

## **Solution**      Section 3.1 – Maxima and Minima

### **Exercise**

Find the absolute maximum and minimum values of the function. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

$$f(x) = \frac{2}{3}x - 5 \quad -2 \leq x \leq 3$$

### **Solution**

$$f'(x) = \frac{2}{3}$$

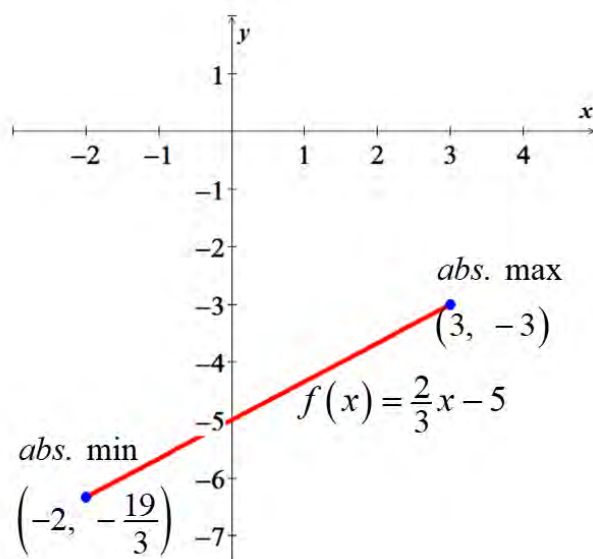
$\therefore$  No **Critical Points (CP)** or **(CN)**.

$$f(-2) = \frac{2}{3}(-2) - 5 = -\frac{19}{3}$$

$$f(3) = \frac{2}{3}(3) - 5 = -3$$

**Absolute Maximum:**  $(3, -3)$

**Absolute Minimum:**  $(-2, -\frac{19}{3})$



### **Exercise**

Find the absolute maximum and minimum values of the function. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

$$f(x) = x^2 - 1 \quad -1 \leq x \leq 2$$

### **Solution**

$$f'(x) = 2x = 0 \Rightarrow \boxed{x=0} \quad (\text{CN})$$

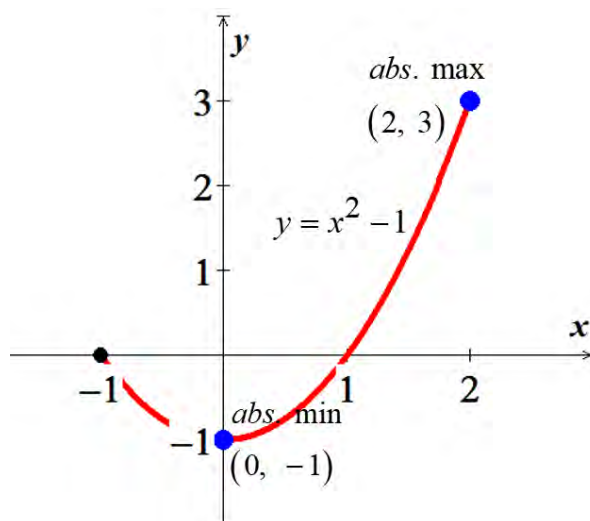
$$f(-1) = (-1)^2 - 1 = 0$$

$$f(0) = (0)^2 - 1 = -1$$

$$f(2) = (2)^2 - 1 = 3$$

**Abs. Maximum:**  $(2, 3)$

**Abs. Minimum:**  $(0, -1)$



## Exercise

Find the absolute maximum and minimum values of the function. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

$$f(x) = -\frac{1}{x^2} \quad 0.5 \leq x \leq 2$$

### Solution

$$f'(x) = \frac{1}{2x^3} \quad \boxed{x=0} \quad \text{Which it is not in the domain}$$

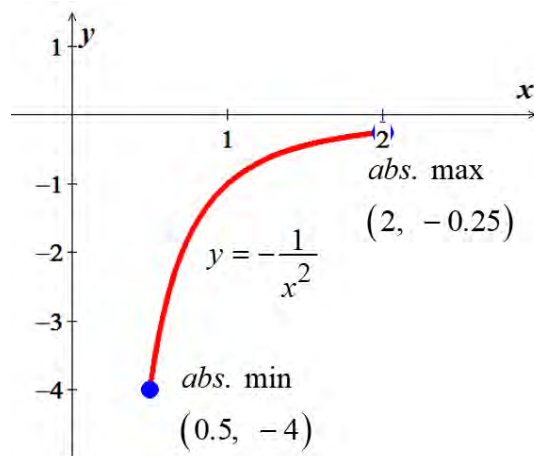
No critical point.

$$f(0.5) = -\frac{1}{(0.5)^2} = -4$$

$$f(2) = -\frac{1}{(2)^2} = -0.25$$

$$\text{Abs. Max: } \underline{(2, -0.25)}$$

$$\text{Abs. Min: } \underline{(0.5, -4)}$$



## Exercise

Find the absolute maximum and minimum values of the function. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

$$f(x) = \sqrt{4-x^2} \quad -2 \leq x \leq 1$$

### Solution

$$f(x) = (4-x^2)^{1/2}$$

$$f'(x) = \frac{1}{2}(4-x^2)^{-1/2}(-2x)$$

$$= \frac{-x}{\sqrt{4-x^2}} = 0 \rightarrow \begin{cases} \boxed{x=0} \\ 4-x^2=0 \Rightarrow \boxed{x=\pm 2} \end{cases}$$

Critical points:  $x = 0, -2$

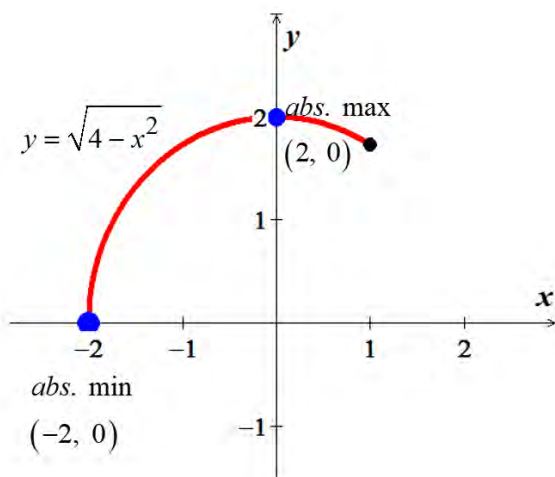
$$f(-2) = \sqrt{4-(-2)^2} = 0$$

$$f(0) = \sqrt{4-(0)^2} = 2$$

$$f(1) = \sqrt{4-(1)^2} = \sqrt{3}$$

$$\text{Abs. Max: } \underline{(0, 2)}$$

$$\text{Abs. Min: } \underline{(-2, 0)}$$



### Exercise

Find the absolute maximum and minimum values of the function. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

$$f(\theta) = \sin \theta \quad -\frac{\pi}{2} \leq \theta \leq \frac{5\pi}{6}$$

### Solution

$$f'(\theta) = \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2} \text{ (CN)}$$

