

Section 2.9 – Rank and the Fundamental Matrix Spaces

The **Reduced Row Echelon Form** (*rref*) is a matrix (R) with each pivot column has only one nonzero entry (the pivots which is always 1).

$$R = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R = \text{rref}(A)$$

Rank of a Matrix

The rank of a matrix A (m by n) is the number of **nonzero rows** in the row-reduced echelon form of A (it is the number of pivot). The common dimension of the row space and column space of a matrix A is called the **rank** of A and is denoted by

$$\text{rank}(A) = r$$

Note:

The rank of a matrix is well defined due to the uniqueness of the row-reduced echelon form. No matter what sequence of elementary row operations is performed to put the given matrix in row-reduced echelon form; there will always be the same number of nonzero rows.

Theorem

The row space and column space of a matrix A have the same dimension

The objective is to connect **rank** and **dimension**.

- The **rank** of a matrix is the number of pivots.
- The **dimension** of a subspace is the number of vectors in a basis.

✓ *A has full row rank if every row has a pivot: $r = m$. No zero in R .*

✓ *A has full column rank if every column has a pivot: $r = n$. No free variables.*

Example

Find the rank of $A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 1 & -3 & 0 & 5 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 1 & -3 & 0 & 5 \end{bmatrix} \quad R_3 - R_1$$

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & -2 & -2 & 4 \end{bmatrix} \quad \begin{array}{l} R_1 + R_2 \\ R_3 + 2R_2 \end{array}$$

$$R = \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix R has 2 nonzero rows, therefore the $\text{rank}(A) = 2$

Example

The columns of A are dependent. $A\vec{x} = \vec{0}$ has a nonzero solution.

$$Ax = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

$$-3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The rank of A is only $r = 2$.

Independent columns would give full column rank $r = n = 3$.

✚ The columns of A are independent exactly when the rank is $r = n$. There are n pivots and no free variables. Only $\vec{x} = \vec{0}$ is the nullspace.

Example

When all rows are multiplying of one pivot row, the rank is $r = 1$:

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix}, \quad \begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \quad [6]$$

Solution

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix} \quad R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix} \quad 3R_2 - 5R_1$$

$$\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \quad \frac{1}{3}R_2$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} \quad 5R_2 - 2R_1 \quad \frac{1}{5}R_1 \quad \rightarrow \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The row-reduced echelon form $R = rref(A)$:

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad [1]$$

These matrices have only one pivot.

Dimension *Theorem* for Matrices

If A is a matrix with n columns, then

$$\boxed{\text{rank}(A) + \text{nullity}(A) = n}$$

Theorem

If A is an $m \times n$ matrix, then

- $\text{rank}(A)$ = the number of leading variables in the general solution of $A\vec{x} = \vec{0}$
- $\text{nullity}(A)$ = the number of parameters in the general solution of $A\vec{x} = \vec{0}$

Theorem

If A is any matrix, then $\text{rank}(A) = \text{rank}(A^T)$

✚ $Ax = 0$ has $n - r$ free variables and special solutions: n columns minus r pivot columns. The null matrix N has $n - r$ columns (the special solutions).

✚ The particular solution solves: $A\vec{x}_p = \vec{b}$

✚ Full column rank $R = \begin{bmatrix} n \text{ by } n \text{ identity matrix} \\ m - n \text{ rows of zeros} \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$

The reduced row echelon form looks like:

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \quad \begin{array}{l} \textcolor{blue}{r} \text{ pivot rows} \\ \textcolor{blue}{m - r} \text{ zero rows} \end{array}$$

The pivot variables in the $n - r$ special columns come by changing F to $-F$:

$$\text{Nullspace matrix: } N = \begin{pmatrix} -F \\ I \end{pmatrix} \quad \begin{array}{l} \textcolor{blue}{r} \text{ pivot variables} \\ \textcolor{blue}{n - r} \text{ free variables} \end{array}$$

➤ Every matrix A with **full column rank** ($r = n$) has all these properties:

1. All columns of A are pivot columns
2. There are no free variables or special solutions.
3. The nullspace $NS(A)$ contains only the zero vector $\vec{x} = \vec{0}$
4. If $A\vec{x} = \vec{b}$ has a solution (might not) then it has only one solution.

