Solution Section 4.2 – Area under Curves

Exercise

Use finite approximations to estimate the area under the graph of the function using

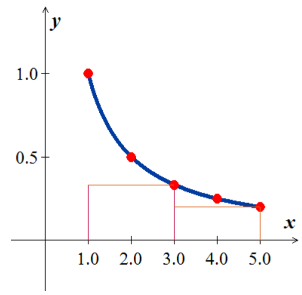
$$f(x) = \frac{1}{x} \quad \textit{between} \quad x = 1 \quad \textit{and} \quad x = 5$$

a) A lower sum with two rectangles of equal width
b) A lower sum with four rectangles of equal width
c) A upper sum with two rectangles of equal width
d) A upper sum with four rectangles of equal width

Solution

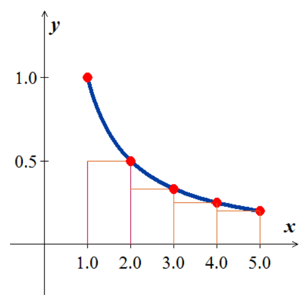
a) Using 2 lower rectangles: $\Delta x = \frac{5-1}{2} = 2$

$$\begin{aligned} A &\approx \Delta x\left(f(x_1) + f(x_2)\right) \\ &\approx 2\cdot\left(f(3) + f(5)\right) \\ &\approx 2\cdot\left(\frac{1}{3} + \frac{1}{5}\right) \\ &\approx \frac{16}{15} \end{aligned}$$



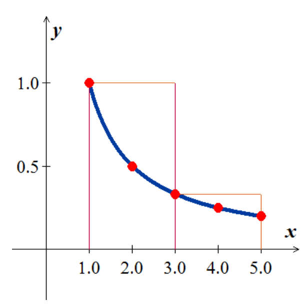
b) Using 4 lower rectangles: $\Delta x = \frac{5-1}{4} = 1$

$$\begin{aligned} A &\approx 1\cdot\left(f(2) + f(3) + f(4) + f(5)\right) \\ &\approx 1\cdot\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) \\ &\approx \frac{77}{60} \end{aligned}$$



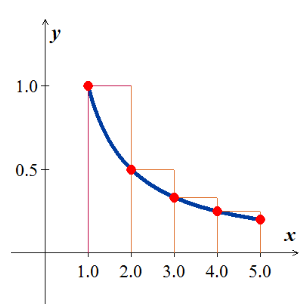
c) Using 2 upper rectangles: $\Delta x = \frac{5-1}{2} = 2$

$$\begin{aligned} A &\approx 2\cdot\left(f(1) + f(3)\right) \\ &\approx 2\cdot\left(1 + \frac{1}{3}\right) \\ &\approx \frac{8}{3} \end{aligned}$$



d) Using 4 lower rectangles: $\Delta x = \frac{5-1}{4} = 1$

$$\begin{aligned} A &\approx 1\cdot\left(f(1) + f(2) + f(3) + f(4)\right) \\ &\approx 1\cdot\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) \\ &\approx \frac{25}{12} \end{aligned}$$



Exercise

Use finite approximations to estimate the area under the graph of the function using

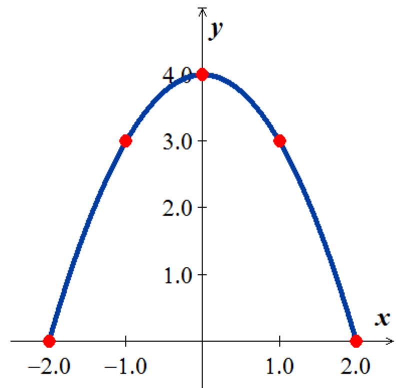
$$f(x) = 4 - x^2 \quad \textit{between} \quad x = -2 \quad \textit{and} \quad x = 2$$

a) A lower sum with two rectangles of equal width
b) A lower sum with four rectangles of equal width
c) A upper sum with two rectangles of equal width
d) A upper sum with four rectangles of equal width

Solution

a) Using 2 lower rectangles: $\Delta x = \frac{2-(-2)}{2} = 2$

$$\begin{aligned} A &\approx \Delta x\left(f\left(x_1\right) + f\left(x_2\right)\right) \\ &\approx 2 \cdot \left(f(-2) + f(2)\right) \\ &\approx 2 \cdot \left[\left(4-(-2)^2\right) + \left(4 - 2^2\right)\right] \\ &= 0 \end{aligned}$$



b) Using 4 lower rectangles: $\Delta x = \frac{2-(-2)}{4} = 1$

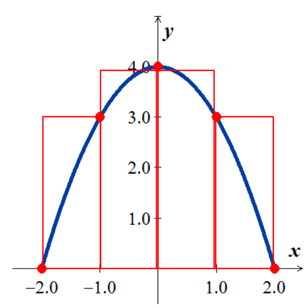
$$\begin{aligned} A &\approx 1 \cdot \left(f(-2) + f(-1) + f(1) + f(2)\right) \\ &\approx 1 \cdot (0 + 3 + 3 + 0) \\ &= 6 \end{aligned}$$

