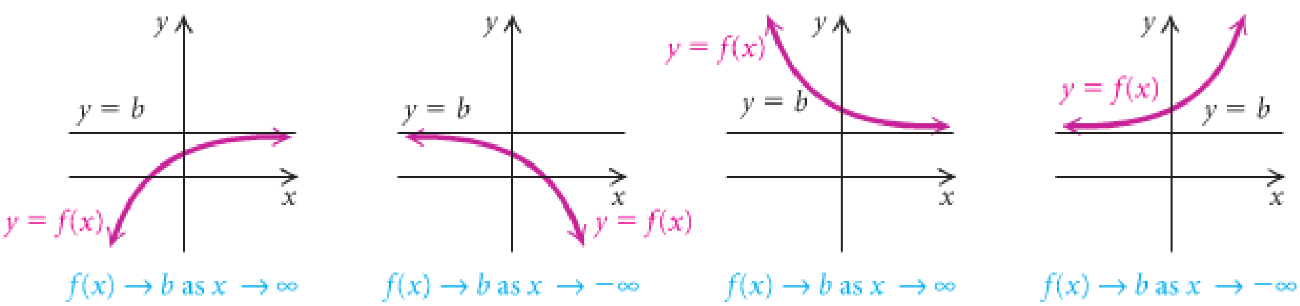
$$y = \frac{2x+1}{4x^2+5} \quad \Rightarrow \boxed{y = 0}$$

2. If the degree of numerator is equal of denominator $(n = m) \Rightarrow y = \dfrac{a_n}{b_m}$

$$y = \frac{2x^2+1}{4x^2+5} \quad \Rightarrow \boxed{y = \frac{2}{4} = \frac{1}{2}}$$

3. If the degree of numerator is greater than of denominator $(n > m) \Rightarrow$ No horizontal asymptote

$$y = \frac{2x^3+1}{4x^2+5} \quad \Rightarrow No \quad HA$$



Slant or *Oblique* Asymptotes

When the degree of the numerator is one greater than the degree of the numerator, the graph has a slant or oblique asymptote and it is a line $y = ax + b, \quad a \neq 0$. To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

$$y = \frac{3x^2 - 1}{x + 2}$$

$$\begin{array}{r|l} & 3x - 6 \\ \hline x+2 & 3x^2 + 0x - 1 \\ & \underline{3x^2 + 6x} \\ & \quad -6x - 1 \\ & \quad \underline{-6x - 12} \\ & \qquad R = 11 \end{array}$$

$$y = \frac{3x^2 - 1}{x + 2} = (3x - 6) + \frac{11}{x + 2}$$

The ***oblique asymptote*** is the line ***y* = 3x - 6**

Example

Find all the asymptotes and sketch the graph of f if $f(x) = \dfrac{x^2 - 9}{2x - 4}$

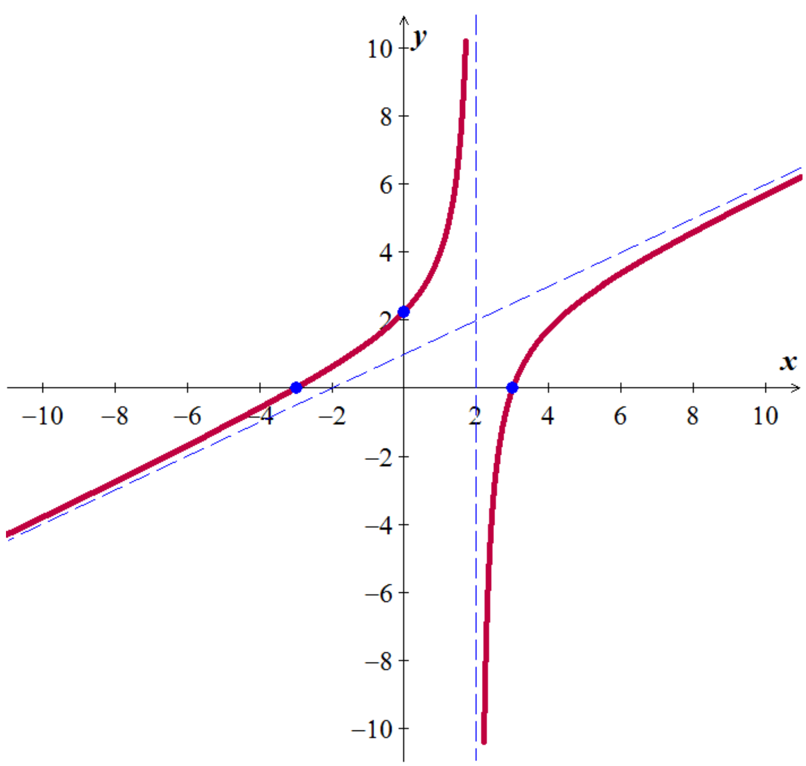
Solution

$$\begin{array}{r|l} & \frac{1}{2}x + 1 \\ \hline 2x-4 & x^2 \qquad - 9 \\ & \underline{x^2 - 2x} \\ & \quad 2x - 9 \\ & \quad \underline{2x - 4} \\ & \qquad \boxed{-5} \end{array}$$

$$f(x) = \frac{x^2 - 9}{2x - 4} = \left(\tfrac{1}{2}x + 1\right) - \frac{5}{2x - 4}$$

VA	$x = 2$
HA	*n/a*
OA	$y = \frac{1}{2}x + 1$

x	*y*
0	2.25
±3	0



Example

Find all asymptotes for the graph of f , if it exists

a) $f(x)=\frac{3x-1}{x^2-x-6}$ *b)* $f(x)=\frac{5x^2+1}{3x^2-4}$ *c)* $f(x)=\frac{2x^4-3x^2+5}{x^2+1}$

Solution

a) $f(x)=\frac{3x-1}{x^2-x-6}$

VA: $x=-2, \quad x=3$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

b) $f(x)=\frac{5x^2+1}{3x^2-4}$

$$3x^2-4=0 \rightarrow 3x^2=4 \rightarrow x^2=\frac{4}{3} \rightarrow \boxed{x=\pm\frac{2}{\sqrt{3}}}$$

VA: $x=\pm\frac{2}{\sqrt{3}}$ ***HA***: $y=\frac{5}{3}$

Hole: n/a ***Oblique asymptote***: n/a

c) $f(x)=\frac{2x^4-3x^2+5}{x^2+1}$

VA: $x=\pm\frac{2}{\sqrt{3}}$ ***HA***: n/a

Hole: n/a ***Oblique asymptote***: $y=2x^2-5$

$$\begin{array}{r|l} & 2x^2-5 \\ \hline x^2+1 & 2x^4-3x^2+5 \\ & \underline{-2x^4-2x^2} \\ & \quad -5x^2+5 \end{array}$$

