

## ***Solution***      **Section 2.3 – Product and Quotient Rules**

### ***Exercise***

Find the derivative of  $y = (x+1)(\sqrt{x}+2)$

### **Solution**

$$\begin{aligned}y' &= (1)\left(x^{1/2} + 2\right) + (x+1)\left(\frac{1}{2}x^{-1/2}\right) \\&= x^{1/2} + 2 + \frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2} \\&= \frac{3}{2}x^{1/2} + \frac{1}{2}x^{-1/2} + 2 \quad \Big| \end{aligned}$$

### ***Exercise***

Find the derivative of  $y = (4x + 3x^2)(6 - 3x)$

### **Solution**

$$\begin{aligned}y' &= \left(4x + 3x^2\right)\frac{d}{dx}(6 - 3x) + (6 - 3x)\frac{d}{dx}(4x + 3x^2) \\&= \left(4x + 3x^2\right)(-3) + (6 - 3x)(4 + 6x) \\&= -12x - 9x^2 + 24 + 36x - 12x - 18x^2 \\&= -27x^2 + 12x + 24 \quad \Big| \end{aligned}$$

$$y = 24x + 6x^2 - 9x^3$$

### ***Exercise***

Find the derivative of  $y = \left(\frac{1}{x} + 1\right)(2x + 1)$

### **Solution**

$$\begin{aligned}y' &= \left(x^{-1} + 1\right)\frac{d}{dx}(2x + 1) + (2x + 1)\frac{d}{dx}\left(x^{-1} + 1\right) \\&= \left(x^{-1} + 1\right)(2) + (2x + 1)\left(-x^{-2}\right) \\&= \frac{2}{x} + 2 + (2x + 1)\left(-\frac{1}{x^2}\right) \\&= \frac{2}{x} + 2 - \frac{2}{x} - \frac{1}{x^2} \\&= 2 - \frac{1}{x^2} \\&= \frac{2x^2 - 1}{x^2} \quad \Big| \end{aligned}$$

### Exercise

Find the derivative of  $y = \frac{3x-2}{x+4}$

### Solution

$$\begin{aligned}y &= \frac{3x-2}{x+4} \\&= \frac{3x-2}{x} \cdot \frac{1}{x+4} \\&= \frac{3x-2}{x^2+4x}\end{aligned}$$

$$\begin{aligned}y' &= \frac{\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} x^2 + 2 \begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} x + \begin{vmatrix} 3 & -2 \\ 4 & 0 \end{vmatrix}}{(x^2+4x)^2} \\&= \frac{-3x^2+4x+8}{x^2(x+4)^2}\end{aligned}$$

**OR**

$$\begin{aligned}y' &= \frac{(x^2+4x)(3) - (3x-2)(2x+4)}{[x(x+4)]^2} \\&= \frac{3x^2+12x-6x^2-12x+4x+8}{x^2(x+4)^2} \\&= \frac{-3x^2+4x+8}{x^2(x+4)^2}\end{aligned}$$

### Exercise

Find the derivative of  $g(x) = \frac{x^2-4x+2}{x^2+3}$

### Solution

$$g'(x) = \frac{\begin{vmatrix} 1 & -4 \\ 1 & 0 \end{vmatrix} x^2 + 2 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} x + \begin{vmatrix} -4 & 2 \\ 0 & 3 \end{vmatrix}}{(x^2+3)^2}$$

$$\frac{d}{dx} \left( \frac{ax^2+bx+c}{dx^2+ex+f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2+ex+f)^2}$$

$$\frac{d}{dx} \left( \frac{ax^2+bx+c}{dx^2+ex+f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2+ex+f)^2}$$

$$= \frac{4x^2 + 2x - 12}{(x^2 + 3)^2}$$

**Or**

$$\begin{aligned} g' &= \frac{(2x-4)(x^2+3) - (x^2-4x+2)(2x)}{(x^2+3)^2} \\ &= \frac{2x^3 + 6x - 4x^2 - 12 - 2x^3 + 8x^2 - 4x}{(x^2+3)^2} \\ &= \frac{4x^2 + 2x - 12}{(x^2+3)^2} \end{aligned}$$