Example

Suppose A is a square invertible matrix, $m = n = r$. What are \vec{x}_p and \vec{x}_n ?

Solution

The particular solution is the one and only solution $A^{-1}\vec{b}$.

There are no special solutions or free variables. $R = I$ has no zero rows.

The only vector in the null space is $\vec{x}_n = \vec{0}$.

The complete solution is

$$\begin{aligned}\vec{x} &= \vec{x}_p + \vec{x}_n \\ &= A^{-1}\vec{b} + \vec{0} \\ &= \underline{A^{-1}\vec{b}}\end{aligned}$$

Example

Compute $N(A)$ for $A: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by $A = (x + y, x, 2x - y)$

Solution

To find $N(A)$, we must solve the equation $A(x, y) = (0, 0, 0)$

$$\begin{pmatrix} x + y \\ x \\ 2x - y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} x + y = 0 &\Rightarrow \boxed{y = 0} \\ \boxed{x = 0} \end{aligned}$$

Thus $NS(A) = \{0\}$, the set that consists solely of the zero vector.



If $A\vec{x} = \vec{0}$ has more unknowns than equations (more columns than rows) then it has nonzero solutions. There must be free columns, without pivots.

Definition

If W is a subspace of \mathbb{R}^n that are orthogonal to every vector in W is called orthogonal complement of W and is denoted by the symbol W^\perp . $N(A)^\perp$ is exactly the row space $C(A^T)$

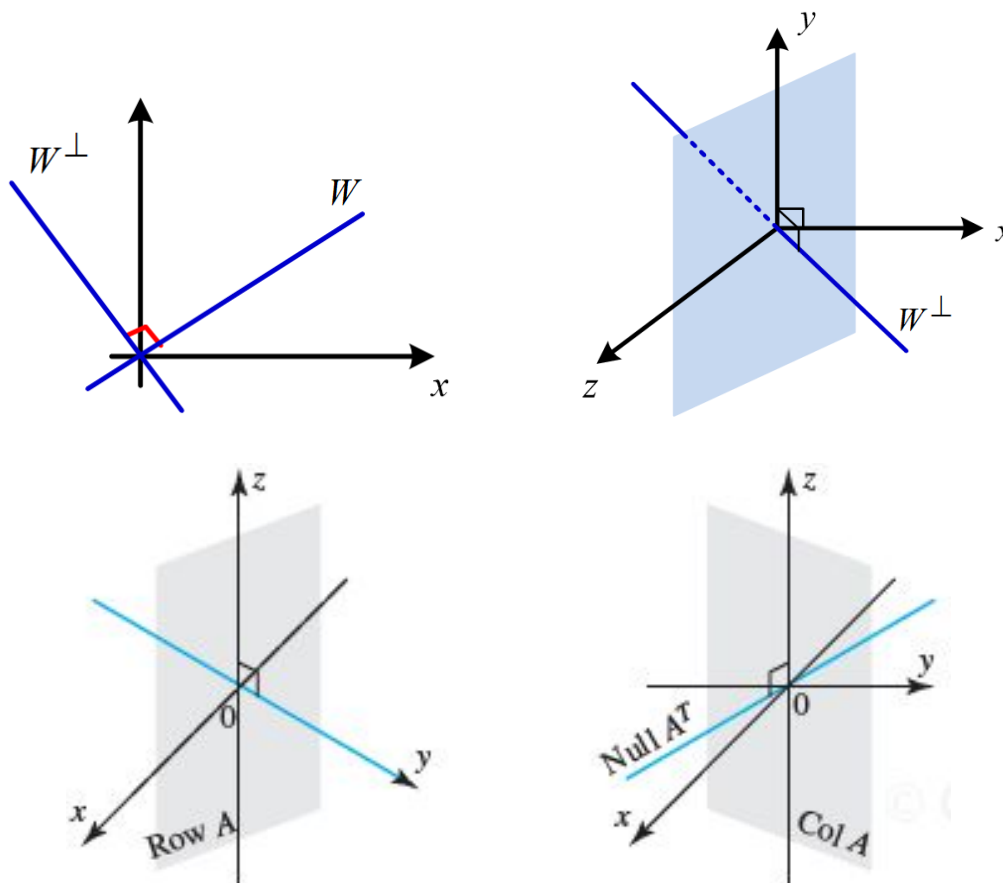
Fundamental Theorem of Linear Algebra

The nullspace is the orthogonal complement of the row space (in \mathbb{R}^n).

