

Solution

Section 1.7 – Properties of Determinants: Cramer’s Rule

Exercise

Use Cramer’s Rule with ratios $\frac{\det B_j}{\det A}$ to solve $A\mathbf{x} = b$. Also find the inverse matrix $A^{-1} = \frac{C^T}{\det A}$. Why

is the solution \mathbf{x} is the first part the same as column 3 of A^{-1} ? Which cofactors are involved in computing that column \mathbf{x} ?

$$Ax = b \quad \text{is} \quad \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Find the volumes of the boxes whose edges are columns of A and then rows of A^{-1} .

Solution

$$|A| = \begin{vmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{vmatrix} = 2$$

$$|B_1| = \begin{vmatrix} 0 & 6 & 2 \\ 0 & 4 & 2 \\ 1 & 9 & 0 \end{vmatrix} = 4$$

$$|B_2| = \begin{vmatrix} 2 & 0 & 2 \\ 1 & 0 & 2 \\ 5 & 1 & 0 \end{vmatrix} = -2$$

$$|B_3| = \begin{vmatrix} 2 & 6 & 0 \\ 1 & 4 & 0 \\ 5 & 9 & 1 \end{vmatrix} = 2$$

$$x = \frac{4}{2} = 2; \quad y = \frac{-2}{2} = -1; \quad z = \frac{2}{2} = 1$$

The solution is: (2, -1, 1)

$$C_{11} = \begin{vmatrix} 4 & 2 \\ 9 & 0 \end{vmatrix} = -18$$

$$C_{12} = -\begin{vmatrix} 1 & 2 \\ 5 & 0 \end{vmatrix} = 10$$

$$C_{13} = \begin{vmatrix} 1 & 4 \\ 5 & 9 \end{vmatrix} = -11$$

$$C_{21} = -\begin{vmatrix} 6 & 2 \\ 9 & 0 \end{vmatrix} = 18$$

$$C_{22} = \begin{vmatrix} 2 & 2 \\ 5 & 0 \end{vmatrix} = -10$$

$$C_{23} = -\begin{vmatrix} 2 & 6 \\ 5 & 9 \end{vmatrix} = 12$$

$$C_{31} = \begin{vmatrix} 6 & 2 \\ 4 & 2 \end{vmatrix} = 4$$

$$C_{32} = -\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = -2$$

$$C_{33} = \begin{vmatrix} 2 & 6 \\ 1 & 4 \end{vmatrix} = 2$$

$$C = \begin{pmatrix} -18 & 10 & -11 \\ 18 & -10 & 12 \\ 4 & -2 & 2 \end{pmatrix}$$

$$C^T = \begin{pmatrix} -18 & 18 & 4 \\ 10 & -10 & -2 \\ -11 & 12 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{2} \begin{pmatrix} -18 & 18 & 4 \\ 10 & -10 & -2 \\ -11 & 12 & 2 \end{pmatrix} \quad A^{-1} = \frac{C^T}{\det A}$$

$$= \begin{pmatrix} -9 & 9 & 2 \\ 5 & -5 & -1 \\ -\frac{11}{2} & 6 & 1 \end{pmatrix}$$

The solution \mathbf{x} is the third column of A^{-1} because $\mathbf{b} = (0, 0, 1)$ is the third column of I .

The volume of the boxes whose edges are columns of $\mathbf{A} = \det(\mathbf{A}) = 2$.

Since $|A^T| = |A|$. The box from rows of A^{-1} has volume $|A^{-1}| = \frac{1}{|A|} = \frac{1}{2}$

Exercise

Verify that $\det(AB) = \det(BA)$ and determine whether the equality $\det(A+B) = \det(A) + \det(B)$ holds

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

Solution

$$AB = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{bmatrix}$$

$$\det(AB) = \begin{vmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{vmatrix}$$

$$= -170$$

$$BA = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -3 & 6 \\ 17 & 11 & 4 \\ 10 & 5 & 2 \end{bmatrix}$$

$$\det(BA) = \begin{vmatrix} -1 & -3 & 6 \\ 17 & 11 & 4 \\ 10 & 5 & 2 \end{vmatrix}$$

$$\underline{= -170}$$

Thus, $\underline{\det(AB) = \det(BA)}$

$$\det(A) = \begin{vmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

$$\underline{= 10}$$

$$\det(B) = \begin{vmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{vmatrix}$$

$$\underline{= -17}$$

$$A+B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 3 \\ 10 & 5 & 2 \\ 5 & 0 & 3 \end{bmatrix}$$

$$\det(A+B) = \begin{vmatrix} 3 & 0 & 3 \\ 10 & 5 & 2 \\ 5 & 0 & 3 \end{vmatrix}$$

$$\underline{= -30}$$

$$\det(A) + \det(B) = 10 - 17$$

$$= -7 \neq -30$$

$$\underline{\neq \det(A+B)}$$

Exercise

Verify that $\det(kA) = k^n \det(A)$ $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$, $k = 2$

Solution

$$\begin{aligned} \det(A) &= \begin{vmatrix} -1 & 2 \\ 3 & 4 \end{vmatrix} \\ &= -10 \end{aligned}$$

$$\begin{aligned} \det(2A) &= \begin{vmatrix} -2 & 4 \\ 6 & 8 \end{vmatrix} \\ &= -40 \\ &= 4(-10) \\ &= 2^2(-10) \\ &= k^2 \det(A) \end{aligned}$$

Exercise

Verify that $\det(kA) = k^n \det(A)$ $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{bmatrix}$, $k = -2$

Solution

$$\begin{aligned} \det(A) &= \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{vmatrix} \\ &= 56 \end{aligned}$$

$$\begin{aligned} \det(-2A) &= \begin{vmatrix} -4 & 2 & -6 \\ -6 & -4 & -2 \\ -2 & -8 & -10 \end{vmatrix} \\ &= -448 \\ &= (-2)^3(56) \\ &= k^3 \det(A) \end{aligned}$$

Exercise

Verify that $\det(kA) = k^n \det(A)$ $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix}$, $k = 3$

Solution

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{vmatrix} \\ = -7$$

$$\det(3A) = \begin{vmatrix} 3 & 3 & 3 \\ 0 & 6 & 9 \\ 0 & 3 & -6 \end{vmatrix} \\ = -189 \\ = 3^3(-7) \\ = k^3 \det(A)$$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$

Solution

$$D = \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = -3 - 4 = -7$$

$$D_x = \begin{vmatrix} -4 & 2 \\ -5 & -1 \end{vmatrix} = 4 - (-10) = 14$$

$$D_y = \begin{vmatrix} 3 & -4 \\ 2 & -5 \end{vmatrix} = -15 - (-8) = -7$$

$$x = \frac{D_x}{D} = \frac{14}{-7} = -2$$

$$y = \frac{D_y}{D} = \frac{-7}{-7} = 1$$

$$\therefore \text{Solution: } \underline{(-2, 1)}$$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$$

Solution

$$D = \begin{vmatrix} 2 & 5 \\ 5 & -2 \end{vmatrix} = -29 \quad D_x = \begin{vmatrix} 7 & 5 \\ -3 & -2 \end{vmatrix} = 1 \quad D_y = \begin{vmatrix} 2 & 7 \\ 5 & -3 \end{vmatrix} = -41$$

$$x = \frac{1}{-29} = -\frac{1}{29} \quad x = \frac{D_x}{D}$$

$$y = \frac{41}{29} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \left(-\frac{1}{29}, \frac{41}{29} \right)$$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$$

Solution

$$D = \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = -7 \quad D_x = \begin{vmatrix} -4 & -5 \\ 2 & -1 \end{vmatrix} = 14 \quad D_y = \begin{vmatrix} 3 & -4 \\ 2 & -5 \end{vmatrix} = -7$$

$$x = -\frac{14}{7} = -2 \quad x = \frac{D_x}{D}$$

$$y = \frac{7}{7} = 1 \quad y = \frac{D_y}{D}$$

$$\text{Solution: } (-2, 1)$$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$$

