Solution   *Section* 2.1 – Functions and Graphs

Exercise

$$f(x)=\begin{cases} 2+x & \text{if } \ x<-4 \\ -x & \text{if } \ -4\le x\le 2 \\ 3x & \text{if } \ x>2 \end{cases}$$

Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

Solution

a) $f(-5)=2-5=-3$

b) $f(-1)=-(-1)=1$

c) $f(0)=-0=0$

d) $f(3)=3(3)=9$

Exercise

$$f(x)=\begin{cases} -2x & \text{if } \ x<-3 \\ 3x-1 & \text{if } \ -3\le x\le 2 \\ -4x & \text{if } \ x>2 \end{cases}$$

Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

Solution

a) $f(-5)=-2(-5)=10$

b) $f(-1)=3(-1)-1=-4$

c) $f(0)=3(0)-1=-1$

d) $f(3)=-4(3)=-12$

Exercise

$$f(x)=\begin{cases} x^3+3 & \text{if } \ -2\le x\le 0 \\ x+3 & \text{if } \ 0<x<1 \\ 4+x-x^2 & \text{if } \ 1\le x\le 3 \end{cases}$$

Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

Solution

a) $f(-5)=\ doesn't\ exist$

b) $f(-1)=(-1)^3+3$

$\quad\quad =2$

c) $f(0)=(0)^3+3$

$\underline{= 3}$

d) $f(3) = 4 + (3) - (3)^2$

$\underline{= -2}$

Exercise

$$h(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & if \quad x \neq 3 \\ 6 & if \quad x = 3 \end{cases}$$

Find: $h(5),\ h(0),\ and\ h(3)$

Solution

a) $h(5) = \dfrac{5^2 - 9}{5 - 3}$

$\underline{= 8}$

b) $h(0) = \dfrac{0^2 - 9}{0 - 3}$

$\underline{= 3}$

c) $\underline{h(3) = 6}$

Exercise

$$f(x) = \begin{cases} 3x + 5 & if \quad x < 0 \\ 4x + 7 & if \quad x \geq 0 \end{cases}$$

Find

a) $f(0)$ *b)* $f(-2)$ *c)* $f(1)$ *d)* $f(3) + f(-3)$ *e)* Graph $f(x)$

Solution

a) $f(0) = 4(0) + 7$

$\underline{= 7}$

b) $f(-2) = 3(-2) + 5$

$\underline{= -1}$

c) $f(1) = 4(1) + 7$

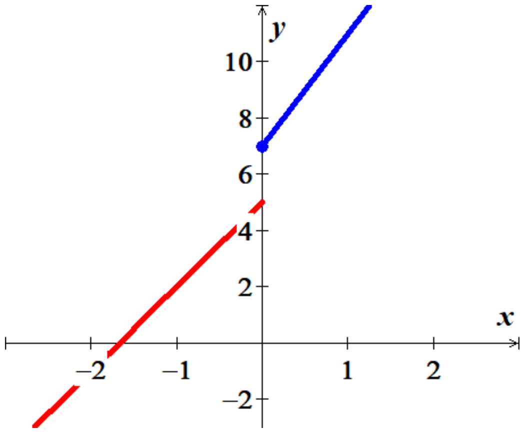
$\underline{= 11}$

d) $f(3) + f(-3) = 4(3) + 7 + 3(-3) + 5$

$= 12 + 7 - 9 + 5$

$\underline{= 15}$

e)



Exercise

$$f(x)=\begin{cases} 6x-1 & if \quad x<0 \\ 7x+3 & if \quad x\ge 0 \end{cases} \quad \text{Find}$$

a) $f(0)$ *b)* $f(-1)$ *c)* $f(4)$ *d)* $f(2)+f(-2)$ *e)* Graph $f(x)$

Solution

a) $f(0)=7(0)+3$

$=3$

b) $f(-2)=6(-1)-1$

$=-7$

c) $f(4)=7(4)+3$

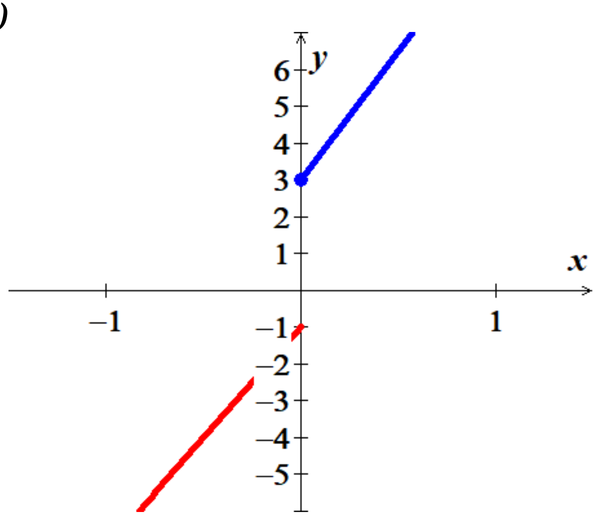
$=31$

d) $f(2)+f(-2)=7(2)+3+6(-2)-1$

$=14+3-12-1$

$=4$

e)



Exercise

$$f(x)=\begin{cases} 2x+1 & if \quad x\le 1 \\ 3x-2 & if \quad x>1 \end{cases} \quad \text{Find}$$

a) $f(0)$ *b)* $f(2)$ *c)* $f(-2)$ *d)* $f(1)+f(-1)$ *e)* Graph $f(x)$

Solution

a) $f(0)=2(0)+1$

$\underline{=1}$

b) $f(2)=3(2)-2$

$\underline{=4}$

c) $f(-2)=2(-2)+1$

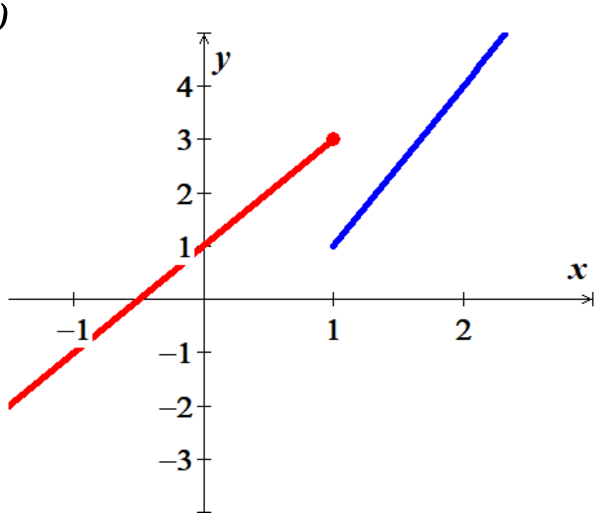
$\underline{=-3}$

d) $f(1)+f(-1)=\big(2(1)+1\big)+\big(2(-1)+1\big)$

$=2+1-2+1$

$\underline{=2}$

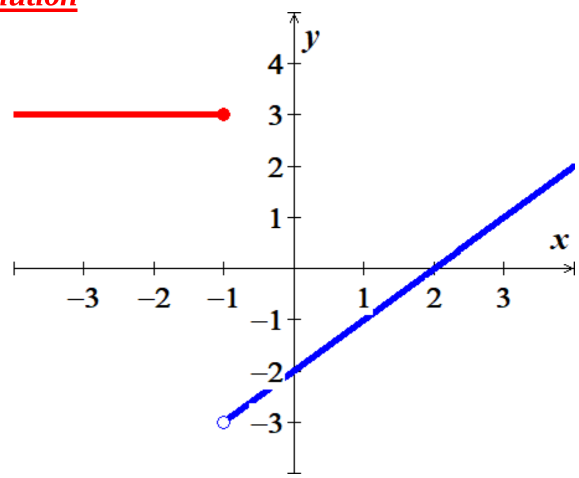
e)



Exercise

Graph the piecewise function defined by $f(x)=\begin{cases} 3 & if \quad x\le -1 \\ x-2 & if \quad x>-1 \end{cases}$

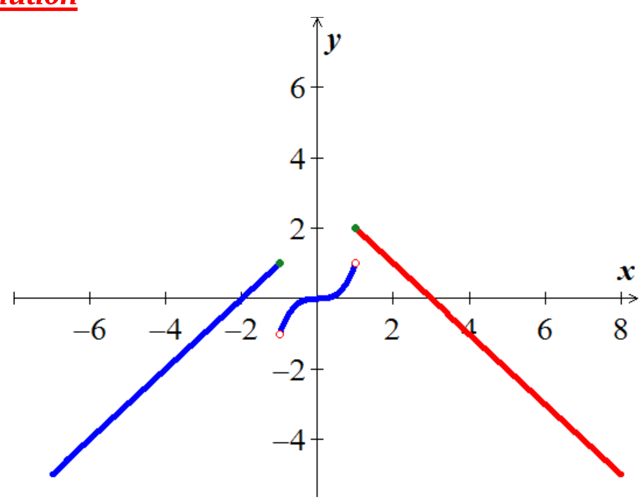
Solution



Exercise

Sketch the graph $f(x)=\begin{cases} x+2 & if \quad x\le -1 \\ x^3 & if \quad -1<x<1 \\ -x+3 & if \quad x\ge 1 \end{cases}$

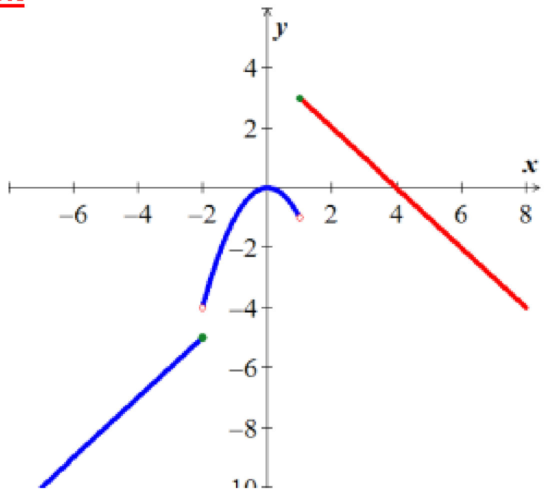
Solution



Exercise

Sketch the graph $f(x)=\begin{cases} x-3 & if \quad x\le -2 \\ -x^2 & if \quad -2<x<1 \\ -x+4 & if \quad x\ge 1 \end{cases}$

Solution



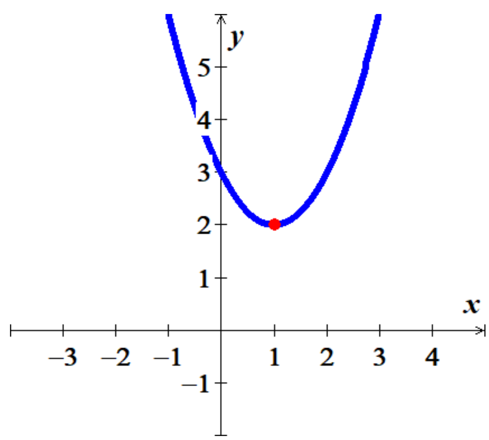
Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x)=x^2-2x+3$$

Solution

Relative Maximum:	*None*
Minimum Point:	$(1,\ 2)$
Increasing:	$(1,\ \infty)$
Decreasing:	$(-\infty,\ 1)$
Domain:	$\mathbb{R}$
Range:	$[2,\ \infty)$



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x) = -x^2 - 2x + 3$$

Solution

Maximum Point: $(-1,\ 4)$

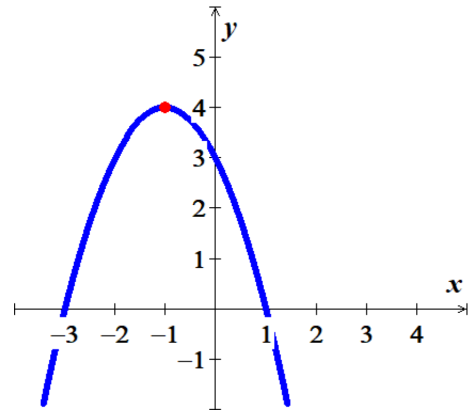
Relative Minimum: *None*

Increasing: $(-\infty,\ -1)$

Decreasing: $(-1,\ \infty)$

Domain: $\mathbb{R}$

Range: $(-\infty,\ 4]$



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x) = -x^3 + 3x^2$$

Solution

Relative Maximum: $(2,\ 4)$

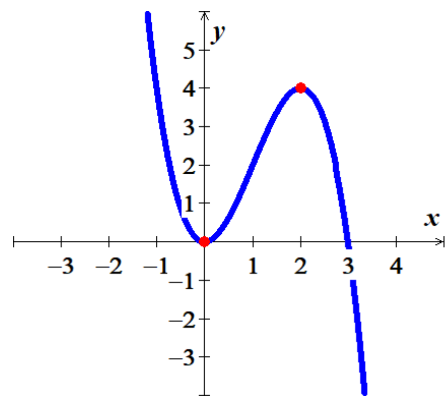
Relative Minimum: $(0,\ 0)$

Increasing: $(0,\ 2)$

Decreasing: $(-\infty,\ 0)\quad(2,\ \infty)$

Domain: $\mathbb{R}$

Range: $\mathbb{R}$



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x) = x^3 - 3x^2$$

Solution

Relative Maximum: $(0,\ 0)$

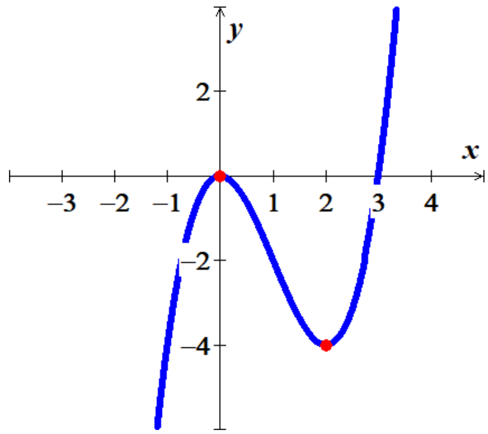
Relative Minimum: $(2,\ -4)$

Increasing: $(-\infty,\ 0)\ (2,\ \infty)$

Decreasing: $(0,\ 2)$

Domain: $\mathbb{R}$

Range: $\mathbb{R}$



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x) = \frac{1}{4}x^4 - 2x^2$$

Solution

Relative Maximum: $(0,\ 0)$

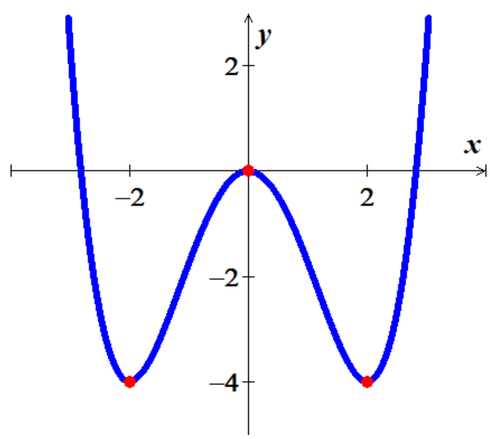
Minimum Points: $(-2,\ -4)\ \&\ (2,\ -4)$

Increasing: $(-2,\ 0) \cup (2,\ \infty)$

Decreasing: $(-\infty,\ -2) \cup (0,\ 2)$

Domain: $\mathbb{R}$

Range: $[-4,\ \infty)$



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x) = \frac{4}{81}x^4 - \frac{8}{9}x^2 + 4$$

Solution

Relative Maximum: $(0,\ 4)$

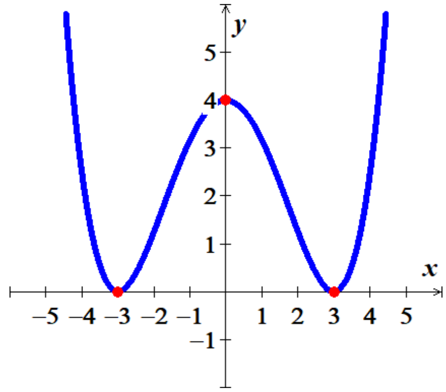
Minimum Points: $(-3,\ 0) \,\&\, (3,\ 0)$

Increasing: $(-3,\ 0) \cup (3,\ \infty)$

Decreasing: $(-\infty,\ -3) \cup (0,\ 3)$

Domain: $\mathbb{R}$

Range: $[0,\ \infty)$



Exercise

The elevation *H*, in *meters*, above sea level at which the boiling point of water is in ***t*** degrees Celsius is given by the function

$$H(t) = 1000(100 - t) + 580(100 - t)^2$$

At what elevation is the boiling point 99.5°.

Solution

$$H(99.5) = 1000(100 - 99.5) + 580(100 - 99.5)^2$$

$$= 645\ m$$

Exercise

A hot-air balloon rises straight up from the ground at a rate of 120 *ft./min*. The balloon is tracked from a rangefinder on the ground at point P, which is 400 *ft*. from the release point Q of the balloon. Let ***d*** = the distance from the balloon to the rangefinder and ***t*** – the time, in minutes, since the balloon was released. Express ***d*** as a function of ***t***.

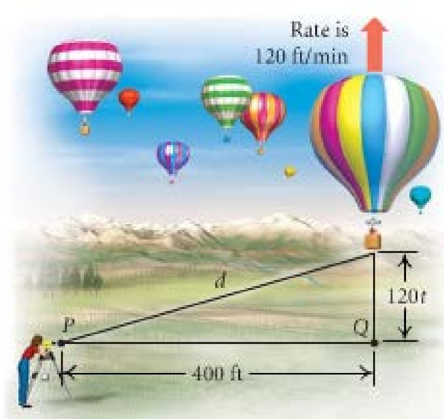
Solution

$$d^2 = (120t)^2 + 400^2$$

$$d = \sqrt{14400t^2 + 160000}$$

$$d = \sqrt{1600\left(9t^2 + 100\right)}$$

$$d(t) = 40\sqrt{9t^2 + 100}$$



Exercise

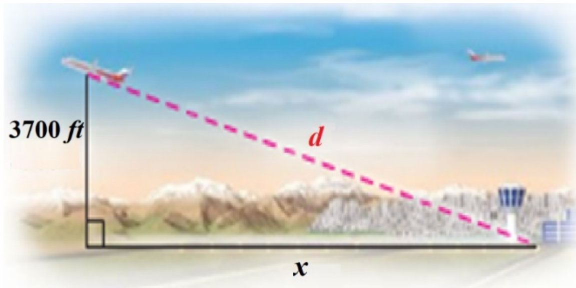
An airplane is flying at an altitude of 3700 *feet*. The slanted distance directly to the airport is d *feet*. Express the horizontal distance x as a function of d.

Solution

$$d^2 = (3,700)^2 + x^2$$

$$h^2 = d^2 - (3700)^2$$

$$\boxed{h(t) = \sqrt{d^2 - (3,700)^2}}$$



Exercise

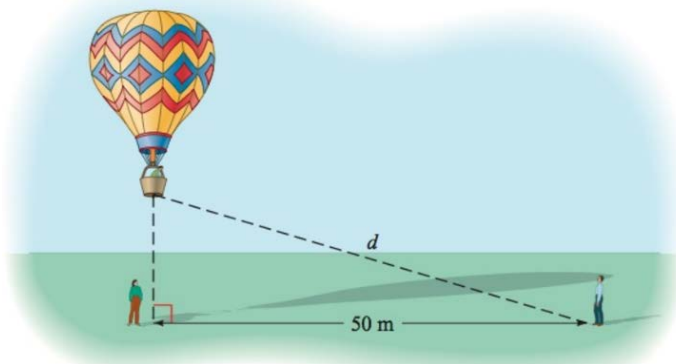
For the first minute of flight, a hot air balloon rises vertically at a rate of 3 *m/sec*. If *t* is the time in *seconds* that the balloon has been airborne, write the distance *d* between the balloon and a point on the ground 50 *meters* from the point to lift off as a function of *t*.

Solution

$$\boxed{h = 3t} \qquad v = \frac{h}{t}$$

$$d^2 = h^2 + 50^2$$

$$\boxed{d(t) = \sqrt{9t^2 + 2,500}}$$



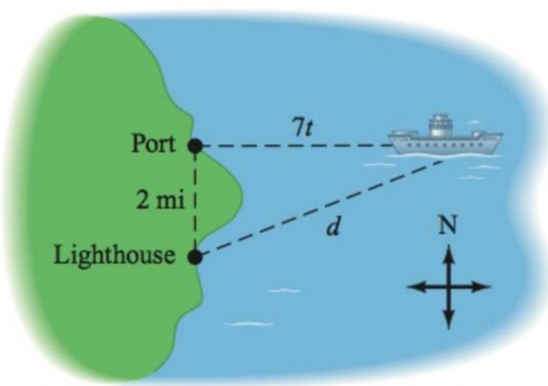
Exercise

A light house is 2 *miles* south of a port. A ship leaves port and sails east at a rate of 7 *miles* per *hour*. Express the distance *d* between the ship and the lighthouse as a function of time, given that the ship has been sailing for *t* hours.

Solution

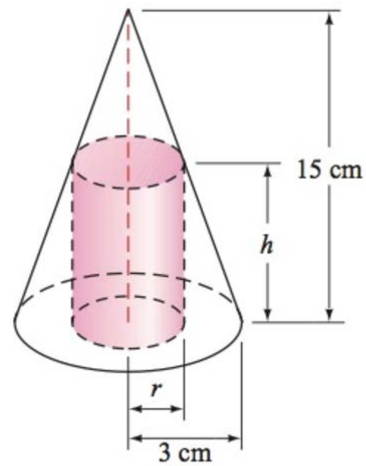
$$d^2 = 4^2 + (7t)^2$$

$$\boxed{d(t) = \sqrt{16 + 49t^2}}$$



Exercise

A cone has an altitude of 15 *cm* and a radius of 3 *cm*. A right circular cylinder of radius *r* and height *h* is inscribed in the cone. Use similar triangles to write *h* as a function of *r*.



Solution

$$\frac{15-h}{15}=\frac{r}{3}$$

$$15-h=5r$$

$$\boxed{h(r)=15-5r}$$

Exercise

Water is flowing into a conical drinking cup with an altitude of 4 *inches* am a radius of 2 *inches*.

a) Write the radius *r* of the surface of the water as a function of its depth *h*.

b) Write the volume *V* of the water as a function of its depth *h*.

Solution

a) $\frac{h}{4}=\frac{r}{2}$

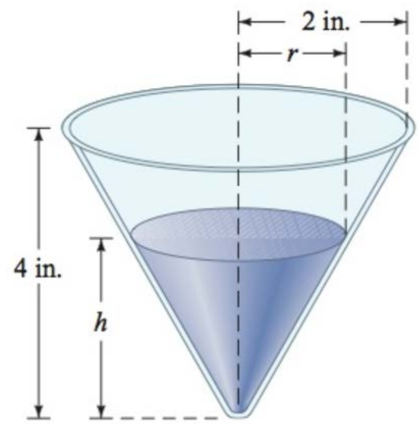
$$\boxed{r(h)=\frac{1}{2}h}$$

b) $Area=\pi r^2$

$$V=\frac{1}{3}\pi r^2 h$$

$$=\frac{1}{3}\pi\left(\frac{h^2}{4}\right)h$$

$$\boxed{=\frac{1}{12}\pi h^3}$$



Exercise

A water tank has the shape of a right circular cone with height 16 *feet* and radius 8 *feet*. Water is running into the tank so that the radius *r* (in *feet*) of the surface of the water is given by $r=1.5t$, where *t* is the time (in *minutes*) that the water has been running.

a) The area *A* of the surface of the water is $A=\pi r^2$. Find $A(t)$ and use it to determine the area of the surface of the water when $t=2\ \ minutes$.

b) The volume *V* of the water is given by $V=\frac{1}{3}\pi r^2 h$. Find $V(t)$ and use it to determine the volume of the water when $t=3\ \ minutes$

Solution

c) $Area = \pi r^2$

$A(t) = \pi\left(\frac{3}{2}t\right)^2$

$= \frac{9\pi}{4}t^2$

d) $\frac{h}{16} = \frac{r}{8}$

$h = 2r$

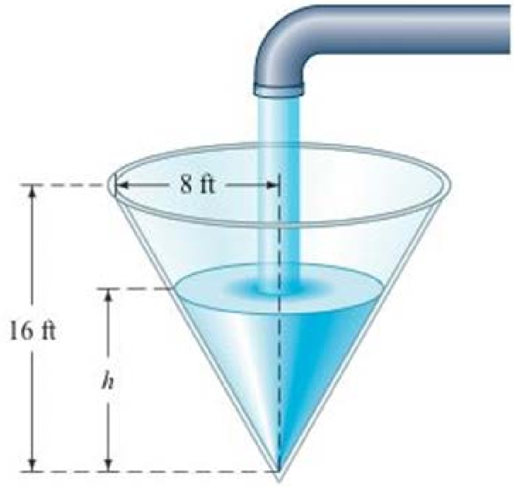
$V(t) = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi r^2(2r)$

$= \frac{2}{3}\pi r^3$

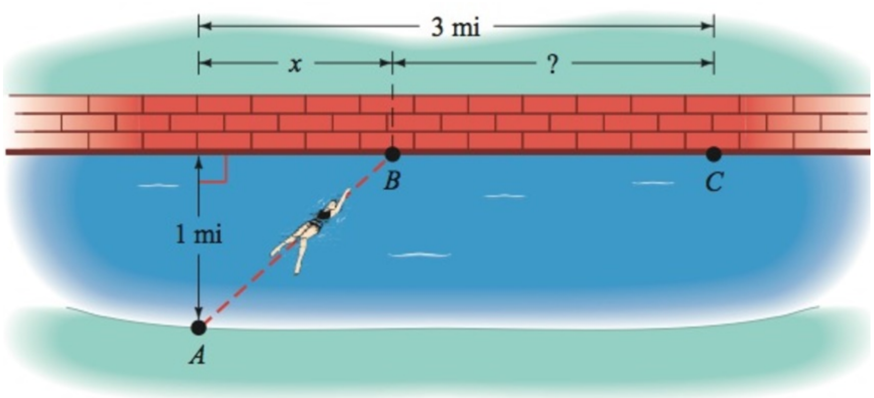
$= \frac{2}{3}\pi\left(\frac{3}{2}t\right)^3$

$= \frac{9}{4}\pi t^3$



Exercise

An athlete swims from point ***A*** to point ***B*** at a rate of 2 *miles* per *hour* and runs from point ***B*** to point ***C*** at a rate of 8 *miles* per *hour*. Use the dimensions in the figure to write the time *t* required to reach point ***C*** as a function of *x*.



Solution

$Swimming\ distance = \sqrt{x^2+1}$

$t_{swim} = \frac{\sqrt{x^2+1}}{2}$ $\qquad t = \frac{d}{v}$

$Running\ distance = 3 - x$

$$t_{run} = \frac{3-x}{8} \qquad\qquad t = \frac{d}{v}$$

$$\boxed{t_{total} = \frac{\sqrt{x^2+1}}{2} + \frac{3-x}{8}}$$

Exercise

A device used in golf to estimate the distance d, in *yards*, to a hole measures the size $\boldsymbol{s}$, in *inches*, that the 7-*foot* pin appears to be in a viewfinder. Express the distance d as a function of $\boldsymbol{s}$.

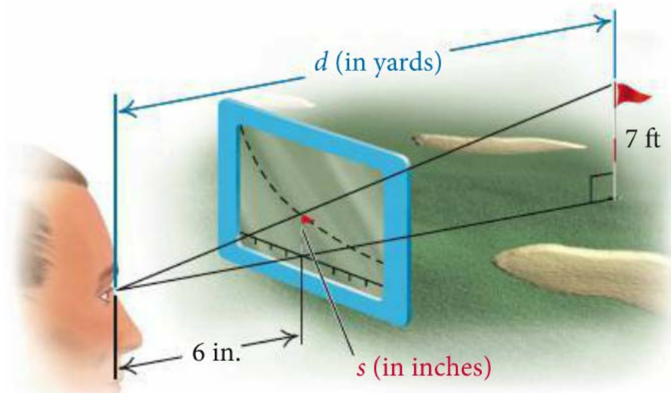
Solution

$$\frac{d}{6} = \frac{7}{s}\frac{ft}{in}$$

$$d = \frac{7}{s}\frac{ft}{in}6in$$

$$d = \frac{42}{s}\,ft\,\frac{1yd}{3\,ft}$$

$$\boxed{d(s) = \frac{14}{s}}$$



Exercise

A *rhombus* is inscribed in a rectangle that is w *meters* wide with a perimeter of 40 m. Each vertex of the rhombus is a midpoint of a side of the rectangle. Express the area of the *rhombus* as a function of the rectangle's width.

Solution

The area of the rhombus = $\frac{1}{2}$ area of the rectangle, since each vertex of the rhombus is a midpoint of a side of the rectangle.

$$Perimter: \quad 2l + 2w = 40 \qquad \textit{Divide both sides by } 2$$

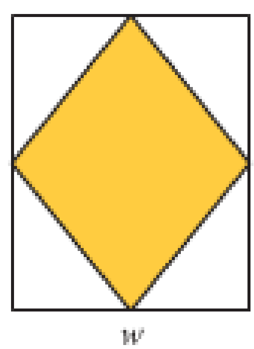
$$l + w = 20$$

$$l = 20 - w$$

$$Area\ of\ the\ rectangle = lw = (20-w)w$$

$$Area\ of\ the\ rhombus = \frac{1}{2}\left(20w - w^2\right)$$

$$\boxed{= -\frac{1}{2}w^2 + 10w}$$



Exercise

The surface area S of a right circular cylinder is given by the formula $S = 2\pi rh + 2\pi r^2$. if the height is twice the radius, find each of the following.

a) A function $S(r)$ for the surface area as a function of r.

b) A function $S(h)$ for the surface area as a function of h.

Solution

Given: $h = 2r$

a) $S = 2\pi rh + 2\pi r^2$

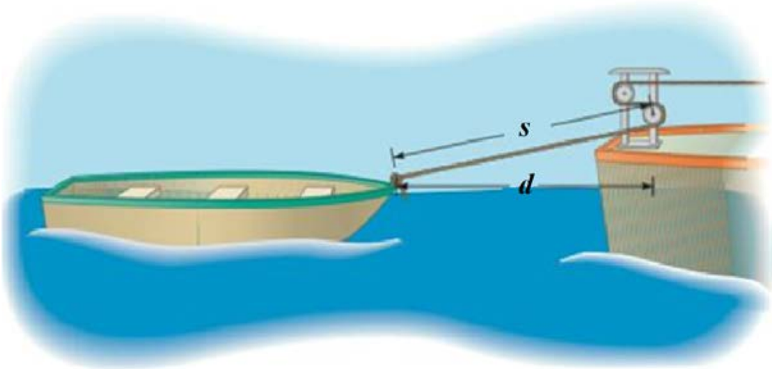
$$S(r) = 2\pi r(2r) + 2\pi r^2$$
$$= 4\pi r^2 + 2\pi r^2$$
$$\boxed{= 6\pi r^2}$$

b) $r = \frac{1}{2}h$

$$S(h) = 2\pi\left(\frac{1}{2}h\right)h + 2\pi\left(\frac{1}{2}h\right)^2$$
$$= \pi h^2 + \frac{1}{2}\pi h^2$$
$$\boxed{= \frac{3}{2}\pi h^2}$$

Exercise

A boat is towed by a rope that runs through a pulley that is 4 *feet* above the point where the rope is tied to the boat. The length (in *feet*) of the rope from the boat to the pulley is given by $s = 48 - t$, where t is the time in *seconds* that the boat has been in tow. The horizontal distance from the pulley to the boat is d.



a) Find $d(t)$

b) Evaluate $s(35)$ and $d(35)$

Solution

a) $s^2 = d^2 + 4^2$

$d^2 = (48-t)^2 - 16$

$d(t) = \sqrt{2,304 - 96t + t^2 - 16}$

$\boxed{= \sqrt{t^2 - 96t + 2,288}}$

b) $s(35) = 48 - 35$

$\boxed{= 13 \ \ feet}$

$d(35) = \sqrt{(48-35)^2 - 16}$

$= \sqrt{13^2 - 16}$

$\boxed{= \sqrt{153} \ \ feet}$

Exercise

The light from a lamppost casts a shadow from a ball that was dropped from a height of 22 *feet* above the ground. The distance *d*, in *feet*, the ball has dropped *t seconds* after it is released is given by $d(t) = 16t^2$. Find the distance *x*, in *feet*, of the shadow from the base of the lamppost as a function of time *t*.

Solution

$\dfrac{22-16t^2}{22} = \dfrac{x-12}{x}$

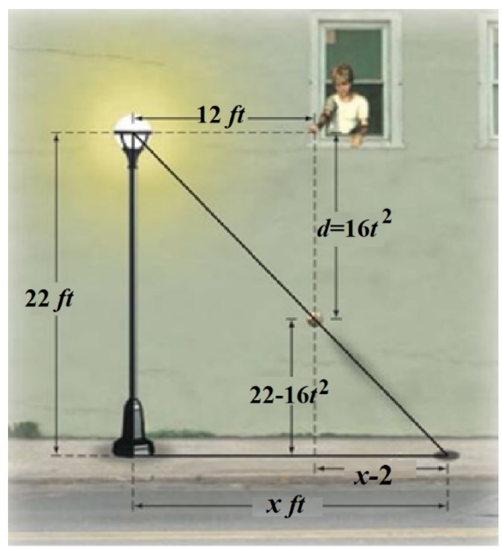
$\left(22-16t^2\right)x = 22(x-12)$

$\left(22-16t^2\right)x = 22x - 264$

$\left(22-16t^2-22\right)x = -264$

$-16t^2 x = -264$

$\boxed{x(t) = \dfrac{33}{2t^2}}$



Exercise

A right circular cylinder of height *h* and a radius *r* is inscribed in a right circular cone with a height of 10 *feet* and a base with radius 6 *feet*.

a) Express the height *h* of the cylinder as a function of *r*.

b) Express the volume *V* of the cylinder as a function of *r*.

c) Express the volume *V* of the cylinder as a function of *h*.

Solution

a) $\dfrac{h}{10}=\dfrac{6-r}{6}$

$$\boxed{h(r)=\frac{5}{3}(6-r)}$$

b) $V=\pi r^2 h$

$$V(r)=\frac{5}{3}\pi r^2(6-r)$$

$$\boxed{=\frac{5}{3}\pi\left(6r^2-r^3\right)}$$

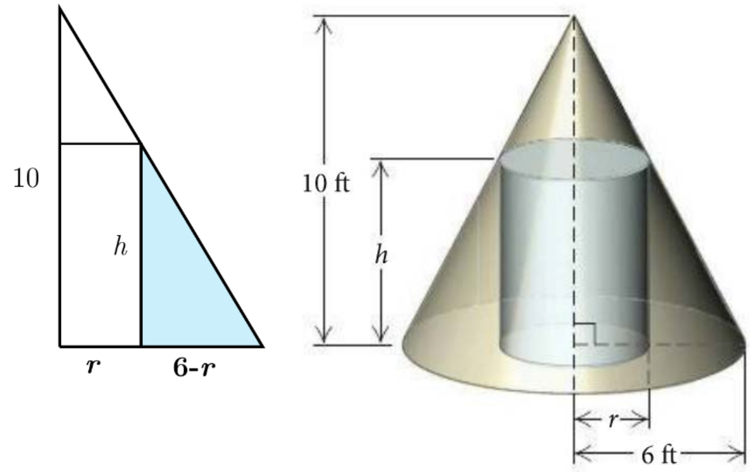
c) $\dfrac{3}{5}h=6-r$

$$r=6-\frac{3}{5}h$$

$$V=\pi r^2 h$$

$$V(h)=\pi\left(\frac{30-3h}{5}\right)^2 h$$

$$\boxed{=\frac{1}{25}\pi h(30-3h)^2}$$



Solution   *Section* 2.2 – Function Operations

Exercise

Find the domain: $f(x) = 7x + 4$

Solution

Domain: $(-\infty, \infty)$

Exercise

Find the domain: $f(x) = |3x - 2|$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain: $f(x) = 3x + \pi$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain: $f(x) = \sqrt{7}x + \frac{1}{2}$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain: $f(x) = -2x^2 + 3x - 5$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain: $f(x) = x^3 - 2x^2 + x - 3$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain: $f(x) = x^2 - 2x - 15$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain $f(x) = 4 - \frac{2}{x}$

Solution

Domain: $x \neq 0$

Exercise

Find the domain $f(x) = \frac{1}{x^4}$

Solution

Domain: $x \neq 0$

Exercise

Find the domain: $g(x) = \frac{3}{x-4}$

Solution

Domain: $x \neq 4$

Exercise

Find the domain $y = \frac{2}{x-3}$

Solution

Domain: $x \neq 3$

Exercise

Find the domain $y = \dfrac{-7}{x-5}$

Solution

Domain: $\underline{x \neq 5}$

Exercise

Find the domain $f(x) = \dfrac{x+5}{2-x}$

Solution

$2 - x \neq 0$

Domain: $\underline{x \neq 2}$

Exercise

Find the domain $f(x) = \dfrac{8}{x+4}$

Solution

$x + 4 \neq 0$

Domain: $\underline{x \neq -4}$

Exercise

Find the domain $f(x) = \dfrac{1}{x+4}$

Solution

Domain: $\underline{x \neq -4}$

Exercise

Find the domain $f(x) = \dfrac{1}{x-4}$

Solution

Domain: $\underline{x \neq 4}$

Exercise

Find the domain $f(x) = \dfrac{3x}{x+2}$

Solution

Domain: $x \neq -2$

Exercise

Find the domain $f(x) = x - \dfrac{2}{x-3}$

Solution

Domain: $x \neq 3$

Exercise

Find the domain $f(x) = x + \dfrac{3}{x-5}$

Solution

Domain: $x \neq 5$

Exercise

Find the domain $f(x) = \dfrac{1}{2}x - \dfrac{8}{x+7}$

Solution

Domain: $x \neq -7$

Exercise

Find the domain $f(x) = \dfrac{1}{x-3} - \dfrac{8}{x+7}$

Solution

Domain: $x \neq -7,\ 3$

Exercise

Find the domain $f(x) = \dfrac{1}{x+4} - \dfrac{2x}{x-4}$

Solution

Domain: $x \neq \pm 4$

Exercise

Fib+cnd the domain $f(x)=\dfrac{3x^2}{x+3}-\dfrac{4x}{x-2}$

Solution

Domain: $\underline{x \neq -3,\ 2\ |}$

Exercise

Find the domain $f(x)=\dfrac{1}{x^2-2x+1}$

Solution

$x^2-2x+1 \neq 0 \qquad a+b+c=0 \ \rightarrow \ x=1,\ \dfrac{c}{a}$

Domain: $\underline{x \neq 1\ |}$

Exercise

Find the domain $f(x)=\dfrac{x}{x^2+3x+2}$

Solution

$x^2+3x+2 \neq 0 \qquad a-b+c=0 \ \rightarrow \ x=-1,\ -\dfrac{c}{a}$

Domain: $\underline{x \neq -1,\ -2\ |}$

Exercise

Find the domain $f(x)=\dfrac{x^2}{x^2-5x+4}$

Solution

$x^2-5x+4 \neq 0 \qquad a+b+c=0 \ \rightarrow \ x=1,\ \dfrac{c}{a}$

Domain: $\underline{x \neq 1,\ 4\ |}$

Exercise

Find the domain $f(x)=\dfrac{1}{x^2-4x-5}$

Solution

$x^2-4x-5 \neq 0 \qquad a-b+c=0 \ \rightarrow \ x=-1,\ -\dfrac{c}{a}$

Domain: $\underline{x \neq -1,\ 5\ |}$

Exercise

Find the domain $g(x) = \dfrac{2}{x^2 + x - 12}$

Solution

$$x^2 + x - 12 \neq 0$$
$$(x+4)(x-3) \neq 0$$

Domain: $\boxed{x \neq -4,\ 3}$ $\boxed{(-\infty, -4) \cup (-4, 3) \cup (3, \infty)}$

Exercise

Find the domain $h(x) = \dfrac{5}{\frac{4}{x} - 1}$

Solution

$x \neq 0$ $\quad \dfrac{4}{x} - 1 \neq 0$

$$\frac{4-x}{x} \neq 0$$
$$4 - x \neq 0$$
$$x \neq 4$$

Domain: $\boxed{x \neq 0,\ 4}$ $\boxed{(-\infty, 0) \cup (0, 4) \cup (4, \infty)}$

Exercise

Find the domain $y = \sqrt{x}$

Solution

$$x \geq 0$$

Domain: $\boxed{x \geq 0}$ $\boxed{[0,\ \infty)}$

Exercise

Find the domain $f(x) = \sqrt{8 - 3x}$

Solution

$$8 - 3x \geq 0$$
$$8 \geq 3x$$

Domain: $\boxed{x \leq \frac{8}{3}}$ $\boxed{\left(-\infty,\ \frac{8}{3}\right]}$

Exercise

Find the domain $\quad y=\sqrt{4x+1}$

Solution

$$4x+1\geq 0 \Rightarrow x\geq -\frac{1}{4}$$

Domain: $x\geq -\frac{1}{4}$ $\quad \left[-\frac{1}{4},\infty\right)$

Exercise

Find the domain $\quad y=\sqrt{7-2x}$

Solution

$$7-2x\geq 0$$

$$-2x\geq -7$$

Domain: $x\leq \frac{7}{2}$ $\quad \left(-\infty,\frac{7}{2}\right]$

Exercise

Find the domain $\quad f(x)=\sqrt{8-x}$

Solution

$$8-x\geq 0$$

Domain: $x\leq -8$ $\quad (-\infty,\ 8]$

Exercise

Find the domain $\quad f(x)=\sqrt{3-2x}$

Solution

Domain: $x\leq \frac{3}{2}$

Exercise

Find the domain $\quad f(x)=\sqrt{3+2x}$

Solution

Domain: $x\geq -\frac{3}{2}$

Exercise

Find the domain $f(x)=\sqrt{5-x}$

Solution

Domain: $\underline{x \le 5}$

Exercise

Find the domain $f(x)=\sqrt{x-5}$

Solution

Domain: $\underline{x \ge 5}$

Exercise

Find the domain $f(x)=\sqrt{6-3x}$

Solution

Domain: $\underline{x \le 2}$

Exercise

Find the domain $f(x)=\sqrt{3x-6}$

Solution

Domain: $\underline{x \ge 2}$

Exercise

Find the domain $f(x)=\sqrt{2x+7}$

Solution

Domain: $\underline{x \ge -\frac{7}{2}}$

Exercise

Find the domain $f(x)=\sqrt{x^2-16}$

Solution

$x^2-16=0$

$x^2 = 16$

$x = \pm 4$

Domain: $\underline{x \le -4 \quad x \ge 4}$

Exercise

Find the domain $\qquad f(x) = \sqrt{16 - x^2}$

Solution

$x = \pm 4$

Domain: $\underline{-4 \le x \le 4}$

Exercise

Find the domain $\qquad f(x) = \sqrt{9 - x^2}$

Solution

$x = \pm 3$

Domain: $\underline{-3 \le x \le 3}$

Exercise

Find the domain $\qquad f(x) = \sqrt{x^2 - 25}$

Solution

$x = \pm 5$

Domain: $\underline{-5 \le x \le 5}$

Exercise

Find the domain $\qquad f(x) = \sqrt{x^2 - 5x + 4}$

Solution

$x^2 - 5x + 4 \qquad\qquad a + b + c = 0 \;\rightarrow\; x = 1, \; \frac{c}{a}$

$x = 1, \; 4$

Domain: $\underline{x \le 1 \quad x \ge 4}$

Exercise

Find the domain $f(x)=\sqrt{x^2+5x+4}$

Solution

$x^2+5x+4 \qquad a-b+c=0 \quad \to \quad x=-1,\ -\frac{c}{a}$

$x=-1,\ -4$

Domain: $\underline{x\leq -4 \quad x\geq -1}$

Exercise

Find the domain $f(x)=\sqrt{x^2+3x+2}$

Solution

$x^2+3x+2 \qquad a-b+c=0 \quad \to \quad x=-1,\ -\frac{c}{a}$

$x=-1,\ -2$

Domain: $\underline{x\leq -2 \quad x\geq -1}$

Exercise

Find the domain $f(x)=\sqrt{x^2-3x+2}$

Solution

$x^2-3x+2 \qquad a+b+c=0 \quad \to \quad x=1,\ \frac{c}{a}$

$x=1,\ 2$

Domain: $\underline{x\leq 1 \quad x\geq 2}$

Exercise

Find the domain $f(x)=\sqrt{x-4}+\sqrt{x+1}$

Solution

$x\geq 4 \quad x\geq -1$

Domain: $\underline{x\geq 4}$

Exercise

Find the domain $f(x)=\sqrt{3-x}+\sqrt{x-2}$

Solution

$x \le 3 \quad x \ge 2$

Domain: $\underline{2 \le x \le 3}$

Exercise

Find the domain $f(x)=\sqrt{1-x}+\sqrt{4-x}$

Solution

$x \le 1 \quad x \le 4$

Domain: $\underline{x \le 1}$

Exercise

Find the domain $f(x)=\sqrt{1-x}-\sqrt{x-3}$

Solution

$x \le 1 \quad x \ge 3$

Domain: $\underline{\varnothing}$

Exercise

Find the domain $f(x)=\sqrt{x+4}-\sqrt{x-1}$

Solution

$x \ge -4 \quad x \ge 1$

Domain: $\underline{x \ge 1}$

Exercise

Find the domain $f(x)=\dfrac{\sqrt{x+1}}{x}$

Solution

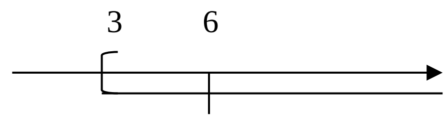
$x+1 \ge 0 \qquad\qquad x \ne 0$

$x \ge -1$

Domain: $\underline{x \ge -1 \quad x \ne 0} \qquad \underline{[-1,\ 0)\cup(0,\ \infty)}$

Exercise

Find the domain $g(x) = \dfrac{\sqrt{x-3}}{x-6}$



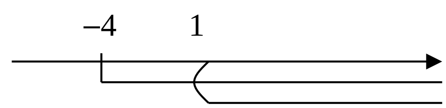
Solution

$$\rightarrow \begin{cases} x \ge 3 \\ x \ne 6 \end{cases}$$

Domain: $x \ge 3 \quad x \ne 6$ $\quad [3,\ 6) \cup (6,\ \infty)$

Exercise

Find the domain $f(x) = \dfrac{\sqrt{x+4}}{\sqrt{x-1}}$



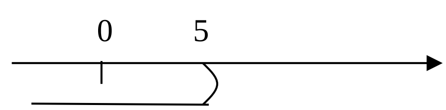
Solution

$$\rightarrow \begin{cases} x \ge -4 \\ x > 1 \end{cases}$$

Domain: $x > 1$ $\quad (1,\ \infty)$

Exercise

Find the domain $f(x) = \dfrac{\sqrt{5-x}}{x}$



Solution

$x \le 5 \quad x \ne 0$

Domain: $x \le 5 \quad x \ne 0$ $\quad (-\infty,\ 0) \cup (0,\ 5]$

Exercise

Find the domain $f(x) = \dfrac{x}{\sqrt{5-x}}$

Solution

Domain: $x < 5$ $\quad (-\infty,\ 5)$

Exercise

Find the domain $f(x) = \frac{1}{x\sqrt{5-x}}$

Solution

$x < 5 \quad x \neq 0$

Domain: $\underline{x < 5 \quad x \neq 0}$ $\underline{(-\infty,\ 0) \cup (0,\ 5)}$

Exercise

Find the domain $f(x) = \frac{x+1}{x^3 - 4x}$

Solution

$x^3 - 4x \neq 0$

$x\left(x^2 - 4\right) \neq 0$

Domain: $\underline{x \neq 0,\ \pm 2}$ $(-\infty, -2) \cup (-2, 0) \cup (0, 2) \cup (2, \infty)$

