Solution   Section 1.7 – Physical Applications

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = 1 + \sin x; \quad 0 \le x \le \pi$

Solution

$$m = \int_0^{\pi} (1 + \sin x)\, dx \qquad\qquad m = \int_a^b \rho(x)\, dx$$

$$= x - \cos x \Big|_0^{\pi}$$

$$= \pi + 1 + 1$$

$$= \pi + 2 \quad \text{unit}$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = 1 + x^3; \quad 0 \le x \le 1$

Solution

$$m = \int_0^{1} \left(1 + x^3\right) dx \qquad\qquad m = \int_a^b \rho(x)\, dx$$

$$= x + \frac{1}{4}x^4 \Big|_0^{1}$$

$$= 1 + \frac{1}{4}$$

$$= \frac{5}{4} \quad \text{unit}$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = 2 - \frac{x}{2}; \quad 0 \le x \le 2$

Solution

$$m = \int_0^{2} \left(2 - \frac{x}{2}\right) dx \qquad\qquad m = \int_a^b \rho(x)\, dx$$

$$= 2x - \frac{1}{4}x^2 \Big|_0^{2}$$

$$= 4 - 1$$

$$= 3 \quad \text{unit}$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = 5e^{-2x}; \quad 0 \le x \le 4$

Solution

$$m = \int_0^4 5e^{-2x}\, dx \qquad\qquad m = \int_a^b \rho(x)\,dx$$

$$= -\frac{5}{2}e^{-2x} \Big|_0^4$$

$$= -\frac{5}{2}\left(e^{-8} - 1\right)$$

$$= \frac{5}{2}\left(1 - e^{-8}\right) \quad unit$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = x\sqrt{2 - x^2}; \quad 0 \le x \le 1$

Solution

$$m = \int_0^1 x\sqrt{2 - x^2}\, dx \qquad\qquad m = \int_a^b \rho(x)\,dx$$

$$= -\frac{1}{2}\int_0^1 \left(2 - x^2\right)^{1/2}\, d\left(2 - x^2\right)$$

$$= -\frac{1}{3}\left(2 - x^2\right)^{3/2} \Big|_0^1$$

$$= -\frac{1}{3}\left(1 - 2\sqrt{2}\right)$$

$$= \frac{1}{3}\left(2\sqrt{2} - 1\right) \quad unit$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = \begin{cases} 1 & if\ \ 0 \le x \le 2 \\ 2 & if\ \ 2 < x \le 3 \end{cases}$

Solution

$$m = \int_0^2 1\, dx + \int_2^3 2\, dx \qquad\qquad m = \int_a^b \rho(x)\,dx$$

$$= x \Big|_0^2 + \left(2x \Big|_2^3\right.$$

$= 2 + (6 - 4)$

$= 4 \quad unit$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = \begin{cases} 1 & if \ \ 0 \le x \le 2 \\ 1 + x & if \ \ 2 < x \le 4 \end{cases}$

Solution

$$m = \int_0^2 1\, dx + \int_2^4 (1 + x)\, dx \qquad\qquad m = \int_a^b \rho(x)\, dx$$

$$= x \Big|_0^2 + \left(x + \frac{1}{2}x^2\right) \Big|_2^4$$

$$= 2 + (4 + 8 - 2 - 2)$$

$$= 10 \quad unit$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = \begin{cases} x^2 & if \ \ 0 \le x \le 1 \\ x(2 - x) & if \ \ 1 < x \le 2 \end{cases}$

Solution

$$m = \int_0^1 x^2\, dx + \int_1^2 \left(2x - x^2\right) dx \qquad\qquad m = \int_a^b \rho(x)\, dx$$

$$= \frac{1}{3}x^3 \Big|_0^1 + \left(x^2 - \frac{1}{3}x^3\right) \Big|_1^2$$

$$= \frac{1}{3} + 4 - \frac{8}{3} - 1 + \frac{1}{3}$$

$$= 1 \quad unit$$

Exercise

Find the mass of a bar on the interval $0 \le x \le 9$ with a density (in *g/cm*) given by $\rho(x) = 3 + 2\sqrt{x}$

Solution

$$m = \int_0^9 \left(3 + 2x^{1/2}\right) dx \qquad\qquad m = \int_a^b \rho(x)\, dx$$

$$= 3x + \frac{4}{3}x^{3/2} \Big|_0^9$$

$$= 27 + 36$$

$$= 63 \quad g$$

Exercise

Find the mass of a 3-*m* bar on the interval $0 \le x \le 3$ with a density (in *g/m*) given by $\rho(x) = 150e^{-x/3}$

Solution

$$m = \int_0^3 150e^{-x/3}\, dx \qquad\qquad m = \int_a^b \rho(x)\,dx$$

$$= -450e^{-x/3} \Big|_0^3$$

$$= -450\left(e^{-1} - 1\right)$$

$$\boxed{= 450\left(1 - \frac{1}{e}\right)\ g}$$

Exercise

Find the mass of a bar on the interval $0 \le x \le 6$ with a density $\rho(x) = \begin{cases} 1 & if \quad 0 \le x < 2 \\ 2 & if \quad 2 \le x < 4 \\ 4 & if \quad 4 \le x \le 6 \end{cases}$

Solution

$$m = \int_0^2 dx + \int_2^4 2\, dx + \int_4^6 4\, dx \qquad\qquad m = \int_a^b \rho(x)\,dx$$

$$= x \Big|_0^2 + 2x \Big|_2^4 + 4x \Big|_4^6$$

$$= 2 + 4 + 8$$

$$\boxed{= 14 \quad unit}$$

Exercise

It takes 50 *J* of work to stretch a spring 0.2 *m* from its equilibrium position. How much work is needed to stretch it an additional 0.5 *m*?

Solution

$$W = \int_0^{0.2} kx\, dx = 50$$

$$\frac{1}{2}kx^2 \Big|_0^{0.2} = 50$$

$$\frac{1}{2}k(0.04) = 50$$

$$k = \frac{50}{.02}$$
$$= 2,500$$

$$W = \int_{0.2}^{0.7} 2,500x\, dx$$
$$= 1,250x^2 \Big|_{0.2}^{0.7}$$
$$= 1,250(0.49 - 0.04)$$
$$= 562.5 \quad J$$

Exercise

It takes 50 *N* of force to stretch a spring 0.2 *m* from its equilibrium position. How much work is needed to stretch it an additional 0.5 *m*?

Solution

$$f(0.2) = 0.2k = 50$$
$$k = \frac{50}{.2}$$
$$= 250$$

$$W = \int_{0.2}^{0.7} 250x\, dx$$
$$= 125x^2 \Big|_{0.2}^{0.7}$$
$$= 125(0.49 - 0.04)$$
$$= 56.25 \quad J$$

Exercise

A cylindrical water tank has a height of 6 *m* and a radius of 4 *m*. how much work is required to empty the full tank by pumping the water to an outflow pipe at the top of the tank?

Solution

$$W = \int_0^6 \pi\rho g 16(16 - y)\, dy$$
$$= 16\pi\rho g\left(16y - \frac{1}{2}y^2\right. \Big|_0^6$$

$$= 16\pi\rho g(96 - 18)$$

$$\boxed{= 288\pi\rho g \quad N}$$

$$\boxed{\approx 8,866,830 \quad N}$$

Exercise

Find the total force on the face of a semicircular dam with a radius of 20 *m* when its reservoir is full of water. The diameter of the semicircle is the top of the dam.

Solution

$$x^2 + (y - 20)^2 = 400$$

$$x^2 = 400 - y^2 + 40y - 400$$

$$= 40y - y^2$$

$$x = \pm\sqrt{40y - y^2}$$

$$F = \rho g \int_0^{20} (20 - y)\left(2\sqrt{40y - y^2}\right) dy$$

$$= \rho g \int_0^{20} \left(40y - y^2\right)^{1/2} d\left(40y - y^2\right)$$

$$= \frac{2}{3}\rho g\left(40y - y^2\right)^{3/2} \Big|_0^{20}$$

$$= \frac{2}{3}\rho g(800 - 400)^{3/2}$$

$$= \frac{2}{3}\rho g\left(20^2\right)^{3/2}$$

$$\boxed{= \frac{16,000}{3}\rho g \quad N}$$

$$\boxed{= 5.2 \times 10^7 \quad N}$$

Exercise

A rock climber is about to haul up 100 *N* (about 22.5 *lb.*) of equipment that has been hanging beneath her on 40 *m* rope that weighs 0.8 *N/m*. How much work will it take? (*Hint*: Solve for the rope and equipment separately, then add)

Solution

Equipment alone: $F_1 = 100\ N$

$$W_1 = \int_0^{40} 100\, dx$$

$$= 100x \Big|_0^{40}$$

$$= 4,000\ J$$

Rope: $F_2 = 0.8(40 - x)$

$$W_2 = 0.8\int_0^{40} (40 - x)\, dx$$

$$= 0.8\left(40x - \frac{1}{2}x^2\right) \Big|_0^{40}$$

$$= 0.8(1,600 - 800)$$

$$= 640\ J$$

$$Total\ Work = 4,000 + 640$$

$$= 4,640\ J$$

Exercise

A 2-oz tennis ball was served at 160 ft/sec. How much work was done on the ball to make it go this fast? (to find the ball's mass from its weight, express the weight in pounds and divide by 32 ft/sec^2, the acceleration of gravity.)

Solution

$$weight = 2\ oz\frac{1\ lb}{16\ oz}$$

$$= \frac{1}{8}\ lb$$

$$mass = \frac{\frac{1}{8}}{32}$$

$$= \frac{1}{256}\ slugs$$

$$W = \frac{1}{2}\left(\frac{1}{256}\ slug\right)\left(160\ \frac{ft}{sec}\right)^2$$

$$\approx 50\ ft - lb$$

Exercise

How many foot-pounds of work does it take to throw a baseball 90 *mph*? A baseball weights 5 *oz*.

Solution

$$weight = 5\ oz \frac{1\ lb}{16\ oz}$$
$$= \frac{5}{16}\ \ lb$$

$$90\ \ mph = 90 \frac{mi}{hr} \cdot \frac{1\ hr}{3600\ \sec} \cdot \frac{5280\ ft}{1\ mi}$$
$$= 132\ \ ft/sec$$

$$mass = \frac{1}{32}\frac{5}{16}$$
$$\boxed{= \frac{5}{512}\ \ slugs}$$

$$W = \frac{1}{2}\left(\frac{5}{512}\ \ slug\right)\left(132\ \ \frac{ft}{sec}\right)^2$$
$$\boxed{= \frac{5,445}{64}\ \ ft\text{-}lb}$$

Exercise

A 1.6-*oz* golf ball is driven off the tee at a speed of $280\ \ ft/sec$. How many foot-pounds of work are done on the ball getting it into the air?

Solution

$$weight = 1.6\ oz \frac{1\ lb}{16\ oz}$$
$$= \frac{1}{10}\ \ lb$$

$$mass = \frac{1}{32} \cdot \frac{1}{10}$$
$$\boxed{= \frac{1}{320}\ \ slugs}$$

$$W = \frac{1}{2}\left(\frac{1}{320}\ \ slug\right)\left(280\ \ \frac{ft}{sec}\right)^2$$
$$\boxed{= \frac{245}{2}\ \ ft\text{-}lb}$$

Exercise

You drove an 800-*gal* tank truck of water from the base of a mountain to the summit and discovered on arrival that the tank was only half full. You started with a full tank, climbed at a steady rate, and accomplished the 4750-*ft* elevation change in 50 *min*. Assuming that the water leaked out at a steady rate, how much work was spent in carrying water to the top? Do not count the work done in getting yourself and the truck there. Water weighs 8 *lb/gal*.

Solution

$$8\frac{lb}{gal}\cdot 800gal = 6,400\ lb$$

$$8\frac{lb}{gal}\cdot 400gal = 3,200\ lb$$

$$F(x) = 6,400\frac{2(4,750)-x}{2(4,750)}$$

$$= 6,400\left(1-\frac{x}{9,500}\right)$$

$$W = \int_0^{4750} 6,400\left(1-\frac{x}{9,500}\right)dx$$

$$= 6,400\left(x-\frac{1}{19,000}x^2\right)\Big|_0^{4,750}$$

$$= 6,400\left(4,750-\frac{4,750^2}{19,000}\right)$$

$$= 6,400(4,750)\left(1-\frac{1}{3}\right)$$

$$= 22,800,000\ ft\text{-}lb$$

Exercise

A force of 200 *N* will stretch a garage door spring 0.8 *m* beyond its unstressed strength. How far will a 300-*N* force stretch the spring? How much work does it take to stretch the spring this far from its unstressed length?

Solution

$$f(0.8) = 0.8k = 200$$

$$k = \frac{200}{0.8}$$

$$= 250$$

$$W = \int_0^{1.2} 250x\ dx$$

$$= 125x^2 \Big|_0^{1.2}$$

$$= 125(1.44)$$

$$\underline{= 180\ J}$$

Exercise

A heavy-duty shock absorber is compressed 2 *cm* from its equilibrium position by a mass of 500 *kg*. How much work is required tocompress the shock absorber 4 *cm* from its equilibrium position? (A mass of 500 *kg* exerts a force (in newtons) of 500 *g*)

Solution

Given: $F(0.02) = 500 \quad g = 9.8\ m/s^2$

$$F(0.02) = 0.02k = 500 \times 9.8 \qquad F(x) = kx = mg$$

$$k = \frac{4900}{0.02}$$

$$\underline{= 245,000}$$

$$W = \int_0^{.04} 245,000x\, dx \qquad W = \int_a^b F(x)\,dx$$

$$= 122,500\, x^2 \Big|_0^{0.04}$$

$$\underline{= 195\ J}$$

Exercise

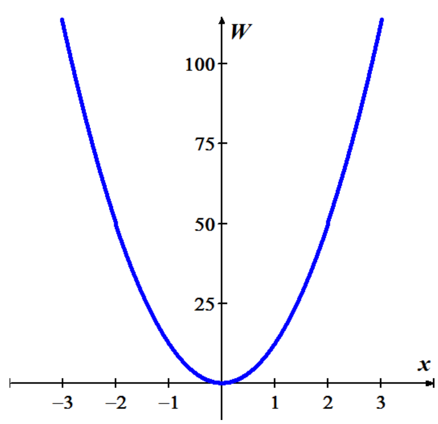
A spring has a restoring force given by $F(x) = 25x$. Let $W(x)$ be the work required to stretch the spring from its equilibrium position $(x = 0)$ to a variable distance x. Graph the work function. Compare the work required to stretch the spring x units from equilibrium to the work required to compress the spring x units from equilibrium.

Solution

$$W = \int_0^x 25t\, dt \qquad m = \int_a^b \rho(x)\,dx$$

$$= \frac{25}{2} t^2 \Big|_0^x$$

$$\underline{= \frac{25}{2} x^2}$$



Since $W(x)$ is an even function.

So that $W(-x)=W(x)$, and thus the work is the same to compress or stretch the spring a given distance from its equilibrium position.

Exercise

A swimming pool has the shape of a box with a base that measures 25 *m* by 15 *m* and a depth of 2.5 *m*. How much work is required to pump the water out of the pool when it is full?

Solution

$$W=\int_0^{2.5} \rho g A(y)(2.5-y)\,dy$$

$$=\int_0^{2.5} (1000)(9.8)(25\times 15)(2.5-y)\,dy$$

$$=3{,}675{,}000\left(2.5y-\frac{1}{2}y^2\right)\Big|_0^{2.5}$$

$$=3{,}675{,}000\left(6.25-\frac{6.25}{2}\right)$$

$$=11{,}484{,}375\ \ J$$

Exercise

Find the fluid force on a rectangular metal sheet measuring 3 *feet* by 4 *feet* that is submerged in 6 *feet* of water.

Solution

Freshwater Weight density: $\rho g=62.4\ \ lb/ft^3$

$$F=62.4(6)(3\times 4)$$

$$=4{,}492.8\ \ lbs$$

Exercise

It took 1800 *J* of work to stretch a spring from its natural length of 2 *m* to a length of 5 *m*. Find the spring's force constant

Solution

$$W=\int_0^{3} F(x)\,dx$$

$$1800 = \int_0^3 kx\, dx$$

$$1800 = \frac{1}{2}k\,x^2 \Big|_0^3$$

$$1800 = \frac{1}{2}k(9-0)$$

$$1800 = \frac{9}{2}k$$

$$1800\left(\frac{2}{9}\right) = k$$

$$\underline{k = 400 \;\; N/m}$$

Exercise

How much work is required to move am object from $x=1$ to $x=5$ (measired in meters) in the presence of a constant force at 5 N acting along the x-*axis* .

Solution

$$W = \int_1^5 5\, dx$$

$$= 5(5-1)$$

$$\underline{= 20 \;\; J}$$

Exercise

How much work is required to move am object from $x=0$ to $x=3$ (measired in meters) with a force (in N) is given by $F(x) = \frac{2}{x^2}$ acting along the x-*axis* .

