Solution Section 3.3 – Gram-Schmidt Process

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\boldsymbol{R}^m$.

a) $\boldsymbol{u}_1 = (1,\ -3),\quad \boldsymbol{u}_2 = (2,\ 2)$

b) $\boldsymbol{u}_1 = (1,\ 0),\quad \boldsymbol{u}_2 = (3,\ -5)$

c) $\{(1,\ 1,\ 1),\ (-1,\ 1,\ 0),\ (1,\ 2,\ 1)\}$

d) $\{(1,\ 1,\ 1),\ (0,\ 1,\ 1),\ (0,\ 0,\ 1)\}$

e) $\{(1,\ 1,\ 1,\ 1),\ (1,\ 2,\ 1,\ 0),\ (1,\ 3,\ 0,\ 0)\}$

f) $\{(0,\ 2,\ -1,\ 1),\ (0,\ 0,\ 1,\ 1),\ (-2,\ 1,\ 1,\ -1)\}$

g) $\boldsymbol{u}_1 = (1,\ 0,\ 0),\quad \boldsymbol{u}_2 = (3,\ 7,\ -2),\quad \boldsymbol{u}_3 = (0,\ 4,\ 1)$

h) $\boldsymbol{u}_1 = (0,2,1,0),\quad \boldsymbol{u}_2 = (1,-1,0,0),\quad \boldsymbol{u}_3 = (1,2,0,-1),\quad \boldsymbol{u}_4 = (1,0,0,1)$

Solution

a) $$v_1 = \frac{u_1}{\|u_1\|} = \frac{(1,\ -3)}{\sqrt{1^2+(-3)^2}} = \frac{(1,\ -3)}{\sqrt{10}} = \left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right)$$

$$\begin{aligned}
w_2 &= u_2 - (u_2.v_1)v_1 \\
&= (2,\ 2) - \left[(2,\ 2).\left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right)\right]\left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right) \\
&= (2,\ 2) - \left[\frac{2}{\sqrt{10}} - \frac{6}{\sqrt{10}}\right]\left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right) \\
&= (2,\ 2) - \left[-\frac{4}{\sqrt{10}}\right]\left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right) \\
&= (2,\ 2) - \left(-\frac{4}{10},\ \frac{12}{10}\right) \\
&= (2,\ 2) - \left(-\frac{2}{5},\ \frac{6}{5}\right) \\
&= \left(\frac{12}{5},\ \frac{4}{5}\right)
\end{aligned}$$

$$\|w_2\| = \sqrt{\left(\frac{12}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{144}{25} + \frac{16}{25}} = \sqrt{\frac{160}{25}} = \frac{\sqrt{16(10)}}{\sqrt{25}} = \frac{4\sqrt{10}}{5}$$

$$v_2 = \frac{w_2}{\|w_2\|} = \frac{4\sqrt{10}}{5}\left(\frac{12}{5},\ \frac{4}{5}\right) = \left(\frac{48\sqrt{10}}{25},\ \frac{16\sqrt{10}}{25}\right)$$

b) $u_1 = (1,\ 0),\ u_2 = (3,\ -5)$

$$v_1 = \frac{u_1}{\|u_1\|} = \frac{(1,\ 0)}{\sqrt{1^2+0^2}} = (1,\ 0)$$

$$w_2 = u_2 - (u_2.v_1)v_1 = (0,\ -5)$$
$$= (3,\ -5) - [(3,\ -5).(1,\ 0)](1,\ 0)$$
$$= (3,\ -5) - [3](1,\ 0)$$
$$= (3,\ -5) - (3,\ 0)$$

$$\|w_2\| = \sqrt{0^2 + (-5)^2} = 5$$

$$v_2 = \frac{w_2}{\|w_2\|} = \frac{1}{5}(0,\ -5) = (0,\ -1)$$

c) $\{(1,\ 1,\ 1),\ (-1,\ 1,\ 0),\ (1,\ 2,\ 1)\}$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(1, 1, 1)}{\sqrt{1^2+1^2+1^2}} = \frac{(1,\ 1,\ 1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$w_2 = v_2 - (v_2.u_1)u_1$$
$$= (-1,\ 1,\ 0) - \left[(-1,\ 1,\ 0)\bullet\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$
$$= (-1,\ 1,\ 0) - \left[-\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + 0\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$
$$= (-1,\ 1,\ 0) - 0\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$
$$= (-1,\ 1,\ 0)$$
$$\|w_2\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{(-1,\ 1,\ 0)}{\sqrt{2}} = \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$v_3\bullet u_1 = (1,\ 2,\ 1)\bullet\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{3}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = \sqrt{3}$$
$$v_3\bullet u_2 = (1,\ 2,\ 1)\bullet\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right) = -\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} + 0 = \frac{1}{\sqrt{2}}$$