The left nullspace is the orthogonal complement of the column space (in \mathbb{R}^m).

If W is a subspace of \mathbb{R}^n

- W^\perp is a subspace of \mathbb{R}^n .
- The only vector common to W and W^\perp is 0.
- The orthogonal complement of W^\perp is W .



Left Nullspace

A matrix A^T has m columns and has r ranks, so the number of free columns of A^T must be $m - r$.

$$\dim N(A^T) = m - r$$

The left nullspace is the collection of vectors \vec{y} for which $A^T \vec{y} = \vec{0}$. Equivalently, $\vec{y}^T A = \vec{0}$, where \vec{y} and $\vec{0}$ are row vectors. We can call “**left nullspace**” because \vec{y}^T is on the left of matrix A in that equation.

To find a basis for the left nullspace we reduce an augmented type of A .

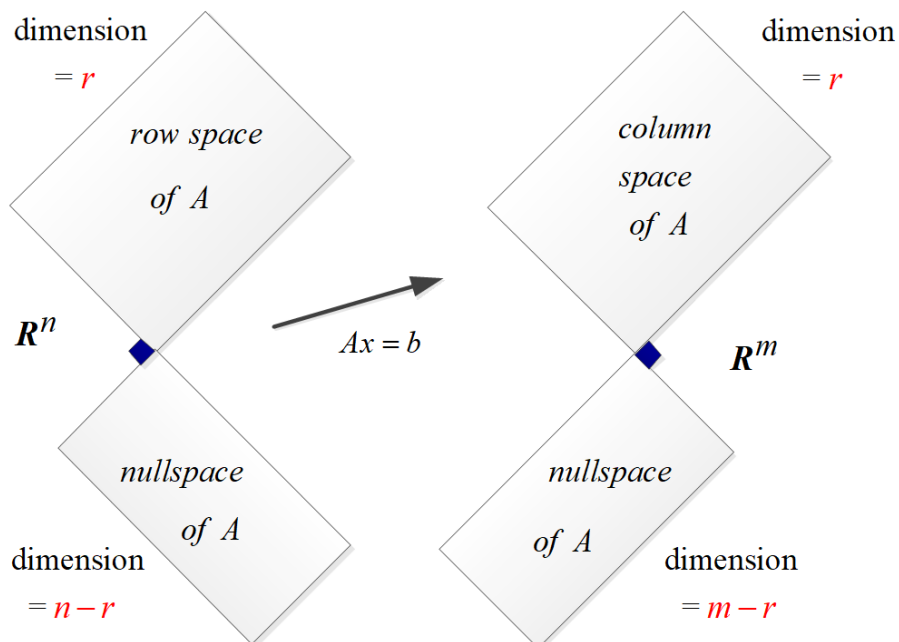
$$\left[A_{m \times n} \mid I_{m \times m} \right] \rightarrow \left[R_{m \times n} \mid E_{m \times m} \right]$$

Where matrix E can be found from $EA = R$.

If matrix A is a square matrix, then $E = A^{-1}$.

The Four Fundamental Subspaces

1. The **row space** is $C(A^T)$, a subspace of \mathbb{R}^n .
2. The **column space** is $C(A)$, a subspace of \mathbb{R}^m .
3. The **null space** is $N(A)$, a subspace of \mathbb{R}^n .
4. The **left null space** is $N(A^T)$, a subspace of \mathbb{R}^m .



Two pairs of orthogonal subspaces.

