Solution Section 4.4 –Eigenvalues & Eigenvectors

Exercise

Find the eigenvalues and eigenvectors of $A,\ A^2,\ A^{-1},\ \ and\ \ A+4I$:

$$A=\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad and \quad A^2=\begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

Check the trace $\lambda_1+\lambda_2$ and the determinant $\lambda_1\lambda_2$ for A and also A^2.

Solution

For A:

$$\det(A-\lambda I)=\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix}$$

$$=(2-\lambda)^2-1$$

$$=\lambda^2-4\lambda+3=0$$

The eigenvalues of A are $\lambda_1=1 \quad and \quad \lambda_2=3$.

The trace of a square matrix A is the sum of the elements on the main diagonal: 2 + 2 agrees with 1+ 3. The det(A) = 3 agrees with the product $\lambda_1\lambda_2$.

The eigenvectors for A are:

$\lambda_1=1$: $\quad (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} 2-1 & -1 \\ -1 & 2-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x-y=0 \\ \end{cases}$$

$$\Rightarrow \ \underline{x=y}$$

Therefore, the eigenvector $V_1=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2=3$: $\quad (A-\lambda_2 I)V_2=0$

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -x-y=0 \\ \end{cases}$$

$$\Rightarrow \ \underline{x=-y}$$

Therefore, the eigenvector $V_2=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

For A^2:

$$\det\left(A^2-\lambda I\right)=\begin{vmatrix}5-\lambda & -4\\ -4 & 5-\lambda\end{vmatrix}$$

$$=(5-\lambda)^2-16$$

$$=\lambda^2-10\lambda+9=0$$

The eigenvalues of A^2 are $\lambda_1=1 \quad and \quad \lambda_2=9$. ***Or*** $\lambda_1=1^2=1 \quad and \quad \lambda_2=3^2=9$

$$\begin{cases} tr(A)=5+5=10\\ \lambda_1+\lambda_2=1+9=10\end{cases}$$

$$\underline{tr(A)=\lambda_1+\lambda_2}$$

$$\begin{cases}\left|A^2\right|=\begin{vmatrix}5 & -4\\ -4 & 5\end{vmatrix}=9\\ \lambda_1\lambda_2=1(9)=9\end{cases}$$

$$\Rightarrow \underline{\left|A^2\right|=\lambda_1\lambda_2}$$

$\lambda_1=1$: $\quad \left(A^2-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix}4 & -4\\ -4 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\rightarrow\begin{cases}4x-4y=0\end{cases}$$

$$\Rightarrow \underline{x=y}$$

Therefore; the eigenvector $V_1=\begin{pmatrix}1\\ 1\end{pmatrix}$

$\lambda_2=9$: $\quad \left(A^2-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix}-4 & -4\\ -4 & -4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\rightarrow\begin{cases}-4x-4y=0\end{cases}$$

$$\Rightarrow \underline{x=-y}$$

Therefore; the eigenvector $V_2=\begin{pmatrix}1\\ -1\end{pmatrix}$

For A^{-1}:

$$A^{-1}=\frac{1}{3}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$$

$$=\begin{pmatrix}\frac{2}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}\end{pmatrix}$$

$$\det\left(A^{-1}-\lambda I\right)=\begin{vmatrix}\frac{2}{3}-\lambda & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}-\lambda\end{vmatrix}$$

$$=\left(\frac{2}{3}-\lambda\right)^2-\frac{1}{9}$$

$$=\lambda^2-\frac{4}{3}\lambda+\frac{1}{3}=0$$

The eigenvalues of A^{-1} are $\lambda_1=1 \quad and \quad \lambda_2=\frac{1}{3}$.

$\lambda_1=1$: $\quad \left(A^{-1}-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix}\frac{2}{3}-1 & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}-1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}-\frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & -\frac{1}{3}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow \begin{cases}-\frac{1}{3}x+\frac{1}{3}y=0\end{cases}$$

$\rightarrow \underline{x=y}$

Therefore; the eigenvector $V_1=\begin{pmatrix}1\\ 1\end{pmatrix}$

$\lambda_2=\frac{1}{3}$: $\quad \left(A^{-1}-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix}\frac{2}{3}-\frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}-\frac{1}{3}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}\frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow \begin{cases}\frac{1}{3}x+\frac{1}{3}y=0\end{cases}$$

$\rightarrow \underline{x=-y}$

Therefore; the eigenvector $V_2=\begin{pmatrix}1\\ -1\end{pmatrix}$

For $A+4I$:

$$A+4I=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}+\begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}$$

$$=\begin{pmatrix}6 & 1\\ 1 & 6\end{pmatrix}$$

$$\det\left(A^{-1}-\lambda I\right)=\begin{vmatrix} 6-\lambda & 1 \\ 1 & 6-\lambda \end{vmatrix}$$

$$=(6-\lambda)^2-1$$

$$=\lambda^2-12\lambda+35=0$$

The eigenvalues of A^{-1} are $\lambda_1=5$ *and* $\lambda_2=7$.

$\lambda_1=5$: $\left(A+4I-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix} 6-5 & 1 \\ 1 & 6-5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x+y=0 \\ \end{cases}$$

$\rightarrow \underline{x=-y}|$

Therefore; the eigenvector $V_1=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2=7$: $\left(A+4I-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \\ x-y=0 \end{cases}$$

$\rightarrow \underline{x=y}|$

Therefore; the eigenvector $V_2=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The eigenvalues $(A)=\lambda$

The eigenvalues $\left(A^2\right)=\lambda^2$

The eigenvalues $\left(A^{-1}\right)=\frac{1}{\lambda}$

Exercise

Show directly that the given vectors are eigenvectors of the given matrix. What are the corresponding eigenvalues?

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$$

Solution

$$A\vec{v}_1 = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ -21 \end{bmatrix}$$

$$= 7\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\underline{= 7\vec{v}_1 \,|}$$

$\vec{v}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue 7

$$A\vec{v}_2 = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= 0\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\underline{= 0\vec{v}_2 \,|}$$

$\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 0

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -2 \\ -3 & 6-\lambda \end{vmatrix}$$

$$= (1-\lambda)(6-\lambda) - 6$$

$$= 6 - 7\lambda + \lambda^2 - 6$$

$$= \lambda^2 - 7\lambda = 0$$

The eigenvalues are: $\lambda_1 = 0$ *and* $\lambda_2 = 7$

Exercise

For which real numbers c does this matrix A have

$$A = \begin{pmatrix} 2 & -c \\ -1 & 2 \end{pmatrix}$$

a) Two real eigenvalues and eigenvectors.

b) A repeated eigenvalue with only one eigenvector

c) Two complex eigenvalues and eigenvectors.

Solution

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 2-\lambda & -c \\ -1 & 2-\lambda \end{vmatrix} \\ &= (2-\lambda)^2 - c \\ &= \lambda^2 - 4\lambda + 4 - c = 0 \end{aligned}$$

$$\lambda = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-c)}}{2(1)}$$

$$\boxed{\lambda_{1,2} = \frac{4 \pm \sqrt{16+4c}}{2}}$$

a) Two real eigenvalues and eigenvectors, when

$$16 + 4c > 0$$

$$4c > -16$$

$$\boxed{c > -4}$$

b) A repeated eigenvalue with only one eigenvector, when

$$16 + 4c = 0$$

$$\boxed{c = -4}$$

c) Two complex eigenvalues and eigenvectors, when

$$16 + 4c < 0$$

$$\boxed{c < -4}$$

Exercise

Find the eigenvalues of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{AB}$, and $\boldsymbol{BA}$:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad AB = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad BA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

a) The eigenvalues of $\boldsymbol{AB}$ (are equal to) (are not equal to) eigenvalues of $\boldsymbol{A}$ times eigenvalues of $\boldsymbol{B}$.

b) The eigenvalues of $\boldsymbol{AB}$ (are equal to) (are not equal to) eigenvalues of $\boldsymbol{BA}$.

Solution

Since $\boldsymbol{A}$ is a lower triangular, then $\lambda_1 = \lambda_2 = 1$

Since $\boldsymbol{B}$ is an upper triangular, then $\lambda_1 = \lambda_2 = 1$

$$\det(AB - I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)(2-\lambda) - 1$$

$$= \lambda^2 - 3\lambda + 1 = 0$$

$$\lambda_{1,2} = \frac{3 \pm \sqrt{5}}{2}$$

$$\det(BA - I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)(1-\lambda) - 1$$

$$= \lambda^2 - 3\lambda + 1 = 0$$

$$\lambda_{1,2} = \frac{3 \pm \sqrt{5}}{2}$$

a) The eigenvalues of $\boldsymbol{AB}$ are ***not*** equal to eigenvalues of $\boldsymbol{A}$ times eigenvalues of $\boldsymbol{B}$.

b) The eigenvalues of $\boldsymbol{AB}$ are equal to the eigenvalues of $\boldsymbol{BA}$.

Exercise

When $a+b=c+d$ show that (1, 1) is an eigenvector and find both eigenvalues of

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Solution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}=\begin{pmatrix} a+b \\ c+d \end{pmatrix}$$

$$=\begin{pmatrix} a+b \\ a+b \end{pmatrix}$$

$$=(a+b)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

If $a+b=c+d=\lambda_1$

$tr(A)=a+d=\lambda_1+\lambda_2$

$$\lambda_2=(a+d)-\lambda_1$$

$$=a+d-(a+b)$$

$$=a+d-a-b$$

$$=d-b \qquad \text{or} \quad =a-c$$

The eigenvalues for λ_2:

$$\begin{pmatrix} a-\lambda_2 & b \\ c & d-\lambda_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a-(a-c) & b \\ c & d-(d-b) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} c & b \\ c & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}\to\{cx+by=0$$

$$cx=-by$$

The eigenvector: $V_2=\begin{pmatrix} b \\ -c \end{pmatrix}$

Exercise

The eigenvalues of A equal to the eigenvalues of A^T. This is because $\det(A-\lambda I)$ equals $\det\left(A^T-\lambda I\right)$.

That is true because ______. Show by an example that the eigenvectors of A and A^T are not the same.

Solution

$$\begin{aligned}\det(A-\lambda I) &= \det(A-\lambda I)^T \\ &= \det\left(A^T-(\lambda I)^T\right) \\ &= \det\left(A^T-\lambda I\right)\end{aligned}$$

Therefore, A and A^T have the same eigenvalues.

Let consider the matrix:

$$A=\begin{pmatrix}0 & 1\\ 4 & 0\end{pmatrix} \quad\Rightarrow\quad A^T=\begin{pmatrix}0 & 4\\ 1 & 0\end{pmatrix}$$

$$\begin{aligned}|A-\lambda I| &= \begin{vmatrix}-\lambda & 1\\ 4 & -\lambda\end{vmatrix} \\ &= \lambda^2-4=0\end{aligned}$$

The eigenvalues of A are: $\boxed{\lambda_{1,2}=\pm 2}$

For $\lambda_1=-2$: $\quad \left(A-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix}2 & 1\\ 4 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\rightarrow\begin{cases}2x+y=0\\ \end{cases}$$

$\boxed{y=-2x}$

$$V_1=\begin{pmatrix}1\\ -2\end{pmatrix}$$

For $\lambda_2=2$: $\quad \left(A-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix}-2 & 1\\ 4 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\rightarrow\begin{cases}-2x+y=0\\ \end{cases}$$

$\boxed{y=2x}$

$$V_2=\begin{pmatrix}1\\ 2\end{pmatrix}$$

For the transpose matrix A^T

$$\left|A^T-\lambda I\right|=\begin{vmatrix}-\lambda & 4\\ 1 & -\lambda\end{vmatrix}$$

$$= \lambda^2 - 4 = 0$$

The eigenvalues of A^T are: $\boxed{\lambda_{1,2} = \pm 2}$

For $\lambda_1 = -2$: $\quad \left(A^T - \lambda_1 I\right) V_3 = 0$

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{cases} \\ x + 2y = 0 \end{cases}$$

$\boxed{x = -2y}$

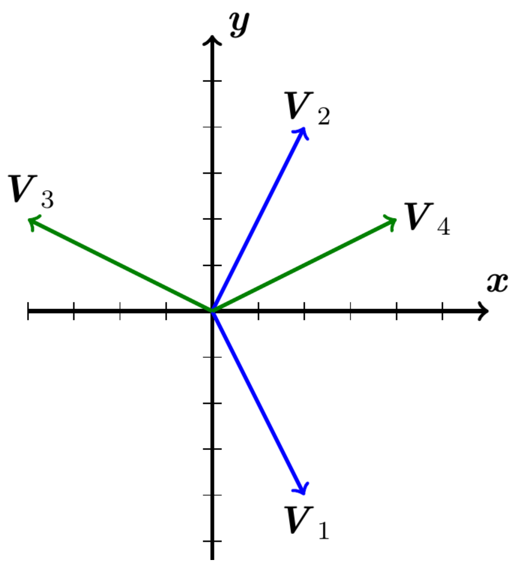
$$V_3 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2$: $\quad \left(A^T - \lambda_2 I\right) V_4 = 0$

$$\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{cases} \\ x - 2y = 0 \end{cases}$$

$\boxed{x = 2y}$

$$V_4 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$



$$V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad V_3 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad V_4 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

The eigenvectors of A and A^T are not the same and from the graph they are not on same line.

Exercise

Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$. Compute the eigenvalues and eigenvectors of A.

Solution

$$\det\left(A - \lambda I\right) = \begin{vmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)^2 + 1 = 0$$

$$(2-\lambda)^2 = -1$$

$$2 - \lambda = \pm\sqrt{-1}$$

$$\boxed{= \pm i}$$

The eigenvalues of A are: $\boxed{\lambda_{1,2} = 2 \pm i}$

For $\lambda_1 = 2 - i \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2-(2-i) & -1 \\ 1 & 2-(2-i) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \\ x + iy = 0 \end{cases}$$

$\underline{x = -iy}|$

The eigenvector is: $V_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

For $\lambda_2 = 2 + i \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \\ x - iy = 0 \end{cases}$$

$\underline{x = iy}|$

The eigenvector is: $V_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

Exercise

Let $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

a) What is the characteristic polynomial for A (i.e. compute $\det(A - \lambda I)$?

b) Verify that 1 is an eigenvalue of A. What is a corresponding eigenvector?

c) What are the other eigenvalues of A?

