Lecture Three – Multiple Integrals

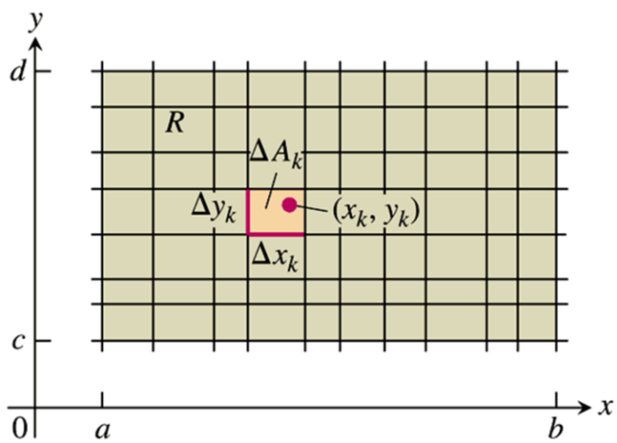
Section 3.1 – Double Integrals over Rectangular Regions

Double Integrals

Consider a function $f(x, y)$ defined on a rectangular region *R*,

$$R: \quad a \le x \le b, \quad c \le y \le d$$

A small rectangular piece of width Δx and height Δy has area $\Delta A = \Delta x \Delta y$.



To form a Riemann sum over *R*, select a point (x_k, y_k) in the kth small rectangle, multiply the value of *f* at that point by the area ΔA_k and add together the products:

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

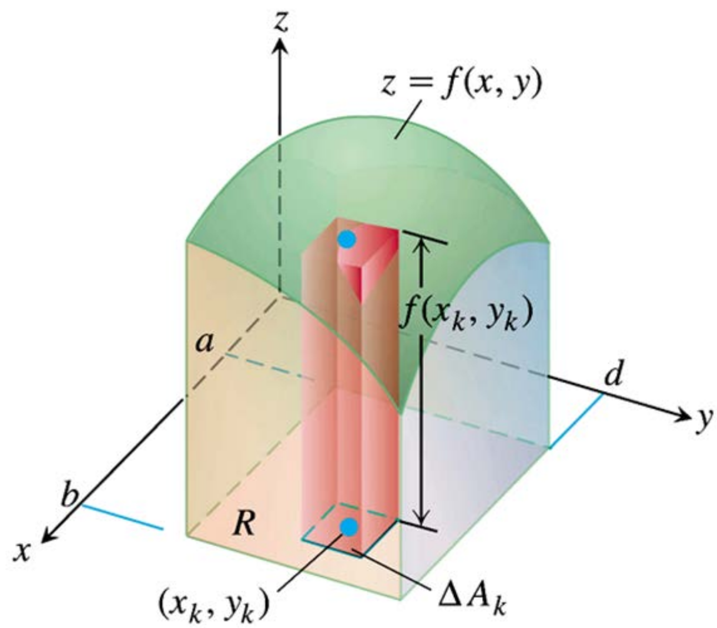
As the rectangles get narrow and short, their number *n* increases, therefore

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

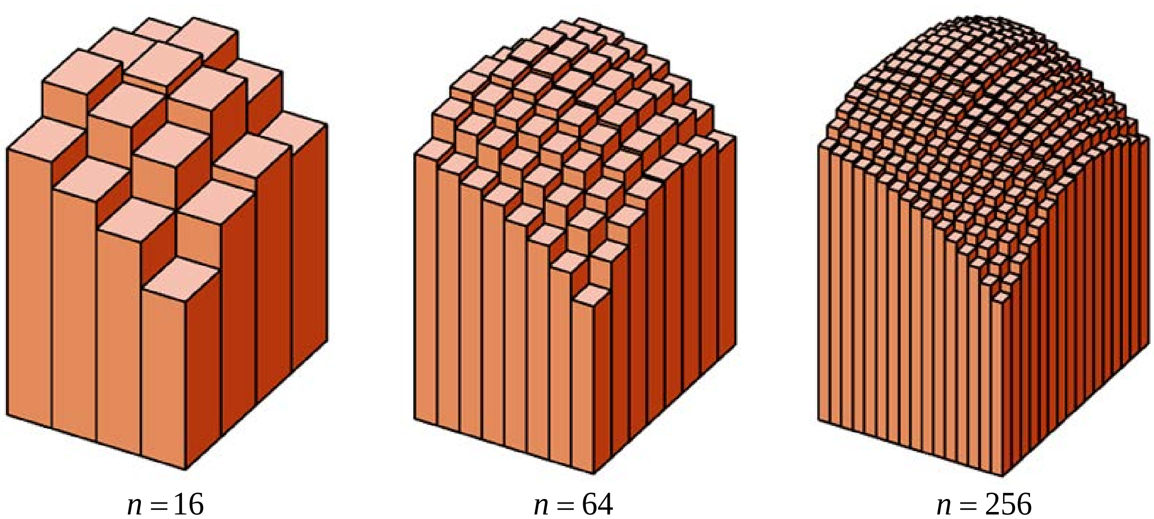
Then the function f is said to be integrable and the limit is called double integral of f over *R*,

$$\iint_R f(x, y)\,dA \quad \text{or} \quad \iint_R f(x, y)\,dxdy$$

Double Integrals as Volumes



$$Volume = \lim_{n\to\infty} S_n = \iint_R f(x,y)\,dA, \ \ where\ \Delta A_k \to 0 \ as\ n \to \infty$$



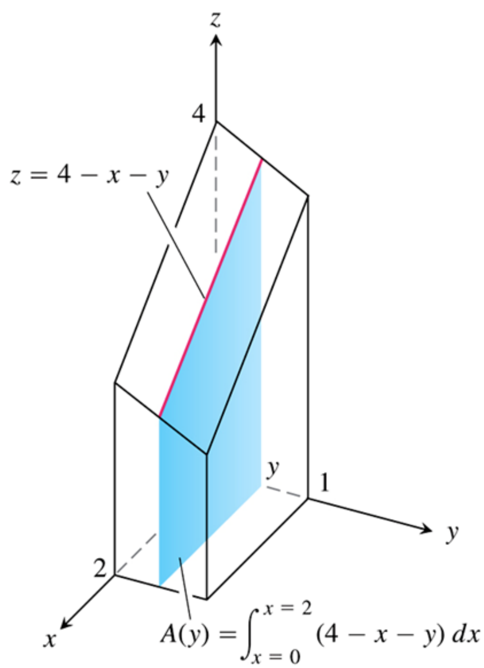
As *n* increases, the ***Riemann sum*** approximations approach the total volume of the solid

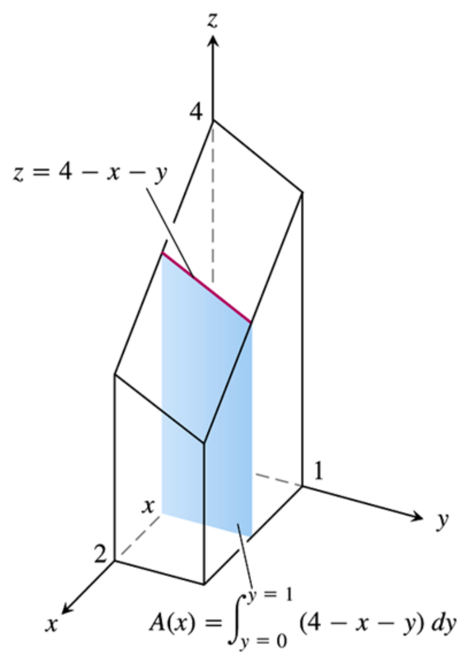
Example

Calculate the volume under the plane $z = 4 - x - y$ over the rectangular region $R: \quad 0 \le x \le 2, \quad 0 \le y \le 1$ in the *xy*-plane.

Solution

$$
\begin{aligned}
Volume &= \int_{x=0}^{x=2} A(x)\,dx \\
&= \int_{x=0}^{x=2} \int_{y=0}^{y=1} (4 - x - y)\,dy\ dx \\
&= \int_{x=0}^{x=2} \left[4y - xy - \frac{1}{2}y^2 \right]_{y=0}^{y=1} dx \\
&= \int_{x=0}^{x=2} \left(4 - x - \frac{1}{2}\right) dx \\
&= \int_{x=0}^{x=2} \left(\frac{7}{2} - x\right) dx \\
&= \left[\frac{7}{2}x - \frac{1}{2}x^2 \right]_0^2 \\
&= 7 - 2 \\
&= 5 \quad unit^3
\end{aligned}
$$





$$
Volume = \int_0^1 \int_0^2 (4 - x - y)\,dx\ dy
$$

Theorem – Fubini's Theorem

If $f(x, y)$ is continuous throughout the rectangular region R: $a \le x \le b,\ c \le y \le d$, then

$$\iint\limits_R f(x, y)\,dA = \int_c^d \int_a^b f(x, y)\,dx\,dy = \int_a^b \int_c^d f(x, y)\,dy\,dx$$

Example

Calculate $\displaystyle\iint\limits_R f(x, y)\,dA$ for $f(x, y) = 100 - 6x^2 y$ *and* $R:\ 0 \le x \le 2,\ \ -1 \le y \le 1$

Solution

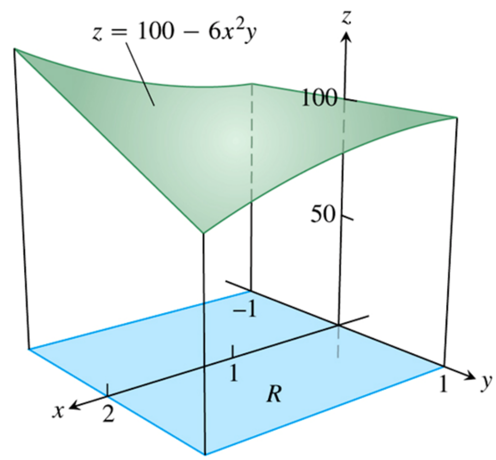
$$\int_{-1}^{1}\int_{0}^{2}\left(100 - 6x^2 y\right)dx\,dy = \int_{-1}^{1}\left(100x - 2x^3 y\ \Big|_0^2\right)dy$$

$$= \int_{-1}^{1}(200 - 16y)\,dy$$

$$= 200y - 8y^2\ \Big|_{-1}^{1}$$

$$= 200 - 8 - (-200 - 8)$$

$$\underline{= 400}$$



Example

Evaluate $\displaystyle\int\int e^{4x} y^3\,dy\,dx$

Solution

$$\int\int e^{4x} y^3\,dy\,dx = \int e^{4x}\,dx \int y^3\,dy$$

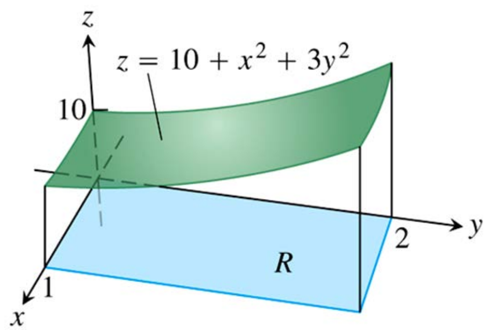
$$= \frac{1}{4}e^{4x}\left(\frac{1}{4}y^4\right) + C$$

$$\underline{= \frac{1}{16}y^4 e^{4x} + C}$$

Example

Find the volume of the region bounded above the elliptical paraboloid $z=10+x^2+3y^2$ and below the rectangle $R:\ 0\le x\le 1,\quad 0\le y\le 2$

Solution



$$Volume=\int_0^1\int_0^2\left(10+x^2+3y^2\right)dydx$$

$$=\int_0^1\left(10y+yx^2+y^3\ \Big|_0^2\ dx\right.$$

$$=\int_0^1\left(2x^2+28\right)dx$$

$$=\frac{2}{3}x^3+28x\ \Big|_0^1$$

$$=\frac{2}{3}+28$$

$$=\boxed{\frac{86}{3}\ unit^3}$$

Example

Evaluate $\int_0^1\int_y^1 ye^{-x^3}dxdy$

Solution

$$y\le x\le 1\quad\to\quad\begin{cases}y=x\\x=1\end{cases}$$

$$0\le y\le 1$$

$$0\le y\le x\quad \&\quad 0\le x\le 1$$

$$\int_0^1\int_y^1 ye^{-x^3}dxdy=\int_0^1\int_0^x ye^{-x^3}dydx$$

$$=\frac{1}{2}\int_0^1 e^{-x^3}y^2\ \Big|_0^x\ dx$$

$$= \frac{1}{2}\int_0^1 x^2 e^{-x^3}\, dx$$

$$= -\frac{1}{6}\int_0^1 e^{-x^3}\, d\left(-x^3\right)$$

$$= -\frac{1}{6}e^{-x^3}\Big|_0^1$$

$$= -\frac{1}{6}\left(e^{-1}-1\right)$$

$$= \frac{1}{6}\left(1-\frac{1}{e}\right)$$

$$= \frac{e-1}{6e}$$

Exercises *Section* 3.1 – Double Integrals over Rectangular Regions

(1 – 18) Evaluate the iterated integral

1. $\displaystyle\int_1^2\int_0^4 2xy\ dydx$

2. $\displaystyle\int_0^2\int_{-1}^1 (x-y)\ dydx$

3. $\displaystyle\int_0^1\int_0^1\left(1-\frac{x^2+y^2}{2}\right)dxdy$

4. $\displaystyle\int_0^3\int_{-2}^0\left(x^2y-2xy\right)dydx$

5. $\displaystyle\int_0^1\int_0^1\frac{y}{1+xy}dxdy$

6. $\displaystyle\int_0^{\ln 2}\int_1^{\ln 5} e^{2x+y}dydx$

7. $\displaystyle\int_0^1\int_1^2 xye^x dydx$

8. $\displaystyle\int_\pi^{2\pi}\int_0^\pi (\sin x+\cos y)dxdy$

9. $\displaystyle\int_1^2\int_1^4\frac{xy}{\left(x^2+y^2\right)^2}dxdy$

10. $\displaystyle\int_1^3\int_1^{e^x}\frac{x}{y}\ dydx$

11. $\displaystyle\int_1^2\int_0^{\ln x} x^3e^y dydx$

12. $\displaystyle\int_1^{10}\int_0^{1/y} ye^{xy}dxdy$

13. $\displaystyle\int_0^1\int_{\sqrt{y}}^{2-\sqrt{y}} xy\ dxdy$

14. $\displaystyle\int_0^1\int_{x^2}^{x}\sqrt{x}\ dydx$

15. $\displaystyle\int_0^{3/2}\int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} ydxdy$

16. $\displaystyle\int_0^2\int_0^{4-x^2} 2x\ dydx$

17. $\displaystyle\int_0^1\int_{2y}^2 4\cos\left(x^2\right)\ dxdy$

18. $\displaystyle\int_0^1\int_{\sqrt[3]{y}}^1\frac{2\pi\sin\pi x^2}{x^2}\ dxdy$

(19 – 26) Evaluate the double integral over the given region *R*.

19. $\displaystyle\iint_R\left(6y^2-2x\right)dA \quad R:\ 0\le x\le 1,\quad 0\le y\le 2$

20. $\displaystyle\iint_R\left(\frac{\sqrt{x}}{y^2}\right)dA \quad R:\ 0\le x\le 4,\quad 1\le y\le 2$

21. $\iint\limits_R y\sin(x+y)\,dA \quad R: \ -\pi \le x \le 0, \quad 0 \le y \le \pi$

22. $\iint\limits_R e^{x-y}\,dA \quad R: \ 0 \le x \le \ln 2, \quad 0 \le y \le \ln 2$

23. $\iint\limits_R \frac{y}{x^2y^2+1}\,dA \quad R: \ 0 \le x \le 1, \quad 0 \le y \le 1$

24. $\iint\limits_R x^{-1/2}e^{y}\,dA$; R is the region bounded by $x=1$, $x=4$, $y=\sqrt{x}$, and $y=0$

25. $\iint\limits_R \left(x^2+y^2\right)dA$; R is the region $\{(x,y): \ 0 \le x \le 2, \ 0 \le y \le x\}$

26. $\iint_R \frac{2y}{\sqrt{x^4+1}}\,dA$; *R is the region bounded by* $x=1, \quad x=2, \quad y=x^{3/2}, \quad y=0$

27. Integrate $f(x,y)=\frac{1}{xy}$ over the ***square*** $1 \le x \le 2, \quad 1 \le y \le 2$

28. Integrate $f(x,y)=y\cos xy$ over the ***rectangle*** $0 \le x \le \pi, \quad 0 \le y \le 1$

29. Find the volume of the region bounded above the paraboloid $z=x^2+y^2$ and below by the square $R: \ -1 \le x \le 1, \quad -1 \le y \le 1$

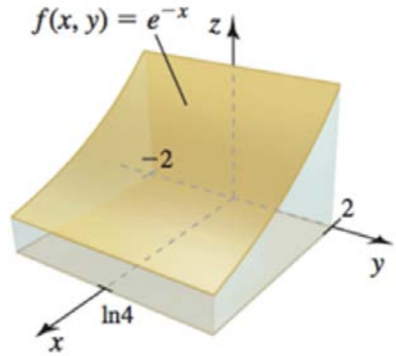
30. Find the volume of the region bounded above the plane $z=\frac{y}{2}$ and below by the rectangle $R: \ 0 \le x \le 4, \quad 0 \le y \le 2$

31. Find the volume of the region bounded above the surface $z=4-y^2$ and below by the rectangle $R: \ 0 \le x \le 1, \quad 0 \le y \le 2$

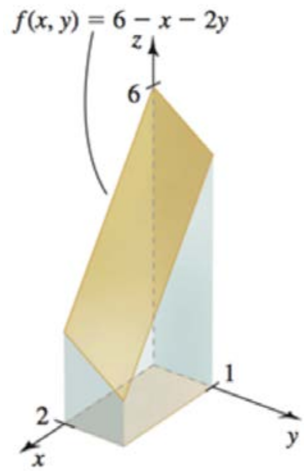
32. Find the volume of the region bounded above the ellipitical paraboloid $z=16-x^2-y^2$ and below by the square $R: \ 0 \le x \le 2, \quad 0 \le y \le 2$

33. Evaluate $\int_0^{1/2}\left(\sin^{-1}[2x]-\sin^{-1}x\right)dx$ by converting it to a double integral.

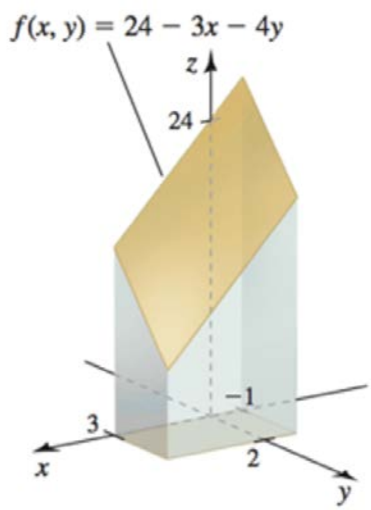
34. Find the volume of the solid beneath the cylinder $f(x, y) = e^{-x}$ and above the region $R = \{(x, y): \ 0 \le x \le \ln 4, \ \ -2 \le y \le 2\}$



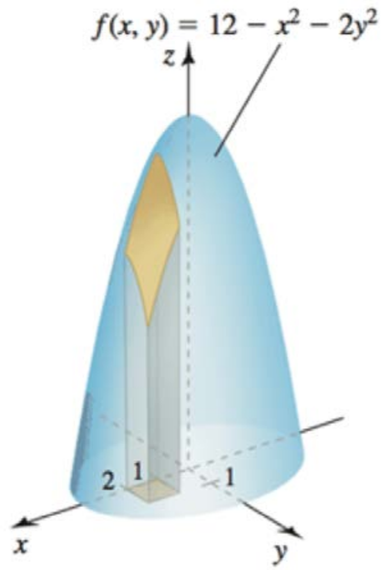
35. Find the volume of the solid beneath the plane $f(x, y) = 6 - x - 2y$ and above the region $R = \{(x, y): \ 0 \le x \le 2, \ \ 0 \le y \le 1\}$



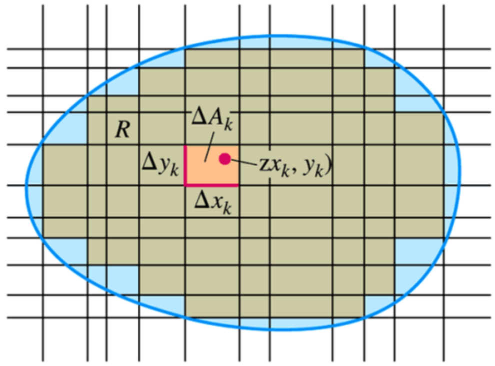
36. Find the volume of the solid beneath the plane $f(x, y) = 24 - 3x - 4y$ and above the region $R = \{(x, y): \ -1 \le x \le 3, \ \ 0 \le y \le 2\}$



37. Find the volume of the solid beneath the paraboloid $f(x, y) = 12 - x^2 - 2y^2$ and above the region $R = \{(x, y):\ 1 \le x \le 2,\ \ 0 \le y \le 1\}$



Section 3.2 – Double Integrals over General Regions



Volumes

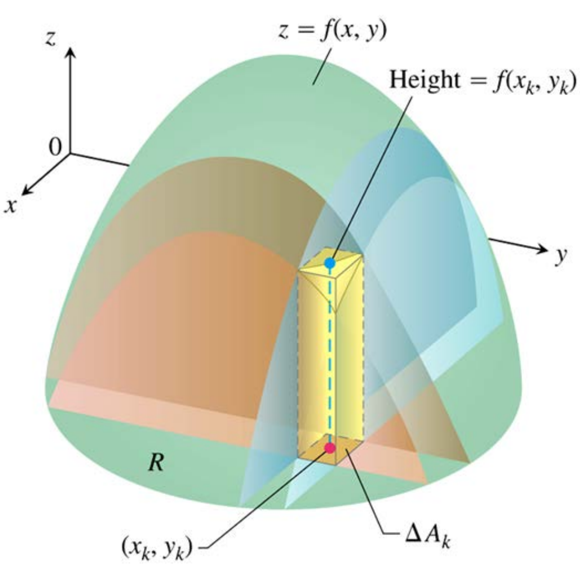
If $f(x,y)$ is positive and continuous over *R*, we define the volume of the solid region between *R* and the surface $z=f(x,y)$ to be $\iint\limits_R f(x,y)\,dA$.

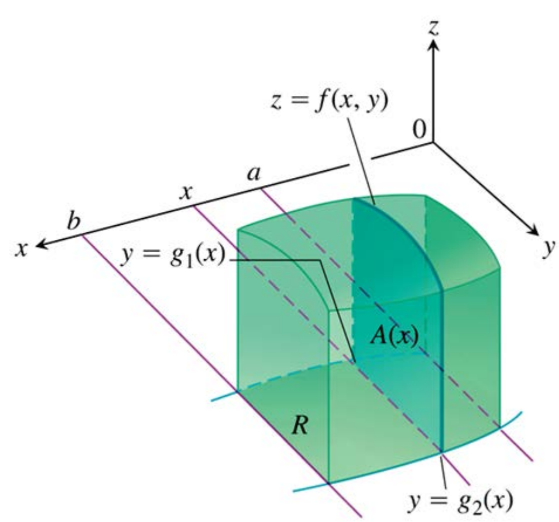
If *R* is a region in the *xy*-plane, bounded ***above*** and ***below*** by the curves $y=g_1(x)$ and $y=g_2(x)$ and on the sides by the lines $x=a,\ x=b$. Calculate the cross-sectional area

$$A(x)=\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy$$

Then integrate $A(x)$ from $x=a$ *to* $x=b$ to get the volume as an iterated integral

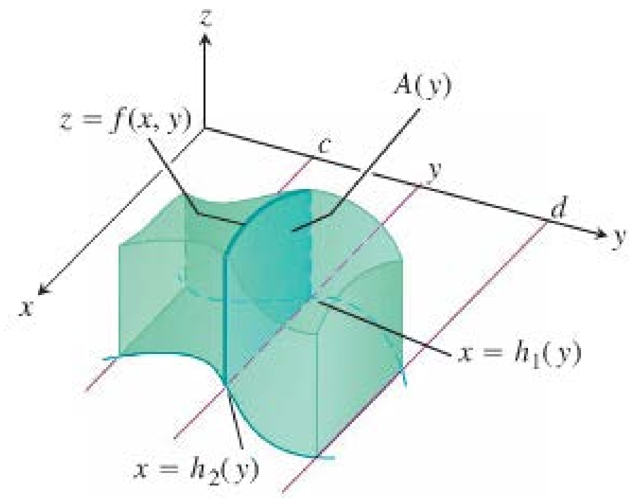
$$V=\int_a^b A(x)\,dx=\int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$





Similarly, if *R* is a region bounded by the curves $x = h_1(y)$ and $x = h_2(y)$ and the lines $y = c,\ y = d$, then the volume calculated by slicing is given by the iterated integral .

$$V = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy$$



$$\int_c^d A(y)\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy$$

$$Volume = \lim \sum f\left(x_k, y_k\right)\Delta A_k = \iint_R f(x,y)\,dA$$

Theorem – Fubini's Theorem

Let $f(x,y)$ is continuous on a region *R*,

1. If *R* is defined by : $a \le x \le b,\ g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on [*a*, *b*], then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dydx$$

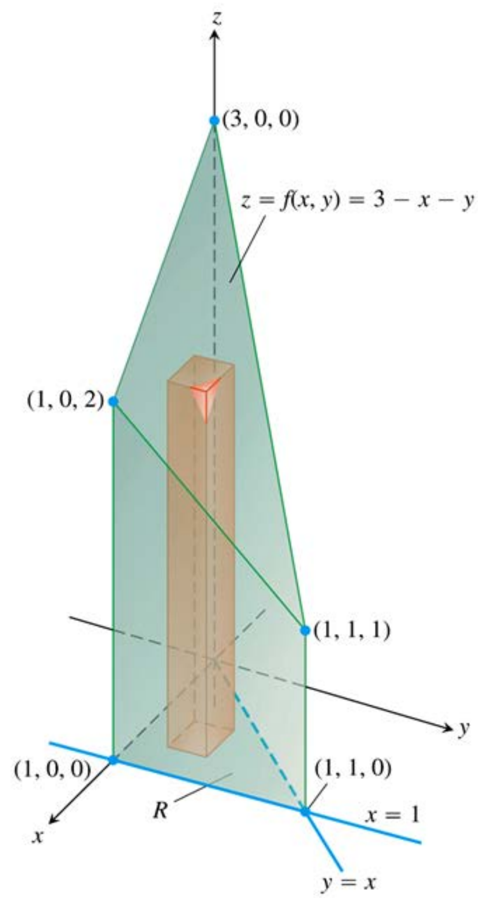
2. If *R* is defined by : $c \le y \le d,\ h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on [*c*, *d*], then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy$$

Example

Find the volume of the prism whose base is the triangle in the *xy*-plane bounded by the *x*-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane $z = f(x, y) = 3 - x - y$

Solution



$0 \le x \le 1, \quad 0 \le y \le x$

$$V = \int_0^1 \int_0^x (3 - x - y)\,dy\,dx$$

$$= \int_0^1 \left[3y - xy - \frac{1}{2}y^2\right]_0^x dx$$

$$= \int_0^1 \left(3x - x^2 - \frac{1}{2}x^2\right) dx$$

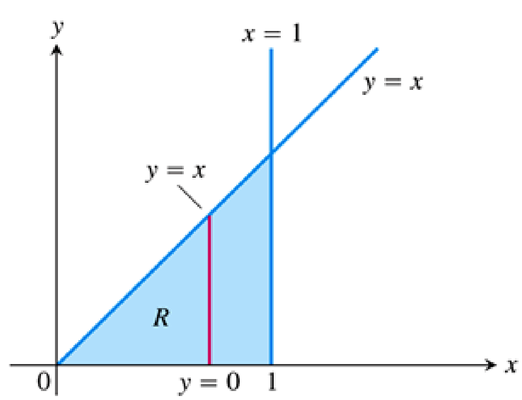
$$= \int_0^1 \left(3x - \frac{3}{2}x^2\right) dx$$

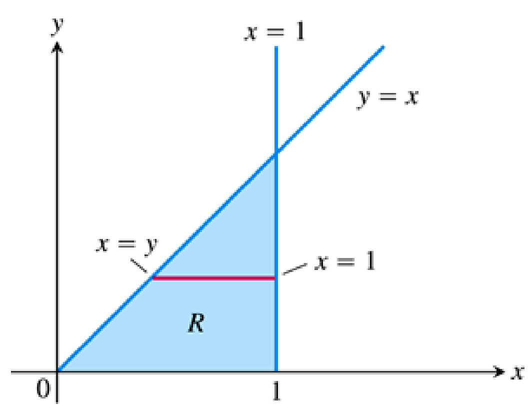
$$= \left[\frac{3}{2}x^2 - \frac{1}{2}x^3\right]_0^1$$

$$= \frac{3}{2} - \frac{1}{2}$$

$$\underline{= 1 \quad unit^3}$$

$$V = \int_0^1 \int_y^1 (3 - x - y)\,dx\,dy = 1$$

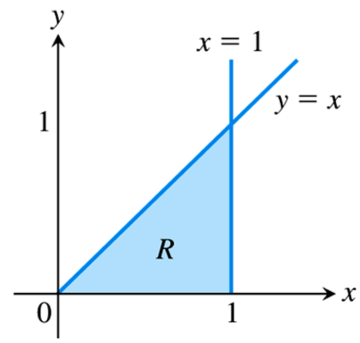




Example

Calculate $\iint\limits_R \frac{\sin x}{x}\,dA$ where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$, and the line $x = 1$.

Solution



$$\int_0^1 \int_0^x \left(\frac{\sin x}{x}\right) dy\ dx = \int_0^1 \left(\frac{\sin x}{x}\, y\right)_0^x dx$$

$$= \int_0^1 \sin x dx$$

$$= -\cos x \Big|_0^1$$

$$= -\cos(1) + 1$$

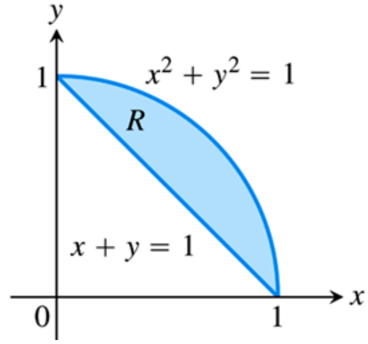
$$\underline{= 1 - \cos 1\,|} \quad \underline{\approx 0.46|}$$

$\int_0^1 \int_y^1 \left(\frac{\sin x}{x}\right) dx\ dy$, we run into a problem because $\int \frac{\sin x}{x} dx$ cannot be expressed in terms of elementary functions.

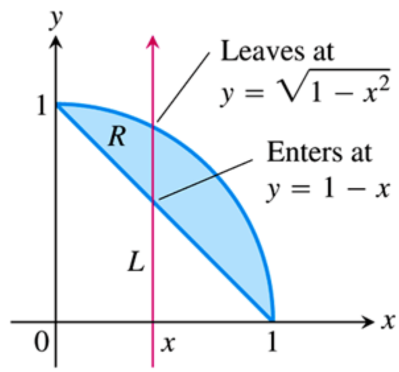
Finding Limits on Intergration

Using Vertical Cross-sections

1. Sketch the region of Integration and label the bounding curves

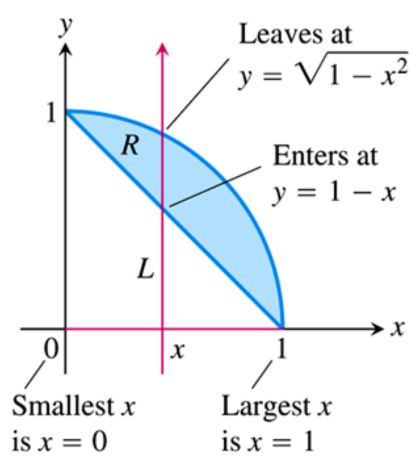


2. *Find the y-limits of integration*. Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x (instead of constants).



3. *Find the x-limits of integration*. Choose x-limits that include all the vertical lines through R. The integral is

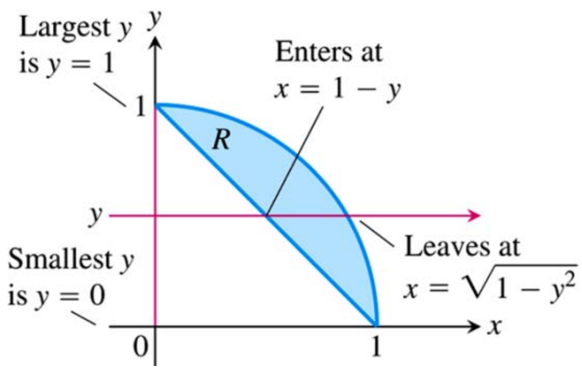
$$\iint_R f(x,y)\,dA = \int_{x=0}^{x=1}\int_{y=1-x}^{y=\sqrt{1-x^2}} f(x,y)\,dy\,dx$$



Using Horizontal Cross-sections

To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines.

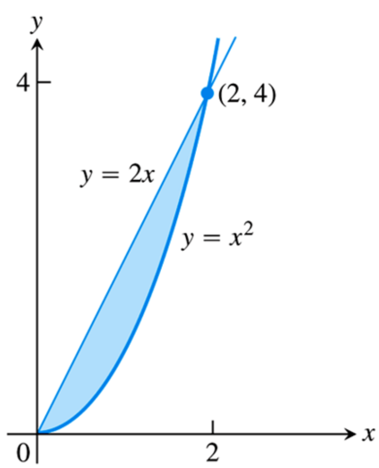
$$\iint_R f(x,y)\,dA = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x,y)\ dxdy$$

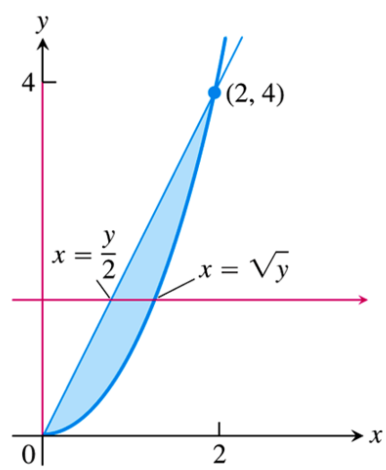


Example

Sketch the region of integration for the integral $\int_0^2 \int_{x^2}^{2x} (4x+2)\,dydx$ and write an equivalent integral with the order of integration reversed.