Example

Sketch the graph of f if $f(x)=\dfrac{3x+4}{2x-5}$

Solution

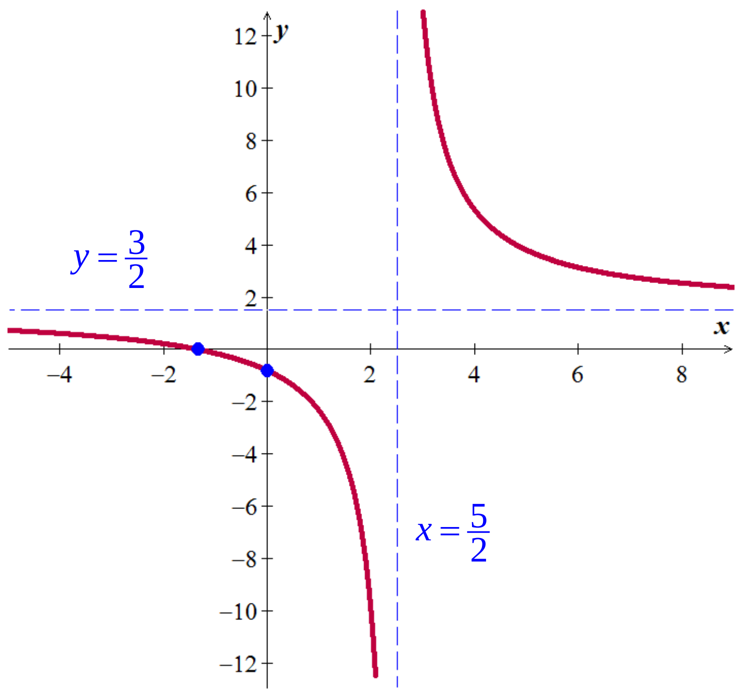
VA: $x=\frac{5}{2}$

HA: $y=-\frac{5}{3}$

Hole: n/a

Oblique asymptote: n/a

x	y
0	$-\frac{4}{5}$
$-\frac{4}{3}$	0
4	5.3



Example

Sketch the graph of f if $f(x)=\dfrac{x^2}{x^2-x-2}$

Solution

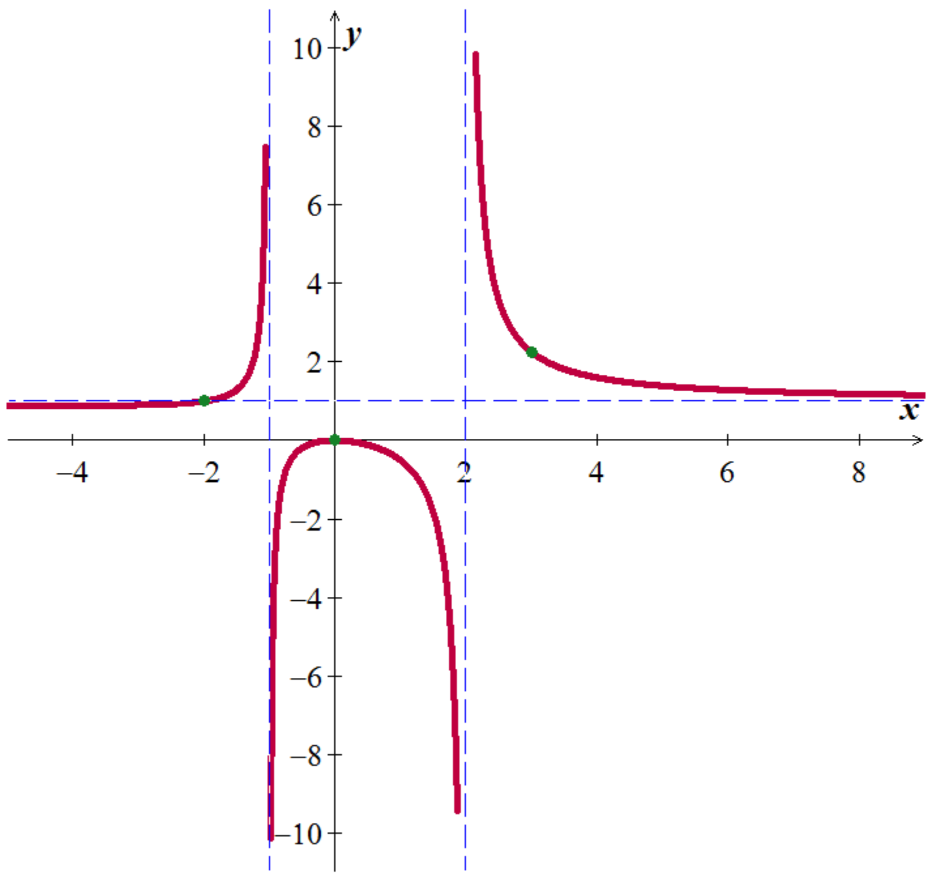
VA: $x=-1,\ 2$

HA: $y=1$

Hole: n/a

Oblique asymptote: n/a

x	***y***
0	0
–4	0.88
–2	1
3	$\frac{9}{4}$



Example

Sketch the graph of f if $f(x)=\dfrac{x-1}{x^2-x-6}$

Solution

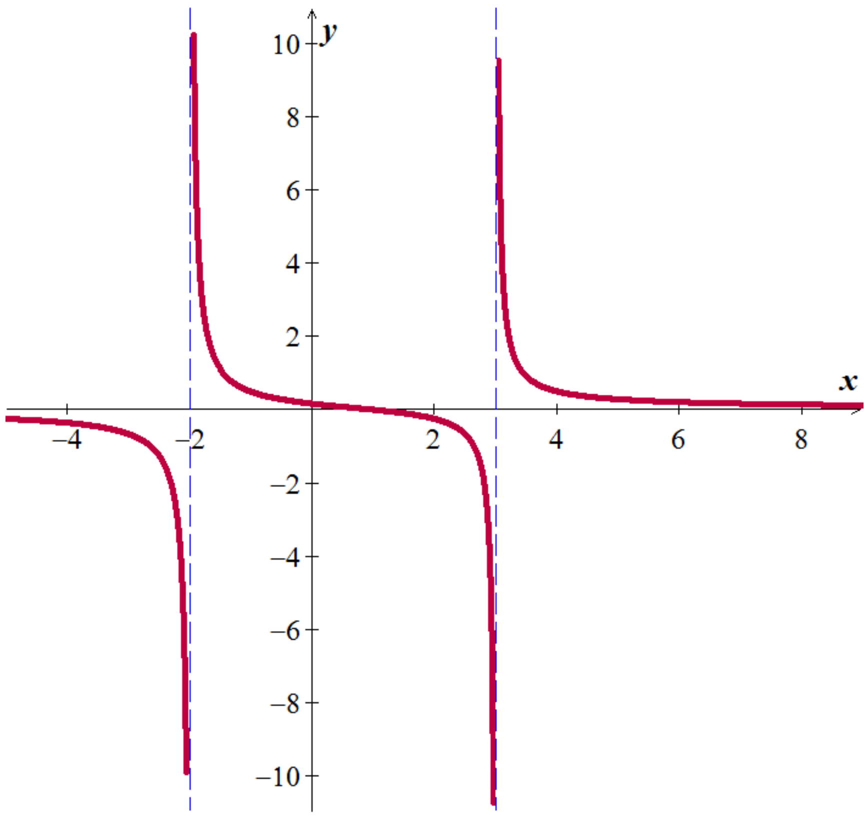
VA: $x=-2,\ 3$

HA: $y=0$

Hole: n/a

Oblique asymptote: n/a

x	y
–4	–.36
–3	–.67
0	$\frac{1}{6}$
1	0
4	.5
5	$\frac{2}{7}$



Exercises Section 1.3 – Rational Functions

Determine all asymptotes of the function

1. $y=\dfrac{3x}{1-x}$
2. $y=\dfrac{x^2}{x^2+9}$
3. $y=\dfrac{x-2}{x^2-4x+3}$
4. $y=\dfrac{3}{x-5}$
5. $y=\dfrac{x^3-1}{x^2+1}$
6. $y=\dfrac{3x^2-27}{(x+3)(2x+1)}$
7. $y=\dfrac{x^3+3x^2-2}{x^2-4}$
8. $y=\dfrac{x-3}{x^2-9}$
9. $y=\dfrac{6}{\sqrt{x^2-4x}}$
10. $y=\dfrac{5x-1}{1-3x}$
11. $f(x)=\dfrac{2x-11}{x^2+2x-8}$
12. $f(x)=\dfrac{x^2-4x}{x^3-x}$
13. $f(x)=\dfrac{x-2}{x^3-5x}$
14. $f(x)=\dfrac{4x}{x^2+10x}$
15. $f(x)=\dfrac{3-x}{(x-4)(x+6)}$
16. $f(x)=\dfrac{x^3}{2x^3-x^2-3x}$
17. $f(x)=\dfrac{3x^2+5}{4x^2-3}$
18. $f(x)=\dfrac{x+6}{x^3+2x^2}$
19. $f(x)=\dfrac{x^2+4x-1}{x+3}$
20. $f(x)=\dfrac{x^2-6x}{x-5}$
21. $f(x)=\dfrac{x^3-x^2+x-4}{x^2+2x-1}$

Determine all asymptotes and sketch the graph of

22. $f(x)=\dfrac{-3x}{x+2}$
23. $f(x)=\dfrac{x+1}{x^2+2x-3}$
24. $f(x)=\dfrac{2x^2-2x-4}{x^2+x-12}$
25. $f(x)=\dfrac{-2x^2+10x-12}{x^2+x}$
26. $f(x)=\dfrac{x^2-x-6}{x+1}$
27. $f(x)=\dfrac{x^3+1}{x-2}$
28. $f(x)=\dfrac{2x^2+x-6}{x^2+3x+2}$
29. $f(x)=\dfrac{x-1}{1-x^2}$
30. $f(x)=\dfrac{x^2+x-2}{x+2}$
31. $f(x)=\dfrac{x^3-2x^2-4x+8}{x-2}$
32. $f(x)=\dfrac{2x^2-3x-1}{x-2}$
33. $f(x)=\dfrac{2x+3}{3x^2+7x-6}$
34. $f(x)=\dfrac{x^2-1}{x^2+x-6}$
35. $f(x)=\dfrac{-2x^2-x+15}{x^2-x-12}$

