Solution   *Section* 3.1 – Mathematical Induction

Exercise

Prove that $1^2+3^2+5^2+\cdots+(2n+1)^2=\frac{1}{3}(n+1)(2n+1)(2n+3)$ whenever n is a nonnegative integer.

Solution

Since n is a nonnegative integer that implies to $n\geq 0$

(1) For $\boldsymbol{n=0} \Rightarrow 1^2=\frac{1}{3}(0+1)(0+1)(0+3)$

$$1=\frac{1}{3}(1)(2)(3)=1 \quad \checkmark$$

Hence P_1 is true.

(1) Assume that $1^2+3^2+\cdots+(2k+1)^2=\frac{1}{3}(k+1)(2k+1)(2k+3)$ is true

$$1^2+3^2+\cdots+(2k+1)^2+(2(k+1)+1)^2=\frac{1}{3}((k+1)+1)(2(k+1)+1)(2(k+1)+3)$$

$$1^2+3^2+\cdots+(2k+1)^2+(2k+3)^2=\frac{1}{3}(k+2)(2k+3)(2k+5)$$

$$\begin{aligned}1^2+3^2+\cdots+(2k+1)^2+(2k+3)^2&=\frac{1}{3}(k+1)(2k+1)(2k+3)+(2k+3)^2\\&=\frac{1}{3}(2k+3)\left[(k+1)(2k+1)+3(2k+3)\right]\\&=\frac{1}{3}(2k+3)\left(2k^2+k+2k+1+6k+9\right)\\&=\frac{1}{3}(2k+3)\left(2k^2+9k+10\right)\\&=\frac{1}{3}(2k+3)(k+2)(2k+5) \quad \checkmark\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that $1\cdot 1!+2\cdot 2!+\cdots+n\cdot n!=(n+1)!-1$ whenever n is a positive integer.

Solution

Since n is a positive integer that implies to $n\geq 1$

(2) For $\boldsymbol{n=1}$

$$1\cdot 1!=(1+1)!-1$$

$$1=1 \quad \checkmark$$

Hence P_1 is true.

(3) Assume that $1\cdot1!+2\cdot2!+\cdots+k\cdot k!=(k+1)!-1$ is true

$$1\cdot1!+2\cdot2!+\cdots+k\cdot k!+(k+1)\cdot(k+1)!=((k+1)+1)!-1=(k+2)!-1$$

$$\begin{aligned}1\cdot1!+2\cdot2!+\cdots+k\cdot k!+(k+1)\cdot(k+1)! &= (k+1)!-1+(k+1)\cdot(k+1)!\\ &= (k+1)\cdot(k+1)!+(k+1)!-1\\ &= (k+1)!(k+1+1)-1\\ &= (k+1)!(k+2)-1\\ &= (k+2)!-1 \quad \checkmark\end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that $3+3\cdot5+3\cdot5^2+\cdots+3\cdot5^n=\frac{3}{4}\left(5^{n+1}-1\right)$ whenever n is a nonnegative integer.

Solution

(1) For $\boldsymbol{n=0} \Rightarrow 3=\frac{3}{4}(5-1)$

$$3=3 \quad \checkmark$$

Hence P_1 is true.

(4) Assume that $3+3\cdot5+3\cdot5^2+\cdots+3\cdot5^k=\frac{3}{4}\left(5^{k+1}-1\right)$ is true

$$3+3\cdot5+3\cdot5^2+\cdots+3\cdot5^k+3\cdot5^{k+1}=\frac{3}{4}\left(5^{k+2}-1\right)$$

$$\begin{aligned}3+3\cdot5+3\cdot5^2+\cdots+3\cdot5^k+3\cdot5^{k+1} &= \frac{3}{4}\left(5^{k+1}-1\right)+3\cdot5^{k+1}\\ &= \frac{3}{4}\left[5^{k+1}-1+4\cdot5^{k+1}\right]\\ &= \frac{3}{4}\left(5\cdot5^{k+1}-1\right)\\ &= \frac{3}{4}\left(5^{k+2}-1\right) \quad \checkmark\end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that $2-2\cdot 7+2\cdot 7^2-\cdots+2\cdot(-7)^n=\dfrac{1-(-7)^{n+1}}{4}$ whenever n is a nonnegative integer.

Solution

(1) For $\boldsymbol{n} = \mathbf{0} \Rightarrow\ 2=\dfrac{1-(-7)^1}{4}$

$$2=\frac{8}{4}=2 \quad \checkmark$$

Hence P_1 is true.

(2) Assume that $2-2\cdot 7+2\cdot 7^2-\cdots+2\cdot(-7)^k=\dfrac{1-(-7)^{k+1}}{4}$ is true

We need to prove that P_{k+1} is also true

$$2-2\cdot 7+2\cdot 7^2-\cdots+2\cdot(-7)^k+2\cdot(-7)^{k+1}=\frac{1-(-7)^{(k+1)+1}}{4}$$

$$2-2\cdot 7+2\cdot 7^2-\cdots+2\cdot(-7)^k+2\cdot(-7)^{k+1}=\frac{1-(-7)^{k+2}}{4}$$

$$\begin{aligned}2-2\cdot 7+2\cdot 7^2-\cdots+2\cdot(-7)^k+2\cdot(-7)^{k+1}&=\frac{1-(-7)^{k+1}}{4}+2\cdot(-7)^{k+1}\\&=\frac{1-(-7)^{k+1}+8\cdot(-7)^{k+1}}{4}\\&=\frac{1-(-7)^{k+1}(1-8)}{4}\\&=\frac{1-(-7)^{k+1}(-7)}{4}\\&=\frac{1-(-7)^{k+2}}{4}\quad \checkmark\end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Find a formula for the sum of the first *n* even positive integers. Prove the formula.

Solution

$$\begin{array}{r} 1+2+\cdots+(n-1)+n \\ n+(n-1)+\cdots+2+1 \\ \hline (n+1)+(n+1)+\cdots+(n+1) \end{array}$$

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

(1) For ***n* = 1**

$$1=\frac{1(2)}{2}$$

$$1=1 \quad \checkmark$$

Hence P_1 is true.

