Solution Section 1.1 – Idea / Definition of Limits

Exercise

Find the average rate of change of the function $f(x)=x^3+1 \quad 2\le x\le 3$

Solution

$$\frac{\Delta f}{\Delta x}=\frac{f(3)-f(2)}{3-2} \qquad\qquad \frac{\Delta y}{\Delta x}=\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

$$=\frac{3^3+1-\left(2^3+1\right)}{1}$$

$$=27+1-(8+1)$$

$$\underline{=19}$$

Exercise

Find the average rate of change of the function $f(x)=x^2 \quad -1\le x\le 1$

Solution

$$\frac{\Delta f}{\Delta x}=\frac{f(1)-f(-1)}{1-(-1)} \qquad\qquad \frac{\Delta y}{\Delta x}=\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

$$=\frac{1^2-(-1)^2}{2}$$

$$=\frac{0}{2}$$

$$\underline{=0}$$

Exercise

Find the average rate of change of the function $f(x)=x^2+1 \quad 0\le x\le 2$

Solution

$$\frac{\Delta f}{\Delta x}=\frac{f(2)-f(0)}{2-(0)} \qquad\qquad \frac{\Delta y}{\Delta x}=\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

$$=\frac{5-1}{2}$$

$$\underline{=2}$$

Exercise

Find the average rate of change of the function $f(x)=x^2+1 \quad -2 \le x \le 2$

Solution

$$\frac{\Delta f}{\Delta x}=\frac{f(2)-f(-2)}{2-(-2)} \qquad \frac{\Delta y}{\Delta x}=\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

$$=\frac{5-5}{4}$$

$$=0$$

Exercise

Find the average rate of change of the function $f(x)=x^3+2 \quad 0 \le x \le 2$

Solution

$$\frac{\Delta f}{\Delta x}=\frac{f(2)-f(0)}{2-(0)} \qquad \frac{\Delta y}{\Delta x}=\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

$$=\frac{10-2}{2}$$

$$=4$$

Exercise

Find the average rate of change of the function $f(x)=\sqrt{x+1} \quad 0 \le x \le 3$

Solution

$$\frac{\Delta f}{\Delta x}=\frac{f(3)-f(0)}{3-(0)} \qquad \frac{\Delta y}{\Delta x}=\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

$$=\frac{2-1}{3}$$

$$=\frac{1}{3}$$

Exercise

Find the average rate of change of the function $f(x)=\frac{1}{x} \quad 1 \le x \le 2$

Solution

$$\frac{\Delta f}{\Delta x} = \frac{f(2) - f(1)}{2 - (1)} \qquad\qquad \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{\frac{1}{2} - 1}{1}$$

$$= -\frac{1}{2}$$

Exercise

Find the average rate of change of the function $f(t) = \cos t \quad 0 \le t \le \frac{\pi}{2}$

Solution

$$\frac{\Delta f}{\Delta t} = \frac{f\left(\frac{\pi}{2}\right) - f(0)}{\frac{\pi}{2} - 0} \qquad\qquad \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{0 - 1}{\frac{\pi}{2}}$$

$$= -\frac{2}{\pi}$$

Exercise

Find the average rate of change of the function $f(t) = \sin t \quad 0 \le t \le \frac{\pi}{2}$

Solution

$$\frac{\Delta f}{\Delta t} = \frac{f\left(\frac{\pi}{2}\right) - f(0)}{\frac{\pi}{2} - 0} \qquad\qquad \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{1 - 0}{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}$$

Exercise

Find the average rate of change of the function $f(t) = 2 + \cos t \quad -\pi \le t \le \pi$

Solution

$$\frac{\Delta f}{\Delta t}=\frac{f(\pi)-f(-\pi)}{\pi-(-\pi)}$$

$$\frac{\Delta y}{\Delta x}=\frac{f\left(x_2\right)-f\left(x_1\right)}{x_2-x_1}$$

$$=\frac{2+\cos\pi-(2+\cos(-\pi))}{2\pi}$$

$$=\frac{2-1-(2-1)}{2}$$

$$=0$$

Exercise

Find the slope of $y=x^2-3$ @ $P(2,\ 1)$ and an equation of the tangent line at this P.

Solution

$$\frac{\Delta y}{\Delta x}=\frac{f(2+h)-f(2)}{h}$$

$$\frac{\Delta y}{\Delta x}=\frac{f\left(x_1+h\right)-f\left(x_1\right)}{h}$$

$$=\frac{(2+h)^2-3-\left(2^2-3\right)}{h}$$

$$=\frac{4+4h+h^2-3-(4-3)}{h}$$

$$=\frac{4h+h^2}{h}$$

$$=4+h$$

As h approaches 0. Then the tangent slope $h+4\to 4=slope$

$$y=4(x-2)+1$$

$$y=m\left(x-x_1\right)+y_1$$

$$y-1+1=4x-8+1$$

$$y=4x-7$$

Exercise

Find the slope of $y=x^2-2x-3$ @ $P(2,\ -3)$ and an equation of the tangent line at this P.

Solution

$$\frac{\Delta y}{\Delta x}=\frac{f(2+h)-f(2)}{h}$$

$$\frac{\Delta y}{\Delta x}=\frac{f\left(x_1+h\right)-f\left(x_1\right)}{h}$$

$$= \frac{(2+h)^2 - 2(2+h) - 3 - \left(2^2 - 2(2) - 3\right)}{h}$$

$$= \frac{4 + 4h + h^2 - 4 - 2h - 3 - (-3)}{h}$$

$$= \frac{2h + h^2}{h}$$

$$= 2 + h$$

As h approaches 0. Then the tangent slope $2 + h \to 2 = slope$

$y + 3 = 2(x - 2)$ $\qquad y = m\left(x - x_1\right) + y_1$

$y = 2x - 4 - 3$

$y = 2x - 7$

Exercise

Find the slope of $y = x^3$ @ $P(2,\ 8)$ and an equation of the tangent line at this *P*.

Solution

$$\frac{\Delta y}{\Delta x} = \frac{f(2+h) - f(2)}{h} \qquad \frac{\Delta y}{\Delta x} = \frac{f\left(x_1 + h\right) - f\left(x_1\right)}{h}$$

$$= \frac{(2+h)^3 - 2^3}{h}$$

$$= \frac{8 + 12h + 6h^2 + h^3 - 8}{h}$$

$$= 12 + 6h + h^2$$

As h approaches 0, the $slope = 12$

$y = 12x - 24 + 8$ $\qquad y = m\left(x - x_1\right) + y_1$

$y = 12x - 16$

Exercise

Find the slope of $y = 2 - \sin x$ @ $P(0,\ 2)$ and an equation of the tangent line at this *P*.

Solution

$$\frac{\Delta y}{\Delta x}=\frac{f(0+h)-f(0)}{h} \qquad\qquad \frac{\Delta y}{\Delta x}=\frac{f\left(x_1+h\right)-f\left(x_1\right)}{h}$$

$$=\frac{2-\sin(h)-2}{h}$$

$$=-\frac{\sin(h)}{h}$$

As h approaches 0, next section we will introduce that $\frac{\sin(h)}{h}=1$

That imply the slope $=-1$

$$y=-(x-0)+2 \qquad\qquad y=m\left(x-x_1\right)+y_1$$

$$y=-x+2$$

Exercise

Find the slope of $y=x^2+1$ @ $x=1$ and an equation of the tangent line at this *P*.

Solution

$$\frac{\Delta y}{\Delta x}=\frac{f(1+h)-f(1)}{h} \qquad\qquad \frac{\Delta y}{\Delta x}=\frac{f\left(x_1+h\right)-f\left(x_1\right)}{h}$$

$$=\frac{(1+h)^2+1-2}{h}$$

$$=\frac{h^2+2h+1-1}{h}$$

$$=\frac{h^2+2h}{h}$$

$$=h+2$$

As h approaches 0, that imply the slope $=2$

$$y=2(x-1)+2 \qquad\qquad y=m\left(x-x_1\right)+y_1$$

$$y=2x$$

Exercise

Find the slope of $y=\frac{1}{x}$ @ $x=1$ and an equation of the tangent line at this *P*.

Solution

$$\frac{\Delta y}{\Delta x} = \frac{f(1+h)-f(1)}{h} \qquad\qquad \frac{\Delta y}{\Delta x} = \frac{f(x_1+h)-f(x_1)}{h}$$

$$= \frac{\frac{1}{1+h}-1}{h}$$

$$= \frac{1-1-h}{h(h+1)}$$

$$= -\frac{h}{h(h+1)}$$

$$= -\frac{1}{h+1}$$

As h approaches 0, that imply the slope $= -1$

$$y = 2(x-1)+2 \qquad\qquad y = m(x-x_1)+y_1$$

$$y = 2x$$

Exercise

Find the slope of $y = \sin x$ @ $x = 0$ and an equation of the tangent line at this *P*.

Solution

$$f(0) = \sin 0 = 0$$

$$\frac{\Delta y}{\Delta x} = \frac{f(0+h)-f(0)}{h} \qquad\qquad \frac{\Delta y}{\Delta x} = \frac{f(x_1+h)-f(x_1)}{h}$$

$$= \frac{\sin(h)}{h}$$

As h approaches 0, next section we will introduce that $\frac{\sin(h)}{h} = 1$

That imply the slope $= 1$

$$y = (x-0)+0 \qquad\qquad y = m(x-x_1)+y_1$$

$$y = x$$

Exercise

Find the slope of $y = 2 - \cos x$ @ $x = -\pi$ and an equation of the tangent line at this *P*.

Solution

$f(-\pi)=2-\cos(-\pi) \ =3$

$$\frac{\Delta y}{\Delta x}=\frac{f(-\pi+h)-f(-\pi)}{h} \qquad \frac{\Delta y}{\Delta x}=\frac{f(x_1+h)-f(x_1)}{h}$$

$$=\frac{2-\cos(h-\pi)-3}{h}$$

$$=-\frac{1+\cos(h-\pi)}{h}$$

$$=-\frac{1+\cos(h)\cos\pi+\sin(h)\sin\pi}{h}$$

$$=-\frac{1-\cos(h)}{h}$$

$$\boxed{=-\frac{1}{h}(1-\cos(h))}$$

As h approaches 0, $1-\cos(h)=0$ that imply the slope $=0$

$y=0(x+\pi)+3 \qquad y=m(x-x_1)+y_1$

$\boxed{y=3}$

Exercise

Evaluate the limit using the form $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ for $\quad f(x)=x^2, \quad x=1$

Solution

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{(1+h)^2-1}{h}$$

$$=\lim_{h\to 0}\frac{1+2h+h^2-1}{h}$$

$$=\lim_{h\to 0}\left(\frac{2h}{h}+\frac{h^2}{h}\right)$$

$$=\lim_{h\to 0}(2+h)$$

$$\boxed{=2}$$

Exercise

Evaluate the limit using the form $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ for $y=x^2-1$ $x=-1$

Solution

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(-1+h)^2-1-0}{h}$$

$$=\lim_{h\to 0}\frac{1-2h+h^2-1}{h}$$

$$=\lim_{h\to 0}\left(-\frac{2h}{h}+\frac{h^2}{h}\right)$$

$$=\lim_{h\to 0}(-2+h)$$

$$=-2$$

Exercise

Evaluate the limit using the form $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ for $f(x)=\sqrt{3x+1}$, $x=0$

Solution

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{3(x+h)+1}-\sqrt{3x+1}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{3x+3h+1}-\sqrt{3x+1}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{3x+3h+1}-\sqrt{3x+1}}{h}\bullet\frac{\sqrt{3x+3h+1}+\sqrt{3x+1}}{\sqrt{3x+3h+1}+\sqrt{3x+1}}$$

$$=\lim_{h\to 0}\frac{3x+3h+1-(3x+1)}{h\left(\sqrt{3x+3h+1}+\sqrt{3x+1}\right)}$$

$$=\lim_{h\to 0}\frac{3x+3h+1-3x-1}{h\left(\sqrt{3x+3h+1}+\sqrt{3x+1}\right)}$$

$$=\lim_{h\to 0}\frac{3h}{h\left(\sqrt{3x+3h+1}+\sqrt{3x+1}\right)}$$

$$=\lim_{h\to 0}\frac{3}{\sqrt{3x+3h+1}+\sqrt{3x+1}}$$

$$=\frac{3}{\sqrt{3(0)+1}+\sqrt{3(0)+1}}\qquad \textit{Given}: x=0$$

$$=\frac{3}{2}$$

Exercise

Evaluate the limit using the form $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ for $\quad f(x)=\dfrac{1}{x+1},\quad x=0$

Solution

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{h+1}-1\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{1-h-1}{h+1}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{-h}{h+1}\right)$$

$$=-\lim_{h\to 0}\left(\frac{1}{h+1}\right)$$

$$=-1$$

Exercise

Make a table of values for the function $f(x)=\dfrac{x+2}{x-2}$ at the points

$$x=1.2,\quad x=\frac{11}{10},\quad x=\frac{101}{100},\quad x=\frac{1001}{1000},\quad x=\frac{10001}{10000},\quad and\ x=1$$

a) Find the average rate of change of $f(x)$ over the intervals $[1,\ x]$ for each $x\neq 1$ in the table

b) Extending the table if necessary, try to determine the rate of change of $f(x)$ at $x=1$.

Solution

a)

x	1.2	1.1	1.01	1.001	1.0001	1
$f(x)$	-4.0	$-3.\overline{4}$	$-3.\overline{04}$	$-3.\overline{004}$	$-3.\overline{004}$	-3

$$\frac{\Delta y}{\Delta x}=\frac{-4-(-3)}{1.2-1}=-5.0$$

$$\frac{\Delta y}{\Delta x}=\frac{-3.\overline{4}-(-3)}{1.1-1}=-4.\overline{4}$$

$$\frac{\Delta y}{\Delta x}=\frac{-3.\overline{04}-(-3)}{1.01-1}=-4.\overline{04}$$

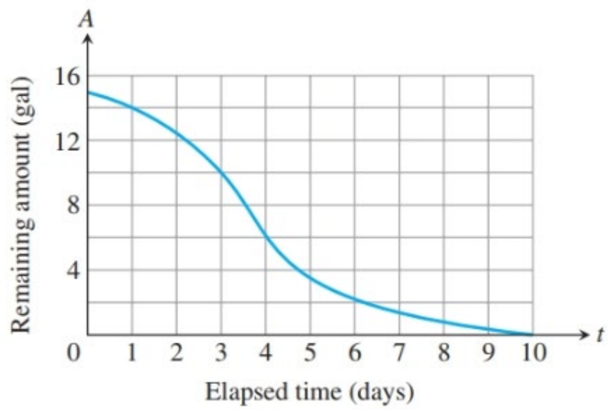
$$\frac{\Delta y}{\Delta x}=\frac{-3.\overline{004}-(-3)}{1.001-1}=-4.\overline{004}$$

$$\frac{\Delta y}{\Delta x} = \frac{-3.\overline{0004} - (-3)}{1.0001 - 1} = -4.\overline{0004}$$

b) The rate of change of $f(x)$ at $x = 1$ is $\underline{-4}$

Exercise

The accompanying graph shows the total amount of gasoline A in the gas tank of an automobile after being driven for t days.



a) Estimate the average rate of gasoline consumption over the time intervals

$[0,\ 3],\ [0,\ 5],\ and\ [7,\ 10]$

b) Estimate the instantaneous rate of gasoline consumption over the time $t = 1,\ t = 4,\ and\ t = 8$

Solution

a) Average rate of gasoline consumption over the time intervals:

$[0,\ 3]$

$$Average\ Rate = \frac{10-15}{3-0}$$

$$= -\frac{5}{3}$$

$$\approx -1.67\ gal/day$$

$[0,\ 5]$

$$Average\ Rate = \frac{3.9-15}{3-0}$$

$$\approx -2.2\ gal/day$$

$[7,\ 10]$

$$Average\ Rate = \frac{0-1.4}{10-7}$$

$$= -\frac{1.4}{3}$$

$$= -\frac{7}{15}$$

$$\boxed{\approx -0.5 \ \ gal/day}$$

b) At $t=1 \quad \to P(1,\ 14)$

At $t=4 \quad \to P(4,\ 6)$

At $t=8 \quad \to P(8,\ 1)$

Exercise

A rock is tossed into the air from ground level with an initial velocity of 15 $m/\sec$. Its height in meters at time *t* seconds is given by the function $h(t)=15t-4.9t^2$

a) Find the average velocity during the first 2 *sec* ?

b) Find the average velocity of the rock over the time intervals $[1,\ 1.05]$?

Solution

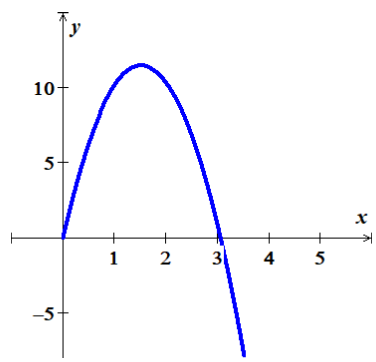
a) Average velocity $= \dfrac{\Delta h}{\Delta t}$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

$$= \frac{h(1)-h(0)}{1-0}$$

$$= 15-4.9$$

$$\boxed{= 10.1 \quad m/\sec}$$



b) Average velocity $= \dfrac{\Delta h}{\Delta t}$

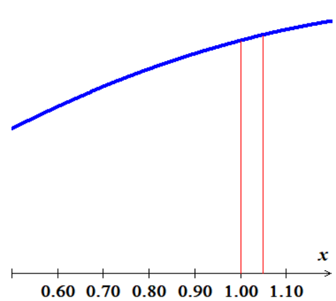
$$= \frac{h(1.05)-h(1)}{1.05-1}$$

$$= \frac{15(1.05)-4.9(1.05)^2-15+4.9}{.05}$$

$$= \frac{15.75-5.40225-15+4.9}{.05}$$

$$= \frac{0.24775}{.05}$$

$$\boxed{= 4.955 \ \ m/\sec}$$



Since, the average velocity is positive then the rock is going up in a positive direction

Exercise

A rock is free dropped from an initial height of 25 ft. Its height in feet at time t seconds is given by the function $h(t)=25+16t^2$

a) Find the average velocity during the first 1 *sec* ?

b) Find the average velocity of the rock over the time intervals $[1,\ 2]$?

c) Find the average velocity of the rock over the time intervals $[1.4,\ 1.5]$?

d) Find the average velocity of the rock over the time intervals $[1.5,\ 1.6]$?

e) Interpret the results.

Solution

a) Average velocity $=\dfrac{\Delta h}{\Delta t}$

$$\frac{\Delta y}{\Delta x}=\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

$$=\frac{h(1)-h(0)}{1-0}$$

$$=25+16(1)-25-16(0)$$

$$=16\ ft/\sec$$

b) $[1,\ 2]$

Average velocity $=\dfrac{\Delta h}{\Delta t}$

$$=\frac{h(2)-h(1)}{2-1}$$

$$=\frac{25+16(2)-25-16(1)}{1}$$

$$=32-16$$

$$=16\ ft/\sec$$

c) $[1.4,\ 1.5]$

Average velocity $=\dfrac{\Delta h}{\Delta t}$

$$=\frac{h(1.5)-h(1.4)}{1.5-1.4}$$

$$=\frac{25+16(1.5)-25-16(1.4)}{0.1}$$

$$=\frac{16(1.5-1.4)}{0.1}$$

$$=\frac{16(0.1)}{0.1}$$

$= 16\ \ ft / \sec$

d) $[1.5,\ 1.6]$

Average velocity $= \dfrac{\Delta h}{\Delta t}$

$= \dfrac{h(1.6) - h(1.5)}{1.6 - 1.5}$

$= \dfrac{25 + 16(1.6) - 25 - 16(1.5)}{0.1}$

$= \dfrac{16(1.6 - 1.5)}{0.1}$

$= \dfrac{16(0.1)}{0.1}$

$= 16\ \ ft / \sec$

e) From the previous, the average velocities are the same $16\ \ ft / \sec$ because it is a free fall.

Exercise

A rocket is launched into the air from ground level. The height, in feet, is given by the function

$h(t) = 860 + 130t - 16t^2$

a) Find the average velocity during the first 1 *sec* ?

b) Find the average velocity of the rocket over the time intervals $[4,\ 4.5]$?

c) Find the average velocity of the rocket over the time intervals $[4,\ 4.1]$?

Solution

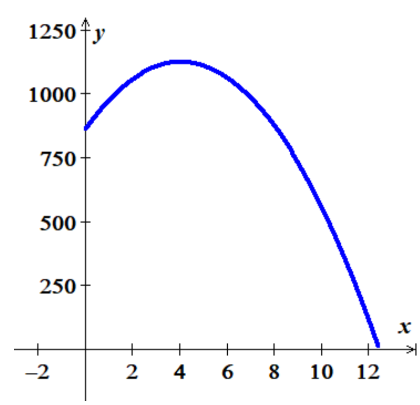
a) Average velocity $= \dfrac{\Delta h}{\Delta t}$

$\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

$= \dfrac{h(1) - h(0)}{1 - 0}$

$= 860 + 130 - 16$

$= 974\ \ ft / \sec$



b) $[4,\ 4.5]$

Average velocity $= \dfrac{\Delta h}{\Delta t}$

$= \dfrac{h(4.5) - h(4)}{4.5 - 4}$

$$= \frac{860+130(4.5)-16(4.5)^2-860-130(4)+16(16)}{.5}$$

$$= \frac{10}{5}\left(130(4.5-4)-16(20.25-16)\right)$$

$$= 2(65-68)$$

$$\underline{\left. = -6 \ \ ft/\sec \right|}$$

c) $[4,\ 4.1]$

$$\text{Average velocity } = \frac{\Delta h}{\Delta t}$$

$$= \frac{h(41)-h(4)}{4.1-4}$$

$$= \frac{860+130(4.1)-16(4.1)^2-860-130(4)+16(16)}{.1}$$

$$= 10\left(130(4.1-4)-16(16.81-16)\right)$$

$$= 10(13-12.96)$$

$$\underline{\left. = 0.4 \ \ ft/\sec \right|}$$

Exercise

An athlete running a 40-*m* dash. The position of the athlete is given by $d(t)=\frac{1}{6}t^3+4t$, where d is the position in meters and t is the time elapsed, measured in seconds.

a) Find the average velocity during the first 1 *sec* ?

b) Find the average velocity between second 1 and *second* 2 ?

c) Find the average velocity of the rocket over the time intervals $[1.95,\ 2.05]$?

Solution

a) $\text{Average velocity } = \frac{\Delta d}{\Delta t}$ $\frac{\Delta d}{\Delta t} = \frac{d(t_2)-f(t_1)}{t_2-t_1}$

$$= \frac{d(1)-d(0)}{1-0}$$

$$= \frac{1}{6}+4$$

$$\underline{\left. = \frac{25}{6} \ \ m/\sec \right|}$$

$$\underline{\left. \approx 4.167 \ \ m/\sec \right|}$$

b) $[1,\ 2]$

$$\text{Average velocity } = \frac{\Delta d}{\Delta t} \qquad\qquad \frac{\Delta d}{\Delta t} = \frac{d\left(t_2\right) - f\left(t_1\right)}{t_2 - t_1}$$

$$= \frac{d(2) - d(1)}{2-1}$$

$$= \frac{1}{6}(8) + 4(2) - \frac{1}{6} - 4$$

$$= \frac{8-1}{6} + 4$$

$$= \frac{7}{6} + 4$$

$$\boxed{= \frac{31}{6} \ \ m / \sec}$$

$$\boxed{\approx 5.167 \ \ m / \sec}$$

c) $[1.95,\ 2.05]$

$$\text{Average velocity } = \frac{\Delta d}{\Delta t}$$

$$= \frac{d(2.05) - d(1.95)}{2.05 - 1.95}$$

$$= \frac{1}{.1}\left(\frac{1}{6}(2.05)^3 + 4(2.05) - \frac{1}{6}(1.95)^3 - 4(1.95)\right)$$

$$= 10\left(\frac{1}{6}\left((2.05)^3 - (1.95)^3\right) + 4(2.05 - 1.95)\right)$$

$$= 10\left(\frac{1}{6}(1.2) + 4(.1)\right)$$

$$\boxed{\approx 6 \ \ m / \sec}$$

Exercise

An object dropped from certain height will free fall and is given by the function $h(t) = 16t^2$, where t in seconds.

a) Find the displacement from 3 to 5 seconds ?

b) Find the average rate of change of the object over the time interval $[3,\ 5]$?

Solution

a) $Displacement = h(5) - h(3)$

$$= 16(25) - 16(9)$$

$$= 16(25 - 9)$$

$$=16^2$$

$$\boxed{=256 \quad ft}$$

b) Average rate of change of the object over the time interval $[3,\ 5]$

$$\text{Average rate of change } = \frac{h(5)-h(3)}{5-3} \qquad \frac{\Delta h}{\Delta t} = \frac{h(t_2)-h(t_1)}{t_2-t_1}$$

$$=\frac{256}{2}$$

$$\boxed{=128 \quad ft\,/\sec}$$

Exercise

A function given by the function $f(x)=\frac{11}{100}x^{1.36}$

Find the average rate of change of the function over the interval $[200,\ 300]$?

Solution

$$\text{Average rate of change } = \frac{f(300)-f(200)}{300-200} \qquad \frac{\Delta y}{\Delta x} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

$$=\frac{1}{100}\left(\frac{11}{100}(300)^{1.36}-\frac{11}{100}(200)^{1.36}\right)$$

$$=\frac{11}{10^4}\left((300)^{1.36}-(200)^{1.36}\right)$$

$$=\frac{11}{10^4}(2,338.2175-1,347.103)$$

$$=\frac{11}{10^4}(991.115)$$

$$\boxed{\approx 1.09}$$

Exercise

A function given by the function $f(x)=3x^2+56x+863$

Find the average rate of change of the function over the interval $[4,\ 17]$?

Solution

$$\text{Average rate of change } = \frac{f(17)-f(4)}{17-4} \qquad \frac{\Delta y}{\Delta x} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

$$=\frac{1}{13}(867+952+863-48-224-863)$$

$$=\frac{1,547}{13}$$

$$\underline{=119}$$

Exercise

The cost of sending an envelope first-class is $\$1.00$ for the first ounce and $\$0.25$ for any additional ounce. If x represents the weight of the envelope, in ounces, the $C(x)$ is the cost of mailing it, where

$$C(x)=\begin{cases}\$1.00 & if \quad 0<x\le 1\\ \$1.25 & if \quad 1<x\le 2\\ \$1.50 & if \quad 2<x\le 3\end{cases}$$

a) Graph the cost function in the intervals $[0,\ 5]$

b) $\lim\limits_{x\to 1^-} C(x)$, $\lim\limits_{x\to 1^+} C(x)$, $\lim\limits_{x\to 1} C(x)$

c) $\lim\limits_{x\to 2^-} C(x)$, $\lim\limits_{x\to 2^+} C(x)$, $\lim\limits_{x\to 2} C(x)$

d) $\lim\limits_{x\to 3^-} C(x)$, $\lim\limits_{x\to 3^+} C(x)$, $\lim\limits_{x\to 3} C(x)$

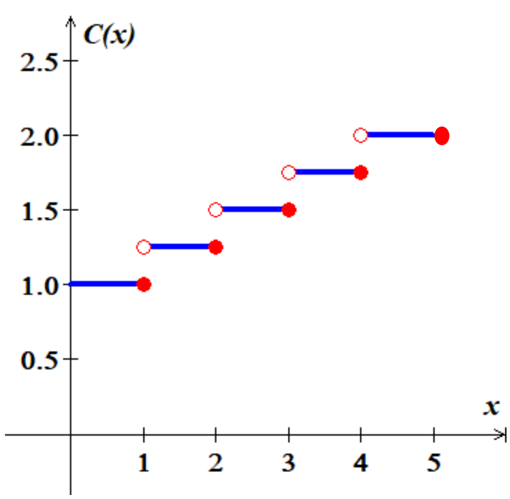
e) $\lim\limits_{x\to 3.5^-} C(x)$, $\lim\limits_{x\to 3.5^+} C(x)$, $\lim\limits_{x\to 3.5} C(x)$

f) $\lim\limits_{x\to 4^+} C(x)$

g) $\lim\limits_{x\to 4.5} C(x)$

Solution

a)



b) $\lim\limits_{x\to 1^-} C(x)=\$1.00$

$\lim\limits_{x\to 1^+} C(x) = \$1.25$

$\lim\limits_{x\to 1} C(x) = \$1.00$

c) $\lim\limits_{x\to 2^-} C(x) = \$1.25$

$\lim\limits_{x\to 2^+} C(x) = \$1.50$

$\lim\limits_{x\to 2} C(x) = \$1.25$

d) $\lim\limits_{x\to 3^-} C(x) = \$1.50$

$\lim\limits_{x\to 3^+} C(x) = \$1.75$

$\lim\limits_{x\to 3} C(x) = \$1.50$

e) $\lim\limits_{x\to 3.5^-} C(x) = \$1.75$

$\lim\limits_{x\to 3.5^+} C(x) = \$1.75$

$\lim\limits_{x\to 3.5} C(x) = \$1.75$

f) $\lim\limits_{x\to 4^+} C(x) = \$2.00$

g) $\lim\limits_{x\to 4.5} C(x) = \$2.00$

Exercise

For the function $f(x) = x^2$, find the different quotient when

a) $x = 3$ & $h = 2$

b) $x = 3$ & $h = 0.1$

c) $x = 3$ & $h = 0.01$

Solution

a)
$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{1}{h}\left((x+h)^2 - x^2\right)\\ &= \frac{1}{h}\left(x^2 + 2hx + h^2 - x^2\right)\\ &= \frac{1}{h}\left(2hx + h^2\right)\end{aligned}$$

$$= 2x + h$$

b) $$\frac{f(x+h)-f(x)}{h} = 2x + h \Big|_{x=3,\ h=2}$$

$$= 6 + 2$$

$$= 8$$

c) $$\frac{f(x+h)-f(x)}{h} = 2x + h \Big|_{x=3,\ h=.1}$$

$$= 6 + .1$$

$$= 6.1$$

d) $$\frac{f(x+h)-f(x)}{h} = 2x + h \Big|_{x=3,\ h=.01}$$

$$= 6 + .01$$

$$= 6.01$$

Exercise

For the function $f(x) = x^3$, find the different quotient when

a) For any x and h.

b) $x = 3 \quad \& \quad h = 2$

c) $x = 3 \quad \& \quad h = 0.1$

d) $x = 3 \quad \& \quad h = 0.01$

Solution

a) $$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\left((x+h)^3 - x^3\right)$$

$$= \frac{1}{h}\left(x^3 + 3x^2h + 3xh^2 + h^3 - x^3\right)$$

$$= \frac{1}{h}\left(3x^2h + 3xh^2 + h^3\right)$$

$$= 3x^2 + 3xh + h^2$$

b) $$\frac{f(x+h)-f(x)}{h} = 3x^2 + 3xh + h^2 \Big|_{x=3,\ h=2}$$

$$= 27 + 18 + 4$$

$$= 49$$

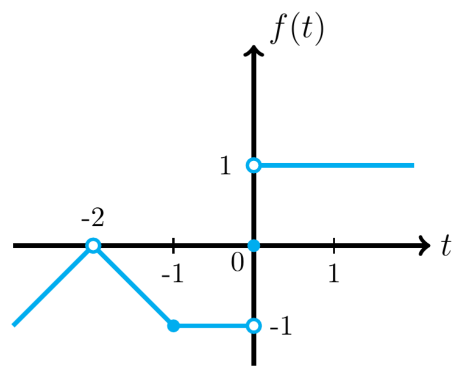
c) $$\frac{f(x+h)-f(x)}{h} = 3x^2 + 3xh + h^2 \Big|_{x=3,\ h=.1}$$

$$= 27 + .9 + .01$$
$$\boxed{= 27.91}$$

d) $$\frac{f(x+h)-f(x)}{h} = 3x^2 + 3xh + h^2 \Big|_{x=3,\ h=.01}$$
$$= 27 + .09 + .001$$
$$\boxed{= 27.091}$$

Exercise

For the function $f(t)$ graphed, find the following limits, or explain why they do not exist.



$a)\ \lim\limits_{t\to -2^-} f(t)$ $b)\ \lim\limits_{t\to -2^+} f(t)$ $c)\ \lim\limits_{t\to -2} f(t)$ $d)\ \lim\limits_{t\to -1^-} f(t)$

$e)\ \lim\limits_{t\to -1^+} f(t)$ $f)\ \lim\limits_{t\to -1} f(t)$ $g)\ \lim\limits_{t\to 0^-} f(t)$ $h)\ \lim\limits_{t\to 0^+} f(t)$

$i)\ \lim\limits_{t\to 0} f(t)$ $j)\ \lim\limits_{t\to -\frac{1}{2}} f(t)$ $k)\ \lim\limits_{t\to \frac{1}{2}} f(t)$

Solution

a) $\boxed{\lim\limits_{t\to -2^-} f(t) = 0}$

b) $\boxed{\lim\limits_{t\to -2^+} f(t) = 0}$

c) $\boxed{\lim\limits_{t\to -2} f(t) = 0}$

d) $\boxed{\lim\limits_{t\to -1^-} f(t) = -1}$

e) $\boxed{\lim\limits_{t\to -1^+} f(t) = -1}$

f) $\lim\limits_{t\to -1} f(t) = -1$

g) $\lim\limits_{t\to 0^-} f(t) = -1$

h) $\lim\limits_{t\to 0^+} f(t) = 1$

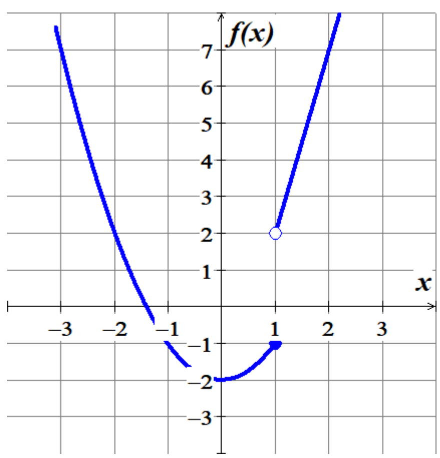
i) $\lim\limits_{t\to 0} f(t) =$ *doesn't exist*

j) $\lim\limits_{t\to -\frac{1}{2}} f(t) = -1$

k) $\lim\limits_{t\to \frac{1}{2}} f(t) = 1$

Exercise

For the function $f(x)$ graphed, find the following limit, or explain why they do not exist.



$a)\ \lim\limits_{x\to -2} f(x)$ $\quad b)\ \lim\limits_{x\to 2} f(x)$ $\quad c)\ \lim\limits_{x\to 1^-} f(x)$ $\quad d)\ \lim\limits_{x\to 1^+} f(x)$ $\quad e)\ \lim\limits_{x\to 1} f(x)$

Solution

a) $\lim\limits_{x\to -2} f(x) = 2$

b) $\lim\limits_{x\to 2} f(x) = 7$

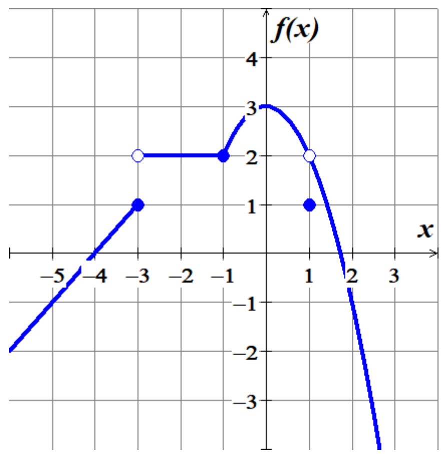
c) $\lim\limits_{x\to 1^-} f(x) = 1$

d) $\lim\limits_{x\to 1^+} f(x) = 0$

e) $\lim\limits_{x\to 1^-} f(x) = \nexists$

Exercise

For the function $f(x)$ graphed, find the following limit, or explain why they do not exist.



$a)\ \lim\limits_{x\to -5} f(x)$ $\quad b)\ \lim\limits_{x\to -3^-} f(x)$ $\quad c)\ \lim\limits_{x\to -3^+} f(x)$ $\quad d)\ \lim\limits_{x\to -3} f(x)$

$e)\ \lim\limits_{x\to -1^-} f(x)$ $\quad f)\ \lim\limits_{x\to -1^+} f(x)$ $\quad g)\ \lim\limits_{x\to -1} f(x)$ $\quad h)\ \lim\limits_{x\to 0} f(x)$

$i)\ \lim\limits_{x\to 1^-} f(x)$ $\quad j)\ \lim\limits_{x\to 1^+} f(x)$ $\quad k)\ \lim\limits_{x\to 1} f(x)$ $\quad l)\ \lim\limits_{x\to 2} f(x)$

Solution

a) $\lim\limits_{x\to -5} f(x) = -1$

b) $\lim\limits_{x\to -3^-} f(x) = 1$

c) $\lim\limits_{x\to -3^+} f(x) = 2$

d) $\lim\limits_{x\to -3} f(x) = \nexists$

e) $\lim\limits_{x\to -1^-} f(x) = 2$

f) $\lim\limits_{x\to -1^+} f(x) = 2$

g) $\lim\limits_{x\to -1} f(x) = 2$

h) $\lim\limits_{x\to 0} f(x) = 3$

i) $\lim\limits_{x\to 1^-} f(x) = 2$

j) $\lim\limits_{x\to 1^+} f(x) = 2$

k) $\lim\limits_{x\to 1} f(x) = \nexists$

Exercise

Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

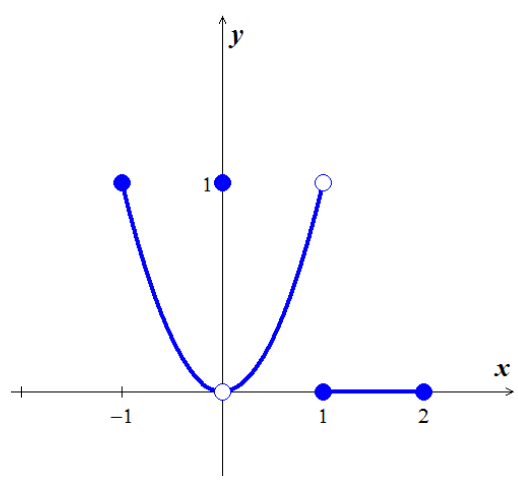
Solution

a) $\lim\limits_{x\to -1^+} f(x) = 1$ ***True***

b) $\lim\limits_{x\to 0^-} f(x) = 0$ ***True***

c) $\lim\limits_{x\to 0^-} f(x) = 1$ ***False***

d) $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x)$ ***True***



e) $\lim\limits_{x\to 0} f(x)$ *exists* ***True***

f) $\lim\limits_{x\to 0} f(x) = 0$ ***True***

g) $\lim\limits_{x\to 0} f(x) = 1$ ***False***

h) $\lim\limits_{x\to 1} f(x) = 1$ ***False***

i) $\lim\limits_{x\to 1} f(x) = 0$ ***False***

j) $\lim\limits_{x\to 2^-} f(x) = 2$ ***False***

k) $\lim\limits_{x \to -1^-} f(x) = 0$ *does not exist* ***True***

l) $\lim\limits_{x \to 2^+} f(x) = 0$ ***False***

Solution Section 1.2 – Techniques of Limits

Exercise

Find the limit: $\lim\limits_{x\to 3}(-1)$

Solution

$$\lim_{x\to 3}(-1) = -1$$

Exercise

Find the limit: $\lim\limits_{x\to -1}(3)$

Solution

$$\lim_{x\to -1}(3) = 3$$

Exercise

Find the limit: $\lim\limits_{x\to 1000} 18\pi^2$

Solution

$$\lim_{x\to 1000} 18\pi^2 = 18\pi^2$$

Exercise

Find the limit: $\lim\limits_{x\to 3} 4t^2$

Solution

$$\lim_{x\to 3} 4t^2 = 4t^2$$

Exercise

Find the limit: $\lim\limits_{x\to 2}(6x-2)$

Solution

$$\lim_{x\to 2} (6x-2) = 10$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \sqrt{5x+6}$

Solution

$$\lim_{x\to 1} \sqrt{5x+6} = \sqrt{11}$$

Exercise

Find the limit: $\lim\limits_{x\to 9} \sqrt{x}$

Solution

$$\lim_{x\to 9} \sqrt{x} = \sqrt{9}$$

$$= 3$$

Exercise

Find the limit: $\lim\limits_{x\to -3} \left(x^2+3x\right)$

Solution

$$\lim_{x\to -3} \left(x^2+3x\right) = (-3)^2 + 3(-3)$$

$$= 9-9$$

$$= 0$$

Exercise

Find the limit: $\lim\limits_{x\to -4} |x-4|$

Solution

$$\lim_{x\to -4} |x-4| = |-4-4|$$

$$= |-8|$$

$$= 8$$

Exercise

Find the limit: $\lim\limits_{x\to 4}(x+2)$

Solution

$$\lim_{x\to 4}(x+2) = 4+2$$
$$= 6$$

Exercise

Find the limit: $\lim\limits_{x\to 4}(x-4)$

Solution

$$\lim_{x\to 4}(x-4) = 4-4$$
$$= 0$$

Exercise

Find the limit: $\lim\limits_{x\to 2}(5x-6)^{3/2}$

Solution

$$\lim_{x\to 2}(5x-6)^{3/2} = (10-6)^{3/2}$$
$$= \sqrt{4^3}$$
$$= 8$$

Exercise

Find the limit: $\lim\limits_{x\to 9}\dfrac{x-9}{\sqrt{x}-3}$

Solution

$$\lim_{x\to 9}\frac{x-9}{\sqrt{x}-3} = \frac{9-9}{3-3} = \frac{0}{0}$$
$$= \lim_{x\to 9}\frac{(\sqrt{x}-3)(\sqrt{x}+3)}{\sqrt{x}-3}$$
$$= \lim_{x\to 9}(\sqrt{x}+3)$$
$$= 6$$

Exercise

Find the limit: $\lim\limits_{x\to 1}(2x+4)$

Solution

$$\lim_{x\to 1}(2x+4) = 2(1)+4$$

$$= 6$$

Exercise

Find the limit: $\lim\limits_{x\to 1}\dfrac{x^2-4}{x-2}$

Solution

$$\lim_{x\to 1}\frac{x^2-4}{x-2} = \frac{1^2-4}{1-2}$$

$$= \frac{-3}{-1}$$

$$= 3$$

Exercise

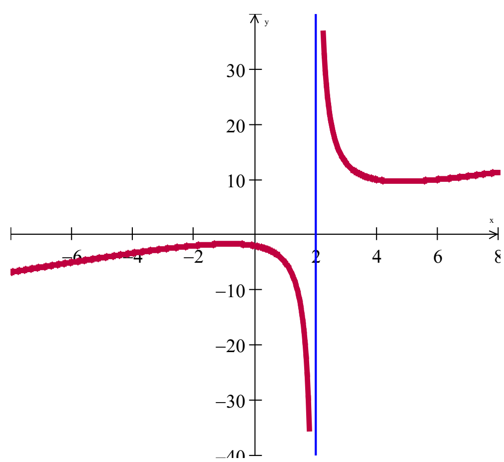
Find the limit: $\lim\limits_{x\to 2}\dfrac{x^2+4}{x-2}$

Solution

$$\lim_{x\to 2}\frac{x^2+4}{x-2} = \frac{2^2+4}{2-2}$$

$$= \frac{8}{0}$$

$$= \infty \qquad (\textit{Doesn't exist})$$



Exercise

Find the limit: $\lim\limits_{x\to 0}\dfrac{|x|}{x}$

Solution

$$\lim_{x\to 0}\frac{|x|}{x} = \frac{0}{0}$$

$$\lim_{x\to 0^-} \frac{|x|}{x} = \frac{x}{-x} = -1$$

$$\lim_{x\to 0^+} \frac{|x|}{x} = \frac{x}{x} = 1$$

Doesn't exist

Exercise

Find: $\lim_{x\to 3} \frac{x^2 - x - 1}{\sqrt{x+1}}$

Solution

$$\lim_{x\to 3} \frac{x^2 - x - 1}{\sqrt{x+1}} = \frac{3^2 - 3 - 1}{\sqrt{3+1}}$$

$$= \frac{5}{2}$$

Exercise

Find: $\lim_{x\to 2} \frac{x^2 + x - 6}{x-2}$

Solution

$$\lim_{x\to 2} \frac{x^2 + x - 6}{x-2} = \frac{2^2 + 2 - 6}{2-2} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{(x+3)(x-2)}{x-2}$$

$$= \lim_{x\to 2} (x+3)$$

$$= 5$$

Exercise

Find the limit: $\lim_{x\to 0} (3x-2)$

Solution

$$\lim_{x\to 0} (3x-2) = 3(0) - 2$$

$$= -2$$

Exercise

Find the limit: $\lim\limits_{x\to 1}\left(2x^2 - x + 4\right)$

Solution

$$\lim_{x\to 1}(2x^2 - x + 4) = 2(1)^2 - (1) + 4$$

$$= 5$$

Exercise

Find the limit: $\lim\limits_{x\to -2}\left(x^3 - 2x^2 + 4x + 8\right)$

Solution

$$\lim_{x\to -2}\left(x^3 - 2x^2 + 4x + 8\right) = (-2)^3 - 2(-2)^2 + 4(-2) + 8$$

$$= -16$$

Exercise

Find the limit: $\lim\limits_{x\to 1}\left(x^4 - 3x^3 + 7x^2 + 9x - 5\right)$

Solution

$$\lim_{x\to 1}\left(x^4 - 3x^3 + 7x^2 + 9x - 5\right) = 1 - 3 + 7 + 9 - 5$$

$$= 9$$

Exercise

Find the limit: $\lim\limits_{x\to 0}\left(x^4 - 3x^3 + 7x^2 + 9x - 5\right)$

Solution

$$\lim_{x\to 0}\left(x^4 - 3x^3 + 7x^2 + 9x - 5\right) = -5$$

Exercise

Find the limit: $\lim\limits_{x\to -1}\left(2x^6-2x^3+7x^2+4x-5\right)$

Solution

$$\lim\limits_{x\to -1}\left(2x^6-2x^3+7x^2+4x-5\right)=2+2+7-4-5$$
$$\underline{=2\,|}$$

Exercise

Find the limit: $\lim\limits_{x\to 2}\left(-2x^5-\frac{3}{2}x^3+\frac{1}{4}x^2+5x-5\right)$

Solution

$$\lim\limits_{x\to 2}\left(-2x^5-\frac{3}{2}x^3+\frac{1}{4}x^2+5x-5\right)=-64-12+4+10-5$$
$$\underline{=-67\,|}$$

Exercise

Find the limit: $\lim\limits_{x\to 2}\frac{x^2-4}{x-2}$

Solution

$$\lim\limits_{x\to 2}\frac{x^2-4}{x-2}=\frac{2^2-4}{2-2}=\frac{0}{0}$$
$$=\lim\limits_{x\to 2}\frac{(x-2)(x+2)}{x-2}$$
$$=\lim\limits_{x\to 2}(x+2)$$
$$\underline{=4\,|}$$

Exercise

Find the limit: $\lim\limits_{x\to 2}\frac{x^3-8}{x-2}$

Solution

$$\lim\limits_{x\to 2}\frac{x^3-8}{x-2}=\frac{0}{0}$$

$$= \lim_{x \to 2} \frac{(x-2)(x^2+2x+4)}{x-2}$$

$$= \lim_{x \to 2} x^2 + 2x + 4$$

$$= 2^2 + 2(2) + 4$$

$$\underline{= 12}$$

Exercise

Find the limit: $\lim_{x \to 3} \frac{x^2+x-12}{x-3}$

Solution

$$\lim_{x \to 3} \frac{x^2+x-12}{x-3} = \frac{0}{0}$$

$$= \lim_{x \to 3} \frac{(x-3)(x+4)}{x-3}$$

$$= \lim_{x \to 3} (x+4)$$

$$\underline{= 7}$$

Exercise

Find the limit: $\lim_{x \to 0} \frac{\sqrt{x+4}-2}{x}$

Solution

$$\lim_{x \to 0} \frac{\sqrt{x+4}-2}{x} = \frac{\sqrt{4}-2}{0} = \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\sqrt{x+4}-2}{x} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$$

$$= \lim_{x \to 0} \frac{x+4-4}{x\left(\sqrt{x+4}+2\right)}$$

$$= \lim_{x \to 0} \frac{x}{x\left(\sqrt{x+4}+2\right)}$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{x+4}+2}$$

$$= \frac{1}{\sqrt{4}+2}$$

$$= \boxed{\frac{1}{4}}$$

Exercise

Find the limit: $\lim\limits_{x\to -2} \dfrac{5}{x+2}$

Solution

$$\lim_{x\to -2} \frac{5}{x+2} = \frac{5}{0}$$

$$= \boxed{\infty}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{3}{\sqrt{3x+1}+1}$

Solution

$$\lim_{x\to 0} \frac{3}{\sqrt{3x+1}+1} = \frac{3}{\sqrt{3(0)+1}+1}$$

$$= \frac{3}{1+1}$$

$$= \boxed{\frac{3}{2}}$$

Exercise

Find the limit: $\lim\limits_{x\to 3} \dfrac{\sqrt{x+1}-1}{x}$

Solution

$$\lim_{x\to 3} \frac{\sqrt{x+1}-1}{x} = \frac{\sqrt{3+1}-1}{3} = \frac{2-1}{3}$$

$$= \boxed{\frac{1}{3}}$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1}$

Solution

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1}$$

$$= \lim_{x\to 1} (x+1)$$

$$= 2$$

Exercise

Find the limit: $\lim\limits_{x\to -2} \dfrac{|x+2|}{x+2}$

Solution

$$\lim_{x\to -2} \frac{|x+2|}{x+2} = \frac{|-2+2|}{-2+2} = \frac{0}{0}$$

$$\lim_{x\to -2^+} \frac{|x+2|}{x+2} = \frac{(x+2)}{(x+2)}$$

$$= 1$$

$$\lim_{x\to -2^-} \frac{|x+2|}{x+2} = \frac{(x+2)}{-(x+2)}$$

$$= -1$$

Doesn't exist

Exercise

Find the limit: $\lim\limits_{x\to 0} (2x-8)^{1/3}$

Solution

$$\lim_{x\to 0} (2x-8)^{1/3} = (2(0)-8)^{1/3}$$

$$= (-8)^{1/3}$$

$$= -2$$

Exercise

Find the limit: $\lim\limits_{x\to 2} \dfrac{x^2-7x+10}{x-2}$

Solution

$$\lim\limits_{x\to 2} \frac{x^2-7x+10}{x-2} = \frac{2^2-7(2)+10}{2-2} = \frac{0}{0}$$

$$= \lim\limits_{x\to 2} \frac{(x-2)(x-5)}{x-2}$$

$$= \lim\limits_{x\to 2} (x-5)$$

$$= 2-5$$

$$\boxed{= -3}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{5x^3+8x^2}{3x^4-16x^2}$

Solution

$$\lim\limits_{x\to 0} \frac{5x^3+8x^2}{3x^4-16x^2} = \frac{0}{0}$$

$$\lim\limits_{x\to 0} \frac{5x^3+8x^2}{3x^4-16x^2} = \lim\limits_{x\to 0} \frac{x^2(5x+8)}{x^2\left(3x^2-16\right)}$$

$$= \lim\limits_{x\to 0} \frac{5x+8}{3x^2-16}$$

$$= \frac{8}{-16}$$

$$\boxed{= -\frac{1}{2}}$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{\frac{1}{x}-1}{x-1}$