Solution

$$D = \begin{vmatrix} 2 & 5 \\ 5 & -2 \end{vmatrix} = -29 \quad D_x = \begin{vmatrix} 7 & 5 \\ -3 & -2 \end{vmatrix} = 1 \quad D_y = \begin{vmatrix} 2 & 7 \\ 5 & -3 \end{vmatrix} = -41$$

$$x = -\frac{1}{29} \quad x = \frac{D_x}{D}$$

$$y = \frac{41}{29} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \left(-\frac{1}{29}, \frac{41}{29} \right)$$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$$

Solution

$$D = \begin{vmatrix} 4 & -7 \\ 2 & 5 \end{vmatrix} = 34 \quad D_x = \begin{vmatrix} -16 & -7 \\ 9 & 5 \end{vmatrix} = -17 \quad D_y = \begin{vmatrix} 4 & -16 \\ 2 & 9 \end{vmatrix} = 68$$

$$x = -\frac{17}{34} = -\frac{1}{2} \quad x = \frac{D_x}{D}$$

$$y = \frac{68}{34} = 2 \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \left(-\frac{1}{2}, 2 \right)$$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$$

Solution

$$D = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = -1 \quad D_x = \begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix} = 2 \quad D_y = \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = -5$$

$$x = -2 \quad x = \frac{D_x}{D}$$

$$y = 5 \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \left(-2, 5 \right)$$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 3x + 4y = 2 \\ 2x + 5y = -1 \end{cases}$$

Solution

$$D = \begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} = 7$$

$$D_x = \begin{vmatrix} 2 & 4 \\ -1 & 5 \end{vmatrix} = 14$$

$$D_y = \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = -7$$

$$x = \frac{14}{7} = 2$$

$$x = \frac{D_x}{D}$$

$$y = -\frac{7}{7} = -1$$

$$y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(2, -1)}$$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 5x - 2y = 4 \\ -10x + 4y = 7 \end{cases}$$

Solution

$$D = \begin{vmatrix} 5 & -2 \\ -10 & 4 \end{vmatrix} = 0$$

$$D_y = \begin{vmatrix} 5 & 4 \\ -10 & 7 \end{vmatrix} = 75 \neq 0$$

\therefore **No Solution**

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} x - 4y = -8 \\ 5x - 20y = -40 \end{cases}$$

Solution

$$D = \begin{vmatrix} 1 & -4 \\ 5 & -20 \end{vmatrix} = 0$$

$$D_y = \begin{vmatrix} 1 & -8 \\ 5 & -40 \end{vmatrix} = 0$$

$$\frac{1}{5} \times \begin{cases} x - 4y = -8 \\ 5x - 20y = -40 \end{cases}$$

$$\begin{cases} x - 4y = -8 \\ x - 4y = -8 \end{cases}$$

$$\therefore \text{Solution: } \underline{(4y - 8, y)}$$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$$

Solution

$$D = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$$

$$D_x = \begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix} = -6$$

$$D_y = \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = 3$$

$$x = \frac{6}{3} = 2$$

$$x = \frac{D_x}{D}$$

$$y = -\frac{3}{3} = -1$$

$$y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(2, -1)}$$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 2x + 10y = -14 \\ 7x - 2y = -16 \end{cases}$$

Solution

$$D = \begin{vmatrix} 2 & 10 \\ 7 & -2 \end{vmatrix} = -74$$

$$D_x = \begin{vmatrix} -14 & 10 \\ -16 & -2 \end{vmatrix} = 188$$

$$D_y = \begin{vmatrix} 2 & -14 \\ 7 & -16 \end{vmatrix} = 66$$

$$x = -\frac{188}{74} = -\frac{94}{37}$$

$$x = \frac{D_x}{D}$$

$$y = -\frac{66}{74} = -\frac{33}{37}$$

$$y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{\left(-\frac{94}{37}, -\frac{33}{37}\right)}$$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$$

Solution

$$D = \begin{vmatrix} 4 & -3 \\ -3 & 9 \end{vmatrix} = 27$$

$$D_x = \begin{vmatrix} 24 & -3 \\ -1 & 9 \end{vmatrix} = 213$$

$$D_y = \begin{vmatrix} 4 & 24 \\ -3 & -1 \end{vmatrix} = 68$$

$$x = \frac{213}{27} = \frac{71}{9}$$

$$x = \frac{D_x}{D}$$

$$y = \frac{68}{27}$$

$$y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{\left(\frac{71}{9}, \frac{68}{27}\right)}$$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$$

Solution

$$D = \begin{vmatrix} 4 & 2 \\ 3 & -2 \end{vmatrix} = -14 \quad D_x = \begin{vmatrix} 12 & 2 \\ 16 & -2 \end{vmatrix} = -56 \quad D_y = \begin{vmatrix} 4 & 12 \\ 3 & 16 \end{vmatrix} = 28$$

$$x = \frac{56}{14} = 4 \quad x = \frac{D_x}{D}$$

$$y = -\frac{28}{14} = -2 \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(4, -2)}$$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$$

Solution

$$D = \begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} = -10 \quad D_x = \begin{vmatrix} -1 & 2 \\ 6 & -2 \end{vmatrix} = -10 \quad D_y = \begin{vmatrix} 1 & -1 \\ 4 & 6 \end{vmatrix} = 10$$

$$x = 1 \quad x = \frac{D_x}{D}$$

$$y = -1 \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(1, -1)}$$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} x - 2y = 5 \\ -10x + 2y = 4 \end{cases}$$

Solution

$$D = \begin{vmatrix} 1 & -2 \\ -10 & 2 \end{vmatrix} = -18 \quad D_x = \begin{vmatrix} 5 & -2 \\ 4 & 2 \end{vmatrix} = 18 \quad D_y = \begin{vmatrix} 1 & 5 \\ -10 & 4 \end{vmatrix} = 54$$

$$x = -1 \quad x = \frac{D_x}{D}$$

$$y = -\frac{54}{18} = -3 \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(-1, -3)}$$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 12x + 15y = -27 \\ 30x - 15y = -15 \end{cases}$$

Solution

$$\begin{aligned} \frac{1}{3} \times & \begin{cases} 12x + 15y = -27 \\ 30x - 15y = -15 \end{cases} \\ \frac{1}{15} \times & \end{aligned}$$

$$\begin{cases} 4x + 5y = -9 \\ 2x - y = -1 \end{cases}$$

$$D = \begin{vmatrix} 4 & 5 \\ 2 & -1 \end{vmatrix} = -14 \quad D_x = \begin{vmatrix} -9 & 5 \\ -1 & -1 \end{vmatrix} = 14 \quad D_y = \begin{vmatrix} 4 & -9 \\ 2 & -1 \end{vmatrix} = 14$$

$$\underline{x = -1} \quad x = \frac{D_x}{D}$$

$$\underline{y = -1} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(-1, -1)}$$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$$

Solution

$$\begin{aligned} \frac{1}{4} \times & \begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases} \\ \frac{1}{4} \times & \end{aligned}$$

$$\begin{cases} x - y = -3 \\ x + y = -5 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \quad D_x = \begin{vmatrix} -3 & -1 \\ -5 & 1 \end{vmatrix} = -8 \quad D_y = \begin{vmatrix} 1 & -3 \\ 1 & -5 \end{vmatrix} = -2$$

$$\underline{x = -4} \quad x = \frac{D_x}{D}$$

$$\underline{y = -1} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(-4, -1)}$$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x + y = 7 \\ x - y = 3 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

$$D_x = \begin{vmatrix} 7 & 1 \\ 3 & -1 \end{vmatrix} = -10 \quad D_y = \begin{vmatrix} 1 & 7 \\ 1 & 3 \end{vmatrix} = -4$$

$$\underline{x = 5} \quad x = \frac{D_x}{D}$$

$$\underline{y = 2} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(5, 2)}$$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$

Solution

$$D = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$$

$$D_x = \begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix} = -6 \quad D_y = \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = 3$$

$$\underline{x = 2} \quad x = \frac{D_x}{D}$$

$$\underline{y = -1} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(2, -1)}$$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 12x + 3y = 15 \\ 2x - 3y = 13 \end{cases}$

