

Solution

Section 7.2 – Sum and Difference Formulas

Exercise

Write the expression as a single trigonometric function $\sin 8x \cos x - \cos 8x \sin x$

Solution

$$\begin{aligned}\sin 8x \cos x - \cos 8x \sin x &= \sin(8x - x) \\ &= \sin 7x\end{aligned}$$

Exercise

Show that $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$

Solution

$$\begin{aligned}\sin\left(x - \frac{\pi}{2}\right) &= \sin x \cos \frac{\pi}{2} - \cos x \sin \frac{\pi}{2} \\ &= \sin x \cdot (0) - \cos x \cdot (1) \\ &= -\cos x\end{aligned}$$

Exercise

If $\sin A = \frac{4}{5}$ with A in QII, and $\cos B = -\frac{5}{13}$ with B in QIII, find

- | | | |
|------------------|------------------|------------------|
| a) $\sin(A + B)$ | b) $\cos(A + B)$ | c) $\tan(A + B)$ |
| d) $\sin(A - B)$ | e) $\cos(A - B)$ | f) $\tan(A - B)$ |

Solution

$$\cos A = -\frac{3}{5} \qquad \sin B = -\frac{12}{13}$$

$$\begin{aligned}a) \quad \sin(A + B) &= \sin A \cos B + \sin B \cos A \\ &= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) + \left(-\frac{12}{13}\right)\left(-\frac{3}{5}\right) \\ &= -\frac{20}{65} + \frac{36}{65} \\ &= \frac{16}{65}\end{aligned}$$

$$\begin{aligned}b) \quad \cos(A + B) &= \cos A \cos B - \sin A \sin B \\ &= \left(-\frac{3}{5}\right)\left(-\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) \\ &= \frac{15}{65} + \frac{48}{65}\end{aligned}$$

$$\underline{= \frac{63}{65}} \quad \Big|$$

$$c) \quad \tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$\underline{= \frac{16}{63}} \quad \Big|$$

$$\begin{aligned} d) \quad \sin(A-B) &= \sin A \cos B - \sin B \cos A \\ &= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) - \left(-\frac{12}{13}\right)\left(-\frac{3}{5}\right) \\ &= -\frac{20}{65} - \frac{36}{65} \end{aligned}$$

$$\underline{= -\frac{56}{65}} \quad \Big|$$

$$\begin{aligned} e) \quad \cos(A-B) &= \cos A \cos B + \sin A \sin B \\ &= \left(-\frac{3}{5}\right)\left(-\frac{5}{13}\right) + \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) \\ &= \frac{15}{65} - \frac{48}{65} \end{aligned}$$

$$\underline{= -\frac{33}{65}} \quad \Big|$$

$$f) \quad \tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)}$$

$$\underline{= \frac{56}{33}} \quad \Big|$$

Exercise

If $\sin A = \frac{3}{5}$ ($A \in QII$), and $\cos B = -\frac{12}{13}$ ($B \in QIII$), find

- | | | |
|----------------|----------------|----------------|
| a) $\sin(A+B)$ | b) $\cos(A+B)$ | c) $\tan(A+B)$ |
| d) $\sin(A-B)$ | e) $\cos(A-B)$ | f) $\tan(A-B)$ |

Solution

$$\cos A = -\frac{4}{5} \quad \sin B = -\frac{5}{13}$$

$$\begin{aligned} a) \quad \sin(A+B) &= \sin A \cos B + \cos A \sin B \\ &= \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) + \left(-\frac{4}{5}\right)\left(-\frac{5}{13}\right) \\ &= \frac{-36+20}{65} \end{aligned}$$

$$\underline{= -\frac{16}{65}} \quad \Big|$$

$$\begin{aligned}
 b) \quad \cos(A + B) &= \cos A \cos B - \sin A \sin B \\
 &= \left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right) - \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right) \\
 &= \frac{48+15}{65} \\
 &= \frac{63}{65}
 \end{aligned}$$

$$\begin{aligned}
 c) \quad \tan(A + B) &= \frac{\sin(A + B)}{\cos(A + B)} \\
 &= -\frac{16}{63}
 \end{aligned}$$

$$\begin{aligned}
 d) \quad \sin(A - B) &= \sin A \cos B - \cos A \sin B \\
 &= \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) - \left(-\frac{4}{5}\right)\left(-\frac{5}{13}\right) \\
 &= \frac{-36-20}{65} \\
 &= -\frac{56}{65}
 \end{aligned}$$

$$\begin{aligned}
 e) \quad \cos(A - B) &= \cos A \cos B + \sin A \sin B \\
 &= \left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right) + \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right) \\
 &= \frac{48-15}{65} \\
 &= \frac{33}{65}
 \end{aligned}$$

$$\begin{aligned}
 f) \quad \tan(A - B) &= \frac{\sin(A - B)}{\cos(A - B)} \\
 &= -\frac{56}{33}
 \end{aligned}$$

