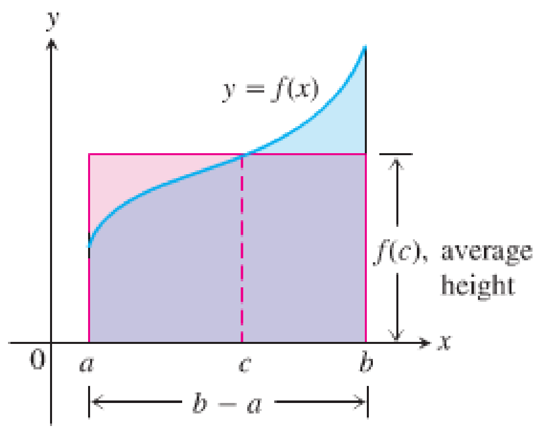
Section 4.4 – Fundamental Theorem of Calculus

Mean Value Theorem for Definite Integrals

If f is continuous on [a, b], then some point c in [a, b],

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$



Theorem – The Fundamental Theorem of Calculus, P-1

If f is continuous on [a, b], then $F(x) = \int_a^x f(t)\,dt$ is continuous on [a, b], and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Theorem – The Fundamental Theorem of Calculus, P-2

If f is continuous at every point in [a, b], then F is any antiderivative of f on [a, b], then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

Example

a) $$\int_0^{\pi} \cos x \, dx = \sin x \Big|_0^{\pi}$$

$$= \sin \pi - \sin 0$$

$$\underline{= 0}$$

b) $$\int_{-\frac{\pi}{4}}^{0} \sec x \tan x \, dx = \sec x \Big|_{-\frac{\pi}{4}}^{0}$$

$$= \sec 0 - \sec\left(-\frac{\pi}{4}\right)$$

$$= \underline{1 - \sqrt{2}}$$

c) $$\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx = \left[x^{3/2} + \frac{4}{x}\right]_1^4$$

$$= \left((4)^{3/2} + \frac{4}{4}\right) - \left((1)^{3/2} + \frac{4}{1}\right)$$

$$= (9) - (5)$$

$$= \underline{4}$$

Theorem – The Net Change Theorem

The net change in a function $F(x)$ over an interval $a \le x \le b$ is the integral of tis rate of change:

$$F(b) - F(a) = \int_a^b F'(x)\,dx$$

Example

Consider the analysis of a heavy rock blown straight up from the ground by a dynamite blast. The velocity of the rock at any time *t* during its motion was given as $v(t) = 160 - 32t \ \ ft/\sec$

a) Find the displacement of the rock during the time period $0 \le t \le 8$

b) Find the total distance traveled during this time period.

Solution

a) $\quad displacement: \quad s(t) = \int_0^8 v(t)\,dt$

$$\begin{aligned} &= \int_0^8 (160 - 32t)\,dt \\ &= \left[160t - 16t^2\right]_0^8 \\ &= \left(160(8) - 16(8)^2\right) - \left(160(0) - 16(0)^2\right) \\ &= 256 \end{aligned}$$

The height if the rock is 256 *ft* above the ground 8 *sec* after the explosion.

b) $\quad v(t) = 160 - 32t = 0 \quad \rightarrow \quad \boxed{t = 5 \ \sec}$

The velocity is positive over the time [0, 5] and negative over [5, 8]

$$\begin{aligned} \int_0^8 |v(t)|\,dt &= \int_0^5 |v(t)|\,dt + \int_5^8 |v(t)|\,dt \\ &= \int_0^5 (160 - 32t)\,dt - \int_5^8 (160 - 32t)\,dt \\ &= \left[160t - 16t^2\right]_0^5 - \left[160t - 16t^2\right]_5^8 \\ &= \left[\left(160(5) - 16(5)^2\right) - \left(160(0) - 16(0)^2\right)\right] \\ &\qquad - \left[\left(160(8) - 16(8)^2\right) - \left(160(5) - 16(5)^2\right)\right] \\ &= 400 - (-144) \\ &= 544 \end{aligned}$$

