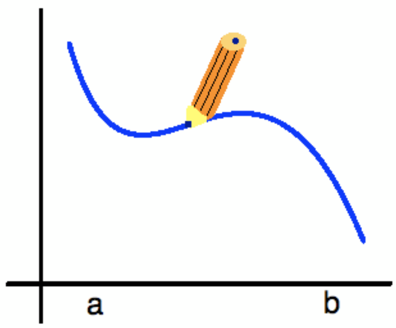
Section 1.5 – Continuity

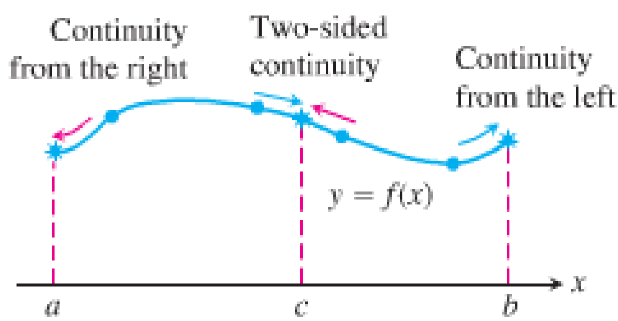
Definition of *Continuity*

Let c be a number in the interval (a, b), and let f be a function whose domain contains the interval (a, b). The function f is continuous at the point c if the following conditions are true.

1. $f(c)$ is defined
2. $\lim\limits_{x\to c} f(x)$ *exists*
3. $\lim\limits_{x\to c} f(x) = f(c)$

If f is continuous at every point in the interval (a, b), then it is continuous on an open interval (a, b)





Definition

Interior point: A function $y = f(x)$ is **continuous at an interior point *c*** of its domain if

$$\lim_{x\to c} f(x) = f(c)$$

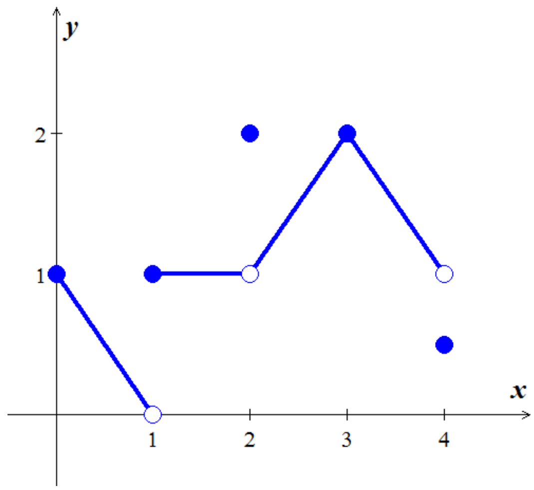
Endpoint: A function $y = f(x)$ is **continuous at a left point *a*** or is **continuous at a right point *b*** of its domain if

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x\to b^-} f(x) = f(b), \quad \textit{respectively}$$

- If a function f is not continuous at a point c, we say that f is ***discontinuous*** at c. (is a ***point of discontinuity***)

Example

Find the points at which the function f is continuous and the points at which f is not continuous



Solution

The function f is continuous at every point in its domain [0, 4] except at $x = 1$, $x = 2$, and $x = 4$. At these points, there are breaks in the graph.

$x = 0$	$\lim\limits_{x \to 0^+} f(x) = f(0) = 1$	f is continuous @ $x = 0$
$x = 1$	$\lim\limits_{x \to 1} f(x)$ *doesn't exist*	f is discontinuous @ $x = 1$
$x = 2$	$\lim\limits_{x \to 2} f(x) = 1,\ but\ \ 1 \ne f(2)$	f is discontinuous @ $x = 2$
$x = 3$	$\lim\limits_{x \to 3} f(x) = f(3) = 2$	f is continuous @ $x = 3$
$x = 4$	$\lim\limits_{x \to 4^-} f(x) = 1,\ but\ \ 1 \ne f(4)$	f is discontinuous @ $x = 4$
$c < 0,\ c > 4$	These points are not in the domain of *f*.	f is discontinuous
$0 < c < 4,\ \ c \ne 1, 2$	$\lim\limits_{x \to c} f(x) = f(c)$	

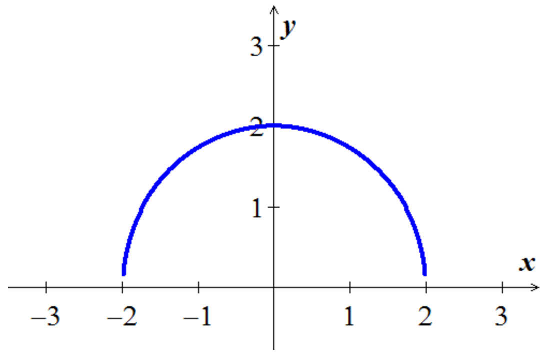
Example

At what points the function $f(x)=\sqrt{4-x^2}$ is continuous?