Solution

a) $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix}$

$$= (2-\lambda)(1-\lambda)(-1-\lambda) - 2 + 9 - 3(1-\lambda) - 3(2-\lambda) + 2(-1-\lambda)$$

$$= \left(2 - 3\lambda + \lambda^2\right)(-1-\lambda) + 7 - 3 + 3\lambda - 6 + 3\lambda - 2 - 2\lambda$$

$$= -2 + 3\lambda - \lambda^2 - 2\lambda + 3\lambda^2 - \lambda^3 + 4\lambda - 4$$

$$\underline{= -\lambda^3 + 2\lambda^2 + 5\lambda - 6}|$$

b) *If* $\lambda = 1 \quad \to -\lambda^3 + 2\lambda^2 + 5\lambda - 6 \overset{?}{=} 0$

$$-1^3 + 2(1)^2 + 5(1) - 6 \overset{?}{=} 0$$

$$-1 + 2 + 5 - 6 \overset{?}{=} 0$$

$$\boxed{0 = 0}$$

1 is an eigenvalue of A.

$$\begin{pmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \to \begin{cases} x - 2y + 3z = 0 \\ x \qquad\quad + z = 0 \\ x + 3y - 2z = 0 \end{cases}$$

$$\begin{cases} \\ \qquad \underline{x = -z} \\ 3y = 2z - x = 2z + z = 3z \quad \Rightarrow \underline{y = z} \end{cases}$$

The eigenvector for $\lambda = 1$ is $V = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

c) $-\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0$

$\underline{\lambda_1 = 1 \quad \lambda_2 = -2 \quad \lambda_3 = 3}$

Exercise

For the matrix: $\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$$

i. $\det\left(A - \lambda I\right) = \begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix}$

$$= (3-\lambda)(-1-\lambda) - 0$$

$$= \lambda^2 - 2\lambda - 3$$

The characteristic equation: $\underline{\lambda^2 - 2\lambda - 3}$

ii. $\lambda^2 - 2\lambda - 3 = 0$

The eigenvalues are $\lambda_1 = -1$ *and* $\lambda_2 = 3$

iii. $\lambda_1 = -1 \rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & 0 \\ 8 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 4x = 0 \\ \end{cases}$$

$\underline{x = 0}$

Therefore, the eigenvector $V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda_2 = 3 \rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 0 & 0 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \\ 8x - 4y = 0 \end{cases}$$

$\underline{2x = y}$

Therefore; the eigenvector $V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

The eigenvectors are given by: $V = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$

Exercise

For the matrix: $\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$$

i. $\det(A - \lambda I) = \begin{vmatrix} 10-\lambda & -9 \\ 4 & -2-\lambda \end{vmatrix}$

$$= (10 - \lambda)(-2 - \lambda) + 36$$
$$= \lambda^2 - 8\lambda + 16$$

The characteristic equation: $\underline{\lambda^2 - 8\lambda + 16 = 0}$

ii. $\lambda^2 - 8\lambda + 16 = 0$

The eigenvalues are $\underline{\lambda_{1,2} = 4}$

iii. $\lambda_1 = 4 \quad \rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 4x - 6y = 0 \end{cases}$$

$\underline{2x = 3y}$

Therefore; the eigenvector $\underline{V_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}}$

For the second eigenvector $V_2 \quad \Rightarrow \quad AV_2 = V_1$

$$\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 4x - 2y = 2 \end{cases}$$

$\underline{y = 2x - 1}$

If $x = 1 \quad \Rightarrow \quad y = 1$

$\underline{V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

Exercise

For the matrix: $\begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix}$$

i. $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 3 \\ 4 & -\lambda \end{vmatrix}$

$$= \lambda^2 - 12$$

The characteristic equation: $\boxed{\lambda^2 - 12 = 0}$

ii. $\lambda^2 - 12 = 0$

The eigenvalues are $\boxed{\lambda_{1,2} = \pm\sqrt{12}}$

iii. For $\lambda_1 = -\sqrt{12} \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} \sqrt{12} & 3 \\ 4 & \sqrt{12} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \sqrt{12}x + 3y = 0 \\ \end{cases}$$

$\rightarrow \boxed{\sqrt{12}x = -3y}$

Therefore; the eigenvector $V_1 = \begin{pmatrix} -3 \\ \sqrt{12} \end{pmatrix}$

For $\lambda_2 = \sqrt{12} \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -\sqrt{12} & 3 \\ 4 & -\sqrt{12} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -\sqrt{12}x + 3y = 0 \\ \end{cases}$$

$\rightarrow \boxed{\sqrt{12}x = 3y}$

Therefore; the eigenvector $V_2 = \begin{pmatrix} 3 \\ \sqrt{12} \end{pmatrix}$

The vectors are given by: $V = \begin{pmatrix} -3 & 3 \\ \sqrt{12} & \sqrt{12} \end{pmatrix}$

Exercise

For the matrix: $\begin{pmatrix} -2 & -7 \\ 1 & 2 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -2 & -7 \\ 1 & 2 \end{pmatrix}$$

i. $\begin{vmatrix} -2-\lambda & -7 \\ 1 & 2-\lambda \end{vmatrix} = (-2-\lambda)(2-\lambda) + 7$

$$= -4 + \lambda^2 + 7$$
$$= \lambda^2 + 3$$

The characteristic equation: $\underline{\lambda^2 + 3 = 0}$

ii. $\lambda^2 = -3$

The eigenvalues are: $\underline{\lambda_{1,2} = \pm i\sqrt{3}}$

iii. For $\lambda_1 = -i\sqrt{3} \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -2 + i\sqrt{3} & -7 \\ 1 & 2 + i\sqrt{3} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \\ x_1 + (2 + i\sqrt{3})y_1 = 0 \end{cases}$$

$$\underline{x_1 = -(2 + i\sqrt{3})y_1}$$

Therefore; the eigenvector $V_1 = \begin{pmatrix} 2 + i\sqrt{3} \\ -1 \end{pmatrix}$

For $\lambda_2 = i\sqrt{3} \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -2 - i\sqrt{3} & -7 \\ 1 & 2 - i\sqrt{3} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \\ x_2 + (2 - i\sqrt{3})y_2 = 0 \end{cases}$$

$$\underline{x_2 = -(2 - i\sqrt{3})y_2}$$

Therefore; the eigenvector $V_2 = \begin{pmatrix} 2 - i\sqrt{3} \\ -1 \end{pmatrix}$

The vectors are given by: $V = \begin{pmatrix} 2 + i\sqrt{3} & 2 - i\sqrt{3} \\ -1 & -1 \end{pmatrix}$

Exercise

For the matrix: $\begin{pmatrix} 12 & 14 \\ -7 & -9 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 12 & 14 \\ -7 & -9 \end{pmatrix}$$

i. $\left|A-\lambda I\right| = \begin{vmatrix} 12-\lambda & 14 \\ -7 & -9-\lambda \end{vmatrix}$

$$= (12-\lambda)(-9-\lambda) - (14)(-7)$$
$$= -108 - 12\lambda + 9\lambda + \lambda^2 + 98$$
$$= \lambda^2 - 3\lambda - 10$$

The characteristic equation: $\boxed{\lambda^2 - 3\lambda - 10 = 0}$

ii. The eigenvalues are: $\boxed{\lambda_1 = -2 \ \ and \ \ \lambda_2 = 5}$

iii. For $\lambda_1 = -2$, we have: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 12+2 & 14 \\ -7 & -9+2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 14 \\ -7 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 14x + 14y = 0 \\ \end{cases}$$

$\boxed{x = -y}$

$$\Rightarrow \quad \boxed{V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = 5$, we have $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 7 & 14 \\ -7 & -14 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 7x + 14y = 0 \\ \end{cases}$$

$\boxed{x = -2y}$

$$\Rightarrow \quad \boxed{V_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$$

The vectors are given by: $V = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$

Exercise

For the matrix: $\begin{pmatrix} -4 & 1 \\ -2 & 1 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -4 & 1 \\ -2 & 1 \end{pmatrix}$$

i. $\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix}$

$$= (-4-\lambda)(1-\lambda) + 2$$

$$= \lambda^2 + 3\lambda - 2$$

The characteristic equation: $\boxed{\lambda^2 + 3\lambda - 2 = 0}$

ii. Thus, the eigenvalues are: $\boxed{\lambda_{1,2} = \frac{-3 \pm \sqrt{17}}{2}}$

iii. For $\lambda_1 = \frac{-3-\sqrt{17}}{2}$: $\quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -4 - \frac{-3-\sqrt{17}}{2} & 1 \\ -2 & 1 - \frac{-3-\sqrt{17}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{-5+\sqrt{17}}{2} & 1 \\ -2 & \frac{5+\sqrt{17}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} \frac{-5+\sqrt{17}}{2} x + y = 0 \\ -2x + \frac{5+\sqrt{17}}{2} y = 0 \end{cases}$$

$$\boxed{x = \left(\frac{5+\sqrt{17}}{4} \right) y}$$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} \frac{5+\sqrt{17}}{4} \\ 1 \end{pmatrix}}$$

For $\lambda_2 = \frac{-3+\sqrt{17}}{2}$: $\left(A - \lambda_2 I\right) V_2 = 0$

$$\begin{pmatrix} -4 - \frac{-3+\sqrt{17}}{2} & 1 \\ -2 & 1 - \frac{-3+\sqrt{17}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{-5-\sqrt{17}}{2} & 1 \\ -2 & \frac{5-\sqrt{17}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} \frac{-5-\sqrt{17}}{2} x + y = 0 \\ -2x + \frac{5-\sqrt{17}}{2} y = 0 \end{cases}$$

$$\boxed{x = \left(\frac{5-\sqrt{17}}{4} \right) y}$$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} \frac{5-\sqrt{17}}{4} \\ 1 \end{pmatrix}}$$

The eigenvectors can be written: $\begin{pmatrix} \frac{5+\sqrt{17}}{4} & \frac{5-\sqrt{17}}{4} \\ 1 & 1 \end{pmatrix}$

Exercise

For the matrix: $\begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$$

i. $\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 3 \\ -6 & -4-\lambda \end{vmatrix}$

$$= (-4-\lambda)(5-\lambda) + 18$$

$$= \lambda^2 - \lambda - 2$$

The characteristic equation: $\lambda^2 - \lambda - 2 = 0$

ii. Thus, the eigenvalues are: $\lambda_1 = -1$ *and* $\lambda_2 = 2$

iii. For $\lambda_1 = -1$: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 6 & 3 \\ -6 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 6x + 3y = 0 \\ -6x - 3y = 0 \end{cases}$$

$y = -2x$

$$\Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

For $\lambda_2 = 2 \;\Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 3 & 3 \\ -6 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 3x + 3y = 0 \\ \end{cases}$$

$y = -x$

$$\Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} -2 & 3 \\ 0 & -5 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -2 & 3 \\ 0 & -5 \end{pmatrix}$$

i. $\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 3 \\ 0 & -5-\lambda \end{vmatrix}$

$$= (-2-\lambda)(-5-\lambda) - 0$$

$$= (2+\lambda)(5+\lambda)$$

The characteristic equation: $(2+\lambda)(5+\lambda) = 0$

ii. Thus, the eigenvalues are: $\lambda_1 = -5 \ \ and \ \ \lambda_2 = -2$

iii. For $\lambda_1 = -5$: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 3x + 3y = 0 \\ \end{cases}$$

$y = -x$

$$\Rightarrow \ V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\lambda_2 = -2$: $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 0 & 3 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 3y = 0 \\ -3y = 0 \end{cases}$$

$y = 0$

$$\Rightarrow \ V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 2 & 0 \\ -4 & -2 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 2 & 0 \\ -4 & -2 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 \\ -4 & -2-\lambda \end{vmatrix}$

$$= (2-\lambda)(-2-\lambda) - 0$$

$$= \lambda^2 - 4$$

The characteristic equation: $\boxed{\lambda^2 - 4 = 0}$

ii. Thus, the eigenvalues are: $\boxed{\lambda_1 = -2 \ \ and \ \ \lambda_2 = 2}$

iii. For $\lambda_1 = -2$: $\quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & 0 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 4x = 0 \\ \end{cases}$$

$\boxed{x = 0}$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = 2$: $\quad (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 0 & 0 \\ -4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} \\ -4x - 4y = 0 \end{cases}$$

$\boxed{x = -y}$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix}$

$$= \lambda^2 - 4\lambda - 5$$

The characteristic equation: $\boxed{\lambda^2 - 4\lambda - 5 = 0}$

ii. The eigenvalues are: $\boxed{\lambda_1 = -1 \ \ and \ \ \lambda_2 = 5}$

iii. For $\lambda_1 = -1$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} 2x + 2y = 0 \\ \end{cases}$$

$\boxed{x = -y}$

$$\Rightarrow \ \boxed{V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

For $\lambda_2 = 5$: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} -4x + 2y = 0 \\ \end{cases}$$

$\boxed{2x = y}$

$$\Rightarrow \ \boxed{V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix}$

$$= \lambda^2 - 5\lambda + 4$$

The characteristic equation: $\underline{\lambda^2 - 5\lambda + 4 = 0}$

ii. The eigenvalues are: $\underline{\lambda_1 = 1 \;\; and \;\; \lambda_2 = 4}$

iii. For $\lambda_1 = 1$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} x + 2y = 0 \\ \end{cases}$$

$\underline{x = -2y}$

$$\Rightarrow \; V_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 4 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} \\ x - y = 0 \end{cases}$$

$\underline{x = y}$

$$\Rightarrow \; V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} -4 & 2 \\ -\frac{5}{2} & 2 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -4 & 2 \\ -\frac{5}{2} & 2 \end{pmatrix}$$

i. $$|A - \lambda I| = \begin{vmatrix} -4-\lambda & 2 \\ -\frac{5}{2} & 2-\lambda \end{vmatrix}$$

$$= \lambda^2 + 2\lambda - 3$$

The characteristic equation: $\boxed{\lambda^2 + 2\lambda - 3 = 0}$

ii. The eigenvalues are: $\boxed{\lambda_1 = -3 \ \textit{and} \ \lambda_2 = 1}$

iii. For $\lambda_1 = -3$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -1 & 2 \\ -\frac{5}{2} & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} -x + 2y = 0 \end{cases}$$

$\boxed{x = 2y}$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = 1$: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -5 & 2 \\ -\frac{5}{2} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} -5x + 2y = 0 \end{cases}$$

$\boxed{5x = 2y}$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} -\frac{5}{2} & 2 \\ \frac{3}{4} & -2 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -\frac{5}{2} & 2 \\ \frac{3}{4} & -2 \end{pmatrix}$$

i. $$|A - \lambda I| = \begin{vmatrix} -\frac{5}{2} - \lambda & 2 \\ \frac{3}{4} & -2 - \lambda \end{vmatrix}$$

$$= \lambda^2 + \frac{9}{2}\lambda + \frac{7}{2}$$

The characteristic equation: $\underline{2\lambda^2 + 9\lambda + 7 = 0}$

ii. The eigenvalues are: $\lambda_1 = -\frac{7}{2}$ *and* $\lambda_2 = -1$

iii. For $\lambda_1 = -\frac{7}{2}$: $\left(A - \lambda_1 I\right) V_1 = 0$

$$\begin{pmatrix} 1 & 2 \\ \frac{3}{4} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} x + 2y = 0 \\ \end{cases}$$

