Solution *Section* 3.4 – Using Laplace Transform to Solve Differential Equations

Exercise

Solve using the Laplace transform: $y' + y = te^t, \quad y(0) = -2$

Solution

$$\mathcal{L}(y' + y) = \mathcal{L}\left(te^t\right)$$

$$\mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}\left(te^t\right)$$

$$sY(s) - y(0) + Y(s) = \frac{1}{(s-1)^2} \qquad y(0) = -2$$

$$(s+1)Y(s) + 2 = \frac{1}{(s-1)^2}$$

$$(s+1)Y(s) = \frac{1}{(s-1)^2} - 2$$

$$Y(s) = \frac{1}{(s+1)(s-1)^2} - \frac{2}{s+1}$$

$$\frac{1}{(s+1)(s-1)^2} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$1 = (A+B)s^2 + (C-2A)s + A - B + C$$

$$\begin{cases} A + B = 0 \\ C - 2A = 0 \\ A - B + C = 1 \end{cases} \Rightarrow A = \frac{1}{4} \quad B = -\frac{1}{4} \quad C = \frac{1}{2}$$

$$Y(s) = \frac{1}{4}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2} - \frac{2}{s+1}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{-\frac{7}{4}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2}\right\}$$

$$y(t) = \underline{-\frac{7}{4}e^{-t} - \frac{1}{4}e^t + \frac{1}{2}te^t}$$

Exercise

Solve using the Laplace transform: $y' - y = 2\cos 5t, \quad y(0) = 0$

Solution

$$\mathcal{L}\{y' - y\} = \mathcal{L}\{2\cos 5t\}$$

$$sY(s) - y(0) - Y(s) = \frac{2s}{s^2 + 25} \qquad y(0) = 0$$

$$(s-1)Y(s) = \frac{2s}{s^2 + 25}$$

$$Y(s) = \frac{2s}{(s-1)\left(s^2 + 25\right)} = \frac{A}{s-1} + \frac{Bs + C}{s^2 + 25}$$

$$2s = As^2 + 25A + Bs^2 - Bs + Cs - C$$

$$\begin{cases} A + B = 0 \\ -B + C = 2 \\ 25A - C = 0 \end{cases} \Rightarrow A = \frac{1}{13} \quad B = -\frac{1}{13} \quad C = \frac{25}{13}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{13}\frac{1}{s-1} - \frac{1}{13}\frac{s}{s^2+25} + \frac{25}{13}\frac{1}{5}\frac{5}{s^2+25}\right\}$$

$$\boxed{y(t) = \frac{1}{13}e^t - \frac{1}{13}\cos 4t + \frac{5}{13}\sin 5t}$$

Exercise

Solve using the Laplace transform: $y' - y = 1 + te^t, \quad y(0) = 0$

Solution

$$\mathcal{L}\{y' - y\} = \mathcal{L}\left\{1 + te^t\right\}$$

$$sY(s) - y(0) - Y(s) = \frac{1}{s} + \frac{1}{(s-1)^2} \qquad y(0) = 0$$

$$(s-1)Y(s) = \frac{s^2 - s + 1}{s(s-1)^2}$$

$$Y(s) = \frac{s^2 - s + 1}{s(s-1)^3} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2} + \frac{D}{(s-1)^3}$$

$$s^2 - s + 1 = As^3 - 3As^2 + 3As - A + Bs^3 - 2Bs^2 + Bs + Cs^2 + -Cs + Ds$$

$$\begin{array}{lll} s^3 & A + B = 0 & \underline{B = 1} \\ s^2 & -3A - 2B + C = 1 & \underline{C = 0} \\ s & 3A + B - C + D = -1 & \underline{D = 1} \\ s^0 & \underline{A = -1} & \end{array}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{-\frac{1}{s} + \frac{1}{s-1} + \frac{1}{(s-1)^3}\right\}$$

$$\boxed{y(t) = -1 + e^t + \frac{1}{2}t^2e^t}$$

Exercise

Solve using the Laplace transform: $y' + 3y = e^{2t}, \quad y(0) = -1$

Solution

$$\mathcal{L}(y' + 3y) = \mathcal{L}\left(e^{2t}\right)$$

$$\mathcal{L}(y') + 3\mathcal{L}(y) = \mathcal{L}\left(e^{2t}\right)$$

$$sY(s) - y(0) + 3Y(s) = \frac{1}{s-2}$$

$$(s+3)Y(s) + 1 = \frac{1}{s-2}$$

$$(s+3)Y(s) = \frac{1}{s-2} - 1$$

$$Y(s) = \frac{1}{(s-2)(s+3)} - \frac{1}{s+3}$$

$$\frac{1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$$

$$1 = (A+B)s + 3A - 2B$$

$$\begin{cases} A + B = 0 \\ 3A - 2B = 1 \end{cases} \Rightarrow A = \frac{1}{5} \quad B = -\frac{1}{5}$$

$$Y(s) = \frac{1}{5}\frac{1}{s-2} - \frac{1}{5}\frac{1}{s+3} - \frac{1}{s+3}$$

$$y(t) = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{6}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{1}{5}e^{2t} - \frac{6}{5}e^{-3t}$$

Exercise

Solve using the Laplace transform: $y' + 4y = \cos t, \quad y(0) = 0$

Solution

$$\mathcal{L}(y' + 4y) = \mathcal{L}(\cos t)$$

$$sY(s) - y(0) + 4Y(s) = \frac{s}{s^2+1} \qquad y(0) = 0$$

$$(s+4)Y(s) = \frac{s}{s^2+1}$$

$$Y(s) = \frac{s}{(s+4)\left(s^2+1\right)}$$

$$\frac{s}{(s+4)\left(s^2+1\right)} = \frac{A}{s+4} + \frac{Bs+C}{s^2+1}$$

$$s = As^2 + A + Bs^2 + 4Bs + Cs + 4C$$

$$s = (A+B)s^2 + (4B+C)s + A + 4C$$

$$\begin{cases} A+B=0 \\ 4B+C=1 \Rightarrow A=-\frac{4}{17} \quad B=\frac{4}{17} \quad C=\frac{1}{17} \\ A+4C=0 \end{cases}$$

$$Y(s) = -\frac{4}{17}\frac{1}{s+4} + \frac{4}{17}\frac{s}{s^2+1} + \frac{1}{17}\frac{1}{s^2+1}$$

$$y(t) = -\frac{4}{17}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} + \frac{4}{17}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{17}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\boxed{= -\frac{4}{17}e^{-4t} + \frac{4}{17}e^{-9t}\cos t + \frac{1}{17}\sin t}$$

Exercise

Solve using the Laplace transform: $y' + 4y = e^{-4t}, \quad y(0) = 2$

Solution

$$\mathcal{L}\{y' + 4y\} = \mathcal{L}\left\{e^{-4t}\right\}$$

$$sY(s) - y(0) + 4Y(s) = \frac{1}{s+4} \qquad y(0) = 2$$

$$(s+4)Y(s) = \frac{1}{s+4} + 2$$

$$Y(s) = \frac{2s+9}{(s+4)^2} = \frac{A}{s+4} + \frac{B}{(s+4)^2}$$

$$2s + 9 = As + 4A + B$$

$$s \qquad \boxed{A = 2}$$

$$s^0 \qquad 4A + B = 9 \quad \boxed{B = 1}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s+4} + \frac{1}{(s+4)^2}\right\}$$

$$\boxed{y(t) = 2e^{-4t} + te^{-4t}}$$

Exercise

Solve using the Laplace transform: $y' - 4y = t^2e^{-2t}, \quad y(0) = 1$

Solution

$$\mathcal{L}(y' - 4y) = \mathcal{L}\left(t^2e^{-2t}\right)$$

$$sY(s) - y(0) - 4Y(s) = \frac{2!}{(s+2)^3} \qquad y(0) = 1$$

$$(s-4)Y(s)-1=\frac{2}{(s+2)^3}$$

$$(s-4)Y(s)=1+\frac{2}{(s+2)^3}$$

$$Y(s)=\frac{1}{s-4}+\frac{2}{(s-4)(s+2)^3}$$

$$\frac{2}{(s-4)(s+2)^3}=\frac{A}{s-4}+\frac{B}{s+2}+\frac{C}{(s+2)^2}+\frac{D}{(s+2)^3}$$

$$2=A\left(s^3+6s^2+12s+8A\right)+B\left(s^2+4s+4\right)(s-4)+C(s-4)(s+2)+D(s-4)$$

$$\begin{aligned}2&=As^3+6As^2+12As+8A+Bs^3+4Bs^2+4Bs-4Bs^2-16Bs-16B\\&\quad+Cs^2-2Cs-8C+Ds-4D\end{aligned}$$

$$2=(A+B)s^3+(6A+C)s^2+(12A-12B-2C+D)s+8A-16B-8C-4D$$

$$\begin{cases}A+B=0\\6A+C=0\\12A-12B-2C+D=0\\8A-16B-8C-4D=2\end{cases}\Rightarrow\begin{matrix}A=\frac{1}{108} & B=-\frac{1}{108}\\C=-\frac{1}{18} & D=-\frac{1}{3}\end{matrix}$$

$$Y(s)=\frac{1}{s-4}+\frac{1}{108}\frac{1}{s-4}-\frac{1}{108}\frac{1}{s+2}-\frac{1}{18}\frac{1}{(s+2)^2}-\frac{1}{3}\frac{1}{(s+2)^3}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{109}{108}\frac{1}{s-4}-\frac{1}{108}\frac{1}{s+2}-\frac{1}{18}\frac{1}{(s+2)^2}-\frac{1}{3}\frac{1}{(s+2)^3}\right\}$$

$$y(t)=\frac{109}{108}e^{4t}-\frac{1}{108}e^{-2t}-\frac{1}{18}te^{-2t}-\frac{1}{6}t^2e^{-2t}$$

Exercise

Solve using the Laplace transform: $y'+9y=e^{-t}$, $y(0)=0$

Solution

$$\mathcal{L}(y'+9y)=\mathcal{L}\left(e^{-t}\right) \qquad Y(s)=\mathcal{L}(y)(s)$$

$$\mathcal{L}(y')+9\mathcal{L}(y)=\mathcal{L}\left(e^{-t}\right)$$

$$sY(s)-y(0)+9Y(s)=\frac{1}{s+1}$$

$$(s+9)Y(s)=\frac{1}{s+1}$$

$$Y(s)=\frac{1}{(s+1)(s+9)}$$

$$\frac{1}{(s+1)(s+9)}=\frac{A}{s+1}+\frac{B}{s+9}$$

$$1=(A+B)s+9A+B$$

$$\begin{cases} A+B=0 \\ 9a+B=1 \end{cases} \Rightarrow A=\frac{1}{8} \quad B=-\frac{1}{8}$$

$$Y(s)=\frac{1}{8}\frac{1}{s+1}-\frac{1}{8}\frac{1}{s+9}$$

$$y(t)=\frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+9}\right\}$$

$$=\frac{1}{8}e^{-t}-\frac{1}{8}e^{-9t}$$

Exercise

Solve using the Laplace transform: $y'+16y=\sin 3t, \quad y(0)=1$

Solution

$$\mathcal{L}\left(y'+16y\right)=\mathcal{L}(\sin 3t)$$

$$sY(s)-y(0)+16Y(s)=\frac{3}{s^2+9} \qquad y(0)=1$$

$$(s+16)Y(s)-1=\frac{3}{s^2+9}$$

$$(s+16)Y(s)=\frac{3}{s^2+9}+1$$

$$Y(s)=\frac{1}{s+16}+\frac{3}{(s+16)\left(s^2+9\right)}$$

$$\frac{3}{(s+16)\left(s^2+9\right)}=\frac{A}{s+16}+\frac{Bs+C}{s^2+9}$$

$$s=As^2+9A+Bs^2+16Bs+Cs+16C$$

$$s=(A+B)s^2+(16B+C)s+9A+16C$$

$$\begin{cases} A+B=0 \\ 16B+C=1 \\ 9A+16C=0 \end{cases} \Rightarrow A=\frac{3}{265} \quad B=-\frac{3}{265} \quad C=\frac{48}{265}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s+16}+\frac{3}{265}\frac{1}{s+16}-\frac{3}{265}\frac{s}{s^2+9}+\frac{48}{265}\frac{1}{s^2+9}\right\}$$

$$y(t)=\frac{268}{265}e^{-16t}-\frac{3}{265}\cos 3t+\frac{16}{265}\sin 3t$$

Exercise

Solve using the Laplace transform: $y'' - y = e^{2t}$; $y(0) = 0$, $y'(0) = 1$

Solution

$$\mathcal{L}\{y'' - y\} = \mathcal{L}\left\{e^{2t}\right\}$$

$$s^2 Y(s) - sy(0) - y'(0) - Y(s) = \frac{1}{s-2} \qquad y(0) = 0, \quad y'(0) = 1$$

$$\left(s^2 - 1\right)Y(s) - 1 = \frac{1}{s-2}$$

$$\left(s^2 - 1\right)Y(s) = \frac{1}{s-2} + 1$$

$$(s-1)(s+1)Y(s) = \frac{s-1}{s-2}$$

$$Y(s) = \frac{1}{(s+1)(s-2)} = \frac{A}{(s+1)} + \frac{B}{(s-2)}$$

$$(A+B)s + B - 2A = 1 \qquad \begin{cases} A + B = 0 \\ -2A + B = 1 \end{cases} \Rightarrow A = -\frac{1}{3}; \ B = \frac{1}{3}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{3}\mathcal{L}^{-1}\left\{-\frac{1}{(s+1)} + \frac{1}{(s-2)}\right\}$$

$$\boxed{y(t) = \frac{1}{3}\left(e^{2t} - e^{-t}\right)}$$

Exercise

Solve using the Laplace transform: $y'' - y = 2t$; $y(0) = 0$, $y'(0) = -1$

Solution

$$\mathcal{L}\left(y'' - y\right) = \mathcal{L}(2t)$$

$$s^2 Y(s) - sy(0) - y'(0) - Y(s) = 2\frac{1}{s^2} \qquad y(0) = 0 \quad y'(0) = -1$$

$$\left(s^2 - 1\right)Y(s) + 1 = \frac{2}{s^2}$$

$$Y(s) = \frac{2}{s^2(s-1)(s+1)} - \frac{1}{(s-1)(s+1)}$$

$$\frac{2}{s^2(s-1)(s+1)} = \frac{A}{s^2} + \frac{B}{s-1} + \frac{C}{s+1}$$

$$2 = As^2 - A + Bs^3 + Bs^2 + Cs^3 - Cs^2$$

$$2 = (B+C)s^3 + (A+B-C)s^2 - A$$

$$\begin{cases} B + C = 0 \\ A + B - C = 0 \\ -A = 2 \end{cases} \Rightarrow A = -2 \quad B = 1 \quad C = -1$$

$$\frac{1}{(s-1)(s+1)} = \frac{D}{s-1} + \frac{E}{s+1}$$

$$\begin{cases} D+E=0 \\ D-E=1 \end{cases} \Rightarrow D = \frac{1}{2} \quad E = -\frac{1}{2}$$

$$Y(s) = -\frac{2}{s^2} + \frac{1}{s-1} - \frac{1}{s+1} - \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1}$$

$$= -\frac{2}{s^2} + \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}$$

$$y(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$\boxed{y(t) = -2t + \frac{1}{2}e^{t} - \frac{1}{2}e^{-t}}$$

Exercise

Solve using the Laplace transform: $y'' - y = t - 2$; $y(2) = 3$, $y'(2) = 0$

Solution

Let: $w(t) = y(t+2) \quad \leftrightarrow \quad y(t) = w(t-2)$

$$\mathcal{L}\{y'' - y\} = \mathcal{L}\{t+2\}$$

$$\mathcal{L}\{w'' - w\} = \mathcal{L}\{t\}$$

$$s^2W(s) - sw(0) - w'(0) - W(s) = \frac{1}{s} \qquad y(2) = w(2-2) = w(0) = 3, \quad y'(2) = w'(0) = 0$$

$$\left(s^2 - 1\right)W(s) = \frac{1}{s} + 3s$$

$$W(s) = \frac{1+3s^2}{s\left(s^2-1\right)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}$$

$$1 + 3s^2 = As^2 - A + Bs^2 + Bs + Cs^2 - Cs$$

$s^2 \quad A + C = 3 \quad \boxed{C = 2}$

$s^1 \quad B - C = 0 \quad \boxed{B = 2}$

$s^0 \quad \boxed{A = -1}$

$$\mathcal{L}^{-1}\{W(s)\} = \mathcal{L}^{-1}\left\{-\frac{1}{s} + \frac{2}{s-1} + \frac{2}{s+1}\right\}$$

$$w(t) = -t + 2e^{t} + 2e^{-t}$$

$$y(t) = w(t-2) = -(t-2) + 2e^{t-2} + 2e^{-(t-2)}$$

$$\boxed{= 2 - t + 2e^{t-2} + 2e^{-t+2}}$$

Exercise

Solve using the Laplace transform: $y'' + y = t$; $y(\pi) = y'(\pi) = 0$

Solution

Let: $w(t) = y(t+\pi) \quad \leftrightarrow \quad y(t) = w(t-\pi)$

$y'' + y = t \quad \to \quad w'' + w = t + \pi$

$\mathcal{L}\{w'' + w\} = \mathcal{L}\{t + \pi\}$

$s^2 W(s) - sw(0) - w'(0) + W(s) = \frac{1}{s^2} + \frac{\pi}{s} \qquad y(\pi) = w(\pi - \pi) = w(0) = 0, \quad y'(\pi) = w'(0) = 0$

$\left(s^2 + 1\right) W(s) = \frac{1 + \pi s}{s^2}$

$W(s) = \frac{1 + \pi s}{s^2\left(s^2 + 1\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 1}$

$1 + \pi s = As^3 + As + Bs^2 + B + Cs^3 + Ds^2$

$s^3 \quad A + C = 0 \quad \underline{C = -\pi}$

$s^2 \quad B + D = 0 \quad \underline{D = -1}$

$s^1 \quad \underline{A = \pi}$

$s^0 \quad \underline{B = 1}$

$\mathcal{L}^{-1}\{W(s)\} = \mathcal{L}^{-1}\left\{\frac{\pi}{s} + \frac{1}{s^2} - \frac{\pi s}{s^2 + 1} - \frac{1}{s^2 + 1}\right\}$

$w(t) = \pi + t - \pi \cos t - \sin t$

$y(t) = w(t - \pi) = \pi + (t - \pi) - \pi\cos(t - \pi) - \sin(t - \pi)$

$= t - \pi(\cos t \cos\pi - \sin t \sin\pi) - (\cos t \sin\pi - \cos\pi \sin t)$

$\underline{= t + \pi\cos t + \sin t}$

Exercise

Solve using the Laplace transform: $y'' - 2y' + 5y = -8e^{\pi - t}$; $y(\pi) = 2, \ y'(\pi) = 12$

Solution

Let: $w(t) = y(t+\pi) \quad \leftrightarrow \quad y(t) = w(t-\pi)$

$y'' - 2y' + 5y = -8e^{\pi - t} \quad \to \quad w'' - 2w' + 5w = -8e^{-t}$

$\mathcal{L}\{w'' - 2w' + 5w\} = \mathcal{L}\left\{-8e^{-t}\right\}$

$s^2 W(s) - sw(0) - w'(0) - 2sW(s) + 2w(0) + 5W(s) = -\frac{8}{s+1} \qquad y(\pi) = w(0) = 2, \quad y'(\pi) = w'(0) = 12$

$$\left(s^2-2s+5\right)W(s)=-\frac{8}{s+1}+2s+12-4$$

$$\left(s^2-2s+5\right)W(s)=\frac{2s^2+10s}{s+1}$$

$$W(s)=\frac{2s^2+10s}{(s+1)\left((s-1)^2+4\right)}=\frac{A}{s+1}+\frac{B(s-1)+C}{(s-1)^2+4}$$

$$2s^2+10s=As^2-2As+5A+Bs^2-B+Cs+C$$

$$s^2 \qquad A+B=2$$

$$s^1 \qquad -2A+C=10$$

$$s^0 \qquad 5A-B+C=0$$

$$\Delta=\begin{vmatrix}1 & 1 & 0\\ -2 & 0 & 1\\ 5 & -1 & 1\end{vmatrix}=8 \quad \Delta_A=\begin{vmatrix}2 & 1 & 0\\ 10 & 0 & 1\\ 0 & -1 & 1\end{vmatrix}=-8$$

$$\underline{A=-1 \quad B=3 \quad C=8}$$

$$\mathcal{L}^{-1}\{W(s)\}=\mathcal{L}^{-1}\left\{-\frac{1}{s+1}+\frac{3(s-1)}{(s-1)^2+2^2}+\frac{4(2)}{(s-1)^2+2^2}\right\}$$

$$w(t)=-e^{-t}+3e^{t}\cos 2t+4e^{t}\sin 2t$$

$$y(t)=w(t-\pi)=-e^{-(t-\pi)}+3e^{t-\pi}\cos 2(t-\pi)+4e^{t-\pi}\sin 2(t-\pi)$$

$$\underline{=-e^{-t+\pi}+3e^{t-\pi}\cos 2t+4e^{t-\pi}\sin 2t}$$

Exercise

Solve using the Laplace transform: $y''+y=t^2+2$; $y(0)=1,\quad y'(0)=-1$

Solution

$$\mathcal{L}\{y''+y\}=\mathcal{L}\left\{t^2+2\right\}$$

$$s^2Y(s)-sy(0)-y'(0)+Y(s)=\frac{2}{s^3}+\frac{2}{s} \qquad y(0)=1;\ y'(0)=-1$$

$$s^2Y(s)-s+1+Y(s)=\frac{2+2s^2}{s^3}$$

$$\left(s^2+1\right)Y(s)=\frac{2+2s^2}{s^3}+s-1$$

$$Y(s)=\frac{2+2s^2+s^4-s^3}{s^3\left(s^2+1\right)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{Ds+E}{s^2+1}$$

$$s^4-s^3+2s^2+2=As^4+As^2+Bs^3+Bs+Cs^2+C+Ds^4+Es^3$$

$$s^4 \quad A+D=1 \quad \underline{D=1}$$

$$s^3 \quad B+E=-1 \quad \underline{E=-1}$$

$$s^2 \quad A+C=2 \quad \underline{A=0}$$

$$s \quad \underline{B=0}$$

$$s^0 \quad \underline{C=2}$$

$$Y(s)=\frac{2}{s^3}+\frac{s}{s^2+1}-\frac{1}{s^2+1}$$

$$\boxed{y(t)=t^2+\cos t-\sin t}$$

Exercise

Solve using the Laplace transform: $\quad y''+y=\sqrt{2}\sin\sqrt{2}t; \quad y(0)=10, \quad y'(0)=0$

Solution

$$\mathcal{L}\{y''+y\}=\mathcal{L}\left\{\sqrt{2}\sin\sqrt{2}t\right\}$$

$$s^2Y(s)-sy(0)-y'(0)+Y(s)=\sqrt{2}\,\frac{\sqrt{2}}{s^2+2} \qquad y(0)=10; \;\; y'(0)=0$$

$$\left(s^2+1\right)Y(s)-10s=\frac{2}{s^2+2}$$

$$\left(s^2+1\right)Y(s)=\frac{2}{s^2+2}+10s$$

$$\left(s^2+1\right)Y(s)=\frac{10s^3+20s+2}{s^2+2}$$

$$Y(s)=\frac{10s^3+20s+2}{\left(s^2+1\right)\left(s^2+2\right)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+2}$$

$$As^3+2As+Bs^2+2B+Cs^3+Cs+Ds^2+D=10s^3+20s+2$$

$$\begin{array}{ll} s^3 & A+C=10 \\ s^2 & B+D=0 \\ s^1 & 2A+C=20 \\ s^0 & 2B+D=2 \end{array} \;\rightarrow\; \begin{array}{l} \begin{cases} A+C=10 \\ 2A+C=20 \end{cases} \rightarrow \underline{A=10,\; C=0} \\ \begin{cases} B+D=0 \\ 2B+D=2 \end{cases} \rightarrow \underline{B=2,\; D=-2} \end{array}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{10s}{s^2+1}+\frac{2}{s^2+1}-\sqrt{2}\,\frac{\sqrt{2}}{s^2+2}\right\}$$

$$\boxed{y(t)=10\cos t+2\sin t-\sqrt{2}\sin\sqrt{2}t}$$

Exercise

Solve using the Laplace transform: $y'' + y = -2\cos 2t$; $y(0)=1, \quad y'(0)=-1$

Solution

$$\mathcal{L}(y''+y)(s) = \mathcal{L}(-2\cos 2t)(s)$$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = -2\frac{s}{s^2+4} \qquad y(0)=1 \quad y'(0)=-1$$

$$s^2Y(s) - s + 1 + Y(s) = -2\frac{s}{s^2+4}$$

$$\left(s^2+1\right)Y(s) = \frac{-2s}{s^2+4} + s - 1$$

$$Y(s) = \frac{-2s}{\left(s^2+4\right)\left(s^2+1\right)} + \frac{s}{s^2+1} - \frac{1}{s^2+1}$$

$$= \frac{As}{s^2+4} + \frac{Bs}{s^2+1} + \frac{s}{s^2+1} - \frac{1}{s^2+1}$$

$$\frac{-2s}{\left(s^2+4\right)\left(s^2+1\right)} = \frac{As}{s^2+4} + \frac{Bs}{s^2+1}$$

$$\Rightarrow \quad -2s = As\left(s^2+1\right) + Bs\left(s^2+4\right)$$

$$-2s = As^3 + As + Bs^3 + 4Bs$$

$$-2s = (A+B)s^3 + (A+4B)s$$

$$\begin{cases} A+B=0 \\ A+4B=-2 \end{cases} \Rightarrow \boxed{A=\frac{2}{3}} \quad \boxed{B=-\frac{2}{3}}$$

$$Y(s) = \frac{2}{3}\frac{s}{s^2+4} - \frac{2}{3}\frac{s}{s^2+1} + \frac{s}{s^2+1} - \frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{3}\frac{s}{s^2+4} + \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{s^2+1}\right\}$$

$$y(t) = \frac{2}{3}\cos 2t + \frac{1}{3}\cos t - \sin t$$

Exercise

Solve using the Laplace transform: $y'' - y' = e^t\cos t$; $y(0)=0, \quad y'(0)=0$

Solution

$$\mathcal{L}\{y''-y'\} = \mathcal{L}\left\{e^t\cos t\right\}$$

$$s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) = \frac{s-1}{(s-1)^2+1} \qquad y(0)=0; \ \ y'(0)=0$$

$$\left(s^2+s\right)Y(s)=\frac{s-1}{s^2-2s+2}$$

$$Y(s)=\frac{s-1}{\left(s^2+s\right)\left(s^2-2s+2\right)}\quad=\frac{A}{s}+\frac{B}{s+1}+\frac{Cs+D}{s^2-2s+2}$$

$$As^3-As^2+2A+Bs^3-2Bs^2+2Bs+Cs^3+Cs^2+Ds^2+Ds=s-1$$

$$\begin{array}{ll} s^3 & A+B+C+D=0 \\ s^2 & -A-2B+C+D=0 \\ s^1 & 2B+D=1 \\ s^0 & 2A=-1\to \boxed{A=-\frac{1}{2}} \end{array} \quad\rightarrow\quad \begin{cases} B+C+D=\frac{1}{2} \\ -2B+C+D=-\frac{1}{2} \\ 2B+D=1 \end{cases} \Rightarrow \boxed{B=\frac{1}{3}\quad C=-\frac{1}{6}\quad D=\frac{1}{3}}$$

$$Y(s)=-\frac{1}{2}\frac{1}{s}+\frac{1}{3}\frac{1}{s+1}-\frac{1}{6}\frac{s-2}{(s-1)^2+1}$$

$$=-\frac{1}{2}\frac{1}{s}+\frac{1}{3}\frac{1}{s+1}-\frac{1}{6}\frac{s-1-1}{(s-1)^2+1}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{-\frac{1}{2}\frac{1}{s}+\frac{1}{3}\frac{1}{s+1}-\frac{1}{6}\frac{s-1}{(s-1)^2+1}+\frac{1}{6}\frac{1}{(s-1)^2+1}\right\}$$

$$\boxed{y(t)=-\frac{1}{2}t+\frac{1}{3}e^{-t}-\frac{1}{6}e^{t}\cos t+\frac{1}{6}e^{t}\sin t}$$

