Ex 1?

$$\begin{cases} z = x^2 + 3y^2 \\ z = 8 - x^2 - y^2 \end{cases}$$

$$x^2 + 3y^2 \le z \le 8 - x^2 - y^2$$

$$z = x^2 + 3y^2 = 8 - x^2 - y^2$$

$$4y^2 = 8 - 2x^2$$

$$y^2 = 2 - \frac{1}{2}x^2$$

$$y = \pm\sqrt{\frac{4-x^2}{2}}$$

$$\int_a^b d\theta = b - a \qquad \theta\Big|_a^b$$

$$y = 0 = 2 - \frac{1}{2}x^2 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

$$V = \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8 - x^2 - y^2 - x^2 - 3y^2)\, dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8 - 2x^2 - 4y^2)\, dy\, dx$$

$$= \int_{-2}^{2} \left(8y - 2x^2 y - \frac{4}{3}y^3\right)\Bigg|_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} dx$$

$$V = \int_{-2}^{2} \Big[2(4-x^2)\sqrt{\frac{4-x^2}{2}} - \frac{4}{3}\left(\frac{4-x^2}{2}\right)^{3/2} + 2(4-x^2)\left(\frac{4-x^2}{2}\right)^{1/2} - \frac{4}{3}\left(\frac{4-x^2}{2}\right)^{3/2} \Big] dx$$

$$= \int_{-2}^{2} \Big[2\left(\frac{2}{\sqrt{2}}\right)(4-x^2)^{3/2} - (2)\frac{4}{3}\frac{(4-x^2)^{3/2}}{2^{3/2}} \Big] dx$$

$\sqrt{2}$, 2 , $\cdot 2^{1/2}$

$$= \left(2\sqrt{2} - \frac{2\sqrt{2}}{3}\right) \int_{-2}^{2} (4-x^2)^{3/2} dx$$

$$= \frac{4\sqrt{2}}{3} \int_{-2}^{2} \left((2\cos u)^2\right)^{3/2} 2\cos u \, du$$

$x = 2\sin u$

$dx = 2\cos u \, du$

$(4-x^2) = 4\cos^2 u$

$\sqrt{4-x^2} = 2\cos u$

$\to$ same.

$$= \frac{64\sqrt{2}}{3} \int_{-2}^{2} \cos^4 u \, du$$

$$= \frac{64\sqrt{2}}{3} \int_{-2}^{2} \left(\frac{1+\cos 2u}{2}\right)^2 du$$

$$= \frac{16\sqrt{2}}{3} \int_{-2}^{2} (1 + 2\cos 2u + \cos^2 2u) du$$

$$= \frac{16\sqrt{2}}{3} \int_{-2}^{2} \left(1 + 2\cos 2u + \frac{1}{2} + \frac{1}{2}\cos 4u\right) du$$

$$= \frac{16\sqrt{2}}{3} \left[\frac{3}{2}u + \sin 2u + \frac{1}{8}\sin 4u \right]_{-\pi/2}^{\pi/2}$$

$\sin u = \frac{1}{2}x$ $\begin{cases} x = 2 \to \sin u = 1 & u = \frac{\pi}{2} \\ x = -2 \to \sin u = -1 & u = \frac{-\pi}{2} \end{cases}$