Solution

$$\lim\limits_{x\to 1} \frac{\frac{1}{x}-1}{x-1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$\lim_{x\to 1}\frac{\frac{1}{x}-1}{x-1}=\lim_{x\to 1}\frac{\frac{1-x}{x}}{x-1}$$

$$=\lim_{x\to 1}\left(\frac{1-x}{x}\right)\left(\frac{1}{x-1}\right)$$

$$=\lim_{x\to 1}\left(\frac{-(x-1)}{x}\right)\left(\frac{1}{x-1}\right)$$

$$=\lim_{x\to 1}\frac{-1}{x}$$

$$=-1$$

Exercise

Find the limit: $\lim_{u\to 1}\frac{u^4-1}{u^3-1}$

Solution

$$\lim_{u\to 1}\frac{u^4-1}{u^3-1}=\lim_{u\to 1}\frac{\left(u^2-1\right)\left(u^2+1\right)}{(u-1)\left(u^2+u+1\right)}$$

$$=\lim_{u\to 1}\frac{(u-1)(u+1)\left(u^2+1\right)}{(u-1)\left(u^2+u+1\right)}$$

$$=\lim_{u\to 1}\frac{(u+1)\left(u^2+1\right)}{u^2+u+1}$$

$$=\frac{(1+1)\left(1^2+1\right)}{1^2+1+1}$$

$$=\frac{4}{3}$$

Exercise

Find the limit: $\lim_{x\to 1}\frac{x-1}{\sqrt{x+3}-2}$

Solution

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x+3}-2}=\frac{1-1}{\sqrt{1+3}-2}$$

$$=\frac{0}{\sqrt{4}-2}=\frac{0}{0}$$

$$= \lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2} \bullet \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}$$

$$= \lim_{x\to 1} \frac{(x-1)\left(\sqrt{x+3}+2\right)}{x+3-4}$$

$$= \lim_{x\to 1} \frac{(x-1)\left(\sqrt{x+3}+2\right)}{x-1}$$

$$= \lim_{x\to 1} \left(\sqrt{x+3}+2\right)$$

$$= \sqrt{1+3}+2$$

$$\boxed{= 4}$$

Exercise

Find the limit: $\lim\limits_{x\to -1} \dfrac{\sqrt{x^2+8}-3}{x+1}$

Solution

$$\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} = \frac{\sqrt{(-1)^2+8}-3}{-1+1} = \frac{\sqrt{9}-3}{0} = \frac{0}{0}$$

$$= \lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} \bullet \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}$$

$$= \lim_{x\to -1} \frac{x^2+8-9}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x\to -1} \frac{x^2-1}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x\to -1} \frac{(x-1)(x+1)}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x\to -1} \frac{(x-1)}{\sqrt{x^2+8}+3}$$

$$= \frac{-2}{\sqrt{9}+3}$$

$$= \frac{-2}{6}$$

$$\boxed{= -\frac{1}{3}}$$

Exercise

Find the limit: $\lim\limits_{x\to -3} \dfrac{2-\sqrt{x^2-5}}{x+3}$

Solution

$$\lim_{x\to -3} \frac{2-\sqrt{x^2-5}}{x+3} = \frac{2-\sqrt{(-3)^2-5}}{-3+3}$$

$$= \frac{2-\sqrt{9-5}}{0}$$

$$= \frac{2-\sqrt{4}}{0} = \frac{0}{0}$$

$$= \lim_{x\to -3} \frac{2-\sqrt{x^2-5}}{x+3} \bullet \frac{2+\sqrt{x^2-5}}{2+\sqrt{x^2-5}}$$

$$= \lim_{x\to -3} \frac{4-\left(x^2-5\right)}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{4-x^2+5}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{9-x^2}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{(x-3)(x+3)}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{(x-3)}{2+\sqrt{x^2-5}}$$

$$= \frac{-6}{2+\sqrt{9-5}}$$

$$= \frac{-6}{2+\sqrt{4}}$$

$$= -\frac{6}{4}$$

$$= -\frac{3}{2}$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{\sqrt{x}-1}{x-1}$

Solution

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} \frac{\sqrt{x}+1}{\sqrt{x}+1}$$

$$= \lim_{x\to 1} \frac{x-1}{(x-1)\left(\sqrt{x}+1\right)}$$

$$= \lim_{x\to 1} \frac{1}{\sqrt{x}+1}$$

$$= \frac{1}{2}$$

$$x-1 = \left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)$$

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} \frac{\sqrt{x}-1}{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}$$

$$= \lim_{x\to 1} \frac{1}{\sqrt{x}+1}$$

$$= \frac{1}{2}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{\sqrt{9+x}-\sqrt{9-x}}{x}$

Solution

$$\lim_{x\to 0} \frac{\sqrt{9+x}-\sqrt{9-x}}{x} = \frac{3-3}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sqrt{9+x}-\sqrt{9-x}}{x} \frac{\sqrt{9+x}+\sqrt{9-x}}{\sqrt{9+x}+\sqrt{9-x}}$$

$$= \lim_{x\to 0} \frac{9+x-9+x}{x\left(\sqrt{9+x}+\sqrt{9-x}\right)}$$

$$= \lim_{x\to 0} \frac{2x}{x\left(\sqrt{9+x}+\sqrt{9-x}\right)}$$

$$= \lim_{x\to 0} \frac{2}{\sqrt{9+x}+\sqrt{9-x}}$$

$$= \frac{2}{3+3}$$

$$= \frac{1}{3}$$

Exercise

Find the limit: $\lim_{x\to 0} \frac{\sqrt{9+2x}-\sqrt{9}}{x}$

Solution

$$\lim_{x\to 0} \frac{\sqrt{9+2x}-\sqrt{9}}{x} = \frac{3-3}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sqrt{9+2x}-\sqrt{9}}{x} \frac{\sqrt{9+2x}+\sqrt{9}}{\sqrt{9+2x}+\sqrt{9}}$$

$$= \lim_{x\to 0} \frac{9+2x-9}{x\left(\sqrt{9+2x}+\sqrt{9}\right)}$$

$$= \lim_{x\to 0} \frac{2x}{x\left(\sqrt{9+2x}+\sqrt{9}\right)}$$

$$= \lim_{x\to 0} \frac{2}{\sqrt{9+2x}+\sqrt{9}}$$

$$= \frac{2}{3+3}$$

$$= \frac{1}{3}$$

Exercise

Find the limit: $\lim_{x\to 1} \frac{x-\sqrt[4]{x}}{x-1}$

Solution

$$\lim_{x\to 1} \frac{x-\sqrt[4]{x}}{x-1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{x-\sqrt[4]{x}}{x-1} \frac{x+\sqrt[4]{x}}{x+\sqrt[4]{x}}$$

$$= \lim_{x\to 1} \frac{x^2-\sqrt{x}}{(x-1)\left(x+\sqrt[4]{x}\right)} = \frac{0}{0}$$

x	$\frac{x-\sqrt[4]{x}}{x-1}$
.9	$\frac{.9-\sqrt[4]{.9}}{.9-1}=\frac{-.074}{-.1}\approx .75$
1.1	$\frac{1.01-\sqrt[4]{1.01}}{1.01-1}=\frac{.0075}{.01}\approx .75$

$$\lim_{x\to 1}\frac{x-\sqrt[4]{x}}{x-1}=\frac{3}{4}$$

Exercise

Find the limit: $\lim_{x\to 0}\frac{6-\sqrt{36-x^2}}{x}$

Solution

$$\lim_{x\to 0}\frac{6-\sqrt{36-x^2}}{x}=\frac{6-6}{0}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{6-\sqrt{36-x^2}}{x}\frac{6+\sqrt{36-x^2}}{6+\sqrt{36-x^2}}$$

$$=\lim_{x\to 0}\frac{36-36+x^2}{x\left(6+\sqrt{36-x^2}\right)}$$

$$=\lim_{x\to 0}\frac{x^2}{x\left(6+\sqrt{36-x^2}\right)}$$

$$=\lim_{x\to 0}\frac{x}{6+\sqrt{36-x^2}}$$

$$=\frac{0}{12}$$

$$=0$$

Exercise

Find the limit: $\lim_{x\to 2}\frac{x^2-4}{\sqrt{x^2+5}-3}$

Solution

$$\lim_{x\to 2}\frac{x^2-4}{\sqrt{x^2+5}-3}=\frac{0}{3-3}=\frac{0}{0}$$

$$=\lim_{x\to 2}\frac{x^2-4}{\sqrt{x^2+5}-3}\,\frac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3}$$

$$=\lim_{x\to 2}\frac{\left(x^2-4\right)\left(\sqrt{x^2+5}+3\right)}{x^2+5-9}$$

$$=\lim_{x\to 2}\frac{\left(x^2-4\right)\left(\sqrt{x^2+5}+3\right)}{x^2-4}$$

$$=\lim_{x\to 2}\left(\sqrt{x^2+5}+3\right)$$

$$=6$$

Exercise

Find the limit: $\lim\limits_{x\to 0}(2\sin x-1)$

Solution

$$\lim_{x\to 0}(2\sin x-1)=2\sin(0)-1$$

$$=0-1$$

$$=-1$$

Exercise

Find the limit: $\lim\limits_{x\to 0}\sin^2 x$

Solution

$$\lim_{x\to 0}\sin^2 x=\sin^2(0)$$

$$=0$$

Exercise

Find the limit: $\lim\limits_{x\to 0}\sec x$

Solution

$$\lim_{x\to 0} \sec x = \sec(0)$$
$$= \frac{1}{\cos(0)}$$
$$\boxed{=1}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{1+x+\sin x}{3\cos x}$

Solution

$$\lim_{x\to 0} \frac{1+x+\sin x}{3\cos x} = \frac{1+0+\sin(0)}{3\cos(0)}$$
$$\boxed{=\frac{1}{3}}$$

Exercise

Find the limit: $\lim\limits_{x\to -\pi} \sqrt{x+4}\,\cos(x+\pi)$

Solution

$$\lim_{x\to -\pi} \sqrt{x+4}\,\cos(x+\pi) = \sqrt{-\pi+4}\,\cos(-\pi+\pi)$$
$$= \sqrt{-\pi+4}\,\cos(0)$$
$$\boxed{=\sqrt{4-\pi}}$$

Exercise

Find $\lim\limits_{x\to -0.5^-} \sqrt{\dfrac{x+2}{x+1}}$

Solution

$$\lim_{x\to -0.5^-} \sqrt{\frac{x+2}{x+1}} = \sqrt{\frac{-0.5+2}{-0.5+1}}$$
$$= \sqrt{\frac{1.5}{0.5}}$$
$$\boxed{=\sqrt{3}}$$

Exercise

Find $\lim\limits_{x\to 1^+} \sqrt{\dfrac{x-1}{x+2}}$

Solution

$$\lim_{x\to 1^+} \sqrt{\frac{x-1}{x+2}} = \sqrt{\frac{1-1}{1+2}}$$

$$= 0$$

Exercise

Find $\lim\limits_{x\to -2^+} \left(\dfrac{x}{x+1}\right)\left(\dfrac{2x+5}{x^2+x}\right)$

Solution

$$\lim_{x\to -2^+} \left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right) = \left(\frac{-2}{-2+1}\right)\left(\frac{2(-2)+5}{(-2)^2+(-2)}\right)$$

$$= \left(\frac{-2}{-1}\right)\left(\frac{1}{2}\right)$$

$$= 1$$

Exercise

Find $\lim\limits_{x\to 0^+} \dfrac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}$

Solution

$$\lim_{x\to 0^+} \frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x} = \frac{\sqrt{5}-\sqrt{5}}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}\,\frac{\sqrt{x^2+4x+5}+\sqrt{5}}{\sqrt{x^2+4x+5}+\sqrt{5}}$$

$$= \lim_{x\to 0^+} \frac{x^2+4x+5-5}{x\left(\sqrt{x^2+4x+5}+\sqrt{5}\right)}$$

$$= \lim_{x\to 0^+} \frac{x^2+4x}{x\left(\sqrt{x^2+4x+5}+\sqrt{5}\right)}$$

$$= \lim_{x\to 0^+} \frac{x(x+4)}{x\left(\sqrt{x^2+4x+5}+\sqrt{5}\right)}$$

$$= \lim_{x\to 0^+} \frac{x+4}{\sqrt{x^2+4x+5}+\sqrt{5}}$$

$$= \frac{0+4}{\sqrt{0^2+4(0)+5}+\sqrt{5}}$$

$$= \frac{4}{\sqrt{5}+\sqrt{5}}$$

$$= \frac{4}{2\sqrt{5}}$$

$$= \frac{2}{\sqrt{5}}$$

Exercise

Find $\lim\limits_{x\to -2^+} (x+3)\dfrac{|x+2|}{x+2}$

Solution

$$\lim_{x\to -2^+} (x+3)\frac{|x+2|}{x+2} = (x+3)\frac{|-2+2|}{-2+2} = \frac{0}{0}$$

Since $x \to -2^+ \Rightarrow x > -2$

$\Rightarrow |x+2| = (x+2)$

$$\lim_{x\to -2^+} (x+3)\frac{|x+2|}{x+2} = \lim_{x\to -2^+} (x+3)\frac{x+2}{x+2}$$

$$= \lim_{x\to -2^+} (x+3)$$

$$= -2+3$$

$$= 1$$

Exercise

Find $\lim\limits_{x\to 1^+} \dfrac{\sqrt{2x}(x-1)}{|x-1|}$

Solution

$$\lim_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|} = \frac{\sqrt{2(1)}(1-1)}{|1-1|} = \frac{0}{0}$$

Since $x \to 1^+ \quad \Rightarrow \quad x > 1$

$$\Rightarrow \quad |x-1| = x-1$$

$$\lim_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|} = \lim_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{x-1}$$

$$= \lim_{x\to 1^+} \sqrt{2x}$$

$$= \sqrt{2}$$

Exercise

Find $\displaystyle \lim_{x\to 0^-} \frac{x}{\sin 3x}$

Solution

$$\lim_{x\to 0^-} \frac{x}{\sin 3x} = \lim_{x\to 0^-} \frac{x}{\sin 3x}\left(\frac{3}{3}\right)$$

$$= \frac{1}{3} \lim_{x\to 0^-} \frac{3x}{\sin 3x}$$

$$= \frac{1}{3} \lim_{x\to 0^-} \frac{1}{\frac{\sin 3x}{3x}}$$

By definition: $\displaystyle \lim_{u\to 0} \frac{\sin u}{u} = 1$

$$= \frac{1}{3}$$

Exercise

Find $\displaystyle \lim_{\theta\to 0} \frac{\sin \sqrt{2}.\theta}{\sqrt{2}.\theta}$

Solution

Let: $\sqrt{2}\theta = x \to 0$

$$\lim_{\theta\to 0} \frac{\sin \sqrt{2}.\theta}{\sqrt{2}.\theta} = \lim_{x\to 0} \frac{\sin x}{x}$$

$$= 1$$

Exercise

Find $\lim\limits_{x\to 0}\frac{\sin 3x}{4x}$

Solution

$$\lim\limits_{x\to 0}\frac{\sin 3x}{4x}=\lim\limits_{x\to 0}\frac{\sin 3x}{4x}\frac{3}{3}$$

$$=\frac{3}{4}\lim\limits_{x\to 0}\frac{\sin 3x}{3x}$$

Let: $3x=u$

$$=\frac{3}{4}\lim\limits_{u\to 0}\frac{\sin u}{u}$$

By definition: $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$

$$=\frac{3}{4}$$

Exercise

Find $\lim\limits_{x\to 0}\frac{\tan 2x}{x}$

Solution

$$\lim\limits_{x\to 0}\frac{\tan 2x}{x}=\lim\limits_{x\to 0}\frac{\frac{\sin 2x}{\cos 2x}}{x}$$

$$=\lim\limits_{x\to 0}\left(\frac{\sin 2x}{x}\cdot\frac{1}{\cos 2x}\right)$$

$$=\lim\limits_{x\to 0}\left(2\frac{\sin 2x}{2x}\right)\lim\limits_{x\to 0}\left(\frac{1}{\cos 2x}\right)$$

$$=2\frac{1}{\cos 0}$$

$$=2$$

Exercise

Find $\lim\limits_{x\to 0}6x^2(\cot x)(\csc 2x)$

Solution

$$\lim\limits_{x\to 0}6x^2(\cot x)(\csc 2x)=\lim\limits_{x\to 0}6x^2\left(\frac{\cos x}{\sin x}\right)\left(\frac{1}{\sin 2x}\right)$$

$$=\lim\limits_{x\to 0}3\cos x\left(\frac{x}{\sin x}\right)\left(\frac{2x}{\sin 2x}\right)$$

$$=3\lim\limits_{x\to 0}(\cos x)\cdot\lim\limits_{x\to 0}\left(\frac{x}{\sin x}\right)\cdot\lim\limits_{2x\to 0}\left(\frac{2x}{\sin 2x}\right)$$

$$=(3)(1)(1)(1)$$
$$\underline{=3}$$

Exercise

Find $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\sin 2\theta}$

Solution

$$\lim_{\theta\to 0}\frac{\sin\theta}{\sin 2\theta}=\lim_{\theta\to 0}\frac{\sin\theta}{\sin 2\theta}\frac{2\theta}{2\theta}$$
$$=\frac{1}{2}\lim_{\theta\to 0}\left(\frac{2\theta}{\sin 2\theta}\cdot\frac{\sin\theta}{\theta}\right)$$
$$=\frac{1}{2}(1)(1)$$
$$\underline{=\frac{1}{2}}$$

Exercise

Find $\lim\limits_{h\to 0}\frac{\sin(\sin h)}{\sin h}$

Solution

Let: $\sin h=\theta$ $\qquad\theta=\sin h\xrightarrow[h\to 0]{}0$

$$\lim_{h\to 0}\frac{\sin(\sin h)}{\sin h}=\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}$$
$$\underline{=1}$$

Exercise

Find $\lim\limits_{\theta\to 0}\frac{\theta\cot 4\theta}{\sin^2\theta\cot^2 2\theta}$

Solution

$$\lim_{\theta\to 0}\frac{\theta\cot 4\theta}{\sin^2\theta\cot^2 2\theta}=\lim_{\theta\to 0}\frac{\theta\frac{\cos 4\theta}{\sin 4\theta}}{\sin^2\theta\frac{\cos^2 2\theta}{\sin^2 2\theta}}$$

$$= \lim_{\theta\to 0} \theta \frac{\cos 4\theta}{2\sin 2\theta \cos 2\theta} \frac{\sin^2 2\theta}{\sin^2\theta \cos^2 2\theta}$$

$$= \lim_{\theta\to 0} \left(\frac{1}{2}\cdot\theta\cdot\cos 4\theta \bullet \frac{2\sin\theta\cos\theta}{\sin^2\theta}\cdot\frac{1}{\cos^3 2\theta} \right)$$

$$= \lim_{\theta\to 0} \left(\cos 4\theta \bullet \frac{\theta}{\sin\theta}\cdot\cos\theta\cdot\frac{1}{\cos^3 2\theta} \right)$$

$$= \lim_{\theta\to 0} (\cos 4\theta) \quad \lim_{\theta\to 0} \left(\frac{\theta}{\sin\theta}\right) \quad \lim_{\theta\to 0} \left(\frac{\cos\theta}{\cos^3 2\theta}\right)$$

$$=(1)(1)(1)$$

$$\underline{=1}$$

Exercise

Find $\displaystyle\lim_{\theta\to\pi/4} \frac{\sin^2\theta-\cos^2\theta}{\sin\theta-\cos\theta}$

Solution

$$\lim_{\theta\to\pi/4} \frac{\sin^2\theta-\cos^2\theta}{\sin\theta-\cos\theta} = \frac{\frac{1}{2}-\frac{1}{2}}{\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}} = \frac{0}{0}$$

$$= \lim_{\theta\to\pi/4} \frac{(\sin\theta-\cos\theta)(\sin\theta+\cos\theta)}{\sin\theta-\cos\theta}$$

$$= \lim_{\theta\to\pi/4} (\sin\theta+\cos\theta)$$

$$= \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}$$

$$\underline{=\sqrt{2}}$$

Exercise

Find $\displaystyle\lim_{x\to\pi/2} \frac{\frac{1}{\sqrt{\sin x}}-1}{x+\frac{\pi}{2}}$

Solution

$$\lim_{x\to\pi/2} \frac{\frac{1}{\sqrt{\sin x}}-1}{x+\frac{\pi}{2}} = \frac{1-1}{\frac{\pi}{2}+\frac{\pi}{2}}$$

$$\underline{=0}$$

Exercise

Find $\lim\limits_{x\to 1} \frac{x^3-7x^2+12x}{4-x}$

Solution

$$\lim\limits_{x\to 1} \frac{x^3-7x^2+12x}{4-x} = \frac{1-7+12}{4-1}$$

$$= 2$$

Exercise

Find $\lim\limits_{x\to 4} \frac{x^3-7x^2+12x}{4-x}$

Solution

$$\lim\limits_{x\to 4} \frac{x^3-7x^2+12x}{4-x} = \frac{64-112+48}{4-4} = \frac{0}{0}$$

$$= \lim\limits_{x\to 4} \frac{x(x-3)(x-4)}{4-x}$$

$$= \lim\limits_{x\to 4} -x(x-3)$$

$$= -4$$

Exercise

Find $\lim\limits_{x\to 1} \frac{1-x^2}{x^2-8x+7}$

Solution

$$\lim\limits_{x\to 1} \frac{1-x^2}{x^2-8x+7} = \frac{0}{0}$$

$$= \lim\limits_{x\to 1} \frac{(1-x)(1+x)}{(x-1)(x-7)}$$

$$= -\lim\limits_{x\to 1} \frac{1+x}{x-7}$$

$$= \frac{1}{3}$$

Exercise

Find $\displaystyle\lim_{x\to 3}\frac{\sqrt{3x+16}-5}{x-3}$

Solution

$$\lim_{x\to 3}\frac{\sqrt{3x+16}-5}{x-3}=\frac{\sqrt{9+16}-5}{3-3}=\frac{5-5}{0}=\frac{0}{0}$$

$$=\lim_{x\to 3}\frac{\sqrt{3x+16}-5}{x-3}\frac{\sqrt{3x+16}+5}{\sqrt{3x+16}+5}$$

$$=\lim_{x\to 3}\frac{3x+16-25}{(x-3)\left(\sqrt{3x+16}+5\right)}$$

$$=\lim_{x\to 3}\frac{3(x-3)}{(x-3)\left(\sqrt{3x+16}+5\right)}$$

$$=\lim_{x\to 3}\frac{3}{\sqrt{3x+16}+5}$$

$$=\frac{3}{5+5}$$

$$=\frac{3}{10}$$

Exercise

Find $\displaystyle\lim_{x\to 3}\frac{1}{x-3}\left(\frac{1}{\sqrt{x+1}}-\frac{1}{2}\right)$

Solution

$$\lim_{x\to 3}\frac{1}{x-3}\left(\frac{1}{\sqrt{x+1}}-\frac{1}{2}\right)=\frac{1}{0}\left(\frac{1}{2}-\frac{1}{2}\right)=\frac{0}{0}$$

$$=\lim_{x\to 3}\frac{1}{x-3}\left(\frac{2-\sqrt{x+1}}{\sqrt{x+1}}\right)\left(\frac{2+\sqrt{x+1}}{2+\sqrt{x+1}}\right)$$

$$=\lim_{x\to 3}\frac{1}{x-3}\left(\frac{4-x-1}{2\sqrt{x+1}+x+1}\right)$$

$$=\lim_{x\to 3}\frac{x-3}{x-3}\left(\frac{-1}{2\sqrt{x+1}+x+1}\right)$$

$$=\lim_{x\to 3}\frac{-1}{2\sqrt{x+1}+x+1}$$

$$=-\frac{1}{8}$$

Exercise

Find $\lim\limits_{x\to 1/3} \dfrac{x-\frac{1}{3}}{(3x-1)^2}$

Solution

$$\lim_{x\to 1/3} \frac{x-\frac{1}{3}}{(3x-1)^2} = \frac{\frac{1}{3}-\frac{1}{3}}{\left(3\frac{1}{3}-1\right)^2} = \frac{0}{0}$$

$$= \lim_{x\to 1/3} \frac{x-\frac{1}{3}}{9\left(x-\frac{1}{3}\right)^2}$$

$$= \lim_{x\to 1/3} \frac{1}{9\left(x-\frac{1}{3}\right)} = \frac{1}{0}$$

$$= \infty$$

Exercise

Find $\lim\limits_{x\to 3} \dfrac{x^4-81}{x-3}$

Solution

$$\lim_{x\to 3} \frac{x^4-81}{x-3} = \frac{81-81}{3-3} = \frac{0}{0}$$

$$= \lim_{x\to 3} \frac{(x-3)(x+3)\left(x^2+9\right)}{x-3} \qquad a^4-b^4=\left(a^2-b^2\right)\left(a^2+b^2\right)=(a-b)(a+b)\left(a^2+b^2\right)$$

$$= \lim_{x\to 3} (x+3)\left(x^2+9\right) = 6(18)$$

$$= 108$$

Exercise

Find $\lim\limits_{x\to 1} \dfrac{x^5-1}{x-1}$

Solution

$$\lim_{x\to 1} \frac{x^5-1}{x-1} = \frac{1-1}{1-1} = \frac{0}{0} \qquad \left(a^5-b^5\right)=(a-b)\left(a^4+a^3b+a^2b^2+ab^3+b^4\right)$$

$$= \lim_{x\to 1} \frac{(x-1)\left(x^4+x^3+x^2+x+1\right)}{x-1}$$

$$= \lim_{x\to 1} \left(x^4+x^3+x^2+x+1\right)$$

$$= 5$$

Exercise

Find $\displaystyle\lim_{x\to 81} \frac{\sqrt[4]{x}-3}{x-81}$

Solution

$$\lim_{x\to 81} \frac{\sqrt[4]{x}-3}{x-81} = \frac{3-3}{81-81} = \frac{0}{0}$$

$$= \lim_{x\to 81} \frac{\sqrt[4]{x}-3}{\left(\sqrt{x}+9\right)\left(\sqrt{x}-9\right)}$$

$$= \lim_{x\to 81} \frac{\sqrt[4]{x}-3}{\left(\sqrt{x}+9\right)\left(\sqrt[4]{x}+3\right)\left(\sqrt[4]{x}-3\right)}$$

$$= \lim_{x\to 81} \frac{1}{\left(\sqrt{x}+9\right)\left(\sqrt[4]{x}+3\right)}$$

$$= \frac{1}{(18)(6)}$$

$$= \frac{1}{108}$$

Exercise

Find the limit: $\displaystyle\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{x-1}$

Solution

$$\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{x-1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\left(\sqrt[3]{x}\right)^3-1^3}$$

$$= \lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\left(\sqrt[3]{x}-1\right)\left(x^{2/3}+\sqrt[3]{x}+1\right)}$$

$$= \lim_{x\to 1} \frac{1}{x^{2/3} + \sqrt[3]{x} + 1}$$

$$= \frac{1}{3}$$

Exercise

Find the limit: $\lim\limits_{x\to 2} \frac{x^5 - 32}{x - 2}$

Solution

$$\begin{array}{c|cccccc} 2 & 1 & 0 & 0 & 0 & 0 & -32 \\ & & 2 & 4 & 8 & 16 & 32 \\ \hline & 1 & 2 & 4 & 8 & 16 & 0 \end{array}$$

$$\lim_{x\to 2} \frac{x^5 - 32}{x - 2} = \frac{2^5 - 32}{2 - 2} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{(x-2)\left(x^4 + 2x^3 + 4x^2 + 8x + 16\right)}{x - 2}$$

$$= \lim_{x\to 2} \left(x^4 + 2x^3 + 4x^2 + 8x + 16\right)$$

$$= 16 + 16 + 16 + 16 + 16$$

$$= 80$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \frac{x^6 - 1}{x - 1}$

Solution

$$\begin{array}{c|ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ & & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}$$

$$\lim_{x\to 1} \frac{x^6 - 1}{x - 1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{(x-1)\left(x^5 + x^4 + x^3 + x^2 + x + 1\right)}{x - 1}$$

$$= \lim_{x\to 1} \left(x^5 + x^4 + x^3 + x^2 + x + 1\right)$$

$$= 6$$

Exercise

Find the limit: $\lim\limits_{x\to -1} \dfrac{x^7+1}{x+1}$

Solution

$$\begin{array}{r|rrrrrrrr} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ & & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 0 \end{array}$$

$$\lim\limits_{x\to -1} \frac{x^7+1}{x+1} = \frac{-1+1}{-1+1} = \frac{0}{0}$$

$$= \lim\limits_{x\to 1} \frac{(x+1)\left(x^6-x^5+x^4-x^3+x^2-x+1\right)}{x+1}$$

$$= \lim\limits_{x\to 1} \left(x^6-x^5+x^4-x^3+x^2-x+1\right)$$

$$= 1$$

Exercise

Find the limit: $\lim\limits_{x\to a} \dfrac{x^5-a^5}{x-a}$

Solution

$$\begin{array}{r|rrrrrr} a & 1 & 0 & 0 & 0 & 0 & -a^5 \\ & & a & a^2 & a^3 & a^4 & a^5 \\ \hline & 1 & a & a^2 & a^3 & a^4 & 0 \end{array}$$

$$\lim\limits_{x\to a} \frac{x^5-a^5}{x-a} = \frac{a^5-a^5}{a-a} = \frac{0}{0}$$

$$= \lim\limits_{x\to a} \frac{(x-a)\left(x^4+ax^3+a^2x^2+a^3x+a^4\right)}{x-a}$$

$$= \lim\limits_{x\to a} \left(x^4+ax^3+a^2x^2+a^3x+a^4\right)$$

$$= a^4+a^4+a^4+a^4+a^4$$

$$= 5a^4$$

Exercise

Find the limit: $\lim\limits_{x\to a} \dfrac{x^n - a^n}{x-a} \quad n \in \mathbb{Z}^+$

Solution

$$\begin{array}{c|ccccccc} a & 1 & 0 & 0 & 0 & \cdots & 0 & -a^n \\ & & a & a^2 & a^3 & \cdots & a^{n-1} & a^n \\ \hline & 1 & a & a^2 & a^3 & \cdots & a^{n-1} & 0 \end{array}$$

$$\lim_{x\to a} \frac{x^n - a^n}{x-a} = \frac{a^n - a^n}{a-a} = \frac{0}{0}$$

$$= \lim_{x\to a} \frac{(x-a)\left(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-2}x + a^{n-1}\right)}{x-a}$$

$$= \lim_{x\to a} \left(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-2}x + a^{n-1}\right)$$

$$= a^{n-1} + a^{n-1} + a^{n-1} + \ldots + a^{n-1} + a^{n-1}$$

$$= na^{n-1}$$

Exercise

Find the limit: $\lim\limits_{h\to 0} \dfrac{100}{(10h-1)^{11} + 2}$

Solution

$$\lim_{h\to 0} \frac{100}{(10h-1)^{11} + 2} = \frac{100}{(-1)^{11} + 2}$$

$$= \frac{100}{-1+2}$$

$$= 100$$

Exercise

Find the limit: $\lim\limits_{h\to 0} \dfrac{(5+h)^2 - 25}{h}$

Solution

$$\lim_{h\to 0} \frac{(5+h)^2 - 25}{h} = \frac{5^2 - 25}{0} = \frac{0}{0}$$

$$= \lim_{h\to 0} \frac{\big((5+h)-5\big)\big((5+h)+5\big)}{h}$$

$$= \lim_{h\to 0} \frac{h(h+10)}{h}$$

$$= \lim_{h\to 0} (h+10)$$

$$= 10$$

Exercise

Find the limit: $\lim\limits_{x\to 3} \dfrac{\frac{1}{x^2+2x}-\frac{1}{15}}{x-3}$

Solution

$$\lim_{x\to 3} \frac{\frac{1}{x^2+2x}-\frac{1}{15}}{x-3} = \frac{\frac{1}{15}-\frac{1}{15}}{0} = \frac{0}{0}$$

$$= \lim_{x\to 3} \frac{1}{x-3}\left(\frac{1}{x(x+2)}-\frac{1}{15}\right)$$

$$= \lim_{x\to 3} \frac{1}{x-3}\left(\frac{15-x^2-2x}{15x(x+2)}\right)$$

$$= \lim_{x\to 3} \frac{-(x-3)(x+5)}{15x(x+2)(x-3)}$$

$$= \lim_{x\to 3} \frac{-(x+5)}{15x(x+2)}$$

$$= -\frac{8}{15(3)(5)}$$

$$= -\frac{8}{225}$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{\sqrt{10x-9}-1}{x-1}$

Solution

$$\lim_{x\to 1} \frac{\sqrt{10x-9}-1}{x-1} = \frac{1-1}{0} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{\sqrt{10x-9}-1}{x-1} \cdot \frac{\sqrt{10x-9}+1}{\sqrt{10x-9}+1}$$

$$= \lim_{x\to 1} \frac{10x-9-1}{(x-1)\left(\sqrt{10x-9}+1\right)}$$

$$= \lim_{x\to 1} \frac{10(x-1)}{(x-1)\left(\sqrt{10x-9}+1\right)}$$

$$= \lim_{x\to 1} \frac{10}{\sqrt{10x-9}+1}$$

$$= \frac{10}{2}$$

$$\underline{=5}\,\rfloor$$

Exercise

Find the limit: $\lim\limits_{x\to 2} \left(\frac{1}{x-2} - \frac{2}{x^2-2x}\right)$

Solution

$$\lim_{x\to 2} \left(\frac{1}{x-2} - \frac{2}{x^2-2x}\right) = \frac{1}{0} - \frac{2}{0} = \infty - \infty$$

$$= \lim_{x\to 2} \left(\frac{1}{x-2} - \frac{2}{x(x-2)}\right)$$

$$= \lim_{x\to 2} \frac{x-2}{x(x-2)}$$

$$= \lim_{x\to 2} \frac{1}{x}$$

$$\underline{= \frac{1}{2}}\,\rfloor$$

Exercise

Find the limit: $\lim\limits_{x\to c} \frac{x^2-2cx+c^2}{x-c}$

Solution

$$\lim_{x\to c} \frac{x^2-2cx+c^2}{x-c} = \frac{c^2-2c^2+c^2}{0} = \frac{0}{0}$$

$$= \lim_{x\to c} \frac{(x-c)^2}{x-c}$$

$$= \lim_{x\to c} (x-c)$$

$$\underline{=0}\,\rfloor$$

Exercise

Find the limit: $\lim\limits_{x\to -c}\dfrac{x^2+5cx+4c^2}{x^2+cx}$

Solution

$$\lim_{x\to -c}\frac{x^2+5cx+4c^2}{x^2+cx}=\frac{c^2-5c^2+4c^2}{c^2-c^2}=\frac{0}{0}$$

$$=\lim_{x\to -c}\frac{(x+c)(x+4c)}{x(x+c)}$$

$$=\lim_{x\to -c}\frac{x+4c}{x}$$

$$=\frac{-c+4c}{-c}$$

$$=\frac{3c}{-c}$$

$$=-3$$

Exercise

Find the limit: $\lim\limits_{x\to 16}\dfrac{\sqrt[4]{x}-2}{x-16}$

Solution

$$\lim_{x\to 16}\frac{\sqrt[4]{x}-2}{x-16}=\frac{\sqrt[4]{16}-2}{16-16}=\frac{2-2}{0}=\frac{0}{0}$$

$$\lim_{x\to 16}\frac{\sqrt[4]{x}-2}{\left(\sqrt[4]{x}\right)^4-2^4}$$

$$a^4-b^4=\left(a^2+b^2\right)(a-b)(a+b)$$

$$=\lim_{x\to 16}\frac{\sqrt[4]{x}-2}{\left(\sqrt{x}+2^2\right)\left(\sqrt[4]{x}+2\right)\left(\sqrt[4]{x}-2\right)}$$

$$=\lim_{x\to 16}\frac{1}{\left(\sqrt{x}+4\right)\left(\sqrt[4]{x}+2\right)}$$

$$=\frac{1}{\left(\sqrt{16}+4\right)\left(\sqrt[4]{16}+2\right)}$$

$$=\frac{1}{(4+4)(2+2)}$$

$$=\frac{1}{(8)(4)}$$

$$=\frac{1}{32}$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{x}-1}$

Solution

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}{\sqrt{x}-1}$$

$$= \lim_{x\to 1} \left(\sqrt{x}+1\right)$$

$$= 2$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{4x+5}-3}$

Solution

$$\lim_{x\to 1} \frac{x-1}{\sqrt{4x+5}-3} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{x-1}{\sqrt{4x+5}-3} \cdot \frac{\sqrt{4x+5}+3}{\sqrt{4x+5}+3}$$

$$= \lim_{x\to 1} \frac{(x-1)\left(\sqrt{4x+5}+3\right)}{4x+5-9}$$

$$= \lim_{x\to 1} \frac{(x-1)\left(\sqrt{4x+5}+3\right)}{4(x-1)}$$

$$= \frac{1}{5} \lim_{x\to 1} \left(\sqrt{4x+5}+3\right)$$

$$= \frac{6}{5}$$

Exercise

Find the limit: $\lim\limits_{x\to 4} \dfrac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}}$

Solution

$$\lim_{x\to 4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}} = \frac{0}{3-3} = \frac{0}{0}$$

$$= \lim_{x\to 4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}} \bullet \frac{3+\sqrt{x+5}}{3+\sqrt{x+5}}$$

$$= 3 \lim_{x\to 4} \frac{(x-4)\left(3+\sqrt{x+5}\right)\sqrt{x+5}}{9-(x+5)}$$

$$= 3 \lim_{x\to 4} \frac{(x-4)\left(3+\sqrt{x+5}\right)\sqrt{x+5}}{4-x}$$

$$= -3 \lim_{x\to 4} \left(3+\sqrt{x+5}\right)\sqrt{x+5}$$

$$= -3\ (6)(3)$$

$$= -54$$

Exercise

Find the limit: $\lim_{x\to 0} \dfrac{x}{\sqrt{ax+1}-1} \quad (a \neq 0)$

Solution

$$\lim_{x\to 0} \frac{x}{\sqrt{ax+1}-1} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{x}{\sqrt{ax+1}-1} \bullet \frac{\sqrt{ax+1}+1}{\sqrt{ax+1}+1}$$

$$= \lim_{x\to 0} \frac{x\left(\sqrt{ax+1}+1\right)}{ax+1-1}$$

$$= \lim_{x\to 0} \frac{x\left(\sqrt{ax+1}+1\right)}{ax}$$

$$= \frac{1}{a} \lim_{x\to 0} \left(\sqrt{ax+1}+1\right)$$

$$= \frac{2}{a}$$

Exercise

Find the limit: $\lim_{x\to \pi} \dfrac{\cos^2 x + 3\cos x + 2}{\cos x + 1}$

Solution

$$\lim_{x\to\pi} \frac{\cos^2 x + 3\cos x + 2}{\cos x + 1} = \frac{1-3+2}{-1+1} = \frac{0}{0}$$

$$= \lim_{x\to\pi} \frac{(\cos x + 1)(\cos x + 2)}{\cos x + 1}$$

$$= \lim_{x\to\pi} (\cos x + 2)$$

$$= -1 + 2$$

$$\underline{= 1}$$

Exercise

Find the limit: $\lim\limits_{x\to\frac{3\pi}{2}} \dfrac{\sin^2 x + 6\sin x + 5}{\sin^2 x - 1}$

Solution

$$\lim_{x\to\frac{3\pi}{2}} \frac{\sin^2 x + 6\sin x + 5}{\sin^2 x - 1} = \frac{1-6+5}{1-1} = \frac{0}{0}$$

$$= \lim_{x\to\frac{3\pi}{2}} \frac{(\sin x + 1)(\sin x + 5)}{(\sin x - 1)(\sin x + 1)}$$

$$= \lim_{x\to\frac{3\pi}{2}} \frac{\sin x + 5}{\sin x - 1}$$

$$= \frac{-1+5}{-1-1}$$

$$\underline{= -2}$$

Exercise

Find the limit: $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\sin x - 1}{\sqrt{\sin x} - 1}$

Solution

$$\lim_{x\to\frac{\pi}{2}} \frac{\sin x - 1}{\sqrt{\sin x} - 1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\left(\sqrt{\sin x} - 1\right)\left(\sqrt{\sin x} + 1\right)}{\sqrt{\sin x} - 1}$$

$$= \lim_{x\to\frac{\pi}{2}} \left(\sqrt{\sin x} + 1\right)$$

$$\underline{= 2}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{\dfrac{1}{2+\sin x}-\dfrac{1}{2}}{\sin x}$

Solution

$$\lim_{x\to 0} \frac{\frac{1}{2+\sin x}-\frac{1}{2}}{\sin x} = \frac{\frac{1}{2}-\frac{1}{2}}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{1}{\sin x} \bullet \frac{2-\sin x-2}{2(2+\sin x)}$$

$$= \frac{1}{2} \lim_{x\to 0} \frac{1}{\sin x} \frac{-\sin x}{(2+\sin x)}$$

$$= -\frac{1}{2} \lim_{x\to 0} \frac{1}{2+\sin x}$$

$$= -\frac{1}{2}\left(\frac{1}{2}\right)$$

$$= -\frac{1}{4}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{e^{2x}-1}{e^{x}-1}$

Solution

$$\lim_{x\to 0} \frac{e^{2x}-1}{e^{x}-1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\left(e^{x}-1\right)\left(e^{x}+1\right)}{e^{x}-1}$$

$$= \lim_{x\to 0} \left(e^{x}+1\right)$$

$$= 2$$

Exercise

Find the limit: $\lim\limits_{x\to \frac{\pi}{4}} \csc x$

Solution

$$\lim_{x\to \frac{\pi}{4}} \csc x = \csc\frac{\pi}{4}$$

$$=\frac{1}{\cos\frac{\pi}{4}}$$

$$=\sqrt{2}$$

Exercise

Find the limit: $\lim\limits_{x\to 4}\frac{x-5}{\left(x^2-10x+24\right)^2}$

Solution

$$\lim_{x\to 4}\frac{x-5}{\left(x^2-10x+24\right)^2}=\frac{-1}{(16-41+24)^2}$$

$$=-1$$

Exercise

Find the limit: $\lim\limits_{x\to 0}\frac{\cos x-1}{\sin^2 x}$

Solution

$$\lim_{x\to 0}\frac{\cos x-1}{\sin^2 x}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{\cos x-1}{(1-\cos x)(1+\cos x)}$$

$$=-\lim_{x\to 0}\frac{1}{1+\cos x}$$

$$=-\frac{1}{2}$$

Exercise

Find the limit: $\lim\limits_{x\to 0}\frac{1-\cos^2 x}{\sin x}$

Solution

$$\lim_{x\to 0}\frac{1-\cos^2 x}{\sin x}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{\sin x}$$

$$= \lim_{x\to 0} \sin x$$

$$\underline{= 0}$$

Exercise

Find $\displaystyle\lim_{x\to 0} \frac{x^3 - 5x^2}{x^2}$

Solution

$$\lim_{x\to 0} \frac{x^3 - 5x^2}{x^2} = \frac{0}{0}$$

$$= \lim_{x\to 0} (x - 5)$$

$$\underline{= -5}$$

Exercise

Find $\displaystyle\lim_{x\to 5} \frac{4x^2 - 100}{x-5}$

Solution

$$\lim_{x\to 5} \frac{4x^2 - 100}{x-5} = \frac{0}{0}$$

$$= \lim_{x\to 5} \frac{4(x-5)(x+5)}{x-5}$$

$$= \lim_{x\to 5} 4(x+5)$$

$$\underline{= 40}$$

Exercise

Find $\displaystyle\lim_{x\to 3} \frac{\sqrt{9 - 6x + x^2}}{x - 3}$

Solution

$$\lim_{x\to 3} \frac{\sqrt{9 - 6x + x^2}}{x - 3} = \frac{\sqrt{9 - 18 + 9}}{3 - 3} = \frac{0}{0}$$

$$= \lim_{x\to 3} \frac{\sqrt{(x-3)^2}}{x - 3}$$

$$= \lim_{x \to 3} \frac{x-3}{x-3}$$

$$\underline{=1}$$

Exercise

Find $\lim_{x \to 3} \frac{\sqrt{9+6x+x^2}}{x-3}$

Solution

$$\lim_{x \to 3} \frac{\sqrt{9+6x+x^2}}{x-3} = \frac{\sqrt{9+18+9}}{3-3}$$

$$= \frac{\sqrt{36}}{0}$$

$$\underline{= \infty}$$

Exercise

Find $\lim_{x \to 3} \frac{\sqrt{x^2-9}}{x-3}$

Solution

$$\lim_{x \to 3} \frac{\sqrt{x^2-9}}{x-3} = \frac{\sqrt{9-9}}{3-3} = \frac{0}{0}$$

$$= \lim_{x \to 3} \frac{\sqrt{(x-3)(x+3)}}{x-3}$$

$$= \lim_{x \to 3} \sqrt{\frac{x+3}{x-3}}$$

$$= \sqrt{\frac{6}{0}}$$

$$\underline{= \infty}$$

Exercise

Find $\lim_{x \to \frac{4\pi}{3}} \sin x$

Solution

$$\lim_{x \to \frac{4\pi}{3}} \sin x = \sin \frac{4\pi}{3}$$

$$= -\frac{\sqrt{3}}{2}$$

Exercise

Find $\lim\limits_{x \to \frac{2\pi}{3}} \cos x$

Solution

$$\lim_{x \to \frac{2\pi}{3}} \cos x = \cos\frac{2\pi}{3}$$

$$= -\frac{1}{2}$$

Exercise

Find $\lim\limits_{x \to \frac{7\pi}{4}} \sin x$

Solution

$$\lim_{x \to \frac{7\pi}{4}} \sin x = \sin\frac{7\pi}{4}$$

$$= -\frac{\sqrt{2}}{2}$$

Exercise

Find $\lim\limits_{x \to 1} \dfrac{\sin\sqrt{1-x}}{\sqrt{1-x^2}}$

Solution

$$\lim_{x \to 1} \frac{\sin\sqrt{1-x}}{\sqrt{1-x^2}} = \frac{\sin 0}{0} = \frac{0}{0}$$

$$= \lim_{x \to 1} \frac{\sin\sqrt{1-x}}{\sqrt{1-x}} \frac{1}{\sqrt{1+x}}$$

$$= \lim_{(1-x) \to 0} \frac{\sin\sqrt{1-x}}{\sqrt{1-x}} \quad \lim_{x \to 1} \frac{1}{\sqrt{1+x}}$$

$$= 1\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}$$

Exercise

Find $\displaystyle\lim_{x\to 2}\frac{\sin\sqrt{2-x}}{\sqrt{4-x^2}}$

Solution

$$\lim_{x\to 2}\frac{\sin\sqrt{2-x}}{\sqrt{4-x^2}}=\frac{\sin 0}{0}=\frac{0}{0}$$

$$=\lim_{x\to 2}\frac{\sin\sqrt{2-x}}{\sqrt{2-x}}\frac{1}{\sqrt{2+x}}$$

$$=\lim_{\sqrt{2-x}\to 0}\frac{\sin\sqrt{2-x}}{\sqrt{2-x}}\lim_{x\to 2}\frac{1}{\sqrt{2+x}}$$

$$=1\left(\frac{1}{2}\right)$$

$$=\frac{1}{2}$$

Exercise

Find $\displaystyle\lim_{x\to 0}\frac{\sin\left(\sqrt{5}\,x\right)}{\sin\left(\sqrt{3}\,x\right)}$

Solution

$$\lim_{x\to 0}\frac{\sin\left(\sqrt{5}\,x\right)}{\sin\left(\sqrt{3}\,x\right)}=\frac{\sin 0}{\sin 0}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{\sqrt{5}\,x}{\sqrt{3}\,x}\frac{\sin\left(\sqrt{5}\,x\right)}{\sqrt{5}\,x}\cdot\frac{1}{\frac{\sin\left(\sqrt{3}\,x\right)}{\sqrt{3}\,x}}$$

$$=\frac{\sqrt{5}}{\sqrt{3}}\lim_{\sqrt{5}\,x\to 0}\frac{\sin\left(\sqrt{5}\,x\right)}{\sqrt{5}\,x}\cdot\frac{1}{\lim_{\sqrt{3}\,x\to 0}\frac{\sin\left(\sqrt{3}\,x\right)}{\sqrt{3}\,x}}$$

$$=\frac{\sqrt{5}}{\sqrt{3}}$$

Exercise

Find $\lim\limits_{x\to 0} \dfrac{\sin\left(\sqrt{15}\ x\right)}{\sin\left(\sqrt{3}\ x\right)}$

Solution

$$\lim_{x\to 0} \frac{\sin\left(\sqrt{15}\ x\right)}{\sin\left(\sqrt{3}\ x\right)} = \frac{\sin 0}{\sin 0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sqrt{15}\ x}{\sqrt{3}\ x} \frac{\sin\left(\sqrt{5}\ x\right)}{\sqrt{15}\ x} \cdot \frac{1}{\frac{\sin\left(\sqrt{3}\ x\right)}{\sqrt{3}\ x}}$$

$$= \sqrt{\frac{15}{3}} \lim_{\sqrt{15}\ x\to 0} \frac{\sin\left(\sqrt{15}\ x\right)}{\sqrt{15}\ x} \cdot \frac{1}{\lim\limits_{\sqrt{3}\ x\to 0} \frac{\sin\left(\sqrt{3}\ x\right)}{\sqrt{3}\ x}}$$

$$= \sqrt{3}$$

Exercise

Find $\lim\limits_{x\to 0^+} \dfrac{x-\sqrt{x}}{\sqrt{\sin x}}$

Solution

$$\lim_{x\to 0^+} \frac{x-\sqrt{x}}{\sqrt{\sin x}} = \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{\frac{x-\sqrt{x}}{\sqrt{\sin x}}}{\sqrt{x}} \cdot \frac{1}{\sqrt{x}}$$

$$= \lim_{x\to 0^+} \frac{1}{\sqrt{\frac{\sin x}{x}}} \lim_{x\to 0^+} \frac{x-\sqrt{x}}{\sqrt{x}}$$

$$= (1) \lim_{x\to 0^+} \left(\frac{x}{\sqrt{x}} - \frac{\sqrt{x}}{\sqrt{x}}\right)$$

$$= \lim_{x\to 0^+} \left(\sqrt{x} - 1\right)$$

$$= -1$$

Exercise

Find $\lim\limits_{x\to 1} \dfrac{x-\sqrt{x}}{\sqrt{\sin x}}$

Solution

$$\lim_{x\to 1} \frac{x-\sqrt{x}}{\sqrt{\sin x}} = \frac{0}{\sqrt{\sin 1}}$$

$$\underline{= 0}$$

Exercise

Find $\lim\limits_{x\to \pi} \dfrac{x-\sqrt{x}}{\sqrt{\sin x}}$

Solution

$$\lim_{x\to \pi} \frac{x-\sqrt{x}}{\sqrt{\sin x}} = \frac{\pi-\sqrt{\pi}}{\sqrt{\sin \pi}}$$

$$= \frac{\pi-\sqrt{\pi}}{0}$$

$$\underline{= \infty}$$

Exercise

Find $\lim\limits_{\theta\to \frac{\pi}{2}} \left(\frac{3}{2}\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta\right)$

Solution

$$\lim_{\theta\to \frac{\pi}{2}} \left(\frac{3}{2}\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta\right) = \frac{3\pi}{4} - \sin\pi + \frac{1}{8}\sin 2\pi$$

$$\underline{= \frac{3\pi}{4}}$$

Exercise

Find $\lim\limits_{\theta\to -\pi} \ln|2+\cos\theta|$

Solution

$$\lim_{\theta\to -\pi} \ln|2+\cos\theta| = \ln|2-1|$$

$$= \ln 1$$

$$\underline{= 0}$$

Exercise

Find $\lim\limits_{x\to 0} e^{x^3}$

Solution

$$\lim_{x\to 0} e^{x^3} = e^0$$

$$\boxed{=1}$$

Exercise

Find $\lim\limits_{x\to 1} e^{x^2}$

Solution

$$\lim_{x\to 1} e^{x^2} = e^1$$

$$\boxed{=e}$$

Exercise

Find $\lim\limits_{x\to 1} e^{x^3-1}$

Solution

$$\lim_{x\to 1} e^{x^3-1} = e^{1-1}$$

$$= e^0$$

$$\boxed{=1}$$

Exercise

Find $\lim\limits_{x\to -1} e^{x^3-1}$

Solution

$$\lim_{x\to -1} e^{x^3-1} = e^{-1-1}$$

$$= e^{-2}$$

$$\boxed{=\frac{1}{e^2}}$$

Exercise

Find $\lim\limits_{x\to 2}\left(e^{x^2}-\ln x\right)$

Solution

$$\lim_{x\to 2}\left(e^{x^2}-\ln x\right)=e^4-\ln 2$$

Exercise

Find $\lim\limits_{x\to 1}\left(e^{x^2}-\ln x\right)$

Solution

$$\lim_{x\to 1}\left(e^{x^2}-\ln x\right)=e-\ln 1$$

$$=e$$

Exercise

Find $\lim\limits_{x\to e}\ln x$

Solution

$$\lim_{x\to e}\ln x=\ln e$$

$$=1$$

Exercise

Find $\lim\limits_{x\to e}\ln x^2$

Solution

$$\lim_{x\to e}\ln x^2=\ln e^2$$

$$=2\ln e$$

$$=2$$

Exercise

Find $\lim\limits_{x\to 0^+} \ln x$

Solution

$$\lim_{x\to 0^+} \ln x = \ln 0^+$$
$$= -\infty$$

Exercise

Find $\lim\limits_{x\to 1} \dfrac{1}{\ln x}$

Solution

$$\lim_{x\to 1} \frac{1}{\ln x} = \frac{1}{\ln 1}$$
$$= \frac{1}{0}$$
$$= \infty$$

Exercise

Find $\lim\limits_{x\to e} \ln e^{2x}$

Solution

$$\lim_{x\to e} \ln e^{2x} = \ln e^{2e}$$
$$= 2e \ln e$$
$$= 2e$$

Exercise

Find $\lim\limits_{x\to 1} \ln e^{x^2}$

Solution

$$\lim_{x\to 1} \ln e^{x^2} = \ln e$$
$$= 1$$

Exercise

Find $\lim\limits_{x\to -1} \ln e^{x^2}$

Solution

$$\lim_{x\to -1} \ln e^{x^2} = \ln e^1$$

$$\boxed{=1}$$

Exercise

Find $\lim\limits_{x\to 1} \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Solution

$$\lim_{x\to 1} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e - e^{-1}}{e + e^{-1}}$$

$$= \frac{e - \frac{1}{e}}{e + \frac{1}{e}}$$

$$\boxed{= \frac{e^2 - 1}{e^2 + 1}}$$

Exercise

Find $\lim\limits_{x\to 0} \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Solution

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1-1}{1+1}$$

$$\boxed{=0}$$

Exercise

Find $\lim\limits_{x\to 3} \frac{1}{3}\ln\left|x^3 + 3x^2 - 6x\right|$

Solution

$$\lim_{x\to 3} \frac{1}{3}\ln\left|x^3 + 3x^2 - 6x\right| = \frac{1}{3}\ln|27 + 27 - 18|$$

$$= \frac{1}{3}\ln 36$$

$$= \ln\left(\sqrt[3]{36}\right)$$

Exercise

Find $\lim\limits_{x\to\ln 2} \arctan e^{x}$

Solution

$$\lim_{x\to\ln 2} \arctan e^{x} = \arctan\left(e^{\ln 2}\right)$$

$$= \arctan(2)$$

Exercise

Find $\lim\limits_{x\to 0} \arctan e^{x}$

Solution

$$\lim_{x\to 0} \arctan e^{x} = \arctan(1)$$

$$= \frac{\pi}{4}$$

Exercise

Find $\lim\limits_{x\to 1} \ln\left|2x^3 + 9x^2 + 12x + 36\right|$

Solution

$$\lim_{x\to 1} \ln\left|2x^3 + 9x^2 + 12x + 36\right| = \ln|2 + 9 + 12 + 36|$$

$$= \ln 59$$

Exercise

Find $\lim\limits_{x\to\frac{\pi}{4}} e^{\sin^2 x}$

Solution

$$\lim_{x\to\frac{\pi}{4}} e^{\sin^2 x} = e^{\sin^2\frac{\pi}{4}}$$

$$= e^{\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$\boxed{= e^{\frac{1}{2}}}$$

Exercise

Suppose $\lim\limits_{x\to c} f(x) = 5$ *and* $\lim\limits_{x\to c} g(x) = -2$. Find

a) $\lim\limits_{x\to c} f(x)g(x)$

b) $\lim\limits_{x\to c} 2f(x)g(x)$

c) $\lim\limits_{x\to c} \left(f(x)+3g(x)\right)$

d) $\lim\limits_{x\to c} \dfrac{f(x)}{f(x)-g(x)}$

Solution

a) $\lim\limits_{x\to c} f(x)g(x) = \lim\limits_{x\to c} f(x) \cdot \lim\limits_{x\to c} g(x)$

$$= (5)(-2)$$

$$\boxed{= -10}$$

b) $\lim\limits_{x\to c} 2f(x)g(x) = 2 \lim\limits_{x\to c} f(x) \cdot \lim\limits_{x\to c} g(x)$

$$= 2(-10)$$

$$\boxed{= -20}$$

c) $\lim\limits_{x\to c} \left(f(x)+3g(x)\right) = \lim\limits_{x\to c} f(x) + 3 \lim\limits_{x\to c} g(x)$

$$= 5 + 3(-2)$$

$$\boxed{= -1}$$

d) $\lim\limits_{x\to c} \dfrac{f(x)}{f(x)-g(x)} = \dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} f(x) - \lim\limits_{x\to c} g(x)}$

$$= \frac{5}{5-(-2)}$$

$$\boxed{= \frac{5}{7}}$$

Exercise

Explain why the limits do not exist for $\lim\limits_{x\to 0} \frac{x}{|x|}$

Solution

$$\lim_{x\to 0} \frac{x}{|x|} = \frac{0}{0}$$

$$\lim_{x\to 0^-} \frac{x}{|x|} = \frac{-x}{x} = -1$$

$$\lim_{x\to 0^+} \frac{x}{|x|} = \frac{x}{x} = 1$$

Doesn't exist

Exercise

If $\lim\limits_{x\to 4} \frac{f(x)-5}{x-2} = 1$, find $\lim\limits_{x\to 4} f(x)$

Solution

$$\lim_{x\to 4} \frac{f(x)-5}{x-2} = 1$$

$$\frac{\lim\limits_{x\to 4} f(x)-5}{4-2} = 1$$

$$\frac{\lim\limits_{x\to 4} f(x)-5}{2} = 1$$

Multiply both sides by 2

$$\lim_{x\to 4} f(x) - 5 = 2$$

Add 5 on both sides

$$\lim_{x\to 4} f(x) = 7$$

Exercise

If $\lim\limits_{x\to 0} \frac{f(x)}{x^2} = 1$, find $\lim\limits_{x\to 0} f(x)$ *and* $\lim\limits_{x\to 0} \frac{f(x)}{x}$

Solution

$$\lim_{x\to 0} \frac{f(x)}{x^2} = 1$$

$$\frac{\lim\limits_{x\to 0} f(x)}{\lim\limits_{x\to 0} x^2} = 1$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2$$

$$\underline{= 0 \,|}$$

$$\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0}\left(\frac{f(x)}{x^2}\bullet x\right)$$

$$= \lim_{x\to 0}\frac{f(x)}{x^2}\bullet \lim_{x\to 0} x$$

$$= 1\bullet 0$$

$$\underline{= 0 \,|}$$

Exercise

If $x^4 \le f(x) \le x^2;\quad -1 \le x \le 1$ and $x^2 \le f(x) \le x^4;\quad x < -1 \;\; and \;\; x > 1$. At what points c do you automatically know $\lim\limits_{x\to c} f(x)$? What can you say about the value of the limits at these points?

