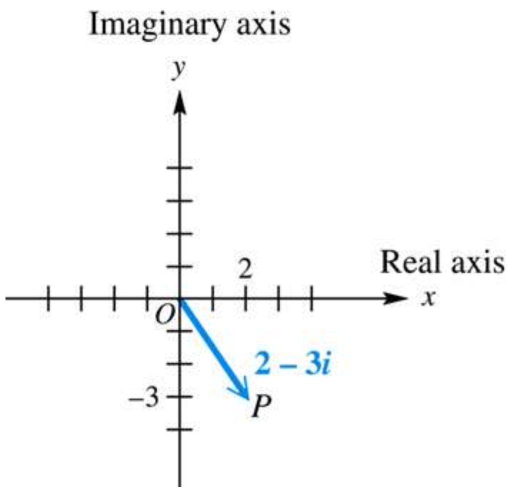
Section 4.4 – Trigonometric Form of Complex Numbers

$\sqrt{-1} = i$

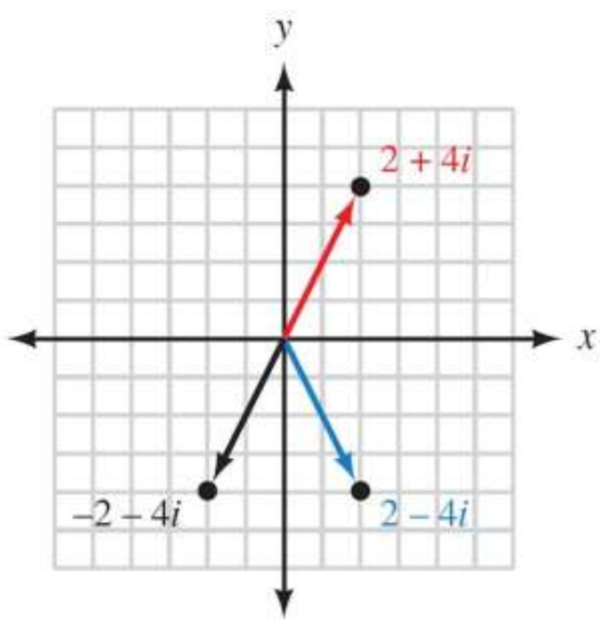
The graph of the complex number *x* = *yi* is a vector (arrow) that extends from the origin out to the point (*x*, *y*)

- Horizontal axis: ***real axis***
- Vertical axis: ***imaginary axis***



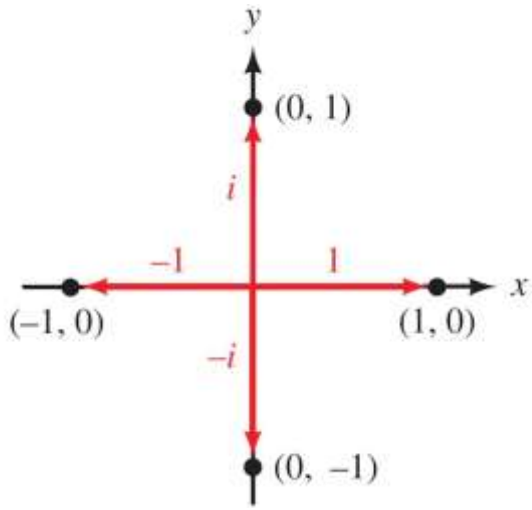
Example

Graph each complex number: $2+4i$, $-2-4i$, and $2-4i$



Example

Graph each complex number: $1,\ i,\ -1,\ and\ -i$



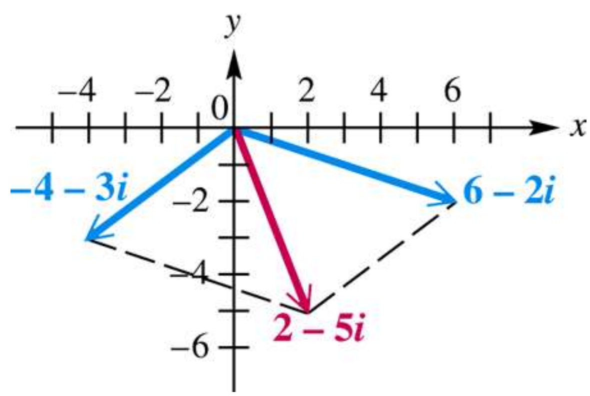
Example

Find the sum of $6 - 2i$ and $-4 - 3i$. Graph both complex numbers and their resultant.