### Exercise

Find the derivative of  $f(x) = \frac{(3-4x)(5x+1)}{7x-9}$

**Solution**

$$f(x) = \frac{-20x^2 + 11x + 3}{7x-9}$$

$$f'(x) = \frac{\begin{vmatrix} -20 & 11 \\ 0 & 7 \end{vmatrix} x^2 + 2 \begin{vmatrix} -20 & 3 \\ 0 & -9 \end{vmatrix} x + \begin{vmatrix} 11 & 3 \\ 7 & -9 \end{vmatrix}}{(7x-9)^2}$$

$$\frac{d}{dx} \left( \frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

$$= \frac{-140x^2 + 360x - 120}{(7x-9)^2}$$

**Or**

$$\begin{aligned} D_x \left[ \frac{(3-4x)(5x+1)}{7x-9} \right] &= \frac{[(-4)(5x+1) + (3-4x)(5)](7x-9) - (3-4x)(5x+1)(7)}{(7x-9)^2} \\ &= \frac{[-20x - 4 + 15 - 20x](7x-9) - (15x + 3 - 20x^2 - 4x)(7)}{(7x-9)^2} \\ &= \frac{(-40x + 11)(7x-9) - 7(-20x^2 + 11x + 3)}{(7x-9)^2} \\ &= \frac{-280x^2 + 360x + 77x - 99 + 140x^2 - 77x - 21}{(7x-9)^2} \end{aligned}$$

$$\boxed{= \frac{-140x^2 + 360x - 120}{(7x - 9)^2}}$$

### Exercise

Find the derivative of  $f(x) = x\left(1 - \frac{2}{x+1}\right)$

### Solution

$$f(x) = x - \frac{2x}{x+1}$$

$$\boxed{f'(x) = 1 - \frac{2}{(x+1)^2}}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

**OR**

$$\left(\frac{2x}{x+1}\right)' \Rightarrow \begin{array}{ll} f = 2x & f' = 2 \\ g = x+1 & g' = 1 \end{array}$$

$$\begin{aligned} f'(x) &= 1 - \frac{2(x+1) - 2x}{(x+1)^2} \\ &= 1 - \frac{2x+2-2x}{(x+1)^2} \\ &= 1 - \frac{2}{(x+1)^2} \end{aligned}$$

### Exercise

Find the derivative of  $f(x) = (\sqrt{x} + 3)(x^2 - 5x)$

### Solution

$$\begin{aligned} f' &= \left(\frac{1}{2}x^{-1/2}\right)(x^2 - 5x) + (\sqrt{x} + 3)(2x - 5) \\ &= \frac{1}{2}x^{3/2} - \frac{5}{2}x^{1/2} + 2x^{3/2} - 5x^{1/2} + 6x - 15 \\ &= \frac{5}{2}x^{3/2} - \frac{15}{2}x^{1/2} + 6x - 15 \\ &= \frac{5}{2}x^{3/2} + 6x - \frac{15}{2}x^{1/2} - 15 \end{aligned}$$

### Exercise

Find the derivative of  $y = (2x + 3)(5x^2 - 4x)$

#### Solution

$$\begin{aligned}y &= (2x + 3)(5x^2 - 4x) = 10x^3 - 8x^2 + 15x^2 - 12x \\&= 10x^3 + 7x^2 - 12x \\y' &= \underline{30x^2 + 14x - 12} \quad | \end{aligned}$$

### Exercise

Find the derivative of  $y = (x^2 + 1)\left(x + 5 + \frac{1}{x}\right)$

#### Solution

$$\begin{aligned}y &= x^3 + 5x^2 + x + x + 5 + \frac{1}{x} \\&= x^3 + 5x^2 + 2x + 5 + x^{-1} \\y' &= 3x^2 + 10x + 2 - x^{-2} \\&= \underline{3x^2 + 10x + 2 - \frac{1}{x^2}} \quad | \end{aligned}$$

### Exercise

Find the derivative of  $y = \frac{x + 4}{5x - 2}$

#### Solution

$$y' = \underline{-\frac{22}{(5x - 2)^2}} \quad |$$

$$\left(\frac{ax + b}{cx + d}\right)' = \frac{ad - bc}{(cx + d)^2}$$

OR

$$\begin{aligned}y' &= \frac{(5x - 2) \frac{d}{dx}[(x + 4)] - (x + 4) \frac{d}{dx}[(5x - 2)]}{(5x - 2)^2} \\&= \frac{(5x - 2)(1) - (x + 4)(5)}{(5x - 2)^2} \\&= \frac{5x - 2 - 5x - 20}{(5x - 2)^2} \\&= \underline{-\frac{22}{(5x - 2)^2}} \quad | \end{aligned}$$

