Lecture Two – Differentiation

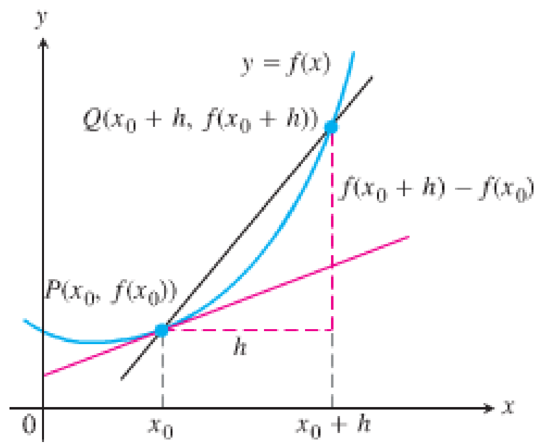
Section 2.1 –Introducing the Derivative

Definition

The slope of the curve $y = f(x)$ at the point $P\left(x_0, f\left(x_0\right)\right)$ is the number

$$m = \lim_{h\to 0} \frac{f\left(x_0 + h\right) - f\left(x_0\right)}{h}, \quad (\lim \exists)$$

The tangent line to the curve at *P* is the line through *P* with this slope.



Example

a) Find the slope of the curve $y = \frac{1}{x}$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?

b) Where does the slope equal $-\frac{1}{4}$?

c) What happens to the tangent to the curve at the point $\left(a, \frac{1}{a}\right)$ as a changes?

Solution

a) The slope of $f(x) = \frac{1}{x}$ at $\left(a, \frac{1}{a}\right)$ is

$$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\frac{a-(a+h)}{a(a+h)}$$

$$= \lim_{h\to 0} \frac{1}{h}\frac{a-a-h}{a(a+h)}$$

$$= \lim_{h\to 0} \frac{1}{h}\frac{-h}{a(a+h)}$$

$$= \lim_{h\to 0} \frac{-1}{a(a+h)}$$

$$= -\frac{1}{a^2}$$

The slope at $x=-1$ is: $= -\frac{1}{(-1)^2}$ $= -1$

b) The slope equals to $x = -\frac{1}{4}$

$\Rightarrow -\frac{1}{a^2} = -\frac{1}{4}$

$a^2 = 4 \to \boxed{a = \pm 2}$

$$\begin{array}{l} x=-2 \Rightarrow y = -\frac{1}{2} \\ x = 2 \Rightarrow y = \frac{1}{2} \end{array} \Rightarrow \left(-2,\ -\frac{1}{2}\right) \text{ and } \left(2,\ \frac{1}{2}\right)$$

c) The slope $\left(-\frac{1}{a^2}\right)$ is always negative if $a \neq 0$

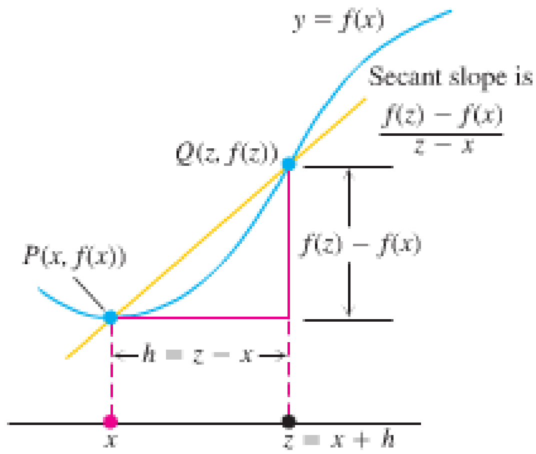
$\lim\limits_{x\to\pm\infty}\left(-\frac{1}{a^2}\right) = 0$ The slope approaches 0 and the tangent becomes horizontal.

$\lim\limits_{x\to 0^-}\left(-\frac{1}{a^2}\right) = -\infty$ The slope approaches $-\infty$ and the tangent increasingly steep.

