SOLUTION Section 3.1 – Sequences

Exercise

Find the values of $a_1, a_2, a_3, \text{ and } a_4$ for $a_n = \dfrac{1-n}{n^2}$

Solution

$$a_1 = \frac{1-1}{1^2} \underline{= 0}$$

$$a_2 = \frac{1-2}{2^2} \underline{= -\frac{1}{4}}$$

$$a_3 = \frac{1-3}{3^2} \underline{= -\frac{2}{9}}$$

$$a_4 = \frac{1-4}{4^2} \underline{= -\frac{3}{16}}$$

Exercise

Find the values of $a_1, a_2, a_3, \text{ and } a_4$ for $a_n = \dfrac{1}{n!}$

Solution

$$a_1 = \frac{1}{1!} \underline{= 1}$$

$$a_2 = \frac{1}{2!} \underline{= \frac{1}{4}}$$

$$a_3 = \frac{1}{3!} \underline{= \frac{1}{6}}$$

$$a_4 = \frac{1}{4!} \underline{= \frac{1}{24}}$$

Exercise

Find the values of $a_1, a_2, a_3, \text{ and } a_4$ for $a_n = \dfrac{(-1)^{n+1}}{2n-1}$

Solution

$$a_1 = \frac{(-1)^{1+1}}{2(1)-1} \underline{= 1}$$

$$a_2 = \frac{(-1)^{2+1}}{2(2)-1} \underline{= -\frac{1}{3}}$$

$$a_3 = \frac{(-1)^{3+1}}{2(3)-1} \underline{= \frac{1}{5}}$$

$$a_4 = \frac{(-1)^{4+1}}{2(4)-1} = -\frac{1}{7}$$

Exercise

Find the values of $a_1,\ a_2,\ a_3,\ and\ a_4$ for $a_n = 2+(-1)^n$

Solution

$$a_1 = 2+(-1)^1 = 1$$

$$a_2 = 2+(-1)^2 = 3$$

$$a_3 = 2+(-1)^3 = 1$$

$$a_4 = 2+(-1)^4 = 3$$

Exercise

Find the values of $a_1,\ a_2,\ a_3,\ and\ a_4$ for $a_n = \dfrac{2^n-1}{2^n}$

Solution

$$a_1 = \frac{2^1-1}{2^1} = \frac{1}{2}$$

$$a_2 = \frac{2^2-1}{2^2} = \frac{3}{4}$$

$$a_3 = \frac{2^3-1}{2^3} = \frac{7}{8}$$

$$a_4 = \frac{2^4-1}{2^5} = \frac{15}{32}$$

Exercise

Write the first ten terms of the sequence $a_1 = 1, \quad a_{n+1} = a_n + \dfrac{1}{2^n}$

Solution

$$a_2 = a_1 + \frac{1}{2^1}$$

$$= 1+\frac{1}{2}$$

$$= \frac{3}{2}$$

$$a_3 = a_2 + \frac{1}{2^2}$$
$$= \frac{3}{2} + \frac{1}{4}$$
$$= \frac{7}{4}$$

$$a_4 = a_3 + \frac{1}{2^3}$$
$$= \frac{7}{4} + \frac{1}{8}$$
$$= \frac{15}{8}$$

$$a_5 = a_4 + \frac{1}{2^4}$$
$$= \frac{15}{8} + \frac{1}{16}$$
$$= \frac{31}{16}$$

$$a_6 = a_5 + \frac{1}{2^5}$$
$$= \frac{31}{16} + \frac{1}{32}$$
$$= \frac{63}{32}$$

$$a_7 = a_6 + \frac{1}{2^6}$$
$$= \frac{63}{32} + \frac{1}{64}$$
$$= \frac{127}{64}$$

$$a_8 = a_7 + \frac{1}{2^7}$$
$$= \frac{127}{64} + \frac{1}{128}$$
$$= \frac{255}{128}$$

$$a_9 = a_8 + \frac{1}{2^8}$$
$$= \frac{255}{128} + \frac{1}{256}$$
$$= \frac{511}{256}$$

$$a_{10} = a_9 + \frac{1}{2^9}$$

$= \frac{511}{256} + \frac{1}{512}$

$= \boxed{\frac{1023}{512}}$

Exercise

Write the first ten terms of the sequence $a_1 = 1, \quad a_{n+1} = \frac{a_n}{n+1}$

Solution

$\boxed{a_1 = 1}$

$a_2 = \frac{1}{1+1} = \boxed{\frac{1}{2}}$

$a_3 = \frac{\frac{1}{2}}{2+1} = \boxed{\frac{1}{6}}$

$a_4 = \frac{\frac{1}{6}}{3+1} = \boxed{\frac{1}{24}}$

$a_5 = \frac{\frac{1}{24}}{4+1} = \boxed{\frac{1}{120}}$

$a_6 = \frac{\frac{1}{120}}{5+1}$

$= \boxed{\frac{1}{720}}$

$a_7 = \frac{\frac{1}{720}}{6+1}$

$= \boxed{\frac{1}{5040}}$

$a_8 = \frac{\frac{1}{5040}}{7+1}$

$= \boxed{\frac{1}{40,320}}$

$a_9 = \frac{\frac{1}{40,320}}{8+1}$

$= \boxed{\frac{1}{362,880}}$

$a_{10} = \frac{\frac{1}{362,880}}{9+1}$

$= \boxed{\frac{1}{3,628,800}}$

Exercise

Write the first ten terms of the sequence $a_1 = 2, \quad a_2 = -1, \quad a_{n+2} = \frac{a_{n+1}}{a_n}$

Solution

$a_1 = 2, \quad a_2 = -1$

$\boxed{a_3 = -\frac{1}{2}}$

$$a_4 = \frac{-\frac{1}{2}}{-1}$$

$$\boxed{= \frac{1}{2}}$$

$$a_5 = \frac{\frac{1}{2}}{-\frac{1}{2}}$$

$$\boxed{= -1}$$

$$a_6 = \frac{-1}{\frac{1}{2}}$$

$$\boxed{= -2}$$

$$a_7 = \frac{-2}{-1}$$

$$\boxed{= 2}$$

$$a_8 = \frac{2}{-2}$$

$$\boxed{= -1}$$

$$a_9 = \frac{-1}{2}$$

$$\boxed{= -\frac{1}{2}}$$

$$a_{10} = \frac{-\frac{1}{2}}{-1}$$

$$\boxed{= \frac{1}{2}}$$

Exercise

Find a formula for the nth term of the sequence $-1, \ 1, \ -1, \ 1, \ -1, \cdots$

Solution

$\boxed{a_n = (-1)^n} \quad n \in \mathbb{N}$

Exercise

Find a formula for the nth term of the sequence $1,\ -\frac{1}{4},\ \frac{1}{9},\ -\frac{1}{16},\ \frac{1}{25},\cdots$

Solution

$a_1 = 1 \quad r = -\frac{1}{4}$

$$\begin{aligned} a_n &= a_1 r \\ &= -\frac{1}{4} \\ &= \frac{(-1)^{n+1}}{n^2} \end{aligned}$$

$$\boxed{a_n = \frac{(-1)^{n+1}}{n^2}} \quad n \in \mathbb{N}$$

Exercise

Find a formula for the nth term of the sequence $\frac{1}{9},\ \frac{2}{12},\ \frac{2^2}{15},\ \frac{2^3}{18},\ \frac{2^4}{21},\cdots$

Solution

$$\boxed{a_n = \frac{2^{n-1}}{3(n+2)}} \quad n \in \mathbb{N}$$

Exercise

Find a formula for the nth term of the sequence $-3,\ -2,\ -1,\ 0,\ 1,\cdots$

Solution

$d = -2 - (-3) = 1$

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= -3 + (n-1)(1) \\ &= -3 + n - 1 \\ &= \boxed{n-4} \quad n \in \mathbb{N} \end{aligned}$$

Exercise

Find a formula for the nth term of the sequence $\frac{1}{25},\ \frac{8}{125},\ \frac{27}{625},\ \frac{64}{3125},\ \frac{125}{15,625},\cdots$

Solution

$$\frac{1}{5^2}, \frac{2^3}{5^3}, \frac{3^3}{5^4}, \frac{4^3}{5^5}, \frac{5^3}{5^6}, \cdots$$

$$\boxed{a_n = \frac{n^3}{5^{n+1}}} \quad n \in \mathbb{N}$$

Exercise

Find a formula for the nth term of the sequence $0, 1, 1, 2, 2, 3, 3, 4, \cdots$

Solution

$$\boxed{a_n = \frac{n-\frac{1}{2}+(-1)^n\left(\frac{1}{2}\right)}{2}} \quad n \in \mathbb{N}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{n+(-1)^n}{n}$$

Solution

$$\lim_{n\to\infty} \frac{n+(-1)^n}{n} = \lim_{n\to\infty}\left(1+\frac{(-1)^n}{n}\right)$$

$$\boxed{=1} \quad \Rightarrow \quad \textit{converges}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{1-2n}{1+2n}$$

Solution

$$\lim_{n\to\infty} \frac{1-2n}{1+2n} = \lim_{n\to\infty}\left(\frac{\frac{1}{n}-2}{\frac{1}{n}+2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{-2}{2}\right)$$

$\boxed{=-1}$ The limit ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{1-n^3}{70-4n^2}$$

Solution

$$\lim_{n\to\infty} \frac{1-n^3}{70-4n^2} = \lim_{n\to\infty} \frac{\frac{1}{n^2}-n}{\frac{70}{n^2}-4}$$

$$\lim_{n\to\infty} \frac{0-n}{0-4}$$

$$\underline{=\infty} \quad \Rightarrow \quad \textit{diverges}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \left(2-\frac{1}{2^n}\right)\left(3+\frac{1}{2^n}\right)$$

Solution

$$\lim_{n\to\infty} \left(2-\frac{1}{2^n}\right)\left(3+\frac{1}{2^n}\right) = (2)(3)$$

$$\underline{=6} \quad \Rightarrow \quad \textit{converges}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = n\pi\cos(n\pi)$$

Solution

$$\lim_{n\to\infty} n\pi\cos(n\pi) = \lim_{n\to\infty} n\pi(-1)^n$$

$$\underline{=\infty} \quad \Rightarrow \quad \textit{diverges}$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = n-\sqrt{n^2-n}$$

Solution

$$\lim_{n\to\infty} n-\sqrt{n^2-n} = \lim_{n\to\infty}\left(n-\sqrt{n^2-n}\right)\frac{n+\sqrt{n^2-n}}{n+\sqrt{n^2-n}}$$

$$= \lim_{n\to\infty} \frac{n^2-\left(n^2-n\right)}{n+\sqrt{n^2-n}}$$

$$= \lim_{n\to\infty} \frac{n}{n+\sqrt{n^2-n}}$$

$$= \lim_{n\to\infty} \frac{1}{1+\sqrt{1-\frac{1}{n}}}$$

$$\boxed{=\frac{1}{2}}$$

The given series *converges*.

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \sqrt{\frac{2n}{n+1}}$$

Solution

$$\lim_{n\to\infty} \sqrt{\frac{2n}{n+1}} = \sqrt{\lim_{n\to\infty} \frac{2}{1+\frac{1}{n}}}$$

$$\boxed{=\sqrt{2}}$$

The given series converges

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\sin^2 n}{2^n}$$

Solution

$0 \le \frac{\sin^2 n}{2^n} \le \frac{1}{2^n}$ By the Sandwich Theorem for sequences

$$\lim_{n\to\infty} \frac{\sin^2 n}{2^n} \boxed{=0}$$

The given series converges

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\ln n}{\ln 2n}$$

Solution

$$\lim_{n\to\infty} \frac{\ln n}{\ln 2n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{2}{2n}}$$

$$\underline{=1}$$

The given series converges

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{3^n \cdot 6^n}{2^{-n} \cdot n!}$$

Solution

$$\lim_{n\to\infty} \frac{3^n \cdot 6^n}{2^{-n} \cdot n!} = \lim_{n\to\infty} \frac{2^n \cdot 3^n \cdot 6^n}{n!}$$

$$= \lim_{n\to\infty} \frac{36^n}{n!}$$

$$\underline{=0}$$

The given series converges

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \sqrt{n}\ \sin\frac{1}{\sqrt{n}}$$

Solution

$$\lim_{n\to\infty} \frac{\sin\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\left(-\frac{1}{2n^{3/2}}\right)\cos\frac{1}{\sqrt{n}}}{-\frac{1}{2n^{3/2}}}$$

$$= \lim_{n\to\infty} \cos\frac{1}{\sqrt{n}}$$

$$= \cos 0$$

$$\underline{=1}$$

The given series converges

$$or \quad \lim_{n\to\infty} \frac{\sin\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{u\to 0} \frac{\sin u}{u} = 1$$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{n^2}{2^n - 1}$$

Solution

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{2^n - 1} \qquad \text{L'Hôpital Rule}$$

$$= \lim_{x\to\infty} \frac{2x}{(\ln 2)\cdot 2^x}$$

$$= \lim_{x\to\infty} \frac{2}{(\ln 2)^2 \cdot 2^x}$$

$$\underline{=0}$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$\{c_n\} = \left\{(-1)^n \frac{1}{n!}\right\}$$

Solution

$$n! = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdots n = 24\cdot \underbrace{5\cdot 6\cdots n}_{n-4}$$

$$2^2 = 2\cdot 2\cdot 2\cdot 2\cdot 2\cdots 2 = 16\cdot \underbrace{2\cdot 2\cdots 2}_{n-4}$$

$$\frac{-1}{2^n} \le (-1)^n \frac{1}{n!} \le \frac{1}{2^n} \qquad n \ge 4$$

By the Squeeze Theorem

$$\lim_{n\to\infty} (-1)^n \frac{1}{n!} \underline{=0}$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{5}{n+2}$$

Solution

$$\boxed{\lim_{n\to\infty} \frac{5}{n+2} = 0}$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = 8 + \frac{5}{n}$$

Solution

$$\boxed{\lim_{n\to\infty} \left(8 + \frac{5}{n}\right) = 8}$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = (-1)^n \left(\frac{n}{n+1}\right)$$

Solution

$$\lim_{n\to\infty} (-1)^n \left(\frac{n}{n+1}\right) \qquad \text{does not exist (oscillates between } -1 \text{ and } 1)$$

The sequence ***diverges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{1+(-1)^n}{n^2}$$

Solution

$$\boxed{\lim_{n\to\infty} \frac{1+(-1)^n}{n^2} = 0}$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{10n^2 + 3n + 7}{2n^2 - 6}$$

Solution

$$\lim_{n\to\infty} \frac{10n^2 + 3n + 7}{2n^2 - 6} = \lim_{n\to\infty} \frac{10n^2}{2n^2}$$

$$= 5$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\sqrt[3]{n}}{\sqrt[3]{n} + 1}$$

Solution

$$\lim_{n\to\infty} \frac{\sqrt[3]{n}}{\sqrt[3]{n} + 1} = 1$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\ln\left(n^3\right)}{2n}$$

Solution

$$\lim_{n\to\infty} \frac{\ln\left(n^3\right)}{2n} = \lim_{n\to\infty} \frac{3\ln(n)}{2n}$$

$$= \lim_{n\to\infty} \frac{3}{2}\frac{\frac{1}{n}}{1}$$

$$= 0$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{5^n}{3^n}$$

Solution

$$\lim_{n\to\infty} \frac{5^n}{3^n} = \lim_{n\to\infty} \left(\frac{5}{3}\right)^n$$

$$= \infty$$

The sequence ***diverges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{(n+1)!}{n!}$$

Solution

$$\lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1)$$

$$= \infty$$

The sequence ***diverges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{(n-2)!}{n!}$$

Solution

$$\lim_{n\to\infty} \frac{(n-2)!}{n!} = \lim_{n\to\infty} \frac{1}{n(n-1)}$$

$$= 0$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{n^p}{e^n}, \quad p > 0$$

Solution

$$\lim_{n\to\infty} \frac{n^p}{e^n} \underline{=0}$$

The sequence ***converges*** $(p>0, \quad n\ge 2)$

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = n\sin\frac{1}{n}$$

Solution

$$\lim_{n\to\infty} n\sin\frac{1}{n} = \lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}} \qquad \text{Let } x=\frac{1}{n}\xrightarrow[n\to\infty]{}0$$

$$= \lim_{x\to 0}\frac{\sin x}{x} \qquad \text{Since } \lim_{x\to 0}\frac{\sin x}{x}=1 \quad \lim_{x\to 0}\frac{\cos x}{1}=1$$

$$\underline{=1}$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = 2^{1/n}$$

Solution

$$\lim_{n\to\infty} 2^{1/n} = 2^0$$

$$\underline{=1}$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = -3^{-n}$$

Solution

$$\lim_{n\to\infty} -3^{-n} = \lim_{n\to\infty}\left(-\frac{1}{3^n}\right)$$

$$\underline{=0}$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\sin n}{n}$$

Solution

$$\lim_{n\to\infty} \frac{\sin n}{n} = \lim_{n\to\infty} \frac{1}{n}(\sin n)$$

$$\underline{= 0} \qquad \text{since } \frac{1}{n} \to 0$$

The sequence ***converges***

Exercise

Determine if the sequence converge or diverge? Then find the limit of the convergent sequence.

$$a_n = \frac{\cos \pi n}{n^2}$$

Solution

$$\lim_{n\to\infty} \frac{\cos \pi n}{n^2} = \lim_{n\to\infty} \frac{1}{n^2}(\cos \pi n)$$

$$\underline{= 0}$$

The sequence ***converges***

SOLUTION Section 3.2 – Infinite Series

Exercise

Find the limit of the following sequences or determine the limit does not exist $a_n = \dfrac{n^3}{n^4+1}$

Solution

$$\lim_{n\to\infty}\frac{n^3}{n^4+1} = \lim_{n\to\infty}\frac{n^3}{n^4}$$

$$= \lim_{n\to\infty}\frac{1}{n}$$

$$\underline{=0}$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $a_n = n^{1/n}$

Solution

$$\lim_{n\to\infty} n^{1/n} = \infty^0$$

$$y = n^{1/n}$$

$$\ln y = \frac{1}{n}\ln n$$

$$\lim_{n\to\infty}\frac{\ln n}{n} = \lim_{x\to\infty}\frac{\ln x}{x}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{x}}{1} \qquad \textit{Using L'Hôpital's rule}$$

$$\underline{=0}$$

$$\ln y = 0 \quad \Rightarrow \quad y = e^0 = 1$$

$$\lim_{n\to\infty} n^{1/n} \underline{=1}$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $\left\{\dfrac{n^{12}}{3n^{12}+4}\right\}$

Solution

$$\lim_{n\to\infty} \frac{n^{12}}{3n^{12}+4} = \lim_{n\to\infty} \frac{n^{12}}{3n^{12}}$$

$$= \frac{1}{3}$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $\left\{\frac{2e^{n+1}}{e^n}\right\}$

Solution

$$\lim_{x\to\infty} \frac{2e^{n+1}}{e^n} = \lim_{x\to\infty} 2e$$

$$= 2e$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $\left\{\frac{\tan^{-1} n}{n}\right\}$

Solution

$$\lim_{n\to\infty} \frac{\tan^{-1} n}{n} = \frac{\frac{\pi}{2}}{\infty}$$

$$= 0$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $\left\{\left(1+\frac{2}{n}\right)^n\right\}$

Solution

$$y = \left(1+\frac{2}{n}\right)^n$$

$$\ln y = n\ln\left(1+\frac{2}{n}\right)$$

$$= \frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}}$$

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{-\frac{2}{n^2}}{1+\frac{2}{n}}}{-\frac{1}{n^2}}$$

Using L'Hôpital's rule

$$= \lim_{n\to\infty} \frac{2}{1+\frac{2}{n}}$$

$$= 2$$

$$\ln y = 2 \quad \to \quad y = e^2$$

$$\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^n = e^2$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $\left\{\left(\frac{n}{n+5}\right)^n\right\}$

Solution

$$y = \left(\frac{n}{n+5}\right)^n$$

$$\ln y = n\ln\left(\frac{n}{n+5}\right)$$

$$\lim_{n\to\infty} n\ln\left(\frac{n}{n+5}\right) = \lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+5}\right)}{\frac{1}{n}}$$

Using L'Hôpital's rule

$$= \lim_{n\to\infty} \frac{\frac{n+5}{n}\,\frac{n+5-n}{(n+5)^2}}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{\frac{5}{n^2+5}}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{-5n^2}{n^2+5}$$

$$= -5$$

$$\ln y = -5 \quad \to \quad y = e^{-5}$$

$$\lim_{n\to\infty} \left(\frac{n}{n+5}\right)^n = e^{-5}$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $\left\{\dfrac{\ln\left(\frac{1}{n}\right)}{n}\right\}$

Solution

$$\lim_{n\to\infty}\frac{1}{n}\ln\left(\frac{1}{n}\right)=\lim_{n\to\infty}\frac{-\ln(n)}{n}$$

Using L'Hôpital's rule

$$=\lim_{n\to\infty}\frac{-\frac{1}{n}}{1}$$

$$=0$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $\{\ln\sin(1/n)+\ln n\}$

Solution

$$\lim_{n\to\infty}\left(\ln\sin(1/n)+\ln n\right)=\lim_{n\to\infty}\left(\ln\frac{\sin(1/n)}{n}\right)$$

$$\lim_{x\to\infty}\frac{\sin(1/x)}{x}=1$$

$$=\ln 1$$

$$=0$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $a_n=\dfrac{n!}{n^n}$

Solution

$$\frac{a_{n+1}}{a_n}=\frac{(n+1)!}{(n+1)^{n+1}}\frac{n^n}{n!}$$

$$=\frac{n^n(n+1)}{(n+1)(n+1)^n}$$

$$=\frac{n^n}{(n+1)^n}$$

$$=\left(\frac{n}{n+1}\right)^n$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$$

$$=\boxed{\frac{1}{e}}$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $\qquad a_n = \dfrac{n^2+4}{\sqrt{4n^4+1}}$

Solution

$$\lim_{n\to\infty} \frac{n^2+4}{\sqrt{4n^4+1}} = \lim_{n\to\infty} \frac{n^2}{2n^2}$$

$$=\boxed{\frac{1}{2}}$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $\qquad a_n = \dfrac{8^n}{n!}$

Solution

$$\frac{a_{n+1}}{a_n} = \frac{8^{n+1}}{(n+1)!}\cdot\frac{n!}{8^n}$$

$$=\frac{8}{n+1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{8}{n+1}$$

$$=\boxed{0}$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $\qquad a_n = \left(1+\dfrac{3}{n}\right)^{2n}$

Solution

$$\ln a_n = \ln\left(1+\frac{3}{n}\right)^{2n}$$

$$=2n\ \ln\left(1+\frac{3}{n}\right)$$

$$=\frac{\ln\left(1+\frac{3}{n}\right)}{\frac{1}{2n}}$$

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{3}{n}\right)}{\frac{1}{2n}} = \frac{0}{0}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+3}\left(-\frac{3}{n^2}\right)\right)\left(-2n^2\right)$$

$$= 6 \lim_{n\to\infty} \left(\frac{n}{n+3}\right)$$

$$\underline{= 6}$$

$$\lim_{n\to\infty} \ln a_n = 6$$

$$\lim_{n\to\infty} a_n = e^6$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $\quad a_n = \sqrt[n]{n}$

Solution

$$a_n = n^{1/n}$$

$$\ln a_n = \ln n^{1/n}$$

$$= \frac{\ln n}{n}$$

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{\ln n}{n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1}$$

$$\underline{= 0}$$

$$\lim_{n\to\infty} \ln a_n = 0$$

$$\lim_{n\to\infty} a_n = e^0$$

$$\underline{= 1}$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $\quad a_n = n - \sqrt{n^2 - 1}$

Solution

$$\lim_{n\to\infty}\left(n-\sqrt{n^2-1}\right)=\lim_{n\to\infty}\left(n-\sqrt{n^2-1}\right)\cdot\frac{n+\sqrt{n^2-1}}{n+\sqrt{n^2-1}}$$

$$=\lim_{n\to\infty}\frac{n^2-n^2+1}{n+\sqrt{n^2-1}}$$

$$=\lim_{n\to\infty}\frac{1}{n+\sqrt{n^2-1}}$$

$$=0$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $a_n=\left(\frac{1}{n}\right)^{1/\ln n}$

Solution

$$\ln a_n=\ln\left(\frac{1}{n}\right)^{1/\ln n}$$

$$=\frac{\ln\frac{1}{n}}{\ln n}$$

$$=-\frac{\ln n}{\ln n}$$

$$=-1$$

$$\lim_{n\to\infty}a_n=e^{-1}$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $a_n=\sin\frac{\pi n}{6}$

Solution

The series $a_n=\sin\frac{\pi n}{6}$ oscillates

Therefore; there is ***no limit***.

Exercise

Find the limit of the following sequences or determine the limit does not exist $a_n=\frac{(-1)^n}{0.9^n}$

Solution

$$a_n = \frac{(-1)^n}{0.9^n}$$

$$= \left(-\frac{1}{0.9}\right)^n$$

$$= \left(-\frac{10}{9}\right)^n$$

Using Gemeometric series Test $|r| = \frac{10}{9} > 1$

Therefore; the given sequence ***diverges***.

Exercise

Find the limit of the following sequences or determine the limit does not exist $\quad a_n = \tan^{-1} n$

Solution

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \tan^{-1} n$$

$$= \tan^{-1} \infty$$

$$\boxed{= \frac{\pi}{2}}$$

Exercise

Find a formula for the *n*th term partial sum of the series and use it to find the series' sum if the series converges $\quad 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{n-1}} + \cdots$

Solution

$$a = 2, \quad r = \frac{1}{3}$$

$$s_n = a\frac{1-r^n}{1-r} = 2\frac{1-\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} 2\frac{1}{1-\frac{1}{3}}$$

$$\boxed{= 3}$$

Exercise

Find a formula for the nth term partial sum of the series and use it to find the series' sum if the series converges $\quad \frac{9}{100}+\frac{9}{100^2}+\frac{9}{100^3}+\cdots+\frac{9}{100^n}+\cdots$

Solution

$$a=\frac{9}{100}, \quad r=\frac{1}{100}$$

$$s_n = a\frac{1-r^n}{1-r}=\frac{9}{100}\frac{1-\left(\frac{1}{100}\right)^n}{1-\frac{1}{100}}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\frac{9}{100}\frac{1-\left(\frac{1}{100}\right)^n}{1-\frac{1}{100}} \qquad \lim_{n\to\infty}\left(\frac{1}{100}\right)^n=0$$

$$=\frac{9}{100}\frac{1}{1-\frac{1}{100}}$$

$$=\frac{9}{100}\frac{1}{\frac{99}{100}}$$

$$=\frac{9}{99}$$

$$=\frac{1}{11}$$

Exercise

Find a formula for the nth term partial sum of the series and use it to find the series' sum if the series converges $\quad 1-2+4-8+\cdots+(-1)^{n-1}2^{n-1}+\cdots$

Solution

$$r=-2 \quad \to \quad |r|>1$$

The series ***diverges***

Exercise

Find a formula for the nth term partial sum of the series and use it to find the series' sum if the series converges $\quad \frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{(n+1)(n+2)}+\cdots$

Solution

$$\frac{1}{(n+1)(n+2)}=\frac{1}{n+1}-\frac{1}{n+2}$$

$$s_n=\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$$

$$= \frac{1}{2} - \frac{1}{n+2} \qquad\qquad \lim_{n\to\infty} \frac{1}{n+2} = 0$$

$$\lim_{n\to\infty} s_n = \frac{1}{2}$$

Exercise

Find a formula for the nth term partial sum of the series and use it to find the series' sum if the series converges $\quad \frac{5}{1\cdot 2} + \frac{5}{2\cdot 3} + \frac{5}{3\cdot 4} + \cdots + \frac{5}{n(n+1)} + \cdots$

Solution

$$\frac{5}{n(n+1)} = \frac{5}{n} - \frac{5}{n+1}$$

$$s_n = 5\left[\left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$$

$$= 5\left(1 - \frac{1}{n+1}\right) \qquad\qquad \lim_{n\to\infty} \frac{1}{n+1} = 0$$

$$\lim_{n\to\infty} s_n = 5$$

Exercise

Write out the first few terms of each series to show how the series starts. Then find the sum of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}$$

Solution

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} = 1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \cdots$$

The sum of geometric series: $\frac{1}{1-\left(-\frac{1}{4}\right)} = \frac{4}{5}$

Exercise

Write out the first few terms of each series to show how the series starts. Then find the sum of the series

$$\sum_{n=0}^{\infty} \frac{1}{4^n}$$

Solution

$$\sum_{n=0}^{\infty} \frac{1}{4^n} = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots$$

The sum of geometric series: $\frac{1}{1-\frac{1}{4}} = \frac{4}{3}$

Exercise

Write out the first few terms of each series to show how the series starts. Then find the sum of the series

$$\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n} \right)$$

Solution

$$\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n} \right) = (5-1) + \left(\frac{5}{2} - \frac{1}{3} \right) + \left(\frac{5}{4} - \frac{1}{9} \right) + \left(\frac{5}{8} - \frac{1}{27} \right) + \cdots$$

The sum of geometric series: $\frac{5}{1-\left(\frac{1}{2}\right)} - \frac{1}{1-\left(\frac{1}{3}\right)} = 10 - \frac{3}{2} \quad = \frac{17}{2}$

Exercise

Write out the first few terms of each series to show how the series starts. Then find the sum of the series

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n}$$

Solution

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n} = 2 + \frac{4}{5} + \frac{8}{25} + \frac{16}{125} + \cdots$$

$$= 2\left(1 + \frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \cdots\right)$$

The sum of geometric series: $2\frac{1}{1-\left(\frac{2}{5}\right)} = 2 \cdot \frac{5}{3}$

$$= \frac{10}{3}$$

Exercise

Determine if the geometric series converges or diverges. If a series converges, find its sum

$$1+\left(\frac{2}{5}\right)+\left(\frac{2}{5}\right)^2+\left(\frac{2}{5}\right)^3+\left(\frac{2}{5}\right)^4+\cdots$$

Solution

$$r=\frac{2}{5}<1$$

$\Rightarrow$ The series is geometric ***converges*** to $\dfrac{1}{1-\left(\frac{2}{5}\right)}$ $=\dfrac{5}{3}$

Exercise

Determine if the geometric series converges or diverges. If a series converges, find its sum

$$1+(-3)+(-3)^2+(-3)^3+(-3)^4+\cdots$$

Solution

$$r=-3 \quad\Rightarrow\quad |r|=|-3|>1$$

$\Rightarrow$ The series is geometric and ***diverges***.

Exercise

Determine if the geometric series converges or diverges. If a series converges, find its sum

$$\left(-\frac{2}{3}\right)^2+\left(-\frac{2}{3}\right)^3+\left(-\frac{2}{3}\right)^4+\left(-\frac{2}{3}\right)^5+\cdots$$

Solution

The series is geometric with $r=-\frac{2}{3} \;\Rightarrow\; \left|-\frac{2}{3}\right|<1$

Converges to $\dfrac{-\frac{2}{3}}{1-\left(-\frac{2}{3}\right)}$ $=-\dfrac{2}{5}$

Exercise

Express each of the numbers as the ratio of two integers (fraction) $\quad 0.\overline{23}=0.23\ 23\ 23\cdots$

Solution

$$0.\overline{23}=0.23+.0023+.000023+\cdots$$

$$=\frac{23}{100}+\frac{23}{10^4}+\frac{23}{10^6}+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{23}{100}\left(\frac{1}{10^2}\right)^n$$

$$=\frac{\frac{23}{100}}{1-\frac{1}{100}}$$

$$=\frac{23}{99}$$

Exercise

Express each of the numbers as the ratio of two integers (fraction) $0.\overline{234}=0.234\ 234\ 234\cdots$

Solution

$$0.\overline{234}=\frac{234}{10^3}+\frac{234}{10^6}+\frac{234}{10^9}+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{234}{1000}\left(\frac{1}{10^3}\right)^n$$

$$=\frac{\frac{234}{1000}}{1-\frac{1}{1000}}$$

$$=\frac{234}{999}$$

Exercise

Express each of the numbers as the ratio of two integers (fraction) $1.\overline{414}=1.414\ 414\ 414\cdots$

Solution

$$1.\overline{414}=1+\frac{414}{10^3}+\frac{414}{10^6}+\frac{414}{10^9}+\cdots$$

$$=1+\sum_{n=0}^{\infty}\frac{414}{1000}\left(\frac{1}{10^3}\right)^n$$

$$=1+\frac{\frac{414}{1000}}{1-\frac{1}{1000}}$$

$$=1+\frac{414}{999}$$

$$=\frac{1413}{999}$$

Exercise

Express each of the numbers as the ratio of two integers (fraction) $1.24\overline{123} = 1.24\ 123\ 123\ 123\cdots$

Solution

$$1.24\overline{123} = 1.24 + \frac{123}{10^3} + \frac{123}{10^6} + \frac{123}{10^9} + \cdots$$

$$= 1.24 + \sum_{n=0}^{\infty} \frac{123}{10^5}\left(\frac{1}{10^3}\right)^n$$

$$= 1.24 + \frac{\frac{123}{10^5}}{1 - \frac{1}{1000}}$$

$$= 1.24 + \frac{\frac{123}{10^5}}{\frac{10^3 - 1}{10^3}}$$

$$= 1.24 + \frac{123}{10^5}\frac{10^3}{999}$$

$$= \frac{124}{100} + \frac{123}{99{,}900}$$

$$= \frac{123{,}999}{99{,}900}$$

$$\boxed{= \frac{41{,}333}{33{,}300}}$$

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n$

Solution

Geometric series ***converges***: $\frac{1}{1 - \frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2} - 1} \cdot \frac{\sqrt{2} + 1}{\sqrt{2} + 1}$

$$= \frac{2 + \sqrt{2}}{2 - 1}$$

$$\boxed{= 2 + \sqrt{2}}$$

Exercise

Use Divergence to determine if the series converges or diverges. $\sum_{k=0}^{\infty} \frac{k}{2k+1}$

Solution

$$a_k = \frac{k}{2k+1}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k}{2k+1}$$

$$= \frac{1}{2} \neq 0$$

By the divergence Test, the given series ***diverges***.

Exercise

Use Divergence to determine if the series converges or diverges. $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$

Solution

$$a_k = \frac{k}{k^2+1}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k}{k^2+1}$$

$$= 0$$

By the divergence Test, the given series ***inconclusive***.