For an $m \times n$ matrix of rank r :

<i>Fundamental Space</i>	<i>Subspace of</i>	<i>Dimension</i>
Nullspace	\mathbb{R}^n	$n - r$
Column Space	\mathbb{R}^m	r
Row space	\mathbb{R}^n	r
Left nullspace	\mathbb{R}^m	$m - r$

Example

Find a basis for each of the four subspaces associated with matrix A :

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix} \quad R_2 - 2R_1$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \quad x_1 = -2x_2 - 4x_3 \leftarrow \text{Row space}$$

1. Basis for **row space**: $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

2. Basis of the **column spaces**: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\text{Rank}(A) = 1$$

$$\text{Dimension of } A = 1$$

The pivots variables are: x_1

The free variables are: x_2, x_3

$$\text{Set } x_2 = 1 \quad x_3 = 0$$

$$\text{The special solution: } s_1 = (-2, 1, 0)$$

$$\text{Set } x_2 = 0 \quad x_3 = 1$$

$$\text{The special solution: } s_2 = (-4, 0, 1)$$

3. Basis of the **Null space**: $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$

$$A^T = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 8 \end{pmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \end{array}$$

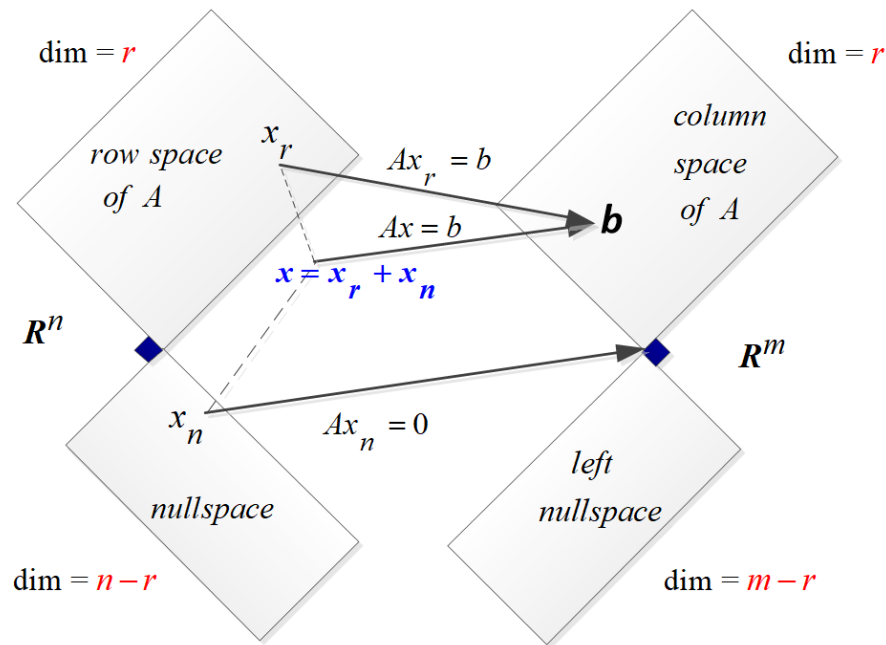
$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad y_1 = -2y_2$$

$$\text{Set } y_2 = 1 \Rightarrow s^*_1 = (-2, 1)$$

4. Basis of the **Left Nullspace**: $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Combining Bases from Subspaces

- Any n linearly independent vectors in \mathbb{R}^n must span \mathbb{R}^n . They are basis. Any n vectors that span \mathbb{R}^n must be independent. They are a basis.
- If the n columns of A are independent, they span \mathbb{R}^n , So $A\vec{x} = \vec{b}$ is solvable,
- If the n columns span \mathbb{R}^n , they are independent. So $A\vec{x} = \vec{b}$ has only one solution.



- When the orthogonal complement of a subspace S is defined to be the subspace whose vectors pairs to zero with the vectors in S . The larger the S is, the more restriction S^\perp has, and hence the smaller S^\perp is.

Theorem – Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.

- a) A is invertible
- b) $A\vec{x} = \vec{0}$ has only the trivial solution
- c) The reduced row echelon form of A is I_n
- d) A is expressible as a product of elementary matrices
- e) $A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ matrix \vec{b}
- f) $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ matrix \vec{b}
- g) $\det(A) \neq 0$
- h) The column vectors of A are linearly independent
- i) The row vectors of A are linearly independent
- j) The column vectors of A span \mathbb{R}^n
- k) The row vectors of A span \mathbb{R}^n
- l) The column vectors of A form a basis for \mathbb{R}^n
- m) The row vectors of A form a basis for \mathbb{R}^n
- n) A has a rank n .
- o) A has nullity 0.
- p) The orthogonal complement of the null space of A is \mathbb{R}^n
- q) The orthogonal complement of the row space of A is $\{0\}$

Exercises

Section 2.9 – Rank and the Fundamental Matrix Spaces

1. Verify that $\text{rank}(A) = \text{rank}(A^T)$

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix}$$

2. Find the rank and nullity of the matrix; then verify that the values obtained satisfy $\text{rank}(A) + N(A) = n$

a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

3. If A is an $m \times n$ matrix, what is the largest possible value for its rank and the smallest possible value of the nullity of A .

