

Section 1.6 – Exponential and Logarithmic Equations

Properties of Logarithms For $M > 0$ and $N > 0$

Product Rule $\log_b MN = \log_b M + \log_b N$

Power Rule $\log_b M^p = p \log_b M$

Quotient Rule $\log_b \frac{M}{N} = \log_b M - \log_b N$

Example

Express $\log_a \frac{x^3 \sqrt{y}}{z^2}$ in terms of logarithms of x , y , and z .

Solution

$$\begin{aligned}\log_a \frac{x^3 \sqrt{y}}{z^2} &= \log_a x^3 y^{1/2} - \log_a z^2 && \text{Quotient Rule} \\ &= \log_a x^3 + \log_a y^{1/2} - \log_a z^2 && \text{Product Rule} \\ &= 3 \log_a x + \frac{1}{2} \log_a y - 2 \log_a z && \text{Power Rule}\end{aligned}$$

Example

Express as one logarithm: $\frac{1}{3} \log_a (x^2 - 1) - \log_a y - 4 \log_a z$

Solution

$$\begin{aligned}\frac{1}{3} \log_a (x^2 - 1) - \log_a y - 4 \log_a z &= \log_a (x^2 - 1)^{1/3} - \log_a y - \log_a z^4 && \text{Power Rule} \\ &= \log_a \sqrt[3]{x^2 - 1} - (\log_a y + \log_a z^4) && \text{Factor } (-) \\ &= \log_a \sqrt[3]{x^2 - 1} - (\log_a yz^4) && \text{Product Rule} \\ &= \log_a \frac{\sqrt[3]{x^2 - 1}}{yz^4} && \text{Quotient Rule}\end{aligned}$$

Exponential Functions are One-to-One

$$b^M = b^N \leftrightarrow M = N \text{ for any } b > 0, \neq 1$$

Example

Solve $8^{x+2} = 4^{x-3}$

Solution

$$(2^3)^{x+2} = (2^2)^{x-3}$$

$$2^{3(x+2)} = 2^{2(x-3)}$$

$$3(x+2) = 2(x-3)$$

$$3x + 6 = 2x - 6$$

$$3x - 2x = -6 - 6$$

$$x = -12$$

Using Natural Logarithms

1. Isolate the exponential expression
2. Take the natural logarithm on both sides of the equation
3. Simplify using one of the following properties: $\ln b^x = x \ln b$ or $\ln e^x = x$
4. Solve for the variable

Example

Solve the equation $3^x = 21$

Solution

1 st method	2 nd method
$3^x = 21$ <i>ln both sides</i> $\ln 3^x = \ln 21$ $x \ln 3 = \ln 21$ $x = \frac{\ln 21}{\ln 3}$	$3^x = 21 \Rightarrow x = \log_3 21$ <i>Convert to log form</i> $x = \frac{\ln 21}{\ln 3}$ <i>Change of base</i>

Example

Solve the equation $5^{2x+1} = 6^{x-2}$

Solution

$$\ln 5^{2x+1} = \ln 6^{x-2}$$

$$(2x+1)\ln 5 = (x-2)\ln 6$$

$$2x\ln 5 + \ln 5 = x\ln 6 - 2\ln 6$$

$$2x\ln 5 - x\ln 6 = -2\ln 6 - \ln 5$$

$$x(2\ln 5 - \ln 6) = -\ln 6^2 - \ln 5$$

$$x(\ln 5^2 - \ln 6) = -(\ln 36 + \ln 5)$$

$$x\left(\ln \frac{25}{6}\right) = -\ln(36 \times 5)$$

$$\underline{x = -\frac{\ln(180)}{\ln \frac{25}{6}} \approx -3.64}$$

Example

Solve the equation $\frac{5^x - 5^{-x}}{2} = 3$

Solution

$$5^x - 5^{-x} = 6 \quad \text{Multiply by 2 both sides}$$

$$5^x 5^x - 5^{-x} 5^x = 6 5^x \quad \text{Multiply by } 5^x \text{ both sides}$$

$$(5^x)^2 - 1 = 6(5^x)$$

$$(5^x)^2 - 6(5^x) - 1 = 0$$

$$5^x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-1)}}{2(1)} = \frac{6 \pm \sqrt{40}}{2} = \frac{6 \pm 2\sqrt{10}}{2} = \begin{cases} 3 + \sqrt{10} \\ 3 - \sqrt{10} < 0 \end{cases}$$