(2) Assume that $1+2+\cdots+k=\frac{k(k+1)}{2}$ is true

We need to prove that P_{k+1} is also true $1+2+\cdots+k+(k+1)=\frac{(k+1)((k+1)+1)}{2}=\frac{(k+1)(k+2)}{2}$

$$\begin{aligned} 1+2+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\ &= \frac{k(k+1)+2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

a) Find a formula for $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}$ by examining the values of this expression for values of this expression for small values of *n*.

b) Prove the formula.

Solution

a) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

b) For $n=1 \quad \Rightarrow \quad \frac{1}{1\cdot 2}=\frac{1}{1+1}$

$$\frac{1}{2}=\frac{1}{2} \quad \surd$$

Hence P_1 is true.

Assume that $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$ is true

We need to prove that P_{k+1} is also true

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{(k+1)+1}=\frac{k+1}{k+2}$$

$$\begin{aligned}\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}&=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}\\&=\frac{k(k+2)+1}{(k+1)(k+2)}\\&=\frac{k^2+2k+1}{(k+1)(k+2)}\\&=\frac{(k+1)^2}{(k+1)(k+2)}\\&=\frac{k+1}{k+2} \quad \surd\end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that $1^2-2^2+3^2-\cdots+(-1)^{n-1}n^2=(-1)^{n-1}\frac{n(n+1)}{2}$ whenever n is a positive integer.

Solution

(1) For ***n* = 1**

$$1^2=(-1)^0\frac{1(2)}{2}$$

$$1=1 \quad \surd$$

Hence P_1 is true.

(2) Assume that $1^2-2^2+3^2-\cdots+(-1)^{k-1}k^2=(-1)^{k-1}\frac{k(k+1)}{2}$ is true

We need to prove that $1^2-2^2+3^2-\cdots+(-1)^{k-1}k^2+(-1)^k(k+1)^2=(-1)^k\frac{(k+1)(k+2)}{2}$ is also true

$$1^2-2^2+3^2-\cdots+(-1)^{k-1}k^2+(-1)^k(k+1)^2=(-1)^{k-1}\frac{k(k+1)}{2}+(-1)^k(k+1)^2$$

$$= (-1)^k (k+1)\left[(-1)^{-1}\frac{1}{2}k + (k+1)\right]$$
$$= (-1)^k (k+1)\left(-\frac{k}{2} + k + 1\right)$$
$$= (-1)^k (k+1)\left(\frac{k}{2} + 1\right)$$
$$= (-1)^k (k+1)\left(\frac{k+2}{2}\right) \quad \surd$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that for very positive integer n $\qquad \displaystyle\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$

Solution

For $\boldsymbol{n = 1} \Rightarrow 1\cdot 2^1 = (1-1)^0\, 2^2 + 2$

$$2 = 2 \quad \surd$$

Hence P_1 is true

Assume that $\displaystyle\sum_{k=1}^{n} k\cdot 2^k = (n-1)2^{n+1} + 2$ is true

We need to prove that $\displaystyle\sum_{k=1}^{n+1} k\cdot 2^k = n\cdot 2^{n+2} + 2$ is also true

$$\sum_{k=1}^{n+1} k\cdot 2^k = \sum_{k=1}^{n} k\cdot 2^k + (n+1)\cdot 2^{n+1}$$
$$= (n-1)\cdot 2^{n+1} + 2 + (n+1)\cdot 2^{n+1}$$
$$= (n-1+n+1)\cdot 2^{n+1} + 2$$
$$= 2n\cdot 2^{n+1} + 2$$
$$= n\cdot 2^{n+2} + 2 \quad \surd$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that for very positive integer n $\quad 1\cdot 2+2\cdot 3+\cdots+n(n+1)=\frac{1}{3}n(n+1)(n+2)$.

Solution

For $\boldsymbol{n}$ **= 1**

$$1\cdot 2=\frac{1}{3}1(1+1)(1+2)$$

$$2=\frac{1}{3}(2)(3)=2 \quad \checkmark$$

Hence P_1 is true

Assume that $1\cdot 2+2\cdot 3+\cdots+k(k+1)=\frac{1}{3}k(k+1)(k+2)$ is true

We need to prove that $1\cdot 2+2\cdot 3+\cdots+k(k+1)+(k+1)(k+2)=\frac{1}{3}(k+1)(k+2)(k+3)$ is also true

$$\begin{aligned} 1\cdot 2+2\cdot 3+\cdots+k(k+1)+(k+1)(k+2) &= \frac{1}{3}k(k+1)(k+2)+(k+1)(k+2) \\ &= (k+1)(k+2)\left(\frac{1}{3}k+1\right) \\ &= (k+1)(k+2)\left(\frac{k+3}{3}\right) \\ &= \frac{1}{3}(k+1)(k+2)(k+3) \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that for very positive integer n $\quad 1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots+n(n+1)(n+2)=\frac{1}{4}n(n+1)(n+2)(n+3)$

Solution

For $\boldsymbol{n}$ **= 1**

$$1\cdot 2\cdot 3=\frac{1}{4}1(1+1)(1+2)(1+3)$$

$$2=\frac{1}{3}(2)(3)=2 \quad \checkmark$$

Hence P_1 is true

Assume that $1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots+k(k+1)(k+2)=\frac{1}{4}k(k+1)(k+2)(k+3)$ is true

$$1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots+k(k+1)(k+2)+(k+1)(k+2)(k+3)=\frac{1}{4}(k+1)(k+2)(k+3)(k+4)$$

$$\begin{aligned} 1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots+k(k+1)(k+2)+(k+1)(k+2)(k+3) & \\ &= \frac{1}{4}k(k+1)(k+2)(k+3)+(k+1)(k+2)(k+3) \\ &= \frac{1}{4}(k+1)(k+2)(k+3)[k+4] \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Let $P(n)$ be the statement that $1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}<2-\frac{1}{n}$ where n is an integer greater than 1.

a) Show is the statement $P(2)$?

b) Show that $P(2)$ is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step.

f) Explain why these steps show that this inequality is true whenever n is an integer greater than 1.

Solution

a) $P(2):\ \ 1+\frac{1}{4}<2-\frac{1}{2}$

b) $1+\frac{1}{4}<2-\frac{1}{2}$

$\frac{5}{4}<\frac{3}{2}$

$10<12$ √

Exercise

Prove that $3^n<n!$ if n is an integer greater than 6.

