

Solution **Section 4.3 – Polar Coordinates**

Exercise

Find the Cartesian coordinates of the following points (given in polar coordinates)

$$a) \left(\sqrt{2}, \frac{\pi}{4} \right) \quad b) (1, 0) \quad c) \left(0, \frac{\pi}{2} \right) \quad d) \left(-\sqrt{2}, \frac{\pi}{4} \right)$$

Solution

$$a) \begin{cases} x = r \cos \theta = \sqrt{2} \cos \frac{\pi}{4} = 1 \\ x = r \sin \theta = \sqrt{2} \sin \frac{\pi}{4} = 1 \end{cases}$$

Cartesian coordinates $(1, 1)$

$$b) \begin{cases} x = r \cos \theta = 1 \cos 0 = 1 \\ x = r \sin \theta = 1 \sin 0 = 0 \end{cases}$$

Cartesian coordinates $(1, 0)$

$$c) \begin{cases} x = r \cos \theta = 0 \cos \frac{\pi}{2} = 0 \\ x = r \sin \theta = 0 \sin \frac{\pi}{2} = 0 \end{cases}$$

Cartesian coordinates $(0, 0)$

$$d) \begin{cases} x = r \cos \theta = -\sqrt{2} \cos \frac{\pi}{4} = -1 \\ x = r \sin \theta = -\sqrt{2} \sin \frac{\pi}{4} = -1 \end{cases}$$

Cartesian coordinates $(-1, -1)$

Exercise

Find the polar coordinates, $0 \leq \theta < 2\pi$ and $r \geq 0$, of the following points given in Cartesian coordinates

$$a) (1, 1) \quad b) (-3, 0) \quad c) (\sqrt{3}, -1) \quad d) (-3, 4)$$

Solution

$$a) \begin{cases} r = \sqrt{1^2 + 1^2} = \sqrt{2} \\ \theta = \tan^{-1} \frac{1}{1} = \frac{\pi}{4} \end{cases}$$

Polar coordinates $\left(\sqrt{2}, \frac{\pi}{4} \right)$

$$b) \begin{cases} r = \sqrt{(-3)^2 + 0^2} = 3 \\ \theta = \tan^{-1} \frac{0}{-3} = \pi \end{cases}$$

Polar coordinates $(3, \pi)$

$$c) \begin{cases} r = \sqrt{\sqrt{3}^2 + (-1)^2} = 2 \\ \theta = \tan^{-1} \frac{-1}{\sqrt{3}} = \frac{11\pi}{6} \end{cases}$$

Polar coordinates $(2, \frac{11\pi}{6})$

$$d) \begin{cases} r = \sqrt{(-3)^2 + 4^2} = 5 \\ \theta = \tan^{-1} \frac{4}{-3} = \pi - \arctan\left(\frac{4}{3}\right) \end{cases}$$

Polar coordinates $(5, \pi - \arctan(\frac{4}{3}))$

Exercise

Find the polar coordinates, $-\pi \leq \theta < \pi$ and $r \geq 0$, of the following points given in Cartesian coordinates

$$a) (-2, -2) \quad b) (0, 3) \quad c) (-\sqrt{3}, 1) \quad d) (5, -12)$$

Solution

$$a) \begin{cases} r = \sqrt{(-2)^2 + (-2)^2} = 2\sqrt{2} \\ \theta = \tan^{-1} \frac{-2}{-2} = -\frac{3\pi}{4} \end{cases}$$

Polar coordinates $(2\sqrt{2}, -\frac{3\pi}{4})$

$$b) \begin{cases} r = \sqrt{0^2 + 3^2} = 3 \\ \theta = \tan^{-1} \frac{3}{0} = \frac{\pi}{2} \end{cases}$$

Polar coordinates $(3, \frac{\pi}{2})$

$$c) \begin{cases} r = \sqrt{(-\sqrt{3})^2 + 1^2} = 2 \\ \theta = \tan^{-1} \frac{1}{-\sqrt{3}} = \frac{5\pi}{6} \end{cases}$$

Polar coordinates $(2, \frac{5\pi}{6})$

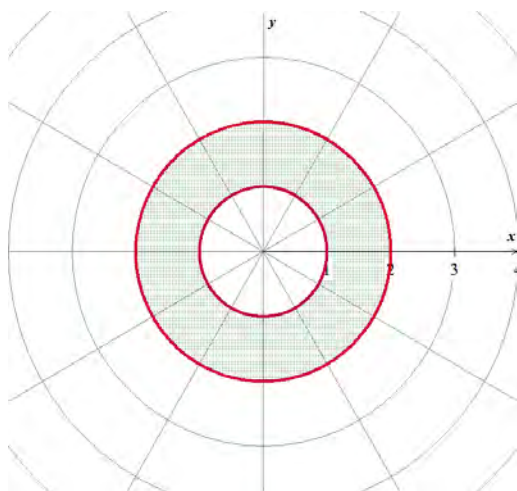
$$d) \begin{cases} r = \sqrt{5^2 + (-12)^2} = 13 \\ \theta = \tan^{-1} \frac{-12}{5} = -\arctan\left(\frac{12}{5}\right) \end{cases}$$

Polar coordinates $\left(13, -\arctan\left(\frac{12}{5}\right)\right)$

Exercise

Graph $1 \leq r \leq 2$

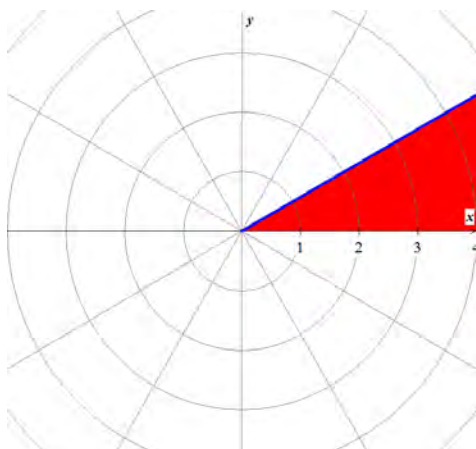
Solution



Exercise

Graph $0 \leq \theta \leq \frac{\pi}{6}, r \geq 0$

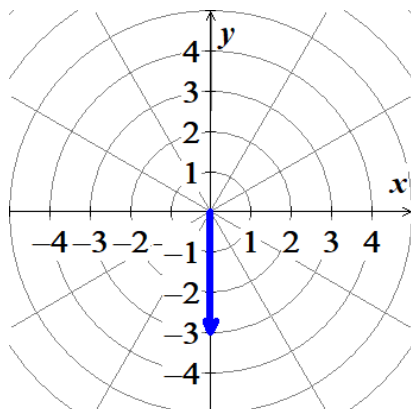
Solution



Exercise

Graph $\theta = \frac{\pi}{2}, \quad r \leq 0$

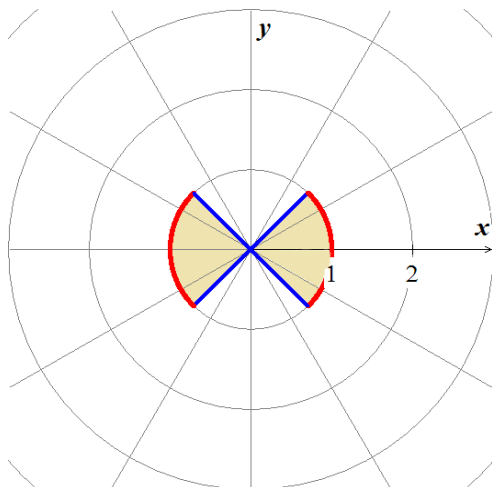
Solution



Exercise

Graph $-\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}, \quad 0 \leq r \leq 1$