Solution





The given inequalities are: $x^2 \le y \le 2x \quad and \quad 0 \le x \le 2$

$$\to \begin{cases} y = x^2 & x = \sqrt{y} \\ y = 2x & x = \dfrac{y}{2} \end{cases} \qquad\qquad \to \begin{cases} x = 0 & y = 0 \\ x = 2 & y = 4 \end{cases}$$

The integral is $\int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\,dx\,dy$

- If $f(x,y)$ and $g(x,y)$ are continuous on the bounded region *R*, then the following properties hold

  1. *Constant Multiple*: $\displaystyle\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA$

  2. *Sum and Difference*: $\displaystyle\iint_R \big(f(x,y) \pm g(x,y)\big)\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$

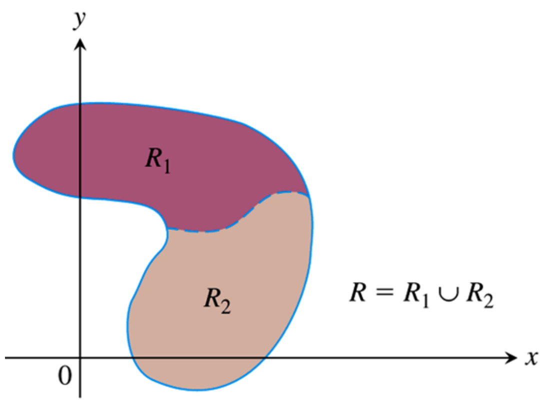
  3. *Domination*:

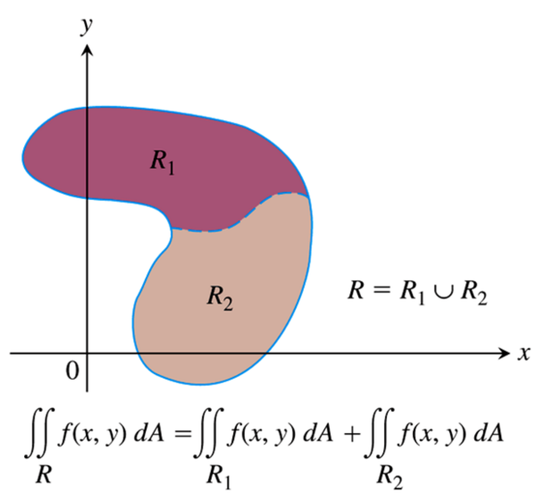
     a) $\displaystyle\iint_R f(x,y)\,dA \ge 0 \quad \text{if} \quad f(x,y) \ge 0 \ \text{on } R$

     b) $\displaystyle\iint_R f(x,y)\,dA \ge \iint_R g(x,y)\,dA \quad \text{if} \quad f(x,y) \ge g(x,y) \ \text{on } R$

  4. *Additivity*: $\displaystyle\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$

     If *R* is the union of two non-overlapping regions R_1 and R_2.

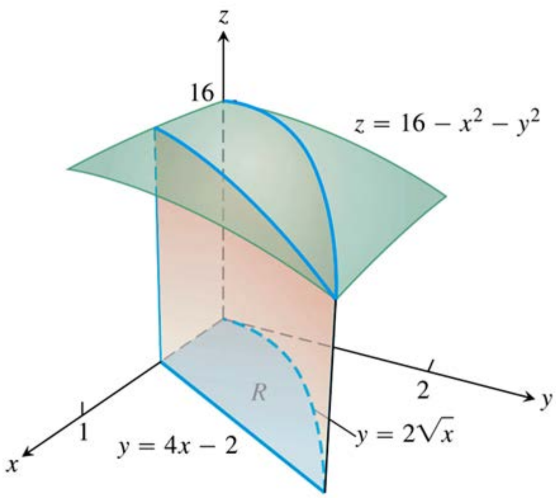


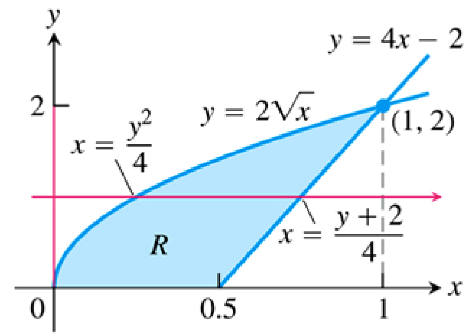


Example

Find the volume of the wedge like solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.

Solution





$$y = 2\sqrt{x} \quad \rightarrow x = \frac{y^2}{4}$$

$$y = 4x - 2 \quad \rightarrow x = \frac{y+2}{4}$$

$$y = 4\frac{y^2}{4} - 2 = y^2 - 2 \quad \rightarrow \quad y^2 - y - 2 = 0 \Rightarrow \underline{y = -1}, \underline{2}$$

$$Volume = \int_0^2 \int_{y^2/4}^{(y+2)/4} \left(16 - x^2 - y^2\right) dx dy$$

$$= \int_0^2 \left(16x - \frac{1}{3}x^3 - y^2 x \Big|_{y^2/4}^{(y+2)/4} dy\right.$$

$$= \int_0^2 \left[\left(16\frac{y+2}{4} - \frac{1}{3}\left(\frac{y+2}{4}\right)^3 - y^2\frac{y+2}{4}\right) - \left(16\frac{y^2}{4} - \frac{1}{3}\frac{y^6}{64} - \frac{y^4}{4}\right)\right] dy$$

$$= \int_0^2 \left[4y + 8 - \frac{1}{192}\left(y^3 + 6y^2 + 12y + 8\right) - \frac{1}{4}y^3 - \frac{1}{2}y^2 - 4y^2 + \frac{1}{192}y^6 + \frac{1}{4}y^4\right] dy$$

$$= \int_0^2 \left(4y + 8 - \frac{1}{192}y^3 - \frac{1}{32}y^2 - \frac{1}{16}y - \frac{1}{24} - \frac{1}{4}y^3 - \frac{9}{2}y^2 + \frac{1}{192}y^6 + \frac{1}{4}y^4\right) dy$$

$$= \int_0^2 \left(\frac{1}{192}y^6 + \frac{1}{4}y^4 - \frac{49}{192}y^3 - \frac{145}{32}y^2 + \frac{63}{16}y + \frac{191}{24}\right) dy$$

$$= \frac{1}{1344}y^7 + \frac{1}{20}y^5 - \frac{49}{768}y^4 - \frac{145}{96}y^3 + \frac{63}{32}y^2 + \frac{191}{24}y \Big|_0^2$$

$$= \frac{2}{21} + \frac{8}{5} - \frac{49}{48} - \frac{145}{12} + \frac{63}{8} + \frac{191}{12}$$

$$= \frac{178}{105} + \frac{513}{48}$$

$$= \frac{62,409}{5,040}\ \ unit^3 \qquad \approx 12.4\ \ unit^3$$

Definition

The area of a closed, bounded plane region R is $\qquad A = \iint\limits_R dA$

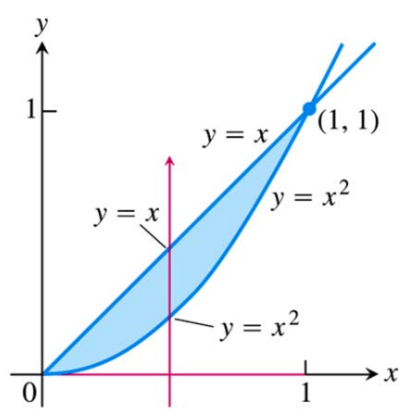
Example

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

Solution

$$y = x = x^2 \quad \rightarrow \quad x = 0,\ 1$$

$$A = \int_0^1 \int_{x^2}^{x} dydx$$



$$= \int_0^1 y \Big|_{x^2}^{x} dx$$

$$= \int_0^1 \left(x - x^2\right) dx$$

$$= \frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_0^1$$

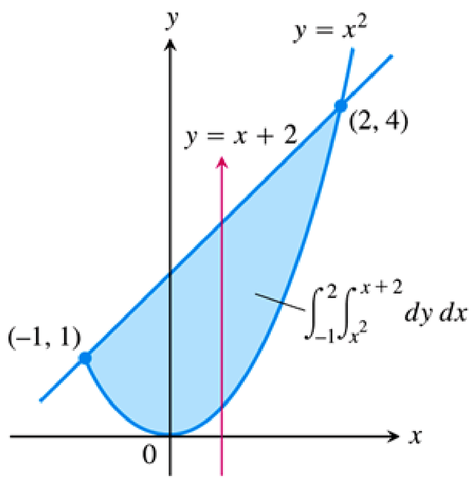
$$= \frac{1}{2} - \frac{1}{3}$$

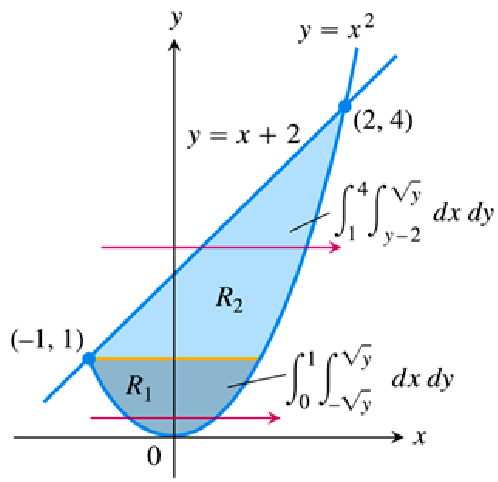
$$= \frac{1}{6}\ \ unit^2$$

Example

Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

Solution





$y = x^2 = x + 2$

$x^2 - x - 2 = 0$

$\underline{x = -1,\ 2}$

$$A = \int_{-1}^{2}\int_{x^2}^{x+2} dydx$$

$$= \int_{-1}^{2} y \Big|_{x^2}^{x+2} dx$$

$$= \int_{-1}^{2} \left(x + 2 - x^2\right) dx$$

$$= \frac{1}{2}x^2 + 2x - \frac{1}{3}x^3 \Big|_{-1}^{2}$$

$$= \frac{1}{2}(4) + 2(4) - \frac{1}{3}(8) - \left(\frac{1}{2}(-1)^2 - 2 + \frac{1}{3}\right)$$

$$\underline{= \frac{9}{2} \quad unit^2}$$

Example

Find the area of the region R between $y = x^2$ and $y^2 = x$.

Solution

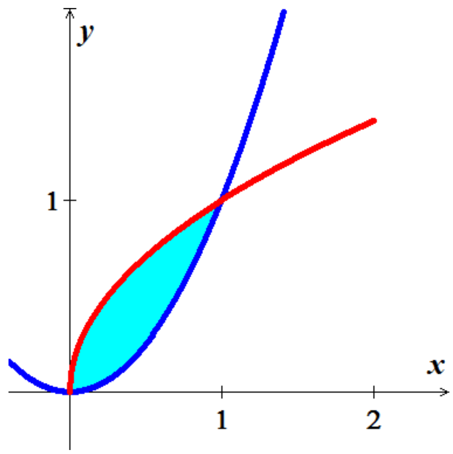
$$y = x^2 = \left(y^2\right)^2$$

$$y = y^4 \quad \rightarrow \quad \underline{y = 0,\ 1|}$$

$$\underline{0 \le y \le 1|}$$

$$y = x^2 \quad \rightarrow \quad x = \sqrt{y}$$

$$\underline{y^2 \le x \le \sqrt{y}\ |}$$



$$Area = \int_0^1 \int_{y^2}^{\sqrt{y}} dxdy$$

$$= \int_0^1 x \Big|_{y^2}^{\sqrt{y}} dy$$

$$= \int_0^1 \left(y^{1/2} - y^2\right) dy$$

$$= \frac{2}{3} y^{3/2} - \frac{1}{3} y^3 \Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{3}$$

$$\underline{= \frac{1}{3}\ unit^2\ |}$$

Average values of f over $\boldsymbol{R}$ $= \dfrac{1}{area\ of\ R} \iint\limits_R f dA$

❖ Average value of f over R $= \dfrac{1}{area\ of\ R} \iint\limits_R f dA \quad \boxed{= \dfrac{2}{\pi}}$

Example

Find the average value of $f(x,y) = x\cos xy$ over the rectangle $R:\ 0 \le x \le \pi, \quad 0 \le y \le 1$.

Solution

$$\int_0^\pi \int_0^1 x\cos xy\ dydx = \int_0^\pi \sin xy \Big|_0^1 dx \qquad\qquad \int x\cos xy\ dy = \sin xy + C$$

$$= \int_0^\pi (\sin x - 0)\ dx$$

$$= \int_0^\pi \sin x\ dx$$

$$= -\cos x \Big|_0^\pi$$

$$= 1 + 1$$

$$\boxed{= 2}$$

Exercises Section 3.2 – Double Integrals over General Regions

(**1 – 4**) Sketch the region of integration and evaluate the integral

1. $\int_0^{\pi}\int_0^{x} x\sin y\,dydx$

2. $\int_0^{\pi}\int_0^{\sin x} y\,dydx$

3. $\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\,dxdy$

4. $\int_1^{4}\int_0^{\sqrt{x}} \frac{3}{2}e^{y/\sqrt{x}}\,dydx$

5. Integrate $f(x,y)=\frac{x}{y}$ over the region in the first quadrant bounded by the lines $y=x,\quad y=2x,\quad x=1,\quad and\quad x=2$

6. Integrate $f(x,y)=x^2+y^2$ over the triangular region with vertices $(0,0),\quad (1,0)\quad and\quad (0,1)$

7. Integrate $f(s,t)=e^s\ln t$ over the region in the first quadrant of the *st*-plane that lies above the curve $s=\ln t$ from $t=1$ to $t=2$.

8. Evaluate $\int_{-2}^{0}\int_{v}^{-v} 2\,dpdv$

9. Evaluate $\int_{-\pi/3}^{\pi/3}\int_0^{\sec t} 3\cos t\;dudt$

(**10 – 13**) Sketch the region of integration, reverse the order of integration, and evaluate the integral

10. $\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\,dydx$

11. $\int_0^{2}\int_x^{2} 2y^2\sin xy\;dydx$

12. $\int_0^{1/16}\int_{y^{1/4}}^{1/2} \cos\left(16\pi x^5\right)dxdy$

13. $\int_0^{2}\int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,dydx$

14. Find the volume of the region bounded above the paraboloid $z=x^2+y^2$ and below by the triangle enclosed by the lines $y=x$, $x=0$, and $x+y=2$ in the *xy*-plane

15. Find the volume of the solid that is bounded above the cylinder $z=x^2$ and below by the region enclosed by the parabola $y=2-x^2$ and the line $y=x$ in the *xy*-plane

16. Find the volume of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2 + y^2 = 4$ and the plane $z + y = 3$

17. Find the volume of the solid that is bounded on the front and back by the planes $x = 2$, and $x = 1$, on the sides by the cylinders $y = \pm\frac{1}{x}$ and above and below the planes $z = x + 1$ and $z = 0$.

18. Find the volume under the parabolic cylinder $z = x^2$ above the region enclosed by the parabola $y = 6 - x^2$ and the line $y = x$ in the *xy*-plane

19. Find the area of the region enclosed by the line $y = 2x + 4$ and the parabola $y = 4 - x^2$ in the *xy*-plane.

20. Find the area of the region enclosed by the coordinate axes and the lin $x = 0$ e $x + y = 2$.

21. Find the area of the region enclosed by the lines , $y = 2x$, and $y = 4$

22. Find the area of the region enclosed by the parabola $x = y - y^2$ and the line $y = -x$.

23. Find the area of the region enclosed by the curve $y = e^x$ and the lines $y = 0$, $x = 0$ and $x = \ln 2$

24. Find the area of the region enclosed by the curve $y = \ln x$ and $y = 2\ln x$ and the lines $x = e$ in the first quadrant.

25. Find the area of the region enclosed by the lines $y = x$, $y = \frac{x}{3}$, and $y = 2$

26. Find the area of the region enclosed by the lines $y = x - 2$ and $y = -x$ and the curve $y = \sqrt{x}$

27. Find the area of the region enclosed by the parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$

28. Find the area of the region bounded by the lines $y = -x - 4$, $y = x$, and $y = 2x - 4$. Make a sketch of the region.

29. Find the area of the region bounded by the lines $y = |x|$ and $y = 20 - x^2$. Make a sketch of the region.

30. Find the area of the region bounded by the lines $y = x^2$ and $y = 1 + x - x^2$. Make a sketch of the region.

(31 – 34) Find the area of the region

31. $\displaystyle\int_0^6 \int_{y^2/3}^{2y} dxdy$

32. $\displaystyle\int_0^{\pi/4} \int_{\sin x}^{\cos x} dydx$

33. $\displaystyle\int_{-1}^2 \int_{y^2}^{y+2} dxdy$

34. $\displaystyle\int_0^2 \int_{x^2-4}^0 dydx + \int_0^4 \int_0^{\sqrt{x}} dydx$

35. Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2, \quad 0 \le y \le 2$

36. Find the average height of $f(x,y) = \dfrac{1}{xy}$ over the square $\ln 2 \le x \le 2\ln 2, \quad \ln 2 \le y \le 2\ln 2$

(**37 – 40**) Evaluate the integral over the given region

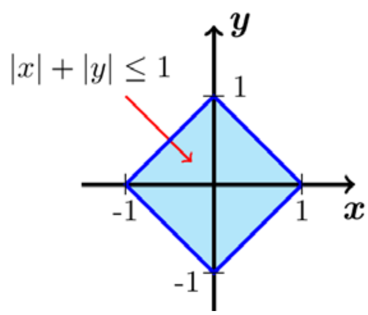
37. $\displaystyle\iint_R y\,dA \quad R = \left\{(x,\ y):\ \ 0 \le x \le \frac{\pi}{3},\ \ 0 \le y \le \sec x\right\}$

38. $\displaystyle\iint_R (x+y)\,dA \quad R \text{ is the region bounded by } y = \frac{1}{x} \text{ and } \ y = \frac{5}{2} - x$

39. $\displaystyle\iint_R \frac{xy}{1+x^2+y^2}\,dA \quad R = \left\{(x,\ y):\ \ 0 \le y \le x,\ \ 0 \le x \le 2\right\}$

40. $\displaystyle\iint_R x\sec^2 y\,dA \quad R = \left\{(x,\ y):\ \ 0 \le y \le x^2,\ \ 0 \le x \le \frac{\sqrt{\pi}}{2}\right\}$

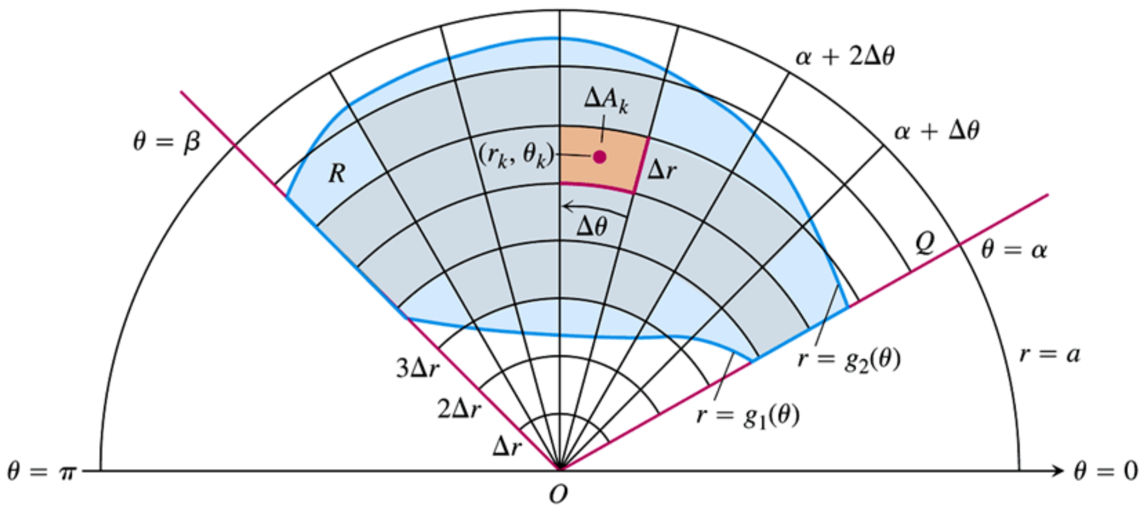
41. Consider the region $\quad R = \left\{(x,\ y):\ \ |x| + |y| \le 1\right\}$



a) Use a double integral to show that the area of *R* is 2.
b) Find the volume of the square column whose base is *R* and whose upper surface is $z = 12 - 3x - 4y$.
c) Find the volume of the solid above *R* and beneath the cylinder $x^2 + z^2 = 1$.
d) Find the volume of the pyramid whose base is *R* and whose vertex is on the *z*-axis at $(0,\ 0,\ 6)$

Section 3.3 – Double Integrals in Polar Coordinates

Integrals in Polar Coordinates



$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\Delta A_k$$

If f is continuous throughout R, this sum will approach a limit as Δr and $\Delta\theta$ go to zero. The limit is called the double integral of f over R.

$$\lim_{n\to\infty} S_n = \iint_R f(r,\theta)\, dA$$

However, the area of a wedge-shaped sector of a circle having radius r and angle θ is

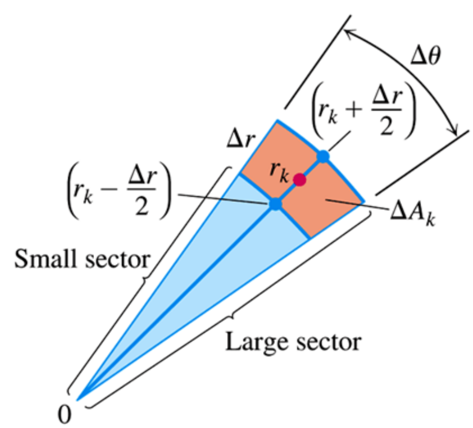
$$A = \frac{1}{2}\theta \cdot r^2$$

Inner radius: $\quad \frac{1}{2}\left(r_k - \frac{\Delta r}{2}\right)^2 \cdot \Delta\theta$

outer radius: $\quad \frac{1}{2}\left(r_k + \frac{\Delta r}{2}\right)^2 \cdot \Delta\theta$

$$\Delta A_k = \left(\begin{matrix} \text{area of} \\ \text{large sector} \end{matrix}\right) - \left(\begin{matrix} \text{area of} \\ \text{small sector} \end{matrix}\right)$$

Leads to the formula: $\Delta A_k = r_k \Delta r \Delta\theta$

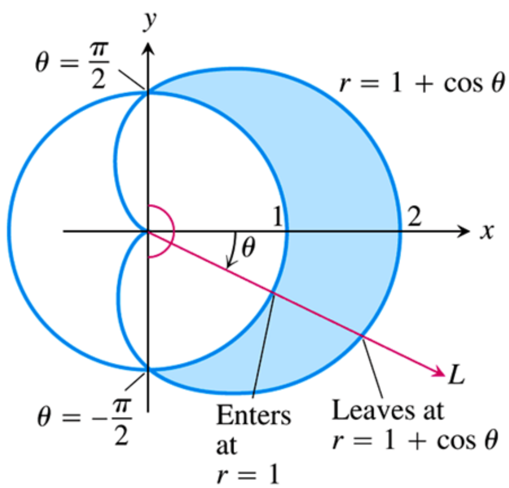


Example

Find the limits of integration for integrating $f(r,\theta)$ over the region *R* that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

Solution

The sketch of the region:



From the graph, we can find the *r - limits of integration*. A typical ray from the origin enters *R* where $r = 1$ and leaves where $r = 1 + \cos\theta$

θ *- limits of integration*: The rays from the origin that intersects *R* run from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$. The integral is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{1+\cos\theta} f(r,\theta) r \ dr \ d\theta$$

Area in Polar Coordinates

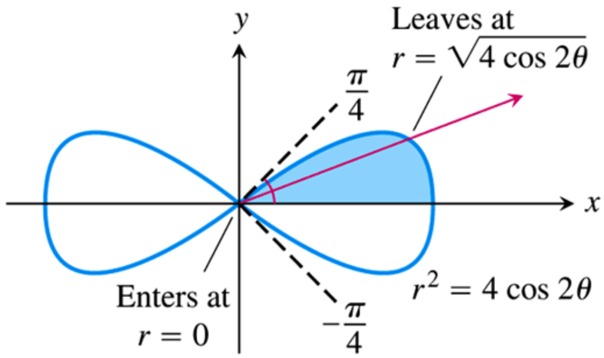
The area of a closed and bounded region *R* in the polar coordinate plane is

$$A = \iint_R r \ dr \ d\theta$$

Example

Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$

Solution



From the graph, we can determine the lemniscate limits of integration, and the total area is 4 times the first-quadrant portion, since it has a form of symmetry.

$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\ dr d\theta$$

$$= 4\int_0^{\pi/4} \frac{1}{2}r^2 \Big|_0^{\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} (2\cos 2\theta)\, d\theta$$

$$= 4\int_0^{\pi/4} \cos 2\theta\ d(2\theta)$$

$$= 4\sin 2\theta \Big|_0^{\pi/4}$$

$$= 4\sin\frac{\pi}{2}$$

$$\underline{= 4\quad unit^2}$$

Changing Cartesian Integrals into Polar Integrals

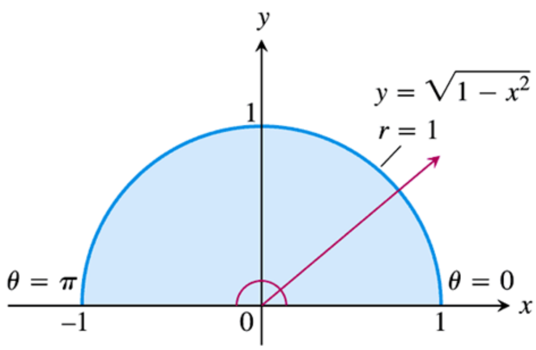
$$\iint_R f(x,y)\,dxdy = \iint_G f(r\cos\theta, r\sin\theta)\,r\ drd\theta$$

Example

Evaluate $\displaystyle\iint_R e^{x^2+y^2}\ dydx$

Where R is the semicircular region bounded by the x-axis and the curve $y=\sqrt{1-x^2}$

Solution



$$\iint_R e^{x^2+y^2}dydx = \int_0^{\pi}\int_0^1 e^{r^2}r\ drd\theta \qquad\qquad d\left(r^2\right)=2rdr$$

$$=\frac{1}{2}\int_0^{\pi} d\theta\ \int_0^1 e^{r^2}\ d\left(r^2\right)$$

$$=\frac{1}{2}\ \theta\ \Big|_0^{\pi}\quad e^{r^2}\ \Big|_0^1$$

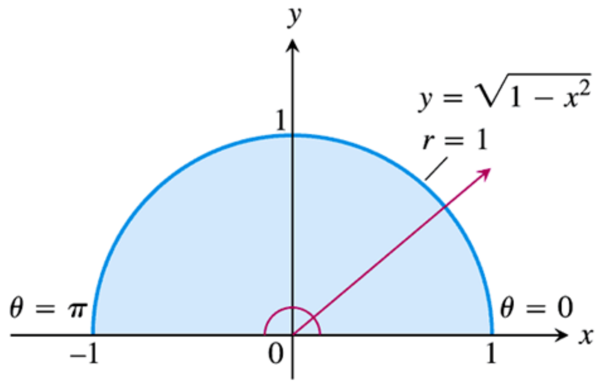
$$=\frac{1}{2}\int_0^{\pi}(e-1)\,d\theta$$

$$=\frac{\pi}{2}(e-1)$$

Example

Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \left(x^2+y^2\right) dydx$

Solution



Since: $0 \le x \le 1 \quad \rightarrow \ \textit{interior of} \ \ x^2+y^2=1 \quad \textit{and in QI}$

Let: $r^2 = x^2 + y^2 \quad \textit{with} \quad 0 \le r \le 1$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \left(x^2+y^2\right) dydx = \int_0^{\pi/2} \int_0^1 \left(r^2\right) r dr d\theta$$

$$= \int_0^{\pi/2} d\theta \int_0^1 r^3 \, dr$$

$$= \theta \Big|_0^{\pi/2} \ \frac{1}{4} r^4 \Big|_0^1$$

$$\boxed{= \frac{\pi}{8}}$$

- Or we can use the integral table to solve it

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \left(x^2+y^2\right) dydx = \int_0^1 \left[x^2\sqrt{1-x^2} + \frac{1}{3}\left(1-x^2\right)^3 \right] dx$$

Example

Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the *xy*-plane.

Solution

The region of integration *R* is the unit circle:

$x^2 + y^2 = 1 \quad \to \quad r = 1, \quad 0 \le \theta \le 2\pi$

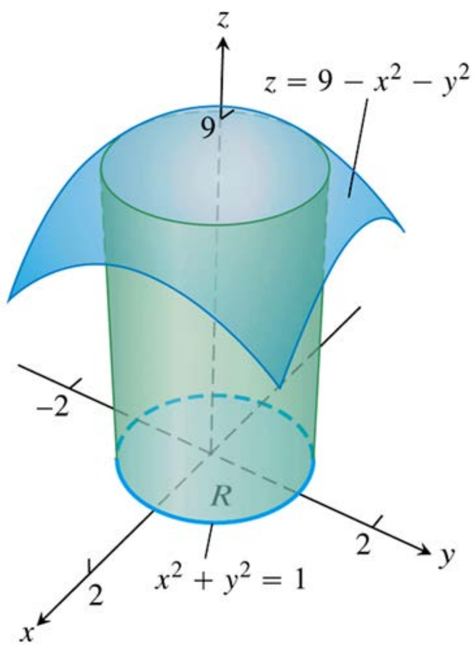
$$Volume = \int_0^{2\pi} \int_0^1 \left(9 - r^2\right) r \; dr d\theta$$

$$= \int_0^{2\pi} d\theta \quad \int_0^1 \left(9r - r^3\right) dr$$

$$= 2\pi \quad \left(\frac{9}{2} r^2 - \frac{1}{4} r^4 \right|_0^1$$

$$= 2\pi \left(\frac{9}{2} - \frac{1}{4}\right)$$

$$\boxed{= \frac{17\pi}{2} \quad unit^3}$$



Example

Using the polar integration, find the area of the region *R* in the *xy*-plane enclosed by the circle $x^2 + y^2 = 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$.

Solution

The $y = \sqrt{3}x$ has a slope of $\sqrt{3} = \tan\theta \;\Rightarrow\; \boxed{\theta = \frac{\pi}{3}}$

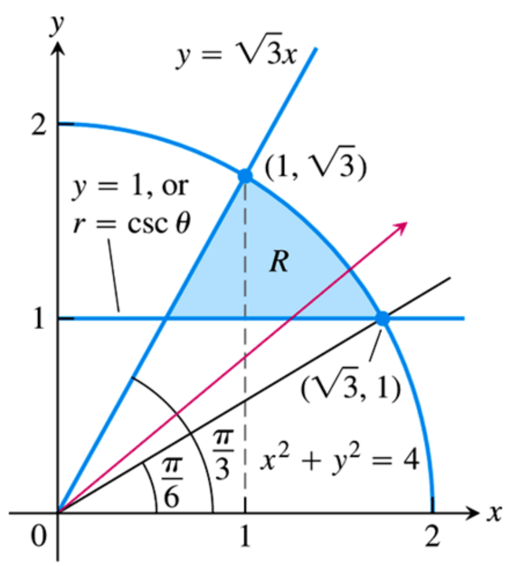
Line $y = 1$ intersects $x^2 + y^2 = 4$

when $x^2 + 1 = 4 \to x = \sqrt{3}$.

A line from origin to $\left(\sqrt{3},\, 1\right)$ has a slope of

$$\frac{1}{\sqrt{3}} = \tan\theta \to \boxed{\theta = \frac{\pi}{6}}$$

$$\therefore \quad \boxed{\frac{\pi}{6} \le \theta \le \frac{\pi}{3}}$$



The polar coordinate *r* varies from the horizontal line $y = 1$ to the circle $x^2 + y^2 = 4$.

Substituting $r\sin\theta$ for *y*:

$$y = 1 \to r\sin\theta = 1$$

$$r = \frac{1}{\sin\theta} = \underline{\csc\theta}\,\big|$$

The radius of the circle is 2.

$$\therefore \quad \underline{\csc\theta \le r \le 2}\,\big|$$

$$Area = \int_{\pi/6}^{\pi/3}\int_{\csc\theta}^{2} r\, dr d\theta$$

$$= \int_{\pi/6}^{\pi/3} \frac{1}{2}r^2 \Big|_{\csc\theta}^{2} d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{\pi/3}\left(4 - \csc^2\theta\right) d\theta$$

$$= \frac{1}{2}(4\theta + \cot\theta \Big|_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2}\left[\frac{4\pi}{3} + \frac{1}{\sqrt{3}} - \left(\frac{4\pi}{6} + \sqrt{3}\right)\right]$$

$$= \frac{1}{2}\left(\frac{2\pi}{3} + \frac{\sqrt{3}}{3} - \sqrt{3}\right)$$

$$= \frac{1}{2}\left(\frac{2\pi - 2\sqrt{3}}{3}\right)$$

$$= \frac{\pi - \sqrt{3}}{3} \quad unit^2$$

Example

Evaluate the double integral: $\iint e^{-x^2 - y^2} dA$

In the first quadrant and bounded by the circle $x^2 + y^2 = a^2$ and the coordinate axes.

Solution

$$x^2 + y^2 = r^2$$

$$\underline{0 \le r \le a}\,\big|$$

In QI: $\underline{0 \le \theta \le \frac{\pi}{2}}\,\big|$

$$\iint e^{-x^2-y^2}\,dA = \int_0^{\frac{\pi}{2}}\int_0^{a} e^{-r^2} r\ dr d\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta \ \int_0^{a} e^{-r^2}\ d\left(-r^2\right)$$

$$= -\frac{1}{2}\theta \Big|_0^{\frac{\pi}{2}} \ \ e^{-r^2}\Big|_0^{a}$$

$$= -\frac{1}{2}\left(\frac{\pi}{2}\right)\left(e^{-a^2} - 1\right)$$

$$\boxed{= \frac{\pi}{4}\left(1-e^{-a^2}\right)}$$

Exercises Section 3.3 – Double Integrals in Polar Coordinates

(1 – 16) Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

1. $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dydx$

2. $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} \left(x^2+y^2\right)dxdy$

3. $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dydx$

4. $\displaystyle\int_{0}^{6}\int_{0}^{y} xdxdy$

5. $\displaystyle\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}dydx$

6. $\displaystyle\int_{0}^{\ln 2}\int_{0}^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}dxdy$

7. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right)dxdy$

8. $\displaystyle\int_{1}^{2}\int_{0}^{\sqrt{2x-x^2}} \frac{1}{\left(x^2+y^2\right)^2}dydx$

9. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2dydx}{\left(1+x^2+y^2\right)^2}$

10. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right)dxdy$

11. $\displaystyle\int_{0}^{\infty}\int_{0}^{\infty} \frac{1}{\left(1+x^2+y^2\right)^2}dxdy$

12. $\displaystyle\int_{0}^{3}\int_{0}^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\ dydx$

13. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(x^2+y^2\right)^{3/2}\ dydx$

14. $\displaystyle\int_{-4}^{4}\int_{0}^{\sqrt{16-y^2}} \left(16-x^2-y^2\right)\ dxdy$

15. $\displaystyle\int_{0}^{\frac{\pi}{4}}\int_{0}^{\sec\theta} r^3\ drd\theta$

16. $\displaystyle\int_{0}^{\frac{\pi}{2}}\int_{1}^{\infty} \frac{\cos\theta}{r^3}r\ drd\theta$

17. Find the area of the region cut from the first quadrant by the curve $r=2(2-\sin 2\theta)^{1/2}$

18. Find the area of the region lies inside the cardioid $r=1+\cos\theta$ and outside the circle $r=1$

19. Find the area enclosed by one leaf of the rose $r=12\cos 3\theta$

20. Find the area of the region common to the interiors of the cardioids $r=1+\cos\theta$ and $r=1-\cos\theta$

21. Find the area of the region bounded by all leaves of the rose $r=3\cos 2\theta$

22. Find the area of the region inside both the circles $r=2$ and $r=4\cos\theta$

23. Find the area of the region that lies inside both the cardioids $r = 2 - 2\cos\theta$ and $r = 2 + 2\cos\theta$

24. Find the area of the annular region $\{(r,\ \theta):\ \ 1 \le r \le 2,\quad 0 \le \theta \le 2\pi\}$

25. Find the area of the region bounded by the cardioid $r = 2(1 - \sin\theta)$

26. Find the area of the region bounded by all leaves of the rose $r = 2\cos 3\theta$

27. Find the area of the region inside both the cardioid $r = 1 - \cos\theta$ and the circle $r = 1$

28. Find the area of the region inside both the cardioid $r = 1 + \sin\theta$ and the cardioid $r = 1 + \cos\theta$

29. Find the area of the region bounded by the spiral $r = 2\theta$, for $0 \le \theta \le \pi$, and the x-axis.

30. Find the area of the region inside the limaçon $r = 1 + \frac{1}{2}\cos\theta$

31. Find the area of the region bounded by $r = 2\sin 2\theta$

32. Find the area of the region bounded by $r^2 = 2\sin 2\theta$

33. Find the area of the region outside the circle $r = 1$ and inside the rose $r = 2\sin 3\theta$ in QI

34. Find the area of the region outside the circle $r = \frac{1}{2}$ and inside the circle $r = 1 + \cos\theta$

35. Integrate $f(x, y) = \dfrac{\ln\left(x^2 + y^2\right)}{\sqrt{x^2 + y^2}}$ over the region $1 \le x^2 + y^2 \le e$

36. The region enclosed by the lemniscates $r^2 = 2\cos 2\theta$ is the base of a solid right cylinder whose top is bounded by the sphere $z = \sqrt{2 - r^2}$. Find the cylinder's volume.

