Section 4.6 – Orthogonal Diagonalization

Definition

A square matrix $\boldsymbol{A}$ is called orthogonally diagonalizable if there is an orthogonal matrix P such that $P^{-1}AP\left(=P^TAP\right)$ is diagonal; the matrix P is said to orthogonally diagonalize A.

$$P^TAP=D$$

We say that A is orthogonally diagonalizable and that P orthogonally diagonalizes A.

Theorem

If $\boldsymbol{A}$ is an n x n matrix, then the following are equivalent.

a) $\boldsymbol{A}$ is orthogonally diagonalizable

b) $\boldsymbol{A}$ has an orthonormal set of n eigenvectors.

c) $\boldsymbol{A}$ is symmetric.

Theorem

If $\boldsymbol{A}$ is symmetric matrix, then:

a) The eigenvalues of $\boldsymbol{A}$ are all real numbers.

b) Eigenvectors from different eigenspaces are orthogonal.

Example

Find an orthogonal matrix P that diagonalizes

$$A=\begin{pmatrix}4&2&2\\2&4&2\\2&2&4\end{pmatrix}$$

Solution

$$\det(A-\lambda I)=\begin{pmatrix}4-\lambda&2&2\\2&4-\lambda&2\\2&2&4-\lambda\end{pmatrix}$$

$$=(4-\lambda)^3+8+8-4(4-\lambda)-4(4-\lambda)-4(4-\lambda)$$

$$=64-48\lambda+12\lambda^2-\lambda^3+16-12(4-\lambda)$$

$$=64-48\lambda+12\lambda^2-\lambda^3+16-48+12\lambda$$

$$=-\lambda^3+12\lambda^2-36\lambda+32=0$$

The eigenvalues are: $\boxed{\lambda = 2 \quad and \quad \lambda = 8}$

For $\lambda_1 = 2$, we have: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 2x_1 + 2y_1 + 2z_1 = 0 \\ 2x_1 + 2y_1 + 2z_1 = 0 \\ 2x_1 + 2y_1 + 2z_1 = 0 \end{cases}$$

If $z_1 = 0 \quad and \quad y_1 = 1 \quad \Rightarrow x_1 = -1$, therefore the eigenvector $V_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda_2 = 2$, we have: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 2x_2 + 2y_2 + 2z_2 = 0 \\ 2x_2 + 2y_2 + 2z_2 = 0 \\ 2x_2 + 2y_2 + 2z_2 = 0 \end{cases}$$

If $y_2 = 0 \quad and \quad z_2 = 1 \quad \Rightarrow x_2 = -1$, therefore the eigenvector $V_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

For $\lambda_3 = 8$, we have: $\left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} -4x_3 + 2y_3 + 2z_3 = 0 \\ 2x_3 - 4y_3 + 2z_3 = 0 \\ 2x_3 + 2y_3 - 4z_3 = 0 \end{cases}$$

$$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} x_3 = z_3 \\ y_3 = z_3 \end{matrix}$$

If $z_3 = 1 \quad \Rightarrow \quad y_3 = 1,\ x_3 = 1$, therefore the eigenvector $V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$u_1 = \frac{v_1}{\left\| v_1 \right\|} = \frac{(-1,1,0)}{\sqrt{(-1)^2 + 1^2 + 0}} = \frac{(-1,1,0)}{\sqrt{2}} = \boxed{\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)}$$

$$w_2 = v_2 - \left(v_2 . u_1\right)u_1 = (-1,0,1) - \left[(-1,0,1) \cdot \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)\right]\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$= (-1,0,1) - \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$= \left(-\frac{1}{2}, -\frac{1}{2}, 1\right)$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{\left(-\frac{1}{2}, -\frac{1}{2}, 1\right)}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}} = \frac{\left(-\frac{1}{2}, -\frac{1}{2}, 1\right)}{\sqrt{\frac{6}{4}}}$$

$$= \frac{2}{\sqrt{6}}\left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)$$

$$= \underline{\left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)}$$

$$w_3 = v_3 - (v_3 . u_1)u_1 - (v_3 . u_2)u_2$$

$$= (1,\ 1,\ 1) - (0)\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) - (0)\left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)$$

$$= (1,\ 1,\ 1)$$

$$u_3 = \frac{w_3}{\|w_3\|} = \frac{1}{\sqrt{1^2+1^2+1^2}}(1,\ 1,\ 1)$$

$$= \underline{\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)}$$

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$P^T A P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Spectral Decomposition

The spectral decomposition of A is:

$$A = \lambda_1 \boldsymbol{u}_1 \boldsymbol{u}_1^T + \lambda_2 \boldsymbol{u}_2 \boldsymbol{u}_2^T + \cdots + \lambda_n \boldsymbol{u}_n \boldsymbol{u}_n^T$$

