Solution   Section 4.4 – Solution about Singular Points

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x^2y'' + 3y' - xy = 0$$

Solution

$$y'' + \frac{3}{x^2}y' - \frac{x}{x^2}y = 0$$

$$P(x) = \frac{3}{x^2} \qquad Q(x) = -\frac{x}{x^2}$$

For $P(x) = \frac{3}{x^2} \quad \rightarrow \quad \underline{x = 0}$

$\therefore\ p(x)$ is analytic except at $\underline{x = 0}$

For $Q(x) = -\frac{x}{x^2} = -\frac{1}{x}$

$\therefore\ q(x)$ is not analytic at $\underline{x = 0}$

The singular point is: $\underline{x = 0}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^2 + x\right)y'' + 3y' - 6xy = 0$$

Solution

$$y'' + \frac{3}{x(x+1)}y' - \frac{6}{x+1}y = 0$$

$$P(x) = \frac{3}{x(x+1)} \qquad Q(x) = -\frac{6x}{x(x+1)}$$

For $P(x) = \frac{3}{x(x+1)} \quad \rightarrow \quad \underline{x = 0, -1}$

$\therefore\ p(x)$ is analytic except at $\underline{x = 0, -1}$

For $q(x) = -\frac{6x}{x(x+1)} \quad \rightarrow \quad x = 0,\ \ -1$

$q(x) = -\frac{6x}{x(x+1)} = -\frac{6}{x+1}$; is actually analytic at $x = 0$

$\therefore\ q(x)$ is analytic except at $\underline{x = -1}$

The singular points are: $\underline{x = 0,\ \ -1}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^2-1\right)y''+(1-x)y'+\left(x^2-2x+1\right)y=0$$

Solution

$$(x-1)(x+1)y''+(1-x)y'+(x-1)^2 y=0$$

$$y''-\frac{1}{x+1}y'+\frac{x-1}{x+1}y=0$$

$$P(x)=\frac{1-x}{x^2-1} \qquad Q(x)=\frac{(x-1)^2}{x^2-1}$$

For $p(x)=\dfrac{1-x}{(x-1)(x+1)} \quad \rightarrow \quad \underline{x=-1,\ 1}$

$p(x)=\dfrac{1-x}{(x-1)(x+1)}=-\dfrac{1}{x+1}$; is actually analytic at $x=1$

$\therefore\ p(x)$ is analytic except at $\underline{x=-1}$

For $q(x)=\dfrac{(x-1)^2}{(x-1)(x+1)} \quad \rightarrow \quad \underline{x=-1,\ 1}$

$q(x)=\dfrac{(x-1)^2}{(x-1)(x+1)}=\dfrac{x-1}{x+1}$; is actually analytic at $x=1$

$\therefore\ q(x)$ is analytic except at $\underline{x=-1}$

The singular point is: $\underline{x=-1}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$e^x y''-\left(x^2-1\right)y'+2xy=0$$

Solution

$$y''-\frac{x^2-1}{e^x}y'+\frac{2x}{e^x}y=0$$

$$P(x)=-\frac{x^2-1}{e^x} \qquad Q(x)=\frac{2x}{e^x}$$

Since $e^x \neq 0$, there are ***no*** singular points.

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\ln(x-1)y''+(\sin 2x)y'-e^{x}y=0$$

Solution

$$y''+\frac{\sin 2x}{\ln(x-1)}y'-\frac{e^{x}}{\ln(x-1)}y=0$$

$$P(x)=\frac{\sin 2x}{\ln(x-1)} \qquad Q(x)=-\frac{e^{x}}{\ln(x-1)}$$

$$\ln(x-1)=0 \quad \to \quad x-1=1 \ \Rightarrow \ \underline{x=2}$$

The singular point is: $\underline{x\le 1,\ x=2}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$xy''+x(1-x)^{-1}y'+(\sin x)y=0$$

Solution

$$y''+\frac{x}{x(1-x)}y'+\frac{\sin x}{x}y=0$$

$$P(x)=\frac{x}{x(1-x)} \qquad Q(x)=\frac{\sin x}{x}$$

For $p(x)=\dfrac{x}{x(1-x)} \quad \to \quad \underline{x=0,\ 1}$

$p(x)=\dfrac{x}{x(1-x)}=\dfrac{1}{1-x}$; is actually analytic at $x=0$

$\therefore\ p(x)$ is analytic except at $\underline{x=1}$

For $q(x)=\dfrac{\sin x}{x}=\dfrac{x-\frac{1}{3!}x^{3}+\cdots}{x}=1-\dfrac{1}{3!}x^{2}+\cdots$ is analytic everywhere ($x=0$ is removable).

The only singular point is $\underline{x=1}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x(x+3)^{2}y''-y=0$$

Solution

$$y'' - \frac{1}{x(x+3)^2} y = 0$$

$$P(x) = 0 \qquad Q(x) = -\frac{1}{x(x+3)^2}$$

For $q(x) = -\frac{1}{x(x+3)^2} \quad \rightarrow \quad \underline{x = 0, \ -3}\,|$, is analytic elsewhere

The *Regular* singular points are $\underline{x = 0, \ -3}\,|$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^2-9\right)^2 y'' + (x+3)y' + 2y = 0$$

Solution

$$y'' + \frac{x+3}{\left(x^2-9\right)^2} y' + \frac{2}{\left(x^2-9\right)^2} y = 0$$

$$P(x) = \frac{x+3}{\left(x^2-9\right)^2} \qquad Q(x) = \frac{2}{\left(x^2-9\right)^2}$$

For $P(x) = \frac{x+3}{\left(x^2-9\right)^2} \quad \rightarrow \quad \underline{x = \pm 3}\,|$

$$p(x) = \frac{x+3}{((x+3)(x-3))^2} = \frac{(x+3)(x-3)}{(x+3)(x-3)^2}$$; is analytic at $x = -3$

For $Q(x) = \frac{2\left(x^2-9\right)^2}{\left(x^2-9\right)^2} \quad \rightarrow \quad \underline{x = \pm 3}\,|$

$\therefore \ q(x)$ is analytic at $\underline{x = \pm 3}\,|$

The *Regular* singular point: $\underline{x = -3}\,|$, and *Irregular* singular point: $\underline{x = 3}\,|$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$y'' - \frac{1}{x} y' + \frac{1}{(x-1)^3} y = 0$$

Solution

$$y'' - \frac{1}{x}y' + \frac{1}{(x-1)^3}y = 0$$

$$P(x) = -\frac{1}{x} \qquad Q(x) = \frac{1}{(x-1)^3}$$

For $P(x) = -\frac{1}{x} \quad \to \quad \underline{x = 0\,}|$

$p(x) = \frac{x}{x} = 1$ is analytic at $\underline{x = 0}|$

For $Q(x) = \frac{1}{(x-1)^3} \quad \to \quad \underline{x = 1\,}|$

$q(x) = \frac{(x-1)^2}{(x-1)^3} = \frac{1}{x-1}$ is *not* an analytic at $x = 1$

The *Regular* singular point: $\underline{x = 0\,}|$, and *Irregular* singular point: $\underline{x = 1\,}|$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^3 + 4x\right)y'' - 2xy' + 6y = 0$$

Solution

$$y'' - \frac{2x}{x\left(x^2+4\right)}y' + \frac{6}{x\left(x^2+4\right)}y = 0$$

$$P(x) = -\frac{2x}{x\left(x^2+4\right)} \qquad Q(x) = \frac{6}{x\left(x^2+4\right)}$$

For $P(x) = -\frac{2x}{x\left(x^2+4\right)} \quad \to \quad x = 0, \ \pm 2i$

$p(x) = -\frac{2}{x^2+4}$ is analytic at $x = \pm 2i$

For $Q(x) = \frac{6}{x\left(x^2+4\right)} \quad \to \quad x = 0, \ \pm 2i$ is analytic

The *Regular* singular points: $\underline{x = 0, \ \pm 2i\,}|$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x^2(x-5)^2 y'' + 4xy' + \left(x^2-25\right)y = 0$$

Solution

$$y'' + \frac{4x}{x^2(x-5)^2}y' + \frac{x^2-25}{x^2(x-5)^2}y = 0$$

$$P(x) = \frac{4x}{x^2(x-5)^2} \qquad Q(x) = \frac{x^2-25}{x^2(x-5)^2}$$

For $P(x) = \dfrac{4x}{x^2(x-5)^2} \quad \to \quad x = 0,\ 5$

$$p(x) = \frac{4x}{x^2(x-5)^2} = x(x-5)\frac{4}{x(x-5)^2} \text{ is not an analytic at } x = 5$$

For $Q(x) = \dfrac{(x-5)(x+5)}{x^2(x-5)^2} \quad \to \quad x = 0,\ 5$

$$q(x) = x^2(x-5)^2\frac{x+5}{x^2(x-5)} \text{ is an analytic at } x = 0,\ 5$$

The *Regular* singular point: $x = 0$, and *Irregular* singular point: $x = 5$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^2+x-6\right)y'' + (x+3)y' + (x-2)y = 0$$

Solution

$$y'' + \frac{x+3}{x^2+x-6}y' + \frac{x-2}{x^2+x-6}y = 0$$

$$P(x) = \frac{x+3}{(x+3)(x-2)} \qquad Q(x) = \frac{x-2}{(x+3)(x-2)}$$

For $P(x) = \dfrac{x+3}{(x+3)(x-2)} \quad \to \quad x = -3,\ 2$

$$p(x) = \frac{1}{x-2} \text{ is an analytic at } x = 2$$

For $Q(x) = \dfrac{x-2}{(x+3)(x-2)} \quad \to \quad x = -3,\ 2$

$$q(x) = \frac{1}{x+3} \text{ is an analytic at } x = -3$$

The *Regular* singular points: $x = -3,\ 2$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x\left(x^2+1\right)^2 y''+y=0$$

Solution

$$y''+\frac{1}{x\left(x^2+1\right)^2}y=0$$

$$P(x)=0 \qquad Q(x)=\frac{1}{x\left(x^2+1\right)^2}$$

For $Q(x)=\dfrac{1}{x\left(x^2+1\right)^2} \quad \rightarrow \quad x=0, \ \pm i$

$$q(x)=x^2\left(x^2+1\right)^2\frac{1}{x\left(x^2+1\right)^2} \text{ is an analytic at } x=0, \ \pm i$$

The *Regular* singular points: $\underline{x=0, \ \pm i}|$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x^3\left(x^2-25\right)(x-2)^2 y''+3x(x-2)y'+7(x+5)y=0$$

Solution

$$y''+\frac{3x(x-2)}{x^3\left(x^2-25\right)(x-2)^2}y'+\frac{7(x+5)}{x^3\left(x^2-25\right)(x-2)^2}y=0$$

$$P(x)=\frac{3x(x-2)}{x^3(x-5)(x+5)(x-2)^2} \qquad Q(x)=\frac{7(x+5)}{x^3(x-5)(x+5)(x-2)^2}$$

For $P(x)=\dfrac{3x(x-2)}{x^3(x-5)(x+5)(x-2)^2} \quad \rightarrow \quad x=0, \ \pm 5, \ 2$

$$p(x)=\frac{3x(x-5)(x+5)(x-2)}{x^2(x-5)(x+5)(x-2)} \text{ is not an analytic at } x=0$$

For $Q(x)=\dfrac{7(x+5)}{x^3(x-5)(x+5)(x-2)^2} \quad \rightarrow \quad x=0, \ \pm 5, \ 2$

$$q(x)=\frac{3x^2(x-5)^2(x+5)^2(x-2)^2}{x^3(x-5)(x-2)^2} \text{ is not an analytic at } x=0$$

The *Regular* singular point: $\underline{x=2,\ \pm 5}$, and *Irregular* singular point: $\underline{x=0}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^3-2x^2-3x\right)^2 y''+x(x-3)^2 y'-(x+1)y=0$$

Solution

$$y''+\frac{x(x-3)^2}{\left(x^3-2x^2-3x\right)^2}y'-\frac{x+1}{\left(x^3-2x^2-3x\right)^2}y=0$$

$$P(x)=\frac{x(x-3)^2}{x^2(x-3)^2(x+1)^2} \qquad Q(x)=-\frac{x+1}{x^2(x-3)^2(x+1)^2}$$

For $P(x)=\dfrac{x(x-3)^2}{x^2(x-3)^2(x+1)^2} \quad \to \quad x=0,\ -1,\ 3$

$$p(x)=\frac{1}{x(x+1)^2} \text{ is not an analytic at } x=-1$$

For $Q(x)=\dfrac{x+1}{x^2(x-3)^2(x+1)^2} \quad \to \quad x=0,\ -1,\ 3$

$$q(x)=\frac{x^2(x-3)^2(x+1)^2}{x^2(x-3)^2(x+1)} \text{ is an analytic at } x=0,\ -1,\ 3$$

The *Regular* singular point: $\underline{x=0,\ 3}$, and *Irregular* singular point: $\underline{x=-1}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(1-x^2\right)y''+(\tan x)y'+x^{5/3}y=0$$

Solution

$$y''+\frac{\tan x}{1-x^2}y'+\frac{x^{5/3}}{1-x^2}y=0$$

$$P(x)=\frac{\tan x}{1-x^2} \qquad Q(x)=\frac{x^{5/3}}{1-x^2}$$

For $P(x)=\dfrac{\tan x}{1-x^2} \quad \to \quad x=\pm 1$

$\tan x=\pm\infty \quad \to \quad x=\pm\frac{\pi}{2}$ (Vertical Asymptotes).

For $Q(x)=\dfrac{x^{5/3}}{1-x^2} \quad \to \quad x=\pm 1$ is not analytic

The second derivatices doesn't exist at $x=0$

The *Regular* singular point: $\underline{x=0,\ \pm 1,\ \pm\frac{\pi}{2}}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x(x-1)^2(x+2)y''+x^2y'-\left(x^3+2x-1\right)y=0$$

Solution

$$y''+\frac{x^2}{x(x-1)^2(x+2)}y'-\frac{x^3+2x-1}{x(x-1)^2(x+2)}y=0$$

$$P(x)=\frac{x}{(x-1)^2(x+2)} \quad \& \quad Q(x)=-\frac{x^3+2x-1}{x(x-1)^2(x+2)}$$

For $P(x)=\dfrac{x}{(x-1)^2(x+2)} \quad \to \quad x=1,\ -2$

$$p_0=\lim_{x\to 1}(x-1)P(x)=\lim_{x\to 1}\frac{x}{(x-1)(x+2)}\underline{=\infty} \quad \text{is not analytic}$$

$$p_0=\lim_{x\to -2}(x+2)P(x)=\lim_{x\to -2}\frac{x}{(x-1)^2}\underline{=-\frac{2}{9}}$$

For $Q(x)=-\dfrac{x^3+2x-1}{x(x-1)^2(x+2)} \quad \to \quad x=0,\ 1,\ -2$

$$q_0=\lim_{x\to 0}x^2Q(x)=\lim_{x\to 0}\frac{x\left(x^3+2x-1\right)}{(x-1)^2(x+2)}\underline{=0}$$

$$q_0=\lim_{x\to 1}(x-1)^2Q(x)=\lim_{x\to 1}\frac{x^3+2x-1}{x(x+2)}\underline{=\frac{2}{3}}$$

$$q_0=\lim_{x\to -2}(x+2)^2Q(x)=-\lim_{x\to -2}\frac{\left(x^3+2x-1\right)(x+2)}{x(x-1)^2}\underline{=0}$$

The *Regular* singular point: $\underline{x=0,\ -2}$, and *Irregular* singular point: $\underline{x=1}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x^4\left(x^2+1\right)(x-1)^2 y''+4x^3(x-1)y'+(x+1)y=0$$

Solution

$$y''+\frac{4x^3(x-1)}{x^4\left(x^2+1\right)(x-1)^2}y'+\frac{x+1}{x^4\left(x^2+1\right)(x-1)^2}y=0$$

$$P(x)=\frac{4}{x\left(x^2+1\right)(x-1)} \quad \& \quad Q(x)=\frac{x+1}{x^4\left(x^2+1\right)(x-1)^2}$$

For $P(x)=\dfrac{4}{x\left(x^2+1\right)(x-1)} \quad \rightarrow \quad \underline{x=0,\ 1,\ \pm i}$

$$p_0=\lim_{x\to 0} xP(x)=\lim_{x\to 0}\frac{4}{\left(x^2+1\right)(x-1)}\ \underline{=-4}$$

$$p_0=\lim_{x\to 1}(x-1)P(x)=\lim_{x\to 1}\frac{4}{x\left(x^2+1\right)}\ \underline{=2}$$

$$p_0=\lim_{x\to i}(x-i)P(x)=\lim_{x\to i}\frac{4}{x(x-1)(x+i)}=-\frac{2}{i-1}=-\frac{2}{i-1}\frac{i+1}{i+1}\ \underline{=i+1}$$

$$p_0=\lim_{x\to -i}(x+i)P(x)=\lim_{x\to -i}\frac{4}{x(x-1)(x-i)}=\frac{2}{i-1}=\frac{2}{i-1}\frac{i+1}{i+1}\ \underline{=-i-1}$$

For $Q(x)=\dfrac{x+1}{x^4\left(x^2+1\right)(x-1)^2} \quad \rightarrow \quad x=0,\ 1,\ \pm i$

$$q_0=\lim_{x\to 0} x^2Q(x)=\lim_{x\to 0}\frac{x+1}{x^2\left(x^2+1\right)(x-1)^2}\ \underline{=\infty} \text{ is not analytic}$$

$$q_0=\lim_{x\to 1}(x-1)^2Q(x)=\lim_{x\to 0}\frac{x+1}{x^4\left(x^2+1\right)}\ \underline{=1}$$

$$q_0=\lim_{x\to \pm i}\left(x^2+1\right)^2Q(x)=\lim_{x\to \pm i}\frac{(x+1)\left(x^2+1\right)}{x^2(x-1)^2}\ \underline{=0}$$

The *Regular* singular point: $\underline{x=0,\ \pm i}$, and *Irregular* singular point: $\underline{x=0}$

Exercise

Determine whether $x=0$ is an ordinary point, singular point, or irregular singular point of the given differential equation $\quad xy''+(1-\cos x)y'+x^2y=0$

Solution

$$y'' + \frac{1-\cos x}{x}y' + xy = 0$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\frac{1-\cos x}{x} = \frac{1}{x}\left(1 - 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right)$$

$$= \boxed{\frac{x}{2} - \frac{x^3}{24} + \frac{x^5}{720} - \cdots}, \quad \text{is analytic at } x = 0$$

$\therefore$ $x = 0$ is an ordinary point of the differential equation.

Exercise

Determine whether $x = 0$ is an ordinary point, singular point, or irregular singular point of the given differential equation $\left(e^x - 1 - x\right)y'' + xy = 0$

Solution

$$x^2 y'' + x^2 \frac{x}{e^x - 1 - x} y = 0$$

$$x^2 y'' + \frac{x^3}{e^x - 1 - x} y = 0$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^x - 1 - x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots - 1 - x$$

$$= \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$\frac{x^3}{e^x - 1 - x} = \frac{1}{\frac{1}{x^3}\left(\frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)}$$

$$= \frac{1}{\frac{1}{2x} + \frac{1}{6} + \frac{x}{24} + \cdots}$$

$\therefore$ $x = 0$ is a regular singular point of the differential equation

Exercise

Find the Frobenius series solutions of $2x^2 y'' + 3xy' - \left(1 + x^2\right)y = 0$

Solution

$$y'' + \frac{3}{2}\frac{1}{x}y' - \frac{1}{2}\frac{1+x^2}{x^2}y = 0$$ ***Divide each term by*** $2x^2$

Therefore, $x = 0$ is a regular singular point, and that $p_0 = \frac{3}{2}, \quad q_0 = -\frac{1}{2}$

$p(x) \equiv \frac{3}{2}, \quad q(x) = -\frac{1}{2} - \frac{1}{2}x^2$ are polynomials.

The Frobenius series will converge for all $x > 0$. The indicial equation is

$$r(r-1) + \frac{3}{2}r - \frac{1}{2} = r^2 + \frac{1}{2}r - \frac{1}{2} = (r+1)\left(r - \frac{1}{2}\right) = 0$$

So the roots are $r_1 = \frac{1}{2}$ *and* $r_2 = -1$.