Exercise

Solve using the Laplace transform: $y''+y'-y=t^3$; $y(0)=1,\quad y'(0)=0$

Solution

$$\mathcal{L}\left(y''-y'-y\right)=\mathcal{L}\left(t^3\right)$$

$$s^2Y(s)-sy(0)-y'(0)+sY(s)-y(0)-Y(s)=\frac{6}{s^4}\qquad\qquad y(0)=1\quad y'(0)=0$$

$$\left(s^2+s-1\right)Y(s)-s-1=\frac{6}{s^4}$$

$$\left(s^2+s-1\right)Y(s)=\frac{6}{s^4}+s+1$$

$$Y(s)=\frac{s^5+s^4+6}{s^4\left(s^2+s-1\right)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s^4}+\frac{E}{s+\frac{1}{2}-\frac{\sqrt{5}}{2}}+\frac{F}{s+\frac{1}{2}+\frac{\sqrt{5}}{2}}$$

$$
\begin{array}{lll}
s^5 & A+E+F=1 & E+F=19 \\
s^4 & A+B+\left(\frac{1+\sqrt{5}}{2}\right)E+\left(\frac{1-\sqrt{5}}{2}\right)F=1 & \left(\frac{1+\sqrt{5}}{2}\right)E+\left(\frac{1-\sqrt{5}}{2}\right)F=31 \\
s^3 & -A+B+C=0 & \underline{A=-18} \\
s^2 & -B+C+D=0 & \underline{B=-12} \\
s^1 & -C+D=0 & \underline{C=-6} \\
s^0 & -D=6 & \underline{D=-6}
\end{array}
$$

$$\underline{F=\frac{19}{2}-\frac{43\sqrt{5}}{10} \quad E=-\frac{19}{2}+\frac{43\sqrt{5}}{10}}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{-\frac{18}{s}-\frac{12}{s^2}-\frac{6}{s^3}-\frac{6}{s^4}+\frac{-\frac{19}{2}+\frac{43\sqrt{5}}{10}}{s+\frac{1}{2}-\frac{\sqrt{5}}{2}}+\frac{\frac{19}{2}-\frac{43\sqrt{5}}{10}}{s+\frac{1}{2}+\frac{\sqrt{5}}{2}}\right\}$$

$$\underline{y(t)=-18-12t-3t^2-t^3+\left(-\frac{19}{2}+\frac{43\sqrt{5}}{10}\right)e^{\left(\frac{-1+\sqrt{5}}{2}\right)t}+\left(\frac{19}{2}-\frac{43\sqrt{5}}{10}\right)e^{-\left(\frac{1+\sqrt{5}}{2}\right)t}}$$

Exercise

Solve using the Laplace transform: $y''-y'-2y=4t^2, \quad y(0)=1, \quad y'(0)=4$

Solution

$$\mathcal{L}\{y''-y'-2y\}(s)=\mathcal{L}\left\{4t^2\right\}(s) \qquad \mathcal{L}\left\{t^n\right\}(s)=\frac{n!}{s^{n+1}}$$

$$s^2Y(s)-sy(0)-y'(0)-sY(s)+y(0)-2Y(s)=\frac{8}{s^3} \qquad y(0)=1, \quad y'(0)=4$$

$$\left(s^2-s-2\right)Y(s)-s-4+1=\frac{8}{s^3}$$

$$(s+1)(s-2)Y(s)=\frac{8}{s^3}+s+3$$

$$Y(s)=\frac{s^4+3s^3+8}{s^3(s+1)(s-2)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s+1}+\frac{E}{s-2}$$

$$s^4+3s^3+8=As^2\left(s^2-s-2\right)+Bs\left(s^2-s-2\right)+Cs^2-Cs-2C+Ds^3(s-2)+Es^3(s+1)$$

$$=As^4-As^3-2As^2+Bs^3-Bs^2-2Bs+Cs^2-Cs-2C+Ds^4-2Ds^3+Es^4+Es^3$$

$$\begin{cases} s^4 & A+D+E=1 & D+E=4 \\ s^3 & -A+B-2D+E=3 & -2D+E=-2 \\ s^2 & -2A-B+C=0 & \to \underline{A=-3} \\ s & -2B-C=0 & \to \underline{B=2} \\ s^0 & -2C=8 & \to \underline{C=-4} \end{cases} \quad \Rightarrow \underline{D=2, \ \ E=2}$$

$$\mathcal{L}^{-1}\{Y(s)\}(t) = \mathcal{L}^{-1}\left\{-\frac{3}{s}+\frac{2}{s^2}-\frac{4}{s^3}+\frac{2}{s+1}+\frac{2}{s-2}\right\}(t)$$

$$\boxed{y(t) = -3+2t-2t^2+2e^{-t}+2e^{2t}}$$

Exercise

Solve using the Laplace transform: $\quad y''-y'-2y=e^{2t}$; $\quad y(0)=-1, \quad y'(0)=0$

Solution

$$\mathcal{L}(y''-y'-2y) = \mathcal{L}\left(e^{2t}\right)$$

$$s^2Y(s)-sy(0)-y'(0)-\left(sY(s)-y(0)\right)-2Y(s)=\frac{1}{s-2} \qquad y(0)=-1 \quad y'(0)=0$$

$$s^2Y(s)+s-sY(s)-1-2Y(s)=\frac{1}{s-2}$$

$$\left(s^2-s-2\right)Y(s)=\frac{1}{s-2}-s+1$$

$$(s+1)(s-2)(Y(s)=\frac{1}{s-2}-s+1$$

$$Y(s)=\frac{1}{(s+1)(s-2)^2}-\frac{s-1}{(s+1)(s-2)}$$

$$=\frac{1-(s-1)(s-2)}{(s+1)(s-2)^2}$$

$$Y(s)=\frac{-s^2+3s-1}{(s+1)(s-2)^2}=\frac{A}{s+1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}$$

$$-s^2+3s-1=As^2-4As+4A+Bs^2-Bs-2B+Cs+C$$

$$-s^2+3s-1=(A+B)s^2+(-4A-B+C)s+4A-2B+C$$

$$\begin{cases} A+B=-1 \\ -4A-B+C=3 \\ 4A-2B+C=-1 \end{cases} \quad \Rightarrow A=-\tfrac{5}{9} \quad B=-\tfrac{4}{9} \quad C=\tfrac{1}{3}$$

$$Y(s)=-\frac{5}{9}\frac{1}{s+1}-\frac{4}{9}\frac{1}{s-2}+\frac{1}{3}\frac{1}{(s-2)^2}$$

$$y(t)=-\frac{5}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}-\frac{4}{9}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}+\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$\boxed{y(t) = -\frac{5}{9}e^{-t} - \frac{4}{9}e^{2t} + \frac{1}{3}te^{2t}}$$

Exercise

Solve using the Laplace transform: $y'' - y' - 2y = 0, \quad y(0) = -2, \quad y'(0) = 5$

Solution

$$\mathcal{L}(y'' - y' - 2y) = 0$$

$$s^2Y(s) - sy(0) - y'(0) - sY(s) + y(0) - 2Y(s) = 0 \qquad y(0) = -2 \quad y'(0) = 5$$

$$\left(s^2 - s - 2\right)Y(s) = 7 - 2s$$

$$Y(s) = \frac{7-2s}{s^2 - s - 2} = \frac{A}{s+1} + \frac{B}{s-2}$$

$$\begin{array}{ll} s & A + B = -2 \\ s^0 & -2A + B = 7 \end{array} \quad \rightarrow \quad \boxed{A = -3, \ B = 1}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{-3}{s+1} + \frac{1}{s-2}\right\}$$

$$\boxed{y(t) = e^{2t} - 3e^{-t}}$$

Exercise

Solve using the Laplace transform: $y'' - y' - 2y = -8\cos t - 2\sin t \ ; \quad y\left(\frac{\pi}{2}\right) = 1, \quad y'\left(\frac{\pi}{2}\right) = 0$

Solution

Let: $w(t) = y\left(t + \frac{\pi}{2}\right) \ \leftrightarrow \ y(t) = w\left(t - \frac{\pi}{2}\right)$

$$y'' - y' - 2y = -8\cos t - 2\sin t \quad \rightarrow \quad w'' - w' - 2w = -8\cos\left(t + \frac{\pi}{2}\right) - 2\sin\left(t + \frac{\pi}{2}\right)$$

$$w'' - w' - 2w = -8\left(\cos t \cos\frac{\pi}{2} - \sin t \sin\frac{\pi}{2}\right) - 2\left(\sin t \cos\frac{\pi}{2} + \cos t \sin\frac{\pi}{2}\right)$$

$$\mathcal{L}\{w'' - w' - 2w\} = \mathcal{L}\{8\sin t - 2\cos t\}$$

$$s^2W(s) - sw(0) - w'(0) - sW(s) + w(0) - 2W(s) = \frac{8}{s^2+1} - \frac{2s}{s^2+1}$$

$$y\left(\frac{\pi}{2}\right) = w(0) = 1, \quad y'\left(\frac{\pi}{2}\right) = w'(0) = 0$$

$$\left(s^2 - s - 2\right)W(s) = \frac{8-2s}{s^2+1} + s - 1$$

$$W(s) = \frac{s^3 - s^2 - s + 7}{(s+1)(s-2)\left(s^2+1\right)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{Cs+D}{s^2+1}$$

$$s^3 - s^2 - s + 7 = As^3 - 2As^2 + As - 2A + Bs^3 + Bs^2 + Bs + B + Cs^3 - Cs^2 - 2Cs + Ds^2 - Ds - 2D$$

$$\begin{cases} s^3 & A+B+C=1 \\ s^2 & -2A+B-C+D=-1 \\ s^1 & A+B-2C-D=-1 \\ s^0 & -2A+B-2D=7 \end{cases} \quad \rightarrow \underline{A=-1,\ B=\frac{3}{5},\ C=\frac{7}{5},\ D=-\frac{11}{5}}$$

$$\mathcal{L}^{-1}\{W(s)\}=\mathcal{L}^{-1}\left\{-\frac{13}{3}\frac{1}{s+1}+\frac{74}{15}\frac{1}{s-2}+\frac{7}{5}\frac{s}{s^2+1}-\frac{11}{5}\frac{1}{s^2+1}\right\}$$

$$w(t)=-e^{-t}+\frac{3}{5}e^{2t}+\frac{7}{5}\cos t-\frac{11}{5}\sin t$$

$$y(t)=w\left(t-\frac{\pi}{2}\right)=-e^{-t+\frac{\pi}{2}}+\frac{3}{5}e^{2\left(t-\frac{\pi}{2}\right)}+\frac{7}{5}\cos\left(t-\frac{\pi}{2}\right)-\frac{11}{5}\sin\left(t-\frac{\pi}{2}\right)$$

$$=-e^{-t+\frac{\pi}{2}}+\frac{3}{5}e^{2t-\pi}+\frac{7}{5}\left(\cos t\cos\frac{\pi}{2}+\sin t\sin\frac{\pi}{2}\right)-\frac{11}{5}\left(\sin t\cos\frac{\pi}{2}-\cos t\sin\frac{\pi}{2}\right)$$

$$\underline{=\frac{3}{5}e^{2t-\pi}-e^{-t+\frac{\pi}{2}}+\frac{7}{5}\sin t+\frac{11}{5}\cos t}$$

Exercise

Solve using the Laplace transform: $x''-x'-6x=0$; $x(0)=2,\quad x'(0)=-1$

Solution

$$\mathcal{L}\{x''-x'-6x\}=0$$

$$s^2X(s)-sx(0)-x'(0)-sX(s)+x(0)-6X(s)=0 \qquad x(0)=2,\ x'(0)=-1$$

$$\left(s^2-s-6\right)X(s)-2s+1+2=0$$

$$\left(s^2-s-6\right)X(s)=2s-3$$

$$X(s)=\frac{2s-3}{s^2-s-6}=\frac{A}{s-3}+\frac{B}{s+2}$$

$$As+2A+Bs-3B=2s-3$$

$$\begin{cases} A+B=2 \\ 2A-3B=-3 \end{cases} \quad \begin{vmatrix}1 & 1\\ 2 & -3\end{vmatrix}=-5 \quad \begin{vmatrix}2 & 1\\ -3 & -3\end{vmatrix}=-3 \quad \rightarrow A=\frac{3}{5},\ \ B=\frac{7}{5}$$

$$\mathcal{L}^{-1}\{X(s)\}=\mathcal{L}^{-1}\left\{\frac{3}{5}\frac{1}{s-3}+\frac{7}{5}\frac{1}{s+2}\right\}$$

$$\underline{x(t)=\frac{3}{5}e^{3t}+\frac{7}{5}e^{-2t}}$$

Exercise

Solve using the Laplace transform: $y'' + 2y' + y = 0,\quad y(0) = 1,\quad y'(0) = 1$

Solution

$$\mathcal{L}\{y'' + 2y' + y\} = 0$$

$$s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + Y(s) = 0 \qquad y(0) = 1 \quad y'(0) = 1$$

$$\left(s^2 + 2s + 1\right)Y(s) - s - 1 - 2 = 0$$

$$Y(s) = \frac{s+3}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$$

$$s + 3 = As + A + B$$

$$s \qquad \underline{A = 1}$$

$$s^0 \qquad A + B = 3 \quad \underline{B = 2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1} + \frac{2}{(s+1)^2}\right\}$$

$$\boxed{y(t) = e^{-t} + 2te^{-t}}$$

Exercise

Solve using the Laplace transform: $y'' + 2y' + y = t\ ,\quad y(0) = -3,\quad y(1) = -1$

Solution

$$\mathcal{L}\{y'' + 2y' + y\}(s) = \mathcal{L}\{t\}(s)$$

$$s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + Y(s) = \frac{1}{s^2} \qquad y(0) = -3$$

$$\left(s^2 + 2s + 1\right)Y(s) + 3s - y'(0) + 6 = \frac{1}{s^2}$$

$$(s+1)^2 Y(s) = \frac{1}{s^2} - 3s - 6 + y'(0)$$

$$Y(s) = \frac{-3s^3 + \left(y'(0) - 6\right)s^2 + 1}{s^2(s+1)^2} \qquad \text{Let} \qquad k = y'(0) - 6$$

$$= \frac{-3s^3 + ks^2 + 1}{s^2(s+1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2}$$

$$-3s^3 + ks^2 + 1 = As^3 + 2As^2 + As + Bs^2 + 2Bs + B + Cs^3 + Cs^2 + Ds^2$$

$$\begin{cases} s^3 & A+C=-3 & \rightarrow \underline{C=-1} \\ s^2 & 2A+B+C+D=k & \rightarrow \underline{D=k+4} \\ s & A+2B=0 & \rightarrow \underline{A=-2} \\ s^0 & \underline{B=1} & \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{-\frac{2}{s} + \frac{1}{s^2} - \frac{1}{s+1} + \frac{k+4}{(s+1)^2}\right\}$$

$$y(t) = -2 + t - e^{-t} + (k+4)te^{-t} \qquad y(1) = -1$$

$$-1 = -1 - e^{-1} + (k+4)e^{-1}$$

$$(k+4)e^{-1} = e^{-1}$$

$$k+4=1 \quad \rightarrow \quad \underline{k=-3}$$

$$\underline{y(t) = -2 + t - e^{-t} + te^{-t}}$$

Exercise

Solve using the Laplace transform: $y'' - 2y' - y = e^{2t} - e^{t}$; $y(0)=1,\quad y'(0)=3$

Solution

$$\mathcal{L}\{y'' - 2y' - y\} = \mathcal{L}\left\{e^{2t} - e^{t}\right\}$$

$$s^2Y(s) - sy(0) - y'(0) - 2sY(s) + 2y(0) - Y(s) = \frac{1}{s-2} - \frac{1}{s-1} \qquad y(0)=1 \quad y'(0)=3$$

$$\left(s^2 - 2s - 1\right)Y(s) = \frac{1}{(s-2)(s-1)} + s + 1$$

$$Y(s) = \frac{s^3 - 2s^2 - s + 3}{(s-2)(s-1)\left(s-1-\sqrt{2}\right)\left(s-1+\sqrt{2}\right)} = \frac{A}{s-2} + \frac{B}{s-1} + \frac{C}{s-1-\sqrt{2}} + \frac{D}{s-1+\sqrt{2}}$$

$$\begin{aligned} s^3 - 2s^2 - s + 3 = {} & A(s-1)\left(s^2-2s-1\right) + B(s-2)\left(s^2-2s-1\right) \\ & + C\left(s-1+\sqrt{2}\right)\left(s^2-3s+2\right) + D\left(s-1-\sqrt{2}\right)\left(s^2-3s+2\right) \end{aligned}$$

$$\begin{array}{ll} s^3 & A+B+C+D=1 \\ s^2 & -3A-4B+\left(-4+\sqrt{2}\right)C+\left(-4-\sqrt{2}\right)D=-2 \\ s^1 & A+3B+\left(5-3\sqrt{2}\right)C+\left(5+3\sqrt{2}\right)D=-1 \\ s^0 & A+2B+\left(-2+2\sqrt{2}\right)C-2\left(1+\sqrt{2}\right)D=3 \end{array}$$

$$\underline{A=-1 \quad B=\tfrac{1}{2} \quad C=\tfrac{3}{4}\left(1+\sqrt{2}\right) \quad D=\tfrac{3}{4}\left(1-\sqrt{2}\right)}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{-1}{s-2} + \frac{1}{2}\frac{1}{s-1} + \frac{3}{4}\left(1+\sqrt{2}\right)\frac{1}{s-1-\sqrt{2}} + \frac{3}{4}\left(1-\sqrt{2}\right)\frac{1}{s-1+\sqrt{2}}\right\}$$

$$\boxed{y(t) = -e^{2t} + \frac{1}{2}e^{t} + \left(\frac{3}{4}+\frac{3\sqrt{2}}{4}\right)e^{\left(1+\sqrt{2}\right)t} + \left(\frac{3}{4}-\frac{3\sqrt{2}}{4}\right)e^{\left(1-\sqrt{2}\right)t}}$$

Exercise

Solve using the Laplace transform: $y'' - 2y' + y = 6t - 2$; $y(-1) = 3,\quad y'(-1) = 7$

Solution

Let: $w(t) = y(t-1) \quad \leftrightarrow \quad y(t) = w(t+1)$

$$w'' - 2w' + w = 6(t-1) - 2 = 6t - 8$$

$$\mathcal{L}\{w'' - 2w' + w\} = \mathcal{L}\{6t - 8\}$$

$$s^2W(s) - sw(0) - w'(0) - 2sW(s) + 2w(0) + W(s) = \frac{6}{s^2} - \frac{8}{s} \qquad y(-1) = w(0) = 3,\quad y'(-1) = w'(0) = 7$$

$$\left(s^2 - 2s + 1\right)W(s) = \frac{6-8s}{s^2} + 3s + 1$$

$$W(s) = \frac{3s^3 + s^2 - 8s + 6}{s^2(s-1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$$

$$3s^3 + s^2 - 8s + 6 = As\left(s^2 - 2s + 1\right) + B\left(s^2 - 2s + 1\right) + Cs^2(s-1) + Ds^2$$

s^3	$A + C = 3$	$\boxed{C = -1}$
s^2	$-2A + B - C + D = 1$	$\boxed{D = 2}$
s^1	$A - 2B = -8$	$\boxed{A = 4}$
s^0	$\boxed{B = 6}$	

$$\mathcal{L}^{-1}\{W(s)\} = \mathcal{L}^{-1}\left\{\frac{4}{s} + \frac{6}{s^2} - \frac{1}{s-1} + \frac{2}{(s-1)^2}\right\}$$

$$\boxed{w(t) = 4 + 6t - e^{t} + 2te^{t}}$$

$$y(t) = w(t+1) = 4 + 6(t+1) - e^{t+1} + 2(t+1)e^{t+1}$$

$$\boxed{= 6t + 10 + 2te^{t+1} + e^{t+1}}$$

Exercise

Solve using the Laplace transform: $y'' - 2y' + y = \cos t - \sin t$; $y(0) = 1,\quad y'(0) = 3$

Solution

$$\mathcal{L}\{y'' - 2y' + y\} = \mathcal{L}\{\cos t - \sin t\}$$

$$s^2Y(s) - sy(0) - y'(0) - 2sY(s) + 2y(0) + Y(s) = \frac{s}{s^2+1} - \frac{1}{s^2+1} \qquad y(0)=1, \quad y'(0)=3$$

$$\left(s^2 - 2s + 1\right)Y(s) = \frac{s-1}{s^2+1} + s + 1$$

$$Y(s) = \frac{s^3 + s^2 + 2s}{\left(s^2+1\right)(s-1)^2} = \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$$

$$s^3 + s^2 + 2s = (As+B)\left(s^2 - 2s + 1\right) + C\left(s^2+1\right)(s-1) + D\left(s^2+1\right)$$

$$\begin{array}{ll} s^3 & A + C = 1 \\ s^2 & -2A + B - C + D = 1 \\ s^1 & A - 2B + C = 2 \\ s^0 & B - C + D = 0 \end{array}$$

$$\boxed{A = -\frac{1}{2} \quad B = -\frac{1}{2} \quad C = \frac{3}{2} \quad D = 2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{-\frac{1}{2}\frac{s}{s^2+1} - \frac{1}{2}\frac{1}{s^2+1} + \frac{3}{2}\frac{1}{s-1} + \frac{2}{(s-1)^2}\right\}$$

$$\boxed{y(t) = -\frac{1}{2}\cos t - \frac{1}{2}\sin t + \frac{3}{2}e^t + 2te^t}$$

Exercise

Solve using the Laplace transform: $y'' - 2y' + 5y = 0, \quad y(0) = 2, \quad y'(0) = 4$

Solution

$$\mathcal{L}\{y'' - 2y' + 5y\} = 0$$

$$s^2Y(s) - sy(0) - y'(0) - 2sY(s) + 2y(0) + 5Y(s) = 0 \qquad y(0) = 2 \quad y'(0) = 4$$

$$\left(s^2 - 2s + 5\right)Y(s) = 2s$$

$$Y(s) = \frac{2s}{(s-1)^2 + 4}$$

$$= \frac{2(s-1) + 2}{(s-1)^2 + 4}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2(s-1)}{(s-1)^2+4} + \frac{2}{(s-1)^2+4}\right\}$$

$$\boxed{y(t) = 2e^t \cos 2t + e^t \sin 2t}$$

Exercise

Solve using the Laplace transform: $y'' - 2y' + 5y = 1 + t$, $y(0) = 0,$ $y'(0) = 0$

Solution

$$\mathcal{L}\{y'' - 2y' + 5y\} = \mathcal{L}\{1 + t\}$$

$$s^2Y(s) - sy(0) - y'(0) - 2sY(s) + 2y(0) + 5Y(s) = \frac{1}{s} + \frac{1}{s^2} \qquad y(0) = 0 \quad y'(0) = 0$$

$$\left(s^2 - 2s + 5\right)Y(s) = \frac{s+1}{s^2}$$

$$Y(s) = \frac{s+1}{s^2\left((s-1)^2 + 4\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C(s-1) + D}{(s-1)^2 + 4}$$

$$s + 1 = As^3 - 2As^2 + 5As + Bs^2 - 2Bs + 5B + Cs^3 - Cs^2 + Ds^2$$

$$\begin{array}{lll} s^3 & A + C = 0 & C = -\frac{7}{25} \\ s^2 & -2A + B - C + D = 0 & D = \frac{2}{25} \\ s^1 & 5A - 2B = 1 & A = \frac{7}{25} \\ s^0 & 5B = 1 & B = \frac{1}{5} \end{array}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{7}{25}\frac{1}{s} + \frac{1}{5}\frac{1}{s^2} - \frac{7}{25}\frac{s-1}{(s-1)^2 + 2^2} + \frac{1}{25}\frac{2}{(s-1)^2 + 2^2}\right\}$$

$$y(t) = \frac{7}{25} + \frac{1}{5}t - \frac{7}{25}e^t\cos 2t + \frac{1}{25}e^t\sin 2t$$

Exercise

Solve using the Laplace transform: $y'' + 3y' = -3t$; $y(0) = -1,$ $y'(0) = 1$

Solution

$$\mathcal{L}(y'' + 3y') = \mathcal{L}(-3t)$$

$$s^2Y(s) - sy(0) - y'(0) + 3\left(sY(s) - y(0)\right) = -3\frac{1}{s^2} \qquad y(0) = -1 \quad y'(0) = 1$$

$$s^2Y(s) + s - 1 + 3sY(s) + 3 = -\frac{3}{s^2}$$

$$\left(s^2 + 3s\right)Y(s) = -\frac{3}{s^2} - s - 2$$

$$Y(s) = -\frac{3}{s^3(s+3)} - \frac{s+2}{s(s+3)}$$

$$\frac{3}{s^3(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+3}$$

$$3 = As^2(s+3) + Bs(s+3) + C(s+3) + Ds^3$$

$$3 = (A+D)s^3 + (3A+B)s^2 + (3B+C) + 3C$$

$$\begin{cases} A+D=0 \\ 3A+B=0 \\ 3B+C=0 \\ 3C=3 \end{cases} \Rightarrow \begin{matrix} A=\frac{1}{9} & B=-\frac{1}{3} \\ C=1 & D=-\frac{1}{9} \end{matrix}$$

$$\frac{s+2}{s(s+3)} = \frac{E}{s} + \frac{F}{s+3}$$

$$\begin{cases} E+F=1 \\ 3E=2 \end{cases} \Rightarrow E=\frac{2}{3} \quad F=\frac{1}{3}$$

$$Y(s) = -\left(\frac{1}{9}\frac{1}{s} - \frac{1}{3}\frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{9}\frac{1}{s+3}\right) - \left(\frac{2}{3}\frac{1}{s} + \frac{1}{3}\frac{1}{s+3}\right)$$

$$= -\frac{1}{9}\frac{1}{s} + \frac{1}{3}\frac{1}{s^2} - \frac{1}{s^3} + \frac{1}{9}\frac{1}{s+3} - \frac{2}{3}\frac{1}{s} - \frac{1}{3}\frac{1}{s+3}$$

$$= -\frac{7}{9}\frac{1}{s} + \frac{1}{3}\frac{1}{s^2} - \frac{1}{s^3} - \frac{2}{9}\frac{1}{s+3}$$

$$y(t) = -\frac{7}{9}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{2!}\frac{2!}{s^3}\right\} - \frac{2}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$\boxed{y(t) = -\frac{7}{9} + \frac{1}{3}t - \frac{1}{2}t^2 - \frac{2}{9}e^{-3t}}$$

Exercise

Solve using the Laplace transform: $y'' + 3y = t^3$; $y(0)=0,\quad y'(0)=0$

Solution

$$\mathcal{L}\{y'' + 3y'\} = \mathcal{L}\{t^3\}$$

$$s^2Y(s) - sy(0) - y'(0) + 3Y(s) = \frac{6}{s^4} \qquad y(0)=0 \quad y'(0)=0$$

$$\left(s^2+3\right)Y(s) = \frac{6}{s^4}$$

$$Y(s) = \frac{6}{s^4\left(s^2+3\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s^4} + \frac{Es+F}{s^2+3}$$

$$6 = As^5 + 3As^3 + Bs^4 + 3Bs^2 + Cs^3 + 3Cs + Ds^2 + 3D + Es^5 + Fs^4$$

$s^5 \quad A+E=0 \quad \boxed{E=0}$

$s^4 \quad B+F=0 \quad \boxed{F=\frac{2}{3}}$

$s^3 \quad 3A+C=0 \quad \boxed{A=0}$

$s^2 \quad 3B+D=0 \quad \boxed{B=-\frac{2}{3}}$

$s^1 \quad 3C=0 \quad \boxed{C=0}$

$s^0 \quad 3D=6 \quad \boxed{D=2}$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{-\frac{2}{3}\frac{1}{s^2}+\frac{2}{s^4}+\frac{2}{3}\frac{1}{s^2+\left(\sqrt{3}\right)^2}\right\}$$

$$y(t)=-\frac{2}{3}t+2\frac{1}{3!}t^3+\frac{2}{3}\frac{1}{\sqrt{3}}\sin\left(\sqrt{3}t\right)$$

$$\boxed{=-\frac{2}{3}t+\frac{1}{3}t^3+\frac{2\sqrt{3}}{9}\sin\left(\sqrt{3}t\right)}$$

Exercise

Solve using the Laplace transform: $\quad y''-3y'+2y=e^{-t}\ ,\quad y(1)=0,\quad y'(1)=0$

Solution

Let $v=t-1 \quad \to \quad t=v+1$

$x(v)=y(t)=y(v+1) \qquad y(1)=x(0)=0 \quad y'(1)=x'(0)=0$

$y''(t)-3y'(t)+2y(t)=e^{-t}$

$\to \quad y''(v+1)-3y'(v+1)+2y(v+1)=e^{-(v+1)}$

$x''(v)-3x'(v)+2x(v)=e^{-(v+1)}$

$\mathcal{L}\{x''-3x'+2x\}=\mathcal{L}\left\{e^{-1}e^{-v}\right\}$

$s^2X(s)-sx(0)-x'(0)-3sX(s)+3x(0)+2X(s)=\dfrac{e^{-1}}{s+1}$

$\left(s^2-3s+2\right)X(s)=\dfrac{e^{-1}}{s+1}$

$X(s)=e^{-1}\dfrac{1}{(s+1)(s-1)(s-2)}$

$\dfrac{1}{(s+1)(s-1)(s-2)}=\dfrac{A}{s+1}+\dfrac{B}{s-1}+\dfrac{C}{s-2}$

$1=As^2-3As+2A+Bs^2-Bs-2B+Cs^2-C$

$$\begin{cases} s^2 & A+B+C=0 \\ s & -3A-B=0 \\ s^0 & 2A-2B-C=1 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ -3 & -1 & 0 \\ 2 & -2 & -1 \end{vmatrix} = 6 \quad \Delta_A = \begin{vmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & -2 & -1 \end{vmatrix} = 1 \quad \Delta_B = \begin{vmatrix} 1 & 0 & 1 \\ -3 & 0 & 0 \\ 2 & 1 & -1 \end{vmatrix} = -3$$

$$\underline{A = \frac{1}{6}, \quad B = -\frac{1}{2}, \quad C = \frac{1}{3}}$$

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{e^{-1}\left(\frac{1}{6}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s-1} + \frac{1}{3}\frac{1}{s-2}\right)\right\}$$

$$x(v) = e^{-1}\left(\frac{1}{6}e^{-v} - \frac{1}{2}e^{v} + \frac{1}{3}e^{2v}\right) \qquad v = t-1$$

$$y(t) = e^{-1}\left(\frac{1}{6}e^{-t+1} - \frac{1}{2}e^{t-1} + \frac{1}{3}e^{2(t-1)}\right)$$

$$\underline{= \frac{1}{6}e^{-t} - \frac{1}{2}e^{t-2} + \frac{1}{3}e^{2t-3}}$$

Exercise

Solve using the Laplace transform: $y'' - 3y' + 2y = \cos t$; $y(0) = 0, \quad y'(0) = -1$