37. Evaluate $\displaystyle\iint_R (x + y)\,dA$; R is the disk bounded by circle $r = 4\sin\theta$

38. Find the volume of the solid bounded above by the paraboloid $z = 2 - x^2 - y^2$ and below by the plane $z = 1$

39. Find the volume of the solid bounded above by the paraboloid $z = 8 - x^2 - 3y^2$ and below by the hyperbolic paraboloid $z = x^2 - y^2$

(40 – 51) Evaluate the integral over R using polar coordinates

40. $\displaystyle\iint_R \left(x^2 + y^2\right) dA;\quad R = \{(r,\ \theta):\ \ 0 \le r \le 4,\quad 0 \le \theta \le 2\pi\}$

41. $\iint\limits_R 2xy\,dA;\quad R=\left\{(r,\ \theta):\ 1\le r\le 3,\quad 0\le\theta\le\frac{\pi}{2}\right\}$

42. $\iint\limits_R 2xy\ dA;\quad R=\left\{(x,\ y):\ x^2+y^2\le 9,\quad y\ge 0\right\}$

43. $\iint\limits_R \frac{dA}{1+x^2+y^2};\quad R=\{(r,\ \theta):\ 1\le r\le 2,\quad 0\le\theta\le\pi\}$

44. $\iint\limits_R \frac{dA}{\sqrt{16-x^2-y^2}};\quad R=\left\{(x,\ y):\ x^2+y^2\le 4,\quad y\ge 0\right\}$

45. $\iint\limits_R \frac{dA}{\sqrt{16-x^2-y^2}};\quad R=\left\{(x,\ y):\ x^2+y^2\le 4,\quad x,\ y\ge 0\right\}$

46. $\iint\limits_R e^{-x^2-y^2}dA;\quad R=\left\{(x,\ y):\ x^2+y^2\le 9\right\}$

47. $\iint\limits_R \sqrt{x^2+y^2}\ dA;\quad R=\{(x,\ y):\ y\le x\le 1,\quad 0\le y\le 1\}$

48. $\iint\limits_R \sqrt{x^2+y^2}\ dA;\quad R=\left\{(x,\ y):\ 1\le x^2+y^2\le 2\right\}$

49. $\iint\limits_R \frac{dA}{\left(x^2+y^2\right)^{5/2}};\quad R=\{(r,\ \theta):\ 1\le r\le\infty,\quad 0\le\theta\le 2\pi\}$

50. $\iint\limits_R e^{-x^2-y^2}dA;\quad R=\left\{(r,\ \theta):\ 0\le r\le\infty,\quad 0\le\theta\le\frac{\pi}{2}\right\}$

51. $\iint\limits_R \frac{dA}{\left(1+x^2+y^2\right)^2};\quad R\in QI$

52. Which bowl holds more water if it is filled to a depth of four units?

a) The paraboloid $z=x^2+y^2$, for $0\le z\le 4$

b) The cone $z=\sqrt{x^2+y^2}$, for $0\le z\le 4$

c) The hyperboloid $z=\sqrt{1+x^2+y^2}$, for $1\le z\le 5$

d) To what weight (above the bottom of the bowl) must the cone and paraboloid bowls be filled to hold the same volume of water as the hyperboloid bowl filled to a depth of 4 units $(1 \le z \le 5)$

53. Consider the surface $z = x^2 - y^2$

a) Find the region in the *xy*-plane in polar coordinates for which $z \ge 0$.

b) Let $R = \left\{(r,\ \theta):\quad 0 \le r \le a,\quad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}\right\}$, which is a sector of a circle of radius a. Find the volume of the region below the hyperbolic paraboloid and above the region *R*.

54. A cake is shaped like a hemisphere of radius 4 with its base on the *xy*-plane. A wedge of the cake is removed by making two slices from the center of the cake outward, perpendicular to the *xy*-plane and separated by an angle of φ.

a) Use a double integral to find the volume of the slice for $\varphi = \frac{\pi}{4}$.

b) Suppose the cake is sliced by a plane perpendicular to the *xy*-plane at $x = a > 0$. Let *D* be the smaller of the two pieces produced. For what value of *a* is the volume of *D* equal to the volume in part (*a*)?

55. Suppose the density of a thin plate represented by the region *R* is $\rho(r,\ \theta)$ (in units of mass per area). The mass of the plate is $\iint\limits_R \rho(r,\ \theta)\,dA$. Find the mass of the thin half annulus $R = \{(r,\ \theta):\quad 1 \le r \le 4,\quad 0 \le \theta \le \pi\}$ with a density $\rho(r,\ \theta) = 4 + r\sin\theta$

56. An important integral in statistics associated with the normal distribution is $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$. It is evaluated in the following steps.

a) Assume that $I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dxdy$

Where we have chosen the variables of integration to be *x* and *y* and then written the product as an iterated integral. Evaluate this integral in polar coordinates and show that $I = \sqrt{\pi}$. Why is the solution $I = -\sqrt{\pi}$ rejected?

b) Evaluate $\int_0^{\infty} e^{-x^2}\,dx$, $\int_0^{\infty} xe^{-x^2}\,dx$, and $\int_0^{\infty} x^2e^{-x^2}\,dx$.

57. For what values of p does the integral $\displaystyle\iint_R \frac{k}{\left(x^2+y^2\right)^p}\,dA$ exist in the following cases?

a) $R=\{(r,\ \theta):\ 1\le r\le\infty,\ \ 0\le\theta\le 2\pi\}$

b) $R=\{(r,\ \theta):\ 0\le r\le 1,\ \ 0\le\theta\le 2\pi\}$

58. Consider the integral $\displaystyle\iint_R \frac{k}{\left(1+x^2+y^2\right)^2}dA$ where $R=\{(x,\ y):\ 0\le x\le 1,\ \ 0\le y\le a\}$

a) Evaluate I for $a=1$.

b) Evaluate I for arbitrary $a>0$.

c) Let $a\to\infty$ in part (b) to find I over the infinite strip $R=\{(x,\ y):\ 0\le x\le 1,\ \ 0\le y\le\infty\}$

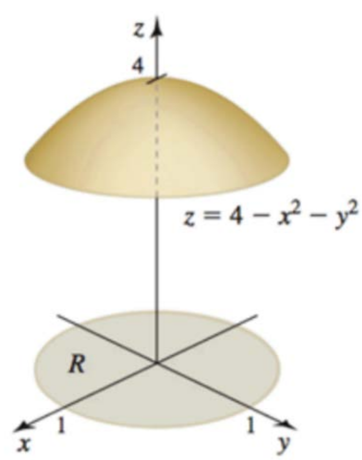
59. In polar coordinates an equation of an ellipse with eccentricity $0<e<1$ and semimajor axis a is

$$r=\frac{a\left(1-e^2\right)}{1+e\cos\theta}$$

a) Write the integral that gives the area of the ellipse.

b) Show that the area of an ellipse is πab, where $b^2=a^2\left(1-e^2\right)$

(**60 – 63**) Find the volume of the solid below the paraboloid $z=4-x^2-y^2$ and above



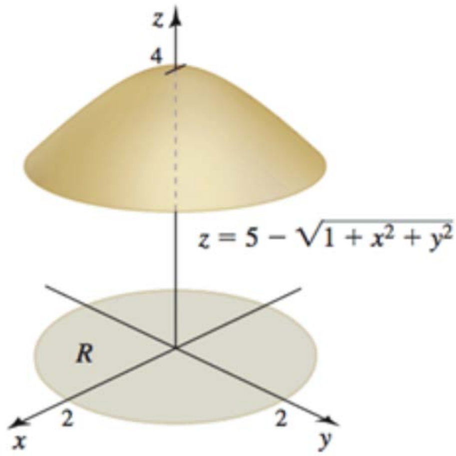
60. $R=\{(r,\ \theta):\ 0\le r\le 1,\ \ 0\le\theta\le 2\pi\}$

61. $R=\{(r,\ \theta):\ 0\le r\le 2,\ \ 0\le\theta\le 2\pi\}$

62. $R=\{(r,\ \theta):\ 1\le r\le 2,\ \ 0\le\theta\le 2\pi\}$

63. $R=\left\{(r,\ \theta):\ 1\le r\le 2,\ \ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\right\}$

(64 – 67) Find the volume of the solid below the hyperboloid $z=5-\sqrt{1+x^2+y^2}$ and above



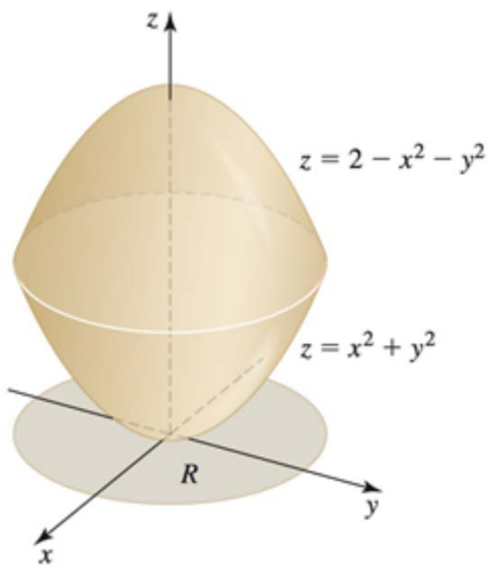
64. $R=\left\{(r,\ \theta):\ \ 0\le r\le 2,\ \ \ 0\le\theta\le 2\pi\right\}$

65. $R=\left\{(r,\ \theta):\ \ 0\le r\le 1,\ \ \ 0\le\theta\le \pi\right\}$

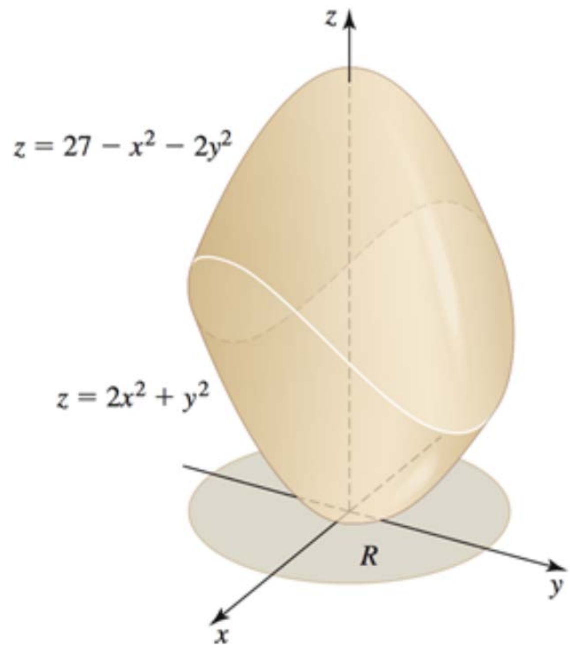
66. $R=\left\{(r,\ \theta):\ \ \sqrt{3}\le r\le 2\sqrt{2},\ \ \ 0\le\theta\le 2\pi\right\}$

67. $R=\left\{(r,\ \theta):\ \ \sqrt{3}\le r\le \sqrt{15},\ \ \ -\frac{\pi}{2}\le\theta\le \pi\right\}$

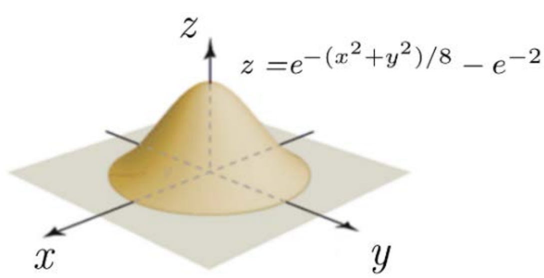
68. Find the volume of the solid between the paraboloids $z=x^2+y^2$ and $z=2-x^2-y^2$



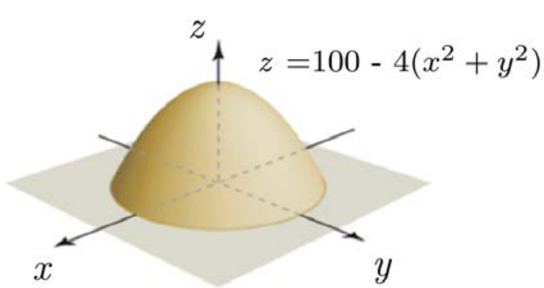
69. Find the volume of the solid between the paraboloids $z = 2x^2 + y^2$ and $z = 27 - x^2 - 2y^2$



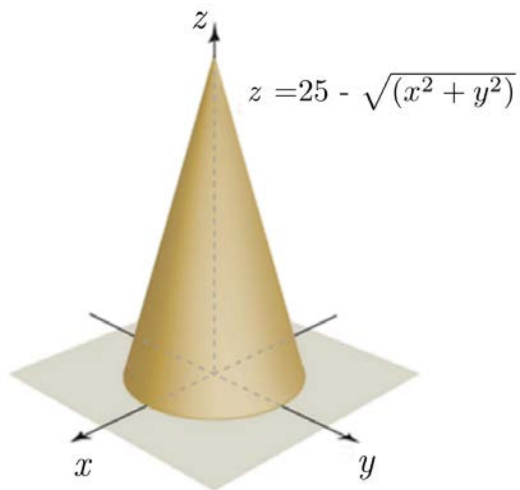
70. Find the volume of island $z = e^{-\left(x^2+y^2\right)/8} - e^{-2}$



71. Find the volume of island $z = 100 - 4\left(x^2 + y^2\right)$



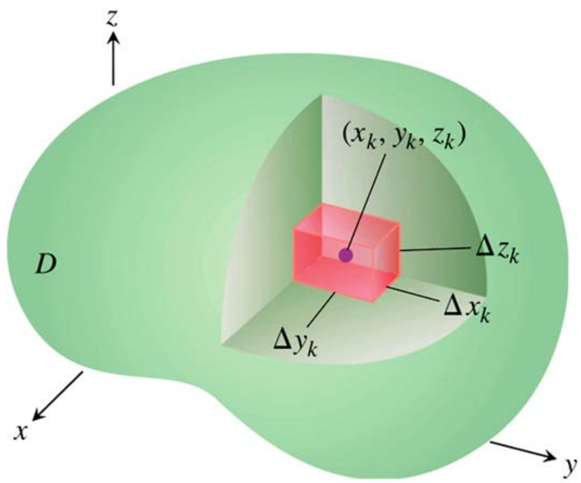
72. Find the volume of island $z = 25 - \sqrt{x^2 + y^2}$



Section 3.4 – Triple Integrals

Triple Integrals

If $F(x, y, z)$ is a function defined on a closed, bounded region D in space, such a solid ball or a lump of clay, then the integral of F over D may be defined in the following way.



$$\Delta V_k = \Delta x_k \Delta y_k \Delta z_k \quad \to \quad S_n = \sum_{k=1}^{n} F\left(x_k, y_k, z_k\right)\Delta V_k$$

The limit of this summation is the triple integral of F over D

$$\lim_{n\to\infty} S_n = \iiint\limits_D F(x, y, z)\ dV \quad or \quad \lim_{\|P\|\to} S_n = \iiint\limits_D F(x, y, z)\ dxdydz$$

Volume of a region in Space

Definition

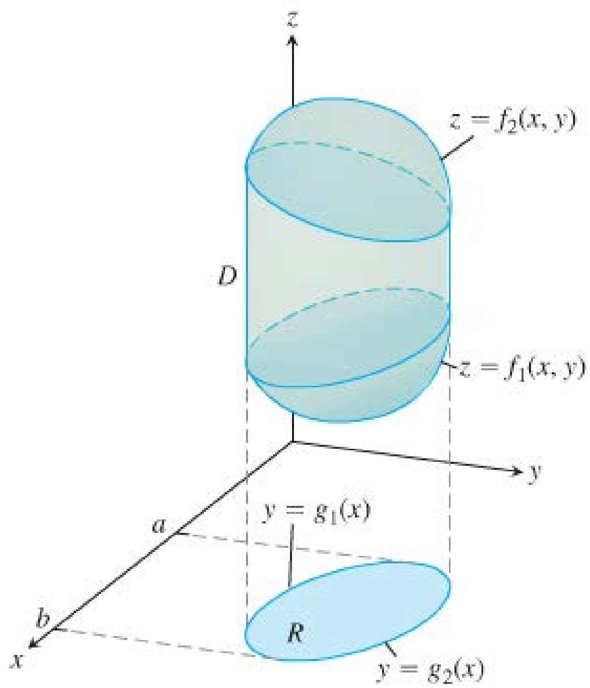
The volume of a closed, bounded region D in space is

$$V = \iiint\limits_D dV$$

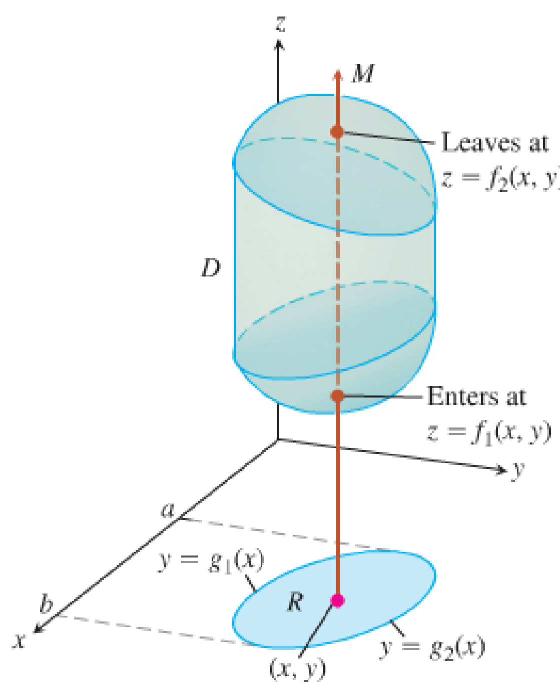
Find Limits of Integration in the Order *dz dy dx*

To evaluate $\displaystyle\iiint_D F(x, y, z)\ dV$

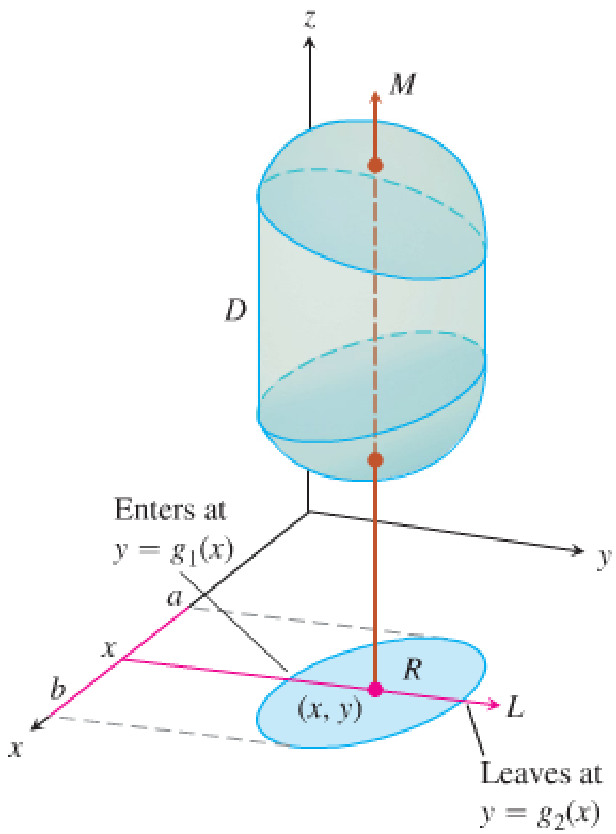
1. ***Sketch***: Sketch the region D along with its "shadow" R (vertical projection) in the *xy*-plane. Label the upper and lower bounding surfaces of D and R.



2. ***Find the z-limits of integration***: Draw a line M passing through (x, y) in R parallel to the z-axis. As z increases, M enters D at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$.



3. ***Find the y-limits of integration***: Draw a line L passing through (x, y) parallel to the y-axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$.



4. ***Find the x-limits of integration***: Choose x-limits that include all lines through R parallel to the y-axis $(x = a \ \ and \ \ x = b)$.

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\ dz\ dy\ dx$$

Example

Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

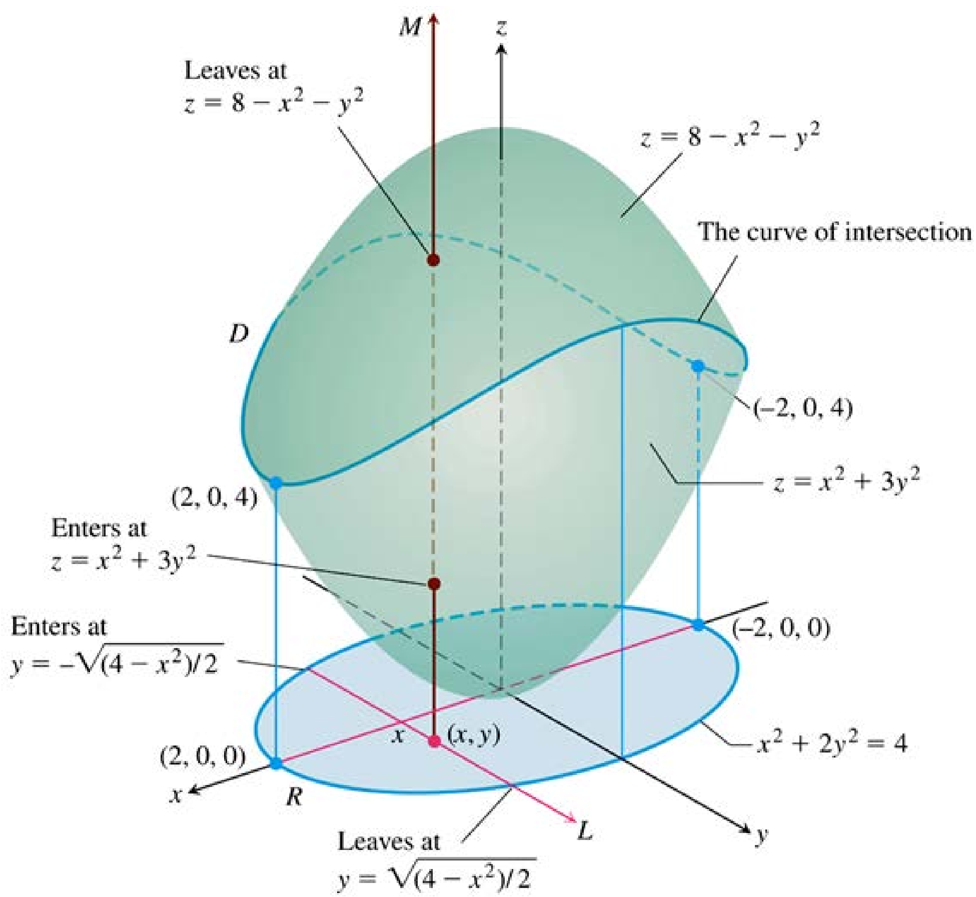
Solution

***z*-limits**: $x^2 + 3y^2 \le z \le 8 - x^2 - y^2$

***y*-limits**: $z = x^2 + 3y^2 = 8 - x^2 - y^2 \quad \to \quad 2x^2 + 4y^2 = 8 \ \Rightarrow \ x^2 + 2y^2 = 4$

$$y^2 = \frac{4-x^2}{2} \quad \Rightarrow y = \pm\sqrt{\frac{4-x^2}{2}} \quad \to -\sqrt{\frac{4-x^2}{2}} \le y \le \sqrt{\frac{4-x^2}{2}}$$

***x*-limits**: $x^2 + 2y^2 = 4 \quad (y = 0) \quad \to x = \pm 2$



$$V = \int_{-2}^{2} \int_{-\sqrt{\left(4-x^2\right)/2}}^{\sqrt{\left(4-x^2\right)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dzdydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{\left(4-x^2\right)/2}}^{\sqrt{\left(4-x^2\right)/2}} z \Big|_{x^2+3y^2}^{8-x^2-y^2} dydx$$

$$= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \left(8-x^2-y^2-x^2-3y^2\right)\,dydx$$

$$= \int_{-2}^{2} \left(\left(8-2x^2\right)y-\frac{4}{3}y^3 \,\Big|_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\,dx\right.$$

$$= \int_{-2}^{2} \left[\left(8-2x^2\right)\sqrt{\frac{4-x^2}{2}}-\frac{4}{3}\left(\frac{4-x^2}{2}\right)^{3/2}+\left(8-2x^2\right)\sqrt{\frac{4-x^2}{2}}-\frac{4}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right]dx$$

$$= \int_{-2}^{2} \left[2\left(8-2x^2\right)\sqrt{\frac{4-x^2}{2}}-\frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right]dx$$

$$= \int_{-2}^{2} \left[2\left(\frac{2}{2}\right)(2)\left(4-x^2\right)\sqrt{\frac{4-x^2}{2}}-\frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right]dx$$

$$= \int_{-2}^{2} \left[8\left(\frac{4-x^2}{2}\right)\left(\frac{4-x^2}{2}\right)^{1/2}-\frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right]dx$$

$$= \int_{-2}^{2} \left[8\left(\frac{4-x^2}{2}\right)^{3/2}-\frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right]dx$$

$$= \frac{16}{3}\int_{-2}^{2} \left(\frac{4-x^2}{2}\right)^{3/2} dx$$

$$= \frac{16}{3(2)^{3/2}}\int_{-2}^{2} \left(4-x^2\right)^{3/2} dx \qquad \frac{16}{3(2)^{3/2}}\frac{2^{1/2}}{2^{1/2}}=\frac{16\sqrt{2}}{3\cdot 4}=\frac{4\sqrt{2}}{3}$$

$$x=2\sin u \quad dx=2\cos u\,du \quad \left(4-x^2=4-4\sin^2 u=4\cos^2 u\right)$$

$$\begin{cases} x=2 & \to u=\sin^{-1}\frac{x}{2}=\sin^{-1}1=\frac{\pi}{2} \\ x=-2 & \to u=\sin^{-1}(-1)=-\frac{\pi}{2} \end{cases}$$

$$= \frac{4\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2}\left(4\cos^2 u\right)^{3/2}\left(2\cos u\,du\right)$$

$$= \frac{4\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} 16(\cos u)^3 (\cos u)\, du$$

$$= \frac{64\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \cos^4 u \ du$$

$$= \frac{64\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(\frac{1+\cos 2u}{2}\right)^2 du$$

$$= \frac{16\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(1 + 2\cos 2u + \cos^2 2u\right) du$$

$$= \frac{16\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(1 + 2\cos 2u + \frac{1}{2} + \frac{1}{2}\cos 4u\right) du$$

$$= \frac{16\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(\frac{3}{2} + 2\cos 2u + \frac{1}{2}\cos 4u\right) du$$

$$= \frac{16\sqrt{2}}{3} \left(\frac{3}{2}u + \sin 2u + \frac{1}{8}\sin 4u \right. \Bigg|_{-\pi/2}^{\pi/2}$$

$$= \frac{16\sqrt{2}}{3}\left[\frac{3\pi}{4} + \sin\pi + \frac{1}{8}\sin 2\pi - \left(-\frac{3\pi}{4} - \sin\pi - \frac{1}{8}\sin 2\pi\right)\right]$$

$$= \frac{16\sqrt{2}}{3}\left(\frac{3\pi}{2}\right)$$

$$\boxed{= 8\pi\sqrt{2} \quad unit^3}$$

Example

Set up the limits of integration for evaluating the triple integral of a function $F(x,y,z)$ over the tetrahedron D with vertices (0, 0, 0), (1, 1, 0), (0, 1, 0), and (0, 1, 1). Use the order of integration $dydzdx$.

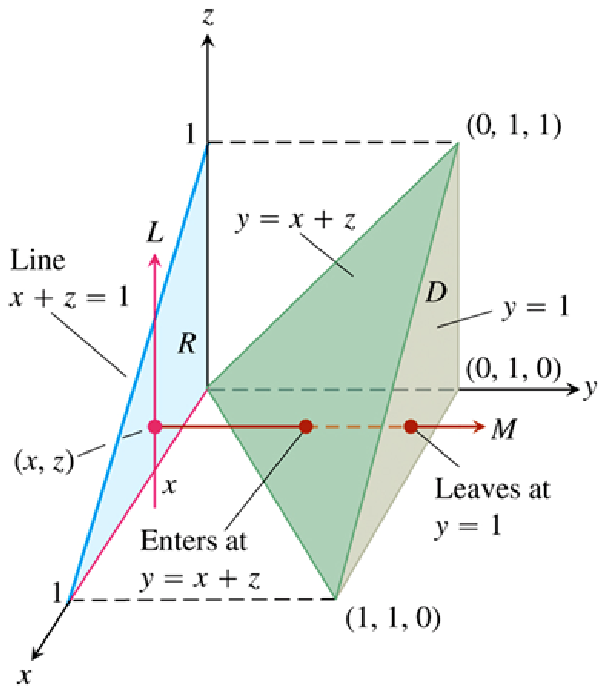
Solution

From the sketch, the upper (right-hand) bounding surface of D lies in the plane $y=1$.

The lower (left-hand) bounding surface lies in the plane $y=x+z$.

The upper boundary of R is the line $z=1-x$.

The lower boundary is the line $z=0$.



***y*-limits**: The line through $(x,\ z)$ in R parallel to the y-axis enters D at $y=x+z$ and leaves at $y=1$.

***z*-limits**: The line through $(x,\ z)$ in R parallel to the z-axis enters R at $z=0$ and leaves at $z=1-x$.

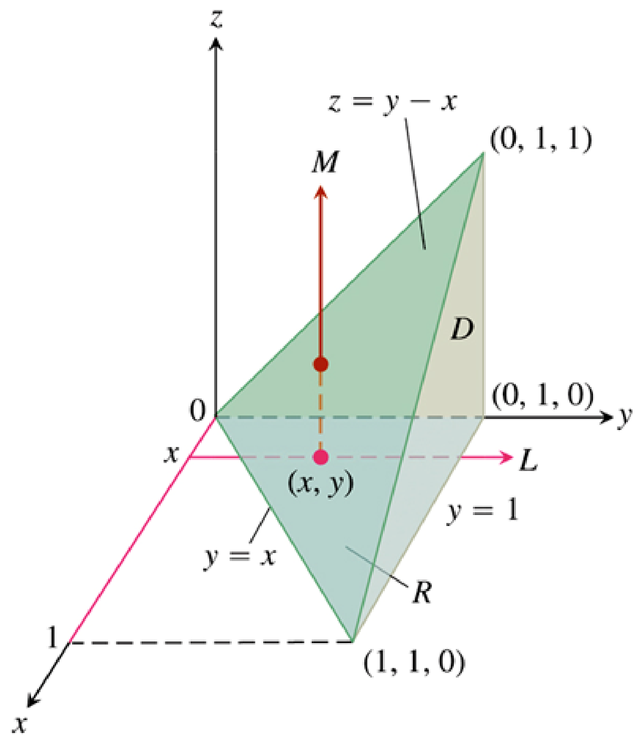
***x*-limits**: $0 \le x \le 1$

$$\int_0^1 \int_0^{1-x} \int_{x+z}^{1} F(x,y,z)\ dydzdx$$

Example

Integrate $F(x,y,z)=1$ over the tetrahedron D in the previous example in the order $dz\ dy\ dx$, and then integrate in the order $dy\ dz\ dx$.