Solution

$$D = \begin{vmatrix} 12 & 3 \\ 2 & -3 \end{vmatrix} = -42$$

$$D_x = \begin{vmatrix} 15 & 3 \\ 13 & -3 \end{vmatrix} = -84 \quad D_y = \begin{vmatrix} 12 & 15 \\ 2 & 13 \end{vmatrix} = 126$$

$$\underline{x = 2} \quad x = \frac{D_x}{D}$$

$$\underline{y = -3} \quad y = \frac{D_y}{D}$$

$$\therefore \textbf{Solution: } \underline{(2, -3)}$$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x - 2y = 5 \\ 5x - y = -2 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & -2 \\ 5 & -1 \end{vmatrix} = 9$$

$$D_x = \begin{vmatrix} 5 & -2 \\ -2 & -1 \end{vmatrix} = -9 \quad D_y = \begin{vmatrix} 1 & 5 \\ 5 & -2 \end{vmatrix} = -27$$

$$\underline{x = -1} \quad x = \frac{D_x}{D}$$

$$\underline{y = -3} \quad y = \frac{D_y}{D}$$

$$\therefore \textbf{Solution: } \underline{(-1, -3)}$$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x - 5y = 17 \\ 2x + 3y = 3 \end{cases}$

Solution

$$D = \begin{vmatrix} 4 & -5 \\ 2 & 3 \end{vmatrix} = 22$$

$$D_x = \begin{vmatrix} 17 & -5 \\ 3 & 3 \end{vmatrix} = 66 \quad D_y = \begin{vmatrix} 4 & 17 \\ 2 & 3 \end{vmatrix} = -22$$

$$\underline{x = 3} \quad x = \frac{D_x}{D}$$

$$\underline{y = -1} \quad y = \frac{D_y}{D}$$

$$\therefore \textbf{Solution: } \underline{(3, -1)}$$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 3x + 2y = 2 \\ 2x + 2y = 3 \end{cases}$$

Solution

$$D = \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = 2$$

$$D_x = \begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} = -2 \quad D_y = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 5$$

$$\underline{x = -1} \quad x = \frac{D_x}{D}$$

$$\underline{y = \frac{5}{2}} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{\left(-1, \frac{5}{2}\right)}$$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} x - 3y = 4 \\ 3x - 4y = 12 \end{cases}$$

Solution

$$D = \begin{vmatrix} 1 & -3 \\ 3 & -4 \end{vmatrix} = 5 \quad D_x = \begin{vmatrix} 4 & -3 \\ 12 & -4 \end{vmatrix} = 20 \quad D_y = \begin{vmatrix} 1 & 4 \\ 3 & 12 \end{vmatrix} = 0$$

$$\underline{x = 4} \quad x = \frac{D_x}{D}$$

$$\underline{y = 0} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(4, 0)}$$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 2x - 9y = 5 \\ 3x - 3y = 11 \end{cases}$$

Solution

$$D = \begin{vmatrix} 2 & -9 \\ 3 & -3 \end{vmatrix} = 21 \quad D_x = \begin{vmatrix} 5 & -9 \\ 11 & -3 \end{vmatrix} = 84 \quad D_y = \begin{vmatrix} 2 & 5 \\ 3 & 11 \end{vmatrix} = 7$$

$$\underline{x = 4} \quad x = \frac{D_x}{D}$$

$$\underline{y = \frac{1}{3}} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{\left(4, \frac{1}{3}\right)}$$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 3x - 4y = 4 \\ x + y = 6 \end{cases}$$

Solution

$$D = \begin{vmatrix} 3 & -4 \\ 1 & 1 \end{vmatrix} = 7$$

$$D_x = \begin{vmatrix} 4 & -4 \\ 6 & 1 \end{vmatrix} = 28 \quad D_y = \begin{vmatrix} 3 & 4 \\ 1 & 6 \end{vmatrix} = 14$$

$$\underline{x = 4} \quad x = \frac{D_x}{D}$$

$$\underline{y = 2} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(4, 2)}$$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 3x = 7y + 1 \\ 2x = 3y - 1 \end{cases}$$

Solution

$$\begin{cases} 3x - 7y = 1 \\ 2x - 3y = -1 \end{cases}$$

$$D = \begin{vmatrix} 3 & -7 \\ 2 & -3 \end{vmatrix} = 5$$

$$D_x = \begin{vmatrix} 1 & -7 \\ -1 & -3 \end{vmatrix} = -10 \quad D_y = \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} = -5$$

$$\underline{x = -2} \quad x = \frac{D_x}{D}$$

$$\underline{y = -1} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(-2, -1)}$$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x = 3y + 2 \\ 5x = 51 - 4y \end{cases}$

Solution

$$\begin{cases} 2x - 3y = 2 \\ 5x + 4y = 51 \end{cases}$$

$$D = \begin{vmatrix} 2 & -3 \\ 5 & 4 \end{vmatrix} = 23$$

$$D_x = \begin{vmatrix} 2 & -3 \\ 51 & 4 \end{vmatrix} = 161 \quad D_y = \begin{vmatrix} 2 & 2 \\ 5 & 51 \end{vmatrix} = 92$$

$$\underline{x = 7} \quad x = \frac{D_x}{D}$$

$$\underline{y = 4} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(7, 4)}$$

Exercise

Use Cramer's rule to solve the system $\begin{cases} y = -4x + 2 \\ 2x = 3y - 1 \end{cases}$

Solution

$$\begin{cases} 4x + y = 2 \\ 2x - 3y = -1 \end{cases}$$

$$D = \begin{vmatrix} 4 & 1 \\ 2 & -3 \end{vmatrix} = -14$$

$$D_x = \begin{vmatrix} 2 & 1 \\ -1 & -3 \end{vmatrix} = -5 \quad D_y = \begin{vmatrix} 4 & 2 \\ 2 & -1 \end{vmatrix} = -8$$

$$\underline{x = \frac{5}{14}} \quad x = \frac{D_x}{D}$$

$$\underline{y = \frac{4}{7}} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{\left(\frac{5}{14}, \frac{4}{7}\right)}$$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 3x = 2 - 3y \\ 2y = 3 - 2x \end{cases}$$

Solution

$$\begin{cases} 3x + 3y = 2 \\ 2x + 2y = 3 \end{cases}$$

$$D = \begin{vmatrix} 3 & 3 \\ 2 & 2 \end{vmatrix} = 0 \qquad D_y = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 5 \neq 0$$

\therefore **No Solution**

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} x + 2y - 3 = 0 \\ 12 = 8y + 4x \end{cases}$$

Solution

$$\begin{cases} x + 2y = 3 \\ 4x + 8y = 12 \end{cases}$$

$$\begin{cases} x + 2y = 3 \\ x + 2y = 3 \end{cases}$$

\therefore **Solution:** $\underline{(3 - 2y, y)}$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 7x - 2y = 3 \\ 3x + y = 5 \end{cases}$$

Solution

$$D = \begin{vmatrix} 7 & -2 \\ 3 & 1 \end{vmatrix} = 13$$

$$D_x = \begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} = 13 \qquad D_y = \begin{vmatrix} 7 & 3 \\ 3 & 5 \end{vmatrix} = 26$$

$$\underline{x = 1} \qquad x = \frac{D_x}{D}$$

$$\underline{y = 2} \qquad y = \frac{D_y}{D}$$

\therefore **Solution:** $\underline{(1, 2)}$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 3x + 2y - z = 4 \\ 3x - 2y + z = 5 \\ 4x - 5y - z = -1 \end{cases}$$

Solution

$$D = \begin{vmatrix} 3 & 2 & -1 \\ 3 & -2 & 1 \\ 4 & -5 & -1 \end{vmatrix} = 42$$

$$D_x = \begin{vmatrix} 4 & 2 & -1 \\ 5 & -2 & 1 \\ -1 & -5 & -1 \end{vmatrix} = 63$$

$$D_y = \begin{vmatrix} 3 & 4 & -1 \\ 3 & 5 & 1 \\ 4 & -1 & -1 \end{vmatrix} = 39$$

$$D_z = \begin{vmatrix} 3 & 2 & 4 \\ 3 & -2 & 5 \\ 4 & -5 & -1 \end{vmatrix} = 99$$

$$x = \frac{63}{42} = \underline{\underline{\frac{3}{2}}} \quad x = \frac{D_x}{D}$$

$$y = \frac{39}{42} = \underline{\underline{\frac{13}{14}}} \quad y = \frac{D_y}{D}$$

$$z = \frac{99}{42} = \underline{\underline{\frac{33}{14}}} \quad z = \frac{D_z}{D}$$

