Lecture Eight – Identities and Solving Trigonometric

Section 8.1 - Proving Identities

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Ratio Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos\theta = \pm\sqrt{1-\sin^2\theta}$$

$$\sin\theta = \pm\sqrt{1-\cos^2\theta}$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Example

Write $\sec\theta \tan\theta$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify.

Solution

$$\sec\theta\tan\theta = \frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\cos^2\theta}$$

Example

Add $\frac{1}{\sin\theta} + \frac{1}{\cos\theta}$

Solution

$$\frac{1}{\sin\theta} + \frac{1}{\cos\theta} = \frac{\cos\theta + \sin\theta}{\sin\theta\cos\theta}$$

$$\frac{1}{\sin\theta}\frac{\cos\theta}{\cos\theta} + \frac{1}{\cos\theta}\frac{\sin\theta}{\sin\theta}$$

Example

Write: $\tan\alpha + \cot\alpha$ in terms of $\sin\alpha$ *and* $\cos\alpha$

Solution

$$\tan\alpha + \cot\alpha = \frac{\sin\alpha}{\cos\alpha} + \frac{\cos\alpha}{\sin\alpha}$$

$$= \frac{\sin\alpha}{\cos\alpha}\frac{\sin\alpha}{\sin\alpha} + \frac{\cos\alpha}{\sin\alpha}\frac{\cos\alpha}{\cos\alpha}$$

$$= \frac{\sin^2\alpha + \cos^2\alpha}{\cos\alpha\sin\alpha}$$

$$= \underline{\frac{1}{\cos\alpha\sin\alpha}\Big|}$$

Example

Prove: $\tan x + \cos x = \sin x(\sec x + \cot x)$

Solution

$$\tan x + \cos x = \frac{\sin x}{\cos x} + \cos x$$

$$= \sin x\frac{1}{\cos x} + \cos x\frac{\sin x}{\sin x}$$

$$= \sin x\sec x + \sin x\frac{\cos x}{\sin x}$$

$$= \sin x(\sec x + \cot x) \quad \surd$$

or

$$\sin x(\sec x + \cot x) = \sin x\left(\frac{1}{\cos x} + \frac{\cos x}{\sin x}\right)$$

$$= \frac{\sin x}{\cos x} + \sin x\frac{\cos x}{\sin x}$$

$$\underline{= \tan x + \cos x\Big|}$$

Example

Prove: $\cot\alpha + 1 = \csc\alpha\left(\cos\alpha + \sin\alpha\right)$

Solution

$$\csc\alpha\left(\cos\alpha + \sin\alpha\right) = \frac{1}{\sin\alpha}\left(\cos\alpha + \sin\alpha\right)$$

$$= \frac{1}{\sin\alpha}\cos\alpha + \frac{1}{\sin\alpha}\sin\alpha$$

$$= \cot\alpha + 1 \quad \checkmark$$

Guidelines for Proving Identities

1. Work on the complicated side first (more trigonometry functions)
2. Look for trigonometry substitutions.
3. Look for algebraic operations
4. If not always change everything to sines and cosines
5. Keep an eye on the side you are not working.

Example

Prove $\dfrac{\cos^4 t - \sin^4 t}{\cos^2 t} = 1 - \tan^2 t$

Solution

$$\frac{\cos^4 t - \sin^4 t}{\cos^2 t} = \frac{\left(\cos^2 t - \sin^2 t\right)\left(\cos^2 t + \sin^2 t\right)}{\cos^2 t} \qquad a^2 - b^2 = (a-b)(a+b)$$

$$= \frac{\left(\cos^2 t - \sin^2 t\right)(1)}{\cos^2 t} \qquad \cos^2 t + \sin^2 t = 1$$

$$= \frac{\cos^2 t - \sin^2 t}{\cos^2 t}$$

$$= \frac{\cos^2 t}{\cos^2 t} - \frac{\sin^2 t}{\cos^2 t}$$

$$= 1 - \tan^2 t \quad \checkmark$$

Example

Prove: $1+\cos\theta = \dfrac{\sin^2\theta}{1-\cos\theta}$

Solution

$$\frac{\sin^2\theta}{1-\cos\theta} = \frac{1-\cos^2\theta}{1-\cos\theta} \qquad \sin^2\theta = 1-\cos^2\theta$$

