Solution   *Section* 3.4 – Using Laplace Transform to Solve Differential Equations

Exercise

Solve using the Laplace transform: $y' + y = te^t, \quad y(0) = -2$

Solution

$$\mathcal{L}(y' + y) = \mathcal{L}\left(te^t\right)$$

$$\mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}\left(te^t\right)$$

$$sY(s) - y(0) + Y(s) = \frac{1}{(s-1)^2} \qquad y(0) = -2$$

$$(s+1)Y(s) + 2 = \frac{1}{(s-1)^2}$$

$$(s+1)Y(s) = \frac{1}{(s-1)^2} - 2$$

$$Y(s) = \frac{1}{(s+1)(s-1)^2} - \frac{2}{s+1}$$

$$\frac{1}{(s+1)(s-1)^2} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$1 = (A+B)s^2 + (C-2A)s + A - B + C$$

$$\begin{cases} A + B = 0 \\ C - 2A = 0 \\ A - B + C = 1 \end{cases} \Rightarrow A = \frac{1}{4} \quad B = -\frac{1}{4} \quad C = \frac{1}{2}$$

$$Y(s) = \frac{1}{4}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2} - \frac{2}{s+1}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{-\frac{7}{4}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2}\right\}$$

$$\boxed{y(t) = -\frac{7}{4}e^{-t} - \frac{1}{4}e^t + \frac{1}{2}te^t}$$

Exercise

Solve using the Laplace transform: $y' - y = 2\cos 5t, \quad y(0) = 0$

Solution

$$\mathcal{L}\{y' - y\} = \mathcal{L}\{2\cos 5t\}$$

$$sY(s) - y(0) - Y(s) = \frac{2s}{s^2 + 25} \qquad y(0) = 0$$

$$(s-1)Y(s) = \frac{2s}{s^2+25}$$

$$Y(s) = \frac{2s}{(s-1)\left(s^2+25\right)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+25}$$

$$2s = As^2 + 25A + Bs^2 - Bs + Cs - C$$

$$\begin{cases} A + B = 0 \\ -B + C = 2 \\ 25A - C = 0 \end{cases} \Rightarrow A = \frac{1}{13} \quad B = -\frac{1}{13} \quad C = \frac{25}{13}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{13}\frac{1}{s-1} - \frac{1}{13}\frac{s}{s^2+25} + \frac{25}{13}\frac{1}{5}\frac{5}{s^2+25}\right\}$$

$$\boxed{y(t) = \frac{1}{13}e^t - \frac{1}{13}\cos 4t + \frac{5}{13}\sin 5t}$$

Exercise

Solve using the Laplace transform: $y' - y = 1 + te^t, \quad y(0) = 0$

Solution

$$\mathcal{L}\{y' - y\} = \mathcal{L}\left\{1 + te^t\right\}$$

$$sY(s) - y(0) - Y(s) = \frac{1}{s} + \frac{1}{(s-1)^2} \qquad y(0) = 0$$

$$(s-1)Y(s) = \frac{s^2 - s + 1}{s(s-1)^2}$$

$$Y(s) = \frac{s^2 - s + 1}{s(s-1)^3} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2} + \frac{D}{(s-1)^3}$$

$$s^2 - s + 1 = As^3 - 3As^2 + 3As - A + Bs^3 - 2Bs^2 + Bs + Cs^2 + -Cs + Ds$$

$$\begin{cases} s^3 & A + B = 0 & \boxed{B = 1} \\ s^2 & -3A - 2B + C = 1 & \boxed{C = 0} \\ s & 3A + B - C + D = -1 & \boxed{D = 1} \\ s^0 & \boxed{A = -1} & \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{-\frac{1}{s} + \frac{1}{s-1} + \frac{1}{(s-1)^3}\right\}$$

$$\boxed{y(t) = -1 + e^t + \frac{1}{2}t^2e^t}$$

Exercise

Solve using the Laplace transform: $\quad y' + 3y = e^{2t}, \quad y(0) = -1$

Solution

$$\mathcal{L}(y'+3y) = \mathcal{L}\left(e^{2t}\right)$$

$$\mathcal{L}(y') + 3\mathcal{L}(y) = \mathcal{L}\left(e^{2t}\right)$$

$$sY(s) - y(0) + 3Y(s) = \frac{1}{s-2}$$

$$(s+3)Y(s) + 1 = \frac{1}{s-2}$$

$$(s+3)Y(s) = \frac{1}{s-2} - 1$$

$$Y(s) = \frac{1}{(s-2)(s+3)} - \frac{1}{s+3}$$

$$\frac{1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$$

$$1 = (A+B)s + 3A - 2B$$

$$\begin{cases} A+B=0 \\ 3A-2B=1 \end{cases} \Rightarrow A = \frac{1}{5} \quad B = -\frac{1}{5}$$

$$Y(s) = \frac{1}{5}\frac{1}{s-2} - \frac{1}{5}\frac{1}{s+3} - \frac{1}{s+3}$$

$$y(t) = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{6}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$\boxed{= \frac{1}{5}e^{2t} - \frac{6}{5}e^{-3t}}$$

Exercise

Solve using the Laplace transform: $\quad y' + 4y = \cos t, \quad y(0) = 0$

Solution

$$\mathcal{L}(y'+4y) = \mathcal{L}(\cos t)$$

$$sY(s) - y(0) + 4Y(s) = \frac{s}{s^2+1} \qquad\qquad y(0) = 0$$

$$(s+4)Y(s) = \frac{s}{s^2+1}$$

$$Y(s) = \frac{s}{(s+4)\left(s^2+1\right)}$$

$$\frac{s}{(s+4)\left(s^2+1\right)} = \frac{A}{s+4} + \frac{Bs+C}{s^2+1}$$

$$s = As^2 + A + Bs^2 + 4Bs + Cs + 4C$$

$$s = (A+B)s^2 + (4B+C)s + A + 4C$$

$$\begin{cases} A+B=0 \\ 4B+C=1 \Rightarrow A=-\frac{4}{17} \quad B=\frac{4}{17} \quad C=\frac{1}{17} \\ A+4C=0 \end{cases}$$

$$Y(s) = -\frac{4}{17}\frac{1}{s+4} + \frac{4}{17}\frac{s}{s^2+1} + \frac{1}{17}\frac{1}{s^2+1}$$

$$y(t) = -\frac{4}{17}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} + \frac{4}{17}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{17}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\boxed{= -\frac{4}{17}e^{-4t} + \frac{4}{17}e^{-9t}\cos t + \frac{1}{17}\sin t}$$

Exercise

Solve using the Laplace transform: $y' + 4y = e^{-4t}$, $y(0) = 2$

Solution

$$\mathcal{L}\{y' + 4y\} = \mathcal{L}\left\{e^{-4t}\right\}$$

$$sY(s) - y(0) + 4Y(s) = \frac{1}{s+4} \qquad y(0) = 2$$

$$(s+4)Y(s) = \frac{1}{s+4} + 2$$

$$Y(s) = \frac{2s+9}{(s+4)^2} = \frac{A}{s+4} + \frac{B}{(s+4)^2}$$

$$2s + 9 = As + 4A + B$$

$$\begin{cases} s & \boxed{A=2} \\ s^0 & 4A+B=9 \quad \boxed{B=1} \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s+4} + \frac{1}{(s+4)^2}\right\}$$

$$\boxed{y(t) = 2e^{-4t} + te^{-4t}}$$

Exercise

Solve using the Laplace transform: $y' - 4y = t^2e^{-2t}$, $y(0) = 1$

Solution

$$\mathcal{L}(y' - 4y) = \mathcal{L}\left(t^2e^{-2t}\right)$$

$$sY(s) - y(0) - 4Y(s) = \frac{2!}{(s+2)^3} \qquad\qquad y(0) = 1$$

$$(s-4)Y(s) - 1 = \frac{2}{(s+2)^3}$$

$$(s-4)Y(s) = 1 + \frac{2}{(s+2)^3}$$

$$Y(s) = \frac{1}{s-4} + \frac{2}{(s-4)(s+2)^3}$$

$$\frac{2}{(s-4)(s+2)^3} = \frac{A}{s-4} + \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3}$$

$$2 = A\left(s^3 + 6s^2 + 12s + 8A\right) + B\left(s^2 + 4s + 4\right)(s-4) + C(s-4)(s+2) + D(s-4)$$

$$2 = As^3 + 6As^2 + 12As + 8A + Bs^3 + 4Bs^2 + 4Bs - 4Bs^2 - 16Bs - 16B$$
$$+ Cs^2 - 2Cs - 8C + Ds - 4D$$

$$2 = (A+B)s^3 + (6A+C)s^2 + (12A - 12B - 2C + D)s + 8A - 16B - 8C - 4D$$

$$\begin{cases} A + B = 0 \\ 6A + C = 0 \\ 12A - 12B - 2C + D = 0 \\ 8A - 16B - 8C - 4D = 2 \end{cases} \Rightarrow \begin{matrix} A = \frac{1}{108} & B = -\frac{1}{108} \\ C = -\frac{1}{18} & D = -\frac{1}{3} \end{matrix}$$

$$Y(s) = \frac{1}{s-4} + \frac{1}{108}\frac{1}{s-4} - \frac{1}{108}\frac{1}{s+2} - \frac{1}{18}\frac{1}{(s+2)^2} - \frac{1}{3}\frac{1}{(s+2)^3}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{109}{108}\frac{1}{s-4} - \frac{1}{108}\frac{1}{s+2} - \frac{1}{18}\frac{1}{(s+2)^2} - \frac{1}{3}\frac{1}{(s+2)^3}\right\}$$

$$\boxed{y(t) = \frac{109}{108}e^{4t} - \frac{1}{108}e^{-2t} - \frac{1}{18}te^{-2t} - \frac{1}{6}t^2e^{-2t}}$$

Exercise

Solve using the Laplace transform: $y' + 9y = e^{-t}$, $y(0) = 0$

Solution

$$\mathcal{L}(y' + 9y) = \mathcal{L}\left(e^{-t}\right) \qquad\qquad Y(s) = \mathcal{L}(y)(s)$$

$$\mathcal{L}(y') + 9\mathcal{L}(y) = \mathcal{L}\left(e^{-t}\right)$$

$$sY(s) - y(0) + 9Y(s) = \frac{1}{s+1}$$

$$(s+9)Y(s)=\frac{1}{s+1}$$

$$Y(s)=\frac{1}{(s+1)(s+9)}$$

$$\frac{1}{(s+1)(s+9)}=\frac{A}{s+1}+\frac{B}{s+9}$$

$$1=(A+B)s+9A+B$$

$$\begin{cases} A+B=0 \\ 9a+B=1 \end{cases} \Rightarrow A=\frac{1}{8} \quad B=-\frac{1}{8}$$

$$Y(s)=\frac{1}{8}\frac{1}{s+1}-\frac{1}{8}\frac{1}{s+9}$$

$$y(t)=\frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+9}\right\}$$

$$=\frac{1}{8}e^{-t}-\frac{1}{8}e^{-9t}$$

Exercise

Solve using the Laplace transform: $y'+16y=\sin 3t, \quad y(0)=1$

Solution

$$\mathcal{L}(y'+16y)=\mathcal{L}(\sin 3t)$$

$$sY(s)-y(0)+16Y(s)=\frac{3}{s^2+9} \qquad y(0)=1$$

$$(s+16)Y(s)-1=\frac{3}{s^2+9}$$

$$(s+16)Y(s)=\frac{3}{s^2+9}+1$$

$$Y(s)=\frac{1}{s+16}+\frac{3}{(s+16)\left(s^2+9\right)}$$

$$\frac{3}{(s+16)\left(s^2+9\right)}=\frac{A}{s+16}+\frac{Bs+C}{s^2+9}$$

$$s=As^2+9A+Bs^2+16Bs+Cs+16C$$

$$s=(A+B)s^2+(16B+C)s+9A+16C$$

$$\begin{cases} A+B=0 \\ 16B+C=1 \\ 9A+16C=0 \end{cases} \Rightarrow A=\frac{3}{265} \quad B=-\frac{3}{265} \quad C=\frac{48}{265}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s+16}+\frac{3}{265}\frac{1}{s+16}-\frac{3}{265}\frac{s}{s^2+9}+\frac{48}{265}\frac{1}{s^2+9}\right\}$$

$$y(t) = \frac{268}{265}e^{-16t} - \frac{3}{265}\cos 3t + \frac{16}{265}\sin 3t$$

Exercise

Solve using the Laplace transform: $y'' - y = e^{2t}; \quad y(0) = 0, \quad y'(0) = 1$

Solution

$$\mathcal{L}\{y'' - y\} = \mathcal{L}\left\{e^{2t}\right\}$$

$$s^2Y(s) - sy(0) - y'(0) - Y(s) = \frac{1}{s-2} \qquad y(0) = 0, \quad y'(0) = 1$$

$$\left(s^2 - 1\right)Y(s) - 1 = \frac{1}{s-2}$$

$$\left(s^2 - 1\right)Y(s) = \frac{1}{s-2} + 1$$

$$(s-1)(s+1)Y(s) = \frac{s-1}{s-2}$$

$$Y(s) = \frac{1}{(s+1)(s-2)} = \frac{A}{(s+1)} + \frac{B}{(s-2)}$$

$$(A+B)s + B - 2A = 1 \qquad \begin{cases} A + B = 0 \\ -2A + B = 1 \end{cases} \Rightarrow A = -\frac{1}{3};\ B = \frac{1}{3}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{3}\mathcal{L}^{-1}\left\{-\frac{1}{(s+1)} + \frac{1}{(s-2)}\right\}$$

$$y(t) = \frac{1}{3}\left(e^{2t} - e^{-t}\right)$$

Exercise

Solve using the Laplace transform: $y'' - y = 2t; \quad y(0) = 0, \quad y'(0) = -1$

Solution

$$\mathcal{L}(y'' - y) = \mathcal{L}(2t)$$

$$s^2Y(s) - sy(0) - y'(0) - Y(s) = 2\frac{1}{s^2} \qquad y(0) = 0 \quad y'(0) = -1$$

$$\left(s^2 - 1\right)Y(s) + 1 = \frac{2}{s^2}$$

$$Y(s) = \frac{2}{s^2(s-1)(s+1)} - \frac{1}{(s-1)(s+1)}$$

$$\frac{2}{s^2(s-1)(s+1)} = \frac{A}{s^2} + \frac{B}{s-1} + \frac{C}{s+1}$$

$$2 = As^2 - A + Bs^3 + Bs^2 + Cs^3 - Cs^2$$

$$2=(B+C)s^3+(A+B-C)s^2-A$$

$$\begin{cases} B+C=0 \\ A+B-C=0 \\ -A=2 \end{cases} \Rightarrow A=-2 \quad B=1 \quad C=-1$$

$$\frac{1}{(s-1)(s+1)}=\frac{D}{s-1}+\frac{E}{s+1}$$

$$\begin{cases} D+E=0 \\ D-E=1 \end{cases} \Rightarrow D=\frac{1}{2} \quad E=-\frac{1}{2}$$

$$Y(s)=-\frac{2}{s^2}+\frac{1}{s-1}-\frac{1}{s+1}-\frac{1}{2}\frac{1}{s-1}+\frac{1}{2}\frac{1}{s+1}$$

$$=-\frac{2}{s^2}+\frac{1}{2}\frac{1}{s-1}-\frac{1}{2}\frac{1}{s+1}$$

$$y(t)=2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}+\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}-\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$\boxed{y(t)=-2t+\frac{1}{2}e^{t}-\frac{1}{2}e^{-t}}$$

Exercise

Solve using the Laplace transform: $y''-y=t-2$; $y(2)=3,\quad y'(2)=0$

Solution

Let: $w(t)=y(t+2) \quad \leftrightarrow \quad y(t)=w(t-2)$

$$\mathcal{L}\{y''-y\}=\mathcal{L}\{t+2\}$$

$$\mathcal{L}\{w''-w\}=\mathcal{L}\{t\}$$

$$s^2W(s)-sw(0)-w'(0)-W(s)=\frac{1}{s} \qquad y(2)=w(2-2)=w(0)=3,\quad y'(2)=w'(0)=0$$

$$\left(s^2-1\right)W(s)=\frac{1}{s}+3s$$

$$W(s)=\frac{1+3s^2}{s\left(s^2-1\right)}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s+1}$$

$$1+3s^2=As^2-A+Bs^2+Bs+Cs^2-Cs$$

$$\begin{cases} s^2 & A+C=3 & \boxed{C=2} \\ s^1 & B-C=0 & \boxed{B=2} \\ s^0 & & \boxed{A=-1} \end{cases}$$

$$\mathcal{L}^{-1}\{W(s)\}=\mathcal{L}^{-1}\left\{-\frac{1}{s}+\frac{2}{s-1}+\frac{2}{s+1}\right\}$$

$$w(t) = -t + 2e^{t} + 2e^{-t}$$

$$y(t) = w(t-2) = -(t-2) + 2e^{t-2} + 2e^{-(t-2)}$$

$$\underline{= 2 - t + 2e^{t-2} + 2e^{-t+2}}$$

Exercise

Solve using the Laplace transform: $y'' + y = t$; $y(\pi) = y'(\pi) = 0$

Solution

Let: $w(t) = y(t+\pi) \quad \leftrightarrow \quad y(t) = w(t-\pi)$

$$y'' + y = t \quad \to \quad w'' + w = t + \pi$$

$$\mathcal{L}\{w'' + w\} = \mathcal{L}\{t + \pi\}$$

$$s^2 W(s) - sw(0) - w'(0) + W(s) = \frac{1}{s^2} + \frac{\pi}{s} \qquad y(\pi) = w(\pi - \pi) = w(0) = 0, \quad y'(\pi) = w'(0) = 0$$

$$\left(s^2 + 1\right) W(s) = \frac{1 + \pi s}{s^2}$$

$$W(s) = \frac{1 + \pi s}{s^2\left(s^2 + 1\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 1}$$

$$1 + \pi s = As^3 + As + Bs^2 + B + Cs^3 + Ds^2$$

$$\begin{cases} s^3 & A + C = 0 \quad \underline{C = -\pi} \\ s^2 & B + D = 0 \quad \underline{D = -1} \\ s^1 & \underline{A = \pi} \\ s^0 & \underline{B = 1} \end{cases}$$

$$\mathcal{L}^{-1}\{W(s)\} = \mathcal{L}^{-1}\left\{\frac{\pi}{s} + \frac{1}{s^2} - \frac{\pi s}{s^2 + 1} - \frac{1}{s^2 + 1}\right\}$$

$$w(t) = \pi + t - \pi \cos t - \sin t$$

$$y(t) = w(t - \pi) = \pi + (t - \pi) - \pi \cos(t - \pi) - \sin(t - \pi)$$

$$= t - \pi(\cos t \cos \pi - \sin t \sin \pi) - (\cos t \sin \pi - \cos \pi \sin t)$$

$$\underline{= t + \pi \cos t + \sin t}$$

Exercise

Solve using the Laplace transform: $y'' - 2y' + 5y = -8e^{\pi - t}$; $y(\pi) = 2, \ y'(\pi) = 12$

Solution

Let: $w(t)=y(t+\pi) \quad \leftrightarrow \quad y(t)=w(t-\pi)$

$y''-2y'+5y=-8e^{\pi-t} \quad \to \quad w''-2w'+5w=-8e^{-t}$

$\mathcal{L}\{w''-2w'+5w\}=\mathcal{L}\left\{-8e^{-t}\right\}$

$s^2W(s)-sw(0)-w'(0)-2sW(s)+2w(0)+5W(s)=-\frac{8}{s+1} \qquad y(\pi)=w(0)=2, \quad y'(\pi)=w'(0)=12$

$\left(s^2-2s+5\right)W(s)=-\frac{8}{s+1}+2s+12-4$

$\left(s^2-2s+5\right)W(s)=\frac{2s^2+10s}{s+1}$

$W(s)=\frac{2s^2+10s}{(s+1)\left((s-1)^2+4\right)}=\frac{A}{s+1}+\frac{B(s-1)+C}{(s-1)^2+4}$

$2s^2+10s=As^2-2As+5A+Bs^2-B+Cs+C$

$$\begin{cases} s^2 & A+B=2 \\ s^1 & -2A+C=10 \\ s^0 & 5A-B+C=0 \end{cases}$$

$$\Delta=\begin{vmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 5 & -1 & 1 \end{vmatrix}=8 \quad \Delta_A=\begin{vmatrix} 2 & 1 & 0 \\ 10 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix}=-8$$

$\underline{A=-1 \quad B=3 \quad C=8}$

$\mathcal{L}^{-1}\{W(s)\}=\mathcal{L}^{-1}\left\{-\frac{1}{s+1}+\frac{3(s-1)}{(s-1)^2+2^2}+\frac{4(2)}{(s-1)^2+2^2}\right\}$

$w(t)=-e^{-t}+3e^{t}\cos 2t+4e^{t}\sin 2t$

$y(t)=w(t-\pi)=-e^{-(t-\pi)}+3e^{t-\pi}\cos 2(t-\pi)+4e^{t-\pi}\sin 2(t-\pi)$

$\qquad =-e^{-t+\pi}+3e^{t-\pi}\cos 2t+4e^{t-\pi}\sin 2t$

Exercise

Solve using the Laplace transform: $\quad y''+y=t^2+2\ ; \quad y(0)=1, \quad y'(0)=-1$

Solution

$\mathcal{L}\{y''+y\}=\mathcal{L}\left\{t^2+2\right\}$

$s^2Y(s)-sy(0)-y'(0)+Y(s)=\frac{2}{s^3}+\frac{2}{s} \qquad y(0)=1; \ \ y'(0)=-1$

$$s^2Y(s) - s + 1 + Y(s) = \frac{2+2s^2}{s^3}$$

$$\left(s^2+1\right)Y(s) = \frac{2+2s^2}{s^3} + s - 1$$

$$Y(s) = \frac{2+2s^2+s^4-s^3}{s^3\left(s^2+1\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds+E}{s^2+1}$$

$$s^4 - s^3 + 2s^2 + 2 = As^4 + As^2 + Bs^3 + Bs + Cs^2 + C + Ds^4 + Es^3$$

$$\begin{cases} s^4 & A+D=1 & \underline{D=1} \\ s^3 & B+E=-1 & \underline{E=-1} \\ s^2 & A+C=2 & \underline{A=0} \\ s & \underline{B=0} & \\ s^0 & \underline{C=2} & \end{cases}$$

$$Y(s) = \frac{2}{s^3} + \frac{s}{s^2+1} - \frac{1}{s^2+1}$$

$$\boxed{y(t) = t^2 + \cos t - \sin t}$$

Exercise

Solve using the Laplace transform: $y'' + y = \sqrt{2}\sin\sqrt{2}t;\quad y(0) = 10,\quad y'(0) = 0$

Solution

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\left\{\sqrt{2}\sin\sqrt{2}t\right\}$$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \sqrt{2}\,\frac{\sqrt{2}}{s^2+2} \qquad y(0)=10;\ \ y'(0)=0$$

$$\left(s^2+1\right)Y(s) - 10s = \frac{2}{s^2+2}$$

$$\left(s^2+1\right)Y(s) = \frac{2}{s^2+2} + 10s$$

$$\left(s^2+1\right)Y(s) = \frac{10s^3+20s+2}{s^2+2}$$

$$Y(s) = \frac{10s^3+20s+2}{\left(s^2+1\right)\left(s^2+2\right)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2}$$

$$As^3 + 2As + Bs^2 + 2B + Cs^3 + Cs + Ds^2 + D = 10s^3 + 20s + 2$$

$$\begin{cases} s^3 & A+C=10 \\ s^2 & B+D=0 \\ s^1 & 2A+C=20 \\ s^0 & 2B+D=2 \end{cases} \rightarrow \begin{array}{l} \begin{cases} A+C=10 \\ 2A+C=20 \end{cases} \rightarrow \boxed{A=10,\ C=0} \\ \begin{cases} B+D=0 \\ 2B+D=2 \end{cases} \rightarrow \boxed{B=2,\ D=-2} \end{array}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{10s}{s^2+1} + \frac{2}{s^2+1} - \sqrt{2}\frac{\sqrt{2}}{s^2+2}\right\}$$

$$\boxed{y(t) = 10\cos t + 2\sin t - \sqrt{2}\sin\sqrt{2}t}$$

Exercise

Solve using the Laplace transform: $y'' + y = -2\cos 2t$; $y(0)=1,\quad y'(0)=-1$

Solution

$$\mathcal{L}(y''+y)(s) = \mathcal{L}(-2\cos 2t)(s)$$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = -2\frac{s}{s^2+4} \qquad y(0)=1 \quad y'(0)=-1$$

$$s^2Y(s) - s + 1 + Y(s) = -2\frac{s}{s^2+4}$$

$$\left(s^2+1\right)Y(s) = \frac{-2s}{s^2+4} + s - 1$$

$$Y(s) = \frac{-2s}{\left(s^2+4\right)\left(s^2+1\right)} + \frac{s}{s^2+1} - \frac{1}{s^2+1}$$

$$= \frac{As}{s^2+4} + \frac{Bs}{s^2+1} + \frac{s}{s^2+1} - \frac{1}{s^2+1}$$

$$\frac{-2s}{\left(s^2+4\right)\left(s^2+1\right)} = \frac{As}{s^2+4} + \frac{Bs}{s^2+1}$$

$$\Rightarrow \quad -2s = As\left(s^2+1\right) + Bs\left(s^2+4\right)$$

$$-2s = As^3 + As + Bs^3 + 4Bs$$

$$-2s = (A+B)s^3 + (A+4B)s$$

$$\begin{cases} A+B=0 \\ A+4B=-2 \end{cases} \Rightarrow \boxed{A=\frac{2}{3}} \quad \boxed{B=-\frac{2}{3}}$$

$$Y(s) = \frac{2}{3}\frac{s}{s^2+4} - \frac{2}{3}\frac{s}{s^2+1} + \frac{s}{s^2+1} - \frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{3}\frac{s}{s^2+4} + \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{s^2+1}\right\}$$

$$\boxed{y(t)=\frac{2}{3}\cos 2t+\frac{1}{3}\cos t-\sin t}$$

Exercise

Solve using the Laplace transform: $y''-y'=e^{t}\cos t$; $y(0)=0,\quad y'(0)=0$

Solution

$$\mathcal{L}\{y''-y'\}=\mathcal{L}\left\{e^{t}\cos t\right\}$$

$$s^{2}Y(s)-sy(0)-y'(0)+sY(s)-y(0)=\frac{s-1}{(s-1)^{2}+1}\qquad y(0)=0;\ y'(0)=0$$

$$\left(s^{2}+s\right)Y(s)=\frac{s-1}{s^{2}-2s+2}$$

$$Y(s)=\frac{s-1}{\left(s^{2}+s\right)\left(s^{2}-2s+2\right)}\quad=\frac{A}{s}+\frac{B}{s+1}+\frac{Cs+D}{s^{2}-2s+2}$$

$$As^{3}-As^{2}+2A+Bs^{3}-2Bs^{2}+2Bs+Cs^{3}+Cs^{2}+Ds^{2}+Ds=s-1$$

$$\begin{cases} s^{3} & A+B+C+D=0\\ s^{2} & -A-2B+C+D=0\\ s^{1} & 2B+D=1\\ s^{0} & 2A=-1\rightarrow \boxed{A=-\frac{1}{2}}\end{cases}$$

$$\rightarrow\begin{cases} B+C+D=\frac{1}{2}\\ -2B+C+D=-\frac{1}{2}\\ 2B+D=1\end{cases}\Rightarrow \boxed{B=\frac{1}{3}\quad C=-\frac{1}{6}\quad D=\frac{1}{3}}$$

$$Y(s)=-\frac{1}{2}\frac{1}{s}+\frac{1}{3}\frac{1}{s+1}-\frac{1}{6}\frac{s-2}{(s-1)^{2}+1}$$

$$=-\frac{1}{2}\frac{1}{s}+\frac{1}{3}\frac{1}{s+1}-\frac{1}{6}\frac{s-1-1}{(s-1)^{2}+1}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{-\frac{1}{2}\frac{1}{s}+\frac{1}{3}\frac{1}{s+1}-\frac{1}{6}\frac{s-1}{(s-1)^{2}+1}+\frac{1}{6}\frac{1}{(s-1)^{2}+1}\right\}$$

$$\boxed{y(t)=-\frac{1}{2}t+\frac{1}{3}e^{-t}-\frac{1}{6}e^{t}\cos t+\frac{1}{6}e^{t}\sin t}$$

Exercise

Solve using the Laplace transform: $y'' + y' - y = t^3$; $y(0) = 1,\quad y'(0) = 0$

Solution

$$\mathcal{L}\left(y'' - y' - y\right) = \mathcal{L}\left(t^3\right)$$

$$s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) - Y(s) = \frac{6}{s^4} \qquad y(0)=1 \quad y'(0)=0$$

$$\left(s^2 + s - 1\right)Y(s) - s - 1 = \frac{6}{s^4}$$

$$\left(s^2 + s - 1\right)Y(s) = \frac{6}{s^4} + s + 1$$

$$Y(s) = \frac{s^5 + s^4 + 6}{s^4\left(s^2 + s - 1\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s^4} + \frac{E}{s + \frac{1}{2} - \frac{\sqrt{5}}{2}} + \frac{F}{s + \frac{1}{2} + \frac{\sqrt{5}}{2}}$$

