Lecture Four

Section 4.1 – Matrix Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

Definition

If *V* and *W* are vector spaces, and if *f* is a function with domain *V* and codomain *W*, then we say that *f* is a transformation from *V* to *W* or that f maps *V* to *W*, which we denote by writing

$$f : V \to W$$

In the special case where *V* = *W*, the transformation is also called an operator on *V*.

Matrix Transformation

$$\begin{aligned} w_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ w_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \quad & \quad \vdots \qquad \quad \vdots \qquad \qquad \vdots \\ w_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned}$$

Which we can write in matrix formation

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\vec{w} = A\vec{x}$$

Although we could view this as a linear system, we will view it instead as a transformation that maps the column vector $\vec{x}$ in $\mathbb{R}^n$ into the column vector $\vec{w}$ in $\mathbb{R}^m$ by multiplying $\vec{x}$ on the left by *A*. We call this a ***matrix transformation*** or ***function*** or ***mapping T*** from $\mathbb{R}^n$ to $\mathbb{R}^m$ (or ***matrix operator*** if *m* = *n*) and we denote it by

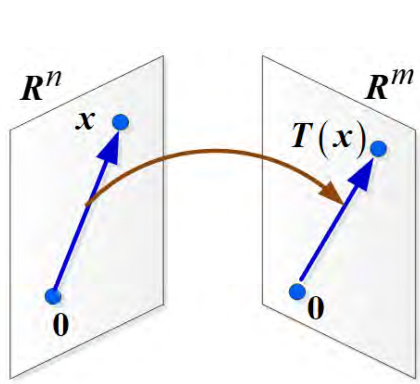
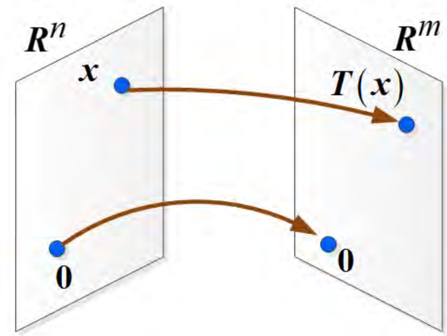
$$T_A : \ \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

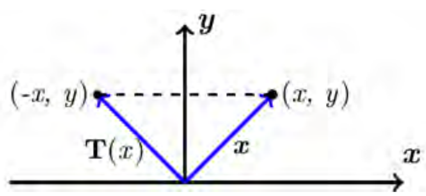
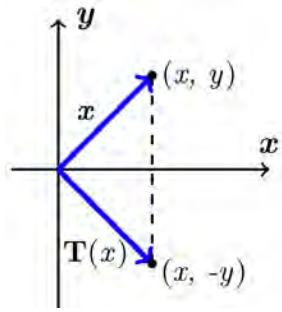
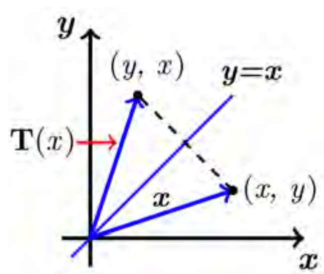
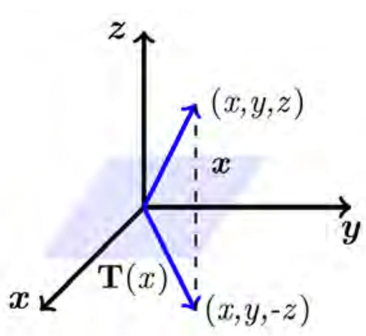
$\mathbb{R}^n$ is called the domain of *T*

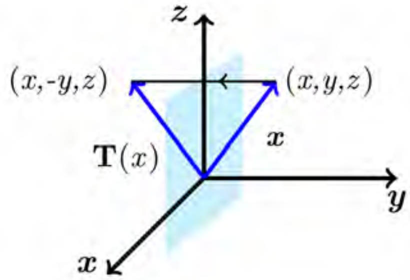
$\mathbb{R}^m$ is called the codomain of *T*

For $\vec{x}$ in $\mathbb{R}^n$, the vector $T(\vec{x})$ in $\mathbb{R}^m$ is called the image of $\vec{x}$ (under the action of *T*)

The set of all images $T(\vec{x})$ is called the range of *T*.