Exercise

If $\sin A = \frac{1}{\sqrt{5}}$ ($A \in QI$), and $\tan B = \frac{3}{4}$ ($B \in QI$), find

- | | | |
|------------------------|------------------------|------------------------|
| $a) \quad \sin(A + B)$ | $b) \quad \cos(A + B)$ | $c) \quad \tan(A + B)$ |
| $d) \quad \sin(A - B)$ | $e) \quad \cos(A - B)$ | $f) \quad \tan(A - B)$ |

Solution

$$\cos A = \sqrt{1 - \frac{1}{5}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}} \quad \sin B = \frac{3}{5}; \quad \cos B = \frac{4}{5}$$

$$a) \quad \sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\begin{aligned}
&= \left(\frac{1}{\sqrt{5}} \right) \left(\frac{4}{5} \right) + \left(\frac{3}{5} \right) \left(\frac{2}{\sqrt{5}} \right) \\
&= \frac{4+6}{5\sqrt{5}} \\
&= \frac{10}{5\sqrt{5}} \\
&= \frac{2}{\sqrt{5}} \quad |
\end{aligned}$$

$$b) \quad \cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\begin{aligned}
&= \left(\frac{2}{\sqrt{5}} \right) \left(\frac{4}{5} \right) - \left(\frac{1}{\sqrt{5}} \right) \left(\frac{3}{5} \right) \\
&= \frac{8-3}{5\sqrt{5}} \\
&= \frac{1}{\sqrt{5}} \quad |
\end{aligned}$$

$$c) \quad \tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= 2 \quad |$$

$$d) \quad \sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\begin{aligned}
&= \left(\frac{1}{\sqrt{5}} \right) \left(\frac{4}{5} \right) - \left(\frac{3}{5} \right) \left(\frac{2}{\sqrt{5}} \right) \\
&= \frac{4-6}{5\sqrt{5}} \\
&= -\frac{2}{5\sqrt{5}} \quad |
\end{aligned}$$

$$e) \quad \cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\begin{aligned}
&= \left(\frac{2}{\sqrt{5}} \right) \left(\frac{4}{5} \right) + \left(\frac{1}{\sqrt{5}} \right) \left(\frac{3}{5} \right) \\
&= \frac{8+3}{5\sqrt{5}} \\
&= \frac{11}{5\sqrt{5}} \quad |
\end{aligned}$$

$$f) \quad \tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)}$$

$$= -\frac{2}{11} \quad |$$

Exercise

If $\sin A = \frac{3}{5}$ ($A \in QII$), and $\cos B = \frac{12}{13}$ ($B \in QIV$), find

a) $\sin(A + B)$ b) $\cos(A + B)$ c) $\tan(A + B)$

d) $\sin(A - B)$ e) $\cos(A - B)$ f) $\tan(A - B)$

Solution

$$\cos A = -\frac{4}{5} \qquad \sin B = -\frac{5}{13}$$

$$\begin{aligned} a) \quad \sin(A + B) &= \sin A \cos B + \cos A \sin B \\ &= \left(\frac{3}{5}\right)\left(\frac{12}{13}\right) + \left(-\frac{4}{5}\right)\left(-\frac{5}{13}\right) \\ &= \frac{36 + 20}{65} \\ &= \frac{56}{65} \end{aligned}$$

$$\begin{aligned} b) \quad \cos(A + B) &= \cos A \cos B - \sin A \sin B \\ &= \left(-\frac{4}{5}\right)\left(\frac{12}{13}\right) - \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right) \\ &= \frac{-48 - 15}{65} \\ &= -\frac{63}{65} \end{aligned}$$

$$c) \quad \tan(A + B) = -\frac{56}{63} \qquad \tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)}$$

$$\begin{aligned} d) \quad \sin(A - B) &= \sin A \cos B - \cos A \sin B \\ &= \left(\frac{3}{5}\right)\left(\frac{12}{13}\right) - \left(-\frac{4}{5}\right)\left(-\frac{5}{13}\right) \\ &= \frac{36 - 20}{65} \\ &= \frac{16}{65} \end{aligned}$$

$$\begin{aligned} e) \quad \cos(A - B) &= \cos A \cos B + \sin A \sin B \\ &= \left(-\frac{4}{5}\right)\left(\frac{12}{13}\right) + \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right) \\ &= \frac{-48 - 15}{65} \\ &= -\frac{63}{65} \end{aligned}$$

$$f) \quad \tan(A - B) = -\frac{16}{63} \qquad \tan(A - B) = \frac{\sin(A - B)}{\cos(A - B)}$$