Exercise

Find the domain $f(x) = \frac{\sqrt{x+5}}{x}$

Solution

$x \geq -5 \quad x \neq 0$

Domain: $\underline{x \geq -5 \quad x \neq 0}$

Exercise

Find the domain $f(x) = \frac{x}{\sqrt{x+5}}$

Solution

$x > -5$

Domain: $\underline{x > -5}$

Exercise

Find the domain $f(x) = \dfrac{1}{x\sqrt{x+5}}$

Solution

$x > -5 \quad x \neq 0$

Domain: $\underline{x > -5 \quad x \neq 0}$

Exercise

Find the domain $f(x) = \dfrac{x+3}{\sqrt{x-3}}$

Solution

Domain: $\underline{x > 3}$

Exercise

Find the domain $f(x) = \dfrac{\sqrt{x+3}}{\sqrt{x-3}}$

Solution

$x \geq -3 \quad x > 3$

Domain: $\underline{x > 3}$

Exercise

Find the domain $f(x) = \dfrac{\sqrt{x-2}}{\sqrt{x+2}}$

Solution

$x \geq 2 \quad x > -2$

Domain: $\underline{x \geq 2}$

Exercise

Find the domain $f(x) = \dfrac{\sqrt{2-x}}{\sqrt{x+2}}$

Solution

$x \leq 2 \quad x > -2$

Domain: $\underline{-2 < x \leq 2}$

Exercise

Find the domain $\quad f(x)=\dfrac{x-4}{\sqrt{x-2}}$

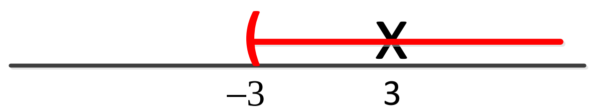
Solution

Domain: $\underline{x>2}|$

Exercise

Find the domain of $f(x)=\dfrac{1}{(x-3)\sqrt{x+3}}$

Solution



$x-3\neq 0 \qquad x+3>0$

$x\neq 3 \qquad x>-3$

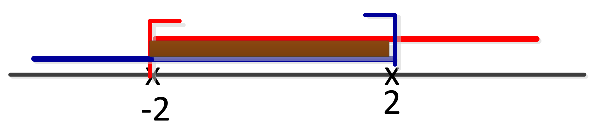
Domain: $\{x \mid x>-3 \ \ and \ \ x\neq 3\}$

$(-3,\ 3)\cup(3,\ \infty)$

Exercise

Find the domain of $f(x)=\sqrt{x+2}+\sqrt{2-x}$

Solution



$x+2\geq 0 \qquad 2-x\geq 0$

$x\geq -2 \qquad -x\geq -2\to x\leq 2$

Domain: $\{x \mid -2\leq x\leq 2\}$

Exercise

Find the domain of $f(x)=\sqrt{(x-2)(x-6)}$

Solution

$x-2\geq 0 \quad x-6\geq 0$

$x\geq 2 \quad x\geq 6$

Domain: $\{x \mid x\leq 2,\ x\geq 6\}$

$\qquad\qquad 2 \qquad\qquad 6$

−	+	+
−	−	+
+	−	+

Exercise

Find the domain of $f(x) = \sqrt{x+3} - \sqrt{4-x}$

Solution

$x \ge -3 \quad x \le 4$

Domain: $-3 \le x \le 4$

Exercise

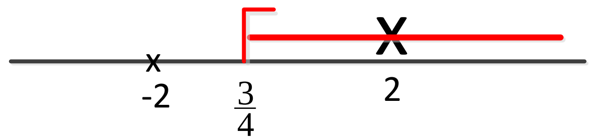
Find the domain of $f(x) = \dfrac{\sqrt{4x-3}}{x^2 - 4}$

Solution

$4x - 3 \ge 0 \qquad x^2 - 4 \ne 0$

$4x \ge 3 \qquad x \ne \pm 2$

$x \ge \dfrac{3}{4}$



Domain: $\left[\dfrac{3}{4},\ 2\right) \cup (2,\ \infty)$

Exercise

Find the domain of $f(x) = \dfrac{4x}{6x^2 + 13x - 5}$

Solution

$6x^2 + 13x - 5 \ne 0$

$$x = \frac{-13 \pm \sqrt{169 + 120}}{12}$$

$$= \begin{cases} \dfrac{-13-17}{12} = -\dfrac{5}{2} \\ \dfrac{-13+17}{12} = \dfrac{1}{3} \end{cases}$$

Domain: $x \ne -\dfrac{5}{2},\ \dfrac{1}{3}$

Exercise

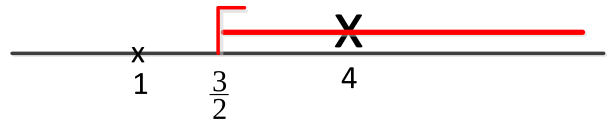
Find the domain of $f(x) = \dfrac{\sqrt{2x-3}}{x^2 - 5x + 4}$

Solution

$2x - 3 \ge 0 \qquad x^2 - 5x + 4 \ne 0$

$2x \ge 3 \qquad x \ne 1,\ 4$

$x \ge \frac{3}{2}$



Domain: $x \ge \frac{3}{2},\ \ x \ne 4$ $\qquad \left[\frac{3}{2},\ 4\right) \cup (4,\ \infty)$

Exercise

Find the domain of $\quad f(x) = \dfrac{x^2}{\sqrt{x^2 - 5x + 4}}$

Solution

$x^2 - 5x + 4 \qquad\qquad a + b + c = 0 \quad \to \quad x = 1,\ \frac{c}{a}$

$x = 1,\ 4$

Domain: $x < 1 \quad x > 4$

Exercise

Find the domain of $\quad f(x) = \dfrac{x + 2}{\sqrt{x^2 + 5x + 4}}$

Solution

$x^2 + 5x + 4 \qquad\qquad a - b + c = 0 \quad \to \quad x = -1,\ -\frac{c}{a}$

$x = -1,\ -4$

Domain: $x < -4 \quad x > -1$

Exercise

Find the domain of $\quad f(x) = \dfrac{\sqrt{x + 2}}{\sqrt{x^2 + 3x + 2}}$

Solution

$x^2 + 3x + 2 \qquad\qquad a - b + c = 0 \quad \to \quad x = -1,\ -\frac{c}{a}$

$x < -2 \quad x > -1$

$\sqrt{x + 2} \quad \to \quad x \ge -2$

Domain: $x > -1$

Exercise

Find the domain of $f(x) = \dfrac{\sqrt{2x+3}}{x^2-6x+5}$

Solution

$x^2-6x+5 \qquad\qquad a+b+c=0 \quad \to \quad x=1, \ \dfrac{c}{a}$

$x \neq 1, \ 5$

$\sqrt{2x+3} \quad \to \quad x \geq -\dfrac{3}{2}$

Domain: $\boxed{x \geq -\dfrac{3}{2} \quad x \neq 1, \ 5}$

Exercise

Let $f(x) = 4x-3$ and $g(x) = 5x+7$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\dfrac{f}{g}\right)(x)$

Solution

a) $(f+g)(x) = 4x-3+5x+7$

$\boxed{=9x+4}$

Domain: $\boxed{\mathbb{R}}$

b) $(f-g)(x) = 4x-3-(5x+7)$

$= 4x-3-5x-7$

$\boxed{=-x-10}$

Domain: $\boxed{\mathbb{R}}$

c) $(fg)(x) = (4x-3)(5x+7)$

$\boxed{=20x^2+13x-21}$

Domain: $\boxed{\mathbb{R}}$

d) $\boxed{\left(\dfrac{f}{g}\right)(x) = \dfrac{4x-3}{5x+7}}$

Domain: $\boxed{x \neq -\dfrac{7}{5}}$

Exercise

Let $f(x) = 2x^2 + 3$ and $g(x) = 3x - 4$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\frac{f}{g}\right)(x)$

Solution

a) $(f+g)(x) = 2x^2 + 3 + 3x - 4$

$$= 2x^2 + 3x - 1$$

Domain: $\mathbb{R}$

b) $(f-g)(x) = 2x^2 + 3 - (3x - 4)$

$$= 2x^2 + 3 - 3x + 4$$

$$= 2x^2 - x + 7$$

Domain: $\mathbb{R}$

c) $(fg)(x) = \left(2x^2 + 3\right)(3x - 4)$

$$= 6x^2 + x - 12$$

Domain: $\mathbb{R}$

d) $\left(\frac{f}{g}\right)(x) = \frac{2x^2 + 3}{3x - 4}$

Domain: $x \neq -\frac{4}{3}$

Exercise

Let $f(x) = x^2 - 2x - 3$ and $g(x) = x^2 + 3x - 2$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\frac{f}{g}\right)(x)$

Solution

a) $(f+g)(x) = x^2 - 2x - 3 + x^2 + 3x - 2$

$$= 2x^2 + x - 5$$

Domain: $\mathbb{R}$

b) $(f-g)(x) = x^2 - 2x - 3 - x^2 - 3x + 2$

$$= -5x - 1$$

Domain: $\mathbb{R}$

c) $(fg)(x) = \left(x^2 - 2x - 3\right)\left(x^2 + 3x - 2\right)$

$$= x^4 + 3x^3 - 2x^2 - 2x^3 - 6x^2 + 4x - 3x^2 - 9x + 6$$

$$= x^4 + x^3 - 11x^2 - 5x + 6$$

Domain: $\mathbb{R}$

d) $\left(\dfrac{f}{g}\right)(x) = \dfrac{x^2 - 2x - 3}{x^2 + 3x - 2}$

Domain: $x \ne \dfrac{-3 \pm \sqrt{17}}{2}$

Exercise

Let $f(x) = \sqrt{4x-1}$ and $g(x) = \dfrac{1}{x}$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\dfrac{f}{g}\right)(x)$

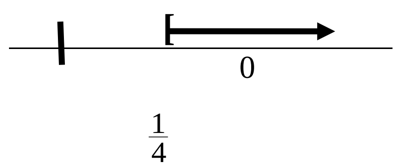
Solution

a) $(f+g)(x)$

$(f+g)(x) = \sqrt{4x-1} + \dfrac{1}{x}$

$4x - 1 \ge 0 \qquad x \ne 0$

$x \ge \dfrac{1}{4}$



Domain: $\left[\dfrac{1}{4}, \infty\right)$

b) $(f-g)(x)$

$(f-g)(x) = \sqrt{4x-1} - \dfrac{1}{x}$

$4x - 1 \ge 0 \qquad x \ne 0$

$x \ge \dfrac{1}{4}$

Domain: $\left[\dfrac{1}{4}, \infty\right)$

c) $(fg)(x) = \sqrt{4x-1}\left(\dfrac{1}{x}\right)$

$$= \frac{\sqrt{4x-1}}{x}$$

$$4x - 1 \ge 0 \qquad\qquad x \ne 0$$

$$x \ge \frac{1}{4}$$

Domain: $\left[\frac{1}{4}, \infty\right)$

d) $\left(\frac{f}{g}\right)(x) = \dfrac{\sqrt{4x-1}}{\frac{1}{x}}$ *Domain:* $x \ne 0$

$$= x\sqrt{4x-1}$$

$$4x - 1 \ge 0$$

$$x \ge \frac{1}{4}$$

Domain: $\left[\frac{1}{4}, \infty\right)$

Exercise

Given that $f(x) = x+1$ and $g(x) = \sqrt{x+3}$

a) Find $(f+g)(x)$

b) Find the domain of $(f+g)(x)$

c) Find: $(f+g)(6)$

Solution

a) $(f+g)(x) = f(x) + g(x)$

$$= x + 1 + \sqrt{x+3}$$

b) $x + 3 \ge 0 \to x \ge -3$

Domain $= [-3, \infty)$

c) $(f+g)(6) = 6 + 1 + \sqrt{6+3}$

$$= 10$$

Exercise

Given that $f(x) = x^2 - 4$ and $g(x) = x + 2$

a) Find $(f+g)(x)$ and its domain

b) Find $(f/g)(x)$ and its domain

Solution

a) $(f+g)(x) = x^2 - 4 + x + 2$

$= x^2 + x - 2$

Domain: $\mathbb{R}$

b) $\dfrac{f(x)}{g(x)} = \dfrac{x^2-4}{x+2}$

$x \neq -2$

Domain: $(-\infty,-2)\cup(-2,\infty)$

Exercise

Let $f(x) = x^2 + 1$ and $g(x) = 3x + 5$. Find $(f+g)(1)$, $(f-g)(-3)$, $(fg)(5)$, and $\left(\frac{f}{g}\right)(0)$

Solution

a) $(f+g)(1) = f(1) + g(1)$

$= 1^2 + 1 + 3(1) + 5$

$= 10$

b) $(f-g)(-3) = f(-3) - g(-3)$

$= (-3)^2 + 1 - (3(-3) + 5)$

$= 10$

c) $(fg)(5) = f(5)\cdot g(5)$

$= \left(5^2 + 1\right)\cdot(3(5) + 5)$

$= (26)\cdot(20)$

$= 520$

d) $\left(\frac{f}{g}\right)(0) = \dfrac{f(0)}{g(0)}$

$= \dfrac{0^2 + 1}{3(0) + 5}$

$= \dfrac{1}{5}$

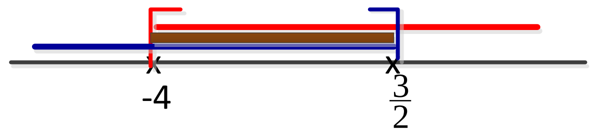
Exercise

Find $(f+g)(x),\ \ (f-g)(x),\ \ (f\cdot g)(x),\ \ and\ \ (f/g)(x)$ and the domain of

$f(x)=\sqrt{3-2x},\quad g(x)=\sqrt{x+4}$

Solution

$$f(x)+g(x)=\sqrt{3-2x}+\sqrt{x+4}$$



$$3-2x\ge 0 \qquad x+4\ge 0$$
$$-2x\ge -3 \qquad x\ge -4$$
$$x\le \frac{3}{2}$$

Domain: $\left\{x\mid -4\le x\le \frac{3}{2}\right\}$

$$f(x)-g(x)=\sqrt{3-2x}-\sqrt{x+4}$$

$$3-2x\ge 0 \qquad x+4\ge 0$$
$$-2x\ge -3 \qquad x\ge -4$$
$$x\le \frac{3}{2}$$

Domain: $\left\{x\mid -4\le x\le \frac{3}{2}\right\}$

$$\begin{aligned}(f\cdot g)(x)&=\left(\sqrt{3-2x}\right)\left(\sqrt{x+4}\right)\\&=\sqrt{(3-2x)(x+4)}\\&=\sqrt{-2x^2-5x+12}\end{aligned}$$

$$3-2x\ge 0 \qquad x+4\ge 0$$
$$-2x\ge -3 \qquad x\ge -4$$
$$x\le \frac{3}{2}$$

Domain: $\left\{x\mid -4\le x\le \frac{3}{2}\right\}$

$$\begin{aligned}(f/g)(x)&=\frac{\sqrt{3-2x}}{\sqrt{x+4}}\,\frac{\sqrt{x+4}}{\sqrt{x+4}}\\&=\frac{\sqrt{-2x^2-5x+12}}{x+4}\end{aligned}$$

$$3-2x\ge 0 \qquad x+4>0$$
$$-2x\ge -3 \qquad x>-4$$
$$x\le \frac{3}{2}$$

Domain: $\left\{x\mid -4< x\le \frac{3}{2}\right\}$ $\qquad \left(-4,\ \frac{3}{2}\right]$

Exercise

Find $(f+g)(x)$, $(f-g)(x)$, $(f\cdot g)(x)$, *and* $(f/g)(x)$ and the domain of

$f(x)=\dfrac{2x}{x-4}$, $g(x)=\dfrac{x}{x+5}$

Solution

$$\begin{aligned}(f+g)(x)&=\frac{2x}{x-4}+\frac{x}{x+5}\\&=\frac{2x(x+5)+x(x-4)}{(x-4)(x+5)}\\&=\frac{2x^2+10x+x^2-4x}{(x-4)(x+5)}\\&=\frac{3x^2+6x}{(x-4)(x+5)}\end{aligned}$$

$x-4\neq 0 \qquad x+5\neq 0$

$x\neq 4 \qquad x\neq -5$

Domain: $\{x \mid x\neq -5,4\}$ $\qquad (-\infty,\ -5)\cup(-5,\ 4)\cup(4,\ \infty)$

$$\begin{aligned}(f-g)(x)&=\frac{2x}{x-4}-\frac{x}{x+5}\\&=\frac{2x(x+5)-x(x-4)}{(x-4)(x+5)}\\&=\frac{2x^2+10x-x^2+4x}{(x-4)(x+5)}\\&=\frac{x^2+14x}{(x-4)(x+5)}\end{aligned}$$

$x\neq 4 \qquad x\neq -5$

Domain: $\{x \mid x\neq -5,4\}$

$$\begin{aligned}(f\bullet g)(x)&=\frac{2x}{x-4}\frac{x}{x+5}\\&=\frac{2x^2}{(x-4)(x+5)}\end{aligned}$$

$x\neq 4 \qquad x\neq -5$

Domain: $\{x \mid x\neq -5,4\}$

$$\begin{aligned}(f/g)(x)&=\frac{2x}{x-4}\div\frac{x}{x+5}\\&=\frac{2x}{x-4}\frac{x+5}{x}\end{aligned}$$

$$= 2\frac{x+5}{x-4}$$

$x \neq 4 \qquad x \neq -5$

Domain: $\{x \mid x \neq -5, 4\}$

Exercise

Find $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, *and* $(f/g)(x)$ of $f(x) = x-5$ and $g(x) = x^2 - 1$

Solution

a) $(f+g)(x) = f(x) + g(x)$

$$= x - 5 + x^2 - 1$$

$$\underline{= x^2 + x - 6}$$

b) $(f-g)(x) = f(x) - g(x)$

$$= x - 5 - (x^2 - 1)$$

$$= x - 5 - x^2 + 1$$

$$\underline{= -x^2 + x - 4}$$

c) $(fg)(x) = f(x)g(x)$

$$= (x-5)(x^2-1)$$

$$= x^3 - x - 5x^2 + 5$$

$$\underline{= x^3 - 5x^2 - x + 5}$$

d) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$$\underline{= \frac{x-5}{x^2-1}}$$

Exercise

For the function *f* given by $f(x) = 9x+5$, find the difference quotient $\frac{f(x+h)-f(x)}{h}$

Solution

$f(x+h) = 9(x+h) + 5 = 9x + 9h + 5$

$$\frac{f(x+h)-f(x)}{h} = \frac{\underbrace{9x+9h+5}_{f(x+h)} \; - \; \underbrace{(9x+5)}_{f(x)}}{h}$$

$$= \frac{9x+9h+5 \; - \; 9x-5}{h}$$

$$= \frac{9h}{h}$$

$$\underline{= 9}$$

Exercise

For the function *f* given by $f(x) = 6x + 2$, find the difference quotient $\dfrac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{6(x+h) + 2 - (6x+2)}{h} \\ &= \frac{6x + 6h + 2 - 6x - 2}{h} \\ &= \frac{6h}{h} \\ &= 6\end{aligned}$$

Exercise

For the function *f* given by $f(x) = 4x + 11$, find the difference quotient $\dfrac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{4(x+h) + 11 - (4x+11)}{h} \\ &= \frac{4x + 4h + 11 - 4x - 11}{h} \\ &= \frac{4h}{h} \\ &= 4\end{aligned}$$

Exercise

For the function *f* given by $f(x) = 3x - 5$, find the difference quotient $\dfrac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{3(x+h) - 5 - 3x + 5}{h} \\ &= \frac{3x + 3h - 5 - 3x + 5}{h} \\ &= \frac{3h}{h} \\ &= 3\end{aligned}$$

Exercise

For the function *f* given by $f(x) = -2x - 3$, find the difference quotient $\dfrac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{-2(x+h)-3+2x+3}{h} \\ &= \frac{-2x-2h-3+2x+3}{h} \\ &= \frac{-2h}{h} \\ &= -2\end{aligned}$$

Exercise

For the function *f* given by $f(x) = -4x + 3$, find the difference quotient $\dfrac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{-4(x+h)+3+4x-3}{h} \\ &= \frac{-4x-4h+3+4x-3}{h} \\ &= \frac{-4h}{h} \\ &= -4\end{aligned}$$

Exercise

For the function *f* given by $f(x) = 3x - 6$, find the difference quotient $\dfrac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{3(x+h)-6-3x+6}{h} \\ &= \frac{3x+3h-6-3x+6}{h} \\ &= \frac{3h}{h} \\ &= 3\end{aligned}$$

Exercise

For the function *f* given by $f(x) = -5x - 7$, find the difference quotient $\dfrac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{-5(x+h)-7+5x+7}{h} \\
&= \frac{-5x-5h-7+5x+7}{h} \\
&= \frac{-5h}{h} \\
&= -5
\end{aligned}$$

Exercise

Given the function: $f(x) = 2x^2$. Find and simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}
f(x+h) &= 2(x+h)^2 \\
&= 2\left(x^2 + 2hx + h^2\right) \\
&= 2x^2 + 4hx + 2h^2
\end{aligned}$$

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{2x^2 + 4hx + 2h^2 - 2x^2}{h} \\
&= \frac{4hx + 2h^2}{h} \\
&= \frac{4hx}{h} + \frac{2h^2}{h} \\
&= 4x + 2h
\end{aligned}$$

Exercise

For the function *f* given by $f(x) = 5x^2$, find the difference quotient $\dfrac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{5(x+h)^2 - 5x^2}{h} \\
&= \frac{5\left(x^2 + 2hx + h^2\right) - 5x^2}{h}
\end{aligned}$$

$$= \frac{5x^2 + 10hx + 5h^2 - 5x^2}{h}$$

$$= \frac{10hx + 5h^2}{h}$$

$$\underline{= 10x + 5h}$$

Exercise

For the function *f* given by $f(x) = 3x^2 - 4x$, find the difference quotient $\dfrac{f(x+h) - f(x)}{h}$

Solution

$$\frac{f(x+h) - f(x)}{h} = \frac{3(x+h)^2 - 4x - 3x^2 + 4x}{h}$$

$$= \frac{3\left(x^2 + 2hx + h^2\right) - 4(x+h) - 3x^2 + 4x}{h}$$

$$= \frac{3x^2 + 6hx + 3h^2 - 4x - 4h - 3x^2 + 4x}{h}$$

$$= \frac{6hx + 3h^2 - 4h}{h}$$

$$\underline{= 6x + 3h - 4}$$

Exercise

For the function *f* given by $f(x) = 2x^2 - 3x$, find the difference quotient $\dfrac{f(x+h) - f(x)}{h}$

Solution

$$f(x+h) = 2(\text{—})^2 - 3(\text{—})$$

$$= 2(x+h)^2 - 3(x+h) \qquad (a+b)^2 = a^2 + 2ab + b^2$$

$$= 2\left(x^2 + 2xh + h^2\right) - 3x - 3h$$

$$= 2x^2 + 4xh + 2h^2 - 3x - 3h$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\overbrace{2x^2 + 4xh + 2h^2 - 3x - 3h}^{f(x+h)} - \overbrace{(2x^2 - 3x)}^{f(x)}}{h}$$

$$= \frac{2x^2 + 4xh + 2h^2 - 3x - 3h - 2x^2 + 3x}{h}$$

$$= \frac{4xh + 2h^2 - 3h}{h}$$

$$= \frac{4xh}{h} + \frac{2h^2}{h} - \frac{3h}{h}$$

$$\underline{= 4x + 2h - 3}$$

Exercise

For the function f given by $f(x) = 2x^2 - x - 3$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$f(x+h) = 2(x+h)^2 - (x+h) - 3$$

$$= 2\left(x^2 + 2hx + h^2\right) - x - h - 3$$

$$= 2x^2 + 4hx + 2h^2 - x - h - 3$$

$$\frac{f(x+h) - f(x)}{h} = \frac{2x^2 + 2h^2 + 4hx - x - h - 3 - \left(2x^2 - x - 3\right)}{h}$$

$$= \frac{2x^2 + 2h^2 + 4hx - x - h - 3 - 2x^2 + x + 3}{h}$$

$$= \frac{2h^2 + 4hx - h}{h}$$

$$= \frac{2h^2}{h} + \frac{4hx}{h} - \frac{h}{h}$$

$$\underline{= 2h + 4x - 1}$$

Exercise

For the given function $f(x) = 2x^2 - x - 3$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\frac{f(x+h) - f(x)}{h} = \frac{2(x+h)^2 - (x+h) - 3 - 2x^2 + x + 3}{h}$$

$$= \frac{2\left(x^2 + 2hx + h^2\right) - x - h - 2x^2 + x}{h}$$

$$= \frac{2x^2 + 4hx + 2h^2 - h - 2x^2}{h}$$

$$= \frac{4hx + 2h^2 - h}{h}$$

$$\underline{= 4x + 2h - 1}$$

Exercise

For the given function $f(x)=x^2-2x+5$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h)-f(x)}{h}&=\frac{(x+h)^2-2(x+h)+5-x^2+2x-5}{h}\\&=\frac{x^2+2hx+h^2-2x-2h-x^2+2x}{h}\\&=\frac{2hx+h^2-2h}{h}\\&=2x+h-2\end{aligned}$$

Exercise

For the given function $f(x)=3x^2-2x+5$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h)-f(x)}{h}&=\frac{3(x+h)^2-2(x+h)+5-3x^2+2x-5}{h}\\&=\frac{3\left(x^2+2hx+h^2\right)-2x-2h-3x^2+2x}{h}\\&=\frac{3x^2+6hx+3h^2-2h-3x^2}{h}\\&=\frac{6hx+3h^2-2h}{h}\\&=6x+3h-2\end{aligned}$$

Exercise

For the given function $f(x)=-2x^2-3x+7$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h)-f(x)}{h}&=\frac{-2(x+h)^2-3(x+h)+7+2x^2+3x-7}{h}\\&=\frac{-2\left(x^2+2hx+h^2\right)-3x-3h+2x^2+3x}{h}\\&=\frac{-2x^2-4hx-2h^2-3h+2x^2}{h}\end{aligned}$$

$$= \frac{-4hx - 2h^2 - 3h}{h}$$

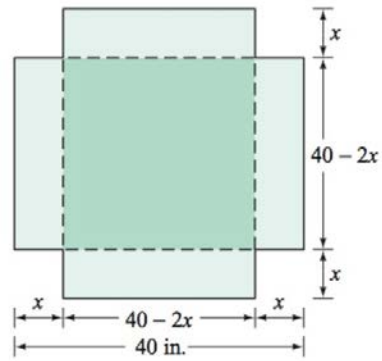
$$\underline{= -4x - 2h - 3}$$

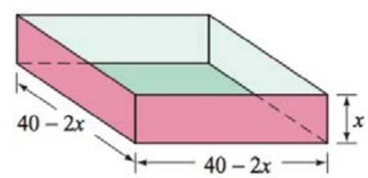
Exercise

An open box is to be made from a square piece of cardboard that measures 40 *inches* on each side, to construct the box, squares that measure *x inches* on each side are cut from each corner of the cardboard.

a) Express the volume *V* of the box as a function of *x*.

b) Determine the domain of *V*.





Solution

a) $V(x) = x(40-2x)^2$

$$= x\left(1600 - 160x + 4x^2\right)$$

$$\underline{= 4x^3 - 160x^2 + 1600x}$$

b) $40 - 2x = 0$

$\underline{x = 20}$

Domain: $\{x \mid \ 0 < x < 20\}$

Exercise

A child 4 *feet* tall is standing near a street lamp that is 12 *feet* high. The light from the lamp casts a shadow.

a) Find the length *l* of the shadow as a function of the distance *d* of the child from the lamppost.

b) What is the domain of the function?

c) What is the length of the shadow when the child is 8 *feet* from the base of the lamppost?

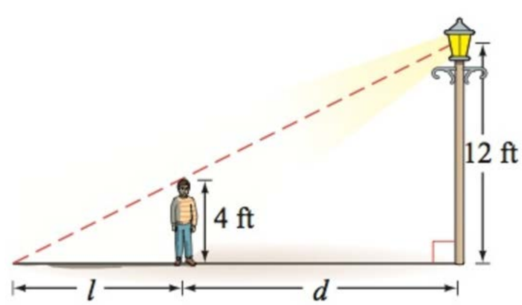
Solution

a) $\dfrac{l+d}{12} = \dfrac{l}{4}$

$l + d = 3l$

$2l = d$

$\underline{l(d) = \frac{1}{2}d}$



b) **Domain**: $\{x \mid \ 0 \le d < \infty\}$

c) **Given**: $d = 8$

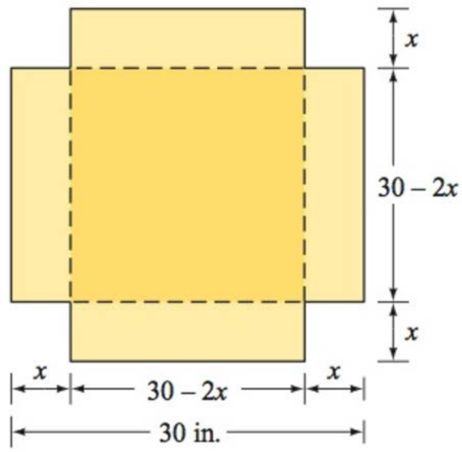
$\underline{l = 4 \textit{ feet}}$

Exercise

An open box is to be made from a square piece of cardboard with the dimensions 30 *inches* by 30 *inches* by cutting out squares of area x^2 from each corner.

a) Express the volume *V* of the box as a function of *x*.

b) Determine the domain of *V*.



Solution

a) $V(x) = x(30-2x)^2$

$$= x\left(900 - 120x + 4x^2\right)$$

$$\underline{= 4x^3 - 120x^2 + 900x}$$

b) $30 - 2x = 0$

$\underline{x = 15}$

Domain: $\{x \mid 0 < x < 15\}$

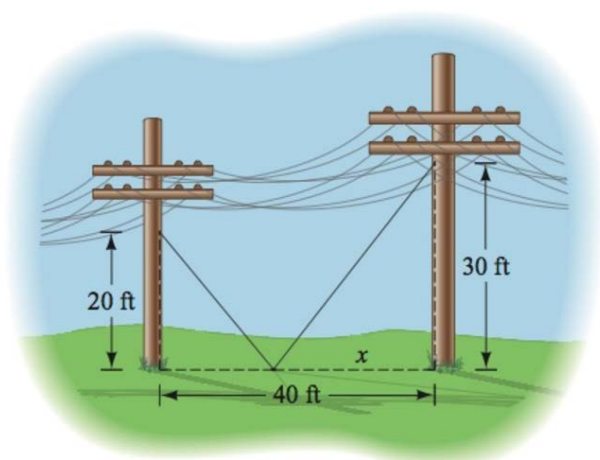
Exercise

Two guy wires are attached to utility poles that are 40 *feet* apart.

a) Find the total length of the two guy wires as a function of *x*.

b) What is the domain of this function?

Solution



a) $\ell_1 = \sqrt{(40-x)^2 + 20^2}$

$$= \sqrt{1,600 - 80x + x^2 + 400}$$

$$= \sqrt{2,000 - 80x + x^2}$$

$$\ell_2 = \sqrt{x^2 + 30^2}$$

$$= \sqrt{x^2 + 900}$$

$$\underline{\ell(x) = \sqrt{2,000 - 80x + x^2} + \sqrt{x^2 + 900}}$$

b) ***Domain***: $[0,\ 40]$

Exercise

A rancher has 360 *yards* of fencing with which to enclose two adjacent rectangular corrals, one for sheep and one for cattle. A river forms one side of the corrals. Suppose the width of each corral is *x* yards.

a) Express the total area of the two corrals as a function of *x*.

b) Find the domain of the function.

Solution

a) $P = 3x + l = 360$

$l = 360 - 3x$

$A = xl$

$= x(360 - 3x)$

$\boxed{A(x) = 360x - 3x^2}$

b) $x(360 - 3x) = 0$

$\boxed{x = 0}$

$360 - 3x = 0$

$3x = 360$

$\Rightarrow \boxed{x = 120}$

Domain: $\boxed{0 < x < 120}$

Exercise

A rectangle is bounded by the *x*- and *y-axis* of $y = -\frac{1}{2}x + 4$

a) Find the area of the rectangle as a function of *x*.

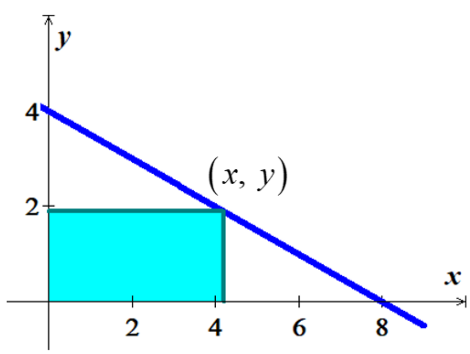
b) What is the domain of this function.

Solution

a) *Area* $= xy$

$A(x) = x\left(-\frac{1}{2}x + 4\right)$

$\boxed{= -\frac{1}{2}x^2 + 4x}$



b) $x\left(-\frac{1}{2}x + 4\right) = 0$

$x = 0 \quad x = 8$

Domain: $\boxed{0 < x < 8}$

Solution *Section* 2.3 – Composition Functions

Exercise

Given that $f(x)=2x-5$ and $g(x)=x^2-3x+8$, find $(f\circ g)(x)$, $(g\circ f)(x)$ and their domain then find $(f\circ g)(7)$

Solution

$$f(g(x))=f\left(x^2-3x+8\right) \qquad \textit{Domain: } (-\infty,\infty)$$

$$=2(\text{------------})-5$$

$$=2\left(2x^2-3x+8\right)-5$$

$$=2x^2-6x+16-5$$

$$=2x^2-6x+11 \qquad \textit{Domain: } (-\infty,\infty)$$

Domain: $\mathbb{R}$

$$g(f(x))=g(2x-5) \qquad \textit{Domain: } (-\infty,\infty)$$

$$=(---)^2-3(---)+8$$

$$=(2x-5)^2-3(2x-5)+8$$

$$=4x^2-20x+25-6x+15+8$$

$$=4x^2-26x+48 \qquad \textit{Domain: } (-\infty,\infty)$$

Domain: $\mathbb{R}$

$$f(g(7))=2(7)^2-6(7)+11=67$$

Exercise

Given that $f(x)=\sqrt{x}$ and $g(x)=x-1$, find

a) $(f\circ g)(x)=f(g(x))$

b) $(g\circ f)(x)=g(f(x))$

c) $(f\circ g)(2)=f(g(2))$

Solution

a) $(f\circ g)(x)=f(g(x))$

$$=f(x-1)$$

$$=\sqrt{x-1}$$

b) $(g \circ f)(x) = g(f(x))$

$$= g\left(\sqrt{x}\right)$$

$$\boxed{= \sqrt{x} - 1}$$

c) $(f \circ g)(2) = f(g(2))$

$$= \sqrt{x-1}$$

$$= \sqrt{2-1}$$

$$\boxed{= 1}$$

Exercise

Given that $f(x) = \dfrac{x}{x+5}$ and $g(x) = \dfrac{6}{x}$, find

a) $(f \circ g)(x) = f(g(x))$

b) $(g \circ f)(x) = g(f(x))$

c) $(f \circ g)(2) = f(g(2))$

Solution

a) $(f \circ g)(x) = f(g(x))$

$$= f\left(\frac{6}{x}\right)$$

$$= \frac{\frac{6}{x}}{\frac{6}{x}+5}$$

$$= \frac{\frac{6}{x}}{\frac{6+5x}{x}}$$

$$\boxed{= \frac{6}{6+5x}}$$

b) $(g \circ f)(x) = g(f(x))$

$$= g\left(\frac{x}{x+5}\right)$$

$$= \frac{6}{\frac{x}{x+5}}$$

$$= \frac{6(x+5)}{x}$$

$$\boxed{= \frac{6x+30}{x}}$$

c) $(f \circ g)(2) = f(g(2))$

$$= \frac{6}{6+5(2)}$$

$$= \frac{6}{16}$$

$$\boxed{= \frac{3}{8}}$$

Exercise

Find $(f \circ g)(x)$, $(g \circ f)(x)$, $f(g(-2))$ *and* $g(f(3))$: $f(x) = 2x^2 + 3x - 4$, $g(x) = 2x - 1$

Solution

$$f(g(x)) = f(2x-1)$$

$$= 2(2x-1)^2 + 3(2x-1) - 4$$

$$= 2\left(4x^2 - 4x + 1\right) + 6x - 3 - 4$$

$$= 8x^2 - 8x + 2 + 6x - 7$$

$$\boxed{= 8x^2 - 2x - 5}$$

$$g(f(x)) = g\left(2x^2 + 3x - 4\right)$$

$$= 2\left(2x^2 + 3x - 4\right) - 1$$

$$= 4x^2 + 6x - 8 - 1$$

$$\boxed{= 4x^2 + 6x - 9}$$

$$f(g(-2)) = 8(-2)^2 - 2(-2) - 5$$

$$\boxed{= 31}$$

$$g(f(3)) = 4(3)^2 + 6(3) - 9$$

$$\boxed{= 45}$$

Exercise

Find $(f \circ g)(x)$, $(g \circ f)(x)$, $f(g(-2))$ *and* $g(f(3))$: $f(x) = x^3 + 2x^2$, $g(x) = 3x$

Solution

$$f(g(x)) = f(3x)$$

$$= (3x)^3 + 2(3x)^2$$

$$= 27x^3 + 18x^2$$

$$g\left(f(x)\right) = g\left(x^3 + 2x^2\right)$$

$$= 3\left(x^3 + 2x^2\right)$$

$$= 3x^3 + 6x^2$$

$$f\left(g(-2)\right) = 27(-2)^3 + 18(-2)^2$$

$$= 288$$

$$g\left(f(3)\right) = 3(3)^3 + 6(3)^2$$

$$= 135$$

Exercise

Find $\left(f \circ g\right)(x)$, $\left(g \circ f\right)(x)$, $f\left(g(-2)\right)$ *and* $g\left(f(3)\right)$: $f(x) = |x|$, $g(x) = -7$

Solution

$$f\left(g(x)\right) = f\left(-7\right)$$

$$= |-7|$$

$$= 7$$

$$g\left(f(x)\right) = g\left(|x|\right)$$

$$= -7$$

$$f\left(g(-2)\right) = 7$$

$$g\left(f(3)\right) = -7$$

Exercise

Given $f(x) = x - 3$ *and* $g(x) = x + 3$

a) Find $\left(f \circ g\right)(x)$ and the domain of $f \circ g$

b) Find $\left(g \circ f\right)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(x + 3\right)$ ***Domain***: $\mathbb{R}$

$$= \left(x - 3\right) + 3$$

$$= x$$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g(f(x)) = g(x-3)$ ***Domain***: $\mathbb{R}$

$= (x+3)-3$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = \frac{2}{3}x$ *and* $g(x) = \frac{3}{2}x$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\frac{3}{2}x\right)$ ***Domain***: $\mathbb{R}$

$= \frac{2}{3}\left(\frac{3}{2}x\right)$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g(f(x)) = g\left(\frac{2}{3}x\right)$ ***Domain***: $\mathbb{R}$

$= \frac{3}{2}\left(\frac{2}{3}x\right)$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = x-1$ *and* $g(x) = 3x^2 - 2x - 1$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(3x^2 - 2x - 1\right)$ ***Domain***: $\mathbb{R}$

$= 3(x-1)^2 - 2(x-1) - 1$

$$= 3\left(x^2 - 2x + 1\right) - 2x + 2 - 1$$

$$= 3x^2 - 6x + 3 - 2x + 1$$

$$= 3x^2 - 8x + 4$$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g\left(f(x)\right) = g(x-1)$ ***Domain***: $\mathbb{R}$

$$= 3x^2 - 2x - 1 - 1$$

$$= 3x^2 - 2x - 2$$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = 3x - 2$ *and* $g(x) = x^2 - 5$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(x^2 - 5\right)$ ***Domain***: $\mathbb{R}$

$$= 3\left(x^2 - 5\right) - 2$$

$$= 3x^2 - 15 - 2$$

$$= 3x^2 - 17$$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g\left(f(x)\right) = g(3x - 2)$ ***Domain***: $\mathbb{R}$

$$= (3x - 2)^2 - 5$$

$$= 9x^2 - 12x + 4 - 5$$

$$= 9x^2 - 12x - 1$$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = x^2 - 2$ *and* $g(x) = 4x - 3$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f(4x-3)$ ***Domain***: $\mathbb{R}$

$$= (4x-3)^2 - 2$$

$$= 16x^2 - 24x + 9 - 2$$

$$= 16x^2 - 24x + 7$$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g(f(x)) = g(x^2 - 2)$ ***Domain***: $\mathbb{R}$

$$= 4(x^2 - 2) - 3$$

$$= 4x^2 - 8 - 3$$

$$= 4x^2 - 11$$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = 4x^2 - x + 10$ *and* $g(x) = 2x - 7$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f(2x-7)$ ***Domain***: $\mathbb{R}$

$$= 4(2x-7)^2 - (2x-7) + 10$$

$$= 4(4x^2 - 28x + 49) - 2x + 7 + 10$$

$$= 16x^2 - 112x + 196 - 2x + 17$$

$$= 16x^2 - 114x + 213$$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g(f(x)) = g(4x^2 - x + 10)$ ***Domain***: $\mathbb{R}$

$$= 2\left(4x^2 - x + 10\right) - 7$$

$$= 8x^2 - 2x + 20 - 7$$

$$\underline{= 8x^2 - 2x + 13}$$ ***Domain***: $\mathbb{R}$

Domain: $\underline{\mathbb{R}}$

Exercise

Given $f(x) = \sqrt{x}$ *and* $g(x) = x + 3$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f(x+3)$ ***Domain***: $\mathbb{R}$

$$\underline{= \sqrt{x+3}}$$ ***Domain***: $x \ge -3$

Domain: $\underline{x \ge -3}$

b) $g(f(x)) = g(\sqrt{x})$ ***Domain***: $x \ge 0$

$$\underline{= \sqrt{x} + 3}$$ ***Domain***: $x \ge 0$

Domain: $\underline{x \ge 0}$

Exercise

Given $f(x) = \sqrt{x}$ *and* $g(x) = 2 - 3x$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f(2-3x)$ ***Domain***: $\mathbb{R}$

$$\underline{= \sqrt{2-3x}}$$ ***Domain***: $x \le \frac{2}{3}$

Domain: $\underline{x \le \frac{2}{3}}$

b) $g(f(x)) = g(\sqrt{x})$ ***Domain***: $x \ge 0$

$$\underline{= 2 - 3\sqrt{x}}$$ ***Domain***: $x \ge 0$

Domain: $\underline{x \ge 0}$

Exercise

Given $f(x) = 3x + 2$ *and* $g(x) = \sqrt{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f(\sqrt{x})$ *Domain*: $x \ge 0$

$= 3\sqrt{x} + 2$ *Domain*: $x \ge 0$

Domain: $x \ge 0$

b) $g(f(x)) = g(3x+2)$ *Domain*: $\mathbb{R}$

$= \sqrt{3x+2}$ *Domain*: $x \ge -\frac{2}{3}$

Domain: $x \ge -\frac{2}{3}$

Exercise

Given $f(x) = x^4$ *and* $g(x) = \sqrt[4]{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f(\sqrt[4]{x})$ *Domain*: $x \ge 0$

$= (\sqrt[4]{x})^4$

$= x$ *Domain*: $\mathbb{R}$

Domain: $x \ge 0$

b) $g(f(x)) = g(x^4)$ *Domain*: $\mathbb{R}$

$= \sqrt[4]{x^4}$

$= x$ *Domain*: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x)=x^n$ *and* $g(x)=\sqrt[n]{x}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(\sqrt[n]{x}\right)$ **Domain**: $\begin{cases} \text{If } n \text{ is even} & \underline{x\geq 0} \\ \text{If } n \text{ is odd} & \underline{\mathbb{R}} \end{cases}$

$=\left(\sqrt[n]{x}\right)^n$

$\underline{=x}$ **Domain**: $\mathbb{R}$

Domain: $\begin{cases} \text{If } n \text{ is even} & \underline{x\geq 0} \\ \text{If } n \text{ is odd} & \underline{\mathbb{R}} \end{cases}$

b) $g(f(x))=g\left(x^n\right)$ **Domain**: $\mathbb{R}$

$=\sqrt[n]{x^n}$

$\underline{=x}$ **Domain**: $\mathbb{R}$

Domain: $\underline{\mathbb{R}}$

Exercise

Given $f(x)=x^2-3x$ *and* $g(x)=\sqrt{x+2}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(\sqrt{x+2}\right)$ $x+2\geq 0\Rightarrow x\geq -2$

$=\left(\sqrt{x+2}\right)^2-3\sqrt{x+2}$

$\underline{=x+2-3\sqrt{x+2}}$ $x+2\geq 0\Rightarrow x\geq -2$

Domain: $\{x\mid x\geq -2\}$

b) $g(f(x))=g\left(x^2-3x\right)$ $\mathbb{R}$

$\underline{=\sqrt{x^2-3x+2}}$ $x^2-3x+2\geq 0\Rightarrow (x=1,\ 2)\leftrightarrow x\leq 1,\ x\geq 2$

Domain: $\{x\mid x\leq 1,\ x\geq 2\}$

Exercise

Given $f(x)=\sqrt{x-2}$ *and* $g(x)=\sqrt{x+5}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(\sqrt{x+5}\right)$ $\quad x+5\ge 0 \Rightarrow x\ge -5$

$=\sqrt{\sqrt{x+5}-2}$ $\quad \sqrt{x+5}-2\ge 0 \Rightarrow \sqrt{x+5}\ge 2$

$x+5\ge 4$

$x\ge -1$

Domain: $\{x \mid x\ge -1\}$

b) $g(f(x))=g\left(\sqrt{x-2}\right)$ $\quad x-2\ge 0 \Rightarrow x\ge 2$

$=\sqrt{\sqrt{x-2}+5}$ $\quad \sqrt{x-2}+5\ge 0 \Rightarrow \sqrt{x-2}\ge -5$ *Always true when* $x\ge 2$

Domain: $\{x \mid x\ge 2\}$

Exercise

Given $f(x)=x^2+2$ *and* $g(x)=\sqrt{3-x}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(\sqrt{3-x}\right)$ $\quad$ ***Domain***: $x\le 3$

$=\left(\sqrt{3-x}\right)^2+2$

$=3-x+2$

$=5-x$ $\quad$ ***Domain***: $\mathbb{R}$

Domain: $x\le 3$

b) $g(f(x))=g\left(x^2+2\right)$ $\quad$ ***Domain***: $\mathbb{R}$

$=\sqrt{3-x^2-2}$

$=\sqrt{1-x^2}$ $\quad$ ***Domain***: $-1\le x\le 1$

Domain: $-1\le x\le 1$

Exercise

Given $f(x) = x^5 - 2$ *and* $g(x) = \sqrt[5]{x+2}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\sqrt[5]{x+2}\right)$ **Domain**: $\mathbb{R}$

$$= \left(\sqrt[5]{x+2}\right)^5 - 2$$

$$= x + 2 - 2$$

$$\underline{= x}$$ **Domain**: $\mathbb{R}$

Domain: $\underline{\mathbb{R}}$

b) $g(f(x)) = g\left(x^5 - 2\right)$ **Domain**: $\mathbb{R}$

$$= \sqrt[5]{x^5 - 2 + 2}$$

$$= \sqrt[5]{x^5}$$

$$\underline{= x}$$ **Domain**: $\mathbb{R}$

Domain: $\underline{\mathbb{R}}$

Exercise

Given $f(x) = 1 - x^2$ *and* $g(x) = \sqrt{x^2 - 25}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\sqrt{x^2 - 25}\right)$ **Domain**: $x \le -5 \quad x \ge 5$

$$= 1 - \left(\sqrt{x^2 - 25}\right)^2$$

$$= 1 - \left(x^2 - 25\right)$$

$$= 1 - x^2 + 25$$

$$\underline{= 26 - x^2}$$ **Domain**: $\mathbb{R}$

Domain: $\underline{x \le -5 \quad x \ge 5}$

b) $g\left(f(x)\right)=g\left(1-x^2\right)$ ***Domain***: $\mathbb{R}$

$$=\sqrt{\left(1-x^2\right)^2-25}$$

$$=\sqrt{1-2x^2+x^4-25}$$

$$\underline{=\sqrt{x^4-2x^2-24}}$$

$$x^2=\frac{2\pm\sqrt{4+96}}{2}$$

$$=\begin{cases}\frac{2-10}{2}=-4 \times \\ \frac{2+10}{2}=6\end{cases}$$

$x^2=6 \rightarrow \underline{x=\pm\sqrt{6}}$ ***Domain***: $x\le-\sqrt{6}\quad x\ge\sqrt{6}$

