Section 2.4 – Partial Fractions

This section shows how to express a rational; function as a sum of simpler functions, called ***partial fractions***.

Example

Evaluate $\displaystyle\int \frac{5x-3}{x^2-2x-3}\,dx$

Solution

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}$$

$$5x-3 = (A+B)x - 3A + B$$

$$x \qquad A+B=5$$

$$x^0 \qquad -3A+B=-3$$

$$\begin{array}{r} A+B=5 \\ -\quad 3A-B=3 \\ \hline 4A=8 \end{array}$$

$$A=2,\quad B=3$$

$$\int \frac{5x-3}{x^2-2x-3}\,dx = \int \left(\frac{2}{x+1} + \frac{3}{x-3}\right)dx$$

$$= 2\ln|x+1| + 3\ln|x-3| + C$$

Example

Use partial fractions to evaluate $\displaystyle\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx$

Solution

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

$$x^2+4x+1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)$$

$$= Ax^2 + 4Ax + 3A + Bx^2 + 2Bx - 3B + Cx^2 - C$$

$$x^2 \qquad A+B+C=1$$

$$x \qquad 4A+2B=4$$

$$x^0 \qquad 3A-3B-C=1$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 3 & -3 & -1 \end{vmatrix} = -16 \qquad \Delta_A = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 1 & -3 & -1 \end{vmatrix} = -12$$

$$\Delta_B = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 4 & 0 \\ 3 & 1 & -1 \end{vmatrix} = -8 \qquad \Delta_C = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 4 \\ 3 & -3 & 1 \end{vmatrix} = 4$$

$$A = \frac{-12}{-16} = \boxed{\frac{3}{4}} \quad B = \frac{-8}{-16} = \boxed{\frac{1}{2}} \quad C = \frac{4}{-16} = \boxed{-\frac{1}{4}}$$

$$\int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)}\, dx = \int \left(\frac{3}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{4}\frac{1}{x+3} \right) dx$$

$$= \boxed{\frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + K}$$

Method of Partial Fractions $\left(f(x)/g(x)\ \textbf{Proper}\right)$

1. Let $(x-r)$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $(x-r)$ that divides $g(x)$. Then,

$$\frac{A_1}{(x-r)}+\frac{A_2}{(x-r)^2}+\cdots+\frac{A_m}{(x-r)^m}$$

2. Let x^2+px+q be an irreducible quadratic function of $g(x)$ has no real roots. Suppose that $\left(x^2+px+q\right)^n$ is the highest power. Then

$$\frac{B_1x+C_1}{\left(x^2+px+q\right)}+\frac{B_2x+C_2}{\left(x^2+px+q\right)^2}+\cdots+\frac{B_nx+C_n}{\left(x^2+px+q\right)^n}$$

3. Set the original fraction $\frac{f(x)}{g(x)}$ equal to the sum of these partial fractions.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Example

Use partial fractions to evaluate $\int \frac{6x+7}{(x+2)^2}dx$

Solution

$$\frac{6x+7}{(x+2)^2}=\frac{A}{x+2}+\frac{B}{(x+2)^2}$$

$$\begin{aligned}6x+7&=A(x+2)+B\\&=Ax+2A+B\end{aligned}$$

$$\to\begin{cases}\boxed{A=6}\\ 2A+B=7 & \to \underline{B=7-12=-5}\end{cases}$$

$$\begin{aligned}\int\frac{6x+7}{(x+2)^2}dx&=\int\left(\frac{6}{x+2}-\frac{5}{(x+2)^2}\right)dx \qquad\qquad d(x+2)=dx\\&=\int\frac{6}{x+2}dx-5\int(x+2)^{-2}\,d(x+2)\\&=6\ln|x+2|+5(x+2)^{-1}+C\\&=6\ln|x+2|+\frac{5}{x+2}+C\end{aligned}$$

Example

Use partial fractions to evaluate $\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx$