Exercise

Use Divergence to determine if the series converges or diverges. $\sum_{k=2}^{\infty} \frac{k}{\ln k}$

Solution

$$a_k = \frac{k}{\ln k}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k}{\ln k} = \frac{\infty}{\infty}$$

$$= \lim_{k\to\infty} \frac{1}{\frac{1}{k}}$$

$$= \lim_{k\to\infty} k$$

$$= \infty$$

By the divergence Test, the given series ***diverges***.

Exercise

Use Divergence to determine if the series converges or diverges. $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{2^k}$

Solution

$$a_k = \frac{k^2}{2^k}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k^2}{2^k} = \frac{\infty}{\infty}$$

$$= \lim_{k\to\infty} \frac{2k}{2^k \ln 2} = \frac{\infty}{\infty}$$

$$= \lim_{k\to\infty} \frac{2}{2^k (\ln 2)^2}$$

$$= 0$$

By the divergence Test, the given series ***inconclusive***.

Exercise

Use Divergence to determine if the series converges or diverges. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k+100}$

Solution

$$a_k = \frac{1}{k+100}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k+100}$$

$$= 0$$

By the divergence Test, the given series ***inconclusive***.

Exercise

Use Divergence to determine if the series converges or diverges. $\displaystyle\sum_{k=0}^{\infty} \frac{k^3}{k^3+1}$

Solution

$$a_k = \frac{k^3}{k^3+1}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k^3}{k^3+1}$$

$$=1 \neq 0$$

By the divergence Test, the given series ***diverges***.

Exercise

Use Divergence to determine if the series converges or diverges. $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k^2+1}}{k}$

Solution

$$a_k = \frac{\sqrt{k^2+1}}{k}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{\sqrt{k^2+1}}{k}$$

$$=1 \neq 0$$

By the divergence Test, the given series ***diverges***.

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} n$

Solution

$$\lim_{n\to\infty} (-1)^{n+1} n \neq 0$$

The given series diverges

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n}$

Solution

$$\cos(n\pi) = (-1)^n$$

$$\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n}$$

$$=\sum_{n=0}^{\infty}\left(-\frac{1}{5}\right)^n$$

It is geometric series ***converges*** with sum $\dfrac{1}{1-\left(-\frac{1}{5}\right)}=\dfrac{5}{6}$

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=0}^{\infty} e^{-2n}$

Solution

$$e^{-2n}=\left(\frac{1}{e^2}\right)^n$$

$\Rightarrow$ It is geometric series ***converges*** with sum $\dfrac{1}{1-\left(\frac{1}{e^2}\right)}=\dfrac{e^2}{e^2-1}$

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=1}^{\infty} \ln\frac{1}{3^n}$

Solution

$$\lim_{n\to\infty}\ln\frac{1}{3^n}=-\infty\ \neq 0$$

The given series diverges

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Solution

$$\lim_{n\to\infty}\frac{n^n}{n!}=\lim_{n\to\infty}\frac{n\cdot n\cdots n}{1\cdot 2\cdot 3\cdots n}$$

$$> \lim_{n\to\infty} n$$

$$\underline{=\infty}$$

The given series diverges

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=1}^{\infty}\frac{2^n+3^n}{4^n}$

Solution

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{2^n+3^n}{4^n} &= \sum_{n=1}^{\infty}\frac{2^n}{4^n}+\sum_{n=1}^{\infty}\frac{3^n}{4^n}\\ &= \sum_{n=1}^{\infty}\left(\frac{2}{4}\right)^n+\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n\\ &= \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n+\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n\end{aligned}$$

Both are geometric series.

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n=\frac{\frac{1}{2}}{1-\frac{1}{2}}=1$$

$$\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n=\frac{\frac{3}{4}}{1-\frac{3}{4}}=3$$

$$\sum_{n=1}^{\infty}\frac{2^n+3^n}{4^n}=3+1\ \underline{=4}$$

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\sum_{n=0}^{\infty} \frac{e^{\pi n}}{\pi^{ne}}$

Solution

$$\sum_{n=0}^{\infty} \frac{e^{\pi n}}{\pi^{ne}} = \sum_{n=0}^{\infty} \left(\frac{e^{\pi}}{\pi^{e}}\right)^{n} \qquad r = \frac{e^{\pi}}{\pi^{e}} \approx 1.03 > 1$$

The geometric series ***diverges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\sum_{n=0}^{\infty} (1.075)^{n}$

Solution

This geometric series: $r = 1.075 > 1$

The geometric series ***diverges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\sum_{n=0}^{\infty} \frac{3^{n}}{1000}$

Solution

This geometric series: $r = 3 > 1$

The geometric series ***diverges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\sum_{n=1}^{\infty} \frac{n+10}{10n+1}$

Solution

$$\lim_{n\to\infty} \frac{n+10}{10n+1} = \boxed{\frac{1}{10}} \neq 0$$

The series ***diverges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=1}^{\infty} \frac{4n+1}{3n-1}$

Solution

$$\lim_{n\to\infty} \frac{4n+1}{3n-1} = \boxed{\frac{4}{3}} \neq 0$$

The series ***diverges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$

Solution

$$\begin{aligned} S_n &= \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+2}\right) \\ &= 1+\frac{1}{2}+\frac{1}{n+1}-\frac{1}{n+2} \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} S_n &= \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{n+1}-\frac{1}{n+2}\right) \\ &= \boxed{\frac{3}{2}} \end{aligned}$$

The series ***converges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

Solution

$$S_n = \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$$

$$= \frac{1}{2}-\frac{1}{n+2}$$

$$\lim_{n\to\infty} S_n = \frac{1}{2}$$

The series ***converges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

Solution

$$\lim_{n\to\infty} \frac{3^n}{n^3} = \lim_{n\to\infty} \frac{3^n \ln 2}{3n^2}$$

$$= \lim_{n\to\infty} \frac{3^n (\ln 2)^2}{6n}$$

$$= \lim_{n\to\infty} \frac{3^n (\ln 2)^3}{6}$$

$$= \infty$$

The series ***diverges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\sum_{n=0}^{\infty} \frac{3}{5^n}$

Solution

$$\sum_{n=0}^{\infty} \frac{3}{5^n} = \sum_{n=0}^{\infty} 3\left(\frac{1}{5}\right)^n$$

This geometric series: $r = \frac{1}{5} < 1$

The geometric series ***converges***

$$S = \frac{3}{1-\frac{1}{5}} = \frac{15}{4}$$

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\sum_{n=2}^{\infty} \frac{n}{\ln n}$

Solution

Since $n > \ln n \Rightarrow \frac{n}{\ln n}$ do not approach 0 as $n \to \infty$

The series ***diverges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\sum_{n=1}^{\infty} \ln \frac{1}{n}$

Solution

$$\sum_{n=1}^{\infty} \ln \frac{1}{n} = -\sum_{n=1}^{\infty} \ln n$$

$$= -(0 + \ln 2 + \ln 3 + \cdots)$$

$$= -\infty$$

The series ***diverges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\sum_{n=1}^{\infty} \left(1+\frac{k}{n}\right)^n$

Solution

$$\sum_{n=1}^{\infty} \left(1+\frac{k}{n}\right)^n = \sum_{n=1}^{\infty} \left(\left(1+\frac{k}{n}\right)^{n/2}\right)^2$$

$$\lim_{n\to\infty} \left(1+\frac{k}{n}\right)^{n/2} = e^k \neq 0$$

The series ***diverges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\sum_{n=1}^{\infty} e^{-n}$

Solution

$$\sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$$

This geometric series: $r = \frac{1}{e} < 1$

The geometric series ***converges***

$$S = \frac{1}{1-\frac{1}{e}}$$

$$= \frac{e}{e-1}$$

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\sum_{n=1}^{\infty} \arctan n$

Solution

$$\lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$$

The series ***diverges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

Solution

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} \left(\ln(n+1) - \ln n\right)$$

$$S_n = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + (\ln(n+1) - \ln)$$
$$= \ln(n+1) - \ln 1$$
$$= \ln(n+1)$$

$$\lim_{n\to\infty} S_n = \infty$$

The series ***diverges***

Exercise

Determine if the series converges or diverges and describe whether they do so monotonically or by oscillation. Give the limit when the sequence converges. $a_n = 0.3^n$

Solution

Since $0.3 < 1$, this sequence ***converges*** to 0, and since $0.3 > 0$, this converge is monotone.

Exercise

Determine if the series converges or diverges and describe whether they do so monotonically or by oscillation. Give the limit when the sequence converges. $a_n = 1.3^n$

Solution

Since $1.2 > 1$, this sequence *diverges* monotonically to ∞.

Exercise

Determine if the series converges or diverges and describe whether they do so monotonically or by oscillation. Give the limit when the sequence converges. $a_n = (-0.6)^n$

Solution

Since $|-0.6| < 1$, this sequence ***converges*** to 0, and since $-0.6 < 0$, this converge is not monotone.

Exercise

Determine if the series converges or diverges and describe whether they do so monotonically or by oscillation. Give the limit when the sequence converges. $a_n = (-1.01)^n$

Solution

Since $|-1.1| > 1$, this sequence ***diverges***, and since $-1.1 < 0$, this diverge is not monotone.

Exercise

Determine if the series converges or diverges and describe whether they do so monotonically or by oscillation. Give the limit when the sequence converges. $a_n = 2^n 3^{-n}$

Solution

$$a_n = \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$$

Since $0 < \frac{2}{3} < 1$, the sequence ***converges*** monotonically to zero.

Exercise

Determine if the series converges or diverges and describe whether they do so monotonically or by oscillation. Give the limit when the sequence converges. $a_n = (-0.003)^n$

Solution

Since $|-0.003| < 1$

This sequence ***converges*** to 0, and since $-0.003 < 0$, this converge is not monotone.

Exercise

Find the limit of the following sequences or state that they diverge $a_n = \frac{\sin n}{2^n}$

Solution

Since $-1 \le \sin n \le 1$ for all n, this sequence satisfies $-\frac{1}{2^n} \le \frac{\sin n}{2^n} \le \frac{1}{2^n}$ and

since both $\pm\frac{1}{2^n} \xrightarrow[n\to\infty]{} 0$ this ***converge*** to 0.

Exercise

Find the limit of the following sequences or state that they diverge $a_n = \frac{\cos\left(\frac{n\pi}{2}\right)}{\sqrt{n}}$

Solution

Since $-1 \le \cos\left(\frac{n\pi}{2}\right) \le 1$ for all n, this sequence satisfies $-\frac{1}{\sqrt{n}} \le \frac{\cos\left(\frac{n\pi}{2}\right)}{\sqrt{n}} \le \frac{1}{\sqrt{n}}$ and since both $\pm\frac{1}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0$ this ***converge*** to 0.

Exercise

Find the limit of the following sequences or state that they diverge $a_n = \frac{2\tan^{-1} n}{n^3+4}$

Solution

$$-\frac{\pi}{2} < \tan^{-1} n < \frac{\pi}{2}$$

$$-\frac{\pi}{n^3+4} < \frac{2\tan^{-1} n}{n^3+4} < \frac{\pi}{n^3+4}$$

By the Squeeze Theorem, the given sequence ***converges*** to zero.

Exercise

Find the limit of the following sequences or state that they diverge $a_n = \frac{n\sin^3 n}{n+1}$

Solution

Let $b_n = \frac{n}{n+1} \xrightarrow[n\to\infty]{} 0$

Then $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{n\sin^3 n}{n+1}\frac{n+1}{n}$

$= \lim\limits_{n\to\infty} \sin^3 n$ doesn't converge.

This sequence ***diverges***.

Exercise

Find the limit of the following sequences or state that they diverge $\sum\limits_{k=1}^{\infty}\left(\frac{9}{10}\right)^k$

Solution

This is a Geometric series with $|r| = \frac{9}{10} < 1$

$k_0 \Rightarrow a_0 = \frac{9}{10}$

$$S = \frac{\frac{9}{10}}{1-\frac{9}{10}}$$

$$= \frac{9}{10}\frac{10}{1}$$

$$\underline{= 9}$$

Exercise

Find the limit of the following sequences or state that they diverge $\displaystyle\sum_{k=1}^{\infty} 3(1.001)^k$

Solution

This is a Geometric series with $|r| = 1.001 \ > 1$

Therefore; by the Geometric series, the given series ***diverges***.

Exercise

Find the limit of the following sequences or state that they diverge $\displaystyle\sum_{k=0}^{\infty} \left(-\frac{1}{5}\right)^k$

Solution

This is a Geometric series with $|r| = \frac{1}{5} \ < 1$

$$S = \frac{1}{1+\frac{1}{5}}$$

$$\underline{= \frac{5}{6}}$$

Exercise

Find the limit of the following sequences or state that they diverge $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

Solution

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

This series is Telescopic series

$$S_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)$$

$$\underline{= 1}$$

Exercise

Find the limit of the following sequences or state that they diverge $\displaystyle\sum_{k=2}^{\infty}\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k-1}}\right)$

Solution

This series is Telescopic series

$$S_n = \frac{1}{\sqrt{2}} - 1 + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n-1}}$$

$$= \frac{1}{\sqrt{n}} - 1$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}-1\right)$$

$$\underline{= -1}$$

Exercise

Find the limit of the following sequences or state that they diverge $\displaystyle\sum_{k=1}^{\infty}\left(\frac{3}{3k-2}-\frac{3}{3k+1}\right)$

Solution

This series is Telescopic series

$$S_n = 3\left(1-\frac{1}{4}+\frac{1}{4}-\frac{1}{7}+\frac{1}{7}-\frac{1}{10}+\cdots+\frac{1}{3n-2}-\frac{1}{3n+1}\right)$$

$$= 3\left(1-\frac{1}{3n+1}\right)$$

$$\lim_{n\to\infty} S_n = 3\lim_{n\to\infty}\left(1-\frac{1}{3n+1}\right)$$

$$\underline{= 3}$$

Exercise

Find the limit of the following sequences or state that they diverge $\displaystyle\sum_{k=1}^{\infty} 4^{-3k}$

Solution

$$\sum_{k=1}^{\infty} 4^{-3k} = \sum_{k=1}^{\infty}\left(4^{-3}\right)^k$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{4^3}\right)^k$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{64}\right)^k$$

This is a Geometric series with $|r| = \frac{1}{64}$ < 1

$k_0 \Rightarrow a_0 = \frac{1}{64}$

$$S = \frac{\frac{1}{64}}{1 - \frac{1}{64}}$$

$$= \frac{1}{64}\frac{64}{63}$$

$$= \frac{1}{63}$$

Exercise

Find the limit of the following sequences or state that they diverge $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{3^{k+2}}$

Solution

$$\sum_{k=1}^{\infty} \frac{2^k}{3^{k+2}} = \sum_{k=1}^{\infty} \frac{2^k}{3^k 3^2}$$

$$= \frac{1}{9}\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k$$

This is a Geometric series with $|r| = \frac{2}{3}$ < 1

$k_0 \Rightarrow a_0 = \frac{1}{64}$

$$S = \frac{1}{9}\frac{\frac{2}{3}}{1 - \frac{2}{3}}$$

$$= \frac{2}{9}$$

Exercise

Find the limit of the following sequences or state that they diverge $\displaystyle\sum_{k=0}^{\infty}\left(\left(\frac{1}{3}\right)^{k}-\left(\frac{2}{3}\right)^{k+1}\right)$

Solution

$$\sum_{k=0}^{\infty}\left(\left(\frac{1}{3}\right)^{k}-\left(\frac{2}{3}\right)^{k+1}\right)=\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^{k}-\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^{k+1}$$

These are Geometric series with $\left|r_1\right|=\frac{1}{3}<1$ and $\left|r_2\right|=\frac{2}{3}<1$

$$S=\frac{\frac{1}{3}}{1-\frac{1}{3}}-\frac{\frac{2}{3}}{1-\frac{2}{3}}$$

$$=\frac{3}{2}-2$$

$$=-\frac{1}{2}$$

Exercise

Consider the series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k(k+2)}=\frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+2}\right)$

a) Write the first four terms of the sequence of partial sums S_1, S_2, S_3, S_4.

b) Write the nth term of the sequence of partial sums S_n.

c) Find $\lim\limits_{n\to\infty} S_n$ and evaluate the series.

Solution

a) $S_1=\frac{1}{1(3)}=\frac{1}{3}$

$S_2=\frac{1}{3}+\frac{1}{8}=\frac{11}{24}$

$S_3=\frac{11}{24}+\frac{1}{15}=\frac{21}{40}$

$S_4=\frac{21}{40}+\frac{1}{24}$

$=\frac{68}{120}$

$=\frac{17}{30}$

b) $S_n = \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{n} - \frac{1}{n+2}\right)$

$$= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$$

∴ Telescopes.

c) $\lim\limits_{n\to\infty} S_n = \frac{1}{2}\lim\limits_{n\to\infty}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$

$$= \frac{1}{2}\left(1 + \frac{1}{2}\right)$$

$$= \frac{1}{2}\left(\frac{3}{2}\right)$$

$$= \frac{3}{4}$$

∴ Which is the sum of the series.

Exercise

Many people take aspirin on a regular basis as a preventive measure for heart disease. Suppose a person take 80 *mg* of aspirin every 24 *hr*. Assume also that aspirin has a half-life of 24 *hr*; that is, every 24 *hr* half of the drug in the blood is eliminated.

a) Find a recurrence relation for the sequence $\{d_n\}$ that gives the amount of drug in the blood after the n^{th} dose, where $d_1 = 80$

b) Find the limit of $\{d_n\}$

Solution

a) After the n^{th} dose is given, the amount of drug in the bloodstream is $d_n = 0.5d_{n-1} + 80$, since the half-life is one day.
The initial value is $d_1 = 80$

b) $Sum = \frac{a}{1-r} = \frac{80}{1-\frac{1}{2}} = 160$

The limit of this sequence is 160 *mg*.

Exercise

Suppose a tank is filled with 100 L of a 40% alcohol solution (by volume). You repeatedly perform the following operation: Remove 2 L of the solution from the tank and replace them with 2 L of 10% alcohol solution

a) Let C_n be the concentration of the solution in the tank after the n^{th} replacement, where $C_0 = 40\%$.

Write the first five terms of the sequence $\{C_n\}$

b) After how many replacements does the alcohol concentration reach 15%?

c) Determine the limiting (steady-state) concentration of the solution that is approached after many replacements.

Solution

a) Let D_n be the total number of liters of alcohol in the mixture after n^{th} replacement.

Next, 2 liters of the 100 liters is removed left with 98 liters (0.98).

D_n liters of alcohol, and then $0.1\times 2 = 0.2$ liters of alcohol are added.

Thus $D_n = 0.98D_{n-1} + 0.2$ *and* let $C_n = \frac{D_n}{100}$

$100C_n = 0.98\left(100C_{n-1}\right) + 0.2 \Rightarrow \underline{C_n = 0.98C_{n-1} + 0.002}$

$C_0 = 0.4$

$C_1 = 0.98C_0 + 0.002 = 0.98(.4) + 0.002 = \underline{0.394}$

$C_2 = 0.98C_1 + 0.002 = 0.98(0.394) + 0.002 = \underline{0.38812}$

$C_3 = 0.98C_2 + 0.002 = 0.98(0.38812) + 0.002 = \underline{0.382358}$

$C_4 = 0.98C_3 + 0.002 = 0.98(0.382358) + 0.002 = \underline{0.376710}$

$C_5 = 0.98C_4 + 0.002 = 0.98(0.376710) + 0.002 = \underline{0.371176}$

b) For $C_n < 0.15$

C	0	0.4													
C	1	0.394		C	26	0.277419		C	51	0.207066		C	76	0.164610
C	2	0.38812		C	27	0.273870		C	52	0.204925		C	77	0.163318
C	3	0.382358		C	28	0.270393		C	53	0.202826		C	78	0.162052
C	4	0.376710		C	29	0.266985		C	54	0.200770		C	79	0.160811
C	5	0.371176		C	30	0.263645		C	55	0.198754		C	80	0.159595
C	6	0.365753		C	31	0.260372		C	56	0.196779		C	81	0.158403
C	7	0.360438		C	32	0.257165		C	57	0.194843		C	82	0.157235
C	8	0.355229		C	33	0.254022		C	58	0.192947		C	83	0.156090
C	9	0.350124		C	34	0.250941		C	59	0.191088		C	84	0.154968
C	10	0.345122		C	35	0.247922		C	60	0.189266		C	85	0.153869
C	11	0.340219		C	36	0.244964		C	61	0.187481		C	86	0.152791
C	12	0.335415		C	37	0.242065		C	62	0.185731		C	87	0.151736
C	13	0.330707		C	38	0.239223		C	63	0.184016		C	88	0.150701

C	14	0.326093	C	39	0.236439	C	64	0.182336	C	89	0.149687
C	15	0.321571	C	40	0.233710	C	65	0.180689			
C	16	0.317139	C	41	0.231036	C	66	0.179076			
C	17	0.312797	C	42	0.228415	C	67	0.177494			
C	18	0.308541	C	43	0.225847	C	68	0.175944			
C	19	0.304370	C	44	0.223330	C	69	0.174425			
C	20	0.300282	C	45	0.220863	C	70	0.172937			
C	21	0.296277	C	46	0.218446	C	71	0.171478			
C	22	0.292351	C	47	0.216077	C	72	0.170048			
C	23	0.288504	C	48	0.213756	C	73	0.168648			
C	24	0.284734	C	49	0.211481	C	74	0.167275			
C	25	0.281039	C	50	0.209251	C	75	0.165929			

c) Assume that the $\lim\limits_{n\to\infty} C_n = L$, then $L = 0.98L + 0.002$

$$0.02L = 0.002$$

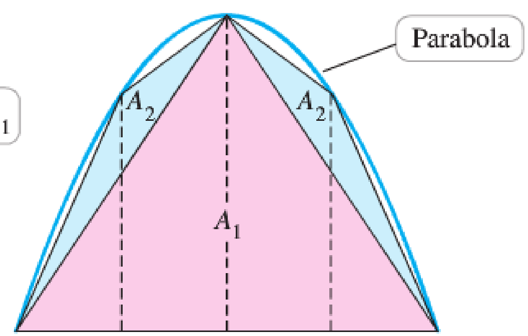
$$L = 0.1 = 10\%$$

Exercise

The Greeks solved several calculus problems almost 2000 years before the discovery of calculus. One example is Archimedes' calculation of the area of the region R bounded by a segment of a parabola, which he did using the "method of exhaustion".

The idea was to fill R with an infinite sequence of triangles. Archimedes began with an isosceles triangle inscribed in the parabola, with an area A_1, and proceeded in stages, with the number of new triangles doubling at each stage. He was able to show (the key to the solution) that at each stage, the area of a new triangle is $\frac{1}{8}$ of the area of a triangle at the previous stage; for example, $A_2 = \frac{1}{8}A_1$, and so forth. Show, as Archimedes did, that the area of R is $\frac{4}{3}$ times the area of A_1.

Solution



At the nth stage, there are 2^{n-1} triangles of area

$$A_n = \frac{1}{8}A_{n-1}$$

$$= \frac{1}{8^{n-1}}A_1$$

So the total area of the triangles formed at the nth stage is

$$\frac{2^{n-1}}{8^{n-1}}A_1 = \left(\frac{1}{4}\right)^{n-1}A_1$$

The total area under the parabola is

$$\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^{n-1} A_1 = A_1\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^{n-1}$$

$$= A_1 \frac{1}{1-\frac{1}{4}}$$

$$\boxed{= \frac{4}{3}A_1}$$

Exercise

a) Evalute the series $\displaystyle\sum_{k=1}^{\infty}\frac{3^k}{\left(3^{k+1}-1\right)\left(3^k-1\right)}$

b) For what values of a does the the series converge, and in those cases, what is its value?

$$\sum_{k=1}^{\infty}\frac{a^k}{\left(a^{k+1}-1\right)\left(a^k-1\right)}$$

Solution

a) $$\frac{3^k}{\left(3^{k+1}-1\right)\left(3^k-1\right)} = \frac{A}{3^{k+1}-1}+\frac{B}{3^k-1}$$

$$3^k = A3^k - A + B3^{k+1} - B$$

$$= A\left(3^k\right) + 3B\left(3^k\right) - A - B$$

$$\begin{cases} A+3B=1 \\ -A-B=0 \end{cases} \quad \rightarrow \quad B=\frac{1}{2}=-A$$

$$= \frac{1}{2}\left(\frac{1}{3^k-1}-\frac{1}{3^{k+1}-1}\right)$$

$$\sum_{k=1}^{\infty}\frac{3^k}{\left(3^{k+1}-1\right)\left(3^k-1\right)} = \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{3^k-1}-\frac{1}{3^{k+1}-1}\right)$$

This series telescopes to gives

$$S_n = \frac{1}{2}\left(\frac{1}{3-1}-\frac{1}{3^{n+1}-1}\right)$$

$$\lim_{n\to\infty} S_n \boxed{= \frac{1}{4}}$$

b) $\dfrac{a^k}{\left(a^{k+1}-1\right)\left(a^k-1\right)}=\dfrac{A}{a^{k+1}-1}+\dfrac{B}{a^k-1}$

$$a^k = Aa^k - A + Ba^{k+1} - B$$
$$= Aa^k + aBa^k - A - B$$

$$\begin{cases} A + aB = 1 \\ -A - B = 0 \end{cases} \quad \to \quad B = \frac{1}{a-1} = -A$$

$$= \frac{1}{a-1}\left(\frac{1}{a^k-1} - \frac{1}{a^{k+1}-1}\right) \quad (a \neq 1)$$

$$\sum_{k=1}^{\infty} \frac{a^k}{\left(a^{k+1}-1\right)\left(a^k-1\right)} = \frac{1}{a-1}\sum_{k=1}^{\infty}\left(\frac{1}{a^k-1} - \frac{1}{a^{k+1}-1}\right)$$

This series telescopes to gives

$$S_n = \frac{1}{a-1}\cdot\left(\frac{1}{a-1} - \frac{1}{a^{n+1}-1}\right)$$

$\lim\limits_{n\to\infty} \dfrac{1}{a^{n+1}-1}$ converges only iff $|a| > 1$

Therefore, the series converges to:

$$\lim_{n\to\infty} S_n = \boxed{\frac{1}{(a-1)^2}} \quad if \quad |a| > 1$$

Exercise

Suppose you borrow \$20,000 for a new car at a monthly interest rate of 0.75%. If you make payments of \$600/month, after how many months will the loan balance be zero? Estimate the answer by graphing the sequence of loan balances and then obtain an exact answer using infinite series.

Solution

$B_0 = 20,000$

Let B_n be the loan after *n* months.

$$B_n = 1.0075 \cdot B_{n-1} - 600$$

$B_{n-1} = 1.0075 \cdot B_{n-2} - 600 \qquad B_n = 1.0075\left(1.0075 \cdot B_{n-2} - 600\right) - 600$

$$= (1.0075)^2 B_{n-2} - 600(1 + 1.0075)$$

$B_{n-2} = 1.0075 \cdot B_{n-3} - 600 \qquad B_n = (1.0075)^2 B_{n-2} - 600(1 + 1.0075)$

$$= (1.0075)^2 \left(1.0075 \cdot B_{n-3} - 600\right) - 600(1 + 1.0075)$$

$$= (1.0075)^3 B_{n-3} - 600\left(1 + 1.0075 + 1.0075^2\right)$$

$$\vdots \quad \vdots \quad \vdots \qquad\qquad\qquad \vdots \quad \vdots \quad \vdots$$

$$= (1.0075)^n B_0 - 600\left(1 + 1.0075 + (1.0075)^2 + \cdots + (1.0075)^{n-1}\right)$$

$$1 + 1.0075 + (1.0075)^2 + \cdots + (1.0075)^{n-1} = \frac{(1.0075)^n - 1}{1.0075 - 1}$$

$$= \frac{(1.0075)^n - 1}{0.0075}$$

$$B_n = 20,000(1.0075)^n - 600\frac{(1.0075)^n - 1}{0.0075} = 0$$

$$150(1.0075)^n - 600(1.0075)^n + 600 = 0$$

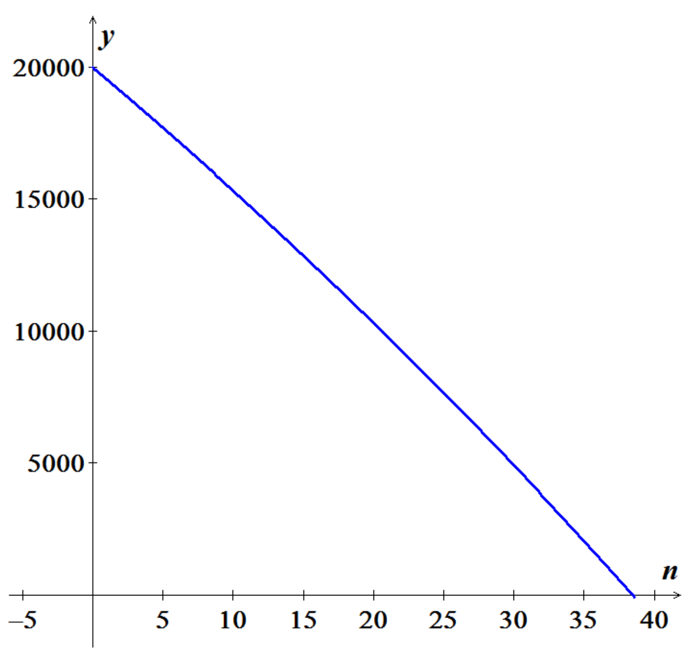
$$450(1.0075)^n = 600$$

$$(1.0075)^n = \frac{60}{45}$$

$$n = \log_{1.0075} \frac{60}{45}$$

$$= 38.501$$

So the loan will be paid off after 39 months.

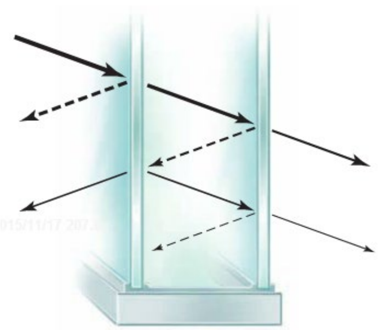


Exercise

An insulated windows consists of two parallel panes of glass with a small spacing between them. Suppose that each pane reflects a fraction p of the incoming light and transmits the remaining light. Considering all reflections of light between the panes, what fraction of the incoming light is ultimately transmitted by the windows? Assume the amount of incoming light is 1.

Solution

Let L_n be the amount of light transmitted through the window the n^{th} time the beam hits the second pane $\Rightarrow \dfrac{L_n}{1-p}$.



$\dfrac{pL_n}{1-p}$: is the reflection to the first pane.

$\dfrac{p^2L_n}{1-p}$: is the reflection back to the second pane.

Thus,

$$L_{n+1} = (1-p)\frac{p^2L_n}{1-p} = p^2L_n$$

The amount of light transmitted through the windows the first time is $(1-p)^2$

Thus the total amount is:

$$\sum_{i=0}^{\infty} p^{2n}(1-p)^2 = \frac{(1-p)^2}{1-p^2}$$

$$= \frac{1-p}{1+p}$$

Exercise

Suppose a rubber ball, when dropped from a given height, returns to a fraction p of that height. In the absence of air resistance, a ball dropped from a height h requires $\sqrt{\dfrac{2h}{g}}$ seconds to fall to the ground, where $g \approx 9.8\ m/s^2$ is the acceleration due to gravity. The time taken to bounce up to a given to fall from that height to the ground. How long does it take a ball dropped from 10 m to come to rest?

Solution

The height after the n^{th} bounce is: $10p^n$.

The total time spent in that bounce is: $2\sqrt{\dfrac{20p^n}{g}}$

The total time before the ball comes to rest:

$$\sqrt{\frac{20}{g}}+2\sum_{n=1}^{\infty}\sqrt{\frac{20p^n}{g}}=\sqrt{\frac{20}{g}}+2\sqrt{\frac{20}{g}}\sum_{n=1}^{\infty}\left(\sqrt{p}\right)^n$$

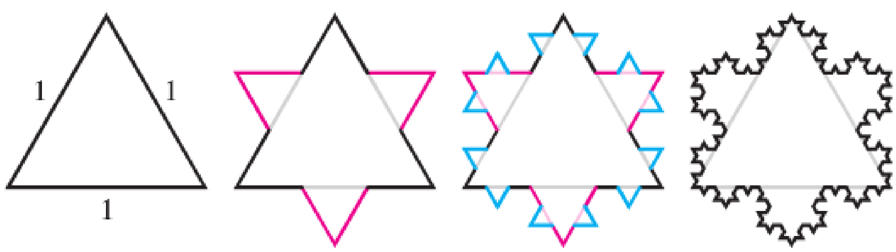
$$=\sqrt{\frac{20}{g}}+2\sqrt{\frac{20}{g}}\left(\frac{\sqrt{p}}{1-\sqrt{p}}\right)$$

$$=\sqrt{\frac{20}{g}}\left(1+\frac{2\sqrt{p}}{1-\sqrt{p}}\right)$$

$$=\sqrt{\frac{20}{g}}\left(\frac{1+\sqrt{p}}{1-\sqrt{p}}\right)\ sec$$

Exercise

The fractal called the snowflake island (or Koch island) is constructed as flows: Let I_0 be an equilateral triangle with sides of length 1. The figure I_1 is obtained by replacing the middle third of each side of I_0 with a new outward equilateral triangle with sides of length $\frac{1}{3}$. The process is repeated where I_{n+1} is obtained by replacing the middle third of each side of I_n with a new outward equilateral triangle with sides of length of $\frac{1}{3^{n+1}}$. The limiting figure as $n\to\infty$ is called the snowflake island.



I_0 I_1 I_2 I_3

a) Let L_n be the perimeter of I_n. Show that $\lim_{n\to\infty} L_n=\infty$

b) Let A_n be the area of I_n. Find $\lim_{n\to\infty} A_n$. It exists!

Solution

a) Triangle has 3 equal sizes, from I_0 to I_1 each side turn into 4, and so on.

I_{n+1} is obtained by I_n by dividing each edge into 3 equal parts, removing the middle part, and adding 2 parts equal to it. Thus 3 equal parts turn into 4, so $L_{n+1}=\frac{4}{3}L_n$

This is a geometric series with a ratio greater than 1, so the n^{th} term grows without bound.

$$\lim_{n\to\infty} L_n = \infty$$

b) From part (*a*) I_n has $3\cdot 4^n$ sides of length $\frac{1}{3^n}$; each of those sides turns into an added triangle in I_{n+1} of sides length $\frac{1}{3^{n+1}}$.

Thus the added area in I_{n+1} consists of $3\cdot 4^n$ equilateral triangles with side $\frac{1}{3^{n+1}}$.