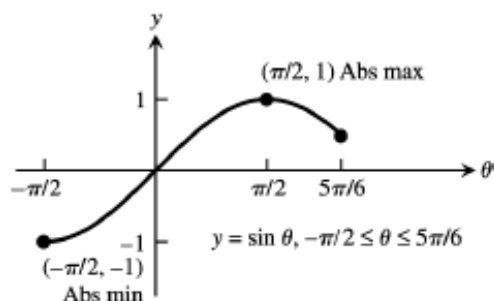
$$f\left(-\frac{\pi}{2}\right) = \sin\left(-\frac{\pi}{2}\right) = -1$$

$$f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

$$f\left(\frac{5\pi}{6}\right) = \sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$$

$$\text{Abs. Min: } \left(-\frac{\pi}{2}, -1\right)$$

$$\text{Abs. Max: } \left(\frac{\pi}{2}, 1\right)$$



### Exercise

Find the absolute maximum and minimum values of the function. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

$$g(x) = \sec x \quad -\frac{\pi}{3} \leq x \leq \frac{\pi}{6}$$

### Solution

$$g'(x) = \sec x \tan x = 0 \Rightarrow \boxed{x = 0} \text{ (CN)}$$

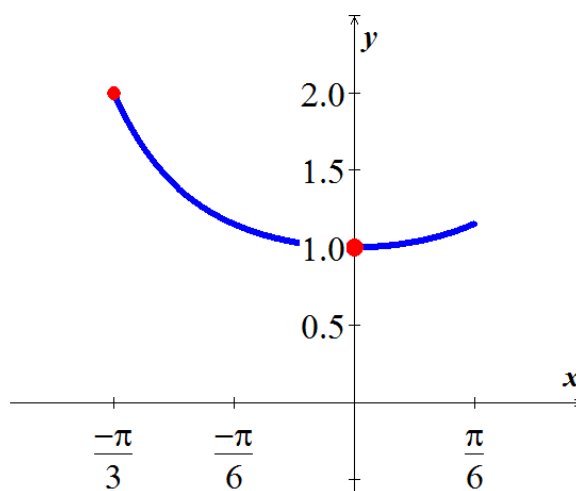
$$g\left(-\frac{\pi}{3}\right) = \sec\left(-\frac{\pi}{3}\right) = 2$$

$$g(0) = \sec(0) = 1$$

$$g\left(\frac{\pi}{6}\right) = \sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}$$

$$\text{Abs. Max: } \left(-\frac{\pi}{3}, 2\right)$$

$$\text{Abs. Min: } (0, 1)$$



### Exercise

Find the absolute maximum and minimum values of  $f(x) = x^{4/3}$ ,  $-1 \leq x \leq 8$

#### Solution

$$f'(x) = \frac{4}{3}x^{1/3} = 0 \Rightarrow \boxed{x=0} \quad (CN)$$

$$f(-1) = 1$$

$$f(0) = 0$$

$$f(8) = 16$$

$$\text{Abs. Max: } \boxed{(8, 16)}$$

$$\text{Abs. Min: } \boxed{(0, 0)}$$

### Exercise

Find the absolute maximum and minimum values of  $f(\theta) = \theta^{3/5}$ ,  $-32 \leq \theta \leq 1$

#### Solution

$$f'(\theta) = \frac{3}{5}\theta^{-2/5} = 0 \Rightarrow \boxed{\theta=0} \quad (CN)$$

$$f(-32) = -8$$

$$f(0) = 0$$

$$f(1) = 1$$

$$\text{Abs. Max: } \boxed{(1, 1)}$$

$$\text{Abs. Min: } \boxed{(-32, -8)}$$

### Exercise

Find the absolute maximum and minimum values of  $f(x) = 2^x \sin x$   $[-2, 6]$

#### Solution

$$\begin{aligned} f'(x) &= 2^x (\ln 2) \sin x + 2^x \cos x \\ &= 2^x (\ln 2 \sin x + \cos x) = 0 \end{aligned}$$

$$\ln 2 \sin x + \cos x = 0; \quad 2^x \neq 0$$

$$\cos x = -\ln 2 \sin x$$

$$\frac{\sin x}{\cos x} = -\frac{1}{\ln 2} = \tan x$$

$$x = \tan^{-1}(-\ln 2) \approx \underline{-0.96468}$$

$$x = -0.96468 + \pi \approx \underline{2.1769}$$

$$x = -0.96468 + 2\pi \approx \underline{5.3185}$$

(since it is *periodic*)

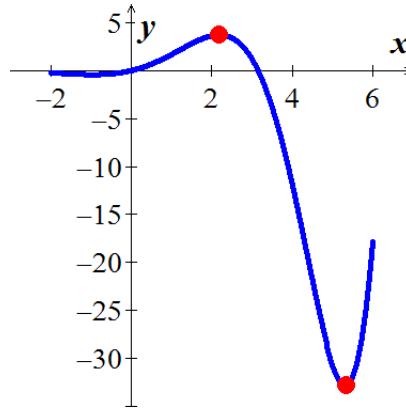
$$f(-2) = -0.227$$

$$f(-.96468) \approx -0.4211$$

$$f(2.1769) \approx 3.7164$$

$$f(5.3185) \approx -32.7968$$

$$f(6) = -17.88$$



$$\text{Abs. Max: } \underline{(2.1769, 3.7164)} \mid$$

$$\text{Abs. Min: } \underline{(5.3185, -32.7968)} \mid$$

### Exercise

Find the absolute maximum and minimum values of  $f(x) = \sec x$   $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

#### Solution

$$f'(x) = \sec x \tan x = 0 \Rightarrow \boxed{x=0} \quad (\text{CN})$$

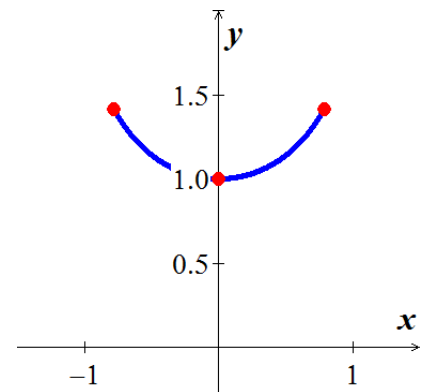
$$f\left(-\frac{\pi}{4}\right) = \sqrt{2}$$

$$f(0) = 1$$

$$f\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$$\text{Abs. Max: } \underline{\left(-\frac{\pi}{4}, \sqrt{2}\right) \& \left(\frac{\pi}{4}, \sqrt{2}\right)} \mid$$

$$\text{Abs. Min: } \underline{(0, 1)} \mid$$



### Exercise

Find the absolute maximum and minimum values of  $f(x) = x^3 e^{-x}$   $[-1, 5]$

### Solution

$$\begin{aligned} f'(x) &= 3x^2 e^{-x} - x^3 e^{-x} \\ &= x^2 e^{-x} (3 - x) = 0 \end{aligned}$$

$$\rightarrow \underline{x = 0, 3} \quad (CN)$$

$$f(-1) = -e \approx -2.718$$

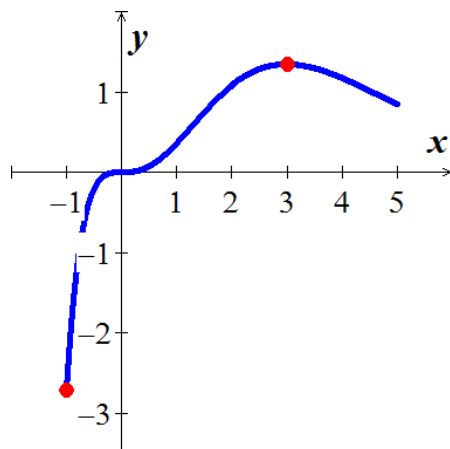
$$f(0) = 0$$

$$f(3) = 27e^{-3} \approx 1.344$$

$$f(5) = 125e^{-5} \approx 0.842$$

$$\text{Abs. Max: } \underline{(3, 27e^{-3})}$$

$$\text{Abs. Min: } \underline{(-1, -e)}$$



### Exercise

Find the absolute maximum and minimum values of  $f(x) = x \ln\left(\frac{x}{5}\right)$   $[0.1, 5]$

