

Solution **Section 2.5 – Numerical Integration**

Exercise

Find the Midpoint Rule approximations to: $\int_0^1 \sin \pi x \, dx$ using $n = 6$ subintervals

Solution

$$\Delta x = \frac{1-0}{6}$$

$$= \frac{1}{6}$$

$$x_k = a + k\Delta x$$

$$x_0 = 0$$

$$x_1 = 0 + \frac{1}{6}$$

$$= \frac{1}{6}$$

$$x_2 = 0 + 2\left(\frac{1}{6}\right)$$

$$= \frac{1}{3}$$

$$x_3 = 0 + 3\left(\frac{1}{6}\right)$$

$$= \frac{1}{2}$$

$$x_4 = 0 + 4\left(\frac{1}{6}\right)$$

$$= \frac{2}{3}$$

$$x_5 = 0 + 5\left(\frac{1}{6}\right)$$

$$= \frac{5}{6}$$

$$x_6 = 0 + 6\left(\frac{1}{6}\right)$$

$$= 1$$

$$m_1 = \frac{1}{2}\left(0 + \frac{1}{6}\right)$$

$$= \frac{1}{12}$$

$$m_2 = \frac{1}{2}\left(\frac{1}{6} + \frac{1}{3}\right)$$

$$\Delta x = \frac{b-a}{n}$$

$$m_k = \frac{x_{k-1} + x_k}{2}$$

$$= \frac{1}{4} \Big|$$

$$m_3 = \frac{1}{2} \left(\frac{1}{3} + \frac{1}{2} \right)$$

$$= \frac{5}{12} \Big|$$

$$m_4 = \frac{1}{2} \left(\frac{1}{2} + \frac{2}{3} \right)$$

$$= \frac{7}{12} \Big|$$

$$m_5 = \frac{1}{2} \left(\frac{2}{3} + \frac{5}{6} \right)$$

$$= \frac{3}{4} \Big|$$

$$m_6 = \frac{1}{2} \left(\frac{5}{6} + 1 \right)$$

$$= \frac{11}{12} \Big|$$

$$M(n) = f(m_1) \Delta x + f(m_2) \Delta x + \cdots + f(m_n) \Delta x$$

$$M(6) = \left(\sin\left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{5\pi}{12}\right) + \sin\left(\frac{7\pi}{12}\right) + \sin\left(\frac{9\pi}{12}\right) + \sin\left(\frac{11\pi}{12}\right) \right) \left(\frac{1}{6} \right)$$

$$\approx 0.6439505509 \Big|$$

Exercise

Find the Midpoint Rule approximations to: $\int_0^{\pi} x^2 \sin x \, dx \quad n = 8 \text{ subintervals}$

Solution

$$\Delta x = \frac{\pi - 0}{8} = \frac{\pi}{8} \Big|$$

$$\Delta x = \frac{b - a}{n}$$

$$x_0 = 0 \Big|$$

$$x_k = x_0 + k \Delta x$$

$$x_1 = 0 + \frac{\pi}{8} = \frac{\pi}{8} \Big|$$

$$x_2 = 0 + 2 \left(\frac{\pi}{8} \right) = \frac{\pi}{4} \Big|$$

$$x_3 = 0 + 3 \left(\frac{\pi}{8} \right) = \frac{3\pi}{8} \Big|$$

$$x_4 = 0 + 4 \left(\frac{\pi}{8} \right) = \frac{\pi}{2} \Big|$$

$$x_5 = 0 + 5\left(\frac{\pi}{8}\right) = \underline{\underline{\frac{5\pi}{8}}}$$

$$x_6 = 0 + 6\left(\frac{\pi}{8}\right) = \underline{\underline{\frac{3\pi}{4}}}$$

$$x_7 = 0 + 7\left(\frac{\pi}{8}\right) = \underline{\underline{\frac{7\pi}{8}}}$$

$$x_8 = 0 + 8\left(\frac{\pi}{8}\right) = \underline{\underline{\pi}}$$

$$m_1 = \frac{1}{2}\left(0 + \frac{\pi}{8}\right) = \underline{\underline{\frac{\pi}{16}}}$$

$$m_k = \frac{x_{k-1} + x_k}{2}$$

$$m_2 = \frac{1}{2}\left(\frac{\pi}{8} + \frac{\pi}{4}\right) = \underline{\underline{\frac{3\pi}{16}}}$$

$$m_3 = \frac{1}{2}\left(\frac{\pi}{4} + \frac{3\pi}{8}\right) = \underline{\underline{\frac{5\pi}{16}}}$$

$$m_4 = \frac{1}{2}\left(\frac{3\pi}{8} + \frac{\pi}{2}\right) = \underline{\underline{\frac{7\pi}{16}}}$$

$$m_5 = \frac{1}{2}\left(\frac{\pi}{2} + \frac{5\pi}{8}\right) = \underline{\underline{\frac{9\pi}{16}}}$$

$$m_6 = \frac{1}{2}\left(\frac{5\pi}{8} + \frac{3\pi}{4}\right) = \underline{\underline{\frac{11\pi}{16}}}$$

$$m_7 = \frac{1}{2}\left(\frac{3\pi}{4} + \frac{7\pi}{8}\right) = \underline{\underline{\frac{13\pi}{16}}}$$

$$m_8 = \frac{1}{2}\left(\frac{7\pi}{8} + \pi\right) = \underline{\underline{\frac{15\pi}{16}}}$$

$$M(n) = f(m_1)\Delta x + f(m_2)\Delta x + \dots + f(m_n)\Delta x$$

$$M(8) = \left(\left(\frac{\pi}{16}\right)^2 \sin \frac{\pi}{16} + \left(\frac{3\pi}{16}\right)^2 \sin \frac{3\pi}{16} + \left(\frac{5\pi}{16}\right)^2 \sin \frac{5\pi}{16} + \left(\frac{7\pi}{16}\right)^2 \sin \frac{7\pi}{16} + \left(\frac{9\pi}{16}\right)^2 \sin \frac{9\pi}{16} \right. \\ \left. + \left(\frac{11\pi}{16}\right)^2 \sin \frac{11\pi}{16} + \left(\frac{13\pi}{16}\right)^2 \sin \frac{13\pi}{16} + \left(\frac{15\pi}{16}\right)^2 \sin \frac{15\pi}{16} \right) \left(\frac{\pi}{8}\right)$$

$$= \left(\sin \frac{\pi}{16} + 9 \sin \frac{3\pi}{16} + 25 \sin \frac{5\pi}{16} + 49 \sin \frac{7\pi}{16} + 81 \sin \frac{9\pi}{16} + 121 \sin \frac{11\pi}{16} \right. \\ \left. + 169 \sin \frac{13\pi}{16} + 225 \sin \frac{15\pi}{16} \right) \frac{\pi^3}{2,048}$$

$$\approx \left(0.19509 + 5.000132 + 20.7867403 + 67.3478925 + 79.4436077 \right) \frac{\pi^3}{2,048}$$

$$\approx \underline{\underline{6.22414635}}$$

Exercise

Find the Midpoint Rule approximations to: $\int_0^1 e^{-\sqrt{x}} dx$ $n = 6$ subintervals

