Section 1.5 – Introduction to Proofs

Some Terminology

A ***theorem*** is a statement that can be shown to be true. Theorems can also be referred to as facts or results. We demonstrate that a theorem is true with a ***proof***. A proof is a valid argument that establishes the truth of a theorem. The statements used in a proof can include ***axioms*** (or ***postulates***), which are statements we assume to be true.

Less important theorems sometimes are called ***propositions***. A less important theorem that is helpful in the proof of other results is called a ***lemma*** (***plural lemmas or lemmata***).

A ***corollary*** is a theorem that can be established directly from a theorem that has been proved.
A ***conjecture*** is a statement that is being proposed to be true statement, usually on the basis of some partial evidence, a heuristic argument, or the intuition of an expert.

Direct Proofs

A direct proof is a conditional statement $p \to q$ is constructed when the first step is the assumption that *p* is true; subsequence steps are constructed using rules of inference, with the final step showing that *q* must be true.
Consider an implication: $p \to q$

- If *p* is false, then the implication is always true.
- Show that if *p* is true then *q* is true.

Definition

The integer *n* is ***even*** if there exists an integer *k* such that $n = 2k$, and *n* is ***odd*** if there exists an integer *k* such that $n = 2k + 1$. Two integers have the ***same parity*** when both are even or both are odd; they have ***opposite parity*** when one is even and the other is odd.

Example

Give a direct proof of the theorem "If *n* is an odd integer, then n^2 is odd"

Solution

This states: $\forall n\ P(n) \to Q(n)$, where

$P(n)$ is "*n* is an odd integer"

$Q(n)$ is "n^2 is an odd"

Using direct proof, we assume that n is odd, is a true statement. By the definition of an odd integer, it follows that $n = 2k + 1$, where k is some integer. We need to show that n^2 is odd.

$$
\begin{aligned}
n^2 &= (2k+1)^2 && \textbf{\textit{Square both sides}} \\
&= 4k^2 + 4k + 1 \\
&= 2\left(2k^2 + 2k\right) + 1 && 2k^2 + 2k = K \\
&= 2K + 1
\end{aligned}
$$

By the definition of an odd integer, we can conclude that n^2 is also an odd integer.

Example

Give a direct proof that if m and n are both perfect squares, then nm is also a perfect square.
(An integer a is a perfect square if there is an integer b such that $a = b^2$.)

Solution

Using direct proof of this theorem, we assume that the hypothesis of this conditional statement is true, namely, we assume that m and n are both perfect squares.
By the definition of a perfect square:

$\exists s \ \ni \ m = s^2$ There is an integer s such that $m = s^2$

$\exists t \ \ni \ n = t^2$

The goal is to show that nm is also a perfect square.

$$mn = s^2t^2 = (ss)(tt) = (st)(st) = (st)^2$$

By the definition of a perfect square, it follows that nm is also a perfect square.

Proof by Contraposition

In logic, ***proof by contrapositive*** is a form of proof that establishes the truth or validity of a proposition by demonstrating the truth or validity of the converse of its negated parts.
To prove by contraposition, consider an implication $p \to q$, prove that $\neg q \to \neg p$,

- If the antecedent $\neg q$ is false, then the contrapositive is always true.
- Show that if $\neg q$ is true, then $\neg p$ is true

To perform an indirect proof, do a direct proof on the contrapositive.

Example

Prove that if *n* is an integer and 3*n* + 2 is odd, then *n* is odd.

Solution

The first step in a proof by contraposition is to assume that the conclusion of the conditional statement "3*n* + 2 is odd, then *n* is odd" is false. Assume that *n* is even, then by the definition of an even integer, $n = 2k$ for some integer *k*.

$$3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$$

This show $3n + 2$ is even, because it is a multiple of 2, therefore not odd. This is the negation of the theorem of the conditional statement implies that the hypothesis is false; the original conditional statement is true.
Our proof by contraposition succeeded; we have proved the theorem "If 3*n* + 2 is odd, then *n* is odd."

Example

Prove that if $n = ab$ where *a* and *b* are positive integers, then $a \le \sqrt{n}$ or $b \le \sqrt{n}$.