### Solution

$$\begin{aligned} f'(x) &= \ln\left(\frac{x}{5}\right) + x\left(\frac{1}{5} \div \frac{x}{5}\right) \\ &= \ln\left(\frac{x}{5}\right) + 1 = 0 \end{aligned}$$

$$\ln\left(\frac{x}{5}\right) = -1 \rightarrow \underline{x = 5e^{-1}} \quad (CN)$$

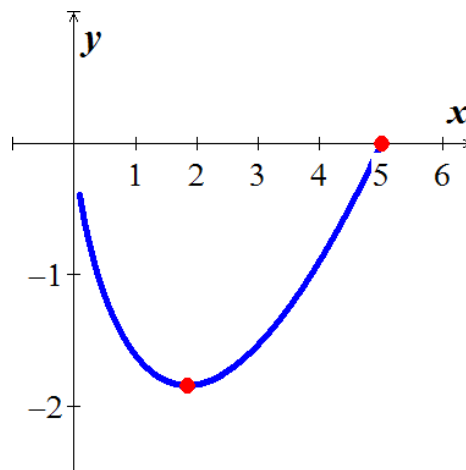
$$f(0.1) = \frac{1}{10} \ln \frac{1}{50} = -\frac{1}{2} \ln 50$$

$$f\left(\frac{5}{e}\right) = \frac{5}{e} \ln \frac{1}{e} = -\frac{5}{e}$$

$$f(5) = 0$$

$$\text{Abs. Max: } \underline{(5, 0)}$$

$$\text{Abs. Min: } \underline{\left(\frac{5}{e}, -\frac{5}{e}\right)}$$



### Exercise

Find the absolute extrema of  $f(x) = x^{8/3} - 16x^{2/3}$   $[-1, 8]$

#### Solution

$$\begin{aligned} f'(x) &= \frac{8}{3}x^{5/3} - \frac{32}{3}x^{-1/3} \\ &= \frac{8}{3}\left(x^{5/3} - \frac{4}{x^{1/3}}\right) \\ &= \frac{8}{3}\left(\frac{x^2 - 4}{x^{1/3}}\right) = 0 \end{aligned}$$

$$CN: \boxed{x = \pm 2}$$

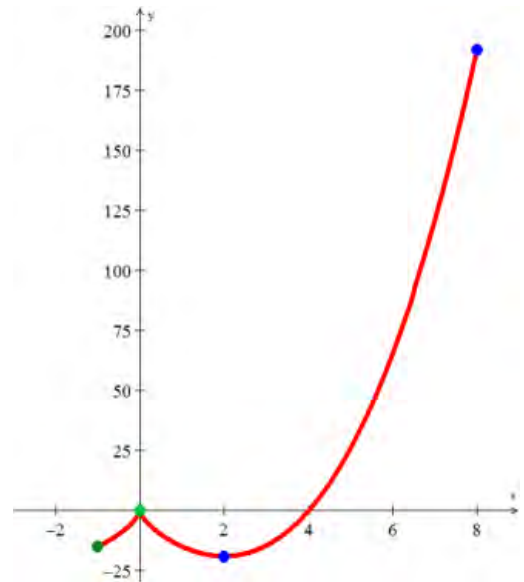
$$x \neq -2 \notin [-1, 8]$$

The derivative is *undefined* at  $\boxed{x = 0}$

$x$	$f(x)$
-1	-15
0	0
2	-19.05
8	192

$$\text{Abs. max: } \boxed{(8, 192)}$$

$$\text{Abs. Min } \boxed{(2, -19.05)}$$



### Exercise

Find the minimum and maximum values of  $f(x) = x^2 - 8x + 10$   $[0, 7]$

#### Solution

$$f'(x) = 2x - 8 = 0$$

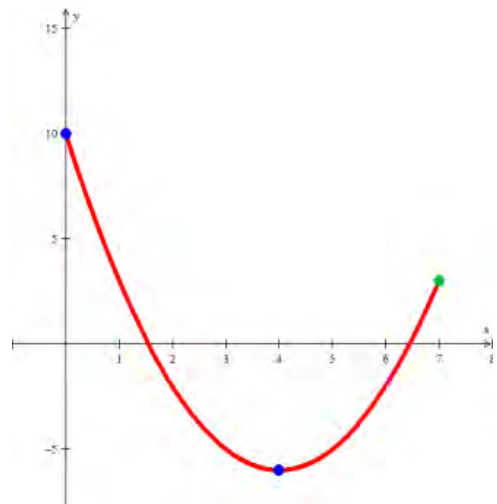
$$\Rightarrow x = 4 \text{ (CN)}$$

$$\rightarrow y = 16 - 32 + 10 = -6$$

$$\begin{cases} x = 0 \rightarrow y = 10 \\ x = 7 \rightarrow y = 3 \end{cases}$$

$$\text{Abs. Maximum } \boxed{(0, 10)}$$

$$\text{Abs. Minimum } \boxed{(4, -6)}$$



### Exercise

Find the absolute extrema of the function on the closed interval  $f(x) = 2(3 - x)$ ,  $[-1, 2]$

#### Solution

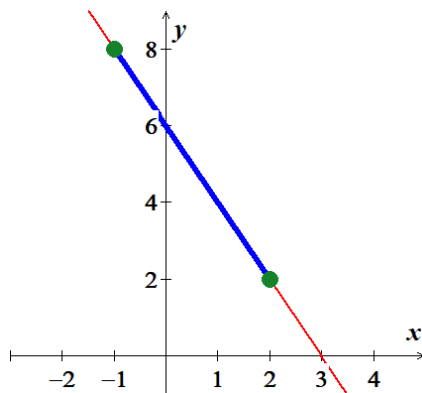
$$f' = -2 \neq 0$$

$$f(-1) = 2(3 - (-1)) = 8$$

$$f(2) = 2(3 - 2) = 2$$

$$\text{Abs. Max: } (-1, 8)$$

$$\text{abs Min: } (2, 2)$$



### Exercise

Find the absolute extrema of the function on the closed interval  $f(x) = x^3 - 3x^2$ ,  $[0, 4]$

#### Solution

$$f'(x) = 3x^2 - 6x = 0$$

$$3x(x - 2) = 0 \rightarrow x_{1,2} = 0, 2$$

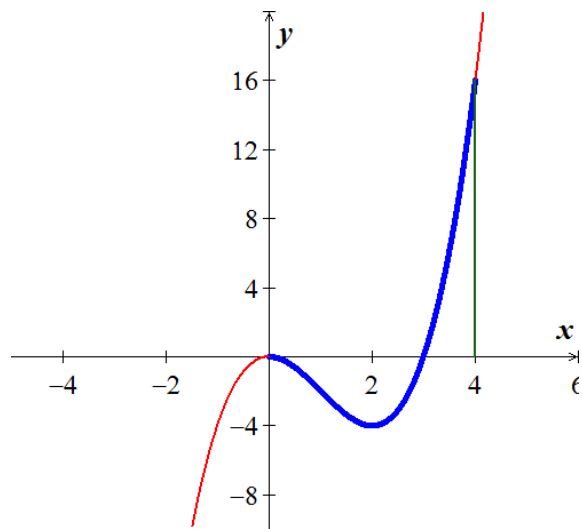
$$f(0) = 0^3 - 3(0)^2 = 0$$

$$f(2) = 2^3 - 3(2)^2 = -4$$

$$f(4) = 4^3 - 3(4)^2 = 16$$

$$\text{Abs. Max: } (4, 16)$$

$$\text{LMIN: } (2, -4)$$



### Exercise

Find the absolute extrema of the function on the closed interval

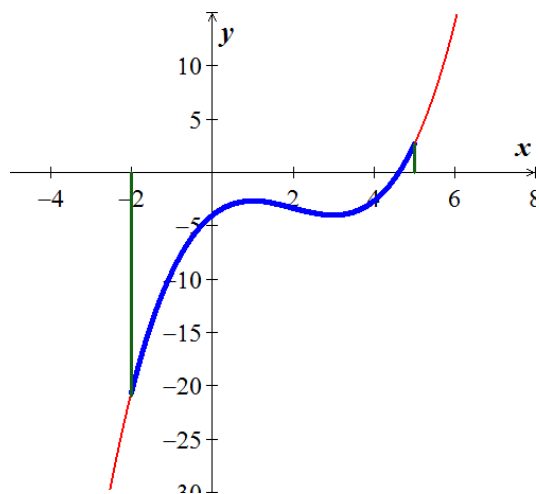
$$f(x) = \frac{1}{3}x^3 - 2x^2 + 3x - 4, \quad [-2, 5]$$

