

Solution **Section 1.1 – Introduction to System of Linear Equations**

Exercise

Find a solution for x, y, z to the system of equations

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3e + 2\sqrt{2} + \pi \\ 6e + 5\sqrt{2} + 4\pi \\ 9e + 8\sqrt{2} + 7\pi \end{pmatrix}$$

Solution

$$\begin{pmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{pmatrix} = \begin{pmatrix} 3e + 2\sqrt{2} + \pi \\ 6e + 5\sqrt{2} + 4\pi \\ 9e + 8\sqrt{2} + 7\pi \end{pmatrix}$$

$$\Rightarrow \begin{cases} x + 2y + 3z = \pi + 2\sqrt{2} + 3e \\ 4x + 5y + 6z = 4\pi + 5\sqrt{2} + 6e \\ 7x + 8y + 9z = 7\pi + 8\sqrt{2} + 9e \end{cases}$$

$$\text{Solution: } \underline{x = \pi \quad y = \sqrt{2} \quad z = e}$$

Exercise

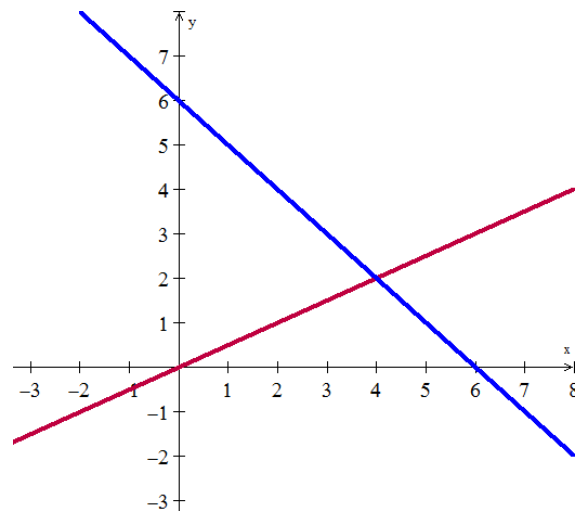
Draw the two pictures in two planes for the equations: $x - 2y = 0$, $x + y = 6$

Solution

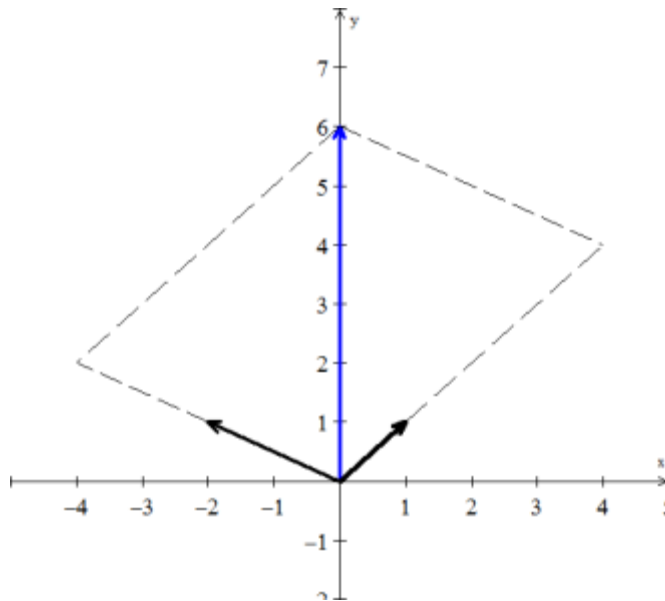
The matrix form of the 2 equations:

$$\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

Row picture is the 2 lines from the given equations and their intersection is the point $(4, 2)$ which is the solution for the system.



Column Picture is the column vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$



$$x \begin{pmatrix} 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} -2 \\ 1 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

The parallelogram shows how the solution vector $\begin{pmatrix} 0 \\ 6 \end{pmatrix}$ can be written as the linear combination of the column vectors.

Exercise

Normally 4 planes in 4-dimensional space meet at a _____. Normally 4 column vectors in 4-dimensional space can combine to produce b . what combinations of

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ produces $b = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 2 \end{pmatrix}$?

What 4 equations for x, y, z, w are you solving?

Solution

Normally 4 planes in 4-dimensional space meet at a ***point***.

The combination of the vectors producing b is:

$$0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 2 \end{pmatrix}$$

$$1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 2 \end{pmatrix}$$

The system of equations that satisfies the given vectors is:

$$\begin{cases} x + y + z + w = 3 \\ y + z + w = 3 \\ z + w = 3 \\ w = 2 \end{cases}$$

Exercise

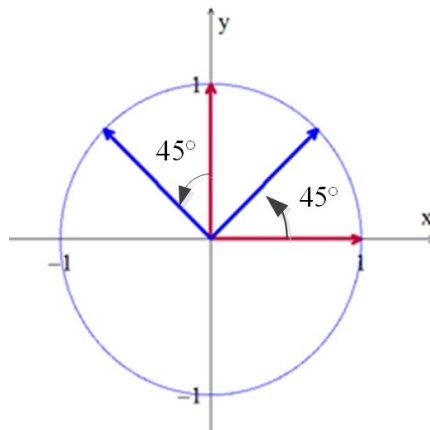
What 2 by 2 matrix A rotates every vector through 45° ?