$x = -2y$

$$\Rightarrow \quad V_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

For $\lambda_1 = -1$: $\left(A - \lambda_1 I\right) V_1 = 0$

$$\begin{pmatrix} -\frac{3}{2} & 2 \\ \frac{3}{4} & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} -\frac{3}{2}x + 2y = 0 \\ \end{cases}$$

$3x = 4y$

$$\Rightarrow \quad V_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 \\ 9 & -3-\lambda \end{vmatrix}$

$$= (3-\lambda)(-3-\lambda) + 9$$

$$= \lambda^2$$

The characteristic equation: $\lambda^2 = 0$

ii. The eigenvalues are: $\lambda_{1,2} = 0$

iii. For $\lambda_1 = 0$: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} 3x - y = 0 \end{cases}$$

$3x = y$

$$\Rightarrow V_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

For the second eigenvector V_2 $\Rightarrow AV_2 = V_1$

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \to \quad \begin{cases} 3x - y = 1 \end{cases}$$

$\to if \ x = 1 \Rightarrow y = 2$

$$\Rightarrow V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}$$

i. $$|A - \lambda I| = \begin{vmatrix} -6-\lambda & 5 \\ -5 & 4-\lambda \end{vmatrix}$$

$$= -24 + 2\lambda + \lambda^2 + 25$$

$$= \lambda^2 + 2\lambda + 1$$

The characteristic equation: $\lambda^2 + 2\lambda + 1 = 0$

ii. The eigenvalues are: $\lambda_{1,2} = -1$

iii. For $\lambda_1 = 0$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -5 & 5 \\ -5 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} -5x + 5y = 0 \\ \end{cases}$$

$x = y$

$$\Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For the second eigenvector $V_2 \quad \Rightarrow \quad AV_2 = V_1$

$$\begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \to \quad \begin{cases} -6x_2 + 5y_2 = 1 \\ \end{cases}$$

$\to if \ x_2 = 0 \quad \to \quad y_2 = \frac{1}{5}$

$$\Rightarrow \quad V_2 = \begin{pmatrix} 0 \\ \frac{1}{5} \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 6 & -1 \\ 5 & 2 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 6 & -1 \\ 5 & 2 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 6-\lambda & -1 \\ 5 & 2-\lambda \end{vmatrix}$

$$= \lambda^2 - 8\lambda + 17$$

The characteristic equation: $\lambda^2 - 8\lambda + 17 = 0$

ii. $\lambda_{1,2} = \dfrac{8 \pm \sqrt{64-68}}{2}$

The eigenvalues are: $\lambda_{1,2} = 4 \pm i$

iii. For $\lambda_1 = 4 - i$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 2-i & -1 \\ 5 & -2-i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} (2-i)x - y = 0 \\ \end{cases}$$

$$(2-i)x = y$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$$

$$\Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 2+i \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ -2 & -1-\lambda \end{vmatrix}$

$$= \lambda^2 + 1$$

The characteristic equation: $\lambda^2 + 1 = 0$

ii. $\lambda^2 = -1$

The eigenvalues are: $\lambda_{1,2} = \pm i$

iii. For $\lambda_1 = -i$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1+i & 1 \\ -2 & -1+i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} (1+i)x + y = 0 \\ \end{cases}$$

$(1+i)x = -y$

$$\Rightarrow \quad V_1 = \begin{pmatrix} -1 \\ 1+i \end{pmatrix}$$

$$\Rightarrow \quad V_2 = \begin{pmatrix} -1 \\ 1-i \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 5-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix}$

$$= 15 - 8\lambda + \lambda^2 + 2$$

$$= \lambda^2 - 8\lambda + 17$$

The characteristic equation: $\boxed{\lambda^2 - 8\lambda + 17 = 0}$

ii. $\lambda_{1,2} = \dfrac{8 \pm \sqrt{64-68}}{2}$

The eigenvalues are: $\boxed{\lambda_{1,2} = 4 \pm i}$

iii. For $\lambda_1 = 4 - i$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1+i & 1 \\ -2 & -1+i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} (1+i)x + y = 0 \\ \end{cases}$$

$$\boxed{(1+i)x = -y}$$

$$\Rightarrow \quad \boxed{V_1 = \begin{pmatrix} -1 \\ 1+i \end{pmatrix}}$$

$$\Rightarrow \quad \boxed{V_2 = \begin{pmatrix} -1 \\ 1-i \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 4 & 5 \\ -2 & 6 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 4 & 5 \\ -2 & 6 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 4-\lambda & 5 \\ -2 & 6-\lambda \end{vmatrix}$

$$= 24 - 10\lambda + \lambda^2 + 10$$

$$= \lambda^2 - 10\lambda + 34$$

The characteristic equation: $\boxed{\lambda^2 - 10\lambda + 34 = 0}$

ii. $\lambda_{1,2} = \dfrac{10 \pm \sqrt{100-136}}{2}$

The eigenvalues are: $\boxed{\lambda_{1,2} = 5 \pm 3i}$

iii. For $\lambda_1 = 5 - 3i$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -1+3i & 5 \\ -2 & 1+3i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} (-1+3i)x + 5y = 0 \\ \end{cases}$$

$$\boxed{(-1+3i)x = -y}$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} -1 \\ -1+3i \end{pmatrix}$$

$$\Rightarrow \quad V_2 = \begin{pmatrix} -1 \\ -1-3i \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 5-\lambda & -4 \\ 2 & -1-\lambda \end{vmatrix}$

$$= -5 - 4\lambda + \lambda^2 + 8$$

$$= \lambda^2 - 4\lambda + 3$$

The characteristic equation: $\underline{\lambda^2 - 4\lambda + 3 = 0}$

ii. The eigenvalues are: $\underline{\lambda_{1,2} = 1,\ 3}$

iii. For $\lambda_1 = 1$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} 4x - 4y = 0 \\ \end{cases}$$

$\underline{x = y}$

$$\Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 3$: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} 2x - 4y = 0 \\ \end{cases}$$

$\underline{x = 2y}$

$$\Rightarrow \quad V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 6 & -6 \\ 4 & -4 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 6 & -6 \\ 4 & -4 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 6-\lambda & -6 \\ 4 & -4-\lambda \end{vmatrix}$

$$= -24 - 2\lambda + \lambda^2 + 24$$

$$= \lambda^2 - 2\lambda$$

The characteristic equation: $\lambda^2 - 2\lambda = 0$

ii. The eigenvalues are: $\lambda_{1,2} = 0,\ 2$

iii. For $\lambda_1 = 0$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 6 & -6 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 6x - 6y = 0 \\ \end{cases}$$

$x = y$

$$\Rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2$: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 4 & -6 \\ 4 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 4x - 6y = 0 \\ \end{cases}$$

$2x = 3y$

$$\Rightarrow V_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 5 & -3 \\ 2 & 0 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 5 & -3 \\ 2 & 0 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 5-\lambda & -3 \\ 2 & -\lambda \end{vmatrix}$

$$= \lambda^2 - 5\lambda + 6$$

The characteristic equation: $\boxed{\lambda^2 - 5\lambda + 6 = 0}$

ii. The eigenvalues are: $\boxed{\lambda_{1,2} = 2,\ 3}$

iii. For $\lambda_1 = 2$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 3x - 3y = 0 \\ \end{cases}$$

$\boxed{x = y}$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = 3$: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 2 & -3 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 2x - 3y = 0 \\ \end{cases}$$

$\boxed{2x = 3y}$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 5 & -4 \\ 3 & -2 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 5 & -4 \\ 3 & -2 \end{pmatrix}$$

i. $$|A - \lambda I| = \begin{vmatrix} 5-\lambda & -4 \\ 3 & -2-\lambda \end{vmatrix}$$

$$= -10 - 3\lambda + \lambda^2 + 12$$

$$= \lambda^2 - 3\lambda + 2$$

The characteristic equation: $\underline{\lambda^2 - 3\lambda + 2 = 0}$

ii. The eigenvalues are: $\underline{\lambda_{1,2} = 1,\ 2}$

iii. For $\lambda_1 = 1$: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & -4 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 4x - 4y = 0 \\ \end{cases}$$

$\underline{x = y}$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

For $\lambda_1 = 2$: $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 3 & -4 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 3x - 4y = 0 \\ \end{cases}$$

$\underline{3x = 4y}$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 6 & -10 \\ 2 & -3 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 6 & -10 \\ 2 & -3 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 6-\lambda & -10 \\ 2 & -3-\lambda \end{vmatrix}$

$$= -18 - 3\lambda + \lambda^2 + 20$$

$$= \lambda^2 - 3\lambda + 2$$

The characteristic equation: $\lambda^2 - 3\lambda + 2 = 0$

ii. The eigenvalues are: $\lambda_{1,2} = 1,\ 2$

iii. For $\lambda_1 = 1$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 5 & -10 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} \\ 2x - 4y = 0 \end{cases}$$

$x = 2y$

$$\Rightarrow \ V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda_1 = 2$: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 4 & -10 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} \\ 2x - 5y = 0 \end{cases}$$

$2x = 5y$

$$\Rightarrow \ V_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 11 & -15 \\ 6 & -8 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 11 & -15 \\ 6 & -8 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 11-\lambda & -15 \\ 6 & -8-\lambda \end{vmatrix}$

$$= -88 - 3\lambda + \lambda^2 + 90$$

$$= \lambda^2 - 3\lambda + 2$$

The characteristic equation: $\boxed{\lambda^2 - 3\lambda + 2 = 0}$

ii. The eigenvalues are: $\boxed{\lambda_{1,2} = 1,\ 2}$

iii. For $\lambda_1 = 1$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 10 & -15 \\ 6 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} \\ 6x - 9y = 0 \end{cases}$$

$\boxed{2x = 3y}$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}}$$

For $\lambda_1 = 2$: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 9 & -15 \\ 6 & -10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} \\ 6x - 10y = 0 \end{cases}$$

$\boxed{3x = 5y}$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

i. $$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= 9 - 6\lambda + \lambda^2 - 1$$

$$= \lambda^2 - 6\lambda + 8$$

The characteristic equation: $\lambda^2 - 6\lambda + 8 = 0$

ii. $$\lambda_{1,2} = \frac{6 \pm \sqrt{36-32}}{2}$$

$$= \frac{6 \pm 2}{2}$$

The eigenvalues are: $\lambda_1 = 2$ & $\lambda_2 = 4$

iii. For $\lambda_1 = 2$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} x + y = 0 \\ \end{cases}$$

$x = -y$

$$\Rightarrow \quad V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 4$: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} -x + y = 0 \\ \end{cases}$$

$x = y$

$$\Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 9 & 2 \\ 2 & 6 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 9 & 2 \\ 2 & 6 \end{pmatrix}$$

i. $$|A - \lambda I| = \begin{vmatrix} 9-\lambda & 2 \\ 2 & 6-\lambda \end{vmatrix}$$

$$= 54 - 15\lambda + \lambda^2 - 4$$

$$= \lambda^2 - 15\lambda + 50$$

The characteristic equation: $\lambda^2 - 15\lambda + 50 = 0$

ii. $$\lambda_{1,2} = \frac{15 \pm \sqrt{225 - 200}}{2}$$

$$= \frac{15 \pm 5}{2}$$

The eigenvalues are: $\lambda_1 = 5$ & $\lambda_2 = 10$

iii. For $\lambda_1 = 5$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} \\ 2x + y = 0 \end{cases}$$

$2x = -y$

$$\Rightarrow \ V_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = 10$: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} -x + 2y = 0 \\ \end{cases}$$

$x = 2y$

$$\Rightarrow \ V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 13 & 4 \\ 4 & 7 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 13 & 4 \\ 4 & 7 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 13-\lambda & 4 \\ 4 & 7-\lambda \end{vmatrix}$

$$= 91 - 20\lambda + \lambda^2 - 16$$

$$= \lambda^2 - 20\lambda + 75$$

The characteristic equation: $\lambda^2 - 20\lambda + 75 = 0$

ii. $\lambda_{1,2} = \dfrac{20 \pm \sqrt{400-300}}{2}$

$$= \frac{20 \pm 10}{2}$$

The eigenvalues are: $\lambda_1 = 5$ & $\lambda_2 = 15$

iii. For $\lambda_1 = 5$: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} \\ 4x + 2y = 0 \end{cases}$$

$2x = -y$

$$\Rightarrow V_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = 15$: $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} -2x + 4y = 0 \\ \end{cases}$$

$x = 2y$

$$\Rightarrow V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 5 & -1 \\ 3 & -1 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 5 & -1 \\ 3 & -1 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 5-\lambda & -1 \\ 3 & -1-\lambda \end{vmatrix}$

$$= -5 - 4\lambda + \lambda^2 + 3$$

$$= \lambda^2 - 4\lambda - 2$$

The characteristic equation: $\lambda^2 - 4\lambda - 2 = 0$

ii. $\lambda_{1,2} = \dfrac{4 \pm \sqrt{16+8}}{2}$

$$= \frac{4 \pm 2\sqrt{6}}{2}$$

The eigenvalues are: $\lambda_{1,2} = 2 \pm \sqrt{6}$

iii. For $\lambda_1 = 2 - \sqrt{6}$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 3+\sqrt{6} & -1 \\ 3 & -3+\sqrt{6} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} \left(3+\sqrt{6}\right)x - y = 0 \\ \end{cases}$$

$\left(3+\sqrt{6}\right)x = y$

$$\Rightarrow V_1 = \begin{pmatrix} 1 \\ 3+\sqrt{6} \end{pmatrix}$$

For $\lambda_2 = 2 + \sqrt{6}$: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 3-\sqrt{6} & -1 \\ 3 & -3-\sqrt{6} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} \left(3-\sqrt{6}\right)x - y = 0 \\ \end{cases}$$

$\left(3-\sqrt{6}\right)x = y$

$$\Rightarrow V_2 = \begin{pmatrix} 1 \\ 3-\sqrt{6} \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$$

i. $$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix}$$

$$= -2 + \lambda + \lambda^2 - 4$$

$$= \lambda^2 + \lambda - 6$$

The characteristic equation: $\lambda^2 + \lambda - 6 = 0$

ii. $$\lambda_{1,2} = \frac{-1 \pm \sqrt{1+24}}{2}$$

The eigenvalues are: $\lambda_1 = -3$ & $\lambda_2 = 2$

iii. For $\lambda_1 = -3$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} 4x + y = 0 \end{cases}$$

$4x = -y$

$$\Rightarrow \quad V_1 = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} -x + y = 0 \end{cases}$$