Solution

$$\mathcal{L}\{y'' - 3y' + 2y\} = \mathcal{L}\{\cos t\}$$

$$s^2Y(s) - sy(0) - y'(0) - 3sY(s) + 3y(0) + 2Y(s) = \frac{s}{s^2+1} \qquad y(0) = 0 \quad y'(0) = -1$$

$$\left(s^2 - 3s + 2\right)Y(s) = \frac{s}{s^2+1} - 1$$

$$Y(s) = \frac{-s^2 + s - 1}{(s-1)(s-2)\left(s^2+1\right)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{Cs+D}{s^2+1}$$

$$\begin{cases} s^3 & A+B+C=0 \\ s^2 & -2A-B-3C+D=-1 \\ s^1 & A+B+2C-3D=1 \\ s^0 & -2A-B+2D=-1 \end{cases} \rightarrow \underline{A = \frac{1}{2}, \ B = -\frac{3}{5}, \ C = \frac{1}{10}, \ D = -\frac{3}{10}}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s-1} - \frac{3}{5}\frac{1}{s-2} + \frac{1}{10}\frac{s}{s^2+1} - \frac{3}{10}\frac{1}{s^2+1}\right\}$$

$$\underline{y(t) = \frac{1}{2}e^{t} - \frac{3}{5}e^{2t} + \frac{1}{10}\cos t - \frac{3}{10}\sin t}$$

Exercise

Solve using the Laplace transform: $y'' - 4y = e^{-t}$; $y(0) = -1, \quad y'(0) = 0$

Solution

$$\mathcal{L}(y'' - 4y) = \mathcal{L}\left(e^{-t}\right)$$

$$s^2Y(s) - sy(0) - y'(0) - 4Y(s) = \frac{1}{s+1} \qquad\qquad y(0) = -1 \quad y'(0) = 0$$

$$\left(s^2 - 4\right)Y(s) + s = \frac{1}{s+1}$$

$$Y(s) = \frac{1}{(s+1)\left(s^2-4\right)} - \frac{s}{s^2-4}$$

$$\frac{1}{(s+1)\left(s^2-4\right)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s+2}$$

$$1 = As^2 - 4A + Bs^2 + 3Bs + 2B + Cs^2 - Cs - 2C$$

$$1 = (A+B+C)s^2 + (3B-C)s + 2B - 4A - 2C$$

$$\begin{cases} A+B+C=0 \\ 3B-C=0 \\ -4A+2B-2C=1 \end{cases} \Rightarrow A = -\frac{1}{3} \quad B = \frac{1}{12} \quad C = \frac{1}{4}$$

$$\frac{s}{s^2-4} = \frac{D}{s-2} + \frac{E}{s+2}$$

$$s = (D+E)s + 2D - 2E$$

$$\begin{cases} D+E=1 \\ 2D-2E=0 \end{cases} \Rightarrow D = \frac{1}{2} \quad E = \frac{1}{2}$$

$$Y(s) = -\frac{1}{3}\frac{1}{s+1} + \frac{1}{12}\frac{1}{s-2} + \frac{1}{4}\frac{1}{s+2} - \frac{1}{2}\frac{1}{s-2} - \frac{1}{2}\frac{1}{s+2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{-\frac{1}{3}\frac{1}{s+1} - \frac{5}{12}\frac{1}{s-2} - \frac{1}{4}\frac{1}{s+2}\right\}$$

$$\boxed{y(t) = -\frac{1}{3}e^{-t} - \frac{5}{12}e^{2t} - \frac{1}{4}e^{-2t}}$$

Exercise

Solve using the Laplace transform: $y'' - 4y' = 6e^{3t} - 3e^{-t}$, $y(0) = 1 \quad y'(0) = -1$

Solution

$$\mathcal{L}\{y'' - 4y'\} = \mathcal{L}\left\{6e^{3t} - 3e^{-t}\right\}$$

$$s^2Y(s) - sy(0) - y'(0) - 4sY(s) + 4y(0) = 0 \qquad\qquad y(0) = 1; \quad y'(0) = -1$$

$$\left(s^2 - 4s\right)Y(s) - s + 1 + 4 = 0$$

$$Y(s) = \frac{s-5}{s(s-4)} \quad = \frac{A}{s} + \frac{B}{s-4}$$

$$As - 4A + Bs = s - 5$$

$$\begin{cases} A + B = 1 \\ -4A = -5 \end{cases} \quad \to \quad \underline{A = \frac{5}{4},\ B = -\frac{1}{4}|}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{5}{4}\frac{1}{s} - \frac{1}{4}\frac{1}{s-4}\right\}$$

$$\underline{y(t) = \frac{5}{4}t - \frac{1}{4}e^{4t}|}$$

Exercise

Solve using the Laplace transform: $y'' - 4y' + 4y = t^3 e^{2t}$; $y(0) = 0,\quad y'(0) = 0$

Solution

$$\mathcal{L}\{y'' - 4y' + 4y\} = \mathcal{L}\left\{t^3 e^{2t}\right\}$$

$$s^2Y(s) - sy(0) - y'(0) - 4sY(s) + 4y(0) + 4Y(s) = \frac{6}{(s-2)^4} \qquad y(0) = 0;\ \ y'(0) = 0$$

$$\left(s^2 - 4s + 4\right)Y(s) = \frac{6}{(s-2)^4}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{6}{(s-2)^6}\right\}$$

$$\underline{y(t) = \frac{1}{20}t^5 e^{2t}|}$$

Exercise

Solve using the Laplace transform: $y'' - 4y' + 4y = t^3,\quad y(0) = 1,\quad y'(0) = 0$

Solution

$$\mathcal{L}\{y'' - 4y' + 4y\} = \mathcal{L}\left\{t^3\right\}$$

$$s^2Y(s) - sy(0) - y'(0) - 4sY(s) + 4y(0) + 4Y(s) = \frac{6}{s^4} \qquad y(0) = 1;\ \ y'(0) = 0$$

$$\left(s^2 - 4s + 4\right)Y(s) = \frac{6}{s^4} + s - 4$$

$$Y(s) = \frac{s^5 - 4s^4 + 6}{s^4(s-2)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s^4} + \frac{E}{s-2} + \frac{F}{(s-2)^2}$$

$$s^5 \qquad A+E=1 \qquad \boxed{E=-\frac{1}{4}}$$

$$s^4 \qquad -4A+B-2E+F=-4 \qquad \boxed{F=-\frac{13}{8}}$$

$$s^3 \qquad 4A-4B+C=0 \qquad \boxed{A=\frac{3}{4}}$$

$$s^2 \qquad 4B-4C+D=0 \qquad \boxed{B=\frac{9}{8}}$$

$$s^1 \qquad 4C-4D=0 \qquad \boxed{C=\frac{3}{2}}$$

$$s^0 \qquad 4D=6 \qquad \boxed{D=\frac{3}{2}}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{3}{4}\frac{1}{s}+\frac{9}{8}\frac{1}{s^2}+\frac{3}{2}\frac{1}{s^3}+\frac{3}{2}\frac{1}{s^4}+\frac{1}{4}\frac{1}{s-2}-\frac{13}{8}\frac{1}{(s-2)^2}\right\}$$

$$\boxed{y(t)=\frac{3}{4}+\frac{9}{8}t+\frac{3}{2}t^2+\frac{3}{2}t^3+\frac{1}{4}e^{2t}-\frac{13}{8}te^{2t}}$$

Exercise

Solve using the Laplace transform: $\quad x''+4x'+4x=t^2; \quad x(0)=x'(0)=0$

Solution

$$\mathcal{L}\{x''+4x'+4x\}=\mathcal{L}\{t^2\}$$

$$s^2X(s)-sx(0)-x'(0)+4sX(s)-4x(0)+4X(s)=\frac{2}{s^3} \qquad x(0)=x'(0)=0$$

$$\left(s^2+4s+4\right)X(s)=\frac{2}{s^3}$$

$$X(s)=\frac{2}{s^3(s+2)^2}=\frac{A}{s^3}+\frac{B}{s^2}+\frac{C}{s}+\frac{D}{s+2}+\frac{E}{(s+2)^2}$$

$$As^2+4As+4A+Bs^3+4Bs^2+4Bs+Cs^4+4Cs^3+4Cs^2+Ds^4+2Ds^3+Es^3=3$$

$$s^4 \qquad C+D=0 \qquad \to \boxed{D=-\frac{9}{16}}$$

$$s^3 \qquad B+4C+2D+E=0 \qquad \to \boxed{E=-\frac{3}{8}}$$

$$s^2 \qquad A+4B+4C=0 \qquad \to \boxed{C=\frac{9}{16}}$$

$$s^1 \qquad 4A+4B=0 \qquad \to \boxed{B=-\frac{3}{4}}$$

$$s^0 \qquad 4A=3 \ \to \ \boxed{A=\frac{3}{4}}$$

$$\mathcal{L}^{-1}\{X(s)\}=\mathcal{L}^{-1}\left\{\frac{3}{4}\frac{1}{s^3}-\frac{3}{4}\frac{1}{s^2}+\frac{9}{16}\frac{1}{s}-\frac{9}{16}\frac{1}{s+2}-\frac{3}{8}\frac{1}{(s+2)^2}\right\}$$

$$\boxed{x(t) = \frac{3}{8}t^2 - \frac{3}{4}t + \frac{9}{16} - \frac{9}{16}e^{-2t} - \frac{3}{8}te^{-2t}}$$

Exercise

Solve using the Laplace transform: $y'' + 4y = 4t^2 - 4t + 10$; $y(0) = 0$, $y'(0) = 3$

Solution

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\left\{4t^2 - 4t + 10\right\}$$

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{8}{s^3} - \frac{4}{s^2} + \frac{10}{s} \qquad y(0) = 0 \;\; y'(0) = 3$$

$$\left(s^2 + 4\right)Y(s) = \frac{8 - 4s + 10s^2}{s^3} + 3$$

$$Y(s) = \frac{3s^3 + 10s^2 - 4s + 8}{s^3\left(s^2 + 4\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds + E}{s^2 + 4}$$

$$\begin{array}{lll} s^4 & A + D = 0 & \boxed{D = -2} \\ s^3 & B + E = 3 & \boxed{E = 4} \\ s^2 & 4A + C = 10 & \boxed{A = 2} \\ s^1 & 4B = -4 & \boxed{B = -1} \\ s^0 & 4C = 8 & \boxed{C = 2} \end{array}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s} - \frac{1}{s^2} + \frac{2}{s^3} - \frac{2s}{s^2 + 2^2} + \frac{4}{s^2 + 2^2}\right\}$$

$$\boxed{y(t) = 2 - t + t^2 - 2\cos 2t + 2\sin 2t}$$

Exercise

Solve using the Laplace transform: $y'' - 4y = 4t - 8e^{-2t}$; $y(0) = 0$, $y'(0) = 5$

Solution

$$\mathcal{L}\{y'' - 4y\} = \mathcal{L}\left\{4t - 8e^{-2t}\right\}$$

$$s^2Y(s) - sy(0) - y'(0) - 4Y(s) = \frac{4}{s^2} - \frac{8}{s + 2} \qquad y(0) = 0 \qquad y'(0) = 5$$

$$\left(s^2 - 4\right)Y(s) = \frac{-8s^2 + 4s + 8}{s^2(s + 2)} + 5$$

$$Y(s) = \frac{5s^3 + 2s^2 + 4s + 8}{s^2(s + 2)^2(s - 2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s + 2} + \frac{D}{(s + 2)^2} + \frac{E}{s - 2}$$

$$5s^3+2s^2+4s+8=A\left(s^2+2s\right)\left(s^2-4\right)+B(s+2)\left(s^2-4\right)+C\left(s^4-4s^2\right)+D\left(s^3-2s^2\right)+Es^2\left(s^2+4s+4\right)$$

$$\begin{cases} s^4 & A+C+E=0 & & C+E=0 \\ s^3 & 2A+B+D+4E=5 & & D+4E=6 \\ s^2 & -4A+2B-4C-2D+4E=2 & & -4C-2D+4E=4 \\ s^1 & -8A-4B=4 & \underline{A=0|} & \\ s^0 & -8B=8 & \underline{B=-1|} & \end{cases}$$

$$\underline{C=-1 \quad D=2 \quad E=1|}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{-\frac{1}{s^2}-\frac{1}{s+2}+\frac{2}{(s+2)^2}+\frac{1}{s-2}\right\}$$

$$\underline{y(t)=-t-e^{-2t}+2te^{-2t}+e^{2t}|}$$

Exercise

Solve using the Laplace transform: $y''+4y'=\cos(t-3)+4t$, $y(3)=0,$ $y'(3)=7$

Solution

$$y''(t)+4y'(t)=\cos(t-3)+4t$$

Let $v=t-3 \quad \to \quad t=v+3$

$$x(v)=y(t)=y(v+3) \qquad y(3)=x(0)=0 \quad y'(3)=x'(0)=7$$

$$x''(v)+4x'(v)=\cos v+4v+12$$

$$\mathcal{L}\{x''+4x'\}=\mathcal{L}\{\cos v+4v+12\}$$

$$s^2X(s)-sx(0)-x'(0)+4sX(s)-4x(0)=\frac{s}{s^2+1}+\frac{4}{s^2}+\frac{12}{s}$$

$$\left(s^2+4s\right)X(s)=\frac{s}{s^2+1}+\frac{4+12s}{s^2}+7$$

$$s(s+4)X(s)=\frac{s}{s^2+1}+\frac{7s^2+12s+4}{s^2}$$

$$X(s)=\frac{1}{(s+4)\left(s^2+1\right)}+\frac{7s^2+12s+4}{s^3(s+4)}$$

$$\frac{1}{(s+4)\left(s^2+1\right)}=\frac{A_1}{s+4}+\frac{A_2s+A_3}{s^2+1}$$

$$1=A_1s^2+A_1+A_2s^2+4A_2s+A_3s+4A_3$$

$$\begin{cases} s^2 & A_1 + A_2 = 0 & \rightarrow A_1 = -A_2 & \boxed{A_1 = \frac{1}{17}} \\ s & 4A_2 + A_3 = 0 & \rightarrow A_3 = -4A_2 & \boxed{A_3 = \frac{4}{17}} \\ s^0 & A_1 + 4A_3 = 1 & \Rightarrow -A_2 - 16A_2 = 1 \ \rightarrow & \boxed{A_2 = -\frac{1}{17}} \end{cases}$$

$$\frac{1}{(s+4)(s^2+1)} = \frac{1}{17}\frac{1}{s+4} + \frac{1}{17}\frac{-s+4}{s^2+1}$$

$$\frac{7s^2+12s+4}{s^3(s+4)} = \frac{B_1}{s} + \frac{B_2}{s^2} + \frac{B_3}{s^3} + \frac{B_4}{s+4}$$

$$7s^2 + 12s + 4 = B_1 s^3 + 4B_1 s^2 + B_2 s^2 + 4B_2 s + B_3 s + 4B_3 + B_4 s^3$$

$$\begin{cases} s^3 & B_1 + B_4 = 0 & \rightarrow \boxed{B_4 = -\frac{17}{16}} \\ s^2 & 4B_1 + B_2 = 7 & \rightarrow \boxed{B_1 = \frac{17}{16}} \\ s & 4B_2 + B_3 = 12 & \rightarrow \boxed{B_2 = \frac{11}{4}} \\ s^0 & 4B_3 = 4 & \rightarrow \boxed{B_3 = 1} \end{cases}$$

$$\frac{7s^2+12s+4}{s^3(s+4)} = \frac{17}{16}\frac{1}{s} + \frac{11}{4}\frac{1}{s^2} + \frac{1}{s^3} - \frac{17}{16}\frac{1}{s+4}$$

$$X(s) = \frac{1}{17}\frac{1}{s+4} + \frac{1}{17}\frac{-s+4}{s^2+1} + \frac{17}{16}\frac{1}{s} + \frac{11}{4}\frac{1}{s^2} + \frac{1}{s^3} - \frac{17}{16}\frac{1}{s+4}$$

$$= \frac{17}{16}\frac{1}{s} + \frac{11}{4}\frac{1}{s^2} + \frac{1}{s^3} - \frac{273}{272}\frac{1}{s+4} - \frac{1}{17}\frac{s}{s^2+1} + \frac{4}{17}\frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{17}{16}\frac{1}{s} + \frac{11}{4}\frac{1}{s^2} + \frac{1}{s^3} - \frac{273}{272}\frac{1}{s+4} - \frac{1}{17}\frac{s}{s^2+1} + \frac{4}{17}\frac{1}{s^2+1}\right\}$$

$$x(v) = \frac{17}{16} + \frac{11}{4}v + \frac{1}{2}v^2 - \frac{273}{272}e^{-4v} - \frac{1}{17}\cos v + \frac{4}{17}\sin v \qquad v = t-3$$

$$y(t) = \frac{17}{16} + \frac{11}{4}(t-3) + \frac{1}{2}(t-3)^2 - \frac{273}{272}e^{-4(t-3)} - \frac{1}{17}\cos(t-3) + \frac{4}{17}\sin(t-3)$$

$$= \frac{17}{16} + \frac{11}{4}t - \frac{33}{4} + \frac{1}{2}t^2 - 3t + \frac{9}{2} - \frac{273}{272}e^{-4(t-3)} + \frac{1}{17}\left(4\sin(t-3) - \cos(t-3)\right)$$

$$\boxed{= \frac{43}{16} - \frac{1}{4}t + \frac{1}{2}t^2 - \frac{273}{272}e^{-4(t-3)} + \frac{1}{17}\left(4\sin(t-3) - \cos(t-3)\right)}$$

Exercise

Solve using the Laplace transform: $y'' + 4y' + 8y = \sin t$, $y(0) = 1$, $y'(0) = 0$

Solution

$$\mathcal{L}\{y'' + 4y' + 8y\}(s) = \mathcal{L}\{\sin t\}(s)$$

$$s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 8Y(s) = \frac{1}{s^2+1} \qquad y(0) = 1, \quad y'(0) = 0$$

$$\left(s^2 + 4s + 8\right)Y(s) - s - 4 = \frac{1}{s^2+1}$$

$$\left(s^2 + 4s + 8\right)Y(s) = \frac{1}{s^2+1} + s + 4$$

$$Y(s) = \frac{s^3 + 4s^2 + s + 5}{\left(s^2+1\right)\left(s^2+4s+8\right)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4s+8}$$

$$s^3 + 4s^2 + s + 5 = As^3 + 4As^2 + 8As + Bs^2 + 4Bs + 8B + Cs^3 + Cs + Ds^2 + D$$

$$\begin{cases} s^3 & A + C = 1 \\ s^2 & 4A + B + D = 4 \\ s & 8A + 4B + C = 1 \\ s^0 & 8B + D = 5 \end{cases} \Rightarrow \quad A = -\frac{4}{65} \quad B = \frac{7}{65} \quad C = \frac{69}{65} \quad D = \frac{269}{65}$$

$$Y(s) = \frac{1}{65}\left(-4\frac{s}{s^2+1} + \frac{7}{s^2+1} + \frac{69(s+2) - 138 + 269}{(s+2)^2+4}\right)$$

$$= \frac{1}{65}\left(-4\frac{s}{s^2+1} + \frac{7}{s^2+1} + \frac{69s}{(s+2)^2+4} + \frac{2}{2}\frac{131}{(s+2)^2+4}\right)$$

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{65}\mathcal{L}^{-1}\left\{-4\frac{s}{s^2+1} + \frac{7}{s^2+1} + \frac{69s}{(s+2)^2+4} + \frac{131}{2}\frac{2}{(s+2)^2+4}\right\}$$

$$\boxed{y(t) = -\frac{4}{65}\cos t + \frac{7}{65}\sin t + \frac{69}{65}e^{-2t}\cos 2t + \frac{131}{130}e^{-2t}\sin 2t}$$

Exercise

Solve using the Laplace transform: $y'' + 5y' - y = e^t - 1$; $y(0) = 1$, $y'(0) = 1$

Solution

$$\mathcal{L}\{y'' + 5y' - y\} = \mathcal{L}\left\{e^t - 1\right\}$$

$$s^2Y(s) - sy(0) - y'(0) + 5sY(s) - 5y(0) - Y(s) = \frac{1}{s-1} - \frac{1}{s} \qquad y(0) = 1 \quad y'(0) = 1$$

$$\left(s^2+5s-1\right)Y(s)=\frac{1}{s(s-1)}+s+6$$

$$Y(s)=\frac{s^3+5s^2-6s+1}{s(s-1)\left(s+\frac{5}{2}-\frac{\sqrt{29}}{2}\right)\left(s+\frac{5}{2}+\frac{\sqrt{29}}{2}\right)}$$

$$=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s+\frac{5}{2}-\frac{\sqrt{29}}{2}}+\frac{D}{s+\frac{5}{2}+\frac{\sqrt{29}}{2}}$$

$$\begin{cases} s^3 & A+C+D=1 \\ s^2 & 4A+B+\left(\frac{3}{2}+\frac{\sqrt{29}}{2}\right)C+\left(\frac{3}{2}-\frac{\sqrt{29}}{2}\right)C=5 \\ s^1 & -6A+5B-\left(\frac{5}{2}+\frac{\sqrt{29}}{2}\right)C-\left(\frac{5}{2}-\frac{\sqrt{29}}{2}\right)D=-6 \\ s^0 & A-B=1 \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{1}{5}\frac{1}{s-1}+\left(-\frac{1}{10}+\frac{3}{10\sqrt{29}}\right)\frac{1}{s+\frac{5}{2}-\frac{\sqrt{29}}{2}}-\left(\frac{1}{10}+\frac{3}{10\sqrt{29}}\right)\frac{1}{s+\frac{5}{2}+\frac{\sqrt{29}}{2}}\right\}$$

$$y(t)=1+\frac{1}{5}e^t+\left(-\frac{1}{10}+\frac{3\sqrt{29}}{290}\right)e^{\frac{-5+\sqrt{29}}{2}t}-\left(\frac{1}{10}+\frac{3\sqrt{29}}{290}\right)e^{\frac{-5-\sqrt{29}}{2}t}$$

Exercise

Solve using the Laplace transform: $y''+5y'-6y=21e^{t-1}$ $\quad y(1)=-1, \quad y'(1)=9$

Solution

Let: $w(t)=y(t+1) \quad \leftrightarrow \quad y(t)=w(t-1)$

$$\mathcal{L}\{w''+5w'-6w\}=\mathcal{L}\left\{21e^t\right\}$$

$$s^2W(s)-sw(0)-w'(0)+5sW(s)-5w(0)-6W(s)=21\frac{1}{s-1} \qquad y(1)=w(0)=-1, \quad y'(1)=w(0)=9$$

$$\left(s^2+5s-6\right)W(s)=\frac{21}{s-1}-s+4$$

$$W(s)=\frac{-s^2+5s+17}{(s+6)(s-1)^2}=\frac{A}{s+6}+\frac{B}{s-1}+\frac{C}{(s-1)^2}$$

$$\begin{array}{ll} s^2 & A+B=-1 \\ s^1 & -2A+5B+C=5 \\ s^0 & A-6B+6C=17 \end{array}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ -2 & 5 & 1 \\ 1 & -6 & 6 \end{vmatrix} = 49 \quad \Delta_A = \begin{vmatrix} -1 & 1 & 0 \\ 5 & 5 & 1 \\ 17 & -6 & 6 \end{vmatrix} = -49 \quad \Delta_B = \begin{vmatrix} 1 & -1 & 0 \\ -2 & 5 & 1 \\ 1 & 17 & 6 \end{vmatrix} = 0 \quad \Delta_C = \begin{vmatrix} 1 & 1 & -1 \\ -2 & 5 & 5 \\ 1 & -6 & 17 \end{vmatrix} = 147$$

$$\boxed{A = -1, \quad B = 0, \quad C = 3}$$

$$\mathcal{L}^{-1}\{W(s)\} = \mathcal{L}^{-1}\left\{\frac{-1}{s+6} + \frac{3}{(s-1)^2}\right\}$$

$$w(t) = -e^{-6t} + 3te^{t}$$

$$y(t) = w(t-1) \; \boxed{= -e^{-6(t-1)} + 3(t-1)e^{(t-1)}}$$

Exercise

Solve using the Laplace transform: $\quad y'' + 5y' + 4y = 0; \quad y(0) = 1, \quad y'(0) = 0$

Solution

$$\mathcal{L}\{y'' + 5y' + 4y\} = 0$$

$$s^2Y(s) - sy(0) - y'(0) + 5sY(s) - 5y(0) + 4Y(s) = 0 \qquad y(0) = 1; \; y'(0) = 0$$

$$\left(s^2 + 5s + 4\right)Y(s) - s - 5 = 0$$

$$Y(s) = \frac{s+5}{s^2 + 5s + 4} \quad = \frac{A}{s+1} + \frac{B}{s+4}$$

$$As + 4A + Bs + B = s + 5$$

$$\begin{cases} A + B = 1 \\ 4A + B = 5 \end{cases} \quad \rightarrow \quad \boxed{A = \tfrac{4}{3}; \; B = -\tfrac{1}{3}}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{4}{3}\frac{1}{s+1} - \frac{1}{3}\frac{1}{s+4}\right\}$$

$$\boxed{y(t) = \tfrac{4}{3}e^{-t} - \tfrac{1}{3}e^{-4t}}$$

Exercise

Solve using the Laplace transform: $\quad y'' + 6y = t^2 - 1\,; \quad y(0) = 0, \quad y'(0) = -1$

Solution

$$\mathcal{L}\{y'' + 6y\} = \mathcal{L}\left\{t^2 - 1\right\}$$

$$s^2Y(s) - sy(0) - y'(0) + 6Y(s) = \frac{2}{s^3} - \frac{1}{s} \qquad y(0) = 0 \quad y'(0) = -1$$

$$\left(s^2+6\right)Y(s)=\frac{2-s^2}{s^3}-1$$

$$Y(s)=\frac{2-s^2-s^3}{s^3\left(s^2+6\right)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{Ds+E}{s^2+6}$$

$$\begin{array}{lll} s^4 & A+D=0 & \boxed{D=\frac{2}{9}} \\ s^3 & B+E=-1 & \boxed{E=-1} \\ s^2 & 6A+C=-1 & \boxed{A=-\frac{2}{9}} \\ s & \boxed{B=0} & \\ s^0 & 6C=2 & \boxed{C=\frac{1}{3}} \end{array}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{-\frac{2}{9}\frac{1}{s}+\frac{1}{3}\frac{1}{s^3}+\frac{1}{9}\frac{s}{s^2+6}-\frac{1}{s^2+6}\right\}$$

$$\boxed{y(t)=-\frac{2}{9}+\frac{1}{6}t^2+\frac{1}{9}\cos\sqrt{6}t-\frac{1}{\sqrt{6}}\sin\sqrt{6}t}$$

Exercise

Solve using the Laplace transform: $y''-6y'+9y=t\ ;\quad y(0)=0,\quad y'(0)=1$

Solution

$$\mathcal{L}\{y''-6y'+9y\}=\mathcal{L}\{t\}$$

$$s^2Y(s)-sy(0)-y'(0)-6sY(s)+6y(0)+9Y(s)=\frac{1}{s^2} \qquad y(0)=0 \quad y'(0)=1$$

$$\left(s^2-6s+9\right)Y(s)=\frac{1}{s^2}+1$$

$$Y(s)=\frac{s^2+1}{s^2(s-3)^2}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-3}+\frac{D}{(s-3)^2}$$

$$\begin{cases} s^3 & A+C=0 & \boxed{C=-\frac{2}{27}} \\ s^2 & -6A+B-3C+D=1 & \boxed{D=\frac{10}{9}} \\ s & 9A-6B=0 & \boxed{A=\frac{2}{27}} \\ s^0 & 9B=1 & \boxed{B=\frac{1}{9}} \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{2}{27}\frac{1}{s}+\frac{1}{9}\frac{1}{s^2}-\frac{2}{27}\frac{1}{s-3}+\frac{10}{9}\frac{1}{(s-3)^2}\right\}$$

$$\boxed{y(t)=\frac{2}{27}+\frac{1}{9}t-\frac{2}{27}e^{3t}+\frac{10}{9}te^{3t}}$$