Example

The matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$

Solution

$$\det(A - \lambda I) = \begin{pmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{pmatrix} = (1-\lambda)(-2-\lambda) - 4$$

$$= \lambda^2 + \lambda - 6 = 0$$

The eigenvalues are: $\boxed{\lambda = -3 \quad and \quad \lambda = 2}$

The corresponding eigenvectors are: $V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad and \quad V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(1,-2)}{\sqrt{1^2 + (-2)^2}} = \underline{\left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right)}$$

$$w_2 = v_2 - (v_2 . u_1) u_1 = (2,1) - \left[(2,1) \cdot \left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right) \right] \left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right)$$

$$= (2,1) - (0) \left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right)$$

$$= (2,\ 1)$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{(2,\ 1)}{\sqrt{5}}$$

$$= \underline{\left(\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}} \right)}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} = \lambda_1 \boldsymbol{u}_1 \boldsymbol{u}_1^T + \lambda_2 \boldsymbol{u}_2 \boldsymbol{u}_2^T$$

$$= -3 \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix} + 2 \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$= -3\begin{pmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{pmatrix} + 2\begin{pmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

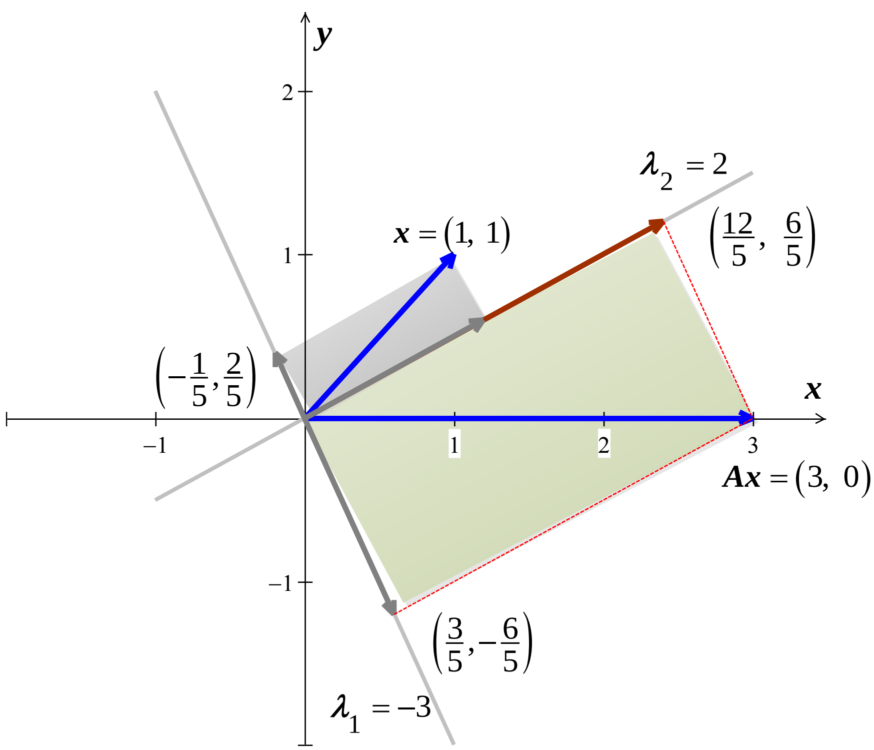
The spectral decomposition about the image of the vector $\boldsymbol{x} = (1,\ 1)$

$$Ax = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = -3\begin{pmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= -3\begin{pmatrix} -\frac{1}{5} \\ \frac{2}{5} \end{pmatrix} + 2\begin{pmatrix} \frac{6}{5} \\ \frac{3}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{5} \\ -\frac{6}{5} \end{pmatrix} + \begin{pmatrix} \frac{12}{5} \\ \frac{6}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$



Example

Consider a 2 by 2 symmetric matrix $A=\begin{pmatrix} a & b \\ b & c \end{pmatrix}$

Solution

The eigenvalues are:

$$\det(A-\lambda I)=\begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix}=(a-\lambda)(c-\lambda)-b^2$$

$$=\lambda^2-(a+c)\lambda+ac-b^2=0$$

$$\lambda=\frac{(a+c)\pm\sqrt{(a+c)^2-4\left(ac-b^2\right)}}{2} \qquad \therefore (a+c)^2-4\left(ac-b^2\right)>0$$