$$w_3 = v_3 - (v_3\bullet u_1)u_1 - (v_3\bullet u_2)u_2$$
$$= (1,\ 2,\ 1) - \sqrt{3}\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) - \sqrt{2}\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$
$$= (1,\ 2,\ 1) - (1,\ 1,\ 1) - (-1,\ 1,\ 0)$$
$$= (1,\ 0,\ 0)$$

$$u_3 = \frac{w_3}{\|w_3\|} = \frac{(1,\ 0,\ 0)}{\sqrt{1^2}} = (1,\ 0,\ 0)$$

d) $\{(1,\ 1,\ 1),\ (0,\ 1,\ 1),\ (0,\ 0,\ 1)\}$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(1,1,1)}{\sqrt{1^2+1^2+1^2}} = \frac{(1,\ 1,\ 1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\begin{aligned}
w_2 &= v_2 - \left(v_2 . u_1\right) u_1 \\
&= (0,\ 1,\ 1) - \left[(0,\ 1,\ 1)\bullet\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) \\
&= (0,\ 1,\ 1) - \left[\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) \\
&= (0,\ 1,\ 1) - \left[\frac{2}{\sqrt{3}}\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) \\
&= (0,\ 1,\ 1) - \left(\tfrac{2}{3},\ \tfrac{2}{3},\ \tfrac{2}{3}\right) \\
&= \left(-\tfrac{2}{3},\ \tfrac{1}{3},\ \tfrac{1}{3}\right)
\end{aligned}$$

$$\|w_2\| = \sqrt{\left(-\tfrac{2}{3}\right)^2 + \left(\tfrac{1}{3}\right)^2 + \left(\tfrac{1}{3}\right)^2} = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{1}{9}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{3}$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{\left(-\tfrac{2}{3},\ \tfrac{1}{3},\ \tfrac{1}{3}\right)}{\frac{\sqrt{6}}{3}} = \frac{3}{\sqrt{6}}\left(-\tfrac{2}{3},\ \tfrac{1}{3},\ \tfrac{1}{3}\right) = \left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$v_3 \bullet u_1 = (0,\ 0,\ 1)\bullet\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}$$

$$v_3 \bullet u_2 = (0,\ 0,\ 1)\bullet\left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right) = \frac{1}{\sqrt{6}}$$

$$\begin{aligned}
w_3 &= v_3 - \left(v_3 \bullet u_1\right) u_1 - \left(v_3 \bullet u_2\right) u_2 \\
&= (0,\ 0,\ 1) - \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) - \frac{1}{\sqrt{6}}\left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right) \\
&= (0,\ 0,\ 1) - \left(\tfrac{1}{3},\ \tfrac{1}{3},\ \tfrac{1}{3}\right) - \left(-\tfrac{1}{3},\ \tfrac{1}{6},\ \tfrac{1}{6}\right) \\
&= \left(0,\ -\tfrac{1}{2},\ \tfrac{1}{2}\right)
\end{aligned}$$

$$u_3 = \frac{w_3}{\|w_3\|} = \frac{\left(0,\ -\tfrac{1}{2},\ \tfrac{1}{2}\right)}{\sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2}}$$

$$= \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\sqrt{\frac{1}{2}}}$$

$$= \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\frac{1}{\sqrt{2}}}$$

$$= \sqrt{2}\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)$$

$$\boxed{= \left(0,\ -\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)}$$

e) $\{(1,\ 1,\ 1,\ 1),\ (1,\ 2,\ 1,\ 0),\ (1,\ 3,\ 0,\ 0)\}$

$$\boldsymbol{v}_1 = \boldsymbol{u}_1\ \boxed{= (1,\ 1,\ 1,\ 1)}$$

$$\boldsymbol{q}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \frac{(1,\ 1,\ 1,\ 1)}{\sqrt{4}} = \left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$\boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\langle \boldsymbol{u}_2, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1$$

