Solution   Section 1.2 – Gaussian Elimination

Exercise

When elimination is applied to the matrix $A=\begin{bmatrix}3&1&0\\6&9&2\\0&1&5\end{bmatrix}$

a) What are the first and second pivots?

b) What is the multiplier l_{21} in the first step (l_{21} times row 1 is subtracted from row 2)?

c) What entry in the 2, 2 position (instead of 9) would force an exchange of rows 2 and 3?

d) What is the multiplier $l_{31}=0$, subtracting 0 times row 1 from row 3?

Solution

a) The first pivot is 3 and when 2 times row 1 is subtracted from row 2, the second pivot is revealed as 7.

$$\begin{bmatrix}3&1&0\\6&9&2\\0&1&5\end{bmatrix}\ \begin{matrix}\textit{subtract 2 times row.1}\\ \textit{from row.2}\end{matrix}\ \begin{bmatrix}3&1&0\\0&\boxed{7}&2\\0&1&5\end{bmatrix}$$

b) The multiplier l_{21} in the first step is $\frac{6}{3}=2$.

c) If we reduce the entry 9 to 2, that drop of 7 in the a_{22} position would force a row exchange.

$$\begin{bmatrix}3&1&0\\6&9&2\\0&1&5\end{bmatrix}\ \begin{matrix}\textit{subtract 7 times row.1}\\ \textit{from row.2}\end{matrix}\ \begin{bmatrix}3&1&0\\-15&\boxed{2}&2\\0&1&5\end{bmatrix}$$

d) The multiplier l_{31} is already zero because $a_{31}=0$ and no needs row elimination.

Exercise

Use elimination to reach upper triangular matrices $\boldsymbol{U}$. Solve by back substitution or explain why this impossible. What are the pivots (never zero)? Exchange equations when necessary. The only difference is the $-x$ in equation (3).

$$\begin{cases}x+y+z=7\\x+y-z=5\\x-y+z=3\end{cases}\qquad\qquad\begin{cases}x+y+z=7\\x+y-z=5\\-x-y+z=3\end{cases}$$

Solution

For the *first* system:

$$\begin{array}{lll} x+y+z=7 & \textit{subtract eqn.1} & x+y+z=7 \\ x+y-z=5 & \textit{from eqn.2} & 0y-2z=-2 \\ x-y+z=3 & \textit{from eqn.3} & -2y-0z=-4 \end{array}$$

$$\begin{array}{lll} x+y+z=7 & & 1x+y+z=7 \\ x+y-z=5 & \textit{Exchange eqn.2} & -2y-0z=-4 \\ x-y+z=3 & \textit{and eqn.3} & -2z=-2 \end{array}$$

The solutions are: $z=1 \quad y=2 \quad x=4$ and the pivots are 1, -2, -2.

For the *second* system:

$$\begin{array}{lll} x+y+z=7 & \textit{Subtract eqn.1} & x+y+z=7 \\ x+y-z=5 & \textit{from eqn.2} & 0y-2z=-2 \\ -x-y+z=3 & \textit{Add eqn.1 to eqn.3} & 0y+2z=10 \end{array}$$

$$\begin{array}{lll} x+y+z=7 & & x+y+z=7 \\ 0y-2z=-2 & \textit{Add eqn.2} & 0y-2z=-2 \\ 0y+2z=10 & \textit{to eqn.3} & \boxed{0z=8} \end{array}$$

The three planes don't meet. But if we change '3' in the last equation to '-5'

$$\begin{array}{lll} x+y+z=7 & \textit{Subtract eqn.1} & x+y+z=7 \\ x+y-z=5 & \textit{from eqn.2} & 0y-2z=-2 \\ -x-y+z=-5 & \textit{Add eqn.1 to eqn.3} & 0y+2z=2 \end{array}$$

$$\begin{array}{ll} x+y+z=7 & x+y=6 \\ 0y-2z=-2 & \\ 0y+2z=10 & z=1 \end{array}$$

There are unique infinite many solutions!

The three planes now meet along a whole line.

Exercise

For which numbers a does the elimination break down (1) permanently (2) temporarily

$$\begin{aligned} ax+3y&=-3 \\ 4x+6y&=6 \end{aligned}$$

Solve for x and y after fixing the second breakdown by a row change.

Solution

The matrix form is: $\begin{pmatrix} a & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$

If $a = 0$, the elimination brakes down temporarily.

$$\begin{pmatrix} 4 & 6 \\ 0 & \boxed{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$$

The system is in upper triangular form and entry row 2 column 2 is not equal to zero, therefore the system has a solution.

If $a \neq 0$,

$$\begin{pmatrix} a & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix} \qquad R_2 - \tfrac{4}{a}R_1$$

$$\begin{pmatrix} a & 3 \\ 0 & 6 - \frac{12}{a} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 + \frac{12}{a} \end{pmatrix}$$

$$6 - \frac{12}{a} = 0$$

$$\frac{12}{a} = 6$$

$$a = \frac{12}{6} \underline{= 2}$$

If $a = 2$,

$$\begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \end{pmatrix}$$, the system will fail and has no solution.

If $a \neq 2$;

$$\begin{pmatrix} a & 3 \\ 0 & 6 - \frac{12}{a} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 + \frac{12}{a} \end{pmatrix}$$, the system has a unique solution.

Exercise

Find the pivots and the solution for these four equations:

$$\begin{aligned} 2x+y \quad\quad &= 0 \\ x+2y+z \quad &= 0 \\ y+2z+t &= 0 \\ z+2t &= 5 \end{aligned}$$

Solution

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix} R_2 - \tfrac{1}{2}R_1$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 1.5 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix} R_3 - \tfrac{2}{3}R_2$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix} R_4 - \tfrac{3}{4}R_3$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 5 \end{pmatrix} \quad \begin{aligned} 2x = -y &\Rightarrow \underline{x = -2\tfrac{1}{2} = -1} \\ \tfrac{3}{2}y + z = 0 &\Rightarrow y = -z\tfrac{2}{3} = -(-3)\tfrac{2}{3} \to \boxed{y=2} \\ \tfrac{4}{3}z + t = 0 &\to \tfrac{4}{3}z = -t \to \underline{z = -4\tfrac{3}{4} = -3} \\ \tfrac{5}{4}t = 5 &\to \boxed{t=4} \end{aligned}$$

The pivots are diagonal entries and the solution is: $(-1,\ 2,\ -3,\ 4)$

Exercise

Look for a matrix that has row sums 4 and 8, and column sums 2 and s.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{matrix} a+b=4 & a+c=2 \\ c+d=8 & b+d=s \end{matrix}$$

The four equations are solvable only if $s =$ ____. Then find two different matrices that have the correct row and column sums.

Solution

$$\begin{array}{rr} & a+b=4 \\ + & c+d=8 \\ \hline & a+c+b+d=12 \end{array}$$

$$2+s=12$$

$$\boxed{s=10}$$

Exercise

Three planes can fail to have an intersection point, even if no planes are parallel. The system is singular if row 3 of A is a _______ of the first two rows. Find a third equation that can't be solved together with $x+y+z=0$ and $x-2y-z=1$

Solution

The system is singular if row 3 of A is a ***linear combination*** of the first two rows.

There are many possible of a third equation that can't be solved together with $x+y+z=0$ and $x-2y-z=1$.

$$\begin{array}{ll} 3 \textit{ times } 1^{st} \textit{ equation} & 3x+3y+3z \\ \qquad \textit{minus } 2nd & -x+2y+z \\ \hline & 2x+5y+4z=1 \end{array}$$

Exercise

Use the Gauss-Jordan method to solve the system
$$\begin{aligned} x - y + 5z &= -6 \\ 3x + 3y - z &= 10 \\ x + 3y + 2z &= 5 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 3 & -1 & 10 \\ -3 & 3 & -15 & 18 \\ \hline 0 & 6 & -16 & 28 \end{array} \qquad \begin{array}{cccc} 1 & 3 & 2 & 5 \\ -1 & 1 & -5 & 6 \\ \hline 0 & 4 & -3 & 11 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 0 & 4 & -3 & 11 \end{array}\right] \begin{array}{l} \\ \frac{1}{6}R_2 \\ \\ \end{array} \qquad 0 \quad 1 \quad -\frac{8}{3} \quad \frac{14}{3}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 4 & -3 & 11 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ \\ R_3 - 4R_2 \end{array}$$

$$\begin{array}{cccc} 0 & 4 & -3 & 11 \\ 0 & -4 & \frac{32}{3} & -\frac{56}{3} \\ \hline 0 & 0 & \frac{23}{3} & -\frac{23}{3} \end{array} \qquad \begin{array}{cccc} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ \hline 1 & 0 & \frac{7}{3} & -\frac{4}{3} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{3} & -\frac{4}{3} \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & \frac{23}{3} & -\frac{23}{3} \end{array}\right] \begin{array}{l} \\ \\ \frac{3}{23}R_3 \end{array} \qquad 0 \quad 0 \quad 1 \quad -1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{3} & -\frac{4}{3} \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} R_1 - \frac{7}{3}R_3 \\ R_2 + \frac{8}{3}R_3 \\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & \frac{7}{3} & -\frac{4}{3} \\ 0 & 0 & -\frac{7}{3} & \frac{7}{3} \\ \hline 1 & 0 & 0 & 1 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & \frac{8}{3} & -\frac{8}{3} \\ \hline 0 & 1 & 0 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