Solution

For $\boldsymbol{n=7} \Rightarrow 3^7<7! \ \Rightarrow\ 2187<5040$; Hence P_7 is true

Assume that $3^k<k!$ is true, we need to prove that $3^{k+1}<(k+1)!$

$$
\begin{aligned}
3^{k+1} &= 3^k 3 \\
&< k!\cdot 3 \qquad \text{Since } k>6 \ \Rightarrow\ 6<k \ \rightarrow\ 3<k+1 \\
&< k!\cdot(k+1) \\
&= (k+1)! \ \checkmark
\end{aligned}
$$

$\therefore$ The statement $3^n<n!$ is true

Exercise

Prove that for every positive integer *n*: $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}>2\left(\sqrt{n+1}-1\right)$

Solution

For $\boldsymbol{n}$ **= 1**

$$1>2\left(\sqrt{1+1}-1\right)$$

$$1>2\left(\sqrt{2}-1\right)\approx 0.828$$

Hence P_1 is true

Assume that $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}}>2\left(\sqrt{k+1}-1\right)$ is true.

We need to prove that $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}>2\left(\sqrt{(k+1)+1}-1\right)=2\left(\sqrt{k+2}-1\right)$

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}>2\left(\sqrt{k+1}-1\right)+\frac{1}{\sqrt{k+1}}$$

$$2\left(\sqrt{k+1}-1\right)+\frac{1}{\sqrt{k+1}}>2\left(\sqrt{k+2}-1\right)$$

$$2\sqrt{k+1}-2+\frac{1}{\sqrt{k+1}}>2\sqrt{k+2}-2$$

$$2\sqrt{k+1}+\frac{1}{\sqrt{k+1}}>2\sqrt{k+2}$$

$$\frac{1}{\sqrt{k+1}}>2\sqrt{k+2}-2\sqrt{k+1}$$

$$\frac{1}{\sqrt{k+1}}>2\left(\sqrt{k+2}-\sqrt{k+1}\right)$$

$$\left(\sqrt{k+2}+\sqrt{k+1}\right)\frac{1}{\sqrt{k+1}}>2\left(\sqrt{k+2}-\sqrt{k+1}\right)\left(\sqrt{k+2}+\sqrt{k+1}\right)$$

$$\frac{\sqrt{k+2}}{\sqrt{k+1}}+1>2\left(k+2-k-1\right)$$

$$\frac{\sqrt{k+2}}{\sqrt{k+1}}+1>2$$

Which is clearly true since $\frac{\sqrt{k+2}}{\sqrt{k+1}}>1$

Exercise

Use mathematical induction to prove that 2 divides n^2+n whenever *n* is a positive integer.

Solution

For $\boldsymbol{n}$ **= 1**

$$1^2+1=2$$

since 2 divides 2;

Hence P_1 is true

Assume that 2 divides k^2+k is true, we need to prove that 2 divides $(k+1)^2+(k+1)$ is true

$$
\begin{aligned}
(k+1)^2+(k+1) &= k^2+2k+1+k+1 \\
&= k^2+k+2k+2 \\
&= k^2+k+2(k+1) \quad \checkmark
\end{aligned}
$$

2 divides k^2+k and certainly 2 divides $2(k+1)$, so 2 divides their sum.

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Use mathematical induction to prove that 3 divides n^3+2n whenever n is a positive integer.

Solution

For ***n* = 1**

$1^3+2(1)=3$

since 3 divides 3 $\checkmark$

Hence P_1 is true

Assume that 3 divides k^3+2k is true.

We need to prove that 3 divides $(k+1)^3+2(k+1)$ is also true

$$
\begin{aligned}
(k+1)^3+2(k+1) &= k^3+3k^2+3k+1+2k+2 \\
&= k^3+2k+3k^2+3k+3 \\
&= k^3+2k+3\left(k^2+k+1\right) \quad \checkmark
\end{aligned}
$$

By the inductive hypothesis, 3 divides k^3+2k and certainly 3 divides $3\left(k^2+k+1\right)$, so 3 divides their sum.

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Use mathematical induction to prove that 5 divides $n^5 - n$ whenever *n* is a positive integer.

Solution

For $\boldsymbol{n = 1}$

$1^5 - 1 = 0$, which is divisible by 5

Hence P_1 is true

Assume that 5 divides $k^5 - k$ is true.

We need to prove that 5 divides $(k+1)^5 - (k+1)$ is also true

$$\begin{aligned}(k+1)^5 - (k+1) &= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 \\ &= k^5 - k + 5k^4 + 10k^3 + 10k^2 + 5k \\ &= k^5 - k + 5\left(k^4 + 2k^3 + 2k^2 + k\right) \qquad \checkmark\end{aligned}$$

By the inductive hypothesis, 5 divides $k^5 - k$ and certainly 5 divides $5\left(k^4 + 2k^3 + 2k^2 + k\right)$, so 5 divides their sum.

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Use mathematical induction to prove that $n^2 - 1$ is divisible by 8 whenever *n* is an odd positive integer.

Solution

For $\boldsymbol{n = 1}$

$1^2 - 1 = 0$, which is divisible by 8

Hence P_1 is true

Assume that 8 divides $k^2 - 1$ is true, than implies to $k^2 - 1 = 8p$.

We need to prove that 8 divides $(k+1)^2 - 1$ is also true

$$\begin{aligned}(k+1)^2 - 1 &= k^2 + 2k + 1 - 1 \\ &= \left(k^2 - 1\right) + 2k + 1\end{aligned}$$

By the inductive hypothesis, 8 divides $k^2 - 1$ and certainly 8 divides $2k + 1$, so 8 divides their sum.

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Use mathematical induction to prove that 21 divides $4^{n+1}+5^{2n-1}$ whenever *n* is a positive integer.

Solution

For $\boldsymbol{n} = \mathbf{1}$

$4^2+5^1=21$, which is divisible by 21

Hence P_1 is true.

Assume that 21 divides $4^{k+1}+5^{2k-1}$ is true.