$x = y$

$$\Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$$

i. $$\left|A - \lambda I\right| = \begin{vmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix}$$

$$= 1 + 2\lambda + \lambda^2 + 4$$

$$= \lambda^2 + 2\lambda + 5$$

The characteristic equation: $\lambda^2 + 2\lambda + 5 = 0$

ii. $$\lambda_{1,2} = \frac{-2 \pm \sqrt{4-20}}{2}$$

The eigenvalues are: $\lambda_{1,2} = -1 \pm 2i$

iii. For $\lambda_1 = -1 - 2i$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 2i & -4 \\ 1 & 2i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \\ x + 2iy = 0 \end{cases}$$

$x = -2iy$

$$\Rightarrow V_1 = \begin{pmatrix} -2i \\ 1 \end{pmatrix}$$

$$\Rightarrow V_2 = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} -1 & 0 & 0 \\ 2 & -5 & -6 \\ -2 & 3 & 4 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -5 & -6 \\ -2 & 3 & 4 \end{pmatrix}$$

i. $$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 0 & 0 \\ 2 & -5-\lambda & -6 \\ -2 & 3 & 4-\lambda \end{vmatrix}$$

$$= (-1-\lambda)(-5-\lambda)(4-\lambda) + 18(-1-\lambda)$$

$$= \left(5 + 6\lambda + \lambda^2\right)(4-\lambda) - 18 - 18\lambda$$

$$= 20 - 5\lambda + 24\lambda - 6\lambda^2 + 4\lambda^2 - \lambda^3 - 18 - 18\lambda$$

$$= -\lambda^3 - 2\lambda^2 + \lambda + 2$$

The characteristic equation: $\boxed{\lambda^3 + 2\lambda^2 - \lambda - 2 = 0}$

ii. $\lambda = 1$

$$\begin{array}{c|cccc} 1 & 1 & 2 & -1 & -2 \\ & & 1 & 3 & 2 \\ \hline & 1 & 3 & 2 & 0 \end{array} \quad \to \quad \lambda^2 + 3\lambda + 2 = 0$$

The eigenvalues are: $\boxed{\lambda_1 = -2 \quad \lambda_2 = -1 \quad and \quad \lambda_3 = 1}$

iii. For $\lambda_1 = -2$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & -6 \\ -2 & 3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \to \begin{cases} \boxed{x = 0} \\ 2x - 3y - 6z = 0 & (1) \\ -2x + 3y + 6z = 0 \end{cases}$$

$(1) \to \quad \boxed{y = -2z}$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = -1 \quad \Rightarrow \left(A - \lambda_2 I\right) V_2 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & -4 & -6 \\ -2 & 3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} 0 = 0 & \\ 2x - 4y - 6z = 0 & (1) \\ -2x + 3y + 5z = 0 & (2) \end{cases}$$

$(1) + (2) \to \quad -y - z = 0 \quad \Rightarrow \underline{y = -z}$

$(1) \to \quad 2x = -4z + 6z \quad \Rightarrow \underline{x = z}$

$$\Rightarrow V_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

For $\lambda_3 = 1 \quad \Rightarrow \left(A - \lambda_3 I\right) V_3 = 0$

$$\begin{pmatrix} -2 & 0 & 0 \\ 2 & -6 & -6 \\ -2 & 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} -2x = 0 & \to \underline{x = 0} \\ 2x - 6y - 6z = 0 & (1) \\ -2x + 3y + 3z = 0 & (2) \end{cases}$$

$(1) + (2) \to \quad -3y - 3z = 0 \quad \Rightarrow \underline{y = -z}$

$$\Rightarrow V_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

The vectors are given by: $V = \begin{pmatrix} 0 & 1 & 0 \\ -2 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

Exercise

For the matrix: $\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

i. $\left|A-\lambda I\right| = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & -1-\lambda \end{vmatrix}$

$$= -\left(1-\lambda^2\right)(2-\lambda)$$

The characteristic equation: $\boxed{\left(1-\lambda^2\right)(2-\lambda) = 0}$

ii. The eigenvalues are: $\boxed{\lambda_1 = -1 \quad \lambda_2 = 1 \quad and \quad \lambda_3 = 2}$

iii. For $\lambda_1 = -1$: $\left(A-\lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 2x+y-z=0 & (1) \\ \boxed{y=0} & \\ & \end{cases}$$

$(1) \rightarrow \boxed{2x = z}$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}}$$

r $\lambda_2 = 1$: $\left(A-\lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} y-z=0 & \boxed{z=0} \\ \boxed{y=0} & \\ & \end{cases}$$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

For $\lambda_3 = 2$: $\left(A-\lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} -x+y-z=0 & (1) \\ & \\ \boxed{y=3z} & \end{cases}$$

$(1) \rightarrow \boxed{x = 2z}$

$$\Rightarrow \boxed{V_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{pmatrix}$$

i. $$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -7 & 0 \\ 5 & 10-\lambda & 4 \\ 0 & 5 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)^2(10-\lambda) - 20(2-\lambda) + 35(2-\lambda)$$

$$= (2-\lambda)\big((10-\lambda)(2-\lambda) + 15\big)$$

$$= (2-\lambda)\left(35 - 12\lambda + \lambda^2\right)$$

The characteristic equation: $(2-\lambda)\left(\lambda^2 - 12\lambda + 35\right) = 0$

ii. $$\lambda = \frac{12 \pm \sqrt{144-140}}{2}$$

$$= \frac{12 \pm 2}{2}$$

The eigenvalues are: $\lambda_1 = 2 \quad \lambda_2 = 5 \quad and \quad \lambda_3 = 7$

iii. For $\lambda_1 = 2$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 0 & -7 & 0 \\ 5 & 8 & 4 \\ 0 & 5 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} y = 0 \\ 5x + 8y + 4z = 0 \quad\quad 5x = -4z \end{cases}$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix}$$

For $\lambda_2 = 5$: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -5 & -7 & 0 \\ 5 & 3 & 4 \\ 0 & 5 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} -5x-7y=0 & \to \ \underline{-5x=7y} \\ \\ 5y-5z=0 & \to \ \underline{z=y} \end{cases}$$

$$\Rightarrow \quad V_2 = \begin{pmatrix} 7 \\ -5 \\ -5 \end{pmatrix}$$

For $\lambda_3 = 7$: $\quad \left(A-\lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} 0 & -7 & 0 \\ 5 & 8 & 4 \\ 0 & 5 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} \underline{y=0} \\ 5x+8y+4z=0 & \to \ \underline{5x=-4z} \\ \end{cases}$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

i. $\quad |A-\lambda I| = \begin{vmatrix} 3-\lambda & -1 & -1 \\ 1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix}$

$$= \left(1-2\lambda+\lambda^2\right)(3-\lambda)+2+2-2\lambda-3+\lambda$$

$$= 3-7\lambda+5\lambda^2-\lambda^3+1-\lambda$$

$$= -\lambda^3+5\lambda^2-8\lambda+4$$

The characteristic equation: $\quad \underline{-\lambda^3+5\lambda^2-8\lambda+4=0}$

ii. $\lambda = 1$

$$\begin{array}{c|cccc} 1 & -1 & 5 & -8 & 4 \\ & & -1 & 4 & -4 \\ \hline & -1 & 4 & -4 & 0 \end{array} \quad \rightarrow \underline{-\lambda^2 + 4\lambda - 4 = 0}|$$

The eigenvalues are: $\underline{\lambda_1 = 1 \ \ and \ \ \lambda_{2,3} = 2}|$

iii. For $\lambda_1 = 1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 2x - y + z = 0 & \\ x - z = 0 & \rightarrow \underline{x = z}| \\ x - y = 0 & \rightarrow \underline{x = y}| \end{cases}$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}|$$

For $\lambda_2 = 2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} x - y - z = 0 \quad (1) \\ \\ \end{cases}$$

$(1) \rightarrow \ \underline{x = y + z}| \quad let \ \ \underline{y = 0}|$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}|$$

$(1) \rightarrow \quad let \ \ \underline{z = 0}| \quad \underline{x = y}|$

$$\Rightarrow \underline{V_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}|$$

Exercise

For the matrix: $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

i. $$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}$$

$$= (-\lambda)(3-\lambda)^2 + 16 + 16 + 16\lambda - 4(3-\lambda) - 4(3-\lambda)$$

$$= -9\lambda + 6\lambda^2 - \lambda^3 + 32 + 16\lambda - 12 + 4\lambda - 12 + 4\lambda$$

$$= -\lambda^3 + 6\lambda^2 + 15\lambda + 8$$

The characteristic equation: $\underline{-\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0}$

ii. $\lambda = -1$

$$\begin{array}{r|rrrr} -1 & -1 & 6 & 15 & 8 \\ & & 1 & -7 & -8 \\ \hline & -1 & 7 & 8 & 0 \end{array} \quad \rightarrow \quad \underline{-\lambda^2 + 7\lambda + 8 = 0}$$

The eigenvalues are: $\underline{\lambda_{1,2} = -1 \ \ and \ \ \lambda_3 = 8}$

iii. For $\lambda_{1,2} = -1 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 2x + y + 2z = 0 & (1) \\ \end{cases}$$

Assume $z = 0 \quad \rightarrow \quad \underline{2x = -y}$

$$\Rightarrow \quad V_1 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

Assume $y = 0 \quad \rightarrow \quad \underline{x = -z}$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

For $\lambda_3 = 8 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 5x - 2y - 4z = 0 & (1) \\ x - 4y + z = 0 & (2) \\ 4x + 2y - 5z = 0 & (3) \end{cases}$$

$(1) + (3) \rightarrow \quad 9x - 9z = 0$

$\underline{x = z}$

$(2) \rightarrow \quad 4y = 2z$

Assume $\quad z = 2 = x \quad \rightarrow \quad \underline{y = 1}$

$$\Rightarrow \underline{V_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ -8 & -5 & -3-\lambda \end{vmatrix}$

$$= \left(1 - 2\lambda + \lambda^2\right)(-3 - \lambda) - 2 + 3 - 3\lambda + 6 + 2\lambda$$

$$= -\lambda^3 - \lambda^2 + 4\lambda + 4$$

The characteristic equation: $\quad \underline{-\lambda^3 - \lambda^2 + 4\lambda + 4 = 0}$

ii. $\lambda = -1$

$$\begin{array}{r|rrrr} -1 & -1 & -1 & 4 & 4 \\ & & 1 & 0 & -4 \\ \hline & -1 & 0 & 4 & 0 \end{array} \quad \to \underline{-\lambda^2 + 4 = 0}$$

The eigenvalues are: $\underline{\lambda_1 = -2,\ \lambda_2 = -1,\ and\ \lambda_3 = 2}$

iii. For $\lambda_1 = -2 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 3 & -1 \\ -8 & -5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} 3x + y + z = 0 \\ 2x + 3y - z = 0 \\ -8x - 5y - z = 0 \end{cases}$$

Assume $z = 1 \quad \to \begin{cases} 3x + y = -1 \\ 2x + 3y = 1 \end{cases}$

$$\Delta = \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} = 7 \quad \Delta_x = \begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix} = -4 \quad \Delta_y = \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} = 5$$

$$x = -\frac{4}{7} \quad y = \frac{5}{7} \quad z = 1$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} -4 \\ 5 \\ 7 \end{pmatrix}}$$

For $\lambda_2 = -1 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ -8 & -5 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} 2x + y + z = 0 \\ 2x + 2y - z = 0 \\ -8x - 5y - 2z = 0 \end{cases}$$

$z = 1 \quad \to \begin{cases} 2x + y = -1 \\ 2x + 2y = 1 \end{cases}$

$$\Delta = \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = 2 \quad \Delta_x = \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} = -3 \quad \Delta_y = \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} = 4$$

$$x = -\frac{3}{2} \quad y = 2 \quad z = 1$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}}$$

For $\lambda_3 = 2 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -8 & -5 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} -x+y+z=0 \\ 2x-y-z=0 \\ -8x-5y-5z=0 \end{cases}$$

$$x=0 \quad \to \quad \begin{cases} y+z=0 \\ -y-z=0 \end{cases}$$

$$\underline{y=-z}$$

$$\Rightarrow \quad V_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

i. $$|A-\lambda I| = \begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix}$$

$$= -\left(1-\lambda^2\right)(2-\lambda)+14-16+8\lambda+1-\lambda-3-3\lambda$$

$$= -\lambda^3+2\lambda^2+5\lambda-6$$

The characteristic equation: $\underline{-\lambda^3+2\lambda^2+5\lambda-6=0}$

ii. $\lambda = 1$

$$\begin{array}{c|cccc} 1 & -1 & 2 & 5 & -6 \\ & & -1 & 1 & 6 \\ \hline & -1 & 1 & 6 & 0 \end{array} \quad \to \quad \underline{-\lambda^2+\lambda+6=0}$$

The eigenvalues are: $\underline{\lambda_1 = -2, \ \lambda_2 = 1, \ and \ \lambda_3 = 3}$

iii. For $\lambda_1 = -2 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} 3x - y + 4z = 0 \\ 3x + 4y - z = 0 \\ 2x + y + z = 0 \end{cases}$$

Assume $z = 1 \quad \to \begin{cases} 3x - y = -4 \\ 3x + 4y = 1 \end{cases}$

$$\Delta = \begin{vmatrix} 3 & -1 \\ 3 & 4 \end{vmatrix} = 15 \quad \Delta_x = \begin{vmatrix} -4 & -1 \\ 1 & 4 \end{vmatrix} = -15 \quad \Delta_y = \begin{vmatrix} 3 & -4 \\ 3 & 1 \end{vmatrix} = 15$$

$\underline{x = -1 \quad y = 1}|$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = 1 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} -y + 4z = 0 \\ 3x + y - z = 0 \\ 2x + y - 2z = 0 \end{cases}$$

Assume $z = 1 \quad \to \begin{cases} \underline{y = 4}| \\ 3x + y = 1 \quad \to \underline{x = -1}| \end{cases}$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}}$$

For $\lambda_3 = 3 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} -2x - y + 4z = 0 \\ 3x - y - z = 0 \\ 2x + y - 4z = 0 \end{cases}$$

Assume $z = 1 \quad \to \begin{cases} -2x - y = -4 \\ 3x - y = 1 \end{cases}$

$$\Delta = \begin{vmatrix} -2 & -1 \\ 3 & -1 \end{vmatrix} = 5 \quad \Delta_x = \begin{vmatrix} -4 & -1 \\ 1 & -1 \end{vmatrix} = 5 \quad \Delta_y = \begin{vmatrix} -2 & -4 \\ 3 & 1 \end{vmatrix} = 10$$

$\underline{x = -1 \quad y = 2}|$

$$\Rightarrow \boxed{V_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix}$

$$= \left(1 - 2\lambda + \lambda^2\right)(3-\lambda) - (3-\lambda)$$

$$= (3-\lambda)\left(\lambda^2 - 2\lambda\right)$$

The characteristic equation: $\underline{(3-\lambda)\left(\lambda^2 - 2\lambda\right) = 0}$

ii. The eigenvalues are: $\underline{\lambda_1 = 0, \ \lambda_2 = 2, \ and \ \lambda_3 = 3}$

iii. For $\lambda_1 = 0 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} x + y = 0 \\ 3z = 0 \end{cases}$$