Example

Shows the graph of $f(x)=x^2-4$ and its mirror image $g(x)=4-x^2$ are reflected across the *x*-axis. For each function, compute

a) The definite integral over the interval [–2, 2]

b) The area between the graph and the *x*-axis over [–2, 2]

Solution

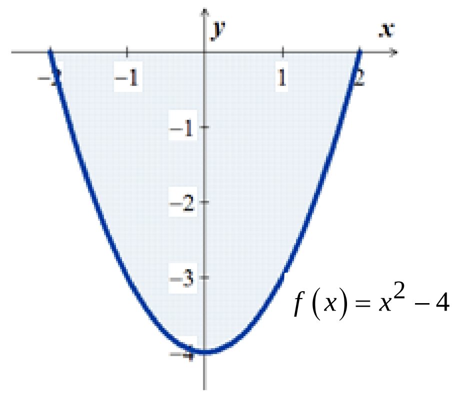
a) $\displaystyle\int_{-2}^{2} f(x)\,dx=\int_{-2}^{2}\left(x^2-4\right)dx$

$$=\left[\frac{x^3}{3}-4x\right]_{-2}^{2}$$

$$=\left[\frac{(2)^3}{3}-4(2)\right]-\left[\frac{(-2)^3}{3}-4(-2)\right]$$

$$=\left(\frac{8}{3}-8\right)-\left(-\frac{8}{3}+8\right)$$

$$=-\frac{32}{3}$$

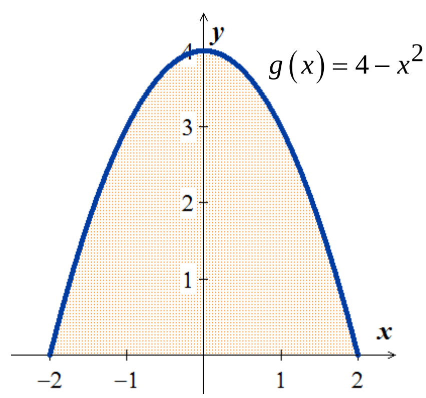


$$\int_{-2}^{2} g(x)\,dx=\int_{-2}^{2}\left(4-x^2\right)dx$$

$$=\left[4x-\frac{x^3}{3}\right]_{-2}^{2}$$

$$=\left[4(2)-\frac{(2)^3}{3}\right]-\left[4(-2)-\frac{(-2)^3}{3}\right]$$

$$=\frac{32}{3}$$



b) In both cases, the area between the curve and the *x*-axis over [–2, 2] is $\frac{32}{3}$ units.

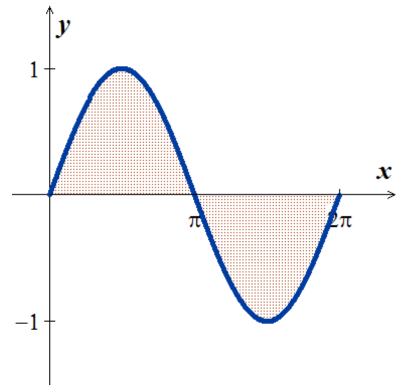
Example

Shows the graph of $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$. Compute

a) The definite integral of $f(x)$ over [0, 2π]

b) The area between the graph and the *x*-axis over [0, 2π]

Solution



a) $\displaystyle\int_0^{2\pi} \sin x\,dx = -\cos x \Big|_0^{2\pi}$

$= -(\cos 2\pi - \cos 0)$

$= -(1-1)$

$\underline{= 0}$

b) The area between the graph and the axis is obtained by adding the absolute values

$$Area = \left|\int_0^{\pi} \sin x\,dx\right| + \left|\int_{\pi}^{2\pi} \sin x\,dx\right|$$

$$= \left|-\cos x\right| \Big|_0^{\pi} + \left|-\cos x\right| \Big|_{\pi}^{2\pi}$$

$$= \left|-(\cos \pi - \cos 0)\right| + \left|-(\cos 2\pi - \cos \pi)\right|$$

$$= \left|-(-1-1)\right| + \left|-(1-(-1))\right|$$

$$= |2| + |-2|$$

$$\underline{= 4}$$

Summary

To find the area between the graph of $y = f(x)$ and the *x*-axis over the interval [*a*, *b*]:

1. Subdivide [*a*, *b*] at the zeros of *f*.

2. Integrate f over each subinterval.

3. Add the absolute values of the integrals.

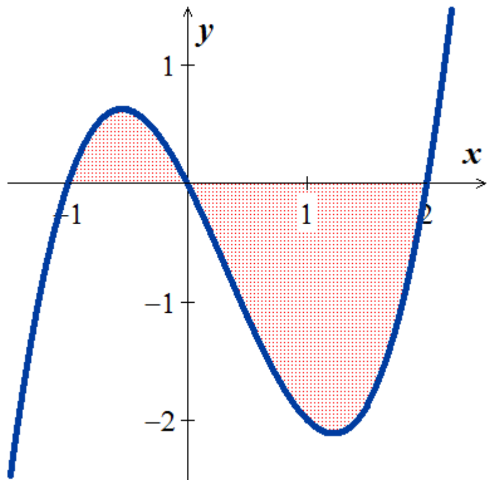
Example

Find the area of the region between the *x*-axis and the graph of $f(x) = x^3 - x^2 - 2x, \quad -1 \le x \le 2$

Solution

The zeros of: $f(x) = x^3 - x^2 - 2x = 0$

$$x\left(x^2 - x - 2\right) = 0 \quad \Rightarrow \quad x = 0, \ -1, \ 2$$



$$\int_{-1}^{0}\left(x^3 - x^2 - 2x\right)dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0}$$

$$= \left[0 - \left(\frac{(-1)^4}{4} - \frac{(-1)^3}{3} - (-1)^2\right)\right]$$

$$= -\left(\frac{1}{4} + \frac{1}{3} - 1\right)$$

$$= \frac{5}{12}$$

$$\int_{0}^{2}\left(x^3 - x^2 - 2x\right)dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2}$$

$$= \left[\left(\frac{(2)^4}{4} - \frac{(2)^3}{3} - (2)^2\right) - 0\right]$$

$$= \left(4 - \frac{8}{3} - 4\right)$$

$$= -\frac{8}{3}$$

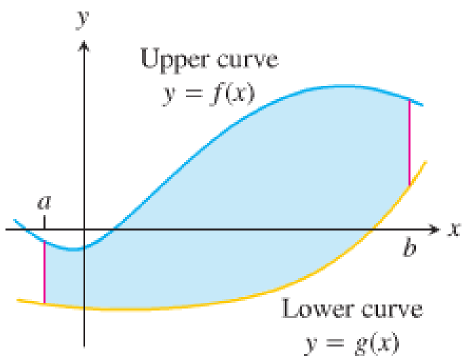
$$Area = \left|\int_{-1}^{0}\left(x^3 - x^2 - 2x\right)dx\right| + \left|\int_{0}^{2}\left(x^3 - x^2 - 2x\right)dx\right|$$

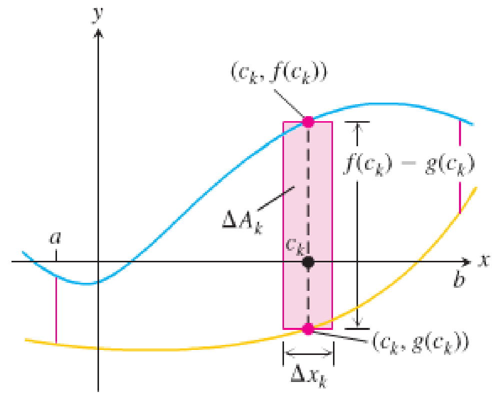
$$= \frac{5}{12} + \left|-\frac{8}{3}\right|$$

