Solution *Section* 3.5 – Inverse Trigonometric Functions

Exercise

Find the exact value of the expression whenever it is defined: $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Solution

$$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$

Exercise

Find the exact value of the expression whenever it is defined: $\arccos\left(\frac{\sqrt{2}}{2}\right)$

Solution

$$\arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

Exercise

Find the exact value of the expression whenever it is defined: $\arctan\left(-\frac{\sqrt{3}}{3}\right)$

Solution

$$\arctan\left(-\frac{\sqrt{3}}{3}\right) = -\arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$$

Exercise

Find the exact value of the expression whenever it is defined: $\sin\left[\arcsin\left(-\frac{3}{10}\right)\right]$

Solution

$$\alpha = \arcsin\left(-\frac{3}{10}\right) \Rightarrow \sin\alpha = -\frac{3}{10}$$

$$\sin\left[\arcsin\left(-\frac{3}{10}\right)\right] = -\frac{3}{10}$$

Exercise

Find the exact value of the expression whenever it is defined: $\tan\left[\arctan(14)\right]$

Solution

$$\tan\left[\arctan(14)\right] = 14$$

Exercise

Find the exact value of the expression whenever it is defined: $\sin\left[\sin^{-1}\left(\frac{2}{3}\right)\right]$

Solution

$$\sin\left[\sin^{-1}\left(\frac{2}{3}\right)\right] = \frac{2}{3}$$

Exercise

Find the exact value of the expression whenever it is defined: $\cos^{-1}\left[\cos\left(\frac{5\pi}{6}\right)\right]$

Solution

$$\cos^{-1}\left[\cos\left(\frac{5\pi}{6}\right)\right] = \frac{5\pi}{6} \qquad 0 \le \frac{5\pi}{6} \le \pi$$

Exercise

Find the exact value of the expression whenever it is defined: $\tan^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right]$

Solution

$$\tan^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right] = -\frac{\pi}{6} \qquad -\frac{\pi}{2} \le -\frac{\pi}{6} \le \frac{\pi}{2}$$

Exercise

Find the exact value of the expression whenever it is defined: $\arcsin\left[\sin\left(-\frac{\pi}{2}\right)\right]$

Solution

$$\arcsin\left[\sin\left(-\frac{\pi}{2}\right)\right] = -\frac{\pi}{2} \qquad -\frac{\pi}{2} \le -\frac{\pi}{2} \le \frac{\pi}{2}$$

Exercise

Find the exact value of the expression whenever it is defined: $\arccos\left[\cos(0)\right]$

Solution

$$\arccos\left[\cos(0)\right] = 0 \qquad 0 \le 0 \le \pi$$

Exercise

Find the exact value of the expression whenever it is defined: $\arctan\left[\tan\left(-\frac{\pi}{4}\right)\right]$

Solution

$$\tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right] \underline{= -\frac{\pi}{4}} \qquad -\frac{\pi}{2} \le -\frac{\pi}{4} \le \frac{\pi}{2}$$

Exercise

Find the exact value of the expression whenever it is defined: $\sin\left[\arcsin\left(\frac{1}{2}\right) + \arccos 0\right]$

Solution

$$\sin\left[\arcsin\left(\frac{1}{2}\right) + \arccos 0\right] = \sin\left(\frac{\pi}{6} + 0\right) = \sin\left(\frac{\pi}{6}\right) \underline{= \frac{1}{2}}$$

Exercise

Find the exact value of the expression whenever it is defined: $\cos\left[\arctan\left(-\frac{3}{4}\right) - \arcsin\frac{4}{5}\right]$

Solution

$$\begin{aligned}\cos\left[\arctan\left(-\frac{3}{4}\right) - \arcsin\frac{4}{5}\right] &= \cos(\alpha - \beta) \\ &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\end{aligned}$$