Solution

$$\Delta x = \frac{1-0}{6} = \frac{1}{6}$$

$$\Delta x = \frac{b-a}{n}$$

$$x_0 = 0$$

$$x_1 = 0 + \frac{1}{6} = \frac{1}{6}$$

$$x_k = x_0 + k\Delta x$$

$$x_2 = 0 + 2\left(\frac{1}{6}\right) = \frac{1}{3}$$

$$x_3 = 0 + 3\left(\frac{1}{6}\right) = \frac{1}{2}$$

$$x_4 = 0 + 4\left(\frac{1}{6}\right) = \frac{2}{3}$$

$$x_5 = 0 + 5\left(\frac{1}{6}\right) = \frac{5}{6}$$

$$x_6 = 0 + 6\left(\frac{1}{6}\right) = 1$$

$$m_1 = \frac{1}{2}\left(0 + \frac{1}{6}\right) = \frac{1}{12}$$

$$m_k = \frac{x_{k-1} + x_k}{2}$$

$$m_2 = \frac{1}{2}\left(\frac{1}{6} + \frac{1}{3}\right) = \frac{1}{4}$$

$$m_3 = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{2}\right) = \frac{5}{12}$$

$$m_4 = \frac{1}{2}\left(\frac{1}{2} + \frac{2}{3}\right) = \frac{7}{12}$$

$$m_5 = \frac{1}{2}\left(\frac{2}{3} + \frac{5}{6}\right) = \frac{3}{4}$$

$$m_6 = \frac{1}{2}\left(\frac{5}{6} + 1\right) = \frac{11}{12}$$

$$M(n) = f(m_1)\Delta x + f(m_2)\Delta x + \cdots + f(m_n)\Delta x$$

$$M(6) = \left(e^{-\sqrt{1/12}} + e^{-\sqrt{1/4}} + e^{-\sqrt{5/12}} + e^{-\sqrt{7/12}} + e^{-\sqrt{3/4}} + e^{-\sqrt{11/12}}\right)\left(\frac{1}{6}\right)$$

$$\approx (.74925557 + .6065306597 + .52440173 + .46591 + .42062 + .383879)\left(\frac{1}{6}\right)$$

$$\approx \underline{0.67787732}$$

Exercise

Find the Midpoint Rule approximations to: $\int_0^1 e^{-x} dx$ using $n = 8$ subintervals

Solution

$$\Delta x = \frac{1-0}{8} = \frac{1}{8}$$

$$x_0 = 0$$

$$x_1 = 0 + \frac{1}{8} = \frac{1}{8}$$

$$x_k = x_0 + k\Delta x$$

$$x_2 = 0 + 2\left(\frac{1}{8}\right) = \frac{1}{4}$$

$$x_3 = 0 + 3\left(\frac{1}{8}\right) = \frac{3}{8}$$

$$x_4 = 0 + 4\left(\frac{1}{8}\right) = \frac{1}{2}$$

$$x_5 = 0 + 5\left(\frac{1}{8}\right) = \frac{5}{8}$$

$$x_6 = 0 + 6\left(\frac{1}{8}\right) = \frac{3}{4}$$

$$x_7 = 0 + 7\left(\frac{1}{8}\right) = \frac{7}{8}$$

$$x_8 = 0 + 8\left(\frac{1}{8}\right) = 1$$

$$m_1 = \frac{1}{2}\left(0 + \frac{1}{8}\right) = \frac{1}{16}$$

$$m_k = \frac{x_{k-1} + x_k}{2}$$

$$m_2 = \frac{1}{2}\left(\frac{1}{8} + \frac{1}{4}\right) = \frac{3}{16}$$

$$m_3 = \frac{1}{2}\left(\frac{1}{4} + \frac{3}{8}\right) = \frac{5}{16}$$

$$m_4 = \frac{1}{2}\left(\frac{3}{8} + \frac{1}{2}\right) = \frac{7}{16}$$

$$m_5 = \frac{1}{2}\left(\frac{1}{2} + \frac{5}{8}\right) = \frac{9}{16}$$

$$m_6 = \frac{1}{2}\left(\frac{5}{8} + \frac{3}{4}\right) = \frac{11}{16}$$

$$m_7 = \frac{1}{2}\left(\frac{3}{4} + \frac{7}{8}\right) = \frac{13}{16}$$

$$m_8 = \frac{1}{2}\left(\frac{7}{8} + 1\right) = \frac{15}{16}$$

$$M(8) = \frac{1}{8}\left(e^{-1/16} + e^{-3/16} + e^{-5/16} + e^{-7/16} + e^{-9/16} + e^{-11/16} + e^{-13/16} + e^{-15/16}\right)$$

$$\approx 0.6317092095$$

Exercise

Estimate the minimum number of subintervals to approximate the integrals with an error of magnitude of

$$10^{-4} \text{ by (a) the Trapezoid Rule and (b) Simpson's Rule. } \int_1^3 (2x-1) dx$$

Solution

$$a) \quad i) \quad \Delta x = \frac{3-1}{4}$$

$$\Delta x = \frac{b-a}{n}$$

$$= \frac{1}{2} \mid$$

$$T = \frac{1}{2} \Delta x \left(m f(x_i) \right)$$

$$= \frac{1}{2} \frac{1}{2} (24)$$

$$= 6 \mid$$

$$f(x) = 2x - 1 \Rightarrow f'(x) = 2$$

$$\Rightarrow f''(x) = 0 = M$$

$$\Rightarrow \text{Error} = 0$$

$$ii) \quad \int_1^3 (2x-1) dx = \left(x^2 - x \right) \Big|_1^3$$

$$= (3^2 - 3) - (1^2 - 1)$$

$$= 6 \mid$$

$$iii) \quad \text{Error} = \frac{|E_T|}{\text{True Value}} \times 100$$

$$= 0\% \mid$$

$$b) \quad i) \quad \Delta x = \frac{3-1}{4}$$

$$\Delta x = \frac{b-a}{n}$$

$$= \frac{1}{2} \mid$$

$$S = \frac{1}{3} \Delta x \left(\sum m f(x_i) \right)$$

$$= \frac{1}{3} \frac{1}{2} (36)$$

$$= 6 \mid$$

$$f(x) = 2x - 1$$

$$\Rightarrow f^{(4)}(x) = 0 = M$$

	x_i	$f(x_i) = 2x_i - 1$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	$1 + \frac{1}{2} = \frac{3}{2}$	2	2	4
x_2	2	3	2	6
x_3	$\frac{5}{2}$	4	2	8
x_4	3	5	1	5
				24

	x_i	$f(x_i) = 2x_i - 1$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	$\frac{3}{2}$	2	4	8
x_2	2	3	2	6
x_3	$\frac{5}{2}$	4	4	16
x_4	3	5	1	5
				36

$$\Rightarrow |E_s| = 0$$

$$ii) \int_1^3 (2x-1) dx = 6$$

$$\begin{aligned} |E_s| &= \int_1^3 (2x-1) dx - S \\ &= 6 - 6 \\ &= 0 \end{aligned}$$

$$iii) \text{ Error} = \frac{|E_T|}{\text{True Value}} \times 100$$

$$= 0\%$$

Exercise

Estimate the minimum number of subintervals to approximate the integrals with an error of magnitude of