$$= (1,\ 2,\ 1,\ 0) - \frac{(1,\ 2,\ 1,\ 0)\bullet(1,\ 1,\ 1,\ 1)}{\|(1,\ 1,\ 1,\ 1)\|^2}(1,\ 1,\ 1,\ 1)$$

$$= (1,\ 2,\ 1,\ 0) - \frac{1+2+1}{4}(1,\ 1,\ 1,\ 1)$$

$$= (1,\ 2,\ 1,\ 0) - \frac{4}{4}(1,\ 1,\ 1,\ 1)$$

$$= (1,\ 2,\ 1,\ 0) - (1,\ 1,\ 1,\ 1)$$

$$\boxed{= (0,\ 1,\ 0,\ -1)}$$

$$\boldsymbol{q}_2 = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|} = \frac{(0,\ 1,\ 0,\ -1)}{\sqrt{1+1}} = \left(0,\ \frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right)$$

$$\boldsymbol{v}_3 = \boldsymbol{u}_3 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2$$

$$= (1,\ 3,\ 0,\ 0) - \frac{(1,\ 3,\ 0,\ 0)\bullet(1,\ 1,\ 1,\ 1)}{\|(1,\ 1,\ 1,\ 1)\|^2}(1,\ 1,\ 1,\ 1) - \frac{(1,\ 3,\ 0,\ 0)\bullet(0,\ 1,\ 0,\ -1)}{\|(0,\ 1,\ 0,\ -1)\|^2}(0,\ 1,\ 0,\ -1)$$

$$= (1,\ 3,\ 0,\ 0) - \frac{4}{4}(1,\ 1,\ 1,\ 1) - \frac{3}{2}(0,\ 1,\ 0,\ -1)$$

$$= (1,\ 3,\ 0,\ 0) - (1,\ 1,\ 1,\ 1) - \left(0,\ \frac{3}{2},\ 0,\ -\frac{3}{2}\right)$$

$$\boxed{= \left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)}$$

$$q_3 = \frac{v_3}{\|v_3\|} = \frac{\left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)}{\sqrt{\frac{1}{4}+1+\frac{1}{4}}} = \frac{\left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)}{\frac{\sqrt{6}}{2}}$$

$$= \frac{2}{\sqrt{6}}\left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)$$

$$\boxed{= \left(0,\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)}$$

f) $\{(0,\ 2,\ -1,\ 1),\ (0,\ 0,\ 1,\ 1),\ (-2,\ 1,\ 1,\ -1)\}$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(0,\ 2,\ -1,\ 1)}{\sqrt{2^2+1^2+1^2}} = \frac{(0,\ 2,\ -1,\ 1)}{\sqrt{6}} = \left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$w_2 = v_2 - (v_2.u_1)u_1$$

$$= (0,\ 0,\ 1,\ 1) - \left[(0,\ 0,\ 1,\ 1)\bullet\left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)\right]\left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$= (0,\ 0,\ 1,\ 1) - \left[-\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}}\right]\left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$= (0,\ 0,\ 1,\ 1) - [0]\left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$= (0,\ 0,\ 1,\ 1)$$

$$\boxed{u_2} = \frac{w_2}{\|w_2\|} = \frac{(0,\ 0,\ 1,\ 1)}{\sqrt{2}} = \left(0,\ 0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

$$v_3\bullet u_1 = (-2,\ 1,\ 1,\ -1)\bullet\left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right) = \frac{2}{\sqrt{6}} - \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}}\ \boxed{=0}$$

$$v_3\bullet u_2 = (-2,\ 1,\ 1,\ -1)\bullet\left(0,\ 0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\ \boxed{=0}$$

$$w_3 = v_3 - (v_3\bullet u_1)u_1 - (v_3\bullet u_2)u_2$$

$$= (-2,\ 1,\ 1,\ -1) - 0 - 0$$

$$= (-2,\ 1,\ 1,\ -1)$$

$$\boxed{u_3} = \frac{w_3}{\|w_3\|} = \frac{(-2,\ 1,\ 1,\ -1)}{\sqrt{(-2)^2+1^2+1^2+(-1)^2}} = \frac{(-2,\ 1,\ 1,\ -1)}{\sqrt{7}}$$