The eigenvectors are:

$$\left(A-\lambda_1 I\right)V_1=\begin{pmatrix} a-\lambda_1 & b \\ b & c-\lambda_1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}\to\begin{cases}\left(a-\lambda_1\right)x_1+by_1=0 \\ bx_1+\left(c-\lambda_1\right)y_1=0\end{cases}\Rightarrow if\ x_1=b \quad y_1=a-\lambda_1$$

$$V_1=\begin{pmatrix} b \\ \lambda_1-a \end{pmatrix}$$

$$\left(A-\lambda_2 I\right)V_2=\begin{pmatrix} a-\lambda_2 & b \\ b & c-\lambda_2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}\to\begin{cases}\left(a-\lambda_2\right)x_2+by_2=0 \\ bx_2+\left(c-\lambda_2\right)y_2=0\end{cases}\Rightarrow if\ y_2=b \quad x_2=c-\lambda_1$$

$$V_2=\begin{pmatrix} \lambda_2-a \\ c \end{pmatrix}$$

$$\lambda_1+\lambda_2=\frac{(a+c)-\sqrt{(a+c)^2-4\left(ac-b^2\right)}+(a+c)+\sqrt{(a+c)^2-4\left(ac-b^2\right)}}{2}$$

$$=\frac{2(a+c)}{2}$$

$$=a+c$$

$$V_1\cdot V_2=b\left(\lambda_1-a\right)+b\left(\lambda_2-c\right)$$

$$=b\left(\lambda_1+\lambda_2-a-c\right)$$

$$=b\left(a+c-a-c\right)$$

$$=0$$

Therefore, these eigenvectors are perpendicular.

Theorem

Orthogonal Eigenvectors: Eigenvectors of a real symmetric matrix (when they correspond to different λ's) are always perpendicular.

Proof

Suppose $Ax = \lambda_1 x$, $Ay = \lambda_2 y$ and $A = A^T$.

The dot products of the first equation with y and the second with x:

$$\left(\lambda_1 x\right)^T y = (Ax)^T y = x^T A^T y = x^T Ay = x^T \lambda_2 y$$

$$\Rightarrow x^T \lambda_1 y = x^T \lambda_2 y$$

Since $\lambda_1 \neq \lambda_2$, this proves that $x^T y = 0$.

The eigenvector x $\left(\text{for } \lambda_1 \right)$ is perpendicular to the eigenvector y $\left(\text{for } \lambda_2 \right)$

Example

Find the λ's and v's for this symmetric matrix with trace zero: $A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$

Solution

$$\begin{aligned} \det\left(A - \lambda I\right) &= \begin{vmatrix} -3-\lambda & 4 \\ 4 & 3-\lambda \end{vmatrix} \\ &= (-3-\lambda)(3-\lambda) - 16 \\ &= -9 + \lambda^2 - 16 \\ &= \lambda^2 - 25 \end{aligned}$$

The eigenvalues are: $\lambda_1 = 5 \quad \lambda_2 = -5$

The eigenvectors are:

$$\left(A - \lambda_1 I\right)v_1 = \begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} -8x_1 + 4y_1 = 0 \\ 4x_1 - 2y_1 = 0 \end{matrix} \Rightarrow 2x_1 = y_1 \rightarrow v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\left(A - \lambda_2 I\right)v_2 = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} 2x_2 + 4y_2 = 0 \\ 4x_2 + 8y_2 = 0 \end{matrix} \Rightarrow x_2 = -2y_2 \rightarrow v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$v_1 \cdot v_2 = (1)(-2) + (2)(1) = -2 + 2 = 0$

Thus, the eigenvectors are perpendicular.

The unit vector of the eigenvectors by dividing by their length $\sqrt{2^2 + 1^2} = \sqrt{5}$

The v_1 *and* v_2 are the columns of Q.

$$Q = \frac{\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}}{\sqrt{5}} \rightarrow Q^{-1} = Q^T = \frac{\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}}{\sqrt{5}}$$

$$A = QDQ^T = \frac{\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}}{\sqrt{5}} \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix} \frac{\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}}{\sqrt{5}}$$

- Every symmetric matrix A has a complete set of orthogonal eigenvectors:

$$A = PDP^{-1} \quad \Rightarrow \quad A = QDQ^T$$

Complex Eigenvalues of Real Matrices

For real matrices, complex λ's and x's come in "conjugate pairs"

$$\text{if} \quad Ax = \lambda x \quad \text{then} \quad A\bar{x} = \bar{\lambda}\bar{x}$$

Example

Given $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Solution

The eigenvalues of A:

$$\det(A - \lambda I) = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix}$$

$$= (\cos\theta - \lambda)(\cos\theta - \lambda) + \sin^2\theta$$

$$= \cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta$$

$$= \lambda^2 - 2\lambda\cos\theta + 1$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \frac{2\cos\theta \pm \sqrt{4\left(\cos^2\theta - 1\right)}}{2} \qquad \cos^2\theta + \sin^2\theta = 1 \to \cos^2\theta - 1 = -\sin^2\theta$$

$$= \frac{2\cos\theta \pm 2\sqrt{-\sin^2\theta}}{2}$$

$$= \cos\theta \pm i\sin\theta$$

The eigenvalues are conjugate to each other.