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/2}\sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{-1}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$2x^2y'' + 3xy' - \left(1 + x^2\right)y = 0$$

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + 3x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty}a_n x^{n+r} - x^2\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(2n+2r-2)a_n x^{n+r} + \sum_{n=0}^{\infty}3(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty}a_n x^{n+r} - \sum_{n=0}^{\infty}a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-2) + 3(n+r) - 1\right]a_n x^{n+r} - \sum_{n=2}^{\infty}a_{n-2}x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r+1) - 1\right]a_n x^{n+r} - \sum_{n=2}^{\infty}a_{n-2}x^{n+r} = 0$$

$$\left(2r^2 + r - 1\right)a_0 + \left(2r^2 + 5r + 2\right)a_1 x +$$

$$\sum_{n=2}^{\infty}\left[(n+r)(2n+2r+1) - 1\right]a_n x^{n+r} - \sum_{n=2}^{\infty}a_{n-2}x^{n+r} = 0$$

$$\left(2r^2+r-1\right)a_0+\left(2r^2+5r+2\right)a_1x+\sum_{n=2}^{\infty}\left(\left[(n+r)(2n+2r+1)-1\right]a_n-a_{n-2}\right)x^{n+r}=0$$

For $n=0 \rightarrow \left(2r^2+r-1\right)a_0=0$

$$r=-1 \quad or \quad r=\tfrac{1}{2} \quad \checkmark$$

For $n=1 \rightarrow \left(2r^2+5r+2\right)a_1=0$

$$\cancel{r=-2 \quad or \quad r=-\tfrac{1}{2}} \quad \text{Therefore } a_1=0$$

$$\left[(n+r)(2n+2r+1)-1\right]a_n-a_{n-2}=0$$

$$a_n=\frac{1}{(n+r)\left(2(n+r)+1\right)-1}a_{n-2}$$

$$=\frac{1}{2(n+r)^2+(n+r)-1}a_{n-2} \qquad for \ \ n\ge 2$$

$r=\frac{1}{2}$

$$a_n=\frac{1}{2\left(n+\frac{1}{2}\right)^2+\left(n+\frac{1}{2}\right)-1}a_{n-2}=\frac{1}{2n^2+3n}a_{n-2}$$

$$a_2=\frac{1}{14}a_0$$

$$a_3=\frac{1}{24}a_1=0$$

$$a_4=\frac{1}{44}a_2=\frac{1}{616}a_0$$

$$a_5=0$$

$$a_6=\frac{1}{90}a_4=\frac{1}{55440}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$r=-1$

$$b_n=\frac{1}{2(n-1)^2+(n-1)-1}b_{n-2}$$

$$b_2=\frac{1}{2}b_0$$

$$b_3=\frac{1}{9}b_1=0$$

$$b_4=\frac{1}{20}b_2=\frac{1}{40}b_0$$

$$b_5=0$$

$$b_6=\frac{1}{54}b_4=\frac{1}{2160}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_1(x)=a_0x^{1/2}\left(1+\frac{x^2}{14}+\frac{x^4}{616}+\frac{x^6}{55,440}+\cdots\right) \qquad y_2(x)=b_0x^{-1}\left(1+\frac{x^2}{2}+\frac{x^4}{40}+\frac{x^6}{2160}+\cdots\right)$$

$$y(x)=C_1\left(1+\sum_{n=0}^{\infty}\frac{4}{n!\,1\cdot 3\cdot 5\cdots(2n-1)}x^n\right)+C_2x^{1/2}\left(1+\sum_{n=0}^{\infty}\frac{4^n}{n!\,3\cdot 5\cdots(2n+1)}x^n\right)$$

Exercise

Find the Frobenius series solutions of $2x^2y'' - xy' + \left(1+x^2\right)y = 0$

Solution

$$y'' - \frac{1}{2}\frac{1}{x}y' + \frac{1}{2}\frac{1+x^2}{x^2}y = 0 \qquad \textbf{\textit{Divide each term by }} 2x^2$$

Therefore, $x = 0$ is a *regular singular* point, and that $p_0 = -\frac{1}{2}, \quad q_0 = \frac{1}{2}$

$p(x) = -\frac{1}{2}, \quad q(x) = \frac{1}{2} + \frac{1}{2}x^2$ are polynomials.

The Frobenius series will converge for all $x > 0$. The indicial equation is

$$r(r-1) - \frac{1}{2}r + \frac{1}{2} = r^2 - \frac{3}{2}r + \frac{1}{2} = \frac{1}{2}\left(2r^2 - 3r + 1\right) = 0$$

So the roots are $\boxed{r_1 = \frac{1}{2} \quad and \quad r_2 = 1}$.

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/2}\sum_{n=0}^{\infty} a_n x^n \quad and \quad y_2(x) = x^1\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$2x^2y'' - xy' + \left(1+x^2\right)y = 0$$

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}a_n x^{n+r} + x^2\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-2) - (n+r) + 1\right]a_n x^{n+r} + \sum_{n=2}^{\infty}a_{n-2}x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-3) + 1\right]a_n x^{n+r} + \sum_{n=2}^{\infty}a_{n-2}x^{n+r} = 0$$

$$\left(2r^2-3r+1\right)a_0+\left((1+r)(2r-1)+1\right)a_1x+$$

$$\sum_{n=2}^{\infty}\left[(n+r)(2n+2r-3)+1\right]a_nx^{n+r}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r}=0$$

$$\left(2r^2-3r+1\right)a_0+\left(2r^2+r\right)a_1x+\sum_{n=2}^{\infty}\left[\left((n+r)(2n+2r-3)+1\right)a_n+a_{n-2}\right]x^{n+r}=0$$

For $n=0 \rightarrow \left(2r^2-3r+1\right)a_0=0 \Rightarrow \boxed{r=1 \;\; or \;\; r=\frac{1}{2}}$ √

For $n=1 \rightarrow \left(2r^2+r\right)a_1=0 \Rightarrow r\; \cancel{=0,-\frac{1}{2}}$ Therefore $\boxed{a_1=0}$

$$\left((n+r)(2n+2r-3)+1\right)a_n+a_{n-2}=0$$

$$\boxed{a_n=-\frac{1}{(n+r)(2n+2r-3)+1}a_{n-2}} \qquad for \;\; n\ge 2$$

$r=\frac{1}{2}$

$$a_n=-\frac{1}{\left(n+\frac{1}{2}\right)(2n-2)+1}a_{n-2}\;\boxed{=-\frac{1}{2n^2-n}a_{n-2}}$$

$n=2 \rightarrow a_2=-\frac{1}{6}a_0$

$n=3 \rightarrow a_3=-\frac{1}{15}a_1=0$

$n=4 \rightarrow a_4=-\frac{1}{28}a_2=\frac{1}{168}a_0$

$n=5 \rightarrow a_5=0$

$n=6 \rightarrow a_6=-\frac{1}{66}a_4=-\frac{1}{11,088}a_0$

$$y_1(x)=a_0x^{1/2}\left(1-\frac{x^2}{6}+\frac{x^4}{168}-\frac{x^6}{11,088}+\cdots\right)$$

$$\boxed{=a_0\left(x^{1/2}-\frac{1}{6}x^{5/2}+\frac{1}{168}x^{9/2}-\frac{1}{11,088}x^{13/2}+\cdots\right)}$$

$r=1$

$$b_n=-\frac{1}{(n+1)(2n-1)+1}b_{n-2}\;\boxed{=-\frac{1}{2n^2+n}b_{n-2}}$$

$n=2 \rightarrow b_2=-\frac{1}{10}b_0$

$n=3 \rightarrow b_3=-\frac{1}{21}b_1=0$

$n=4 \rightarrow b_4=-\frac{1}{36}b_2=\frac{1}{360}b_0$

$n=5 \rightarrow b_5=0$

$n=6 \rightarrow b_6=-\frac{1}{78}b_4=-\frac{1}{28,080}b_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_2(x) = b_0 x\left(1 - \frac{x^2}{10} + \frac{x^4}{360} - \frac{x^6}{28,080} + \cdots\right)$$

$$\boxed{= b_0\left(x - \frac{1}{10}x^3 + \frac{1}{360}x^5 - \frac{1}{28,080}x^7 + \cdots\right)}$$

$$\boxed{y(x) = a_0\left(x^{1/2} - \frac{1}{6}x^{5/2} + \frac{1}{168}x^{9/2} - \frac{1}{11,088}x^{13/2} + \cdots\right) + b_0\left(x - \frac{1}{10}x^3 + \frac{1}{360}x^5 - \frac{1}{28,080}x^7 + \cdots\right)}$$

Exercise

Find the general solution to the equation $2xy'' + (1+x)y' + y = 0$

Solution

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$2xy'' + (1+x)y' + y = 0$$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + (1+x)\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty}c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty}c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)c_n x^{n-1}x^r + \sum_{n=0}^{\infty}(n+r)c_n x^{n-1}x^r + \sum_{n=0}^{\infty}(n+r)c_n x^n x^r + \sum_{n=0}^{\infty}c_n x^n x^r = 0$$

$$x^r\left(\sum_{n=0}^{\infty}(n+r)(2n+2r-2+1)c_n x^{n-1} + \sum_{n=0}^{\infty}(n+r+1)c_n x^n\right) = 0$$

$$x^r\left(\sum_{n=0}^{\infty}(n+r)(2n+2r-1)c_n x^{n-1} + \sum_{n=0}^{\infty}(n+r+1)c_n x^n\right) = 0$$

$$x^r\left(c_0 r(2r-1)x^{-1}+\underbrace{\sum_{n=1}^{\infty}c_n(n+r)(2n+2r-1)x^{n-1}}_{k=n-1}+\underbrace{\sum_{n=0}^{\infty}(n+r+1)c_n x^n}_{k=n}\right)=0$$

$$x^r\left(c_0 r(2r-1)x^{-1}+\sum_{k=0}^{\infty}c_{k+1}(r+k+1)(2k+2r+1)x^k+\sum_{k=0}^{\infty}c_k(r+k+1)x^k\right)=0$$

$$x^r\left(c_0 r(2r-1)x^{-1}+\sum_{k=0}^{\infty}\left[c_{k+1}(r+k+1)(2k+2r+1)+c_k(r+k+1)\right]x^k\right)=0$$

$$\begin{cases} c_0 r(2r-1)=0 & \Rightarrow \quad \boxed{r=0} \quad \boxed{r=\frac{1}{2}} \\ c_{k+1}(r+k+1)(2k+2r+1)+c_k(r+k)=0 & \Rightarrow \quad \boxed{c_{k+1}=-\frac{r+k+1}{(r+k+1)(2k+2r+1)}c_k} \end{cases}$$

$r=0$

$$c_{k+1}=-\frac{1}{2k+1}c_k$$

$$c_1=-\frac{1}{1}c_0$$

$$c_2=-\frac{1}{3}c_1=\frac{1}{3}c_0$$

$$c_3=-\frac{1}{5}c_2=-\frac{1}{1\cdot 3\cdot 5}c_0$$

$$c_4=-\frac{1}{7}c_3=\frac{1}{1\cdot 3\cdot 5\cdot 7}c_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$c_n=\frac{(-1)^n}{1\cdot 3\cdot 5\cdots(2n-1)}c_0$$

$r=\frac{1}{2}$

$$c_{k+1}=-\frac{k+\frac{3}{2}}{\left(k+\frac{3}{2}\right)(2k+2)}c_k=-\frac{1}{2(k+1)}c_k$$

$$c_1=-\frac{1}{2}c_0$$

$$c_2=-\frac{1}{2\cdot 2}c_1=\frac{1}{2\cdot 2\cdot 2}c_0$$

$$c_3=-\frac{1}{2\cdot 3}c_2=-\frac{1}{2^3(2\cdot 3)}c_0=-\frac{1}{2^3\cdot 3!}c_0$$

$$c_4=-\frac{1}{2\cdot 4}c_3=\frac{1}{2^4\cdot 4!}c_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$c_n=\frac{(-1)^n}{2^n n!}c_0$$

$$y_1(x)=c_0\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{1\cdot 3\cdot 5\cdots(2n-1)}x^n\right)$$

$$y_2(x)=c_0 x^{1/2}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n n!}x^n\right)$$

$$y(x)=C_1 y_1(x)+C_2 y_2(x)$$

$$=C_1\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{1\cdot 3\cdot 5\cdots(2n-1)}x^n\right)+C_2\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}x^{n+1/2}\right)$$

Exercise

Find the Frobenius series solutions of $xy''+2y'+xy=0$

Solution

$$\frac{1}{x}\left(xy''+2y'+xy=0\right)$$

$$y''+2\frac{1}{x}y'+\frac{x^2}{x^2}y=0$$

$\therefore\ x=0$ is a regular singular point with $p_0=2$ and $q_0=0$

The indicial equation is: $r(r-1)+2r=r(r+1)=0 \quad\to\quad \underline{r=0,\ -1}|$

The two possible Frobenius series solutions are then of the forms

$$y_1(x)=x^{1/2}\sum_{n=0}^{\infty}a_n x^n \quad \text{and} \quad y_2(x)=x^1\sum_{n=0}^{\infty}b_n x^n$$

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$xy''+2y'+xy=0$$

$$x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+x\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1}+\sum_{n=0}^{\infty}2(n+r)a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r+1}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)+2(n+r)\right]a_n x^{n+r-1}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r-1}=0$$

$$r(r+1)a_0x^{r-1}+(r+1)(r+2)a_1x^r+\sum_{n=2}^{\infty}(n+r)(n+r+1)a_n x^{n+r-1}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r-1}=0$$

For $n=0 \quad\to\quad r(r+1)a_0=0 \quad\Rightarrow\quad \underline{r=0\ \text{ or }\ r=-1}|$ $\surd$

For $n=1 \quad\to\quad (r+1)(r+2)a_1=0 \quad\Rightarrow\quad \underline{\cancel{r=-1,\ -2}}| \qquad \therefore\ \underline{a_1=0}|$

$$(n+r)(n+r+1)a_n+a_{n-2}=0$$

$$\boxed{a_n = -\frac{1}{(n+r)(n+r+1)}a_{n-2}}$$

$r=0 \quad \to \quad \boxed{a_n = -\frac{1}{n(n+1)}a_{n-2}}$

$n=2 \ \to \ a_2 = -\frac{1}{2\cdot 3}a_0$ $\qquad$ $n=3 \ \to \ a_3 = -\frac{1}{12}a_1 = 0$

$n=4 \ \to \ a_4 = -\frac{1}{4\cdot 5}a_2 = \frac{1}{5!}a_0$ $\qquad$ $n=5 \ \to \ a_5 = 0$

$n=6 \ \to \ a_6 = -\frac{1}{6\cdot 7}a_4 = -\frac{1}{7!}a_0$ $\qquad$ $\vdots \quad \vdots \quad \vdots \quad \vdots$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_1(x) = a_0\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right)$$

$$\boxed{= \frac{a_0}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)}$$

$r=-1 \quad \to \quad \boxed{b_n = -\frac{1}{n(n-1)}b_{n-2}}$

$n=2 \ \to \ b_2 = -\frac{1}{2\cdot 1}b_0$ $\qquad$ $n=3 \ \to \ b_3 = -\frac{1}{6}b_1 = 0$

$n=4 \ \to \ b_4 = -\frac{1}{4\cdot 3}b_2 = \frac{1}{4!}b_0$ $\qquad$ $n=5 \ \to \ b_5 = 0$

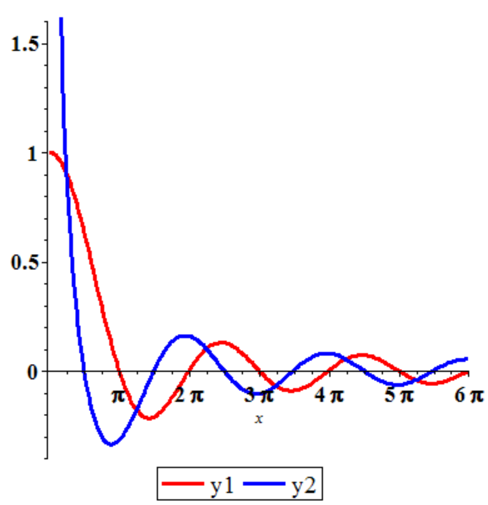
$n=6 \ \to \ b_6 = -\frac{1}{6\cdot 5}b_4 = -\frac{1}{6!}b_0$ $\qquad$ $\vdots \quad \vdots \quad \vdots \quad \vdots$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_2(x) = b_0 x^{-1}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)}$$

$$y(x) = \frac{a_0}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) + \frac{b_0}{x}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$\boxed{= a_0\frac{\sin x}{x} + b_0\frac{\cos x}{x}}$$



Exercise

Find the Frobenius series solutions of $2xy'' - y' + 2y = 0$

Solution

$$\left(\frac{x}{2}\right)2xy'' - \left(\frac{x}{2}\right)y' + \left(\frac{x}{2}\right)2y = 0$$

$$x^2 y'' - \frac{1}{2}xy' + xy = 0$$

That implies to $p(x) = -\frac{1}{2}$ and $q(x) = x$, both are analytic.

Hence, $x_0 = 0$ is a regular point

The indicial equation is: $r(r-1) - \frac{1}{2}r = r\left(r - \frac{3}{2}\right) = 0 \quad \to \quad \underline{r = 0,\ \frac{3}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{3/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2xy'' - y' + 2y = 0$$

$$2x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[2(n+r)(n+r-1) - (n+r)\right] a_n x^{n+r-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r-3)a_n + 2a_{n-1}\right] x^{n+r-1} = 0$$

$$(n+r)(2n+2r-3)a_n + 2a_{n-1} = 0$$

$$\boxed{a_n = -\frac{2}{(n+r)(2n+2r-3)}a_{n-1}}$$

$$r=0 \quad \to \quad \boxed{a_n = -\frac{2}{n(2n-3)}a_{n-1}}$$

$n=1 \ \to \ a_1 = 2a_0$

$n=2 \ \to \ a_2 = -a_1 = -2a_0$

$n=3 \ \to \ a_3 = -\frac{2}{9}a_2 = \frac{4}{9}a_0$

$n=4 \ \to \ a_4 = -\frac{1}{10}a_3 = -\frac{2}{45}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0\left(1+2x-2x^2+\frac{4}{9}x^3-\frac{2}{45}x^4+\cdots\right)}$$

$$r=\frac{3}{2} \quad \to \quad \boxed{b_n = -\frac{1}{n\left(n+\frac{3}{2}\right)}b_{n-1}}$$

$n=1 \ \to \ b_1 = -\frac{2}{5}b_0$

$n=2 \ \to \ b_2 = -\frac{1}{7}b_1 = \frac{2}{35}b_0$

$n=3 \ \to \ b_3 = -\frac{2}{27}b_2 = -\frac{4}{945}b_0$

$n=4 \ \to \ b_4 = -\frac{1}{22}b_3 = \frac{2}{20,790}b_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_2(x) = b_0x^{3/2}\left(1-\frac{2}{5}x+\frac{2}{35}x^2-\frac{4}{945}x^3+\frac{2}{20,790}x^4-\cdots\right)$$

$$= b_0\sqrt{x}\left(x-\frac{2}{5}x^2+\frac{2}{35}x^3-\frac{4}{945}x^4+\frac{2}{20,790}x^5-\cdots\right)$$

$$\boxed{y(x) = a_0\left(1+2x-2x^2+\frac{4}{9}x^3-\frac{2}{45}x^4+\cdots\right)+b_0\sqrt{x}\left(x-\frac{2}{5}x^2+\frac{2}{35}x^3-\frac{4}{945}x^4+\frac{2}{20,790}x^5-\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2xy''+5y'+xy=0$

Solution

$$\left(\frac{x}{2}\right)2xy''+5\left(\frac{x}{2}\right)y'+\left(\frac{x}{2}\right)xy=0$$

$$x^2y''+\frac{5}{2}xy'+\frac{1}{2}x^2y=0$$

That implies to $p(x)=\frac{5}{2}$ and $q(x)=\frac{1}{2}x^2$, both are analytic.