### Exercise

Find the derivative of  $z = \frac{4-3x}{3x^2+x}$

### Solution

$$\underline{z' = \frac{9x^2 - 24x - 4}{(3x^2 + x)^2}} \quad \begin{array}{ccc} 0 & -3 & 4 \\ 3 & 1 & 0 \end{array} \quad \left( \frac{ax^2 + bx + c}{dx^2 + ex + f} \right)' = \frac{(ae - bd)x^2 + 2(af - cd)x + bf - ce}{(dx^2 + ex + f)^2}$$

### OR

$$u = 4 - 3x \quad v = 3x^2 + x$$

$$u' = -3 \quad v' = 6x + 1$$

$$\begin{aligned} z' &= \frac{-3(3x^2 + x) - (6x + 1)(4 - 3x)}{(3x^2 + x)^2} \\ &= \frac{-9x^2 - 3x - (24x - 18x^2 + 4 - 3x)}{(3x^2 + x)^2} \\ &= \frac{-9x^2 - 3x - 24x + 18x^2 - 4}{(3x^2 + x)^2} \\ &= \frac{9x^2 - 24x - 4}{(3x^2 + x)^2} \end{aligned}$$

$$z' = \frac{u'v - v'u}{u^2}$$

### Exercise

Find the derivative of  $y = (2x - 7)^{-1}(x + 5)$

### Solution

$$y = \frac{x+5}{2x-7}$$

$$\underline{y' = -\frac{17}{(2x-7)^2}}$$

$$\left( \frac{ax+b}{cx+d} \right)' = \frac{ad-bc}{(cx+d)^2}$$

### OR

$$\begin{aligned} y' &= -(2x-7)^{-2}(2)(x+5) + (2x-7)^{-1} \\ &= -(2x-7)^{-2}(2x+10) + (2x-7)^{-1} \end{aligned}$$

$$\begin{aligned}
&= \left[ -(2x-7)^{-2}(2x+10) + (2x-7)^{-1} \right] \frac{(2x-7)^2}{(2x-7)^2} \\
&= \frac{-2x-10+2x-7}{(2x-7)^2} \\
&= \frac{-17}{(2x-7)^2}
\end{aligned}$$

### Exercise

Find the derivative of  $f(x) = \frac{\sqrt{x}-1}{\sqrt{x}+1}$

### Solution

$$\begin{aligned}
f'(x) &= \frac{\frac{1}{2}(1+1)x^{-1/2}}{(\sqrt{x}+1)^2} & \left( \frac{ax^n+b}{cx^n+d} \right)' &= \frac{n(ad-bc)x^{n-1}}{(cx+d)^2} \\
&= \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}
\end{aligned}$$

OR

$$\begin{aligned}
f'(x) &= \frac{\frac{1}{2}x^{-1/2}(x^{1/2}+1) - \frac{1}{2}x^{-1/2}(x^{1/2}-1)}{(\sqrt{x}+1)^2} & u &= x^{1/2}-1 & v &= x^{1/2}+1 \\
&= \frac{\frac{1}{2}1+x^{-1/2}-1+x^{-1/2}}{(\sqrt{x}+1)^2} & u' &= \frac{1}{2}x^{-1/2} & v' &= \frac{1}{2}x^{-1/2} \\
&= \frac{\frac{1}{2}2x^{-1/2}}{(\sqrt{x}+1)^2} \\
&= \frac{1}{x^{1/2}(\sqrt{x}+1)^2} \\
&= \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}
\end{aligned}$$

### Exercise

Find the derivative of  $y = \frac{1}{(x^2-1)(x^2+x+1)}$

### Solution

$$y = \frac{1}{x^4 + x^3 + x^2 - x^2 - x - 1}$$

$$= \frac{1}{x^4 + x^3 - x - 1}$$

$$y' = \frac{-(4x^3 + 3x^2 - 1)}{(x^4 + x^3 - x - 1)^2} \quad \left(\frac{1}{v}\right)' = -\frac{v'}{v^2}$$