Find an equation of a rational function f that satisfies the given conditions

36. $\begin{cases} \textit{vertical asymptote}: \ x=4 \\ \textit{horizontal asymptote}: \ y=-1 \\ x-\textit{intercept}: \ 3 \end{cases}$

37. $\begin{cases} \textit{vertical asymptote}: \ x=-3, x=1 \\ \textit{horizontal asymptote}: \ y=0 \\ x-\textit{intercept}: \ -1, \ f(0)=-2 \\ \textit{hole at} \ \ x=2 \end{cases}$

38. $\begin{cases} vertical\ asymptote:\ x=-4, x=5 \\ horizontal\ asymptote:\ y=\dfrac{3}{2} \\ x-intercept:\ -2 \end{cases}$

39. $\begin{cases} vertical\ asymptote:\ x=5 \\ horizontal\ asymptote:\ y=-1 \\ x-intercept:\ 2 \end{cases}$

40. $\begin{cases} vertical\ asymptote:\ x=-2,\ x=0 \\ horizontal\ asymptote:\ y=0 \\ x-intercept:\ 2, \quad f(3)=1 \end{cases}$

41. $\begin{cases} vertical\ asymptote:\ x=-3,\ x=1 \\ horizontal\ asymptote:\ y=0 \\ x-intercept:\ -1, \quad f(0)=-2 \\ hole:\ x=2 \end{cases}$

42. $\begin{cases} vertical\ asymptote:\ x=-1,\ x=3 \\ horizontal\ asymptote:\ y=2 \\ x-intercept:\ -2,\ 1 \\ hole:\ x=0 \end{cases}$

Section 1.4 – Inverse Functions

Inverse Relations

Interchanging the first and second coordinates of each ordered pair in a relation produces the inverse relation.

If a relation is defined by an equation, interchanging the variables produces an equation of the inverse relation

One-to-One Function

A function f is one-to-one (1 – 1) if different inputs have different outputs that is,

$$if \;\; a \neq b, \quad then \quad f(a) \neq f(b)$$

A function f is one-to-one (1 – 1) if different outputs the same, the inputs are the same – that is,

$$if \;\; f(a) = f(b), \quad then \;\; a = b$$

Example

Given the function f described by $f(x) = 2x - 3$, prove that f is one-to-one.

Solution

$f(a) = f(b)$

$2a - 3 = 2b - 3$ ***Add 3 on both sides***

$2a = 2b$ ***Divide by 2***

$a = b$

f is one-to-one

Example

If $g(x) = x^2 - 3$, prove that g is not one-to-one.

Solution

$g(-1) \neq g(1)$

$1^2 - 3 \overset{?}{\neq} (-1)^2 - 3$

$-2 = -2$

g is not one-to-one. In fact, since g is an even function that implies to $g(-a) = g(a)$.

Theorem

A function that is increasing throughout its domain is one-to-one.
A function that is decreasing throughout its domain is one-to-one.

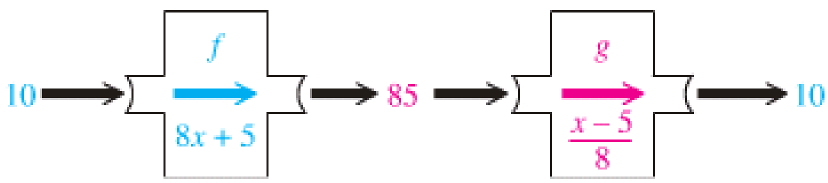
Definition of Inverse Function

Let f be one-to-one function with domain D and range R. A function g with domain R and range D is the ***inverse function*** of f, provided the following condition is true for every x in D and every y in R:

$$y = f(x) \quad \textit{iff} \quad x = g(y)$$

Let f and g be two functions such that: $\quad f(g(x)) = x \quad$ *and* $\quad g(f(x)) = x$

$$x \underset{g=f^{-1}}{\overset{f}{\rightleftarrows}} f(x) \qquad\qquad g(f(x)) = f^{-1}(f(x)) = x$$



If the inverse of a function f is also a function, it is named f^{-1} read "f – inverse"

The -1 in f^{-1} is not an exponent! And is not equal to $\cancel{\frac{1}{f(x)}}$

Definition

If a function f is one-to-one, then f^{-1} is the unique function such that each of the following holds.

$\boxed{\left(f^{-1} \circ f\right)(x) = f^{-1}(f(x)) = x}$ for each x in the domain of f, and

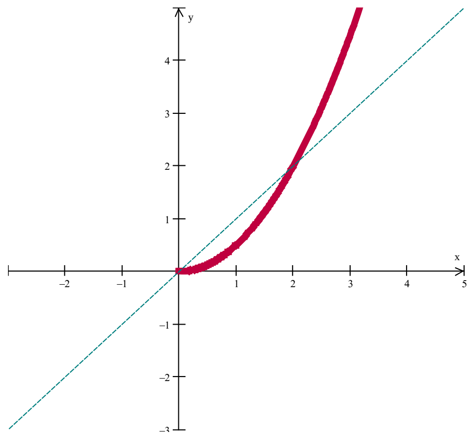
$\boxed{\left(f \circ f^{-1}\right)(x) = f(f^{-1}(x)) = x}$ for each x in the domain of f^{-1}

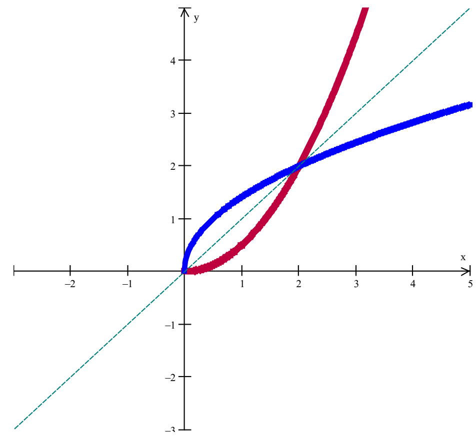
The condition that f is one-to-one in the definition of inverse function is important; otherwise, g will not define a function

Domain and _Range_ of f *and* f^{-1}

$$\text{domain of } f^{-1} = \text{range of } f$$

$$\text{range of } f^{-1} = \text{domain of } f$$





Example

Show that each function is the inverse of the other: $f(x) = 4x - 7$ *and* $g(x) = \dfrac{x+7}{4}$

Solution

$$\begin{aligned} f(g(x)) &= f\left(\frac{x+7}{4}\right) \\ &= 4\left(\frac{x+7}{4}\right) - 7 \\ &= x + 7 - 7 \\ &= x \end{aligned}$$

$$\begin{aligned} g(f(x)) &= g(4x-7) \\ &= \frac{4x - 7 + 7}{4} \\ &= \frac{4x}{4} \\ &= x \end{aligned}$$

Finding the *Inverse Function*

	Example
Finding an Inverse Function	$f\left(x\right)=2x+7$
1. Replace $f(x)$ with y	$y=2x+7$
2. Interchange x and y	$x=2y+7$
3. Solve for y	$x-7=2y$ $\frac{x-7}{2}=y$
4. Replace y with $f^{-1}(x)$	$f^{-1}(x)=\frac{x-7}{2}$

Guidelines for Finding f^{-1} in Simple Cases

1. Verify that f is a one-to-one function throughout its domain.

2. Solve the equation $y=f(x)$ for x in terms of y, obtaining an equation of the form $x=f^{-1}(y)$.

3. Verify the following two conditions:

$f^{-1}(f(x))=x$ for every x in the domain of f, and

$f(f^{-1}(x))=x$ for every x in the domain of f^{-1}

Example

Let $f(x)=x^2-3$ for $x\geq 0$. Find the inverse function of f.

