Solution ***Section* 3.1 – Proving Identities**

Exercise

Prove the identity $\cos\theta\cot\theta+\sin\theta=\csc\theta$

Solution

$$\begin{aligned}
\cos\theta\cot\theta+\sin\theta &= \cos\theta\frac{\cos\theta}{\sin\theta}+\sin\theta \\
&= \frac{\cos^2\theta}{\sin\theta}+\sin\theta \\
&= \frac{\cos^2\theta}{\sin\theta}+\sin\theta\frac{\sin\theta}{\sin\theta} \\
&= \frac{\cos^2\theta}{\sin\theta}+\frac{\sin^2\theta}{\sin\theta} \\
&= \frac{\cos^2\theta+\sin^2\theta}{\sin\theta} \\
&= \frac{1}{\sin\theta} \\
&= \csc\theta
\end{aligned}$$

Exercise

Prove the identity $\sec\theta\cot\theta-\sin\theta=\dfrac{\cos^2\theta}{\sin\theta}$

Solution

$$\begin{aligned}
\sec\theta\cot\theta-\sin\theta &= \frac{1}{\cos\theta}\frac{\cos\theta}{\sin\theta}-\sin\theta \\
&= \frac{1}{\sin\theta}-\sin\theta \\
&= \frac{1-\sin^2\theta}{\sin\theta} \\
&= \frac{\cos^2\theta}{\sin\theta}
\end{aligned}$$

Exercise

Prove the identity $\dfrac{\csc\theta\tan\theta}{\sec\theta}=1$

Solution

$$\begin{aligned}\frac{\csc\theta\tan\theta}{\sec\theta}&=\csc\theta\tan\theta\frac{1}{\sec\theta}\\&=\frac{1}{\sin\theta}\frac{\sin\theta}{\cos\theta}\cos\theta\\&=1\end{aligned}$$

Exercise

Prove the identity $(\sin\theta+\cos\theta)^2=1+2\sin\theta\cos\theta$

Solution

$$\begin{aligned}(\sin\theta+\cos\theta)^2&=\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta\\&=1+2\sin\theta\cos\theta\end{aligned}$$

Exercise

Prove the identity $\sin\theta(\sec\theta+\cot\theta)=\tan\theta+\cos\theta$

Solution

$$\begin{aligned}\sin\theta(\sec\theta+\cot\theta)&=\sin\theta\left(\frac{1}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)\\&=\sin\theta\frac{1}{\cos\theta}+\sin\theta\frac{\cos\theta}{\sin\theta}\\&=\frac{\sin\theta}{\cos\theta}+\cos\theta\\&=\tan\theta+\cos\theta\end{aligned}$$

Exercise

Prove the identity $\cos\theta(\csc\theta+\tan\theta)=\cot\theta+\sin\theta$

Solution

$$\begin{aligned}\cos\theta(\csc\theta+\tan\theta)&=\cos\theta\frac{1}{\sin\theta}+\cos\theta\frac{\sin\theta}{\cos\theta}\\&=\cot\theta+\sin\theta\end{aligned}$$

Exercise

Prove the identity $\cos\theta(\sec\theta-\cos\theta)=\sin^2\theta$

Solution

$$\cos\theta(\sec\theta-\cos\theta)=\cos\theta\frac{1}{\cos\theta}-\cos^2\theta$$
$$=1-\cos^2\theta$$
$$=\sin^2\theta$$

Exercise

Prove the identity $\cot\theta+\tan\theta=\csc\theta\sec\theta$

Solution

$$\cot\theta+\tan\theta=\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}$$
$$=\frac{\cos^2\theta+\sin^2\theta}{\sin\theta\cos\theta}$$
$$=\frac{1}{\sin\theta\cos\theta}$$
$$=\frac{1}{\sin\theta}\frac{1}{\cos\theta}$$
$$=\csc\theta\sec\theta$$

Exercise

Prove $\tan x(\cos x+\cot x)=\sin x+1$

Solution

$$\tan x(\cos x+\cot x)=\frac{\sin x}{\cos x}\left(\cos x+\frac{\cos x}{\sin x}\right)$$
$$=\cos x\frac{\sin x}{\cos x}+\frac{\sin x}{\cos x}\frac{\cos x}{\sin x}$$
$$=\sin x+1$$

Exercise

Prove $\quad \dfrac{1-\cos^4\theta}{1+\cos^2\theta}=\sin^2\theta$

Solution

$$\frac{1-\cos^4\theta}{1+\cos^2\theta}=\frac{(1+\cos^2\theta)(1-\cos^2\theta)}{1+\cos^2\theta}$$

$$=1-\cos^2\theta$$

$$=\sin^2\theta$$

Exercise

Prove $\quad \dfrac{1-\sec x}{1+\sec x}=\dfrac{\cos x-1}{\cos x+1}$

Solution

$$\frac{1-\sec x}{1+\sec x}=\frac{1-\frac{1}{\cos x}}{1+\frac{1}{\cos x}}$$

$$=\frac{\frac{\cos x-1}{\cos x}}{\frac{\cos x+1}{\cos x}}$$

$$=\frac{\cos x-1}{\cos x+1}$$

Exercise

Prove $\quad \dfrac{\cos x}{1-\sin x}-\dfrac{1-\sin x}{\cos x}=0$

Solution

$$\frac{\cos x}{1+\sin x}-\frac{1-\sin x}{\cos x}=\frac{\cos x}{\cos x}\frac{\cos x}{1-\sin x}-\frac{1+\sin x}{1+\sin x}\frac{1-\sin x}{\cos x}$$