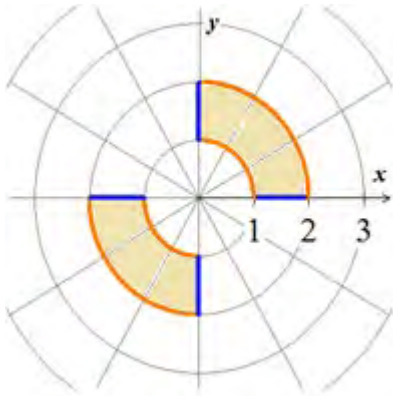
Solution



Exercise

Graph $0 \leq \theta \leq \frac{\pi}{2}, 1 \leq |r| \leq 2$

Solution



Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r \cos \theta = 2$

Solution

$$r \cos \theta = 2 \Rightarrow x = 2, \text{ vertical line}$$

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r \sin \theta = -1$

Solution

$$r \sin \theta = -1 \Rightarrow y = -1, \text{ horizontal line}$$

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r = -3 \sec \theta$

Solution

$$r = -3 \sec \theta = -\frac{3}{\cos \theta} \Rightarrow r \cos \theta = -3$$
$$x = -3, \text{ vertical line through } (-3, 0)$$

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r \cos \theta + r \sin \theta = 1$

Solution

$$r \cos \theta + r \sin \theta = 1 \Rightarrow x + y = 1, \text{ line with slope } -1$$

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r^2 = 4r \sin \theta$

Solution

$$\begin{aligned}r^2 = 4r \sin \theta &\Rightarrow x^2 + y^2 = 4y \\x^2 + y^2 - 4y &= 0 \\x^2 + y^2 - 4y + 4 &= 4 \\x^2 + (y - 2)^2 &= 4\end{aligned}$$

It is a circle with a center $C = (0, 2)$ and radius $r = 2$.

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r = \frac{5}{\sin \theta - 2 \cos \theta}$

Solution

$$\begin{aligned}r &= \frac{5}{\sin \theta - 2 \cos \theta} \\r \sin \theta - 2r \cos \theta &= 5 \\y - 2x &= 5 \\y &= 2x + 5\end{aligned}$$

It is a line with slope $m = 2$ and intercept $b = 5$

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r = 4 \tan \theta \sec \theta$

Solution

$$\begin{aligned}r &= 4 \tan \theta \sec \theta \\&= 4 \frac{\sin \theta}{\cos \theta} \frac{1}{\cos \theta} \\&= 4 \frac{\sin \theta}{\cos^2 \theta} \\r \cos^2 \theta &= 4 \sin \theta \\r^2 \cos^2 \theta &= 4r \sin \theta \\x^2 &= 4y \\y &= \frac{1}{4} x^2\end{aligned}$$

It is a parabola with vertex $(0, 0)$.

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph

$$r \sin \theta = \ln r + \ln \cos \theta$$

Solution

$$\begin{aligned} r \sin \theta &= \ln r + \ln \cos \theta \\ &= \ln r \cos \theta \end{aligned}$$

Power Rule

$$y = \ln x$$

Graph of the natural logarithm function

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $\cos^2 \theta = \sin^2 \theta$

Solution

$$\begin{aligned} \cos^2 \theta &= \sin^2 \theta \\ r^2 \cos^2 \theta &= r^2 \sin^2 \theta \\ x^2 &= y^2 \\ y &= \pm x \end{aligned}$$

The graph is 2 perpendicular lines through the origin with slopes -1 and 1 ,

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r = 2 \cos \theta + 2 \sin \theta$

Solution

$$\begin{aligned} r &= 2 \cos \theta + 2 \sin \theta \\ r^2 &= 2r \cos \theta + 2r \sin \theta \\ x^2 + y^2 &= 2x + 2y \\ x^2 - 2x + y^2 - 2y &= 0 \\ x^2 - 2x + 1 + y^2 - 2y + 1 &= 1 + 1 \\ (x-1)^2 + (y-1)^2 &= 2 \end{aligned}$$

It is a circle with a center $C = (1, 1)$ and radius $r = \sqrt{2}$.

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r \sin\left(\frac{2\pi}{3} - \theta\right) = 5$

Solution

$$r \sin\left(\frac{2\pi}{3} - \theta\right) = 5$$

$$r\left(\sin \frac{2\pi}{3} \cos \theta - \cos \frac{2\pi}{3} \sin \theta\right) = 5$$

$$\frac{\sqrt{3}}{2} r \cos \theta + \frac{1}{2} r \sin \theta = 5$$

$$\frac{\sqrt{3}}{2} x + \frac{1}{2} y = 5$$

$$\sqrt{3}x + y = 10$$

It is a line with slope $m = -\sqrt{3}$ and intercept $b = 10$

Exercise

Replace the polar equation with equivalent Cartesian equation and identify the graph $r = \frac{4}{2\cos\theta - \sin\theta}$

Solution

$$2r \cos \theta - r \sin \theta = 4$$

$$2x - y = 4$$

The graph: Line $2x - y = 4$ with slope $m = 2$.

Exercise

Replace the Cartesian equation with equivalent polar equation $x = y$

Solution

$$x = y$$

$$r \cos \theta = r \sin \theta$$

$$\cos \theta = \sin \theta$$

$$\theta = \frac{\pi}{4}$$

Exercise

Replace the Cartesian equation with equivalent polar equation $x^2 - y^2 = 1$

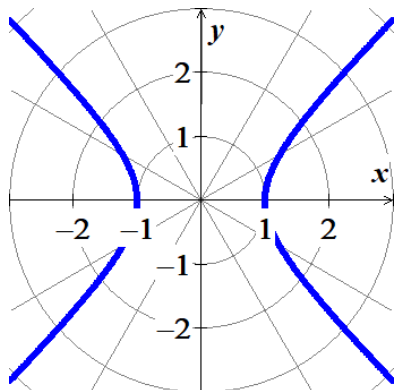
Solution

$$x^2 - y^2 = 1$$

$$r^2 \cos^2 \theta - r^2 \sin^2 \theta = 1$$

$$r^2 (\cos^2 \theta - \sin^2 \theta) = 1$$

$$\boxed{r^2 \cos 2\theta = 1}$$



Exercise

Replace the Cartesian equation with equivalent polar equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Solution