Exercise

If $\sin A = \frac{7}{25}$ ($A \in QII$), and $\cos B = -\frac{8}{17}$ ($B \in QIII$), find

a) $\sin(A + B)$ b) $\cos(A + B)$ c) $\tan(A + B)$

d) $\sin(A - B)$ e) $\cos(A - B)$ f) $\tan(A - B)$

Solution

$$\cos A = -\frac{24}{25} \quad \sin B = -\frac{15}{17}$$

$$\begin{aligned} \text{a) } \sin(A + B) &= \sin A \cos B + \cos A \sin B \\ &= \left(\frac{7}{25}\right)\left(-\frac{8}{17}\right) + \left(-\frac{24}{25}\right)\left(-\frac{15}{17}\right) \\ &= \frac{-56 + 360}{425} \\ &= \frac{304}{425} \end{aligned}$$

$$\begin{aligned} \text{b) } \cos(A + B) &= \cos A \cos B - \sin A \sin B \\ &= \left(-\frac{24}{25}\right)\left(-\frac{8}{17}\right) - \left(\frac{7}{25}\right)\left(-\frac{15}{17}\right) \\ &= \frac{192 + 105}{425} \\ &= \frac{297}{425} \end{aligned}$$

$$\text{c) } \tan(A + B) = \frac{304}{297} \quad \tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)}$$

$$\begin{aligned} \text{d) } \sin(A - B) &= \sin A \cos B - \cos A \sin B \\ &= \left(\frac{7}{25}\right)\left(-\frac{8}{17}\right) - \left(-\frac{24}{25}\right)\left(-\frac{15}{17}\right) \\ &= \frac{-56 - 360}{425} \\ &= -\frac{416}{425} \end{aligned}$$

$$\begin{aligned} \text{e) } \cos(A - B) &= \cos A \cos B + \sin A \sin B \\ &= \left(-\frac{24}{25}\right)\left(-\frac{8}{17}\right) + \left(\frac{7}{25}\right)\left(-\frac{15}{17}\right) \\ &= \frac{192 - 105}{425} \\ &= -\frac{87}{425} \end{aligned}$$

$$\text{f) } \tan(A - B) = -\frac{416}{87} \quad \tan(A - B) = \frac{\sin(A - B)}{\cos(A - B)}$$

Exercise

If $\cos A = -\frac{4}{5}$ ($A \in QII$), and $\sin B = \frac{24}{25}$ ($B \in QII$), find

a) $\sin(A + B)$ b) $\cos(A + B)$ c) $\tan(A + B)$

d) $\sin(A - B)$ e) $\cos(A - B)$ f) $\tan(A - B)$

Solution

$$\sin A = \frac{3}{5} \qquad \cos B = -\frac{7}{25}$$

$$\begin{aligned} a) \quad \sin(A + B) &= \sin A \cos B + \cos A \sin B \\ &= \left(\frac{3}{5}\right)\left(-\frac{7}{25}\right) + \left(-\frac{4}{5}\right)\left(\frac{24}{25}\right) \\ &= \frac{-21-96}{125} \\ &= -\frac{117}{125} \end{aligned}$$

$$\begin{aligned} b) \quad \cos(A + B) &= \cos A \cos B - \sin A \sin B \\ &= \left(-\frac{4}{5}\right)\left(-\frac{7}{25}\right) - \left(\frac{3}{5}\right)\left(\frac{24}{25}\right) \\ &= \frac{28-72}{125} \\ &= -\frac{44}{125} \end{aligned}$$

$$c) \quad \tan(A + B) = \frac{117}{44} \qquad \tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)}$$

$$\begin{aligned} d) \quad \sin(A - B) &= \sin A \cos B - \cos A \sin B \\ &= \left(\frac{3}{5}\right)\left(-\frac{7}{25}\right) - \left(-\frac{4}{5}\right)\left(\frac{24}{25}\right) \\ &= \frac{-21+96}{125} \\ &= \frac{75}{125} \end{aligned}$$

$$\begin{aligned} e) \quad \cos(A - B) &= \cos A \cos B + \sin A \sin B \\ &= \left(-\frac{4}{5}\right)\left(-\frac{7}{25}\right) + \left(\frac{3}{5}\right)\left(\frac{24}{25}\right) \\ &= \frac{28+72}{125} \\ &= \frac{100}{125} \end{aligned}$$

$$f) \quad \tan(A - B) = \frac{75}{100} \qquad \tan(A - B) = \frac{\sin(A - B)}{\cos(A - B)}$$