$s^5 \qquad A + E + F = 1 \qquad\qquad E + F = 19$

$s^4 \qquad A + B + \left(\frac{1+\sqrt{5}}{2}\right)E + \left(\frac{1-\sqrt{5}}{2}\right)F = 1 \qquad \left(\frac{1+\sqrt{5}}{2}\right)E + \left(\frac{1-\sqrt{5}}{2}\right)F = 31$

$s^3 \qquad -A + B + C = 0 \qquad\qquad \underline{A = -18}$

$s^2 \qquad -B + C + D = 0 \qquad\qquad \underline{B = -12}$

$s^1 \qquad -C + D = 0 \qquad\qquad \underline{C = -6}$

$s^0 \qquad -D = 6 \qquad\qquad \underline{D = -6}$

$$F = \frac{19}{2} - \frac{43\sqrt{5}}{10} \qquad E = -\frac{19}{2} + \frac{43\sqrt{5}}{10}$$

$$\mathcal{L}^{-1}\left\{Y(s)\right\} = \mathcal{L}^{-1}\left\{-\frac{18}{s} - \frac{12}{s^2} - \frac{6}{s^3} - \frac{6}{s^4} + \frac{-\frac{19}{2} + \frac{43\sqrt{5}}{10}}{s + \frac{1}{2} - \frac{\sqrt{5}}{2}} + \frac{\frac{19}{2} - \frac{43\sqrt{5}}{10}}{s + \frac{1}{2} + \frac{\sqrt{5}}{2}}\right\}$$

$$y(t) = -18 - 12t - 3t^2 - t^3 + \left(-\frac{19}{2} + \frac{43\sqrt{5}}{10}\right)e^{\left(\frac{-1+\sqrt{5}}{2}\right)t} + \left(\frac{19}{2} - \frac{43\sqrt{5}}{10}\right)e^{-\left(\frac{1+\sqrt{5}}{2}\right)t}$$

Exercise

Solve using the Laplace transform: $y'' - y' - 2y = 4t^2,\quad y(0) = 1,\quad y'(0) = 4$

Solution

$$\mathcal{L}\left\{y'' - y' - 2y\right\}(s) = \mathcal{L}\left\{4t^2\right\}(s) \qquad\qquad \mathcal{L}\left\{t^n\right\}(s) = \frac{n!}{s^{n+1}}$$

$$s^2Y(s)-sy(0)-y'(0)-sY(s)+y(0)-2Y(s)=\frac{8}{s^3} \qquad y(0)=1, \quad y'(0)=4$$

$$\left(s^2-s-2\right)Y(s)-s-4+1=\frac{8}{s^3}$$

$$(s+1)(s-2)Y(s)=\frac{8}{s^3}+s+3$$

$$Y(s)=\frac{s^4+3s^3+8}{s^3(s+1)(s-2)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s+1}+\frac{E}{s-2}$$

$$s^4+3s^3+8=As^2\left(s^2-s-2\right)+Bs\left(s^2-s-2\right)+Cs^2-Cs-2C+Ds^3(s-2)+Es^3(s+1)$$

$$=As^4-As^3-2As^2+Bs^3-Bs^2-2Bs+Cs^2-Cs-2C+Ds^4-2Ds^3+Es^4+Es^3$$

$$\begin{cases} s^4 & A+D+E=1 & D+E=4 \\ s^3 & -A+B-2D+E=3 & -2D+E=-2 \\ s^2 & -2A-B+C=0 & \to \underline{A=-3} \\ s & -2B-C=0 & \to \underline{B=2} \\ s^0 & -2C=8 & \to \underline{C=-4} \end{cases} \qquad \Rightarrow \underline{D=2, \ E=2}$$

$$\mathcal{L}^{-1}\{Y(s)\}(t)=\mathcal{L}^{-1}\left\{-\frac{3}{s}+\frac{2}{s^2}-\frac{4}{s^3}+\frac{2}{s+1}+\frac{2}{s-2}\right\}(t)$$

$$\boxed{y(t)=-3+2t-2t^2+2e^{-t}+2e^{2t}}$$

Exercise

Solve using the Laplace transform: $y''-y'-2y=e^{2t}$; $y(0)=-1, \quad y'(0)=0$

Solution

$$\mathcal{L}\left(y''-y'-2y\right)=\mathcal{L}\left(e^{2t}\right)$$

$$s^2Y(s)-sy(0)-y'(0)-\left(sY(s)-y(0)\right)-2Y(s)=\frac{1}{s-2} \qquad y(0)=-1 \quad y'(0)=0$$

$$s^2Y(s)+s-sY(s)-1-2Y(s)=\frac{1}{s-2}$$

$$\left(s^2-s-2\right)Y(s)=\frac{1}{s-2}-s+1$$

$$(s+1)(s-2)(Y(s)=\frac{1}{s-2}-s+1$$

$$Y(s)=\frac{1}{(s+1)(s-2)^2}-\frac{s-1}{(s+1)(s-2)}$$

$$=\frac{1-(s-1)(s-2)}{(s+1)(s-2)^2}$$

$$Y(s)=\frac{-s^2+3s-1}{(s+1)(s-2)^2}=\frac{A}{s+1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}$$

$$-s^2+3s-1=As^2-4As+4A+Bs^2-Bs-2B+Cs+C$$

$$-s^2+3s-1=(A+B)s^2+(-4A-B+C)s+4A-2B+C$$

$$\begin{cases} A+B=-1 \\ -4A-B+C=3 \\ 4A-2B+C=-1 \end{cases} \Rightarrow A=-\frac{5}{9} \quad B=-\frac{4}{9} \quad C=\frac{1}{3}$$

$$Y(s)=-\frac{5}{9}\frac{1}{s+1}-\frac{4}{9}\frac{1}{s-2}+\frac{1}{3}\frac{1}{(s-2)^2}$$

$$y(t)=-\frac{5}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}-\frac{4}{9}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}+\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$\boxed{y(t)=-\frac{5}{9}e^{-t}-\frac{4}{9}e^{2t}+\frac{1}{3}te^{2t}}$$

Exercise

Solve using the Laplace transform: $y''-y'-2y=0, \quad y(0)=-2, \quad y'(0)=5$

Solution

$$\mathcal{L}(y''-y'-2y)=0$$

$$s^2Y(s)-sy(0)-y'(0)-sY(s)+y(0)-2Y(s)=0 \qquad y(0)=-2 \quad y'(0)=5$$

$$\left(s^2-s-2\right)Y(s)=7-2s$$

$$Y(s)=\frac{7-2s}{s^2-s-2}=\frac{A}{s+1}+\frac{B}{s-2}$$

$$\begin{cases} s & A+B=-2 \\ s^0 & -2A+B=7 \end{cases} \rightarrow \boxed{A=-3, \ B=1}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{-3}{s+1}+\frac{1}{s-2}\right\}$$

$$\boxed{y(t)=e^{2t}-3e^{-t}}$$

Exercise

Solve using the Laplace transform: $y''-y'-2y=-8\cos t-2\sin t \ ; \quad y\left(\frac{\pi}{2}\right)=1, \quad y'\left(\frac{\pi}{2}\right)=0$

Solution

Let: $w(t) = y\left(t+\frac{\pi}{2}\right) \quad \leftrightarrow \quad y(t) = w\left(t-\frac{\pi}{2}\right)$

$$y'' - y' - 2y = -8\cos t - 2\sin t \quad \rightarrow \quad w'' - w' - 2w = -8\cos\left(t+\frac{\pi}{2}\right) - 2\sin\left(t+\frac{\pi}{2}\right)$$

$$w'' - w' - 2w = -8\left(\cos t\cos\frac{\pi}{2} - \sin t\sin\frac{\pi}{2}\right) - 2\left(\sin t\cos\frac{\pi}{2} + \cos t\sin\frac{\pi}{2}\right)$$

$$\mathcal{L}\{w'' - w' - 2w\} = \mathcal{L}\{8\sin t - 2\cos t\}$$

$$s^2W(s) - sw(0) - w'(0) - sW(s) + w(0) - 2W(s) = \frac{8}{s^2+1} - \frac{2s}{s^2+1}$$

$$y\left(\frac{\pi}{2}\right) = w(0) = 1, \quad y'\left(\frac{\pi}{2}\right) = w'(0) = 0$$

$$\left(s^2 - s - 2\right)W(s) = \frac{8-2s}{s^2+1} + s - 1$$

$$W(s) = \frac{s^3 - s^2 - s + 7}{(s+1)(s-2)\left(s^2+1\right)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{Cs+D}{s^2+1}$$

$$s^3 - s^2 - s + 7 = As^3 - 2As^2 + As - 2A + Bs^3 + Bs^2 + Bs + B + Cs^3 - Cs^2 - 2Cs + Ds^2 - Ds - 2D$$

$$\begin{cases} s^3 & A + B + C = 1 \\ s^2 & -2A + B - C + D = -1 \\ s^1 & A + B - 2C - D = -1 \\ s^0 & -2A + B - 2D = 7 \end{cases} \quad \rightarrow \quad \boxed{A = -1,\ B = \frac{3}{5},\ C = \frac{7}{5},\ D = -\frac{11}{5}}$$

$$\mathcal{L}^{-1}\{W(s)\} = \mathcal{L}^{-1}\left\{-\frac{13}{3}\frac{1}{s+1} + \frac{74}{15}\frac{1}{s-2} + \frac{7}{5}\frac{s}{s^2+1} - \frac{11}{5}\frac{1}{s^2+1}\right\}$$

$$w(t) = -e^{-t} + \frac{3}{5}e^{2t} + \frac{7}{5}\cos t - \frac{11}{5}\sin t$$

$$y(t) = w\left(t-\frac{\pi}{2}\right) = -e^{-t+\frac{\pi}{2}} + \frac{3}{5}e^{2\left(t-\frac{\pi}{2}\right)} + \frac{7}{5}\cos\left(t-\frac{\pi}{2}\right) - \frac{11}{5}\sin\left(t-\frac{\pi}{2}\right)$$

$$= -e^{-t+\frac{\pi}{2}} + \frac{3}{5}e^{2t-\pi} + \frac{7}{5}\left(\cos t\cos\frac{\pi}{2} + \sin t\sin\frac{\pi}{2}\right) - \frac{11}{5}\left(\sin t\cos\frac{\pi}{2} - \cos t\sin\frac{\pi}{2}\right)$$

$$\boxed{= \frac{3}{5}e^{2t-\pi} - e^{-t+\frac{\pi}{2}} + \frac{7}{5}\sin t + \frac{11}{5}\cos t}$$

Exercise

Solve using the Laplace transform: $\quad x'' - x' - 6x = 0\ ; \quad x(0) = 2, \quad x'(0) = -1$

Solution

$$\mathcal{L}\{x'' - x' - 6x\} = 0$$

$$s^2X(s)-sx(0)-x'(0)-sX(s)+x(0)-6X(s)=0 \qquad x(0)=2,\ \ x'(0)=-1$$

$$\left(s^2-s-6\right)X(s)-2s+1+2=0$$

$$\left(s^2-s-6\right)X(s)=2s-3$$

$$X(s)=\frac{2s-3}{s^2-s-6}\ =\frac{A}{s-3}+\frac{B}{s+2}$$

$$As+2A+Bs-3B=2s-3$$

$$\begin{cases} A+B=2 \\ 2A-3B=-3 \end{cases} \qquad \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}=-5 \quad \begin{vmatrix} 2 & 1 \\ -3 & -3 \end{vmatrix}=-3 \ \rightarrow A=\frac{3}{5},\ \ B=\frac{7}{5}$$

$$\mathcal{L}^{-1}\{X(s)\}=\mathcal{L}^{-1}\left\{\frac{3}{5}\frac{1}{s-3}+\frac{7}{5}\frac{1}{s+2}\right\}$$

$$x(t)=\frac{3}{5}e^{3t}+\frac{7}{5}e^{-2t}$$

Exercise

Solve using the Laplace transform: $y''+2y'+y=0,\quad y(0)=1,\quad y'(0)=1$

Solution

$$\mathcal{L}\{y''+2y'+y\}=0$$

$$s^2Y(s)-sy(0)-y'(0)+2sY(s)-2y(0)+Y(s)=0 \qquad y(0)=1 \quad y'(0)=1$$

$$\left(s^2+2s+1\right)Y(s)-s-1-2=0$$

$$Y(s)=\frac{s+3}{(s+1)^2}=\frac{A}{s+1}+\frac{B}{(s+1)^2}$$

$$s+3=As+A+B$$

$$\begin{cases} s & \underline{A=1} \\ s^0 & A+B=3 \quad \underline{B=2} \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s+1}+\frac{2}{(s+1)^2}\right\}$$

$$y(t)=e^{-t}+2te^{-t}$$

Exercise

Solve using the Laplace transform: $y''+2y'+y=t\ ,\quad y(0)=-3,\quad y(1)=-1$

Solution

$$\mathcal{L}\{y''+2y'+y\}(s)=\mathcal{L}\{t\}(s)$$

$$s^2Y(s)-sy(0)-y'(0)+2sY(s)-2y(0)+Y(s)=\frac{1}{s^2} \qquad y(0)=-3$$

$$\left(s^2+2s+1\right)Y(s)+3s-y'(0)+6=\frac{1}{s^2}$$

$$(s+1)^2Y(s)=\frac{1}{s^2}-3s-6+y'(0)$$

$$Y(s)=\frac{-3s^3+\left(y'(0)-6\right)s^2+1}{s^2(s+1)^2} \qquad \text{Let} \quad k=y'(0)-6$$

$$=\frac{-3s^3+ks^2+1}{s^2(s+1)^2}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+1}+\frac{D}{(s+1)^2}$$

$$-3s^3+ks^2+1=As^3+2As^2+As+Bs^2+2Bs+B+Cs^3+Cs^2+Ds^2$$

$$\begin{cases} s^3 & A+C=-3 & \rightarrow \underline{C=-1} \\ s^2 & 2A+B+C+D=k & \rightarrow \underline{D=k+4} \\ s & A+2B=0 & \rightarrow \underline{A=-2} \\ s^0 & \underline{B=1} & \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{-\frac{2}{s}+\frac{1}{s^2}-\frac{1}{s+1}+\frac{k+4}{(s+1)^2}\right\}$$

$$y(t)=-2+t-e^{-t}+(k+4)te^{-t} \qquad y(1)=-1$$

$$-1=-1-e^{-1}+(k+4)e^{-1}$$

$$(k+4)e^{-1}=e^{-1}$$

$$k+4=1 \rightarrow \underline{k=-3}$$

$$\boxed{y(t)=-2+t-e^{-t}+te^{-t}}$$

Exercise

Solve using the Laplace transform: $y''-2y'-y=e^{2t}-e^{t}$; $y(0)=1, \quad y'(0)=3$

Solution

$$\mathcal{L}\{y''-2y'-y\}=\mathcal{L}\left\{e^{2t}-e^{t}\right\}$$

$$s^2Y(s)-sy(0)-y'(0)-2sY(s)+2y(0)-Y(s)=\frac{1}{s-2}-\frac{1}{s-1} \qquad y(0)=1 \quad y'(0)=3$$

$$\left(s^2-2s-1\right)Y(s)=\frac{1}{(s-2)(s-1)}+s+1$$

$$Y(s)=\frac{s^3-2s^2-s+3}{(s-2)(s-1)\left(s-1-\sqrt{2}\right)\left(s-1+\sqrt{2}\right)}=\frac{A}{s-2}+\frac{B}{s-1}+\frac{C}{s-1-\sqrt{2}}+\frac{D}{s-1+\sqrt{2}}$$

$$s^3-2s^2-s+3=A(s-1)\left(s^2-2s-1\right)+B(s-2)\left(s^2-2s-1\right)$$
$$+C\left(s-1+\sqrt{2}\right)\left(s^2-3s+2\right)+D\left(s-1-\sqrt{2}\right)\left(s^2-3s+2\right)$$

$$\begin{cases} s^3 & A+B+C+D=1 \\ s^2 & -3A-4B+\left(-4+\sqrt{2}\right)C+\left(-4-\sqrt{2}\right)D=-2 \\ s^1 & A+3B+\left(5-3\sqrt{2}\right)C+\left(5+3\sqrt{2}\right)D=-1 \\ s^0 & A+2B+\left(-2+2\sqrt{2}\right)C-2\left(1+\sqrt{2}\right)D=3 \end{cases}$$

$$\boxed{A=-1 \quad B=\frac{1}{2} \quad C=\frac{3}{4}\left(1+\sqrt{2}\right) \quad D=\frac{3}{4}\left(1-\sqrt{2}\right)}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{-1}{s-2}+\frac{1}{2}\frac{1}{s-1}+\frac{3}{4}\left(1+\sqrt{2}\right)\frac{1}{s-1-\sqrt{2}}+\frac{3}{4}\left(1-\sqrt{2}\right)\frac{1}{s-1+\sqrt{2}}\right\}$$

$$\boxed{y(t)=-e^{2t}+\frac{1}{2}e^{t}+\left(\frac{3}{4}+\frac{3\sqrt{2}}{4}\right)e^{\left(1+\sqrt{2}\right)t}+\left(\frac{3}{4}-\frac{3\sqrt{2}}{4}\right)e^{\left(1-\sqrt{2}\right)t}}$$

Exercise

Solve using the Laplace transform: $y''-2y'+y=6t-2$; $y(-1)=3,\quad y'(-1)=7$

Solution

Let: $w(t)=y(t-1) \quad \leftrightarrow \quad y(t)=w(t+1)$

$$w''-2w'+w=6(t-1)-2=6t-8$$

$$\mathcal{L}\{w''-2w'+w\}=\mathcal{L}\{6t-8\}$$

$$s^2W(s)-sw(0)-w'(0)-2sW(s)+2w(0)+W(s)=\frac{6}{s^2}-\frac{8}{s} \qquad y(-1)=w(0)=3,\quad y'(-1)=w'(0)=7$$

$$\left(s^2-2s+1\right)W(s)=\frac{6-8s}{s^2}+3s+1$$

$$W(s)=\frac{3s^3+s^2-8s+6}{s^2(s-1)^2}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-1}+\frac{D}{(s-1)^2}$$

$$3s^3+s^2-8s+6=As\left(s^2-2s+1\right)+B\left(s^2-2s+1\right)+Cs^2(s-1)+Ds^2$$

$$\begin{cases} s^3 & A+C=3 & \underline{C=-1} \\ s^2 & -2A+B-C+D=1 & \underline{D=2} \\ s^1 & A-2B=-8 & \underline{A=4} \\ s^0 & \underline{B=6} & \end{cases}$$

$$\mathcal{L}^{-1}\{W(s)\} = \mathcal{L}^{-1}\left\{\frac{4}{s}+\frac{6}{s^2}-\frac{1}{s-1}+\frac{2}{(s-1)^2}\right\}$$

$$\underline{w(t)=4+6t-e^t+2te^t}$$

$$y(t)=w(t+1)=4+6(t+1)-e^{t+1}+2(t+1)e^{t+1}$$

$$\underline{=6t+10+2te^{t+1}+e^{t+1}}$$

Exercise

Solve using the Laplace transform: $y''-2y'+y=\cos t-\sin t$; $y(0)=1,\quad y'(0)=3$

Solution

$$\mathcal{L}\{y''-2y'+y\}=\mathcal{L}\{\cos t-\sin t\}$$

$$s^2Y(s)-sy(0)-y'(0)-2sY(s)+2y(0)+Y(s)=\frac{s}{s^2+1}-\frac{1}{s^2+1} \qquad y(0)=1,\quad y'(0)=3$$

$$\left(s^2-2s+1\right)Y(s)=\frac{s-1}{s^2+1}+s+1$$

$$Y(s)=\frac{s^3+s^2+2s}{\left(s^2+1\right)(s-1)^2}=\frac{As+B}{s^2+1}+\frac{C}{s-1}+\frac{D}{(s-1)^2}$$

$$s^3+s^2+2s=(As+B)\left(s^2-2s+1\right)+C\left(s^2+1\right)(s-1)+D\left(s^2+1\right)$$

$$\begin{cases} s^3 & A+C=1 \\ s^2 & -2A+B-C+D=1 \\ s^1 & A-2B+C=2 \\ s^0 & B-C+D=0 \end{cases}$$

$$\underline{A=-\frac{1}{2}\quad B=-\frac{1}{2}\quad C=\frac{3}{2}\quad D=2}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{-\frac{1}{2}\frac{s}{s^2+1}-\frac{1}{2}\frac{1}{s^2+1}+\frac{3}{2}\frac{1}{s-1}+\frac{2}{(s-1)^2}\right\}$$

$$\underline{y(t)=-\frac{1}{2}\cos t-\frac{1}{2}\sin t+\frac{3}{2}e^t+2te^t}$$

Exercise

Solve using the Laplace transform: $y'' - 2y' + 5y = 0,\quad y(0) = 2,\quad y'(0) = 4$

Solution

$$\mathcal{L}\{y'' - 2y' + 5y\} = 0$$

$$s^2Y(s) - sy(0) - y'(0) - 2sY(s) + 2y(0) + 5Y(s) = 0 \qquad y(0) = 2 \quad y'(0) = 4$$

$$\left(s^2 - 2s + 5\right)Y(s) = 2s$$

$$Y(s) = \frac{2s}{(s-1)^2 + 4}$$

$$= \frac{2(s-1) + 2}{(s-1)^2 + 4}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2(s-1)}{(s-1)^2 + 4} + \frac{2}{(s-1)^2 + 4}\right\}$$

$$\boxed{y(t) = 2e^t \cos 2t + e^t \sin 2t}$$

Exercise

Solve using the Laplace transform: $y'' - 2y' + 5y = 1 + t\ ,\quad y(0) = 0,\quad y'(0) = 0$

Solution

$$\mathcal{L}\{y'' - 2y' + 5y\} = \mathcal{L}\{1 + t\}$$

$$s^2Y(s) - sy(0) - y'(0) - 2sY(s) + 2y(0) + 5Y(s) = \frac{1}{s} + \frac{1}{s^2} \qquad y(0) = 0 \quad y'(0) = 0$$

$$\left(s^2 - 2s + 5\right)Y(s) = \frac{s+1}{s^2}$$

$$Y(s) = \frac{s+1}{s^2\left((s-1)^2 + 4\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C(s-1) + D}{(s-1)^2 + 4}$$

$$s + 1 = As^3 - 2As^2 + 5As + Bs^2 - 2Bs + 5B + Cs^3 - Cs^2 + Ds^2$$

$$\begin{cases} s^3 & A + C = 0 & \boxed{C = -\frac{7}{25}} \\ s^2 & -2A + B - C + D = 0 & \boxed{D = \frac{2}{25}} \\ s^1 & 5A - 2B = 1 & \boxed{A = \frac{7}{25}} \\ s^0 & 5B = 1 & \boxed{B = \frac{1}{5}} \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{7}{25}\frac{1}{s}+\frac{1}{5}\frac{1}{s^2}-\frac{7}{25}\frac{s-1}{(s-1)^2+2^2}+\frac{1}{25}\frac{2}{(s-1)^2+2^2}\right\}$$

$$\boxed{y(t)=\frac{7}{25}+\frac{1}{5}t-\frac{7}{25}e^t\cos 2t+\frac{1}{25}e^t\sin 2t}$$

Exercise

Solve using the Laplace transform: $y''+3y'=-3t$; $y(0)=-1,\quad y'(0)=1$

Solution

$$\mathcal{L}(y''+3y')=\mathcal{L}(-3t)$$

$$s^2Y(s)-sy(0)-y'(0)+3\big(sY(s)-y(0)\big)=-3\frac{1}{s^2} \qquad y(0)=-1 \quad y'(0)=1$$

$$s^2Y(s)+s-1+3sY(s)+3=-\frac{3}{s^2}$$

$$\left(s^2+3s\right)Y(s)=-\frac{3}{s^2}-s-2$$

$$Y(s)=-\frac{3}{s^3(s+3)}-\frac{s+2}{s(s+3)}$$

$$\frac{3}{s^3(s+3)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s+3}$$

$$3=As^2(s+3)+Bs(s+3)+C(s+3)+Ds^3$$

$$3=(A+D)s^3+(3A+B)s^2+(3B+C)+3C$$

$$\begin{cases}A+D=0\\3A+B=0\\3B+C=0\\3C=3\end{cases}\Rightarrow\quad \begin{matrix}A=\frac{1}{9} & B=-\frac{1}{3}\\ C=1 & D=-\frac{1}{9}\end{matrix}$$

$$\frac{s+2}{s(s+3)}=\frac{E}{s}+\frac{F}{s+3}$$

$$\begin{cases}E+F=1\\3E=2\end{cases}\Rightarrow E=\frac{2}{3}\quad F=\frac{1}{3}$$

$$Y(s)=-\left(\frac{1}{9}\frac{1}{s}-\frac{1}{3}\frac{1}{s^2}+\frac{1}{s^3}-\frac{1}{9}\frac{1}{s+3}\right)-\left(\frac{2}{3}\frac{1}{s}+\frac{1}{3}\frac{1}{s+3}\right)$$

$$=-\frac{1}{9}\frac{1}{s}+\frac{1}{3}\frac{1}{s^2}-\frac{1}{s^3}+\frac{1}{9}\frac{1}{s+3}-\frac{2}{3}\frac{1}{s}-\frac{1}{3}\frac{1}{s+3}$$

$$=-\frac{7}{9}\frac{1}{s}+\frac{1}{3}\frac{1}{s^2}-\frac{1}{s^3}-\frac{2}{9}\frac{1}{s+3}$$

$$y(t) = -\frac{7}{9}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{2!}\frac{2!}{s^3}\right\} - \frac{2}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$\boxed{y(t) = -\frac{7}{9} + \frac{1}{3}t - \frac{1}{2}t^2 - \frac{2}{9}e^{-3t}}$$

Exercise

Solve using the Laplace transform: $y'' + 3y = t^3$; $y(0) = 0,\quad y'(0) = 0$

Solution

$$\mathcal{L}\{y'' + 3y'\} = \mathcal{L}\left\{t^3\right\}$$

$$s^2Y(s) - sy(0) - y'(0) + 3Y(s) = \frac{6}{s^4} \qquad y(0) = 0 \quad y'(0) = 0$$

$$\left(s^2 + 3\right)Y(s) = \frac{6}{s^4}$$

$$Y(s) = \frac{6}{s^4\left(s^2 + 3\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s^4} + \frac{Es + F}{s^2 + 3}$$

$$6 = As^5 + 3As^3 + Bs^4 + 3Bs^2 + Cs^3 + 3Cs + Ds^2 + 3D + Es^5 + Fs^4$$

s^5	$A + E = 0$	$\boxed{E = 0}$
s^4	$B + F = 0$	$\boxed{F = \frac{2}{3}}$
s^3	$3A + C = 0$	$\boxed{A = 0}$
s^2	$3B + D = 0$	$\boxed{B = -\frac{2}{3}}$
s^1	$3C = 0$	$\boxed{C = 0}$
s^0	$3D = 6$	$\boxed{D = 2}$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{-\frac{2}{3}\frac{1}{s^2} + \frac{2}{s^4} + \frac{2}{3}\frac{1}{s^2 + \left(\sqrt{3}\right)^2}\right\}$$

$$y(t) = -\frac{2}{3}t + 2\frac{1}{3!}t^3 + \frac{2}{3}\frac{1}{\sqrt{3}}\sin\left(\sqrt{3}t\right)$$

$$\boxed{= -\frac{2}{3}t + \frac{1}{3}t^3 + \frac{2\sqrt{3}}{9}\sin\left(\sqrt{3}t\right)}$$

Exercise

Solve using the Laplace transform: $y'' - 3y' + 2y = e^{-t}$, $y(1) = 0,\quad y'(1) = 0$

Solution

Let $v = t - 1 \quad \rightarrow \quad t = v + 1$

$x(v) = y(t) = y(v+1) \qquad\qquad y(1) = x(0) = 0 \quad y'(1) = x'(0) = 0$

$y''(t) - 3y'(t) + 2y(t) = e^{-t}$

$\rightarrow \quad y''(v+1) - 3y'(v+1) + 2y(v+1) = e^{-(v+1)}$

$x''(v) - 3x'(v) + 2x(v) = e^{-(v+1)}$

$\mathcal{L}\{x'' - 3x' + 2x\} = \mathcal{L}\left\{e^{-1}e^{-v}\right\}$

$s^2X(s) - sx(0) - x'(0) - 3sX(s) + 3x(0) + 2X(s) = \dfrac{e^{-1}}{s+1}$

$\left(s^2 - 3s + 2\right)X(s) = \dfrac{e^{-1}}{s+1}$

$X(s) = e^{-1}\dfrac{1}{(s+1)(s-1)(s-2)}$

$\dfrac{1}{(s+1)(s-1)(s-2)} = \dfrac{A}{s+1} + \dfrac{B}{s-1} + \dfrac{C}{s-2}$

$1 = As^2 - 3As + 2A + Bs^2 - Bs - 2B + Cs^2 - C$

$$\begin{cases} s^2 & A + B + C = 0 \\ s & -3A - B = 0 \\ s^0 & 2A - 2B - C = 1 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ -3 & -1 & 0 \\ 2 & -2 & -1 \end{vmatrix} = 6 \quad \Delta_A = \begin{vmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & -2 & -1 \end{vmatrix} = 1 \quad \Delta_B = \begin{vmatrix} 1 & 0 & 1 \\ -3 & 0 & 0 \\ 2 & 1 & -1 \end{vmatrix} = -3$$