We need to prove that 21 divides $4^{(k+1)+1}+5^{2(k+1)-1}$ is also true

$$\begin{aligned}4^{(k+1)+1}+5^{2(k+1)-1} &= 4\cdot 4^{(k+1)}+5^{2k+2-1}\\ &= 4\cdot 4^{(k+1)}+5^2\cdot 5^{2k-1}\\ &= 4\cdot 4^{(k+1)}+25\cdot 5^{2k-1}\\ &= 4\cdot 4^{(k+1)}+(4+21)\cdot 5^{2k-1}\\ &= 4\cdot 4^{(k+1)}+4\cdot 5^{2k-1}+21\cdot 5^{2k-1}\\ &= 4\cdot\left(4^{(k+1)}+5^{2k-1}\right)+21\cdot 5^{2k-1}\end{aligned}$$

By the inductive hypothesis, 21 divides $4^{k+1}+5^{2k-1}$ and certainly 21 divides 5^{2k-1}, so 21 divides their sum.

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer *n*. $1+2.2+3.2^2+\ldots+n.2^{n-1}=1+(n-1).2^n$

Solution

For $\boldsymbol{n} = \mathbf{1}$

$1\overset{?}{=}1+(1-1)2^1=1-0=1$

Hence P_1 is true.

$1+2.2+3.2^2+\ldots+k.2^{k-1}=1+(k-1).2^k$ is true

$1+2.2+3.2^2+\ldots+k.2^{k-1}+(k+1).2^{(k+1)-1}=1+((k+1)-1).2^{k+1}$?

$$\begin{aligned}1+2.2+3.2^2+\ldots+k.2^{k-1}+(k+1).2^{(k+1)-1} &= 1+(k-1).2^k+(k+1).2^{k+1-1}\\ &= 1+k.2^k-1.2^k+(k+1).2^k\end{aligned}$$

$$=1+k.2^{k}-1.2^{k}+k.2^{k}+1.2^{k}$$
$$=1+2^{1}k.2^{k}$$
$$=1+(k+0).2^{k+1}$$
$$=1+((k+1)-1).2^{k+1} \quad \surd$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. $1^2+2^2+3^2+\ldots+n^2=\dfrac{n(n+1)(2n+1)}{6}$

Solution

For $\boldsymbol{n}$ **= 1**

$$1^2 \stackrel{?}{=} \frac{1(1+1)(2(1)+1)}{6}=\frac{1(2)(3)}{6}=\frac{6}{6}=1 \quad \surd$$

Hence P_1 is true.

$$1^2+2^2+3^2+\ldots+k^2=\frac{k(k+1)(2k+1)}{6} \text{ is true}$$

$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}?$$

$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$$
$$=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$
$$=\frac{(k+1)\left[k(2k+1)+6(k+1)\right]}{6}$$
$$=\frac{(k+1)\left[2k^2+k+6k+6\right]}{6}$$
$$=\frac{(k+1)\left[2k^2+7k+6\right]}{6}$$
$$=\frac{(k+1)((k+2)(2k+3))}{6}$$
$$=\frac{(k+1)((k+1+1)(2k+2+1))}{6}$$
$$=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \quad \surd$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

Solution

For $\boldsymbol{n = 1}$

$$\frac{1}{1.2}\overset{?}{=}\frac{1}{1+1}=\frac{1}{2}=\frac{1}{1.2}\ \checkmark$$

Hence P_1 is true.

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}\text{ is true}$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{(k+1)+1}?$$

$$\begin{aligned}\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}&=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}\\ &=\frac{k(k+2)+1}{(k+1)(k+2)}\\ &=\frac{k^2+2k+1}{(k+1)(k+2)}\\ &=\frac{(k+1)(k+1)}{(k+1)(k+2)}\\ &=\frac{k+1}{(k+1+1)}\\ &=\frac{k+1}{(k+1)+1}\ \checkmark\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$

Solution

For $\boldsymbol{n = 1}$

$$\frac{1}{2}\overset{?}{=}1-\frac{1}{2}=\frac{1}{2}\ \checkmark$$

Hence, P_1 is true.

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}=1-\frac{1}{2^k} \text{ is true}$$

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^{k+1}}?$$

$$\begin{aligned}\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}}&=1-\frac{1}{2^k}+\frac{1}{2^{k+1}}\\&=1-\frac{1}{2^k}+\frac{1}{2^k\cdot 2}\\&=\frac{2^{k+1}-2+1}{2^{k+1}}\\&=\frac{2^{k+1}-1}{2^{k+1}}\\&=\frac{2^{k+1}}{2^{k+1}}-\frac{1}{2^{k+1}}\\&=1-\frac{1}{2^{k+1}}\ \checkmark\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots+\frac{1}{(3n-2)\cdot(3n+1)}=\frac{n}{3n+1}$

Solution

For ***n* = 1**

$$\frac{1}{1\cdot 4}\overset{?}{=}\frac{1}{3(1)+1}=\frac{1}{4}\ \checkmark$$

Hence, P_1 is true.

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1} \text{ is true}$$

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3(k+1)-2)(3(k+1)+1)}\overset{?}{=}\frac{k+1}{3(k+1)+1}$$

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3(k+1)-2)(3(k+1)+1)}=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$$

$$=\frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$=\frac{(3k+1)(k+1)}{(3k+1)(3k+3+1)}$$

$$=\frac{k+1}{3(k+1)+1} \quad \surd$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $\frac{4}{5}+\frac{4}{5^2}+\frac{4}{5^3}+\cdots+\frac{4}{5^n}=1-\frac{1}{5^n}$

Solution

For $\boldsymbol{n = 1}$

$$\frac{4}{5}\overset{?}{=}1-\frac{1}{5}=\frac{4}{5} \quad \surd$$

Hence, P_1 is true.

$\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}=1-\frac{1}{5^k}$ is true

$$\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}+\frac{4}{5^{k+1}}\overset{?}{=}1-\frac{1}{5^{k+1}}$$

$$\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}+\frac{4}{5^{k+1}}=1-\frac{1}{5^k}+\frac{4}{5^{k+1}}$$

$$=1-\left(\frac{1}{5^k}-\frac{4}{5^{k+1}}\right)$$

$$=1-\frac{5-4}{5^{k+1}}$$

$$=1-\frac{1}{5^{k+1}} \quad \surd$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $1^3+2^3+3^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}$

Solution

For $\boldsymbol{n = 1}$

$$1^3 \stackrel{?}{=} \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1 \quad \surd$$

Hence, P_1 is true.