$\alpha = \arctan\left(-\frac{3}{4}\right) \Rightarrow \tan\alpha = -\frac{3}{4}$ $r = \sqrt{3^2 + 4^2} = 5$ $\sin\alpha = -\frac{3}{5} \qquad \cos\alpha = \frac{4}{5}$	$\beta = \arcsin\frac{4}{5} \Rightarrow \sin\beta = \frac{4}{5}$ $\Rightarrow \cos\beta = \frac{3}{5}$

$$\cos\left[\arctan\left(-\frac{3}{4}\right) - \arcsin\frac{4}{5}\right] = \frac{4}{5}\frac{3}{5} + \left(-\frac{3}{5}\right)\frac{4}{5} \underline{= 0}$$

Exercise

Find the exact value of the expression whenever it is defined: $\tan\left[\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)\right]$

Solution

$$\tan\left[\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)\right] = \tan\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \tan\left(\frac{\pi}{3}\right) \underline{= \sqrt{3}}$$

Exercise

Find the exact value of the expression whenever it is defined: $\sin\left[2\arccos\left(-\frac{3}{5}\right)\right]$

Solution

$$\sin\left[2\arccos\left(-\frac{3}{5}\right)\right] = \sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\alpha = \arccos\left(-\frac{3}{5}\right) \to \cos\alpha = -\frac{3}{5}$$

$$\sin\alpha = \frac{3}{5}$$

$$\sin\left[2\arccos\left(-\frac{3}{5}\right)\right] = 2\frac{3}{5}\left(-\frac{3}{5}\right) \underline{\left. = -\frac{18}{25} \right|}$$

Exercise

Find the exact value of the expression whenever it is defined: $\cos\left[2\sin^{-1}\left(\frac{15}{17}\right)\right]$

Solution

$$\cos\left[2\sin^{-1}\left(\frac{15}{17}\right)\right] = \cos 2\alpha = 1 - 2\sin^2\alpha$$

$$\alpha = \sin^{-1}\left(\frac{15}{17}\right) \to \sin\alpha = \frac{15}{17}$$

$$\cos\left[2\sin^{-1}\left(\frac{15}{17}\right)\right] = 1 - 2\left(\frac{15}{17}\right)^2$$

$$= 1 - \frac{450}{289}$$

$$\underline{\left. = -\frac{161}{289} \right|}$$

Exercise

Find the exact value of the expression whenever it is defined: $\tan\left[2\tan^{-1}\left(\frac{3}{4}\right)\right]$

Solution

$$\tan\left[2\tan^{-1}\left(\frac{3}{4}\right)\right] = \tan 2\alpha \Rightarrow \alpha = \tan^{-1}\left(\frac{3}{4}\right) \quad \to \quad \tan\alpha = \frac{3}{4}$$

$$\tan\left[2\tan^{-1}\left(\frac{3}{4}\right)\right] = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$= \frac{2\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^2}$$

$$= \frac{\frac{3}{2}}{1-\frac{9}{16}}$$

$$= \frac{3}{2}\frac{16}{7}$$

$$\boxed{= \frac{24}{7}}$$

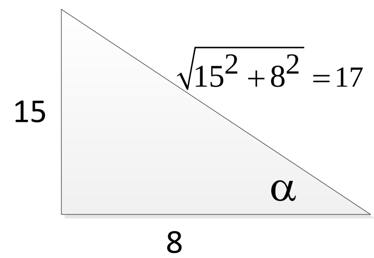
Exercise

Find the exact value of the expression whenever it is defined: $\cos\left[\frac{1}{2}\tan^{-1}\left(\frac{8}{15}\right)\right]$

Solution

$$\cos\left[\frac{1}{2}\tan^{-1}\left(\frac{8}{15}\right)\right] = \cos\left(\frac{1}{2}\alpha\right)$$

$$\to \alpha = \tan^{-1}\left(\frac{8}{15}\right) \Rightarrow \tan\alpha = \frac{8}{15}$$



$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1}{2}(1+\cos\alpha)}$$

$$= \sqrt{\frac{1}{2}\left(1+\frac{8}{17}\right)}$$

$$= \sqrt{\frac{25}{34}}$$

$$\boxed{= \frac{5}{\sqrt{34}}} \quad \textit{or} \quad \frac{5\sqrt{34}}{34}$$

Exercise

Evaluate without using a calculator: $\cos\left(\cos^{-1}\frac{3}{5}\right)$

Solution

$$\cos\left(\cos^{-1}\frac{3}{5}\right) \boxed{= \frac{3}{5}}$$

Exercise

Evaluate without using a calculator: $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