$$V' = \frac{16\pi^2}{3}\left(\frac{\alpha\pi}{4} + \frac{3\pi}{11}\right)$$

$$= 8\pi\,6.25 \text{ unit}^3$$

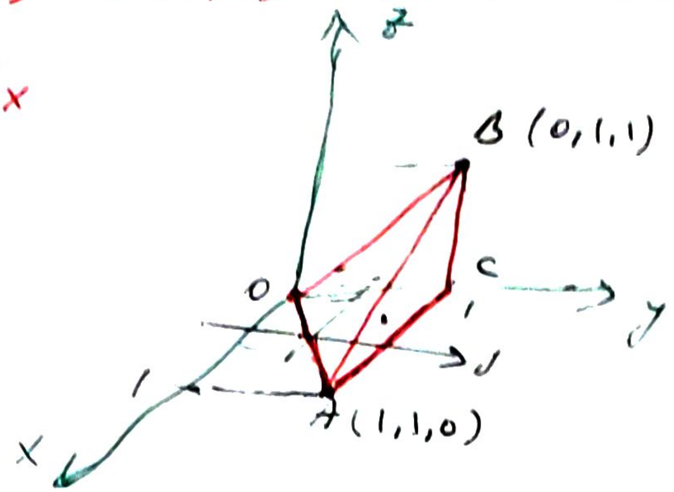
Ex $f(x,y,z) = 1$

$(0,0,0)$, $(1,1,0)$ $(0,1,0)$ $(0,1,1)$

$dz\,dy\,dx$

Tethrahedron

$0 \le x \le 1$



Plane OAB

$ax + by + cz = 0$ $\to d = 0$ $(0,0,0)$

A $a + b = 0 \to a = -b$

B $b + c = 0 \to c = -b$

$-bx + by - bz = 0$

$-x + y - z = 0$

$z = -x + y$

$0 \le z \le y - x$

$x \le y \le 1$

$$V = \int_0^1 \int_x^1 \int_0^{y-x} dz\,dy\,dx$$

$$= \int_0^1 \int_x^1 (y - x)\,dy\,dx$$

$$= \int_0^1 \int_x^1 \left(\frac{1}{2}y^2 - xy \Big|_x^1\right) dx$$

$$V = \int_0^1 \left(\frac{1}{2} - x - \frac{1}{2}x^2 + x^2 \right) dx$$

$$= \int_0^1 \left(\frac{1}{2} - x + \frac{1}{2}x^2 \right) dx$$

$$= \frac{1}{2}x - \frac{1}{2}x^2 + \frac{1}{6}x^3 \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{6}$$

$$= \boxed{\frac{1}{6} \text{ unit}^3}$$

Ex $\int_0^1 \int_x^{x^2} \int_{xy}^{x^2y^3} xy \, dz \, dy \, dx$

$$= \int_0^1 \int_x^{x^2} xyz \Big|_{xy}^{x^2y^3} dy \, dx$$

$$= \int_0^1 \int_x^{x^2} xy\,(x^2y^3 - xy)\, dy \, dx$$

$$= \int_0^1 \int_x^{x^2} (x^3y^4 - x^2y^2)\, dy \, dx$$

$$= \int_0^1 \left(\frac{1}{5}x^3y^5 - \frac{1}{3}x^2y^3\right)\Big|_x^{x^2} dx$$

$$= \int_0^1 \Big(\frac{1}{5}x^{13} \underbrace{- \frac{1}{3}x^8 - \frac{1}{5}x^8}_{-\frac{8}{15}} - \frac{1}{3}x^5\Big) dx$$

$$= \frac{1}{70}x^{14} - \frac{8}{135}x^9 - \frac{1}{18}x^6 \Big|_0^1$$

$$= \frac{1}{70} - \frac{8}{135} - \frac{1}{18}$$

$$= \frac{2}{189}$$

Ex

$$\int_0^a \int_0^{a-z} \int_0^{a-y-z} yz\,dx\,dy\,dz$$

$$= \int_0^a \int_0^{a-z} yz(a-y-z)\,dy\,dz$$

$$= \int_0^a \int_0^{a-z} (ayz - y^2 z - yz^2)\,dy\,dz$$

$$= \int_0^a \left(\tfrac{1}{2}az\underline{y^2} - \tfrac{1}{3}zy^3 - \tfrac{1}{2}z^2\underline{y^2}\right)\Big|_0^{a-z} dz$$

$$= \int_0^a \left(\left(\tfrac{1}{2}az - \tfrac{1}{2}z^2\right)(a-z)^2 - \tfrac{1}{3}z(a-z)^3\right) dz$$