Exercise

Solve using the Laplace transform: $\quad y''-6y'+15y=2\sin 3t\,,\quad y(0)=-1,\quad y'(0)=-4$

Solution

$$\mathcal{L}\{y''-6y'+15y\}(s)=\mathcal{L}\{2\sin 3t\}(s)$$

$$s^2Y(s)-sy(0)-y'(0)-6sY(s)+6y(0)+15Y(s)=\frac{6}{s^2+9} \qquad y(0)=-1,\quad y'(0)=-4$$

$$\left(s^2-6s+15\right)Y(s)+s+4-6=\frac{6}{s^2+9}$$

$$\left(s^2-6s+15\right)Y(s)=\frac{6}{s^2+9}-s+2$$

$$Y(s)=\frac{-s^3+2s^2-9s+24}{\left(s^2+9\right)\left(s^2-6s+15\right)}=\frac{As+B}{s^2+9}+\frac{Cs+D}{s^2-6s+15}$$

$$-s^3+2s^2-9s+24=As^3-6As^2+15As+Bs^2-6Bs+15B+Cs^3+9Cs+Ds^2+9D$$

$$\begin{cases} s^3 & A+C=-1 \\ s^2 & -6A+B+D=2 \\ s & 15A-6B+9C=-9 \\ s^0 & 15B+9D=24 \end{cases} \quad\rightarrow\quad \begin{matrix} A=\frac{1}{10} & B=\frac{1}{10} \\ C=-\frac{11}{10} & D=\frac{5}{2} \end{matrix}$$

$$Y(s)=\frac{1}{10}\frac{s}{s^2+9}+\frac{1}{10}\frac{1}{s^2+9}+\frac{1}{10}\frac{-11(s-3)-33+25}{(s-3)^2-9+15}$$

$$=\frac{1}{10}\frac{s}{s^2+9}+\frac{1}{10}\frac{1}{s^2+9}\frac{3}{3}-\frac{11}{10}\frac{s-3}{(s-3)^2+6}-\frac{1}{10}\frac{8}{(s-3)^2+6}\frac{\sqrt{6}}{\sqrt{6}}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{10}\frac{s}{s^2+9}+\frac{1}{30}\frac{3}{s^2+9}-\frac{11}{10}\frac{s-3}{(s-3)^2+6}-\frac{8}{10\sqrt{6}}\frac{\sqrt{6}}{(s-3)^2+6}\right\}$$

$$\boxed{y(t)=\frac{1}{10}\cos 3t+\frac{1}{30}\sin 3t-\frac{11}{10}e^{3t}\cos\sqrt{6}t-\frac{8}{10\sqrt{6}}e^{3t}\sin\sqrt{6}t}$$

Exercise

Solve using the Laplace transform: $\quad y''-6y'+13y=0\,;\quad y(0)=0,\quad y'(0)=-3$

Solution

$$\mathcal{L}\{y''-6y'+13y\}=0$$

$$s^2Y(s) - sy(0) - y'(0) - 6sY(s) + 6y(0) + 13Y(s) = 0 \qquad y(0)=0 \quad y'(0)=-3$$

$$\left(s^2 - 6s + 13\right)Y(s) = -3$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{-3}{(s-3)^2+4}\right\}$$

$$\boxed{y(t) = -\frac{3}{2}e^{3t}\sin 2t}$$

Exercise

Solve using the Laplace transform: $y'' + 6y' + 9y = 0, \quad y(0) = -1, \quad y'(0) = 6$

Solution

$$\mathcal{L}\{y'' + 6y' + 9y\} = 0$$

$$s^2Y(s) - sy(0) - y'(0) + 6Y(s) - 6y(0) + 9Y(s) = 0 \qquad y(0)=-1 \quad y'(0)=6$$

$$\left(s^2 + 6s + 9\right)Y(s) = -s$$

$$Y(s) = -\frac{s}{(s+3)^2} = \frac{A}{s+3} + \frac{B}{(s+3)^2}$$

$$\begin{cases} s & \underline{A=-1} \\ s^0 & 3A + B = 0 \quad \underline{B=3} \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{-1}{s+3} + \frac{3}{(s+3)^2}\right\}$$

$$\boxed{y(t) = -e^{-3t} + 3te^{-3t}}$$

Exercise

Solve using the Laplace transform: $y'' + 6y' + 5y = 12e^t, \quad y(0) = -1, \quad y'(0) = 7$

Solution

$$\mathcal{L}\{y'' + 6y' + 5y\} = \mathcal{L}\left\{12e^t\right\}$$

$$s^2Y(s) - sy(0) - y'(0) + 6Y(s) - 6y(0) + 5Y(s) = \frac{12}{s-1} \qquad y(0)=-1 \quad y'(0)=7$$

$$\left(s^2 + 6s + 5\right)Y(s) = \frac{12}{s-1} - s + 1$$

$$Y(s) = \frac{-s^2 + 2s + 11}{(s+1)(s+5)(s-1)} = \frac{A}{s+1} + \frac{B}{s+5} + \frac{C}{s-1}$$

$$\begin{cases} s^2 & A+B+C=-1 \\ s & 4A+6C=2 \\ s^0 & -5A-B+5C=11 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 0 & 6 \\ -5 & -1 & 5 \end{vmatrix} = -48 \quad \Delta_A = \begin{vmatrix} -1 & 1 & 1 \\ 2 & 0 & 6 \\ 11 & -1 & 5 \end{vmatrix} = 48 \quad \Delta_B = \begin{vmatrix} 1 & -1 & 1 \\ 4 & 2 & 6 \\ -5 & 11 & 5 \end{vmatrix} = 48$$

$$\underline{A=-1 \quad B=-1 \quad C=1}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{-1}{s+1} - \frac{1}{s+5} + \frac{1}{s-1}\right\}$$

$$\underline{y(t) = -e^{-t} - e^{-5t} + e^{t}}$$

Exercise

Solve using the Laplace transform: $\quad y'' - 7y' + 10y = 9\cos t + 7\sin t \ ; \quad y(0)=5, \quad y'(0)=-4$

Solution

$$\mathcal{L}\{y'' - 7y' + 10y\} = \mathcal{L}\{9\cos t + 7\sin t\}$$

$$s^2Y(s) - sy(0) - y'(0) - 7sY(s) + 7y(0) + 10Y(s) = \frac{9s}{s^2+1} + \frac{7}{s^2+1} \qquad y(0)=5, \quad y'(0)=-4$$

$$\left(s^2 - 7s + 10\right)Y(s) = \frac{9s+7}{s^2+1} + 5s - 39$$

$$Y(s) = \frac{5s^3 - 39s^2 + 14s - 32}{(s-2)(s-5)\left(s^2+1\right)} = \frac{A}{s-2} + \frac{B}{s-5} + \frac{Cs+D}{s^2+1}$$

$$\begin{cases} s^3 & A+B+C=5 \\ s^2 & -5A-2B-7C+D=-39 \\ s & A+B+10C-7D=14 \\ s^0 & -5A-2B+10D=-32 \end{cases} \rightarrow \underline{A=8, \ B=-4, \ C=1, \ D=0}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{8}{s-2} - \frac{4}{s-5} + \frac{s}{s^2+1}\right\}$$

$$\underline{y(t) = 8e^{2t} - 4e^{5t} + \cos t}$$

Exercise

Solve using the Laplace transform: $\quad y'' + 8y' + 25y = 0 \ , \quad y(\pi)=0, \quad y'(\pi)=6$

Solution

Let: $\quad w(t) = y(t+\pi) \quad \leftrightarrow \quad y(t) = w(t-\pi)$

$$y''(t)+8y'(t)+25y(t)=0$$

$$\mathcal{L}\{w''+8w'+25w\}=0$$

$$s^2W(s)-sw(0)-w'(0)+8sW(s)-8w(0)+25W(s)=0 \qquad y(\pi)=w(0)=0, \quad y'(\pi)=w(0)=6$$

$$\left(s^2+8s+25\right)W(s)-6=0$$

$$W(s)=\frac{6}{(s+4)^2-16+25}$$

$$\mathcal{L}^{-1}\{W(s)\}=\mathcal{L}^{-1}\left\{\frac{2\ (3)}{(s+4)^2+9}\right\}$$

$$w(t)=2e^{-4t}\sin 3t \qquad y(t)=w(t-\pi)$$

$$y(t)=2e^{-4(t-\pi)}\sin 3(t-\pi) \qquad \sin(3t-3\pi)=\sin 3t\cos 3\pi-\cos 3t\sin 3\pi=\sin 3t(-1)-0=-\sin 3t$$

$$=-2e^{-4(t-\pi)}\sin 3t$$

Exercise

Solve using the Laplace transform: $y''+9y=2\sin 2t;\quad y(0)=0,\quad y'(0)=-1$

Solution

$$\mathcal{L}(y''+9y)=\mathcal{L}(2\sin 2t)$$

$$s^2Y(s)-sy(0)-y'(0)+9Y(s)=2\frac{2}{s^2+2^2} \qquad y(0)=0 \quad y'(0)=-1$$

$$\left(s^2+9\right)Y(s)+1=\frac{4}{s^2+4}$$

$$Y(s)=\frac{4}{\left(s^2+9\right)\left(s^2+4\right)}-\frac{1}{s^2+9}$$

$$\frac{4}{\left(s^2+9\right)\left(s^2+4\right)}=\frac{A}{s^2+9}+\frac{B}{s^2+4}$$

$$4=(A+B)s^2+4A+9B$$

$$\begin{cases} A+B=0 \\ 4A+9B=4 \end{cases} \Rightarrow A=-\frac{4}{5} \quad B=\frac{4}{5}$$

$$Y(s)=-\frac{4}{5}\frac{1}{s^2+9}+\frac{4}{5}\frac{1}{s^2+4}-\frac{1}{s^2+9}$$

$$=\frac{4}{5}\frac{1}{s^2+4}-\frac{9}{5}\frac{1}{s^2+9}$$

$$=\frac{4}{5}\frac{1}{2}\frac{2}{s^2+4}-\frac{3}{5}\frac{3}{s^2+9}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{5}\frac{2}{s^2+4} - \frac{3}{5}\frac{3}{s^2+9}\right\}$$

$$\boxed{y(t) = \frac{2}{5}\sin 2t - \frac{3}{5}\sin 3t}$$

Exercise

Solve using the Laplace transform: $y'' + 9y = 3\sin 2t$; $y(0) = 0$, $y'(0) = -1$

Solution

$$\mathcal{L}(y'' + 9y)(s) = \mathcal{L}(3\sin 2t)(s)$$

$$s^2Y(s) - sy(0) - y'(0) + 9Y(s) = 3\frac{1}{s^2+4} \qquad y(0) = 0 \quad y'(0) = -1$$

$$s^2Y(s) + 1 + Y(s) = \frac{3}{s^2+4}$$

$$\left(s^2+1\right)Y(s) = \frac{3}{s^2+4} - 1$$

$$\left(s^2+1\right)Y(s) = \frac{-s^2-1}{s^2+4}$$

$$Y(s) = \frac{-s^2-1}{\left(s^2+4\right)\left(s^2+1\right)} \quad = \frac{A}{s^2+4} + \frac{B}{s^2+1}$$

$$As^2 + A + Bs^2 + 4B = -s^2 - 1$$

$$\begin{cases} s^2 \\ s^0 \end{cases} \begin{cases} A + B = -1 \\ A + 4B = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{2}{3} \\ B = -\frac{5}{3} \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{3}\frac{1}{s^2+4} - \frac{5}{3}\frac{1}{s^2+1}\right\}$$

$$\boxed{y(t) = \frac{2}{3}\sin 2t - \frac{5}{3}\sin t}$$

Exercise

Solve using the Laplace transform: $y'' + 16y = 2\sin 4t$; $y(0) = -\frac{1}{2}$, $y'(0) = 0$

Solution

$$\mathcal{L}\{y'' + 16y\}(s) = \mathcal{L}\{2\sin 4t\}(s)$$

$$s^2Y(s) - sy(0) - y'(0) + 16Y(s) = \frac{8}{s^2+16}$$

$$\left(s^2+16\right)Y(s)+\frac{s}{2}=\frac{8}{s^2+16}$$

$$\left(s^2+16\right)Y(s)=\frac{8}{s^2+16}-\frac{s}{2}$$

$$Y(s)=\frac{8}{\left(s^2+16\right)^2}-\frac{1}{2}\frac{s}{s^2+16}$$

$$\mathcal{L}^{-1}\ \frac{2a^3}{\left(s^2+a^2\right)^2}=\sin(at)-at\cos(at)$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{8}{128}\frac{128}{\left(s^2+4^2\right)^2}-\frac{1}{2}\frac{s}{s^2+16}\right\}$$

$$\boxed{y(t)=\frac{1}{16}(\sin 4t-4t\cos 4t)-\frac{1}{2}\cos 4t}$$

Exercise

Solve using the Laplace transform: $y''-10y'+9y=5t$; $y(0)=-1,\quad y'(0)=2$

Solution

$$\mathcal{L}\{y''-10y'+9y\}(s)=\mathcal{L}\{5t\}(s)$$

$$s^2Y(s)-sy(0)-y'(0)-10sY(s)+10y(0)+9Y(s)=\frac{5}{s^2}$$

$$\left(s^2-10s+9\right)Y(s)=\frac{5}{s^2}-s+12$$

$$Y(s)=\frac{-s^3+12s^2+5}{s^2(s-9)(s-1)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-9}+\frac{D}{s-1}$$

$$-s^3+12s^2+5=As^3-10As^2+9As+Bs^2-10Bs+9B+Cs^3-Cs^2+Ds^3-9Ds^2$$

$$\begin{cases} s^3 & A+C+D=-1 \\ s^2 & -10A+B-C-9D=-12 \\ s^1 & 9A-10B=0 & \rightarrow \boxed{A=\frac{50}{81}} \\ s^0 & 9B=5 & \rightarrow \boxed{B=\frac{5}{9}} \end{cases}$$

$$\begin{cases} C+D=-\frac{131}{81} \\ C-9D=-\frac{517}{81} \end{cases} \rightarrow \boxed{C=\frac{31}{81}} \quad \boxed{D=-2}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{50}{81}\frac{1}{s}+\frac{5}{9}\frac{1}{s^2}+\frac{31}{81}\frac{1}{s-9}-\frac{2}{s-1}\right\}$$

$$\boxed{y(t)=\frac{50}{81}+\frac{5}{9}t+\frac{31}{81}e^{9t}-2e^t}$$

Exercise

Solve using the Laplace transform: $2y'' + 3y' - 2y = te^{-2t}, \quad y(0) = 0, \quad y'(0) = -2$

Solution

$$\mathcal{L}\{2y'' + 3y' - 2y\}(s) = \mathcal{L}\{te^{-2t}\}(s)$$

$$2s^2Y(s) - 2sy(0) - 2y'(0) + 3sY(s) - 3y(0) - 2Y(s) = \frac{1}{(s+2)^2}$$

$$\left(2s^2 + 3s - 2\right)Y(s) + 4 = \frac{1}{(s+2)^2}$$

$$(2s-1)(s+2)Y(s) = \frac{1}{(s+2)^2} - 4$$

$$Y(s) = \frac{-4s^2 - 16s - 15}{(2s-1)(s+2)^3} = \frac{A}{2s-1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3}$$

$$-4s^2 - 16s - 15 = As^3 + 6As^2 + 12As + 8A + (2Bs - B)\left(s^2 + 4s + 4\right) + 2Cs^2 + 2Cs - 2C + 2Ds - D$$

$$\begin{cases} s^3 & A + 2B = 0 \\ s^2 & 6A + 7B + 2C = -4 \\ s^1 & 12A + 4B + 3C + 2D = -16 \\ s^0 & 8A - 4B - 2C - D = -15 \end{cases}$$

$$\boxed{A = -\frac{192}{125} \quad B = \frac{96}{125} \quad C = -\frac{2}{25} \quad D = -\frac{1}{5}}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{-\frac{192}{125}\frac{1}{2\left(s - \frac{1}{2}\right)} + \frac{96}{125}\frac{1}{s+2} - \frac{2}{25}\frac{1}{(s+2)^2} - \frac{1}{5}\frac{1}{(s+2)^3}\right\}$$

$$\boxed{y(t) = -\frac{96}{125}e^{t/2} + \frac{96}{125}e^{-2t} - \frac{2}{25}te^{-2t} - \frac{1}{5}t^2e^{-2t}}$$

Exercise

Solve using the Laplace transform: $2y'' + 20y' + 51y = 0, \quad y(0) = 2, \quad y'(0) = 0$

Solution

$$\mathcal{L}\{2y'' + 20y' + 51y\} = 0$$

$$2s^2Y(s) - 2sy(0) - 2y'(0) + 20sY(s) - 20y(0) + 51Y(s) = 0 \qquad y(0) = 2 \quad y'(0) = 0$$

$$\left(2s^2 + 20s + 51\right)Y(s) = 4s + 40$$

$$Y(s)=\frac{4s+40}{2\left(s^2+10s+\frac{51}{2}\right)}$$

$$=\frac{2s+20}{(s+5)^2+\frac{1}{2}}$$

$$=\frac{2s}{(s+5)^2+\left(\frac{1}{\sqrt{2}}\right)^2}+\frac{20}{\sqrt{2}}\frac{\frac{1}{\sqrt{2}}}{(s+5)^2+\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{2s}{(s+5)^2+\left(\frac{\sqrt{2}}{2}\right)^2}+10\sqrt{2}\frac{\frac{1}{\sqrt{2}}}{(s+5)^2+\left(\frac{\sqrt{2}}{2}\right)^2}\right\}$$

$$\boxed{y(t)=2e^{-5t}\cos\frac{\sqrt{2}t}{2}+10\sqrt{2}e^{-5t}\sin\frac{\sqrt{2}t}{2}}$$

Exercise

Solve using the Laplace transform: $y'''+y'=e^t$, $y(0)=y'(0)=y''(0)=0$

Solution

$$\mathcal{L}\{y'''+y'\}=\mathcal{L}\{e^t\}$$

$$s^3Y(s)-s^2y(0)-sy'(0)-y''(0)+sY(s)-y(0)=\frac{1}{s-1}$$

$$\left(s^3+s\right)Y(s)=\frac{1}{s-1}$$

$$Y(s)=\frac{1}{s(s-1)\left(s^2+1\right)}=\frac{A}{s}+\frac{B}{s-1}+\frac{Cs+D}{s^2+1}$$

$$1=As^3-As^2+As-A+Bs^3+Bs+Cs^3-Cs^2+Ds^2-Ds$$

$$\begin{cases} s^3 & A+B+C=0 & B+C=1 \\ s^2 & -A-C+D=0 & -C+D=-1 \\ s & A+B-D=0 & B-D=1 \\ s^0 & -A=1 & \rightarrow \boxed{A=-1} \end{cases} \Rightarrow B=\frac{1}{2}\quad C=\frac{1}{2}\quad D=-\frac{1}{2}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{-\frac{1}{s}+\frac{1}{2}\frac{B}{s-1}+\frac{1}{2}\frac{s}{s^2+1}-\frac{1}{2}\frac{1}{s^2+1}\right\}$$

$$\boxed{y(t)=-1+\frac{1}{2}e^t+\frac{1}{2}\cos t-\frac{1}{2}\sin t}$$

Exercise

Solve using the Laplace transform: $2y^{(3)} + 3y'' - 3y' - 2y = e^{-t}; \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1$

Solution

$$\mathcal{L}\left\{2y^{(3)} + 3y'' - 3y' - 2y\right\} = \mathcal{L}\left\{e^{-t}\right\}$$

$$2s^3Y(s) - 2s^2y(0) - 2sy'(0) - 2y''(0) + 3s^2Y(s) - 3sy(0) - 3y'(0) - 3sY(s) + 3y(0) - 2Y(s) = \frac{1}{s+1}$$

$$\left(2s^3 + 3s^2 - 3s - 2\right)Y(s) - 2 = \frac{1}{s+1} \qquad y(0) = y'(0) = 0, \;\; y''(0) = 1$$

$$(s-1)(2s+1)(s+2)Y(s) = \frac{1}{s+1} + 2$$

$$Y(s) = \frac{2s+3}{(s-1)(2s+1)(s+2)(s+1)} = \frac{A}{s-1} + \frac{B}{2s+1} + \frac{C}{s+2} + \frac{D}{s+1}$$

$$A(2s+1)\left(s^2+3s+2\right) + B(s+2)\left(s^2-1\right) + C(2s+1)\left(s^2-1\right) + D(2s+1)\left(s^2+s-2\right) = 2s+3$$

$$\begin{cases} s^3 & 2A + B + 2C + 2D = 0 \\ s^2 & 7A + 2B + C + 3D = 0 \\ s^1 & 7A - B - 2C - 3D = 2 \\ s^0 & 2A - 2B - C - 2D = 3 \end{cases} \qquad \Rightarrow A = \frac{5}{18} \quad B = -\frac{16}{9} \quad C = \frac{1}{9} \quad D = \frac{1}{2}$$

$$Y(s) = \frac{5}{18}\frac{1}{s-1} - \frac{16}{9}\frac{1}{2s+1} + \frac{1}{9}\frac{1}{s+2} + \frac{1}{2}\frac{1}{s+1}$$

$$\mathcal{L}^{-1}\left\{Y(s)\right\} = \mathcal{L}^{-1}\left\{\frac{5}{18}\frac{1}{s-1} - \frac{8}{9}\frac{1}{s+\frac{1}{2}} + \frac{1}{9}\frac{1}{s+2} + \frac{1}{2}\frac{1}{s+1}\right\}$$

$$y(t) = \frac{5}{18}e^{t} - \frac{8}{9}e^{-t/2} + \frac{1}{9}e^{-2t} + \frac{1}{2}e^{-t}$$

Exercise

Solve using the Laplace transform: $y^{(3)} + 2y'' - y' - 2y = \sin 3t; \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1$

Solution

$$\mathcal{L}\left\{y^{(3)} + 2y'' - y' - 2y\right\} = \mathcal{L}\{\sin 3t\}$$

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) + 2s^2Y(s) - 2sy(0) - 2y'(0) - sY(s) + y(0) - 2Y(s) = \frac{3}{s^2+9}$$

$$\left(s^3 + 2s^2 - s - 2\right)Y(s) - 1 = \frac{3}{s^2+9} \qquad y(0) = y'(0) = 0, \;\; y''(0) = 1$$

$$(s-1)(s+1)(s+2)Y(s) = \frac{3}{s^2+9} + 1$$

$$Y(s)=\frac{s^2+12}{(s-1)(s+1)(s+2)\left(s^2+9\right)} \quad =\frac{A}{s-1}+\frac{B}{s+1}+\frac{C}{s+2}+\frac{Ds+E}{s^2+9}$$

$$A\left(s^2+3s+2\right)\left(s^2+9\right)+B\left(s^2+s-2\right)\left(s^2+9\right)+C\left(s^2-1\right)\left(s^2+9\right)+(Ds+E)\left(s^3+2s^2-s-2\right)$$

$$\begin{cases} s^4 & A+B+C+D=0 \\ s^3 & 3A+B+2D+E=0 \\ s^2 & 11A+7B+8C-D+2E=1 \\ s^1 & 27A+9B-2D-E=0 \\ s^0 & 18A-18B-9C-2E=12 \end{cases} \rightarrow \begin{matrix} A=\frac{13}{60} & B=-\frac{13}{20} & C=\frac{16}{39} \\ D=\frac{3}{130} & E=-\frac{3}{65} & \end{matrix}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{13}{60}\frac{1}{s-1}-\frac{13}{20}\frac{1}{s+1}+\frac{16}{39}\frac{1}{s+2}+\frac{3}{130}\frac{s}{s^2+9}-\frac{1}{65}\frac{3}{s^2+9}\right\}$$

$$\boxed{y(t)=\frac{13}{60}e^t-\frac{13}{20}e^{-t}+\frac{16}{39}e^{-2t}+\frac{3}{130}\cos 3t-\frac{1}{65}\sin 3t}$$

Exercise

Solve using the Laplace transform: $y^{(3)}-y''+y'-y=0$; $y(0)=1,\quad y'(0)=1,\quad y''(0)=3$

Solution

$$s^3Y(s)-s^2y(0)-sy'(0)-y''(0)-s^2Y(s)+sy(0)+y'(0)+sY(s)-y(0)-Y(s)=0$$

$$\left(s^3-s^2+s-1\right)Y(s)-s^2-s-3+s=0 \qquad y(0)=1 \quad y'(0)=1 \quad y''(0)=3$$

$$s^3-s^2+s-1=s^2(s-1)+(s-1)$$

$$Y(s)=\frac{s^2+3}{(s-1)\left(s^2+1\right)}=\frac{A}{s-1}+\frac{Bs+C}{s^2+1}$$

$$\begin{cases} s^2 & A+B=1 \\ s & -B+C=0 \\ s^0 & A-C=3 \end{cases} \quad \begin{matrix} A+C=1 \\ A-C=3 \end{matrix} \quad \boxed{A=2,\ C=-1,\ B=-1}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{2}{s-1}-\frac{s}{s^2+1}-\frac{1}{s^2+1}\right\}$$

$$\boxed{y(t)=2e^t-\cos t-\sin t}$$

Exercise

Solve using the Laplace transform: $y^{(3)}+4y''+y'-6y=-12$; $y(0)=1,\quad y'(0)=4,\quad y''(0)=-2$

Solution

$$s^3Y(s)-s^2y(0)-sy'(0)-y''(0)+4s^2Y(s)-4sy(0)-4y'(0)+sY(s)-y(0)-6Y(s)=-\frac{12}{s}$$

$$\left(s^3+4s^2+s-6\right)Y(s)=s^2-8s-15-\frac{12}{s}$$

$$s^3+4s^2+s-6=(s-1)\left(s^2+5s+6\right)$$

$$\begin{array}{c|cccc} 1 & 1 & 4 & 1 & -6 \\ & & 1 & 5 & 6 \\ \hline & 1 & 5 & 6 & 0 \end{array}$$

$$Y(s)=\frac{s^3-8s^2-15s-12}{s(s-1)(s+2)(s+3)}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s+2}+\frac{D}{s+3}$$

$$\begin{cases} s^3 & A+B+C+D=1 \\ s^2 & 4A+5B+2C+D=-8 \\ s & A+6B-3C-2D=-15 \\ s^0 & -6A=-12 \end{cases}$$

$$\underline{A=2,\quad B=1,\quad C=-3,\quad D=1}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{2}{s}+\frac{1}{s-1}-\frac{3}{s+2}+\frac{1}{s+3}\right\}$$

$$\underline{y(t)=2+e^{t}-3e^{-2t}+e^{-3t}}$$

Exercise

Solve using the Laplace transform: $y^{(3)}+3y''+3y'+y=0$; $y(0)=-4,\quad y'(0)=4,\quad y''(0)=-2$

Solution

$$\mathcal{L}\left\{y^{(3)}+3y''+3y'+y\right\}(s)=0$$

$$s^3Y(s)-s^2y(0)-sy'(0)-y''(0)+3s^2Y(s)-3sy(0)-3y'(0)+3sY(s)-3y(0)+Y(s)=0$$

$$\left(s^3+3s^2+3s+1\right)Y(s)=-4s^2-8s-2$$

$$Y(s)=\frac{-4s^2-8s-2}{(s+1)^3}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{C}{(s+1)^3}$$

$$\begin{cases} s^2 & \underline{A=-4} \\ s & 2A+B=-8 \quad \underline{B=0} \\ s^0 & A+B+C=-2 \quad \underline{C=2} \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{-4}{s+1} + \frac{2}{(s+1)^3}\right\}$$

$$\boxed{y(t) = -4e^{-t} + t^2e^{-t}}$$

Exercise

Solve using the Laplace transform: $y^{(3)} - 3y'' + 3y' - y = t^2e^t$, $y(0)=1,\quad y'(0)=2,\quad y''(0)=3$

Solution

$$\mathcal{L}\left\{y^{(3)} - 3y'' + 3y' - y\right\}(s) = \mathcal{L}\left\{t^2e^t\right\}(s) \qquad \mathcal{L}\left\{t^ne^{-at}\right\}(s) = \frac{n!}{(s+a)^{n+1}}$$