$\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}=1-\frac{1}{5^k}$ is true

$$1^3+2^3+\cdots+k^3+(k+1)^3 \stackrel{?}{=} \frac{(k+1)^2\left((k+1)+1\right)^2}{4}$$

$$\begin{aligned}
1^3+2^3+\cdots+k^3+(k+1)^3 &= \frac{k^2(k+1)^2}{4}+(k+1)^3 \\
&= \frac{k^2(k+1)^2+4(k+1)^3}{4} \\
&= \frac{(k+1)^2\left[k^2+4(k+1)\right]}{4} \\
&= \frac{(k+1)^2\left(k^2+4k+4\right)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \frac{(k+1)^2\left((k+1)+1\right)^2}{4} \quad \surd
\end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. $3+3^2+3^3+\ldots+3^n=\frac{3}{2}\left(3^n-1\right)$

Solution

For $\boldsymbol{n}$ **= 1**

$$3 \stackrel{?}{=} \frac{3}{2}\left(3^1-1\right)=\frac{3}{2}2=3 \quad \surd$$

Hence, P_1 is true.

$3+3^2+\cdots+3^k=\frac{3}{2}\left(3^k-1\right)$ is true

$$3+3^2+\cdots+3^k+3^{k+1} \stackrel{?}{=} \frac{3}{2}\left(3^{k+1}-1\right)$$

$$3+3^2+\cdots+3^k+3^{k+1}=\frac{3}{2}\left(3^k-1\right)+3^{k+1}$$

$$=\frac{1}{2}3^{k+1}-\frac{3}{2}+3^{k+1}$$

$$=\frac{3}{2}3^{k+1}-\frac{3}{2}$$

$$=\frac{3}{2}\left(3^{k+1}-1\right) \quad \surd$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $x^{2n}+x^{2n-1}y+\cdots+xy^{2n-1}+y^{2n}=\dfrac{x^{2n+1}-y^{2n+1}}{x-y}$

Solution

For ***n* = 1**

$$x^2+xy+y^2\overset{?}{=}\frac{x^3-y^3}{x-y}$$

$$=\frac{(x-y)\left(x^2+xy+y^2\right)}{x-y}$$

$$=x^2+xy+y^2 \quad \surd$$

Hence, P_1 is true.

$x^{2k}+x^{2k-1}y+\cdots+xy^{2k-1}+y^{2k}=\dfrac{x^{2k+1}-y^{2k+1}}{x-y}$ is true

$$x^{2(k+1)}+x^{2(k+1)-1}y+\cdots+xy^{2(k+1)-1}+y^{2(k+1)}\overset{?}{=}\frac{x^{2(k+1)+1}-y^{2(k+1)+1}}{x-y}$$

$$x^{2k+2}+x^{2k+1}y+\cdots+xy^{2k+1}+y^{2k+2}=x^2\left(x^{2k}+x^{2k-1}y+\cdots+y^{2k}\right)+xy^{2k+1}+y^{2k+2}$$

$$=x^2\left(\frac{x^{2k+1}-y^{2k+1}}{x-y}\right)+xy^{2k+1}+y^{2k+2}$$

$$=\frac{x^{2k+3}-x^2y^{2k+1}+x^2y^{2k+1}+xy^{2k+2}-xy^{2k+2}-y^{2(k+1)+1}}{x-y}$$

$$=\frac{x^{2(k+1)+1}-y^{2(k+1)+1}}{x-y} \quad \surd$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $5\cdot 6+5\cdot 6^2+5\cdot 6^3+\cdots+5\cdot 6^n=6\left(6^n-1\right)$

Solution

For $\boldsymbol{n}$ **= 1**

$$5\cdot 6\overset{?}{=}6\left(6^1-1\right)=6(5)\ \checkmark$$

Hence, P_1 is true.

$$5\cdot 6+5\cdot 6^2+\cdots+5\cdot 6^k=6\left(6^k-1\right)\ \text{is true}$$

$$5\cdot 6+5\cdot 6^2+\cdots+5\cdot 6^k+5\cdot 6^{k+1}\overset{?}{=}6\left(6^{k+1}-1\right)$$

$$\begin{aligned}5\cdot 6+5\cdot 6^2+\cdots+5\cdot 6^k+5\cdot 6^{k+1}&=6\left(6^k-1\right)+5\cdot 6^{k+1}\\&=6^{k+1}-6+5\cdot 6^{k+1}\\&=6^{k+1}(1+5)-6\\&=6\cdot 6^{k+1}-6\\&=6\left(6^{k+1}-1\right)\ \checkmark\end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $7\cdot 8+7\cdot 8^2+7\cdot 8^3+\cdots+7\cdot 8^n=8\left(8^n-1\right)$

Solution

For $\boldsymbol{n}$ **= 1**

$$7\cdot 8\overset{?}{=}8\left(8^1-1\right)=8(7)\ \checkmark$$

Hence, P_1 is true.

$$7\cdot 8+7\cdot 8^2+\cdots+7\cdot 8^k=8\left(8^k-1\right)\ \text{is true}$$

$$7\cdot 8+7\cdot 8^2+\cdots+7\cdot 8^k+7\cdot 8^{k+1}\overset{?}{=}8\left(8^{k+1}-1\right)$$

$$\begin{aligned}7\cdot 8+7\cdot 8^2+\cdots+7\cdot 8^k+7\cdot 8^{k+1}&=8\left(8^k-1\right)+7\cdot 8^{k+1}\\&=8^{k+1}-8+7\cdot 8^{k+1}\end{aligned}$$

$$=8^{k+1}(1+7)-8$$
$$=8\left(8^{k+1}-1\right) \quad \surd$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $3+6+9+\cdots+3n=\dfrac{3n(n+1)}{2}$

Solution

For ***n* = 1**

$$3\overset{?}{=}\frac{3(1)(1+1)}{2}=3 \quad \surd$$

Hence, P_1 is true.