Exercise

If $\cos A = \frac{15}{17}$ ($A \in QI$), and $\cos B = -\frac{12}{13}$ ($B \in QII$), find

a) $\sin(A + B)$ b) $\cos(A + B)$ c) $\tan(A + B)$

d) $\sin(A - B)$ e) $\cos(A - B)$ f) $\tan(A - B)$

Solution

$$\sin A = \frac{8}{17} \qquad \sin B = \frac{5}{13}$$

$$\begin{aligned} \text{a) } \sin(A + B) &= \sin A \cos B + \cos A \sin B \\ &= \left(\frac{8}{17}\right)\left(-\frac{12}{13}\right) + \left(\frac{15}{17}\right)\left(\frac{5}{13}\right) \\ &= \frac{-96 + 75}{221} \\ &= -\frac{21}{221} \end{aligned}$$

$$\begin{aligned} \text{b) } \cos(A + B) &= \cos A \cos B - \sin A \sin B \\ &= \left(\frac{15}{17}\right)\left(-\frac{12}{13}\right) - \left(\frac{8}{17}\right)\left(\frac{5}{13}\right) \\ &= \frac{-180 - 40}{221} \\ &= -\frac{220}{221} \end{aligned}$$

$$\text{c) } \tan(A + B) = \frac{21}{221} \qquad \tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)}$$

$$\begin{aligned} \text{d) } \sin(A - B) &= \sin A \cos B - \cos A \sin B \\ &= \left(\frac{8}{17}\right)\left(-\frac{12}{13}\right) - \left(\frac{15}{17}\right)\left(\frac{5}{13}\right) \\ &= \frac{-96 - 75}{221} \\ &= -\frac{171}{221} \end{aligned}$$

$$\begin{aligned} \text{e) } \cos(A - B) &= \cos A \cos B + \sin A \sin B \\ &= \left(\frac{15}{17}\right)\left(-\frac{12}{13}\right) + \left(\frac{8}{17}\right)\left(\frac{5}{13}\right) \\ &= \frac{-180 + 40}{221} \\ &= -\frac{140}{221} \end{aligned}$$

$$\text{f) } \tan(A - B) = \frac{171}{140} \qquad \tan(A - B) = \frac{\sin(A - B)}{\cos(A - B)}$$

Exercise

If $\sin A = -\frac{3}{5}$ ($A \in QIV$), and $\sin B = \frac{7}{25}$ ($B \in QII$), find

a) $\sin(A + B)$ b) $\cos(A + B)$ c) $\tan(A + B)$

d) $\sin(A - B)$ e) $\cos(A - B)$ f) $\tan(A - B)$

Solution

$$\cos A = -\frac{4}{5} \qquad \cos B = -\frac{24}{25}$$

a) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

$$\begin{aligned} &= \left(-\frac{3}{5}\right)\left(-\frac{24}{25}\right) + \left(-\frac{4}{5}\right)\left(\frac{7}{25}\right) \\ &= \frac{72 - 28}{125} \\ &= \frac{44}{125} \end{aligned}$$

b) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

$$\begin{aligned} &= \left(-\frac{4}{5}\right)\left(-\frac{24}{25}\right) - \left(-\frac{3}{5}\right)\left(\frac{7}{25}\right) \\ &= \frac{96 + 21}{125} \\ &= \frac{117}{125} \end{aligned}$$

c) $\tan(A + B) = \frac{44}{117}$

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)}$$

d) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

$$\begin{aligned} &= \left(-\frac{3}{5}\right)\left(-\frac{24}{25}\right) - \left(-\frac{4}{5}\right)\left(\frac{7}{25}\right) \\ &= \frac{72 + 28}{125} \\ &= \frac{100}{125} \end{aligned}$$

e) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

$$\begin{aligned} &= \left(-\frac{4}{5}\right)\left(-\frac{24}{25}\right) + \left(-\frac{3}{5}\right)\left(\frac{7}{25}\right) \\ &= \frac{96 - 21}{125} \\ &= \frac{75}{125} \end{aligned}$$

f) $\tan(A - B) = \frac{100}{75}$

$$\tan(A - B) = \frac{\sin(A - B)}{\cos(A - B)}$$

Exercise

If $\sec A = \sqrt{5}$ with A in QI , and $\sec B = \sqrt{10}$ with B in QI , find $\sec(A + B)$

Solution

$$\sec(A + B) = \frac{1}{\cos(A + B)}$$

$$\sec A = \sqrt{5}$$

$$\cos A = \frac{1}{\sqrt{5}} \quad \sin A = \frac{2}{\sqrt{5}}$$

$$\sec B = \sqrt{10}$$

$$\cos B = \frac{1}{\sqrt{10}} \quad \sin B = \sqrt{1 - \frac{1}{10}} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$= \frac{1}{\sqrt{5}} \frac{1}{\sqrt{10}} - \frac{2}{\sqrt{5}} \frac{3}{\sqrt{10}}$$