S***olution***: $\underline{(1,\ 2,\ -1)}$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} 2x - y + 4z = -3 \\ x - 2y - 10z = -6 \\ 3x \qquad + 4z = 7 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & -1 & 4 & -3 \\ 1 & -2 & -10 & -6 \\ 3 & 0 & 4 & 7 \end{array}\right] \quad \tfrac{1}{2}R_1 \qquad\qquad 1 \quad -\tfrac{1}{2} \quad 2 \quad -\tfrac{3}{2}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 1 & -2 & -10 & -6 \\ 3 & 0 & 4 & 7 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 3R_1 \end{array} \qquad \begin{array}{cccc} 1 & -2 & -10 & -6 \\ -1 & \frac{1}{2} & -2 & \frac{3}{2} \\ \hline 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \end{array} \qquad \begin{array}{cccc} 3 & 0 & 4 & 7 \\ -3 & \frac{3}{2} & -6 & \frac{9}{2} \\ \hline 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \\ 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}\right] \quad -\tfrac{2}{3}R_2 \qquad\qquad 0 \quad 1 \quad 8 \quad 3$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & 1 & 8 & 3 \\ 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}\right] \begin{array}{l} R_1 + \frac{1}{2}R_2 \\ \\ R_3 - \frac{3}{2}R_2 \end{array} \qquad \begin{array}{cccc} 0 & \frac{3}{2} & -2 & \frac{23}{2} \\ 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \\ \hline 0 & 0 & -14 & 7 \end{array} \qquad \begin{array}{cccc} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & \frac{1}{2} & 4 & \frac{3}{2} \\ \hline 1 & 0 & 6 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & -14 & 7 \end{array}\right] \quad -\tfrac{1}{14}R_3 \qquad\qquad 0 \quad 0 \quad 1 \quad -\tfrac{1}{2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right] \begin{array}{l} R_1 - 6R_3 \\ R_2 - 8R_3 \\ \\ \end{array} \qquad \begin{array}{cccc} 1 & 0 & 6 & 0 \\ 0 & 0 & -6 & 3 \\ \hline 1 & 0 & 0 & 3 \end{array} \qquad \begin{array}{cccc} 0 & 1 & 8 & 3 \\ 0 & 0 & -8 & 4 \\ \hline 0 & 1 & 0 & 7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right]$$

Solution: $\left(3,\ 7,\ -\frac{1}{2}\right)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} 4x+3y-5z=-29 \\ 3x-7y-z=-19 \\ 2x+5y+2z=-10 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 4 & 3 & -5 & -29 \\ 3 & -7 & -1 & -19 \\ 2 & 5 & 2 & -10 \end{array}\right] \quad \tfrac{1}{4}R_1$$

$$1 \quad \tfrac{3}{4} \quad -\tfrac{5}{4} \quad -\tfrac{29}{4}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 3 & -7 & -1 & -19 \\ 2 & 5 & 2 & -10 \end{array}\right] \begin{array}{l} \\ R_2-3R_1 \\ R_3-2R_1 \end{array}$$

$$\begin{array}{cccc} 3 & -7 & -1 & -19 \\ -3 & -\frac{9}{4} & \frac{15}{4} & \frac{87}{4} \\ \hline 0 & -\frac{37}{4} & \frac{11}{4} & \frac{11}{4} \end{array} \qquad \begin{array}{cccc} 2 & 5 & 2 & -10 \\ -2 & -\frac{3}{2} & \frac{5}{2} & \frac{29}{2} \\ \hline 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & -\frac{37}{4} & \frac{11}{4} & \frac{11}{4} \\ 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}\right] \begin{array}{l} \\ -\frac{4}{37}R_2 \\ \\ \end{array}$$

$$0 \quad 1 \quad -\tfrac{11}{37} \quad -\tfrac{11}{37}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}\right] \begin{array}{l} R_1-\frac{3}{4}R_2 \\ \\ R_3-\frac{7}{2}R_2 \end{array}$$

$$\begin{array}{cccc} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & -\frac{3}{4} & \frac{33}{148} & \frac{33}{148} \\ \hline 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \end{array} \qquad \begin{array}{cccc} 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \\ 0 & -\frac{7}{2} & \frac{77}{72} & \frac{77}{72} \\ \hline 0 & 0 & \frac{401}{72} & \frac{401}{72} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & \frac{401}{72} & \frac{401}{72} \end{array}\right] \begin{array}{l} \\ \\ \frac{72}{401}R_3 \end{array}$$

$$0 \quad 0 \quad 1 \quad 1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1+\frac{38}{37}R_3 \\ R_2+\frac{11}{37}R_3 \\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 0 & \frac{38}{37} & \frac{38}{37} \\ \hline 1 & 0 & 0 & -6 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & \frac{11}{37} & \frac{11}{37} \\ \hline 0 & 1 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Solution: $\underline{(-6,\ 0,\ 1)}$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} x+2y-3z=-15 \\ 2x-3y+4z=18 \\ -3x+y+z=1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 2 & -3 & 4 & 18 \\ -3 & 1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 2 & -3 & 4 & 18 \\ -3 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} R_2-2R_1 \\ R_3+3R_1 \end{array}$$

$$\begin{array}{cccc} -2 & -4 & 6 & 30 \\ 2 & -3 & 4 & 18 \\ \hline 0 & -7 & 10 & 48 \end{array} \qquad \begin{array}{cccc} 3 & 6 & -9 & -45 \\ -3 & 1 & 1 & 1 \\ \hline 0 & 7 & -8 & -44 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 0 & -7 & 10 & 48 \\ 0 & 7 & -8 & -44 \end{array}\right] -\frac{1}{7}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 7 & -8 & -44 \end{array}\right] \begin{array}{l} R_1-2R_2 \\ \\ R_3-7R_2 \end{array}$$

$$\begin{array}{cccc} 1 & 2 & -3 & -15 \\ 0 & -2 & \frac{20}{7} & \frac{96}{7} \\ \hline 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \end{array} \qquad \begin{array}{cccc} 0 & -7 & 10 & 48 \\ 0 & 7 & -8 & -44 \\ \hline 0 & 0 & 2 & 4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & 2 & 4 \end{array}\right] \frac{1}{2}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1+\frac{1}{7}R_3 \\ R_2+\frac{10}{7}R_3 \end{array}$$

$$\begin{array}{cccc} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 0 & \frac{1}{7} & \frac{2}{7} \\ \hline 1 & 0 & 0 & -1 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & \frac{10}{7} & \frac{20}{7} \\ \hline 0 & 1 & 0 & -4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solution: $(-1,\ -4,\ 2)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} x+2y+3z=10 \\ 4x+5y+6z=11 \\ 7x+8y+9z=12 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 4 & 5 & 6 & 11 \\ 7 & 8 & 9 & 12 \end{array}\right] \begin{array}{l} \\ R_2 - 4R_1 \\ R_3 - 7R_1 \end{array}$$

$$\begin{array}{cccc} -4 & -8 & -12 & -40 \\ 4 & 5 & 6 & 11 \\ \hline 0 & -3 & -6 & -29 \end{array} \qquad \begin{array}{cccc} -7 & -14 & -21 & -70 \\ 7 & 8 & 9 & 12 \\ \hline 0 & -6 & -12 & -58 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & -3 & -6 & -29 \\ 0 & -6 & -12 & -58 \end{array}\right] \ \tfrac{1}{3}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & \frac{29}{3} \\ 0 & -6 & -12 & -58 \end{array}\right] \ R_3 + 6R_2$$

$$\begin{array}{cccc} 0 & -6 & -12 & -58 \\ 0 & 6 & 12 & 58 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & \frac{29}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