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

$$4x^2 + 9y^2 = 36$$

$$\boxed{4r^2 \cos^2 \theta + 9r^2 \sin^2 \theta = 36}$$

Exercise

Replace the Cartesian equation with equivalent polar equation $xy = 1$

Solution

$$xy = 1$$

$$r^2 \cos \theta \sin \theta = 1$$

$$\sin 2\theta = 2 \cos \theta \sin \theta$$

$$r^2 \frac{1}{2} \sin 2\theta = 1$$

$$\boxed{r^2 \sin 2\theta = 2}$$

Exercise

Replace the Cartesian equation with equivalent polar equation $x^2 + xy + y^2 = 1$

Solution

$$x^2 + xy + y^2 = 1$$

$$r^2 + r^2 \cos \theta \sin \theta = 1$$

$$\underline{r^2 (1 + \cos \theta \sin \theta) = 1}$$

Exercise

Replace the Cartesian equation with equivalent polar equation $x^2 + (y - 2)^2 = 4$

Solution

$$x^2 + (y - 2)^2 = 4$$

$$x^2 + y^2 - 4y + 4 = 4$$

$$x^2 + y^2 - 4y = 0$$

$$r^2 - 4r \sin \theta = 0$$

$$r^2 = 4r \sin \theta$$

$$\underline{r = 4 \sin \theta}$$

Exercise

Replace the Cartesian equation with equivalent polar equation $(x + 2)^2 + (y - 5)^2 = 16$

Solution

$$(x + 2)^2 + (y - 5)^2 = 16$$

$$x^2 + 4x + 4 + y^2 - 10y + 25 = 16$$

$$x^2 + 4x + y^2 - 10y = -13$$

$$r^2 + 4r \cos \theta - 10r \sin \theta = -13$$

$$\underline{r^2 = -4r \cos \theta + 10r \sin \theta - 13}$$

Exercise

- a) Show that every vertical line in the xy -plane has a polar equation of the form $r = a \sec \theta$
b) Find the analogous polar equation for horizontal lines in the xy -plane.

Solution

$$a) \quad x = a \Rightarrow r \cos \theta = a$$

$$r = \frac{a}{\cos \theta}$$

$$= a \sec \theta$$

$$b) \quad y = b$$

$$r \sin \theta = b$$

$$r = \frac{b}{\sin \theta}$$

$$= b \csc \theta$$

Exercise

Identify the symmetries of the curve. Then sketch the curve. $r = 2 - 2 \cos \theta$

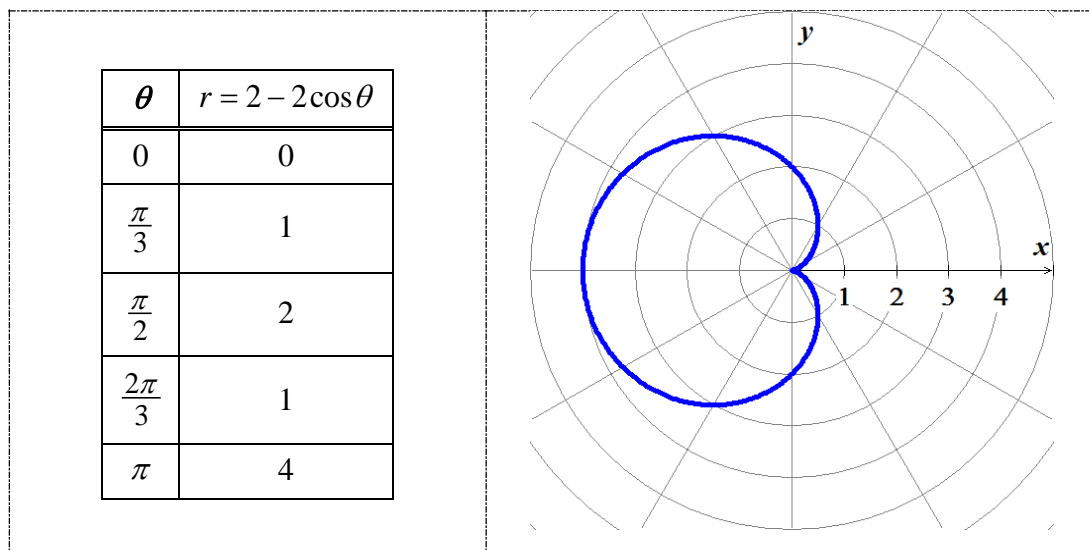
Solution

$$2 - 2 \cos(-\theta) = 2 - 2 \cos \theta = r$$

Symmetric about the x -axis

$$\begin{cases} 2 - 2 \cos(-\theta) \neq -r \\ 2 - 2 \cos(\pi - \theta) = 2 + 2 \cos \theta \neq r \end{cases} \Rightarrow \text{It is not symmetric about the } y\text{-axis}$$

Therefore; it is *not* symmetric about the origin.



Exercise

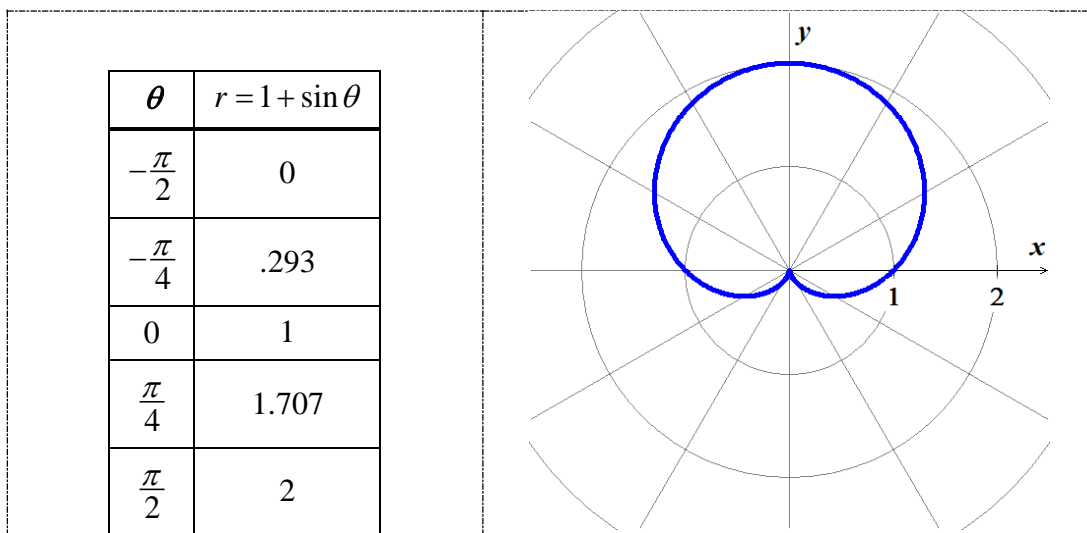
Identify the symmetries of the curve. Then sketch the curve. $r = 1 + \sin \theta$

Solution

$$\begin{cases} 1 + \sin(-\theta) = 1 - \sin \theta \neq r \\ 1 + \sin(\pi - \theta) = 1 + \sin \theta \neq -r \end{cases} \Rightarrow \text{It is not symmetric about the } x\text{-axis}$$

$$1 + \sin(\pi - \theta) = 1 + \sin \theta = r \Rightarrow \text{It is symmetric about the y-axis}$$

Therefore; it is not symmetric about the origin.