	
T *maps vectors to vectors*	***T*** *maps points to points*

Reflection about the y-axis $T(x,y)=(-x,y)$		$T(e_1)=T(1,0)=(-1,0)$ $T(e_2)=T(0,1)=(0,1)$	$\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$
Reflection about the x-axis $T(x,y)=(x,-y)$		$T(e_1)=T(1,0)=(1,0)$ $T(e_2)=T(0,1)=(0,-1)$	$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$
Reflection about the line $y=x$ $T(x,y)=(x,-y)$		$T(e_1)=T(1,0)=(0,1)$ $T(e_2)=T(0,1)=(1,0)$	$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$
Reflection about the xy-plane $T(x,y,z)=(x,y,-z)$		$T(e_1)=T(1,0,0)=(1,0,0)$ $T(e_2)=T(0,1,0)=(0,1,0)$ $T(e_3)=T(0,0,1)=(0,0,-1)$	$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$

<table>
<tr>
<td>Reflection about the xy-plane
$T(x,y,z)=(x,-y,z)$</td>
<td>
</td>
<td>$T(e_1)=T(1,0,0)=(1,0,0)$
$T(e_2)=T(0,1,0)=(0,-1,0)$
$T(e_3)=T(0,0,1)=(0,0,1)$</td>
<td>$\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$</td>
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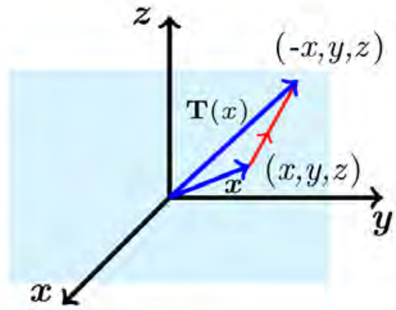
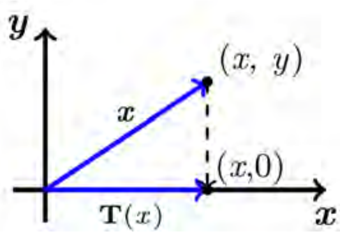
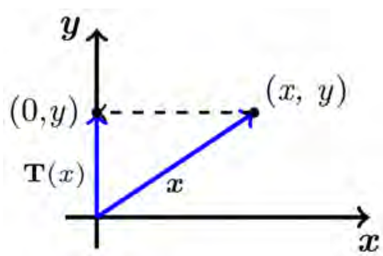
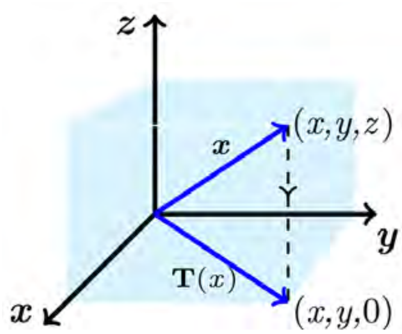
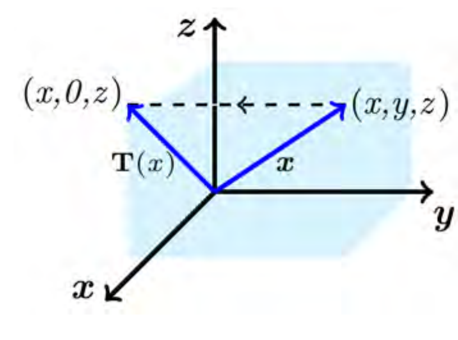
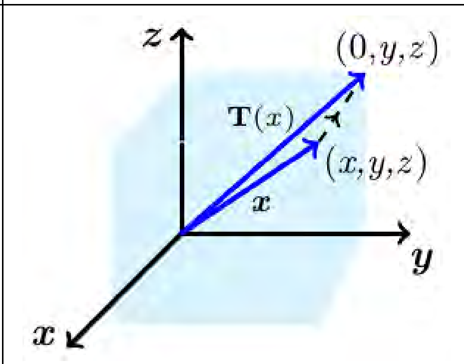
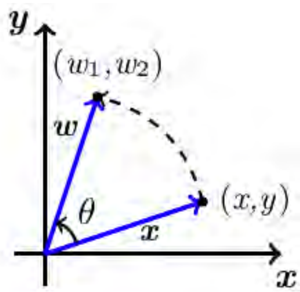
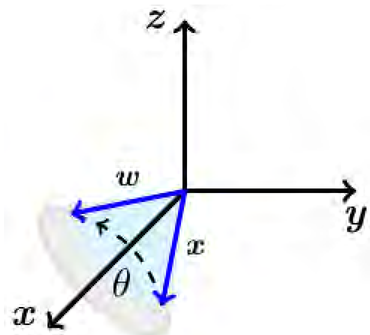