let z be the variable

From Row 1 $\Rightarrow y + 2z = \frac{29}{3}$

$$\underline{y = \frac{29}{3} - 2z}$$

From Row 1 $\Rightarrow x + 2y + 3z = 10$

$$x = 10 - 2y - 3z$$

$$x = 10 - 2\left(\frac{29}{3} - 2z\right) - 3z$$

$$x = 10 - \frac{58}{3} + 4z - 3z$$

$$\underline{x = z - \frac{28}{3}}$$

Solution: $\underline{\left(z - \frac{28}{3},\ \frac{29}{3} - 2z,\ z\right)}$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} 2x+y+2z=4 \\ 2x+2y \qquad =5 \\ 2x-y+6z=2 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 4 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array}\right] \begin{array}{l} \frac{1}{2}R_1 \\ \\ \\ \end{array} \qquad \begin{array}{cccc} 1 & \frac{1}{2} & 1 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-2R_1 \end{array} \qquad \begin{array}{cccc} 2 & 2 & 0 & 5 \\ -2 & -1 & -2 & -4 \\ \hline 0 & 1 & -2 & 1 \end{array} \qquad \begin{array}{cccc} 2 & -1 & 6 & 2 \\ -2 & -1 & -2 & -4 \\ \hline 0 & -2 & 4 & -2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 4 & -2 \end{array}\right] \begin{array}{l} R_1-\frac{1}{2}R_2 \\ \\ R_3+2R_2 \end{array} \qquad \begin{array}{cccc} 1 & \frac{1}{2} & 1 & 2 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ \hline 1 & 0 & 2 & \frac{3}{2} \end{array} \qquad \begin{array}{cccc} 0 & -2 & 4 & -2 \\ 0 & 2 & -4 & 2 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & \frac{3}{2} \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From Row 3: 0 = 0 is a true statement. Let z be the variable.

From Row 2: $y-2z=1$

$$\underline{y=1+2z}$$

From Row 1: $x+2z=\frac{3}{2}$

$$\underline{x=-2z+\frac{3}{2}}$$

$\therefore$ ***Solution***: $\underline{\left(-2z+\frac{3}{2},\ 2z+1,\ z\right)}$

Exercise

Use the Gauss-Jordan method to solve the system
$$\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} -1 & -2 & 3 & 1 \\ 1 & 1 & 2 & 8 \\ \hline 0 & -1 & 5 & 9 \end{array} \qquad \begin{array}{cccc} 3 & -7 & 4 & 10 \\ -3 & -3 & -6 & -24 \\ \hline 0 & -10 & -2 & -14 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right] -R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 10R_2 \end{array}$$

$$\begin{array}{cccc} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ \hline 1 & 0 & 7 & 17 \end{array} \qquad \begin{array}{cccc} 0 & -10 & -2 & -14 \\ 0 & 10 & -50 & -90 \\ \hline 0 & 0 & -52 & -104 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right] -\frac{1}{52}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 - 7R_3 \\ R_2 + 5R_3 \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & 7 & 17 \\ 0 & 0 & -7 & -14 \\ \hline 1 & 0 & 0 & 3 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -5 & -9 \\ 0 & 0 & 5 & 10 \\ \hline 0 & 1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$\therefore$ ***Solution***: $(3,\ 1,\ 2)$

Exercise

Use augmented elimination to solve linear system

$$\begin{aligned} 2x-5y+3z&=1\\ x-2y-2z&=8 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8\\ 2 & -5 & 3 & 1 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -5 & 3 & 1\\ -2 & 4 & 4 & -16\\ \hline 0 & -1 & 7 & -15 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8\\ 0 & -1 & 7 & -15 \end{array}\right] \begin{array}{l} \\ -R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8\\ 0 & 1 & -7 & 15 \end{array}\right] \begin{array}{l} R_1+2R_2 \\ \end{array}$$

$$\begin{array}{cccc} 1 & -2 & -2 & 8\\ 0 & 2 & -14 & 30\\ \hline 1 & 0 & -16 & 38 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -16 & 38\\ 0 & 1 & -7 & 15 \end{array}\right] \begin{array}{l} \rightarrow\ x-16z=38\\ \rightarrow\ y-7z=15 \end{array}$$

$$\begin{cases} x=16z+38\\ y=7z+15 \end{cases}$$

$\therefore$ ***Solution***: $\underline{(16z+38,\ 7z+15,\ z)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x+y+z=2\\ 2x+y-z=5\\ x-y+z=-2 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2\\ 2 & 1 & -1 & 5\\ 1 & -1 & 1 & -2 \end{array}\right] \begin{array}{l} \\ R_2-2R_1\\ R_3-R_1 \end{array}$$

$$\begin{array}{cccc} 2 & 1 & -1 & 5\\ -2 & -2 & -2 & -4\\ \hline 0 & -1 & -3 & 1 \end{array} \qquad \begin{array}{cccc} 1 & -1 & 1 & -2\\ -1 & -1 & -1 & -2\\ \hline 0 & -2 & 0 & -4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2\\ 0 & -1 & -3 & 1\\ 0 & -2 & 0 & -4 \end{array}\right] \begin{array}{l} (2)\\ (1)\\ -2y=-4 \end{array}$$

$\underline{y=2}$

$(1)\rightarrow\quad -y-3z=1$

$3z=-1-2$

$\underline{z=-1}$

$(2) \rightarrow \quad x + y + z = 2$

$x = 2 - 2 + 1$

$\underline{= 1}|$

$\therefore$ ***Solution***: $\underline{(1,\ 2,\ -1)}\ |$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ -1 & -1 & 1 & 1 \\ 3 & -1 & 1 & 9 \end{array}\right] \begin{array}{l} \\ 2R_2 + R_1 \\ 2R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} -2 & -2 & 2 & 2 \\ 2 & 1 & 1 & 9 \\ \hline 0 & -1 & 3 & 11 \end{array} \qquad \begin{array}{cccc} 6 & -2 & 2 & 18 \\ -6 & -3 & -3 & -27 \\ \hline 0 & -5 & -1 & -9 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ 0 & -1 & 3 & 11 \\ 0 & -5 & -1 & -9 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 5R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -5 & -1 & -9 \\ 0 & 5 & -15 & -55 \\ \hline 0 & 0 & -16 & -64 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ 0 & -1 & 3 & 11 \\ 0 & 0 & -16 & -64 \end{array}\right] \begin{array}{c} (2) \\ (1) \\ -16z = -64 \end{array}$$

$\underline{z = 4}|$

$(1) \rightarrow \quad -y + 3z = 11$

$y = 12 - 11$

$\underline{= 1}|$

$(2) \rightarrow \quad 2x + y + z = 9$

$2x = 9 - 1 - 4$

$\underline{x = 2}|$

$\therefore$ ***Solution***: $\underline{(2,\ 1,\ 4)}\ |$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ -3 & 6 & 2 & 11 \end{array}\right] \quad R_3 + 3R_1$$

$$\begin{array}{cccc} -3 & 6 & 2 & 11 \\ 3 & 15 & -3 & -12 \\ \hline 0 & 21 & -1 & -1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ 0 & 21 & -1 & -1 \end{array}\right] \quad R_3 - 7R_2$$

$$\begin{array}{cccc} 0 & 21 & -1 & -1 \\ 0 & -21 & 7 & 7 \\ \hline 0 & 0 & 6 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ 0 & 0 & 6 & 6 \end{array}\right] \quad \begin{array}{ll} \to x + 5y - z = -4 & (2) \\ \to 3y - z = -1 & (1) \\ \to 6z = 6 & \end{array}$$

$\underline{z = 1}|$

$(1) \to \quad 3y = -1 + 1$

$\qquad \underline{y = 0}|$

$(2) \to \quad x = -4 + 1$

$\qquad \underline{x = -3}|$

$\therefore$ ***Solution***: $\underline{(-3,\ 0,\ 1)}\,|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 2 & -3 & 2 & 10 \\ 3 & -1 & 1 & 9 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -3 & 2 & 10 \\ -2 & -6 & -8 & -28 \\ \hline 0 & -9 & -6 & -18 \end{array} \qquad \begin{array}{cccc} 3 & -1 & 1 & 9 \\ -3 & -9 & -12 & -42 \\ \hline 0 & -10 & -11 & -33 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & -9 & -6 & -18 \\ 0 & -10 & -11 & -33 \end{array}\right] \quad 9R_3 - 10R_2$$

$$\begin{array}{cccc} 0 & -90 & -99 & -297 \\ 0 & 90 & 60 & 180 \\ \hline 0 & 0 & -39 & -117 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & -9 & -6 & -18 \\ 0 & 0 & -39 & -117 \end{array}\right] \quad \begin{array}{ll} x + 3y + 4z = 14 & (3) \\ -9y - 6z = -18 & (2) \\ -39z = -117 & (1) \end{array}$$