$$5^x = 3 + \sqrt{10}$$

$$\ln 5^x = \ln(3 + \sqrt{10})$$

$$x\ln 5 = \ln(3 + \sqrt{10})$$

$$\underline{x = \frac{\ln(3 + \sqrt{10})}{\ln 5} \approx 1.13}$$

Logarithmic Equations

1. Express the equation in the form $\log_b M = c$
2. Use the definition of a logarithm to rewrite the equation in exponential form:

$$\log_b M = c \Rightarrow b^c = M$$

3. Solve for the variable
4. Check proposed solution in the original equation. Include only the set for $M > 0$

Example

Solve: $\log x + \log(x - 3) = 1$

Solution

$$\log[x(x - 3)] = 1$$

Product Rule

$$x(x - 3) = 10^1$$

Convert to exponential form

$$x^2 - 3x = 10$$

$$x^2 - 3x - 10 = 0$$

Solve for x

$$\Rightarrow x = -2, 5$$

Check: $x = -2 \Rightarrow \log(-2) + \log(-2 - 3) = 1$
 $x = 5 \Rightarrow \log(5) + \log(5 - 3) = 1$

Example

Solve the equation $\log_2 x + \log_2 (x + 2) = 3$

Solution

$$\log_2 [x(x + 2)] = 3$$

Product Rule

$$x(x + 2) = 2^3$$

Change to exponential form

$$x^2 + 2x - 8 = 0$$

Solve for x

$$x = -4 \quad x = 2$$

Check: $\log_2 (-4) + \log_2 (-4 + 2) = 3$ Not a solution (negative inside the log)
 $\log_2 (2) + \log_2 (2 + 2) = 3$ Only solution

Property of Logarithmic Equality

The logarithmic function with base b is 1-1. Thus the following equivalent conditions are satisfied for positive real numbers M and N .

For any $M > 0, N > 0, b > 0, \neq 1$

$$\text{If } \log_b M = \log_b N \Rightarrow M = N$$

$$\text{If } M \neq N \Rightarrow \log_b M \neq \log_b N$$

Example

Solve the equation $\log_6(4x - 5) = \log_6(2x + 1)$

Solution

$$\log_6(4x - 5) = \log_6(2x + 1)$$

$$4x - 5 = 2x + 1$$

$$4x - 2x = 5 + 1$$

$$2x = 6$$

$$x = 3$$

Check: $\log_6(4(3) - 5) = \log_6(2(3) + 1)$

$$\log_6(7) = \log_6(7) \quad \text{True statement}$$

$x = 3$ is a solution

Example

Solve the equation $\ln(x + 6) - \ln 10 = \ln(x - 1) - \ln 2$

Solution

$$\ln(x + 6) - \ln 10 = \ln(x - 1) - \ln 2$$

$$\ln(x + 6) - \ln(x - 1) = \ln 10 - \ln 2$$

$$\ln\left(\frac{x+6}{x-1}\right) = \ln\frac{10}{2}$$

$$\frac{x+6}{x-1} = 5$$

$$x + 6 = 5(x - 1)$$

$$x + 6 = 5x - 5$$

$$x - 5x = -5 - 6$$

$$-4x = -11$$

$$\boxed{x = \frac{11}{4}}$$

Check: $\ln\left(\frac{11}{4} + 6\right) - \ln 10 = \ln\left(\frac{11}{4} - 1\right) - \ln 2$

$$\ln\left(\frac{35}{4}\right) - \ln 10 = \ln\left(\frac{7}{4}\right) - \ln 2$$

$x = \frac{11}{4}$ is the solution

Example

Solve the equation $\log \sqrt[3]{x} = \sqrt{\log x}$ for x .

Solution

$$\log x^{1/3} = \sqrt{\log x}$$

$$\left(\frac{1}{3}\log x\right)^2 = \left(\sqrt{\log x}\right)^2$$

$$\frac{1}{9}(\log x)^2 = \log x$$

$$(\log x)^2 = 9\log x$$

$$(\log x)^2 - 9\log x = 0$$

$$\log x(\log x - 9) = 0$$

$$\log x = 0$$

$$\boxed{x = 1}$$

$$\log x - 9 = 0$$

$$\log x = 9$$

$$\boxed{x = 10^9}$$

Check: $x = 1 \Rightarrow \log \sqrt[3]{1} = \sqrt{\log 1} \rightarrow 0 = 0$

$$x = 10^9 \Rightarrow \log \sqrt[3]{10^9} = \sqrt{\log 10^9} \rightarrow 3 = 3$$

The equation has two solutions: $\boxed{x = 1, 10^9}$

Example (hyperbolic secant function)

Solve the equation $y = \frac{2}{e^x + e^{-x}}$ for x in terms of y .