Solution

By using proof by contraposition, let assume that the conclusion of the conditional statement "if $n = ab$ where *a* and *b* are positive integers, then $a \le \sqrt{n}$ or $b \le \sqrt{n}$" is false.
$\left(a \le \sqrt{n}\right) \vee \left(b \le \sqrt{n}\right)$ is false.

Using the meaning of disjunction together with De Morgan's law, that implies that both $a \le \sqrt{n}$ and $b \le \sqrt{n}$ are false $\Rightarrow$ $a > \sqrt{n}$ *and* $b > \sqrt{n}$

Then $\quad ab > \sqrt{n}\sqrt{n} = n \quad \Rightarrow \quad ab \ne n$, which contradicts the statement $n = ab$.
This is the negation of the theorem of the conditional statement implies that the hypothesis is false; the original conditional statement is true. Our proof by contraposition succeeded; we have proved the theorem "If $n = ab$ where *a* and *b* are positive integers, then $a \le \sqrt{n}$ or $b \le \sqrt{n}$."

Vacuous and Trivial Proofs

If *p* is a conjunction of other hypotheses and we know one or more of these hypotheses is false, then *p* is false and so $p \to q$ is ***vacuously*** true regardless of the truth value of *q*.

If we know *q* is true then $p \to q$ is true regardless of the truth value of *p*, this called ***Trivial Proofs***.

Example

Show that the proposition $P(0)$ is true, where $P(n)$ is "If $n > 1$, then $n^2 > n$" and the domain consists of all integers.

Solution

Using a vacuous proof; $P(0)$ is "If $0 > 1$, then $0^2 > 0$". Indeed, the hypothesis $0 > 1$ is false. This tells us that $P(0)$ is automatically true.

Example

Let $P(n)$ be "If *a* and *b* are positive integers with $a \ge b$, then $a^n \ge b^n$," where the domain consists of all nonnegative integers. Show that $P(0)$ is true.

Solution

The proposition $P(0)$ is "If $a \ge b$, then $a^0 \ge b^0$." Because $a^0 = b^0 = 1$, the conclusion of the conditional statement is true. Hence, this conditional statement, which is $P(0)$, is true.

Definition

The real number *r* is rational if there exist integers *p* and *q* with $q \ne 0$ such that $r = \frac{p}{q}$. A real number that is not rational is called irrational.

Example

Prove that the sum of two rational numbers is rational. (Note that if we include the implicit quantifiers here, the theorem we want to prove is "For every real number *r* and every real number *s*, if *r* and *s* are rational numbers, the $r + s$ is rational.)

Solution

From the definition of a rational number, that there exist integers *p* and *q* with $q \ne 0$ such that $r = \frac{p}{q}$, and integers *t* and *u* with $u \ne 0$ such that $s = \frac{t}{u}$.

$$r+s=\frac{p}{q}+\frac{t}{u}=\frac{pu+qt}{qu}$$

Because $q\neq 0$ and $u\neq 0$, it follows that $qu\neq 0$. Therefore; we have $r + s$ is rational.

Example

Prove that *n* is an integer and n^2 is odd, then *n* is odd

Solution

Suppose that *n* is an integer and n^2 is odd. Then, $\exists k\in\mathbb{Z} \ni n^2=2k+1$.

$\Rightarrow n=\pm\sqrt{2k+1}$ (which is not useful).

By using proof by contraposition, the statement *n* is not odd, that means *n* is even.

That implies that $\exists k\in\mathbb{Z} \ni n=2k$.

To prove the theorem, we need to show that this hypothesis implies the conclusion that n^2 is not odd, that means n^2 is even.

$n^2=(2k)^2=4k^2=2\left(2k^2\right)$, which implies that n^2 is even.

We have proved that *n* is an integer and n^2 is odd, then *n* is odd by a proof of contraposition.

Proofs by Contradiction

The basic ides of a proof of contradiction is to assume that the statement we want to prove is false. That is, the supposition that p is false followed necessarily by the conclusion q from not $\neg p$, where q is false, which implies that p is true.

Given a statement p, assume it is false, assume $\neg p$

- Prove that $\neg p$ cannot occur
  - A contradiction exists
  - Given a statement of the form $p \rightarrow q$
  - To assume it's false, you only have to consider the case where p is true and q is false

Example

Show that at least four of any 22 days must fall on the same day of the week.