#### Solution

$$f'(x) = x^2 - 4x + 3 = 0 \rightarrow x_{1,2} = 1, 3$$

$$f(-2) = -\frac{8}{3} - 8 - 6 - 4 = -\frac{62}{3}$$

$$f(1) = \frac{1}{3}(1)^3 - 2(1)^2 + 3(1) - 4 = -\frac{8}{3}$$





$$f(3) = \frac{1}{3}(3)^3 - 2(3)^2 + 3(3) - 4 = -4$$

$$f(5) = \frac{1}{3}(5)^3 - 2(5)^2 + 3(5) - 4 = \frac{8}{3}$$

$$\text{Abs. max: } \left( 5, \frac{8}{3} \right)$$

$$\text{Abs. min: } \left( -2, -\frac{62}{3} \right)$$

### Exercise

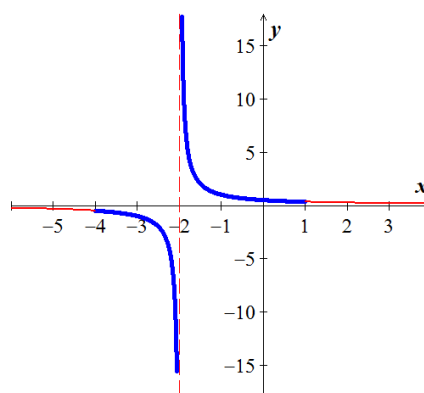
Find the absolute extrema of the function on the closed interval  $f(x) = \frac{1}{x+2}$ ,  $[-4, 1]$

#### Solution

$$x + 2 \neq 0 \rightarrow x \neq -2 \quad (\text{Asymptote})$$

$$f'(x) = -\frac{1}{(x+2)^2} \neq 0$$

There is **no** Relative Extrema.



### Exercise

Find the absolute extrema of the function on the closed interval  $f(x) = (x^2 + 4)^{2/3}$ ,  $[-2, 2]$

#### Solution

$$f'(x) = \frac{2}{3}(2x)(x^2 + 4)^{2/3-1}$$

$$= \frac{4x}{3}(x^2 + 4)^{-1/3}$$

$$f' = \frac{4x}{3}(x^2 + 4)^{-1/3} = 0; \quad x^2 + 4 \neq 0$$

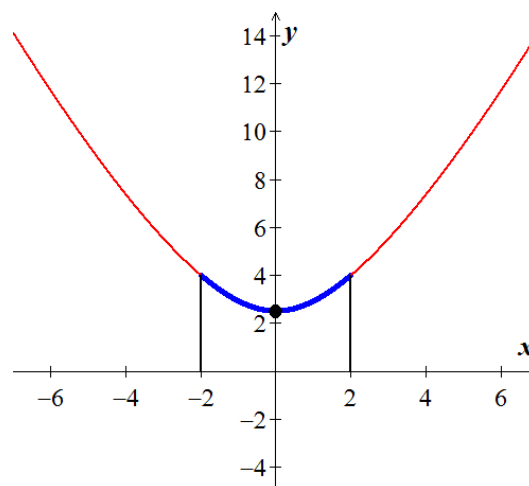
$$\text{CN: } x = 0$$

$$f(x = -2) = ((-2)^2 + 4)^{2/3} = 4$$

$$f(x = 0) = ((0)^2 + 4)^{2/3} = \sqrt[3]{16}$$

$$f(x = 2) = ((2)^2 + 4)^{2/3} = 4$$

$$\text{RMAX: } (-2, 4) \cup (2, 4) \quad \text{RMIN: } (0, \sqrt[3]{16})$$



### Exercise

Find the absolute maximum and minimum values of each function (if they exist).

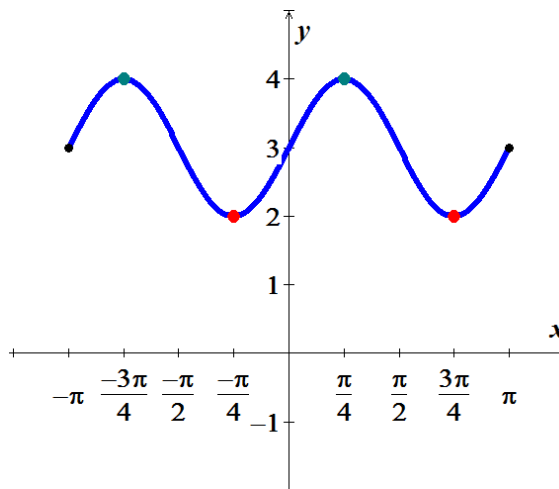
$$f(x) = \sin 2x + 3 \quad \text{on} \quad [-\pi, \pi]$$

### Solution

$$f'(x) = 2 \cos 2x = 0 \rightarrow 2x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}$$

$$x = \pm \frac{\pi}{4}, \pm \frac{3\pi}{4} \quad (CN)$$

$x$	$f(x)$
$-\pi$	3
$-\frac{3\pi}{4}$	4
$-\frac{\pi}{4}$	2
$\frac{\pi}{4}$	4
$\frac{3\pi}{4}$	2
$\pi$	3



$$\text{Abs. Min: } \left( -\frac{\pi}{4}, 2 \right) \left( \frac{3\pi}{4}, 2 \right)$$

$$\text{Abs. Max: } \left( -\frac{3\pi}{4}, 4 \right) \left( \frac{\pi}{4}, 4 \right)$$

### Exercise

Find the absolute maximum and minimum values of each function (if they exist).

$$f(x) = 2x^3 - 3x^2 - 36x + 12 \quad \text{on} \quad (-\infty, \infty)$$

### Solution

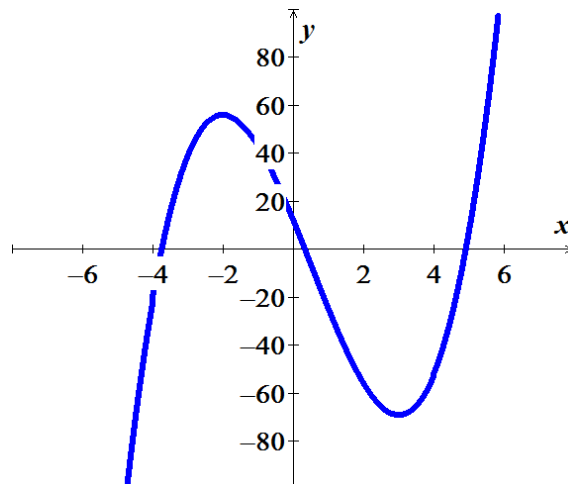
$$f'(x) = 6x^2 - 6x - 36 = 0$$

$$x^2 - x - 3 = 0$$

$$CN: x = -2, 3$$

There is no **absolute Max.** or **Min.**

$$\text{since } \lim_{x \rightarrow \pm\infty} f(x) = \pm\infty$$



### Exercise

Find the absolute maximum and minimum values of each function (if they exist).