c) Using 2 upper rectangles: $\Delta x = \frac{2-(-2)}{2} = 2$

$$\begin{aligned} A &\approx 2 \cdot \left(f(0) + f(0)\right) \\ &\approx 2 \cdot (4 + 4) \\ &= 16 \end{aligned}$$

d) Using 4 lower rectangles: $\Delta x = \frac{2-(-2)}{4} = 1$

$$\begin{aligned} A &\approx 1 \cdot \left(f(-1) + f(0) + f(1) + f(2)\right) \\ &\approx 1 \cdot (3 + 4 + 4 + 3) \\ &= 14 \end{aligned}$$

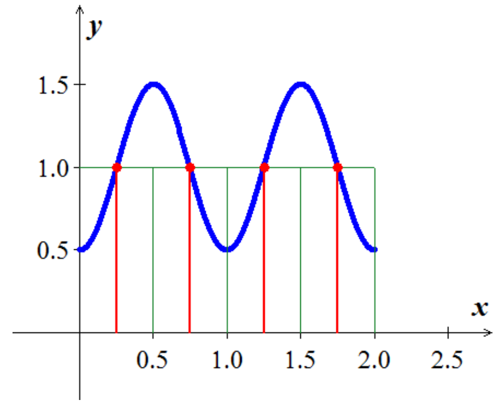


Exercise

Use finite approximations to estimate the average value of *f* on the given interval by partitioning the interval into four subintervals of equal length and evaluating *f* at the subinterval midpoints.

$$f(t)=\frac{1}{2}+\sin^2 \pi t \quad on \quad [0,\ 2]$$

Solution



$$\Delta x=\frac{2-0}{4}=0.5$$

$$f(t=.25)=\frac{1}{2}+\sin^2(.25\pi)=1$$

$$f(t=.75)=\frac{1}{2}+\sin^2(.75\pi)=1$$

$$f(t=1.25)=\frac{1}{2}+\sin^2(1.25\pi)=1$$

$$f(t=1.75)=\frac{1}{2}+\sin^2(1.75\pi)=1$$

$$\begin{aligned}A&\approx .5\cdot\big(f(.25)+f(.75)+f(1.25)+f(1.75)\big)\\&=.5(1+1+1+1)\\&=2\end{aligned}$$

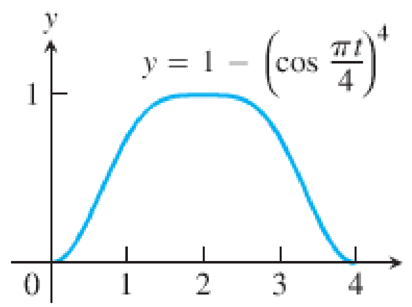
$$\begin{aligned}\text{Average value } &\approx \frac{Area}{Length\ [0,\ 2]}\\&=\frac{2}{2}\\&=1\end{aligned}$$

Exercise

Use finite approximations to estimate the average value of *f* on the given interval by partitioning the interval into four subintervals of equal length and evaluating *f* at the subinterval midpoints.

$$f(t)=1-\left(\cos\frac{\pi t}{4}\right)^4 \quad on \quad [0,\ 4]$$

Solution



$$\Delta x=\frac{4-0}{4}=1$$

$$f(t=0.5)=1-\left(\cos\frac{0.5\pi}{4}\right)^4=0.27145$$

$$f(t=1.5)=1-\left(\cos\frac{1.5\pi}{4}\right)^4=0.97855$$

$$f(t=2.5)=1-\left(\cos\frac{2.5\pi}{4}\right)^4=0.97855$$

$$f\left(t=3.5\right)=1-\left(\cos\frac{3.5\pi}{4}\right)^4=0.27145$$

$$A\approx 1\cdot\left(f\left(.5\right)+f\left(1.5\right)+f\left(2.5\right)+f\left(3.5\right)\right)$$

$$=1\left(0.27145+0.97855+0.97855+0.27145\right)$$

$$\underline{=2.5}$$

$$\text{Average value }\approx\frac{Area}{Length\ \left[0,\ 2\right]}$$

$$=\frac{2.5}{4}$$

$$=\underline{0.625}$$

Exercise

Write the sums without sigma notation. Then evaluate: $\displaystyle\sum_{k=1}^{2}\frac{6k}{k+1}$

Solution

$$\sum_{k=1}^{2}\frac{6k}{k+1}=\frac{6}{2}+\frac{12}{3}\ \underline{=7}$$

Exercise

Write the sums without sigma notation. Then evaluate: $\displaystyle\sum_{k=1}^{3}\frac{k-1}{k}$