Solution

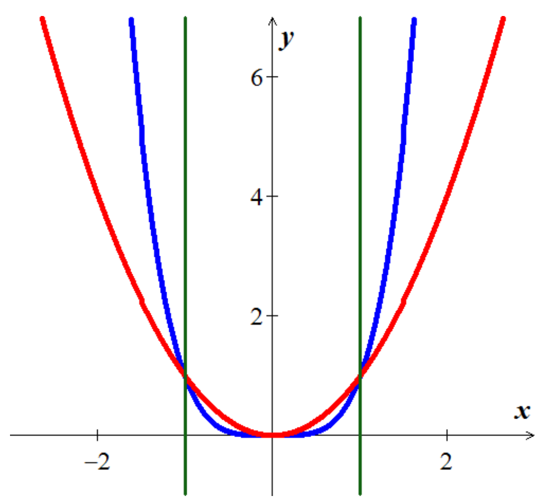
$$\lim_{x\to c} x^4 = \lim_{x\to c} x^2 \quad \Rightarrow \quad c^4 = c^2$$

$$c^4 - c^2 = 0$$

$$c^2\left(c^2 - 1\right) = 0$$

$$c^2 = 0 \qquad\qquad c^2 - 1 = 0$$

$$\boxed{c = 0} \qquad\qquad \boxed{c = \pm 1}$$



$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2$$

$$\underline{= 0 \,|}$$

$$\lim_{x\to -1} f(x) = \lim_{x\to 1} f(x)$$

$$\underline{= 1 \,|}$$

Exercise

Given the piecewise function: $f(x)=\begin{cases} 3x-1 & for \quad x\le -1 \\ x^2 & for \quad x>-1 \end{cases}$

Find

a) $\lim\limits_{x\to -5} f(x)$ *b)* $\lim\limits_{x\to -1} f(x)$ *c)* $\lim\limits_{x\to 1} f(x)$

Solution

a) $\lim\limits_{x\to -5} f(x)=3(-5)-1$

$$=-6$$

b) $\lim\limits_{x\to -1} f(x)=3(-1)-1$

$$=-4$$

c) $\lim\limits_{x\to 1} f(x) = 1$

Exercise

Find the limit: $\lim\limits_{x\to 0} f(x)$ $f(x)=\begin{cases} x^2+1 & x<0 \\ 2x+1 & x>0 \end{cases}$

Solution

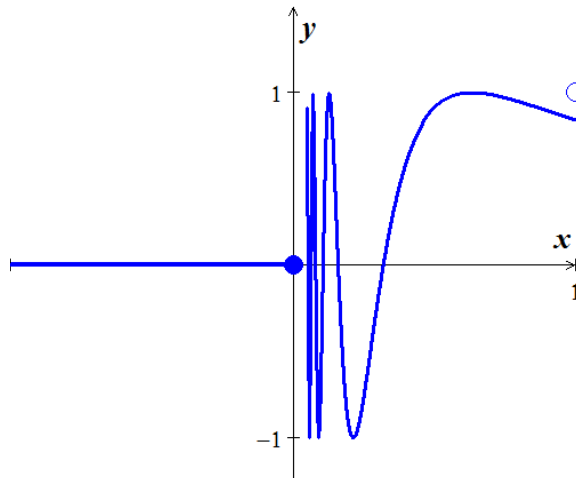
$$\lim_{x\to 0^-} x^2+1 = 1$$

$$\lim_{x\to 0+} 2x+1 = 1$$

$$\lim_{x\to 0} f(x) = 1$$

Exercise

Let $f(x)=\begin{cases} 0, & x \le 0 \\ \sin\frac{1}{x}, & x > 0 \end{cases}$



a) Does $\lim\limits_{x\to 0^+} f(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim\limits_{x\to 0^-} f(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim\limits_{x\to 0} f(x)$ exist? If so, what is it? If not, why not?

Solution

a) $\lim\limits_{x\to 0^+} f(x)$ doesn't exist,

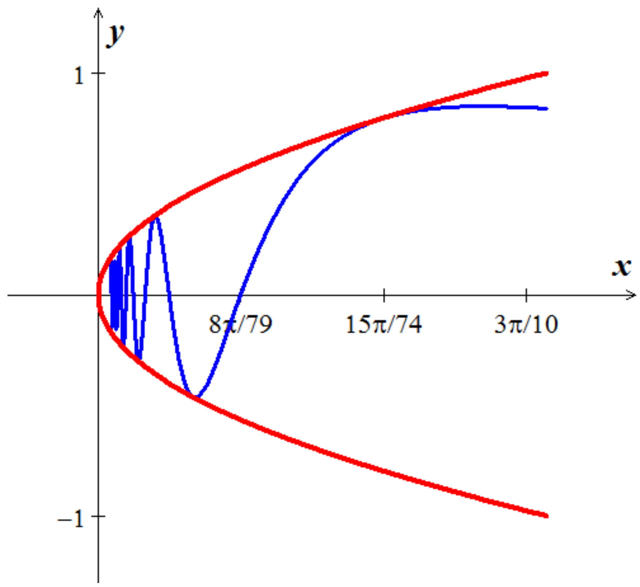
Since $\sin\left(\frac{1}{x}\right)$ doesn't approach any single value as $x \to 0$

b) $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} 0 = 0$

c) $\lim\limits_{x\to 0} f(x)$ doesn't exist, since $\lim\limits_{x\to 0^+} f(x)$ doesn't exist

Exercise

Let $g(x)=\sqrt{x}\ \sin\frac{1}{x}$



a) Does $\lim\limits_{x\to 0^+} g(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim\limits_{x\to 0^-} g(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim\limits_{x\to 0} g(x)$ exist? If so, what is it? If not, why not?

Solution

a) $\lim\limits_{x\to 0^+} g(x)$ exists, by the sandwich theorem $-\sqrt{x} \le g(x) \le \sqrt{x}.$ *for* $x>0$

b) $\lim\limits_{x\to 0^-} g(x)$ doesn't exist, since $\sqrt{x}$ is not defined for $x<0$

c) $\lim\limits_{x\to 0} g(x)$ doesn't exist, since $\lim\limits_{x\to 0^-} g(x)$ doesn't exist.

Solution Section 1.3 – Infinite Limits

Exercise

Find $\lim\limits_{x\to 5} \dfrac{x-7}{x(x-5)^2}$

Solution

$$\lim_{x\to 5} \frac{x-7}{x(x-5)^2} = \frac{-2}{0}$$

$$= \infty$$

Exercise

Find $\lim\limits_{x\to -5^+} \dfrac{x-5}{x+5}$

Solution

$$\lim_{x\to -5^+} \frac{x-5}{x+5} = \frac{-10}{0^+}$$

$$= -\infty$$

Exercise

Find $\lim\limits_{x\to 3^-} \dfrac{x-4}{x^2-3x}$

Solution

$$\lim_{x\to 3^-} \frac{x-4}{x^2-3x} = \frac{-1}{0^-}$$

$$= \infty$$

Exercise

Find $\lim\limits_{x\to 0^+} \dfrac{1}{3x}$

Solution

$$\lim_{x\to 0^+} \frac{1}{3x} = \frac{1}{0^+}$$

$$= \infty$$

Exercise

Find $\displaystyle\lim_{x\to -5^-} \frac{3x}{2x+10}$

Solution

$$\lim_{x\to -5^-} \frac{3x}{2x+10} = \lim_{x\to -5^-} \frac{3}{2+\frac{10}{x}}$$

$$= \infty$$

Exercise

Find $\displaystyle\lim_{x\to 0} \frac{1}{x^{2/3}}$

Solution

$$\lim_{x\to 0} \frac{1}{x^{2/3}} = \lim_{x\to 0} \frac{1}{\left(x^{1/3}\right)^2}$$

$$= \infty$$

Exercise

Find $\displaystyle\lim_{x\to 0^-} \frac{1}{3x^{1/3}}$

Solution

$$\lim_{x\to 0^-} \frac{1}{3x^{1/3}} = \frac{1}{0^-}$$

$$= -\infty$$

Exercise

Find $\displaystyle\lim_{x\to \left(-\frac{\pi}{2}\right)^+} \sec x$

Solution

$$\lim_{x\to \left(-\frac{\pi}{2}\right)^+} \sec x = \frac{1}{\cos \frac{\pi}{2}^+}$$

$$= \infty$$

Exercise

Find $\lim\limits_{\theta\to 0^-}(1+\csc\theta)$

Solution

$$\lim\limits_{\theta\to 0^-}(1+\csc\theta)=\lim\limits_{\theta\to 0^-}\left(1+\frac{1}{\sin\theta}\right)$$

$$\underline{=-\infty}$$

Exercise

Find $\lim\limits_{\theta\to 0^+}\csc\theta$

Solution

$$\lim\limits_{\theta\to 0^+}\csc\theta=\lim\limits_{\theta\to 0^+}\frac{1}{\sin\theta}$$

As $\theta\to 0^+ \quad \sin\theta>0$

$$\underline{=+\infty}$$

Exercise

Find $\lim\limits_{x\to 0+}(-10\cot x)$

Solution

$$\lim\limits_{x\to 0^+}(-10\cot x)=-10\lim\limits_{x\to 0^+}\frac{\cos\theta}{\sin\theta}=-10\left(\frac{1}{0}\right)$$

As $x\to 0^+ \quad \cos\theta>0; \quad \sin\theta>0$

$$\underline{=-\infty}$$

Exercise

Find $\lim\limits_{\theta\to \frac{\pi}{2}^+}\frac{1}{3}\tan\theta$

Solution

$$\lim\limits_{\theta\to \frac{\pi}{2}^+}\frac{1}{3}\tan\theta=\frac{1}{3}\lim\limits_{\theta\to \frac{\pi}{2}^+}\frac{\sin\theta}{\cos\theta}=\frac{1}{3}\left(-\frac{1}{0}\right)$$

As $\theta\to \frac{\pi}{2}^+ \quad \cos\theta<0; \quad \sin\theta>0$

$$\underline{=-\infty}$$

Exercise

Find $\displaystyle\lim_{x\to 2^+} \frac{1}{x-2}$

Solution

$$\lim_{x\to 2^+} \frac{1}{x-2} = \frac{1}{2^+-2} = \frac{1}{0^+}$$

$$\underline{= \infty}$$

Exercise

Find $\displaystyle\lim_{x\to 2^-} \frac{1}{x-2}$

Solution

$$\lim_{x\to 2^-} \frac{1}{x-2} = \frac{1}{2^--2} = \frac{1}{0^-}$$

$$\underline{= -\infty}$$

Exercise

Find $\displaystyle\lim_{x\to 2} \frac{1}{x-2}$

Solution

$$\lim_{x\to 2} \frac{1}{x-2} = \frac{1}{0}$$

$$\underline{= \infty}$$

Exercise

Find $\displaystyle\lim_{x\to 3^+} \frac{2}{(x-3)^3}$

Solution

$$\lim_{x\to 3^+} \frac{2}{(x-3)^3} = \frac{2}{0^+}$$

$$\underline{= \infty}$$

Exercise

Find $\lim\limits_{x \to 3^-} \dfrac{2}{(x-3)^3}$

Solution

$$\lim_{x \to 3^-} \frac{2}{(x-3)^3} = \frac{2}{0^-}$$

$$= -\infty$$

Exercise

Find $\lim\limits_{x \to 3} \dfrac{2}{(x-3)^3}$

Solution

$$\lim_{x \to 3} \frac{2}{(x-3)^3} = \frac{2}{0}$$

$$= \infty$$

Exercise

Find $\lim\limits_{x \to 4^+} \dfrac{x-5}{(x-4)^2}$

Solution

$$\lim_{x \to 4^+} \frac{x-5}{(x-4)^2} = \frac{-1}{0}$$

$$= -\infty$$

Exercise

Find $\lim\limits_{x \to 4^-} \dfrac{x-5}{(x-4)^2}$

Solution

$$\lim_{x \to 4^-} \frac{x-5}{(x-4)^2} = \frac{-1}{0}$$

$$= -\infty$$

Exercise

Find $\displaystyle\lim_{x\to 4} \frac{x-5}{(x-4)^2}$

Solution

$$\lim_{x\to 4^-} \frac{x-5}{(x-4)^2} = \frac{-1}{0}$$

$$= -\infty$$

Exercise

Find $\displaystyle\lim_{x\to 1^+} \frac{x-2}{(x-1)^3}$

Solution

$$\lim_{x\to 1^+} \frac{x-2}{(x-1)^3} = \frac{-1}{0^+}$$

$$= -\infty$$

Exercise

Find $\displaystyle\lim_{x\to 1^-} \frac{x-2}{(x-1)^3}$

Solution

$$\lim_{x\to 1^-} \frac{x-2}{(x-1)^3} = \frac{-1}{0^-}$$

$$= \infty$$

Exercise

Find $\displaystyle\lim_{x\to 1} \frac{x-2}{(x-1)^3}$

Solution

$$\lim_{x\to 1} \frac{x-2}{(x-1)^3} = \frac{-1}{0^+}$$

$$= \nexists$$

Exercise

Find $\lim\limits_{x\to 3^+} \dfrac{(x-1)(x-2)}{x-3}$

Solution

$$\lim_{x\to 3^+} \frac{(x-1)(x-2)}{x-3} = \frac{2}{0}$$

$$\boxed{=\infty}$$

Exercise

Find $\lim\limits_{x\to 3^-} \dfrac{(x-1)(x-2)}{x-3}$

Solution

$$\lim_{x\to 3^-} \frac{(x-1)(x-2)}{x-3} = \frac{2}{0^-}$$

$$\boxed{=-\infty}$$

Exercise

Find $\lim\limits_{x\to 3} \dfrac{(x-1)(x-2)}{x-3}$

Solution

$$\lim_{x\to 3^-} \frac{(x-1)(x-2)}{x-3} = \frac{2}{0^-}$$

$$\boxed{=-\infty}$$

$$\lim_{x\to 3^+} \frac{(x-1)(x-2)}{x-3} \boxed{=\infty}$$

$$\lim_{x\to 3} \frac{(x-1)(x-2)}{x-3} \boxed{=\nexists}$$

Exercise

Find $\lim\limits_{x\to -2^+} \dfrac{x-4}{x(x+2)}$

Solution

$$\lim_{x\to -2^+} \frac{x-4}{x(x+2)} = \frac{-6}{-0^+}$$

$$\underline{= \infty}$$

Exercise

Find $\displaystyle\lim_{x\to -2^-} \frac{x-4}{x(x+2)}$

Solution

$$\lim_{x\to -2^-} \frac{x-4}{x(x+2)} = \frac{-6}{0^+}$$

$$\underline{= -\infty}$$

Exercise

Find $\displaystyle\lim_{x\to -2} \frac{x-4}{x(x+2)}$

Solution

$$\lim_{x\to -2^+} \frac{x-4}{x(x+2)} = \infty$$

$$\lim_{x\to -2^-} \frac{x-4}{x(x+2)} = -\infty$$

$$\lim_{x\to -2} \frac{x-4}{x(x+2)} \underline{= \nexists}$$

Exercise

Find $\displaystyle\lim_{x\to 2^+} \frac{x^2-4x+3}{(x-2)^2}$

Solution

$$\lim_{x\to 2^+} \frac{x^2-4x+3}{(x-2)^2} = \frac{-1}{0^+}$$

$$\underline{= -\infty}$$

Exercise

Find $\lim\limits_{x\to 2^-} \dfrac{x^2-4x+3}{(x-2)^2}$

Solution

$$\lim_{x\to 2^-} \frac{x^2-4x+3}{(x-2)^2} = \frac{-1}{0^+}$$

$$\underline{= -\infty}$$

Exercise

Find $\lim\limits_{x\to 2} \dfrac{x^2-4x+3}{(x-2)^2}$

Solution

$$\lim_{x\to 2} \frac{x^2-4x+3}{(x-2)^2} = \frac{-1}{0}$$

$$\underline{= -\infty}$$

Exercise

Find $\lim\limits_{x\to -2^+} \dfrac{x^3-5x^2+6x}{x^4-4x^2}$

Solution

$$\lim_{x\to -2^+} \frac{x^3-5x^2+6x}{x^4-4x^2} = \lim_{x\to -2^+} \frac{x(x-2)(x-3)}{x^2(x-2)(x+2)}$$

$$= \lim_{x\to -2^+} \frac{x-3}{x(x+2)} \qquad \frac{-}{-(+)}$$

$$\underline{= \infty}$$

Exercise

Find $\lim\limits_{x\to -2^-} \dfrac{x^3-5x^2+6x}{x^4-4x^2}$

Solution

$$\lim_{x\to -2^-} \frac{x^3-5x^2+6x}{x^4-4x^2} = \lim_{x\to -2^-} \frac{x(x-2)(x-3)}{x^2(x-2)(x+2)}$$

$$= \lim_{x\to -2^-} \frac{x-3}{x(x+2)} \quad \frac{-}{-(-)}$$

$$\underline{= -\infty}$$

Exercise

Find $\lim_{x\to -2} \dfrac{x^3-5x^2+6x}{x^4-4x^2}$

Solution

$$\lim_{x\to -2} \frac{x^3-5x^2+6x}{x^4-4x^2} = \frac{-8-20-12}{16-16}$$

$$= \frac{-40}{0}$$

$$\underline{= -\infty}$$

Exercise

Find $\lim_{u\to 0^+} \dfrac{u-1}{\sin u}$

Solution

$$\lim_{u\to 0^+} \frac{u-1}{\sin u} = \frac{-1}{0^+}$$

$$\underline{= -\infty}$$

Exercise

Find $\lim_{x\to 0^-} \dfrac{2}{\tan x}$

Solution

$$\lim_{x\to 0^-} \frac{2}{\tan x} = \frac{2}{0^-}$$

$$\underline{= -\infty}$$

Exercise

Find $\lim_{x\to 1^+} \dfrac{x^2-5x+6}{x-1}$

Solution

$$\lim_{x \to 1^+} \frac{x^2 - 5x + 6}{x - 1} = \frac{2}{0^+}$$

$$\underline{= \infty}$$

Exercise

Find $\lim_{x \to 2\pi^-} \csc x$

Solution

$$\lim_{x \to 2\pi^-} \csc x = \frac{1}{\sin\left(2\pi^-\right)} = \frac{1}{0^-}$$

$$\underline{= -\infty}$$

Exercise

Find $\lim_{x \to 0^+} e^{\sqrt{x}}$

Solution

$$\lim_{x \to 0^+} e^{\sqrt{x}} \underline{= 1}$$

Exercise

Find $\lim_{x \to \frac{\pi}{2}^-} \frac{1 + \sin x}{\cos x}$

Solution

$$\lim_{x \to \frac{\pi}{2}^-} \frac{1 + \sin x}{\cos x} = \frac{2}{0^+}$$

$$\underline{= \infty}$$

Exercise

Find $\lim_{x \to \frac{\pi}{2}^+} \frac{1 + \sin x}{\cos x}$

Solution

$$\lim_{x \to \frac{\pi}{2}^+} \frac{1+\sin x}{\cos x} = \frac{2}{0^-}$$

$$= -\infty$$

Exercise

Find $\quad \lim_{x \to 0^-} \dfrac{e^x}{1+e^x}$

Solution

$$\lim_{x \to 0^-} \frac{e^x}{1-e^x} = \frac{1}{0^+}$$

$$= \infty$$

Exercise

Find $\quad \lim_{x \to 0^+} \dfrac{e^x}{1-e^x}$

Solution

$$\lim_{x \to 0^+} \frac{e^x}{1-e^x} = \frac{1}{0^-}$$

$$= -\infty$$

Exercise

Find $\quad \lim_{x \to 1^-} \dfrac{x}{\ln x}$

Solution

$$\lim_{x \to 1^-} \frac{x}{\ln x} = \frac{1}{0^-}$$

$$= -\infty$$

Exercise

Find $\quad \lim_{x \to 0^+} \dfrac{x}{\ln x}$

Solution

$$\lim_{x\to 0^+} \frac{x}{\ln x} = \frac{0}{-\infty}$$

$$= 0$$

Exercise

Find $\displaystyle \lim_{x\to 0^-} \frac{2e^x + 5e^{3x}}{e^{2x} - e^{3x}}$

Solution

$$\lim_{x\to 0^-} \frac{2e^x + 5e^{3x}}{e^{2x} - e^{3x}} = \lim_{x\to 0^-} \frac{2e^x + 5e^{3x}}{e^{2x}\left(1 - e^x\right)}$$

$$= \frac{7}{0}$$

$$= \infty$$

Exercise

Find $\displaystyle \lim_{x\to 0^+} \frac{2e^x + 5e^{3x}}{e^{2x} - e^{3x}}$

Solution

$$\lim_{x\to 0^+} \frac{2e^x + 5e^{3x}}{e^{2x} - e^{3x}} = \lim_{x\to 0^+} \frac{2e^x + 5e^{3x}}{e^{2x}\left(1 - e^x\right)}$$

$$= \frac{7}{0^-}$$

$$= -\infty$$

Exercise

Find $\displaystyle \lim_{x\to 1^-} \frac{\ln x}{\sin^{-1} x}$

Solution

$$\lim_{x\to 1^-} \frac{\ln x}{\sin^{-1} x} = \frac{\ln 1}{\sin^{-1} 1}$$

$$= \frac{0}{\frac{\pi}{2}}$$

$$= 0$$

Exercise

Find $\lim\limits_{x\to 0} \dfrac{e^x}{\sin x}$

Solution

$$\lim_{x\to 0} \frac{e^x}{\sin x} = \frac{e^0}{\sin 0}$$
$$= \frac{1}{0}$$
$$\boxed{= \infty}$$

Exercise

Find $\lim\limits_{x\to 0^-} \dfrac{e^x}{\sin x}$

Solution

$$\lim_{x\to 0^-} \frac{e^x}{\sin x} = \frac{e^0}{\sin 0^-}$$
$$= \frac{1}{0^-}$$
$$\boxed{= -\infty}$$

Exercise

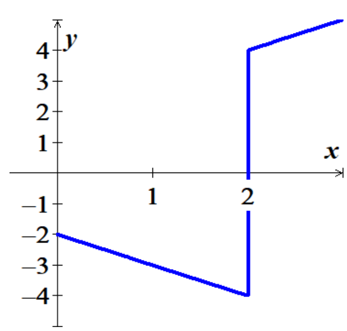
Find $\lim\limits_{x\to 2^-} \dfrac{|x-3|}{x-2}$

Solution

$$\lim_{x\to 2^-} \frac{|x-3|}{x-2} = \frac{|-1|}{0^+}$$
$$\boxed{= \infty}$$

Exercise

Find $\lim\limits_{x\to 2^-} \dfrac{|x^2-4|}{x-2}$



Solution

$$\lim_{x\to 2^-} \frac{|x^2-4|}{x-2} = \frac{|4-4|}{0^-} = \frac{0}{0}$$

$$= \lim_{x\to 2^-} \frac{|(x-2)(x+2)|}{x-2}$$

$$= 4 \lim_{x\to 2^-} \frac{|x-2|}{x-2}$$

$$\underline{= -4}$$

Exercise

Find $\displaystyle \lim_{x\to 2^+} \frac{|x^2-4|}{x-2}$

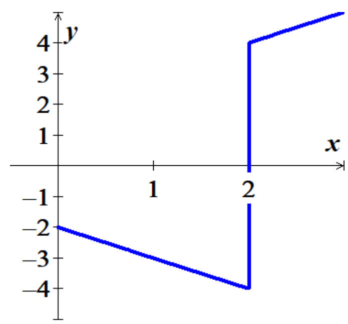
Solution

$$\lim_{x\to 2^+} \frac{|x^2-4|}{x-2} = \frac{0}{0}$$

$$= \lim_{x\to 2^+} \frac{|(x-2)(x+2)|}{x-2}$$

$$= 4 \lim_{x\to 2^+} \frac{|x-2|}{x-2}$$

$$\underline{= 4}$$



Exercise

Find $\displaystyle \lim_{x\to 3^-} \frac{1}{(x-3)^4}$

Solution

$$\lim_{x\to 3^-} \frac{1}{(x-3)^4} = \frac{1}{0^+}$$

$$\underline{= \infty}$$

Exercise

Find $\displaystyle \lim_{x\to 3^+} \frac{1}{(x-3)^4}$

Solution

$$\lim_{x\to 3^+} \frac{1}{(x-3)^4} = \frac{1}{0^+}$$

$$\underline{= \infty}$$

Exercise

Find $\displaystyle\lim_{x\to -2^-} \frac{1}{(x+2)^3}$

Solution

$$\lim_{x\to -2^-} \frac{1}{(x+2)^3} = \frac{1}{0^-}$$

$$= -\infty$$

Exercise

Find $\displaystyle\lim_{x\to -2^+} \frac{1}{(x+2)^3}$

Solution

$$\lim_{x\to -2^+} \frac{1}{(x+2)^3} = \frac{1}{0^+}$$

$$= \infty$$

Exercise

Find $\displaystyle\lim_{x\to \frac{\pi}{2}} \frac{e^x}{\cos x}$

Solution

$$\lim_{x\to \frac{\pi}{2}} \frac{e^x}{\cos x} = \frac{e^{\frac{\pi}{2}}}{\cos\frac{\pi}{2}}$$

$$= \frac{e^{\frac{\pi}{2}}}{0}$$

$$= \infty$$

Exercise

Find $\displaystyle\lim_{x\to 0} \frac{e^x}{\ln(x)}$

Solution

$$\lim_{x\to 0} \frac{e^x}{\ln(x)} = \frac{e^0}{\ln 0}$$

$$= \frac{1}{-\infty}$$

$$\underline{= 0}$$

Exercise

Find $\lim\limits_{x \to 1} \frac{1}{\ln x}$

Solution

$$\lim_{x \to 1} \frac{1}{\ln x} = \frac{1}{\ln 1}$$

$$= \frac{1}{0}$$

$$\underline{= \infty}$$

Exercise

Find $\lim\limits_{x \to \pi} \frac{x}{\sin x}$

Solution

$$\lim_{x \to \pi} \frac{x}{\sin x} = \frac{\pi}{\sin \pi}$$

$$= \frac{\pi}{0}$$

$$\underline{= \infty}$$

Exercise

Find $\lim\limits_{x \to \frac{\pi}{2}} \frac{\ln\left(\frac{1}{x}\right)}{\cos x}$

Solution

$$\lim_{x \to \frac{\pi}{2}} \frac{\ln\left(\frac{1}{x}\right)}{\cos x} = \frac{\ln\left(\frac{2}{\pi}\right)}{\cos \frac{\pi}{2}}$$

$$= \frac{\ln 2 - \ln \pi}{0}$$

$$\underline{= -\infty}$$

Exercise

Let $f(x)=\dfrac{x^2-7x+12}{x-a}$

a) For what values of a, if any, does $\lim\limits_{x\to a^+} f(x)$ equal a finite number?

b) For what values of a, if any, does $\lim\limits_{x\to a^+} f(x)=\infty$?

c) For what values of a, if any, does $\lim\limits_{x\to a^+} f(x)=-\infty$?

Solution

$$f(x)=\frac{x^2-7x+12}{x-a}=\frac{(x-3)(x-4)}{x-a}$$

a) If $a=3$, then

$$\lim_{x\to 3}\frac{(x-3)(x-4)}{x-3}=\lim_{x\to 3}(x-4)$$
$$\underline{=-1}\,|$$

If $a=4$, then

$$\lim_{x\to 4}\frac{(x-3)(x-4)}{x-4}=\lim_{x\to 4}(x-1)$$
$$\underline{=1}\,|$$

b) $\lim\limits_{x\to a^+} f(x)=\infty$ for any number other than 3 or 4.

As $x\to a^+$, then $(x-a)$ is always positive.

$(x-3)(x-4)>0 \quad \Rightarrow \quad (-\infty,\ 3)\cup(4,\ \infty)$

c) $\lim\limits_{x\to a^+} f(x)=-\infty$ for any number other than 3 or 4.

As $x\to a^+$, then $(x-a)$ is always positive, and $(3,\ 4)$

Exercise

Analyze $\lim\limits_{x\to 1^+}\sqrt{\dfrac{x-1}{x-3}}$ and $\lim\limits_{x\to 1^-}\sqrt{\dfrac{x-1}{x-3}}$

Solution

$$\lim_{x\to 1^+}\sqrt{\frac{x-1}{x-3}}=\sqrt{\frac{0^+}{-2}}\quad \nexists$$

$$\lim_{x\to 1^-} \sqrt{\frac{x-1}{x-3}} = \sqrt{\frac{0^-}{-2}}$$

$$\underline{= 0\;|}$$

Solution Section 1.4 – Limits at Infinity

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $h(x) = \dfrac{-5 + \frac{7}{x}}{3 - \frac{1}{x^2}}$

Solution

$$\lim_{x \to \infty} \frac{-5 + \frac{7}{x}}{3 - \frac{1}{x^2}} = -\frac{5}{3}$$

$$\lim_{x \to -\infty} \frac{-5 + \frac{7}{x}}{3 - \frac{1}{x^2}} = -\frac{5}{3}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{2x+3}{5x+7}$

Solution

$$\lim_{x \to \infty} \frac{2x+3}{5x+7} = \lim_{x \to \infty} \frac{2 + \frac{3}{x}}{5 + \frac{7}{x}}$$

$$= \frac{2}{5}$$

$$\lim_{x \to -\infty} \frac{2x+3}{5x+7} = \lim_{x \to -\infty} \frac{2 + \frac{3}{x}}{5 + \frac{7}{x}}$$

$$= \frac{2}{5}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{2x^3 + 7}{x^3 - x^2 + x + 7}$

Solution

$$\lim_{x\to\infty} \frac{2x^3+7}{x^3-x^2+x+7} = \lim_{x\to\infty} \frac{2+\frac{7}{x^3}}{1-\frac{1}{x}+\frac{1}{x^2}+\frac{7}{x^3}}$$

$$= 2$$

$$\lim_{x\to-\infty} \frac{2x^3+7}{x^3-x^2+x+7} = \lim_{x\to-\infty} \frac{2+\frac{7}{x^3}}{1-\frac{1}{x}+\frac{1}{x^2}+\frac{7}{x^3}}$$

$$= 2$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \frac{x+1}{x^2+3}$

Solution

$$\lim_{x\to\infty} \frac{x+1}{x^2+3} = \lim_{x\to\infty} \frac{\frac{x}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{3}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x^2}}$$

$$= 0$$

$$\lim_{x\to-\infty} \frac{x+1}{x^2+3} = \lim_{x\to-\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x^2}}$$

$$= 0$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \frac{7x^3}{x^3-3x^2+6x}$

Solution

$$\lim_{x\to\infty} \frac{7x^3}{x^3-3x^2+6x} = \lim_{x\to\infty} \frac{7}{1-\frac{3}{x}+\frac{6}{x^2}}$$

$$= 7$$

$$\lim_{x\to-\infty} \frac{7x^3}{x^3-3x^2+6x} = \lim_{x\to-\infty} \frac{7}{1-\frac{3}{x}+\frac{6}{x^2}}$$

$$\boxed{=7}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{9x^4+x}{2x^4+5x^2-x+6}$

Solution

$$\lim_{x\to\infty} \frac{9x^4+x}{2x^4+5x^2-x+6} = \lim_{x\to\infty} \frac{\frac{9x^4}{x^4}+\frac{x}{x^4}}{\frac{2x^4}{x^4}+\frac{5x^2}{x^4}-\frac{x}{x^4}+\frac{6}{x^4}}$$

$$= \lim_{x\to\infty} \frac{9+\frac{1}{x^3}}{2+\frac{5}{x^2}-\frac{1}{x^3}+\frac{6}{x^4}}$$

$$\boxed{=\frac{9}{2}}$$

$$\lim_{x\to-\infty} \frac{9x^4+x}{2x^4+5x^2-x+6} = \lim_{x\to-\infty} \frac{9+\frac{1}{x^3}}{2+\frac{5}{x^2}-\frac{1}{x^3}+\frac{6}{x^4}}$$

$$\boxed{=\frac{9}{2}}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{-2x^3-2x+3}{3x^3+3x^2-5x}$

Solution

$$\lim_{x\to\infty} \frac{-2x^3-2x+3}{3x^3+3x^2-5x} = \lim_{x\to\infty} \frac{-2-\frac{2}{x^2}+\frac{3}{x^3}}{3+\frac{3}{x}-\frac{5}{x^2}}$$

$$\boxed{=-\frac{2}{3}}$$

$$\lim_{x\to-\infty} \frac{-2x^3-2x+3}{3x^3+3x^2-5x} = \lim_{x\to-\infty} \frac{-2-\frac{2}{x^2}+\frac{3}{x^3}}{3+\frac{3}{x}-\frac{5}{x^2}}$$

$$\boxed{= -\frac{2}{3}}$$

Exercise

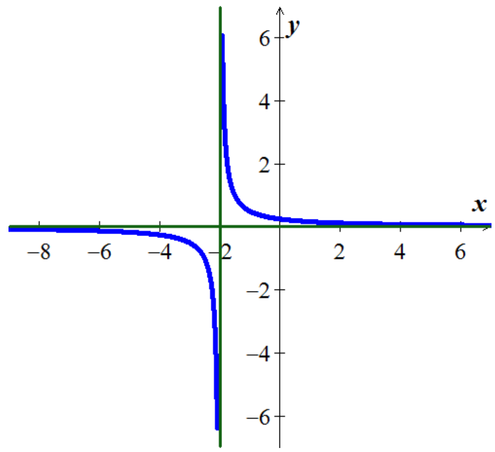
Graph the rational function $y = \dfrac{1}{2x+4}$. Include the equations of the asymptotes.

Solution

VA: $2x = 4 = 0$

$\boxed{x = -2}$

HA: $\boxed{y = 0}$



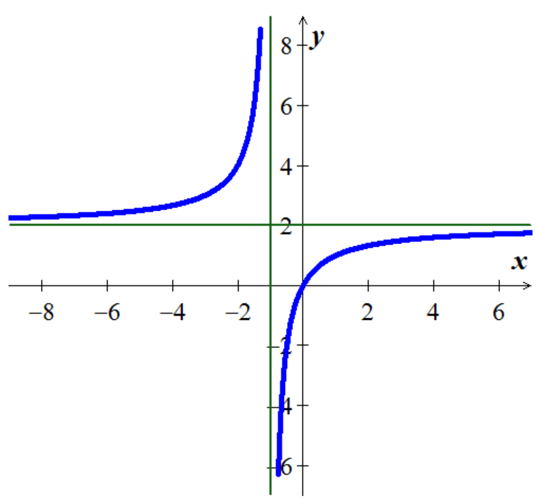
Exercise

Graph the rational function $y = \dfrac{2x}{x+1}$. Include the equations of the asymptotes.

Solution

VA: $\boxed{x = -1}$

HA: $\boxed{y = 2}$



Exercise

Graph the rational function $y=\dfrac{x^2}{x-1}$. Include the equations of the asymptotes.

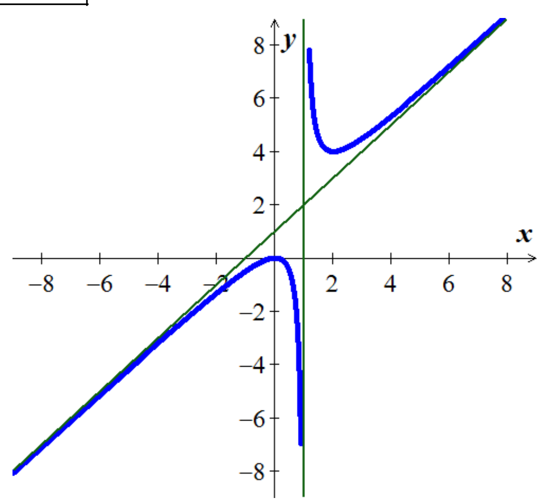
Solution

$$\begin{array}{r|l} & x+1 \\ \hline x-1 & x^2 \\ & \underline{x^2-x} \\ & \quad x \\ & \quad \underline{x-1} \\ & \qquad \boxed{1} \end{array}$$

$$y=\frac{x^2}{x-1}$$
$$=x+1+\frac{1}{x-1}$$

VA: $\underline{x=1}$

Oblique Asymptote: $\underline{y=x+1}$



Exercise

Graph the rational function $y=\dfrac{x^3+1}{x^2}$. Include the equations of the asymptotes.

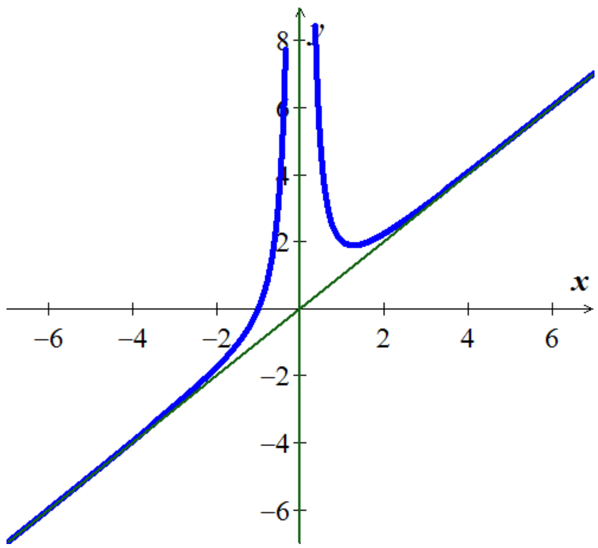
Solution

$$\begin{array}{r|l} & x \\ \hline x^2 & x^3+1 \\ & \underline{x^3} \\ & \quad \boxed{1} \end{array}$$

$$y=\frac{x^3+1}{x^2}=x+\frac{1}{x^2}$$

VA: $\boxed{x=0}$

Oblique Asymptote: $\boxed{y=x}$



Exercise

Let $\quad f(x)=\dfrac{x^2-5x+6}{x^2-2x}$

a) Analyze $\lim\limits_{x\to 0^-} f(x)$, $\lim\limits_{x\to 0^+} f(x)$, $\lim\limits_{x\to 2^-} f(x)$, and $\lim\limits_{x\to 2^+} f(x)$

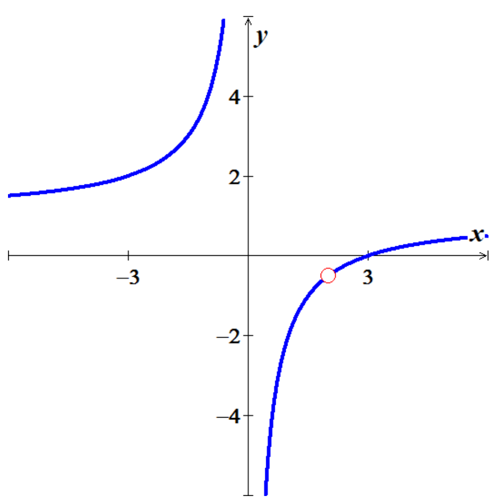
b) Does the graph of *f* have any vertical asymptotes? Explain?