Solution

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

$$\begin{array}{r} 2x \\ x^2 - 2x - 3 \overline{\big) 2x^3 - 4x^2 - x - 3} \\ \underline{2x^3 - 4x^2 - 6x} \\ 5x - 3 \end{array}$$

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x+1} + \frac{B}{x-3}$$

$$5x - 3 = (A + B)x - 3A + B$$

$$x \qquad A + B = 5$$

$$x^0 \qquad -3A + B = -3$$

$$\begin{array}{r} A + B = 5 \\ -\quad 3A - B = 3 \\ \hline 4A = 8 \end{array}$$

$$\boxed{A = 2, \quad B = 3}$$

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx = \int 2x dx + \int \frac{5x - 3}{x^2 - 2x - 3} dx$$

$$= \int 2x dx + \int \frac{2}{x+1} dx + \int \frac{3}{x-3} dx$$

$$\boxed{= x^2 + 2\ln|x+1| + 3\ln|x-3| + C}$$

Example

Use partial fractions to evaluate $\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} dx$

Solution

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}$$

$$-2x + 4 = (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1)$$

$$= (Ax + B)(x^2 - 2x + 1) + C(x^3 - x^2 + x - 1) + Dx^2 + D$$

$$= (A + C)x^3 + (-2A + B - C + D)x^2 + (A - 2B + C)x + B - C + D$$

$x^3 \qquad A + C = 0 \qquad \to C = -A$

$x^2 \qquad -2A + B - C + D = 0$

$x \qquad A - 2B + C = -2 \qquad \to -2B = -2 \quad \underline{B = 1}$

$x^0 \qquad B - C + D = 4$

$$\begin{array}{r} -A + D = -1 \\ \underline{A + D = 3} \\ 2D = 2 \end{array} \quad \to \quad \underline{D = 1 \quad A = 2}$$

$\to \underline{C = -2}$

$$\int \frac{-2x+4}{\left(x^2+1\right)(x-1)^2}dx = \int \left(\frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right)dx$$

$$= \int \left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right)dx$$

$$= \ln\left(x^2+1\right) + \tan^{-1}x - 2\ln|x-1| - \frac{1}{x-1} + K$$

Example

Use partial fractions to evaluate $\displaystyle\int \frac{dx}{x\left(x^2+1\right)^2}$

Solution

$$\frac{1}{x\left(x^2+1\right)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{\left(x^2+1\right)^2}$$

$$1 = A\left(x^2+1\right)^2 + (Bx+C)x\left(x^2+1\right) + x(Dx+E)$$

$$1 = (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

$$\begin{cases} x^4 & A + B = 0 & \to \underline{B = -1} \\ x^3 & \underline{C = 0} & \\ x^2 & 2A + B + D = 0 & \to \underline{D = -1} \\ x & C + E = 0 & \to \underline{E = 0} \\ x^0 & \underline{A = 1} & \end{cases}$$

$$\int \frac{dx}{x\left(x^2+1\right)^2} = \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{\left(x^2+1\right)^2}$$

$$= \int \frac{dx}{x} - \frac{1}{2}\int \frac{1}{x^2+1}\,d\left(x^2+1\right) - \frac{1}{2}\int \frac{1}{\left(x^2+1\right)^2}\,d\left(x^2+1\right)$$

$$= \ln|x| - \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{2}\frac{1}{x^2+1} + K$$

$$= \ln|x| - \ln\sqrt{x^2+1} + \frac{1}{2}\frac{1}{x^2+1} + K$$

$$= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2\left(x^2+1\right)} + K$$