Solution

$$y = \frac{2}{e^x + e^{-x}}$$

$$y(e^x + e^{-x}) = 2$$

$$ye^x + ye^{-x} = 2$$

$$ye^x e^x + ye^{-x} e^x = 2e^x$$

$$y(e^x)^2 - 2e^x + y = 0$$

$$e^x = \frac{2 \pm \sqrt{4 - 4y^2}}{2y}$$

$$= \frac{2 \pm \sqrt{4(1 - y^2)}}{2y}$$

$$= \frac{2 \pm 2\sqrt{1 - y^2}}{2y}$$

$$= \frac{1 \pm \sqrt{1 - y^2}}{y}$$

$$\ln e^x = \ln \left(\frac{1 \pm \sqrt{1 - y^2}}{y} \right)$$

$$\underline{x = \ln \frac{1 \pm \sqrt{1 - y^2}}{y}}$$

Exercises Section 1.6 – Exponential and Logarithmic Equations

(1 – 31) Express the following in terms of sums and differences of logarithms

1. $\log_3(ab)$

2. $\log_7(7x)$

3. $\log \frac{x}{1000}$

4. $\log_5 \left(\frac{125}{y} \right)$

5. $\log_b x^7$

6. $\ln \sqrt[7]{x}$

7. $\log_a \frac{x^2 y}{z^4}$

8. $\log_b \frac{x^2 y}{b^3}$

9. $\log_b \left(\frac{x^3 y}{z^2} \right)$

10. $\log_b \left(\frac{\sqrt[3]{x} y^4}{z^5} \right)$

11. $\log \left(\frac{100x^3 \sqrt[3]{5-x}}{3(x+7)^2} \right)$

12. $\log_a \sqrt[4]{\frac{m^8 n^{12}}{a^3 b^5}}$

13. $\log_p \sqrt[3]{\frac{m^5 n^4}{t^2}}$

14. $\log_b \sqrt[n]{\frac{x^3 y^5}{z^m}}$

15. $\log_a \sqrt[3]{\frac{a^2 b}{c^5}}$

16. $\log_b \left(x^4 \sqrt[3]{y} \right)$

17. $\log_5 \left(\frac{\sqrt{x}}{25y^3} \right)$

18. $\log_a \frac{x^3 w}{y^2 z^4}$

19. $\log_a \frac{\sqrt{y}}{x^4 \sqrt[3]{z}}$

20. $\ln 4 \sqrt[4]{\frac{x^7}{y^5 z}}$

21. $\ln x \sqrt[3]{\frac{y^4}{z^5}}$

22. $\log_b \sqrt[5]{\frac{m^4 n^5}{x^2 ab^{10}}}$

23. $\log_b \frac{a^5 b^{10}}{c^2 \sqrt[4]{d^3}}$

24. $\ln \left(x^2 \sqrt{x^2 + 1} \right)$

25. $\ln \frac{x^2}{x^2 + 1}$

26. $\ln \left(\frac{x^2 (x+1)^3}{(x+3)^{1/2}} \right)$

27. $\ln \sqrt{\frac{(x+1)^5}{(x+2)^{20}}}$

28. $\ln \frac{(x^2 + 1)^5}{\sqrt{1-x}}$

29. $\ln \left(\sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}} \right)$