Definition of the Derivative

The derivative of a function f at a point x_0, denoted $f'(x_0)$ is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}, \quad (\lim \exists)$$



If f' exists at a particular x, we say that f is ***differentiable*** (has a ***derivative***) at x.

If f' exists at every point in the domain of f, we call f ***differentiable***

The process of finding derivatives is called ***differentiation***.

Notations

Some common alternative notations for the derivative are

$$f'(x), \quad f', \quad \frac{d}{dx}[f(x)], \quad \frac{d}{dx} f, \quad \frac{dy}{dx}, \quad y', \quad \dot{y}, \text{ and } D_x[y]$$

Example

Differentiate $f(x) = \frac{x}{x-1}$

Solution

$$f(x+h) = \frac{(x+h)}{(x+h)-1}$$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} \\ &= \lim_{h \to 0} \frac{1}{h} \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} \end{aligned}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{x^2 - x + hx - h - x^2 - hx + x}{(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{-h}{(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)}$$

$$= \frac{-1}{(x-1)(x-1)}$$

$$\boxed{= \frac{-1}{(x-1)^2}}$$

Example

Find the derivative of $f(x) = x^2$

Solution

$$f(x+h) = (x+h)^2$$
$$= x^2 + 2hx + h^2$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{2hx + h^2}{h}$$

$$= \lim_{h\to 0} (2x+h)$$

$$\boxed{= 2x}$$

Example

a) Find the derivative of $f(x)=\sqrt{x}$ for $x>0$

b) Find the tangent line to the curve $y=\sqrt{x}$ *at* $x=4$

Solution

a) $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$=\lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$=\frac{1}{\sqrt{x}+\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x}}$$

b) The slope of the curve at $x = 4$ is: $f'(4)=\frac{1}{2\sqrt{4}}=\frac{1}{4}$

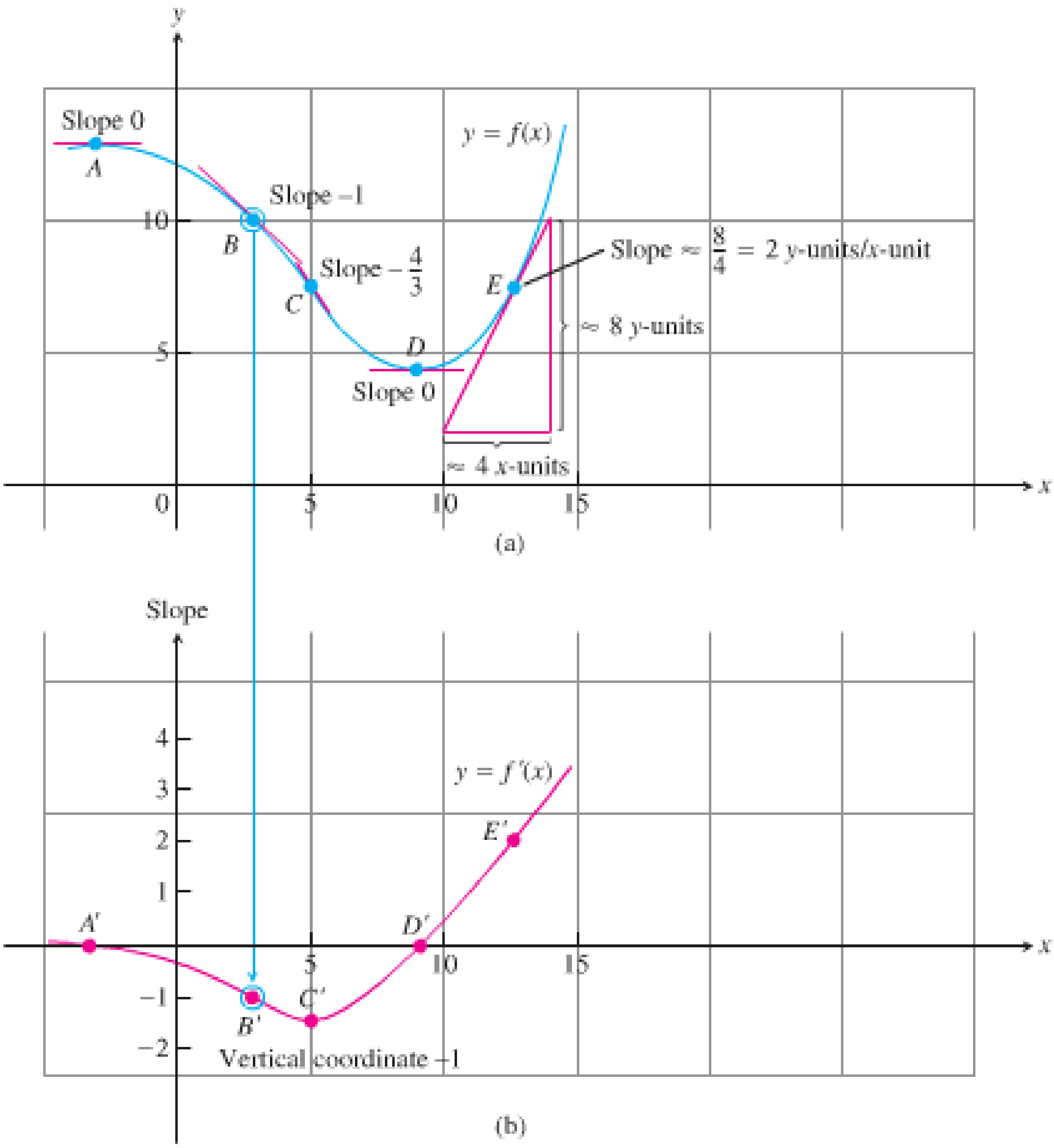
The tangent is the line through the point (4, 2) with slope $\frac{1}{4}$:

$$y-2=\frac{1}{4}(x-4)$$

$$y=\frac{1}{4}x-1+2$$

$$\boxed{y=\frac{1}{4}x+1}$$

Graphing



- ✓ The rate of change of *f* is positive, negative, or zero
- ✓ The rough size of the growth rate at any *x* and its size in relation to the size of $f(x)$
- ✓ Where the rate of change itself is increasing or decreasing.

Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

Theorem **– Differentiability Implies Continuity**

If f has a derivative at $x = c$, then f is continuous at $x = c$

Proof

Given that $f'(c)$ exists, we must show that $\lim_{x\to c} f(x) = f(c)$, or equivalently, that $\lim_{h\to 0} f(c+h) = f(c)$. If $h \neq 0$, then

$$f(c+h) = f(c) + \left(f(c+h) - f(c)\right)$$

$$= f(c) + \frac{f(c+h) - f(c)}{h} \cdot h$$

Take the limits as $h \to 0$.

$$\lim_{h\to 0} f(c+h) = \lim_{h\to 0} f(c) + \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \cdot \lim_{h\to 0} h$$

$$= f(c) + f'(c) \cdot 0$$

$$= f(c)$$

Summary

The following are all interpretations for the limit of the difference quotient, $\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$

1. The slope of the graph of $y = f(x)$ @ $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ @ $x = x_0$
3. The rate of change of $f(x)$ with respect to x @ $x = x_0$
4. The derivative $f'(x_0)$ at a point

Exercises *Section* 2.1 – Introducing the Derivative

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at *P*; then graph the curve and the tangent line.