$$=\frac{\cos^2 x-(1-\sin^2 x)}{\cos x(1+\sin x)}$$

$$=\frac{\cos^2 x-1+\sin^2 x)}{\cos x(1+\sin x)}$$

$$=\frac{1-1}{\cos x(1+\sin x)}$$

$$=\frac{0}{\cos x(1+\sin x)}$$

$$=0$$

Exercise

Prove $\quad \dfrac{1+\cot^3 t}{1+\cot t} = \csc^2 t - \cot t$

Solution

$$\frac{1+\cot^3 t}{1+\cot t} = \frac{1+\frac{\cos^3 t}{\sin^3 t}}{1+\frac{\cos t}{\sin t}}$$

$$= \frac{\frac{\sin^3 t+\cos^3 t}{\sin^3 t}}{\frac{\sin t+\cos t}{\sin t}}$$

$$= \frac{\sin^3 t+\cos^3 t}{\sin^3 t} \cdot \frac{\sin t}{\sin t+\cos t}$$

$$= \frac{(\sin t+\cos t)(\sin^2 t-\sin t\cos t+\cos^2 t)}{\sin^2 t} \cdot \frac{1}{\sin t+\cos t}$$

$$= \frac{1-\sin t\cos t}{\sin^2 t}$$

$$= \frac{1}{\sin^2 t} - \frac{\sin t\cos t}{\sin^2 t}$$

$$= \csc^2 t - \frac{\cos t}{\sin t}$$

$$= \csc^2 t - \cot t$$

Exercise

Prove: $\quad \tan x + \cot x = \sec x \csc x$

Solution

$$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$$

$$= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x}$$

$$= \frac{1}{\cos x \sin x}$$

$$= \frac{1}{\cos x}\frac{1}{\sin x}$$

$$= \sec x \csc x$$

Exercise

Prove: $\dfrac{\tan x-\cot x}{\sin x\cos x}=\sec^2 x-\csc^2 x$

Solution

$$\frac{\tan x-\cot x}{\sin x\cos x}=\frac{\tan x}{\sin x\cos x}-\frac{\cot x}{\sin x\cos x}$$

$$=\tan x\frac{1}{\sin x\cos x}-\cot x\frac{1}{\sin x\cos x}$$

$$=\frac{\sin x}{\cos x}\frac{1}{\sin x\cos x}-\frac{\cos x}{\sin x}\frac{1}{\sin x\cos x}$$

$$=\frac{1}{\cos^2 x}-\frac{1}{\sin^2 x}$$

$$=\sec^2 x-\csc^2 x$$

Exercise

Prove: $\dfrac{\sec x+\tan x}{\sec x-\tan x}=\dfrac{1+2\sin x+\sin^2 x}{\cos^2 x}$

Solution

$$\frac{\sec x+\tan x}{\sec x-\tan x}=\frac{\frac{1}{\cos x}+\frac{\sin x}{\cos x}}{\frac{1}{\cos x}-\frac{\sin x}{\cos x}}\,\frac{\cos x}{\cos x}$$

$$=\frac{\frac{1}{\cos x}\cos x+\frac{\sin x}{\cos x}\cos x}{\frac{1}{\cos x}\cos x-\frac{\sin x}{\cos x}\cos x}$$

$$=\frac{1+\sin x}{1-\sin x}$$

$$=\frac{1+\sin x}{1-\sin x}\,\frac{1+\sin x}{1+\sin x}$$

$$=\frac{(1+\sin x)^2}{1-\sin^2 x}$$

$$=\frac{1+2\sin x+\sin^2 x}{\cos^2 x}$$

Exercise

Prove the identity: $\sin^2 x - \cos^2 x = 2\sin^2 x - 1$

Solution

$$\begin{aligned}\sin^2 x - \cos^2 x &= \sin^2 x - \left(1 - \sin^2 x\right)\\ &= \sin^2 x - 1 + \sin^2 x\\ &= 2\sin^2 x - 1\end{aligned}$$

Exercise

Prove the identity: $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$

Solution

$$\begin{aligned}\sin^4 x - \cos^4 x &= \left(\sin^2 x + \cos^2 x\right)\left(\sin^2 x - \cos^2 x\right)\\ &= (1)\left(\sin^2 x - \cos^2 x\right)\\ &= \sin^2 x - \cos^2 x\end{aligned}$$

Exercise

Prove the identity: $\dfrac{\cos\alpha}{1+\sin\alpha} = \sec\alpha - \tan\alpha$

Solution

$$\begin{aligned}\frac{\cos\alpha}{1+\sin\alpha} &= \frac{\cos\alpha}{1+\sin\alpha}\,\frac{1-\sin\alpha}{1-\sin\alpha}\\ &= \frac{\cos\alpha - \cos\alpha\sin\alpha}{1-\sin^2\alpha}\\ &= \frac{\cos\alpha - \cos\alpha\sin\alpha}{\cos^2\alpha}\\ &= \frac{\cos\alpha}{\cos^2\alpha} - \frac{\cos\alpha\sin\alpha}{\cos^2\alpha}\\ &= \frac{1}{\cos\alpha} - \frac{\sin\alpha}{\cos\alpha}\\ &= \sec\alpha - \tan\alpha\end{aligned}$$

Exercise

Prove the identity: $\frac{\sin\alpha}{1-\sin\alpha}-\frac{\cos\alpha}{1-\sin\alpha}=\frac{1-\cot\alpha}{\csc\alpha-1}$