Solution

Let p be the proposition "at least four of any 22 days must fall on the same day of the week"

Suppose that $\neg p$ is true $\Rightarrow$ "at most three of the 22 days fall on the same day of the week".

There are 7 days per week $\Rightarrow$ at most 3 of the chosen days could fall on that day. That contradicts the premise that we have 22 days under consideration.

If r is the statement that 22 days are chosen, that we have shown that $\neg p \rightarrow (r \wedge \neg r)$.

We know that p is true. We have proved that at least four of any 22 days must fall on the same day of the week.

Example

Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Solution

Let p be the proposition "$\sqrt{2}$ is irrational". Suppose that $\neg p$ is true $\Rightarrow$ "$\sqrt{2}$ is rational".

If $\sqrt{2}$ is rational, $\exists a \text{ and } b \ \ni \sqrt{2} = \frac{a}{b}$

$$\left(\sqrt{2}\right)^2 = \left(\frac{a}{b}\right)^2 \quad \Rightarrow \quad 2 = \frac{a^2}{b^2} \ \rightarrow \ \underline{2b^2 = a^2}$$

It follows that a^2 is even, that implies a must also be even. Therefore, by the definition of an even integer then we can let $a = 2c$ for some integer c. Thus, $2b^2 = 4c^2 \quad \Rightarrow \quad b^2 = 2c^2$

By the definition of even, this means that b^2 is even, that implies b must also be even as well.

The assumption of $\neg p$ leads to the equation $\sqrt{2} = \frac{a}{b}$, where a and b have no common factors, but both a and b are even, that is, 2 divides both a and b. However, our assumption $\neg p$ leads to the contradiction that 2 divides both a and b and 2 doesn't divide both a and b, $\neg p$ must be false.

That is, the statement p "$\sqrt{2}$ is irrational" is true.

Proofs of Equivalence

To prove a theorem that is a biconditional statement, that is, a statement of the form $p \leftrightarrow q$, we must show that $p \to q$ and $q \to p$ are both true. The validity of this approach is based on the tautology

$$(p \leftrightarrow q) \leftrightarrow (p \to q) \land (q \to p)$$

Example

Prove the theorem "If *n* is an integer, then *n* is odd if and only if n^2 is odd"

Solution

Let: *p* is "*n* is odd" and *q* is "n^2 is odd".
The theorem has the form: "p *iff* q". To prove this theorem, we need to show that $p \to q$ and $q \to p$ are both true.

Using direct proof, we assume that *n* is odd, is a true statement. By the definition of an odd integer, it follows that $n = 2k+1$, where *k* is some integer. We need to show that n^2 is odd.

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2\left(2k^2 + 2k\right) + 1 = 2K + 1 \qquad 2k^2 + 2k = K$$

By the definition of an odd integer, we can conclude that n^2 is also an odd integer. Therefore, $p \to q$ is true.

Suppose that *n* is an integer and n^2 is odd. Then, $\exists k \in \mathbb{Z} \ni n^2 = 2k+1 \Rightarrow n = \pm\sqrt{2k+1}$ (which is not useful). By using proof by contraposition, the statement *n* is not odd, that means *n* is even. That implies that $\exists k \in \mathbb{Z} \ni n = 2k$.

To prove the theorem, we need to show that this hypothesis implies the conclusion that n^2 is not odd, that means n^2 is even. $n^2 = (2k)^2 = 4k^2 = 2\left(2k^2\right)$, which implies that n^2 is even.

We have proved that *n* is an integer and n^2 is odd, then *n* is odd by a proof of contraposition. Therefore, $q \to p$ is true.

Because $p \to q$ and $q \to p$ are both true, we have shown that the theorem is true.

Example

Show that these statements about the integer *n* are equivalent:

p_1 : *n is even*

p_2 : $n-1$ *is odd*

p_3 : n^2 *is even*

Solution

We will show that these 3 statements are equivalent by showing that the condition statements $p_1 \to p_2$, $p_2 \to p_3$, $p_3 \to p_1$ are true.

Using a direct proof to show that $p_1 \to p_2$.

Suppose that *n* is even, then $n = 2k$ for some $k \in \mathbb{Z}$.

$$\begin{aligned} n-1 &= 2k-1 \\ &= 2(k-1)+1 \end{aligned}$$

This means that $n-1$ is odd because it is of the form $2m+1$, where m is the integer $k-1$. Therefore the statement $p_1 \to p_2$ is true.