$$= \frac{5}{12} + \frac{8}{3}$$

$$= \frac{37}{12}$$

Areas between Curves

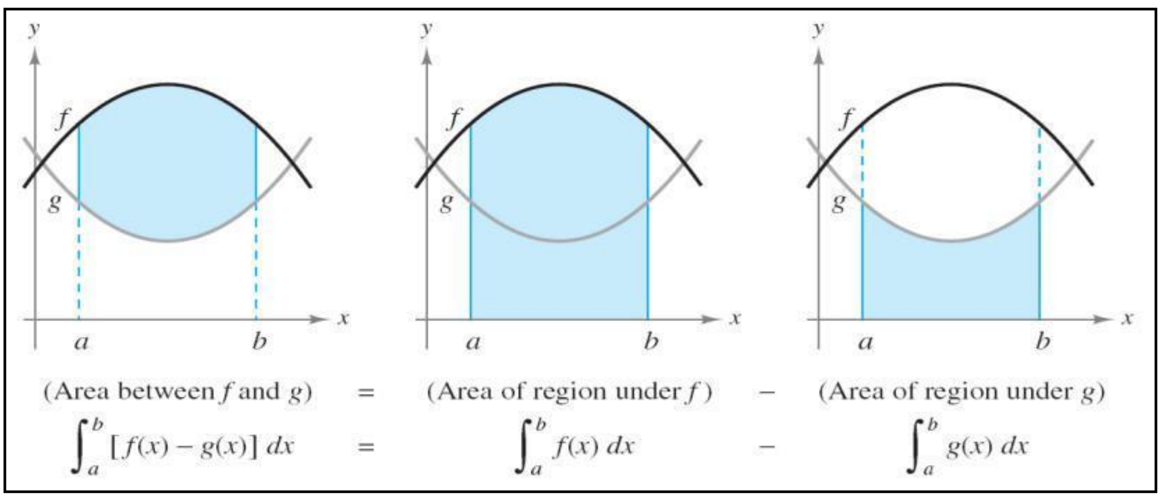




Definition

If f and g are continuous with $f(x) \ge g(x)$ throughout [a, b], then the ***area of the region between the curves*** $y = f(x)$ *and* $y = g(x)$ **from *a* to *b*** is:

$$A = \int_a^b \left[f(x) - g(x) \right] dx$$



Example

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution

The limits of integrations are found by letting:

$2 - x^2 = -x \qquad \Rightarrow x^2 - x - 2 = 0 \quad \rightarrow \quad x = -1, \ 2$

$$A = \int_{-1}^{2} \left[f(x) - g(x) \right] dx$$

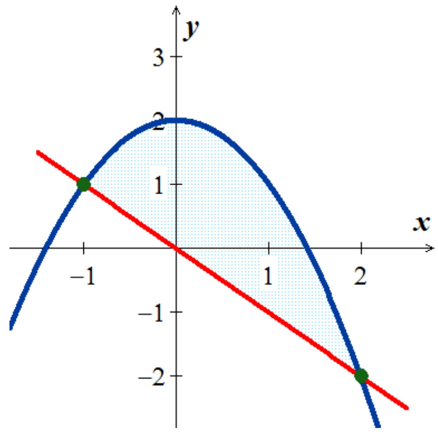
$$= \int_{-1}^{2} \left[2 - x^2 - (-x) \right] dx$$

$$= \int_{-1}^{2} \left(2 - x^2 + x \right) dx$$

$$= \left[2x - \frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^{2}$$

$$= \left(4 - \frac{8}{3} + \frac{4}{2} \right) - \left(-2 + \frac{1}{3} + \frac{1}{2} \right)$$

$$= \frac{9}{2}$$



Example

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below the x-axis and the line $y = x - 2$

Solution

$$\left(y = \sqrt{x} \right) \cap \left(y = 0 \right) \quad \rightarrow \quad (0,\ 0)$$

$$\left(y = \sqrt{x} \right) \cap \left(y = x - 2 \right) \quad \rightarrow \quad \sqrt{x} = x - 2$$