$$= \frac{(1-\cos\theta)(1+\cos\theta)}{1-\cos\theta} \qquad a^2-b^2 = (a-b)(a+b)$$

$$= 1+\cos\theta \quad \surd$$

Example

Prove: $\tan^2\alpha\left(1+\cot^2\alpha\right) = \dfrac{1}{1-\sin^2\alpha}$

Solution

$$\tan^2\alpha\left(1+\cot^2\alpha\right) = \tan^2\alpha + \tan^2\alpha\cot^2\alpha$$

$$= \tan^2\alpha + \tan^2\alpha\frac{1}{\tan^2\alpha}$$

$$= \tan^2\alpha + 1 \qquad \tan^2\alpha+1 = \sec^2\alpha$$

$$= \sec^2\alpha$$

$$= \frac{1}{\cos^2\alpha} \qquad \cos^2\alpha = 1-\sin^2\alpha$$

$$= \frac{1}{1-\sin^2\alpha} \quad \surd$$

Example

Prove: $\dfrac{\sin\alpha}{1+\cos\alpha} + \dfrac{1+\cos\alpha}{\sin\alpha} = 2\csc\alpha$

Solution

$$\frac{\sin\alpha}{1+\cos\alpha} + \frac{1+\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{\sin\alpha}\cdot\frac{\sin\alpha}{1+\cos\alpha} + \frac{1+\cos\alpha}{\sin\alpha}\cdot\frac{1+\cos\alpha}{1+\cos\alpha}$$

$$= \frac{\sin^2\alpha + (1+\cos\alpha)^2}{\sin\alpha(1+\cos\alpha)}$$

$$= \frac{\sin^2\alpha + 1 + \cos^2\alpha + 2\cos\alpha}{\sin\alpha(1+\cos\alpha)}$$

$$= \frac{2+2\cos\alpha}{\sin\alpha(1+\cos\alpha)}$$

$$= \frac{2(1+\cos\alpha)}{\sin\alpha(1+\cos\alpha)}$$

$$= \frac{2}{\sin\alpha}$$

$$= 2\csc\alpha \quad \checkmark$$

Example

Prove $\dfrac{1+\sin t}{\cos t} = \dfrac{\cos t}{1-\sin t}$

Solution

$$\frac{1+\sin t}{\cos t} = \frac{1+\sin t}{\cos t} \cdot \frac{1-\sin t}{1-\sin t}$$

$$= \frac{1-\sin^2 t}{\cos t(1-\sin t)}$$

$$= \frac{\cos^2 t}{\cos t(1-\sin t)}$$

$$= \frac{\cos t}{1-\sin t} \quad \checkmark$$

Example

Show that $\cot^2\theta + \cos^2\theta = \cot^2\theta\cos^2\theta$ is not an identity by finding a counterexample

Solution

$$\cot^2\frac{\pi}{4} + \cos^2\frac{\pi}{4} = \cot^2\frac{\pi}{4}\cos^2\frac{\pi}{4}$$

$$1^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1^2\left(\frac{1}{\sqrt{2}}\right)^2$$