Solution

$y=x^2-3$

$y+3=x^2$

$x^2=y+3$

$x=\pm\sqrt{y+3}$ *Since* $x\geq 0$

$x=\sqrt{y+3}$

$\boxed{f^{-1}\left(x\right)=\sqrt{x+3}}$

Exercises *Section* 1.4 – Inverse Functions

Determine whether the function is one-to-one

1. $f(x)=3x-7$
2. $f(x)=x^2-9$
3. $f(x)=\sqrt{x}$
4. $f(x)=\sqrt[3]{x}$
5. $f(x)=|x|$
6. $f(x)=\frac{2}{x+3}$
7. $f(x)=(x-2)^3$
8. $y=x^2+2$
9. $f(x)=\frac{x+1}{x-3}$

Prove the f *and* g are inverse functions of each other, and sketch the graphs of f *and* g

10. $f(x)=3x-2 \quad g(x)=\frac{x+2}{3}$

11. $f(x)=x^2+5, x\le 0 \quad g(x)=-\sqrt{x-5}, x\ge 5$

12. $f(x)=x^3-4; \quad g(x)=\sqrt[3]{x+4}$

Determine the domain and range of f^{-1} (Hint: first find the domain and range of *f*)

13. $f(x)=-\frac{2}{x-1}$

14. $f(x)=\frac{5}{x+3}$

15. $f(x)=\frac{4x+5}{3x-8}$

For the given functions

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

16. $f(x)=3x+5$

17. $f(x)=\frac{1}{3x-2}$

18. $f(x)=\frac{3x+2}{2x-5}$

19. $f(x)=2-3x^2; \quad x\le 0$

20. $f(x)=\frac{4x}{x-2}$

21. $f(x)=2x^3-5$

22. $f(x)=\sqrt{3-x}$

23. $f(x)=\sqrt[3]{x}+1$

24. $f(x)=\left(x^3+1\right)^5$

25. $f(x)=x^2-6x; \quad x\ge 3$

26. $f(x)=(x-2)^3$

27. $f(x)=\frac{x+1}{x-3}$

28. $f(x)=\frac{2x+1}{x-3}$

29. Let $f(x)=x^3-1$ and $g(x)=\sqrt[3]{x+1}$, is g the inverse function of *f*?

30. Given that $f(x)=5x+8$, use composition of functions to show that $f^{-1}(x)=\frac{x-8}{5}$

31. Given the function $f(x)=(x+8)^3$

a) Find $f^{-1}(x)$

b) Graph f and f^{-1} in the same rectangular coordinate system

c) Find the domain and the range of f and f^{-1}

Section 1.5 – Exponential Functions

Definition

The exponential function f with base $\boldsymbol{b}$ is defined by

$$f(x) = b^x \qquad \text{or} \qquad y = b^x$$

Base

where $b > 0$, $b \neq 1$ and $\boldsymbol{x}$ is any real number.

$$f(x) = 2^x \qquad f(x) = \left(\tfrac{1}{2}\right)^{2x+1} \qquad f(x) = 3^{-x} \qquad \cancel{f(x) = (-2)^x}$$

Example

If $f(x) = 2^x$, find each of the following. $f(-1)$, $f(3)$, $f\left(\frac{5}{2}\right)$

Solution

a) $f(-1) = 2^{-1} = 0.5$

b) $f(3) = 2^3 = 8$

c) $f\left(\frac{5}{2}\right) = 2^{\frac{5}{2}} = 5.6569$

Theorem

Exponential Functions are One-to-One

The exponential function f given by:

$$f(x) = a^x \text{ for } 0 < a < 1 \quad \text{or} \quad a > 1$$

is one to one. Thus the following equivalent conditions are satisfied for ream numbers x_1 *and* x_2

If $x_1 \neq x_2$, *then* $a^{x_1} \neq a^{x_2}$

If $a^{x_1} = a^{x_2}$, *then* $x_1 = x_2$

Graphing Exponential

1. Define the Horizontal Asymptote $f(x) = b^x \pm d$

   $y = 0 \pm d$

   The exponential function always equals to 0

   $x \to \infty$ *or* $x \to -\infty \Rightarrow f(x) \to 0$

2. Define/Make a table

   (Force your exponential to = 0, then solve for *x*)

x	***f(x)***
$x-2$	
$x-1$	
x	
$x+1$	
$x+2$	

Domain: $(-\infty, \infty)$

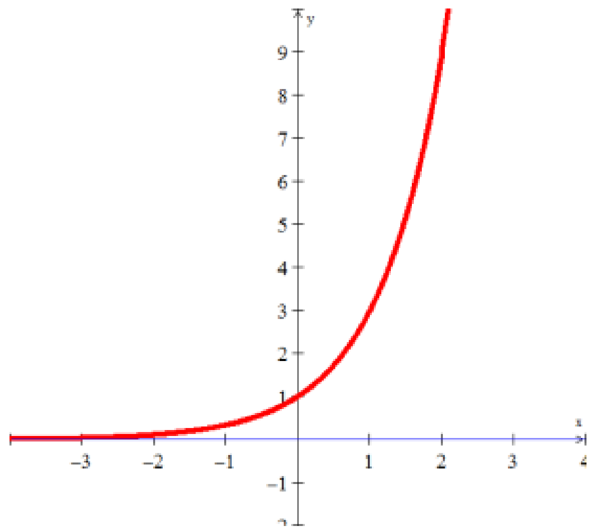
Range: (d, ∞)

Example

$f(x) = 3^x$

Asymptote: $y = 0$

x	*f(x)*
–2	1/9
–1	1/3
0	1
1	3
2	9



Example

Sketch $f(x) = \left(\frac{1}{3}\right)^x$

Solution

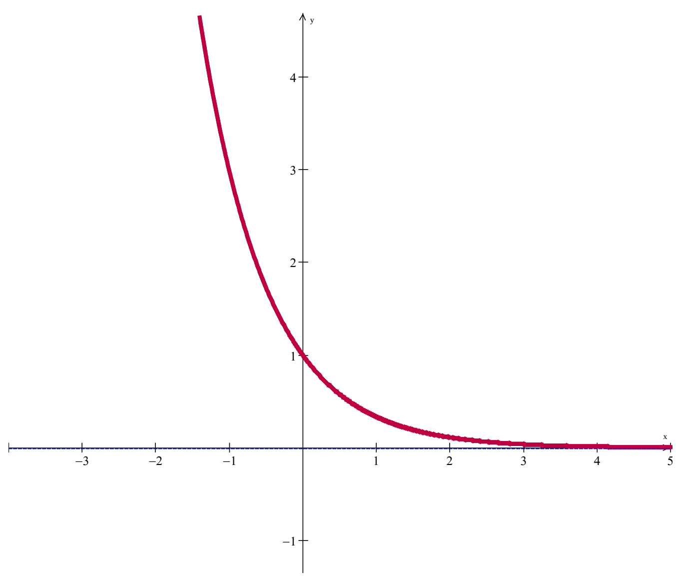
$$f(x) = \left(3^{-1}\right)^x$$
$$= 3^{-x}$$