Solution

$$f(x)=\frac{x^2-5x+6}{x^2-2x}$$

$$=\frac{(x-2)(x-3)}{x(x-2)}$$

$$=\frac{x-3}{x}$$



a) $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}\dfrac{x-3}{x}$

$$=\frac{-3}{0^-}$$

$$\boxed{=\infty}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x-3}{x}$$
$$= \frac{-3}{0^+}$$
$$\boxed{= -\infty}$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x-3}{x}$$
$$= \frac{2-3}{2}$$
$$\boxed{= -\frac{1}{2}}$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x-3}{x}$$
$$= \frac{2-3}{2}$$
$$\boxed{= -\frac{1}{2}}$$

b) ***VA***: $x = 0$ ***Hole***: $x = 2 \to f(2) = -\frac{1}{2}$

HA: $y = 1$ ***OA***: n/a

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y = \frac{3x}{1-x}$

Solution

VA: $x = 1$, ***Hole***: n/a, ***HA***: $y = -3$, ***OA***: n/a

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y = \frac{x^2}{x^2+9}$

Solution

VA: n/a; ***Hole***: n/a; ***HA***: $y = 1$; ***OA***: n/a

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y = \dfrac{x-2}{x^2-4x+3}$

Solution

$VA:\ x = 1,\ 3;\quad Hole:\ n/a;\quad HA:\ y = 0;\quad OA:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y = \dfrac{5x-1}{1-3x}$

Solution

$VA:\ x = \frac{1}{3};\quad Hole:\ n/a;\quad HA:\ y = -\frac{5}{3};\quad OA:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y = \dfrac{3}{x-5}$

Solution

$VA:\ x = 5,\quad Hole:\ n/a,\quad HA:\ y = 0,\quad OA:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y = \dfrac{x^3-1}{x^2+1}$

Solution

$$\begin{array}{r|l} & x \\ \hline x^2+1 & x^3 - 1 \\ & \underline{x^3 + x} \\ & \boxed{-x-1} \end{array}$$

$$\begin{aligned} y &= \frac{x^3-1}{x^2+1} \\ &= x + \frac{-x-1}{x^2+1} \\ &= x - \frac{x+1}{x^2+1} \end{aligned}$$

$VA:\ n/a,\quad Hole:\ n/a,\quad HA:\ n/a,\quad OA:\ y = x$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y = \dfrac{3x^2 - 27}{(x+3)(2x+1)}$

Solution

$$\textbf{VA}: \ x = -3, \ -\frac{1}{2}; \quad \textbf{Hole}: \ n/a; \quad \textbf{HA}: y = \frac{3}{2}; \quad \textbf{OA}: \ n/a$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y = \dfrac{x^3 + 3x^2 - 2}{x^2 - 4}$

Solution

$$\begin{array}{r|l} & x+3 \\ \hline x^2 - 4 & x^3 + 3x^2 - 2 \\ & \underline{x^3 \qquad - 4x} \\ & \quad 3x^2 + 4x - 2 \end{array}$$

$$\begin{aligned} y &= \frac{x^3 + 3x^2 - 2}{x^2 - 4} \\ &= x + 3 + \frac{4x + 10}{x^2 - 4} \end{aligned}$$

$$\textbf{VA}: \ x = \pm 2, \quad \textbf{Hole}: \ n/a, \quad \textbf{HA}: \ n/a, \quad \textbf{OA}: \ y = x + 3$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y = \dfrac{x - 3}{x^2 - 9}$

Solution

$$\textbf{VA}: \ x = -3; \quad \textbf{Hole}: \ x = 3; \quad \textbf{HA}: \ y = 0; \quad \textbf{OA}: \ n/a$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y = \dfrac{6}{\sqrt{x^2 - 4x}}$

Solution

$$\textbf{VA}: \ x = 0, \ 4; \quad \textbf{Hole}: \ n/a; \quad \textbf{HA}: \ y = 0; \quad \textbf{OA}: \ n/a$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{4x^3+1}{1-x^3}$

Solution

$VA:\ x=1;\quad Hole:\ n/a;\quad HA:\ y=-4;\quad OA:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{x+1}{\sqrt{9x^2+x}}$

Solution

$VA:\ x=0,\ -\frac{1}{9};\quad Hole:\ n/a;\quad HA:\ y=\frac{1}{3};\quad OA:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=1-e^{-2x}$

Solution

$VA:\ n/a;\quad Hole:\ n/a;\quad HA:\ y=1;\quad OA:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{1}{\ln x^2}$

Solution

$VA:\ x=0;\quad Hole:\ n/a\ ;\quad HA:\ y=0;\quad OA:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{1}{\tan^{-1}x}$

Solution

$VA:\ x=0;\quad Hole:\ n/a;\quad HA:\ y=\frac{1}{\frac{\pi}{2}}=\frac{2}{\pi};\quad OA:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{2x^2+6}{2x^2+3x-2}$

Solution

$$VA:\ x=-2,\ \frac{1}{2};\quad Hole:\ n/a;\quad HA:\ y=1;\quad OA:\ n/a$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{3x^2+2x-1}{4x+1}$

Solution

$$\begin{array}{r|l} & \frac{3}{4}x+\frac{5}{16} \\ \hline 4x+1 & 3x^2+2x-1 \\ & \underline{3x^2+\frac{3}{4}x} \\ & \qquad \frac{5}{4}x-1 \end{array}$$

$$VA:\ x=-\frac{1}{4};\quad Hole:\ n/a;\quad HA:\ n/a;\quad OA:\ y=\frac{3}{4}x+\frac{5}{16}$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{9x^2+4}{(2x-1)^2}$

Solution

$$VA:\ x=\frac{1}{2};\quad Hole:\ n/a;\quad HA:\ y=\frac{9}{4};\quad OA:\ n/a$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{1+x-2x^2-x^3}{x^2+1}$

Solution

$$\begin{array}{r|l} & \qquad\quad -x-2 \\ \hline x^2+1 & -x^3-2x^2+x+1 \\ & \underline{-x^3 \qquad\quad -x} \\ & \quad -2x^2+2x \end{array}$$

$VA:\ n/a;\quad Hole:\ n/a;\quad HA:\ n/a;\quad OA:\ y=-x-2$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{x(x+2)^3}{3x^2-4x}$

Solution

$$f(x)=\frac{x\left(x^3+6x^2+12x+8\right)}{x(3x-4)}$$

$$=\frac{x^3+6x^2+12x+8}{3x-4}$$

$$\begin{array}{r|l}
 & \frac{1}{3}x^2+\frac{22}{9}x+\frac{196}{27} \\ \hline
3x-4 & x^3+6x^2+12x+8 \\
 & \underline{x^3-\frac{4}{3}x^2} \\
 & \quad \frac{22}{3}x^2+12x \\
 & \quad \underline{\frac{22}{3}x^2-\frac{88}{9}x} \\
 & \qquad \frac{196}{9}x
\end{array}$$

$VA:\ x=\frac{4}{3};\quad Hole:\ (0,\ -2);\quad HA:\ n/a;\quad OA:\ y=\frac{1}{3}x^2+\frac{22}{9}x+\frac{196}{27}$

Exercise

Find $\displaystyle\lim_{x\to\infty} x^{12}$

Solution

$$\lim_{x\to\infty} x^{12}\ \underline{=\infty}\,|$$

Exercise

Find $\displaystyle\lim_{x\to-\infty} 3x^9$

Solution

$$\lim_{x\to-\infty} 3x^9\ \underline{=-\infty}\,|$$

Exercise

Find $\lim_{x\to-\infty} x^{-8}$

Solution

$$\lim_{x\to-\infty} x^{-8} = \frac{1}{(-\infty)^8}$$
$$= 0$$

Exercise

Find $\lim_{x\to-\infty} x^{-9}$

Solution

$$\lim_{x\to-\infty} x^{-9} = \frac{1}{(-\infty)^9}$$
$$= 0$$

Exercise

Find $\lim_{x\to-\infty} 2x^{-6}$

Solution

$$\lim_{x\to-\infty} 2x^{-6} = \frac{2}{\infty}$$
$$= 0$$

Exercise

Find $\lim_{x\to\infty} \left(3x^{12} - 9x^7\right)$

Solution

$$\lim_{x\to\infty} \left(3x^{12} - 9x^7\right) = \infty$$

Exercise

Find $\lim_{x\to-\infty} \left(3x^7 + x^2\right)$

Solution

$$\lim_{x\to-\infty}\left(3x^7+x^2\right)=\lim_{x\to-\infty}x^2\left(3x^5+1\right)$$

$$\underline{=-\infty\,}|$$

Exercise

Find $\quad \lim_{x\to-\infty}\left(-2x^{16}+2\right)$

Solution

$$\lim_{x\to-\infty}\left(-2x^{16}+2\right)\underline{=-\infty\,}|$$

Exercise

Find $\quad \lim_{x\to-\infty}\left(2x^{-6}+4x^5\right)$

Solution

$$\lim_{x\to-\infty}\left(2x^{-6}+4x^5\right)=\lim_{x\to-\infty}x^{-6}\left(2+4x^{11}\right) \qquad +\infty(-\infty)$$

$$\underline{=-\infty\,}|$$

Exercise

Find $\quad \lim_{x\to-\infty}\frac{\cos x}{3x}$

Solution

$$-\frac{1}{3x}\le\frac{\cos x}{3x}\le\frac{1}{3x}, \qquad -1\le\cos x\le 1$$

$$\lim_{x\to-\infty}\frac{\cos x}{3x}\underline{=0\,}| \qquad \textit{By the Sandwich Theorem}$$

Exercise

Find $\quad \lim_{x\to\infty}\frac{x+\sin x}{2x+7-5\sin x}$

Solution

$$\lim_{x\to\infty} \frac{x+\sin x}{2x+7-5\sin x} = \lim_{x\to\infty} \frac{1+\frac{\sin x}{x}}{2+\frac{7}{x}-\frac{5\sin x}{x}}$$

$$= \frac{1+0}{2+0-0}$$

$$= \frac{1}{2}$$

Exercise

Find $\lim_{x\to\infty} \sqrt{\frac{8x^2-3}{2x^2+x}}$

Solution

$$\lim_{x\to\infty} \sqrt{\frac{8x^2-3}{2x^2+x}} = \lim_{x\to\infty} \sqrt{\frac{8-\frac{3}{x^2}}{2+\frac{1}{x}}}$$

$$= \sqrt{\frac{8}{2}}$$

$$= 2$$

Exercise

Find $\lim_{x\to-\infty} \left(\frac{x^2+x-1}{8x^2-3}\right)^{1/3}$

Solution

$$\lim_{x\to-\infty} \left(\frac{x^2+x-1}{8x^2-3}\right)^{1/3} = \lim_{x\to-\infty} \left(\frac{1+\frac{1}{x}-\frac{1}{x^2}}{8-\frac{3}{x^2}}\right)^{1/3}$$

$$= \left(\frac{1}{8}\right)^{1/3}$$

$$= \frac{1}{2}$$

Exercise

Find $\lim_{x\to\infty} \frac{2\sqrt{x}+x^{-1}}{3x-7}$

Solution

$$\lim_{x\to\infty}\frac{2\sqrt{x}+x^{-1}}{3x-7}=\lim_{x\to\infty}\frac{\frac{2\sqrt{x}}{x}+\frac{x^{-1}}{x}}{3-\frac{7}{x}}$$

$$=\lim_{x\to\infty}\frac{\frac{2}{x^{1/2}}+\frac{1}{x^2}}{3-\frac{7}{x}}$$

$$\boxed{=0}$$

Exercise

Find $\quad \lim_{x\to\infty}\frac{x^{-1}+x^{-4}}{x^{-2}+x^{-3}}$

Solution

$$\lim_{x\to\infty}\frac{x^{-1}+x^{-4}}{x^{-2}+x^{-3}}=\lim_{x\to\infty}\frac{\frac{x^{-1}}{x^{-2}}+\frac{x^{-4}}{x^{-2}}}{\frac{x^{-2}}{x^{-2}}+\frac{x^{-3}}{x^{-2}}}$$

$$=\lim_{x\to\infty}\frac{x+\frac{1}{x^2}}{1+\frac{1}{x}}$$

$$\boxed{=\infty}$$

Exercise

Find $\quad \lim_{x\to-\infty}\frac{4-3x^3}{\sqrt{x^6+9}}$

Solution

$$\lim_{x\to-\infty}\frac{4-3x^3}{\sqrt{x^6+9}}=\lim_{x\to-\infty}\frac{\frac{4-3x^3}{\sqrt{x^6}}}{\frac{\sqrt{x^6+9}}{\sqrt{x^6}}}$$

$$=\lim_{x\to-\infty}\frac{\frac{4-3x^3}{x^3}}{\sqrt{\frac{x^6+9}{x^6}}}$$

$$=\lim_{x\to-\infty}\frac{\frac{4}{x3}-3}{\sqrt{1+\frac{9}{x^6}}}$$

$$= \frac{-3}{\sqrt{1}}$$

$$\underline{= -3}$$

Exercise

Find $\lim\limits_{x\to\infty} \frac{2x-3}{4x+10}$

Solution

$$\lim_{x\to\infty} \frac{2x-3}{4x+10} \underline{= \frac{1}{2}}$$

Exercise

Find $\lim\limits_{x\to\infty} \frac{x^4-1}{x^5+2}$

Solution

$$\lim_{x\to\infty} \frac{x^4-1}{x^5+2} \underline{= 0}$$

Exercise

Find $\lim\limits_{x\to-\infty} \left(-3x^3+5\right)$

Solution

$$\lim_{x\to-\infty} \left(-3x^3+5\right) \underline{= \infty}$$

Exercise

Find $\lim\limits_{x\to\infty} \left(e^{-2x}+\frac{2}{x}\right)$

Solution

$$\lim_{x\to\infty} \left(e^{-2x}+\frac{2}{x}\right) = e^{-\infty}+0$$

$$\underline{= 0}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{1}{\ln x+1}$

Solution

$$\lim_{x\to\infty} \frac{1}{\ln x+1} = \frac{1}{\infty}$$

$$= 0$$

Exercise

Find $\lim\limits_{x\to\infty} \left(3+\dfrac{10}{x^2}\right)$

Solution

$$\lim_{x\to\infty} \left(3+\frac{10}{x^2}\right) = 3+0$$

$$= 3$$

Exercise

Find $\lim\limits_{x\to\infty} \left(5+\dfrac{1}{x}+\dfrac{10}{x^2}\right)$

Solution

$$\lim_{x\to\infty} \left(5+\frac{1}{x}+\frac{10}{x^2}\right) = 5+0+0$$

$$= 5$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{4x^2+2x+3}{x^2}$

Solution

$$\lim_{x\to\infty} \frac{4x^2+2x+3}{x^2} = \lim_{x\to\infty} \frac{4x^2}{x^2}$$

$$= 4$$

Exercise

Find $\lim\limits_{x\to\infty}\left(5+\dfrac{100}{x}+\dfrac{\sin^4 x^3}{x^2}\right)$

Solution

$$-1\le \sin\theta \le 1$$

$$0\le \sin^4\theta \le 1$$

$$0\le \frac{\sin^4\theta}{x^2}\le \frac{1}{x^2}\ \to 0$$

$$\lim_{x\to\infty}\left(5+\frac{100}{x}+\frac{\sin^4 x^3}{x^2}\right) = 5$$

Exercise

Find $\lim\limits_{\theta\to\infty}\dfrac{\cos\theta}{\theta^2}$

Solution

$$-1\le \cos\theta \le 1$$

$$-\frac{1}{\theta^2}\le \frac{\cos\theta}{\theta^2}\le \frac{1}{\theta^2}\ \to 0$$

$$\lim_{\theta\to\infty}\frac{\cos\theta}{\theta^2} = 0$$

Exercise

Find $\lim\limits_{\theta\to\infty}\dfrac{\cos\theta^5}{\sqrt{\theta}}$

Solution

$$-1\le \cos\theta^5 \le 1$$

$$-\frac{1}{\sqrt{\theta}}\le \frac{\cos\theta^5}{\sqrt{\theta}}\le \frac{1}{\sqrt{\theta}}\ \to 0$$

$$\lim_{\theta\to\infty}\frac{\cos\theta^5}{\sqrt{\theta}} = 0$$

Exercise

Find $\quad \lim\limits_{x\to\infty} \dfrac{4x}{20x+1}$

Solution

$$\lim_{x\to\infty} \frac{4x}{20x+1} = \frac{4}{20}$$

$$= \frac{1}{5}$$

Exercise

Find $\quad \lim\limits_{x\to-\infty} \dfrac{4x}{20x+1}$

Solution

$$\lim_{x\to-\infty} \frac{4x}{20x+1} = \lim_{x\to-\infty} \frac{4x}{20x}$$

$$= \frac{1}{5}$$

Exercise

Find $\quad \lim\limits_{x\to\infty} \dfrac{3x^2-7}{x^2+5x}$

Solution

$$\lim_{x\to\infty} \frac{3x^2-7}{x^2+5x} = 3$$

Exercise

Find $\quad \lim\limits_{x\to-\infty} \dfrac{3x^2-7}{x^2+5x}$

Solution

$$\lim_{x\to-\infty} \frac{3x^2-7}{x^2+5x} = \lim_{x\to-\infty} \frac{3x^2}{x^2}$$

$$= 3$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{6x^2-9x+8}{3x^2+2}$

Solution

$$\lim_{x\to\infty} \frac{6x^2-9x+8}{3x^2+2} = \lim_{x\to\infty} \frac{6x^2}{3x^2}$$

$$= \frac{6}{3}$$

$$\underline{= 2}$$

Exercise

Find $\lim\limits_{x\to-\infty} \dfrac{6x^2-9x+8}{3x^2+2}$

Solution

$$\lim_{x\to-\infty} \frac{6x^2-9x+8}{3x^2+2} = \lim_{x\to-\infty} \frac{6x^2}{3x^2}$$

$$= \frac{6}{3}$$

$$\underline{= 2}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{4x^2-7}{8x^2+5x+2}$

Solution

$$\lim_{x\to\infty} \frac{4x^2-7}{8x^2+5x+2} = \lim_{x\to\infty} \frac{4x^2}{8x^2}$$

$$= \frac{4}{8}$$

$$\underline{= \frac{1}{2}}$$

Exercise

Find $\lim\limits_{x\to-\infty} \dfrac{4x^2-7}{8x^2+5x+2}$

Solution

$$\lim_{x\to-\infty} \frac{4x^2-7}{8x^2+5x+2} = \lim_{x\to-\infty} \frac{4x^2}{8x^2}$$

$$= \frac{4}{8}$$

$$\boxed{= \frac{1}{2}}$$

Exercise

Find $\lim_{x\to\infty} \frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4}$

Solution

$$\lim_{x\to\infty} \frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4} = \lim_{x\to\infty} \frac{\sqrt{16x^4}+x^2}{2x^2}$$

$$= \lim_{x\to\infty} \frac{4x^2+x^2}{2x^2}$$

$$= \lim_{x\to\infty} \frac{5x^2}{2x^2}$$

$$\boxed{= \frac{5}{2}}$$

Exercise

Find $\lim_{x\to-\infty} \frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4}$

Solution

$$\lim_{x\to-\infty} \frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4} = \lim_{x\to-\infty} \frac{\sqrt{16x^4}+x^2}{2x^2}$$

$$= \lim_{x\to-\infty} \frac{4x^2+x^2}{2x^2}$$

$$= \lim_{x\to-\infty} \frac{5x^2}{2x^2}$$

$$\boxed{= \frac{5}{2}}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{3x^4+3x^3-36x^2}{x^4-25x^2+144}$

Solution

$$\lim_{x\to\infty} \frac{3x^4+3x^3-36x^2}{x^4-25x^2+144} = \lim_{x\to\infty} \frac{3x^4}{x^4}$$

$$\underline{=3}$$

Exercise

Find $\lim\limits_{x\to-\infty} \dfrac{3x^4+3x^3-36x^2}{x^4-25x^2+144}$

Solution

$$\lim_{x\to-\infty} \frac{3x^4+3x^3-36x^2}{x^4-25x^2+144} = \lim_{x\to-\infty} \frac{3x^4}{x^4}$$

$$\underline{=3}$$

Exercise

Find $\lim\limits_{x\to-\infty} \left(\sqrt{x^2+3}+x\right)$

Solution

$$\lim_{x\to-\infty} \left(\sqrt{x^2+3}+x\right) = \lim_{x\to-\infty} \left(\sqrt{x^2+3}+x\right)\frac{\sqrt{x^2+3}-x}{\sqrt{x^2+3}-x}$$

$$= \lim_{x\to-\infty} \frac{x^2+3-x^2}{\sqrt{x^2+3}-x}$$

$$= \lim_{x\to-\infty} \frac{3}{\sqrt{x^2+3}-x}$$

$$= \lim_{x\to-\infty} \frac{\frac{3}{x}}{\sqrt{\frac{x^2}{x^2}+\frac{3}{x^2}}-\frac{x}{x}}$$

$$= \lim_{x\to-\infty} \frac{\frac{3}{x}}{\sqrt{1+\frac{3}{x^2}}+1}$$

$$= \frac{0}{\sqrt{1}+1}$$

$$\underline{= 0}$$

Exercise

Find $\lim\limits_{x\to\infty} \left(\sqrt{x^2+3x} - \sqrt{x^2-2x}\right)$

Solution

$$\lim_{x\to\infty} \left(\sqrt{x^2+3x} - \sqrt{x^2-2x}\right) = \lim_{x\to\infty} \left(\sqrt{x^2+3x} - \sqrt{x^2-2x}\right)\frac{\sqrt{x^2+3x}+\sqrt{x^2-2x}}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$$

$$= \lim_{x\to\infty} \frac{\left(x^2+3x\right)-\left(x^2-2x\right)}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$$

$$= \lim_{x\to\infty} \frac{x^2+3x-x^2+2x}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$$

$$= \lim_{x\to\infty} \frac{5x}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$$

$$= \lim_{x\to\infty} \frac{\frac{5x}{\sqrt{x^2}}}{\sqrt{\frac{x^2}{x^2}+\frac{3x}{x^2}}+\sqrt{\frac{x^2}{x^2}-\frac{2x}{x^2}}}$$

$$= \lim_{x\to\infty} \frac{5}{\sqrt{1+\frac{3}{x}}+\sqrt{1-\frac{2}{x}}}$$

$$= \frac{5}{\sqrt{1}+\sqrt{1}}$$

$$\underline{= \frac{5}{2}}$$

Exercise

Find $\lim\limits_{x\to\infty} 16x^2\left(4x^2 - \sqrt{16x^4+1}\right)$

Solution

$$\lim_{x\to\infty} 16x^2\left(4x^2 - \sqrt{16x^4+1}\right) = \infty - \infty$$

$$= \lim_{x\to\infty} 16x^2\left(4x^2 - \sqrt{16x^4+1}\right)\cdot\frac{4x^2+\sqrt{16x^4+1}}{4x^2+\sqrt{16x^4+1}}$$

$$= \lim_{x\to\infty} 16x^2\,\frac{16x^4-16x^4-1}{4x^2+\sqrt{16x^4+1}}$$

$$= \lim_{x\to\infty} 16x^2\,\frac{-1}{4x^2+\sqrt{16x^4+1}}$$

$$= \lim_{x\to\infty} \frac{-16x^2}{4x^2+4x^2}$$

$$= \lim_{x\to\infty} \frac{-16x^2}{8x^2}$$

$$\underline{= -2}$$

Exercise

Find $\lim_{x\to-\infty} 16x^2\left(4x^2 - \sqrt{16x^4+1}\right)$

Solution

$$\lim_{x\to-\infty} 16x^2\left(4x^2 - \sqrt{16x^4+1}\right) = \infty - \infty$$

$$= \lim_{x\to-\infty} 16x^2\left(4x^2 - \sqrt{16x^4+1}\right)\cdot\frac{4x^2+\sqrt{16x^4+1}}{4x^2+\sqrt{16x^4+1}}$$

$$= \lim_{x\to-\infty} 16x^2\,\frac{16x^4-16x^4-1}{4x^2+\sqrt{16x^4+1}}$$

$$= \lim_{x\to-\infty} 16x^2\,\frac{-1}{4x^2+\sqrt{16x^4+1}}$$

$$= \lim_{x\to-\infty} \frac{-16x^2}{4x^2+4x^2}$$

$$= \lim_{x\to-\infty} \frac{-16x^2}{8x^2}$$

$$\underline{= -2}$$

Exercise

Find the limit $\lim\limits_{x\to-\infty}\left(x+\sqrt{x^2-4x+2}\right)$

Solution

$$\lim_{x\to-\infty}\left(x+\sqrt{x^2-4x+2}\right)=-\infty+\infty$$

$$=\lim_{x\to-\infty}\left(x+\sqrt{x^2-4x+2}\right)\cdot\frac{x-\sqrt{x^2-4x+2}}{x-\sqrt{x^2-4x+2}}$$

$$=\lim_{x\to-\infty}\frac{x^2-x^2+4x-2}{x-\sqrt{x^2-4x+2}}$$

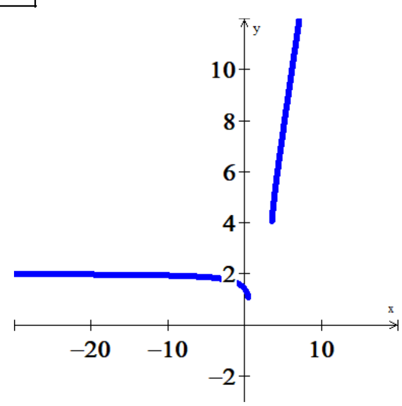
$$=\lim_{x\to-\infty}\frac{4x-2}{x-\sqrt{x^2-4x+2}}$$

$$=\lim_{x\to-\infty}\frac{4x-2}{x-|x|} \qquad x\to-\infty\quad (x<0)\to\quad |x|=-x$$

$$=\lim_{x\to-\infty}\frac{4x}{x+x}$$

$$=\lim_{x\to-\infty}\frac{4x}{2x}$$

$$=2$$



Exercise

Find the limit $\lim\limits_{x\to\infty}\left(\sqrt{x^2+2x}-\sqrt{x^2-2x}\right)$

Solution

$$\lim_{x\to\infty}\left(\sqrt{x^2+2x}-\sqrt{x^2-2x}\right)=\infty-\infty$$

$$= \lim_{x\to\infty} \left(\sqrt{x^2+2x}-\sqrt{x^2-2x}\right)\frac{\sqrt{x^2+2x}+\sqrt{x^2-2x}}{\sqrt{x^2+2x}+\sqrt{x^2-2x}}$$

$$= \lim_{x\to\infty} \frac{x^2+2x-x^2+2x}{\sqrt{x^2+2x}+\sqrt{x^2-2x}}$$

$$= \lim_{x\to\infty} \frac{4x}{\sqrt{x^2+2x}+\sqrt{x^2-2x}}$$

$$= \lim_{x\to\infty} \frac{4x}{\sqrt{x^2}+\sqrt{x^2}}$$

$$= \lim_{x\to\infty} \frac{4x}{|x|+|x|}$$

$$= \lim_{x\to\infty} \frac{4x}{2x}$$

$$\underline{= 2}$$

Exercise

Find the limit $\lim\limits_{x\to-\infty} \left(\sqrt{x^2+2x}-\sqrt{x^2-2x}\right)$

Solution

$$\lim_{x\to-\infty} \left(\sqrt{x^2+2x}-\sqrt{x^2-2x}\right) = \lim_{x\to-\infty} \left(\sqrt{x^2}-\sqrt{x^2}\right)$$

$$= \infty - \infty$$

$$= \lim_{x\to-\infty} \left(\sqrt{x^2+2x}-\sqrt{x^2-2x}\right)\frac{\sqrt{x^2+2x}+\sqrt{x^2-2x}}{\sqrt{x^2+2x}+\sqrt{x^2-2x}}$$

$$= \lim_{x\to-\infty} \frac{x^2+2x-x^2+2x}{\sqrt{x^2+2x}+\sqrt{x^2-2x}}$$

$$= \lim_{x\to-\infty} \frac{4x}{\sqrt{x^2+2x}+\sqrt{x^2-2x}}$$

$$= \lim_{x\to-\infty} \frac{4x}{\sqrt{x^2}+\sqrt{x^2}}$$

$$= \lim_{x\to-\infty} \frac{4x}{|x|+|x|}$$

$$= \lim_{x\to-\infty} \frac{4x}{2|x|}$$

$$\underline{= -2}$$

Exercise

Find the limit $\lim\limits_{x\to\infty}\left(x+\sqrt{x^2-4x+1}\right)$

Solution

$$\begin{aligned}\lim_{x\to\infty}\left(x+\sqrt{x^2-4x+1}\right) &= \lim_{x\to\infty}\left(x+\sqrt{x^2}\right)\\ &= \lim_{x\to\infty}\left(x+|x|\right)\\ &= \infty\end{aligned}$$

Exercise

Find the limit $\lim\limits_{x\to-\infty}\left(x+\sqrt{x^2-4x+2}\right)$

Solution

$$\begin{aligned}\lim_{x\to-\infty}\left(x+\sqrt{x^2-4x+2}\right) &= \lim_{x\to-\infty}\left(x+\sqrt{x^2}\right)\\ &= \lim_{x\to-\infty}\left(x+|x|\right)\\ &= -\infty+|-\infty|\\ &= -\infty+\infty\end{aligned}$$

$$\lim_{x\to-\infty}\left(x+\sqrt{x^2-4x+2}\right)\frac{x-\sqrt{x^2-4x+2}}{x-\sqrt{x^2-4x+2}}$$

$$\begin{aligned}&= \lim_{x\to-\infty}\frac{x^2-x^2+4x-2}{x-\sqrt{x^2-4x+2}}\\ &= \lim_{x\to-\infty}\frac{4x-2}{x-\sqrt{x^2-4x+2}}\\ &= \lim_{x\to-\infty}\frac{4x}{x-|x|}\\ &= \lim_{x\to-\infty}\frac{4x}{-|x|-|x|}\\ &= \lim_{x\to-\infty}\frac{4|x|}{-2|x|}\\ &= -2\end{aligned}$$

Exercise

Find the limit $\lim\limits_{x\to\infty}\left(\sqrt{4x^2-2x+1}-2x\right)$

Solution

$$\lim_{x\to\infty}\left(\sqrt{4x^2-2x+1}-2x\right)=\infty-\infty$$

$$\begin{aligned}
&=\lim_{x\to\infty}\left(\sqrt{4x^2-2x+1}-2x\right)\frac{\sqrt{4x^2-2x+1}+2x}{\sqrt{4x^2-2x+1}+2x}\\
&=\lim_{x\to\infty}\frac{4x^2-2x+1-4x^2}{\sqrt{4x^2-2x+1}+2x}\\
&=\lim_{x\to\infty}\frac{-2x+1}{\sqrt{4x^2-2x+1}+2x}\\
&=\lim_{x\to\infty}\frac{-2x}{2x+2x}\\
&=\lim_{x\to\infty}\frac{-2x}{4x}\\
&=-\frac{1}{2}
\end{aligned}$$

Exercise

Find the limit $\lim\limits_{x\to-\infty}\left(\sqrt{4x^2-2x+1}-2x\right)$

Solution

$$\begin{aligned}
\lim_{x\to-\infty}\left(\sqrt{4x^2-2x+1}-2x\right)&=\lim_{x\to-\infty}\left(\sqrt{4x^2}-2x\right)\\
&=\lim_{x\to-\infty}\left(2|x|-2x\right)\\
&=\infty+\infty\\
&=\infty
\end{aligned}$$

Exercise

Find the limit $\lim\limits_{x\to\infty}x\left(\sqrt{4x^2+1}-2x\right)$

Solution

$$\lim_{x\to\infty}x\left(\sqrt{4x^2+1}-2x\right)=\infty(\infty-\infty)$$

$$= \lim_{x\to\infty} x\left(\sqrt{4x^2+1}-2x\right)\frac{\sqrt{4x^2+1}+2x}{\sqrt{4x^2+1}+2x}$$

$$= \lim_{x\to\infty} \frac{x\left(4x^2+1-4x^2\right)}{\sqrt{4x^2+1}+2x}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{4x^2+1}+2x}$$

$$= \lim_{x\to\infty} \frac{x}{2x+2x}$$

$$= \lim_{x\to\infty} \frac{x}{4x}$$

$$\boxed{=\frac{1}{4}}$$

Exercise

Find the limit $\displaystyle\lim_{x\to-\infty} x\left(\sqrt{4x^2+1}+2x\right)$

Solution

$$\lim_{x\to-\infty} x\left(\sqrt{4x^2+1}+2x\right) = \lim_{x\to-\infty}\left(x\sqrt{4x^2+1}+2x^2\right)$$

$$= \lim_{x\to-\infty}\left(2x|x|+2x^2\right)$$

$$= -\infty+\infty$$

$$= \lim_{x\to-\infty} x\left(\sqrt{4x^2+1}+2x\right)\frac{\sqrt{4x^2+1}-2x}{\sqrt{4x^2+1}-2x}$$

$$= \lim_{x\to-\infty} x\cdot\frac{4x^2+1-4x^2}{\sqrt{4x^2+1}-2x}$$

$$= \lim_{x\to-\infty} \frac{x}{\sqrt{4x^2+1}-2x}$$

$$= \lim_{x\to-\infty} \frac{x}{2|x|-2x}$$

$$= \lim_{x\to\infty} \frac{x}{2|x|+2|x|}$$

$$= \lim_{x\to\infty} \frac{-|x|}{4|x|}$$

$$\boxed{=-\frac{1}{4}}$$

Exercise

Find the limit $\lim\limits_{x\to-\infty} x\left(\sqrt{x^2+4}+x\right)$

Solution

$$\lim\limits_{x\to-\infty} x\left(\sqrt{x^2+4}+x\right) = \lim\limits_{x\to-\infty} x\left(|x|+x\right)$$

$$= \infty - \infty$$

$$= \lim\limits_{x\to-\infty} x\left(\sqrt{x^2+4}+x\right)\frac{\sqrt{x^2+4}-x}{\sqrt{x^2+4}-x}$$

$$= \lim\limits_{x\to-\infty} x\cdot\frac{x^2+4-x^2}{\sqrt{x^2+4}-x}$$

$$= \lim\limits_{x\to-\infty} \frac{4x}{\sqrt{x^2+4}-x}$$

$$= \lim\limits_{x\to\infty} \frac{-4|x|}{|x|+|x|}$$

$$= \lim\limits_{x\to\infty} \frac{-4\ |x|}{2\ |x|}$$

$$\underline{= -2\ |}$$

Exercise

Find the limit $\lim\limits_{x\to-\infty} \left(\sqrt{5x^2+4x+7}-\sqrt{5x^2+x+3}\right)$

Solution

$$\lim\limits_{x\to-\infty} \left(\sqrt{5x^2+4x+7}-\sqrt{5x^2+x+3}\right) = \infty - \infty$$

$$= \lim\limits_{x\to-\infty} \left(\sqrt{5x^2+4x+7}-\sqrt{5x^2+x+3}\right)\frac{\sqrt{5x^2+4x+7}+\sqrt{5x^2+x+3}}{\sqrt{5x^2+4x+7}+\sqrt{5x^2+x+3}}$$

$$= \lim\limits_{x\to-\infty} \frac{5x^2+4x+7-5x^2-x-3}{\sqrt{5x^2+4x+7}+\sqrt{5x^2+x+3}}$$

$$= \lim\limits_{x\to-\infty} \frac{3x+7}{\sqrt{5x^2+4x+7}+\sqrt{5x^2+x+3}}$$

$$= \lim\limits_{x\to-\infty} \frac{3x}{\sqrt{5x^2}+\sqrt{5x^2}}$$

$$= \lim\limits_{x\to-\infty} \frac{3x}{2\sqrt{5}\ |x|}$$

$$= \lim_{x\to\infty} \frac{-3\,|x|}{2\sqrt{5}\,|x|}$$

$$\boxed{= \frac{-3}{2\sqrt{5}}}$$

Exercise

Find the limit $\lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right)$

Solution

$$\lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) = \infty - \infty$$

$$= \lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty} \frac{x^2+1-x^2}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty} \frac{1}{|x|+x}$$

$$= \frac{1}{\infty}$$

$$\boxed{= 0}$$

Exercise

Find $\lim_{x\to\infty} \frac{x-1}{x^{2/3}-1}$

Solution

$$\lim_{x\to\infty} \frac{x-1}{x^{2/3}-1} = \lim_{x\to\infty} \frac{x}{x^{2/3}}$$

$$= \lim_{x\to\infty} x^{1/3}$$

$$\boxed{= \infty}$$

Exercise

Find $\lim\limits_{x\to-\infty} \dfrac{x-1}{x^{2/3}-1}$

Solution

$$\lim_{x\to-\infty} \frac{x-1}{x^{2/3}-1} = \lim_{x\to-\infty} \frac{x}{x^{2/3}}$$

$$= \lim_{x\to-\infty} x^{1/3}$$

$$\underline{= -\infty}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+2x+6}-3}{x-1}$

Solution

$$\lim_{x\to\infty} \frac{\sqrt{x^2+2x+6}-3}{x-1} = \lim_{x\to\infty} \frac{\sqrt{x^2}}{x}$$

$$= \lim_{x\to\infty} \frac{x}{x}$$

$$\underline{= 1}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{\left|1-x^2\right|}{x(x+1)}$

Solution

$$\lim_{x\to\infty} \frac{\left|1-x^2\right|}{x(x+1)} = \lim_{x\to\infty} \frac{x^2-1}{x^2+1}$$

$$= \lim_{x\to\infty} \frac{x^2}{x^2}$$

$$\underline{= 1}$$

Exercise

Find $\lim\limits_{x\to\infty}\left(\sqrt{|x|}-\sqrt{|x-1|}\right)$

Solution

$$\lim\limits_{x\to\infty}\left(\sqrt{|x|}-\sqrt{|x-1|}\right)=\infty-\infty \qquad x\to\infty \Rightarrow |x|=x \;\; \& \;\; |x-1|=x-1$$

$$=\lim\limits_{x\to\infty}\left(\sqrt{x}-\sqrt{x-1}\right)\cdot\frac{\sqrt{x}+\sqrt{x-1}}{\sqrt{x}+\sqrt{x-1}}$$

$$=\lim\limits_{x\to\infty}\frac{x-x+1}{\sqrt{x}+\sqrt{x-1}}$$

$$=\lim\limits_{x\to\infty}\frac{1}{\sqrt{x}+\sqrt{x-1}}$$

$$=\frac{1}{\infty}$$

$$=0$$

Exercise

Find $\lim\limits_{x\to\infty}\dfrac{\tan^{-1}x}{x}$

Solution

$$-\frac{\pi}{2}\le\tan^{-1}x\le\frac{\pi}{2}$$

$$-\frac{\pi}{2x}\le\frac{\tan^{-1}x}{x}\le\frac{\pi}{2x} \quad \to \; 0$$

$$\lim\limits_{x\to\infty}\frac{\tan^{-1}x}{x}=0$$

Exercise

Find $\lim\limits_{x\to\infty}\dfrac{\cos x}{e^{3x}}$

Solution

$$-1\le\cos x\le 1$$

$$-\frac{1}{e^{3x}}\le\frac{\cos x}{e^{3x}}\le\frac{1}{e^{3x}} \quad \to \; 0$$

$$\lim\limits_{x\to\infty}\frac{\cos x}{e^{3x}}=0$$

Exercise

Find $\lim\limits_{x\to 0} \dfrac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}}$

Solution

$$\lim_{x\to 0} \frac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}} = \frac{2+10}{1+1}$$

$$\underline{= 6}$$

Exercise

Find $\lim\limits_{x\to \infty} \dfrac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}}$

Solution

$$\lim_{x\to \infty} \frac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}} = \lim_{x\to \infty} \frac{2e^{x}}{e^{x}} \qquad \lim_{x\to \infty} e^{-x} = 0$$

$$\underline{= 2}$$

Exercise

Find $\lim\limits_{x\to -\infty} \dfrac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}}$

Solution

$$\lim_{x\to \infty} \frac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}} = \lim_{x\to \infty} \frac{10e^{-x}}{e^{-x}} \qquad \lim_{x\to -\infty} e^{x} = 0$$

$$\underline{= 10}$$

Exercise

Find the limit $\lim\limits_{x\to 0} \dfrac{x^2-4x+4}{x^3+5x^2-14x}$

Solution

$$\lim_{x\to 0} \frac{x^2-4x+4}{x^3+5x^2-14x} = \frac{4}{0}$$

$$\underline{= \infty}$$

Exercise

Find the limit $\lim\limits_{x\to 2} \dfrac{x^2-4x+4}{x^3+5x^2-14x}$

Solution

$$\lim_{x\to 2} \frac{x^2-4x+4}{x^3+5x^2-14x} = \frac{4-8+4}{8+20-28} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{(x-2)(x-2)}{x(x-2)(x+7)}$$

$$= \lim_{x\to 2} \frac{x-2}{x(x+7)}$$

$$= \frac{0}{18}$$

$$= 0$$

Exercise

Find the limit $\lim\limits_{x\to a} \dfrac{x^2-a^2}{x^4-a^4}$

Solution

$$\lim_{x\to a} \frac{x^2-a^2}{x^4-a^4} = \frac{a^2-a^2}{a^4-a^4} = \frac{0}{0}$$

$$= \lim_{x\to a} \frac{x^2-a^2}{\left(x^2-a^2\right)\left(x^2+a^2\right)}$$

$$= \lim_{x\to a} \frac{1}{x^2+a^2}$$

$$= \frac{1}{a^2+a^2}$$

$$= \frac{1}{2a^2}$$

Exercise

Find the limit $\lim\limits_{x\to 0} \dfrac{(x+h)^2-x^2}{h}$

Solution

$$\lim_{x\to 0} \frac{(x+h)^2-x^2}{h} = \frac{h^2}{h}$$

$$= h$$

Exercise

Find the limit $\displaystyle\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$

Solution

$$\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \frac{x^2 - x^2}{0} = \frac{0}{0}$$

$$= \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{2hx + h^2}{h}$$

$$= \lim_{h\to 0} (2x + h)$$

$$= 2x$$

Exercise

Find the limit $\displaystyle\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}$

Solution

$$\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}$$

$$= \lim_{x\to 1} \frac{1}{1+\sqrt{x}}$$

$$= \frac{1}{2}$$

$$\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} = \lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}}$$

$$= \lim_{x\to 1} \frac{1-x}{(1-x)(1+\sqrt{x})}$$

$$= \lim_{x\to 1} \frac{1}{1+\sqrt{x}}$$

$$=\boxed{\frac{1}{2}}$$

Exercise

Find the limit $\lim\limits_{x\to 0} \dfrac{\frac{1}{2+x}-\frac{1}{2}}{x}$

Solution

$$\lim_{x\to 0} \frac{\frac{1}{2+x}-\frac{1}{2}}{x} = \frac{\frac{1}{2}-\frac{1}{2}}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{1}{x}\left(\frac{2-2-x}{2(2+x)}\right)$$

$$= \frac{1}{2}\lim_{x\to 0} \frac{1}{x}\left(\frac{-x}{2+x}\right)$$

$$= -\frac{1}{2}\lim_{x\to 0} \frac{1}{2+x}$$

$$= -\frac{1}{2}\left(\frac{1}{2}\right)$$

$$= \boxed{-\frac{1}{4}}$$

Exercise

Find the limit $\lim\limits_{x\to 1} \dfrac{x^{1/3}-1}{\sqrt{x}-1}$

Solution

$$\lim_{x\to 1} \frac{x^{1/3}-1}{\sqrt{x}-1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{x^{1/3}-1}{\sqrt{x}-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}$$

$$= \lim_{x\to 1} \frac{\left(x^{1/3}-1\right)\left(\sqrt{x}+1\right)}{x-1}$$

$$= \lim_{x\to 1} \frac{\left(x^{1/3}-1\right)\left(\sqrt{x}+1\right)}{\left(x^{1/3}\right)^{3}-1^{3}} \qquad a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$$

$$= \lim_{x\to 1} \frac{\left(x^{1/3}-1\right)\left(\sqrt{x}+1\right)}{\left(x^{1/3}-1\right)\left(x^{2/3}+x^{1/3}+1\right)}$$

$$= \lim_{x\to 1} \frac{\sqrt{x}+1}{x^{2/3}+x^{1/3}+1}$$

$$= \frac{2}{3}$$

Exercise

Find the limit $\displaystyle\lim_{x\to 64} \frac{x^{2/3}-16}{\sqrt{x}-8}$

Solution

$$\lim_{x\to 64} \frac{x^{2/3}-16}{\sqrt{x}-8} = \frac{\left(4^3\right)^{2/3}-16}{8-8}$$

$$= \frac{16-16}{0} = \frac{0}{0}$$

$$= \lim_{x\to 64} \frac{\left(x^{1/3}\right)^2-16}{\sqrt{x}-8} \cdot \frac{\sqrt{x}+8}{\sqrt{x}+8}$$

$$= \lim_{x\to 64} \frac{\left(x^{1/3}-4\right)\left(x^{1/3}+4\right)\left(\sqrt{x}+8\right)}{x-64}$$

$$= \lim_{x\to 64} \frac{\left(x^{1/3}-4\right)\left(x^{1/3}+4\right)\left(\sqrt{x}+8\right)}{\left(x^{1/3}\right)^3-4^3} \qquad a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$$

$$= \lim_{x\to 64} \frac{\left(x^{1/3}-4\right)\left(x^{1/3}+4\right)\left(\sqrt{x}+8\right)}{\left(x^{1/3}-4\right)\left(x^{2/3}+4x^{1/3}+16\right)}$$

$$= \lim_{x\to 64} \frac{\left(x^{1/3}+4\right)\left(\sqrt{x}+8\right)}{x^{2/3}+4x^{1/3}+16}$$

$$= \frac{(4+4)(8+8)}{16+16+16}$$

$$= \frac{8}{3}$$

Exercise

Find the limit $\quad \lim\limits_{x\to 0} \dfrac{\tan(2x)}{\tan(\pi x)}$

Solution

$$\lim_{x\to 0} \frac{\tan(2x)}{\tan(\pi x)} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin 2x}{\cos 2x} \cdot \frac{\cos(\pi x)}{\sin(\pi x)}$$

$$= \lim_{x\to 0} \frac{\cos(\pi x)}{\cos 2x} \cdot \frac{\sin 2x}{2x} \cdot \frac{2x}{\pi x} \cdot \frac{\pi x}{\sin(\pi x)}$$

$$= \frac{2}{\pi} \frac{\cos 0}{\cos 0} \cdot \lim_{2x\to 0} \frac{\sin 2x}{2x} \lim_{\pi x\to 0} \frac{1}{\frac{\sin \pi x}{\pi x}}$$

$$= \frac{2}{\pi}$$

Exercise

Find the limit $\quad \lim\limits_{x\to \pi^-} \csc x$

Solution

$$\lim_{x\to \pi^-} \csc x = \frac{1}{\sin \pi^-}$$

$$= \frac{1}{0^-}$$

$$= -\infty$$

Exercise

Find the limit $\quad \lim\limits_{x\to \pi} \sin\left(\frac{x}{2} + \sin x\right)$

Solution

$$\lim_{x\to \pi} \sin\left(\frac{x}{2} + \sin x\right) = \sin\left(\frac{\pi}{2} + \sin \pi\right)$$

$$= \sin\frac{\pi}{2}$$

$$= 1$$

Exercise

Find the limit $\lim\limits_{x\to\pi} \cos^2(x - \tan x)$

Solution

$$\lim\limits_{x\to\pi} \cos^2(x - \tan x) = \cos^2(\pi - \tan\pi)$$
$$= \cos^2(\pi)$$
$$= (-1)^2$$
$$= 1$$

Exercise

Find the limit $\lim\limits_{x\to 0} \dfrac{8x}{3\sin x - x}$

Solution

$$\lim\limits_{x\to 0} \frac{8x}{3\sin x - x} = \frac{0}{0}$$

$$= \lim\limits_{x\to 0} \frac{8\frac{x}{x}}{3\frac{\sin x}{x} - \frac{x}{x}}$$

$$= \frac{8}{3\lim\limits_{x\to 0} \frac{\sin x}{x} - 1} \qquad \lim\limits_{x\to 0} \frac{\sin x}{x} = 1$$

$$= \frac{8}{3-1}$$

$$= 4$$

Exercise

Find the limit $\lim\limits_{x\to 0} \dfrac{\cos 2x - 1}{\sin x}$

Solution

$$\lim\limits_{x\to 0} \frac{\cos 2x - 1}{\sin x} = \frac{0}{0}$$

$$= \lim\limits_{x\to 0} \frac{1 - 2\sin^2 x - 1}{\sin x}$$

$$= \lim_{x\to 0} \frac{-2\sin^2 x}{\sin x}$$
$$= -2 \lim_{x\to 0} \sin x$$
$$\underline{= 0}$$

Exercise

Find the limit $\lim\limits_{x\to -\infty} \dfrac{4-3x^3}{\sqrt{x^6+9}}$

Solution

$$\lim_{x\to -\infty} \frac{4-3x^3}{\sqrt{x^6+9}} = \lim_{x\to -\infty} \frac{3x^3}{\sqrt{x^6}}$$
$$= \lim_{x\to -\infty} \frac{3x^3}{x^3}$$
$$\underline{= 3}$$

Exercise

Find the limit $\lim\limits_{x\to -\infty} \dfrac{x^2-4x+8}{3x^3}$

Solution

$$\lim_{x\to -\infty} \frac{x^2-4x+8}{3x^3} = \lim_{x\to -\infty} \frac{x^2}{3x^3}$$
$$= \lim_{x\to -\infty} \frac{1}{3x}$$
$$\underline{= 0}$$

Exercise

Find the limit $\lim\limits_{x\to -\infty} \dfrac{2x^2+3}{5x^2+7}$

Solution

$$\boxed{\lim_{x\to -\infty} \frac{2x^2+3}{5x^2+7} = \frac{2}{5}}$$

Exercise

Find the limit $\displaystyle\lim_{x\to\infty}\frac{x^4+x^3}{12x^3+128}$

Solution

$$\lim_{x\to\infty}\frac{x^4+x^3}{12x^3+128}=\infty$$

Exercise

Find the limit $\displaystyle\lim_{x\to-\infty}\frac{2+\sqrt{x}}{2-\sqrt{x}}$

Solution

Since $x\to-\infty$ and inside the square root cannot be negative

$$\lim_{x\to-\infty}\frac{2+\sqrt{x}}{2-\sqrt{x}}=\nexists$$

Exercise

Find the limit $\displaystyle\lim_{x\to\infty}\frac{2+\sqrt{x}}{2-\sqrt{x}}$

Solution

$$\lim_{x\to\infty}\frac{2+\sqrt{x}}{2-\sqrt{x}}=\lim_{x\to\infty}\frac{\sqrt{x}}{-\sqrt{x}}$$

$$=-1$$

Exercise

Find the limit $\displaystyle\lim_{x\to-\infty}\frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}}$

Solution

$$\lim_{x\to-\infty}\frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}}=\lim_{x\to-\infty}\frac{\sqrt[3]{x}}{\sqrt[3]{x}}$$

$$=1$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{\frac{1}{x}+\frac{1}{x^4}}{\frac{1}{x^2}-\frac{1}{x^3}}$

Solution

$$\lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^4}}{\frac{1}{x^2}-\frac{1}{x^3}} = \frac{0}{0}$$

$$= \lim_{x\to\infty} \frac{\frac{x^3+1}{x^4}}{\frac{x-1}{x^3}}$$

$$= \lim_{x\to\infty} \frac{x^3+1}{x-1} \cdot \frac{x^3}{x^4}$$

$$= \lim_{x\to\infty} \frac{x^3+1}{x(x-1)}$$

$$= \lim_{x\to\infty} \frac{x^3}{x^2}$$

$$= \infty$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{2x^{5/3}-x^{1/3}+7}{x^{8/5}+3x+\sqrt{x}}$

Solution

$$\lim_{x\to\infty} \frac{2x^{5/3}-x^{1/3}+7}{x^{8/5}+3x+\sqrt{x}} = \lim_{x\to\infty} \frac{2x^{5/3}}{x^{8/5}}$$

$$= \lim_{x\to\infty} 2x^{\left(\frac{5}{3}-\frac{8}{5}\right)}$$

$$= \lim_{x\to\infty} 2x^{\frac{1}{15}}$$

$$= \infty$$

Exercise

Find the limit $\lim\limits_{x\to 2^+} \ln(x-2)$

Solution

$$\lim_{x\to 2^+} \ln(x-2) = \ln\left(0^+\right)$$
$$\boxed{= -\infty}$$

Exercise

Find the limit $\lim\limits_{x\to 1} x^2 \ln\left(2-\sqrt{x}\right)$

Solution

$$\lim_{x\to 1} x^2 \ln\left(2-\sqrt{x}\right) = \ln(2-1)$$
$$= \ln 1$$
$$\boxed{= 0}$$

Exercise

Find the limit $\lim\limits_{\theta\to 0^+} \sqrt{\theta}\, e^{\cos\frac{\pi}{\theta}}$

Solution

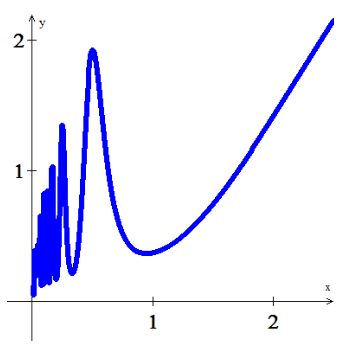
$$\lim_{\theta\to 0^+} \sqrt{\theta}\, e^{\cos\frac{\pi}{\theta}} = 0 \cdot e^{\cos\infty}$$

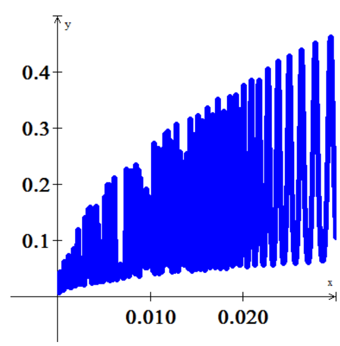
$$-1 \le \cos\frac{\pi}{\theta} \le 1$$

$$e^{-1} \le e^{\cos\frac{\pi}{\theta}} \le e$$

$$0 \cdot \frac{1}{e} \le 0 \cdot e^{\cos\frac{\pi}{\theta}} \le 0 \cdot e$$

$$\boxed{\lim_{\theta\to 0^+} \sqrt{\theta}\, e^{\cos\frac{\pi}{\theta}} = 0}$$





Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{2x-3}{5x+6}$

Solution

$$\lim_{x\to\infty} \frac{2x-3}{5x+6} = \lim_{x\to\infty} \frac{2x}{5x}$$

$$= \frac{2}{5}$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{2x^2-3}{5x^2+6}$

Solution

$$\lim_{x\to\infty} \frac{2x^2-3}{5x^2+6} = \lim_{x\to\infty} \frac{2x^2}{5x^2}$$

$$= \frac{2}{5}$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{2x-3}{5x^3+6}$

Solution

$$\lim_{x\to\infty} \frac{2x-3}{5x^3+6} = \lim_{x\to\infty} \frac{2x}{5x^3}$$

$$= 0$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{1}{5x^2-3x+6}$

Solution

$$\lim_{x\to\infty} \frac{1}{5x^2-3x+6} = \lim_{x\to\infty} \frac{1}{5x^2}$$

$$= 0$$

Exercise

Find the limit $\displaystyle\lim_{\theta\to 0}\frac{\theta\ \cot 4\theta}{\sin^2\theta\ \cot^2 2\theta}$

Solution

$$\lim_{\theta\to 0}\frac{\theta\ \cot 4\theta}{\sin^2\theta\ \cot^2 2\theta}=\frac{0}{0}$$

$$=\lim_{\theta\to 0}\frac{1}{\frac{\sin\theta}{\theta}}\cdot\frac{1}{\sin\theta}\cdot\frac{\cos 4\theta}{\sin 4\theta}\cdot\frac{\sin^2 2\theta}{\cos^2 2\theta}$$

$$=\lim_{\theta\to 0}\frac{1}{\frac{\sin\theta}{\theta}}\ \lim_{\theta\to 0}\frac{\cos 4\theta}{\cos^2 2\theta}\ \lim_{\theta\to 0}\frac{1}{\sin\theta}\cdot\frac{\sin 2\theta\ \sin 2\theta}{2\sin 2\theta\ \cos 2\theta}$$

$$=(1)(1)\ \lim_{\theta\to 0}\frac{1}{\sin\theta}\cdot\frac{2\sin\theta\cos\theta}{2\cos 2\theta}$$

$$=\lim_{\theta\to 0}\frac{\cos\theta}{\cos 2\theta}$$

$$=1$$

Exercise

Find the limit $\displaystyle\lim_{x\to 0^+}\frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}$

Solution

$$\lim_{x\to 0^+}\frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}=\frac{\sqrt{5}-\sqrt{5}}{0}=\frac{0}{0}$$

$$=\lim_{x\to 0^+}\frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}\cdot\frac{\sqrt{x^2+4x+5}+\sqrt{5}}{\sqrt{x^2+4x+5}+\sqrt{5}}$$

$$=\lim_{x\to 0^+}\frac{x^2+4x+5-5}{x\left(\sqrt{x^2+4x+5}+\sqrt{5}\right)}$$

$$=\lim_{x\to 0^+}\frac{x(x+4)}{x\left(\sqrt{x^2+4x+5}+\sqrt{5}\right)}$$

$$=\lim_{x\to 0^+}\frac{x+4}{\sqrt{x^2+4x+5}+\sqrt{5}}$$

$$=\frac{4}{\sqrt{5}+\sqrt{5}}$$

$$= \frac{4}{2\sqrt{5}}$$

$$\boxed{= \frac{2}{\sqrt{5}}}$$

Exercise

Find the limit $\lim\limits_{x \to 2} \frac{x^4 - 16}{x - 2}$

Solution

$$\lim_{x \to 2} \frac{x^4 - 16}{x - 2} = \frac{16 - 16}{2 - 2} = \frac{0}{0}$$

$$= \lim_{x \to 2} \frac{(x-2)(x+2)\left(x^2+4\right)}{x-2} \qquad a^4 - b^4 = (a-b)(a+b)\left(a^2+b^2\right)$$

$$= \lim_{x \to 2} (x+2)\left(x^2+4\right)$$

$$= (4)(8)$$

$$\boxed{= 32}$$

Exercise

Find the limit $\lim\limits_{x \to 2} \frac{x^3 - 8}{x - 2}$

Solution

$$\lim_{x \to 2} \frac{x^3 - 8}{x - 2} = \frac{0}{0}$$

$$= \lim_{x \to 2} \frac{(x-2)\left(x^2+2x+4\right)}{x-2} \qquad a^3 - b^3 = (a-b)\left(a^2+ab+b^2\right)$$

$$= \lim_{x \to 2} \left(x^2+2x+4\right)$$

$$= 4 + 4 + 4$$

$$\boxed{= 12}$$

Exercise

Find the limit $\lim\limits_{x\to-\infty} \dfrac{\sqrt[3]{x}-5x+3}{2x+x^{2/3}-4}$

Solution

$$\lim_{x\to-\infty} \frac{\sqrt[3]{x}-5x+3}{2x+x^{2/3}-4} = \lim_{x\to-\infty} \frac{-5x}{2x}$$

$$= -\frac{5}{2}$$

Exercise

Find the limit $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x^2+1}}{x+1}$

Solution

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x+1} = \lim_{x\to-\infty} \frac{\sqrt{x^2}}{x}$$

$$= \lim_{x\to-\infty} \frac{|x|}{x}$$

$$= -1$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+1}}{x+1}$

Solution

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x+1} = \lim_{x\to\infty} \frac{\sqrt{x^2}}{x}$$

$$= \lim_{x\to\infty} \frac{|x|}{x}$$

$$= 1$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{x-3}{\sqrt{4x^2+25}}$