$$= \frac{-4x^3 - 3x^2 + 1}{(x^4 + x^3 - x - 1)^2}$$

### Exercise

Find the derivative of  $f(x) = \frac{x^{3/2}(x^2 + 1)}{x + 1}$

#### Solution

$$f(x) = \frac{x^{7/2} + x^{3/2}}{x + 1}$$

$$u = x^{7/2} + x^{3/2} \quad v = x + 1$$

$$u' = \frac{7}{2}x^{5/2} + \frac{3}{2}x^{1/2} \quad v' = 1$$

$$f'(x) = \frac{\frac{7}{2}x^{7/2} + \frac{3}{2}x^{3/2} + \frac{7}{2}x^{5/2} + \frac{3}{2}x^{1/2} - x^{7/2} - x^{3/2}}{(x + 1)^2}$$

$$= \frac{\frac{1}{2}(5x^{7/2} + x^{3/2} + 7x^{5/2} + 3x^{1/2})}{(x + 1)^2}$$

### Exercise

Find the derivative of  $f(x) = \frac{x^3 - 4x^2 + x}{x - 2}$

#### Solution

$$f'(x) = \frac{3x^3 - 8x^2 + x - 6x^2 + 16x - 2 - x^3 + 4x^2 - x}{(x - 2)^2}$$

$$u = x^3 - 4x^2 + x \quad v = x - 2$$

$$u' = 3x^2 - 8x + 1 \quad v' = 1$$

$$= \frac{2x^3 - 10x^2 + 16x - 2}{(x - 2)^2}$$



### Exercise

Find the derivative of  $g(x) = \frac{x(3-x)}{2x^2}$

#### Solution

$$g(x) = \frac{1}{2} \frac{3-x}{x}$$

$$g' = -\frac{3}{2x^2}$$

$$\left( \frac{ax+b}{cx+d} \right)' = \frac{ad-bc}{(cx+d)^2}$$

**OR**

$$u = 3-x \quad v = x$$

$$u' = -1 \quad v' = 1$$

$$g'(x) = \frac{1}{2} \frac{-x-3+x}{x^2}$$

$$= -\frac{3}{2x^2}$$

### Exercise

Find the derivative of  $y = \frac{2x^2}{3x+1}$

#### Solution

$$y' = \frac{6x^2+4x}{(3x+1)^2}$$

$$\begin{array}{ccc} 2 & 0 & 0 \\ 0 & 3 & 1 \end{array} \quad \left( \frac{ax^2+bx+c}{dx^2+ex+f} \right)' = \frac{(ae-bd)x^2 + 2(af-cd)x + bf-ce}{(dx^2+ex+f)^2}$$

**OR**

$$u = x^2 \quad v = 3x+1$$

$$u' = 2x \quad v' = 3$$

$$y' = 2 \frac{6x^2+2x-3x^2}{(3x+1)^2}$$

$$= \frac{6x^2+4x}{(3x+1)^2}$$

### Exercise

Find the derivative of  $f(x) = \frac{x^9 + x^8 + 4x^5 - 7x}{x^4 - 3x^2 + 2x + 1}$

### Solution

$$u = x^9 + x^8 + 4x^5 - 7x \quad v = x^4 - 3x^2 + 2x + 1$$

$$u' = 9x^8 + 8x^7 + 20x^4 - 7 \quad v' = 4x^3 - 6x + 2$$

$x^{12}$	$x^{11}$	$x^{10}$	$x^9$	$x^8$	$x^7$	$x^6$	$x^5$	$x^4$	$x^3$	$x^2$	$x$	$x^0$
				9								
			18	16				20				
9	8	-27	-24	20	8	-60	40	-7		21	-14	-7
-12	-4	6	6			24	-8			42	14	
			-2	-16				-28				
				-2								

$$f'(x) = \frac{-3x^{12} + 4x^{11} - 21x^{10} - 2x^9 + 27x^8 + 8x^7 - 36x^6 + 32x^5 - 15x^4 + 63x^2 - 7}{(x^4 - 3x^2 + 2x + 1)^2}$$

### Exercise

Find the derivative of  $f(x) = \frac{x}{1+x^2}$

### Solution

$f' = \frac{1-x^2}{(x^2+1)^2}$	$\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}$	$\left( \frac{ax^2 + bx + c}{dx^2 + ex + f} \right)' = \frac{(ae - bd)x^2 + 2(af - cd)x + bf - ce}{(dx^2 + ex + f)^2}$
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**OR**

$$u = x \quad v = 1 + x^2$$

$$u' = 1 \quad v' = 2x$$

$$f'(x) = \frac{1+x^2 - 2x^2}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2}$$