Solution

$$W = \int_1^3 \frac{2}{x^2}\, dx$$

$$= -2\frac{1}{x}\Big|_1^3$$

$$= -2\left(\frac{1}{3}-1\right)$$

$$\underline{= \frac{4}{3} \;\; J}$$

Exercise

A spring on a horizontal surface can be stretched and held 0.5 *m* from its equilibrium position with a force of 50 *N*.

a) How much work is done in stretching the spring 1.5 *m* from its equilibrium position?

b) How much work is done in compressing the spring 0.5 *m* from its equilibrium position?

Solution

a) $f(x) = kx$

$f(0.5) = 50 = 0.5k$

$\rightarrow \underline{k = 100}$

$$W = \int_0^{1.5} 100x\, dx$$

$$= 50x^2 \Big|_0^{1.5}$$

$$= 112.5 \ \ J$$

b) $$W = \int_0^{-.5} 100x\, dx$$

$$= 50x^2 \Big|_0^{-0.5}$$

$$= 12.5 \ \ J$$

Exercise

Suppose a force of 10 *N* is required to stretch a spring 0.1 *m* from its equilibrium position and hold it in that position.

a) Assuming that the spring obeys Hooke's law, find the spring constant *k*.

b) How much work is needed to ***compress*** the spring 0.5 *m* from its equilibrium position?

c) How much work is needed to ***stretch*** the spring 0.25 *m* from its equilibrium position?

d) How much additional work is required to stretch the spring 0.25 *m* if it has already been stretched 0.1 *m* from its equilibrium position?

Solution

a) $F(0.1) = k(0.1) = 10$

$$k = \frac{10}{0.1}$$

$$= 100 \ \ N/m$$

Therfore, Hooke's law for this spring: $F(x) = 100x$

b) Work is needed to ***compress*** the spring

$$W = \int_0^{-0.5} 100x\, dx$$

$$= 50x^2 \Big|_0^{-0.5}$$

$$= 50(-0.5)^2$$

$$\underline{= 12.5 \;\; J}$$

c) Work is needed to ***stretch*** the spring

$$W = \int_0^{0.25} 100x\, dx$$

$$= 50x^2 \Big|_0^{0.25}$$

$$= 50(.25)^2$$

$$\underline{= 3.125 \;\; J}$$

d) Work is required to ***stretch*** the spring

$$W = \int_{0.1}^{0.35} 100x\, dx$$

$$= 50\, x^2 \Big|_{0.1}^{0.35}$$

$$= 50\left(0.35^2 - 0.1^2\right)$$

$$\underline{= 5.625 \;\; J}$$

Exercise

A force of 200 N will stretch a garage door spring 0.8-*m* beyond its unstressed length.

a) How far will a 300-N-force stretch the spring?
b) How much work does it take to stretch the spring this far?

Solution

$$k = \frac{F}{x} = \frac{200}{0.8}$$

$$\underline{= 250 \;\; N/m}$$

a) $300 = 250x$

$$\underline{x = 1.2 \;\; m}$$

b) $W = \int_0^{1.2} 250x\, dx$

$$= 125x^2 \Big|_0^{1.2}$$

$$\boxed{= 180 \quad J \ (N-m)}$$

Exercise

A spring has a natural length of 10 *in*. An 800-*lb* force stretches the spring to 14 *in*.

a) Find the force constant.

b) How much work is done in stretching the spring from 10 *in* to 12 *in*?

c) How far beyond its natural length will a 1600-*lb* force stretch the spring?

Solution

a) $k = \frac{F}{x}$

$$= \frac{800}{14-10}$$

$$= \frac{800}{4}$$

$$\boxed{= 200 \ \text{lb}/in}$$

b) $\Delta x = 12 - 10 = 2 \ in$

$$W = k\int_0^2 x\,dx$$

$$= 200\left(\frac{1}{2}x^2\right)\Big|_0^2$$

$$= 100(4-0)$$

$$= 400 \ in.lb$$

$$= 400\frac{1\,ft}{12\,in} \ in.lb$$

$$\boxed{= 33.3 \ ft \cdot lb}$$

c) $F = 200x$

$$1600 = 200x$$

$$\frac{1600}{200} = x$$

$$\boxed{x = 8 \ in}$$

Exercise

It takes a force of 21,714 *lb* to compress a coil spring assembly on a Transit Authority subway car from its free height of 8 *in*. to its fully compressed height of 5 *in*.

a) What is the assembly's force constant?

b) How much work does it take to compress the assembly the first half inch? The second half inch? Answer to the nearest *in-lb*.

Solution

a) $F = kx$

$$21714 = k(8-5)$$

$$21714 = 3k$$

$$k \underline{= 7238 \ \text{lb}/in}$$

b) $W = k\int_0^{0.5} x\,dx$

$$= 7{,}238 \left(\tfrac{1}{2}x^2 \right|_0^{0.5}$$

$$= 7{,}238\left(\tfrac{1}{2}(0.5)^2\right)$$

$$\underline{= 905 \ \ in \cdot lb}$$

$$W = 7{,}238\int_{0.5}^{1} x\,dx$$

$$= 7238 \left(\tfrac{1}{2}x^2 \right|_{0.5}^{1}$$

$$= 3{,}619(1-.25)$$

$$\underline{= 2{,}714 \ \ in \cdot lb}$$

Exercise

A mountain climber is about to haul up a 50 *m* length of hanging rope. How much work will it take if the rope weighs 0.624 *N/m*?

Solution

$$W = 0.624\int_0^{50} x\,dx$$

$$= \frac{624}{1{,}000}\left(\tfrac{1}{2}x^2 \right|_0^{50}$$

$$= \frac{156}{500}(2,500)$$

$$= 780 \quad J$$

Exercise

A bag of sand originally weighing 144 *lb* was lifted a constant rate. As it rose, sand also leaked out at a constant rate. The sand was half gone by the time the bag had been lifted to 18 *ft*. How much work was done lifting the sand this far? (Neglect the weight of the bag and lifting equipment.)

Solution

The weight of sands decreases by $\frac{1}{2}144 = 72\ lb$ over the 18 ft. at rate $\frac{72}{18} = 4\ lb\,/\,ft$

$$F(x) = 144 - 4x$$

$$W = \int_0^{18} (144 - 4x)\,dx$$

$$= 144x - 2x^2 \Big|_0^{18}$$

$$= 144(18) - 2(18)^2 - (0)$$

$$= 1944 \quad ft \cdot lb$$

Exercise

An electric elevator with a motor at the top has a multistrand cable weighing 4.5 *lb/ft*. When the car is at the first floor, 180 *feet* of cable are paid out, and effectively 0 *foot* are out when the car is at the top floor. How much work does the motor do just lifting the cable when it takes the car from the first floor to the top?

Solution

$$F(x) = k\Delta x = 4.5(180 - x)$$

$$W = \int_0^{180} 4.5(180 - x)\,dx$$

$$= 4.5\left(180x - \frac{1}{2}x^2\right.\Big|_0^{180}$$

$$= 4.5\left[180(180) - \frac{1}{2}(180)^2 - 0\right]$$

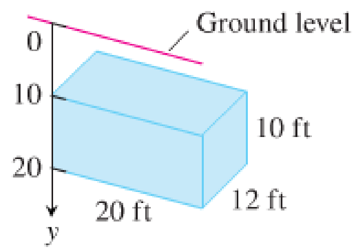
$$= 72,900 \quad ft \cdot lb$$

Exercise

The rectangular cistern (storage tank for rainwater) shown has its top 10 *ft* below ground level. The cistern, currently full, is to be emptied for inspection by pumping its contents to ground level. Assume that the water weighs $62.4\ lb/ft^3$

a) How much work will it take to empty the cistern?
b) How long will it take a 1/-*hp* pump, rated at 275 *ft-lb/sec*, to pump the tank dry?
c) How long will it take the pump in part (*b*) to empty the tank hallway? (It will be less than half the time required to empty the tank completely)
d) What are the answers to parts (*a*) through (*c*) in a location where water weighs 62.6 lb/ft^3? 62.59 lb/ft^3?

Solution



a) $\Delta V = (20)(12)\Delta y = 240\Delta y$

$$F = 62.4(\Delta V)$$
$$= (62.4)240\Delta y$$
$$= 14976\Delta y$$

$$\Delta W = force \times distance$$
$$= 14976\ \Delta y \times y$$

$$W = 14976\int_{10}^{20} y\,dy$$
$$= 14976\left(\frac{1}{2}y^2\right)\Big|_{10}^{20}$$
$$= \frac{14976}{2}\left(20^2 - 10^2\right)$$
$$\boxed{= 2{,}246{,}400\ \ ft\cdot lb}$$

b) $t = \dfrac{W}{275\frac{ft.lb}{\sec}}$

$$= \frac{2{,}246{,}400\ ft.lb}{275}\frac{\sec}{ft.lb}$$
$$\approx 8{,}168.73\ sec\frac{1\ \ hr}{3600\ sec}$$
$$\boxed{\approx 2.27\ \ hrs}\quad 2\ hrs\ \ \&\ \ 16.1\min$$

c) $W = 14976\displaystyle\int_{10}^{15} y\,dy$

$$= 14976\left(\frac{1}{2}y^2\right)\Big|_{10}^{15}$$

$$=\frac{14976}{2}\left(15^2-10^2\right)$$

$$\underline{=936,000\ \ ft\cdot lb}$$

$$t=\frac{W}{275\frac{ft.lb}{\sec}}$$

$$=\frac{936,000\ ft.lb}{275}\frac{\sec}{ft.lb}$$

$$\approx 3403.64\ sec\frac{1\ \ \min}{60\ sec}$$

$$\underline{\approx 56.7\ \ min}$$

d) ***Water weighs*** $62.26\ lb/ft^3$

$$W=(62.26)(240)(150)$$

$$\underline{=2,214,360\ \ ft\cdot lb}$$

$$t=\frac{W}{275\frac{ft.lb}{\sec}}$$

$$=\frac{2,241,360\ ft.lb}{275}\frac{\sec}{ft.lb}$$

$$\approx 8,150.4\ sec\frac{1\ \ hr}{3600\ sec}$$

$$\underline{\approx 2.264\ \ hrs}\quad 2\ hrs\ \ \&\ \ 15.8\ min$$

$$W=(62.26)(240)\left(\frac{150}{2}\right)$$

$$\underline{=933,900\ \ ft\cdot lb}$$

$$t=\frac{W}{275\frac{ft.lb}{\sec}}$$

$$=\frac{933,900\ ft.lb}{275}\frac{\sec}{ft.lb}$$

$$\approx 3396\ sec\frac{1\ \ \min}{60\ sec}$$

$$\underline{\approx 56.6\ \ min}$$

Water weighs $62.59\ lb/ft^3$

$$W=(62.59)(240)(150)$$

$$\underline{=2,253,240\ \ ft\cdot lb}$$

$$t=\frac{W}{275\frac{ft.lb}{\sec}}$$

$$=\frac{2,253,240\ ft.lb}{275}\frac{\sec}{ft.lb}$$

$$\approx 8{,}193.60\ sec \frac{1\ \ hr}{3600\ sec}$$

$$\underline{\approx 2.276\ \ hrs} \quad 2\ hrs\ \ \&\ \ 16.56\,\text{min}$$

$$W = (62.59)(240)\left(\frac{150}{2}\right)$$

$$\underline{= 938{,}850\ \ \ ft \cdot lb}$$

$$t = \frac{W}{275\frac{ft.lb}{\text{sec}}}$$

$$= \frac{938{,}850\ ft.lb}{275}\frac{\text{sec}}{ft.lb}$$

$$\approx 3414\ sec \frac{1\ \ \text{min}}{60\ sec}$$

$$\underline{\approx 56.9\ \ min}$$

Exercise

When a particle of mass m is at $(x, 0)$, it is attracted toward the origin with a force whose magnitude is $\frac{k}{x^2}$.

If the particle starts from rest at $x = b$ and is acted on by no other forces, find the work done on it by the time reaches $x = a$, $0 < a < b$.

Solution

$$F(x) = -\frac{k}{x^2}$$

$$W = \int_a^b -\frac{k}{x^2}\,dx$$

$$= -k\int_a^b \frac{1}{x^2}\,dx \qquad\qquad \int \frac{1}{x^2}\,dx = -\frac{1}{x}$$

$$= k\ \frac{1}{x}\Big|_a^b$$

$$= k\left(\frac{1}{b} - \frac{1}{a}\right)$$

$$\underline{= \frac{k(a-b)}{ab}}$$

Exercise

The strength of Earth's gravitation field varies with the distance r from Earth's center, and the magnitude of the gravitational force experienced by a satellite of mass m during and after launch is

$$F(r)=\frac{mMG}{r^2}$$

Here, $M=5.975\times 10^{24}\ kg$ is Earth's mass, $G=6.6720\times 10^{-11}\ N\cdot m^2 kg^{-2}$ is the universal gravitational constant, and r is measured in meters. The work it takes to lift a 1000-*kg* satellite from Earth's surface to a circular orbit 35,780 *km* above Earth's center is therefore given by the integral

$$W=\int_{6,370,000}^{35,780,000}\frac{1000MG}{r^2}\,dr\ \ joules$$

Evaluate the integral. The lower limit of integration is Earth's radius in meters at the launch site. (This calculation does not take into account energy spend lifting the launch vehicle or energy spent bringing the satellite to orbit velocity.)

Solution

$$\begin{aligned}W&=1000MG\int_{6,370,000}^{35,780,000}\frac{dr}{r^2}\\&=1000MG\left(-\frac{1}{r}\right|_{6,370,000}^{35,780,000}\\&=1000\left(5.975\times 10^{24}\right)\left(6.6720\times 10^{-11}\right)\left(\frac{1}{6,370,000}-\frac{1}{35,780,000}\right)\\&\approx 5.144\times 10^{10}\ \ J\end{aligned}$$

Exercise

You drove an 800-*gal* truck from the base of Mt. Washington to the summit and discovered on arrival that the tank was only half full. You started with a full tank, climbed at a steady rate, and accomplished the 4750-*ft* elevation change in 50 *minutes*.

Assuming that the water leaked out at a steady rate, how much work was spent in carrying the water to the top? Do not count the work done in getting yourself and the truck there. Water weighs 8- *lb./gal*.

Solution

The force required to lift the water is equal to the water's weight which varies $8(800)\ lbs.$ to $8(400)\ lbs.$ over the 4750 *ft* change in elevation. Since it looses half of the water when the truck reaches its destination, it would loose all of the water if it went twice the distance. When the truck is x feet of the base of Mt. Washington, the water's weight is the following proportion.

$$F(x)=8(800)\left(\frac{2(4750)-x}{2(4750)}\right)=6400\left(1-\frac{x}{9500}\right)$$

$$= 6400\left(1-\frac{x}{9500}\right)$$

$$W = 6,400\int_0^{4750}\left(1-\frac{x}{9,500}\right)dx$$

$$= 6,400\left(x-\frac{x^2}{19,000}\right.\Bigg|_0^{4750}$$

$$\underline{= 22,800,000 \quad ft-lbs}$$

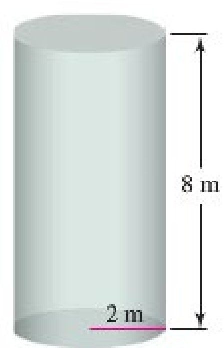
Exercise

A cylindrical water tank has height 8 *m* and radius 2 *m*

a) If the tank is full of water, how much work is required to pump the water to the level of the top of the tank and out of the tank?

b) Is it true it takes half as much work to pump the water out of the tank when it is half full as when it is full? Explain.

Solution



a) $W = \rho g\int_0^a (a-y)w(y)dy$

$$= 1,000(9.8)\int_0^8 (8-y)\left(2^2\pi\right)dy$$

$$= 39,200\pi\left(8y-\frac{1}{2}y^2\right.\Bigg|_0^8$$

$$= 39,200\pi(64-32)$$

$$= 125,400\pi$$

$$\underline{\approx 3.941\times 10^6 \quad J}$$

b) The work done pumping the water from a half-full tank

$$W = 9,800(4\pi)\int_4^8 (8-y)dy$$

$$= 39,200\pi\left(8y-\frac{1}{2}y^2\right.\Bigg|_4^8$$

$$= 39,200\pi(32-32+8)$$

$$\underline{\approx 985,203 \quad J}$$

To empty a half-full tank, the work is

$$W = 39,200\pi \left(8y - \frac{1}{2}y^2\right|_0^4$$

$$= 39,200\pi(32-16)$$

$$\approx 2.9556 \times 10^6 \ J$$

NO, it is not true.