Solution

$$\frac{\sin\alpha}{1-\sin\alpha}-\frac{\cos\alpha}{1-\sin\alpha}=\frac{\sin\alpha-\cos\alpha}{1-\sin\alpha}$$

$$=\frac{\frac{\sin\alpha-\cos\alpha}{\sin\alpha}}{\frac{1-\sin\alpha}{\sin\alpha}}$$

$$=\frac{\frac{\sin\alpha}{\sin\alpha}-\frac{\cos\alpha}{\sin\alpha}}{\frac{1}{\sin\alpha}-\frac{\sin\alpha}{\sin\alpha}}$$

$$=\frac{1-\cot\alpha}{\csc\alpha-1}$$

Exercise

Prove the identity: $\frac{\frac{1}{\tan x}+\cot x}{\frac{1}{\tan x}+\tan x}=\frac{2}{\sec^2 x}$

Solution

$$\frac{\frac{1}{\tan x}+\cot x}{\frac{1}{\tan x}+\tan x}=\frac{\frac{1}{\tan x}+\frac{1}{\tan x}}{\frac{1+\tan^2 x}{\tan x}}$$

$$=\frac{\frac{2}{\tan x}}{\frac{\sec^2 x}{\tan x}}$$

$$=\frac{2}{\sec^2 x}$$

Exercise

Prove the following equation is an identity: $\frac{\cot^2\theta+3\cot\theta-4}{\cot\theta+4}=\cot\theta-1$

Solution

$$\frac{\cot^2\theta+3\cot\theta-4}{\cot\theta+4}=\frac{(\cot\theta+4)(\cot\theta-1)}{\cot\theta+4}$$

$$=\cot\theta-1$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin\theta}{1+\cos\theta}=\dfrac{1-\cos\theta}{\sin\theta}$

Solution

$$\frac{\sin\theta}{1+\cos\theta}=\frac{\sin\theta}{1+\cos\theta}\frac{1-\cos\theta}{1-\cos\theta}$$

$$=\frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta}$$

$$=\frac{\sin\theta(1-\cos\theta)}{\sin^2\theta}$$

$$=\frac{1-\cos\theta}{\sin\theta}$$

Exercise

Prove the following equation is an identity: $\tan x(\csc x-\sin x)=\cos x$

Solution

$$\tan x(\csc x-\sin x)=\frac{\sin x}{\cos x}\left(\frac{1}{\sin x}-\sin x\right)$$

$$=\frac{\sin x}{\cos x}\left(\frac{1-\sin^2 x}{\sin x}\right)$$

$$=\frac{1}{\cos x}\left(\frac{\cos^2 x}{1}\right)$$

$$=\cos x$$

Exercise

Prove the following equation is an identity: $\sin x(\tan x\cos x-\cot x\cos x)=1-2\cos^2 x$

Solution

$$\sin x(\tan x\cos x-\cot x\cos x)=\sin x\cos x\left(\frac{\sin x}{\cos x}-\frac{\cos x}{\sin x}\right)$$

$$=\sin x\cos x\left(\frac{\sin^2 x-\cos^2 x}{\cos x\sin x}\right)$$

$$=1-\cos^2 x-\cos^2 x$$

$$=1-2\cos^2 x$$

Exercise

Prove the following equation is an identity: $(1+\tan x)^2+(\tan x-1)^2=2\sec^2 x$

Solution

$$\begin{aligned}(1+\tan x)^2+(\tan x-1)^2 &= 1+2\tan x+\tan^2 x+1-2\tan x+\tan^2 x \\ &= 2+2\tan^2 x \\ &= 2\left(1+\tan^2 x\right) \\ &= 2\sec^2 x\end{aligned}$$

Exercise

Prove the following equation is an identity: $\sec x+\tan x=\dfrac{\cos x}{1-\sin x}$

Solution

$$\begin{aligned}\sec x+\tan x &= \frac{1}{\cos x}+\frac{\sin x}{\cos x} \\ &= \frac{1+\sin x}{\cos x}\frac{1-\sin x}{1-\sin x} \\ &= \frac{1-\sin^2 x}{\cos x(1-\sin x)} \\ &= \frac{\cos^2 x}{\cos x(1-\sin x)} \\ &= \frac{\cos x}{1-\sin x}\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\tan x-1}{\tan x+1}=\dfrac{1-\cot x}{1+\cot x}$

Solution

$$\begin{aligned}\frac{\tan x-1}{\tan x+1} &= \frac{\frac{1}{\cot x}-1}{\frac{1}{\cot x}+1} \\ &= \frac{\frac{1-\cot x}{\cot x}}{\frac{1+\cot x}{\cot x}} \\ &= \frac{1-\cot x}{1+\cot x}\end{aligned}$$

Exercise

Prove the following equation is an identity: $7\csc^2 x - 5\cot^2 x = 2\csc^2 x + 5$

Solution

$$\begin{aligned} 7\csc^2 x - 5\cot^2 x &= 7\csc^2 x - 5\left(\csc^2 x - 1\right) \\ &= 7\csc^2 x - 5\csc^2 x + 5 \\ &= 2\csc^2 x + 5 \end{aligned}$$

Exercise

Prove the following equation is an identity: $1 - \dfrac{\cos^2 x}{1 - \sin x} = -\sin x$

Solution

$$\begin{aligned} 1 - \frac{\cos^2 x}{1 - \sin x} &= 1 - \frac{1 - \sin^2 x}{1 - \sin x} \\ &= 1 - \frac{(1 - \sin x)(1 + \sin x)}{1 - \sin x} \\ &= 1 - (1 + \sin x) \\ &= 1 - 1 - \sin x \\ &= -\sin x \end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{1 - \cos x}{1 + \cos x} = \dfrac{\sec x - 1}{\sec x + 1}$