**Exercise**

Find the derivative of  $y = \frac{x^2 - 2ax + a^2}{x - a}$

**Solution**

$$y = \frac{(x-a)^2}{x-a}$$

$$= x - a$$

$$\underline{y' = 1}$$

**Exercise**

Find the derivative of  $f(x) = \frac{x^2 + 4x^{1/2}}{x^2}$

**Solution**

$$f(x) = 1 + 4x^{-3/2}$$

$$\underline{f'(x) = -6x^{-5/2}}$$

**Exercise**

Find the derivative of  $f(x) = (2x+1)(3x^2+2)$

**Solution**

$$f'(x) = 2(3x^2+2) + (6x)(2x+1)$$

$$= 6x^2 + 4 + 12x^2 + 6x$$

$$\underline{= 18x^2 + 6x + 4}$$

**Exercise**

Find the derivative of  $f(x) = \frac{x^2-1}{x^2+1}$

**Solution**

$$\underline{f' = \frac{-2x^3 + 2x^2 + 4x}{(x^2+1)^2}}$$

$$\begin{array}{ccc} 1 & 0 & -1 \\ 1 & 0 & 1 \end{array} \quad \left( \frac{ax^2+bx+c}{dx^2+ex+f} \right)' = \frac{(ae-bd)x^2 + 2(af-cd)x + bf-ce}{(dx^2+ex+f)^2}$$

**OR**

$$u = x^2 - 1 \quad v = x^2 + 1$$

$$u' = 2x \quad v' = 2x$$

$$f'(x) = \frac{2x^2 + 2x - 2x^3 + 2x}{(x^2 + 1)^2}$$

$$= \frac{-2x^3 + 2x^2 + 4x}{(x^2 + 1)^2}$$

### Exercise

Find the derivative of  $y = \frac{4x^3 + 3x + 1}{2x^5}$

#### Solution

$$y = 2x^{-2} + \frac{3}{2}x^{-4} + \frac{1}{2}x^{-5}$$

$$y' = -4x^{-3} - 6x^{-5} - \frac{5}{2}x^{-6}$$

$$= -\frac{1}{2}x^{-6}(8x^3 + 12x + 5)$$

$$= -\frac{8x^3 + 12x + 5}{2x^6}$$

### Exercise

Find the derivative of  $y = \frac{4}{3-x}$

#### Solution

$$y' = \frac{4}{(3-x)^2}$$

$$\left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$$

### Exercise

Find the derivative of  $y = \frac{2}{1-x^2}$

#### Solution

$$y' = \frac{4x}{(1-x^2)^2}$$

$$\left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$$

### Exercise

Find the derivative of  $f(x) = \frac{\pi}{2 - \pi x}$

#### Solution

$$\underline{f'(x) = \frac{\pi^2}{(2 - \pi x)^2}} \quad \left( \frac{1}{u} \right)' = -\frac{u'}{u^2}$$

### Exercise

Find the derivative of  $y = \frac{x-4}{5x-2}$

#### Solution

$$\underline{y' = \frac{1(-2) - (-4)(5)}{(5x-2)^2} = \frac{18}{(5x-2)^2}} \quad \left( \frac{ax+b}{cx+d} \right)' = \frac{ad-bc}{(cx+d)^2}$$

### Exercise

Find the derivative of  $y = \frac{3x-4}{2x-1}$

#### Solution

$$\underline{y' = \frac{-3+8}{(2x-1)^2} = \frac{5}{(2x-1)^2}} \quad \left( \frac{ax+b}{cx+d} \right)' = \frac{ad-bc}{(cx+d)^2}$$

### Exercise

Find the derivative of  $y = \frac{3x+4}{2x+1}$

#### Solution

$$\underline{y' = \frac{3-8}{(2x+1)^2} = \frac{-5}{(2x+1)^2}} \quad \left( \frac{ax+b}{cx+d} \right)' = \frac{ad-bc}{(cx+d)^2}$$

### ***Exercise***

Find the derivative of  $y = \frac{-3x+4}{2x+1}$

#### **Solution**

$$y' = \frac{-3-8}{(2x+1)^2}$$
$$= -\frac{11}{(2x+1)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

### ***Exercise***

Find the derivative of  $y = \frac{-3x-4}{2x-1}$

#### **Solution**

$$y' = \frac{3+8}{(2x-1)^2}$$
$$= \frac{11}{(2x-1)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