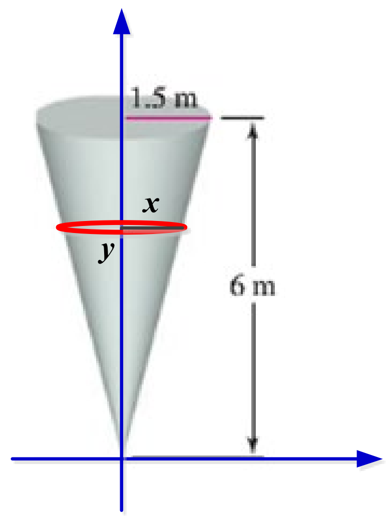
Exercise

A water tank is shaped like an inverted cone with height 6 *m* and base radius 1.5 *m*.

a) If the tank is full of water, how much work is required to pump the water to the level of the top of the tank and out of the tank?

b) Is it true it takes half as much work to pump the water out of the tank when it is half full as when it is full? Explain.

Solution



$$\frac{x}{1.5} = \frac{y}{6} \quad \to \quad x = \frac{y}{4}$$

$$Area = \pi x^2$$

$$= \frac{\pi}{16}y^2$$

a) $W = \rho g \int_0^a A(y)(a-y)\,dy$

$$= 9,800 \int_0^6 \frac{\pi}{16}y^2(6-y)\,dy$$

$$= 612.5\pi \int_0^6 \left(6y^2 - y^3\right)dy$$

$$= 612.5\pi \left(2y^3 - \frac{1}{4}y^4\right|_0^6$$

$$= 612.5\pi(432-324)$$

$$= 66,150\pi \ J$$

b) $W = 9,800 \int_0^3 \frac{\pi}{16}y^2(6-y)\,dy$

$$= 612.5\pi \left(2y^3 - \frac{1}{4}y^4\right|_0^3$$

$$= 612.5\pi(54-20.25)$$

$$\approx 20,672\pi \ J$$

The work done is less than half the half amount from part (*a*).

It is not true, while the water must be raised further than water in the top half, due to the shape of the tank, there is far less water in the bottom half than in the top.

Exercise

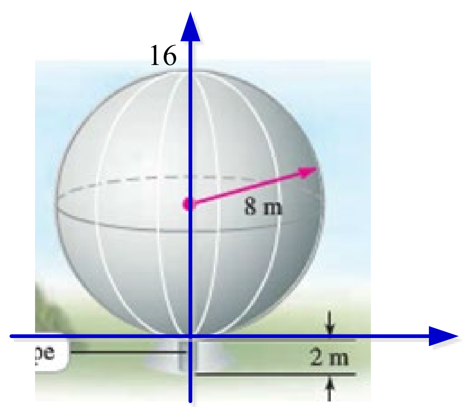
A spherical water tank with an inner radius of 8 *m* has its lowest point 2 *m* above the ground. It is filled by a pipe that feed the tank at its lowest point.

a) Neglecting the volume of the inflow pipe, how much work is required to fill the tank if it is initially empty?

b) Now assume that the inflow pipe feeds the tank at the top of the tank. Neglecting the volume of the inflow pipe, how much work is required to fill the tank if it is initially empty?

Solution

a) Equation of the tank:



$$x^2+(y-8)^2=64$$

$$x^2=16y-y^2$$

$$A(y)=\pi x^2$$

$$=\pi\left(16y-y^2\right)$$

$$W=\rho g\pi\int_0^{16}\left(16y-y^2\right)dy$$

$$=9,800\pi\left(8y^2-\frac{1}{3}y^3\right)\Big|_0^{16}$$

$$=9800\pi\left(16^2\right)\left(8-\frac{16}{3}\right)$$

$$\approx 2.102\times 10^8\ \ J$$

b) The toal weight of the water lifted up for 18 *m* is

$$W=\frac{4\pi}{3}R^3\rho gh$$

$$=\frac{4\pi}{3}8^3(9800)(18)$$

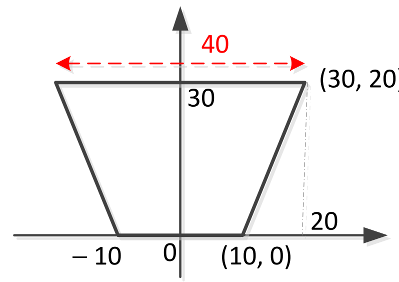
$$=120,422,400\,\pi\ \ J$$

$$\approx 3.783\times 10^8\ \ J$$

Exercise

A large vertical dam in the shape of a symmetric trapezoid has a height of 30 *m*. a width of 20 *m*. at its base, and a width of 40 *m*. at the top. What is the total force on the face of the dam when the reservoir is full?

$\left(\rho = 1000\dfrac{kg}{m^3},\ g = 9.8\dfrac{m}{s^2}\right)$



Solution

$$y - 0 = \frac{30}{10}(x - 10)$$

$$y = 3x - 30$$

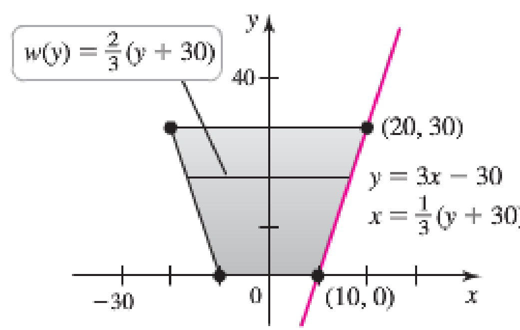
$$x = \frac{1}{3}(y + 30)$$

$$Width:\ w(y) = 2x$$

$$= \frac{2}{3}(y + 30)$$

$$Depth:\ 30 - y$$

$$Boundary:\ 0 \le y \le 30$$



$$F = \rho g \int_0^a (a - y)\, w(y)\, dy$$

$$= \left(10^3\right)(9.8) \int_0^{30} (30 - y)\frac{2}{3}(y + 30)\, dy$$

$$= \frac{19,600}{3} \int_0^{30} \left(900 - y^2\right) dy$$

$$= \frac{19,600}{3} \left(900y - \frac{1}{3}y^3\right) \Big|_0^{30}$$

$$= \frac{19600}{3}(27000 - 9000)$$

$$= 1.176 \times 10^8\ kg$$

Exercise

A vertical gate in a dam has the shape of an isosceles trapezoid 8 *feet* across the top and 6 *feet* across the bottom. With a height of 5 *feet*. What is the fluid force on the gate if the top of the gate is 4 *feet* below the surface of the water?

Solution

Freshwater Weight density: $\rho g = 62.4\ \ lb/ft^3$

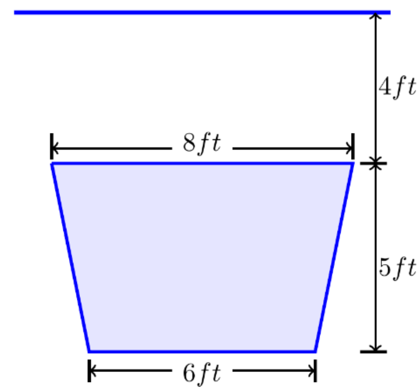
$$y = \frac{-9+4}{3-4}(x-4) - 4$$
$$= 5(x-4) - 4$$
$$= 5x - 24$$

$$x = \frac{1}{5}(y+24)$$

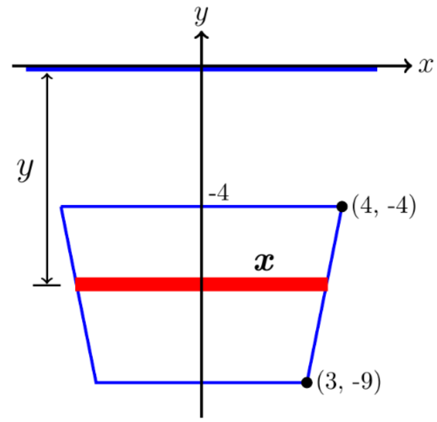
Width: $w(y) = 2x$
$$= \frac{2}{5}(y+24)$$

Depth: $-y$

Boundary: $-9 \le y \le -4$



$$F = 62.4\int_{-9}^{-4}(-y)\frac{2}{5}(y+24)\,dy$$
$$= -\frac{624}{25}\int_{-9}^{-4}\left(y^2+24y\right)dy$$
$$= -\frac{624}{25}\left(\frac{1}{3}y^3 + 12y^2\right.\Big|_{-9}^{-4}$$
$$= -\frac{624}{25}\left(-\frac{64}{3} + 192 + 243 - 972\right)$$
$$= -\frac{624}{25}\left(-\frac{64}{3} - 537\right)$$
$$= \frac{624}{25}\left(\frac{1,675}{3}\right)$$
$$= 13,936 \quad lbs$$



Exercise

Pumping water from a lake 15-*feet* below the bottom of the tank can fill the cylindrical tank shown here.

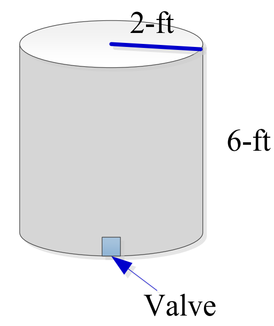
There are two ways to go about it. One is to pump the water through a hose attached to a valve in the bottom of the tank. The other is to attach the hose to the rim of the tank and let the water pour in. Which way will be faster? Give reason for answer

Solution

The water is being pumped from a lake that is 15-*feet* below the tank. That is the distance it takes to get the water from the lake to the valve, but this is not the total distance that the water is moved. We are forcing the water up into the tank, so the water travels a distance of y in the tank.

The total distance the water travels is $15 + y$.

The cross-section is a circle: $A(y) = \pi r^2 = 4\pi$



$$W = 62.4\int_0^6 (4\pi)(15+y)\,dy$$

$$= 249.6\,\pi\left(15y + \frac{1}{2}y^2\right|_0^6$$

$$= 249.6\pi(90+18)$$

$$\approx 84,687.3\ \ ft-lbs$$

Now we are pumping the water to the top of the tank and letting it pour in.
Therefore, the distance that the water is pumped is $15+6=21\,ft$

$$W = 62.4\int_0^6 21(4\pi)\,dy$$

$$= 16,466.97\ \left(y\right|_0^6$$

$$\approx 98,801.83\ \ ft-lbs$$

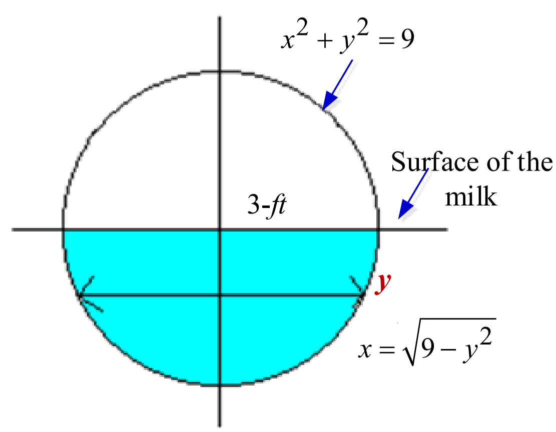
Exercise

A tank truck hauls milk in a 6-*feet* diameter horizontal right circular cylindrical tank. How must force does the milk exert on each end of the tank when the tank is half full?

Solution

Diameter = 6 $\rightarrow r = 3$

Circular cylinder: $x^2 + y^2 = 9$



$$L(y) = 2x$$

$$= 2\sqrt{9-y^2}$$

Weight density of milk is $64.5\ \ lbs\,/\,ft^3$

$$F = 64.5\int_{-3}^{0} 2\sqrt{9-y^2}\,(0-y)\,dy$$

$$= 64.5\int_{-3}^{0}\left(9-y^2\right)^{1/2}\ d\left(9-y^2\right)$$

$$= 64.5\left(\frac{2}{3}\right)\left(9-y^2\right)^{3/2}\Big|_{-3}^{0}$$

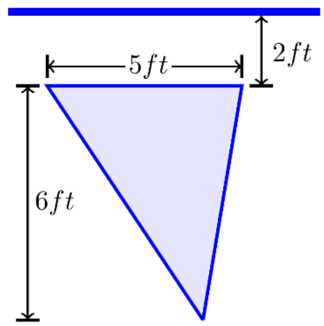
$$= 43(27)$$

$$= 1,161\ \ lbs$$

Exercise

A triangular plate, base 5 *feet*, height 6 *feet*, is submerged in water, vertex down, plane vertical, and 2 *feet* below the surface. Find the total force on one face of the plate.

Solution



Freshwater Weight density: $\rho g = 62.4\ \ lb/ft^3$

$\frac{L}{5} = \frac{y}{6}$

Width: $L(y) = \frac{5}{6}y$

Depth: $8 - y$

Boundary: $0 \le y \le 6$

$$F = 62.4\int_0^6 (8-y)\left(\frac{5}{6}y\right)dy$$

$$F = \int_0^a \rho g(a-y)L(y)\,dy$$

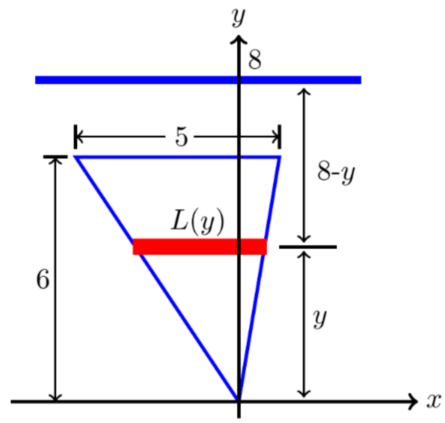
$$= 10.4(5)\int_0^6 \left(8y - y^2\right)dy$$

$$= 52\left(4y^2 - \frac{1}{3}y^3\right)\Big|_0^6$$

$$= 52(144 - 72)$$

$$= 52(72)$$

$$\boxed{= 3,744\ \ lbs}$$



Exercise

The vertical triangular plate shown here is the end plate of a trough full of water. What is the fluid force against the plate?

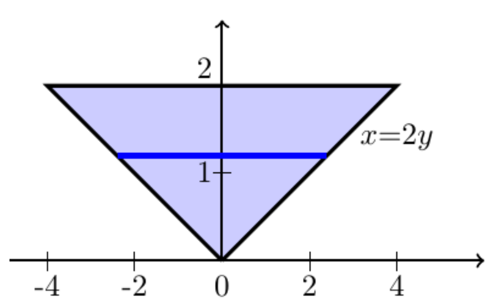
Solution

$L(y) = 2x = 4y$

$$F = 62.4\int_0^2 (2-y)\cdot(4y)\,dy$$

$$= 249.6\int_0^2 \left(2y - y^2\right)dy$$

$$= 249.6\left(y^2 - \frac{1}{3}y^3\right)\Big|_0^2$$



$$= 249.6\left(4-\frac{8}{3}\right)$$

$$\underline{= 332.8 \quad lbs}$$

Exercise

A cylindrical gasoline tank is placed so that the axis of the cylinder is horizontal. Find the fluid force on a circular end of the tank if the tank is ***half full*** assuming that the diameter is 3 *feet* and the gasoline weighs 42 *pounds* per *cubic foot*.

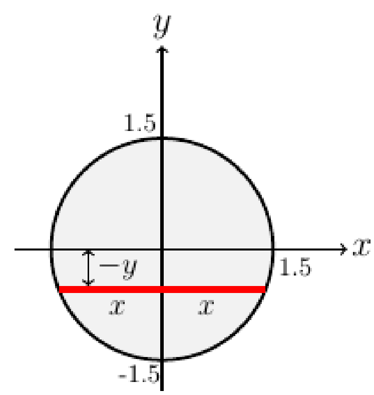
Solution

Weight density: $42\ lb/ft^3$

$$x^2+y^2=\left(\frac{3}{2}\right)^2$$

$$x=\sqrt{\frac{9}{4}-y^2}$$

$$=\frac{1}{2}\sqrt{9-4y^2}$$



Width: $L(y)=2x$

$$\boxed{=\sqrt{9-4y^2}}$$

Depth: $\boxed{-y}$

Boundary: $-\frac{3}{2}\le y\le 0$

$$F=42\int_{-\frac{3}{2}}^{0}(-y)\left(\sqrt{9-4y^2}\right)dy$$

$$=\frac{42}{8}\int_{-\frac{3}{2}}^{0}\left(9-4y^2\right)^{1/2}d\left(9-4y^2\right)$$

$$=\frac{21}{4}\left(\frac{2}{3}\right)\left(9-4y^2\right)^{3/2}\Big|_{-\frac{3}{2}}^{0}$$

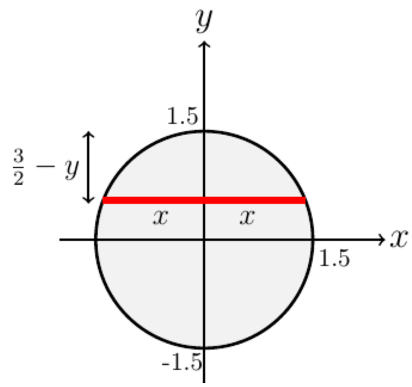
$$=\frac{7}{2}(27-0)$$

$$\underline{=\frac{189}{2}\ lb} \qquad \underline{=94.5\ lb}$$

Exercise

A cylindrical gasoline tank is placed so that the axis of the cylinder is horizontal. Find the fluid force on a circular end of the tank if the tank is ***full*** assuming that the diameter is 3 *feet* and the gasoline weighs 42 *pounds* per *cubic foot*.

