5.7 Mathematical Induction

sum of n (>0) $\quad \frac{n(n+1)}{2}$.

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

① $n=1 \Rightarrow \quad 1 \stackrel{?}{=} \frac{1(1+1)}{2}$

$1 = 1$ ✓ $\quad P_1$ is true

② Assume $P_k : 1+2+\cdots+k = \frac{k(k+1)}{2}$ is true

is $P_{k+1} : \underbrace{1+\cdots+k+(k+1)}_{\text{copy}} = \frac{(k+1)(k+2)}{2}$? is true

$$\underbrace{1+\cdots+k}_{\text{Replace}} + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= (k+1)\left(\frac{k}{2}+1\right)$$

$$= (k+1)\left(\frac{k+2}{2}\right) ✓$$

P_{k+1} is also true.

$\therefore$ By the mathematical Induction, the proof is completed

Ex $1^2+3^2+\cdots+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$

① $n=1 \Rightarrow 1^2 \overset{?}{=} \frac{1(1)(3)}{3}$?

$1=1$ ✓ P_1 is true

Assume P_k : $1^2+\cdots+(2k-1)^2=\frac{k(2k-1)(2k+1)}{3}$

Is P_{k+1} : $1^2+\cdots+(2k-1)^2+(2(k+1)-1)^2=\frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$

$1^2+\cdots+(2k-1)^2+(2k+1)^2=\frac{(k+1)(2k+1)(2k+3)}{3}$?

$1^2+\cdots+(2k-1)^2+(2k+1)^2=\frac{1}{3}k(2k-1)(2k+1)+(2k+1)^2$

$=(2k+1)\left(\frac{1}{3}k(2k-1)+\frac{3}{3}(2k+1)\right)$

$=\frac{1}{3}(2k+1)(2k^2-k+6k+3)$

$=\frac{1}{3}(2k+1)(2k^2+5k+3)$

$=\frac{1}{3}(2k+1)(k+1)(2k+3)$ ✓

P_{k+1} is also true

∴ By the mathematical induction,

the proof is completed

Ex. 2 is a factor of n^2+5n $(n>0)$

n is integer (+) $n \in \mathbb{Z}^+$

$n=1 \longrightarrow 1^2+5(1) = 6$

$= 2(3)$ P_1 is true.

Assume P_k: $k^2+5k = 2p$ is true

is P_{k+1} $(k+1)^2+5(k+1)$ 2 is a factor.

$$(k+1)^2+5(k+1) = k^2+2k+1+5k+5$$
$$= 2p+2k+6$$
$$= 2(p+k+3)$$

P_{k+1} is also true

$\therefore$ By the mathematical induction, the proof is completed.

Ex. $a \in \mathbb{R} - \{0\}$ $a > -1$

$$(1+a)^n > 1 + na \quad (n \geqslant 2)$$

$n = 2 \Rightarrow (1+a)^2 \overset{?}{>} 1 + 2a$

$$1 + 2a + a^2 > 1 + 2a \quad (a^2 > 0)$$

P_2 is true.

Assume: $(1+a)^k > 1 + ka$ is true

Is P_{k+1}: $(1+a)^{k+1} > 1 + (k+1)a$?

$$\begin{aligned}(1+a)^{k+1} &= (1+a)^k (1+a) \\ &> (1+ka)(1+a) \\ &= 1 + a + ka + ka^2 \\ &= 1 + (1+k)a + \underline{ka^2} \\ &> 1 + (k+1)a \end{aligned}$$

$k \geqslant 2$
$a^2 > 0$

P_{k+1} is also true

$\therefore$ By the mathematical induction, the proof is completed.

#5 $1+2\cdot 2+3\cdot 2^2+\cdots+n\cdot 2^{n-1}=1+(n-1)2^n$

$n=1 \Rightarrow \quad 1 = 1+0(2)$

$1=1$ ✓ $\quad P_1$ is true.