### ***Exercise***

Find the derivative of  $y = \frac{2x-3}{x+1}$

#### **Solution**

$$y' = \frac{2+3}{(x+1)^2}$$
$$= \frac{5}{(x+1)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

### ***Exercise***

Find the derivative of  $y = \frac{3x}{3x-2}$

#### **Solution**

$$y' = \frac{-6}{(3x-2)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

### Exercise

Find the derivative of  $y = \frac{x-3}{2x+5}$

#### Solution

$$y' = \frac{11}{(2x+5)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

### Exercise

Find the derivative of  $y = \frac{5x-3}{2x+5}$

#### Solution

$$y' = \frac{31}{(2x+5)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

### Exercise

Find the derivative of  $y = \frac{6x-8}{2x-3}$

#### Solution

$$y' = -\frac{2}{(2x-3)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

### Exercise

Find the derivative of  $y = \frac{x^2-4}{5x^2-2}$

#### Solution

$$y' = \frac{2(-2+20)x}{(5x^2-2)^2} \\ = \frac{36x}{(5x^2-2)^2}$$

$$\left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{(cx^n+d)^2}$$

### Exercise

Find the derivative of  $y = \frac{3x^2-4}{2x^2-1}$

#### Solution

$$y' = \frac{2(-3+8)x}{(2x^2-1)^2}$$

$$= \frac{10x}{(2x^2-1)^2}$$

$$\left( \frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

### Exercise

Find the derivative of  $y = \frac{3x^2 + 4}{2x^2 + 1}$

#### Solution

$$y' = \frac{2(3-8)x}{(2x^2+1)^2}$$

$$= -\frac{10x}{(2x^2+1)^2}$$

$$\left( \frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

### Exercise

Find the derivative of  $y = \frac{2x^2 - 3}{x^2 + 1}$

#### Solution

$$y' = \frac{2(2+3)x}{(x^2+1)^2}$$

$$= \frac{10x}{(x^2+1)^2}$$

$$\left( \frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

### Exercise

Find the derivative of  $y = \frac{3x^2}{3x^2 - 2}$

#### Solution

$$y' = -\frac{12x}{(3x^2-2)^2}$$

$$\left( \frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$



### Exercise

Find the derivative of  $y = \frac{5x^2 - 3}{2x^2 + 5}$

#### Solution

$$y' = \frac{2(25 + 6)x}{(2x^2 + 5)^2}$$
$$= \frac{62x}{(2x^2 + 5)^2}$$

$$\left( \frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

### Exercise

Find the derivative of  $y = \frac{6x^2 - 8}{2x^2 + 1}$

#### Solution

$$y' = \frac{2(6 + 16)x}{(2x^2 + 1)^2}$$
$$= \frac{44x}{(2x^2 + 1)^2}$$

$$\left( \frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

### Exercise

Find the derivative of  $y = \frac{6x^3 + 8}{2x^3 + 1}$

#### Solution

$$y' = \frac{3(6 - 16)x^2}{(2x^3 + 1)^2}$$
$$= -\frac{30x^2}{(2x^3 + 1)^2}$$

$$\left( \frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

### Exercise

Find the derivative of  $y = \frac{5x^3 - 3}{2x^3 + 5}$

#### Solution

$$y' = \frac{3(25+6)x^2}{(2x^3+5)^2}$$

$$= \frac{93x^2}{(2x^3+1)^2}$$

$$\left( \frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

### Exercise

Find the derivative of  $y = \frac{x^3}{3x^3 - 2}$

#### Solution

$$y' = -\frac{6x^2}{(3x^3 - 2)^2}$$

$$\left( \frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

### Exercise

Find the derivative of  $y = \frac{2x^3 - 3}{2x^3 + 1}$

#### Solution

$$y' = \frac{24x^2}{(2x^3 + 1)^2}$$

$$\left( \frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

### Exercise

Find the derivative of  $y = \frac{2x^4 - 3}{2x^4 + 1}$

#### Solution

$$y' = \frac{4(2+6)x^3}{(2x^4+1)^2}$$

$$= \frac{32x^3}{(2x^4+1)^2}$$

$$\left( \frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

### Exercise

Find the derivative of  $y = \frac{x^2 - 4x + 1}{5x^2 - 2x - 1}$

### Solution

$$y' = \frac{\begin{vmatrix} 1 & -4 \\ 5 & -2 \end{vmatrix} x^2 + 2 \begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} x + \begin{vmatrix} -4 & 1 \\ -2 & -1 \end{vmatrix}}{(5x^2 - 2x - 1)^2}$$
$$= \frac{18x^2 - 12x + 6}{(5x^2 - 2x - 1)^2}$$