Hence, $x_0 = 0$ is a regular point

The indicial equation is: $r(r-1)+\frac{5}{2}r = r^2+\frac{3}{2}r = 0 \quad \to \quad \boxed{r = 0,\ -\frac{3}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{-3/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$2xy''+5y'+xy=0$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\sum_{n=0}^{\infty}5(n+r)a_n x^{n+r-1}+x\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r-1}+\sum_{n=0}^{\infty}5(n+r)a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r+1}=0$$

$$\sum_{n=0}^{\infty}\left[2(n+r)(n+r-1)+5(n+r)\right]a_n x^{n+r-1}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r-1}=0$$

$$r(2r+3)a_0+(r+1)(2r+5)a_1+\sum_{n=2}^{\infty}(n+r)(2n+2r+3)a_n x^{n+r-1}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r-1}=0$$

$$r(2r+3)a_0+(r+1)(2r+5)a_1+\sum_{n=2}^{\infty}\left[(n+r)(2n+2r+3)a_n+a_{n-2}\right]x^{n+r-1}=0$$

For $n=0 \quad \to \quad r(2r+3)a_0=0 \quad \Rightarrow \quad \boxed{r=0 \ \text{ or } \ r=-\frac{3}{2}}$ √

For $n=1 \quad \to \quad (r+1)(2r+5)a_1=0 \quad \Rightarrow \quad \boxed{\cancel{r=-1,\ -\frac{5}{2}}} \quad \to \quad \boxed{a_1=0}$

$$(n+r)(2n+2r+3)a_n+a_{n-2}=0$$

$$\boxed{a_n=-\frac{1}{(n+r)(2n+2r+3)}a_{n-2}}$$

$$r=0 \quad \to \quad a_n = -\frac{1}{n(2n+3)} a_{n-2}$$

$$n=2 \ \to \ a_2 = -\frac{1}{14} a_0 \qquad\qquad n=3 \ \to \ a_3 = -\frac{1}{27} a_1 = 0$$

$$n=4 \ \to \ a_4 = -\frac{1}{88} a_2 = \frac{1}{616} a_0 \qquad\qquad n=5 \ \to \ a_5 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_1(x) = a_0 \left(1 - \frac{1}{14}x^2 + \frac{1}{616}x^4 - \cdots\right)$$

$$r = -\frac{3}{2} \quad \to \quad b_n = -\frac{1}{2n\left(n-\frac{3}{2}\right)} b_{n-2} = -\frac{1}{n(2n-3)} b_{n-2}$$

$$n=2 \ \to \ b_2 = -\frac{1}{2} b_0 \qquad\qquad n=3 \ \to \ b_3 = -\frac{1}{9} b_1 = 0$$

$$n=4 \ \to \ b_4 = -\frac{1}{20} b_2 = \frac{1}{40} b_0 \qquad\qquad n=5 \ \to \ b_5 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{-3/2}\left(1 - \frac{1}{2}x^2 + \frac{1}{40}x^3 - \cdots\right)$$

$$= b_0 \left(x^{-3/2} - \frac{1}{2}x^{1/2} + \frac{1}{40}x^{3/2} - \cdots\right)$$

$$y(x) = a_0 \left(1 - \frac{1}{14}x^2 + \frac{1}{616}x^4 - \cdots\right) + b_0 \left(x^{-3/2} - \frac{1}{2}x^{1/2} + \frac{1}{40}x^{3/2} - \cdots\right)$$

Exercise

Find the Frobenius series solutions of $4xy'' + \frac{1}{2}y' + y = 0$

Solution

$$\left(\frac{x}{4}\right)4xy'' + \frac{1}{2}\left(\frac{x}{4}\right)y' + \left(\frac{x}{4}\right)y = 0$$

$$x^2y'' + \frac{1}{8}xy' + \frac{1}{4}x^2y = 0$$

$$y'' + \frac{1}{8x}y' + \frac{1}{4}y = 0$$

That implies to $p(x) = \frac{1}{8x}$ and $q(x) = \frac{1}{4}$

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{1}{8x} = \frac{1}{8}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \frac{1}{8} = 0$$

The indicial equation is: $r(r-1) + \frac{1}{8}r = r^2 - \frac{7}{8}r = 0 \quad \to \quad r = 0,\ \frac{7}{8}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{7/8} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$8xy'' + y' + 2y = 0$$

$$8x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + 2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 8(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[8(n+r)(n+r-1) + (n+r)\right] a_n x^{n+r-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1} = 0$$

$$r(8r-7)a_0 + \sum_{n=0}^{\infty} (n+r)(8n+8r-7) a_n x^{n+r-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1} = 0$$

$$r(8r-7)a_0 + \sum_{n=0}^{\infty} \left[(n+r)(8n+8r-7) a_n + 2a_{n-1}\right] x^{n+r-1} = 0$$

For $n = 0 \rightarrow r(8r-7)a_0 = 0 \Rightarrow \underline{r = 0, \ \frac{7}{8}}$ ✓

$$(n+r)(8n+8r-7)a_n + 2a_{n-1} = 0$$

$$\underline{a_n = -\frac{2}{(n+r)(8n+8r-7)} a_{n-1}}$$

$$r = 0 \rightarrow \underline{a_n = -\frac{2}{n(8n-7)} a_{n-1}}$$

$n = 1 \rightarrow a_1 = -2a_0$

$n = 2 \rightarrow a_2 = -\frac{1}{9} a_1 = \frac{2}{9} a_0$

$n = 3 \rightarrow a_3 = -\frac{2}{51} a_2 = -\frac{4}{459} a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0\left(1 - 2x + \frac{2}{9}x^2 - \frac{4}{459}x^3 + \cdots\right)}$$

$$r = \frac{7}{8} \quad \to \quad b_n = -\frac{2}{\left(n + \frac{7}{8}\right)(8n)} b_{n-1} = \boxed{-\frac{2}{n(8n+7)} b_{n-1}}$$

$$n = 1 \;\to\; b_1 = -\frac{2}{15} b_0$$

$$n = 2 \;\to\; b_2 = -\frac{1}{23} b_1 = \frac{2}{345} b_0$$

$$n = 3 \;\to\; b_3 = -\frac{2}{93} b_2 = -\frac{4}{32,085} b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{7/8}\left(1 - \frac{2}{15}x + \frac{2}{345}x^2 - \frac{4}{32,085}x^3 + \cdots\right)}$$

$$\boxed{y(x) = a_0\left(1 - 2x + \frac{2}{9}x^2 - \frac{4}{459}x^3 + \cdots\right) + b_0 x^{7/8}\left(1 - \frac{2}{15}x + \frac{2}{345}x^2 - \frac{4}{32,085}x^3 + \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2x^2y'' - xy' + \left(x^2 + 1\right)y = 0$

Solution

$$\frac{1}{2}2x^2y'' - \frac{1}{2}xy' + \frac{1}{2}\left(x^2 + 1\right)y = 0$$

$$x^2y'' - \frac{1}{2}xy' + \frac{1}{2}\left(x^2 + 1\right)y = 0$$

$$y'' - \frac{1}{2x}y' + \left(\frac{1}{2} + \frac{1}{2x^2}\right)y = 0$$

That implies to $p(x) = -\frac{1}{2x}$ and $q(x) = \frac{1}{2} + \frac{1}{2x^2}$.

$$p_0 = \lim_{x \to 0} xp(x) = -\lim_{x \to 0} x\frac{1}{2x} = \boxed{-\frac{1}{2}}$$

$$q_0 = \lim_{x \to 0} x^2q(x) = \lim_{x \to 0} x^2\left(\frac{1}{2} + \frac{1}{2x^2}\right) = \lim_{x \to 0}\left(\frac{1}{2}x^2 + \frac{1}{2}\right) = \boxed{\frac{1}{2}}$$

The indicial equation is: $r(r-1) - \frac{1}{2}r + \frac{1}{2} = r^2 - \frac{3}{2}r + \frac{1}{2} = 0 \quad \to \quad \boxed{r = 1,\ \frac{1}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^1\sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{1/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$2x^2y''-xy'+\left(x^2+1\right)y=0$$

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}-x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\left(x^2+1\right)\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r}-\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}+\sum_{n=0}^{\infty}a_n x^{n+r+2}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left(2(n+r)(n+r-1)-(n+r)+1\right)a_n x^{n+r}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r}$$

$$\left(r(2r-3)+1\right)a_0+\left((r+1)(2r-1)+1\right)a_1+\sum_{n=2}^{\infty}\left((n+r)(2n+2r-3)+1\right)a_n x^{n+r}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r}=0$$

$$\left(2r^2-3r+1\right)a_0+\left(2r^2+r\right)a_1+\sum_{n=2}^{\infty}\left[\left((n+r)(2n+2r-3)+1\right)a_n+a_{n-2}\right]x^{n+r}=0$$

For $n=0 \to \left(2r^2-3r+1\right)a_0=0 \Rightarrow \boxed{r=1,\ \frac{1}{2}}$ √

For $n=1 \to \left(2r^2+r\right)a_1=0 \Rightarrow \boxed{\cancel{r=0,\ -\frac{1}{2}}} \to \boxed{a_1=0}$

$$\left((n+r)(2n+2r-3)+1\right)a_n+a_{n-2}=0$$

$$\boxed{a_n=-\frac{1}{(n+r)(2n+2r-3)+1}a_{n-2}}$$

$r=1 \to a_n=-\frac{1}{(n+1)(2n-1)+1}a_{n-2}=\boxed{-\frac{1}{2n^2+n}a_{n-2}}$

$n=2 \to a_2=-\frac{1}{10}a_0$ $\qquad$ $n=3 \to a_3=-\frac{1}{21}a_1=0$

$n=4 \to a_4=-\frac{1}{36}a_2=\frac{1}{360}a_0$ $\qquad$ $n=5 \to a_5=0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_1(x) = a_0 x\left(1 - \frac{1}{10}x^2 + \frac{1}{360}x^4 - \cdots\right)$$

$$= a_0\left(x - \frac{1}{10}x^3 + \frac{1}{360}x^5 - \cdots\right)$$

$$r = \frac{1}{2} \quad \to \quad b_n = -\frac{1}{\left(n+\frac{1}{2}\right)(2n-2)+1} b_{n-2} = -\frac{1}{2n^2 - n} b_{n-2}$$

$$n = 2 \ \to \ b_2 = -\frac{1}{6}b_0 \qquad\qquad n = 3 \ \to \ a_3 = -\frac{1}{15}a_1 = 0$$

$$n = 4 \ \to \ b_4 = -\frac{1}{28}b_2 = \frac{1}{168}b_0 \qquad\qquad n = 5 \ \to \ a_5 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{1/2}\left(1 - \frac{1}{6}x^2 + \frac{1}{168}x^4 - \cdots\right)$$

$$y_1(x) = a_0\left(x - \frac{1}{10}x^3 + \frac{1}{360}x^5 - \cdots\right) + b_0 x^{1/2}\left(1 - \frac{1}{6}x^2 + \frac{1}{168}x^4 - \cdots\right)$$

Exercise

Find the Frobenius series solutions of $3xy'' + (2-x)y' - y = 0$

Solution

$$\frac{x}{3}3xy'' + \frac{x}{3}(2-x)y' - \frac{x}{3}y = 0$$

$$x^2y'' + \left(\frac{2}{3}x - \frac{1}{3}x^2\right)y' - \frac{x}{3}y = 0$$

$$y'' + \left(\frac{2}{3x} - \frac{1}{3}\right)y' - \frac{1}{3x}y = 0$$

That implies to $p(x) = \frac{2}{3x} - \frac{1}{3}$ and $q(x) = -\frac{1}{3x}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(\frac{2}{3x} - \frac{1}{3}\right) = \lim_{x\to 0}\left(\frac{2}{3} - \frac{1}{3}x\right) = \frac{2}{3}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = -\lim_{x\to 0} x^2 \frac{1}{3x} = \lim_{x\to 0}\frac{x}{3} = 0$$

The indicial equation is: $r(r-1) + \frac{2}{3}r = r^2 - \frac{1}{3}r = 0 \quad \to \quad r = 0, \ \frac{1}{3}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/3}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$3xy'' + (2-x)y' - y = 0$$

$$3x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + (2-x)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}3(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}2(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(3n+3r-3) + 2(n+r)\right]a_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r+1)a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(3n+3r-1)a_n x^{n+r-1} - \sum_{n=1}^{\infty}(n+r)a_{n-1} x^{n+r-1} = 0$$

$$r(3r-1)a_0 + \sum_{n=1}^{\infty}(n+r)(3n+3r-1)a_n x^{n+r-1} - \sum_{n=1}^{\infty}(n+r)a_{n-1} x^{n+r-1} = 0$$

$$r(3r-1)a_0 + \sum_{n=1}^{\infty}\left[(n+r)(3n+3r-1)a_n - (n+r)a_{n-1}\right]x^{n+r-1} = 0$$

For $n = 0 \quad \rightarrow \quad r(3r-1)a_0 = 0 \quad \Rightarrow \quad \boxed{r = 0, \ \frac{1}{3}}$ $\checkmark$

$$(n+r)(3n+3r-1)a_n - (n+r)a_{n-1} = 0$$

$$\boxed{a_n = \frac{1}{3n+3r-1}a_{n-1}}$$

$$r = 0 \quad \rightarrow \quad \boxed{a_n = \frac{1}{3n-1}a_{n-1}}$$

$$n = 1 \ \rightarrow \ a_1 = \frac{1}{2}a_0$$

$$n = 2 \ \rightarrow \ a_2 = \frac{1}{5}a_1 = \frac{1}{10}a_0$$

$$n = 3 \ \rightarrow \ a_3 = \frac{1}{8}a_2 = \frac{1}{80}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0\left(1+\frac{1}{2}x+\frac{1}{10}x^2+\frac{1}{80}x^3+\cdots\right)}$$

$$r=\frac{1}{3} \quad \to \quad \boxed{b_n = \frac{1}{3n}b_{n-1}}$$

$$n=1 \ \to \ b_1 = \frac{1}{3}b_0$$

$$n=2 \ \to \ b_2 = \frac{1}{6}b_1 = \frac{1}{18}b_0$$

$$n=3 \ \to \ b_3 = \frac{1}{9}b_2 = \frac{1}{162}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{1/3}\left(1+\frac{1}{3}x+\frac{1}{18}x^2+\frac{1}{162}x^3+\cdots\right)}$$

$$\boxed{y(x) = a_0\left(1+\frac{1}{2}x+\frac{1}{10}x^2+\frac{1}{80}x^3+\cdots\right)+b_0 x^{1/3}\left(1+\frac{1}{3}x+\frac{1}{18}x^2+\frac{1}{162}x^3+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2xy'' - (3+2x)y' + y = 0$

Solution

$$\frac{x}{2}2xy'' - \frac{x}{2}(3+2x)y' + \frac{x}{2}y = 0$$

$$x^2y'' - \left(\frac{3}{2}x + x^2\right)y' + \frac{1}{2}xy = 0$$

$$y'' - \left(\frac{3}{2x}+1\right)y' + \frac{1}{2x}y = 0$$

That implies to $p(x) = -\frac{3}{2x} - 1$ and $q(x) = \frac{1}{2x}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(-\frac{3}{2x}-1\right) = \lim_{x\to 0}\left(-\frac{3}{2}-x\right) \boxed{= -\frac{3}{2}}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\frac{1}{2x} = \lim_{x\to 0}\frac{x}{2} \boxed{=0}$$

The indicial equation is: $r(r-1) - \frac{3}{2}r = r^2 - \frac{5}{2}r = 0 \quad \to \quad \boxed{r = 0, \ \frac{5}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{5/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2xy'' - (3+2x) y' + y = 0$$

$$2x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} - (3+2x) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} - \sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[2(n+r)(n+r-1) - 3(n+r) \right] a_n x^{n+r-1} - \sum_{n=0}^{\infty} (2n+2r-1) a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(2n+2r-5) a_n x^{n+r-1} - \sum_{n=1}^{\infty} (2n+2r-3) a_{n-1} x^{n+r-1} = 0$$

$$r(2r-5) a_0 + \sum_{n=1}^{\infty} (n+r)(2n+2r-5) a_n x^{n+r-1} - \sum_{n=1}^{\infty} (2n+2r-3) a_{n-1} x^{n+r-1} = 0$$

$$r(2r-5) a_0 + \sum_{n=1}^{\infty} \left[(n+r)(2n+2r-5) a_n - (2n+2r-3) a_{n-1} \right] x^{n+r-1} = 0$$

For $n = 0 \quad \rightarrow \quad r(2r-5) a_0 = 0 \quad \Rightarrow \quad \boxed{r = 0, \ \frac{5}{2}}$ √

$$(n+r)(2n+2r-5) a_n - (2n+2r-3) a_{n-1} = 0$$

$$\boxed{a_n = \frac{2n+2r-3}{(n+r)(2n+2r-5)} a_{n-1}}$$

$$\boxed{r = 0 \quad \rightarrow \quad a_n = \frac{2n-3}{n(2n-5)} a_{n-1}}$$

$$n = 1 \rightarrow a_1 = \frac{1}{3} a_0$$

$$n = 2 \rightarrow a_2 = -\frac{1}{2} a_1 = -\frac{1}{6} a_0$$

$$n = 3 \rightarrow a_3 = -a_2 = -\frac{1}{6} a_0$$

$$n = 4 \rightarrow a_4 = \frac{5}{12} a_3 = -\frac{5}{72} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0\left(1+\frac{1}{3}x-\frac{1}{6}x^2-\frac{1}{6}x^3-\frac{5}{72}x^4-\cdots\right)}$$

$$r=\frac{5}{2} \quad\rightarrow\quad b_n = \frac{2n+2}{2n\left(n+\frac{5}{2}\right)}b_{n-1} \;=\boxed{\frac{2n+2}{n(2n+5)}b_{n-1}}$$

$$n=1 \;\rightarrow\; b_1 = \frac{4}{7}b_0$$

$$n=2 \;\rightarrow\; b_2 = \frac{1}{3}b_1 = \frac{4}{21}b_0$$

$$n=3 \;\rightarrow\; b_3 = \frac{8}{33}b_2 = \frac{32}{693}b_0$$

$$n=4 \;\rightarrow\; b_4 = \frac{5}{26}b_3 = \frac{80}{9,009}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = b_0x^{5/2}\left(1+\frac{4}{7}x+\frac{4}{21}x^2+\frac{32}{693}x^3+\frac{80}{9,009}x^4+\cdots\right)}$$

$$\boxed{y(x) = a_0\left(1+\frac{1}{3}x-\frac{1}{6}x^2-\frac{1}{6}x^3-\frac{5}{72}x^4-\cdots\right)+b_0x^{5/2}\left(1+\frac{4}{7}x+\frac{4}{21}x^2+\frac{32}{693}x^3+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $xy''+(x-6)y'-3y=0$

Solution

$$xxy''+x(x-6)y'-3xy=0$$

$$x^2y''+\left(x^2-6x\right)y'-3xy=0$$

$$y''+\left(1-\frac{6}{x}\right)y'-\frac{3}{x}y=0$$

That implies to $p(x)=1-\frac{6}{x}$ and $q(x)=-\frac{3}{x}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(1-\frac{6}{x}\right) = \lim_{x\to 0}(x-6) \;\underline{=-6}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = -\lim_{x\to 0} x^2\frac{3}{x} = -\lim_{x\to 0} 3x \;\underline{=0}$$

The indicial equation is: $r(r-1)-6r=r^2-7r=0 \quad\rightarrow\quad \underline{r=0,\ 7}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x)=x^0\sum_{n=0}^{\infty}a_nx^n \quad \textit{and} \quad y_2(x)=x^7\sum_{n=0}^{\infty}b_nx^n$$

$$y=\sum_{n=0}^{\infty}a_nx^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$xy'' + (x-6)y' - 3y = 0$$

$$x\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + (x-6)\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - 3\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} 6(n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} 3a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) - 6(n+r)\right] a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r-3) a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-7) a_n x^{n+r-1} + \sum_{n=1}^{\infty} (n+r-4) a_{n-1} x^{n+r-1} = 0$$

$$r(r-7)a_0 + \sum_{n=1}^{\infty} (n+r)(n+r-7) a_n x^{n+r-1} + \sum_{n=1}^{\infty} (n+r-4) a_{n-1} x^{n+r-1} = 0$$

$$r(r-7)a_0 + \sum_{n=1}^{\infty} \left[(n+r)(n+r-7) a_n + (n+r-4) a_{n-1}\right] x^{n+r-1} = 0$$