$$\boxed{= \left(-\frac{2}{\sqrt{7}},\ \frac{1}{\sqrt{7}},\ \frac{1}{\sqrt{7}},\ -\frac{1}{\sqrt{7}}\right)}$$

g) $\boldsymbol{u}_1 = (1,\ 0,\ 0), \quad \boldsymbol{u}_2 = (3,\ 7,\ -2), \quad \boldsymbol{u}_3 = (0,\ 4,\ 1)$

$$v_1 = \frac{u_1}{\|u_1\|} = \frac{(1,\ 0,\ 0)}{\sqrt{1^2+0^2+0^2}} = (1,\ 0,\ 0)$$

$$\begin{aligned} w_2 &= u_2 - (u_2 . v_1)v_1 \\ &= (3,7,-2) - \left[(3,7,-2)\bullet(1,0,0)\right](1,0,0) \\ &= (3,7,-2) - 3(1,0,0) \\ &= (0,\ 7,\ -2) \end{aligned}$$

$$\boxed{v_2} = \frac{w_2}{\|w_2\|} = \frac{(0,\ 7,\ -2)}{\sqrt{7^2+(-2)^2}} = \frac{1}{\sqrt{53}}(0,\ 7,\ -2) = \left(0,\ \frac{7}{\sqrt{53}},\ -\frac{2}{\sqrt{53}}\right)$$

$$u_3 \bullet v_1 = (0,\ 4,\ 1)\bullet(1,\ 0,\ 0)\ \boxed{=0}$$

$$u_3 \bullet v_2 = (0,\ 4,\ 1)\bullet\left(0,\ \frac{7}{\sqrt{53}},\ -\frac{2}{\sqrt{53}}\right)\ \boxed{=\frac{26}{\sqrt{53}}}$$

$$\begin{aligned} w_3 &= u_3 - (u_3 \bullet v_1)v_1 - (u_3 \bullet v_2)v_2 \\ &= (0,\ 4,\ 1) - 0 - \frac{26}{\sqrt{53}}\left(0,\ \frac{7}{\sqrt{53}},\ -\frac{2}{\sqrt{6}}\right) \\ &= (0,\ 4,\ 1) - \left(0,\ \frac{182}{53},\ -\frac{52}{53}\right) \\ &= \left(0,\ \frac{30}{53},\ \frac{105}{53}\right) \end{aligned}$$

$$\begin{aligned} \boxed{v_3} = \frac{w_3}{\|w_3\|} &= \frac{\left(0,\ \frac{30}{53},\ \frac{105}{53}\right)}{\sqrt{\left(\frac{30}{53}\right)^2 + \left(\frac{105}{53}\right)^2}} \\ &= \frac{53}{\sqrt{11925}}\left(0,\ \frac{30}{53},\ \frac{105}{53}\right) \\ &= \frac{53}{15\sqrt{53}}\left(0,\ \frac{30}{53},\ \frac{105}{53}\right) \\ &\boxed{= \left(0,\ \frac{2}{\sqrt{15}},\ \frac{7}{\sqrt{15}}\right)} \end{aligned}$$

h) $\boldsymbol{u}_1=(0,\ 2,\ 1,\ 0),\quad \boldsymbol{u}_2=(1,\ -1,\ 0,\ 0),\quad \boldsymbol{u}_3=(1,\ 2,\ 0,\ -1),\quad \boldsymbol{u}_4=(1,\ 0,\ 0,\ 1)$

$$v_1=\frac{u_1}{\|u_1\|}=\frac{(0,\,2,\,1,\,0)}{\sqrt{0^2+2^2+1^2+0^2}}=\frac{(0,\ 2,\ 1,\ 0)}{\sqrt{5}}=\left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)$$

$$w_2=u_2-(u_2.v_1)v_1$$

$$=(1,-1,\ 0,\ 0)-\left[(1,-1,\ 0,\ 0)\bullet\left(0,\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}},0\right)\right]\left(0,\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}},0\right)$$

$$=(1,-1,\ 0,\ 0)-\left(-\frac{2}{\sqrt{5}}\right)\left(0,\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}},0\right).$$