4. Discuss how the rank of A varies with t .

a) $A = \begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix}$

b) $A = \begin{bmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{bmatrix}$

5. Are there values of r and s for which

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

Has rank 1? Has rank 2? If so, find those values.

6. Find the row reduced form R and the rank r of A (those depend on c).
Which are the pivot columns of A ? Which variables are free? What are the special solutions and the nullspace matrix N (always depending on c)?

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & c \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} c & c \\ c & c \end{bmatrix}$$

7. Find the row reduced form R and the rank r of A (those depend on c).
Which are the pivot columns of A ? Which variables are free? What are the special solutions and the nullspace matrix N (always depending on c)?

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}$$

8. If A has a rank r , then it has an r by r sub-matrix S that is invertible. Remove $m - r$ rows and $n - r$ columns to find an invertible sub-matrix S inside each A (you could keep the pivot rows and pivot columns of A).

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

9. Suppose that column 3 of 4×6 matrix is all zero. Then x_3 must be a _____ variable. Give one special solution for this matrix.

10. Fill in the missing numbers to make A rank 1, rank 2, rank 3. (your solution should be 3 matrices)

$$A = \begin{pmatrix} & -3 & \\ 1 & 3 & -1 \\ & 9 & -3 \end{pmatrix}$$

11. Fill out these matrices so that they have rank 1:

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & & \\ 4 & & \end{pmatrix} \quad B = \begin{pmatrix} 2 & & \\ 1 & & \\ 2 & 6 & -3 \end{pmatrix} \quad M = \begin{pmatrix} a & b \\ c & \end{pmatrix}$$

12. Suppose A and B are n by n matrices, and $AB = I$. Prove from $\text{rank}(AB) \leq \text{rank}(A)$ that the $\text{rank}(A) = n$. So A is invertible and B must be its two-sided inverse. Therefore $BA = I$ (which is not so obvious!).

13. Every m by n matrix of rank r reduces to $(m \text{ by } r)$ times $(r \text{ by } n)$:

$$A = (\text{pivot columns of } A) (\text{first } r \text{ rows of } R) = (COL)(ROW)^T$$

Write the 3 by 4 matrix $A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{pmatrix}$ as the product of the 3 by 2 from the pivot columns and the 2 by 4 matrix from R .

14. Suppose R is m by n matrix of rank r , with pivot columns first: $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

- What are the shapes of those 4 blocks?
- Find the right-inverse B with $RB = I$ if $r = m$.
- Find the right-inverse C with $CR = I$ if $r = n$.
- What is the reduced row echelon form of R^T (with shapes)?
- What is the reduced row echelon form of $R^T R$ (with shapes)?

Prove that $R^T R$ has the same nullspace as R . Then show that $A^T A$ always has the same nullspace as A (a value fact).

- Suppose you allow elementary column operations on A as well as elementary row operations (which get to R). What is the “row-and-column reduced form” for an m by n matrix of rank r ?

15. True or False (check addition or give a counterexample)

- The symmetric matrices in M (with $A^T = A$) form a subspace.
- The skew-symmetric matrices in M (with $A^T = -A$) form a subspace.
- The un-symmetric matrices in M (with $A^T \neq A$) form a subspace.
- Invertible matrices
- Singular matrices

16. Let $A = \begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 2 & 4 & -3 & 7 & 0 \\ 3 & 6 & -5 & 10 & -2 \\ 5 & 10 & -9 & 16 & 0 \end{pmatrix}$

- Reduce A to row-reduced echelon form.
- What is the rank of A ?
- What are the pivots?
- What are the free variables?
- Find the special solutions. What is the nullspace $N(A)$?
- Exhibit an $r \times r$ submatrix of A which is invertible, where $r = \text{rank}(A)$. (An $r \times r$ submatrix of A is obtained by keeping r rows and r columns of A)

17. Let $A = \begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 2 & 1 & 0 & 0 & -15 \\ 6 & -1 & -8 & -1 & -47 \\ 0 & 2 & 4 & 3 & 16 \end{pmatrix}$

- Reduce A to row-reduced echelon form.
- What is the rank of A ?
- What the pivots?
- What are the free variables?
- Find the special solutions. What is the nullspace $N(A)$?

f) Give the complete solution to $Ax = b$, where $b = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

18. Let $A = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

- Reduce A to row-reduced echelon form.
- What is the rank of A ?
- What the pivots?
- What are the free variables?
- Find the special solutions.
- What is the nullspace $N(A)$?