Solution

$$\lim_{x\to\infty} \frac{x-3}{\sqrt{4x^2+25}} = \lim_{x\to\infty} \frac{x}{2|x|}$$

$$= \frac{1}{2}$$

Exercise

Find the limit $\lim_{x\to-\infty} \frac{4-3x^3}{\sqrt{x^6+9}}$

Solution

$$\lim_{x\to-\infty} \frac{4-3x^3}{\sqrt{x^6+9}} = \lim_{x\to-\infty} \frac{3x^3}{x^3}$$

$$= 3$$

Exercise

Find the limit $\lim_{x\to\infty} \frac{x^4-x}{15x^3+4}$

Solution

$$\lim_{x\to\infty} \frac{x^4-x}{15x^3+4} = \infty$$

Exercise

Find the limit $\lim_{x\to\infty} \frac{x+\sin x+2\sqrt{x}}{x+\sin x}$

Solution

$$-1 \le \sin x \le 1$$

$$\lim_{x\to\infty} \frac{x+\sin x+2\sqrt{x}}{x+\sin x} = \lim_{x\to\infty} \frac{x}{x}$$

$$= 1$$

Exercise

Find the limit $\lim_{x\to\infty} \frac{x^{2/3}-x^{-1}}{x^{2/3}+\cos^2 x}$

Solution

$$-1 \le \cos x \le 1$$

$$0 \le \cos^2 x \le 1$$

$$\lim_{x\to\infty} \frac{x^{2/3} - \frac{1}{x}}{x^{2/3} + \cos^2 x} = \lim_{x\to\infty} \frac{x^{2/3}}{x^{2/3}}$$

$$\boxed{=1}$$

Exercise

Find the limit $\lim_{x\to\infty} \frac{\sin 2x}{x}$

Solution

$$-1 \le \sin 2x \le 1$$

$$-\lim_{x\to\infty} \frac{1}{x} \le \lim_{x\to\infty} \frac{\sin 2x}{x} \le \lim_{x\to\infty} \frac{1}{x}$$

$$0 \le \lim_{x\to\infty} \frac{\sin 2x}{x} \le 0$$

$$\boxed{\lim_{x\to\infty} \frac{\sin 2x}{x} = 0}$$

Exercise

Find the limit $\lim_{x\to 0} \frac{\sin 5x}{3x}$

Solution

$$\lim_{x\to 0} \frac{\sin 5x}{3x} = \lim_{5x\to 0} \frac{5}{3} \cdot \frac{\sin 5x}{5x}$$

$$\boxed{=\frac{5}{3}}$$

Exercise

Find the limit $\lim_{x\to -\infty} \frac{\cos x}{2x}$

Solution

$$-1 \le \cos x \le 1$$

$$-\lim_{x\to\infty} \frac{1}{2x} \le \lim_{x\to\infty} \frac{\cos x}{2x} \le \lim_{x\to\infty} \frac{1}{2x}$$

$$0 \le \lim_{x\to\infty} \frac{\cos x}{2x} \le 0$$

$$\lim_{x\to\infty} \frac{\cos x}{2x} = 0$$

Exercise

Find the limit $\lim_{x\to-\infty} \left(\frac{x^2+x-1}{8x^2-3}\right)^{1/3}$

Solution

$$\lim_{x\to-\infty} \left(\frac{x^2+x-1}{8x^2-3}\right)^{1/3} = \lim_{x\to-\infty} \left(\frac{x^2}{8x^2}\right)^{1/3}$$

$$= \left(\frac{1}{2^3}\right)^{1/3}$$

$$= \frac{1}{2}$$

Exercise

Find the limit $\lim_{x\to-1} \frac{\sqrt{x^2+8}-3}{x+1}$

Solution

$$\lim_{x\to-1} \frac{\sqrt{x^2+8}-3}{x+1} = \frac{3-3}{-1+1} = \frac{0}{0}$$

$$= \lim_{x\to-1} \frac{\sqrt{x^2+8}-3}{x+1} \cdot \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}$$

$$= \lim_{x\to-1} \frac{x^2+8-9}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x\to-1} \frac{x^2-1}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x\to-1} \frac{(x-1)(x+1)}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x\to -1} \frac{x+1}{\sqrt{x^2+8}+3}$$

$$= \frac{0}{6}$$

$$= 0$$

Exercise

Find the limit $\displaystyle \lim_{x\to -\infty} \left(\frac{1-x^3}{x^2+7x}\right)^5$

Solution

$$\lim_{x\to -\infty} \left(\frac{1-x^3}{x^2+7x}\right)^5 = \lim_{x\to -\infty} \left(\frac{-x^3}{x^2}\right)^5$$

$$= \lim_{x\to -\infty} \left(-x^5\right)$$

$$= \infty$$

Exercise

Find the limit $\displaystyle \lim_{x\to \infty} \sqrt{\frac{x^2-5x}{x^3+x-2}}$

Solution

$$\lim_{x\to \infty} \sqrt{\frac{x^2-5x}{x^3+x-2}} = \lim_{x\to \infty} \sqrt{\frac{x^2}{x^3}}$$

$$= \lim_{x\to \infty} \frac{1}{\sqrt{x}}$$

$$= 0$$

Exercise

Find the limit $\displaystyle \lim_{x\to \infty} \frac{2\sqrt{x}+x^{-1}}{3x-7}$

Solution

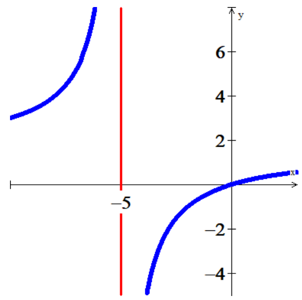
$$\lim_{x\to \infty} \frac{2\sqrt{x}+x^{-1}}{3x-7} = \lim_{x\to \infty} \frac{2\sqrt{x}}{3x}$$

$$= \lim_{x\to \infty} \frac{2}{3\sqrt{x}}$$

$$= 0$$

Exercise

Find the limit $\lim\limits_{x \to -5^-} \dfrac{3x}{2x+10}$



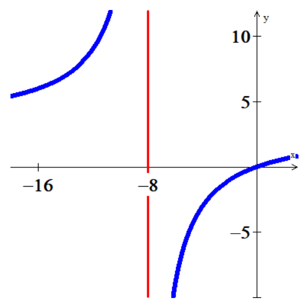
Solution

$$\lim_{x \to -5^-} \frac{3x}{2x+10} = \frac{-15}{0^-}$$

$$= \infty$$

Exercise

Find the limit $\lim\limits_{x \to -8^+} \dfrac{3x}{x+8}$



Solution

$$\lim_{x \to -8^+} \frac{3x}{x+8} = \frac{-24}{0^+}$$

$$= -\infty$$

Exercise

Find the limit $\lim\limits_{x \to 0} \dfrac{-1}{x^2(x+1)}$

Solution

$$\lim_{x \to 0} \frac{-1}{x^2(x+1)} = -\frac{1}{0}$$

$$= -\infty$$

Exercise

Find the limit $\lim\limits_{x \to 7} \dfrac{4}{(x-7)^2}$

Solution

$$\lim_{x \to 7} \frac{4}{(x-7)^2} = \frac{4}{0}$$

$$= \infty$$

Exercise

Find the limit $\displaystyle\lim_{x \to 0} \frac{1}{x^{2/3}}$

Solution

$$\lim_{x \to 0} \frac{1}{x^{2/3}} = \infty$$

Exercise

Find the limit $\displaystyle\lim_{x \to 2} \frac{x-2}{6x^2-10x-4}$

Solution

$$\lim_{x \to 2} \frac{x-2}{6x^2-10x-4} = \frac{0}{0}$$

$$= \lim_{x \to 2} \frac{x-2}{2(x-2)(3x+1)}$$

$$= \lim_{x \to 2} \frac{1}{2(3x+1)}$$

$$= \frac{1}{14}$$

Exercise

Find the limit $\displaystyle\lim_{x \to 6} \frac{\sqrt{x-2}-2}{x^2-36}$

Solution

$$\lim_{x \to 6} \frac{\sqrt{x-2}-2}{x^2-36} = \frac{2-2}{36-36} = \frac{0}{0}$$

$$= \lim_{x \to 6} \frac{\sqrt{x-2}-2}{(x-6)(x+6)} \frac{\sqrt{x-2}+2}{\sqrt{x-2}+2}$$

$$= \lim_{x \to 6} \frac{x-2-4}{(x-6)(x+6)\left(\sqrt{x-2}+2\right)}$$

$$= \lim_{x \to 6} \frac{x-6}{(x-6)(x+6)\left(\sqrt{x-2}+2\right)}$$

$$= \lim_{x \to 6} \frac{1}{(x+6)\left(\sqrt{x-2}+2\right)}$$

$$\boxed{= \frac{1}{48}}$$

Exercise

Find the limit $\lim_{x \to 0} \frac{\sqrt{x+9}-3}{x}$

Solution

$$\lim_{x \to 0} \frac{\sqrt{x+9}-3}{x} = \frac{3-3}{0} = \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\sqrt{x+9}-3}{x} \frac{\sqrt{x+9}+3}{\sqrt{x+9}+3}$$

$$= \lim_{x \to 0} \frac{x+9-9}{x\left(\sqrt{x+9}+3\right)}$$

$$= \lim_{x \to 0} \frac{x}{x\left(\sqrt{x+9}+3\right)}$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{x+9}+3}$$

$$\boxed{= \frac{1}{6}}$$

Exercise

Find the limit $\lim_{x \to 0} \frac{\sin 3x}{\sin x}$

Solution

$$\boxed{\lim_{x \to 0} \frac{\sin 3x}{\sin x} = 3}$$

Exercise

Find the limit $\lim_{x \to 0} \frac{\frac{1}{x}}{\sin \frac{\pi}{x}}$

Solution

$$\lim_{x\to 0}\frac{\frac{1}{x}}{\sin\frac{\pi}{x}}=\frac{\infty}{\sin\infty}$$

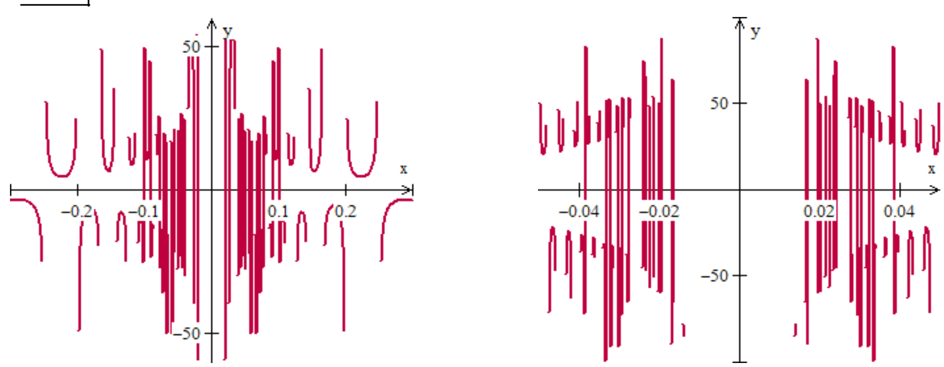
Since $-1\le \sin\frac{\pi}{x}\le 1$

$$-x\le x\sin\frac{\pi}{x}\le x$$

$$-\frac{1}{x}\le \frac{1}{x\sin\frac{\pi}{x}}\le \frac{1}{x}$$

$$\lim_{x\to 0}\frac{\frac{1}{x}}{\sin\frac{\pi}{x}}=\lim_{x\to 0}\frac{1}{x\sin\frac{\pi}{x}}$$

$$=\nexists$$



Exercise

Find the limit $\lim_{x\to 0}\frac{\sin 2x-2\sin x}{\sin 3x-3\sin x}$

Solution

$$\lim_{x\to 0}\frac{\sin 2x-2\sin x}{\sin 3x-3\sin x}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{2\sin x\cos x-2\sin x}{\sin(x+2x)-3\sin x}$$

$$=\lim_{x\to 0}\frac{2\sin x(\cos x-1)}{\sin x\cos 2x+\cos x\sin 2x-3\sin x}$$

$$=\lim_{x\to 0}\frac{2\sin x(\cos x-1)}{\sin x\left(2\cos^2 x-1\right)+2\cos^2 x\sin x-3\sin x}$$

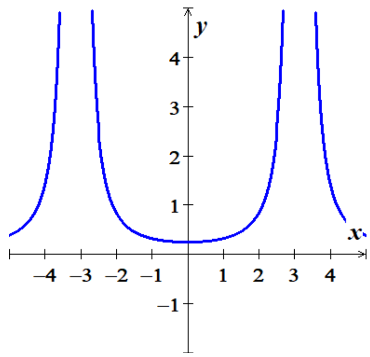
$$=\lim_{x\to 0}\frac{2\sin x(\cos x-1)}{\sin x\left(2\cos^2 x-1+2\cos^2 x-3\right)}$$

$$= \lim_{x \to 0} \frac{2(\cos x - 1)}{4\cos^2 x - 4}$$

$$= \lim_{x \to 0} \frac{2(\cos x - 1)}{4(\cos x - 1)(\cos x + 1)}$$

$$= \frac{1}{2} \lim_{x \to 0} \frac{1}{\cos x + 1}$$

$$= \frac{1}{4}$$



Exercise

Find the limit $\lim_{x \to 1} \frac{\frac{1}{x+1} - \frac{1}{2}}{x-1}$

Solution

$$\lim_{x \to 1} \frac{\frac{1}{x+1} - \frac{1}{2}}{x-1} = \frac{\frac{1}{2} - \frac{1}{2}}{0} = \frac{0}{0}$$

$$= \lim_{x \to 1} \frac{1}{x-1}\left(\frac{1}{x+1} - \frac{1}{2}\right)$$

$$= \lim_{x \to 1} \frac{1}{x-1}\left(\frac{2-x-1}{2(x+1)}\right)$$

$$= \lim_{x \to 1} \frac{1}{x-1}\left(\frac{-x+1}{2(x+1)}\right)$$

$$= -\frac{1}{2} \lim_{x \to 1} \frac{1}{x+1}$$

$$= -\frac{1}{4}$$

Exercise

Find the limit $\lim\limits_{x\to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right)$

Solution

$$\lim_{x\to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right)=\infty+\frac{5}{0(-5)}$$
$$=\infty-\infty$$

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right) &= \lim_{x\to 0}\frac{x-5+5}{x(x-5)}\\ &= \lim_{x\to 0}\frac{x}{x(x-5)}\\ &= \lim_{x\to 0}\frac{1}{x-5}\\ &= -\frac{1}{5}\end{aligned}$$

Exercise

Find the limit $\lim\limits_{x\to 3}\left(\frac{1}{x-3}-\frac{4}{x^2-2x-3}\right)$

Solution

$$\lim_{x\to 3}\left(\frac{1}{x-3}-\frac{4}{x^2-2x-3}\right)=\frac{1}{0}-\frac{4}{0}$$
$$=\infty-\infty$$

$$\begin{aligned}\lim_{x\to 3}\left(\frac{1}{x-3}-\frac{4}{x^2-2x-3}\right) &= \lim_{x\to 3}\left(\frac{1}{x-3}-\frac{4}{(x+1)(x-3)}\right)\\ &= \lim_{x\to 3}\frac{x+1-4}{(x+1)(x-3)}\\ &= \lim_{x\to 3}\frac{x-3}{(x+1)(x-3)}\\ &= \lim_{x\to 3}\frac{1}{x+1}\\ &= \frac{1}{4}\end{aligned}$$

Exercise

Find the limit $\lim\limits_{x\to 2^-} \sqrt{x-2}$

Solution

$$\lim_{x\to 2^-} \sqrt{x-2} = \sqrt{0^-}$$

$$\boxed{= \nexists}$$

Exercise

Find the limit $\lim\limits_{x\to 2^+} \sqrt{x-2}$

Solution

$$\lim_{x\to 2^+} \sqrt{x-2} = \sqrt{0^+}$$

$$\boxed{= 0}$$

Exercise

Find the limit $\lim\limits_{x\to -2} \sqrt{x^2-6x+3}$

Solution

$$\lim_{x\to -2} \sqrt{x^2-6x+3} = \sqrt{19}$$

Exercise

Find the limit $\lim\limits_{x\to \infty} \dfrac{\sqrt{2x^2+1}}{3x-5}$

Solution

$$\lim_{x\to \infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to \infty} \frac{\sqrt{2}\ |x|}{3x}$$

$$= \frac{\sqrt{2}}{3} \lim_{x\to \infty} \frac{x}{x}$$

$$\boxed{= \frac{\sqrt{2}}{3}}$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{1}{\sqrt{x^2-2x}-x}$

Solution

$$\lim_{x\to\infty} \frac{1}{\sqrt{x^2-2x}-x} = \lim_{x\to\infty} \frac{1}{|x|-x}$$

$$= \frac{1}{\infty-\infty}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x^2-2x}-x}\,\frac{\sqrt{x^2-2x}+x}{\sqrt{x^2-2x}+x}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2-2x}+x}{x^2-2x-x^2}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2-2x}+x}{-2x}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2}+x}{-2x}$$

$$= \lim_{x\to\infty} \frac{|x|+x}{-2x}$$

$$= \lim_{x\to\infty} \frac{x+x}{-2x}$$

$$= \lim_{x\to\infty} \frac{2x}{-2x}$$

$$= -1$$

Exercise

Find the limit $\lim\limits_{x\to-\infty} \dfrac{1}{\sqrt{x^2-2x}-x}$

Solution

$$\lim_{x\to-\infty} \frac{1}{\sqrt{x^2-2x}-x} = \lim_{x\to-\infty} \frac{1}{\sqrt{x^2}-x}$$

$$= \lim_{x\to-\infty} \frac{1}{|x|-x}$$

$$= \frac{1}{\infty-(-\infty)}$$

$$=\frac{1}{\infty}$$

$$\underline{=0}$$

Exercise

Find the limit $\lim\limits_{x\to\infty}\frac{1+2x-3x^2}{x^2+x^3}$

Solution

$$\lim_{x\to\infty}\frac{1+2x-3x^2}{x^2+x^3}=\lim_{x\to\infty}\frac{3x^2}{x^3}$$

$$\underline{=0}$$

Exercise

Find the limit $\lim\limits_{x\to\infty}\frac{1-2x+3x^5}{x^2-3x^3}$

Solution

$$\lim_{x\to\infty}\frac{1-2x+3x^5}{x^2-3x^3}=\lim_{x\to\infty}\frac{3x^5}{-3x^3}$$

$$\underline{=\infty}$$

Exercise

Find the limit $\lim\limits_{x\to 1}\frac{x-3}{x^2-2x+1}$

Solution

$$\lim_{x\to 1}\frac{x-3}{x^2-2x+1}=\frac{-2}{0}$$

$$\underline{=-\infty}$$

Exercise

Find the limit $\lim\limits_{x\to\infty}\frac{x\sqrt{x+1}\left(1-\sqrt{2x+3}\right)}{4x^2-6x+7}$

Solution

$$\lim_{x\to\infty}\frac{x\sqrt{x+1}\left(1-\sqrt{2x+3}\right)}{4x^2-6x+7}=\lim_{x\to\infty}\frac{x\sqrt{x}\left(-\sqrt{2x}\right)}{4x^2}$$

$$=-\frac{\sqrt{2}}{4}\lim_{x\to\infty}\frac{x^2}{x^2}$$

$$=-\frac{\sqrt{2}}{4}$$

Exercise

Find the limit $\lim\limits_{x\to\infty}\left(\frac{x^2}{x+1}-\frac{x^2}{x-1}\right)$

Solution

$$\lim_{x\to\infty}\left(\frac{x^2}{x+1}-\frac{x^2}{x-1}\right)=\lim_{x\to\infty}(x-x)$$

$$=\infty-\infty$$

$$=\lim_{x\to\infty}\frac{x^3-x^2-x^3-x^2}{x^2-1}$$

$$=\lim_{x\to\infty}\frac{-2x^2}{x^2}$$

$$=-2$$

Exercise

Find the limit $\lim\limits_{x\to-\infty}\frac{2x-5}{|3x+2|}$

Solution

$$\lim_{x\to-\infty}\frac{2x-5}{|3x+2|}=\lim_{x\to-\infty}\frac{2x}{3|x|}$$

$$=\frac{2}{3}\frac{-}{|-|}$$

$$=-\frac{2}{3}$$

Exercise

Find the limit $\lim\limits_{x\to\infty}\frac{2x-5}{|3x+2|}$

Solution

$$\lim_{x\to\infty} \frac{2x-5}{|3x+2|} = \lim_{x\to\infty} \frac{2x}{3|x|}$$

$$= \frac{2}{3}\frac{+}{|+|}$$

$$= \frac{2}{3}$$

Exercise

Find the limit $\lim_{x\to\infty} \frac{3x+2\sqrt{x}}{1-x}$

Solution

$$\lim_{x\to\infty} \frac{3x+2\sqrt{x}}{1-x} = \lim_{x\to\infty} \frac{3x}{-x}$$

$$= -3$$

Exercise

Find the limit $\lim_{x\to\infty} \frac{2x+1}{\sqrt{3x^2+x-1}}$

Solution

$$\lim_{x\to\infty} \frac{2x+1}{\sqrt{3x^2+x-1}} = \lim_{x\to\infty} \frac{2x}{\sqrt{3x^2}}$$

$$= \frac{2}{\sqrt{3}}$$

Exercise

Find the limit $\lim_{x\to\infty} \frac{1-x^2}{3x^2-x-1}$

Solution

$$\lim_{x\to\infty} \frac{1-x^2}{3x^2-x-1} = \lim_{x\to\infty} \frac{-x^2}{3x^2}$$

$$= -\frac{1}{3}$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{x^3-2}{x^2+5}$

Solution

$$\lim_{x\to\infty} \frac{x^3-2}{x^2+5} = \lim_{x\to\infty} \frac{x^3}{x^2}$$

$$= \infty$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \sqrt{x}$

Solution

$$\lim_{x\to\infty} \sqrt{x} = \infty$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{1}{\sqrt{x^2+1}}$

Solution

$$\lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{1}{x}$$

$$= 0$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{x^3+\sin x}{x^3+\cos x}$

Solution

$$-1 \geq \ \sin x,\ \cos x\ \leq 1$$

$$\lim_{x\to\infty} \frac{x^3+\sin x}{x^3+\cos x} = \lim_{x\to\infty} \frac{x^3}{x^3}$$

$$= 1$$

Exercise

Find the limit $\lim\limits_{x\to 1^-} \dfrac{1}{x-1}$

Solution

$$\lim_{x\to 1^-} \frac{1}{x-1} = \frac{1}{0^-}$$

$$\boxed{= -\infty}$$

Exercise

Find the limit $\lim\limits_{x\to 1^+} \dfrac{1}{x-1}$

Solution

$$\lim_{x\to 1^+} \frac{1}{x-1} = \frac{1}{0^+}$$

$$\boxed{= \infty}$$

Exercise

Find the limit $\lim\limits_{x\to 0} \sin\dfrac{1}{x^2}$

Solution

$$\lim_{x\to 0} \sin\frac{1}{x^2} = \lim_{x\to 0} \sin\infty$$

$$\boxed{= \nexists}$$

Exercise

Find the limit $\lim\limits_{\theta\to \infty} \sin\theta$

Solution

$$\lim_{\theta\to \infty} \sin\theta = \lim_{\theta\to \infty} \sin\infty$$

$$\boxed{= \nexists}$$

Exercise

Find the limit $\lim\limits_{\theta\to\infty} \dfrac{\cos\theta}{\theta}$

Solution

$$-1 \le \cos\theta \le 1$$

$$-\frac{1}{\theta} \le \frac{\cos\theta}{\theta} \le \frac{1}{\theta}$$

$$\lim_{\theta\to\infty} \frac{\cos\theta}{\theta} = \lim_{\theta\to\infty} \frac{1}{\theta}$$

$$\underline{=0}$$

Exercise

Find the limit $\lim\limits_{\theta\to 0} \theta^2\cos(2\pi\theta)$

Solution

$$\lim_{\theta\to 0} \theta^2\cos(2\pi\theta) = 0\ \cos 0$$

$$= 0\cdot 1$$

$$\underline{=0}$$

Exercise

Find the limit $\lim\limits_{\theta\to\frac{\pi}{2}} \dfrac{\cot\theta}{\cos\theta}$

Solution

$$\lim_{\theta\to\frac{\pi}{2}} \frac{\cot\theta}{\cos\theta} = \frac{0}{0}$$

$$= \lim_{\theta\to\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta}\frac{1}{\cos\theta}$$

$$= \lim_{\theta\to\frac{\pi}{2}} \frac{1}{\sin\theta}$$

$$= \frac{1}{\sin\frac{\pi}{2}}$$

$$\underline{=1}$$

Exercise

Find the limit $\lim\limits_{\theta \to \pi} \dfrac{\sin\theta}{\tan\theta}$

Solution

$$\lim_{\theta \to \pi} \frac{\sin\theta}{\tan\theta} = \frac{\sin\pi}{\tan\pi} = \frac{0}{0}$$

$$= \lim_{\theta \to \pi} \sin\theta \frac{\cos\theta}{\sin\theta}$$

$$= \lim_{\theta \to \pi} \cos\theta$$

$$= \cos\pi$$

$$= -1$$

Exercise

Find the limit $\lim\limits_{x \to -\infty} e^{x^2}$

Solution

$$\lim_{x \to -\infty} e^{x^2} = e^{\infty}$$

$$= \infty$$

Exercise

Find the limit $\lim\limits_{x \to -\infty} e^{x^3}$

Solution

$$\lim_{x \to -\infty} e^{x^3} = e^{-\infty}$$

$$= \frac{1}{e^{\infty}}$$

$$= 0$$

Exercise

Find the limit $\lim\limits_{x \to -\infty} \ln|x|$

Solution

$$\lim_{x \to -\infty} \ln|x| = \ln|-\infty|$$
$$= \ln \infty$$
$$\boxed{= \infty}$$

Exercise

Find the limit $\lim\limits_{x \to \frac{2\pi}{3}} \sin x$

Solution

$$\lim_{x \to \frac{2\pi}{3}} \sin x = \sin \frac{2\pi}{3}$$
$$\boxed{= \frac{1}{2}}$$

Exercise

Find the limit $\lim\limits_{x \to \frac{5\pi}{4}} \cos x$

Solution

$$\lim_{x \to \frac{5\pi}{4}} \cos x = \cos \frac{5\pi}{4}$$
$$\boxed{= -\frac{\sqrt{2}}{2}}$$

Exercise

Find the limit $\lim\limits_{x \to 0} \dfrac{\sin(2\pi x)}{\sin(3\pi x)}$

Solution

$$\lim_{x \to 0} \frac{\sin(2\pi x)}{\sin(3\pi x)} = \frac{0}{0}$$
$$= \lim_{x \to 0} \frac{2\pi x}{3\pi x} \frac{\sin(2\pi x)}{2\pi x} \frac{1}{\frac{\sin(3\pi x)}{3\pi x}}$$
$$= \frac{2}{3} \lim_{2\pi x \to 0} \frac{\sin(2\pi x)}{2\pi x} \frac{1}{\lim\limits_{3\pi x \to 0} \frac{\sin(3\pi x)}{3\pi x}}$$

$$= \frac{2}{3}$$

Exercise

Find the limit $\lim\limits_{x\to 0^+} \dfrac{x-\sqrt{x}}{\sqrt{\sin x}}$

Solution

$$\lim_{x\to 0^+} \frac{x-\sqrt{x}}{\sqrt{\sin x}} = \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{\sqrt{x}\left(\sqrt{x}-1\right)}{\sqrt{\sin x}}$$

$$= \lim_{x\to 0^+} \frac{\left(\sqrt{x}-1\right)}{\frac{\sqrt{\sin x}}{\sqrt{x}}}$$

$$= \lim_{x\to 0^+} \frac{\left(\sqrt{x}-1\right)}{\sqrt{\frac{\sin x}{x}}}$$

$$= \frac{0-1}{1}$$

$$= -1$$

Exercise

Find the limit $\lim\limits_{x\to 0^+} \dfrac{\sin\sqrt{1-x}}{\sqrt{1-x^2}}$

Solution

$$\lim_{x\to 0^+} \frac{\sin\sqrt{1-x}}{\sqrt{1-x^2}} = \frac{\sin 1}{1}$$

$$= 1$$

Exercise

Find the limit $\lim\limits_{x\to 0} e^{x^2}$

Solution

$$\lim_{x \to 0} e^{x^2} = e^0$$
$$\underline{=1}$$

Exercise

Find the limit $\lim_{x \to 1} e^{x^2-1}$

Solution

$$\lim_{x \to 1} e^{x^2-1} = e^0$$
$$\underline{=1}$$

Exercise

Find the limit $\lim_{x \to 1} \ln x$

Solution

$$\lim_{x \to 1} \ln x = \ln 1$$
$$\underline{=0}$$

Exercise

Find the limit $\lim_{x \to 2} \left(e^x - \ln x\right)$

Solution

$$\lim_{x \to 2} \left(e^x - \ln x\right) = e^2 - \ln 2$$

Exercise

Find the limit $\lim_{x \to 1} \frac{1}{\ln x}$

Solution

$$\lim_{x \to 1} \frac{1}{\ln x} = \frac{1}{\ln 1}$$
$$= \frac{1}{0}$$
$$\underline{= \infty}$$

Exercise

Find the limit $\lim\limits_{x\to 4}\left(x^2-4x+1\right)$

Solution

$$\lim_{x\to 4}\left(x^2-4x+1\right)=16-16+1$$
$$=1$$

Exercise

Find the limit $\lim\limits_{x\to 1}\frac{x+3}{x+6}$

Solution

$$\lim_{x\to 1}\frac{x+3}{x+6}=\frac{4}{7}$$

Exercise

Find the limit $\lim\limits_{x\to 1}\frac{x^2-1}{x+1}$

Solution

$$\lim_{x\to 1}\frac{x^2-1}{x+1}=\frac{0}{2}$$
$$=0$$

Exercise

Find the limit $\lim\limits_{x\to 3}\frac{x^2-6x+9}{x^2-9}$

Solution

$$\lim_{x\to 3}\frac{x^2-6x+9}{x^2-9}=\frac{9-18+9}{9-9}=\frac{0}{0}$$
$$=\lim_{x\to 3}\frac{(x-3)^2}{(x-3)(x+3)}$$
$$=\lim_{x\to 3}\frac{x-3}{x+3}$$
$$=\frac{0}{6}$$
$$=0$$

Exercise

Find the limit $\lim\limits_{x \to 2} \dfrac{1}{4-x^2}$

Solution

$$\lim_{x \to 2} \frac{1}{4-x^2} = \frac{1}{0}$$

$$= \infty$$

Exercise

Find the limit $\lim\limits_{x \to 9} \dfrac{\sqrt{x}-3}{x-9}$

Solution

$$\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9} = \frac{3-3}{9-9} = \frac{0}{0}$$

$$= \lim_{x \to 9} \frac{\sqrt{x}-3}{\left(\sqrt{x}-3\right)\left(\sqrt{x}+3\right)}$$

$$= \lim_{x \to 9} \frac{1}{\sqrt{x}+3}$$

$$= \frac{1}{6}$$

Exercise

Find the limit $\lim\limits_{x \to \pi} \dfrac{(x-\pi)^2}{\pi x}$

Solution

$$\lim_{x \to \pi} \frac{(x-\pi)^2}{\pi x} = \frac{0}{\pi^2}$$

$$= 0$$

Exercise

Find the limit $\lim\limits_{x \to 2} \dfrac{\sqrt{4-4x+x^2}}{x-2}$

Solution

$$\lim_{x\to 2} \frac{\sqrt{4-4x+x^2}}{x-2} = \frac{\sqrt{4-8+4}}{2-2} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{\sqrt{(x-2)^2}}{x-2}$$

$$= \lim_{x\to 2} \frac{x-2}{x-2}$$

$$\underline{=1}$$

Exercise

Find the limit $\lim_{x\to 2} \frac{(x-2)^2}{x^2-4}$

Solution

$$\lim_{x\to 2} \frac{(x-2)^2}{x^2-4} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{(x-2)^2}{(x-2)(x+2)}$$

$$= \lim_{x\to 2} \frac{x-2}{x+2}$$

$$= \frac{0}{4}$$

$$\underline{=0}$$

Exercise

Find the limit $\lim_{x\to 3} \frac{x-3}{x^2-9}$

Solution

$$\lim_{x\to 3} \frac{x-3}{x^2-9} = \frac{0}{0}$$

$$= \lim_{x\to 3} \frac{x-3}{(x-3)(x+3)}$$

$$= \lim_{x\to 3} \frac{1}{x+3}$$

$$\underline{= \frac{1}{6}}$$

Exercise

Find the limit $\lim\limits_{x\to 2} \dfrac{x+2}{x^2+5x+6}$

Solution

$$\lim_{x\to 2} \frac{x+2}{x^2+5x+6} = \frac{4}{4+10+6}$$

$$= \frac{1}{5}$$

Exercise

Find the limit $\lim\limits_{x\to -3} (5-x)^{4/3}$

Solution

$$\lim_{x\to -3} (5-x)^{4/3} = 8^{4/3}$$

$$= \left(2^3\right)^{4/3}$$

$$= 2^4$$

$$= 16$$

Exercise

Find the limit $\lim\limits_{x\to 0} \sqrt{7+\sec^2 x}$

Solution

$$\lim_{x\to 0} \sqrt{7+\sec^2 x} = \sqrt{7+1}$$

$$= \sqrt{8}$$

$$= 2\sqrt{2}$$

Exercise

Find the limit $\lim\limits_{x\to 4} \dfrac{4-x}{5-\sqrt{x^2+9}}$

Solution

$$\lim_{x\to 4}\frac{4-x}{5-\sqrt{x^2+9}}=\frac{0}{5-5}=\frac{0}{0}$$

$$=\lim_{x\to 4}\frac{4-x}{5-\sqrt{x^2+9}}\frac{5+\sqrt{x^2+9}}{5+\sqrt{x^2+9}}$$

$$=\lim_{x\to 4}\frac{(4-x)\left(5+\sqrt{x^2+9}\right)}{25-x^2-9}$$

$$=\lim_{x\to 4}\frac{(4-x)\left(5+\sqrt{x^2+9}\right)}{16-x^2}$$

$$=\lim_{x\to 4}\frac{(4-x)\left(5+\sqrt{x^2+9}\right)}{(4-x)(4+x)}$$

$$=\lim_{x\to 4}\frac{5+\sqrt{x^2+9}}{4+x}$$

$$=\frac{5+5}{8}$$

$$=\frac{5}{4}$$

Exercise

Find the limit $\lim_{x\to -3}\frac{2-\sqrt{x^2-5}}{x+3}$

Solution

$$\lim_{x\to -3}\frac{2-\sqrt{x^2-5}}{x+3}=\frac{2-2}{0}=\frac{0}{0}$$

$$=\lim_{x\to -3}\frac{2-\sqrt{x^2-5}}{x+3}\frac{2+\sqrt{x^2-5}}{2+\sqrt{x^2-5}}$$

$$=\lim_{x\to -3}\frac{4-x^2+5}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$=\lim_{x\to -3}\frac{9-x^2}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{(3-x)(3+x)}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{3-x}{2+\sqrt{x^2-5}}$$

$$= \frac{6}{2+2}$$

$$= \frac{3}{2}$$

Exercise

Find the limit $\lim\limits_{x\to 0} \dfrac{x^2+4x}{\sqrt{x^3+x^2}}$

Solution

$$\lim_{x\to 0} \frac{x^2+4x}{\sqrt{x^3+x^2}} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{x(x+4)}{|x|\sqrt{x+1}}$$

$$= \lim_{x\to 0} \frac{x+4}{\sqrt{x+1}}$$

$$= 4$$

Exercise

Find the limit $\lim\limits_{x\to 0} \sin e^{-1/x^2}$

Solution

$$\lim_{x\to 0} \sin e^{-1/x^2} = \sin\left(e^{-\infty}\right)$$

$$= \sin\left(\frac{1}{e^{\infty}}\right)$$

$$= \sin(0)$$

$$= 0$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \sin e^{-1/x^2}$

Solution

$$\lim_{x\to\infty} \sin e^{-1/x^2} = \sin\left(e^0\right)$$

$$= \sin 1$$

Exercise

Find the limit $\lim\limits_{x\to 2^-} \dfrac{-x}{\sqrt{4-x^2}}$

Solution

$$\lim_{x\to 2^-} \frac{-x}{\sqrt{4-x^2}} = \frac{-2}{0^+}$$

$$= -\infty$$

Exercise

Find the limit $\lim\limits_{x\to -\infty} \dfrac{-x}{\sqrt{4+x^2}}$

Solution

$$\lim_{x\to -\infty} \frac{-x}{\sqrt{4+x^2}} = \lim_{x\to -\infty} \frac{-x}{\sqrt{x^2}}$$

$$= \lim_{x\to -\infty} \frac{-x}{|x|}$$

$$= \frac{+}{+}\infty$$

$$= \infty$$

Exercise

Find the limit $\lim\limits_{x\to -\infty} \dfrac{x}{\sqrt{4+x^2}}$

Solution

$$\lim_{x\to -\infty} \frac{x}{\sqrt{4+x^2}} = \lim_{x\to -\infty} \frac{x}{\sqrt{x^2}}$$

$$= \lim_{x\to-\infty} \frac{x}{|x|}$$

$$= \frac{-}{+}\infty$$

$$= -\infty$$

Exercise

Find the limit $\displaystyle\lim_{x\to-\infty} \frac{\ln(x+2)}{\ln\left(x^2+x-2\right)}$

Solution

$$\lim_{x\to-\infty} \frac{\ln(x+2)}{\ln\left(x^2+x-2\right)} = \lim_{x\to-\infty} \frac{\ln(-\infty)}{}$$

$$= \nexists$$

Exercise

Find the limit $\displaystyle\lim_{x\to\infty} \frac{\ln(x+2)}{\ln\left(x^2+x-2\right)}$

Solution

$$\lim_{x\to\infty} \frac{\ln(x+2)}{\ln\left(x^2+x-2\right)} = \lim_{x\to\infty} \frac{\ln(x)}{\ln\left(x^2\right)}$$

$$= \lim_{x\to\infty} \frac{\ln(x)}{2\ln(x)}$$

$$= \frac{1}{2}$$

Exercise

Find the limit $\displaystyle\lim_{x\to-1^-} \left(x^2-2x-3\right)^{-2/3}$

Solution

$$\lim_{x\to-1^-} \left(x^2-2x-3\right)^{-2/3} = (0)^{-2/3}$$

$$= 0$$

Exercise

Find the limit $\lim\limits_{x\to 1^+} \dfrac{1}{\sqrt[3]{x^2-3x+2}}$

Solution

$$\lim_{x\to 1^+} \frac{1}{\sqrt[3]{x^2-3x+2}} = \frac{1}{\sqrt[3]{0^-}}$$

$$= \frac{1}{0^-}$$

$$= -\infty$$

Exercise

Find the limit $\lim\limits_{x\to -1^-} \dfrac{1}{\sqrt[4]{2x^2+5x+3}}$

Solution

$$2x^2+5x+3=0 \quad\Rightarrow\quad x=-1,\ -\frac{3}{2}$$

$$-\frac{3}{2} < x = -1^- < -1 \quad\Rightarrow\quad 2x^2+5x+3<0$$

$$\lim_{x\to -1^-} \frac{1}{\sqrt[4]{2x^2+5x+3}} = \frac{1}{\sqrt[4]{0^-}}$$

$$= \nexists$$

Since, no negative inside the fourth root.

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{x^3-x+6}{3x^3+2x-5}$

Solution

$$\lim_{x\to\infty} \frac{x^3-x+6}{3x^3+2x-5} = \lim_{x\to\infty} \frac{x^3}{3x^3}$$

$$= \frac{1}{3}$$

Exercise

Find the limit $\lim\limits_{\theta\to\infty} \cot^{-1}\theta$

Solution

$$\lim_{\theta \to \infty} \cot^{-1}\theta = \cot^{-1}(\infty)$$

$$= 0^{+}$$

Exercise

Find the limit $\lim_{\theta \to \infty} \sec^{-1}\theta$

Solution

$$\lim_{\theta \to \infty} \sec^{-1}\theta = \sec^{-1}(\infty)$$

$$= \frac{\pi}{2}^{-}$$

Exercise

Find the limit $\lim_{x \to \infty} \left(e^{-3x}\cos 2x\right)$

Solution

$$-1 \le \cos 2x \le 1$$

$$-\frac{1}{e^{3x}} \le e^{-3x}\cos 2x \le \frac{1}{e^{3x}}$$

$$\lim_{x \to \infty} \left(e^{-3x}\cos 2x\right) = \lim_{x \to \infty} \frac{1}{e^{3x}}$$

$$= \frac{1}{\infty}$$

$$= 0$$

Exercise

Find the limit $\lim_{x \to -\infty} \left(e^{-3x}\cos 2x\right)$

Solution

$$-1 \le \cos 2x \le 1$$

$$-\frac{1}{e^{3x}} \le e^{-3x}\cos 2x \le \frac{1}{e^{3x}}$$

$$\lim_{x \to -\infty} \left(e^{-3x}\cos 2x\right) = \lim_{x \to -\infty} e^{-3x}$$

$$= e^{\infty}$$

$$\underline{= \infty}$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \sin\left(\tan^{-1} x\right)$

Solution

$$\lim_{x\to\infty} \sin\left(\tan^{-1} x\right) = \sin\left(\tan^{-1} \infty\right)$$

$$= \sin\left(\frac{\pi}{2}\right)$$

$$\underline{= 1}$$

Exercise

Find the limit $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2}$

Solution

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1-1}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{1-\cos x}{x^2} \frac{1+\cos x}{1+\cos x}$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x^2(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \frac{\sin x}{x} \frac{1}{(1+\cos x)}$$

$$= (1)(1)\left(\frac{1}{2}\right)$$

$$\underline{= \frac{1}{2}}$$

Exercise

Find the limit $\lim\limits_{x\to 0} \dfrac{1-\cos x^3}{x^6}$

Solution

$$\lim_{x\to 0}\frac{1-\cos x^3}{x^6}=\frac{1-1}{0}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{1-\cos x^3}{\left(x^3\right)^2}\frac{1+\cos x^3}{1+\cos x^3}$$

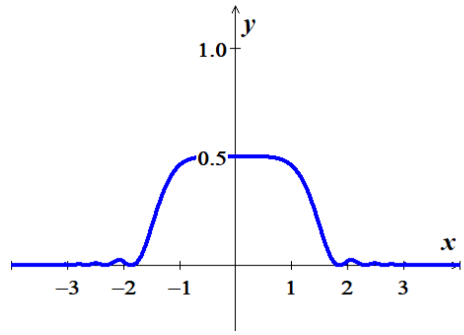
$$=\lim_{x\to 0}\frac{1-\cos^2\left(x^3\right)}{\left(x^3\right)^2\left(1+\cos x^3\right)}$$

$$=\lim_{x\to 0}\frac{\sin^2\left(x^3\right)}{\left(x^3\right)^2\left(1+\cos x^3\right)}$$

$$=\lim_{x^3\to 0}\frac{\sin x^3}{x^3}\cdot\frac{\sin x^3}{x^3}\cdot\frac{1}{1+\cos x^3}$$

$$=(1)(1)\left(\frac{1}{2}\right)$$

$$=\frac{1}{2}$$



Exercise

Find the limit $\lim_{x\to 0}\frac{1-\cos x^4}{x^8}$

Solution

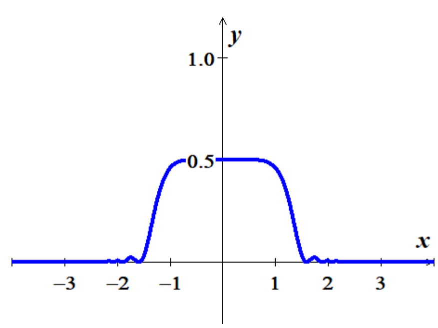
$$\lim_{x\to 0}\frac{1-\cos x^4}{x^8}=\frac{1-1}{0}=\frac{0}{0}$$

$$=\lim_{x^4\to 0}\frac{1-\cos x^4}{\left(x^4\right)^2}\frac{1+\cos x^4}{1+\cos x^4}$$

$$=\lim_{x^4\to 0}\frac{1-\cos^2\left(x^4\right)}{\left(x^4\right)^2\left(1+\cos x^4\right)}$$

$$=\lim_{x^4\to 0}\frac{\sin^2\left(x^4\right)}{\left(x^4\right)^2\left(1+\cos x^4\right)}$$

$$=\lim_{x^4\to 0}\frac{\sin x^4}{x^4}\cdot\frac{\sin x^4}{x^4}\cdot\frac{1}{1+\cos x^4}$$



$$=(1)(1)\left(\frac{1}{2}\right)$$

$$\underline{=\frac{1}{2}}$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{\ln\left(3+e^{2x}\right)}{\ln\left(1+e^{x}\right)}$

Solution

$$\lim_{x\to\infty} \frac{\ln\left(3+e^{2x}\right)}{\ln\left(1+e^{x}\right)} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{\ln\left(e^{2x}\right)}{\ln\left(e^{x}\right)}$$

$$= \lim_{x\to\infty} \frac{2\ln\left(e^{x}\right)}{\ln\left(e^{x}\right)}$$

$$\underline{=2}$$

Exercise

Find the limit $\lim\limits_{x\to\frac{\pi}{2}} e^{-\tan x}$

Solution

$$\lim_{x\to\frac{\pi}{2}} e^{-\tan x} = e^{-\tan\frac{\pi}{2}}$$

$$= e^{-\infty}$$

$$\underline{=0}$$

Exercise

Find the limit $\lim\limits_{x\to 0^+} \tan^{-1}(\ln x)$

Solution

$$\lim_{x\to 0^+} \tan^{-1}(\ln x) = \tan^{-1}\left(\ln\left(0^+\right)\right)$$
$$= \tan^{-1}(-\infty)$$
$$\underline{= -\frac{\pi}{2}}$$

Exercise

Find the limit $\lim_{x\to -\infty} 4\tan^{-1} x - 1$

Solution

$$\lim_{x\to -\infty} 4\tan^{-1} x - 1 = 4\tan^{-1}(-\infty) - 1$$
$$= 4\left(-\frac{\pi}{2}\right) - 1$$
$$\underline{= -2\pi - 1}$$

Exercise

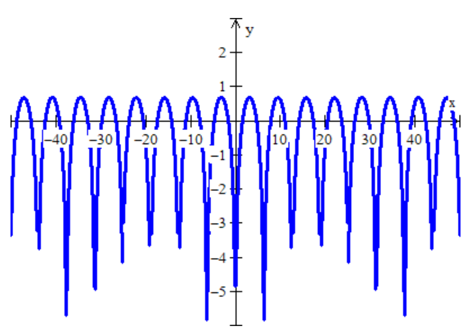
Find the limit $\lim_{x\to \infty} 4\tan^{-1} x - 1$

Solution

$$\lim_{x\to \infty} 4\tan^{-1} x - 1 = 4\tan^{-1}(\infty) - 1$$
$$= 4\left(\frac{\pi}{2}\right) - 1$$
$$\underline{= 2\pi - 1}$$

Exercise

Find the limit $\lim_{x\to \infty} \ln(1-\cos x)$



Solution

$$\lim_{x\to \infty} \ln(1-\cos x) = \ln(1-\infty)$$
$$= \ln(-\infty)$$
$$\underline{= \nexists}$$

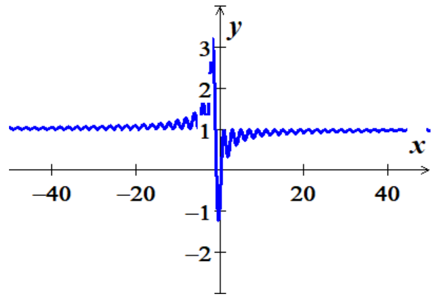
Exercise

Find the limit $\lim\limits_{x\to\infty} \frac{x-\cos \pi x}{x+1}$

Solution

$$\lim\limits_{x\to\infty} \frac{x-\cos \pi x}{x+1} = \lim\limits_{x\to\infty} \frac{x}{x}$$

$$= 1$$



Exercise

Find the limit $\lim\limits_{x\to -1} \frac{x-\cos \pi x}{x+1}$

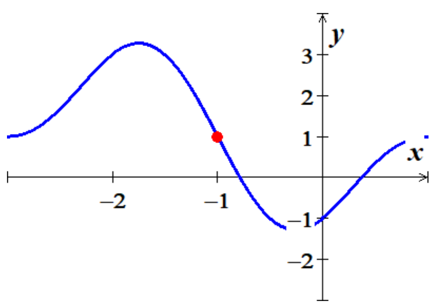
Solution

$$\lim\limits_{x\to -1} \frac{x-\cos \pi x}{x+1} = \frac{-1+1}{0} = \frac{0}{0}$$

$$= \lim\limits_{x\to -1} \frac{x-\cos(-\pi)}{x+1}$$

$$= \lim\limits_{x\to -1} \frac{x+1}{x+1}$$

$$= 1$$



Exercise

Find the limit $\lim\limits_{x\to\infty} \frac{60x^{-0.4}+50}{3x^{-0.4}+5}$

Solution

$$\lim\limits_{x\to\infty} \frac{60x^{-0.4}+50}{3x^{-0.4}+5} = \lim\limits_{x\to\infty} \frac{60\,x^{-0.4}}{3\,x^{-0.4}}$$

$$= \frac{60}{3}$$

$$= 20$$

Exercise

Find the limit $\lim\limits_{t \to 2^-} \dfrac{t^2}{t^2 - 4}$

Solution

$$t^2 - 4 < 0 \quad \Rightarrow \quad -2 \le t \le 2$$

$$\lim_{t \to 2^-} \frac{t^2}{t^2 - 4} = \frac{4}{0^-}$$

$$\underline{= -\infty}$$

Exercise

Find the limit $\lim\limits_{t \to 2^-} \dfrac{t}{t^2 - 4}$

Solution

$$t^2 - 4 < 0 \quad \Rightarrow \quad -2 \le t \le 2$$

$$\lim_{t \to 2^-} \frac{t}{t^2 - 4} = \frac{2}{0^-}$$

$$\underline{= -\infty}$$

Exercise

Find the limit $\lim\limits_{t \to 2^+} \dfrac{t^2}{t^2 - 4}$

Solution

$$t^2 - 4 > 0 \quad \Rightarrow \quad t \le -2, \quad t \ge 2$$

$$\lim_{t \to 2^+} \frac{t^2}{t^2 - 4} = \frac{4}{0^+}$$

$$\underline{= \infty}$$

Exercise

Find the limit $\lim\limits_{t \to -2^-} \dfrac{t^2}{t^2 - 4}$

Solution

$$t^2 - 4 > 0 \quad \Rightarrow \quad t \le -2, \quad t \ge 2$$

$$\lim_{t \to -2^-} \frac{t^2}{t^2-4} = \frac{4}{0^+}$$

$$\underline{= \infty}$$

Exercise

Find the limit $\lim\limits_{t \to -2^+} \frac{t^2}{t^2-4}$

Solution

$$t^2 - 4 < 0 \quad \Rightarrow \quad -2 \le t \le 2$$

$$\lim_{t \to -2^+} \frac{t^2}{t^2-4} = \frac{4}{0^-}$$

$$\underline{= -\infty}$$

Exercise

Find the limit $\lim\limits_{t \to 1} \frac{t^3-1}{t^2+t+1}$

Solution

$$\lim_{t \to 1} \frac{t^3-1}{t^2+t+1} = \frac{0}{3}$$

$$\underline{= 0}$$

Exercise

Find the limit $\lim\limits_{t \to 1^-} \frac{t^2+t+1}{t^3-1}$

Solution

$$\lim_{t \to 1^-} \frac{t^2+t+1}{t^3-1} = \frac{3}{0^-}$$

$$\underline{= -\infty}$$

Exercise

Find the limit $\lim\limits_{t \to 1^+} \frac{t^2+t+1}{t^3-1}$

Solution

$$\lim_{t\to 1^+} \frac{t^2+t+1}{t^3-1} = \frac{3}{0^+}$$

$$= \infty$$

Exercise

A 25-*foot* ladder is leaning against a house. If the base of the ladder is pulled away from the house at the rate of $2\ ft/\sec$, the top will move down the wall at a rate of

$$h(x) = \frac{2x}{\sqrt{625-x^2}} \quad ft/\sec$$

a) Find the rate when x is 7 *feet*.

b) Find the rate when x is 15 *feet*.

c) Find the limit of h as $x \to 25^-$.

Solution

a) $h(7) = \dfrac{2(7)}{\sqrt{625-7^2}}$

$$= \frac{14}{\sqrt{576}}$$

$$= \frac{14}{24}$$

$$= \frac{7}{12}\ ft$$

b) $h(15) = \dfrac{30}{\sqrt{625-225}}$

$$= \frac{30}{\sqrt{400}}$$

$$= \frac{30}{20}$$

$$= \frac{3}{2}\ ft$$

c) $\displaystyle\lim_{x\to 25^-} h(x) = \lim_{x\to 25^-} \frac{2x}{\sqrt{625-x^2}}$

$$= \lim_{x\to 25^-} \frac{50}{\sqrt{625-625^-}}$$

$$= \frac{50}{0}$$

$$= \infty$$

Which means that the ladder is lay flat on the ground.