$$\frac{d}{dx} \left( \frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

### Exercise

Find the derivative of  $y = \frac{3x^2 - 4x + 2}{2x^2 + x - 1}$

### Solution

$$y' = \frac{\begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} x^2 + 2 \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} x + \begin{vmatrix} -4 & 2 \\ 1 & -1 \end{vmatrix}}{(2x^2 + x - 1)^2}$$
$$= \frac{11x^2 - 14x + 6}{(2x^2 + x - 1)^2}$$

$$\frac{d}{dx} \left( \frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

### Exercise

Find the derivative of  $y = \frac{3x^2 + x - 4}{2x^2 + 1}$

### Solution

$$y' = \frac{\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} x^2 + 2 \begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} x + \begin{vmatrix} 1 & -4 \\ 0 & 1 \end{vmatrix}}{(2x^2 + 1)^2}$$
$$= \frac{2x^2 + 22x + 1}{(2x^2 + 1)^2}$$

$$\frac{d}{dx} \left( \frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

### Exercise

Find the derivative of  $y = \frac{2x^2 - 3}{x^2 + 5x + 1}$

#### Solution

$$y' = \frac{\begin{vmatrix} 2 & 0 \\ 1 & 5 \end{vmatrix} x^2 + 2 \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} x + \begin{vmatrix} 0 & -3 \\ 5 & 1 \end{vmatrix}}{(x^2 + 5x + 1)^2}$$
$$= \frac{10x^2 + 10x + 15}{(x^2 + 5x + 1)^2}$$

$$\frac{d}{dx} \left( \frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

### Exercise

Find the derivative of  $y = \frac{3x^2}{3x^2 + 6x - 8}$

#### Solution

$$y' = \frac{\begin{vmatrix} 3 & 0 \\ 3 & 6 \end{vmatrix} x^2 + 2 \begin{vmatrix} 3 & 0 \\ 3 & -8 \end{vmatrix} x + \begin{vmatrix} 0 & 0 \\ 6 & -8 \end{vmatrix}}{(3x^2 + 6x - 8)^2}$$
$$= \frac{18x^2 - 48x}{(3x^2 + 6x - 8)^2}$$

$$\frac{d}{dx} \left( \frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

### Exercise

Find the derivative of  $y = \frac{x^2 + 2x}{2x^2 + x - 5}$

#### Solution

$$y' = \frac{\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} x^2 + 2 \begin{vmatrix} 1 & 0 \\ 2 & -5 \end{vmatrix} x + \begin{vmatrix} 2 & 0 \\ 1 & -5 \end{vmatrix}}{(2x^2 + x - 5)^2}$$
$$= \frac{-3x^2 - 10x - 10}{(2x^2 + x - 5)^2}$$

$$\frac{d}{dx} \left( \frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

### Exercise

Find the derivative of  $y = \frac{x^2 + 5x + 1}{x^2}$

#### Solution

$$y' = \frac{\begin{vmatrix} 1 & 5 \\ 1 & 0 \end{vmatrix} x^2 + 2 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} x + \begin{vmatrix} 5 & 1 \\ 0 & 0 \end{vmatrix}}{x^4}$$

$$= \frac{-5x^2 - 4x}{x^4}$$

$$= \frac{-5x - 4}{x^3}$$

$$\frac{d}{dx} \left( \frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

### Exercise

Find the derivative of  $y = \frac{x^2 - 3x + 1}{x^2 - 8x + 5}$

#### Solution

$$y' = \frac{\begin{vmatrix} 1 & -3 \\ 1 & -8 \end{vmatrix} x^2 + 2 \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} x + \begin{vmatrix} -3 & 1 \\ -8 & 5 \end{vmatrix}}{(x^2 - 8x + 5)^2}$$

