

Section 2.4 – Properties of Division

Long Division

Divide $(x^3 + 2x^2 - 5x - 6) \div (x + 1)$

$$\begin{array}{r}
 \text{Quotient} \\
 \overline{x^2 + x - 6} \\
 x + 1 \overline{) x^3 + 2x^2 - 5x - 6} \quad \leftarrow \text{Dividend} \\
 \underline{x^3 + x^2} \\
 x^2 - 5x \\
 \underline{x^2 - x} \\
 -6x - 6 \\
 \underline{-6x - 6} \\
 0 \quad \leftarrow \text{Remainder}
 \end{array}$$

Divisor

$$\underline{Q(x) = x^2 + x - 6}$$

$$\underline{R(x) = 0}$$

Example

Use the long division to find the quotient and the remainder: $(x^4 - 16) \div (x^2 + 3x + 1)$

Solution

$$\begin{array}{r}
 x^2 - 3x + 8 \\
 x^2 + 3x + 1 \overline{) x^4 + 0x^3 + 0x^2 + 0x - 16} \\
 \underline{x^4 + 3x^3 + x^2} \\
 -3x^3 - x^2 \\
 \underline{-3x^3 - 9x^2 - 3x} \\
 8x^2 + 3x - 16 \\
 \underline{8x^2 + 24x + 8} \\
 -21x - 24
 \end{array}$$

$$\frac{x^4 - 16}{x^2 + 3x + 1} = x^2 - 3x + 8 + \frac{-21x - 24}{x^2 + 3x + 1}$$

$$\underline{x^4 - 16 = (x^2 + 3x + 1)(x^2 - 3x + 8) + (-21x - 24)}$$

Remainder Theorem

If a number c is substituted for x in the polynomial $f(x)$, then the result $f(c)$ is the remainder that would be obtained by dividing $f(x)$ by $x - c$.

That is, if $f(x) = (x - c)Q(x) + R(x)$ then $f(c) = R$

Example

If $f(x) = x^3 - 3x^2 + x + 5$, use the remainder theorem to find $f(2)$

Solution

$$\begin{array}{r} x^2 - x - 1 \\ x - 2 \overline{) x^3 - 3x^2 + x + 5} \\ \underline{x^3 - 2x^2} \\ -x^2 + x \\ \underline{-x^2 + 2x} \\ -x + 5 \\ \underline{-x + 2} \\ 3 \end{array}$$

$$\underline{f(2) = 3}$$

Factor Theorem

A polynomial $f(x)$ has a factor $x - c$ if and only if $f(c) = 0$

Example

Show that $x - 2$ is a factor of $f(x) = x^3 - 4x^2 + 3x + 2$.

Solution

$$\begin{aligned} \text{Since } f(2) &= (2)^3 - 4(2)^2 + 3(2) \\ &= 0 \end{aligned}$$

From the factor theorem; $x - 2$ is a factor of $f(x)$.

Synthetic Division

Use synthetic division to find the quotient and the remainder of $(4x^3 - 3x^2 + x + 7) \div (x - 2)$

$$\begin{array}{r|rrrr}
 & x^3 & x^2 & x^1 & x^0 \\
 2 & 4 & -3 & 1 & 7 \\
 & & 8 & 10 & 22 \\
 \hline
 & 4 & 5 & 11 & 29
 \end{array}$$

$x^2 \quad x^1 \quad x^0$

Quotient : $Q(x) = 4x^2 + 5x + 11$

Remainder : $R(x) = 29$

Example

If $f(x) = 3x^5 - 38x^3 + 5x^2 - 1$, use the synthetic division to find $f(4)$.

Solution

$$\begin{array}{r|rrrrrr}
 4 & 3 & 0 & -38 & 5 & 0 & -1 \\
 & & 12 & 48 & 40 & 180 & 720 \\
 \hline
 & 3 & 12 & 10 & 45 & 180 & 719
 \end{array}$$

$f(4) = 719$

Example

Show that -11 is a zero of the polynomial $f(x) = x^3 + 8x^2 - 29x + 44$

Solution

$$\begin{array}{r|rrrr}
 -11 & 1 & 8 & -29 & 44 \\
 & & -11 & 33 & -44 \\
 \hline
 & 1 & -3 & 4 & 0
 \end{array}$$

Thus, $f(-11) = 0$, and -11 is a zero of f .