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) - 3s^2Y(s) + 3sy(0) + 3y'(0) + 3sY(s) - 3y(0) - Y(s) = \frac{2}{(s-1)^3}$$

$$\left(s^3 - 3s^2 + 3s - 1\right)Y(s) - s^2 - 2s - 3 + 3s + 6 - 3 = \frac{2}{(s-1)^3}$$

$$(s-1)^3Y(s) = \frac{2}{(s-1)^3} + s^2 - s$$

$$Y(s) = \frac{2 + \left(s^2 - s\right)\left(s^3 - 3s^2 + 3s - 1\right)}{(s-1)^6}$$

$$= \frac{s^5 - 4s^4 + 6s^3 - 4s^2 + s + 2}{(s-1)^6} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} + \frac{D}{(s-1)^4} + \frac{E}{(s-1)^5} + \frac{F}{(s-1)^6}$$

$$s^5 - 4s^4 + 6s^3 - 4s^2 + s + 2 = A(s-1)^5 + B(s-1)^4 + C(s-1)^3 + D(s-1)^2 + E(s-1) + F$$

$$(s-1)^5 = s^5 - 5s^4 + 10s^3 - 10s^2 + 5s - 1 \qquad (s-1)^4 = s^4 - 4s^3 + 6s^2 - 4s + 1$$

$$\begin{cases} s^4 & \boxed{A = 1} & \\ s^4 & -5A + B = -4 & \rightarrow \boxed{B = 1} \\ s^3 & 10A - 4B + C = 6 & \rightarrow \boxed{C = 0} \\ s^2 & -10A + 6B - 3C + D = -4 & \rightarrow \boxed{D = 0} \\ s^1 & 5A - 4B + 3C - 2D + E = 1 & \rightarrow \boxed{E = 0} \\ s^0 & -A + B - C + D - E + F = 2 & \rightarrow \boxed{F = 2} \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1} + \frac{1}{(s-1)^2} + \frac{2}{(s-1)^6}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{1}{(s+a)^{n+1}}\right\} = \frac{1}{n!}t^ne^{-at}$$

$$y(t) = e^t + te^t + \frac{2}{5!}t^5e^t \quad \boxed{= e^t\left(1 + t + \frac{1}{60}t^5\right)}$$

Exercise

Solve using the Laplace transform: $y^{(3)} + y'' + 3y' - 5y = 16e^{-t}$; $y(0) = 0, \quad y'(0) = 2, \quad y''(0) = -4$

Solution

$$\mathcal{L}\left\{y^{(3)} + y'' + 3y' - 5y\right\}(s) = \mathcal{L}\left\{16e^{-t}\right\}(s)$$

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) + s^2Y(s) - sy(0) - y'(0) + 3sY(s) - 3y(0) - 5Y(s) = \frac{16}{s+1}$$

$$\left(s^3 + s^2 + 3s - 5\right)Y(s) = \frac{16}{s+1} + 2s - 2$$

$$s^3 + s^2 + 3s - 5 = (s-1)\left(s^2 + 2s + 5\right)$$

$$\begin{array}{r|rrrr} 1 & 1 & 1 & 3 & -5 \\ & & 1 & 2 & 5 \\ \hline & 1 & 2 & 5 & 0 \end{array}$$

$$Y(s) = \frac{2s^2 + 14}{(s+1)(s-1)\left(s^2 + 2s + 5\right)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C(s+1) + D}{(s+1)^2 + 4}$$

$$\begin{cases} s^3 & A + B + C = 0 \\ s^2 & A + 3B + C + D = 2 \\ s & 3A + 7B - C = 0 \\ s^0 & -5A + 5B - C - D = 14 \end{cases}$$

$$\underline{A = -2, \quad B = 1, \quad C = 1, \quad D = 0|}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{-2}{s+1} + \frac{1}{s-1} + \frac{(s+1)}{(s+1)^2 + 4}\right\}$$

$$\underline{y(t) = -2e^{-t} + e^{t} + e^{-t}\cos 2t \;|}$$

Exercise

Solve using the Laplace transform: $y''' + 4y'' + 5y' + 2y = 10\cos t$, $y(0) = y'(0) = 0, \quad y''(0) = 3$

Solution

$$\mathcal{L}\{y''' + 4y'' + 5y' + 2y\} = \mathcal{L}\{10\cos t\}$$

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) + 4s^2Y(s) - 4sy(0) - 4y'(0) + 5sY(s) - 5y(0) + 2Y(s) = \frac{10s}{s^2+1}$$

$$\left(s^3 + 4s^2 + 5s + 2\right)Y(s) = \frac{10s}{s^2+1} + 3$$

$$\begin{array}{r|rrrr} -1 & 1 & 4 & 5 & 2 \\ & & -1 & -3 & -2 \\ \hline & 1 & 3 & 2 & 0 \end{array} \rightarrow \quad s^2 + 3s + 2 = 0 \qquad \underline{s = -1, \ -1, \ -2|}$$

$$Y(s)=\frac{3s^2+10s+3}{(s+2)\left(s^2+1\right)(s+1)^2}=\frac{A}{s+2}+\frac{Bs+C}{s^2+1}+\frac{D}{s+1}+\frac{E}{(s+1)^2}$$

$$\begin{aligned}3s^2+10s+3&=A\left(s^2+1\right)\left(s^2+2s+1\right)+(Bs+C)(s+2)\left(s^2+2s+1\right)\\&\quad+D(s+1)(s+2)\left(s^2+1\right)+E\left(s^2+1\right)(s+2)\\&=A\left(s^2+1\right)\left(s^2+2s+1\right)+\left(Bs^2+2Bs+Cs+2C\right)\left(s^2+2s+1\right)\\&\quad+D\left(s^2+3s+2\right)\left(s^2+1\right)+Es^3+2Es^2+Es+2E\\&=As^4+2As^3+2As^2+2As+A+Bs^4+4Bs^3+5Bs^2+2Bs+Cs^3+4Cs^2\\&\quad+5Cs+2C+Ds^4+3Ds^3+3Ds^2+3Ds+2D+Es^2+3Es+2E\end{aligned}$$

$$\begin{cases}s^4 & A+B+D=0\\ s^3 & 2A+4B+C+3D+E=0\\ s^2 & 2A+5B+4C+3D+2E=3\\ s^1 & 2A+2B+5C+3D+E=10\\ s^0 & A+2C+2D+2E=3\end{cases}\quad\rightarrow\quad\begin{array}{lll}A=-1 & B=-1 & C=2\\ D=2 & E=-2\end{array}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{-1}{s+2}-\frac{s}{s^2+1}+\frac{2}{s^2+1}+\frac{2}{s+1}-\frac{2}{(s+1)^2}\right\}$$

$$\boxed{y(t)=-e^{-2t}-\cos t+2\sin t+2e^{-t}-2te^{-t}}$$

Exercise

Solve using the Laplace transform: $y^{(4)}+2y''+y=4te^t;\quad y(0)=y'(0)=y''(0)=y^{(3)}(0)=0$

Solution

$$\mathcal{L}\left\{y^{(4)}+2y''+y\right\}=\mathcal{L}\left\{4te^t\right\}$$

$$s^4Y(s)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)+2s^2Y(s)-2sy(0)-2y'(0)+Y(s)=\frac{4}{(s-1)^2}$$

$$\left(s^4+2s^2+1\right)Y(s)=\frac{4}{(s-1)^2}\qquad y(0)=y'(0)=y''(0)=y^{(3)}(0)=0$$

$$\left(s^2+1\right)^2Y(s)=\frac{4}{(s-1)^2}$$

$$Y(s)=\frac{4}{(s-1)^2\left(s^2+1\right)^2}=\frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{Cs+D}{s^2+1}+\frac{Es+F}{\left(s^2+1\right)^2}$$

$$A(s-1)\left(s^4+2s^2+1\right)+B\left(s^4+2s^2+1\right)+(Cs+D)\left(s^2-2s+1\right)\left(s^2+1\right)+(Es+F)\left(s^2-2s+1\right)=4$$

$$\begin{cases} s^5 & A+C=0 \\ s^4 & -A-2C+D=0 \\ s^3 & 2A+C-2D+E=0 \\ s^2 & -2A+2B-2C+2D-2E+F=0 \\ s^1 & A+C-2D+E-2F=0 \\ s^0 & -A+B+D+F=4 \end{cases} \rightarrow \begin{matrix} A=-\frac{16}{17} & B=\frac{28}{17} & C=\frac{16}{17} \\ D=\frac{16}{17} & E=\frac{48}{17} & F=\frac{8}{17} \end{matrix}$$

$$Y(s)=-\frac{16}{17}\frac{1}{s-1}+\frac{28}{17}\frac{1}{(s-1)^2}+\frac{16}{17}\frac{s}{s^2+1}+\frac{16}{17}\frac{1}{s^2+1}+\frac{48}{17}\frac{s}{\left(s^2+1\right)^2}+\frac{8}{17}\frac{1}{\left(s^2+1\right)^2}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{-\frac{16}{17}\frac{1}{s-1}+\frac{28}{17}\frac{1}{(s-1)^2}+\frac{16}{17}\frac{s}{s^2+1}+\frac{16}{17}\frac{1}{s^2+1}+\frac{24}{17}\frac{2s}{\left(s^2+1\right)^2}+\frac{4}{17}\frac{2}{\left(s^2+1\right)^2}\right\}$$

$$\boxed{y(t)=-\frac{16}{17}e^t+\frac{28}{17}te^t+\frac{16}{17}\cos t+\frac{16}{17}\sin t+\frac{27}{17}t\sin t+\frac{4}{17}\sin t-t\cos t}$$

Exercise

Solve using the Laplace transform: $y^{(4)}-y=0$; $y(0)=1,\quad y'(0)=0,\quad y''(0)=0,\quad y^{(3)}(0)=0$

Solution

$$\mathcal{L}\left\{y^{(4)}-y\right\}(s)=0$$

$$s^4Y(s)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)-Y(s)=0$$

$$\left(s^4-1\right)Y(s)=s^3$$

$$Y(s)=\frac{s^3}{(s-1)(s+1)\left(s^2+1\right)}=\frac{A}{s-1}+\frac{B}{s+1}+\frac{Cs+D}{s^2+1}$$

$$s^3=As^3+As^2+As+A+Bs^3-Bs^2+Bs-B++Cs^3-Cs+Ds^2-D$$

$$\begin{cases} s^3 & A+B+C=1 \\ s^2 & A-B+D=0 \\ s & A+B-C=0 \\ s^0 & A-B-D=0 \end{cases} \rightarrow \begin{matrix} A=\frac{1}{4} & B=\frac{1}{4} \\ C=\frac{1}{2} & D=0 \end{matrix}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s-1}+\frac{1}{4}\frac{1}{s+1}+\frac{1}{2}\frac{s}{s^2+1}\right\}$$

$$\boxed{y(t)=\frac{1}{4}e^{t}+\frac{1}{4}e^{-t}+\frac{1}{2}\cos t}$$

Exercise

Solve using the Laplace transform: $y^{(4)}-4y=0$; $y(0)=1,\quad y'(0)=0,\quad y''(0)=-2,\quad y^{(3)}(0)=0$

Solution

$$\mathcal{L}\left\{y^{(4)}-4y\right\}=0$$

$$s^4Y(s)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)-4Y(s)=0 \qquad y(0)=1,\ y'(0)=0,\ y''(0)=-2,\ y^{(3)}(0)=0$$

$$\left(s^4-4\right)Y(s)-s^3+2s=0$$

$$Y(s)=\frac{s\left(s^2-2\right)}{\left(s^2-2\right)\left(s^2+2\right)}$$

$$=\frac{s}{s^2+2}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\}$$

$$\boxed{y(t)=\cos\sqrt{2}t}$$

Exercise

Solve using the Laplace transform:

$$y^{(4)}-4y^{(3)}+6y''-4y'+y=0\ ;\quad y(0)=0,\quad y'(0)=1,\quad y''(0)=0,\quad y^{(3)}(0)=1$$

Solution

$$\mathcal{L}\left\{y^{(4)}-4y^{(3)}+6y''-4y'+y=0\right\}=0$$

$$s^4Y(s)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)-4s^3Y(s)+4s^2y(0)$$

$$+4sy'(0)+4y''(0)+6s^2Y(s)-6sy(0)-6y'(0)-4sY(s)+4y(0)+Y(s)=0$$

$$\left(s^4-4s^3+6s^2-4s+1\right)Y(s)-s^2-1+4s-6=0$$

$$(s+1)^4Y(s)=s^2-4s+7$$

$$Y(s)=\frac{s^2-4s+7}{(s+1)^4}\quad=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{C}{(s+1)^3}+\frac{D}{(s+1)^4}$$

$$As^3+3As^2+3As+A+Bs^2+2Bs+B+Cs+C+D=s^2-4s+7$$

$$\begin{cases} s^3 & A=0 \\ s^2 & 3A+B=1 \\ s^1 & 3A+2B+C=-4 \\ s^0 & A+B+C+D=7 \end{cases} \quad \Rightarrow B=1 \quad C=-6 \quad D=13$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2} - \frac{6}{(s+1)^3} + \frac{13}{(s+1)^4}\right\}$$

$$y(t) = te^t - 3t^2e^t + \frac{13}{6}t^3e^t$$

$$= \left(t - 3t^2 + \frac{13}{6}t^3\right)e^t$$

Exercise

Given: $y'' - 4y' + 3y = 0, \quad y(0) = 1 \quad y'(0) = -1$

a) Show that the general solution is: $y(t) = C_1e^{3t} + C_2e^t$ and find C_1 *and* C_2

b) Use Laplace transform to solve the system

Solution

a) $\lambda^2 - 4\lambda + 3 = 0 \quad \Rightarrow \quad \boxed{\lambda = 3,\ 1}$

That implies to the general solution: $y = C_1e^{3t} + C_2e^t$

$$1 = C_1e^{3(0)} + C_2e^{(0)}$$

$$1 = C_1 + C_2$$

$$y' = 3C_1e^{3t} + C_2e^t$$

$$-1 = 3C_1e^{3(0)} + C_2e^{(0)}$$

$$-1 = 3C_1 + C_2$$

$$\begin{cases} C_1 + C_2 = 1 \\ 3C_1 + C_2 = -1 \end{cases} \quad \Rightarrow \quad \boxed{C_1 = -1} \quad \boxed{C_2 = 2}$$

Therefore; the general solution is: $\quad y(t) = -e^{3t} + 2e^t$

b) $\mathcal{L}(y'' - 4y' + 3y)(s) = 0$

$$s^2Y(s) - sy(0) - y'(0) - 4\big(sY(s) - y(0)\big) + 3Y(s) = 0$$

$$s^2Y(s) - s + 1 - 4\big(sY(s) - 1\big) + 3Y(s) = 0$$

$$s^2Y(s) - s + 1 - 4sY(s) + 4 + 3Y(s) = 0$$

$$\left(s^2-4s+3\right)Y(s)=s-5$$

$$Y(s)=\frac{s-5}{s^2-4s+3}$$

$$=\frac{s-5}{(s-1)(s-3)}$$

$$=\frac{A}{s-1}+\frac{B}{s-3}=\frac{(A+B)s-3A-B}{(s-1)(s-3)}$$

$$\begin{cases} A+B=1 \\ -3A-B=-5 \end{cases} \Rightarrow \boxed{A=2} \quad \boxed{B=-1}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{2}{s-1}-\frac{1}{s-3}\right\}$$

$$\underline{y(t)=2e^t-e^{3t}}$$

Exercise

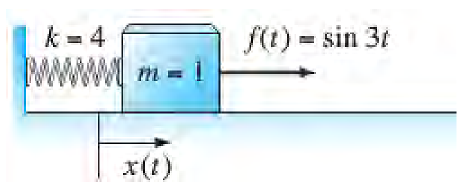
Solve the initial value problem $x''+4x=\sin 3t; \quad x(0)=x'(0)=0.$

Such problem arises in the motion of a mass-and-spring system with external force as shown below.

Solution

$$\mathcal{L}\{x''+4x\}=\mathcal{L}\{\sin 3t\}$$

$$s^2X(s)-sx(0)-x'(0)+4X(s)=\frac{3}{s^2+9}$$



$$\left(s^2+4\right)X(s)=\frac{3}{s^2+9}$$

$$X(s)=\frac{3}{\left(s^2+4\right)\left(s^2+9\right)}=\frac{As+B}{s^2+4}+\frac{Cs+D}{s^2+9}$$

$$As^3+9AS+Bs^2+9B+Cs^3+4CS+Ds^2+4C=3$$

$$\begin{cases} s^3 & A+C=0 \\ s^2 & B+D=0 \\ s^1 & 9A+4C=0 \\ s^0 & 9B+4D=3 \end{cases} \to \; A=C=0 \quad 5B=3 \Rightarrow B=\frac{3}{5} \quad D=-\frac{3}{5}$$

$$X(s)=\frac{3}{5}\frac{1}{s^2+4}-\frac{3}{5}\frac{1}{s^2+9}$$

$$\mathcal{L}^{-1}\{X(s)\}=\mathcal{L}^{-1}\left\{\frac{3}{10}\frac{2}{s^2+4}-\frac{1}{5}\frac{3}{s^2+9}\right\}$$

$$\underline{x(t)=\frac{3}{10}\sin 2t-\frac{1}{5}\sin 3t}$$

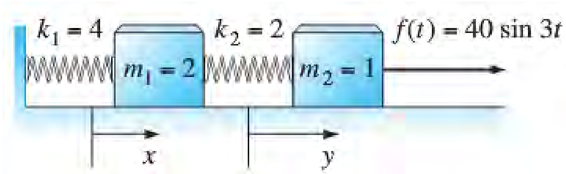
Exercise

Solve the system $\begin{cases} 2x'' = -6x + 2y \\ y'' = 2x - 2y + 40\sin 3t \end{cases}$

Subject to the initial conditions $x(0) = x'(0) = y(0) = y'(0) = 0$

Thus the force $f(t) = 40\sin 3t$ is applied to the second mass as shown below, beginning at time $t = 0$ when the system is at rest in its equilibrium position.

Solution



$$\begin{cases} \mathcal{L}\{2x''\} = \mathcal{L}\{-6x + 2y\} \\ \mathcal{L}\{y''\} = \mathcal{L}\{2x - 2y + 40\sin 3t\} \end{cases}$$

$$\begin{cases} 2s^2 X(s) - 2sx(0)) - 2x'(0) = -6X(s) + 2Y(s) \\ s^2 Y(s) - sy(0)) - y'(0) = 2X(s) - 2Y(s) + \dfrac{120}{s^2+9} \end{cases}$$

Given: $x(0) = x'(0) = y(0) = y'(0) = 0$

$$\begin{cases} 2s^2 X(s) = -6X(s) + 2Y(s) \\ s^2 Y(s) = 2X(s) - 2Y(s) + \dfrac{120}{s^2+9} \end{cases}$$

$$\begin{cases} \left(s^2+3\right) X(s) - Y(s) = 0 & (1) \\ -2X(s) \quad + \left(s^2+2\right) Y(s) = \dfrac{120}{s^2+9} & (2) \end{cases}$$

$$\begin{vmatrix} s^2+3 & -1 \\ -2 & s^2+2 \end{vmatrix} = s^4 + 5s^2 + 4 = \left(s^2+1\right)\left(s^2+4\right)$$

$$\begin{vmatrix} 0 & -1 \\ \dfrac{120}{s^2+9} & s^2+2 \end{vmatrix} = \frac{120}{s^2+9} \quad \rightarrow \quad X(s) = \frac{120}{\left(s^2+1\right)\left(s^2+4\right)\left(s^2+9\right)}$$

$$\begin{vmatrix} s^2+3 & 0 \\ -2 & \dfrac{120}{s^2+9} \end{vmatrix} = 120\frac{s^2+3}{s^2+9} \quad \rightarrow \quad Y(s) = \frac{120\left(s^2+3\right)}{\left(s^2+1\right)\left(s^2+4\right)\left(s^2+9\right)}$$

$$X(s) = \frac{120}{\left(s^2+1\right)\left(s^2+4\right)\left(s^2+9\right)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} + \frac{Es+F}{s^2+9}$$

$$(As+B)\left(s^2+4\right)\left(s^2+9\right) + (Cs+D)\left(s^2+1\right)\left(s^2+9\right) + (Es+F)\left(s^2+1\right)\left(s^2+4\right) = 120$$

$$(As+B)\left(s^4+13s^2+36\right) + (Cs+D)\left(s^4+10s^2+9\right) + (Es+F)\left(s^4+5s^2+4\right) = 120$$

$$\begin{cases} s^5 & A+C+E=0 \\ s^4 & B+D+F=0 \\ s^3 & 13A+10C+5E=0 \\ s^2 & 13B+10D+5F=0 \\ s & 36A+9C+4E=0 \\ & 36B+9D+4F=120 \end{cases}$$

$$\Rightarrow \begin{cases} A+C+E=0 \\ 13A+10C+5E=0 \\ 36A+9C+4E=0 \end{cases} \rightarrow A+C+E=0$$

$$\Rightarrow \begin{cases} B+D+F=0 \\ 13B+10D+5F=0 \\ 36B+9D+4F=120 \end{cases} \rightarrow B=5;\ D=-8;\ F=3$$

$$Y(s)=\frac{120\left(s^2+3\right)}{\left(s^2+1\right)\left(s^2+4\right)\left(s^2+9\right)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+4}+\frac{Es+F}{s^2+9}$$

$$(As+B)\left(s^2+4\right)\left(s^2+9\right)+(Cs+D)\left(s^2+1\right)\left(s^2+9\right)+(Es+F)\left(s^2+1\right)\left(s^2+4\right)=120s^2+360$$

$$\begin{cases} s^5 & A+C+E=0 \\ s^4 & B+D+F=0 \\ s^3 & 13A+10C+5E=0 \\ s^2 & 13B+10D+5F=120 \\ s & 36A+9C+4E=0 \\ & 36B+9D+4F=360 \end{cases}$$

$$\Rightarrow \begin{cases} A+C+E=0 \\ 13A+10C+5E=0 \\ 36A+9C+4E=0 \end{cases} \rightarrow A+C+E=0$$

$$\Rightarrow \begin{cases} B+D+F=0 \\ 13B+10D+5F=120 \\ 36B+9D+4F=360 \end{cases} \rightarrow B=10;\ D=8;\ F=-18$$

$$X(s)=\frac{5}{s^2+1}-\frac{8}{s^2+4}+\frac{3}{s^2+9}$$

$$\mathcal{L}^{-1}\{X(s)\}=\mathcal{L}^{-1}\left\{5\frac{1}{s^2+1}-4\frac{2}{s^2+4}+\frac{3}{s^2+9}\right\}$$

$$\underline{x(t)=5\sin t-4\sin 2t+\sin 3t}$$

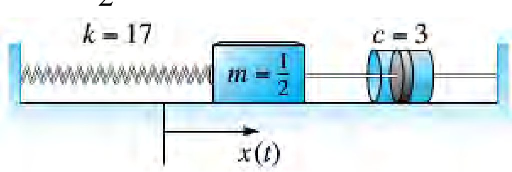
$$Y(s)=\frac{10}{s^2+1}+\frac{8}{s^2+4}-\frac{18}{s^2+9}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{10\frac{1}{s^2+1}+4\frac{2}{s^2+4}-6\frac{3}{s^2+9}\right\}$$

$$\underline{y(t)=10\sin t+4\sin 2t-6\sin 3t}$$

Exercise

Consider a mass-spring system with $m=\frac{1}{2},\quad k=17,\quad and\ c=3$.



Let $x(t)$ be the displacement of the mass m from its equilibrium position. If the mass is set in motion with $x(0)=3\ and\ x'(0)=1$, find $x(t)$ for the resulting damped free oscillations.

Solution

$$\frac{1}{2}x''+3x'+17x=0 \qquad mx''+cx'+kx=0$$

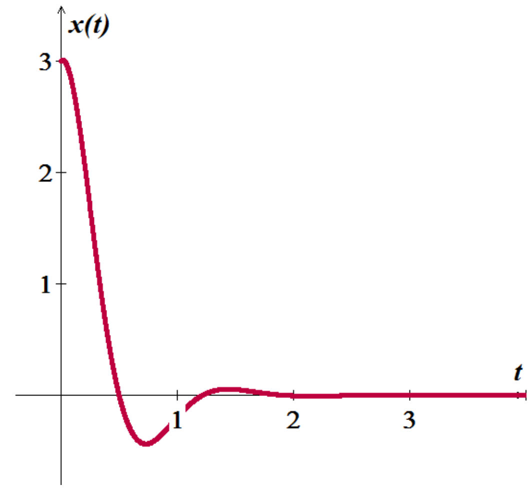
$$x''+6x'+34x=0 \qquad x(0)=3;\ x'(0)=1$$

$\mathcal{L}\{x'' + 6x' + 34x\} = 0$

$s^2 X(s) - sx(0) - x'(0) + 6sX(s) - 6x(0) + 34X(s) = 0$

$s^2 X(s) - 3s - 1 + 6sX(s) - 18 + 34X(s) = 0$

$\left(s^2 + 6s + 34\right) X(s) = 3s + 19$



$$X(s) = \frac{3s+19}{s^2 + 6s + 34}$$

$$= \frac{3s+19}{(s+3)^2 + 25}$$

$$= \frac{3s+9+10}{(s+3)^2 + 25}$$

$$= \frac{3(s+3)}{(s+3)^2 + 25} + \frac{10}{(s+3)^2 + 25}$$

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{3 \cdot \frac{(s+3)}{(s+3)^2 + 25} + 5 \cdot \frac{2}{(s+3)^2 + 25}\right\}$$

$$\boxed{x(t) = (3\cos 5t + 2\sin 5t)e^{-3t}}$$

Exercise

A $4-lb$ weight stretches a spring 2 *feet*. The weight is released from rest 18 *inches* above the equilibrium position, and the resulting motion takes place in a medium offering a damping force numerically equal to $\frac{7}{8}$ times the instantaneous velocity. Use the Laplace transform to find the equation of motion $x(t)$.

Solution

$m = \frac{4}{32} = \frac{1}{8} \qquad (w = mg)$

$k = \frac{4}{2} = 2 \qquad (xk = mg)$

$c = \frac{7}{8}$

$\frac{1}{8}x'' + \frac{7}{8}x' + 2x = 0 \qquad\qquad mx'' + cx' + kx = f(t)$

$x'' + 7x' + 16x = 0\ ; \quad x(0) = -\frac{18}{12} = -\frac{3}{2}, \quad x'(0) = 0$

$\mathcal{L}\{x'' + 7x' + 16x\} = 0$

$s^2 X(s) - sx(0) - x'(0) + 7sX(s) - 7x(0) + 16X(s) = 0$

$\left(s^2 + 7s + 16\right) X(s) = -\frac{3}{2}s - \frac{21}{2}$

$$X(s) = -\frac{3}{2}\frac{s+7}{\left(s+\frac{7}{2}\right)^2+\frac{15}{4}}$$

$$\frac{s+7}{\left(s+\frac{7}{2}\right)^2+\frac{15}{4}} = \frac{A\left(s+\frac{7}{2}\right)}{\left(s+\frac{7}{2}\right)^2+\frac{15}{4}} + \frac{B}{\left(s+\frac{7}{2}\right)^2+\frac{15}{4}}$$

$$\begin{cases} s & \underline{A=1} \\ s^0 & \frac{7}{2}A+B=7 \quad \underline{B=\frac{7}{2}} \end{cases}$$

$$\{X(s)\} = -\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{s+\frac{7}{2}}{\left(s+\frac{7}{2}\right)^2+\left(\frac{\sqrt{15}}{2}\right)^2}\right\} - \frac{3}{2}\frac{7}{2}\frac{2}{\sqrt{15}}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{15}}{2}}{\left(s+\frac{7}{2}\right)^2+\left(\frac{\sqrt{15}}{2}\right)^2}\right\}$$

$$\underline{x(t) = -\frac{3}{2}e^{7t/2}\cos\frac{\sqrt{15}}{2}t - \frac{7\sqrt{15}}{10}e^{7t/2}\sin\frac{\sqrt{15}}{2}t}$$

Exercise

Consider a mass-spring-dashpot system with $m=\frac{1}{2}$, $k=17$, $c=3$, *and* $f(t)=15\sin 2t$ with initial conditions $x(0)=x'(0)=0$. Let $x(t)$ be the displacement of the mass *m* from its equilibrium position. Find the resulting transient motion and steady periodic motion of the mass..