Solution

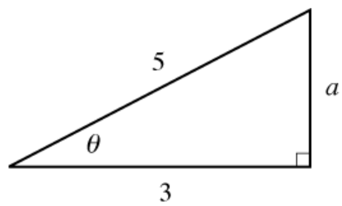
$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) \boxed{= \frac{5\pi}{6}}$$

Exercise

Evaluate without using a calculator: $\tan\left(\cos^{-1}\frac{3}{5}\right)$

Solution

$\tan\left(\cos^{-1}\frac{3}{5}\right)$



$5^2 = 3^2 + a^2 \to a = 4$

$\tan\left(\cos^{-1}\frac{3}{5}\right) = \tan\theta$

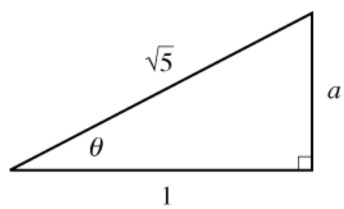
$= \boxed{\frac{4}{3}}$

Exercise

Evaluate without using a calculator: $\sin\left(\cos^{-1}\frac{1}{\sqrt{5}}\right)$

Solution

$\sin\left(\cos^{-1}\frac{1}{\sqrt{5}}\right)$



$\left(\sqrt{5}\right)^2 = 1^2 + a^2 \to a^2 = 5 - 1 \to a = 2$

$\sin\left(\cos^{-1}\frac{1}{\sqrt{5}}\right) = \sin\theta$

$= \boxed{\frac{2}{\sqrt{5}}}$

Exercise

Evaluate without using a calculator: $\cos\left(\sin^{-1}\frac{1}{2}\right)$

Solution

$\cos\left(\sin^{-1}\frac{1}{2}\right)$

$\sin\frac{\pi}{6} = \frac{1}{2} \Rightarrow \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$\cos\left(\sin^{-1}\frac{1}{2}\right) = \cos\frac{\pi}{6}$

$= \boxed{\frac{\sqrt{3}}{2}}$

Exercise

Evaluate without using a calculator: $\sin\left(\sin^{-1}\frac{3}{5}\right)$

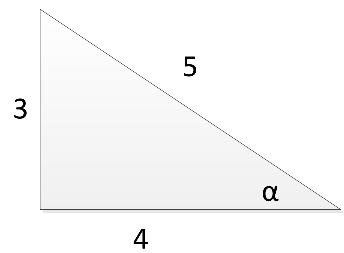
Solution

$$\sin\left(\sin^{-1}\frac{3}{5}\right) = \frac{3}{5}$$

Exercise

Evaluate without using a calculator: $\cos\left(\tan^{-1}\frac{3}{4}\right)$

Solution



$$\alpha = \tan^{-1}\frac{3}{4} \Rightarrow \tan\alpha = \frac{3}{4}$$

$$r = \sqrt{3^2 + 4^2} = 5$$

$$\Rightarrow \boxed{\cos\alpha = \frac{4}{5}}$$

Exercise

Evaluate without using a calculator: $\tan\left(\sin^{-1}\frac{3}{5}\right)$

Solution

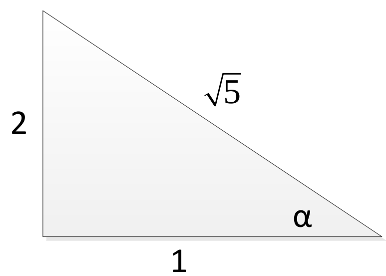
$$\sin\alpha = \frac{3}{5}$$

$$\boxed{\tan\left(\sin^{-1}\frac{3}{5}\right) = \frac{3}{4}}$$

Exercise

Evaluate without using a calculator: $\sec\left(\cos^{-1}\frac{1}{\sqrt{5}}\right)$