$(1) \to \quad z = \frac{117}{39}$

$\qquad \underline{= 3}|$

$(2) \rightarrow \quad 9y = 18 - 6(3)$

$9y = 0$

$\underline{y = 0}|$

$(3) \rightarrow \quad x = 14 - 12$

$\underline{x = 2}|$

$\therefore$ ***Solution***: $\underline{(2,\ 0,\ 3)}\,|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 3 & 2 & 1 & 8 \\ 2 & -3 & 2 & -16 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 2 & 1 & 8 \\ -3 & -12 & 3 & -60 \\ \hline 0 & -10 & 4 & -52 \end{array} \qquad \begin{array}{cccc} 2 & -3 & 2 & -16 \\ -2 & -8 & 2 & -40 \\ \hline 0 & -11 & 4 & -56 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 0 & -10 & 4 & -52 \\ 0 & -11 & 4 & -56 \end{array}\right] \begin{array}{l} \\ \\ 10R_3 - 11R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -110 & 40 & -560 \\ 0 & 110 & -44 & 572 \\ \hline 0 & 0 & -4 & 12 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 0 & -10 & 4 & -52 \\ 0 & 0 & -4 & 12 \end{array}\right] \quad \begin{array}{rl} x + 4y - z = 20 & (3) \\ -10y + 4z = -52 & (2) \\ -4z = 12 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z = -3}|$

$(2) \rightarrow \quad -10y = -52 + 12$

$-10y = -40$

$\underline{y = 4}|$

$(3) \rightarrow \quad x = 20 - 16 - 3$

$\underline{x = 1}|$

$\therefore$ ***Solution***: $\underline{(1,\ 4,\ -3)}\,|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 17 \\ 0 & 2 & -1 & 7 \\ 2 & -3 & 2 & -1 \end{array}\right] R_3 - 2R_1 \qquad \begin{array}{cccc} 2 & -3 & 2 & -1 \\ -2 & -4 & -2 & -34 \\ \hline 0 & -7 & 0 & -35 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 17 \\ 0 & 2 & -1 & 7 \\ 0 & -7 & 0 & -35 \end{array}\right] \begin{array}{ll} x + 2y + z = 17 & (3) \\ 2y - z = 7 & (2) \\ -7y = -35 & (1) \end{array}$$

$(1) \to \quad \underline{y = 5}$

$(2) \to \quad z = 10 - 7$

$\qquad \underline{= 3}$

$(3) \to \quad x = 17 - 10 - 3$

$\qquad \underline{= 4}$

$\therefore$ ***Solution***: $\underline{(4,\ 5,\ 3)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ -4 & 5 & 3 & 7 \\ -6 & 3 & 5 & -4 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \qquad \begin{array}{cccc} -4 & 5 & 3 & 7 \\ 4 & -12 & -14 & -6 \\ \hline 0 & -7 & -11 & 1 \end{array} \qquad \begin{array}{cccc} -6 & 3 & 5 & -4 \\ 6 & -18 & -21 & -9 \\ \hline 0 & -15 & -16 & -13 \end{array}$$

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ 0 & -7 & -11 & 1 \\ 0 & -15 & -16 & -13 \end{array}\right] 7R_3 - 15R_1 \qquad \begin{array}{cccc} 0 & -105 & -112 & -91 \\ 0 & 105 & 165 & -15 \\ \hline 0 & 0 & 53 & -106 \end{array}$$

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ 0 & -7 & -11 & 1 \\ 0 & 0 & 53 & -106 \end{array}\right] \begin{array}{ll} -2x + 6y + 7z = 3 & (3) \\ -7y - 11z = 1 & (2) \\ 53z = -106 & (1) \end{array}$$

$(1) \to \quad \underline{z = -2}$

$(2) \rightarrow \quad -7y = 1 - 22$

$$-7y = -21$$

$$\underline{y = 3}|$$

$(3) \rightarrow \quad -2x = 3 - 18 + 14$

$$-2x = -1$$

$$\underline{x = \frac{1}{2}}|$$

$\therefore$ ***Solution***: $\left(\frac{1}{2},\ 3,\ -2\right)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 3 & -3 & 4 & 5 \\ 4 & -2 & 3 & 4 \end{array}\right] \quad \begin{array}{l} 2R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\begin{array}{cccc} 6 & -6 & 8 & 10 \\ -6 & 3 & -3 & -3 \\ \hline 0 & -3 & 5 & 7 \end{array} \qquad \begin{array}{cccc} 4 & -2 & 3 & 4 \\ -4 & 2 & -2 & -2 \\ \hline 0 & 0 & 1 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 0 & -3 & 5 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \begin{array}{ll} 2x - y + z = 1 & (2) \\ -3y + 5z = 7 & (1) \\ \underline{z = 2}| & \end{array}$$

$(1) \rightarrow \quad -3y = 7 - 10$

$$-3y = -3$$

$$\underline{y = 1}|$$

$(3) \rightarrow \quad 2x = 1 + 1 - 2$

$$\underline{x = 0}|$$

$\therefore$ ***Solution***: $(0,\ 1,\ 2)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x-4y+4z=7 \\ x-y-2z=2 \\ 2x-3y+6z=5 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 3 & -4 & 4 & 7 \\ 2 & -3 & 6 & 5 \end{array}\right] \begin{array}{l} \\ R_2-3R_1 \\ R_3-2R_1 \end{array}$$

$$\begin{array}{cccc} 3 & -4 & 4 & 7 \\ -3 & 3 & 6 & -6 \\ \hline 0 & -1 & 10 & 1 \end{array} \qquad \begin{array}{cccc} 2 & -3 & 6 & 5 \\ -2 & 2 & 4 & -4 \\ \hline 0 & -1 & 10 & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & -1 & 10 & 1 \\ 0 & -1 & 10 & 1 \end{array}\right] \begin{array}{l} \to x-y-2z=2 \quad (2) \\ \to -y+10z=1 \quad (1) \\ R_3=R_2 \end{array}$$

$(1) \to \quad \underline{y=10z-1}|$

$(2) \to \quad x=2+10z-1+2z$

$\qquad\qquad \underline{=12z+1}|$

$\therefore$ ***Solution***: $\underline{(12z+1,\ \ 10z-1,\ \ z)}|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x-2y-z=2 \\ 2x-y+z=4 \\ -x+y+z=4 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 2 & -1 & 1 & 4 \\ -1 & 1 & 1 & 4 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3+R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -1 & 1 & 4 \\ -2 & 4 & 2 & -4 \\ \hline 0 & 3 & 3 & 0 \end{array} \qquad \begin{array}{cccc} -1 & 1 & 1 & 4 \\ 1 & -2 & -1 & 2 \\ \hline 0 & -1 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 3 & 3 & 0 \\ 0 & -1 & 0 & 6 \end{array}\right] \begin{array}{ll} x-2y-z=2 & (3) \\ 3y+3z=0 & (2) \\ -y=6 & (1) \end{array}$$

$(1) \to \quad \underline{y=-6}|$

$(2) \to \quad z=-y$

$\qquad\qquad \underline{=6}|$

$(3) \to \quad x=2-12+6$

$\qquad\qquad \underline{=-4}|$

$\therefore$ ***Solution***: $\underline{(-4,\ \ -6,\ \ 6)}|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x+y+z=3 \\ -y+2z=1 \\ -x+z=0 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ -1 & 0 & 1 & 0 \end{array}\right] \quad R_3+R_1 \qquad \begin{array}{cccc} -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 3 \\ \hline 0 & 1 & 2 & 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 1 & 2 & 3 \end{array}\right] \quad R_3+R_2 \qquad \begin{array}{cccc} 0 & 1 & 2 & 3 \\ 0 & -1 & 2 & 1 \\ \hline 0 & 0 & 4 & 4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 4 & 4 \end{array}\right] \quad \begin{array}{rl} x+y+z=3 & (3) \\ -y+2z=1 & (2) \\ 4z=4 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z=1}$

$(2) \rightarrow \quad -y=1-2$

$\qquad \underline{y=1}$

$(3) \rightarrow \quad x=3-1-1$

$\qquad \underline{=1}$

$\therefore$ ***Solution***: $\underline{(1,\ 1,\ 1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x+y+3z=14 \\ 7x+5y+8z=37 \\ x+3y+2z=9 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 3 & 1 & 3 & 14 \\ 7 & 5 & 8 & 37 \end{array}\right] \quad \begin{array}{l} R_2-3R_1 \\ R_3-7R_1 \end{array} \qquad \begin{array}{cccc} 3 & 1 & 3 & 14 \\ -3 & -9 & -6 & -27 \\ \hline 0 & -8 & -3 & -13 \end{array} \qquad \begin{array}{cccc} 7 & 5 & 8 & 37 \\ -7 & -21 & -14 & -63 \\ \hline 0 & -16 & -6 & -26 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 0 & -8 & -3 & -13 \\ 0 & -16 & -6 & -26 \end{array}\right] \quad R_3-2R_2 \qquad \begin{array}{cccc} 0 & -16 & -6 & -26 \\ 0 & 16 & 6 & 26 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 0 & -8 & -3 & -13 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{ll} x+3y+2z=9 & (2) \\ -8y-3z=-13 & (1) \end{array}$$