Exercise

After an injection, the concentration of a drug in a muscle varies according to a function of time $f(t)$. Suppose that t is measured in hours and $f(t) = e^{-0.02t} - e^{-0.42t}$.

a) Find the $\lim\limits_{t\to 0} f(t)$

b) Find the $\lim\limits_{t\to \infty} f(t)$

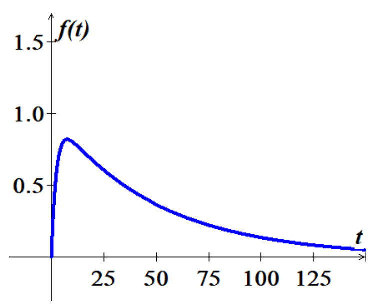
c) Interpret both limits in terms of the concentration of the drug.

Solution

a) $$\lim_{t\to 0} f(t) = \lim_{t\to 0} e^{-0.02t} - e^{-0.42t}$$
$$= e^0 - e^0$$
$$= 1 - 1$$
$$\underline{= 0}$$

b) $$\lim_{t\to \infty} f(t) = \lim_{t\to \infty} e^{-0.02t} - e^{-0.42t}$$
$$= e^{-\infty} - e^{-\infty}$$
$$\underline{= 0}$$

c) At time zero, the drug will inject to the body, it will take time to show the drug is working. As the time goes on the drug will have less effect to the body.



Exercise

Suppose an object with initial velocity $v_0 = 0$ $ft/\sec$ and a mass m slugs (constant) is accelerated by a constant force F (pounds) for t seconds.

According to Newton's laws of motion, the object's speed will ne $v_N = \dfrac{Ft}{m}$.

According to Einstein's theory of relativity, the object's speed will be $v_E = \dfrac{Fct}{\sqrt{mc^2 + F^2t^2}}$

Where c is the speed of light.

a) Compute $\lim\limits_{t\to\infty} v_N$

b) Compute $\lim\limits_{t\to\infty} v_E$

Solution

a) $\lim\limits_{t\to\infty} v_N = \lim\limits_{t\to\infty} \frac{F}{m}t$

$= \infty$

b) $\lim\limits_{t\to\infty} v_E = \lim\limits_{t\to\infty} \frac{Fc\,t}{\sqrt{mc^2 + F^2t^2}}$

$= \lim\limits_{t\to\infty} \frac{Fc\,t}{\sqrt{F^2t^2}}$

$= \lim\limits_{t\to\infty} \frac{Fc\,t}{F\,t}$

$= c$

Exercise

In relativity theory, the length of an object, appears to an observer to depend on the speed at which the object is traveling with respect to the observer. If the observer measures the object's length as L_0 at rest. Then at speed v the length will appear to be

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}}$$

This equation is the Lorentz contraction formula. Where c is the speed of light in a vacuum, $\approx 3\times 10^8\ m/\sec$.

a) What happens to L as v increases?

b) Find the $\lim\limits_{v\to c^-} L$

c) Find the $\lim\limits_{v\to c^+} L$

Solution

Given: $c \approx 3\times 10^8\ m/\sec$

$$L = L_0\sqrt{1 - \frac{v^2}{9\times 10^{16}}}$$

a) As v increases then the $\frac{v^2}{c^2}$ value will decrease. So, the value inside the square root will get smaller. Therefore, the value of L gets smaller to the zero value when $v = c$

b) $\lim\limits_{v\to c^-} L = \lim\limits_{v\to c^-} L_0\sqrt{1-\dfrac{v^2}{c^2}}$

$= L_0\sqrt{1-\dfrac{\left(c^-\right)^2}{c^2}}$

$= L_0\sqrt{1-1^-}$

$\boxed{= 0}$

c) $\lim\limits_{v\to c^+} L = \lim\limits_{v\to c^+} L_0\sqrt{1-\dfrac{v^2}{c^2}}$

$= L_0\sqrt{1-\dfrac{\left(c^+\right)^2}{c^2}}$

$= L_0\sqrt{1-1^+}$

$= L_0\sqrt{-0^+}$

$\boxed{= \nexists}$

Exercise

A quadratic equation is given by $ax^2+2x-1=0$, where a is a constant and r_i are the roots.

a) Find the limit for each root as $a\to -1^+$.

b) Find the limit for each root as $a\to 0$.

Solution

$ax^2+2x-1=0$

$r_{1,2} = \dfrac{-2\pm\sqrt{4+4a}}{2a}$

$= \dfrac{-2\pm 2\sqrt{1+a}}{2a}$

$\boxed{= \dfrac{-1\pm\sqrt{1+a}}{a}}$

$r_1(a) = \dfrac{-1-\sqrt{1+a}}{a}$ & $r_2(a) = \dfrac{-1+\sqrt{1+a}}{a}$

a) $\lim\limits_{a\to -1^+} r_1(a) = \lim\limits_{a\to -1^+} \dfrac{-1-\sqrt{1+a}}{a}$

$$= \frac{-1 - \sqrt{1 + \left(-1^+\right)}}{-1}$$

$$= \frac{-1 - \sqrt{0^+}}{-1}$$

$$\boxed{= 1}$$

$$\lim_{a \to -1^+} r_2(a) = \lim_{a \to -1^+} \frac{-1 + \sqrt{1+a}}{a}$$

$$= \frac{-1 + \sqrt{1 + \left(-1^+\right)}}{-1}$$

$$= \frac{-1 + \sqrt{0^+}}{-1}$$

$$\boxed{= 1}$$

b) $$\lim_{a \to 0} r_1(a) = \lim_{a \to 0} \frac{-1 - \sqrt{1+0}}{0}$$

$$= \frac{-1-1}{0}$$

$$\boxed{= \nexists}$$

$$\lim_{a \to 0} r_2(a) = \lim_{a \to 0} \frac{-1 + \sqrt{1+0}}{0}$$

$$= \frac{0}{0}$$

$$= \lim_{a \to 0} \frac{-1 + \sqrt{1+a}}{a} \frac{-1 - \sqrt{1+a}}{-1 - \sqrt{1+a}}$$

$$= \lim_{a \to 0} \frac{1 - 1 - a}{a\left(-1 - \sqrt{1+a}\right)}$$

$$= \lim_{a \to 0} \frac{-a}{a\left(-1 - \sqrt{1+a}\right)}$$

$$= \lim_{a \to 0} \frac{-1}{-1 - \sqrt{1+a}}$$

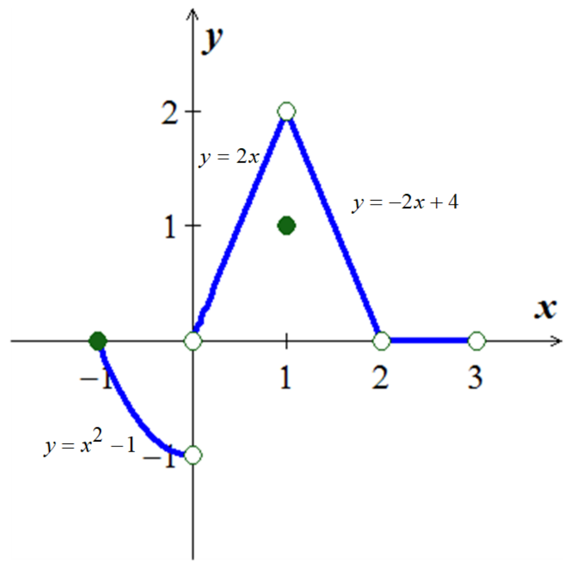
$$= \frac{-1}{-1-1}$$

$$\boxed{= \frac{1}{2}}$$

Solution — Section 1.5 – Continuity

Exercise

Given the graphed function $f(x)$



a) Does $f(-1)$ exist?

b) Does $\lim\limits_{x \to -1^+} f(x)$ exist?

c) Does $\lim\limits_{x \to -1^+} f(x) = f(-1)$?

d) Is f continuous at $x = -1$?

e) Does $f(1)$ exist?

f) Does $\lim\limits_{x \to 1} f(x)$ exist?

g) Does $\lim\limits_{x \to 1} f(x) = f(1)$?

h) Is f continuous at $x = 1$?

Solution

a) Yes $f(-1) = 0$

b) Yes, $\lim\limits_{x \to -1^+} f(x) = 0$

c) Yes

d) Yes

e) Yes, $f(1) = 1$

f) Yes, $\lim\limits_{x \to 1} f(x) = 2$

g) No

h) No

Exercise

At what points is the function $y = \dfrac{1}{x-2} - 3x$ continuous?

Solution

$x - 2 = 0 \Rightarrow x = 2$

The function is continuous everywhere except when $x = 2$

Exercise

At what points is the function $y = \dfrac{x+3}{x^2 - 3x - 10}$ continuous?

Solution

The function is continuous everywhere except when $x^2 - 3x - 10 = 0 \Rightarrow x = -2,\ 5$

Exercise

At what points is the function $f(x) = \dfrac{x^2}{x^2 + 4}$ continuous?

Solution

Since $x^2 + 4 \neq 0$

The function is continuous everywhere.

Exercise

At what points is the function $f(x) = \dfrac{x+1}{x^2 - 5x + 4}$ continuous?

Solution

$x^2 - 5x + 4 = 0 \quad \Rightarrow \quad x = 1,\ 4$

The function is continuous everywhere except when $x = 1,\ 4$

Exercise

At what points is the function $f(x) = \dfrac{2}{(x+3)(x-3)}$ continuous?

Solution

$(x+3)(x-3) = 0 \quad \Rightarrow \quad x = \pm 3$

The function is continuous everywhere except when $x = \pm 3$

Exercise

At what points is the function $f(x) = \dfrac{1}{x^2 - 4}$ continuous?

Solution

$x^2 - 4 = 0 \quad \Rightarrow \quad x = \pm 2$

The function is continuous everywhere except when $x = \pm 2$

Exercise

At what points is the function $f(x)=x^2-3x+4$ continuous?

Solution

The function is continuous everywhere.

Exercise

At what points is the function $f(x)=\cos(\pi x)$ continuous?

Solution

The function is continuous everywhere.

Exercise

At what points is the function $y=|x-1|+\sin x$ continuous?

Solution

The function is continuous everywhere

Exercise

At what points is the function $y=\dfrac{x+2}{\cos x}$ continuous?

Solution

$$\cos x=0 \quad \Rightarrow \quad x=\frac{\pi}{2}+n\pi, \quad n\in\mathbb{Z}$$

The function is continuous everywhere except when $x=\frac{\pi}{2}+n\pi$

Exercise

At what points is the function $y=\tan\dfrac{\pi x}{2}$ continuous?

Solution

$$\tan\frac{\pi x}{2}=\infty$$

$$\frac{\pi x}{2}=\pm\frac{\pi}{2}, \ \pm\frac{3\pi}{2}, \ \ldots, \ \frac{(2n+1)\pi}{2}$$

$$\frac{\pi x}{2}=\frac{(2n+1)\pi}{2}$$

$\underline{x = (2n+1)}\,\big|$

The function is continuous everywhere except when $x = 2n+1, \quad n \in \mathbb{Z}$

Exercise

At what points is the function $y = \dfrac{x \tan x}{x^2+1}$ continuous?

Solution

The function is continuous everywhere except when $x = (2n-1)\frac{\pi}{2}, \quad n \in \mathbb{Z}$

Exercise

At what points is the function $y = \dfrac{\sqrt{x^4+1}}{1+\sin^2 x}$ continuous?

Solution

The function is continuous everywhere

Exercise

At what points is the function $y = \sqrt{2x+3}$ continuous?

Solution

The function is continuous on the interval $2x+3 \ge 0 \quad \to \quad x \ge -\frac{3}{2} \Rightarrow \left[-\frac{3}{2}, \infty\right)$, and discontinuous when $x < -\frac{3}{2}$

Exercise

At what points is the function $y = \sqrt[4]{3x-1}$ continuous?

Solution

The function is continuous on the interval $3x-1 \ge 0 \to \left[\frac{1}{3}, \infty\right)$, and discontinuous when $x < \frac{1}{3}$

Exercise

At what points is the function $y = (2-x)^{1/5}$ continuous?

Solution

The function is continuous everywhere $\forall x$

Exercise

At what points is the function $f(x) = \sqrt{x^2 - 9}$ continuous?

Solution

$$x^2 - 9 \ge 0 \quad \Rightarrow \quad x \le -3 \quad x \ge 3$$

The function is continuous everywhere except when $x \in (-3,\ 3)$

Exercise

At what points is the function $f(x) = \sqrt{9 - x^2}$ continuous?

Solution

$$9 - x^2 \ge 0 \quad \Rightarrow \quad -3 \le x \le 3$$

The function is continuous everywhere except when $x < -3$ & $x > 3$

Exercise

At what points is the function $f(x) = \dfrac{1}{\sqrt{4 - x^2}}$ continuous?

Solution

$$4 - x^2 > 0 \quad \Rightarrow \quad -2 < x < 2$$

The function is continuous everywhere except when $x \le 2$ & $x \ge 2$

Exercise

At what points is the function $f(x) = \dfrac{1}{\sqrt{x^2 - 4}}$ continuous?

Solution

$$x^2 - 4 > 0 \quad \Rightarrow \quad x < -3 \quad x > 3$$

The function is continuous everywhere except when $-2 \le x \le 2$

Exercise

Find $\lim\limits_{x\to\pi} \sin\left(x-\sin x\right)$, then is the function continuous at the point being approached?

Solution

$$\begin{aligned}\lim_{x\to\pi} \sin\left(x-\sin x\right) &= \sin\left(\pi-\sin\pi\right)\\ &= \sin\left(\pi-0\right)\\ &= \sin\left(\pi\right)\\ &= 0\end{aligned}$$

The function is continuous at $x=\pi$

Exercise

Find $\lim\limits_{x\to 0} \tan\left(\frac{\pi}{4}\cos\left(\sin x^{1/3}\right)\right)$, then is the function continuous at the point being approached?

Solution

$$\begin{aligned}\lim_{x\to 0} \tan\left(\frac{\pi}{4}\cos\left(\sin x^{1/3}\right)\right) &= \tan\left(\frac{\pi}{4}\cos\left(\sin\left(0\right)^{1/3}\right)\right)\\ &= \tan\left(\frac{\pi}{4}\cos\left(0\right)\right)\\ &= \tan\left(\frac{\pi}{4}\right)\\ &= 1\end{aligned}$$

The function is continuous at $x=0$

Exercise

Find $\lim\limits_{t\to 0} \cos\left(\frac{\pi}{\sqrt{19-3\sec 2t}}\right)$, then is the function continuous at the point being approached?

Solution

$$\begin{aligned}\lim_{t\to 0} \cos\left(\frac{\pi}{\sqrt{19-3\sec 2t}}\right) &= \cos\left(\frac{\pi}{\sqrt{19-3\sec 2\left(0\right)}}\right)\\ &= \cos\left(\frac{\pi}{\sqrt{19-3}}\right)\\ &= \cos\left(\frac{\pi}{\sqrt{16}}\right)\\ &= \cos\left(\frac{\pi}{4}\right)\end{aligned}$$

$$= \frac{\sqrt{2}}{2}$$

$\therefore$ The function is continuous at $t = 0$

Exercise

Explain why the equation $\cos x = x$ has at least one solution.

Solution

$$\cos x - x = 0$$

$$\begin{cases} \text{if } \ x = -\frac{\pi}{2} & \rightarrow \cos\left(-\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right) > 0 \\ \text{if } \ x = \frac{\pi}{2} & \rightarrow \cos\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\right) < 0 \end{cases}$$

$$\Rightarrow \cos x - x = 0$$

for some x between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

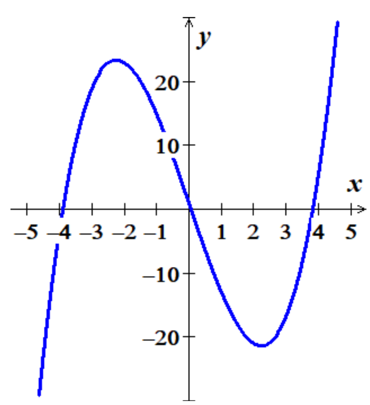
According to the Intermediate Value Theorem, and the function $\cos x = x$ is continuous and has at least one solution.

Exercise

Show that the equation has three solutions in the interval: $x^3 - 15x + 1 = 0; \quad [-4,\ 4]$

Solution

x	$f(x)$
–4	–3
–3	19
–2	23
–1	15
0	1
1	–13
2	–21
3	–17
4	5



By the Intermediate Value Theorem, $f(x)=0$ for some x in each of the intervals $-4<x<-1$, $-1<x<1$, and $1<x<4$.

Thus, $x^3-15x+1=0$ has three solutions in [–4, 4].

Since the polynomial of degree 3 can have at most 3 solutions, these are the solutions.

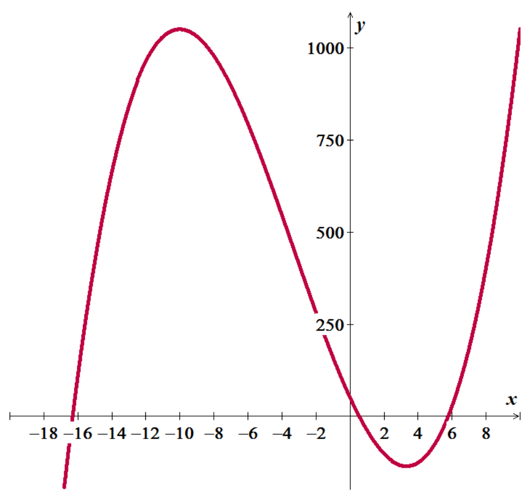
Exercise

Show that the equation has three solutions in the given interval

$$x^3+10x^2-100x+50=0; \quad (-20,\ 10)$$

Solution

x	y
–19	–1299
–18	–742
–17	–273
–16	114
–15	425
–14	666
–13	962
–12	1029
–10	1050
–9	1031
–8	978
–7	897
–6	794
–5	675
–4	546
–3	413
–2	282
–1	159
0	50
1	–39
2	–102
3	–133
4	–126
5	–75



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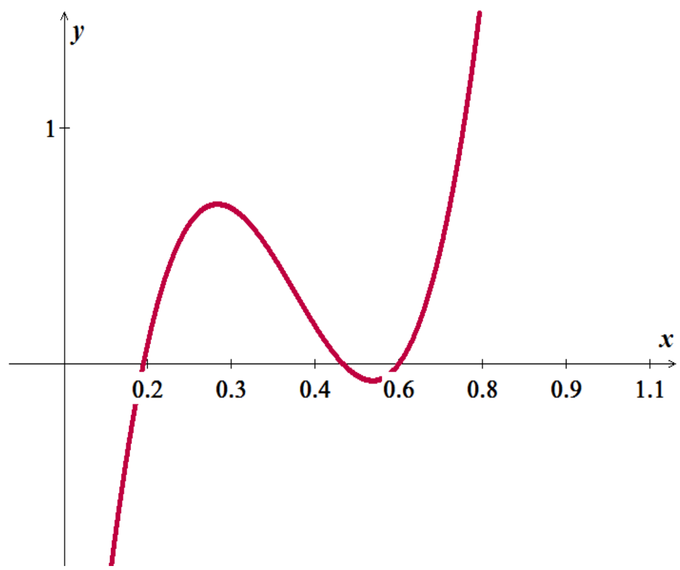
By the Intermediate Value Theorem, $f(x)=0$ for some x in each of the intervals $-17 < x < -16$, $0 < x < 1$, and $5 < x < 6$.

Exercise

Show that the equation has three solutions in the given interval $70x^3 - 87x^2 + 32x - 3 = 0; \quad (0, 1)$

Solution

x	y
.05	−1.6
.1	−0.6
.15	0.08
.2	.48
.25	.656
.3	.66
.35	.543
.4	.36
.45	.161
.5	0
.55	−.07
.6	0
.65	.266
.7	.78
.75	1.6
.8	2.76
.85	4.33
.9	6.36
.95	8.9



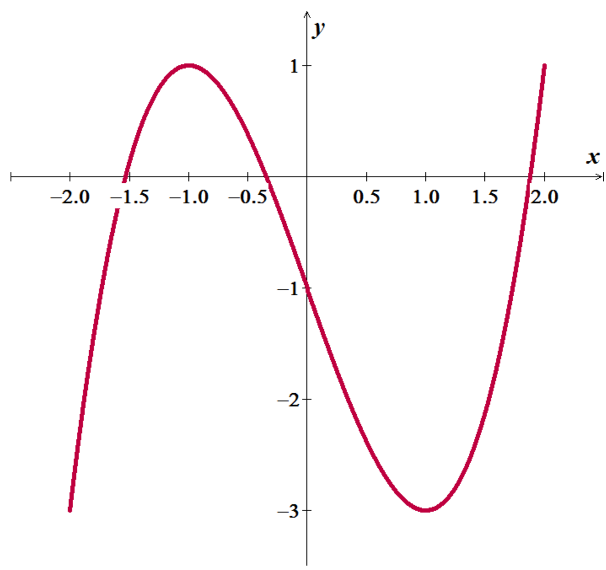
By the Intermediate Value Theorem, $f(x)=0$ for some x in each of the intervals $0.1 < x < 0.15$, $0.5 < x < 0.55$, and $0.55 < x < 0.6$.

Exercise

Show that the equation has three solutions in the given interval $x^3 - 3x - 1 = 0; \quad [-2,\ 2]$

Solution

x	y
–2	–3.0
–1.75	–1.109
–1.5	0.125
–1.25	0.797
–1.0	1
–0.75	0.828
–0.5	0.375
–0.25	–0.266
0	–1.0
0.5	–2.375
1.0	–3.0
1.5	–2.12
1.75	–0.89
2.	1.0



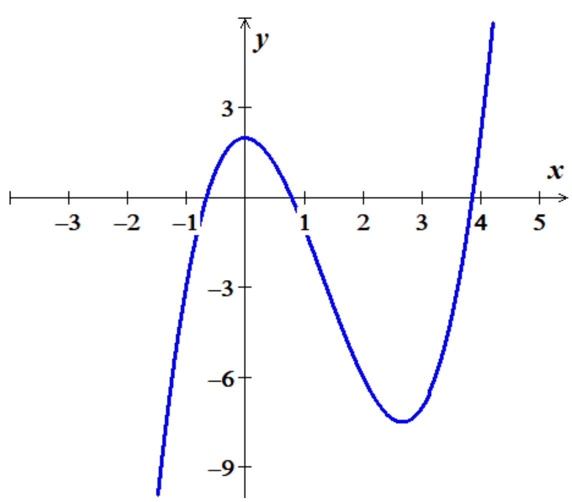
By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-1.75 < x < -1.5$, $-0.5 < x < -0.25$, and $1.75 < x < 2$.

Exercise

Show that the equation has three solutions in the given interval $x^3 - 4x^2 + 2 = 0; \quad (-3,\ 5)$

Solution

x	$f(x)$
–2	–22
–1	–3
0	2
1	–1
2	–6
3	–7
4	2



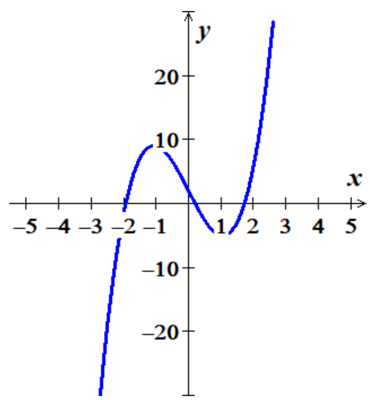
By the Intermediate Value Theorem, $f(x)=0$ for some x in each of the intervals $-1<x<0$, $0<x<1$, and $3<x<4$.

Exercise

Show that the equation has three solutions in the given interval $3x^3-10x+2=0;\ [-4,\ 4]$

Solution

x	$f(x)$
–4	–150
–3	–49
–2	–2
–1	9
0	2
1	–5
2	6
3	53
4	154



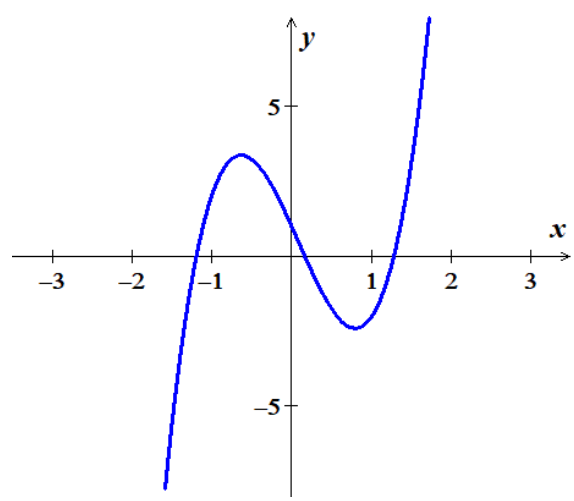
By the Intermediate Value Theorem, $f(x)=0$ for some x in each of the intervals $-2<x<-1$, $0<x<1$, and $1<x<2$.

Exercise

Show that the equation has three solutions in the given interval $4x^3-x^2-6x+1=0,\ (-3,\ 3)$

Solution

x	$f(x)$
–2	–2
–1	9
0	2
1	–5
2	6



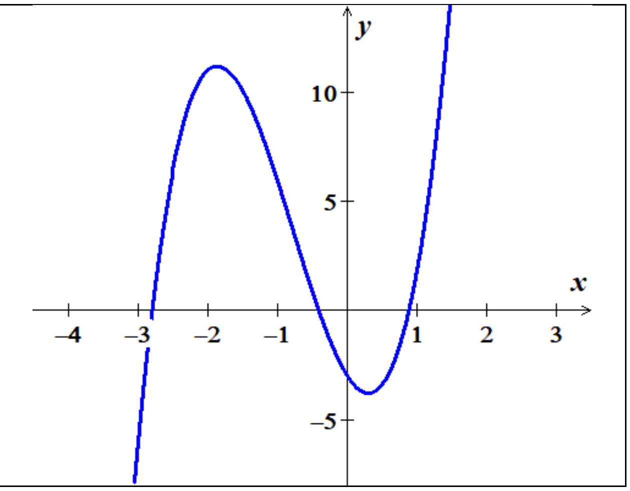
By the Intermediate Value Theorem, $f(x)=0$ for some x in each of the intervals $-2<x<-1$, $0<x<1$, and $1<x<2$.

Exercise

Show that the equation has three solutions in the given interval $3x^3+7x^2-5x-3=0, \quad (-4,\ 2)$

Solution

x	$f(x)$
-3	-6
-2	11
-1	6
0	-3
1	2
2	39



By the Intermediate Value Theorem, $f(x)=0$ for some x in each of the intervals $-3<x<-2$, $-1<x<0$, and $0<x<1$.

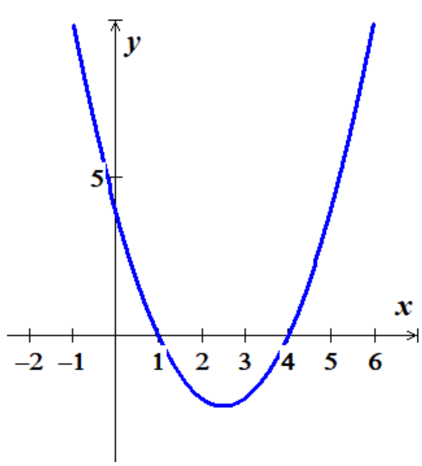
Exercise

Use the Intermediate Value Theorem to find the zeros in the given interval

$x^2-5x+4=0; \quad [-2,\ 5]$

Solution

x	$f(x)$
-2	18
-1	10
0	4
1	0
2	-2
3	-2
4	0
5	4



By the Intermediate Value Theorem, $f(x)=0$ when $x=1,\ 4$

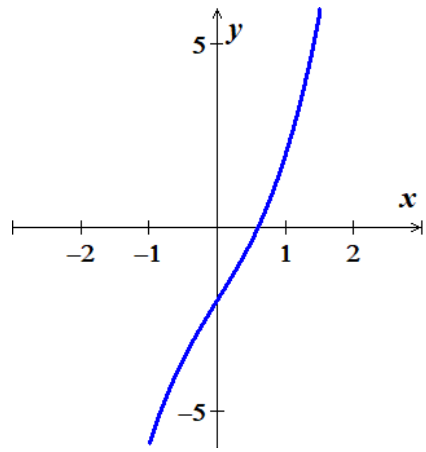
Exercise

Use the Intermediate Value Theorem to find the zeros in the given interval

$$x^3+3x-2=0;\quad [-2,\ 2]$$

Solution

x	$f(x)$
–2	–16
–1	–6
0	–2
1	2
2	12



By the Intermediate Value Theorem, $f(x)=0$ for one x in the interval $0<x<1$.

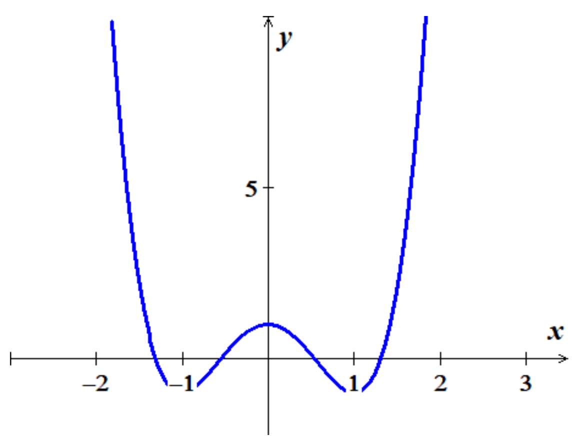
Exercise

Use the Intermediate Value Theorem to find the zeros in the given interval

$$2x^4-4x^2+1=0;\quad [-2,\ 2]$$

Solution

x	$f(x)$
–2	17
–1	–1
0	1
1	–1
2	17



By the Intermediate Value Theorem, $f(x)=0$ for ***four*** x in the interval $-2<x<-1$, $-1<x<0$, $0<x<1$, $1<x<2$.

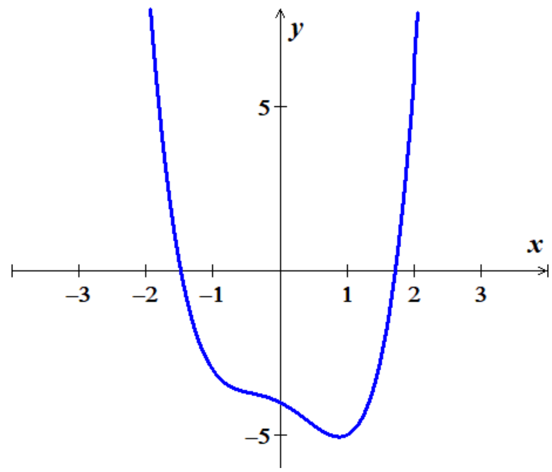
Exercise

Use the Intermediate Value Theorem to find the zeros in the given interval

$$x^4 - x^2 - x - 4 = 0, \quad (-3,\ 3)$$

Solution

x	$f(x)$
-2	10
-1	-3
0	-4
1	-5
2	6



By the Intermediate Value Theorem, $f(x) = 0$ for ***two*** x in the interval $-2 < x < -1$, $1 < x < 2$.

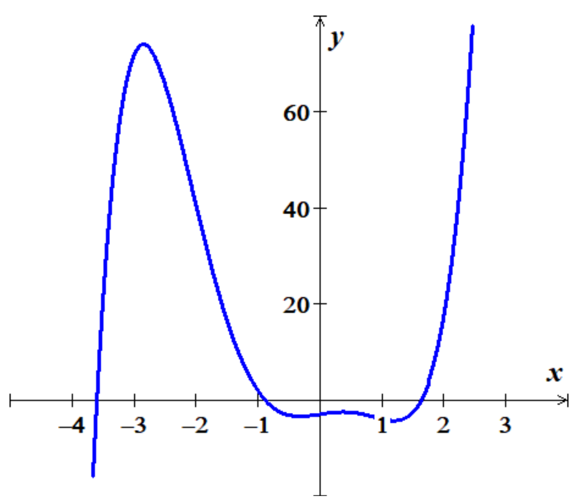
Exercise

Use the Intermediate Value Theorem to find the zeros in the given interval

$$x^5 + 2x^4 - 6x^3 + 2x - 3 = 0; \quad [-4,\ 2]$$

Solution

x	$f(x)$
-4	-139
-3	72
-2	41
-1	2
0	-3
1	-4
2	17



By the Intermediate Value Theorem, $f(x) = 0$ for ***three*** x in the interval $-4 < x < -3$, $-1 < x < 0$, $1 < x < 2$.

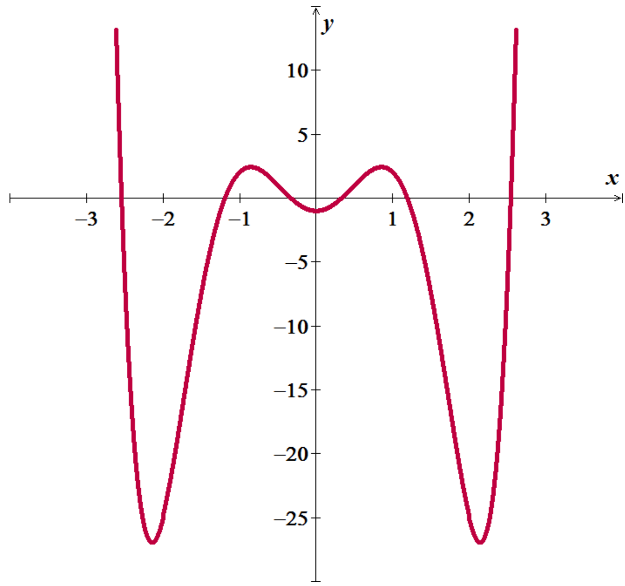
Exercise

Use the Intermediate Value Theorem to find the zeros in the given interval

$$x^6 - 8x^4 + 10x^2 - 1 = 0; \quad [-3, 3]$$

Solution

x	y
–3.0	170.0
–2.5	–6.86
–2.0	–25.0
–1.5	–7.61
–1.0	2.0
–0.5	1.02
0.0	–1.0
0.5	1.01
1.0	2.0
1.5	–7.6
2.0	–25.0
2.5	–6.86
3.0	170.0



By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-3.0 < x < -2.5$, $-1.5 < x < -1.0$, $-0.5 \le x \le 0$, $-0.0 \le x \le 0.5$, $1.0 \le x \le 1.5$ and $2.5 < x < 3.0$.

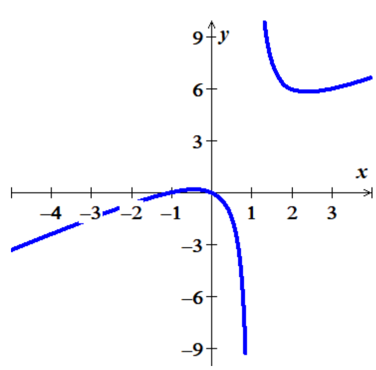
Exercise

Use the Intermediate Value Theorem to find the zeros in the given interval

$$\frac{x^2 + x}{x - 1} = 0; \quad [-4, 1)$$

Solution

x	$f(x)$
–4	$-\frac{12}{5}$
–3	$-\frac{3}{2}$
–2	$-\frac{2}{3}$



-1	0
0	0

By the Intermediate Value Theorem, $f(x)=0$ when $x=-1,\ 0$

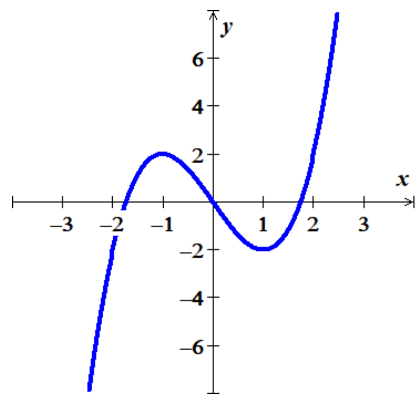
Exercise

Use the Intermediate Value Theorem to find the zeros in the given interval

$$f(x)=x^3-3x;\quad [-2,\ 2]$$

Solution

x	$f(x)$
-2	-2
-1	2
0	0
1	-2
2	2



By the Intermediate Value Theorem, $f(x)=0$ for ***two*** x in the interval $-2<x<-1$, $1<x<2$, and $x=0$.

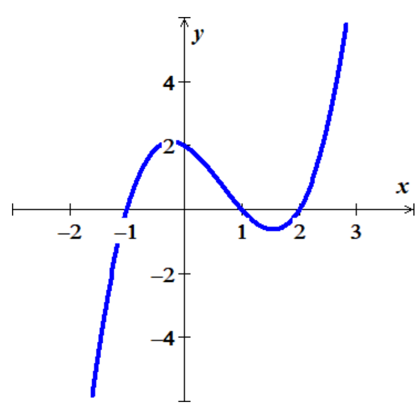
Exercise

Use the Intermediate Value Theorem to find the zeros in the given interval

$$f(x)=x^3-2x^2-x+2;\quad [-2,\ 3]$$

Solution

x	$f(x)$
-2	-12
-1	0
0	2
1	0
2	0
3	8



By the Intermediate Value Theorem, $f(x)=0$ when $x=-1,\ 1,\ 2.$

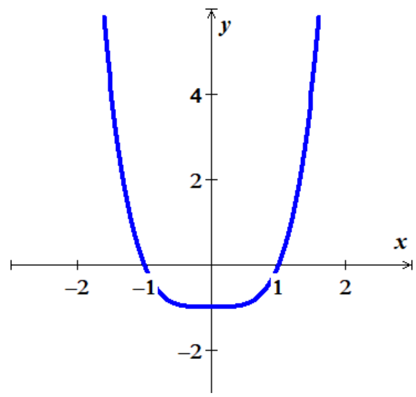
Exercise

Use the Intermediate Value Theorem to find the zeros in the given interval

$$f(x) = x^4 - 1; \quad [-2, 2]$$

Solution

x	$f(x)$
-2	15
-1	0
0	-1
1	0
2	15



By the Intermediate Value Theorem, $f(x) = 0$ when $x = -1,\ 1.$

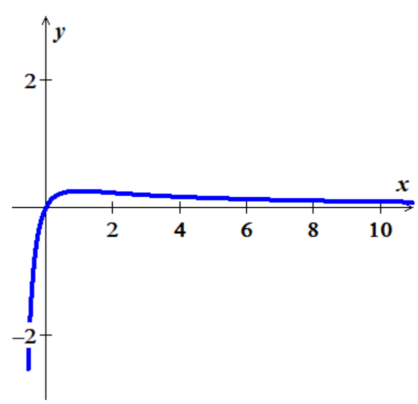
Exercise

Use the Intermediate Value Theorem to find the zeros in the given interval

$$f(x) = \frac{x}{(x+1)^2}; \quad [1, 8]$$

Solution

x	$f(x)$
1	$\frac{1}{4}$
2	$\frac{2}{9}$
3	$\frac{3}{16}$
4	$\frac{4}{25}$
5	$\frac{5}{36}$
6	$\frac{6}{49}$
7	$\frac{7}{49}$
8	$\frac{8}{64}$



By the Intermediate Value Theorem, $f(x) = 0$ has ***no*** zero in the given interval

Exercise

Use the Intermediate Value Theorem to find the zeros in the given interval

$$f(x)=\frac{x^3-1}{x-1}\ ;\quad [-2,\ 2]$$

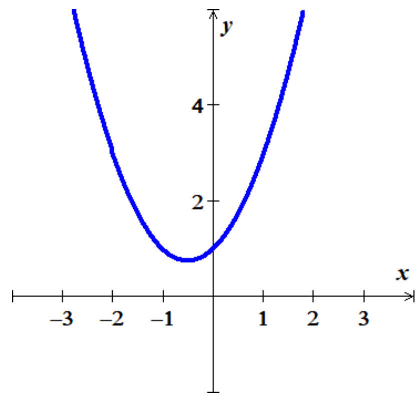
Solution

$$f(x)=\frac{x^3-1}{x-1}$$

$$=\frac{(x-1)\left(x^2+x+1\right)}{x-1}$$

$$\underline{=x^2+x+1}$$

x	$f(x)$
-2	3
-1	1
0	1
1	3
2	6



By the Intermediate Value Theorem, $f(x)=0$ has ***no*** zero in the given interval

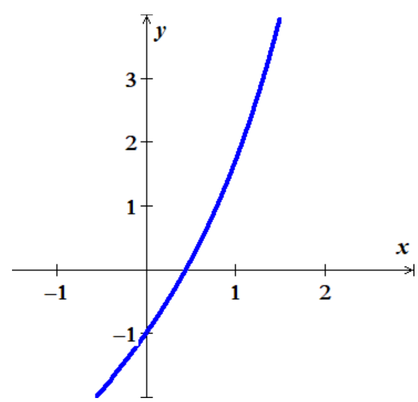
Exercise

Use the Intermediate Value Theorem to find the zeros in the given interval

$$f(x)=e^x+x-2\ ;\quad [0,\ 2]$$

Solution

x	$f(x)$
0	-1
1	$e-1$
2	e^2



By the Intermediate Value Theorem, $f(x)=0$ for ***one*** x in the interval $0<x<1$.

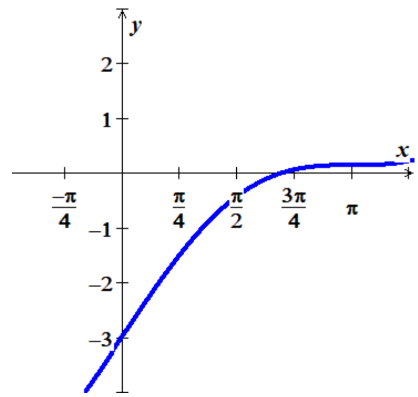
Exercise

Use the Intermediate Value Theorem to find the zeros in the given interval

$$f(x) = \sin x + x - 3\ ; \quad [0,\ \pi]$$

Solution

x	$f(x)$
0	-3
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} + \frac{\pi}{4} - 3\ < 0$
$\frac{\pi}{2}$	$1 + \frac{\pi}{2} - 3\ < 0$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2} + \frac{3\pi}{4} - 3\ > 0$
π	$\pi - 3\ > 0$



By the Intermediate Value Theorem, $f(x) = 0$ for ***one*** x in the interval $\frac{\pi}{2} < x < \frac{3\pi}{4}$,

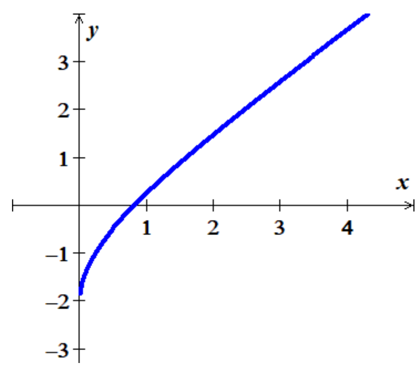
Exercise

Use the Intermediate Value Theorem to find the zeros in the given interval

$$f(x) = \sqrt{x^2 + 4x} - 2\ ; \quad [0,\ 3]$$

Solution

x	$f(x)$
0	-2
1	$\sqrt{5} - 2$
2	$2\sqrt{3} - 2$
3	$\sqrt{21} - 2$



By the Intermediate Value Theorem, $f(x) = 0$ for ***one*** x in the interval $0 < x < 1$.

Exercise

If functions $f(x)$ and $g(x)$ are continuous for $0 \le x \le 1$, could $\dfrac{f(x)}{g(x)}$ possibly be discontinuous at a point of [0, 1]? Give reason for your answer.

Solution

Yes, if we can get a value of $g(x)$ is between [0, 1], $x = \frac{1}{2} \quad \Rightarrow g(x) = 2x - 1$ and $f(x) = x$.

Then $\dfrac{f(x)}{g(x)} = \dfrac{x}{2x-1}$

$\dfrac{f(x)}{g(x)}$ is discontinuous at $x = \dfrac{1}{2}$

Exercise

Suppose that a function f is continuous on the closed interval [0, 1] and that $0 \le f(x) \le 1$ for every x in [0, 1]. Show that there must exist a number c in [0, 1] such that $f(c) = c$ (**c** is called a ***fixed point*** of f).

Solution

Let $f(x) = x \Rightarrow f(0) = 0 \quad or \quad f(1) = 1$. In these cases, $c = 0$ or $c = 1$.

Let $f(0) = a > 0 \quad and \quad f(1) = b < 1$ because $0 \le f(x) \le 1$.

Define $g(x) = f(x) - x \Rightarrow g$ is continuous on [0, 1].

$$\Rightarrow \begin{cases} g(0) = f(0) - 0 = a > 0 \\ g(1) = f(1) - 1 = b - 1 < 0 \end{cases}$$

By the Intermediate Value Theorem there is a number c in [0, 1] such that

$g(c) = 0$

$f(c) - c = 0$

$f(c) = c$

Exercise

Use the Intermediate Value Theorem to show that the equation $x^5 + 7x + 5 = 0$ has a solution in the interval $(-1,\ 0)$.

Solution

$f(-1) = -1 - 7 + 5 = -3 < 0$

$f(0) = 5 > 0$

By Intermediate value theorem, the function has a solution in $(-1,\ 0)$

Exercise

Determine whether the following functions are continuous at a. $f(x) = \dfrac{1}{x-5}$; $a = 5$

Solution

$f(5)\ \nexists$

The function is continuous everywhere except @ $x = 5$

Exercise

Determine whether the following functions are continuous at a. $h(x) = \sqrt{x^2 - 9}$; $a = 3$

Solution

$\lim\limits_{x \to 3^-} h(x)\ \nexists \quad \therefore\ h$ is discontinuous @ 3

Exercise

Determine whether the following functions are continuous at a.

$$g(x) = \begin{cases} \dfrac{x^2 - 16}{x - 4} & \text{if} \quad x \neq 4 \\ 9 & \text{if} \quad x = 4 \end{cases}; \quad a = 4$$

Solution

$$\begin{aligned} \lim_{x \to 4} g(x) &= \lim_{x \to 4} \frac{(x-4)(x+4)}{x-4} \\ &= \lim_{x \to 4} (x+4) = 8 \neq 9 = g(4) \end{aligned}$$

$\therefore\ g$ is discontinuous @ 4

Exercise

Determine whether the following functions are continuous at a.

$$f(x)=\begin{cases}\dfrac{x^2-4}{x-2} & if \quad x\neq 2\\ 5 & if \quad x=2\end{cases};\quad a=2$$

Solution

$$\lim_{x\to 2} f(x)=\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right)=\frac{0}{0}$$

$$=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}$$

$$=\lim_{x\to 2}(x+2)$$

$$=2+2$$

$$=4$$

$$\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right)\neq 5$$

$\therefore\ f(x)$ is discontinuous at $a=2$

Exercise

Determine whether the following functions are continuous at a.

$$f(x)=\begin{cases}\frac{1}{2}x+1 & if \quad x<4\\ -x+7 & if \quad x>4\end{cases};\quad a=4$$

Solution

$$\lim_{x\to 4} f(x)=\lim_{x\to 4}\left(\tfrac{1}{2}x+1\right)$$

$$=2+1$$

$$=3$$

$$\lim_{x\to 4} f(x)=\lim_{x\to 4}(-x+7)$$

$$=-4+7$$

$$=3$$

$$\lim_{x\to 4}\left(\tfrac{1}{2}x+1\right)=\lim_{x\to 4}(-x+7)$$

$\therefore\ f(x)$ is continuous at $a=4$

Exercise

Determine whether the following functions are continuous at a.

$$f(x)=\begin{cases}2x+5 & if \quad x\le 2\\ 4x+1 & if \quad x>2\end{cases}; \quad a=2$$

Solution

$$\lim_{x\to 2} f(x)=\lim_{x\to 2}(2x+5)$$
$$=4+5$$
$$=9$$

$$\lim_{x\to 2} f(x)=\lim_{x\to 2}(4x+1)$$
$$=8+1$$
$$=9$$

$$\lim_{x\to 2}(2x+5)=\lim_{x\to 2}(4x+1)$$

$\therefore\ f(x)$ is continuous at $a=2$

Exercise

Determine whether the following functions are continuous at a.

$$f(x)=\begin{cases}2x & if \quad x\le 0\\ 2x+1 & if \quad x>0\end{cases}; \quad a=0$$

Solution

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}(2x)$$
$$=0$$

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}(2x+1)$$
$$=0+1$$
$$=1$$

$$\lim_{x\to 0}(2x)\ne\lim_{x\to 0}(2x+1)$$

$\therefore\ f(x)$ is discontinuous at $a=0$

Exercise

Determine whether the following functions are continuous at *a*.

$$f(x)=\begin{cases} 3x-1 & if \quad x<1 \\ 4 & if \quad x=1\,; \\ 2x & if \quad x>1 \end{cases} \quad a=1$$

Solution

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} (3x-1)$$

$$= 3-1$$

$$\underline{= 2}\,|$$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} (4)$$

$$\underline{= 4}\,|$$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} (2x)$$

$$\underline{= 2}\,|$$

$$\lim_{x\to 1} (3x-1) = \lim_{x\to 1} (2x) \neq \lim_{x\to 1} (4)$$

$\therefore\ f(x)$ is discontinuous at $a=1$

Exercise

Determine whether the following functions are continuous at *a*.

$$f(x)=\begin{cases} x^2 & if \quad x<-1 \\ 2 & if \quad x=-1\,; \\ -3x+2 & if \quad x>-1 \end{cases} \quad a=-1$$

Solution

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} \left(x^2\right)$$

$$= (-1)^2$$

$$\underline{= 1}\,|$$

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} (2)$$

$$\underline{= 2}\,|$$

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} (-3x+2)$$

$$= -3(-1) + 2$$

$$= 5$$

$$\lim_{x \to -1} \left(x^2\right) \neq \lim_{x \to -1} (2) \neq \lim_{x \to -1} (-3x + 2)$$

$\therefore\ f(x)$ is discontinuous at $a = 1$

Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $\quad f(x) = \sqrt{x^2 - 5}$

Solution

$$\sqrt{x^2 - 5 \geq 0} \quad \Rightarrow \quad x \leq -5 \ \ \& \ \ x \geq 5$$

The function is continuous at -5 to the left and right of $x = 5$

Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $\quad f(x) = e^{\sqrt{x-2}}$

Solution

The function is continuous at and to the right of $x = 2$

Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $\quad f(x) = \dfrac{2x}{x^3 - 25x}$

Solution

The function is continuous everywhere except at $x = 0, \ \pm 5$

The function is continuous to the left of -5, then to the right of -5 to the left of 0, then to the right of 0 thru the left of 5 then to the tight of 5.

Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $\qquad f(x)=\cos e^{x}$

Solution

e^{x} is continuous everywhere.

$\therefore$ The function is continuous everywhere.