30. $\ln \left(\sqrt{\frac{1}{x(x+1)}} \right)$

31. $\ln \left(\sqrt{(x^2 + 1)(x-1)^2} \right)$

(32 – 55) Write the expression as a single logarithm and simplify if necessary

32. $\log(x+5) + 2\log x$

33. $3\log_b x - \frac{1}{3}\log_b y + 4\log_b z$

34. $\frac{1}{2}\log_b(x+5) - 5\log_b y$

35. $\ln(x^2 - y^2) - \ln(x - y)$

36. $\ln(xz) - \ln(x\sqrt{y}) + 2\ln \frac{y}{z}$

37. $\log(x^2 y) - \log z$

38. $\log(z^2 \sqrt{y}) - \log z^{1/2}$

39. $2\log_a x + \frac{1}{3}\log_a(x-2) - 5\log_a(2x+3)$

40. $5\log_a x - \frac{1}{2}\log_a (3x-4) - 3\log_a (5x+1)$
41. $\log(x^3 y^2) - 2\log(x\sqrt[3]{y}) - 3\log\left(\frac{x}{y}\right)$
42. $\ln y^3 + \frac{1}{3}\ln(x^3 y^6) - 5\ln y$
43. $2\ln x - 4\ln\left(\frac{1}{y}\right) - 3\ln(xy)$
44. $4\ln x + 7\ln y - 3\ln z$
45. $\frac{1}{3}\left[5\ln(x+6) - \ln x - \ln(x^2 - 25)\right]$
46. $\frac{2}{3}\left[\ln(x^2 - 4) - \ln(x+2)\right] + \ln(x+y)$
47. $\frac{1}{2}\log_b m + \frac{3}{2}\log_b 2n - \log_b m^2 n$
48. $\frac{1}{2}\log_y p^3 q^4 - \frac{2}{3}\log_y p^4 q^3$
49. $\frac{1}{2}\log_a x + 4\log_a y - 3\log_a x$
50. $\frac{2}{3}\left[\ln(x^2 - 9) - \ln(x+3)\right] + \ln(x+y)$
51. $\frac{1}{4}\log_b x - 2\log_b 5 - 10\log_b y$
52. $2\ln(x+4) - \ln x - \ln(x^2 - 3)$
53. $\ln x + \ln(y+3) + \ln(y+2) - \ln(y^2 + 5y + 6)$
54. $\ln x + \ln(x+4) + \ln(x+1) - \ln(x^2 + 5x + 4)$
55. $\ln(x^2 - 25) - 2\ln(x+5) + \ln(x-5)$

(56 – 169) Solve the equations

56. $2^x = 128$
57. $3^x = 243$
58. $5^x = 70$
59. $6^x = 50$
60. $5^x = 134$
61. $7^x = 12$
62. $9^x = \frac{1}{\sqrt[3]{3}}$
63. $49^x = \frac{1}{343}$
64. $2^{5x+3} = \frac{1}{16}$
65. $\left(\frac{2}{5}\right)^x = \frac{8}{125}$
66. $2^{3x-7} = 32$
67. $4^{2x-1} = 64$
68. $3^{1-x} = \frac{1}{27}$
69. $2^{-x^2} = 5$
70. $2^{-x} = 8$
71. $\left(\frac{1}{3}\right)^x = 81$
72. $3^{-x} = 120$
73. $27 = 3^{5x} 9^{x^2}$
74. $4^{x+3} = 3^{-x}$
75. $2^{x+4} = 8^{x-6}$
76. $8^{x+2} = 4^{x-3}$
77. $7^x = 12$
78. $5^{x+4} = 4^{x+5}$
79. $5^{x+2} = 4^{1-x}$
80. $3^{2x-1} = 0.4^{x+2}$
81. $4^{3x-5} = 16$
82. $4^{x+3} = 3^{-x}$
83. $7^{2x+1} = 3^{x+2}$
84. $3^{x-1} = 7^{2x+5}$

85. $4^{x-2} = 2^{3x+3}$
86. $3^{5x-8} = 9^{x+2}$
87. $3^{x+4} = 2^{1-3x}$
88. $3^{2-3x} = 4^{2x+1}$
89. $4^{x+3} = 3^{-x}$
90. $7^{x+6} = 7^{3x-4}$
91. $2^{-100x} = (0.5)^{x-4}$
92. $4^x \left(\frac{1}{2}\right)^{3-2x} = 8 \cdot (2^x)^2$
93. $5^x + 125(5^{-x}) = 30$
94. $4^x - 3(4^{-x}) = 8$
95. $5^{3x-6} = 125$
96. $e^x = 15$
97. $e^{x+1} = 20$
98. $9e^x = 107$
99. $e^{x \ln 3} = 27$
100. $e^{x^2} = e^{7x-12}$
101. $f(x) = xe^x + e^x$
102. $f(x) = x^3(4e^{4x}) + 3x^2e^{4x}$
103. $e^{2x} - 2e^x - 3 = 0$
104. $e^{0.08t} = 2500$
105. $e^{x^2} = 200$
106. $e^{2x+1} \cdot e^{-4x} = 3e$
107. $e^{2x} - 8e^x + 7 = 0$
108. $e^{2x} + 2e^x - 15 = 0$
109. $e^x + e^{-x} - 6 = 0$
110. $e^{1-3x} \cdot e^{5x} = 2e$
111. $6 \ln(2x) = 30$
112. $\log_5(x-7) = 2$
113. $\log_4(5+x) = 3$
114. $\log(4x-18) = 1$
115. $\log_3 x = -2$
116. $\log(x^2 + 19) = 2$
117. $\ln(x^2 - 12) = \ln x$
118. $\log(2x^2 + 3x) = \log(10x + 30)$
119. $\log_5(2x+3) = \log_5 11 + \log_5 3$
120. $\log_3 x - \log_9(x+42) = 0$
121. $\log_5 x + \log_5(4x-1) = 1$
122. $\log x - \log(x+3) = 1$
123. $\log x + \log(x-9) = 1$
124. $\log_2(x+1) + \log_2(x-1) = 3$
125. $\log_8(x+1) - \log_8 x = 2$
126. $\ln(x+8) + \ln(x-1) = 2 \ln x$
127. $\ln(4x+6) - \ln(x+5) = \ln x$
128. $\ln(5+4x) - \ln(x+3) = \ln 3$
129. $\ln \sqrt[4]{x} = \sqrt{\ln x}$
130. $\sqrt{\ln x} = \ln \sqrt{x}$
131. $\log x^2 = (\log x)^2$
132. $\log x^3 = (\log x)^2$
133. $\log(\log x) = 1$
134. $\log(\log x) = 2$