The area of an equilateral triangle with side *x* is $\frac{\sqrt{3}}{4}x^2$.

$$\text{Thus } A_{n+1} = A_n + 3\cdot 4^n \frac{\sqrt{3}}{4}\left(\frac{1}{3^{n+1}}\right)^2$$

$$= A_n + 3\cdot\frac{4^n}{9\cdot 3^{2n}}\frac{\sqrt{3}}{4}$$

$$= A_n + \frac{\sqrt{3}}{12}\cdot\left(\frac{4}{9}\right)^n$$

$$A_0 = \frac{\sqrt{3}}{4}$$

$$A_{n+1} = A_0 + \frac{\sqrt{3}}{12}\sum_{k=0}^{n}\left(\frac{4}{9}\right)^k$$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}\sum_{k=0}^{n}\left(\frac{4}{9}\right)^k$$

$$\lim_{n\to\infty} A_{n+1} = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}\cdot\frac{1}{1-\frac{4}{9}}$$

$$= \frac{\sqrt{3}}{4}\left(1+\frac{1}{3}\cdot\frac{9}{5}\right)$$

$$= \boxed{\frac{2\sqrt{3}}{5}}$$

Exercise

Imagine a stack of hemispherical soap bubbles with decreasing radii $r_1 = 1,\ r_2,\ r_3, \ldots$ Let h_n be the distance between the diameters of bubble n and bubble $n+1$, and let H_n be the total height of the stack with n bubbles.

a) Use the Pythagorean theorem to show that in a stack with n bubbles $h_1^2 = r_1^2 - r_2^2$, $h_2^2 = r_2^2 - r_3^2$, and so forth. Note that for the last bubble $h_n = r_n$.

b) Use part (a) to show that the height of a stack with n bubbles is

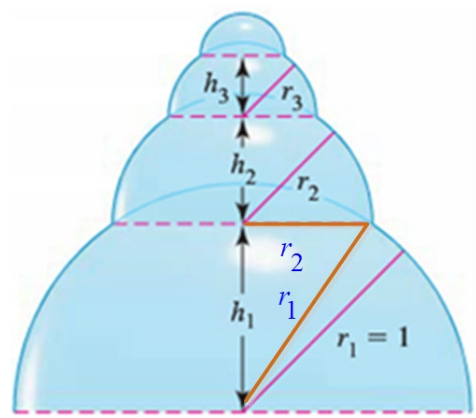
$$H_n = \sqrt{r_1^2 - r_2^2} + \sqrt{r_2^2 - r_3^2} + \cdots + \sqrt{r_{n-1}^2 - r_n^2} + r_n$$

c) The height of a stack of bubbles depends on how the radii decrease. Suppose that $r_1 = 1,\ r_2 = a,\ r_3 = a^2,\ \ldots,\ r_n = a^{n-1}$ where $0 < a < 1$ is a fixed real number. In terms of a, find the height H_n of a stack with n bubbles.

d) Suppose the stack in part (c) is extended indefinitely $(n \to \infty)$. In terms of a, how high would the stack be?

Solution

a) Using the Pythagorean theorem; let $k < n$:



$$r_1^2 = h_1^2 + r_2^2 \quad \Rightarrow \quad h_1^2 = r_1^2 - r_2^2$$

$$r_2^2 = h_2^2 + r_3^2 \quad \Rightarrow \quad h_2^2 = r_2^2 - r_3^2$$

$$\vdots \quad \Rightarrow \quad \vdots$$

$$\Rightarrow \quad h_k^2 = r_k^2 - r_{k+1}^2$$

b) Since $h_1^2 = r_1^2 - r_2^2,\ \ h_2^2 = r_2^2 - r_3^2$ *and* $h_k^2 = r_k^2 - r_{k+1}^2$

$$H_n = h_1 + h_2 + \cdots + h_{n-1} + r_n$$

$$= \sqrt{r_1^2 - r_2^2} + \sqrt{r_2^2 - r_3^2} + \cdots + \sqrt{r_{n-1}^2 - r_n^2} + r_n$$

$$= r_n + \sum_{i=1}^{n-1} \sqrt{r_i^2 - r_{i+1}^2}$$

c) ***Given***: $r_1 = 1,\ r_2 = a,\ r_3 = a^2,\ \ldots,\ r_n = a^{n-1}$

$$H_n = r_n + \sum_{i=1}^{n-1} \sqrt{r_i^2 - r_{i+1}^2}$$ Since given $r_i = a^{i-1}$

$$= a^{n-1} + \sum_{i=1}^{n-1} \sqrt{a^{2i-2} - a^{2i}}$$

$$= a^{n-1} + \sum_{i=1}^{n-1} \sqrt{a^{2i-2}\left(1-a^2\right)}$$

$$= a^{n-1} + \sum_{i=1}^{n-1} a^{i-1}\sqrt{1-a^2}$$

$$= a^{n-1} + \sqrt{1-a^2}\ \sum_{i=1}^{n-1} a^{i-1}$$ Since $0 < a < 1$

$$\boxed{= a^{n-1} + \sqrt{1-a^2}\left(\frac{1-a^{n-1}}{1-a}\right)}$$ $\sum_{k=1}^{n} r^{k-1} = \frac{1-r^n}{1-r}$

d) $$\lim_{n\to\infty} H_n = \lim_{n\to\infty}\left[a^{n-1} + \sqrt{1-a^2}\left(\frac{1-a^{n-1}}{1-a}\right)\right]$$ $\lim_{n\to\infty} a^{n-1} = 0$ $(0 < a < 1)$

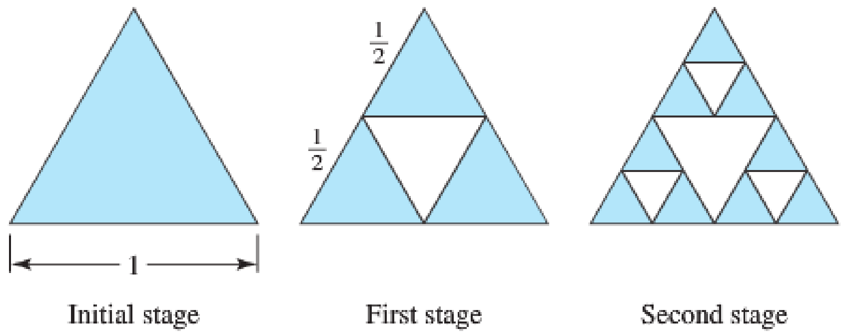
$$= 0 + \sqrt{1-a^2}\left(\frac{1}{1-a}\right)$$

$$= \frac{\sqrt{1-a^2}}{1-a}$$

$$\boxed{= \sqrt{\frac{1+a}{1-a}}}$$

Exercise

The fractal called the *Sierpinski triangle* is the limit of a sequence of figures. Starting with the equilateral triangle with sides of length 1, an inverted equilateral triangle with sides of length $\frac{1}{2}$ is removed. Then, the three inverted equilateral triangles with sides of length $\frac{1}{4}$ are removed from this figure.



The process continues in this way. Let T_n be the total area of the removed triangles after stage *n* of the process. The area on equilateral triangle with side length *L* is $A=\frac{\sqrt{3}}{4}L^2$.

a) Find T_1 and T_2 the total area of the removed triangles after stages 1 and 2, respectively.

b) Find T_n for $n=1,\ 2,\ 3,\ \ldots$

c) Find $\lim\limits_{n\to\infty} T_n$

d) What is the area of the original triangle that remains as $n\to\infty$?

Solution

a) $T_1 = \frac{\sqrt{3}}{4}\left(\frac{1}{2}\right)^2$

$= \frac{\sqrt{3}}{16}$

$T_2 = T_1 + 3A$

$= \frac{\sqrt{3}}{16} + 3\frac{\sqrt{3}}{4}\left(\frac{1}{4}\right)^2$

$= \frac{7\sqrt{3}}{64}$

b) At stage *n*, there are 3^{n-1} triangles of side length $\frac{1}{2^n}$ are removed.

Each of those triangles has an area of $\frac{\sqrt{3}}{4\cdot 4^n} = \frac{\sqrt{3}}{4^{n+1}}$

Total $= 3^{n-1}\frac{\sqrt{3}}{4^{n+1}} = \frac{\sqrt{3}}{16}\left(\frac{3}{4}\right)^{n-1}$

$$T_n = \frac{\sqrt{3}}{16}\sum_{k=1}^{n}\left(\frac{3}{4}\right)^{k-1}$$

$$= \frac{\sqrt{3}}{16}\sum_{k=0}^{n}\left(\frac{3}{4}\right)^{k}$$

$$= \frac{\sqrt{3}}{16}\cdot\frac{1-\left(\frac{3}{4}\right)^n}{1-\frac{3}{4}}$$

$$= \frac{\sqrt{3}}{4}\left(1-\left(\frac{3}{4}\right)^n\right)$$

c) $\lim\limits_{n\to\infty} T_n = \lim\limits_{n\to\infty}\frac{\sqrt{3}}{4}\left(1-\left(\frac{3}{4}\right)^n\right)$ $\qquad \left(\frac{3}{4}\right)^n \xrightarrow[n\to\infty]{} 0$

$$= \frac{\sqrt{3}}{4}$$

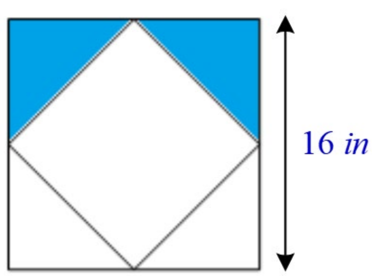
d) The area of the triangle was originally $\frac{\sqrt{3}}{4}$, so none of the original area is left.

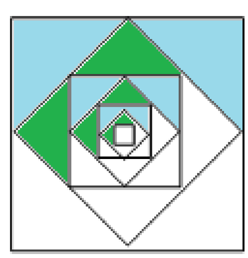
Exercise

The sides of a ***square*** are 16 *inches* in length. A new square is formed by connecting the midpoints of the sides of the original square, and two of the triangles outside the second square are shaded.
Determine the area of the shaded regions

a) When this process is continued five more times

b) When this pattern of shading is continued infinitely.





Solution

a) The first process: $A_1 = 2\left(\frac{1}{2}8^2\right) = 64 = 2^6$

$$A_2 = 2\left(\frac{1}{2}\left(4\sqrt{2}\right)^2\right) = 32 = 2^5$$

$$A_3 = 2\left(\frac{1}{2}(4)^2\right) = 16 = 2^4$$

$$A_4 = 2\left(\frac{1}{2}\left(2\sqrt{2}\right)^2\right) = 8 = 2^3$$

$$A_k = 64(2)^{-k}$$

$$A = 64+32+16+8+4+2$$

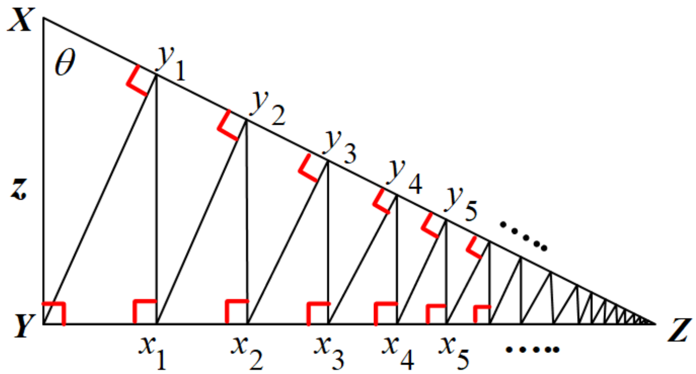
$$= 126\ in^2$$

b) $\displaystyle\sum_{n=0}^{\infty} 64\left(\frac{1}{2}\right)^n = \frac{64}{1-\frac{1}{2}}$ $\qquad\qquad$ $S = \dfrac{a_0}{1-r}$

$$\boxed{= 128 \quad in^2}$$

Exercise

A right triangle XYZ is shown below where $|XY| = z$ and $\angle X = \theta$. Line segments are continually drawn to be perpendicular to the triangle.



a) Find the total length of the perpendicular line segments $|Yy_1| + |x_1y_1| + |x_1y_2| + \cdots$ in terms of z and θ.

b) Find the total length of the perpendicular line segments when $z = 1$ and $\theta = \frac{\pi}{6}$

Solution

a) $\sin\theta = \dfrac{|Yy_1|}{z} \quad \Rightarrow \quad |Yy_1| = z\sin\theta$

$\sin\theta = \dfrac{|x_1y_1|}{|Yy_1|} \quad \Rightarrow \quad |x_1y_1| = |Yy_1|\sin\theta = z\sin^2\theta$

$\sin\theta = \dfrac{|x_1y_2|}{|x_1y_1|} \quad \Rightarrow \quad |x_1y_2| = |x_1y_1|\sin\theta = z\sin^3\theta$

$\vdots \qquad\qquad\qquad \vdots$

Total Length $= z\sin\theta + z\sin^2\theta + z\sin^3\theta + \cdots$

$= z\left(\sin\theta + \sin^2\theta + \sin^3\theta + \cdots\right)$ $\qquad\qquad$ $|\sin\theta| < 1 \quad S = \dfrac{a_0}{1-r}$

$$\boxed{= z\,\frac{\sin\theta}{1-\sin\theta}}$$

b) *Given*: $z=1$ and $\theta=\frac{\pi}{6}$

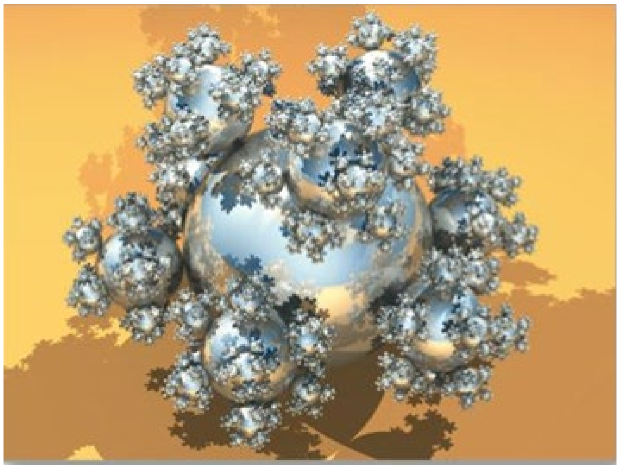
$$\text{Total Length } = 1\,\frac{\sin\frac{\pi}{6}}{1-\sin\frac{\pi}{6}}$$

$$=1\,\frac{\frac{1}{2}}{1-\frac{1}{2}}$$

$$=1$$

Exercise

The sphereflake is a computer–generated fractal that was created by Eric Haines. The radius of the large sphere is 1. To the large sphere, nine spheres of radius $\frac{1}{3}$ are attached. To each of these, nine spheres of radius $\frac{1}{9}$ are attached. This process is continued infinitely.



Prove that the sphereflake has an infinite surface area.

Solution

$$\text{Surface area } = 4\pi(1)^2 + 9\times 4\pi\left(\frac{1}{3}\right)^2 + 9^2\times 4\pi\left(\frac{1}{9}\right)^2 + \cdots$$

$$= 4\pi(1+1+1+\cdots)$$

$$= \infty$$

SOLUTION Section 3.3 – Integral Test

Exercise

Use the ***Integral Test*** to determine if the series converge or diverge. $\sum_{n=1}^{\infty} \frac{1}{n^{0.2}}$

Solution

$$f(x) = \frac{1}{x^{0.2}}$$

$$\int_1^\infty \frac{dx}{x^{0.2}} = \int_1^\infty x^{-0.2}\, dx$$

$$= \frac{1}{0.8} x^{0.8} \Big|_1^\infty$$

$$= \frac{1}{0.8}(\infty - 1)$$

$$\underline{= \infty}$$

By the *Integral Test*, the given series ***diverges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\sum_{n=1}^{\infty} \frac{1}{n^2 + 4}$

Solution

$$f(x) = \frac{1}{x^2 + 4}$$

$$\int_1^\infty \frac{dx}{x^2 + 4} = \frac{1}{2} \tan^{-1} \frac{x}{2} \Big|_1^\infty$$

$$= \frac{1}{2}\left(\tan^{-1} \infty - \tan^{-1} \frac{1}{2}\right)$$

$$\underline{= \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1} \frac{1}{2}\right)}$$

By the *Integral Test*, the given series ***converges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\sum_{n=1}^{\infty} e^{-2n}$

Solution

$$f(x) = e^{-2x}$$

$$\int_1^{\infty} e^{-2x} dx = -\frac{1}{2} e^{-2x} \Big|_1^{\infty}$$

$$= -\frac{1}{2}\left(e^{-\infty} - e^{-2}\right)$$

$$= -\frac{1}{2}\left(\frac{1}{e^{\infty}} - \frac{1}{e^2}\right)$$

$$= -\frac{1}{2e^2}$$

By the *Integral Test*, the given series *converges*.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Solution

Function is positive, continuous, and decreasing for $x \ge 2$.

$$f(x) = \frac{1}{x(\ln x)^2}$$

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \int_2^{\infty} \frac{d(\ln x)}{(\ln x)^2} \qquad d(\ln x) = \frac{1}{x} dx$$

$$= -\frac{1}{\ln x} \Big|_2^{\infty}$$

$$= -\left(\frac{1}{\ln \infty} - \frac{1}{\ln 2}\right)$$

$$= \frac{1}{\ln 2}$$

By the *Integral Test*, the given series *converges*.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}}$

Solution

$f(x) = \dfrac{x^2}{e^{x/3}}$ is positive, continuous for $x \geq 1$.

$$f'(x) = \frac{2xe^{x/3} - \frac{1}{3}x^2 e^{x/3}}{\left(e^{x/3}\right)^2}$$

$$= \frac{6x - x^2}{3e^{x/3}}$$

$$= \frac{-x(x-6)}{3e^{x/3}} < 0 \quad \textit{for} \quad x > 6$$

		$\int e^{-x/3}\, dx$
+	x^2	$-3e^{-x/3}$
−	$2x$	$9e^{-x/3}$
+	2	$-27e^{-x/3}$

$$\int_7^{\infty} \frac{x^2}{e^{x/3}}\, dx = -\frac{3x^2}{e^{x/3}} - \frac{18x}{e^{x/3}} - \frac{54}{e^{x/3}} \Bigg|_7^{\infty}$$

$$= \frac{3}{e^{x/3}}(49 + 42 + 18)$$

$$= \frac{327}{e^{7/3}}$$

By the *Integral Test*, the given series *converges*.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\displaystyle\sum_{n=1}^{\infty} \frac{n-4}{n^2 - 2n + 1}$

Solution

$$f(x) = \frac{x-4}{x^2 - 2x + 1}$$

$$= \frac{x-4}{(x-1)^2} \quad \text{is continuous for } x \ge 2, \text{ and positive } x > 4.$$

$$f'(x) = \frac{(x-1)^2 - 2(x-1)(x-4)}{(x-1)^4}$$

$$= \frac{(x-1)(x-1-2x+8)}{(x-1)^4}$$

$$= \frac{-x+7}{(x-1)^3} < 0 \quad \text{for} \quad x > 7$$

$$\int_8^\infty \frac{x-4}{(x-1)^2}dx = \lim_{b\to\infty}\left[\int_8^b \frac{x-1}{(x-1)^2}dx - \int_b^\infty \frac{3}{(x-1)^2}dx\right] \qquad d(x-1) = dx$$

$$= \lim_{b\to\infty}\left[\int_8^b \frac{1}{x-1}dx - \int_b^\infty \frac{3}{(x-1)^2}d(x-1)\right]$$

$$= \lim_{b\to\infty}\left(\ln|x-1|\right)\Big|_8^b - \lim_{b\to\infty}\left(\frac{3}{(x-1)}\right)\Big|_8^b$$

$$= \lim_{b\to\infty}\left[\ln|b-1| - \ln 7 - \frac{3}{b-1} + \frac{3}{7}\right]$$

$$\underline{= \infty}$$

By the *Integral Test*, the given series *diverges*.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Solution

Let $f(x) = \frac{1}{x \ln x}$

$$\int_2^\infty \frac{1}{x\ln x}dx = \int_2^\infty \frac{1}{\ln x}d(\ln x)$$

$$= \ln|\ln x|\Big|_2^\infty$$

$$= \infty - 0$$

$$\underline{= \infty}$$

Therefore; by the *Integral Test*, the given series ***diverges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\sum_{k=1}^{\infty} \frac{k}{\sqrt{k^2+4}}$

Solution

Let $f(x)=\frac{x}{\sqrt{x^2+4}}$, $f(x)$ is continuous for $x \ge 1$.

$$f'(x)=\frac{\sqrt{x^2+4}-x^2\left(x^2+4\right)^{-1/2}}{\left(\sqrt{x^2+4}\right)^2}$$

$$=\frac{4}{\left(\sqrt{x^2+4}\right)^3} > 0$$

Thus $f(x)$ is increasing, and the conditions of the Integral Test are not satisfied.

Therefore; by the *Integral Test*, the given series ***diverges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\sum_{k=1}^{\infty} ke^{-2k^2}$

Solution

Let $f(x)=x\cdot e^{-2x^2}$, $f(x)$ is continuous for $x \ge 1$.

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} xe^{-2x^2}\,dx$$

$$=-\frac{1}{4}\int_1^{\infty} e^{-2x^2}\,d\left(-2x^2\right)$$

$$=-\frac{1}{4}e^{-2x^2}\Big|_1^{\infty}$$

$$=-\frac{1}{4}\left(0-e^{-2}\right)$$

$$=\frac{1}{4e^2}$$

Therefore; by the *Integral Test*, the given series ***converges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{\sqrt{k+8}}$

Solution

Let $f(x) = \dfrac{1}{\sqrt{x+8}}$, $f(x)$ is continuous and decreasing for $x \ge 1$.

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{\sqrt{x+8}}\,dx$$

$$= 2\sqrt{x+8}\,\Big|_1^{\infty}$$

$$\underline{= \infty}$$

Therefore; by the *Integral Test*, the given series ***diverges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \ln k \ln(\ln k)}$

Solution

Let $f(x) = \dfrac{1}{x \ln x \ln(\ln x)}$, $f(x)$ is ***not continuous*** at $x = e$.

Therefore; the *Integral Test* does not apply.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

Solution

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \int_1^{\infty} \frac{1}{x^2+1}\,d\left(x^2+1\right)$$

$$= \ln\left(x^2+1\right)\Big|_1^{\infty}$$

$$\underline{= \infty}$$

Therefore; by the *Integral Test*, the given series ***diverges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Solution

$$\int_1^{\infty} \frac{1}{x^2+1} dx = \arctan x \Big|_1^{\infty}$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Therefore; by the *Integral Test*, the given series ***converges***.

Exercise

Use the *Integral Test* st to determine if the series converge or diverge. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Solution

$$\int_1^{\infty} \frac{1}{x^3} dx = -\frac{1}{2}\frac{1}{x^2} \Big|_1^{\infty}$$

$$= -\frac{1}{2}(0-1)$$

$$= \frac{1}{2}$$

Therefore; by the *Integral Test*, the given series ***converges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

Solution

$$\int_1^{\infty} x^{-1/2} dx = \frac{1}{2}\sqrt{x} \Big|_1^{\infty}$$

$$= \infty$$

Therefore; by the *Integral Test*, the given series ***diverges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\sum_{n=1}^{\infty} \frac{1}{n^{1/4}}$

Solution

Let $f(x) = \frac{1}{x^{1/4}}$

$$\int_1^{\infty} x^{-1/4} dx = \frac{4}{3} x^{3/4} \Big|_1^{\infty}$$

$$\boxed{= \infty}$$

Therefore; by the *Integral Test*, the given series ***diverges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\sum_{n=1}^{\infty} \frac{1}{n^5}$

Solution

Let $f(x) = \frac{1}{x^5}$

$$\int_1^{\infty} x^{-5} dx = -\frac{1}{4} \frac{1}{x^4} \Big|_1^{\infty}$$

$$= -\frac{1}{4}(0 - 1)$$

$$\boxed{= \frac{1}{4}}$$

Therefore; by the *Integral Test*, the given series ***converges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\sum_{n=2}^{\infty} \frac{1}{e^n}$

Solution

Let $f(x) = \frac{1}{e^x}$

$$\int_2^{\infty} e^{-x} dx = -e^{-x} \Big|_2^{\infty}$$

$$= -e^{-\infty} + e^{-2}$$

$$= \boxed{\frac{1}{e^2}}$$

Therefore; by the *Integral Test*, the given series ***converges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$

Solution

Let $f(x) = \dfrac{x}{e^x}$

		$\int e^{-x}$
+	x	$-e^{-x}$
−	1	e^{-x}

$$\int_1^{\infty} xe^{-x}dx = e^{-x}(-x-1)\Big|_1^{\infty}$$

$$= 0 - e^{-1}(-2)$$

$$= \boxed{\frac{2}{e}}$$

Therefore; by the *Integral Test*, the given series ***converges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\displaystyle\sum_{k=1}^{\infty} \frac{|\sin k|}{k^2}$

Solution

$-1 \le \sin k \le 1$

$0 \le |\sin k| \le 1$

The *Integral Test* does not apply, because the series is not decreasing.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{\left(k^2+1\right)^3}$

Solution

Let $f(x)=\dfrac{x}{\left(x^2+1\right)^3}$

$$\int_1^{\infty}\frac{x}{\left(x^2+1\right)^3}\,dx=\frac{1}{2}\int_1^{\infty}\left(x^2+1\right)^{-3}\,d\left(x^2+1\right)$$

$$=-\frac{1}{4}\left(x^2+1\right)^{-2}\Big|_1^{\infty}$$

$$=-\frac{1}{4}\left(0-\frac{1}{8}\right)$$

$$=\frac{1}{16}$$

Therefore; by the *Integral Test*, the given series ***converges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{\sqrt[3]{k+10}}$

Solution

Let $f(x)=\dfrac{1}{\sqrt[3]{x+10}}$

$$\int_1^{\infty}\frac{1}{\sqrt[3]{x+10}}\,dx=\int_1^{\infty}(x+10)^{-1/3}\,d(x+10)$$

$$=\frac{3}{2}(x+10)^{2/3}\Big|_1^{\infty}$$

$$=\infty$$

Therefore; by the *Integral Test*, the given series ***diverges***.

Exercise

Use the ***p-series*** *Test* to determine if the series converge or diverge. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^9}$

Solution

Which is *p-series* with $p=9>1$

Therefore; by the ***p-series*** *Test*, the given series ***converges***.

Exercise

Use the ***p-series*** *Test* to determine if the series converge or diverge. $\sum_{k=1}^{\infty} \frac{1}{k^6}$

Solution

Which is *p-series* with $p = 6 > 1$

Therefore; by the ***p-series*** *Test*, the given series ***converges***.

Exercise

Use the ***p-series*** *Test* to determine if the series converge or diverge. $\sum_{k=1}^{\infty} \frac{1}{k^{1/9}}$

Solution

v $p = \frac{1}{9} \le 1$

Therefore; by the ***p-series*** *Test*, the given series ***diverges***.

Exercise

Use the ***p-series*** *Test* to determine if the series converge or diverge. $\sum_{k=1}^{\infty} k^{-2}$

Solution

$$\sum_{k=1}^{\infty} k^{-2} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Which is *p-series* with $p = 2 > 1$

Therefore; by the ***p-series*** *Test*, the given series ***converges***.

Exercise

Use the ***p-series*** *Test* to determine if the series converge or diverge. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k}}$

Solution

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/4}}$$

Which is *p-series* with $p = \frac{1}{4} \le 1$

Therefore; by the ***p-series*** *Test*, the given series ***diverges***.

Exercise

Use the ***p-series*** *Test* to determine if the series converge or diverge. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{16k^2}}$

Solution

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{16k^2}} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$$

Which is *p-series* with $p = \frac{1}{2} \le 1$

Therefore; by the ***p-series*** *Test*, the given series ***diverges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^8}$

Solution

Which is *p-series* with $p = 8 > 1$

Therefore; by the ***p-series*** *Test*, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3^k}$

Solution

Let $f(x) = \frac{1}{3^x}$

$$\int_1^{\infty} \frac{1}{3^x}\,dx = \int_1^{\infty} 3^{-x}\,dx$$

$$= -(\ln 3)\ 3^{-x}\Big|_1^{\infty}$$

$$= -(\ln 3)\left(0 - \frac{1}{3}\right)$$

$$= \boxed{\frac{1}{3}\ln 3}$$

Therefore; by the *Integral Test*, the given series ***converges***.

$$\sum_{k=1}^{\infty} \frac{1}{3^k} = \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k$$

By Geometric series $r = \frac{1}{3} < 1$

Therefore; by the *Geometric Test*, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{5/2}}$

Solution

Which is *p-series* with $p = \frac{5}{2} > 1$

Therefore; by the ***p-series*** *Test*, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

Solution

Let $f(x) = xe^{-x^2}$

$$\int_1^{\infty} xe^{-x^2}\, dx = -\frac{1}{2}\int_1^{\infty} e^{-x^2}\, d\left(-x^2\right)$$

$$= -\frac{1}{2}e^{-x^2}\Big|_1^{\infty}$$

$$= -\frac{1}{2}(0-1)$$

$$= \boxed{\frac{1}{2}}$$

Therefore; by the *Integral Test*, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=0}^{\infty} \frac{10}{n^2+9}$

Solution

Let $f(x) = \dfrac{10}{x^2+9}$

$$\int_0^{\infty} \frac{10}{x^2+9}\, dx = \frac{10}{3}\tan^{-1}\frac{x}{3}\Big|_0^{\infty}$$

$$= \frac{10}{3}\left(\tan^{-1}\infty - \tan^{-1}0\right)$$

$$= \frac{10}{3}\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{5\pi}{3}$$

Therefore; by the *Integral Test*, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n+1)(3n+4)}$

Solution

Let $f(x) = \dfrac{1}{(3x+1)(3x+4)}$

$$= \frac{A}{3x+1} + \frac{B}{3x+4}$$

$$3Ax + 4A + 3Bx + B = 1$$

$$\begin{cases} x & 3A + 3B = 0 \\ x^0 & 4A + B = 1 \end{cases}$$

$$\Delta = \begin{vmatrix} 3 & 3 \\ 4 & 1 \end{vmatrix} = -9 \qquad \Delta_A = \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} = -3 \qquad \Delta_B = \begin{vmatrix} 3 & 0 \\ 4 & 1 \end{vmatrix} = 3$$

$$A = \frac{1}{3} \quad B = -\frac{1}{3}$$

$$\int_1^{\infty} \frac{1}{(3x+1)(3x+4)}\, dx = \int_1^{\infty} \left(\frac{1}{3}\frac{1}{3x+1} - \frac{1}{3}\frac{1}{3x+4}\right) dx$$

$$= \frac{1}{9}\int_1^{\infty} \frac{1}{3x+1}\, d(3x+1) - \frac{1}{9}\int_1^{\infty} \frac{1}{3x+4}\, d(3x+4)$$

$$= \frac{1}{9}\left(\ln(3x+1) - \ln(3x+4)\right) \Big|_1^{\infty}$$

$$= \frac{1}{9}\left(\ln\left(\frac{3x+1}{3x+4}\right)\right) \Big|_1^{\infty}$$

$$= \frac{1}{9}\left(\ln 1 - \ln\frac{4}{7}\right)$$

$$= -\frac{1}{9}\ln\frac{4}{7}$$

Therefore; by the *Integral Test*, the given series ***converges***.

This is a telescoping series with

$$S_n = \frac{1}{3}\left(\frac{1}{4} - \frac{1}{3n+4}\right)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{3}\left(\frac{1}{4} - \frac{1}{3n+4}\right)$$

$$= \frac{1}{12}$$

Exercise

Determine if the series converge or diverge $\sum_{n=1}^{\infty} e^{-n}$

Solution

The series is a geometric series with $r = \frac{1}{e} < 1$

Therefore; it ***converges***.

Exercise

Determine if the series converge or diverge $\sum_{n=1}^{\infty} \frac{n}{n+1}$

Solution

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$$

Therefore; the given series ***diverges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty}\frac{3}{\sqrt{n}}$

Solution

$$\sum_{n=1}^{\infty}\frac{3}{\sqrt{n}}=3\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$$

Which is a divergent p-series $\left(p=\frac{1}{2}\le 1\right)$.

Therefore; the given series ***diverges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty}\frac{-8}{n}$

Solution

$$\sum_{n=1}^{\infty}\frac{-8}{n}=-8\sum_{n=1}^{\infty}\frac{1}{n},$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n}\ \text{diverges.}$$

Therefore; the given series ***diverges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=2}^{\infty}\frac{\ln}{n}$

Solution

By the Integral Test:

$$\int_{2}^{\infty}\frac{\ln x}{x}dx=\int_{2}^{\infty}\ln x\, d(\ln x)$$

$$=\frac{1}{2}\ln^2 x\Big|_2^{\infty}$$

$$=\frac{1}{2}\left(\ln^2\infty-\ln^2 2\right)$$

$$\underline{=\infty}$$

Therefore; the given series ***diverges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty}\frac{5^n}{4^n+3}$

Solution

Using L'Hôpital rule:

$$\sum_{n=1}^{\infty}\frac{5^n}{4^n+3}=\sum_{n=1}^{\infty}\frac{5^n\ln 5}{4^n\ln 4}$$

$$=\frac{\ln 5}{\ln 4}\sum_{n=1}^{\infty}\left(\frac{5}{4}\right)^n$$

Using the Geometric series: $|r|=\frac{5}{4}\geq 1$ which diverges.

Therefore; by Geoemtric test, the given series ***diverges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty}n\sin\frac{1}{n}$

Solution

$$\lim_{n\to\infty}n\sin\frac{1}{n}=\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}}$$

$$=\lim_{x\to 0}\frac{\sin x}{x}$$

$$\underline{=1\neq 0}$$

Therefore; the given series ***diverges***.

Exercise

Determine if the series converge or diverge $\sum_{n=1}^{\infty} \frac{1}{n^{10}}$

Solution

This is a p-series with $p = 10$.

Therefore; by the p-series, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\sum_{n=2}^{\infty} \frac{n^e}{n^\pi}$

Solution

$$\sum_{n=2}^{\infty} \frac{n^e}{n^\pi} = \sum_{n=2}^{\infty} \frac{1}{n^{\pi-e}}$$

$$= \sum_{n=2}^{\infty} \frac{1}{n^{3.1416-2.71828}}$$

$$= \sum_{n=2}^{\infty} \frac{1}{n^{0.42331}}$$

$0.42331 < 1$

Therefore; the given series ***diverges***.

Exercise

Determine if the series converge or diverge $\sum_{n=3}^{\infty} \frac{1}{(n-2)^4}$

Solution

$$\sum_{n=3}^{\infty} \frac{1}{(n-2)^4} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

This is a p-series with $p = 4$.