$(1) \to \quad -8y = 3z - 13$

$$\underline{y = -\frac{3}{8}z + \frac{13}{8}}$$

$(3) \to \quad x = 9 - 3\left(\frac{13}{8} - \frac{3}{8}z\right) - 2z$

$$= 9 - \frac{39}{8} + \frac{9}{8}z - 2z$$

$$\underline{= \frac{33}{8} - \frac{7}{8}z}$$

$\therefore$ ***Solution***: $\underline{\left(\frac{33}{8} - \frac{7}{8}z,\ \ \frac{13}{8} - \frac{3}{8}z,\ \ z\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 4x-2y+z=7 \\ x+y+z=-2 \\ 4x+2y+z=3 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 4 & -2 & 1 & 7 \\ 4 & 2 & 1 & 3 \end{array}\right] \quad \begin{array}{l} \\ R_2 - 4R_1 \\ R_3 - 4R_1 \end{array} \qquad \begin{array}{cccc} 4 & -2 & 1 & 7 \\ -4 & -4 & -4 & 8 \\ \hline 0 & -6 & -3 & 15 \end{array} \qquad \begin{array}{cccc} 4 & 2 & 1 & 3 \\ -4 & -4 & -4 & 8 \\ \hline 0 & -2 & -3 & 11 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -6 & -3 & 15 \\ 0 & -2 & -3 & 11 \end{array}\right] \quad \begin{array}{l} \\ \\ -3R_3 + R_2 \end{array} \qquad \begin{array}{cccc} 0 & 6 & 9 & -33 \\ 0 & -6 & -3 & 15 \\ \hline 0 & 0 & 6 & -18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -6 & -3 & 15 \\ 0 & 0 & 6 & -18 \end{array}\right] \quad \begin{array}{ll} x+y+z=-2 & (3) \\ -6y-3z=15 & (2) \\ 6z=-18 & (1) \end{array}$$

$(1) \to \quad \underline{z = -3}$

$(2) \to \quad -6y = 15 - 9$

$$\underline{y = -1}$$

$(3) \to \quad x = -2 + 1 + 3$

$$\underline{= 2}$$

$\therefore$ ***Solution***: $\underline{(2,\ -1,\ -3)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x-2y+z=-4 \\ 6x+4y-3z=-24 \\ x-2y+2z=1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 2 & -2 & 1 & -4 \\ 6 & 4 & -3 & -24 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-6R_1 \end{array} \qquad \begin{array}{cccc} 2 & -2 & 1 & -4 \\ -2 & 4 & -4 & -2 \\ \hline 0 & 2 & -3 & -6 \end{array} \qquad \begin{array}{cccc} 6 & 4 & -3 & -24 \\ -6 & 12 & -12 & -6 \\ \hline 0 & 16 & -15 & -30 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 2 & -3 & -6 \\ 0 & 16 & -15 & -30 \end{array}\right] \begin{array}{l} \\ \\ R_3-8R_2 \end{array} \qquad \begin{array}{cccc} 0 & 16 & -15 & -30 \\ 0 & -16 & 24 & 48 \\ \hline 0 & 0 & 9 & 18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 2 & -3 & -6 \\ 0 & 0 & 9 & 18 \end{array}\right] \quad \begin{array}{rl} x-2y+2z=1 & (3) \\ 2y-3z=-6 & (2) \\ 9z=18 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z=2}$

$(2) \rightarrow \quad 2y=-6+6$

$\qquad \underline{y=0}$

$(3) \rightarrow \quad x=1-4$

$\qquad \underline{=-3}$

$\therefore$ ***Solution***: $\underline{(-3,\ 0,\ 2)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 9x+3y+z=4 \\ 16x+4y+z=2 \\ 25x+5y+z=2 \end{cases}$

Solution

$$\begin{cases} z+9x+3y=4 \\ z+16x+4y=2 \\ z+25x+5y=2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 1 & 16 & 4 & 2 \\ 1 & 25 & 5 & 2 \end{array}\right] \begin{array}{l} \\ R_2-R_1 \\ R_3-R_1 \end{array} \qquad \begin{array}{cccc} 1 & 16 & 4 & 2 \\ -1 & -9 & -3 & -4 \\ \hline 0 & 7 & 1 & -2 \end{array} \qquad \begin{array}{cccc} 1 & 25 & 5 & 2 \\ -1 & -9 & -3 & -4 \\ \hline 0 & 16 & 2 & -2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 0 & 7 & 1 & -2 \\ 0 & 16 & 2 & -2 \end{array}\right] \quad 7R_3 - 16R_3$$

$$\begin{array}{cccc} 0 & 112 & 14 & -14 \\ 0 & -112 & -16 & 32 \\ \hline 0 & 0 & -2 & 18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 0 & 7 & 1 & -2 \\ 0 & 0 & -2 & 18 \end{array}\right] \quad \begin{array}{rl} z + 9x + 3y = 4 & (3) \\ 7x + y = -2 & (2) \\ -2y = 18 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{y = -9}|$

$(2) \rightarrow \quad 7x = -2 + 9$

$\qquad \underline{= 1}|$

$(3) \rightarrow \quad z = 4 - 9 + 27$

$\qquad \underline{= 22}|$

$\therefore$ ***Solution***: $\underline{(1,\ -9,\ 22)}\ |$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x - y + 2z = -8 \\ x + 2y - 3z = 9 \\ 3x - y - 4z = 3 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 2 & -8 \\ 3 & -1 & -4 & 3 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -1 & 2 & -8 \\ -2 & -4 & 6 & -18 \\ \hline 0 & -5 & 8 & -26 \end{array} \qquad \begin{array}{cccc} 3 & -1 & -4 & 3 \\ -3 & -6 & 9 & -27 \\ \hline 0 & -7 & 5 & -24 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 0 & -7 & 5 & -24 \end{array}\right] \quad 5R_3 - 7R_2$$

$$\begin{array}{cccc} 0 & -35 & 25 & -120 \\ 0 & 35 & -56 & 182 \\ \hline 0 & 0 & -31 & 62 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 0 & 0 & -31 & 62 \end{array}\right] \quad \begin{array}{rl} x + 2y - 3z = 9 & (3) \\ -5y + 8z = -26 & (2) \\ -31z = 62 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z = -2}|$

$(2) \rightarrow \quad -5y = -26 + 16$

$\qquad -5y = 10$

$\qquad \underline{y = 2}|$

$(3) \rightarrow \quad x = 9 - 4 - 6$

$\underline{=-1}|$

$\therefore$ ***Solution***: $\underline{(-1,\ 2,\ -2)}|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x \quad -3z=-5 \\ 2x-y+2z=16 \\ 7x-3y-5z=19 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 2 & -1 & 2 & 16 \\ 7 & -3 & -5 & 19 \end{array}\right] \quad \begin{array}{l} \\ R_2-2R_1 \\ R_3-7R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -1 & 2 & 16 \\ -2 & 0 & 6 & 10 \\ \hline 0 & -1 & 8 & 26 \end{array} \qquad \begin{array}{cccc} 7 & -3 & -5 & 19 \\ -7 & 0 & 21 & 35 \\ \hline 0 & -3 & 16 & 54 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 0 & -1 & 8 & 26 \\ 0 & -3 & 16 & 54 \end{array}\right] \quad \begin{array}{l} \\ \\ R_3-3R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -3 & 16 & 54 \\ 0 & 3 & -24 & -78 \\ \hline 0 & 0 & -8 & -24 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 0 & -1 & 8 & 26 \\ 0 & 0 & -8 & -24 \end{array}\right] \quad \begin{array}{rl} x-3z=-5 & (3) \\ -y+8z=26 & (2) \\ -8z=-24 & (1) \end{array}$$

$(1)\rightarrow \quad \underline{z=3}|$

$(2)\rightarrow \quad -y=26-24$

$\underline{y=-2}|$

$(3)\rightarrow \quad x=-5+9$

$\underline{=4}|$

$\therefore$ ***Solution***: $\underline{(4,\ -2,\ 3)}|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x+2y-z=5 \\ 2x-y+3z=0 \\ \quad 2y+z=1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 2 & -1 & 3 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right] \quad R_2-2R_1$$

$$\begin{array}{cccc} 2 & -1 & 3 & 0 \\ -2 & -4 & 2 & -10 \\ \hline 0 & -5 & 5 & -10 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 5 & -10 \\ 0 & 2 & 1 & 1 \end{array}\right] \quad 5R_3 + 2R_2 \qquad \begin{array}{cccc} 0 & 10 & 5 & 5 \\ 0 & -10 & 10 & -20 \\ \hline 0 & 0 & 15 & -15 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 5 & -10 \\ 0 & 0 & 15 & -15 \end{array}\right] \quad \begin{array}{rl} x+2y-z=5 & (3) \\ -5y+5z=-10 & (2) \\ 15z=-15 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z=-1}$