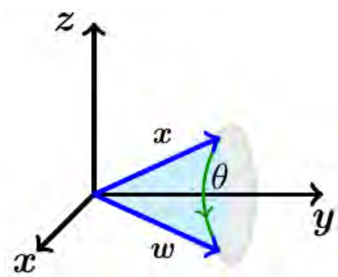
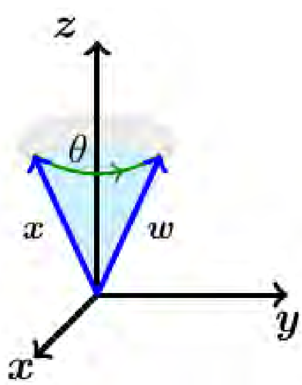
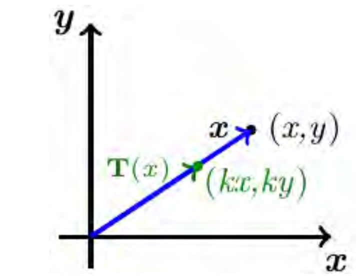
<tr>
<td>Reflection about the yz-plane
$T(x,y,z)=(-x,y,z)$</td>
<td>
</td>
<td>$T(e_1)=T(1,0,0)=(-1,0,0)$
$T(e_2)=T(0,1,0)=(0,1,0)$
$T(e_3)=T(0,0,1)=(0,0,1)$</td>
<td>$\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}$</td>
</tr>
<tr>
<td>Orthogonal projection on the x-axis
$T(x,y)=(x,0)$</td>
<td>
</td>
<td>$T(e_1)=T(1,0)=(1,0)$
$T(e_2)=T(0,1)=(0,0)$</td>
<td>$\begin{bmatrix}1&0\\0&0\end{bmatrix}$</td>
</tr>
<tr>
<td>Orthogonal projection on the y-axis
$T(x,y)=(0,y)$</td>
<td>
</td>
<td>$T(e_1)=T(1,0)=(0,0)$
$T(e_2)=T(0,1)=(0,1)$</td>
<td>$\begin{bmatrix}0&0\\0&1\end{bmatrix}$</td>
</tr>
<tr>
<td>Orthogonal projection on the xy-Plane
$T(x,y,z)=(x,y,0)$</td>
<td>
</td>
<td>$T(1,0,0)=(1,0,0)$
$T(0,1,0)=(0,1,0)$
$T(0,0,1)=(0,0,0)$</td>
<td>$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$</td>
</tr>
</table>

Orthogonal projection on the *xz*-Plane $T(x,y,z)=(x,0,z)$		$T(1,0,0)=(1,0,0)$ $T(0,1,0)=(0,0,0)$ $T(0,0,1)=(0,0,1)$	$\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}$
Orthogonal projection on the *yz*-Plane $T(x,y,z)=(0,y,z)$		$T(1,0,0)=(0,0,0)$ $T(0,1,0)=(0,1,0)$ $T(0,0,1)=(0,0,1)$	$\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}$