Solution



Weight density: 42 lb/ft^3

$$x^2+y^2=\left(\frac{3}{2}\right)^2$$

$$x=\sqrt{\frac{9}{4}-y^2}$$

$$=\frac{1}{2}\sqrt{9-4y^2}$$

Width: $L(y)=2x$

$$=\sqrt{9-4y^2}$$

Depth: $\frac{3}{2}-y$

Boundary: $-\frac{3}{2}\le y\le\frac{3}{2}$

$$F=42\int_{-\frac{3}{2}}^{\frac{3}{2}}\left(\frac{3}{2}-y\right)\left(\sqrt{9-4y^2}\right)dy$$

$$=63\int_{-\frac{3}{2}}^{\frac{3}{2}}\left(9-4y^2\right)^{1/2}dy-\frac{21}{4}\int_{-\frac{3}{2}}^{\frac{3}{2}}\left(9-4y^2\right)^{1/2}d\left(9-4y^2\right)$$

$\int_{-r}^{r}\sqrt{r^2-y^2}\,dy=$ *Area of a circle*

$$=63\pi\left(\frac{3}{2}\right)^2-\frac{7}{2}\left(9-4y^2\right)^{3/2}\Big|_{-\frac{3}{2}}^{\frac{3}{2}}$$

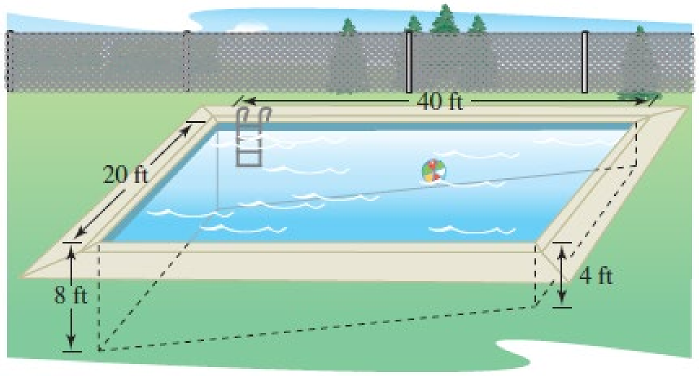
$$=\frac{567}{4}\pi-\frac{7}{2}(0-0)$$

$$=\frac{567}{4}\pi\ \ lb$$

$$\approx 445.32\ \ lb$$

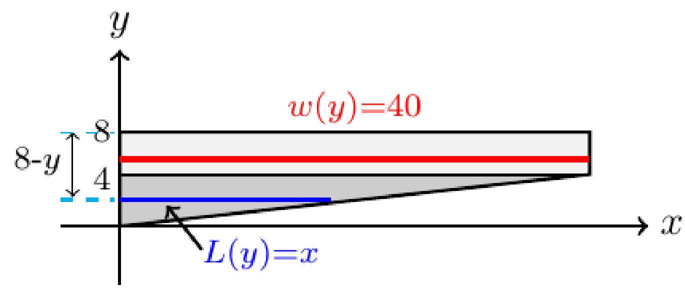
Exercise

A swimming pool is 20 *feet* wide, 40 *feet* long, 4 *feet* deep at one end, and 8 *feet* deep at the other end. The bottom is an inclined plane. Find the fluid force on each vertical wall.



Solution

Depth: $\underline{8-y|}$



From 0–4 *ft*:

$$y=\frac{4}{40}x$$

$$x=10y$$

Width: $L(y)=10y$

Boundary: $0\le y\le 4$

From 4–8 *ft*:

Width: $w(y)=40$

Boundary: $4\le y\le 8$

$$F=62.4\int_0^4(8-y)(10y)\,dy\ +\ 62.4\int_4^8(8-y)(40)\,dy$$

$$=624\int_0^4\left(8y-y^2\right)dy\ +\ 2,496\int_4^8(8-y)\,dy$$

$$=624\left(4y^2-\frac{1}{3}y^3\right.\Big|_0^4\ +\ 2,496\left(8y-\frac{1}{2}y^2\right.\Big|_4^8$$

$$=624\left(64-\frac{64}{3}\right)+2,496(64-32-32+8)$$

$$=39,936\left(1-\frac{1}{3}\right)+2,496(8)$$

$$=26,624+19,968$$

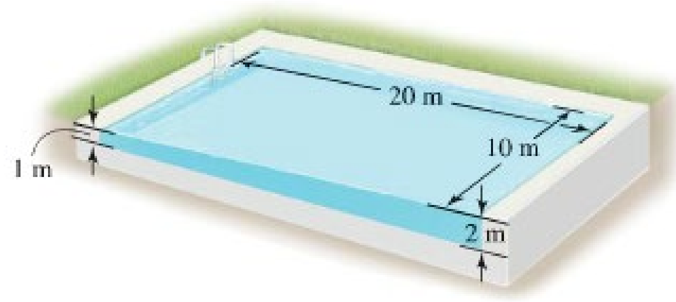
$$\underline{=46,592\quad lb\,|}$$

Exercise

A swimming pool is 20 *m* leg long and 10 *m* wide, with a bottom that slopes uniformly from a depth of 1 *m* at one end to a depth of 2 *m* at the other end. Assuming the pool is full, how much work is required to pump the water to a level 0.2 *m* above the top of the pool?

Solution

Depth: $(2+.2)-y=2.2-y$



From 0–1 *m*:

$$y=\frac{1}{20}(10-x)$$

$$10-x=20y$$

$$A(y)=10(20y)$$

$$\underline{=200y}$$

From 1–2 *m*:

$$A(y)=10(20)$$

$$\underline{=200}$$

$$W=\rho g\int_0^1 200y(2.2-y)\,dy+\rho g\int_1^2 200(2.2-y)\,dy$$

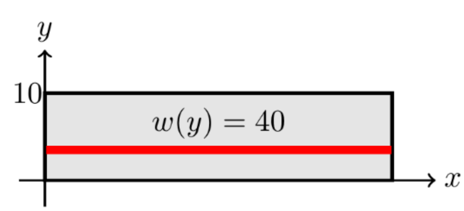
$$=(200\rho g)\left\{\left(1.1y^2-\frac{1}{3}y^3\right|_0^1+\left(2.2y-\frac{1}{2}y^2\right|_1^2\right\}$$

$$=\left(1.96\times10^6\right)\left(1.1-\frac{1}{3}+4.4-2-2.2+\frac{1}{2}\right)$$

$$\underline{\approx 2.875\times10^6\ J}$$

Exercise

Find the total force on the face of the given dam



Solution

Freshwater Weight density: $9.8\times10^3\ kg/m^3$

$$F=\int_0^{10}\left(9.8\times10^3\right)(10-y)(40)\,dy$$

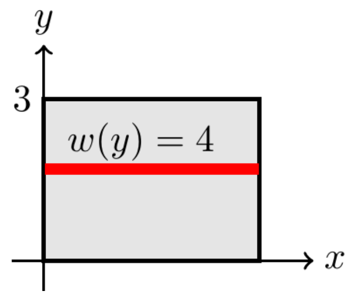
$$=392\times10^3\left(10y-\frac{1}{2}y^2\right|_0^{10}$$

$$=392\times10^3(100-50)$$

$$\underline{=196\times10^5\ N}$$

Exercise

Find the total force on the face of the given dam



Solution

Freshwater Weight density: $62.4\ lb/ft^3$

$$F = 62.4\int_0^3 (3-y)(4)\,dy$$

$$= 249.6\left(3y - \frac{1}{2}y^2\right)\Big|_0^3$$

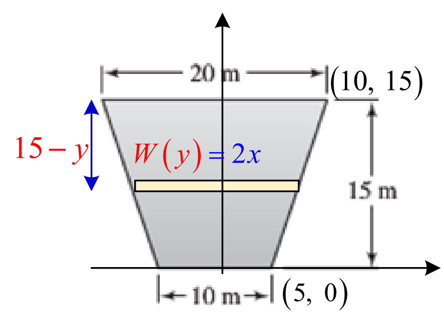
$$= 249.6\left(9 - \frac{9}{2}\right)$$

$$= 249.6\left(\frac{9}{2}\right)$$

$$\boxed{= 1{,}123.2\ \ lbs}$$

Exercise

Find the total force on the face of the given dam



Solution

Freshwater Weight density: $\rho = 10^3\ kg/m^3$

$$y = \frac{15-0}{10-5}(x-5)$$

$$= 3x - 15$$

$$x = \frac{1}{3}(y+15)$$

$$\Rightarrow\ \boxed{2x = \frac{2}{3}(y+15)}$$

$$F = \left(10^3\right)(9.8)\int_0^{15}(15-y)\frac{2}{3}(y+15)\,dy$$

$$F = \int_0^a \rho g(a-y)w(y)\,dy$$

$$= \frac{19.6}{3}\times 10^3 \int_0^{15}\left(225 - y^2\right)dy$$

$$= \frac{19.6}{3}\times 10^3\left(225y - \frac{1}{3}y^3\right)\Big|_0^{15}$$

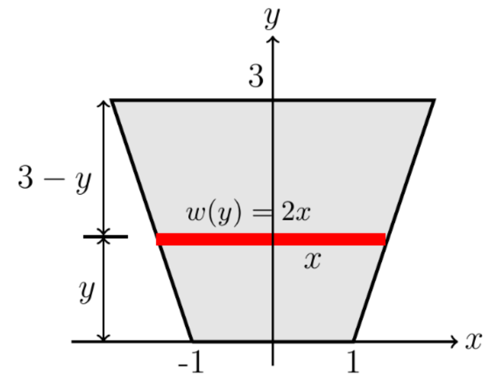
$$= \frac{19.6}{3}\times 10^3\left(15^5 - \frac{1}{3}15^3\right)$$

$$\boxed{= 1.47\times 10^7\ \ N}$$

Exercise

Find the total force on the face of the given dam

Solution



Freshwater Weight density: 62.4 lb/ft^3

$$y = \frac{3-0}{2-1}(x-1)$$
$$= 3x - 3$$
$$x = \frac{1}{3}(y+3)$$

Width: $w(y) = 2x$

$$= \frac{2}{3}(y+3)$$

Depth: $3 - y$

Boundary: $0 \le y \le 3$

$$F = 62.4\int_0^3 (3-y)\frac{2}{3}(y+3)\,dy \qquad\qquad F = \rho g \int_0^a (a-y)w(y)\,dy$$

$$= 41.6\int_0^3 \left(9 - y^2\right)dy$$

$$= 41.6\left(9y - \frac{1}{3}y^3\right.\Big|_0^3$$

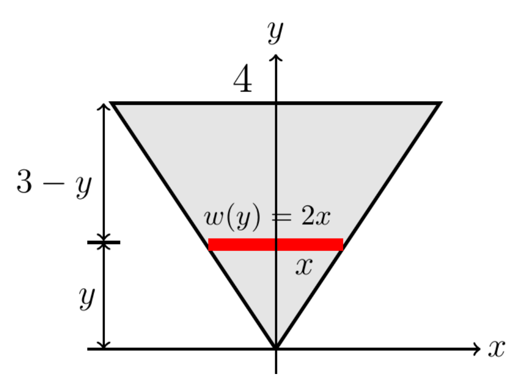
$$= 41.6(27 - 9)$$

$$= 748.8 \ lb$$

Exercise

Find the total force on the face of the given dam

Solution



Freshwater Weight density: 62.4 lb/ft^3

$$\frac{2x}{4} = \frac{y}{3}$$

Width: $w(y) = 2x$

$$= \frac{4}{3}y$$

Depth: $3 - y$

Boundary: $0 \le y \le 3$

$$F = 62.4\int_0^3 (3-y)\left(\frac{4}{3}y\right)dy \qquad\qquad F = \rho g\int_0^a (a-y)\,w(y)\,dy$$

$$= 83.2\int_0^3 \left(3y - y^2\right)dy$$

$$= 83.2\left(\frac{3}{2}y^2 - \frac{1}{3}y^3\right.\Big|_0^3$$

$$= 83.2\left(\frac{27}{2} - 9\right)$$

$$= 83.2\left(\frac{9}{2}\right)$$

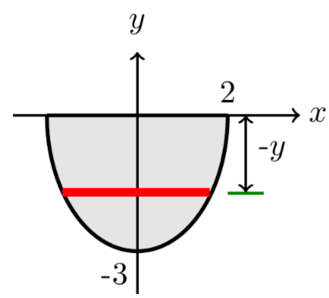
$$\underline{= 374.4\ \ lb}$$

Exercise

Find the total force on the face of the given dam $\quad y = -\frac{1}{2}\sqrt{36 - 9x^2}$

Solution

Freshwater Weight density: $62.4\ lb/ft^3$



$$2y = \sqrt{36 - 9x^2}$$

$$4y^2 = 36 - 9x^2$$

$$9x^2 = 36 - 4y^2$$

$$x^2 = \frac{4}{9}\left(9 - y^2\right)$$

$$x = \frac{2}{3}\sqrt{9 - y^2}$$

Width: $w(y) = 2x$

$$\underline{= \frac{4}{3}\sqrt{9 - y^2}}$$

Depth: $\underline{-y}$

Boundary: $-3 \le y \le 0$

$$F = 62.4\int_{-3}^0 (-y)\left(\frac{4}{3}\sqrt{9 - y^2}\right)dy \qquad\qquad F = \rho g\int_0^a (a-y)\,w(y)\,dy$$

$$= 41.6\int_{-3}^{0}\left(9-y^2\right)^{1/2} d\left(9-y^2\right)$$

$$= \frac{83.2}{3}\left(9-y^2\right)^{3/2}\Big|_{-3}^{0}$$

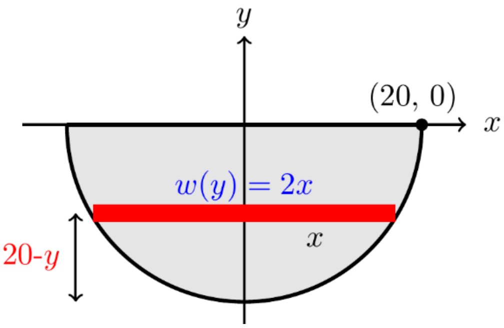
$$= \frac{83.2}{3}(27-0)$$

$$= 83.2(9)$$

$$\underline{= 748.8\ lb}$$

Exercise

Find the total force on the face of the given dam



Solution

Freshwater Weight density: $\rho = 10^3\ kg/m^3$

$$x^2 + (y-20)^2 = 20^2$$

$$x = \sqrt{400-\left(y^2-40y+400\right)}$$

$$\rightarrow\ 2x = 2\sqrt{40y-y^2}$$

$$F = \left(10^3\right)(9.8)\int_0^{20}(20-y)(2)\sqrt{40y-y^2}\ dy \qquad F = \rho g\int_0^a (a-y)w(y)\,dy$$

$$= 9.8\times10^3\int_0^{20}\left(40y-y^2\right)^{1/2} d\left(40y-y^2\right).$$

$$= \frac{19.6}{3}\times10^3\left(40y-y^2\right)^{3/2}\Big|_0^{20}$$

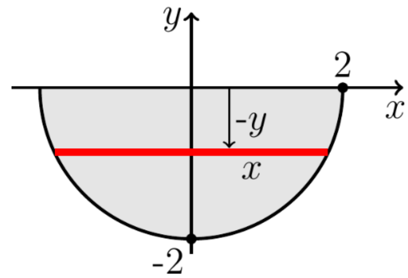
$$= \frac{19.6}{3}\times10^3(800-400)^{3/2}$$

$$= \frac{19.6}{3}\times10^3(20)^3$$

$$\underline{= 5.227\times10^7\ \ N}$$

Exercise

Find the total force on the face of the given dam



Solution

Freshwater Weight density: 62.4 $lb\,/\,ft^3$

$$x^2 + y^2 = 4$$

$$x = \sqrt{4 - y^2}$$

Width: $w(y) = 2x$

$$\boxed{= 2\sqrt{4 - y^2}}$$

Depth: $\boxed{-y}$

Boundary: $-2 \le y \le 0$

$$F = 62.4 \int_{-2}^{0} (-y)\left(2\sqrt{4 - y^2}\right) dy \qquad F = \int_0^a \rho g (a - y) w(y)\, dy$$

$$= 62.4 \int_{-2}^{0} \left(4 - y^2\right)^{1/2} d\left(4 - y^2\right)$$

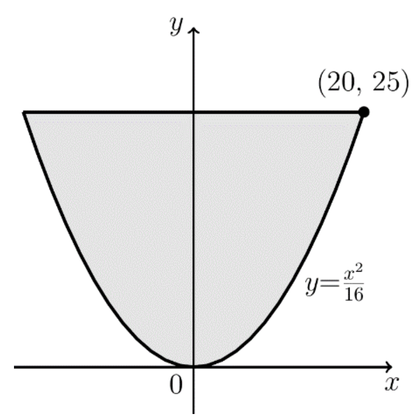
$$= 62.4\left(\frac{2}{3}\right) \left(4 - y^2\right)^{3/2} \Big|_{-2}^{0}$$

$$= 41.6(8 - 0)$$

$$\boxed{= 332.8 \quad lb}$$

Exercise

Find the total force on the face of the given dam



Solution

Freshwater Weight density: $\rho g = 9.8 \times 10^3 \ kg\,/\,m^3$

$$x^2 = 16y$$

$$x = 4\sqrt{y}$$

$$\Rightarrow \ \boxed{2x = 8\sqrt{y}}$$

Depth: $\boxed{25 - y}$

Boundary: $0 \le y \le 25$

$$F = \left(10^3\right)(9.8)\int_0^{25} (25-y)(8)\sqrt{y}\ dy \qquad F = \rho g \int_0^a (a-y)\,w(y)\,dy$$

$$= 78.4\times 10^3 \int_0^{25} \left(25y^{1/2} - y^{3/2}\right)dy$$

$$= 78.4\times 10^3 \left(\frac{50}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right)\Big|_0^{25}$$