$\boxed{A = \frac{1}{6},\quad B = -\frac{1}{2},\quad C = \frac{1}{3}}$

$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{e^{-1}\left(\frac{1}{6}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s-1} + \frac{1}{3}\frac{1}{s-2}\right)\right\}$

$x(v) = e^{-1}\left(\frac{1}{6}e^{-v} - \frac{1}{2}e^{v} + \frac{1}{3}e^{2v}\right) \qquad v = t - 1$

$y(t) = e^{-1}\left(\frac{1}{6}e^{-t+1} - \frac{1}{2}e^{t-1} + \frac{1}{3}e^{2(t-1)}\right)$

$\boxed{= \frac{1}{6}e^{-t} - \frac{1}{2}e^{t-2} + \frac{1}{3}e^{2t-3}}$

Exercise

Solve using the Laplace transform: $y'' - 3y' + 2y = \cos t$; $y(0) = 0, \quad y'(0) = -1$

Solution

$$\mathcal{L}\{y'' - 3y' + 2y\} = \mathcal{L}\{\cos t\}$$

$$s^2Y(s) - sy(0) - y'(0) - 3sY(s) + 3y(0) + 2Y(s) = \frac{s}{s^2+1} \qquad y(0) = 0 \quad y'(0) = -1$$

$$\left(s^2 - 3s + 2\right)Y(s) = \frac{s}{s^2+1} - 1$$

$$Y(s) = \frac{-s^2 + s - 1}{(s-1)(s-2)\left(s^2+1\right)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{Cs+D}{s^2+1}$$

$$\begin{cases} s^3 & A + B + C = 0 \\ s^2 & -2A - B - 3C + D = -1 \\ s^1 & A + B + 2C - 3D = 1 \\ s^0 & -2A - B + 2D = -1 \end{cases} \rightarrow \boxed{A = \frac{1}{2},\ B = -\frac{3}{5},\ C = \frac{1}{10},\ D = -\frac{3}{10}}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s-1} - \frac{3}{5}\frac{1}{s-2} + \frac{1}{10}\frac{s}{s^2+1} - \frac{3}{10}\frac{1}{s^2+1}\right\}$$

$$\boxed{y(t) = \frac{1}{2}e^t - \frac{3}{5}e^{2t} + \frac{1}{10}\cos t - \frac{3}{10}\sin t}$$

Exercise

Solve using the Laplace transform: $y'' - 4y = e^{-t}$; $y(0) = -1, \quad y'(0) = 0$

Solution

$$\mathcal{L}(y'' - 4y) = \mathcal{L}\left(e^{-t}\right)$$

$$s^2Y(s) - sy(0) - y'(0) - 4Y(s) = \frac{1}{s+1} \qquad y(0) = -1 \quad y'(0) = 0$$

$$\left(s^2 - 4\right)Y(s) + s = \frac{1}{s+1}$$

$$Y(s) = \frac{1}{(s+1)\left(s^2-4\right)} - \frac{s}{s^2-4}$$

$$\frac{1}{(s+1)\left(s^2-4\right)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s+2}$$

$$1 = As^2 - 4A + Bs^2 + 3Bs + 2B + Cs^2 - Cs - 2C$$

$$1 = (A + B + C)s^2 + (3B - C)s + 2B - 4A - 2C$$

$$\begin{cases} A+B+C=0 \\ 3B-C=0 \\ -4A+2B-2C=1 \end{cases} \Rightarrow A=-\frac{1}{3} \quad B=\frac{1}{12} \quad C=\frac{1}{4}$$

$$\frac{s}{s^2-4}=\frac{D}{s-2}+\frac{E}{s+2}$$

$$s=(D+E)s+2D-2E$$

$$\begin{cases} D+E=1 \\ 2D-2E=0 \end{cases} \Rightarrow D=\frac{1}{2} \quad E=\frac{1}{2}$$

$$Y(s)=-\frac{1}{3}\frac{1}{s+1}+\frac{1}{12}\frac{1}{s-2}+\frac{1}{4}\frac{1}{s+2}-\frac{1}{2}\frac{1}{s-2}-\frac{1}{2}\frac{1}{s+2}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{-\frac{1}{3}\frac{1}{s+1}-\frac{5}{12}\frac{1}{s-2}-\frac{1}{4}\frac{1}{s+2}\right\}$$

$$\boxed{y(t)=-\frac{1}{3}e^{-t}-\frac{5}{12}e^{2t}-\frac{1}{4}e^{-2t}}$$

Exercise

Solve using the Laplace transform: $\quad y''-4y'=6e^{3t}-3e^{-t}, \quad y(0)=1 \quad y'(0)=-1$

Solution

$$\mathcal{L}\{y''-4y'\}=\mathcal{L}\left\{6e^{3t}-3e^{-t}\right\}$$

$$s^2Y(s)-sy(0)-y'(0)-4sY(s)+4y(0)=0 \qquad y(0)=1; \ y'(0)=-1$$

$$\left(s^2-4s\right)Y(s)-s+1+4=0$$

$$Y(s)=\frac{s-5}{s(s-4)} \quad =\frac{A}{s}+\frac{B}{s-4}$$

$$As-4A+Bs=s-5$$

$$\begin{cases} A+B=1 \\ -4A=-5 \end{cases} \rightarrow \boxed{A=\frac{5}{4}, \ B=-\frac{1}{4}}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{5}{4}\frac{1}{s}-\frac{1}{4}\frac{1}{s-4}\right\}$$

$$\boxed{y(t)=\frac{5}{4}t-\frac{1}{4}e^{4t}}$$

Exercise

Solve using the Laplace transform: $\quad y''-4y'+4y=t^3e^{2t} \ ; \quad y(0)=0, \quad y'(0)=0$

Solution

$$\mathcal{L}\{y''-4y'+4y\}=\mathcal{L}\{t^3e^{2t}\}$$

$$s^2Y(s)-sy(0)-y'(0)-4sY(s)+4y(0)+4Y(s)=\frac{6}{(s-2)^4} \qquad y(0)=0;\ \ y'(0)=0$$

$$\left(s^2-4s+4\right)Y(s)=\frac{6}{(s-2)^4}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{6}{(s-2)^6}\right\}$$

$$\boxed{y(t)=\frac{1}{20}t^5e^{2t}}$$

Exercise

Solve using the Laplace transform: $y''-4y'+4y=t^3,\quad y(0)=1,\quad y'(0)=0$

Solution

$$\mathcal{L}\{y''-4y'+4y\}=\mathcal{L}\{t^3\}$$

$$s^2Y(s)-sy(0)-y'(0)-4sY(s)+4y(0)+4Y(s)=\frac{6}{s^4} \qquad y(0)=1;\ \ y'(0)=0$$

$$\left(s^2-4s+4\right)Y(s)=\frac{6}{s^4}+s-4$$

$$Y(s)=\frac{s^5-4s^4+6}{s^4(s-2)^2}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s^4}+\frac{E}{s-2}+\frac{F}{(s-2)^2}$$

s^5	$A+E=1$	$\boxed{E=-\frac{1}{4}}$
s^4	$-4A+B-2E+F=-4$	$\boxed{F=-\frac{13}{8}}$
s^3	$4A-4B+C=0$	$\boxed{A=\frac{3}{4}}$
s^2	$4B-4C+D=0$	$\boxed{B=\frac{9}{8}}$
s^1	$4C-4D=0$	$\boxed{C=\frac{3}{2}}$
s^0	$4D=6$	$\boxed{D=\frac{3}{2}}$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{3}{4}\frac{1}{s}+\frac{9}{8}\frac{1}{s^2}+\frac{3}{2}\frac{1}{s^3}+\frac{3}{2}\frac{1}{s^4}+\frac{1}{4}\frac{1}{s-2}-\frac{13}{8}\frac{1}{(s-2)^2}\right\}$$

$$\boxed{y(t)=\frac{3}{4}+\frac{9}{8}t+\frac{3}{2}t^2+\frac{3}{2}t^3+\frac{1}{4}e^{2t}-\frac{13}{8}te^{2t}}$$

Exercise

Solve using the Laplace transform: $x'' + 4x' + 4x = t^2;\quad x(0) = x'(0) = 0$

Solution

$$\mathcal{L}\left\{x'' + 4x' + 4x\right\} = \mathcal{L}\left\{t^2\right\}$$

$$s^2 X(s) - sx(0) - x'(0) + 4sX(s) - 4x(0) + 4X(s) = \frac{2}{s^3} \qquad x(0) = x'(0) = 0$$

$$\left(s^2 + 4s + 4\right)X(s) = \frac{2}{s^3}$$

$$X(s) = \frac{2}{s^3(s+2)^2} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s+2} + \frac{E}{(s+2)^2}$$

$$As^2 + 4As + 4A + Bs^3 + 4Bs^2 + 4Bs + Cs^4 + 4Cs^3 + 4Cs^2 + Ds^4 + 2Ds^3 + Es^3 = 3$$

$$\begin{cases} s^4 & C + D = 0 & \rightarrow \boxed{D = -\frac{9}{16}} \\ s^3 & B + 4C + 2D + E = 0 & \rightarrow \boxed{E = -\frac{3}{8}} \\ s^2 & A + 4B + 4C = 0 & \rightarrow \boxed{C = \frac{9}{16}} \\ s^1 & 4A + 4B = 0 & \rightarrow \boxed{B = -\frac{3}{4}} \\ s^0 & 4A = 3 \ \rightarrow\ \boxed{A = \frac{3}{4}} & \end{cases}$$

$$\mathcal{L}^{-1}\left\{X(s)\right\} = \mathcal{L}^{-1}\left\{\frac{3}{4}\frac{1}{s^3} - \frac{3}{4}\frac{1}{s^2} + \frac{9}{16}\frac{1}{s} - \frac{9}{16}\frac{1}{s+2} - \frac{3}{8}\frac{1}{(s+2)^2}\right\}$$

$$\boxed{x(t) = \frac{3}{8}t^2 - \frac{3}{4}t + \frac{9}{16} - \frac{9}{16}e^{-2t} - \frac{3}{8}te^{-2t}}$$

Exercise

Solve using the Laplace transform: $y'' + 4y = 4t^2 - 4t + 10\ ;\quad y(0) = 0,\quad y'(0) = 3$

Solution

$$\mathcal{L}\left\{y'' + 4y\right\} = \mathcal{L}\left\{4t^2 - 4t + 10\right\}$$

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{8}{s^3} - \frac{4}{s^2} + \frac{10}{s} \qquad y(0) = 0\ \ y'(0) = 3$$

$$\left(s^2 + 4\right)Y(s) = \frac{8 - 4s + 10s^2}{s^3} + 3$$

$$Y(s)=\frac{3s^3+10s^2-4s+8}{s^3\left(s^2+4\right)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{Ds+E}{s^2+4}$$

$$\begin{cases} s^4 & A+D=0 & \underline{D=-2} \\ s^3 & B+E=3 & \underline{E=4} \\ s^2 & 4A+C=10 & \underline{A=2} \\ s^1 & 4B=-4 & \underline{B=-1} \\ s^0 & 4C=8 & \underline{C=2} \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{2}{s}-\frac{1}{s^2}+\frac{2}{s^3}-\frac{2s}{s^2+2^2}+\frac{4}{s^2+2^2}\right\}$$

$$\boxed{y(t)=2-t+t^2-2\cos 2t+2\sin 2t}$$

Exercise

Solve using the Laplace transform: $\quad y''-4y=4t-8e^{-2t}; \quad y(0)=0, \quad y'(0)=5$

Solution

$$\mathcal{L}\{y''-4y\}=\mathcal{L}\left\{4t-8e^{-2t}\right\}$$

$$s^2Y(s)-sy(0)-y'(0)-4Y(s)=\frac{4}{s^2}-\frac{8}{s+2} \qquad y(0)=0 \qquad y'(0)=5$$

$$\left(s^2-4\right)Y(s)=\frac{-8s^2+4s+8}{s^2(s+2)}+5$$

$$Y(s)=\frac{5s^3+2s^2+4s+8}{s^2(s+2)^2(s-2)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+2}+\frac{D}{(s+2)^2}+\frac{E}{s-2}$$

$$5s^3+2s^2+4s+8=A\left(s^2+2s\right)\left(s^2-4\right)+B(s+2)\left(s^2-4\right)$$
$$+C\left(s^4-4s^2\right)+D\left(s^3-2s^2\right)+Es^2\left(s^2+4s+4\right)$$

$$\begin{cases} s^4 & A+C+E=0 & & C+E=0 \\ s^3 & 2A+B+D+4E=5 & & D+4E=6 \\ s^2 & -4A+2B-4C-2D+4E=2 & & -4C-2D+4E=4 \\ s^1 & -8A-4B=4 & \underline{A=0} & \\ s^0 & -8B=8 & \underline{B=-1} & \end{cases}$$

$$\underline{C=-1 \quad D=2 \quad E=1}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{-\frac{1}{s^2} - \frac{1}{s+2} + \frac{2}{(s+2)^2} + \frac{1}{s-2}\right\}$$

$$\boxed{y(t) = -t - e^{-2t} + 2te^{-2t} + e^{2t}}$$

Exercise

Solve using the Laplace transform: $y'' + 4y' = \cos(t-3) + 4t$, $y(3) = 0$, $y'(3) = 7$

Solution

$$y''(t) + 4y'(t) = \cos(t-3) + 4t$$

Let $v = t - 3 \quad \rightarrow \quad t = v + 3$

$$x(v) = y(t) = y(v+3) \qquad y(3) = x(0) = 0 \quad y'(3) = x'(0) = 7$$

$$x''(v) + 4x'(v) = \cos v + 4v + 12$$

$$\mathcal{L}\{x'' + 4x'\} = \mathcal{L}\{\cos v + 4v + 12\}$$

$$s^2X(s) - sx(0) - x'(0) + 4sX(s) - 4x(0) = \frac{s}{s^2+1} + \frac{4}{s^2} + \frac{12}{s}$$

$$\left(s^2 + 4s\right)X(s) = \frac{s}{s^2+1} + \frac{4+12s}{s^2} + 7$$

$$s(s+4)X(s) = \frac{s}{s^2+1} + \frac{7s^2+12s+4}{s^2}$$

$$X(s) = \frac{1}{(s+4)\left(s^2+1\right)} + \frac{7s^2+12s+4}{s^3(s+4)}$$

$$\frac{1}{(s+4)\left(s^2+1\right)} = \frac{A_1}{s+4} + \frac{A_2s + A_3}{s^2+1}$$

$$1 = A_1s^2 + A_1 + A_2s^2 + 4A_2s + A_3s + 4A_3$$

$$\begin{cases} s^2 & A_1 + A_2 = 0 & \rightarrow A_1 = -A_2 & \boxed{A_1 = \frac{1}{17}} \\ s & 4A_2 + A_3 = 0 & \rightarrow A_3 = -4A_2 & \boxed{A_3 = \frac{4}{17}} \\ s^0 & A_1 + 4A_3 = 1 & \Rightarrow -A_2 - 16A_2 = 1 & \rightarrow \boxed{A_2 = -\frac{1}{17}} \end{cases}$$

$$\frac{1}{(s+4)\left(s^2+1\right)} = \frac{1}{17}\frac{1}{s+4} + \frac{1}{17}\frac{-s+4}{s^2+1}$$

$$\frac{7s^2+12s+4}{s^3(s+4)} = \frac{B_1}{s} + \frac{B_2}{s^2} + \frac{B_3}{s^3} + \frac{B_4}{s+4}$$

$$7s^2+12s+4=B_1s^3+4B_1s^2+B_2s^2+4B_2s+B_3s+4B_3+B_4s^3$$

$$\begin{cases} s^3 & B_1+B_4=0 & \rightarrow \boxed{B_4=-\frac{17}{16}} \\ s^2 & 4B_1+B_2=7 & \rightarrow \boxed{B_1=\frac{17}{16}} \\ s & 4B_2+B_3=12 & \rightarrow \boxed{B_2=\frac{11}{4}} \\ s^0 & 4B_3=4 & \rightarrow \boxed{B_3=1} \end{cases}$$

$$\frac{7s^2+12s+4}{s^3(s+4)}=\frac{17}{16}\frac{1}{s}+\frac{11}{4}\frac{1}{s^2}+\frac{1}{s^3}-\frac{17}{16}\frac{1}{s+4}$$

$$X(s)=\frac{1}{17}\frac{1}{s+4}+\frac{1}{17}\frac{-s+4}{s^2+1}+\frac{17}{16}\frac{1}{s}+\frac{11}{4}\frac{1}{s^2}+\frac{1}{s^3}-\frac{17}{16}\frac{1}{s+4}$$

$$=\frac{17}{16}\frac{1}{s}+\frac{11}{4}\frac{1}{s^2}+\frac{1}{s^3}-\frac{273}{272}\frac{1}{s+4}-\frac{1}{17}\frac{s}{s^2+1}+\frac{4}{17}\frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\{X(s)\}=\mathcal{L}^{-1}\left\{\frac{17}{16}\frac{1}{s}+\frac{11}{4}\frac{1}{s^2}+\frac{1}{s^3}-\frac{273}{272}\frac{1}{s+4}-\frac{1}{17}\frac{s}{s^2+1}+\frac{4}{17}\frac{1}{s^2+1}\right\}$$

$$x(v)=\frac{17}{16}+\frac{11}{4}v+\frac{1}{2}v^2-\frac{273}{272}e^{-4v}-\frac{1}{17}\cos v+\frac{4}{17}\sin v \qquad v=t-3$$

$$y(t)=\frac{17}{16}+\frac{11}{4}(t-3)+\frac{1}{2}(t-3)^2-\frac{273}{272}e^{-4(t-3)}-\frac{1}{17}\cos(t-3)+\frac{4}{17}\sin(t-3)$$

$$=\frac{17}{16}+\frac{11}{4}t-\frac{33}{4}+\frac{1}{2}t^2-3t+\frac{9}{2}-\frac{273}{272}e^{-4(t-3)}+\frac{1}{17}\big(4\sin(t-3)-\cos(t-3)\big)$$

$$\boxed{=\frac{43}{16}-\frac{1}{4}t+\frac{1}{2}t^2-\frac{273}{272}e^{-4(t-3)}+\frac{1}{17}\big(4\sin(t-3)-\cos(t-3)\big)}$$

Exercise

Solve using the Laplace transform: $y''+4y'+8y=\sin t$, $y(0)=1$, $y'(0)=0$

Solution

$$\mathcal{L}\{y''+4y'+8y\}(s)=\mathcal{L}\{\sin t\}(s)$$

$$s^2Y(s)-sy(0)-y'(0)+4sY(s)-4y(0)+8Y(s)=\frac{1}{s^2+1} \qquad y(0)=1, \quad y'(0)=0$$

$$\left(s^2+4s+8\right)Y(s)-s-4=\frac{1}{s^2+1}$$

$$\left(s^2+4s+8\right)Y(s)=\frac{1}{s^2+1}+s+4$$

$$Y(s)=\frac{s^3+4s^2+s+5}{\left(s^2+1\right)\left(s^2+4s+8\right)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+4s+8}$$

$$s^3+4s^2+s+5=As^3+4As^2+8As+Bs^2+4Bs+8B+Cs^3+Cs+Ds^2+D$$

$$\begin{cases} s^3 & A+C=1 \\ s^2 & 4A+B+D=4 \\ s & 8A+4B+C=1 \\ s^0 & 8B+D=5 \end{cases} \Rightarrow \begin{matrix} A=-\frac{4}{65} & B=\frac{7}{65} \\ C=\frac{69}{65} & D=\frac{269}{65} \end{matrix}$$

$$Y(s)=\frac{1}{65}\left(-4\frac{s}{s^2+1}+\frac{7}{s^2+1}+\frac{69(s+2)-138+269}{(s+2)^2+4}\right)$$

$$=\frac{1}{65}\left(-4\frac{s}{s^2+1}+\frac{7}{s^2+1}+\frac{69s}{(s+2)^2+4}+\frac{2}{2}\frac{131}{(s+2)^2+4}\right)$$

$$\mathcal{L}^{-1}\{Y(s)\}=\frac{1}{65}\mathcal{L}^{-1}\left\{-4\frac{s}{s^2+1}+\frac{7}{s^2+1}+\frac{69s}{(s+2)^2+4}+\frac{131}{2}\frac{2}{(s+2)^2+4}\right\}$$

$$\boxed{y(t)=-\frac{4}{65}\cos t+\frac{7}{65}\sin t+\frac{69}{65}e^{-2t}\cos 2t+\frac{131}{130}e^{-2t}\sin 2t}$$

Exercise

Solve using the Laplace transform: $y''+5y'-y=e^t-1$; $y(0)=1$, $y'(0)=1$

Solution

$$\mathcal{L}\{y''+5y'-y\}=\mathcal{L}\left\{e^t-1\right\}$$

$$s^2Y(s)-sy(0)-y'(0)+5sY(s)-5y(0)-Y(s)=\frac{1}{s-1}-\frac{1}{s} \qquad y(0)=1 \quad y'(0)=1$$

$$\left(s^2+5s-1\right)Y(s)=\frac{1}{s(s-1)}+s+6$$

$$Y(s)=\frac{s^3+5s^2-6s+1}{s(s-1)\left(s+\frac{5}{2}-\frac{\sqrt{29}}{2}\right)\left(s+\frac{5}{2}+\frac{\sqrt{29}}{2}\right)}$$

$$=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s+\frac{5}{2}-\frac{\sqrt{29}}{2}}+\frac{D}{s+\frac{5}{2}+\frac{\sqrt{29}}{2}}$$

$$\begin{cases} s^3 & A+C+D=1 \\ s^2 & 4A+B+\left(\frac{3}{2}+\frac{\sqrt{29}}{2}\right)C+\left(\frac{3}{2}-\frac{\sqrt{29}}{2}\right)C=5 \\ s^1 & -6A+5B-\left(\frac{5}{2}+\frac{\sqrt{29}}{2}\right)C-\left(\frac{5}{2}-\frac{\sqrt{29}}{2}\right)D=-6 \\ s^0 & A-B=1 \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{1}{5}\frac{1}{s-1}+\left(-\frac{1}{10}+\frac{3}{10\sqrt{29}}\right)\frac{1}{s+\frac{5}{2}-\frac{\sqrt{29}}{2}}-\left(\frac{1}{10}+\frac{3}{10\sqrt{29}}\right)\frac{1}{s+\frac{5}{2}+\frac{\sqrt{29}}{2}}\right\}$$

$$\boxed{y(t)=1+\frac{1}{5}e^{t}+\left(-\frac{1}{10}+\frac{3\sqrt{29}}{290}\right)e^{\frac{-5+\sqrt{29}}{2}t}-\left(\frac{1}{10}+\frac{3\sqrt{29}}{290}\right)e^{\frac{-5-\sqrt{29}}{2}t}}$$

Exercise

Solve using the Laplace transform: $y''+5y'-6y=21e^{t-1}$ $y(1)=-1,\quad y'(1)=9$

Solution

Let: $w(t)=y(t+1) \quad \leftrightarrow \quad y(t)=w(t-1)$

$$\mathcal{L}\{w''+5w'-6w\}=\mathcal{L}\{21e^{t}\}$$

$$s^2W(s)-sw(0)-w'(0)+5sW(s)-5w(0)-6W(s)=21\frac{1}{s-1} \qquad y(1)=w(0)=-1,\quad y'(1)=w(0)=9$$

$$\left(s^2+5s-6\right)W(s)=\frac{21}{s-1}-s+4$$

$$W(s)=\frac{-s^2+5s+17}{(s+6)(s-1)^2}=\frac{A}{s+6}+\frac{B}{s-1}+\frac{C}{(s-1)^2}$$

$$\begin{cases} s^2 & A+B=-1 \\ s^1 & -2A+5B+C=5 \\ s^0 & A-6B+6C=17 \end{cases}$$

$$\Delta=\begin{vmatrix}1&1&0\\-2&5&1\\1&-6&6\end{vmatrix}=49 \quad \Delta_A=\begin{vmatrix}-1&1&0\\5&5&1\\17&-6&6\end{vmatrix}=-49 \quad \Delta_B=\begin{vmatrix}1&-1&0\\-2&5&1\\1&17&6\end{vmatrix}=0 \quad \Delta_C=\begin{vmatrix}1&1&-1\\-2&5&5\\1&-6&17\end{vmatrix}=147$$

$$\boxed{A=-1,\quad B=0,\quad C=3}$$

$$\mathcal{L}^{-1}\{W(s)\}=\mathcal{L}^{-1}\left\{\frac{-1}{s+6}+\frac{3}{(s-1)^2}\right\}$$

$$w(t)=-e^{-6t}+3te^{t}$$

$$y(t)=w(t-1)\ \boxed{=-e^{-6(t-1)}+3(t-1)e^{(t-1)}}$$

Exercise

Solve using the Laplace transform: $y'' + 5y' + 4y = 0;\quad y(0) = 1,\quad y'(0) = 0$

Solution

$$\mathcal{L}\{y'' + 5y' + 4y\} = 0$$

$$s^2Y(s) - sy(0) - y'(0) + 5sY(s) - 5y(0) + 4Y(s) = 0 \qquad y(0) = 1;\ \ y'(0) = 0$$

$$\left(s^2 + 5s + 4\right)Y(s) - s - 5 = 0$$

$$Y(s) = \frac{s+5}{s^2 + 5s + 4} = \frac{A}{s+1} + \frac{B}{s+4}$$

$$As + 4A + Bs + B = s + 5$$

$$\begin{cases} A + B = 1 \\ 4A + B = 5 \end{cases} \rightarrow \boxed{A = \frac{4}{3};\ B = -\frac{1}{3}}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{4}{3}\frac{1}{s+1} - \frac{1}{3}\frac{1}{s+4}\right\}$$

$$\boxed{y(t) = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}}$$

Exercise

Solve using the Laplace transform: $y'' + 6y = t^2 - 1;\quad y(0) = 0,\quad y'(0) = -1$

Solution

$$\mathcal{L}\{y'' + 6y\} = \mathcal{L}\left\{t^2 - 1\right\}$$

$$s^2Y(s) - sy(0) - y'(0) + 6Y(s) = \frac{2}{s^3} - \frac{1}{s} \qquad y(0) = 0 \quad y'(0) = -1$$

$$\left(s^2 + 6\right)Y(s) = \frac{2 - s^2}{s^3} - 1$$

$$Y(s) = \frac{2 - s^2 - s^3}{s^3\left(s^2 + 6\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds + E}{s^2 + 6}$$

$$\begin{cases} s^4 & A + D = 0 & \boxed{D = \frac{2}{9}} \\ s^3 & B + E = -1 & \boxed{E = -1} \\ s^2 & 6A + C = -1 & \boxed{A = -\frac{2}{9}} \\ s & \boxed{B = 0} & \\ s^0 & 6C = 2 & \boxed{C = \frac{1}{3}} \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{-\frac{2}{9}\frac{1}{s} + \frac{1}{3}\frac{1}{s^3} + \frac{1}{9}\frac{s}{s^2+6} - \frac{1}{s^2+6}\right\}$$

$$\boxed{y(t) = -\frac{2}{9} + \frac{1}{6}t^2 + \frac{1}{9}\cos\sqrt{6}t - \frac{1}{\sqrt{6}}\sin\sqrt{6}t}$$

Exercise

Solve using the Laplace transform: $y'' - 6y' + 9y = t$; $y(0) = 0,$ $y'(0) = 1$

Solution

$$\mathcal{L}\{y'' - 6y' + 9y\} = \mathcal{L}\{t\}$$

$$s^2Y(s) - sy(0) - y'(0) - 6sY(s) + 6y(0) + 9Y(s) = \frac{1}{s^2} \qquad y(0) = 0 \quad y'(0) = 1$$

$$\left(s^2 - 6s + 9\right)Y(s) = \frac{1}{s^2} + 1$$

$$Y(s) = \frac{s^2+1}{s^2(s-3)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3} + \frac{D}{(s-3)^2}$$

$$\begin{cases} s^3 & A + C = 0 & \boxed{C = -\frac{2}{27}} \\ s^2 & -6A + B - 3C + D = 1 & \boxed{D = \frac{10}{9}} \\ s & 9A - 6B = 0 & \boxed{A = \frac{2}{27}} \\ s^0 & 9B = 1 & \boxed{B = \frac{1}{9}} \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{27}\frac{1}{s} + \frac{1}{9}\frac{1}{s^2} - \frac{2}{27}\frac{1}{s-3} + \frac{10}{9}\frac{1}{(s-3)^2}\right\}$$

$$\boxed{y(t) = \frac{2}{27} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{10}{9}te^{3t}}$$