Solution

The function is continuous at every point of its domain [–2, 2].

Including $x=-2$, where f is right-continuous, and $x=2$, where f is left-continuous.



Continuous Functions

A function is ***continuous on an interval*** iff it is continuous at every point of the interval. A ***continuous function*** is one that is continuous at every point of its domain. A continuous function need not be continuous on every interval.

Example

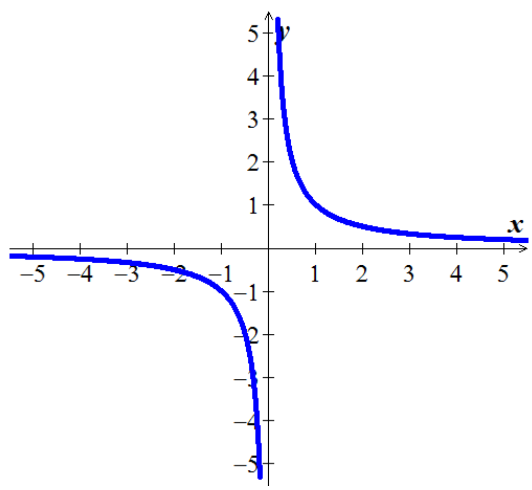
Determine at which points do the function $f(x)=\dfrac{1}{x}$ is continuous and discontinuous

Solution

The function $f(x)$ is a continuous function because it is continuous at every point of its domain.

It has a point of discontinuity at $x=0$, however, because it is not defined.

It is discontinuous on any interval containing $x=0$



Theorem – Properties of Continuous Functions

If the functions f *and* g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

Sums and Differences	$f \pm g$
Constant multiples	$k \cdot g$, for any number k.
Products	$f \cdot g$
Quotients	$\dfrac{f}{g}$
Powers	f^n ***n*** *a positive integer*
Roots	$\sqrt[n]{f}$, provided it is defined on an open interval containing c, where n is a positive integer

Proof

$$\lim_{x\to c}(f+g)(x) = \lim_{x\to c}\left(f(x)+g(x)\right)$$

$$= \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$

$$= f(c) + g(c)$$

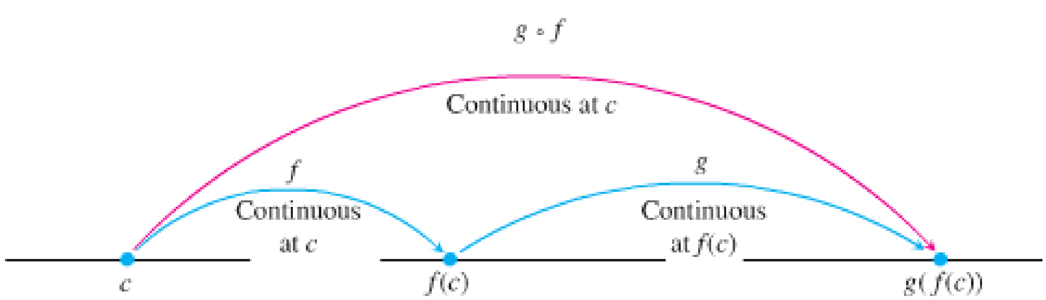
$$= (f+g)(c)$$

This shows that $f + g$ is continuous

Composites

All composites of continuous functions are continuous.

If $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$, then $g \circ f$ is continuous at $x = c$



Example

Show that $y=\sqrt{x^2-2x-5}$ is continuous everywhere on its domain

Solution

Let $\begin{cases} f(x)=x^2-2x-5, & Domain:\mathbb{R} \\ g(x)=\sqrt{x} & Domain:[0,\ \infty) \end{cases}$

$\therefore$ The function y is continuous on $[0,\ \infty)$

Example

Show that $y=\left|\dfrac{x\sin x}{x^2+2}\right|$ is continuous everywhere on its domain

Solution

Let $\begin{cases} x\sin x & Domain:\mathbb{R} \\ x^2+2 & Domain:\mathbb{R} \end{cases}$

$\therefore$ The function is the composite of a quotient continuous functions with the continuous absolute value function.