Solution



z-limits of integration: A line *M* parallel to the *z*-axis through a typical point (*x, y*) in the *xy*-plane "shadow" enters the tetrahedron at *z* = 0 and exists through the upper plane where $z = y - x.\ \ 0 \le z \le y - x$

Line is given by: $ax + by + cz = 0$ passes through the 2 points:

$(1,\ 1,\ 0) \rightarrow a + b = 0 \quad \Rightarrow \quad a = -b$

and $(0,\ 1,\ 1) \rightarrow b + c = 0 \quad \Rightarrow \quad c = -b$

$\rightarrow -bx + by - bz = 0$

$-x + y - z = 0 \quad \Rightarrow \quad z = y - x$

y-limits of integration: On the *xy*-plane, where *z* = 0, the sloped side of the tetrahedron crosses the plane along the line $y = x$. A line *L* through (*x, y*) parallel to the *y*-axis enters the shadow in the *xy*-plane at $y = x$ and exists at *y* = 1. $x \le y \le 1$

x-limits of integration: A line *L* parallel to the *y*-axis through a typical point (*x, y*) in the *xy*-plane sweeps out the shadow, where $0 \le x \le 1$ at the point $(1,\ 1,\ 0)$

The integral is: $$\int_0^1 \int_x^1 \int_0^{y-x} F(x,y,z)\ dzdydx$$

$$V=\int_0^1\int_x^1\int_0^{y-x} dzdydx$$

$$=\int_0^1\int_x^1 z\Big|_0^{y-x} dydx$$

$$=\int_0^1\int_x^1 (y-x)\,dydx$$

$$=\int_0^1 \left(\frac{1}{2}y^2-xy\ \Big|_x^1\ dx\right.$$

$$=\int_0^1 \left[\frac{1}{2}-x-\left(\frac{1}{2}x^2-x^2\right)\right] dx$$

$$=\int_0^1 \left(\frac{1}{2}-x+\frac{1}{2}x^2\right)dx$$

$$=\left(\frac{1}{2}x-\frac{1}{2}x^2+\frac{1}{6}x^3\ \Big|_0^1\right.$$

$$=\frac{1}{2}-\frac{1}{2}+\frac{1}{6}$$

$$\boxed{=\frac{1}{6}\quad unit^3}$$

$$V=\int_0^1\int_0^{1-x}\int_{x+z}^1 dydzdx$$

$$=\int_0^1\int_0^{1-x} y\Big|_{x+z}^1\ dzdx$$

$$=\int_0^1\int_0^{1-x} (1-x-z)\,dzdx$$

$$=\int_0^1 \left(z-xz-\frac{1}{2}z^2\ \Big|_0^{1-x}\ dx\right.$$

$$=\int_0^1 \left(1-x-x(1-x)-\frac{1}{2}(1-x)^2\right) dx$$

$$=\int_0^1 \left((1-x)(1-x)-\frac{1}{2}(1-x)^2\right) dx$$

$$= \int_0^1 \left((1-x)^2 - \frac{1}{2}(1-x)^2 \right) dx$$

$$= \frac{1}{2}\int_0^1 (1-x)^2 \, dx$$

$$= -\frac{1}{6}(1-x)^3 \Big|_0^1$$

$$= \frac{1}{6} \quad unit^3$$

Example

Evaluate the integral $\displaystyle\int_0^1 \int_x^{x^2} \int_{xy}^{x^2y^3} xy \; dzdydx$

Solution

$$\int_0^1 \int_x^{x^2} \int_{xy}^{x^2y^3} xy \; dzdydx = \int_0^1 \int_x^{x^2} xyz \Big|_{xy}^{x^2y^3} dydx$$

$$= \int_0^1 \int_x^{x^2} xy\left(x^2y^3 - xy\right) dydx$$

$$= \int_0^1 \int_x^{x^2} \left(x^3y^4 - x^2y^2\right) dydx$$

$$= \int_0^1 \left(\frac{1}{5}x^3y^5 - \frac{1}{3}x^2y^3 \right) \Big|_x^{x^2} dx$$

$$= \int_0^1 \left(\frac{1}{5}x^{13} - \frac{1}{3}x^8 - \frac{1}{5}x^8 + \frac{1}{3}x^5 \right) dx$$

$$= \int_0^1 \left(\frac{1}{5}x^{13} - \frac{8}{15}x^8 + \frac{1}{3}x^5 \right) dx$$

$$= \left(\frac{1}{70}x^{14} - \frac{8}{135}x^9 + \frac{1}{18}x^6 \right|_0^1$$

$$= \frac{1}{70} - \frac{8}{135} + \frac{1}{18}$$

$$= \frac{2}{189}$$

Example

Evaluate the integral $\int_0^a \int_0^{a-z} \int_0^{a-y-z} yz\ dxdydz$

Solution

$$\int_0^a \int_0^{a-z} \int_0^{a-y-z} yz\ dxdydz = \int_0^a \int_0^{a-z} yzx \Big|_0^{a-y-z}\ dydz$$

$$= \int_0^a \int_0^{a-z} \left(ayz - y^2 z - yz^2\right)\ dydz$$

$$= \int_0^a \left(\frac{1}{2}azy^2 - \frac{1}{3}zy^3 - \frac{1}{2}z^2y^2\ \Big|_0^{a-z}\ dz\right.$$

$$= \int_0^a \left(\frac{1}{2}\left(az - z^2\right)(a-z)^2 - \frac{1}{3}z(a-z)^3\right)\ dz$$

$$= \frac{1}{6}\int_0^a (a-z)^2\left(3\left(az - z^2\right) - 2z(a-z)\right)\ dz$$

$$= \frac{1}{6}\int_0^a (a-z)^2\left(3az - 3z^2 - 2az + 2z^2\right)\ dz$$

$$= \frac{1}{6}\int_0^a (a-z)^2\left(az - z^2\right)\ dz$$

$$= \frac{1}{6}\int_0^a z(a-z)^3\ dz$$

$$= \frac{1}{6}\int_0^a \left(a^3z - 3a^2z^2 + 3az^3 - z^4\right)\ dz$$

$$= \frac{1}{6}\left(\frac{1}{2}a^3z^2 - a^2z^3 + \frac{3}{4}az^4 - \frac{1}{5}z^5\ \Big|_0^a\right.$$

$$= \frac{1}{6}\left(\frac{1}{2}a^5 - a^5 + \frac{3}{4}a^5 - \frac{1}{5}a^5\right)$$

$$= \frac{a^5}{6}\left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5}\right)$$

$$= \frac{a^5}{6}\left(\frac{1}{20}\right)$$

$$\boxed{= \frac{a^5}{120}}$$

Example

Find the volume bounded by the cylinder $z = \dfrac{4}{y^2+1}$, bounded by the planes

$y = x, \quad y = 3, \quad x = 0, \quad z = 0$

Solution

$$0 \le z \le \frac{4}{y^2+1}$$

$$0 \le x \le y$$

$$0 \le y \le 3$$

$$V = \int_0^3 \int_0^y \int_0^{\frac{4}{y^2+1}} dzdxdy$$

$$= \int_0^3 \int_0^y z \Big|_0^{\frac{4}{y^2+1}} dxdy$$

$$= \int_0^3 \int_0^y \frac{4}{y^2+1} dxdy$$

$$= \int_0^3 \frac{4}{y^2+1} x \Big|_0^y dy$$

$$= \int_0^3 \frac{4y}{y^2+1} dy$$

$$= 4\int_0^3 \frac{1}{y^2+1} d\left(y^2+1\right)$$

$$= 4 \ln\left(y^2+1\right) \Big|_0^3$$

$$\boxed{= 2 \ln(10) \ \ unit^3}$$

Average Value of a Function in Space

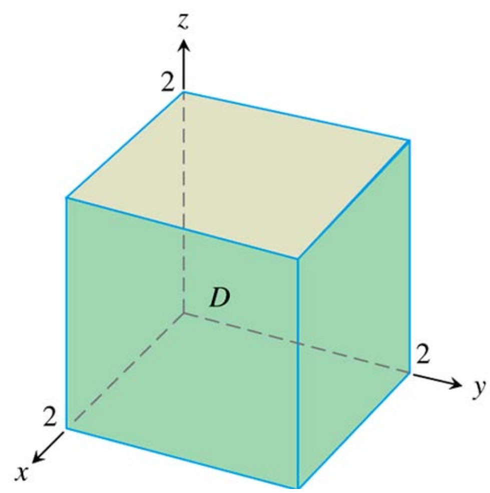
The average value of a function F over a region D in space is defined by the formula

$$\textbf{\textit{Average value}} \text{ of } F \text{ over } D = \frac{1}{\text{volume of } D} \iiint_D F\,dV$$

Example

Find the average of $F(x,y,z) = xyz$ throughout the cubical region D bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$ in the first octant.

Solution



$$Volume = 2 \cdot 2 \cdot 2$$
$$= 8 \ \ unit^3$$

The value of the integral of F over the cube is

$$V = \int_0^2 \int_0^2 \int_0^2 xyz\ dx\ dy\ dz$$
$$= \int_0^2 zdz \int_0^2 ydy \int_0^2 xdx$$
$$= \left(\frac{1}{2}z^2 \right|_0^2 \left(\frac{1}{2}y^2 \right|_0^2 \left(\frac{1}{2}x^2 \right|_0^2$$
$$= \frac{1}{8}(4)(4)(4)$$
$$= 8 \ \ unit^3$$

$$\textbf{\textit{Average value}} \text{ of } xyz \text{ over } cube = \frac{1}{\text{volume of } D} \iiint_{cube} xyz\ dV$$
$$= \left(\frac{1}{8}\right)(8)$$
$$= 1$$

Exercises *Section* 3.4 – Triple Integrals

(1 – 31) Evaluate the integral

1. $\int_0^1\int_0^1\int_0^1 \left(x^2+y^2+z^2\right)dzdydx$

2. $\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dzdxdy$

3. $\int_0^{\pi/6}\int_0^1\int_{-2}^3 y\sin z\ dxdydz$

4. $\int_{-1}^1\int_0^1\int_0^2 (x+y+z)dydxdz$

5. $\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2}} dzdydx$

6. $\int_0^1\int_0^{1-x^2}\int_0^{4-x^2-y} xdzdydx$

7. $\int_0^\pi\int_0^\pi\int_0^\pi \cos(u+v+w)dudvdw$

8. $\int_0^{\pi/4}\int_0^{\ln\sec v}\int_{-\infty}^{2t} e^x dxdtdv$

9. $\int_0^1\int_{-z}^z\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dydxdz$

10. $\int_0^\pi\int_0^y\int_0^{\sin x} dzdxdy$

11. $\int_0^9\int_0^1\int_{2y}^2 \frac{4\sin x^2}{\sqrt{z}}dxdydz$

12. $\int_0^\pi\int_0^\pi\int_0^\pi \cos(x+y+z)dxdydz$

13. $\int_1^e\int_1^x\int_0^z \frac{2y}{z^3}dydzdx$

14. $\int_{\ln 6}^{\ln 7}\int_0^{\ln 2}\int_{\ln 4}^{\ln 5} e^{(x+y+z)}dzdydx$

15. $\int_0^1\int_0^{x^2}\int_0^{x+y} (2x-y-z)dzdydx$

16. $\int_{-2}^2\int_3^6\int_0^2 dxdydz$

17. $\int_{-1}^1\int_{-1}^2\int_0^1 6xyz\ dydxdz$

18. $\int_{-2}^2\int_1^2\int_1^e \frac{xy^2}{z}dzdxdy$

19. $\int_0^{\ln 4}\int_0^{\ln 3}\int_0^{\ln 2} e^{-x+y+z}dxdydz$

20. $\int_0^{\frac{\pi}{2}}\int_0^1\int_0^{\frac{\pi}{2}} \sin\pi x\cos y\sin 2z\ dydxdz$

21. $\int_0^2\int_1^2\int_0^1 yze^x dxdzdy$

22. $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dzdydx$

23. $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} 2xz\ dzdydx$

24. $\int_0^4\int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}}\int_0^{16-\frac{1}{4}x^2-y^2} dzdxdy$

25. $\displaystyle\int_1^6\int_0^{4-\frac{2}{3}y}\int_0^{12-2y-3z}\frac{1}{y}\,dxdzdy$

26. $\displaystyle\int_0^3\int_0^{\sqrt{9-z^2}}\int_0^{\sqrt{1+x^2+z^2}}dydxdz$

27. $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{\sin x}\sin y\;dzdxdy$

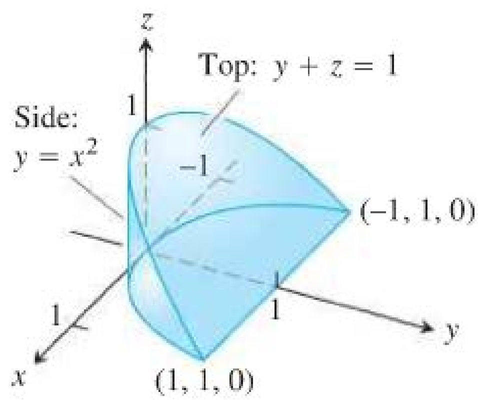
28. $\displaystyle\int_0^{\ln 8}\int_1^{\sqrt{z}}\int_{\ln y}^{\ln 2y}e^{x+y^2-z}dxdydz$

29. $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{2-x}4yz\;dzdydx$

30. $\displaystyle\int_0^2\int_0^4\int_{y^2}^4\sqrt{x}\;dzdxdy$

31. $\displaystyle\int_0^1\int_y^{2-y}\int_0^{2-x-y}xy\;dzdxdy$

32. Here is the region of integration of the integral $\displaystyle\int_{-1}^1\int_{x^2}^1\int_0^{1-y}dzdydx$



a) *dydzdx* ***b***) *dydxdz* ***c***) *dxdydz* ***d***) *dxdzdy* ***e***) *dzdxdy*

(**33 – 37**) Use another order to evaluate

33. $\displaystyle\int_0^5\int_{-1}^0\int_0^{4x+4}dydxdz$

34. $\displaystyle\int_0^1\int_{-2}^2\int_0^{\sqrt{4-y^2}}dzdydx$

35. $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}}dydzdx$

36. $\displaystyle\int_0^4\int_0^{\sqrt{16-x^2}}\int_0^{\sqrt{16-x^2-z^2}}dydzdx$

37. $\displaystyle\int_1^4\int_z^{4z}\int_0^{\pi^2}\frac{\sin\sqrt{yz}}{x^{3/2}}dydxdz$

(**38 – 39**) Evaluate

38. $\iiint\limits_D (xy + xz + yz)\,dV$; $\qquad D = \{(x,\ y,\ z):\ -1 \le x \le 1,\quad -2 \le y \le 2,\quad -3 \le z \le 3\}$

39. $\iiint\limits_D xyze^{-x^2-y^2}\,dV$; $D = \{(x,\ y,\ z):\ 0 \le x \le \sqrt{\ln 2},\quad 0 \le y \le \sqrt{\ln 4},\quad 0 \le z \le 1\}$

40. Let $D = \{(x, y, z):\ 0 \le x \le y^2,\ 0 \le y \le z^3,\ 0 \le z \le 2\}$

a) Use a triple integral to find the volume of *D*.

b) In theory, how many other possible orderings of the variables (besides the one used in part (*a*)) can be used to find the volume of *D*? Verify the result of part (*a*) using one of these other ordering.

c) What is the volume of the region $D = \{(x, y, z):\ 0 \le x \le y^p,\ 0 \le y \le z^q,\ 0 \le z \le 2\}$, where *p* and *q* are positive real numbers?

41. Find the volume the parallelepiped (slanted box) with vertices $(0,\ 0,\ 0)$, $(1,\ 0,\ 0)$, $(0,\ 1,\ 0)$, $(1,\ 1,\ 0)$, $(0,\ 1,\ 1)$, $(1,\ 1,\ 1)$, $(0,\ 2,\ 1)$, $(1,\ 2,\ 1)$

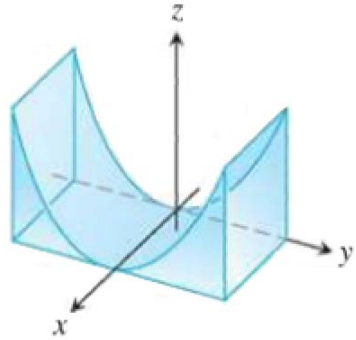
42. Find the volume the larger of two solids formed when the parallelepiped with vertices $(0,\ 0,\ 0)$, $(2,\ 0,\ 0)$, $(0,\ 2,\ 0)$, $(2,\ 2,\ 0)$, $(0,\ 1,\ 1)$, $(2,\ 1,\ 1)$, $(0,\ 3,\ 1)$, $(2,\ 3,\ 1)$ is sliced by the plane $y = 2$.

43. Find the volume of the pyramid with vertices $(0,\ 0,\ 0)$, $(2,\ 0,\ 0)$, $(2,\ 2,\ 0)$, $(0,\ 2,\ 0)$, $(0,\ 0,\ 4)$

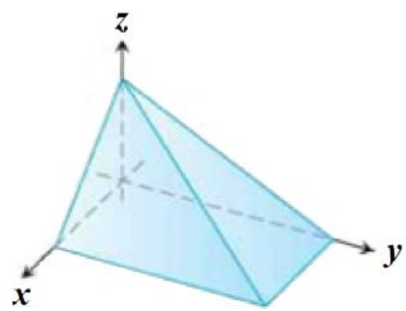
44. Two different tetrahedrons fill the region in the first octant bounded by the coordinate planes and the plane $x + y + z = 4$. Both solids have densities that vary in the *z*-direction between $\rho = 4$ and $\rho = 8$, according to the functions $\rho_1 = 8 - z$ and $\rho_2 = 4 + z$. Find the mass of each solid

45. Suppose a wedge of cheese fills the region in the first octant bounded by the planes $y = z$, $y = 4$ and $x = 4$. You could divide the wedge into two equal pieces (by volume) if you sliced the wedge with the plane $x = 2$. Instead find *a* with $0 < a < 1$ such that slicing the wedge with the plane $y = a$ divides the wedge into two pieces of equal volume

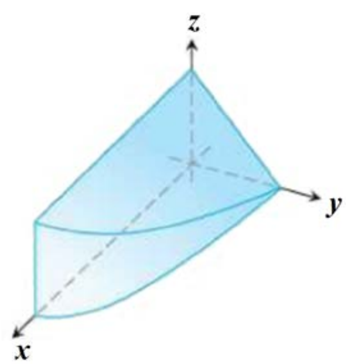
46. Find the volumes of the region between the cylinder $z = y^2$ and the *xy*-plane that is bounded by the planes $x = 0, \quad x = 1, \quad y = -1, \quad y = 1$



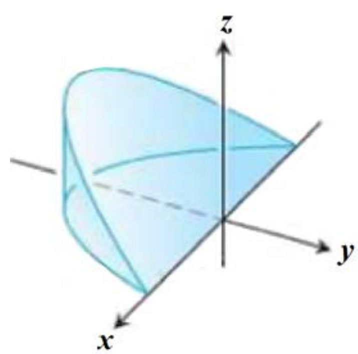
47. Find the volumes of the region in the first octant bounded by the coordinate planes and the planes $x + z = 1, \quad y + 2z = 2$



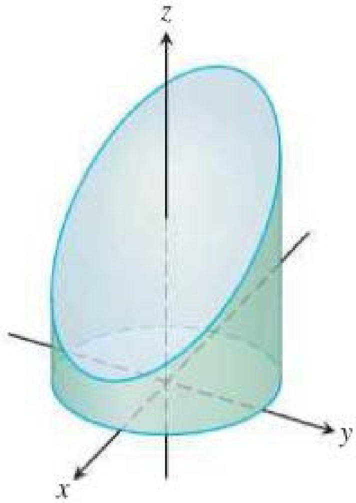
48. Find the volumes of the region in the first octant bounded by the coordinate planes and the plane $y + z = 2$, and the cylinder $x = 4 - y^2$



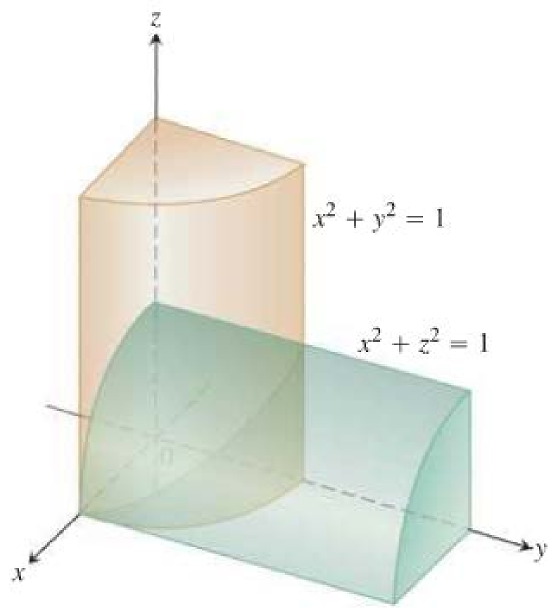
49. Find the volumes of the wedge cut from the cylinder $x^2 + y^2 = 1$ by the planes $z = -y, \quad z = 0$



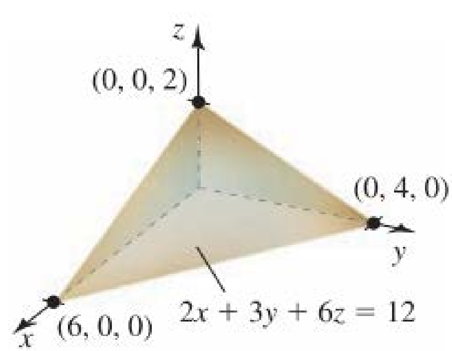
50. Find the volumes of the region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$



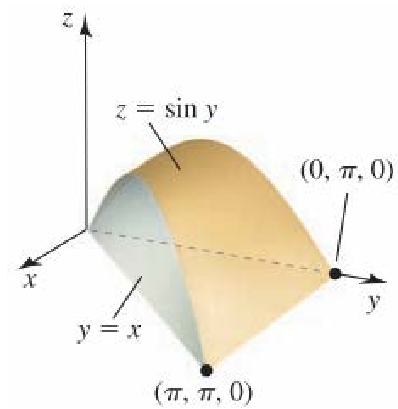
51. Find the volumes of the region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$, one-eighth of which is shown below



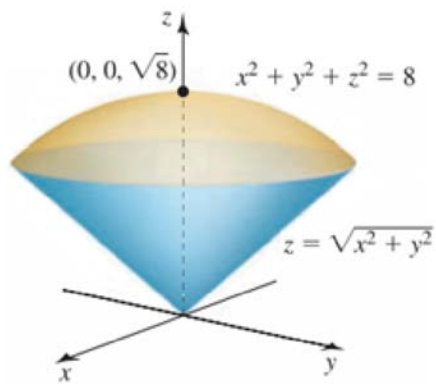
52. Find the volume of the solid in the first octant bounded by the plane $2x + 3y + 6z = 12$ and the coordinate planes



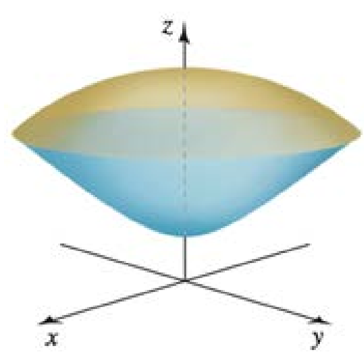
53. Find the volume of the solid in the first octant formed when the cylinder $z = \sin y$, for $0 \le y \le \pi$, is sliced by the planes $y = x$ and $x = 0$



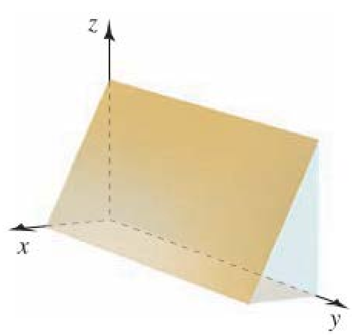
54. Find the volume of the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and bounded above the sphere $x^2 + y^2 + z^2 = 8$



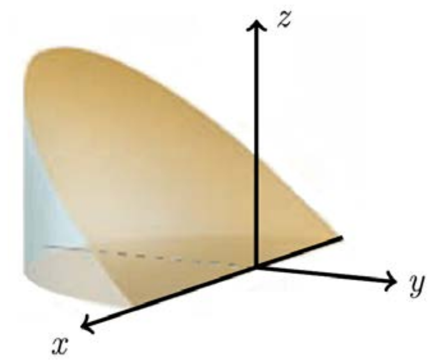
55. The solid between the sphere $x^2 + y^2 + z^2 = 19$ and the hyperboloid $z^2 - x^2 - y^2 = 1$, for $z > 0$



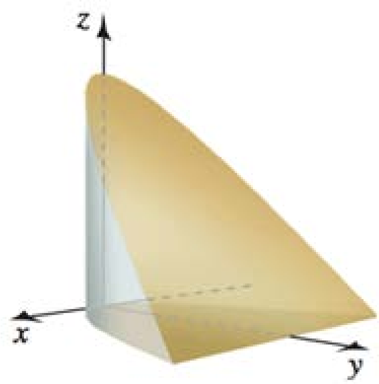
56. Find the volume of the prism in the first octant bounded below by $z = 2 - 4x$ and $y = 8$



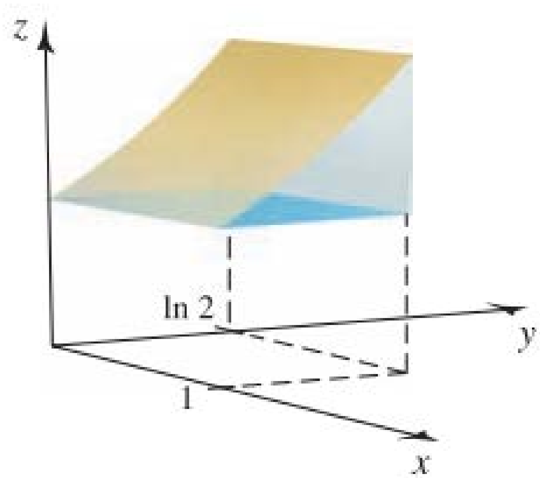
57. Find the volume of the wedge above the *xy*-plane formed when the cylinder $x^2 + y^2 = 4$ is cut by the planes $z = 0$ and $y = -z$



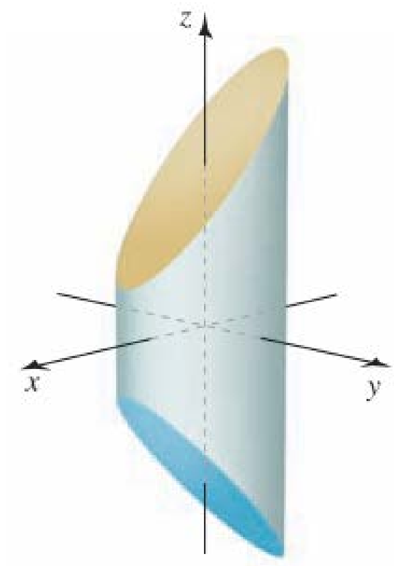
58. The wedge bounded by the parabolic cylinder $y = x^2$ and the planes $z = 3 - y$ and $z = 0$



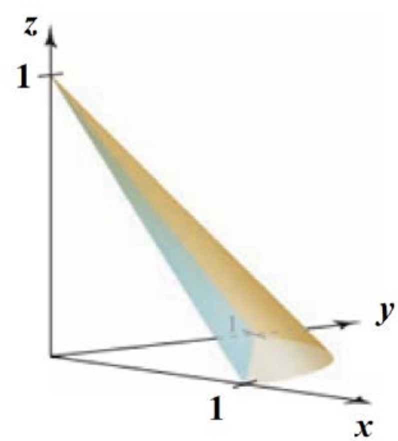
59. Find the volume of the solid bounded by the surfaces $z = e^y$ and $z = 1$ over the rectangle $\{(x, y):\ 0 \le x \le 1,\ 0 \le y \le \ln 2\}$



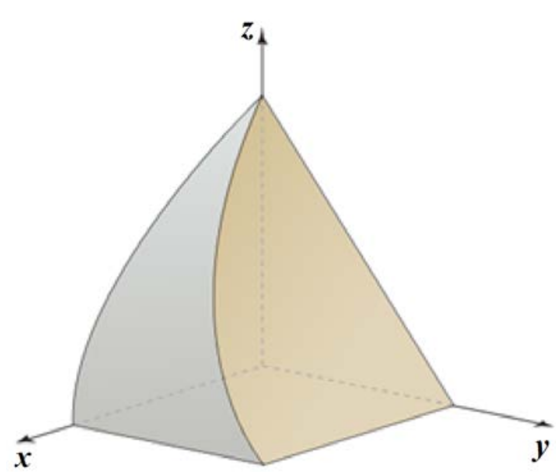
60. Find the volume of the wedge of the cylinder $x^2 + 4y^2 = 4$ created by the planes $z = 3 - x$ and $z = x - 3$



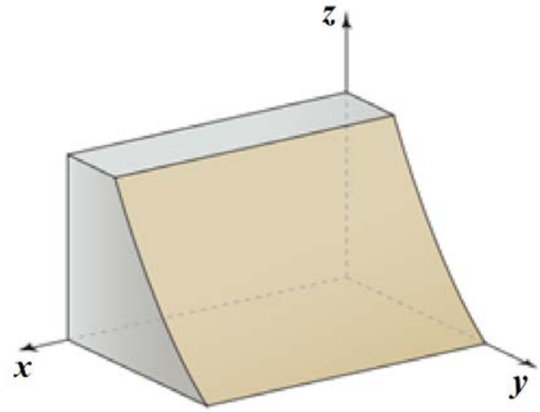
61. Find the volume of the solid in the first octant bounded by the cone $z = 1 - \sqrt{x^2 + y^2}$ and the plane $x + y + z = 1$



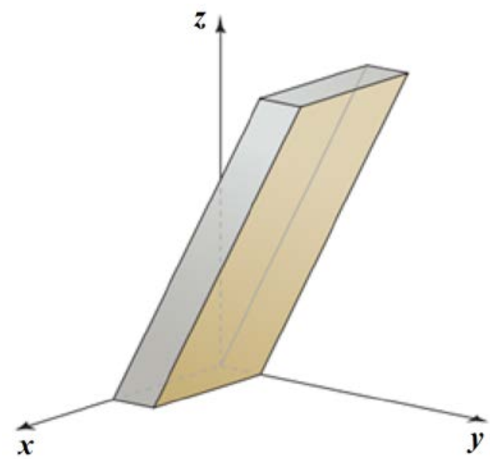
62. Find the volume of the solid bounded by $x = 0$, $x = 1 - z^2$, $y = 0$, $z = 0$, and $z = 1 - y$



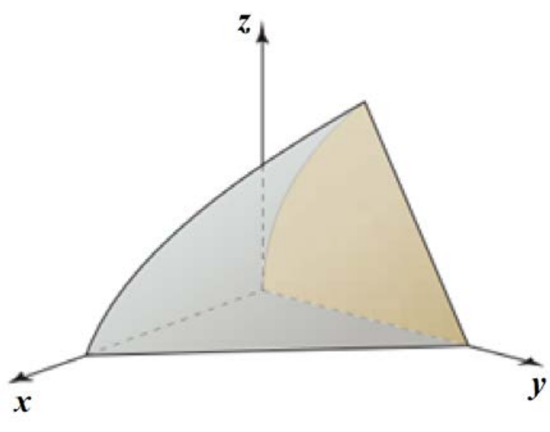
63. Find the volume of the solid bounded by $x=0$, $x=2$, $y=0$, $y=e^{-z}$, $z=0$, and $z=1$



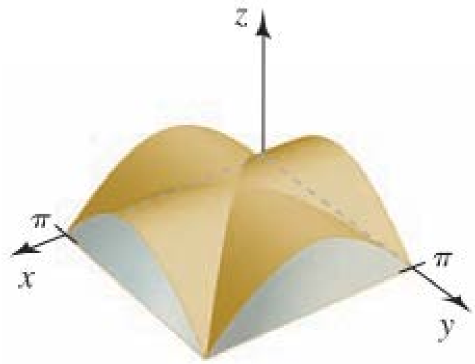
64. Find the volume of the solid bounded by $x=0$, $x=2$, $y=z$, $y=z+1$, $z=0$, and $z=4$



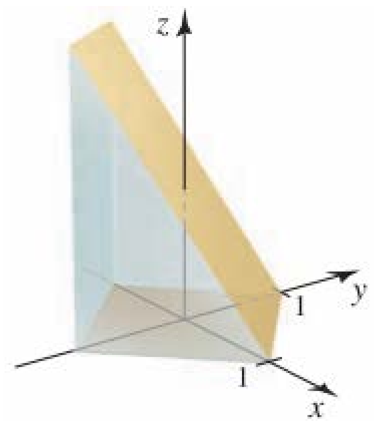
65. Find the volume of the solid bounded by $x=0$, $y=z^2$, $z=0$, and $z=2-x-y$



66. Find the volume of the solid common to the cylinders $z = \sin x$ and $z = \sin y$ over the square $R = \{(x, y):\ 0 \le x \le \pi,\ 0 \le y \le \pi\}$



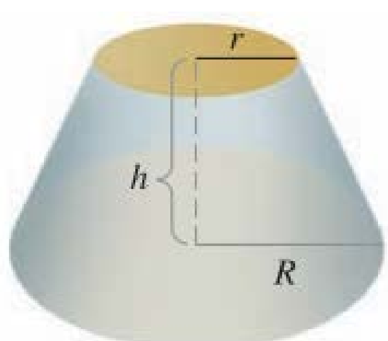
67. Find the volume of the wedge of the square column $|x| + |y| = 1$ created by the planes $z = 0$ and $x + y + z = 1$



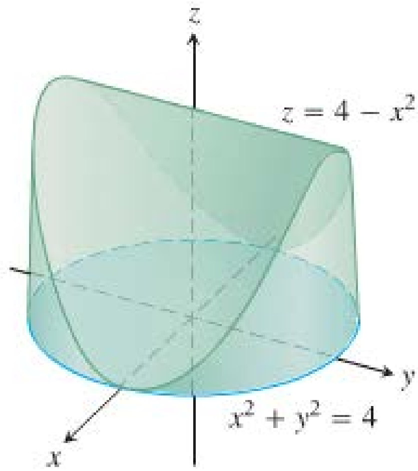
68. Find the volume of a right circular cone with height h and base radius r.

69. Find the volume of a tetrahedron whose vertices are located at $(0,\ 0,\ 0)$, $(a,\ 0,\ 0)$, $(0,\ b,\ 0)$, and $(0,\ 0,\ c)$

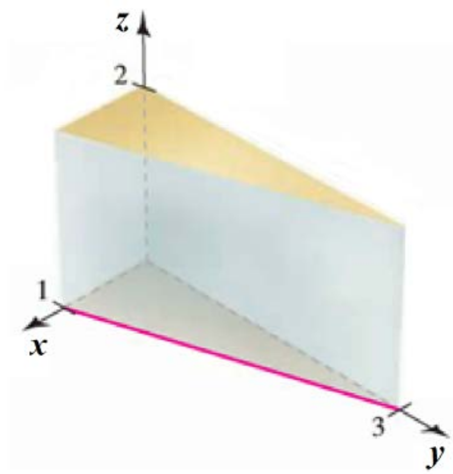
70. Find the volume of a truncated cone of height h whose ends have radii r and R.



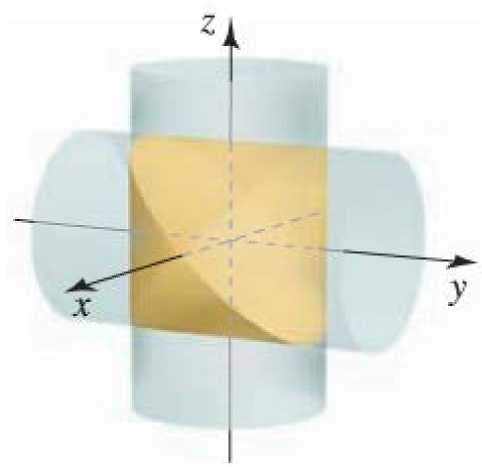
71. Find the volume of the solid that is bounded above by the cylinder $z = 4 - x^2$, on the sides by the cylinder $x^2 + y^2 = 4$, and below by the *xy*-plane.



