Section 5.4 – Infinite Sequences and Summation Notation

An arbitrary *infinite sequence* may be denoted as follows:

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

An infinite sequence is a function whose domain is the set of positive integers.

Example

Find the first four terms and the tenth term of the sequence: $\left\{\dfrac{n}{n+1}\right\}$

Solution

$$n = 1 \quad \rightarrow \quad \frac{1}{1+1} = \frac{1}{2}$$

$$n = 2 \quad \rightarrow \quad \frac{2}{2+1} = \frac{2}{3}$$

$$n = 3 \quad \rightarrow \quad \frac{3}{3+1} = \frac{3}{4}$$

$$n = 4 \quad \rightarrow \quad \frac{4}{4+1} = \frac{4}{5}$$

$$n = 10 \quad \Rightarrow \quad \frac{10}{11}$$

Example

Find the first four terms and the tenth term of the sequence: $\left\{2 + (0.1)^n\right\}$

Solution

$$n = 1 \quad \rightarrow \quad 2 + 0.1 = 2.1$$

$$n = 2 \quad \rightarrow \quad 2 + 0.1^2 = 2.01$$

$$n = 3 \quad \rightarrow \quad 2 + 0.1^3 = 2.001$$

$$n = 4 \quad \rightarrow \quad 2 + 0.1^4 = 2.0001$$

$$n = 10 \quad \Rightarrow \quad 2.0000000001$$

Example

Find the first four terms and the tenth term of the sequence: $\left\{(-1)^{n+1}\dfrac{n^2}{3n-1}\right\}$

Solution

$n=1 \quad \rightarrow \quad (-1)^2\dfrac{1^2}{3(1)-1}=\dfrac{1}{2}$

$n=2 \quad \rightarrow \quad (-1)^3\dfrac{2^2}{3(2)-1}=-\dfrac{4}{5}$

$n=3 \quad \rightarrow \quad (-1)^4\dfrac{3^2}{3(3)-1}=\dfrac{9}{8}$

$n=4 \quad \rightarrow \quad (-1)^5\dfrac{4^2}{3(4)-1}=-\dfrac{16}{11}$

$n=10 \quad \Rightarrow \quad \underline{-\dfrac{100}{29}}$

Example

Find the first four terms and the tenth term of the sequence: $\{4\}$

Solution

$n=1 \quad \rightarrow \quad 4$

$n=2 \quad \rightarrow \quad 4$

$n=3 \quad \rightarrow \quad 4$

$n=4 \quad \rightarrow \quad 4$

$n=10 \quad \Rightarrow \quad 4$

Example

Find the first four terms of the recursively defined infinite sequence $a_1=3, \quad a_{n+1}=(n+1)a_n$

Solution

$\underline{a_1=3}$

$n=1 \quad \rightarrow \quad a_2=(1+1)a_1=2(3)\underline{=6}$

$n=2 \quad \rightarrow \quad a_3=(2+1)a_2=3(6)\underline{=18}$

$n=3 \quad \rightarrow \quad a_4=(3+1)a_3=4(18)\underline{=72}$

Summation Notation

To find the sum of many terms of an infinite sequence, it is easy to express using summation notation.

$$\underbrace{\sum_{\underbrace{n=1}_{\text{First value of } n}}^{\overbrace{5}^{\text{Last value of } n}} 2n+3} \quad \leftarrow \textit{Formula for each term}$$

Example

Find the sum: $\displaystyle\sum_{k=1}^{4} k^2(k-3)$

Solution

$$\sum_{k=1}^{4} k^2(k-3) = 1^2(1-3) + 2^2(2-3) + 3^2(3-3) + 4^2(4-3)$$

$$= -2 - 4 + 0 + 16$$

$$= 10$$

Theorem on the Sum of a *Constant*

$$\text{(1)} \quad \sum_{k=1}^{n} c = nc \qquad \text{(2)} \quad \sum_{k=m}^{n} c = (n-m+1)c$$