Exercise

Identify the symmetries of the curve. Then sketch the curve.

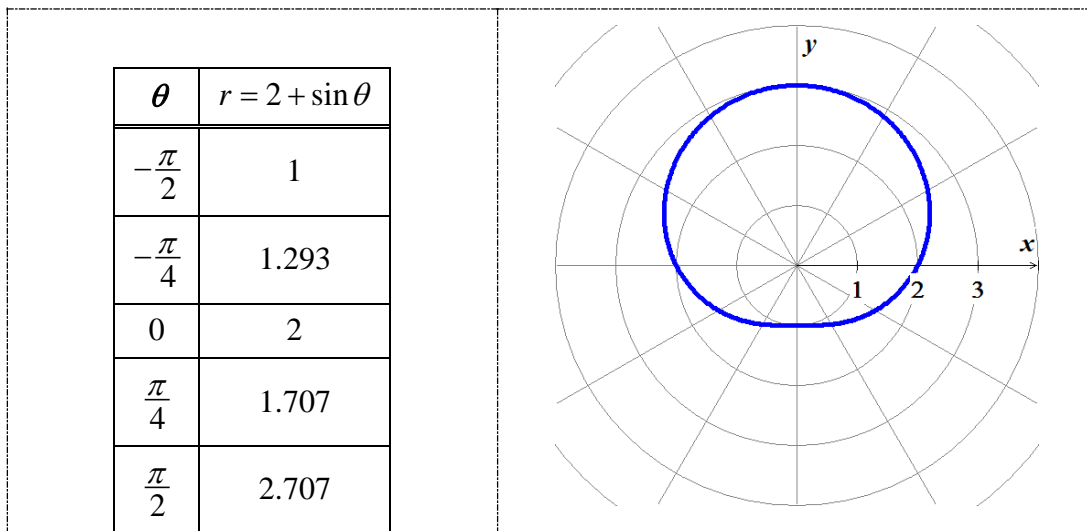
$$r = 2 + \sin \theta$$

Solution

$$\begin{cases} 2 + \sin(-\theta) = 2 - \sin \theta \neq r \\ 2 + \sin(\pi - \theta) = 2 + \sin \theta \neq -r \end{cases} \Rightarrow \text{It is not symmetric about the x-axis}$$

$$2 + \sin(\pi - \theta) = 2 + \sin \theta = r \Rightarrow \text{It is symmetric about the y-axis}$$

Therefore; it is not symmetric about the origin.



Exercise

Identify the symmetries of the curve. Then sketch the curve.

$$r^2 = \sin \theta$$

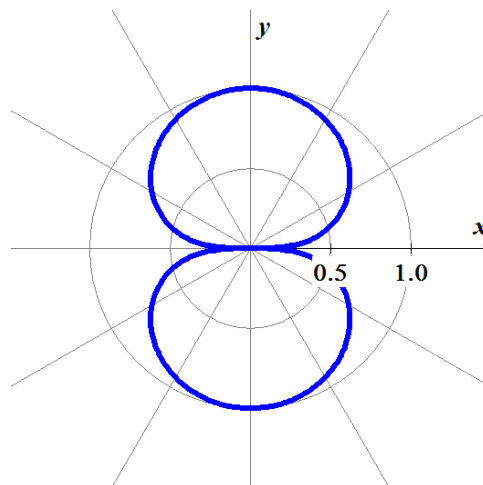
Solution

$\sin(\pi - \theta) = \sin \theta = r^2 \Rightarrow$ It is symmetric about the x -axis

$\sin(\pi - \theta) = \sin \theta = r^2 \Rightarrow$ It is symmetric about the y -axis

Therefore; it is symmetric about the origin.

θ	$r = \sqrt{\sin \theta}$
0	0
$\frac{\pi}{6}$	0.707
$\frac{\pi}{4}$	0.84
$\frac{\pi}{3}$	0.93
$\frac{\pi}{2}$	1



Exercise

Identify the symmetries of the curve. Then sketch the curve.

$$r^2 = -\sin \theta$$

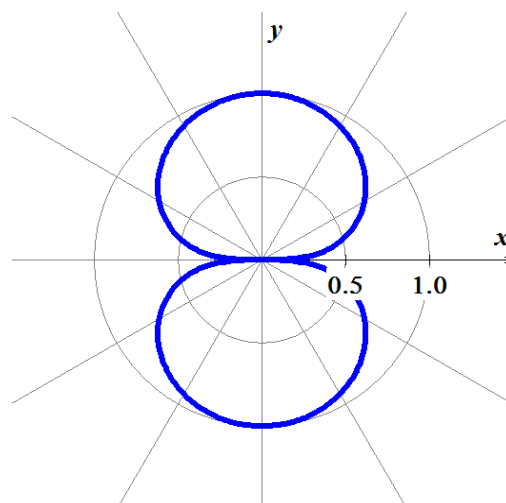
Solution

$-\sin(\pi - \theta) = -\sin \theta = r^2 \Rightarrow$ It is symmetric about the x -axis

$-\sin(\pi - \theta) = -\sin \theta = r^2 \Rightarrow$ It is symmetric about the y -axis

Therefore; it is symmetric about the origin

θ	$r^2 = -\sin \theta$
0	0
$\frac{\pi}{6}$	0.707
$\frac{\pi}{4}$	0.84
$\frac{\pi}{3}$	0.93
$\frac{\pi}{2}$	1



Exercise

Identify the symmetries of the curve. Then sketch the curve.

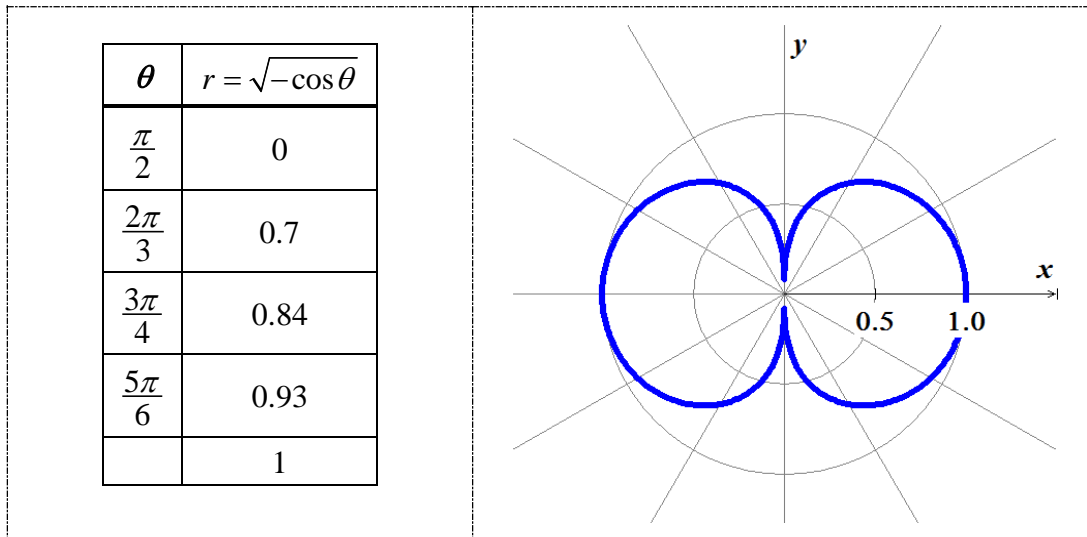
$$r^2 = -\cos \theta$$

Solution

$-\cos(-\theta) = -\cos \theta = r^2 \Rightarrow$ It is symmetric about the x -axis