Theorem

If g is continuous at the point b and $\lim\limits_{x\to c} f(x)=b$, then

$$\lim_{x\to c} g(f(x))=g(b)=g\left(\lim_{x\to c} f(x)\right)$$

Proof

Let $\varepsilon>0$ be given. Since g is continuous at b, there exists a number $\delta_1>0$ such that

$$|g(y)-g(b)|<\varepsilon \quad whenever \quad 0<|y-b|<\delta_1$$

$\lim\limits_{x\to c} f(x)=b,\ \exists\ \delta>0 \ni\ |f(x)-b|<\delta_1 \quad whenever \quad 0<|x-c|<\delta$

If we let $y=f(x)$, we then have that $\quad |y-b|<\delta_1 \quad whenever \quad 0<|x-c|<\delta$

Which implies from the first statement that $|g(y)-g(b)|=|g(f(x))-g(b)|<\varepsilon$ whenever $0<|x-c|<\delta$. From the definition of the limit, this proves that $\lim\limits_{x\to c} g(f(x))=g(b)$

Example

Find the $\lim\limits_{x\to\frac{\pi}{2}} \cos\left(2x+\sin\left(\frac{3\pi}{2}+x\right)\right)$

Solution

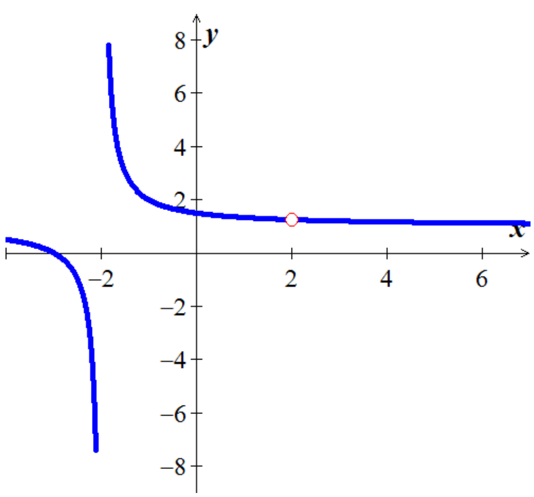
$$\begin{aligned}\lim_{x\to\frac{\pi}{2}} \cos\left(2x+\sin\left(\frac{3\pi}{2}+x\right)\right) &= \cos\left(\lim_{x\to\frac{\pi}{2}} 2x + \lim_{x\to\frac{\pi}{2}} \sin\left(\frac{3\pi}{2}+x\right)\right)\\ &= \cos(\pi+\sin 2\pi)\\ &= \cos(\pi+0)\\ &= \cos(\pi)\\ &= -1\end{aligned}$$

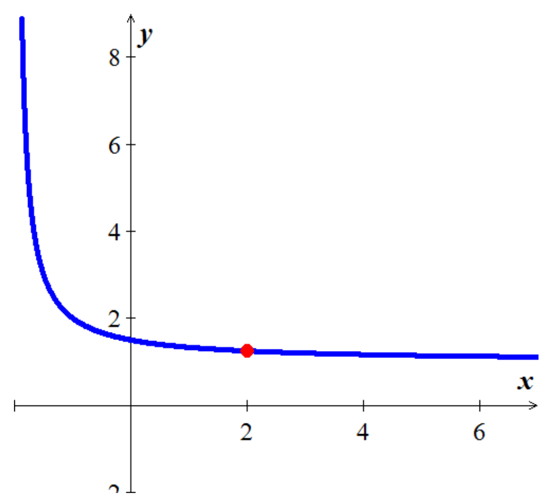
Example

Show that $f(x)=\dfrac{x^2+x-6}{x^2-4}, \quad x\neq 2$ has a continuous extension to $x=2$, and find that extension.

Solution

$$f(x)=\frac{x^2+x-6}{x^2-4}=\frac{(x-2)(x+3)}{(x-2)(x+2)}=\frac{x+3}{x+2}$$





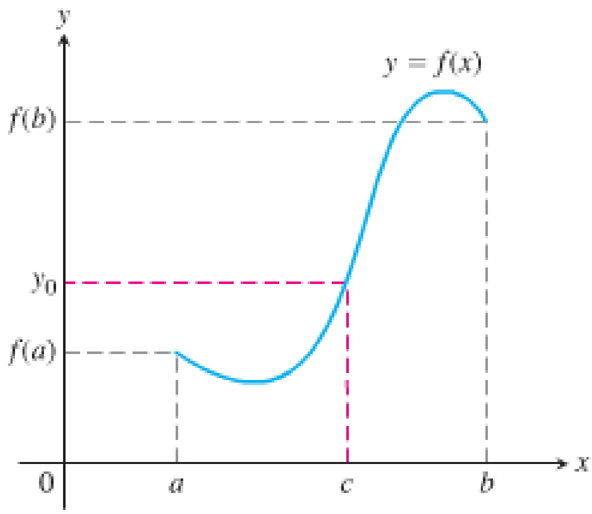
After simplification the function is continuous at $x=2$