Therefore; by the p-series, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} 2n^{-3/2}$

Solution

$$\sum_{n=1}^{\infty} 2n^{-3/2} = 2\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

This is a p-series with $p = \frac{3}{2}$

Therefore; by the p-series, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{k=0}^{\infty} \frac{k}{\sqrt{k^2+4}}$

Solution

$$\lim_{k\to\infty} \frac{k}{\sqrt{k^2+4}} = \lim_{k\to\infty} \frac{k}{\sqrt{k^2}}$$

$$= \lim_{k\to\infty} \frac{k}{k}$$

$$= 1 \neq 0$$

Therefore; the given series ***diverges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{k=1}^{\infty} \frac{2^k+3^k}{4^k}$

Solution

$$\int_1^{\infty} \left(\frac{2^x}{4^x} + \frac{3^x}{4^x}\right) dx = \int_1^{\infty} \left(\left(\frac{1}{2}\right)^x + \left(\frac{3}{4}\right)^x\right) dx$$

$$= -\frac{1}{\ln(2)}\left(\frac{1}{2}\right)^x - \frac{1}{\ln(3/4)}\left(\frac{3}{4}\right)^x \Big|_1^{\infty}$$

$$= 0 - \left(-\frac{1}{2\ln(2)} - \frac{3}{4\ln(3/4)}\right)$$

$$\approx 3.3284 < \infty$$

Therefore; by the integral test, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{k=2}^{\infty} \frac{4}{k \ln^2 k}$

Solution

$$\int_2^{\infty} \frac{4}{x \ln^2 x} dx = 4 \int_2^{\infty} \frac{1}{\ln^2 x} d(\ln x)$$

$$= -4 \frac{1}{\ln x} \Big|_2^{\infty}$$

$$\boxed{= 4 \frac{1}{\ln 2} < \infty}$$

Therefore; by the integral test, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$

Solution

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/5}}$$

This is a p-series with $p = \frac{1}{5} < 1$

Therefore; by the p-series, the given series ***diverges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$

Solution

This is a p-series with $p = \frac{5}{3} > 1$

Therefore; by the p-series, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$

Solution

This is a p-series with $p = 1.04 > 1$

Therefore; by the p-series, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$

Solution

This is a p-series with $p = \pi > 1$

Therefore; by the p-series, the given series ***converges***.

Exercise

Determine if the series converge or diverge $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots$

Solution

$$1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots = \sum_{1}^{\infty} \frac{1}{n^{3/2}}$$

This is a p-series with $p = \frac{3}{2} > 1$

Therefore; by the p-series, the given series ***converges***.

Exercise

Determine if the series converge or diverge $1 + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{9}} + \frac{1}{\sqrt[3]{16}} + \frac{1}{\sqrt[3]{25}} + \cdots$

Solution

$$1 + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{9}} + \frac{1}{\sqrt[3]{16}} + \frac{1}{\sqrt[3]{25}} + \cdots = \sum_{1}^{\infty} \frac{1}{n^{2/3}}$$

This is a p-series with $p = \frac{2}{3} < 1$

Therefore; by the p-series, the given series ***diverges***.

Exercise

Consider the series $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$, where p is a real number.

a) Use the Integral Test to determine the values of p for which this series converges.

b) Does this series converge faster for $p=2$ or $p=3$? Explain.

Solution

a) Let $f(x) = \frac{1}{x(\ln x)^p}$

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx = \int_2^{\infty} (\ln x)^{-p}\; d(\ln x)$$

$$= \frac{1}{1-p}(\ln x)^{1-p}\,\Big|_2^{\infty}$$

In order for the integral to exist doe the given series to converge, then the value(s) of p:

$1-p<0 \quad \rightarrow \quad \underline{p>1}$

b) Since series converges for $p>1$

For $p=2$

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \int_2^{\infty} \frac{1}{(\ln x)^2}\; d(\ln x)$$

$$= -\frac{1}{\ln x}\,\Big|_2^{\infty}$$

$$= -\left(0 - \frac{1}{\ln 2}\right)$$

$$= \frac{1}{\ln 2}$$

For $p=3$

$$\int_2^{\infty} \frac{1}{x(\ln x)^3}\,dx = \int_2^{\infty} (\ln x)^{-3}\; d(\ln x)$$

$$= -\frac{1}{2}\frac{1}{(\ln x)^2}\,\Big|_2^{\infty}$$

$$= -\frac{1}{2}\left(0 - \frac{1}{(\ln 2)^2}\right)$$

$$= \frac{1}{2}\frac{1}{\ln^2 2}$$

$p=2$	$p=3$
$\frac{1}{\ln 2}$	$\frac{1}{2}\frac{1}{\ln^2 2}$
1	$\frac{1}{2}\frac{1}{\ln 2}$

From the table, the value of $p=3$ is smaller than $p=2$

Therefore; the series converges faster for $p=3$.

Exercise

Consider the series $\sum_{k=2}^{\infty}\frac{1}{k(\ln k)\ (\ln(\ln k))^{p}}$, where p is a real number.

a) For what values of p does this series converge?

b) Which he following series converge faster? Explain.

$$\sum_{k=2}^{\infty}\frac{1}{k(\ln k)^2} \quad \text{or} \quad \sum_{k=2}^{\infty}\frac{1}{k(\ln k)\ (\ln(\ln k))^{2}}$$

Solution

a) Let $f(x)=\frac{1}{x\ln x\ (\ln(\ln x))^{p}}$

$$\int_2^{\infty}\frac{1}{x\ln x\ (\ln(\ln x))^{p}}=\int_2^{\infty}(\ln(\ln x))^{-p}\ d(\ln(\ln x)) \qquad d(\ln(\ln x))=\frac{1}{\ln x}\frac{1}{x}dx$$

$$=\frac{1}{1-p}(\ln(\ln x))^{1-p}\ \Big|_2^{\infty}$$

In order for the integral to exist doe the given series to converge, then the value(s) of p:

$1-p<0 \quad \rightarrow \quad \underline{p>1}$

b) $$\int_2^{\infty}\frac{1}{x\ln x\ (\ln(\ln x))^{2}}=-\frac{1}{\ln\ln x}\ \Big|_2^{\infty}$$

$$=-\left(0-\frac{1}{\ln\ln 2}\right)$$

$$=\frac{1}{\ln\ln 2}$$

$$\int_2^{\infty}\frac{1}{x(\ln x)^2}\,dx=\int_2^{\infty}\frac{1}{(\ln x)^2}\,d(\ln x)$$

$$= -\frac{1}{\ln x}\Bigg|_{2}^{\infty}$$

$$= -\left(0 - \frac{1}{\ln 2}\right)$$

$$= \boxed{\frac{1}{\ln 2}}$$

$\ln 2 > \ln \ln 2$

$\dfrac{1}{\ln 2} < \dfrac{1}{\ln \ln 2}$

Therefore; the first series converges faster because the terms get smaller faster.

Exercise

Consider a wedding cake of infinite height, each layer of which is a right circular cylinder of height 1. The bottom layer of the cake has a radius of 1, the second layer has a radius of $\frac{1}{2}$, the third layer has a radius of $\frac{1}{3}$, and the n^{th} layer has a radius of $\frac{1}{n}$.

a) To determine how much frosting is needed to cover the cake, find the area of the lateral (vertical sides of the wedding cake. What is the area of the horizontal surfaces of the cake?

b) Determine the volume of the cake.

c) Comment on your answer to parts (a) and (b)

Solution

a) The circumference of the k^{th} layer is: $2\pi \frac{1}{k}$, so its area $\frac{2\pi}{k}$

The total vertical surface area:

$$\sum_{k=1}^{\infty} \frac{2\pi}{k} = 2\pi \sum_{k=1}^{\infty} \frac{1}{k}$$

$$\boxed{= \infty}$$

Looking at the cake from above, the horizontal area

$$Area = \pi r^2 = \pi \cdot 1^2$$

$$\boxed{= \pi}$$

b) The volume of a cylinder $= \pi r^2 h$

Volume of the k^{th} layer $= \pi \frac{1}{k^2} \cdot 1 = \frac{\pi}{k^2}$

Thus the volume of the cake is:

$$\sum_{k=1}^{\infty} \frac{\pi}{k^2} = \pi \sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \quad (Leonhard\ Euler)$$

$$= \frac{\pi^3}{6}$$

$$\approx 5.168$$

c) This cake has infinite area, it has finite volume.

Exercise

The Riemann zeta function is the subject of extensive research and is associated with several renowned unsolved problems. Is its defined by $\zeta(x) = \sum_{k=1}^{\neq} \frac{1}{k^x}$, when x is a real number, the zeta function becomes a p-series. For even positive integers ρ, the value of $\zeta(\rho)$ is known exactly. For example,

$$\sum_{k=1}^{\neq} \frac{1}{k^2} = \frac{\pi^2}{6}, \quad \sum_{k=1}^{\neq} \frac{1}{k^4} = \frac{\pi^4}{90}, \quad \text{and} \quad \sum_{k=1}^{\neq} \frac{1}{k^6} = \frac{\pi^6}{945}, \quad \ldots$$

a) Use the estimation techniques to approximate $\zeta(3)$ and $\zeta(5)$ (whose values are not known exactly) with a remainder less than 10^{-3}.

b) Determine the sum of the reciprocals of the squares of the odd positive integers by rearranging the terms of the series $(x = 2)$ without changing the value of the series.

Solution

a) $\zeta(m) = \int_n^{\infty} \frac{1}{x^m} dx = \frac{1}{m-1} x^{1-m} \Big|_n^{\infty} = \frac{1}{m-1} n^{1-m}$

For $\zeta(3) = \frac{1}{2} n^{-2} < 10^{-3}$

$$\frac{1}{2n^2} < \frac{1}{10^3}$$

$$2n^2 > 10^3$$

$$n > \sqrt{500} \approx 23$$

$$\sum_{k=1}^{23} \frac{1}{k^3} \approx 1.201151955$$

```
for i = 1:n
    kk = 1 / (i^x);
    k = k + kk;
end
```

The true value is ≈ 1.202056903

For $\zeta(5) = \frac{1}{4} n^{-4} < 10^{-3}$

$$\frac{1}{4n^4} < \frac{1}{10^3}$$

$$4n^4 > 10^3$$

$n > (250)^{1/4} \approx 4$

$$\sum_{k=1}^{4} \frac{1}{k^5} \approx 1.0363417888$$

The true value is ≈ 1.036927755

b) $$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

$$= \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

$$\frac{3}{4}\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{3}{4}\frac{\pi^2}{6}$$

$$= \frac{\pi^2}{8}$$

Exercise

Consider a set of identical dominoes that are 2 inches long. The dominoes are stacked on top of each other with their long edges aligned so that each domino overhangs the one beneath is as far as possible

a) If there are n dominoes in the stack, what is the greatest distance that the top domino can be made to overhang the bottom domino? (*Hint*: Put the n^{th} domino beneath the previous $n-1$ dominoes.)

b) If we allow for infinitely many dominoes in the stack, what is the greatest distance that the top domino can be made to overhang the bottom domino?

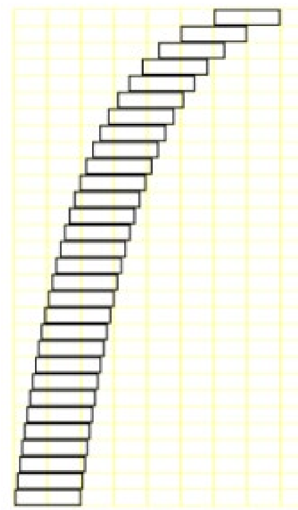
Solution

a) The center of gravity of any stack of dominoes is the average of the locations of their centers.

Define the midpoint of the zeroth (top) domino to be $x = 0$, and stack additional dominoes down and to its right (to increasingly positive x-coordinates).

Let $m(n)$ be the x-coordinate of the midpoint of the n^{th} domino. Then in order for the stack not to fall over, the left edge of the n^{th} domino must be placed directly under the center of gravity of dominos 0 through $n-1$, which is $\frac{1}{n}\sum_{i=0}^{n-1} m(i)$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$$



So $m(n) = 1+\frac{1}{n}\sum_{i=0}^{n-1} m(i) = \sum_{k=1}^{n}\frac{1}{k}$

Proof by induction;

For $n=1 \Rightarrow m(1)=1$ ✓ P_1 is true

Let P_j is true $m(j)=\sum_{k=1}^{j}\frac{1}{k} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{j}$;

we need to prove it is also true for P_{j+1}

$$m(j+1) = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{j}+\frac{1}{j+1}$$

$$= \sum_{k=1}^{j}\frac{1}{k}+\frac{1}{j+1}$$

$$= \sum_{k=1}^{j+1}\frac{1}{k}$$

Therefore; the formula is clearly true by mathematical induction.

b) For infinite number of dominos, because the overhang is the harmonic series, the distance is potentially infinite. This series diverges so with enough dominoes.

Exercise

A theorem states that the sequence of prime numbers $\{p_k\}$ satisfies $\lim_{k\to\infty}\frac{p_k}{k\ln k}$.

Show that $\sum_{k=2}^{\infty}\frac{1}{k\ln k}$ diverges, which implies that the series $\sum_{k=1}^{\infty}\frac{1}{p_k}$

(A prime number is a positive integer number that is divisible only by 1 and itself).

Solution

Let $f(x)=\dfrac{1}{x\ln x}$

$$\int_2^{\infty}\frac{1}{x\ln x}dx=\int_2^{\infty}\frac{1}{\ln x}d(\ln x)$$

$$=\ln|\ln x|\,\Big|_2^{\infty}$$

$$=\infty-\ln\ln 2$$

$$\underline{=\infty}\,\Big|$$

Therefore; by the *Integral Test*, the given series ***diverges***.

SOLUTION *Section* 3.4 – Comparison Tests

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{n^2+30}$

Solution

Comparing with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series, since $p = 2 > 1$.

Both series have nonnegative terms for $n \ge 1$

$$n^2 \le n^2 + 30$$

$$\frac{1}{n^2} \ge \frac{1}{n^2+30}$$

Then, by Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n^2+30}$ ***converges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n-1}{n^4+2}$

Solution

Comparing with $\sum_{n=1}^{\infty} \frac{1}{n^3}$, which is a convergent p-series, since $p = 3 > 1$.

Both series have nonnegative terms for $n \ge 1$

$$n^4 \le n^4 + 2 \quad \Rightarrow \quad \frac{1}{n^4} \ge \frac{1}{n^4+2}$$

$$\frac{n}{n^4} \ge \frac{n}{n^4+2} \ge \frac{n-1}{n^4+2}$$

$$\frac{1}{n^3} \ge \frac{n}{n^4+2} \ge \frac{n-1}{n^4+2}$$

Then, by *Comparison Test*, the given series ***converges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$

Solution

Comparing with $\sum_{n=1}^{\infty} \frac{1}{n}$, which is a divergent p-series, since $p = 1 \le 1$.

Both series have nonnegative terms for $n \ge 2$

$$n^2 - n \le n^2 \quad \Rightarrow \quad \frac{1}{n^2-n} \ge \frac{1}{n^2}$$

$$\frac{n}{n^2-n} \ge \frac{n}{n^2} = \frac{1}{n}$$

Then, by *Comparison Test*, the given series ***diverges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$

Solution

Comparing with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which is a convergent p-series, since $p = \frac{3}{2} > 1$.

Both series have nonnegative terms for $n \ge 1$

$$0 \le \cos^2 n \le 1$$

$$0 \le \frac{\cos^2 n}{n^{3/2}} \le \frac{1}{n^{3/2}}$$

Then, by *Comparison Test*, the given series ***converges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$

Solution

Comparing with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series, since $p = \frac{1}{2} < 1$.

Both series have nonnegative terms for $n \ge 1$

$\sqrt{n} \ge 1 \quad \Rightarrow \quad 2\sqrt{n} \ge 2$

$$2\sqrt{n} + 1 \ge 3$$

Then, by *Comparison Test*, the given series ***diverges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$

Solution

Comparing with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$, which is a convergent series.

Then $0 < \dfrac{1}{2^n + 1} < \dfrac{1}{2^n}$

Therefore, the given series ***converges*** by *comparison Test*.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{3n+1}{n^3 + 1}$

Solution

$$\frac{3n+1}{n^3+1} \xrightarrow[n\to\infty]{} \frac{3}{n^2}$$

$$\begin{aligned}\frac{3n+1}{n^3+1} &= \frac{3n}{n^3+1} + \frac{1}{n^3+1} \\ &< \frac{3n}{n^3} + \frac{1}{n^3} \\ &< \frac{3}{n^2} + \frac{1}{n^2} \\ &= \frac{4}{n^2}\end{aligned}$$

Therefore, by *Comparison Test,* the given series ***converges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

Solution

$< \ln n < n$

$\frac{1}{\ln n} > \frac{1}{n}$

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ ***diverges*** to infinity (it is a harmonic series),

Therefore; by *Comparison Test* the series $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ ***diverges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

Solution

$2n - 1 < 2n$

$\frac{1}{2n-1} > \frac{1}{2n} \qquad \textit{for } n \geq 1$

$\sum_{n=1}^{\infty} \frac{1}{2n-1} > \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$

By the ***p****-series* the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $p = 1$

Therefore; by *Comparison Test,* the given series ***diverges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{3n^2 + 2}$

Solution

$3n^2 + 2 > 3n^2 \quad \Rightarrow \quad \frac{1}{3n^2+2} < \frac{1}{3n^2}$

By the ***p**-series* the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $p > 1$

Therefore; by *Comparison Test,* the given series ***converges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

Solution

$$\sqrt{n} - 1 < \sqrt{n}$$

$$\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}} \qquad \text{for } n \ge 2$$

By the ***p**-series* the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges $p = \frac{1}{2} < 1$

Therefore; by *Comparison Test,* the given series ***diverges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=0}^{\infty} \frac{4^n}{5^n + 3}$

Solution

$$\frac{4^n}{5^n + 3} < \frac{4^n}{5^n} = \left(\frac{4}{5}\right)^n$$

By the geometric series: $r = \frac{4}{5} < 1$ converges

Therefore; by *Comparison Test,* the given series ***converges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=2}^{\infty} \frac{\ln n}{n+1}$

Solution

$$\frac{\ln n}{n+1} > \frac{1}{n+1} \qquad \text{(and by integral test)}$$

The given series ***converges*** by *Comparison Test* with the divergent series.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3+1}}$

Solution

$$\frac{1}{\sqrt{n^3+1}}<\frac{1}{\sqrt{n^3}}$$

By the ***p****-series* the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges $p=\frac{3}{2}>1$

Therefore; by *Comparison Test,* the given series ***converges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}$

Solution

$$\frac{1}{n^2}>\frac{1}{n!}\quad \text{For } n>3$$

By the ***p****-series* the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges $p=2>1$

Therefore; by *Comparison Test,* the given series ***converges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{4\sqrt[3]{n}-1}$

Solution

$$\frac{1}{4\sqrt[3]{n}-1}>\frac{1}{4\sqrt[3]{n}}$$

By the ***p****-series* the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{1/3}}$ diverges $p=\frac{1}{3}<1$

Therefore; by *Comparison Test,* the given series ***diverges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=0}^{\infty} e^{-n^2}$

Solution

$$\frac{1}{e^{n^2}} \le \frac{1}{e^n}$$

Geometric series: $r = \frac{1}{e} < 1$ converges

Therefore; by *Comparison Test,* the given series ***converges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{3^n}{2^n - 1}$

Solution

$$\frac{3^n}{2^n - 1} > \left(\frac{3}{2}\right)^n$$

Geometric series: $r = \frac{3}{2} > 1$ *diverges*

Therefore; by *Comparison Test,* the given series ***diverges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n-2}{n^3 - n^2 + 3}$

Solution

Comparing with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent *p*-series, since $p = 2 > 1$.

$$a_n = \frac{n-2}{n^3 - n^2 + 3} \quad \Rightarrow \quad b_n = \frac{1}{n^2}$$

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n-2}{n^3 - n^2 + 3} \cdot \frac{n^2}{1} \\ &= \lim_{n\to\infty} \frac{n^3 - 2n^2}{n^3 - n^2 + 3} \qquad \text{or } \textbf{\textit{L'Hopital Rule}} \\ &= 1 > 0 \end{aligned}$$

Therefore; by *Limit Comparison Test,* the given series ***converges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=2}^{\infty}\frac{n(n+1)}{(n^2+1)(n-1)}$

Solution

Comparing with $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n}$, which is a divergent p-series, since $p=1\le 1$.

$$a_n=\frac{n(n+1)}{(n^2+1)(n-1)} \quad\Rightarrow\quad b_n=\frac{n^2}{n^3}=\frac{1}{n}$$

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{n(n+1)}{(n^2+1)(n-1)}\cdot\frac{n}{1}\\ &= \lim_{n\to\infty}\frac{n^3+n^2}{n^3-n^2+n-1}\\ &= \underline{1>0}\,\big|\end{aligned}$$

Therefore; by *Limit Comparison Test,* the given series ***diverges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{3+4^n}$

Solution

Comparing with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n}$, which is a convergent geometric, since $|r|=\frac{1}{2}<1$.

$$a_n=\frac{2^n}{3+4^n} \quad\Rightarrow\quad b_n=\frac{1}{2^n}$$

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{2^n}{3+4^n}\cdot\frac{2^n}{1}\\ &= \lim_{n\to\infty}\frac{2^{2n}}{3+4^n}\\ &= \lim_{n\to\infty}\frac{4^n}{3+4^n}\\ &= \lim_{n\to\infty}\frac{4^n\ln 4}{4^n\ln 4}\end{aligned}$$

$= 1 > 0$

Therefore; by *Limit Comparison Test,* the given series ***converges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}\,4^n}$

Solution

Comparing with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series, since $p = \frac{1}{2} < 1$.

$$a_n = \frac{5^n}{\sqrt{n}\,4^n} \quad \Rightarrow \quad b_n = \frac{1}{\sqrt{n}}$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{5^n}{\sqrt{n}\,4^n} \cdot \frac{\sqrt{n}}{1} \\
&= \lim_{n\to\infty} \frac{5^n}{4^n} \\
&= \lim_{n\to\infty} \left(\frac{5}{4}\right)^n \\
&= \infty
\end{aligned}$$

Therefore; by *Limit Comparison Test,* the given series ***diverges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4}\right)^n$

Solution

Comparing with $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$, which is a convergent geometric, since $|r| = \frac{2}{5} < 1$.

$$a_n = \left(\frac{2n+3}{5n+4}\right)^n \quad \Rightarrow \quad b_n = \left(\frac{2}{5}\right)^n$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \left(\frac{2n+3}{5n+4}\right)^n \cdot \left(\frac{5}{2}\right)^n \\
&= \lim_{n\to\infty} \left(\frac{10n+15}{10n+8}\right)^n \\
&= \lim_{n\to\infty} \left(\frac{10n}{10n}\right)^n
\end{aligned}$$

$$= \lim_{n\to\infty} 1^n$$

$$\underline{= 1 > 0}$$

Therefore; by *Limit Comparison Test,* the given series ***converges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

Solution

Comparing with $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$, which is a divergent p-series, since $p = 1 \le 1$.

$$a_n = \frac{1}{\ln n} \quad \Rightarrow \quad b_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\ln n} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{n}{\ln n}$$

$$= \lim_{n\to\infty} \frac{1}{1/n}$$

$$= \lim_{n\to\infty} n$$

$$\underline{= \infty}$$

Therefore; by *Limit Comparison Test,* the given series ***diverges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$

Solution

Let $b_n = \dfrac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{1}{1+\sqrt{n}} \Big/ \frac{1}{\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}}$$

$$\underline{= 1}$$

Since the ***p****-series* diverges to infinity $\left(p = \frac{1}{2}\right)$

Therefore; by *Limit Comparison Test,* the given series ***diverges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n+5}{n^3-2n+3}$

Solution

Let $b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{n+5}{n^3-2n+3} \Big/ \frac{1}{n^2} = \lim_{n\to\infty} \frac{n^3+5n^2}{n^3-2n+3}$$

$$\underline{=1<\infty}$$

Since the ***p**-series* converges $(p=2)$

Therefore; by *Limit Comparison Test,* the given series ***converges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

Solution

$b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{n}{n^2+1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^2}{n^2+1}$$

$$\underline{=1<\infty}$$

Since the ***p**-series* diverges to infinity $(p=1)$

Therefore; by *Limit Comparison Test,* the given series ***diverges***.

Exercise

Use the Limit Comparison Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{5}{4^n+1}$

Solution

$b_n = \frac{1}{4^n} = \left(\frac{1}{4}\right)^n$

$$\lim_{n\to\infty} \frac{5}{4^n+1} \cdot \frac{4^n}{1} = 5$$

By geometric series $\left(r = \frac{1}{4} < 1\right)$ converges

Therefore; by *Limit Comparison Test,* the given series ***converges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$

Solution

$$b_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} \cdot \frac{n}{1} = 1 < \infty$$

Since the ***p***-*series* diverges to infinity $(p = 1)$

Therefore; by *Limit Comparison Test,* the given series ***diverges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{2^n+1}{5^n+1}$

Solution

$$b_n = \left(\frac{2}{5}\right)^n$$

$$\lim_{n\to\infty} \frac{2^n+1}{5^n+1} \cdot \frac{5^n}{2^n} = 1$$

By geometric series $\left(r = \frac{2}{5} < 1\right)$ converges

Therefore; by *Limit Comparison Test,* the given series ***converges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{2n^2-1}{3n^5+2n+1}$

Solution

Let $b_n = \frac{1}{n^3}$

$$\lim_{n\to\infty} \frac{2n^2-1}{3n^5+2n+1} \Big/ \frac{1}{n^3} = \lim_{n\to\infty} \frac{2n^5-n^3}{3n^5+2n+1}$$

$$= \frac{2}{3} < \infty$$

Since the ***p***-*series* converges $(p = 3)$

Therefore; by *Limit Comparison Test,* the given series ***converges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{n^2(n+3)}$

Solution

Let $b_n = \frac{1}{n^3}$ By the ***p**-series* converges $(p=3)$

$$\lim_{n\to\infty} \frac{1}{n^2(n+3)} \bigg/ \frac{1}{n^3} = \lim_{n\to\infty} \frac{n^3}{n^2(n+3)}$$

$$\underline{= 1 < \infty}$$

Therefore; by *Limit Comparison Test,* the given series ***converges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$

Solution

Let $b_n = \frac{1}{n^2}$ By the ***p**-series* converges $(p=2)$,

$$\lim_{n\to\infty} \frac{1}{n\sqrt{n^2+1}} \cdot \frac{n^2}{1} \quad \underline{= 1 < \infty}$$

Therefore; by *Limit Comparison Test,* the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n} + \sqrt[3]{n}}$

Solution

Comparing with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series, since $p = \frac{1}{2} < 1$.

$$a_n = \frac{1}{2\sqrt{n} + \sqrt[3]{n}} \quad \Rightarrow \quad b_n = \frac{1}{\sqrt{n}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{2\sqrt{n} + \sqrt[3]{n}} \cdot \frac{\sqrt{n}}{1}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{2\sqrt{n} + \sqrt[3]{n}}$$

$$= \lim_{n\to\infty} \frac{1}{2+n^{1/3-1/2}}$$

$$= \lim_{n\to\infty} \frac{1}{2+n^{-1/6}}$$

$$= \frac{1}{2} > 0$$

Then, by Comparison Test, the given series ***diverges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$

Solution

Comparing with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$, which is a convergent geometric, since $|r| = \frac{1}{2} < 1$.

By the Direct Comparison Test: $\dfrac{\sin^2 n}{2^n} \le \dfrac{1}{2^n}$

The given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$

Solution

Comparing with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which is a convergent p-series, since $p = \frac{3}{2} > 1$.

$$a_n = \frac{n+1}{n^2\sqrt{n}} \quad \Rightarrow \quad b_n = \frac{1}{n^{3/2}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+1}{n^2\sqrt{n}} \cdot \frac{n^{3/2}}{1}$$

$$= \lim_{n\to\infty} \frac{n+1}{n}$$

$$= 1 > 0$$

Then, by Comparison Test, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$

Solution

Comparing with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series, since $p = 2 > 1$.

$$a_n = \frac{10n+1}{n(n+1)(n+2)} \quad \Rightarrow \quad b_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{10n+1}{n(n+1)(n+2)} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{10n^3 + n^2}{n\left(n^2 + 3n + 2\right)}$$

$$= \lim_{n\to\infty} \frac{10n^3 + n^2}{n^3 + 3n^2 + 2}$$

$$= 10 > 0$$

Then, by Comparison Test, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$

Solution

By the Direct Comparison Test: $\left(\frac{n}{3n+1}\right)^n < \left(\frac{n}{3n}\right)^n = \left(\frac{1}{3}\right)^n$

$\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$, which is a convergent geometric, since $|r| = \frac{1}{3} < 1$.

Therefore, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$

Solution

Comparing with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent *p*-series, since $p = 2 > 1$.

$$a_n = \frac{(\ln n)^2}{n^3} \quad \Rightarrow \quad b_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(\ln n)^2}{n^3} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{(\ln n)^2}{n}$$

$$= \lim_{n\to\infty} \frac{2\ln n\left(\frac{1}{n}\right)}{1}$$

$$= 2 \lim_{n\to\infty} \frac{\ln n}{n}$$

$$\underline{= 0 \,|}$$

Then, by Comparison Test, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=1}^{\infty} \frac{1+\sin n}{n^2}$

Solution

Let $b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{1+\sin n}{n^2} \Big/ \frac{1}{n^2} = \lim_{n\to\infty} (1+\sin n) \text{ which does not exist.}$$

$\frac{1+\sin n}{n^2} \le \frac{2}{n^2}$, then the given series converges by comparison test

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

Solution

$$a_n = \frac{1}{2+3^n} < \frac{1}{3^n} = b_n$$

So, by the Direct Comparison Test, the series converges.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$

Solution

$$2+\sqrt{n} < n \quad \Rightarrow \quad \frac{1}{2+\sqrt{n}} \ge \frac{1}{n}$$

By the ***p***-*series* the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $p=1$

The given series ***diverges*** by Comparison Test using ***p***-*series.*

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{1}{an+b}$

Solution

$$an+b < n \quad \Rightarrow \quad \frac{1}{an+b} \ge \frac{1}{n} \qquad a,\ b > 0$$

By the ***p***-*series* the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $p=1$

The given series ***diverges*** by Comparison Test using ***p***-*series.*

$$\lim_{n\to\infty} \frac{1}{an+b} \cdot \frac{n}{1} = \frac{1}{a} > 0$$

The given series ***diverges*** by Limit Comparison Test

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

Solution

Let $b_n = \dfrac{1}{n^{3/2}}$ By the ***p***-*series* converges $\left(p = \frac{3}{2} > 1\right)$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n^2+1} \cdot \frac{n^{3/2}}{1} = 1$$

Therefore, the given series ***converges*** by the limit comparison test.

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$

Solution

$$\frac{\sqrt[3]{n}}{n} = \frac{1}{n^{2/3}}$$

By the ***p****-series* the series $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges $p = \frac{2}{3} < 1$

The given series ***diverges*** by comparison test using ***p****-series.*

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=0}^{\infty} 5\left(-\frac{4}{3}\right)^n$

Solution

$$|r| = \frac{4}{3} > 1$$

The given series ***diverges*** by Geometric series

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=1}^{\infty} \frac{1}{5^n + 1}$

Solution

$$\frac{1}{5^n + 1} < \left(\frac{1}{5}\right)^n$$

The given series converges by a Direct Comparison with the convergent geometric series $\left(r = \frac{1}{5} < 1\right)$

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=2}^{\infty} \frac{1}{n^3 - 8}$

Solution

Let $b_n = \frac{1}{n^3}$ By the ***p****-series* converges $(p = 3)$

$$\lim_{n\to\infty} \frac{1}{n^3-8} \cdot \frac{n^3}{1} = 1$$

Therefore, the given series ***converges*** by the limit comparison test with ***p***-*series*.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{3n-2}$

Solution

$$\lim_{n\to\infty} \frac{2n}{3n-2} = \frac{2}{3} \neq 0$$

The given series ***diverges*** by the Limit.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

Solution

$$\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots$$

$$= \frac{1}{2}$$

The given series ***converges*** by telescoping series.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\left(n^2+1\right)^2}$

Solution

$$\int_1^{\infty} \frac{x}{\left(x^2+1\right)^2} dx = \frac{1}{2} \int_1^{\infty} \frac{1}{\left(x^2+1\right)^2} d\left(x^2+1\right)$$

$$= -\frac{1}{2} \frac{1}{x^2+1} \Big|_1^{\infty}$$

$$= -\frac{1}{2}\left(0 - \frac{1}{2}\right)$$

$$=\boxed{\frac{1}{4}}$$

The given series ***converges*** by the *Integral Test*.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty}\frac{n2^n}{4n^3+1}$

Solution

$$b_n=\frac{n^2}{2^n}$$

$$a_n=\frac{n2^n}{4n^3+1}$$

$$\lim_{k\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n2^n}{4n^3+1}\cdot\frac{n^2}{2^n}$$

$$=\boxed{\frac{1}{4}}$$

Therefore; by the *Limit Comparison Test*, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{k=1}^{\infty}\frac{|\sin k|}{k^2}$

Solution

$$|\sin k|\le 1$$

$$\frac{|\sin k|}{k^2}\le\frac{1}{k^2}$$

$$\sum_{k=1}^{\infty}\frac{1}{k^2}\quad\text{converges by } \textbf{\textit{p-series}}\ (p=2\ >1)$$

Therefore; by the *Comparison Test*, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{k=1}^{\infty}\frac{\sin^2 k}{k^2}$

Solution

$$0 \le \sin^2 k \le 1$$

$$0 \le \frac{\sin^2 k}{k^2} \le \frac{1}{k^2}$$

$\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by ***p**-series* $(p = 2\ > 1)$

Therefore; by the *Comparison Test*, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\sum_{k=1}^{\infty} \sin^2 \frac{1}{k}$

Solution

Let $b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by ***p**-series* $(p = 2\ > 1)$

$$a_k = \sin^2 \frac{1}{k}$$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\sin^2 \frac{1}{k}}{\frac{1}{k^2}} = \frac{0}{0}$$

$$= \lim_{k\to\infty} \left(\frac{\sin \frac{1}{k}}{\frac{1}{k}} \right)^2 \qquad \text{Let } x = \frac{1}{k} \ \to 0$$

$$= \lim_{x\to 0} \left(\frac{\sin x}{x} \right)^2$$

$$= 1$$

Therefore; by the *Limit Comparison Test*, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\sum_{k=1}^{\infty} \sin \frac{1}{k}$

Solution

Let $b_k = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges by ***p**-series* $(p = 1\ \le 1)$

$a_k = \sin\frac{1}{k}$

$$\lim_{k\to\infty}\frac{a_k}{b_k} = \lim_{k\to\infty}\frac{\sin\frac{1}{k}}{\frac{1}{k}} = \frac{0}{0} \qquad \text{Let } x = \frac{1}{k} \to 0$$

$$= \lim_{x\to 0}\frac{\sin x}{x}$$

$$= 1$$

Therefore; by the *Limit Comparison Test*, the given series ***diverges***.