Solution: $\left(\frac{3}{2}, \frac{13}{14}, \frac{33}{14} \right)$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$$

Solution

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = -6$$

$$D_x = \begin{vmatrix} 2 & 1 & 1 \\ 5 & 1 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -6$$

$$D_y = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & -1 \\ 1 & -2 & 1 \end{vmatrix} = -12$$

$$D_z = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 5 \\ 1 & -2 & 2 \end{vmatrix} = 6$$

$$x = \underline{\underline{1}} \quad x = \frac{D_x}{D}$$

$$y = \underline{\underline{2}} \quad y = \frac{D_y}{D}$$

$$z = \underline{\underline{-1}} \qquad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \underline{(1, 2, -1)}$$

Exercise

Use Cramer's rule to solve the system

$$\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$$

Solution

$$D = \begin{vmatrix} 2 & 1 & 1 \\ -1 & -1 & 1 \\ 3 & -1 & 1 \end{vmatrix} \begin{matrix} 2 & 1 \\ -1 & -1 \\ 3 & -1 \end{matrix} = -2 + 3 + 1 + 3 + 2 + 1$$

$$= \underline{\underline{8}}$$

$$D_x = \begin{vmatrix} 9 & 1 & 1 \\ 1 & -1 & 1 \\ 9 & -1 & 1 \end{vmatrix} \begin{matrix} 9 & 1 \\ 1 & -1 \\ 9 & -1 \end{matrix}$$

$$= -9 + 9 - 1 + 9 + 9 - 1$$

$$= \underline{\underline{16}}$$

$$D_y = \begin{vmatrix} 2 & 9 & 1 \\ -1 & 1 & 1 \\ 3 & 9 & 1 \end{vmatrix} \begin{matrix} 2 & 9 \\ -1 & 1 \\ 3 & 9 \end{matrix}$$

$$= 2 + 27 - 9 - 3 - 18 + 9$$

$$= \underline{\underline{8}}$$

$$D_z = \begin{vmatrix} 2 & 1 & 9 \\ -1 & -1 & 1 \\ 3 & -1 & 9 \end{vmatrix} \begin{matrix} 2 & 1 \\ -1 & -1 \\ 3 & -1 \end{matrix}$$

$$= -18 + 3 + 9 + 27 + 2 + 9$$

$$= \underline{\underline{32}}$$

$$x = \underline{\underline{2}} \qquad x = \frac{D_x}{D}$$

$$y = \underline{\underline{1}} \qquad y = \frac{D_y}{D}$$

$$z = \frac{32}{8} = \underline{\underline{4}} \qquad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \underline{(2, 1, 4)}$$

Exercise

Use Cramer's rule to solve the system

$$\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$$

Solution

$$D = \begin{vmatrix} 0 & 3 & -1 \\ 1 & 5 & -1 \\ -3 & 6 & 2 \end{vmatrix} \begin{matrix} 0 & 3 \\ 1 & 5 \\ -3 & 6 \end{matrix}$$
$$= 9 - 6 - 15 - 6$$
$$= \underline{-18}$$

$$D_x = \begin{vmatrix} -1 & 3 & -1 \\ -4 & 5 & -1 \\ 11 & 6 & 2 \end{vmatrix} \begin{matrix} -1 & 3 \\ -4 & 5 \\ 11 & 6 \end{matrix}$$
$$= -10 - 33 + 24 + 55 - 6 + 24$$
$$= \underline{54}$$

$$D_y = \begin{vmatrix} 0 & -1 & -1 \\ 1 & -4 & -1 \\ -3 & 11 & 2 \end{vmatrix} \begin{matrix} 0 & -1 \\ 1 & -4 \\ -3 & 11 \end{matrix}$$
$$= -3 - 11 + 12 + 2$$
$$= \underline{0}$$

$$D_z = \begin{vmatrix} 0 & 3 & -1 \\ 1 & 5 & -4 \\ -3 & 6 & 11 \end{vmatrix} \begin{matrix} 0 & 3 \\ 1 & 5 \\ -3 & 6 \end{matrix}$$
$$= 36 - 6 - 15 - 33$$
$$= \underline{-18}$$

$$x = \underline{-3} \quad x = \frac{D_x}{D}$$

$$y = \underline{0} \quad y = \frac{D_y}{D}$$

$$z = \underline{1} \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \underline{(-3, 0, 1)}$$

Exercise

Use Cramer's rule to solve the system

$$\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$$

Solution

$$D = \begin{vmatrix} 1 & 3 & 4 \\ 2 & -3 & 2 \\ 3 & -1 & 1 \end{vmatrix} \begin{vmatrix} 1 & 3 \\ 2 & -3 \\ 3 & -1 \end{vmatrix}$$
$$= -3 + 18 - 8 + 36 + 2 - 6$$
$$= \underline{39}$$

$$D_x = \begin{vmatrix} 14 & 3 & 4 \\ 10 & -3 & 2 \\ 9 & -1 & 1 \end{vmatrix} \begin{vmatrix} 14 & 3 \\ 10 & -3 \\ 9 & -1 \end{vmatrix}$$
$$= -42 + 54 - 40 + 108 + 28 - 30$$
$$= \underline{78}$$

$$D_y = \begin{vmatrix} 1 & 14 & 4 \\ 2 & 10 & 2 \\ 3 & 9 & 1 \end{vmatrix} \begin{vmatrix} 1 & 14 \\ 2 & 10 \\ 3 & 9 \end{vmatrix}$$
$$= 10 + 84 + 72 - 120 - 18 - 28$$
$$= \underline{0}$$

$$D_z = \begin{vmatrix} 1 & 3 & 14 \\ 2 & -3 & 10 \\ 3 & -1 & 9 \end{vmatrix} \begin{vmatrix} 1 & 3 \\ 2 & -3 \\ 3 & -1 \end{vmatrix}$$
$$= -27 + 90 - 28 + 126 + 10 - 54$$
$$= \underline{117}$$

$$x = \frac{78}{39} = \underline{2} \quad x = \frac{D_x}{D}$$

$$y = \underline{0} \quad y = \frac{D_y}{D}$$

$$z = \frac{117}{39} = \underline{3} \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \underline{(2, 0, 3)}$$

Exercise

Use Cramer's rule to solve the system

$$\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$$

Solution

$$D = \begin{vmatrix} 1 & 4 & -1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix} \begin{vmatrix} 1 & 4 \\ 3 & 2 \\ 2 & -3 \end{vmatrix}$$
$$= 4 + 8 + 9 + 4 + 3 - 24$$
$$= \underline{4}$$

$$D_x = \begin{vmatrix} 20 & 4 & -1 \\ 8 & 2 & 1 \\ -16 & -3 & 2 \end{vmatrix} \begin{vmatrix} 20 & 4 \\ 8 & 2 \\ -16 & -3 \end{vmatrix}$$
$$= 80 - 64 + 24 - 32 + 60 - 64$$
$$= \underline{4}$$

$$D_y = \begin{vmatrix} 1 & 20 & -1 \\ 3 & 8 & 1 \\ 2 & -16 & 2 \end{vmatrix} \begin{vmatrix} 1 & 20 \\ 3 & 8 \\ 2 & -16 \end{vmatrix}$$
$$= 16 + 40 + 48 + 16 + 16 - 120$$
$$= \underline{16}$$

$$D_z = \begin{vmatrix} 1 & 4 & 20 \\ 3 & 2 & 8 \\ 2 & -3 & -16 \end{vmatrix} \begin{vmatrix} 1 & 4 \\ 3 & 2 \\ 2 & -3 \end{vmatrix}$$
$$= -32 + 64 - 180 - 80 + 24 + 192$$
$$= \underline{-12}$$

$$x = \frac{4}{4} = \underline{1} \quad x = \frac{D_x}{D}$$

$$y = \frac{16}{4} = \underline{4} \quad y = \frac{D_y}{D}$$

$$z = -\frac{12}{4} = \underline{-3} \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \underline{(1, 4, -3)}$$