Solution

$$\begin{aligned} \frac{1 - \cos x}{1 + \cos x} &= \frac{\frac{1}{\cos x} - \frac{\cos x}{\cos x}}{\frac{1}{\cos x} + \frac{\cos x}{\cos x}} \\ &= \frac{\sec x - 1}{\sec x + 1} \end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\sec x - 1}{\tan x} = \dfrac{\tan x}{\sec x + 1}$

Solution

$$\frac{\sec x-1}{\tan x}=\frac{\sec x-1}{\tan x}\frac{\sec x+1}{\sec x+1}$$

$$=\frac{\sec^2 x-1}{\tan x(\sec x+1)}$$

$$=\frac{\tan^2 x}{\tan x(\sec x+1)}$$

$$=\frac{\tan x}{\sec x+1}$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos x}{\cos x-\sin x}=\dfrac{1}{1-\tan x}$

Solution

$$\frac{\cos x}{\cos x-\sin x}=\frac{\frac{\cos x}{\cos x}}{\frac{\cos x}{\cos x}-\frac{\sin x}{\cos x}}$$

$$=\frac{1}{1-\tan x}$$

Exercise

Prove the following equation is an identity: $(\sec x+\tan x)^2=\dfrac{1+\sin x}{1-\sin x}$

Solution

$$(\sec x+\tan x)^2=\left(\frac{1}{\cos x}+\frac{\sin x}{\cos x}\right)^2$$

$$=\left(\frac{1+\sin x}{\cos x}\right)^2$$

$$=\frac{(1+\sin x)^2}{\cos^2 x}$$

$$=\frac{(1+\sin x)^2}{1-\sin^2 x}$$

$$=\frac{(1+\sin x)^2}{(1-\sin x)(1+\sin x)}$$

$$=\frac{1+\sin x}{1-\sin x}$$

Exercise

Prove the following equation is an identity: $\frac{\cos x}{1+\tan x} - \frac{\sin x}{1+\cot x} = \cos x - \sin x$

Solution

$$\begin{aligned}
\frac{\cos x}{1+\tan x} - \frac{\sin x}{1+\cot x} &= \frac{\cos x}{1+\frac{\sin x}{\cos x}} - \frac{\sin x}{1+\frac{\cos x}{\sin x}} \\
&= \frac{\cos x}{\frac{\cos x+\sin x}{\cos x}} - \frac{\sin x}{\frac{\sin x+\cos x}{\sin x}} \\
&= \frac{\cos^2 x}{\cos x+\sin x} - \frac{\sin^2 x}{\sin x+\cos x} \\
&= \frac{\cos^2 x-\sin^2 x}{\cos x+\sin x} \\
&= \frac{(\cos x+\sin x)(\cos x-\sin x)}{\cos x+\sin x} \\
&= \cos x - \sin x
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\cot x+\csc x-1}{\cot x-\csc x+1} = \csc x+\cot x$

Solution

$$\begin{aligned}
\frac{\cot x+\csc x-1}{\cot x-\csc x+1} &= \frac{\cot x+\csc x-\left(\csc^2 x-\cot^2 x\right)}{\cot x-\csc x+1} \\
&= \frac{\cot x+\csc x-(\csc x-\cot x)(\csc x+\cot x)}{\cot x-\csc x+1} \\
&= \frac{(\csc x+\cot x)(1-(\csc x-\cot x))}{\cot x-\csc x+1} \\
&= \frac{(\csc x+\cot x)(1-\csc x+\cot x)}{\cot x-\csc x+1} \\
&= \csc x+\cot x
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\tan x+\cot x}{\tan x-\cot x}=\dfrac{1}{\sin^2 x-\cos^2 x}$

Solution

$$\frac{\tan x+\cot x}{\tan x-\cot x}=\frac{\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}}{\frac{\sin x}{\cos x}-\frac{\cos x}{\sin x}}$$

$$=\frac{\frac{\sin^2 x+\cos^2 x}{\cos x\sin x}}{\frac{\sin^2 x-\cos^2 x}{\cos x\sin x}}$$

$$=\frac{1}{\sin^2 x-\cos^2 x}$$

Exercise

Prove the following equation is an identity: $\dfrac{1-\cot^2 x}{1+\cot^2 x}+1=2\sin^2 x$

Solution

$$\frac{1-\cot^2 x}{1+\cot^2 x}+1=\frac{1-\cot^2 x+1+\cot^2 x}{1+\cot^2 x}$$

$$=\frac{2}{\csc^2 x}$$

$$=2\sin^2 x$$

Exercise

Prove the following equation is an identity: $\dfrac{1+\cos x}{1-\cos x}-\dfrac{1-\cos x}{1+\cos x}=4\cot x\csc x$

Solution

$$\frac{1+\cos x}{1-\cos x}-\frac{1-\cos x}{1+\cos x}=\frac{(1+\cos x)^2-(1-\cos x)^2}{1-\cos^2 x}$$

$$=\frac{(1+\cos x+1-\cos x)(1+\cos x-1+\cos x)}{\sin^2 x} \qquad a^2-b^2=(a-b)(a+b)$$

$$=\frac{(2)(2\cos x)}{\sin^2 x}$$

$$=4\frac{\cos x}{\sin x}\frac{1}{\sin x}$$

$$=4\cot x\csc x$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin^3 x - \cos^3 x}{\sin x - \cos x} = 1 + \sin x \cos x$