Rotation Operators			
Rotation through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
Counterclockwise rotation about the positive x-axis through an angle θ		$w_1 = x$ $w_2 = y\cos\theta - z\sin\theta$ $w_3 = y\sin\theta + z\cos\theta$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$
Counterclockwise rotation about the positive y-axis through an angle θ		$w_1 = x\cos\theta + z\sin\theta$ $w_2 = y$ $w_3 = -x\sin\theta + z\cos\theta$	$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$
Counterclockwise rotation about the positive z-axis through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$ $w_3 = z$	$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Contractions and Dilations

<table>
<tr>
<td>Contraction with factor k on $\mathbb{R}^2$
$(0 \le k < 1)$</td>
<td>
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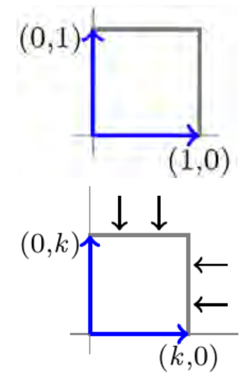
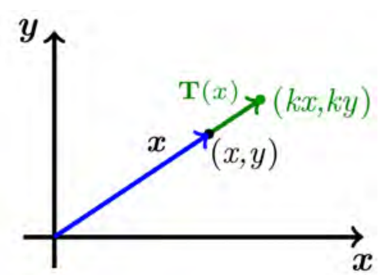
<td>
</td>
<td rowspan="2">$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$</td>
</tr>
<tr>
<td>Dilation with factor k on $\mathbb{R}^2$