Exercise

Use Cramer's rule to solve the system

$$\begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$$

Solution

$$D = \begin{vmatrix} -2 & 6 & 7 \\ -4 & 5 & 3 \\ -6 & 3 & 5 \end{vmatrix} \begin{matrix} -2 & 6 \\ -4 & 5 \\ -6 & 3 \end{matrix}$$
$$= -50 - 108 - 84 + 210 + 18 + 120$$
$$= \underline{106}$$

$$D_x = \begin{vmatrix} 3 & 6 & 7 \\ 7 & 5 & 3 \\ -4 & 3 & 5 \end{vmatrix} \begin{matrix} 3 & 6 \\ 7 & 5 \\ -4 & 3 \end{matrix}$$
$$= 75 - 72 + 147 + 140 - 27 - 210$$
$$= \underline{53}$$

$$D_y = \begin{vmatrix} -2 & 3 & 7 \\ -4 & 7 & 3 \\ -6 & -4 & 5 \end{vmatrix} \begin{matrix} -2 & 3 \\ -4 & 7 \\ -6 & -4 \end{matrix}$$
$$= -70 - 54 + 112 + 294 - 24 + 60$$
$$= \underline{318}$$

$$D_z = \begin{vmatrix} -2 & 6 & 3 \\ -4 & 5 & 7 \\ -6 & 3 & -4 \end{vmatrix} \begin{matrix} -2 & 6 \\ -4 & 5 \\ -6 & 3 \end{matrix}$$
$$= 40 - 252 - 36 + 90 + 42 - 96$$
$$= \underline{-212}$$

$$x = \frac{53}{106} = \underline{\frac{1}{2}} \quad x = \frac{D_x}{D}$$

$$y = \frac{318}{106} = \underline{3} \quad y = \frac{D_y}{D}$$

$$z = -\frac{212}{106} = \underline{-2} \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \underline{\left(\frac{1}{2}, 3, -2\right)}$$

Exercise

Use Cramer's rule to solve the system

$$\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$$

Solution

$$D = \begin{vmatrix} 2 & -1 & 1 \\ 3 & -3 & 4 \\ 4 & -2 & 3 \end{vmatrix} \begin{vmatrix} 2 & -1 \\ 3 & -3 \\ 4 & -2 \end{vmatrix}$$
$$= -18 - 16 - 6 + 12 + 16 + 9$$
$$= -3$$

$$D_x = \begin{vmatrix} 1 & -1 & 1 \\ 5 & -3 & 4 \\ 4 & -2 & 3 \end{vmatrix} \begin{vmatrix} 1 & -1 \\ 5 & -3 \\ 4 & -2 \end{vmatrix}$$
$$= -9 - 16 - 10 + 12 + 8 + 15$$
$$= 0$$

$$D_y = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 5 & 4 \\ 4 & 4 & 3 \end{vmatrix} \begin{vmatrix} 2 & 1 \\ 3 & 5 \\ 4 & 4 \end{vmatrix}$$
$$= 30 + 16 + 12 - 20 - 32 - 9$$
$$= -3$$

$$D_z = \begin{vmatrix} 2 & -1 & 1 \\ 3 & -3 & 5 \\ 4 & -2 & 4 \end{vmatrix} \begin{vmatrix} 2 & -1 \\ 3 & -3 \\ 4 & -2 \end{vmatrix}$$
$$= -24 - 20 - 6 + 12 + 20 + 12$$
$$= -6$$

$$x = -\frac{0}{-3} = 0 \quad x = \frac{D_x}{D}$$

$$y = \frac{-3}{-3} = 1 \quad y = \frac{D_y}{D}$$

$$z = \frac{-6}{-3} = 2 \quad z = \frac{D_z}{D}$$

\therefore **Solution:** $(0, 1, 2)$

Exercise

Use Cramer's rule to solve the system

$$\begin{cases} 3x - 4y + 4z = 7 \\ x - y - 2z = 2 \\ 2x - 3y + 6z = 5 \end{cases}$$

Solution

$$D = \begin{vmatrix} 3 & -4 & 4 \\ 1 & -1 & -2 \\ 2 & -3 & 6 \end{vmatrix} \begin{vmatrix} 3 & -4 \\ 1 & -1 \\ 2 & -3 \end{vmatrix}$$
$$= -18 + 16 - 12 + 8 - 18 + 24$$
$$= 0$$

$$D_z = \begin{vmatrix} 3 & -4 & 7 \\ 1 & -1 & 2 \\ 2 & -3 & 5 \end{vmatrix} \begin{vmatrix} 3 & -4 \\ 1 & -1 \\ 2 & -3 \end{vmatrix}$$
$$= -15 - 16 - 21 + 14 + 18 + 20$$
$$= 0$$

$$\begin{array}{l} -3 \times (2) \quad \begin{cases} -3x + 3y + 6z = -6 \\ 2x - 3y + 6z = 5 \end{cases} \\ \hline -x + 12z = -1 \end{array}$$

$$x = 12z + 1$$

$$(2) \rightarrow y = 12z + 1 - 2z - 2$$
$$= 10z - 1$$

$$\therefore \text{Solution: } (12z + 1, 10z - 1, z)$$

Exercise

Use Cramer's rule to solve the system

$$\begin{cases} x - 2y - z = 2 \\ 2x - y + z = 4 \\ -x + y + z = 4 \end{cases}$$

Solution

$$D = \begin{vmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} \begin{vmatrix} 1 & -2 \\ 2 & -1 \\ -1 & 1 \end{vmatrix}$$
$$= -1 + 2 - 2 + 1 - 1 + 4$$
$$= 3$$

$$D_x = \begin{vmatrix} 2 & -2 & -1 \\ 4 & -1 & 1 \\ 4 & 1 & 1 \end{vmatrix} \begin{vmatrix} 2 & -2 \\ 4 & -1 \\ 4 & 1 \end{vmatrix}$$

$$= -2 - 8 - 4 - 4 - 2 + 8$$

$$= -12$$

$$D_y = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ -1 & 4 & 1 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 2 & 4 \\ -1 & 4 \end{vmatrix}$$

$$= 4 - 2 - 8 - 4 - 4 - 4$$

$$= -18$$

$$D_z = \begin{vmatrix} 1 & -2 & 2 \\ 2 & -1 & 4 \\ -1 & 1 & 4 \end{vmatrix} \begin{vmatrix} 1 & -2 \\ 2 & -1 \\ -1 & 1 \end{vmatrix}$$

$$= -4 + 8 + 4 - 2 - 4 + 16$$

$$= 18$$

$$x = -\frac{12}{3} = -4 \quad x = \frac{D_x}{D}$$

$$y = -\frac{18}{3} = -6 \quad y = \frac{D_y}{D}$$

$$z = \frac{18}{3} = 6 \quad z = \frac{D_z}{D}$$

∴ Solution: $(-4, -6, 6)$

Exercise

Use Cramer's rule to solve the system

$$\begin{cases} x + y + z = 3 \\ -y + 2z = 1 \\ -x + z = 0 \end{cases}$$

Solution

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{vmatrix}$$

$$= -4$$

$$D_x = \begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 0 \end{vmatrix}$$

$$\underline{\underline{=-4}}$$

$$D_y = \begin{vmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{vmatrix}$$

$$\underline{\underline{=-4}}$$

$$D_z = \begin{vmatrix} 1 & 1 & 3 \\ 0 & -1 & 1 \\ -1 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{vmatrix}$$

$$\underline{\underline{=-4}}$$

$$x = \frac{4}{4} \underline{\underline{=1}} \quad x = \frac{D_x}{D}$$

$$y = \frac{4}{4} \underline{\underline{=1}} \quad y = \frac{D_y}{D}$$

$$z = \frac{4}{4} \underline{\underline{=1}} \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \underline{(1, 1, 1)}$$

Exercise

Use Cramer's rule to solve the system

$$\begin{cases} 3x + y + 3z = 14 \\ 7x + 5y + 8z = 37 \\ x + 3y + 2z = 9 \end{cases}$$