Solution

$$\frac{\sin^3 x - \cos^3 x}{\sin x - \cos x} = \frac{(\sin x - \cos x)\left(\sin^2 x + \sin x \cos x + \cos^2 x\right)}{\sin x - \cos x} \qquad a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$$

$$= 1 + \sin x \cos x$$

Exercise

Prove the following equation is an identity: $1 + \sec^2 x \sin^2 x = \sec^2 x$

Solution

$$1 + \sec^2 x \sin^2 x = 1 + \frac{1}{\cos^2 x}\sin^2 x$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

Exercise

Prove the following equation is an identity: $\dfrac{1 + \csc x}{\sec x} = \cos x + \cot x$

Solution

$$\frac{1 + \csc x}{\sec x} = \frac{1}{\sec x} + \frac{\csc x}{\sec x}$$

$$= \cos x + \frac{\frac{1}{\sin x}}{\frac{1}{\cos x}}$$

$$= \cos x + \frac{\cos x}{\sin x}$$

$$= \cos x + \cot x$$

Exercise

Prove the following equation is an identity: $\tan^2 x = \sec^2 x - \sin^2 x - \cos^2 x$

Solution

$$\begin{aligned}\sec^2 x - \sin^2 x - \cos^2 x &= \frac{1}{\cos^2 x} - \left(\sin^2 x + \cos^2 x\right) \\ &= \frac{1}{\cos^2 x} - 1 \\ &= \frac{1-\cos^2 x}{\cos^2 x} \\ &= \frac{\sin^2 x}{\cos^2 x} \\ &= \tan^2 x\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\sin x}{1-\cos x} + \frac{\sin x}{1+\cos x} = 2\csc x$

Solution

$$\begin{aligned}\frac{\sin x}{1-\cos x} + \frac{\sin x}{1+\cos x} &= \sin x\left(\frac{1}{1-\cos x} + \frac{1}{1+\cos x}\right) \\ &= \sin x\left(\frac{1+\cos x + 1 - \cos x}{1-\cos^2 x}\right) \\ &= \sin x\left(\frac{2}{\sin^2 x}\right) \\ &= \frac{2}{\sin x} \\ &= 2\csc x\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cos^2(\alpha-\beta) - \cos^2(\alpha+\beta) = \sin^2(\alpha+\beta) - \sin^2(\alpha-\beta)$

Solution

$$\begin{aligned}\cos^2(\alpha-\beta) - \cos^2(\alpha+\beta) &= 1 - \sin^2(\alpha-\beta) - \left[1 - \sin^2(\alpha+\beta)\right] \\ &= 1 - \sin^2(\alpha-\beta) - 1 + \sin^2(\alpha+\beta) \\ &= \sin^2(\alpha+\beta) - \sin^2(\alpha-\beta)\end{aligned}$$

Exercise

Prove the following equation is an identity: $\tan x \csc x - \sec^2 x \cos x = 0$

Solution

$$\tan x \csc x - \sec^2 x \cos x = \frac{\sin x}{\cos x}\frac{1}{\sin x} - \frac{1}{\cos^2 x}\cos x$$

$$= \frac{1}{\cos x} - \frac{1}{\cos x}$$

$$= 0$$

Exercise

Prove the following equation is an identity: $(1+\tan x)^2 - 2\tan x = \frac{1}{(1-\sin x)(1+\sin x)}$

Solution

$$(1+\tan x)^2 - 2\tan x = 1 + 2\tan x + \tan^2 x - 2\tan x$$

$$= 1 + \tan^2 x$$

$$= \sec^2 x$$

$$= \frac{1}{\cos^2 x}$$

$$= \frac{1}{1-\sin^2 x}$$

$$= \frac{1}{(1-\sin x)(1+\sin x)}$$

Exercise

Prove the following equation is an identity: $\frac{3\csc^2 x - 5\csc x - 28}{\csc x - 4} = \frac{3}{\sin x} + 7$

Solution

$$\frac{3\csc^2 x - 5\csc x - 28}{\csc x - 4} = \frac{(3\csc x + 7)(\csc x - 4)}{\csc x - 4}$$

$$= 3\csc x + 7$$

$$= \frac{3}{\sin x} + 7$$

Exercise

Prove the following equation is an identity: $\left(\sec^2 x-1\right)\left(\sec^2 x+1\right)=\tan^4 x+2\tan^2 x$

Solution

$$\left(\sec^2 x-1\right)\left(\sec^2 x+1\right)=\sec^4 x-1 \qquad (a-b)(a+b)=a^2-b^2 \quad a=\sec^2 x$$

$$=\left(\sec^2 x\right)^2-1$$

$$=\left(1+\tan^2 x\right)^2-1$$

$$=1+2\tan^2 x+\tan^4 x-1$$

$$=\tan^4 x+2\tan^2 x$$

Exercise

Prove the following equation is an identity: $\dfrac{\csc x}{\cot x}-\dfrac{\cot x}{\csc x}=\dfrac{\sin x}{\cot x}$

Solution

$$\frac{\csc x}{\cot x}-\frac{\cot x}{\csc x}=\frac{\csc^2 x-\cot^2 x}{\cot x\csc x}$$

$$=\frac{\csc^2 x-\left(\csc^2 x-1\right)}{\cot x\csc x}$$

$$=\frac{\csc^2 x-\csc^2 x+1}{\cot x\csc x}$$

$$=\frac{1}{\cot x\frac{1}{\sin x}}$$

$$=\frac{\sin x}{\cot x}$$

Exercise

Prove the following equation is an identity: $\dfrac{1-\cos^2 x}{1+\cos x}=\dfrac{\sec x-1}{\sec x}$