Exercise

Use any method to determine if the series converges or diverges $\sum_{k=1}^{\infty}\frac{1}{k}\sin\frac{1}{k}$

Solution

Let $b_k = \sum_{k=1}^{\infty}\frac{1}{k^2}$ converges by ***p**-series* $(p = 2 > 1)$

$a_k = \frac{1}{k}\sin\frac{1}{k}$

$$\lim_{k\to\infty}\frac{a_k}{b_k} = \lim_{k\to\infty}\frac{\frac{1}{k}\sin\frac{1}{k}}{\frac{1}{k^2}}$$

$$= \lim_{k\to\infty}\frac{\sin\frac{1}{k}}{\frac{1}{k}} \qquad \text{Let } x = \frac{1}{k} \to 0$$

$$= \lim_{x\to 0}\frac{\sin x}{x}$$

$$= 1$$

Therefore; by the *Limit Comparison Test*, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\sum_{k=1}^{\infty}\frac{1}{k^2}\sin\frac{1}{k}$

Solution

$-1 \le \sin\frac{1}{k} \le 1$

$$-\frac{1}{k^2} \le \frac{\sin\frac{1}{k}}{k^2} \le \frac{1}{k^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \quad \text{converges by } \boldsymbol{p}\textit{-series } (p = 2 > 1)$$

Therefore; by the *Comparison Test*, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{k=1}^{\infty} (-1)^k k \sin\frac{1}{k}$

Solution

$$-1 \le \sin\frac{1}{k} \le 1$$

$$-k \le k\sin\frac{1}{k} \le k$$

$$\lim_{k\to\infty} k = \infty$$

Therefore; by the *Comparison Test*, the given series ***diverges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}\sin\frac{\pi}{2k}$

Solution

$$-1 \le \sin\frac{\pi}{2k} \le 1$$

$$-\frac{1}{k^2} \le \frac{1}{k^2}\sin\frac{\pi}{2k} \le \frac{1}{k^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \quad \text{converges by } \boldsymbol{p}\textit{-series } (p = 2 > 1)$$

Therefore; by the *Comparison Test*, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{k=1}^{\infty} \tan\frac{1}{k}$

Solution

Let $b_k = \displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges by ***p**-series* $(p = 1 \le 1)$

$a_k = \tan\dfrac{1}{k}$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\tan\frac{1}{k}}{\frac{1}{k}}$$

$$= \lim_{x\to 0} \frac{\tan x}{x} \qquad \text{Let } x = \frac{1}{k} \to 0$$

$$= 1$$

Therefore; by the *Limit Comparison Test*, the given series ***diverges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{k=1}^{\infty} (-1)^k \tan^{-1} k$

Solution

$$\lim_{k\to\infty} \tan^{-1} k = \tan^{-1} \infty$$

$$= \frac{\pi}{2} \ne 0$$

Therefore; by the *Divergence Test*, the given series ***diverges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^3}$

Solution

$|\cos n| \le 1$

$\dfrac{|\cos n|}{n^3} \le \dfrac{1}{n^3}$

$\displaystyle\sum \frac{1}{n^3}$ converges by ***p**-series* $(p = 3 > 1)$

Therefore; by the *Comparison Test*, the given series ***converges*** (absolutely)

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{k=2}^{\infty} \frac{k}{\ln k}$

Solution

$$\lim_{k\to\infty} \frac{k}{\ln k} = \frac{\infty}{\infty}$$

$$= \lim_{k\to\infty} \frac{1}{\frac{1}{k}}$$

$$= \lim_{k\to\infty} k$$

$$\underline{= \infty}$$

Therefore; by the *Divergence Test*, the given series ***diverges***.

Exercise

Use any method to determine if the series converges or diverges $\dfrac{1}{1+\sqrt{1}} + \dfrac{1}{1+\sqrt{2}} + \dfrac{1}{1+\sqrt{3}} + \cdots$

Solution

$$\frac{1}{1+\sqrt{1}} + \frac{1}{1+\sqrt{2}} + \frac{1}{1+\sqrt{3}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$$

$$\sqrt{n} < n$$

$$1+\sqrt{n} < n$$

$$\frac{1}{1+\sqrt{n}} > \frac{1}{n}$$

$\sum \frac{1}{n}$ diverges by ***p**-series* $(p = 1 \ \le 1)$

Therefore; by the *Comparison Test*, the given series ***diverges***.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n!}$

Solution

$$\lim_{n\to\infty}\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{n!}{2^n}} = \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}$$

$$= \lim_{n\to\infty}\frac{2}{(n+1)}$$

$$\underline{= 0 < 1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\frac{2^{n+1}}{n3^{n-1}}$

Solution

$$\lim_{n\to\infty}\frac{2^{n+2}}{(n+1)3^n}\cdot\frac{n3^{n-1}}{2^{n+1}} = \lim_{n\to\infty}\frac{2n}{3(n+1)}$$

$$= \lim_{n\to\infty}\frac{2n}{3n}$$

$$\underline{= \frac{2}{3} < 1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=2}^{\infty}\frac{3^{n+2}}{\ln n}$

Solution

$$\lim_{n\to\infty}\frac{3^{n+3}}{\ln(n+1)}\cdot\frac{\ln n}{3^{n+2}} = \lim_{n\to\infty}\frac{3\ln n}{\ln(n+1)}$$

$$= 3\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)}$$

$$= 3 \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}}$$

$$= 3 \lim_{n\to\infty} \frac{n+1}{n}$$

$$\boxed{= 3 > 1}$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2(n+2)!}{n!3^{2n}}$

Solution

$$\lim_{n\to\infty} \frac{(n+1)^2(n+3)!}{(n+1)!\ 3^{2(n+1)}} \cdot \frac{n!\ 3^{2n}}{n^2(n+2)!} = \lim_{n\to\infty} \frac{(n+1)^2(n+3)}{n^2(n+1)\ \cdot 3^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)(n+3)}{n^2\ \cdot 3^2}$$

$$= \lim_{n\to\infty} \frac{n^2+4n+3}{9n^2}$$

$$\boxed{= \frac{1}{9} < 1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{n5^n}{(2n+3)\ln(n+1)}$

Solution

$$\lim_{n\to\infty} \frac{(n+1)\cdot 5^{n+1}}{(2n+5)\ln(n+2)} \cdot \frac{(2n+3)\ln(n+1)}{n\cdot 5^n} = \lim_{n\to\infty} \frac{5(n+1)(2n+3)\ln(n+1)}{n(2n+5)\ln(n+2)}$$

$$= \lim_{n\to\infty} \frac{10n^2+10n+6}{2n^2+5n} \cdot \lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n+2)}$$

$$= 5 \cdot \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n+2}}$$

$$= 5 \cdot \lim_{n \to \infty} \frac{n+2}{n+1}$$

$$\underline{= 5 > 1}$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{99^n}{n!}$

Solution

$$\rho = \lim_{n \to \infty} \frac{99^{n+1}}{(n+1)!} \Big/ \frac{99^n}{n!} \qquad \rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n \to \infty} \frac{99}{n+1}$$

$$\underline{= 0 < 1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{2^n}$

Solution

$$\rho = \lim_{n \to \infty} \frac{(n+1)^5}{2^{n+1}} \Big/ \frac{n^5}{2^n} \qquad \rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n \to \infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^5$$

$$\underline{= \frac{1}{2} < 1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \Big/ \frac{n!}{n^n} \qquad\qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$= \frac{1}{e} < 1$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(2(n+1))!}{((n+1)!)^2} \Big/ \frac{(2n)!}{(n!)^2} \qquad\qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{(2(n+1))!}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{4n^2}{n^2}$$

$$= 4 > 1$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{1}{5^{n+1}} \cdot \frac{5^n}{1} \qquad \rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \frac{1}{5} < 1$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{n!}$

Solution

$$\rho = \lim_{n\to\infty} \frac{1}{(n+1)!} \cdot \frac{n!}{1} \qquad \rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \frac{1}{n+1}$$

$$= 0 < 1$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\sum_{n=0}^{\infty} \frac{n!}{3^n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!} \qquad \rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \frac{n+1}{3}$$

$$= \infty$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

Solution

$$\rho = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \qquad \rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \frac{2}{n+1}$$

$$\boxed{= 0 < 1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{6}{5}\right)^n$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)\left(\frac{6}{5}\right)^{n+1}}{n\left(\frac{6}{5}\right)^n} \qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{n+1}{n}\left(\frac{6}{5}\right)$$

$$\boxed{= \frac{6}{5} > 1}$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{7}{8}\right)^n$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)\left(\frac{7}{8}\right)^{n+1}}{n\left(\frac{7}{8}\right)^n} \qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{n+1}{n}\left(\frac{7}{8}\right)$$

$$\boxed{= \frac{7}{8} < 1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n}{4^n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{n+1}{4^{n+1}} \cdot \frac{4^n}{n} \qquad\qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{n+1}{4n}$$

$$= \frac{1}{4} < 1$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{5^n}{n^4}$

Solution

$$\rho = \lim_{n\to\infty} \frac{5^{n+1}}{(n+1)^4} \cdot \frac{n^4}{5^n} \qquad\qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} 5\left(\frac{n}{n+1}\right)^4$$

$$= 5 > 1$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} \qquad\qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{1}{3}\left(\frac{n+1}{n}\right)^3$$

$$= \frac{1}{3} < 1$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n+2)}{n(n+1)}$

Solution

$$\rho = \lim_{n\to\infty} \frac{n+3}{(n+1)(n+2)}\cdot\frac{n(n+1)}{(n+2)} \qquad\qquad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty}\frac{n(n+3)}{(n+2)^2}$$

$$=1$$

$$\lim_{n\to\infty}\left|a_n\right| = \lim_{n\to\infty}\frac{(n+2)}{n(n+1)}$$

$$=0$$

Therefore; the given series ***converges*** ***Conditionally*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{n!}$

Solution

$$\rho = \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n} \qquad\qquad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty}\frac{2}{n+1}$$

$$=0<1$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\left(\frac{3}{2}\right)^n}{n^2}$

Solution

$$\rho = \lim_{n\to\infty}\frac{\left(\frac{3}{2}\right)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{\left(\frac{3}{2}\right)^n} \qquad\qquad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty}\frac{3}{2}\left(\frac{n}{n+1}\right)^2$$

$$= \boxed{\frac{3}{2} > 1}$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n3^n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)3^{n+1}} \cdot \frac{n3^n}{n!} \qquad\qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{n}{3}$$

$$= \boxed{\infty}$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(2n+2)!}{(n+1)^5} \cdot \frac{n^5}{(2n)!} \qquad\qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} (2n+1)(2n+2)\left(\frac{n}{n+1}\right)^5$$

$$= \boxed{\infty}$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\frac{4^n}{(3n)^n}} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{4}{3n}\right)^n}$$

$$= \lim_{n\to\infty} \frac{4}{3n}$$

$$\boxed{= 0 < 1}$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5}\right)^n$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{4n+3}{3n-5}\right)^n} = \lim_{n\to\infty} \left(\frac{4n+3}{3n-5}\right)$$

$$\boxed{= \frac{4}{3} > 1}$$

Therefore; the given series ***diverges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \left(\ln\left(e^2 + \frac{1}{n}\right)\right)^{n+1}$

Solution

$$\lim_{n\to\infty} \sum_{n=1}^{\infty} \sqrt[n]{\left(\ln\left(e^2 + \frac{1}{n}\right)\right)^{n+1}} = \lim_{n\to\infty} \sum_{n=1}^{\infty} \left(\ln\left(e^2 + \frac{1}{n}\right)\right)^{1+\frac{1}{n}}$$

$$= \ln\left(e^2\right)$$

$$\boxed{= 2 > 1}$$

Therefore; the given series ***diverges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\sin^n\left(\frac{1}{\sqrt{n}}\right)} = \lim_{n\to\infty} \sin\left(\frac{1}{\sqrt{n}}\right)$$

$$= \sin(0)$$

$$\underline{= 0 < 1}$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^{n^2}$

Solution

$$\lim_{n\to\infty}\sqrt[n]{\left(1-\frac{1}{n}\right)^{n^2}} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n}$$

$$\underline{= e^{-1} < 1}$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\frac{e^{2n}}{n^n}$

Solution

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{e^{2n}}{n^n}\right|} = \lim_{n\to\infty}\frac{e^2}{n}$$

$$\underline{= 0 < 1}$$

Therefore; the given series ***converges absolutely*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{5^n}$

Solution

$$\lim_{n\to\infty}\sqrt[n]{\frac{1}{5^n}} \underline{= \frac{1}{5} < 1}$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{n^n}$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n^n}\right|} = \lim_{n\to\infty} \frac{1}{n}$$

$$\underline{= 0 < 1}$$

Therefore; the given series ***converges*** *absolutely* by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n}{2n+1}\right)^n\right|} = \lim_{n\to\infty} \frac{n}{2n+1}$$

$$\underline{= \frac{1}{2} < 1}$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{2n}{n+1}\right)^n\right|} = \lim_{n\to\infty} \frac{2n}{n+1}$$

$$\underline{= 2 > 1}$$

Therefore; the given series ***diverges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \left(\frac{3n+2}{n+3}\right)^n$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{3n+2}{n+3}\right)^n\right|} = \lim_{n\to\infty} \frac{3n+2}{n+3}$$

$$= 3 > 1$$

Therefore; the given series ***diverges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n-2}{5n+1}\right)^n$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n-2}{5n+1}\right)^n\right|} = \lim_{n\to\infty} \left|\frac{n-2}{5n+1}\right|$$

$$= \frac{1}{5} < 1$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{(\ln n)^n}$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n}{(\ln n)^n}\right|} = \lim_{n\to\infty} \left|\frac{1}{\ln n}\right|$$

$$= 0 < 1$$

Therefore; the given series ***converges*** by the Root Test.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{-3n}{2n+1}\right)^{3n}$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{-3n}{2n+1}\right)^{3n}\right|} = \lim_{n\to\infty} \left|\left(\frac{-3n}{2n+1}\right)^3\right|$$

$$= \left(\frac{3}{2}\right)^3$$

$$\boxed{=\frac{27}{8}>1}$$

Therefore; the given series ***diverges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\left(2\sqrt[n]{n}+1\right)^n$

Solution

$$\lim_{n\to\infty}\sqrt[n]{\left|\left(2\sqrt[n]{n}+1\right)^n\right|}=\lim_{n\to\infty}\left|\left(2\sqrt[n]{n}+1\right)\right|$$

$$\text{Let } x=\lim_{n\to\infty}\sqrt[n]{n} \quad\Rightarrow\quad \ln x=\lim_{n\to\infty}\ln\sqrt[n]{n}$$

$$\begin{aligned}\ln x&=\lim_{n\to\infty}\frac{\ln n}{n}\\&=\lim_{n\to\infty}\frac{1/n}{1}\\&=0\end{aligned}$$

$$x=e^0=1=\lim_{n\to\infty}\sqrt[n]{n}$$

$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{\left|\left(2\sqrt[n]{n}+1\right)^n\right|}&=2(1)+1\\&=3>1\end{aligned}$$

Therefore; the given series ***diverges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=0}^{\infty}e^{-3n}$

Solution

$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{\left|e^{-3n}\right|}&=\lim_{n\to\infty}\left|e^{-3}\right|\\&=\frac{1}{e^3}<1\end{aligned}$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n}{3^n}$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n}{3^n}\right|} = \lim_{n\to\infty} \left|\frac{\sqrt[n]{n}}{3}\right|$$

Let $x = \sqrt[n]{n} \quad \Rightarrow \quad \ln x = \ln \sqrt[n]{n} = \frac{\ln}{n}$

$$\lim_{n\to\infty} \ln x = \lim_{n\to\infty} \frac{\ln}{n}$$

$$= \lim_{n\to\infty} \frac{1/n}{1}$$

$$= 0$$

$$x = e^0 = 1 = \lim_{n\to\infty} \sqrt[n]{n}$$

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n}{3^n}\right|} = \frac{1}{3} < 1$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \left(\frac{n}{500}\right)^n$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n}{500}\right)^n\right|} = \lim_{n\to\infty} \left|\frac{n}{500}\right|$$

$$= \infty > 1$$

Therefore; the given series ***diverges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{1}{n}-\frac{1}{n^2}\right)^n\right|} = \lim_{n\to\infty} \left|\frac{1}{n}-\frac{1}{n^2}\right|$$

$$\underline{=0<1}$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\ln n}{n}\right)^n$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{\ln n}{n}\right)^n\right|} = \lim_{n\to\infty} \left|\frac{\ln n}{n}\right|$$

$$\underline{=0<1}$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=2}^{\infty}\frac{n}{(\ln n)^n}$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n}{(\ln n)^n}\right|} = \lim_{n\to\infty} \left|\frac{n^{1/n}}{\ln n}\right|$$

$$\underline{=0<1}$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\frac{n^{\sqrt{2}}}{2^n}$

Solution

$$\lim_{n\to\infty} \frac{(n+1)^{\sqrt{2}}}{2^{n+1}} \cdot \frac{2^n}{n^{\sqrt{2}}} = \lim_{n\to\infty} \frac{1}{2} \cdot \frac{(n+1)^{\sqrt{2}}}{n^{\sqrt{2}}}$$

$$= \frac{1}{2} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{\sqrt{2}}$$

$$= \frac{1}{2} < 1$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)^2}{e^{n+1}} \cdot \frac{e^n}{n^2} \qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{1}{e} \cdot \left(\frac{n+1}{n}\right)^2$$

$$= \frac{1}{e} < 1$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} \qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{n+1}{10}$$

$$= \infty > 1$$

Therefore; the given series ***converges*** by the Ratio Test.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\frac{(\ln n)^n}{n^n}} = \lim_{n\to\infty} \frac{\ln n}{n} \qquad \textit{Hopital Rule}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1}$$

$$= \lim_{n\to\infty} \frac{1}{n}$$

$$\boxed{= 0 < 1}$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{n2^n (n+1)!}{3^n n!}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)2^{n+1}(n+2)!}{3^{n+1}(n+1)!} \cdot \frac{3^n n!}{n2^n (n+1)!} \qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{2}{3} \frac{(n+1)(n+2)}{n(n+1)}$$

$$= \lim_{n\to\infty} \frac{2}{3}\left(\frac{n+2}{n}\right)$$

$$\boxed{= \frac{2}{3} < 1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

Solution

$$\rho = \lim_{n\to\infty} \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} \qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n+2)}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{2(2n+1)(n+1)}$$

$$= \lim_{n\to\infty} \frac{n+1}{4n+2}$$

$= \frac{1}{4} < 1$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

Solution

Using comparison method:

$$\frac{1}{n} = \frac{n^2}{n^3} < \frac{n^2+1}{n^3+1}$$

Since $\dfrac{n^2+1}{n^3+1} > \dfrac{1}{n}$

Therefore; the given series ***diverges*** by the *Comparison Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \left|\sin\frac{1}{n^2}\right|$

Solution

For $x \ge 0 \quad \Rightarrow \quad \sin x \le x$

$$\left|\sin\frac{1}{n^2}\right| = \sin\frac{1}{n^2} \le \frac{1}{n^2}$$

Therefore; the given series ***converges*** by *Comparison Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=8}^{\infty} \frac{1}{\pi^n+5}$

Solution

The given series converges by comparison with $\displaystyle\sum_{n=8}^{\infty} \left(\frac{1}{\pi}\right)^n$

Since $0 < \dfrac{1}{\pi^n+5} < \dfrac{1}{\pi^n}$

Therefore; the given series ***converges*** by *Comparison Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{(\ln n)^3}$

Solution

Since $(\ln n)^3 < n \quad \Rightarrow \quad \dfrac{1}{(\ln n)^3} > \dfrac{1}{n}$

Therefore; the given series ***diverges*** by *Comparison Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\pi^n - n^\pi}$

Solution

$$a_n = \frac{1}{\pi^n - n^\pi} \quad \Rightarrow \quad b_n = \frac{1}{\pi^n} = \left(\frac{1}{\pi}\right)^n$$

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{\pi^n}$ converges geometric since $|r| = \dfrac{1}{\pi} < 1$

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{1}{\pi^n - n^\pi}\cdot\frac{1}{\frac{1}{\pi^n}}\\ &= \lim_{n\to\infty}\frac{1}{1-\frac{n^\pi}{\pi^n}}\\ &= 1\end{aligned}$$

Therefore; the given series ***converges*** by *Comparison Test* with geometric series

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=0}^{\infty}\frac{1+n}{2+n}$

Solution

$$\lim_{n\to\infty}\frac{1+n}{2+n} = 1 > 0$$

Therefore; the given series ***diverges*** by the divergence series.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1+n^{4/3}}{2+n^{5/3}}$

Solution

Let $b_n = \frac{1}{n^{1/3}}$

$$\lim_{x\to\infty} \frac{1+n^{4/3}}{2+n^{5/3}} \Big/ \frac{1}{n^{1/3}} = \lim_{x\to\infty} \frac{n^{1/3}+n^{5/3}}{2+n^{5/3}}$$

$$= \lim_{x\to\infty} \frac{n^{5/3}}{n^{5/3}}$$

$$= 1$$

Therefore; the given series ***diverges*** to infinity by *Comparison Test* with divergent ***p****-series*

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n^2}{1+n\sqrt{n}}$

Solution

$$\lim_{n\to\infty} \frac{n^2}{1+n\sqrt{n}} = \lim_{n\to\infty} \frac{n^2}{n^{3/2}}$$

$$= \infty$$

Therefore; the given series ***diverges*** to infinity.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=2}^{\infty} \frac{1}{n\ln n(\ln\ln n)^2}$

Solution

$$\int_2^{\infty} \frac{1}{x\ln x(\ln\ln x)^2}dx = \int_2^{\infty} \frac{d(\ln\ln x)}{(\ln\ln x)^2}$$

$$= -\frac{1}{\ln(\ln x)}\Bigg|_2^{\infty}$$

$$= \frac{1}{\ln(\ln 2)} < \infty$$

Therefore; the given series ***converges*** by *Integral Test*

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n \sqrt{\ln \ln n}}$

Solution

$$\int_3^{\infty} \frac{1}{x \ln x \sqrt{\ln \ln x}} dx = \int_3^{\infty} \frac{d(\ln \ln x)}{(\ln \ln x)^{1/2}}$$

$$= 2\sqrt{\ln(\ln x)} \Big|_3^{\infty}$$

$$= \infty$$

Therefore; the given series ***diverges*** by *Integral Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{1+(-1)^n}{\sqrt{n}}$

Solution

$$\sum_{n=1}^{\infty} \frac{1+(-1)^n}{\sqrt{n}} = 0 + \frac{2}{\sqrt{2}} + 0 + \frac{2}{\sqrt{4}} + 0 + \frac{2}{\sqrt{6}}$$

$$= 2\sum_{k=1}^{\infty} \frac{1}{\sqrt{2k}}$$

$$= \sqrt{2}\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

$$= \sqrt{2}\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$$

Therefore; the given series ***diverges*** to infinity by p-series $\left(p = \frac{1}{2} < 1\right)$

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^2 e^n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^2 e^{n+1}} \cdot \frac{n^2 e^n}{n!}$$

$$= \lim_{n\to\infty} \frac{1}{e} \cdot \frac{n^2}{n+1}$$

$$\underline{= \infty}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{(2n)!6^n}{(3n)!}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(2(n+1))!6^{n+1}}{(3(n+1))!} \cdot \frac{(3n)!}{(2n)!6^n}$$

$$= 6 \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(3n+1)(3n+2)(3n+3)}$$

$$= \lim_{n\to\infty} \frac{4n^2}{27n^3}$$

$$\underline{= 0}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{3^n \ln n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{\sqrt{n+1}}{3^{n+1} \ln(n+1)} \cdot \frac{3^n \ln n}{\sqrt{n}}$$

$$= \frac{1}{3} \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} \frac{\ln n}{\ln(n+1)}$$

$$= \frac{1}{3} \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}} \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)}$$

$$\underline{= \frac{1}{3} < 1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=0}^{\infty} \frac{n^{100}2^n}{\sqrt{n!}}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)^{100}\, 2^{n+1}}{\sqrt{(n+1)!}} \cdot \frac{\sqrt{n!}}{n^{100}2^n}$$

$$= 2\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{100} \frac{1}{\sqrt{n+1}}$$

$$\underline{=0}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1+n!}{(1+n)!}$

Solution

$$1+n! > n!$$

$$\frac{1+n!}{(1+n)!} > \frac{n!}{(1+n)!} = \frac{1}{n+1}$$

Therefore; the given series ***diverges*** by *Comparison Test* with the Harmonic

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{2^n}{3^n - n^3}$

Solution

$$\rho = \lim_{n\to\infty} \frac{2^{n+1}}{3^{n+1} - (n+1)^3} \cdot \frac{3^n - n^3}{2^n}$$

$$= \frac{2}{3}\lim_{n\to\infty} \frac{3^n - n^3}{3^n - \frac{1}{3}(n+1)^3}$$

$$= \frac{2}{3}\lim_{n\to\infty} \frac{1 - \frac{n^3}{3^n}}{1 - \frac{(n+1)^3}{3^{n+1}}}$$

$$= \boxed{\frac{2}{3} < 1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{\pi^n n!}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{\pi^{n+1}(n+1)!} \cdot \frac{\pi^n n!}{n^n}$$

$$= \frac{1}{\pi} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$= \frac{1}{\pi} \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

$$= \boxed{\frac{e}{\pi} < 1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

Solution

$$\rho = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}$$

$$= \lim_{n\to\infty} \frac{2}{n+1}$$

$$= \boxed{0 < 1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)^2\, 2^{n+2}}{3^{n+1}} \cdot \frac{3^n}{n^2\, 2^{n+1}}$$

$$= \lim_{n\to\infty} \frac{2}{3}\left(\frac{n+1}{n}\right)^2$$

$$= \frac{2}{3} < 1$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}$$

$$= \lim_{n\to\infty} \frac{n+1}{n+1}\left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

$$= e > 1$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{100}{n}$

Solution

$$\sum_{n=1}^{\infty} \frac{100}{n} = 100 \sum_{n=1}^{\infty} \frac{1}{n}$$

Therefore; the given series ***diverges*** by *harmonic series*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$

Solution

$$\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}} = 1\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

Therefore; the given series ***converges*** by ***p****-series* $\left(p = \frac{3}{2} > 1\right)$

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \left(\frac{2\pi}{3}\right)^n$

Solution

$$|r| = \frac{2\pi}{3} > 1$$

Therefore; the given series ***diverges*** by *Geometric series*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{5n}{2n-1}$

Solution

$$\lim_{n\to\infty} \frac{5n}{2n-1} = \frac{5}{2} \neq 0$$

Therefore; the given series ***diverges*** by n*th*-Term Test.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$

Solution

$$\lim_{n\to\infty} \frac{n}{2n^2+1} = \frac{1}{2} > 0$$

Therefore; the given series ***diverges*** by *harmonic series*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} (-1)^n \frac{3^{n-2}}{2^n}$

Solution

$$\sum_{n=1}^{\infty} (-1)^n \frac{3^{n-2}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{9}\left(-\frac{3}{2}\right)^n$$

$$|r| = \frac{3}{2} > 1$$

Therefore; the given series ***diverges*** by *Geometric series*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{10}{3\sqrt{n^3}}$

Solution

$$\sum_{n=1}^{\infty} \frac{10}{3\sqrt{n^3}} = \frac{10}{3}\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

Therefore; the given series ***converges*** by ***p**-series* $\left(p = \frac{3}{2} > 1\right)$

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{10n+3}{n2^n}$

Solution

$$b_n = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$$

$$\lim_{n\to\infty} \frac{10n+3}{n2^n} \cdot \frac{2^n}{1} = \lim_{n\to\infty} \frac{10n+3}{n}$$

$$= 10 \qquad \textit{converges}$$

Therefore; the given series ***converges*** by Limit Comparison Test with Geometric series $\left(|r| = \frac{1}{2} < 1\right)$

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{2^n}{4n^2-1}$

Solution

$$\lim_{n\to\infty} \frac{2^n}{4n^2-1} = \lim_{n\to\infty} \frac{2^n(\ln 2)}{8n}$$

$$= \lim_{n\to\infty} \frac{2^n(\ln 2)^2}{8}$$

$$= \infty$$

Therefore; the given series ***diverges*** by nth-Term Test.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{\cos n}{3^n}$

Solution

$$\left|\frac{\cos n}{3^n}\right| \le \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$$

Therefore; the given series ***converges*** by *Direct Comparison Test* with Geometric series $\left(|r| = \frac{1}{3} < 1\right)$

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n!}{n\,7^n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)7^{n+1}} \cdot \frac{n7^n}{n!}$$

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{1}{7}n$$

$$= \infty$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

Solution

$$\frac{\ln n}{n^2} \le \frac{1}{n^{3/2}}$$

Therefore; the given series ***converges*** by *Comparison Test* with ***p**-series* $\left(p = \frac{3}{2} > 1\right)$

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$

Solution

Let $f(x) = \frac{1}{x(\ln x)^2}$

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \int_2^{\infty} \frac{1}{(\ln x)^2} d(\ln x)$$

$$= -\frac{1}{\ln x} \Big|_2^{\infty}$$

$$= -\left(0 - \frac{1}{\ln 2}\right)$$

$$= \frac{1}{\ln 2}$$

Therefore; the given series ***converges*** by *Integral Test*

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=1}^{\infty} \left(\frac{1}{\ln(k+1)}\right)^k$

Solution

$$\lim_{k\to\infty} \sqrt[k]{\left(\frac{1}{\ln(k+1)}\right)^k} = \lim_{k\to\infty} \frac{1}{\ln(k+1)}$$

$$= \frac{1}{\infty}$$

$$= 0$$

Therefore; the given series ***converges*** by *Root Test*

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=2}^{\infty} \frac{1}{k^2(\ln k)^2}$

Solution

$k \ln k > k$

$(k \ln k)^2 > k^2$

$\frac{1}{(k \ln k)^2} < \frac{1}{k^2}$

$\sum \frac{1}{k^2}$ converges by ***p****-series* $(p = 2 > 1)$

Therefore; the given series also ***converges*** by *Comparison Test*

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=3}^{\infty} \frac{1}{\ln k}$

Solution

Let $\quad b_k = \frac{1}{k}$

$a_k = \frac{1}{\ln k}$

$$\begin{aligned}
\lim_{k\to\infty} \frac{a_k}{b_k} &= \lim_{k\to\infty} \frac{1}{\ln k} \cdot \frac{k}{1} \\
&= \lim_{k\to\infty} \frac{k}{\ln k} = \frac{\infty}{\infty} \\
&= \lim_{k\to\infty} \frac{1}{\frac{1}{k}} \\
&= \lim_{k\to\infty} k \\
&= \infty
\end{aligned}$$

Therefore; the given series also ***diverges*** by *Limit Comparison Test*

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=2}^{\infty} \frac{5 \ln k}{k}$

Solution

$$\int_2^{\infty} \frac{5\ln x}{x}\,dx = 5\int_2^{\infty} \ln x \; d(\ln x)$$

$$= \frac{5}{2}(\ln x)^2 \Big|_2^{\infty}$$

$$= \frac{5}{2}\left(\infty - (\ln 2)^2\right)$$

$$= \infty$$

Therefore; the given series ***diverges*** by *Integral Test*

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=1}^{\infty} \ln\left(\frac{k+2}{k+1}\right)$

Solution

$$\sum_{k=1}^{\infty} \ln\left(\frac{k+2}{k+1}\right) = \sum_{k=1}^{\infty} \left(\ln(k+2) - \ln(k+1)\right) \qquad \textit{Telescopic series}$$

$$= (\ln 3 - \ln 2) + (\ln 4 - \ln 3) + (\ln 5 - \ln 4) + \cdots + \left(\ln(n+2) - \ln(n+1)\right)$$

$$= \ln(n+2) - \ln 2$$

$$\lim_{k\to\infty} \ln\left(\frac{k+2}{k+1}\right) = \lim_{k\to\infty} \left(\ln(k+2) - \ln 2\right)$$

$$= \infty$$

Therefore; the given series ***diverges***.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2 \ln k}$

Solution

$$k^2 \ln k > k^2$$

$$\frac{1}{k^2 \ln k} < \frac{1}{k^2}$$

$\displaystyle\sum \frac{1}{k^2}$ converges by ***p**-series* $(p = 2 > 1)$

Therefore; the given series also ***converges*** by *Comparison Test.*

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^{\ln k}}$

Solution

$\ln k > 2$ For large k.

$k^{\ln k} > k^2$

$\dfrac{1}{k^{\ln k}} < \dfrac{1}{k^2}$

$\sum \dfrac{1}{k^2}$ converges by ***p****-series* $(p = 2 > 1)$

Therefore; the given series also ***converges*** by *Comparison Test*

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Solution

Let $a_n = \dfrac{n!}{n^n}$

$$\rho = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \qquad \rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \frac{n+1}{n+1}\left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-n}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$$

$$= \lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^n\right)^{-1} \qquad \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$$

$$= e^{-1}$$

$$= \frac{1}{e} < 1$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\frac{1+\sqrt{2}}{2}+\frac{1+\sqrt{3}}{4}+\frac{1+\sqrt{4}}{8}+\cdots$

Solution

$$\frac{1+\sqrt{2}}{2}+\frac{1+\sqrt{3}}{4}+\frac{1+\sqrt{4}}{8}+\cdots=\sum_{n=1}^{\infty}\frac{1+\sqrt{n+1}}{2^n}$$

Let $\quad a_n=\frac{1+\sqrt{n+1}}{2^n}$

$$\rho=\lim_{n\to\infty}\frac{1+\sqrt{n+2}}{2^{n+1}}\cdot\frac{2^n}{1+\sqrt{n+1}} \qquad\qquad \rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$=\frac{1}{2}\lim_{n\to\infty}\frac{1+\sqrt{n+2}}{1+\sqrt{n+1}}$$

$$=\frac{1}{2}\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}}$$

$$=\frac{1}{2}<1$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty}\frac{(-3)^n}{3\cdot5\cdot7\cdots(2n+1)}$

Solution

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{(-3)^{n+2}}{3\cdot5\cdot7\cdots(2n+1)\cdot(2n+3)}\cdot\frac{3\cdot5\cdot7\cdots(2n+1)}{(-3)^n}$$

$$=\lim_{n\to\infty}\frac{9}{2n+3}$$

$$=0$$

Therefore; the given series ***converges*** by the Ratio Test.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\frac{3\cdot5\cdot7\cdots(2n+1)}{18^n(2n-1)n!}$

Solution

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3\cdot5\cdot7\cdots(2n+1)(2n+3)}{18^{n+1}(2n-1)(2n+1)(n+1)!}\cdot\frac{18^n(2n-1)n!}{3\cdot5\cdot7\cdots(2n+1)}$$

$$= \lim_{n\to\infty}\frac{1}{18}\frac{2n+3}{(2n+1)(n+1)}$$

$$\underline{=0\,|}$$

Therefore; the given series ***converges*** by the Ratio Test.