$$=\left(1,-\tfrac{1}{5},\ \tfrac{2}{5},\ 0\right)$$

$$v_2=\frac{w_2}{\|w_2\|}=\frac{\left(1,-\frac{1}{5},\ \frac{2}{5},\ 0\right)}{\sqrt{1+\frac{1}{25}+\frac{4}{25}+0}}=\frac{5}{\sqrt{30}}\left(1,-\frac{1}{5},\ \frac{2}{5},\ 0\right)=\left(\frac{5}{\sqrt{30}},-\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right)$$

$$u_3\bullet v_1=(1,\ 2,\ 0,\ -1)\bullet\left(0,\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)=\frac{4}{\sqrt{5}}$$

$$u_3\bullet v_2=(1,\ 2,\ 0,\ -1)\bullet\left(\frac{5}{\sqrt{30}},-\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right)=\frac{5}{\sqrt{30}}-\frac{2}{\sqrt{30}}=\frac{3}{\sqrt{30}}$$

$$w_3=u_3-(u_3\bullet v_1)v_1-(u_3\bullet v_2)v_2$$

$$=(1,\ 2,\ 0,\ -1)-\left(\frac{4}{\sqrt{5}}\right)\left(0,\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)-\left(\frac{3}{\sqrt{30}}\right)\left(\frac{5}{\sqrt{30}},-\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right)$$

$$=(1,\ 2,\ 0,\ -1)-\left(0,\tfrac{8}{5},\ \tfrac{4}{5},\ 0\right)-\left(\tfrac{1}{2},-\tfrac{1}{10},\ \tfrac{1}{5},\ 0\right)$$

$$=\left(\tfrac{1}{2},\ \tfrac{1}{2},\ -1,\ -1\right)$$

$$v_3=\frac{w_3}{\|w_3\|}=\frac{\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)}{\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+(-1)^2+(-1)^2}}$$

$$=\frac{1}{\sqrt{\frac{5}{2}}}\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)$$

$$=\frac{\sqrt{2}}{\sqrt{5}}\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)=\frac{2}{\sqrt{10}}\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)$$

$$=\left(\frac{1}{\sqrt{10}},\ \frac{1}{\sqrt{10}},\ -\frac{2}{\sqrt{10}},\ -\frac{2}{\sqrt{10}}\right)$$

$$u_4 \cdot v_1 = (1,0,0,1)\cdot\left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right) = 0$$

$$u_4 \cdot v_2 = (1,0,0,1)\cdot\left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0\right) = \frac{5}{\sqrt{30}}$$

$$u_4 \cdot v_3 = (1,0,0,1)\cdot\left(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, -\frac{2}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right) = -\frac{1}{\sqrt{10}}$$

$$\begin{aligned} w_4 &= u_4 - (u_4 \cdot v_1)v_1 - (u_4 \cdot v_2)v_2 - (u_4 \cdot v_3)v_3 \\ &= (1,2,0,-1) - (0) - \left(\frac{5}{\sqrt{30}}\right)\left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0\right) + \left(\frac{1}{\sqrt{10}}\right)\left(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, -\frac{2}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right) \\ &= (1,2,0,-1) - \left(\frac{5}{6}, -\frac{1}{6}, \frac{1}{3}, 0\right) + \left(\frac{1}{10}, \frac{1}{10}, -\frac{1}{5}, -\frac{1}{5}\right) \\ &= \left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right) \end{aligned}$$

$$v_4 = \frac{w_4}{\|w_4\|} = \frac{\left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right)}{\sqrt{\left(\frac{4}{15}\right)^2 + \left(\frac{4}{15}\right)^2 + \left(-\frac{8}{15}\right)^2 + \left(\frac{4}{5}\right)^2}}$$

$$= \frac{1}{\sqrt{\frac{240}{225}}}\left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right)$$

$$= \frac{1}{\frac{4}{\sqrt{15}}}\left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right)$$

$$= \frac{15}{4\sqrt{15}}\left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right)$$

$$= \left(\frac{1}{\sqrt{15}}, \frac{1}{\sqrt{15}}, -\frac{2}{\sqrt{15}}, \frac{3}{\sqrt{15}}\right)$$