Exercise

Let $\qquad g(x)=\begin{cases} 5x-2 & if \quad x<1 \\ a & if \quad x=1 \\ ax^{2}+bx & if \quad x>1 \end{cases}$

Determine the values of the constants a and b for which $g(x)$ is continuous at $x=1$

Solution

$$\lim_{x\to 1^{-}} g(x)=g(1)$$

$$=5-2$$

$$\underline{=3=a}$$

$$\lim_{x\to 1^{-}} g(x)=g(1)$$

$$=a+b$$

$$=3+b=3$$

$$\to \quad \underline{b=0}$$

Exercise

Let $\qquad f(x)=\begin{cases} \dfrac{x^{3}-3x^{2}-4x+12}{x-3} & if \quad x\neq 3 \\ a & if \quad x=3 \end{cases}$

Determine the value of the constant a for which $f(x)$ is continuous at $x=3$

Solution

$$\lim_{x\to 3} f(x)=\lim_{x\to 3}\frac{x^{3}-3x^{2}-4x+12}{x-3}$$

$$=\frac{27-27-12+12}{3-3}=\frac{0}{0}$$

$$= \lim_{x\to 3} \frac{(x-3)\left(x^2-4\right)}{x-3}$$

$$= \lim_{x\to 3} \left(x^2-4\right)$$

$$= 9-4$$

$$= 5$$

$\therefore$ For $f(x)$ is continuous at $x=3$, then $a=5$

Exercise

Let $f(x)=\begin{cases} \dfrac{\sqrt{2x+5}-\sqrt{x+7}}{x-2} & \text{if} \quad x \ge -\frac{5}{2},\ x \ne 2 \\ a & \text{if} \quad x=2 \end{cases}$

Determine the value of the constant a for which $f(x)$ is continuous at $x=2$

Solution

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}$$

$$= \frac{\sqrt{9}-\sqrt{9}}{2-2} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2} \frac{\sqrt{2x+5}+\sqrt{x+7}}{\sqrt{2x+5}+\sqrt{x+7}}$$

$$= \lim_{x\to 2} \frac{2x+5-x-7}{(x-2)\left(\sqrt{2x+5}+\sqrt{x+7}\right)}$$

$$= \lim_{x\to 2} \frac{x-2}{(x-2)\left(\sqrt{2x+5}+\sqrt{x+7}\right)}$$

$$= \lim_{x\to 2} \frac{1}{\sqrt{2x+5}+\sqrt{x+7}}$$

$$= \frac{1}{3+3}$$

$$= \frac{1}{6}$$

$\therefore$ For $f(x)$ is continuous at $x=2$, then $a=\frac{1}{6}$

Exercise

A Factory sells candy by the pound, charging \$1.50 per pound for the quantities up to and including 20 pounds. Above 20 pounds, the Factory charges \$1.25 per pound for the entire quantity, plus a quantity surcharge a. If x represents the number of pounds, the price function is

$$p(x) = \begin{cases} 1.50x & if \quad x \le 20 \\ 1.25x + a & if \quad x > 20 \end{cases}$$

Determine the value of the constant a for which $p(x)$ is continuous at $x = 20$

Solution

$$\lim_{x\to 20} p(x) = \lim_{x\to 20} (1.5x)$$

$$= 1.5(20)$$

$$= 30$$

$$\lim_{x\to 20} (1.25x + a) = 30$$

$$1.25(20) + a = 30$$

$$25 + a = 30$$

$$a = 30 - 25$$

$$= 5$$

$\therefore$ For $p(x)$ is continuous at $x = 20$, then $a = 5$

Exercise

The amount of an antibiotic (in *mg*) in the blood t hours after an intravenous line is opened is given by

$$m(t) = 100\left(e^{-0.1t} - e^{-0.3t}\right)$$

a) Use the Intermediate Value Theorem to show that the amount of drug is 30 *mg* at some time in the interval $[0,\ 5]$ and again at some time in the interval $[5,\ 15]$

b) Estimate the times at which $m = 30\ mg$

c) Is the amount of drug in the blood ever 50 *mg*?

Solution

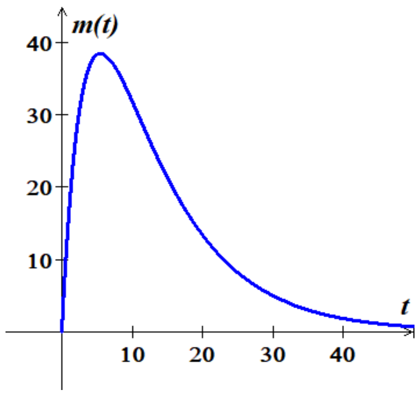
a) $m(0) = 100(1-1) = 0$

$m(5) \approx 38.34 > 30$

$m(15) \approx 21.2 < 30$

30 is an intermediate value between for both $[0,\ 5]$ and $[5,\ 15]$.

b) $m(t) = 100\left(e^{-0.1t} - e^{-0.3t}\right) = 30$



$$e^{-0.1t} - e^{-0.3t} = 0.3 \xrightarrow{\textit{software}} \begin{cases} \boxed{t_1 \approx 2.4} \\ \boxed{t_2 \approx 10.8} \end{cases}$$

c) No, peak is 38.5 (using the graph)

Solution — Section 1.6 – Precise Definition of Limits

Exercise

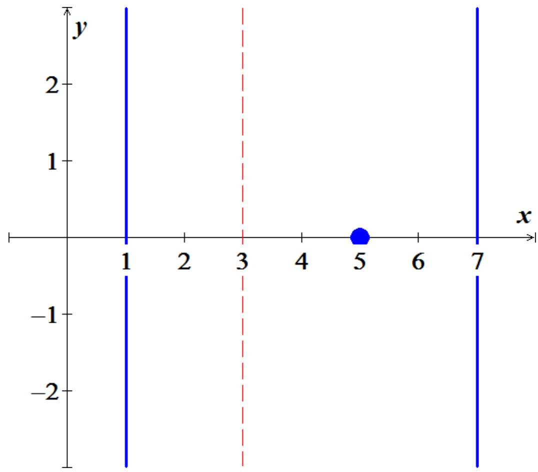
Sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \Rightarrow a < x < b$ for $a = 1,\quad b = 7,\quad x_0 = 5$

Solution

$$|x-5| < \delta$$
$$-\delta < x-5 < \delta$$
$$-\delta + 5 < x < \delta + 5$$

$$-\delta + 5 = 1 \Rightarrow \underline{\delta = 4}$$

$$\delta + 5 = 7 \Rightarrow \underline{\delta = 2}$$



Exercise

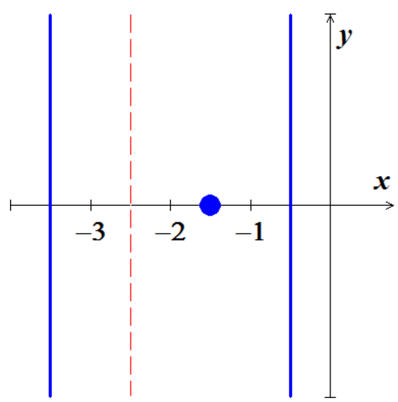
Sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \Rightarrow a < x < b$ for $a = -\frac{7}{2},\quad b = -\frac{1}{2},\quad x_0 = -\frac{3}{2}$

Solution

$$\left|x + \frac{3}{2}\right| < \delta$$
$$-\delta < x + \frac{3}{2} < \delta$$
$$-\delta - \frac{3}{2} < x < \delta - \frac{3}{2}$$

$$-\delta - \frac{3}{2} = -\frac{7}{2}$$
$$\delta = \frac{7}{2} - \frac{3}{2}$$
$$\underline{= 2}$$

$$\delta - \frac{3}{2} = -\frac{1}{2}$$
$$\delta = \frac{1}{2} - \frac{3}{2}$$
$$\underline{= -1}$$



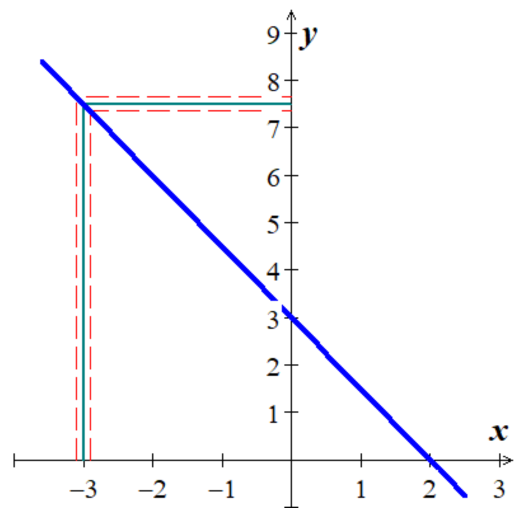
Exercise

Use the graph to find a $\delta > 0$ such that for all x $\quad 0 < |x - x_0| < \delta \ \Rightarrow\ |f(x) - L| < \varepsilon$

$$f(x) = -\frac{3}{2}x + 3 \quad x_0 = -3 \quad L = 7.5 \quad \varepsilon = 0.15$$

Solution

Given: $\quad a = -3.1, \quad b = -2.9, \quad x_0 = -3$



$|x + 3| < \delta$

$-\delta < x + 3 < \delta$

$-\delta - 3 < x < \delta - 3$

$-\delta - 3 = -3.1$

$\delta = 3.1 - 3$

$\underline{= 0.1}$

$\delta - 3 = -2.9$

$\delta = 3 - 2.9$

$\underline{= 0.1}$

Exercise

Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

$$f(x) = x + 1, \qquad L = 5, \quad x_0 = 4, \quad \varepsilon = 0.01$$

Solution

$|(x + 1) - 5| < .01$

$|x - 4| < .01$

$-.01 < x - 4 < .01$

$-.01 + 4 < x - 4 + 4 < .01 + 4$

$3.99 < x < 4.01$

$|x - 4| < \delta$

$-\delta < x - 4 < \delta$

$-\delta + 4 < x < \delta + 4$

$-\delta + 4 = 3.99$

$\delta = 4 - 3.99$

$\underline{= 0.01}$

$\delta + 4 = 4.01$

$\delta = 4.01 - 4$

$\underline{= 0.01}$

$\Rightarrow \underline{\delta = .01}$

Exercise

Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

$$f(x) = 2x - 1, \qquad L = 3, \quad x_0 = 2, \quad \varepsilon = 0.1$$

Solution

$|2x - 1 - 3| < .1$

$|2x - 4| < .1$

$-.1 < 2x - 4 < .1$

$-.1 + 4 < 2x - 4 + 4 < .1 + 4$

$3.9 < 2x < 4.1$

$\frac{3.9}{2} < x < \frac{4.1}{2}$

$1.95 < x < 2.05$

$|x - 2| < \delta$

$-\delta < x - 2 < \delta$

$-\delta + 2 < x < \delta + 2$

$-\delta + 2 = 1.95$

$\delta = 2 - 1.95$

$\underline{= 0.05}$

$\delta + 2 = 2.05$

$\delta = 2.05 - 2$

$\underline{= 0.05}$

$\Rightarrow \underline{\delta = .05}$

Exercise

Find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

$$f(x)=x+2, \qquad L=3, \quad x_0=1, \quad \varepsilon=0.001$$

Solution

$$|x+2-3|<.001$$

$$|x-1|<.001$$

$$-.001<x-1<.001$$

$$-.001+1<x-1+1<.001+1$$

$$0.999<x<1.001$$

$$|x-1|<\delta$$

$$-\delta<x-1<\delta$$

$$-\delta+1<x<\delta+1$$

$$-\delta+1=.999$$

$$\delta=1-.999$$

$$\underline{=0.001}$$

$$\delta+1=1.001$$

$$\delta=1.001-1$$

$$\underline{=0.001}$$

$$\Rightarrow \underline{\delta=.001}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds.

Then give a value for $\delta>0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

$$f(x)=3x+2, \qquad L=2, \quad x_0=0, \quad \varepsilon=0.1$$

Solution

$$|3x+2-2|<.1$$

$$|3x|<.1$$

$$-.1<3x<.1$$

$$-\frac{.1}{3}<x<\frac{.1}{3}$$

$$-\frac{1}{30} < x < \frac{1}{30}$$

$$|x-0| < \delta$$

$$-\delta < x < \delta$$

$$-\delta = -\frac{1}{30}$$

$$\boxed{\delta = \frac{1}{30}}$$

$$\Rightarrow \boxed{\delta = \frac{1}{30}}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x)-L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x)-L| < \varepsilon$ holds.

$$f(x) = 2x-4, \qquad L=-2, \quad x_0 = 1, \quad \varepsilon = 0.1$$

Solution

$$|2x-4+2| < .1$$

$$|2x-2| < .1$$

$$-.1 < 2x-2 < .1$$

$$1.9 < 2x < 2.1$$

$$\frac{19}{20} < x < \frac{21}{20}$$

$$|x-1| < \delta$$

$$-\delta < x-1 < \delta$$

$$1-\delta < x < 1+\delta$$

$$\frac{19}{20} < x < \frac{21}{20}$$

$$1-\delta = \frac{19}{20}$$

$$\boxed{\delta = \frac{1}{20}}$$

$$1+\delta = \frac{21}{20}$$

$$\boxed{\delta = \frac{1}{20}}$$

$$\Rightarrow \boxed{\delta = \frac{1}{20}}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

$$f(x)=2x+1, \qquad L=3, \quad x_0=1, \quad \varepsilon=0.01$$

Solution

$$|2x+1-3|<.01$$

$$|2x-2|<\frac{1}{100}$$

$$-\frac{1}{100}<2x-2<\frac{1}{100}$$

$$\frac{199}{100}<2x<\frac{201}{100}$$

$$\frac{199}{200}<x<\frac{201}{200}$$

$$|x-1|<\delta$$

$$-\delta<x-1<\delta$$

$$1-\delta<x<1+\delta$$

$$\frac{199}{200}<x<\frac{201}{200}$$

$$1-\delta=\frac{199}{200}$$

$$\delta=\frac{1}{200}$$

$$1+\delta=\frac{201}{200}$$

$$\delta=\frac{1}{200}$$

$$\Rightarrow \ \delta=\frac{1}{200}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

$$f(x)=3-4x, \qquad L=-1, \quad x_0=1, \quad \varepsilon=0.05$$

Solution

$$|3-4x+1|<.05$$

$$|4-4x|<\frac{5}{100}$$

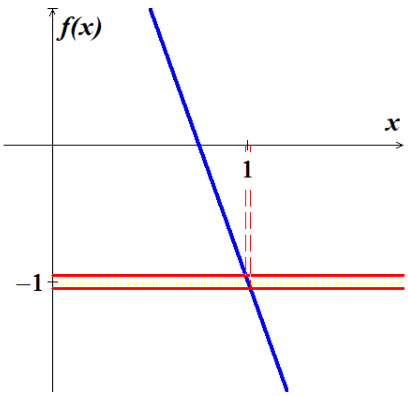
$$-\frac{1}{20}< 4-4x <\frac{1}{20}$$

$$-4-\frac{1}{20}< -4x <-4+\frac{1}{20}$$

$$-\frac{81}{20}< -4x <-\frac{79}{20}$$

$$\frac{79}{20}< 4x <\frac{81}{20}$$

$$\frac{79}{80}< x <\frac{81}{80}$$



$$|x-1|<\delta$$

$$-\delta<x-1<\delta$$

$$1-\delta<x<1+\delta$$

$$\frac{79}{80}< x <\frac{81}{80}$$

$$1-\delta=\frac{79}{80}$$

$$\delta=\frac{1}{80}$$

$$1+\delta=\frac{81}{80}$$

$$\delta=\frac{1}{80}$$

$$\Rightarrow\ \delta=\frac{1}{80}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds.

Then give a value for $\delta>0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds. $f(x)=\sqrt{x+1}$, $L=1$, $x_0=0$, $\varepsilon=0.1$

Solution

$$\left|\sqrt{x+1}-1\right|<0.1$$

$$-0.1<\sqrt{x+1}-1<0.1$$

$$-0.1+1<\sqrt{x+1}-1+1<0.1+1$$

$$.9<\sqrt{x+1}<1.1$$

$$(.9)^2 < \left(\sqrt{x+1}\right)^2 < (1.1)^2$$
$$.81 < x+1 < 1.21$$
$$.81 - 1 < x+1-1 < 1.21-1$$
$$-0.19 < x < 0.21$$

$$|x-0| < \delta$$
$$-\delta < x < \delta$$

$$-\delta = -0.19$$
$$\delta = 0.19$$
$$\delta = 0.21$$
$$\delta = 0.19$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x)-L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x-x_0| < \delta$ the inequality $|f(x)-L| < \varepsilon$ holds.

$$f(x) = \sqrt{x-7}, \qquad L = 4, \quad x_0 = 23, \quad \varepsilon = 1$$

Solution

$$\left|\sqrt{x-7} - 4\right| < 1$$
$$-1 < \sqrt{x-7} - 4 < 1$$
$$3 < \sqrt{x-7} < 5$$
$$(3)^2 < \left(\sqrt{x-7}\right)^2 < (5)^2$$
$$9 < x-7 < 25$$
$$9+7 < x-7+7 < 25+7$$
$$16 < x < 32$$

$$|x-23| < \delta$$
$$-\delta < x-23 < \delta$$
$$-\delta + 23 < x < \delta + 23$$

$$-\delta + 23 = 16$$
$$\delta = 23 - 16$$
$$= 7$$

$$\delta + 23 = 32$$

$$\delta = 32 - 23$$
$$= 9$$

$$\Rightarrow \quad \delta = 7$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

$$f(x) = \sqrt{x-4}, \qquad L = 2, \quad x_0 = 8, \quad \varepsilon = 0.1$$

Solution

$$\left|\sqrt{x-4} - 2\right| < .1$$

$$-\frac{1}{10} < \sqrt{x-4} - 2 < \frac{1}{10}$$

$$2 - \frac{1}{10} < \sqrt{x-4} < 2 + \frac{1}{10}$$

$$\frac{19}{10} < \sqrt{x-4} < \frac{21}{10}$$

$$\frac{361}{100} < x - 4 < \frac{441}{100}$$

$$4 + \frac{361}{100} < x < 4 + \frac{441}{100}$$

$$\frac{761}{100} < x < \frac{841}{100}$$

$$|x - 8| < \delta$$

$$-\delta < x - 8 < \delta$$

$$8 - \delta < x < 8 + \delta$$

$$\frac{761}{100} < x < \frac{841}{100}$$

$$8 - \delta = \frac{761}{100}$$

$$\delta = 8 - \frac{761}{100}$$

$$\delta = \frac{39}{100}$$

$$8 + \delta = \frac{841}{100}$$

$$\delta = \frac{841}{100} - 8$$

$$\boxed{\delta = \frac{41}{100}}$$

$$\Rightarrow \ \boxed{\delta = \frac{39}{100}}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

$$f(x) = \sqrt{x+3}, \qquad L = 2, \quad x_0 = 1, \quad \varepsilon = 0.05$$

Solution

$$\left|\sqrt{x+3} - 2\right| < 0.05$$

$$-\frac{5}{100} < \sqrt{x+4} - 2 < \frac{5}{100}$$

$$2 - \frac{1}{20} < \sqrt{x+3} < 2 + \frac{1}{20}$$

$$\frac{39}{20} < \sqrt{x+3} < \frac{41}{20}$$

$$\frac{1,521}{400} < x + 3 < \frac{1,681}{400}$$

$$\frac{1,521}{400} - 3 < x < \frac{1,681}{400} - 3$$

$$\frac{321}{400} < x < \frac{481}{400}$$

$$|x - 1| < \delta$$

$$-\delta < x - 1 < \delta$$

$$1 - \delta < x < 1 + \delta$$

$$\frac{321}{400} < x < \frac{481}{400}$$

$$1 - \delta = \frac{321}{400}$$

$$\delta = 1 - \frac{321}{400}$$

$$\boxed{\delta = \frac{79}{400}}$$

$$1 + \delta = \frac{481}{400}$$

$$\delta = \frac{481}{400} - 1$$

$$\boxed{\delta = \frac{81}{400}}$$

$$\Rightarrow \boxed{\delta = \frac{79}{400}}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

$$f(x) = x^2, \quad L = 3, \quad x_0 = \sqrt{3}, \quad \varepsilon = 0.1$$

Solution

$|x^2 - 3| < 0.1$

$$-0.1 < x^2 - 3 < 0.1$$
$$2.9 < x^2 < 3.1$$
$$\sqrt{2.9} < x < \sqrt{3.1}$$

$|x - \sqrt{3}| < \delta$

$$-\delta < x - \sqrt{3} < \delta$$
$$-\delta + \sqrt{3} < x < \delta + \sqrt{3}$$

$-\delta + \sqrt{3} = \sqrt{2.9}$

$$\delta = \sqrt{3} - \sqrt{2.9}$$
$$\boxed{\approx .029}$$

$\delta + \sqrt{3} = \sqrt{3.1}$

$$\delta = \sqrt{3.1} - \sqrt{3}$$
$$\boxed{\approx .029}$$

$$\Rightarrow \boxed{\delta = .029}$$

Exercise

Find an open interval about x_0 on which the inequality $\left|f(x)-L\right|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<\left|x-x_0\right|<\delta$ the inequality $\left|f(x)-L\right|<\varepsilon$ holds.

$$f(x)=x^2+1, \qquad L=5, \quad x_0=2, \quad \varepsilon=0.1$$

Solution

$$\left|x^2+1-5\right|<0.1$$

$$-\frac{1}{10}<x^2-4<\frac{1}{10}$$

$$4-\frac{1}{10}<x^2<4+\frac{1}{10}$$

$$\frac{39}{10}<x^2<\frac{41}{10}$$

$$\sqrt{\frac{39}{10}}<x<\sqrt{\frac{41}{10}}$$

$$\left|x-2\right|<\delta$$

$$-\delta<x-2<\delta$$

$$2-\delta<x<2+\delta$$

$$\sqrt{\frac{39}{10}}<x<\sqrt{\frac{41}{10}}$$

$$2-\delta=\sqrt{\frac{39}{10}}$$

$$\delta=2-\sqrt{\frac{39}{10}}$$

$$2+\delta=\sqrt{\frac{41}{10}}$$

$$\delta=2-\sqrt{\frac{41}{10}}$$

$$\delta=\left|2-\sqrt{\frac{41}{10}}\right|$$

$$\Rightarrow \quad \delta=\sqrt{\frac{41}{10}}-2$$

Exercise

Find an open interval about x_0 on which the inequality $\left|f(x)-L\right|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<\left|x-x_0\right|<\delta$ the inequality $\left|f(x)-L\right|<\varepsilon$ holds.

$$f(x)=x^2+1, \qquad L=5, \quad x_0=2, \quad \varepsilon=0.05$$

Solution

$$\left|x^2+1-5\right|<0.05$$

$$-\frac{5}{100}<x^2-4<\frac{5}{100}$$

$$4-\frac{5}{100}<x^2<4+\frac{5}{100}$$

$$\frac{395}{100}<x^2<\frac{405}{100}$$

$$\frac{\sqrt{395}}{10}<x<\frac{\sqrt{405}}{10}$$

$$\left|x-2\right|<\delta$$

$$-\delta<x-2<\delta$$

$$2-\delta<x<2+\delta$$

$$\frac{\sqrt{395}}{10}<x<\frac{\sqrt{405}}{10}$$

$$2-\delta=\frac{\sqrt{395}}{10}$$

$$\delta=2-\frac{\sqrt{395}}{10} \quad 0.012539$$

$$2+\delta=\frac{\sqrt{405}}{10}$$

$$\delta=2-\frac{\sqrt{405}}{10}$$

$$\delta=\left|2-\frac{\sqrt{405}}{10}\right|$$

$$\Rightarrow \quad \delta=\frac{\sqrt{405}}{10}-2$$

Exercise

Find an open interval about x_0 on which the inequality $\left|f(x)-L\right|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<\left|x-x_0\right|<\delta$ the inequality $\left|f(x)-L\right|<\varepsilon$ holds.

$$f(x)=x^3, \qquad L=1, \quad x_0=1, \quad \varepsilon=0.01$$

Solution

$$\left|x^3-1\right|<0.01$$

$$-\frac{1}{100}<x^3-1<\frac{1}{100}$$

$$1-\frac{1}{100}<x^3<1+\frac{1}{100}$$

$$\frac{99}{100}<x^3<\frac{101}{100}$$

$$\sqrt[3]{\frac{99}{100}}<x^3<\sqrt[3]{\frac{101}{100}}$$

$$|x-1|<\delta$$

$$-\delta<x-1<\delta$$

$$1-\delta<x<1+\delta$$

$$\sqrt[3]{\frac{99}{100}}<x^3<\sqrt[3]{\frac{101}{100}}$$

$$1-\delta=\sqrt[3]{\frac{99}{100}}$$

$$\boxed{\delta=1-\sqrt[3]{\frac{99}{100}}}$$

$$1+\delta=\sqrt[3]{\frac{101}{100}}$$

$$\boxed{\delta=1-\sqrt[3]{\frac{101}{100}}}$$

$$\delta=\left|1-\sqrt[3]{\frac{101}{100}}\right|$$

$$\Rightarrow\ \boxed{\delta=1-\sqrt[3]{\frac{101}{100}}}$$

Exercise

Find an open interval about x_0 on which the inequality $\left|f(x)-L\right|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<\left|x-x_0\right|<\delta$ the inequality $\left|f(x)-L\right|<\varepsilon$ holds.

$$f(x)=\frac{120}{x}, \qquad L=5, \quad x_0=24, \quad \varepsilon=1$$

Solution

$$\left|\frac{120}{x}-5\right|<0.1$$

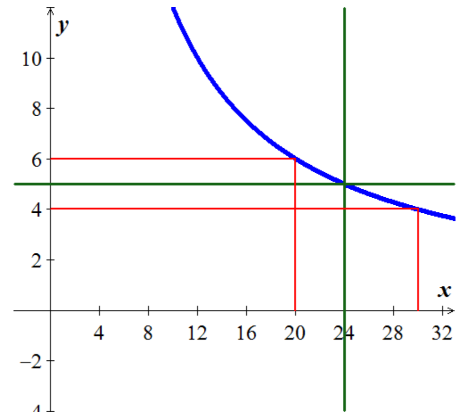
$$-1<\frac{120}{x}-5<1$$

$$4<\frac{120}{x}<6$$

$$\frac{1}{6}<\frac{x}{120}<\frac{1}{4}$$

$$\frac{1}{6}(120)<x<\frac{1}{4}(120)$$

$$20<x<30$$



$$|x-24|<\delta$$

$$-\delta<x-24<\delta$$

$$-\delta+24<x<\delta+24$$

$$-\delta+24=20$$

$$\delta=24-20$$

$$=4$$

$$\delta+24=30$$

$$\delta=30-24$$

$$=6$$

$$\Rightarrow \quad \delta=4$$

Exercise

Find an open interval about x_0 on which the inequality $\left|f(x)-L\right|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<\left|x-x_0\right|<\delta$ the inequality $\left|f(x)-L\right|<\varepsilon$ holds.

$$f(x)=\frac{x+2}{x^2}, \qquad L=3, \quad x_0=1, \quad \varepsilon=0.1$$

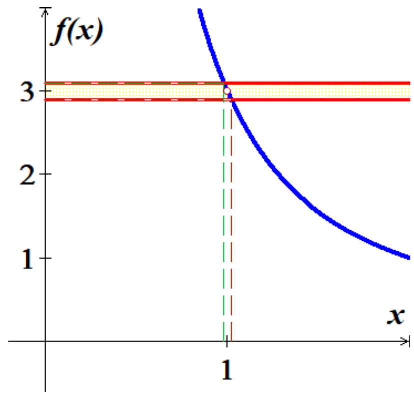
Solution

$\left|\frac{x+2}{x^2}-3\right|<0.1$

$$-\frac{1}{10}<\frac{-3x^2+x+2}{x^2}<\frac{1}{10}$$

$$-\frac{1}{10}<\frac{-(3x+2)(x-2)}{x^2}<\frac{1}{10}$$

$$-\frac{1}{10}<\frac{(3x+2)(x-1)}{x^2}<\frac{1}{10}$$



$|x-1|<\delta$

$$-\delta<x-1<\delta$$

Let assume: $C=\frac{3x+2}{x^2}$

$$-\frac{1}{10}<C(x-1)<\frac{1}{10}$$

$$-\frac{1}{10C}<x-1<\frac{1}{10C}$$

$$C=\left.\frac{3x+2}{x^2}\right|_{x=1}$$

$$=5$$

$$-\frac{1}{50}<x-1<\frac{1}{50}$$

$$-\delta<x-1<\delta$$

$$\Rightarrow \boxed{\delta=\frac{1}{50}}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

$$f(x)=\frac{x^2}{x+2}, \qquad L=1, \quad x_0=2, \quad \varepsilon=0.1$$

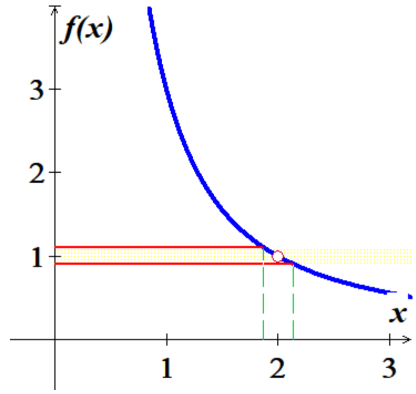
Solution

$\left|\frac{x+2}{x^2}-1\right|<0.1$

$$-\frac{1}{10}<\frac{-x^2+x+2}{x^2}<\frac{1}{10}$$

$$-\frac{1}{10} < \frac{-(x+1)(x-2)}{x^2} < \frac{1}{10}$$

$$-\frac{1}{10} < \frac{(x+1)(x-2)}{x^2} < \frac{1}{10}$$



$$|x-2| < \delta$$

$$-\delta < x-2 < \delta$$

Let assume: $C = \frac{x+1}{x^2}$

$$-\frac{1}{10} < C(x-2) < \frac{1}{10}$$

$$-\frac{1}{10C} < x-2 < \frac{1}{10C}$$

$$C = \left.\frac{x+1}{x^2}\right|_{x=2}$$

$$= \frac{3}{4}$$

$$-\frac{4}{30} < x-2 < \frac{4}{30}$$

$$-\frac{2}{15} < x-2 < \frac{2}{15}$$

$$-\delta < x-2 < \delta$$

$$\Rightarrow \quad \delta = \frac{2}{15}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x)-L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x-x_0| < \delta$ the inequality $|f(x)-L| < \varepsilon$ holds.

$$f(x) = \sin 2x, \qquad L = \frac{1}{2}, \quad x_0 = \frac{\pi}{12}, \quad \varepsilon = 0.1$$

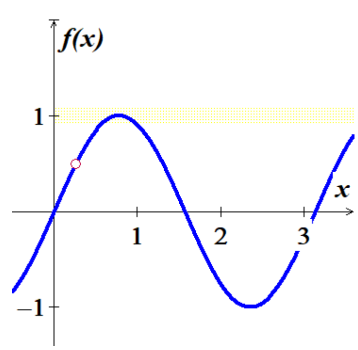
Solution

$$\left|\sin 2x - \frac{1}{2}\right| < 0.1$$

$$-\frac{1}{10} < \sin 2x - \frac{1}{2} < \frac{1}{10}$$

$$\frac{1}{2} - \frac{1}{10} < \sin 2x < \frac{1}{2} + \frac{1}{10}$$

$$\frac{2}{5} < \sin 2x < \frac{3}{5}$$



$$\left|x-\frac{\pi}{12}\right|<\delta$$

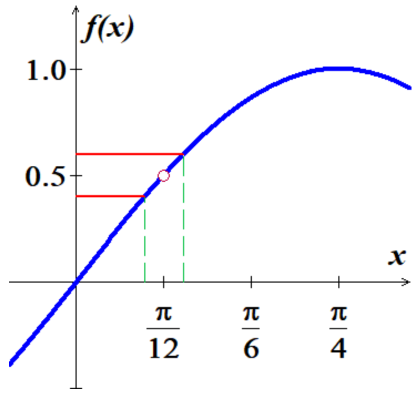
$$-\delta < x-\frac{\pi}{12} < \delta$$

$$\frac{\pi}{12}-\delta < x < \frac{\pi}{12}+\delta$$

$$\frac{\pi}{6}-2\delta < 2x < \frac{\pi}{6}+2\delta$$

$$\sin\left(\frac{\pi}{6}-2\delta\right) < \sin 2x < \sin\left(\frac{\pi}{6}+2\delta\right)$$

$$\frac{2}{5} < \sin 2x < \frac{3}{5}$$



$$\sin\left(\frac{\pi}{6}-2\delta\right)=\frac{2}{5}$$

$$\frac{\pi}{6}-2\delta=\sin^{-1}\left(\frac{2}{5}\right)$$

$$2\delta=\frac{\pi}{6}-\sin^{-1}\left(\frac{2}{5}\right)$$

$$\boxed{\delta=\frac{\pi}{12}-\frac{1}{2}\sin^{-1}\left(\frac{2}{5}\right)}$$

$$\boxed{\approx .056}$$

$$\sin\left(\frac{\pi}{6}+2\delta\right)=\frac{3}{5}$$

$$\frac{\pi}{6}+2\delta=\sin^{-1}\left(\frac{3}{5}\right)$$

$$2\delta=\sin^{-1}\left(\frac{3}{5}\right)-\frac{\pi}{6}$$

$$\boxed{\delta=\frac{1}{2}\sin^{-1}\left(\frac{3}{5}\right)-\frac{\pi}{12}}$$

$$\boxed{\approx .06}$$

$$\Rightarrow \boxed{\delta=\frac{\pi}{12}-\frac{1}{2}\sin^{-1}\left(\frac{2}{5}\right)}$$

Exercise

Find an open interval about x_0 on which the inequality $\left|f(x)-L\right|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<\left|x-x_0\right|<\delta$ the inequality $\left|f(x)-L\right|<\varepsilon$ holds.

$$f(x)=\cos x,\qquad L=\frac{\sqrt{3}}{2},\quad x_0=\frac{\pi}{6},\quad \varepsilon=0.05$$

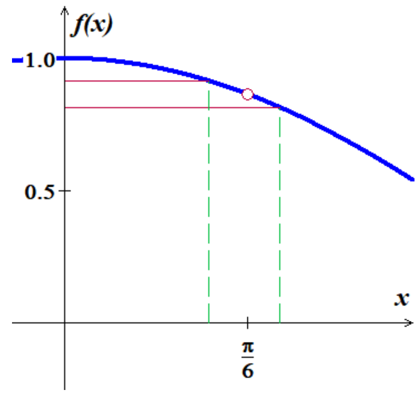
Solution

$$\left|\cos x - \frac{\sqrt{3}}{2}\right| < 0.05$$

$$-\frac{5}{100} < \cos x - \frac{\sqrt{3}}{2} < \frac{5}{100}$$

$$\frac{\sqrt{3}}{2} - \frac{1}{20} < \cos x < \frac{\sqrt{3}}{2} + \frac{1}{20}$$

$$\frac{10\sqrt{3}-1}{20} < \cos x < \frac{10\sqrt{3}+1}{20}$$



$$\left|x - \frac{\pi}{6}\right| < \delta$$

$$-\delta < x - \frac{\pi}{6} < \delta$$

$$\frac{\pi}{6} - \delta < x < \frac{\pi}{6} + \delta$$

$$\cos\left(\frac{\pi}{6} - \delta\right) < \cos x < \cos\left(\frac{\pi}{6} + \delta\right)$$

$$\frac{10\sqrt{3}-1}{20} < \cos x < \frac{10\sqrt{3}+1}{20}$$

$$\cos\left(\frac{\pi}{6} - \delta\right) = \frac{10\sqrt{3}-1}{20}$$

$$\frac{\pi}{6} - \delta = \cos^{-1}\left(\frac{10\sqrt{3}-1}{20}\right)$$

$$\delta = \frac{\pi}{6} - \cos^{-1}\left(\frac{10\sqrt{3}-1}{20}\right)$$

$$\approx -0.093$$

$$\cos\left(\frac{\pi}{6} + \delta\right) = \frac{10\sqrt{3}+1}{20}$$

$$\frac{\pi}{6} + \delta = \cos^{-1}\left(\frac{10\sqrt{3}+1}{20}\right)$$

$$\delta = \cos^{-1}\left(\frac{10\sqrt{3}+1}{20}\right) - \frac{\pi}{6}$$

$$\approx -0.11$$

$$\Rightarrow \quad \delta = \cos^{-1}\left(\frac{10\sqrt{3}-1}{20}\right) - \frac{\pi}{6}$$

Exercise

Find an open interval about x_0 on which the inequality $\left|f(x)-L\right|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<\left|x-x_0\right|<\delta$ the inequality $\left|f(x)-L\right|<\varepsilon$ holds.

$$f(x)=\cos 3x, \qquad L=\frac{1}{2}, \quad x_0=\frac{\pi}{9}, \quad \varepsilon=0.01$$

Solution

$$\left|\cos 3x-\frac{1}{2}\right|<0.01$$

$$-\frac{1}{100}<\cos 3x-\frac{1}{2}<\frac{1}{100}$$

$$\frac{1}{2}-\frac{1}{100}<\cos 3x<\frac{1}{2}+\frac{1}{100}$$

$$\frac{49}{100}<\cos 3x<\frac{51}{100}$$

$$\left|x-\frac{\pi}{9}\right|<\delta$$

$$-\delta<x-\frac{\pi}{9}<\delta$$

$$\frac{\pi}{9}-\delta<x<\frac{\pi}{9}+\delta$$

$$\frac{\pi}{3}-3\delta<3x<\frac{\pi}{3}+3\delta$$

$$\cos\left(\frac{\pi}{3}-3\delta\right)<\cos(3x)<\cos\left(\frac{\pi}{3}+3\delta\right)$$

$$\frac{49}{100}<\cos 3x<\frac{51}{100}$$

$$\cos\left(\frac{\pi}{3}-3\delta\right)=\frac{49}{100}$$

$$\frac{\pi}{3}-3\delta=\cos^{-1}\left(\frac{49}{100}\right)$$

$$3\delta=\frac{\pi}{3}-\cos^{-1}\left(\frac{49}{100}\right)$$

$$\delta=\frac{\pi}{9}-\frac{1}{3}\cos^{-1}\left(\frac{49}{100}\right)$$

$$\approx -0.004$$

$$\cos\left(\frac{\pi}{3}+3\delta\right)=\frac{51}{100}$$

$$\frac{\pi}{3}+3\delta=\cos^{-1}\left(\frac{51}{100}\right)$$

$$3\delta=\cos^{-1}\left(\frac{51}{100}\right)-\frac{\pi}{3}$$

$$\boxed{\delta = \frac{1}{3}\cos^{-1}\left(\frac{51}{100}\right) - \frac{\pi}{9}}$$

$$\boxed{\approx -0.003}$$

$$\Rightarrow \quad \boxed{\delta = \frac{\pi}{9} - \frac{1}{3}\cos^{-1}\left(\frac{51}{100}\right)}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

$$f(x) = e^x, \qquad L = 1, \quad x_0 = 0, \quad \varepsilon = 0.01$$

Solution

$$|e^x - 1| < 0.01$$

$$-\frac{1}{100} < e^x - 1 < \frac{1}{100}$$

$$1 - \frac{1}{100} < e^x < 1 + \frac{1}{100}$$

$$\frac{99}{100} < e^x < \frac{101}{100}$$

$$|x - 0| < \delta$$

$$-\delta < x < \delta$$

$$e^{-\delta} < e^x < e^{\delta}$$

$$\frac{99}{100} < e^x < \frac{101}{100}$$

$$e^{-\delta} = \frac{99}{100}$$

$$-\delta = \ln\left(\frac{99}{100}\right)$$

$$\delta = \ln\left(\frac{100}{99}\right)$$

$$\boxed{\delta = \ln\left(\frac{100}{99}\right)}$$

$$\boxed{\approx 0.01}$$

$$e^{\delta} = \frac{101}{100}$$

$$\boxed{\delta = \ln\left(\frac{101}{100}\right)}$$

$$\boxed{\approx 0.0099}$$

$\Rightarrow \quad \boxed{\delta = \ln\left(\frac{101}{100}\right)}$

Exercise

Prove that $\quad \lim\limits_{x\to 4}(9-x)=5$

Solution

$|9-x-5|<\varepsilon$

$-\varepsilon < 4-x < \varepsilon$

$-\varepsilon-4 < -x < \varepsilon-4 \qquad$ *divide by* $(-)$.

$\varepsilon+4 > x > 4-\varepsilon$

$4-\varepsilon < x < \varepsilon+4$

$|x-4|<\delta$

$-\delta < x-4 < \delta$

$-\delta+4 < x < \delta+4$

$-\delta+4 = 4-\varepsilon$

$-\delta = -\varepsilon$

$\boxed{\delta=\varepsilon}$

$\delta+4=\varepsilon+4$

$\boxed{\delta=\varepsilon}$

$\Rightarrow \quad \boxed{\delta=\varepsilon}$

Exercise

Prove that $\quad \lim\limits_{x\to 1}\frac{1}{x}=1$

Solution

$\left|\frac{1}{x}-1\right|<\varepsilon$

$-\varepsilon < \frac{1}{x}-1 < \varepsilon$

$-\varepsilon+1 < \frac{1}{x} < \varepsilon+1$

$\frac{1}{\varepsilon+1} > x > \frac{1}{-\varepsilon+1}$

$\frac{1}{1+\varepsilon} < x < \frac{1}{1-\varepsilon}$

$$|x-1|<\delta$$

$$-\delta < x-1 < \delta$$

$$1-\delta < x < 1+\delta$$

$$1-\delta = \frac{1}{1+\varepsilon}$$

$$\delta = 1+\frac{1}{1+\varepsilon}$$

$$= \boxed{\frac{2+\varepsilon}{1+\varepsilon}}$$

$$1+\delta = \frac{1}{1-\varepsilon}$$

$$\delta = \frac{1}{1-\varepsilon} - 1$$

$$= \boxed{\frac{\varepsilon}{1-\varepsilon}}$$

The smallest : $\boxed{\delta = \frac{\varepsilon}{1-\varepsilon}}$

Exercise

Prove that $\lim\limits_{x\to 5} \frac{x^2-25}{x-5} = 10$

Solution

$$\left|\frac{x^2-25}{x-5}-10\right| < \varepsilon$$

$$-\varepsilon < \frac{(x-5)(x+5)}{x-5} - 10 < \varepsilon$$

$$-\varepsilon + 10 < x+5 < \varepsilon + 10$$

$$5-\varepsilon < x < 5+\varepsilon$$

$$|x-5| < \delta$$

$$-\delta < x-5 < \delta$$

$$5-\delta < x < 5+\delta$$

$$5-\varepsilon < x < 5+\varepsilon$$

$$5-\delta = 5-\varepsilon$$

$$\boxed{\delta = \varepsilon}$$

$$5+\delta = \varepsilon + 5$$

$$\boxed{\delta = \varepsilon}$$

$Therefore: \ \underline{\delta = \varepsilon}$

Exercise

Prove that $\lim_{x\to 3} \frac{x^2-9}{x-3} = 6$

Solution

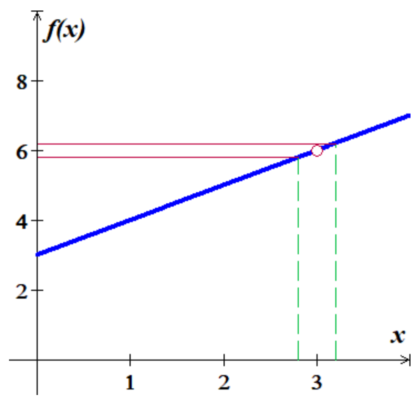
$\left| \frac{x^2-9}{x-3} - 6 \right| < \varepsilon$

$-\varepsilon < \frac{(x-3)(x+3)}{x-3} - 6 < \varepsilon$

$-\varepsilon < x+3-6 < \varepsilon$

$-\varepsilon < x-3 < \varepsilon$

$3-\varepsilon < x < 3+\varepsilon$



$|x-3| < \delta$

$-\delta < x-3 < \delta$

$3-\delta < x < 3+\delta$

$3-\delta = 3-\varepsilon$

$\underline{\delta = \varepsilon}$

$3+\delta = 3+\varepsilon$

$\underline{\delta = \varepsilon}$

$Therefore: \ \underline{\delta = \varepsilon}$

Exercise

Prove that $\lim_{x\to 2} \frac{2x^2-3x-2}{x-2} = 5$

Solution

$\left| \frac{2x^2-3x-2}{x-2} - 5 \right| < \varepsilon$

$-\varepsilon < \frac{(x-2)(2x+1)}{x-2} - 5 < \varepsilon$

$-\varepsilon < 2x+1-5 < \varepsilon$

$-\varepsilon < 2x-4 < \varepsilon$

$4-\varepsilon < 2x < 4+\varepsilon$

$2-\frac{1}{2}\varepsilon < x < 2+\frac{1}{2}\varepsilon$

$|x-2|<\delta$

$$-\delta < x-2 < \delta$$
$$2-\delta < x < 2+\delta$$
$$2-\frac{1}{2}\varepsilon < x < 2+\frac{1}{2}\varepsilon$$

$$2-\delta = 2-\frac{1}{2}\varepsilon$$
$$\boxed{\delta = \frac{1}{2}\varepsilon}$$

$$2+\delta = 2+\frac{1}{2}\varepsilon$$
$$\boxed{\delta = \frac{1}{2}\varepsilon}$$

Therefore: $\boxed{\delta = \frac{1}{2}\varepsilon}$

Exercise

Prove that $\lim\limits_{x\to 1} \dfrac{2x^2+2x-4}{x-1} = 6$

Solution

$$\left|\frac{2x^2+2x-4}{x-1}-6\right| < \varepsilon$$
$$-\varepsilon < \frac{(x-1)(2x+4)}{x-1} - 6 < \varepsilon$$
$$-\varepsilon < 2x+4-6 < \varepsilon$$
$$-\varepsilon < 2x-2 < \varepsilon$$
$$2-\varepsilon < 2x < 2+\varepsilon$$
$$1-\frac{1}{2}\varepsilon < x < 1+\frac{1}{2}\varepsilon$$

$|x-1|<\delta$

$$-\delta < x-1 < \delta$$
$$1-\delta < x < 1+\delta$$
$$1-\frac{1}{2}\varepsilon < x < 1+\frac{1}{2}\varepsilon$$

$$1-\delta = 1-\frac{1}{2}\varepsilon$$
$$\boxed{\delta = \frac{1}{2}\varepsilon}$$

$$1+\delta = 1+\frac{1}{2}\varepsilon$$

$$\delta = \frac{1}{2}\varepsilon$$

Therefore: $\delta = \frac{1}{2}\varepsilon$

Exercise

Prove that $\lim\limits_{x\to 2} (5x+8) = 18$

Solution

$$|5x+8-18| < \varepsilon$$

$$-\varepsilon < 5x-10 < \varepsilon$$

$$10-\varepsilon < 5x < 10+\varepsilon$$

$$2-\frac{1}{5}\varepsilon < x < 2+\frac{1}{5}\varepsilon$$

$$|x-2| < \delta$$

$$-\delta < x-2 < \delta$$

$$2-\delta < x < 2+\delta$$

$$2-\frac{1}{5}\varepsilon < x < 2+\frac{1}{5}\varepsilon$$

$$2-\delta = 2-\frac{1}{5}\varepsilon$$

$$\delta = \frac{1}{5}\varepsilon$$

$$2+\delta = 2+\frac{1}{5}\varepsilon$$

$$\delta = \frac{1}{5}\varepsilon$$

Therefore: $\delta = \frac{1}{5}\varepsilon$

Exercise

Prove that $\lim\limits_{x\to 1} (5x-2) = 3$

Solution

$$|(5x-2)-3| < \varepsilon$$

$$-\varepsilon < 5x - 5 < \varepsilon$$
$$5 - \varepsilon < 5x < \varepsilon + 5$$
$$1 - \frac{1}{5}\varepsilon < x < 1 + \frac{1}{5}\varepsilon$$

$$|x - 3| < \delta$$
$$-\delta < x - 3 < \delta$$
$$3 - \delta < x < 3 + \delta$$

$$3 - \delta = 1 - \frac{1}{5}\varepsilon$$
$$\delta = \frac{1}{5}\varepsilon + 2$$

$$3 + \delta = 1 + \frac{1}{5}\varepsilon$$
$$\delta = \frac{1}{5}\varepsilon - 2$$

The smallest: $\delta = \frac{1}{5}\varepsilon - 2$

Exercise

Prove that $\lim\limits_{x \to 0} x^4 = 0$

Solution

$$|x^4 - 0| < \varepsilon$$
$$-\varepsilon < x^4 < \varepsilon$$
$$-\sqrt[4]{\varepsilon} < x < \sqrt[4]{\varepsilon}$$
$$|x - 0| < \delta$$
$$-\delta < x < \delta$$
$$-\sqrt[4]{\varepsilon} < x < \sqrt[4]{\varepsilon}$$

$$-\delta = -\sqrt[4]{\varepsilon}$$
$$\delta = \sqrt[4]{\varepsilon}$$

$$\delta = \sqrt[4]{\varepsilon}$$

The smallest: $\delta = \sqrt[4]{\varepsilon}$

Exercise

Prove that $\lim\limits_{x\to 2} x^2 = 4$

Solution

$$\left|x^2-4\right|<\varepsilon$$

$$-\varepsilon < x^2 - 4 < \varepsilon$$

$$4-\varepsilon < x^2 < 4+\varepsilon$$

$$\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}$$

$$|x-2| < \delta$$

$$-\delta < x < \delta$$

$$2-\delta < x < 2+\delta$$

$$\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}$$

$$2-\delta = \sqrt{4-\varepsilon}$$

$$\delta = 2-\sqrt{4-\varepsilon} \quad 0.2679$$

$$2+\delta = \sqrt{4+\varepsilon}$$

$$\delta = 2-\sqrt{4+\varepsilon}$$

$$\textit{The smallest}: \ \delta = \sqrt{4+\varepsilon}-2$$

Exercise

Prove that $\lim\limits_{x\to 2} \dfrac{1}{(x-2)^4} = \infty$

Solution

Let $N>0$ and let $\delta = \dfrac{1}{\sqrt[4]{N}}$

Suppose that $0<|x-2|<\delta$

$$|x-2| < \delta = \frac{1}{\sqrt[4]{N}}$$

$$\frac{1}{|x-2|} > \sqrt[4]{N}$$

$$\frac{1}{(x-2)^4} > N$$

Exercise

Prove that $\lim\limits_{x\to 2}\left(x^2+2x\right)=8$

Solution

$$\left|x^2+2x-8\right|<\varepsilon$$

$$-\varepsilon<x^2+2x+1-9<\varepsilon$$

$$9-\varepsilon<(x+1)^2<9+\varepsilon$$

$$\sqrt{9-\varepsilon}<x+1<\sqrt{9+\varepsilon}$$

$$\sqrt{9-\varepsilon}-1<x<\sqrt{9+\varepsilon}-1$$

$$|x-2|<\delta$$

$$-\delta<x-2<\delta$$

$$2-\delta<x<2+\delta$$

$$\sqrt{9-\varepsilon}-1<x<\sqrt{9+\varepsilon}-1$$

$$2-\delta=\sqrt{9-\varepsilon}-1$$

$$\delta=3-\sqrt{9-\varepsilon}$$

$$2+\delta=\sqrt{9+\varepsilon}-1$$

$$\delta=3-\sqrt{9+\varepsilon}$$

The smallest: $\boxed{\delta=\sqrt{9+\varepsilon}-3}$

Exercise

Prove that $\lim\limits_{x\to 0} f(x)=0$ *if* $f(x)=\begin{cases}2x, & x<0\\ \frac{x}{2}, & x\ge 0\end{cases}$

Solution

For $x<0$: $|2x-0|<\varepsilon$

$$-\varepsilon<2x<0$$

$$-\frac{\varepsilon}{2}<x<0$$

For $x\ge 0$: $\left|\frac{x}{2}-0\right|<\varepsilon$

$$0\le\frac{x}{2}<\varepsilon$$

$$0\le x<2\varepsilon$$

$$|x-0|<\delta$$
$$-\delta<x<\delta$$
$$-\delta=-\frac{\varepsilon}{2}$$
$$\delta=\frac{\varepsilon}{2}$$
$$\delta=2\varepsilon$$

The smallest: $\boxed{\delta=\frac{\varepsilon}{2}}$

Exercise

Prove that $\lim\limits_{x\to 0^+} f(x)=-2 \quad if \quad f(x)=\begin{cases} 8x-3, & x<0 \\ 4x-2, & x\geq 0 \end{cases}$

Solution

For $x<0$: $|8x-3+2|<\varepsilon$

$$-\varepsilon<8x-1<0$$
$$1-\varepsilon<8x<1$$
$$\frac{1-\varepsilon}{8}<x<\frac{1}{8}$$

For $x\geq 0$: $|4x-2+2|<\varepsilon$

$$0\leq 4x<\varepsilon$$
$$0\leq x<\frac{\varepsilon}{4}$$

$$\left|x-0^+\right|<\delta$$
$$-\delta<x<\delta$$
$$-\delta=\frac{1-\varepsilon}{8}$$
$$\boxed{\delta=\frac{\varepsilon-1}{8}}$$
$$\boxed{\delta=\frac{\varepsilon}{4}}$$

$$\frac{\varepsilon-1}{8} \ ? \ \frac{\varepsilon}{4}$$
$$|\varepsilon-1| > 2\varepsilon \text{ since } \varepsilon<1$$

The smallest: $\boxed{\delta=\frac{\varepsilon}{4}}$

Exercise

Prove that $\lim\limits_{x\to 1^-} f(x)=3 \quad \text{if} \quad f(x)=\begin{cases} 5x-2, & x<1 \\ 7x-1, & x\geq 1 \end{cases}$

Solution

For $x<1$: $|5x-2-3|<\varepsilon$

$$-\varepsilon<5x-5<1$$

$$5-\varepsilon<5x<6$$

$$\frac{5-\varepsilon}{5}<x<\frac{6}{5}$$

For $x\geq 1$: $|7x-1-3|<\varepsilon$

$$1\leq 7x-4<\varepsilon$$

$$5\leq 7x<4+\varepsilon$$

$$\frac{7}{5}\leq x<\frac{4+\varepsilon}{7}$$

$$\left|x-1^-\right|<\delta$$

$$-\delta<x-1<\delta$$

$$1-\delta<x<1+\delta$$

$$1-\delta=\frac{5-\varepsilon}{5}$$

$$1-\delta=1-\frac{\varepsilon}{5}$$

$$\delta=\frac{\varepsilon}{5}$$

$$1+\delta=\frac{4+\varepsilon}{7}$$

$$\delta=\frac{4+\varepsilon}{7}-1$$

$$=\frac{\varepsilon-3}{7}$$

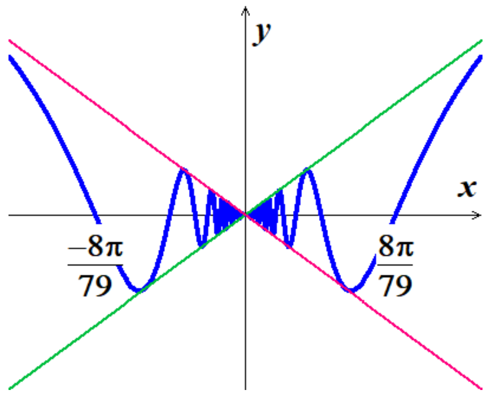
$$\frac{\varepsilon-3}{7} \ ? \ \frac{\varepsilon}{5}$$

$|5\varepsilon-15| > 7\varepsilon$ since $\varepsilon<1$

The smallest: $\delta=\frac{\varepsilon}{5}$

Exercise

Prove that $\lim\limits_{x\to 0} x\dfrac{1}{\sin x} = 0$



Solution

$$\begin{cases} -x \le x\sin\dfrac{1}{x} \le x, & \forall x > 0 \\ -x \ge x\sin\dfrac{1}{x} \ge x, & \forall x < 0 \end{cases}$$

$$\lim_{x\to 0}(-x) = \lim_{x\to 0}(x) = 0$$

Then by the sandwich theorem, $\lim\limits_{x\to 0} x\sin\left(\dfrac{1}{x}\right) = 0$

Solution Lecture 1 – Review

Exercise

Find the slope of the parabola $y = x^2 + 3$ at the point $P(3,\ 12)$. Write an equation for the tangent to the parabola at this point.