Domain: $\underline{x\le-\sqrt{6}\quad x\ge\sqrt{6}}$

Exercise

Given $f(x)=2x+3$ *and* $g(x)=\dfrac{x-3}{2}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f\left(g(x)\right)=f\left(\frac{x-3}{2}\right)$ ***Domain***: $\mathbb{R}$

$$=2\left(\frac{x-3}{2}\right)+3$$

$$=x-3+3$$

$\underline{=x}$ ***Domain***: $\mathbb{R}$

Domain: $\underline{\mathbb{R}}$

b) $g\left(f(x)\right)=g(2x+3)$ ***Domain***: $\mathbb{R}$

$$=\frac{1}{2}(2x+3-3)$$

$\underline{=x}$ ***Domain***: $\mathbb{R}$

Domain: $\underline{\mathbb{R}}$

Exercise

Given $f(x) = 4x - 5$ *and* $g(x) = \dfrac{x+5}{4}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\dfrac{x+5}{4}\right)$ ***Domain***: $\mathbb{R}$

$= 4\left(\dfrac{x+5}{4}\right) - 5$

$= x + 5 - 5$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g(f(x)) = g(4x - 5)$ ***Domain***: $\mathbb{R}$

$= \dfrac{1}{4}(4x - 5 + 5)$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = \dfrac{4}{1-5x}$ *and* $g(x) = \dfrac{1}{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\dfrac{1}{x}\right)$ ***Domain***: $x \ne 0$

$= \dfrac{4}{1 - 5\frac{1}{x}}$

$= \dfrac{4x}{x-5}$ ***Domain***: $x \ne 5$

Domain: $x \ne 0,\ 5$

b) $g(f(x)) = g\left(\dfrac{4}{1-5x}\right)$ ***Domain***: $x \ne \dfrac{1}{5}$

$= \dfrac{1-5x}{4}$ ***Domain***: $\mathbb{R}$

Domain: $x \ne \dfrac{1}{5}$

Exercise

Given $f(x)=\dfrac{1}{x-2}$ *and* $g(x)=\dfrac{x+2}{x}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(\dfrac{x+2}{x}\right)$ **Domain**: $x\neq 0$

$$=\dfrac{1}{\frac{x+2}{x}-2}$$

$$=\dfrac{1}{\frac{x+2-2x}{x}}$$

$$=\dfrac{x}{2-x}$$ **Domain**: $x\neq 2$

Domain: $x\neq 0,\ 2$ $(-\infty,\ 0)\cup(0,\ 2)\cup(2,\ \infty)$

b) $g(f(x))=g\left(\dfrac{1}{x-2}\right)$ **Domain**: $x\neq 2$

$$=\dfrac{\frac{1}{x-2}+2}{\frac{1}{x-2}}$$

$$=\dfrac{\frac{1+2x-4}{x-2}}{\frac{1}{x-2}}$$

$$=2x-3$$ **Domain**: $\mathbb{R}$

Domain: $x\neq 2$ $(-\infty,\ 2)\cup(2,\ \infty)$

Exercise

Given $f(x)=\dfrac{3x+5}{2}$ *and* $g(x)=\dfrac{2x-5}{3}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(\dfrac{2x-5}{3}\right)$ **Domain**: $\mathbb{R}$

$$=\dfrac{3\frac{2x-5}{3}+5}{2}$$

$$= \frac{2x-5+5}{2}$$

$$= \frac{2x}{2}$$

$$\underline{= x \rfloor}$$

Domain: $\underline{\mathbb{R} \rfloor}$

b) $g\left(f(x)\right) = g\left(\frac{3x+5}{2}\right)$ ***Domain***: $\mathbb{R}$

$$= \frac{2\frac{3x+5}{2} - 5}{3}$$

$$= \frac{3x+5-5}{3}$$

$$= \frac{3x}{3}$$

$$\underline{= x \rfloor}$$

Domain: $\underline{\mathbb{R} \rfloor}$

Exercise

Given $f(x) = \frac{1}{1+x}$ *and* $g(x) = \frac{1-x}{x}$

a) Find $\left(f \circ g\right)(x)$ and the domain of $f \circ g$

b) Find $\left(g \circ f\right)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(\frac{1-x}{x}\right)$ ***Domain***: $x \neq 0$

$$= \frac{1}{1+\frac{1-x}{x}}$$

$$= \frac{x}{x+1-x}$$

$\underline{= x \rfloor}$ ***Domain***: $\mathbb{R}$

Domain: $\underline{x \neq 0 \rfloor}$

b) $g\left(f(x)\right) = g\left(\frac{1}{x+1}\right)$ ***Domain***: $x \neq -1$

$$= \frac{1-\frac{1}{x+1}}{\frac{1}{x+1}}$$

$$= x+1-1$$

$\underline{= x \rfloor}$ ***Domain***: $\mathbb{R}$

Domain: $\underline{\mathbb{R} \rfloor}$

Exercise

Given $f(x) = \dfrac{x-1}{x-2}$ *and* $g(x) = \dfrac{x-3}{x-4}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(\dfrac{x-3}{x-4}\right)$ ***Domain***: $x \neq 4$

$$= \dfrac{\dfrac{x-3}{x-4}-1}{\dfrac{x-3}{x-4}-2}$$

$$= \dfrac{\dfrac{x-3-(x-4)}{x-4}}{\dfrac{x-3-2(x-4)}{x-4}}$$

$$= \dfrac{x-3+x+4}{x-3-2x+8}$$

$$= \boxed{\dfrac{2x+1}{-x+5}}$$ ***Domain***: $x \neq 5$

Domain: $\{x \mid x \neq 4,\ 5\}$

b) $g\left(f(x)\right) = g\left(\dfrac{x-1}{x-2}\right)$ ***Domain***: $x \neq 2$

$$= \dfrac{\dfrac{x-1}{x-2}-3}{\dfrac{x-1}{x-2}-4}$$

$$= \dfrac{x-1-3(x-2)}{x-1-4(x-2)}$$

$$= \dfrac{x-1-3x+6}{x-1-4x+8}$$

$$= \boxed{\dfrac{-2x+5}{-3x+7}}$$ ***Domain***: $x \neq \dfrac{7}{3}$

Domain: $\left\{x \mid x \neq 2,\ \dfrac{7}{3}\right\}$

Exercise

Given $f(x)=\dfrac{6}{x-3}$ *and* $g(x)=\dfrac{1}{x}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $(f\circ g)(x)$

$$f(g(x))=f\left(\frac{1}{x}\right) \qquad \textbf{Domain: } x\neq 0$$

$$=\frac{6}{\frac{1}{x}-3}$$

$$=\frac{6}{\frac{1-3x}{x}}$$

$$=\frac{6x}{1-3x} \qquad \textbf{Domain: } x\neq \frac{1}{3}$$

Domain: $x\neq 0,\ \frac{1}{3}$ $\qquad (-\infty,0)\cup\left(0,\frac{1}{3}\right)\cup\left(\frac{1}{3},\infty\right)$

b) $(g\circ f)(x)$

$$g(f(x))=g\left(\frac{6}{x-3}\right) \qquad \textbf{Domain: } x\neq 3$$

$$=\frac{1}{\frac{6}{x-3}}$$

$$=\frac{x-3}{6} \qquad \textbf{Domain: } (-\infty,\infty)$$

Domain: $x\neq 3$ $\qquad (-\infty,3)\cup(3,\infty)$

Exercise

Given $f(x)=\dfrac{6}{x}$ *and* $g(x)=\dfrac{1}{2x+1}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(\dfrac{1}{2x+1}\right) \qquad$ **Domain:** $x\neq -\frac{1}{2}$

$$=\frac{6}{\frac{1}{2x+1}}$$

$$=12x+6 \qquad \textbf{Domain: } \mathbb{R}$$

Domain: $\boxed{x \neq -\frac{1}{2}}$

b) $g\left(f(x)\right) = g\left(\frac{6}{x}\right)$ **Domain:** $x \neq 0$

$= \frac{1}{2\frac{6}{x}+1}$

$= \boxed{\frac{x}{12+x}}$ **Domain:** $x \neq -12$

Domain: $\boxed{x \neq -12,\ 0}$

Exercise

Given $f(x) = 3x - 7$ *and* $g(x) = \frac{x+7}{3}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(\frac{x+7}{3}\right)$ **Domain:** $\mathbb{R}$

$= 3\frac{x+7}{3} - 7$

$= x + 7 - 7$

$= \boxed{x}$ **Domain:** $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

b) $g\left(f(x)\right) = g(3x-7)$ **Domain:** $\mathbb{R}$

$= \frac{3x-7+7}{3}$

$= \boxed{x}$ **Domain:** $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

Exercise

Given $f(x) = \frac{2x+3}{x-4}$ *and* $g(x) = \frac{4x+3}{x-2}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(\dfrac{4x+3}{x-2}\right)$ ***Domain***: $x \ne 2$

$$= \dfrac{2\,\dfrac{4x+3}{x-2}+3}{\dfrac{4x+3}{x-2}-4}$$

$$= \dfrac{8x+6+3x-6}{4x+3-4x+8}$$

$$= \dfrac{11x}{11}$$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $x \ne 2$

b) $g\left(f(x)\right) = g\left(\dfrac{2x+3}{x-4}\right)$ ***Domain***: $x \ne 4$

$$= \dfrac{4\,\dfrac{2x+3}{x-4}+3}{\dfrac{2x+3}{x-4}-2}$$

$$= \dfrac{8x+12+3x-4}{2x+3-2x+8}$$

$$= \dfrac{11x}{11}$$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $x \ne 4$

Exercise

Given $f(x) = \dfrac{2x+3}{x+4}$ *and* $g(x) = \dfrac{-4x+3}{x-2}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(\dfrac{-4x+3}{x-2}\right)$ ***Domain***: $x \ne 2$

$$= \dfrac{2\,\dfrac{-4x+3}{x-2}+3}{\dfrac{4x+3}{x-2}+4}$$

$$= \dfrac{-8x+6+3x-6}{4x+3+4x-8}$$

$$= \dfrac{-5x}{-5}$$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{x \ne 2}$

b) $g\left(f(x)\right) = g\left(\dfrac{2x+3}{x+4}\right)$ ***Domain***: $x \ne -4$

$$= \frac{-4\frac{2x+3}{x+4}+3}{\frac{2x+3}{x+4}-2}$$

$$= \frac{-8x-12+3x+12}{2x+3-2x-8}$$

$$= \frac{-5x}{-5}$$

$$\boxed{= x}$$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{x \ne -4}$

Exercise

Given $f(x) = x+1$ *and* $g(x) = x^3 - 5x^2 + 3x + 7$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(x^3 - 5x^2 + 3x + 7\right)$ ***Domain***: $\mathbb{R}$

$$= x^3 - 5x^2 + 3x + 7 + 1$$

$$\boxed{= x^3 - 5x^2 + 3x + 8}$$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

b) $g\left(f(x)\right) = g(x+1)$ ***Domain***: $\mathbb{R}$

$$= (x+1)^3 - 5(x+1)^2 + 3(x+1) + 7$$

$$= x^3 + 3x^2 + 3x + 1 - 5\left(x^2 + 2x + 1\right) + 3x + 3 + 7$$

$$= x^3 + 3x^2 + 6x + 11 - 5x^2 - 10x - 5$$

$$\boxed{= x^3 - 2x^2 - 4x + 6}$$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

Exercise

Given $f(x) = x - 1 \quad and \quad g(x) = x^3 + 2x^2 - 3x - 9$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(x^3 + 2x^2 - 3x - 9\right)$ ***Domain***: $\mathbb{R}$

$$= x^3 + 2x^2 - 3x - 9 - 1$$

$$= x^3 + 2x^2 - 3x - 10$$

Domain: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g\left(f(x)\right) = g(x - 1)$ ***Domain***: $\mathbb{R}$

$$= (x-1)^3 + 2(x-1)^2 - (x-1) - 9$$

$$= x^3 - 3x^2 + 3x - 1 + 2\left(x^2 - 2x + 1\right) - 3x + 3 - 9$$

$$= x^3 - 3x^2 - 7 + 2x^2 - 4x + 2$$

$$= x^3 - x^2 - 4x - 5$$

Domain: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3 \quad and \quad g(x) = x^2 - 5x: \quad (f \circ g)(4)$

Solution

$$\begin{aligned}(f \circ g)(4) &= f\left(g(4)\right) \\ &= f(16 - 20) \\ &= f(-4) \\ &= -8 - 3 \\ &= -11\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3 \quad and \quad g(x) = x^2 - 5x: \quad (g \circ f)(4)$

Solution

$$(g \circ f)(4) = g\left(f(4)\right)$$

$$= g(8-3)$$
$$= g(5)$$
$$= 25-25$$
$$\underline{= 0}$$

Exercise

Evaluate each composite function, where $f(x)=2x-3$ *and* $g(x)=x^2-5x$: $(f\circ g)(-2)$

Solution

$$\begin{aligned}(f\circ g)(-2) &= f(g(-2))\\ &= f(4+10)\\ &= f(14)\\ &= 28-3\\ &\underline{= 25}\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x)=2x-3$ *and* $g(x)=x^2-5x$: $(g\circ f)(-2)$

Solution

$$\begin{aligned}(g\circ f)(-2) &= g(f(-2))\\ &= g(-4-3)\\ &= g(-7)\\ &= 49+35\\ &\underline{= 84}\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x)=2x-3$ *and* $g(x)=x^2-5x$: $(f\circ f)(-3)$

Solution

$$\begin{aligned}(f\circ f)(-3) &= f(f(-3))\\ &= f(-6-3)\\ &= f(-9)\\ &= -18-3\\ &\underline{= -21}\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ *and* $g(x) = x^2 - 5x$: $(g \circ g)(7)$

Solution

$$\begin{aligned}(g \circ g)(7) &= g(g(7)) \\ &= g(49 - 35) \\ &= g(14) \\ &= 196 - 70 \\ &= 126\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ *and* $g(x) = x^2 - 5x$: $(f \circ g)(\sqrt{2})$

Solution

$$\begin{aligned}(f \circ g)(\sqrt{2}) &= f\left(g\left(\sqrt{2}\right)\right) \\ &= f\left(2 - 5\sqrt{2}\right) \\ &= 2\left(2 - 5\sqrt{2}\right) - 3 \\ &= 4 - 10\sqrt{2} - 3 \\ &= 1 - 10\sqrt{2}\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ *and* $g(x) = x^2 - 5x$: $(g \circ f)(\sqrt{3})$

Solution

$$\begin{aligned}(g \circ f)(\sqrt{3}) &= g\left(f\left(\sqrt{3}\right)\right) \\ &= g\left(2\sqrt{3} - 3\right) \\ &= \left(2\sqrt{3} - 3\right)^2 - 5\left(2\sqrt{3} - 3\right) \\ &= 12 - 12\sqrt{3} + 9 - 10\sqrt{3} + 15 \\ &= 36 - 22\sqrt{3}\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ *and* $g(x) = x^2 - 5x$: $(f \circ g)(2a)$

Solution

$$\begin{aligned}(f \circ g)(2a) &= f(g(2a)) \\ &= f\left(4a^2 - 10a\right) \\ &= 2\left(4a^2 - 10a\right) - 3 \\ &= 8a^2 - 20a - 3\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ *and* $g(x) = x^2 - 5x$: $(g \circ f)(3b)$

Solution

$$\begin{aligned}(g \circ f)(3b) &= g(f(3b)) \\ &= g(6b - 3) \\ &= (6b - 3)^2 - 5(6b - 3) \\ &= 36b^2 - 36b + 9 - 30b + 15 \\ &= 36b^2 - 66b + 24\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ *and* $g(x) = x^2 - 5x$: $(f \circ g)(k+1)$

Solution

$$\begin{aligned}(f \circ g)(k+1) &= f(g(k+1)) \\ &= f\left((k+1)^2 - 5k - 5\right) \\ &= 2\left((k+1)^2 - 5k - 5\right) - 3 \\ &= 2\left(k^2 + 2k + 1\right) - 10k - 10 - 3 \\ &= 2k^2 + 4k + 2 - 10k - 13 \\ &= 2k^2 - 6k - 11\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x)=2x-3$ *and* $g(x)=x^2-5x$: $(g\circ f)(k-1)$

Solution

$$
\begin{aligned}
(g\circ f)(k-1) &= g\left(f(k-1)\right)\\
&= g(2k-2-3)\\
&= g(2k-5)\\
&= (2k-5)^2-5(2k-5)\\
&= 4k^2-20k+25-10k+25\\
&= 4k^2-30k+50
\end{aligned}
$$

Solution   *Section* 2.4 – Properties of Division

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=2x^4-x^3+7x-12;\quad p(x)=x^2-3$

Solution

$$\begin{array}{r}
2x^2-x+6 \\
x^2-3\overline{\big)2x^4-x^3+0x^2+7x-12} \\
\underline{2x^4 \qquad -6x^2} \qquad\qquad \\
-x^3+6x^2+7x \qquad \\
\underline{-x^3 \qquad +3x} \qquad \\
6x^2+4x-12 \\
\underline{6x^2 \qquad -18} \\
4x+6
\end{array}$$

$$\boxed{Q(x)=2x^2-x+6;\quad R(x)=4x+6}$$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=3x^3+2x-4;\quad p(x)=2x^2+1$

Solution

$$\begin{array}{r}
\frac{3}{2}x \qquad \\
2x^2+1\overline{\big)3x^3+0x^2+2x-4} \\
\underline{3x^3 \qquad +\frac{3}{2}x \qquad} \\
\frac{1}{2}x-4
\end{array}$$

$$\boxed{Q(x)=\frac{3}{2}x;\quad R(x)=\frac{1}{2}x-4}$$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=7x+2;\quad p(x)=2x^2-x-4$

Solution

$$
\begin{array}{r|l}
 & \frac{2}{7}x-\frac{11}{49} \\
\hline
7x+2 & 2x^2-x-4 \\
 & 2x^2+\frac{4}{7}x \\
\hline
 & -\frac{11}{7}x-4 \\
 & -\frac{11}{7}x-\frac{22}{49} \\
\hline
 & -\frac{174}{49}
\end{array}
$$

$$\boxed{Q(x)=\frac{2}{7}x-\frac{11}{49}} \quad \boxed{R(x)=-\frac{174}{49}}$$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=9x+4;\quad p(x)=2x-5$

Solution

$$
\begin{array}{r|l}
 & \frac{9}{2} \\
\hline
2x-5 & 9x+4 \\
 & 9x-\frac{45}{2} \\
\hline
 & -\frac{37}{2}
\end{array}
$$

$$\boxed{Q(x)=\frac{9}{2};\quad R(x)=-\frac{37}{2}}$$

Exercise

Use the remainder theorem to find $f(c)$: $f(x)=x^4-6x^2+4x-8;\quad c=-3$

Solution

$$f(-3)=(-3)^4-6(-3)^2+4(-3)-8$$

$$\boxed{=7}$$

Exercise

Use the remainder theorem to find $f(c)$: $f(x)=x^4+3x^2-12; \quad c=-2$

Solution

$$f(-2)=(-2)^4+3(-2)^2-12$$
$$\underline{=16}$$

Exercise

Use the factor theorem to show that $x-c$ is a factor of $f(x)$: $f(x)=x^3+x^2-2x+12; \quad c=-3$

Solution

$$f(-3)=(-3)^3+(-3)^2-2(-3)+12$$
$$\underline{=0}$$

From the factor theorem; $x+3$ is a factor of $f(x)$.

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $2x^3-3x^2+4x-5; \quad x-2$

Solution

$$\begin{array}{r|rrrr} 2 & 2 & -3 & 4 & -5 \\ & & 4 & 2 & 12 \\ \hline & 2 & 1 & 6 & \boxed{7} \end{array}$$

$$\underline{Q(x)=2x^2+x+6 \qquad R(x)=7}$$

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $5x^3-6x^2+15; \quad x-4$

Solution

$$\begin{array}{r|rrrr} 4 & 5 & -6 & 0 & 15 \\ & & 20 & 56 & 224 \\ \hline & 5 & 14 & 56 & \boxed{239} \end{array}$$

$$\underline{Q(x)=5x^2+14x+56 \qquad R(x)=239}$$

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $9x^3 - 6x^2 + 3x - 4;\quad x - \frac{1}{3}$

Solution

$$\begin{array}{r|rrrr} \frac{1}{3} & 9 & -6 & 3 & -4 \\ & & 3 & -1 & \frac{2}{3} \\ \hline & 9 & -3 & 2 & \boxed{-\frac{10}{3}} \end{array}$$

$$\boxed{Q(x) = 9x^2 - 3x + 2 \qquad R(x) = -\frac{10}{3}}$$

Exercise

Use the synthetic division to find $f(c)$: $f(x) = 2x^3 + 3x^2 - 4x + 4;\quad c = 3$

Solution

$$\begin{array}{r|rrrr} 3 & 2 & 3 & -4 & 4 \\ & & 9 & 36 & 93 \\ \hline & 3 & 12 & 32 & \boxed{97} \end{array}$$

$$\boxed{f(3) = 97}$$

Exercise

Use the synthetic division to find $f(c)$: $f(x) = 8x^5 - 3x^2 + 7;\quad c = \frac{1}{2}$

Solution

$$\begin{array}{r|rrrrrr} \frac{1}{2} & 8 & 0 & 0 & -3 & 0 & 7 \\ & & 4 & 2 & 1 & -1 & -\frac{1}{2} \\ \hline & 8 & 4 & 2 & -2 & -1 & \boxed{\frac{13}{2}} \end{array}$$

$$\boxed{f\left(\frac{1}{2}\right) = \frac{13}{2}}$$

Exercise

Use the synthetic division to find $f(c)$: $f(x)=x^3-3x^2-8;\quad c=1+\sqrt{2}$

Solution

$$\begin{array}{r|rrrr} 1+\sqrt{2} & 3 & -3 & 0 & -8 \\ & & 3+3\sqrt{2} & 6+3\sqrt{2} & 12+9\sqrt{2} \\ \hline & 3 & 3\sqrt{2} & 6+3\sqrt{2} & \boxed{4+9\sqrt{2}} \end{array}$$

$$f\left(1+\sqrt{2}\right)=4+9\sqrt{2}$$

Exercise

Use the synthetic division to show that c is a zero of $f(x)$: $f(x)=3x^4+8x^3-2x^2-10x+4;\quad c=-2$

Solution

$$\begin{array}{r|rrrrr} -2 & 3 & 8 & -2 & -10 & 4 \\ & & -6 & -4 & 12 & -4 \\ \hline & 3 & 2 & -6 & 2 & \boxed{0} \end{array}$$

$$f(-2)=0$$

Exercise

Use the synthetic division to show that c is a zero of $f(x)$: $f(x)=27x^4-9x^3+3x^2+6x+1;\quad c=-\frac{1}{3}$

Solution

$$\begin{array}{r|rrrrr} -\frac{1}{3} & 27 & -9 & 3 & 6 & 1 \\ & & -9 & 6 & -3 & -1 \\ \hline & 27 & -18 & 9 & 3 & \boxed{0} \end{array}$$

$$f\left(-\tfrac{1}{3}\right)=0$$

Exercise

Find all values of k such that $f(x)$ is divisible by the given linear polynomial:

$f(x)=kx^3+x^2+k^2x+3k^2+11;\quad x+2$

Solution

$$\begin{array}{r|cccc} -2 & k & 1 & k^2 & 3k^2+11 \\ & & -2k & 4k-2 & -2k^2-8k+4 \\ \hline & k & 1-2k & k^2+4k-2 & k^2-8k+15 \end{array}$$

$k^2-8k+15=0 \Rightarrow \boxed{k=3,\ 5}$

Exercise

Find all solutions of the equation: $x^3-x^2-10x-8=0$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{8}{1}\right\}=\pm\{1,\ 2,\ 4,\ 8\}$

$$\begin{array}{r|cccc} -1 & 1 & -1 & -10 & -8 \\ & & -1 & 2 & 8 \\ \hline & 1 & -2 & -8 & 0 \end{array} \rightarrow\ x^2-2x-8=0$$

The solutions are: $x=-1,\ -2,\ 4$

Exercise

Find all solutions of the equation: $x^3+x^2-14x-24=0$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{24}{1}\right\}=\pm\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24\}$

$$\begin{array}{r|cccc} -2 & 1 & 1 & -14 & -24 \\ & & -2 & 2 & 24 \\ \hline & 1 & -1 & -12 & 0 \end{array} \rightarrow\ x^2-x-12=0$$

The solutions are: $x=-2,\ -3,\ 4$

Exercise

Find all solutions of the equation: $2x^3-3x^2-17x+30=0$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{30}{2}\right\}=\pm\left\{1,\ 2,\ 3,\ 5,\ 6,\ 10,\ 15,\ 30,\ \frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ \frac{15}{2}\right\}$

$$\begin{array}{c|cccc} 2 & 2 & -3 & -17 & 30 \\ & & 4 & 2 & -30 \\ \hline & 2 & 1 & -15 & \boxed{0} \end{array} \rightarrow \quad 2x^2 + x - 15 = 0$$

The solutions are: $\boxed{x = 2, \ -3, \ \frac{5}{2}}$

Exercise

Find all solutions of the equation: $\quad 12x^3 + 8x^2 - 3x - 2 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{2}{12}\right\} = \pm\left\{1, \ 2, \ \frac{1}{2}, \ \frac{1}{3}, \ \frac{2}{3}, \ \frac{1}{2}, \ \frac{1}{6}, \ \frac{1}{12}\right\}$

$$\begin{array}{c|cccc} \frac{1}{2} & 12 & 8 & -3 & -2 \\ & & 6 & 7 & 2 \\ \hline & 12 & 14 & 4 & \boxed{0} \end{array} \rightarrow \quad 12x^2 + 14x + 4 = 0$$

The solutions are: $\boxed{x = \frac{1}{2}, \ -\frac{1}{2}, \ -\frac{2}{3}}$

Exercise

Find all solutions of the equation: $\quad x^3 + x^2 - 6x - 8 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{8}{1}\right\} = \pm\{1, \ 2, \ 4, \ 8\}$

$$\begin{array}{c|cccc} -2 & 1 & 1 & -6 & -8 \\ & & -2 & 2 & 8 \\ \hline & 1 & -1 & -4 & \boxed{0} \end{array} \rightarrow \quad x^2 - x - 4 = 0$$

$$x = \frac{1 \pm \sqrt{1+16}}{2}$$

The solutions are: $\boxed{x = -2, \ \frac{1 \pm \sqrt{17}}{2}}$

Exercise

Find all solutions of the equation: $\quad x^3 - 19x - 30 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{30}{1}\right\} = \pm\{1, \ 2, \ 3, \ 5, \ 6, \ 15, \ 30\}$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -19 & -30 \\ & & -2 & 4 & 30 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array} \rightarrow \quad x^2 - 2x - 15$$

$$x = \frac{2 \pm \sqrt{4+60}}{2}$$

$$= \begin{cases} \frac{2-8}{2} = -3 \\ \frac{2+8}{2} = 5 \end{cases}$$

The solutions are: $\underline{x = -2, \ -3, \ 5}|$

Exercise

Find all solutions of the equation: $\quad 2x^3 + x^2 - 25x + 12 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{12}{2} \right\} = \pm \left\{ 1, \ 2, \ 3, \ 4, \ 6, \ 12, \ \frac{1}{2}, \ \frac{3}{2} \right\}$

$$\begin{array}{r|rrrr} 3 & 2 & 1 & -25 & 12 \\ & & 6 & 21 & -12 \\ \hline & 2 & 7 & -4 & \boxed{0} \end{array} \rightarrow \quad 2x^2 + 7x - 4$$

$$x = \frac{-7 \pm \sqrt{49+32}}{4}$$

$$= \begin{cases} \frac{-7-9}{4} = -4 \\ \frac{-7+9}{4} = \frac{1}{2} \end{cases}$$

The solutions are: $\underline{x = -4, \ \frac{1}{2}, \ 3}|$

Exercise

Find all solutions of the equation: $\quad 3x^3 + 11x^2 - 6x - 8 = 0$

Solution

possibilities for $\frac{c}{d} : \pm \left\{ \frac{8}{3} \right\} = \pm \left\{ 1, \ 2, \ 4, \ 8, \ \frac{1}{3}, \ \frac{2}{3}, \ \frac{4}{3}, \ \frac{8}{3} \right\}$

$$\begin{array}{r|rrrr} 1 & 3 & 11 & -6 & -8 \\ & & 3 & 14 & 8 \\ \hline & 3 & 14 & 8 & \boxed{0} \end{array} \rightarrow \quad 3x^2 + 14x + 8$$

$$x = \frac{-14 \pm \sqrt{196-96}}{6}$$

$$=\begin{cases}\frac{-14-10}{6}=-4\\ \frac{-14+10}{6}=-\frac{2}{3}\end{cases}$$

The solutions are: $\underline{x=-4,\ -\frac{2}{3},\ 1}$

Exercise

Find all solutions of the equation: $2x^3+9x^2-2x-9=0$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{9}{2}\right\}=\pm\left\{1,\ 3,\ 9,\ \frac{1}{2},\ \frac{3}{2},\ \frac{9}{2}\right\}$

$$\begin{array}{r|rrrr} 1 & 2 & 9 & -2 & -9 \\ & & 2 & 11 & 9 \\ \hline & 2 & 11 & 9 & \boxed{0} \end{array} \rightarrow\ 2x^2+11x+9$$

$\underline{x=-1,\ -\frac{9}{2}}$ $\quad a-b+c=0\ \rightarrow\ x=-1,\ -\frac{c}{a}$

The solutions are: $\underline{x=-\frac{9}{2},\ -1,\ 1}$

Exercise

Find all solutions of the equation: $x^3+3x^2-6x-8=0$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{8}{1}\right\}=\pm\{1,\ 2,\ 4,\ 8\}$

$$\begin{array}{r|rrrr} -1 & 1 & 3 & -6 & -8 \\ & & -1 & -2 & 8 \\ \hline & 1 & 2 & -8 & \boxed{0} \end{array} \rightarrow\ x^2+2x-8=0$$

$$x=\frac{-2\pm\sqrt{4+32}}{2}$$

$$=\begin{cases}\frac{-2-6}{2}=-4\\ \frac{-2+6}{2}=2\end{cases}$$

The solutions are: $\underline{x=-4,\ -1,\ 2}$

Exercise

Find all solutions of the equation: $3x^3 - x^2 - 6x + 2 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{2}{3}\right\} = \pm\left\{1,\ 2,\ \frac{1}{3},\ \frac{2}{3}\right\}$

$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & -1 & -6 & 2 \\ & & 1 & 0 & -2 \\ \hline & 3 & 0 & -6 & \boxed{0} \end{array} \quad \to \quad 3x^2 - 6 = 0$$

$x^2 = 2$

The solutions are: $\underline{x = \frac{1}{3},\ \pm\sqrt{2}}$

Exercise

Find all solutions of the equation: $x^3 - 8x^2 + 8x + 24 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{24}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24\}$

$$\begin{array}{r|rrrr} 6 & 1 & -8 & 8 & 24 \\ & & 6 & -12 & -24 \\ \hline & 1 & -2 & -4 & \boxed{0} \end{array} \quad \to \quad x^2 - 2x - 4 = 0$$

$$x = \frac{2 \pm \sqrt{4+16}}{2}$$

$$= \frac{2 \pm 2\sqrt{5}}{2}$$

The solutions are: $\underline{x = 6,\ 1 \pm \sqrt{5}}$

Exercise

Find all solutions of the equation: $x^3 - 7x^2 - 7x + 69 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{69}{1}\right\} = \pm\{1,\ 3,\ 23,\ 69\}$

$$\begin{array}{r|rrrr} -3 & 1 & -7 & -7 & 69 \\ & & -3 & 30 & -69 \\ \hline & 1 & -10 & 23 & \boxed{0} \end{array} \quad \to \quad x^2 - 10x + 23 = 0$$

$$x = \frac{10 \pm \sqrt{100 - 92}}{2}$$

$$= \frac{10 \pm 2\sqrt{2}}{2}$$

The solutions are: $\underline{x = -3,\ 5 \pm \sqrt{2}}$

Exercise

Find all solutions of the equation: $x^3 - 3x - 2 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -3 & -2 \\ & & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array} \quad \rightarrow \quad x^2 - x - 2 = 0$$

$\underline{x = -1,\ 2}$ $\qquad a - b + c = 0 \ \rightarrow\ x = -1,\ -\frac{c}{a}$

The solutions are: $\underline{x = -1,\ \ -1,\ 2}$

Exercise

Find all solutions of the equation: $x^3 - 2x + 1 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\{1\}$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -2 & 1 \\ & & 1 & 1 & 1 \\ \hline & 1 & 1 & -1 & \boxed{0} \end{array} \quad \rightarrow \quad x^2 + x - 1 = 0$$

$$x = \frac{-1 \pm \sqrt{5}}{2}$$

The solutions are: $\underline{x = 1,\ \frac{-1 \pm \sqrt{5}}{2}}$

Exercise

Find all solutions of the equation: $x^3-2x^2-11x+12=0$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{12}{1}\right\}=\pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -11 & 12 \\ & & 1 & -1 & 12 \\ \hline & 1 & -1 & -12 & \boxed{0} \end{array} \quad \to \quad x^2-x-12=0$$

$$x=\frac{1\pm\sqrt{1+48}}{2}$$

$$=\begin{cases}\frac{1-7}{2}=-3 \\ \frac{1+7}{2}=4\end{cases}$$

The solutions are: $x=-3,\ 1,\ 4$

Exercise

Find all solutions of the equation: $x^3-2x^2-7x-4=0$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{4}{1}\right\}=\pm\{1,\ 2,\ 4\}$

$$\begin{array}{r|rrrr} -1 & 1 & -2 & -7 & -4 \\ & & -1 & 3 & 4 \\ \hline & 1 & -3 & -4 & \boxed{0} \end{array} \quad \to \quad x^2-3x-4=0$$

$x=-1,\ 4$ $\qquad a-b+c=0 \ \to\ x=-1,\ -\frac{c}{a}$

The solutions are: $x=-1,\ -1,\ 4$

Exercise

Find all solutions of the equation: $x^3-10x-12=0$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{12}{1}\right\}=\pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -10 & -12 \\ & & -2 & 4 & 12 \\ \hline & 1 & -2 & -6 & \boxed{0} \end{array} \quad \to \quad x^2 - 2x - 6 = 0$$

$$\begin{aligned} x &= \frac{2 \pm \sqrt{4+24}}{2} \\ &= \frac{2 \pm 2\sqrt{7}}{2} \end{aligned}$$

The solutions are: $\underline{x = -2,\ 1 \pm \sqrt{7}}$

Exercise

Find all solutions of the equation: $x^3 - 5x^2 + 17x - 13 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{13}{1}\right\} = \pm\{1,\ 13\}$

$$\begin{array}{r|rrrr} 1 & 1 & -5 & 17 & -13 \\ & & 1 & -4 & 13 \\ \hline & 1 & -4 & 13 & \boxed{0} \end{array} \quad \to \quad x^2 - 4x + 13 = 0$$

$$\begin{aligned} x &= \frac{4 \pm \sqrt{16-52}}{2} \\ &= \frac{4 \pm 6i}{2} \end{aligned}$$

The solutions are: $\underline{x = 1,\ 2 \pm 3i}$

Exercise

Find all solutions of the equation: $6x^3 + 25x^2 - 24x + 5 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{5}{6}\right\} = \pm\left\{1,\ 5,\ \frac{1}{2},\ \frac{5}{2},\ \frac{1}{3},\ \frac{5}{3},\ \frac{1}{6},\ \frac{5}{6}\right\}$

$$\begin{array}{r|rrrr} -5 & 6 & 25 & -24 & 5 \\ & & -30 & 25 & -5 \\ \hline & 6 & -5 & 1 & \boxed{0} \end{array} \quad \to \quad 6x^2 - 5x + 1 = 0$$

$$x = \frac{5 \pm \sqrt{25-24}}{12}$$

$$= \begin{cases} \frac{5-1}{12} = \frac{1}{3} \\ \frac{5+1}{12} = \frac{1}{2} \end{cases}$$

The solutions are: $x = -5,\ \frac{1}{3},\ \frac{1}{2}$

Exercise

Find all solutions of the equation: $8x^3 + 18x^2 + 45x + 27 = 0$

Solution

$$possibilities: \ \pm\left\{\frac{27}{8}\right\} = \pm\left\{\frac{1,\ 3,\ 9,\ 27}{1,\ 2,\ 4,\ 8}\right\}$$

$$= \pm\left\{1,\ 3,\ 9,\ 27,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{3}{2},\ \frac{3}{4},\ \frac{3}{8},\ \frac{9}{2},\ \frac{9}{4},\ \frac{9}{8},\ \frac{27}{2},\ \frac{27}{4},\ \frac{27}{8}\right\}$$

$$\begin{array}{r|rrrr} -\frac{3}{4} & 8 & 18 & 45 & 27 \\ & & -6 & -9 & -27 \\ \hline & 8 & 12 & 36 & \boxed{0} \end{array} \quad \rightarrow \ 8x^2 + 12x + 36 = 0$$

The solutions are: $x = -\frac{3}{4},\ -\frac{3}{4} \pm i\frac{3\sqrt{7}}{4}$

Exercise

Find all solutions of the equation: $3x^3 - x^2 + 11x - 20 = 0$

Solution

$$possibilities: \ \pm\left\{\frac{20}{3}\right\} = \pm\left\{\frac{1,\ 2,\ 4,\ 5,\ 10,\ 20}{1,\ 3}\right\}$$

$$= \pm\left\{1,\ 2,\ 4,\ 5,\ 10,\ 20,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3},\ \frac{5}{3},\ \frac{10}{3},\ \frac{20}{3}\right\}$$

$$\begin{array}{r|rrrr} \frac{4}{3} & 3 & -1 & 11 & -20 \\ & & 4 & 4 & 20 \\ \hline & 3 & 3 & 15 & \boxed{0} \end{array} \quad \rightarrow \ 3x^2 + 3x + 15 = 0$$

$$x = \frac{-3 \pm \sqrt{9-180}}{6}$$

$$= \frac{-3 \pm 3\sqrt{19}}{6}$$

The solutions are: $x = \frac{4}{3},\ -\frac{1}{2} \pm i\frac{\sqrt{19}}{2}$

Exercise

Find all solutions of the equation: $x^4 - x^3 - 9x^2 + 3x + 18 = 0$

Solution

possibilities for $\frac{c}{d}: \pm\left\{\frac{18}{1}\right\} = \pm\{1,\ 2,\ 3,\ 6,\ 9,\ 18\}$

$$\begin{array}{r|rrrrr}
-2 & 1 & -1 & -9 & 3 & 18 \\
 & & -2 & 6 & 6 & -18 \\
\hline
3 & 1 & -3 & -3 & 9 & 0 \\
 & & 3 & 0 & -9 & \\
\hline
 & 1 & 0 & -3 & 0 &
\end{array}$$

$\to x^3 - 3x^2 - 3x + 9 = 0 \to \pm\left\{\frac{9}{1}\right\} = \pm\{1,\ 3,\ 9\}$

$\to x^2 - 3 = 0 \Rightarrow \underline{x = \pm\sqrt{3}}$

The solutions are: $x = -2,\ 3,\ \pm\sqrt{3}$

Exercise

Find all solutions of the equation: $2x^4 - 9x^3 + 9x^2 + x - 3 = 0$

Solution

possibilities for $\frac{c}{d}: \pm\left\{\frac{3}{2}\right\} = \pm\left\{1,\ 3,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$$\begin{array}{r|rrrrr}
1 & 2 & -9 & 9 & 1 & -3 \\
 & & 2 & -7 & 2 & 3 \\
\hline
1 & 2 & -7 & 2 & 3 & 0 \\
 & & 2 & -5 & -3 & \\
\hline
 & 2 & -5 & -3 & 0 &
\end{array}$$

$\to 2x^3 - 7x^2 + 2x + 3 = 0 \to \pm\left\{\frac{3}{2}\right\} = \pm\left\{1,\ 3,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$\to 2x^2 - 5x - 3 = 0$

$$x = \frac{5 \pm \sqrt{25 + 24}}{4}$$

$$= \begin{cases} \frac{5-7}{4} = -\frac{1}{2} \\ \frac{5+7}{4} = 3 \end{cases}$$

The solutions are: $x = 1,\ 1,\ -\frac{1}{2},\ 3$

Exercise

Find all solutions of the equation: $6x^4+5x^3-17x^2-6x=0$

Solution

$$x\left(6x^3+5x^2-17x-6\right)=0 \quad \to \quad \underline{x=0}$$

$$possibilities: \ \pm\left\{\frac{6}{6}\right\}=\pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{3},\ \frac{2}{3}\right\}$$

$$\begin{array}{r|rrrr} -2 & 6 & 5 & -17 & -6 \\ & & -12 & 14 & 6 \\ \hline & 6 & -7 & -3 & \boxed{0} \end{array} \quad \to \ 6x^2-7x-3=0$$

$$x=\frac{7\pm\sqrt{49+72}}{12}$$

$$=\begin{cases} \frac{7-11}{12}=-\frac{1}{3} \\ \frac{7+11}{12}=\frac{3}{2} \end{cases}$$

The solutions are: $x=0,\ -2,\ -\frac{1}{3},\ \frac{3}{2}$

Exercise

Find all solutions of the equation: $x^4-2x^2-16x-15=0$

Solution

$$possibilities: \ \pm\left\{\frac{15}{1}\right\}=\pm\{1,\ 3,\ 5,\ 15\}$$

$$\begin{array}{r|rrrrr} -1 & 1 & 0 & -2 & -16 & -15 \\ & & -1 & 1 & 1 & 15 \\ \hline 3 & 1 & -1 & -1 & -15 & 0 \\ & & 3 & 6 & 15 & \\ \hline & 1 & 2 & 5 & 0 & \end{array}$$

$\to \ x^3-x^2-x-15=0 \ \to \ \pm\left\{\frac{15}{1}\right\}=\pm\{1,\ 3,\ 5,\ 15\}$

$\to \ x^2+2x+5=0$

$$x=\frac{-2\pm\sqrt{-16}}{2}$$

$$=-1\pm 2i$$

The solutions are: $x=-1,\ 3,\ -1\pm 2i$

Exercise

Find all solutions of the equation: $x^4 - 2x^3 - 5x^2 + 8x + 4 = 0$

Solution

possibilities : $\pm\left\{\frac{4}{1}\right\} = \pm\{1,\ 2,\ 4\}$

$$\begin{array}{r|rrrrr}
2 & 1 & -2 & -5 & 8 & 4 \\
 & & 2 & 0 & -10 & -4 \\
\hline
\overline{-2} & 1 & 0 & -5 & -2 & 0 \\
 & & -2 & 4 & 2 & \\
\hline
 & 1 & -2 & -1 & 0 &
\end{array}$$

$\rightarrow x^3 - 5x - 2 = 0 \rightarrow \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$

$\rightarrow x^2 - 2x - 1 = 0$

$$x = \frac{2 \pm \sqrt{8}}{2}$$

$$= \frac{2 \pm 2\sqrt{2}}{2}$$

The solutions are: $\underline{x = -2,\ 2,\ 1 \pm \sqrt{2}}|$

Exercise

Find all solutions of the equation: $2x^4 - 17x^3 + 4x^2 + 35x - 24 = 0$

Solution

possibilities : $\pm\left\{\frac{24}{2}\right\} = \pm\left\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$$\begin{array}{r|rrrrr}
1 & 2 & -17 & 4 & 35 & -24 \\
 & & 2 & -15 & -11 & 24 \\
\hline
\overline{1} & 2 & -15 & -11 & 24 & 0 \\
 & & 2 & -13 & 24 & \\
\hline
 & 2 & -13 & -24 & 0 &
\end{array}$$

$\rightarrow 2x^3 - 15x^2 - 11x + 24 = 0 \rightarrow \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$

$\rightarrow 2x^2 - 13x - 24 = 0$

$$x = \frac{13 \pm \sqrt{169 + 192}}{4}$$

$$= \begin{cases} \frac{13 - 19}{4} = -\frac{3}{2} \\ \frac{13 + 19}{4} = 8 \end{cases}$$

The solutions are: $\underline{x = -\frac{3}{2},\ 1,\ 1,\ 8}|$

Exercise

Find all solutions of the equation: $x^4 + x^3 - 3x^2 - 5x - 2 = 0$

Solution

$$possibilities: \ \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$$

$$\begin{array}{r|rrrrr}
-1 & 1 & 1 & -3 & -5 & -2 \\
 & & -1 & 0 & 3 & 2 \\
\hline
-1 & 1 & 0 & -3 & -2 & 0 \\
 & & -1 & 1 & 2 & \\
\hline
 & 1 & -1 & -2 & 0 &
\end{array}$$

$\to x^3 - 3x - 2 = 0 \to \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$

$\to x^2 - x - 2 = 0$

$$x = \frac{1 \pm \sqrt{9}}{2}$$

$$= \begin{cases} \frac{1-3}{2} = -1 \\ \frac{1+3}{2} = 2 \end{cases}$$

The solutions are: $\underline{x = -1,\ \ -1,\ \ -1,\ 2}$

Exercise

Find all solutions of the equation: $6x^4 - 17x^3 - 11x^2 + 42x = 0$

Solution

$$x\left(6x^3 - 17x^2 - 11x + 42\right) = 0$$

$$x = 0 \quad 6x^3 - 17x^2 - 11x + 42 = 0$$

$$possibilities: \ \pm\left\{\frac{42}{6}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ 7,\ 14,\ 21,\ 42,\ \frac{1}{2},\ \frac{3}{2},\ \frac{7}{2},\ \frac{21}{2},\ \frac{1}{3},\ \frac{2}{3},\ \frac{7}{3},\ \frac{14}{3},\ \frac{1}{6},\ \frac{7}{6},\ \frac{21}{6}\right\}$$

$$\begin{array}{r|rrrr}
2 & 6 & -17 & -11 & 42 \\
 & & 12 & -10 & -42 \\
\hline
 & 6 & -5 & -21 & \boxed{0}
\end{array}$$

$\to 6x^2 - 5x - 21 = 0$

$$x = \frac{5 \pm \sqrt{25 + 504}}{12}$$

$$= \begin{cases} \frac{5-23}{12} = -\frac{3}{2} \\ \frac{5+23}{12} = \frac{7}{3} \end{cases}$$

The solutions are: $\underline{x = -\frac{3}{2},\ 0,\ 2,\ \frac{7}{3}}$

Exercise

Find all solutions of the equation: $x^4 - 5x^2 - 2x = 0$

Solution

$$x\left(x^3 - 5x - 2\right) = 0$$

$$x = 0 \quad x^3 - 5x - 2 = 0$$

possibilities : $\pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -5 & -2 \\ & & -2 & 4 & 2 \\ \hline & 1 & -2 & -1 & \boxed{0} \end{array} \quad \to x^2 - 2x - 1 = 0$$

$$x = \frac{2 \pm \sqrt{4+4}}{2}$$

$$= \frac{2 \pm 2\sqrt{2}}{2}$$

The solutions are: $x = -2,\ 0,\ 1 \pm \sqrt{2}$

Exercise

Find all solutions of the equation: $3x^4 - 4x^3 - 11x^2 + 16x - 4 = 0$

Solution

possibilities : $\pm\left\{\frac{4}{3}\right\} = \pm\left\{1,\ 2,\ 4,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3}\right\}$

$$\begin{array}{r|rrrrr} 1 & 3 & -4 & -11 & 16 & -4 \\ & & 3 & -1 & -12 & 4 \\ \hline \overline{2} & 3 & -1 & -12 & 4 & 0 \\ & & 6 & 10 & -4 & \\ \hline & 3 & 5 & -2 & 0 & \end{array}$$

$\to 3x^3 - x^2 - 12x + 4 = 0 \to \pm\left\{\frac{4}{3}\right\} = \pm\left\{1,\ 2,\ 4,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3}\right\}$

$\to 3x^2 + 5x - 2 = 0$

$$x = \frac{-5 \pm \sqrt{25+24}}{6}$$

$$= \begin{cases} \frac{-5-7}{6} = -2 \\ \frac{-5+7}{6} = \frac{1}{3} \end{cases}$$