Exercise

Find the QR–decomposition of

a) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

Solution

a) Since $\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5 \neq 0$, The matrix is invertible

$$u_1(1,\ 2), \quad u_2 = (-1,\ 3)$$

$$v_1 = u_1 = (1,\ 2)$$

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{(1,\ 2)}{\sqrt{1^2+2^2}} = \frac{(1,\ 2)}{\sqrt{5}} \ \underline{= \left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right)}$$

$$\begin{aligned} v_2 &= u_2 - \left(u_2 . v_1\right) v_1 \\ &= (-1,\ 3) - \left[(-1,\ 3).\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)\right]\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \\ &= (-1,\ 3) - \left(\frac{5}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \\ &= (-1,\ 3) - (1,\ 2) \\ &= (-2,\ 1) \end{aligned}$$

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{(-2,1)}{\sqrt{(-2)^2+1^2}} \ \underline{= \left(-\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right)}$$

$$\left\langle \boldsymbol{u}_1, \boldsymbol{q}_1 \right\rangle = (1,2) \bullet \left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right) = \frac{5}{\sqrt{5}} = \sqrt{5}$$

$$\left\langle \boldsymbol{u}_2, \boldsymbol{q}_1 \right\rangle = (-1,\ 3) \bullet \left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right) = \frac{5}{\sqrt{5}} = \sqrt{5}$$

$$\left\langle \boldsymbol{u}_2, \boldsymbol{q}_2 \right\rangle = (-1,\ 3) \bullet \left(-\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right) = \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{5}} = \sqrt{5}$$

$$R = \begin{bmatrix} \left\langle \boldsymbol{u}_1, \boldsymbol{q}_1 \right\rangle & \left\langle \boldsymbol{u}_2, \boldsymbol{q}_1 \right\rangle \\ 0 & \left\langle \boldsymbol{u}_2, \boldsymbol{q}_2 \right\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}$$

The QR-decomposition of the matrix is

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}$$

b) The column vectors of are: $\boldsymbol{u}_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$$\boldsymbol{v}_1 = \boldsymbol{u}_1 \ = (3, \ -4)$$

$$\boldsymbol{q}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \frac{(3, \ -4)}{\sqrt{9+16}} = \boxed{\left(\frac{3}{5}, \ -\frac{4}{5}\right)}$$

$$\boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\langle \boldsymbol{u}_2, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2} \boldsymbol{v}_1$$

$$= (5, \ 0) - \frac{(5, \ 0) \bullet (3, \ -4)}{25}(3, \ -4)$$

$$= (5, \ 0) - \frac{15}{25}(3, \ -4)$$

$$= (5, \ 0) - \frac{3}{5}(3, \ -4)$$

$$= (5, \ 0) - \left(\frac{9}{5}, \ -\frac{12}{5}\right)$$

$$= \left(\frac{16}{5}, \ \frac{12}{5}\right)$$

$$\boldsymbol{q}_2 = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|} = \frac{\left(\frac{16}{5}, \frac{12}{5}\right)}{\sqrt{\frac{256}{25} + \frac{144}{25}}} = \frac{1}{\sqrt{\frac{400}{25}}}\left(\frac{16}{5}, \ \frac{12}{5}\right) = \frac{1}{\sqrt{16}}\left(\frac{16}{5}, \ \frac{12}{5}\right) = \frac{1}{4}\left(\frac{16}{5}, \ \frac{12}{5}\right) = \boxed{\left(\frac{4}{5}, \ \frac{3}{5}\right)}$$

$$R = \begin{bmatrix} \langle \boldsymbol{u}_1, \boldsymbol{q}_1 \rangle & \langle \boldsymbol{u}_2, \boldsymbol{q}_1 \rangle \\ 0 & \langle \boldsymbol{u}_2, \boldsymbol{q}_2 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} 3\left(\frac{3}{5}\right) - 4\left(-\frac{4}{5}\right) & 5\left(\frac{3}{5}\right) + 0\left(-\frac{4}{5}\right) \\ 0 & 5\left(\frac{4}{5}\right) - 0\left(\frac{3}{5}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 \\ 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \cdot \begin{bmatrix} 5 & 3 \\ 0 & 4 \end{bmatrix}$$

$$\boldsymbol{A} \quad = \quad \boldsymbol{Q} \qquad \boldsymbol{R}$$

c) Since the column vectors $\boldsymbol{u}_1(1,\ 0,\ 1),\quad \boldsymbol{u}_2 = (2,\ 1,\ 4)$ are linearly independent, so has a QR–decomposition.