Solution

$$\frac{1-\cos^2 x}{1+\cos x}=\frac{(1-\cos x)(1+\cos x)}{1+\cos x}$$

$$=1-\cos x$$

$$= 1 - \frac{1}{\sec x}$$

$$= \frac{\sec x - 1}{\sec x}$$

Exercise

Prove the following equation is an identity: $\frac{\cos x}{1+\cos x} = \frac{\sec x - 1}{\tan^2 x}$

Solution

$$\frac{\cos x}{1+\cos x} = \frac{\cos x}{1+\cos x}\frac{1-\cos x}{1-\cos x}$$

$$= \frac{\cos x - \cos^2 x}{\cos^2 x - 1}$$

$$= \frac{\cos x - \cos^2 x}{\sin^2 x}\frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x}}$$

$$= \frac{\frac{1}{\cos x} - 1}{\frac{\sin^2 x}{\cos^2 x}}$$

$$= \frac{\sec x - 1}{\tan^2 x}$$

Exercise

Prove the following equation is an identity: $\frac{1 - 2\sin^2 x}{1 + 2\sin x \cos x} = \frac{\cos x - \sin x}{\cos x + \sin x}$

Solution

$$\frac{1 - 2\sin^2 x}{1 + 2\sin x \cos x} = \frac{\cos^2 x + \sin^2 x - 2\sin^2 x}{\cos^2 x + \sin^2 x + 2\sin x \cos x}$$

$$= \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2}$$

$$= \frac{(\cos x - \sin x)(\cos x + \sin x)}{(\cos x + \sin x)^2}$$

$$= \frac{\cos x - \sin x}{\cos x + \sin x}$$

Exercise

Prove the following equation is an identity: $(\cos x - \sin x)^2 + (\cos x + \sin x)^2 = 2$

Solution

$$\begin{aligned}(\cos x - \sin x)^2 + (\cos x + \sin x)^2 &= \cos^2 x - 2\sin x\cos x + \sin^2 x + \cos^2 x + 2\sin x\cos x + \sin^2 x \\ &= \cos^2 x + \sin^2 x + \cos^2 x + \sin^2 x \\ &= 1 + 1 \\ &= 2\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin x}{1+\cos x} + \dfrac{1+\cos x}{\sin x} = 2\csc x$

Solution

$$\begin{aligned}\frac{\sin x}{1+\cos x} + \frac{1+\cos x}{\sin x} &= \frac{\sin x\sin x + (1+\cos x)(1+\cos x)}{(1+\cos x)\sin x} \\ &= \frac{\sin^2 x + 1 + 2\cos x + \cos^2 x}{(1+\cos x)\sin x} \\ &= \frac{1+1+2\cos x}{(1+\cos x)\sin x} \\ &= \frac{2+2\cos x}{(1+\cos x)\sin x} \\ &= \frac{2(1+\cos x)}{(1+\cos x)\sin x} \\ &= \frac{2}{\sin x} \\ &= 2\csc x\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin x + \tan x}{\cot x + \csc x} = \sin x\tan x$

Solution

$$\begin{aligned}\frac{\sin x + \tan x}{\cot x + \csc x} &= \frac{\sin x + \tan x}{\frac{1}{\tan x} + \frac{1}{\sin x}} \\ &= \frac{\sin x + \tan x}{\frac{\sin x + \tan x}{\tan x\sin x}} \\ &= (\sin x + \tan x)\frac{\tan x\sin x}{\sin x + \tan x} \\ &= \tan x\sin x\end{aligned}$$

Exercise

Prove the following equation is an identity: $\csc^2 x \sec^2 x = \sec^2 x + \csc^2 x$

Solution

$$\begin{aligned}
\csc^2 x \sec^2 x &= \frac{1}{\sin^2 x}\frac{1}{\cos^2 x} \\
&= \frac{1}{\sin^2 x \cos^2 x} \\
&= \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} \\
&= \frac{\sin^2 x}{\sin^2 x \cos^2 x} + \frac{\cos^2 x}{\sin^2 x \cos^2 x} \\
&= \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \\
&= \sec^2 x + \csc^2 x
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cos^2 x + 1 = 2\cos^2 x + \sin^2 x$

Solution

$$\begin{aligned}
\cos^2 x + 1 &= \cos^2 x + \cos^2 x + \sin^2 x \\
&= 2\cos^2 x + \sin^2 x
\end{aligned}$$

Exercise

Prove the following equation is an identity: $1 - \dfrac{\cos^2 x}{1+\sin x} = \sin x$

Solution

$$\begin{aligned}
1 - \frac{\cos^2 x}{1+\sin x} &= 1 - \frac{1-\sin^2 x}{1+\sin x} \\
&= 1 - \frac{(1-\sin x)(1+\sin x)}{1+\sin x} \\
&= 1 - (1-\sin x) \\
&= 1 - 1 + \sin x \\
&= \sin x
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cot^2 x = (\csc x - 1)(\csc x + 1)$

Solution

$$\cot^2 x = \csc^2 x - 1$$

$$= (\csc x - 1)(\csc x + 1)$$

Exercise

Prove the following equation is an identity: $\dfrac{\sec x - 1}{\tan x} = \dfrac{\tan x}{\sec x + 1}$