For $n=0 \rightarrow r(r-7)a_0 = 0 \Rightarrow \underline{r = 0,\ 7|}$ √

$$(n+r)(n+r-7)a_n + (n+r-4)a_{n-1} = 0$$

$$\boxed{a_n = -\frac{n+r-4}{(n+r)(n+r-7)} a_{n-1}}$$

$$\boxed{r = 0 \rightarrow a_n = -\frac{n-4}{n(n-7)} a_{n-1}}$$

$n = 1 \rightarrow a_1 = -\frac{1}{2} a_0$

$n = 2 \rightarrow a_2 = -\frac{1}{5} a_1 = \frac{1}{10} a_0$

$n = 3 \rightarrow a_3 = -\frac{1}{12} a_2 = -\frac{1}{120} a_0$

$n = 4 \rightarrow a_4 = 0 a_3 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0\left(1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3\right)}$$

$$\boxed{r = 7 \quad \to \quad b_n = -\frac{n+3}{n(n+7)}b_{n-1}}$$

$$n = 1 \;\to\; b_1 = -\frac{1}{2}b_0$$

$$n = 2 \;\to\; b_2 = -\frac{5}{18}b_1 = \frac{5}{36}b_0$$

$$n = 3 \;\to\; b_3 = -\frac{1}{5}b_2 = -\frac{1}{36}b_0$$

$$n = 4 \;\to\; b_4 = -\frac{7}{44}b_3 = \frac{7}{1,584}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^7\left(1 - \frac{1}{2}x + \frac{5}{36}x^2 - \frac{1}{36}x^3 + \frac{7}{1,584}x^4 - \cdots\right)}$$

$$\boxed{y(x) = a_0\left(1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3\right) + b_0 x^7\left(1 - \frac{1}{2}x + \frac{5}{36}x^2 - \frac{1}{36}x^3 + \frac{7}{1,584}x^4 - \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $x(x-1)y'' + 3y' - 2y = 0$

Solution

$$\frac{1}{x}x(x-1)y'' + 3\frac{1}{x}y' - 2\frac{1}{x}y = 0$$

$$(x-1)y'' + \frac{3}{x}y' - \frac{2}{x}y = 0$$

That implies to $p(x) = \frac{3}{x}$ and $q(x) = -\frac{2}{x}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{3}{x} \;\underline{= 3}|$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = -\lim_{x\to 0} x^2\frac{2}{x} = -\lim_{x\to 0} 2x \;\underline{= 0}|$$

The indicial equation is: $-r(r-1) + 3r = -r^2 + 4r = 0 \quad \to \quad \underline{r = 0,\ 4}|$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^4\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x(x-1)y'' + 3y' - 2y = 0$$

$$\left(x^2 - x\right) \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + 3\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1}$$

$$+ \sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) - 2\right] a_n x^{n+r} - \sum_{n=0}^{\infty} \left[(n+r)(n+r-1) - 3(n+r)\right] a_n x^{n+r-1} = 0$$

$$\sum_{n=1}^{\infty} \left[(n-1+r)(n+r-2) - 2\right] a_{n-1} x^{n+r-1} - \sum_{n=0}^{\infty} (n+r)(n+r-4) a_n x^{n+r-1} = 0$$

$$\sum_{n=1}^{\infty} \left[(n-1+r)(n+r-2) - 2\right] a_{n-1} x^{n+r-1} - r(r-4)a_0 - \sum_{n=1}^{\infty} (n+r)(n+r-4) a_n x^{n+r-1} = 0$$

$$-r(r-4)a_0 + \sum_{n=1}^{\infty} \left[\left((n+r-1)(n+r-2) - 2\right) a_{n-1} - (n+r)(n+r-4) a_n\right] x^{n+r-1} = 0$$

For $n = 0 \quad \rightarrow \quad -r(r-4)a_0 = 0 \quad \Rightarrow \quad \underline{r = 0, \quad 4|}$ √

$$\left((n+r-1)(n+r-2) - 2\right) a_{n-1} - (n+r)(n+r-4) a_n = 0$$

$$\boxed{a_n = \frac{(n+r-1)(n+r-2) - 2}{(n+r)(n+r-4)} a_{n-1}}$$

$$\boxed{r = 0 \quad \rightarrow \quad a_n = \frac{(n-1)(n-2) - 2}{n(n-4)} a_{n-1}}$$

$$n = 1 \ \rightarrow \ a_1 = \frac{2}{3} a_0$$

$$n = 2 \ \rightarrow \ a_2 = \frac{1}{2} a_1 = \frac{1}{3} a_0$$

$$n = 3 \;\rightarrow\; a_3 = \frac{0}{3}a_2 = 0$$

$$n = 4 \;\rightarrow\; a_4 = 0a_3 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0\left(1 + \frac{2}{3}x + \frac{1}{3}x^2\right)}$$

$$\boxed{r = 4 \quad \rightarrow \quad b_n = \frac{(n+3)(n+2)-2}{n(n+4)}b_{n-1}}$$

$$n = 1 \;\rightarrow\; b_1 = 2b_0$$

$$n = 2 \;\rightarrow\; b_2 = \frac{3}{2}b_1 = 3b_0$$

$$n = 3 \;\rightarrow\; b_3 = \frac{28}{21}b_2 = 4b_0$$

$$n = 4 \;\rightarrow\; b_4 = \frac{5}{4}b_3 = 5b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^4\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots\right)}$$

$$\boxed{y(x) = a_0\left(1 + \frac{2}{3}x + \frac{1}{3}x^2\right) + b_0\left(x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $x^2 y'' - \left(x - \frac{2}{9}\right)y = 0$

Solution

$$x^2 y'' - \left(x - \frac{2}{9}\right)y = 0$$

$$y'' - \left(\frac{1}{x} - \frac{2}{9x^2}\right)y = 0$$

That implies to $p(x) = 0$ and $q(x) = \frac{2}{9x^2} - \frac{1}{x}$.

$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2\left(\frac{2}{9x^2} - \frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{2}{9} - x\right) = \boxed{\frac{2}{9}}$$

The indicial equation is: $r(r-1) + \frac{2}{9} = r^2 - r + \frac{2}{9} = 0$

$$9r^2 - 9r + 2 = 0 \quad \rightarrow \quad r = \frac{9 \pm 3}{18} = \boxed{\frac{1}{3},\ \frac{2}{3}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/3}\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{2/3}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x^2 y'' - \left(x - \tfrac{2}{9}\right) y = 0$$

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} - \left(x - \tfrac{2}{9}\right)\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} + \sum_{n=0}^{\infty} \tfrac{2}{9} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) + \tfrac{2}{9}\right] a_n x^{n+r} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0$$

$$\left(r^2 - r + \tfrac{2}{9}\right) a_0 + \sum_{n=1}^{\infty} \left[(n+r)(n+r-1) + \tfrac{2}{9}\right] a_n x^{n+r} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0$$

$$\left(r^2 - r + \tfrac{2}{9}\right) a_0 + \sum_{n=1}^{\infty} \left[\left((n+r)(n+r-1) + \tfrac{2}{9}\right) a_n - a_{n-1}\right] x^{n+r} = 0$$

For $n = 0 \rightarrow \left(r^2 - r + \tfrac{2}{9}\right) a_0 = 0 \Rightarrow \boxed{r = \tfrac{1}{3},\ \tfrac{2}{3}}$ √

$$\left((n+r)(n+r-1) + \tfrac{2}{9}\right) a_n - a_{n-1} = 0$$

$$\boxed{a_n = \frac{1}{(n+r)(n+r-1) + \frac{2}{9}} a_{n-1}}$$

$$r = \frac{1}{3} \rightarrow \quad a_n = \frac{1}{\left(n + \frac{1}{3}\right)\left(n - \frac{2}{3}\right) + \frac{2}{9}} a_{n-1}$$

$$= \frac{1}{n^2 - \frac{1}{3} n} a_{n-1}$$

$$= \boxed{\frac{3}{3n^2 - n} a_{n-1}}$$

$n = 1 \;\to\; a_1 = \frac{3}{2} a_0$

$n = 2 \;\to\; a_2 = \frac{3}{10} a_1 = \frac{9}{20} a_0$

$n = 3 \;\to\; a_3 = \frac{1}{8} a_2 = \frac{9}{160} a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0 x^{1/3}\left(1 + \frac{3}{2}x + \frac{9}{20}x^2 + \frac{9}{160}x^3 + \cdots\right)}$$

$r = \frac{2}{3} \quad \to \quad b_n = \frac{1}{\left(n + \frac{2}{3}\right)\left(n - \frac{1}{3}\right) + \frac{2}{9}} b_{n-1}$

$$= \boxed{\frac{3}{3n^2 + n} b_{n-1}}$$

$n = 1 \;\to\; b_1 = \frac{3}{4} b_0$

$n = 2 \;\to\; b_2 = \frac{3}{14} b_1 = \frac{9}{56} b_0$

$n = 3 \;\to\; b_3 = \frac{1}{10} b_2 = \frac{9}{560} b_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_2(x) = b_0 x^{2/3}\left(1 + \frac{3}{4}x + \frac{9}{56}x^2 + \frac{9}{560}x^3 + \cdots\right)}$$

$$\boxed{y(x) = a_0 x^{1/3}\left(1 + \frac{3}{2}x + \frac{9}{20}x^2 + \frac{9}{160}x^3 + \cdots\right) + b_0 x^{2/3}\left(1 + \frac{3}{4}x + \frac{9}{56}x^2 + \frac{9}{560}x^3 + \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $x^2 y'' + x(3 + x) y' - 3y = 0$

Solution

$$\frac{1}{x^2} x^2 y'' + \frac{1}{x^2} x(3 + x) y' - 3\frac{1}{x^2} y = 0$$

$$y'' + \left(\frac{3}{x} + 1\right) y' - \frac{3}{x^2} y = 0$$

That implies to $p(x) = \frac{3}{x} + 1$ and $q(x) = -\frac{3}{x^2}$.

$$p_0 = \lim_{x \to 0} x p(x) = \lim_{x \to 0} x\left(\frac{3}{x} + 1\right) = \lim_{x \to 0} (3 + x) \; \underline{= 3|}$$

$$q_0 = \lim_{x \to 0} x^2 q(x) = -\lim_{x \to 0} x^2 \frac{3}{x^2} \; \underline{= -3|}$$

The indicial equation is: $r(r-1)+3r-3=r^2+2r-3=0 \quad \rightarrow \quad \underline{r=1,\ -3|}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x)=x\sum_{n=0}^{\infty}a_n x^n \quad \textit{and} \quad y_2(x)=x^{-3}\sum_{n=0}^{\infty}b_n x^n$$

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$x^2y''+x(3+x)y'-3y=0$$

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\left(3x+x^2\right)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}-3\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}+\sum_{n=0}^{\infty}3(n+r)a_n x^{n+r}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1}-\sum_{n=0}^{\infty}3a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)+3(n+r)-3\right]a_n x^{n+r}+\sum_{n=1}^{\infty}(n+r-1)a_{n-1}x^{n+r}=0$$

$$\left(r^2+2r-3\right)a_0+\sum_{n=1}^{\infty}\left[(n+r)(n+r+2)-3\right]a_n x^{n+r}+\sum_{n=1}^{\infty}(n+r-1)a_{n-1}x^{n+r}=0$$

$$\left(r^2+2r-3\right)a_0+\sum_{n=1}^{\infty}\left[\left((n+r)(n+r+2)-3\right)a_n+(n+r-1)a_{n-1}\right]x^{n+r}=0$$

For $n=0 \quad \rightarrow \quad \left(r^2+2r-3\right)a_0=0 \quad \Rightarrow \quad \underline{r=1,\ -3|}$ √

$$\left((n+r)(n+r+2)-3\right)a_n+(n+r-1)a_{n-1}=0$$

$$\boxed{a_n=-\frac{n+r-1}{(n+r)(n+r+2)-3}a_{n-1}}$$

$$r=1 \quad \rightarrow \quad a_n=-\frac{n}{(n+1)(n+3)-3}a_{n-1}$$

$$\boxed{=-\frac{n}{n^2+4n}a_{n-1}}$$

$$n=1 \;\rightarrow\; a_1 = -\tfrac{1}{5}a_0$$

$$n=2 \;\rightarrow\; a_2 = -\tfrac{2}{12}a_1 = \tfrac{1}{30}a_0$$

$$n=3 \;\rightarrow\; a_3 = -\tfrac{3}{21}a_2 = -\tfrac{1}{210}a_0$$

$$n=4 \;\rightarrow\; a_4 = -\tfrac{4}{32}a_3 = \tfrac{1}{1,680}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0 x\left(1-\tfrac{1}{5}x+\tfrac{1}{30}x^2-\tfrac{1}{210}x^3+\tfrac{1}{1,680}x^4-\cdots\right)}$$

$$r=-3 \;\rightarrow\; b_n = -\frac{n-4}{(n-3)(n-1)-3}b_{n-1}$$

$$\boxed{= -\frac{n-4}{n^2-4n}b_{n-1}}$$

$$n=1 \;\rightarrow\; b_1 = -b_0$$

$$n=2 \;\rightarrow\; b_2 = -\tfrac{2}{4}b_1 = \tfrac{1}{2}b_0$$

$$n=3 \;\rightarrow\; b_3 = -\tfrac{1}{3}b_2 = -\tfrac{1}{6}b_0$$

$$n=4 \;\rightarrow\; b_4 = 0b_3 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{-3}\left(1-x+\tfrac{1}{2}x^2-\tfrac{1}{6}x^3\right)}$$

$$\boxed{y(x) = a_0 x\left(1-\tfrac{1}{5}x+\tfrac{1}{30}x^2-\tfrac{1}{210}x^3+\tfrac{1}{1,680}x^4-\cdots\right)+b_0 x^{-3}\left(1-x+\tfrac{1}{2}x^2-\tfrac{1}{6}x^3\right)}$$

Exercise

Find the Frobenius series solutions of $x^2y''+\left(x^2-2x\right)y'+2y=0$

Solution

$$\frac{1}{x^2}x^2y''+\frac{1}{x^2}\left(x^2-2x\right)y'+2\frac{1}{x^2}y=0$$

$$y''+\left(1-\frac{2}{x}\right)y'+\frac{2}{x^2}y=0$$

That implies to $p(x)=1-\frac{2}{x}$ and $q(x)=\frac{2}{x^2}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(1-\frac{2}{x}\right) = \lim_{x\to 0}(x-2) \; \underline{=-2}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \frac{2}{x^2} = \underline{2|}$$

The indicial equation is: $r(r-1) - 2r + 2 = r^2 - 3r + 2 = 0 \quad \to \quad \underline{r = 1,\ 2|}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^2 \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x^2 y'' + \left(x^2 - 2x\right) y' + 2y = 0$$

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \left(x^2 - 2x\right) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r+1} - \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) - 2(n+r) + 2\right] a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-3) + 2\right] a_n x^{n+r} + \sum_{n=1}^{\infty} (n+r-1) a_{n-1} x^{n+r} = 0$$

$$\left(r^2 - 3r + 2\right) a_0 + \sum_{n=1}^{\infty} \left[(n+r)(n+r-3) + 2\right] a_n x^{n+r} + \sum_{n=1}^{\infty} (n+r-1) a_{n-1} x^{n+r} = 0$$

$$\left(r^2 - 3r + 2\right) a_0 + \sum_{n=1}^{\infty} \left[\left((n+r)(n+r-3) + 2\right) a_n + (n+r-1) a_{n-1}\right] x^{n+r} = 0$$

For $n = 0 \quad \to \quad \left(r^2 - 3r + 2\right) a_0 = 0 \quad \Rightarrow \quad \underline{r = 1,\ 2|} \quad \surd$

$$\left((n+r)(n+r-3) + 2\right) a_n + (n+r-1) a_{n-1} = 0$$

$$a_n = -\frac{n+r-1}{(n+r-1)(n+r-2)}a_{n-1} \underline{= -\frac{1}{n+r-2}a_{n-1}}$$

$$r=2 \quad \to \quad a_n == -\frac{1}{n}a_{n-1}$$

$$n=1 \ \to \ a_1 = -a_0$$

$$n=2 \ \to \ a_2 = -\frac{1}{2}a_1 = \frac{1}{2}a_0$$

$$n=3 \ \to \ a_3 = -\frac{1}{3}a_2 = -\frac{1}{3!}a_0$$

$$n=4 \ \to \ a_4 = -\frac{1}{4}a_3 = \frac{1}{4!}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 x\left(1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \cdots\right)}$$

$$r=1 \quad \to \quad b_n = -\frac{1}{n-1}b_{n-1}$$

Since $n \neq 1$

$$y(x) = a_0 x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \cdots\right) + b_0 x\ln x\left(-x + x^2 - \frac{x^3}{2} + \frac{x^4}{6} - \cdots\right)$$
$$+ x\ln x\left(1 - x + \frac{x^3}{4} - \frac{5}{36}x^4 + \cdots\right)$$

Exercise

Find the Frobenius series solutions of $x^2y'' + (x^2 + 2x)y' - 2y = 0$

Solution

$$\frac{1}{x^2}x^2y'' + \frac{1}{x^2}(x^2 + 2x)y' - 2\frac{1}{x^2}y = 0$$

$$y'' + \left(1 + \frac{2}{x}\right)y' - \frac{2}{x^2}y = 0$$

That implies to $p(x) = 1 + \frac{2}{x}$ and $q(x) = -\frac{2}{x^2}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(1 + \frac{2}{x}\right) = \lim_{x\to 0}(x+2) \ \underline{= 2}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = -\lim_{x\to 0} x^2\frac{2}{x^2} \ \underline{= -2}$$

The indicial equation is: $r(r-1) + 2r - 2 = r^2 + r - 2 = 0 \quad \to \quad \underline{r = 1, \ -2}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x\sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{-2}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x^2 y'' + \left(x^2 + 2x\right) y' - 2y = 0$$

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \left(x^2 + 2x\right)\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r+1} + \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r} - 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) + 2(n+r) - 2\right] a_n x^{n+r} + \sum_{n=1}^{\infty} (n+r-1) a_{n-1} x^{n+r} = 0$$

$$\left(r^2 + r - 2\right) a_0 + \sum_{n=1}^{\infty} \left[(n+r)(n+r+1) - 2\right] a_n x^{n+r} + \sum_{n=1}^{\infty} (n+r-1) a_{n-1} x^{n+r} = 0$$

$$\left(r^2 + r - 2\right) a_0 + \sum_{n=0}^{\infty} \left[\left((n+r)(n+r+1) - 2\right) a_n + (n+r-1) a_{n-1}\right] x^{n+r} = 0$$

For $n = 0 \quad \rightarrow \quad \left(r^2 + r - 2\right) a_0 = 0 \quad \Rightarrow \quad \underline{r = 1,\ 2} \quad \checkmark$

$$\left((n+r)^2 + (n+r) - 2\right) a_n + (n+r-1) a_{n-1} = 0$$

$$a_n = -\frac{n+r-1}{(n+r-1)(n+r+2)} a_{n-1} = -\frac{1}{n+r+2} a_{n-1}$$

$$r = 1 \quad \rightarrow \quad a_n == -\frac{1}{n+3} a_{n-1}$$

$$n = 1 \rightarrow a_1 = -\tfrac{1}{4} a_0$$

$$n = 2 \rightarrow a_2 = -\tfrac{1}{5} a_1 = \tfrac{1}{20} a_0$$

$$n = 3 \rightarrow a_3 = -\tfrac{1}{6} a_2 = -\tfrac{1}{120} a_0$$

$n=4 \rightarrow a_4 = -\frac{1}{7}a_3 = \frac{1}{840}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_1(x) = a_0 x\left(1 - \frac{1}{4}x + \frac{1}{20}x^2 - \frac{1}{120}x^3 + \frac{1}{840}x^4 - \cdots\right)$$

$r=2 \rightarrow b_n = = -\frac{1}{n+4}b_{n-1}$

$n=1 \rightarrow b_1 = -\frac{1}{5}b_0$

$n=2 \rightarrow b_2 = -\frac{1}{6}b_1 = \frac{1}{30}b_0$

$n=3 \rightarrow b_3 = -\frac{1}{7}b_2 = -\frac{1}{210}b_0$

$n=4 \rightarrow b_4 = -\frac{1}{8}b_3 = \frac{1}{1,680}b_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_2(x) = b_0 x^2\left(1 - \frac{x}{5} + \frac{x^2}{30} - \frac{x^3}{210} + \frac{x^4}{1,680} - \cdots\right)$$

$$y(x) = a_0 x\left(1 - \frac{x}{4} + \frac{x^2}{20} - \frac{x^3}{120} + \frac{x^4}{840} - \cdots\right) + b_0 x^2\left(1 - \frac{x}{5} + \frac{x^2}{30} - \frac{x^3}{210} + \frac{x^4}{1,680} - \cdots\right)$$