Solution

$$\frac{1}{2}x''+3x'+17x=15\sin 2t \qquad mx''+cx'+kx=0$$

$$x''+6x'+34x=30\sin 2t \qquad x(0)=x'(0)=0$$

$$\mathcal{L}\{x''+6x'+34x\} = \mathcal{L}\{30\sin 2t\}$$

$$s^2X(s)-sx(0)-x'(0)+6sX(s)-6x(0)+34X(s) = \frac{60}{s^2+4}$$

$$\left(s^2+6s+34\right)X(s) = \frac{60}{s^2+4}$$

$$X(s) = \frac{60}{\left(s^2+4\right)\left((s+3)^2+25\right)} = \frac{As+B}{s^2+4}+\frac{Cs+D}{(s+3)^2+25}$$

$$As^3+6As^2+34sA+Bs^2+6sB+34B+Cs^3+4Cs+Ds^2+4D=60$$

$$\begin{cases} s^3 & A+C=0 \\ s^2 & 6A+B+D=0 \\ s^1 & 34A+6B+4C=0 \\ s^0 & 34B+4D=60 \end{cases}$$

$$\begin{cases} C=-A & 6A-\frac{15}{2}B=-15 \\ & 30A+6B=0 \\ D=15-\frac{17}{2}B & \end{cases} \rightarrow \begin{cases} -30A+\frac{75}{2}B=75 \\ 30A+6B=0 \end{cases} \quad \boxed{B=\frac{150}{87}=\frac{50}{29}}$$

$$A=-\frac{10}{29};\quad B=\frac{50}{29};\quad C=\frac{10}{29};\quad D=\frac{10}{29}$$

$$X(s)=\frac{10}{29}\left(\frac{-s+5}{s^2+4}+\frac{s+1}{(s+3)^2+25}\right)$$

$$=\frac{10}{29}\left(\frac{5}{s^2+4}-\frac{s}{s^2+4}+\frac{s+3}{(s+3)^2+25}-\frac{2}{(s+3)^2+25}\right)$$

$$\mathcal{L}^{-1}\{X(s)\}=\frac{10}{29}\mathcal{L}^{-1}\left(\frac{5}{2}\cdot\frac{2}{s^2+4}-\frac{s}{s^2+4}+\frac{s+3}{(s+3)^2+25}-\frac{2}{5}\cdot\frac{5}{(s+3)^2+25}\right)$$

$$x(t)=\frac{10}{29}\left(\frac{5}{2}\sin 2t-\cos 2t+e^{-3t}\left(\cos 5t-\frac{2}{5}\sin 5t\right)\right)$$

$$\boxed{=\frac{5}{29}(5\sin 2t-2\cos 2t)+\frac{2}{29}e^{-3t}(5\cos 5t-2\sin 5t)}$$

Exercise

A 8-*kg* mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by 3 *N-sec/m* and the spring constant is 40 *N/m*. If the mass is driven by an external force equal to $f(t)=2\sin 2t\cos 2t$ *N*. Find the solution.

Solution

Given: $m=8 \quad k=40 \quad c=3$

$$my''+cy'+ky=F(t)$$

$$8y''+3y'+40y=2\sin 2t\cos 2t$$
$$=\sin 4t$$

$$8y''+3y'+40y=\sin 4t\ ;\quad y(0)=0,\quad y'(0)=0$$

$$\mathcal{L}\{8y''+3y'+40y\}=\mathcal{L}\{\sin 4t\}$$

$$8s^2Y(s)-8sy(0)-8y'(0)+3sY(s)-3y(0)+40Y(s)=\frac{4}{s^2+16}$$

$$\left(8s^2+3s+40\right)Y(s)=\frac{4}{s^2+16}$$

$$Y(s)=\frac{4}{8\left(s^2+\frac{3}{8}s+5\right)\left(s^2+16\right)}$$

$$=\frac{\frac{1}{2}}{\left(\left(s+\frac{3}{16}\right)^2+\frac{1271}{256}\right)\left(s^2+16\right)}=\frac{A\left(s+\frac{3}{16}\right)+B}{\left(s+\frac{3}{16}\right)^2+\frac{1271}{256}}+\frac{Cs+D}{s^2+16}$$

$$\begin{array}{ll} s^3 & A+C=0 \\ s^2 & \frac{3}{16}A+B+\frac{3}{8}C+D=0 \\ s & 16A+5C+\frac{3}{8}D=0 \\ s^0 & 3A+16B+5D=\frac{1}{2} \end{array}$$

$$A=\frac{3}{1972}\quad B=\frac{1417}{31552}\quad C=-\frac{3}{1972}\quad D=-\frac{22}{493}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\left\{\frac{3}{1972}\frac{s+\frac{3}{16}}{\left(s+\frac{3}{16}\right)^2+\frac{1271}{256}}+\frac{1417}{31552}\frac{1}{\left(s+\frac{3}{16}\right)^2+\frac{1271}{256}}-\frac{3}{1972}\frac{s}{s^2+16}-\frac{22}{493}\frac{1}{s^2+16}\right\}$$

$$y(t)=\frac{3}{1972}e^{-3t/16}\cos\frac{\sqrt{1271}}{16}t+\frac{1417}{31552}\frac{16}{\sqrt{1271}}e^{-3t/16}\sin\frac{\sqrt{1271}}{16}t-\frac{3}{1972}\cos 4t-\frac{22}{493}\frac{1}{4}\sin 4t$$

$$=\frac{3}{1972}e^{-3t/16}\cos\frac{\sqrt{1271}}{16}t+\frac{1417}{1972\sqrt{1271}}e^{-3t/16}\sin\frac{\sqrt{1271}}{16}t-\frac{3}{1972}\cos 4t-\frac{11}{986}\sin 4t$$

Exercise

A 2-*kg* mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by $c=8$ *kg/sec* and the spring constant is $k=80$ *N/m*. At time $t=0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t)=20e^{-t}$ N. (*t* in *seconds*). Find the equation of motion.

Solution

$$2y''+8y'+80y=20e^{-t}\ ;\quad y(0)=0,\quad y'(0)=0 \qquad\qquad my''+cy'+ky=F(t)$$

$$\mathcal{L}^{-1}\{2y''+8y'+80y\}=\mathcal{L}^{-1}\left\{20e^{-t}\right\}$$

$$2s^2Y(s)-2sy(0)-2y'(0)+8sY(s)-8y(0)+80Y(s)=\frac{20}{s+1}$$

$$2\left(s^2+4s+40\right)Y(s)=\frac{20}{s+1}$$

$$Y(s)=\frac{10}{(s+1)\left((s+2)^2+36\right)}=\frac{A}{s+1}+\frac{B(s+2)+C}{(s+2)^2+36}$$

$$\begin{cases} s^2 & A+B=0 \\ s & 4A+3B+C=0 \\ s^0 & 40A+2B+C=10 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ 4 & 3 & 1 \\ 40 & 2 & 1 \end{vmatrix} = 37 \quad \Delta_A = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 3 & 1 \\ 10 & 2 & 1 \end{vmatrix} = 10$$

$$\boxed{A = \frac{10}{37} \quad B = -\frac{10}{37} \quad C = -\frac{10}{37}}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{10}{37}\frac{1}{s+1} - \frac{10}{37}\frac{(s+2)}{(s+2)^2+6^2} - \frac{10}{37}\frac{1}{6}\frac{6}{(s+2)^2+6^2}\right\}$$

$$\boxed{y(t) = \frac{10}{37}e^{-t} - \left(\frac{10}{37}\cos 6t + \frac{5}{111}\sin 6t\right)e^{-2t}}$$

Exercise

A 10-*kg* mass is attached to a spring having a spring constant of 140 *N/m*. The mass is started in motion initially from the equilibrium position with an initial velocity 1 *m/sec* in the upward direction and with an applied external force $F(t) = 5\sin t$. If the force due to air resistance is $-90y'$ *N*. Find the equation motion of the mass.

Solution

$$10y'' + 90y' + 140y = 5\sin t$$

$$y'' + 9y' + 14y = \frac{1}{2}\sin t\ ; \quad y(0) = 0, \quad y'(0) = -1$$

$$s^2Y(s) - sy(0) - y'(0) + 9sY(s) - 9y(0) + 14Y(s) = \frac{1}{2}\frac{1}{s^2+1}$$

$$\left(s^2 + 9s + 14\right)Y(s) = \frac{1}{2}\frac{1}{s^2+1} - 1$$

$$Y(s) = \frac{-s^2 - \frac{1}{2}}{(s+2)(s+7)\left(s^2+1\right)} = \frac{A}{s+2} + \frac{B}{s+7} + \frac{Cs+D}{s^2+1}$$

$$\begin{cases} s^3 & A+B+C=0 \\ s^2 & 7A+2B+9C+D=-1 \\ s & A+B+14C+9D=0 \\ s^0 & 7A+2B+14D=-\frac{1}{2} \end{cases}$$

$$\boxed{A = -\frac{9}{50} \quad B = \frac{99}{500} \quad C = -\frac{9}{500} \quad D = \frac{13}{500}}$$

$$\{Y(s)\} = \left\{-\frac{9}{50}\frac{1}{s+2} + \frac{99}{500}\frac{1}{s+7} - \frac{9}{500}\frac{s}{s^2+1} + \frac{13}{500}\frac{1}{s^2+1}\right\}$$

$$y(t) = -\frac{9}{50}e^{-2t} + \frac{99}{500}e^{-7t} - \frac{9}{500}\cos t + \frac{13}{500}\sin t$$

$$\boxed{= \frac{1}{500}\left(99e^{-7t} - 90e^{-2t} + 13\sin t - 9\cos t\right)}$$

Exercise

A 128-*lb* weight is attached to a spring having a spring constant of 64 *lb/ft*. The weight is started in motion initially by displacing it 6 *in* above the equilibrium position with no initial velocity and with an applied external force $F(t) = 8\sin 4t$. Assume no air resistance. Find the equation motion of the mass.

Solution

$$\underline{m = \frac{128}{32} = 4}$$

$$4y'' + 64y = 8\sin 4t$$

$$y'' + 16y = 2\sin 4t\ ; \quad y(0) = -\frac{6}{12} = -\frac{1}{2}, \quad y'(0) = 0$$

$$s^2Y(s) - sy(0) - y'(0) + 16Y(s) = \frac{8}{s^2+16}$$

$$\left(s^2+16\right)Y(s) = \frac{8}{s^2+16} - \frac{1}{2}s$$

$$Y(s) = \frac{8}{\left(s^2+16\right)^2} - \frac{1}{2}\frac{s}{s^2+16}$$

$$\frac{8}{\left(s^2+16\right)^2} = \frac{As+B}{s^2+16} + \frac{C\left(s^2-16\right)}{\left(s^2+16\right)^2} + \frac{Ds}{\left(s^2+16\right)^2}$$

$s^3 \quad A = 0$

$s^2 \quad B + C = 0$

$s \quad 16A + D = 0$

$s^0 \quad 16B - 16C = 8$

$$\underline{A = 0 \quad B = \frac{1}{4} \quad C = -\frac{1}{4} \quad D = 0}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s^2+16} - \frac{1}{4}\frac{s^2-16}{\left(s^2+16\right)^2} - \frac{1}{2}\frac{s}{s^2+16}\right\}$$

$$\underline{y(t) = \frac{1}{16}\sin 4t - \frac{1}{4}t\cos 4t - \frac{1}{2}\cos 4t}$$

Exercise

Find the motion of a damped mass-and-spring system with $m = 1$, $c = 2$, and $k = 26$ under the influence of an external force $F(t) = 82\cos 4t$ with $x(0) = 6$ and $x'(0) = 0$.

Solution

Given: $m = 1$, $c = 2$, $k = 26$, and $F(t) = 82\cos 4t \qquad x(0) = 6;\ x'(0) = 0$

$$x'' + 2x' + 26x = 82\cos 4t \qquad mx'' + cx' + kx = F(t)$$

$$\mathcal{L}\{x''+2x'+26x\}=\mathcal{L}\{82\cos 4t\}$$

$$s^2X(s)-sx(0)-x'(0)+2sX(s)-2x(0)+26X(s)=\frac{82s}{s^2+16}$$

$$\left(s^2+2s+26\right)X(s)=\frac{82s}{s^2+16}+6s+12$$

$$X(s)=\frac{6s^3+12s^2+178s+192}{\left(s^2+16\right)\left((s+1)^2+25\right)}=\frac{As+B}{s^2+16}+\frac{C(s+1)+D}{(s+1)^2+25}$$

$$\begin{cases} s^3 & A+C=6 \\ s^2 & 2A+B+C+D=12 \\ s & 26A+2B+16C=178 \\ s^0 & 26B+16C+16D=192 \end{cases} \qquad \underline{A=5 \quad B=16 \quad C=1 \quad D=-15}$$

$$\mathcal{L}^{-1}\{X(s)\}=\mathcal{L}^{-1}\left\{\frac{5s}{s^2+16}+\frac{16}{s^2+16}+\frac{s+1}{(s+1)^2+25}-\frac{15}{(s+1)^2+5^2}\right\}$$

$$\underline{x(t)=5\cos 4t+4\sin 4t+e^{-t}(\cos 5t-3\sin 5t)}$$

Exercise

A spring with a mass of 2-*kg* has natural length 0.5 *m*. A force of 25.6 *N* is required to maintain it stretched to a length of 0.7 *m*. The spring is immersed in a fluid with damping constant $c=40$. If the spring is started from the equilibrium position and is given a push to start it with initial velocity 0.6 *m/s*. Find the position of the mass at any time *t*.

Solution

$$k=\frac{25.6}{0.7-0.5}\ \underline{=128} \qquad\qquad k\left(x_2-x_1\right)=F$$

$$2x''+40x+128=0\ ; \quad x(0)=0, \quad x'(0)=0.6$$

$$\mathcal{L}\{x''+20x+64\}=0$$

$$s^2X(s)-sx(0)-x'(0)+20sX(s)-20x(0)+64X(s)=0$$

$$\left(s^2+20s+64\right)X(s)=\frac{6}{10}$$

$$X(s)=\frac{3}{5}\frac{1}{(s+16)(s+4)}=\frac{3}{5}\left(\frac{A}{s+16}+\frac{B}{s+4}\right)$$

$$\begin{matrix} s & A+B=0 \\ s^0 & 4A+16B=1 \end{matrix} \quad \rightarrow \quad \underline{A=-\frac{1}{12},\ B=\frac{1}{12}}$$

$$\mathcal{L}^{-1}\{X(s)\}=\frac{3}{5}\mathcal{L}^{-1}\left\{-\frac{1}{12}\frac{1}{s+16}+\frac{1}{12}\frac{1}{s+4}\right\}$$

$$x(t)=\frac{3}{5}\left(-\frac{1}{12}e^{-16t}+\frac{1}{12}e^{-t}\right)$$

$$=\frac{1}{20}e^{-4t}-\frac{1}{20}e^{-16t}$$

Exercise

A spring with a mass of 3-*kg* is held stretched 0.6 *m* beyond its natural length by a force of 20 *N*. If the spring begins at its equilibrium and with initial velocity 1.2 *m/s*. Find the position of the mass.

Solution

$$k=\frac{20}{0.6}=\frac{100}{3} \qquad kx=F$$

$$3x''+\frac{100}{3}x=0\ ; \quad x(0)=0, \quad x'(0)=1.2=\frac{6}{5} \qquad mx''+cx'+kx=F(t)$$

$$\mathcal{L}\{9x''+100x\}=0$$

$$9s^2X(s)-9sx(0)-9x'(0)+100X(s)=0$$

$$\left(9s^2+100\right)X(s)=\frac{36}{5}$$

$$\mathcal{L}^{-1}\{X(s)\}=\frac{6}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+\frac{100}{9}}\right\}$$

$$x(t)=\frac{6}{5}\frac{3}{10}\sin\frac{10}{3}t$$

$$=\frac{9}{25}\sin\frac{10}{3}t$$

Exercise

A spring with a mass of 2-*kg* is held stretched 0.5 *m*, has damping constant 14, and a force of 6 *N*. If the spring is stretched 1 *m* beyond at its equilibrium and with no initial velocity. Find the position of the mass at any time *t*.

Solution

$$k=\frac{6}{.5}=12 \qquad kx=F$$

$$2x''+14x'+12x=0\ ; \quad x(0)=1, \quad x'(0)=0 \qquad mx''+cx'+kx=F(t)$$

$$\mathcal{L}\{x''+7x'+6x\}=0$$

$$s^2X(s)-sx(0)-x'(0)+7sX(s)-7x(0)+6X(s)=0$$

$$\left(s^2+7s+6\right)X(s)=s+7$$

$$X(s)=\frac{s+7}{(s+1)(s+6)}=\frac{A}{s+1}+\frac{B}{s+6}$$

$$s + 7 = As + 6A + Bs + B$$

$$\begin{cases} s & A + B = 1 \\ s^0 & 6A + B = 7 \end{cases} \rightarrow \boxed{A = \frac{6}{5},\ B = -\frac{1}{5}}$$

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{6}{5}\frac{1}{s+1} - \frac{1}{5}\frac{1}{s+6}\right\}$$

$$\boxed{x(t) = \frac{6}{5}e^{-t} - \frac{1}{5}e^{-6t}}$$

Exercise

Find the charge $q(t)$ on the capacitor in an LRC–series circuit when $L = 0.25\ H$, $R = 10\ \Omega$, $C = 0.001\ F$, $E(t) = 0$, $q(0) = q_0\ C$, and $i(0) = 0$.

Solution

$$0.25q'' + 10q' + \frac{1}{0.001}q = 0 \qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$\mathcal{L}\{q'' + 40q' + 4{,}000q\} = 0$$

$$s^2Q(s) - sq(0) - q'(0) + 40sQ(s) - 40q(0) + 4{,}000Q(s) = 0 \qquad q(0) = q_0 \quad q'(0) = 0$$

$$\left(s^2 + 40s + 4000\right)Q(s) = sq_0 + 40q_0$$

$$Q(s) = q_0\frac{s + 40}{(s+20)^2 + 3600}$$

$$\mathcal{L}^{-1}\{Q(s)\} = \mathcal{L}^{-1}\left\{q_0\frac{s+20}{(s+20)^2 + 60^2} + q_0\frac{20}{(s+20)^2 + 60^2}\right\}$$

$$\boxed{q(t) = q_0e^{-20t}\left(\cos 60t + \frac{1}{3}\sin 60t\right)}$$

Exercise

Find the charge $q(t)$ on the capacitor in an LRC–series circuit at $t = 0.01\ sec$ when $L = 0.05\ h$, $R = 2\ \Omega$, $C = 0.01\ f$, $E(t) = 0$, $q(0) = 5\ C$, and $i(0) = 0\ A$.

Solution

$$0.05q'' + 2q' + \frac{1}{0.01}q = 0 \qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 40q' + 2{,}000q = 0$$

$$s^2Q(s) - sq(0) - q'(0) + 40sQ(s) - 40q(0) + 2{,}000Q(s) = 0 \qquad q(0) = 5 \quad q'(0) = 0$$

$$\left(s^2 + 40s + 2{,}000\right)Q(s) = 5s + 200$$

$$Q(s)=\frac{5(s+40)}{(s+20)^2+1600}$$

$$\mathcal{L}^{-1}\{Q(s)\}=5\mathcal{L}^{-1}\left\{\frac{s+20}{(s+20)^2+40^2}+\frac{20}{(s+20)^2+40^2}\right\}$$

$$\boxed{q(t)=\left(5\cos 40t+\frac{5}{2}\sin 40t\right)e^{-20t}}$$

Exercise

Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit when $L=\frac{5}{3}\ h$, $R=10\ \Omega$, $C=\frac{1}{30}\ f$, $E(t)=0$, $q(0)=4\ C$, and $i(0)=0\ A$.

Solution

$$\frac{5}{3}q''+10q'+30q=0 \qquad Lq''+Rq'+\frac{1}{C}q=E(t)$$

$$q''+6q'+18q=0$$

$$s^2Q(s)-sq(0)-q'(0)+6sQ(s)-6q(0)+18Q(s)=0 \qquad q(0)=4 \quad q'(0)=0$$

$$\left(s^2+6s+18\right)Q(s)=4s+24$$

$$Q(s)=\frac{4(s+6)}{(s+3)^2+9}$$

$$\mathcal{L}^{-1}\{Q(s)\}=4\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+3^2}+\frac{3}{(s+3)^2+3^2}\right\}$$

$$\boxed{q(t)=e^{-3t}(4\cos 3t+4\sin 3t)}$$

Exercise

Find the current $i(t)$ in an *LRC*–series circuit when $L=1\ h$, $R=20\ \Omega$, $C=0.005\ f$, $E(t)=150\ V$, $q(0)=0\ C$, and $i(0)=0\ A$.

Solution

$$q''+20q'+\frac{1}{.005}q=150\ ; \quad q(0)=0 \quad q'(0)=0 \qquad Lq''+Rq'+\frac{1}{C}q=E(t)$$

$$\mathcal{L}\{q''+20q'+200q\}=\mathcal{L}\{150\}$$

$$s^2Q(s)-sq(0)-q'(0)+20sQ(s)-20q(0)+200Q(s)=\frac{150}{s}$$

$$\left(s^2+20s+200\right)Q(s)=\frac{150}{s}$$

$$Q(s)=\frac{150}{s\left((s+10)^2+100\right)}=\frac{A}{s}+\frac{B(s+10)+C}{(s+10)^2+100}$$

$$\begin{cases} s^2 & A+B=0 \\ s & 20A+10B+C=0 \\ s^0 & 200A=150 \end{cases} \quad\rightarrow\quad \boxed{A=\frac{3}{4},\ B=-\frac{3}{4},\ C=-\frac{15}{2}}$$

$$Q(s)=\frac{3}{4}\frac{1}{s}-\frac{3}{4}\frac{s+10}{(s+10)^2+10^2}-\frac{15}{2}\frac{1}{(s+10)^2+10^2}$$

$$\boxed{q(t)=\frac{3}{4}-\frac{3}{4}e^{-10t}\cos 10t-\frac{3}{4}e^{-10t}\sin 10t}$$

$$i(t)=q'(t)=\frac{15}{2}e^{-10t}\cos 10t+\frac{15}{2}e^{-10t}\sin 10t+\frac{15}{2}e^{-10t}\sin 10t-\frac{15}{2}e^{-10t}\cos 10t$$

$$\boxed{=15e^{-10t}\sin 10t}$$

Exercise

A resistor $R=20\ \Omega$ and a capacitor of $C=0.1\ F$ are joined in series with an electronic force (emf) $E=E(t)$ and no charge on the capacitor at $t=0$. Find the ensuing charge on the capacitor at time t for the given $E(t)=100\sin 2t$

Solution

$$R\frac{dQ}{dt}+\frac{1}{C}Q=E$$

$$20Q'+\frac{1}{0.1}Q=100\sin 2t$$

$$\mathcal{L}(Q'+0.5Q)=5\mathcal{L}(\sin 2t)$$

$$sQ(s)-Q(0)+0.5Q(s)=5\frac{2}{s^2+4} \qquad Q(0)=0$$

$$(s+0.5)Q(s)=10\frac{1}{s^2+4}$$

$$Q(s)=10\frac{1}{(s+0.5)\left(s^2+4\right)}$$

$$\frac{1}{(s+0.5)\left(s^2+4\right)}=\frac{A}{s+0.5}+\frac{Bs+C}{s^2+4}$$

$$1=As^2+4A+Bs^2+0.5Bs+Cs+0.5C$$

$$1=(A+B)s^2+(0.5B+C)s+4A+0.5C$$

$$\begin{cases} A+B=0 \\ 0.5B+C=0 \\ 4A+0.5C=1 \end{cases} \Rightarrow \boxed{A=\frac{4}{17}} \quad \boxed{B=-\frac{4}{17}} \quad \boxed{C=\frac{2}{17}}$$

$$Q(s) = 10\left(\frac{4}{17}\frac{1}{s+\frac{1}{2}} - \frac{4}{17}\frac{s}{s^2+4} + \frac{2}{17}\frac{1}{s^2+4} \right)$$

$$= \frac{1}{17}\left(40\frac{1}{s+\frac{1}{2}} - 40\frac{s}{s^2+4} + 10\frac{2}{s^2+4} \right)$$

$$Q(t) = \frac{1}{17}\left(40\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\} - 40\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 10\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \right)$$

$$= \frac{1}{17}\left(40e^{-t/2} - 40\cos 2t + 10\sin 2t \right)$$

Exercise

A resistor $R = 20\ \Omega$ and a capacitor of $C = 0.1\ F$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge on the capacitor at time *t* for the given $E(t) = 100e^{-0.1t}$

Solution

$$20Q' + \frac{1}{0.1}Q = 100e^{-0.1t}$$

$$\mathcal{L}\left(Q' + 0.5Q\right) = 5\mathcal{L}\left(e^{-0.1t}\right)$$

$$sQ(s) - Q(0) + 0.5Q(s) = 5\frac{1}{s+0.1} \qquad Q(0) = 0$$

$$(s+0.5)Q(s) = 5\frac{1}{s+0.1}$$

$$Q(s) = 5\frac{1}{(s+0.5)(s+0.1)}$$

$$\frac{1}{(s+0.5)(s+0.1)} = \frac{A}{s+0.5} + \frac{B}{s+0.1}$$

$$1 = (A+B)s + 0.1A + 0.5B$$

$$\begin{cases} A + B = 0 \\ 0.1A + 0.5B = 1 \end{cases} \Rightarrow A = -\frac{5}{2} \quad B = \frac{5}{2}$$

$$Q(s) = 5\left(-\frac{5}{2}\frac{1}{s+\frac{1}{2}} + \frac{5}{2}\frac{1}{s+0.1} \right) = \frac{25}{2}\left(-\frac{1}{s+\frac{1}{2}} + \frac{1}{s+\frac{1}{10}} \right)$$

$$Q(t) = \frac{25}{2}\left(\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{10}}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\} \right)$$

$$= \frac{25}{2}\left(e^{-t/10} - e^{-t/2} \right)$$

Exercise

A resistor $R = 20\ \Omega$ and a capacitor of $C = 0.1\ F$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge on the capacitor at time t for the given $E(t) = 100\left(1 - e^{-0.1t}\right)$

Solution

$$20Q' + \frac{1}{0.1}Q = 100\left(1 - e^{-0.1t}\right)$$

$$\mathcal{L}\left(Q' + 0.5Q\right) = 5\mathcal{L}\left(1 - e^{-0.1t}\right)$$

$$sQ(s) - Q(0) + 0.5Q(s) = 5\left(\frac{1}{s} - \frac{1}{s+0.1}\right) \qquad Q(0) = 0$$

$$(s+0.5)Q(s) = 5\left(\frac{1}{s} - \frac{1}{s+0.1}\right)$$

$$Q(s) = 5\frac{1}{s+0.5}\left(\frac{s+0.1-s}{s(s+0.1)}\right)$$

$$Q(s) = 0.5\frac{1}{s(s+0.5)(s+0.1)}$$

$$\frac{1}{s(s+0.5)(s+0.1)} = \frac{A}{s} + \frac{B}{s+0.5} + \frac{C}{s+0.1}$$

$$1 = A\left(s^2 + 0.6s + 0.05\right) + B\left(s^2 + 0.1s\right) + C\left(s^2 + 0.5s\right)$$

$$1 = (A+B+C)s^2 + (0.6A + 0.1B + 0.5C)s + 0.05A$$

$$\begin{cases} A+B+C = 0 \\ 0.6A + 0.1B + 0.5C = 0 \\ 0.05A = 1 \end{cases} \Rightarrow A = \frac{1}{0.05} = 20 \quad B = 5 \quad C = -25$$

$$Q(s) = \frac{1}{2}\left(20\frac{1}{s} + 5\frac{1}{s+0.5} - 25\frac{1}{s+0.1}\right)$$

$$= 10\frac{1}{s} + \frac{5}{2}\frac{1}{s+\frac{1}{2}} - \frac{25}{2}\frac{1}{s+\frac{1}{10}}$$

$$Q(t) = \left(10\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\} - \frac{25}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{10}}\right\}\right)$$

$$\boxed{= 10 + \frac{5}{2}e^{-t/2} - \frac{25}{2}e^{-t/10}}$$

Exercise

A resistor $R = 20\ \Omega$ and a capacitor of $C = 0.1\ F$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge on the capacitor at time t for the given $E(t) = 100\cos 3t$

Solution

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E$$

$$20Q' + \frac{1}{0.1}Q = 100\cos 3t$$

$$Q' + \frac{1}{0.1(20)}Q = \frac{100}{20}\cos 3t$$

$$\mathcal{L}\left(Q' + 0.5Q\right) = 5\mathcal{L}\left(\cos 3t\right)$$

$$sQ(s) - Q(0) + 0.5Q(s) = 5\frac{s}{s^2+9} \qquad Q(0) = 0$$

$$(s+0.5)Q(s) = 5\frac{s}{s^2+9}$$

$$Q(s) = 5\frac{s}{(s+0.5)\left(s^2+9\right)}$$

$$\frac{s}{(s+0.5)\left(s^2+9\right)} = \frac{A}{s+0.5} + \frac{Bs+C}{s^2+9}$$

$$s = As^2 + 9A + Bs^2 + 0.5Bs + Cs + 0.5C$$

$$s = (A+B)s^2 + (0.5B + C)s + 9A + 0.5C$$

$$\begin{cases} A + B = 0 \\ 0.5B + C = 1 \\ 9A + 0.5C = 0 \end{cases} \Rightarrow A = -\frac{2}{37} \quad B = \frac{2}{37} \quad C = \frac{36}{37}$$

$$Q(s) = 5\left(-\frac{2}{37}\frac{1}{s+\frac{1}{2}} + \frac{2}{37}\frac{s}{s^2+9} + \frac{36}{37}\frac{1}{s^2+9}\right)$$

$$= \frac{1}{37}\left(-10\frac{1}{s+\frac{1}{2}} + 10\frac{s}{s^2+9} + \frac{5(36)}{3}\frac{3}{s^2+9}\right)$$

$$Q(t) = \frac{1}{37}\left(-10\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\} + 10\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + 60\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}\right)$$

$$\boxed{= \frac{1}{37}\left(-10e^{-t/2} + 10\cos 3t + 60\sin 3t\right)}$$

Exercise

An inductor $(L = 1\ H)$ and a resistor $(R = 0.1\ \Omega)$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge current in the current at time t for the given $E(t) = 10 - 2t$