$\underline{x = -y}$ $\underline{z = 0}$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}$$

For $\lambda_2 = 2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} -x + y = 0 \quad \rightarrow \underline{x = y} \\ \underline{z = 0} \end{cases}$$

$$\Rightarrow \quad \boxed{V_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$$

For $\lambda_3 = 3 \quad \Rightarrow \left(A - \lambda_3 I\right) V_3 = 0$

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 2x = y \\ x = 2y \\ \end{cases}$$

$\boxed{x = y = 0}$

$$\Rightarrow \quad \boxed{V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} -\frac{5}{2} & 1 & 1 \\ 1 & -\frac{5}{2} & 1 \\ 1 & 1 & -\frac{5}{2} \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -\frac{5}{2} & 1 & 1 \\ 1 & -\frac{5}{2} & 1 \\ 1 & 1 & -\frac{5}{2} \end{pmatrix}$$

i. $\left|A - \lambda I\right| = \begin{vmatrix} -\frac{5}{2} - \lambda & 1 & 1 \\ 1 & -\frac{5}{2} - \lambda & 1 \\ 1 & 1 & -\frac{5}{2} - \lambda \end{vmatrix}$

$$= -\left(\frac{5}{2} + \lambda\right)^3 + 2 + 3\left(\frac{5}{2} + \lambda\right)$$

$$= -\frac{49}{8} - \frac{63}{4}\lambda - \frac{15}{2}\lambda^2 - \lambda^3$$

$$-\frac{49}{8}-\frac{63}{4}\lambda-\frac{15}{2}\lambda^2-\lambda^3=0$$

The characteristic equation: $\underline{8\lambda^3+60\lambda^2+126\lambda+49=0}$

ii. $\lambda=-\frac{1}{2}$

$$\begin{array}{c|cccc} -\frac{1}{2} & 8 & 60 & 126 & 49 \\ & & -4 & -28 & -49 \\ \hline & 8 & 56 & 98 & 0 \end{array} \quad \to \underline{8\lambda^2+56\lambda+98=0}$$

$$4\lambda^2+28\lambda+49=0$$

$$\lambda=\frac{-28\pm\sqrt{784-784}}{8}$$

The eigenvalues are: $\underline{\lambda_1=-\frac{1}{2} \quad \& \quad \lambda_{2,3}=-\frac{7}{2}}$

iii. For $\lambda_1=-\frac{7}{2} \quad \Rightarrow \left(A-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \quad \to x+y+z=1$$

Assume $z=0 \quad \to x+y=1$

$y=1 \quad \Rightarrow \underline{x=-1}$

$$\to \underline{V_1=\begin{pmatrix}-1\\1\\0\end{pmatrix}}$$

Assume $y=0 \quad \to x+z=1$

$z=1 \quad \Rightarrow \underline{x=-1}$

$$\to \underline{V_2=\begin{pmatrix}-1\\0\\1\end{pmatrix}}$$

For $\lambda_3=-\frac{1}{2} \quad \Rightarrow \left(A-\lambda_3 I\right)V_3=0$

$$\begin{pmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \quad \to \begin{array}{r}-2x+y+z=0\\x-2y+z=0\\x+y-2z=0\end{array}$$

Assume $z=1 \quad \rightarrow \begin{cases} -2x+y=-1 \\ x-2y=-1 \end{cases}$

$$\Delta = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 3 \quad \Delta_x = \begin{vmatrix} -2 & -1 \\ 1 & -1 \end{vmatrix} = 3$$

$$\underline{x=1} \quad y=-1+2 \ \underline{=1}$$

$$\rightarrow \ V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

i. $\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix}$

$$=(4-\lambda)(1-\lambda)(1-\lambda)+2(1-\lambda)$$

$$=(1-\lambda)\left[(4-\lambda)(1-\lambda)+2\right]$$

$$=(1-\lambda)\left(\lambda^2-5\lambda+6\right)$$

The characteristic equation: $\underline{-\lambda^3+6\lambda^2-11\lambda+6=0}$

ii. $-\lambda^3+6\lambda^2-11\lambda+6=0 \quad \rightarrow \quad \lambda_1=1$

$$\begin{array}{c|cccc} 1 & -1 & 6 & -11 & 6 \\ & & -1 & 5 & -6 \\ \hline & -1 & 5 & -6 & 0 \end{array} \quad \rightarrow \quad \underline{-\lambda^2+5\lambda-6=0}$$

Therefore; the eigenvalues are: $\boxed{\lambda_{1,2,3} = 1,\ 2,\ 3}$

iii. For $\lambda_1 = 1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 3x_1 + x_3 = 0 & (1) \\ -2x_1 = 0 & (2) \end{cases}$$

$(2) \quad \Rightarrow \boxed{x_1 = 0}$

$(1) \quad \Rightarrow \boxed{x_3 = x_1 = 0}$

Therefore; the eigenvector $\boxed{V_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$

For $\lambda_2 = 2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} 2x_1 + x_3 = 0 & \Rightarrow \ x_3 = -2x_1 \\ -2x_1 - x_2 = 0 & \Rightarrow \ x_2 = -2x_1 \\ -2x_1 - x_3 = 0 \end{cases}$$

Therefore; the eigenvector $\boxed{V_2 = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}}$

For $\lambda_3 = 3 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} x_1 + x_3 = 0 & \Rightarrow \ x_3 = -x_1 \\ -2x_1 - 2x_2 = 0 & \Rightarrow \ x_2 = -x_1 \\ -2x_1 - 2x_3 = 0 \end{cases}$$

Therefore; the eigenvector $\boxed{V_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}}$

Exercise

For the matrix: $\begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$$

i. $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & -5 \\ \frac{1}{5} & -1-\lambda & 0 \\ 1 & 1 & -2-\lambda \end{vmatrix}$

$$= (3-\lambda)(-1-\lambda)(-2-\lambda) - 1 + 5(-1-\lambda)$$

$$= (3-\lambda)\left(\lambda^2 + 3\lambda + 2\right) - 1 - 5 - 5\lambda$$

$$= 3\lambda^2 + 9\lambda + 6 - \lambda^3 - 3\lambda^2 - 2\lambda - 6 - 5\lambda$$

$$= -\lambda^3 + 2\lambda$$

The characteristic equation: $-\lambda^3 + 2\lambda = 0$

ii. $-\lambda\left(\lambda^2 - 2\right) = 0$

Therefore; the eigenvalues are: $\lambda_{1,2,3} = 0, \ \pm\sqrt{2}$

iii. For $\lambda_1 = -\sqrt{2} \quad \Rightarrow \quad \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 3+\sqrt{2} & 0 & -5 \\ \frac{1}{5} & -1+\sqrt{2} & 0 \\ 1 & 1 & -2+\sqrt{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \left(3+\sqrt{2}\right)x_1 - 5x_3 = 0 & \Rightarrow \quad x_3 = \frac{3+\sqrt{2}}{5}x_1 \\ \frac{1}{5}x_1 + \left(-1+\sqrt{2}\right)x_2 = 0 & \Rightarrow \quad x_2 = -\frac{1}{5\left(-1+\sqrt{2}\right)}x_1 \\ x_1 + x_2 + \left(-2+\sqrt{2}\right)x_3 = 0 \end{cases}$$

Therefore; the eigenvector $V_1 = \begin{pmatrix} 1 \\ \frac{1}{5\left(1-\sqrt{2}\right)} \\ \frac{3+\sqrt{2}}{5} \end{pmatrix}$

For $\lambda_2 = 0 \quad \Rightarrow \left(A - \lambda_2 I\right) V_2 = 0$

$$\begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 3x_1 - 5x_3 = 0 & \Rightarrow \quad x_3 = \frac{3}{5} x_1 \\ \frac{1}{5} x_1 - x_2 = 0 & \Rightarrow \quad x_2 = \frac{1}{5} x_1 \\ x_1 + x_2 - 2x_3 = 0 & \end{cases} \Rightarrow \begin{cases} x_3 = \frac{3}{5} x_1 \\ x_2 = \frac{1}{5} x_1 \end{cases}$$

Therefore; the eigenvector $V_2 = \begin{pmatrix} 5 \\ \frac{1}{5} \\ \frac{3}{5} \end{pmatrix}$

For $\lambda_3 = \sqrt{2} \quad \Rightarrow \left(A - \lambda_3 I\right) V_3 = 0$

$$\begin{pmatrix} 3-\sqrt{2} & 0 & -5 \\ \frac{1}{5} & -1-\sqrt{2} & 0 \\ 1 & 1 & -2-\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \left(3-\sqrt{2}\right) x_1 - 5x_3 = 0 & \Rightarrow \quad x_3 = \frac{3-\sqrt{2}}{5} x_1 \\ \frac{1}{5} x_1 + \left(-1-\sqrt{2}\right) x_2 = 0 & \Rightarrow \quad x_2 = \frac{1}{5\left(1+\sqrt{2}\right)} x_1 \\ x_1 + x_2 + \left(-2-\sqrt{2}\right) x_3 = 0 & \end{cases}$$

Therefore; the eigenvector $V_3 = \begin{pmatrix} 1 \\ \frac{1}{5\left(1+\sqrt{2}\right)} \\ \frac{3-\sqrt{2}}{5} \end{pmatrix}$

Exercise

For the matrix: $\begin{pmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{pmatrix}$$

i. $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & -5 \\ \frac{1}{5} & -1-\lambda & 0 \\ 1 & 1 & -2-\lambda \end{vmatrix}$

$$= (3-\lambda)(-1-\lambda)(-2-\lambda) - 1 + 5(-1-\lambda)$$

$$= (3-\lambda)\left(\lambda^2 + 3\lambda + 2\right) - 1 - 5 - 5\lambda$$

$$= 3\lambda^2 + 9\lambda + 6 - \lambda^3 - 3\lambda^2 - 2\lambda - 6 - 5\lambda$$

$$= -\lambda^3 + 2\lambda$$

The characteristic equation: $-\lambda^3 \ 2\lambda = 0$

ii. $-\lambda\left(\lambda^2 - 2\right) = 0$

Therefore; the eigenvalues are: $\lambda_{1,2,3} = 0, \ \pm\sqrt{2}$

iv. For $\lambda_1 = -\sqrt{2} \quad \Rightarrow \ \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 3+\sqrt{2} & 0 & -5 \\ \frac{1}{5} & -1+\sqrt{2} & 0 \\ 1 & 1 & -2+\sqrt{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \left(3+\sqrt{2}\right)x_1 - 5x_3 = 0 & \Rightarrow \quad x_3 = \frac{3+\sqrt{2}}{5}x_1 \\ \frac{1}{5}x_1 + \left(-1+\sqrt{2}\right)x_2 = 0 & \Rightarrow \quad x_2 = -\frac{1}{5\left(-1+\sqrt{2}\right)}x_1 \\ x_1 + x_2 + \left(-2+\sqrt{2}\right)x_3 = 0 \end{cases}$$

Therefore; the eigenvector $V_1 = \begin{pmatrix} 1 \\ \frac{1}{5\left(1-\sqrt{2}\right)} \\ \frac{3+\sqrt{2}}{5} \end{pmatrix}$

For $\lambda_2 = 0 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 3x_1 - 5x_3 = 0 & \Rightarrow \underline{x_3 = \frac{3}{5}x_1} \\ \frac{1}{5}x_1 - x_2 = 0 & \Rightarrow \underline{x_2 = \frac{1}{5}x_1} \\ x_1 + x_2 - 2x_3 = 0 \end{cases}$$

Therefore; the eigenvector $V_2 = \begin{pmatrix} 5 \\ \frac{1}{5} \\ \frac{3}{5} \end{pmatrix}$

For $\lambda_3 = \sqrt{2} \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} 3-\sqrt{2} & 0 & -5 \\ \frac{1}{5} & -1-\sqrt{2} & 0 \\ 1 & 1 & -2-\sqrt{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \left(3-\sqrt{2}\right)x_1 - 5x_3 = 0 & \Rightarrow \underline{x_3 = \frac{3-\sqrt{2}}{5}x_1} \\ \frac{1}{5}x_1 + \left(-1-\sqrt{2}\right)x_2 = 0 & \Rightarrow \underline{x_2 = \frac{1}{5\left(1+\sqrt{2}\right)}x_1} \\ x_1 + x_2 + \left(-2-\sqrt{2}\right)x_3 = 0 \end{cases}$$

Therefore; the eigenvector $V_3 = \begin{pmatrix} 1 \\ \frac{1}{5\left(1+\sqrt{2}\right)} \\ \frac{3-\sqrt{2}}{5} \end{pmatrix}$

Exercise

For the matrix: $\begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$$

i. $\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 0 & 1 \\ -1 & 3-\lambda & 0 \\ -4 & 13 & -1-\lambda \end{vmatrix}$

$$= (-1-\lambda)^2(3-\lambda) - 13 + 4(3-\lambda)$$
$$= \left(\lambda^2 + 2\lambda + 1\right)(3-\lambda) - 13 + 12 - 4\lambda$$
$$= 3\lambda^2 + 6\lambda + 3 - \lambda^3 - 2\lambda^2 - \lambda - 1 - 4\lambda$$
$$= -\lambda^3 + \lambda^2 + \lambda + 2$$

The characteristic equation: $\underline{-\lambda^3 + \lambda^2 + \lambda + 2 = 0}$

ii. $\lambda = 2$

$$\begin{array}{c|cccc} 2 & -1 & 1 & 1 & 2 \\ & & -2 & -2 & 2 \\ \hline & -1 & -1 & -1 & 0 \end{array} \quad \rightarrow \quad \underline{-\lambda^2 - \lambda - 1 = 0}$$

Therefore; the eigenvalues are: $\underline{\lambda_{1,2,3} = 2, \ -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}}$

iii. For $\lambda_1 = 2$, we have: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -3 & 0 & 1 \\ -1 & 1 & 0 \\ -4 & 13 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -3x_1 + z_1 = 0 & \Rightarrow \quad \underline{z_1 = 3x_1} \\ -x_1 + y_1 = 0 & \Rightarrow \quad \underline{y_1 = x_1} \\ -4x_1 + 13y_1 - 3z_1 = 0 \end{cases}$$

Therefore, the eigenvector $V_1 = \boxed{\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}}$