72. Find the volume of the prism in the first octant bounded by the planes $y = 3 - 3x$ *and* $z = 2$



73. Find the volume of the prism in the first octant bounded by the planes $x^2 + y^2 = 4$ *and* $x^2 + z^2 = 4$

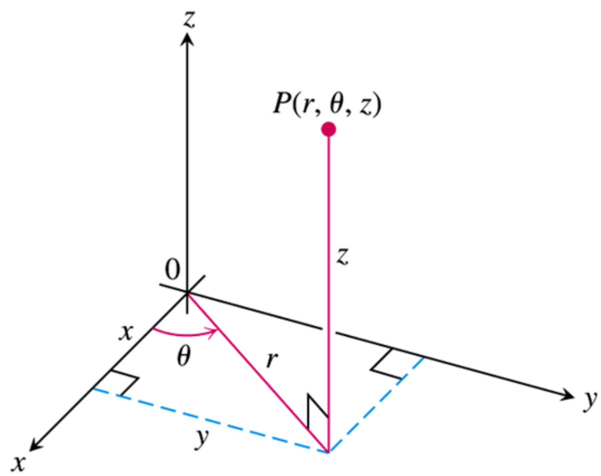


Section 3.5 – Triple Integrals in Cylindrical and Spherical Coordinates

Integration in Cylindrical Coordinates

Definition

Cylindrical coordinates represent a point P in space by ordered triples (r, θ, z) in which



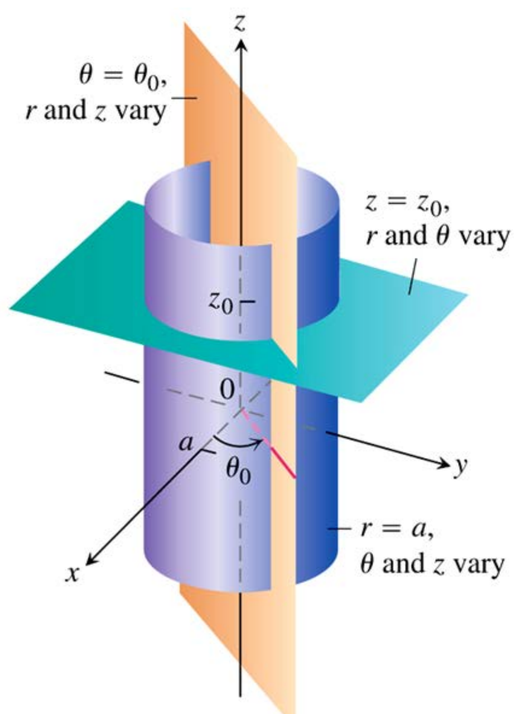
1. r *and* θ are polar coordinates for the vertical projection of P on the xy-plane

2. z is the rectangular vertical coordinate.

Equations Reating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$



The triple integral of a function f over D is obtained by taking a limit of such Riemann sums with partitions whose norms approach zero:

$$\lim_{n\to\infty} S_n = \iiint_D f\ dV = \iiint_D f\ dz\ rdrd\theta$$

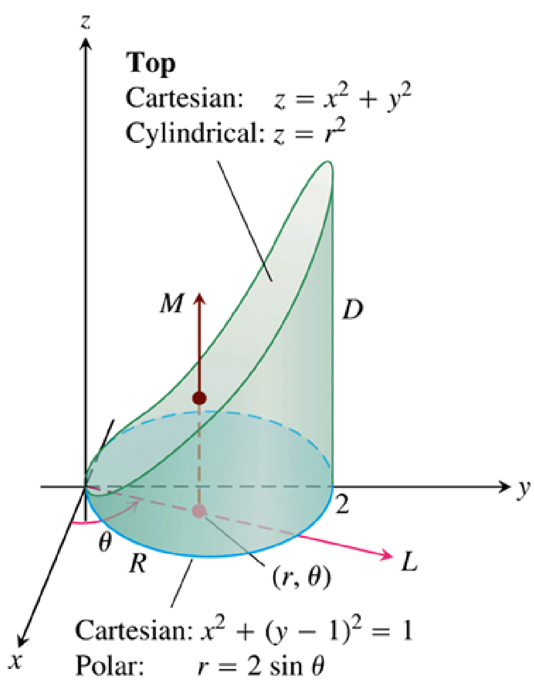
Example

Find the limits of integration in cylindrical coordinates for integrating a function $f(r,\theta,z)$ over the region D bounded below by the plane $z=0$, laterally by the circular cylinder $x^2+(y-1)^2=1$, and above by the paraboloid $z=x^2+y^2$.

Solution

Base of D is the region's projection R on the xy-plane.

The boundary of R is the circle $x^2+(y-1)^2=1$.



The polar coordinate equation is

$$x^2+(y-1)^2=1$$

$$x^2+y^2-2y+1=1$$

$$r^2-2r\sin\theta=0$$

$$r(r-2\sin\theta)=0$$

$$r=2\sin\theta$$

***z*–limits**: A line M through a typical point $(r,\ \theta)$ in

R // z-axis enters D at $z=0$ and leaves at $z=x^2+y^2=r^2$

***r*–limits**: starts at $r=0$ and ends at $r=2\sin\theta$

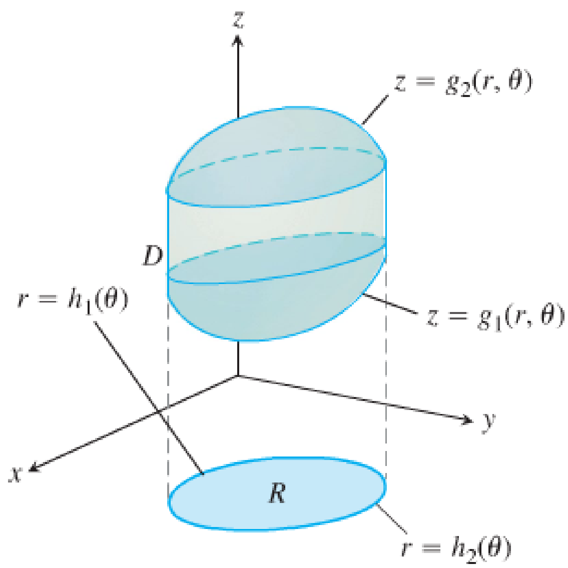
***θ*–limits**: From $\theta=0$ to $\theta=\pi$

$$\iiint_D f\ dz\ rdrd\theta = \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2} f(r,\theta,z)\,dz\ r\ drd\theta$$

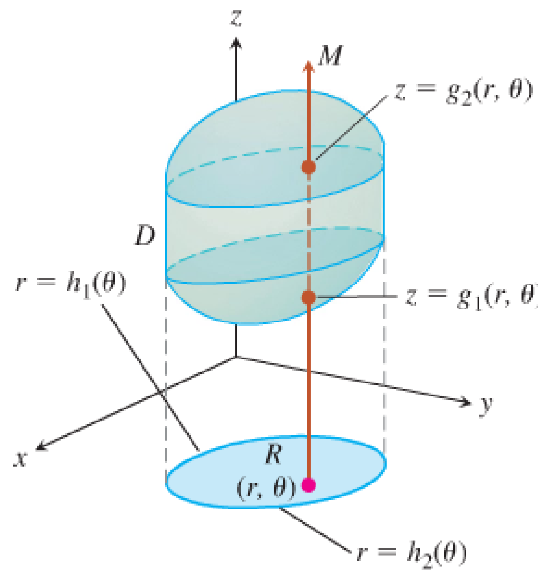
How to integrate in Cylindrical Coordinates

To evaluate $\iiint_D F(r,\ \theta,\ z)\ dV$

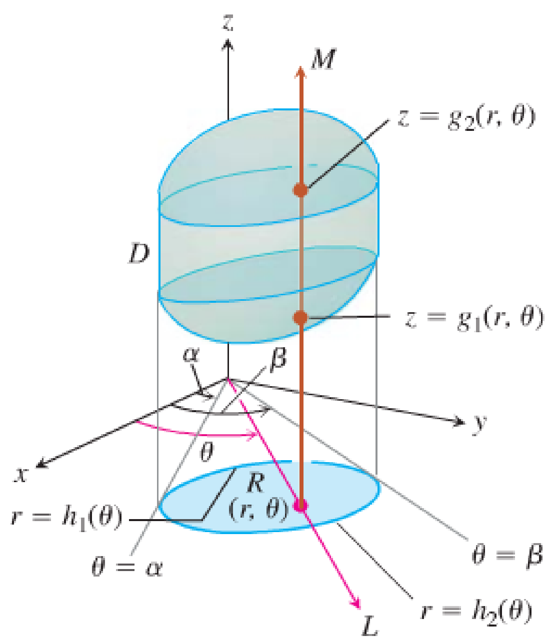
1. ***Sketch***: Sketch the region D along with its projection R on the xy-plane. Label the upper and lower bounding surfaces of D and R.



2. ***Find the z–limits of integration***: Draw a line M passing through (r,θ) in R // z-axis. As z increases, M enters D at $z=g_1(r,\theta)$ to $z=g_2(r,\theta)$.



3. ***Find the r–limits of integration***: Draw a line L passing through (r,θ) from the origin. From $r=h_1(\theta)$ to $r=h_2(\theta)$.



4. ***Find the θ–limits of integration***: As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta=\alpha$ *and* $\theta=\beta$.

$$\iiint_D F(r,\ \theta,\ z)\ dV = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r=h_2(\theta)}\int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} F(r,\theta,z)\,dz\ r dr d\theta$$

Example

Find the volume bounded by the sphere $x^2+y^2+z^2=9$ and the paraboloid $x^2+y^2=8z$

Solution

$$x^2+y^2=r^2$$

$$\begin{cases} r^2+z^2=9 \\ r^2=8z \end{cases}$$

$$\begin{cases} z=\sqrt{9-r^2} \\ z=\frac{1}{8}r^2 \end{cases} \rightarrow \quad \frac{1}{8}r^2 \le z \le \sqrt{9-r^2}$$

$$r^2=9-z^2=8z$$

$$z^2+8z-9=0 \quad \rightarrow \quad \underline{z=1,\ \cancel{-9}}$$

$$z=1 \Rightarrow \quad r^2=8z=8$$

$$r=2\sqrt{2} \quad \rightarrow \quad \underline{0 \le r \le 2\sqrt{2}}$$

$$\underline{0 \le \theta \le 2\pi}$$

$$V=\int_0^{2\pi}\int_0^{2\sqrt{2}}\int_{\frac{1}{8}r^2}^{\sqrt{9-r^2}} r\ dzdrd\theta$$

$$=\int_0^{2\pi} d\theta \quad \int_0^{2\sqrt{2}} r\ z \Big|_{\frac{1}{8}r^2}^{\sqrt{9-r^2}} dr$$

$$=2\pi\int_0^{2\sqrt{2}} r\left(\sqrt{9-r^2}-\frac{1}{8}r^2\right) dr$$

$$=2\pi\int_0^{2\sqrt{2}} r\left(9-r^2\right)^{1/2} dr-\frac{\pi}{4}\int_0^{2\sqrt{2}} r^3\ dr$$

$$=-\pi\int_0^{2\sqrt{2}}\left(9-r^2\right)^{1/2} d\left(9-r^2\right)-\frac{\pi}{16}r^4 \Big|_0^{2\sqrt{2}}$$

$$=-\frac{2\pi}{3}\left(9-r^2\right)^{3/2} \Big|_0^{2\sqrt{2}} -\frac{\pi}{16}(64)$$

$$=-\frac{2\pi}{3}(1-27)-4\pi$$

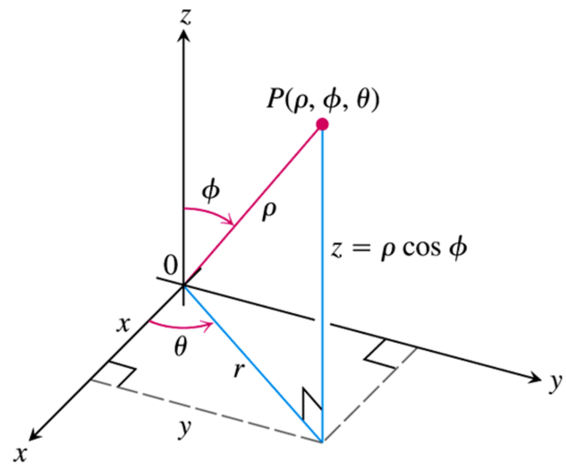
$$= \frac{52\pi}{3} - 4\pi$$

$$= \frac{40\pi}{3} \quad unit^3$$

Definition

Spherical coordinates represent a point P in space by ordered triple $(\rho,\ \phi,\ \theta)$ in which

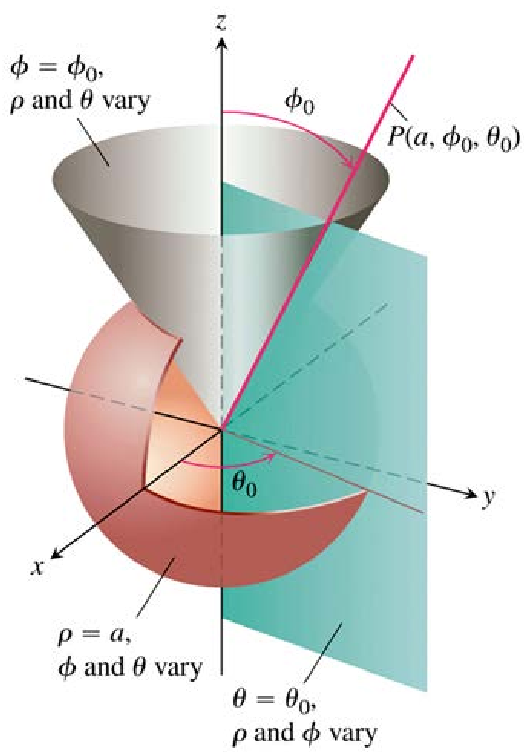
1. ρ is the distance from P to the origin
2. ϕ is the angle $\overrightarrow{OP}$ makes with positive z-axis $(0 \le \phi \le \pi)$.
3. θ is the angle from the cylindrical coordinates $(0 \le \theta \le 2\pi)$



Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$r = \rho \sin\phi, \quad x = r\cos\theta = \rho\sin\phi\cos\theta,$$
$$z = \rho\cos\phi, \quad y = r\sin\theta = \rho\sin\phi\sin\theta,$$

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$$



Example

Find a spherical coordinate equation for the sphere $x^2 + y^2 + (z-1)^2 = 1$

Solution

$$x^2 + y^2 + (z-1)^2 = 1$$

$$\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta + (\rho\cos\phi - 1)^2 = 1$$

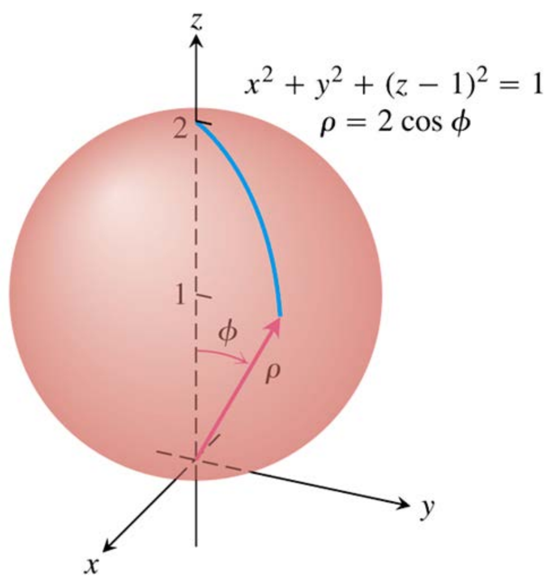
$$\rho^2 \sin^2\phi \left(\cos^2\theta + \sin^2\theta\right) + \rho^2\cos^2\phi - 2\rho\cos\phi + 1 = 1 \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\rho^2\left(\sin^2\phi + \cos^2\phi\right) - 2\rho\cos\phi = 0$$

$$\rho^2 - 2\rho\cos\phi = 0$$

$$\rho(\rho - 2\cos\phi) = 0 \qquad \rho > 0$$

$$\boxed{\rho = 2\cos\phi}$$



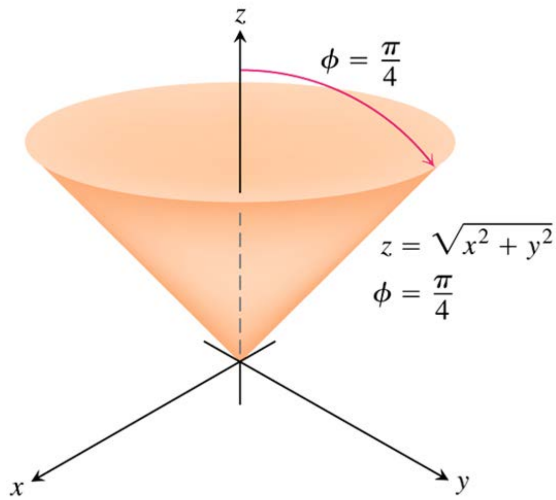
The angle ϕ varies from 0 to the north pole of the sphere to $\frac{\pi}{2}$ at the south pole; the angle θ doesn't appear in the expression for ρ, reflecting the symmetry about the z-axis.

Example

Find a spherical coordinate equation for the sphere $z=\sqrt{x^2+y^2}$

Solution

The cone is symmetric with respect to the *z*-axis and cuts the first quadrant of the yz-plane along the line $z=y$. The angle between the cone and the positive *z*-axis is therefore $\frac{\pi}{4}$ rad. The cone consists of the points whose spherical coordinates have $\phi=\frac{\pi}{4}$.



$$z=\sqrt{x^2+y^2}$$

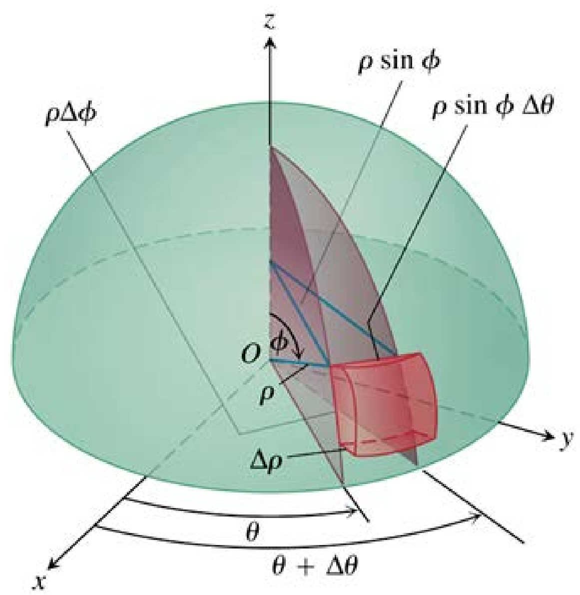
$$\rho\cos\phi=\sqrt{\rho^2\sin^2\phi}$$

$$\rho\cos\phi=\rho\sin\phi$$

$$\cos\phi=\sin\phi \quad \rightarrow \quad \boxed{\phi=\frac{\pi}{4}}$$

Volume Differential in Spherical Coordinates

$$dV=\rho^2\sin\phi\; d\rho\; d\phi\; d\theta$$



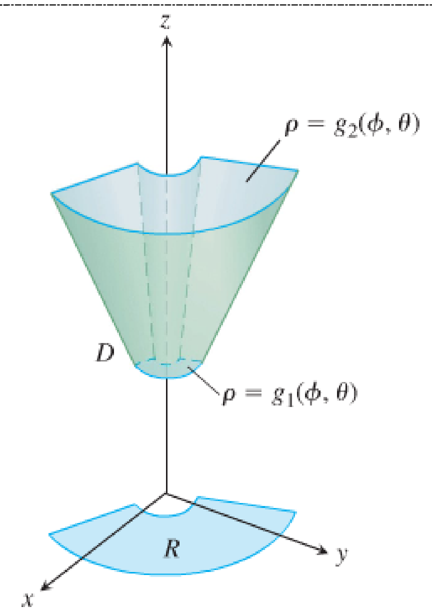
$$dV = \; d\rho\cdot\rho d\phi\cdot\rho\sin\phi d\theta \; = \; \rho^2\sin\phi\; d\rho\; d\phi\; d\theta$$

How to integrate in Spherical Coordinates

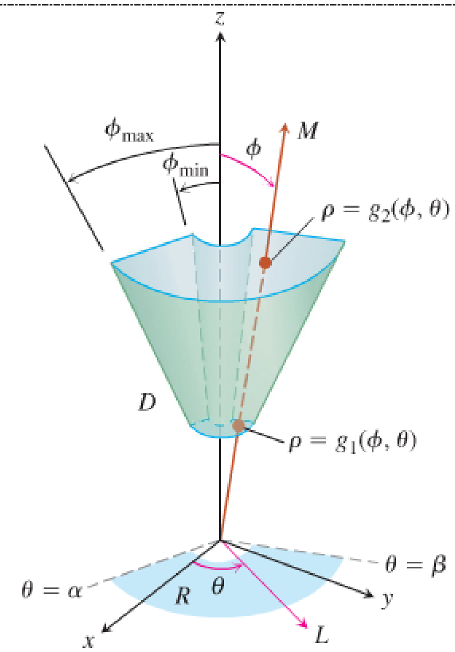
To evaluate $\iiint\limits_D F(\rho,\ \phi,\ \theta)\ dV$

1. ***Sketch***: Sketch the region D along its projection R on the xy-plane. Label the surface that bound of D.

2. ***Find the ρ–limits of integration***: Draw a ray M from the origin through D making an angle ϕ with the positive z-axis. Also draw the projection of M on the xy-plane (call the projection L). The ray L makes an angle θ with the positive x-axis. As ρ increases, M enters D at $\rho = g_1(\phi,\theta)$ to $\rho = g_2(\phi,\theta)$.



3. ***Find the ϕ–limits of integration***: For the given θ, the angle ϕ that M makes with the z-axis runs $\phi = \phi_{\min}$ to $\phi = \phi_{\max}$.



5. ***Find the θ–limits of integration***: As L sweeps over R as θ runs from α to β.

$$\iiint\limits_D f(\rho,\ \phi,\ \theta)\,dV = \int_{\theta=\alpha}^{\theta=\beta}\int_{\phi=\phi_{\min}}^{\phi=\phi_{\max}}\int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} f(\rho,\ \phi,\ \theta)\ \rho^2\ \sin\phi\ d\rho\, d\phi\, d\theta$$

Example

Find the volume of the "ice cream cone" D cut from the solid sphere $\rho \le 1$ by the cone $\phi = \frac{\pi}{3}$

Solution

$$f(\rho,\ \phi,\ \theta) = 1$$

$$V = \iiint_D \rho^2 \sin\phi \ d\rho d\phi d\theta$$

ρ–limits: Draw a ray M from the origin through D making an angle ϕ with the positive z-axis. And L, the projection of M on the xy-plane, along with the angle θ that L makes with the positive x-axis. Ray M enters D form $\rho = 0$ to $\rho = 1$

ϕ–limits: The cone $\phi = \frac{\pi}{3}$ makes with the positive z-axis. $0 \le \phi \le \frac{\pi}{3}$

θ–limits: $0 \le \theta \le 2\pi$

$$V = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi \ d\rho d\phi d\theta$$

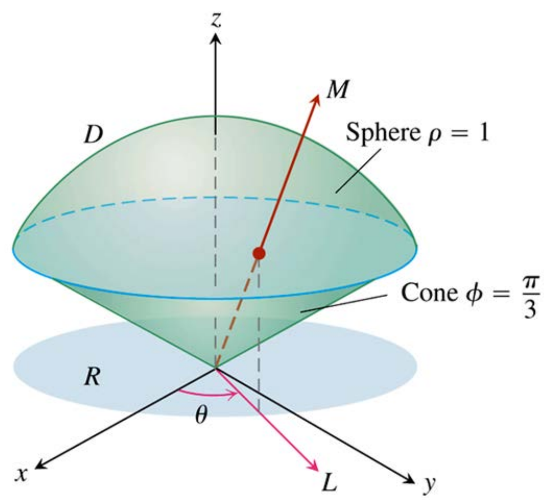
$$= \int_0^{2\pi} d\theta \int_0^{\pi/3} \sin\phi \ d\phi \ \left(\frac{1}{3}\rho^3 \ \Big|_0^1\right.$$

$$= \frac{2\pi}{3} \ \left(-\cos\phi \ \Big|_0^{\pi/3}\right.$$

$$= -\frac{2\pi}{3}\left(\frac{1}{2} - 1\right)$$

$$= -\frac{2\pi}{3}\left(-\frac{1}{2}\right)$$

$$\boxed{= \frac{\pi}{3} \quad unit^3}$$



Example

Find the volume cut from the cone $x^2+y^2-z^2=0$, by the sphere $x^2+y^2+(z-2)^2=4$

Solution

$x^2+y^2=z^2$

$x^2+y^2=4-(z-2)^2=z^2$

$4-z^2+4z-4=z^2$

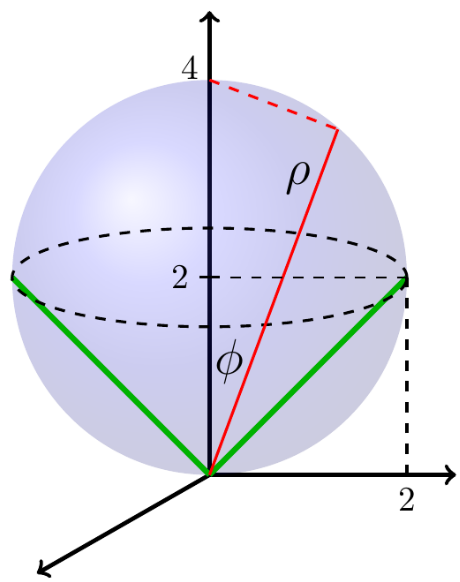
$2z^2-4z=0 \quad \to \quad \underline{z=0,\ 2}$

$0+y^2=z^2 \quad \to \quad \underline{y=z=2}$

$\phi=\tan^{-1}\frac{2}{2}=\frac{\pi}{4}$

$\underline{0\le\phi\le\frac{\pi}{4}}$

$\underline{0\le\theta\le 2\pi}$



$\cos\phi=\frac{\rho}{4} \quad \to \quad \rho=4\cos\phi$

$\underline{0\le\rho\le 4\cos\phi}$

$$V=4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^{4\cos\phi}\rho^2\sin\phi\ d\rho d\phi d\theta$$

$$=\frac{4}{3}\int_0^{\frac{\pi}{2}}d\theta\ \int_0^{\frac{\pi}{4}}(\sin\phi)\rho^3\ \Big|_0^{4\cos\phi}\ d\phi$$

$$=\frac{128\pi}{3}\int_0^{\frac{\pi}{4}}(\sin\phi)\cos^3\phi\ d\phi$$

$$=-\frac{128\pi}{3}\int_0^{\frac{\pi}{4}}\cos^3\phi\ d(\cos\phi)$$

$$=-\frac{32\pi}{3}\ \cos^4\phi\ \Big|_0^{\frac{\pi}{4}}$$

$$=-\frac{32\pi}{3}\left(\frac{1}{4}-1\right)$$

$$=-\frac{32\pi}{3}\left(-\frac{3}{4}\right)$$

$$\underline{=8\pi\ \ unit^3}$$

Example

Evaluate the integral $\iiint\limits_D \left(x^2+y^2+z^2\right)^{-3/2}\, dV$ over the region D.

Where the region D in the first octant between 2 spheres of radius 1 and 2 centered at the origin.

Solution

$$D=\left\{(\rho,\ \varphi,\ \theta):\ 1\le\rho\le 2;\quad 0\le\varphi\le\frac{\pi}{2};\quad 0\le\theta\le\frac{\pi}{2}\right\}$$

$$\iiint\limits_D \left(x^2+y^2+z^2\right)^{-3/2}\, dV=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_1^2 \left(\rho^2\right)^{-3/2}\rho^2\sin\varphi\; d\rho d\varphi d\theta$$

$$=\int_0^{\frac{\pi}{2}} d\theta\int_0^{\frac{\pi}{2}}\sin\varphi d\varphi\int_1^2 \rho^{-1}\, d\rho$$

$$=\frac{\pi}{2}\left(-\cos\varphi\ \Big|_0^{\frac{\pi}{2}}\right)\left(\ln\rho\ \Big|_1^2\right)$$

$$=\frac{\pi}{2}(1)\ (\ln 2)$$

$$=\boxed{\frac{\pi\ln 2}{2}}$$

Example

Evaluate the integral $\int_0^a\int_0^{\sqrt{a^2-z^2}}\int_0^{\sqrt{a^2-y^2-z^2}} \sqrt{x^2+y^2+z^2}\; dxdydz$

Solution

$$\sqrt{x^2+y^2+z^2}=\rho$$

$$0\le x\le\sqrt{a^2-y^2-z^2}$$

$$0\le y\le\sqrt{a^2-z^2}$$

$$\Rightarrow\quad \boxed{0\le\theta\le\frac{\pi}{2}}$$

$$a=\rho\quad\to\quad \boxed{0\le\rho\le a}$$

$$z=a\quad\to\quad \varphi=\frac{\pi}{2}$$

$$\Rightarrow \quad \boxed{0 \le \varphi \le \frac{\pi}{2}}$$

$$\int_0^a \int_0^{\sqrt{a^2-z^2}} \int_0^{\sqrt{a^2-y^2-z^2}} \sqrt{x^2+y^2+z^2}\ dxdydz$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a \rho\ \rho^2 \sin\varphi\ d\rho d\varphi d\theta$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} \sin\varphi d\varphi \int_0^a \rho^3\ d\rho$$

$$= \frac{\pi}{2}\left(-\cos\varphi \Big|_0^{\frac{\pi}{2}}\right)\left(\frac{1}{4}\rho^4 \Big|_0^a\right)$$

$$\boxed{= \frac{\pi a^4}{8}}$$

Coordinate Conversion Formulas

Cylindrical to Rectangular	***Spherical to Rectangular***	***Spherical to Cylindrical***
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$
\begin{aligned}
dV &= dx\ dy\ dz \\
&= dz\ rdr\ d\theta \\
&= \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta
\end{aligned}
$$

Exercises *Section* 3.5 – Triple Integrals in Cylindrical and Spherical Coordinates

(**1 – 16**) Evaluate the cylindrical coordinate integral

1. $\displaystyle\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\ r dr\ d\theta$

2. $\displaystyle\int_0^{2\pi}\int_0^{\theta/(2\pi)}\int_r^{3+24r^2} dz\ r dr\ d\theta$

3. $\displaystyle\int_0^{\pi}\int_0^{\theta/\pi}\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} z dz\ r dr\ d\theta$

4. $\displaystyle\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2}\left(r^2\sin^2\theta+z^2\right) dz\ r dr\ d\theta$

5. $\displaystyle\int_0^{2\pi}\int_0^3\int_0^{z/3} r^3 dr\ dz\ d\theta$

6. $\displaystyle\int_0^1\int_0^{\sqrt{z}}\int_0^{2\pi}\left(r^2\cos^2\theta+z^2\right) r\ d\theta\ dr dz$

7. $\displaystyle\int_0^2\int_{r-2}^{\sqrt{4-r^2}}\int_0^{2\pi}\left(r\sin\theta+1\right) r\ d\theta\ dz\ dr$

8. $\displaystyle\int_{-1}^5\int_0^{\frac{\pi}{2}}\int_0^3 r\cos\theta\ dr d\theta dz$

9. $\displaystyle\int_0^{\frac{\pi}{4}}\int_0^6\int_0^{6-r} rz\ dz dr d\theta$

10. $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{2\cos^2\theta}\int_0^{4-r^2} r\sin\theta\ dz dr d\theta$

11. $\displaystyle\int_0^4\int_0^z\int_0^{\frac{\pi}{2}} re^r\ d\theta dr dz$

12. $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^3\int_0^{e^{-r^2}} r\ dz dr d\theta$

13. $\displaystyle\int_0^{2\pi}\int_0^{\sqrt{5}}\int_0^{5-r^2} r\ dz\ dr\ d\theta$

14. $\displaystyle\int_0^{\pi}\int_0^{\cos\theta}\int_{2r^2}^{2r\cos\theta} r\ dz\ dr\ d\theta$

15. $\displaystyle\int_0^{\pi}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\ dz\ dr\ d\theta$

16. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a\cos\theta}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\ dz\ dr\ d\theta$

17. Convert $\displaystyle\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} 3dz\ r dr d\theta, \qquad r\ge 0$

a) Rectangular coordinates with order of integration dzd*x*d*y*.

b) Spherical coordinates

c) Evaluate one of the integrals.

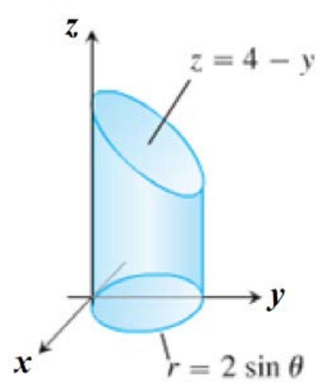
18. Convert the integral $\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{x}\left(x^2+y^2\right)dzdxdy$ to an equivalent integral in cylindrical coordinates and evaluate the result.

19. Set up an integral in rectangular coordinates equivalent to the integral

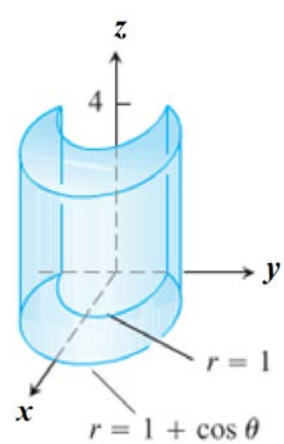
$$\int_{0}^{\pi/2}\int_{1}^{\sqrt{3}}\int_{1}^{\sqrt{4-r^2}} r^3(\sin\theta\cos\theta)z^2\ dzdrd\theta$$

Arrange the order of integration to be *z* first, then *y*, then *x*.

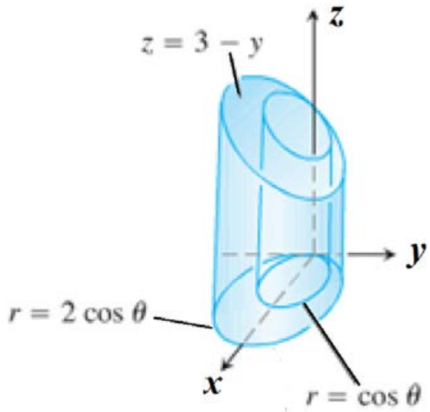
20. Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)dzdrd\theta$ over the region *D* that is the right circular cylinder whose base is the circle $r=2\sin\theta$ in the *xy*-plane and whose top lies in the plane $z=4-y$



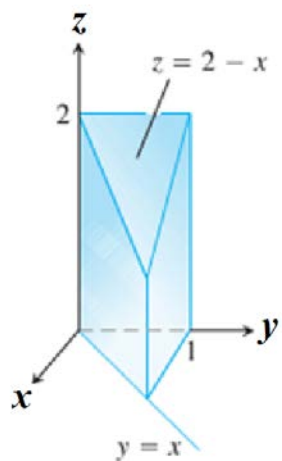
21. Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)dzdrd\theta$ over the region *D* which is the solid right cylinder whose base is the region in the *xy*-plane that lies inside the cardioid $r=1+\cos\theta$ and outside the circle $r=1$ and whose top lies in the plane $z=4$



22. Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\,dz\,dr\,d\theta$ over the region D which is the solid right cylinder whose base is the region between the circles $r=\cos\theta$ and $r=2\cos\theta$ and whose top lies in the plane $z=3-y$



23. Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\,dz\,dr\,d\theta$ over the region D which is the prism whose base is the triangle in the *xy*-plane bounded by the *y*-axis and the lines $y=x$ and $y=1$ and whose top lies in the plane $z=2-x$



(24 – 25) Evaluate the integrals in cylindrical coordinates.