The solutions are: $x = -2,\ \frac{1}{3},\ 1,\ 2$

Exercise

Find all solutions of the equation: $6x^4 + 23x^3 + 19x^2 - 8x - 4 = 0$

Solution

possibilities : $\pm\left\{\frac{4}{6}\right\} = \pm\left\{1,\ 2,\ 4,\ \frac{1}{6},\ \frac{2}{3},\ \frac{2}{3}\right\}$

$$\begin{array}{r|rrrrr}
-2 & 6 & 23 & 19 & -8 & -4 \\
 & & -12 & -22 & 6 & 4 \\ \hline
-2 & 6 & 11 & -3 & -2 & 0 \\
 & & -12 & 2 & 2 & \\ \hline
 & 6 & -1 & -1 & 0 &
\end{array}$$

$\rightarrow 6x^3 + 11x^2 - 3x - 2 = 0 \rightarrow \pm\left\{\frac{2}{6}\right\} = \pm\left\{1,\ 2,\ \frac{1}{6},\ \frac{1}{3}\right\}$

$\rightarrow 6x^2 - x - 1 = 0$

$$x = \frac{1 \pm \sqrt{25}}{12}$$

$$= \begin{cases} \frac{1-5}{12} = -\frac{1}{3} \\ \frac{1+5}{12} = \frac{1}{2} \end{cases}$$

The solutions are: $x = -2,\ -2,\ -\frac{1}{3},\ \frac{1}{2}$

Exercise

Find all solutions of the equation: $4x^4 - 12x^3 + 3x^2 + 12x - 7 = 0$

Solution

possibilities : $\pm\left\{\frac{7}{4}\right\} = \pm\left\{1,\ 7,\ \frac{1}{2},\ \frac{7}{2},\ \frac{1}{4},\ \frac{7}{4}\right\}$

$$\begin{array}{r|rrrrr}
1 & 4 & -12 & 3 & 12 & -7 \\
 & & 4 & -8 & -5 & 7 \\ \hline
-1 & 4 & -8 & -5 & 7 & 0 \\
 & & -4 & 12 & -7 & \\ \hline
 & 4 & -12 & 7 & 0 &
\end{array}$$

$\rightarrow 4x^3 - 8x^2 - 5x + 7 = 0 \rightarrow \pm\left\{\frac{7}{4}\right\} = \pm\left\{1,\ 7,\ \frac{1}{2},\ \frac{7}{2},\ \frac{1}{4},\ \frac{7}{4}\right\}$

$\rightarrow 4x^2 - 12x + 7 = 0$

$$x = \frac{12 \pm \sqrt{144 - 112}}{8}$$

$$= \frac{12 \pm 4\sqrt{2}}{8}$$

The solutions are: $x = -1,\ 1,\ \frac{3 \pm \sqrt{2}}{2}$

Exercise

Find all solutions of the equation: $2x^4 - 9x^3 - 2x^2 + 27x - 12 = 0$

Solution

possibilities : $\pm\left\{\frac{12}{2}\right\} = \pm\left\{1,\ 2,\ 3,\ 4,\ 6,\ 12,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$$\begin{array}{r|rrrrr}
4 & 2 & -9 & -2 & 27 & -12 \\
 & & 8 & -4 & -24 & 12 \\ \hline
\frac{1}{2} & 2 & -1 & -6 & 3 & 0 \\
 & & 1 & 0 & -3 & \\ \hline
 & 2 & 0 & -6 & 0 &
\end{array}$$

$\rightarrow 2x^3 - x^2 - 6x + 3 = 0 \ \rightarrow \ \pm\left\{\frac{3}{2}\right\} = \pm\left\{1,\ 3,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$\rightarrow 2x^2 - 6 = 0$

$x^2 = 3$

The solutions are: $\boxed{x = \frac{1}{2},\ 4,\ \pm\sqrt{3}}$

Exercise

Find all solutions of the equation: $2x^4 - 19x^3 + 51x^2 - 31x + 5 = 0$

Solution

possibilities : $\pm\left\{\frac{5}{2}\right\} = \pm\left\{1,\ 5,\ \frac{1}{2},\ \frac{5}{2}\right\}$

$$\begin{array}{r|rrrrr}
5 & 2 & -19 & 51 & -31 & 5 \\
 & & 10 & -45 & 30 & -5 \\ \hline
\frac{1}{2} & 2 & -9 & 6 & -1 & 0 \\
 & & 1 & -4 & 1 & \\ \hline
 & 2 & -8 & 2 & 0 &
\end{array}$$

$\rightarrow 2x^3 - 9x^2 + 6x - 1 = 0 \ \rightarrow \ \pm\left\{\frac{1}{2}\right\}$

$\rightarrow 2x^2 - 8x + 2 = 0$

$$x = \frac{8 \pm \sqrt{64 - 16}}{4}$$

$$= \frac{8 \pm 4\sqrt{3}}{4}$$

The solutions are: $\boxed{x = \frac{1}{2},\ 5,\ 2 \pm \sqrt{3}}$

Exercise

Find all solutions of the equation: $4x^4 - 35x^3 + 71x^2 - 4x - 6 = 0$

Solution

possibilities : $\pm\left\{\frac{6}{4}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{4},\ \frac{3}{4}\right\}$

$$\begin{array}{r|rrrrr}
3 & 4 & -35 & 71 & -4 & -6 \\
 & & 12 & -69 & 6 & 6 \\ \hline
-\frac{1}{4} & 4 & -23 & 2 & 2 & 0 \\
 & & -1 & 6 & -2 & \\ \hline
 & 4 & -24 & 8 & 0 &
\end{array}$$

$\to 4x^3 - 23x^2 + 2x + 2 = 0 \to \pm\left\{\frac{2}{4}\right\} = \pm\left\{1,\ 2,\ \frac{1}{2},\ \frac{1}{4}\right\}$

$\to 4x^2 - 24x + 8 = 0$

$x^2 - 6x + 2 = 0$

$x = \dfrac{6 \pm \sqrt{36-8}}{2}$

$= \dfrac{6 \pm 2\sqrt{7}}{4}$

The solutions are: $\underline{x = -\frac{1}{4},\ 3,\ 3 \pm \sqrt{7}}$

Exercise

Find all solutions of the equation: $2x^4 + 3x^3 - 4x^2 - 3x + 2 = 0$

Solution

possibilities : $\pm\left\{\frac{2}{2}\right\} = \pm\left\{1,\ 2,\ \frac{1}{2}\right\}$

$$\begin{array}{r|rrrrr}
1 & 2 & 3 & -4 & -3 & 2 \\
 & & 2 & 5 & 1 & -2 \\ \hline
-1 & 2 & 5 & 1 & -2 & 0 \\
 & & -2 & -3 & 2 & \\ \hline
 & 2 & 3 & -2 & 0 &
\end{array}$$

$\to 2x^3 - 23x^2 + 2x - 2 = 0 \to \pm\left\{\frac{2}{2}\right\} = \pm\left\{1,\ 2,\ \frac{1}{2}\right\}$

$\to 2x^2 + 3x - 2 = 0$

$x = \dfrac{-3 \pm \sqrt{9+16}}{4}$

$= \begin{cases} \dfrac{-3-5}{4} = -2 \\ \dfrac{-3+5}{4} = \dfrac{1}{2} \end{cases}$

The solutions are: $\underline{x = -2,\ \ -1,\ \frac{1}{2},\ 1}$

Exercise

Find all solutions of the equation: $x^4 + 3x^3 - 30x^2 - 6x + 56 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{56}{1}\right\} = \pm\{1,\ 2,\ 4,\ 7,\ 8,\ 14,\ 28,\ 56\}$

$$
\begin{array}{r|rrrrr}
4 & 1 & 3 & -3 & -6 & 56 \\
 & & 4 & 28 & -8 & -56 \\
\hline
-7 & 1 & 7 & -2 & -14 & 0 \\
 & & -7 & 0 & 14 & \\
\hline
 & 1 & 0 & -2 & &
\end{array}
$$

$\rightarrow \quad x^3 + 7x^2 - 2x - 14 = 0 \quad \Rightarrow \quad \frac{c}{d} = \pm\left\{\frac{14}{1}\right\} = \pm\{1,\ 2,\ 7,\ 14\}$

$\rightarrow \quad x^2 - 2 = 0$

$\Rightarrow \underline{x = \pm\sqrt{2}|}$

The solutions are: $\underline{x = 4,\ \ -7,\ \ \pm\sqrt{2}\ |}$

Exercise

Find all solutions of the equation: $\quad 3x^5 - 10x^4 - 6x^3 + 24x^2 + 11x - 6 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{6}{3}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$

$$
\begin{array}{r|rrrrrr}
-1 & 3 & -10 & -6 & 24 & 11 & -6 \\
 & & -3 & 13 & -7 & -17 & 6 \\
\hline
-1 & 3 & -13 & 7 & 17 & -6 & 0 \\
 & & -3 & 16 & -23 & 6 & \\
\hline
2 & 3 & -16 & 23 & -6 & 0 & \\
 & & 6 & 20 & 6 & & \\
\hline
 & 3 & -10 & 3 & 0 & &
\end{array}
$$

$x^4 - 13x^3 + 7x^2 + 17x - 6 = 0 \rightarrow \pm\left\{\frac{6}{3}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$

$3x^3 - 16x^2 + 26x - 6 = 0 \rightarrow \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$

$3x^2 - 10x + 3 = 0$

$$x = \frac{10 \pm \sqrt{100 - 36}}{6}$$

$$= \begin{cases} \frac{10 - 8}{6} = \frac{1}{3} \\ \frac{10 + 8}{6} = 3 \end{cases}$$

The solutions are: $\underline{x = -1,\ \ -1,\ \ \frac{1}{3},\ \ 2,\ \ 3\ |}$

Exercise

Find all solutions of the equation: $\quad 6x^5 + 19x^4 + x^3 - 6x^2 = 0$

Solution

$x^2\left(6x^3 + 19x^2 + x - 6\right) = 0 \quad \rightarrow \quad \underline{x = 0,\ 0|}$

$6x^3 + 19x^2 + x - 6 = 0$

possibilities for $\frac{c}{d}: \pm\left\{\frac{6}{6}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{3},\ \frac{2}{3}\right\}$

$$\begin{array}{r|rrrr}
-3 & 6 & 19 & 1 & -6 \\
 & & -18 & -3 & 6 \\
\hline
 & 6 & 1 & -2 & \boxed{0}
\end{array} \quad 6x^2 + x - 2 = 0$$

$$x = \frac{-1 \pm \sqrt{1+48}}{12}$$

$$= \begin{cases} \frac{-1-7}{12} = -\frac{2}{3} \\ \frac{-1+7}{12} = \frac{1}{2} \end{cases}$$

The solutions are: $x = 0,\ 0,\ -\frac{2}{3},\ -3,\ \frac{1}{2}$

Exercise

Find all solutions of the equation: $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 = 0$

Solution

$x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 = (x+1)^5 = 0$

possibilities for $\frac{c}{d}: \pm\{1\}$

$$\begin{array}{r|rrrrrr}
-1 & 1 & 5 & 10 & 10 & 5 & 1 \\
 & & -1 & -4 & -6 & -4 & -1 \\
\hline
-1 & 1 & 4 & 6 & 4 & 1 & 0 \\
 & & -1 & -3 & -3 & -1 & \\
\hline
-1 & 1 & 3 & 3 & 1 & 0 & \\
 & & -1 & -2 & -1 & & \\
\hline
 & 1 & 2 & 1 & 0 & &
\end{array}
\quad
\begin{array}{l}
\to x^4 + 4x^3 + 6x^2 + 4x + 1 = 0 \to \pm\{1\} \\
\to x^3 + 3x^2 + 3x + 1 = 0 \to \pm\{1\} \\
\to x^2 + 2x + 1 = 0
\end{array}$$

$x^2 + 2x + 1 = (x+1)^2$

The solutions are: $x = -1,\ -1,\ -1,\ -1,\ -1$

Exercise

Find all solutions of the equation: $x^5 - x^4 - 7x^3 + 7x^2 + 12x - 12 = 0$

Solution

possibilities for $\frac{c}{d}: \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$$\begin{array}{r|rrrrrr}
1 & 1 & -1 & -7 & 7 & 12 & -12 \\
 & & 1 & 0 & -7 & 0 & 12 \\
\hline
2 & 1 & 0 & -7 & 0 & 12 & 0 \\
 & & 2 & 4 & -6 & -12 & \\
\hline
-2 & 1 & 2 & -3 & -6 & 0 & \\
 & & -2 & 0 & 6 & & \\
\hline
 & 1 & 0 & -3 & 0 & &
\end{array}$$

$\rightarrow x^4 - 7x^2 - 12 = 0 \rightarrow \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$\rightarrow x^3 + 2x^2 - 3x - 6 = 0 \rightarrow \pm\{1,\ 2,\ 3,\ 6\}$

$\rightarrow x^2 - 3 = 0$

$x^2 = 3$

The solutions are: $\underline{x = -2,\ 1,\ 2,\ \pm\sqrt{3}}$

Exercise

Find all solutions of the equation: $x^5 - 2x^3 - 8x = 0$

Solution

$x\left(x^4 - 2x^2 - 8\right) = 0$

$\underline{x = 0}$

$x^4 - 2x^2 - 8 = 0$

$$x^2 = \frac{2 \pm \sqrt{4+32}}{2}$$

$$= \begin{cases} \frac{2-6}{2} = -2 \\ \frac{2+6}{2} = 4 \end{cases}$$

$$\begin{cases} x^2 = -2 & \rightarrow & x = \pm i\sqrt{2} \\ x^2 = 4 & \rightarrow & x = \pm 2 \end{cases}$$

The solutions are: $\underline{x = 0,\ \pm 2,\ \pm i\sqrt{2}}$

Exercise

Find all solutions of the equation: $x^5 - 32 = 0$

Solution

possibilities for $\frac{c}{d}: \pm\{1,\ 2,\ 4,\ 8,\ 16,\ 32\}$

$$\begin{array}{c|cccccc} 2 & 1 & 0 & 0 & 0 & 0 & -32 \\ & & 2 & 4 & 8 & 16 & 32 \\ \hline & 1 & 2 & 4 & 8 & 16 & 0 \end{array} \quad \to \quad x^4 + 2x^3 + 4x^2 + 8x + 16 = 0 \to \pm\{1,\ 2,\ 4,\ 8,\ 16\}$$

$x^4 + 2x^3 + 4x^2 + 8x + 16 = 0$

Cannot be solved using rational zero theorem.
Therefore; using program

The solutions are: $x = 2,\ \dfrac{-1-\sqrt{5} \pm i\sqrt{2}\sqrt{5-\sqrt{5}}}{2},\ \dfrac{-1+\sqrt{5} \pm i\sqrt{2}\sqrt{5-\sqrt{5}}}{2}$

Exercise

Find all solutions of the equation: $3x^6 - 10x^5 - 29x^4 + 34x^3 + 50x^2 - 24x - 24 = 0$

Solution

possibilities for $\frac{c}{d}: \pm\left\{\frac{24}{3}\right\} = \pm\left\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3},\ \frac{8}{3}\right\}$

$$\begin{array}{c|ccccccc} 1 & 3 & -10 & -29 & 34 & 50 & -24 & -24 \\ & & 3 & -7 & -36 & -2 & 48 & 24 \\ \hline -1 & 3 & -7 & -36 & -2 & 48 & 24 & 0 \\ & & -3 & 10 & 26 & -24 & -24 & \\ \hline -2 & 3 & -10 & -26 & 24 & 24 & 0 & \\ & & -6 & 32 & -12 & -24 & & \\ \hline -\frac{2}{3} & 3 & -16 & 6 & 12 & 0 & & \\ & & -2 & 12 & -12 & & & \\ \hline & 3 & -18 & 18 & 0 & & & \end{array}$$

$\to\ 3x^5 - 7x^4 - 36x^3 - 2x^2 + 48x + 24 = 0$

$\to\ 3x^4 - 10x^3 - 26x^2 + 24x + 24 = 0$

$\to\ 3x^3 - 16x^2 + 12x - 12 = 0$

$\to\ 3x^2 - 18x + 18 = 0$

$x^2 - 6x + 6 = 0$

$$x = \frac{6 \pm \sqrt{36-24}}{2}$$

$$= \frac{6 \pm 2\sqrt{3}}{2}$$

The solutions are: $x = -2,\ -1,\ 1,\ -\frac{2}{3},\ 3 \pm \sqrt{3}$

Exercise

Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T(k)=\frac{1}{6}\left(k^3+3k^2+2k\right)$$

Where k is the number of levels in the pyramid. If 220 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

Solution

$$\frac{1}{6}\left(k^3+3k^2+2k\right)=220$$

$$k^3+3k^2+2k-1,320=0$$

$$\begin{array}{r|rrrr} 10 & 1 & 3 & 2 & -1320 \\ & & 10 & 130 & 1320 \\ \hline & 1 & 13 & 132 & 0 \end{array} \quad \to \quad k^2+13k+132=0$$

$$k=\frac{-13\pm\sqrt{-359}}{2} \quad \mathbb{C}$$

The are **10** levels in the pyramid.

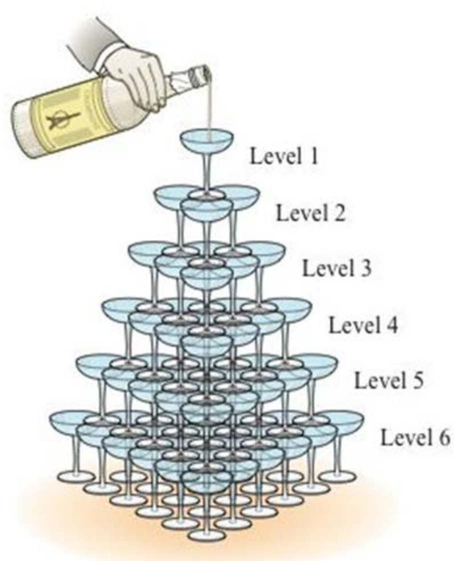
Exercise

Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T(k)=\frac{1}{6}\left(2k^3+3k^2+k\right)$$

Where k is the number of levels in the pyramid. If 140 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

Solution



$$\frac{1}{6}\left(2k^3+3k^2+k\right)=150$$

$$2k^3+3k^2+k-840=0$$

$$\begin{array}{r|rrrr} 7 & 2 & 3 & 1 & -840 \\ & & 14 & 119 & 840 \\ \hline & 2 & 17 & 120 & 0 \end{array} \quad \to \quad 2k^2+17k+120=0$$

$$k=\frac{-17\pm\sqrt{-671}}{4} \quad \mathbb{C}$$

The are **7** levels in the pyramid.

Exercise

A carbon dioxide cartridge for a paintball rifle has the shape of a right circular cylinder with a hemisphere at each end. The cylinder is 4 *inches* long, and the volume of the cartridge is $2\pi \ \ in^3$.
The common interior radius of the cylinder and the hemispheres is denoted by *x*. Estimate the length of the radius *x*.

Solution

Volume of the Cartridge = 2 × (Volume of Hemisphere) + Volume of Cylinder

$$Volume\ of\ \ Sphere = \frac{4}{3}\pi x^3$$

$$Volume\ of\ \ Cylinder = 4\pi x^2$$

$$Volume\ of\ \ Cartridge = \frac{4}{3}\pi x^3 + 4\pi x^2$$

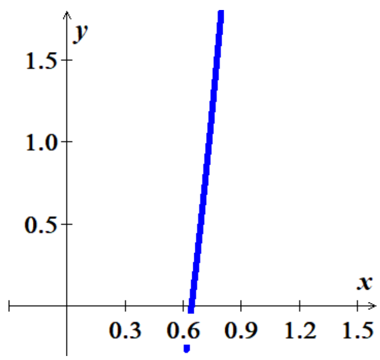
$$2\pi = \frac{4}{3}\pi x^3 + 4\pi x^2$$

$$2x^3 + 6x^2 = 3$$

$$2x^3 + 6x^2 - 3 = 0$$

Using Graph:

$\underline{x \approx 0.64 \ \ in.}$

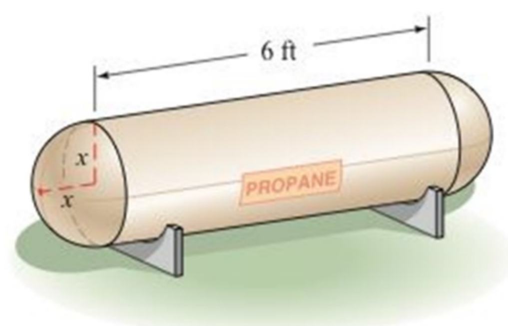


Exercise

A propane tank has the shape of a circular cylinder with a hemisphere at each end. The cylinder is 6 *feet* long and the volume of the tank is $9\pi \ \ ft^3$. Find the length of the radius *x*.

Solution

Volume of the Cartridge = 2 × (Volume of Hemisphere) + Volume of Cylinder



$$Volume\ of\ \ Sphere = \frac{4}{3}\pi x^3$$

$$Volume\ of\ \ Cylinder = 6\pi x^2$$

$$Volume\ of\ \ Cartridge = \frac{4}{3}\pi x^3 + 6\pi x^2$$

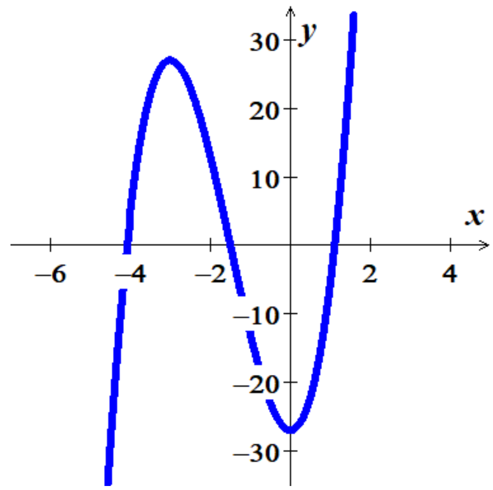
$$9\pi = \frac{4}{3}\pi x^3 + 6\pi x^2$$

$$27 = 4x^3 + 18x^2$$

$$4x^3 + 18x^2 - 27 = 0$$

$$possibilities\ for\ \ \frac{c}{d} : \pm\left\{\frac{27}{4}\right\} = \pm\left\{1,\ 3,\ 9,\ 27,\ \frac{1}{2},\ \frac{3}{2},\ \frac{9}{2},\ \frac{27}{2},\ \frac{1}{4},\ \frac{3}{4},\ \frac{9}{4},\ \frac{27}{4}\right\}$$

$$\begin{array}{r|rrrr} -\frac{3}{2} & 4 & 18 & 0 & -27 \\ & & -6 & -18 & 27 \\ \hline & 4 & 12 & -18 & 0 \end{array} \quad \rightarrow \quad 4x^2 + 12x - 18 = 0$$



$$2x^2 + 6x - 9 = 0$$

$$x = \frac{-6 \pm \sqrt{36+72}}{4}$$

$$= \frac{-6 \pm 6\sqrt{3}}{4}$$

$$= \frac{-3 \pm 3\sqrt{3}}{2}$$

$$x = -\frac{3}{2}, \ \frac{-3-3\sqrt{3}}{2}, \ \frac{-3+3\sqrt{3}}{2}$$

$\therefore$ the length of the radius x is $\frac{-3+3\sqrt{3}}{2} \approx 1.1$ *foot*

Exercise

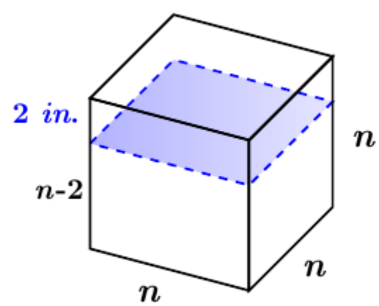
A cube measures n inches on each edge. If a slice 2 *inches* thick is cut from one face of the cube, the resulting solid has a volume of $567\ in^3$. Find n.

Solution

$$Volume = n^2(n-2)$$

$$n^3 - 2n^2 = 567$$

$$n^3 - 2n^2 - 567 = 0$$



$$possibilities\ for\ \frac{c}{d} := \pm\{1,\ 3,\ 7,\ 9,\ 21,\ 27,\ 63,\ 81,\ 189,\ 567\}$$

$$\begin{array}{r|rrrr} 9 & 1 & -2 & 0 & -567 \\ & & 9 & 63 & 567 \\ \hline & 1 & 7 & 63 & 0 \end{array} \quad \rightarrow \quad n^2 + 7n + 63 = 0$$

$$n = \frac{-7 \pm \sqrt{49-252}}{2}$$

$$= \frac{-7 \pm i\sqrt{203}}{2} \quad \times$$

$$\therefore \quad n = 9$$

Exercise

A cube measures n inches on each edge. If a slice 1 *inch* thick is cut from one face of the cube and then a slice 3 inches thick is cut from another face of the cube, the resulting solid has a volume of $1560\ in^3$. Find the dimensions of the original cube.

Solution

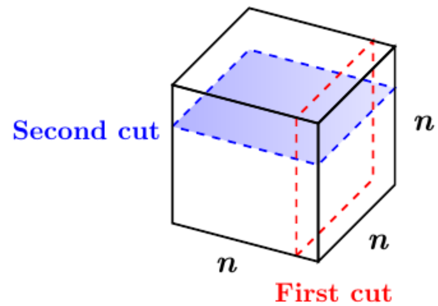
$Volume = n(n-1)(n-3)$

$n^3 - 4n^2 + 3n = 1560$

$n^3 - 4n^2 + 3n - 1560 = 0$

$possibilities\ for\ \frac{c}{d} := \pm\left\{\begin{array}{l}1,\ 2,\ 4,\ 5,\ 6,\ 8,\ 10,\ 12,\ 13,\ 15,\ 20,\ 24,\ 26,\ 30,\ 39, \\ 40,\ 52,\ 60,\ 65,\ 78,\ 104,\ 120,\ 130,\ 156,\ 195,\ 260,\ 312,\ 390,\ 780,\ 1560\end{array}\right\}$

$$\begin{array}{r|rrrr} 13 & 1 & -4 & 3 & -1560 \\ & & 13 & 117 & 1560 \\ \hline & 1 & 9 & 120 & 0 \end{array} \quad \to \quad n^2 + 9n + 120 = 0$$



$n = \dfrac{-9 \pm \sqrt{81 - 480}}{2}$

$= \dfrac{-9 \pm i\sqrt{399}}{2}$ ✗

$\therefore\ \ \underline{n = 13}$

Exercise

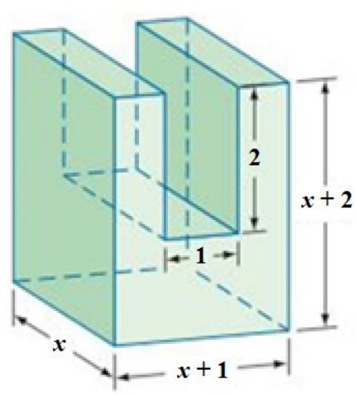
For what value of x will the volume of the following solid be $112\ in^3$

Solution

Volume of the bottom portion $= x^2(x+1)$

Volume of one side portion $= 2x\left(\frac{1}{2}x\right)$

$= x^2$



Total Volume $= x^2(x+1) + 2x^2$

$112 = x^3 + 3x^2$

$x^3 + 3x^2 - 112 = 0$

$possibilities\ for\ \frac{c}{d} := \pm\{1,\ 2,\ 4,\ 8,\ 14,\ 28,\ 56,\ 112\}$

$$\begin{array}{r|rrrr} 4 & 1 & 3 & 0 & -112 \\ & & 4 & 28 & 112 \\ \hline & 1 & 7 & 28 & 0 \end{array} \rightarrow \quad x^2+7x+28=0$$

$$x=\frac{-7\pm\sqrt{49-112}}{2}$$

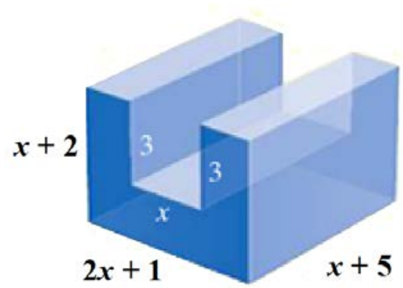
$$=\frac{-7\pm 3i\sqrt{7}}{2} \quad \times$$

$$\therefore \quad \boxed{x=4}$$

Exercise

For what value of x will the volume of the following solid be $208 \ in^3$

Solution



Volume of the bottom portion $=(2x+1)(x+5)(x+2-3)$

$$=\left(2x^2+11x+5\right)(x-1)$$

$$=2x^3+11x^2+5x-2x^2-11x-5$$

$$=2x^3+9x^2-6x-5$$

Volume of one side portion $=(3)\frac{1}{2}(2x+1-x)(x+5)$

$$=\frac{3}{2}(x+1)(x+5)$$

$$=\frac{3}{2}\left(x^2+6x+5\right)$$

Total Volume $=2x^3+9x^2-6x-5+2\left(\frac{3}{2}\right)\left(x^2+6x+5\right)$

$$208=2x^3+9x^2-6x-5+3x^2+18x+15$$

$$2x^3+12x^2+12x-198=0$$

$$x^3+6x^2+6x-99=0$$

possibilities for $\frac{c}{d} := \pm\{1,\ 3,\ 9,\ 11,\ 33,\ 99\}$

$$\begin{array}{r|rrrr} 3 & 1 & 6 & 6 & -99 \\ & & 3 & 27 & 99 \\ \hline & 1 & 9 & 33 & 0 \end{array} \rightarrow \quad x^2+9x+33=0$$

$$x=\frac{-9\pm\sqrt{81-132}}{2}$$

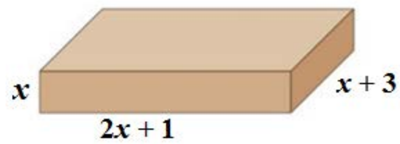
$$=\frac{-9\pm i\sqrt{51}}{2} \quad \times$$

$$\therefore \quad \boxed{x=3}$$

Exercise

The length of rectangular box is 1 *inch* more than twice the height of the box, and the width is 3 *inches* more than the height. If the volume of the box is 126 in^3, find the dimensions of the box.

Solution



$$Volume = x(2x+1)(x+3)$$

$$2x^3+7x^2+3x=126$$

$$2x^3+7x^2+3x-126=0$$

$$possibilities\ for\ \frac{c}{d} := \pm\left\{\frac{126}{2}\right\}$$

$$= \pm\left\{1,\ 2,\ 3,\ 6,\ 9,\ 14,\ 21,\ 42,\ 63,\ 126,\ \frac{1}{2},\ \frac{3}{2},\ \frac{9}{2},\ \frac{21}{2},\ \frac{63}{2}\right\}$$

$$\begin{array}{c|cccc} 3 & 2 & 7 & 3 & -126 \\ & & 6 & 39 & 126 \\ \hline & 2 & 13 & 42 & 0 \end{array} \rightarrow\ 2x^2+13x+42=0$$

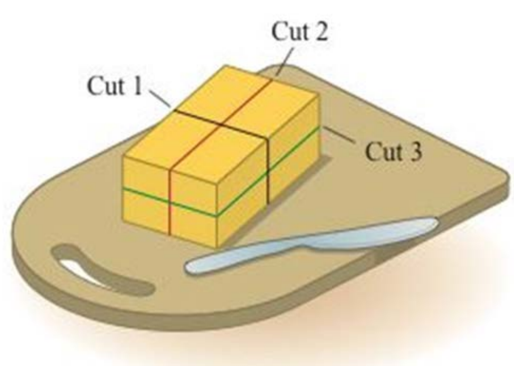
$$x=\frac{-13\pm\sqrt{169-336}}{4}$$

$$=\frac{-13\pm i\sqrt{167}}{4} \quad \times$$

$$\therefore\ \ x=3$$

Exercise

One straight cut through a thick piece of cheese produces two pieces. Two straight cuts can produce a maximum of four pieces. Two straight cuts can produce a maximum of four pieces. Three straight cuts can produce a maximum of eight pieces.



You might be inclined to think that every additional cut double number of pieces. However, for four straight cuts, you get a maximum of 15 pieces. The maximum number of pieces P that can be produced by n straight cuts is given by

$$P(n)=\frac{n^3+5n+6}{6}$$

a) Determine number of pieces that can be produces by five straight cuts.

b) What is the fewest number of straight cuts that are needed to produce 64 pieces?

Solution

a) $P(5) = \frac{5^3 + 25 + 6}{6}$

$= 26$

b) $\frac{n^3 + 5n + 6}{6} = 64$

$n^3 + 5n + 6 = 384$

$n^3 + 5n - 378 = 0$

possibilities for $\frac{c}{d} := \pm\{378\}$

$= \pm\{1,\ 2,\ 3,\ 6,\ 7,\ 9,\ 14,\ 18,\ 21,\ 27,\ 42,\ 54,\ 63,\ 126,\ 189,\ 378\}$

$$\begin{array}{r|rrrr} 7 & 1 & 0 & 5 & -378 \\ & & 7 & 49 & 378 \\ \hline & 1 & 7 & 54 & 0 \end{array} \quad \rightarrow \quad n^2 + 7n + 54 = 0$$

$n = \frac{-7 \pm \sqrt{49 - 216}}{2}$

$= \frac{-7 \pm i\sqrt{167}}{2}$ ✗

$\therefore\ n = 7$

Exercise

The number of ways one can select three cards from a group of n cards (the order of the selection matters), where $n \ge 3$, is given by $P(n) = n^3 - 3n^2 + 2n$. For a certain card trick, a magician has determined that there are exactly 504 *ways* to choose three cards from a given group. How many cards are in the group?

Solution

$P(n) = n^3 - 3n^2 + 2n = 504$

$n^3 - 3n^2 + 2n - 504 = 0$

possibilities for $\frac{c}{d} := \pm\{504\}$

$$= \pm\left\{\begin{array}{l} 1,\ 2,\ 3,\ 4,\ 6,\ 7,\ 8,\ 9,\ 12,\ 14,\ 18,\ 21, \\ 24,\ 28,\ 36,\ 42,\ 56,\ 63,\ 72,\ 84,\ 126,\ 168,\ 252,\ 504 \end{array}\right\}$$

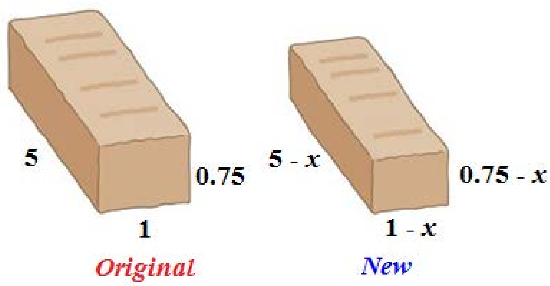
$$\begin{array}{c|cccc} 9 & 1 & -3 & 2 & -504 \\ & & 9 & 54 & 504 \\ \hline & 1 & 6 & 56 & 0 \end{array} \to \quad n^2+6n+56=0$$

$$\begin{aligned} n &= \frac{-6\pm\sqrt{36-224}}{2} \\ &= -3\pm i\sqrt{47} \quad \times \end{aligned}$$

$\therefore \quad \underline{n=9}$

Exercise

A nutrition bar in the shape of a rectangular solid measure 0.75 *in*. by 1 *in*. by 5 *inches*.



To reduce costs, the manufacturer has decided to decrease each of the dimensions of the nutrition bar by x *inches*, what value of x will produce a new bar with a volume that is $0.75\ in^3$ less than the present bar's volume.

Solution

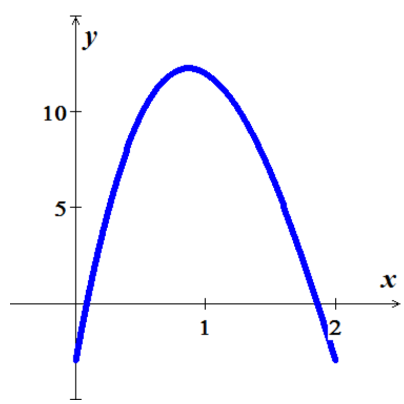
$$\begin{aligned} V_{original} &= (5)(1)\left(\frac{3}{4}\right) \\ &= \frac{15}{4} \end{aligned}$$

$$V_{new} = (5-x)(1-x)\left(\frac{3}{4}-x\right) \qquad \left(x<\frac{3}{4}\right)$$

$$\left(5-6x+x^2\right)\left(\frac{3-4x}{4}\right)=\frac{15}{4}-\frac{3}{4}$$

$$15-20x-18x+24x^2+3x^2-4x^3=4(3)$$

$$4x^3-27x^2+38x-3=0$$



From graph table:

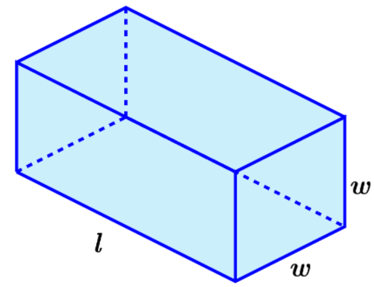
```
0.08200   -0.06334
0.08400    0.00386
```

$\underline{x\approx 0.083\ \ in.}$

Exercise

A rectangular box is square on two ends and has length plus girth of 81 *inches*. (Girth: distance *around* the box). Determine the possible lengths $l\ (l > w)$ of the box if its volume is $4900\ \ in^3$.

Solution



$$81 = l + 4w$$
$$l = 81 - 4w$$
$$V = lw^2$$
$$= (81 - 4w)w^2$$

$$-4w^3 + 81w^2 = 4900$$
$$4w^3 - 81w^2 + 4900 = 0$$

$$\textit{possibilities for } \frac{c}{d} := \pm\left\{\frac{4900}{4}\right\} = \pm\left\{\begin{array}{l}1,\ 2,\ 4,\ 7,\ 10,\ 14,\ 20,\ 28,\ 49, \\ \ \ 100,\ 175,\ 245,\ 350,\ 490,\ 700,\ 1225,\ 2450,\ 4900,\cdots\end{array}\right\}$$

$$\begin{array}{r|rrrr} 14 & 4 & -81 & 0 & 4900 \\ & & 56 & -350 & -4900 \\ \hline & 4 & -25 & -350 & 0 \end{array} \quad \to \quad 4w^2 - 25w - 350 = 0$$

$$w = \frac{25 \pm \sqrt{625 + 5600}}{8}$$
$$= \frac{25 \pm 5\sqrt{249}}{8}$$
$$= \begin{cases} \dfrac{25 - 5\sqrt{249}}{8} < 0 \\ \dfrac{25 + 5\sqrt{249}}{8} \approx 13 \end{cases}$$

$$l = 81 - 4(14) = 25$$
$$l = 81 - 4(13) = 29$$

$\therefore$ the possible lengths l are around **25** *in*. ***or*** **29** *in*.

Solution   *Section* 2.5 – Polynomial Functions

Exercise

Determine the end behavior of the graph of the polynomial function $f(x)=5x^3+7x^2-x+9$

Solution

Leading term: $5x^3$ with 3^{rd} degree (n is ***odd***)

$x\to -\infty \quad \Rightarrow \quad f(x)\to -\infty \qquad f(x)$ falls left

$x\to \infty \quad \Rightarrow \quad f(x)\to \infty \qquad f(x)$ rises right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x)=11x^3-6x^2+x+3$

Solution

Leading term: $11x^3$ with 3^{rd} degree (n is ***odd***)

$x\to -\infty \quad \Rightarrow \quad f(x)\to -\infty \qquad f(x)$ falls left

$x\to \infty \quad \Rightarrow \quad f(x)\to \infty \qquad$ rises right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x)=-11x^3-6x^2+x+3$

Solution

Leading term: $-11x^3$ with 3^{rd} degree (n is ***odd***)

$x\to -\infty \quad \Rightarrow \quad f(x)\to \infty \qquad f(x)$ rises left

$x\to \infty \quad \Rightarrow \quad f(x)\to -\infty \qquad f(x)$ falls right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x)=2x^3+3x^2-23x-42$

Solution

Leading term: $2x^3$ with 3^{rd} degree (n is ***odd***)

$x\to -\infty \quad \Rightarrow \quad f(x)\to -\infty \qquad f(x)$ falls left

$x\to \infty \quad \Rightarrow \quad f(x)\to \infty \qquad f(x)$ rises right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x)=5x^4+7x^2-x+9$

Solution

Leading term: $5x^4$ with 4^{rd} degree (n is ***even***)

$x\to-\infty \;\Rightarrow\; f(x)\to\infty$ $\quad f(x)$ rises left

$x\to\infty \;\Rightarrow\; f(x)\to\infty$ $\quad f(x)$ rises right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x)=11x^4-6x^2+x+3$

Solution

Leading term: $11x^4$ with 4^{rd} degree (n is ***even***)

$x\to-\infty \;\Rightarrow\; f(x)\to\infty$ $\quad f(x)$ rises left

$x\to\infty \;\Rightarrow\; f(x)\to\infty$ $\quad f(x)$ rises right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x)=-5x^4+7x^2-x+9$

Solution

Leading term: $-5x^4$ with 4^{rd} degree (n is ***even***)

$x\to-\infty \;\Rightarrow\; f(x)\to-\infty$ $\quad f(x)$ falls left

$x\to\infty \;\Rightarrow\; f(x)\to-\infty$ $\quad f(x)$ falls right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x)=-11x^4-6x^2+x+3$

Solution

Leading term: $-11x^4$ with 4^{rd} degree (n is ***even***)

$x\to-\infty \;\Rightarrow\; f(x)\to-\infty$ $\quad f(x)$ falls left

$x\to\infty \;\Rightarrow\; f(x)\to-\infty$ $\quad f(x)$ falls right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = 5x^5 - 16x^2 - 20x + 64$

Solution

Leading term: $5x^5$ with 5^{th} degree (n is ***odd***)

$x \to -\infty \;\Rightarrow\; f(x) \to -\infty$ $\qquad f(x)$ falls left

$x \to \infty \;\Rightarrow\; f(x) \to \infty$ $\qquad f(x)$ rises right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = -5x^5 - 16x^2 - 20x + 64$

Solution

Leading term: $-5x^5$ with 5^{th} degree (n is ***odd***)

$x \to -\infty \;\Rightarrow\; f(x) \to \infty$ $\qquad f(x)$ rises left

$x \to \infty \;\Rightarrow\; f(x) \to -\infty$ $\qquad f(x)$ falls right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = -3x^6 - 16x^3 + 64$

Solution

Leading term: $-3x^6$ with 6^{th} degree (n is ***even***)

$x \to -\infty \;\Rightarrow\; f(x) \to -\infty$ $\qquad f(x)$ falls left

$x \to \infty \;\Rightarrow\; f(x) \to -\infty$ $\qquad f(x)$ falls right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = 3x^6 - 16x^3 + 4$

Solution

Leading term: $3x^6$ with 6^{th} degree (n is ***even***)

$x \to -\infty \;\Rightarrow\; f(x) \to \infty$ $\qquad f(x)$ rises left

$x \to \infty \;\Rightarrow\; f(x) \to \infty$ $\qquad f(x)$ rises right

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = x^3 - x - 1;\quad \textit{between}\ \ 1\ \textit{and}\ \ 2$

Solution

$$f(1) = (1)^3 - (1) - 1$$

$$\underline{= -1}$$

$$f(2) = (2)^3 - (2) - 1$$

$$\underline{= 5}$$

Since $f(1)$ *and* $f(2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 1 and 2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = x^3 - 4x^2 + 2;\quad \textit{between}\ \ 0\ \textit{and}\ 1$

Solution

$$f(0) = (0)^3 - 4(0)^2 + 2$$

$$\underline{= 2}$$

$$f(1) = (1)^3 - 4(1)^2 + 2$$

$$\underline{= -1}$$

Since $f(0)$ *and* $f(1)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 0 and 1.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = 2x^4 - 4x^2 + 1;\quad \textit{between}\ \ -1\ \textit{and}\ \ 0$

Solution

$$f(-1) = 2(-1)^4 - 4(-1)^2 + 1$$

$$\underline{= -1}$$

$$f(0) = 2(0)^4 - 4(0)^2 + 1$$

$$\underline{= 1}$$

Since $f(0)$ *and* $f(-1)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between -1 and 0.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = x^4 + 6x^3 - 18x^2;$ *between* 2 *and* 3

Solution

$$f(2) = (2)^4 + 6(2)^3 - 18(2)^2$$

$$= -8$$

$$f(3) = (3)^4 + 6(3)^3 - 18(3)^2$$

$$= 81$$

Since $f(2)$ *and* $f(3)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 2 and 3.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = x^3 + x^2 - 2x + 1;$ *between* -3 *and* -2

Solution

$$f(-3) = (-3)^3 + (-3)^2 - 2(-3) + 1$$

$$= -11$$

$$f(-2) = (-2)^3 + (-2)^2 - 2(-2) + 1$$

$$= 1$$

Since $f(-3)$ *and* $f(-2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between -2 and -3.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = x^5 - x^3 - 1;$ *between* 1 *and* 2

Solution

$$f(1)=(1)^5-(1)^3-1$$
$$\underline{=-1}$$
$$f(2)=(2)^5-(2)^3-1$$
$$\underline{=23}$$

Since $f(1)$ *and* $f(2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 1 and 2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x)=3x^3-10x+9;\quad \textit{between} \ -3 \ \textit{and} \ -2$

Solution

$$f(-3)=3(-3)^3-10(-3)+9$$
$$\underline{=-42}$$
$$f(-2)=3(-2)^3-10(-2)+9$$
$$\underline{=5}$$

Since $f(-3)$ *and* $f(-2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between –3 and –2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x)=3x^3-8x^2+x+2;\quad \textit{between} \ 2 \ \textit{and} \ 3$

Solution

$$f(2)=3(2)^3-8(2)^2+(2)+2$$
$$\underline{=-4}$$
$$f(3)=3(3)^3-8(3)^2+(3)+2$$
$$\underline{=14}$$

Since $f(2)$ *and* $f(3)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 2 and 3.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = 3x^3 - 8x^2 + x + 2;$ *between* 1 *and* 2

Solution

$$f(1) = 3(1)^3 - 8(1)^2 + (1) + 2$$
$$\underline{= -2}$$

$$f(2) = 3(2)^3 - 8(2)^2 + (2) + 2$$
$$\underline{= -4}$$

Since $f(1)$ *and* $f(2)$ have same signs.

Therefore, ***cannot be determined***.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3;$ *between* 0 *and* 1

Solution

$$f(0) = (0)^5 + 2(0)^4 - 6(0)^3 + 2(0) - 3$$
$$\underline{= -3}$$

$$f(1) = (1)^5 + 2(1)^4 - 6(1)^3 + 2(1) - 3$$
$$\underline{= -4}$$

Since $f(0)$ *and* $f(1)$ have same signs.

Therefore, ***cannot be determined***.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 2x^3 + 3x^2 - 23x - 42,\quad a = 3,\quad b = 4$

Solution

$$P(3) = 54 + 27 - 69 - 42$$
$$\underline{= -30}$$

$$P(4) = 128 + 48 - 92 - 42$$
$$\underline{= 90}$$

Since $P(3)$ *and* $P(4)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 3 and 4.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 4x^3 - x^2 - 6x + 1, \quad a = 0, \quad b = 1$

Solution

$P(0) = 1$

$P(1) = 4 - 1 - 6 + 1$

$= -2$

Since $P(0)$ *and* $P(1)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 0 and 1.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 3x^3 + 7x^2 + 3x + 7, \quad a = -3, \quad b = -2$

Solution

$P(-3) = -81 + 63 - 9 + 7$

$= -20$

$P(-2) = -24 + 28 - 6 + 7$

$= 5$

Since $P(-3)$ *and* $P(-2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between –3 and –2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 2x^3 - 21x^2 - 2x + 25, \quad a = 1, \quad b = 2$

Solution

$P(1) = 2 - 21 - 2 + 25$

$= 4$

$P(2) = 16 - 84 - 4 + 25$

$\quad\quad \underline{= -47}$

Since $P(1)$ *and* $P(2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 1 and 2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 4x^4 + 7x^3 - 11x^2 + 7x - 15, \quad a = 1, \quad b = \frac{3}{2}$

Solution

$P(1) = 4 + 7 - 11 + 7 - 15$

$\quad\quad \underline{= -8}$

$$P\left(\frac{3}{2}\right) = 81 + \frac{189}{8} - \frac{99}{4} + \frac{21}{2} - 15$$

$$= 66 + \frac{189 - 198 + 84}{8}$$

$$= 66 + \frac{75}{8}$$

$$\underline{= \frac{603}{8}}$$

Since $P(1)$ *and* $P\left(\frac{3}{2}\right)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 1 and $\frac{3}{2}$.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 5x^3 - 16x^2 - 20x + 64, \quad a = 3, \quad b = \frac{7}{2}$

Solution

$P(3) = 135 - 144 - 60 + 64$

$\quad\quad \underline{= -5}$

$$P\left(\frac{7}{2}\right) = \frac{1715}{8} - 196 - 70 + 64$$

$$= \frac{1715}{8} - 202$$

$$\underline{= \frac{99}{8}}$$

Since $P(3)$ *and* $P\left(\frac{7}{2}\right)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 3 and $\frac{7}{2}$.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x)=x^4-x^2-x-4, \quad a=1, \quad b=2$

Solution

$$P(1)=1-1-1-4$$
$$\underline{=-5}$$
$$P(2)=16-4-2-4$$
$$\underline{=6}$$

Since $P(1)$ *and* $P(2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 1 and 2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x)=x^3-x-8, \quad a=2, \quad b=3$

Solution

$$P(2)=8-2-8$$
$$\underline{=-2}$$
$$P(3)=27-3-8$$
$$\underline{=16}$$

Since $P(2)$ *and* $P(3)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 2 and 3.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x)=x^3-x-8, \quad a=0, \quad b=1$

Solution

$$\underline{P(0)=-8}$$

$$P(1) = 1 - 1 - 8$$

$$\underline{= -8}$$

Since $P(0)$ *and* $P(1)$ have same sign.