10^{-4} by (a) the Trapezoid Rule and (b) Simpson's Rule. $\int_{-1}^1 (x^2 + 1) dx$

Solution

$$\begin{aligned} a) \quad i) \quad \Delta x &= \frac{b-a}{n} \\ &= \frac{1-(-1)}{4} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} T &= \frac{1}{2} \Delta x (m f(x_i)) \\ &= \frac{1}{2} \frac{1}{2} (11) \\ &= \frac{11}{4} \end{aligned}$$

$$f(x) = x^2 + 1$$

$$f'(x) = 2x$$

$$f''(x) = 2 = M$$

$$\begin{aligned} |E_T| &= \frac{1-(-1)}{12} \left(\frac{1}{2}\right)^2 (2) \\ &= \frac{1}{12} \\ &= 0.0833... \end{aligned}$$

	x_i	$f(x_i)$	m	$m f(x_i)$
x_0	-1	2	1	2
x_1	$-\frac{1}{2}$	$\frac{5}{4}$	2	$\frac{5}{2}$
x_2	0	1	2	2
x_3	$\frac{1}{2}$	$\frac{5}{4}$	2	$\frac{5}{2}$
x_4	1	2	1	2
				11

$$\begin{aligned}
 ii) \int_{-1}^1 (x^2 + 1) dx &= \left(\frac{1}{3}x^3 + x \right) \Big|_{-1}^1 \\
 &= \left(\frac{1}{3} + 1 \right) - \left(-\frac{1}{3} - 1 \right) \\
 &= \underline{\frac{8}{3}}
 \end{aligned}$$

$$\begin{aligned}
 E_T &= \int_{-1}^1 (x^2 + 1) dx - T \\
 &= \frac{8}{3} - \frac{11}{4} \\
 &= \underline{-\frac{1}{12}}
 \end{aligned}$$

$$\begin{aligned}
 iii) \text{ Error} &= \frac{|E_T|}{\text{True Value}} \times 100 \\
 &= \frac{\frac{1}{12}}{\frac{8}{3}} \\
 &= \underline{\frac{100}{32}} \quad \underline{\approx 3\%}
 \end{aligned}$$

$$\begin{aligned}
 b) \ i) \Delta x &= \frac{b-a}{n} \\
 &= \frac{-1 - (-1)}{4} \\
 &= \underline{\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 S &= \frac{1}{3} \Delta x \left(\sum m f(x_i) \right) \\
 &= \frac{1}{3} \frac{1}{2} (16) \\
 &= \underline{\frac{8}{3}}
 \end{aligned}$$

$$f(x) = x^2 + 1$$

$$f^{(4)}(x) = 0 = M$$

$$\Rightarrow |E_s| = 0$$

$$ii) \int_{-1}^1 (x^2 + 1) dx = \underline{\frac{8}{3}}$$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	-1	2	1	2
x_1	$-\frac{1}{2}$	$\frac{5}{4}$	4	5
x_2	0	1	2	2
x_3	$\frac{1}{2}$	$\frac{5}{4}$	4	5
x_4	1	2	1	2
				16

$$\begin{aligned}
 E_S &= \int_{-1}^1 (x^2 + 1) dx - S \\
 &= \frac{8}{3} - \frac{8}{3} \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \text{iii) Error} &= \frac{|E_T|}{\text{True Value}} \times 100 \\
 &= 0\%
 \end{aligned}$$

Exercise

Estimate the minimum number of subintervals to approximate the integrals with an error of magnitude of

$$10^{-4} \text{ by (a) the Trapezoid Rule and (b) Simpson's Rule. } \int_2^4 \frac{1}{(s-1)^2} ds$$

Solution

$$\begin{aligned}
 \text{a) } \Delta x &= \frac{4-2}{4} & \Delta x &= \frac{b-a}{n} \\
 &= \frac{1}{2}
 \end{aligned}$$

$$x_0 = 2$$

$$x_1 = 2 + \frac{1}{2} = \frac{5}{2}$$

$$x_2 = 2 + 2\left(\frac{1}{2}\right) = 3$$

$$x_3 = 2 + 3\left(\frac{1}{2}\right) = \frac{7}{2}$$

$$x_4 = 4$$

$$\begin{aligned}
 T &= \frac{1}{2} \Delta x \left(m f(x_i) \right) \\
 &= \frac{1}{2} \frac{1}{2} \left(\frac{1}{(2-1)^2} + 2 \frac{1}{\left(\frac{5}{2}-1\right)^2} + 2 \frac{1}{(3-1)^2} + 2 \frac{1}{\left(\frac{7}{2}-1\right)^2} + \frac{1}{(4-1)^2} \right) \\
 &= \frac{1}{4} \left(\frac{1}{1} + \frac{8}{9} + \frac{1}{2} + \frac{8}{25} + \frac{1}{9} \right) \\
 &= \frac{1269}{1800} \\
 &\approx 0.705
 \end{aligned}$$

$$f(s) = \frac{1}{(s-1)^2}$$

$$f'(s) = -\frac{2}{(s-1)^3}$$

$$f''(s) = \frac{6}{(s-1)^4} \Rightarrow \underline{M=6}$$

$$\begin{aligned} \int_2^4 \frac{1}{(s-1)^2} ds &= -\frac{1}{s-1} \Big|_2^4 \\ &= -\left(\frac{1}{3} - 1\right) \\ &= \underline{\frac{2}{3}} \end{aligned}$$

$$\begin{aligned} \text{The percentage error: } &\approx \frac{|0.705 - .6667|}{.6667} \\ &\approx \underline{0.0575} \quad \underline{5.75\%} \end{aligned}$$

$$\begin{aligned} b) \quad \Delta x &= \frac{4-2}{4} & \Delta x &= \frac{b-a}{n} \\ &= \underline{\frac{1}{2}} \end{aligned}$$

$$x_0 = 2$$

$$x_1 = 2 + \frac{1}{2} = \frac{5}{2}$$

$$x_2 = 2 + 2\left(\frac{1}{2}\right) = 3$$

$$x_3 = 2 + 3\left(\frac{1}{2}\right) = \frac{7}{2}$$

$$x_4 = 4$$

$$S = \frac{1}{3} \Delta x \left(m f(x_i) \right)$$

$$= \frac{1}{3} \frac{1}{2} \left(\frac{1}{(2-1)^2} + 4 \frac{1}{\left(\frac{5}{2}-1\right)^2} + 2 \frac{1}{(3-1)^2} + 4 \frac{1}{\left(\frac{7}{2}-1\right)^2} + \frac{1}{(4-1)^2} \right)$$

$$= \frac{1}{6} \left(\frac{1}{1} + \frac{16}{9} + \frac{1}{2} + \frac{16}{25} + \frac{1}{9} \right)$$

$$= \underline{\frac{1813}{450}}$$

$$\approx \underline{0.67148}$$

$$\left| \int_2^4 \frac{1}{(s-1)^2} ds = \frac{2}{3} \right|$$

$$\text{The percentage error:} = \frac{|0.67148 - .6667|}{.6667} \\ \approx 0.0072 \quad | \quad 0.72\%$$