Exercise

Find the Frobenius series solutions of $2xy'' + 3y' - y = 0$

Solution

$\frac{x}{2}2xy'' + 3\frac{x}{2}y' - \frac{x}{2}y = 0$

$x^2y'' + \frac{3}{2}xy' - \frac{1}{2}xy = 0$

$y'' + \frac{3}{2x}y' - \frac{1}{2x}y = 0$

That implies to $p(x) = \frac{3}{2x}$ and $q(x) = -\frac{1}{2x}$

$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{3}{2x} = \frac{3}{2}$

$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\frac{1}{2x} = 0$

The indicial equation is: $r(r-1) + \frac{3}{2}r = r^2 + \frac{1}{2}r = 0 \quad \rightarrow \quad r = 0, \ -\frac{1}{2}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{-1/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y=\sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$2xy''+3y'-y=0$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\sum_{n=0}^{\infty}3(n+r)a_n x^{n+r-1}-\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r-1}+\sum_{n=0}^{\infty}3(n+r)a_n x^{n+r-1}-\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left[2(n+r)(n+r-1)a_n+3(n+r)\right]x^{n+r-1}-\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}=0$$

$$r(2r+1)a_0+\sum_{n=1}^{\infty}(n+r)(2n+2r+1)a_n x^{n+r-1}-\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}=0$$

$$r(2r+1)a_0+\sum_{n=1}^{\infty}\left[(n+r)(2n+2r+1)a_n-a_{n-1}\right]x^{n+r-1}=0$$

For $n=0 \quad \rightarrow \quad r(2r+1)a_0=0 \quad \Rightarrow \quad \boxed{r=0,\ -\frac{1}{2}}$ √

$$(n+r)(2n+2r+1)a_n-a_{n-1}=0$$

$$\boxed{a_n=\frac{1}{(n+r)(2n+2r+1)}a_{n-1}}$$

$$\boxed{r=0 \quad \rightarrow \quad a_n=\frac{1}{n(2n+1)}a_{n-1}}$$

$n=1 \ \rightarrow \ a_1=\frac{1}{3}a_0$

$n=2 \ \rightarrow \ a_2=\frac{1}{15}a_1=\frac{1}{30}a_0$

$n=3 \ \rightarrow \ a_3=\frac{1}{21}a_2=\frac{1}{630}a_0$

$n=4 \ \rightarrow \ a_4=\frac{1}{36}a_3=\frac{1}{22,680}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0\left(1+\frac{1}{3}x+\frac{1}{30}x^2+\frac{1}{630}x^3+\frac{1}{22,680}x^4+\cdots\right)}$$

$$\boxed{r=-\frac{1}{2} \quad \rightarrow \quad b_n = \frac{1}{n(2n-1)}b_{n-1}}$$

$$n=1 \ \rightarrow \ b_1 = b_0$$

$$n=2 \ \rightarrow \ b_2 = \frac{1}{6}b_1 = \frac{1}{6}b_0$$

$$n=3 \ \rightarrow \ b_3 = \frac{1}{15}b_2 = \frac{1}{90}b_0$$

$$n=4 \ \rightarrow \ b_4 = \frac{1}{28}b_3 = \frac{1}{2,520}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0x^{-1/2}\left(1+x+\frac{1}{6}x^2+\frac{1}{90}x^3+\frac{1}{2,520}x^4+\cdots\right)}$$

$$\boxed{y(x) = a_0\left(1+\frac{1}{3}x+\frac{1}{30}x^2+\frac{1}{630}x^3+\frac{1}{22,680}x^4+\cdots\right)+b_0x^{-1/2}\left(1+x+\frac{1}{6}x^2+\frac{1}{90}x^3+\frac{1}{2,520}x^4+\cdots\right)}$$

$$\boxed{y(x) = a_0\sum_{n=0}^{\infty}\frac{x^n}{n!(2n+1)!!}+\frac{b_0}{\sqrt{x}}\left(1+\sum_{n=0}^{\infty}\frac{x^n}{n!(2n-1)!!}\right)}$$

Exercise

Find the Frobenius series solutions of $2xy'' - y' - y = 0$

Solution

$$\frac{1}{2x}2xy'' - \frac{1}{2x}y' - \frac{1}{2x}y = 0$$

$$y'' - \frac{1}{2x}y' - \frac{1}{2x}y = 0$$

That implies to $p(x) = -\frac{1}{2x}$ and $q(x) = \frac{1}{2x}$

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{-1}{2x} = -\frac{1}{2}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\frac{-1}{2x} = -\lim_{x\to 0}\frac{x}{2} = 0$$

The indicial equation is: $r(r-1)-\frac{1}{2}r = r^2 - \frac{3}{2}r = 0 \quad \rightarrow \quad \boxed{r = 0, \ \frac{3}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty}a_nx^n \quad \textit{and} \quad y_2(x) = x^{3/2}\sum_{n=0}^{\infty}b_nx^n$$

$$y=\sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$2xy''-y'-y=0$

$$2x\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}-\sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}-\sum_{n=0}^{\infty} a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r-1}-\sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}-\sum_{n=0}^{\infty} a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty} \left[2(n+r)(n+r-1)a_n-(n+r)\right]x^{n+r-1}-\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}=0$$

$$r(2r-3)a_0+\sum_{n=1}^{\infty} (n+r)(2n+2r-3)a_n x^{n+r-1}-\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}=0$$

$$r(2r-3)a_0+\sum_{n=1}^{\infty} \left[(n+r)(2n+2r-3)a_n-a_{n-1}\right]x^{n+r-1}=0$$

For $n=0 \quad \rightarrow \quad r(2r-3)a_0=0 \quad \Rightarrow \quad \boxed{r=0,\ \frac{3}{2}}$ √

$$(n+r)(2n+2r-3)a_n-a_{n-1}=0$$

$$\boxed{a_n=\frac{1}{(n+r)(2n+2r-3)}a_{n-1}}$$

$$\boxed{r=0 \quad \rightarrow \quad a_n=\frac{1}{n(2n-3)}a_{n-1}}$$

$n=1 \ \rightarrow \ a_1=-a_0$

$n=2 \ \rightarrow \ a_2=\frac{1}{2}a_1=-\frac{1}{2}a_0$

$n=3 \ \rightarrow \ a_3=\frac{1}{9}a_2=-\frac{1}{18}a_0$

$n=4 \ \rightarrow \ a_4=\frac{1}{20}a_3=-\frac{1}{360}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_1(x) = a_0\left(1 - x - \frac{1}{2}x^2 - \frac{1}{18}x^3 - \frac{1}{360}x^4 - \cdots\right)$$

$$r = \frac{3}{2} \quad \rightarrow \quad b_n = \frac{1}{n(2n+3)}b_{n-1}$$

$$n = 1 \ \rightarrow \ b_1 = \frac{1}{5}b_0$$

$$n = 2 \ \rightarrow \ b_2 = \frac{1}{14}b_1 = \frac{1}{70}b_0$$

$$n = 3 \ \rightarrow \ b_3 = \frac{1}{27}b_2 = \frac{1}{1890}b_0$$

$$n = 4 \ \rightarrow \ b_4 = \frac{1}{44}b_3 = \frac{1}{83{,}160}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{3/2}\left(1 + \frac{1}{5}x + \frac{1}{70}x^2 + \frac{1}{1890}x^3 + \frac{1}{83{,}160}x^4 + \cdots\right)$$

$$y(x) = a_0\left(1 - x - \frac{1}{2}x^2 - \frac{1}{18}x^3 - \frac{1}{360}x^4 + \cdots\right) + b_0 x^{3/2}\left(1 + \frac{1}{5}x + \frac{1}{70}x^2 + \frac{1}{1{,}890}x^3 + \frac{1}{83{,}160}x^4 - \cdots\right)$$

Exercise

Find the Frobenius series solutions of $2xy'' + (1+x)y' + y = 0$

Solution

$$\frac{1}{2x}2xy'' + \frac{1}{2x}(1+x)y' + \frac{1}{2x}y = 0$$

$$y'' + \left(\frac{1}{2x} + \frac{1}{2}\right)y' + \frac{1}{2x}y = 0$$

That implies to $p(x) = \frac{1}{2x} + \frac{1}{2}$ and $q(x) = \frac{1}{2x}$

$$p_0 = \lim_{x \to 0} xp(x) = \lim_{x \to 0} x\left(\frac{1}{2x} + \frac{1}{2}\right) = \lim_{x \to 0}\left(\frac{1}{2} + \frac{1}{2}x\right) = \frac{1}{2}$$

$$q_0 = \lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} x^2 \frac{1}{2x} = \lim_{x \to 0} \frac{x}{2} = 0$$

The indicial equation is: $r(r-1) + \frac{1}{2}r = r^2 - \frac{1}{2}r = 0 \quad \rightarrow \quad r = 0, \ \frac{1}{2}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$2xy'' + (1+x)y' + y = 0$$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + (1+x)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[2(n+r)(n+r-1)+(n+r)\right]a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r+1)a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-1)\right]a_n x^{n+r-1} + \sum_{n=1}^{\infty}(n+r)a_{n-1} x^{n+r-1} = 0$$

$$r(2r-1)a_0 + \sum_{n=1}^{\infty}(n+r)(2n+2r-1)a_n x^{n+r-1} + \sum_{n=1}^{\infty}(n+r)a_{n-1} x^{n+r-1} = 0$$

$$r(2r-1)a_0 + \sum_{n=1}^{\infty}\left[(n+r)(2n+2r-1)a_n + (n+r)a_{n-1}\right]x^{n+r-1} = 0$$

For $n=0 \quad \rightarrow \quad r(2r-1)a_0 = 0 \quad \Rightarrow \quad \underline{r = 0,\ \tfrac{1}{2}}\ \surd$

$$(n+r)(2n+2r-1)a_n + (n+r)a_{n-1} = 0$$

$$\boxed{a_n = -\frac{1}{2n+2r-1}a_{n-1}}$$

$$\boxed{r = 0 \quad \rightarrow \quad a_n = -\frac{1}{2n-1}a_{n-1}}$$

$n=1 \ \rightarrow \ a_1 = -a_0$

$n=2 \ \rightarrow \ a_2 = \frac{1}{3}a_1 = -\frac{1}{3}a_0$

$n=3 \ \rightarrow \ a_3 = -\frac{1}{5}a_2 = \frac{1}{15}a_0$

$n=4 \ \rightarrow \ a_4 = -\frac{1}{7}a_3 = -\frac{1}{105}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0\left(1 - x + \frac{1}{3}x^2 - \frac{1}{15}x^3 + \frac{1}{105}x^4 - \cdots\right)}$$

$$\boxed{r = \frac{1}{2} \quad \to \quad b_n = -\frac{1}{2n}b_{n-1}}$$

$$n = 1 \ \to \ b_1 = -\frac{1}{2}b_0$$

$$n = 2 \ \to \ b_2 = -\frac{1}{4}b_1 = \frac{1}{8}b_0$$

$$n = 3 \ \to \ b_3 = -\frac{1}{6}b_2 = -\frac{1}{48}b_0$$

$$n = 4 \ \to \ b_4 = -\frac{1}{8}b_3 = -\frac{1}{384}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{1/2}\left(1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4 + \cdots\right)}$$

$$\boxed{y(x) = a_0\left(1 - x + \frac{1}{3}x^2 - \frac{1}{15}x^3 + \frac{1}{105}x^4 - \cdots\right) + b_0 x^{1/2}\left(1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4 + \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2xy'' + \left(1 - 2x^2\right)y' - 4xy = 0$

Solution

$$\frac{x}{2}2xy'' + \frac{x}{2}\left(1 - 2x^2\right)y' - \frac{x}{2}4xy = 0$$

$$x^2y'' + \left(\frac{1}{2}x - x^3\right)y' + 2x^2y = 0$$

$$y'' + \left(\frac{1}{2x} - x\right)y' + 2y = 0$$

That implies to $p(x) = \frac{1}{2x} - x$ and $q(x) = 2$

$$p_0 = \lim_{x \to 0} xp(x) = \lim_{x \to 0} x\left(\frac{1}{2x} - x\right) = \lim_{x \to 0}\left(\frac{1}{2} - x^2\right) = \frac{1}{2}$$

$$q_0 = \lim_{x \to 0} x^2q(x) = \lim_{x \to 0} 2x^2 = 0$$

The indicial equation is: $r(r-1) + \frac{1}{2}r = r^2 - \frac{1}{2}r = 0 \quad \to \quad \boxed{r = 0, \ \frac{1}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{1/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$2xy''+\left(1-2x^2\right)y'-4xy=0$$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\left(1-2x^2\right)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}-4x\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r-1}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}-\sum_{n=0}^{\infty}2(n+r)a_n x^{n+r+1}-\sum_{n=0}^{\infty}4a_n x^{n+r+1}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-2)+(n+r)\right]a_n x^{n+r-1}-\sum_{n=0}^{\infty}(2n+2r+4)a_n x^{n+r+1}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-2)+(n+r)\right]a_n x^{n+r-1}-\sum_{n=2}^{\infty}(2n+2r)a_{n-2} x^{n+r-1}=0$$

$$r(2r-1)a_0+(r+2)(2r+1)a_1+\sum_{n=2}^{\infty}(n+r)(2n+2r-1)a_n x^{n+r-1}-\sum_{n=2}^{\infty}(2n+2r)a_{n-2} x^{n+r-1}=0$$

$$r(2r-1)a_0+(r+2)(2r+1)a_1+\sum_{n=2}^{\infty}\left[(n+r)(2n+2r-1)a_n-2(n+r)a_{n-2}\right]x^{n+r-1}=0$$

For $n=0 \quad\to\quad r(2r+1)a_0=0 \quad\Rightarrow\quad \boxed{r=0,\ -\tfrac{1}{2}}$ √

For $n=1 \quad\to\quad (r+2)(2r+1)a_1=0 \quad\Rightarrow\quad \boxed{\cancel{r=-2,\ -\tfrac{1}{2}}} \qquad \boxed{a_1=0}$

$$(n+r)(2n+2r-1)a_n-2(n+r)a_{n-2}=0$$

$$\boxed{a_n=\frac{2}{2n+2r-1}a_{n-2}}$$

$$\boxed{r=0 \quad\to\quad a_n=\frac{2}{2n-1}a_{n-2}}$$

$n=2 \ \to\ a_2=\frac{2}{3}a_0 \qquad\qquad n=3 \ \to\ a_3=\frac{2}{5}a_1=0$

$n=4 \ \to\ a_4=\frac{2}{7}a_2=\frac{4}{21}a_0 \qquad\qquad n=5 \ \to\ a_5=\frac{2}{9}a_3=0$

$n=6 \to a_6 = \frac{2}{11}a_4 = \frac{8}{231}a_0$ $\quad n=7 \to a_7 = 0$

$n=8 \to a_8 = \frac{2}{15}a_6 = \frac{16}{3,465}a_0$ $\quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0\left(1+\frac{2}{3}x^2+\frac{4}{21}x^4+\frac{8}{231}x^6+\frac{16}{3465}x^8+\cdots\right)}$$

$$\boxed{r=\frac{1}{2} \quad \to \quad b_n = \frac{1}{n}b_{n-2}}$$

$n=2 \to b_2 = \frac{1}{2}b_0$ $\quad n=3 \to b_3 = \frac{1}{3}b_1 = 0$

$n=4 \to b_4 = \frac{1}{4}b_2 = \frac{1}{8}b_0$ $\quad n=5 \to b_5 = \frac{1}{5}b_3 = 0$

$n=6 \to b_6 = \frac{1}{6}b_4 = \frac{1}{48}b_0$ $\quad n=7 \to b_7 = 0$

$n=8 \to b_8 = \frac{1}{8}b_6 = \frac{1}{384}b_0$ $\quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_2(x) = b_0x^{1/2}\left(1+\frac{1}{2}x^2+\frac{1}{8}x^4+\frac{1}{48}x^6+\frac{1}{384}x^8+\cdots\right)}$$

$$\boxed{y(x) = a_0\left(1+\frac{2}{3}x^2+\frac{4}{21}x^4+\frac{8}{231}x^6+\frac{16}{3465}x^8+\cdots\right)+b_0x^{1/2}\left(1+\frac{1}{2}x^2+\frac{1}{8}x^4+\frac{1}{48}x^6+\frac{1}{384}x^8+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2x^2y''+xy'-\left(1+2x^2\right)y=0$

Solution

$$\frac{1}{2}2x^2y''+\frac{1}{2}xy'-\frac{1}{2}\left(1+2x^2\right)y=0$$

$$x^2y''+\frac{1}{2}xy'-\frac{1}{2}\left(1+2x^2\right)y=0$$

$$y''+\frac{1}{2x}y'-\left(\frac{1}{2x^2}+1\right)y=0$$

That implies to $p(x)=\frac{1}{2x}$ and $q(x)=-\frac{1}{2x^2}-1$

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{1}{2x} = \lim_{x\to 0}\frac{1}{2} = \frac{1}{2}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\left(-\frac{1}{2x^2}-1\right) = \lim_{x\to 0} x^2\left(-\frac{1}{2}-x^2\right) = -\frac{1}{2}$$

The indicial equation is: $r(r-1)+\frac{1}{2}r-\frac{1}{2}=r^2-\frac{1}{2}r-\frac{1}{2}=0 \quad \to \quad \boxed{r=1,\ -\frac{1}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^1 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{-1/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x^2 y'' + xy' - \left(1+2x^2\right) y = 0$$

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \left(1+2x^2\right) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} 2a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r-2)+(n+r)-1\right] a_n x^{n+r} - \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r} = 0$$

$$\left(2r^2-r-1\right)a_0 + \left((r+1)(2r+1)-1\right)a_1 + \sum_{n=2}^{\infty} \left[(n+r)(2n+2r-1)-1\right] a_n x^{n+r} - \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r} = 0$$

$$\left(2r^2-r-1\right)a_0 + \left(2r^2+3r\right)a_1 + \sum_{n=2}^{\infty} \left[\left((n+r)(2n+2r-1)-1\right)a_n - 2a_{n-2}\right] x^{n+r} = 0$$

For $n=0 \quad \rightarrow \quad \left(2r^2-r-1\right)a_0 = 0 \quad \Rightarrow \quad \boxed{r = 1, \ -\frac{1}{2}}$ √

For $n=1 \quad \rightarrow \quad r(2r+3)a_1 = 0 \quad \Rightarrow \quad \cancel{r = 0, \ -\frac{3}{2}} \qquad \boxed{a_1 = 0}$

$$\left((n+r)(2n+2r-1)-1\right)a_n - 2a_{n-2} = 0$$

$$\boxed{a_n = \frac{2}{(n+r)(2n+2r-1)-1} a_{n-2}}$$

$$r = 1 \quad \rightarrow \quad \boxed{a_n = \frac{2}{(n+1)(2n+1)-1} a_{n-2} = \frac{2}{2n^2+3n} a_{n-2}}$$

$n=2 \;\rightarrow\; a_2 = \frac{2}{14}a_0 = \frac{1}{7}a_0$ $\qquad$ $n=3 \;\rightarrow\; a_3 = \frac{2}{27}a_1 = 0$

$n=4 \;\rightarrow\; a_4 = \frac{2}{44}a_2 = \frac{1}{154}a_0$ $\qquad$ $n=5 \;\rightarrow\; a_5 = \frac{2}{65}a_3 = 0$

$n=6 \;\rightarrow\; a_6 = \frac{2}{90}a_4 = \frac{1}{6,390}a_0$ $\qquad$ $n=7 \;\rightarrow\; a_7 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0 x\left(1+\frac{1}{7}x^2+\frac{1}{154}x^4+\frac{1}{6,930}x^6+\cdots\right)}$$

$$\boxed{r=-\frac{1}{2} \;\rightarrow\; b_n = \frac{2b_{n-2}}{\left(n-\frac{1}{2}\right)(2n-2)-1} = \frac{2}{2n^2-3n}b_{n-2}}$$

$n=2 \;\rightarrow\; b_2 = \frac{2}{2}b_0 = b_0$ $\qquad$ $n=3 \;\rightarrow\; b_3 = \frac{2}{9}b_1 = 0$

$n=4 \;\rightarrow\; b_4 = \frac{2}{20}b_2 = \frac{1}{10}b_0$ $\qquad$ $n=5 \;\rightarrow\; b_5 = \frac{2}{35}b_3 = 0$

$n=6 \;\rightarrow\; b_6 = \frac{2}{54}b_4 = \frac{1}{270}b_0$ $\qquad$ $n=7 \;\rightarrow\; b_7 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_2(x) = b_0 x^{-1/2}\left(1+x^2+\frac{1}{10}x^4+\frac{1}{270}x^6+\cdots\right)}$$

$$\boxed{y(x) = a_0 x\left(1+\frac{1}{7}x^2+\frac{1}{154}x^4+\frac{1}{6,930}x^6+\cdots\right)+b_0 x^{-1/2}\left(1+x^2+\frac{1}{10}x^4+\frac{1}{270}x^6+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2x^2y''+xy'-\left(3-2x^2\right)y=0$