Assume P_k : $1+2\cdot 2+\cdots+k\cdot 2^{k-1}=1+(k-1)2^k$ true

is P_{k+1} : $1+\cdots+k\cdot 2^{k-1}+(k+1)\cdot 2^k=1+k\,2^{k+1}$?

$$\begin{aligned}1+\cdots+k\cdot 2^{k-1}+(k+1)\cdot 2^k &= 1+(k-1)2^k+(k+1)\cdot 2^k\\ &= 1+(k-1+k+1)2^k\\ &= 1+(2k)2^k\\ &= 1+k(2\cdot 2^k)\\ &= 1+k\,2^{k+1} \checkmark\end{aligned}$$

P_{k+1} is also true.

By the mathematical induction, the proof is completed

#6 $1^2+2^2+3^2+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$

$n=1 \Rightarrow \quad 1^2 = \dfrac{1(2)(3)}{6}$

$1 = 1$ ✓ P_1 is true.

Let P_k: $1^2+\cdots+k^2 = \dfrac{k(k+1)(2k+1)}{6}$ is true

is P_{k+1}: $1^2+\cdots+k^2+(k+1)^2 = \frac{1}{6}(k+1)(k+2)(2k+3)$?

$$\begin{aligned}
1^2+\cdots+k^2+(k+1)^2 &= \tfrac{1}{6}k(k+1)(2k+1)+(k+1)^2 \\
&= (k+1)\left(\tfrac{1}{6}k(2k+1)+k+1\right) \\
&= \tfrac{1}{6}(k+1)(2k^2+k+6k+6) \\
&= \tfrac{1}{6}(k+1)\ (2k+3)(k+2) \ ✓
\end{aligned}$$

P_{k+1} is also true.

∴ By the mathematical induction,
the proof is completed.

#11: $1^3+2^3+3^3+\cdots+n^3=\frac{1}{4}n^2(n+1)^2$

$n=1 \Rightarrow 1^3 \stackrel{?}{=} \frac{1}{4}(1)(2)^2$

$1=1$ ✓ P_1 is true.

$k+1+1$

Let $P_k: 1^3+\cdots+k^3=\frac{1}{4}k^2(k+1)^2$

Is $P_{k+1}: 1^3+\cdots+k^3+(k+1)^3=\frac{1}{4}(k+1)^2(k+2)^2$?

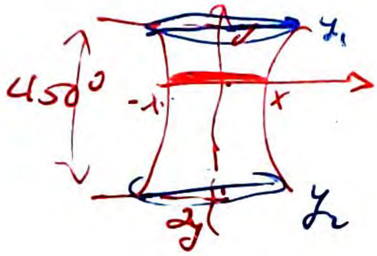
$$\begin{aligned}1^3+\cdots+k^3+(k+1)^3 &= \frac{1}{4}k^2(k+1)^2+(k+1)^3\\ &=(k+1)^2\left(\frac{1}{4}k^2+k+1\right)\\ &=\frac{1}{4}(k+1)^2(k^2+4k+4)\\ &=\frac{1}{4}(k+1)^2(k+2)^2 \checkmark\end{aligned}$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

5.4 soln Hwk

$$\frac{x^2}{90^2} - \frac{y^2}{130^2} = 1$$



$y_2 = 2y_1$

$y_1 + y_2 = 450$

$y_1 + 2y_1 = 450$

$3y_1 = 450 \Rightarrow \boxed{y_1 = 150}$

$\boxed{y_2 = 300}$

x_1 ? $\quad \dfrac{x_1^2}{90^2} = 1 + \left(\dfrac{150}{130}\right)^2$

$$x_1^2 = 90^2\left(1 + \frac{15^2}{13^2}\right)$$

$$= \frac{90^2}{13^2}(169 + 225)$$

$$x_1 = \frac{90}{13}\sqrt{394}$$

Top diameter is $2x_1 = \boxed{\frac{180}{13}\sqrt{394}\ ft}$

x_2 ? $\quad \dfrac{x_2^2}{90^2} = 1 + \left(\dfrac{30\cancel{0}}{13\cancel{0}}\right)^2$

$$x_2^2 = \frac{90^2}{13^2}(169 + 900)$$

$$x_2 = \frac{90}{13}\sqrt{1069}$$

Bottom diameter $2x_2 = \boxed{\frac{180}{13}\sqrt{1069}\ ft}$