Exercise

Find the **Trapezoid & Simpson's** Rule approximations and error: $\int_0^1 \sin \pi x \, dx \quad n = 6 \text{ subintervals}$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.1666666667	0.5000000000	1.0000000000
2	0.3333333333	0.8660254000	1.7320508000
3	0.5000000000	1.0000000000	2.0000000000
4	0.6666666667	0.8660254000	1.7320508000
5	0.8333333333	0.5000000000	1.0000000000
6	1.0000000000	0.0000000000	0.0000000000

Trapezoid Rule approximation ≈ 0.62200847

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.1666666667	0.5000000000	2.0000000000
2	0.3333333333	0.8660254000	1.7320508000
3	0.5000000000	1.0000000000	2.0000000000
4	0.6666666667	0.8660254000	1.7320508000
5	0.8333333333	0.5000000000	1.0000000000
6	1.0000000000	0.0000000000	0.0000000000

Simpson's Rule approximation ≈ 0.63689453

Exact	Trapezoid	Simpson
Value: 0.63661977	0.62200847	0.63689453
Error:	2.2951 %	0.0432 %

Exercise

Find the *Trapezoid* & *Simpson's* Rule approximations to and error to $\int_0^1 e^{-x} dx$ $n = 8$ subintervals

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.1250000000	0.8824969000	1.7649938000
2	0.2500000000	0.7788007800	1.5576015600
3	0.3750000000	0.6872892800	1.3745785600
4	0.5000000000	0.6065306600	1.2130613200
5	0.6250000000	0.5352614300	1.0705228600
6	0.7500000000	0.4723665500	0.9447331000
7	0.8750000000	0.4168620200	0.8337240400
8	1.0000000000	0.3678794400	0.3678794400

Trapezoid Rule approximation ≈ 0.63294342

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.1250000000	0.8824969000	3.5299876000
2	0.2500000000	0.7788007800	1.5576015600
3	0.3750000000	0.6872892800	2.7491571200
4	0.5000000000	0.6065306600	1.2130613200
5	0.6250000000	0.5352614300	2.1410457200
6	0.7500000000	0.4723665500	0.9447331000
7	0.8750000000	0.4168620200	1.6674480800
8	1.0000000000	0.3678794400	0.3678794400

Simpson's Rule approximation ≈ 0.63212141

<i>Exact</i>	<i>Trapezoid</i>	<i>Simpson</i>
Value: 0.63212056	0.63294342	0.63212141
Error:	0.1302 %	0.0001 %

Exercise

Find the *Trapezoid* & *Simpson's* Rule approximations and error to:

$$\int_1^5 (3x^2 - 2x) dx \quad n = 8 \text{ subintervals}$$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	1.5000000000	3.7500000000	7.5000000000
2	2.0000000000	8.0000000000	16.0000000000
3	2.5000000000	13.7500000000	27.5000000000
4	3.0000000000	21.0000000000	42.0000000000
5	3.5000000000	29.7500000000	59.5000000000
6	4.0000000000	40.0000000000	80.0000000000
7	4.5000000000	51.7500000000	103.5000000000
8	5.0000000000	65.0000000000	65.0000000000

Trapezoid Rule approximation ≈ 100.50000000

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	1.5000000000	3.7500000000	15.0000000000
2	2.0000000000	8.0000000000	16.0000000000
3	2.5000000000	13.7500000000	55.0000000000
4	3.0000000000	21.0000000000	42.0000000000
5	3.5000000000	29.7500000000	119.0000000000
6	4.0000000000	40.0000000000	80.0000000000
7	4.5000000000	51.7500000000	207.0000000000
8	5.0000000000	65.0000000000	65.0000000000

Simpson's Rule approximation ≈ 100.00000000

<i>Exact</i>	<i>Trapezoid</i>	<i>Simpson</i>

Value: 100.000000	100.500000	100.00000000

Error:	0.5000%	0.0000 %

Exercise

Find the *Trapezoid* & *Simpson's* Rule approximations and error: $\int_0^{\pi/4} 3 \sin 2x \, dx \quad n = 8 \text{ subintervals}$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.0981747704	0.5852709700	1.1705419400
2	0.1963495408	1.1480503000	2.2961006000
3	0.2945243113	1.6667107000	3.3334214000
4	0.3926990817	2.1213203400	4.2426406800
5	0.4908738521	2.4944088400	4.9888176800
6	0.5890486225	2.7716386000	5.5432772000
7	0.6872233930	2.9423558400	5.8847116800
8	0.7853981634	3.0000000000	3.0000000000

Trapezoid Rule approximation ≈ 1.49517776

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.0981747704	0.5852709700	2.3410838800
2	0.1963495408	1.1480503000	2.2961006000
3	0.2945243113	1.6667107000	6.6668428000
4	0.3926990817	2.1213203400	4.2426406800
5	0.4908738521	2.4944088400	9.9776353600
6	0.5890486225	2.7716386000	5.5432772000
7	0.6872233930	2.9423558400	11.7694233600
8	0.7853981634	3.0000000000	3.0000000000

Simpson's Rule approximation ≈ 1.50001244

<i>Exact</i>	<i>Trapezoid</i>	<i>Simpson</i>

Value: 1.500000	1.49517776	1.50001244

Error:	0.3215 %	0.0008 %

Exercise

Find the *Trapezoid* & *Simpson's* Rule approximations and error: $\int_0^8 e^{-2x} dx$ $n = 8$ subintervals

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	1.0000000000	0.1353352800	0.2706705600
2	2.0000000000	0.0183156400	0.0366312800
3	3.0000000000	0.0024787500	0.0049575000
4	4.0000000000	0.0003354600	0.0006709200
5	5.0000000000	0.0000454000	0.0000908000
6	6.0000000000	0.0000061400	0.0000122800
7	7.0000000000	0.0000008300	0.0000016600
8	8.0000000000	0.0000001100	0.0000001100