Exercise

Use the integral test to show that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{1+n^2}$ converges. Show that the sum s of the series is less than $\frac{\pi}{2}$

Solution

$$\int_0^{\infty} f(x)\,dx = \int_0^{\infty}\frac{1}{1+x^2}\,dx$$

$$= \tan^{-1}x\Big|_0^{\infty}$$

$$\underline{=\frac{\pi}{2}\,|}$$

Therefore; the given series ***converges*** by the *Integral Test* and its sum is less than $\frac{\pi}{2}$

Exercise

Use the root test to show that $\displaystyle\sum_{n=1}^{\infty}\frac{2^{n+1}}{n^n}$ converges

Solution

$$\lim_{n\to\infty}\sqrt[n]{\frac{2^{n+1}}{n^n}} = \lim_{n\to\infty}\frac{2^{(n+1)/n}}{n}$$

$$= \lim_{n\to\infty}\frac{2\times2^{1/n}}{n}$$

$$\underline{=0\,|}$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use the root test to test that $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$ converges

Solution

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^{n^2/n}$$

$$= \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^{\frac{1}{n}}$$

$$\boxed{=\frac{1}{e}<1}$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Try to use the ratio test to determine whether $\sum_{n=1}^{\infty}\frac{2^{2n}(n!)^2}{(2n)!}$ converges. What happen?

Now observe that $\frac{2^{2n}(n!)^2}{(2n)!}=\frac{\left[2n(2n-2)(2n-4)\ \cdots\ 6\times4\times2\right]^2}{2n(2n-1)(2n-2)\ \cdots\ 3\times2\times1}$

$$=\frac{2n}{2n-1}\times\frac{2n-2}{2n-3}\times\frac{4}{3}\times\frac{2}{1}$$

Does the given series converges? Why or why not?

Solution

$$\lim_{n\to\infty}\frac{2^{2n+2}\left((n+1)!\right)^2}{(2n+2)!}\cdot\frac{(2n)!}{2^{2n}(n!)^2}=\lim_{n\to\infty}\frac{2^2(n+1)^2}{(2n+2)(2n+1)}$$

$$=\lim_{n\to\infty}\frac{4n^2}{4n^2}$$

$$\boxed{=1}$$

Therefore; the ratio test provides no information

However from the given:

$$\frac{2^{2n}(n!)^2}{(2n)!}=\frac{\left[2n(2n-2)(2n-4)\ \cdots\ 6\times4\times2\right]^2}{2n(2n-1)(2n-2)\ \cdots\ 3\times2\times1}$$

$$= \frac{2n}{2n-1} \times \frac{2n-2}{2n-3} \times \frac{4}{3} \times \frac{2}{1} \;\underline{>1}\,|$$

Therefore; the given series ***diverges*** to infinity.

Exercise

Suppose $a_n > 0$ and $\frac{a_{n+1}}{a_n} > \frac{n}{n+1}$ for all n. Show that $\sum_{n=1}^{\infty} a_n$ diverges.

$\left(a_n \geq \frac{K}{n} \ \textit{for some constant } K \right)$

Solution

If $a_n > 0$ and $\frac{a_{n+1}}{a_n} > \frac{n}{n+1}$ for all n.

Then, by using induction

$$\frac{a_2}{a_1} > \frac{1}{2} \quad \Rightarrow \quad a_2 > \frac{1}{2} a_1$$

$$\frac{a_3}{a_2} > \frac{2}{3} \quad \Rightarrow \quad a_3 > \frac{2}{3} a_2 > \frac{1}{3} a_1$$

$$\frac{a_4}{a_3} > \frac{3}{4} \quad \Rightarrow \quad a_4 > \frac{3}{4} a_3 > \frac{1}{4} a_1$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\frac{a_n}{a_{n-1}} > \frac{n-1}{n} \quad \Rightarrow \quad a_n > \frac{1}{n} a_{n-1}$$

Therefore; the given series ***diverges*** by comparison with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$

Exercise

Working in the early 1600s, the mathematicians Wallis, Pascal, and Fermat were calculating the area of the region under the curve $y = x^p$ between $x = 0$ and $x = 1$, where p is the positive integer. Using arguments that predated the Fundamental Theorem of Calculus, they were able to prove that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{k}{n} \right)^p = \frac{1}{p+1}$$

Use Riemann sums and integrals to verify this limit.

Solution

The sum on the left is simply the left Riemann sum over n equal intervals between 0 and 1 for $f(x) = x^p$.

The limit of the sum is:

$$\int_0^1 x^p\,dx = \frac{1}{p+1}x^{p+1}\Big|_0^1$$

$$= \frac{1}{p+1} \qquad (p > 0)$$

Exercise

Complete the following steps to find the values of $p > 0$ for which the series $\sum_{k=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2k-1)}{p^k k!}$ converges

a) Use the Ratio Test to show that $\sum_{k=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2k-1)}{p^k k!}$ converges for $p > 2$.

b) Use Stirling's formula, $k! = \sqrt{2\pi k}\ k^k e^{-k}$ for large k, to determine whether the series converges when $p = 2$.

$$\left(Hint:\ 1\cdot 3\cdot 5\cdots(2k-1) = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2k-1)2k}{2\cdot 4\cdot 6\cdots 2k} \right)$$

Solution

a) Using the Ratio Test

$$\frac{a_{k+1}}{a_k} = \frac{1\cdot 3\cdot 5\cdots(2k-1)\cdot(2(k+1)-1)}{p^{k+1}(k+1)!}\cdot\frac{p^k k!}{1\cdot 3\cdot 5\cdots(2k-1)}$$

$$= \frac{2k+1}{(k+1)p}$$

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\frac{2k+1}{(k+1)p}$$

$$= \frac{2}{p}$$

Therefore; the given series converges for $p > 2$.

b) When $p = 2$

Given: $1\cdot 3\cdot 5\cdots(2k-1) = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2k-1)2k}{2\cdot 4\cdot 6\cdots 2k} = \frac{(2k)!}{2\cdot 4\cdot 6\cdots 2k}$

$$\sum_{k=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^k k!} = \sum_{k=1}^{\infty} \frac{(2k)!}{2^k k!(2\cdot 4\cdot 6\cdots 2k)}$$

$$= \sum_{k=1}^{\infty} \frac{(2k)!}{\left(2^k\right)^2 (k!)^2}$$

Given: $k! = \sqrt{2\pi k}\ k^k e^{-k}$

$$\rightarrow\ (2k)! = \sqrt{2\pi(2k)}\ (2k)^{2k}\ e^{-2k}$$

$$= 2\sqrt{\pi}\sqrt{k}\ \left(2^k\right)^2 \left(k^k\right)^2\ e^{-2k}$$

$$\sum_{k=1}^{\infty} \frac{(2k)!}{\left(2^k\right)^2 (k!)^2} = \sum_{k=1}^{\infty} \frac{2\sqrt{\pi}\sqrt{k}\ \left(2^k\right)^2 \left(k^k\right)^2 e^{-2k}}{\left(2^k\right)^2 \left(\sqrt{2\pi k}\ k^k e^{-k}\right)^2}$$

$$= \sum_{k=1}^{\infty} \frac{2\sqrt{\pi}\sqrt{k}}{\left(\sqrt{2\pi k}\ \right)^2}$$

$$= \sum_{k=1}^{\infty} \frac{1}{\sqrt{\pi}\sqrt{k}}$$

$$= \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{1}{k^{1/2}} \quad \left(p = \tfrac{1}{2} < 1\right)$$

Therefore; the given series diverges for $p = 2$ by the *Limit Comparison Test* with ***p***-series $\left(p = \tfrac{1}{2} < 1\right)$

SOLUTION *Section* 3.6 – Alternating Series, Absolute and Conditional Convergence

Exercise

Determine if the alternating series converges or diverges $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$

Solution

$$n \ge 1 \Rightarrow n+1 \ge n$$

$$\sqrt{n+1} \ge \sqrt{n}$$

$$\frac{1}{\sqrt{n+1}} \le \frac{1}{\sqrt{n}} \Rightarrow u_{n+1} \le u_n$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$$

Therefore; the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$ ***converges*** by *Alternating Convergence Test*.

Exercise

Determine if the alternating series converges or diverges $\displaystyle\sum_{n=2}^{\infty}(-1)^{n}\frac{4}{(\ln n)^2}$

Solution

$$n \ge 1 \Rightarrow n+1 \ge n$$

$$\ln(n+1) \ge \ln$$

$$(\ln(n+1))^2 \ge (\ln)^2$$

$$\frac{1}{(\ln(n+1))^2} \le \frac{1}{(\ln n)^2}$$

$$\frac{4}{(\ln(n+1))^2} \le \frac{4}{(\ln n)^2} \quad \Rightarrow \quad u_{n+1} \le u_n$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{(\ln n)^2} = 0$$

Therefore; the series $\displaystyle\sum_{n=2}^{\infty}(-1)^{n}\frac{4}{(\ln n)^2}$ ***converges*** by *Alternating Series Test*.

Exercise

Determine if the alternating series converges or diverges $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1}$

Solution

$$n \ge 1 \Rightarrow n^2 + n \ge n^2 + n + 1$$

$$2n^2 + 2n \ge n^2 + n + 1$$

$$n^3 + 2n^2 + 2n \ge n^3 + n^2 + n + 1$$

$$n\left(n^2 + 2n + 2\right) \ge \left(n^2 + 1\right)(n+1)$$

$$n\left((n+1)^2 + 1\right) \ge \left(n^2 + 1\right)(n+1)$$

$$\frac{n}{n^2+1} \ge \frac{n+1}{(n+1)^2+1}$$

$$\underline{u_n \ge u_{n+1}}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n^2+1}$$

$$\underline{= 0}$$

Therefore; the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1}$ ***converges*** by *Alternating Series Test*.

Exercise

Determine if the alternating series converges or diverges $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n^2+5}{n^2+4}$

Solution

$$\lim_{n\to\infty} \frac{n^2+5}{n^2+4} = 1$$

$$\lim_{n\to\infty} (-1)^n \frac{n^2+5}{n^2+4} = doesn't\ exist$$

The given series ***diverges*** by nth Term Test for Divergence.

Exercise

Determine if the alternating series converges or diverges $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{10}\right)^n$

Solution

$$\lim_{n\to\infty}\left(\frac{n}{10}\right)^n \neq 0$$

$$\lim_{n\to\infty}(-1)^{n+1}\left(\frac{n}{10}\right)^n \quad \textit{diverges}$$

Therefore; the given series ***diverges*** by n*th Term Test* for Divergence.

Exercise

Determine if the alternating series converges or diverges $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}$

Solution

$$f(x)=\frac{\sqrt{x}+1}{x+1}$$

$$f'(x)=\frac{\frac{1}{2}x^{-1/2}(x+1)-(1)\left(\sqrt{x}+1\right)}{(x+1)^2}$$

$$=\frac{x+1-2\sqrt{x}\left(\sqrt{x}+1\right)}{2\sqrt{x}(x+1)^2}$$

$$=\frac{x+1-2x-2\sqrt{x}}{2\sqrt{x}(x+1)^2}$$

$$=\frac{1-x-2\sqrt{x}}{2\sqrt{x}(x+1)^2}<0 \quad \to \quad f(x) \;\; \textit{is} \;\; \textit{decreasing}$$

$$u_n \geq u_{n+1}$$

Therefore; the given series ***converges*** by *Alternating Series Test*.

Exercise

Determine if the alternating series converges or diverges $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+1}$

Solution

$$u_{n+1} = \frac{1}{n+2} < \frac{1}{n+1} = u_n$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n+2}$$

$$= 0$$

Therefore; the given series ***converges*** by *Alternating Series Test*.

Exercise

Determine if the alternating series converges or diverges $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{3n+2}$

Solution

$$\lim_{n\to\infty} \frac{n}{3n+2} = \frac{1}{3} < 1$$

Therefore; the given series ***converges*** by n*th*-Term Test.

Exercise

Determine if the alternating series converges or diverges $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n}$

Solution

$$u_{n+1} = \frac{1}{3^{n+1}} < \frac{1}{3^n} = u_n$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{3^n}$$

$$= 0$$

Therefore; the given series ***converges*** by Alternating Series Test. (*Geometric series too*)

Exercise

Determine if the alternating series converges or diverges $\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n}$

Solution

$$u_{n+1} = \frac{1}{e^{n+1}} < \frac{1}{e^n} = u_n$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{e^n}$$

$$= 0$$

Therefore; the given series ***converges*** by Alternating Series Test. (*Geometric series too*)

Exercise

Determine if the alternating series converges or diverges $\sum_{n=1}^{\infty}(-1)^n \frac{5n-1}{4n+1}$

Solution

$$\lim_{n\to\infty} \frac{5n-1}{4n+1} = \frac{5}{4} > 1$$

Therefore; the given series ***diverges*** by n*th*-Term Test.

Exercise

Determine if the alternating series converges or diverges $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n}{n^2+5}$

Solution

$$(n+1)^2+5 > n^2+5$$

or $\quad f(x) = \frac{x}{x^2+5}$

$$f'(x) = -\frac{x^2-5}{\left(x^2+5\right)^2} < 0 \quad \text{for } x \ge 3$$

$$\frac{1}{(n+1)^2+5} < \frac{1}{n^2+5}$$

$$u_{n+1} = \frac{n+1}{(n+1)^2+5} < \frac{n}{n^2+5} = u_n$$

$$\lim_{n\to\infty} \frac{n}{n^2+5} = 0$$

Therefore; the given series ***converges*** by *Alternating Series Test*.

Exercise

Determine if the alternating series converges or diverges $\sum_{n=1}^{\infty}(-1)^n \frac{n}{\ln(n+1)}$

Solution

$$\lim_{n\to\infty} \frac{n}{\ln(n+1)} = \infty$$

Therefore; the given series ***diverges*** by n*th*-Term Test.

Exercise

Determine if the alternating series converges or diverges $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

Solution

$$u_{n+1} = \frac{1}{\ln(n+2)} < \frac{1}{\ln(n+1)} = u_n$$

$$\lim_{n\to\infty} \frac{1}{\ln(n+1)} \underline{= 0}$$

Therefore; the given series ***converges*** by *Alternating Series Test*.

Exercise

Determine if the alternating series converges or diverges $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Solution

$$u_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = u_n$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \underline{= 0}$$

Therefore; the given series ***converges*** by *Alternating Series Test*.

Exercise

Determine if the alternating series converges or diverges $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^2+4}$

Solution

$$\lim_{n\to\infty} \frac{n^2}{n^2+4} \underline{= 1}$$

Therefore; the given series ***converges*** by n*th*-Term Test.

Exercise

Determine if the alternating series converges or diverges $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{\ln(n+1)}$

Solution

$$\lim_{n\to\infty} \frac{n+1}{\ln(n+1)} = \lim_{n\to\infty} \frac{1}{\frac{1}{n+1}}$$

$$= \lim_{n\to\infty} (n+1)$$

$$\underline{= \infty}$$

Therefore; the given series ***diverges*** by n*th*-Term Test.

Exercise

Determine if the alternating series converges or diverges $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln(n+1)}{n+1}$

Solution

$$u_{n+1} = \frac{\ln(n+2)}{n+2} < \frac{\ln(n+1)}{n+1} = u_n$$

$$\lim_{n\to\infty} \frac{\ln(n+1)}{n+1} = 0$$

Therefore; the given series ***converges*** by *Alternating Series Test*.

Exercise

Determine if the alternating series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \sin\frac{(2n-1)\pi}{2}$

Solution

$$\sum_{n=1}^{\infty} \sin\frac{(2n-1)\pi}{2} = \sum_{n=1}^{\infty} (-1)^{n+1}$$

Therefore; the given series ***diverges*** by n*th*-Term Test.

Exercise

Determine if the alternating series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\cos n\pi$

Solution

$$\sum_{n=1}^{\infty} \frac{1}{n}\cos n\pi = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$u_{n+1} = \frac{1}{n+1} < \frac{1}{n} = u_n$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Therefore; the given series ***converges*** by *Alternating Series Test*.

Exercise

Determine if the alternating series converges or diverges $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

Solution

$$u_{n+1} = \frac{1}{(n+1)!} < \frac{1}{n!} = u_n$$

$$\lim_{n\to\infty} \frac{1}{n!} = 0$$

Therefore; the given series ***converges*** by *Alternating Series Test*.

Exercise

Determine if the alternating series converges or diverges $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$

Solution

$$u_{n+1} = \frac{1}{(2(n+1)+1)!} < \frac{1}{(2n+1)!} = u_n$$

$$\lim_{n\to\infty} \frac{1}{2n+1} = 0$$

Therefore; the given series ***converges*** by *Alternating Series Test*.

Exercise

Determine if the alternating series converges or diverges $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{n+2}$

Solution

$$u_{n+1} = \frac{\sqrt{n+1}}{(n+1)+2} < \frac{\sqrt{n}}{n+2} = u_n$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n+2} = 0$$

Therefore; the given series ***converges*** by *Alternating Series Test*.

Exercise

Determine if the alternating series converges or diverges $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{\sqrt[3]{n}}$

Solution

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt[3]{n}} = \lim_{n\to\infty} n^{2/3}$$

$$= \infty$$

Therefore; the given series ***diverges*** by n*th*-Term Test.

Exercise

Determine if the series converge absolutely and if it converges or diverges $\sum_{n=1}^{\infty} (-1)^{n+1} (0.1)^n$

Solution

$(0.1)^n$ converges geometric since $r = 0.1 < 1$

The given series ***converges*** absolutely.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.1)^n}{n}$

Solution

$$\left| (-1)^{n+1} \frac{(0.1)^n}{n} \right| = \frac{1}{n(10)^n}$$

$$< \frac{1}{(10)^n}$$

$$= \left(\frac{1}{10}\right)^n \qquad \text{converges geometric } \left(|r| = \frac{1}{10} < 1\right)$$

The given series ***converges*** absolutely by *Direct Comparison Test*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}$

Solution

$$\frac{1}{\sqrt{n}} > \frac{1}{1+\sqrt{n+1}} > 0$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0 \quad \Rightarrow \; converges$$

The given series ***converges*** conditionally, but $\sum\limits_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a divergent p-series.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum\limits_{n=1}^{\infty} (-1)^n \frac{\sin n}{n^2}$

Solution

By Direct Comparison Test $\left|\frac{\sin n}{n^2}\right| \le \frac{1}{n^2}$

The given series ***converges*** *absolutely*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{3+n}{5+n}$

Solution

$$\lim_{n\to\infty} \frac{3+n}{5+n} = 1 \neq 0$$

The given series ***diverges*** by the n^{th}-Term Test.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum\limits_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n}$

Solution

$$f(x) = \frac{1}{x \ln x}$$

$$f'(x) = -\frac{\ln x + 1}{(x \ln x)^2} < 0 \quad \rightarrow f(x) \text{ is decreasing}$$

$$u_n > u_{n+1} > 0 \quad for \quad n \ge 2$$

$$\lim_{n\to\infty} \frac{1}{n \ln n} = 0 \quad \Rightarrow \; converges$$

But by the Integral Test:

$$\int_2^{\infty} \frac{dx}{x\ln x} = \int_2^{\infty} \frac{d(\ln x)}{\ln x}$$

$$= \ln(\ln x) \Big|_2^{\infty}$$

$$\underline{= \infty \,\big|}$$

$$\Rightarrow \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n\ln n} \text{ ***diverges***.}$$

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+2n+1}$

Solution

$$\frac{1}{n^2+2n+1} < \frac{1}{n^2}$$

Which is a convergent p-series, since $p = 2 > 1$.

The given series ***converges absolutely*** by Direct Comparison Test.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$

Solution

Let $a_n = \dfrac{(-1)^{n-1}}{2n-1}$

$b_n = \dfrac{1}{2n-1} > \dfrac{1}{n}$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n-1}}{2n-1} \right| \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n}{2n-1}$$

$$\underline{= \frac{1}{2} > 0 \,\big|}$$

Since the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ ***diverges*** to infinity, then the given series doesn't converge absolutely.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty}\frac{n\cos(n\pi)}{2^n}$

Solution

$$\rho = \lim_{n\to\infty}\left|\frac{(n+1)\cos((n+1)\pi)}{2^{n+1}}\cdot\frac{2^n}{n\cos(n\pi)}\right|$$

$$= \lim_{n\to\infty}\frac{n+1}{2n}$$

$$\boxed{=\frac{1}{2}<1}$$

Therefore; by the Ratio Test, the given series ***converges absolutely***.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$

Solution

$\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$ converges by the alternating series test, since the terms alternate in sign (decrease in size) and approach 0.

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ ***diverges*** to infinity, then the series converge conditionally only.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+\ln n}$

Solution

$$n^2+\ln n \ge n^2$$

$$\frac{1}{n^2+\ln n}\le\frac{1}{n^2}$$

$\displaystyle\left|\frac{(-1)^n}{n^2+\ln n}\right|\le\frac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges,

Therefore; the given series ***converges absolutely***.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\left(n^2-1\right)}{n^2+1}$

Solution

$$\lim_{n\to\infty}\left|\frac{(-1)^n\left(n^2-1\right)}{n^2+1}\right|=\lim_{n\to\infty}\frac{n^2}{n^2}$$

$$\underline{=1|}$$

The given series ***diverges*** (since its terms do not approach 0.)

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{(n+1)\ln(n+1)}$

Solution

$$\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{(n+1)\ln(n+1)}=\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)\ln(n+1)}$$ converges by alternating series test.

let $x=n$, then

$$\int_1^{\infty}\frac{dx}{(x+1)\ln(x+1)}=\ln\left(\ln(x+1)\right)\Big|_1^{\infty}$$

$$=\ln\left(\ln\infty\right)-\ln\left(\ln 2\right)$$

$$\underline{=\infty|}$$

The series ***converges conditionally*** since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n+1)\ln(n+1)}$ diverges to infinity by the *Integral Test*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{n!}$

Solution

$$\lim_{n\to\infty}\left|\frac{(-2)^{n+1}}{(n+1)!}\cdot\frac{n!}{(-2)^n}\right|=2\lim_{n\to\infty}\frac{1}{n+1}$$

$$\underline{=0|}$$

Therefore; the given series ***converges*** absolutely by the *Ratio Test*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi^n}$

Solution

$\left|\frac{(-1)^n}{n\pi^n}\right| \le \frac{1}{\pi^n}$, and since $\sum \frac{1}{\pi^n}$ is convergent geometric series, then the given series ***converges*** absolutely.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum_{n=1}^{\infty} \frac{100\cos(n\pi)}{2n+3}$

Solution

$$\sum_{n=1}^{\infty} \frac{100\cos(n\pi)}{2n+3} = \sum_{n=1}^{\infty} \frac{100(-1)^n}{2n+3}$$

$$\lim_{n\to\infty} \left|\frac{100(-1)^n}{2n+3}\right| = \lim_{n\to\infty} \frac{1}{2n+3}$$

$$= 0$$

The series converges by alternating series test but only conditionally.

The given series ***diverges*** to infinity.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum_{n=1}^{\infty} \frac{n!}{(-100)^n}$

Solution

$$\lim_{n\to\infty} \left|\frac{n!}{(-100)^n}\right| = \lim_{n\to\infty} \frac{n!}{100^n}$$

$$\lim_{n\to\infty} \frac{(n+1)!}{100^{n+1}} \cdot \frac{100^n}{n!} = \lim_{n\to\infty} \frac{n+1}{100}$$

$$= \infty$$

The given series ***diverges***.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=10}^{\infty} \frac{\sin\left(n+\frac{1}{2}\right)\pi}{\ln \ln n}$

Solution

$$\sum_{n=10}^{\infty} \frac{\sin\left(n+\frac{1}{2}\right)\pi}{\ln \ln n} = \sum_{n=10}^{\infty} \frac{(-1)^n}{\ln \ln n}$$

$$0 < \ln(\ln n) < n$$

$$\frac{1}{\ln(\ln n)} > \frac{1}{n} \qquad \text{for } n \geq 10$$

Since $\displaystyle\sum_{n=10}^{\infty} \frac{1}{n}$ diverges to infinity (it is a harmonic series), so does $\displaystyle\sum_{n=10}^{\infty} \frac{1}{\ln(\ln n)}$ by comparison.

The series converges conditionally by the *Alternating Series Test*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$

Solution

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges Geometric series

The given series ***converges*** *absolutely*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

Solution

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ***p***-series

The given series ***converges*** *absolutely*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

Solution

$$\frac{1}{n!} < \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{converges } \boldsymbol{p}\text{-series}$$

The given series ***converges*** *absolutely*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+3}$

Solution

$$\frac{1}{n+3} < \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges by } \textit{Comparison}$$

$$u_{n+1} = \frac{1}{(n+1)+3} < \frac{1}{n+3} = u_n$$

$$\lim_{n\to\infty} \frac{1}{n+3} = 0 \quad \text{converges by } \textit{Alternating Series Test}$$

The given series ***converges*** *absolutely*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

Solution

$$u_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = u_n$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0 \quad \text{converges by } \textit{Alternating Series Test}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \qquad \text{diverges by } \boldsymbol{p}\text{-series } \left(p = \frac{1}{2} < 1\right)$$

The given series ***converges*** *conditionally*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$

Solution

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} \qquad \text{converges by } \boldsymbol{p}\text{-series } \left(p = \frac{3}{2} > 1\right)$$

The given series ***converges*** *conditionally*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{(n+1)^2}$

Solution

$$\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$$

The given series ***diverges*** by the nth-Term Test.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+3}{n+10}$

Solution

$$\lim_{n\to\infty} \frac{2n+3}{n+10} = 2$$

The given series ***diverges*** by the nth-Term Test.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$

Solution

$$\int_2^{\infty} \frac{dx}{x \ln x} = \int_2^{\infty} \frac{d(\ln x)}{\ln x}$$

$$= \ln(\ln x) \Big|_2^{\infty}$$

$$\underline{= \infty} \quad \text{By the Integral Test, the series diverges}$$

$$u_{n+1} = \frac{1}{(n+1)\ln(n+1)} < \frac{1}{n \ln n} = u_n$$

$$\lim_{n \to \infty} \frac{1}{n \ln n} \underline{= 0} \quad \text{converges by } \textit{Alternating Series Test}$$

The given series ***converges*** *conditionally*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=0}^{\infty} (-1)^n e^{-n^2}$

Solution

$$b_n = \left(\frac{1}{e}\right)^n \quad \text{Converges by geometric series } \left(r = \frac{1}{e} < 1\right)$$

$$\left(\frac{1}{e}\right)^{n^2} < \left(\frac{1}{e}\right)^n \quad \text{converges by Comparison Test}$$

The given series ***converges*** *absolutely*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{n}{n^3 - 5}$

Solution

$$\frac{n}{n^3 - 5} = \frac{n}{n^3} = \frac{1}{n^2}$$

$$b_n = \frac{1}{n^2} \quad \text{converges by } p\text{-series } (p = 2 > 1)$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^3-5}\frac{n^2}{1}$$

$\underline{=1\,|}$ converges by Limit Comparison Test

The given series ***converges*** *absolutely*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{4/3}}$

Solution

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ converges by p-series $\left(p = \frac{4}{3} > 1\right)$

$$u_{n+1} = \frac{1}{(n+1)^{4/3}} < \frac{1}{n^{4/3}} = u_n$$

$\displaystyle\lim_{n\to\infty} \frac{1}{n^{4/3}} \underline{=0\,|}$ converges by *Alternating Series Test*

The given series ***converges*** *absolutely*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+4}}$

Solution

$$u_{n+1} = \frac{1}{\sqrt{(n+1)+4}} < \frac{1}{\sqrt{n+4}} = u_n$$

$\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n+4}} \underline{=0\,|}$ converges by *Alternating Series Test*

$b_n = \dfrac{1}{\sqrt{n}}$ diverges by ***p***-series $\left(p = \frac{1}{2} < 1\right)$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n+4}}\frac{\sqrt{n}}{1}$$

$\underline{=1\,|}$ diverges by *Limit Comparison Test* using ***p***-series

The given series ***converges*** *conditionally*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1}$

Solution

$$\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

$$u_{n+1} = \frac{1}{(n+1)+1} < \frac{1}{n+1} = u_n$$

$$\lim_{n\to\infty} \frac{1}{n+1} = 0 \qquad \text{converges by Alternating Series Test}$$

$$\sum_{n=0}^{\infty} \frac{|\cos n\pi|}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} \qquad \text{diverges by a Limit Comparison to the divergent harmonic series}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{|\cos n\pi|}{n+1} \cdot \frac{n}{1}$$

$$= 1$$

The given series ***converges*** *conditionally*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \arctan n$

Solution

$$\lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$$

The given series ***diverges*** by the n*th*-Term Test.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}$

Solution

$$\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$u_{n+1} = \frac{1}{(n+1)^2} < \frac{1}{n^2} = u_n$$

$$\lim_{n\to\infty} \frac{1}{n^2} = 0$$ converges by Alternating Series Test

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$ converges by p-series $(p = 2 > 1)$

The given series ***converges*** *absolutely*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\left[(n-1)\frac{\pi}{2}\right]}{n}$

Solution

$$\sum_{n=1}^{\infty} \frac{\sin\left[(n-1)\frac{\pi}{2}\right]}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

$$u_{n+1} = \frac{1}{n+1} < \frac{1}{n} = u_n$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$ converges by Alternating Series Test

$$\sum_{n=1}^{\infty} \frac{1}{n}$$ diverges by ***p***-series $(p = 1)$

The given series ***converges*** *conditionally*.

Exercise

For what values of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n\, 2^n}$ converge absolutely? Converge conditionally?

Solution

$$\lim_{n\to\infty} \left| \frac{(x-5)^{n+1}}{(n+1)\, 2^{n+1}} \frac{n\, 2^n}{(x-5)^n} \right| = \lim_{n\to\infty} \frac{n}{n+1} \left| \frac{x-5}{2} \right|$$

$$= \left| \frac{x-5}{2} \right|$$

$$\left| \frac{x-5}{2} \right| < 1 \quad \to |x-5| < 2$$

$\Rightarrow -2 < x-5 < 2$

$\Rightarrow \;\; 3 < x < 7$ Then the series converges absolutely

$\left|\dfrac{x-5}{2}\right| > 1 \quad \to |x-5| > 2 \quad \Rightarrow x-5 < -2 \quad and \quad x-5 > 2$

$\Rightarrow \quad x < 3 \quad and \quad x > 7$ Then the series diverges (the term does approach zero)

$\left|\dfrac{x-5}{2}\right| = 1 \quad \to |x-5| = 2 \quad \Rightarrow x-5 = -2 \quad and \quad x-5 = 2$

If $x=3$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n\ 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges conditionally (it is an alternating harmonic series).

If $x=7$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(2)^n}{n\ 2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ the series is harmonic which diverges.

Hence, the series ***converges absolutely*** on the open interval (3, 7), **converges conditionally** at $x=3$, and ***diverges*** everywhere else.

Exercise

For what values of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2\ 2^{2n}}$ converge absolutely? Converge conditionally? Diverge?

Solution

Using the ratio test

$$\rho = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2\ 2^{2n+2}} \frac{n^2\ 2^{2n}}{(x-2)^n} \right|$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 \frac{|x-2|}{4}$$

$$= \frac{|x-2|}{4} < 1$$

$$\frac{|x-2|}{4} < 1$$

$|x-2| < 4$

$-4 < x-2 < 4$

$\Rightarrow \;\; -2 < x < 6$ Then the series converges absolutely

$$\frac{|x-2|}{4} > 1$$

$|x-2| > 4$

$x-2<-4 \;\; and \;\; x-2>4$

$\Rightarrow \quad x<-2 \quad and \;\; x>6$ Then the series diverges (the term does approach zero)

$\frac{|x-2|}{4}=1$

$|x-2|=4$

$x-2=-4 \;\; and \;\; x-2=4$

If $x=-2$, the series:

$$\sum_{n=1}^{\infty}\frac{(x-2)^n}{n^2\ 2^{2n}}=\sum_{n=1}^{\infty}\frac{(-4)^n}{n^2\ 2^{2n}}$$

$$=\sum_{n=1}^{\infty}\frac{\left(-2^2\right)^n}{n^2\ 2^{2n}}$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

which converges absolutely (it is an alternating harmonic series).

If $x=6$, the series

$$\sum_{n=1}^{\infty}\frac{(x-2)^n}{n^2\ 2^{2n}}=\sum_{n=1}^{\infty}\frac{(4)^n}{n^2\ 2^{2n}}$$

$$=\sum_{n=1}^{\infty}\frac{2^{2n}}{n^2\ 2^{2n}}$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

the series converges absolutely.

Hence, the series ***converges*** absolutely if $-2\le x\le 6$ and diverges everywhere else.

Exercise

For what values of *x* does the series $\sum_{n=1}^{\infty}(n+1)^2\left(\frac{x}{x+2}\right)^n$ converge absolutely? Converge conditionally? Diverge?

Solution

Using the ratio test

$$\lim_{n\to\infty}\left|\frac{(n+2)^2\left(\frac{x}{x+2}\right)^{n+1}}{(n+1)^2\left(\frac{x}{x+2}\right)^{n}}\right| = \lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)^2\left|\frac{x}{x+2}\right|$$

$$=\left|\frac{x}{x+2}\right|$$

$$\left|\frac{x}{x+2}\right|=1 \Rightarrow \frac{x}{x+2}=1 \qquad \frac{x}{x+2}=-1$$

$$x=x+2 \quad (\textit{impossible}) \qquad x=-x-2$$

$$\boxed{x=-1}$$

If $\left|\frac{x}{x+2}\right|<1 \Rightarrow -2<x<0$.

Hence $x>-1$ the series converges absolutely.