Therefore, ***cannot be determined***.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = x^3 - x - 8, \quad a = 2.1, \quad b = 2.2$

Solution

$$P(2.1) = P\left(\frac{21}{10}\right)$$

$$= \frac{9261}{1000} - \frac{21}{10} - 8$$

$$= \frac{9261 - 2100 - 8000}{1000}$$

$$\underline{= -\frac{839}{1,000}}$$

$$P(2.2) = P\left(\frac{2.2}{10}\right)$$

$$= \frac{10,648}{1000} - \frac{22}{10} - 8$$

$$= \frac{10,648 - 2,200 - 8,000}{1000}$$

$$= \frac{448}{1,000}$$

$$\underline{= 0.448}$$

Since $P(2.1)$ *and* $P(2.2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 2.1 and 2.2.

Solution Section 2.6 – Rational Functions

Exercise

Determine all asymptotes of the function: $y = \dfrac{3x}{1-x}$

Solution

VA: $x = 1$ ***HA***: $y = -3$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \dfrac{x^2}{x^2+9}$

Solution

VA: n/a $x^2 + 9 \neq 0$ ***HA***: $y = 1$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \dfrac{x-2}{x^2-4x+3}$

Solution

$$x^2 - 4x + 3 = 0 \;\Rightarrow\; x = 1,\; 3$$

$$y = \frac{x}{x^2} \to 0$$

VA: $x = 1,\;\; x = 3$ ***HA***: $y = 0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \dfrac{3}{x-5}$

Solution

VA: $x = 5$ ***HA***: $y = 0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y=\dfrac{x^3-1}{x^2+1}$

Solution

VA: *none*

HA: *none*

Hole: n/a

Oblique asymptote: $y=x$

$$\begin{array}{r} x \\ x^2+1\overline{)\,x^3-1} \\ \underline{-x^3-x} \\ -x-1 \end{array}$$

$$y=x-\frac{x+1}{x^2+1}$$

Exercise

Determine all asymptotes of the function: $y=\dfrac{x^3+3x^2-2}{x^2-4}$

Solution

VA: $x=\pm 2$

HA: n/a

Hole: n/a

Oblique asymptote: $y=x+3$

$$\begin{array}{r} x+3 \\ x^2-4\overline{)\,x^3+3x^2-2} \\ \underline{-x^3+4x} \\ 3x^2+4x-2 \\ \underline{-3x^2\quad+12} \\ 4x+10 \end{array}$$

$$y=x+3+\frac{4x+10}{x^2-4}$$

Exercise

Determine all asymptotes of the function: $y=\dfrac{3x^2-27}{(x+3)(2x+1)}$

Solution

$$y=\frac{3x^2-27}{(x+3)(2x+1)}=\frac{3\left(x^2-9\right)}{(x+3)(2x+1)}=\frac{3(x+3)(x-3)}{(x+3)(2x+1)}=\frac{3(x-3)}{(2x+1)}$$

VA: $x=-3,\ -\frac{1}{2}$ ***HA***: $y=\frac{3}{2}$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y=\dfrac{x-3}{x^2-9}$

Solution

$$x^2-9=0 \to \boxed{x=\pm 3}$$

$$y=\frac{x-3}{(x-3)(x+3)}$$

$$=\frac{1}{x+3}$$

VA: $x=3$ ***HA***: $y=0$

Hole: $x=3 \to y=\frac{1}{6}$ ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y=\dfrac{6}{\sqrt{x^2-4x}}$

Solution

$$x^2-4x=0$$

$$\Rightarrow x(x-4)=0 \to \boxed{x=0,4}$$

VA: $x=0,\ x=4$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y=\dfrac{5x-1}{1-3x}$

Solution

VA: $x=\frac{1}{3}$ ***HA***: $y=-\frac{5}{3}$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $f(x)=\dfrac{2x-11}{x^2+2x-8}$

Solution

VA: $x=2,\ x=-4$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $f(x)=\dfrac{x^2-4x}{x^3-x}$

Solution

$$f(x)=\frac{x(x-4)}{x\left(x^2-1\right)}$$
$$=\frac{x-4}{x^2-1}$$

VA: $x=-1,\ x=1$ ***HA***: $y=0$

Hole: $x=0\rightarrow y=4$ ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $f(x)=\dfrac{x-2}{x^3-5x}$

Solution

VA: $x=0,\ x=\pm\sqrt{5}$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{4x}{x^2+10x}$

Solution

$$x^2+10x=0 \quad\rightarrow\quad x=0,\ -10$$

Domain: $(-\infty,\ -10)\cup(-10,\ 0)\cup(0,\ \infty)$

$$f(x)=\frac{4x}{x(x+10)}$$
$$=\frac{4}{x+10}$$

VA: $x=-10$ ***HA***: $y=0$

Hole: $x=0 \quad\rightarrow\quad y=\frac{4}{10}\Rightarrow$ *hole* $\left(0,\ \frac{2}{5}\right)$ ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{3-x}{(x-4)(x+6)}$

Solution

VA: $x=-6$ *and* $x=4$ $\quad$ HA: $y=0$

$Hole$: n/a $\quad$ $Oblique\ asymptote$: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^3}{2x^3-x^2-3x}$

Solution

$$2x^3-x^2-3x=x\left(2x^2-x-3\right)=0 \quad \to \quad x=0,\ -1,\ \frac{3}{2}$$

$$f(x)=\frac{x^3}{2x^3-x^2-3x}$$

$$=\frac{x^3}{x\left(2x^2-x-3\right)}$$

$$=\frac{x^2}{2x^2-x-3}$$

VA: $x=-1$ *and* $x=\frac{3}{2}$ $\quad$ HA: $y=\frac{1}{2}$

$Hole$: $x=0 \ \to \ y=0 \Rightarrow hole\ (0,\ 0)$ $\quad$ $Oblique\ asymptote$: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{3x^2+5}{4x^2-3}$

Solution

$$4x^2-3=0 \quad \to \quad x=\pm\frac{\sqrt{3}}{2}$$

$Domain$: $\left(-\infty,\ -\frac{\sqrt{3}}{2}\right)\cup\left(-\frac{\sqrt{3}}{2},\ \frac{\sqrt{3}}{2}\right)\cup\left(\frac{\sqrt{3}}{2},\ \infty\right)$

VA: $x=-\frac{\sqrt{3}}{2}$ *and* $x=\frac{\sqrt{3}}{2}$ $\quad$ HA: $y=\frac{3}{4}$

$Hole$: n/a $\quad$ $Oblique\ asymptote$: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x+6}{x^3+2x^2}$

Solution

$$x^3+2x^2=x^2(x+2)=0 \quad \to \quad x=0,-2$$

Domain: $(-\infty,\ -2)\cup(-2,\ 0)\cup(0,\ \infty)$

VA: $x=0$ *and* $x=2$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^2+4x-1}{x+3}$

Solution

VA: $x=-3$

HA: n/a

Hole: n/a

Oblique asymptote: $y=x+1$

$$\begin{array}{r|l} & x+1 \\ \hline x+3 & x^2+4x-1 \\ & \underline{-x^2-3x} \\ & \quad\ x-1 \\ & \quad\ \underline{-x-3} \\ & \quad\quad -4 \end{array}$$

$$f(x)=\frac{x^2+4x-1}{x+3}=x+1-\frac{4}{x+3}$$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^2-6x}{x-5}$

Solution

$x-5=0 \quad \to \quad x=5$

Domain: $(-\infty,\ 5)\cup(5,\ \infty)$

$$\begin{array}{r|l} & x-1 \\ \hline x-5 & x^2-6x \\ & \underline{-x^2+5x} \\ & \quad\ -x \\ & \quad\ \underline{x-5} \\ & \quad\ -5 \end{array}$$

$$\begin{aligned} f(x)&=\frac{x^2-6x}{x-5} \\ &=x-1-\frac{5}{x-5} \end{aligned}$$

VA: $x=5$

HA: N/A

Hole: N/A

Oblique asymptote: $y=x-1$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^3-x^2+x-4}{x^2+2x-1}$

Solution

$x^2+2x-1=0 \quad \to \quad x=-1\pm\sqrt{2}$

Domain: $\left(-\infty,\ -1-\sqrt{2}\right)\cup\left(-1-\sqrt{2},\ -1+\sqrt{2}\right)\cup\left(-1+\sqrt{2},\ \infty\right)$

$$\begin{aligned} f(x)&=\frac{x^3-x^2+x-4}{x^2+2x-1} \\ &=x-3+\frac{8x-7}{x^2+2x-1} \end{aligned}$$

$$\begin{array}{r} x-3 \\ x^2+2x-1 \overline{\big) x^3-x^2+x-4} \\ \underline{-x^3-2x^2+x} \\ -3x^2+2x-4 \\ \underline{3x^2+6x-3} \\ 8x-7 \end{array}$$

VA: $x=-1\pm\sqrt{2}$

HA: n/a

Hole: n/a

Oblique asymptote: $y=x-3$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{4x}{x^2+10x}$

Solution

$x^2+10x=0 \quad \to \quad x=0,\ -10$ **Domain**: $(-\infty,\ -10)\cup(-10,\ 0)\cup(0,\ \infty)$

$f(x)=\dfrac{4x}{x(x+10)}=\dfrac{4}{x+10}$

VA: $x=-10$ ***HA***: $y=0$

Hole: $x=0 \quad \to \quad y=\frac{4}{10} \Rightarrow$ *hole* $\left(0,\ \frac{2}{5}\right)$

Oblique asymptote: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{3-x}{(x-4)(x+6)}$

Solution

Domain: $(-\infty,\ -6)\cup(-6,\ 4)\cup(4,\ \infty)$

VA: $x=-6$ *and* $x=4$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^3}{2x^3-x^2-3x}$

Solution

$$2x^3-x^2-3x=x\left(2x^2-x-3\right)=0 \quad\to\quad x=0,\ -1,\ \tfrac{3}{2}$$

Domain: $(-\infty,\ -1)\cup(-1,\ 0)\cup\left(0,\ \frac{3}{2}\right)\cup\left(\frac{3}{2},\ \infty\right)$

$$f(x)=\frac{x^3}{2x^3-x^2-3x}=\frac{x^3}{x\left(2x^2-x-3\right)}=\frac{x^2}{2x^2-x-3}$$

VA: $x=-1$ *and* $x=\frac{3}{2}$ ***HA***: $y=\frac{1}{2}$

Hole: $x=0 \quad\to\quad y=0 \Rightarrow$ *hole* $(0,\ 0)$

Oblique asymptote: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{3x^2+5}{4x^2-3}$

Solution

$4x^2-3=0 \quad\to\quad x=\pm\frac{\sqrt{3}}{2}$ ***Domain***: $\left(-\infty,\ -\frac{\sqrt{3}}{2}\right)\cup\left(-\frac{\sqrt{3}}{2},\ \frac{\sqrt{3}}{2}\right)\cup\left(\frac{\sqrt{3}}{2},\ \infty\right)$

VA: $x=-\frac{\sqrt{3}}{2}$ *and* $x=\frac{\sqrt{3}}{2}$ ***HA***: $y=\frac{3}{4}$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x+6}{x^3+2x^2}$

Solution

$x^3+2x^2=x^2(x+2)=0 \quad\to\quad x=0,-2$ ***Domain***: $(-\infty,\ -2)\cup(-2,\ 0)\cup(0,\ \infty)$

VA: $x=0$ *and* $x=2$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^2+4x-1}{x+3}$

Solution

$x+3=0 \quad \to \quad x=-3$ **Domain**: $(-\infty,\ -3)\cup(-3,\ \infty)$

$$\begin{array}{r} x+1 \\ x+3\overline{\smash{)}x^2+4x-1} \\ \underline{-x^2-3x} \\ x-1 \\ \underline{-x-3} \\ -4 \end{array}$$

$$f(x)=\frac{x^2+4x-1}{x+3}=x+1-\frac{4}{x+3}$$

VA: $x=-3$ **HA**: n/a

Hole: n/a **Oblique asymptote**: $y=x+1$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^2-6x}{x-5}$

Solution

$x-5=0 \quad \to \quad x=5$ **Domain**: $(-\infty,\ 5)\cup(5,\ \infty)$

$$\begin{array}{r} x-1 \\ x-5\overline{\smash{)}x^2-6x} \\ \underline{-x^2+5x} \\ -x \\ \underline{x-5} \\ -5 \end{array}$$

$$f(x)=\frac{x^2-6x}{x-5}=x-1-\frac{5}{x-5}$$

VA: $x=5$

HA: N/A

Hole: N/A

Oblique asymptote: $y=x-1$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^3-x^2+x-4}{x^2+2x-1}$

Solution

$x^2+2x-1=0 \quad \to \quad x=-1\pm\sqrt{2}$

Domain: $\left(-\infty,\ -1-\sqrt{2}\right)\cup\left(-1-\sqrt{2},\ -1+\sqrt{2}\right)\cup\left(-1+\sqrt{2},\ \infty\right)$

$$\begin{array}{r}
x-3 \\
x^2+2x-1\overline{\big)\,x^3-x^2+x-4} \\
\underline{-x^3-2x^2+x} \\
-3x^2+2x-4 \\
\underline{3x^2+6x-3} \\
8x-7
\end{array}$$

VA: $x=-1\pm\sqrt{2}$

HA: N/A

Hole: N/A

Oblique asymptote: $y=x-3$

$$f(x)=\frac{x^3-x^2+x-4}{x^2+2x-1}$$

$$=x-3+\frac{8x-7}{x^2+2x-1}$$

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

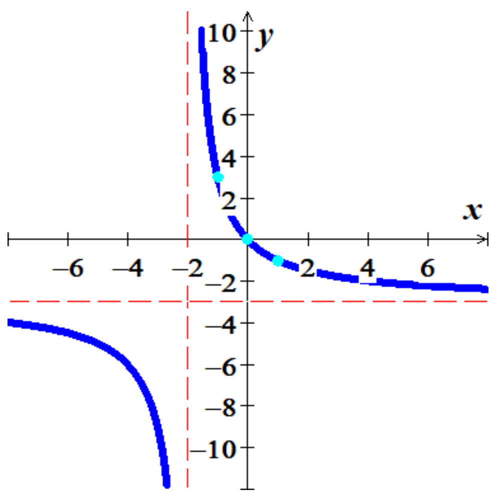
$$f(x)=\frac{-3x}{x+2}$$

Solution

VA: $x=-2$ ***HA***: $y=-3$

Hole: n/a ***OA***: n/a

x	y
0	0
1	–1
–1	3



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

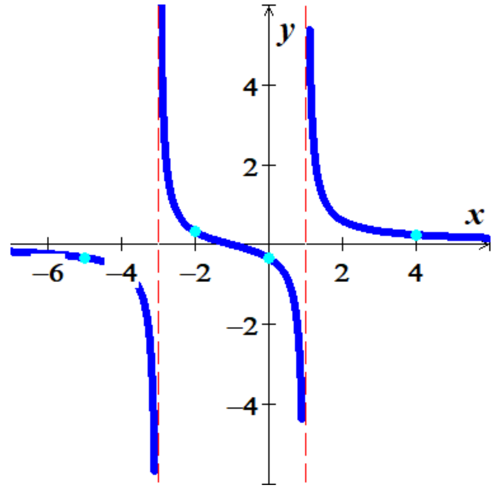
$$f(x) = \frac{x+1}{x^2 + 2x - 3}$$

Solution

VA: $x = 1,\ x = -3$ ***HA***: $y = 0$

Hole: n/a ***Oblique asymptote***: n/a

x	y
–5	–0.33
–2	0.33
0	–1/3
4	0.24



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

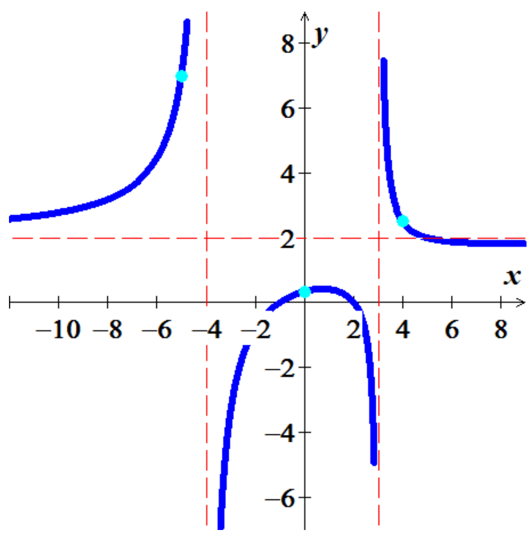
$$f(x) = \frac{2x^2 - 2x - 4}{x^2 + x - 12}$$

Solution

VA: $x = -4,\ 3$ ***HA***: $y = 2$

Hole: n/a ***OA***: n/a

x	y
–5	7
–2	–0.8
0	1/3
4	2.5
5	2



Exercise

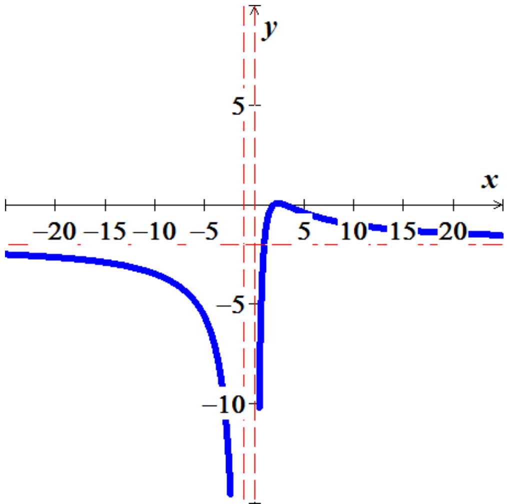
Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

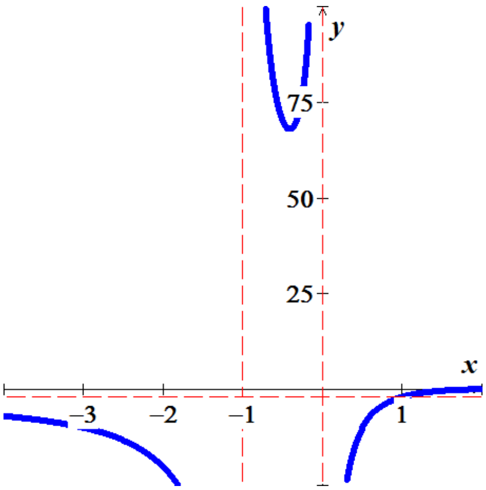
$$f(x) = \frac{-2x^2 + 10x - 12}{x^2 + x}$$

Solution

VA: $x = -1,\ 0$ ***HA***: $y = -2$

Hole: n/a ***OA***: n/a





Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x) = \frac{x^2 - x - 6}{x + 1}$$

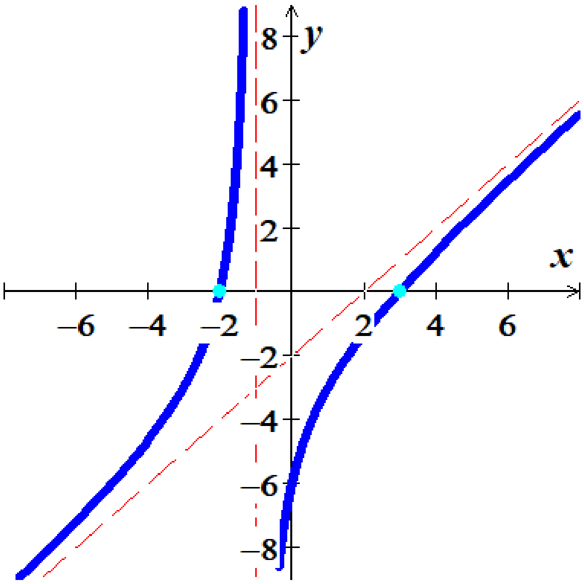
Solution

$$\begin{array}{r|l} & x - 2 \\ \hline x+1 & x^2 - x - 6 \\ & \underline{x^2 + x} \\ & -2x - 6 \\ & \underline{-2x - 2} \\ & \boxed{-4} \end{array}$$

VA: $x = -1$ ***HA***: n/a

Hole: n/a ***OA***: $y = x - 2$

x	***y***
2	0
–2	0
0	–6



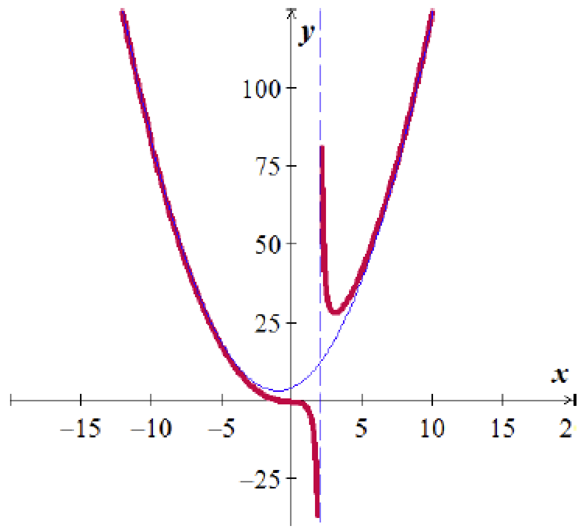
Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x) = \frac{x^3+1}{x-2}$$

Solution

$$\begin{array}{r|l} & x^2+2x+4 \\ \hline x-2 & x^3 \qquad -1 \\ & \underline{x^3-2x^2} \\ & \quad 2x^2 \\ & \quad \underline{2x^2-4x} \\ & \qquad\quad 4x-1 \\ & \qquad\quad \underline{4x-8} \\ & \qquad\qquad 7 \end{array}$$



VA: $x = 2$ ***HA***: n/a

Hole: n/a ***OA***: $y = x^2+2x+4$

Exercise

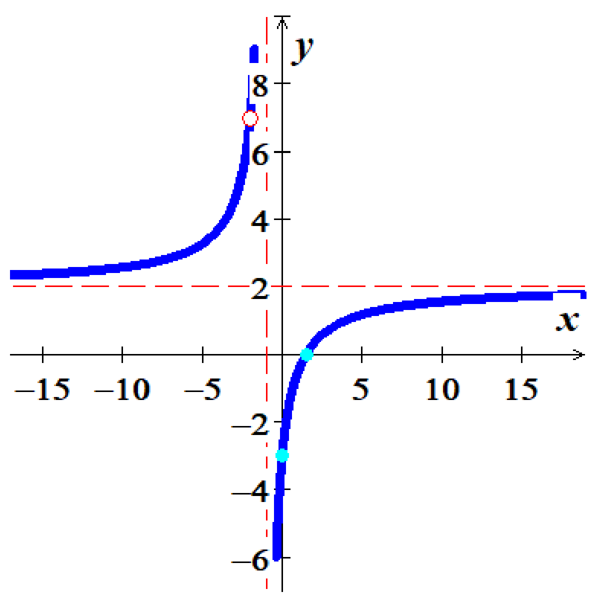
Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x) = \frac{2x^2+x-6}{x^2+3x+2}$$

Solution

$$f(x) = \frac{(2x-3)(x+2)}{(x+1)(x+2)}$$

$$= \frac{2x-3}{x+1}$$



VA: $x = -1$ ***HA***: $y = 2$

Hole: $(-2,\ 7)$ ***OA***: n/a

x	***y***
0	−3
$-\frac{3}{2}$	0

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x) = \frac{x-1}{1-x^2}$$

Solution

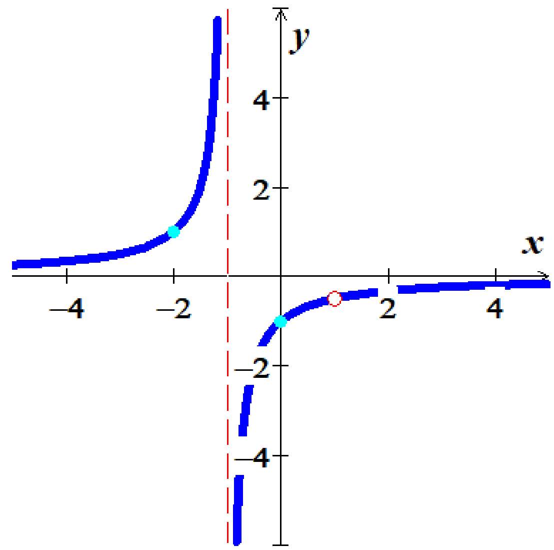
$$f(x) = \frac{x-1}{(x+1)(1-x)}$$

$$= -\frac{1}{x+1}$$

VA: $x = -1$ ***HA***: $y = 0$

Hole: $\left(1,\ -\frac{1}{2}\right)$ ***OA***: n/a

x	y
0	–1
–2	1



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x) = \frac{x^2+x-2}{x+2}$$

Solution

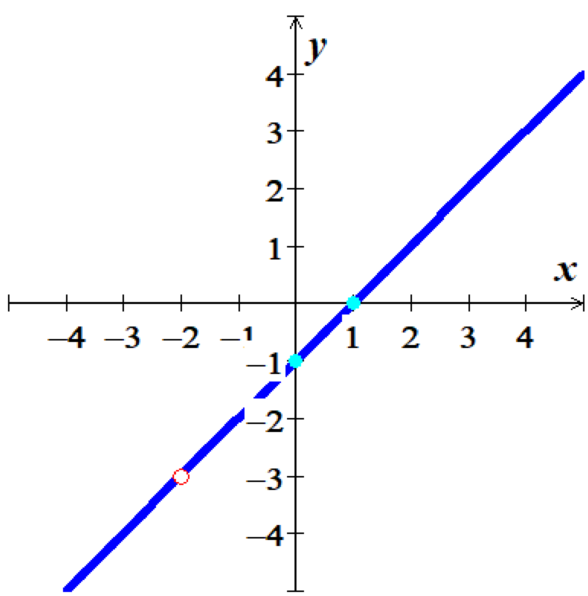
$$f(x) = \frac{(x+2)(x-1)}{x+2}$$

$$= x-1$$

VA: n/a ***HA***: n/a

Hole: $(-2,\ -3)$ ***OA***: n/a

x	y
0	–1
1	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x) = \frac{x^3 - 2x^2 - 4x + 8}{x - 2}$$

Solution

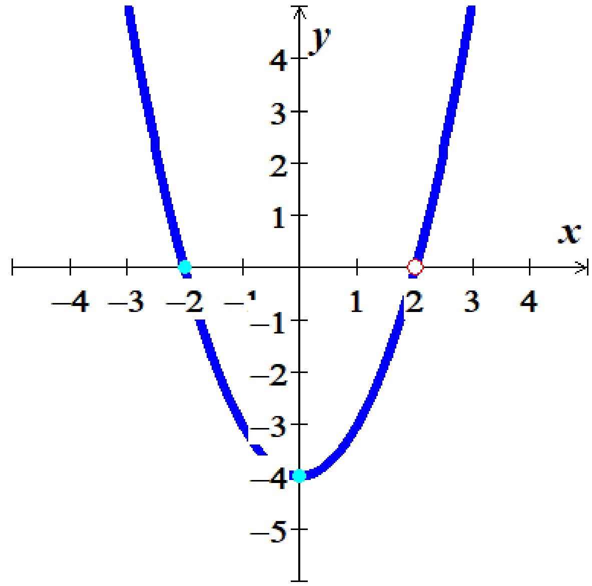
$$f(x) = \frac{\left(x^2 - 4\right)(x - 2)}{x - 2}$$

$$= x^2 - 4$$

VA: n/a ***HA***: n/a

Hole: $(2,\ 0)$ ***OA***: n/a

x	y
0	–4
–2	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

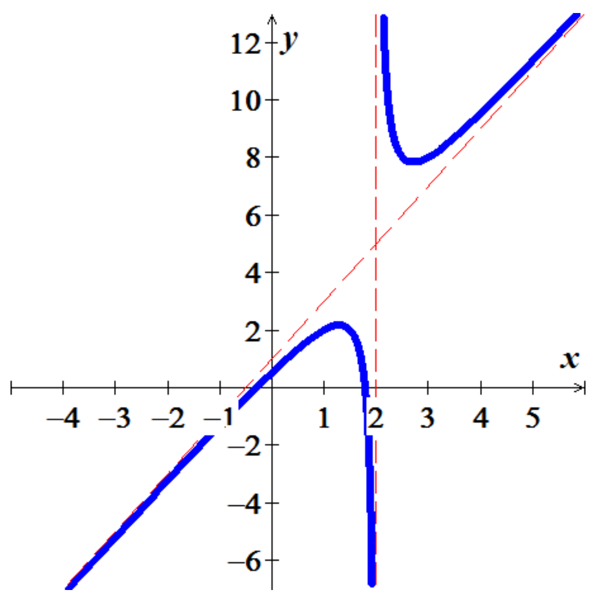
$$f(x) = \frac{2x^2 - 3x - 1}{x - 2}$$

Solution

$$\begin{array}{r} 2x+1 \\ x-2 \overline{\big) 2x^2 - 3x - 1} \\ \underline{-2x^2 + 4x} \\ x - 1 \\ \underline{-x + 2} \\ 1 \end{array}$$

$$f(x) = \frac{2x^2 - 3x - 1}{x - 2}$$

$$= (2x + 1) + \frac{1}{x - 2}$$



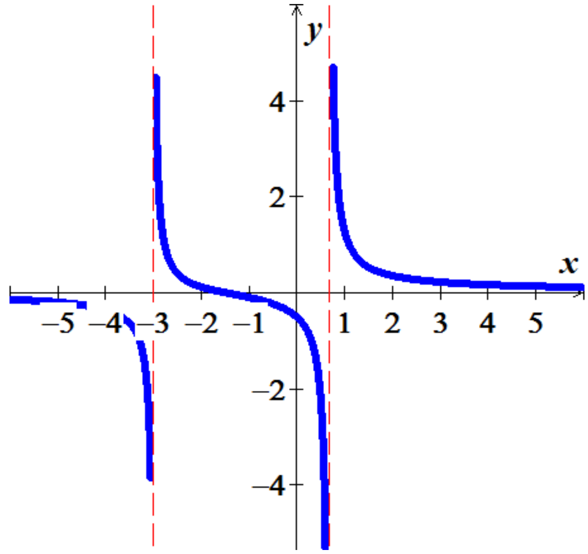
VA: $x = 2$ ***HA***: $y = 1$

Hole: n/a ***OA***: $y = 2x + 1$

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{2x+3}{3x^2+7x-6}$$



Solution

$$3x^2+7x-6=0 \quad\Rightarrow\quad x=-3,\ \frac{2}{3}$$

VA: $x=-3$ *and* $x=\frac{2}{3}$

HA: $y=0$

Hole: n/a

OA: n/a

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{x^2-1}{x^2+x-6}$$

Solution

$$x^2+x-6=0 \quad\Rightarrow\quad \underline{x=-3,\ 2|}$$

VA: $x=-3$ *and* $x=2$ ***HA***: $y=1$

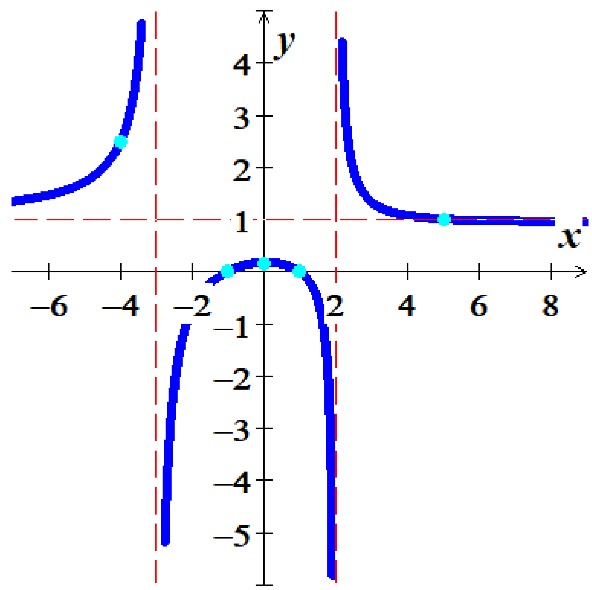
Hole: n/a ***OA:*** n/a

$$1=\frac{x^2-1}{x^2+x-6}$$

$$x^2+x-6=x^2-1$$

$$\underline{x=5|}$$

x	**y**
0	$\frac{1}{6}$
5	1
±1	0
−4	$\frac{5}{2}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{-2x^2-x+15}{x^2-x-12}$$

Solution

$x^2-x-12=0 \quad \Rightarrow \quad \underline{x=-3,\ 4}|$

Domain: $(-\infty,-3)\cup(-3,4)\cup(4,\infty)$

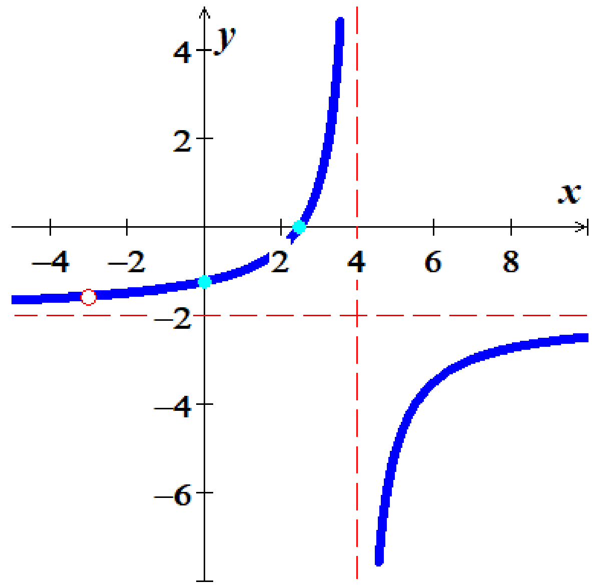
$$f(x)=\frac{(-2x+5)(x+3)}{(x-4)(x+3)}$$

$$=\frac{-2x+5}{x-4}$$

VA: $x=4$ ***HA***: $y=-2$

Hole: $\left(-3,\ -\frac{11}{7}\right)$ ***OA:*** n/a

x	y
0	$-\frac{5}{4}$
$\frac{5}{2}$	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

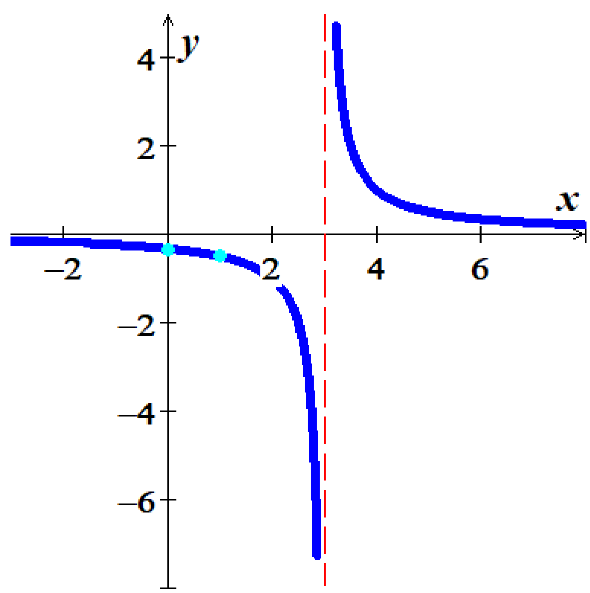
$$f(x)=\frac{1}{x-3}$$

Solution

VA: $x=3$ ***HA***: $y=0$

Hole: n/a ***OA:*** n/a

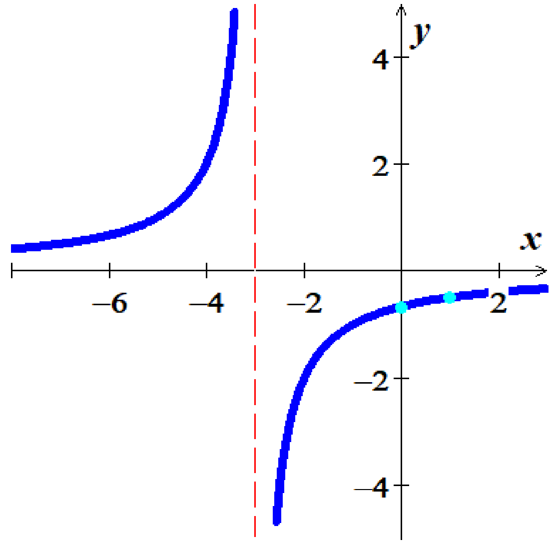
x	y
0	$-\frac{1}{3}$
1	$-\frac{1}{2}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{-2}{x+3}$$



Solution

VA: $x=-3$ ***HA***: $y=0$

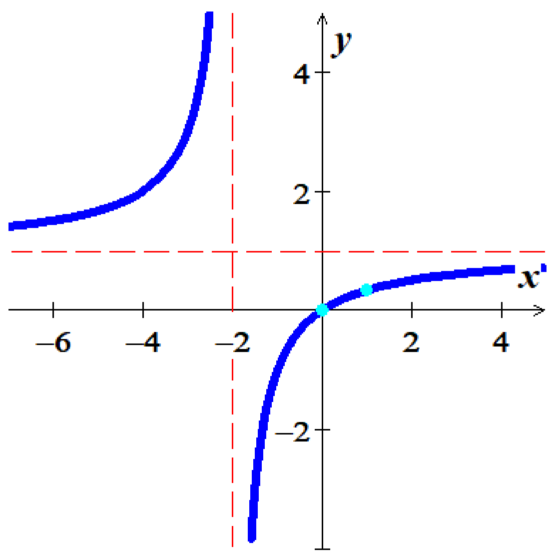
Hole: n/a ***OA:*** n/a

x	y
0	$-\frac{2}{3}$
1	$-\frac{1}{2}$

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x}{x+2}$$



Solution

VA: $x=-2$ ***HA***: $y=1$

Hole: n/a ***OA:*** n/a

x	y
0	0
1	$\frac{1}{3}$

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

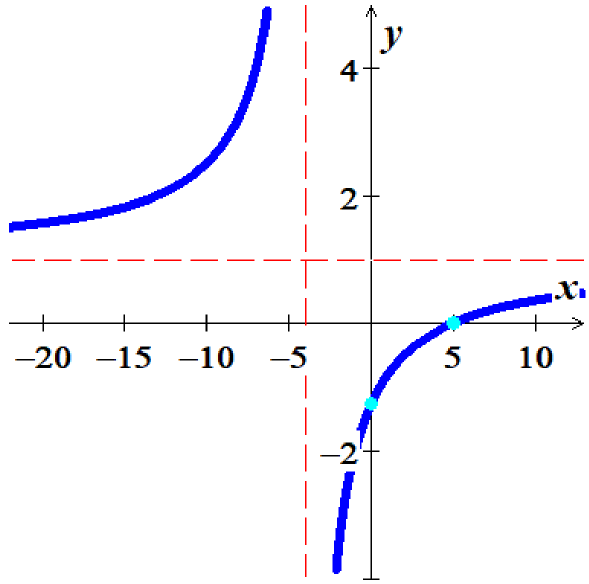
$$f(x)=\frac{x-5}{x+4}$$

Solution

VA: $x=-4$ ***HA***: $y=1$

Hole: n/a ***OA:*** n/a

x	***y***
0	$-\frac{5}{4}$
5	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{2x^2-2}{x^2-9}$$

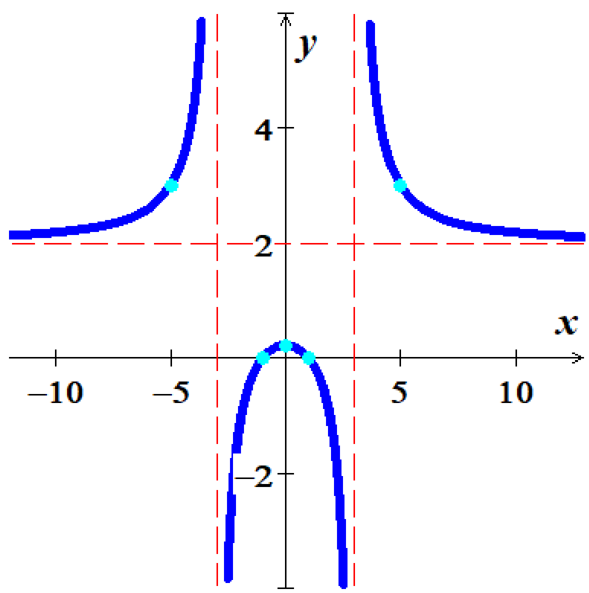
Solution

$x^2=9 \quad \to \quad \underline{x=\pm 3}$

VA: $x=\pm 3$ ***HA***: $y=2$

Hole: n/a ***OA:*** n/a

x	***y***
0	$\frac{2}{9}$
± 1	0
± 5	3



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

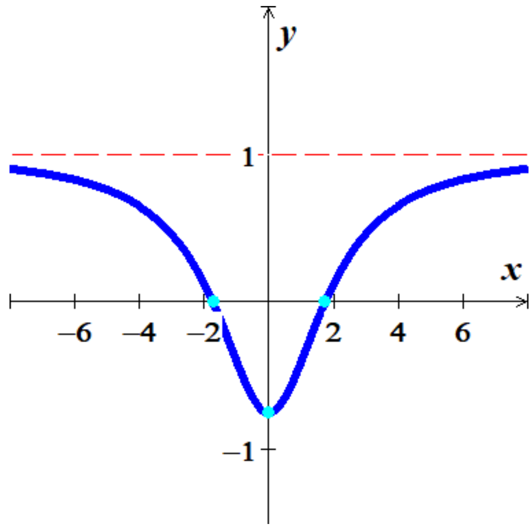
$$f(x)=\frac{x^2-3}{x^2+4}$$

Solution

VA: n/a ***HA***: $y=1$

Hole: n/a ***OA:*** n/a

x	y
0	$-\frac{3}{4}$
$\pm\sqrt{3}$	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2+4}{x^2-3}$$

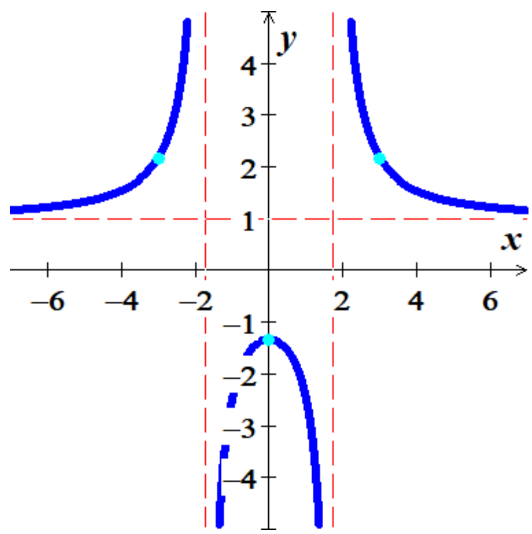
Solution

$x^2-3=0 \quad \rightarrow \quad x=\pm\sqrt{3}$

VA: $x=\pm\sqrt{3}$ ***HA***: $y=1$

Hole: n/a ***OA:*** n/a

x	y
0	$-\frac{4}{3}$
± 3	$\frac{13}{6}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2}{x^2-6x+9}$$

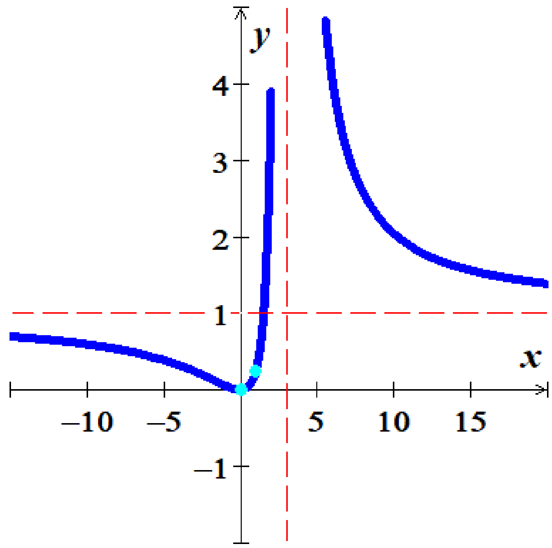
Solution

$x^2-6x+9=0 \quad \to \quad x=3$

VA: $x=3$ ***HA***: $y=1$

Hole: n/a ***OA:*** n/a

x	y
0	0
1	$\frac{1}{4}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2+x+4}{x^2+2x-1}$$

Solution

$x^2+2x-1=0$

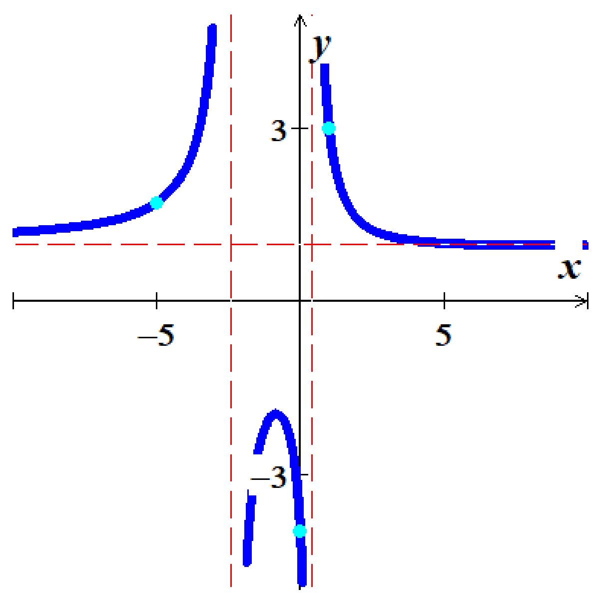
$x=\frac{-2\pm\sqrt{8}}{2}$

$\quad =-1\pm\sqrt{2}$

VA: $x=-1\pm\sqrt{2}$ ***HA***: $y=1$

Hole: n/a ***OA:*** n/a

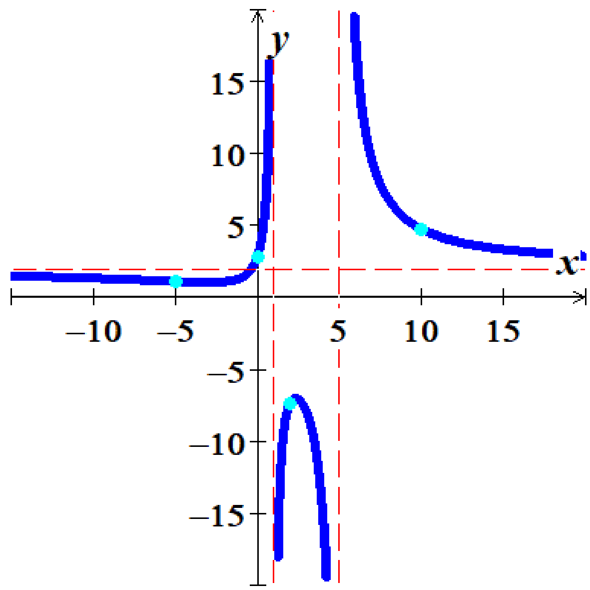
x	y
0	–4
1	3
–5	$\frac{12}{7}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{2x^2+14}{x^2-6x+5}$$