19. Let $A = \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix}$

- Reduce A to row-reduced echelon form.
- What is the rank of A ?
- What the pivots?
- What are the free variables?
- Find the special solutions.
- What is the nullspace $N(A)$?

20. The 3 by 3 matrix A has rank 2.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 5x_4 &= b_1 \\ A\vec{x} = \vec{b} \quad \text{is} \quad 2x_1 + 4x_2 + 8x_3 + 12x_4 &= b_2 \\ 3x_1 + 6x_2 + 7x_3 + 13x_4 &= b_3 \end{aligned}$$

- Reduce $\begin{bmatrix} A & \vec{b} \end{bmatrix}$ to $\begin{bmatrix} U & \vec{c} \end{bmatrix}$, so that $A\vec{x} = \vec{b}$ becomes triangular system $U\vec{x} = \vec{c}$.
- Find the condition on (b_1, b_2, b_3) for $A\vec{x} = \vec{b}$ to have a solution
- Describe the column space of A . Which plane in \mathbb{R}^3 ?
- Describe the nullspace of A . Which special solutions in \mathbb{R}^4 ?
- Find a particular solution to $A\vec{x} = (0, 6, -6)$ and then complete solution.

21. Find the special solutions and describe the complete solution to $A\vec{x} = \vec{0}$ for

$$A_1 = 3 \text{ by } 4 \text{ zero matrix} \quad A_2 = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \quad A_3 = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$$

Which are the pivot columns? Which are the free variables? What is the R (Reduced Row Echelon matrix) in each case?

22. Create a 3 by 4 matrix whose special solutions to $A\vec{x} = \vec{0}$ are \vec{s}_1 and \vec{s}_2 :

$$\vec{s}_1 = \begin{pmatrix} -3 \\ 2 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{s}_2 = \begin{pmatrix} -2 \\ 0 \\ -6 \\ 1 \end{pmatrix}$$

You could create the matrix A in row reduced form R . Then describe all possible matrices A with the required Nullspace $N(A) = \text{all combinations of } \vec{s}_1 \text{ and } \vec{s}_2$.

23. The plane $x - 3y - z = 12$ is parallel to the plane $x - 3y - z = 0$. One particular point on this plane is $(12, 0, 0)$. All points on the plane have the form (fill the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Construct a matrix whose column space contains $(1, 1, 5)$ and $(0, 3, 1)$ and whose Nullspace contains $(1, 1, 2)$.
- Construct a matrix whose column space contains $(1, 1, 0)$ and $(0, 1, 1)$ and whose Nullspace contains $(1, 0, 1)$ and $(0, 0, 1)$.

26. Construct a matrix whose column space contains $(1, 1, 1)$ and whose Nullspace contains $(1, 1, 1, 1)$.
27. How is the Nullspace $N(C)$ related to the spaces $N(A)$ and $N(B)$, if $C = \begin{bmatrix} A \\ B \end{bmatrix}$?
28. Why does no 3 by 3 matrix have a nullspace that equals its column space?
29. If $AB = 0$ then the column space B is contained in the _____ of A . Give an example of A and B .
30. True or false (with reason if true or example to show it is false)
- A square matrix has no free variables.
 - An invertible matrix has no free variables.
 - An m by n matrix has no more than n pivot variables.
 - An m by n matrix has no more than m pivot variables.
31. Suppose an m by n matrix has r pivots. The number of special solutions is _____.
 The Nullspace contains only $x = 0$ when $r =$ _____.
 The column space is all of \mathbb{R}^m when $r =$ _____.
32. Find the complete solution in the form $\vec{x}_p + \vec{x}_n$ to these full rank system:
- $x + y + z = 4$
 - $\begin{cases} x + y + z = 4 \\ x - y + z = 4 \end{cases}$
33. Find the complete solution in the form $\vec{x}_p + \vec{x}_n$ to the system:
- $$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$
34. If A is 3 x 7 matrix, its largest possible rank is _____. In this case, there is a pivot in every _____ of U and R , the solution to $A\vec{x} = \vec{b}$ _____ (always exists or is unique), and the column space of A is _____. Construct an example of such a matrix A .
35. If A is 6 x 3 matrix, its largest possible rank is _____. In this case, there is a pivot in every _____ of U and R , the solution to $A\vec{x} = \vec{b}$ _____ (always exists or is unique), and the nullspace of A is _____. Construct an example of such a matrix A .
36. Find the rank of $A, A^T A$ and AA^T for $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix}$