Exercises *Section* 2.4 – Partial Fractions

Express the integrand as a sum of partial fractions and evaluate the integrals

1. $\displaystyle\int \frac{dx}{x^2+2x}$

2. $\displaystyle\int \frac{2x+1}{x^2-7x+12}\,dx$

3. $\displaystyle\int \frac{x+3}{2x^3-8x}\,dx$

4. $\displaystyle\int \frac{x^2}{(x-1)\left(x^2+2x+1\right)}\,dx$

5. $\displaystyle\int \frac{8x^2+8x+2}{\left(4x^2+1\right)^2}\,dx$

6. $\displaystyle\int \frac{x^2+x}{x^4-3x^2-4}\,dx$

7. $\displaystyle\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{\left(\theta^2+1\right)^3}\,d\theta$

8. $\displaystyle\int \frac{x^4}{x^2-1}\,dx$

9. $\displaystyle\int \frac{16x^3}{4x^2-4x+1}\,dx$

10. $\displaystyle\int \frac{e^{4x}+2e^{2x}-e^x}{e^{2x}+1}\,dx$

11. $\displaystyle\int \frac{\sin\theta\,d\theta}{\cos^2\theta+\cos\theta-2}$

12. $\displaystyle\int \frac{(x-2)^2\tan^{-1}(2x)-12x^3-3x}{\left(4x^2+1\right)(x-2)^2}\,dx$

13. $\displaystyle\int \frac{\sqrt{x+1}}{x}\,dx$

14. $\displaystyle\int \frac{x^3-2x^2+3x-4}{x^2+1}\,dx$

15. $\displaystyle\int \frac{4x^2+2x+4}{x+1}\,dx$

16. $\displaystyle\int \frac{3x^2+7x-2}{x^3-x^2-2x}\,dx$

17. $\displaystyle\int \frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)}\,dx$

18. $\displaystyle\int \frac{5x^2-3x+2}{x^3-2x^2}\,dx$

19. $\displaystyle\int \frac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}\,dx$

20. $\displaystyle\int \frac{1}{x^2-5x+6}\,dx$

21. $\displaystyle\int \frac{1}{x^2-5x+5}\,dx$

22. $\displaystyle\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx$

23. $\displaystyle\int \frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}\,dx$