Solution

$$\frac{\Delta y}{\Delta x} = \frac{f(3+h) - f(3)}{h}$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1 + h) - f(x_1)}{h}$$

$$= \frac{(3+h)^2 + 3 - 12}{h}$$

$$= \frac{9 + 6h + h^2 + 3 - 12}{h}$$

$$= \frac{6h + h^2}{h}$$

$$= 6 + h$$

As h approaches 0. Then the tangent slope $h + 6 \ \rightarrow \ 6 = slope$

$$y = 6(x-3) + 12$$

$$y = m(x - x_1) + y_1$$

$$y = 6x - 18 + 12$$

$$y = 6x - 6$$

Exercise

Find the slope of the parabola $y = e^x$ @ $x = 1$. Write an equation for the tangent to the parabola at this point.

Solution

$$\frac{\Delta y}{\Delta x} = \frac{f(1+h) - f(1)}{h}$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1 + h) - f(x_1)}{h}$$

$$= \frac{e^{h+1} - e}{h}$$

$$= \frac{e^h e - e}{h}$$

$$= e\ \frac{e^h - 1}{h}$$

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

$\lim\limits_{h\to 0}\dfrac{e^h-1}{h}=1$, in lecture 3 will find the limit, the easy way, by using $L'H\hat{o}pital$ Rule.

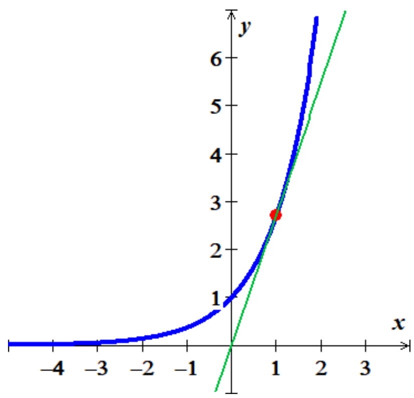
$\boxed{=e}$

Then the tangent $slope = e$

$y = e(x-1)+e$ $\qquad\qquad y = m(x-x_1)+y_1$

$y = ex - e + e$

$\boxed{y = ex}$



Exercise

Prove that $\lim\limits_{x\to 0}\dfrac{x}{\sin x}=1$

Solution

$$\lim_{x\to 0}\frac{x}{\sin x}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{1}{\frac{\sin x}{x}}$$

$$=\frac{1}{\lim\limits_{x\to 0}\frac{\sin x}{x}} \qquad \lim_{x\to 0}\frac{\sin x}{x}=1$$

$$=\frac{1}{1}$$

$$\boxed{=1}$$

Exercise

Prove that $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x} = 0$

Solution

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{1-\cos x}{x}\,\frac{1+\cos x}{1+\cos x}$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} \qquad \sin^2 x + \cos^2 x = 1$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin x}{x}\,\frac{\sin x}{1+\cos x} \qquad \lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$= \lim_{x\to 0} \frac{\sin x}{x}\,\lim_{x\to 0} \frac{\sin x}{1+\cos x}$$

$$= (1)\left(\frac{0}{2}\right)$$

$$\underline{= 0}$$

Exercise

Prove that $\lim\limits_{x\to 0} x^2\cos(2\pi x) = 0$

Solution

$$-1 \le \cos(2\pi x) \le 1$$

$$-x^2 \le x^2\cos(2\pi x) \le x^2$$

Since $\lim\limits_{x\to 0} x^2 = 0$

$$\therefore \lim_{x\to 0} x^2\cos(2\pi x) = 0 \qquad \checkmark$$

Exercise

Prove that $\lim\limits_{x\to 0} x^3\sin\left(\dfrac{\pi}{x}\right) = 0$

Solution

$$-1 \le \sin\left(\frac{\pi}{x}\right) \le 1$$

$$-x^3 \le x^3 \sin\left(\frac{\pi}{x}\right) \le x^3$$

Since, $\lim\limits_{x\to 0} x^3 = 0$

$$\therefore \quad \lim_{x\to 0} x^3 \sin\left(\frac{\pi}{x}\right) = 0 \qquad \checkmark$$

Exercise

Prove that $\lim\limits_{x\to\infty} \dfrac{1}{x \sin x} = 0$

Solution

$$-1 \le \sin x \le 1$$

$$-x \le x\sin x \le x$$

$$-\frac{1}{x} \le \frac{1}{x\sin x} \le \frac{1}{x}$$

Since, $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$

$$\therefore \quad \lim_{x\to\infty} \frac{1}{x \sin x} = 0 \qquad \checkmark$$

Exercise

Prove that $\lim\limits_{x\to\infty} \dfrac{\cos x}{x} = 0$

Solution

$$-1 \le \cos x \le 1$$

$$-\frac{1}{x} \le \frac{\cos x}{x} \le \frac{1}{x}$$

Since $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$

$$\therefore \quad \lim_{x\to\infty} \frac{\cos x}{x} = 0 \qquad \checkmark$$

Exercise

Prove that $\displaystyle \lim_{x\to\infty} \frac{\sin^2 x}{2^x} = 0$

Solution

$-1 \le \sin x \le 1$

$0 \le \sin^2 x \le 1$

$0 \le \dfrac{\sin^2 x}{2^x} \le \dfrac{1}{2^x}$

Since, $\displaystyle \lim_{x\to\infty} \frac{1}{2^x} = \frac{1}{\infty}$

$= 0$

$\therefore \displaystyle \lim_{x\to\infty} \frac{\sin^2 x}{2^x} = 0$ ✓

Exercise

Prove that $\displaystyle \lim_{x\to\infty} \frac{\cos \pi x}{5^x} = \frac{5}{6}$

Solution

$-1 \le \cos \pi x \le 1$

$-\dfrac{1}{5^x} \le \dfrac{\cos \pi x}{5^x} \le \dfrac{1}{5^x}$

Since, $\displaystyle \lim_{x\to\infty} \frac{1}{5^x} = \frac{1}{\infty}$

$= 0$

$\therefore \displaystyle \lim_{x\to\infty} \frac{\cos \pi x}{5^x} = \frac{5}{6}$ ✓

Exercise

Find the limit $\displaystyle \lim_{x\to 2} \left(t^2 - 3t + 5\right)$

Solution

$\displaystyle \lim_{x\to 2} \left(t^2 - 3t + 5\right) = t^2 - 3t + 5$

Exercise

Find the limit $\lim\limits_{x\to 2}\left(\pi^2\right)$

Solution

$$\lim_{x\to 2}\left(\pi^2\right)=\pi^2$$

Exercise

Find the limit $\lim\limits_{\theta\to \frac{\pi}{6}}\cos\theta$

Solution

$$\lim_{\theta\to \frac{\pi}{6}}\cos\theta=\cos\frac{\pi}{6}$$

$$=\frac{\sqrt{3}}{2}$$

Exercise

Find the limit $\lim\limits_{\theta\to \frac{\pi}{3}}\sin\theta$

Solution

$$\lim_{\theta\to \frac{\pi}{3}}\sin\theta=\sin\frac{\pi}{3}$$

$$=\frac{\sqrt{3}}{2}$$

Exercise

Find the limit $\lim\limits_{x\to 4}\left(x^2-3x+5\right)$

Solution

$$\lim_{x\to 4}\left(x^2-3x+5\right)=16-12+5$$

$$=9$$

Exercise

Find the limit $\lim\limits_{t\to 2}\,(2t-3)$

Solution

$$\lim_{t\to 2}\,(2t-3)=4-3$$

$$\underline{=1}$$

Exercise

Find the limit $\lim\limits_{x\to -2}\,(3x+2)$

Solution

$$\lim_{x\to -2}\,(3x+2)=-6+2$$

$$\underline{=-4}$$

Exercise

Find the limit $\lim\limits_{x\to 2}\dfrac{x^2-5x+6}{x-2}$

Solution

$$\lim_{x\to 2}\frac{x^2-5x+6}{x-2}=\frac{4-10+6}{0}=\frac{0}{0}$$

$$=\lim_{x\to 2}\frac{(x-2)(x-3)}{x-2}$$

$$=\lim_{x\to 2}\,(x-3)$$

$$=2-3$$

$$\underline{=-1}$$

Exercise

Find the limit $\lim\limits_{x\to 1}\dfrac{x^2-6x+5}{x+2}$

Solution

$$\lim_{x\to 1}\frac{x^2-6x+5}{x+2}=\frac{1-6+5}{3}$$

$$=\frac{0}{3}$$

$$\underline{=0}$$

Exercise

Find the limit $\displaystyle\lim_{x\to 1}\frac{x^2+6x+5}{x-1}$

Solution

$$\lim_{x\to 1}\frac{x^2+6x+5}{x-1}=\frac{1+6+5}{1-1}$$

$$=\frac{12}{0}$$

$$\underline{=\infty}$$

Exercise

Find the limit $\displaystyle\lim_{x\to 0^+}\frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}$

Solution

$$\lim_{x\to 0^+}\frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}=\frac{\sqrt{5}-\sqrt{5}}{0}=\frac{0}{0}$$

$$=\lim_{x\to 0^+}\frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}\frac{\sqrt{x^2+4x+5}+\sqrt{5}}{\sqrt{x^2+4x+5}+\sqrt{5}}$$

$$=\lim_{x\to 0^+}\frac{x^2+4x+5-5}{x\left(\sqrt{x^2+4x+5}+\sqrt{5}\right)}$$

$$=\lim_{x\to 0^+}\frac{x^2+4x}{x\left(\sqrt{x^2+4x+5}+\sqrt{5}\right)}$$

$$=\lim_{x\to 0^+}\frac{x+4}{\sqrt{x^2+4x+5}+\sqrt{5}}$$

$$=\frac{4}{\sqrt{5}+\sqrt{5}}$$

$$=\frac{4}{2\sqrt{5}}$$

$$\underline{=\frac{2}{\sqrt{5}}}$$

Exercise

Find the limit $\lim\limits_{x\to 0} \dfrac{\sin 5x}{3x}$

Solution

$$\boxed{\lim_{x\to 0} \frac{\sin 5x}{3x} = \frac{5}{3}} \qquad\qquad \lim_{x\to 0} \frac{\sin ax}{bx} = \frac{a}{b}$$

Exercise

Find the limit $\lim\limits_{\theta\to 0} \dfrac{\theta \cot 4\theta}{\sin^2\theta \cot^2 2\theta}$

Solution

$$\lim_{\theta\to 0} \frac{\theta \cot 4\theta}{\sin^2\theta \cot^2 2\theta} = \frac{0}{0}$$

$$= \lim_{\theta\to 0} \frac{\theta}{\sin\theta} \frac{1}{\sin\theta} \frac{\cos 4\theta}{\sin 4\theta} \frac{\sin^2 2\theta}{\cos^2 2\theta}$$

$$= \lim_{\theta\to 0} \frac{1}{\frac{\sin\theta}{\theta}} \quad \lim_{\theta\to 0} \frac{\cos 4\theta}{\cos^2 2\theta} \quad \lim_{\theta\to 0} \frac{1}{\sin\theta} \frac{\sin^2 2\theta}{\sin 4\theta}$$

$$= (1)(1) \quad \lim_{\theta\to 0} \frac{1}{\sin\theta} \frac{\sin^2 2\theta}{2\sin 2\theta \cos 2\theta}$$

$$= \frac{1}{2} \lim_{\theta\to 0} \frac{1}{\cos 2\theta} \quad \lim_{\theta\to 0} \frac{1}{\sin\theta} \frac{\sin^2 2\theta}{\sin 2\theta}$$

$$= \frac{1}{2}(1) \quad \lim_{\theta\to 0} \frac{\sin 2\theta}{\sin\theta}$$

$$= \frac{1}{2} \lim_{\theta\to 0} \frac{2\sin\theta \cos\theta}{\sin\theta}$$

$$= \lim_{\theta\to 0} \cos\theta$$

$$\boxed{= 1}$$

Exercise

Find the limit $\lim\limits_{x\to a} \dfrac{x^2 - a^2}{x^4 - a^4}$

Solution

$$\lim_{x\to a} \frac{x^2 - a^2}{x^4 - a^4} = \frac{a^2 - a^2}{a^4 - a^4} = \frac{0}{0}$$

$$= \lim_{x \to a} \frac{x^2 - a^2}{\left(x^2 - a^2\right)\left(x^2 + a^2\right)}$$

$$= \lim_{x \to a} \frac{1}{x^2 + a^2}$$

$$= \frac{1}{a^2 + a^2}$$

$$= \frac{1}{2a^2}$$

Exercise

Find the limit $\lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x}$

Solution

$$\lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x} = \frac{1 - 1}{1 - 1} = \frac{0}{0}$$

$$= \lim_{x \to 1} \frac{1 - \sqrt{x}}{\left(1 - \sqrt{x}\right)\left(1 + \sqrt{x}\right)}$$

$$= \lim_{x \to 1} \frac{1}{1 + \sqrt{x}}$$

$$= \frac{1}{1 + 1}$$

$$= \frac{1}{2}$$

$$\lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x} = \lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x} \frac{1 + \sqrt{x}}{1 + \sqrt{x}}$$

$$= \lim_{x \to 1} \frac{1 - x}{(1 - x)\left(1 + \sqrt{x}\right)}$$

$$= \lim_{x \to 1} \frac{1}{1 + \sqrt{x}}$$

$$= \frac{1}{2}$$

Exercise

Find the limit $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{x-2}$

Solution

$$\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \frac{4+2-6}{2-2} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{(x-2)(x+3)}{x-2}$$

$$= \lim_{x\to 2} (x+3)$$

$$\underline{=5}$$

Exercise

Find the limit $\lim\limits_{x\to 2} \dfrac{x^2-4x+4}{x^3+5x^2-14x}$

Solution

$$\lim_{x\to 2} \frac{x^2-4x+4}{x^3+5x^2-14x} = \frac{4-4+4}{8+20-28} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{(x-2)^2}{x(x-2)(x+7)}$$

$$= \lim_{x\to 2} \frac{x-2}{x(x+7)}$$

$$= \frac{0}{18}$$

$$\underline{=0}$$

Exercise

Find the limit $\lim\limits_{x\to 2} \dfrac{x^3-8}{x-2}$

Solution

$$\lim_{x\to 2} \frac{x^3-8}{x-2} = \frac{8-8}{2-2} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{(x-2)\left(x^2+2x+4\right)}{x-2}$$

$$= \lim_{x\to 2} \left(x^2+2x+4\right)$$

$$= 4 + 4 + 4$$

$$\boxed{= 12}$$

Exercise

Find the limit $\lim\limits_{x\to 0} \dfrac{\frac{1}{2+x} - \frac{1}{2}}{x}$

Solution

$$\lim_{x\to 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = \frac{\frac{1}{2} - \frac{1}{2}}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{1}{x}\left(\frac{2-2-x}{2(2+x)}\right)$$

$$= \lim_{x\to 0} \frac{1}{x}\left(\frac{-x}{2(2+x)}\right)$$

$$= -\frac{1}{2} \lim_{x\to 0} \frac{1}{2+x}$$

$$= -\frac{1}{2}\left(\frac{1}{2}\right)$$

$$\boxed{= -\frac{1}{4}}$$

Exercise

Find the limit $\lim\limits_{x\to -1} \dfrac{\sqrt{x^2+8} - 3}{x+1}$

Solution

$$\lim_{x\to -1} \frac{\sqrt{x^2+8} - 3}{x+1} = \frac{3-3}{-1+1} = \frac{0}{0}$$

$$= \lim_{x\to -1} \frac{\sqrt{x^2+8} - 3}{x+1} \frac{\sqrt{x^2+8} + 3}{\sqrt{x^2+8} + 3}$$

$$= \lim_{x\to -1} \frac{x^2+8-9}{(x+1)\left(\sqrt{x^2+8} + 3\right)}$$

$$= \lim_{x\to -1} \frac{x^2-1}{(x+1)\left(\sqrt{x^2+8} + 3\right)}$$

$$= \lim_{x\to -1} \frac{(x-1)(x+1)}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x\to -1} \frac{x+1}{\sqrt{x^2+8}+3}$$

$$= \frac{-1+1}{\sqrt{9}+3}$$

$$= \frac{0}{6}$$

$$\boxed{= 0}$$

Exercise

Find the limit $\displaystyle \lim_{x\to 0} \frac{\tan(\pi x)}{\tan(3x)}$

Solution

$$\lim_{x\to 0} \frac{\tan(\pi x)}{\tan(3x)} = \frac{\tan 0}{\tan 0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin(\pi x)}{\cos(\pi x)} \frac{\cos(3x)}{\sin(3x)}$$

$$= \lim_{x\to 0} \frac{\cos(3x)}{\cos(\pi x)} \lim_{x\to 0} \frac{\pi x}{3x} \frac{\sin(\pi x)}{\pi x} \frac{3x}{\sin(3x)}$$

$$= \frac{\pi}{3} \frac{\cos(0)}{\cos(0)} \lim_{\pi x\to 0} \frac{\sin(\pi x)}{\pi x} \lim_{3x\to 0} \frac{1}{\frac{\sin(3x)}{3x}} \qquad \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

$$\boxed{= \frac{\pi}{3}}$$

Exercise

Find the limit $\displaystyle \lim_{x\to 0} \frac{\cos 2x - 1}{\sin x}$

Solution

$$\lim_{x\to 0} \frac{\cos 2x - 1}{\sin x} = \frac{1-1}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{1 - 2\sin^2 x - 1}{\sin x}$$

$$= \lim_{x\to 0} \frac{-2\sin^2 x}{\sin x}$$

$$= -2 \lim_{x\to 0} \sin x$$

$$\underline{= 0}$$

Exercise

Find the limit $\displaystyle\lim_{x\to 0^+}\left(x - \frac{1}{x^3}\right)$

Solution

$$\lim_{x\to 0^+}\left(x - \frac{1}{x^3}\right) = 0 - \frac{1}{0^+}$$

$$\underline{= -\infty}$$

Exercise

Find the limit $\displaystyle\lim_{x\to -2^+}\frac{2x^2+x+1}{x+2}$

Solution

$$\lim_{x\to -2^+}\frac{2x^2+x+1}{x+2} = \frac{8+2+1}{-2^+ + 2}$$

$$= \frac{11}{0^+}$$

$$\underline{= \infty}$$

Exercise

Find the limit $\displaystyle\lim_{x\to -2^-}\frac{2x^2+x+1}{x+2}$

Solution

$$\lim_{x\to -2^-}\frac{2x^2+x+1}{x+2} = \frac{8+2+1}{-2^- + 2}$$

$$= \frac{11}{0^-}$$

$$\underline{= -\infty}$$

Exercise

Find the limit $\lim\limits_{x\to 1^+} \dfrac{x^2-3x}{x-1}$

Solution

$$\lim_{x\to 1^+} \frac{x^2-3x}{x-1} = \frac{1-3}{1^+-1}$$

$$= \frac{-3}{0^+}$$

$$\boxed{= -\infty}$$

Exercise

Find the limit $\lim\limits_{x\to 0} \dfrac{\sqrt{4+x}-2}{x}$

Solution

$$\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x} = \frac{2-2}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sqrt{4+x}-2}{x} \frac{\sqrt{4+x}+2}{\sqrt{4+x}+2} \qquad (a-b)(a+b) = a^2-b^2$$

$$= \lim_{x\to 0} \frac{4+x-4}{x\left(\sqrt{4+x}+2\right)}$$

$$= \lim_{x\to 0} \frac{x}{x\left(\sqrt{4+x}+2\right)}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{4+x}+2}$$

$$= \frac{1}{\sqrt{4}+2}$$

$$\boxed{= \frac{1}{4}}$$

Exercise

Find the limit $\lim\limits_{x\to 4} \dfrac{16\left(\sqrt{x}-2\right)}{x-4}$

Solution

$$\lim_{x\to 4} \frac{16\left(\sqrt{x}-2\right)}{x-4} = \frac{16(2-2)}{4-4} = \frac{0}{0}$$

$$= \lim_{x\to 4} \frac{16\left(\sqrt{x}-2\right)}{x-4}\,\frac{\sqrt{x}+2}{\sqrt{x}+2}$$

$$(a-b)(a+b)=a^2-b^2$$

$$= \lim_{x\to 4} \frac{16(x-4)}{(x-4)\left(\sqrt{x}+2\right)}$$

$$= \lim_{x\to 4} \frac{16}{\sqrt{x}+2}$$

$$= \frac{16}{2+2}$$

$$\boxed{=4}$$

$$\lim_{x\to 4} \frac{16\left(\sqrt{x}-2\right)}{x-4} = \lim_{x\to 4} \frac{16\left(\sqrt{x}-2\right)}{\left(\sqrt{x}+2\right)\left(\sqrt{x}-2\right)}$$

$$a^2-b^2=(a-b)(a+b)$$

$$= \lim_{x\to 4} \frac{16}{\sqrt{x}+2}$$

$$\boxed{=4}$$

Exercise

Find the limit $\displaystyle \lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3}$

Solution

$$\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} = \frac{2-2}{3-3} = \frac{0}{0}$$

$$= \lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3}\,\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}$$

$$(a-b)(a+b)=a^2-b^2$$

$$= \lim_{x\to 3} \frac{x+1-4}{(x-3)\left(\sqrt{x+1}+2\right)}$$

$$= \lim_{x\to 3} \frac{x-3}{(x-3)\left(\sqrt{x+1}+2\right)}$$

$$= \lim_{x\to 3} \frac{1}{\sqrt{x+1}+2}$$

$$= \frac{1}{2+2}$$

$$\boxed{=\frac{1}{4}}$$

Exercise

Find the limit $\lim\limits_{x\to 5} \dfrac{\sqrt{x-1}-2}{x-5}$

Solution

$$\lim_{x\to 5} \frac{\sqrt{x-1}-2}{x-5} = \frac{2-2}{5-5} = \frac{0}{0}$$

$$= \lim_{x\to 5} \frac{\sqrt{x-1}-2}{x-5}\,\frac{\sqrt{x-1}+2}{\sqrt{x-1}+2}$$

$(a-b)(a+b) = a^2 - b^2$

$$= \lim_{x\to 5} \frac{x-1-4}{(x-5)\left(\sqrt{x-1}+2\right)}$$

$$= \lim_{x\to 5} \frac{x-5}{(x-5)\left(\sqrt{x-1}+2\right)}$$

$$= \lim_{x\to 5} \frac{1}{\sqrt{x-1}+2}$$

$$= \frac{1}{2+2}$$

$$= \frac{1}{4}$$

Exercise

Find the limit $\lim\limits_{x\to \infty} \dfrac{\sqrt{x+1}-2}{x-3}$

Solution

$$\lim_{x\to \infty} \frac{\sqrt{x+1}-2}{x-3} = \lim_{x\to \infty} \frac{\sqrt{x}}{x}$$

$$= \lim_{x\to \infty} \frac{1}{\sqrt{x}}$$

$$= \frac{1}{\infty}$$

$$= 0$$

Exercise

Find the limit $\lim\limits_{x\to -\infty} \dfrac{\sqrt{x+1}-2}{x-3}$

Solution

$$\lim_{x\to -\infty} \frac{\sqrt{x+1}-2}{x-3} = \lim_{x\to -\infty} \frac{\sqrt{x}}{x}$$

$$= \frac{\sqrt{-}}{\infty}$$

$$= \nexists$$

Exercise

Find the limit $\displaystyle\lim_{x\to\infty} \frac{x+\sin x+2\sqrt{x}}{x+\sin x}$

Solution

Since $-1 \le \sin x \le 1$

$$\lim_{x\to\infty} \frac{x+\sin x+2\sqrt{x}}{x+\sin x} = \lim_{x\to\infty} \frac{x}{x}$$

$$= 1$$

Exercise

Find the limit $\displaystyle\lim_{x\to-\infty} \left(\frac{x^2+x-1}{8x^2-3}\right)^{1/3}$

Solution

$$\lim_{x\to-\infty} \left(\frac{x^2+x-1}{8x^2-3}\right)^{1/3} = \lim_{x\to-\infty} \left(\frac{x^2}{8x^2}\right)^{1/3}$$

$$= \lim_{x\to-\infty} \left(\frac{1}{2^3}\right)^{1/3}$$

$$= \frac{1}{2}$$

Exercise

Find the limit $\displaystyle\lim_{x\to-\infty} \frac{4-3x^3}{\sqrt{x^6+9}}$

Solution

$$\lim_{x\to-\infty} \frac{4-3x^3}{\sqrt{x^6+9}} = \lim_{x\to-\infty} \frac{-3x^3}{\sqrt{x^6}}$$

$$= \lim_{x\to-\infty} \frac{-3x^3}{x^3}$$

$$= -3$$

Exercise

Find the limit $\lim\limits_{x\to-\infty} \dfrac{x^2-4x+8}{3x^3}$

Solution

$$\lim_{x\to-\infty} \frac{x^2-4x+8}{3x^3} = \lim_{x\to-\infty} \frac{x^2}{3x^3}$$

$$= \lim_{x\to-\infty} \frac{1}{3x}$$

$$= 0$$

Exercise

Find the limit $\lim\limits_{x\to-\infty} \dfrac{2x^2+3}{5x^2+7}$

Solution

$$\lim_{x\to-\infty} \frac{2x^2+3}{5x^2+7} = \lim_{x\to-\infty} \frac{2x^2}{5x^2}$$

$$= \frac{2}{5}$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{x^4+x^3}{12x^3+128}$

Solution

$$\lim_{x\to\infty} \frac{x^4+x^3}{12x^3+128} = \lim_{x\to\infty} \frac{x^4}{12x^3}$$

$$= \infty$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{\sqrt[5]{x^{15}-2x^4+x^3}}{12x^3+128}$

Solution

$$\lim_{x\to\infty} \frac{\sqrt[5]{x^{15}-2x^4+x^3}}{12x^3+128} = \lim_{x\to\infty} \frac{\sqrt[5]{x^{15}}}{12x^3}$$

$$= \lim_{x\to\infty} \frac{x^3}{12x^3}$$

$$= \frac{1}{12}$$

Exercise

Find the limit $\lim_{\theta\to 0} \frac{1-\cos\theta}{\sin\theta}$

Solution

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\sin\theta} = \frac{1-1}{0} = \frac{0}{0}$$

$$= \lim_{\theta\to 0} \frac{1-\cos\theta}{\sin\theta} \frac{1+\cos\theta}{1+\cos\theta} \qquad (a-b)(a+b) = a^2 - b^2$$

$$= \lim_{\theta\to 0} \frac{1-\cos^2\theta}{\sin\theta\,(1+\cos\theta)} \qquad \cos^2\theta + \sin^2\theta = 1$$

$$= \lim_{\theta\to 0} \frac{\sin^2\theta}{\sin\theta\,(1+\cos\theta)}$$

$$= \lim_{\theta\to 0} \frac{\sin\theta}{1+\cos\theta}$$

$$= \frac{0}{2}$$

$$= 0$$

Exercise

Find the limit $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta}$

Solution

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = \frac{1-1}{0} = \frac{0}{0}$$

$$= \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} \frac{1+\cos\theta}{1+\cos\theta} \qquad (a-b)(a+b) = a^2 - b^2$$

$$= \lim_{\theta\to 0} \frac{1-\cos^2\theta}{\theta\,(1+\cos\theta)} \qquad \cos^2\theta + \sin^2\theta = 1$$

$$= \lim_{\theta\to 0} \frac{\sin^2\theta}{\theta\,(1+\cos\theta)}$$

$$= \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \frac{\sin\theta}{1+\cos\theta} \qquad \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

$$=(1)\ \frac{0}{2}$$

$$\boxed{=0}$$

Exercise

Find the limit $\lim\limits_{\theta\to\frac{\pi}{4}} \frac{\cos\theta}{\cot\theta}$

Solution

$$\lim_{\theta\to\frac{\pi}{4}} \frac{\cos\theta}{\cot\theta} = \frac{\cos\frac{\pi}{4}}{\cot\frac{\pi}{4}}$$

$$= \frac{\frac{\sqrt{2}}{2}}{1}$$

$$\boxed{= \frac{\sqrt{2}}{2}}$$

Exercise

Find the limit $\lim\limits_{\theta\to\frac{\pi}{2}} \frac{\cos\theta}{\cot\theta}$

Solution

$$\lim_{\theta\to\frac{\pi}{4}} \frac{\cos\theta}{\cot\theta} = \frac{\cos\frac{\pi}{2}}{\cot\frac{\pi}{2}}$$

$$= \frac{0}{0}$$

$$= \lim_{\theta\to\frac{\pi}{2}} \cos\theta\ \frac{\sin\theta}{\cos\theta}$$

$$= \lim_{\theta\to\frac{\pi}{2}} \sin\theta$$

$$= \sin\frac{\pi}{2}$$

$$\boxed{=1}$$

Exercise

Find the limit $\lim\limits_{\theta \to \frac{\pi}{2}} (\sec\theta + \tan\theta)$

Solution

$$\lim_{\theta \to \frac{\pi}{2}} (\sec\theta + \tan\theta) = \sec\frac{\pi}{2} + \tan\frac{\pi}{2}$$

$$= \infty + \infty$$

$$\underline{= \infty}$$

Exercise

Find the limit $\lim\limits_{\theta \to \frac{\pi}{2}} \dfrac{\sec\theta}{\tan\theta}$

Solution

$$\lim_{\theta \to \frac{\pi}{2}} \frac{\sec\theta}{\tan\theta} = \frac{\sec\frac{\pi}{2}}{\tan\frac{\pi}{2}}$$

$$= \frac{\infty}{\infty}$$

$$= \lim_{\theta \to \frac{\pi}{2}} \frac{1}{\cos\theta} \frac{\cos\theta}{\sin\theta}$$

$$= \lim_{\theta \to \frac{\pi}{2}} \frac{1}{\sin\theta}$$

$$= \frac{1}{\sin\frac{\pi}{2}}$$

$$\underline{= 1}$$

Exercise

Find the limit $\lim\limits_{x \to \infty} \ln x$

Solution

$$\lim_{x \to \infty} \ln x = \ln(\infty)$$

$$\underline{= \infty}$$

Exercise

Find the limit $\lim\limits_{x \to e} \ln x$

Solution

$$\lim_{x \to e} \ln x = \ln e$$

$$\underline{= 1}$$

Exercise

Find the limit $\lim\limits_{x \to -\infty} e^{x^2}$

Solution

$$\lim_{x \to -\infty} e^{x^2} = e^{(-\infty)^2}$$

$$= e^{\infty}$$

$$\underline{= \infty}$$

Exercise

Find the limit $\lim\limits_{x \to \infty} e^{x^2}$

Solution

$$\lim_{x \to \infty} e^{x^2} = e^{(\infty)^2}$$

$$= e^{\infty}$$

$$\underline{= \infty}$$

Exercise

Find the limit $\lim\limits_{x \to 3} e^{\ln x}$

Solution

$$\lim_{x \to 3} e^{\ln x} = e^{\ln 3}$$

$$\underline{= 3}$$

Exercise

Find the limit $\lim\limits_{x\to -1} \ln\left(e^{x^2}\right)$

Solution

$$\lim_{x\to -1} \ln\left(e^{x^2}\right) = \ln e$$

$$= 1$$

Exercise

Find the limit $\lim\limits_{x\to 1} \dfrac{e^x - e^{-x}}{2}$

Solution

$$\lim_{x\to 1} \frac{e^x - e^{-x}}{2} = \frac{e - e^{-1}}{2}$$

$$= \frac{e - \frac{1}{e}}{2}$$

$$= \frac{e^2 - 1}{2e}$$

Exercise

Find the limit $\lim\limits_{x\to 0} \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Solution

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^0 - e^0}{e^0 + e^0}$$

$$= \frac{1-1}{1+1}$$

$$= \frac{0}{2}$$

$$= 0$$

Exercise

Find the limit $\lim\limits_{x\to \infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Solution

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to\infty} \frac{e^x}{e^x}$$

$$\underline{=1}$$

Exercise

Find the limit $\lim_{x\to 0} \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

Solution

$$\lim_{x\to 0} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{1+1}{1-1}$$

$$= \frac{2}{0}$$

$$\underline{= \infty}$$

Exercise

Find the limit $\lim_{x\to 1} \ln\left(e^x - e^{-x}\right)$

Solution

$$\lim_{x\to 1} \ln\left(e^x - e^{-x}\right) = e - e^{-1}$$

$$= e - \frac{1}{e}$$

$$\underline{= \frac{e^2 - 1}{e}}$$

Exercise

Find the limit $\lim_{x\to 0} \left(\ln\left(e^x\right) + e^{-x}\right)$

Solution

$$\lim_{x\to 0} \left(\ln\left(e^x\right) + e^{-x}\right) = \ln e^0 + e^0$$

$$= \ln 1 + 1$$

$$= 0 + 1$$

$$\underline{= 1}$$

Exercise

Find the limit $\lim\limits_{x\to 0} \ln(\sec x)$

Solution

$$\lim\limits_{x\to 0} \ln(\sec x) = \ln(\sec 0)$$

$$= \ln 1$$

$$\boxed{= 0}$$

Exercise

If $\lim\limits_{x\to 4} \dfrac{f(x)-5}{x-2} = 1$, find $\lim\limits_{x\to 4} f(x)$

Solution

$$\lim\limits_{x\to 4} \frac{f(x)-5}{x-2} = 1$$

$$\frac{\lim\limits_{x\to 4} f(x)-5}{4-2} = 1$$

$$\frac{\lim\limits_{x\to 4} f(x)-5}{2} = 1$$

$$\lim\limits_{x\to 4} f(x)-5 = 2$$

$$\boxed{\lim\limits_{x\to 4} f(x) = 7}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $h(x) = \dfrac{-5+\frac{7}{x}}{3-\frac{1}{x^2}}$

Solution

$$h(x) = \frac{-5+\frac{7}{x}}{3-\frac{1}{x^2}}$$

$$= \frac{-5x+7}{x} \frac{1}{\frac{3x^2-1}{x^2}}$$

$$= \frac{-5x+7}{x} \frac{x^2}{3x^2-1}$$

$$= \frac{x(-5x+7)}{3x^2-1}$$

$$= \frac{-5x^2+7x}{3x^2-1}$$

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{-5x^2+7x}{3x^2-1}$$

$$= \lim_{x\to\infty} \frac{-5x^2}{3x^2}$$

$$= -\frac{5}{3}$$

$$\lim_{x\to-\infty} h(x) = \lim_{x\to\infty} \frac{-5x^2}{3x^2}$$

$$= -\frac{5}{3}$$

Exercise

At what points is the function $f(x) = |x-1| + \sin x$ continuous?

Solution

The function $f(x)$ everywhere $\forall x \in \mathbb{R}$

Exercise

At what points is the function $f(x) = \dfrac{x-2}{x^2-5x+4}$ continuous?

Solution

$$x^2 - 5x + 4 = 0$$

$$x = 1,\ 4$$

The function $f(x)$ everywhere except when $x = 1,\ 4$

Exercise

At what points is the function $f(x)=\dfrac{x-2}{x^2+3x+2}$ continuous?

Solution

$x^2+3x+2=0 \quad\Rightarrow\quad x=-1,\ -2$

The function $f(x)$ everywhere except when $x=-1,\ -2$

Exercise

At what points is the function $f(x)=\ln(x)$ continuous?

Solution

Since, inside $\ln\ >0$

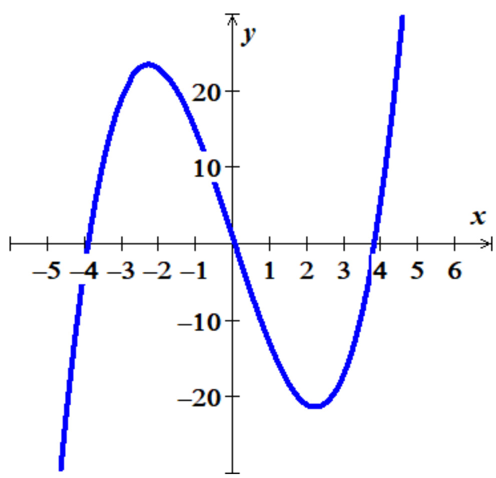
The function $f(x)$ everywhere except when $x\le 0$

Exercise

Show that the equation $x^3-15x+1=0$ has ***three*** solutions in the interval [–4, 4]

Solution

x	$f(x)$
–4	–3
–3	19
–2	23
–1	15
0	1
1	–13
2	–21
3	–17
4	5



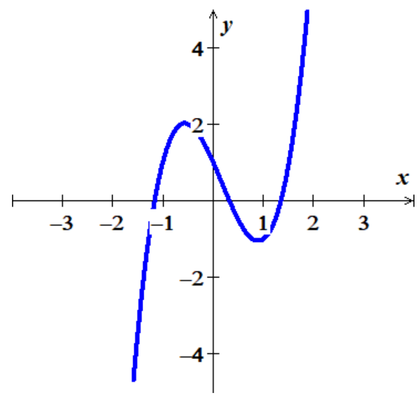
By the Intermediate Value Theorem, $f(x)=0$ for some x in each of the intervals $-4<x<-3$, $0<x<1$, and $3<x<4$.

Exercise

Show that the equation $2x^3 - x^2 - 3x + 1 = 0$ has ***three*** solutions in the interval $(-3,\ 3)$

Solution

x	$f(x)$
-2	-13
-1	1
0	1
1	-1
2	7



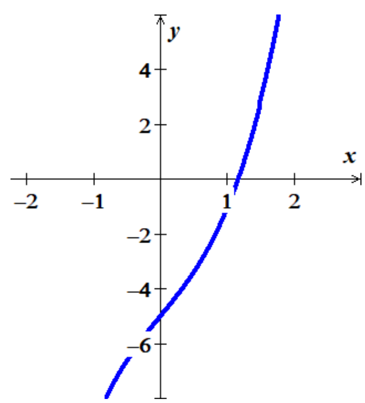
By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-2 < x < -1$, $0 < x < 1$, and $1 < x < 2$.

Exercise

Show that the equation $x^3 + 3x - 5 = 0$ has ***one*** solution in the interval $[-2,\ 2]$

Solution

x	$f(x)$
-2	-19
-1	-9
0	-5
1	-1
2	9



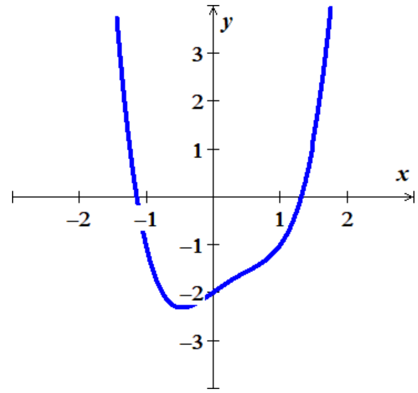
By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $1 < x < 2$.

Exercise

Show that the equation $x^4 - x^3 + x - 2 = 0$ has ***two*** solutions in the interval $[-2,\ 2]$

Solution

x	$f(x)$
-2	20
-1	-1
0	-2
1	-1
2	8



By the Intermediate Value Theorem, $f(x)=0$ for some x in each of the intervals $-2<x<-1$, and $1<x<2$.

Exercise

Find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

$$f(x)=\sqrt{x-7}, \quad L=4, \quad x_0=23, \quad \varepsilon=1$$

Solution

$$\left|\sqrt{x-7}-4\right| < 1$$

$$-1 < \sqrt{x-7}-4 < 1$$

$$-1+4 < \sqrt{x-7}-4+4 < 1+4$$

$$3 < \sqrt{x-7} < 5$$

$$(3)^2 < \left(\sqrt{x-7}\right)^2 < (5)^2$$

$$9 < x-7 < 25$$

$$9+7 < x-7+7 < 25+7$$

$$16 < x < 32$$

$$|x-23| < \delta$$

$$-\delta < x-23 < \delta$$

$$23-\delta < x < 23+\delta$$

$$16 < x < 32$$

$$23-\delta=16$$

$$\underline{\delta=7\,|}$$

$$23+\delta=32$$

$$\underline{\delta=9\,|}$$

$$\boxed{\delta = 7}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

$$f(x) = 4x - 3, \quad L = 1, \quad x_0 = 1, \quad \varepsilon = .01$$

Solution

$$|4x - 3 - 1| < .01$$

$$-\frac{1}{100} < 4x - 4 < \frac{1}{100}$$

$$-\frac{1}{100} + 4 < 4x - 4 + 4 < \frac{1}{100} + 4$$

$$\frac{309}{100} < 4x < \frac{401}{100}$$

$$\frac{309}{400} < x < \frac{401}{400}$$

$$|x - 1| < \delta$$

$$-\delta < x - 1 < \delta$$

$$1 - \delta < x < 1 + \delta$$

$$\frac{309}{400} < x < \frac{401}{400}$$

$$1 - \delta = \frac{309}{400}$$

$$\delta = 1 - \frac{309}{400}$$

$$\boxed{\delta = \frac{1}{400}}$$

$$1 + \delta = \frac{401}{400}$$

$$\delta = 1 - \frac{401}{400}$$

$$\boxed{\delta = -\frac{1}{400}}$$

$$\boxed{\delta = \frac{1}{400}}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

$$f(x)=x^2-1, \quad L=3, \quad x_0=2, \quad \varepsilon=.1$$

Solution

$$|x^2-1-3|<0.1$$

$$-\frac{1}{10}<x^2-4<\frac{1}{10}$$

$$4-\frac{1}{10}<x^2<\frac{1}{10}+4$$

$$\frac{39}{10}<x^2<\frac{41}{10}$$

$$\sqrt{\frac{39}{10}}<x<\sqrt{\frac{41}{10}}$$

$$|x-2|<\delta$$

$$-\delta<x-2<\delta$$

$$2-\delta<x<2+\delta$$

$$\sqrt{\frac{39}{10}}<x<\sqrt{\frac{41}{10}}$$

$$2-\delta=\sqrt{\frac{39}{10}}$$

$$\delta=2-\sqrt{\frac{39}{10}}$$

$$2+\delta=\sqrt{\frac{41}{10}}$$

$$\delta=2-\sqrt{\frac{41}{10}}$$

$$\delta=\left|2-\sqrt{\frac{41}{10}}\right|$$

$$\Rightarrow \quad \boxed{\delta=\sqrt{\frac{41}{10}}-2}$$

Exercise

Find the vertical, horizontal, hole and oblique asymptotes (if any) of $y = \dfrac{x-2}{x^2-4x+3}$

Solution

$$x^2 - 4x + 3 = 0 \quad \Rightarrow \quad x = 1,\ 3$$

$$\textbf{VA}:\ x = 1,\ 3; \quad \textbf{Hole}:\ n/a; \quad \textbf{HA}:\ y = 0; \quad \textbf{OA}:\ n/a$$

Exercise

Find the vertical, horizontal, hole and oblique asymptotes (if any) of $f(x) = \dfrac{x^2-x-2}{x^2-2x+1}$

Solution

$$x^2 - 2x + 1 = 0 \quad \Rightarrow \quad x = 1$$

$$f(1) = \frac{-2}{0}$$

$$\lim_{x\to\infty} \frac{x^2-x-2}{x^2-2x+1} = \lim_{x\to\infty} \frac{x^2}{x^2}$$

$$= 1$$

$$\textbf{VA}:\ x = 1; \quad \textbf{Hole}:\ n/a; \quad \textbf{HA}:\ y = 1; \quad \textbf{OA}:\ n/a$$

Exercise

Find the vertical, horizontal, hole and oblique asymptotes (if any) of $f(x) = \dfrac{x^3+3x^2-2}{x^2-4}$

Solution

$$x^2 - 4 = 0 \quad \Rightarrow \quad x = \pm 2$$

$$x^3 + 3x^2 - 2 \,\Big|_{x=-2} = -8 + 8 - 2 \neq 0$$

$$x^3 + 3x^2 - 2 \,\Big|_{x=2} = 8 + 8 - 2 \neq 0$$

$$\begin{array}{r|l} & x+3 \\ \hline x^2-4 & x^3+3x^2-2 \\ & \underline{x^3-4x} \\ & \quad 3x^2+4x-2 \\ & \quad \underline{3x^2-12} \\ & \quad\quad 4x-14 \end{array}$$

$VA:\ x=\pm 2$; $Hole:\ n/a$; $HA:\ n/a$; $OA:\ y=x+3$

Exercise

Find the vertical, horizontal, hole and oblique asymptotes (if any) of $f(x)=\dfrac{x^3-2x^2-4x+8}{x-2}$

Solution

$x-2=0 \quad \Rightarrow \quad x=2$

$x^3-2x^2-4x+8\,\Big|_{x=2}=8-8-8+8=0$

$$\begin{array}{r|rrrr} 2 & 1 & -2 & -4 & 8 \\ & & 2 & 0 & -8 \\ \hline & 1 & 0 & -4 & 0 \end{array}$$

$$f(x)=\frac{(x-2)\left(x^2-4\right)}{x-2}$$

$$=x^2-4$$

$x^2-4\,\Big|_{x=2}=4-4=0$

$VA:\ x=n/a$; $Hole:\ (2,\ 0)$; $HA:\ n/a$; $OA:\ n/a$

Exercise

Find the vertical, horizontal, hole and oblique asymptotes (if any) of $f(x)=\dfrac{x^2+4}{x-3}$

Solution

$x-3=0 \quad \Rightarrow \quad x=3$

$x^2+4\,\Big|_{x=3}=9+4\neq 0$

$$\begin{array}{r|rrr} 3 & 1 & 0 & 4 \\ & & 3 & 9 \\ \hline & 1 & 3 & 13 \end{array}$$

$$f(x) = \frac{x^2+4}{x-3}$$

$$= x + 3 + \frac{13}{x-3}$$

VA: $x = 3$; ***Hole***: n/a; ***HA***: n/a; ***OA***: $y = x + 3$

Exercise

Find the vertical, horizontal, hole and oblique asymptotes (if any) of $y = \dfrac{\sqrt{x^2+4}}{x}$

Solution

$$\lim_{x\to\infty} \frac{\sqrt{x^2+4}}{x} = \lim_{x\to\infty} \frac{\sqrt{x^2}}{x}$$

$$= \lim_{x\to\infty} \frac{x}{x}$$

$$= 1$$

VA: $x = 0$; ***Hole***: n/a; ***HA***: $y = 1$; ***OA***: n/a

Exercise

The motion of a spring is the result by given by the steady-state function.

$$x(t) = -\frac{1}{625} e^{-t} \left(24\cos 4t + 7\sin 4t\right)$$

a) Find the limit as *t* approaches infinity

b) Graph the steady-state function.

c) Compare the part(a) with the graph.

Solution

a) $\displaystyle \lim_{t\to\infty} x(t) = \lim_{t\to\infty} \left(-\frac{1}{625} e^{-t} \left(24\cos 4t + 7\sin 4t\right)\right)$

Since $-1 \le \sin\alpha \le 1$ & $-1 \le \cos\alpha \le 1$

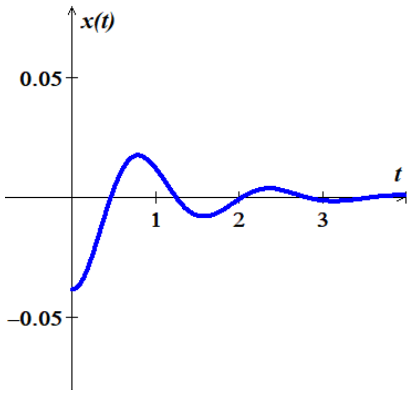
$$\lim_{t\to\infty} x(t) = -\frac{1}{625} \lim_{t\to\infty} \left(e^{-t}\right)$$

$$= -\frac{1}{625}\, e^{-\infty}$$

$$= \frac{1}{e^{\infty}}$$

$$\underline{= 0}$$

b)



c) From part (*b*), the graph, as *t* increases the function oscillated and approaches *t*-axis as the limit from part (*a*).

Exercise

The motion of a spring is the result by given by the transient function.

$$x(t) = \frac{1}{625}(24 + 100t)e^{-t}$$

a) Find the limit as *t* approaches infinity
b) Graph the steady-state function.
c) Compare the part(a) with the graph.

Solution

a) $$\lim_{t\to\infty} x(t) = \lim_{t\to\infty}\left(\frac{1}{625}(24 + 100t)e^{-t}\right)$$

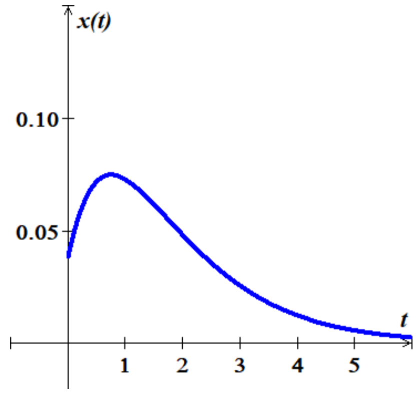
$$= \lim_{t\to\infty}\left(\frac{100}{625}\, t\, e^{-t}\right)$$

$$= \lim_{t\to\infty}\left(e^{-t}\right)$$

$$= \frac{1}{e^{\infty}}$$

$$\underline{= 0}$$

b)



d) From part (b), the graph, as t increases the function approaches t-axis as the limit from part (a).

Exercise

A mass of 64 *pounds* is attached to a spring with a spring constant 32 *lb/ft* and then comes to rest in the equilibrium position. Neglect the damping. The results are given by the two given functions

Steady-State: $\quad x_p(t) = e^{-2t}\left(\frac{1}{2}\cos 4t - 2\sin 4t\right)$

Transient: $\quad x_h(t) = -\frac{1}{2}\cos 4t + \frac{9}{4}\sin 4t$

a) Find $\lim\limits_{t\to\infty} x_p(t)$

b) Find $\lim\limits_{t\to\infty} x_h(t)$

c) Find $\lim\limits_{t\to\infty} \left(x_h(t) + x_p(t)\right)$

Solution

a) $\lim\limits_{t\to\infty} x_p(t) = \lim\limits_{t\to\infty} e^{-2t}\left(\frac{1}{2}\cos 4t - 2\sin 4t\right)$

Since $-1 \le \sin\alpha \le 1$ & $-1 \le \cos\alpha \le 1$

$$= \lim_{t\to\infty}\left(e^{-2t}\right)$$

$$= e^{-\infty}$$

$$= \frac{1}{e^{\infty}}$$

$$= 0$$

b) $\lim\limits_{t\to\infty} x_h(t) = \lim\limits_{t\to\infty}\left(-\frac{1}{2}\cos 4t + \frac{9}{4}\sin 4t\right)$

Since $-1 \le \sin\alpha \le 1$ & $-1 \le \cos\alpha \le 1$

$$= \nexists$$

c) $\displaystyle \lim_{t\to\infty}\left(x_h(t)+x_p(t)\right)=\lim_{t\to\infty}\left(e^{-2t}\left(\frac{1}{2}\cos 4t-2\sin 4t\right)-\frac{1}{2}\cos 4t+\frac{9}{4}\sin 4t\right)$

$$= 0+\nexists$$

$$= \nexists$$