$\begin{cases} -\cos(-\theta) = -\cos \theta = r^2 \\ (-r)^2 = r^2 = -\cos \theta \end{cases} \Rightarrow$ It is symmetric about the y -axis

Therefore; it is symmetric about the origin

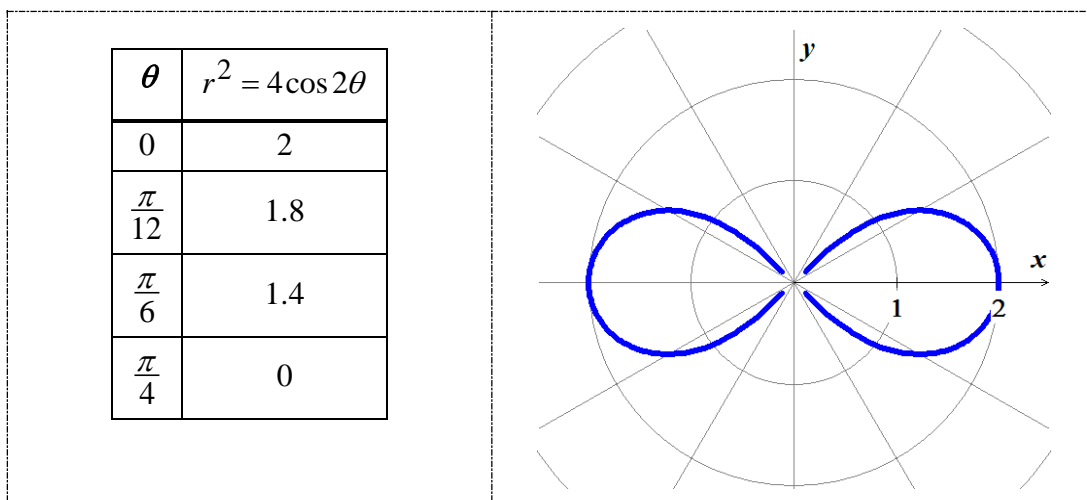


Exercise

Graph the lemniscate. What symmetries do these curves have?

$$r^2 = 4 \cos 2\theta$$

Solution

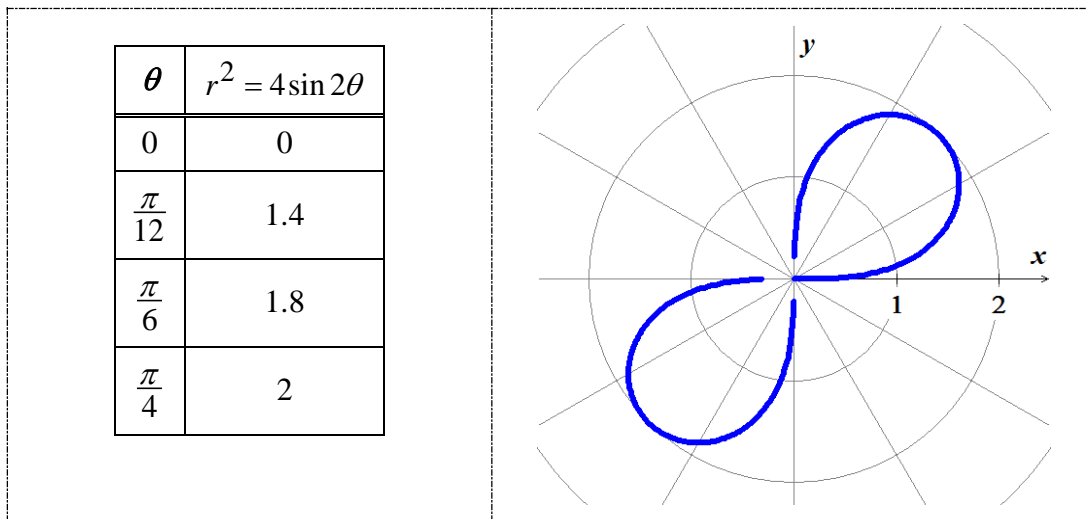


$(\pm r)^2 = 4 \cos 2(-\theta) \Rightarrow r^2 = 4 \cos 2\theta$ The graph is symmetric about the x -axis and the y -axis
 \Rightarrow The graph is symmetric about the origin.

Exercise

Graph the lemniscate. What symmetries do these curves have? $r^2 = 4\sin 2\theta$

Solution



$(\pm r)^2 = 4\sin 2\theta \Rightarrow r^2 = 4\sin 2\theta$ The graph is symmetric about the origin.

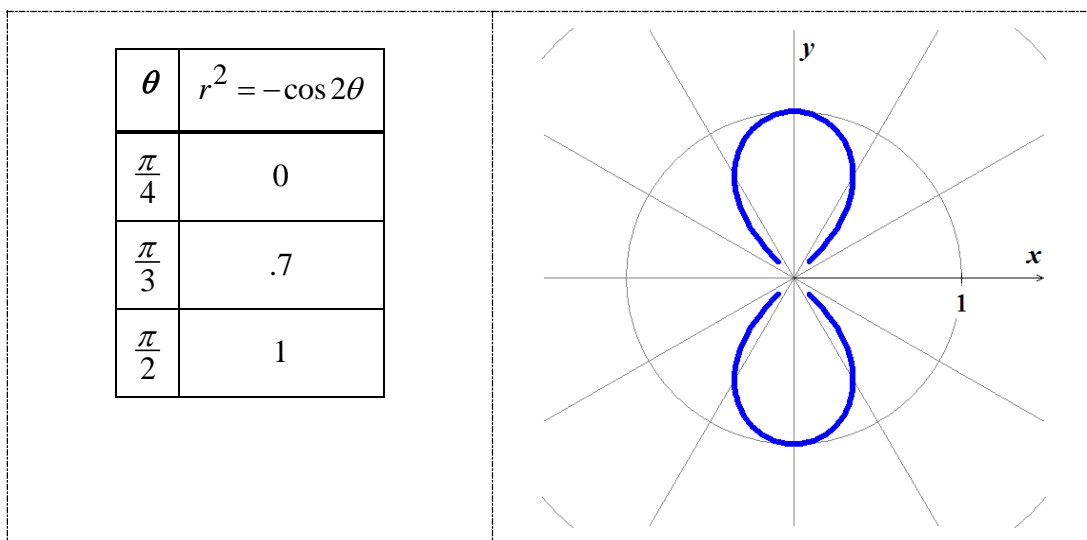
$4\sin 2(-\theta) = -4\sin 2\theta \neq r^2 \Rightarrow$ The graph is *not* symmetric about the x -axis

$4\sin 2(\pi - \theta) = 4\sin(2\pi - 2\theta) = 4\sin(-2\theta) = -4\sin 2\theta \neq r^2 \Rightarrow$ The graph is *not* symmetric about the y -axis.