24. $\displaystyle\int \frac{8x^3+13x}{\left(x^2+2\right)^2}\,dx$

25. $\displaystyle\int \frac{\sin x}{\cos x+\cos^2 x}\,dx$

26. $\displaystyle\int \frac{5\cos x}{\sin^2 x+3\sin x-4}\,dx$

27. $\displaystyle\int \frac{e^x}{\left(e^x-1\right)\left(e^x+4\right)}\,dx$

28. $\displaystyle\int \frac{e^x}{\left(e^{2x}+1\right)\left(e^x-1\right)}\,dx$

29. $\displaystyle\int \frac{\sqrt{x}}{x-4}\,dx$

30. $\displaystyle\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$

31. $\displaystyle\int \frac{dx}{1+\sin x}$

32. $\displaystyle\int \frac{dx}{2+\cos x}$

33. $\displaystyle\int \frac{dx}{1-\cos x}$

34. $\displaystyle\int \frac{dx}{1+\sin x+\cos x}$

35. $\displaystyle\int \frac{1}{x^2-9}\,dx$

36. $\displaystyle\int \frac{5}{x^2+3x-4}\,dx$

37. $\displaystyle\int \frac{2}{9x^2-1}\,dx$

38. $\displaystyle\int \frac{3-x}{3x^2-2x-1}\,dx$

39. $\displaystyle\int \frac{x^2+12x+12}{x^3-4x}\,dx$

40. $\displaystyle\int \frac{x^3-x+3}{x^2+x-2}\,dx$

41. $\displaystyle\int \frac{5x-2}{(x-2)^2}\,dx$

42. $\displaystyle\int \frac{2x^3-4x^2-15x+4}{x^2-2x-8}\,dx$

43. $\displaystyle\int \frac{x+2}{x^2+5x}\,dx$

44. $\displaystyle\int \frac{\sec^2 x}{\tan^2 x+5\tan x+6}\,dx$

45. $\displaystyle\int \frac{\sec^2 x}{\tan x(\tan x+1)}\,dx$

46. $\displaystyle\int \frac{x\;dx}{x^2+4x+3}$

47. $\displaystyle\int \frac{x+1}{x^2(x-1)}\,dx$

48. $\displaystyle\int \frac{2x^3+x^2-21x+24}{x^2+2x-8}\,dx$

49. $\displaystyle\int \frac{8x+5}{2x^2+3x+1}\,dx$

50. $\displaystyle\int \frac{2x^2+7x+4}{x^3+2x^2+2x}\,dx$

51. $\displaystyle\int \frac{3x^3+4x^2+6x}{(x+1)^2(x^2+4)}\,dx$

52. $\displaystyle\int \frac{x^2-4}{x^2+4}\,dx$

53. $\displaystyle\int \frac{dx}{x^2-2x-15}$

54. $\displaystyle\int \frac{3x^2+x-3}{x^2-1}\,dx$

55. $\displaystyle\int \frac{2x^2-4x}{x^2-4}\,dx$

56. $\displaystyle\int \frac{dx}{x^3-2x^2}$

57. $\displaystyle\int \frac{dx}{x^2-x-2}$

58. $\displaystyle\int \frac{4x^2+13x-9}{x^3+2x^2-3x}\,dx$

59. $\displaystyle\int \frac{3x^3-18x^2+29x-4}{(x+1)(x-2)^3}\,dx$

60. $\displaystyle\int \frac{x^2-x-21}{2x^3-x^2+8x-4}\,dx$

61. $\displaystyle\int \frac{5x^3-3x^2+7x-3}{\left(x^2+1\right)^2}\,dx$

62. $\displaystyle\int \frac{2x^4-2x^3+6x^2-5x+1}{x^3-x^2+x-1}\,dx$

63. $\displaystyle\int \frac{81}{x^3-9x^2}\,dx$

64. $\displaystyle\int \frac{10x}{x^2-2x-24}\,dx$

65. $\displaystyle\int \frac{x+1}{x^2\left(x^2+4\right)}\,dx$

66. $\displaystyle\int \frac{1+x^2}{(x+1)^3}\,dx$

67. $\displaystyle\int \frac{6}{x^2-1}\,dx$

68. $\displaystyle\int \frac{21x^2}{x^3-x^2-12x}\,dx$

69. $\displaystyle\int \frac{x+1}{x^3+3x^2-18x}\,dx$

70. $\displaystyle\int \frac{x^2+12x-4}{x^3-4x}\,dx$

71. $\displaystyle\int \frac{6x^2}{x^4-5x^2+4}\,dx$

72. $\displaystyle\int \frac{4x-2}{x^3-x}\,dx$

73. $\displaystyle\int \frac{16x^2}{(x-6)(x+2)^2}\,dx$

74. $\displaystyle\int \frac{8\left(x^2+4\right)}{x\left(x^2+8\right)}\,dx$

75. $\displaystyle\int \frac{x^2+x+2}{(x+1)\left(x^2+1\right)}\,dx$