24. $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^3 \left(x^2+y^2\right)^{3/2} dzdydz$

25. $\displaystyle\int_{-2}^2\int_{-1}^1\int_0^{\sqrt{1-z^2}} \frac{1}{\left(1+x^2+z^2\right)^2}\,dxdzdy$

(26 – 41) Evaluate the spherical coordinate integral

26. $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta$

27. $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} (\rho\cos\phi)\rho^2 \sin\phi \; d\rho \; d\phi \; d\theta$

28. $\displaystyle\int_0^{3\pi/2}\int_0^{\pi}\int_0^{1} 5\rho^3 \sin^3\phi \; d\rho \; d\phi \; d\theta$

29. $\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\cos\varphi} \rho^2 \sin\varphi \; d\rho d\varphi d\theta$

30. $\displaystyle\int_0^{\pi}\int_0^{\pi/4}\int_{2\sec\varphi}^{4\sec\varphi} \rho^2 \sin\varphi \; d\rho d\varphi d\theta$

31. $\displaystyle\int_0^{2}\int_{-\pi}^{0}\int_{\pi/4}^{\pi/2} \rho^3 \sin 2\phi \; d\phi \; d\theta \; d\rho$

32. $\displaystyle\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\csc\phi}^{2} 5\rho^4 \sin^3\phi \; d\rho \; d\theta \; d\phi$

33. $\displaystyle\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{3} \rho^2 \sin\phi \; d\rho d\phi d\theta$

34. $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{3} \rho^3 \cos\phi \sin\phi \; d\rho d\phi d\theta$

35. $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{\sin\theta} 2\cos\phi \; \rho^2 \; d\rho d\theta d\phi$

36. $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{2} e^{-\rho^3} \rho^2 \; d\rho d\theta d\varphi$

37. $\displaystyle\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\varphi} \rho^2 \sin\phi \; d\rho d\phi d\theta$

38. $\displaystyle\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\int_0^{\cos\theta} \rho^2 \sin\varphi \cos\varphi \; d\rho d\theta d\varphi$

39. $\displaystyle\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^{4} \rho^2 \sin\varphi \; d\rho d\varphi d\theta$

40. $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^{5} \rho^2 \sin\varphi \; d\rho d\varphi d\theta$

41. $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{\sin\theta} 2\cos\varphi \; \rho^2 \; d\rho d\theta d\varphi$

(42 – 45) Evaluate the integrals

42. $\displaystyle\int_0^{4}\int_0^{\frac{\sqrt{2}}{2}}\int_x^{\sqrt{1-x^2}} e^{-x^2-y^2} \; dydxdz$

43. $\displaystyle\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{\sqrt{x^2+y^2}}^{4} dzdydx$

44. $\displaystyle\int_0^{3}\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{x^2+y^2}} \left(x^2+y^2\right)^{-1/2} dzdydx$

45. $\displaystyle\int_{-1}^{1}\int_0^{\frac{1}{2}}\int_{\sqrt{3}y}^{\sqrt{1-y^2}} \sqrt{x^2+y^2} \; dxdydz$

46. Evaluate $\displaystyle\iiint_D \left(x^2+y^2+z^2\right)^{5/2} dV$; D is the unit ball.

47. Evaluate $\displaystyle\iiint_D e^{-\left(x^2+y^2+z^2\right)^{3/2}} dV$; D is the unit ball.

48. Evaluate $\displaystyle\iiint_D \frac{1}{\left(x^2+y^2+z^2\right)^{3/2}} dV$; D is the solid between the spheres of radius 1 and 2 centered at the origin.

49. Evaluate $\displaystyle\iiint_D \left(x^2+y^2+z^2\right) dV$, where D is the region in the first octant between two spheres of radius 1 and 2 centered at the origin.

50. Evaluate $\displaystyle\iiint_D x^2 dV$; $D=\{(r,\ \theta,\ z):\ \ 0\le r\le 1,\ \ 0\le z\le 2r,\ \ 0\le\theta\le 2\pi\}$

51. Evaluate $\displaystyle\iiint_D dV$; $D=\left\{(r,\ \theta,\ z):\ \ 0\le r\le 1,\ \ -\sqrt{4-r^2}\le z\le\sqrt{4-r^2},\ \ 0\le\theta\le 2\pi\right\}$

52. Evaluate $\displaystyle\iiint_D dV$; $D=\left\{(r,\ \theta,\ z):\ \ 0\le r\le 1,\ \ r\le z\le\sqrt{2-r^2},\ \ 0\le\theta\le 2\pi\right\}$

53. Evaluate $\displaystyle\iiint_D dV$; $D=\left\{(r,\ \theta,\ z):\ \ 0\le r\le 1,\ \ r^2\le z\le\sqrt{2-r^2},\ \ 0\le\theta\le 2\pi\right\}$

54. Evaluate $\displaystyle\iiint_D dV$; $D=\left\{(r,\ \theta,\ z):\ \ 0\le r\le 4,\ \ 2r\le z\le 24-r^2,\ \ 0\le\theta\le 2\pi\right\}$

55. Evaluate $\displaystyle\iiint_D y^2z^2 dV$; $D=\left\{(\rho,\ \varphi,\ \theta):\ \ 0\le\rho\le 1,\ \ 0\le\varphi\le\frac{\pi}{3},\ \ 0\le\theta\le 2\pi\right\}$

56. Evaluate $\displaystyle\iiint_D \left(x^2+y^2\right) dV$; $D=\{(\rho,\ \varphi,\ \theta):\ \ 2\le\rho\le 3,\ \ 0\le\varphi\le\pi,\ \ 0\le\theta\le 2\pi\}$

57. Evaluate $\displaystyle\iiint_D y^2 dV$; $D=\{(\rho,\ \varphi,\ \theta):\ \ 0\le\rho\le 3,\ \ 0\le\varphi\le\pi,\ \ 0\le\theta\le\pi\}$

58. Evaluate $\iiint\limits_D xe^{x^2+y^2+z^2}\,dV$; $D=\left\{(\rho,\ \varphi,\ \theta):\ \ 0\le\rho\le1,\ \ 0\le\varphi\le\frac{\pi}{2},\ \ 0\le\theta\le\frac{\pi}{2}\right\}$

59. Evaluate $\iiint\limits_D \sqrt{x^2+y^2+z^2}\,dV$; $D=\left\{(\rho,\ \varphi,\ \theta):\ \ 1\le\rho\le2,\ \ 0\le\varphi\le\frac{\pi}{4},\ \ 0\le\theta\le2\pi\right\}$

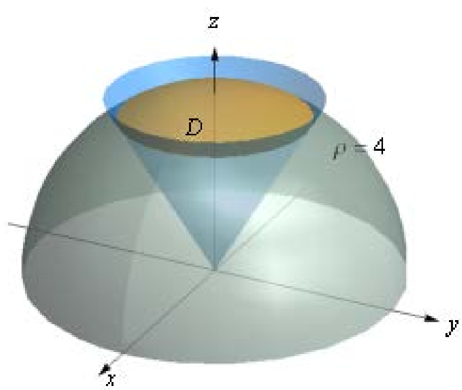
60. Find the volume of the solid whose height is 4 and whose base is the disk $\{(r,\ \theta):\ \ 0\le r\le 2\cos\theta\}$

61. Find the volume of the solid in the first octant bounded by the cylinder $r=1$ and the plane $z=x$

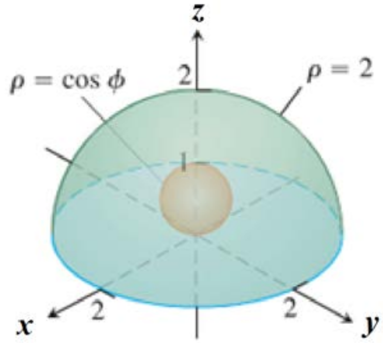
62. Find the volume of the solid bounded by the cylinder $r=1$ and $r=2$ and the planes $z=4-x-y$ and $z=0$

63. Find the volume of the solid D between the cone $z=\sqrt{x^2+y^2}$ and the inverted paraboloid $z=12-x^2-y^2$

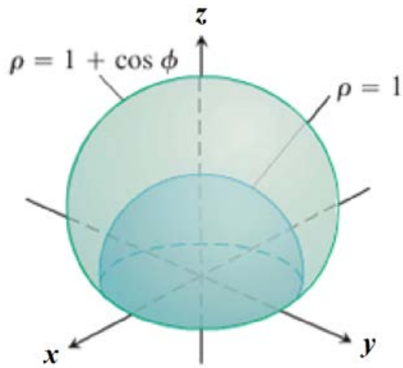
64. Find the volume of the solid region D that lies inside the cone $\phi=\frac{\pi}{6}$ and inside the sphere $\rho=4$



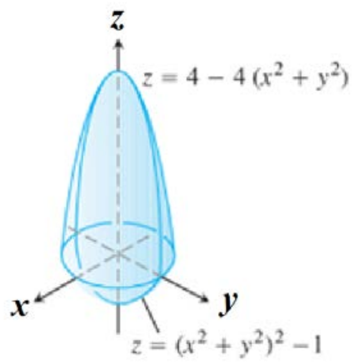
65. Find the volume of the solid between the sphere $\rho=\cos\phi$ and the hemisphere $\rho=2,\ \ z\ge0$



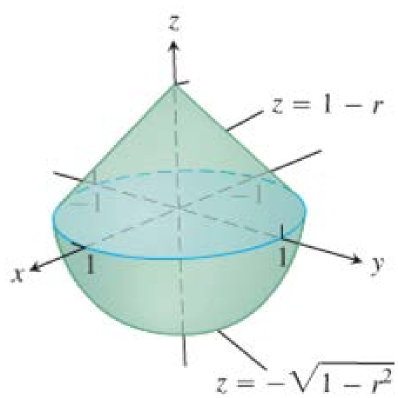
66. Find the volume of the solid bounded below by the hemisphere $\rho = 1,\ z \geq 0$, and above the cardioid of revolution $\rho = 1 + \cos\phi$



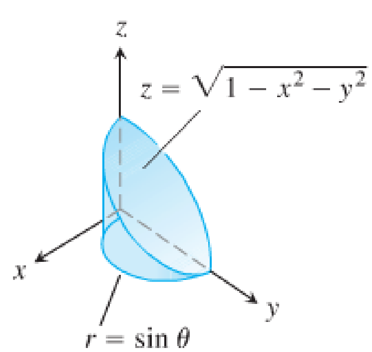
67. Find the volume of the solid



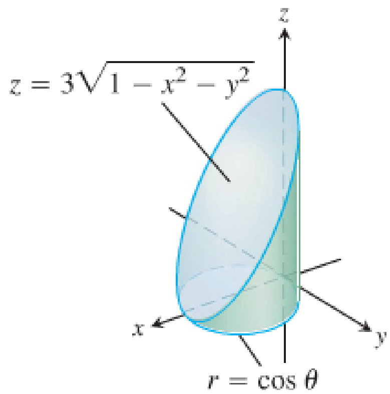
68. Find the volume of the solid



69. Find the volume of the solid



70. Find the volume of the solid

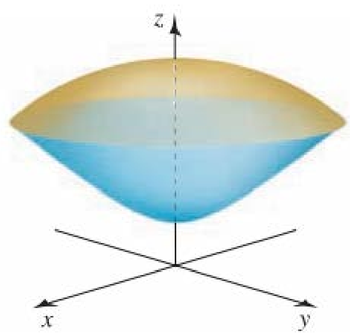


71. Find the volume of the smaller region cut from the solid sphere $\rho \le 2$ by the plane $z = 1$

72. Find the volume of the region bounded below by the paraboloid $z = x^2 + y^2$, laterally by the cylinder $x^2 + y^2 = 1$, and above by the paraboloid $z = x^2 + y^2 + 1$

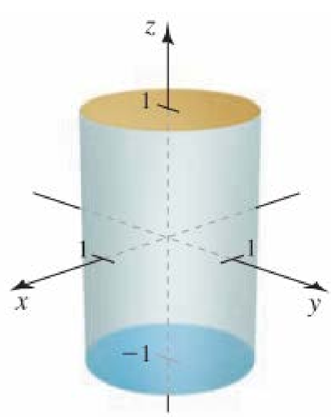
73. Find the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$

74. Find the volume of the solid between the sphere $x^2 + y^2 + z^2 = 19$ and the hyperboloid $z^2 - x^2 - y^2 = 1$ for $z > 0$

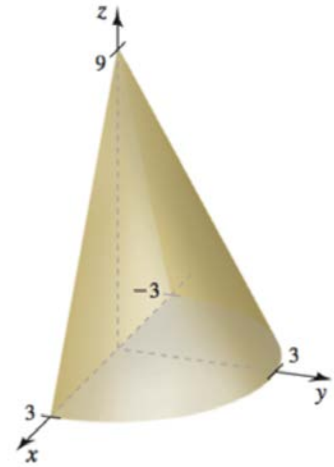


75. Evaluate the integral in cylindrical coordinates

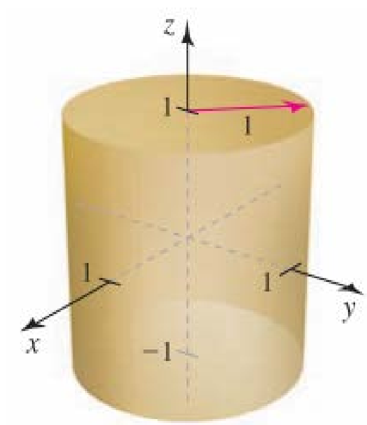
$$\int_0^{2\pi} \int_0^1 \int_{-1}^1 r \, dz dr d\theta$$



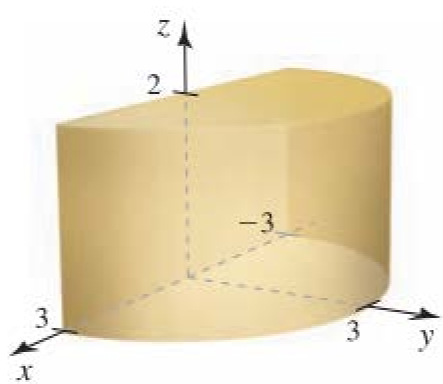
76. Evaluate the integral in cylindrical coordinates $$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_0^{9-3\sqrt{x^2+y^2}} dz\,dx\,dy$$



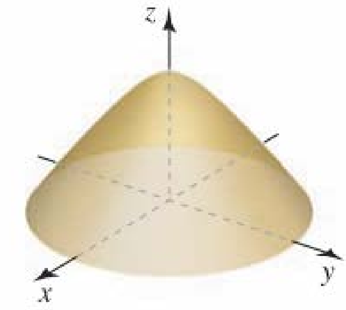
77. Evaluate the integral in cylindrical coordinates $$\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{-1}^1 \left(x^2+y^2\right)^{3/2} dz\,dx\,dy$$



78. Evaluate the integral in cylindrical coordinates $$\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \int_0^2 \frac{1}{1+x^2+y^2}\, dz\,dy\,dx$$

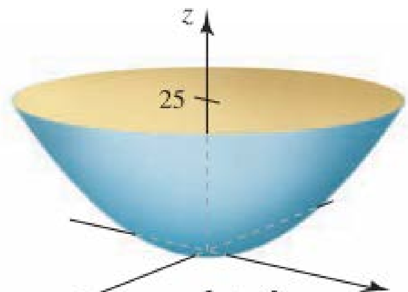


79. Evaluate the integral in cylindrical coordinates to find the volume of the solid bounded by the plane $z = 0$ and the hyperboloid $z = \sqrt{17} - \sqrt{1 + x^2 + y^2}$



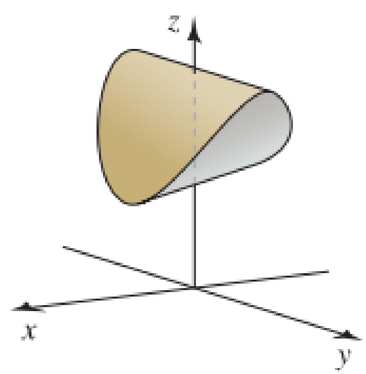
$z = \sqrt{17} - \sqrt{1 + x^2 + y^2}$

80. Evaluate the integral in cylindrical coordinates to find the volume of the solid bounded by the plane $z = 25$ and the paraboloid $z = x^2 + y^2$

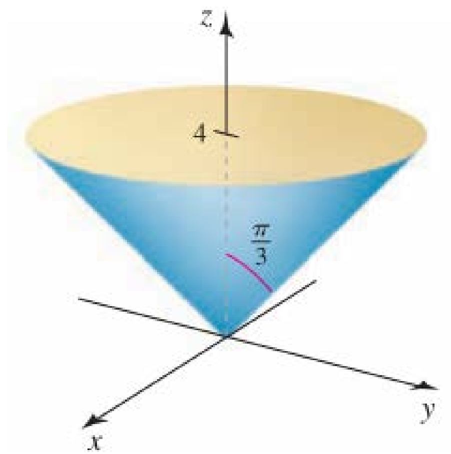


$z = x^2 + y^2$

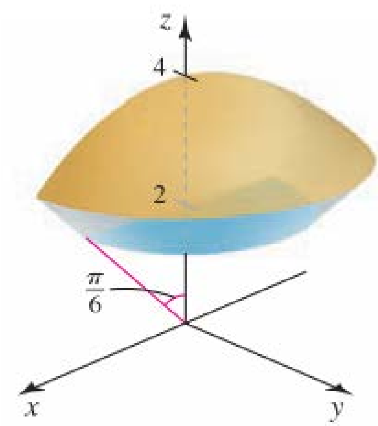
81. Evaluate the integral in cylindrical coordinates to find the volume of the solid bounded by the parabolic cylinders $z = y^2 + 1$ and $z = 2 - x^2$



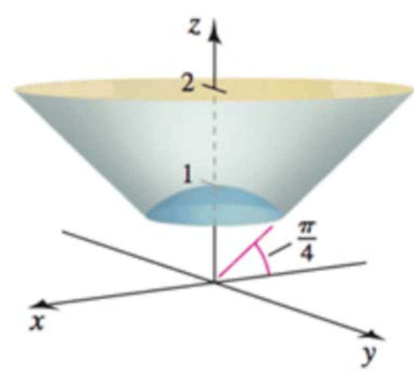
82. Evaluate the integral $\int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\sec\varphi} \rho^2 \sin\varphi \; d\rho d\varphi d\theta$



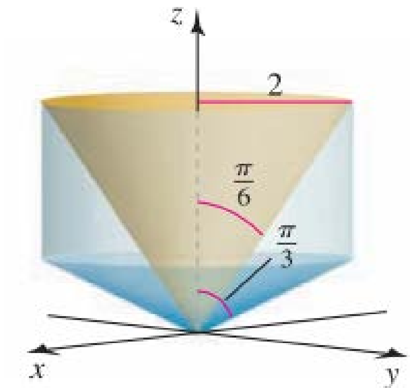
83. Evaluate the integral $\int_0^{\pi}\int_0^{\pi/6}\int_{2\sec\varphi}^{4} \rho^2 \sin\varphi \; d\rho d\varphi d\theta$



84. Evaluate the integral $\int_0^{2\pi}\int_0^{\pi/4}\int_1^{2\sec\varphi} \left(\rho^{-3}\right)\rho^2 \sin\varphi \; d\rho d\varphi d\theta$

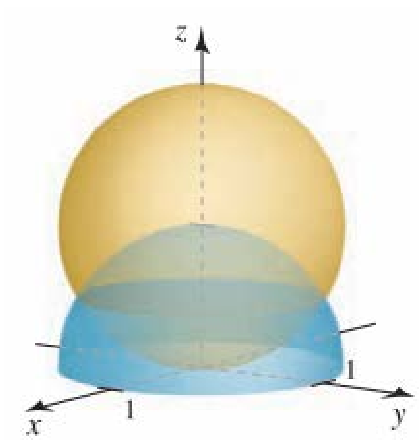


85. Evaluate the integral $\int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_0^{2\csc\varphi}\rho^2\sin\varphi\, d\rho d\varphi d\theta$



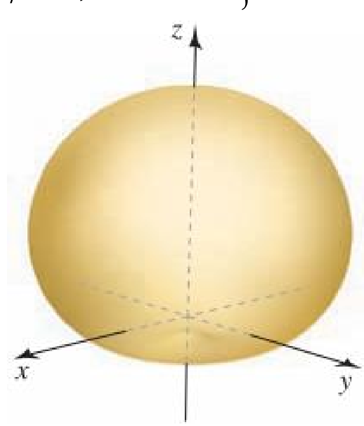
86. Use the spherical coordinates to find the volume of a ball of radius $a > 0$

87. Use the spherical coordinates to find the volume of the solid bounded by the sphere $\rho = 2\cos\varphi$ and the hemisphere $\rho = 1,\ z \geq 0$

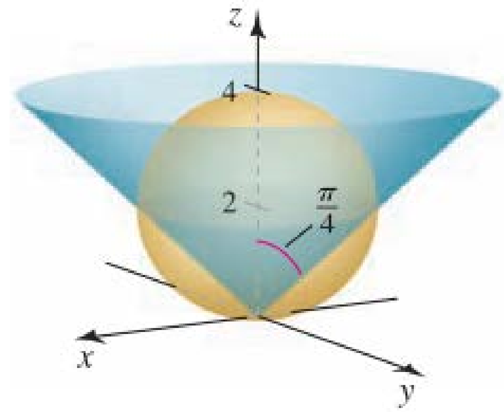


88. Use the spherical coordinates to find the volume of the solid of revolution

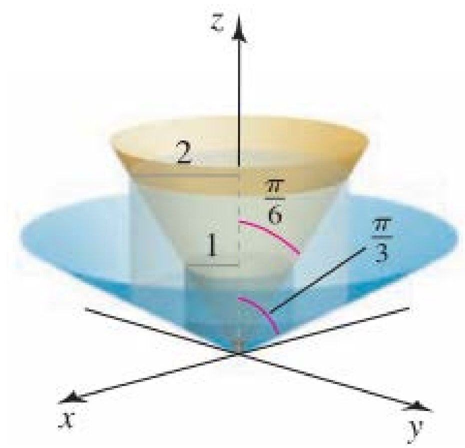
$$D = \{(\rho,\varphi,\theta):\ 0 \leq \rho \leq 1+\cos\varphi,\ 0 \leq \varphi \leq \pi,\ 0 \leq \theta \leq 2\pi\}$$



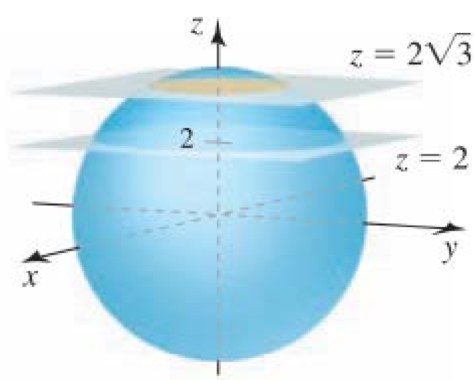
89. Use the spherical coordinates to find the volume of the solid outside the cone $\varphi = \frac{\pi}{4}$ and inside the sphere $\rho = 4\cos\varphi$



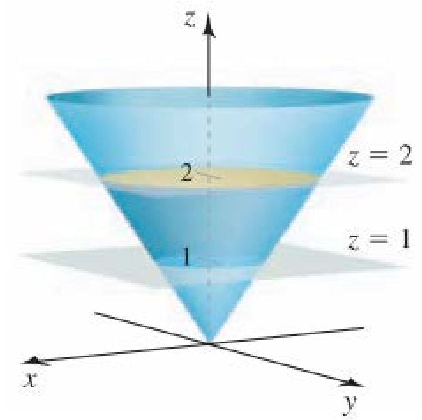
90. Use the spherical coordinates to find the volume of the solid bounded by the cylinders $r = 1$ and $r = 2$, and the cone $\varphi = \frac{\pi}{6}$ and $\varphi = \frac{\pi}{3}$



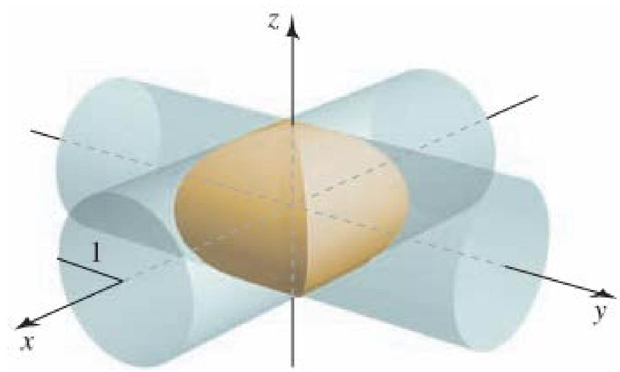
91. Use the spherical coordinates to find the volume of the ball $\rho \leq 4$ that lies between the planes $z = 2$ and $z = 2\sqrt{3}$



92. Use the spherical coordinates to find the volume of the solid inside the cone $z = \left(x^2 + y^2\right)^{1/2}$ that lies between the planes $z = 1$ and $z = 2$

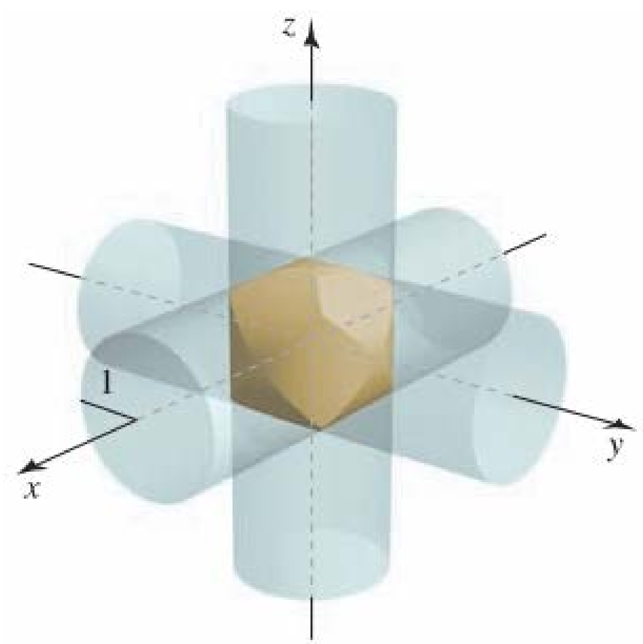


93. The *x*- and *y*-axes from the axes of two right circular cylinders with radius 1.



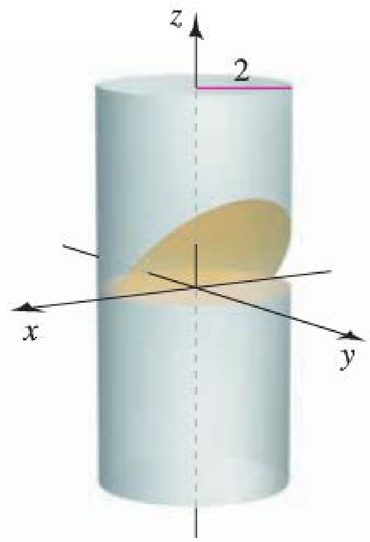
Find the volume of the solid that is common to the two cylinders.

94. The coordinate axes from the axes of three right circular cylinders with radius 1.

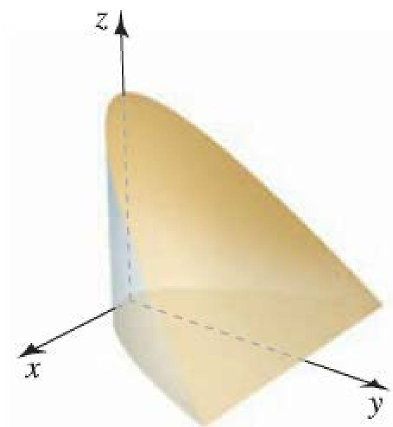


Find the volume of the solid that is common to the three cylinders.

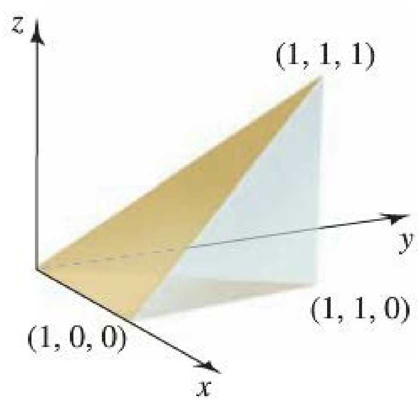
95. Find the volume of one of the wedges formed when the cylinder $x^2 + y^2 = 4$ is cut by the planes $z = 0$ and $y = z$



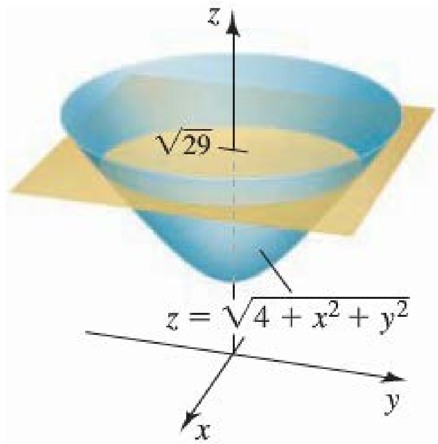
96. Find the volume of the region inside the parabolic cylinder $y = x^2$ between the planes $z = 3 - y$ and $z = 0$



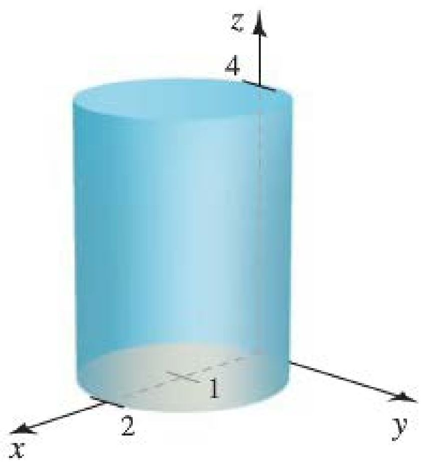
97. Find the volume of the tetrahedron with vertices $(0,\ 0,\ 0)$, $(1,\ 0,\ 0)$, $(1,\ 1,\ 0)$, and $(1,\ 1,\ 1)$



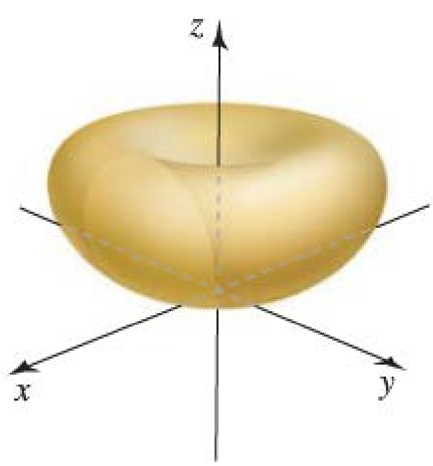
98. Find the volume of the region bounded by the plane $z=\sqrt{29}$ and the hyperboloid $z=\sqrt{4+x^2+y^2}$. Use integration in cylindrical coordinates.



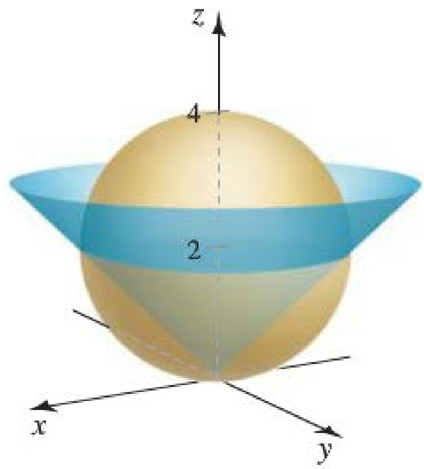
99. Find the volume of the solid cylinder whose height is 4 and whose base is the disk $\{(r,\theta):\ 0\le r\le 2\cos\theta\}$. Use integration in cylindrical coordinates



100. Use integration in spherical coordinates to find the volume of the rose petal of revolution $D=\left\{(\rho,\varphi,\theta):\ 0\le\rho\le 4\sin 2\varphi,\ 0\le\varphi\le\frac{\pi}{2},\ 0\le\theta\le 2\pi\right\}$

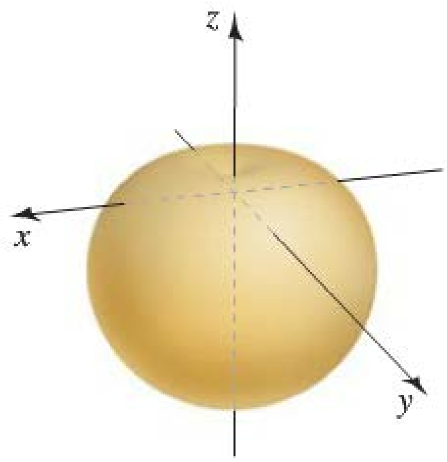


101. Use integration in spherical coordinates to find the volume of the region above the cone $\varphi = \frac{\pi}{4}$ and inside the sphere $\rho = 4\cos\varphi$.



102. Find the volume of the cardioid of revolution

$$D = \left\{ (\rho,\varphi,\theta) :\ 0 \le \rho \le \frac{1-\cos\varphi}{2},\ 0 \le \varphi \le \pi,\ 0 \le \theta \le 2\pi \right\}$$



103. A cake is shaped like a solid cone with radius 4 and height 2, with its base on the *xy*-plane. A wedge of the cake is removed by making two slices from the axis of the cone outward, perpendicular to the *xy*-plane separated by an angle of *Q* radians, where $0 < Q < 2\pi$

a) Find the volume of the slice for $Q = \frac{\pi}{4}$. Use geometry to check your answer.

b) Find the volume of the slice for $0 < Q < 2\pi$. Use geometry to check your answer.

104. A spherical fish tank with a radius of 1 *ft* is filled with water to a level 6 *in*. below the top of the tank.

a) Determine the volume and weight of the water in the fish tank. (The weight density of water is about 62.5 lb/ft^3.)

b) How much additional water must be added to completely fill the tank?

105. A spherical cloud of electric charge has known charge density $Q(\rho)$, where ρ is the spherical coordinate. Find the total charge in the cloud in the following cases.

a) $Q(\rho)=\dfrac{2\times10^{-4}}{\rho^4},\quad 1\le\rho<\infty$

b) $Q(\rho)=\dfrac{2\times10^{-4}}{1+\rho^3},\quad 1\le\rho<\infty$

c) $Q(\rho)=2\times10^{-4}e^{-0.01\rho^3},\quad 0\le\rho<\infty$

106. A point mass *m* is a distance *d* from the center of a thin spherical shell of mass *M* and radius *R*. The magnitude of the gravitational force on the point mass is given by the integral

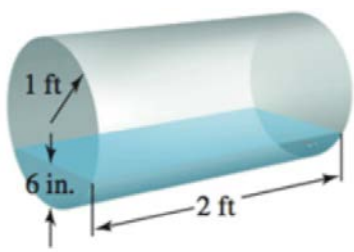
$$F(d)=\frac{GMm}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{(d-R\cos\phi)\sin\phi}{\left(R^2+d^2-2Rd\cos\phi\right)^{3/2}}\,d\phi d\theta$$

Where *G* is the gravitational constant.

a) Use the change of variable $x=\cos\phi$ to evaluate the integral and show that if $d>R$, then $F(d)=\dfrac{GMm}{d^2}$, which means the force is the same as if the mass of the shell were concentrated at its center.

b) Show that is $d<r$ (the point mass is inside the shell), then $F=0$.