$$f(x) = 4x^{1/2} - x^{5/2} \quad \text{on } [0, 4]$$

### Solution

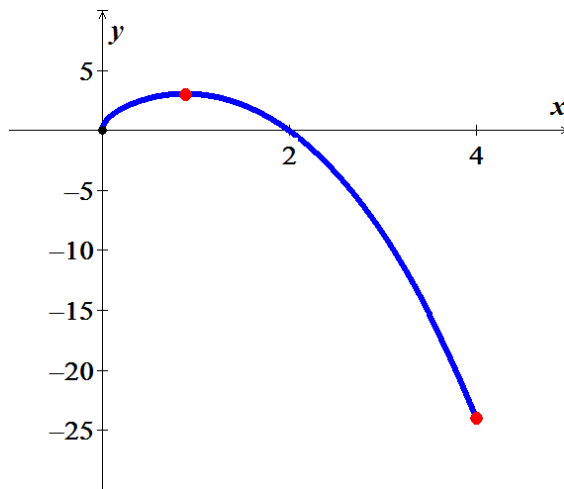
$$f'(x) = 2x^{-1/2} - \frac{5}{2}x^{3/2} = 0$$

$$\left(2x^{1/2} \times\right) 2x^{-1/2} - \frac{5}{2}x^{3/2} = 0 \quad (x \neq 0)$$

$$4 - 5x^2 = 0$$

$$\text{CN: } x = \pm \frac{2}{\sqrt{5}}, 0$$

$x$	$f(x)$
0	0
$\frac{2}{\sqrt{5}}$	$4\left(\frac{2}{\sqrt{5}}\right)^{1/2} - \left(\frac{2}{\sqrt{5}}\right)^{5/2} = 4\left(\frac{2}{\sqrt{5}}\right)^{1/2} - \frac{4}{5}\left(\frac{2}{\sqrt{5}}\right)^{1/2} = \frac{16}{5}\left(\frac{2}{\sqrt{5}}\right)^{1/2}$
4	$4(4)^{1/2} - (4)^{5/2} = 8 - 32 = -24$



$$\text{Abs. Min: } (4, -24)$$

$$\text{Abs. Max: } \left( \frac{2}{\sqrt{5}}, \frac{16}{5} \left( \frac{2}{\sqrt{5}} \right)^{1/2} \right)$$

### Exercise

Find the absolute maximum and minimum values of each function (if they exist).

$$f(x) = 2x \ln x + 10 \quad \text{on } (0, 4)$$

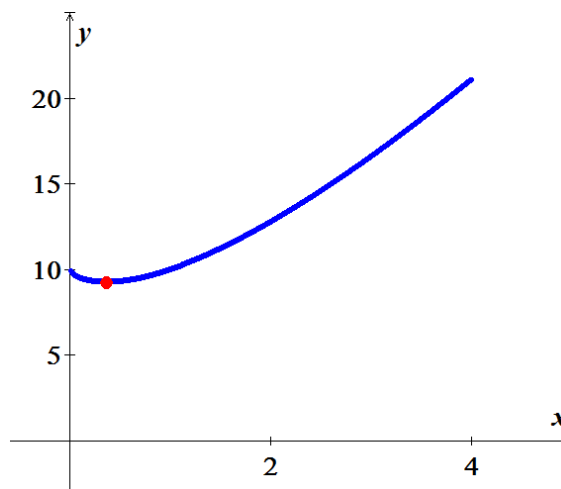
#### Solution

$$f'(x) = 2 \ln x + 2 = 0 \rightarrow \ln x = -1$$

$$\text{CN: } x = e^{-1}$$

$$f\left(\frac{1}{e}\right) = \frac{2}{e} \ln e^{-1} + 10 = 10 - \frac{2}{e}$$

$$\text{Abs. Min: } \left(\frac{1}{e}, 10 - \frac{2}{e}\right)$$



### Exercise

Find the absolute maximum and minimum values of each function (if they exist).

$$f(x) = x \sin^{-1} x \quad \text{on } [-1, 1]$$

#### Solution

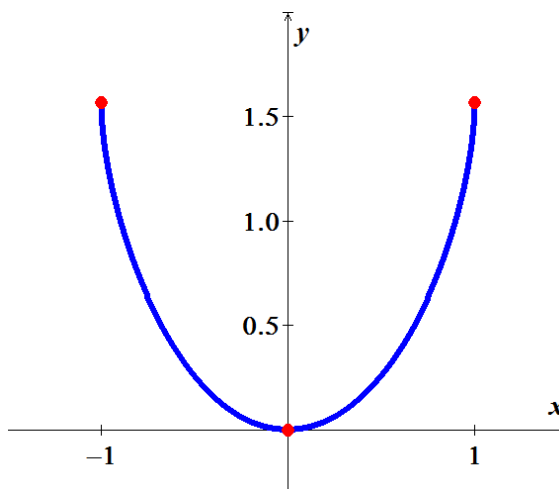
$$f'(x) = \sin^{-1} x + \frac{x}{\sqrt{1-x^2}} = 0$$

$$\text{CN: } x = 0$$

$x$	$f(x)$
-1	$-\sin^{-1}(-1) = \frac{\pi}{2}$
0	0
1	$\sin^{-1}(1) = \frac{\pi}{2}$

$$\text{Abs. Min: } (0, 0)$$

$$\text{Abs. Max: } \left(\pm 1, \frac{\pi}{2}\right)$$



### Exercise

Determine all critical points of  $y = x^2 - 6x + 7$

#### Solution

$$y' = 2x - 6 = 0 \Rightarrow \boxed{x = 3} \quad (\text{CN})$$

$$y|_{x=3} = 3^2 - 6(3) + 7 = -2$$

**Critical point:**  $\underline{(3, -2)}$

### Exercise

Determine all critical points of  $g(x) = (x-1)^2(x-3)^2$

#### Solution

$$\begin{aligned} g'(x) &= 2(x-1)(x-3)^2 + 2(x-1)^2(x-3) & (uv)' &= u'v + v'u \\ &= 2(x-1)(x-3)(x-3+x-1) \\ &= 2(x-1)(x-3)(2x-4) \end{aligned}$$

The **critical numbers** are:  $x = 1, 2, 3$

$$g(1) = 0 \quad g(2) = 1 \quad g(3) = 0$$

**Critical points:**  $\underline{(1, 0), (2, 1) \text{ and } (3, 0)}$

### Exercise

Determine all critical points of  $f(x) = \frac{x^2}{x-2}$

#### Solution

$$\begin{aligned} f'(x) &= \frac{2x(x-2) - x^2}{(x-2)^2} & \left(\frac{u}{v}\right)' &= \frac{u'v - v'u}{v^2} \\ &= \frac{x^2 - 4x}{(x-2)^2} = 0 \end{aligned}$$

$x = 2$  is *not* in the domain.