$$= \int_0^a \left[\tfrac{1}{2}z(a-z)^3 - \tfrac{1}{3}z(a-z)^3\right] dz$$

$$= \frac{1}{6}\int_0^a z(a-z)^3\,dz$$

$$= \frac{1}{6}\int_0^a (a^3 z - 3a^2 z^2 + 3az^3 - z^4)\,dz$$

$$= \frac{1}{6}\left[\tfrac{1}{2}a^3 z^2 - a^2 z^3 + \tfrac{3}{4}az^4 - \tfrac{1}{5}z^5\right|_0^a$$

$$= \frac{1}{6}\left(\tfrac{1}{2}a^5 - a^5 + \tfrac{3}{4}a^5 - \tfrac{1}{5}a^5\right)$$

$$= \frac{a^5}{6}\left(\tfrac{1}{2} - 1 + \tfrac{3}{4} - \tfrac{1}{5}\right)$$

$$= \frac{a^5}{120}$$

Ex $\quad V?$ $\quad$ 13 $\quad z = \dfrac{4}{y^2+1} \;\Big|\; \begin{array}{l} y = x \\ y = 3 \\ x = 0 \\ z = 0 \end{array}$

$$0 \le z \le \frac{4}{y^2+1}$$

$$0 \le x \le y \qquad\qquad \cancel{x} \le \cancel{y} \le 3$$

$$0 \le y \le 3 \qquad\qquad 0 \le x \le 3$$

$$V = \int_0^3 \int_0^y \int_0^{\frac{4}{y^2+1}} dz\,dx\,dy$$

$$= \int_0^3 \int_0^y \frac{4}{y^2+1}\,dx\,dy$$

$$= \int_0^3 \frac{4y}{y^2+1}\,dy$$

$$= \int_0^3 \frac{2}{y^2+1}\,d(y^2+1)$$

$$= 2\ln(y^2+1)\Big|_0^3$$

$$= 2(\ln 10 - \ln 1)$$

$$= \underline{2\ln 10}\;\Big| \quad \text{unit}^3$$

$$(x, y, z) \to (r, \theta, z)$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad x^2+y^2=r^2, \quad \tan\theta = \frac{y}{x}$$

Ex V?
$$\begin{cases} x^2+y^2+z^2=9 \to r^2+z^2=9 & (1) \\ x^2+y^2=8z \to r^2=8z & (2) \end{cases}$$

$$(1) \to z = \sqrt{9-r^2}$$
$$(2) \to z = \frac{1}{8}r^2$$
$$\Big\} \ \frac{1}{8}r^2 \le z \le \sqrt{9-r^2}$$

$$\boxed{0 \le \theta \le 2\pi}$$

$$r^2 = 9 - z^2 = 8z$$
$$z^2+8z-9=0 \Rightarrow z = \begin{cases} 1 \\ -9 \ \# \end{cases}$$
$$r^2 = 8z = 8 \Rightarrow r = 2\sqrt{2}$$
$$0 \le r \le 2\sqrt{2}$$

$$V = \int_0^{2\pi} d\theta \int_0^{2\sqrt{2}} \int_{\frac{1}{8}r^2}^{\sqrt{9-r^2}} dz \, r \, dr$$

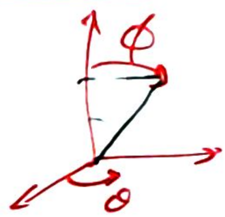
$$= 2\pi \int_0^{2\sqrt{2}} r\left(\sqrt{9-r^2} - \frac{1}{8}r^2\right) dr$$

$$= 2\pi \left[-\frac{1}{2}\int_0^{2\sqrt{2}} (9-r^2)^{1/2} d(9-r^2) - \frac{1}{8}\int_0^{2\sqrt{2}} r^3 dr \right]$$

$$V = 2\pi\left[-\frac{1}{3}(9-r^2)^{3/2} - \frac{1}{32}r^4\right|_0^{2\sqrt{2}} \quad 2^{3/2}$$

