1/ $D = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -1 \end{bmatrix} \qquad D^T = \begin{bmatrix} 1 & -3 & 5 \\ -2 & 4 & -1 \end{bmatrix}$

2/ $A = \begin{bmatrix} -1 & 1 & -2 \\ 2 & 0 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} -1 & 2 \\ 1 & 0 \\ -2 & 1 \end{bmatrix}$

$B = \begin{bmatrix} -3 & 0 \\ 1 & 2 \\ 1 & -1 \end{bmatrix} \qquad B^T = \begin{bmatrix} -3 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix}$

$$\underset{2\times 2 \quad 3\times 2}{AB} = \begin{pmatrix} -1 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 \\ -5 & -1 \end{pmatrix}$$

$$(AB)^T = \begin{pmatrix} 2 & -5 \\ 4 & -1 \end{pmatrix}$$

$$B^T A^T = \begin{pmatrix} -3 & 1 & 1 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -5 \\ 4 & -1 \end{pmatrix}$$

$$\Rightarrow (AB)^T = B^T A^T = \begin{pmatrix} 2 & -5 \\ 4 & -1 \end{pmatrix}$$

3/ $A = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix}$

$A^T = \begin{pmatrix} 4 & 0 \\ 2 & 2 \\ 1 & -1 \end{pmatrix}$

a) $A^T A = \begin{pmatrix} 4 & 0 \\ 2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 4 & 2 & 1 \\ 0 & 2 & -1 \end{pmatrix}$

$= \begin{pmatrix} 16 & 8 & 4 \\ 8 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}$

b) $\underset{2\times 3}{A}\,\underset{3\times 2}{A^T} = \begin{pmatrix} 4 & 2 & 1 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 2 & 2 \\ 1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 21 & 3 \\ 3 & 5 \end{pmatrix}$

4/ $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}^{16} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

5/ $B = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} \rightarrow B^2 = \begin{pmatrix} 9^2 & 0 \\ 0 & 4^2 \end{pmatrix}$

$= \begin{pmatrix} 81 & 0 \\ 0 & 16 \end{pmatrix}$

#6 $(A+B)(A-B) = AA - AB + BA - BB$
$= A^2 - AB + BA - B^2$

since AB is not commutative, then it's not necessary that $AB = BA$.

Therefore, $(A+B)(A-B) \neq A^2 - B^2$

#7 $(A+B)(A+B) = AA + AB + BA + BB$
$= A^2 + AB + BA + B^2$

since $AB \neq BA$ (not necessarily)

then $AB \neq BA$

$\therefore (A+B)(A+B) \neq A^2 + 2AB + B^2$

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8. $A : m \times n$

Proof: AA^T & A^TA are symmetric matrices

$B^T = B$

let $B = AA^T$

1. we need to prove $(AA^T)^T = AA^T$?

$(AA^T)^T = (A^T)^T A^T$
$= AA^T$ ✓.

$AA^T \stackrel{?}{=} (AA^T)^T$

$AA^T = (A^T)^T A^T$ $\quad\quad A = (A^T)^T$

$= (AA^T)^T$ ✓.

Same: $(A^TA)^T = A^T(A^T)^T$
$= A^TA$ ✓

$A^TA = A^T(A^T)^T$
$= (AA^T)^T$ ✓

$\therefore AA^T$ & A^TA are symmetric matrices

9 Given: A + B nxn symmetric

$\Rightarrow A^T = A$ and $B^T = B$

a) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$

$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} +1 & 0 \\ -1 & 1 \end{pmatrix}$

$\therefore$ AB is not symmetric

b) AB symmetric iff $AB = \cancel{A}BA$

i) If AB symmetric $\Rightarrow (AB)^T = AB$

$AB = (AB)^T = \cancel{B^T}$

$= B^T A^T$ A + B are symmetric

$= BA$ ✓

ii) if $AB = BA$

$(AB)^T = \{B^T A\}^T$ A + B symmetric

$= \cancel{A}BA$

$= AB$ ✓