The vector $(1, 0)$ goes to $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. The vector $(0, 1)$ goes to $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Those determine the matrix. Draw these particular vectors in the xy -plane and find A .

Solution

$$A = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$



Exercise

What two vectors are obtained by rotating the plane vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by 30° (cw) ?

Write a matrix A such that for every vector v in the plane, Av is the vector obtained by rotating v clockwise by 30° .

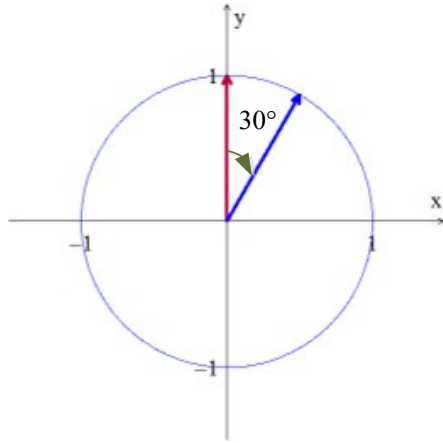
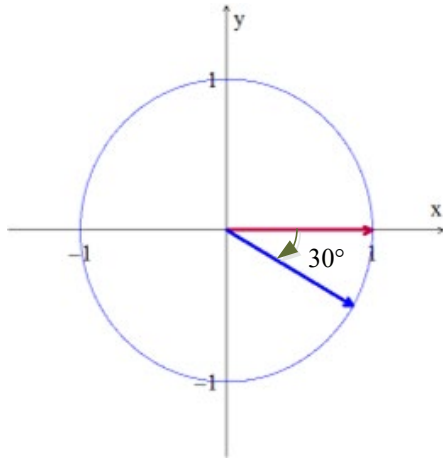
Find a matrix B such that for every 3-dimensional vector v , the vector Bv is the reflection of v through the plane $x + y + z = 0$. *Hint* : $v = (1, 0, 0)$

Solution

Rotating the vectors by 30° (cw) yields:

$$\text{For the vector } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ yields to } \begin{pmatrix} \cos(-30^\circ) \\ \sin(-30^\circ) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}$$

And for the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ yields to $\begin{pmatrix} \sin(30^\circ) \\ \cos(30^\circ) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$



The desired matrix is: $A = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

To get 1 from $\frac{\sqrt{3}}{2}$ is to multiply by $\frac{2}{\sqrt{3}} = 2\frac{1}{\sqrt{3}}$

The unit vector to the plane $x + y + z = 0$ is $\hat{u} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$$\begin{aligned} Bv &= B \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{2}{\sqrt{3}} \hat{u} \\ &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{2}{\sqrt{3}} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
B \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{\sqrt{3}} \hat{u} \\
&= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
B \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{2}{\sqrt{3}} \hat{u} \\
&= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}
\end{aligned}$$

$$\text{The solution: } \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

Exercise

Find a system of linear equation corresponding to the given augmented matrix

$$\begin{bmatrix} 0 & 3 & -1 & -1 & -1 \\ 5 & 2 & 0 & -3 & -6 \end{bmatrix}$$

Solution

$$\begin{cases} 3x_2 - x_3 - x_4 = -1 \\ 5x_1 + 2x_2 - 3x_4 = -6 \end{cases}$$

Exercise

Find a system of linear equation corresponding to the given augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \\ 5 & -6 & 1 & 1 \\ -8 & 0 & 0 & 3 \end{bmatrix}$$

Solution

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ -4x_1 - 3x_2 - 2x_3 = -1 \\ 5x_1 - 6x_2 + x_3 = 1 \\ -8x_1 = 3 \end{cases}$$

Exercise

Find the augmented matrix for the given system of linear equations.

$$\begin{cases} -2x_1 = 6 \\ 3x_1 = 8 \\ 9x_1 = -3 \end{cases}$$

Solution

$$\left[\begin{array}{c|c} -2 & 6 \\ 3 & 8 \\ 9 & -3 \end{array} \right]$$

Exercise

Find the augmented matrix for the given system of linear equations.

$$\begin{cases} 3x_1 - 2x_2 = -1 \\ 4x_1 + 5x_2 = 3 \\ 7x_1 + 3x_2 = 2 \end{cases}$$

Solution

$$\left[\begin{array}{cc|c} 3 & -2 & -1 \\ 4 & 5 & 3 \\ 7 & 3 & 2 \end{array} \right]$$

Exercise

Find the augmented matrix for the given system of linear equations.

$$\begin{cases} 2x_1 + 2x_3 = 1 \\ 3x_1 - x_2 + 4x_3 = 7 \\ 6x_1 + x_2 - x_3 = 0 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 1 \\ 3 & -1 & 4 & 7 \\ 6 & 1 & -1 & 0 \end{array} \right]$$