$3+6+9+\cdots+3k=\dfrac{3k(k+1)}{2}$ is true

$$3+6+9+\cdots+3k+3(k+1)\overset{?}{=}\frac{3(k+1)(k+2)}{2}$$

$$\begin{aligned}3+6+9+\cdots+3k+3(k+1)&=\frac{3k(k+1)}{2}+3(k+1)\\&=\frac{3k(k+1)+6(k+1)}{2}\\&=\frac{(k+1)(3k+6)}{2}\\&=\frac{3(k+1)(k+2)}{2} \quad \surd\end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $5+10+15+\cdots+5n=\dfrac{5n(n+1)}{2}$

Solution

For ***n* = 1**

$$5\overset{?}{=}\frac{5(1)(1+1)}{2}=5 \quad \surd$$

Hence, P_1 is true.

$$5+10+15+\cdots+5k=\frac{5k(k+1)}{2} \text{ is true}$$

$$5+10+15+\cdots+5k+5(k+1)\overset{?}{=}\frac{5(k+1)(k+2)}{2}$$

$$\begin{aligned}5+10+15+\cdots+5k+5(k+1)&=\frac{5k(k+1)}{2}+5(k+1)\\&=\frac{5k(k+1)+10(k+1)}{2}\\&=\frac{(k+1)(5k+10)}{2}\\&=\frac{5(k+1)(k+2)}{2} \quad \checkmark\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $1+3+5+\cdots+(2n-1)=n^2$

Solution

For ***n* = 1**

$$1\overset{?}{=}1^2=1 \quad \checkmark$$

Hence, P_1 is true.

$$1+3+5+\cdots+(2k-1)=k^2 \text{ is true}$$

$$1+3+5+\cdots+(2k-1)+(2(k+1)-1)\overset{?}{=}(k+1)^2$$

$$\begin{aligned}1+3+5+\cdots+(2k-1)+(2(k+1)-1)&=k^2+2k+2-1\\&=k^2+2k+1\\&=(k+1)^2 \quad \checkmark\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $4+7+10+\cdots+(3n+1)=\dfrac{n(3n+5)}{2}$

Solution

For $\boldsymbol{n}$ **= 1**

$$4 \stackrel{?}{=} \frac{1(3+5)}{2} = 4 \quad \checkmark$$

Hence, P_1 is true.

$4+7+10+\cdots+(3k+1)=\dfrac{k(3k+5)}{2}$ is true

$$4+7+10+\cdots+(3k+1)+(3(k+1)+1) \stackrel{?}{=} \frac{(k+1)(3(k+1)+5)}{2} = \frac{(k+1)(3k+8)}{2}$$

$$\begin{aligned}
4+7+10+\cdots+(3k+1)+(3k+4) &= \frac{k(3k+5)}{2}+3k+4 \\
&= \frac{3k^2+5k+6k+8}{2} \\
&= \frac{3k^2+5k+3k+3k+8}{2} \\
&= \frac{k(3k+8)+(3k+8)}{2} \\
&= \frac{(3k+8)(k+1)}{2} \quad \checkmark
\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction:

$$2+4+6+\cdots+2(n-1)+2n=n(n+1)$$

Solution

For $\boldsymbol{n}$ **= 1**

$$2 \stackrel{?}{=} 1(1+1)$$

$$2=2 \quad \checkmark$$

Hence, P_1 is true.

For $\boldsymbol{k}$: $2+4+6+\cdots+2(k-1)+2k=k(k+1)$

$$2+4+\cdots+2k+2(k+1)=(k+1)(k+2)$$

$$2+4+\cdots+2k+2(k+1)=k(k+1)+2(k+1)$$

$$=(k+1)(k+2) \quad \surd$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction:

$$1+(1+2)+(1+2+3)+\ \cdots\ +(1+2+\cdots+n)=\frac{n(n+1)(n+2)}{6}$$

Solution

For $\boldsymbol{n}$ **= 1**

$$1\overset{?}{=}\frac{1(1+1)(1+2)}{6}$$

$$1\overset{?}{=}\frac{(2)(3)}{6}$$

$$1=1 \quad \surd$$

Hence, P_1 is true.

For $\boldsymbol{k}$: $\quad 1+(1+2)+\ \cdots\ +(1+2+\cdots+k)=\dfrac{k(k+1)(k+2)}{6}$

Is P_{k+1}: $\quad 1+(1+2)+\ \cdots\ +(1+2+\cdots+k)+\big(1+2+\cdots+k+(k+1)\big)\overset{?}{=}\dfrac{(k+1)(k+2)(k+3)}{6}$

$$1+(1+2)+\ \cdots\ +(1+2+\cdots+k)+\big(1+2+\cdots+k+(k+1)\big)=\frac{k(k+1)(k+2)}{6}+\big(1+2+\cdots+k+(k+1)\big)$$

$$1+2+\cdots+n=\frac{1}{2}n(n+1)$$

$$1+2+\cdots+k+(k+1)=\frac{1}{2}k(k+1)+(k+1)$$

$$=(k+1)\left(\frac{1}{2}k+1\right)$$

$$=\frac{1}{2}(k+1)(k+2)$$

$$1+(1+2)+\ \cdots\ +(1+2+\cdots+k)+\big(1+2+\cdots+k+(k+1)\big)=\frac{k(k+1)(k+2)}{6}+\frac{1}{2}(k+1)(k+2)$$

$$=(k+1)(k+2)\left(\frac{k}{6}+\frac{1}{2}\right)$$

$$=\frac{(k+1)(k+2)(k+3)}{6} \quad \surd$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction:

$$1+2+3+\cdots+n<\frac{(2n+3)^2}{7}$$

Solution

For $\boldsymbol{n}$ **= 1**

$$1\overset{?}{<}\frac{(2+3)^2}{7}$$

$$1<\frac{25}{7} \quad >1 \quad \checkmark$$

Hence, P_1 is true.