The critical numbers are:  $x = 0, 4$

$$f(0) = 0 \quad f(4) = 8$$

**Critical points:**  $\underline{(0, 0), (4, 8)}$

### Exercise

Determine all critical points of  $g(x) = x^2 - 32\sqrt{x}$

### Solution

$$g'(x) = 2x - \frac{16}{\sqrt{x}} = \frac{2x^{3/2} - 16}{\sqrt{x}} = 0$$

$$\begin{cases} 2x^{3/2} - 16 = 0 \Rightarrow x^{3/2} = 8 \rightarrow \boxed{x = 4} \\ \sqrt{x} = 0 \Rightarrow \boxed{x = 0} \end{cases}$$

The critical numbers are:  $x = 0, 4$

$$g(0) = 0 \qquad g(4) = 16 - 32\sqrt{4} = 48$$

**Critical points:**  $\boxed{(0,0), (4,48)}$

### Exercise

Find the extreme values (absolute and local) of the function and where they occur  $y = x^3 - 2x + 4$

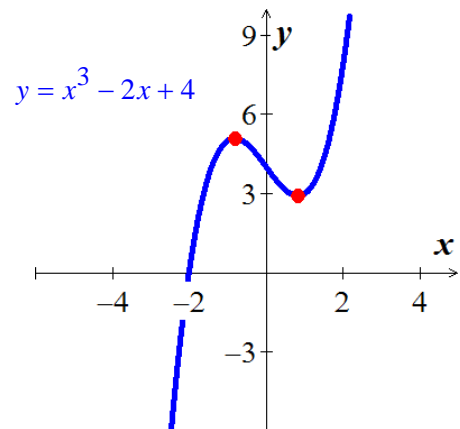
### Solution

$$y' = 3x^2 - 2 = 0 \Rightarrow \boxed{x = \pm\sqrt{\frac{2}{3}}}$$

$$x = -\sqrt{\frac{2}{3}} \Rightarrow y = \left(-\sqrt{\frac{2}{3}}\right)^3 - 2\left(-\sqrt{\frac{2}{3}}\right) + 4 = 5.089$$

$$x = \sqrt{\frac{2}{3}} \Rightarrow y = \left(\sqrt{\frac{2}{3}}\right)^3 - 2\left(\sqrt{\frac{2}{3}}\right) + 4 = 2.911$$

**LMAX:**  $\boxed{(-.816, 5.089)}$       **LMIN:**  $\boxed{(.816, 2.911)}$



### Exercise

Find the extreme values (absolute and local) of the function and where they occur  $y = \sqrt{x^2 - 1}$

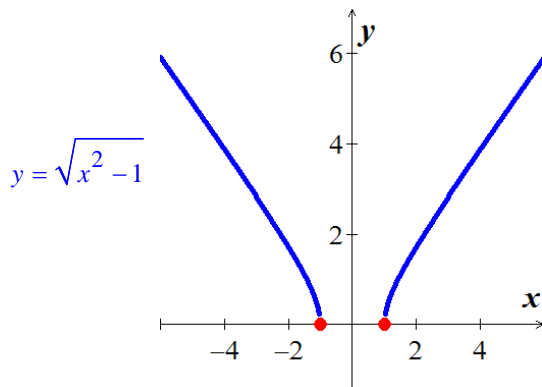
### Solution

**Domain:**  $x \leq -1 \quad x \geq 1$

$$y' = \frac{x}{\sqrt{x^2 - 1}} = 0 \Rightarrow x = \cancel{0}, \pm 1 \text{ (CN)}$$

$$y = \sqrt{(\pm 1)^2 - 1} = 0$$

**LMIN:**  $\boxed{(-1, 0) \text{ \& } (1, 0)}$



### Exercise

Find the extreme values (absolute and local) of the function and where they occur  $y = \frac{1}{\sqrt[3]{1-x^2}}$

### Solution

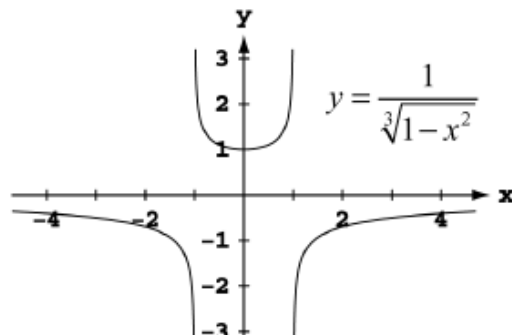
VA:  $x = \pm 1$

$$y = (1 - x^2)^{-1/3}$$

$$y' = -\frac{1}{3}(1 - x^2)^{-4/3}(-2x) = \frac{2}{3} \frac{x}{(1 - x^2)^{4/3}} = 0$$

$$\boxed{x = 0, \cancel{x = \pm 1}} \text{ (CN)}$$

$$\text{LMIN } \underline{(0, 1)}$$



### Exercise

Find the extreme values (absolute and local) of the function and where they occur  $y = x^2\sqrt{3-x}$

### Solution

$$y' = 2x\sqrt{3-x} + \frac{1}{2}\left(\frac{-1}{\sqrt{3-x}}\right)x^2 \quad (uv)' = u'v + v'u$$

$$= \frac{4x(3-x) - x^2}{2\sqrt{3-x}}$$

$$= \frac{4x(3-x) - x^2}{2\sqrt{3-x}}$$

$$= \frac{12x - 5x^2}{2\sqrt{3-x}} = 0$$

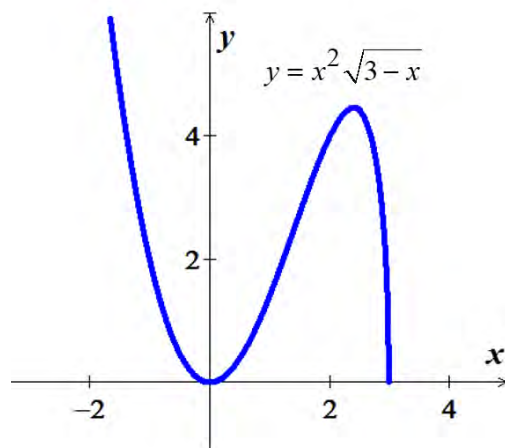
$$\text{CN: } \underline{x = \frac{5}{12}, 0, 3}$$

$$y \Big|_{x=\frac{5}{12}} = 0.279$$

$$y \Big|_{x=0} = 0$$

$$y \Big|_{x=3} = 0$$

$$\text{LMAX: } \underline{\left(\frac{5}{12}, 0.279\right)} \quad \text{LMIN: } \underline{(0, 0) \cup (3, 0)}$$



### Exercise

Find the extreme values (absolute and local) of the function and where they occur  $y = \frac{x+1}{x^2+2x+2}$

#### Solution

$$y' = \frac{x^2+2x+2-(2x+2)(x+1)}{(x^2+2x+2)^2} \quad \left(\frac{u}{v}\right)' = \frac{u'v-v'u}{v^2}$$

$$= \frac{x^2+2x+2-2x^2-2x-2x-2}{(x^2+2x+2)^2}$$

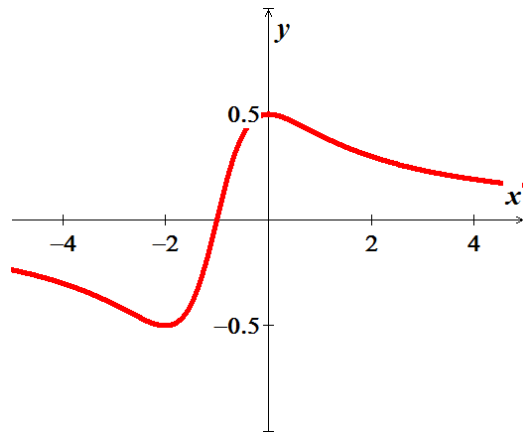
$$= \frac{-x^2-2x}{(x^2+2x+2)^2} = 0$$

CN:  $x=0, -2$

$$y|_{x=0} = \frac{1}{2}$$

$$y|_{x=-2} = -2$$

LMAX:  $\left(0, \frac{1}{2}\right)$       LMIN:  $\left(-2, -\frac{1}{2}\right)$



### Exercise

Find the extreme values (absolute and local) of the function and where they occur  $y = x^{2/3}(x+2)$