The Rational Zeros *Theorem*

If the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ has integer coefficients and if $\frac{c}{d}$ is a rational zero of $f(x)$ such that c and d have no common prime factor, then

1. The numerator c of the zero is a factor of the constant term a_0
2. The denominator d of the zero is a factor of the leading coefficient a_n

$$\text{possible rational zeros} = \frac{\text{factors of the constant term } a_0}{\text{factors of the leading coefficient } a_n} = \frac{\text{possibilities for } a_0}{\text{possibilities for } a_n}$$

Example

Find all rational solutions of the equation: $3x^4 + 14x^3 + 14x^2 - 8x - 8 = 0$

Solution

possibilities for a_0	$\pm 1, \pm 2, \pm 4, \pm 8$
possibilities for a_n	$\pm 1, \pm 3$
possibilities for c/d	$\pm 1, \pm 2, \pm 4, \pm 8, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}, \pm \frac{8}{3}$

Using the calculator, the result will show that -2 is a zero.

$$\begin{array}{r|rrrrr} -2 & 3 & 14 & 14 & -8 & -8 \\ & & -6 & -16 & 4 & 8 \\ \hline & 3 & 8 & -2 & -4 & \boxed{0} \end{array}$$

We have the factorization of: $(x+2)(3x^3 + 8x^2 - 2x - 4) = 0$

$$\text{For } 3x^3 + 8x^2 - 2x - 4 \Rightarrow \frac{c}{d} = \frac{\pm 1, \pm 2, \pm 4}{\pm 1, \pm 3}$$

$x = -\frac{2}{3}$ is another solution.

$$\begin{array}{r|rrrr} -\frac{2}{3} & 3 & 8 & -2 & -4 \\ & & -2 & -4 & 4 \\ \hline & 3 & 6 & -6 & \boxed{0} \end{array}$$

We have the factorization of: $(x+2)\left(x + \frac{2}{3}\right)(3x^2 + 6x - 6) = 0$

By applying quadratic formula to solve: $3x^2 + 6x - 6 = 0 \Rightarrow x = -1 \pm \sqrt{3}$

Hence, the polynomial has two rational roots $x = -2$ and $-\frac{2}{3}$ and two irrational roots $x = -1 \pm \sqrt{3}$.

Exercises **Section 2.4 – Properties of Division**

1. Find the quotient and remainder if $f(x)$ is divided by $p(x)$:

$$f(x) = 2x^4 - x^3 + 7x - 12; \quad p(x) = x^2 - 3$$

- (2 – 4) Find the quotient and remainder if $f(x)$ is divided by $p(x)$

2. $f(x) = 3x^3 + 2x - 4; \quad p(x) = 2x^2 + 1$

3. $f(x) = 7x + 2; \quad p(x) = 2x^2 - x - 4$

4. $f(x) = 9x + 4; \quad p(x) = 2x - 5$

5. Use the remainder theorem to find $f(c)$: $f(x) = x^4 - 6x^2 + 4x - 8; \quad c = -3$

6. Use the remainder theorem to find $f(c)$: $f(x) = x^4 + 3x^2 - 12; \quad c = -2$

7. Use the factor theorem to show that $x - c$ is a factor of $f(x)$: $f(x) = x^3 + x^2 - 2x + 12; \quad c = -3$

8. Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $2x^3 - 3x^2 + 4x - 5; \quad x - 2$

9. Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $5x^3 - 6x^2 + 15; \quad x - 4$

10. Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $9x^3 - 6x^2 + 3x - 4; \quad x - \frac{1}{3}$

- (11 – 13) Use the synthetic division to find $f(c)$:

11. $f(x) = 2x^3 + 3x^2 - 4x + 4; \quad c = 3$

12. $f(x) = 8x^5 - 3x^2 + 7; \quad c = \frac{1}{2}$

13. $f(x) = x^3 - 3x^2 - 8; \quad c = 1 + \sqrt{2}$

14. Use the synthetic division to show that c is a zero of $f(x)$:

$$f(x) = 3x^4 + 8x^3 - 2x^2 - 10x + 4; \quad c = -2$$

15. Use the synthetic division to show that c is a zero of $f(x)$:

$$f(x) = 27x^4 - 9x^3 + 3x^2 + 6x + 1; \quad c = -\frac{1}{3}$$

16. Find all values of k such that $f(x)$ is divisible by the given linear polynomial:

$$f(x) = kx^3 + x^2 + k^2x + 3k^2 + 11; \quad x + 2$$

(17 – 62) Find all solutions of the equation

17. $x^3 - x^2 - 10x - 8 = 0$

18. $x^3 + x^2 - 14x - 24 = 0$

19. $2x^3 - 3x^2 - 17x + 30 = 0$

20. $12x^3 + 8x^2 - 3x - 2 = 0$

21. $x^3 + x^2 - 6x - 8 = 0$

22. $x^3 - 19x - 30 = 0$

23. $2x^3 + x^2 - 25x + 12 = 0$

24. $3x^3 + 11x^2 - 6x - 8 = 0$

25. $2x^3 + 9x^2 - 2x - 9 = 0$

26. $x^3 + 3x^2 - 6x - 8 = 0$

27. $3x^3 - x^2 - 6x + 2 = 0$

28. $x^3 - 8x^2 + 8x + 24 = 0$

29. $x^3 - 7x^2 - 7x + 69 = 0$

30. $x^3 - 3x - 2 = 0$

31. $x^3 - 2x + 1 = 0$

32. $x^3 - 2x^2 - 11x + 12 = 0$

33. $x^3 - 2x^2 - 7x - 4 = 0$

34. $x^3 - 10x - 12 = 0$

35. $x^3 - 5x^2 + 17x - 13 = 0$

36. $6x^3 + 25x^2 - 24x + 5 = 0$

37. $8x^3 + 18x^2 + 45x + 27 = 0$

38. $3x^3 - x^2 + 11x - 20 = 0$

39. $x^4 - x^3 - 9x^2 + 3x + 18 = 0$

40. $2x^4 - 9x^3 + 9x^2 + x - 3 = 0$

41. $6x^4 + 5x^3 - 17x^2 - 6x = 0$

42. $x^4 - 2x^2 - 16x - 15 = 0$

43. $x^4 - 2x^3 - 5x^2 + 8x + 4 = 0$

44. $2x^4 - 17x^3 + 4x^2 + 35x - 24 = 0$

45. $x^4 + x^3 - 3x^2 - 5x - 2 = 0$

46. $6x^4 - 17x^3 - 11x^2 + 42x = 0$

47. $x^4 - 5x^2 - 2x = 0$

48. $3x^4 - 4x^3 - 11x^2 + 16x - 4 = 0$

49. $6x^4 + 23x^3 + 19x^2 - 8x - 4 = 0$

50. $4x^4 - 12x^3 + 3x^2 + 12x - 7 = 0$