$(k > 1)$</td>
<td>
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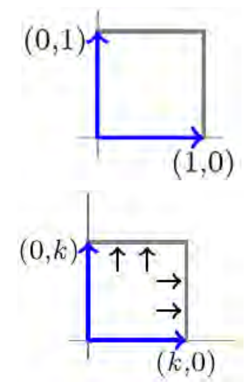
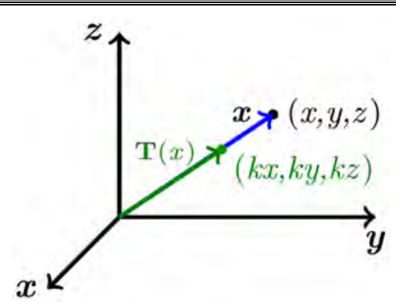
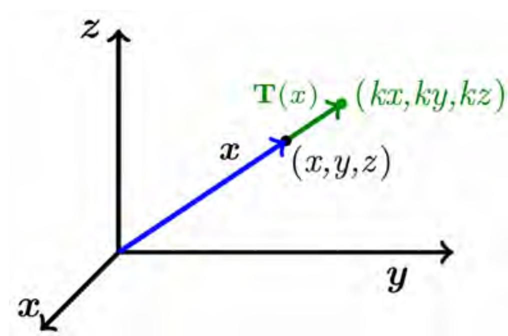
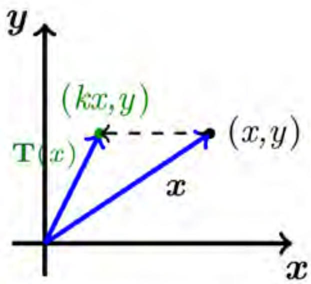
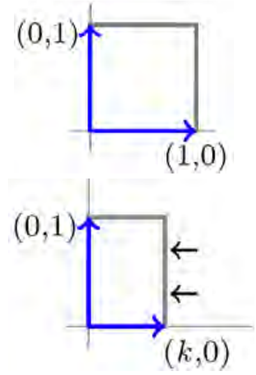
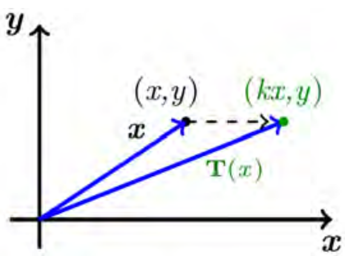
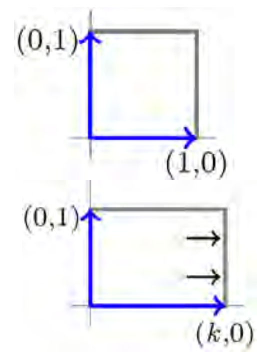
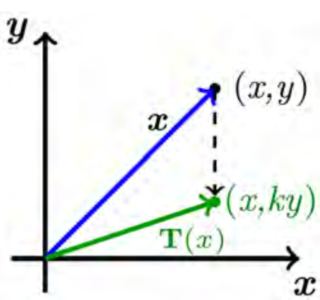
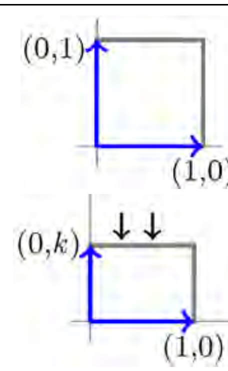
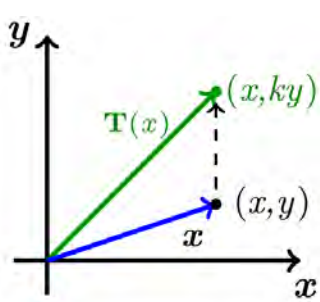
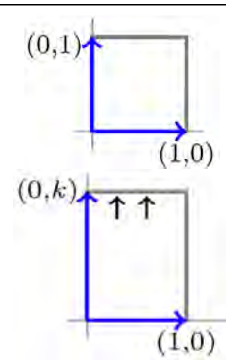
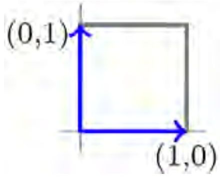
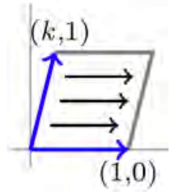
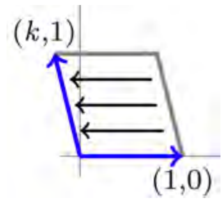
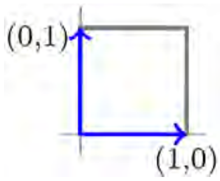
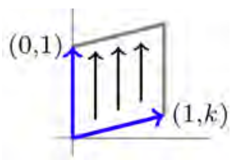
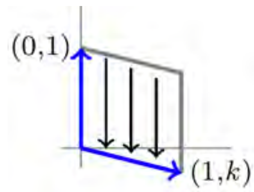
<td>
</td>
</tr>
<tr>
<td>Contraction with factor k on $\mathbb{R}^2$
$(0 \le k \le 1)$</td>
<td colspan="2">
</td>
<td rowspan="2">$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$</td>
</tr>
<tr>
<td>Dilation with factor k on $\mathbb{R}^2$
$(k \ge 1)$</td>
<td colspan="2">
</td>
</tr>
</table>