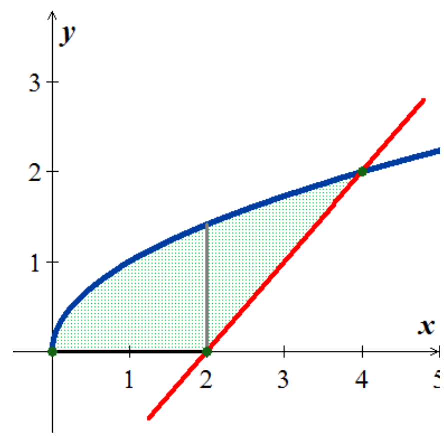
$$\left(\sqrt{x} \right)^2 = (x-2)^2$$

$$x = x^2 - 4x + 4$$

$$x^2 - 5x + 4 = 0$$

$$\rightarrow \quad x = \cancel{1},\ 4$$

$$\left(y = 0 \right) \cap \left(y = x - 2 \right) \quad \rightarrow \quad x = 2$$



$$\text{Total Area } = \int_{0}^{2} \left[\sqrt{x} - 0 \right] dx + \int_{2}^{4} \left[\sqrt{x} - (-x + 2) \right] dx$$

$$= \left[\frac{2}{3} x^{3/2} \right]_{0}^{2} + \left[\frac{2}{3} x^{3/2} - \frac{x^2}{2} + 2x \right]_{2}^{4}$$

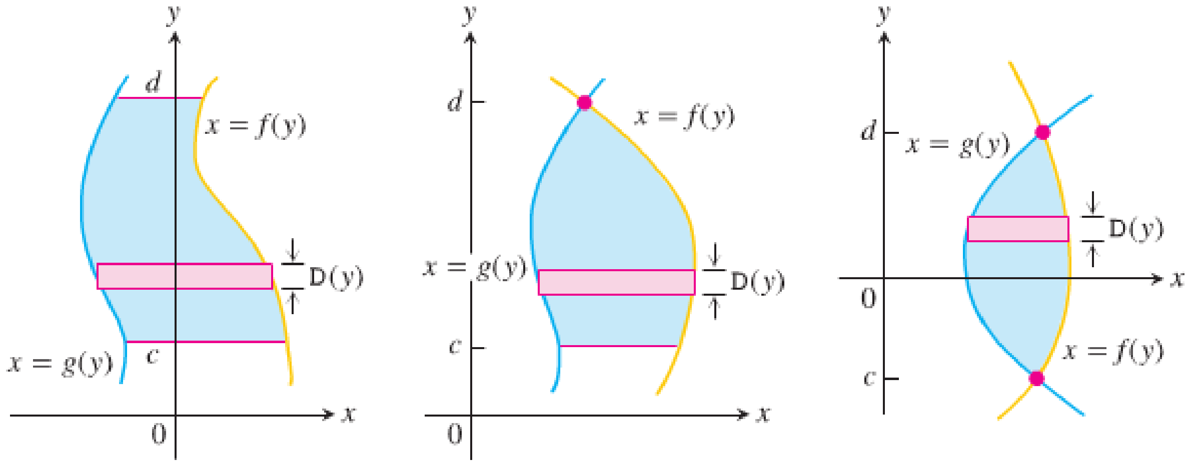
$$= \left[\frac{2}{3} \left(2^{3/2} \right) - 0 \right] + \left(\frac{2}{3} 4^{3/2} - \frac{4^2}{2} + 2(4) \right) - \left(\frac{2}{3} 2^{3/2} - \frac{2^2}{2} + 2(2) \right)$$

$$= \frac{2}{3} \left(2^{3/2} \right) + \frac{2}{3} 4^{3/2} - \frac{16}{2} + 8 - \frac{2}{3} 2^{3/2} + \frac{4}{2} - 4$$