Exercise

Graph the lemniscate. What symmetries do these curves have? $r^2 = -\cos 2\theta$

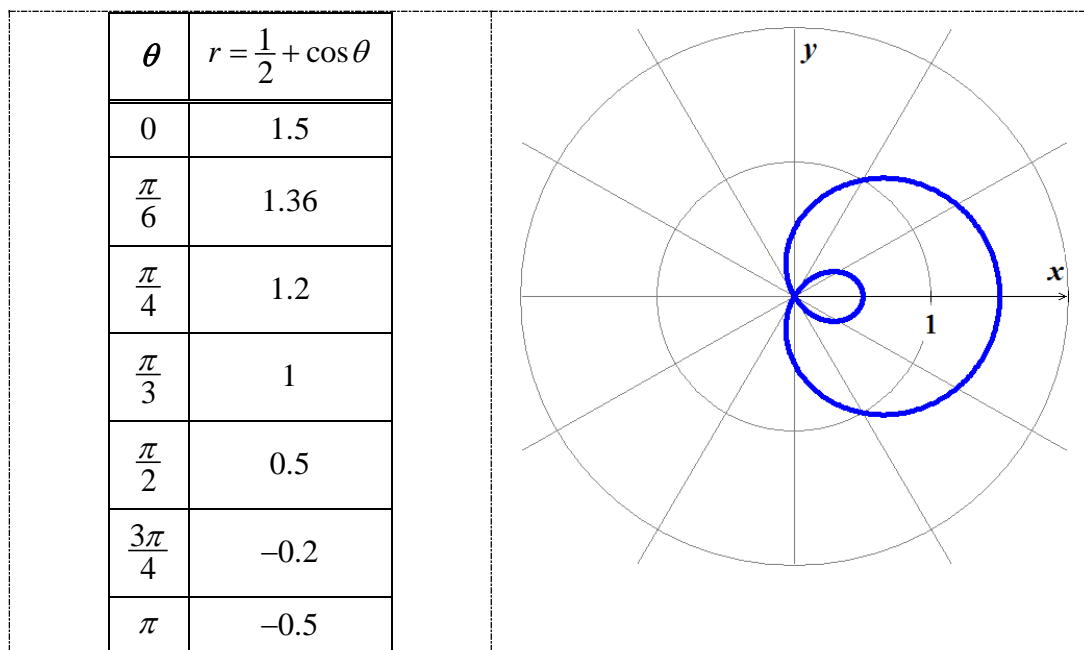
Solution



Exercise

Graph the limaçons is Old French for “snail”. Equations for limaçons have the form $r = \frac{1}{2} + \cos \theta$

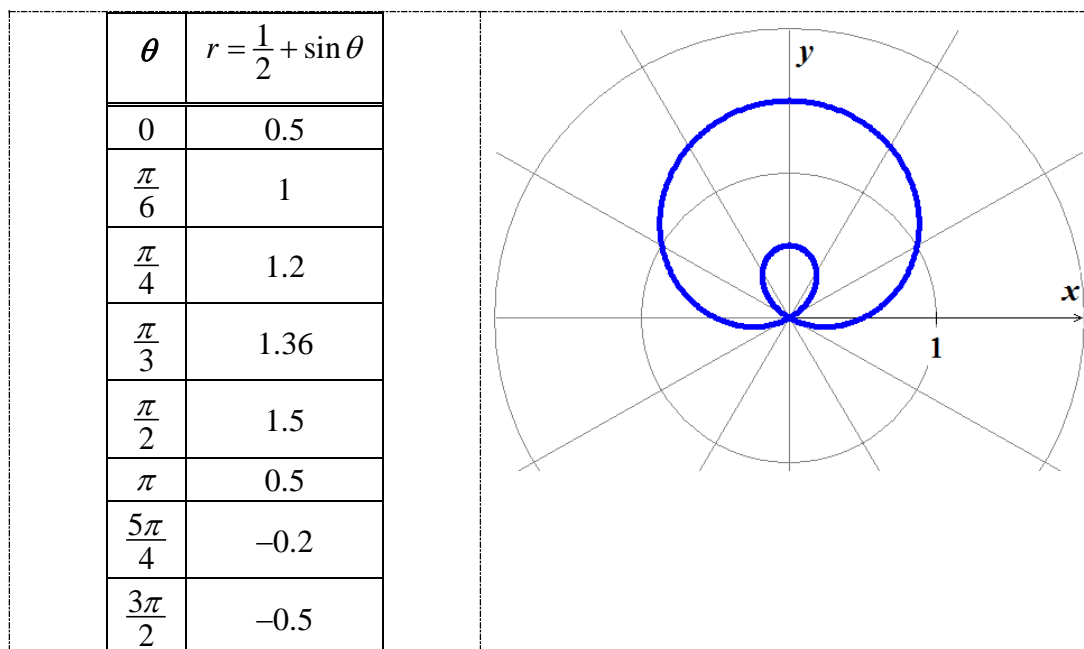
Solution



Exercise

Graph the limaçons is Old French for “snail”. Equations for limaçons have the form $r = \frac{1}{2} + \sin \theta$

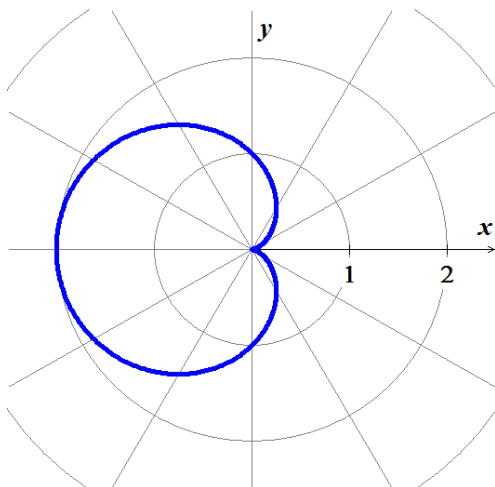
Solution



Exercise

Graph the limaçon is Old French for “snail”. Equations for limaçons have the form $r = 1 - \cos \theta$

Solution

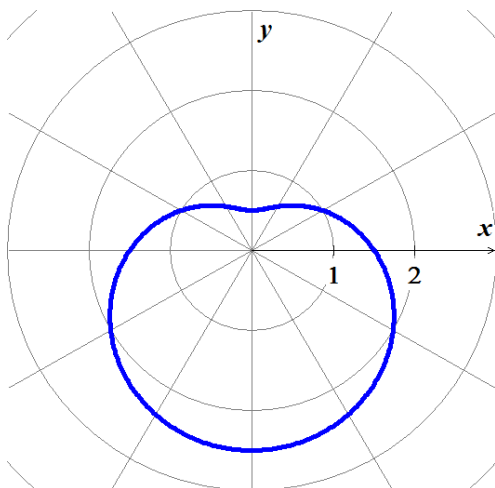


Exercise

Graph the limaçon is Old French for “snail”.