Solution

$$L\frac{dI}{dt} + RI = E(t)$$

$$\frac{dI}{dt} + 0.1I = 10 - 2t$$

$$\mathcal{L}\left(I' + 0.1I\right) = \mathcal{L}\left(10 - 2t\right)$$

$$sI(s) - I(0) + 0.1I(s) = \frac{10}{s} - \frac{2}{s^2} \qquad I(0) = 0$$

$$\left(s + 0.1\right)I(s) = \frac{10s - 2}{s^2}$$

$$I(s) = \frac{10s - 2}{s^2\left(s + 0.1\right)}$$

$$\frac{10s - 2}{s^2\left(s + 0.1\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s + 0.1}$$

$$10s - 2 = As(s + 0.1) + B(s + 0.1) + Cs^2$$

$$10s - 2 = \left(A + C\right)s^2 + \left(B + 0.1A\right)s + 0.1B$$

$$\begin{cases} A + C = 0 \\ 0.1A + B = 10 \\ 0.1B = -2 \end{cases} \Rightarrow A = 300 \quad B = -20 \quad C = -300$$

$$I(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s + 0.1}$$

$$= 300\frac{1}{s} - 20\frac{1}{s^2} - 300\frac{1}{s + \frac{1}{10}}$$

$$I(t) = \left(3000\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 20\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - 300\mathcal{L}^{-1}\left\{\frac{1}{s + \frac{1}{10}}\right\}\right)$$

$$\underline{= 300 - 20t - 300e^{-t/10}}$$

Exercise

An inductor $(L = 1\ H)$ and a resistor $(R = 0.1\ \Omega)$ are joined in series with an electronic force (*emf*) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing current in the current at time *t* for the given $E(t) = 4\cos 3t$

Solution

$$L\frac{dI}{dt} + RI = E(t)$$

$$\frac{dI}{dt} + 0.1I = 4\cos 3t$$

$$\mathcal{L}\left(I' + 0.1I\right) = 4\mathcal{L}\left(\cos 3t\right)$$

$$sI(s) - I(0) + 0.1I(s) = 4\frac{s}{s^2+9} \qquad I(0) = 0$$

$$\left(s+0.1\right)I(s) = 4\frac{s}{s^2+9}$$

$$I(s) = 4\frac{s}{\left(s+0.1\right)\left(s^2+9\right)}$$

$$\frac{s}{\left(s+0.1\right)\left(s^2+9\right)} = \frac{A}{s+0.1} + \frac{Bs+C}{s^2+9}$$

$$s = As^2 + 9A + Bs^2 + 0.1Bs + Cs + 0.1C$$

$$s = \left(A+B\right)s^2 + \left(0.1B+C\right)s + 9A + 0.1C$$

$$\begin{cases} A+B=0 \\ 0.1B+C=1 \\ 9A+0.1C=0 \end{cases} \Rightarrow A = -\frac{10}{901} \quad B = \frac{10}{901} \quad C = \frac{900}{901}$$

$$I(s) = 4\left(\frac{A}{s+0.1} + \frac{Bs}{s^2+9} + \frac{C}{s^2+9}\right)$$

$$= 4\left(-\frac{10}{901}\frac{1}{s+0.1} + \frac{10}{901}\frac{s}{s^2+9} + \frac{900}{901}\frac{1}{3}\frac{3}{s^2+9}\right)$$

$$I(t) = \frac{1}{901}\left(-40\mathcal{L}^{-1}\left\{\frac{1}{s+0.1}\right\} + 40\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + 1200\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}\right)$$

$$\boxed{= \frac{1}{901}\left(-40e^{-t/10} + 40\cos 3t + 1200\sin 3t\right)}$$

Exercise

An inductor $(L = 1\ H)$ and a resistor $(R = 0.1\ \Omega)$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing current in the current at time *t* for the given $E(t) = 4\sin 2\pi t$

Solution

$$L\frac{dI}{dt} + RI = E(t)$$

$$\frac{dI}{dt} + 0.1I = 4\sin 2\pi t$$

$$\mathcal{L}(I' + 0.1I) = 4\mathcal{L}(\sin 2\pi t)$$

$$sI(s) - I(0) + 0.1I(s) = 4\frac{2\pi}{s^2 + 4\pi^2} \qquad I(0) = 0$$

$$(s + 0.1)I(s) = 8\pi\frac{1}{s^2 + 4\pi^2}$$

$$I(s) = 8\pi\frac{1}{(s + 0.1)(s^2 + 4\pi^2)}$$

$$\frac{1}{(s + 0.1)(s^2 + 4\pi^2)} = \frac{A}{s + 0.1} + \frac{Bs + C}{s^2 + 4\pi^2}$$

$$s = As^2 + 4\pi^2 A + Bs^2 + 0.1Bs + Cs + 0.1C$$

$$s = (A + B)s^2 + (0.1B + C)s + 4\pi^2 A + 0.1C$$

$$\begin{cases} A + B = 0 \\ 0.1B + C = 0 \\ 4\pi^2 A + 0.1C = 1 \end{cases} \Rightarrow A = \frac{100}{1 + 400\pi^2} \quad B = -\frac{100}{1 + 400\pi^2} \quad C = \frac{10}{1 + 400\pi^2}$$

$$I(s) = 8\pi\left(\frac{A}{s + 0.1} + \frac{Bs}{s^2 + 4\pi^2} + \frac{C}{s^2 + 4\pi^2}\right)$$

$$= 8\pi\left(\frac{100}{1 + 400\pi^2}\frac{1}{s + 0.1} - \frac{100}{1 + 400\pi^2}\frac{s}{s^2 + 4\pi^2} + \frac{10}{1 + 400\pi^2}\frac{1}{s^2 + 4\pi^2}\right)$$

$$I(t) = \frac{8}{1 + 400\pi^2}\left(100\pi\mathcal{L}^{-1}\left\{\frac{1}{s + 0.1}\right\} - 100\pi\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 4\pi^2}\right\} + 10\pi\frac{1}{2\pi}\mathcal{L}^{-1}\left\{\frac{2\pi}{s^2 + 4\pi^2}\right\}\right)$$

$$\boxed{= \frac{8}{1 + 400\pi^2}\left(100\pi e^{-t/10} - 100\pi\cos 2\pi t + 5\sin 2\pi t\right)}$$

Exercise

Solve the general initial value problem modeling the *RC* circuit

$$R\frac{dQ}{dt}+\frac{1}{C}Q=E, \quad Q(0)=0$$

Where E is a constant source of emf

Solution

$$\mathcal{L}\left(\frac{dQ}{dt}+\frac{1}{RC}Q\right)=\mathcal{L}\left(\frac{E}{R}\right)$$

$$sQ(s)-Q(0)+\frac{1}{RC}Q(s)=\frac{E}{R}\frac{1}{s} \qquad Q(0)=0$$

$$\left(s+\frac{1}{RC}\right)Q(s)=\frac{E}{R}\frac{1}{s}$$

$$Q(s)=\frac{E}{R}\frac{1}{s\left(s+\frac{1}{RC}\right)} \qquad \frac{1}{s\left(s+\frac{1}{RC}\right)}=\frac{A}{s}+\frac{B}{s+\frac{1}{RC}}=\frac{As+\frac{1}{RC}A+Bs}{s\left(s+\frac{1}{RC}\right)}$$

$$\begin{cases} A+B=0 \\ \frac{1}{RC}A=1 \end{cases} \rightarrow \quad A=RC \quad B=-RC$$

$$Q(t)=\frac{E}{R}\left(RC\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-RC\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{RC}}\right\}\right)$$

$$=\frac{E}{R}\left(RC-RCe^{-t/RC}\right)$$

$$=EC\left(1-e^{-t/RC}\right)$$

Exercise

Solve the general initial value problem modeling the *LR* circuit $L\frac{dI}{dt}+RI=E, \quad I(0)=I_0$

Where E is a constant source of emf

Solution

$$\mathcal{L}\left(\frac{dI}{dt}+\frac{R}{L}I\right)=\mathcal{L}\left(\frac{E}{L}\right)$$

$$sI(s)-I(0)+\frac{R}{L}I(s)=\frac{E}{L}\frac{1}{s} \qquad I(0)=I_0$$

$$\left(s+\frac{R}{L}\right)I(s)=\frac{E}{L}\frac{1}{s}+I_0$$

$$I(s)=\frac{E}{L}\frac{1}{s\left(s+\frac{R}{L}\right)}+I_0\frac{1}{s+\frac{R}{L}}$$

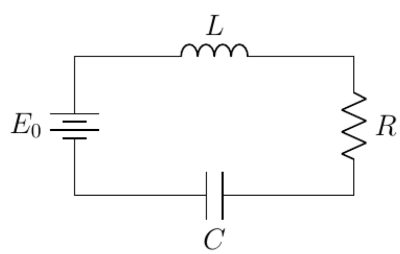
$$\frac{1}{s\left(s+\frac{R}{L}\right)}=\frac{A}{s}+\frac{B}{s+\frac{R}{L}}=\frac{As+\frac{R}{L}A+Bs}{s\left(s+\frac{R}{L}\right)}$$

$$\begin{cases} A+B=0 \\ \frac{R}{L}A=1 \end{cases} \quad \rightarrow \quad A=\frac{L}{R} \quad B=-\frac{L}{R}$$

$$\begin{aligned}
I(t)&=\frac{E}{L}\left(\frac{L}{R}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-\frac{L}{R}\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{R}{L}}\right\}\right)+I_0\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{R}{L}}\right\}\\
&=\frac{E}{L}\left(\frac{L}{R}-\frac{L}{R}e^{-Rt/L}\right)+I_0e^{-Rt/L}\\
&=\frac{E}{R}-\frac{E}{R}e^{-Rt/L}+I_0e^{-Rt/L}\\
&=\frac{1}{R}\left(E-Ee^{-Rt/L}+RI_0e^{-Rt/L}\right)\\
&=\frac{1}{R}\left(E+\left(RI_0-E\right)e^{-Rt/L}\right)
\end{aligned}$$

Exercise

Consider a battery of constant voltage E_0 that charges the capacitor. $L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=E(t)$



Divide the given equation by L and define $2\lambda=\frac{R}{L}$ and $\omega^2=\frac{1}{LC}$.

a) Use the Laplace transform to show that the solution $q(t)$ of $q''+2\lambda q'+\omega^2q=\frac{E_0}{L}$ subject to $q(0)=0$, $i(0)=0$ is

$$q(t)=\begin{cases} E_0C\left[1-e^{-\lambda t}\left(\cosh\sqrt{\lambda^2-\omega^2}\,t+\frac{\lambda}{\sqrt{\lambda^2-\omega^2}}\sinh\sqrt{\lambda^2-\omega^2}\,t\right)\right] & \lambda>\omega \\ E_0C\left[1-e^{-\lambda t}(1+\lambda t)\right] & \lambda=\omega \\ E_0C\left[1-e^{-\lambda t}\left(\cos\sqrt{\omega^2-\lambda^2}\,t+\frac{\lambda}{\sqrt{\omega^2-\lambda^2}}\sin\sqrt{\omega^2-\lambda^2}\,t\right)\right] & \lambda<\omega \end{cases}$$

b) Use the Laplace transform to find the charge $q(t)$ in an *RC* series when $q(0)=0$ and $E(t)=E_0e^{-kt}$, $k>0$. Consider *two* cases: $k\neq\frac{1}{RC}$ and $k=\frac{1}{RC}$

Solution

a) $\mathcal{L}\left\{q'' + 2\lambda q' + \omega^2 q\right\} = \mathcal{L}\left\{\frac{E_0}{L}\right\}$

$$s^2Q(s) - sq(0) - q'(0) + 2\lambda\left(sQ(s) - q(0)\right) + \omega^2 Q(s) = \frac{E_0}{L}\frac{1}{s} \qquad q(0) = 0,\ q'(0) = 0$$

$$\left(s^2 + 2s\lambda + \omega^2\right)Q(s) = \frac{E_0}{L}\frac{1}{s}$$

$$Q(s) = \frac{E_0}{L}\frac{1}{s\left(s^2 + 2s\lambda + \omega^2\right)}$$

$$\frac{1}{s\left(s^2 + 2s\lambda + \omega^2\right)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s\lambda + \omega^2}$$

$$As^2 + 2A\lambda s + \omega^2 A + Bs^2 + Cs = 1$$

$s^2 \quad A + B = 0 \qquad \lfloor B = -\frac{1}{\omega^2}$

$s^1 \quad 2\lambda A + C = 0 \qquad C = -\frac{2\lambda}{\omega^2}$

$s^0 \quad \omega^2 A = 1 \quad \rightarrow A = \frac{1}{\omega^2}$

$$Q(s) = \frac{E_0}{L}\left(\frac{1}{\omega^2}\frac{1}{s} - \frac{1}{\omega^2}\frac{s + 2\lambda}{s^2 + 2\lambda s + \omega^2}\right)$$

For $\lambda > \omega$, then $s^2 + 2\lambda s + \omega^2 = s^2 + 2\lambda s + \lambda^2 - \lambda^2 + \omega^2 = (s+\lambda)^2 - \left(\lambda^2 - \omega^2\right)$

$$Q(s) = \frac{E_0}{L\omega^2}\left(\frac{1}{s} - \frac{s + \lambda + \lambda}{(s+\lambda)^2 - \left(\lambda^2 - \omega^2\right)}\right) \qquad \omega^2 = \frac{1}{LC}$$

$$\mathcal{L}^{-1}\{Q(s)\} = E_0 C\,\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+\lambda}{(s+\lambda)^2 - \left(\lambda^2 - \omega^2\right)} - \frac{\lambda}{(s+\lambda)^2 - \left(\lambda^2 - \omega^2\right)}\right\}$$

$$q(t) = E_0 C\left(1 - e^{-\lambda t}\cosh\sqrt{\lambda^2 - \omega^2}\,t - \frac{\lambda}{\sqrt{\lambda^2 - \omega^2}}e^{-\lambda t}\sinh\sqrt{\lambda^2 - \omega^2}\,t\right)$$

For $\lambda < \omega$, then $s^2 + 2\lambda s + \omega^2 = (s+\lambda)^2 + \left(\omega^2 - \lambda^2\right)$

$$\mathcal{L}^{-1}\{Q(s)\} = E_0 C\,\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+\lambda}{(s+\lambda)^2 + \left(\omega^2 - \lambda^2\right)} - \frac{\lambda}{(s+\lambda)^2 + \left(\omega^2 - \lambda^2\right)}\right\}$$

$$q(t) = E_0 C\left(1 - e^{-\lambda t}\left(\cos\sqrt{\omega^2 - \lambda^2}\,t - \frac{\lambda}{\sqrt{\omega^2 - \lambda^2}}\sin\sqrt{\omega^2 - \lambda^2}\,t\right)\right)$$

For $\lambda = \omega$, then $s^2 + 2\lambda s + \omega^2 = s^2 + 2\lambda s + \lambda^2 = (s+\lambda)^2$

$$\frac{1}{s(s+\lambda)^2} = \frac{A}{s} + \frac{B}{s+\lambda} + \frac{C}{(s+\lambda)^2}$$

$$As^2 + 2A\lambda s + \lambda^2 A + Bs^2 + B\lambda s + Cs = 1$$

$$s^2 \qquad A + B = 0 \qquad \boxed{B = -\frac{1}{\lambda^2}}$$

$$s^1 \qquad 2\lambda A + B\lambda + C = 0 \quad C = -\frac{2}{\lambda} + \frac{1}{\lambda} \quad \boxed{C = -\frac{1}{\lambda}}$$

$$s^0 \qquad \lambda^2 A = 1 \qquad \to \boxed{A = \frac{1}{\lambda^2}}$$

$$\mathcal{L}^{-1}\{Q(s)\} = \frac{E_0}{L}\,\mathcal{L}^{-1}\left\{\frac{1}{\lambda^2}\frac{1}{s} - \frac{1}{\lambda^2}\frac{1}{s+\lambda} - \frac{1}{\lambda}\frac{1}{(s+\lambda)^2}\right\}$$

$$q(t) = \frac{E_0}{L}\frac{1}{\lambda^2}\left(1 - e^{-\lambda t} - \lambda t e^{-\lambda t}\right) \boxed{= E_0 C\left(1 - e^{-\lambda t} - \lambda t e^{-\lambda t}\right)} \qquad \omega^2 = \lambda^2 = \frac{1}{LC}$$

b) $R\frac{dq}{dt} + \frac{1}{C}q = E_0 e^{-kt}$

$$R\left(sQ(s) - q(0)\right) + \frac{1}{C}Q(s) = E_0\,\frac{1}{s+k} \qquad q(0) = 0$$

$$\left(Rs + \frac{1}{C}\right)Q(s) = E_0\,\frac{1}{s+k}$$

$$\left(\frac{RCs+1}{C}\right)Q(s) = E_0\,\frac{1}{s+k}$$

$$Q(s) = E_0 C\frac{1}{(s+k)(RCs+1)} = E_0 C\left(\frac{A}{s+k} + \frac{B}{RCs+1}\right)$$

$$RCAs + A + Bs + kB = 1$$

$$\begin{array}{ll} s^1 & (RC)A + B = 0 \\ s^0 & A + kB = 1 \end{array} \to (1 - kRC)A = 1 \Rightarrow A = \frac{1}{1-kRC} \qquad B = -\frac{RC}{1-kRC}$$

$$Q(s) = E_0 C\left(\frac{1}{1-kRC}\frac{1}{s+k} - \frac{1}{1-kRC}\frac{RC}{RCs+1}\right)$$

$$= \frac{E_0 C}{1-kRC}\left(\frac{1}{s+k} - \frac{1}{s+\frac{1}{RC}}\right)$$

When $k \neq \frac{1}{RC}$

$$\mathcal{L}\{Q(s)\} = \frac{E_0 C}{1-kRC}\mathcal{L}\left(\frac{1}{s+k} - \frac{1}{s+\frac{1}{RC}}\right)$$

$$\boxed{q(t) = \frac{E_0 C}{1-kRC}\left(e^{-kt} - e^{-t/RC}\right)}$$

When $k = \frac{1}{RC}$ $\quad \Rightarrow Q(s) = E_0 C\frac{1}{\left(s+\frac{1}{RC}\right)(RCs+1)}$

$$= E_0 RC^2 \frac{1}{(RCs+1)^2} = E_0 RC^2 \left(\frac{A}{RCs+1} + \frac{B}{(RCs+1)^2} \right)$$

$$RCAs + A + B = 1$$

$$s^1 \quad (RC)A = 0 \rightarrow \underline{A = 0}|$$

$$s^0 \quad A + B = 1 \rightarrow \underline{B = 1}|$$

$$\mathcal{L}\{Q(s)\} = E_0 RC^2 \mathcal{L}\left\{ \frac{1}{(RCs+1)^2} \right\}$$

$$= E_0 RC^2 \mathcal{L}\left\{ \frac{1}{(RC)^2} \frac{1}{\left(s + \frac{1}{RC}\right)^2} \right\}$$

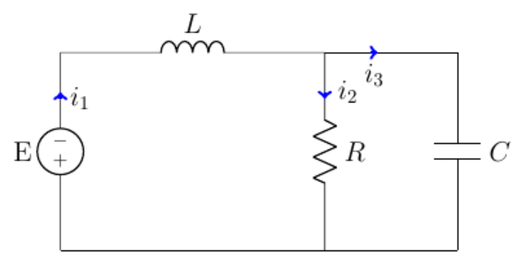
$$\boxed{q(t) = \frac{E_0}{R} te^{-t/RC}}$$

Exercise

Solve the system under the conditions $E(t) = 60\ V$, $L = 1\ h$, $R = 50\ \Omega$, $C = 10^{-4}\ f$, and the currents i_1 and i_2 are initially zero.

Solution

$$\begin{cases} L\frac{di_1}{dt} + Ri_2 = E(t) \\ RC\frac{di_2}{dt} + i_2 - i_1 = 0 \end{cases}$$



$$\begin{cases} \frac{di_1}{dt} + 50i_2 = 60 \\ 50 \times 10^{-4} \frac{di_2}{dt} + i_2 - i_1 = 0 \end{cases} \qquad i_2(0) = i_1(0) = 0$$

$$\begin{cases} \mathcal{L}\left\{ \frac{di_1}{dt} + 50i_2 \right\} = \mathcal{L}\{60\} \\ \mathcal{L}\left\{ 50 \times 10^{-4} \frac{di_2}{dt} + i_2 - i_1 \right\} = 0 \end{cases}$$

$$\begin{cases} sI_1(s) - i_1(0) + 50I_2(s) = \frac{60}{s} \\ \frac{1}{200}\left(sI_2(s) - i_2(0)\right) + I_2(s) - I_1(s) = 0 \end{cases}$$

$$\begin{cases} sI_1(s)+50I_2(s)=\dfrac{60}{s} \\ -200I_1(s)+(s+200)I_2(s)=0 \end{cases} \rightarrow \begin{cases} 200sI_1(s)+10^4 I_2(s)=\dfrac{12000}{s} \\ -200sI_1(s)+(s+200)sI_2(s)=0 \end{cases}$$

$$\left(s^2+200s+10^4\right)I_2(s)=\frac{12000}{s} \quad \Rightarrow I_2(s)=\frac{12000}{s(s+100)^2}$$

$$sI_1(s)=\frac{60}{s}-50\frac{12,000}{s(s+100)^2}=\frac{60s^2+12,000s-6\times10^5-6\times10^5}{s(s+100)^2} \quad \Rightarrow I_1(s)=\frac{60s+12,000}{s(s+100)^2}$$

$$I_1(s)=\frac{60s+12,000}{s(s+100)^2}=\frac{A}{s}+\frac{B}{s+100}+\frac{C}{(s+100)^2}$$

$$As^2+200As+10,000A+Bs^2+100Bs+Cs=60s+12,000$$

$$A+B=0 \qquad \boxed{B=-\frac{6}{5}}$$

$$200A+100B+C=60 \qquad \boxed{C=60-240+120=-60}$$

$$10,000A=12,000 \qquad \boxed{A=\frac{6}{5}}$$

$$\mathcal{L}^{-1}\{I_1(s)\}=\mathcal{L}^{-1}\left\{\frac{6}{5}\frac{1}{s}-\frac{6}{5}\frac{1}{s+100}-\frac{60}{(s+100)^2}\right\}$$

$$\boxed{i_1(t)=\frac{6}{5}-\frac{6}{5}e^{-100t}-60te^{-100t}}$$

$$I_2(s)=\frac{12000}{s(s+100)^2}=\frac{A}{s}+\frac{B}{s+100}+\frac{C}{(s+100)^2}$$

$$As^2+200As+10,000A+Bs^2+100Bs+Cs=12,000$$

$$A+B=0 \qquad \boxed{B=-\frac{6}{5}}$$

$$200A+100B+C=0 \qquad \boxed{C=-240+120=-120}$$

$$10,000A=12,000 \qquad \boxed{A=\frac{6}{5}}$$

$$\mathcal{L}^{-1}\{I_2(s)\}=\mathcal{L}^{-1}\left\{\frac{6}{5}\frac{1}{s}-\frac{6}{5}\frac{1}{s+100}-\frac{120}{(s+100)^2}\right\}$$

$$\boxed{i_2(t)=\frac{6}{5}-\frac{6}{5}e^{-100t}-120te^{-100t}}$$

Exercise

Solve
$$\begin{aligned} x_1'' + 10x_1 \quad - 4x_2 &= 0 \\ -4x_1 + x_2'' + 4x_2 &= 0 \end{aligned}$$

Subject to $x_1(0)=0,\ \ x_1'(0)=1,\ \ x_2(0)=0,\ \ x_2'(0)=-1$

Solution

$$s^2X_1(s) - sx_1(0) - x_1'(0) + 10X_1(s) - 4X_2(s) = 0$$

$$-4X_1(s) + s^2X_2(s) - sx_2(0) - x_2'(0) + 4X_2(s) = 0$$

$$\left(s^2+10\right)X_1(s) - 4X_2(s) = 1$$

$$-4X_1(s) + \left(s^2+4\right)X_2(s) = -1$$

$$\Delta = \begin{vmatrix} s^2+10 & -4 \\ -4 & s^2+4 \end{vmatrix} = \left(s^2+10\right)\left(s^2+4\right) - 16 \quad \Delta_1 = \begin{vmatrix} 1 & -4 \\ -1 & s^2+4 \end{vmatrix} = s^2 \quad \Delta_2 = \begin{vmatrix} s^2+10 & 1 \\ -4 & -1 \end{vmatrix} = -s^2-6$$

$$X_1(s) = \frac{s^2}{s^4+14s^2+24} = \frac{As+B}{s^2+2} + \frac{Cs+D}{s^2+12}$$

$$\begin{array}{ll} s^3 & A+C=0 \\ s^2 & B+D=1 \\ s & 12A+2C=0 \\ s^0 & 12B+2D=0 \end{array} \qquad \boxed{A=0 \quad B=-\frac{1}{5} \quad C=0 \quad D=\frac{6}{5}}$$

$$\mathcal{L}^{-1}\left\{X_1(s)\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{5\sqrt{2}}\frac{\sqrt{2}}{s^2+2} + \frac{6}{5\sqrt{12}}\frac{\sqrt{12}}{s^2+12}\right\}$$

$$\boxed{x_1(t) = -\frac{\sqrt{2}}{10}\sin\sqrt{2}t + \frac{\sqrt{3}}{5}\sin 2\sqrt{3}t}$$

$$X_2(s) = \frac{-s^2-6}{\left(s^2+2\right)\left(s^2+12\right)} = \frac{As+B}{s^2+2} + \frac{Cs+D}{s^2+12}$$

$$\begin{array}{ll} s^3 & A+C=0 \\ s^2 & B+D=-1 \\ s & 12A+2C=0 \\ s^0 & 12B+2D=-6 \end{array} \qquad \boxed{A=0 \quad B=-\frac{2}{5} \quad C=0 \quad D=-\frac{3}{5}}$$

$$\mathcal{L}^{-1}\left\{X_2(s)\right\} = \mathcal{L}^{-1}\left\{-\frac{2}{5\sqrt{2}}\frac{\sqrt{2}}{s^2+2} - \frac{3}{5\sqrt{12}}\frac{2\sqrt{3}}{s^2+12}\right\}$$

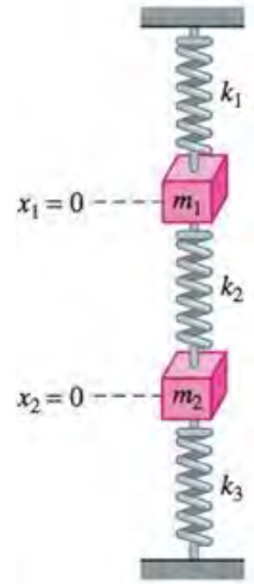
$$\boxed{x_1(t) = -\frac{\sqrt{2}}{5}\sin\sqrt{2}t - \frac{\sqrt{3}}{10}\sin 2\sqrt{3}t}$$

Exercise

Derive the system of differential equations describing the straight-line vertical motion of the coupled springs. Use the Laplace transform to solve the system when

$k_1 = 1,\ k_2 = 1,\ k_3 = 1,\ m_1 = 1,\ m_2 = 1$ and

$x_1(0) = 0,\ x_1'(0) = -1,\ x_2(0) = 0,\ x_2'(0) = 1$



Solution

$$x_1'' + 2x_1 - x_2 = 0 \qquad m_1 x_1'' = -k_1 x_1 + k_2\left(x_2 - x_1\right)$$

$$x_2'' + 2x_2 - x_1 = 0 \qquad m_2 x_2'' = -k_2\left(x_2 - x_1\right) - k_3 x_2$$

$$s^2 X_1(s) - s x_1(0) - x_1'(0) + 2X_1(s) - X_2(s) = 0$$

$$s^2 X_2(s) - s x_2(0) - x_2'(0) + 2X_2(s) - X_1(s) = 0$$

$$\left(s^2 + 2\right) X_1(s) - X_2(s) = -1$$

$$-X_1(s) + \left(s^2 + 2\right) X_2(s) = 1$$

$$\Delta = \begin{vmatrix} s^2+2 & -1 \\ -1 & s^2+2 \end{vmatrix} = \left(s^2+2\right)^2 - 1 \quad \Delta_1 = \begin{vmatrix} -1 & -1 \\ 1 & s^2+2 \end{vmatrix} = -s^2 - 1 \quad \Delta_2 = \begin{vmatrix} s^2+2 & -1 \\ -1 & 1 \end{vmatrix} = s^2 + 1$$

$$X_1(s) = \frac{-\left(s^2+1\right)}{s^4 + 4s^2 + 3}$$

$$= -\frac{s^2+1}{\left(s^2+1\right)\left(s^2+3\right)}$$

$$\mathcal{L}^{-1}\left\{X_1(s)\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{s^2+3}\right\}$$

$$\boxed{x_1(t) = -\sin\sqrt{3}t}$$

$$X_2(s) = \frac{s^2+1}{\left(s^2+1\right)\left(s^2+3\right)} = \frac{1}{s^2+3}$$

$$\mathcal{L}^{-1}\left\{X_1(s)\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{s^2+3}\right\}$$

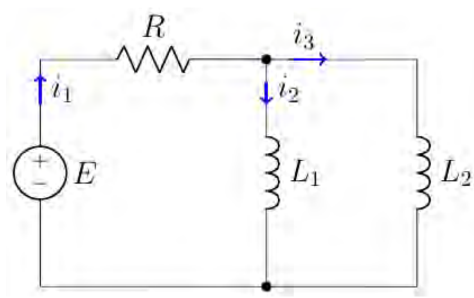
$$\boxed{x_1(t) = -\sin\sqrt{3}t}$$

Exercise

Solve the currents $i_1(t)$, $i_2(t)$ and $i_3(t)$ in the given electrical network.