$$\lim_{x\to 2}\frac{x^2+x-6}{x^2-4}=\lim_{x\to 2}\frac{x+3}{x+2}=\frac{5}{4}$$

The new function is the function f with its point of discontinuity at $x=2$ removed.

Theorem **– the Intermediate Value Theorem for Continuous Functions**

If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



A Consequence for Root Finding

We call a solution of the equation $f(x) = 0$ a ***root*** of the equation or zero of the function f. The Intermediate Value Theorem said that if f is continuous, then any interval on which f changes sign contains a zero of the function.

Example

Show that there is a root of the equation $x^3 - x - 1$ between 1 and 2.

Solution

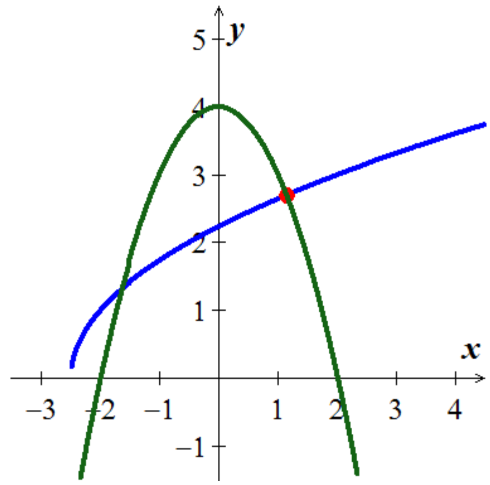
$$f(1) = 1^3 - 1 - 1 = -1 < 0$$

$$f(2) = 2^3 - 2 - 1 = 5 > 0$$

Since f is continuous, the Intermediate Value Theorem says there is a zero of f between 1 and 2.

Example

Use the Intermediate Value Theorem to prove that the equation $\sqrt{2x+5} = 4 - x^2$ has a solution.



Solution

The function $g(x) = \sqrt{2x+5}$ is continuous on the interval $\left[-\frac{5}{2},\ \infty\right)$ since it is the composite of the square root function with nonnegative linear function $y = 2x+5$. Then the function $f(x) = \sqrt{2x+5} + x^2$ is the sum of the function $g(x)$ and $y = x^2$. It follows that $f(x)$ is continuous on the interval $\left[-\frac{5}{2},\ \infty\right)$.

By trial and error:

$$f(0) = \sqrt{2(0)+5} + 0^2 = \sqrt{5} > 0$$

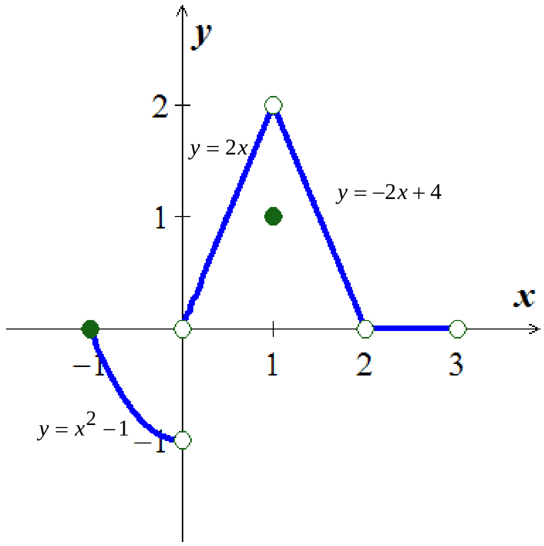
$$f(2) = \sqrt{2(2)+5} + 2^2 = \sqrt{9} + 4 = 7 > 0$$

f is continuous on the interval $[0,\ 2] \subset \left[-\frac{5}{2},\ \infty\right)$.

Since the value $y_0 = 4$ is between $\sqrt{5}$ and 7, by the Intermediate Value Theorem there is a number $c \in [0,\ 2] \ni f(c) = 4$. That is, the number c solves the original equation.