Solution

$$D = \begin{vmatrix} 3 & 1 & 3 \\ 7 & 5 & 8 \\ 1 & 3 & 2 \end{vmatrix} \begin{vmatrix} 3 & 1 \\ 7 & 5 \\ 1 & 3 \end{vmatrix}$$

$$= 30 + 8 + 62 - 15 - 72 - 14$$

$$\underline{\underline{=0}}$$

$$D_z = \begin{vmatrix} 3 & 1 & 14 \\ 7 & 5 & 37 \\ 1 & 3 & 9 \end{vmatrix} \begin{vmatrix} 3 & 1 \\ 7 & 5 \\ 1 & 3 \end{vmatrix}$$

$$= 135 + 37 + 294 - 70 - 333 - 63$$

$$\underline{\underline{=0}}$$

$$\begin{array}{l} -3 \times (1) \\ (3) \end{array} \begin{cases} -9x - 3y - 9z = -42 \\ x + 3y + 2z = 9 \end{cases}$$

$$-8x - 7z = -33$$

$$\underline{x = -\frac{7}{8}z + \frac{33}{8}}$$

$$\begin{aligned} (1) \rightarrow y &= 14 - 3z - 3\left(-\frac{7}{8}z + \frac{33}{8}\right) \\ &= \frac{13}{8} - \frac{3}{8}z \end{aligned}$$

$$\therefore \text{Solution: } \underline{\left(\frac{33}{8} - \frac{7}{8}z, \frac{13}{8} - \frac{3}{8}z, z\right)}$$

Exercise

Use Cramer's rule to solve the system

$$\begin{cases} 4x - 2y + z = 7 \\ x + y + z = -2 \\ 4x + 2y + z = 3 \end{cases}$$

Solution

$$\begin{aligned} D &= \begin{vmatrix} 4 & -2 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix} \begin{vmatrix} 4 & -2 \\ 1 & 1 \\ 4 & 2 \end{vmatrix} \\ &= -12 \end{aligned}$$

$$\begin{aligned} D_x &= \begin{vmatrix} 7 & -2 & 1 \\ -2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} \begin{vmatrix} 7 & -2 \\ -2 & 1 \\ 3 & 2 \end{vmatrix} \\ &= -24 \end{aligned}$$

$$\begin{aligned} D_y &= \begin{vmatrix} 4 & 7 & 1 \\ 1 & -2 & 1 \\ 4 & 3 & 1 \end{vmatrix} \begin{vmatrix} 4 & 7 \\ 1 & -2 \\ 4 & 3 \end{vmatrix} \\ &= 12 \end{aligned}$$

$$\begin{aligned} D_z &= \begin{vmatrix} 4 & -2 & 7 \\ 1 & 1 & -2 \\ 4 & 2 & 3 \end{vmatrix} \begin{vmatrix} 4 & -2 \\ 1 & 1 \\ 4 & 2 \end{vmatrix} \\ &= 36 \end{aligned}$$

$$x = \frac{24}{-12} = -2 \quad x = \frac{D_x}{D}$$

$$y = -\frac{12}{12} = -1 \quad y = \frac{D_y}{D}$$

$$z = -\frac{36}{12} = -3 \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \underline{(2, -1, -3)}$$

Exercise

Use Cramer's rule to solve the system

$$\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$$

Solution

$$D = \begin{vmatrix} 0 & 2 & -1 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix} \begin{vmatrix} 0 & 2 \\ 1 & 2 \\ 2 & 3 \end{vmatrix}$$
$$= 1$$

$$D_x = \begin{vmatrix} 7 & 2 & -1 \\ 17 & 2 & 1 \\ -1 & 3 & 2 \end{vmatrix} \begin{vmatrix} 7 & 2 \\ 17 & 2 \\ -1 & 3 \end{vmatrix}$$
$$= -116$$

$$D_y = \begin{vmatrix} 0 & 7 & -1 \\ 1 & 17 & 1 \\ 2 & -1 & 2 \end{vmatrix} \begin{vmatrix} 0 & 7 \\ 1 & 17 \\ 2 & -1 \end{vmatrix}$$
$$= 35$$

$$D_z = \begin{vmatrix} 0 & 2 & 7 \\ 1 & 2 & 17 \\ 2 & 3 & -1 \end{vmatrix} \begin{vmatrix} 0 & 2 \\ 1 & 2 \\ 2 & 3 \end{vmatrix}$$
$$= 63$$

$$x = \frac{-116}{D} \quad x = \frac{D_x}{D}$$

$$y = \frac{35}{D} \quad y = \frac{D_y}{D}$$

$$z = \frac{63}{D} \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \underline{(-116, 35, 63)}$$

Exercise

Use Cramer's rule to solve the system

$$\begin{cases} 2x - 2y + z = -4 \\ 6x + 4y - 3z = -24 \\ x - 2y + 2z = 1 \end{cases}$$

Solution

$$D = \begin{vmatrix} 2 & -2 & 1 \\ 6 & 4 & -3 \\ 1 & -2 & 2 \end{vmatrix} \begin{vmatrix} 2 & -2 \\ 6 & 4 \\ 1 & -2 \end{vmatrix}$$

$$\underline{= 18}$$

$$D_x = \begin{vmatrix} -4 & -2 & 1 \\ -24 & 4 & -3 \\ 1 & -2 & 2 \end{vmatrix} \begin{vmatrix} -4 & -2 \\ -24 & 4 \\ 1 & -2 \end{vmatrix}$$

$$\underline{= -54}$$

$$D_y = \begin{vmatrix} 2 & -4 & 1 \\ 6 & -24 & -3 \\ 1 & 1 & 2 \end{vmatrix} \begin{vmatrix} 2 & -4 \\ 6 & -24 \\ 1 & 1 \end{vmatrix}$$

$$\underline{= 0}$$

$$D_z = \begin{vmatrix} 2 & -2 & -4 \\ 6 & 4 & -24 \\ 1 & -2 & 1 \end{vmatrix} \begin{vmatrix} 2 & -2 \\ 6 & 4 \\ 1 & -2 \end{vmatrix}$$

$$\underline{= 36}$$

$$x = -\frac{54}{18} \quad x = \frac{D_x}{D}$$

$$\underline{= -3}$$

$$y = 0 \quad y = \frac{D_y}{D}$$

$$z = 2 \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \underline{(-3, 0, 2)}$$

Exercise

Use Cramer's rule to solve the system

$$\begin{cases} 9x + 3y + z = 4 \\ 16x + 4y + z = 2 \\ 25x + 5y + z = 2 \end{cases}$$

Solution

$$D = \begin{vmatrix} 9 & 3 & 1 \\ 16 & 4 & 1 \\ 25 & 5 & 1 \end{vmatrix} \begin{matrix} 9 & 3 \\ 16 & 4 \\ 25 & 5 \end{matrix}$$
$$= -2$$

$$D_x = \begin{vmatrix} 4 & 3 & 1 \\ 2 & 4 & 1 \\ 2 & 5 & 1 \end{vmatrix} \begin{matrix} 4 & 3 \\ 2 & 4 \\ 2 & 5 \end{matrix}$$
$$= -2$$

$$D_y = \begin{vmatrix} 9 & 4 & 1 \\ 16 & 2 & 1 \\ 25 & 2 & 1 \end{vmatrix} \begin{matrix} 9 & 4 \\ 16 & 2 \\ 25 & 2 \end{matrix}$$
$$= 18$$

$$D_z = \begin{vmatrix} 9 & 3 & 4 \\ 16 & 4 & 2 \\ 25 & 5 & 2 \end{vmatrix} \begin{matrix} 9 & 3 \\ 16 & 4 \\ 25 & 5 \end{matrix}$$
$$= -44$$

$$x = \frac{-2}{-2} \qquad x = \frac{D_x}{D}$$
$$= 1$$

$$y = \frac{18}{-2} \qquad y = \frac{D_y}{D}$$
$$= -9$$

$$z = \frac{-44}{-2} \qquad z = \frac{D_z}{D}$$
$$= 22$$

$$\therefore \text{Solution: } \underline{(1, -9, 22)}$$

Exercise

Use Cramer's rule to solve the system

$$\begin{cases} 2x - y + 2z = -8 \\ x + 2y - 3z = 9 \\ 3x - y - 4z = 3 \end{cases}$$