Solution



$$\alpha = \cos^{-1}\frac{1}{\sqrt{5}} \quad \rightarrow \quad \cos\alpha = \frac{1}{\sqrt{5}}$$

$$\sec\alpha = \frac{1}{\cos\alpha} = \frac{1}{\frac{1}{\sqrt{5}}} = \sqrt{5}$$

Exercise

Evaluate without using a calculator: $\cot\left(\tan^{-1}\frac{1}{2}\right)$

Solution

$$\alpha = \tan^{-1}\frac{1}{2} \Rightarrow \tan\alpha = \frac{1}{2}$$

$$\cot\alpha = \frac{1}{\tan\alpha} = 2$$

Exercise

Write an equivalent expression that involves x only for $\cos\left(\cos^{-1}x\right)$

Solution

$$\alpha = \cos^{-1}x \Rightarrow \cos\alpha = x$$

$$\cos\left(\cos^{-1}x\right) = \cos\alpha = x$$

Exercise

Write an equivalent expression that involves x only for $\tan\left(\cos^{-1}x\right)$

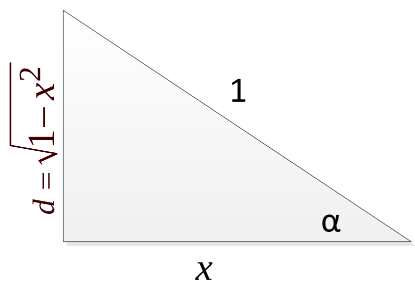
Solution

$$\alpha = \cos^{-1}x \Rightarrow \cos\alpha = x = \frac{x}{1}$$

$$x^2 + d^2 = 1 \Rightarrow d^2 = 1 - x^2$$

$$d = \sqrt{1-x^2}$$

$$\tan\left(\cos^{-1}x\right) = \tan\alpha = \frac{\sqrt{1-x^2}}{x}$$



Exercise

Write an equivalent expression that involves x only for $\csc\left(\sin^{-1}\frac{1}{x}\right)$

Solution

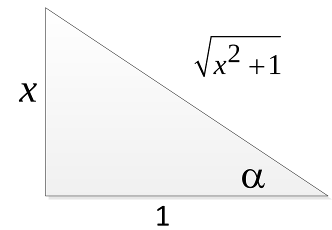
$$\alpha = \sin^{-1}\frac{1}{x} \Rightarrow \sin\alpha = \frac{1}{x}$$

$$\csc\left(\sin^{-1}x\right) = \csc\alpha = \frac{1}{\sin\alpha} = x$$

Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\sin\left(\tan^{-1} x\right)$

Solution



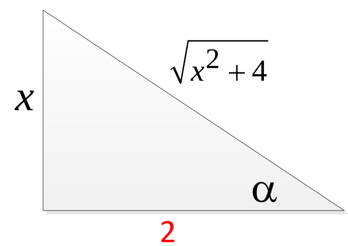
$$\sin\left(\tan^{-1} x\right) = \sin\alpha \Rightarrow \alpha = \tan^{-1} x \to \tan\alpha = x$$

$$\sin\left(\tan^{-1} x\right) = \frac{x}{\sqrt{x^2+1}}$$

Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\sec\left(\sin^{-1}\frac{x}{\sqrt{x^2+4}}\right)$

Solution



$$\alpha = \sin^{-1}\frac{x}{\sqrt{x^2+4}} \Rightarrow \sin\alpha = \frac{x}{\sqrt{x^2+4}}$$

$$\sqrt{\left(\sqrt{x^2+4}\right)^2 - x^2} = \sqrt{x^2+4-x^2} = \sqrt{4} = 2$$

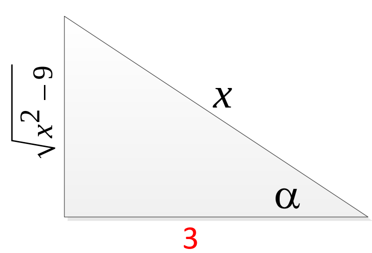
$$\sec\left(\sin^{-1}\frac{x}{\sqrt{x^2+4}}\right) = \frac{1}{\cos\alpha}$$

$$= \frac{2}{\sqrt{x^2+4}}$$

Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\cot\left(\sin^{-1}\frac{\sqrt{x^2-9}}{x}\right)$