37. Explain why these are all false:

- a) The complete solution is any linear combination of \vec{x}_p and \vec{x}_n .
- b) A system $A\vec{x} = \vec{b}$ has at most one particular solution.
- c) The solution \vec{x}_p with all free variables zero is the shortest solution (minimum length $\|\vec{x}\|$). Find a 2 by 2 counterexample.
- d) If A is invertible there is no solution \vec{x}_n in the null space.

38. Write down all known relation between r and m and n if $A\vec{x} = \vec{b}$ has

- a) No solution for some \vec{b} .
- b) Infinitely many solutions for every \vec{b} .
- c) Exactly one solution for some \vec{b} , no solution for other \vec{b} .
- d) Exactly one solution for every \vec{b} .

39. Find a basis for its row space, find a basis for its column space, and determine its rank

$$a) \begin{bmatrix} 0 & 2 & -3 & 4 & 1 & 2 & 1 & 7 \\ 0 & 0 & 3 & -2 & 0 & 4 & -5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 6 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad b) \begin{bmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{bmatrix}$$

40. Find a basis for the row space, find a basis for the null space, find $\dim RS$, find $\dim NS$, and verify $\dim RS + \dim NS = n$

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ 3 & 1 & -3 & -1 \\ 5 & -3 & 5 & 1 \end{bmatrix}$$

41. Determine if \vec{b} lies in the column space of the given matrix. If it does, express \vec{b} as linear combination of the column.

$$\begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

42. Find the transition matrix from B to C and find $[\vec{x}]_C$

$$a) B = \{(3, 1), (-1, -2)\}, \quad C = \{(1, -3), (5, 0)\}, \quad [x]_B = [-1 \quad -2]^T$$

$$b) B = \{(1, 1, 1), (-2, -1, 0), (2, 1, 2)\}, \quad C = \{(-6, -2, 1), (-1, 1, 5), (-1, -1, 1)\},$$

$$[\vec{x}]_B = [-3 \quad 2 \quad 4]^T$$

43. Does A and A^T have the same number of pivots.

(44 – 49) For the given matrix A , which is given in row reduction echelon form

- a) What is the rank of A ?
- b) What is the dimension of A ?
- c) What are the pivots?
- d) What are the free variables?
- e) Find the special (homogeneous) solutions.
- f) What is the nullspace $N(A)$?
- g) Find the particular solution $Ax = b$
- h) Give the complete solution.

$$44. \quad A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{where } b = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$45. \quad A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{where } b = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$46. \quad A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{where } b = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$47. \quad A = \begin{pmatrix} 1 & 0 & 0 & \frac{13}{11} \\ 0 & 1 & 0 & -\frac{17}{11} \\ 0 & 0 & 1 & \frac{6}{11} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{where } b = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$48. \quad A = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{where } b = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$49. \quad A = \begin{pmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{where } b = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

(50 – 55) Find a basis for each of the four subspaces associated with each given matrix

$$50. \quad A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{pmatrix}$$

$$53. \quad D = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix}$$

$$51. \quad B = \begin{pmatrix} 1 & 3 & 0 & 5 \\ 2 & 6 & 1 & 16 \\ 5 & 15 & 0 & 25 \end{pmatrix}$$

$$54. \quad M = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 3 & 1 & -3 & -1 \\ 5 & -3 & 5 & 1 \end{pmatrix}$$

$$52. \quad C = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$55. \quad N = \begin{pmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{pmatrix}$$