Solution

$$D = \begin{vmatrix} 2 & -1 & 2 \\ 1 & 2 & -3 \\ 3 & -1 & -4 \end{vmatrix} \begin{vmatrix} 2 & -1 \\ 1 & 2 \\ 3 & -1 \end{vmatrix} \\ = -31$$

$$D_x = \begin{vmatrix} -8 & -1 & 2 \\ 9 & 2 & -3 \\ 3 & -1 & -4 \end{vmatrix} \begin{vmatrix} -8 & -1 \\ 9 & 2 \\ 3 & -1 \end{vmatrix} \\ = 31$$

$$D_y = \begin{vmatrix} 2 & -8 & 2 \\ 1 & 9 & -3 \\ 3 & 3 & -4 \end{vmatrix} \begin{vmatrix} 2 & -8 \\ 1 & 9 \\ 3 & 3 \end{vmatrix} \\ = -62$$

$$D_z = \begin{vmatrix} 2 & -1 & -8 \\ 1 & 2 & 9 \\ 3 & -1 & 3 \end{vmatrix} \begin{vmatrix} 2 & -1 \\ 1 & 2 \\ 3 & -1 \end{vmatrix} \\ = 62$$

$$x = -\frac{31}{-31} \qquad x = \frac{D_x}{D} \\ = 1$$

$$y = \frac{62}{-62} \qquad y = \frac{D_y}{D} \\ = -1$$

$$z = -\frac{62}{62} \qquad z = \frac{D_z}{D} \\ = -1$$

$$\therefore \text{Solution: } \underline{(-1, -1, -1)}$$

Exercise

Use Cramer's rule to solve the system

$$\begin{cases} x - 3z = -5 \\ 2x - y + 2z = 16 \\ 7x - 3y - 5z = 19 \end{cases}$$

Solution

$$D = \begin{vmatrix} 1 & 0 & -3 \\ 2 & -1 & 2 \\ 7 & -3 & -5 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 2 & -1 \\ 7 & -3 \end{vmatrix} \\ = 8$$

$$D_x = \begin{vmatrix} -5 & 0 & -3 \\ 16 & -1 & 2 \\ 19 & -3 & -5 \end{vmatrix} \begin{vmatrix} -5 & 0 \\ 16 & -1 \\ 19 & -3 \end{vmatrix} \\ = 32$$

$$D_y = \begin{vmatrix} 1 & -5 & -3 \\ 2 & 16 & 2 \\ 7 & 19 & -5 \end{vmatrix} \begin{vmatrix} 1 & -5 \\ 2 & 16 \\ 7 & 19 \end{vmatrix} \\ = -16$$

$$D_z = \begin{vmatrix} 1 & 0 & -5 \\ 2 & -1 & 16 \\ 7 & -3 & 19 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 2 & -1 \\ 7 & -3 \end{vmatrix} \\ = 24$$

$$x = \frac{32}{8} \qquad x = \frac{D_x}{D} \\ = 4$$

$$y = -\frac{16}{8} \qquad y = \frac{D_y}{D} \\ = -2$$

$$z = \frac{24}{8} \qquad z = \frac{D_z}{D} \\ = 3$$

$$\therefore \text{Solution: } \underline{(4, -2, 3)}$$

Exercise

Use Cramer's rule to solve the system

$$\begin{cases} x + 2y - z = 5 \\ 2x - y + 3z = 0 \\ 2y + z = 1 \end{cases}$$

Solution

$$D = \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 0 & 2 & 1 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 2 & -1 \\ 0 & 2 \end{vmatrix}$$
$$= -15$$

$$D_x = \begin{vmatrix} 5 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 2 & 1 \end{vmatrix} \begin{vmatrix} 5 & 2 \\ 0 & -1 \\ 1 & 2 \end{vmatrix}$$
$$= -30$$

$$D_y = \begin{vmatrix} 1 & 5 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 1 \end{vmatrix} \begin{vmatrix} 1 & 5 \\ 2 & 0 \\ 0 & 1 \end{vmatrix}$$
$$= -15$$

$$D_z = \begin{vmatrix} 1 & 2 & 5 \\ 2 & -1 & 0 \\ 0 & 2 & 1 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 2 & -1 \\ 0 & 2 \end{vmatrix}$$
$$= 15$$

$$x = \frac{30}{-15}$$
$$= -2$$
$$x = \frac{D_x}{D}$$

$$y = \frac{-15}{-15}$$
$$= 1$$
$$y = \frac{D_y}{D}$$

$$z = \frac{15}{-15}$$
$$= -1$$
$$z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \underline{(2, 1, -1)}$$

Exercise

Use Cramer's rule to solve the system

$$\begin{cases} x + y + z = 6 \\ 3x + 4y - 7z = 1 \\ 2x - y + 3z = 5 \end{cases}$$

Solution

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 4 & -7 \\ 2 & -1 & 3 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 3 & 4 \\ 2 & -1 \end{vmatrix} \\ = -29$$

$$D_x = \begin{vmatrix} 6 & 1 & 1 \\ 1 & 4 & -7 \\ 5 & -1 & 3 \end{vmatrix} \begin{vmatrix} 6 & 1 \\ 1 & 4 \\ 5 & -1 \end{vmatrix} \\ = -29$$

$$D_y = \begin{vmatrix} 1 & 6 & 1 \\ 3 & 1 & -7 \\ 2 & 5 & 3 \end{vmatrix} \begin{vmatrix} 1 & 6 \\ 3 & 1 \\ 2 & 5 \end{vmatrix} \\ = -87$$

$$D_z = \begin{vmatrix} 1 & 2 & 6 \\ 2 & -1 & 1 \\ 0 & 2 & 5 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 2 & -1 \\ 0 & 2 \end{vmatrix} \\ = -58$$

$$x = \frac{29}{29} \qquad x = \frac{D_x}{D} \\ = 1$$

$$y = \frac{87}{29} \qquad y = \frac{D_y}{D} \\ = 3$$

$$z = \frac{58}{29} \qquad z = \frac{D_z}{D} \\ = 2$$

$$\therefore \text{Solution: } (1, 3, 2)$$

Exercise

Use Cramer's rule to solve the system

$$\begin{cases} 3x + 2y + 3z = 3 \\ 4x - 5y + 7z = 1 \\ 2x + 3y - 2z = 6 \end{cases}$$

Solution

$$D = \begin{vmatrix} 3 & 2 & 3 \\ 4 & -5 & 7 \\ 2 & 3 & -2 \end{vmatrix} \begin{vmatrix} 3 & 2 \\ 4 & -5 \\ 2 & 3 \end{vmatrix} \\ = 77 \mid$$

$$D_x = \begin{vmatrix} 3 & 2 & 3 \\ 1 & -5 & 7 \\ 6 & 3 & -2 \end{vmatrix} \begin{vmatrix} 3 & 2 \\ 1 & -5 \\ 6 & 3 \end{vmatrix} \\ = 154 \mid$$

$$D_y = \begin{vmatrix} 3 & 3 & 3 \\ 4 & 1 & 7 \\ 2 & 6 & -2 \end{vmatrix} \begin{vmatrix} 3 & 3 \\ 4 & 1 \\ 2 & 6 \end{vmatrix} \\ = 0 \mid$$

$$D_z = \begin{vmatrix} 3 & 2 & 3 \\ 4 & -5 & 1 \\ 2 & 3 & 6 \end{vmatrix} \begin{vmatrix} 3 & 2 \\ 4 & -5 \\ 2 & 3 \end{vmatrix} \\ = -77 \mid$$

$$x = \frac{154}{77} = 2 \mid \\ = 2 \mid$$

$$x = \frac{D_x}{D}$$

$$y = 0 \mid$$

$$y = \frac{D_y}{D}$$

$$z = -\frac{77}{77} \\ = -1 \mid$$

$$z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } (2, 0, -1) \mid$$

Exercise

Use Cramer's rule to solve the system

$$\begin{cases} 4x + 5y = 2 \\ 11x + y + 2z = 3 \\ x + 5y + 2z = 1 \end{cases}$$

Solution

$$D = \begin{vmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} \begin{matrix} 4 & 5 \\ 11 & 1 \\ 1 & 5 \end{matrix}$$
$$= -132$$

$$D_x = \begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} \begin{matrix} 2 & 5 \\ 3 & 1 \\ 1 & 5 \end{matrix}$$
$$= -36$$

$$D_y = \begin{vmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} \begin{matrix} 4 & 2 \\ 11 & 3 \\ 1 & 1 \end{matrix}$$
$$= -24$$

$$D_z = \begin{vmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix} \begin{matrix} 4 & 5 \\ 11 & 1 \\ 1 & 5 \end{matrix}$$
$$= 12$$

$$x = \frac{36}{132}$$
$$= \frac{3}{11}$$
$$x = \frac{D_x}{D}$$

$$y = \frac{24}{132}$$
$$= \frac{2}{11}$$
$$y = \frac{D_y}{D}$$

$$z = -\frac{12}{132}$$
$$= -\frac{1}{11}$$
$$z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \left(\frac{3}{11}, \frac{2}{11}, -\frac{1}{11} \right)$$