1. $y = 4 - x^2; \quad P(-1,\ 3)$

2. $y = \frac{1}{x^2}; \quad P(-1,\ 1)$

3. $f(x) = 2\sqrt{x}; \quad P(1,\ 2)$

4. $f(x) = x^3 + 3x; \quad P(1,\ 4)$

5. $f(x) = 4x^2 - 7x + 5; \quad P(2,\ 7)$

6. $f(x) = 5x^3 + x; \quad P(1,\ 6)$

7. $f(x) = \frac{x+3}{2x+1}; \quad P(0,\ 3)$

8. $f(x) = \frac{1}{2\sqrt{3x+1}}; \quad P\left(0,\ \frac{1}{2}\right)$

9. Find the slope of the curve $y = 1 - x^2$ at the point $x = 2$

10. Find the slope of the curve $y = \frac{1}{x-1}$ at the point $x = 3$

11. Find the slope of the curve $y = \frac{x-1}{x+1}$ at the point $x = 0$

12. Find equations of all lines having slope –1 that are tangent to the curve $y = \frac{1}{x-1}$

13. What is the rate of change of the area of a circle $\left(A = \pi r^2\right)$ with respect to the radius when the radius is $r = 3$?

14. Find the slope of the tangent to the curve $y = \frac{1}{\sqrt{x}}$ at the point where $x = 4$

15. Find the values of the derivatives of the function $f(x) = 4 - x^2$. Then find the values of $f'(-3),\ f'(0),\ f'(1)$

16. Find the values of the derivatives of the function $r(s) = \sqrt{2s+1}$. Then find the values of $r'(0),\ r'\left(\frac{1}{2}\right),\ r'(1)$

17. Find the derivative of $f(x) = 3x^2 - 2x$

18. Find the derivative of *y* with the respect to *t* for the function $y = \frac{4}{t}$

19. Find the derivative of $\frac{dy}{dx}$ *if* $y = 2x^3$

20. Find the equation of the tangent line to $f(x) = x^2 + 1$ that is parallel to $2x + y = 0$

21. Differentiate the function $y = \frac{x+3}{1-x}$ and find the slope of the tangent line at the given value of the independent variable.

22. Use the definition of limits to find the derivative: $f(x) = \frac{3}{\sqrt{x}}$

23. Use the definition of limits to find the derivative: $f(x) = \sqrt{x+2}$

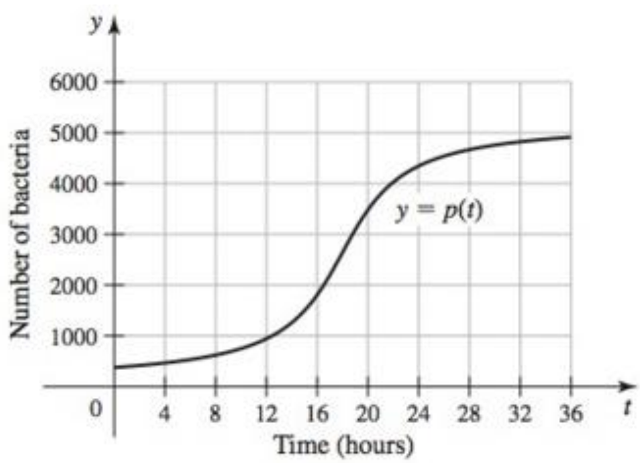
24. Suppose the height *s* of an object (in *m*) above the ground after *t* seconds is approximated by the function $s = -4.9t^2 + 25t + 1$

a) Make a table showing the average velocities of the object from time $t = 1$ to $t = 1 + h$, for $h = 0.01,\ 0.001,\ 0.0001,\ \textit{and}\ 0.00001$.

b) Use the table in part (*a*) to estimate the instantaneous velocity of the object at $t = 1$.

c) Use limits to verify your estimate in part (*b*).

25. Suppose the following graph represents the number of bacteria in a culture *t* hours after the start of an experiment.



a) At approximately what time is the instantaneous growth rate the greatest, for $0 \le t \le 36$? Estimate the growth rate at this time.

b) At approximately what time is the instantaneous growth rate the least, for $0 \le t \le 36$? Estimate the growth rate at this time.

c) What is the average growth rate over the interval $0 \le t \le 36$?