Reflected across y-axis

Asymptote: $y = 0$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$



Example

Sketch $f(x) = 3^{x-2}$

Solution

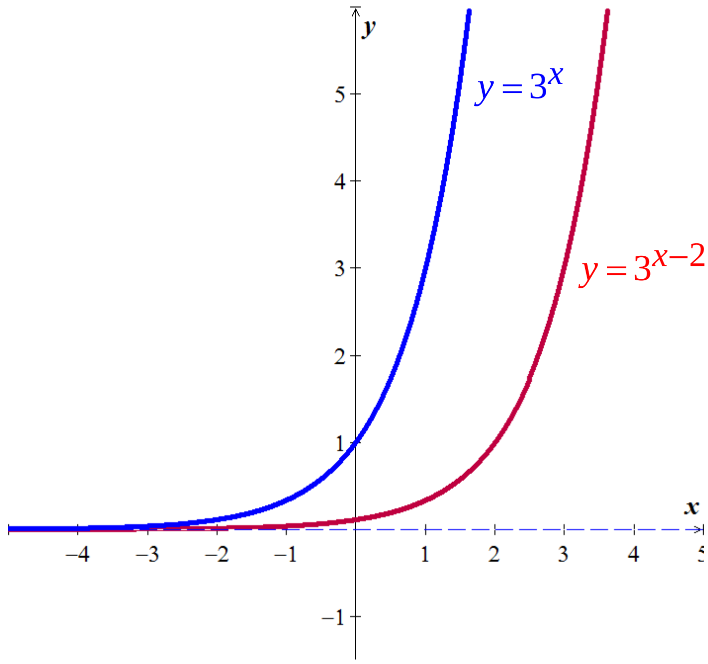
Shift right 2 unit

Asymptote: $y = 0$

x	$f(x)$
1	1/3
2	1
3	3
4	9

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$



Example

Sketch the graph of $f(x) = 2^{-x^2}$

Solution

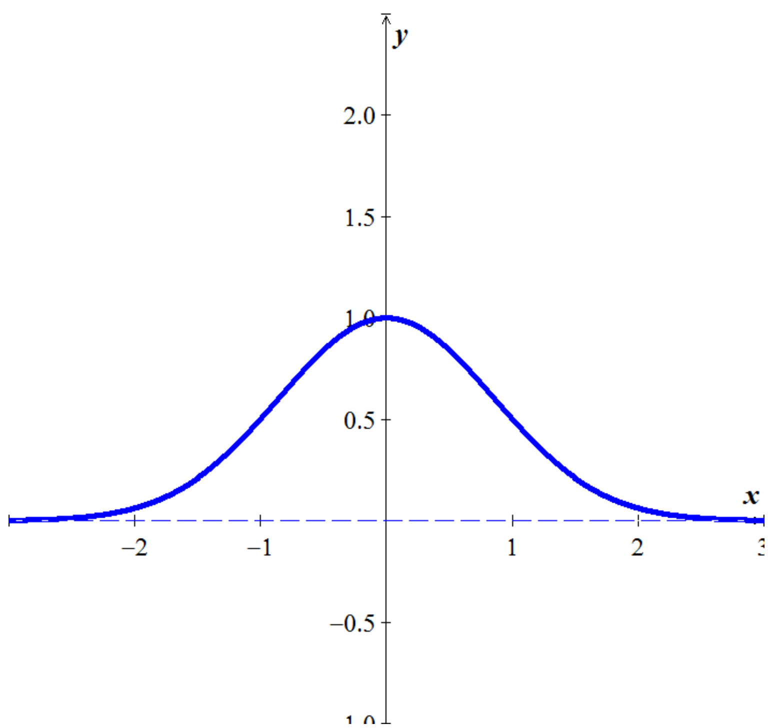
$f(x) = \dfrac{1}{2^{x^2}}$

Asymptote: $y = 0$

x	$f(x)$
± 0	1
± 1	$\frac{1}{2}$
± 2	$\frac{1}{16}$

Function is increasing $(-\infty,\ 0)$

Function is decreasing $(0,\ \infty)$



The Number e

If n is a positive integer, then

$$\left(1+\frac{1}{n}\right)^n \rightarrow e \approx 2.71828 \quad as \quad n \rightarrow \infty$$

Natural Base e

The irrational number e is called natural base

$f(x) = e^x$ is called natural exponential function

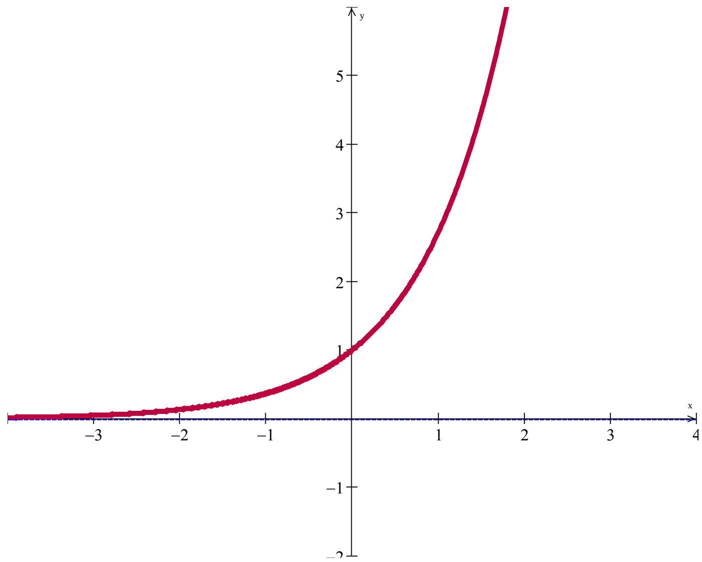
Example

Sketch $f(x) = e^x$

Solution

Asymptote: $y = 0$

x	$f(x)$
–2	.14
–1	.4
0	1
1	2.7
2	7.4



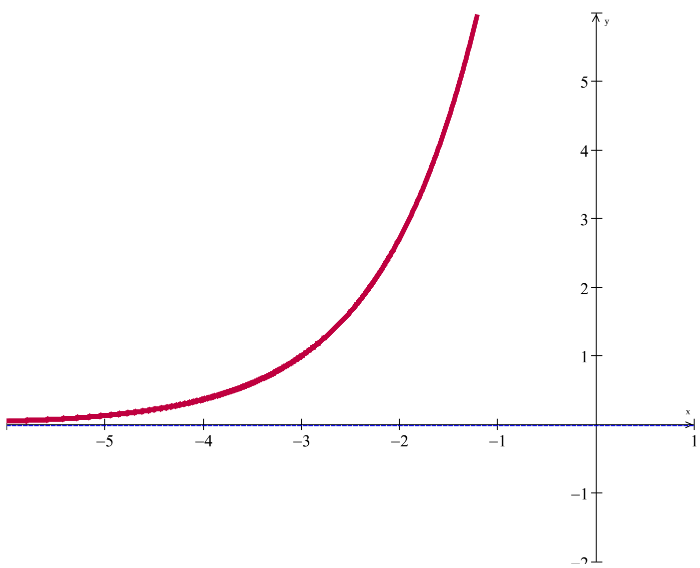
Example

Sketch $f(x) = e^{x+3}$

Solution

Shifted left 3 units

Asymptote: $y = 0$



Exercises Section 1.5 – Exponential Functions

Sketch the graph

1. $f(x)=2^{x}+3$

2. $f(x)=2^{3-x}$

3. $f(x)=\left(\frac{2}{5}\right)^{-x}$

4. $f(x)=e^{x+4}$

5. $f(x)=-\left(\frac{1}{2}\right)^{x}+4$

6. Simplify the expression $\dfrac{\left(e^{x}+e^{-x}\right)\left(e^{x}+e^{-x}\right)-\left(e^{x}-e^{-x}\right)\left(e^{x}-e^{-x}\right)}{\left(e^{x}+e^{-x}\right)^{2}}$

7. Simplify the expression $\dfrac{\left(e^{x}-e^{-x}\right)^{2}-\left(e^{x}+e^{-x}\right)^{2}}{\left(e^{x}+e^{-x}\right)^{2}}$

8. The exponential function $f(x)=1066e^{0.042x}$ models the gray wolf population of the Western Great Lakes, $f(x)$, in billions, x years after 1978. Project the gray population in the recovery area in 2012.

9. The function $f(x)=6.4e^{0.0123x}$ describes world population, $f(x)$, in billions, x years after 2004 subject to a growth rate of 1.23% annually. Use the function to predict world population in 2050.

Section 1.6 – Logarithmic Functions and Properties

Logarithmic Function (*Definition)*

For $x > 0$ and $b > 0$, $b \neq 1$

$$y = \log_b x \quad \text{is equivalent to} \quad x = b^y$$

$$y = \log_b x \iff x = b^y$$

Base

The function $f(x) = \log_b x$ is the logarithmic function with base b.

$\log_b x$: *read* log base b of x $\qquad$ $\log x$ ***means*** $\log_{10} x$

Example

Write each equation in its equivalent exponential form:

a) $3 = \log_7 x \quad \Rightarrow x = 7^3$

b) $2 = \log_b 25 \quad \Rightarrow 25 = b^2$

Write each equation in its equivalent logarithmic form:

a) $2^5 = x \quad \Rightarrow 5 = \log_2 x$

b) $27 = b^3 \quad \Rightarrow 3 = \log_b 27$

Example

The number N of bacteria in a certain culture after t hours is given by $N = (1000)2^t$. Express t as logarithmic function of N with base 2.

Solution

$$\frac{N}{1000} = 2^t \Rightarrow t = \log_2 \frac{N}{1000}$$

Basic Logarithmic Properties

$\log_b b = 1 \quad \rightarrow \quad b = b^1$

$\log_b 1 = 0 \quad \rightarrow \quad 1 = b^0$

Inverse Properties

$\log_b b^x = x \qquad\qquad \log_7 7^8 = 8$

$b^{\log_b x} = x \qquad\qquad 3^{\log_3 17} = 17$

Example

Find the number, if possible

$\log_{10} 100 = \log_{10} 10^2 = 2$

$\log_9 3 \qquad \log_9 3 = \log_9 \sqrt{9} = \log_9 9^{1/2} = \frac{1}{2}$

$\log_2 \frac{1}{32} \qquad \log_2 \frac{1}{32} = \log_2 \frac{1}{2^5} = \log_2 2^{-5} = -5$

Natural Logarithms

Definition

$$f(x) = \log_e x = \ln x$$

The logarithmic function with base ***e*** is called natural logarithmic function.