$$1 + \frac{1}{2} = \frac{1}{2}$$

$$\frac{3}{2} \neq \frac{1}{2} \quad \checkmark$$

Exercises *Section* 8.1 – Proving Identities

(**1–80**) Prove the identity

1. $\cos\theta\cot\theta+\sin\theta=\csc\theta$

2. $\sec\theta\cot\theta-\sin\theta=\dfrac{\cos^2\theta}{\sin\theta}$

3. $\dfrac{\csc\theta\tan\theta}{\sec\theta}=1$

4. $(\sin\theta+\cos\theta)^2=1+2\sin\theta\cos\theta$

5. $\sin\theta(\sec\theta+\cot\theta)=\tan\theta+\cos\theta$

6. $\cos\theta(\csc\theta+\tan\theta)=\cot\theta+\sin\theta$

7. $\cot\theta+\tan\theta=\csc\theta\sec\theta$

8. $\tan x(\cos x+\cot x)=\sin x+1$

9. $\dfrac{1-\cos^4\theta}{1+\cos^2\theta}=\sin^2\theta$

10. $\dfrac{1-\sec x}{1+\sec x}=\dfrac{\cos x-1}{\cos x+1}$

11. $\dfrac{\cos x}{1+\sin x}-\dfrac{1-\sin x}{\cos x}=0$

12. $\dfrac{1+\cot^3 t}{1+\cot t}=\csc^2 t-\cot t$

13. $\tan x+\cot x=\sec x\csc x$

14. $\dfrac{\tan x-\cot x}{\sin x\cos x}=\sec^2 x-\csc^2 x$

15. $\dfrac{\sec x+\tan x}{\sec x-\tan x}=\dfrac{1+2\sin x+\sin^2 x}{\cos^2 x}$

16. $\sin^2 x-\cos^2 x=2\sin^2 x-1$

17. $\sin^4 x-\cos^4 x=\sin^2 x-\cos^2 x$

18. $\dfrac{\cos\alpha}{1+\sin\alpha}=\sec\alpha-\tan\alpha$

19. $\dfrac{\sin\alpha}{1-\sin\alpha}-\dfrac{\cos\alpha}{1-\sin\alpha}=\dfrac{1-\cot\alpha}{\csc\alpha-1}$

20. $\dfrac{\dfrac{1}{\tan x}+\cot x}{\dfrac{1}{\tan x}+\tan x}=\dfrac{2}{\sec^2 x}$

21. $\dfrac{\cot^2\theta+3\cot\theta-4}{\cot\theta+4}=\cot\theta-1$

22. $\dfrac{\sin\theta}{1+\cos\theta}=\dfrac{1-\cos\theta}{\sin\theta}$

23. $\tan x(\csc x-\sin x)=\cos x$

24. $\sin x(\tan x\cos x-\cot x\cos x)=1-2\cos^2 x$

25. $(1+\tan x)^2+(\tan x-1)^2=2\sec^2 x$

26. $\sec x+\tan x=\dfrac{\cos x}{1-\sin x}$

27. $\dfrac{\tan x-1}{\tan x+1}=\dfrac{1-\cot x}{1+\cot x}$

28. $7\csc^2 x-5\cot^2 x=2\csc^2 x+5$

29. $1-\dfrac{\cos^2 x}{1-\sin x}=-\sin x$

30. $\dfrac{1-\cos x}{1+\cos x}=\dfrac{\sec x-1}{\sec x+1}$

31. $\dfrac{\sec x-1}{\tan x}=\dfrac{\tan x}{\sec x+1}$

32. $\dfrac{\cos x}{\cos x-\sin x}=\dfrac{1}{1-\tan x}$

33. $(\sec x+\tan x)^2=\dfrac{1+\sin x}{1-\sin x}$

34. $\dfrac{\cos x}{1+\tan x}-\dfrac{\sin x}{1+\cot x}=\cos x-\sin x$

35. $\dfrac{\cot x+\csc x-1}{\cot x-\csc x+1}=\csc x+\cot x$

36. $\dfrac{\tan x+\cot x}{\tan x-\cot x}=\dfrac{1}{\sin^2 x-\cos^2 x}$

37. $\dfrac{1-\cot^2 x}{1+\cot^2 x}+1=2\sin^2 x$

38. $\dfrac{1+\cos x}{1-\cos x}-\dfrac{1-\cos x}{1+\cos x}=4\cot x\csc x$

39. $\dfrac{\sin^3 x-\cos^3 x}{\sin x-\cos x}=1+\sin x\cos x$

40. $1+\sec^2 x\sin^2 x=\sec^2 x$

41. $\dfrac{1+\csc x}{\sec x}=\cos x+\cot x$

42. $\tan^2 x=\sec^2 x-\sin^2 x-\cos^2 x$

43. $\dfrac{\sin x}{1-\cos x}+\dfrac{\sin x}{1+\cos x}=2\csc x$

44. $\dfrac{\tan x+\sec x}{\sec x}-\dfrac{\tan x+\sec x}{\tan x}=-\cos x\cot x$

45. $\cos^2(\alpha-\beta)-\cos^2(\alpha+\beta)=\sin^2(\alpha+\beta)-\sin^2(\alpha-\beta)$