Proof:

$$\sum_{k=1}^{n} c = \underbrace{c + c + \ldots + c}_{n} = nc$$

Example

Find the sum: $\displaystyle\sum_{k=10}^{20} 5$

Solution

$$\sum_{k=10}^{20} 5 = (20 - 10 + 1)5$$

$$= 55$$

Theorem on Sums

If a_1, a_2, a_3, ..., a_n, ... and b_1, b_2, b_3, ..., b_n, ... are infinite sequences, then for every positive integer n,

$$(1) \quad \sum_{k=1}^{n}\left(a_k + b_k\right) = \sum_{k=1}^{n}a_k + \sum_{k=1}^{n}b_k$$

$$(2) \quad \sum_{k=1}^{n}\left(a_k - b_k\right) = \sum_{k=1}^{n}a_k - \sum_{k=1}^{n}b_k$$

$$(3) \quad \sum_{k=1}^{n}ca_k = c\left(\sum_{k=1}^{n}a_k\right)$$

Proof

$$\sum_{k=1}^{n}\left(a_k + b_k\right) = \left(a_1 + b_1\right) + \left(a_2 + b_2\right) + ... + \left(a_n + b_n\right)$$

$$= \left(a_1 + a_2 + ... + a_n\right) + \left(b_1 + b_2 + ... + b_n\right)$$

$$= \sum_{k=1}^{n}a_k + \sum_{k=1}^{n}b_k$$

Example

Express the sum using summation notation $\quad 2^1 + 2^2 + 2^3 + \cdots + 2^{16}$

Solution

$$2^1 + 2^2 + 2^3 + \cdots + 2^{16} = \sum_{k=1}^{16}2^k$$

Exercises   *Section* **5.4 – Infinite Sequences and Summation Notation**

(1 – 13)   Find the first four terms and the eight term of the sequence:

1. $\{12 - 3n\}$

2. $\left\{\dfrac{3n-2}{n^2+1}\right\}$

3. $\{9\}$

4. $\left\{(-1)^{n-1}\dfrac{n+7}{2n}\right\}$

5. $\left\{\dfrac{2^n}{n^2+2}\right\}$

6. $\left\{(-1)^{n-1}\dfrac{n}{2n-1}\right\}$

7. $\left\{\dfrac{2^n}{3^n+1}\right\}$

8. $\left\{\dfrac{n^2}{2^n}\right\}$

9. $\left\{\dfrac{n}{e^n}\right\}$

10. $\{c_n\} = \left\{(-1)^{n+1}n^2\right\}$

11. $\{c_n\} = \left\{\dfrac{(-1)^n}{(n+1)(n+2)}\right\}$

12. $\{c_n\} = \left\{\left(\dfrac{4}{3}\right)^n\right\}$

13. $\{b_n\} = \left\{\dfrac{3^n}{n}\right\}$

14. Graph the sequence $\left\{\dfrac{1}{\sqrt{n}}\right\}$

15. Find the first four terms of the sequence of partial sums for the given sequence. $\left\{3+\tfrac{1}{2}n\right\}$

(16 – 27)   Find the first five terms of the recursively defined infinite sequence

16. $a_1 = 2,\quad a_{k+1} = 3a_k - 5$

17. $a_1 = -3,\quad a_{k+1} = a_k^2$

18. $a_1 = 5,\quad a_{k+1} = ka_k$

19. $a_1 = 2,\quad a_n = 3 + a_{n-1}$

20. $a_1 = 5,\quad a_n = 2a_{n-1}$

21. $a_1 = \sqrt{2},\quad a_n = \sqrt{2 + a_{n-1}}$

22. $a_1 = 2,\quad a_{n+1} = 7 - 2a_n$

23. $a_1 = 128,\quad a_{n+1} = \tfrac{1}{4}a_n$

24. $a_1 = 2,\quad a_{n+1} = (a_n)^n$

25. $a_1 = A,\quad a_n = a_{n-1} + d$

26. $a_1 = A,\quad a_n = ra_{n-1},\quad r \neq 0$

27. $a_1 = 2,\quad a_2 = 2;\quad a_n = a_{n-1}\cdot a_{n-2}$

(28 – 37)   Express each sum using summation notation

28. $1 + 2 + 3 + \ldots + 20$

29. $1 + 2 + 3 + \ldots + 40$

30. $1^3 + 2^3 + 3^3 + \ldots + 8^3$

31. $1^2 + 2^2 + 3^2 + \ldots + 15^2$

32. $2^2 + 2^3 + 2^4 + \ldots + 2^{11}$

33. $\dfrac{1}{2} + \dfrac{2}{3} + \dfrac{3}{4} + \cdots + \dfrac{13}{14}$

34. $1 - \dfrac{1}{3} + \dfrac{1}{9} - \dfrac{1}{27} + \cdots + (-1)^6\dfrac{1}{3^6}$