$$\boldsymbol{v}_1 = \boldsymbol{u}_1 = (1,\ 0,\ 1)$$

$$\boldsymbol{q}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \frac{(1,\ 0,\ 1)}{\sqrt{1^2+0+1^2}} = \frac{(1,\ 0,\ 1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$\begin{aligned} v_2 &= u_2 - (u_2 . v_1) v_1 \\ &= (2,1,4) - \left[(2,1,4).\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\right]\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) \\ &= (2,1,4) - \left(\frac{6}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) \\ &= (2,1,4) - (3,\ 0,\ 3) \\ &= (-1,\ 1,\ 1) \end{aligned}$$

$$\boldsymbol{q}_2 = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|} = \frac{(-1,\ 1,\ 1)}{\sqrt{(-1)^2+1^2+1^2}} = \left(-\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\langle \boldsymbol{u}_1, \boldsymbol{q}_1 \rangle = (1,\ 0,\ 1) \bullet \left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\langle \boldsymbol{u}_2, \boldsymbol{q}_1 \rangle = (2,\ 1,\ 4) \bullet \left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} + \frac{4}{\sqrt{2}} = 3\sqrt{2}$$

$$\langle \boldsymbol{u}_2, \boldsymbol{q}_2 \rangle = (2,\ 1,\ 4) \bullet \left(-\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) = -\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{4}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

$$\begin{aligned} R &= \begin{bmatrix} \langle \boldsymbol{u}_1, \boldsymbol{q}_1 \rangle & \langle \boldsymbol{u}_2, \boldsymbol{q}_1 \rangle \\ 0 & \langle \boldsymbol{u}_2, \boldsymbol{q}_2 \rangle \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix} \end{aligned}$$

The QR–decomposition of the matrix is $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$

d) Since $\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{vmatrix} = -4 \neq 0$, The matrix is invertible, so it has a *QR*-decomposition.

$$u_1\left(1,\ 1,\ 0\right), \quad u_2 = \left(2,\ 1,\ 3\right), \quad u_3 = \left(1,\ 1,\ 1\right)$$

$$v_1 = u_1 = \left(1,\ 1,\ 0\right)$$

$$q_1 = \frac{v_1}{\left\|v_1\right\|} = \frac{(1,\ 1,\ 0)}{\sqrt{1^2 + 1^2 + 0}} = \frac{(1,\ 1,\ 0)}{\sqrt{2}} \boxed{= \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)}$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \left(\vec{u}_2 \bullet \vec{v}_1\right)\vec{v}_1 \\
&= \left(2,\ 1,\ 3\right) - \left[\left(2,\ 1,\ 3\right).\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)\right]\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right) \\
&= \left(2,\ 1,\ 3\right) - \frac{3}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right) \\
&= \left(2,\ 1,\ 3\right) - \left(\frac{3}{2},\ \frac{3}{2},\ 0\right) \\
&= \left(\frac{1}{2},\ -\frac{1}{2},\ 3\right)
\end{aligned}$$

$$\begin{aligned}
q_2 &= \frac{v_2}{\left\|v_2\right\|} = \frac{\left(\frac{1}{2},\ -\frac{1}{2},\ 3\right)}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + 3^2}} \\
&= \frac{\left(\frac{1}{2},\ -\frac{1}{2},\ 3\right)}{\sqrt{\frac{19}{2}}} \\
&= \frac{\sqrt{2}}{\sqrt{19}}\left(\frac{1}{2},\ -\frac{1}{2},\ 3\right) \\
&= \boxed{\left(\frac{1}{\sqrt{38}},\ -\frac{1}{\sqrt{38}},\ \frac{6}{\sqrt{38}}\right)} \qquad \boxed{= \left(\frac{\sqrt{2}}{2\sqrt{19}},\ -\frac{\sqrt{2}}{2\sqrt{19}},\ \frac{3\sqrt{2}}{\sqrt{19}}\right)}
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \left(\vec{u}_3 \bullet \vec{v}_1\right)\vec{v}_1 - \left(\vec{u}_3 \bullet \vec{v}_2\right)\vec{v}_2 \\
&= (1,1,1) - \left[(1,1,1).\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)\right]\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \\
&\qquad - \left[(1,1,1).\left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}}\right)\right]\left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}}\right) \\
&= (1,1,1) - \frac{2}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) - \frac{6}{\sqrt{38}}\left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}}\right) \\
&= (1,1,1) - (1,1,0) - \left(\frac{3}{19}, -\frac{3}{19}, \frac{18}{19}\right) \\
&= \left(-\frac{3}{19},\ \frac{3}{19},\ \frac{1}{19}\right)
\end{aligned}$$