$$135. \ln(\ln x) = 2$$

$$136. \ln\left(e^{x^2}\right) = 64$$

$$137. e^{\ln(x-1)} = 4$$

$$138. 10^{\log(2x+5)} = 9$$

$$139. \log\sqrt{x^3-9} = 2$$

$$140. \log\sqrt{x^3-17} = \frac{1}{2}$$

$$141. \log_4 x = \log_4 (8-x)$$

$$142. \log_7 (x-5) = \log_7 (6x)$$

$$143. \ln x^2 = \ln(12-x)$$

$$144. \log_2 (x+7) + \log_2 x = 3$$

$$145. \ln x = 1 - \ln(x+2)$$

$$146. \ln x = 1 + \ln(x+1)$$

$$147. \log_6 (2x-3) = \log_6 12 - \log_6 3$$

$$148. \log(3x+2) + \log(x-1) = 1$$

$$149. \log_5 (x+2) + \log_5 (x-2) = 1$$

$$150. \log_2 x + \log_2 (x-4) = 2$$

$$151. \log_3 x + \log_3 (x+6) = 3$$

$$152. \log_3 (x+3) + \log_3 (x+5) = 1$$

$$153. \ln x = \frac{1}{2} \ln\left(2x + \frac{5}{2}\right) + \frac{1}{2} \ln 2$$

$$154. \ln(-4-x) + \ln 3 = \ln(2-x)$$

$$155. \log_4 x + \log_4 (x-2) = \log_4 (15)$$

$$156. \ln(x-5) - \ln(x+4) = \ln(x-1) - \ln(x+2)$$

$$157. \ln(4-x) = \ln(x+8) + \ln(2x+13)$$

$$158. \log(x^2+4) - \log(x+2) = 2 + \log(x-2)$$

$$159. \log_3 (x-2) = \log_3 27 - \log_3 (x-4) - 5^{\log_5 1}$$

$$160. \log_2 (x+3) = \log_2 (x-3) + \log_3 9 + 4^{\log_4 3}$$

$$161. \frac{10^x - 10^{-x}}{2} = 20$$

$$166. \frac{e^x - e^{-x}}{2} = 15$$

$$162. \frac{10^x + 10^{-x}}{2} = 8$$

$$167. \frac{1}{e^x - e^{-x}} = 4$$

$$163. \frac{10^x + 10^{-x}}{10^x - 10^{-x}} = 5$$

$$168. \frac{e^x + e^{-x}}{e^x - e^{-x}} = 3$$

$$164. \frac{10^x + 10^{-x}}{10^x - 10^{-x}} = 2$$

$$169. \frac{e^x - e^{-x}}{e^x + e^{-x}} = 6$$

$$165. \frac{e^x + e^{-x}}{2} = 15$$

(170 – 173) Use common logarithms to solve for x in terms of y

170. $y = \frac{10^x + 10^{-x}}{2}$

172. $y = \frac{e^x - e^{-x}}{2}$

171. $y = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}$

173. $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

174. Solve for t using logarithms with base a : $2a^{t/3} = 5$

175. Solve for t using logarithms with base a : $K = H - Ca^t$