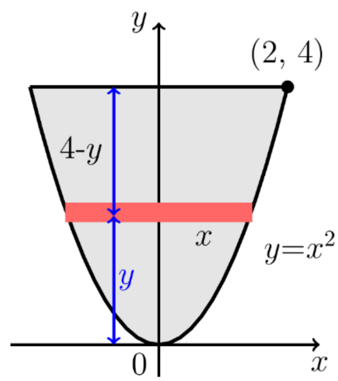
$$= 78.4\times 10^3 \left(\frac{2}{3}5^5 - \frac{2}{5}5^5\right)$$

$$= 78.4\times 5^5 \times 10^3 \left(\frac{4}{15}\right)$$

$$\underline{= 6.533\times 10^7 \ \ N}$$

Exercise

Find the total force on the face of the given dam:



Solution

Freshwater Weight density: $62.4\ lb/ft^3$

Width: $\underline{w(y) = 2\sqrt{y}}$

Depth: $\underline{4-y}$

Boundary: $0 \le y \le 4$

$$F = 62.4\int_0^4 (4-y)\left(2\sqrt{y}\right)dy \qquad F = \rho g \int_0^a (a-y)\,w(y)\,dy$$

$$= 124.8\int_0^4 \left(4y^{1/2} - y^{3/2}\right)dy$$

$$= 124.8\left(\frac{8}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right)\Big|_0^{2^2}$$

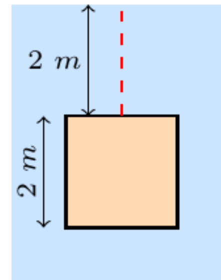
$$= 124.8\left(\frac{64}{3} - \frac{64}{5}\right)$$

$$= 7,987.2\left(\frac{2}{15}\right)$$

$$\underline{= 1,064.96 \ \ lb}$$

Exercise

Find the fluid force on the vertical plate submerged in water



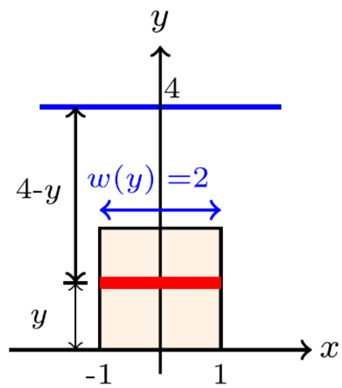
Solution

Freshwater Weight density: $= 9{,}800 \ \ kg / m^3$

Width: $\underline{w(y) = 2}$

Depth: $\underline{4 - y}$

Boundary: $\quad 0 \le y \le 2$



$$F = 9{,}800 \int_0^2 (4 - y)(2)\,dy$$

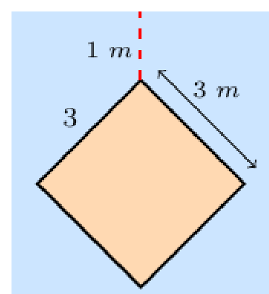
$$= 19{,}600 \left(4y - \frac{1}{2}y^2\right)\Big|_0^2$$

$$= 19{,}600(8 - 2)$$

$$\underline{= 117{,}600 \ \ N}$$

Exercise

Find the fluid force on the vertical plate submerged in water



Solution

Freshwater Weight density $= 9{,}800 \ \ kg / m^3$

$$|OA| = \sqrt{3^2 + 3^2}$$

$$= \sqrt{18}$$

$$\underline{= 3\sqrt{2}}$$

$$A = \left(0, \ 3\sqrt{2}\right)$$

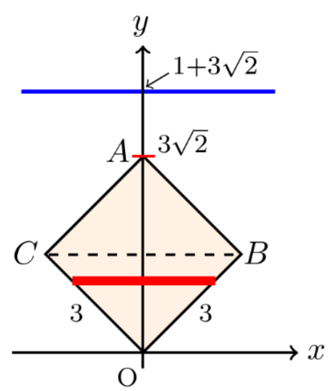
Depth: $\quad \underline{1 + 3\sqrt{2} - y}$

The area below the dashed BC line:

Line OB: $y = x$

Width: $\quad \underline{L(y) = 2y}$

Boundary: $\quad 0 \le y \le \dfrac{3\sqrt{2}}{2}$



The area above the dashed BC line:

$$B = \left(\frac{3\sqrt{2}}{2}, \ \frac{3\sqrt{2}}{2}\right)$$

$$\text{Line } AB\text{: } y = \frac{\frac{3\sqrt{2}}{2} - 3\sqrt{2}}{\frac{3\sqrt{2}}{2}} x + 3\sqrt{2}$$

$$= -x + 3\sqrt{2}$$

$$x = 3\sqrt{2} - y$$

$$\textit{Width}: \quad L(y) = 2x$$

$$\boxed{= 2\left(3\sqrt{2} - y\right)}$$

$$\textit{Boundary}: \quad \frac{3\sqrt{2}}{2} \le y \le 3\sqrt{2}$$

$$F = 19,600 \int_0^{\frac{3\sqrt{2}}{2}} \left(1 + 3\sqrt{2} - y\right)(y)\,dy \ + \ 19,600 \int_{\frac{3\sqrt{2}}{2}}^{3\sqrt{2}} \left(1 + 3\sqrt{2} - y\right)\left(3\sqrt{2} - y\right)dy$$

$$= 19,600 \int_0^{\frac{3\sqrt{2}}{2}} \left(\left(1 + 3\sqrt{2}\right)y - y^2\right)dy \ + \ 19,600 \int_{\frac{3\sqrt{2}}{2}}^{3\sqrt{2}} \left(3\sqrt{2} + 18 - y - 6\sqrt{2}y + y^2\right)dy$$

$$= 19,600 \left(\frac{1}{2}\left(1 + 3\sqrt{2}\right)y^2 - \frac{1}{3}y^3 \ \Big|_0^{\frac{3\sqrt{2}}{2}}\right) \ + \ 19,600 \left(3\sqrt{2}y + 18y - \frac{1}{2}y^2 - 3\sqrt{2}y^2 + \frac{1}{3}y^3 \ \Big|_{\frac{3\sqrt{2}}{2}}^{3\sqrt{2}}\right)$$

$$= 19,600\left(\frac{9}{4} + \frac{27\sqrt{2}}{4} - \frac{9\sqrt{2}}{4} + 18 + 54\sqrt{2} - 9 - 54\sqrt{2} + 18\sqrt{2} - 9 - 27\sqrt{2} + \frac{9}{4} + \frac{27\sqrt{2}}{2} - \frac{9\sqrt{2}}{4}\right)$$

$$= 19,600\left(\frac{18}{4} + \frac{9\sqrt{2}}{4} - 9\sqrt{2} + \frac{27\sqrt{2}}{2}\right)$$

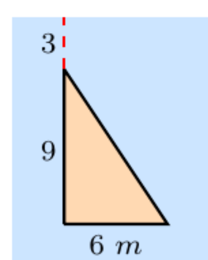
$$= 176,400\left(\frac{2}{4} + \frac{\sqrt{2}}{4} - \sqrt{2} + \frac{3\sqrt{2}}{2}\right)$$

$$= 176,400\left(\frac{2 + \sqrt{2} - 4\sqrt{2} + 6\sqrt{2}}{4}\right)$$

$$\boxed{= 44,100\left(2 + 3\sqrt{2}\right) \ N}$$

Exercise

Find the fluid force on the vertical plate submerged in water



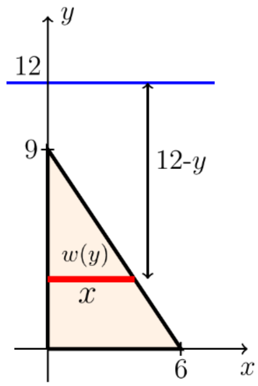
Solution

Freshwater Weight density: $= 9,800 \ \ kg / m^3$

$$y = \frac{9-0}{0-6}x + 9$$

$$= -\frac{3}{2}x + 9$$

$$\frac{3}{2}x = 9 - y$$



Width: $\underline{L(y) = \frac{2}{3}(9-y)}|$

Depth: $\underline{12 - y}|$

Boundary: $0 \le y \le 9$

$$F = 9,800\left(\frac{2}{3}\right)\int_0^9 (12-y)(9-y)\,dy$$

$$= \frac{19,600}{3}\int_0^9 \left(108 - 21y + y^2\right)dy$$

$$= \frac{19,600}{3}\left(108y - \frac{21}{2}y^2 + \frac{1}{3}y^3\right)\Big|_0^9$$

$$= \frac{19,600}{3}\left(972 - \frac{1,701}{2} + 243\right)$$

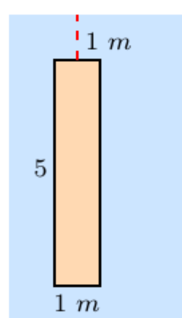
$$= \frac{19,600}{3}\left(1,215 - \frac{1,701}{2}\right)$$

$$= \frac{19,600}{3}\left(\frac{729}{2}\right)$$

$$\underline{= 2,381,400\ \ N}|$$

Exercise

Find the fluid force on the vertical plate submerged in water



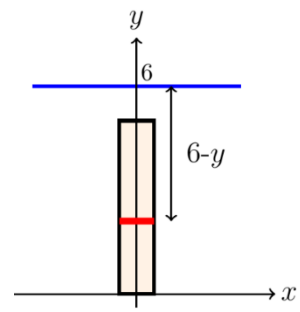
Solution

Freshwater Weight density: $= 9,800\ \ kg/m^3$

Width: $\underline{L(y) = 1}|$

Depth: $\underline{6 - y}|$

Boundary: $0 \le y \le 5$



$$F = 9,800\int_0^5 (6-y)\,dy$$

$$= 9,800 \left(6y - \frac{1}{2}y^2\right.\Big|_0^5$$

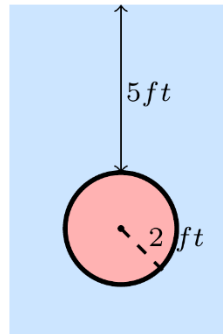
$$= 9,800\left(30 - \frac{25}{2}\right)$$

$$= 9,800\left(\frac{35}{2}\right)$$

$$\underline{= 171,500 \ N}$$

Exercise

Find the fluid force on the vertical plate submerged in water



Solution

Freshwater Weight density: $62.4 \ lb / ft^3$

$$x^2 + y^2 = 4$$

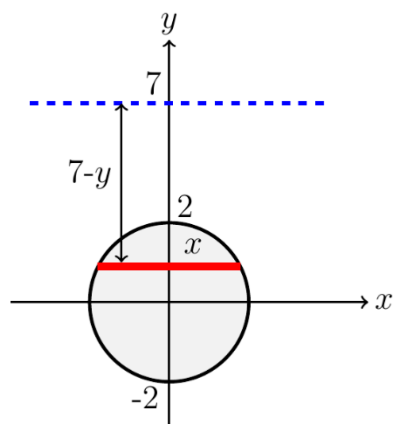
$$x = \sqrt{4 - y^2}$$

Width: $w(y) = 2x$

$$\underline{= 2\sqrt{4 - y^2}}$$

Depth: $\underline{7 - y}$

Boundary: $-2 \le y \le 2$



$$F = 62.4\int_{-2}^{2}(7 - y)\left(2\sqrt{4 - y^2}\right)dy$$

$$= 124.8\int_{-2}^{2} 7\sqrt{4 - y^2}\, dy \ - \ 124.8\int_{-2}^{2} y\sqrt{4 - y^2}\, dy$$

$\int\sqrt{9 - y^2}\,dy$ = *Area of a semicircle*

$$= 873\left(\frac{1}{2}\pi\left(2^2\right)\right) \ + \ 62.4\int_{-2}^{2}\left(4 - y^2\right)^{1/2} d\left(4 - y^2\right)$$

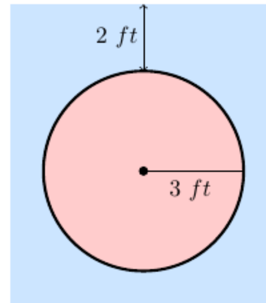
$$= 1,747.2\pi \ + \ \frac{124.8}{3}\left(4 - y^2\right)^{3/2} \Big|_{-2}^{2}$$

$$= 1,747.2\pi + 41.6(0 - 0)$$

$$\underline{= 1,747.2\pi \ \ lb}$$

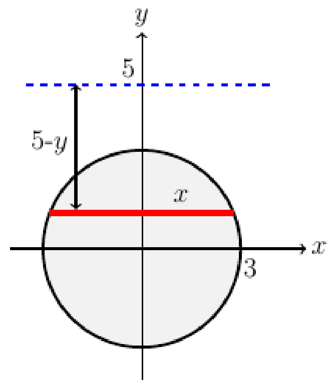
Exercise

Find the fluid force on the vertical plate submerged in water



Solution

Freshwater Weight density: $62.4\ lb\,/\,ft^3$



$$x^2 + y^2 = 9$$

$$x = \sqrt{9 - y^2}$$

Width: $w(y) = 2x$

$$\boxed{= 2\sqrt{9 - y^2}}$$

Depth: $\boxed{5 - y}$

Boundary: $-3 \le y \le 3$

$$F = 62.4\int_{-3}^{3}(5 - y)\left(2\sqrt{9 - y^2}\right)dy \qquad F = \rho g\int_0^a (a - y)\,w(y)\,dy$$

$$= 124.8\int_{-3}^{3} 5\sqrt{9 - y^2}\ dy - 124.8\int_{-3}^{3} y\sqrt{9 - y^2}\ dy \qquad \int\sqrt{9 - y^2}\,dy = \text{Area of a semicircle}$$

$$= 624\left(\frac{1}{2}\pi\left(3^2\right)\right) + 62.4\int_{-3}^{3}\left(9 - y^2\right)^{1/2} d\left(9 - y^2\right)$$

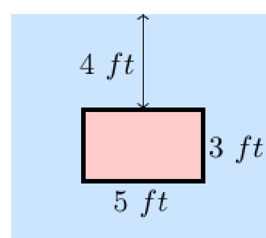
$$= 2{,}808\pi + \frac{124.8}{3}\left(9 - y^2\right)^{3/2}\Big|_{-3}^{3}$$

$$= 2{,}808\pi + 41.6(0 - 0)$$

$$\boxed{= 2{,}808\pi\ \ lb}$$

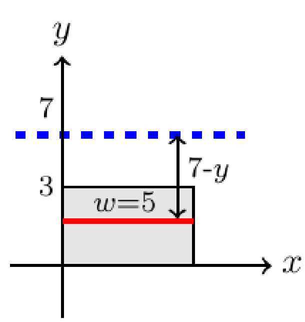
Exercise

Find the fluid force on the vertical plate submerged in water



Solution

Freshwater Weight density: $62.4\ lb\,/\,ft^3$



Width: $\boxed{w(y) = 5}$

Depth: $\boxed{7 - y}$

Boundary: $0 \le y \le 3$

$$F = 62.4\int_0^3 (7-y)(5)\,dy \qquad\qquad F = \rho g\int_0^a (a-y)\,w(y)\,dy$$

$$= 312\left(7y - \frac{1}{2}y^2\right)\Big|_0^3$$

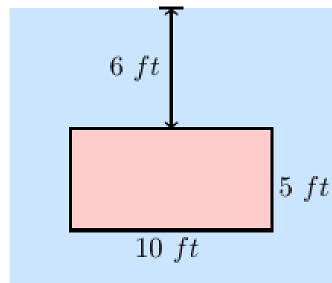
$$= 312\left(21 - \frac{9}{2}\right)$$

$$= 312\left(\frac{33}{2}\right)$$

$$\underline{= 5{,}148\ \ lb}$$

Exercise

Find the fluid force on the vertical plate submerged in water



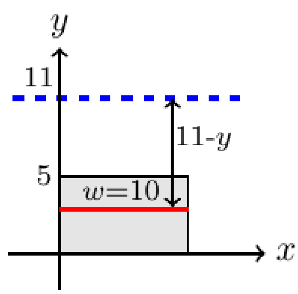
Solution

Freshwater Weight density: $62.4\ lb/ft^3$

Width: $\underline{w(y) = 10}$

Depth: $\underline{11 - y}$

Boundary: $0 \le y \le 5$



$$F = 62.4\int_0^5 (11-y)(10)\,dy$$

$$= 624\left(11y - \frac{1}{2}y^2\right)\Big|_0^5$$

$$= 624\left(55 - \frac{25}{2}\right)$$

$$= 624\left(\frac{85}{2}\right)$$

$$\underline{= 26{,}520\ \ lb}$$

Exercise

A rectangular plate of height ***h*** *feet* and base ***b*** *feet* is submerged vertically in a tank of fluid that weighs ***w*** *pounds per cubic foot*. The center is ***k*** *feet* below the surface of the fluid, where $h \le \frac{k}{2}$. Show that the fluid force on the surface of the plate is $F = wkhb$