Solution

$$\frac{\sec x - 1}{\tan x} = \frac{\sec x - 1}{\tan x}\frac{\sec x + 1}{\sec x + 1}$$

$$= \frac{\sec^2 x - 1}{\tan x(\sec x + 1)}$$

$$= \frac{\tan^2 x}{\tan x(\sec x + 1)}$$

$$= \frac{\tan x}{\sec x + 1}$$

Exercise

Prove the following equation is an identity: $10\csc^2 x - 6\cot^2 x = 4\csc^2 x + 6$

Solution

$$10\csc^2 x - 6\cot^2 x = 10\csc^2 x - 6\left(\csc^2 x - 1\right)$$

$$= 10\csc^2 x - 6\csc^2 x + 6$$

$$= 4\csc^2 x + 6$$

Exercise

Prove the following equation is an identity: $\dfrac{\csc x + \cot x}{\tan x + \sin x} = \csc x \cot x$

Solution

$$\frac{\csc x + \cot x}{\tan x + \sin x} = \frac{\csc x + \cot x}{\frac{1}{\cot x} + \frac{1}{\csc x}}$$

$$= \frac{\csc x + \cot x}{\frac{\csc x + \cot x}{\cot x \csc x}}$$

$$= \csc x + \cot x \frac{\cot x \csc x}{\csc x + \cot x}$$

$$= \cot x \csc x$$

Exercise

Prove the following equation is an identity: $\frac{1-\sec x}{\tan x} + \frac{\tan x}{1-\sec x} = -2\csc x$

Solution

$$\frac{1-\sec x}{\tan x} + \frac{\tan x}{1-\sec x} = \frac{(1-\sec x)(1-\sec x) + \tan^2 x}{\tan x(1-\sec x)}$$

$$= \frac{(1-\sec x)^2 + \sec^2 x - 1}{\tan x(1-\sec x)}$$

$$= \frac{(1-\sec x)^2 + (\sec x + 1)(\sec x - 1)}{\tan x(1-\sec x)}$$

$$= \frac{(1-\sec x)^2 - (\sec x + 1)(1-\sec x)}{\tan x(1-\sec x)}$$

$$= \frac{(1-\sec x)\left[(1-\sec x) - (\sec x + 1)\right]}{\tan x(1-\sec x)}$$

$$= \frac{1-\sec x - \sec x - 1}{\tan x}$$

$$= \frac{-2\sec x}{\tan x}$$

$$= -2\frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}}$$

$$= -2\frac{1}{\sin x}$$

$$= -2\csc x$$

Exercise

Prove the following equation is an identity: $\csc x - \sin x = \cos x \cot x$

Solution

$$\begin{aligned}\csc x - \sin x &= \frac{1}{\sin x} - \sin x \\ &= \frac{1-\sin^2 x}{\sin x} \\ &= \frac{\cos^2 x}{\sin x} \\ &= \cos x \frac{\cos x}{\sin x} \\ &= \cos x \cot x\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\tan x + \sec x}{\sec x} - \frac{\tan x + \sec x}{\tan x} = -\cos x \cot x$

Solution

$$\begin{aligned}\frac{\tan x + \sec x}{\sec x} - \frac{\tan x + \sec x}{\tan x} &= \frac{(\tan x + \sec x)\tan x - \sec x(\tan x + \sec x)}{\sec x \tan x} \\ &= \frac{\tan^2 x + \sec x \tan x - \sec x \tan x - \sec^2 x}{\sec x \tan x} \\ &= \frac{\tan^2 x - \sec^2 x}{\sec x \tan x} \qquad\qquad 1+\tan^2\alpha = \sec^2\alpha \\ &= \frac{-1}{\sec x \tan x} \\ &= -\frac{1}{\sec x}\frac{1}{\tan x} \\ &= -\cos x \cot x\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cot^3 x = \cot x\left(\csc^2 x - 1\right)$

Solution

$$\begin{aligned}\cot^3 x &= \cot x \cot^2 x \\ &= \cot x\left(\csc^2 x - 1\right)\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\cot^2 x}{\csc x - 1} = \frac{1 + \sin x}{\sin x}$

Solution

$$\begin{aligned}\frac{\cot^2 x}{\csc x - 1} &= \frac{\csc^2 x - 1}{\csc x - 1}\\ &= \frac{(\csc x - 1)(\csc x + 1)}{\csc x - 1}\\ &= \csc x + 1\\ &= \frac{1}{\sin x} + 1\\ &= \frac{1 + \sin x}{\sin x}\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cot^2 x + \csc^2 x = 2\csc^2 x - 1$

Solution

$$\begin{aligned}\cot^2 x + \csc^2 x &= \csc^2 x - 1 + \csc^2 x\\ &= 2\csc^2 x - 1\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\cot^2 x}{1 + \csc x} = \csc x - 1$

Solution

$$\begin{aligned}\frac{\cot^2 x}{1 + \csc x} &= \frac{\csc^2 x - 1}{1 + \csc x}\\ &= \frac{(\csc x - 1)(\csc x + 1)}{1 + \csc x}\\ &= \csc x - 1\end{aligned}$$

Exercise

Prove the following equation is an identity: $\sec^4 x - \tan^4 x = \sec^2 x + \tan^2 x$

Solution

$$\begin{aligned}\sec^4 x - \tan^4 x &= \left(\sec^2 x + \tan^2 x\right)\left(\sec^2 x - \tan^2 x\right) \qquad a^2 - b^2 = (a-b)(a+b)\\ &= \left(\sec^2 x + \tan^2 x\right)(1)\\ &= \sec^2 x + \tan^2 x\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\cos x}{1+\sin x}+\frac{1+\sin x}{\cos x}=2\sec x$