Solution

$$\sum_{k=1}^{3}\frac{k-1}{k}=\frac{0}{1}+\frac{1}{2}+\frac{2}{3}\ \underline{=\frac{7}{6}}$$

Exercise

Write the sums without sigma notation. Then evaluate: $\displaystyle\sum_{k=1}^{5}\sin k\pi$

Solution

$$\sum_{k=1}^{5}\sin k\pi=\sin\pi+\sin 2\pi+\sin 3\pi+\sin 4\pi+\sin 5\pi=0+0+0+0\ \underline{=0}$$

Exercise

Write the sums without sigma notation. Then evaluate: $\sum_{k=1}^{4}(-1)^k \cos k\pi$

Solution

$$\sum_{k=1}^{4}(-1)^k \cos k\pi = -\cos\pi + \cos 2\pi - \cos 3\pi + \cos 4\pi = -(-1)+1-(-1)+1 \underline{= 4}$$

Exercise

Write the following expression 1 + 2 + 4 + 8 + 16 + 32 in sigma notation

Solution

$$1+2+4+8+16+32 = \sum_{k=1}^{6} 2^{k-1}$$

$$1+2+4+8+16+32 = \sum_{k=0}^{5} 2^{k}$$

Exercise

Write the following expression 1 – 2 + 4 – 8 + 16 – 32 in sigma notation

Solution

$$1-2+4-8+16-32 = \sum_{k=1}^{6} (-2)^{k-1}$$

$$1-2+4-8+16-32 = \sum_{k=0}^{2} (-1)^k 2^{k}$$

Exercise

Write the following expression $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$ in sigma notation

Solution

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} = \frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4} = \sum_{k=1}^{4}\frac{1}{2^k}$$

Exercise

Write the following expression $-\frac{1}{5}+\frac{2}{5}-\frac{3}{5}+\frac{4}{5}-\frac{5}{5}$ in sigma notation

Solution

$$-\frac{1}{5}+\frac{2}{5}-\frac{3}{5}+\frac{4}{5}-\frac{5}{5}=\sum_{k=1}^{5}(-1)^k\frac{k}{5}$$

Exercise

Suppose that $\sum_{k=1}^{n}a_k=-5$ *and* $\sum_{k=1}^{n}b_k=6$. Find the value of $\sum_{k=1}^{n}\left(b_k-2a_k\right)$

Solution

$$\begin{aligned}\sum_{k=1}^{n}\left(b_k-2a_k\right)&=\sum_{k=1}^{n}b_k-2\sum_{k=1}^{n}a_k\\&=6-2(-5)\\&=16\end{aligned}$$

Exercise

Evaluate the sums $\sum_{k=1}^{10}k^3$

Solution

$$\begin{aligned}\sum_{k=1}^{10}k^3&=\left(\frac{10(10+1)}{2}\right)^2\\&=55^2\\&=3025\end{aligned}$$

Exercise

Evaluate the sums $\sum_{k=1}^{7}(-2k)$

Solution

$$\sum_{k=1}^{7}(-2k) = -2\sum_{k=1}^{7} k$$

$$= -2\left(\frac{7(7+1)}{2}\right)$$

$$\underline{= -56}$$

Exercise

Evaluate the sums $\displaystyle\sum_{k=1}^{5} \frac{\pi k}{15}$

Solution

$$\sum_{k=1}^{5} \frac{\pi k}{15} = \frac{\pi}{15}\sum_{k=1}^{5} k$$

$$= \frac{\pi}{15}\left(\frac{5(5+1)}{2}\right)$$

$$\underline{= \pi}$$

Exercise

Evaluate the sums $\displaystyle\sum_{k=1}^{5} k(3k+5)$

Solution

$$\sum_{k=1}^{5} k(3k+5) = \sum_{k=1}^{5}\left(3k^2+5k\right)$$

$$= 3\sum_{k=1}^{5} k^2 + 5\sum_{k=1}^{5} k$$

$$= 3\left(\frac{5(5+1)(2(5)+1)}{6}\right) + 5\frac{5(5+1)}{2}$$

$$\underline{= 240}$$

Exercise

Evaluate the sums $\displaystyle\sum_{k=1}^{5}\frac{k^3}{225}+\left(\sum_{k=1}^{5}k\right)^3$