Solution



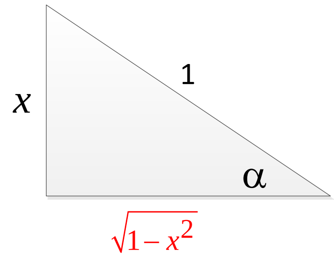
$$\alpha = \sin^{-1}\frac{\sqrt{x^2-9}}{x} \Rightarrow \sin\alpha = \frac{\sqrt{x^2-9}}{x}$$

$$\cot\left(\sin^{-1}\frac{\sqrt{x^2-9}}{x}\right) = \cot\alpha = \frac{3}{\sqrt{x^2-9}}$$

Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\sin\left(2\sin^{-1}x\right)$

Solution



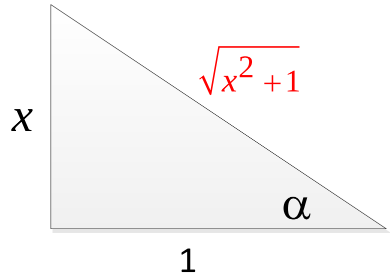
$$\alpha = \sin^{-1}x \rightarrow \sin\alpha = x$$

$$\begin{aligned}\sin\left(2\sin^{-1}x\right) &= \sin 2\alpha \\ &= 2\sin\alpha\cos\alpha \\ &= \boxed{2x\sqrt{1-x^2}}\end{aligned}$$

Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\cos\left(2\tan^{-1}x\right)$

Solution



$$\alpha = \tan^{-1}x \rightarrow \tan\alpha = x$$

$$\begin{aligned}\cos\left(2\tan^{-1}x\right) &= \cos(2\alpha) \\ &= 2\cos^2\alpha - 1 \\ &= 2\left(\frac{1}{\sqrt{x^2+1}}\right)^2 - 1 \\ &= \frac{2}{x^2+1} - 1 \\ &= \boxed{\frac{-x^2+1}{x^2+1}}\end{aligned}$$

Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\cos\left(\frac{1}{2}\arccos x\right)$

Solution

$$\alpha = \arccos x \Rightarrow \cos\alpha = x$$

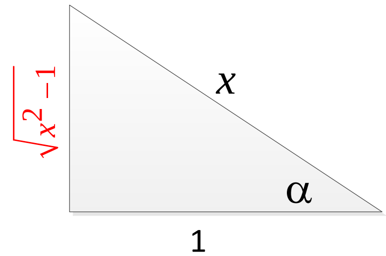
$$\begin{aligned}\cos\left(\frac{1}{2}\arccos x\right) &= \cos\left(\frac{\alpha}{2}\right) == \sqrt{\frac{1+\cos\alpha}{2}} \\ &= \boxed{\sqrt{\frac{1+x}{2}}}\end{aligned}$$

Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\tan\left(\frac{1}{2}\cos^{-1}\frac{1}{x}\right)$

Solution

$$\alpha = \cos^{-1}\frac{1}{x} \quad \Rightarrow \quad \cos\alpha = \frac{1}{x}$$



$$\tan\left(\frac{1}{2}\cos^{-1}\frac{1}{x}\right) = \tan\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{\sin\alpha}$$

$$= \frac{1-\frac{1}{x}}{\frac{\sqrt{x^2-1}}{x}} = \frac{\frac{x-1}{x}}{\frac{\sqrt{x^2-1}}{x}}$$

$$= \frac{x-1}{\sqrt{x^2-1}}$$

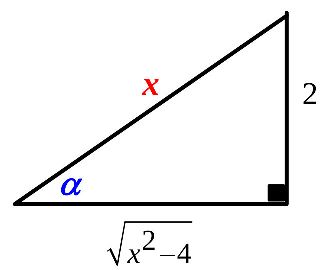
Exercise

Write the expression as an algebraic expression in x: $\sec\left(\tan^{-1}\frac{2}{\sqrt{x^2-4}}\right) \quad x > 0$