Solution

$$\frac{1}{2x^2}2x^2y''+\frac{1}{2x^2}xy'-\frac{1}{2x^2}\left(3-2x^2\right)y=0$$

$$y''+\frac{1}{2x}y'-\frac{1}{2x^2}\left(3-2x^2\right)y=0$$

That implies to $p(x)=\frac{1}{2x}$ and $q(x)=-\frac{1}{2x^2}\left(3-2x^2\right)$

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{1}{2x} \;\boxed{=\frac{1}{2}}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = -\lim_{x\to 0} x^2\left(\frac{1}{2x^2}\left(3-2x^2\right)\right) = -\lim_{x\to 0}\left(\frac{1}{2}\left(3-2x^2\right)\right) \;\boxed{=\frac{3}{2}}$$

The indicial equation is: $r(r-1)+\frac{1}{2}r-\frac{3}{2} = r^2-\frac{1}{2}r-\frac{3}{2}=0 \;\rightarrow\; \boxed{r=-1,\ \frac{3}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{-1}\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{3/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x^2 y'' + xy' - \left(3 - 2x^2\right) y = 0$$

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + x\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \left(3 - 2x^2\right)\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} 3a_n x^{n+r} + \sum_{n=0}^{\infty} 2a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r-2) + (n+r) - 3\right] a_n x^{n+r} + \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r} = 0$$

$$\left(2r^2 - r - 3\right) a_0 + \left((r+1)(2r+1) - 3\right) a_1 + \sum_{n=2}^{\infty} \left[(n+r)(2n+2r-1) - 3\right] a_n x^{n+r} + \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r} = 0$$

$$\left(2r^2 - r - 3\right) a_0 + \left(2r^2 + 3r - 2\right) a_1 + \sum_{n=2}^{\infty} \left[\left((n+r)(2n+2r-1) - 3\right) a_n + 2a_{n-2}\right] x^{n+r} = 0$$

For $n = 0 \quad \rightarrow \quad \left(2r^2 - r - 3\right) a_0 = 0 \quad \Rightarrow \quad \boxed{r = -1, \ \frac{3}{2}}$ √

For $n = 1 \quad \rightarrow \quad \left(2r^2 + 3r - 2\right) a_1 = 0 \quad \Rightarrow \quad \boxed{r = \cancel{-2, \ \frac{1}{2}}} \qquad \boxed{a_1 = 0}$

$$\left((n+r)(2n+2r-1) - 3\right) a_n + 2a_{n-2} = 0$$

$$\boxed{a_n = -\frac{2}{(n+r)(2n+2r-1) - 3} a_{n-2}}$$

$$\boxed{r = -1 \quad \rightarrow \quad a_n = -\frac{2}{(n-1)(2n-3) - 3} a_{n-2} = -\frac{2}{2n^2 - 5n} a_{n-2}}$$

$n = 2 \rightarrow a_2 = a_0 \qquad\qquad n = 3 \rightarrow a_3 = -\frac{2}{3} a_1 = 0$

$$n=4 \rightarrow a_4 = -\frac{2}{12}a_2 = -\frac{1}{6}a_0 \qquad n=5 \rightarrow a_5 = -\frac{2}{25}a_3 = 0$$

$$n=6 \rightarrow a_6 = -\frac{2}{252}a_4 = \frac{1}{126}a_0 \qquad n=7 \rightarrow a_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0 x^{-1}\left(1 + x^2 - \frac{1}{6}x^4 + \frac{1}{126}x^6 - \cdots\right)}$$

$$\boxed{r = \frac{3}{2} \quad \rightarrow \quad b_n = -\frac{2b_{n-2}}{\left(n+\frac{3}{2}\right)(2n+2)-3} = -\frac{2}{2n^2+5n}b_{n-2}}$$

$$n=2 \rightarrow b_2 = -\frac{2}{18}b_0 = -\frac{1}{9}b_0 \qquad n=3 \rightarrow b_3 = -\frac{2}{33}b_1 = 0$$

$$n=4 \rightarrow b_4 = -\frac{2}{568}b_2 = \frac{1}{234}b_0 \qquad n=5 \rightarrow b_5 = 0$$

$$n=6 \rightarrow b_6 = \frac{2}{102}b_4 = \frac{1}{11{,}934}b_0 \qquad n=7 \rightarrow b_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{3/2}\left(1 - \frac{1}{9}x^2 + \frac{1}{234}x^4 - \frac{1}{11{,}934}x^6 + \cdots\right)}$$

$$\boxed{y(x) = a_0 x^{-1}\left(1 + x^2 - \frac{1}{6}x^4 + \frac{1}{126}x^6 - \cdots\right) + b_0 x^{3/2}\left(1 - \frac{1}{9}x^2 + \frac{1}{234}x^4 - \frac{1}{11{,}934}x^6 + \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $3xy'' + 2y' + 2y = 0$

Solution

$$\frac{x}{3}3xy'' + 2\frac{x}{3}y' + 2\frac{x}{3}y = 0$$

$$x^2y'' + \frac{2}{3}xy' + \frac{2}{3}xy = 0$$

$$y'' + \frac{2}{3x}y' + \frac{2}{3x}y = 0$$

That implies to $p(x) = \frac{2}{3x}$ and $q(x) = \frac{2}{3x}$

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{2}{3x} = \boxed{\frac{2}{3}}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\frac{2}{3x} = \lim_{x\to 0} \frac{2}{3}x = \boxed{0}$$

The indicial equation is: $r(r-1) + \frac{2}{3}r = r^2 - \frac{1}{3}r = 0 \quad \rightarrow \quad \boxed{r = 0, \ \frac{1}{3}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/3} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$3xy'' + 2y' + 2y = 0$$

$$3x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + 2 \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + 2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 3(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(3n+3r-3) + 2(n+r)\right] a_n x^{n+r-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1} = 0$$

$$r(3r-1)a_n + \sum_{n=1}^{\infty} (n+r)(3n+3r-1) a_n x^{n+r-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1} = 0$$

$$r(3r-1)a_n + \sum_{n=1}^{\infty} \left[(n+r)(3n+3r-1) a_n + 2a_{n-1}\right] x^{n+r-1} = 0$$

For $n = 0 \;\rightarrow\; r(3r-1)a_0 = 0 \;\Rightarrow\; \boxed{r = 0,\ \frac{1}{3}}$ √

$$(n+r)(3n+3r-1)a_n + 2a_{n-1} = 0$$

$$\boxed{a_n = -\frac{2}{(n+r)(3n+3r-1)} a_{n-1}}$$

$$\boxed{r = 0 \;\rightarrow\; a_n = -\frac{2}{3n^2 - n} a_{n-1}}$$

$n = 1 \;\rightarrow\; a_1 = -a_0$

$n = 2 \;\rightarrow\; a_2 = -\frac{1}{5} a_1 = \frac{1}{5} a_0$

$n = 3 \;\rightarrow\; a_3 = -\frac{2}{24} a_2 = -\frac{1}{60} a_0$

$$n=4 \rightarrow a_4 = -\frac{2}{44}a_3 = \frac{1}{1320}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0 x^0\left(1 - x + \frac{1}{5}x^2 - \frac{1}{60}x^3 + \frac{1}{1320}x^4 - \cdots\right)}$$

$$\boxed{r = \frac{1}{3} \rightarrow b_n = -\frac{2}{3n^2+n}b_{n-1}}$$

$$n=1 \rightarrow b_1 = -\frac{1}{2}b_0$$

$$n=2 \rightarrow b_2 = -\frac{2}{14}b_1 = \frac{1}{14}b_0$$

$$n=3 \rightarrow b_3 = -\frac{2}{30}b_2 = -\frac{1}{210}b_0$$

$$n=4 \rightarrow b_4 = -\frac{2}{52}b_3 = \frac{1}{5460}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{1/3}\left(1 - \frac{1}{2}x + \frac{1}{14}x^2 - \frac{1}{210}x^3 + \frac{1}{5460}x^4 - \cdots\right)}$$

$$\boxed{y(x) = a_0\left(1 - x + \frac{1}{5}x^2 - \frac{1}{60}x^3 + \frac{1}{1320}x^4 - \cdots\right) + b_0 x^{1/3}\left(1 - \frac{1}{2}x + \frac{1}{14}x^2 - \frac{1}{210}x^3 + \frac{1}{5460}x^4 - \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $3x^2y'' + 2xy' + x^2y = 0$

Solution

$$\frac{1}{3}3x^2y'' + 2\frac{1}{3}xy' + \frac{1}{3}x^2y = 0$$

$$x^2y'' + \frac{2}{3}xy' + \frac{1}{3}x^2y = 0$$

$$y'' + \frac{2}{3x}y' + \frac{1}{3}y = 0$$

That implies to $p(x) = \frac{2}{3x}$ and $q(x) = \frac{1}{3}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{2}{3x} \boxed{= \frac{2}{3}}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\frac{1}{3} \boxed{= 0}$$

The indicial equation is: $r(r-1) + \frac{2}{3}r = r^2 - \frac{1}{3}r = 0 \quad \rightarrow \quad \boxed{r = 0,\ \frac{1}{3}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/3}\sum_{n=0}^{\infty} b_n x^n$$

$$y=\sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$3x^2 y''+2xy'+x^2 y=0$$

$$3x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+2x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+x^2\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}3(n+r)(n+r-1)a_n x^{n+r}+\sum_{n=0}^{\infty}2(n+r)a_n x^{n+r}+\sum_{n=0}^{\infty}a_n x^{n+r+2}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(3n+3r-3)+2(n+r)\right]a_n x^{n+r}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r}=0$$

$$r(3r-1)a_0+(r+1)(3r+2)a_1+\sum_{n=2}^{\infty}(n+r)(3n+3r-1)a_n x^{n+r}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r}=0$$

$$r(3r-1)a_0+(r+1)(3r+2)a_1+\sum_{n=2}^{\infty}\left[(n+r)(3n+3r-1)a_n+a_{n-2}\right]x^{n+r}=0$$

For $n=0 \quad \rightarrow \quad r(3r-1)a_0=0 \quad \Rightarrow \quad \boxed{r=0,\ \frac{1}{3}}$ √

For $n=1 \quad \rightarrow \quad (r+1)(3r+2)a_1=0 \quad \Rightarrow \quad \cancel{r=-1,\ -\frac{2}{3}}$ $\quad \boxed{a_1=0}$

$$(n+r)(3n+3r-1)a_n+a_{n-2}=0$$

$$\boxed{a_n=-\frac{1}{(n+r)(3n+3r-1)}a_{n-2}}$$

$$\boxed{r=0 \quad \rightarrow \quad a_n=-\frac{1}{n(3n-1)}a_{n-2}}$$

$n=2 \ \rightarrow \ a_2=-\frac{1}{10}a_0$ $\qquad$ $n=3 \ \rightarrow \ a_3=-\frac{1}{24}a_1=0$

$n=4 \ \rightarrow \ a_4=-\frac{1}{44}a_2=\frac{1}{440}a_0$ $\qquad$ $n=5 \ \rightarrow \ a_5=-\frac{1}{70}a_3=0$

$n=6 \ \rightarrow \ a_6=-\frac{1}{102}a_4=-\frac{1}{44,880}a_0$ $\qquad$ $n=7 \ \rightarrow \ a_7=0$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 x^0\left(1 - \frac{1}{10}x^2 + \frac{1}{440}x^4 - \frac{1}{44,880}x^6 + \cdots\right)}$$

$$r = \frac{1}{3} \quad \rightarrow \quad b_n = -\frac{1}{\left(n+\frac{1}{3}\right)(3n)}b_{n-2} = -\frac{1}{n(3n+1)}b_{n-2}$$

$$n = 2 \ \rightarrow \ b_2 = -\frac{1}{14}b_0 \qquad\qquad n = 3 \ \rightarrow \ b_3 = -\frac{1}{30}b_1 = 0$$

$$n = 4 \ \rightarrow \ b_4 = -\frac{1}{72}b_2 = \frac{1}{728}b_0 \qquad\qquad n = 5 \ \rightarrow \ b_5 = 0$$

$$n = 6 \ \rightarrow \ b_6 = -\frac{1}{114}b_4 = -\frac{1}{82,992}b_0 \qquad\qquad n = 7 \ \rightarrow \ b_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_2(x) = b_0 x^{1/3}\left(1 - \frac{1}{14}x^2 + \frac{1}{728}x^4 - \frac{1}{82,992}x^6 + \cdots\right)}$$

$$\underline{y(x) = a_0\left(1 - \frac{1}{10}x^2 + \frac{1}{440}x^4 - \frac{1}{44,880}x^6 + \cdots\right) + b_0 x^{1/3}\left(1 - \frac{1}{14}x^2 + \frac{1}{728}x^4 - \frac{1}{82,992}x^6 + \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $3x^2y'' - xy' + y = 0$

Solution

$$\frac{1}{3}3x^2y'' - \frac{1}{3}xy' + \frac{1}{3}y = 0$$

$$x^2y'' - \frac{1}{3}xy' + \frac{1}{3}y = 0$$

$$y'' - \frac{1}{3x}y' + \frac{1}{3x^2}y = 0$$

That implies to $p(x) = -\frac{1}{3x}$ and $q(x) = \frac{1}{3x^2}$.

$$p_0 = \lim_{x\to 0} xp(x) = -\lim_{x\to 0} x\frac{1}{3x} = -\frac{1}{3}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\frac{1}{3x^2} = \frac{1}{3}$$

The indicial equation is: $r(r-1) - \frac{1}{3}r + \frac{1}{3} = r^2 - \frac{4}{3}r + \frac{1}{3} = 0 \quad \rightarrow \quad r = 1, \ \frac{1}{3}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x\sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{1/3}\sum_{n=0}^{\infty} b_n x^n$$

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$3x^2y''-xy'+y=0$

$$3x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}-x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}3(n+r)(n+r-1)a_n x^{n+r}-\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(3n+3r-3)-(n+r)+1\right]a_n x^{n+r}=0$$

Since neither of λ, then let assume $a_n=0,\quad n\geq 1$

$$y_1(x)=x\sum_{n=0}^{\infty}a_n x^n\ \underline{=a_0x|}$$

$$y_2(x)=x^{1/3}\sum_{n=0}^{\infty}b_n x^n\ \underline{=b_0x^{1/3}|}$$

$$\underline{y(x)=a_0x+b_0x^{1/3}|}$$

Exercise

Find the Frobenius series solutions of $4xy''+2y'+y=0$

Solution

$$\frac{x}{4}4xy''+2\frac{x}{4}y'+\frac{x}{4}y=0$$

$$x^2y''+\frac{1}{2}xy'+\frac{x}{4}y=0$$

$$y''+\frac{1}{2x}y'+\frac{1}{4x}y=0$$

That implies to $p(x)=\frac{1}{2x}$ and $q(x)=\frac{1}{4x}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{1}{2x} = \frac{1}{2}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2\frac{1}{4x} = \lim_{x\to 0}\frac{x}{4} = 0$$

The indicial equation is: $r(r-1)+\frac{1}{2}r = r^2 - \frac{1}{2}r = 0 \quad \to \quad r = 0,\ \frac{1}{2}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{1/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$4xy'' + 2y' + y = 0$$

$$4x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + 2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}4(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}2(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(4n+4r-4)+2(n+r)\right]a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

$$r(4r-2)a_n + \sum_{n=1}^{\infty}(n+r)(4n+4r-2)a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

$$2r(2r-1)a_n + \sum_{n=1}^{\infty}\left[2(n+r)(2n+2r-1)a_n + a_{n-1}\right]x^{n+r-1} = 0$$

For $n = 0 \quad \to \quad 2r(2r-1)a_0 = 0 \quad \Rightarrow \quad r = 0,\ \frac{1}{2}$ $\checkmark$

$$2(n+r)(2n+2r-1)a_n + a_{n-1} = 0$$

$$a_n = -\frac{1}{2(n+r)(2n+2r-1)}a_{n-1}$$

$$\boxed{r=0 \quad \to \quad a_n = -\frac{1}{2n(2n-1)}a_{n-1}}$$

$$n=1 \ \to\ a_1 = -\frac{1}{2}a_0$$

$$n=2 \ \to\ a_2 = -\frac{1}{12}a_1 = \frac{1}{24}a_0$$

$$n=3 \ \to\ a_3 = -\frac{1}{30}a_2 = -\frac{1}{720}a_0$$

$$n=4 \ \to\ a_4 = -\frac{1}{42}a_3 = \frac{1}{30,240}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0x^0\left(1-\frac{1}{2}x+\frac{1}{24}x^2-\frac{1}{720}x^3+\frac{1}{30,240}x^4-\cdots\right)}$$

$$\boxed{r=\frac{1}{2} \quad \to \quad b_n = -\frac{1}{2\left(n+\frac{1}{2}\right)(2n)}b_{n-1} = -\frac{1}{4n^2+2n}b_{n-1}}$$

$$n=1 \ \to\ b_1 = -\frac{1}{6}b_0$$

$$n=2 \ \to\ b_2 = -\frac{1}{20}b_1 = \frac{1}{120}b_0$$

$$n=3 \ \to\ b_3 = -\frac{1}{42}b_2 = -\frac{1}{5040}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0x^{1/2}\left(1-\frac{1}{6}x+\frac{1}{120}x^2-\frac{1}{5,040}x^3+\cdots\right)}$$

$$\boxed{y(x) = a_0\left(1-\frac{1}{2}x+\frac{1}{24}x^2-\frac{1}{720}x^3+\frac{1}{30,240}x^4-\cdots\right)+b_0x^{1/2}\left(1-\frac{1}{6}x+\frac{1}{120}x^2-\frac{1}{5,040}x^3+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $6x^2y''+7xy'-\left(x^2+2\right)y=0$

Solution

$$\frac{1}{6}6x^2y''+\frac{1}{6}7xy'-\frac{1}{6}\left(x^2+2\right)y=0$$

$$x^2y''+\frac{7}{6}xy'-\frac{1}{6}\left(x^2+2\right)y=0$$

$$y''+\frac{7}{6x}y'-\frac{1}{6x^2}\left(x^2+2\right)y=0$$

That implies to $p(x)=\frac{7}{6x}$ and $q(x)=-\frac{1}{6x^2}\left(x^2+2\right)$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{7}{6x} = \underline{\frac{7}{6}}\Big|$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = -\lim_{x\to 0} x^2 \frac{1}{6x^2}\left(x^2+2\right) = -\lim_{x\to 0}\left(\frac{1}{6}x^2+\frac{1}{3}\right) = \underline{-\frac{1}{3}}\Big|$$

The indicial equation is: $\quad r(r-1)+\frac{7}{6}r-\frac{1}{3} = r^2+\frac{1}{6}r-\frac{1}{3} = 0$

$$6r^2+r-2=0 \quad\to\quad r = \frac{-1\pm 7}{12} \quad \underline{r = \frac{1}{2},\ -\frac{2}{3}}\Big|$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/2}\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{-2/3}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$6x^2y''+7xy'-\left(x^2+2\right)y = 0$$

$$6x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + 7x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - x^2\sum_{n=0}^{\infty}a_n x^{n+r} - 2\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}6(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}7(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty}a_n x^{n+r+2} - \sum_{n=0}^{\infty}2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[6(n+r)(n+r-1)+7(n+r)-2\right]a_n x^{n+r} - \sum_{n=2}^{\infty}a_{n-2}x^{n+r} = 0$$

$$\left(6r^2+r-2\right)a_0 + \left((r+1)(6r+7)-2\right)a_1 + \sum_{n=2}^{\infty}\left[(n+r)(6n+6r+1)-2\right]a_n x^{n+r} - \sum_{n=2}^{\infty}a_{n-2}x^{n+r} = 0$$

$$\left(6r^2+r-2\right)a_0 + \left(6r^2+13r+5\right)a_1 + \sum_{n=2}^{\infty}\left[\left((n+r)(6n+6r+1)-2\right)a_n - a_{n-2}\right]x^{n+r} = 0$$

For $\ n=0 \quad\to\quad \left(6r^2+r-2\right)a_0 = 0 \quad\Rightarrow\quad \underline{r=\frac{1}{2},\ -\frac{2}{3}}\Big| \quad \surd$