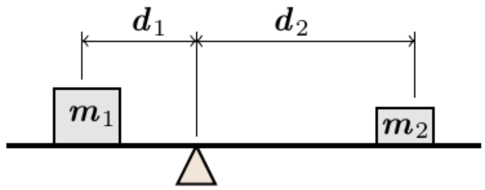
107. Before a gasoline-powered engine is started, water must be drained from the bottom of the fuel tank. Suppose the tank is a right circular cylinder on its side with a length of 2 *ft* and a radius of 1 *ft*. If the water level is 6 *in*. above the lowest part of the tank, determine how much water must be drained from the tank.



Section 3.6 – Integrals for Mass Calculations

One-Dimensional Center of Mass

If two objects with masses m_1 and m_2 sit at distances d_1 and d_2 from the pivot point (with no mass), then the balances provided $m_1 d_1 = m_2 d_2$



Solving the equation for $\overline{x}$, the balance point or center of mass of the two-mass system is located at

$$\overline{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

The quantities $m_1 x_1$ and $m_2 x_2$ are called moments about the origin (or just moments). The location of the center of mass is the sum of moments divided by the sum of the masses.

$$\overline{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

Mass and Moment Calculations

We treat coil springs and wires as masses distributed along smooth curves in space. The distribution is described by a continuous density function $\delta(x, y, z)$ representing mass per unit length. When a curve C is parametrized by $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}, \quad a \le t \le b$, the density is the function $\delta(x(t), y(t), z(t))$, and then the arc length differential is given by

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

The formula of mass is

$$M = m = \int_a^b \delta(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

Mass and moment formulas for coil springs, wires, and thin rods lying along a smooth curve C in space

Mass: $m = \int_C \delta ds$ $\delta = \delta(x,y,z)$ *is the density at* (x,y,z)
First moments about the coordinates planes: $M_{yz} = \int_C x\delta ds, \quad M_{xz} = \int_C y\delta ds, \quad M_{xy} = \int_C z\delta ds$
Coordinates of the center of mass: $\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}$
Moments of inertia about axes and other lines: $I_x = \int_C \left(y^2+z^2\right)\delta ds, \quad I_y = \int_C \left(x^2+z^2\right)\delta ds, \quad I_z = \int_C \left(x^2+y^2\right)\delta ds$ $I_L = \int_C r^2 \delta ds$ $r(x,y,z) =$ *distance from the point* (x,y,z) *to the line L*

Example

A slender metal arch, denser at the bottom than top, lies along the semicircle $z^2+y^2=1, \ \ z \ge 0$, in the *yz*-plane. Find the center of the arch's mass if the density at the point (x,y,z) on the arch is $\delta(x,y,z) = 2 - z$

Solution

$\bar{x} = 0$ *and* $\bar{y} = 0$, because the arch lies in the *yz*-plane with its mass distributed symmetrically about the *z*-axis.

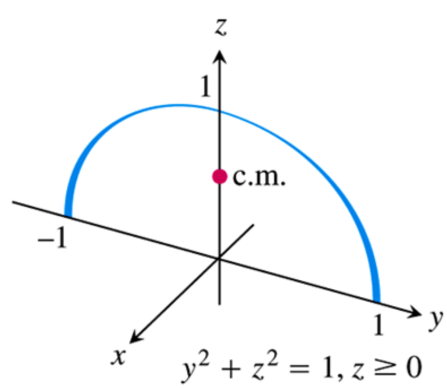
$$\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{k}, \quad 0 \le t \le \pi$$

$$|v(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$= \sqrt{(0)^2 + (-\sin t)^2 + (\cos t)^2}$$

$$= \sqrt{\sin^2 t + \cos^2 t}$$

$$= 1$$



$$\Rightarrow ds = |v|\,dt = dt$$

$$m = \int_0^{\pi} (2 - z)\ dt$$

$$= \int_0^{\pi} (2 - \sin t)\ dt$$

$$= [2t + \cos t]_0^{\pi}$$

$$= 2\pi + \cos\pi - \cos 0$$

$$\underline{= 2\pi - 2}$$

$$M_{xy} = \int_C z\delta ds$$

$$= \int_C z(2 - z) ds$$

$$= \int_0^{\pi} (\sin t)(2 - \sin t)\ dt$$

$$= \int_0^{\pi} \left(2\sin t - \sin^2 t\right)\ dt$$

$$= \left[-2\cos t - \frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{\pi}$$

$$= -2(-1) - \frac{\pi}{2} + 2$$

$$= 4 - \frac{\pi}{2}$$

$$\underline{= \frac{8 - \pi}{2}}$$

$$\bar{z} = \frac{M_{xy}}{m}$$

$$= \frac{8 - \pi}{2} \cdot \frac{1}{2\pi - 2}$$

$$= \frac{8 - \pi}{4\pi - 4} \qquad \underline{\approx 0.57}$$

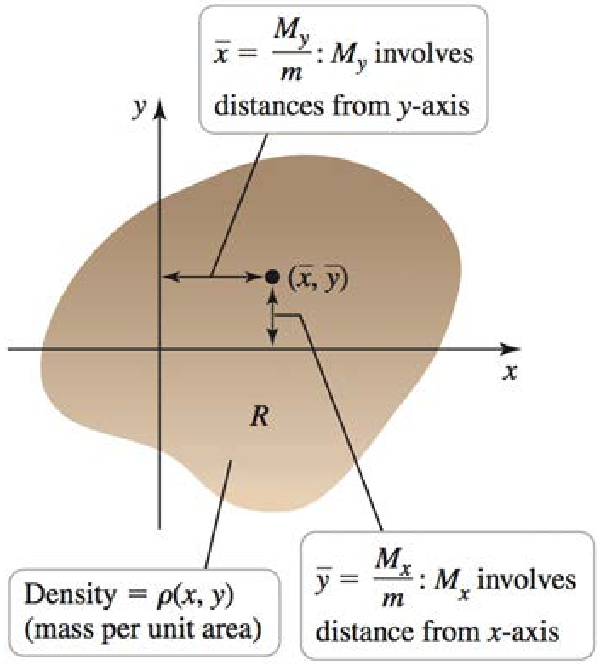
The center mass is $\underline{\left(0,\ 0,\ \frac{8 - \pi}{4\pi - 4}\right)}$

Two-Dimensional Objects

Definition

Let ρ be an integrable area density function defined over a closed bounded region $\boldsymbol{R}$ in $\mathbb{R}^2$. The coordinates of the center of mass of the object represented by $\boldsymbol{R}$ are:

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_R x\rho(x,y)\,dA \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_R y\rho(x,y)\,dA$$



Where $m = \iint_R \rho(x,y)\,dA$ is the mass, and M_y and M_x are the moments with respect to the y-axis and x-axis, respectively. If ρ is constant, the center of mass is called the ***centroid*** and is independent of the density,

Example

Find the centroid (center of mass) of the constant density, dart-shaped region bounded by the *y*-axis and the curves $y = e^{-x} - \frac{1}{2}$ *and* $y = \frac{1}{2} - e^{-x}$

Solution

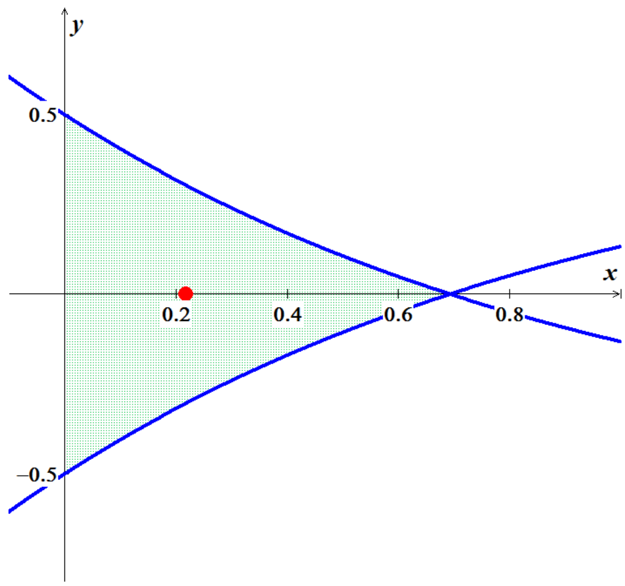
Assume: $\rho = 1$

$y = e^{-x} - \frac{1}{2} = \frac{1}{2} - e^{-x}$

$2e^{-x} = 1$

$-x = \ln\frac{1}{2}$

$\underline{x = \ln 2}$



$$m = \int_0^{\ln 2} \int_{\frac{1}{2}-e^{-x}}^{e^{-x}-\frac{1}{2}} 1\,dydx$$

$$= \int_0^{\ln 2} y \Big|_{\frac{1}{2}-e^{-x}}^{e^{-x}-\frac{1}{2}} dx$$

$$= \int_0^{\ln 2} \left(2e^{-x} - 1\right) dx$$

$$= \left[-2e^{-x} - x\right]_0^{\ln 2}$$

$$= -2e^{-\ln 2} - \ln 2 + 2$$

$$= -2\left(\frac{1}{2}\right) - \ln 2 + 2$$

$$\underline{= 1 - \ln 2} \quad \underline{\approx 0.307}$$

$$M_y = \int_0^{\ln 2} \int_{\frac{1}{2}-e^{-x}}^{e^{-x}-\frac{1}{2}} x\,dydx$$

$$= \int_0^{\ln 2} xy \Big|_{\frac{1}{2}-e^{-x}}^{e^{-x}-\frac{1}{2}} dx$$

$$= \int_0^{\ln 2} x\left(e^{-x} - \frac{1}{2} - \frac{1}{2} + e^{-x}\right) dx$$

		$\int e^{-x}$
+	x	$-e^{-x}$
−	1	e^{-x}

$$= \int_0^{\ln 2} x\left(2e^{-x} - 1\right)dx \qquad \int xe^{ax}dx = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right)$$

$$= -2xe^{-x} - 2e^{-x} - \frac{1}{2}x^2 \Bigg|_0^{\ln 2}$$

$$= -2(\ln 2)\left(\tfrac{1}{2}\right) - 2\left(\tfrac{1}{2}\right) - \tfrac{1}{2}(\ln 2)^2 + 0 + 2 + 0$$

$$\boxed{= 1 - \ln 2 - \tfrac{1}{2}(\ln 2)^2 \quad \approx 0.067}$$

$$\bar{x} = \frac{1 - \ln 2 - \frac{1}{2}(\ln 2)^2}{1 - \ln 2} \qquad \bar{x} = \frac{M_y}{m}$$

$$\boxed{= \frac{1}{2}\frac{2 - 2\ln 2 - (\ln 2)^2}{1 - \ln 2}}$$

$$\boxed{\approx 0.217}$$

The center of mass is located approximately at $\left(\dfrac{2 - 2\ln 2 - (\ln 2)^2}{2 - 2\ln 2},\ \ 0\right)$ $\quad (0.217,\ \ 0)$

Three-Dimensional Objects

Definition

Let ρ be an integrable area density function defined over a closed bounded region ***D*** in $\mathbb{R}^3$. The coordinates of the center of mass of the object represented by ***D*** are:

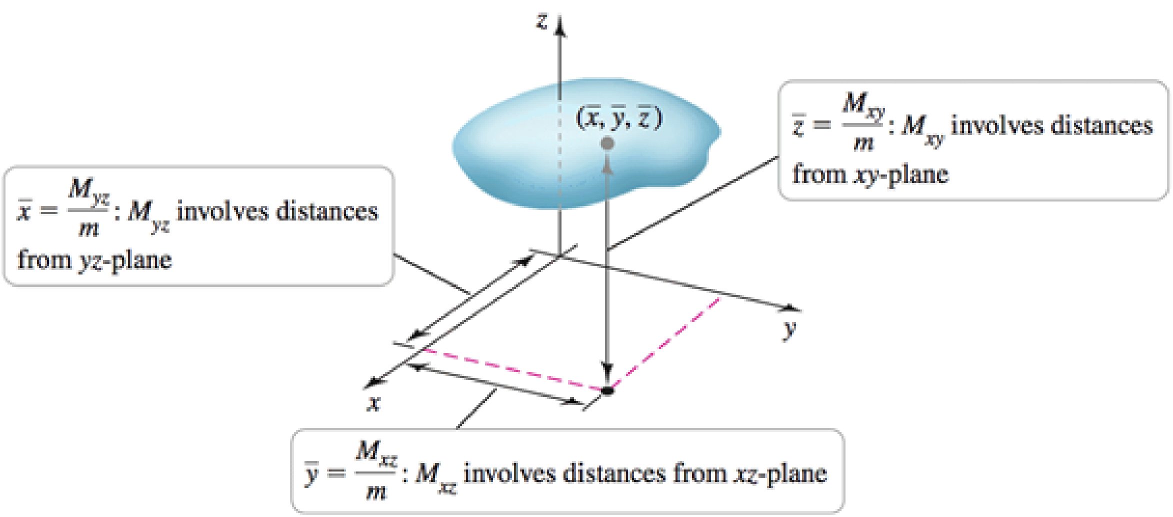
$$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m}\iiint_D x\,\rho(x,y,z)\,dV$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{1}{m}\iiint_D y\,\rho(x,y,z)\,dV$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m}\iiint_D z\,\rho(x,y,z)\,dV$$

Where $m = \iiint_D \rho(x,y,z)\,dV$ is the mass.

M_{yz}, M_{xz}, *and* M_{xy} are the moments with respect to the coordinates planes.



Example

Find the center of mass of the constant density solid cone ***D*** bounded by the surface

$$z = 4 - \sqrt{x^2 + y^2} \quad \text{and} \quad z = 0$$

Solution

The one is symmetric about the z-axis and has uniform density, the center of mass lies on the z-axis, that is, $\bar{x} = 0$ *and* $\bar{y} = 0$.

The disk has a radius of 4 and centered at the origin. Therefore, the cone has height 4 and radius 4; by the volume formula is $\frac{1}{3}\pi h r^2 = \frac{1}{3}\pi 4\left(4^2\right) = \frac{64\pi}{3}$.

The cone has a constant density, so we assume that $\rho = 1$ and its mass is $m = \frac{64\pi}{3}$

$$z = 4 - \sqrt{x^2 + y^2} = 4 - r$$

$$M_{xy} = \int_0^{2\pi}\int_0^4\int_0^{4-r} z\, dz\, r dr d\theta$$

$$= \int_0^{2\pi}\int_0^4 r\left(\frac{1}{2}z^2 \Big|_0^{4-r}\right) dr d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^4 r(4-r)^2\, dr d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^4 \left(16r - 8r^2 + r^3\right) dr d\theta$$

$$= \frac{1}{2}\left(8r^2 - \frac{8}{3}r^3 + \frac{1}{4}r^4 \Big|_0^4\right)\left(\theta \Big|_0^{2\pi}\right)$$

$$= \frac{1}{2}\left(128 - \frac{512}{3} + 64\right)(2\pi)$$

$$= \frac{64\pi}{3}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{64\pi/3}{64\pi/3}$$

$$= 1$$

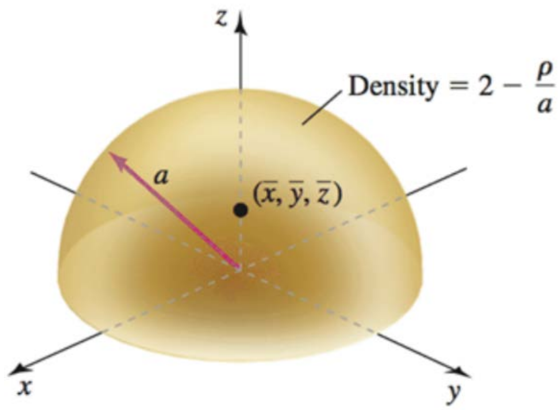
$\therefore$ The center of mass is located at $(0,\ 0,\ 1)$

Example

Find the center of mass of the interior of the hemisphere ***D*** of a radius a with its base on the xy-plane. The density of the objects is $f(\rho,\ \phi,\ \theta) = 2 - \frac{\rho}{a}$ (heavy near the center and light near the outer surface.)

Solution

The one is symmetric about the z-axis and has uniform density, the center of mass lies on the z-axis, that is $\overline{x} = 0$ *and* $\overline{y} = 0$.



$$m = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{a} \left(2 - \frac{\rho}{a}\right)\rho^2 \sin\phi \ d\rho \ d\phi \ d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\phi d\phi \int_0^{a} \left(2\rho^2 - \frac{1}{a}\rho^3\right) d\rho$$

$$= \theta \Big|_0^{2\pi} \left(-\cos\phi \Big|_0^{\pi/2}\right) \left(\frac{2}{3}\rho^3 - \frac{1}{4a}\rho^4 \Big|_0^{a}\right.$$

$$= (2\pi)(1)\left(\frac{2}{3}a^3 - \frac{1}{4}a^3\right)$$

$$\boxed{= \frac{5\pi}{6}a^3}$$

In spherical coordinate: $z = \rho\cos\phi$

$$M_{xy} = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{a} \rho\cos\phi\left(2 - \frac{\rho}{a}\right)\rho^2 \sin\phi \ d\rho \ d\phi \ d\theta$$

$$M_{xy} = \iiint_D z\ \rho(x, y, z) dV$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/2} \frac{1}{2}\sin 2\phi d\phi \int_0^{a} \left(2\rho^3 - \frac{1}{a}\rho^4\right) d\rho$$

$$2\sin\phi\cos\phi = \sin 2\phi$$

$$= \theta \Big|_0^{2\pi} \left(-\frac{1}{4}\cos 2\phi \Big|_0^{\pi/2}\right) \left(\frac{1}{2}\rho^4 - \frac{1}{5a}\rho^5 \Big|_0^{a}\right.$$

$$= -\frac{1}{4}(2\pi)(-2)\left(\frac{1}{2}a^4 - \frac{1}{5}a^4\right)$$

$$\boxed{= \frac{3\pi}{10}a^4}$$

$$\overline{z} = \frac{M_{xy}}{m} = \frac{\frac{3\pi a^4}{10}}{\frac{5\pi a^3}{6}}$$

$$\boxed{= \frac{9a}{25}}$$

However; the center of mass of a uniform-density hemisphere solid of radius a is $\frac{3a}{8} = 0.375a$ units above the base. In this particular case, the variable density shifts the center of mass.

Example

Find the moment of inertia about the x-axis of the curve: $4x = 2y^2 - \ln y$ from $y = 2$ to $y = 4$

Solution

$$x = \frac{1}{2}y^2 - \frac{1}{4}\ln y$$

$$\frac{dx}{dy} = y - \frac{1}{4y}$$

$$ds = \sqrt{\left(y - \frac{1}{4y}\right)^2 + 1} \qquad\qquad ds = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}$$

$$= \sqrt{y^2 - \frac{1}{2} + \frac{1}{16y^2} + 1}$$

$$= \sqrt{y^2 + \frac{1}{2} + \frac{1}{16y^2}}$$

$$= \sqrt{\left(y + \frac{1}{4y}\right)^2}$$

$$= y + \frac{1}{4y}$$

$$I_x = \int_2^4 y^2\left(y + \frac{1}{4y}\right)dy$$

$$= \int_2^4 \left(y^3 + \frac{1}{4}y\right)dy$$

$$= \frac{1}{4}y^4 + \frac{1}{8}y^2 \Big|_2^4$$

$$= 64 + 2 - 4 - \frac{1}{2}$$

$$= \frac{123}{2}$$

Exercises Section 3.6 – Integrals for Mass Calculations

Find the location of the center of mass

1. $m_1 = 10\ kg$ located at $x = 3\ m$; $m_2 = 3\ kg$ located at $x = -1\ m$

2. $m_1 = 8\ kg$ located at $x = 2\ m$; $m_2 = 4\ kg$ located at $x = -4\ m$; $m_2 = 1\ kg$ located at $x = 0\ m$

(**3 – 6**) Find the mass of the following objects with given density functions

3. The solid cylinder $D = \{(r,\ \theta,\ z):\ \ 0 \le r \le 4,\ \ \ 0 \le z \le 10\}$ with density $\rho(r,\ \theta,\ z) = 1 + \frac{z}{2}$

4. The solid cylinder $D = \{(r,\ \theta,\ z):\ \ 0 \le r \le 3,\ \ \ 0 \le z \le 2\}$ with density $\rho(r,\ \theta,\ z) = 5e^{-r^2}$

5. The solid cylinder $D = \{(r,\ \theta,\ z):\ \ 0 \le r \le 6,\ \ \ 0 \le z \le 6 - r\}$ with density $\rho(r,\ \theta,\ z) = 7 - z$

6. The solid cylinder $D = \left\{(r,\ \theta,\ z):\ \ 0 \le r \le 3,\ \ \ 0 \le z \le 9 - r^2\right\}$ with density $\rho(r,\ \theta,\ z) = 1 + \frac{z}{9}$

(**7 – 12**) Find the mass and center of mass of the thin rods with the following density functions.

7. $\rho(x) = 1 + \sin x \quad for \ \ 0 \le x \le \pi$

8. $\rho(x) = 1 + x^3 \quad for \ \ 0 \le x \le 1$

9. $\rho(x) = 2 - \frac{x^2}{16} \quad for \ \ 0 \le x \le 4$

10. $\rho(x) = 2 + \cos x \quad for \ \ 0 \le x \le \pi$

11. $\rho(x) = \begin{cases} x^2 & if \ \ 0 \le x \le 1 \\ 2x - x^2 & if \ \ 1 \le x \le 2 \end{cases}$

12. $\rho(x) = \begin{cases} 1 & if \ \ 0 \le x \le 2 \\ 1 + x & if \ \ 2 < x \le 4 \end{cases}$

(**13 – 27**) Find the mass and centroid (center of mass) of the following thin plates, assuming a constant density. Sketch the region corresponding to the plate and indicate the location of the center of mass. Use symmetry when possible to simplify your work.

13. The region bounded by $y = \sin x$ and $y = 1 - \sin x$ between $x = \frac{\pi}{4} \quad and \quad x = \frac{3\pi}{4}$

14. The region bounded by $y = 1 - |x|$ and the x-axis

15. The region bounded by $y = e^x, \quad y = e^{-x}, \quad x = 0, \quad and \quad x = \ln 2$

16. The region bounded by $y = \ln x, \quad x - axis, \quad and \quad x = e$

17. The region bounded by $x^2 + y^2 = 1 \quad and \quad x^2 + y^2 = 9, \quad for \ \ y \ge 0$

18. The region bounded by $y = \sin x$ and $y = 0$ between $x = 0$ and $x = \pi$.

19. The region bounded by $y = x^3$ and $y = x^2$ between $x = 0$ and $x = 1$.

20. The half annulus $\{(r, \theta):\ 2 \le r \le 4,\ 0 \le \theta \le \pi\}$

21. The region bounded by $y = x^2$ and $y = a^2 - x^2$

22. The semicircular disk $R = \{(r,\ \theta):\quad 0 \le r \le 2, \quad 0 \le \theta \le \pi\}$

23. The quarter-circular disk $R = \left\{(r,\ \theta):\quad 0 \le r \le 2, \quad 0 \le \theta \le \frac{\pi}{2}\right\}$

24. The region bounded by the cardioid $r = 1 + \cos\theta$

25. The region bounded by the cardioid $r = 3 - 3\cos\theta$

26. The region bounded by one leaf of the rose $r = \sin 2\theta$ for $0 \le \theta \le \frac{\pi}{2}$

27. The region bounded by the limaçon $r = 2 + \cos\theta$

(**28 – 42**) Find the coordinates of the center of mass of the following plane regions with variable density. Describe the distribution of mass in the region

28. $R = \{(x,\ y):\ 0 \le x \le 4,\ 0 \le y \le 2\}; \quad \rho(x,\ y) = 1 + \frac{x}{2}$

29. $R = \{(x,\ y):\ -1 \le x \le 1,\ 0 \le y \le 1\}; \quad \rho(x,\ y) = 2 - y$

30. $R = \{(x,\ y):\ 0 \le x \le 1,\ 0 \le y \le 5\}; \quad \rho(x,\ y) = 2e^{-y/2}$

31. $R = \{(x,\ y,\ z):\ 0 \le x \le 4,\ 0 \le y \le 1,\ 0 \le z \le 1\}; \quad \rho(x,\ y,\ z) = 1 + \frac{x}{2}$

32. The triangular plate in the first quadrant bounded by $x + y = 4$ with $\rho(x,\ y) = 1 + x + y$

33. The upper half $(y \ge 0)$ of the disk bounded by the circle $x^2 + y^2 = 4$ with $\rho(x,\ y) = 1 + \frac{y}{2}$

34. The upper half $(y \ge 0)$ of the disk bounded by the ellipse $x^2 + 9y^2 = 9$ with $\rho(x,\ y) = 1 + y$

35. The quarter disk in the first quadrant bounded by $x^2 + y^2 = 4$ with $\rho(x, y) = 1 + x^2 + y^2$

36. The upper half of a ball $\left\{(\rho, \varphi, \theta):\ 0 \le \rho \le 16,\ 0 \le \varphi \le \frac{\pi}{2},\ 0 \le \theta \le 2\pi\right\}$ with density

$f(\rho,\ \varphi,\ \theta) = 1 + \frac{\rho}{4}$

37. The region bounded by the upper half of the sphere $\rho = 6$ and $z = 0$ with density

$f(\rho,\ \varphi,\ \theta) = 1 + \frac{\rho}{4}$

38. The cube in the first octant bounded by the planes $x = 2$, $y = 2$, $z = 2$, with $\rho(x, y, z) = 1 + x + y + z$

39. The interior of the cube in the first octant formed by the planes $x = 1$, $y = 1$, $z = 1$ with $\rho(x,\ y,\ z) = 2 + x + y + z$

40. The region bounded by the paraboloid $z = 4 - x^2 - y^2$ and $z = 0$ with $\rho(x, y, z) = 5 - z$

41. The interior of the prism formed by $x=1$, $y=4$, $z=x$, and the coordinate planes with $\rho(x,\ y,\ z)=2+y$

42. The region bounded by the cone $z=9-r$ and $z=0$ with $\rho(r,\ \theta,\ z)=1+z$

(**43 – 51**) Find the center of mass of the following solids, assuming a constant density of 1. Sketch the region and indicate the location of the centroid. Use symmetry when possible and choose a convenient coordinate system.

43. The upper half of the ball $x^2+y^2+z^2\le 16 \quad (\textit{for } z\ge 0)$

44. The region bounded by the paraboloid $z=x^2+y^2$ and the plane $z=25$

45. The tetrahedron in the first octant bounded by $z=1-x-y$ and the coordinate planes

46. The solid bounded by the cone $z=16-r$ and the plane $z=0$

47. The paraboloid bowl bounded by $z=x^2+y^2$ and $z=36$

48. The tetrahedron bounded by $z=4-x-2y$ and the coordinate planes.

49. The solid bounded by the cone $z=4-\sqrt{x^2+y^2}$ and the plane $z=0$

50. The sliced solid cylinder bounded by $x^2+y^2=1,\ z=0,\ \textit{and}\ \ y+z=1$

51. The solid bounded by the upper half $(z\ge 0)$ of the ellipsoid $4x^2+4y^2+z^2=16$

(**52 – 60**) Consider the following two- and three- dimensional regions. Compute the center of mass assuming constant density. All parameters are positive real numbers.

52. A region is bounded by a paraboloid with a circular base of radius R and height h. How far from the base is the center of mass?

53. Let R be the region enclosed by an equilateral triangle with sides of length s. what is the perpendicular distance between the center of mass of R and the edges of R?

54. An isosceles triangle has two sides of length s and a base of length b. how far from the base is the center of mass of the region enclosed by the triangle?

55. A tetrahedron is bounded by the coordinate planes and the plane $x+\frac{y}{2}+\frac{z}{3}=1$. What are the coordinates of the center of mass?

56. A solid box has sides of length a, b, and c. Where is the center of mass relative to the faces of the box?

57. A solid cone has a base with a radius of r and a height of h. How far from the base is the center of mass?

58. A solid is enclosed by a hemisphere of radius a. How far from the base is the center of mass?

59. A tetrahedron is bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{a}+\frac{z}{a}=1$. What are the coordinates.

60. A solid is enclosed by the upper half of an ellipsoid with a circular base of radius *r* and a height of *a*. How far from the base is the center of mass?

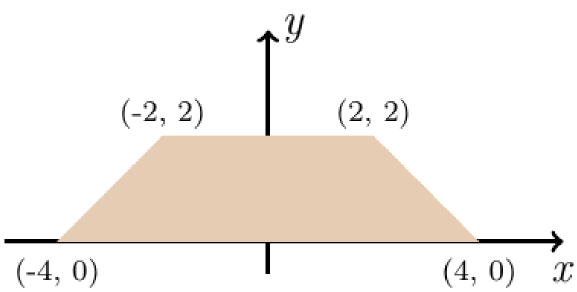
61. A thin (one-dimensional) wire of constant density is bent into the shape of a semicircular of radius *r*. Find the location of its center of mass.

62. A thin plate of constant density occupies the region between the parabola $y = ax^2$ and the horizontal line $y = b$, where $a > 0$ and $b > 0$. Show that the center of mass is $\left(0, \frac{3b}{5}\right)$, independent of *a*.

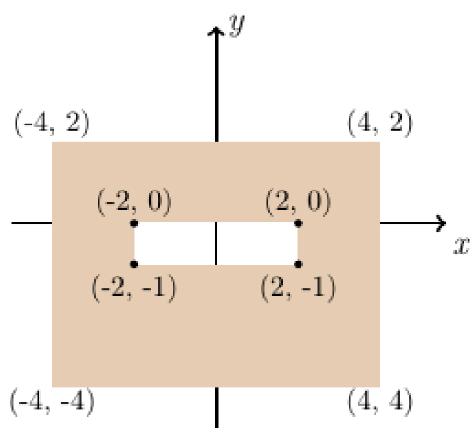
63. Find the center of mass of the region in the first quadrant bounded by the circle $x^2 + y^2 = a^2$ and the lines $x = a$ and $y = a$, where $a > 0$

Find the mass and center of mass of the thin constant-density of the plate

64.



65.



66. A thin rod of length *L* has a linear density given by $\rho(x) = 2e^{-x/3}$ on the interval $0 \le x \le L$. Find the mass and center of mass of the rod. How does the center of mass change as $L \to \infty$?

67. A thin rod of length *L* has a linear density given by $\rho(x) = \frac{10}{1+x^2}$ on the interval $0 \le x \le L$. Find the mass and center of mass of the rod. How does the center of mass change as $L \to \infty$?

68. A thin plate is bounded by the graphs of $y = e^{-x}$, $y = -e^{-x}$, $x = 0$, and $x = L$. Find its center of mass. How does the center of mass change as $L \to \infty$?

69. Consider the thin constant-density plate $\{(r,\ \theta):\ a \le r \le 1,\ 0 \le \theta \le \pi\}$ bounded by two semicircles and the x-axis.

a) Find the graph the y-coordinate of the center of mass of the plate as a function of a.

b) For what value of a is the center of mass on the edge of the plate?

70. Consider the thin constant-density plate $\left\{(\rho,\ \phi,\ \theta):\ 0 < a \le \rho \le 1,\ 0 \le \phi \le \frac{\pi}{2},\ 0 \le \theta \le 2\pi\right\}$ bounded by two hemispheres and the xy-axis.

a) Find the graph the z-coordinate of the center of mass of the plate as a function of a.

b) For what value of a is the center of mass on the edge of the solid?

71. A cylindrical soda can has a radius of 4 *cm* and a height of 12 *cm*. When the can is full of soda, the center of mass of the contents of the can is 6 cm above the base on the axis of the can (halfway along the axis of the can). As the can is drained, the center of mass descends for a while. However, when the can is empty (filled only with air), the center of mass is once again 6 *cm* above the base on the axis of the can. Find the depth of soda in the can for which the center of mass is at its lowest point. Neglect the mass of the can, and assume the density of the soda is $1\ g / cm^3$ and the density of air is $0.001\ g / cm^3$.

72. For $0 \le r \le 1$, the solid bounded by the cone $z = 4 - 4r$ and the solid bounded by the paraboloid $z = 4 - 4r^2$ have the same base in the xy-plane and the same height. Which object has the greater mass if the density of both objects is $\rho(r,\ \theta,\ z) = 10 - 2z$

73. For $0 \le r \le 1$, the solid bounded by the cone $z = 4 - 4r$ and the solid bounded by the paraboloid $z = 4 - 4r^2$ have the same base in the xy-plane and the same height. Which object has the greater mass if the density of both objects is $\rho(r,\ \theta,\ z) = \frac{8}{\pi} e^{-z}$

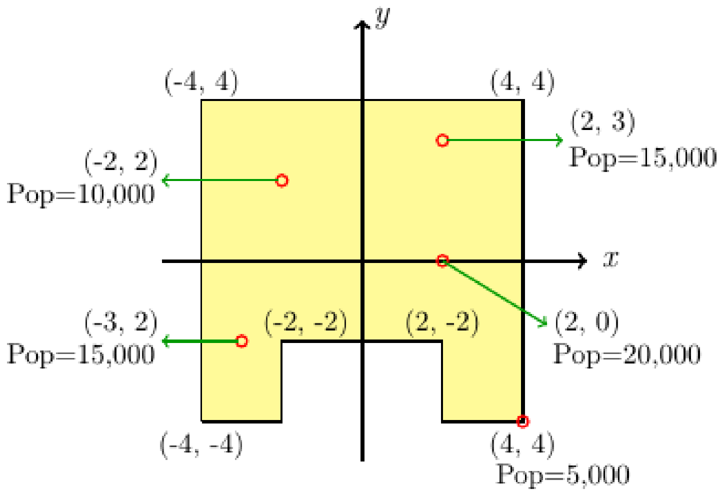
74. A right circular cylinder with height 8 *cm* and radius 2 *cm* is filled with water. A heated filament running along its axis produces a variable density in the water given by $\rho(r) = 1 - 0.05e^{-0.01r^2}$ g/cm^3 (ρ stands for density, not the radial spherical coordinate). Find the mass of the water in the cylinder. Neglect the volume of the filament.

75. A triangular region has a base that connects the vertices $(0,\ 0)$ and $(b,\ 0)$, and a third vertex at $(a,\ h)$, where $a > 0,\ b > 0$, and $h > 0$

a) Show that the centroid of the triangle is $\left(\frac{a+b}{3},\ \frac{h}{3}\right)$

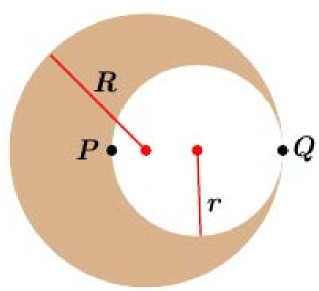
b) Recall that the three medians of a triangle extend from each vertex to the midpoint of the opposite side. Knowing that the medians of a triangle intersect in a point M and that each median bisects the triangle, conclude that the centroid of the triangle is M.