Solution

$$\tan\alpha = \frac{2}{\sqrt{x^2-4}}$$

$$\sec\alpha = \frac{x}{\sqrt{x^2-4}}$$



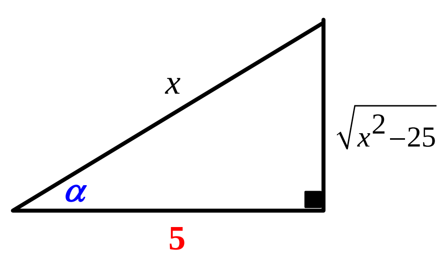
Exercise

Write the expression as an algebraic expression in x: $\sec\left(\sin^{-1}\frac{\sqrt{x^2-25}}{x}\right) \quad x > 0$

Solution

$$\sin\alpha = \frac{\sqrt{x^2-25}}{x}$$

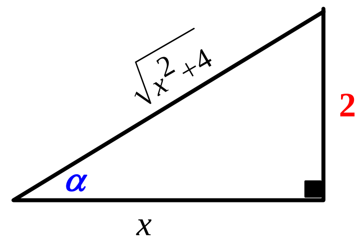
$$\sec\alpha = \frac{x}{5}$$



Exercise

Write the expression as an algebraic expression in x: $\sin\left(\cos^{-1}\dfrac{x}{\sqrt{x^2+4}}\right) \quad x>0$

Solution



$$\cos\alpha = \frac{x}{\sqrt{x^2+4}}$$

$$\boxed{\sin\alpha = \frac{2}{\sqrt{x^2+4}}}$$

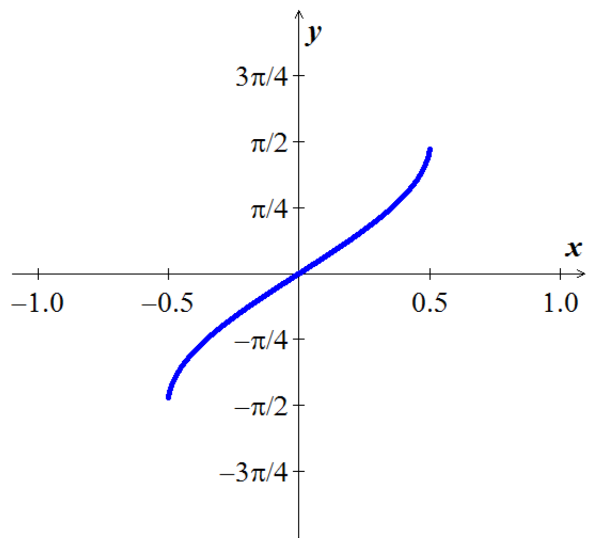
Exercise

Sketch he graph of the equation: $y=\sin^{-1}2x$

Solution

$$-\frac{\pi}{2} \le y \le \frac{\pi}{2} \quad \textit{and} \quad -1 \le 2x \le 1$$

$$-\frac{1}{2} \le x \le \frac{1}{2}$$



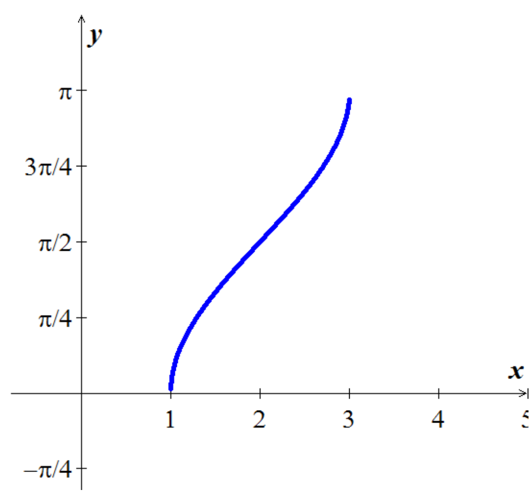
Exercise

Sketch he graph of the equation: $y=\sin^{-1}(x-2)+\frac{\pi}{2}$

Solution

$$-\frac{\pi}{2}+\frac{\pi}{2} \le y \le \frac{\pi}{2}+\frac{\pi}{2} \quad \textit{and} \quad -1 \le x-2 \le 1$$