51. $2x^4 - 9x^3 - 2x^2 + 27x - 12 = 0$

52. $2x^4 - 19x^3 + 51x^2 - 31x + 5 = 0$

53. $4x^4 - 35x^3 + 71x^2 - 4x - 6 = 0$

54. $2x^4 + 3x^3 - 4x^2 - 3x + 2 = 0$

55. $x^4 + 3x^3 - 30x^2 - 6x + 56 = 0$

56. $6x^5 + 19x^4 + x^3 - 6x^2 = 0$

57. $3x^5 - 10x^4 - 6x^3 + 24x^2 + 11x - 6 = 0$

58. $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 = 0$

59. $x^5 - x^4 - 7x^3 + 7x^2 + 12x - 12 = 0$

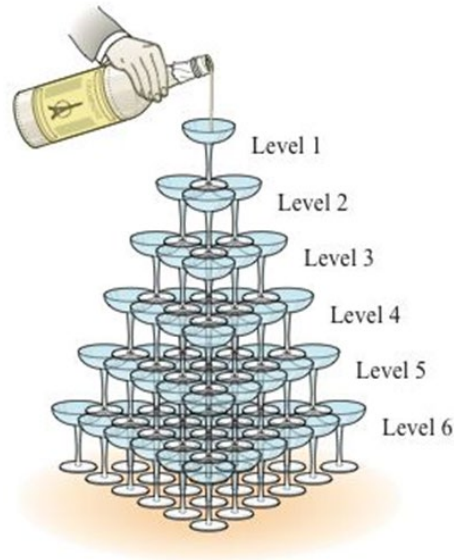
60. $x^5 - 2x^3 - 8x = 0$

61. $x^5 - 32 = 0$

62. $3x^6 - 10x^5 - 29x^4 + 34x^3 + 50x^2 - 24x - 24 = 0$

63. Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

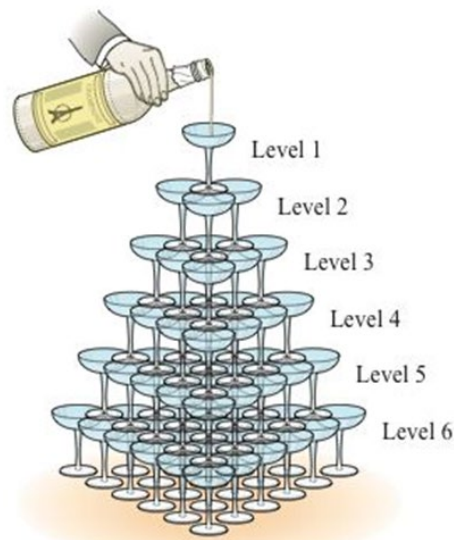
$$T(k) = \frac{1}{6}(k^3 + 3k^2 + 2k)$$



Where k is the number of levels in the pyramid. If 220 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

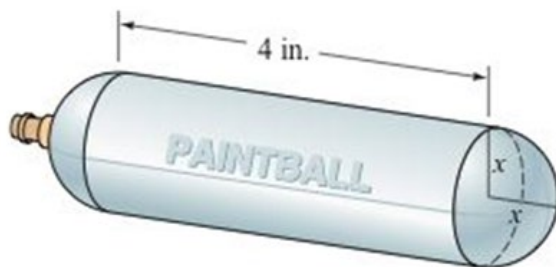
64. Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T(k) = \frac{1}{6}(2k^3 + 3k^2 + k)$$



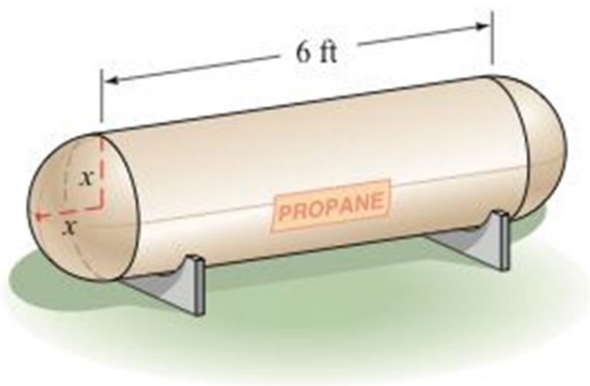
Where k is the number of levels in the pyramid. If 140 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

65. A carbon dioxide cartridge for a paintball rifle has the shape of a right circular cylinder with a hemisphere at each end. The cylinder is 4 inches long, and the volume of the cartridge is $2\pi \text{ in}^3$.

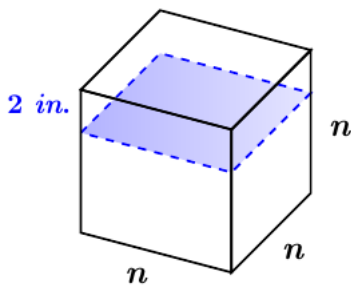


The common interior radius of the cylinder and the hemispheres is denoted by x . Estimate the length of the radius x .

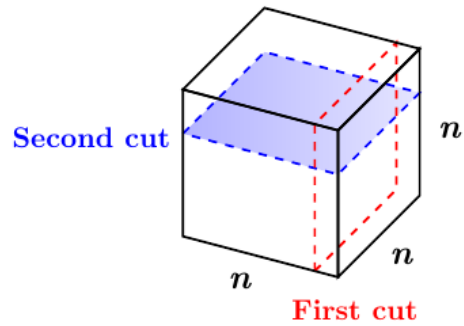
66. A propane tank has the shape of a circular cylinder with a hemisphere at each end. The cylinder is 6 feet long and the volume of the tank is $9\pi \text{ ft}^3$. Find the length of the radius x .