$$= \frac{2}{3}(8) - 2$$

$$= \frac{10}{3}$$

Integration with Respect to *y*



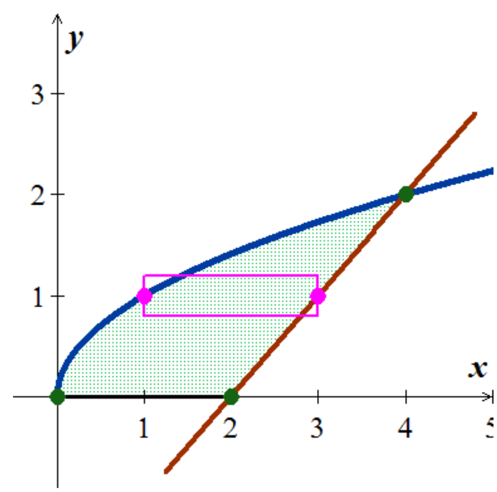
$$A = \int_c^d \left[f(y) - g(y) \right] dy$$

(*From right hand to left* hand)

Example

Find the area of the region by integrating with respect to y, in the first quadrant that is bounded above by $y = \sqrt{x}$ and below the x-axis and the line $y = x - 2$.

Solution



$y = \sqrt{x} \quad \rightarrow \quad x = y^2$

$y = x - 2 \quad \rightarrow \quad x = y + 2$

$\left(x = y^2\right) \cap (y = 0) \quad \rightarrow \quad (0,\ 0)$

$\left(x = y^2\right) \cap (x = y + 2) \quad \rightarrow \quad y^2 = y + 2$

$y^2 - y - 2 = 0 \quad \rightarrow \quad y = -1,\ 2$

$(y = 0) \cap (x = y + 2) \quad \rightarrow \quad y = 0$

$$A = \int_0^2 \left[y + 2 - y^2 \right] dy$$

$$= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_0^2$$

$$= \frac{2^2}{2} + 2(2) - \frac{2^3}{3} - 0$$

$$= \frac{10}{3}$$

Exercises *Section* 4.4 – Fundamental Theorem of Calculus

Evaluate the integrals

1. $\int_0^3 (2x+1)dx$

2. $\int_0^2 x(x-3)dx$

3. $\int_0^4 \left(3x - \frac{x^3}{4}\right)dx$

4. $\int_{-2}^2 \left(x^3 - 2x + 3\right)dx$

5. $\int_0^1 \left(x^2 + \sqrt{x}\right)dx$

6. $\int_0^{\pi/3} 4\sec u \tan u \ du$

7. $\int_{\pi/4}^{3\pi/4} \csc\theta \cot\theta d\theta$

8. $\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right)dt$

9. $\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} dy$

10. $\int_1^8 \frac{\left(x^{1/3}+1\right)\left(2-x^{2/3}\right)}{x^{1/3}} dx$

11. $\int_{\pi/2}^{\pi} \frac{\sin 2x}{2\sin x} dx$

12. $\int_0^{\pi/3} (\cos x + \sec x)^2 dx$

13. $\int_0^{\pi} \frac{1}{2}\left(\cos x + |\cos x|\right)dx$

14. $\int_0^1 2x\left(4 - x^2\right)dx$

15. $\int_0^4 (8-2x)dx$

16. $\int_0^4 \frac{1}{\sqrt{16-x^2}} dx$

17. $\int_{-4}^2 (2x+4)dx$

18. $\int_0^2 (1-x)dx$

19. $\int_0^2 \left(x^2 - 2\right)dx$

20. $\int_0^{\pi/2} \cos x \ dx$

21. $\int_1^7 \frac{dx}{x}$

22. $\int_4^9 3\sqrt{x} \ dx$

23. $\int_{-2}^3 \left(x^2 - x - 6\right)dx$

24. $\int_0^1 \left(1 - \sqrt{x}\right)dx$

25. $\int_0^{\pi/4} 2\cos x \ dx$

26. $\int_{-\pi/4}^{7\pi/4} (\sin x + \cos x)dx$

27. $\int_0^{\ln 8} e^x dx$

28. $\int_1^4 \left(\frac{x-1}{x}\right)dx$

29. $\int_{-2}^{-1} \left(3e^{3x} + \frac{2}{x}\right)dx$

30. $\int_0^2 \frac{dx}{x^2+4}$

Find the total area between the region between the given graph and the *x*-axis

31. $y = -x^2 - 2x, \quad -3 \le x \le 2$

32. $y = x^3 - 3x^2 + 2x, \quad 0 \le x \le 2$

33. $y = x^{1/3} - x, \quad -1 \le x \le 8$

34. $f(x) = x^2 + 1, \quad 2 \le x \le 3$

35. Find the area of the region between the graph of $y = 4x - 8$ and the *x-axis*, for $-4 \le x \le 8$

36. Find the area of the region between the graph of $y = -3x$ and the *x-axis*, for $-2 \le x \le 2$