For $\lambda_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, we have: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -\frac{1}{2}+i\frac{\sqrt{3}}{2} & 0 & 1 \\ -1 & \frac{7}{2}+i\frac{\sqrt{3}}{2} & 0 \\ -4 & 13 & -\frac{1}{2}+i\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow$$

$$\begin{cases} \left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x_2 + z_2 = 0 & \Rightarrow \quad z_2 = -\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x_2 \\ -x_2 + \left(\frac{7}{2}+i\frac{\sqrt{3}}{2}\right)y_2 = 0 & \Rightarrow \quad y_2 = \left(\frac{2}{7+i\sqrt{3}}\right)x_2 \\ -4x_2 + 13y_2 + \left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)z_2 = 0 & \end{cases}$$

Therefore; the eigenvector $V_2 = \begin{pmatrix} 1 \\ \frac{1-i\sqrt{3}}{2} \\ \frac{2}{7+i\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1-i\sqrt{3}}{2} \\ \frac{7-i\sqrt{3}}{26} \end{pmatrix}$

$$V_3 = \begin{pmatrix} 1 \\ \frac{1+i\sqrt{3}}{2} \\ \frac{2}{7-i\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1+i\sqrt{3}}{2} \\ \frac{7+i\sqrt{3}}{26} \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ -8 & -4 & -3 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ -8 & -4 & -3 \end{pmatrix}$$

i. $\left|A - \lambda I\right| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 4 & 3-\lambda & 2 \\ -8 & -4 & -3-\lambda \end{vmatrix}$

$$= (1-\lambda)(3-\lambda)(-3-\lambda) + 8(1-\lambda)$$

$$= -9 + 9\lambda + \lambda^2 - \lambda^3 + 8 - 8\lambda$$

$$= -\lambda^3 + \lambda^2 + \lambda - 1$$

The characteristic equation: $-\lambda^3 + \lambda^2 + \lambda - 1 = 0$

ii. $\lambda = 1$

$$\begin{array}{c|cccc} 1 & -1 & 1 & 1 & -1 \\ & & -1 & 0 & 1 \\ \hline & -1 & 0 & 1 & 0 \end{array} \quad \rightarrow \quad -x^2 + 1 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2,3} = 1,\ 1,\ -1$

iii. For $\lambda_1 = 1 \quad \Rightarrow \quad \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 4 & 2 & 2 \\ -8 & -4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 0 = 0 & \\ 4x + 2y + 2z = 0 & \Rightarrow \quad 2x = -y - z \\ -8x - 4y - 4z = 0 & \Rightarrow \quad 2x = -y - z \end{cases}$$

If $x = 0 \quad \Rightarrow \quad y = -z$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}\Big|$$

For $\lambda_2 = -1 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 2 & 0 & 0 \\ 4 & 4 & 2 \\ -8 & -4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x = 0 & \Rightarrow \quad \underline{x = 0}\,| \\ 4x + 4y + 2z = 0 & \Rightarrow \quad \underline{z = -2y}\,| \\ -8x - 4y - 2z = 0 \end{cases}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}}\Big|$$

For $\lambda_3 = -1$

$$AV_3 = V_2$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ -8 & -4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$$\begin{cases} \underline{x = 0}\,| \\ 4x + 3y + 2z = 1 & \Rightarrow \quad \underline{2z = 1 - 3y}\,| \\ -8x - 4y - 3z = -2 & \Rightarrow \end{cases}$$

$$\Rightarrow \underline{V_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}\Big|$$

Exercise

For the matrix: $\begin{pmatrix} -1 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & -12 & 2 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -1 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & -12 & 2 \end{pmatrix}$$

i. $$\left|A - \lambda I\right| = \begin{vmatrix} -1-\lambda & -4 & -2 \\ 0 & 1-\lambda & 1 \\ -6 & -12 & 2-\lambda \end{vmatrix}$$

$$= (-1-\lambda)(1-\lambda)(2-\lambda) + 24 - 12(1-\lambda) + 12(-1-\lambda)$$

$$= -\lambda^3 + 2\lambda^2 + \lambda - 2 + 24 - 12 + 12\lambda - 12 - 12\lambda$$

$$= -\lambda^3 + 2\lambda^2 + \lambda - 2$$

The characteristic equation: $\underline{-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0}$

ii. $\lambda = 1$

$$\begin{array}{c|cccc} 1 & -1 & 2 & 1 & -2 \\ & & -1 & 1 & 2 \\ \hline & -1 & 1 & 2 & 0 \end{array} \quad \to \quad \underline{-x^2 + x + 2 = 0}$$

Thus, the eigenvalues are: $\underline{\lambda_1 = -1 \quad \lambda_2 = 1 \quad and \quad \lambda_3 = 2}$

iii. For $\lambda_1 = -1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 0 & -4 & -2 \\ 0 & 2 & 1 \\ -6 & -12 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -4y - 2z = 0 & & \\ 2y + z = 0 & \to \ 2y = -z & \Rightarrow \ \underline{y = -\frac{1}{2}z} \\ -6x - 12y + 3z = 0 & \to \ -6x = 12y - 3z & \underline{x = \frac{3}{2}z} \end{cases}$$

$$\Rightarrow V_1 = \boxed{\begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}}$$

For $\lambda_2 = 1 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -2 & -4 & -2 \\ 0 & 0 & 1 \\ -6 & -12 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -2x - 4y - 2z = 0 & \Rightarrow \quad -2x - 4y = 0 \\ \qquad \boxed{z = 0} & \\ -6x - 12y + 2z = 0 & \end{cases}$$

$\boxed{x = -2y}$

$$\Rightarrow V_2 = \boxed{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}$$

For $\lambda_3 = 2 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -3 & -4 & -2 \\ 0 & -1 & 1 \\ -6 & -12 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} -3x - 4y - 2z = 0 & \Rightarrow \quad -3x = 6z \\ \quad -y + z = 0 & \Rightarrow \quad \boxed{y = z} \\ -6x - 12y = 0 & \end{cases}$$

$\boxed{x = -2z}$

$$\Rightarrow V_3 = \boxed{\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$$

i. $$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{vmatrix}$$

$$= (3-\lambda)(4-\lambda)(-1-\lambda) - 4 - 8 + 4(4-\lambda) + 4(3-\lambda) + 2\lambda + 2$$

$$= -\lambda^3 + 6\lambda^2 - 5\lambda - 12 - 12 + 16 - 4\lambda + 12 - 4\lambda + 2\lambda + 2$$

$$= -\lambda^3 + 6\lambda^2 - 11\lambda + 6$$

The characteristic equation: $\underline{-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0}$

ii. $\lambda = 1$

$$\begin{array}{r|rrrr} 1 & -1 & 6 & -11 & 6 \\ & & -1 & 5 & -6 \\ \hline & -1 & 5 & -6 & 0 \end{array} \quad \rightarrow \quad \underline{-\lambda^2 + 5\lambda - 6 = 0}$$

Thus, the eigenvalues are: $\underline{\lambda_1 = 1 \quad \lambda_2 = 2 \quad and \quad \lambda_3 = 3}$

iii. For $\lambda_1 = 1 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x + 2y + 2z = 0 & (1) \\ x + 3y + z = 0 & (2) \\ -2x - 4y - 2z = 0 & (3) \end{cases}$$

$(1) + (3) \quad \rightarrow \quad \underline{y = 0}$

$$\begin{cases} (1) & 2x + 2z = 0 \\ (2) & \underline{x + z = 0} \\ & 3x + 3z = 0 \end{cases} \quad \Rightarrow \quad \underline{x = -z}$$

$$\Rightarrow \quad \underline{V_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = 2 \quad \Rightarrow \quad (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x + 2y + 2z = 0 & (1) \\ x + 2y + z = 0 & (2) \\ -2x - 4y - 3z = 0 & (3) \end{cases}$$

$$\begin{cases} 2 \times (2) & 2x + 4y + 2z = 0 \\ (3) & -2x - 4y - 3z = 0 \end{cases}$$
$$\underline{z = 0}$$

$$(1) \rightarrow \quad x + 2y = 0 \quad \rightarrow \quad \underline{x = -2y}$$

$$\Rightarrow \quad \underline{V_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}$$

For $\lambda_3 = 3 \quad \Rightarrow (A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 & 1 \\ -2 & -4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2y + 2z = 0 & \rightarrow \quad \underline{y = -z} \\ x + y + z = 0 & \qquad\qquad \rightarrow \quad \underline{x = 0} \\ -2x - 4y - 4z = 0 \end{cases}$$

$$\Rightarrow \quad \underline{V_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} -6 & 4 & 4 \\ -4 & 2 & 4 \\ -10 & 8 & 4 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -6 & 4 & 4 \\ -4 & 2 & 4 \\ -10 & 8 & 4 \end{pmatrix}$$

i. $$\left|A - \lambda I\right| = \begin{vmatrix} -6-\lambda & 4 & 4 \\ -4 & 2-\lambda & 4 \\ -10 & 8 & 4-\lambda \end{vmatrix}$$

$$= (-6-\lambda)(2-\lambda)(4-\lambda) - 160 - 128 + 40(2-\lambda) + 32(6+\lambda) + 16(4-\lambda)$$

$$= -\lambda^3 + 4\lambda$$

The characteristic equation: $\underline{-\lambda^3 + 4\lambda = 0}$

ii. $-\lambda\left(\lambda^2 - 4\right) = 0$

Thus, the eigenvalues are: $\underline{\lambda_1 = 0 \quad \lambda_2 = -2 \quad \textit{and} \quad \lambda_3 = 2}$

iii. For $\lambda_1 = 0 \quad \Rightarrow \quad \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -6 & 4 & 4 \\ -4 & 2 & 4 \\ -10 & 8 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -6x + 4y + 4z = 0 & (1) \\ -4x + 2y + 4z = 0 & (2) \\ -10x + 8y + 4z = 0 & (3) \end{cases}$$

$$(1) - (2) \quad \rightarrow \quad -2x + 2y = 0$$

$$\underline{x = y}$$

$$\begin{cases} (1) & -2x + 4z = 0 \\ (2) & -2x + 4z = 0 \end{cases} \quad \rightarrow \quad \underline{x = 2z}$$

$$\Rightarrow \quad \boxed{V_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = -2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -4 & 4 & 4 \\ -4 & 4 & 4 \\ -10 & 8 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -4x + 4y + 4z = 0 & (1) \\ -4x + 4y + 4z = 0 & (2) \\ -10x + 8y + 6z = 0 & (3) \end{cases}$$

$$\begin{cases} -x + y + z = 0 & (4) \\ -5x + 4y + 3z = 0 & (5) \end{cases}$$

$$5 \times (4) - (5) \quad \rightarrow \quad y + 2z = 0$$

$$\boxed{y = -2z}$$

$$(5) \quad \rightarrow \quad -5x - 8z + 3z = 0$$

$$\boxed{x = -z}$$

$$\Rightarrow \quad \boxed{V_2 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}}$$

For $\lambda_3 = 2 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -8 & 4 & 4 \\ -4 & 0 & 4 \\ -10 & 8 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -8x + 4y + 4z = 0 \\ -4x + 4z = 0 \quad \Rightarrow \quad \boxed{x = z} \\ -10x + 8y + 2z = 0 \end{cases}$$

$$\begin{cases} (1) & 4y - 4z = 0 \\ (3) & 8y - 8z = 0 \end{cases} \quad \rightarrow \quad \boxed{y = z}$$

$$\Rightarrow \quad \boxed{V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

i. $$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & 2 & 0 \\ 1 & -\lambda & 1 & 0 \\ 0 & 1 & -2-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} -\lambda & 0 & 2 \\ 1 & -\lambda & 1 \\ 0 & 1 & -2-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left(\lambda^2(-2-\lambda) + 2 + \lambda\right)$$

$$= (1-\lambda)\left(-\lambda^3 - 2\lambda^2 + \lambda + 2\right)$$

$$= \lambda^4 + \lambda^3 - 3\lambda^2 - \lambda + 2$$

The characteristic equation: $\lambda^4 + \lambda^3 - 3\lambda^2 - \lambda + 2 = 0$

ii. $\lambda = 1$

$$\begin{array}{r|rrrrr} 1 & 1 & 1 & -3 & -1 & 2 \\ & & 1 & 2 & -1 & -2 \\ \hline 1 & 1 & 2 & -1 & -2 & 0 \\ & & 1 & 3 & 2 & \\ \hline & 1 & 3 & 2 & 0 & \end{array}$$

$\to \lambda^3 + 2\lambda^2 - \lambda + 2 = 0$

$\to \lambda^2 + 3\lambda + 2 = 0$

Thus, the eigenvalues are: $\lambda_{1,2,3,4} = -2,\ -1,\ 1,\ 1$

iii. For $\lambda_1 = -2 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & 0 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x_1 + 2x_3 = 0 \\ x_1 + 2x_2 + x_3 = 0 \\ x_2 = 0 \\ x_4 = 0 \end{cases}$$

$$\rightarrow \begin{cases} x_1 = -x_3 \\ x_2 = 0 \\ x_4 = 0 \end{cases}$$

Therefore; the eigenvector $V_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda_2 = -1 \quad \Rightarrow \quad (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 + 2x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \\ x_2 - x_3 = 0 \\ x_4 = 0 \end{cases}$$

$$\rightarrow \begin{cases} x_1 = -2x_3 \\ x_2 = x_3 \end{cases}$$

Therefore; the eigenvector $V_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda_3 = 1 \quad \Rightarrow \quad (A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -x_1 + 2x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_2 - 3x_3 = 0 \end{cases}$$

$$\rightarrow \begin{cases} \underline{x_1 = 2x_3} \\ \underline{x_2 = 3x_3} \\ \forall\, x_4 \end{cases}$$

Therefore; the eigenvector $V_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} -x_1 + 2x_3 = 2 \\ x_1 - x_2 + x_3 = 3 \\ x_2 - 3x_3 = 1 \end{cases}$$

$$\begin{cases} x_1 = 2x_3 - 2 \\ \\ x_2 = 1 + 3x_3 \end{cases}$$

Therefore; the eigenvector $V_4 = \begin{pmatrix} 0 \\ 4 \\ 1 \\ 1 \end{pmatrix} \qquad V_4 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Exercise