Exercise

Solve using the Laplace transform: $y'' - 6y' + 15y = 2\sin 3t$, $y(0) = -1,$ $y'(0) = -4$

Solution

$$\mathcal{L}\{y'' - 6y' + 15y\}(s) = \mathcal{L}\{2\sin 3t\}(s)$$

$$s^2Y(s) - sy(0) - y'(0) - 6sY(s) + 6y(0) + 15Y(s) = \frac{6}{s^2+9} \qquad y(0) = -1, \quad y'(0) = -4$$

$$\left(s^2 - 6s + 15\right)Y(s) + s + 4 - 6 = \frac{6}{s^2+9}$$

$$\left(s^2-6s+15\right)Y(s)=\frac{6}{s^2+9}-s+2$$

$$Y(s)=\frac{-s^3+2s^2-9s+24}{\left(s^2+9\right)\left(s^2-6s+15\right)}=\frac{As+B}{s^2+9}+\frac{Cs+D}{s^2-6s+15}$$

$$-s^3+2s^2-9s+24=As^3-6As^2+15As+Bs^2-6Bs+15B+Cs^3+9Cs+Ds^2+9D$$

$$\begin{cases} s^3 & A+C=-1 \\ s^2 & -6A+B+D=2 \\ s & 15A-6B+9C=-9 \\ s^0 & 15B+9D=24 \end{cases} \quad \rightarrow \quad \begin{matrix} A=\frac{1}{10} & B=\frac{1}{10} \\ C=-\frac{11}{10} & D=\frac{5}{2} \end{matrix}$$

$$Y(s)=\frac{1}{10}\frac{s}{s^2+9}+\frac{1}{10}\frac{1}{s^2+9}+\frac{1}{10}\frac{-11(s-3)-33+25}{(s-3)^2-9+15}$$

$$=\frac{1}{10}\frac{s}{s^2+9}+\frac{1}{10}\frac{1}{s^2+9}\frac{3}{3}-\frac{11}{10}\frac{s-3}{(s-3)^2+6}-\frac{1}{10}\frac{8}{(s-3)^2+6}\frac{\sqrt{6}}{\sqrt{6}}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{10}\frac{s}{s^2+9}+\frac{1}{30}\frac{3}{s^2+9}-\frac{11}{10}\frac{s-3}{(s-3)^2+6}-\frac{8}{10\sqrt{6}}\frac{\sqrt{6}}{(s-3)^2+6}\right\}$$

$$\boxed{y(t)=\frac{1}{10}\cos 3t+\frac{1}{30}\sin 3t-\frac{11}{10}e^{3t}\cos\sqrt{6}t-\frac{8}{10\sqrt{6}}e^{3t}\sin\sqrt{6}t}$$

Exercise

Solve using the Laplace transform: $y''-6y'+13y=0$; $y(0)=0,\quad y'(0)=-3$

Solution

$$\mathcal{L}\{y''-6y'+13y\}=0$$

$$s^2Y(s)-sy(0)-y'(0)-6sY(s)+6y(0)+13Y(s)=0 \qquad y(0)=0 \quad y'(0)=-3$$

$$\left(s^2-6s+13\right)Y(s)=-3$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{-3}{(s-3)^2+4}\right\}$$

$$\boxed{y(t)=-\frac{3}{2}e^{3t}\sin 2t}$$

Exercise

Solve using the Laplace transform: $y'' + 6y' + 9y = 0, \quad y(0) = -1, \quad y'(0) = 6$

Solution

$$\mathcal{L}\{y'' + 6y' + 9y\} = 0$$

$$s^2Y(s) - sy(0) - y'(0) + 6Y(s) - 6y(0) + 9Y(s) = 0 \qquad y(0) = -1 \quad y'(0) = 6$$

$$\left(s^2 + 6s + 9\right)Y(s) = -s$$

$$Y(s) = -\frac{s}{(s+3)^2} = \frac{A}{s+3} + \frac{B}{(s+3)^2}$$

$$\begin{cases} s & \underline{A = -1} \\ s^0 & 3A + B = 0 \quad \underline{B = 3} \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{-1}{s+3} + \frac{3}{(s+3)^2}\right\}$$

$$\boxed{y(t) = -e^{-3t} + 3te^{-3t}}$$

Exercise

Solve using the Laplace transform: $y'' + 6y' + 5y = 12e^t, \quad y(0) = -1, \quad y'(0) = 7$

Solution

$$\mathcal{L}\{y'' + 6y' + 5y\} = \mathcal{L}\left\{12e^t\right\}$$

$$s^2Y(s) - sy(0) - y'(0) + 6Y(s) - 6y(0) + 5Y(s) = \frac{12}{s-1} \qquad y(0) = -1 \quad y'(0) = 7$$

$$\left(s^2 + 6s + 5\right)Y(s) = \frac{12}{s-1} - s + 1$$

$$Y(s) = \frac{-s^2 + 2s + 11}{(s+1)(s+5)(s-1)} = \frac{A}{s+1} + \frac{B}{s+5} + \frac{C}{s-1}$$

$$\begin{cases} s^2 & A + B + C = -1 \\ s & 4A + 6C = 2 \\ s^0 & -5A - B + 5C = 11 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 0 & 6 \\ -5 & -1 & 5 \end{vmatrix} = -48 \quad \Delta_A = \begin{vmatrix} -1 & 1 & 1 \\ 2 & 0 & 6 \\ 11 & -1 & 5 \end{vmatrix} = 48 \quad \Delta_B = \begin{vmatrix} 1 & -1 & 1 \\ 4 & 2 & 6 \\ -5 & 11 & 5 \end{vmatrix} = 48$$

$$\boxed{A = -1 \quad B = -1 \quad C = 1}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{-1}{s+1} - \frac{1}{s+5} + \frac{1}{s-1}\right\}$$

$$\boxed{y(t) = -e^{-t} - e^{-5t} + e^{t}}$$

Exercise

Solve using the Laplace transform: $y'' - 7y' + 10y = 9\cos t + 7\sin t$; $y(0) = 5,\quad y'(0) = -4$

Solution

$$\mathcal{L}\{y'' - 7y' + 10y\} = \mathcal{L}\{9\cos t + 7\sin t\}$$

$$s^2Y(s) - sy(0) - y'(0) - 7sY(s) + 7y(0) + 10Y(s) = \frac{9s}{s^2+1} + \frac{7}{s^2+1} \qquad y(0) = 5,\quad y'(0) = -4$$

$$\left(s^2 - 7s + 10\right)Y(s) = \frac{9s+7}{s^2+1} + 5s - 39$$

$$Y(s) = \frac{5s^3 - 39s^2 + 14s - 32}{(s-2)(s-5)\left(s^2+1\right)} = \frac{A}{s-2} + \frac{B}{s-5} + \frac{Cs+D}{s^2+1}$$

$$\begin{cases} s^3 & A + B + C = 5 \\ s^2 & -5A - 2B - 7C + D = -39 \\ s & A + B + 10C - 7D = 14 \\ s^0 & -5A - 2B + 10D = -32 \end{cases} \rightarrow \boxed{A = 8,\ B = -4,\ C = 1,\ D = 0}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{8}{s-2} - \frac{4}{s-5} + \frac{s}{s^2+1}\right\}$$

$$\boxed{y(t) = 8e^{2t} - 4e^{5t} + \cos t}$$

Exercise

Solve using the Laplace transform: $y'' + 8y' + 25y = 0$, $y(\pi) = 0,\quad y'(\pi) = 6$

Solution

Let: $\quad w(t) = y(t+\pi) \quad \leftrightarrow \quad y(t) = w(t-\pi)$

$$y''(t) + 8y'(t) + 25y(t) = 0$$

$$\mathcal{L}\{w'' + 8w' + 25w\} = 0$$

$$s^2W(s) - sw(0) - w'(0) + 8sW(s) - 8w(0) + 25W(s) = 0 \qquad y(\pi) = w(0) = 0,\quad y'(\pi) = w(0) = 6$$

$$\left(s^2 + 8s + 25\right)W(s) - 6 = 0$$

$$W(s)=\frac{6}{(s+4)^2-16+25}$$

$$\mathcal{L}^{-1}\{W(s)\}=\mathcal{L}^{-1}\left\{\frac{2\ (3)}{(s+4)^2+9}\right\}$$

$$w(t)=2e^{-4t}\sin 3t \qquad y(t)=w(t-\pi)$$

$$y(t)=2e^{-4(t-\pi)}\sin 3(t-\pi) \qquad \sin(3t-3\pi)=\sin 3t\cos 3\pi-\cos 3t\sin 3\pi=\sin 3t(-1)-0=-\sin 3t$$

$$\boxed{=-2e^{-4(t-\pi)}\sin 3t}$$

Exercise

Solve using the Laplace transform: $y''+9y=2\sin 2t;\quad y(0)=0,\quad y'(0)=-1$

Solution

$$\mathcal{L}(y''+9y)=\mathcal{L}(2\sin 2t)$$

$$s^2Y(s)-sy(0)-y'(0)+9Y(s)=2\frac{2}{s^2+2^2} \qquad y(0)=0 \quad y'(0)=-1$$

$$(s^2+9)Y(s)+1=\frac{4}{s^2+4}$$

$$Y(s)=\frac{4}{(s^2+9)(s^2+4)}-\frac{1}{s^2+9}$$

$$\frac{4}{(s^2+9)(s^2+4)}=\frac{A}{s^2+9}+\frac{B}{s^2+4}$$

$$4=(A+B)s^2+4A+9B$$

$$\begin{cases} A+B=0 \\ 4A+9B=4 \end{cases} \Rightarrow A=-\frac{4}{5} \quad B=\frac{4}{5}$$

$$Y(s)=-\frac{4}{5}\frac{1}{s^2+9}+\frac{4}{5}\frac{1}{s^2+4}-\frac{1}{s^2+9}$$

$$=\frac{4}{5}\frac{1}{s^2+4}-\frac{9}{5}\frac{1}{s^2+9}$$

$$=\frac{4}{5}\frac{1}{2}\frac{2}{s^2+4}-\frac{3}{5}\frac{3}{s^2+9}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{2}{5}\frac{2}{s^2+4}-\frac{3}{5}\frac{3}{s^2+9}\right\}$$

$$\boxed{y(t)=\frac{2}{5}\sin 2t-\frac{3}{5}\sin 3t}$$

Exercise

Solve using the Laplace transform: $y'' + 9y = 3\sin 2t$; $y(0) = 0,\quad y'(0) = -1$

Solution

$$\mathcal{L}(y'' + 9y)(s) = \mathcal{L}(3\sin 2t)(s)$$

$$s^2 Y(s) - sy(0) - y'(0) + 9Y(s) = 3\frac{1}{s^2+4} \qquad y(0) = 0 \quad y'(0) = -1$$

$$s^2 Y(s) + 1 + Y(s) = \frac{3}{s^2+4}$$

$$\left(s^2+1\right)Y(s) = \frac{3}{s^2+4} - 1$$

$$\left(s^2+1\right)Y(s) = \frac{-s^2-1}{s^2+4}$$

$$Y(s) = \frac{-s^2-1}{\left(s^2+4\right)\left(s^2+1\right)} \quad = \frac{A}{s^2+4} + \frac{B}{s^2+1}$$

$$As^2 + A + Bs^2 + 4B = -s^2 - 1$$

$$\begin{cases} s^2 \\ s^0 \end{cases} \begin{cases} A + B = -1 \\ A + 4B = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{2}{3} \\ B = -\frac{5}{3} \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{3}\frac{1}{s^2+4} - \frac{5}{3}\frac{1}{s^2+1}\right\}$$

$$\boxed{y(t) = \frac{2}{3}\sin 2t - \frac{5}{3}\sin t}$$

Exercise

Solve using the Laplace transform: $y'' + 16y = 2\sin 4t$; $y(0) = -\frac{1}{2},\quad y'(0) = 0$

Solution

$$\mathcal{L}\{y'' + 16y\}(s) = \mathcal{L}\{2\sin 4t\}(s)$$

$$s^2 Y(s) - sy(0) - y'(0) + 16Y(s) = \frac{8}{s^2+16}$$

$$\left(s^2+16\right)Y(s) + \frac{s}{2} = \frac{8}{s^2+16}$$

$$\left(s^2+16\right)Y(s) = \frac{8}{s^2+16} - \frac{s}{2}$$

$$Y(s) = \frac{8}{\left(s^2+16\right)^2} - \frac{1}{2}\frac{s}{s^2+16} \qquad \mathcal{L}^{-1}\frac{2a^3}{\left(s^2+a^2\right)^2} = \sin(at) - at\cos(at)$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{8}{128}\frac{128}{\left(s^2+4^2\right)^2} - \frac{1}{2}\frac{s}{s^2+16}\right\}$$

$$\boxed{y(t) = \frac{1}{16}(\sin 4t - 4t\cos 4t) - \frac{1}{2}\cos 4t}$$

Exercise

Solve using the Laplace transform: $y'' - 10y' + 9y = 5t$; $y(0) = -1,\quad y'(0) = 2$

Solution

$$\mathcal{L}\{y'' - 10y' + 9y\}(s) = \mathcal{L}\{5t\}(s)$$

$$s^2Y(s) - sy(0) - y'(0) - 10sY(s) + 10y(0) + 9Y(s) = \frac{5}{s^2}$$

$$\left(s^2 - 10s + 9\right)Y(s) = \frac{5}{s^2} - s + 12$$

$$Y(s) = \frac{-s^3 + 12s^2 + 5}{s^2(s-9)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

$$-s^3 + 12s^2 + 5 = As^3 - 10As^2 + 9As + Bs^2 - 10Bs + 9B + Cs^3 - Cs^2 + Ds^3 - 9Ds^2$$

$$\begin{cases} s^3 & A + C + D = -1 \\ s^2 & -10A + B - C - 9D = -12 \\ s^1 & 9A - 10B = 0 & \to \boxed{A = \frac{50}{81}} \\ s^0 & 9B = 5 & \to \boxed{B = \frac{5}{9}} \end{cases}$$

$$\begin{cases} C + D = -\frac{131}{81} \\ C - 9D = -\frac{517}{81} \end{cases} \to \quad \boxed{C = \frac{31}{81}} \quad \boxed{D = -2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{50}{81}\frac{1}{s} + \frac{5}{9}\frac{1}{s^2} + \frac{31}{81}\frac{1}{s-9} - \frac{2}{s-1}\right\}$$

$$\boxed{y(t) = \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}e^{9t} - 2e^t}$$

Exercise

Solve using the Laplace transform: $2y'' + 3y' - 2y = te^{-2t},\quad y(0) = 0,\quad y'(0) = -2$

Solution

$$\mathcal{L}\{2y''+3y'-2y\}(s)=\mathcal{L}\left\{te^{-2t}\right\}(s)$$

$$2s^2Y(s)-2sy(0)-2y'(0)+3sY(s)-3y(0)-2Y(s)=\frac{1}{(s+2)^2}$$

$$\left(2s^2+3s-2\right)Y(s)+4=\frac{1}{(s+2)^2}$$

$$(2s-1)(s+2)Y(s)=\frac{1}{(s+2)^2}-4$$

$$Y(s)=\frac{-4s^2-16s-15}{(2s-1)(s+2)^3}=\frac{A}{2s-1}+\frac{B}{s+2}+\frac{C}{(s+2)^2}+\frac{D}{(s+2)^3}$$

$$-4s^2-16s-15=As^3+6As^2+12As+8A+(2Bs-B)\left(s^2+4s+4\right)+2Cs^2+2Cs-2C+2Ds-D$$

$$\begin{cases} s^3 & A+2B=0 \\ s^2 & 6A+7B+2C=-4 \\ s^1 & 12A+4B+3C+2D=-16 \\ s^0 & 8A-4B-2C-D=-15 \end{cases}$$

$$\boxed{A=-\frac{192}{125}\quad B=\frac{96}{125}\quad C=-\frac{2}{25}\quad D=-\frac{1}{5}}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{-\frac{192}{125}\frac{1}{2\left(s-\frac{1}{2}\right)}+\frac{96}{125}\frac{1}{s+2}-\frac{2}{25}\frac{1}{(s+2)^2}-\frac{1}{5}\frac{1}{(s+2)^3}\right\}$$

$$\boxed{y(t)=-\frac{96}{125}e^{t/2}+\frac{96}{125}e^{-2t}-\frac{2}{25}te^{-2t}-\frac{1}{5}t^2e^{-2t}}$$

Exercise

Solve using the Laplace transform: $2y''+20y'+51y=0,\quad y(0)=2,\quad y'(0)=0$

Solution

$$\mathcal{L}\{2y''+20y'+51y\}=0$$

$$2s^2Y(s)-2sy(0)-2y'(0)+20sY(s)-20y(0)+51Y(s)=0 \qquad y(0)=2 \quad y'(0)=0$$

$$\left(2s^2+20s+51\right)Y(s)=4s+40$$

$$Y(s)=\frac{4s+40}{2\left(s^2+10s+\frac{51}{2}\right)}$$

$$=\frac{2s+20}{(s+5)^2+\frac{1}{2}}$$

$$= \frac{2s}{(s+5)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} + \frac{20}{\sqrt{2}} \frac{\frac{1}{\sqrt{2}}}{(s+5)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{ \frac{2s}{(s+5)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} + 10\sqrt{2} \frac{\frac{1}{\sqrt{2}}}{(s+5)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} \right\}$$

$$\boxed{y(t) = 2e^{-5t} \cos \frac{\sqrt{2}t}{2} + 10\sqrt{2}e^{-5t} \sin \frac{\sqrt{2}t}{2}}$$

Exercise

Solve using the Laplace transform: $y''' + y' = e^t$, $y(0) = y'(0) = y''(0) = 0$

Solution

$$\mathcal{L}\{y''' + y'\} = \mathcal{L}\{e^t\}$$

$$s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0) + sY(s) - y(0) = \frac{1}{s-1}$$

$$\left(s^3 + s\right) Y(s) = \frac{1}{s-1}$$

$$Y(s) = \frac{1}{s(s-1)\left(s^2+1\right)} = \frac{A}{s} + \frac{B}{s-1} + \frac{Cs+D}{s^2+1}$$

$$1 = As^3 - As^2 + As - A + Bs^3 + Bs + Cs^3 - Cs^2 + Ds^2 - Ds$$

$$\begin{cases} s^3 & A+B+C=0 & B+C=1 \\ s^2 & -A-C+D=0 & -C+D=-1 \\ s & A+B-D=0 & B-D=1 \\ s^0 & -A=1 & \to \boxed{A=-1} \end{cases} \Rightarrow B = \frac{1}{2} \quad C = \frac{1}{2} \quad D = -\frac{1}{2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{ -\frac{1}{s} + \frac{1}{2}\frac{B}{s-1} + \frac{1}{2}\frac{s}{s^2+1} - \frac{1}{2}\frac{1}{s^2+1} \right\}$$

$$\boxed{y(t) = -1 + \frac{1}{2}e^t + \frac{1}{2}\cos t - \frac{1}{2}\sin t}$$

Exercise

Solve using the Laplace transform: $2y^{(3)} + 3y'' - 3y' - 2y = e^{-t}$; $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$

Solution

$$\mathcal{L}\left\{2y^{(3)}+3y''-3y'-2y\right\}=\mathcal{L}\left\{e^{-t}\right\}$$

$$2s^3Y(s)-2s^2y(0)-2sy'(0)-2y''(0)+3s^2Y(s)-3sy(0)-3y'(0)-3sY(s)+3y(0)-2Y(s)=\frac{1}{s+1}$$

$$\left(2s^3+3s^2-3s-2\right)Y(s)-2=\frac{1}{s+1} \qquad y(0)=y'(0)=0,\ \ y''(0)=1$$

$$(s-1)(2s+1)(s+2)Y(s)=\frac{1}{s+1}+2$$

$$Y(s)=\frac{2s+3}{(s-1)(2s+1)(s+2)(s+1)}=\frac{A}{s-1}+\frac{B}{2s+1}+\frac{C}{s+2}+\frac{D}{s+1}$$

$$A(2s+1)\left(s^2+3s+2\right)+B(s+2)\left(s^2-1\right)+C(2s+1)\left(s^2-1\right)+D(2s+1)\left(s^2+s-2\right)=2s+3$$

$$\begin{cases} s^3 & 2A+B+2C+2D=0 \\ s^2 & 7A+2B+C+3D=0 \\ s^1 & 7A-B-2C-3D=2 \\ s^0 & 2A-2B-C-2D=3 \end{cases} \qquad \underline{\Rightarrow A=\frac{5}{18}\quad B=-\frac{16}{9}\quad C=\frac{1}{9}\quad D=\frac{1}{2}}$$

$$Y(s)=\frac{5}{18}\frac{1}{s-1}-\frac{16}{9}\frac{1}{2s+1}+\frac{1}{9}\frac{1}{s+2}+\frac{1}{2}\frac{1}{s+1}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{5}{18}\frac{1}{s-1}-\frac{8}{9}\frac{1}{s+\frac{1}{2}}+\frac{1}{9}\frac{1}{s+2}+\frac{1}{2}\frac{1}{s+1}\right\}$$

$$\underline{y(t)=\frac{5}{18}e^{t}-\frac{8}{9}e^{-t/2}+\frac{1}{9}e^{-2t}+\frac{1}{2}e^{-t}}$$

Exercise

Solve using the Laplace transform: $y^{(3)}+2y''-y'-2y=\sin 3t;\quad y(0)=0,\quad y'(0)=0,\quad y''(0)=1$

Solution

$$\mathcal{L}\left\{y^{(3)}+2y''-y'-2y\right\}=\mathcal{L}\{\sin 3t\}$$

$$s^3Y(s)-s^2y(0)-sy'(0)-y''(0)+2s^2Y(s)-2sy(0)-2y'(0)-sY(s)+y(0)-2Y(s)=\frac{3}{s^2+9}$$

$$\left(s^3+2s^2-s-2\right)Y(s)-1=\frac{3}{s^2+9} \qquad y(0)=y'(0)=0,\ \ y''(0)=1$$

$$(s-1)(s+1)(s+2)Y(s)=\frac{3}{s^2+9}+1$$

$$Y(s)=\frac{s^2+12}{(s-1)(s+1)(s+2)\left(s^2+9\right)}=\frac{A}{s-1}+\frac{B}{s+1}+\frac{C}{s+2}+\frac{Ds+E}{s^2+9}$$

$$A\left(s^2+3s+2\right)\left(s^2+9\right)+B\left(s^2+s-2\right)\left(s^2+9\right)+C\left(s^2-1\right)\left(s^2+9\right)+(Ds+E)\left(s^3+2s^2-s-2\right)$$

$$\begin{cases} s^4 & A+B+C+D=0 \\ s^3 & 3A+B+2D+E=0 \\ s^2 & 11A+7B+8C-D+2E=1 \\ s^1 & 27A+9B-2D-E=0 \\ s^0 & 18A-18B-9C-2E=12 \end{cases} \quad \to \quad \begin{matrix} A=\frac{13}{60} & B=-\frac{13}{20} & C=\frac{16}{39} \\ D=\frac{3}{130} & E=-\frac{3}{65} \end{matrix}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{13}{60}\frac{1}{s-1}-\frac{13}{20}\frac{1}{s+1}+\frac{16}{39}\frac{1}{s+2}+\frac{3}{130}\frac{s}{s^2+9}-\frac{1}{65}\frac{3}{s^2+9}\right\}$$

$$\boxed{y(t)=\frac{13}{60}e^t-\frac{13}{20}e^{-t}+\frac{16}{39}e^{-2t}+\frac{3}{130}\cos 3t-\frac{1}{65}\sin 3t}$$

Exercise

Solve using the Laplace transform: $y^{(3)}-y''+y'-y=0$; $y(0)=1,\quad y'(0)=1,\quad y''(0)=3$

Solution

$$s^3Y(s)-s^2y(0)-sy'(0)-y''(0)-s^2Y(s)+sy(0)+y'(0)+sY(s)-y(0)-Y(s)=0$$

$$\left(s^3-s^2+s-1\right)Y(s)-s^2-s-3+s=0 \qquad y(0)=1 \quad y'(0)=1 \quad y''(0)=3$$

$$s^3-s^2+s-1=s^2(s-1)+(s-1)$$

$$Y(s)=\frac{s^2+3}{(s-1)\left(s^2+1\right)}=\frac{A}{s-1}+\frac{Bs+C}{s^2+1}$$

$$\begin{cases} s^2 & A+B=1 \\ s & -B+C=0 \\ s^0 & A-C=3 \end{cases} \quad \begin{matrix} A+C=1 \\ A-C=3 \end{matrix} \quad \boxed{A=2,\ C=-1,\ B=-1}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{2}{s-1}-\frac{s}{s^2+1}-\frac{1}{s^2+1}\right\}$$

$$\boxed{y(t)=2e^t-\cos t-\sin t}$$

Exercise

Solve using the Laplace transform: $y^{(3)}+4y''+y'-6y=-12$; $y(0)=1,\quad y'(0)=4,\quad y''(0)=-2$

Solution

$$s^3Y(s)-s^2y(0)-sy'(0)-y''(0)+4s^2Y(s)-4sy(0)-4y'(0)+sY(s)-y(0)-6Y(s)=-\frac{12}{s}$$

$$\left(s^3+4s^2+s-6\right)Y(s)=s^2-8s-15-\frac{12}{s}$$

$$s^3+4s^2+s-6=(s-1)\left(s^2+5s+6\right)$$

$$\begin{array}{r|rrrr} 1 & 1 & 4 & 1 & -6 \\ & & 1 & 5 & 6 \\ \hline & 1 & 5 & 6 & 0 \end{array}$$

$$Y(s)=\frac{s^3-8s^2-15s-12}{s(s-1)(s+2)(s+3)}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s+2}+\frac{D}{s+3}$$

$$\begin{cases} s^3 & A+B+C+D=1 \\ s^2 & 4A+5B+2C+D=-8 \\ s & A+6B-3C-2D=-15 \\ s^0 & -6A=-12 \end{cases} \qquad \underline{A=2, \quad B=1, \quad C=-3, \quad D=1}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{2}{s}+\frac{1}{s-1}-\frac{3}{s+2}+\frac{1}{s+3}\right\}$$

$$\underline{y(t)=2+e^{t}-3e^{-2t}+e^{-3t}}$$

Exercise

Solve using the Laplace transform: $y^{(3)}+3y''+3y'+y=0$; $y(0)=-4, \quad y'(0)=4, \quad y''(0)=-2$

Solution

$$\mathcal{L}\left\{y^{(3)}+3y''+3y'+y\right\}(s)=0$$

$$s^3Y(s)-s^2y(0)-sy'(0)-y''(0)+3s^2Y(s)-3sy(0)-3y'(0)+3sY(s)-3y(0)+Y(s)=0$$

$$\left(s^3+3s^2+3s+1\right)Y(s)=-4s^2-8s-2$$

$$Y(s)=\frac{-4s^2-8s-2}{(s+1)^3}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{C}{(s+1)^3}$$

$$\begin{cases} s^2 & \underline{A=-4} \\ s & 2A+B=-8 \quad \underline{B=0} \\ s^0 & A+B+C=-2 \quad \underline{C=2} \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{-4}{s+1}+\frac{2}{(s+1)^3}\right\}$$

$$\underline{y(t)=-4e^{-t}+t^2e^{-t}}$$

Exercise

Solve using the Laplace transform: $y^{(3)} - 3y'' + 3y' - y = t^2 e^t$, $y(0)=1,\quad y'(0)=2,\quad y''(0)=3$

Solution

$$\mathcal{L}\left\{y^{(3)} - 3y'' + 3y' - y\right\}(s) = \mathcal{L}\left\{t^2 e^t\right\}(s) \qquad \mathcal{L}\left\{t^n e^{-at}\right\}(s) = \frac{n!}{(s+a)^{n+1}}$$

$$s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0) - 3s^2 Y(s) + 3sy(0) + 3y'(0) + 3sY(s) - 3y(0) - Y(s) = \frac{2}{(s-1)^3}$$

$$\left(s^3 - 3s^2 + 3s - 1\right)Y(s) - s^2 - 2s - 3 + 3s + 6 - 3 = \frac{2}{(s-1)^3}$$

$$(s-1)^3 Y(s) = \frac{2}{(s-1)^3} + s^2 - s$$

$$Y(s) = \frac{2 + \left(s^2 - s\right)\left(s^3 - 3s^2 + 3s - 1\right)}{(s-1)^6}$$

$$= \frac{s^5 - 4s^4 + 6s^3 - 4s^2 + s + 2}{(s-1)^6} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} + \frac{D}{(s-1)^4} + \frac{E}{(s-1)^5} + \frac{F}{(s-1)^6}$$

$$s^5 - 4s^4 + 6s^3 - 4s^2 + s + 2 = A(s-1)^5 + B(s-1)^4 + C(s-1)^3 + D(s-1)^2 + E(s-1) + F$$

$$(s-1)^5 = s^5 - 5s^4 + 10s^3 - 10s^2 + 5s - 1 \qquad (s-1)^4 = s^4 - 4s^3 + 6s^2 - 4s + 1$$

$$\begin{cases} s^4 & \underline{A=1} & \\ s^4 & -5A + B = -4 & \rightarrow \underline{B=1} \\ s^3 & 10A - 4B + C = 6 & \rightarrow \underline{C=0} \\ s^2 & -10A + 6B - 3C + D = -4 & \rightarrow \underline{D=0} \\ s^1 & 5A - 4B + 3C - 2D + E = 1 & \rightarrow \underline{E=0} \\ s^0 & -A + B - C + D - E + F = 2 & \rightarrow \underline{F=2} \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1} + \frac{1}{(s-1)^2} + \frac{2}{(s-1)^6}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{1}{(s+a)^{n+1}}\right\} = \frac{1}{n!}t^n e^{-at}$$

$$y(t) = e^t + te^t + \frac{2}{5!}t^5 e^t \quad \boxed{= e^t\left(1 + t + \frac{1}{60}t^5\right)}$$