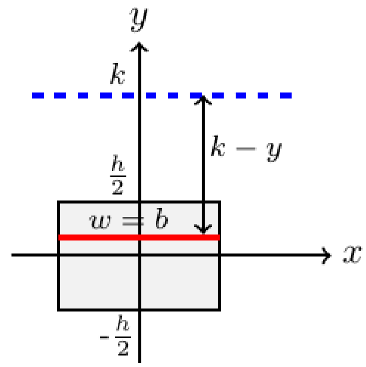
Solution

Fluid Weight density: $w\ lb/ft^3$

Width: $w(y)=b$

Depth: $k-y$

Boundary: $-\frac{h}{2}\le y\le \frac{h}{2}$



$$F=w\int_{-\frac{h}{2}}^{\frac{h}{2}}(k-y)(b)\,dy$$

$$=bw\left(ky-\frac{1}{2}y^2\right)\Big|_{-\frac{h}{2}}^{\frac{h}{2}}$$

$$=bw\left(\frac{1}{2}hk-\frac{1}{2}h^2+\frac{1}{2}kh+\frac{1}{2}h^2\right)$$

$$=bhkw \;\; lb$$

Exercise

A circular plate of radius ***r*** *feet* is submerged vertically in a tank of fluid that weighs ***w*** *pounds per cubic foot*. The center of the circle is $k\ (k>r)$ *feet* below the surface of the fluid. Show that the fluid force on the surface of the plate is $F=\pi wkr^2$.

Solution

Weight density: $w\ lb/ft^3$

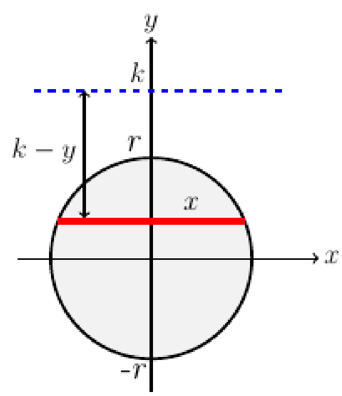
$x^2+y^2=r^2$

$x=\sqrt{r^2-y^2}$

Width: $L(y)=2x$

$=2\sqrt{r^2-y^2}$

Depth: $k-y$



Boundary: $-r\le y\le r$

$$F=w\int_{-3}^{3}(k-y)\left(2\sqrt{r^2-y^2}\right)dy$$

$$=2kw\int_{-r}^{r}\sqrt{r^2-y^2}\,dy\;-\;2w\int_{-r}^{r}y\sqrt{r^2-y^2}\,dy$$

$\int_{-r}^{r}\sqrt{r^2-y^2}\,dy=$ *Area of a semicircle*

$$=2kw\left(\frac{1}{2}\pi\left(r^2\right)\right)\;+\;2w\int_{-r}^{r}\left(r^2-y^2\right)^{1/2}d\left(r^2-y^2\right)$$

$$= \pi wkr^2 + \frac{4w}{3}\left(r^2 - y^2\right)^{3/2} \Big|_{-r}^{r}$$

$$= \pi wkr^2 + \frac{4w}{3}(0-0)$$

$$\underline{= \pi wkr^2 \ \ lb}|$$

Exercise

A large building shaped like a box is 50 *m* high with a face that is 80 *m* wide. A strong wind blows directly at the face of the building, exerting a pressure of 150 N/m^2 at the ground and increasing with height according to $P(y) = 150 + 2y$, where *y* is the height above the ground. Calculate the total force on the building, which is a meaure of the resistance that must be included in the design of the building.

Solution

$$F = \int_0^{50} (150+2y)(80)\ dy \qquad\qquad F = \int_0^a P(y)w(y)dy$$

$$= 80\left(150y + y^2\right)\Big|_0^{50}$$

$$= 80(7500 + 2500)$$

$$\underline{= 8\times 10^5 \ \ N}|$$

Exercise

Adiving pool that is 4 *m* deep full of water has a viewing window on one of its vertical walls. Find the force of the a square window, 0.5 *m* on side, with the lower edge of the window on the bottom of the pool.

Solution

Freshwater Weight density: $\rho = 10^3\ kg/m^3$

$$F = 10^3(9.8)\int_0^{0.5} (4-y)(0.5)\ dy \qquad\qquad F = \int_0^a \rho g(a-y)w(y)dy$$

$$= 4.9\times 10^3\left(4y - \frac{1}{2}y^2\right. \Big|_0^{0.5}$$

$$= 4.9\times 10^3\left(2 - \frac{1}{8}\right)$$

$$= 4.9\times 10^3\left(\frac{15}{8}\right)$$

$$\underline{= 9187.5 \ \ N}|$$

Exercise

Adiving pool that is 4 m deep full of water has a viewing window on one of its vertical walls. Find the force of the a square window, 0.5 m on side, with the lower edge of the window 1 m from the bottom of the pool.

Solution

Freshwater Weight density: $\rho = 10^3\ kg/m^3$

$$F = 10^3(9.8)\int_1^{1.5}(4-y)(0.5)\ dy \qquad\qquad F = \rho g\int_0^a (a-y)w(y)\,dy$$

$$= 4.9\times10^3\left(4y-\frac{1}{2}y^2\right.\Big|_1^{1.5}$$

$$= 4.9\times10^3\left(6-\frac{9}{8}-4+\frac{1}{2}\right)$$

$$= 4.9\times10^3\left(2-\frac{5}{8}\right)$$

$$= 4.9\times10^3\left(\frac{11}{8}\right)$$

$$= 6737.5\ \ N$$

Exercise

Adiving pool that is 4 m deep full of water has a viewing window on one of its vertical walls. Find the force of the a circle window, with a radius of 0.5 m, tangent to the bottom of the pool.

Solution

Equation of the circle: $x^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4}$

$$x^2 = \frac{1}{4}-y^2+y-\frac{1}{4}$$

$$= y-y^2$$

$$x = \sqrt{y-y^2}$$

$$L(y) = 2x = 2\sqrt{y-y^2}$$

$$= 2\sqrt{y-y^2}$$

$$F = 10^3(9.8)\int_0^1 (4-y)\left(2\sqrt{y-y^2}\right)\ dy$$

$$= 19.6\times10^3\int_0^1\left(\frac{7}{2}+\frac{1}{2}-y\right)\sqrt{y-y^2}\ dy \qquad\qquad d\left(y-y^2\right)=1-2y=2\left(\frac{1}{2}-y\right)$$

$$= 19.6\times 10^3 \int_0^1 \frac{7}{2}\sqrt{y-y^2}\, dy \; + \; 19.6\times 10^3 \int_0^1 \left(\frac{1}{2}-y\right)\sqrt{y-y^2}\, dy$$

$$= 7\times 98\times 10^2 \int_0^1 \sqrt{y-y^2}\, dy \; + \; 392\times 10^2 \int_0^1 \left(y-y^2\right)^{1/2}\, d\left(y-y^2\right) \qquad \text{Area of semicircle: } \frac{1}{2}\pi r^2$$

$$= 7\times 98\times 10^2 \left(\frac{1}{2}\frac{\pi}{4}\right) \; + \; 392\times 10^2 \left(\frac{2}{3}\right) \left(y-y^2\right)^{3/2} \Big|_0^1$$

$$= \frac{343\pi}{4}\times 10^2 + \frac{784}{3}\times 10^2 (0)$$

$$= \frac{343\pi}{4}\times 10^2 \quad N$$

Exercise

A rigid body with a mass of 2 *kg* moves along a line due to a force that produces a position function $x(t)=4t^2$, where x is measured in *meters* and t is measured in *seconds*. Find the work done during the first 5 *sec*. in two ways.

a) Note that $x''(t)=8$; then use Newton's second law, $\left(F=ma=mx''(t)\right)$ to evaluate the work integral $W=\int_{x_0}^{x_f} F(x)dx$, where x_0 and x_f are the initial and final positions, repectively.

b) Change variables in the work integral and integrate with respect to t.

Solution

a) $$W=\int_{x_0}^{x_f} mx''\, dx$$

$$=\int_0^{x(5)} 2\times 8\, dx \qquad x(5)=4(5)^2=100$$

$$=16x\Big|_0^{100}$$

$$=1600 \;\; J$$

b) $$W=\int_{x_0}^{x_f} mx''\, dx$$

$$=\int_0^5 2\times 8\, \frac{dx}{dt}\, dt$$

$$= 16\int_0^5 (8t)\,dt$$

$$= 64t^2 \Big|_0^5$$

$$\underline{= 1600\ J}$$

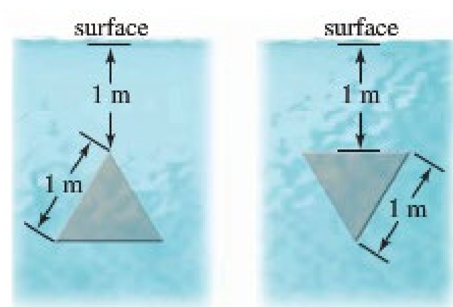
Exercise

A plate shaped like an equilateral triangle 1 *m* on a side is placed on a vetical wall 1 *m* below the surface of a pool filled with water. On which plate in the figure is the force is greater

Solution

The left picture has more force than the right, because of the bottom part is wider side in the pool.

For the left side plate:



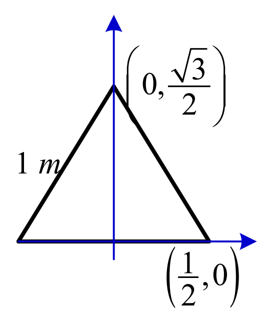
$$y = h = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}$$

Line segement: $y = \dfrac{0-\frac{\sqrt{3}}{2}}{\frac{1}{2}-0}\left(x-\frac{1}{2}\right)$

$$= -\sqrt{3}\left(x-\frac{1}{2}\right)$$

$$x = \frac{1}{2} - \frac{1}{\sqrt{3}}y$$

$$\underline{2x = 1 - \frac{2}{\sqrt{3}}y}$$



$$F = \rho g \int_0^{\sqrt{3}/2} \left(1+\frac{\sqrt{3}}{2}-y\right)\left(1-\frac{2}{\sqrt{3}}y\right)dy$$

$$= \rho g \int_0^{\sqrt{3}/2} \left(1+\frac{\sqrt{3}}{2}-y-\frac{2\sqrt{3}}{3}y-y+\frac{2\sqrt{3}}{3}y^2\right)dy$$

$$= \rho g \int_0^{\sqrt{3}/2} \left(1+\frac{\sqrt{3}}{2}-\frac{2}{3}\left(3+\sqrt{3}\right)y+\frac{2\sqrt{3}}{3}y^2\right)dy$$

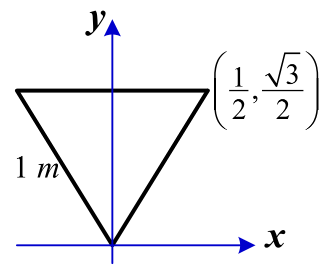
$$= \rho g \left(\left(1+\frac{\sqrt{3}}{2}\right)y-\frac{1}{3}\left(3+\sqrt{3}\right)y^2+\frac{2\sqrt{3}}{9}y^3 \Big|_0^{\sqrt{3}/2}\right.$$

$$= 9{,}800\left(\frac{\sqrt{3}}{2}+\frac{3}{4}-\frac{3}{4}-\frac{\sqrt{3}}{4}+\frac{1}{4}\right) \qquad \rho = 1000\ \frac{kg}{m^3} \quad g = 9.8\ m/s^2$$

$$\underline{= 2{,}450\left(1+\sqrt{3}\right)\ N}$$

For the right side plate:

Line segement: $y = \dfrac{\frac{\sqrt{3}}{2}-0}{\frac{1}{2}-0}(x-0) = \sqrt{3}x$



$x = \dfrac{1}{\sqrt{3}}y$

$\boxed{2x = \dfrac{2}{\sqrt{3}}y}$

$$F = \rho g \int_0^{\sqrt{3}/2} \left(1 + \frac{\sqrt{3}}{2} - y\right)\left(\frac{2}{\sqrt{3}}y\right) dy$$

$$= \rho g \int_0^{\sqrt{3}/2} \left(\frac{2\sqrt{3}}{3}y + y - \frac{2\sqrt{3}}{3}y^2\right) dy$$

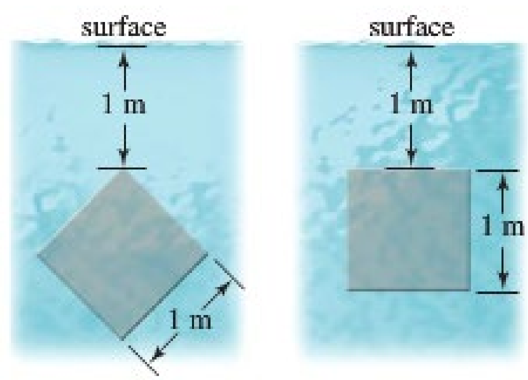
$$= \rho g \left(\frac{\sqrt{3}}{3}y^2 + \frac{1}{2}y^2 - \frac{2\sqrt{3}}{9}y^3 \right|_0^{\sqrt{3}/2}$$

$$= 9{,}800\left(\frac{\sqrt{3}}{4} + \frac{3}{8} - \frac{1}{4}\right)$$

$$\boxed{= 2{,}450\left(1+2\sqrt{3}\right)\ N} \quad \boxed{\approx 10{,}937\ N}$$

Exercise

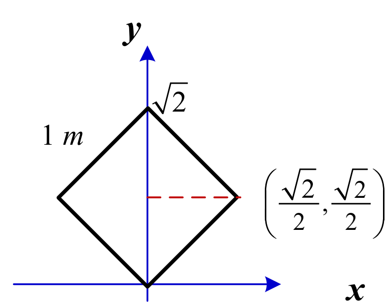
A square plate 1 m on a side is placed on a vetical wall 1 m below the surface of a pool filled with water.