$$= \frac{1 - 6}{\sqrt{50}}$$

$$= \frac{-5}{\sqrt{50}}$$

$$= \frac{-5}{5\sqrt{2}}$$

$$= \frac{-1}{\sqrt{2}}$$

$$\sec(A + B) = \frac{1}{\frac{-1}{\sqrt{2}}}$$

$$= \sqrt{2} \quad |$$

Exercise

Prove the identity $\frac{\sin(A - B)}{\cos A \cos B} = \tan A - \tan B$

Solution

$$\frac{\sin(A - B)}{\cos A \cos B} = \frac{\sin A \cos B - \sin B \cos A}{\cos A \cos B}$$

$$= \frac{\sin A \cos B}{\cos A \cos B} - \frac{\sin B \cos A}{\cos A \cos B}$$

$$= \frac{\sin A}{\cos A} - \frac{\sin B}{\cos B}$$

$$= \tan A - \tan B \quad | \quad \checkmark$$

Exercise

Prove the identity $\sec(A + B) = \frac{\cos(A - B)}{\cos^2 A - \sin^2 B}$

Solution

$$\begin{aligned}\sec(A + B) &= \frac{1}{\cos(A + B)} \\&= \frac{1}{\cos A \cos B - \sin A \sin B} \\&= \frac{1}{\cos A \cos B - \sin A \sin B} \frac{\cos(A - B)}{\cos(A - B)} \\&= \frac{1}{\cos A \cos B - \sin A \sin B} \frac{\cos(A - B)}{\cos A \cos B + \sin A \sin B} \\&= \frac{\cos(A - B)}{\cos^2 A \cos^2 B - \sin^2 A \sin^2 B} \\&= \frac{\cos(A - B)}{\cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B} \\&= \frac{\cos(A - B)}{\cos^2 A - \cos^2 A \sin^2 B - \sin^2 B + \cos^2 A \sin^2 B} \\&= \frac{\cos(A - B)}{\cos^2 A - \sin^2 B} \quad \checkmark\end{aligned}$$

Exercise

Prove the identity $\frac{\cos 4\alpha}{\sin \alpha} - \frac{\sin 4\alpha}{\cos \alpha} = \frac{\cos 5\alpha}{\sin \alpha \cos \alpha}$

Solution

$$\begin{aligned}\frac{\cos 4\alpha}{\sin \alpha} - \frac{\sin 4\alpha}{\cos \alpha} &= \frac{\cos 4\alpha \cos \alpha - \sin 4\alpha \sin \alpha}{\sin \alpha \cos \alpha} \\&= \frac{\cos(4\alpha + \alpha)}{\sin \alpha \cos \alpha} \\&= \frac{\cos 5\alpha}{\sin \alpha \cos \alpha} \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\cos(x+y)}{\cos(x-y)} = \frac{\cot y - \tan x}{\cot y + \tan x}$

Solution

$$\begin{aligned}\frac{\cos(x+y)}{\cos(x-y)} &= \frac{\cos x \cos y - \sin x \sin y}{\cos x \cos y + \sin x \sin y} \\&= \frac{\frac{\cos x \cos y}{\cos x \sin y} - \frac{\sin x \sin y}{\cos x \sin y}}{\frac{\cos x \cos y}{\cos x \sin y} + \frac{\sin x \sin y}{\cos x \sin y}} \\&= \frac{\cot y - \tan x}{\cot y + \tan x} \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\sin(x+y)}{\sin(x-y)} = \frac{\cot y + \cot x}{\cot y - \cot x}$

Solution

$$\begin{aligned}\frac{\sin(x+y)}{\sin(x-y)} &= \frac{\sin x \cos y + \sin y \cos x}{\sin x \cos y - \sin y \cos x} \\&= \frac{\frac{\sin x \cos y}{\sin x \sin y} + \frac{\sin y \cos x}{\sin x \sin y}}{\frac{\sin x \cos y}{\sin x \sin y} - \frac{\sin y \cos x}{\sin x \sin y}} \\&= \frac{\cot y + \cot x}{\cot y - \cot x} \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\cos(x+y)}{\cos(x-y)} = \frac{\cot y - \tan x}{\cot y + \tan x}$

Solution

$$\begin{aligned}\frac{\cos(x+y)}{\cos(x-y)} &= \frac{\cos x \cos y - \sin x \sin y}{\cos x \cos y + \sin x \sin y} \\&= \frac{\frac{\cos x \cos y}{\cos x \sin y} - \frac{\sin x \sin y}{\cos x \sin y}}{\frac{\cos x \cos y}{\cos x \sin y} + \frac{\sin x \sin y}{\cos x \sin y}} \\&= \frac{\cot y - \tan x}{\cot y + \tan x} \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\sin(x-y)}{\sin x \cos y} = 1 - \cot x \tan y$