Also using a direct proof to show that $p_2 \to p_3$.

Suppose that $n-1$ is even, then $n-1 = 2k+1$ for some $k \in \mathbb{Z}$.

$$n = 2k+2$$

$$\begin{aligned} n^2 &= (2k+2)^2 \\ &= 4k^2+8k+4 \\ &= 2\left(2k^2+4k+2\right) \end{aligned}$$

Hence, $n-1$ is even. Therefore the statement $p_2 \to p_3$ is true.

Using a proof by contraposition to prove $p_3 \to p_1$. That is, we have to prove that if n is not even, then n^2 is not even.

To prove the theorem, we need to show that this hypothesis implies the conclusion that n^2 is not odd, that means n^2 is even. $n^2 = (2k)^2 = 4k^2 = 2\left(2k^2\right)$, which implies that n^2 is even.

We have proved that *n* is an integer and n^2 is odd, then *n* is odd by a proof of contraposition. Therefore, $p_3 \to p_1$ is true.

This completes the proof.

Counterexamples

To show that a statement of the form $\forall x\, P(x)$ is false, we need only find a ***counterexample***, that is, an example of *x* for which $P(x)$ is false.

Example

Show that the statement "Every positive integer is the sum of the squares of two integers" is false.

Solution

To show that this statement is false, we look for a counterexample, which is a particular integer that is not the sum of the squares of two integers.

To choose a counterexample, we can select 3 because it cannot be written as the sum of the squares of two integers. Let use 0 and 1 which implies $0^2+1^2=0+1=1\neq 3$. Therefore, we can't get 3 as the sum of two terms of which is 0 or 1.

Consequently, we have shown that "Every positive integer is the sum of the squares of two integers" is false.

Mistakes in Proofs

Each step of a mathematical proof needs to be correct and the conclusion needs to follow logically from the steps that precede it. Many mistakes result from the introduction of steps that do not logically follow from those that precede it.

Example

What is wrong with this "proof?": If n^2 is positive, then *n* is positive.

Proof: Suppose that n^2 is positive. Because the conditional statement "If *n* is positive, then n^2 is positive" is true, we can conclude that *n* is positive.

Solution

Let $P(n)$ be "*n* is positive" and $Q(n)$ be "n^2 is positive."

The statement can be written: $\forall n\left(P(n)\rightarrow Q(n)\right)$.

A counterexample is supplied by $n=-1 \Rightarrow n^2=1$ is positive, but *n* is negative.

Exercises *Section* 1.5 – Introduction to Proofs

1. Show that the square of an even number is an even number

2. Prove that if n is an integer and n^3+5 is odd, then n is even

3. Show that $m^2=n^2$ if and only if $m=n \quad or \quad m=-n$

4. Use a direct proof to show that the sum of two odd integers is even.

5. Use a direct proof to show that the sum of two even integers is even.

6. Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.

7. Prove or disprove that the product of two irrational numbers is irrational.

8. Prove that if x is irrational, then $\frac{1}{x}$ is irrational.

9. Prove that if x is rational and $x \neq 0$, then $\frac{1}{x}$ is rational.

10. Prove the proposition $P(0)$, where $P(n)$ is the proposition "If n is a positive integer greater than 1, then $n^2 > n$." What kind of proof did you use?

11. Let $P(n)$ be the proposition "If a and b are positive real numbers, then $(a+b)^n \ge a^n + b^n$." Prove that $P(1)$ is true. What kind of proof did you use?

12. Show that these statements about the integer x are equivalent:

$\boldsymbol{i})\ 3x+2 \ \textit{ is even} \quad \boldsymbol{ii})\ x+5 \ \textit{ is odd} \quad \boldsymbol{iii})\ x^2 \ \textit{ is even}$

13. Show that these statements about the real number x are equivalent:

$\boldsymbol{i})\ x \ \textit{ is irrational} \quad \boldsymbol{ii})\ 3x+2 \ \textit{ is irrational} \quad \boldsymbol{iii})\ \frac{x}{2} \ \textit{ is irrational}$

14. Prove that at least one of the real numbers $a_1,\ a_2,\ \ldots,\ a_n$ is greater than or equal to the average of these numbers. What kind of proof did you use?