Equations for limaçons have the form $r = \frac{3}{2} - \sin \theta$

Solution

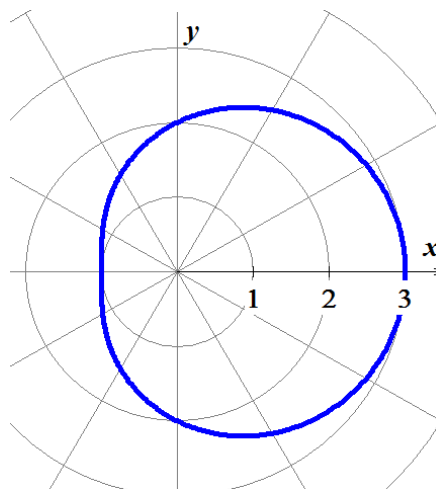


Exercise

Graph the limaçon is Old French for “snail”. Equations for limaçons have the form $r = 2 + \cos \theta$

Solution

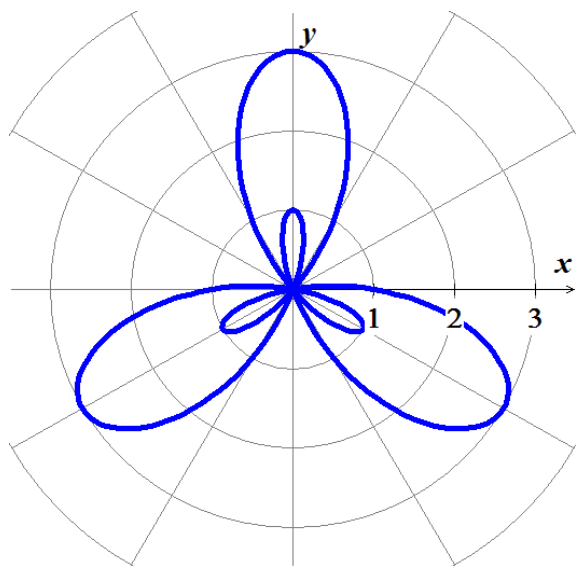
θ	$r = 2 + \cos \theta$
0	3
$\frac{\pi}{6}$	≈ 1.866
$\frac{\pi}{4}$	≈ 1.7
$\frac{\pi}{2}$	2
$\frac{3\pi}{4}$	≈ 1.29
π	1



Exercise

Graph the equation $r = 1 - 2 \sin 3\theta$

Solution



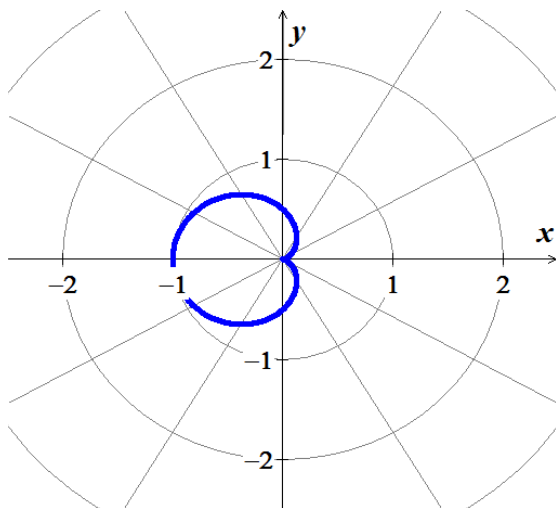
Exercise

Graph the equation $r = \sin^2 \frac{\theta}{2}$

Solution

$$\sin^2 \left(-\frac{\theta}{2} \right) = \sin^2 \left(\frac{\theta}{2} \right) = r \Rightarrow \text{It is symmetric about the } x\text{-axis}$$

θ	$r = \sin^2 \frac{\theta}{2}$
0	0
$\frac{\pi}{3}$	0.25
$\frac{\pi}{2}$	0.5
$\frac{2\pi}{3}$	0.75
π	1

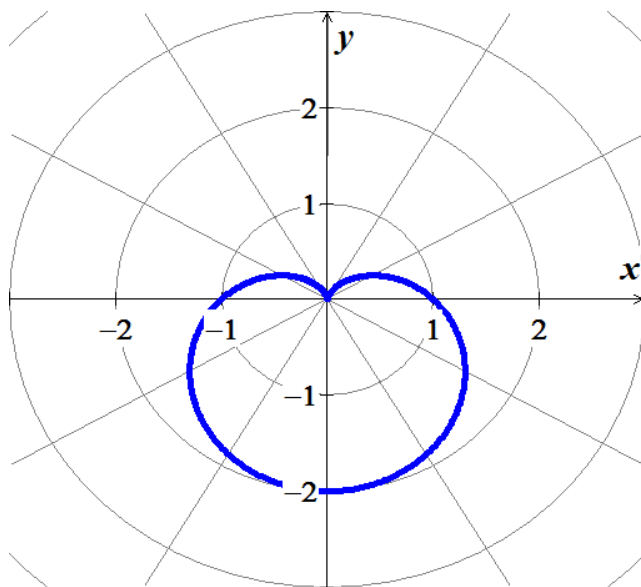


Exercise

Graph the equation $r = 1 - \sin \theta$

Solution

θ	$r = 1 - \sin \theta$
0	1
$\frac{\pi}{6}$	0.5
$\frac{\pi}{4}$	≈ 0.3
$\frac{\pi}{2}$	0
$\frac{3\pi}{4}$	≈ 0.3
π	1
$\frac{7\pi}{6}$	1.5
$\frac{3\pi}{2}$	2



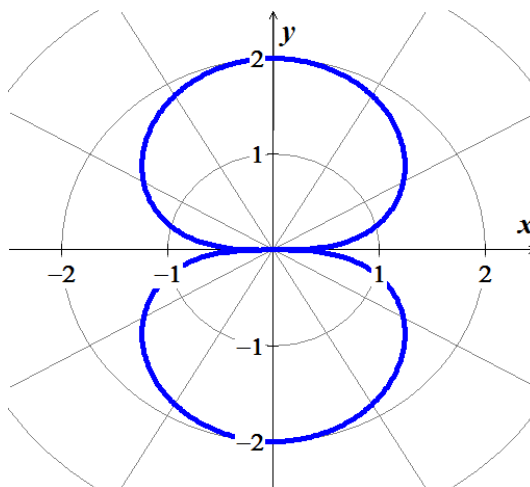
Exercise

Graph the equation $r^2 = 4 \sin \theta$

Solution

$4 \sin(\pi - \theta) = 4 \sin \theta = r \Rightarrow$ It is symmetric about the y-axis