Solution

$$\begin{aligned}\frac{\sin(x-y)}{\sin x \cos y} &= \frac{\sin x \cos y - \cos x \sin y}{\sin x \cos y} \\ &= \frac{\sin x \cos y}{\sin x \cos y} - \frac{\cos x \sin y}{\sin x \cos y} \\ &= 1 - \cot x \tan y \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\sin(x-y)}{\sin x \sin y} = \cot y - \cot x$

Solution

$$\begin{aligned}\frac{\sin(x-y)}{\sin x \sin y} &= \frac{\sin x \cos y - \cos x \sin y}{\sin x \sin y} \\ &= \frac{\sin x \cos y}{\sin x \sin y} - \frac{\cos x \sin y}{\sin x \sin y} \\ &= \frac{\cos y}{\sin y} - \frac{\cos x}{\sin x} \\ &= \cot y - \cot x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\cos(x+y)}{\cos x \sin y} = \cot y - \tan x$

Solution

$$\begin{aligned}\frac{\cos(x+y)}{\cos x \sin y} &= \frac{\cos x \cos y - \sin x \sin y}{\cos x \sin y} \\ &= \frac{\cos x \cos y}{\cos x \sin y} - \frac{\sin x \sin y}{\cos x \sin y} \\ &= \frac{\cos y}{\sin y} - \frac{\sin x}{\cos x} \\ &= \cot y - \tan x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\sin(x+y)}{\cos(x-y)} = \frac{1 + \cot x \tan y}{\cot x + \tan y}$

Solution

$$\begin{aligned}\frac{\sin(x+y)}{\cos(x-y)} &= \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y + \sin x \sin y} \\&= \frac{\frac{\sin x \cos y}{\sin x \cos y} + \frac{\cos x \sin y}{\sin x \cos y}}{\frac{\cos x \cos y}{\sin x \cos y} + \frac{\sin x \sin y}{\sin x \cos y}} \\&= \frac{1 + \cot x \tan y}{\cot x + \tan y} \quad \checkmark\end{aligned}$$

Exercise

Prove the identity $\sin\left(\frac{\pi}{4} + x\right) + \sin\left(\frac{\pi}{4} - x\right) = \sqrt{2} \cos x$

Solution

$$\begin{aligned}\sin\left(\frac{\pi}{4} + x\right) + \sin\left(\frac{\pi}{4} - x\right) &= \sin \frac{\pi}{4} \cos x + \sin x \cos \frac{\pi}{4} + \sin \frac{\pi}{4} \cos x - \sin x \cos \frac{\pi}{4} \\&= \sin \frac{\pi}{4} \cos x + \sin \frac{\pi}{4} \cos x \\&= 2 \sin \frac{\pi}{4} \cos x \\&= 2 \frac{\sqrt{2}}{2} \cos x \\&= \sqrt{2} \cos x \quad \checkmark\end{aligned}$$

Exercise

Prove the identity $\cos(A+B) + \cos(A-B) = 2 \cos A \cos B$

Solution

$$\begin{aligned}\cos(A+B) + \cos(A-B) &= \cos A \cos B - \sin A \sin B + \cos A \cos B + \sin A \sin B \\&= \cos A \cos B + \cos A \cos B \\&= 2 \cos A \cos B \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\sin(x - y) - \sin(y - x) = 2\sin x \cos y - 2\cos x \sin y$

Solution

$$\begin{aligned}\sin(x - y) - \sin(y - x) &= \sin x \cos y - \sin y \cos x - (\sin y \cos x - \sin x \cos y) \\ &= \sin x \cos y - \sin y \cos x - \sin y \cos x + \sin x \cos y \\ &= \underline{2\sin x \cos y - 2\sin y \cos x} \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cos(x - y) + \cos(y - x) = 2\cos x \cos y + 2\sin x \sin y$

Solution

$$\begin{aligned}\cos(x - y) + \cos(y - x) &= \cos x \cos y + \sin x \sin y + \cos y \cos x + \sin y \sin x \\ &= \underline{2\cos x \cos y + 2\sin x \sin y} \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\tan(x + y) \tan(x - y) = \frac{\tan^2 x - \tan^2 y}{1 - \tan^2 x \tan^2 y}$

Solution

$$\begin{aligned}\tan(x + y) \tan(x - y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} \frac{\tan x - \tan y}{1 + \tan x \tan y} & (a + b)(a - b) &= a^2 - b^2 \\ &= \underline{\frac{\tan^2 x - \tan^2 y}{1 - \tan^2 x \tan^2 y}} \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\cos(\alpha - \beta)}{\sin(\alpha + \beta)} = \frac{1 - \tan \alpha \tan \beta}{\tan \alpha - \tan \beta}$

