Solution Section 1.7 – Sets

Exercise

List the members of these sets

a) $\left\{ x \middle| x \text{ is a real number such that } x^2 = 1 \right\}$

b) $\{x | x \text{ is a positive integer less than } 12\}$

c) $\{x | x \text{ is the square of an integer and } x < 100\}$

d) $\left\{ x \middle| x \text{ is an integer such that } x^2 = 2 \right\}$

Solution

a) $\{-1,\ 1\}$

b) $\{1,2,3,4,5,6,7,8,9,10,11,12\}$

c) $\{0,1,4,9,16,25,36,49,64,81\}$

d) $\varnothing$ $\quad \left\{\sqrt{2} \text{ is not an integer}\right\}$

Exercise

Determine whether each these pairs of sets are equal.

a) $\{1,3,3,3,5,5,5,5,5\}, \quad \{5,3,1\}$

b) $\{\{1\}\}, \quad \{1,\ \{1\}\}$

c) $\varnothing, \quad \{\varnothing\}$

Solution

a) Yes; order and repetition do not matter.

b) No; the first set has one element, and the second has two elements.

c) No; the first set has no elements, and the second has one element (namely the empty set).

Exercise

For each of the following sets, determine whether 2 is an element of that set.

a) $\{x \in \mathbb{R} | x \text{ is an integer greater than } 1\}$

b) $\{x \in \mathbb{R} | x \text{ is the square of an integer}\}$

c) $\{2,\ \{2\}\}$

d) $\{\{2\},\ \{\{2\}\}\}$

e) $\{\{2\},\ \{2,\ \{2\}\}\}$

f) $\{\{\{2\}\}\}$

Solution

a) Since 2 is an integer greater than 1, 2 is an element of this set.
b) Since 2 is not a perfect square, 2 is not an element of this set.
c) This set has two elements, and clearly one of those elements is 2.
d) This set has two elements, and clearly neither of those elements is 2. Both of the elements of this set are seats; 2 is a number, not a set.
e) This set has two elements, and clearly neither of those elements is 2. Both of the elements of this set are seats; 2 is a number, not a set.
f) This set has just one element, namely the set $\{\{2\}\}$. So 2 is not an element of this set. Note $\{2\} \neq \{\{2\}\}$

Exercise

Determine whether each of these statements is true or false

a) $0 \in \varnothing$
b) $\varnothing \in \{0\}$
c) $\{0\} \subset \varnothing$
d) $\varnothing \subset \{0\}$
e) $\{0\} \in \{0\}$
f) $\{0\} \subset \{0\}$
g) $\{\varnothing\} \subseteq \{\varnothing\}$
h) $x \in \{x\}$
i) $\{x\} \subseteq \{x\}$
j) $\{x\} \in \{x\}$
k) $\{x\} \in \{\{x\}\}$
l) $\varnothing \subseteq \{x\}$
m) $\varnothing \in \{x\}$

Solution

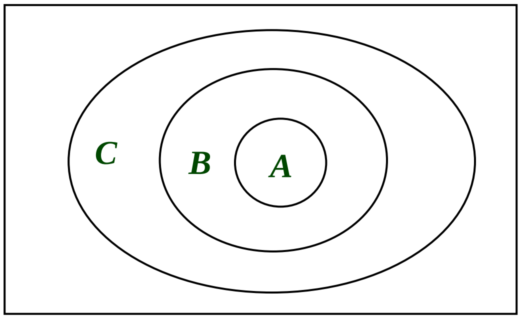
a) *False*, since the empty set has no elements.
b) *False*, the set on the right has only one element, namely 0, not the empty set.
c) *False*, the empty set has no proper subsets.
d) *True*, every element of the set on the left, is vacuously, an element of the set on the right; and the set on the right contains an element, namely 0, that is not the set on the left.
e) *False*, the set on the right has only one element, namely 0, not the set containing the number 0.

f) *False*, for one set to be a proper subset of another, the two sets cannot be equal.
g) *True*, every set is a subset of itself.
h) *True*, x is the only element.
i) *True*, every set is a subset of itself.
j) *False*, the only element of $\{x\}$ is a letter, not a set.
k) *True*, $\{x\}$ is the only element
l) *True*, the empty set is a subset of every set.
m) *False*, the only element of $\{x\}$ is a letter, not a set.

Exercise

Use a Venn Diagram to illustrate the relationships $A \subset B$ and $B \subset C$.

Solution



Exercise

Use a Venn Diagram to illustrate the relationships $A \subset B$ and $A \subset C$.

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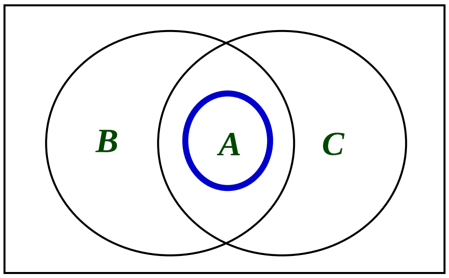

Since no information about the relationship between *B* and *C*, then *B* and *C* can be overlap.
The set *A* must be a subset of each of them.

Exercise

Suppose that *A*, *B*, and *C* are sets such that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$

Solution

Let $x \in A$, then since $A \subseteq B$, we can conclude that $x \in B$. Furthermore, since $B \subseteq C$, the fact that $x \in B$ implies that $x \in C$. Therefore, $A \subseteq C$

Exercise

What is the cardinality of each of these sets?

a) $\{a\}$

b) $\{\{a\}\}$

c) $\{a, \{a\}\}$

d) $\{a, \{a\}, \{a, \{a\}\}\}$

Solution

a) 1

b) 1

c) 2

d) 3

Exercise

How many elements does each of these sets have where *a* and *b* are distinct elements?

a) $\mathcal{P}(\{a, b, \{a, b\}\})$

b) $\mathcal{P}(\{\emptyset, a, \{a\}, \{\{a\}\}\})$

c) $\mathcal{P}(\mathcal{P}(\emptyset))$

Solution

a) Since the set has 3 elements, the power of the set has $2^3 = 8$ elements

b) Since the set has 4 elements, the power of the set has $2^4 = 16$ elements

c) Since the set has 0 elements, the power of the set has $2^0 = 1$ elements. The power of this set therefore has $2^1 = 2$ elements.

Exercise

What is the Cartesian product $A \times B \times C$, where *A* is the set of all airlines and *B* and *C* are both the set of all cities in the United States? Give an example of how this Cartesian product can be used.

Solution

This is the set of triples $(a,\ b,\ c)$, where *a* is an airline and *b* and *c* are cities. A useful subset of this set of triples $(a,\ b,\ c)$ for which *a* flies between *b* and *c*. For example, Continental, Houston, Chicago) is in this subset.

Exercise

What is the Cartesian product $A \times B$, where *A* is the set of all courses offered by the mathematics department and *B* is the set of mathematics professors at this university? Give an example of how this Cartesian product can be used.

Solution

By definition it is the set of all ordered pairs $(c,\ p)$ such that *c* is a course and *p* is a professor. The elements of this set are the possible teaching assignments for the mathematics department.

Exercise

Let *A* be a set. Show that $\varnothing \times A = A \times \varnothing = \varnothing$

Solution

By definition, $\varnothing \times A$ consists of al pairs $(x,\ a)$ such that $x \in \varnothing$ and $a \in A$. Since there are no elements $x \in \varnothing$. There are no such pairs, so $\varnothing \times A = \varnothing$. Similar reasoning $A \times \varnothing = \varnothing$.