For the matrix: $\begin{pmatrix} 10 & -9 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 10 & -9 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

i. $\det(A - \lambda I) = \begin{vmatrix} 10-\lambda & -9 & 0 & 0 \\ 4 & -2-\lambda & 0 & 0 \\ 0 & 0 & -2-\lambda & -7 \\ 0 & 0 & 1 & 2-\lambda \end{vmatrix}$

$$= (10-\lambda)\begin{vmatrix} -2-\lambda & 0 & 0 \\ 0 & -2-\lambda & -7 \\ 0 & 1 & 2-\lambda \end{vmatrix} + 9\begin{vmatrix} 4 & 0 & 0 \\ 0 & -2-\lambda & -7 \\ 0 & 1 & 2-\lambda \end{vmatrix}$$

$$= (10-\lambda)\left[(-2-\lambda)^2(2-\lambda) + 7(-2-\lambda)\right] + 9\left[(4)(-2-\lambda)(2-\lambda) + 28\right]$$

$$= (10-\lambda)(-2-\lambda)\left(3+\lambda^2\right) + 9\left(4\lambda^2 + 12\right)$$

$$= \left(3+\lambda^2\right)\left(-8\lambda + \lambda^2 + 16\right)$$

$$= \left(3+\lambda^2\right)(\lambda - 4)^2$$

$\Rightarrow$ The characteristic equation: $\left(3+\lambda^2\right)(\lambda-4)^2 = 0$

ii. The eigenvalues are $\lambda_{1,2,3,4} = 4,\ 4,\ \pm i\sqrt{3}$

iii. For $\lambda_1 = 4 \quad \Rightarrow \quad \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 6 & -9 & 0 & 0 \\ 4 & -6 & 0 & 0 \\ 0 & 0 & -6 & -7 \\ 0 & 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 6x_1 - 9x_2 = 0 & \rightarrow \quad \underline{2x_1 = 3x_2}| \\ 4x_1 - 6x_2 = 0 & \\ -6x_3 - 7x_4 = 0 & (1) \\ x_3 - 2x_4 = 0 & (2) \end{cases}$$

$$(1)\&(2) \quad \rightarrow \underline{x_3 = x_4 = 0}|$$

Therefore; the eigenvector $V_1 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 6 & -9 & 0 & 0 \\ 4 & -6 & 0 & 0 \\ 0 & 0 & -6 & -7 \\ 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 6x_1 - 9x_2 = 3 & (3) \\ 4x_1 - 6x_2 = 2 & (4) \\ -6x_3 - 7x_4 = 0 & (5) \\ x_3 - 2x_4 = 0 & (6) \end{cases}$$

$$\begin{cases} 6x_1 - 9x_2 = 3 \\ 4x_1 - 6x_2 = 2 \end{cases} \qquad \Delta = \begin{vmatrix} 6 & -9 \\ 4 & -6 \end{vmatrix} = 0 \quad \Delta_4 = \begin{vmatrix} 6 & 3 \\ 4 & 2 \end{vmatrix} = 0$$

$$(5)\&(6) \quad \rightarrow \underline{x_3 = x_4 = 0}|$$

Therefore; the eigenvector $V_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

For $\lambda_3 = -i\sqrt{3} \quad \Rightarrow \quad (A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} 10+i\sqrt{3} & -9 & 0 & 0 \\ 4 & -2+i\sqrt{3} & 0 & 0 \\ 0 & 0 & -2+i\sqrt{3} & -7 \\ 0 & 0 & 1 & 2+i\sqrt{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \left(10+i\sqrt{3}\right)x_1 - 9x_2 = 0 & \rightarrow \quad x_1 = \dfrac{9}{10+i\sqrt{3}}x_2 \\ 4x_1 + \left(-2+i\sqrt{3}\right)x_2 = 0 & \rightarrow \quad x_1 = \dfrac{-2+i\sqrt{3}}{4}x_2 \\ \left(-2+i\sqrt{3}\right)x_3 - 7x_4 = 0 & \rightarrow \quad (6) \\ x_3 + \left(2+i\sqrt{3}\right)x_4 = 0 & \rightarrow \quad \underline{x_3 = -\left(2+i\sqrt{3}\right)x_4|} \end{cases}$$

$$(6) \quad \rightarrow \quad \frac{7}{-2+i\sqrt{3}}\left(\frac{-2-i\sqrt{3}}{-2-i\sqrt{3}}\right) = -\left(2+i\sqrt{3}\right)$$

$$\frac{9}{10+i\sqrt{3}} \overset{?}{=} \frac{-2+i\sqrt{3}}{4}$$

$$36 \neq \left(-2+i\sqrt{3}\right)\left(10+i\sqrt{3}\right)$$

$$\Rightarrow \quad \underline{x_1 = x_2 = 0|}$$

Therefore; the eigenvector $V_3 = \begin{pmatrix} 0 \\ 0 \\ -\left(2+i\sqrt{3}\right) \\ 1 \end{pmatrix}$

Therefore; the eigenvector $V_4 = \begin{pmatrix} 0 \\ 0 \\ -2+i\sqrt{3} \\ 1 \end{pmatrix}$

Exercise

Find the eigenvalues of A^9 for $A = \begin{pmatrix} 1 & 3 & 7 & 11 \\ 0 & \frac{1}{2} & 3 & 8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

Solution

Since the matrix is an upper triangular, then the eigenvalues are: $\lambda = 1, \ \frac{1}{2}, \ 0, \ 2$

The eigenvalues of A^9 are: $1^9 \ \underline{=1|} \qquad \left(\frac{1}{2}\right)^9 \ \underline{=\frac{1}{512}|}$

$$0^9 \ \underline{=0|} \qquad 2^9 \ \underline{=512|}$$

Exercise

Given: $A=\begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}$. Compute A^{11}

Solution

$$\det(A-\lambda I)=\begin{vmatrix} -1-\lambda & 7 & -1 \\ 0 & 1-\lambda & 0 \\ 0 & 15 & -2-\lambda \end{vmatrix}$$

$$=(-1-\lambda)(1-\lambda)(-2-\lambda)$$

The eigenvalues are: $\lambda_{1,2,3}=-1,\ 1,\ -2$

For $\lambda_1=-1$, we have: $(A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} 0 & 7 & -1 \\ 0 & 2 & 0 \\ 0 & 15 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 7y_1-z_1=0 & \rightarrow & z_1=7y_1 \\ 2y_1=0 & \rightarrow & y_1=0 \\ 15y_1-z_1=0 \end{cases}$$

The eigenvector $V_1=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

For $\lambda_2=1$, we have: $(A-I)V_2=0$

$$\begin{pmatrix} -2 & 7 & -1 \\ 0 & 0 & 0 \\ 0 & 15 & -3 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -2x_2+7y_2-z_2=0 & \rightarrow & 2x_2=7y_2-z_2 \\ 15y_2-3z_2=0 & \rightarrow & 5y_2=z_2 \end{cases}$$

$$2x_2=7y_2-5y_2$$

$$x_2=y_2$$

The eigenvector $\boxed{V_2 = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}}$

For $\lambda_3 = -2$, we have: $(A+2I)V_3 = 0$

$$\begin{pmatrix} 1 & 7 & -1 \\ 0 & 3 & 0 \\ 0 & 15 & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_3 + 7y_3 - z_3 = 0 & \rightarrow \quad \boxed{x_3 = z_3} \\ 3y_3 = 0 & \rightarrow \quad \boxed{y_3 = 0} \\ 15y_3 = 0 & \end{cases}$$

The eigenvector $\boxed{V_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$D^{11} = \begin{pmatrix} (-1)^{11} & 0 & 0 \\ 0 & 1^{11} & 0 \\ 0 & 0 & (-2)^{11} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2048 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 5 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 5R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -5 & 1 \end{array}\right) \begin{array}{l} R_1 - R_3 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 4 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -5 & 1 \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} 1 & 4 & -1 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

$$A^{11} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2048 \end{pmatrix}\begin{pmatrix} 1 & 4 & -1 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & -2048 \\ 0 & 1 & 0 \\ 0 & 5 & -2048 \end{pmatrix}\begin{pmatrix} 1 & 4 & -1 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 10237 & 2047 \\ 0 & 1 & 0 \\ 0 & 10245 & -2048 \end{pmatrix}$$

Exercise

Find the eigenvalues of the matrices

$$A = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0.61 & 0.52 \\ 0.39 & 0.48 \end{pmatrix}, \quad A^{\infty} = \begin{pmatrix} 0.57143 & 0.57143 \\ 0.42857 & 0.42857 \end{pmatrix}, \quad and \quad B = \begin{pmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{pmatrix}$$

Solution

The eigenvalues for A:

$$\begin{vmatrix} 0.7-\lambda & 0.4 \\ 0.3 & 0.6-\lambda \end{vmatrix} = (0.7-\lambda)(0.6-\lambda) - .12$$

$$= \lambda^2 - 1.3\lambda + .3 = 0$$

$$\lambda_{1,2} = \frac{1.3 \pm \sqrt{1.69 - 1.2}}{2}$$

$$= .65 \pm \frac{\sqrt{.49}}{2}$$

$$= 0.65 \pm 0.35$$

The eigenvalues are: $\lambda_1 = 1 \quad \lambda_2 = 0.3$

The eigenvalues for A^2:

$$\lambda_1 = 1^2 \boxed{= 1}$$

$$\lambda_2 = 0.3^2 \boxed{= 0.09}$$

The eigenvalues for A^∞:

$$\lambda^2 - \lambda = 0$$

$$\lambda_1 = 1^2 \boxed{= 1}$$

$$\lambda_2 = 0.3^\infty \boxed{= 0}$$

The eigenvalues for B:

$$\begin{vmatrix} 0.3-\lambda & 0.6 \\ 0.7 & 0.4-\lambda \end{vmatrix} = \lambda^2 - .7\lambda - .3 = 0$$

$$\lambda_{1,2} = 0.35 \pm 0.65$$

The eigenvalues are: $\boxed{\lambda_1 = 1 \quad \lambda_2 = -0.3}$

Exercise

Given the matrix $\begin{bmatrix} -1 & -3 \\ -3 & 7 \end{bmatrix}$

a) Find the characteristic polynomial.
b) Find the eigenvalues
c) Find the bases for its eigenspaces
d) Graph the eigenspaces
e) Verify directly that $A\vec{v} = \lambda\vec{v}$, for all associated eigenvectors and eigenvalues.

Solution

a) $\begin{vmatrix} -1-\lambda & -3 \\ -3 & 7-\lambda \end{vmatrix} = (-1-\lambda)(7-\lambda) - 9$

$$= -7 - 6\lambda + \lambda^2 - 9$$

$$= \lambda^2 - 6\lambda - 16$$

The characteristic polynomial is $\boxed{\lambda^2 - 6\lambda - 16 = 0}$

b) $\lambda^2 - 6\lambda - 16 = 0 \quad \Rightarrow \quad \lambda_1 = -2 \quad \textit{and} \quad \lambda_2 = 8$

c) For $\lambda_1 = -2$, we have: $(A + 2I)V_1 = 0$

$$\begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 - 3y_1 = 0 \\ -3x_1 + 9y_1 = 0 \end{cases} \Rightarrow \underline{x_1 = 3y_1}$$

Therefore, the eigenvector $V_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

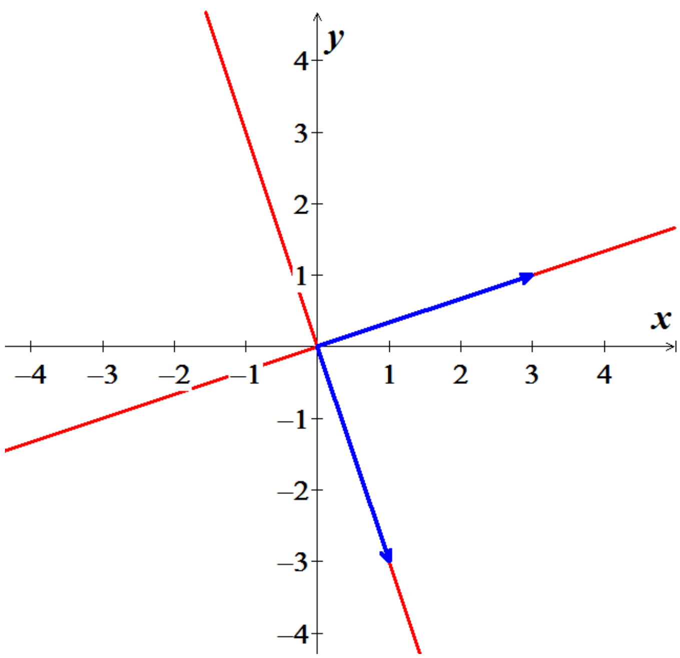
For $\lambda_2 = 8$, we have: $(A + 8I)V_2 = 0$

$$\begin{pmatrix} -9 & -3 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -9x_2 - 3y_2 = 0 \\ -3x_2 - y_2 = 0 \end{cases} \Rightarrow \underline{y_2 = -3x_2}$$

Therefore, the eigenvector $V_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

d)



e) $AV_1 = \lambda_1 V_1$

$$\begin{pmatrix} -1 & -3 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = -2\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -6 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \end{pmatrix} \checkmark$$

$$AV_2 = \lambda_2 V_2$$

$$\begin{pmatrix} -1 & -3 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = 8\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 8 \\ -24 \end{pmatrix} = \begin{pmatrix} -6 \\ -24 \end{pmatrix} \checkmark$$

Exercise

Given the matrix $\begin{bmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{bmatrix}$

a) Find the characteristic polynomial.

b) Find the eigenvalues

c) Find the bases for its eigenspaces

d) Graph the eigenspaces

e) Verify directly that $A\vec{v} = \lambda\vec{v}$, for all associated eigenvectors and eigenvalues.