Exercise

Solve using the Laplace transform: $y^{(3)} + y'' + 3y' - 5y = 16e^{-t}$; $y(0)=0,\quad y'(0)=2,\quad y''(0)=-4$

Solution

$$\mathcal{L}\left\{y^{(3)}+y''+3y'-5y\right\}(s)=\mathcal{L}\left\{16e^{-t}\right\}(s)$$

$$s^3Y(s)-s^2y(0)-sy'(0)-y''(0)+s^2Y(s)-sy(0)-y'(0)+3sY(s)-3y(0)-5Y(s)=\frac{16}{s+1}$$

$$\left(s^3+s^2+3s-5\right)Y(s)=\frac{16}{s+1}+2s-2$$

$$s^3+s^2+3s-5=(s-1)\left(s^2+2s+5\right)$$

$$\begin{array}{r|rrrr} 1 & 1 & 1 & 3 & -5 \\ & & 1 & 2 & 5 \\ \hline & 1 & 2 & 5 & 0 \end{array}$$

$$Y(s)=\frac{2s^2+14}{(s+1)(s-1)\left(s^2+2s+5\right)}=\frac{A}{s+1}+\frac{B}{s-1}+\frac{C(s+1)+D}{(s+1)^2+4}$$

$$\begin{cases} s^3 & A+B+C=0 \\ s^2 & A+3B+C+D=2 \\ s & 3A+7B-C=0 \\ s^0 & -5A+5B-C-D=14 \end{cases} \qquad \underline{A=-2, \quad B=1, \quad C=1, \quad D=0}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{-2}{s+1}+\frac{1}{s-1}+\frac{(s+1)}{(s+1)^2+4}\right\}$$

$$\underline{y(t)=-2e^{-t}+e^{t}+e^{-t}\cos 2t}$$

Exercise

Solve using the Laplace transform: $y'''+4y''+5y'+2y=10\cos t$, $y(0)=y'(0)=0, \quad y''(0)=3$

Solution

$$\mathcal{L}\{y'''+4y''+5y'+2y\}=\mathcal{L}\{10\cos t\}$$

$$s^3Y(s)-s^2y(0)-sy'(0)-y''(0)+4s^2Y(s)-4sy(0)-4y'(0)+5sY(s)-5y(0)+2Y(s)=\frac{10s}{s^2+1}$$

$$\left(s^3+4s^2+5s+2\right)Y(s)=\frac{10s}{s^2+1}+3$$

$$\begin{array}{r|rrrr} -1 & 1 & 4 & 5 & 2 \\ & & -1 & -3 & -2 \\ \hline & 1 & 3 & 2 & 0 \end{array} \quad \to \quad s^2+3s+2=0 \qquad \underline{s=-1, \ -1, \ -2}$$

$$Y(s)=\frac{3s^2+10s+3}{(s+2)\left(s^2+1\right)(s+1)^2}=\frac{A}{s+2}+\frac{Bs+C}{s^2+1}+\frac{D}{s+1}+\frac{E}{(s+1)^2}$$

$$3s^2+10s+3 = A\left(s^2+1\right)\left(s^2+2s+1\right)+(Bs+C)(s+2)\left(s^2+2s+1\right)$$
$$+D(s+1)(s+2)\left(s^2+1\right)+E\left(s^2+1\right)(s+2)$$

$$= A\left(s^2+1\right)\left(s^2+2s+1\right)+\left(Bs^2+2Bs+Cs+2C\right)\left(s^2+2s+1\right)$$
$$+D\left(s^2+3s+2\right)\left(s^2+1\right)+Es^3+2Es^2+Es+2E$$

$$= As^4+2As^3+2As^2+2As+A+Bs^4+4Bs^3+5Bs^2+2Bs+Cs^3+4Cs^2$$
$$+5Cs+2C+Ds^4+3Ds^3+3Ds^2+3Ds+2D+Es^2+3Es+2E$$

$$\begin{cases} s^4 & A+B+D=0 \\ s^3 & 2A+4B+C+3D+E=0 \\ s^2 & 2A+5B+4C+3D+2E=3 \\ s^1 & 2A+2B+5C+3D+E=10 \\ s^0 & A+2C+2D+2E=3 \end{cases} \quad \rightarrow \quad \begin{array}{l} A=-1 \quad B=-1 \quad C=2 \\ D=2 \quad E=-2 \end{array}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{-1}{s+2}-\frac{s}{s^2+1}+\frac{2}{s^2+1}+\frac{2}{s+1}-\frac{2}{(s+1)^2}\right\}$$

$$\underline{y(t) = -e^{-2t}-\cos t+2\sin t+2e^{-t}-2te^{-t}}$$

Exercise

Solve using the Laplace transform: $y^{(4)}+2y''+y=4te^t; \quad y(0)=y'(0)=y''(0)=y^{(3)}(0)=0$

Solution

$$\mathcal{L}\left\{y^{(4)}+2y''+y\right\} = \mathcal{L}\left\{4te^t\right\}$$

$$s^4Y(s)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)+2s^2Y(s)-2sy(0)-2y'(0)+Y(s) = \frac{4}{(s-1)^2}$$

$$\left(s^4+2s^2+1\right)Y(s) = \frac{4}{(s-1)^2} \qquad y(0)=y'(0)=y''(0)=y^{(3)}(0)=0$$

$$\left(s^2+1\right)^2 Y(s) = \frac{4}{(s-1)^2}$$

$$Y(s) = \frac{4}{(s-1)^2\left(s^2+1\right)^2} = \frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{Cs+D}{s^2+1}+\frac{Es+F}{\left(s^2+1\right)^2}$$

$$A(s-1)\left(s^4+2s^2+1\right)+B\left(s^4+2s^2+1\right)+(Cs+D)\left(s^2-2s+1\right)\left(s^2+1\right)+(Es+F)\left(s^2-2s+1\right)=4$$

$$\begin{cases} s^5 & A+C=0 \\ s^4 & -A-2C+D=0 \\ s^3 & 2A+C-2D+E=0 \\ s^2 & -2A+2B-2C+2D-2E+F=0 \\ s^1 & A+C-2D+E-2F=0 \\ s^0 & -A+B+D+F=4 \end{cases} \quad \rightarrow \quad \begin{matrix} A=-\frac{16}{17} & B=\frac{28}{17} & C=\frac{16}{17} \\ D=\frac{16}{17} & E=\frac{48}{17} & F=\frac{8}{17} \end{matrix}$$

$$Y(s)=-\frac{16}{17}\frac{1}{s-1}+\frac{28}{17}\frac{1}{(s-1)^2}+\frac{16}{17}\frac{s}{s^2+1}+\frac{16}{17}\frac{1}{s^2+1}+\frac{48}{17}\frac{s}{\left(s^2+1\right)^2}+\frac{8}{17}\frac{1}{\left(s^2+1\right)^2}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{-\frac{16}{17}\frac{1}{s-1}+\frac{28}{17}\frac{1}{(s-1)^2}+\frac{16}{17}\frac{s}{s^2+1}+\frac{16}{17}\frac{1}{s^2+1}+\frac{24}{17}\frac{2s}{\left(s^2+1\right)^2}+\frac{4}{17}\frac{2}{\left(s^2+1\right)^2}\right\}$$

$$\boxed{y(t)=-\frac{16}{17}e^t+\frac{28}{17}te^t+\frac{16}{17}\cos t+\frac{16}{17}\sin t+\frac{27}{17}t\sin t+\frac{4}{17}\sin t-t\cos t}$$

Exercise

Solve using the Laplace transform: $y^{(4)}-y=0$; $y(0)=1,\quad y'(0)=0,\quad y''(0)=0,\quad y^{(3)}(0)=0$

Solution

$$\mathcal{L}\left\{y^{(4)}-y\right\}(s)=0$$

$$s^4Y(s)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)-Y(s)=0$$

$$\left(s^4-1\right)Y(s)=s^3$$

$$Y(s)=\frac{s^3}{(s-1)(s+1)\left(s^2+1\right)}=\frac{A}{s-1}+\frac{B}{s+1}+\frac{Cs+D}{s^2+1}$$

$$s^3=As^3+As^2+As+A+Bs^3-Bs^2+Bs-B++Cs^3-Cs+Ds^2-D$$

$$\begin{cases} s^3 & A+B+C=1 \\ s^2 & A-B+D=0 \\ s & A+B-C=0 \\ s^0 & A-B-D=0 \end{cases} \quad \rightarrow \quad \begin{matrix} A=\frac{1}{4} & B=\frac{1}{4} \\ C=\frac{1}{2} & D=0 \end{matrix}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s-1}+\frac{1}{4}\frac{1}{s+1}+\frac{1}{2}\frac{s}{s^2+1}\right\}$$

$$\boxed{y(t)=\frac{1}{4}e^t+\frac{1}{4}e^{-t}+\frac{1}{2}\cos t}$$

Exercise

Solve using the Laplace transform: $y^{(4)} - 4y = 0$; $y(0)=1,\quad y'(0)=0,\quad y''(0)=-2,\quad y^{(3)}(0)=0$

Solution

$$\mathcal{L}\left\{y^{(4)} - 4y\right\} = 0$$

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - 4Y(s) = 0 \qquad y(0)=1,\ y'(0)=0,\ y''(0)=-2,\ y^{(3)}(0)=0$$

$$\left(s^4-4\right)Y(s) - s^3 + 2s = 0$$

$$Y(s) = \frac{s\left(s^2-2\right)}{\left(s^2-2\right)\left(s^2+2\right)}$$

$$= \frac{s}{s^2+2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\}$$

$$\boxed{y(t) = \cos\sqrt{2}t}$$

Exercise

Solve using the Laplace transform:

$$y^{(4)} - 4y^{(3)} + 6y'' - 4y' + y = 0\ ; \quad y(0)=0,\quad y'(0)=1,\quad y''(0)=0,\quad y^{(3)}(0)=1$$

Solution

$$\mathcal{L}\left\{y^{(4)} - 4y^{(3)} + 6y'' - 4y' + y = 0\right\} = 0$$

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - 4s^3Y(s) + 4s^2y(0)$$

$$+4sy'(0) + 4y''(0) + 6s^2Y(s) - 6sy(0) - 6y'(0) - 4sY(s) + 4y(0) + Y(s) = 0$$

$$\left(s^4 - 4s^3 + 6s^2 - 4s + 1\right)Y(s) - s^2 - 1 + 4s - 6 = 0$$

$$(s+1)^4Y(s) = s^2 - 4s + 7$$

$$Y(s) = \frac{s^2-4s+7}{(s+1)^4} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^4}$$

$$As^3 + 3As^2 + 3As + A + Bs^2 + 2Bs + B + Cs + C + D = s^2 - 4s + 7$$

$$\begin{cases} s^3 & A = 0 \\ s^2 & 3A + B = 1 \\ s^1 & 3A + 2B + C = -4 \\ s^0 & A + B + C + D = 7 \end{cases} \quad \Rightarrow \boxed{B = 1 \quad C = -6 \quad D = 13}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2} - \frac{6}{(s+1)^3} + \frac{13}{(s+1)^4}\right\}$$

$$y(t) = te^{t} - 3t^2e^{t} + \frac{13}{6}t^3e^{t}$$

$$\boxed{= \left(t - 3t^2 + \frac{13}{6}t^3\right)e^{t}}$$

Exercise

Given: $y'' - 4y' + 3y = 0, \quad y(0) = 1 \quad y'(0) = -1$

a) Show that the general solution is: $y(t) = C_1e^{3t} + C_2e^{t}$ and find C_1 *and* C_2

b) Use Laplace transform to solve the system

Solution

a) $\lambda^2 - 4\lambda + 3 = 0 \quad \Rightarrow \quad \boxed{\lambda = 3,\ 1}$

That implies to the general solution: $y = C_1e^{3t} + C_2e^{t}$

$$1 = C_1e^{3(0)} + C_2e^{(0)}$$

$$\boxed{1 = C_1 + C_2}$$

$$y' = 3C_1e^{3t} + C_2e^{t}$$

$$-1 = 3C_1e^{3(0)} + C_2e^{(0)}$$

$$\boxed{-1 = 3C_1 + C_2}$$

$$\begin{cases} C_1 + C_2 = 1 \\ 3C_1 + C_2 = -1 \end{cases} \quad \Rightarrow \quad \boxed{C_1 = -1} \quad \boxed{C_2 = 2}$$

Therefore; the general solution is: $\boxed{y(t) = -e^{3t} + 2e^{t}}$

b) $\mathcal{L}(y'' - 4y' + 3y)(s) = 0$

$$s^2Y(s) - sy(0) - y'(0) - 4\left(sY(s) - y(0)\right) + 3Y(s) = 0$$

$$s^2Y(s) - s + 1 - 4\left(sY(s) - 1\right) + 3Y(s) = 0$$

$$s^2Y(s)-s+1-4sY(s)+4+3Y(s)=0$$

$$\left(s^2-4s+3\right)Y(s)=s-5$$

$$Y(s)=\frac{s-5}{s^2-4s+3}$$

$$=\frac{s-5}{(s-1)(s-3)}$$

$$=\frac{A}{s-1}+\frac{B}{s-3}=\frac{(A+B)s-3A-B}{(s-1)(s-3)}$$

$$\begin{cases} A+B=1 \\ -3A-B=-5 \end{cases} \Rightarrow \underline{A=2|} \quad \underline{B=-1|}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{2}{s-1}-\frac{1}{s-3}\right\}$$

$$\underline{y(t)=2e^{t}-e^{3t}}$$

Exercise

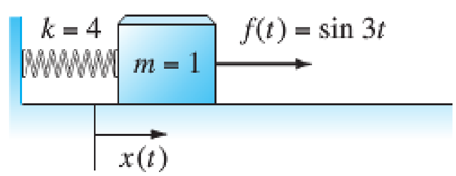
Solve the initial value problem $x''+4x=\sin 3t; \quad x(0)=x'(0)=0.$

Such problem arises in the motion of a mass-and-spring system with external force as shown below.

Solution

$$\mathcal{L}\{x''+4x\}=\mathcal{L}\{\sin 3t\}$$

$$s^2X(s)-sx(0)-x'(0)+4X(s)=\frac{3}{s^2+9}$$



$$\left(s^2+4\right)X(s)=\frac{3}{s^2+9}$$

$$X(s)=\frac{3}{\left(s^2+4\right)\left(s^2+9\right)}=\frac{As+B}{s^2+4}+\frac{Cs+D}{s^2+9}$$

$$As^3+9AS+Bs^2+9B+Cs^3+4CS+Ds^2+4C=3$$

$$\begin{cases} s^3 \quad A+C=0 \\ s^2 \quad B+D=0 \\ s^1 \quad 9A+4C=0 \\ s^0 \quad 9B+4D=3 \end{cases} \rightarrow A=C=0 \quad 5B=3 \Rightarrow B=\frac{3}{5} \quad D=-\frac{3}{5}$$

$$X(s)=\frac{3}{5}\frac{1}{s^2+4}-\frac{3}{5}\frac{1}{s^2+9}$$

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{10}\frac{2}{s^2+4} - \frac{1}{5}\frac{3}{s^2+9}\right\}$$

$$\boxed{x(t) = \frac{3}{10}\sin 2t - \frac{1}{5}\sin 3t}$$

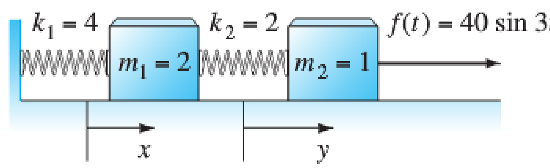
Exercise

Solve the system $\begin{cases} 2x'' = -6x + 2y \\ y'' = 2x - 2y + 40\sin 3t \end{cases}$

Subject to the initial conditions $x(0) = x'(0) = y(0) = y'(0) = 0$

Thus the force $f(t) = 40\sin 3t$ is applied to the second mass as shown below, beginning at time $t = 0$ when the system is at rest in its equilibrium position.

Solution



$$\begin{cases} \mathcal{L}\{2x''\} = \mathcal{L}\{-6x + 2y\} \\ \mathcal{L}\{y''\} = \mathcal{L}\{2x - 2y + 40\sin 3t\} \end{cases}$$

$$\begin{cases} 2s^2X(s) - 2sx(0)) - 2x'(0) = -6X(s) + 2Y(s) \\ s^2Y(s) - sy(0)) - y'(0) = 2X(s) - 2Y(s) + \dfrac{120}{s^2+9} \end{cases}$$

Given: $x(0) = x'(0) = y(0) = y'(0) = 0$

$$\begin{cases} 2s^2X(s) = -6X(s) + 2Y(s) \\ s^2Y(s) = 2X(s) - 2Y(s) + \dfrac{120}{s^2+9} \end{cases}$$

$$\begin{cases} (s^2+3)X(s) - Y(s) = 0 & (1) \\ -2X(s) \quad + (s^2+2)Y(s) = \dfrac{120}{s^2+9} & (2) \end{cases}$$

$$\begin{vmatrix} s^2+3 & -1 \\ -2 & s^2+2 \end{vmatrix} = s^4 + 5s^2 + 4 = (s^2+1)(s^2+4)$$

$$\begin{vmatrix} 0 & -1 \\ \dfrac{120}{s^2+9} & s^2+2 \end{vmatrix} = \frac{120}{s^2+9} \quad \to \quad X(s) = \frac{120}{(s^2+1)(s^2+4)(s^2+9)}$$

$$\begin{vmatrix} s^2+3 & 0 \\ -2 & \dfrac{120}{s^2+9} \end{vmatrix} = 120\frac{s^2+3}{s^2+9} \quad \to \quad Y(s) = \frac{120(s^2+3)}{(s^2+1)(s^2+4)(s^2+9)}$$

$$X(s)=\frac{120}{\left(s^2+1\right)\left(s^2+4\right)\left(s^2+9\right)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+4}+\frac{Es+F}{s^2+9}$$

$$(As+B)\left(s^2+4\right)\left(s^2+9\right)+(Cs+D)\left(s^2+1\right)\left(s^2+9\right)+(Es+F)\left(s^2+1\right)\left(s^2+4\right)=120$$

$$(As+B)\left(s^4+13s^2+36\right)+(Cs+D)\left(s^4+10s^2+9\right)+(Es+F)\left(s^4+5s^2+4\right)=120$$

$$\begin{cases} s^5 & A+C+E=0 \\ s^4 & B+D+F=0 \\ s^3 & 13A+10C+5E=0 \\ s^2 & 13B+10D+5F=0 \\ s & 36A+9C+4E=0 \\ & 36B+9D+4F=120 \end{cases}$$

$$\Rightarrow\begin{cases} A+C+E=0 \\ 13A+10C+5E=0 \\ 36A+9C+4E=0 \end{cases} \rightarrow A+C+E=0$$

$$\Rightarrow\begin{cases} B+D+F=0 \\ 13B+10D+5F=0 \\ 36B+9D+4F=120 \end{cases} \rightarrow B=5;\ D=-8;\ F=3$$

$$Y(s)=\frac{120\left(s^2+3\right)}{\left(s^2+1\right)\left(s^2+4\right)\left(s^2+9\right)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+4}+\frac{Es+F}{s^2+9}$$

$$(As+B)\left(s^2+4\right)\left(s^2+9\right)+(Cs+D)\left(s^2+1\right)\left(s^2+9\right)+(Es+F)\left(s^2+1\right)\left(s^2+4\right)=120s^2+360$$

$$\begin{cases} s^5 & A+C+E=0 \\ s^4 & B+D+F=0 \\ s^3 & 13A+10C+5E=0 \\ s^2 & 13B+10D+5F=120 \\ s & 36A+9C+4E=0 \\ & 36B+9D+4F=360 \end{cases}$$

$$\Rightarrow\begin{cases} A+C+E=0 \\ 13A+10C+5E=0 \\ 36A+9C+4E=0 \end{cases} \rightarrow A+C+E=0$$

$$\Rightarrow\begin{cases} B+D+F=0 \\ 13B+10D+5F=120 \\ 36B+9D+4F=360 \end{cases} \rightarrow B=10;\ D=8;\ F=-18$$

$$X(s)=\frac{5}{s^2+1}-\frac{8}{s^2+4}+\frac{3}{s^2+9}$$

$$\mathcal{L}^{-1}\{X(s)\}=\mathcal{L}^{-1}\left\{5\frac{1}{s^2+1}-4\frac{2}{s^2+4}+\frac{3}{s^2+9}\right\}$$

$$\boxed{x(t)=5\sin t-4\sin 2t+\sin 3t}$$

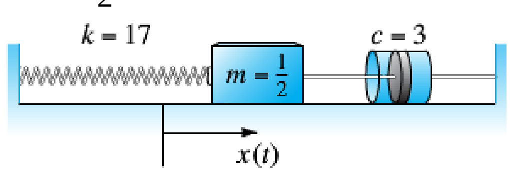
$$Y(s)=\frac{10}{s^2+1}+\frac{8}{s^2+4}-\frac{18}{s^2+9}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{10\frac{1}{s^2+1}+4\frac{2}{s^2+4}-6\frac{3}{s^2+9}\right\}$$

$$\boxed{y(t)=10\sin t+4\sin 2t-6\sin 3t}$$

Exercise

Consider a mass-spring system with $m=\frac{1}{2},\quad k=17,\quad and\ c=3.$



Let $x(t)$ be the displacement of the mass m from its equilibrium position. If the mass is set in motion with $x(0)=3\ and\ x'(0)=1$, find $x(t)$ for the resulting damped free oscillations.

Solution

$$\frac{1}{2}x''+3x'+17x=0 \qquad\qquad mx''+cx'+kx=0$$

$$x''+6x'+34x=0 \qquad x(0)=3;\ \ x'(0)=1$$

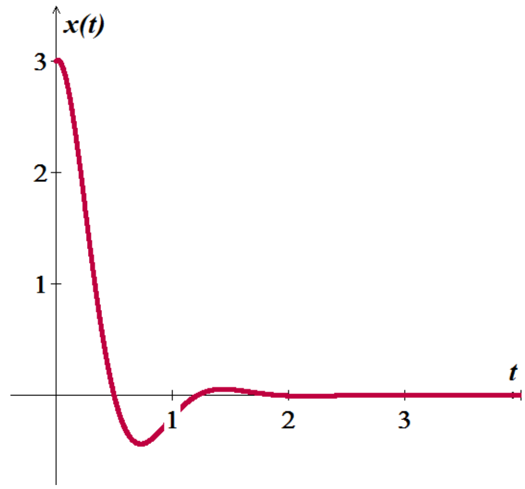
$$\mathcal{L}\{x''+6x'+34x\}=0$$

$$s^2X(s)-sx(0)-x'(0)+6sX(s)-6x(0)+34X(s)=0$$

$$s^2X(s)-3s-1+6sX(s)-18+34X(s)=0$$

$$\left(s^2+6s+34\right)X(s)=3s+19$$

$$\begin{aligned}X(s)&=\frac{3s+19}{s^2+6s+34}\\&=\frac{3s+19}{(s+3)^2+25}\\&=\frac{3s+9+10}{(s+3)^2+25}\\&=\frac{3(s+3)}{(s+3)^2+25}+\frac{10}{(s+3)^2+25}\end{aligned}$$



$$\mathcal{L}^{-1}\{X(s)\}=\mathcal{L}^{-1}\left\{3\cdot\frac{(s+3)}{(s+3)^2+25}+5\cdot\frac{2}{(s+3)^2+25}\right\}$$

$$\boxed{x(t)=(3\cos 5t+2\sin 5t)e^{-3t}}$$

Exercise

A $4-lb$ weight stretches a spring 2 *feet*. The weight is released from rest 18 *inches* above the equilibrium position, and the resulting motion takes place in a medium offering a damping force numerically equal to $\frac{7}{8}$ times the instantaneous velocity. Use the Laplace transform to find the equation of motion $x(t)$.

Solution

$$m=\frac{4}{32}=\frac{1}{8} \qquad (w=mg)$$

$$k=\frac{4}{2}=2 \qquad (xk=mg)$$

$$c=\frac{7}{8}$$

$$\frac{1}{8}x''+\frac{7}{8}x'+2x=0 \qquad\qquad mx''+cx'+kx=f(t)$$

$$x'' + 7x' + 16x = 0 \ ; \quad x(0) = -\frac{18}{12} = -\frac{3}{2}, \quad x'(0) = 0$$

$$\mathcal{L}\{x'' + 7x' + 16x\} = 0$$

$$s^2 X(s) - sx(0) - x'(0) + 7sX(s) - 7x(0) + 16X(s) = 0$$

$$\left(s^2 + 7s + 16\right) X(s) = -\frac{3}{2}s - \frac{21}{2}$$

$$X(s) = -\frac{3}{2}\frac{s+7}{\left(s+\frac{7}{2}\right)^2 + \frac{15}{4}}$$

$$\frac{s+7}{\left(s+\frac{7}{2}\right)^2 + \frac{15}{4}} = \frac{A\left(s+\frac{7}{2}\right)}{\left(s+\frac{7}{2}\right)^2 + \frac{15}{4}} + \frac{B}{\left(s+\frac{7}{2}\right)^2 + \frac{15}{4}}$$

$$\begin{cases} s & \underline{A = 1|} \\ s^0 & \frac{7}{2}A + B = 7 \quad \underline{B = \frac{7}{2}|} \end{cases}$$

$$\{X(s)\} = -\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{s+\frac{7}{2}}{\left(s+\frac{7}{2}\right)^2 + \left(\frac{\sqrt{15}}{2}\right)^2}\right\} - \frac{3}{2}\frac{7}{2}\frac{2}{\sqrt{15}}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{15}}{2}}{\left(s+\frac{7}{2}\right)^2 + \left(\frac{\sqrt{15}}{2}\right)^2}\right\}$$

$$\underline{x(t) = -\frac{3}{2}e^{7t/2}\cos\frac{\sqrt{15}}{2}t - \frac{7\sqrt{15}}{10}e^{7t/2}\sin\frac{\sqrt{15}}{2}t \ |}$$

Exercise

Consider a mass-spring-dashpot system with $m = \frac{1}{2}, \quad k = 17, \quad c = 3, \quad and\ f(t) = 15\sin 2t$ with initial conditions $x(0) = x'(0) = 0$. Let $x(t)$ be the displacement of the mass *m* from its equilibrium position. Find the resulting transient motion and steady periodic motion of the mass..

Solution

$$\frac{1}{2}x'' + 3x' + 17x = 15\sin 2t \qquad mx'' + cx' + kx = 0$$

$$x'' + 6x' + 34x = 30\sin 2t \qquad x(0) = x'(0) = 0$$

$$\mathcal{L}\{x'' + 6x' + 34x\} = \mathcal{L}\{30\sin 2t\}$$

$$s^2 X(s) - sx(0) - x'(0) + 6sX(s) - 6x(0) + 34X(s) = \frac{60}{s^2 + 4}$$

$$\left(s^2 + 6s + 34\right) X(s) = \frac{60}{s^2 + 4}$$

$$X(s)=\frac{60}{\left(s^2+4\right)\left((s+3)^2+25\right)}=\frac{As+B}{s^2+4}+\frac{Cs+D}{(s+3)^2+25}$$

$$As^3+6As^2+34sA+Bs^2+6sB+34B+Cs^3+4Cs+Ds^2+4D=60$$

$$\begin{cases} s^3 & A+C=0 \\ s^2 & 6A+B+D=0 \\ s^1 & 34A+6B+4C=0 \\ s^0 & 34B+4D=60 \end{cases}$$

$$\begin{cases} C=-A & 6A-\frac{15}{2}B=-15 \\ & 30A+6B=0 \\ D=15-\frac{17}{2}B & \end{cases} \to \begin{cases} -30A+\frac{75}{2}B=75 \\ 30A+6B=0 \end{cases} \quad \boxed{B=\frac{150}{87}=\frac{50}{29}}$$

$$\boxed{A=-\frac{10}{29};\quad B=\frac{50}{29};\quad C=\frac{10}{29};\quad D=\frac{10}{29}}$$

$$X(s)=\frac{10}{29}\left(\frac{-s+5}{s^2+4}+\frac{s+1}{(s+3)^2+25}\right)$$

$$=\frac{10}{29}\left(\frac{5}{s^2+4}-\frac{s}{s^2+4}+\frac{s+3}{(s+3)^2+25}-\frac{2}{(s+3)^2+25}\right)$$

$$\mathcal{L}^{-1}\{X(s)\}=\frac{10}{29}\mathcal{L}^{-1}\left(\frac{5}{2}\cdot\frac{2}{s^2+4}-\frac{s}{s^2+4}+\frac{s+3}{(s+3)^2+25}-\frac{2}{5}\cdot\frac{5}{(s+3)^2+25}\right)$$

$$x(t)=\frac{10}{29}\left(\frac{5}{2}\sin 2t-\cos 2t+e^{-3t}\left(\cos 5t-\frac{2}{5}\sin 5t\right)\right)$$

$$\boxed{=\frac{5}{29}(5\sin 2t-2\cos 2t)+\frac{2}{29}e^{-3t}(5\cos 5t-2\sin 5t)}$$

Exercise

A 8-kg mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by 3 $N\text{-}sec/m$ and the spring constant is 40 N/m. If the mass is driven by an external force equal to $f(t)=2\sin 2t\cos 2t\ N$. Find the solution.