Solution

$$\begin{aligned}\frac{\cos(\alpha - \beta)}{\sin(\alpha + \beta)} &= \frac{\cos \alpha \cos \beta + \sin \alpha \sin \beta}{\sin \alpha \cos \beta + \sin \beta \cos \alpha} \\ &= \frac{\frac{\cos \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}}{\frac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\sin \beta \cos \alpha}{\cos \alpha \cos \beta}} \\ &= \underline{\frac{1 + \tan \alpha \tan \beta}{\tan \alpha + \tan \beta}} \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\sec(x+y) = \frac{\cos x \cos y + \sin x \sin y}{\cos^2 x - \sin^2 y}$

Solution

$$\begin{aligned}\sec(x+y) &= \frac{1}{\cos(x+y)} \frac{\cos(x-y)}{\cos(x-y)} \\&= \frac{\cos x \cos y + \sin x \sin y}{(\cos x \cos y - \sin x \sin y)(\cos x \cos y + \sin x \sin y)} \\&= \frac{\cos x \cos y + \sin x \sin y}{\cos^2 x \cos^2 y - \sin^2 x \sin^2 y} \\&= \frac{\cos x \cos y + \sin x \sin y}{\cos^2 x \cos^2 y - \sin^2 x \sin^2 y} \\&= \frac{\cos x \cos y + \sin x \sin y}{\cos^2 x (1 - \sin^2 y) - (1 - \cos^2 x) \sin^2 y} \\&= \frac{\cos x \cos y + \sin x \sin y}{\cos^2 x - \cos^2 x \sin^2 y - \sin^2 y + \cos^2 x \sin^2 y} \\&= \frac{\cos x \cos y + \sin x \sin y}{\cos^2 x - \sin^2 y} \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\csc(x-y) = \frac{\sin x \cos y + \cos x \sin y}{\sin^2 x - \sin^2 y}$

Solution

$$\begin{aligned}\csc(x-y) &= \frac{1}{\sin(x-y)} \frac{\sin x \cos y + \cos x \sin y}{\sin x \cos y + \cos x \sin y} \\&= \frac{\sin x \cos y + \cos x \sin y}{(\sin x \cos y - \cos x \sin y)(\sin x \cos y + \cos x \sin y)} \\&= \frac{\sin x \cos y + \cos x \sin y}{\sin^2 x \cos^2 y - \cos^2 x \sin^2 y} \\&= \frac{\sin x \cos y + \cos x \sin y}{\sin^2 x (1 - \sin^2 y) - (1 - \sin^2 x) \sin^2 y} \\&= \frac{\sin x \cos y + \cos x \sin y}{\sin^2 x - \sin^2 x \sin^2 y - \sin^2 y + \sin^2 x \sin^2 y} \\&= \frac{\sin x \cos y + \cos x \sin y}{\sin^2 x - \sin^2 y} \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\tan(x+y)\tan(x-y) = \frac{\tan^2 x - \tan^2 y}{1 - \tan^2 x \tan^2 y}$

Solution

$$\begin{aligned}\tan(x+y)\tan(x-y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} \cdot \frac{\tan x - \tan y}{1 + \tan x \tan y} & (a+b)(a-b) = a^2 - b^2 \\ &= \frac{\tan^2 x + \tan^2 y}{1 - \tan^2 x \tan^2 y} \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\cos(\alpha - \beta)}{\sin(\alpha + \beta)} = \frac{1 - \tan \alpha \tan \beta}{\tan \alpha - \tan \beta}$

Solution

$$\begin{aligned}\frac{\cos(\alpha - \beta)}{\sin(\alpha + \beta)} &= \frac{\cos \alpha \cos \beta + \sin \alpha \sin \beta}{\sin \alpha \cos \beta + \sin \beta \cos \alpha} \\ &= \frac{\frac{\cos \alpha \cos \beta}{\sin \alpha \cos \beta} + \frac{\sin \alpha \sin \beta}{\sin \beta \cos \alpha}}{\frac{\cos \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} \\ &= \frac{1 + \tan \alpha \tan \beta}{\tan \alpha + \tan \beta} \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\sec(x+y) = \frac{\cos x \cos y + \sin x \sin y}{\cos^2 x - \sin^2 y}$

Solution

$$\begin{aligned}\sec(x+y) &= \frac{1}{\cos(x+y)} \frac{\cos(x-y)}{\cos(x-y)} \\ &= \frac{\cos x \cos y + \sin x \sin y}{(\cos x \cos y - \sin x \sin y)(\cos x \cos y + \sin x \sin y)} \\ &= \frac{\cos x \cos y + \sin x \sin y}{\cos^2 x \cos^2 y - \sin^2 x \sin^2 y} \\ &= \frac{\cos x \cos y + \sin x \sin y}{\cos^2 x \cos^2 y - \sin^2 x \sin^2 y} \\ &= \frac{\cos x \cos y + \sin x \sin y}{\cos^2 x (1 - \sin^2 y) - (1 - \cos^2 x) \sin^2 y}\end{aligned}$$

$$\begin{aligned}
&= \frac{\cos x \cos y + \sin x \sin y}{\cos^2 x - \cos^2 x \sin^2 y - \sin^2 y + \cos^2 x \sin^2 y} \\
&= \frac{\cos x \cos y + \sin x \sin y}{\cos^2 x - \sin^2 y} \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\csc(x - y) = \frac{\sin x \cos y + \cos x \sin y}{\sin^2 x - \sin^2 y}$