For $\boldsymbol{k}$: $\quad 1+2+\cdots+k<\dfrac{(2k+3)^2}{7}$

Is P_{k+1}: $\quad 1+2+\cdots+k+(k+1)<\dfrac{(2(k+1)+3)^2}{7}$

$$<\frac{(2k+5)^2}{7} \; ? \qquad \frac{4k^2+20k+25}{7}$$

$$\begin{aligned}
1+2+\cdots+k+(k+1) &< \frac{(2k+3)^2}{7}+(k+1)\\
&=\frac{4k^2+12k+9+7k+7}{7}\\
&=\frac{1}{7}\left(4k^2+19k+16+k+9-k-9\right)\\
&=\frac{1}{7}\left(4k^2+20k+25-(k+9)\right)\\
&=\frac{(2k+5)^2}{7}-\frac{k+9}{7}\\
&<\frac{(2k+5)^2}{7} \quad \checkmark
\end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction:

$$\frac{1}{2n} \le \frac{1\cdot 3\cdot 5\cdots(2n-3)\cdot(2n-1)}{2\cdot 4\cdot 6\cdots(2n-2)\cdot(2n)}$$

Solution

For $\boldsymbol{n}$ **= 1**

$$\frac{1}{2} \le \frac{1}{2} \ \checkmark$$

Hence, P_1 is true.

For $\boldsymbol{k}$: $\quad \frac{1}{2k} \le \frac{1\cdots(2k-3)\cdot(2k-1)}{2\cdot\ \cdots\ (2k-2)\cdot(2k)}$

Is P_{k+1}: $\quad \frac{1}{2(k+1)} \le \frac{1\cdots(2k-1)\cdot(2k+1)}{2\cdot\ \cdots\ (2k)\cdot(2k+2)}$

$$\frac{1\cdots(2k-1)\cdot(2k+1)}{2\cdot\ \cdots\ (2k)\cdot(2k+2)} \ge \frac{1}{2(k+1)} \quad ?$$

$$\begin{aligned}
\frac{1\cdots(2k-1)\cdot(2k+1)}{2\cdot\ \cdots\ (2k)\cdot(2k+2)} &= \frac{1\cdots(2k-1)}{2\cdot\ \cdots\ (2k)}\frac{2k+1}{2k+2} \\
&\ge \frac{1}{2k}\cdot\frac{2k+1}{2k+2} \\
&= \frac{2k+1}{2k}\cdot\frac{1}{2(k+1)} \\
&= \left(1+\frac{1}{2k}\right)\cdot\frac{1}{2(k+1)} \\
&\ge \frac{1}{2(k+1)} \quad \checkmark
\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction: $\quad \frac{2n+1}{2n+2} \le \frac{\sqrt{n+1}}{\sqrt{n+2}}$

Solution

For $\boldsymbol{n}$ **= 1**

$$\frac{2+1}{2+2} \overset{?}{\le} \frac{\sqrt{1+1}}{\sqrt{1+2}}$$

$\frac{3}{4} \overset{?}{\le} \frac{\sqrt{2}}{\sqrt{3}}$

$3\sqrt{3} \quad ? \quad 4\sqrt{2}$ *Square both sides*

$27 \quad \le \quad 32 \quad \surd$

Hence, P_1 is true.

For ***k***: $\frac{2k+1}{2k+2} \le \frac{\sqrt{k+1}}{\sqrt{k+2}}$

$(2k+1)\sqrt{k+2} \le (2k+2)\sqrt{k+1}$

Is P_{k+1}: $\frac{2k+3}{2k+4} \overset{?}{\le} \frac{\sqrt{k+2}}{\sqrt{k+3}}$

$(2k+3)\sqrt{k+3} \overset{?}{\le} (2k+4)\sqrt{k+2}$

$(2k+3) \le (2k+4)$

$\underline{\sqrt{k+3} \le \sqrt{k+2}}$

$(2k+3)\sqrt{k+3} \le (2k+4)\sqrt{k+2} \quad \surd$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction: $n! < n^n \quad for \quad n > 1$

Solution

For ***n* = 2**

$2! \overset{?}{<} 2^2$

$2 < 4 \quad \surd$

Hence, P_1 is true.

For ***k***: $k! < k^k$

Is P_{k+1}: $(k+1)! < (k+1)^{k+1}$

$(k+1)! = k!\,(k+1)$

$< k^k\,(k+1)$

$k < k+1$

$k^k < (k+1)^k$

$$< (k+1)^k \ (k+1)$$
$$= (k+1)^{k+1} \quad \checkmark$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction: $\left(a^m\right)^n = a^{mn}$ (a and m are constant)

Solution

For ***n*** **= 1**

$$\left(a^m\right)^1 \stackrel{?}{=} a^{m(1)}$$
$$a^m = a^m \quad \checkmark$$

Hence, P_1 is true.

$\left(a^m\right)^k = a^{mk}$ is true

$$\left(a^m\right)^{(k+1)} \stackrel{?}{=} a^{m(k+1)}$$

$$\begin{aligned}\left(a^m\right)^{(k+1)} &= \left(a^m\right)^k a^m \\ &= a^{km} a^m \\ &= a^{km+m} \\ &= a^{m(k+1)} \quad \checkmark\end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. $n < 2^n$

Solution

For ***n*** **= 1**

$$1 \stackrel{?}{<} 2^1 \quad \checkmark$$

Hence, P_1 is true.

Assume that P_k is true $k < 2^k$

We need to prove that P_{k+1} is true, that is $k+1 < 2^{k+1}$

$$\begin{aligned} k+1 < k+k &= 2k \\ &< 2\cdot 2^k \\ &= 2^{k+1} \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. 3 is a factor of $n^3 - n + 3$

Solution

For ***n* = 1**

$1^3 - 1 + 3 = 3 = 3(1)$ ✓

Hence, P_1 is true.

Assume that P_k is true 3 is a factor of $k^3 - k + 3$

We need to prove that P_{k+1} is true, that is $(k+1)^3 - (k+1) + 3$

$$\begin{aligned} (k+1)^3 - (k+1) + 3 &= k^3 + 3k^2 + 3k + 1 - k - 1 + 3 \\ &= \left(k^3 - k + 3\right) + 3k^2 + 3k \qquad\qquad k^3 - k + 3 = 3K \\ &= 3K + 3k^2 + 3k \\ &= 3\left(K + k^2 + k\right) \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. 4 is a factor of $5^n - 1$

Solution

For ***n* = 1**

$5^1 - 1 = 4 = 4(1)$ ✓

Hence, P_1 is true.