Solution

Given: $m=8 \quad k=40 \quad c=3$

$$my''+cy'+ky=F(t)$$

$$8y''+3y'+40y=2\sin 2t\cos 2t$$
$$=\sin 4t$$

$$8y''+3y'+40y=\sin 4t\ ; \quad y(0)=0, \quad y'(0)=0$$

$$\mathcal{L}\{8y''+3y'+40y\}=\mathcal{L}\{\sin 4t\}$$

$$8s^2Y(s)-8sy(0)-8y'(0)+3sY(s)-3y(0)+40Y(s)=\frac{4}{s^2+16}$$

$$\left(8s^2+3s+40\right)Y(s)=\frac{4}{s^2+16}$$

$$Y(s)=\frac{4}{8\left(s^2+\frac{3}{8}s+5\right)\left(s^2+16\right)}$$

$$=\frac{\frac{1}{2}}{\left(\left(s+\frac{3}{16}\right)^2+\frac{1271}{256}\right)\left(s^2+16\right)}=\frac{A\left(s+\frac{3}{16}\right)+B}{\left(s+\frac{3}{16}\right)^2+\frac{1271}{256}}+\frac{Cs+D}{s^2+16}$$

$$\begin{cases} s^3 & A+C=0 \\ s^2 & \frac{3}{16}A+B+\frac{3}{8}C+D=0 \\ s & 16A+5C+\frac{3}{8}D=0 \\ s^0 & 3A+16B+5D=\frac{1}{2} \end{cases}$$

$$\boxed{A=\frac{3}{1972}\quad B=\frac{1417}{31552}\quad C=-\frac{3}{1972}\quad D=-\frac{22}{493}}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\left\{\frac{3}{1972}\frac{s+\frac{3}{16}}{\left(s+\frac{3}{16}\right)^2+\frac{1271}{256}}+\frac{1417}{31552}\frac{1}{\left(s+\frac{3}{16}\right)^2+\frac{1271}{256}}-\frac{3}{1972}\frac{s}{s^2+16}-\frac{22}{493}\frac{1}{s^2+16}\right\}$$

$$y(t)=\frac{3}{1972}e^{-3t/16}\cos\frac{\sqrt{1271}}{16}t+\frac{1417}{31552}\frac{16}{\sqrt{1271}}e^{-3t/16}\sin\frac{\sqrt{1271}}{16}t-\frac{3}{1972}\cos 4t-\frac{22}{493}\frac{1}{4}\sin 4t$$

$$\boxed{=\frac{3}{1972}e^{-3t/16}\cos\frac{\sqrt{1271}}{16}t+\frac{1417}{1972\sqrt{1271}}e^{-3t/16}\sin\frac{\sqrt{1271}}{16}t-\frac{3}{1972}\cos 4t-\frac{11}{986}\sin 4t}$$

Exercise

A 2-*kg* mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by $c=8$ *kg/sec* and the spring constant is $k=80$ *N/m*. At time $t=0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t)=20e^{-t}$ *N*. (*t* in *seconds*). Find the equation of motion.

Solution

$$2y''+8y'+80y=20e^{-t}\ ;\quad y(0)=0,\quad y'(0)=0 \qquad\qquad my''+cy'+ky=F(t)$$

$$\mathcal{L}^{-1}\{2y''+8y'+80y\}=\mathcal{L}^{-1}\left\{20e^{-t}\right\}$$

$$2s^2Y(s)-2sy(0)-2y'(0)+8sY(s)-8y(0)+80Y(s)=\frac{20}{s+1}$$

$$2\left(s^2+4s+40\right)Y(s)=\frac{20}{s+1}$$

$$Y(s)=\frac{10}{(s+1)\left((s+2)^2+36\right)}=\frac{A}{s+1}+\frac{B(s+2)+C}{(s+2)^2+36}$$

$$\begin{cases} s^2 & A+B=0 \\ s & 4A+3B+C=0 \\ s^0 & 40A+2B+C=10 \end{cases}$$

$$\Delta=\begin{vmatrix} 1 & 1 & 0 \\ 4 & 3 & 1 \\ 40 & 2 & 1 \end{vmatrix}=37 \quad \Delta_A=\begin{vmatrix} 0 & 1 & 0 \\ 0 & 3 & 1 \\ 10 & 2 & 1 \end{vmatrix}=10$$

$$\boxed{A=\frac{10}{37} \quad B=-\frac{10}{37} \quad C=-\frac{10}{37}}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{10}{37}\frac{1}{s+1}-\frac{10}{37}\frac{(s+2)}{(s+2)^2+6^2}-\frac{10}{37}\frac{1}{6}\frac{6}{(s+2)^2+6^2}\right\}$$

$$\boxed{y(t)=\frac{10}{37}e^{-t}-\left(\frac{10}{37}\cos 6t+\frac{5}{111}\sin 6t\right)e^{-2t}}$$

Exercise

A 10-*kg* mass is attached to a spring having a spring constant of 140 *N/m*. The mass is started in motion initially from the equilibrium position with an initial velocity 1 *m/sec* in the upward direction and with an applied external force $F(t)=5\sin t$. If the force due to air resistance is $-90y'$ *N*. Find the equation motion of the mass.

Solution

$$10y''+90y'+140y=5\sin t$$

$$y''+9y'+14y=\frac{1}{2}\sin t\ ; \quad y(0)=0, \quad y'(0)=-1$$

$$s^2Y(s)-sy(0)-y'(0)+9sY(s)-9y(0)+14Y(s)=\frac{1}{2}\frac{1}{s^2+1}$$

$$\left(s^2+9s+14\right)Y(s)=\frac{1}{2}\frac{1}{s^2+1}-1$$

$$Y(s)=\frac{-s^2-\frac{1}{2}}{(s+2)(s+7)\left(s^2+1\right)}=\frac{A}{s+2}+\frac{B}{s+7}+\frac{Cs+D}{s^2+1}$$

$$\begin{cases} s^3 & A+B+C=0 \\ s^2 & 7A+2B+9C+D=-1 \\ s & A+B+14C+9D=0 \\ s^0 & 7A+2B+14D=-\frac{1}{2} \end{cases} \qquad \underline{A=-\frac{9}{50} \quad B=\frac{99}{500} \quad C=-\frac{9}{500} \quad D=\frac{13}{500}}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{-\frac{9}{50}\frac{1}{s+2}+\frac{99}{500}\frac{1}{s+7}-\frac{9}{500}\frac{s}{s^2+1}+\frac{13}{500}\frac{1}{s^2+1}\right\}$$

$$y(t)=-\frac{9}{50}e^{-2t}+\frac{99}{500}e^{-7t}-\frac{9}{500}\cos t+\frac{13}{500}\sin t$$

$$\underline{=\frac{1}{500}\left(99e^{-7t}-90e^{-2t}+13\sin t-9\cos t\right)}$$

Exercise

A 128-*lb* weight is attached to a spring having a spring constant of 64 *lb/ft*. The weight is started in motion initially by displacing it 6 *in* above the equilibrium position with no initial velocity and with an applied external force $F(t)=8\sin 4t$. Assume no air resistance. Find the equation motion of the mass.

Solution

$$m=\frac{128}{32}\underline{=4}$$

$$4y''+64y=8\sin 4t$$

$$y''+16y=2\sin 4t\ ; \quad y(0)=-\frac{6}{12}=-\frac{1}{2}, \quad y'(0)=0$$

$$s^2Y(s)-sy(0)-y'(0)+16Y(s)=\frac{8}{s^2+16}$$

$$\left(s^2+16\right)Y(s)=\frac{8}{s^2+16}-\frac{1}{2}s$$

$$Y(s)=\frac{8}{\left(s^2+16\right)^2}-\frac{1}{2}\frac{s}{s^2+16}$$

$$\frac{8}{\left(s^2+16\right)^2}=\frac{As+B}{s^2+16}+\frac{C\left(s^2-16\right)}{\left(s^2+16\right)^2}+\frac{Ds}{\left(s^2+16\right)^2}$$

$$\begin{cases} s^3 & A=0 \\ s^2 & B+C=0 \\ s & 16A+D=0 \\ s^0 & 16B-16C=8 \end{cases}$$

$$\underline{A=0 \quad B=\frac{1}{4} \quad C=-\frac{1}{4} \quad D=0}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s^2+16}-\frac{1}{4}\frac{s^2-16}{\left(s^2+16\right)^2}-\frac{1}{2}\frac{s}{s^2+16}\right\}$$

$$\underline{y(t)=\frac{1}{16}\sin 4t-\frac{1}{4}t\cos 4t-\frac{1}{2}\cos 4t}$$

Exercise

Find the motion of a damped mass-and-spring system with $m=1$, $c=2$, and $k=26$ under the influence of an external force $F(t)=82\cos 4t$ with $x(0)=6$ and $x'(0)=0$.

Solution

Given: $m=1$, $c=2$, $k=26$, and $F(t)=82\cos 4t$ $\quad x(0)=6;\ x'(0)=0$

$x''+2x'+26x=82\cos 4t \qquad mx''+cx'+kx=F(t)$

$$\mathcal{L}\{x''+2x'+26x\}=\mathcal{L}\{82\cos 4t\}$$

$$s^2X(s)-sx(0)-x'(0)+2sX(s)-2x(0)+26X(s)=\frac{82s}{s^2+16}$$

$$\left(s^2+2s+26\right)X(s)=\frac{82s}{s^2+16}+6s+12$$

$$X(s)=\frac{6s^3+12s^2+178s+192}{\left(s^2+16\right)\left((s+1)^2+25\right)}=\frac{As+B}{s^2+16}+\frac{C(s+1)+D}{(s+1)^2+25}$$

$$\begin{cases} s^3 & A+C=6 \\ s^2 & 2A+B+C+D=12 \\ s & 26A+2B+16C=178 \\ s^0 & 26B+16C+16D=192 \end{cases} \qquad \underline{A=5 \quad B=16 \quad C=1 \quad D=-15}$$

$$\mathcal{L}^{-1}\{X(s)\}=\mathcal{L}^{-1}\left\{\frac{5s}{s^2+16}+\frac{16}{s^2+16}+\frac{s+1}{(s+1)^2+25}-\frac{15}{(s+1)^2+5^2}\right\}$$

$$\underline{x(t)=5\cos 4t+4\sin 4t+e^{-t}(\cos 5t-3\sin 5t)}$$

Exercise

A spring with a mass of 2-*kg* has natural length 0.5 *m*. A force of 25.6 *N* is required to maintain it stretched to a length of 0.7 *m*. The spring is immersed in a fluid with damping constant $c=40$. If the spring is started from the equilibrium position and is given a push to start it with initial velocity 0.6 *m/s*. Find the position of the mass at any time *t*.

Solution

$$k = \frac{25.6}{0.7 - 0.5} \ \boxed{= 128} \qquad\qquad k\left(x_2 - x_1\right) = F$$

$$2x'' + 40x + 128 = 0 \ ; \quad x(0) = 0, \quad x'(0) = 0.6$$

$$\mathcal{L}\left\{x'' + 20x + 64\right\} = 0$$

$$s^2 X(s) - sx(0) - x'(0) + 20sX(s) - 20x(0) + 64X(s) = 0$$

$$\left(s^2 + 20s + 64\right) X(s) = \frac{6}{10}$$

$$X(s) = \frac{3}{5} \frac{1}{(s+16)(s+4)} = \frac{3}{5}\left(\frac{A}{s+16} + \frac{B}{s+4}\right)$$

$$\begin{cases} s & A + B = 0 \\ s^0 & 4A + 16B = 1 \end{cases} \to \boxed{A = -\frac{1}{12}, \ B = \frac{1}{12}}$$

$$\mathcal{L}^{-1}\left\{X(s)\right\} = \frac{3}{5}\mathcal{L}^{-1}\left\{-\frac{1}{12}\frac{1}{s+16} + \frac{1}{12}\frac{1}{s+4}\right\}$$

$$x(t) = \frac{3}{5}\left(-\frac{1}{12}e^{-16t} + \frac{1}{12}e^{-t}\right)$$

$$\boxed{= \frac{1}{20}e^{-4t} - \frac{1}{20}e^{-16t}}$$

Exercise

A spring with a mass of 3-*kg* is held stretched 0.6 *m* beyond its natural length by a force of 20 *N*. If the spring begins at its equilibrium and with initial velocity 1.2 *m/s*. Find the position of the mass.

Solution

$$k = \frac{20}{0.6} = \frac{100}{3} \qquad\qquad kx = F$$

$$3x'' + \frac{100}{3}x = 0 \ ; \quad x(0) = 0, \quad x'(0) = 1.2 = \frac{6}{5} \qquad mx'' + cx' + kx = F(t)$$

$$\mathcal{L}\left\{9x'' + 100x\right\} = 0$$

$$9s^2 X(s) - 9sx(0) - 9x'(0) + 100X(s) = 0$$

$$\left(9s^2 + 100\right) X(s) = \frac{36}{5}$$

$$\mathcal{L}^{-1}\left\{X(s)\right\} = \frac{6}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2 + \frac{100}{9}}\right\}$$

$$x(t) = \frac{6}{5}\frac{3}{10}\sin\frac{10}{3}t$$

$$\boxed{= \frac{9}{25}\sin\frac{10}{3}t}$$

Exercise

A spring with a mass of 2-*kg* is held stretched 0.5 *m*, has damping constant 14, and a force of 6 *N*. If the spring is stretched 1 *m* beyond at its equilibrium and with no initial velocity. Find the position of the mass at any time *t*.

Solution

$$k = \frac{6}{.5} = 12 \qquad kx = F$$

$$2x'' + 14x' + 12x = 0\ ; \quad x(0) = 1, \quad x'(0) = 0 \qquad mx'' + cx' + kx = F(t)$$

$$\mathcal{L}\{x'' + 7x' + 6x\} = 0$$

$$s^2X(s) - sx(0) - x'(0) + 7sX(s) - 7x(0) + 6X(s) = 0$$

$$\left(s^2 + 7s + 6\right)X(s) = s + 7$$

$$X(s) = \frac{s+7}{(s+1)(s+6)} = \frac{A}{s+1} + \frac{B}{s+6}$$

$$s + 7 = As + 6A + Bs + B$$

$$\begin{cases} s & A + B = 1 \\ s^0 & 6A + B = 7 \end{cases} \rightarrow \boxed{A = \frac{6}{5},\ B = -\frac{1}{5}}$$

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{6}{5}\frac{1}{s+1} - \frac{1}{5}\frac{1}{s+6}\right\}$$

$$\boxed{x(t) = \frac{6}{5}e^{-t} - \frac{1}{5}e^{-6t}}$$

Exercise

Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit when $L = 0.25\ H$, $R = 10\ \Omega$, $C = 0.001\ F$, $E(t) = 0$, $q(0) = q_0\ C$, and $i(0) = 0$.

Solution

$$0.25q'' + 10q' + \frac{1}{0.001}q = 0 \qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$\mathcal{L}\{q'' + 40q' + 4{,}000q\} = 0$$

$$s^2Q(s) - sq(0) - q'(0) + 40sQ(s) - 40q(0) + 4{,}000Q(s) = 0 \qquad q(0) = q_0 \quad q'(0) = 0$$

$$\left(s^2 + 40s + 4000\right)Q(s) = sq_0 + 40q_0$$

$$Q(s) = q_0\frac{s+40}{(s+20)^2 + 3600}$$

$$\mathcal{L}^{-1}\{Q(s)\} = \mathcal{L}^{-1}\left\{q_0\frac{s+20}{(s+20)^2 + 60^2} + q_0\frac{20}{(s+20)^2 + 60^2}\right\}$$

$$\boxed{q(t) = q_0 e^{-20t}\left(\cos 60t + \frac{1}{3}\sin 60t\right)}$$

Exercise

Find the charge $q(t)$ on the capacitor in an LRC–series circuit at $t = 0.01\ sec$ when $L = 0.05\ h$, $R = 2\ \Omega$, $C = 0.01\ f$, $E(t) = 0$, $q(0) = 5\ C$, and $i(0) = 0\ A$.

Solution

$$0.05q'' + 2q' + \frac{1}{0.01}q = 0 \qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 40q' + 2{,}000q = 0$$

$$s^2Q(s) - sq(0) - q'(0) + 40sQ(s) - 40q(0) + 2{,}000Q(s) = 0 \qquad q(0) = 5 \quad q'(0) = 0$$

$$\left(s^2 + 40s + 2{,}000\right)Q(s) = 5s + 200$$

$$Q(s) = \frac{5(s+40)}{(s+20)^2 + 1600}$$

$$\mathcal{L}^{-1}\{Q(s)\} = 5\mathcal{L}^{-1}\left\{\frac{s+20}{(s+20)^2 + 40^2} + \frac{20}{(s+20)^2 + 40^2}\right\}$$

$$\boxed{q(t) = \left(5\cos 40t + \frac{5}{2}\sin 40t\right)e^{-20t}}$$

Exercise

Find the charge $q(t)$ on the capacitor in an LRC–series circuit when $L = \frac{5}{3}\ h$, $R = 10\ \Omega$, $C = \frac{1}{30}\ f$, $E(t) = 0$, $q(0) = 4\ C$, and $i(0) = 0\ A$.

Solution

$$\frac{5}{3}q'' + 10q' + 30q = 0 \qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 6q' + 18q = 0$$

$$s^2Q(s) - sq(0) - q'(0) + 6sQ(s) - 6q(0) + 18Q(s) = 0 \qquad q(0) = 4 \quad q'(0) = 0$$

$$\left(s^2 + 6s + 18\right)Q(s) = 4s + 24$$

$$Q(s) = \frac{4(s+6)}{(s+3)^2 + 9}$$

$$\mathcal{L}^{-1}\{Q(s)\} = 4\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+3^2} + \frac{3}{(s+3)^2+3^2}\right\}$$

$$\boxed{q(t) = e^{-3t}(4\cos 3t + 4\sin 3t)}$$

Exercise

Find the current $i(t)$ in an *LRC*–series circuit when $L=1\ h$, $R=20\ \Omega$, $C=0.005\ f$, $E(t)=150\ V$, $q(0)=0\ C$, and $i(0)=0\ A$.

Solution

$$q''+20q'+\frac{1}{.005}q=150\ ; \quad q(0)=0 \quad q'(0)=0 \qquad\qquad Lq''+Rq'+\frac{1}{C}q=E(t)$$

$$\mathcal{L}\{q''+20q'+200q\} = \mathcal{L}\{150\}$$

$$s^2Q(s)-sq(0)-q'(0)+20sQ(s)-20q(0)+200Q(s)=\frac{150}{s}$$

$$\left(s^2+20s+200\right)Q(s)=\frac{150}{s}$$

$$Q(s)=\frac{150}{s\left((s+10)^2+100\right)}=\frac{A}{s}+\frac{B(s+10)+C}{(s+10)^2+100}$$

$$\begin{cases} s^2 & A+B=0 \\ s & 20A+10B+C=0 \\ s^0 & 200A=150 \end{cases} \rightarrow \quad \boxed{A=\frac{3}{4},\ B=-\frac{3}{4},\ C=-\frac{15}{2}}$$

$$Q(s)=\frac{3}{4}\frac{1}{s}-\frac{3}{4}\frac{s+10}{(s+10)^2+10^2}-\frac{15}{2}\frac{1}{(s+10)^2+10^2}$$

$$\boxed{q(t)=\frac{3}{4}-\frac{3}{4}e^{-10t}\cos 10t-\frac{3}{4}e^{-10t}\sin 10t}$$

$$i(t)=q'(t)=\frac{15}{2}e^{-10t}\cos 10t+\frac{15}{2}e^{-10t}\sin 10t+\frac{15}{2}e^{-10t}\sin 10t-\frac{15}{2}e^{-10t}\cos 10t$$

$$\boxed{=15e^{-10t}\sin 10t}$$

Exercise

A resistor $R=20\ \Omega$ and a capacitor of $C=0.1\ F$ are joined in series with an electronic force (emf) $E=E(t)$ and no charge on the capacitor at $t=0$. Find the ensuing charge on the capacitor at time t for the given $E(t)=100\sin 2t$

Solution

$$R\frac{dQ}{dt}+\frac{1}{C}Q=E$$

$$20Q'+\frac{1}{0.1}Q=100\sin 2t$$

$$\mathcal{L}\left(Q'+0.5Q\right)=5\mathcal{L}\left(\sin 2t\right)$$

$$sQ(s)-Q(0)+0.5Q(s)=5\frac{2}{s^2+4} \qquad Q(0)=0$$

$$\left(s+0.5\right)Q(s)=10\frac{1}{s^2+4}$$

$$Q(s)=10\frac{1}{(s+0.5)\left(s^2+4\right)}$$

$$\frac{1}{(s+0.5)\left(s^2+4\right)}=\frac{A}{s+0.5}+\frac{Bs+C}{s^2+4}$$

$$1=As^2+4A+Bs^2+0.5Bs+Cs+0.5C$$

$$1=\left(A+B\right)s^2+\left(0.5B+C\right)s+4A+0.5C$$

$$\begin{cases} A+B=0 \\ 0.5B+C=0 \\ 4A+0.5C=1 \end{cases} \Rightarrow \boxed{A=\frac{4}{17}} \quad \boxed{B=-\frac{4}{17}} \quad \boxed{C=\frac{2}{17}}$$

$$Q(s)=10\left(\frac{4}{17}\frac{1}{s+\frac{1}{2}}-\frac{4}{17}\frac{s}{s^2+4}+\frac{2}{17}\frac{1}{s^2+4}\right)$$

$$=\frac{1}{17}\left(40\frac{1}{s+\frac{1}{2}}-40\frac{s}{s^2+4}+10\frac{2}{s^2+4}\right)$$

$$Q(t)=\frac{1}{17}\left(40\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\}-40\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+10\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}\right)$$

$$\boxed{=\frac{1}{17}\left(40e^{-t/2}-40\cos 2t+10\sin 2t\right)}$$

Exercise

A resistor $R=20\ \Omega$ and a capacitor of $C=0.1\ F$ are joined in series with an electronic force (emf) $E=E(t)$ and no charge on the capacitor at $t=0$. Find the ensuing charge on the capacitor at time t for the given $E(t)=100e^{-0.1t}$

Solution

$$20Q'+\frac{1}{0.1}Q=100e^{-0.1t}$$

$$\mathcal{L}(Q' + 0.5Q) = 5\mathcal{L}\left(e^{-0.1t}\right)$$

$$sQ(s) - Q(0) + 0.5Q(s) = 5\frac{1}{s+0.1} \qquad Q(0) = 0$$

$$(s+0.5)Q(s) = 5\frac{1}{s+0.1}$$

$$Q(s) = 5\frac{1}{(s+0.5)(s+0.1)}$$

$$\frac{1}{(s+0.5)(s+0.1)} = \frac{A}{s+0.5} + \frac{B}{s+0.1}$$

$$1 = (A+B)s + 0.1A + 0.5B$$

$$\begin{cases} A + B = 0 \\ 0.1A + 0.5B = 1 \end{cases} \Rightarrow A = -\frac{5}{2} \quad B = \frac{5}{2}$$

$$Q(s) = 5\left(-\frac{5}{2}\frac{1}{s+\frac{1}{2}} + \frac{5}{2}\frac{1}{s+0.1}\right) \quad = \frac{25}{2}\left(-\frac{1}{s+\frac{1}{2}} + \frac{1}{s+\frac{1}{10}}\right)$$

$$Q(t) = \frac{25}{2}\left(\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{10}}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\}\right)$$

$$\boxed{= \frac{25}{2}\left(e^{-t/10} - e^{-t/2}\right)}$$

Exercise

A resistor $R = 20\ \Omega$ and a capacitor of $C = 0.1\ F$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge on the capacitor at time t for the given $E(t) = 100\left(1 - e^{-0.1t}\right)$

Solution

$$20Q' + \frac{1}{0.1}Q = 100\left(1 - e^{-0.1t}\right)$$

$$\mathcal{L}(Q' + 0.5Q) = 5\mathcal{L}\left(1 - e^{-0.1t}\right)$$

$$sQ(s) - Q(0) + 0.5Q(s) = 5\left(\frac{1}{s} - \frac{1}{s+0.1}\right) \qquad Q(0) = 0$$

$$(s+0.5)Q(s) = 5\left(\frac{1}{s} - \frac{1}{s+0.1}\right)$$

$$Q(s) = 5\frac{1}{s+0.5}\left(\frac{s+0.1-s}{s(s+0.1)}\right)$$

$$Q(s) = 0.5\frac{1}{s(s+0.5)(s+0.1)}$$

$$\frac{1}{s(s+0.5)(s+0.1)} = \frac{A}{s} + \frac{B}{s+0.5} + \frac{C}{s+0.1}$$

$$1 = A\left(s^2 + 0.6s + 0.05\right) + B\left(s^2 + 0.1s\right) + C\left(s^2 + 0.5s\right)$$

$$1 = (A+B+C)s^2 + (0.6A + 0.1B + 0.5C)s + 0.05A$$

$$\begin{cases} A+B+C=0 \\ 0.6A+0.1B+0.5C=0 \\ 0.05A=1 \end{cases} \Rightarrow A = \frac{1}{0.05} = 20 \quad B=5 \quad C=-25$$

$$Q(s) = \frac{1}{2}\left(20\frac{1}{s} + 5\frac{1}{s+0.5} - 25\frac{1}{s+0.1}\right)$$

$$= 10\frac{1}{s} + \frac{5}{2}\frac{1}{s+\frac{1}{2}} - \frac{25}{2}\frac{1}{s+\frac{1}{10}}$$

$$Q(t) = \left(10\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\} - \frac{25}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{10}}\right\}\right)$$

$$\boxed{= 10 + \frac{5}{2}e^{-t/2} - \frac{25}{2}e^{-t/10}}$$

Exercise

A resistor $R = 20\ \Omega$ and a capacitor of $C = 0.1\,F$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge on the capacitor at time *t* for the given $E(t) = 100\cos 3t$

Solution

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E$$

$$20Q' + \frac{1}{0.1}Q = 100\cos 3t$$

$$Q' + \frac{1}{0.1(20)}Q = \frac{100}{20}\cos 3t$$

$$\mathcal{L}(Q' + 0.5Q) = 5\mathcal{L}(\cos 3t)$$

$$sQ(s) - Q(0) + 0.5Q(s) = 5\frac{s}{s^2+9} \qquad Q(0) = 0$$

$$(s+0.5)Q(s) = 5\frac{s}{s^2+9}$$

$$Q(s) = 5\frac{s}{(s+0.5)\left(s^2+9\right)}$$

$$\frac{s}{(s+0.5)\left(s^2+9\right)} = \frac{A}{s+0.5} + \frac{Bs+C}{s^2+9}$$

$$s = As^2 + 9A + Bs^2 + 0.5Bs + Cs + 0.5C$$

$$s = (A+B)s^2 + (0.5B + C)s + 9A + 0.5C$$

$$\begin{cases} A + B = 0 \\ 0.5B + C = 1 \\ 9A + 0.5C = 0 \end{cases} \Rightarrow A = -\frac{2}{37} \quad B = \frac{2}{37} \quad C = \frac{36}{37}$$

$$Q(s) = 5\left(-\frac{2}{37}\frac{1}{s+\frac{1}{2}} + \frac{2}{37}\frac{s}{s^2+9} + \frac{36}{37}\frac{1}{s^2+9}\right)$$

$$= \frac{1}{37}\left(-10\frac{1}{s+\frac{1}{2}} + 10\frac{s}{s^2+9} + \frac{5(36)}{3}\frac{3}{s^2+9}\right)$$

$$Q(t) = \frac{1}{37}\left(-10\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\} + 10\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + 60\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}\right)$$

$$\boxed{= \frac{1}{37}\left(-10e^{-t/2} + 10\cos 3t + 60\sin 3t\right)}$$

Exercise

An inductor $(L = 1\ H)$ and a resistor $(R = 0.1\ \Omega)$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge current in the current at time t for the given $E(t) = 10 - 2t$