76. Geographers measure the geographical center of a country (which is the centroid) and the population center of a country (which is the center of mass computed with the population density).A hypothetical country is shown below with the location and population of five towns.



Assuming no one lives outside the towns, find the geographical center of the country and the population center of the country,

77. A disk radius r is removed from a larger disk of radius R to from an earring. Assume the earing is a thin plate of uniform density.



a) Find the center of mass of the earring in terms of r and R. (Hint: Place the origin of a coordinate system either at the center of the larger disk or at Q; either way, the earring is symmetric about the x-axis.)

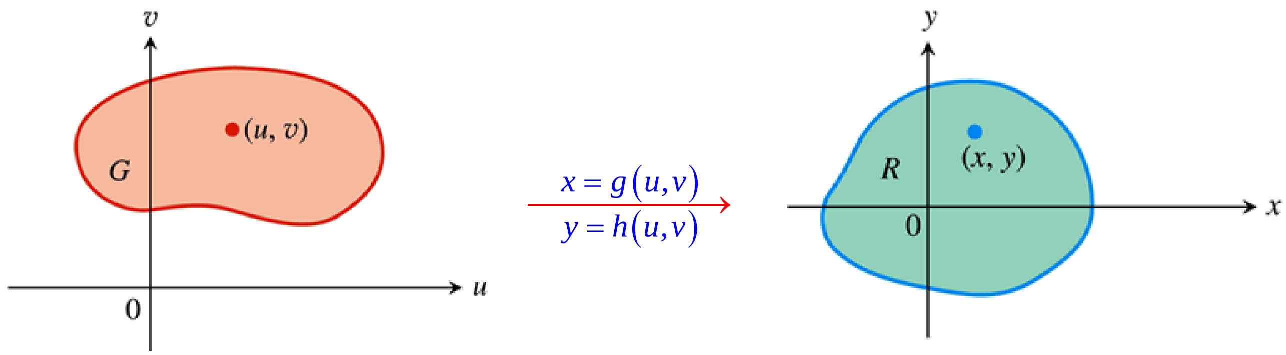
b) Show that the ratio $\frac{R}{r}$ such that the center of mass lies at the point P (on the edge of the inner disk) is the golden mean $\frac{1+\sqrt{5}}{2}$.

Section 3.7 – Change of variables in Multiple Integrals

Substitution in Double Integrals

Suppose that a region G in the uv-plane is transformed one-to-one into the region R in the xy-plane by equations of the form

$$x = g(u,v), \quad y = h(u,v)$$



Cartesian *uv*-plane ***Cartesian*** *xy* - plane

R is the image of G under the transformation, and G the ***preimage*** of R.

$$\iint_R f(x,y)\,dxdy = \iint_G f\big(g(u,v),h(u,v)\big)\,\left|J(u,v)\right|\,dudv$$

Definition

The ***Jacobian determinant*** or ***Jacobian*** of the coordinate transformation $x = g(u,v), \quad y = h(u,v)$ is

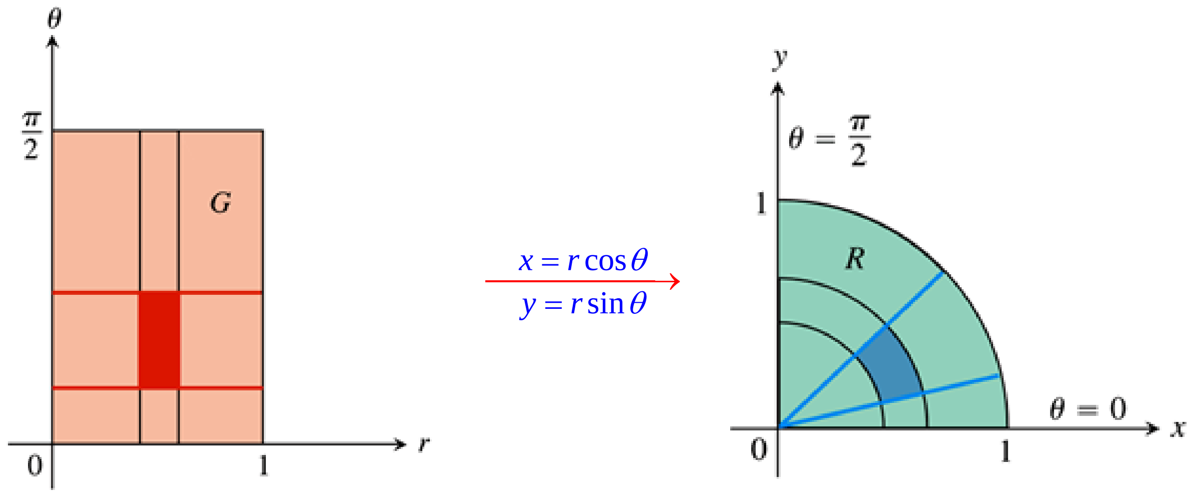
$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Example

Find the Jacobian for the polar coordinate transformation $x = r\cos\theta, \quad y = r\sin\theta$, write the Cartesian integral $\iint_R f(x, y)\,dxdy$ as a polar integral.

Solution



Cartesian $r\theta$–plane

Cartesian xy –plane

$x = r\cos\theta, \quad y = r\sin\theta$ transform the rectangle G: $0 \le r \le 1, \ 0 \le \theta \le 2\pi$, into the quarter circle R bounded by $x^2 + y^2 = 1$ in QI.

$$J(r, \theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r\left(\cos^2\theta + \sin^2\theta\right)$$

$$\underline{= r}$$

Example

Evaluate $\displaystyle\int_0^4 \int_{\frac{1}{2}y}^{\frac{1}{2}y+1} \frac{2x-y}{2}\,dxdy$ by applying the transformation $u=\dfrac{2x-y}{2}$, $v=\dfrac{y}{2}$ and integrating over an appropriate region in the *uv*-plane.

Solution

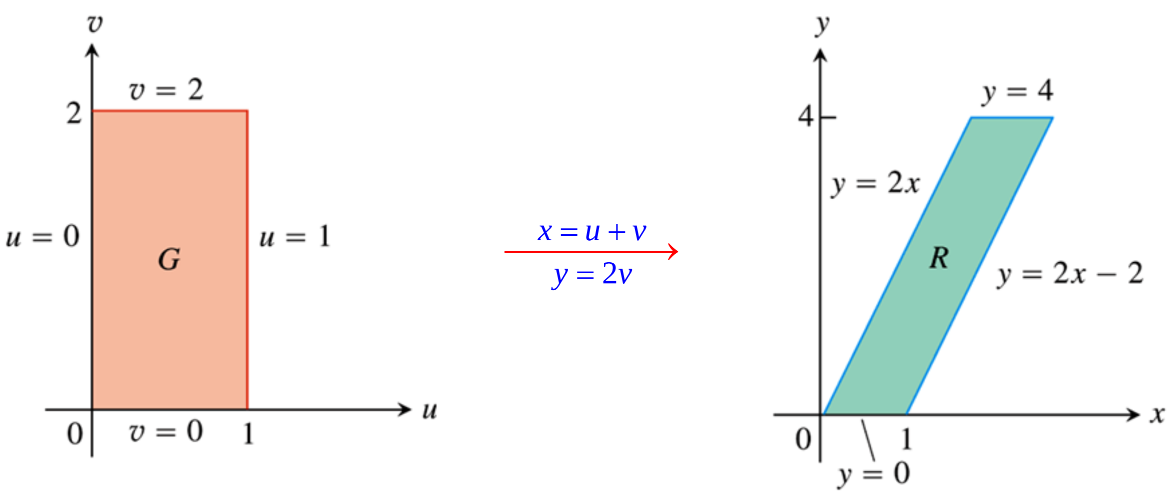
$\underline{y=2v}|$

$2u=2x-y$

$x=\dfrac{2u+y}{2}$

$\quad =\dfrac{2u+2v}{2}$

$\quad \underline{=u+v}|$



xy-eqns for the boundary of *R*	Corresponding *uv*-eqns. for the boundary of *G*	Simplified *uv-eqns*.
$x=\dfrac{y}{2}$	$u+v=\dfrac{2v}{2}=v$	$u=0$
$x=\dfrac{y}{2}+1$	$u+v=\dfrac{2v}{2}+1=v+1$	$u=1$
$y=0$	$2v=0$	$v=0$
$y=4$	$2v=4$	$v=2$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial u}(u+v) & \frac{\partial}{\partial v}(u+v) \\ \frac{\partial}{\partial u}(2v) & \frac{\partial}{\partial v}(2v) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}$$

$$\underline{= 2}$$

$$\int_0^4 \int_{\frac{1}{2}y}^{\frac{1}{2}y+1} \frac{2x-y}{2}\, dxdy = \int_0^{v=2} \int_{u=0}^{u=1} u\left|J(u,v)\right|\, dudv$$

$$= \int_0^2 dv \int_{u=0}^{1} (u)(2)du$$

$$= v \Big|_0^2 \quad u^2 \Big|_0^1$$

$$= (2)(1)$$

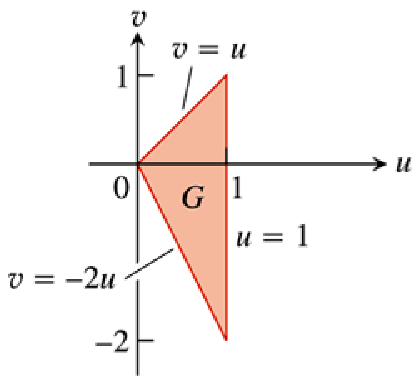
$$\underline{= 2}$$

Example

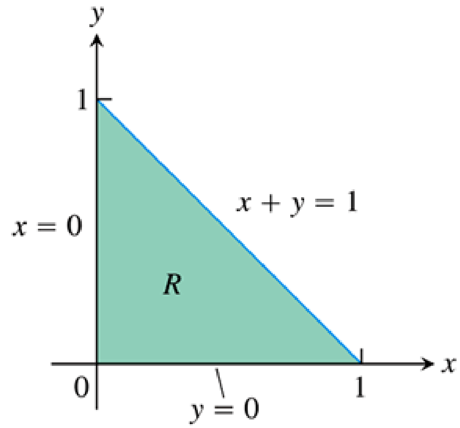
Evaluate $\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dydx$

Solution

$$\begin{cases} u = x+y \\ v = y-2x \end{cases} \to \begin{cases} x = \frac{u}{3} - \frac{v}{3} \\ y = \frac{2u}{3} + \frac{v}{3} \end{cases}$$



$$\xrightarrow[y = \frac{2u}{3} + \frac{v}{3}]{x = \frac{u}{3} - \frac{v}{3}}$$



xy*-eqns for the boundary of *R	**Corresponding *uv*-eqns. for the boundary of *G***	**Simplified *uv*-eqns.**
$x+y=1$	$\left(\frac{u}{3}-\frac{v}{3}\right)+\left(\frac{2u}{3}+\frac{v}{3}\right)=1$	$u=1$
$y=0$	$\frac{2u}{3}+\frac{v}{3}=1$	$v=-2u$
$x=0$	$\frac{u}{3}-\frac{v}{3}=0$	$v=u$
$x=1$	$u=3+v$	$y=2+v\Big\vert_{v=0}=2>1$

$$J(u,v) = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} \qquad J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \frac{1}{3}$$

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dydx = \int_{u=0}^{u=1} \int_{v=-2u}^{v=u} u^{1/2} v^2 \left|J(u,v)\right| dvdu$$

$$= \int_0^1 \int_{-2u}^{u} u^{1/2} v^2 \left(\frac{1}{3}\right) dv du$$

$$= \int_0^1 u^{1/2} \left(\frac{1}{9} v^3 \Big|_{-2u}^{u}\right) du$$

$$= \frac{1}{9} \int_0^1 u^{1/2} \left(u^3 + 8u^3\right) du$$

$$= \int_0^1 u^{7/2} du$$

$$= \frac{2}{9} u^{9/2} \Big|_0^1$$

$$\boxed{= \frac{2}{9}}$$

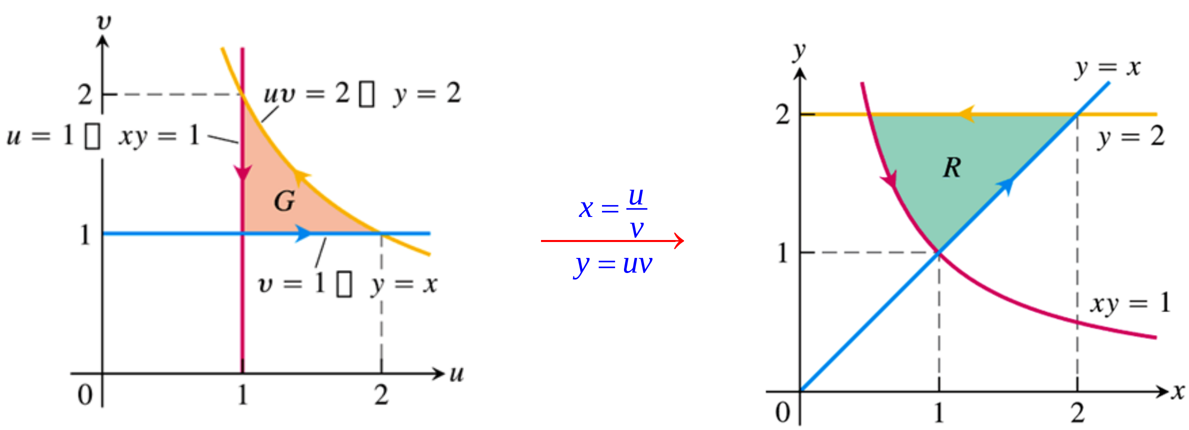
Example

Evaluate $\displaystyle\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}} dxdy$

Solution

$$\begin{cases} u = \sqrt{xy} \\ v = \sqrt{\dfrac{y}{x}} \end{cases} \rightarrow \begin{cases} u^2 = xy \\ v^2 = \dfrac{y}{x} \end{cases}$$

$$\rightarrow \boxed{x = \frac{u}{v}, \quad y = uv}$$



$$J(u,v)=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{1}{v} & -\frac{u}{v^2}\\ v & u\end{vmatrix}$$

$$=\frac{2u}{v}$$

xy*-eqns for the boundary of *R	**Corresponding *uv*-eqns. for the boundary of *G***	**Simplified *uv*-eqns.**
$x=y$	$\frac{u}{v}=uv$	$v=1$
$x=\frac{1}{y}$	$\frac{u}{v}=\frac{1}{uv}$	$u=1$
$y=1$	$uv=1$	
$y=2$	$uv=2$	$u=2 \quad v=\frac{2}{u}$

$$\int_1^2\int_{1/y}^{y}\sqrt{\frac{y}{x}}\,e^{\sqrt{xy}}dxdy=\int_1^2\int_1^{2/u}2u\ e^u dvdu$$

$$=2\int_1^2 u\ e^u\ v\ \Big|_0^{2/u}\ du$$

$$=2\int_1^2 u\ e^u\left(\frac{2}{u}-1\right)du$$

$$=2\int_1^2(2-u)e^u du$$

		e^u
$(+)$	$2-u$	e^u
$(-)$	-1	e^u
	0	

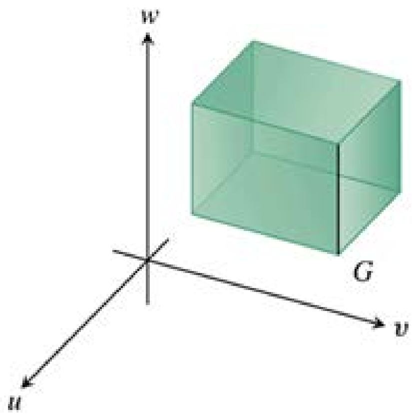
$$=2\left((2-u+1)e^u\ \Big|_1^2\right.$$

$$=2\left[(1)e^2-2e\right]$$

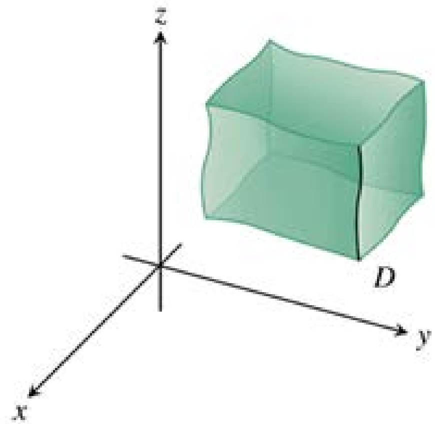
$$=2e(e-2)$$

Substitutions in Triple Integrals

$$x = g(u,v,w), \quad y = h(u,v,w), \quad z = k(u,v,w)$$



$x=g(u,v,w)$
$y=h(u,v,w)$
$z=k(u,v,w)$ $\longrightarrow$



Cartesian *uvw* - plane

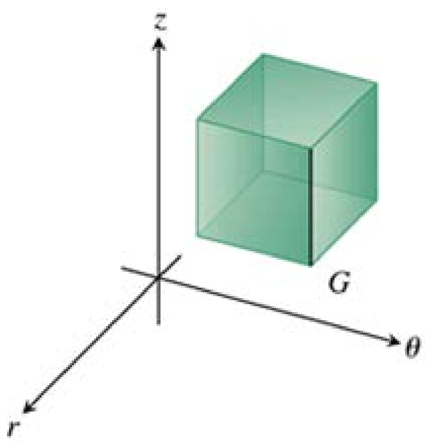
Cartesian *xyz* - plane

$$\iiint_R f(x,y,z)\, dxdydz = \iiint_R H(u,v,w)\, \left|J(u,v,w)\right|\, dudvdw$$

The ***Jacobian determinant*** is

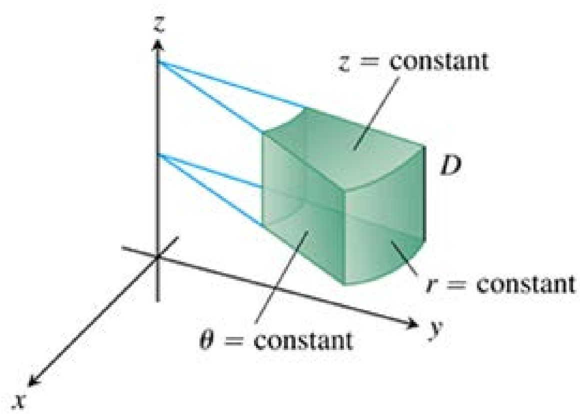
$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

Cube with sides parallel to the axes



$x=r\cos\theta$
$y=r\sin\theta$
$z=z$ $\longrightarrow$

Cube with sides parallel to the axes



Cartesian *rθz* - plane

Cartesian *xyz* - plane

$$J(r,\theta,z)=\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$=\begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$=r\cos^2\theta+r\sin^2\theta$$

$$\underline{=r}$$

For spherical coordinates, ρ, ϕ, and θ take the place of u, v, and w. The transformation from Cartesian $\rho\phi\theta$-space to Cartesian xyz –space is given by

$$x=\rho\sin\phi\cos\theta,\quad y=\rho\sin\phi\sin\theta,\quad z=\rho\cos\phi$$

The Jacobian of the transformation

$$J(\rho,\phi,\theta)=\begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}$$

$$=\begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$=\rho^2\cos^2\phi\sin\phi\cos^2\theta+\rho^2\sin^3\phi\sin^2\theta+\rho^2\sin\phi\cos^2\phi\sin^2\theta+\rho^2\sin^3\phi\cos^2\theta$$

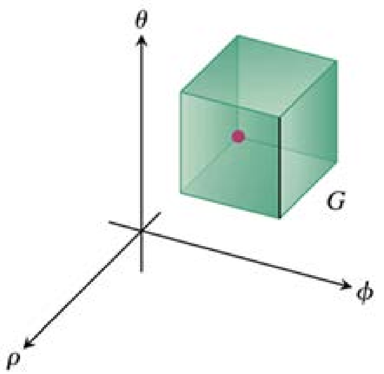
$$=\rho^2\cos^2\phi\sin\phi\left(\cos^2\theta+\sin^2\theta\right)+\rho^2\sin^3\phi\left(\sin^2\theta+\cos^2\theta\right)$$

$$=\rho^2\cos^2\phi\sin\phi+\rho^2\sin^3\phi$$

$$=\rho^2\sin\phi\left(\cos^2\phi+\sin^2\phi\right)$$

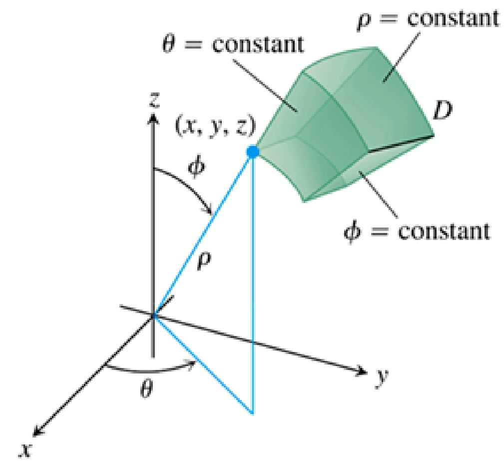
$$\underline{=\rho^2\sin\phi}$$

$$\iiint_D F(x,y,z)\,dxdydz=\iiint_G H(\rho,\phi,\theta)\left|\rho^2\sin\phi\right|\,d\rho d\phi d\theta$$



Cartesian $r\theta z$ - plane

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$



Cartesian xyz - plane

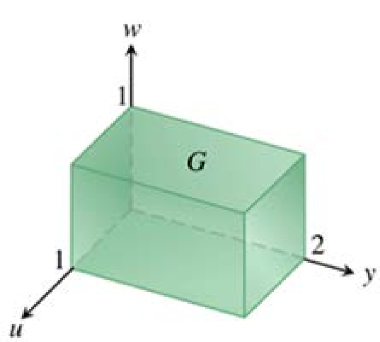
Example

Evaluate $\displaystyle\int_0^3 \int_0^4 \int_{\frac{1}{2}y}^{\frac{1}{2}y+1} \left(\frac{2x-y}{2} + \frac{z}{3}\right) dxdydz$ by applying the transformation

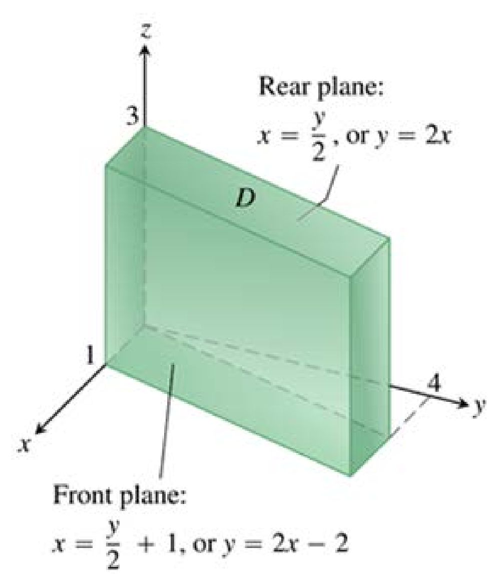
$u = \dfrac{2x-y}{2}, \quad v = \dfrac{y}{2}, \quad w = \dfrac{z}{3}$ and integrating over an appropriate region in the *uvw*-plane.

Solution

$$\rightarrow \begin{cases} u = \dfrac{2x-y}{2} \rightarrow x = u + \dfrac{y}{2} = u + v \\ \quad\quad v = \dfrac{y}{2} \rightarrow y = 2v \\ \quad\quad w = \dfrac{z}{3} \rightarrow z = 3w \end{cases}$$



$$x = u + v$$
$$y = 2v$$
$$z = 3w$$



xyz-*eqns* for the boundary of ***D***	Corresponding *uvw*- *eqns*. for the boundary of ***G***	Simplified *uvw*- *eqns*.
$x=\frac{y}{2}$	$u+v=v$	$u=0$
$x=\frac{y}{2}+1$	$u+v=v+1$	$u=1$
$y=0$	$2v=0$	$v=0$
$y=4$	$2v=4$	$v=2$
$z=0$	$3w=0$	$w=0$
$z=3$	$3w=3$	$w=1$

$$J(u,v,w)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

$$\underline{=6}$$

$$\int_0^3\int_0^4\int_{\frac{1}{2}y}^{\frac{1}{2}y+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dxdydz=\int_0^1\int_0^2\int_0^1 (u+w)\left|J(u,v,w)\right|\,dudvdw$$

$$=6\int_0^1\int_0^2\int_0^1 (u+w)\,dudvdw$$

$$=6\int_0^1\int_0^2 \left(\frac{u^2}{2}+wu\,\Big|_0^1\right. dvdw$$

$$=6\int_0^1\int_0^2 \left(\frac{1}{2}+w\right) dvdw$$

$$=6\int_0^1 \left(\frac{1}{2}v+wv\,\Big|_0^2\right. dw$$

$$= 6\int_0^1 (1+2w)\,dw$$

$$= 6\left(w + w^2 \right|_0^1$$

$$= 6(1+1)$$

$$\underline{= 12}$$

Example

Evaluate $\iiint_D xz\ dV$: D is bounded by the planes: $y - x = 0,\ \ y = 2 + x,\ \ z - y = 0,\ \ z - y = 2,\ \ z = 0,$ and $z = 3$

Solution

$$\begin{cases} y - x = 0 \\ y - x = 2 \end{cases} \quad let \quad \underline{u = y - x}$$

$$\Rightarrow \quad \underline{0 \le u \le 2}$$

$$\begin{cases} z - y = 0 \\ z - y = 2 \end{cases} \quad let \quad \underline{v = z - y}$$

$$\Rightarrow \quad \underline{0 \le v \le 2}$$

$$\begin{cases} z = 0 \\ z = 3 \end{cases} \quad let \quad \underline{w = z}$$

$$\Rightarrow \quad \underline{0 \le w \le 3}$$

$$\begin{cases} \underline{z = w} \\ y - x = u \qquad\qquad \rightarrow \qquad \underline{x = -u - v + w} \\ z - y = v \quad \rightarrow \quad \underline{y = w - v} \end{cases}$$

$$J(u,\ v,\ w) = \begin{vmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \qquad\qquad J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\underline{= 1}$$

$$\iiint_D xz\,dV = \int_0^3\int_0^2\int_0^2 (w-u-v)(w)\ dudvdw$$

$$= \int_0^3\int_0^2\int_0^2 \left(w^2 - uw - vw\right)\ dudvdw$$

$$= \int_0^3\int_0^2 \left(w^2u - \frac{1}{2}wu^2 - vwu\ \Big|_0^2\ dvdw\right.$$

$$= \int_0^3\int_0^2 \left(2w^2 - 2w - 2vw\right)\ dvdw$$

$$= \int_0^3 \left(2w^2v - 2wv - wv^2\ \Big|_0^2\ dw\right.$$

$$= \int_0^3 \left(4w^2 - 4w - 4w\right)\ dw$$

$$= \int_0^3 \left(4w^2 - 8w\right)\ dw$$

$$= \frac{4}{3}w^3 - 4w^2\ \Big|_0^3$$

$$= 36 - 36$$

$$\underline{= 0}$$

Exercises Section 3.7 – Change of Variables in Multiple Integrals

Let $S = \{0 \le u \le 1,\ 0 \le v \le 1\}$ be a unit square in the *uv*-plane. Find the image of *S* in the *xy*-plane under the following transformations.

1. $T: \quad x = v,\ y = u$

2. $T: \quad x = -v,\ y = u$

3. $T: \quad x = \frac{u+v}{2},\ \ y = \frac{u-v}{2}$

4. $T: \quad x = u,\ \ y = 2v + 2$

5. *a*) Solve the system $u = x - y,\quad v = 2x + y$ for *x* and *y* in terms of *u* and *v*. Then find the value of the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$

b) Find the image under the transformation $u = x - y,\ \ v = 2x + y$ of the triangular region with vertices (0, 0), (1, 1), and (1, −2) in the *xy*-plane. Sketch the transformed region in the *uv*-plane.

6. Let *R* be the region in the first quadrant of the *xy*-plane bounded by the hyperbolas $xy = 1,\ xy = 9$ and the lines $y = x,\ y = 4x$. Use the transformation $x = \frac{u}{v},\ y = uv$ with $u > 0,\ \text{and}\ v > 0$ to rewrite

$$\iint\limits_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dxdy$$

As an integral over an appropriate region *G* in the uv-plane. Then evaluate the *uv*-integral over *G*.

7. The area πab of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be found by integrating the function $f(x,y) = 1$ over the region bounded by the ellipse in the *xy*-plane. Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate the integral is to use the transformation $x = au,\ y = bv$ and evaluate the transformed integral over the disk *G*: $u^2 + v^2 \le 1$ in the *uv*-plane. Find the area this way.

8. Use the transformation $x = u + \frac{1}{2}v,\ \ y = v$ to evaluate the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3 (2x - y) e^{(2x-y)^2} dxdy$$

By first writing it as an integral over a region *G* in the *uv*-plane.

9. Use the transformation $x = \frac{u}{v},\ \ y = uv$ to evaluate the integral

$$\int_1^2 \int_{1/y}^{y} \left(x^2 + y^2\right) dxdy + \int_2^4 \int_{y/4}^{4/y} \left(x^2 + y^2\right) dxdy$$

10. Find the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$ of the transformation

a) $x = u\cos v, \quad y = u\sin v$

b) $x = u\sin v, \quad y = u\cos v$

11. Find the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$ of the transformation

a) $x = u\cos v, \quad y = u\sin v, \quad z = w$

b) $x = 2u - 1, \quad y = 3v - 4, \quad z = \frac{1}{2}(w - 4)$

12. Evaluate the appropriate determinant to show that the Jacobian of the transformation from Cartesian $\rho\phi\theta$–space to Cartesian *xyz*–space is $\rho^2 \sin\phi$

13. How can substitutions in single definite integrals be viewed as transformations of regions? What is the Jacobian in such a case? Illustrate with an example.

14. Find the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

(***Hint***: Let $x = au,\ y = bv,\ and\ z = cw$. Then find the volume of an appropriate region in *uvw*-space)

15. Use the transformation $x = u^2 - v^2,\ y = 2uv$ to evaluate the integral

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2}\ dydx$$

(***Hint***: Show that the image of the triangular region ***G*** with vertices (0, 0), (1, 0), (1, 1) in the *uv*-plane is the region of integration ***R*** in the *xy*-plane defined by the limits of integration.)

16. Evaluate $\iint\limits_R y^4 dA$; R is the region bounded by the hyperbolas $xy = 1$ and $xy = 4$ and the lines $\dfrac{y}{x} = 1$, and $\dfrac{y}{x} = 3$

17. Evaluate $\iint\limits_R \left(y^2+xy-2x^2\right)dA$; R is the region bounded by the lines $y=x$, $y=x-3$, $y=-2x+3$, and $y=-2x-3$

18. Evaluate $\iint\limits_D x\ dV$; R is bounded by the planes $y-2x=0$, $y-2x=1$, $z-3y=0$, $z-3y=1$, $z-4x=0$ and $z-4x=3$

19. Let R be the region bounded by the lines $x+y=1$; $x+y=4$; $x-2y=0$; $x-2y=-4$

Evaluate the integral $\iint\limits_R 3xydA$

20. Let R be the region bounded by the square with vertices (0, 1), (1, 2), (2, 1), & (1, 0).

Evaluate the integral $\iint\limits_R (x+y)^2\sin^2(x-y)dA$

21. Evaluate $\iiint_D yzdV$ D is bounded by the planes: $x+2y=1$, $x+2y=2$, $x-z=0$, $x-z=2$, $2y-z=0$, and $2y-z=3$

22. Evaluate $\iint\limits_R xy\ dA$; R is the square with vertices $(0,\ 0)$, $(1,\ 1)$, $(2,\ 0)$, and $(1,\ -1)$

23. Evaluate $\iint\limits_R x^2y\ dA$; $R=\{(x,\ y):\ \ 0\le x\le 2,\quad x\le y\le x+4\}$

24. Evaluate $\iint\limits_R x^2\sqrt{x+2y}\ dA$; $R=\left\{(x,\ y):\ \ 0\le x\le 2,\quad -\frac{x}{2}\le y\le 1-x\right\}$

25. Evaluate $\iint\limits_R xy\ dA$; where R is bounded by the ellipse $9x^2+4y^2=36$.

26. Evaluate $\int_0^1\int_y^{y+2}\sqrt{x-y}\ dxdy$

27. Evaluate $\iint\limits_R \sqrt{y^2 - x^2}\ dA$; where R is the diamond bounded by $y - x = 0$, $y - x = 2$, $y + x = 0$, and $y + x = 2$

28. Evaluate $\iint\limits_R \left(\dfrac{y - x}{y + 2x + 1}\right)^4 dA$; where R is the parallelogram bounded by $y - x = 1$, $y - x = 2$, $y + 2x = 0$, and $y + 2x = 4$

29. Evaluate $\iint\limits_R e^{xy}\ dA$; where R is the region bounded by $xy = 1$, $xy = 4$, $\dfrac{y}{x} = 1$, and $\dfrac{y}{x} = 3$

30. Evaluate $\iint\limits_R xy\ dA$; where R is the region bounded by the hyperbolas $xy = 1$, $xy = 4$, $y = 1$, and $y = 3$

31. Evaluate $\iint\limits_R (x - y)\sqrt{x - 2y}\ dA$; where R is the triangular region bounded by $y = 0$, $x - 2y = 0$, and $x - y = 1$

32. Evaluate $\iiint_D xy\ dV$: D is bounded by the planes: $y - x = 0$, $y - x = 2$, $z - y = 0$, $z - y = 2$, $z = 0$, and $z = 3$

33. Evaluate $\iiint_D dV$: D is bounded by the planes: $y - 2x = 0$, $y - 2x = 1$, $z - 3y = 0$, $z - 3y = 1$, $z - 4x = 0$, and $z - 4x = 3$

34. Evaluate $\iiint_D z\ dV$: D is bounded by the paraboloid $z = 16 - x^2 - 4y^2$ and the xy-plane.

35. Evaluate $\iiint_D dV$: D is bounded by the upper half of the ellipsoid $\dfrac{x^2}{9} + \dfrac{y^2}{4} + z^2 = 1$ and the xy-plane.

36. Evaluate $\iiint_D xz\ dV$: D is bounded by the planes: $y = x$, $y = x + 2$, $x - z = 0$, $z = x + 3$, $z = 0$, and $z = 4$

(37 – 41) Let R be the region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>0$ and $b>0$ are real numbers.

37. Find the area of R.

38. Evaluates $\iint_R |xy|\, dA$

39. Find the center of mass of the upper half of R $(y \geq 0)$ assuming it has a constant density.

40. Find the average square of the distance between points of R and the origin.

41. Find the average distance between points in the upper half of R and the x-axis.

(42 – 45) Let D be the region bounded by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, where $a>0$, $b>0$ and $c>0$ are real numbers.

42. Find the Volume of D.

43. Evaluates $\iint_D |xyz|\, dV$

44. Find the center of mass of the upper half of D $(z \geq 0)$ assuming it has a constant density.

45. Find the average square of the distance between points of D and the origin.