$\ln x$ *read* "**el en of *x***"

log(-1) = *doesn't exist*	ln(-1) = *doesn't exist*
log0 = *doesn't exist*	ln0 = *doesn't exist*
$\log 0.5 \approx -0.3010$	$\ln 0.5 \approx -0.6931$
log1 = 0	ln1 = 0
$\log 2 \approx 0.3010$	$\ln 2 \approx 0.6931$
log10 = 1	ln*e* = 1

Change-of-Base Logarithmic

$$\log_b M = \frac{\log_a M}{\log_a b}$$

$$\log_b M = \frac{\log M}{\log b} \quad \textbf{or} \quad \log_b M = \frac{\ln M}{\ln b}$$

Evaluate

$$\log_7 2506 = \frac{\log 2506}{\log 7} \approx 4.02$$

log(2506**)** / log(7)

Or

$$\log_7 2506 = \frac{\ln 2506}{\ln 7} \approx 4.02$$

ln(2506**)** / ln(7)

Domain

The domain of a logarithmic function of the form $f(x) = \log_b x$ is the set of all positive real numbers. (*Inside* the log has to be > 0)

Range: $(-\infty, \infty)$

Example

Find the domain of

a) $f(x) = \log_4 (x-5)$

$x - 5 > 0 \quad \Rightarrow \quad x > 5$ ***Domain:*** $(5, \infty)$

b) $f(x) = \ln(4-x)$

$4 - x > 0$

$\Rightarrow x < 4$ ***Domain***: $(-\infty, 4)$

c) $h(x) = \ln(x^2)$

$x^2 > 0 \Rightarrow$ all real numbers except 0.

Domain: $\{x \mid x \neq 0\}$ or $(-\infty, 0) \cup (0, \infty)$

Graphs of *Logarithmic* Functions

Example

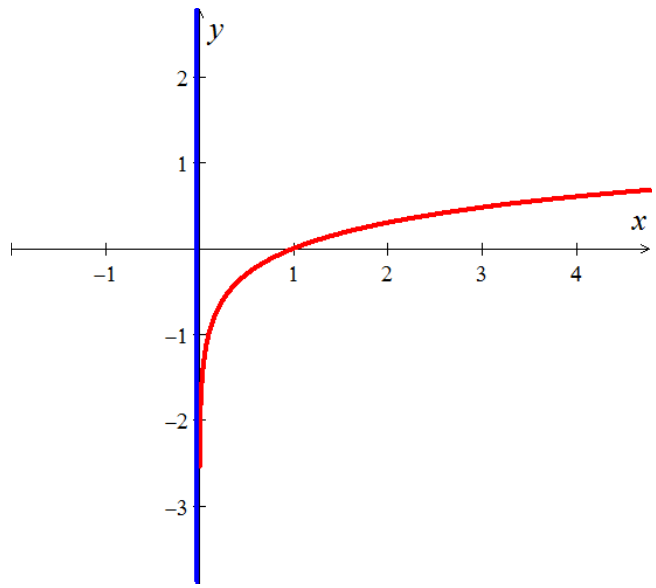
Graph $g(x) = \log x$

Solution

Asymptote: $x = 0$

(*Force inside log to be equal to zero, then solve for* x)

x	$g(x)$
0	
0.5	$-.3$
1	0
2	.3
3	.5



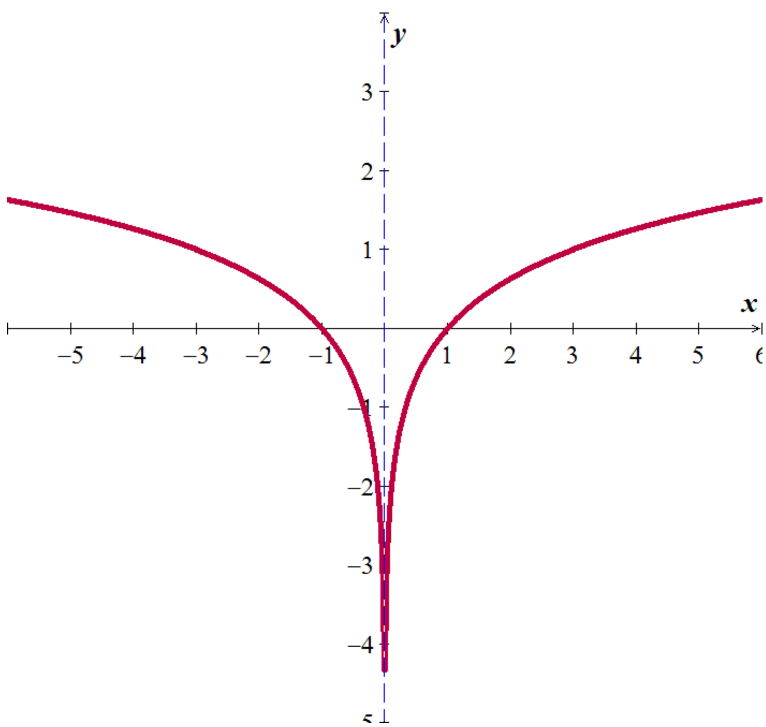
Example

Graph $f(x) = \log_3 |x| \quad for\ x \neq 0$

Solution

$$f(-x) = \log_3 |-x| = \log_3 |x| = f(x)$$

Therefore; the graph is symmetric with respect to the y-axis.



Properties of Logarithms

Product Rule

$$\log_b MN = \log_b M + \log_b N$$

For M > 0 and N > 0

Proof

$$\begin{cases} \log_b M = x & \Rightarrow & M = b^x \\ \log_b N = y & \Rightarrow & N = b^y \end{cases} \Rightarrow \text{MN} = b^x b^y = b^{x+y}$$

Convert back to logarithmic form: $\log_b MN = x + y$

$$\log_b MN = \log_b M + \log_b N$$

Power Rule

$$\log_b M^{p} = p \log_b M$$

Quotient Rule

$$\log_b \frac{M}{N} = \log_b M - \log_b N$$

Example

Express $\log_a \frac{x^3\sqrt{y}}{z^2}$ in terms of logarithms of x, y, and z.

Solution

$$\log_a \frac{x^3\sqrt{y}}{z^2} = \log_a x^3 y^{1/2} - \log_a z^2 \qquad \textit{Quotient Rule}$$

$$= \log_a x^3 + \log_a y^{1/2} - \log_a z^2 \qquad \textit{Product Rule}$$

$$= 3\log_a x + \frac{1}{2}\log_a y - 2\log_a z \qquad \textit{Power Rule}$$

Example

Express as one logarithm: $\frac{1}{3}\log_a\left(x^2-1\right)-\log_a y-4\log_a z$

Solution

$$\frac{1}{3}\log_a\left(x^2-1\right)-\log_a y-4\log_a z=\log_a\left(x^2-1\right)^{1/3}-\log_a y-\log_a z^4 \qquad \textbf{\textit{Power Rule}}$$

$$=\log_a\sqrt[3]{x^2-1}-\left(\log_a y+\log_a z^4\right) \qquad \textbf{\textit{Factor (–)}}$$

$$=\log_a\sqrt[3]{x^2-1}-\left(\log_a yz^4\right) \qquad \textbf{\textit{Product Rule}}$$

$$=\log_a\frac{\sqrt[3]{x^2-1}}{yz^4} \qquad \textbf{\textit{Quotient Rule}}$$

Exercises Section 1.6 – Logarithmic Functions and Properties

Change to logarithm form

1. $4^3 = 64$

2. $4^{-3} = \frac{1}{64}$

3. $3^x = 4 - t$

4. $5^{7t} = \frac{a+b}{a}$

5. $10^x = y + 1$

6. $e^7 = p$

7. $e^{2t} = 3 - x$

Change to exponential form

8. $\log_2 32 = 5$

9. $\log_3 \frac{1}{243} = -5$

10. $\log_3 (x+2) = 5$

11. $\log_2 m = 3x + 4$

12. $\log x = 50$

13. $\ln(z-2) = \frac{1}{6}$

14. $\ln w = 4 + 3x$

Find the number

15. $\log_5 1$

16. $\log_7 7^2$

17. $3^{\log_3 8}$

18. $10^{\log 3}$

19. $e^{2+\ln 3}$

20. $\ln e^{-3}$

21. Find $\log_5 8$ using common logarithms

Evaluate using the change of base formula (without a calculator)