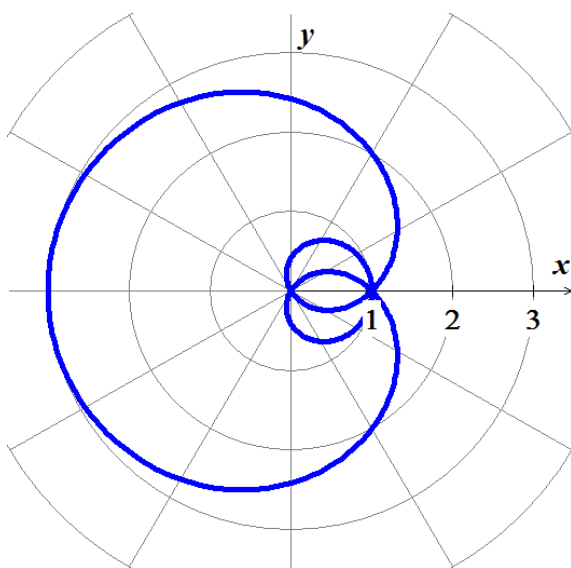
θ	$r = \pm 2\sqrt{\sin \theta}$
0	0
$\frac{\pi}{6}$	$\pm\sqrt{2} \approx \pm 1.4$
$\frac{\pi}{4}$	$\approx \pm 1.7$
$\frac{\pi}{3}$	$\approx \pm 1.9$
$\frac{\pi}{2}$	± 2



Exercise

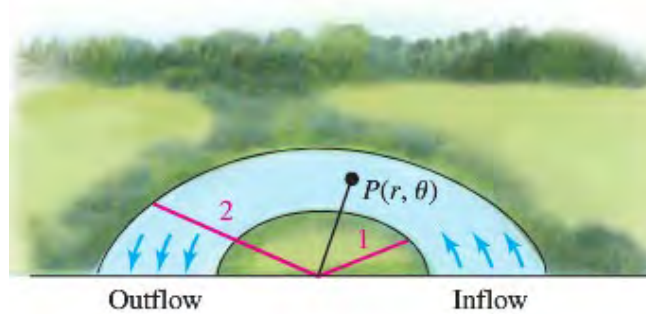
Graph the nephroid of Freeth equation $r = 1 + 2\sin \frac{\theta}{2}$

Solution



Exercise

Water flows in a shallow semicircular channel with inner and outer radii of 1 m and 2 m. At a point $P(r, \theta)$ in the channel, the flow is in the tangential direction (counterclockwise along circles), and it depends only on r , the distance from the center of the semicircles.



- Express the region formed by the channel as a set in polar coordinates.
- Express the inflow and outflow regions of the channel as sets in polar coordinates.
- Suppose the tangential velocity of the water in m/s is given by $v(r) = 10r$, for $1 \leq r \leq 2$. Is the velocity greater at $\left(1.5, \frac{\pi}{4}\right)$ or $\left(1.2, \frac{3\pi}{4}\right)$? Explain.
- Suppose the tangential velocity of the water is given by $v(r) = \frac{20}{r}$, for $1 \leq r \leq 2$. Is the velocity greater at $\left(1.8, \frac{\pi}{6}\right)$ or $\left(1.3, \frac{2\pi}{3}\right)$? Explain.
- The total amount of water that flows through the channel (across a cross section of the channel $\theta = \theta_0$) is proportional to $\int_1^2 v(r) dr$. Is the total flow through the channel greater for the flow in part (c) or (d)?

Solution

- The region is given by $\{(r, \theta): 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$
- The inflow is given by $\{(r, \theta): 1 \leq r \leq 2, \theta = 0\}$
The outflow is given by $\{(r, \theta): 1 \leq r \leq 2, \theta = \pi\}$
- The tangential velocity at $\left(1.5, \frac{\pi}{4}\right)$ is

$$\begin{aligned} v(1.5) &= 10(1.5) \\ &= 15 \text{ m/s} \end{aligned}$$

At $\left(1.2, \frac{3\pi}{4}\right)$ is

$$\begin{aligned} v(1.2) &= 10(1.2) \\ &= 12 \text{ m/s} \end{aligned}$$

So it is greater at 1.5.

d) The tangential velocity at $\left(1.8, \frac{\pi}{6}\right)$ is

$$v(1.8) = \frac{20}{1.8}$$

$$\approx 11.11 \text{ m/s}$$

At $\left(1.3, \frac{2\pi}{3}\right)$

$$v(1.3) = \frac{20}{1.3}$$

$$\approx 15.38 \text{ m/s}$$

So, it is greater at 1.3.

$$e) \int_1^2 v(r) dr = \int_1^2 10r dr$$

$$= 5r^2 \Big|_1^2$$

$$= 15$$

$$\int_1^2 v(r) dr = \int_1^2 \frac{20}{r} dr$$

$$= 20 \ln r \Big|_1^2$$

$$= 20 \ln 2 \approx 13.86$$

So the flow in part (c) is greater.

Exercise

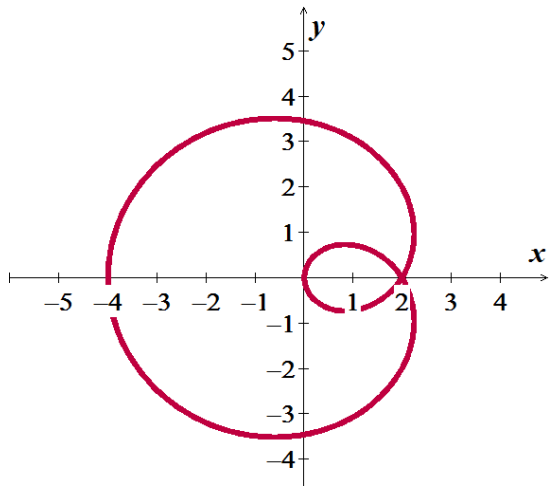
A simplified model assumes that the orbits of Earth and Mars are circular with radii of 2 and 3, respectively, and that Earth completes one orbit in one year while Mars takes two years. When $t = 0$, Earth is at $(2, 0)$ and Mars is at $(3, 0)$; both orbit the Sun (at $(0, 0)$) in the counterclockwise direction. The position of Mars relative to Earth is given by the parametric equations

$$x = (3 - 4 \cos \pi t) \cos \pi t + 2, \quad y = (3 - 4 \cos \pi t) \sin \pi t$$

- Graph the parametric equations, for $0 \leq t \leq 2$
- Letting $r = 3 - 4 \cos \pi t$, explain why the path of Mars relative to Earth is a limaçon.

Solution

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- b)** $r = 3 - 4 \cos \pi t$ is a limaçon, and $x - 2 = r \cos \pi t$ and $y = r \sin \pi t$ is a circle, and the composition of a limaçon and a circle is a limaçon.