$-\infty$	-2	-1	0 ∞
+	–	–	–
+	–	+	+
		–	–

If $\frac{|x|}{|x+2|}>1 \Rightarrow x<-1$, the series diverges.

If $x=-1$, the series is $\sum_{n=1}^{\infty}(-1)^n(n+1)^2$ which diverges

The series converges absolutely for $x>-1$, converges conditionally nowhere, and diverges for $x\le -1$

Exercise

For what values of x does the series $\sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^n}{2n+3}$ converge absolutely? Converge conditionally? Diverge?

Solution

Using the ratio test

$$\rho=\lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{2(n+1)+3}\cdot\frac{2n+3}{(x-1)^n}\right|$$

$$=\lim_{n\to\infty}\frac{2n+3}{2n+5}|x-1| \qquad \lim_{n\to\infty}\frac{2n+3}{2n+5}=\lim_{n\to\infty}\frac{2n}{2n}=1$$

$$=|x-1|$$

If $|x-1|<1 \quad -1<x-1<1 \Rightarrow 0<x<2$ Then the series converges absolutely

If $|x-1|>1 \Rightarrow x<0 \quad \textit{and} \quad x>2$ Then the series diverges

If $|x-1|=1 \Rightarrow x=0 \quad \textit{and} \quad x=2$

If $x=0$, the series

$$\sum_{n=0}^{\infty}(-1)^n \frac{(0-1)^n}{2n+3} = \sum_{n=0}^{\infty}\frac{(-1)^{2n}}{2n+3}$$

$$= \sum_{n=0}^{\infty}\frac{1}{2n+3}$$ is harmonic which diverges.

If $x = 2$, the series

$$\sum_{n=0}^{\infty}(-1)^n \frac{(2-1)^n}{2n+3} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+3}$$

which converges absolutely (it is an alternating harmonic series).

Therefore; the series converges absolutely if and converges conditionally if $x = 2$ and diverges everywhere else.

Exercise

For what values of x does the series $\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{3x+2}{-5}\right)^n$ converge absolutely? Converge conditionally? Diverge?

Solution

Using the ratio test

$$\rho = \lim_{n\to\infty}\left|\frac{1}{2(n+1)-1}\left(\frac{3x+2}{-5}\right)^{n+1}\cdot(2n-1)\left(\frac{3x+2}{-5}\right)^{-n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2n-1}{2n+1}\cdot\frac{3x+2}{-5}\right| \qquad \lim_{n\to\infty}\frac{2n-1}{2n+1} = \lim_{n\to\infty}\frac{2n}{2n} = 1$$

$$= \frac{1}{5}|3x+2|$$

If $\frac{1}{5}|3x+2| < 1$

$$-5 < 3x+2 < 5$$

$$-\frac{7}{3} < x < 1$$

Then the series converges absolutely

If $|3x+2| > 5$

$$3x+2 < -5 \quad \textit{and} \quad 3x+2 > 5$$

$$x < -\frac{7}{3} \quad \textit{and} \quad x > 1.$$

Then the series diverges

If $x=-\frac{7}{3}$, the series $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{2n-1}(1)^n$

$$= \sum_{n=1}^{\infty} \frac{1}{2n-1} \qquad \text{is harmonic which diverges.}$$

If $x=1$, the series $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ converges absolutely (it is an alternating harmonic series).

Therefore; the series converges absolutely if and $-\frac{7}{3} < x < 1$, converges conditionally if $x=1$ and diverges everywhere else.

Exercise

For what values of x does the series $\sum\limits_{n=2}^{\infty} \frac{x^n}{2^n \ln n}$ converge absolutely? Converge conditionally? Diverge?

Solution

Using the ratio test

$$\rho = \lim_{n\to\infty} \left| \frac{x^{n+1}}{2^{n+1}\ln(n+1)} \cdot \frac{2^n \ln n}{x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x}{2} \cdot \frac{\ln n}{\ln(n+1)} \right| \qquad \lim_{n\to\infty} \left| \frac{\ln n}{\ln(n+1)} \right| = \lim_{n\to\infty} \frac{1/n}{1/(n+1)} = \lim_{n\to\infty} \frac{n+1}{n} = 1 \quad (L'H\hat{o}pital\ rule)$$

$$= \frac{|x|}{2}$$

If $\frac{|x|}{2} < 1 \quad \Rightarrow |x| < 2 \quad \rightarrow \quad -2 < x < 2$, the given series converges absolutely.

If $x=-2$, the series

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{(-2)^n}{2^n \ln n}$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} \quad \text{converges absolutely (it is an alternating harmonic series).}$$

If $x=2$, the series

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{(2)^n}{2^n \ln n}$$

$$= \sum_{n=2}^{\infty} \frac{1}{\ln n} \qquad \text{is harmonic which diverges}$$

Therefore; the series converges absolutely if and $-2 < x < 2$, converges conditionally if $x = -2$ and diverges everywhere else.

Exercise

For what values of x does the series $\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^3}$ converge absolutely? Converge conditionally? Diverge?

Solution

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^3}$ by using the ratio test

$$\rho = \lim_{n\to\infty} \left| \frac{(4x+1)^{n+1}}{(n+1)^3} \cdot \frac{n^3}{(4x+1)^n} \right|$$

$$= |4x+1| \lim_{n\to\infty} \left| \frac{n^3}{(n+1)^3} \right|$$

$$\lim_{n\to\infty} \left| \frac{n^3}{(n+1)^3} \right| = \lim_{n\to\infty} \left| \left(\frac{n}{n+1} \right)^3 \right| = 1$$

$$= |4x+1|$$

If $|4x+1| < 1$

$$-1 < 4x+1 < 1$$

$$-\frac{1}{2} < x < 0$$

The given series converges absolutely.

If $x = -\frac{1}{2}$, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ converges absolutely.

If $x = 0$, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges absolutely (p-series)

Therefore; the series ***converges absolutely*** if $-\frac{1}{2} \le x \le 0$ and ***diverges*** everywhere else.

Exercise

For what values of x does the series $\sum_{n=1}^{\infty} \frac{(2x+3)^n}{n^{1/3}4^n}$ converge absolutely? Converge conditionally? Diverge?

Solution

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(2x+3)^n}{n^{1/3}4^n}$ by using the *Ratio Test*

$$\rho = \lim_{n\to\infty} \left| \frac{(2x+3)^{n+1}}{(n+1)^{1/3}4^{n+1}} \cdot \frac{n^{1/3}4^n}{(2x+3)^n} \right|$$

$$= \frac{|2x+3|}{4} \lim_{n\to\infty} \left| \frac{n^{1/3}}{(n+1)^{1/3}} \right|$$

$$= \frac{|2x+3|}{4} \lim_{n\to\infty} \left| \frac{n^{1/3}}{n^{1/3}} \right|$$

$$= \frac{|2x+3|}{4}$$

If $|2x+3| < 4$

$-4 < 2x+3 < 4$

$-\frac{7}{2} < x < \frac{1}{2}$

The given series converges absolutely.

If $x = -\frac{7}{2}$

The series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}4^n}$ converges conditionally (Alternating test).

If $x = \frac{1}{2}$

The series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}4^n}$ diverges.

Therefore; the series converges absolutely if $-\frac{7}{2} \le x \le \frac{1}{2}$, $x = -\frac{7}{2}$ ***converges conditionally*** and ***diverges*** everywhere else.

Exercise

For what values of x does the series $\sum_{n=1}^{\infty} \frac{1}{n}\left(1+\frac{1}{x}\right)^{n}$ converge absolutely? Converge conditionally? Diverge?

Solution

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}\left(1+\frac{1}{x}\right)^{n}$ by using the ratio test

$$\rho = \lim_{n\to\infty} \left| \frac{1}{n+1}\left(1+\frac{1}{x}\right)^{n+1} \cdot n\left(1+\frac{1}{x}\right)^{-n} \right|$$

$$= \left|1+\frac{1}{x}\right| \lim_{n\to\infty} \left|\frac{n}{n+1}\right| \qquad \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = 1$$

$$= \left|1+\frac{1}{x}\right| < 1$$

If $-1 < 1+\frac{1}{x} < 1$

$$\Rightarrow \ -\frac{1}{2} < x < 0$$

The given series converges absolutely.

If $x = -\frac{1}{2}$, the series

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally (Alternating test).

Therefore; the series converges absolutely if $-\frac{1}{2} < x < 0$, $x = -\frac{1}{2}$ converges conditionally, diverges everywhere else, and undefined at $x = 0$.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$

Solution

$$\lim_{n\to\infty} \frac{\sqrt{n+1}}{n+2}\frac{n+1}{\sqrt{n}} = \lim_{n\to\infty} \sqrt{\frac{n+1}{n}}\left(\frac{n+1}{n+2}\right)$$

$$= 1$$

$$\lim_{n\to\infty} \frac{\sqrt{n+1}}{n+2} = \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n+1}}}{1}$$

$$\underline{=0}$$

Therefore; the given series ***converges*** by the *Alternating Series Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n}$

Solution

$$a_{n+1} = \frac{1}{(n+1)\ln(n+1)} \le \frac{1}{n\ln n} = a_n$$

$$\lim_{n\to\infty} \frac{1}{n\ln n} \underline{=0}$$

Therefore; the given series ***converges*** by *Alternating Series Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{5}{n}$

Solution

$$a_{n+1} = \frac{5}{n+1} \le \frac{5}{n} = a_n$$

$$\lim_{n\to\infty} \frac{5}{n} \underline{=0}$$

Therefore; the given series ***converges*** by *Alternating Series Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{3^{n-1}}{n!}$

Solution

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^n}{(n+1)!}\cdot\frac{n!}{3^{n-1}}\right|$$

$$= \lim_{n\to\infty} \frac{3}{n+1}$$

$$\underline{=0}$$

Therefore; the given series ***converges absolutely*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3^n}{n2^n}$

Solution

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n2^n}{3^n}\right|$$

$$= \lim_{n\to\infty}\frac{3}{2}\frac{n}{n+1}$$

$$\boxed{=\frac{3}{2}>1}$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$

Solution

$$a_{n+1}=\frac{1}{n+1}\le\frac{1}{n}=a_n$$

$$\lim_{n\to\infty}\frac{1}{n}\ \boxed{=0}$$

Therefore; the given series ***converges*** by *Alternating Series Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty}\frac{n}{(-2)^{n-1}}$

Solution

$$\frac{1}{2}\le\frac{n}{n+1}$$

$$\frac{2^{n-1}}{2^n}\le\frac{n}{n+1}$$

$$a_{n+1}=\frac{n+1}{2^n}\le\frac{n}{2^{n-1}}=a_n$$

$$\lim_{n\to\infty}\frac{n}{2^{n-1}}=\lim_{n\to\infty}\frac{1}{2^{n-1}(\ln 2)}\ \boxed{=0}$$

Therefore; the given series ***converges*** by *Alternating Series Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{10}{n^{3/2}}$

Solution

Which is *p-series* with $p = \frac{3}{2} > 1$

Therefore; the given series ***converges*** by ***p****-series Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{2}{n^2+5}$

Solution

$$b_n = \frac{1}{n^2}$$

$$\frac{2}{n^2+5} < \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{2}{n^2+5} \underline{= 0}$$

Therefore; the given series ***converges*** by the *Limit Comparison Test* with ***p****-series*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{3^n}{n^2}$

Solution

$$\lim_{x\to\infty} \frac{3^{n+1}}{(n+1)^2} \cdot \frac{n^2}{3^n} = \lim_{x\to\infty} 3\left(\frac{n}{n+1}\right)^2$$

$$\underline{= 3 > 1}$$

Therefore; the given series ***diverges*** by the Ratio Test.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

Solution

$$\frac{1}{2^n+1} < \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$$

Therefore; the given series ***converges*** by the *Limit Comparison Test* with Geometric series $r = \frac{1}{2} < 1$

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} 5\left(\frac{7}{8}\right)^n$

Solution

Therefore; the given series ***converges*** by *Geometric series* $|r| = \frac{7}{8} < 1$

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{3n^2}{2n^2+1}$

Solution

$$\lim_{n\to\infty} \frac{3n^2}{2n^2+1} = \frac{3}{2}$$

Therefore; the given series ***diverges***.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} 100e^{-\pi/2}$

Solution

$$\sum_{n=1}^{\infty} 100e^{-\pi/2} = 100\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{e}}\right)^n$$

Therefore; the given series ***converges*** by *Geometric series* $|r| = \frac{1}{\sqrt{e}} < 1$

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+4}$

Solution

$$a_{n+1} = \frac{1}{(n+1)+4} < \frac{1}{n+4} = a_n$$

$$\lim_{n\to\infty} \frac{1}{n+4} = 0$$

Therefore; the given series ***converges*** conditionally by *Alternating Series Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{4}{3n^2-1}$

Solution

$$3(n+1)^2 - 1 > 3n^2 - 1$$

$$a_{n+1} = \frac{4}{3(n+1)^2 - 1} < \frac{4}{3n^2-1} = a_n$$

$$\lim_{n\to\infty} \frac{4}{3n^2-1} = 0$$

Therefore; the given series ***converges*** absolutely by Alternating Series Test.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=2}^{\infty} \frac{\ln n}{n}$

Solution

$$\int_1^{\infty} \frac{\ln x}{x} dx = \int_1^{\infty} \ln x \, d(\ln x)$$

$$= \frac{1}{2}(\ln x)^2 \Big|_1^{\infty}$$

$$= \infty$$

Therefore; the given series ***diverges*** by *Integral Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=1}^{\infty} \frac{2}{k^{3/2}}$

Solution

Which is *p-series* with $p = \frac{3}{2} > 1$

Therefore; the given series ***converges*** by ***p****-series Test*.

Let $f(x) = 2x^{-3/2}$

$$\int_1^\infty 2x^{-3/2}\,dx = -4x^{-1/2} \Big|_1^\infty$$

$$= -\frac{4}{\sqrt{x}} \Big|_1^\infty$$

$$= -4(0-1)$$

$$= 4$$

Therefore; the given series ***converges*** by *Integral Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=1}^{\infty} k^{-2/3}$

Solution

$$\sum_{k=1}^{\infty} k^{-2/3} = \sum_{k=1}^{\infty} \frac{1}{k^{2/3}}$$

It is *p-series* with $p = \frac{2}{3} < 1$ which diverges.

Therefore; the given series ***diverges*** by ***p****-series Test*.

Let $f(x) = x^{-2/3}$

$$\int_1^\infty x^{-2/3}\,dx = 3x^{1/3} \Big|_1^\infty$$

$$= 3(\infty - 1)$$

$$= \infty$$

Therefore; the given series ***diverges*** by *Integral Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=1}^{\infty} \frac{2k^2+1}{\sqrt{k^3+2}}$

Solution

$$\lim_{k\to\infty} \frac{2k^2+1}{\sqrt{k^3+2}} = \lim_{k\to\infty} \frac{2k^2}{k^{3/2}}$$

$$= \lim_{k\to\infty} 2k^{1/2}$$

$$\underline{= \infty}$$

Therefore; the given series ***diverges*** by *Divergence Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{e^k}$

Solution

$$\sum_{k=1}^{\infty} \frac{2^k}{e^k} = \sum_{k=1}^{\infty} \left(\frac{2}{e}\right)^k$$

This is a Geometric series with $\left(|r| = \frac{2}{e} < 1\right)$, which converges.

$$a_0 = \frac{2}{e}$$

$$S = \frac{\frac{2}{e}}{1-\frac{2}{e}}$$

$$\underline{= \frac{2}{e-2}}$$

Therefore; the given series ***converges*** by *Geometric series Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k}{k+3}\right)^{2k}$

Solution

$$\sum_{k=1}^{\infty}\left(\frac{k}{k+3}\right)^{2k}=\sum_{k=1}^{\infty}\left(\left(\frac{k}{k+3}\right)^{k}\right)^{2}$$

$$\lim_{k\to\infty}\left(\frac{k}{k+3}\right)^{k}=\lim_{k\to\infty}\left(\frac{1}{1+\frac{3}{k}}\right)^{k}$$

$$=\lim_{k\to\infty}\frac{1}{\left(1+\frac{3}{k}\right)^{k}}$$

$$=\frac{1}{e^{3}}\neq 0$$

Therefore; the given series ***diverges*** by *Divergence Test*.

$$\sum_{k=1}^{\infty}\left(\frac{k}{k+3}\right)^{2k}$$

$$\rho=\lim_{k\to\infty}\sqrt[k]{\left(\frac{k}{k+3}\right)^{2k}}$$

$$=\lim_{k\to\infty}\left(\frac{k}{k+3}\right)^{2}$$

$$=1$$

Because the limit is $\rho = 1$, we can't decide from the *Ratio Test*

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=1}^{\infty}\frac{2^{k}k!}{k^{k}}$

Solution

$$\frac{a_{k+1}}{a_k}=\frac{2^{k+1}(k+1)!}{(k+1)^{k+1}}\cdot\frac{k^{k}}{2^{k}k!}$$

$$=\frac{2(k+1)}{k+1}\left(\frac{k}{k+1}\right)^{k}$$

$$=2\left(\frac{k}{k+1}\right)^{k}$$

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k}=2\lim_{k\to\infty}\left(\frac{k}{k+1}\right)^{k}$$

$$= 2 \lim_{k\to\infty} \frac{1}{\left(1+\frac{1}{k}\right)^k}$$

$$= \frac{2}{e} < 1$$

Therefore; the given series ***converges*** by *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}\sqrt{k+1}}$

Solution

$$a_k = \frac{1}{\sqrt{k^2+k}}$$

Let $b_k = \frac{1}{k}$

$\sum b_k$ diverges by ***p****–series* $(p = 1 \le 1)$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1}{\sqrt{k^2+k}} \cdot \frac{k}{1}$$

$$= \lim_{k\to\infty} \frac{k}{k}$$

$$= 1$$

Therefore; the given series ***diverges*** by *Comparison Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=1}^{\infty} \frac{3}{2+e^k}$

Solution

Using Comparison test:

$$2 + e^k < e^k$$

$$\frac{1}{2+e^k} < \frac{1}{e^k}$$

$$\frac{3}{2+e^k} < \frac{3}{e^k}$$

Let $b_k = \frac{3}{e^k}$

$$= 3\left(\frac{1}{e}\right)^k$$

b_k converges by Geometric series with $\left(r = \frac{1}{e} < 1\right)$

Therefore; the given series ***converges*** by *Comparison Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=1}^{\infty} k \sin\frac{1}{k}$

Solution

$$\lim_{k\to\infty} k\sin\frac{1}{k} = \lim_{k\to\infty} \frac{\sin\frac{1}{k}}{\frac{1}{k}} \qquad \left(x = \frac{1}{k} \to 0\right)$$

$$= \lim_{x\to 0} \frac{\sin x}{x}$$

$$= 1 \neq 0$$

Therefore; the given series ***diverges*** by *Divergence Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[k]{k}}{k^3}$

Solution

$$\sum_{k=1}^{\infty} \frac{\sqrt[k]{k}}{k^3} = \sum_{k=1}^{\infty} \frac{k^{1/k}}{k^3}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k^{3-1/k}}$$

$$3 - \frac{1}{k} < 3$$

$$k^{3-\frac{1}{k}} < k^3$$

$$\frac{1}{k^{3-1/k}} < \frac{1}{k^3}$$

Let $b_k = \frac{1}{k^3}$

$\sum b_k$ converges by ***p*–*series*** $(p = 3 > 1)$

Therefore; the given series ***converges*** by *Comparison Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{1+\ln k}$

Solution

$\ln k < k$

$\dfrac{1}{\ln k} > \dfrac{1}{k}$

Let $b_k = \dfrac{1}{k}$

$\sum b_k$ diverges by ***p*–*series*** $(p = 1 \le 1)$

Therefore; the given series ***diverges*** by *Comparison Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=1}^{\infty} k^5 e^{-k}$

Solution

Using Ratio Test:

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)^5 e^{-k-1}}{k^5 e^{-k}}$$

$$= \frac{1}{e}\left(\frac{k+1}{k}\right)^5$$

$$\rho = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$$

$$= \lim_{k\to\infty} \frac{1}{e}\left(\frac{k+1}{k}\right)^5$$

$$= \frac{1}{e} < 1$$

Therefore; the given series ***converges*** by *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=4}^{\infty} \frac{1}{k^2-10}$

Solution

$$k^2-10 > (k-1)^2$$

$$\frac{2}{k^2-10} < \frac{1}{(k-1)^2}$$

$\sum b_k$ converges by ***p****–series* $(p=2>1)$

Therefore; the given series ***converges*** by *Comparison Test*.

Let $\quad f(x)=\dfrac{1}{x^2-10}$

$$x=\sqrt{10}\sec\theta \qquad x^2-10=10\tan^2\theta$$

$$dx=\sqrt{10}\sec\theta\tan\theta\, d\theta$$

$$\int_4^{\infty}\frac{1}{x^2-10}dx=\int_4^{\infty}\frac{1}{10\tan^2\theta}\sqrt{10}\sec\theta\tan\theta\, d\theta$$

$$=\frac{1}{\sqrt{10}}\int_4^{\infty}\frac{\sec\theta}{\tan\theta}\, d\theta$$

$$=\frac{1}{\sqrt{10}}\int_4^{\infty}\frac{1}{\sin\theta}\, d\theta$$

$$=\frac{1}{\sqrt{10}}\int_4^{\infty}\csc\theta\, d\theta$$

$$=\frac{1}{\sqrt{10}}\int_4^{\infty}\csc\theta\frac{\csc\theta+\cot\theta}{\csc\theta+\cot\theta}\, d\theta$$

$$=\frac{1}{\sqrt{10}}\int_4^{\infty}\frac{\csc^2\theta+\csc\theta\cot\theta}{\csc\theta+\cot\theta}\, d\theta$$

$$=-\frac{1}{\sqrt{10}}\int_4^{\infty}\frac{1}{\csc\theta+\cot\theta}\, d(\csc\theta+\cot\theta)$$

$$=-\frac{1}{\sqrt{10}}\ln\left|\csc\theta+\cot\theta\right|\Big|_4^{\infty}$$

$$= -\frac{1}{\sqrt{10}} \ln\left|\frac{1}{\sin\theta} + \frac{1}{\tan\theta}\right| \Bigg|_4^\infty$$

$$= -\frac{1}{\sqrt{10}} \ln\left|\frac{\sec\theta}{\tan\theta} + \frac{1}{\tan\theta}\right| \Bigg|_4^\infty$$

$$= -\frac{1}{\sqrt{10}} \ln\left|\frac{\sec\theta + 1}{\tan\theta}\right| \Bigg|_4^\infty$$

$$= -\frac{1}{\sqrt{10}} \ln\left|\frac{\frac{x}{\sqrt{10}} + 1}{\sqrt{\frac{x^2-10}{10}}}\right| \Bigg|_4^\infty$$

$$= -\frac{1}{\sqrt{10}} \ln\left|\frac{x+\sqrt{10}}{\sqrt{x^2-10}}\right| \Bigg|_4^\infty$$

$$= -\frac{1}{\sqrt{10}}\left(\ln 1 - \ln\frac{4+\sqrt{10}}{\sqrt{6}}\right)$$

$$\boxed{= \frac{1}{\sqrt{10}} \ln\left(\frac{4+\sqrt{10}}{\sqrt{6}}\right)}$$

Therefore; the given series ***converges*** by *Integral Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k^2}{k^2}$

Solution

$$a_k = \frac{\ln k^2}{k^2}$$

$$= \frac{2\ln k}{k^2}$$

$$< \frac{2k^{1/2}}{k^2}$$

$$= \frac{2}{k^{3/2}} = b_k$$

$\sum b_k$ converges by $\boldsymbol{p}$–*series* $\left(p = \frac{3}{2} > 1\right)$

Therefore; the given series ***converges*** by *Comparison Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=1}^{\infty} ke^{-k}$

Solution

Using Ratio Test:

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)e^{-k-1}}{ke^{-k}}$$

$$= \frac{1}{e}\left(\frac{k+1}{k}\right)$$

$$= \frac{1}{e}\left(1+\frac{1}{k}\right)$$

$$\rho = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$$

$$= \lim_{k\to\infty} \frac{1}{e}\left(1+\frac{1}{k}\right)$$

$$= \frac{1}{e} < 1$$

Therefore; the given series ***converges*** by *Ratio Test*.

Let $f(x) = xe^{-x}$

		$\int e^{-x}$
+	x	$-e^{-x}$
−	1	e^{-x}

$$\int_1^{\infty} xe^{-x}dx = -e^{-x}(x+1)\Big|_1^{\infty}$$

$$= -0 + e^{-1}(2)$$

$$= \frac{2}{e}$$

Therefore; the given series ***converges*** by *Integral Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=0}^{\infty} \frac{2 \cdot 4^k}{(2k+1)!}$

Solution

Using Ratio Test:

$$\frac{a_{k+1}}{a_k} = \frac{2 \cdot 4^{k+1}}{(2k+3)!} \cdot \frac{(2k+1)!}{2 \cdot 4^k}$$

$$= \frac{4}{(2k+2)(2k+3)}$$

$$\rho = \lim_{k \to \infty} \frac{4}{(2k+2)(2k+3)} \qquad \rho = \lim_{k \to \infty} \frac{a_{k+1}}{a_k}$$

$$= 0 < 1$$

Therefore; the given series ***converges*** by *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=0}^{\infty} \frac{9^k}{(2k)!}$

Solution

Using Ratio Test:

$$\frac{a_{k+1}}{a_k} = \frac{9^{k+1}}{(2k+2)!} \cdot \frac{(2k)!}{9^k}$$

$$= \frac{9}{(2k+1)(2k+2)}$$

$$\rho = \lim_{k \to \infty} \frac{9}{(2k+1)(2k+2)} \qquad \rho = \lim_{k \to \infty} \frac{a_{k+1}}{a_k}$$

$$= 0 < 1$$

Therefore; the given series ***converges*** by *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=1}^{\infty} \frac{\coth k}{k}$

Solution

$$a_k = \frac{\coth k}{k}$$

Let $b_k = \frac{1}{k}$ diverges by $\boldsymbol{p}$*–series* $(p = 1 \le 1)$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\coth k}{k} \cdot \frac{k}{1}$$

$$= \lim_{k\to\infty} \coth k$$

$$= \lim_{k\to\infty} \frac{e^k + e^{-k}}{e^k - e^{-k}}$$

$$= \lim_{k\to\infty} \frac{e^k}{e^k}$$

$$\underline{= 1 > 0}$$

Therefore; the given series ***diverges*** by *Limit Comparison Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sinh k}$

Solution

$$a_k = \frac{1}{\sinh k}$$

Let $b_k = \frac{1}{e^k}$

$$= \left(\frac{1}{e}\right)^k$$

$\sum b_k$ converges by Geometric series with $\left(r = \frac{1}{e} < 1\right)$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1}{\sinh k} \cdot \frac{e^k}{1}$$

$$= \lim_{k\to\infty} \frac{2}{e^k - e^{-k}} \cdot e^k$$

$$= \lim_{k\to\infty} \frac{2}{1 - e^{-2k}}$$

$$\underline{= 2}$$

Therefore; the given series ***converges*** by *Limit Comparison Test*.

Let $f(x) = \frac{1}{\sinh x}$

$$= \frac{2}{e^x - e^{-x}}$$

$$\int_1^\infty \frac{2}{e^x - e^{-x}} \cdot \frac{e^x}{e^x}\, dx = \int_1^\infty \frac{2e^x}{\left(e^x\right)^2 - 1} dx$$

$$= \int_1^\infty \frac{2}{\left(e^x - 1\right)\left(e^x + 1\right)}\, d\left(e^x\right)$$

$$= \int_1^\infty \left(\frac{1}{e^x - 1} - \frac{1}{e^x + 1}\right) d\left(e^x\right)$$

$$= \int_1^\infty \frac{1}{e^x - 1}\, d\left(e^x - 1\right) - \int_1^\infty \frac{1}{e^x + 1}\, d\left(e^x + 1\right)$$

$$= \ln\left(e^x - 1\right) - \ln\left(e^x + 1\right) \Big|_1^\infty$$

$$= \ln\left(\frac{e^x - 1}{e^x + 1}\right) \Big|_1^\infty$$

$$= \ln 1 - \ln\frac{e-1}{e+1}$$

$$= -\ln\frac{e-1}{e+1}$$

Therefore; the given series ***converges*** by *Integral Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=1}^{\infty} \tanh k$

Solution

$$\lim_{k\to\infty} \tanh k = \lim_{k\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$= 1 \neq 0$$

Therefore; the given series ***diverges*** by *Divergence Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=0}^{\infty} \operatorname{sech} k$

Solution

$$a_k = \operatorname{sech} k$$

Let $b_k = \frac{1}{e^k}$

$$= \left(\frac{1}{e}\right)^k$$

$\sum b_k$ converges by Geometric series with $\left(r = \frac{1}{e} < 1\right)$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{2}{e^k + e^{-k}} \cdot e^k$$

$$= \lim_{k\to\infty} \frac{2}{1+e^{-2k}}$$

$$= 2$$

Therefore; the given series ***converges*** by *Limit Comparison Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=2}^{\infty} \frac{(-1)^k}{k^2-1}$

Solution

$$a_k = \frac{(-1)^k}{k^2-1}$$

$$\left|a_k\right| = \frac{1}{k^2-1}$$

Let $b_k = \frac{1}{k^2}$ converges by $\boldsymbol{p}$–series $(p = 2 > 1)$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1}{k^2-1} \cdot k^2$$

$$= 1$$

Therefore; the given series ***converges absolutely*** by *Limit Comparison Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=1}^{\infty}(-1)^{k+1}\dfrac{k^2+4}{2k^2+1}$

Solution

$$\lim_{k\to\infty}\left|\frac{k^2+4}{2k^2+1}\right| \underline{=\frac{1}{2}\neq 0 |}$$

Therefore; the given series ***diverges*** by *Alternating series*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=1}^{\infty}(-1)^{k}\ ke^{-k}$

Solution

Using Ratio Test:

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{(k+1)e^{-k-1}}{ke^{-k}}$$

$$=\frac{1}{e}\lim_{k\to\infty}\frac{k+1}{k}$$

$$\underline{=\frac{1}{e}<1 |}$$

Therefore; the given series ***converges absolutely*** by *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=1}^{\infty}\dfrac{(-1)^{k}}{\sqrt{k^2+1}}$

Solution

$$b_k = \frac{1}{\sqrt{k^2}}$$

$$=\frac{1}{k}$$

$$\lim_{k\to\infty}\left|\frac{a_k}{b_k}\right| = \lim_{k\to\infty}\frac{1}{\sqrt{k^2+1}}\cdot k$$

$$=\lim_{k\to\infty}\frac{k}{\sqrt{k^2}}$$

$= 1 \neq 0$

Therefore; the given series ***diverges*** by *Limit Comparison Test*.

But the series is *decreasing*, therefore; it is ***conditionally convergent***.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{10^k}{k!}$

Solution

Using Ratio Test:

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{10^{k+1}}{(k+1)!} \cdot \frac{k!}{10^k}$$

$$= \lim_{k\to\infty} \frac{10}{k+1}$$

$$= 0$$

Therefore; the given series ***converges absolutely*** by *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=1}^{\infty} \frac{(-2)^{k+1}}{k^2}$

Solution

$$\frac{(-2)^{k+1}}{k^2} = \frac{(-1)^{k+1}\, 2^{k+1}}{k^2}$$

$$\left|a_k\right| = \frac{2^{k+1}}{k^2}$$

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left(\frac{2^{k+2}}{(k+1)^2} \cdot \frac{k^2}{2^{k+1}}\right)$$

$$= \lim_{k\to\infty} 2\left(\frac{k}{k+1}\right)^2$$

$$= 2 > 1$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=0}^{\infty} \frac{(-1)^k}{e^k + e^{-k}}$

Solution

$$\left|a_k\right| = \frac{1}{e^k + e^{-k}}$$

Using Limit Comparison Test

$$\text{let } \left|b_k\right| = \frac{1}{e^k}$$

$$= \left(\frac{1}{e}\right)^k$$

b_k converges by Geometric series $\left(r = \frac{1}{e} < 1\right)$

$$\lim_{k\to\infty} \left|\frac{a_k}{b_k}\right| = \lim_{k\to\infty} \left(\frac{1}{e^k + e^{-k}} \cdot \frac{e^k}{1}\right)$$

$$= \lim_{k\to\infty} \frac{e^k}{e^k + e^{-k}}$$

$$= 1$$

Therefore; the given series ***converges absolutely*** by the *Limit Comparison Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln k}$

Solution

$$\left|a_k\right| = \frac{1}{k \ln k}$$

$$\text{Let } \quad f(x) = \frac{1}{x \ln x}$$

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \int_2^{\infty} \frac{1}{\ln x} d(\ln x)$$

$$= \ln(\ln x) \Big|_2^{\infty}$$

$$= \ln(\ln \infty) - \ln(\ln 2)$$

$$= \infty$$

Therefore; the given series ***diverges absolutely*** by *Integral Test*.

However;

$$\lim_{k\to\infty} \left|a_k\right| = \lim_{k\to\infty} \frac{1}{k \ln k}$$

$$\underline{= 0}\,\big|$$

This series ***converges conditionally*** by the *Divergence Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=2}^{\infty} (-1)^k \frac{\ln k}{k^2}$

Solution

Let $\quad f(x) = \dfrac{\ln x}{x^2}$

$$f'(x) = \frac{\frac{1}{x}\left(x^2\right) - 2x \ln x}{x^4}$$

$$= \frac{x - 2x \ln x}{x^4}$$

$$= \frac{1 - 2\ln x}{x^3}$$

As x gets larger, $f'(x) < 0$

The given series decreases.

$$\lim_{k\to\infty} \frac{\ln k}{k^2} = \frac{\infty}{\infty}$$

$$= \lim_{k\to\infty} \frac{\frac{1}{k}}{2k}$$

$$= \lim_{k\to\infty} \frac{1}{2k^2}$$

$$\underline{= 0}\,\big|$$

Therefore; the given series ***converges*** by *Alternating Series Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln^2 k}$

Solution

$$k \ln^2 k < (k+1)\ln^2 (k+1)$$

$$\frac{1}{k\ln^2 k} > \frac{1}{(k+1)\ln^2(k+1)} \quad \checkmark$$

$$\lim_{k\to\infty} \frac{1}{k\ln^2 k} = 0$$

Therefore; the given series ***converges*** by *Alternating Series Test*.