22. $\dfrac{\log_5 16}{\log_5 4}$

23. $\dfrac{\log_7 243}{\log_7 3}$

Sketch the graph of

24. $f(x) = \log_4 (x-2)$

25. $f(x) = \log_4 |x|$

26. $f(x) = \left(\log_4 x\right) - 2$

Find the domain of

27. $\log_5 (x+4)$

28. $\log_5 (x+6)$

29. $\log(2-x)$

30. $\log(7-x)$

31. $\ln(x-2)^2$

32. $\ln(x-7)^2$

33. $\log\left(x^2 - 4x - 12\right)$

34. $\log\left(\frac{x-2}{x+5}\right)$

35. Express $\log_a \dfrac{x^3 w}{y^2 z^4}$ in terms of logarithms of *x*, *y*, *z*, and *w*.

36. Express $\log_a \dfrac{\sqrt{y}}{x^4\sqrt[3]{z}}$ in terms of logarithms of *x*, *y*, and *z*.

37. Express $\ln\sqrt[4]{\dfrac{x^7}{y^5 z}}$ in terms of logarithms of *x*, *y*, and *z*.

38. Express $\ln x\sqrt[3]{\dfrac{y^4}{z^5}}$ in terms of logarithms of *x*, *y*, and *z*.

Express the following in terms of sums and differences of logarithms

39. $\log_b\left(\dfrac{x^3 y}{z^2}\right)$

40. $\log_b\left(\dfrac{\sqrt[3]{x}\,y^4}{z^5}\right)$

41. $\log\left(\dfrac{100x^3\ \sqrt[3]{5-x}}{3(x+7)^2}\right)$

42. $\log_a \sqrt[4]{\dfrac{m^8\ n^{12}}{a^3\ b^5}}$

43. $\log_p \sqrt[3]{\dfrac{m^5 n^4}{t^2}}$

44. $\log_b \sqrt[n]{\dfrac{x^3 y^5}{z^m}}$

45. $\log_a \sqrt[3]{\dfrac{a^2\ b}{c^5}}$

46. $\log_b\left(x^4\sqrt[3]{y}\right)$

47. $\log_5\left(\dfrac{\sqrt{x}}{25y^3}\right)$

Write the expression as a single logarithm

48. $4\ln x + 7\ln y - 3\ln z$

49. $\frac{1}{3}\left[5\ln(x+6) - \ln x - \ln(x^2-25)\right]$

50. $\frac{2}{3}\left[\ln\left(x^2-4\right) - \ln\left(x+2\right)\right] + \ln(x+y)$

51. $\frac{1}{2}\log_b m + \frac{3}{2}\log_b 2n - \log_b m^2 n$

52. $\frac{1}{2}\log_y p^3q^4 - \frac{2}{3}\log_y p^4q^3$

53. $\frac{1}{2}\log_a x + 4\log_a y - 3\log_a x$

54. $\frac{2}{3}\left[\ln\left(x^2-9\right) - \ln\left(x+3\right)\right] + \ln\left(x+y\right)$

55. $\dfrac{1}{4}\log_b x - 2\log_b 5 - 10\log_b y$

56. $2\log_a x + \frac{1}{3}\log_a\left(x-2\right) - 5\log_a\left(2x+3\right)$

57. $5\log_a x - \frac{1}{2}\log_a\left(3x-4\right) - 3\log_a\left(5x+1\right)$

58. $\log\left(x^3y^2\right) - 2\log\left(x\sqrt[3]{y}\right) - 3\log\left(\dfrac{x}{y}\right)$

59. $\ln y^3 + \frac{1}{3}\ln\left(x^3y^6\right) - 5\ln y$

60. $2\ln x - 4\ln\left(\dfrac{1}{y}\right) - 3\ln\left(xy\right)$

61. On a study by psychologists Bornstein and Bornstein, it was found that the average walking speed *w*, in feet per second, of a person living in a city of population *P*, in ***thousands***, is given by the function:

$$w(P) = 0.37\ln P + 0.05$$

a) The population is 124,848. Find the average walking speed of people living in Hartford.
b) The population is 1,236,249. Find the average walking speed of people living in San Antonio.

62. The loudness of sounds is measured in a unit called a decibel. To measure with this unit, we first assign an intensity of I_0 to a very faint sound, called the threshold sound. If a particular sound has intensity *I*, then the decibel rating of this louder sound is

$$d = 10 \log \frac{I}{I_0}$$

Find the exact decibel rating of a sound with intensity $10{,}000 I_0$

63. Students in an accounting class took a final exam and then took equivalent forms of the exam at monthly intervals thereafter. The average score $S(t)$, as a percent, after *t* months was found to be given by the function

$$S(t) = 78 - 15 \log(t + 1), \quad t \geq 0$$

a) What was the average score when the students initially took the test, $t = 0$?

b) What was the average score after 4 months? 24 months?

Section 1.7 – Exponential and Logarithmic Equations

Exponential Functions are One-to-One

$$b^M = b^N \quad \leftrightarrow \quad M = N \quad \text{for any b} > 0, \neq 1$$

Example

Solve $8^{x+2} = 4^{x-3}$

Solution

$$\left(2^3\right)^{x+2} = \left(2^2\right)^{x-3}$$

$$2^{3(x+2)} = 2^{2(x-3)}$$

$$3(x+2) = 2(x-3)$$

$$3x + 6 = 2x - 6$$

$$3x - 2x = -6 - 6$$

$$\boxed{x = -12}$$

Using *Natural Logarithms*

1. Isolate the exponential expression
2. Take the natural logarithm on both sides of the equation
3. Simplify using one of the following properties: $\ln b^x = x \ln b$ *or* $\ln e^x = x$
4. Solve for the variable

Example

Solve the equation $3^x = 21$

Solution

1st *method*	2nd *method*
$3^x = 21$ ***ln both sides*** $\ln 3^x = \ln 21$ $x \ln 3 = \ln 21$ $x = \dfrac{\ln 21}{\ln 3}$	$3^x = 21 \Rightarrow x = \log_3 21$ ***Convert to log form*** $x = \dfrac{\ln 21}{\ln 3}$ ***Change of base***

Example

Solve the equation $5^{2x+1} = 6^{x-2}$

Solution

$$\ln 5^{2x+1} = \ln 6^{x-2}$$

$$(2x+1)\ln 5 = (x-2)\ln 6$$

$$2x\ln 5 + \ln 5 = x\ln 6 - 2\ln 6$$

$$2x\ln 5 - x\ln 6 = -2\ln 6 - \ln 5$$

$$x(2\ln 5 - \ln 6) = -\ln 6^2 - \ln 5$$

$$x\left(\ln 5^2 - \ln 6\right) = -(\ln 36 + \ln 5)$$

$$x\left(\ln \frac{25}{6}\right) = -\ln(36 \times 5)$$

$$\underline{\left|x = -\frac{\ln(180)}{\ln\frac{25}{6}} \approx -3.64\right|}$$

Example

Solve the equation $\dfrac{5^x - 5^{-x}}{2} = 3$

Solution

$$5^x - 5^{-x} = 6 \qquad \textit{Multiply by 2 both sides}$$

$$5^x 5^x - 5^{-x} 5^x = 6 5^x \qquad \textit{Multiply by } 5^x \textit{ both sides}$$

$$\left(5^x\right)^2 - 1 = 6\left(5^x\right)$$

$$\left(5^x\right)^2 - 6\left(5^x\right) - 1 = 0$$

$$5^x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-1)}}{2(1)} = \frac{6 \pm \sqrt{40}}{2} = \frac{6 \pm 2\sqrt{10}}{2} = \begin{cases} 3 + \sqrt{10} \\ 3 - \sqrt{10} < 0 \end{cases}$$

$$5^x = 3 + \sqrt{10}$$

$$\ln 5^x = \ln\left(3 + \sqrt{10}\right)$$

$$x\ln 5 = \ln\left(3 + \sqrt{10}\right)$$

$$\underline{\left|x = \frac{\ln\left(3+\sqrt{10}\right)}{\ln 5} \approx 1.13\right|}$$

Logarithmic Equations

1. Express the equation in the form $\log_b M = c$
2. Use the definition of a logarithm to rewrite the equation in exponential form:

$$\log_b M = c \quad \Rightarrow \quad b^c = M$$

3. Solve for the variable
4. Check proposed solution in the original equation. Include only the set for $M > 0$

Example

Solve: $\log x + \log(x-3) = 1$

Solution

$\log\left[x(x-3)\right] = 1$ ***Product Rule***

$x(x-3) = 10^1$ ***Convert to exponential form***

$x^2 - 3x = 10$

$x^2 - 3x - 10 = 0$ ***Solve for x***

$\Rightarrow x = -2, 5$

Check: $x = -2 \Rightarrow \cancel{\log(-2) + \log(x-3) = 1}$

$x = 5 \Rightarrow \log(5) + \log(5-3) = 1$

Example

Solve the equation $\log_2 x + \log_2 (x+2) = 3$

Solution

$\log_2\left[x(x+2)\right] = 3$ ***Product Rule***

$x(x+2) = 2^3$ ***Change to exponential form***

$x^2 + 2x - 8 = 0$ ***Solve for x***

$x = -4 \quad x = 2$

Check: $\log_2(-4) + \log_2(-4+2) = 3$ Not a solution (negative inside the log)

$\log_2(2) + \log_2(2+2) = 3$ Only solution

Property of Logarithmic Equality

The logarithmic function with base ***b*** is 1-1. Thus the following equivalent conditions are satisfied for positive real numbers M and N.