Solution

$$\begin{aligned}
\csc(x - y) &= \frac{1}{\sin(x - y)} \cdot \frac{\sin x \cos y + \cos x \sin y}{\sin x \cos y + \cos x \sin y} \\
&= \frac{\sin x \cos y + \cos x \sin y}{(\sin x \cos y - \cos x \sin y)(\sin x \cos y + \cos x \sin y)} \\
&= \frac{\sin x \cos y + \cos x \sin y}{\sin^2 x \cos^2 y - \cos^2 x \sin^2 y} \\
&= \frac{\sin x \cos y + \cos x \sin y}{\sin^2 x (1 - \sin^2 y) - (1 - \sin^2 x) \sin^2 y} \\
&= \frac{\sin x \cos y + \cos x \sin y}{\sin^2 x - \sin^2 x \sin^2 y - \sin^2 y + \sin^2 x \sin^2 y} \\
&= \frac{\sin x \cos y + \cos x \sin y}{\sin^2 x - \sin^2 y} \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\tan(x + y) + \tan(x - y) = \frac{2 \tan x}{\cos^2 y (1 - \tan^2 x \tan^2 y)}$

Solution

$$\begin{aligned}
\tan(x + y) + \tan(x - y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} + \frac{\tan x - \tan y}{1 + \tan x \tan y} \\
&= \frac{(\tan x + \tan y)(1 + \tan x \tan y) + (\tan x - \tan y)(1 - \tan x \tan y)}{(1 - \tan x \tan y)(1 + \tan x \tan y)} \\
&= \frac{\tan x + \tan^2 x \tan y + \tan y + \tan x \tan^2 y + \tan x - \tan^2 x \tan y - \tan y + \tan x \tan^2 y}{(1 - \tan^2 x \tan^2 y)} \\
&= \frac{2 \tan x + 2 \tan x \tan^2 y}{(1 - \tan^2 x \tan^2 y)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{2 \tan x (1 + \tan^2 y)}{(1 - \tan^2 x \tan^2 y)} \\
&= \frac{2 \tan x \sec^2 y}{(1 - \tan^2 x \tan^2 y)} \\
&= \frac{2 \tan x}{\cos^2 y (1 - \tan^2 x \tan^2 y)} \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\cos(x-y)}{\cos(x+y)} = \frac{1 + \tan x \tan y}{1 - \tan x \tan y}$

Solution

$$\begin{aligned}
\frac{\cos(x-y)}{\cos(x+y)} &= \frac{\cos x \cos y + \sin x \sin y}{\cos x \cos y - \sin x \sin y} \\
&= \frac{\frac{\cos x \cos y}{\cos x \cos y} + \frac{\sin x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y}{\cos x \cos y} - \frac{\sin x \sin y}{\cos x \cos y}} \\
&= \frac{1 + \tan x \tan y}{1 - \tan x \tan y} \quad \checkmark
\end{aligned}$$

Exercise

Common household current is called **alternating current** because the current alternates direction within the wires. The voltage V in a typical 115-volt outlet can be expressed by the function $V(t) = 163 \sin \omega t$ where ω is the angular speed (in *radians per second*) of the rotating generator at the electrical plant, and t is time measured in seconds.

- It is essential for electric generators to rotate at precisely 60 cycles per second so household appliances and computers will function properly. Determine ω for these electric generators.
- Determine a value of ϕ so that the graph of $V(t) = 163 \cos(\omega t - \phi)$ is the same as the graph of $V(t) = 163 \sin \omega t$

Solution

$$\begin{aligned}
a) \quad \omega &= 60 \frac{\text{cycles}}{\text{sec}} \frac{2\pi \text{ rad}}{\text{cycles}} \\
&= 120\pi \frac{\text{rad}}{\text{sec}}
\end{aligned}$$

$$b) \quad V(t) = 163 \cos(\omega t - \phi) = 163 \sin \omega t$$

$$\cos(120\pi t)\cos\phi + \sin(120\pi t)\sin\phi = \sin 120\pi t$$

$$\begin{cases} \cos(120\pi t)\cos\phi = 0 \\ \sin\phi = 1 \end{cases}$$

$$\underline{\phi = \frac{\pi}{2}} \quad \Bigg|$$