Assume that P_k is true 4 is a factor of $5^k - 1$

We need to prove that P_{k+1} is true, that is $5^{k+1} - 1$

$$5^{k+1} - 1 = 5^k 5^1 - 5 + 4$$
$$= 5\left(5^k - 1\right) + 4$$
$$= 5\left(5^k - 1\right) + 4$$

By the induction hypothesis, 4 is a factor of $5^k - 1$ and 4 is a factor of 4, so 4 is a factor of the $(k+1)$ term. √

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: $2^n > 2n \quad if \quad n \geq 3$

Solution

For ***n* = 3**

$2^3 \geq 2(3)$

$8 \geq 6$ √

Hence, P_3 is true.

Assume that P_k is true: $2^k > 2k$

We need to prove that P_{k+1}: $2^{k+1} > 2(k+1)$ is true

$$2^k > 2k$$
$$2^k \cdot 2 > 2k \cdot 2$$
$$2^{k+1} > 4k = 2k + 2k \qquad k \geq 3$$
$$> 2k + 2$$
$$= 2(k+1)$$ √

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: If $0 < a < 1$, then $a^n < a^{n-1}$

Solution

For $\boldsymbol{n = 1}$

$a^1 < a^{1-1}$

$a < 1$ √

since $0 < a < 1 \Rightarrow P_1$ is true.

Assume that P_k is true: $a^k < a^{k-1}$

We need to prove that P_{k+1}: $a^{k+1} < a^k$ is true

$$a^k < a^{k-1} \quad \rightarrow \quad a^k \cdot a < a^{k-1} \cdot a$$

$$a^{k+1} < a^k \quad \checkmark$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: If $n \geq 4$, then $n! > 2^n$

Solution

For $\boldsymbol{n = 4}$

$4! > 2^4$

$24 > 16$ √

Hence, P_4 is true.

Assume that P_k is true: $k! > 2^k$

We need to prove that P_{k+1}: $(k+1)! > 2^{k+1}$ is true

$$\begin{aligned}(k+1)! &= k! \cdot (k+1) \\ &> 2^k (k+1) && \textit{Since } k \geq 4 \Rightarrow k+1 > 2 \\ &> 2^k \cdot 2 \\ &= 2^{k+1} \quad \checkmark\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: $3^n > 2n+1 \quad if \quad n \ge 2$

Solution

For $\boldsymbol{n} = \mathbf{2}$

$3^2 > 2(2)+1$

$9 > 5$ √

Hence, P_2 is true.

Assume that P_k is true: $3^k > 2k+1;$

We need to prove that $P_{k+1}:\ 3^{k+1} > 2(k+1)+1$ is true

$$3^k > 2k+1 \quad \Rightarrow \quad 3^k \cdot 3 > (2k+1)\cdot 3$$

$$\begin{aligned} 3^{k+1} &> 6k+3 \\ &> 2k+2+1 \\ &= 2(k+1)+1 \end{aligned}$$ √

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: $2^n > n^2 \quad for \quad n > 4$

Solution

For $\boldsymbol{n} = \mathbf{5}$

$2^5 > 5^2$

$32 > 25$ √

Hence, P_5 is true.

Assume that P_k is true: $2^k > k^2$

We need to prove that $P_{k+1}:\ 2^{k+1} > (k+1)^2$ is true

$$2^k > k^2$$

$$2^k \cdot 2 > k^2 \cdot 2$$

$$\begin{aligned} 2^{k+1} &> 2k^2 \\ &= k^2 + k^2 \\ &> k^2 + 2k+1 \end{aligned}$$

$k < k+1 \ \Rightarrow \ k \cdot k > k+k+1 \Rightarrow \ k^2 > 2k+1$

$$=(k+1)^2 \quad \checkmark$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: $4^n > n^4 \quad for \quad n \geq 5$

Solution

For $\boldsymbol{n} = \mathbf{5}$

$$4^5 > 5^4$$

$$1024 > 625 \quad \checkmark$$

Hence, P_5 is true.

Assume that P_k is true: $4^k > k^4$

We need to prove that P_{k+1} : $4^{k+1} > (k+1)^4$ is true

$$4^k > k^4$$

$$4^k \cdot 4 > k^4 \cdot 4$$

$$4^{k+1} > 4k^4$$

$$k < k+1$$

$$4k > k+1$$

$$4k^4 > (k+1)^4$$

$$> (k+1)^4 \quad \checkmark$$

Hence P_{k+1} is true.

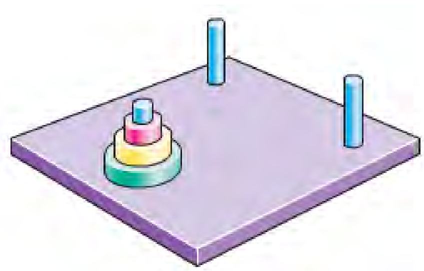
$\therefore$ By the mathematical induction, the given statement is true.

Exercise

A pile of *n* rings, each smaller than the one below it, is on a peg on board. Two other pegs are attached to the board. In the game called the Tower of Hanoi puzzle, all the rings must be moved, one at a time, to a different peg with no ring ever placed on top of a smaller ring.

Find the least number of moves that would be required.

Prove your result by mathematical induction.



Solution

With 1 ring, 1 move is required.

With 2 rings, 3 moves are required $\Rightarrow \quad 3 = 2 + 1$

With 3 rings, 7 moves are required $\Rightarrow \quad 7 = 2^2 + 2 + 1$

With n rings, $2^{n-1} + \cdots + 2^2 + 2^1 + 2^0 = 2^n - 1$ moves are required

For ***n* = 1**

$$2^0 = 2^1 - 1 = 1 \; \checkmark$$

Hence, P_1 is true.

Assume that P_k is true: $2^{k-1} + \cdots + 2^2 + 2^1 + 2^0 = 2^k - 1$

$$2^k + 2^{k-1} + \cdots + 2^2 + 2^1 + 1 \overset{?}{=} 2^{k+1} - 1$$

$$\begin{aligned} 2^k + 2^{k-1} + \cdots + 2^2 + 2^1 + 1 &= 2^k + 2^k - 1 \\ &= 2 \cdot 2^k - 1 \\ &= 2^{k+1} - 1 \quad \checkmark \end{aligned}$$

$\therefore$ By the mathematical induction, the given statement is true.