Expansion or Compression

Compression of $\mathbb{R}^2$ in the x–direction with factor k $(0 \le k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Expansion of $\mathbb{R}^2$ in the x–direction with factor k $(k > 1)$			
Compression of $\mathbb{R}^2$ in the y–direction with factor k $(0 \le k < 1)$			$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
Expansion of $\mathbb{R}^2$ in the y–direction with factor k $(k > 1)$			

Shear

Shear of $\mathbb{R}^2$ in the x–direction with factor k $T(x,\ y)=(x+ky,\ y)$		 $(k>0)$	 $(k<0)$	$\begin{bmatrix}1 & k\\0 & 1\end{bmatrix}$
Shear of $\mathbb{R}^2$ in the y–direction with factor k $T(x,\ y)=(x,\ y+kx)$		 $(k>0)$	 $(k<0)$	$\begin{bmatrix}1 & 0\\k & 1\end{bmatrix}$

Orthogonal Projections on Lines through the Origin

$$T(e_1) = \begin{bmatrix} \cos^2\theta \\ \sin\theta\cos\theta \end{bmatrix} \quad T(e_2) = \begin{bmatrix} \sin\theta\cos\theta \\ \sin^2\theta \end{bmatrix}$$

$$P_0 = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$$

Example

Find the orthogonal projection of the vector $\vec{x} = (1, 5)$ on the line through the origin that makes an angle of $\frac{\pi}{6}\ (=30°)$ with the x-axis

Solution

$$\begin{aligned} P_0 &= \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\left(\frac{\pi}{6}\right) & \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right) & \sin^2\left(\frac{\pi}{6}\right) \end{pmatrix} \\ &= \begin{pmatrix} \left(\frac{\sqrt{3}}{2}\right)^2 & \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\ \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) & \left(\frac{1}{2}\right)^2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} P_0\vec{x} &= \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 1 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} \frac{3+5\sqrt{3}}{4} \\ \frac{\sqrt{3}+5}{4} \end{pmatrix} \\ &\approx \begin{pmatrix} 2.91 \\ 1.68 \end{pmatrix} \end{aligned}$$

Example

Define a linear transformation $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$T(\vec{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

Find the images under T of $\vec{u} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\vec{u} + \vec{v} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$

Solution

$$T(\vec{u}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

$$T(\vec{v}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

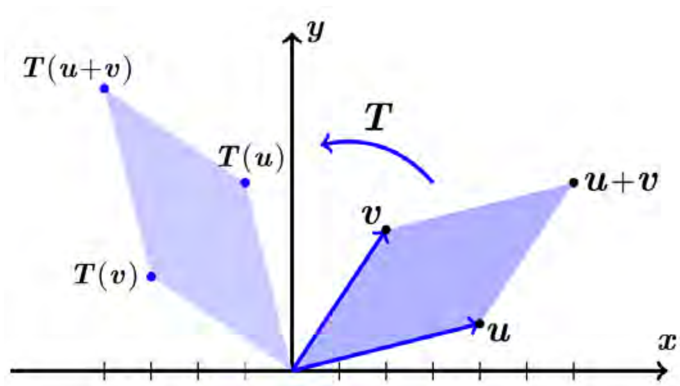
$$= \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$T(\vec{u} + \vec{v}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ 6 \end{pmatrix}$$

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$$

$$\begin{pmatrix} -4 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$



Four Fundamental Subspaces

1. The ***row space*** is $C\left(A^T\right)$, a subspace of $\mathbb{R}^n$.
2. The ***column space*** is $C(A)$, a subspace of $\mathbb{R}^m$.
3. The ***nullspace*** is $N(A)$, a subspace of $\mathbb{R}^n$.
4. The ***left nullspace*** is $N\left(A^T\right)$, a subspace of $\mathbb{R}^m$.

The Four Subspaces for R

Consider the matrix 3 by 5:

$$\begin{bmatrix} 1 & 3 & 5 & 0 & 9 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} m=3 & \textit{pivot rows } 1 \textit{ and } 2 \\ n=5 & \\ r=2 & \textit{pivot columns } 1 \textit{ and } 4 \end{matrix}$$

1. Rows 1 and 2 are a basis. The row space contains combination of all 3 rows.

   The ***row space*** of $\mathbb{R}$ has dimension 2 (= ***rank***).
   The dimension of the row space is r. The nonzero rows of ***R*** form a basis.

2. The ***column space*** of ***R*** has dimension *r* = 2.

   The pivot columns 1 and 4 form a basis. They are independent because they start with the *r* by *r* identity matrix.
   There are 3 special solutions:

$$\begin{aligned} C_2 &= 3C_1 && \textit{The special solution is } (-3,\ 1,\ 0,\ 0,\ 0) \\ C_3 &= 5C_1 && \textit{The special solution is } (-5,\ 0,\ 1,\ 0,\ 0) \\ C_5 &= 9C_1 + 8C_2 && \textit{The special solution is } (-9,\ 0,\ 0,\ -8,\ 1) \end{aligned}$$

   The dimension of the column space is r. The pivot columns form a basis.