Solution

a) $\begin{vmatrix} 5-\lambda & 0 & -4 \\ 0 & -3-\lambda & 0 \\ -4 & 0 & -1-\lambda \end{vmatrix} = (5-\lambda)(-3-\lambda)(-1-\lambda) - 16(-3-\lambda)$

$$= (5-\lambda)\left(3+4\lambda+\lambda^2\right) + 48 + 16\lambda$$

$$= 15 + 20\lambda + 5\lambda^2 - 3\lambda - 4\lambda^2 - \lambda^3 + 48 + 16\lambda$$

$$= -\lambda^3 + \lambda^2 + 33\lambda + 63$$

The characteristic polynomial is $\underline{-\lambda^3 + \lambda^2 + 33\lambda + 63 = 0}$

b) $\lambda = -3$

-3	-1	1	33	63
		3	-12	-63
	-1	4	21	0

$\to \underline{-\lambda^2 + 4\lambda + 21 = 0}$

$$\lambda = \frac{-4 \pm \sqrt{16+84}}{-2}$$

The eigenvalues are: $\underline{\lambda_{1,2,3} = -3,\ -3,\ 7}$

c) For $\lambda_{1,2} = -3$, we have: $(A+3I)V_1 = 0$

$$\begin{pmatrix} 8 & 0 & -4 \\ 0 & 0 & 0 \\ -4 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 8x_1 - 4z_1 = 0 & \rightarrow \quad \underline{z_1 = 2x_1} \\ -4x_1 + 2z_1 = 0 \end{cases}$$

Therefore, the eigenvector $V_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ *and* $V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

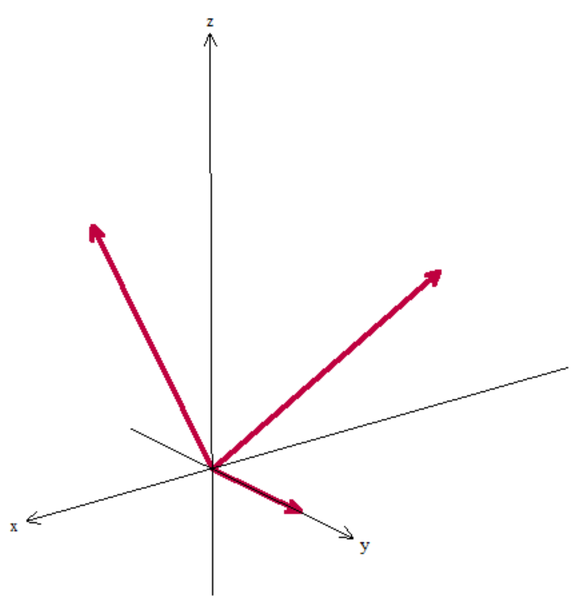
For $\lambda_3 = 7$, we have: $(A - 7I)V_3 = 0$

$$\begin{pmatrix} -2 & 0 & -4 \\ 0 & -10 & 0 \\ -4 & 0 & -8 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -2x_3 - 4z_3 = 0 & \rightarrow \quad \underline{x_3 = -2z_3} \\ -10y_3 = 0 & \rightarrow \quad \underline{y_3 = 0} \\ -4x_3 - 8z_3 = 0 \end{cases}$$

Therefore, the eigenvector $V_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

d)



e) $AV_1 = \lambda_1 V_1$

$$\begin{pmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} -3 \\ 0 \\ -6 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ -6 \end{pmatrix} \checkmark$$

$$AV_2 = \lambda_2 V_2$$

$$\begin{pmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} \checkmark$$

$$AV_3 = \lambda_3 V_3$$

$$\begin{pmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = 7\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -14 \\ 0 \\ 7 \end{pmatrix} = \begin{pmatrix} -14 \\ 0 \\ 7 \end{pmatrix} \checkmark$$

Exercise

Explain why a 2×2 matrix can have at most two distinct eigenvalues. Explain why an $n\times n$ matrix can have at most *n* distinct eigenvalues

Solution

A 2×2 matrix has only 2 entries in the main diagonal. Then, Lambda exists only twice in those entries. By using the determinant, the product will produce a power two characteristics equation. A second–degree equation will produce 2 real distinct eigenvalues, or 2 repeated eigenvalues, or 2 complex eigenvalues. Therefore, a 2×2 matrix can have at most two distinct eigenvalues.

The same for an $n\times n$ matrix, the matrix has *n* entries in the main diagonal with lambda. Then the product of n^{th} lambda will produce a characteristic equation with *n* power. That means that will have *n* real distinct eigenvalues, or *n* repeated eigenvalues, or *n* complex eigenvalues.
Therefore, a $n\times n$ matrix can have at most *n* distinct eigenvalues.

Exercise

Construct an example of a 2×2 matrix with only one distinct eigenvalue.

Solution

A 2×2 matrix with only one distinct eigenvalue, which means that we have repeated lambda. To do so, the other diagonal has a zero and the main diagonal has the same value.

Example for one zero in the diagonal.

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

$$\begin{vmatrix} a-\lambda & b \\ 0 & a-\lambda \end{vmatrix} = (a-\lambda)^2 = 0$$

$$\underline{\lambda_{1,2} = a}|$$

Example: $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

Exercise

Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}.

Solution

Since matrix A in invertible, then $AA^{-1} = A^{-1}A = I$

Let λ be an eigenvalue of an invertible matrix A, then there is a nonzero eigenvector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$

$$A^{-1}A\vec{v} = A^{-1}\lambda\vec{v}$$

$$I\vec{v} = \lambda\left(A^{-1}\vec{v}\right)$$

$$\vec{v} = \lambda\left(A^{-1}\vec{v}\right)$$

Since $\vec{v} \neq \vec{0}$ and λ cannot be zero. Then

$$\frac{1}{\lambda}\vec{v} = A^{-1}\vec{v}$$

$$\lambda^{-1}\vec{v} = A^{-1}\vec{v}$$

That will prove that λ^{-1} is an eigenvalue of A^{-1}

Exercise

Show that if A^2 is the zero matrix, then the only eigenvalue of A is 0

Solution

Assume that A^2 is the zero matrix.

If $A\vec{v} = \lambda\vec{v}$

$AA\vec{v} = A\lambda\vec{v}$

$A^2\,\vec{v} = \lambda\left(A\vec{v}\right)$

$A^2\,\vec{v} = \lambda\left(\lambda\vec{v}\right)$

$A^2\,\vec{v} = \lambda^2\,\vec{v}$

Since $\vec{v} \neq \vec{0}$ and A^2 is the zero matrix. Then λ must be zero.

Therefore, each eigenvalue of A is zero.

Exercise

Show that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T.

Solution

Suppose that λ is an eigenvalue of A, then $A - \lambda I = 0$

$$\left(A - \lambda I\right)^T = A^T - \lambda I^T$$

$$= A^T - \lambda I = 0$$

This will result that matrix and its transpose have the same characteristic equation.

Thus, λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T

Exercise

For $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, find one eigenvalue, without calculation. Justify your answer.

Solution

Since the matrix A has the row then matrix A in not invertible (Columns are linearly dependent).
Therefore, the eigenvalue is zero of the matrix.

Exercise

For $A=\begin{pmatrix}2&2&2\\2&2&2\\2&2&2\end{pmatrix}$, find one eigenvalue, and two linearly independent eigenvectors, without calculation.

Justify your answer.

Solution

Since the matrix A has the row then matrix A in not invertible (Columns are linearly dependent). Therefore, the eigenvalue is zero of the matrix.

For $\lambda=0$, then the eigenvector is given by $(A-\lambda I)V=0$

Since $\lambda=0$, that implies to $AV=0$

Since matrix A is nonzero matrix that it will imply to $2x+2y+2z=0$ all rows are the same.

Which it will result to: $x+y+z=0$

The two linearly independent eigenvectors:

$$V_1=\begin{pmatrix}1\\1\\-2\end{pmatrix}\quad V_2=\begin{pmatrix}0\\1\\-1\end{pmatrix}$$

Exercise

Consider an $n\times n$ matrix A with the property that the row sums all equal the same number ***s***. Show that ***s*** is an eigenvalue of A.

Solution

Let consider a 2×2 matrix with all ones as entries

$$A=\begin{pmatrix}1&2\\1&2\end{pmatrix}$$

$$\underline{s=1+2=3}|$$

$$\begin{vmatrix}1-\lambda&2\\1&2-\lambda\end{vmatrix}=(1-\lambda)(2-\lambda)-2$$

$$=\lambda^2-3\lambda=0$$

One of the eigenvalues is: $\underline{\lambda=3=s}|$

Let $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$

With $\begin{cases}a+b=s\\c+d=s\end{cases}$

$$\begin{pmatrix} a \\ c \end{pmatrix} + \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} s \\ s \end{pmatrix}$$

$$1 \cdot \begin{pmatrix} a \\ c \end{pmatrix} + 1 \cdot \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} s \\ s \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A\vec{v} = s\vec{v} \quad \checkmark$$

For $n \times n$ matrix A:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

Where $s = \sum_{i=1}^{n} a_{1i} = \ldots = \sum_{i=1}^{n} a_{ni}$

$$\begin{pmatrix} a_{11} + \cdots + a_{1n} \\ \vdots \qquad \vdots \\ a_{n1} + \cdots + a_{nn} \end{pmatrix} = \begin{pmatrix} s \\ \vdots \\ s \end{pmatrix}$$

$$1 \cdot \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} + \cdots + 1 \cdot \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} = s\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = s\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$A\vec{v} = s\vec{v} \quad \checkmark$$

That prove that s is an eigenvalue of A.

Exercise

Consider an $n \times n$ matrix A with the property that the column sums all equal the same number $\boldsymbol{s}$. Show that $\boldsymbol{s}$ is an eigenvalue of A.

Solution

Given that the column sums of an $n \times n$ matrix A all equal the same number $\boldsymbol{s}$.

Then the transpose of the matrix A will imply that A^T has the row sums all equal the same number $\boldsymbol{s}$. In addition, the matrix A and A^T have the same eigenvalues.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \qquad \text{Where } s = \sum_{i=1}^{n} a_{i1} = \ldots = \sum_{i=1}^{n} a_{in}$$

$$A^T = \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{pmatrix} \qquad \text{Where } s = \sum_{i=1}^{n} a_{i1} = \ldots = \sum_{i=1}^{n} a_{in}$$

$$\begin{pmatrix} a_{11} + \cdots + a_{n1} \\ \vdots \qquad \vdots \\ a_{1n} + \cdots + a_{nn} \end{pmatrix} = \begin{pmatrix} s \\ \vdots \\ s \end{pmatrix}$$

$$1 \cdot \begin{pmatrix} a_{11} \\ \vdots \\ a_{1n} \end{pmatrix} + \cdots + 1 \cdot \begin{pmatrix} a_{n1} \\ \vdots \\ a_{nn} \end{pmatrix} = s \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} + \cdots + a_{n1} \\ \vdots \qquad \vdots \\ a_{1n} + \cdots + a_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = s \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$A^T \vec{v} = s\vec{v} \quad \checkmark$$

That show that $\boldsymbol{s}$ is an eigenvalue of A^T and since A and A^T have the same eigenvalues.
The prove is completed that $\boldsymbol{s}$ is an eigenvalue of A.

Exercise

Let A be the matrix of the linear transformation T on $\mathbb{R}^2$

T: reflects points across some line through the origin.

Without writing A, find an eigenvalue of A and describe the eigenspace.

Solution

Given T reflects points across some line through the origin in $\mathbb{R}^2$, which implies that the coordinates are equal $(x = \lambda y)$.

The linear transformation can be written in the form: $T(\vec{v}) = \vec{v}$

This line more likely is the scalar nonzero product of the eigenvectors $\vec{v}$.

$$A\vec{v} = \lambda\vec{v}$$

Since A be the matrix of the linear transformation T on $\mathbb{R}^2$, then $A\vec{v} = \vec{v}$.

Thus, the eigenvalue $\lambda = 1$ of the matrix A which will result to the corresponding eigenvector $\vec{v}$.

The other eigenvector $\vec{u}$ can be generated by applying the orthogonal to the line and which leads to the eigenvalue $\lambda = -1$. The result form that each vector on the line through $\vec{u}$ can be transformed into the opposite sign of that vector.

Exercise

Let A be the matrix of the linear transformation T on $\mathbb{R}^2$

T: reflects points about some line through the origin.

Without writing A, find an eigenvalue of A and describe the eigenspace.

Solution

Given T reflects points *about* some line through the origin.

If $\vec{v} \in \mathbb{R}^2$ lines on the line, then the linear transformation can be written in the form:

$$T(\vec{v}) = A\vec{v} = \vec{v}$$

That implies to T rotates points around a given line, the points on the line are not moved at all.

Thus, the eigenvalue $\lambda = 1$ of the matrix A which will result to the corresponding eigenvector $\vec{v}$.

The corresponding eigenspace is either just the line if T doesn't rotate full rotation $(2\pi k)$.

Therefore, the corresponding eigenspace is the line the points are being rotated around.

Exercise

Show that if $\vec{v}$ is an eigenvector of the matrix product AB and $B\vec{v} \neq \vec{0}$, then $B\vec{v}$ is an eigenvector of BA

Solution

Since $\vec{v}$ is an eigenvector of the matrix product AB, that must be some eigenvalue λ to satisfy.

Such that $AB\vec{v} = \lambda\vec{v}$ and $\vec{v} \neq \vec{0}$.

Since $B\vec{v} \neq \vec{0}$, then we can rewrite

$AB\vec{v} = \lambda\vec{v}$

$A(B\vec{v}) = \lambda\vec{v}$ Multiply both sides by matrix *B*.

$BA(B\vec{v}) = B\lambda\vec{v}$

$BA(B\vec{v}) = \lambda(B\vec{v})$

Therefore, since $B\vec{v} \neq \vec{0}$, that is clearly that $B\vec{v}$ is an eigenvector of BA.

Exercise

Explain and demonstrate that the eigenspace of a matrix *A* corresponding to some eigenvalue λ is a subspace.

Solution

λ is an eigenvalue of a square matrix $(n \times n)$, then $A\vec{v} = \lambda\vec{v}$ and $\vec{v}$ is a non-zero vector.

That implies to: $(A - \lambda I)\vec{v} = \vec{0}$.

The eigenspace consists of the zero vector and all the eigenvectors $\vec{v}$ corresponding to the eigenvalue λ.

This is equivalent to the null space of $A - \lambda I$ which includes the trivial (zero vector) solution of $(A - \lambda I)\vec{v} = \vec{0}$ as well as the non-trivial (non-zero) solutions. As the null space is definitely a subspace, and the eigenspace is essentially the same, then the eigenspace is a sunspace too.

Is the eigenspace is closed under addition?

Suppose that $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors corresponding to λ.

Let assume that $A\vec{v}_1 = \lambda\vec{v}_1$ and $A\vec{v}_2 = \lambda\vec{v}_2$

$$
\begin{aligned}
A(\vec{v}_1 + \vec{v}_2) &= A\vec{v}_1 + A\vec{v}_2 \\
&= \lambda\vec{v}_1 + \lambda\vec{v}_2 \\
&= \lambda(\vec{v}_1 + \vec{v}_2)
\end{aligned}
$$

Therefore, $(\vec{v}_1 + \vec{v}_2)$ is in the eigenspace of λ under addition.

Is the eigenspace is closed under scalar multiplication?

Let $\vec{v}_1$ be an eigenvector corresponding to λ and c be any real scalar.

$$
\begin{aligned}
cA\left(\vec{v}_1\right) &= A\left(c\vec{v}_1\right) \\
&= c\left(\lambda\vec{v}_1\right) \\
&= \lambda\left(c\vec{v}_1\right)
\end{aligned}
$$

Therefore, $c\vec{v}_1$ is in the eigenspace of λ under scalar multiplication.

Therefore, the eigenspace of a matrix A corresponding to some eigenvalue λ is a subspace.

Exercise

If λ is an eigenvalue of the matrix A, prove that λ^2 is an eigenvalue of A^2.

Solution

Since λ is an eigenvalue of the matrix A, then $A\vec{v} = \lambda\vec{v}$ where $\vec{v} \neq \vec{0}$.

$$A\vec{v} = \lambda\vec{v}$$

$$A\left(A\vec{v}\right) = A\left(\lambda\vec{v}\right)$$

$$
\begin{aligned}
A^2\,\vec{v} &= \lambda\left(A\,\vec{v}\right) \qquad\qquad A\vec{v} = \lambda\vec{v} \\
&= \lambda\left(\lambda\,\vec{v}\right) \\
&= \lambda^2\,\vec{v}
\end{aligned}
$$

Therefore, λ^2 is an eigenvalue of A^2