Section 4.2 – **Representing Relations**

Representing Relations Using Matrices

A relation between finite sets can be represented using a zero-one matrix. Suppose that R is a relation from $A = \{a_1, a_2, a_3, \ldots, a_m\}$ to $B = \{b_1, b_2, b_3, \ldots, b_n\}$. The relation R can be represented by the matrix $M_a = \{m_{ij}\}$ where

$$m_{ij} = \begin{cases} 1 & \text{if } \left(a_i, b_j\right) \in R \\ 0 & \text{if } \left(a_i, b_j\right) \notin R \end{cases}$$

Example

Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R is $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and $b_1 = 1$, $b_2 = 2$?

Solution

$$R = \{(2, 1), (3, 1), (3, 2)\} \qquad M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Example

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} ?$$
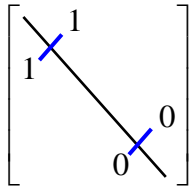
Solution

$$R = \left\{ \left(a_1, b_2\right), \left(a_2, b_1\right), \left(a_2, b_3\right), \left(a_2, b_4\right), \left(a_3, b_1\right), \left(a_3, b_3\right), \left(a_3, b_5\right) \right\}$$
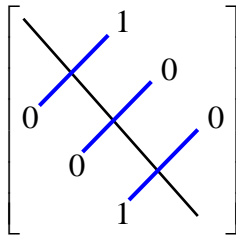
A relation R on A is **reflexive** if $(a, a) \in R$ whenever $a \in A$

$$M_R = \left(M_R\right)^t \qquad \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

A relation R on A is **symmetric**

$$\begin{bmatrix} \diagdown & 1 \\ 1 & & \\ & & 0 \\ & 0 & \diagdown \end{bmatrix}$$

A relation R on A is **antisymmetric** iff $(a,\, b) \in R$ and $(b,\, a) \in R \implies a = b$

$$\begin{bmatrix} & 1 \\ 0 & & 0 \\ & 0 & \\ & 1 & \end{bmatrix}$$

Example

Suppose that the relation R on the set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R reflexive, symmetric, and/or antisymmetric?

Solution

Because the diagonal elements are equal to 1, R is reflexive.

M_R is symmetric and it is not antisymmetric.

Relations Using Diagraphs

Definition

A directed **graph**, or **diagraph**, consists of a set V of **vertices** (or *nodes*) together with a set E ordered pairs of elements of V called **edges** (or *arcs*). The vertex a is called the **initial** vertex of the edge (a, b), and the vertex b is called the **terminal** vertex of this edge.

Example

Draw the directed graph with vertices *a, b, c,* and *d,* and edges (*a, b*), (*a, d*), (*b, b*), (*b, d*), (*c, a*), (*c, b*), and (*d, b*)
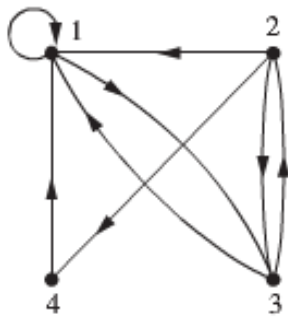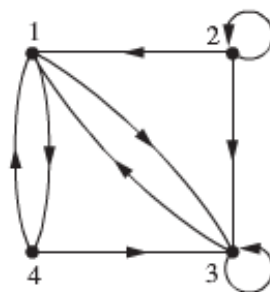
Solution
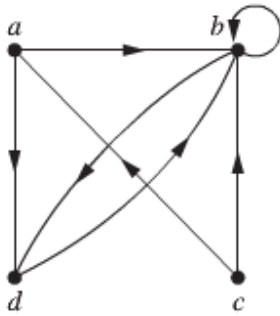
Example

Draw the directed graph of the relation $R = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$ on the set $\{1, 2, 3, 4\}$

Solution

Example

What are the ordered pairs in the relation R represented by the directed graph shown below

Solution

$R = \{(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)\}$

Exercises   *Section* **4.2 – Representing Relations**

1. Represent each of these relations on $\{1, 2, 3\}$ with a matrix (with the elements of this set listed in increasing order). Then draw the directed graphs representing each relation

   a) $\{(1, 1), (1, 2), (1, 3)\}$
   b) $\{(1, 2), (2, 1), (2, 2), (3, 3)\}$
   c) $\{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$
   d) $\{(1, 3), (3, 1)\}$

2. Represent each of these relations on $\{1, 2, 3, 4\}$ with a matrix (with the elements of this set listed in increasing order). Then draw the directed graphs representing each relation

   a) $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
   b) $\{(1, 1), (1, 4), (2, 2), (3, 3), (4, 1)\}$
   c) $\{ (1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)\}$
   d) $\{(2, 4), (3, 1), (3, 2), (3, 4)\}$

3. List the ordered pairs in the relations on $\{1, 2, 3\}$ corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order). Then draw the directed graphs representing each relation

   a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

4. List the ordered pairs in the relations on $\{1, 2, 3, 4\}$ corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order). Then draw the directed graphs representing each relation

   a) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

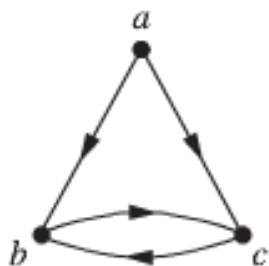5. Let R be the relation represented by the matrix

   $$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$
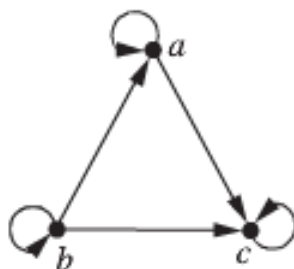
   Find: a) R^2 b) R^3 c) R^4

6. Draw the directed graph that represents the relation $\{(a, a), (a, b), (b, c), (c, b), (c, d), (d, a), (d, b)\}$

7. Determine whether the relations represented by the directed graphs are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive
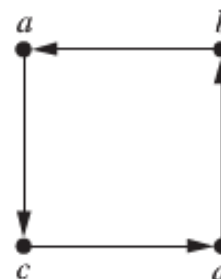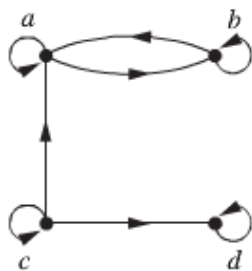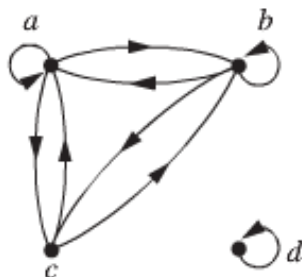
a)

b)

c)

d)

e)

f)