For $n=1 \quad \rightarrow \quad \left(6r^2+13r+5\right)a_1=0 \quad \Rightarrow r=\frac{-13\pm 7}{12}=\cancel{-\frac{1}{2}, -\frac{5}{3}}$ $\quad \boxed{a_1=0}$

$$\left((n+r)(6n+6r+1)-2\right)a_n - a_{n-2}=0$$

$$\boxed{a_n=\frac{1}{(n+r)(6n+6r+1)-2}a_{n-2}}$$

$$\boxed{r=\frac{1}{2} \quad \rightarrow \quad a_n=\frac{1}{n(6n+7)}a_{n-2}}$$

$n=2 \rightarrow a_2=\frac{1}{38}a_0$ $\qquad n=3 \rightarrow a_3=\frac{1}{75}a_1=0$

$n=4 \rightarrow a_4=\frac{1}{124}a_2=\frac{1}{4,712}a_0$ $\qquad n=5 \rightarrow a_5=\frac{1}{185}a_3=0$

$n=6 \rightarrow a_6=\frac{1}{258}a_4=\frac{1}{1,215,696}a_0$ $\qquad n=7 \rightarrow a_7=0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x)=a_0x^{1/2}\left(1+\frac{1}{38}x^2+\frac{1}{4,712}x^4+\frac{1}{1,215,696}x^6+\cdots\right)}$$

$$\boxed{r=-\frac{2}{3} \quad \rightarrow \quad b_n=\frac{1}{n(6n-7)}b_{n-2}}$$

$n=2 \rightarrow b_2=\frac{1}{10}b_0$ $\qquad n=3 \rightarrow b_3=\frac{1}{33}b_1=0$

$n=4 \rightarrow b_4=\frac{1}{68}b_2=\frac{1}{680}b_0$ $\qquad n=5 \rightarrow b_5=0$

$n=6 \rightarrow b_6=\frac{1}{174}b_4=\frac{1}{118,320}b_0$ $\qquad n=7 \rightarrow b_7=0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_2(x)=b_0x^{-2/3}\left(1+\frac{1}{10}x^2+\frac{1}{680}x^4+\frac{1}{118,320}x^6+\cdots\right)}$$

$$\boxed{y(x)=a_0x^{1/2}\left(1+\frac{x^2}{38}+\frac{x^4}{4,712}+\frac{x^6}{1,215,696}+\cdots\right)+b_0x^{-2/3}\left(1+\frac{x^2}{10}+\frac{x^4}{680}+\frac{x^6}{118,320}+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $xy''+y'+2y=0$

Solution

$x\times \quad xy''+y'+2y=0$

$$x^2y''+xy'+2xy=0$$

$$y''+\frac{1}{x}y'+\frac{2}{x}y=0$$

That implies to $p(x)=\frac{1}{x}$ and $q(x)=\frac{2}{x}$.

$$p_0=\lim_{x\to 0} xp(x)=\lim_{x\to 0} x\frac{1}{x}\ \underline{=1}|$$

$$q_0=\lim_{x\to 0} x^2q(x)=\lim_{x\to 0} x^2\frac{2}{x}=\lim_{x\to 0} 2x\ \underline{=0}|$$

The indicial equation is: $r^2+(1-1)r=0 \quad\to\quad \underline{r_{1,2}=0}|$

The two possible Frobenius series solutions are then of the forms

$$y_1(x)=\sum_{n=0}^{\infty}a_n x^n \quad \text{and} \quad y_2(x)=y_1(x)\ln|x|+\sum_{n=0}^{\infty}c_n x^n$$

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}=\sum_{n=0}^{\infty}a_n x^n \qquad (r=r_1=0)$$

$$y'=\sum_{n=0}^{\infty}na_n x^{n+r-1}=\sum_{n=0}^{\infty}na_n x^{n-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}=\sum_{n=0}^{\infty}n(n-1)a_n x^{n-2}$$

$xy''+y'+2y=0$

$$x\sum_{n=0}^{\infty}n(n-1)a_n x^{n-2}+\sum_{n=0}^{\infty}na_n x^{n-1}+\sum_{n=0}^{\infty}2a_n x^n=0$$

$$\sum_{n=0}^{\infty}n(n-1)a_n x^{n-1}+\sum_{n=0}^{\infty}na_n x^{n-1}+\sum_{n=0}^{\infty}2a_n x^n=0$$

$$\sum_{n=0}^{\infty}n^2a_n x^{n-1}+\sum_{n=0}^{\infty}2a_n x^n=0$$

$$\sum_{n=0}^{\infty}n^2a_n x^{n-1}+\sum_{n=1}^{\infty}2a_{n-1} x^{n-1}=0$$

$$\sum_{n=1}^{\infty}n^2a_n x^{n-1}+\sum_{n=1}^{\infty}2a_{n-1} x^{n-1}=0$$

$$\sum_{n=1}^{\infty}\left[n^2a_n+2a_{n-1}\right]x^{n-1}=0$$

$$n^2a_n+2a_{n-1}=0 \quad\Rightarrow\quad \underline{a_n=-\frac{2}{n^2}a_{n-1}}|$$

$$n=1 \to a_1=-2a_0$$

$$n=2 \;\rightarrow\; a_2 = -\frac{2}{2^2}a_1 = a_0 = \frac{2^2}{2^2}a_0$$

$$n=3 \;\rightarrow\; a_3 = -\frac{2}{9}a_2 = -\frac{2^3}{(2\cdot 3)^2}a_0 = -\frac{2}{9}a_0$$

$$n=4 \;\rightarrow\; a_4 = -\frac{2}{4^2}a_3 = \frac{2^4}{(4!)^2}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_n = \frac{(-1)^n 2^n}{(n!)^2}a_0$$

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{(n!)^2}x^n$$

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n \quad (a_0 = 1)$$

$$y_2(x) = y_1(x)\ln x + \sum_{n=0}^{\infty} c_n x^n$$

$$y_2(x) = y_1(x)\ln x + x^{r_1}\sum_{n=0}^{\infty} c_n x^n$$

$$xy_2'' + y_2' + 2y_2 = 0$$

$$x\left(y_1(x)\ln x + \sum_{n=0}^{\infty} c_n x^n\right)'' + \left(y_1(x)\ln x + \sum_{n=0}^{\infty} c_n x^n\right)' + 2\left(y_1(x)\ln x + \sum_{n=0}^{\infty} c_n x^n\right) = 0$$

$$x\left(y_1'\ln x + \frac{1}{x}y_1\right)' + x\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + y_1'\ln x + \frac{1}{x}y_1 + \sum_{n=0}^{\infty} nc_n x^{n-1} + 2y_1\ln x + \sum_{n=0}^{\infty} 2c_n x^n = 0$$

$$x\left(y_1''\ln x + \frac{1}{x}y_1' - \frac{1}{x^2}y_1 + \frac{1}{x}y_1'\right) + y_1'\ln x + \frac{1}{x}y_1 + 2y_1\ln x$$

$$+\sum_{n=0}^{\infty} n(n-1)c_n x^{n-1} + \sum_{n=0}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} 2c_n x^n = 0$$

$$xy_1''\ln x + 2y_1' - \frac{1}{x}y_1 + \frac{1}{x}y_1 + \left(y_1' + 2y_1\right)\ln x + \sum_{n=0}^{\infty} n^2 c_n x^{n-1} + \sum_{n=0}^{\infty} 2c_n x^n = 0$$

$$\left(xy_1'' + y_1' + 2y_1\right)\ln x + 2y_1' + \sum_{n=0}^{\infty} n^2 c_n x^{n-1} + \sum_{n=1}^{\infty} 2c_{n-1} x^{n-1} = 0$$

Since: $xy_1'' + y_1' + 2y_1 = 0$

$$2y_1' + \sum_{n=1}^{\infty} n^2 c_n x^{n-1} + \sum_{n=1}^{\infty} 2c_{n-1} x^{n-1} = 0$$

$$2y_1' + \sum_{n=1}^{\infty}\left(n^2c_n + 2c_{n-1}\right)x^{n-1} = 0$$

$$2\sum_{n=1}^{\infty}(-1)^n \frac{2^n}{(n!)^2} nx^{n-1} + \sum_{n=1}^{\infty}\left(n^2c_n + 2c_{n-1}\right)x^{n-1} = 0$$

$$\sum_{n=1}^{\infty}\left[(-1)^n \frac{2^{n+1}}{(n-1)!n!} + n^2c_n + 2c_{n-1}\right]x^{n-1} = 0$$

$$n^2c_n + 2c_{n-1} + (-1)^n \frac{2^{n+1}}{(n-1)!n!} = 0$$

$$\boxed{c_n = -\frac{2}{n^2}c_{n-1} + (-1)^{n+1}\frac{2^{n+1}}{(n-1)!n!\,n^2}}$$

$n = 1 \;\to\; c_1 = -2c_0 + 4$ $\qquad\qquad c_0 = 0 \;\to\; c_1 = 4$

$n = 2 \;\to\; c_2 = -\frac{1}{2}c_1 - 1$ $\qquad\qquad c_2 = -3$

$n = 3 \;\to\; c_3 = -\frac{2}{9}c_2 + \frac{16}{12(9)}$ $\qquad\qquad c_3 = \frac{22}{27}$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_2(x) = y_1(x)\ln|x| + 4x - 3x^2 + \frac{22}{27}x^3 - \cdots}$$

$$y_2(x) = y_1(x)\ln x + x^{r_1}\sum_{n=0}^{\infty}c_nx^n$$

Exercise

Find the Frobenius series solutions of $xy'' - y = 0$

Solution

$x\times \quad xy'' - y = 0$

$x^2y'' - xy = 0$

$y'' - \frac{1}{x}y = 0$

That implies to $p(x) = 0$ and $q(x) = -\frac{1}{x}$.

$p_0 = \lim\limits_{x\to 0} xp(x) \;\underline{= 0}$

$q_0 = \lim\limits_{x\to 0} x^2q(x) = \lim\limits_{x\to 0} x^2\left(-\frac{1}{x}\right) = -\lim\limits_{x\to 0} x \;\underline{= 0}$

The indicial equation is: $r^2 - r = 0 \;\to\; \underline{r_1 = 1,\, r_2 = 0}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1} \quad \textit{and} \quad y_2(x) = \alpha y_1(x)\ln|x| + \sum_{n=0}^{\infty} c_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=0}^{\infty} a_n x^{n+1} \qquad (r = r_1 = 1)$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n+r-1} = \sum_{n=0}^{\infty} n a_n x^n$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} = \sum_{n=0}^{\infty} n(n+1) a_n x^{n-1}$$

$xy'' - y = 0$

$$x\sum_{n=0}^{\infty} n(n+1) a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} n(n+1) a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} n(n+1) a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\sum_{n=1}^{\infty} n(n+1) a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\sum_{n=1}^{\infty} \left[n(n+1) a_n - a_{n-1} \right] x^n = 0$$

$$n(n+1) a_n - a_{n-1} = 0 \quad \Rightarrow \quad \boxed{a_n = \frac{1}{n(n+1)} a_{n-1}}$$

$$n = 1 \ \rightarrow \ a_1 = \tfrac{1}{2} a_0$$

$$n = 2 \ \rightarrow \ a_2 = \tfrac{1}{6} a_1 = a_0 = \frac{1}{(2)3!} a_0$$

$$n = 3 \ \rightarrow \ a_3 = \frac{1}{3\cdot 4} a_2 = \frac{1}{(2\cdot 3)4!} a_0$$

$$n = 4 \ \rightarrow \ a_4 = \frac{1}{4\cdot 5} a_3 = \frac{1}{4!5!} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_n = \frac{1}{n!(n+1)!} a_0}$$

$$y_1(x) = x\sum_{n=1}^{\infty}\frac{1}{n!(n+1)!}x^n = \sum_{n=1}^{\infty}\frac{1}{n!(n+1)!}x^{n+1}$$

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty}a_n x^n \quad (a_0 = 1)$$

$$y_2(x) = \alpha y_1 \ln x + \sum_{n=0}^{\infty}d_n x^n$$

$$y_2(x) = \alpha y_1(x)\ln x + \sum_{n=0}^{\infty}d_n x^n \quad (d_0 = 1)$$

$$y_2' = \alpha y_1' \ln x + \frac{\alpha}{x}y_1 + \sum_{n=0}^{\infty}nd_n x^{n-1}$$

$$xy_2'' - y_2 = 0$$

$$x\left(\alpha y_1' \ln x + \frac{\alpha}{x}y_1 + \sum_{n=0}^{\infty}nd_n x^{n-1}\right)' - \left(\alpha y_1 \ln x + \sum_{n=0}^{\infty}d_n x^n\right) = 0$$

$$x\left(\alpha y_1' \ln x + \frac{\alpha}{x}y_1\right)' + x\sum_{n=0}^{\infty}n(n-1)d_n x^{n-2} - \alpha y_1 \ln x - \sum_{n=0}^{\infty}d_n x^n = 0$$

$$x\left(\alpha y_1'' \ln x + \frac{\alpha}{x}y_1' - \frac{\alpha}{x^2}y_1 + \frac{\alpha}{x}y_1'\right) - \alpha y_1(x)\ln x + \sum_{n=0}^{\infty}n(n-1)d_n x^{n-1} - \sum_{n=0}^{\infty}d_n x^n = 0$$

$$\alpha\left(2y_1' - \frac{1}{x}y_1\right) + \alpha\left(xy_1'' - y_1\right)\ln x + \sum_{n=0}^{\infty}n(n+1)d_{n+1}x^n - \sum_{n=0}^{\infty}d_n x^n = 0$$

Since: $xy_1'' - y_1 = 0$

$$y_1(x) = \sum_{n=1}^{\infty}\frac{1}{n!(n+1)!}x^{n+1} \quad \to \quad y_1' = \sum_{n=1}^{\infty}\frac{n+1}{n!(n+1)!}x^n$$

$$\alpha\left(2y_1' - \frac{1}{x}y_1\right) + \sum_{n=0}^{\infty}\left[n(n+1)d_{n+1} - d_n\right]x^n = 0$$

$$\alpha\left(2\sum_{n=1}^{\infty}\frac{n+1}{n!(n+1)!}x^n - \frac{1}{x}\sum_{n=1}^{\infty}\frac{1}{n!(n+1)!}x^{n+1}\right) + \sum_{n=0}^{\infty}\left[n(n+1)d_{n+1} - d_n\right]x^n = 0$$

$$\alpha\left(\sum_{n=1}^{\infty}\frac{2n+2}{n!(n+1)!}x^n - \sum_{n=1}^{\infty}\frac{1}{n!(n+1)!}x^n\right) + \sum_{n=0}^{\infty}\left[n(n+1)d_{n+1} - d_n\right]x^n = 0$$

$$\alpha\sum_{n=1}^{\infty}\frac{2n+1}{n!(n+1)!}x^n + \sum_{n=0}^{\infty}\left[n(n+1)d_{n+1} - d_n\right]x^n = 0$$

$$\sum_{n=0}^{\infty}\left[n(n+1)d_{n+1}-d_n\right]x^n=-\alpha\sum_{n=1}^{\infty}\frac{2n+1}{n!(n+1)!}x^n$$

$$\boxed{n(n+1)d_{n+1}-d_n=-\alpha\frac{2n+1}{n!(n+1)!}} \qquad (d_0=1)$$

$$n=0 \rightarrow -d_0=-\alpha \qquad \Rightarrow \boxed{\alpha=d_0=1}$$

$$d_{n+1}=\frac{1}{n(n+1)}\left(d_n-\frac{2n+1}{n!(n+1)!}\right)$$

$$n=1 \rightarrow d_2=\frac{1}{2}\left(d_1-\frac{3}{2}\right)=\boxed{\frac{1}{2}d_1-\frac{3}{4}}$$

$$n=2 \rightarrow d_3=\frac{1}{6}\left(d_2-\frac{5}{12}\right)=\frac{1}{6}\left(\frac{1}{2}d_1-\frac{3}{4}-\frac{5}{12}\right)=\boxed{\frac{1}{12}d_1-\frac{7}{36}}$$

$$n=3 \rightarrow d_4=\frac{1}{12}\left(d_3-\frac{7}{144}\right)=\frac{1}{12}\left(\frac{1}{12}d_1-\frac{7}{36}-\frac{7}{144}\right)=\boxed{\frac{1}{144}d_1-\frac{35}{1{,}728}}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

If we let $d_1=0$

$$\boxed{y_2(x)=y_1(x)\ln x+1-\frac{3}{4}x^2-\frac{7}{36}x^3-\cdots} \qquad y_2(x)=y_1(x)\ln x+\sum_{n=0}^{\infty}d_nx^n$$

Exercise

Find the Frobenius series solutions of $2x(1-x)y''+(1+x)y'-y=0$

Solution

$$xy''+\frac{x+1}{2(1-x)}y'-\frac{1}{2(1-x)}y=0$$

$$x^2y''+\frac{1}{2}\frac{x(x+1)}{1-x}y'-\frac{x}{2(1-x)}y=0$$

$$y''+\frac{1}{2}\frac{x+1}{x(1-x)}y'-\frac{1}{2x(1-x)}y=0$$

That implies to $p(x)=\frac{1}{2}\frac{x+1}{x(1-x)}$ and $q(x)=-\frac{1}{2x(1-x)}$.

$$p_0=\lim_{x\to 0}xp(x)=\frac{1}{2}\lim_{x\to 0}x\frac{x+1}{x(1-x)}=\frac{1}{2}\lim_{x\to 0}\frac{x+1}{1-x}\ \boxed{=\frac{1}{2}}$$

$$q_0=\lim_{x\to 0}x^2q(x)=-\frac{1}{2}\lim_{x\to 0}\frac{x}{1-x}\ \boxed{=0}$$

The indicial equation is: $r^2+\left(\frac{1}{2}-1\right)r=r^2-\frac{1}{2}r=0 \rightarrow \boxed{r=0,\ \frac{1}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x)=\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x)=x^{1/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y=\sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$2x(1-x)y''+(1+x)y'-y=0$$

$$\left(2x-2x^2\right)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+(1+x)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}-\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r-1}-\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}-\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(2n+2r-1)a_n x^{n+r-1}+\sum_{n=0}^{\infty}\left[-2(n+r)(n+r-1)+n+r-1\right]a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(2n+2r-1)a_n x^{n+r-1}+\sum_{n=0}^{\infty}(n+r-1)(-2(n+r)+1)a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(2n+2r-1)a_n x^{n+r-1}+\sum_{n=1}^{\infty}(n+r-2)(-2n-2r+3)a_{n-1} x^{n+r-1}=0$$

$$r(2r-1)a_0 x^{r-1}+\sum_{n=1}^{\infty}(n+r)(2n+2r-1)a_n x^{n+r-1}-\sum_{n=1}^{\infty}(n+r-2)(2n+2r-3)a_{n-1} x^{n+r-1}=0$$

$$r(2r-1)a_0 x^{r-1}+\sum_{n=1}^{\infty}\left[(n+r)(2n+2r-1)a_n-(n+r-2)(2n+2r-3)a_{n-1}\right]x^{n+r-1}=0$$

For $n=0 \;\rightarrow\; r(2r-1)a_0=0 \;\Rightarrow\; r=0,\ \tfrac{1}{2}$

$$(n+r)(2n+2r-1)a_n-(n+r-2)(2n+2r-3)a_{n-1}=0$$

$$a_n = \frac{(n+r-2)(2n+2r-3)}{(n+r)(2n+2r-1)} a_{n-1}$$

$$r = 0 \quad \rightarrow \quad a_n = \frac{(n-2)(2n-3)}{n(2n-1)} a_{n-1}$$

$n = 1 \ \rightarrow \ a_1 = a_0$

$n = 2 \ \rightarrow \ a_2 = 0a_1 = 0$

$n = 3 \ \rightarrow \ a_3 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_1(x) = a_0(1+x)$$

$$r = \frac{1}{2} \quad \rightarrow \quad b_n = \frac{\left(n-\frac{3}{2}\right)(2n-2)}{2n\left(n+\frac{1}{2}\right)} b_{n-1} = \frac{(2n-3)(n-1)}{n(2n+1)} b_{n-1}$$

$n = 1 \ \rightarrow \ b_1 = 0b_0 = 0$

$n = 2 \ \rightarrow \ b_2 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_2(x) = b_0 x^{1/2}$$

$$y(x) = a_0(1+x) + b_0\sqrt{x}$$

Exercise

Find the Frobenius series solutions of $x^2y'' + \left(x^2 + \frac{1}{2}x\right)y' + xy = 0$

Solution

$$y'' + \left(1 + \frac{1}{2x}\right)y' + \frac{1}{x}y = 0$$

That implies to $p(x) = 1 + \frac{1}{2x}$ and $q(x) = \frac{1}{x}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(1+\frac{1}{2x}\right) = \lim_{x\to 0}\left(x+\frac{1}{2}\right) = \frac{1}{2}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x = 0$$