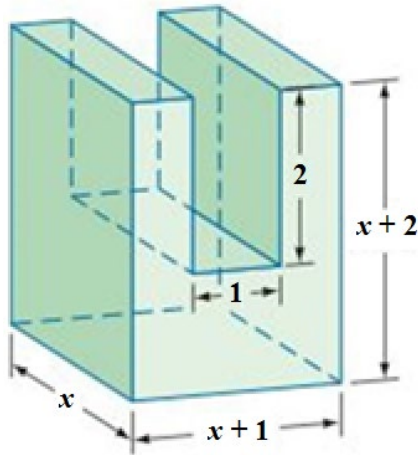
67. A cube measures n inches on each edge. If a slice 2 inches thick is cut from one face of the cube, the resulting solid has a volume of 567 in^3 . Find n .



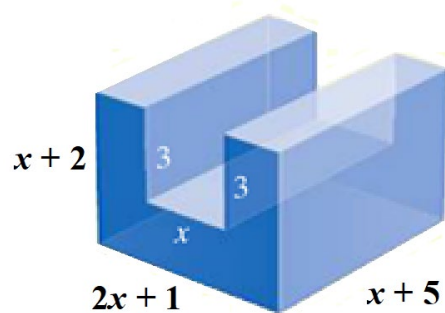
68. A cube measures n inches on each edge. If a slice 1 inch thick is cut from one face of the cube and then a slice 3 inches thick is cut from another face of the cube, the resulting solid has a volume of 1560 in^3 . Find the dimensions of the original cube.



69. For what value of x will the volume of the following solid be 112 in^3



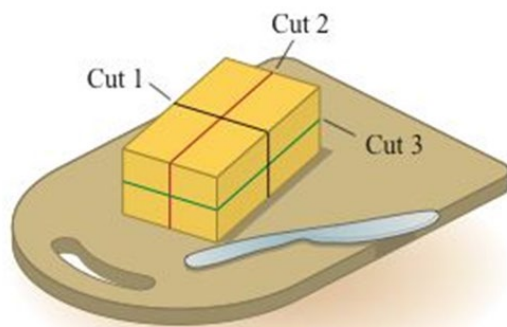
70. For what value of x will the volume of the following solid be 208 in^3



71. The length of rectangular box is 1 inch more than twice the height of the box, and the width is 3 inches more than the height. If the volume of the box is 126 in^3 , find the dimensions of the box.



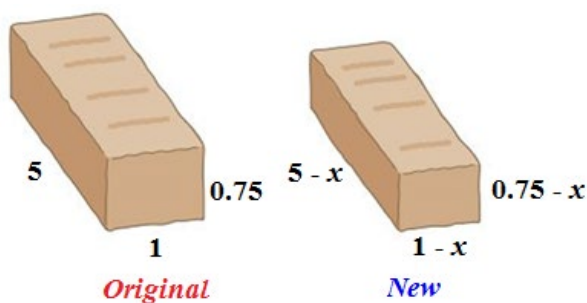
72. One straight cut through a thick piece of cheese produces two pieces. Two straight cuts can produce a maximum of four pieces. Two straight cuts can produce a maximum of four pieces. Three straight cuts can produce a maximum of eight pieces.



You might be inclined to think that every additional cut double numbers of pieces. However, for four straight cuts, you get a maximum of 15 pieces. The maximum number of pieces P that can be produced by n straight cuts is given by

$$P(n) = \frac{n^3 + 5n + 6}{6}$$

- Determine number of pieces that can be produced by five straight cuts.
 - What is the fewest number of straight cuts that are needed to produce 64 pieces?
73. The number of ways one can select three cards from a group of n cards (the order of the selection matters), where $n \geq 3$, is given by $P(n) = n^3 - 3n^2 + 2n$. For a certain card trick, a magician has determined that there are exactly 504 ways to choose three cards from a given group. How many cards are in the group?
74. A nutrition bar in the shape of a rectangular solid measure 0.75 in. by 1 in. by 5 inches.



To reduce costs, the manufacturer has decided to decrease each of the dimensions of the nutrition bar by x inches, what value of x will produce a new bar with a volume that is 0.75 in^3 less than the present bar's volume.

75. A rectangular box is square on two ends and has length plus girth of 81 *inches*. (Girth: distance *around* the box). Determine the possible lengths l ($l > w$) of the box if its volume is 4900 in^3 .