37. Find the area of the region between the graph of $y = 3x + 6$ and the *x-axis*, for $0 \le x \le 6$

38. Find the area of the region between the graph of $y = 1 - |x|$ and the *x-axis*, for $-2 \le x \le 2$

39. Find the area of the region above the *x-axis* bounded by $y = 4 - x^2$

40. Find the area of the region above the *x-axis* bounded by $y = x^4 - 16$

41. Find the area of the region between the graph of $y = 6\cos x$ and the *x-axis*, for $-\frac{\pi}{2} \le x \le \pi$

42. Find the area of the region between the graph of $f(x) = \frac{1}{x}$ and the *x-axis*, for $-2 \le x \le -1$

43. Find the area of the region bounded by the graph of $f(x) = x^2 - 4x + 3$ *x-axis on* $0 \le x \le 3$

44. Find the area of the region bounded by the graph of $f(x) = x^2 + 4x + 3$ *x-axis on* $-3 \le x \le 0$

45. Find the area of the region bounded by the graph of $f(x) = x^2 - 3x + 2$ *x-axis on* $0 \le x \le 2$

46. Find the area of the region bounded by the graph of $f(x) = x^2 + 3x + 2$ *x-axis on* $-2 \le x \le 0$

47. Find the area of the region bounded by the graph of $f(x) = 2x^2 - 4x + 2$ *x-axis on* $0 \le x \le 2$

48. Find the area of the region bounded by the graph of $f(x) = 2x^2 + 4x + 2$ *x-axis on* $-1 \le x \le 1$

49. Find the area of the region bounded by the graphs of $x = y^2 - y$ and $x = 2y^2 - 2y - 6$

50. Find the area of the region bounded by the graphs of $y = x^2 - 4$ & $y = -x^2 - 2x$

Compute the area of the region bounded by the graph of f and the *x-axis* on the given interval.

51. $f(x) = \frac{1}{x^2 + 1}$ *on* $\left[-1,\ \sqrt{3}\right]$

52. $f(x) = 2\sin\frac{x}{4}$ *on* $[0,\ 2\pi]$

53. Archimedes, inventor, military engineer, physicist, and the greatest mathematician of classical times in the Western world, discovered that the area under a parabolic arch is two-thirds the base times the height. Sketch the parabolic arch $y = h - \left(\frac{4h}{b^2}\right)x^2 \quad -\frac{b}{2} \le x \le \frac{b}{2}$, assuming that h and b are positive.

Then use calculus to find the area of the region enclosed between the arch and the x-axis

54. Suppose that a company's marginal revenue from the manufacture and sale of eggbeaters is

$$\frac{dr}{dx} = 2 - \frac{2}{(x+1)^2}$$

Where r is measured in thousands of dollars and x in thousands of units. How much money should the company expect from a production run of $x = 3$ thousand eggbeaters? To find out, integrate the marginal revenue from $x = 0$ to $x = 3$.

55. The height H (ft) of a palm tree after growing for t years is given by

$$H = \sqrt{t+1} + 5t^{1/3} \quad \textit{for} \quad 0 \le t \le 8$$

a) Find the tree's height when $t = 0$, $t = 4$, and $t = 8$.

b) Find the tree's average height for $0 \le t \le 8$