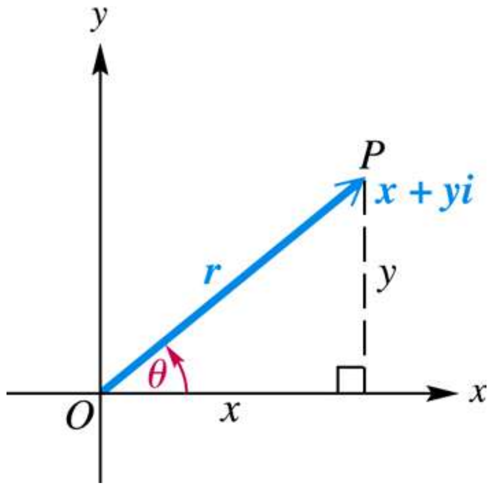
<u>*Solution*</u>

$(6 - 2i) + (-4 - 3i) = 6 - 4 - 2i - 3i$

$= 2 - 5i$



Definition



The *absolute value* or ***modulus*** of the complex number $z = x + yi$ is the distance from the origin to the point (x, y). If this distance is denoted by r, then

$$r = |z| = |x + yi| = \sqrt{x^2 + y^2}$$

The ***argument*** of the complex number $z = x + yi$ denoted $\arg(z)$ is the smallest possible angle θ from the positive real axis to the graph of z.

$$\cos\theta = \frac{x}{r} \qquad \Rightarrow x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \qquad \Rightarrow y = r\sin\theta$$

$$\begin{aligned} z &= x + yi \\ &= r\cos\theta + (r\sin\theta)\, i \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$

$\rightarrow$ is called the *trigonometric* from of z.

Definition

If $z = x + y\,i$ is a complex number in standard form then the ***trigonometric form*** for z is given by

$$z = r(\cos\theta + i\ \sin\theta) = r\ cis\theta$$

Where $\boldsymbol{r}$ is the modulus or absolute value of z and

$\boldsymbol{\theta}$ is the argument of z.

We can convert back and forth between standard form and trigonometric form by using the relationships that follow

For $z = x + y\,i = r(\cos\theta + i\sin\theta) = r\ cis\theta$

$$r = \sqrt{x^2 + y^2}$$

$$\cos\theta = \frac{x}{r},\ \sin\theta = \frac{y}{r},\ and\ \ \tan\theta = \frac{y}{x}$$

Example

Write $z = -1 + i$ in trigonometric form

<u>*Solution*</u>

The modulus r:

$$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$$\cos\theta = \frac{x}{r} = \frac{-1}{\sqrt{2}}$$

$$\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}}$$

$$\to \theta = 135^\circ$$

$$z = x + y\,i$$

$$= \sqrt{2}(\cos 135^\circ + i\sin 135^\circ)$$

$$= \sqrt{2}\ cis 135^\circ$$

In radians: $z = \sqrt{2}\ cis\left(\frac{3\pi}{4}\right)$

Example

Write $z = 2\,cis\,60°$ in rectangular form.

<u>*Solution*</u>

$$z = 2\,cis\,60°$$
$$= 2(\cos 60° + i\sin 60°)$$
$$= 2\left(\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right)$$
$$= 1 + i\sqrt{3}$$

Example

Express $2\left(\cos 300° + i\sin 300°\right)$ in rectangular form.