Solution

VA: $x=1,\ 5$ ***HA***: $y=2$

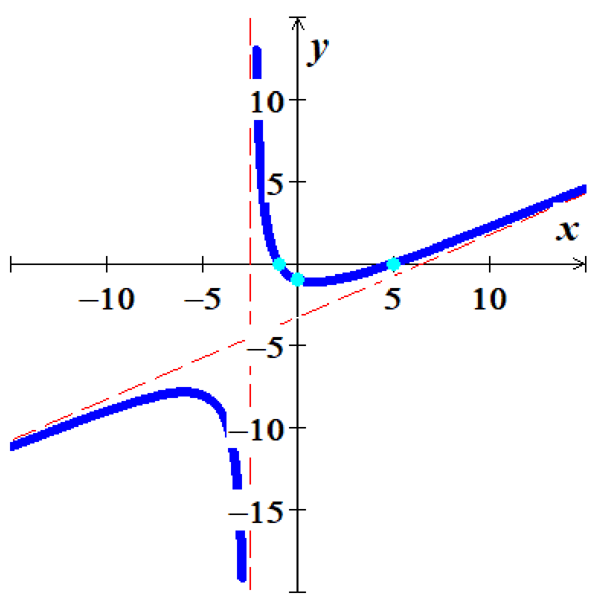
Hole: n/a ***OA:*** n/a

x	**y**
0	$\frac{14}{5}$
2	$-\frac{22}{3}$
–5	$\frac{16}{15}$
10	$\frac{214}{45}$

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2-4x-5}{2x+5}$$



Solution

$$\begin{array}{r} \frac{1}{2}x-\frac{13}{4} \\ 2x+5\overline{\big)\,x^2-4x-5} \\ \underline{x^2+\frac{5}{2}x} \\ -\frac{13}{2}x-5 \end{array}$$

VA: $x=-\frac{5}{2}$ ***HA***: n/a

Hole: n/a ***OA:*** $y=\frac{1}{2}x-\frac{13}{2}$

x	**y**
0	–1
–1, 5	0

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x-3}{x^2-3x+2}$$

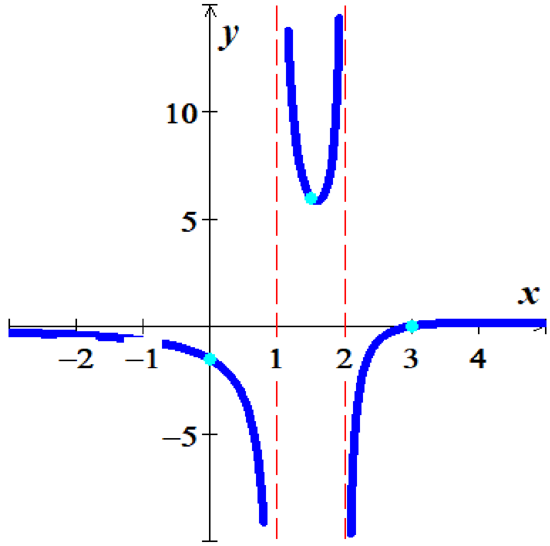
Solution

$x^2-3x+2 \rightarrow \underline{x=1,\ 2|}$

VA: $x=1,\ 2$ ***HA***: $y=0$

Hole: n/a ***OA:*** n/a

x	y
0	$-\frac{3}{2}$
3	0
$\frac{3}{2}$	6



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2+2}{x^2+3x+2}$$

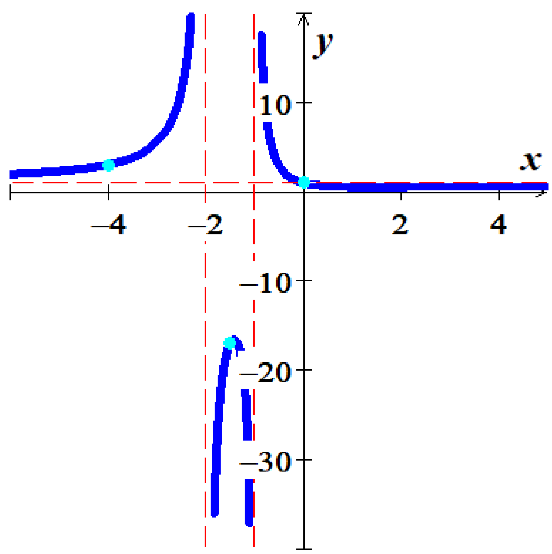
Solution

$x^2+3x+2 \rightarrow \underline{x=-1,\ -2|}$

VA: $x=-1,\ -2$ ***HA***: $y=1$

Hole: n/a ***OA:*** n/a

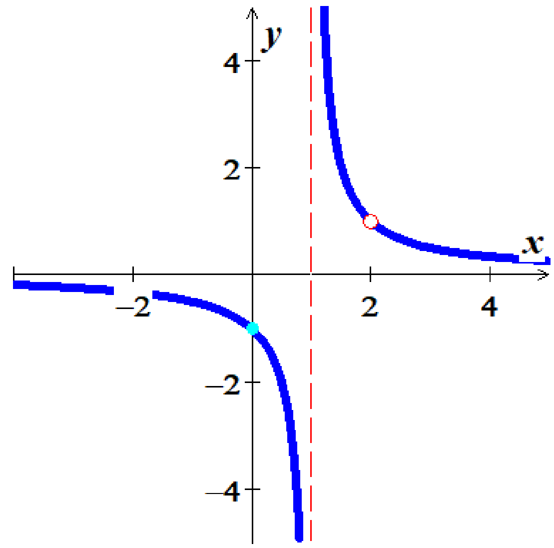
x	y
0	1
$-\frac{3}{2}$	-17
-4	3



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x-2}{x^2-3x+2}$$



Solution

$$x^2-3x+2 \quad\to\quad \underline{x=1,\ 2|}$$

$$f(x)=\frac{x-2}{(x-2)(x-1)}$$

$$=\frac{1}{x-1}$$

VA: $x=1$ ***HA***: $y=0$

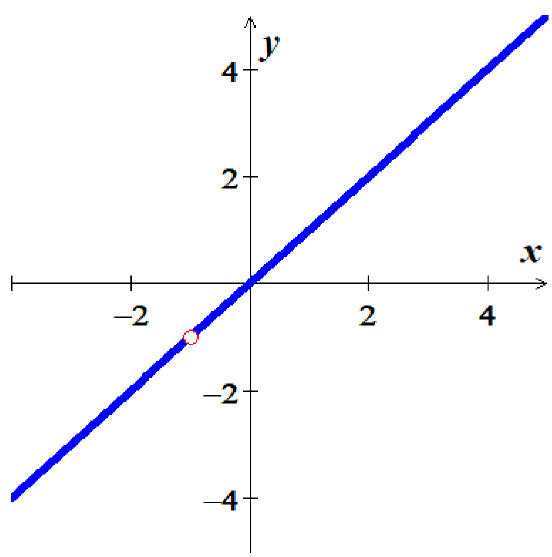
Hole: $(2,\ 1)$ ***OA:*** n/a

x	**y**
0	–1

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2+x}{x+1}$$



Solution

$$f(x)=\frac{x(x+1)}{x+1}$$

$$\underline{=x\,|}$$

VA: n/a ***HA***: n/a

Hole: $(-1,\ -1)$ ***OA:*** n/a

Hole: $\left(-3,\ -\frac{11}{7}\right)$ ***OA:*** n/a

x	**y**
0	0

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2-2x}{x-2}$$

Solution

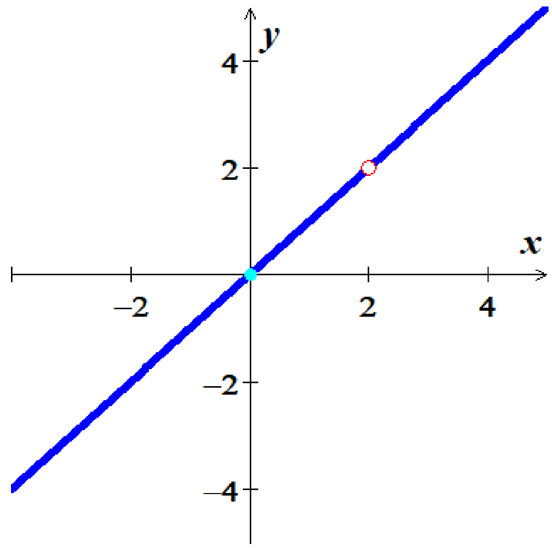
$$f(x)=\frac{x(x-2)}{x-2}$$
$$=x$$

VA: n/a ***HA***: n/a

Hole: $(2,\ 2)$ ***OA:*** n/a

Hole: $\left(-3,\ -\frac{11}{7}\right)$ ***OA:*** n/a

x	***y***
0	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2-3x}{x+3}$$

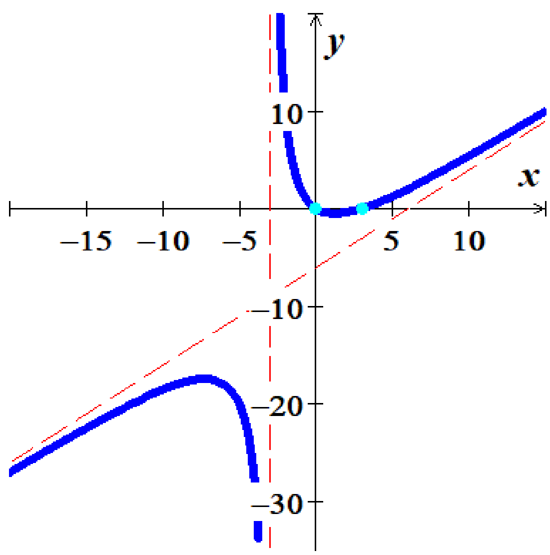
Solution

$$\begin{array}{r|l} & x-6 \\ \hline x+3 & x^2-3x \\ & \underline{x^2+3x} \\ & -6x-5 \end{array}$$

VA: $x=-3$ ***HA***: n/a

Hole: n/a ***OA:*** $y=x-6$

x	***y***
0	0
3	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x^3 + 3x^2 - 4x + 6}{x + 2}$$

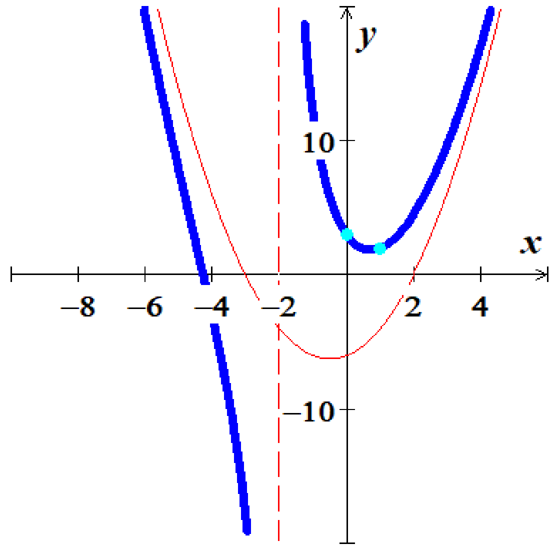
Solution

$$\begin{array}{r|l} & x^2 + x - 6 \\ x+2 & x^3 + 3x^2 - 4x + 6 \\ & \underline{x^3 + 2x^2} \\ & x^2 - 4x \\ & \underline{x^2 + 2x} \\ & -6x + 6 \end{array}$$

VA: $x = -2$ ***HA***: n/a

Hole: n/a ***OA:*** $y = x^2 + x - 6$

x	y
0	3
1	2



Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} \text{vertical asymptote}: \ x = 4 \\ \text{horizontal asymptote}: \ y = -1 \\ x - \text{intercept}: \ 3 \end{cases}$$

Solution

Vertical Asymptote: $f(x) = \frac{\quad}{x - 4}$

Horizontal Asymptote: $f(x) = \frac{-x + a}{x - 4}$

x-intercept: $f(x = 3) = \frac{-3 + a}{3 - 4} = 0 \quad \Rightarrow \quad \underline{a = 3}$

$$\boxed{f(x) = \frac{-x + 3}{x - 4}}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote:\ x=-4, x=5 \\ horizontal\ asymptote:\ y=\dfrac{3}{2} \\ x-intercept:\ -2 \end{cases}$$

Solution

Vertical Asymptote: $\quad f(x)=\dfrac{}{(x+4)(x-5)}$

Horizontal Asymptote: $\quad f(x)=\dfrac{3}{2}\dfrac{(x+a)(x+b)}{(x+4)(x-5)}$

x-intercept: $\quad f(x=-2)=\dfrac{3}{2}\dfrac{(-2+a)(-2+b)}{}$

$$0=(-2+a)(-2+b)$$

$$\boxed{a=b=2}$$

$$f(x)=\frac{3}{2}\frac{(x-2)^2}{x^2-x-20}$$

$$=\frac{3x^2-12x+12}{2x^2-2x-40}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote:\ x=5 \\ horizontal\ asymptote:\ y=-1 \\ x-intercept:\ 2 \end{cases}$$

Solution

Vertical Asymptote: $\quad f(x)=\dfrac{}{x-5}$

x-intercept : $\quad f(x)=\dfrac{x-2}{x-5}$

Horizontal Asymptote: $\quad f(x)=-\dfrac{x-2}{x-5}$

$$f(x)=-\frac{x-2}{x-5}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote: \ x=-2,\ x=0 \\ horizontal\ asymptote: \ y=0 \\ x-intercept: \ 2, \quad f(3)=1 \end{cases}$$

Solution

Vertical Asymptote: $\qquad f(x)=\dfrac{}{x(x+2)}$

x-intercept : $\qquad f(x)=\dfrac{x-2}{x(x+2)}$

Horizontal Asymptote: $\qquad f(x)=\dfrac{a(x-2)}{x(x+2)}$

$f(3)=1 \quad \rightarrow \dfrac{a(1)}{(3)(5)}=1 \quad \Rightarrow \underline{a=15}$

$$\boxed{f(x)=\frac{15x-30}{x^2+2x}}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote: \ x=-3,\ x=1 \\ horizontal\ asymptote: \ y=0 \\ x-intercept: \ -1, \quad f(0)=-2 \\ \quad hole: \ x=2 \end{cases}$$

Solution

Vertical Asymptote: $\qquad f(x)=\dfrac{}{(x+3)(x-1)}$

x-intercept : $\qquad f(x)=\dfrac{(x+1)}{(x+3)(x-1)}$

Horizontal Asymptote: $\qquad f(x)=\dfrac{a(x+1)}{(x+3)(x-1)}$

$f(0)=-2 \quad \rightarrow \dfrac{a}{-3}=-2 \quad \Rightarrow \underline{a=6}$

Hole at $x=2$**:** $\qquad f(x)=\dfrac{6(x+1)(x-2)}{\left(x^2+2x-3\right)(x-2)}$

$$\boxed{f(x)=\frac{6x^2-6x-12}{x^3-7x+6}}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote: \ x=-1, \ \ x=3 \\ horizontal\ asymptote: \ y=2 \\ x-intercept: \ -2, \ \ 1 \\ \ \ hole: \ \ x=0 \end{cases}$$

Solution

Vertical Asymptote: $f(x)=\dfrac{}{(x+1)(x-3)}$

Horizontal Asymptote: $f(x)=\dfrac{2}{(x+1)(x-3)}$

x-intercept : $f(x)=\dfrac{2(x+2)(x-1)}{(x+1)(x-3)}$

Hole at $x=0$: $f(x)=\dfrac{2x(x+2)(x-1)}{x(x+1)(x-3)}$

$$f(x)=\dfrac{2x^3+2x^2-4x}{x^3-2x^2-3x}$$

Solution   Section 2.7 – Ellipses

Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $\frac{x^2}{9}+\frac{y^2}{4}=1$

Solution

$$\begin{cases} a^2 = 9 \rightarrow a = 3 \\ b^2 = 4 \rightarrow b = 2 \end{cases}$$

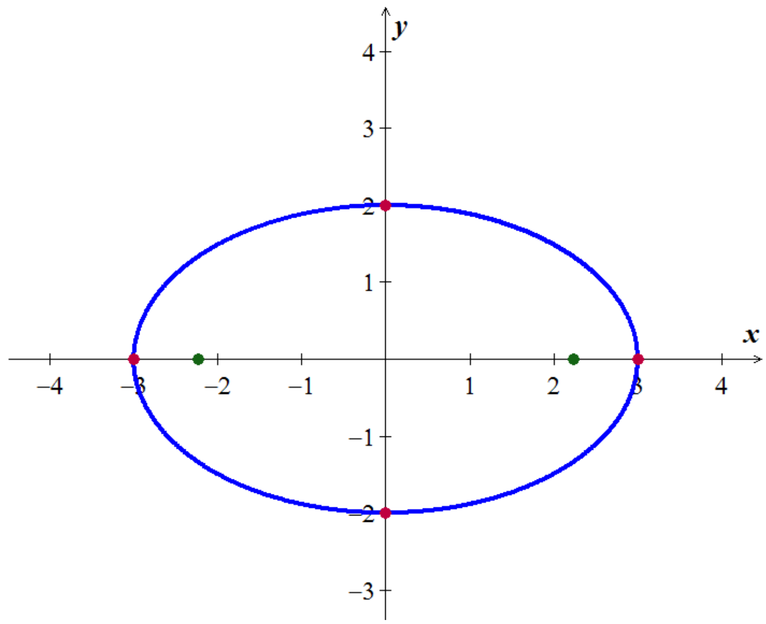
$$c = \sqrt{a^2 - b^2} = \sqrt{9-4} = \sqrt{5}$$

Center: $C(0,\ 0)$

Vertices: $V(\pm 3,\ 0)$

Minor $M(0,\ \pm 2)$

Foci $F(\pm\sqrt{5},\ 0)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $\frac{x^2}{16}+\frac{y^2}{36}=1$

Solution

$$\begin{cases} a^2 = 36 \rightarrow a = 6 \\ b^2 = 16 \rightarrow b = 4 \end{cases}$$

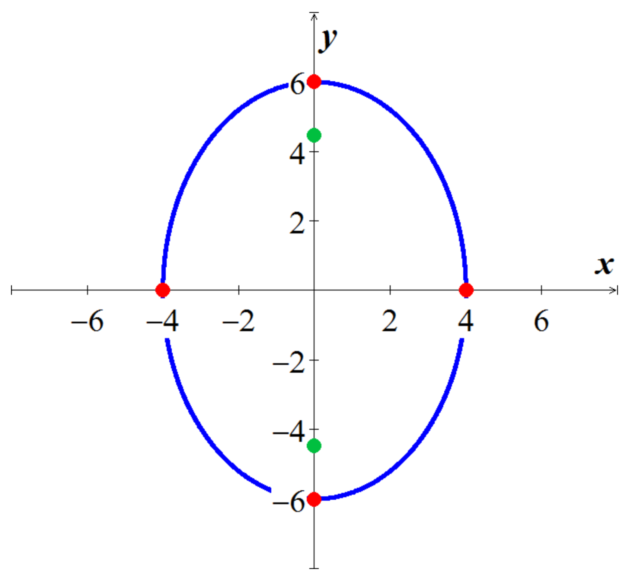
$$c = \sqrt{a^2 - b^2} = \sqrt{36-16} = 2\sqrt{5}$$

Center: $C(0,\ 0)$

Vertices: $V(0,\ \pm 6)$

Minors $M(\pm 4,\ 0)$

Foci $F(0,\ \pm 2\sqrt{5})$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $\frac{x^2}{15}+\frac{y^2}{16}=1$

Solution

$$\begin{cases} a^2=16 \to a=4 \\ b^2=15 \to b=\sqrt{15} \end{cases}$$

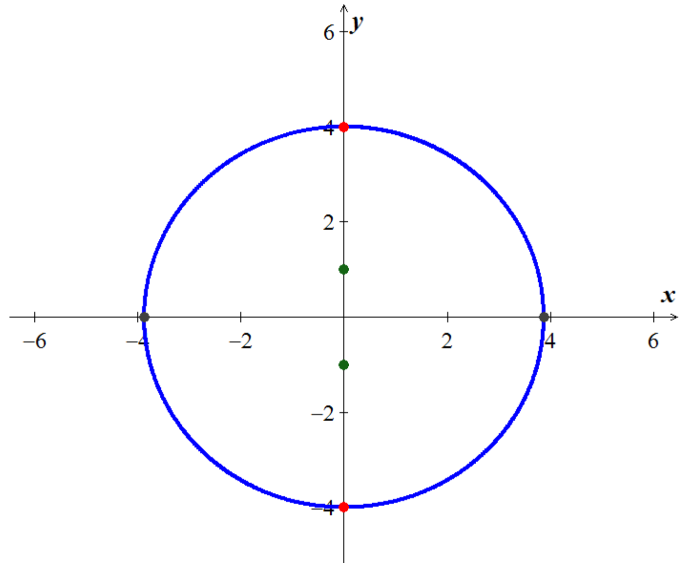
$$c=\sqrt{a^2-b^2}=\sqrt{16-15}=1$$

Center: $C(0,\ 0)$

Vertices: $V(0,\ \pm 4)$

Minors $M\left(\pm\sqrt{15}, 0\right)$

Foci $F(0,\ \pm 1)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $\frac{25x^2}{36}+\frac{64y^2}{9}=1$

Solution

$$\frac{x^2}{\frac{36}{25}}+\frac{y^2}{\frac{9}{64}}=1$$

$$\begin{cases} a^2=\frac{36}{25} \to a=\frac{6}{5} \\ b^2=\frac{9}{64} \to b=\frac{3}{8} \end{cases}$$

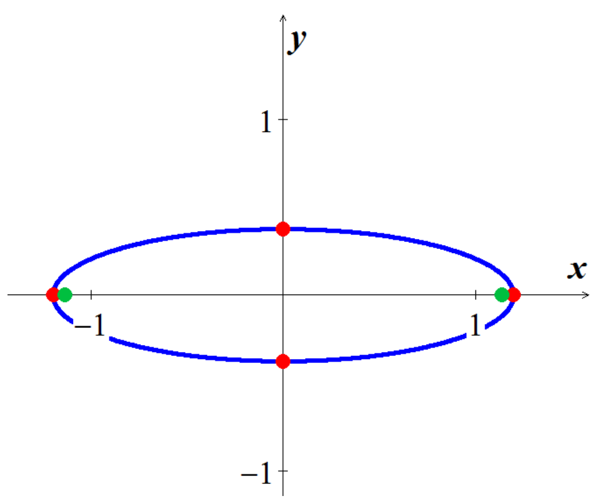
$$c=\sqrt{a^2-b^2}=\sqrt{\frac{36}{25}-\frac{9}{64}}=\sqrt{\frac{2079}{1600}}=\frac{3\sqrt{231}}{40}$$

Center: $C(0,\ 0)$

Vertices: $V\left(\pm\frac{6}{5},\ 0\right)$

Minor $M\left(0,\ \pm\frac{3}{8}\right)$

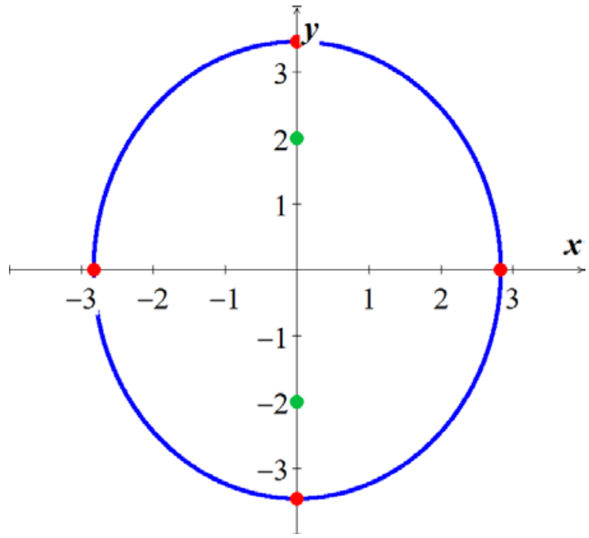
Foci $F\left(\pm\frac{3\sqrt{231}}{40},\ 0\right)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $12x^2 + 8y^2 = 96$

Solution



$$\frac{12}{96}x^2 + \frac{8}{96}y^2 = \frac{96}{96}$$

$$\frac{x^2}{8} + \frac{y^2}{12} = 1$$

$$\to \begin{cases} a^2 = 12 \to a = 2\sqrt{3} \\ b^2 = 8 \to b = 2\sqrt{2} \end{cases}$$

$$c = \sqrt{a^2 - b^2} = \sqrt{12 - 8} = 2$$

Center: $C(0,\ 0)$

Vertices: $V\left(0,\ \pm 2\sqrt{3}\right)$

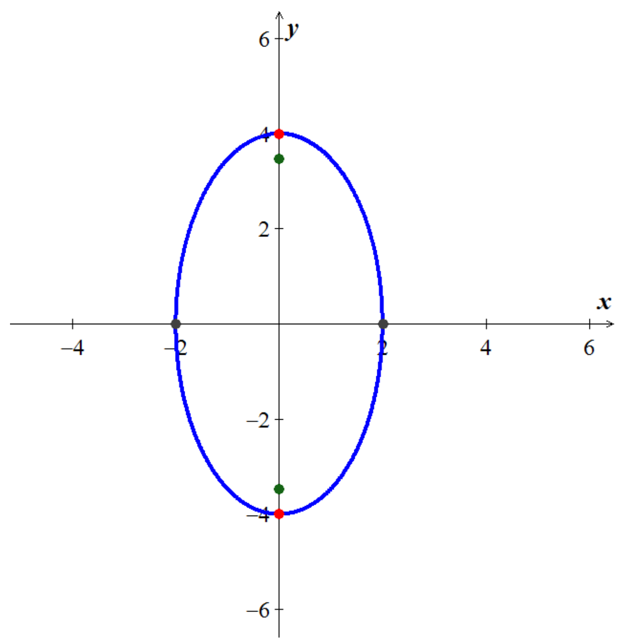
Minors $M\left(\pm 2\sqrt{2}, 0\right)$

Foci $F(0,\ \pm 2)$

Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $4x^2 + y^2 = 16$

Solution



$$\frac{1}{16}4x^2 + \frac{1}{16}y^2 = \frac{1}{16}16$$

$$\frac{x^2}{4} + \frac{y^2}{16} = 1$$

$$\to \begin{cases} a^2 = 16 \to a = 4 \\ b^2 = 4 \to b = 2 \end{cases}$$

$$c = \sqrt{a^2 - b^2} = \sqrt{16 - 4} = \sqrt{12} = 2\sqrt{3}$$

Center: $C(0,\ 0)$

Vertices: $V(0,\ \pm 4)$

Minors $M(\pm 2, 0)$

Foci $F\left(0,\ \pm 2\sqrt{3}\right)$

Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $4x^2 + 25y^2 = 1$

Solution

$$\frac{x^2}{\frac{1}{4}} + \frac{y^2}{\frac{1}{25}} = 1$$

$$\begin{cases} a^2 = \frac{1}{4} \rightarrow a = \frac{1}{2} \\ b^2 = \frac{1}{25} \rightarrow b = \frac{1}{5} \end{cases}$$

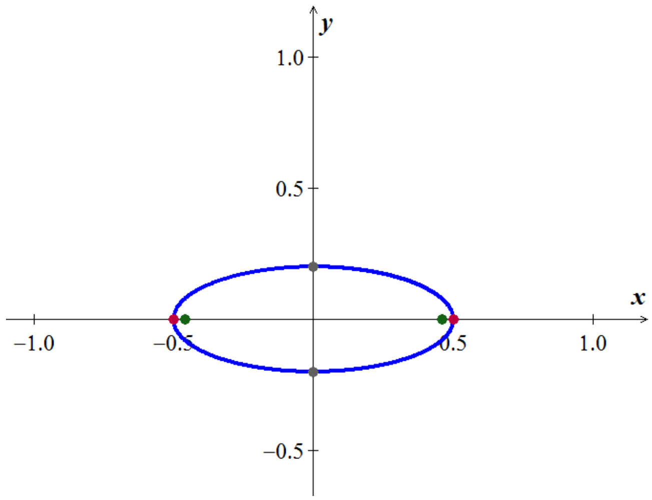
$$c = \sqrt{a^2 - b^2} = \sqrt{\frac{1}{4} - \frac{1}{25}} = \sqrt{\frac{21}{100}} = \frac{\sqrt{21}}{10}$$

Center: $C(0,\ 0)$

Vertices: $V\left(\pm\frac{1}{2},\ 0\right)$

Minor $M\left(0,\ \pm\frac{1}{5}\right)$

Foci $F\left(\pm\frac{\sqrt{21}}{10},\ 0\right)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of

$$\frac{(x-3)^2}{16} + \frac{(y+4)^2}{9} = 1$$

Solution

$$\begin{cases} a^2 = 16 \rightarrow a = 4 \\ b^2 = 9 \rightarrow b = 3 \end{cases}$$

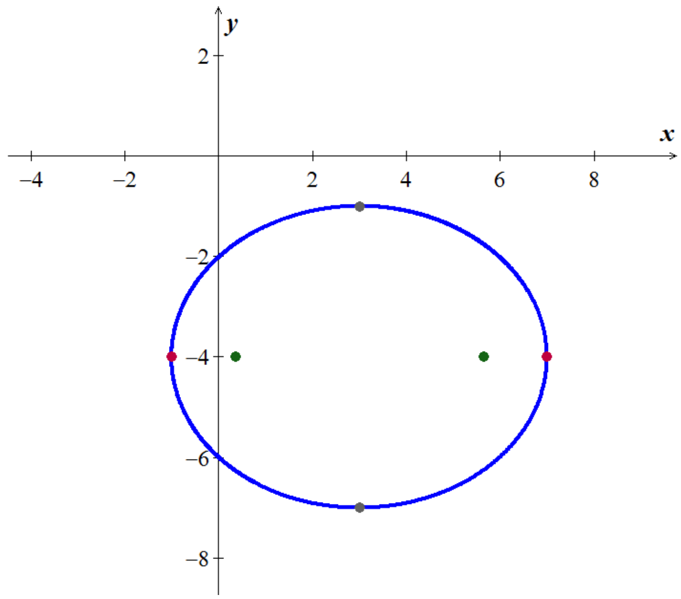
$$c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$$

Center: $C(3, \ -4)$

Vertices: $V(3 \pm 4, \ -4)$

Minor $M(3, \ -4 \pm 3)$

Foci $F\left(3 \pm \sqrt{7}, \ -4\right)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of

$$\frac{(x+3)^2}{16}+\frac{(y-2)^2}{36}=1$$

Solution

$$\begin{cases} a^2=36 \rightarrow a=6 \\ b^2=16 \rightarrow b=4 \end{cases}$$

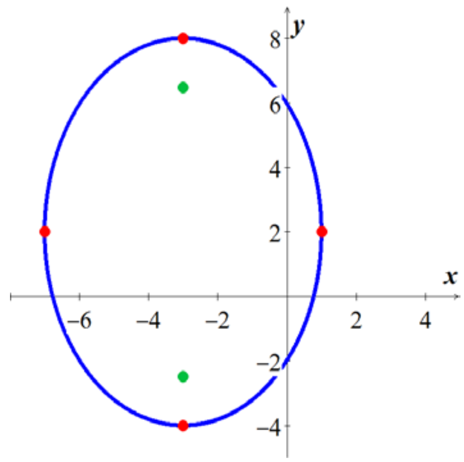
$$c=\sqrt{36-16}=2\sqrt{5}$$

Center: $C(-3,\ 2)$

Vertices: $V(-3,\ 2\pm 6)$

Minor $M(-3\pm 4,\ 2)$

Foci $F\left(-3,\ 2\pm 2\sqrt{5}\right)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of

$$\frac{(x+1)^2}{64}+\frac{(y-2)^2}{49}=1$$

Solution

$$\begin{cases} a^2=64 \rightarrow a=8 \\ b^2=49 \rightarrow b=7 \end{cases}$$

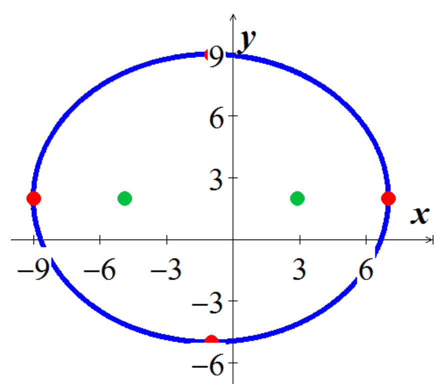
$$c=\sqrt{a^2-b^2}=\sqrt{64-49}=\sqrt{15}$$

Center: $C(-1,\ 2)$

Vertices: $V(-1\pm 8,\ 2)$

Minor $M(-1,\ 2\pm 7)$

Foci $F\left(-1\pm\sqrt{15},\ 2\right)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of
$4x^2+9y^2-32x-36y+64=0$

Solution

$$4\left(x^2-8x+\left(\frac{8}{2}\right)^2\right)+9\left(y^2-4y+\left(\frac{4}{2}\right)^2\right)=-64+4(16)+9(4)$$

$$4(x-4)^2+9(y-2)^2=36$$

$$\frac{(x-4)^2}{9}+\frac{(y-2)^2}{4}=1$$

$$\begin{cases} a^2=9\to a=3 \\ b^2=4\to b=2 \end{cases}$$

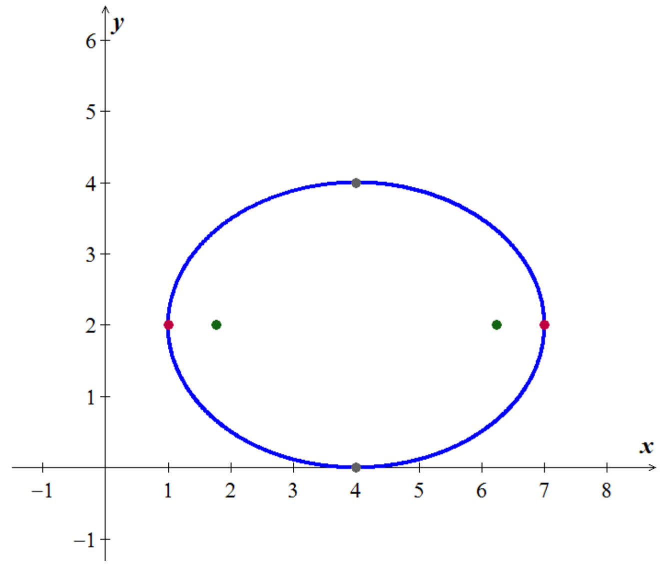
$$c=\sqrt{a^2-b^2}=\sqrt{9-4}=\sqrt{5}$$

Center: $C(4,\ 2)$

Vertices: $(4\pm 3,\ 2)\ V'(1,\ 2)\quad V(7,\ 2)$

Minor $(4,\ 2\pm 2)\quad M'(4,\ 0)\quad M(4,\ 4)$

Foci $F\left(4\pm\sqrt{5},\ 2\right)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of
$x^2+2y^2+2x-20y+43=0$

Solution

$$\left(x^2+2x+\left(\frac{2}{2}\right)^2\right)+2\left(y^2-10y+\left(\frac{10}{2}\right)^2\right)=-43+1+2(100)$$

$$(x+1)^2+2(y-5)^2=8$$

$$\frac{(x+1)^2}{8}+\frac{(y-5)^2}{4}=1$$

$$\begin{cases} a^2=8\to a=2\sqrt{2} \\ b^2=4\to b=2 \end{cases}$$

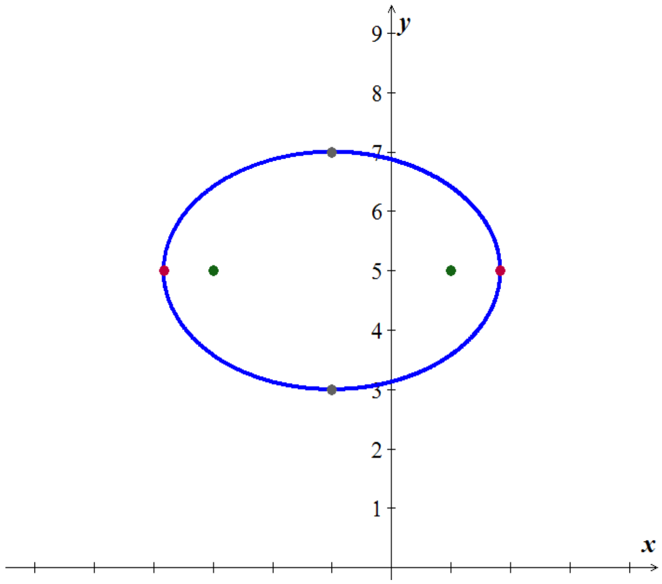
$$c=\sqrt{a^2-b^2}=\sqrt{8-4}=2$$

Center: $C(-1,\ 5)$

Vertices: $V\left(-1 \pm 2\sqrt{2},\ 5\right)$

Minor $(-1,\ 5 \pm 2) \quad \to \quad M'(-1,\ 3) \quad M(-1,\ 7)$

Foci $(-1 \pm 2,\ 5) \quad \to \quad F'(-3,\ 5) \quad F(1,\ 5)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $25x^2 + 4y^2 - 250x - 16y + 541 = 0$

Solution

$$25\left(x^2 - 10x + \left(\frac{10}{2}\right)^2\right) + 4\left(y^2 - 4y + \left(\frac{4}{2}\right)^2\right) = -541 + 25(25) + 4(4)$$

$$25(x-5)^2 + 4(y-2)^2 = 100$$

$$\frac{(x-5)^2}{4} + \frac{(y-2)^2}{25} = 1$$

$$\begin{cases} a^2 = 25 \to a = 5 \\ b^2 = 4 \to b = 2 \end{cases}$$

$$c = \sqrt{a^2 - b^2} = \sqrt{25-4}$$

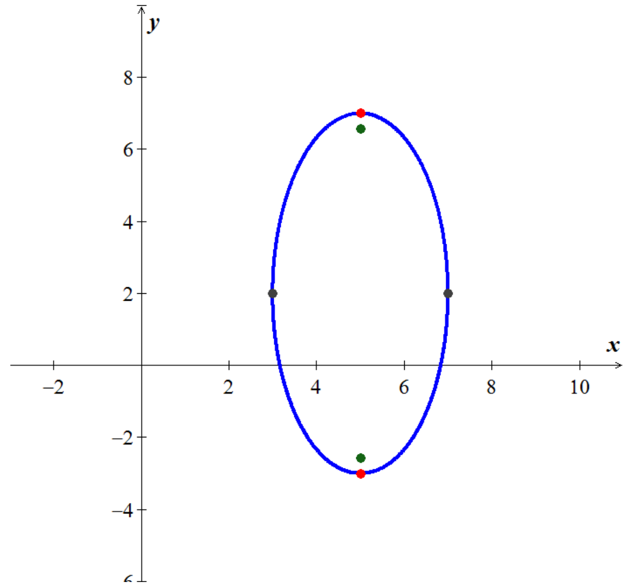
$$= \sqrt{21}$$

Center: $C(5,\ 2)$

Vertices: $(5,\ 2 \pm 5) \quad \to \quad V'(5,\ -3) \quad V(5,\ 7)$

$\boldsymbol{Minor}$ $(5 \pm 2,\ 2) \to \quad M(3,\ 2) \quad M(7,\ 2)$

$\boldsymbol{Foci}$ $F\left(5,\ 2 \pm \sqrt{21}\right)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of $4x^2 + y^2 = 2y$

Solution

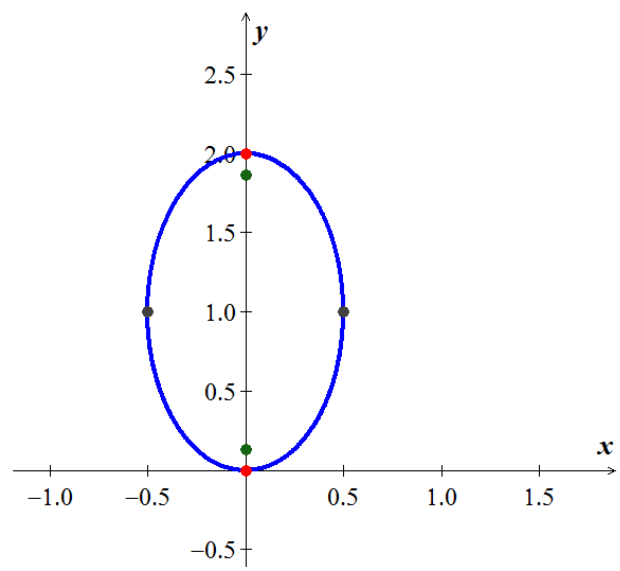
$$4x^2 + y^2 - 2y = 0$$

$$4x^2 + \left(y^2 - 2y + \left(\frac{2}{2}\right)^2\right) = (1)^2$$

$$4x^2 + (y-1)^2 = 1$$

$$\frac{x^2}{\frac{1}{4}} + \frac{(y-1)^2}{1} = 1$$

$$\begin{cases} a^2 = 1 \to a = 1 \\ b^2 = \frac{1}{4} \to b = \frac{1}{2} \end{cases}$$



$$c = \sqrt{a^2 - b^2} = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

Center: $C(0,\ 1)$

Vertices: $(0,\ 1 \pm 1) \to \quad V'(0,\ 0) \quad V(0,\ 2)$

$\boldsymbol{Minor}$ $\left(0 \pm \frac{1}{2},\ 1\right) \to \quad M'\left(-\frac{1}{2},\ 1\right) \quad M\left(\frac{1}{2},\ 1\right)$

$\boldsymbol{Foci}$ $F\left(0,\ 1 \pm \frac{\sqrt{3}}{2}\right)$

Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse Sketch the graph: $2x^2+3y^2-8x+6y+5=0$

Solution

$$2x^2-8x+3y^2+6y=-5$$

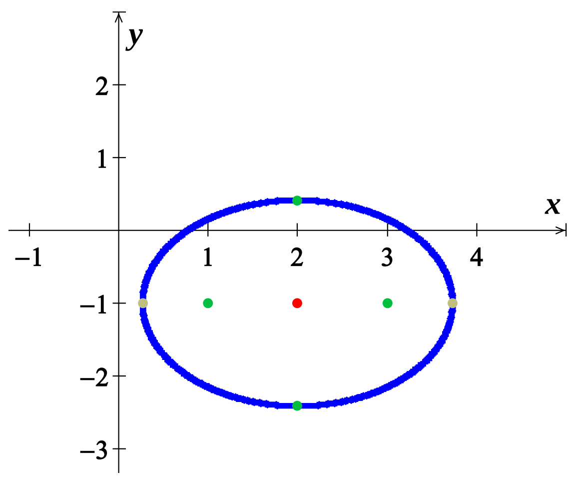
$$2\left(x^2-4x+\left(\frac{-4}{2}\right)^2\right)+3\left(y^2+2y+\left(\frac{2}{2}\right)^2\right)=-5+2\left(\frac{-4}{2}\right)^2+3\left(\frac{2}{2}\right)^2$$

$$2(x-2)^2+3(y+1)^2=-5+8+3$$

$$2(x-2)^2+3(y+1)^2=6$$

$$\frac{2(x-2)^2}{6}+\frac{3(y+1)^2}{6}=1$$

$$\frac{(x-2)^2}{3}+\frac{(y+1)^2}{2}=1$$



$$\begin{cases} a^2=3\to a=\pm\sqrt{3} \\ b^2=2\to b=\pm\sqrt{2} \\ \hline c=\sqrt{a^2-b^2}=\sqrt{1}=1 \end{cases}$$

Center: $(2,\ -1)$

Vertices: $V\left(2\pm\sqrt{3},\ -1\right)$

Minor $M\left(2,\ -1\pm\sqrt{2}\right)$

Foci $(2\pm1,\ -1)\to\quad F'=(1,\ -1)\quad F=(3,\ -1)$

Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of

$4x^2 + 3y^2 + 8x - 6y - 5 = 0$

Solution

$$4x^2 + 8x + 3y^2 - 6y = 5$$

$$4\left(x^2 + 2x + \left(\frac{2}{2}\right)^2\right) + 3\left(y^2 - 2y + \left(\frac{-2}{2}\right)^2\right) = 5 + 4\left(\frac{2}{2}\right)^2 + 3\left(\frac{-2}{2}\right)^2$$

$$4(x+1)^2 + 3(y-1)^2 = 5 + 4 + 3$$

$$4(x+1)^2 + 3(y-1)^2 = 12$$

$$\frac{4(x+1)^2}{12} + \frac{3(y-1)^2}{12} = 1$$

$$\frac{(x+1)^2}{3} + \frac{(y-1)^2}{4} = 1$$

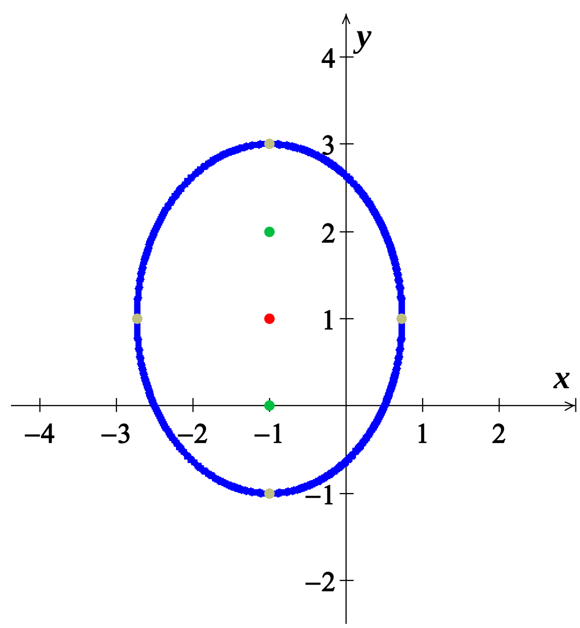
$$\begin{cases} a^2 = 4 \to a = \pm 2 \\ b^2 = 3 \to b = \pm\sqrt{3} \\ c = \pm\sqrt{a^2 - b^2} = \pm\sqrt{4-3} = \pm 1 \end{cases}$$

Center: $(-1,\ 1)$

Vertices: $(-1,\ 1 \pm 2) \to \quad V'(-1,\ -1) \quad V(-1,\ 3)$

Minor $M\left(-1 \pm \sqrt{3},\ 1\right)$

Foci $(-1,\ 1 \pm 1) \to \quad F' = (-1,\ 0) \quad F = (-1,\ 2)$



Exercise

Find the ***center, vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of

$9x^2 + 4y^2 - 18x + 16y - 11 = 0$

Solution

$$9x^2 - 18x + 4y^2 + 16y = 11$$

$$9\left(x^2 - 2x + \left(\frac{-2}{2}\right)^2\right) + 4\left(y^2 + 4y + \left(\frac{4}{2}\right)^2\right) = 11 + 9\left(\frac{-2}{2}\right)^2 + 4\left(\frac{4}{2}\right)^2$$

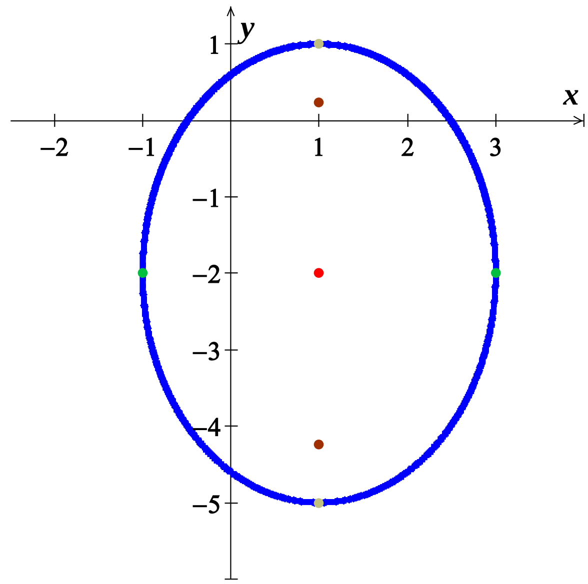
$$9(x-1)^2 + 4(y+2)^2 = 11 + 9 + 16$$

$$9(x-1)^2 + 4(y+2)^2 = 36$$

$$\frac{9(x-1)^2}{36} + \frac{4(y+2)^2}{36} = 1$$

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$$

$$\begin{cases} a^2 = 9 \rightarrow a = \pm 3 \\ b^2 = 4 \rightarrow b = \pm 2 \\ c = \pm\sqrt{a^2 - b^2} = \pm\sqrt{9-4} = \pm\sqrt{5} \end{cases}$$



Center: $(1,\ -2)$

Vertices: $(1,\ -2 \pm 3) \rightarrow \quad V'(1,\ -5) \quad V(1,\ 1)$

Minor: $(1 \pm 2,\ -2) \rightarrow \quad M'(-1,\ -2) \quad M(3,\ -2)$

Foci $\left(1,\ -2 \pm \sqrt{5}\right)$

Exercise

Find an equation for an ellipse with: $x-$*intercepts:* ± 4; *foci* $(-2,\ 0)$ *and* $(2,\ 0)$

Solution

The ellipse is centered at $(0,\ 0)$

Major axis: $a = 4$

Foci: $(\pm 2,\ 0) \quad \Rightarrow \quad c = 2$

$b^2 = a^2 - c^2 = 16 - 4 = 12$

The equation is: $\quad \dfrac{x^2}{16} + \dfrac{y^2}{12} = 1$

Exercise

Find an equation for an ellipse with: *Endpoints of major axis at* $(6,\ 0)$ *and* $(-6,\ 0)$; $c=4$

Solution

The ellipse is centered at $(0,\ 0)$ between the endpoint of the major axis

Major axis: $a=6$

$b^2=a^2-c^2=36-16=20$

The equation is: $\frac{x^2}{36}+\frac{y^2}{20}=1$

Exercise

Find an equation for an ellipse with: Center $(3,-2)$; $a=5$; $c=3$; major axis vertical

Solution

The ellipse is centered at $(3,-2)$

$b^2=a^2-c^2=25-9=16$

The equation is: $\boxed{\frac{(x-3)^2}{25}+\frac{(y+2)^2}{16}=1}$

Exercise

Find an equation for an ellipse with: *major axis of length* 6; *foci* $(0,\ 2)$ *and* $(0,\ -2)$

Solution

The ellipse is centered between the foci at $(0,\ 0)$

Major axis is the vertical with $a=3$

Foci: $(0,\ \pm 2)\ \Rightarrow\ c=2$

$b^2=a^2-c^2=9-4=5$

The equation is: $\boxed{\frac{y^2}{9}+\frac{x^2}{5}=1}$

Exercise

A patient's kidney stone is placed 12 *units* away from the source of the shock waves of a lithotripter. The lithotripter is based on an ellipse with a minor axis that measures 16 *units*. Find an equation of an ellipse that would satisfy this situation.