The eigenvectors:

$$(A - \lambda_1 I)v_1 = \begin{bmatrix} \cos\theta - (\cos\theta + i\sin\theta) & -\sin\theta \\ \sin\theta & \cos\theta - (\cos\theta + i\sin\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -i\sin\theta x_1 - \sin\theta y_1 = 0 \\ \sin\theta x_1 - i\sin\theta y_1 = 0 \end{cases}$$

$$\Rightarrow x_1 - iy_1 = 0 \Rightarrow x_1 = iy_1 \to v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\to v_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

The vector $x = v_1$

$$Ax = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = (\cos\theta + i\sin\theta) \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$A\overline{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \left(\cos\theta - i\sin\theta\right) \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$|\lambda| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

This fact holds for the eigenvalues of every orthogonal matrix.

Theorem – *Equivalent Statements*

If *A* is an $n \times n$ matrix, then the following statements are equivalent.

a) *A* is invertible
b) A***x*** = **0** has only the trivial solution
c) The reduced row echelon form of *A* is I_n
d) A is expressible as a product of elementary matrices
e) A***x*** = ***b*** is consistent for every $n \times 1$ matrix ***b***
f) A***x*** = ***b*** has exactly one solution for every $n \times 1$ matrix ***b***
g) $\det(A) \neq 0$
h) The column vectors of *A* are linearly independent
i) The row vectors of *A* are linearly independent
j) The column vectors of *A* span $\boldsymbol{R}^n$
k) The row vectors of *A* span $\boldsymbol{R}^n$
l) The column vectors of *A* form a basis for $\boldsymbol{R}^n$
m) The row vectors of *A* form a basis for $\boldsymbol{R}^n$
n) *A* has a rank *n*.
o) *A* has nullity 0.
p) The orthogonal complement of the null space of *A* is $\boldsymbol{R}^n$
q) The orthogonal complement of the row space of *A* is $\{0\}$
r) The range of T_A is $\boldsymbol{R}^n$
s) T_A is one-to-one.
t) $\lambda = 0$ is not an eigenvalue of A.
u) $A^T A$ is invertible,

Exercises Section 4.6 – Orthogonal Diagonalization

1. Find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$

a) $A=\begin{pmatrix}3&1\\1&3\end{pmatrix}$

b) $A=\begin{pmatrix}6&2\sqrt{3}\\2\sqrt{3}&7\end{pmatrix}$

c) $A=\begin{pmatrix}-2&0&-36\\0&-3&0\\-36&0&-23\end{pmatrix}$

d) $A=\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix}$

e) $A=\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}$

f) $A=\begin{pmatrix}-7&24&0&0\\24&7&0&0\\0&0&-7&24\\0&0&24&7\end{pmatrix}$

g) $A=\begin{bmatrix}3&1&0&0\\1&3&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$

2. Find the eigenvalues of A and B and check the Orthogonality of their first two eigenvectors. Graph these eigenvectors to see the discrete sines and cosines:

$$A=\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}\qquad B=\begin{bmatrix}1&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&1\end{bmatrix}$$

The –1, 2, –1 pattern in both matrices is a "second derivative. Then $Ax=\lambda x$ and $Bx=\lambda x$ are like $\frac{d^2x}{dt^2}=\lambda x$. This has eigenvectors $x=\sin kt$ and $x=\cos kt$ that are the bases for Fourier series. The matrices lead to "discrete sines" and "discrete cosines" that are the bases for the discrete Fourier Transform. This DFT is absolutely central to all areas of digital signal processing.

3. Suppose $Ax=\lambda x$ and $Ay=0y$ and $\lambda\neq 0$. Then y is in the nullspace and x is in the column space. They are perpendicular because ________. (why are these subspaces orthogonal?) If the second eigenvalue is a nonzero number β, apply this argument to $A-\beta I$. The eigenvalue moves to zero and the eigenvectors stay the same – so they are perpendicular.

4. True or false. Give a reason or a counterexample.

a) A matrix with real eigenvalues and eigenvectors is symmetric.
b) A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.
c) The inverse of a symmetric matrix is symmetric
d) The eigenvector matrix S of a symmetric matrix is symmetric.
e) A complex symmetric matrix has real eigenvalues.
f) If A is symmetric, then e^{iA} is symmetric.
g) If A is Hermitian, then e^{iA} is Hermitian.

5. Find a symmetric matrix $\begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$ that has a negative eigenvalue.

a) How do you know it must have a negative pivot?

b) How do you know it can't have two negative eigenvalues?

6. Which of these classes of matrices do A and B belong to: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad B = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Which of these factorizations are possible for A and B: LU, QR, ADP^{-1}, QDQ^T ?

7. Construct a 3 by 3 matrix A with no zero entries whose columns are mutually perpendicular. Compute $A^T A$. Why is it a diagonal matrix?

8. Assuming that $b \neq 0$, find a matrix that orthogonally diagonalizes $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$