$(2) \rightarrow \quad -5y=-10+5$

$\qquad \underline{y=1}$

$(3) \rightarrow \quad x=5-2-1$

$\qquad \underline{=2}$

$\therefore$ ***Solution***: $\underline{(2,\ 1,\ -1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x+y+z=6 \\ 3x+4y-7z=1 \\ 2x-y+3z=5 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & 4 & -7 & 1 \\ 2 & -1 & 3 & 5 \end{array}\right] \quad \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array} \qquad \begin{array}{cccc} 3 & 4 & -7 & 1 \\ -3 & -3 & -3 & -18 \\ \hline 0 & 1 & -10 & -17 \end{array} \qquad \begin{array}{cccc} 2 & -1 & 3 & 5 \\ -2 & -2 & -2 & -12 \\ \hline 0 & -3 & 1 & -7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -10 & -17 \\ 0 & -3 & 1 & -7 \end{array}\right] \quad R_3 + 3R_2 \qquad \begin{array}{cccc} 0 & -3 & 1 & -7 \\ 0 & 3 & -30 & -51 \\ \hline 0 & 0 & -29 & -58 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -10 & -17 \\ 0 & 0 & -29 & -58 \end{array}\right] \quad \begin{array}{rl} x+y+z=6 & (3) \\ y-10z=-17 & (2) \\ -29z=-58 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z=2}$

$(2) \rightarrow \quad y=-17+20$

$\qquad \underline{=3}$

$(3) \rightarrow \quad x=6-3-2$

$=1$

$\therefore$ ***Solution***: $(1,\ 3,\ 2)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x+2y+3z=3 \\ 4x-5y+7z=1 \\ 2x+3y-2z=6 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 4 & -5 & 7 & 1 \\ 2 & 3 & -2 & 6 \end{array}\right] \begin{array}{l} \\ 3R_2-4R_1 \\ 3R_3-2R_1 \end{array}$$

$$\begin{array}{cccc} 12 & -15 & 21 & 3 \\ -12 & -8 & -12 & -12 \\ \hline 0 & -23 & 9 & -9 \end{array} \qquad \begin{array}{cccc} 6 & 9 & -6 & 18 \\ -6 & -4 & -6 & -6 \\ \hline 0 & 5 & -12 & 12 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 0 & -23 & 9 & -9 \\ 0 & 5 & -12 & 12 \end{array}\right] \begin{array}{l} \\ \\ 23R_3+5R_2 \end{array}$$

$$\begin{array}{cccc} 0 & 115 & -276 & 276 \\ 0 & -115 & 45 & -45 \\ \hline 0 & 0 & -231 & 231 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 0 & -23 & 9 & -9 \\ 0 & 0 & -231 & 231 \end{array}\right] \begin{array}{rl} 3x+2y+3z=3 & (3) \\ -23y+9z=-9 & (2) \\ -231z=231 & (1) \end{array}$$

$(1)\rightarrow\quad z=-1$

$(2)\rightarrow\quad -23y=-9+9$

$y=0$

$(3)\rightarrow\quad 3x=3+3$

$x=2$

$\therefore$ ***Solution***: $(2,\ 0,\ -1)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x-3y+z=2 \\ 4x-12y+4z=8 \\ -2x+6y-2z=-4 \end{cases}$

Solution

$$\begin{cases} & x-3y+z=2 \\ \frac{1}{4}\times & 4x-12y+4z=8 \\ -\frac{1}{2}\times & -2x+6y-2z=-4 \end{cases}$$

$$\begin{cases} x-3y+z=2 \\ x-3y+z=2 \\ x-3y+z=2 \end{cases}$$

Since all three equations are the same.

$\therefore$ ***Solution***: is the plane $x-3y+z=2$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x-2y+z=-1 \\ x+2y-z=2 \\ 6x+4y+3z=5 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & -2 & 1 & -1 \\ 6 & 4 & 3 & 5 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-6R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -2 & 1 & -1 \\ -2 & -4 & 2 & -4 \\ \hline 0 & -6 & 3 & -5 \end{array} \qquad \begin{array}{cccc} 6 & 4 & 3 & 5 \\ -6 & -12 & 6 & -12 \\ \hline 0 & -8 & 9 & -7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -6 & 3 & -5 \\ 0 & -8 & 9 & -7 \end{array}\right] \begin{array}{l} \\ \\ 3R_3-4R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -24 & 27 & -21 \\ 0 & 24 & -12 & 20 \\ \hline 0 & 0 & 15 & -1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -6 & 3 & -5 \\ 0 & 0 & 15 & -1 \end{array}\right] \begin{array}{ll} x+2y-z=2 & (3) \\ -6y+3z=-5 & (2) \\ 15z=-1 & (1) \end{array}$$

$(1)\rightarrow \quad z=-\frac{1}{15}$

$(2)\rightarrow \quad -6y=-5+\frac{1}{5}$

$-6y=-\frac{24}{5}$

$$\boxed{y=\frac{4}{5}}$$

$(3)\to \quad x = 2-\frac{8}{5}-\frac{1}{15}$

$$\boxed{=\frac{1}{3}}$$

$\therefore$ ***Solution***: $\boxed{\left(\frac{1}{3},\ \frac{4}{5},\ -\frac{1}{15}\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 \\ x_2 - 3x_3 - x_4 = 0 \\ 3x_1 \quad\quad + 2x_3 - x_4 = 6 \\ -4x_1 + x_2 + 4x_3 + 2x_4 = -3 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 3 & 0 & 2 & -1 & 6 \\ -4 & 1 & 4 & 2 & -3 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 3R_1 \\ R_4 + 4R_1 \end{array}$$

$$\begin{array}{rrrrr} 3 & 0 & 2 & -1 & 6 \\ -3 & 15 & -6 & 6 & -12 \\ \hline 0 & 15 & -4 & 5 & -6 \end{array} \qquad \begin{array}{rrrrr} -4 & 1 & 4 & 2 & -3 \\ 4 & -20 & 8 & -8 & 16 \\ \hline 0 & -19 & 12 & -6 & 13 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 15 & -4 & 5 & -6 \\ 0 & -19 & 12 & -6 & 13 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 15R_2 \\ R_4 + 19R_2 \end{array}$$

$$\begin{array}{rrrrr} 0 & 15 & -4 & 5 & -6 \\ 0 & -15 & 45 & 15 & 0 \\ \hline 0 & 0 & 41 & 20 & -6 \end{array} \qquad \begin{array}{rrrrr} 0 & -19 & 12 & -6 & 13 \\ 0 & 19 & -57 & -19 & 0 \\ \hline 0 & 0 & -45 & -25 & 13 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 41 & 20 & -6 \\ 0 & 0 & -45 & -25 & 13 \end{array}\right] \begin{array}{l} \\ \\ \\ 41R_4 + 45R_2 \end{array}$$

$$\begin{array}{rrrrr} 0 & 0 & -1845 & -1025 & 533 \\ 0 & 0 & 1845 & 900 & -270 \\ \hline 0 & 0 & 0 & -125 & 263 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 41 & 20 & -6 \\ 0 & 0 & 0 & -125 & 263 \end{array}\right] \quad \begin{array}{rl} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 & (4) \\ x_2 - 3x_3 - x_4 = 0 & (3) \\ 41x_3 + 20x_4 = -6 & (2) \\ -125x_4 = 263 & (1) \end{array}$$