Solution

The patient and the emitter are 12 units apart $\Rightarrow$ these represent the foci of an ellipse, so $c = 6$.

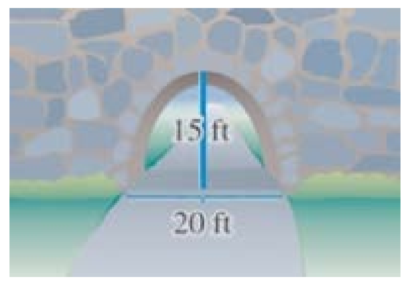
The minor axis: 16 units $\Rightarrow b = 8$.

$\therefore\ a^2 = b^2 + c^2 = 64 + 36 = 100$.

The equation is: $\frac{x^2}{100} + \frac{y^2}{64} = 1$

Exercise

A one-way road passes under an overpass in the form of half of an ellipse 15 *feet* high at the center and 20 *feet* wide. Assuming that a truck is 12 *feet* wide, what is the height of the tallest truck that can pass under the overpass?



Solution

Using a vertical major axis $\Rightarrow a = 15$.

The minor axis: 20 *ft*. $\Rightarrow b = 10$.

The equation is: $\frac{y^2}{225} + \frac{x^2}{100} = 1$

Assuming the truck drives through the middle, we want to find y when $x = 6$

$$\frac{y^2}{225} = 1 - \frac{6^2}{100} = \frac{64}{100}$$

$$\Rightarrow \quad y^2 = 225\frac{64}{100}$$

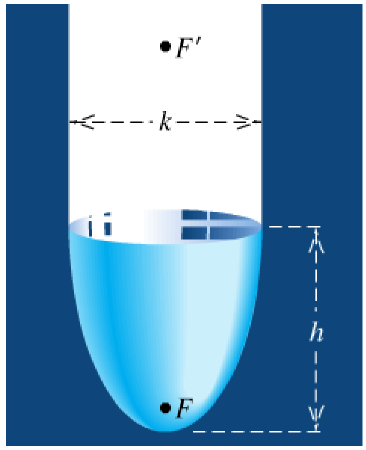
$$y = \sqrt{\frac{225(64)}{100}}$$

$$\underline{= 12}$$

The truck must be just under 12 *feet* high to pass through.

Exercise

The basic shape of an elliptical reflector is a hemi-ellipsoid of height h and diameter k. Waves emitted from focus F will reflect off the surface into focus F'



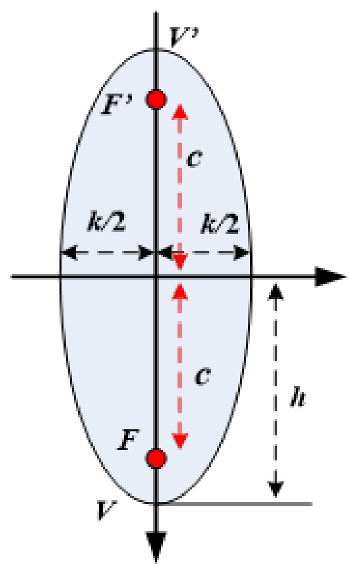
a) Express the distance $d(V,\ F)$ and $d(V,\ F')$ in terms of h and k.

b) An elliptical reflector of height 17 *cm* is to be constructed so that waves emitted from F are reflected to a point F' that is 32 *cm* from V. Find the diameter of the reflector and the location of F.

Solution

Given: $b = \frac{k}{2}, \quad a = h$

$$c^2 = a^2 - b^2 = h^2 - \left(\frac{k}{2}\right)^2$$



a) $d(V,\ F) = h - c$

$$= h - \sqrt{h^2 - \frac{1}{4}k^2}$$

$$d(V,\ F') = h + c$$

$$= h + \sqrt{h^2 - \frac{1}{4}k^2}$$

b) ***Given***: $h = 17\ cm, \quad h + c = 32\ cm$

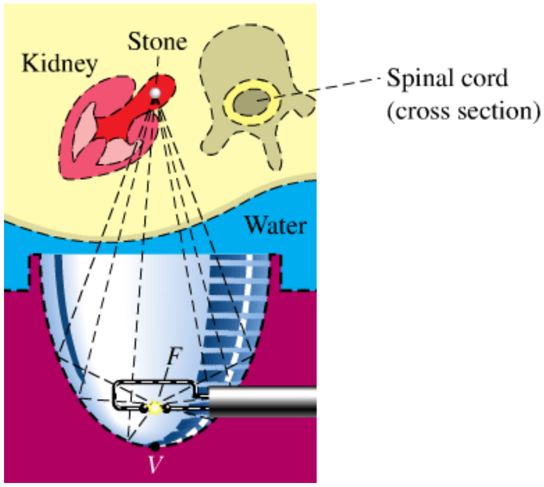
$$c = 32 - h$$
$$= 32 - 17$$
$$= 15\ cm$$

$$d(V,\ F) = h - c$$
$$= 17 - 15$$
$$= 2\ cm$$

The location of F is $16\ cm;\ \ 2\ cm\ \ from\ V'$

Exercise

A lithotripter of height 15 *cm* and diameter 18 *cm* is to be constructed. High-energy underwater shock waves will be emitted from the focus *F* that is closest to the vertex *V*.



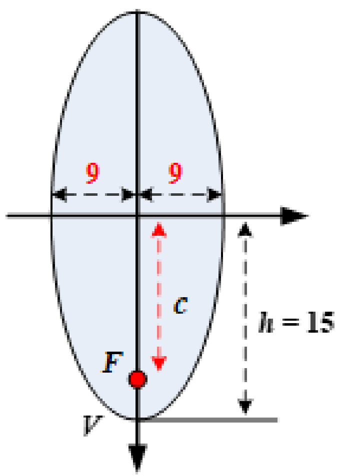
a) Find the distance from *V* to *F*.

b) How far from *V* (in the vertical direction) should a kidney stone located?

Solution

Given: $b = \frac{18}{2} = 9, \quad a = h = 15$

$$c = \sqrt{a^2 - b^2}$$
$$= \sqrt{15^2 - 9^2}$$
$$= 12\ cm$$



a) $d(V,\ F) = h - c$

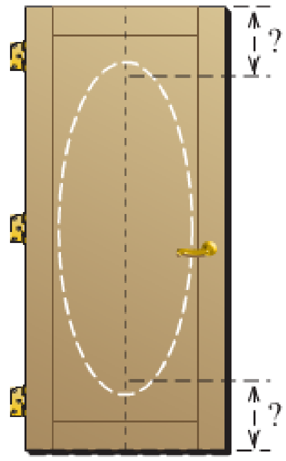
$$= 15 - 12$$
$$= 3\ cm$$

b) $h + c = 15 + 12$

$$= 27\ cm$$

Exercise

An Artist plans to create an elliptical design with major axis 60″ and minor axis 24″, centered on a door that measures 80″ by 36″.



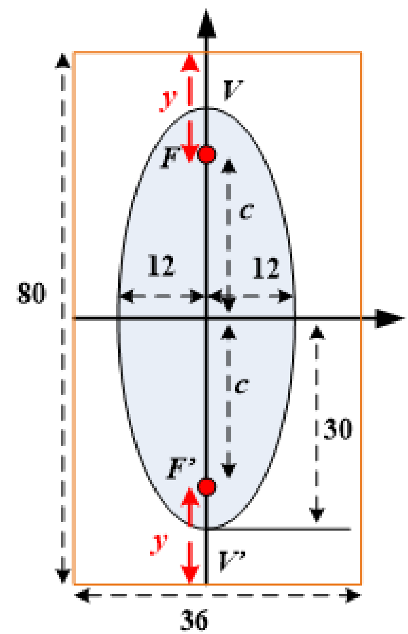
On a vertical line that dissects the door, approximately how far from each end of the door should the push-pins be inserted? How long should the string be?

Solution

Given: $b=\frac{24}{2}=12''$, $a=\frac{60}{2}=30''$

$$c=\sqrt{a^2-b^2}$$
$$=\sqrt{15^2-9^2}$$
$$=27.5$$

$$2y+2c=80$$
$$y=\frac{80-2c}{2}$$
$$=\frac{80-2\sqrt{756}}{2}$$
$$\approx 12.5''$$

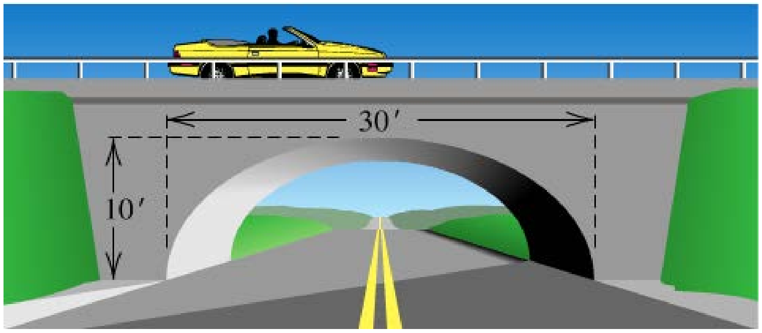


Therefore, the distance from each end of the door should the push-pins be inserted, is 12.5 *in.*

The string should be $=30+30=60$ *in.*

Exercise

An arch of a bridge is semi-elliptical, with major axis horizontal. The base of the arch is 30 *feet*. across, and the highest part of the arch is 10 *feet*. above the horizontal roadway. Find the height of the arch 6 *feet*. from the center of the base.

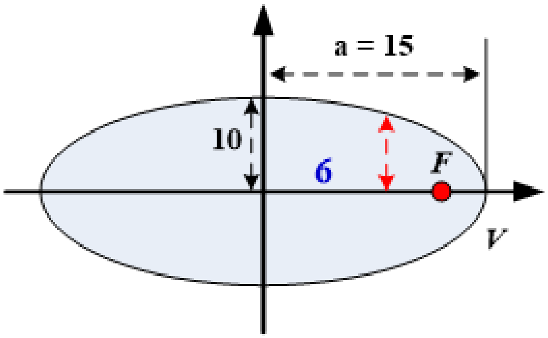


Solution

Given: $b=10',\quad a=\frac{30}{2}=15'$

$$\begin{aligned}c&=\sqrt{a^2-b^2}\\&=\sqrt{15^2-10^2}\\&=\sqrt{125}\\&=5\sqrt{5}\end{aligned}$$

$$\frac{x^2}{225}+\frac{y^2}{100}=1 \qquad\qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



$$\frac{y^2}{100}=1-\frac{6^2}{225}$$

$$y^2=100\left(1-\frac{36}{225}\right)$$

$$\begin{aligned}y&=\sqrt{100\left(1-\frac{36}{225}\right)}\\&=\sqrt{84}\ \approx 9.2\ \ ft\end{aligned}$$

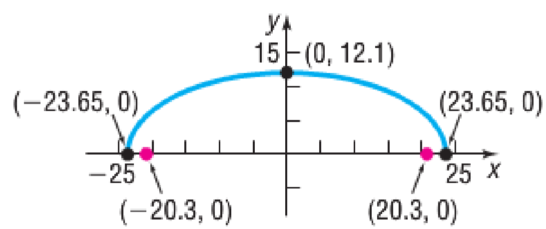
Exercise

The whispering gallery in the Museum of Science and Industry in Chicago is 47.3 *feet* long. The distance from the center of the room to the foci is 20.3 *feet*. Find an equation that describes the shape of the room. How high is the room at its center?

Solution

Set up a rectangular coordinate so that the center of the ellipse is at the origin and the major axis along the x-axis. The equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



Length of the room: 47.3 *ft*.

Distance from the center of the room to each vertex:

$$a=\frac{47.3}{2}$$
$$=\underline{23.65}$$

Distance from the center of the room to each focus is $c=20.3\ ft$

$$b^2=a^2-c^2$$
$$=23.65^2-20.3^2$$
$$=\underline{147.2325}$$

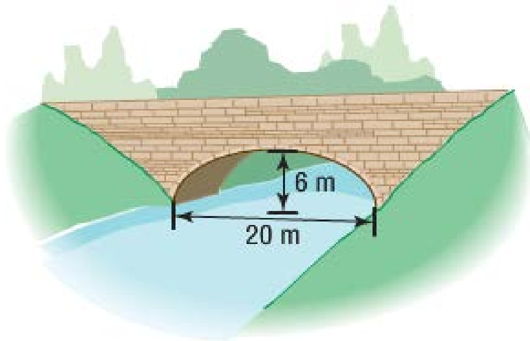
Therefore, the equation is given: $\frac{x^2}{559.3225}+\frac{y^2}{147.2325}=1$

The Height of the room: $b=\sqrt{147.2325}$

$$\approx 12.1\ ft$$

Exercise

An arch in the shape of the upper half of an ellipse is used to support a bridge that is to span a river 20 *meters* wide. The center of the arch is 6 *meters* above the center of the river. Write an equation for the ellipse in which the *x*-axis coincides with the water level and the *y*-axis passes through the center of the arch.



Solution

The center of the ellipse is $(0,\ 0)$. The length of the major axis is 20, so $a=10$.

The length of the half minor axis is 6, so $b=6$.

The ellipse is situated with its major axis on the *x*-axis.

The equation: $\dfrac{x^2}{10^2}+\dfrac{y^2}{6^2}=1$

$$\boxed{\dfrac{x^2}{100}+\dfrac{y^2}{36}=1}$$

Exercise

A bridge is built in the shape of a semielliptical arch. The bridge has a span of 120 *feet* and a maximum height of 25 *feet*. Choose a rectangular coordinate system and find the height of the arch at distances of 10, 30, and 50 *feet* from the center.

Solution

Since the bridge has a span of 120 feet, the length of the major axis is $120=2a \ \rightarrow\ a=60$

The maximum height of the bridge is 25 feet, so $b=25$.

The equation: $\dfrac{x^2}{60^2}+\dfrac{y^2}{25^2}=1$

$$\boxed{\dfrac{x^2}{3600}+\dfrac{y^2}{625}=1}$$

At distance 10 *feet*:

$$\dfrac{10^2}{3600}+\dfrac{y^2}{625}=1$$

$$\dfrac{y^2}{625}=1-\dfrac{100}{3600}$$

$y^2 = 625\left(1 - \frac{1}{36}\right)$

$y = \sqrt{625\left(\frac{35}{36}\right)}$

The height from the center is $y \approx 24.65\ ft$

At distance **30** *feet*:

$\frac{30^2}{3600} + \frac{y^2}{625} = 1$

$\frac{y^2}{625} = 1 - \frac{900}{3600}$

$y^2 = 625\left(1 - \frac{9}{36}\right)$

$y = \sqrt{625\left(\frac{27}{36}\right)}$

The height from the center is $y \approx 21.65\ ft$

At distance **50** *feet*:

$\frac{50^2}{3600} + \frac{y^2}{625} = 1$

$\frac{y^2}{625} = 1 - \frac{2500}{3600}$

$y^2 = 625\left(1 - \frac{25}{36}\right)$

$y = \sqrt{625\left(\frac{11}{36}\right)}$

The height from the center is $y \approx 13.82\ ft$

Exercise

A bridge is built in the shape of a semielliptical arch. The bridge has a span of 100 *feet*. The height of the arch is 10 *feet*. Find the height of the arch at its center.

Solution

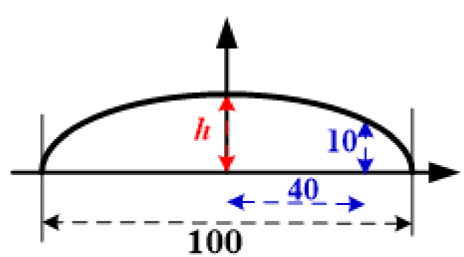
Since the bridge has a span of 100 *feet*.

Length of the major axis is $100 = 2a \ \rightarrow \ a = 50$

The maximum height of the bridge is 25 *feet*, so $b = 25$.

The equation: $\frac{x^2}{2500} + \frac{y^2}{h^2} = 1$

The height of the arch 40 *feet* from the center is 10 *feet*.



So $(40, 10)$ is a point on the ellipse.

$$\frac{40^2}{2500}+\frac{10^2}{h^2}=1$$

$$\frac{10^2}{h^2}=1-\frac{1600}{2500}$$

$$\frac{100}{h^2}=1-\frac{16}{25}$$

$$\frac{100}{h^2}=\frac{9}{25}$$

$$h^2=\frac{100\cdot 25}{9}$$

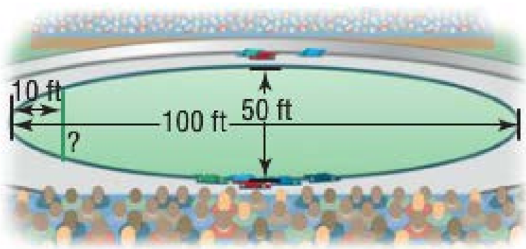
$$h=\sqrt{\frac{100\cdot 25}{9}}$$

$$h\approx 16.67$$

The height of the arch at its center is 16.67 *feet*.

Exercise

A racetrack is in the shape of an ellipse, 100 *feet* long and 50 *feet* wide. What is the width 10 *feet* from a vertex?



Solution

Length of the major axis is $100=2a \;\to\; a=50$

The maximum height of the bridge is $50=2b \;\to\; b=25$.

The equation: $\frac{x^2}{2500}+\frac{y^2}{625}=1$

We need to find y at $x=50-10=40$

$$\frac{40^2}{2500}+\frac{y^2}{625}=1$$

$$\frac{y^2}{625}=1-\frac{1600}{2500}$$

$$y^2=625\frac{9}{25}$$

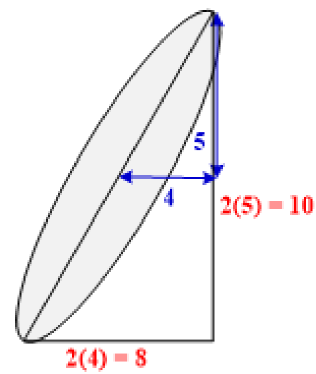
$$y=15\ \ ft$$

The width of the ellipse at 10 *feet* from a vertex $x = 40$ is $\boxed{2 \times 15 = 30\ ft}$

Exercise

A homeowner is putting in a fireplace that has a 4-*inch* radius vent pipe. He needs to cut an elliptical hole in his roof to accommodate the pipe. If the pitch of his roof is $\frac{5}{4}$ (a rise of 5, run of 4) what are the dimensions of the hole?

Solution



The length of the major axis can be determined from the pitch
By using Pythagorean Theorem:

$a = \sqrt{4^2 + 5^2} = \sqrt{41}$

The length of the major axis $2a \ \boxed{= 2\sqrt{41}\ in}$

The length of the minor axis:

$2b = 2(4) = \boxed{= 8\ \ in}$

Exercise

A football is in the shape of a ***prolate spheroid***, which is simply a solid obtained by rotating an ellipse about its major axis. An inflated NFL football averages 11.125 *inches* in length and 28.25 *inches* in center circumference. If the volume of a prolate spheroid is $\frac{4}{3}\pi ab^2$, how much air does the football contain? (Neglect material thickness)

Solution

The length of the football is $2a = 11.125 \quad \Rightarrow \quad a = 5.5625$

The center circumference is $28.25 = 2\pi b \quad \Rightarrow \quad b = \frac{28.25}{2\pi}$

The volume is:

$$V = \frac{4}{3}\pi ab^2$$

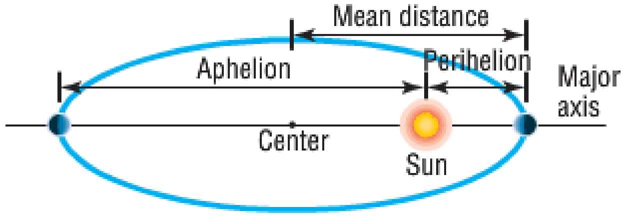
$$= \frac{4}{3}\pi(5.5625)\left(\frac{28.25}{2\pi}\right)^2$$

$$\boxed{\approx 472\ \ in^3}$$

The football contains approximately 471 *cubic inches* of air.

Exercise

The fact that the orbit of a planet about the Sun is an ellipse with the Sun at one focus. The ***aphelion*** of a planet is its greatest distance from the Sun, and the ***perihelion*** is its shortest distance. The ***mean distance*** of a planet from the Sun is the length of the semi-major axis of the elliptical orbit.



a) The mean distance of Earth from the Sun is 93 million *miles*. If the aphelion of Earth is 94.5 million *miles*, what is the perihelion? Write an equation for the orbit of Earth around the Sun.

b) The mean distance of Mars from the Sun is 142 million *miles*. If the perihelion of Mars is 128.5 million *miles*, what is the aphelion? Write an equation for the orbit of Mars about the Sun.

c) The aphelion of Jupiter is 507 million *miles*. If the distance from the center of it elliptical orbit to the Sun is 23.2 million *miles*, what is the perihelion? What is the mean distance? Write an equation for the orbit of Jupiter around the Sun.

d) The perihelion of Pluto is 4551 million *miles*, and the distance from the center of its elliptical orbit to the Sun is 897.5 million *miles*. Find the aphelion of Pluto. What is the mean distance of Pluto from the Sun? Write an equation for the orbit of Pluto about the Sun.

Solution

a) The mean distance is 93 million *miles* $\Rightarrow \underline{a = 93}$

The length of the major axis is 186 million

The perihelion is 186 – 94.5 = 91.5 million *miles*

Distance from the ellipse center to the sun is the focus: c = 93 – 91.5 = 1.5 million *miles*.

$$b^2 = a^2 - c^2$$
$$= 93^2 - 1.5^2$$
$$b = \sqrt{93^2 - 1.5^2}$$
$$\underline{= 92.99 \ \ million}$$

Therefore: $a = 93 \times 10^6$ *and* $b = 92.99 \times 10^6$

The equation is given by: $\dfrac{x^2}{\left(93 \times 10^6\right)^2} + \dfrac{y^2}{\left(92.99 \times 10^6\right)^2} = 1$

Let x and y in millions miles: $\dfrac{x^2}{93^2} + \dfrac{y^2}{92.99^2} = 1$ $\quad$ $(in\ millions\ miles)$

The equation of the orbit is: $\underline{\dfrac{x^2}{8649} + \dfrac{y^2}{8647.14} = 1}$

b) The mean distance is 142 million *miles* $\Rightarrow \underline{a=142}$

The length of the major axis is 284 million

The perihelion is 284 – 128.5 = 155.5 million *miles*

Distance from the ellipse center to the sun is the focus: c = 142 – 128.5 = 13.5 million *miles*.

$b^2 = a^2 - c^2 = 142^2 - 13.5^2 = 19{,}981.75$

$b = \sqrt{142^2 - 13.5^2} \ \underline{= 141.36 \ \ million}$

Let x and y in millions miles: $\frac{x^2}{142^2} + \frac{y^2}{141.36^2} = 1 \qquad (in\ millions\ miles)$

The equation of the orbit is: $\underline{\frac{x^2}{20{,}164} + \frac{y^2}{19{,}981.75} = 1}$

c) The mean distance is 507 – 23.2 = 483.8 million *miles* $\Rightarrow \underline{a = 483.8}$

The perihelion is 483.8 – 23.2 = 460.6 million *miles*

Distance from the ellipse center to the sun is the focus: c = 23.2 million *miles*.

$b^2 = a^2 - c^2 = 438.8^2 - 23.2^2 = 233{,}524.2$

$b = \sqrt{438.8^2 - 23.2^2} \ \underline{= 483.2 \ \ million}$

Let x and y in millions miles: $\frac{x^2}{483.8^2} + \frac{y^2}{483.2^2} = 1 \qquad (in\ millions\ miles)$

The equation of the orbit is: $\underline{\frac{x^2}{234{,}062.44} + \frac{y^2}{233{,}524.2} = 1}$

d) The mean distance is 4551 + 897.5 = 5448.5 million *miles* $\Rightarrow \underline{a = 5448.5}$

The aphelion is 5448.5 + 897.5 = 6346 million *miles*

Distance from the ellipse center to the sun is the focus: c = 897.5 million *miles*.

$b^2 = a^2 - c^2 = 5448.5^2 - 897.5^2 = 28{,}880{,}646$

$b = \sqrt{5448.5^2 - 897.5^2} \ \underline{= 5374.07 \ \ million}$

Let x and y in millions miles: $\frac{x^2}{5448.5^2} + \frac{y^2}{5374.07^2} = 1 \qquad (in\ millions\ miles)$

The equation of the orbit is: $\underline{\frac{x^2}{29{,}686{,}152.25} + \frac{y^2}{28{,}880{,}646} = 1}$

Exercise

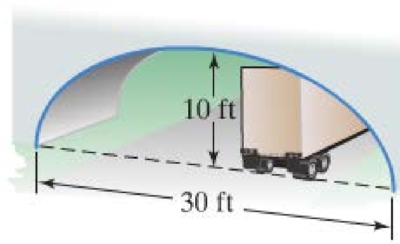
Will a truck that is 8 *feet* wide carrying a load that reaches 7 *feet* above the ground the semielliptical arch on the one-way road that passes under the bridge?

Solution

Given: $a = 15, \quad b = 10$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{225} + \frac{y^2}{100} = 1$$



Given: $x = 4$

$$\frac{y^2}{100} = 1 - \frac{16}{225}$$

$$y = \pm\sqrt{100\left(1 - \frac{16}{225}\right)}$$

$$= \sqrt{\frac{836}{9}}$$

$$\approx 9.64$$

Yes, the truck will clear about $9.6 - 7 = 7.6 \ \ ft$.

Exercise

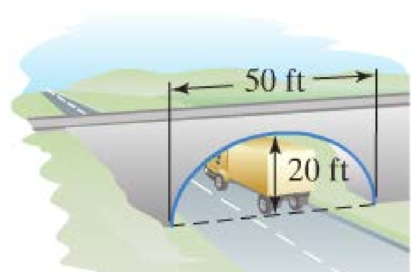
A semielliptic archway has a height of 20 *feet* and a width of 50 *feet* and a width of 50 *feet*. Can a truck 14 *feet* high and 10 *feet* wide drive under the archway without going into the other lane?

Solution

Given: $a = 25, \quad b = 20$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{625} + \frac{y^2}{400} = 1$$



Given: $x = 10$

$$\frac{y^2}{400} = 1 - \frac{100}{625}$$

$$y^2 = 400\left(\frac{500}{625}\right)$$

$$y = \sqrt{400\left(\frac{500}{625}\right)}$$

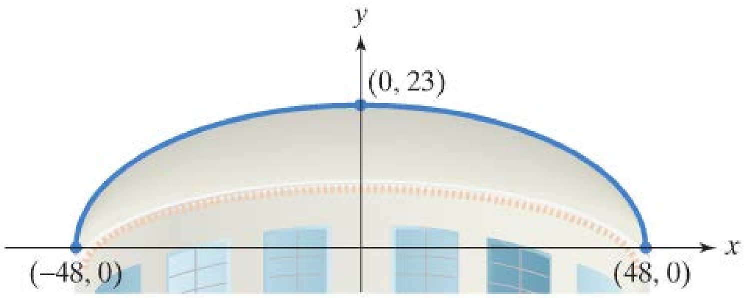
$$= \sqrt{320}$$

$$\approx 17.9$$

Yes, the truck will clear about $17.9 - 14 = 3.9 \ \ ft$.

Exercise

The elliptical ceiling in Statuary Hall is 96 *feet* long and 23 *feet* tall.



a) Using the rectangular coordinate system in the figure shown, write the standard form of the equation of the elliptical ceiling.

b) John Quincy Adams discovered that he could overhear the conversations of opposing party leaders near the left side of the chamber if he situated his desk at the focus at the right side of the chamber. How far from the center of the ellipse along the major axis did Adams situate his desk?

Solution

a) ***Given***: $a = 48, \quad b = 23$

$$\frac{x^2}{48^2} + \frac{y^2}{23^2} = 1 \qquad\qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\boxed{\frac{x^2}{2304} + \frac{y^2}{529} = 1}$$

b) $c^2 = a^2 - b^2$

$$= 2304 - 529$$

$$= 1775$$

$$c = \sqrt{1775}$$

$$\boxed{\approx 42.13}$$

He situated desk about 42 *feet* from the center of the ellipse, along the major axis.

Solution   *Section* 4.8 – Hyperbolas

Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $\frac{x^2}{9} - \frac{y^2}{4} = 1$

Solution

$$\rightarrow \begin{cases} a^2 = 9 \rightarrow a = 3 \\ b^2 = 4 \rightarrow b = 2 \end{cases}$$

$$\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{9+4} = \sqrt{13}$$

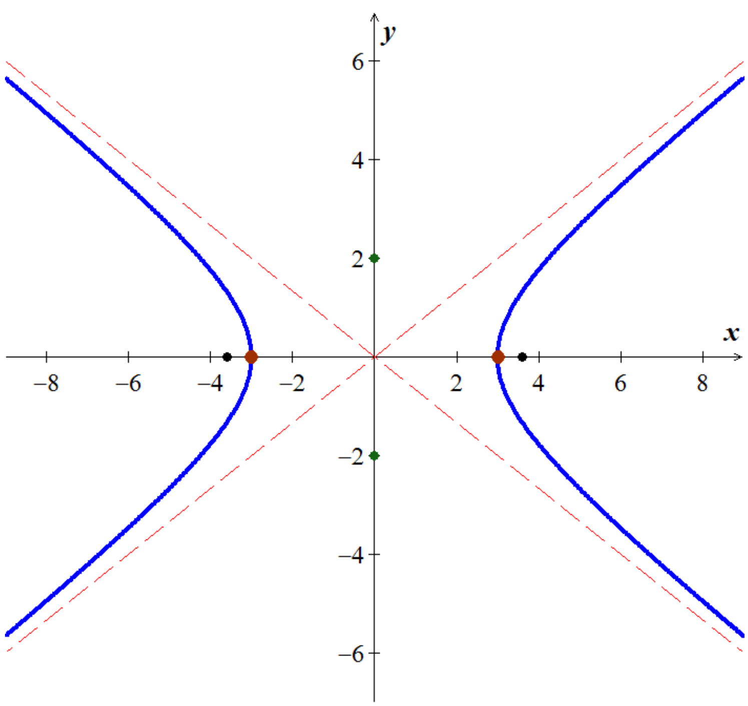
Center: $C = (0,\ 0)$

Vertices: $V = (\pm 3,\ 0)$

Endpoints: $W = (0,\ \pm 2)$

Foci: $F = \left(\pm\sqrt{13},\ 0\right)$

*Equations of the **asymptotes***: $y = \pm\frac{2}{3}x$ $y = \pm\frac{b}{a}x$



Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $\frac{y^2}{9} - \frac{x^2}{4} = 1$

Solution

$$\rightarrow \begin{cases} a^2 = 9 \rightarrow a = 3 \\ b^2 = 4 \rightarrow b = 2 \end{cases}$$

$$\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{9+4} = \sqrt{13}$$

Center: $C = (0,\ 0)$

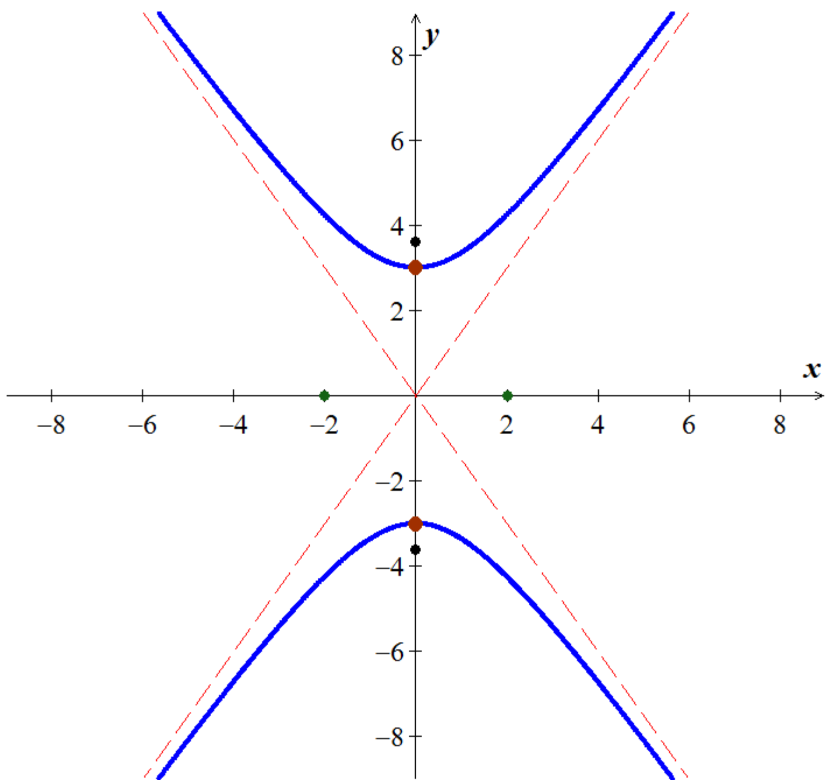
Vertices: $V = (0,\ \pm 3)$

Endpoints: $W = (\pm 2,\ 0)$

Foci: $F = \left(0,\ \pm\sqrt{13}\right)$

Equations of the ***asymptotes***:

$\underline{y = \pm \frac{3}{2} x}$ $\qquad$ $y = \pm \frac{a}{b} x$



Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $x^2 - \frac{y^2}{24} = 1$

Solution

$$\rightarrow \begin{cases} a^2 = 1 \rightarrow a = 1 \\ b^2 = 24 \rightarrow b = 2\sqrt{6} \end{cases}$$

$$\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{1+24} = 5$$

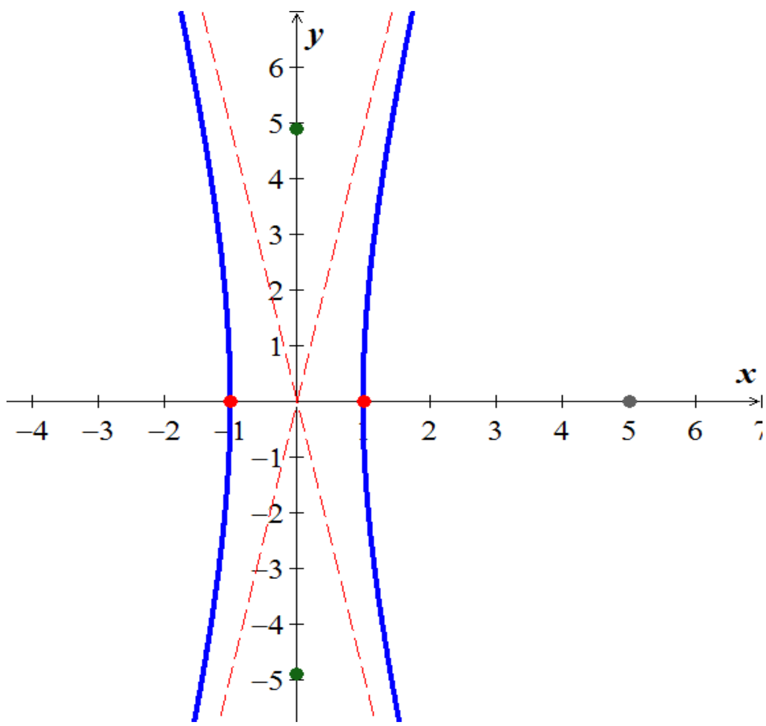
Center: $C = (0,\ 0)$

Vertices: $V = (\pm 1,\ 0)$

Endpoints: $W = \left(0,\ \pm 2\sqrt{6}\right)$

Foci: $F = (\pm 5,\ 0)$

Equations of the ***asymptotes***: $y = \pm 4\sqrt{3}x$ $\quad y = \pm \frac{b}{a} x$



Exercise

Find the ***center***, ***vertices***, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $y^2 - 4x^2 = 16$

Solution

$$\frac{y^2}{16} - \frac{x^2}{4} = 1$$

$$\begin{cases} a^2 = 16 \to a = 4 \\ b^2 = 4 \to b = 2 \end{cases}$$

$$\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{16+4} = 2\sqrt{5}$$

Center: $C = (0,\ 0)$

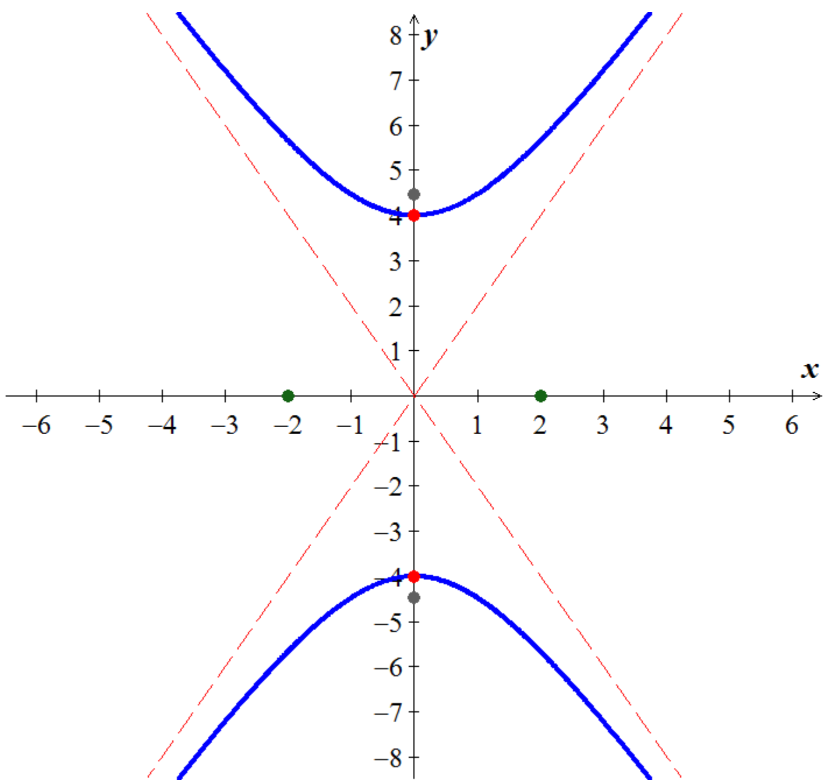
Vertices: $V = (0,\ \pm 4)$

Endpoints: $W = (\pm 2,\ 0)$

Foci: $F = \left(0,\ \pm 2\sqrt{5}\right)$

Equations of the ***asymptotes***:

$y = \pm \frac{4}{2}x \ \underline{= \pm 2x}$ $\qquad\qquad y = \pm \frac{a}{b}x$



Exercise

Find the ***center***, ***vertices***, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $16x^2 - 36y^2 = 1$

Solution

$$\frac{x^2}{\frac{1}{16}} - \frac{y^2}{\frac{1}{36}} = 1$$

$$\rightarrow \begin{cases} a^2 = \frac{1}{16} \rightarrow a = \frac{1}{4} \\ b^2 = \frac{1}{36} \rightarrow b = \frac{1}{6} \end{cases}$$

$$\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{\frac{1}{16} + \frac{1}{36}} = \sqrt{\frac{9+4}{144}} = \pm\frac{\sqrt{13}}{12}$$

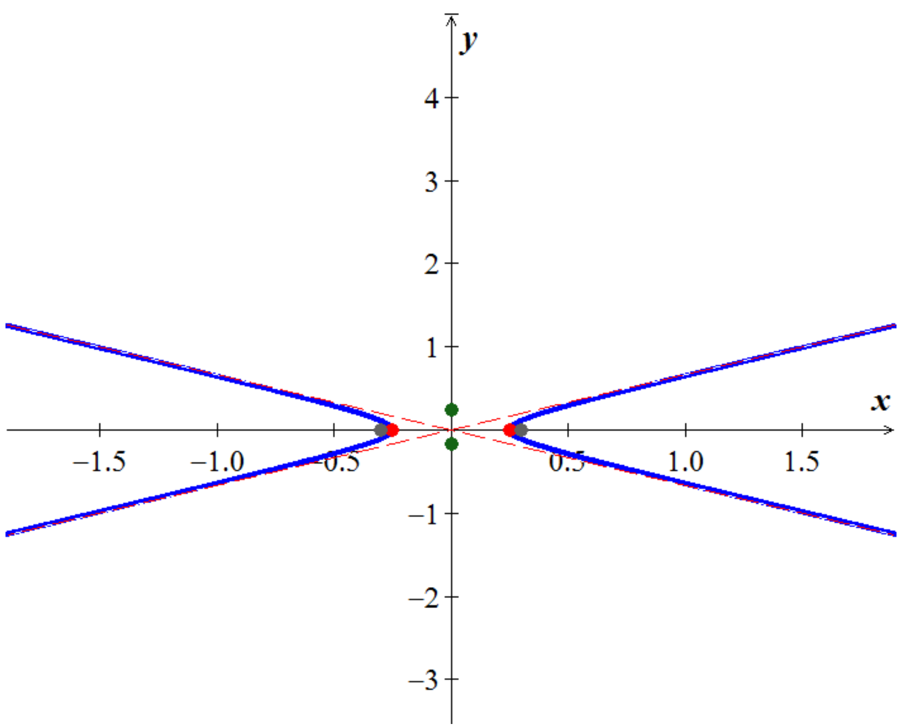
Center: $C = (0,\ 0)$

Vertices: $V = \left(\pm\frac{1}{4},\ 0\right)$

Endpoints: $W = \left(0,\ \pm\frac{1}{6}\right)$

Foci: $F = \left(\pm\frac{\sqrt{13}}{12},\ 0\right)$

Equations of the ***asymptotes***: $y = \pm\frac{\frac{1}{6}}{\frac{1}{4}}x = \pm\frac{4}{6}x \ \underline{= \pm\frac{2}{3}x}$ $\qquad y = \pm\frac{b}{a}x$



Exercise

Find the ***center***, ***vertices***, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $\frac{(y+2)^2}{9}-\frac{(x+2)^2}{4}=1$

Solution

$$\rightarrow\begin{cases} a^2=9\rightarrow a=3 \\ b^2=4\rightarrow b=2 \end{cases}$$

$$\Rightarrow c=\sqrt{a^2+b^2}=\sqrt{9+4}=\pm\sqrt{13}$$

Center: $C=(-2,\ -2)$

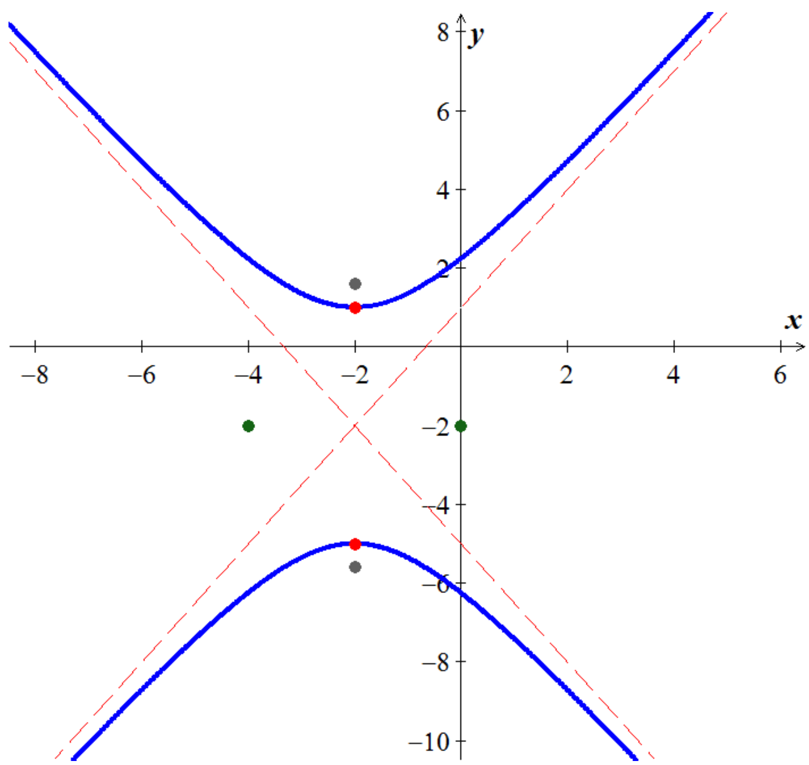
Vertices: $V=(-2,\ -2\pm 3)$

Endpoints: $W=(-2\pm 2,\ -2)$

Foci: $F=\left(-2,\ -2\pm\sqrt{13}\right)$

*Equations of the **asymptotes***: $y+2=\pm\frac{a}{b}(x+2)=\pm\frac{3}{2}(x+2)$

$y+2=-\frac{3}{2}(x+2)$	$y+2=\frac{3}{2}(x+2)$
$y+2=-\frac{3}{2}x-3$	$y+2=\frac{3}{2}x+3$
$y=-\frac{3}{2}x-5$	$y=\frac{3}{2}x+1$



Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $\frac{(x-2)^2}{4}-\frac{(y+3)^2}{9}=1$

Solution

$$\begin{cases} a^2=4 \to a=\pm 2 \\ b^2=9 \to b=\pm 3 \\ \hline c=\pm\sqrt{a^2+b^2}=\pm\sqrt{9+4}=\pm\sqrt{13} \end{cases}$$