$$= 2\pi\left(-\frac{1}{3} - \overset{2}{\overline{2+9}}\right)$$

$$\boxed{= \frac{40\pi}{3}\ \text{unit}^3}$$



$$(\rho, \phi, \theta)$$

$$r = \rho \sin\phi$$

$$x = \underline{r\cos\theta} = \rho \sin\phi \cos\theta$$

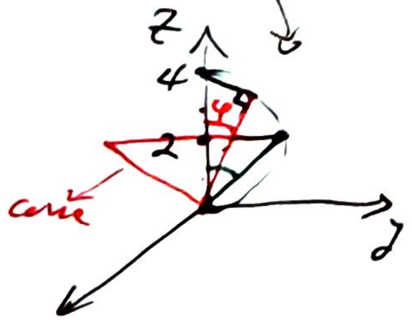
$$y = r\sin\theta = \rho\sin\phi\sin\theta$$

$$z = \rho\cos\phi$$

$$dV = \underline{\rho^2 \sin\phi}\, d\rho\, d\phi\, d\theta$$

Ex V? $x^2+y^2-z^2=0 \rightarrow z^2=x^2+y^2$

$x^2+y^2+(z-2)^2=4$



$0 \le \theta \le 2\pi$

$0 \le \phi \le \frac{\pi}{4}$

$\cos\phi = \frac{\rho}{4}$

$0 \le \rho \le 4\cos\phi$

$$V = \int_0^{2\pi} d\theta \int_0^{\pi/4} \int_0^{4\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi$$

$$= \frac{2\pi}{3} \int_0^{\pi/4} \sin\phi \, \rho^3 \Big|_0^{4\cos\phi} d\phi$$

$$= \frac{128\pi}{3} \int_0^{\pi/4} \sin\phi \cos^3\phi \, d\phi$$

$$= -\frac{128\pi}{3} \int_0^{\pi/4} \cos^3\phi \, d(\cos\phi)$$

$$= -\frac{32\pi}{3} \cos^4\phi \Big|_0^{\pi/4}$$

$$= -\frac{32\pi}{3}\left(\frac{1}{4}-1\right)$$

$$= 8\pi \text{ unit}^3$$

Ex

$$\iiint_D (x^2+y^2+z^2)^{-3/2}\, dV$$

$$\begin{cases} 1^{st} \text{ octant } & x, y, z \geq 0 \\ 2 \text{ spheres } & r = 1, 2 \quad @\, O. \end{cases}$$

$$D = \{(\rho, \phi, \theta) : 1 \leq \rho \leq 2,\ 0 \leq \phi \leq \frac{\pi}{2}$$

$$\theta \in QI \to 0 \leq \theta \leq \frac{\pi}{2}\}$$

$$V = \int_0^{\pi/2} d\theta \int_0^{\pi/2} \sin\phi\, d\phi \int_1^2 (\rho^2)^{-3/2} \rho^2\, d\rho$$

$$= \left(\frac{\pi}{2}\right) \left(-\cos\phi \Big|_0^{\pi/2}\right) \int_1^2 \frac{d\rho}{\rho}$$

$$= \frac{\pi}{2} \ln\rho \Big|_1^2$$

$$= \boxed{\frac{\pi}{2} \ln 2}$$

Ex
$$\int_0^a \int_0^{\sqrt{a^2-z^2}} \int_0^{\sqrt{a^2-y^2-z^2}} \sqrt{x^2+y^2+z^2}\, dx\,dy\,dz$$

$$\sqrt{x^2+y^2+z^2} = \rho$$

$$0 \le x \le \sqrt{a^2-y^2-z^2}$$
$$0 \le y \le \sqrt{a^2-z^2}$$
$$0 \le z \le a$$
$$\boxed{0 \le \rho \le a}$$

$$0 \le \theta \le \frac{\pi}{2}$$

$$0 \le \phi \le \frac{\pi}{2}$$

$$\int_0^{\pi/2} d\theta \int_0^{\pi/2} \sin\phi\, d\phi \int_0^a \underbrace{\rho\, \rho^2}_{\rho^3}\, d\rho$$

$$= \frac{\pi}{2} \left(-\cos\phi\right|_0^{\pi/2} \frac{1}{4} \rho^4 \Big|_0^a$$

$$= \frac{\pi}{8} a^4$$