<u>*Solution*</u>

$$2\left(\cos 300° + i\sin 300°\right) = 2\left(\frac{1}{2} - i\,\frac{\sqrt{3}}{2}\right)$$
$$= 1 - i\sqrt{3}$$

Example

Find the modulus of each of the complex numbers $5i$, 7, and $3 + 4i$

<u>*Solution*</u>

For $z = 5i = 0 + 5i \Rightarrow r = |z| = \sqrt{0^2 + 5^2} = 5$

For $z = 7 = 7 + 0i \Rightarrow r = |z| = \sqrt{7^2 + 0^2} = 7$

For $3 + 4i \Rightarrow r = \sqrt{3^2 + 4^2} = 5$

Product Theorem

If $r_1\left(\cos\theta_1 + i\sin\theta_1\right)$ and $r_2\left(\cos\theta_2 + i\sin\theta_2\right)$ are any two complex numbers, then

$$\left[r_1\left(\cos\theta_1 + i\sin\theta_1\right)\right]\left[r_2\left(\cos\theta_2 + i\sin\theta_2\right)\right] = r_1 r_2\left[\cos\left(\theta_1+\theta_2\right) + i\sin\left(\theta_1+\theta_2\right)\right]$$

$$\left(r_1 cis\theta_1\right)\left(r_2 cis\theta_2\right) = r_1 r_2 cis\left(\theta_1+\theta_2\right)$$

$$\boxed{(a+bi)(a-bi) = a^2+b^2}$$

$$\boxed{\left(\sqrt{a}+\sqrt{b}i\right)\left(\sqrt{a}-\sqrt{b}i\right) = a+b}$$

Example

Find the product of $3(\cos 45° + i\sin 45°)$ and $2(\cos 135° + i\sin 135°)$. Write the result in rectangular form.

<u>*Solution*</u>

$$\left[3(\cos 45° + i\sin 45°)\right]\left[2(\cos 135° + i\sin 135°)\right]$$

$$= (3)(2)\left[\cos(45° + 135°) + i\sin(45° + 135°)\right]$$

$$= 6(\cos 180° + i\sin 180°)$$

$$= 6(-1 + i.0)$$

$$= -6$$

Quotient Theorem

If $r_1\left(\cos\theta_1 + i\sin\theta_1\right)$ and $r_2\left(\cos\theta_2 + i\sin\theta_2\right)$ are any two complex numbers, then

$$\frac{r_1\left(\cos\theta_1 + i\sin\theta_1\right)}{r_2\left(\cos\theta_2 + i\sin\theta_2\right)} = \frac{r_1}{r_2}\left[\cos\left(\theta_1 - \theta_2\right) + i\sin\left(\theta_1 - \theta_2\right)\right]$$

$$\frac{r_1\, cis\theta_1}{r_2\, cis\theta_2} = \frac{r_1}{r_2} cis\left(\theta_1 - \theta_2\right)$$

Example

Find the quotient $\dfrac{10cis\left(-60^\circ\right)}{5cis\left(150^\circ\right)}$. Write the result in rectangular form.

<u>*Solution*</u>

$$\begin{aligned}\frac{10cis\left(-60^\circ\right)}{5cis\left(150^\circ\right)} &= \frac{10}{5} cis\left(-60^\circ - 150^\circ\right)\\ &= 2cis\left(-210^\circ\right)\\ &= 2\left[\cos\left(-210^\circ\right) + i\sin\left(-210^\circ\right)\right]\\ &= 2\left[-\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right]\\ &= -\sqrt{3} + i\end{aligned}$$

Exercises *Section* 4.4 – Trigonometric Form of Complex Numbers

1. Write $-\sqrt{3}+i$ in trigonometric form. (Use radian measure)

2. Write $3-4i$ in trigonometric form.

3. Write $-21-20i$ in trigonometric form.

4. Write $11+2i$ in trigonometric form.

5. Write $4\left(\cos 30°+i\sin 30°\right)$ in standard form.

6. Write $\sqrt{2}\ cis\frac{7\pi}{4}$ in standard form.

7. Find the quotient $\dfrac{20cis\left(75°\right)}{4cis\left(40°\right)}$. Write the result in rectangular form.

8. Divide $z_1=1+i\sqrt{3}$ *by* $z_2=\sqrt{3}+i$. Write the result in rectangular form.