Trapezoid Rule approximation ≈ 0.65651755

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	1.0000000000	0.1353352800	0.5413411200
2	2.0000000000	0.0183156400	0.0366312800
3	3.0000000000	0.0024787500	0.0099150000
4	4.0000000000	0.0003354600	0.0006709200
5	5.0000000000	0.0000454000	0.0001816000
6	6.0000000000	0.0000061400	0.0000122800
7	7.0000000000	0.0000008300	0.0000033200
8	8.0000000000	0.0000001100	0.0000001100

Simpson's Rule approximation ≈ 0.52958521

<i>Exact</i>	<i>Trapezoid</i>	<i>Simpson</i>

Value: 0.49999994	0.65651755	0.52958521

Error:	31.3035 %	5.9171 %

Exercise

Find the *Trapezoid* & *Simpson's* Rule approximations and error: $\int_{-1}^1 \sqrt{x^2 + 1} \, dx \quad n = 8 \text{ subintervals}$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	-1.0000000000	1.4142135600	1.4142135600
1	-0.7500000000	1.2500000000	2.5000000000
2	-0.5000000000	1.1180339900	2.2360679800
3	-0.2500000000	1.0307764100	2.0615528200
4	0.0000000000	1.0000000000	2.0000000000
5	0.2500000000	1.0307764100	2.0615528200
6	0.5000000000	1.1180339900	2.2360679800
7	0.7500000000	1.2500000000	2.5000000000
8	1.0000000000	1.4142135600	1.4142135600

Trapezoid Rule approximation ≈ 2.30295859

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	-1.0000000000	1.4142135600	1.4142135600
1	-0.7500000000	1.2500000000	5.0000000000
2	-0.5000000000	1.1180339900	2.2360679800
3	-0.2500000000	1.0307764100	4.1231056400
4	0.0000000000	1.0000000000	2.0000000000
5	0.2500000000	1.0307764100	4.1231056400
6	0.5000000000	1.1180339900	2.2360679800
7	0.7500000000	1.2500000000	5.0000000000
8	1.0000000000	1.4142135600	1.4142135600

Simpson's Rule approximation ≈ 2.29556453

	<i>Exact</i>	<i>Trapezoid</i>	<i>Simpson</i>
Value:	2.29558715	2.30295859	2.29556453
Error:		0.3211 %	0.0010 %

Exercise

Find the *Trapezoid* & *Simpson's* Rule approximations and error: $\int_0^{1/2} \sin(x^2) dx$ $n = 4$ subintervals

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.1250000000	0.0156243642	0.0312487284
2	0.2500000000	0.0624593178	0.1249186357
3	0.3750000000	0.1401619723	0.2803239447
4	0.5000000000	0.2474039593	0.2474039593

Trapezoid Rule approximation ≈ 0.0427434543

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.1250000000	0.0156243642	0.0624974569
2	0.2500000000	0.0624593178	0.1249186357
3	0.3750000000	0.1401619723	0.5606478894
4	0.5000000000	0.2474039593	0.2474039593

Simpson Rule approximation ≈ 0.0414778309

<i>Exact</i>	<i>Trapezoid</i>	<i>Simpson</i>
Value: 0.0414810243	0.0427434543	0.0414778309
Error:	3.04339%	0.00770 %

Exercise

Find the *Trapezoid* & *Simpson's* Rule approximations and error: $\int_{\pi/2}^{\pi} \frac{\sin x}{x} dx \quad n = 6 \text{ subintervals}$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	1.5708000000	0.6366182800	0.6366182800
1	1.8323333333	0.5271932200	1.0543864400
2	2.0938666667	0.4137271600	0.8274543200
3	2.3554000000	0.3004450800	0.6008901600
4	2.6169333333	0.1914141900	0.3828283800
5	2.8784666667	0.0903606800	0.1807213600
6	3.1400000000	0.0005072100	0.0005072100

Trapezoid Rule approximation ≈ 0.48166674

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	1.5708000000	0.6366182800	0.6366182800
1	1.8323333333	0.5271932200	2.1087728800
2	2.0938666667	0.4137271600	0.8274543200
3	2.3554000000	0.3004450800	1.2017803200
4	2.6169333333	0.1914141900	0.3828283800
5	2.8784666667	0.0903606800	0.3614427200
6	3.1400000000	0.0005072100	0.0005072100

Simpson's Rule approximation ≈ 0.48116938

	<i>Exact</i>	<i>Trapezoid</i>	<i>Simpson</i>
Value:	0.48117214	0.48166674	0.48116938
Error:		0.1028 %	0.0006 %

Exercise

Find the **Trapezoid** & **Simpson's** Rule approximations and error: $\int_0^{\pi/4} x \tan x \, dx \quad n = 6 \text{ subintervals}$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.1308996939	0.0172332716	0.0344665433
2	0.2617993878	0.0701489345	0.1402978691
3	0.3926990817	0.1626612856	0.3253225711
4	0.5235987756	0.3022998940	0.6045997881
5	0.6544984695	0.5022143392	1.0044286785
6	0.7853981634	0.7853981634	0.7853981634

Trapezoid Rule approximation ≈ 0.1894454730

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.1308996939	0.0172332716	0.0689330865
2	0.2617993878	0.0701489345	0.1402978691
3	0.3926990817	0.1626612856	0.6506451423
4	0.5235987756	0.3022998940	0.6045997881
5	0.6544984695	0.5022143392	2.0088573569
6	0.7853981634	0.7853981634	0.7853981634

Simpson Rule approximation ≈ 0.1858222125

<i>Exact</i>	<i>Trapezoid</i>	<i>Simpson</i>
Value: 0.1857845357	0.1894454730	0.1858222125
Error:	1.97053%	0.02028 %

Exercise

Find the *Trapezoid* & *Simpson's* Rule approximations and error: $\int_0^1 e^{-x^2} dx$ $n = 10$ subintervals

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.1000000000	0.9900498337	1.9800996675
2	0.2000000000	0.9607894392	1.9215788783
3	0.3000000000	0.9139311853	1.8278623705
4	0.4000000000	0.8521437890	1.7042875779
5	0.5000000000	0.7788007831	1.5576015661
6	0.6000000000	0.6976763261	1.3953526521
7	0.7000000000	0.6126263942	1.2252527884
8	0.8000000000	0.5272924240	1.0545848481
9	0.9000000000	0.4448580662	0.8897161324
10	1.0000000000	0.3678794412	0.3678794412

Trapezoid Rule approximation ≈ 0.7462107961

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.1000000000	0.9900498337	3.9601993350
2	0.2000000000	0.9607894392	1.9215788783
3	0.3000000000	0.9139311853	3.6557247411
4	0.4000000000	0.8521437890	1.7042875779
5	0.5000000000	0.7788007831	3.1152031323
6	0.6000000000	0.6976763261	1.3953526521
7	0.7000000000	0.6126263942	2.4505055767
8	0.8000000000	0.5272924240	1.0545848481
9	0.9000000000	0.4448580662	1.7794322649
10	1.0000000000	0.3678794412	0.3678794412