3. The ***nullspace*** has dimension *n* – *r* = 5 – 2 = 3 (free variables). Here x_2, x_3, x_5 are free (no pivots in those columns). They yield the three special solutions to $R\vec{x} = 0$. Set a free variable to 1, and solve for x_1 and x_4.

$$s_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad s_3 = \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad s_5 = \begin{bmatrix} -9 \\ 0 \\ 0 \\ -8 \\ 1 \end{bmatrix}$$

$Rx = 0$ has the complete solution: $x = x_2 s_2 + x_3 s_3 + x_5 s_5$

The nullspace has dimension $n - r$. The special solutions form a basis.

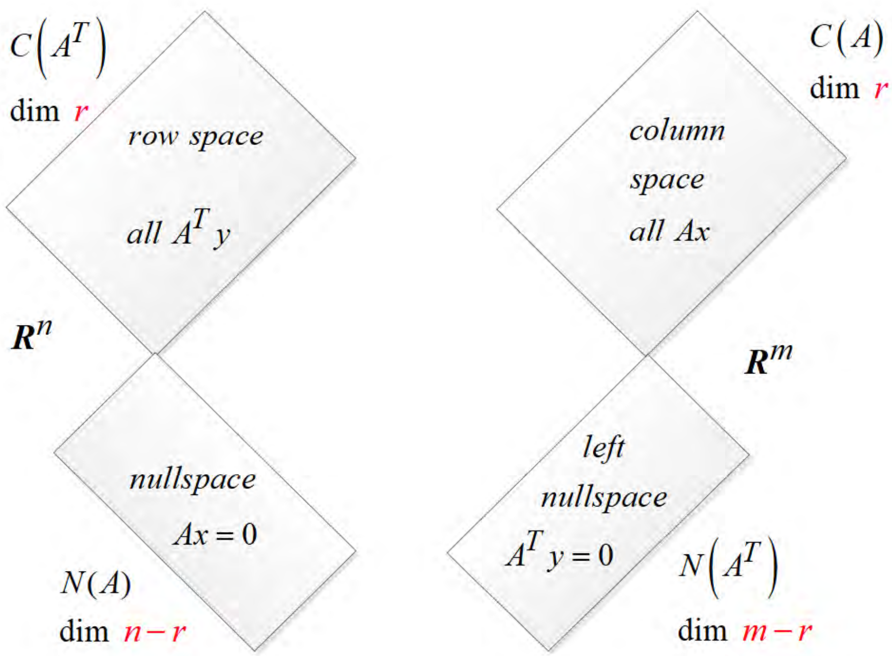
4. The ***nullspace*** of R^T has dimension $m - r = 3 - 2 = 1$

The equation $R^T y = 0$: $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 0 & 0 \\ 0 & 1 & 0 \\ 9 & 8 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} y_1 = 0 \\ y_2 = 0 \\ y_3 \ \textit{anything} \end{cases}$

The nullspace of R^T contains all vectors $y = (0,\ 0,\ y_3)$ and it is the line of the basis vector $(0,\ 0,\ 1)$.

The left nullspace has dimension m – r. The solutions are $y = (0, \ldots, y_{r+1}, \ldots, y_m)$

- In $\mathbb{R}^n$ the row space and nullspace have dimensions r and $n - r$ (adding to n)
- In $\mathbb{R}^m$ the column space and left nullspace have dimensions r and $m - r$ (total m)



The Four Subspaces for A

The subspace dimensions for A are the same as for R.

These matrices are connected by an invertible matrix E. $EA = R$ *and* $A = E^{-1}R$

1. ***A*** *has the same row space as* ***R***. Same dimension *r* and same basis
   Every row of ***A*** is a combination of the rows of ***R***. Also every row of ***R*** is a combination of the rows of ***A***.

2. *The column space of* ***A*** *has dimension* r. The number of independent columns equals the number of independent rows.