#### Solution

$$y' = \frac{2}{3}x^{-1/3}(x+2) + x^{2/3} \quad (uv)' = u'v + v'u$$

$$= \frac{2}{3} \frac{x+2}{x^{1/3}} + x^{2/3}$$

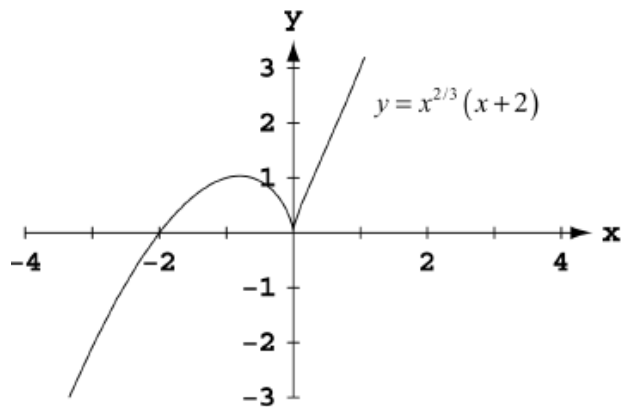
$$= \frac{2x+4+3x}{x^{1/3}}$$

$$= \frac{5x+4}{\sqrt[3]{x}} = 0 \quad \text{CN: } x=0, -\frac{4}{5}$$

$$y|_{x=-\frac{4}{5}} = \left(-\frac{4}{5}\right)^{2/3} \left(-\frac{4}{5} + 2\right) = 1.034$$

$$y|_{x=0} = 0$$

LMAX:  $\left(-\frac{4}{5}, 1.034\right)$       LMIN:  $(0, 0)$





### Exercise

Find the extreme values (absolute and local) of the function and where they occur  $y = x\sqrt{4-x^2}$

#### Solution

$$y' = \sqrt{4-x^2} + \left( \frac{1}{2} \frac{-2x}{\sqrt{4-x^2}} \right) (x) \quad (uv)' = u'v + v'u$$

$$= \frac{4-x^2-x^2}{\sqrt{4-x^2}}$$

$$= \frac{4-2x^2}{\sqrt{4-x^2}} = 0$$

$$\begin{cases} 4-2x^2 = 0 \\ 4-x^2 = 0 \end{cases}$$

$$\text{CN: } \underline{x = \pm\sqrt{2}, \pm 2}$$

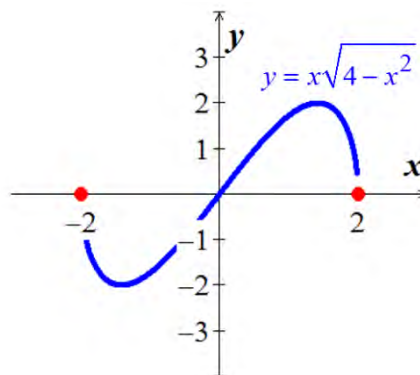
$$y \Big|_{x=-\sqrt{2}} = -\sqrt{2}\sqrt{2} = -2$$

$$y \Big|_{x=\sqrt{2}} = 2$$

$$y \Big|_{x=\pm 2} = 0$$

$$\text{LMAX: } \underline{(\sqrt{2}, 2)}$$

$$\text{LMIN: } \underline{(-\sqrt{2}, -2)}$$



### Exercise

Find the extreme values (absolute and local) of the function and where they occur  $f(x) = \frac{e^x + e^{-x}}{2}$

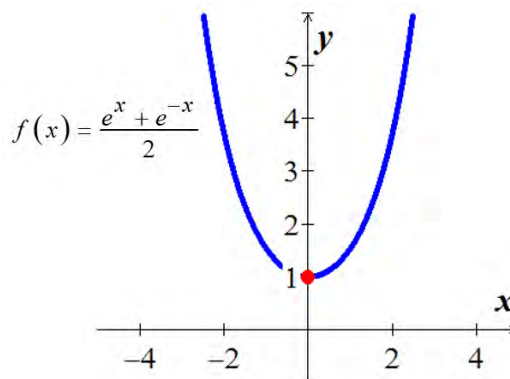
#### Solution

$$f'(x) = \frac{1}{2}(e^x - e^{-x}) = 0$$

$$e^x = e^{-x} \rightarrow \underline{x=0} \quad (\text{CN})$$

$$f(0) = \frac{1+1}{2} = 1$$

$$\text{LMIN: } \underline{(0, 1)}$$



### Exercise

Find the extreme values (absolute and local) of the function and where they occur

$$f(x) = \frac{1}{8}x^3 - \frac{1}{2}x \quad [-1, 3]$$

### Solution

$$f'(x) = \frac{3}{8}x^2 - \frac{1}{2} = 0$$

$$x^2 = \frac{4}{3} \Rightarrow x = \cancel{-\frac{2}{\sqrt{3}}} (< -1), \quad \frac{2}{\sqrt{3}}$$

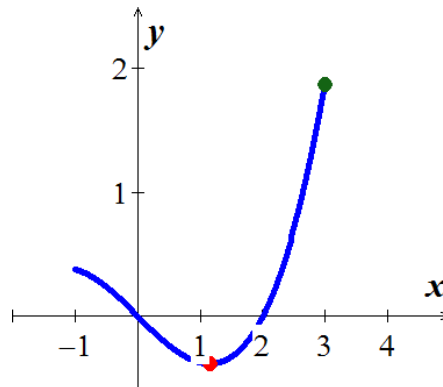
$$(CN): \quad x = \frac{2}{\sqrt{3}} \quad |$$

$$f(-1) = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8}$$

$$f(3) = \frac{27}{8} - \frac{3}{2} = \frac{15}{8}$$

$$f\left(\frac{2}{\sqrt{3}}\right) = \frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}} = -\frac{2}{3\sqrt{3}}$$

$$LMIN: \left( \frac{2}{\sqrt{3}}, -\frac{2}{3\sqrt{3}} \right) \quad | \quad abs. Max: \left( 3, \frac{15}{8} \right) \quad |$$



### Exercise

Find the extreme values (absolute and local) of the function and where they occur  $f(x) = \frac{1}{x} - \ln x$

### Solution

$$\begin{aligned} f(x) &= -\frac{1}{x^2} - \frac{1}{x} \\ &= -\frac{1+x}{x^2} = 0 \end{aligned}$$

$$x = 0, -1 \quad (CN)$$

Since the critical number are not within the domain; inside the log has to be positive.

No abs or local extreme

### Exercise

Find the extreme values (absolute and local) of the function and where they occur

$$f(x) = \sin x \cos x \quad [0, 2\pi]$$

### Solution

$$f'(x) = \cos^2 x - \sin^2 x = \cos 2x = 0$$

$$2x = \frac{\pi}{2} + k\pi$$

$$\text{CN: } x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$f(0) = 0$$

$$f\left(\frac{\pi}{4}\right) = \frac{1}{2}$$

$$f\left(\frac{3\pi}{4}\right) = -\frac{1}{2}$$

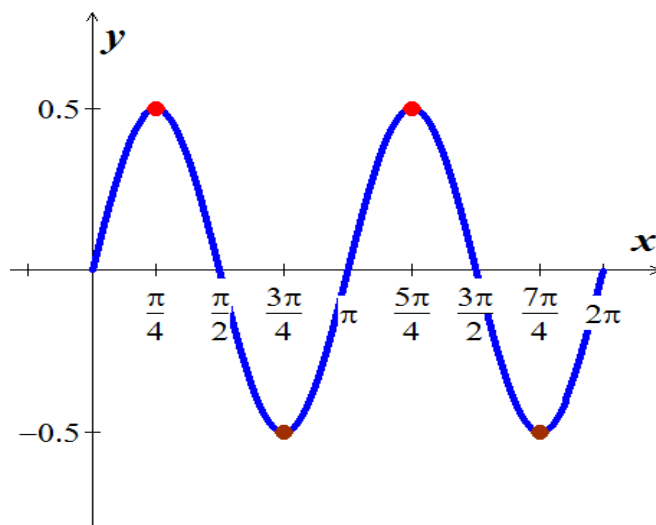
$$f\left(\frac{5\pi}{4}\right) = \frac{1}{2}$$

$$f\left(\frac{7\pi}{4}\right) = -\frac{1}{2}$$

$$f(2\pi) = 0$$

$$\text{LMIN: } \left(\frac{3\pi}{4}, -\frac{1}{2}\right) \text{ \& } \left(\frac{7\pi}{4}, -\frac{1}{2}\right)$$

$$\text{LMAX: } \left(\frac{\pi}{4}, \frac{1}{2}\right) \text{ \& } \left(\frac{5\pi}{4}, \frac{1}{2}\right)$$