$$q_3 = \frac{v_3}{\|v_3\|} = \frac{\left(-\frac{3}{19},\ \frac{3}{19},\ \frac{1}{19}\right)}{\sqrt{\left(-\frac{3}{19}\right)^2 + \left(\frac{3}{19}\right)^2 + \left(\frac{1}{19}\right)^2}}$$

$$= \frac{19}{\sqrt{19}}\left(-\frac{3}{19},\ \frac{3}{19},\ \frac{1}{19}\right)$$

$$= \left(-\frac{3}{\sqrt{19}},\ \frac{3}{\sqrt{19}},\ \frac{1}{\sqrt{19}}\right)$$

$$\langle \boldsymbol{u}_1, \boldsymbol{q}_1 \rangle = (1,\ 1,\ 0) \bullet \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\langle \boldsymbol{u}_2, \boldsymbol{q}_1 \rangle = (2,\ 1,\ 3) \bullet \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right) = \frac{3}{\sqrt{2}}$$

$$\langle \boldsymbol{u}_2, \boldsymbol{q}_2 \rangle = (2,\ 1,\ 3) \bullet \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}}\right) = \frac{2-1+18}{\sqrt{38}} = \frac{19}{\sqrt{38}} = \frac{19}{\sqrt{2}\sqrt{19}} = \frac{\sqrt{19}}{\sqrt{2}}$$

$$\langle \boldsymbol{u}_3, \boldsymbol{q}_1 \rangle = (1,\ 1,\ 1) \bullet \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\langle \boldsymbol{u}_3, \boldsymbol{q}_2 \rangle = (1,\ 1,\ 1) \bullet \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}}\right) = \frac{1-1+6}{\sqrt{38}} = \frac{6}{\sqrt{2}\sqrt{19}} \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{\sqrt{19}}$$

$$\langle \boldsymbol{u}_3, \boldsymbol{q}_3 \rangle = (1,\ 1,\ 1) \bullet \left(-\frac{3}{\sqrt{19}},\ \frac{3}{\sqrt{19}},\ \frac{1}{\sqrt{19}}\right) = \frac{-3+3+1}{\sqrt{19}} = \frac{1}{\sqrt{19}}$$

$$R = \begin{bmatrix} \langle \boldsymbol{u}_1, \boldsymbol{q}_1 \rangle & \langle \boldsymbol{u}_2, \boldsymbol{q}_1 \rangle & \langle \boldsymbol{u}_3, \boldsymbol{q}_1 \rangle \\ 0 & \langle \boldsymbol{u}_2, \boldsymbol{q}_2 \rangle & \langle \boldsymbol{u}_3, \boldsymbol{q}_2 \rangle \\ 0 & 0 & \langle \boldsymbol{u}_3, \boldsymbol{q}_3 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix}$$

The QR–decomposition of the matrix is $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix}$

e) $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The matrix is linearly dependent, so doesn't have a *QR*–decomposition.

Exercise

Verify that the Cauchy-Schwarz inequality holds for the given vectors using the Euclidean inner product

$$\boldsymbol{u} = (0,\ -2,\ 2,\ 1), \quad \boldsymbol{v} = (-1,\ -1,\ 1,\ 1)$$

Solution

$$\langle \boldsymbol{u},\ \boldsymbol{v} \rangle = 0 - 2(-1) + 2(1) + 1(1) = 5$$

$$\left\| \langle \boldsymbol{u}, \boldsymbol{v} \rangle \right\| = \sqrt{5}$$

$$\begin{aligned} \|\boldsymbol{u}\|.\|\boldsymbol{v}\| &= \sqrt{0+4+4+1}\sqrt{1+1+1+1} \\ &= \sqrt{9}\sqrt{4} \\ &= 6 \end{aligned}$$

$$\sqrt{5} < 6 \quad \Rightarrow \quad \left\| \langle \boldsymbol{u}, \boldsymbol{v} \rangle \right\| \le \|\boldsymbol{u}\|.\|\boldsymbol{v}\|$$