Exercise

Use Cramer's rule to solve the system

$$\begin{cases} x - 4y + z = 6 \\ 4x - y + 2z = -1 \\ 2x + 2y - 3z = -20 \end{cases}$$

Solution

$$D = \begin{vmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{vmatrix} \begin{vmatrix} 1 & -4 \\ 4 & -1 \\ 2 & 2 \end{vmatrix} \\ = -55$$

$$D_x = \begin{vmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{vmatrix} \begin{vmatrix} 6 & -4 \\ -1 & -1 \\ -20 & 2 \end{vmatrix} \\ = 144$$

$$D_y = \begin{vmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{vmatrix} \begin{vmatrix} 1 & 6 \\ 4 & -1 \\ 2 & -20 \end{vmatrix} \\ = 61$$

$$D_z = \begin{vmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{vmatrix} \begin{vmatrix} 1 & -4 \\ 4 & -1 \\ 2 & 2 \end{vmatrix} \\ = -230$$

$$x = -\frac{144}{55} \quad x = \frac{D_x}{D}$$

$$y = -\frac{61}{55} \quad y = \frac{D_y}{D}$$

$$z = \frac{230}{55} \quad z = \frac{D_z}{D}$$

$$= \frac{46}{11}$$

$$\therefore \text{Solution: } \left(-\frac{144}{55}, -\frac{61}{55}, \frac{46}{11} \right)$$

Exercise

Use Cramer's rule to solve the system

$$\begin{cases} 2x - y + z = -1 \\ 3x + 4y - z = -1 \\ 4x - y + 2z = -1 \end{cases}$$

Solution

$$D = \begin{vmatrix} 2 & -1 & 1 \\ 3 & 4 & -1 \\ 4 & -1 & 2 \end{vmatrix} \begin{vmatrix} 2 & -1 \\ 3 & 4 \\ 4 & -1 \end{vmatrix} \\ = 5$$

$$D_x = \begin{vmatrix} -1 & -1 & 1 \\ -1 & 4 & -1 \\ -1 & -1 & 2 \end{vmatrix} \begin{vmatrix} -1 & -1 \\ -1 & 4 \\ -1 & -1 \end{vmatrix} \\ = -5$$

$$D_y = \begin{vmatrix} 2 & -1 & 1 \\ 3 & -1 & -1 \\ 4 & -1 & 2 \end{vmatrix} \begin{vmatrix} 2 & -1 \\ 3 & -1 \\ 4 & -1 \end{vmatrix} \\ = 5$$

$$D_z = \begin{vmatrix} 2 & -1 & -1 \\ 3 & 4 & -1 \\ 4 & -1 & -1 \end{vmatrix} \begin{vmatrix} 2 & -1 \\ 3 & 4 \\ 4 & -1 \end{vmatrix} \\ = 10$$

$$x = \frac{-5}{5} \qquad x = \frac{D_x}{D} \\ = -1$$

$$y = \frac{5}{5} \qquad y = \frac{D_y}{D} \\ = 1$$

$$z = \frac{10}{5} \qquad z = \frac{D_z}{D} \\ = 2$$

$$\therefore \text{Solution: } \underline{(-1, 1, 2)}$$

Exercise

Use Cramer's rule to solve the system

$$\begin{cases} -x_1 - 4x_2 + 2x_3 + x_4 = -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 = 14 \\ -x_1 + x_2 + 3x_3 + x_4 = 11 \\ -x_1 - 2x_2 + x_3 - 4x_4 = -4 \end{cases}$$

Solution

$$D = \begin{vmatrix} -1 & -4 & 2 & 1 \\ 2 & -1 & 7 & 9 \\ -1 & 1 & 3 & 1 \\ -1 & -2 & 1 & -4 \end{vmatrix} \\ = -243$$

$$D_1 = \begin{vmatrix} -32 & -4 & 2 & 1 \\ 14 & -1 & 7 & 9 \\ 11 & 1 & 3 & 1 \\ -4 & -2 & 1 & -4 \end{vmatrix} \\ = -2115$$

$$D_2 = \begin{vmatrix} -1 & -32 & 2 & 1 \\ 2 & 14 & 7 & 9 \\ -1 & 11 & 3 & 1 \\ -1 & -4 & 1 & -4 \end{vmatrix} \\ = -1834$$

$$D_3 = \begin{vmatrix} -1 & -4 & -32 & 1 \\ 2 & -1 & 14 & 9 \\ -1 & 1 & 11 & 1 \\ -1 & -2 & -4 & -4 \end{vmatrix} \\ = -1279$$

$$D_4 = \begin{vmatrix} -1 & -4 & 2 & -32 \\ 2 & -1 & 7 & 14 \\ -1 & 1 & 3 & 11 \\ -1 & -2 & 1 & -4 \end{vmatrix} \\ = 883$$

$$x_1 = \frac{-2115}{-243} \\ = \frac{235}{27}$$

$$x_2 = \frac{-1834}{-243}$$

$$= \frac{1834}{243}$$

$$x_3 = \frac{-1279}{-243}$$

$$= \frac{1279}{243}$$

$$x_4 = -\frac{883}{243}$$

$$\therefore \text{Solution: } \left(\frac{235}{27}, \frac{1834}{243}, \frac{1279}{243}, -\frac{883}{243} \right)$$

Exercise

Show that the matrix A is invertible for all values of θ , then find A^{-1} using $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$

$$A = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

$$\begin{aligned} \det(A) &= \begin{vmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \cos^2 \theta + \sin^2 \theta \\ &= 1 \end{aligned}$$

$\Rightarrow A$ is invertible

$$\begin{aligned} C_{11} &= \begin{vmatrix} \cos \theta & 0 \\ 0 & 1 \end{vmatrix} \\ &= \cos \theta \end{aligned}$$

$$\begin{aligned} C_{12} &= -\begin{vmatrix} -\sin \theta & 0 \\ 0 & 1 \end{vmatrix} \\ &= \sin \theta \end{aligned}$$

$$\begin{aligned} C_{13} &= \begin{vmatrix} -\sin \theta & \cos \theta \\ 0 & 0 \end{vmatrix} \\ &= 0 \end{aligned}$$

$$\begin{aligned} C_{21} &= -\begin{vmatrix} \sin \theta & 0 \\ 0 & 1 \end{vmatrix} \\ &= -\sin \theta \end{aligned}$$

$$\begin{aligned} C_{22} &= \begin{vmatrix} \cos \theta & 0 \\ 0 & 1 \end{vmatrix} \\ &= \cos \theta \end{aligned}$$

$$\begin{aligned} C_{23} &= -\begin{vmatrix} \cos \theta & \sin \theta \\ 0 & 0 \end{vmatrix} \\ &= 0 \end{aligned}$$

$$\begin{aligned} C_{31} &= \begin{vmatrix} \sin \theta & 0 \\ \cos \theta & 0 \end{vmatrix} \\ &= 0 \end{aligned}$$

$$\begin{aligned} C_{32} &= -\begin{vmatrix} \cos \theta & 0 \\ -\sin \theta & 0 \end{vmatrix} \\ &= 0 \end{aligned}$$

$$\begin{aligned} C_{33} &= \begin{vmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{vmatrix} \\ &= 1 \end{aligned}$$

$$\text{adj}(A) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} A^{-1} &= \frac{1}{\det(A)} \text{adj}(A) \\ &= \frac{1}{1} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$