Solution

$$L\frac{dI}{dt} + RI = E(t)$$

$$\frac{dI}{dt} + 0.1I = 10 - 2t$$

$$\mathcal{L}(I' + 0.1I) = \mathcal{L}(10 - 2t)$$

$$sI(s) - I(0) + 0.1I(s) = \frac{10}{s} - \frac{2}{s^2} \qquad I(0) = 0$$

$$(s + 0.1)I(s) = \frac{10s - 2}{s^2}$$

$$I(s) = \frac{10s - 2}{s^2(s + 0.1)}$$

$$\frac{10s - 2}{s^2(s+0.1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s + 0.1}$$

$$10s - 2 = As(s + 0.1) + B(s + 0.1) + Cs^2$$

$$10s-2=(A+C)s^2+(B+0.1A)s+0.1B$$

$$\begin{cases} A+C=0 \\ 0.1A+B=10 \\ 0.1B=-2 \end{cases} \Rightarrow A=300 \quad B=-20 \quad C=-300$$

$$I(s)=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+0.1}$$

$$=300\frac{1}{s}-20\frac{1}{s^2}-300\frac{1}{s+\frac{1}{10}}$$

$$I(t)=\left(3000\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-20\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}-300\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{10}}\right\}\right)$$

$$\boxed{=300-20t-300e^{-t/10}}$$

Exercise

An inductor $(L=1\ H)$ and a resistor $(R=0.1\ \Omega)$ are joined in series with an electronic force (*emf*) $E=E(t)$ and no charge on the capacitor at $t=0$. Find the ensuing current in the current at time t for the given $E(t)=4\cos 3t$

Solution

$$L\frac{dI}{dt}+RI=E(t)$$

$$\frac{dI}{dt}+0.1I=4\cos 3t$$

$$\mathcal{L}(I'+0.1I)=4\mathcal{L}(\cos 3t)$$

$$sI(s)-I(0)+0.1I(s)=4\frac{s}{s^2+9} \qquad I(0)=0$$

$$(s+0.1)I(s)=4\frac{s}{s^2+9}$$

$$I(s)=4\frac{s}{(s+0.1)(s^2+9)}$$

$$\frac{s}{(s+0.1)(s^2+9)}=\frac{A}{s+0.1}+\frac{Bs+C}{s^2+9}$$

$$s=As^2+9A+Bs^2+0.1Bs+Cs+0.1C$$

$$s=(A+B)s^2+(0.1B+C)s+9A+0.1C$$

$$\begin{cases} A+B=0 \\ 0.1B+C=1 \\ 9A+0.1C=0 \end{cases} \Rightarrow A=-\frac{10}{901} \quad B=\frac{10}{901} \quad C=\frac{900}{901}$$

$$I(s)=4\left(\frac{A}{s+0.1}+\frac{Bs}{s^2+9}+\frac{C}{s^2+9}\right)$$

$$=4\left(-\frac{10}{901}\frac{1}{s+0.1}+\frac{10}{901}\frac{s}{s^2+9}+\frac{900}{901}\frac{1}{3}\frac{3}{s^2+9}\right)$$

$$I(t)=\frac{1}{901}\left(-40\mathcal{L}^{-1}\left\{\frac{1}{s+0.1}\right\}+40\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}+1200\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}\right)$$

$$=\frac{1}{901}\left(-40e^{-t/10}+40\cos 3t+1200\sin 3t\right)$$

Exercise

An inductor $(L=1\ H)$ and a resistor $(R=0.1\ \Omega)$ are joined in series with an electronic force (emf) $E=E(t)$ and no charge on the capacitor at $t=0$. Find the ensuing current in the current at time t for the given $E(t)=4\sin 2\pi t$

Solution

$$L\frac{dI}{dt}+RI=E(t)$$

$$\frac{dI}{dt}+0.1I=4\sin 2\pi t$$

$$\mathcal{L}\left(I'+0.1I\right)=4\mathcal{L}\left(\sin 2\pi t\right)$$

$$sI(s)-I(0)+0.1I(s)=4\frac{2\pi}{s^2+4\pi^2} \qquad I(0)=0$$

$$(s+0.1)I(s)=8\pi\frac{1}{s^2+4\pi^2}$$

$$I(s)=8\pi\frac{1}{(s+0.1)\left(s^2+4\pi^2\right)}$$

$$\frac{1}{(s+0.1)\left(s^2+4\pi^2\right)}=\frac{A}{s+0.1}+\frac{Bs+C}{s^2+4\pi^2}$$

$$s=As^2+4\pi^2A+Bs^2+0.1Bs+Cs+0.1C$$

$$s=(A+B)s^2+(0.1B+C)s+4\pi^2A+0.1C$$

$$\begin{cases} A+B=0 \\ 0.1B+C=0 \\ 4\pi^2 A+0.1C=1 \end{cases}$$

$$\Rightarrow A=\frac{100}{1+400\pi^2} \quad B=-\frac{100}{1+400\pi^2} \quad C=\frac{10}{1+400\pi^2}$$

$$I(s)=8\pi\left(\frac{A}{s+0.1}+\frac{Bs}{s^2+4\pi^2}+\frac{C}{s^2+4\pi^2}\right)$$

$$=8\pi\left(\frac{100}{1+400\pi^2}\frac{1}{s+0.1}-\frac{100}{1+400\pi^2}\frac{s}{s^2+4\pi^2}+\frac{10}{1+400\pi^2}\frac{1}{s^2+4\pi^2}\right)$$

$$I(t)=\frac{8}{1+400\pi^2}\left(100\pi\mathcal{L}^{-1}\left\{\frac{1}{s+0.1}\right\}-100\pi\mathcal{L}^{-1}\left\{\frac{s}{s^2+4\pi^2}\right\}+10\pi\frac{1}{2\pi}\mathcal{L}^{-1}\left\{\frac{2\pi}{s^2+4\pi^2}\right\}\right)$$

$$\boxed{=\frac{8}{1+400\pi^2}\left(100\pi e^{-t/10}-100\pi\cos 2\pi t+5\sin 2\pi t\right)}$$

Exercise

Solve the general initial value problem modeling the *RC* circuit

$$R\frac{dQ}{dt}+\frac{1}{C}Q=E, \quad Q(0)=0$$

Where E is a constant source of emf

Solution

$$\mathcal{L}\left(\frac{dQ}{dt}+\frac{1}{RC}Q\right)=\mathcal{L}\left(\frac{E}{R}\right)$$

$$sQ(s)-Q(0)+\frac{1}{RC}Q(s)=\frac{E}{R}\frac{1}{s} \qquad Q(0)=0$$

$$\left(s+\frac{1}{RC}\right)Q(s)=\frac{E}{R}\frac{1}{s}$$

$$Q(s)=\frac{E}{R}\frac{1}{s\left(s+\frac{1}{RC}\right)}$$

$$\frac{1}{s\left(s+\frac{1}{RC}\right)}=\frac{A}{s}+\frac{B}{s+\frac{1}{RC}}=\frac{As+\frac{1}{RC}A+Bs}{s\left(s+\frac{1}{RC}\right)}$$

$$\begin{cases} A+B=0 \\ \frac{1}{RC}A=1 \end{cases} \rightarrow \quad A=RC \quad B=-RC$$

$$Q(t)=\frac{E}{R}\left(RC\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-RC\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{RC}}\right\}\right)$$

$$=\frac{E}{R}\left(RC-RCe^{-t/RC}\right)$$

$$\boxed{=EC\left(1-e^{-t/RC}\right)}$$

Exercise

Solve the general initial value problem modeling the LR circuit $L\frac{dI}{dt}+RI=E, \quad I(0)=I_0$

Where E is a constant source of emf

Solution

$$\mathcal{L}\left(\frac{dI}{dt}+\frac{R}{L}I\right)=\mathcal{L}\left(\frac{E}{L}\right)$$

$$sI(s)-I(0)+\frac{R}{L}I(s)=\frac{E}{L}\frac{1}{s} \qquad I(0)=I_0$$

$$\left(s+\frac{R}{L}\right)I(s)=\frac{E}{L}\frac{1}{s}+I_0$$

$$I(s)=\frac{E}{L}\frac{1}{s\left(s+\frac{R}{L}\right)}+I_0\frac{1}{s+\frac{R}{L}}$$

$$\frac{1}{s\left(s+\frac{R}{L}\right)}=\frac{A}{s}+\frac{B}{s+\frac{R}{L}}=\frac{As+\frac{R}{L}A+Bs}{s\left(s+\frac{R}{L}\right)}$$

$$\begin{cases} A+B=0 \\ \frac{R}{L}A=1 \end{cases} \rightarrow \quad A=\frac{L}{R} \quad B=-\frac{L}{R}$$

$$I(t)=\frac{E}{L}\left(\frac{L}{R}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-\frac{L}{R}\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{R}{L}}\right\}\right)+I_0\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{R}{L}}\right\}$$

$$=\frac{E}{L}\left(\frac{L}{R}-\frac{L}{R}e^{-Rt/L}\right)+I_0e^{-Rt/L}$$

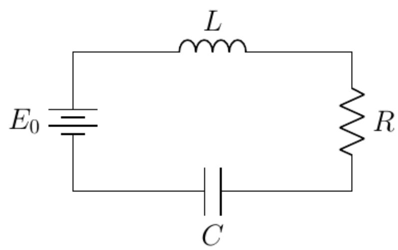
$$=\frac{E}{R}-\frac{E}{R}e^{-Rt/L}+I_0e^{-Rt/L}$$

$$=\frac{1}{R}\left(E-Ee^{-Rt/L}+RI_0e^{-Rt/L}\right)$$

$$\boxed{=\frac{1}{R}\left(E+\left(RI_0-E\right)e^{-Rt/L}\right)}$$

Exercise

Consider a battery of constant voltage E_0 that charges the capacitor. $L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=E(t)$



Divide the given equation by L and define $2\lambda=\frac{R}{L}$ and $\omega^2=\frac{1}{LC}$.

a) Use the Laplace transform to show that the solution $q(t)$ of $q''+2\lambda q'+\omega^2 q=\frac{E_0}{L}$ subject to $q(0)=0$, $i(0)=0$ is

$$q(t)=\begin{cases} E_0C\left[1-e^{-\lambda t}\left(\cosh\sqrt{\lambda^2-\omega^2}\,t+\frac{\lambda}{\sqrt{\lambda^2-\omega^2}}\sinh\sqrt{\lambda^2-\omega^2}\,t\right)\right] & \lambda>\omega \\ E_0C\left[1-e^{-\lambda t}(1+\lambda t)\right] & \lambda=\omega \\ E_0C\left[1-e^{-\lambda t}\left(\cos\sqrt{\omega^2-\lambda^2}\,t+\frac{\lambda}{\sqrt{\omega^2-\lambda^2}}\sin\sqrt{\omega^2-\lambda^2}\,t\right)\right] & \lambda<\omega \end{cases}$$

b) Use the Laplace transform to find the charge $q(t)$ in an *RC* series when $q(0)=0$ and $E(t)=E_0e^{-kt}$, $k>0$. Consider *two* cases: $k\neq\frac{1}{RC}$ and $k=\frac{1}{RC}$

Solution

a) $\mathcal{L}\left\{q''+2\lambda q'+\omega^2 q\right\}=\mathcal{L}\left\{\frac{E_0}{L}\right\}$

$$s^2Q(s)-sq(0)-q'(0)+2\lambda\left(sQ(s)-q(0)\right)+\omega^2Q(s)=\frac{E_0}{L}\frac{1}{s} \qquad q(0)=0,\ q'(0)=0$$

$$\left(s^2+2s\lambda+\omega^2\right)Q(s)=\frac{E_0}{L}\frac{1}{s}$$

$$Q(s)=\frac{E_0}{L}\frac{1}{s\left(s^2+2s\lambda+\omega^2\right)}$$

$$\frac{1}{s\left(s^2+2s\lambda+\omega^2\right)}=\frac{A}{s}+\frac{Bs+C}{s^2+2s\lambda+\omega^2}$$

$$As^2+2A\lambda s+\omega^2A+Bs^2+Cs=1$$

$s^2 \quad A+B=0 \qquad \boxed{B=-\frac{1}{\omega^2}}$

$s^1 \quad 2\lambda A+C=0 \qquad C=-\frac{2\lambda}{\omega^2}$

$s^0 \quad \omega^2 A=1 \quad \rightarrow A=\frac{1}{\omega^2}$

$$Q(s)=\frac{E_0}{L}\left(\frac{1}{\omega^2}\frac{1}{s}-\frac{1}{\omega^2}\frac{s+2\lambda}{s^2+2\lambda s+\omega^2}\right)$$

For $\lambda>\omega$, then $s^2+2\lambda s+\omega^2=s^2+2\lambda s+\lambda^2-\lambda^2+\omega^2=(s+\lambda)^2-\left(\lambda^2-\omega^2\right)$

$$Q(s)=\frac{E_0}{L\omega^2}\left(\frac{1}{s}-\frac{s+\lambda+\lambda}{(s+\lambda)^2-\left(\lambda^2-\omega^2\right)}\right) \qquad \omega^2=\frac{1}{LC}$$

$$\mathcal{L}^{-1}\{Q(s)\}=E_0C\,\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{s+\lambda}{(s+\lambda)^2-\left(\lambda^2-\omega^2\right)}-\frac{\lambda}{(s+\lambda)^2-\left(\lambda^2-\omega^2\right)}\right\}$$

$$q(t)=E_0C\left(1-e^{-\lambda t}\cosh\sqrt{\lambda^2-\omega^2}\,t-\frac{\lambda}{\sqrt{\lambda^2-\omega^2}}e^{-\lambda t}\sinh\sqrt{\lambda^2-\omega^2}\,t\right)$$

For $\lambda<\omega$, then $s^2+2\lambda s+\omega^2=(s+\lambda)^2+\left(\omega^2-\lambda^2\right)$

$$\mathcal{L}^{-1}\{Q(s)\}=E_0C\,\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{s+\lambda}{(s+\lambda)^2+\left(\omega^2-\lambda^2\right)}-\frac{\lambda}{(s+\lambda)^2+\left(\omega^2-\lambda^2\right)}\right\}$$

$$q(t)=E_0C\left(1-e^{-\lambda t}\left(\cos\sqrt{\omega^2-\lambda^2}\,t-\frac{\lambda}{\sqrt{\omega^2-\lambda^2}}\sin\sqrt{\omega^2-\lambda^2}\,t\right)\right)$$

For $\lambda=\omega$, then $s^2+2\lambda s+\omega^2=s^2+2\lambda s+\lambda^2=(s+\lambda)^2$

$$\frac{1}{s(s+\lambda)^2}=\frac{A}{s}+\frac{B}{s+\lambda}+\frac{C}{(s+\lambda)^2}$$

$$As^2+2A\lambda s+\lambda^2A+Bs^2+B\lambda s+Cs=1$$

$s^2 \quad A+B=0 \qquad \boxed{B=-\frac{1}{\lambda^2}}$

$s^1 \quad 2\lambda A+B\lambda+C=0 \quad C=-\frac{2}{\lambda}+\frac{1}{\lambda} \qquad \boxed{C=-\frac{1}{\lambda}}$

$s^0 \quad \lambda^2 A=1 \qquad \rightarrow \boxed{A=\frac{1}{\lambda^2}}$

$$\mathcal{L}^{-1}\{Q(s)\}=\frac{E_0}{L}\,\mathcal{L}^{-1}\left\{\frac{1}{\lambda^2}\frac{1}{s}-\frac{1}{\lambda^2}\frac{1}{s+\lambda}-\frac{1}{\lambda}\frac{1}{(s+\lambda)^2}\right\}$$

$$q(t)=\frac{E_0}{L}\frac{1}{\lambda^2}\left(1-e^{-\lambda t}-\lambda t e^{-\lambda t}\right) \qquad \omega^2=\lambda^2=\frac{1}{LC}$$

$$\boxed{=E_0 C\left(1-e^{-\lambda t}-\lambda t e^{-\lambda t}\right)}$$

b) $R\frac{dq}{dt}+\frac{1}{C}q=E_0 e^{-kt}$

$$R\left(sQ(s)-q(0)\right)+\frac{1}{C}Q(s)=E_0\,\frac{1}{s+k} \qquad q(0)=0$$

$$\left(Rs+\frac{1}{C}\right)Q(s)=E_0\,\frac{1}{s+k}$$

$$\left(\frac{RCs+1}{C}\right)Q(s)=E_0\,\frac{1}{s+k}$$

$$Q(s)=E_0 C\frac{1}{(s+k)(RCs+1)}=E_0 C\left(\frac{A}{s+k}+\frac{B}{RCs+1}\right)$$

$$RCAs+A+Bs+kB=1$$

$$s^1 \quad (RC)A+B=0$$

$$\rightarrow (1-kRC)A=1 \Rightarrow A=\frac{1}{1-kRC} \qquad B=-\frac{RC}{1-kRC}$$

$$s^0 \quad A+kB=1$$

$$Q(s)=E_0 C\left(\frac{1}{1-kRC}\frac{1}{s+k}-\frac{1}{1-kRC}\frac{RC}{RCs+1}\right)$$

$$=\frac{E_0 C}{1-kRC}\left(\frac{1}{s+k}-\frac{1}{s+\frac{1}{RC}}\right)$$

When $k\neq\frac{1}{RC}$

$$\mathcal{L}\{Q(s)\}=\frac{E_0 C}{1-kRC}\mathcal{L}\left(\frac{1}{s+k}-\frac{1}{s+\frac{1}{RC}}\right)$$

$$\boxed{q(t)=\frac{E_0 C}{1-kRC}\left(e^{-kt}-e^{-t/RC}\right)}$$

When $k=\frac{1}{RC}$ $\quad \Rightarrow Q(s)=E_0 C\frac{1}{\left(s+\frac{1}{RC}\right)(RCs+1)}$

$$=E_0 RC^2\frac{1}{(RCs+1)^2}=E_0 RC^2\left(\frac{A}{RCs+1}+\frac{B}{(RCs+1)^2}\right)$$

$$RCAs+A+B=1$$

$$s^1 \quad (RC)A=0\rightarrow \boxed{A=0}$$

$$s^0 \quad A+B=1\rightarrow \boxed{B=1}$$

$$\mathcal{L}\{Q(s)\}=E_0 RC^2\mathcal{L}\left\{\frac{1}{(RCs+1)^2}\right\}$$

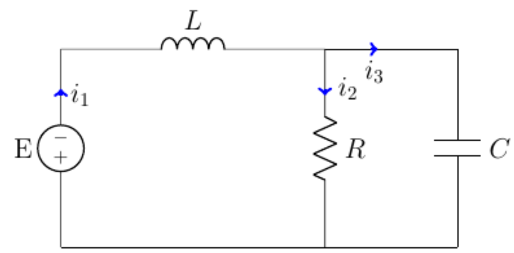
$$= E_0 RC^2 \mathcal{L}\left\{\frac{1}{(RC)^2}\frac{1}{\left(s+\frac{1}{RC}\right)^2}\right\}$$

$$\boxed{q(t) = \frac{E_0}{R} te^{-t/RC}}$$

Exercise

Solve the system under the conditions $E(t) = 60\ V$, $L = 1\ h$, $R = 50\ \Omega$, $C = 10^{-4}\ f$, and the currents i_1 and i_2 are initially zero.

Solution



$$\begin{cases} L\frac{di_1}{dt} + Ri_2 = E(t) \\ RC\frac{di_2}{dt} + i_2 - i_1 = 0 \end{cases}$$

$$\begin{cases} \frac{di_1}{dt} + 50i_2 = 60 \\ 50\times 10^{-4}\frac{di_2}{dt} + i_2 - i_1 = 0 \end{cases} \qquad i_2(0) = i_1(0) = 0$$

$$\begin{cases} \mathcal{L}\left\{\frac{di_1}{dt} + 50i_2\right\} = \mathcal{L}\{60\} \\ \mathcal{L}\left\{50\times 10^{-4}\frac{di_2}{dt} + i_2 - i_1\right\} = 0 \end{cases}$$

$$\begin{cases} sI_1(s) - i_1(0) + 50I_2(s) = \frac{60}{s} \\ \frac{1}{200}\left(sI_2(s) - i_2(0)\right) + I_2(s) - I_1(s) = 0 \end{cases}$$

$$\begin{cases} sI_1(s) + 50I_2(s) = \frac{60}{s} \\ -200I_1(s) + (s+200)I_2(s) = 0 \end{cases}$$

$$\to \begin{cases} 200sI_1(s) + 10^4 I_2(s) = \frac{12000}{s} \\ -200sI_1(s) + (s+200)sI_2(s) = 0 \end{cases}$$

$$\left(s^2 + 200s + 10^4\right)I_2(s) = \frac{12000}{s} \quad \Rightarrow I_2(s) = \frac{12000}{s(s+100)^2}$$

$$sI_1(s) = \frac{60}{s} - 50\frac{12,000}{s(s+100)^2}$$

$$= \frac{60s^2 + 12,000s - 6\times 10^5 - 6\times 10^5}{s(s+100)^2}$$

$$\Rightarrow I_1(s) = \frac{60s + 12,000}{s(s+100)^2}$$

$$I_1(s) = \frac{60s + 12,000}{s(s+100)^2} = \frac{A}{s} + \frac{B}{s+100} + \frac{C}{(s+100)^2}$$

$$As^2 + 200As + 10,000A + Bs^2 + 100Bs + Cs = 60s + 12,000$$

$$A + B = 0 \qquad \boxed{B = -\frac{6}{5}}$$

$$200A + 100B + C = 60 \qquad \boxed{C = 60 - 240 + 120 = -60}$$

$$10,000A = 12,000 \qquad \boxed{A = \frac{6}{5}}$$

$$\mathcal{L}^{-1}\left\{I_1(s)\right\} = \mathcal{L}^{-1}\left\{\frac{6}{5}\frac{1}{s} - \frac{6}{5}\frac{1}{s+100} - \frac{60}{(s+100)^2}\right\}$$

$$\boxed{i_1(t) = \frac{6}{5} - \frac{6}{5}e^{-100t} - 60te^{-100t}}$$

$$I_2(s) = \frac{12000}{s(s+100)^2} = \frac{A}{s} + \frac{B}{s+100} + \frac{C}{(s+100)^2}$$

$$As^2 + 200As + 10,000A + Bs^2 + 100Bs + Cs = 12,000$$

$$A + B = 0 \qquad \boxed{B = -\frac{6}{5}}$$

$$200A + 100B + C = 0 \qquad \boxed{C = -240 + 120 = -120}$$

$$10,000A = 12,000 \qquad \boxed{A = \frac{6}{5}}$$

$$\mathcal{L}^{-1}\left\{I_2(s)\right\} = \mathcal{L}^{-1}\left\{\frac{6}{5}\frac{1}{s} - \frac{6}{5}\frac{1}{s+100} - \frac{120}{(s+100)^2}\right\}$$

$$\boxed{i_2(t) = \frac{6}{5} - \frac{6}{5}e^{-100t} - 120te^{-100t}}$$

Exercise

Solve
$$\begin{aligned} x_1'' + 10x_1 \quad - 4x_2 &= 0 \\ -4x_1 + x_2'' + 4x_2 &= 0 \end{aligned}$$

Subject to $x_1(0) = 0,\ \ x_1'(0) = 1,\ \ x_2(0) = 0,\ \ x_2'(0) = -1$

Solution

$$s^2X_1(s)-sx_1(0)-x_1'(0)+10X_1(s)-4X_2(s)=0$$

$$-4X_1(s)+s^2X_2(s)-sx_2(0)-x_2'(0)+4X_2(s)=0$$

$$\left(s^2+10\right)X_1(s)-4X_2(s)=1$$

$$-4X_1(s)+\left(s^2+4\right)X_2(s)=-1$$

$$\Delta=\begin{vmatrix} s^2+10 & -4 \\ -4 & s^2+4 \end{vmatrix}=\left(s^2+10\right)\left(s^2+4\right)-16$$

$$\Delta_1=\begin{vmatrix} 1 & -4 \\ -1 & s^2+4 \end{vmatrix}=s^2 \qquad \Delta_2=\begin{vmatrix} s^2+10 & 1 \\ -4 & -1 \end{vmatrix}=-s^2-6$$

$$X_1(s)=\frac{s^2}{s^4+14s^2+24}=\frac{As+B}{s^2+2}+\frac{Cs+D}{s^2+12}$$

$$\begin{cases} s^3 & A+C=0 \\ s^2 & B+D=1 \\ s & 12A+2C=0 \\ s^0 & 12B+2D=0 \end{cases} \qquad \boxed{A=0 \quad B=-\frac{1}{5} \quad C=0 \quad D=\frac{6}{5}}$$

$$\mathcal{L}^{-1}\left\{X_1(s)\right\}=\mathcal{L}^{-1}\left\{-\frac{1}{5\sqrt{2}}\frac{\sqrt{2}}{s^2+2}+\frac{6}{5\sqrt{12}}\frac{\sqrt{12}}{s^2+12}\right\}$$

$$\boxed{x_1(t)=-\frac{\sqrt{2}}{10}\sin\sqrt{2}t+\frac{\sqrt{3}}{5}\sin 2\sqrt{3}t}$$

$$X_2(s)=\frac{-s^2-6}{\left(s^2+2\right)\left(s^2+12\right)}=\frac{As+B}{s^2+2}+\frac{Cs+D}{s^2+12}$$

$$\begin{matrix} s^3 & A+C=0 \\ s^2 & B+D=-1 \\ s & 12A+2C=0 \\ s^0 & 12B+2D=-6 \end{matrix} \qquad \boxed{A=0 \quad B=-\frac{2}{5} \quad C=0 \quad D=-\frac{3}{5}}$$

$$\mathcal{L}^{-1}\left\{X_2(s)\right\}=\mathcal{L}^{-1}\left\{-\frac{2}{5\sqrt{2}}\frac{\sqrt{2}}{s^2+2}-\frac{3}{5\sqrt{12}}\frac{2\sqrt{3}}{s^2+12}\right\}$$

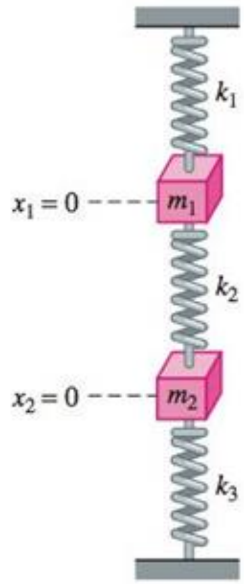
$$\boxed{x_1(t)=-\frac{\sqrt{2}}{5}\sin\sqrt{2}t-\frac{\sqrt{3}}{10}\sin 2\sqrt{3}t}$$

Exercise

Derive the system of differential equations describing the straight-line vertical motion of the coupled springs. Use the Laplace transform to solve the system when

$k_1 = 1,\ k_2 = 1,\ k_3 = 1,\ m_1 = 1,\ m_2 = 1$ and

$x_1(0) = 0,\ \ x_1'(0) = -1,\ \ x_2(0) = 0,\ \ x_2'(0) = 1$



Solution

$$x_1'' + 2x_1 - x_2 = 0 \qquad\qquad m_1 x_1'' = -k_1 x_1 + k_2\left(x_2 - x_1\right)$$

$$x_2'' + 2x_2 - x_1 = 0 \qquad\qquad m_2 x_2'' = -k_2\left(x_2 - x_1\right) - k_3 x_2$$

$$s^2 X_1(s) - s x_1(0) - x_1'(0) + 2X_1(s) - X_2(s) = 0$$

$$s^2 X_2(s) - s x_2(0) - x_2'(0) + 2X_2(s) - X_1(s) = 0$$

$$\left(s^2 + 2\right) X_1(s) - X_2(s) = -1$$

$$-X_1(s) + \left(s^2 + 2\right) X_2(s) = 1$$

$$\Delta = \begin{vmatrix} s^2 + 2 & -1 \\ -1 & s^2 + 2 \end{vmatrix} = \left(s^2 + 2\right)^2 - 1$$

$$\Delta_1 = \begin{vmatrix} -1 & -1 \\ 1 & s^2 + 2 \end{vmatrix} = -s^2 - 1 \qquad \Delta_2 = \begin{vmatrix} s^2 + 2 & -1 \\ -1 & 1 \end{vmatrix} = s^2 + 1$$

$$X_1(s) = \frac{-\left(s^2 + 1\right)}{s^4 + 4s^2 + 3}$$

$$= -\frac{s^2 + 1}{\left(s^2 + 1\right)\left(s^2 + 3\right)}$$

$$\mathcal{L}^{-1}\left\{X_1(s)\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{s^2 + 3}\right\}$$

$$\boxed{x_1(t) = -\sin\sqrt{3}t}$$

$$X_2(s) = \frac{s^2 + 1}{\left(s^2 + 1\right)\left(s^2 + 3\right)} = \frac{1}{s^2 + 3}$$

$$\mathcal{L}^{-1}\left\{X_1(s)\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{s^2 + 3}\right\}$$

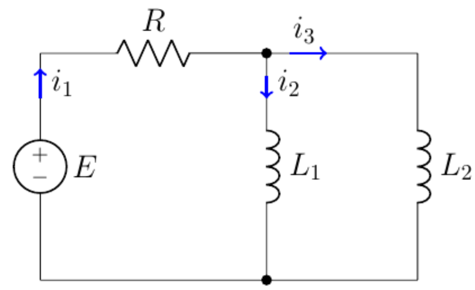
$$\boxed{x_1(t) = -\sin\sqrt{3}t}$$

Exercise

Solve the currents $i_1(t)$, $i_2(t)$ and $i_3(t)$ in the given electrical network.