$(1)\to \quad \boxed{x_4 = -\frac{263}{125}}$

$(2)\to \quad 41x_3 = -6 + \frac{1,052}{25}$

$$= \frac{902}{25}$$

$$\boxed{x_3 = \frac{22}{25}}$$

$$(3) \rightarrow \quad x_2 = \frac{66}{25} - \frac{263}{125}$$

$$= \boxed{\frac{67}{125}}$$

$$(4) \rightarrow \quad x_1 = 4 + \frac{67}{25} - \frac{44}{25} - \frac{526}{125}$$

$$= 4 + \frac{23}{25} - \frac{526}{125}$$

$$= \frac{500 + 115 - 526}{125}$$

$$= \boxed{\frac{89}{125}}$$

$\therefore$ ***Solution***: $\boxed{\left(\frac{89}{125}, \frac{67}{125}, \frac{22}{25}, -\frac{263}{125}\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x_1 + x_2 + x_3 + x_4 = 5 \\ x_1 + 2x_2 - x_3 - 2x_4 = -1 \\ x_1 - 3x_2 - 3x_3 - x_4 = -1 \\ 2x_1 - x_2 + 2x_3 - x_4 = -2 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 1 & 2 & -1 & -2 & -1 \\ 1 & -3 & -3 & -1 & -1 \\ 2 & -1 & 2 & -1 & -2 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - 2R_1 \end{array}$$

$$\begin{array}{rrrrr} 1 & 2 & -1 & -2 & -1 \\ -1 & -1 & -1 & -1 & -5 \\ \hline 0 & 1 & -2 & -3 & -6 \end{array} \qquad \begin{array}{rrrrr} 1 & -3 & -3 & -1 & -1 \\ -1 & -1 & -1 & -1 & -5 \\ \hline 0 & -4 & -4 & -2 & -6 \end{array} \qquad \begin{array}{rrrrr} 2 & -1 & 2 & -1 & -2 \\ -2 & -2 & -2 & -2 & -10 \\ \hline 0 & -3 & 0 & -3 & -12 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & -4 & -4 & -2 & -6 \\ 0 & -3 & 0 & -3 & -12 \end{array}\right] \begin{array}{l} \\ \\ R_3 + 4R_2 \\ R_4 + 3R_2 \end{array}$$

$$\begin{array}{ccccc} 0 & -4 & -4 & -2 & -6 \\ 0 & 4 & -8 & -12 & -24 \\ \hline 0 & 0 & -12 & -14 & -30 \end{array}$$

$$\begin{array}{ccccc} 0 & -3 & 0 & -3 & -12 \\ 0 & 3 & -6 & -9 & -18 \\ \hline 0 & 0 & -6 & -12 & -30 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & 0 & -12 & -14 & -30 \\ 0 & 0 & -6 & -12 & -30 \end{array}\right] \begin{array}{l} \\ \\ \\ -2R_4 + R_3 \end{array}$$

$$\begin{array}{ccccc} 0 & 0 & 12 & 24 & 60 \\ 0 & 0 & -12 & -14 & -30 \\ \hline 0 & 0 & 0 & 10 & 30 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & 0 & -12 & -14 & -30 \\ 0 & 0 & 0 & 10 & 30 \end{array}\right] \quad \begin{array}{rl} x_1 + x_2 + x_3 + x_4 = 5 & (4) \\ x_2 - 2x_3 - 3x_4 = -6 & (3) \\ -12x_3 - 14x_4 = -30 & (2) \\ 10x_4 = 30 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{x_4 = 3}$

$(2) \rightarrow \quad -12x_3 = -30 + 42$

$= 12$

$\underline{x_3 = -1}$

$(3) \rightarrow \quad x_2 = -6 - 2 + 9$

$\underline{= 1}$

$(4) \rightarrow \quad x_1 = 5 - 1 + 1 - 3$

$\underline{= 2}$

$\therefore$ ***Solution***: $\underline{(2,\ 1,\ -1,\ 3)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x+\ 8y-\ z+\ w=0 \\ 4x+16y-3z-w=-10 \\ -2x+4y-z+3w=-6 \\ -6x+2y+5z+w=3 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 4 & 16 & -3 & -1 & -10 \\ -2 & 4 & -1 & 3 & -6 \\ -6 & 2 & 5 & 1 & 3 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3+R_1 \\ R_4+3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4-\frac{13}{6}R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \quad \textit{Interchange } R_2 \textit{ and } R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & \frac{19}{3} & -\frac{14}{3} & 16 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4+\frac{19}{3}R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & 0 & -\frac{71}{3} & -\frac{142}{3} \end{array}\right] \begin{array}{ll} 2x+8y-z+w=0 & (3) \\ 12y-2z+4w=-6 & (2) \\ -z-3w=-10 & (1) \\ -\frac{71}{3}w=-\frac{142}{3} & \to \ \underline{w=2} \end{array}$$

$(1)\to \quad z=10-3w\ \underline{=4}$

$(2)\to \quad 12y=2z-4w-6$

$\underline{y=-\frac{1}{2}}$

$(3)\to \quad 2x=-8y+z-w$

$2x=4+4-2$

$\underline{x=3}$

$\therefore$ ***Solution***: $\underline{\left(3,\ -\frac{1}{2},\ 4,\ 2\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 + 2x_2 \qquad = 0 \\ \qquad x_2 + x_3 = 0 \end{cases}$

Solution

$$\begin{cases} x_1 = -2x_2 \\ x_3 = -x_2 \end{cases}$$

$$2x_1 + x_2 + 3x_3 = 0$$

$$-4x_2 + x_2 - 3x_2 = 0 \quad \rightarrow \quad \underline{x_2 = 0}$$

$\therefore$ ***Solution***: $\underline{(0,\ 0,\ 0)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x + 2y + 4z \qquad = 0 \\ \quad - y - 3z + w = 0 \\ 3x + y + z + 2w = 0 \\ x + 3y - 2z - 2w = 0 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & -1 & -3 & 1 & 0 \\ 3 & 1 & 1 & 2 & 0 \\ 1 & 3 & -2 & -2 & 0 \end{array}\right] \begin{array}{c} \\ -R_2 \\ 2R_3 - 3R_1 \\ 2R_4 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & -4 & -10 & 4 & 0 \\ 0 & 4 & -8 & -4 & 0 \end{array}\right] \begin{array}{c} \\ \\ R_3 + 4R_2 \\ R_4 - 4R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & -20 & 0 & 0 \end{array}\right] \begin{array}{ll} 2x + 2y - 4z = 0 & (1) \\ y + 3z - w = 0 & (2) \\ \rightarrow \underline{z = 0} & \\ & \end{array}$$

$$(2) \quad \rightarrow \quad \underline{y = w}$$

$$(1) \quad \rightarrow 2x = -2y \quad \underline{x = -w}$$

$\therefore$ ***Solution***: $\underline{(-w,\ w,\ 0,\ w)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x \qquad + z + w = 5 \\ \quad y \qquad - w = -1 \\ 3x \qquad - z - w = 0 \\ 4x + y + 2z + w = 9 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 3 & 0 & -1 & -1 & 0 \\ 4 & 1 & 2 & 1 & 9 \end{array}\right] \begin{array}{l} \\ \\ 2R_3 - 3R_1 \\ 2R_4 - 4R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -5 & -5 & -15 \\ 0 & 2 & 0 & -2 & -2 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - 2R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -5 & -5 & -15 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{lr} 2x + z + w = 5 & (1) \\ y - w = -1 & (2) \\ -5z - 5w = -15 & (3) \\ & \end{array}$$

$(2) \rightarrow \qquad \underline{y = 1 + w}$

$(3) \rightarrow \qquad \underline{z = 3 - w}$

$(1) \rightarrow \quad 2x = 5 - (3 - w) - w \quad \Rightarrow \quad \underline{x = 1}$

$\therefore$ ***Solution***: $\underline{(1,\ 1 + w,\ 3 - w,\ w)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} \quad 4y + z = 20 \\ 2x - 2y + z = 0 \\ x \qquad + z = 5 \\ x + y - z = 10 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 10 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 1 & 5 \\ 0 & 4 & 1 & 20 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - R_1 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 10 \\ 0 & -4 & 3 & -20 \\ 0 & -1 & 2 & -5 \\ 0 & 4 & 1 & 20 \end{array}\right] \begin{array}{l} \\ \\ 4R_3 - R_2 \\ R_4 + R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 10 \\ 0 & -4 & 3 & -20 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \begin{array}{r} x + y = 10 \\ \to -4y = -20 \\ \to z = 0 \\ \\ \end{array}$$

$\therefore$ ***Solution***: $\underline{(5,\ 5,\ 0)}|$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x \qquad\qquad - 3w = -3 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ -1 & 3 & -2 & 1 \\ 3 & 4 & -7 & 10 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 5 & -1 & 9 \\ 0 & -2 & -10 & -14 \end{array}\right] \begin{array}{l} \\ \\ 5R_3 + 2R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 5 & -1 & 9 \\ 0 & 0 & -52 & -52 \end{array}\right] \begin{array}{rl} x + 2y + z = 8 & (3) \\ 5y - z = 9 & (2) \\ -52z = -52 & (1) \end{array}$$