Exercises *Section* 1.5 – Continuity

1. Given the graphed function $f(x)$



a) Does $f(-1)$ exist?

b) Does $\lim\limits_{x\to -1^+} f(x)$ exist?

c) Does $\lim\limits_{x\to -1^+} f(x) = f(-1)$?

d) Is f continuous at $x=-1$?

e) Does $f(1)$ exist?

f) Does $\lim\limits_{x\to 1} f(x)$ exist?

g) Does $\lim\limits_{x\to 1} f(x) = f(1)$?

h) Is f continuous at $x=1$?

(2 – 11) At what point(s) is the given function continuous?

2. $y = \dfrac{1}{x-2} - 3x$

3. $y = \dfrac{x+3}{x^2-3x-10}$

4. $y = |x-1| + \sin x$

5. $y = \dfrac{x+2}{\cos x}$

6. $y = \tan \dfrac{\pi x}{2}$

7. $y = \dfrac{x \tan x}{x^2+1}$

8. $y = \dfrac{\sqrt{x^4+1}}{1+\sin^2 x}$

9. $y = \sqrt{2x+3}$

10. $y = \sqrt[4]{3x-1}$

11. $y = (2-x)^{1/5}$

12. Find $\lim\limits_{x\to\pi} \sin(x - \sin x)$, then is the function continuous at the point being approached?

13. Find $\lim\limits_{x\to 0} \tan\left(\dfrac{\pi}{4}\cos\left(\sin x^{1/3}\right)\right)$, then is the function continuous at the point being approached?

14. Find $\lim\limits_{t\to 0} \cos\left(\dfrac{\pi}{\sqrt{19-3\sec 2t}}\right)$, then is the function continuous at the point being approached?

15. Explain why the equation $\cos x = x$ has at least one solution.

(16 – 19) Show that the equation has three solutions in the given interval

16. $x^3 - 15x + 1 = 0;\quad [-4,\ 4]$

17. $x^3 + 10x^2 - 100x + 50 = 0;\quad (-20,\ 10)$

18. $70x^3 - 87x^2 + 32x - 3 = 0;\quad (0,\ 1)$

19. $x^3 - 3x - 1 = 0;\quad [-2,\ 2]$

20. Show that the equation has six solutions in the given interval $x^6 - 8x^4 + 10x^2 - 1 = 0$; $[-3,\ 3]$

21. If functions $f(x)$ and $g(x)$ are continuous for $0 \le x \le 1$, could $\dfrac{f(x)}{g(x)}$ possibly be discontinuous at a point of [0, 1]? Give reason for your answer.

22. Suppose that a function f is continuous on the closed interval [0, 1] and that $0 \le f(x) \le 1$ for every x in [0, 1]. Show that there must exist a number c in [0, 1] such that $f(c) = c$ (**c** is called a ***fixed point*** of f).

23. Use the Intermediate Value Theorem to show that the equation $x^5 + 7x + 5 = 0$ has a solution in the interval $(-1,\ 0)$.

24. The amount of an antibiotic (in *mg*) in the blood *t* hours after an intravenous line is opened is given by

$$m(t) = 100\left(e^{-0.1t} - e^{-0.3t}\right)$$

a) Use the Intermediate Value Theorem to show that the amount of drug is 30 *mg* at some time in the interval $[0,\ 5]$ and again at some time in the interval $[5,\ 15]$

b) Estimate the times at which $m = 30$ *mg*

c) Is the amount of drug in the blood ever 50 *mg*?

(25 – 27) Determine whether the following functions are continuous at a.

25. $f(x) = \dfrac{1}{x-5}$; $a = 5$

26. $h(x) = \sqrt{x^2 - 9}$; $a = 3$

27. $g(x) = \begin{cases} \dfrac{x^2-16}{x-4} & if \quad x \ne 4 \\ 8 & if \quad x = 4 \end{cases}$; $a = 4$

(28 – 31) Find the intervals on which the following functions are continuous. Specify right- or left-continuity at the endpoints

28. $f(x) = \sqrt{x^2 - 5}$ **29.** $f(x) = e^{\sqrt{x-2}}$ **30.** $f(x) = \dfrac{2x}{x^3 - 25x}$ **31.** $f(x) = \cos e^x$

32. Let $g(x) = \begin{cases} 5x - 2 & if \quad x < 1 \\ a & if \quad x = 1 \\ ax^2 + bx & if \quad x > 1 \end{cases}$

Determine values of the constants a and b for which $g(x)$ is continuous at $x = 1$