Given $R = 5\ \Omega \quad L_1 = 0.01\ h, \quad L_2 = 0.0125\ h, \quad E = 100\ V$ and $i_2(0) = 0 \quad i_3(0) = 0$

Solution



$$i_1 = i_2 + i_3$$

$$\begin{cases} Ri_1 + L_1 i_2' = E(t) \\ Ri_1 + L_2 i_3' = E(t) \end{cases}$$

$$\begin{cases} Ri_2 + Ri_3 + L_1 i_2' = E(t) \\ Ri_2 + Ri_3 + L_2 i_3' = E(t) \end{cases}$$

$$\begin{cases} 5i_2 + 5i_3 + .01i_2' = 100 \\ 5i_2 + 5i_3 + .0125i_3' = 100 \end{cases}$$

$$\begin{cases} 5I_2(s) + 5I_3(s) + .01sI_2(s) - i_2(0) = 100 \\ 5I_2(s) + 5I_3(s) + .0125sI_3(s) - i_3(0) = 100 \end{cases}$$

$$\begin{cases} \left(5 + \frac{1}{100}s\right)I_2(s) + 5I_3(s) = \frac{100}{s} \\ 5I_2(s) + \left(5 + \frac{1}{80}s\right)I_3(s) = \frac{100}{s} \end{cases}$$

$$\begin{cases} \left(500s + s^2\right)I_2(s) + 500sI_3(s) = 10^4 \\ 400sI_2(s) + \left(400s + s^2\right)I_3(s) = 8 \times 10^3 \end{cases}$$

$$\Delta = \begin{vmatrix} s^2 + 500s & 500s \\ 400s & s^2 + 400s \end{vmatrix} = s^4 + 900s^3 \qquad \Delta_2 = \begin{vmatrix} 10^4 & 500s \\ 8000 & s^2 + 400s \end{vmatrix} = 10^4 s^2$$

$$\Delta_3 = \begin{vmatrix} s^2 + 500s & 10^4 \\ 400s & 8 \times 10^3 \end{vmatrix} = 8 \times 10^3 s^2$$

$$I_2(s) = \frac{10^4}{s(s+900)} = 10^4\left(\frac{A}{s} + \frac{B}{s+900}\right)$$

$$\begin{array}{ll} s & A + B = 0 \\ s^0 & 900A = 1 \end{array} \quad \rightarrow \quad A = \frac{1}{900}, \quad B = -\frac{1}{900}$$

$$\mathcal{L}^{-1}\left\{I_2(s)\right\} = \mathcal{L}^{-1}\left\{\frac{100}{9}\frac{1}{s} - \frac{100}{9}\frac{1}{s+900}\right\}$$

$$i_2(t) = \frac{100}{9} - \frac{100}{9}e^{-900t}$$

$$I_3(s)=\frac{8,000}{s(s+900)}=\frac{C}{s}+\frac{D}{s+900}$$

$$\begin{array}{ll} s & C+D=0 \\ s^0 & 900C=8000 \end{array} \quad \to \quad \boxed{C=\frac{80}{9}, \quad D=-\frac{80}{9}}$$

$$\mathcal{L}^{-1}\left\{I_2(s)\right\}=\mathcal{L}^{-1}\left\{\frac{80}{9}\frac{1}{s}-\frac{80}{9}\frac{1}{s+900}\right\}$$

$$\boxed{i_3(t)=\frac{80}{9}-\frac{80}{9}e^{-900t}}$$

$$i_1(t)=i_2(t)+i_3(t)$$

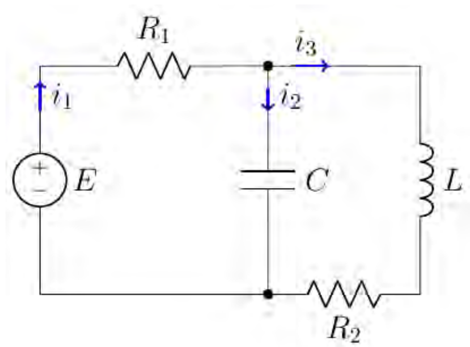
$$\boxed{=20-20e^{-900t}}$$

Exercise

Find the charge on the capacitor $q(t)$ and the current $i_3(t)$ in the given electrical network.

Given: $R_1=1\,\Omega, \quad R_2=1\,\Omega, \quad L=1\,h, \quad C=1\,f \quad \& \quad q(0)=0, \quad i_3(0)=0$

$$E(t)=\begin{cases} 0, & 0<t<1 \\ 50e^{-t}, & t\ge 1 \end{cases}$$



Solution

$$i_1=i_2+i_3 \qquad i_2=q'$$

$$\begin{cases} R_1 i_1+\frac{1}{C}q=E(t) \\ R_1 i_1+Li_3'+R_2 i_3=E(t) \end{cases} \quad \to \quad R_1 i_1=E(t)-\frac{1}{C}q$$

$$\begin{cases} R_1\left(q'+i_3\right)+\frac{1}{C}q=E(t) \\ E(t)-\frac{1}{C}q+Li_3'+R_2 i_3=E(t) \end{cases}$$

$$\begin{cases} q'+q+i_3=E(t) \\ -q+i_3'+i_3=0 \end{cases} \qquad E(t)=50e^{-t}u(t-1)=50e^{-1}e^{-(t-1)}u(t-1)$$

$$\begin{cases} sQ(s)-q(0)+Q(s)+I_3(s)=\frac{50}{e}\frac{e^{-s}}{s+1} \\ -Q(s)+sI_3(s)-i_{3(0)}+I_3(s)=0 \end{cases} \qquad q(0)=0, \quad i_3(0)=0$$

$$\begin{cases} (s+1)Q(s)+I_3(s)=\frac{50}{e}\frac{e^{-s}}{s+1} \\ -Q(s)+(s+1)I_3(s)=0 \end{cases}$$

$$\Delta = \begin{vmatrix} s+1 & 1 \\ -1 & s+1 \end{vmatrix} = s^2+2s+2 \qquad \Delta_q = \begin{vmatrix} \frac{50}{e}\frac{e^{-s}}{s+1} & 1 \\ 0 & s+1 \end{vmatrix} = 50e^{-s-1} \qquad \Delta_{i_3} = \begin{vmatrix} s+1 & \frac{50}{e}\frac{e^{-s}}{s+1} \\ -1 & 0 \end{vmatrix} = \frac{50}{e}\frac{e^{-s}}{s+1}$$

$$\mathcal{L}^{-1}\{Q(s)\} = \mathcal{L}^{-1}\left\{\frac{50e^{-1}e^{-s}}{(s+1)^2+1}\right\}$$

$$q(t) = 50e^{-1}e^{-(t-1)}\sin(t-1)u(t-1)$$

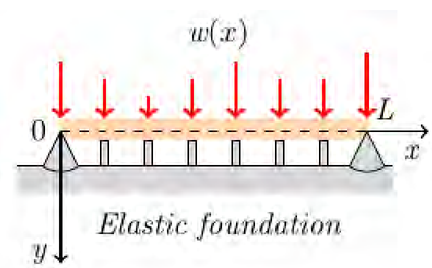
$$\boxed{= 50e^{-t}\sin(t-1)u(t-1)}$$

$$i_3(t) = q' = \left(50e^{-t}\cos(t-1) - 50e^{-t}\sin(t-1)\right)u(t-1)$$

$$\boxed{= 50e^{-t}\left(\cos(t-1)-\sin(t-1)\right)u(t-1)}$$

Exercise

When a uniform beam is supported by an elastic foundation, the differential equation for its deflection $y(x)$ is



$$EI\frac{d^4y}{dx^4} + ky = w(x)$$

Where k is the modulus of the foundation and $-ky$ is the restoring force of the foundation that acts in the direction opposite to that of the load $w(x)$. For algebraic convenience, suppose that the differential equation is written as

$$\frac{d^4y}{dx^4} + 4a^4y = \frac{w(x)}{EI}$$

Where $a = \left(\frac{k}{4EI}\right)^{1/4}$. Assume $L = \pi$ and $a = 1$. Find the deflection $y(x)$ of a beam that is supported on an elastic foundation when

a) The beam is simply supported at both ends and a constant load w_0 is uniformly distributed along its length,

b) The bean is embedded at both ends and $w(x)$ is concentrated load w_0 applied at $x = \frac{\pi}{2}$

Solution

a) $y(0) = y''(0) = 0 \quad \& \quad y(\pi) = y''(\pi) = 0$

Let: $y'(0) = c_1 \quad y'(0) = c_2$

$$\mathcal{L}\left\{\frac{d^4y}{dx^4} + 4y\right\} = \mathcal{L}\left\{\frac{w(x)}{EI}\right\}$$

$$s^4Y(s)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)+4Y(s)=\frac{w_0}{EI}\frac{1}{s}$$

$$\left(s^4+4\right)Y(s)=\frac{w_0}{EI}\frac{1}{s}+c_1s^2+c_2$$

$$Y(s)=\frac{w_0}{EI}\frac{1}{s\left(s^4+4\right)}+\frac{c_1s^2}{s^4+4}+\frac{c_2}{s^4+4}$$

$$\frac{1}{s\left(s^4+4\right)}=\frac{A}{s}+\frac{B(s-1)+C}{(s-1)^2+1}+\frac{D(s+1)+E}{(s+1)^2+1}$$

$$\begin{array}{ll} s^4 & A+B+D=0 \\ s^3 & B+C-D+E=0 \\ s^2 & 2C-2E=0 \\ s & -2B+2C+2D+2E=0 \\ s^0 & 4A=1 \end{array} \qquad \boxed{A=\frac{1}{4},\ B=-\frac{1}{8},\ C=0,\ D=-\frac{1}{8},\ E=0}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{w_0}{EI}\left(\frac{1}{4}\frac{1}{s}-\frac{1}{8}\frac{s-1}{(s-1)^2+1}-\frac{1}{8}\frac{s+1}{(s+1)^2+1}\right)+\frac{c_1}{2}\frac{2s^2}{s^4+4}+\frac{c_2}{4}\frac{4}{s^4+4}\right\}$$

$$y(x)=\frac{w_0}{EI}\left(\frac{1}{4}-\frac{1}{8}e^x\cos x-\frac{1}{8}e^{-x}\cos x\right)+\frac{c_1}{2}(\sin x\cosh x+\cos x\sinh x)$$
$$+\frac{c_2}{4}(\sin x\cosh x-\cos x\sinh x)$$

$$y(x)=\frac{w_0}{4EI}(1-\cos x\cosh x)+\frac{c_1}{2}(\sin x\cosh x+\cos x\sinh x)+\frac{c_2}{4}(\sin x\cosh x-\cos x\sinh x)$$

$$y(\pi)=\frac{w_0}{4EI}(1+\cosh\pi)-\frac{1}{2}c_1\sinh\pi+\frac{1}{4}c_2\sinh\pi=0$$

$$\boxed{2c_1\sinh\pi-c_2\sinh\pi=\frac{w_0}{EI}(1+\cosh\pi)}$$

$$y'=\frac{w_0}{4EI}(\sin x\cosh x-\cos x\sinh x)+c_1\cos x\cosh x+\frac{1}{2}c_2\sin x\sinh x$$

$$y''=\frac{w_0}{2EI}\sin x\sinh x+c_1(-\sin x\cosh x+\cos x\sinh x)+\frac{1}{2}c_2(\cos x\sinh x+\sin x\cosh x)$$

$$y''(\pi)=-c_1\sinh\pi-\frac{1}{2}c_2\sinh\pi=0$$

$$\boxed{c_1=-\frac{1}{2}c_2}$$

$$2c_1\sinh\pi+2c_1\sinh\pi=\frac{w_0}{EI}(1+\cosh\pi)$$

$$\boxed{c_1=\frac{w_0}{4EI}(1+\cosh\pi)\operatorname{csch}\pi\quad c_2=-\frac{w_0}{2EI}(1+\cosh\pi)\operatorname{csch}\pi}$$

$$y(x)=\frac{w_0}{4EI}(1-\cos x\cosh x)+\frac{w_0}{8EI}(1+\cosh\pi)\operatorname{csch}\pi(\sin x\cosh x+\cos x\sinh x)$$
$$-\frac{w_0}{4EI}(1+\cosh\pi)\operatorname{csch}\pi(\sin x\cosh x-\cos x\sinh x)$$

b) $\mathcal{L}\left\{\frac{d^4y}{dx^4}+4y\right\}=\mathcal{L}\left\{\delta\left(t-\frac{\pi}{2}\right)\right\}$

$$s^4Y(s)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)+4Y(s)=\frac{w_0}{EI}e^{-\pi s/2}$$

$$Y(s)=\frac{w_0}{4EI}\frac{4}{s^4+4}e^{-\pi s/2}+\frac{c_1}{2}\frac{2s^2}{s^4+4}+\frac{c_2}{4}\frac{4}{s^4+4}$$

$$y(x)=\frac{w_0}{4EI}\left(\sin\left(x-\frac{\pi}{2}\right)\cosh\left(x-\frac{\pi}{2}\right)-\cos\left(x-\frac{\pi}{2}\right)\sinh\left(x-\frac{\pi}{2}\right)\right)u\left(x-\frac{\pi}{2}\right)$$
$$+\frac{c_1}{2}\sin x\sinh x+\frac{c_2}{4}(\sin x\cosh x-\cos x\sinh x)$$

$$y(\pi)=\frac{w_0}{4EI}\cosh\frac{\pi}{2}+\frac{c_2}{4}\sinh\pi=0\quad\to\quad\boxed{c_2=-\frac{w_0}{EI}\frac{\cosh\frac{\pi}{2}}{\sinh\pi}}$$

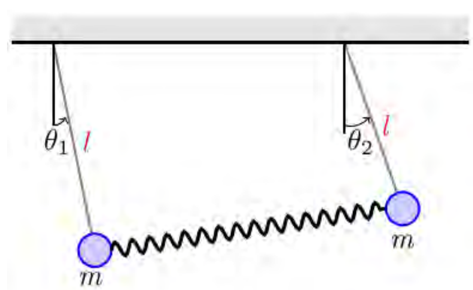
$$\boxed{c_1=\frac{w_0}{EI}\frac{\sinh\frac{\pi}{2}}{\sinh\pi}}$$

$$y(x)=\frac{w_0}{4EI}\left(\sin\left(x-\frac{\pi}{2}\right)\cosh\left(x-\frac{\pi}{2}\right)-\cos\left(x-\frac{\pi}{2}\right)\sinh\left(x-\frac{\pi}{2}\right)\right)u\left(x-\frac{\pi}{2}\right)$$
$$+\frac{w_0}{2EI}\frac{\sinh\frac{\pi}{2}}{\sinh\pi}\sin x\sinh x-\frac{w_0}{4EI}\frac{\cosh\frac{\pi}{2}}{\sinh\pi}(\sin x\cosh x-\cos x\sinh x)$$

Exercise

Suppose two identical pendulums are coupled by means of a spring with constant *k*. when the displacement angles $\theta_1(t)$ and $\theta_2(t)$ are small, the system of linear differential equations describing the motion is

$$\begin{cases}\theta_1''+\frac{g}{l}\theta_1=-\frac{k}{m}(\theta_1-\theta_2)\\ \theta_2''+\frac{g}{l}\theta_2=\frac{k}{m}(\theta_1-\theta_2)\end{cases}$$



a) Use Laplace transform to solve the system when

$$\theta_1'(0)=0\quad\theta_1(0)=\theta_0\quad\theta_2'(0)=0\quad\theta_2(0)=\psi_0$$

Where θ_0 and ψ_0 constants. Let $\omega^2=\frac{g}{l},\ K=\frac{k}{m}$

b) Use the solution in part (*a*) to discuss the motion of the coupled pendulums in the special case when the initial conditions are $\theta_1'(0)=0,\quad \theta_1(0)=\theta_0,\quad \theta_2'(0)=\theta_0,\quad \theta_2(0)=0$

c) Use the solution in part (*a*) to discuss the motion of the coupled pendulums in the special case when the initial conditions are $\theta_1'(0)=0,\quad \theta_1(0)=\theta_0,\quad \theta_2'(0)=-\theta_0,\quad \theta_2(0)=0$

Solution

a) $$\begin{cases} \theta_1'' + \omega^2\theta_1 = -K\theta_1 + K\theta_2 \\ \theta_2'' + \omega^2\theta_2 = K\theta_1 - K\theta_2 \end{cases}$$

$$\begin{cases} s^2\theta_1(s) - s\theta_1(0) - \theta_1'(0) + \omega^2\theta_1(s) + K\theta_1(s) = +K\theta_2(s) \\ s^2\theta_2(s) - s\theta_2(0) - \theta_2'(0) + \omega^2\theta_2(s) + K\theta_2(s) = K\theta_1(s) \end{cases}$$

$$\begin{cases} \left(s^2+\omega^2+K\right)\theta_1(s) - K\theta_2(s) = s\theta_0 \\ -K\theta_1(s) + \left(s^2+\omega^2+K\right)\theta_2(s) = s\psi_0 \end{cases}$$

$$\Delta = \begin{vmatrix} s^2+\omega^2+K & -K \\ -K & s^2+\omega^2+K \end{vmatrix} = \left(s^2+\omega^2+K\right)^2 - K^2 = \left(s^2+\omega^2\right)\left(s^2+\omega^2+2K\right)$$

$$\Delta_1 = \begin{vmatrix} s\theta_0 & -K \\ s\psi_0 & s^2+\omega^2+K \end{vmatrix} = s^3\theta_0 + \left(\omega^2\theta_0 + K\theta_0 + K\psi_0\right)s$$

$$\Delta_2 = \begin{vmatrix} s^2+\omega^2+K & s\theta_0 \\ -K & s\psi_0 \end{vmatrix} = s^3\psi_0 + \left(\omega^2\psi_0 + K\psi_0 + K\theta_0\right)s$$

$$\theta_1(s) = \frac{s^3\theta_0 + \left(\omega^2\theta_0 + K\theta_0 + K\psi_0\right)s}{\left(s^2+\omega^2\right)\left(s^2+\omega^2+2K\right)} = \frac{As+B}{s^2+\omega^2} + \frac{Cs+D}{s^2+\left(\omega^2+2K\right)}$$

$s^3 \qquad A + C = \theta_0$

$s^2 \qquad B + D = 0$

$s \qquad \left(\omega^2+2K\right)A + \omega^2 C = \omega^2\theta_0 + K\theta_0 + K\psi_0$

$s^0 \qquad \left(\omega^2+2K\right)B + \omega^2 D = 0$

$$\left(\omega^2+2K\right)A + \omega^2\left(\theta_0 - A\right) = \omega^2\theta_0 + K\theta_0 + K\psi_0$$

$$2KA = K\theta_0 + K\psi_0 \quad\rightarrow\quad \underline{A = \tfrac{1}{2}\left(\theta_0+\psi_0\right)}$$

$$C = \theta_0 - A \quad\rightarrow\quad \underline{C = \tfrac{1}{2}\left(\theta_0-\psi_0\right)}$$

$$\underline{B = D = 0}$$

$$\left\{\theta_1(s)\right\} = \frac{1}{2}\left(\theta_0 + \psi_0\right)\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} + \frac{1}{2}\left(\theta_0 - \psi_0\right)\mathcal{L}^{-1}\frac{s}{s^2+\left(\sqrt{\omega^2+2K}\right)^2}$$

$$\boxed{\theta_1(t) = \frac{1}{2}\left(\theta_0 + \psi_0\right)\cos\omega t + \frac{1}{2}\left(\theta_0 - \psi_0\right)\cos\sqrt{\omega^2+2K}\, t}$$

$$\theta_2(s) = \frac{s^3\psi_0 + \left(\omega^2\psi_0 + K\psi_0 + K\theta_0\right)s}{\left(s^2+\omega^2\right)\left(s^2+\omega^2+2K\right)} = \frac{as+b}{s^2+\omega^2} + \frac{cs+d}{s^2+\left(\omega^2+2K\right)}$$

$s^3 \quad a + c = \psi_0$

$s^2 \quad b + d = 0$

$s \quad \left(\omega^2+2K\right)a + \omega^2 c = \omega^2\psi_0 + K\psi_0 + K\theta_0$

$s^0 \quad \left(\omega^2+2K\right)b + \omega^2 d = 0$

$$\left(\omega^2+2K\right)a + \omega^2\left(\psi_0 - a\right) = \omega^2\psi_0 + K\psi_0 + K\theta_0$$

$$2Ka = K\theta_0 + K\psi_0 \quad \rightarrow \quad \boxed{a = \frac{1}{2}\left(\theta_0 + \psi_0\right)}$$

$$C = \psi_0 - A \quad \rightarrow \quad \boxed{C = -\frac{1}{2}\left(\theta_0 - \psi_0\right)}$$

$$\boxed{b = d = 0}$$

$$\boxed{\theta_2(t) = \frac{1}{2}\left(\theta_0 + \psi_0\right)\cos\omega t - \frac{1}{2}\left(\theta_0 - \psi_0\right)\cos\sqrt{\omega^2+2K}\, t}$$

b) $\theta_1'(0) = 0, \quad \theta_1(0) = \theta_0, \quad \theta_2'(0) = \theta_0, \quad \theta_2(0) = 0$

$\Rightarrow \boxed{\psi_0 = \theta_0}$

$\boxed{\theta_1(t) = \theta_0\cos\omega t} \quad \& \quad \boxed{\theta_2(t) = \theta_0\cos\omega t}$

∴ This means that both pendulums swing in the same direction (free) and the spring exerts no influemce on the motion.

c) $\theta_1'(0) = 0, \quad \theta_1(0) = \theta_0, \quad \theta_2'(0) = -\theta_0, \quad \theta_2(0) = 0$

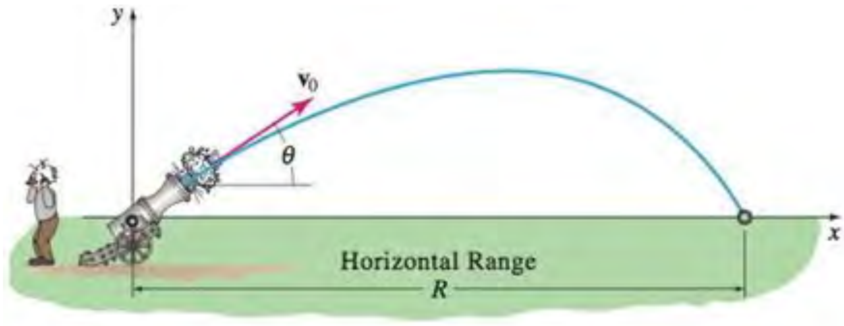
$\Rightarrow \boxed{\psi_0 = -\theta_0}$

$\boxed{\theta_1(t) = \theta_0\cos\sqrt{\omega^2+2K}\, t} \quad \& \quad \boxed{\theta_2(t) = -\theta_0\cos\sqrt{\omega^2+2K}\, t}$

∴ This means that both pendulums swing in the opposite directions, stretching and compressing the spring. The amplitude of both displacements is $\left|\theta_0\right|$. Which the psring is stretched to its maximum.

Exercise

A projectile, such as the canon ball, has weight $w = mg$ and initial velocity $\boldsymbol{v}_0$ that is tangent to its path of motion.



If air resistance and all other forces except its weight are ignored, that motion of the projectile is describe by the system of linear differential equations:

$$\begin{cases} m\dfrac{d^2x}{dt^2} = 0 \\ m\dfrac{d^2y}{dt^2} = -mg \end{cases}$$

a) Use Laplace transform to solve the system when

$$x(0) = 0 \quad x'(0) = v_0 \cos\theta \quad y(0) = 0 \quad y'(0) = v_0 \sin\theta$$

Where $v_0 = |v|$ is constant and θ is the constant angle of elevation.

The solutions $x(t)$ and $y(t)$ are parametric equations of the trajectory of the projectile.

b) Use $x(t)$ in part (*a*) to eliminate the parameter *t* in $y(t)$. Use the resulting equation for *y* to show that the horizontal range *R* of the projectile is given by

$$R = \frac{v_0^2}{g}\sin 2\theta$$

c) From the formula in part (*b*), we see that *R* is a maximum when $\sin 2\theta = 1$ or when $\theta = \frac{\pi}{4}$. Show that the same range – less than the maximum– can be obtained by firing the gun at either of two complementary angles θ and $\frac{\pi}{2} - \theta$. The only difference is that the smaller angle results in a low trajectory whereas the larger angle fives a high trajectory.

d) Suppose $g = 32$ *ft/s*^{2}, $\theta = 30°$, and $v_0 = 300$ *ft/s*. Use part (*b*) to find the horizontal range of the projectile.

e) Find the time when the projectile hits the ground.

f) Use the parametric equations $x(t)$ and $y(t)$ in part (*a*) along with the numerical data in part (*d*) to plot the ballistic curve of the projectile.

g) Repeat with $\theta = 52°$ and $v_0 = 300$ *ft/s*.

h) Superimpose both curves part (*f*) & (*g*) on the same coordinate system.

Solution

a) $\frac{d^2x}{dt^2}=0$ $\quad\frac{d^2y}{dt^2}=-g \quad \rightarrow \quad \begin{cases} s^2X(s)-sx(0)-x'(0)=0 \\ s^2Y(s)-sy(0)-y'(0)=-\frac{g}{s} \end{cases}$

$x(0)=0 \quad x'(0)=v_0\cos\theta \quad y(0)=0 \quad y'(0)=v_0\sin\theta$

$X(s)=v_0\cos\theta\frac{1}{s^2}$

$\boxed{x(t)=(v_0\cos\theta)t}$

$Y(s)=v_0\sin\theta\frac{1}{s^2}-\frac{g}{s^3}$ 150t

$\boxed{y(t)=(v_0\sin\theta)t-\frac{1}{2}gt^2}$

b) $t=\frac{x}{v_0\cos\theta}$

$$\begin{aligned} y(x)&=(v_0\sin\theta)t-\frac{1}{2}gt^2 \\ &=(v_0\sin\theta)\frac{x}{v_0\cos\theta}-\frac{1}{2}g\frac{x^2}{v_0^2\cos^2\theta} \\ &=\frac{1}{2v_0^2\cos^2\theta}\left(2v_0^2\cos\theta\sin\theta x-gx^2\right) \\ &=\frac{1}{2v_0^2\cos^2\theta}\left(v_0^2\sin 2\theta-gx\right)x=0 \end{aligned}$$

At $x=y=0$, the projectile hits the ground.

$v_0^2\sin 2\theta-gx=0$

$\boxed{x=R(\theta)=\frac{1}{g}v_0^2\sin 2\theta}$

c) $$\begin{aligned} R\left(\frac{\pi}{2}-\theta\right)&=\frac{1}{g}v_0^2\sin(\pi-2\theta) \\ &=\frac{1}{g}v_0^2\sin 2\theta \\ &=R(\theta) \end{aligned}$$

$\sin(\pi-\alpha)=\sin\pi\cos\alpha-\cos\pi\sin\alpha$

d) Given: $g=32\ ft/s^2$, $\theta=30°$, and $v_0=300\ ft/s$

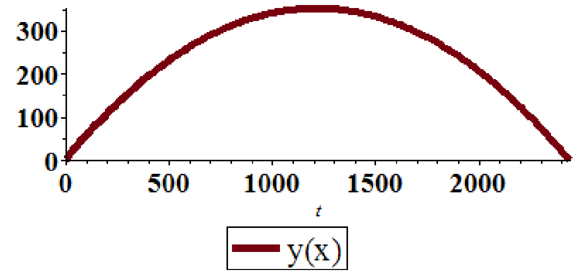
$R(30°)=\frac{1}{32}(300)^2\sin 60° \ \boxed{\approx 2{,}436\ ft}$

e) $x=(v_0\cos\theta)t=2{,}436$

$t=\frac{2{,}436}{300\cos 30°} \ \boxed{\approx 9.38\ sec}$

f) $y(x) = \dfrac{1}{2v_0^2\cos^2\theta}\left(\left(v_0^2\sin 2\theta\right)x - gx^2\right)$

$\underline{= 0.57735x - 0.000237x^2}$



g) Given: $g = 32\ ft/s^2$, $\theta = 52°$, and $v_0 = 300\ ft/s$

$R(30°) = \frac{1}{32}(300)^2\sin 104° \approx \underline{2729\ ft}$

h) $y(x) = \dfrac{1}{2v_0^2\cos^2\theta}\left(\left(v_0^2\sin 2\theta\right)x - gx^2\right)$

$\underline{= 1.2799x - 0.000469x^2}$