On which plate in the figure is the force is greater

Solution

For the plate on left side :



$2x^2 = 1 \to x = \dfrac{\sqrt{2}}{2}$

Line segment OA: $\boxed{y = x}$

$\to 2x = 2y$

Line segment AB:

$$y=\frac{\sqrt{2}-\frac{\sqrt{2}}{2}}{0-\frac{\sqrt{2}}{2}}x+\sqrt{2}$$

$$\underline{=-x+\sqrt{2}\,|}$$

$$\rightarrow\ \underline{x=\sqrt{2}-y\,|}$$

$$F=\rho g\int_0^{\sqrt{2}/2}\left(1+\sqrt{2}-y\right)(2y)\,dy\ +\ \rho g\int_{\sqrt{2}/2}^{\sqrt{2}}\left(1+\sqrt{2}-y\right)(2)\left(\sqrt{2}-y\right)dy$$

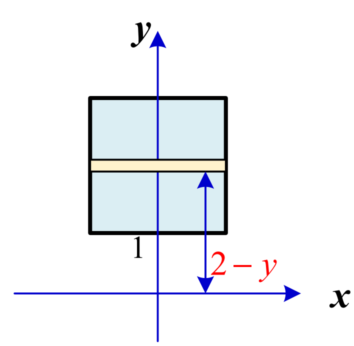
$$=2\rho g\int_0^{\sqrt{2}/2}\left(y+\sqrt{2}y-y^2\right)dy\ +2\rho g\int_{\sqrt{2}/2}^{\sqrt{2}}\left(\sqrt{2}+2-y-2\sqrt{2}y+y^2\right)dy$$

$$=2\rho g\left(\frac{1}{2}y^2+\frac{\sqrt{2}}{2}y^2-\frac{1}{3}y^3\right.\left|\begin{array}{l}\sqrt{2}/2\\0\end{array}\right.\ +2\rho g\left(\sqrt{2}y+2y-\frac{1}{2}y^2-\sqrt{2}y^2+\frac{1}{3}y^3\right.\left|\begin{array}{l}\sqrt{2}\\\sqrt{2}/2\end{array}\right.$$

$$=2\rho g\left(\frac{1}{4}+\frac{\sqrt{2}}{4}-\frac{\sqrt{2}}{12}+2+2\sqrt{2}-2-2\sqrt{2}+\frac{2\sqrt{2}}{3}-1-\sqrt{2}+\frac{1}{4}+\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{12}\right)$$

$$=2\rho g\left(\frac{1}{2}+\frac{\sqrt{2}}{4}\right)\qquad \rho=1000\ \frac{kg}{m^3}\quad g=9.8\ m/s^2$$

$$\underline{=9,800\left(1+\frac{\sqrt{2}}{2}\right)\ N\,|}\quad \underline{\approx 16,730\ N|}$$



For the plate on right side:

$$F=\rho g\int_0^1(2-y)(1)\,dy$$

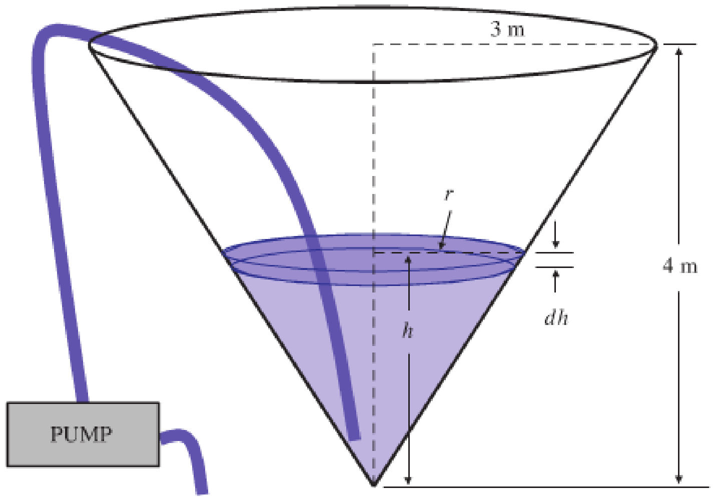
$$=\rho g\left(2y-\frac{1}{2}y^2\right.\left|\begin{array}{l}1\\0\end{array}\right.$$

$$=9,800\left(\frac{3}{2}\right)$$

$$\underline{=14,700\ N\,|}$$

Exercise

Water fills a tank in the shape of a right-circular cone with top radius 3 *m* and depth 4 *m*. How much work must be done (against gravity) to pump all the water out of the tank over the top edge of the tank?



Solution

$$\frac{r}{3}=\frac{h}{4} \quad \Rightarrow \quad r=\frac{3}{4}h \ \text{(similar triangles)}$$

The volume of this slice is:

$$dV=\pi r^2 dh=\frac{9}{16}\pi h^2 dh$$

And its weight (the force of gravity on the mass of water in the slice) is

$$dF=\delta g dV=\frac{9}{16}\delta g\pi h^2 dh$$

The water in this disk must be raised (against gravity) a distance $(4-h)$ *m* by the pump. The work required to do this is

$$dw=\frac{9}{16}\delta g\pi(4-h)h^2 dh$$

$$W=\int_0^4 \frac{9}{16}\delta g\pi\left(4h^2-h^3\right)dh$$

$$=\frac{9}{16}\delta g\pi\int_0^4\left(4h^2-h^3\right)dh$$

$$=\frac{9}{16}\delta g\pi\left(\frac{4}{3}h^3-\frac{1}{4}h^4\right)\Big|_0^4$$

$$=\frac{9}{16}\delta g\pi\left(\frac{4}{3}4^3-\frac{1}{4}4^4\right)$$

$$=\frac{9}{16}(1{,}000)(9.8)(\pi)\left(\frac{64}{3}\right)$$

$$\approx 3.69\times 10^5 \quad N.m$$

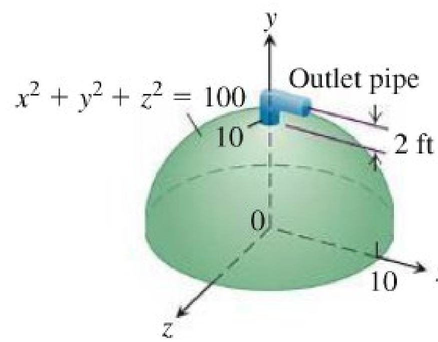
Exercise

You are in charge of the evacuation and repair of the storage tank.

The tank is a hemisphere of radius 10 *feet* and is full of benzene weighing 56 lb/ft^3. A firm you contacted says it can empty the tank for $\frac{1}{2}$ ¢ per foot-pound of work. Find the work required to empty the tank by pumping the benzene to an outlet 2 *feet* above the top of the tank. If you have \$5,000 budget for the job, can you afford to hire the firm?

Solution

$$x^2 + y^2 = 100 \quad \to \quad x = \sqrt{100 - y^2} = r$$



$$\Delta V = \pi (radius)^2 (thnickness)$$

$$= \pi \left(\sqrt{100 - y^2} \right)^2 \Delta y$$

$$\boxed{= \pi \left(100 - y^2\right) \Delta y}$$

The force is:

$$F(y) = 56 \frac{lb}{ft^3} \cdot \Delta V$$

$$\boxed{= 56\pi \left(100 - y^2\right) \Delta y}$$

The distance thorugh which $F(y)$ must act to lift the slab to the level of 2 *feet* above the top of the tank is about $(12 - y)\ ft$, so the work done is

$$\Delta W = 56\pi \left(100 - y^2\right)(2 - y)\Delta y.$$

And on the interval from $y = 0$ to $y = 10$

$$W = \int_0^{10} 56\pi \left(100 - y^2\right)(2 - y) dy$$

$$= 56\pi \int_0^{10} \left(1,200 - 100y - 12y^2 + y^3\right) dy$$

$$= 56\pi \left(1,200y - 50y^2 - 4y^3 + \frac{1}{4}y^4 \right|_0^{10}$$

$$= 56\pi (12,000 - 5,000 - 4,000 + 2,500)$$

$$\boxed{= 308\pi \times 10^3 \ \ ft\text{-}lb}$$

$$Cost = \left(\frac{1}{2}\right) 308\pi \times 10^3$$

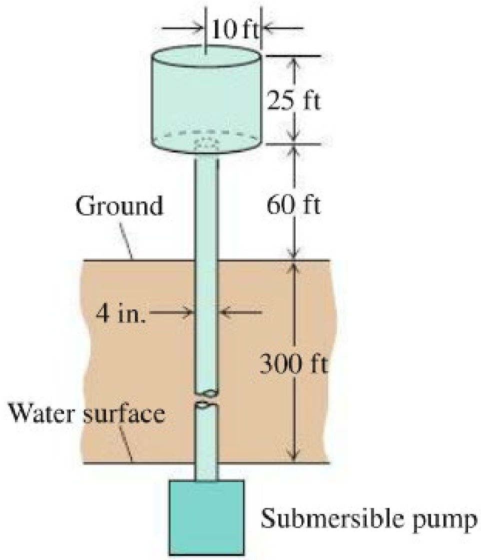
$$= 154\pi \times 10^3 \ ¢$$

$= \$1,540\pi$

$= \$4838.05$

Exercise

You decided to drill a well to increase a water supply.
You have determined that a water tower will be necessary to provide the pressure needed for distribution



The water is to be pumped from a 300-*ft* well through a vertical 4-*in*. pipe into the base of a cylindrical tank 20 *feet* in diameter and 25 *feet* high. The base of the tank will be 60 *feet* above ground. The pump is a 3-*hp* pump, rated at $1,650\ ft \cdot lb/sec$. How long will it take to fill the tank the first time? (Include the time it takes to fill the pipe). Assume that water weighs $62.4\ lb/ft^3$

Solution

$$\Delta V = \pi(radius)^2(thnickness)$$

$$= \pi(10)^2\,\Delta y$$

$$= 100\pi\ \Delta y$$

The force is:

$$F(y) = 62.4\frac{lb}{ft^3}\cdot\Delta V$$

$$= 6,240\pi\ \Delta y$$

The distance thorugh which $F(y)$ must act to lift the slab is $y\ ft$, so the work done is $\Delta W_1 = 6,240\pi y\ \Delta y$.

$$W_1 = \int_{360}^{385} 6,240\pi y\ dy$$

$$= 3{,}120\pi\, y^2 \Big|_{360}^{385}$$

$$= 3{,}120\pi\left(385^2 - 360^2\right)$$

$$= 3{,}120\pi(18{,}625)$$

$$\boxed{= 5{,}811\pi \times 10^4 \ \ ft\text{-}lb}$$

To fill the pipe, the reuired work by taking the radius

$$r = 2in \cdot \frac{1\,ft}{12\ in} = \frac{1}{6}\ ft$$

$$\Delta V = \pi\left(\frac{1}{6}\right)^2 \Delta y$$

$$\boxed{= \frac{\pi}{36}\ \Delta y}$$

The force is:

$$F(y) = 62.4\frac{lb}{ft^3} \cdot \Delta V$$

$$= \frac{624\pi}{360}\ \Delta y$$

$$\boxed{= \frac{26\pi}{15}\ \Delta y}$$

The distance thorugh which $F(y)$ must act to lift the slab is $y\,ft$, so the work done is

$$\Delta W_2 = \frac{26\pi}{15} y\ \Delta y\,.$$

$$W_2 = \frac{26\pi}{15}\int_0^{360} y\ dy$$

$$= \frac{13\pi}{15} y^2 \Big|_0^{360}$$

$$= \frac{13\pi}{15}\pi\left(360^2\right)$$

$$\boxed{= 112{,}320\pi \ \ ft\text{-}lb}$$

$$W = W_1 + W_2$$

$$= 5{,}811\pi \times 10^4 + 112{,}320\pi$$

$$\boxed{= 58{,}222{,}320\pi \ \ ft\text{-}lb}$$

To fill the tank the first time, it will take

$$\frac{W}{1{,}650} = \frac{58{,}222{,}320\pi}{1{,}650}$$

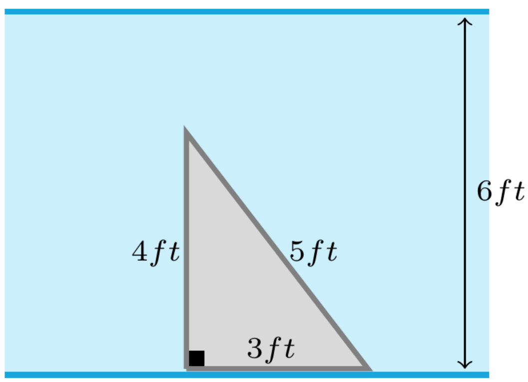
$$\boxed{= \frac{5,822,232}{165}\pi \ \ sec}$$

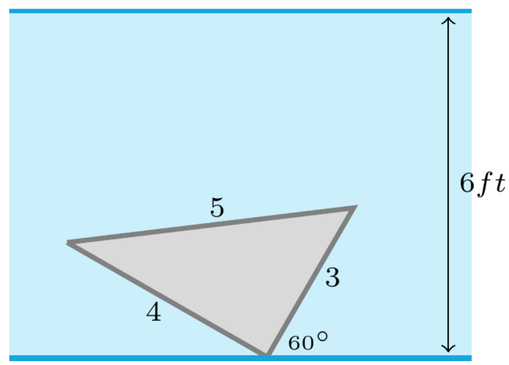
$$\boxed{\approx 110,855 \ \ sec}$$

$$\boxed{\approx 31 \ \ hrs}$$

Exercise

Calculate the fluid force on one side of a right-triangular plate with edges 3 *feet*, 4 *feet*, and 5 *feet* if the plate sits at the bottom of the pool filled with water to a depth of 6 *feet* on its 3-*feet* edge and titlted at $60°$ to the bottom of the pool.

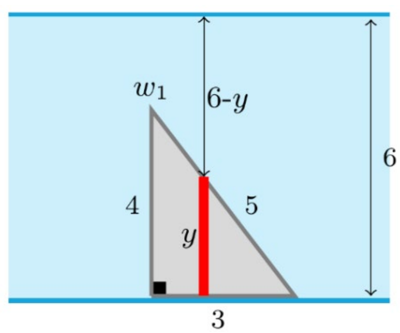


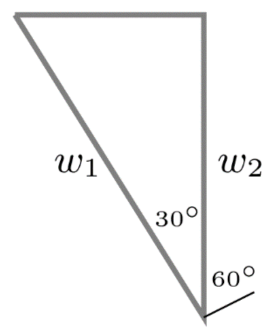


Solution

Freshwater Weight density: $\rho g = 62.4 \ \ lb/ft^3$

$w_1 = 4$





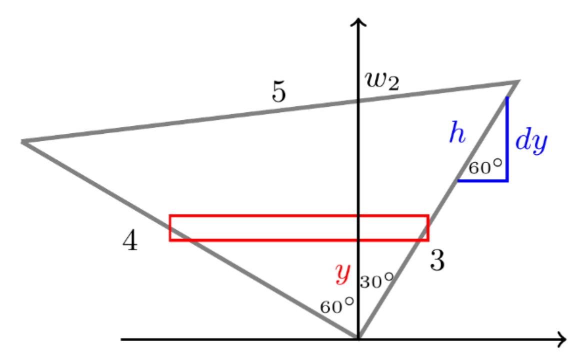
$$\cos 30° = \frac{w_2}{w_1}$$

$$w_2 = 4\frac{\sqrt{3}}{2}$$

$$\boxed{= 2\sqrt{3}}$$

Boundary: $\boxed{0 \le y \le 2\sqrt{3}}$

Depth: $\boxed{6 - y}$



$$\sin 60° = \frac{dy}{h}$$

$$\frac{\sqrt{3}}{2} = \frac{dy}{h}$$

$$\boxed{h = \frac{2}{\sqrt{3}}dy}$$

$$\frac{L}{3} = \frac{w_2 - y}{4}$$

Width: $L(y) = \frac{3}{4}\left(2\sqrt{3} - y\right)$

$$F = 62.4\int_0^{2\sqrt{3}} (6-y)\frac{3}{4}\left(2\sqrt{3}-y\right)\frac{2}{\sqrt{3}}dy \qquad\qquad F = \rho g\int_0^a (a-y)L(y)dy$$

$$= \frac{624}{10}\frac{3}{4}\frac{2}{\sqrt{3}}\int_0^{2\sqrt{3}} \left(12\sqrt{3} - 6y - 2\sqrt{3}y + y^2\right)dy$$

$$= \frac{156\sqrt{3}}{5}\left(12\sqrt{3}y - 3y^2 - \sqrt{3}y^2 + \frac{1}{3}y^3\right)\Big|_0^{2\sqrt{3}}$$

$$= \frac{156\sqrt{3}}{5}\left(72 - 36 - 12\sqrt{3} + 8\sqrt{3}\right)$$

$$= \frac{156\sqrt{3}}{5}\left(36 - 4\sqrt{3}\right)$$

$$= \frac{624}{5}\left(9\sqrt{3} - 3\right)$$

$$= \frac{1,872}{5}\left(3\sqrt{3} - 1\right)\ lb$$

Exercise

Two electrons r *meters* apart repel each other with a force of

$F = \dfrac{23\times 10^{-29}}{r^2}$ *newtons*

a) Suppose one electron is held fixed at the point $(1,\ 0)$ on the x-axis (units in *meters*). How much work does it take to move a second electron along x-axis from the point $(-1,\ 0)$ to the origin?

b) Suppose one electron is held fixed at the point $(-1,\ 0)$ and $(1,\ 0)$. How much work does it take to move a third electron along x-axis from the point $(5,\ 0)$ to $(3,\ 0)$?

Solution

a) Let ρ be the x-coordinate of the second electron.