Simpson Rule approximation ≈ 0.7468249483

<i>Exact</i>	<i>Trapezoid</i>	<i>Simpson</i>
Value: 0.7468241328	0.7462107961	0.7468249483
Error:	0.08213%	0.00011 %

Exercise

Find the *Trapezoid* & *Simpson's* Rule approximations and error: $\int_0^2 \frac{1}{\sqrt{1+x^2}} dx$ $n = 10$ subintervals

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.2000000000	0.9805806800	1.9611613600
2	0.4000000000	0.9284766900	1.8569533800
3	0.6000000000	0.8574929300	1.7149858600
4	0.8000000000	0.7808688100	1.5617376200
5	1.0000000000	0.7071067800	1.4142135600
6	1.2000000000	0.6401844000	1.2803688000
7	1.4000000000	0.5812381900	1.1624763800
8	1.6000000000	0.5299989400	1.0599978800
9	1.8000000000	0.4856429300	0.9712858600
10	2.0000000000	0.4472136000	0.4472136000

Trapezoid Rule approximation ≈ 1.44303943

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.2000000000	0.9805806800	3.9223227200
2	0.4000000000	0.9284766900	1.8569533800
3	0.6000000000	0.8574929300	3.4299717200
4	0.8000000000	0.7808688100	1.5617376200
5	1.0000000000	0.7071067800	2.8284271200
6	1.2000000000	0.6401844000	1.2803688000
7	1.4000000000	0.5812381900	2.3249527600
8	1.6000000000	0.5299989400	1.0599978800
9	1.8000000000	0.4856429300	1.9425717200
10	2.0000000000	0.4472136000	0.4472136000

Simpson's Rule approximation ≈ 1.44363449

	<i>Exact</i>	<i>Trapezoid</i>	<i>Simpson</i>
Value:	1.44363548	1.44303943	1.44363449
Error:		0.0413 %	0.0001 %

Exercise

Find the *Trapezoid* & *Simpson's* Rule approximations and error: $\int_0^{1/2} \sin(e^{x/2}) dx \quad n = 8 \text{ subintervals}$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.8414709848	0.8414709848
1	0.0625000000	0.8581952249	1.7163904498
2	0.1250000000	0.8745438796	1.7490877592
3	0.1875000000	0.8904281963	1.7808563927
4	0.2500000000	0.9057510229	1.8115020459
5	0.3125000000	0.9204063003	1.8408126006
6	0.3750000000	0.9342785616	1.8685571232
7	0.4375000000	0.9472424468	1.8944848937
8	0.5000000000	0.9591622435	0.9591622435

Trapezoid Rule approximation ≈ 0.4519476404

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.8414709848	0.8414709848
1	0.0625000000	0.8581952249	3.4327808996
2	0.1250000000	0.8745438796	1.7490877592
3	0.1875000000	0.8904281963	3.5617127853
4	0.2500000000	0.9057510229	1.8115020459
5	0.3125000000	0.9204063003	3.6816252012
6	0.3750000000	0.9342785616	1.8685571232
7	0.4375000000	0.9472424468	3.7889697874
8	0.5000000000	0.9591622435	0.9591622435

Simpson Rule approximation ≈ 0.4519764340

<i>Exact</i>	<i>Trapezoid</i>	<i>Simpson</i>
Value: 0.4519764600	0.4519476404	0.4519764340
Error:	0.00638%	0.00001%

Exercise

Find the *Trapezoid* & *Simpson's* Rule approximations and error: $\int_2^3 \frac{1}{\ln x} dx$ $n = 10$ subintervals

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	2.0000000000	1.4426950400	1.4426950400
1	2.1000000000	1.3478227100	2.6956454200
2	2.2000000000	1.2682994000	2.5365988000
3	2.3000000000	1.2006111700	2.4012223400
4	2.4000000000	1.1422452400	2.2844904800
5	2.5000000000	1.0913566700	2.1827133400
6	2.6000000000	1.0465599400	2.0931198800
7	2.7000000000	1.0067940700	2.0135881400
8	2.8000000000	0.9712326500	1.9424653000
9	2.9000000000	0.9392222400	1.8784444800
10	3.0000000000	0.9102392300	0.9102392300

Trapezoid Rule approximation ≈ 1.11906112

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	2.0000000000	1.4426950400	1.4426950400
1	2.1000000000	1.3478227100	5.3912908400
2	2.2000000000	1.2682994000	2.5365988000
3	2.3000000000	1.2006111700	4.8024446800
4	2.4000000000	1.1422452400	2.2844904800
5	2.5000000000	1.0913566700	4.3654266800
6	2.6000000000	1.0465599400	2.0931198800
7	2.7000000000	1.0067940700	4.0271762800
8	2.8000000000	0.9712326500	1.9424653000
9	2.9000000000	0.9392222400	3.7568889600
10	3.0000000000	0.9102392300	0.9102392300

Simpson's Rule approximation ≈ 1.11842787

	<i>Exact</i>	<i>Trapezoid</i>	<i>Simpson</i>
Value:	1.11842481	1.11906112	1.11842787
Error:		0.0569 %	0.0003 %

Exercise

Find the *Trapezoid* & *Simpson's* Rule approximations and error: $\int_1^2 e^{1/x} dx$ $n = 4$ subintervals

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	1.0000000000	2.7182818300	2.7182818300
1	1.2500000000	2.2255409300	4.4510818600
2	1.5000000000	1.9477340400	3.8954680800
3	1.7500000000	1.7707949500	3.5415899000
4	2.0000000000	1.6487212700	1.6487212700

Trapezoid Rule approximation ≈ 2.03189287

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	1.0000000000	2.7182818300	2.7182818300
1	1.2500000000	2.2255409300	8.9021637200
2	1.5000000000	1.9477340400	3.8954680800
3	1.7500000000	1.7707949500	7.0831798000
4	2.0000000000	1.6487212700	1.6487212700

Simpson's Rule approximation ≈ 2.02065122

	<i>Exact</i>	<i>Trapezoid</i>	<i>Simpson</i>
Value:	2.02005862	2.03189287	2.02065122
Error:		0.5858 %	0.0293 %

Exercise

Find the *Trapezoid* & *Simpson's* Rule approximations and error: $\int_0^1 \ln(1 + e^x) dx$ $n = 8$ subintervals

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.6931471800	0.6931471800
1	0.1250000000	0.7575990400	1.5151980800
2	0.2500000000	0.8259394200	1.6518788400
3	0.3750000000	0.8981232600	1.7962465200
4	0.5000000000	0.9740769800	1.9481539600
5	0.6250000000	1.0537006800	2.1074013600
6	0.7500000000	1.1368710100	2.2737420200
7	0.8750000000	1.2234445800	2.4468891600
8	1.0000000000	1.3132616900	1.3132616900