$$= \frac{-5x^2 + 8x - 7}{(x^2 - 8x + 5)^2}$$

$$\frac{d}{dx} \left( \frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

### Exercise

Find the **first** and **second** derivative  $y = \frac{x^2 + 5x - 1}{x^2}$

#### Solution

$$y' = \frac{(2x + 5)x^2 - 2x(x^2 + 5x - 1)}{x^4}$$

$$= \frac{(2x + 5)x^2 - 2x(x^2 + 5x - 1)}{x^4}$$

$$= x \frac{(2x + 5)x - 2(x^2 + 5x - 1)}{x^4}$$

$$\left( \frac{u}{v} \right)' = \frac{u'v - v'u}{v^2}$$

$$u = x^2 + 5x - 1 \quad v = x^2$$

$$u' = 2x + 5 \quad v' = 2x$$

$$= \frac{2x^2 + 5x - 2x^2 - 10x + 2}{x^3}$$

$$= \frac{-5x + 2}{x^3}$$

$$u = -5x + 2 \quad v = x^3$$

$$u' = -5 \quad v' = 3x^2$$

$$y'' = \frac{(-5)x^3 - 3x^2(-5x + 2)}{x^6}$$

$$= x^2 \frac{-5x^3 + 15x - 6}{x^6}$$

$$= \frac{-5x^3 + 15x - 6}{x^4}$$

### Exercise

Find  $y'$ ,  $y''$ ,  $y'''$ :  $y = (x-3)\sqrt{x+2}$

### Solution

$$y' = \sqrt{x+2} + \frac{1}{2}(x-3)(x+2)^{-1/2}$$

$$y'' = \frac{1}{2}(x+2)^{-1/2} + \frac{1}{2}(x+2)^{-1/2} - \frac{1}{4}(x-3)(x+2)^{-3/2}$$

$$= (x+2)^{-1/2} - \frac{1}{4}(x-3)(x+2)^{-3/2}$$

$$y''' = -\frac{1}{2}(x+2)^{-3/2} - \frac{1}{4}(x+2)^{-3/2} + \frac{3}{8}(x-3)(x+2)^{-5/2}$$

$$= -\frac{3}{4}(x+2)^{-3/2} + \frac{3}{8}(x-3)(x+2)^{-5/2}$$

### Exercise

For what value(s) of  $x$  is the line tangent to the curve  $y = x\sqrt{6-x}$  horizontal? Vertical?

### Solution

$$y' = \sqrt{6-x} - \frac{x}{2\sqrt{6-x}}$$

$$= \frac{12-3x}{2\sqrt{6-x}} = 0$$

$$12-3x=0$$

$$\underline{x = 4, \quad y = 4\sqrt{2} \quad |}$$

∴ Point  $(4, 4\sqrt{2})$  is a horizontal tangent line.

∴ The vertical tangent line inside the square root of  $y$ .  $\Rightarrow \underline{x = 6 \quad |}$

$$\begin{aligned} \lim_{x \rightarrow 6} y' &= \lim_{x \rightarrow 6} \frac{12 - 3x}{2\sqrt{6 - x}} \\ &= \frac{-6}{0} \\ &= \underline{-\infty \quad |} \end{aligned}$$

### Exercise

Find an equation of the tangent line to the graph of  $y = \frac{x^2 - 4}{2x + 5}$  when  $x = 0$

#### Solution

$$\underline{f' = \frac{2x^2 + 10x + 8}{(2x + 5)^2} \quad |} \quad \begin{array}{ccc} 1 & 0 & -4 \\ 0 & 2 & 5 \end{array} \quad \left( \frac{ax^2 + bx + c}{dx^2 + ex + f} \right)' = \frac{(ae - bd)x^2 + 2(af - cd)x + bf - ce}{(dx^2 + ex + f)^2}$$

**OR**

$$\begin{aligned} y' &= \frac{(2x + 5)(2x) - (x^2 - 4)(2)}{(2x + 5)^2} \\ &= \frac{4x^2 + 10x - 2x^2 + 8}{(2x + 5)^2} \\ &= \underline{\frac{2x^2 + 10x + 8}{(2x + 5)^2} \quad |} \end{aligned}$$

$$\left( \frac{u}{v} \right)' = \frac{u'v - v'u}{v^2}$$

$$\Rightarrow x = 0 \rightarrow y' = \frac{8}{25} = m$$

$$x = 0 \rightarrow y = \frac{x^2 - 4}{2x + 5} = -\frac{4}{5}$$

$$y = \frac{8}{25}(x - 0) - \frac{4}{5}$$

$$y = m(x - x_1) + y_1$$

$$\underline{y = \frac{8}{25}x - \frac{4}{5} \quad |}$$