The indicial equation is: $r^2 + \left(\frac{1}{2} - 1\right)r = r^2 - \frac{1}{2}r = 0 \quad \rightarrow \quad r = 0, \ \frac{1}{2}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{1/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x^2 y'' + \left(x^2 + \tfrac{1}{2} x\right) y' + xy = 0$$

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \left(x^2 + \tfrac{1}{2} x\right) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + x \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r+1} \sum_{n=0}^{\infty} \tfrac{1}{2}(n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty} (n+r+1) a_n x^{n+r+1} + \sum_{n=0}^{\infty} \left[(n+r)(n+r-1) + \tfrac{1}{2}(n+r)\right] a_n x^{n+r} = 0$$

$$\sum_{n=1}^{\infty} (n+r) a_{n-1} x^{n+r} + \sum_{n=0}^{\infty} (n+r)\left(n+r-\tfrac{1}{2}\right) a_n x^{n+r} = 0$$

$$\sum_{n=1}^{\infty} (n+r) a_{n-1} x^{n+r} + r\left(r-\tfrac{1}{2}\right) a_0 + \sum_{n=1}^{\infty} (n+r)\left(n+r-\tfrac{1}{2}\right) a_n x^{n+r} = 0$$

$$r\left(r-\tfrac{1}{2}\right) a_0 + \sum_{n=1}^{\infty} \left[(n+r)\left(n+r-\tfrac{1}{2}\right) a_n + (n+r) a_{n-1}\right] x^{n+r} = 0$$

For $n=0 \quad \rightarrow \quad r(2r-1) a_0 = 0 \quad \Rightarrow \quad \boxed{r = 0,\ \tfrac{1}{2}}$ $\surd$

$$(n+r)\left(n+r-\tfrac{1}{2}\right) a_n + (n+r) a_{n-1} = 0$$

$$\boxed{a_n = -2 \frac{(n+r)}{(n+r)(2n+2r-1)} a_{n-1}}$$

$$\boxed{r = 0 \quad \rightarrow \quad a_n = -\frac{2n}{n(2n-1)} a_{n-1} = -\frac{2}{2n-1} a_{n-1}}$$

$$n = 1 \rightarrow a_1 = -2a_0$$

$$n=2 \rightarrow a_2 = -\frac{2}{3}a_1 = \frac{4}{3}a_0$$

$$n=3 \rightarrow a_3 = -\frac{2}{5}a_2 = -\frac{8}{15}a_0$$

$$n=4 \rightarrow a_4 = -\frac{2}{7}a_3 = \frac{16}{105}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0\left(1-2x+\frac{4}{3}x^2-\frac{8}{15}x^3+\frac{16}{105}x^4-\cdots\right)}$$

$$\boxed{r=\frac{1}{2} \rightarrow b_n = -\frac{2\left(n+\frac{1}{2}\right)}{\left(n+\frac{1}{2}\right)(2n)}b_{n-1} = -\frac{1}{n}b_{n-1}}$$

$$n=1 \rightarrow b_1 = -b_0$$

$$n=2 \rightarrow b_2 = -\frac{1}{2}b_1 = \frac{1}{2}b_0$$

$$n=3 \rightarrow b_3 = -\frac{1}{3}b_2 = -\frac{1}{6}b_0$$

$$n=4 \rightarrow b_4 = -\frac{1}{4}b_3 = \frac{1}{24}b_0$$

$$n=5 \rightarrow b_5 = -\frac{1}{5}b_4 = -\frac{1}{120}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{1/2}\left(1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{120}x^5+\cdots\right)}$$

$$\boxed{y(x) = a_0\left(1-2x+\frac{4}{3}x^2-\frac{8}{15}x^3+\frac{16}{105}x^4-\cdots\right)+b_0\sqrt{x}\left(1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{120}x^5+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $18x^2y''+3x(x+5)y'-(10x+1)y=0$

Solution

$$y''+\frac{x+5}{6x}y'-\frac{10x+1}{18x^2}y=0$$

That implies to $p(x)=\frac{x+5}{6x}$ and $q(x)=-\frac{10x+1}{18x^2}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(\frac{x+5}{6x}\right) = \lim_{x\to 0}\frac{x+5}{6} = \boxed{\frac{5}{6}}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = -\lim_{x\to 0}\frac{10x+1}{18} = \boxed{-\frac{1}{18}}$$

The indicial equation is: $r^2+\left(\frac{5}{6}-1\right)r-\frac{1}{18} = r^2-\frac{1}{6}r-\frac{1}{18}=0$

$$18r^2 - 3r - 1 = 0 \quad \rightarrow \quad \underline{r = -\frac{1}{6},\ \frac{1}{3}}\Big|$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{-1/6}\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/3}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$18x^2 y'' + 3x(x+5)y' - (10x+1)y = 0$$

$$18x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \left(3x^2 + 15x\right)\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - (10x+1)\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 18(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r+1} + \sum_{n=0}^{\infty} 15(n+r) a_n x^{n+r}$$

$$- \sum_{n=0}^{\infty} 10 a_n x^{n+r+1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[18(n+r)(n+r-1) + 15(n+r) - 1\right] a_n x^{n+r} + \sum_{n=0}^{\infty} (3n+3r-10) a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(18n+18r-3) - 1\right] a_n x^{n+r} + \sum_{n=1}^{\infty} (3n+3r-13) a_{n-1} x^{n+r} = 0$$

$$\left(r(18r-3)-1\right) a_0 x^r + \sum_{n=1}^{\infty} \left[(n+r)(18n+18r-3) - 1\right] a_n x^{n+r} + \sum_{n=1}^{\infty} (3n+3r-13) a_{n-1} x^{n+r} = 0$$

$$\left(r(18r-3)-1\right) a_0 x^r + \sum_{n=1}^{\infty} \left[\left((n+r)(18n+18r-3) - 1\right) a_n + (3n+3r-13) a_{n-1}\right] x^{n+r} = 0$$

For $n = 0 \quad \rightarrow \quad \left(18r^2 - 3r - 1\right) a_0 = 0 \quad \Rightarrow \quad \underline{r = -\frac{1}{6},\ \frac{1}{3}}\Big| \quad \checkmark$

$$\left((n+r)(18n+18r-3) - 1\right) a_n + (3n+3r-13) a_{n-1} = 0$$

$$\boxed{a_n = -\frac{3n+3r-13}{(n+r)(18n+18r-3)-1}a_{n-1}}$$

$$\boxed{r=-\frac{1}{6} \quad \to \quad a_n = -\frac{3n-\frac{1}{2}-13}{\left(n-\frac{1}{6}\right)(18n-6)-1}a_{n-1} = -\frac{1}{2}\frac{6n-27}{(6n-1)(3n-1)-1}a_{n-1}}$$

$$n=1 \ \to \ a_1 = -\frac{1}{2}\frac{-21}{9}a_0 = \frac{7}{6}a_0$$

$$n=2 \ \to \ a_2 = -\frac{1}{2}\frac{-15}{54}a_1 = \frac{5}{36}\frac{7}{6}a_0 = \frac{35}{216}a_0$$

$$n=3 \ \to \ a_3 = -\frac{1}{2}\frac{-9}{135}a_2 = \frac{1}{30}\frac{35}{216}a_0 = \frac{7}{1,296}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0 x^{-1/6}\left(1+\frac{7}{6}x+\frac{35}{216}x^2+\frac{7}{1,296}x^3+\cdots\right)}$$

$$\boxed{r=\frac{1}{3} \quad \to \quad b_n = -\frac{3n-12}{\left(n+\frac{1}{3}\right)(18n+3)-1}b_{n-1} = -\frac{3(n-4)}{(3n+1)(6n+1)-1}b_{n-1}}$$

$$n=1 \ \to \ b_1 = -\frac{-9}{27}b_0 = \frac{1}{3}b_0$$

$$n=2 \ \to \ b_2 = \frac{6}{90}b_1 = \frac{1}{15}\frac{1}{3}b_0 = \frac{1}{45}b_0$$

$$n=3 \ \to \ b_3 = \frac{3}{189}b_2 = \frac{1}{63}\frac{1}{45}b_0 = \frac{1}{2,835}b_0$$

$$n=4 \ \to \ b_4 = 0b_3 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{1/3}\left(1+\frac{1}{3}x+\frac{1}{45}x^2+\frac{1}{2835}x^3\right)}$$

$$\boxed{y(x) = a_0\frac{1}{x^{1/6}}\left(1+\frac{7}{6}x+\frac{35}{216}x^2+\frac{7}{1296}x^3+\cdots\right)+b_0 x^{1/3}\left(1+\frac{1}{3}x+\frac{1}{45}x^2+\frac{1}{2835}x^3\right)}$$

Exercise

Find the Frobenius series solutions of $2x^2y'' + 7x(x+1)y' - 3y = 0$

Solution

$$y'' + \frac{7}{2}\frac{x+1}{x}y' - \frac{3}{2x^2}y = 0$$

That implies to $p(x) = \frac{7}{2}\frac{x+1}{x}$ and $q(x) = -\frac{3}{2x^2}$.

$$p_0 = \lim_{x\to 0} xp(x) = \frac{7}{2}\lim_{x\to 0} x\left(\frac{x+1}{x}\right) = \frac{7}{2}\lim_{x\to 0}(x+1) = \boxed{\frac{7}{2}}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = -\lim_{x\to 0} \frac{3}{2} = \boxed{-\frac{3}{2}}$$

The indicial equation is: $r^2 + \left(\frac{7}{2} - 1\right) r - \frac{3}{2} = r^2 + \frac{5}{2} r - \frac{3}{2} = 0$

$$2r^2 + 5r - 3 = 0 \quad \to \quad \boxed{r = -3,\ \frac{1}{2}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{-3} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{1/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x^2 y'' + 7x(x+1) y' - 3y = 0$$

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \left(7x^2 + 7x\right) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - 3 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} 7(n+r) a_n x^{n+r+1} + \sum_{n=0}^{\infty} 7(n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} 3a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r-2) + 7(n+r) - 3\right] a_n x^{n+r} + \sum_{n=0}^{\infty} 7(n+r) a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r+5) - 3\right] a_n x^{n+r} + \sum_{n=1}^{\infty} 7(n+r-1) a_{n-1} x^{n+r} = 0$$

$$\left(2r^2 + 5r - 3\right) a_0 x^r + \sum_{n=1}^{\infty} \left[(n+r)(2n+2r+5) - 3\right] a_n x^{n+r} + \sum_{n=1}^{\infty} 7(n+r-1) a_{n-1} x^{n+r} = 0$$

$$\left(2r^2 + 5r - 3\right) a_0 x^r + \sum_{n=1}^{\infty} \left[\left((n+r)(2n+2r+5) - 3\right) a_n + 7(n+r-1) a_{n-1}\right] x^{n+r} = 0$$

For $n = 0 \quad \to \quad \left(2r^2 + 5r - 3\right) a_0 = 0 \quad \Rightarrow \quad \boxed{r = -3,\ \frac{1}{2}}$ $\checkmark$

$$\left((n+r)(2n+2r+5) - 3\right) a_n + 7(n+r-1) a_{n-1} = 0$$

$$\boxed{a_n = -\frac{7(n+r-1)}{(n+r)(2n+2r+5)-3}a_{n-1}}$$

$$\boxed{r=-3 \quad \rightarrow \quad a_n = -\frac{7(n-4)}{(n-3)(2n-1)-3}a_{n-1}}$$

$$n=1 \;\rightarrow\; a_1 = -\frac{21}{5}a_0$$

$$n=2 \;\rightarrow\; a_2 = -\frac{14}{6}a_1 = -\frac{7}{3}\left(-\frac{21}{5}\right)a_0 = \frac{49}{5}a_0$$

$$n=3 \;\rightarrow\; a_3 = -\frac{-7}{-3}a_2 = -\frac{7}{3}\frac{49}{5}a_0 = -\frac{343}{15}a_0$$

$$n=4 \;\rightarrow\; a_4 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0 x^{-3}\left(1-\frac{21}{5}x+\frac{49}{5}x^2-\frac{343}{15}x^3\right)}$$

$$\boxed{r=\frac{1}{2} \quad \rightarrow \quad b_n = -\frac{7\left(n-\frac{1}{2}\right)}{\left(n+\frac{1}{2}\right)(2n+6)-3}b_{n-1} = -\frac{7}{2}\frac{2n-1}{(2n+1)(n+3)-3}b_{n-1}}$$

$$n=1 \;\rightarrow\; b_1 = -\frac{7}{2}\frac{1}{9}b_0 = -\frac{7}{18}b_0$$

$$n=2 \;\rightarrow\; b_2 = -\frac{7}{2}\frac{3}{22}b_1 = -\frac{21}{44}\frac{-7}{18}b_0 = \frac{49}{264}b_0$$

$$n=3 \;\rightarrow\; b_3 = -\frac{7}{2}\frac{5}{39}b_2 = -\frac{35}{78}\frac{49}{264}b_0 = -\frac{1,715}{20,592}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{1/2}\left(1-\frac{7}{18}x+\frac{49}{264}x^2-\frac{1,715}{20,592}x^3+\cdots\right)}$$

$$\boxed{y(x) = a_0\frac{1}{x^3}\left(1-\frac{21}{5}x+\frac{49}{5}x^2-\frac{343}{15}x^3\right)+b_0\sqrt{x}\left(1-\frac{7}{18}x+\frac{49}{264}x^2-\frac{1,715}{20,592}x^3+\cdots\right)}$$

Exercise

Find the Frobenius series solutions: $x(1-x)y'' + \left[c-(a+b+1)x\right]y' - aby = 0$ (*Gauss' Hypergeometric*)

Solution

$$y'' + \frac{c-(a+b+1)x}{x(1-x)}y' - \frac{ab}{x(1-x)}y = 0$$

That implies to $p(x) = \dfrac{c-(a+b+1)x}{x(1-x)}$ and $q(x) = -\dfrac{ab}{x(1-x)}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(\frac{c-(a+b+1)x}{x(1-x)}\right) = \lim_{x\to 0}\left(\frac{c-(a+b+1)x}{1-x}\right) \boxed{=c}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = -\lim_{x\to 0} x^2 \frac{ab}{x(1-x)} = -\lim_{x\to 0}\frac{abx}{1-x} \underline{= 0|}$$

$$p_1 = \lim_{x\to 1}(x-1)\,p(x) = \lim_{x\to 1}(x-1)\left(\frac{c-(a+b+1)x}{x(1-x)}\right) = \lim_{x\to 1}\left(-\frac{c-(a+b+1)x}{x}\right) \underline{= a+b+1-c|}$$

$$q_1 = \lim_{x\to 1}(x-1)^2 q(x) = -\lim_{x\to 1}(x-1)^2 \frac{ab}{x(1-x)} = \lim_{x\to 1}\frac{ab}{x}(x-1) \underline{= 0|}$$

The *Regular* singular points: $\underline{x = 0,\ 1\,|}$

The indicial equation is: $r(r-1) - cr = r^2 + (c-1)r = 0 \quad \rightarrow \quad \underline{r = 0,\ 1-c\,|}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1-c}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x(1-x)y'' + \left[c-(a+b+1)x\right]y' - aby = 0$$

$$\left(x-x^2\right)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \left[c-(a+b+1)x\right]\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - ab\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}c(n+r)a_n x^{n+r-1}$$

$$-\sum_{n=0}^{\infty}(a+b+1)(n+r)a_n x^{n+r} - ab\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1) + c(n+r)\right]a_n x^{n+r-1}$$

$$-\sum_{n=0}^{\infty}\left[(n+r)(n+r-1) + (a+b+1)(n+r) + ab\right]a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1+c)a_n x^{n+r-1}-\sum_{n=1}^{\infty}\left[(n+r-1)(n+r-2+a+b+1)+ab\right]a_{n-1}x^{n+r-1}=0$$

$$r(r+c-1)a_0x^{r-1}+\sum_{n=1}^{\infty}(n+r)(n+r-1+c)a_n x^{n+r-1}$$

$$-\sum_{n=1}^{\infty}\left[(n+r-1)(n+r-1+a+b)+ab\right]a_{n-1}x^{n+r-1}=0$$

$$r(r+c-1)a_0x^{r-1}+\sum_{n=1}^{\infty}\left[(n+r)(n+r-1+c)a_n-\left((n+r-1)(n+r-1+a+b)+ab\right)a_{n-1}\right]x^{n+r-1}=0$$

For $n=0 \quad \rightarrow \quad r(r+c-1)a_0=0 \quad \Rightarrow \quad \underline{r=0,\ 1-c}\,\rfloor$ √

$$(n+r)(n+r-1+c)a_n-\left((n+r-1)(n+r-1+a+b)+ab\right)a_{n-1}=0$$

$$(n+r)(n+r-1+c)a_n=\left((n+r-1)(n+r-1+a+b)+ab\right)a_{n-1}$$

$$\underline{a_n=\frac{(n+r-1)(n+r-1+a+b)+ab}{(n+r)(n+r-1+c)}a_{n-1}}\,\rfloor$$

$$\underline{r=0 \quad \rightarrow \quad a_n=\frac{(n-1)(n-1+a+b)+ab}{n(n-1+c)}a_{n-1}}\,\rfloor$$

$$n=1 \rightarrow a_1=\frac{ab}{c}a_0$$

$$n=2 \rightarrow a_2=\frac{1+a+b+ab}{2\cdot(c+1)}a_1=\frac{(a+1)(b+1)}{2\cdot(c+1)}a_1=\frac{ab(a+1)(b+1)}{2\cdot c\cdot(c+1)}a_0$$

$$n=3 \rightarrow a_3=\frac{4+2a+2b+ab}{3\cdot(c+2)}a_2=\frac{(a+2)(b+2)}{3\cdot(c+2)}a_2=\frac{a(a+1)(a+2)\cdot b(b+1)(b+2)}{2\cdot3\cdot c(c+1)(c+2)}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\rightarrow a_n=\frac{a(a+1)(a+2)\cdots(a+n-1)\bullet b(b+1)(b+2)\cdots(b+n-1)}{n!\cdot c(c+1)(c+2)\cdots(c+n-1)}a_0$$

$$y_1(x)=a_0\left(1+\frac{ab}{c}x+\frac{ab(a+1)(b+1)}{2\cdot c\cdot(c+1)}x^2+\frac{a(a+1)(a+2)\cdot b(b+1)(b+2)}{2\cdot3\cdot c(c+1)(c+2)}x^3+\cdots\right)$$

$$\underline{=a_0\left(1+\sum_{n=0}^{\infty}\frac{a(a+1)\cdots(a+n-1)\bullet b(b+1)\cdots(b+n-1)}{n!\cdot c(c+1)\cdots(c+n-1)}x^n\right)}\,\rfloor$$

$$\underline{r=1-c \quad \rightarrow \quad b_n=\frac{(n-c)(n-c+a+b)+ab}{n(n+1-c)}b_{n-1}}\,\rfloor$$

$$n = 1 \;\rightarrow\; b_1 = \frac{(1-c)(1-c+a+b)+ab}{2-c}b_0$$

$$n = 2 \;\rightarrow\; b_2 = \frac{(2-c)(2-c+a+b)+ab}{2(3-c)}b_1 = \frac{(1-c)(1-c+a+b)+ab}{2-c}b_0$$

$$= \frac{((2-c)(2-c+a+b)+ab)((1-c)(1-c+a+b)+ab)}{2(2-c)(3-c)}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\rightarrow b_n = \frac{((n-c)(n-c+a+b)+ab)\cdots((1-c)(1-c+a+b)+ab)}{2(2-c)(3-c)\ \cdots(n+1-c)}b_0$$

$$y_2(x) = b_0 x^{1-c}\left(1+\frac{(1-c)(1-c+a+b)+ab}{2-c}x + \frac{((2-c)(2-c+a+b)+ab)((1-c)(1-c+a+b)+ab)}{2(2-c)(3-c)}x^2+\cdots\right)$$

$$y(x) = a_0\left(1+\sum_{n=0}^{\infty}\frac{a(a+1)\cdots(a+n-1)\bullet b(b+1)\cdots(b+n-1)}{n!\cdot c(c+1)\cdots(c+n-1)}x^n\right) + b_0 x^{1-c}\left(1+\sum_{n=0}^{\infty}\frac{((n-c)(n-c+a+b)+ab)\cdots((1-c)(1-c+a+b)+ab)}{2(2-c)(3-c)\ \cdots(n+1-c)}x^n\right)$$