$$0 \le y \le \pi \quad \textit{and} \quad 1 \le x \le 3$$



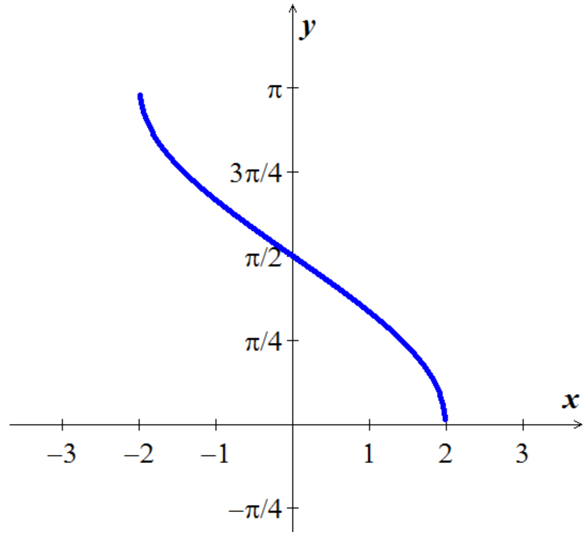
Exercise

Sketch he graph of the equation: $y = \cos^{-1}\frac{1}{2}x$

Solution

$$0 \le y \le \pi \quad \textit{and} \quad -1 \le \frac{1}{2}x \le 1$$

$$-2 \le x \le 2$$



Exercise

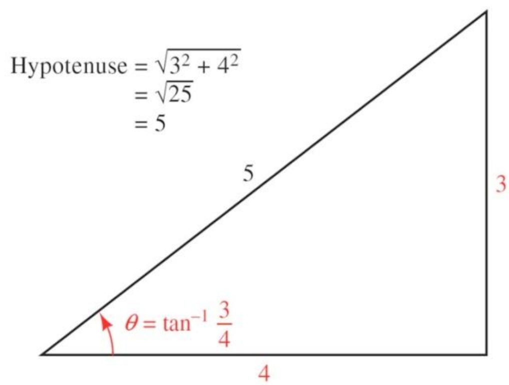
Evaluate $\sin\left(\tan^{-1}\frac{3}{4}\right)$ without using a calculator

Solution

$$\theta = \tan^{-1}\frac{3}{4} \Rightarrow \tan\theta = \frac{3}{4} \rightarrow 0° < \theta < 90°$$

$$\sin\left(\tan^{-1}\frac{3}{4}\right) = \sin\theta$$

$$= \frac{3}{5}$$



Exercise

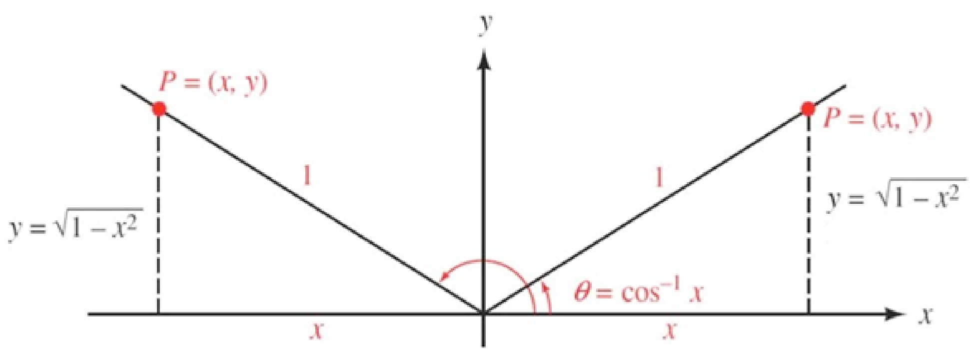
Evaluate $\sin(\cos^{-1}x)$ as an equivalent expression in x only

Solution

$$\sin(\theta) = \frac{y}{r}$$

$$= \frac{\sqrt{1-x^2}}{1}$$

$$= \sqrt{1-x^2}$$



$$\sin(\cos^{-1}x) = \sin\theta$$

$$= \sqrt{1-x^2}$$