Center: $C=(2,\ -3)$

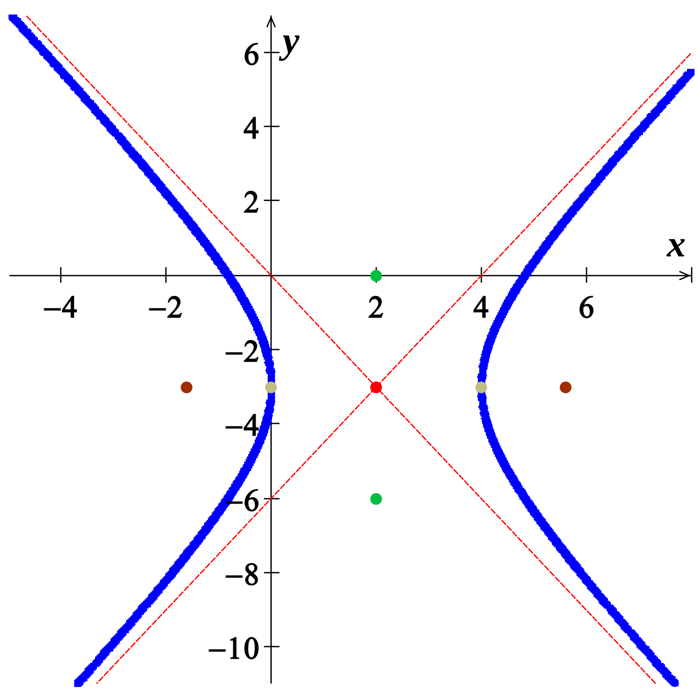
Vertices: $(2\pm 2,\ -3) \to \ V'=(0,\ -3) \quad V=(4,\ -3)$

Endpoints: $(2,\ -3\pm 3) \to \ W'(2,\ -6) \quad W=(2,\ 0)$

Foci: $F=\left(2\pm\sqrt{13},\ -3\right)$

Equations of the ***asymptotes***: $y+3=\pm\frac{b}{a}(x-2)=\pm\frac{3}{2}(x-2)$

$y+3=-\frac{3}{2}(x-2)$	$y+3=\frac{3}{2}(x-2)$
$y+3=-\frac{3}{2}x+3$	$y+3=\frac{3}{2}x-3$
$y=-\frac{3}{2}x$	$y=\frac{3}{2}x-6$



Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $(y-2)^2-4(x+2)^2=4$

Solution

$$\frac{(y-2)^2}{4}-\frac{4(x+2)^2}{4}=1$$

$$\frac{(y-2)^2}{4}-\frac{(x+2)^2}{1}=1$$

$$\begin{cases} a^2=4 \to a=\pm 2 \\ b^2=1 \to b=\pm 1 \\ c=\pm\sqrt{a^2+b^2}=\pm\sqrt{4+1}=\pm\sqrt{5} \end{cases}$$

Center: $C=(-2,\ 2)$

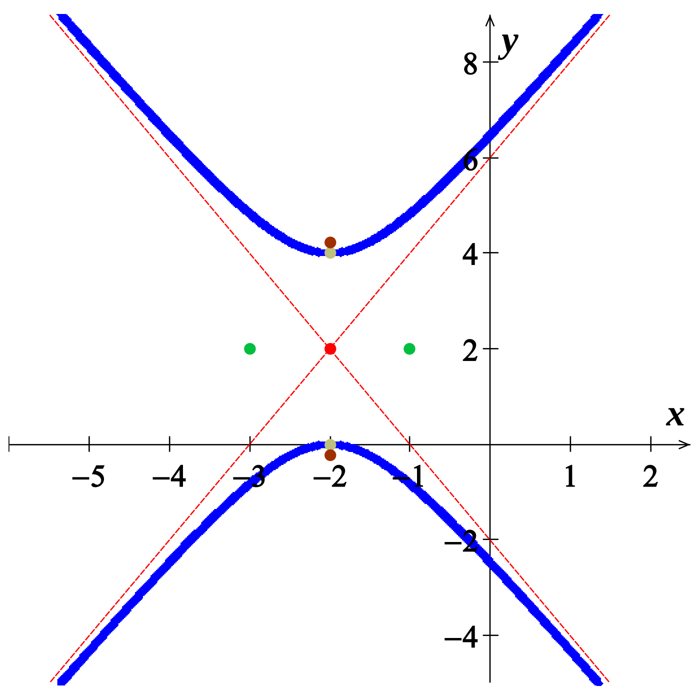
Vertices: $(-2,\ 2\pm 2)\to\ V'=(-2,\ 0)\quad V=(-2,\ 4)$

Endpoints: $(-2\pm 1,\ 2)\to\ W'=(-3,\ 2)\quad W=(-1,\ 2)$

Foci: $F=\left(-2,\ 2\pm\sqrt{5}\right)$

Equations of the ***asymptotes***: $y-2=\pm\frac{a}{b}(x+2)=\pm\frac{2}{1}(x+2)$

$y-2=-2(x+2)$	$y-2=2(x+2)$
$y-2=-2x-4$	$y-2=2x+4$
$y=-2x-2$	$y=2x+6$



Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $(x+4)^2-9(y-3)^2=9$

Solution

$$\frac{(x+4)^2}{9}-\frac{9(y-3)^2}{9}=1$$

$$\frac{(x+4)^2}{9}-\frac{(y-3)^2}{1}=1$$

$$\begin{cases} a^2=9 \to a=\pm 3 \\ b^2=1 \to b=\pm 1 \\ c=\pm\sqrt{a^2+b^2}=\pm\sqrt{9+1}=\pm\sqrt{10} \end{cases}$$

Center: $C=(-4,\ 3)$

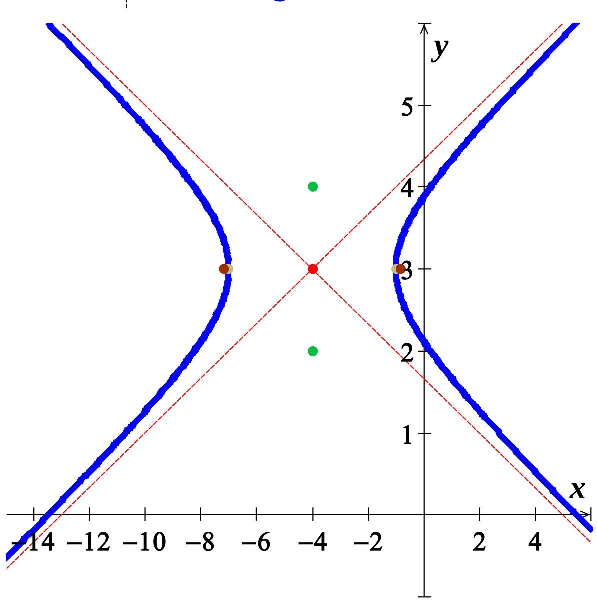
Vertices: $(-4\pm 3,\ 3)\to\ V'=(-7,\ 3)\quad V=(-1,\ 3)$

Endpoints: $(-4,\ 3\pm 1)\to\ W'(-4,\ 2)\quad W=(-4,\ 4)$

Foci: $F=\left(-4\pm\sqrt{10},\ 3\right)$

Equations of the ***asymptotes***: $y-3=\pm\frac{b}{a}(x+4)=\pm\frac{1}{3}(x+4)$

$y-3=-\frac{1}{3}(x+4)$	$y-3=\frac{1}{3}(x+4)$
$y-3=-\frac{1}{3}x-\frac{4}{3}$	$y-3=\frac{1}{3}x+\frac{4}{3}$
$y=-\frac{1}{3}x+\frac{5}{3}$	$y=\frac{1}{3}x+\frac{13}{3}$



Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $144x^2 - 25y^2 + 864x - 100y - 2404 = 0$

Solution

$$144\left(x^2 + 6x + \left(\frac{6}{2}\right)^2\right) - 25\left(y^2 + 4y + \left(\frac{4}{2}\right)^2\right) = 2404 + 144(4) - 25(4)$$

$$144(x+3)^2 - 25(y+2)^2 = 3600$$

$$\frac{(x+3)^2}{25} - \frac{(y+2)^2}{144} = 1$$

$$\rightarrow \begin{cases} a^2 = 25 \rightarrow a = 5 \\ b^2 = 144 \rightarrow b = 12 \end{cases}$$

$$\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{25 + 144}$$
$$\underline{= \pm 13|}$$

Center: $C = (-3,\ -2)$

Vertices: $V = (-3 \pm 5,\ -2)$

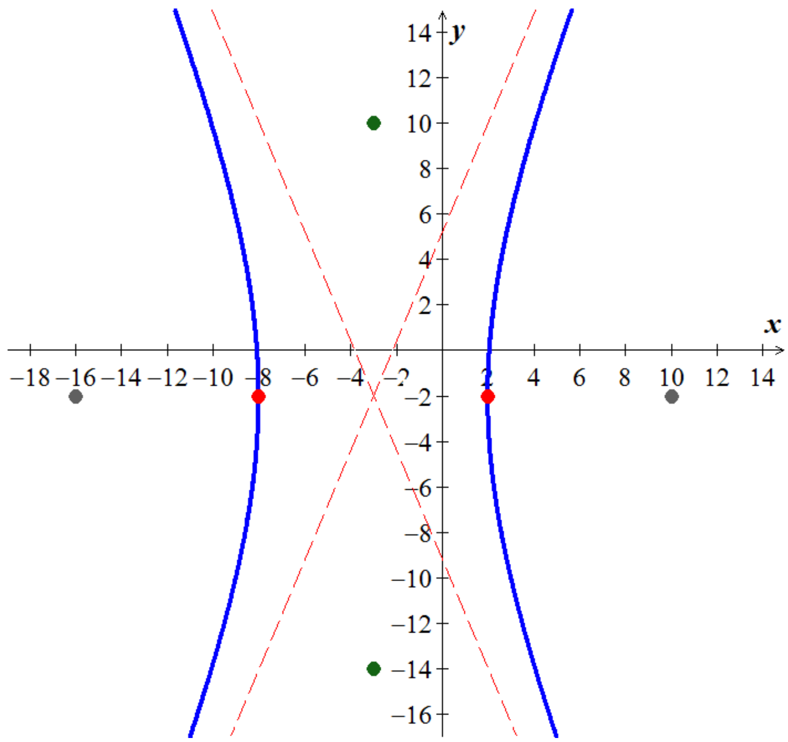
Endpoints: $W = (-3,\ -2 \pm 12)$

Foci: $F = (-3 \pm 13,\ -2)$

Equations of the ***asymptotes***:

$$y + 2 = \pm\frac{b}{a}(x+3) = \pm\frac{12}{5}(x+3)$$

$y + 2 = -\frac{12}{5}(x+3)$	$y + 2 = \frac{12}{5}(x+3)$
$y + 2 = -\frac{12}{5}x - \frac{36}{5}$	$y + 2 = \frac{12}{5}x + \frac{36}{5}$
$y = -\frac{12}{5}x - \frac{46}{5}$	$y = \frac{12}{5}x + \frac{26}{5}$



Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $4y^2 - x^2 + 40y - 4x + 60 = 0$

Solution

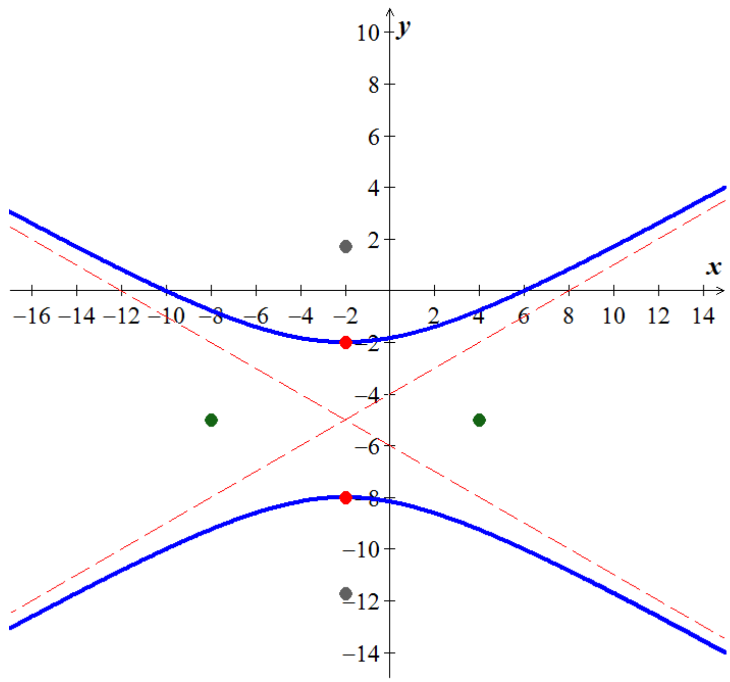
$$4\left(y^2 + 10y + \left(\tfrac{10}{2}\right)^2\right) - \left(x^2 + 4x + \left(\tfrac{4}{2}\right)^2\right) = -60 + 4(25) - (4)$$

$$4(y+5)^2 - (x+2)^2 = 36$$

$$\frac{(y+5)^2}{9} - \frac{(x+2)^2}{36} = 1$$

$$\rightarrow \begin{cases} a^2 = 9 \rightarrow a = 3 \\ b^2 = 36 \rightarrow b = 6 \end{cases}$$

$$\Rightarrow c = \pm\sqrt{a^2 + b^2} = \pm\sqrt{9+36}$$
$$= \pm\sqrt{45}$$
$$= \pm 3\sqrt{5}$$



Center: $C = (-5, \ -2)$

Vertices: $V = (-2, \ -5 \pm 3)$

Endpoints: $W = (-2 \pm 6, \ -5)$

Foci: $F = \left(-2, \ -5 \pm 3\sqrt{5}\right)$

*Equations of the **asymptotes***:

$$y + 5 = \pm\frac{a}{b}(x+2) = \pm\frac{3}{6}(x+2) = \pm\frac{1}{2}(x+2)$$

$y + 5 = -\frac{1}{2}(x+2)$	$y + 5 = \frac{1}{2}(x+2)$
$y + 5 = -\frac{1}{2}x - 1$	$y + 5 = \frac{1}{2}x + 1$
$y = -\frac{1}{2}x - 6$	$y = \frac{1}{2}x - 4$

Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $4x^2 - 16x - 9y^2 + 36y = -16$

Solution

$$4\left(x^2 - 4x\right) - 9\left(y^2 - 4y\right) = -16$$

$$4\left(x^2 - 4x + 2^2\right) - 9\left(y^2 - 4y + 2^2\right) = -16 + 4\left(2^2\right) - 9\left(2^2\right)$$

$$4(x-2)^2 - 9(y-2)^2 = -16 + 16 - 36$$

$$4(x-2)^2 - 9(y-2)^2 = -36$$

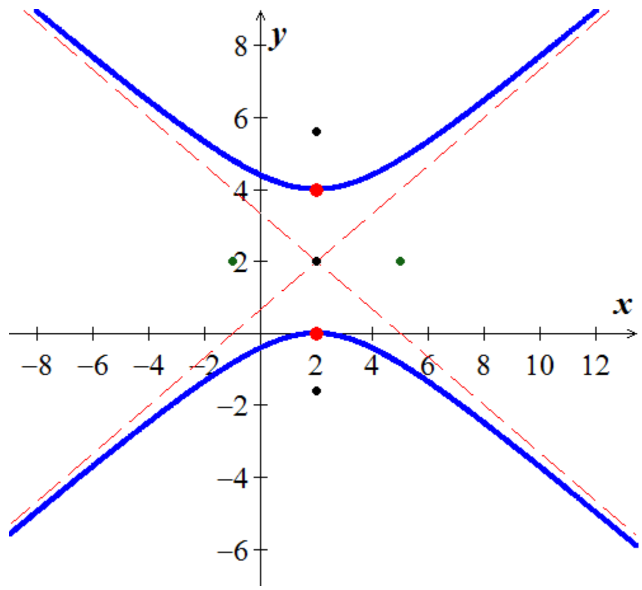
$$\frac{4(x-2)^2}{-36} - \frac{9(y-2)^2}{-36} = 1$$

$$-\frac{4(x-2)^2}{36} + \frac{9(y-2)^2}{36} = 1$$

$$\frac{9(y-2)^2}{36} - \frac{4(x-2)^2}{36} = 1$$

$$\frac{(y-2)^2}{\frac{36}{9}} - \frac{(x-2)^2}{\frac{36}{4}} = 1$$

$$\frac{(y-2)^2}{4} - \frac{(x-2)^2}{9} = 1$$



$$\rightarrow \begin{cases} a^2 = 4 \rightarrow a = \pm 2 \\ b^2 = 9 \rightarrow b = \pm 3 \end{cases}$$

$$\Rightarrow c = \mp\sqrt{a^2 + b^2} = \pm\sqrt{9+4} = \pm\sqrt{13}$$

Center: $(2,\ 2)$

The ***endpoints***: $(2 \pm 3,\ -2) \Rightarrow (-1,\ 2) \quad (5,\ 2)$

The ***vertices***: $(2,\ 2 \pm 2) \Rightarrow (2,\ 0) \quad (2,\ 4)$

The ***foci*** are $\left(2,\ 2 \pm \sqrt{13}\right)$

The equations of the ***asymptotes*** are: $y - 2 = \pm\frac{a}{b}(x-2) \ \Rightarrow \underline{y = \pm\frac{2}{3}(x-2) + 2}$

Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $2x^2 - y^2 + 4x + 4y = 4$

Solution

$$2\left(x^2 + 2x + \left(\frac{2}{2}\right)^2\right) - \left(y^2 - 4y + \left(\frac{-4}{2}\right)^2\right) = 4 + 2\left(\frac{2}{2}\right)^2 + (-1)\left(\frac{-4}{2}\right)^2$$

$$2(x+1)^2 - (y-2)^2 = 4 + 2 - 4$$

$$2(x+1)^2 - (y-2)^2 = 2$$

$$\frac{(x+1)^2}{1} - \frac{(y-2)^2}{2} = 1$$

$$\begin{cases} a^2 = 1 \to a = \pm 1 \\ b^2 = 2 \to b = \pm\sqrt{2} \\ \hline c = \pm\sqrt{a^2 + b^2} = \pm\sqrt{1+2} = \pm\sqrt{3} \end{cases}$$

Center: $C = (-1,\ 2)$

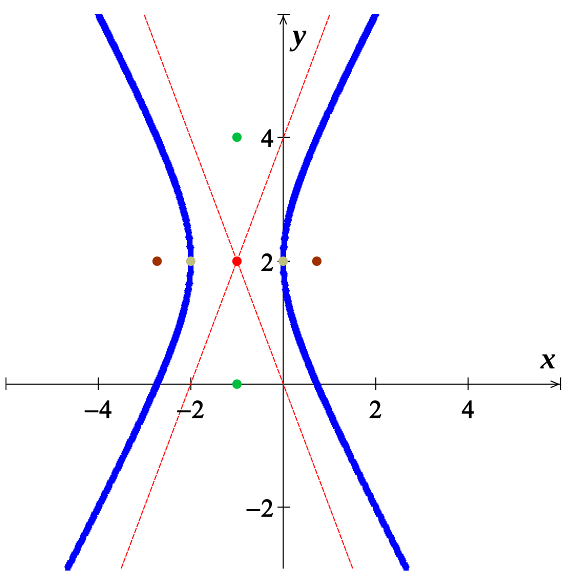
Vertices: $(-1 \pm 1,\ 2) \to \quad V' = (-2,\ 2) \quad V = (0,\ 2)$

Endpoints: $(-1,\ 2 \pm 2) \to \quad W' = (-1,\ 0) \quad W = (-1,\ 4)$

Foci: $F = \left(-1 \pm \sqrt{3},\ 2\right)$

Equations of the ***asymptotes***: $y - 2 = \pm\frac{b}{a}(x+1) = \pm\frac{2}{1}(x+1)$

$y - 2 = -2(x+1)$	$y - 2 = 2(x+1)$
$y - 2 = -2x - 2$	$y - 2 = 2x + 2$
$y = -2x$	$y = 2x + 4$



Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $2y^2 - x^2 + 2x + 8y + 3 = 0$

Solution

$$2y^2 + 8y - x^2 + 2x = -3$$

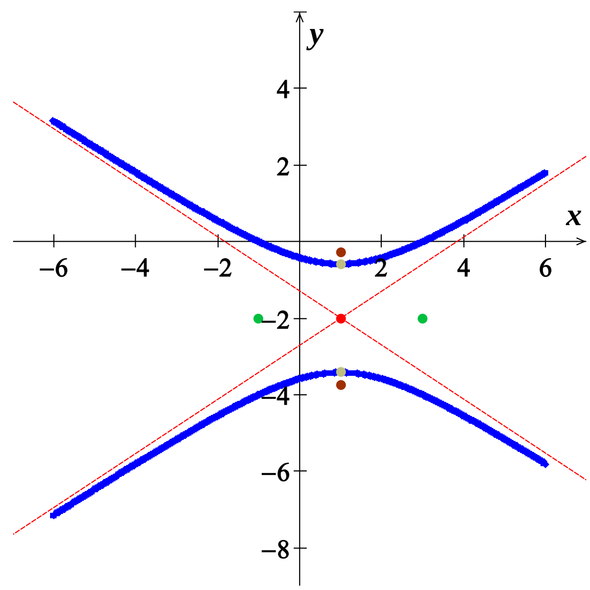
$$2\left(y^2 + 4y + \left(\frac{4}{2}\right)^2\right) - \left(x^2 - 2x + \left(\frac{-2}{2}\right)^2\right) = -3 + 2\left(\frac{4}{2}\right)^2 + (-1)\left(\frac{-2}{2}\right)^2$$

$$2(y+2)^2 - (x-1)^2 = -3 + 8 - 1$$

$$2(y+2)^2 - (x-1)^2 = 4$$

$$\frac{(y+2)^2}{2} - \frac{(x-1)^2}{4} = 1$$

$$\begin{cases} a^2 = 2 \to a = \pm\sqrt{2} \\ b^2 = 4 \to b = \pm 2 \\ c = \pm\sqrt{a^2 + b^2} = \pm\sqrt{2+4} = \pm\sqrt{6} \end{cases}$$



Center: $C = (1, \ -2)$

Vertices: $V = \left(1, \ -2 \pm \sqrt{2}\right)$

Endpoints: $(1 \pm 2, \ -2) \to \ W' = (-1, \ -2) \quad W = (3, \ -2)$

Foci: $F = \left(1, \ -2 \pm \sqrt{3}\right)$

Equations of the ***asymptotes***: $y + 2 = \pm\frac{a}{b}(x-1) = \pm\frac{\sqrt{2}}{2}(x-1)$

$$y + 2 = -\frac{\sqrt{2}}{2}(x-1) \qquad y + 2 = \frac{\sqrt{2}}{2}(x-1)$$

$$y = -\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2} - 2 \qquad y = \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2} - 2$$

Exercise

Find the ***center***, **vertices**, ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci. $2y^2 - 4x^2 - 16x - 2y - 19 = 0$

Solution

$$2y^2 - 2y - 4x^2 - 16x = 19$$

$$2\left(y^2 - y + \left(\frac{-1}{2}\right)^2\right) - 4\left(x^2 + 4x + \left(\frac{4}{2}\right)^2\right) = 19 + 2\left(\frac{-1}{2}\right)^2 - 4\left(\frac{4}{2}\right)^2$$

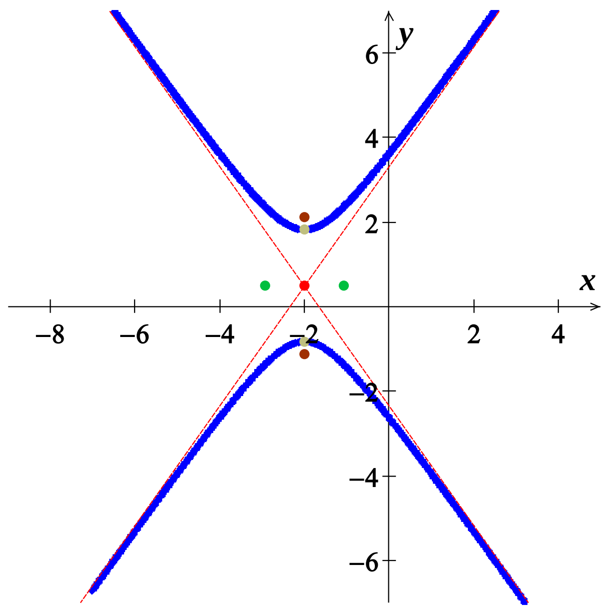
$$2\left(y - \frac{1}{2}\right)^2 - 4(x+2)^2 = 19 + \frac{1}{2} - 16$$

$$2\left(y - \frac{1}{2}\right)^2 - 4(x+2)^2 = \frac{7}{2}$$

$$\frac{2\left(y - \frac{1}{2}\right)^2}{\frac{7}{2}} - \frac{4(x+2)^2}{\frac{7}{2}} = 1$$

$$\frac{\left(y - \frac{1}{2}\right)^2}{\frac{7}{4}} - \frac{4(x+2)^2}{\frac{7}{8}} = 1$$

$$\begin{cases} a^2 = \frac{7}{4} \to a = \pm\frac{\sqrt{7}}{2} \\ b^2 = \frac{7}{8} \to b = \pm\frac{\sqrt{7}}{2\sqrt{2}} = \pm\frac{\sqrt{14}}{4} \\ c = \pm\sqrt{a^2 + b^2} = \pm\sqrt{\frac{7}{4} + \frac{7}{8}} = \pm\sqrt{\frac{21}{8}} \end{cases}$$



Center: $C = \left(-2,\ \frac{1}{2}\right)$

Vertices: $V = \left(1,\ -2 \pm \frac{\sqrt{7}}{2}\right)$

Endpoints: $W\left(-2 \pm \frac{\sqrt{14}}{4},\ \frac{1}{2}\right)$

Foci: $F = \left(-2,\ \frac{1}{2} \pm \sqrt{\frac{21}{8}}\right)$

*Equations of the **asymptotes***: $y - \frac{1}{2} = \pm\frac{a}{b}(x+2) = \pm\frac{\frac{\sqrt{7}}{2}}{\frac{\sqrt{7}}{2\sqrt{2}}}(x+2) = \pm\sqrt{2}(x+2)$

$y - \frac{1}{2} = -\sqrt{2}(x+2)$	$y - \frac{1}{2} = \sqrt{2}(x+2)$
$y = -\sqrt{2}x - 2\sqrt{2} + \frac{1}{2}$	$y = \sqrt{2}x + 2\sqrt{2} + \frac{1}{2}$

Exercise

Suppose a hyperbola has center at the origin, foci at $F'(-c,\ 0)$ and $F(c,\ 0)$, and equation $d(P,\ F') - d(P,\ F) = 2a$. Let $b^2 = c^2 - a^2$, and show that an equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Solution

$$d(P,\ F') - d(P,\ F) = 2a \qquad\qquad d = \sqrt{\left(x - x_0\right)^2 + \left(y - y_0\right)^2}$$

$$\sqrt{(x+c)^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2} = 2a$$

$$\sqrt{x^2 + 2cx + c^2 + y^2} - \sqrt{x^2 - 2cx + c^2 + y^2} = 2a$$

$$\sqrt{x^2 + 2cx + c^2 + y^2} = 2a + \sqrt{x^2 - 2cx + c^2 + y^2}$$

$$\left(\sqrt{x^2 + 2cx + c^2 + y^2}\right)^2 = \left(2a + \sqrt{x^2 - 2cx + c^2 + y^2}\right)^2$$

Square both sides

$$x^2 + 2cx + c^2 + y^2 = 4a^2 + x^2 - 2cx + c^2 + y^2 + 4a\sqrt{x^2 - 2cx + c^2 + y^2}$$

$$4cx - 4a^2 = 4a\sqrt{x^2 - 2cx + c^2 + y^2}$$

Divide by 4

$$cx - a^2 = a\sqrt{x^2 - 2cx + c^2 + y^2}$$

$$\left(cx - a^2\right)^2 = \left(a\sqrt{x^2 - 2cx + c^2 + y^2}\right)^2$$

Square both sides

$$c^2x^2 - 2a^2cx + a^4 = a^2\left(x^2 - 2cx + c^2 + y^2\right)$$

$$c^2x^2 - 2a^2cx + a^4 = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2$$

$$c^2x^2 - a^2x^2 - a^2y^2 = a^2c^2 - a^4$$

$$\left(c^2 - a^2\right)x^2 - a^2y^2 = a^2\left(c^2 - a^2\right)$$

$$\frac{\left(c^2 - a^2\right)x^2}{a^2\left(c^2 - a^2\right)} - \frac{a^2y^2}{a^2\left(c^2 - a^2\right)} = 1$$

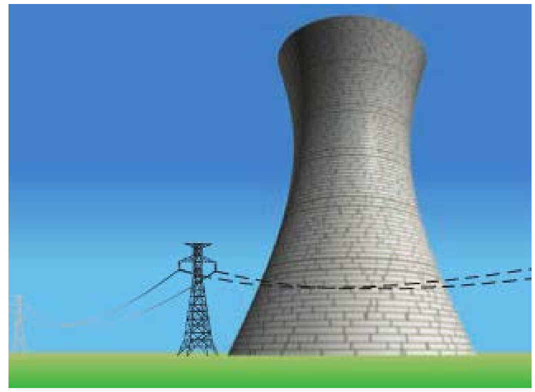
$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1 \qquad\qquad b^2 = c^2 - a^2$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Exercise

A cooling tower is a hydraulic structure. Suppose its base diameter is 100 *meters* and its smallest diameter of 48 *meters* occurs 84 *meters* from the base. If the tower is 120 *meters* high approximate its diameter at the top.

Solution



Given: $a = \frac{48}{2} = 24$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \rightarrow \quad \frac{x^2}{24^2} - \frac{y^2}{b^2} = 1$$

At the point $(50, \ -84)$:

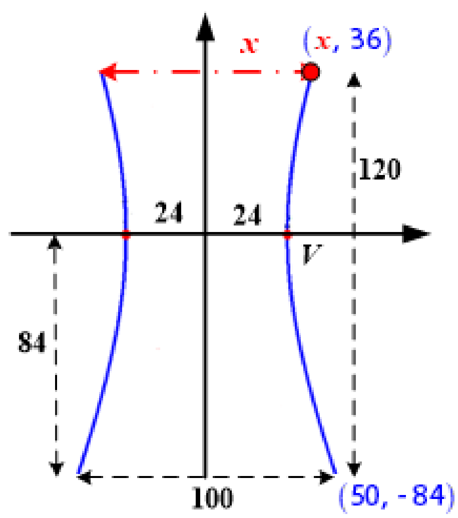
$$\frac{50^2}{24^2} - \frac{(-84)^2}{b^2} = 1$$

$$\frac{50^2}{24^2} - 1 = \frac{84^2}{b^2}$$

$$\frac{50^2 - 24^2}{24^2} = \frac{84^2}{b^2}$$

$$b^2 = \frac{84^2 \cdot 24^2}{50^2 - 24^2} = 2112.4$$

$$\Rightarrow \quad \frac{x^2}{576} - \frac{y^2}{2112.4} = 1$$



At the point $(x, \ 36)$:

$$\frac{x^2}{576} - \frac{36^2}{2112.4} = 1$$

$$\frac{x^2}{576} = 1 + \frac{1296}{2112.4}$$

$$\frac{x^2}{576} = 1.61$$

$$x^2 = 929.45$$

$$x = \sqrt{929.45} \approx 30.49$$

The diameter at the top: $= 2x = \underline{60.97 \ m.}$

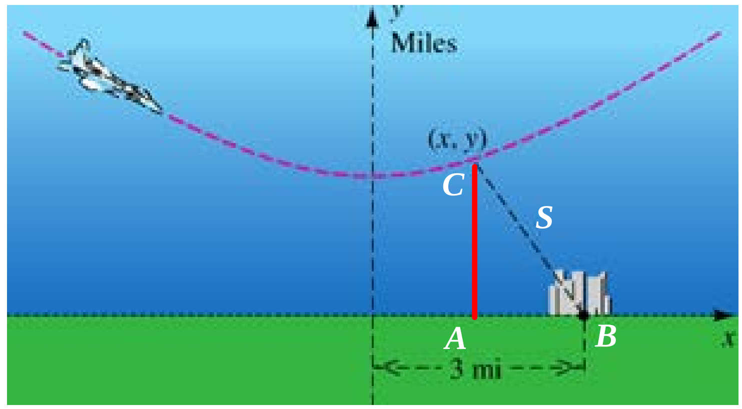
Exercise

An airplane is flyting along the hyperbolic path. If an equation of the path is $2y^2 - x^2 = 8$, determine how close the airplane comes to town located at (3, 0). (Hunt: Let S denote the square of the distance from a point (x, y) on the path to (3, 0), and find the minimum value of S.)

Solution

$$2y^2 - x^2 = 8$$

$$y^2 = \frac{1}{2}x^2 + 4$$



$$\begin{aligned} S^2 &= (3-x)^2 + y^2 \\ &= 9 - 6x + x^2 + \frac{1}{2}x^2 + 4 \\ &= \frac{3}{2}x^2 - 6x + 13 \end{aligned}$$

The vertex point of S^2

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{-6}{2\left(\frac{3}{2}\right)} \\ &= 2 \end{aligned}$$

$$\begin{aligned} S^2 &= \frac{3}{2}(2)^2 - 6(2) + 13 \\ &= 7 \end{aligned}$$

Therefore the close the town to the airplane is $S = \sqrt{7}\ miles$

Exercise

A ship is traveling a course that is 100 *miles* from, and parallel tom a straight shoreline. The ship sends out a distress signal that is received by two Coast Guard stations A and B, located 200 *miles* apart. By measuring the difference in signal reception times, it is determined that the ship is 160 *miles* closer to B than to A. Where is the ship?

Solution

Given: $c = 100$ *and* $BC = AC - 160$

$$d_1 - d_2 = 160 = 2a$$

$$a = 80$$

$$\begin{aligned} b^2 &= c^2 - a^2 \\ &= 100^2 - 80^2 \\ &= 3600 \end{aligned}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \to \quad \frac{x^2}{6400} - \frac{y^2}{3600} = 1$$

At point C(x, 100):

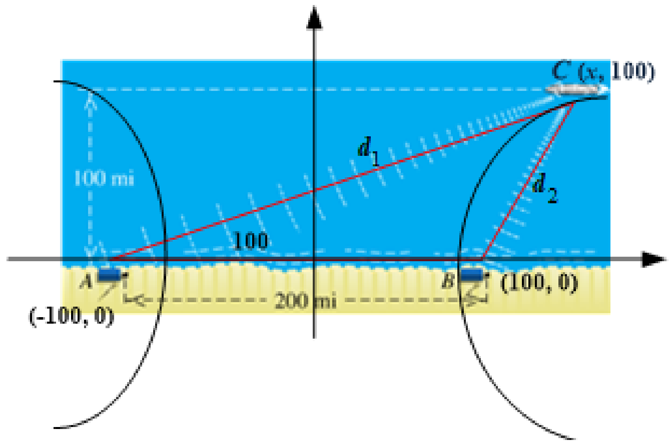
$$\frac{x^2}{6400} - \frac{100^2}{3600} = 1$$

$$\frac{x^2}{6400} = 1 + \frac{100^2}{3600}$$

$$x^2 = 6400\left(1 + \frac{100^2}{3600}\right)$$

$$x = 80\sqrt{\frac{13600}{3600}}$$

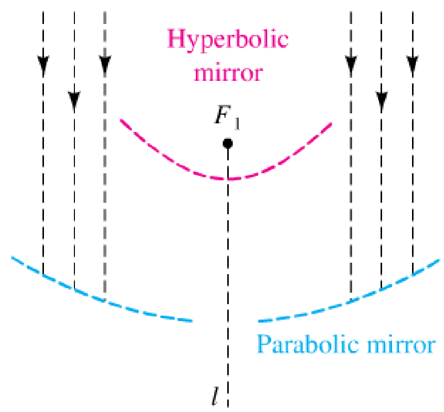
$$= \frac{80}{3}\sqrt{34}$$



The ship position is $\left(\frac{80}{3}\sqrt{34},\ 100\right) = \underline{(155.5,\ 100)}$

Exercise

The Cassegrain telescope design (dating back to 1672) makes use of the reflective properties of both the parabola and the hyperbola. The figure shows a (***split***) parabolic mirror, with one focus at F_1 and axis along the line l, and a hyperbolic mirror, with one focus also at F_1 and transverse axis along l. Where do incoming light waves parallel to the common axis finally collect?



Solution

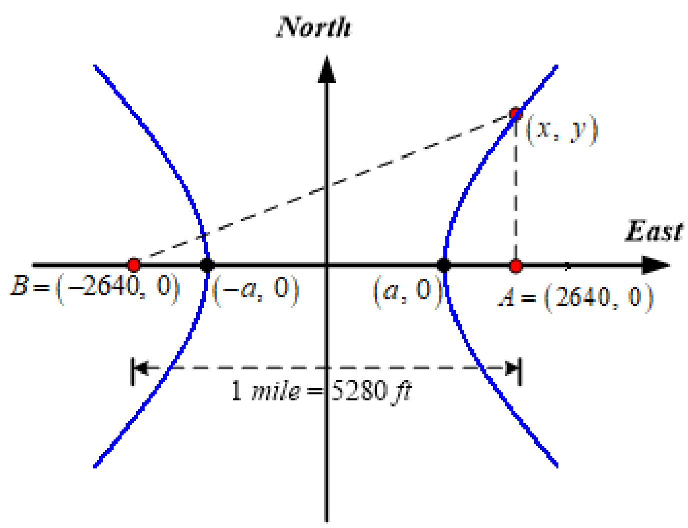
Exterior focus of hyperbolic mirror (below parabolic mirror)

Exercise

Suppose that two people standing 1 *mile* apart both see a flash of lightning. After a period of time, the person standing at point *A* hears the thunder. One second later, the person standing at point *B* hears the thunder. If the person at *B* is due west of the person at *A* and the lightning strike is known to occur due north of the person standing at point *A*, where did the lightning strike occur?
(Sound travels at 1100 ft / sec and 1 $mile = 5280\ ft$)

Solution

Person ***A*** is 1100 *feet* closer to the lightning strike than the person at point ***B***.
Distance from (x, y) to ***B*** ***minus*** distance from (x, y) to ***A*** is 1100.
The point (x, y) lies on a hyperbola whose foci are at ***A*** and ***B***.



$2a = 1100 \quad \Rightarrow \quad \underline{a = 550}$

$2c = 5280 \quad \Rightarrow \quad \underline{c = 2640}$

$$\begin{aligned} b^2 &= c^2 - a^2 \\ &= 2640^2 - 550^2 \\ &= 6{,}667{,}100 \end{aligned}$$

An equation of the hyperbola:

$$\frac{x^2}{550^2} - \frac{y^2}{6{,}667{,}100} = 1 \qquad\qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

At point $A = (2640,\ 0)$, let $x = 2640$, and solve for *y* at that *x* value:

$$\frac{2640^2}{550^2} - \frac{y^2}{6{,}667{,}100} = 1$$

$$y^2 = 6{,}667{,}100\left(\frac{2640^2}{550^2} - 1\right)$$

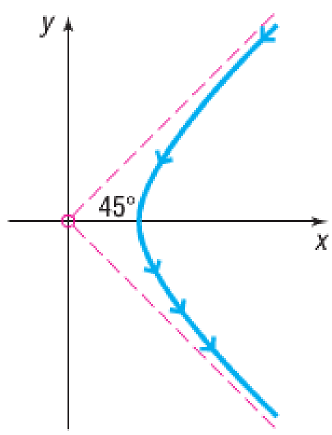
$$y = \sqrt{6{,}667{,}100\left(\frac{2640^2}{550^2} - 1\right)}$$

$$\boxed{= 12{,}122}$$

he lightning strike occurred 12,122 *feet*. north of the person standing at point *A*.

Exercise

Ernest Rutherford published a paper that he described the motion of alpha particles as they are shot at a piece of gold foil 0.00004 *cm* thick. Before conducting this experiment, Rutherford expected that the alpha particles would shoot through the foil just as a bullet would shoot through the foil just as a bullet would shoot through snow. Instead, a small fraction of the alpha particles bounced off the foil. This led to the conclusion that the nucleus of an atom is dense, while the remainder of the atom is sparse. Only the density of the nucleus could cause the alpha particles to deviate from their path. The figure shows a diagram from Rutherford's paper that indicates that the deflected alpha particles follow the path of one branch of a hyperbola.



a) Find an equation of the asymptotes under this scenario.

b) If the vertex of the path of the alpha particles is 10 *cm* from the center of the hyperbola, find a model that describes the path of the particle.

Solution

a) Since the particles are deflected at a 45° angle, the asymptotes will be $\boxed{y = \pm x}$

b) Since the vertex is 10 *cm* from the center of the hyperbola, so $a = 10$

The slope of the asymptotes is given by $\pm\frac{b}{a}$

Therfore: $\frac{b}{a} = 1 \quad \rightarrow \quad b = a = 10$

The equation of the particle path is: $\frac{x^2}{100} - \frac{y^2}{100} = 1 \quad (x \ge 0)$

Exercise

Hyperbolas have interesting reflective properties that make them useful for lenses and mirrors. For example, if a ray of light strikes a convex hyperbolic mirror on a line that would (theoretically) pass through its rear focus, it is reflected through its rear focus, it is reflected through the front focus. This property and that of the parabola were used to develop the *Cassegrain* telescope in 1672. The focus of the parabolic mirror and the rear focus of the hyperbolic mirror are the same point. The rays are collected by the parabolic mirror, reflected toward the common focus, and thus are reflected by the hyperbolic mirror through the opening to its front focus, where the eyepiece is located. If the equation of the hyperbola is $\frac{y^2}{9}-\frac{x^2}{16}=1$ and the focal length (distance from the vertex to the focus) of the parabola is 6, find the equation of the parabola.

Solution

Assume the origin lies at the center of the hyperbola. The foci of the hyperbola are located on *y*-axis at $(0,\ \pm c)$, since the hyperbola has a transverse axis that is parallel to the *y*-axis.

Given: $a^2=9 \quad and \quad b^2=16$

$c^2=a^2+b^2=9+6=25$

$c=\sqrt{25}\ \underline{=5}|$

Therefore, the foci of the foci of the hyperbola are at $(0,\ -5)$ & $(0,\ 5)$

Assume that he parabola opens up, the common focus is at $(0,\ 5)$.

The equation of the parabola: $x^2=4a(y-k)$

The focal length of the parabola is given as $a=6$

The distance focus of the parabola is located at $(0,\ k+a)=(0,\ 5)$

$$k+6=5 \quad \Rightarrow \quad k=1$$

The equation of the parabola becomes $x^2=4(8)(y-(-1))$

$\underline{x^2=24(y+1)}|$ *or* $y=\frac{1}{24}x^2-1$

Exercise

The ***eccentricity e*** of a hyperbola is defined as the number $\frac{c}{a}$, where *a* is the distance of a vertex from the center and *c* is the distance of a focus from the center. Because *c* > *a*, it follows that $e>1$. Describe the general shape of a hyperbola whose eccentricity is close to 1. What is the shape if *e* is very large?

Solution

Assume $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

If the eccentricity is close to 1, then $c\approx a$ and $b\approx 0$.

When b is close to 0, the hyperbola is very narrow, because the slopes of asymptotes are close to 0.

If the eccentricity is very large, then c is much larger than a and b. The result is a hyperbola is very wide, because the slopes of the asymptotes are very large.

For $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the opposite is true.

When the eccentricity is close to 1, the hyperbola is very wide because the slopes of the asymptotes are close to 0.

When the eccentricity is very large, the hyperbola is very narrow because the slopes of asymptotes are very large.

Exercise

An explosive is recorded by two microphone that are 1 *mile* apart. Microphone M_1 received the sound 2 *seconds* before microphone M_2. Assuming sound travels at 1,100 *feet* per *second*, determine the possible locations of the explosion relative to the location of the microphones.

Solution

$$\begin{aligned}\left|d_2-d_1\right| &= 2a\\ &= 2(1100)\\ &= 2,200\ ft\end{aligned}$$

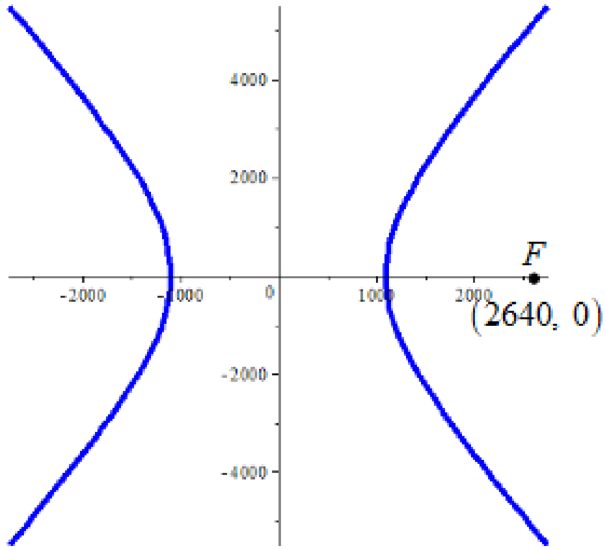
$$a=1,100\ ft$$

$$2c=1\ mi=5,280\ ft$$

$$c=2,640\ ft$$

$$\begin{aligned}b^2 &= c^2-a^2\\ &= 2640^2-1100^2\\ &= 5,759,600\end{aligned}$$

$$\frac{x^2}{1,210,000}-\frac{y^2}{5,759,600}=1 \qquad \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$



If M_1 is located 2640 *feet* to the right of the origin on the *x-axis*, the explosive is located on the right branch of the hyperbola given by the driven equation.

Exercise

Radio towers *A* and *B*, 200 *km* apart, are situated along the coast, with *A* located due west of *B*. Simultaneous radio signals are sent from each tower to a ship, with the signal from *B* received $500\ \mu\text{sec}$ before the signal from *A*.

a) Assuming that the radio signals travel $300\ m/\mu\text{sec}$, determine the equation of the hyperbola on which the ship is located.

b) If the ship lies due north of tower *B*, how far out at sea is it?

Solution

a) $2c = 200\ km \quad \rightarrow \quad c = 100\ km$

$$\left|d_2 - d_1\right| = (500\ \mu\text{sec})\left(300\ \frac{m}{\mu\text{sec}}\right)$$

$$2a = 150,000\ m$$

$$a = \underline{75,000\ m = 75\ km}$$

$$b^2 = c^2 - a^2$$

$$= 100^2 - 75^2$$

$$= \underline{4,375}$$

$$\frac{x^2}{(75)^2} - \frac{y^2}{4375} = 1 \qquad\qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\boxed{\frac{x^2}{5625} - \frac{y^2}{4375} = 1}$$

b) ***Given***: $x = 100\ km$

$$\frac{100^2}{5625} - \frac{y^2}{4375} = 1$$

$$y^2 = 4375\left(\frac{10,000}{5625} - 1\right)$$

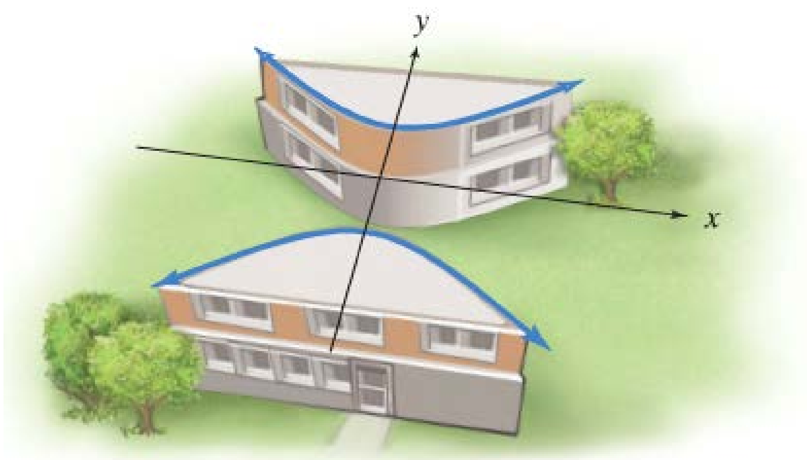
$$y = \pm\sqrt{4375\left(\frac{10,000}{5625} - 1\right)}$$

$$\approx \pm 58.3$$

The ship is about 58.3 *km* from the coast.

Exercise

An architect designs two houses that are shaped and positioned like a part of the branches of the hyperbola whose equation is $625y^2 - 400x^2 = 250,000$, where x and y are in yards. How far apart are the houses at their closest point?



Solution

$$625y^2 - 400x^2 = 250,000$$

$$\frac{625}{250,000}y^2 - \frac{400}{250,000}x^2 = 1$$

$$\frac{y^2}{400} - \frac{x^2}{625} = 1$$

$$a^2 = 400 \rightarrow a = 20$$

$$2a = 40$$

The houses are 40 *yards* apart at their closest point.