For any $M > 0$, $N > 0$, $b > 0, \neq 1$

If $\log_b M = \log_b N \Rightarrow M = N$

If $M \neq N \Rightarrow \log_b M \neq \log_b N$

Example

Solve the equation $\log_6 (4x-5) = \log_6 (2x+1)$

Solution

$$\log_6 (4x-5) = \log_6 (2x+1)$$

$$4x - 5 = 2x + 1$$

$$4x - 2x = 5 + 1$$

$$2x = 6$$

$$x = 3$$

Check: $\log_6 (4(3)-5) = \log_6 (2(3)+1)$

$\log_6 (7) = \log_6 (7)$ ***True statement***

$\boxed{x = 3}$ is a solution

Example

Solve the equation $\ln(x+6) - \ln 10 = \ln(x-1) - \ln 2$

Solution

$$\ln(x+6) - \ln 10 = \ln(x-1) - \ln 2$$

$$\ln(x+6) - \ln(x-1) = \ln 10 - \ln 2$$

$$\ln\left(\frac{x+6}{x-1}\right) = \ln\frac{10}{2}$$

$$\frac{x+6}{x-1} = 5$$

$$x + 6 = 5(x-1)$$

$$x + 6 = 5x - 5$$

$x - 5x = -5 - 6$

$-4x = -11$

$x = \frac{-11}{-4} = \frac{11}{4}$

Check: $\ln\left(\frac{11}{4} + 6\right) - \ln 10 = \ln\left(\frac{11}{4} - 1\right) - \ln 2$

$\ln\left(\frac{35}{4}\right) - \ln 10 = \ln\left(\frac{7}{4}\right) - \ln 2$

$\boxed{x = \frac{11}{4}}$ is the solution

Example

Solve the equation $\log \sqrt[3]{x} = \sqrt{\log x}$ for x.

Solution

$\log x^{1/3} = \sqrt{\log x}$

$\left(\frac{1}{3}\log x\right)^2 = \left(\sqrt{\log x}\right)^2$

$\frac{1}{9}(\log x)^2 = \log x$

$(\log x)^2 = 9\log x$

$(\log x)^2 - 9\log x = 0$

$\log x(\log x - 9) = 0$

$\log x = 0 \qquad \log x - 9 = 0$

$\boxed{x = 1} \qquad \log x = 9$

$\boxed{x = 10^9}$

Check: $x = 1 \quad \Rightarrow \quad \log \sqrt[3]{1} = \sqrt{\log 1} \to 0 = 0$

$x = 10^9 \quad \Rightarrow \quad \log \sqrt[3]{10^9} = \sqrt{\log 10^9} \to 3 = 3$

The equation has two solutions: $\boxed{x = 1,\ 10^9}$

Example *(hyperbolic secant function)*

Solve the equation $y = \dfrac{2}{e^{x} + e^{-x}}$ for x in terms of y.

Solution

$$y = \frac{2}{e^{x} + e^{-x}}$$

$$y\left(e^{x} + e^{-x}\right) = 2$$

$$ye^{x} + ye^{-x} = 2$$

$$ye^{x}e^{x} + ye^{-x}e^{x} = 2e^{x}$$

$$y\left(e^{x}\right)^{2} - 2e^{x} + y = 0$$

$$\begin{aligned} e^{x} &= \frac{2 \pm \sqrt{4 - 4y^{2}}}{2y} \\ &= \frac{2 \pm \sqrt{4\left(1 - y^{2}\right)}}{2y} \\ &= \frac{2 \pm 2\sqrt{1 - y^{2}}}{2y} \\ &= \frac{1 \pm \sqrt{1 - y^{2}}}{y} \end{aligned}$$

$$\ln e^{x} = \ln\left(\frac{1 \pm \sqrt{1 - y^{2}}}{y}\right)$$

$$\boxed{x = \ln \frac{1 \pm \sqrt{1 - y^{2}}}{y}}$$

Exercises *Section* 1.7 – Exponential and Logarithmic Equations

Solve

1. $3^{5x-8} = 9^{x+2}$
2. $7^{x+6} = 7^{3x-4}$
3. $2^{-100x} = (0.5)^{x-4}$
4. $4^x\left(\frac{1}{2}\right)^{3-2x} = 8.\left(2^x\right)^2$
5. $5^{3x-6} = 125$
6. $e^{x^2} = e^{7x-12}$
7. $f(x) = xe^x + e^x$
8. $f(x) = x^3\left(4e^{4x}\right) + 3x^2e^{4x}$
9. $3^{x+4} = 2^{1-3x}$
10. $3^{2-3x} = 4^{2x+1}$
11. $7^{2x+1} = 3^{x+2}$
12. $4^{x+3} = 3^{-x}$
13. $2^{-x^2} = 5$
14. $2^{-x} = 8$
15. $\log_4 x = \log_4 (8-x)$
16. $\log_7 (x-5) = \log_7 (6x)$
17. $\ln x^2 = \ln(12-x)$
18. $e^{x\ln 3} = 27$
19. $e^{2x} + 2e^x - 15 = 0$
20. $\log_3 x - \log_9 (x+42) = 0$
21. $\ln\sqrt[4]{x} = \sqrt{\ln x}$
22. $\sqrt{\ln x} = \ln\sqrt{x}$
23. $\log\left(x^2+4\right) - \log(x+2) = 2 + \log(x-2)$
24. $5^x + 125\left(5^{-x}\right) = 30$
25. $4^x - 3\left(4^{-x}\right) = 8$
26. $\log x^2 = (\log x)^2$
27. $\log(\log x) = 2$
28. $\log\sqrt{x^3-9} = 2$
29. $\ln(-4-x) + \ln 3 = \ln(2-x)$
30. $\log_6 (2x-3) = \log_6 12 - \log_6 3$
31. $\log_2 (x+7) + \log_2 x = 3$
32. $\log_3 (x+3) + \log_3 (x+5) = 1$
33. $\ln x = 1 - \ln(x+2)$
34. $\ln x = 1 + \ln(x+1)$
35. $\log_3 (x-2) = \log_3 27 - \log_3 (x-4) - 5^{\log_5 1}$
36. $\log_2 (x+3) = \log_2 (x-3) + \log_3 9 + 4^{\log_4 3}$
37. $\log_5 (x-7) = 2$
38. $\log_5 x + \log_5 (4x-1) = 1$
39. $\log x + \log(x-3) = 1$
40. $\log x - \log(x+3) = 1$
41. $\log_3 x = -2$
42. $\log(3x+2) + \log(x-1) = 1$
43. $\log_5 (x+2) + \log_5 (x-2) = 1$
44. $\log x + \log(x-9) = 1$

45. $\log_2(x+1)+\log_2(x-1)=3$

46. $\log_8(x+1)-\log_8 x=2$

47. $\log(x+6)-\log(x+2)=\log x$

48. $\ln(x+8)+\ln(x-1)=2\ln x$

49. $\ln(4x+6)-\ln(x+5)=\ln x$

50. $\ln(5+4x)-\ln(x+3)=\ln 3$

51. $\ln(x-5)-\ln(x+4)=\ln(x-1)-\ln(x+2)$

52. $\ln(x-3)=\ln(7x-23)-\ln(x+1)$

53. $\log_4(5+x)=3$

54. $\log_5(2x+3)=\log_5 11+\log_5 3$

Use common logarithms to solve for ***x*** in terms of ***y***

55. $y=\dfrac{10^x+10^{-x}}{2}$

56. $y=\dfrac{10^x-10^{-x}}{10^x+10^{-x}}$

57. $y=\dfrac{e^x-e^{-x}}{2}$

58. $y=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$

59. Solve for t using logarithms with base a: $2a^{t/3}=5$

60. Solve for t using logarithms with base a: $K=H-Ca^t$