46. $\tan x\csc x-\sec^2 x\cos x=0$

47. $(1+\tan x)^2-2\tan x=\dfrac{1}{(1-\sin x)(1+\sin x)}$

48. $\dfrac{3\csc^2 x-5\csc x-28}{\csc x-4}=\dfrac{3}{\sin x}+7$

49. $\left(\sec^2 x-1\right)\left(\sec^2 x+1\right)=\tan^4 x+2\tan^2 x$

50. $\dfrac{\csc x}{\cot x}-\dfrac{\cot x}{\csc x}=\dfrac{\sin x}{\cot x}$

51. $\dfrac{1-\cos^2 x}{1+\cos x}=\dfrac{\sec x-1}{\sec x}$

52. $\dfrac{\cos x}{1+\cos x}=\dfrac{\sec x-1}{\tan^2 x}$

53. $\dfrac{1-2\sin^2 x}{1+2\sin x\cos x}=\dfrac{\cos x-\sin x}{\cos x+\sin x}$

54. $(\cos x-\sin x)^2+(\cos x+\sin x)^2=2$

55. $\dfrac{\sin x}{1+\cos x}+\dfrac{1+\cos x}{\sin x}=2\csc x$

56. $\dfrac{\sin x+\tan x}{\cot x+\csc x}=\sin x\tan x$

57. $\csc^2 x\sec^2 x=\sec^2 x+\csc^2 x$

58. $\cos^2 x+1=2\cos^2 x+\sin^2 x$

59. $1-\dfrac{\cos^2 x}{1+\sin x}=\sin x$

60. $\cot^2 x=(\csc x-1)(\csc x+1)$

61. $\dfrac{\csc x+\cot x}{\tan x+\sin x}=\csc x\cot x$

62. $\dfrac{1-\sec x}{\tan x}+\dfrac{\tan x}{1-\sec x}=-2\csc x$

63. $\csc x-\sin x=\cos x\cot x$

64. $\cot^3 x=\cot x\left(\csc^2 x-1\right)$

65. $\dfrac{\cot^2 x}{\csc x-1}=\dfrac{1+\sin x}{\sin x}$

66. $\cot^2 x+\csc^2 x=2\csc^2 x-1$

67. $\dfrac{\cot^2 x}{1+\csc x}=\csc x-1$

68. $\sec^4 x-\tan^4 x=\sec^2 x+\tan^2 x$

69. $\dfrac{\cos x}{1+\sin x}+\dfrac{1+\sin x}{\cos x}=2\sec x$

70. $\dfrac{\sin x+\cos x}{\sin x-\cos x}=\dfrac{1+2\sin x\cos x}{2\sin^2 x-1}$

71. $\dfrac{\csc x-1}{\csc x+1}=\dfrac{\cot^2 x}{\csc^2 x+2\csc x+1}$

72. $\csc^4 x-\cot^4 x=\csc^2 x+\cot^2 x$

73. $\tan\left(\dfrac{\pi}{4}+x\right)=\cot\left(\dfrac{\pi}{4}-x\right)$

74. $\dfrac{\sin\theta}{1+\sin\theta}-\dfrac{\sin\theta}{1-\sin\theta}=-2\tan^2\theta$

75. $\csc^2 x-\cos^2 x\csc^2 x=1$

76. $1-2\sin^2 x=2\cos^2 x-1$

77. $\csc^2 x-\cos x\sec x=\cot^2 x$

78. $(\sec x-\tan x)(\sec x+\tan x)=1$

79. $\left(1+\tan^2 x\right)\left(1-\sin^2 x\right)=1$

80. $10\csc^2 x-6\cot^2 x=4\csc^2 x+6$