3. ***A*** *has the same nullspace as* ***R***. Dimension $n - r$ and same basis.

   (*dimension of column space*) + (*dimension of nullspace*) = *dimension of* $\boldsymbol{R}^n$

4. *The left nullspace* ***A*** (the nullspace of A^T) *has dimension* $m - r$.

Fundamental Theorem of Linear Algebra, (Part 1)

The column space and row space both have dimension *r*.

The nullspaces have dimensions $n - r$ and $m - r$.

Example

Consider $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

A has $m = 1$, $n = 3$, and rank: $r = 1$.

The row space is a line in $\mathbb{R}^3$.

The nullspace is the plane $Ax = x_1 + 2x_2 + 3x_3 = 0$. This plane has dimension 2 (which is 3 – 1).

The columns of this is 1 by 3 matrix are in $\mathbb{R}^1$. The column space is all of $\mathbb{R}^1$.

The left nullspace contains only the zero vector.

The only solution to $A^T y = 0$ *is* $y = 0$, the only combination of the row that gives the zero row.

Thus, $N\left(A^T\right)$ *is* $\mathbb{Z}$, the zero space with dimension 0 ($m - r$). In $\mathbb{R}^m$ the dimensions (1 + 0) = 1.

Example

Consider $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

A has m = 2, n = 3, and rank: r = 1.

The row space is a line in $\mathbb{R}^3$.

The nullspace is the plane $x_1 + 2x_2 + 3x_3 = 0$. This plane has dimension 3 (1 + 2).

The columns are multiples of the first column (1, 1).

The left nullspace contains more than one zero vector. The solution to $A^T \vec{y} = 0$ has the solution $y = (1, \ -1)$.

The column space and nullspace are perpendicular lines in $\mathbb{R}^2$. Their dimensions are 1 and 1 = 2.

Column space = line through $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Left nullspace = line through $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Exercises Section 4.1 – Matrix Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

1. Find the standard matrix for the transformation defined by the equations

a) $\begin{cases} w_1 = 2x_1 - 3x_2 + 4x_4 \\ w_2 = 3x_1 + 5x_2 - x_4 \end{cases}$

b) $\begin{cases} w_1 = 7x_1 + 2x_2 - 8x_3 \\ w_2 = \quad\quad - x_2 + 5x_3 \\ w_3 = 4x_1 + 7x_2 - x_3 \end{cases}$

c) $\begin{cases} w_1 = x_1 \\ w_2 = x_1 + x_2 \\ w_3 = x_1 + x_2 + x_3 \\ w_4 = x_1 + x_2 + x_3 + x_4 \end{cases}$

(2 – 8) Find the standard matrix for the operator T defined by the formula

2. $T\left(x_1, x_2\right) = \left(2x_1 - x_2,\ x_1 + x_2\right)$

3. $T\left(x_1, x_2, x_3\right) = \left(x_1 + 2x_2 + x_3,\ x_1 + 5x_2,\ x_3\right)$

4. $T\left(x_1, x_2, x_3\right) = \left(4x_1,\ 7x_2,\ -8x_3\right)$

5. $T\left(x_1, x_2\right) = \left(x_2,\ -x_1,\ x_1 + 3x_2,\ x_1 - x_2\right)$

6. $T\left(x_1, x_2, x_3\right) = (0,\ 0,\ 0,\ 0,\ 0)$

7. $T\left(x_1, x_2, x_3, x_4\right) = \left(x_4,\ x_1,\ x_3,\ x_2,\ x_1 - x_3\right)$

8. $T\left(x_1, x_2, x_3, x_4\right) = \left(7x_1 + 2x_2 - x_3 + x_4,\ x_2 + x_3,\ -x_1\right)$

(9 – 8) Plot $\vec{u} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and their images under the given transformation T

9. $T(\vec{x}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

10. $T(\vec{x}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

11. $T\left(\vec{x}\right)=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$

12. $T\left(\vec{x}\right)=\begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$

13. $T\left(\vec{x}\right)=\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$