76. $\displaystyle\int \frac{2}{x\left(x^2+1\right)^2}\,dx$

77. $\displaystyle\int \frac{1}{(x+1)\left(x^2+2x+2\right)^2}\,dx$

78. $\displaystyle\int \frac{2-x}{x^2+x}\,dx$

79. $\displaystyle\int \frac{3x+11}{(x+2)(x+3)}\,dx$

80. $\displaystyle\int \frac{1}{x^2-a^2}\,dx$

81. $\displaystyle\int \frac{1}{x^2+5x+6}\,dx$

82. $\displaystyle\int \frac{x^3+6x^2+3x+6}{x^3+2x^2}\,dx$

83. $\displaystyle\int \frac{2x^2+5x-1}{x^3+x^2-2x}\,dx$

84. $\displaystyle\int \frac{3x+6}{x^3+2x^2-3x}\,dx$

85. $\displaystyle\int \frac{3x^2+2x-2}{x^3-1}\,dx$

86. $\displaystyle\int \frac{x^3+5x^2+2x-4}{x^4-1}\,dx$

87. $\displaystyle\int \frac{x^2+4x}{\left(x^2+4\right)(x-2)^2}\,dx$

88. $\displaystyle\int \frac{x^2+2x+3}{(x-1)(x+1)^2}\,dx$

89. $\displaystyle\int \frac{x^4-x^3+3x^2-x+2}{(x-1)\left(x^2+2\right)^2}\,dx$

90. $\displaystyle\int \frac{-x^2+11x+18}{(x-1)(x+1)\left(x^2+3x+3\right)}\,dx$

91. $\displaystyle\int \frac{x^3+5x^2+2x-4}{x\left(x^2+4\right)^2}\,dx$

92. $\displaystyle\int_{-1}^{2} \frac{5x}{x^2-x-6}\,dx$

93. $\displaystyle\int_{0}^{5} \frac{2}{x^2-4x-32}\,dx$

94. $\displaystyle\int_{0}^{1} \frac{dx}{(x+1)\left(x^2+1\right)}$

95. $\displaystyle\int_{-1/2}^{1/2} \frac{x^2+1}{x^2-1}\,dx$

96. $\displaystyle\int_{0}^{2} \frac{3}{4x^2+5x+1}\,dx$

97. $\displaystyle\int_{1}^{5} \frac{x-1}{x^2(x+1)}\,dx$

98. $\displaystyle\int_{0}^{1} \frac{x^2-x}{x^2+x+1}\,dx$

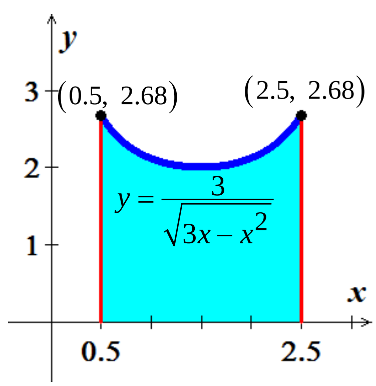
99. $\displaystyle\int_{4}^{8} \frac{y\,dy}{y^2-2y-3}$

100. $\displaystyle\int_{1}^{\sqrt{3}} \frac{3x^2+x+4}{x^3+x}\,dx$

101. $\displaystyle\int_{0}^{\pi/2} \frac{dx}{\sin x+\cos x}$

102. $\displaystyle\int_{0}^{\pi/3} \frac{\sin\theta}{1-\sin\theta}\,d\theta$

103. Find the volume of the solid generated by the revolving the shaded region about x-axis



Find the area of the region bounded by the graphs of

104. $y=\dfrac{12}{x^2+5x+6},\quad y=0,\quad x=0,\quad and\quad x=1$

105. $y=\dfrac{7}{16-x^2}\quad and\quad y=1$

106. Consider the region bounded by the graphs $y = \dfrac{2x}{x^2+1}, \quad y=0, \quad x=0, \quad and\ x=3$.

a) Find the volume of the solid generated by revolving the region about the *x-axis*

b) Find the centroid of the region.

107. Consider the region bounded by the graph $y^2 = \dfrac{(2-x)^2}{(1+x)^2} \quad 0 \le x \le 1$.

Find the volume of the solid generated by revolving this region about the *x-axis*.

108. A single infected individual enters a community of *n* susceptible individuals. Let *x* be the number of newly infected individuals at time *t*. The common epidemic model assumes that the disease spreads at a rate proportional to the product of the total number infected and the number not yet infected. So,

$\dfrac{dx}{dt} = k(x+1)(n-x)$ and you obtain

$$\int \frac{1}{(x+1)(n-x)}\, dx = \int k\ dt$$

Solve for *x* as a function of *t*.

109. Evaluate $\displaystyle\int_0^1 \frac{x}{1+x^4}\, dx$ in ***two*** different ways.