Trapezoid Rule approximation ≈ 0.98411993

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.6931471800	0.6931471800
1	0.1250000000	0.7575990400	3.0303961600
2	0.2500000000	0.8259394200	1.6518788400
3	0.3750000000	0.8981232600	3.5924930400
4	0.5000000000	0.9740769800	1.9481539600
5	0.6250000000	1.0537006800	4.2148027200
6	0.7500000000	1.1368710100	2.2737420200
7	0.8750000000	1.2234445800	4.8937783200
8	1.0000000000	1.3132616900	1.3132616900

Simpson's Rule approximation ≈ 0.98381891

	<i>Exact</i>	<i>Trapezoid</i>	<i>Simpson</i>

Value:	0.98381904	0.98411993	0.98381891

Error:		0.0306 %	0.0000 %

Exercise

Find the *Trapezoid* & *Simpson's* Rule approximations and error: $\int_0^1 x^5 e^x dx$ $n = 10$ subintervals

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.1000000000	0.0000110500	0.0000221000
2	0.2000000000	0.0003908500	0.0007817000
3	0.3000000000	0.0032801600	0.0065603200
4	0.4000000000	0.0152762800	0.0305525600
5	0.5000000000	0.0515225400	0.1030450800
6	0.6000000000	0.1416879600	0.2833759200
7	0.7000000000	0.3384514200	0.6769028400
8	0.8000000000	0.7292652500	1.4585305000
9	0.9000000000	1.4523710400	2.9047420800
10	1.0000000000	2.7182818300	2.7182818300

Trapezoid Rule approximation ≈ 0.40913975

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.1000000000	0.0000110500	0.0000442000
2	0.2000000000	0.0003908500	0.0007817000
3	0.3000000000	0.0032801600	0.0131206400
4	0.4000000000	0.0152762800	0.0305525600
5	0.5000000000	0.0515225400	0.2060901600
6	0.6000000000	0.1416879600	0.2833759200
7	0.7000000000	0.3384514200	1.3538056800
8	0.8000000000	0.7292652500	1.4585305000
9	0.9000000000	1.4523710400	5.8094841600
10	1.0000000000	2.7182818300	2.7182818300

Simpson's Rule approximation ≈ 0.39580225

	<i>Exact</i>	<i>Trapezoid</i>	<i>Simpson</i>
Value:	0.39559955	0.40913975	0.39580225
Error:		3.4227 %	0.0512 %

Exercise

Find the **Trapezoid** & **Simpson's** Rule approximations and error: $\int_0^4 \sqrt{x} \sin x \, dx \quad n = 8 \text{ subintervals}$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.5000000000	0.3390050500	0.6780101000
2	1.0000000000	0.8414709800	1.6829419600
3	1.5000000000	1.2216768700	2.4433537400
4	2.0000000000	1.2859407500	2.5718815000
5	2.5000000000	0.9462675500	1.8925351000
6	3.0000000000	0.2444270200	0.4888540400
7	3.5000000000	-0.6562553300	-1.3125106600
8	4.0000000000	-1.5136049900	-1.5136049900

Trapezoid Rule approximation ≈ 1.73286520

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.5000000000	0.3390050500	1.3560202000
2	1.0000000000	0.8414709800	1.6829419600
3	1.5000000000	1.2216768700	4.8867074800
4	2.0000000000	1.2859407500	2.5718815000
5	2.5000000000	0.9462675500	3.7850702000
6	3.0000000000	0.2444270200	0.4888540400
7	3.5000000000	-0.6562553300	-2.6250213200
8	4.0000000000	-1.5136049900	-1.5136049900

Simpson's Rule approximation ≈ 1.77214151

	Exact	Trapezoid	Simpson

Value:	1.76874870	1.73286520	1.77214151

Error:		2.0288 %	0.1918 %

Exercise

Find the *Trapezoid* & *Simpson's* Rule approximations and error: $\int_0^3 \frac{1}{1+x^5} dx$ $n = 6$ subintervals

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.5000000000	0.9696969700	1.9393939400
2	1.0000000000	0.5000000000	1.0000000000
3	1.5000000000	0.1163636400	0.2327272800
4	2.0000000000	0.0303030300	0.0606060600
5	2.5000000000	0.0101362100	0.0202724200
6	3.0000000000	0.0040983600	0.0040983600

Trapezoid Rule approximation ≈ 1.06427452

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.5000000000	0.9696969700	3.8787878800
2	1.0000000000	0.5000000000	1.0000000000
3	1.5000000000	0.1163636400	0.4654545600
4	2.0000000000	0.0303030300	0.0606060600
5	2.5000000000	0.0101362100	0.0405448400
6	3.0000000000	0.0040983600	0.0040983600

Simpson's Rule approximation ≈ 1.07491528

<i>Exact</i>	<i>Trapezoid</i>	<i>Simpson</i>
Value: 1.06587854	1.06427452	1.07491528
Error:	0.150488%	.84782%

Exercise

Find the *Trapezoid* & *Simpson's* Rule approximations and error: $\int_1^4 \frac{e^x}{x} dx$ $n = 10$ subintervals

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	1.0000000000	2.7182818300	2.7182818300
1	1.3000000000	2.8225359000	5.6450718000
2	1.6000000000	3.0956452700	6.1912905400
3	1.9000000000	3.5188918100	7.0377836200
4	2.2000000000	4.1022788600	8.2045577200
5	2.5000000000	4.8729975800	9.7459951600
6	2.8000000000	5.8730881300	11.7461762600
7	3.1000000000	7.1606294500	14.3212589000
8	3.4000000000	8.8129706000	17.6259412000
9	3.7000000000	10.9317038800	21.8634077600
10	4.0000000000	13.6495375100	13.6495375100

Trapezoid Rule approximation ≈ 17.81239534

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	1.0000000000	2.7182818300	2.7182818300
1	1.3000000000	2.8225359000	11.2901436000
2	1.6000000000	3.0956452700	6.1912905400
3	1.9000000000	3.5188918100	14.0755672400
4	2.2000000000	4.1022788600	8.2045577200
5	2.5000000000	4.8729975800	19.4919903200
6	2.8000000000	5.8730881300	11.7461762600
7	3.1000000000	7.1606294500	28.6425178000
8	3.4000000000	8.8129706000	17.6259412000
9	3.7000000000	10.9317038800	43.7268155200
10	4.0000000000	13.6495375100	13.6495375100

Simpson's Rule approximation ≈ 17.73628195

	<i>Exact</i>	<i>Trapezoid</i>	<i>Simpson</i>
Value:	17.73575665	17.81239534	17.73628195
Error:		0.4321 %	0.0030 %