## Exercise

Find the extreme values (absolute and local) of the function and where they occur  $f(x) = x - \tan^{-1} x$

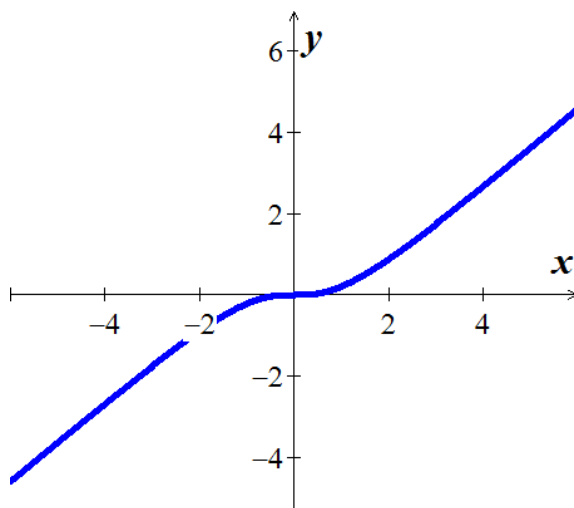
### Solution

$$f'(x) = 1 - \frac{1}{1+x^2} = 0$$

$$1 + x^2 = 1$$

$$\rightarrow x = 0, 0$$

No extreme values



### Exercise

Let  $f(x) = (x-2)^{2/3}$

- a) Does  $f'(2)$  exist?
- b) Show the only local extreme value of  $f$  occurs at  $x = 2$ .
- c) Does the result in part (b) contradict the Extreme Value Theorem?

### Solution

a)  $f'(x) = \frac{2}{3}(x-2)^{-1/3}$  is undefined at  $x = 2$

b)  $f(x=2) = (2-2)^{2/3} = 0$  and  $f(x) > 0 \quad \forall x \neq 2$

- c) No,  $f(x)$  domain is all real numbers and doesn't need to have a global maximum. Any restriction of  $f$  to a closed interval of the form  $[a, b]$  would have a maximum and minimum value on the interval.

### Exercise

When a telephone wire is hung between two poles, the wire forms a U-shape curve called a Catenary. For instance, the function  $y = 30\left(e^{x/60} + e^{-x/60}\right) \quad -30 \leq x \leq 30$  models the shape of the telephone wire strung between two poles that are 60 ft. apart ( $x$  &  $y$  are measured in ft.). Show that the lowest point on the wire is midway between two poles. How much does the wire sag between the two poles?

### Solution

$$\begin{aligned} y' &= 30\left(\frac{1}{60}e^{x/60} - \frac{1}{60}e^{-x/60}\right) \\ &= \frac{1}{2}\left(e^{x/60} - e^{-x/60}\right) \end{aligned}$$

Critical number(s)

$$y' = 0$$

$$\frac{1}{2}\left(e^{x/60} - e^{-x/60}\right) = 0$$

$$e^{x/60} - e^{-x/60} = 0$$

$$e^{x/60} = e^{-x/60}$$

$$\frac{x}{60} = -\frac{x}{60}$$

$$\Rightarrow x = 0$$

$$y(x=-30) = 30\left(e^{-30/60} + e^{-(-30)/60}\right) \approx 67.7 \text{ ft}$$

$$y(x=0) = 30\left(e^0 + e^0\right) = 30(2) = 60 \text{ ft}$$

$$y(x=30) = 30 \left( e^{30/60} + e^{-(30)/60} \right) \approx 67.7 \text{ ft}$$

Sag : 67.7 ft

### Exercise

You are sitting in a classroom next to the wall looking at the blackboard at the front of the room. The blackboard is 12 feet long and starts 3 feet from the wall you are sitting next to.

- Show that your viewing angle is  $\alpha = \cot^{-1} \frac{x}{15} - \cot^{-1} \frac{x}{3}$  If you are  $x$  feet from the front wall
- Find  $x$  so that  $\alpha$  is as large as possible

### Solution

$$a) \cot(\text{wall}) = \frac{x}{3} \Rightarrow \angle \text{wall} = \cot^{-1} \left( \frac{x}{3} \right)$$

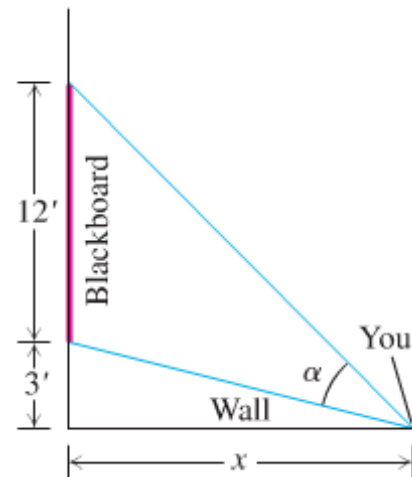
$$\cot(\triangle) = \frac{x}{15} \Rightarrow \angle \triangle = \cot^{-1} \left( \frac{x}{15} \right)$$

$\alpha$  = Angle of the large triangle – Wall triangle angle

$$\alpha = \cot^{-1} \frac{x}{15} - \cot^{-1} \frac{x}{3}$$

$$\begin{aligned} b) \frac{d\alpha}{dx} &= -\frac{\frac{1}{15}}{1+\left(\frac{x}{15}\right)^2} + \frac{\frac{1}{3}}{1+\left(\frac{x}{3}\right)^2} \\ &= -\frac{1}{15} \frac{1}{1+\frac{x^2}{225}} + \frac{1}{3} \frac{1}{1+\frac{x^2}{9}} \\ &= -\frac{1}{15} \frac{225}{225+x^2} + \frac{1}{3} \frac{9}{9+x^2} \\ &= -\frac{15}{225+x^2} + \frac{3}{9+x^2} \\ &= \frac{-15(9+x^2) + 3(225+x^2)}{(225+x^2)(9+x^2)} \\ &= \frac{-135-15x^2+675+3x^2}{(225+x^2)(9+x^2)} \\ &= \frac{-12x^2+540}{(225+x^2)(9+x^2)} = 0 \end{aligned}$$

$$-12x^2 + 540 = 0$$



$$x^2 = \frac{540}{12}$$

$$= 45$$

$$\underline{x = \pm 3\sqrt{5} \mid}$$

$$x = 3\sqrt{5} \approx 6.7082$$

$$\alpha\left(x = 3\sqrt{5}\right) = \cot^{-1} \frac{3\sqrt{5}}{15} - \cot^{-1} \frac{3\sqrt{5}}{3}$$

$$\approx 0.729728$$

$$\underline{\approx 41.8103^\circ \mid}$$

Local maximum of  $41.8103^\circ$  when  $x \approx 6.7082 \text{ ft.}$