35. $\dfrac{2}{3} - \dfrac{4}{9} + \dfrac{8}{27} - \cdots + (-1)^{12}\left(\dfrac{2}{3}\right)^{11}$

36. $\dfrac{1}{2} + \dfrac{2}{3} + \dfrac{3}{4} + \cdots + \dfrac{14}{14+1}$

37. $\dfrac{1}{e} + \dfrac{2}{e^2} + \dfrac{3}{e^3} + \cdots + \dfrac{n}{e^n}$

(38 – 52) Find the sum

38. $\displaystyle\sum_{k=1}^{5}(2k-7)$

43. $\displaystyle\sum_{k=1}^{40}k$

48. $\displaystyle\sum_{k=1}^{16}\left(k^2-4\right)$

39. $\displaystyle\sum_{k=0}^{5}k(k-2)$

44. $\displaystyle\sum_{k=1}^{5}(3k)$

49. $\displaystyle\sum_{k=1}^{6}(10-3k)$

40. $\displaystyle\sum_{k=1}^{5}(-3)^{k-1}$

45. $\displaystyle\sum_{k=1}^{10}\left(k^3+1\right)$

50. $\displaystyle\sum_{k=1}^{10}\left[1+(-1)^k\right]$

41. $\displaystyle\sum_{k=253}^{571}\left(\frac{1}{3}\right)$

46. $\displaystyle\sum_{k=1}^{24}\left(k^2-7k+2\right)$

51. $\displaystyle\sum_{k=1}^{6}\frac{3}{k+1}$

42. $\displaystyle\sum_{k=1}^{50}8$

47. $\displaystyle\sum_{k=6}^{20}\left(4k^2\right)$

52. $\displaystyle\sum_{k=137}^{428}2.1$

(53 – 56) Write out each sum

53. $\displaystyle\sum_{k=1}^{n}(k+2)$

55. $\displaystyle\sum_{k=2}^{n}(-1)^k\ln k$

57. $\displaystyle\sum_{k=0}^{n}\frac{1}{3^k}$

54. $\displaystyle\sum_{k=1}^{n}k^2$

56. $\displaystyle\sum_{k=3}^{n}(-1)^{k+1}2^k$

58. Fred has a balance of $3,000 on his card which charges 1% interest per month on any unpaid balance. Fred can afford to pay $100 toward the balance each month. His balance each month after making a $100 payment is given by the recursively defined sequence

$$B_0=\$3{,}000 \quad B_n=1.01B_{n-1}-100$$

Determine Fred's balance after making the first payment. That is, determine B_1

59. A pond currently has 2,000 trout in it. A fish hatchery decides to add an additional 20 trout each month. Is it also known that the trout population is grwoing at a rate of 3% per month. The size of the population after n months is given bu the recursively defined sequence

$$P_0=2{,}000 \quad P_n=1.03P_{n-1}+20$$

How many trout are in the pond after 2 months? That is, what is P_2?

60. Fred bought a car by taking out a loan for $18,500 at 0.5% interest per month. Fred's normal monthly payment is $434.47 per month, but he decides that he can afford to pay $100 extra toward the balance each month. His balance each month is given by the recursively defined sequence

$$B_0 = \$18,500 \quad B_n = 1.005 B_{n-1} - 534.47$$

Determine Fred's balance after making the first payment. That is, determine B_1

61. The Environmental Protection Agency (EPA) determines that Maple Lake has 250 *tons* of pollutant as a result of industrial waste and that 10% of the pollutant present is neuttralized by solar oxidation every year. The EPA imposes new pollution control laws that result in 15 *tons* of new pollutant entering the lake each year. The amount of pollutant in the lake after n years is given by the recursively defined sequence

$$P_0 = 250 \quad P_n = 0.9 P_{n-1} + 15$$

Determine the amount of pollutant in the lake after 2 years? That is, what is P_2?

62. Let $$u_n = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n \sqrt{5}}$$

Define the nth term of a sequence

a) Show that $u_1 = 1$ and $u_2 = 1$

b) Show that $u_{n+2} = u_{n+1} + u_n$

c) Draw the conclusion that $\{u_n\}$ is a Fibonacci sequence

d) Find the first ten terms of the sequence from part (*c*)