Lecture One – First Order Equations

Section 1.1 – Differential Equations & Solutions

Ordinary Differential Equations

Involve an unknown function of a single variable with one or more of its derivatives.

$$\frac{dy}{dt} = y - t$$

y: $y(t)$ is unknown function

t: independent variable

Some other example:

$$y' = y^2 - t$$
$$ty' = y$$
$$y' + 4y = e^{-3t}$$
$$yy'' + t^2 y = \cos t$$
$$y' = \cos(ty)$$

$\therefore$ The order of a differential equation is the order of the highest derivative that occurs in the equation.

y'' : *second order*

$$\frac{\partial^2 \omega}{\partial t^2} = c^2 \frac{\partial^2 \omega}{\partial x^2}$$ is not an ODE (ω is dependent on x and t)

This equation is called a ***partial differential equation***.

Definition

A first-order differential equation of the form $\frac{dy}{dt} = y' = f(t, y)$ is said to be in normal form.

$y^{(n)} = f\left(t, y, y', \dots, y^{(n-1)}\right)$ is said to be in normal form.

f: is a given function of 2 variables t & y (***rate function***)

Solutions

A solution of the first-order, ordinary differential equation $f(t, y, y') = 0$ is a differentiable function $y(t)$ such that $f(t, y(t), y'(t)) = 0$ for all t in the interval where $y(t)$ is defined.

1. Can be found in explicit and implicit form by applying manipulation (integration)
2. No real solution.

Example

Show that $y(t) = Ce^{-t^2}$ is a solution of the 1st order equation $y' = -2ty$

Solution

$$y(t) = Ce^{-t^2} \Rightarrow y' = -2tCe^{-t^2}$$

$$y' = -2tCe^{-t^2}$$

$$y' = -2t\, y(t) \qquad \text{True; it is a solution}$$

$y(t)$ is called the ***general solution***.

The solutions from the graph are called ***solution curves***.

Example

Is the function $y(t) = \cos t$ a solution to the differential equation $y' = 1 + y^2$

Solution

$$y' = -\sin t$$

$$y' = 1 + y^2 = -\sin t$$

$$1 + \cos^2 t \stackrel{?}{=} -\sin t \qquad \text{False; it is not a solution.}$$

Exercises *Section* 1.1 – Differential Equations & Solutions

1. Show that $y(t)=Ce^{-(1/2)t^2}$ is a solution of the 1st order equation $y'=-ty$ *for* $-3\le C\le 3$

2. Show that $y(t)=\dfrac{4}{1+Ce^{-4t}}$ is a solution of the 1st order equation $y'=y(4-y)$

3. Show that $y(x)=x^{-3/2}$ is a solution of $4x^2y''+12xy'+3y=0$ for $x>0$

4. A general solution may fail to produce all solutions of a differential equation $y(t)=\dfrac{4}{1+Ce^{-4t}}$. Show that $y=0$ is a solution of the differential equation, but no value of *C* in the given general solution will produce this solution.

5. Use the given general solution to find a solution of the differential equation having the given initial condition. $ty'+y=t^2, \quad y(t)=\frac{1}{3}t^2+\frac{C}{t}, \quad y(1)=2$

6. Show that $y(t)=2t-2+Ce^{-t}$ is a solution of the 1st order equation $y'+y=2t$ *for* $-3\le C\le 3$

7. Use the given general solution to find a solution of the differential equation having the given initial condition. $y'+4y=\cos t, \quad y(t)=\frac{4}{17}\cos t+\frac{1}{17}\sin t+Ce^{-4t}, \quad y(0)=-1$

8. Use the given general solution to find a solution of the differential equation having the given initial condition. $ty'+(t+1)y=2te^{-t}, \quad y(t)=e^{-t}\left(t+\frac{C}{t}\right), \quad y(1)=\frac{1}{e}$

9. Use the given general solution to find a solution of the differential equation having the given initial condition. $y'=y(2+y), \quad y(t)=\dfrac{2}{-1+Ce^{-2t}}, \quad y(0)=-3$

10. Find the values of ***m*** so that the function $y=e^{mx}$ is a solution of the given differential equation

a) $y'+2y=0$

b) $5y'-2y=0$

c) $y''-5y'+6y=0$

d) $2y''+7y'-4y=0$

11. Let $x=c_1\cos t+c_2\sin t$ is 2-parameter family solutions of the second order differential equation of $x''+x=0$. Find a solution of the second-order consisting of this differential equation and the given initial conditions.

a) $x(0)=-1, \quad x'(0)=8$

b) $x\left(\frac{\pi}{2}\right)=0, \quad x'\left(\frac{\pi}{2}\right)=1$

c) $x\left(\frac{\pi}{6}\right)=\frac{1}{2}, \quad x'\left(\frac{\pi}{6}\right)=0$

d) $x\left(\frac{\pi}{4}\right)=\sqrt{2}, \quad x'\left(\frac{\pi}{4}\right)=2\sqrt{2}$

12. Find values of *r* such that $y(x)=x^r$ is a solution of $x^2y''-4xy'+6y=0$

Solve the differential equation:

13. $y' = 3x^2 - 2x + 4$

14. $y'' = 2x + \sin 2x$

15. Given the differential equation $x^2 y'' - 2xy' + 2y = 4x^3$, is the given equation a solution?

a) $y = 2x^3 + x^2$

b) $y = 2x + x^2$