$(1) \quad \Rightarrow \quad z = 1$

$(2) \quad \Rightarrow \quad 5y = 9 + 1 = 10 \to y = 2$

$(3) \quad \Rightarrow \quad x = 8 - 4 - 1 = 3$

$\therefore$ ***Solution***: $\underline{(3,\ 2,\ 1)}|$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} 2u-3v+\ \ w-\ \ x+y=\ \ 0 \\ 4u-6v+2w-3x-y=-5 \\ -2u+3v-2w+2x-y=\ \ 3 \end{cases}$$

Solution

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 4 & -6 & 2 & -3 & -1 & -5 \\ -2 & 3 & -2 & 2 & -1 & 3 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3+R_1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -3 & -5 \\ 0 & 0 & -1 & 1 & 0 & 3 \end{array}\right] \qquad \begin{array}{rl} 2u-3v+w-x+y=0 & (3) \\ -x-3y=-5 & (2) \\ -w+x=3 & (1) \end{array}$$

$(2) \Rightarrow x=5-3y$

$(1) \Rightarrow w=x-3=2-3y$

$(3) \Rightarrow 2u=3v-2+3y+5-3y-y=3v-y+3$

$$u=\frac{3}{2}v-\frac{1}{2}y+\frac{3}{2}$$

$\therefore$ ***Solution***: $\left(\frac{3}{2}v-\frac{1}{2}y+\frac{3}{2}, v, 2-3y, 5-3y, y\right)$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} 6x_3+2x_4-4x_5-8x_6=8 \\ 3x_3+x_4-2x_5-4x_6=4 \\ 2x_1-3x_2+x_3+4x_4-7x_5+x_6=2 \\ 6x_1-9x_2+11x_4-19x_5+3x_6=1 \end{cases}$$

Solution

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 6 & -9 & 0 & 11 & -19 & 3 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4-3R_1 \end{array}$$

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & -3 & -1 & 2 & 0 & -5 \end{array}\right] \begin{array}{l} \\ \\ R_3-2R_2 \\ R_4+R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & -1 \end{array}\right] \qquad \begin{array}{r} 2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 = 2 - x_6 \\ 3x_3 + x_4 - 2x_5 = 4 + 4x_6 \\ \\ \to \underline{x_6 = \frac{1}{4}} \end{array}$$

$$\begin{cases} 3x_3 = 5 - x_4 + 2x_5 \\ 2x_1 = \frac{7}{4} + 3x_2 - \frac{1}{3}\left(5 - x_4 + 2x_5\right) - 4x_4 + 7x_5 \end{cases}$$

$$\begin{cases} \underline{x_3 = \frac{5}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5} \\ 2x_1 = \frac{1}{12} + 3x_2 - \frac{11}{3}x_4 + \frac{19}{3}x_5 \end{cases}$$

$\therefore$ ***Solution***: $\underline{\left(\frac{1}{24} + \frac{3}{2}x_2 - \frac{11}{6}x_4 + \frac{19}{6}x_5,\ x_2,\ \frac{5}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5,\ x_4,\ x_5,\ \frac{1}{4}\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x_1 + 2x_2 - x_3 = -15 \\ 5x_1 + 3x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 3x_3 = 11 \\ -6x_1 - 4x_2 + 2x_3 = 30 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ -6 & -4 & 2 & 30 \end{array}\right] \begin{array}{l} \\ 3R_2 - 5R_1 \\ R_3 - R_1 \\ R_4 + 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & -1 & 4 & 26 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & 0 & -7 & -49 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{rl} 3x_1 + 2x_2 - x_3 = -15 & (3) \\ -x_2 + 11x_3 = 75 & (2) \\ -7x_3 = -49 & (1) \end{array}$$

$(1) \to x_3 = 7$

$(2) \;\rightarrow\; x_2 = 77 - 75 = 2$

$(1) \;\rightarrow\; 3x_1 = -15 - 4 + 7 = 12 \quad \Rightarrow \quad x_1 = -4$

$\therefore$ ***Solution***: $\underline{(-4, 2, 7)}$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} x_1 + 3x_2 - 2x_3 \qquad\qquad + 2x_5 \qquad = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1 \\ \qquad\qquad 5x_3 + 10x_4 \qquad + 15x_6 = 5 \\ 2x_1 + 6x_2 \qquad + 8x_4 + 4x_5 + 18x_6 = 6 \end{cases}$$

Solution

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ \\ R_4 - 2R_1 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \begin{array}{l} \\ -R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 5R_2 \\ R_4 - 4R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right] \begin{array}{l} \\ \\ \\ \frac{1}{6}R_4 \quad \textit{then interchanging row3 and row4} \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_3 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \to \begin{cases} x_1 + 3x_2 \quad + 4x_4 + 2x_5 = 0 \\ x_3 + 2x_4 = 0 \\ + x_6 = \frac{1}{3} \end{cases}$$

The general solution of the system: $x_6 = \frac{1}{3}, \quad x_3 = -2x_4, \quad x_1 = -3x_2 - 4x_4 - 2x_5$

∴ ***Solution***: $\left(-3x_2 - 4x_4 - 2x_5, \ x_2, \ -2x_4, \ x_4, \ x_5, \ \frac{1}{3}\right)$

Exercise

Add 3 times the second row to the first of $\begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$

Solution

$$E = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$EA = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 27 & 8 & -1 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix} \quad R_1 + 3R_2 \quad = \begin{bmatrix} 27 & 8 & -1 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$$

Exercise

For what value(s) of k, if any, does the system $\begin{cases} x+y-z=1 \\ 2x+3y+kz=3 \\ x+ky+3z=2 \end{cases}$ have

a) A unique solution?
b) Infinitely many solutions?
c) No solution?

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & k & 3 \\ 1 & k & 3 & 2 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & k+2 & 1 \\ 0 & k-1 & 4 & 1 \end{array}\right] \; R_3-(k-1)R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & k+2 & 1 \\ 0 & 0 & 4-(k-1)(k+2) & 2-k \end{array}\right] \quad \begin{array}{l} x=1-y+z \\ y=1-(k+2)z \\ \to \left(6-k^2-k\right)z=-(k-2) \end{array}$$

$$\begin{cases} z=-\dfrac{k-2}{-(k-2)(k+3)}=\dfrac{1}{k+3} \quad \color{red}{(k\neq 2,-3)} \\ y=1-\dfrac{k+2}{k+3}=\dfrac{1}{k+3} \\ \underline{x=\left|\dfrac{k+2}{k+3}+\dfrac{1}{k+3}=1\right|} \end{cases}$$

a) Unique solution if $k\neq 2,-3$

b) Infinitely solution if $k=2$

c) No solution if $k=-3$

Exercise

Choose a coefficient b that makes the system singular.

$$\begin{cases} 3x+4y=16 \\ 4x+by=g \end{cases}$$

Then choose a right-hand side g that makes it solvable.
Find 2 solutions in that singular case.

Solution

$$\left(\begin{array}{cc|c} 3 & 4 & 16 \\ 4 & b & g \end{array}\right)_{3R_2 - 4R_1} \rightarrow \left(\begin{array}{cc|c} 3 & 4 & 16 \\ 0 & 3b-16 & 3g-64 \end{array}\right)$$

So, the system is singular if

$$3b - 16 = 0 \quad \Rightarrow \quad \underline{b = \frac{16}{3}}$$

$$\& \ 3g - 64 = 0 \quad \Rightarrow \quad \underline{g = \frac{64}{3}}$$

$$\begin{cases} 3x + 4y = 16 \\ 4x + \frac{16}{3}y = \frac{64}{3} \end{cases} \rightarrow \underline{3x + 4y = 16}$$

$$\text{If} \quad \begin{cases} x = 0 \quad \rightarrow \quad \underline{y = 4} \\ x = 4 \quad \rightarrow \quad \underline{y = 1} \end{cases}$$

Exercise

This system us not linear, in some sense,

$$\begin{cases} 2\sin\alpha - \cos\beta + 3\tan\theta = 3 \\ 4\sin\alpha + 2\cos\beta - 2\tan\theta = 10 \\ 6\sin\alpha - 3\cos\beta + \tan\theta = 9 \end{cases}$$

Does the system have a solution?

Solution

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 4 & 2 & -2 & 10 \\ 6 & -3 & 1 & 9 \end{array}\right] \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 0 & 4 & -8 & 4 \\ 0 & 0 & -8 & 0 \end{array}\right] \quad \begin{array}{c} 2\sin\alpha = 3 + \cos\beta \\ 4\cos\beta = 4 \\ \tan\theta = 0 \end{array}$$

$$\begin{cases} \sin\alpha = \frac{3}{2} + \frac{1}{2} = 2 \\ \cos\beta = 1 \\ \tan\theta = 0 \end{cases}$$

The system has *no solution* since $\sin\alpha$ cannot be equal 2. $(-1 \le \sin\alpha \le 1)$