Solution

$$\sum_{k=1}^{5}\frac{k^3}{225}+\left(\sum_{k=1}^{5}k\right)^3=\frac{1}{225}\sum_{k=1}^{5}k^3+\left(\sum_{k=1}^{5}k\right)^3$$

$$=\frac{1}{225}\left(\frac{5(5+1)}{2}\right)^2+\left(\frac{5(5+1)}{2}\right)^3$$

$$\underline{=3376}$$

Exercise

Evaluate the sums $\displaystyle\sum_{k=1}^{500}7$

Solution

$$\sum_{k=1}^{500}7=7(500)\ \underline{=3500}$$

Exercise

Evaluate the sums $\displaystyle\sum_{k=18}^{71}k(k-1)$

Solution

Let $n=(k-18)+1=k-17\begin{cases}k=18 & \to n=1\\ k=71 & \to n=54\end{cases}\quad\Rightarrow\quad k=n+17$

$$\sum_{k=18}^{71}k(k-1)=\sum_{n=1}^{54}(n+17)(n+17-1)$$

$$=\sum_{n=1}^{54}(n+17)(n+16)$$

$$=\sum_{n=1}^{54}\left(n^2+33n+272\right)$$

$$= \sum_{n=1}^{54} n^2 + 33\sum_{n=1}^{54} n + \sum_{n=1}^{54} 272$$

$$= \frac{54(54+1)(54(2)+1)}{6} + 33\cdot\frac{54(54+1)}{6} + 272(54)$$

$$\underline{= 117648}$$

Exercise

Evaluate the sums $\displaystyle\sum_{k=1}^{n}\left(\frac{1}{n}+2n\right)$

Solution

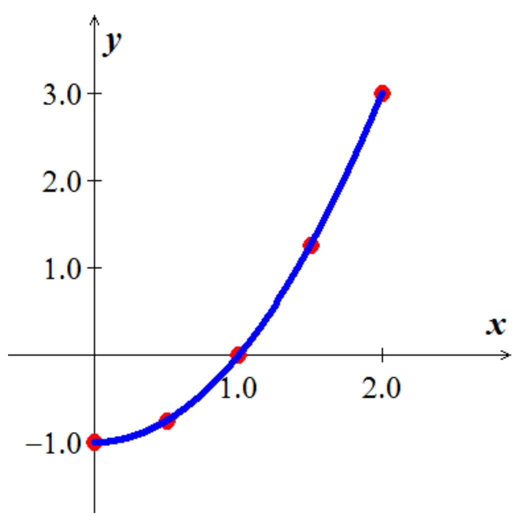
$$\sum_{k=1}^{n}\left(\frac{1}{n}+2n\right) = n\cdot\left(\frac{1}{n}+2n\right) \underline{= 1+2n^2}$$

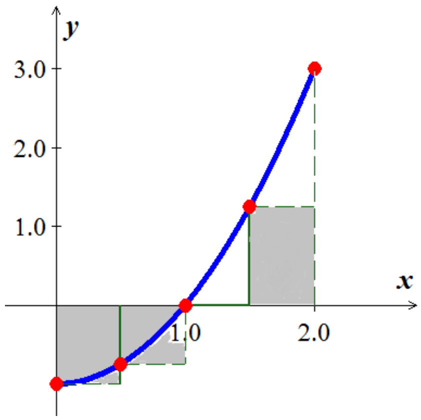
Exercise

Graph the function $f(x) = x^2 - 1$ over the given interval [0, 2]. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\displaystyle\sum_{k=1}^{4} f(c_k)\Delta x_k$, given c_k is the

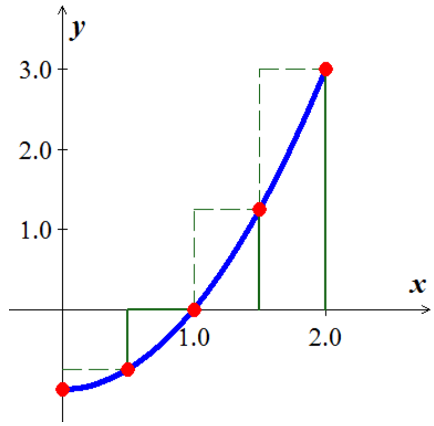
a) Left-hand endpoint
b) Right-hand endpoint
c) Midpoint of kth subinterval.

Solution

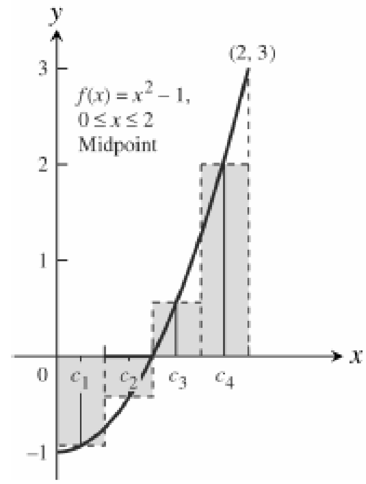




y
3.0
2.0
1.0
x
1.0
2.0



y
3.0
2.0
1.0
x
1.0
2.0



y
(2, 3)
3
$f(x) = x^2 - 1,$
$0 \le x \le 2$
Midpoint
2
1
x
0
c_1
c_2
c_3
c_4
−1