Then, $r^2 = (\rho - 1)^2$

$$W = \int_{-1}^{0} \frac{23\times 10^{-29}}{r^2}d\rho$$

$$= 23\times 10^{-29}\int_{-1}^{0} \frac{d\rho}{(\rho-1)^2}$$

$$= 23\times10^{-29}\int_{-1}^{0}\frac{d(\rho-1)}{(\rho-1)^2}$$

$$= 23\times10^{-29}\left(-\frac{1}{\rho-1}\right|_{-1}^{0}$$

$$= -23\times10^{-29}\left(-1+\frac{1}{2}\right)$$

$$\underline{= \frac{23}{2}\times10^{-29}\ J}$$

b) Let W_1 : is the work done against the field of the first electron

W_2 : is the work done against the field of the second electron

Then $W = W_1 + W_2$

Let ρ be the x-coordinate of the third electron

Then, $r_1^2 = (\rho-1)^2 \quad and \quad r_2^2 = (\rho+1)^2$

$$W_1 = \int_3^5 \frac{23\times10^{-29}}{r_1^2}\,d\rho$$

$$= 23\times10^{-29}\int_3^5\frac{d\rho}{(\rho-1)^2}$$

$$= 23\times10^{-29}\int_3^5\frac{d(\rho-1)}{(\rho-1)^2}$$

$$= 23\times10^{-29}\left(-\frac{1}{\rho-1}\right|_3^5$$

$$= -23\times10^{-29}\left(\frac{1}{4}-\frac{1}{2}\right)$$

$$\underline{= \frac{23}{4}\times10^{-29}}$$

$$W_2 = \int_3^5 \frac{23\times10^{-29}}{r_2^2}\,d\rho$$

$$= 23\times10^{-29}\int_3^5\frac{d\rho}{(\rho+1)^2}$$

$$= 23\times10^{-29}\int_3^5\frac{d(\rho+1)}{(\rho+1)^2}$$

$$= 23\times 10^{-29}\left(-\frac{1}{\rho+1}\right|_3^5$$

$$= -23\times 10^{-29}\left(\frac{1}{6}-\frac{1}{4}\right)$$

$$= -23\times 10^{-29}\left(-\frac{1}{12}\right)$$

$$= \frac{23}{12}\times 10^{-29}$$

$$W = W_1 + W_2$$

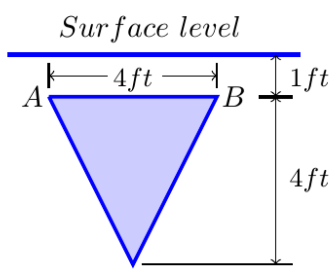
$$= \frac{23}{4}\times 10^{-29} + \frac{23}{12}\times 10^{-29}$$

$$= 23\times 10^{-29}\left(\frac{1}{4}+\frac{1}{12}\right)$$

$$= \frac{23}{3}\times 10^{-29}\ J$$

Exercise

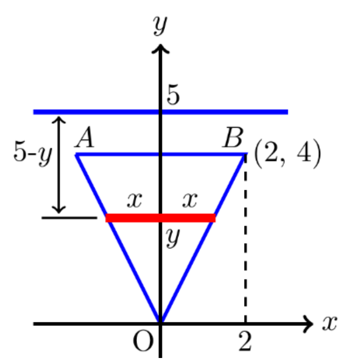
The isosceles triangular plate is submerged vertically 1 *feet* below the surface of a freshwater lake.



a) Find the fluid force against one face of the plate.

b) What would be the fluid force on one side of the plate if the water were seawater instead of freshwater?

Solution



Line OB: $y = \frac{4}{2}x$

$$y = 2x \quad \to \quad x = \frac{1}{2}y$$

Width: $L(y) = 2x$

$$= y$$

Depth: y

a) Freshwater Weight density: $\rho g = 62.4\ \ lb/ft^3$

$$F = 62.4\int_0^4 (5-y)(y)\,dy \qquad\qquad F = \rho g\int_0^a (a-y)L(y)\,dy$$

$$= 62.4\int_0^4 \left(5y - y^2\right)dy$$

$$= \frac{624}{10}\left(\frac{5}{2}y^2 - \frac{1}{3}y^3\right|_0^4$$

$$= \frac{312}{5}\left(40 - \frac{64}{3}\right)$$

$$= \frac{312}{15}(56)$$

$$\underline{= \frac{5,842}{5}\ \ lb}$$

b) Seawater Weight density: $\rho g = 64\ \ lb/ft^3$

$$F = 64\int_0^4 (5-y)(y)\,dy \qquad\qquad F = \rho g\int_0^a (a-y)L(y)\,dy$$

$$= 64\int_0^4 \left(5y - y^2\right)dy$$

$$= 64\left(\frac{5}{2}y^2 - \frac{1}{3}y^3\right|_0^4$$

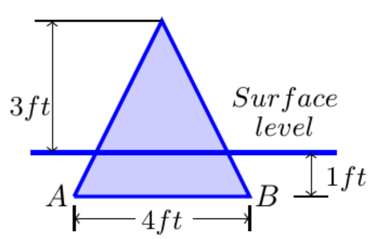
$$= 64\left(40 - \frac{64}{3}\right)$$

$$= 64\left(\frac{56}{3}\right)$$

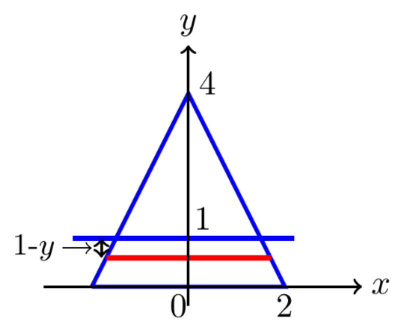
$$\underline{= \frac{3,584}{3}\ \ lb}$$

Exercise

The isosceles triangular plate is submerged vertically 3 *feet* above the surface of a freshwater lake. What force does the water exert on one face of the plate now?



Solution



Freshwater Weight density: $\rho g = 62.4\ \ lb/ft^3$

Line: $y = \frac{4-0}{0-2}x + 4$

$$y=-2x+4 \quad \to \quad x=\frac{1}{2}(4-y)$$

Width: $L(y)=2x$

$$\boxed{=4-y}$$

Depth: $\boxed{1-y}$

$$F=62.4\int_0^1 (1-y)(4-y)\,dy \qquad\qquad F=\rho g\int_0^a (a-y)L(y)\,dy$$

$$=62.4\int_0^1 \left(4-5y+y^2\right)dy$$

$$=\frac{624}{10}\left(4y-\frac{5}{2}y^2+\frac{1}{3}y^3\right)\Big|_0^1$$

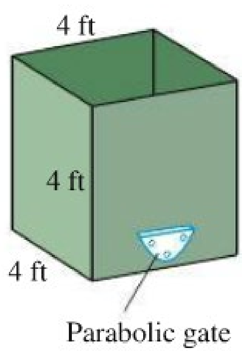
$$=\frac{312}{5}\left(4-\frac{5}{2}+\frac{1}{3}\right)$$

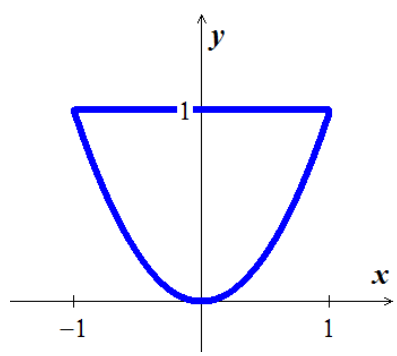
$$=\frac{312}{5}\left(\frac{24-15+2}{6}\right)$$

$$\boxed{=\frac{572}{5}\ lb}$$

Exercise

The cubical metal tank has parabolic gate held in place by bolts and designed to withstand a fluid force of 160 *lb*. without rupturing. The liquid you plan to store has a weight-density of 50 lb/ft^3.





a) What is the fluid force on the gate when the liquid is 2 *feet* deep?

b) What is the maximum height to which the container can be filled without exceeding the gate's design limitation?

Solution

From the graph:

$$y=x^2 \quad \to \quad x=\sqrt{y} \qquad (right\ side)$$

Boundary: $0\le y\le 1$

Width: $L(y) = 2x$

$$= \boxed{2\sqrt{y}}$$

Depth: $\boxed{2 - y}$

Given: weight-density of $50 \ \ lb/ft^3$

a) $F = 50\int_0^1 (2-y)\left(2\sqrt{y}\right)dy$ $\qquad\qquad F = \rho g\int_0^a (a-y)L(y)\,dy$

$$= 50\int_0^1 \left(4y^{1/2} - 2y^{3/2}\right)dy$$

$$= 100\left(\frac{4}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right|_0^1$$

$$= 100\left(\frac{4}{3} - \frac{2}{5}\right)$$

$$= 100\left(\frac{20-6}{15}\right)$$

$$\boxed{= \frac{280}{3} \ \ lb}$$

b) $F = 50\int_0^1 (h-y)\left(2\sqrt{y}\right)dy$

$$\frac{16}{5} = \int_0^1 \left(2hy^{1/2} - 2y^{3/2}\right)dy$$

$$\int_0^1 \left(hy^{1/2} - y^{3/2}\right)dy = \frac{8}{5}$$

$$\frac{2}{3}hy^{3/2} - \frac{2}{5}y^{5/2}\Big|_0^1 = \frac{8}{5}$$

$$\frac{2}{3}h - \frac{2}{5} = \frac{8}{5}$$

$$\frac{2}{3}h = \frac{8}{5} + \frac{2}{5}$$

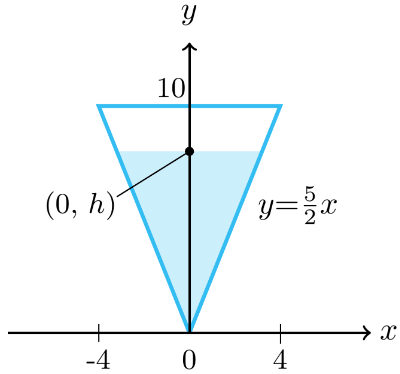
$$\frac{2}{3}h = 2$$

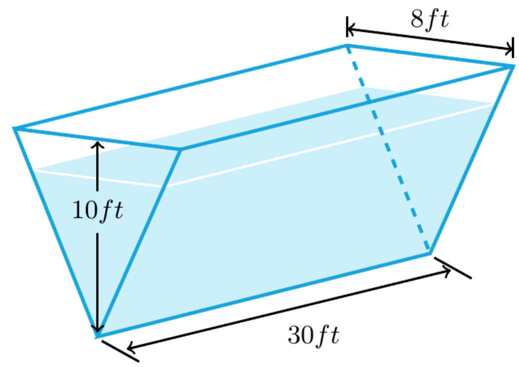
$$\boxed{h = 3 \ \ ft}$$

The maximum height to which the container can be filled without exceeding the gate's design limitation is $3 \ \ feet$

Exercise

The end plates of the trough were designed to withstand a fluid force of 6,667 *lb*.





a) What is the value of *h*?

b) How many cubic feet of water can the tank hold without exceeding this limitation?

Solution

$$y=\frac{5}{2}x \quad \to \quad x=\frac{2}{5}y$$

Boundary: $0 \le y \le h$

Width: $L(y)=2x$

$$=\frac{4}{5}y$$

Depth: $h-y$

Freshwater Weight density: $\rho g = 62.4 \ \ lb/ft^3$

a) $F=62.4\int_0^h (h-y)\left(\frac{4}{5}y\right)dy$ $\qquad F=\rho g\int_0^a (a-y)L(y)\,dy$

$$6,667\frac{5}{4(62.4)}=\int_0^h \left(hy-y^2\right)dy$$

$$6,667\frac{50}{4(624)}=\left(\frac{1}{2}hy^2-\frac{1}{3}y^3\right|_0^h$$

$$\frac{166,675}{1,248}=\frac{1}{2}h^3-\frac{1}{3}h^3$$

$$\frac{1}{6}h^3=\frac{166,675}{1,248}$$

$$h^3=\frac{166,675}{208}$$

$$h=\sqrt[3]{\frac{166,675}{208}}$$

$$\approx 9.29 \ \ ft$$

b) $Volume = \frac{1}{2} \times Base \times Height \times width$

$$\frac{1}{2} \times Base = x$$

$$= \frac{2}{5} y$$

$$\boxed{= \frac{2}{5} h}$$

$$Volume = \left(\frac{2}{5} h\right)(h)(30)$$

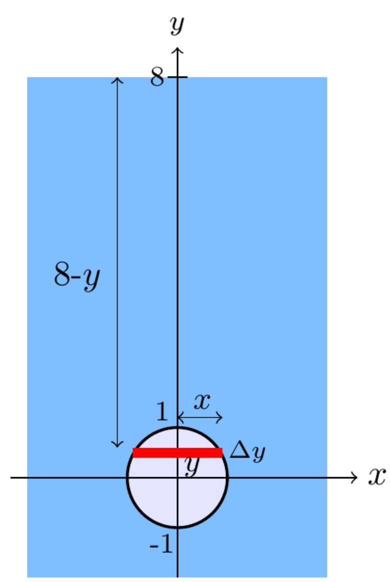
$$= 12(9.29)^2$$

$$\boxed{\approx 1035 \ ft^3}$$

Exercise

A circular observation window on a marine science ship has a radius of 1 *foot*, and the center of the window is 8 *feet* below water level. What is the fluid force on the window?

Solution



$$x^2 + y^2 = 1$$

$$x = \sqrt{1 - y^2}$$

$$Width: \ L(y) = 2x$$

$$\boxed{= 2\sqrt{1 - y^2}}$$

$Depth$: $\boxed{8 - y}$

$Boundary$: $\boxed{-1 \le y \le 1}$

Seawater Weight density: $\rho g = 64 \ \ lb/ft^3$

$$F = 64 \int_{-1}^{1} (8 - y)(2)\sqrt{1 - y^2} \ dy$$

$$= 128 \int_{-1}^{1} 8\sqrt{1 - y^2} \ dy - 128 \int_{-1}^{1} y\sqrt{1 - y^2} \ dy$$

$$= 1{,}024 \int_{-1}^{1} \sqrt{1 - y^2} \ dy + 64 \int_{-1}^{1} \left(1 - y^2\right)^{1/2} \ d\left(1 - y^2\right)$$

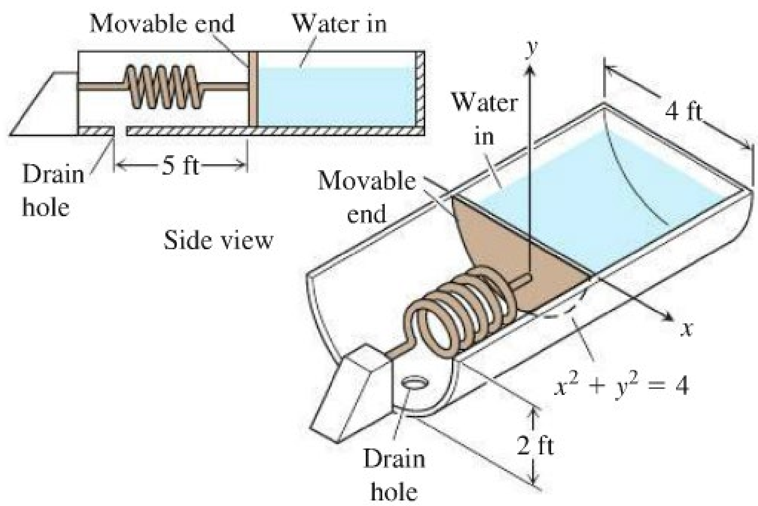
$$= 1{,}024\left(Area \ of \ a \ semicircle \ w/ \ r = 1\right) + \frac{128}{3}\left(1 - y^2\right)^{3/2} \Big|_{-1}^{1}$$

$$= 1{,}024\left(\frac{1}{2}\pi(1)^2\right) + \frac{128}{3}(1 - 1)$$

$$\boxed{= 512\pi \ \ lbs}$$

Exercise

Water pours into the tank at the rate of 4 ft^3/min. The tank's cross-sections are 4-*ft*-diameter semicircles. One end of the tank is movable, but moving it to increase the volume compresses a spring. The spring constant is $k = 100$ lb/ft. If the end of the tank moves 5 feet against the spring, the water will drain out of a safety hole in the bottom at the rate of 5 ft^3/min. Will the movable end reach the hole before the tank overflows?



Solution

By Hooke's law: $F_{spring} = kz$

$F_{spring} = F_{fluid} = kz$

Freshwater Weight density: $\rho g = 62.4$ lb/ft^3

$x^2 + y^2 = 4 \quad \rightarrow \quad x = \sqrt{4 - y^2}$

Width: $w(y) = 2x$

$$= 2\sqrt{4 - y^2}$$

Depth: $-y$

Boundary: $-2 \le y \le d$

$$F_{fluid} = 62.4 \int_{-2}^{d} (-y)(2)\sqrt{4 - y^2}\, dy$$

$$= 62.4 \int_{-2}^{d} \left(4 - y^2\right)^{1/2} d\left(4 - y^2\right)$$

$$= 41.6\left(4 - y^2\right)^{3/2} \Big|_{-2}^{d}$$

$$\boxed{= 41.6\left(4 - d^2\right)^{3/2}}$$

$$z = \frac{1}{k} F_{fluid}$$

$$= \frac{41.6}{100}\left(4 - d^2\right)^{3/2}$$

$$\boxed{= 0.416\left(4 - d^2\right)^{3/2}}$$

$$z(0) = 0.416(4)^{3/2}$$

$$= 0.416(8)$$

$$\boxed{= 3.328}$$

Since the tank is full when $d = 0$; therefore, the tank will overflow and the spring will have compressed $3.328\ ft$