Solution

$$\begin{aligned}
\frac{\cos x}{1+\sin x}+\frac{1+\sin x}{\cos x} &= \frac{\cos^2 x+(1+\sin x)^2}{(1+\sin x)\cos x} \\
&= \frac{\cos^2 x+1+2\sin x+\sin^2 x}{(1+\sin x)\cos x} \\
&= \frac{2+2\sin x}{(1+\sin x)\cos x} \\
&= \frac{2(1+\sin x)}{(1+\sin x)\cos x} \\
&= \frac{2}{\cos x} \\
&= 2\sec x
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\sin x+\cos x}{\sin x-\cos x}=\frac{1+2\sin x\cos x}{2\sin^2 x-1}$

Solution

$$\begin{aligned}
\frac{\sin x+\cos x}{\sin x-\cos x} &= \frac{\sin x+\cos x}{\sin x-\cos x}\,\frac{\sin x+\cos x}{\sin x+\cos x} \\
&= \frac{\sin^2 x+2\sin x\cos x+\cos^2 x}{\sin^2 x-\cos^2 x} \\
&= \frac{1+2\sin x\cos x}{\sin^2 x-\left(1-\sin^2 x\right)} \\
&= \frac{1+2\sin x\cos x}{\sin^2 x-1+\sin^2 x} \\
&= \frac{1+2\sin x\cos x}{2\sin^2 x-1}
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\csc x-1}{\csc x+1}=\frac{\cot^2 x}{\csc^2 x+2\csc x+1}$

Solution

$$\frac{\csc x-1}{\csc x+1}=\frac{\csc x-1}{\csc x+1}\frac{\csc x+1}{\csc x+1}$$

$$=\frac{\csc^2 x-1}{\csc^2 x+2\csc x+1}$$

$$=\frac{\cot^2 x}{\csc^2 x+2\csc x+1}$$

Exercise

Prove the following equation is an identity: $\csc^4 x-\cot^4 x=\csc^2 x+\cot^2 x$

Solution

$$\csc^4 x-\cot^4 x=\left(\csc^2 x+\cot^2 x\right)\left(\csc^2 x-\cot^2 x\right)$$

$$=\left(\csc^2 x+\cot^2 x\right)(1)$$

$$=\csc^2 x+\cot^2 x$$

Exercise

Prove the following equation is an identity: $\tan\left(\frac{\pi}{4}+x\right)=\cot\left(\frac{\pi}{4}-x\right)$

Solution

$$\tan\left(\frac{\pi}{4}+x\right)=\cot\left[\frac{\pi}{2}-\left(\frac{\pi}{4}+x\right)\right]$$

$$=\cot\left[\frac{\pi}{2}-\frac{\pi}{4}-x\right]$$

$$=\cot\left(\frac{\pi}{4}-x\right)$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin\theta}{1+\sin\theta}-\dfrac{\sin\theta}{1-\sin\theta}=-2\tan^2\theta$

Solution

$$\begin{aligned}\frac{\sin\theta}{1+\sin\theta}-\frac{\sin\theta}{1-\sin\theta}&=\sin\theta\left[\frac{1-\sin\theta-(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}\right]\\&=\sin\theta\left[\frac{1-\sin\theta-1-\sin\theta}{1-\sin^2\theta}\right]\\&=\sin\theta\left(\frac{-2\sin\theta}{\cos^2\theta}\right)\\&=-2\frac{\sin^2\theta}{\cos^2\theta}\\&=-2\tan^2\theta\end{aligned}$$

Exercise

Prove the following equation is an identity: $\csc^2 x-\cos^2 x\csc^2 x=1$

Solution

$$\begin{aligned}\csc^2 x-\cos^2 x\csc^2 x&=\csc^2 x\left(1-\cos^2 x\right)\\&=\frac{1}{\sin^2 x}\left(\sin^2 x\right)\\&=1\end{aligned}$$

Exercise

Prove the following equation is an identity: $1-2\sin^2 x=2\cos^2 x-1$

Solution

$$\begin{aligned}1-2\sin^2 x&=1-2\left(1-\cos^2 x\right)\\&=1-2+2\cos^2 x\\&=2\cos^2 x-1\end{aligned}$$

Exercise

Prove the following equation is an identity: $\csc^2 x - \cos x \sec x = \cot^2 x$

Solution

$$\begin{aligned}\csc^2 x - \cos x \sec x &= \frac{1}{\sin^2 x} - \cos x \frac{1}{\cos x}\\ &= \frac{1}{\sin^2 x} - 1\\ &= \frac{1-\sin^2 x}{\sin^2 x}\\ &= \frac{\cos^2 x}{\sin^2 x}\\ &= \cot^2 x\end{aligned}$$

Exercise

Prove the following equation is an identity: $(\sec x - \tan x)(\sec x + \tan x) = 1$

Solution

$$\begin{aligned}(\sec x - \tan x)(\sec x + \tan x) &= \sec^2 x - \tan^2 x\\ &= 1 + \tan^2 x - \tan^2 x\\ &= 1\end{aligned}$$

Exercise

Prove the following equation is an identity: $\left(1+\tan^2 x\right)\left(1-\sin^2 x\right) = 1$

Solution

$$\begin{aligned}\left(1+\tan^2 x\right)\left(1-\sin^2 x\right) &= \sec^2 x \cos^2 x\\ &= \frac{1}{\cos^2 x}\cos^2 x\\ &= 1\end{aligned}$$