Exercise

Find the *Trapezoid* & *Simpson's* Rule approximations and error: $\int_1^2 \frac{dx}{x}$ $n = 10$ subintervals

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	1.0000000000	1.0000000000	1.0000000000
1	1.1000000000	0.9090909100	1.8181818200
2	1.2000000000	0.8333333300	1.6666666600
3	1.3000000000	0.7692307700	1.5384615400
4	1.4000000000	0.7142857100	1.4285714200
5	1.5000000000	0.6666666700	1.3333333400
6	1.6000000000	0.6250000000	1.2500000000
7	1.7000000000	0.5882352900	1.1764705800
8	1.8000000000	0.5555555600	1.1111111200
9	1.9000000000	0.5263157900	1.0526315800
10	2.0000000000	0.5000000000	0.5000000000

Trapezoid Rule approximation ≈ 0.69377140

Simpson's Rule Method

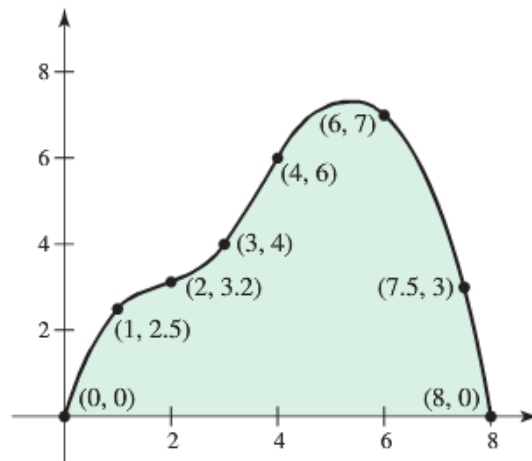
n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	1.0000000000	1.0000000000	1.0000000000
1	1.1000000000	0.9090909100	3.6363636400
2	1.2000000000	0.8333333300	1.6666666600
3	1.3000000000	0.7692307700	3.0769230800
4	1.4000000000	0.7142857100	1.4285714200
5	1.5000000000	0.6666666700	2.6666666800
6	1.6000000000	0.6250000000	1.2500000000
7	1.7000000000	0.5882352900	2.3529411600
8	1.8000000000	0.5555555600	1.1111111200
9	1.9000000000	0.5263157900	2.1052631600
10	2.0000000000	0.5000000000	0.5000000000

Simpson's Rule approximation ≈ 0.69315023

<i>Exact</i>	<i>Trapezoid</i>	<i>Simpson</i>
Value: 0.69314718	0.69377140	0.69315023
Error:	0.0901%	0.0004%

Exercise

A piece of wood paneling must be cut in the shape shown below. The coordinates of several point on its curved surface are also shown (with units of inches).



- Estimate the surface area of the paneling using the Trapezoid Rule
- Estimate the surface area of the paneling using a left Riemann sum.
- Could two identical pieces be cut from a 9-in by 9-in piece of wood?

Solution

- a) The *trapezoid* Rule gives

$$\frac{(0 + 2.5) \cdot 1}{2} + \frac{(2.5 + 3.2) \cdot 1}{2} + \frac{(3.2 + 4) \cdot 1}{2} + \frac{(4 + 6) \cdot 1}{2} + \frac{(6 + 7) \cdot 2}{2} + \frac{(7 + 5.3) \cdot 1.5}{2} + \frac{(3 + 0) \cdot 0.5}{2} \\ = 35.675$$

- b) The left *Riemann* sum gives

$$0 \cdot 1 + 2.5 \cdot 1 + 3.2 \cdot 1 + 4 \cdot 1 + 6 \cdot 2 + 7 \cdot 1.5 + 5.3 \cdot 0.5 = 34.85$$

- c) Although the surface area of the piece appears to be less than half of $81 = 9^2$ (area of 9×9 piece of wood), the shape prohibits the creation of two identical pieces.

Exercise

The region bounded by the curves $y = \frac{1}{1 + e^{-x}}$, $x = 0$ and $x = 10$ is rotated about x -axis. Use Simpson's Rule with $n = 10$ to estimate the volume of the resulting solid.

Solution

Using *Disk* method:

$$V = \pi \int_0^{10} \frac{1}{(1 + e^{-x})^2} dx$$

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.2500000000	0.2500000000
1	1.0000000000	0.5344466454	2.1377865816
2	2.0000000000	0.7758034926	1.5516069851
3	3.0000000000	0.9073974671	3.6295898684
4	4.0000000000	0.9643510838	1.9287021676
5	5.0000000000	0.9866590924	3.9466363696
6	6.0000000000	0.9950608676	1.9901217351
7	7.0000000000	0.9981787276	3.9927149105
8	8.0000000000	0.9993294122	1.9986588244
9	9.0000000000	0.9997532261	3.9990129043
10	10.0000000000	0.9999092063	0.9999092063

Simpson Rule approximation ≈ 8.8082465177

$$\begin{aligned} V &= \pi \int_0^{10} \frac{1}{(1 + e^{-x})^2} dx \\ &\approx \pi(8.8082465177) \\ &\approx \underline{27.6719226 \text{ unit}^3} \end{aligned}$$

Exercise

A pendulum with length L that makes a maximum angle θ_0 with the vertical. Using Newton's Second Law it can be shown that the period T (the time for one complete swing) is given by

$$T = 4 \sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

Where $k = \sin\left(\frac{1}{2}\theta_0\right)$ and g is the acceleration due to gravity. If $L = 1 \text{ m}$ and $\theta_0 = 42^\circ$, use Simpson's Rule with $n = 10$ to find the period.

Solution

$$\begin{aligned} T &= 4 \sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} \\ &= 4 \sqrt{\frac{1}{9.8}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - \sin^2\left(\frac{1}{2}42^\circ\right) \sin^2 x}} \end{aligned}$$

$$= \frac{4}{\sqrt{9.8}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - \sin^2(21^\circ) \sin^2 x}}$$

Simpson's Rule Method $\int_0^{\pi/2} \frac{dx}{\sqrt{1 - \sin^2(21^\circ) \sin^2 x}}$

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.1570796327	1.0031527554	4.0126110216
2	0.3141592654	1.0124160101	2.0248320201
3	0.4712388980	1.0271895774	4.1087583096
4	0.6283185307	1.0464308046	2.0928616093
5	0.7853981634	1.0686201540	4.2744806162
6	0.9424777961	1.0917709315	2.1835418629
7	1.0995574288	1.1135333115	4.4541332459
8	1.2566370614	1.1314314233	2.2628628466
9	1.4137166941	1.1432291699	4.5729166795
10	1.5707963268	1.1473515974	1.1473515974

Simpson Rule approximation ≈ 1.6825506215

$$T = \frac{4}{\sqrt{9.8}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - \sin^2(21^\circ) \sin^2 x}}$$

$$\approx \frac{4}{\sqrt{9.8}} (1.6825506215)$$

$$\approx 2.149884$$