Given $R = 5\ \Omega \quad L_1 = 0.01\ h, \quad L_2 = 0.0125\ h, \quad E = 100\ V$ and $i_2(0) = 0 \quad i_3(0) = 0$

Solution



$$i_1 = i_2 + i_3$$

$$\begin{cases} Ri_1 + L_1 i_2' = E(t) \\ Ri_1 + L_2 i_3' = E(t) \end{cases}$$

$$\begin{cases} Ri_2 + Ri_3 + L_1 i_2' = E(t) \\ Ri_2 + Ri_3 + L_2 i_3' = E(t) \end{cases}$$

$$\begin{cases} 5i_2 + 5i_3 + .01 i_2' = 100 \\ 5i_2 + 5i_3 + .0125 i_3' = 100 \end{cases}$$

$$\begin{cases} 5I_2(s) + 5I_3(s) + .01sI_2(s) - i_2(0) = 100 \\ 5I_2(s) + 5I_3(s) + .0125sI_3(s) - i_3(0) = 100 \end{cases}$$

$$\begin{cases} \left(5 + \frac{1}{100}s\right)I_2(s) + 5I_3(s) = \frac{100}{s} \\ 5I_2(s) + \left(5 + \frac{1}{80}s\right)I_3(s) = \frac{100}{s} \end{cases}$$

$$\begin{cases} \left(500s + s^2\right)I_2(s) + 500sI_3(s) = 10^4 \\ 400sI_2(s) + \left(400s + s^2\right)I_3(s) = 8\times 10^3 \end{cases}$$

$$\Delta = \begin{vmatrix} s^2 + 500s & 500s \\ 400s & s^2 + 400s \end{vmatrix} = s^4 + 900s^3 \qquad \Delta_2 = \begin{vmatrix} 10^4 & 500s \\ 8000 & s^2 + 400s \end{vmatrix} = 10^4 s^2$$

$$\Delta_3 = \begin{vmatrix} s^2 + 500s & 10^4 \\ 400s & 8\times 10^3 \end{vmatrix} = 8\times 10^3 s^2$$

$$I_2(s) = \frac{10^4}{s(s+900)} = 10^4\left(\frac{A}{s} + \frac{B}{s+900}\right)$$

$$\begin{array}{ll} s & A + B = 0 \\ s^0 & 900A = 1 \end{array} \quad \to \quad \boxed{A = \frac{1}{900}, \quad B = -\frac{1}{900}}$$

$$\mathcal{L}^{-1}\left\{I_2(s)\right\} = \mathcal{L}^{-1}\left\{\frac{100}{9}\frac{1}{s} - \frac{100}{9}\frac{1}{s+900}\right\}$$

$$\boxed{i_2(t) = \frac{100}{9} - \frac{100}{9}e^{-900t}}$$

$$I_3(s)=\frac{8,000}{s(s+900)}=\frac{C}{s}+\frac{D}{s+900}$$

$$\begin{array}{ll} s & C+D=0 \\ s^0 & 900C=8000 \end{array} \rightarrow \boxed{C=\frac{80}{9},\ \ D=-\frac{80}{9}}$$

$$\mathcal{L}^{-1}\left\{I_2(s)\right\}=\mathcal{L}^{-1}\left\{\frac{80}{9}\frac{1}{s}-\frac{80}{9}\frac{1}{s+900}\right\}$$

$$\boxed{i_3(t)=\frac{80}{9}-\frac{80}{9}e^{-900t}}$$

$$i_1(t)=i_2(t)+i_3(t)$$

$$\boxed{=20-20e^{-900t}}$$

Exercise

Find the charge on the capacitor $q(t)$ and the current $i_3(t)$ in the given electrical network.

Given: $R_1=1\,\Omega$, $R_2=1\,\Omega$, $L=1\,h$, $C=1\,f$ & $q(0)=0$, $i_3(0)=0$

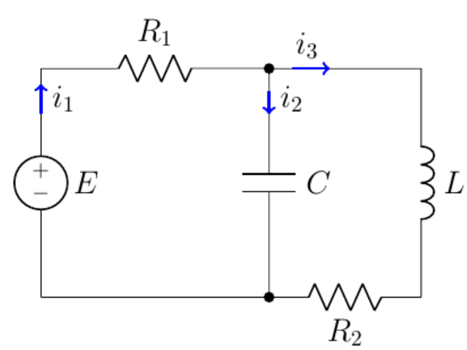
$$E(t)=\begin{cases} 0, & 0<t<1 \\ 50e^{-t}, & t\geq 1 \end{cases}$$

Solution

$$i_1=i_2+i_3 \qquad i_2=q'$$

$$\begin{cases} R_1i_1+\frac{1}{C}q=E(t) \\ R_1i_1+Li_3'+R_2i_3=E(t) \end{cases} \rightarrow R_1i_1=E(t)-\frac{1}{C}q$$

$$\begin{cases} R_1\left(q'+i_3\right)+\frac{1}{C}q=E(t) \\ E(t)-\frac{1}{C}q+Li_3'+R_2i_3=E(t) \end{cases}$$



$$\begin{cases} q'+q+i_3=E(t) \\ -q+i_3'+i_3=0 \end{cases}$$

$$E(t)=50e^{-t}u(t-1)=50e^{-1}e^{-(t-1)}u(t-1)$$

$$\begin{cases} sQ(s)-q(0)+Q(s)+I_3(s)=\frac{50}{e}\frac{e^{-s}}{s+1} \\ -Q(s)+sI_3(s)-i_{3(0)}+I_3(s)=0 \end{cases} \qquad q(0)=0,\ \ i_3(0)=0$$

$$\begin{cases}(s+1)Q(s)+I_3(s)=\frac{50}{e}\frac{e^{-s}}{s+1}\\ -Q(s)+(s+1)I_3(s)=0\end{cases}$$

$$\Delta=\begin{vmatrix}s+1 & 1\\ -1 & s+1\end{vmatrix}=s^2+2s+2 \quad \Delta_q=\begin{vmatrix}\frac{50}{e}\frac{e^{-s}}{s+1} & 1\\ 0 & s+1\end{vmatrix}=50e^{-s-1} \quad \Delta_{i_3}=\begin{vmatrix}s+1 & \frac{50}{e}\frac{e^{-s}}{s+1}\\ -1 & 0\end{vmatrix}=\frac{50}{e}\frac{e^{-s}}{s+1}$$

$$\mathcal{L}^{-1}\{Q(s)\}=\mathcal{L}^{-1}\left\{\frac{50e^{-1}e^{-s}}{(s+1)^2+1}\right\}$$

$$q(t)=50e^{-1}e^{-(t-1)}\sin(t-1)u(t-1)$$

$$\boxed{=50e^{-t}\sin(t-1)u(t-1)}$$

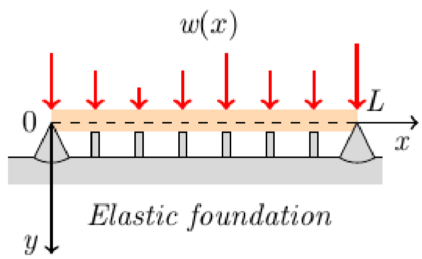
$$i_3(t)=q'=\left(50e^{-t}\cos(t-1)-50e^{-t}\sin(t-1)\right)u(t-1)$$

$$\boxed{=50e^{-t}\left(\cos(t-1)-\sin(t-1)\right)u(t-1)}$$

Exercise

When a uniform beam is supported by an elastic foundation, the differential equation for its deflection $y(x)$ is

$$EI\frac{d^4y}{dx^4}+ky=w(x)$$



Where k is the modulus of the foundation and $-ky$ is the restoring force of the foundation that acts in the direction opposite to that of the load $w(x)$. For algebraic convenience, suppose that the differential equation is written as

$$\frac{d^4y}{dx^4}+4a^4y=\frac{w(x)}{EI}$$

Where $a=\left(\frac{k}{4EI}\right)^{1/4}$. Assume $L=\pi$ and $a=1$. Find the deflection $y(x)$ of a beam that is supported on an elastic foundation when

a) The beam is simply supported at both ends and a constant load w_0 is uniformly distributed along its length,

b) The bean is embedded at both ends and $w(x)$ is concentrated load w_0 applied at $x=\frac{\pi}{2}$

Solution

a) $y(0)=y''(0)=0$ & $y(\pi)=y''(\pi)=0$

Let: $y'(0)=c_1 \quad y'(0)=c_2$

$$\mathcal{L}\left\{\frac{d^4y}{dx^4}+4y\right\}=\mathcal{L}\left\{\frac{w(x)}{EI}\right\}$$

$$s^4Y(s)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)+4Y(s)=\frac{w_0}{EI}\frac{1}{s}$$

$$\left(s^4+4\right)Y(s)=\frac{w_0}{EI}\frac{1}{s}+c_1s^2+c_2$$

$$Y(s)=\frac{w_0}{EI}\frac{1}{s\left(s^4+4\right)}+\frac{c_1s^2}{s^4+4}+\frac{c_2}{s^4+4}$$

$$\frac{1}{s\left(s^4+4\right)}=\frac{A}{s}+\frac{B(s-1)+C}{(s-1)^2+1}+\frac{D(s+1)+E}{(s+1)^2+1}$$

$$\begin{array}{ll} s^4 & A+B+D=0 \\ s^3 & B+C-D+E=0 \\ s^2 & 2C-2E=0 \\ s & -2B+2C+2D+2E=0 \\ s^0 & 4A=1 \end{array}$$

$$\boxed{A=\frac{1}{4},\ B=-\frac{1}{8},\ C=0,\ D=-\frac{1}{8},\ E=0}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{w_0}{EI}\left(\frac{1}{4}\frac{1}{s}-\frac{1}{8}\frac{s-1}{(s-1)^2+1}-\frac{1}{8}\frac{s+1}{(s+1)^2+1}\right)+\frac{c_1}{2}\frac{2s^2}{s^4+4}+\frac{c_2}{4}\frac{4}{s^4+4}\right\}$$

$$y(x)=\frac{w_0}{EI}\left(\frac{1}{4}-\frac{1}{8}e^{x}\cos x-\frac{1}{8}e^{-x}\cos x\right)+\frac{c_1}{2}(\sin x\cosh x+\cos x\sinh x)$$
$$+\frac{c_2}{4}(\sin x\cosh x-\cos x\sinh x)$$

$$y(x)=\frac{w_0}{4EI}(1-\cos x\cosh x)+\frac{c_1}{2}(\sin x\cosh x+\cos x\sinh x)+\frac{c_2}{4}(\sin x\cosh x-\cos x\sinh x)$$

$$y(\pi)=\frac{w_0}{4EI}(1+\cosh\pi)-\frac{1}{2}c_1\sinh\pi+\frac{1}{4}c_2\sinh\pi=0$$

$$\boxed{2c_1\sinh\pi-c_2\sinh\pi=\frac{w_0}{EI}(1+\cosh\pi)}$$

$$y'=\frac{w_0}{4EI}(\sin x\cosh x-\cos x\sinh x)+c_1\cos x\cosh x+\frac{1}{2}c_2\sin x\sinh x$$

$$y''=\frac{w_0}{2EI}\sin x\sinh x+c_1(-\sin x\cosh x+\cos x\sinh x)+\frac{1}{2}c_2(\cos x\sinh x+\sin x\cosh x)$$

$$y''(\pi)=-c_1\sinh\pi-\frac{1}{2}c_2\sinh\pi=0$$

$$\boxed{c_1=-\frac{1}{2}c_2}$$

$$2c_1 \sinh\pi + 2c_1 \sinh\pi = \frac{w_0}{EI}(1+\cosh\pi)$$

$$\boxed{c_1 = \frac{w_0}{4EI}(1+\cosh\pi)\operatorname{csch}\pi \quad c_2 = -\frac{w_0}{2EI}(1+\cosh\pi)\operatorname{csch}\pi}$$

$$y(x) = \frac{w_0}{4EI}(1-\cos x\cosh x) + \frac{w_0}{8EI}(1+\cosh\pi)\operatorname{csch}\pi(\sin x\cosh x + \cos x\sinh x)$$
$$- \frac{w_0}{4EI}(1+\cosh\pi)\operatorname{csch}\pi(\sin x\cosh x - \cos x\sinh x)$$

b) $\mathcal{L}\left\{\frac{d^4y}{dx^4} + 4y\right\} = \mathcal{L}\left\{\delta\left(t-\frac{\pi}{2}\right)\right\}$

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) + 4Y(s) = \frac{w_0}{EI}e^{-\pi s/2}$$

$$Y(s) = \frac{w_0}{4EI}\frac{4}{s^4+4}e^{-\pi s/2} + \frac{c_1}{2}\frac{2s^2}{s^4+4} + \frac{c_2}{4}\frac{4}{s^4+4}$$

$$y(x) = \frac{w_0}{4EI}\left(\sin\left(x-\frac{\pi}{2}\right)\cosh\left(x-\frac{\pi}{2}\right) - \cos\left(x-\frac{\pi}{2}\right)\sinh\left(x-\frac{\pi}{2}\right)\right)u\left(x-\frac{\pi}{2}\right)$$
$$+ \frac{c_1}{2}\sin x\sinh x + \frac{c_2}{4}(\sin x\cosh x - \cos x\sinh x)$$

$$y(\pi) = \frac{w_0}{4EI}\cosh\frac{\pi}{2} + \frac{c_2}{4}\sinh\pi = 0 \quad \to \quad \boxed{c_2 = -\frac{w_0}{EI}\frac{\cosh\frac{\pi}{2}}{\sinh\pi}}$$

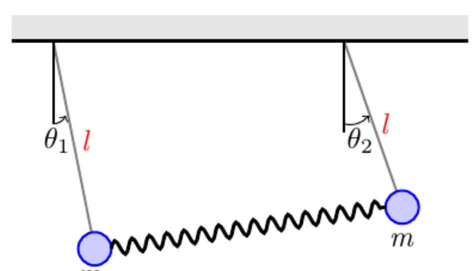
$$\boxed{c_1 = \frac{w_0}{EI}\frac{\sinh\frac{\pi}{2}}{\sinh\pi}}$$

$$y(x) = \frac{w_0}{4EI}\left(\sin\left(x-\frac{\pi}{2}\right)\cosh\left(x-\frac{\pi}{2}\right) - \cos\left(x-\frac{\pi}{2}\right)\sinh\left(x-\frac{\pi}{2}\right)\right)u\left(x-\frac{\pi}{2}\right)$$
$$+ \frac{w_0}{2EI}\frac{\sinh\frac{\pi}{2}}{\sinh\pi}\sin x\sinh x - \frac{w_0}{4EI}\frac{\cosh\frac{\pi}{2}}{\sinh\pi}(\sin x\cosh x - \cos x\sinh x)$$

Exercise

Suppose two identical pendulums are coupled by means of a spring with constant k. when the displacement angles $\theta_1(t)$ and $\theta_2(t)$ are small, the system of linear differential equations describing the motion is

$$\begin{cases} \theta_1'' + \frac{g}{l}\theta_1 = -\frac{k}{m}(\theta_1 - \theta_2) \\ \theta_2'' + \frac{g}{l}\theta_2 = \frac{k}{m}(\theta_1 - \theta_2) \end{cases}$$



a) Use Laplace transform to solve the system when

$$\theta_1'(0)=0 \quad \theta_1(0)=\theta_0 \quad \theta_2'(0)=0 \quad \theta_2(0)=\psi_0$$

Where θ_0 and ψ_0 constants. Let $\omega^2=\frac{g}{l},\ K=\frac{k}{m}$

b) Use the solution in part (*a*) to discuss the motion of the coupled pendulums in the special case when the initial conditions are $\theta_1'(0)=0,\quad \theta_1(0)=\theta_0,\quad \theta_2'(0)=\theta_0,\quad \theta_2(0)=0$

c) Use the solution in part (*a*) to discuss the motion of the coupled pendulums in the special case when the initial conditions are $\theta_1'(0)=0,\quad \theta_1(0)=\theta_0,\quad \theta_2'(0)=-\theta_0,\quad \theta_2(0)=0$

Solution

a) $$\begin{cases}\theta_1''+\omega^2\theta_1=-K\theta_1+K\theta_2\\ \theta_2''+\omega^2\theta_2=K\theta_1-K\theta_2\end{cases}$$

$$\begin{cases}s^2\theta_1(s)-s\theta_1(0)-\theta_1'(0)+\omega^2\theta_1(s)+K\theta_1(s)=+K\theta_2(s)\\ s^2\theta_2(s)-s\theta_2(0)-\theta_2'(0)+\omega^2\theta_2(s)+K\theta_2(s)=K\theta_1(s)\end{cases}$$

$$\begin{cases}\left(s^2+\omega^2+K\right)\theta_1(s)-K\theta_2(s)=s\theta_0\\ -K\theta_1(s)+\left(s^2+\omega^2+K\right)\theta_2(s)=s\psi_0\end{cases}$$

$$\Delta=\begin{vmatrix}s^2+\omega^2+K & -K\\ -K & s^2+\omega^2+K\end{vmatrix}=\left(s^2+\omega^2+K\right)^2-K^2=\left(s^2+\omega^2\right)\left(s^2+\omega^2+2K\right)$$

$$\Delta_1=\begin{vmatrix}s\theta_0 & -K\\ s\psi_0 & s^2+\omega^2+K\end{vmatrix}=s^3\theta_0+\left(\omega^2\theta_0+K\theta_0+K\psi_0\right)s$$

$$\Delta_2=\begin{vmatrix}s^2+\omega^2+K & s\theta_0\\ -K & s\psi_0\end{vmatrix}=s^3\psi_0+\left(\omega^2\psi_0+K\psi_0+K\theta_0\right)s$$

$$\theta_1(s)=\frac{s^3\theta_0+\left(\omega^2\theta_0+K\theta_0+K\psi_0\right)s}{\left(s^2+\omega^2\right)\left(s^2+\omega^2+2K\right)}=\frac{As+B}{s^2+\omega^2}+\frac{Cs+D}{s^2+\left(\omega^2+2K\right)}$$

$$\begin{cases}s^3 & A+C=\theta_0\\ s^2 & B+D=0\\ s & \left(\omega^2+2K\right)A+\omega^2C=\omega^2\theta_0+K\theta_0+K\psi_0\\ s^0 & \left(\omega^2+2K\right)B+\omega^2D=0\end{cases}$$

$$\left(\omega^2+2K\right)A+\omega^2\left(\theta_0-A\right)=\omega^2\theta_0+K\theta_0+K\psi_0$$

$$2KA = K\theta_0 + K\psi_0 \quad \rightarrow \quad \boxed{A = \frac{1}{2}\left(\theta_0 + \psi_0\right)}$$

$$C = \theta_0 - A \quad \rightarrow \quad \boxed{C = \frac{1}{2}\left(\theta_0 - \psi_0\right)}$$

$$\boxed{B = D = 0}$$

$$\left\{\theta_1(s)\right\} = \frac{1}{2}\left(\theta_0 + \psi_0\right)\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} + \frac{1}{2}\left(\theta_0 - \psi_0\right)\mathcal{L}^{-1}\frac{s}{s^2 + \left(\sqrt{\omega^2 + 2K}\right)^2}$$

$$\boxed{\theta_1(t) = \frac{1}{2}\left(\theta_0 + \psi_0\right)\cos\omega t + \frac{1}{2}\left(\theta_0 - \psi_0\right)\cos\sqrt{\omega^2 + 2K}\, t}$$

$$\theta_2(s) = \frac{s^3\psi_0 + \left(\omega^2\psi_0 + K\psi_0 + K\theta_0\right)s}{\left(s^2+\omega^2\right)\left(s^2+\omega^2+2K\right)} = \frac{as+b}{s^2+\omega^2} + \frac{cs+d}{s^2+\left(\omega^2+2K\right)}$$

$$\begin{cases} s^3 & a + c = \psi_0 \\ s^2 & b + d = 0 \\ s & \left(\omega^2 + 2K\right)a + \omega^2 c = \omega^2\psi_0 + K\psi_0 + K\theta_0 \\ s^0 & \left(\omega^2 + 2K\right)b + \omega^2 d = 0 \end{cases}$$

$$\left(\omega^2 + 2K\right)a + \omega^2\left(\psi_0 - a\right) = \omega^2\psi_0 + K\psi_0 + K\theta_0$$

$$2Ka = K\theta_0 + K\psi_0 \quad \rightarrow \quad \boxed{a = \frac{1}{2}\left(\theta_0 + \psi_0\right)}$$

$$C = \psi_0 - A \quad \rightarrow \quad \boxed{C = -\frac{1}{2}\left(\theta_0 - \psi_0\right)}$$

$$\boxed{b = d = 0}$$

$$\boxed{\theta_2(t) = \frac{1}{2}\left(\theta_0 + \psi_0\right)\cos\omega t - \frac{1}{2}\left(\theta_0 - \psi_0\right)\cos\sqrt{\omega^2 + 2K}\, t}$$

b) $\theta_1'(0) = 0, \quad \theta_1(0) = \theta_0, \quad \theta_2'(0) = \theta_0, \quad \theta_2(0) = 0$

$$\Rightarrow \quad \boxed{\psi_0 = \theta_0}$$

$$\boxed{\theta_1(t) = \theta_0 \cos\omega t} \qquad \& \qquad \boxed{\theta_2(t) = \theta_0 \cos\omega t}$$

∴ This means that both pendulums swing in the same direction (free) and the spring exerts no influemce on the motion.

c) $\theta_1'(0) = 0, \quad \theta_1(0) = \theta_0, \quad \theta_2'(0) = -\theta_0, \quad \theta_2(0) = 0$

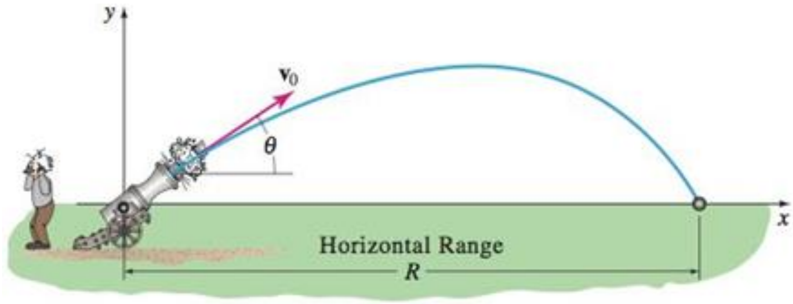
$$\Rightarrow \quad \boxed{\psi_0 = -\theta_0}$$

$$\boxed{\theta_1(t) = \theta_0 \cos\sqrt{\omega^2 + 2K}\ t} \quad \& \quad \boxed{\theta_2(t) = -\theta_0 \cos\sqrt{\omega^2 + 2K}\ t}$$

$\therefore$ This means that both pendulums swing in the opposite directions, stretching and compressing the spring. The amplitude of both displacements is $\left|\theta_0\right|$. Which the psring is stretched to its maximum.

Exercise

A projectile, such as the canon ball, has weight $w = mg$ and initial velocity $\boldsymbol{v}_0$ that is tangent to its path of motion.



If air resistance and all other forces except its weight are ignored, that motion of the projectile is describe by the system of linear differential equations:

$$\begin{cases} m\dfrac{d^2x}{dt^2} = 0 \\ m\dfrac{d^2y}{dt^2} = -mg \end{cases}$$

a) Use Laplace transform to solve the system when

$$x(0) = 0 \quad x'(0) = v_0 \cos\theta \quad y(0) = 0 \quad y'(0) = v_0 \sin\theta$$

Where $v_0 = |v|$ is constant and θ is the constant angle of elevation.

The solutions $x(t)$ and $y(t)$ are parametric equations of the trajectory of the projectile.

b) Use $x(t)$ in part (*a*) to eliminate the parameter *t* in $y(t)$. Use the resulting equation for *y* to show that the horizontal range *R* of the projectile is given by

$$R = \frac{v_0^2}{g}\sin 2\theta$$

c) From the formula in part (*b*), we see that *R* is a maximum when $\sin 2\theta = 1$ or when $\theta = \frac{\pi}{4}$. Show that the same range – less than the maximum– can be obtained by firing the gun at either of two complementary angles θ and $\frac{\pi}{2} - \theta$. The only difference is that the smaller angle results in a low trajectory whereas the larger angle fives a high trajectory.

d) Suppose $g = 32\ ft/s^2$, $\theta = 30°$, and $v_0 = 300\ ft/s$. Use part (*b*) to find the horizontal range of the projectile.

e) Find the time when the projectile hits the ground.

f) Use the parametric equations $x(t)$ and $y(t)$ in part (*a*) along with the numerical data in part (*d*) to plot the ballistic curve of the projectile.

g) Repeat with $\theta = 52°$ and $v_0 = 300\ ft/s$.

h) Superimpose both curves part (*f*) & (*g*) on the same coordinate system.

Solution

a)
$$\frac{d^2x}{dt^2} = 0 \qquad \frac{d^2y}{dt^2} = -g \quad \to \quad \begin{cases} s^2X(s) - sx(0) - x'(0) = 0 \\ s^2Y(s) - sy(0) - y'(0) = -\frac{g}{s} \end{cases}$$

$$x(0) = 0 \quad x'(0) = v_0\cos\theta \quad y(0) = 0 \quad y'(0) = v_0\sin\theta$$

$$X(s) = v_0\cos\theta\frac{1}{s^2}$$

$$\boxed{x(t) = (v_0\cos\theta)t}$$

$$Y(s) = v_0\sin\theta\frac{1}{s^2} - \frac{g}{s^3}\ \text{150t}$$

$$\boxed{y(t) = (v_0\sin\theta)t - \frac{1}{2}gt^2}$$

b) $t = \dfrac{x}{v_0\cos\theta}$

$$\begin{aligned} y(x) &= (v_0\sin\theta)t - \frac{1}{2}gt^2 \\ &= (v_0\sin\theta)\frac{x}{v_0\cos\theta} - \frac{1}{2}g\frac{x^2}{v_0^2\cos^2\theta} \\ &= \frac{1}{2v_0^2\cos^2\theta}\left(2v_0^2\cos\theta\sin\theta x - gx^2\right) \\ &= \frac{1}{2v_0^2\cos^2\theta}\left(v_0^2\sin 2\theta - gx\right)x = 0 \end{aligned}$$

At $x = y = 0$, the projectile hits the ground.

$$v_0^2\sin 2\theta - gx = 0$$

$$x = \boxed{R(\theta) = \frac{1}{g}v_0^2\sin 2\theta}$$

c)
$$\begin{aligned} R\left(\frac{\pi}{2} - \theta\right) &= \frac{1}{g}v_0^2\sin(\pi - 2\theta) \\ &= \frac{1}{g}v_0^2\sin 2\theta \\ &= R(\theta) \end{aligned}$$

$$\sin(\pi - \alpha) = \sin\pi\cos\alpha - \cos\pi\sin\alpha$$

d) Given: $g = 32\ ft/s^2$, $\theta = 30°$, and $v_0 = 300\ ft/s$

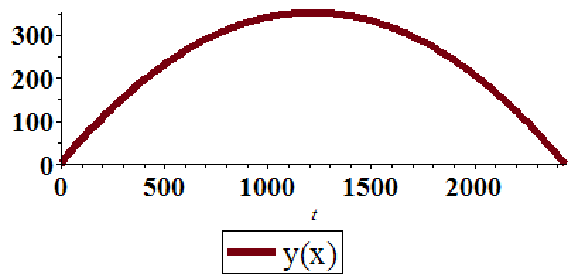
$$R(30°) = \frac{1}{32}(300)^2 \sin 60° \approx 2,436\ ft$$

e) $x = (v_0 \cos\theta)t = 2,436$

$$t = \frac{2,436}{300\cos 30°} \approx 9.38\ sec$$

f) $$y(x) = \frac{1}{2v_0^2\cos^2\theta}\left(\left(v_0^2 \sin 2\theta\right)x - gx^2\right)$$

$$= 0.57735x - 0.000237x^2$$



g) Given: $g = 32\ ft/s^2$, $\theta = 52°$, and $v_0 = 300\ ft/s$

$$R(30°) = \frac{1}{32}(300)^2 \sin 104° \approx 2729\ ft$$

h) $$y(x) = \frac{1}{2v_0^2\cos^2\theta}\left(\left(v_0^2 \sin 2\theta\right)x - gx^2\right)$$

$$= 1.2799x - 0.000469x^2$$