Let $f(x) = \frac{1}{x\ln^2 x}$

$$\int_2^\infty \frac{1}{x\ln^2 x}\,dx = \int_2^\infty \frac{1}{\ln^2 x}\,d(\ln x)$$

$$= -\frac{1}{\ln x}\Big|_2^\infty$$

$$= -\left(\frac{1}{\ln\infty} - \frac{1}{\ln 2}\right)$$

$$= -\left(0 - \frac{1}{\ln 2}\right)$$

$$= \frac{1}{\ln 2}$$

Therefore; the given series ***converges absolutely*** by *Integral Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}$

Solution

$$\ln k < \ln(k+1)$$

$$\frac{1}{\ln k} > \frac{1}{\ln(k+1)} \quad \checkmark$$

$$\lim_{k\to\infty} \frac{1}{\ln k} = 0$$

Therefore; the given series ***converges*** by *Alternating Series Test*.

However,

$$\ln k < k$$

$$\frac{1}{\ln k} > \frac{1}{k}$$

$\sum \frac{1}{k}$ ***diverges*** by ***p**-series* $(p = 1 \le 1)$

Therefore; the given series ***converges conditionally.***

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=2}^{\infty} 3e^{-k}$

Solution

$$\sum_{k=1}^{\infty} 3e^{-k} = 3\sum_{k=1}^{\infty} \left(\frac{1}{e}\right)^k$$

This is Geometric series with $r = \frac{1}{e} < 1$

$$a_2 = \frac{3}{e^2} = a$$

$$S = \frac{\frac{3}{e^2}}{1 - \frac{1}{e}}$$

$$= \frac{3}{e(e-1)}$$

Therefore; the given series ***converges*** by *Geometric Series.*

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=1}^{\infty} \frac{2^k}{e^k - 1}$

Solution

$$a_k = \frac{2^k}{e^k - 1}$$

Let $b_k = \frac{2^k}{e^k}$

$$= \left(\frac{2}{e}\right)^k$$

$\sum b_k$ is Geometric series with $r = \frac{2}{e} < 1$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \left(\frac{2^k}{e^k - 1} \cdot \frac{e^k}{2^k}\right)$$

$$= \lim_{k\to\infty} \frac{e^k}{e^k - 1}$$

$$= 1$$

Therefore; the given series ***converges*** by *Limit Comparison Test.*

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=1}^{\infty}(-1)^{k+1}\dfrac{e^k}{(k+1)!}$

Solution

$$\left|a_k\right| = \frac{e^k}{(k+1)!}$$

Using the Ratio Test:

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\left(\frac{e^{k+1}}{(k+2)!}\cdot\frac{(k+1)!}{e^k}\right)$$

$$= \lim_{k\to\infty}\frac{e}{k+2}$$

$$\boxed{= 0}$$

Therefore; the given series ***converges absolutely*** by *Ratio Test.*

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=1}^{\infty}\dfrac{k}{\left(k^2+1\right)^3}$

Solution

Let $f(x) = \dfrac{x}{\left(x^2+1\right)^3}$

$$\int_1^{\infty}\frac{x}{\left(x^2+1\right)^3}dx = \frac{1}{2}\int_1^{\infty}\left(x^2+1\right)^{-3}d\left(x^2+1\right)$$

$$= -\frac{1}{4}\left(x^2+1\right)^{-2}\Big|_1^{\infty}$$

$$= -\frac{1}{4}\frac{1}{\left(x^2+1\right)^2}\Big|_1^{\infty}$$

$$= -\frac{1}{4}\left(0-\frac{1}{4}\right)$$

$$\boxed{= \frac{1}{16}}$$

Therefore; the given series ***converges*** by *Integral Test.*

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=2}^{\infty} \frac{k^e}{k^{\pi}}$

Solution

$$\sum_{k=2}^{\infty} \frac{k^e}{k^{\pi}} = \sum_{k=2}^{\infty} \frac{1}{k^{\pi-e}}$$

$$\begin{aligned} \pi - e &\approx 3.141 - 2.718 \\ &\approx .42 < 1 \end{aligned}$$

Therefore; the given series ***diverges*** by *p-series* $(p = \pi - e < 1)$.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=3}^{\infty} \frac{1}{(k-2)^4}$

Solution

$$\sum_{k=3}^{\infty} \frac{1}{(k-2)^4} = \sum_{k=1}^{\infty} \frac{1}{k^4}$$

Therefore; the given series ***converges*** by *p-series* $(p = 4 > 1)$.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=1}^{\infty} \left(\frac{2k}{k+1}\right)^k$

Solution

$$\begin{aligned} \lim_{k\to\infty} \sqrt[k]{\left(\frac{2k}{k+1}\right)^k} &= \lim_{k\to\infty} \frac{2k}{k+1} \\ &= 2 > 1 \end{aligned}$$

Therefore; the given series ***diverges*** by *Root Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{2^k}$

Solution

$$\lim_{k\to\infty} \sqrt[k]{\frac{k^2}{2^k}} = \lim_{k\to\infty} \frac{k^{2/k}}{2}$$

$$= \frac{1}{2} < 1$$

Therefore; the given series ***converges*** by *Root Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=1}^{\infty} \frac{k^2-1}{k^3+4}$

Solution

$$a_k = \frac{k^2-1}{k^3+4}$$

$$b_k = \frac{k^2}{k^3}$$

$$= \frac{1}{k}$$

$\sum b_k$ *diverges* by ***p****-series* with $(p = 1 \le 1)$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \left(\frac{k^2-1}{k^3+4} \cdot \frac{k}{1}\right)$$

$$= \lim_{k\to\infty} \frac{k^3-k}{k^3+4}$$

$$= 1$$

Therefore; the given series ***diverges*** by *Limit Comparison Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} \sin\frac{\pi}{2k}$

Solution

$$-1 \le \sin\frac{\pi}{2k} \le 1$$

$$-\frac{1}{k^2} \le \frac{1}{k^2}\sin\frac{\pi}{2k} \le \frac{1}{k^2}$$

$b_k = \frac{1}{k^2}$ *converges* by ***p**-series* with $(p = 2 \ge 1)$

Therefore; the given series ***converges*** by *Comparison Test.*

Let $f(x) = \frac{1}{x^2}\sin\frac{\pi}{2x}$

$$\int_1^\infty \frac{1}{x^2}\sin\frac{\pi}{2x}\,dx = -\frac{2}{\pi}\int_1^\infty \sin\frac{\pi}{2x}\,d\left(\frac{\pi}{2x}\right)$$

$$= \frac{2}{\pi}\cos\frac{\pi}{2x}\Big|_1^\infty$$

$$= \frac{2}{\pi}\left(\cos 0 - \cos\frac{\pi}{2}\right)$$

$$= \frac{2}{\pi}(1-0)$$

$$= \frac{2}{\pi}$$

Therefore; the given series ***converges*** by *Integral Test.*

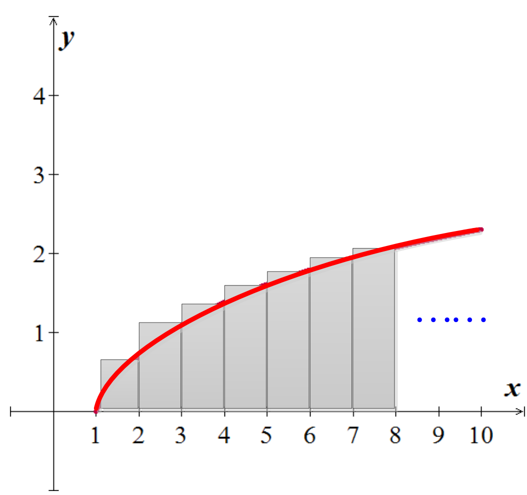
Exercise

Use a Riemann sum argument to show that $\ln n! \ge \int_1^n \ln t\, dt = n\ln n - n + 1$

Then for what values of x does the series $\sum_{n=1}^{\infty}\frac{n!x^n}{n^n}$ converge absolutely? Converge conditionally?

Diverge? (*Use the ratio test first*)

Solution



$\ln n! = \ln 1 + \ln 2 + \ln 3 + \cdots + \ln n$

$\qquad$ = Sum of area of the shaded rectangles

$$> \int_1^n \ln t\, dt$$

$$= t\ln t - t\Big|_1^n$$

$$= n\ln n - n + 1$$

Using the ratio test

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n!x^n}{n^n}$

$$\rho = \lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n!x^n} \right|$$

$$= |x| \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \qquad \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = e$$

$$\boxed{= \frac{|x|}{e} < 1}$$

If $|x| < e$

$-e < x < e$

The given series converges absolutely.

If $x = \pm e$, then

$$\ln\left|\frac{n!x^n}{n^n}\right| = \ln n! + \ln\left|x^n\right| - \ln n^n$$

$$= \ln n! + \ln e^n - \ln n^n$$

$$> n\ln n - n + 1 + n - \ln n^n$$

$$= \ln n^n + 1 - \ln n^n$$

$$= 1$$

$$\Rightarrow \left|\frac{n!x^n}{n^n}\right| > e$$

Hence, the given series ***converges*** absolutely if $-e < x < e$ and ***diverges*** *elsewhere*.

Exercise

Let S_n be the *n*th partial sum of $\sum_{k=1}^{\infty} a_k = 8$. Find the $\lim_{k\to\infty} a_k$ and $\lim_{n\to\infty} S_n$

Solution

Since the series converges to 8, then $\lim_{k\to\infty} a_k = 0$

Therefore; the partial sums converges to 8.

$$\lim_{n\to\infty} S_n = 8$$

Exercise

It can be proved that if a series converges absolutely, then its terms may be summed in any order without changing the value of the series. However, if a series converges conditionally, then the value of the series depends on the order of summation. For example, the (conditionally convergent) alternating harmonic series has the value

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots=\ln 2$$

Show that by rearranging the terms (so the sign pattern is ++–),

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\ldots=\frac{3}{2}\ln 2$$

Solution

$$\begin{aligned}S&=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\\&=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\left(\frac{1}{7}-\frac{1}{8}\right)+\ldots\end{aligned}$$

$$\begin{aligned}\frac{1}{2}S&=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\right)\\&=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots\end{aligned}$$

$$+\begin{cases}S=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\left(\frac{1}{7}-\frac{1}{8}\right)+\ldots=\ln 2\\ \frac{1}{2}S=\quad\frac{1}{2}\quad-\quad\frac{1}{4}\quad+\quad\frac{1}{6}\quad-\quad\frac{1}{8}\quad+\cdots\quad=\frac{1}{2}\ln 2\end{cases}$$

$$\begin{aligned}\frac{3}{2}S&=1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots\\&=\frac{3}{2}\ln 2\end{aligned}$$

Exercise

A crew of workers is constructing a tunnel through a mountain. Understandably, the rate of construction decreases because rocks and earth must be removed a greater distance as the tunnel gets longer. Suppose that each week the crew digs 0.95 of the distance it dug the previous week. In the first week, the crew constructed 100 *m* of tunnel.

a) How far does the crew dig in 10 *weeks*? 20 *weeks*? *N weeks*?

b) What is the longest tunnel the crew can build at this rate?

c) The time required to dig 100 *m* increases by 10% each week, starting with 1 *week* to dig the first 100 *m*. Can the crew complete a 1.5 *km* tunnel in 10 *weeks*? Explain.

Solution

a) Let T_n be the amount of additional tunnel dug during week *n*. Then

$T_0=100$

$$T_n = 0.95\, T_{n-1}$$
$$= (0.95)^n\ T_0$$
$$= 100(0.95)^n$$

So, the total distance dug in N weeks is

$$S_N = 100 \sum_{k=0}^{N-1} (0.95)^k$$
$$= 100\left(\frac{1-(0.95)^N}{1-0.95}\right)$$
$$= 2000\left(1-(0.95)^N\right)$$

For 10 weeks: $S_{10} = 2000\left(1-(0.95)^{10}\right)$

$$\approx 802.5\ m$$

For 20 weeks: $S_{20} = 2000\left(1-(0.95)^{20}\right)$

$$\approx 1283.03\ m$$

b) The longest possible tunnel is

$$S_\infty = 100 \sum_{k=0}^{\infty} (0.95)^k$$
$$= \frac{100}{1-0.95}$$
$$= 2000\ m$$

c) The time required to dig $t_n = 100(n-1)$ through $n \cdot 100$

$$T_n = 1.1\, T_{n-1}$$
$$= (1.1)^{n-1}\ T_1$$
$$= (1.1)^{n-1}\ \textit{weeks}$$

The time required to dig 1500 m is:

$$\sum_{k=1}^{15} t_k = \sum_{k=1}^{15} (1.1)^{k-1}$$
$$= \frac{1-1.1^{15}}{1-1.1}$$

$$\approx 31.77 \quad weeks$$

So, it is ***not*** possible.

Exercise

Consider the alternating series

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k \quad where \quad a_k = \begin{cases} \dfrac{4}{k+1} & if\ k\ is\ odd \\ \dfrac{2}{k} & if\ k\ is\ even \end{cases}$$

a) Write out the first ten terms of the series, group them in pairs, and show that the even partial sums of the series form the (divergent) harmonic series.

b) Show that $\lim\limits_{k\to\infty} a_k = 0$

c) Explain why the series diverges even though the terms of the series approach zero.

Solution

a) The first ten terms of the series are:

$$(2-1)+\left(1-\frac{1}{2}\right)+\left(\frac{2}{3}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{2}{5}-\frac{1}{5}\right)$$

Suppose that $\begin{cases} for\ even & k=2i \\ for\ odd & k=2i-1 \end{cases}$

Then the sum of the $(k-1)st$ term and the kth term is

$$\frac{4}{k}-\frac{2}{k}=\frac{2}{k}$$

$$=\frac{2}{2i}$$

$$=\frac{1}{i}$$

Then the sum of the even partial sums of the given series is $\sum\limits_{i-1}^{n}\frac{1}{i}$

b) $\lim\limits_{k\to\infty}\frac{4}{k+1}=\lim\limits_{k\to\infty}\frac{4}{k}=0$

Given $\varepsilon>0$, $\exists N_1$ so that for $k>N_1$ we have $\frac{4}{k+1}<\varepsilon$.

Also $\exists N_2$ so that for $k>N_2$, $\frac{2}{k}<\varepsilon$.

Let N be the larger of N_1 or N_2. Then for $k>N$, we have $a_k<\varepsilon$ as desired.

c) The series can be seen to diverge because the even partial sums have limit ∞.
This does not contradict the alternating series test because the terms a_k are not nonincreasing.

Exercise

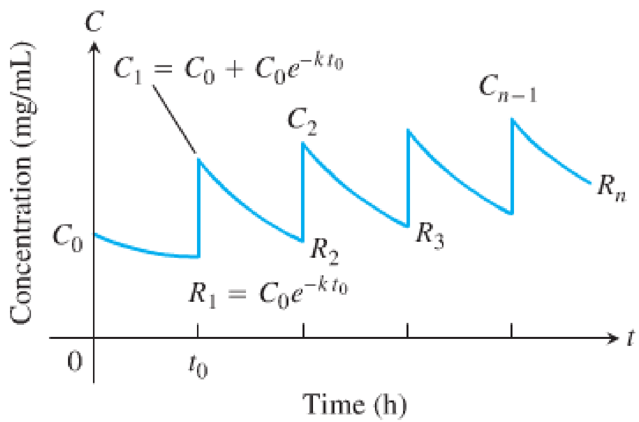
The concentration in the blood resulting from a single dose of a drug normally decreases with time as the drug is eliminated from the body. Doses may therefore need to be repeated periodically to keep the concentration from dropping below some particular level. One model for the effect of repeated doses gives the residual concentration just before the $(n+1)st$ does as

$$R_n = C_0 e^{-kt_0} + C_0 e^{-2kt_0} + \cdots + C_0 e^{-nkt_0}$$

Where C_0 = the change in concentration achievable by a single dose (mg/mL),

$k =$ the elimination constant $\left(h^{-1}\right)$, and

$t_0 =$ time between doses (h).



a) Write R_n in closed form as a single fraction, and find $R = \lim\limits_{n\to\infty} R_n$

b) Calculate R_1 and R_{10} for $C_0 = 1\ mg/\text{mL}$, $k = 0.1\ h^{-1}$, and $t_0 = 10\ h$. How good as estimate of R is R_{10}

c) If $k = 0.01\ h^{-1}$ and $t_0 = 10\ h$, find the smallest n such that $R_n > \frac{1}{2}R$.

Solution

a) $R_n = C_0 e^{-kt_0} + C_0 e^{-2kt_0} + \cdots + C_0 e^{-nkt_0}$

$$= C_0\left(e^{-kt_0} + \left(e^{-kt_0}\right)^2 + \cdots + \left(e^{-kt_0}\right)^n\right)$$

$$= C_0\frac{e^{-kt_0}\left(1 - e^{-nkt_0}\right)}{1 - e^{-kt_0}}$$

$R = \lim\limits_{n\to\infty} R_n$

$$= \lim_{n\to\infty} C_0\frac{e^{-kt_0}\left(1 - e^{-nkt_0}\right)}{1 - e^{-kt_0}} \qquad \lim_{n\to\infty} e^{-nkt_0} = 0$$

$$= \frac{C_0 e^{-kt_0}}{1-e^{-kt_0}}$$

$$\boxed{= \frac{C_0}{e^{kt_0}-1}}$$

b) Given: $C_0 = 1\ mg\,/\,\mathrm{mL}$, $k = 0.1\ h^{-1}$, and $t_0 = 10\ h$

$$R_n = C_0 \frac{e^{-kt_0}\left(1-e^{-nkt_0}\right)}{1-e^{-kt_0}}$$

$$= 1\frac{e^{-(0.1)(10)}\left(1-e^{-n(0.1)(10)}\right)}{1-e^{-(0.1)(10)}}$$

$$\boxed{= \frac{e^{-1}\left(1-e^{-n}\right)}{1-e^{-1}}}$$

$$R_1 = \frac{e^{-1}\left(1-e^{-1}\right)}{1-e^{-1}} = e^{-1}$$

$$\boxed{\approx 0.36787944}$$

$$R_{10} = \frac{e^{-1}\left(1-e^{-10}\right)}{1-e^{-1}}$$

$$\boxed{\approx 0.58195028}$$

c) Given: $k = 0.01\ h^{-1}$ and $t_0 = 10\ h$

$$R_n = C_0 \frac{e^{-0.1}\left(1-e^{-0.1n}\right)}{1-e^{-0.1}}$$

$$R = \frac{C_0}{e^{kt_0}-1}$$

$$= \frac{C_0}{e^{0.1}-1}$$

$$\frac{R}{2} = \frac{1}{2}\frac{C_0}{e^{0.1}-1}$$

$$R_n > \frac{1}{2}R \quad \xrightarrow{n=?} \quad C_0 \frac{e^{-0.1}\left(1-e^{-0.1n}\right)}{1-e^{-0.1}} > \frac{1}{2}\frac{C_0}{e^{0.1}-1}$$

$$\frac{\left(1-e^{-0.1n}\right)}{e^{0.1}-1} > \frac{1}{2}\frac{1}{e^{0.1}-1}$$

$$1-e^{-0.1n} > \frac{1}{2}$$

$$e^{-0.1n} < \frac{1}{2}$$

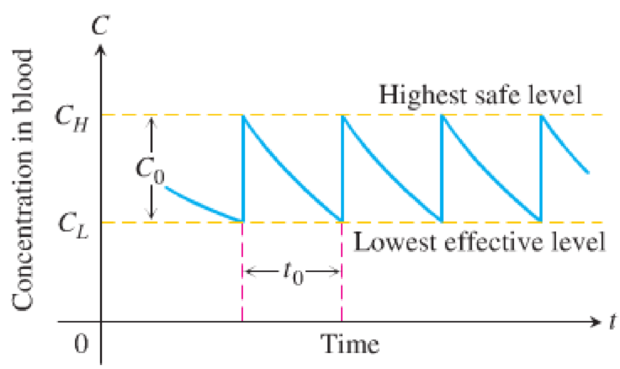
$$-0.1n < \ln\frac{1}{2}$$

$$n > 6.93$$

$$\Rightarrow \quad \underline{n=7}$$

Exercise

If a drug is known to be ineffective below a concentration C_L and harmful above some higher concentration C_H, one needs to find values of C_0 and t_0 that will produce a concentration that is safe (not above C_H) but effective (not below C_L).



We therefore want to find values for C_0 and t_0 for which

$$R = C_L \quad \text{and} \quad C_0 + R = C_H$$

Thus $C_0 = C_H - C_L$. The resulting equation simplifies to

$$t_0 = \frac{1}{k}\ln\frac{C_H}{C_L}$$

To reach an effective level rapidly, one might administer a "loading" dose that would produce a concentration of $C_H\ mg/mL$. This could be followed every t_0 hours by a dose that raises the concentration by $C_0 = C_H - C_L\ \ mg/mL$.

a) Verify the preceding equation for t_0.

b) If $k = 0.05\ h^{-1}$ and the highest safe concentration is *e* times the lowest effective concentration, find the length of time between doses that will assure safe and effective concentrations.

c) Given $C_H = 2\ mg/mL,\ C_L = 0.5\ mg/mL,$ and $k = 0.02\ h^{-1},$ determine a scheme for administering the drug.

d) Suppose that $k = 0.2\ h^{-1}$ and the smallest effective concentration is 0.03 *mg/mL*. A single dose that produces a concentration of 0.1 *mg/mL* is administered. About how long will the drug remain effective?

Solution

a) $R = \dfrac{C_0}{e^{kt_0} - 1}$ $\qquad\qquad (R = C_L)$

$Re^{kt_0} = R + C_0 = C_H$

$e^{kt_0} = \dfrac{C_H}{C_L}$

$\boxed{t_0 = \dfrac{1}{k}\ln\dfrac{C_H}{C_L}}$

b) $t_0 = \dfrac{1}{0.05}\ln e$

$\quad \boxed{= 20\ hrs}$

c) Given $C_H = 2\ mg/mL,\ C_L = 0.5\ mg/mL,$ and $k = 0.02\ h^{-1}$

$t_0 = \dfrac{1}{k}\ln\dfrac{C_H}{C_L}$

$\quad = \dfrac{1}{0.02}\ln\left(\dfrac{2}{0.5}\right)$

$\quad \boxed{\approx 69.31\ hrs}$

A dose raises every 69.31 *hrs.* the concentration by 1.5 mg/mL

d) $t_0 = \dfrac{1}{0.2}\ln\dfrac{0.1}{0.03}$

$\quad \boxed{\approx 6\ hrs}$

SOLUTION Section 3.7 – Power Series

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of *x* does the series converge (***b***) absolutely, (***c***) conditionally?

$$\sum_{n=0}^{\infty} x^n$$

Solution

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{x^n}\right| = |x| < 1 \quad \Rightarrow \quad -1 < x < 1$$

When $x = 1 \quad \Rightarrow \quad \sum_{n=0}^{\infty} 1$

and $x = -1 \quad \Rightarrow \quad \sum_{n=0}^{\infty} (-1)^n$ the series diverges.

a) The radius is 1; the interval of converges $-1 < x < 1$

b) The interval of absolute convergence is $-1 < x < 1$

c) There are no values for which the series converges conditionally

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of *x* does the series converge (***b***) absolutely, (***c***) conditionally?

$$\sum_{n=0}^{\infty} (x+5)^n$$

Solution

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(x+5)^{n+1}}{(x+5)^n}\right|$$

$$= |x+5| < 1$$

$$-6 < x < -4$$

When $x = -6 \quad \Rightarrow \quad \sum_{n=0}^{\infty} (-1)^n$

and $x=-4 \quad \Rightarrow \quad \sum_{n=0}^{\infty} 1$ the series diverges.

a) The radius is 1; the interval of converges $-6<x<-4$

b) The interval of absolute convergence is $-6<x<-4$

c) There are no values for which the series converges conditionally.

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of *x* does the series converge (***b***) absolutely, (***c***) conditionally?

$$\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$$

Solution

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(3x-2)^{n+1}}{n+1} \cdot \frac{n}{(3x-2)^n}\right|$$

$$= \frac{n}{n+1}|3x-2| < 1$$

$$\lim_{n\to\infty} \frac{n}{n+1}|3x-2| < 1$$

$$|3x-2| < 1$$

$$-1 < 3x-2 < 1$$

$$1 < 3x < 3$$

$$\boxed{\frac{1}{3} < x < 1}$$

When $x=\frac{1}{3} \quad \Rightarrow \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ which is the alternating harmonic series and is conditionally convergent.

$x=1 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \frac{1}{n}$ the series diverges harmonic.

a) The radius is $\frac{1}{3}$; the interval of converges $\frac{1}{3} \le x < 1$

b) The interval of absolute convergence is $\frac{1}{3} < x < 1$

c) The series converges conditionally at $x=\frac{1}{3}$

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of x does the series converge
(***b***) absolutely,
(***c***) conditionally?

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$$

Solution

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(x-2)^{n+1}}{10^{n+1}} \cdot \frac{10^n}{(x-2)^n}\right|$$

$$= \frac{|x-2|}{10} < 1$$

$$-1 < \frac{x-2}{10} < 1$$

$$-10 < x-2 < 10$$

$$-8 < x < 12$$

When $x = -8 \quad \Rightarrow \quad \sum_{n=0}^{\infty} (-1)^n$ which is a divergent series

$x = 12 \quad \Rightarrow \quad \sum_{n=0}^{\infty} 1$ the series diverges

a) The radius is 10; the interval of converges $-8 < x < 12$
b) The interval of absolute convergence is $-8 < x < 12$
c) There are no values for which the series converges conditionally

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of x does the series converge
(***b***) absolutely,
(***c***) conditionally?

$$\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$$

Solution

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{3^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n x^n}\right|$$

$$= \frac{3}{n+1}\left|x^n\right| < 1$$

$$3|x| \lim_{n\to\infty} \frac{1}{n+1} < 1 \quad \Rightarrow \ \forall x$$

a) The radius is ∞; the series converges for all x.

b) The series convergence absolutely for all x.

c) There are no values for which the series converges conditionally

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of x does the series converge

(***b***) absolutely,

(***c***) conditionally?

$$\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2+3}}$$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{\sqrt{(n+1)^2+3}} \cdot \frac{\sqrt{n^2+3}}{x^n}\right|$$

$$= |x| \lim_{n\to\infty} \frac{n^2+3}{n^2+2n+4} < 1$$

$$|x| < 1 \quad \Rightarrow \quad -1 < x < 1$$

When $x=-1 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+3}}$ which is a convergent conditionally series

$x=1 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+3}}$ the series diverges

a) The radius is 1; the series converges for $-1 \le x < 1$.

b) The series convergence absolutely for $-1 < x < 1$.

c) The series convergence conditionally for $x=-1$

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of x does the series converge (***b***) absolutely, (***c***) conditionally?

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{\sqrt{n+3}}$$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+2}}{\sqrt{n+1}+3}\cdot\frac{\sqrt{n}+3}{x^{n+1}}\right|$$

$$=|x|\lim_{n\to\infty}\frac{\sqrt{n}+3}{\sqrt{n+1}+3}<1$$

$|x|<1 \quad\Rightarrow\quad -1<x<1$

When $x=-1 \quad\Rightarrow\quad \sum_{n=0}^{\infty}\frac{1}{\sqrt{n}+3}$ which is a divergent series

$x=1 \quad\Rightarrow\quad \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}+3}$ the series converges conditionally

a) The radius is 1; the series converges for $-1<x\le 1$.

b) The series convergence absolutely for $-1<x<1$.

c) The series convergence conditionally for $x=1$

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of *x* does the series converge (***b***) absolutely, (***c***) conditionally?

$$\sum_{n=1}^{\infty}\sqrt[n]{n}\,(2x+5)^n$$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt[n+1]{n+1}\,(2x+5)^{n+1}}{\sqrt[n]{n}\,(2x+5)^n}\right|$$

$$=|2x+5|\lim_{n\to\infty}\frac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}}<1$$

$$=|2x+5|\frac{\lim_{m\to\infty}\sqrt[m]{m}}{\lim_{n\to\infty}\sqrt[n]{n}}<1$$

$$=|2x+5|<1$$

$|2x+5|<1$

$-1<2x+5<1$

$-3<x<-2$

When $x=-3 \quad \Rightarrow \quad \sum_{n=0}^{\infty}(-1)^n \sqrt[n]{n}$ which is a divergent series

$x=-0 \quad \Rightarrow \quad \sum_{n=0}^{\infty}\sqrt[n]{n}$ which is a divergent series

a) The radius is 1; the series converges for $-3<x<-2$.

b) The series convergence absolutely for $-3<x<-2$.

c) There are no values for which the series convergence conditionally

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of x does the series converge
(***b***) absolutely,
(***c***) conditionally?

$$\sum_{n=1}^{\infty}\left(2+(-1)^n\right)\cdot(x+1)^{n-1}$$

Solution

$$\sum_{n=1}^{\infty}\left(2+(-1)^n\right)\cdot(x+1)^{n-1}=\sum_{n=1}^{\infty}2(x+1)^{n-1} \;+\; \sum_{n=1}^{\infty}(-1)^n(x+1)^{n-1}$$

For the series $\sum_{n=1}^{\infty}2(x+1)^{n-1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{2(x+1)^n}{2(x+1)^{n-1}}\right|$$

$$=|x+1|<1$$

$$-1<x+1<1$$
$$-2<x<0$$

For the series $\sum_{n=1}^{\infty}(-1)^n(x+1)^{n-1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x+1)^n}{(x+1)^{n-1}}\right|$$

$$=|x+1|<1$$

$$-1<x+1<1$$

$-2 < x < 0$

When $x = -2 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \left(2 + (-1)^n\right) \cdot (-1)^{n-1}$ which is a divergent series

$x = 0 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \left(2 + (-1)^n\right)$ which is a divergent series

a) The radius is $\frac{1}{2}$; the series converges for $-2 < x < 0$.

b) The series convergence absolutely for $-2 < x < 0$.

c) There are no values for which the series convergence conditionally

Exercise

Find the radius of convergence of the power series $\sum_{n=0}^{\infty} n!\, x^n$

Solution

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right|$$

$$= |x| \lim_{n\to\infty} (n+1)$$

$$= \infty$$

$\rightarrow \quad R = \frac{1}{\infty} = 0$

By the Ratio Test, the series diverges for $|x| > 0$ and converges only at its center, 0.

Therefore; the radius of convergence is $R = 0$.

Exercise

Find the radius of convergence of the power series $\sum_{n=0}^{\infty} 3(x-2)^n$

Solution

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{3(x-2)^{n+1}}{3(x-2)^n}\right|$$

$$= |x-2|$$

By the Ratio Test, the series converges for $|x-2| < 1$ and diverges for $|x-2| > 1$.

Therefore; the radius of convergence is $R = 1$.

Exercise

Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right|$$

$$= x^2 \lim_{n\to\infty} \frac{1}{(2n+2)(2n+3)}$$

$$\underline{= 0}$$

$$\to \quad R = \frac{1}{0} = \infty$$

By the Ratio Test, the series converges for all x. Therefore; the radius of convergence is $R = \infty$.

Exercise

Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+2}\cdot\frac{n+1}{x^n}\right|$$

$$= |x| \lim_{n\to\infty} \frac{n+1}{n+2}$$

$$\underline{= |x|}$$

$$\to \quad R = 1$$

By the Ratio Test, the series converges for $|x| < 1$ and diverges for $|x| > 1$.

Therefore; the radius of convergence is $R = 1$.

Exercise

Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} (3x)^n$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(3x)^{n+1}}{(3x)^n}\right|$$

$$= 3|x|$$

$$\to \quad 3|x| < 1 \;\Rightarrow\; R = \frac{1}{3}$$

By the Ratio Test, the series converges for $|x| < \frac{1}{3}$ and diverges for $|x| > \frac{1}{3}$.

Therefore; the radius of convergence is $R = \frac{1}{3}$.

Exercise

Find the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(4x)^n}{n^2}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(4x)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(4x)^n}\right|$$

$$= |4x| \lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\right|$$

$$= 4|x|$$

$$\to \quad 4|x| < 1 \;\Rightarrow\; R = \frac{1}{4}$$

By the Ratio Test, the series converges for $|x| < \frac{1}{4}$ and diverges for $|x| > \frac{1}{4}$.

Therefore; the radius of convergence is $R = \frac{1}{4}$.

Exercise

Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{5^n}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{5^{n+1}}\cdot\frac{5^n}{x^n}\right|$$

$$= \frac{|x|}{5}$$

$$\to \quad \frac{|x|}{5} < 1 \;\Rightarrow\; R = 5$$

By the Ratio Test, the series converges for $|x| < 5$ and diverges for $|x| > 5$.

Therefore; the radius of convergence is $R = 5$.

Exercise

Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right|$$

$$= x^2 \lim_{n\to\infty} \frac{1}{(2n+1)(2n+1)}$$

$$\underline{=0}$$

$$R = \frac{1}{0} = \infty$$

By the Ratio Test, the series converges for all x. Therefore; the radius of convergence is $R=\infty$.

Exercise

Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!x^{2n}}{n!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!\,x^{2n+2}}{(n+1)!}\cdot\frac{n!}{(2n)!x^{2n}}\right|$$

$$= x^2 \lim_{n\to\infty}\left(\frac{(2n+1)(2n+2)}{n+1}\right)$$

$$\underline{=\infty}$$

$$\to \quad R = \frac{1}{\infty} = 0$$

By the Ratio Test, the series diverges for $|x|>0$ and converges only at its center, 0.

Therefore; the radius of convergence is $R=0$.

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right|$$

$$= |x|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)$$

$$\underline{= |x|}$$

$\to \quad R = 1$

So, by the Ratio Test, the radius of convergence is $R = 1$.

The series centered at 0, it converges in the interval $(-1,\ 1)$

When $x = 1$ $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ *diverges*

When $x = -1$ $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \cdots$ *converges*

Therefore; the interval of convergence $[-1,\ 1)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(x+1)^n}{2^n}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{2^{n+1}}\cdot\frac{2^n}{(x+1)^n}\right|$$

$$\underline{= \frac{1}{2}|x+1|}$$

$|x+1| < 2 \to \quad R = 2$

$$\begin{cases} x+1 = -2 & x = -3 \\ x+1 = 2 & x = 1 \end{cases}$$

So, by the Ratio Test, the radius of convergence is $R = 2$.

The series centered at -1, it converges in the interval $(-3,\ 1)$

When $x = -3$ $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(-2)^n}{2^n} = \sum_{n=0}^{\infty}1$ *diverges*

When $x = 1$ $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(2)^n}{2^n} = \sum_{n=0}^{\infty}(-1)^n$ *diverges*

Therefore; the interval of convergence $(-3,\ 1)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2}\cdot\frac{n^2}{x^n}\right|$$

$$= |x| \lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\right|$$

$$= |x|$$

$\rightarrow \quad R = 1$

So, by the Ratio Test, the radius of convergence is $R = 1$.

The series centered at 0, it converges in the interval $(-1,\ 1)$

When $x = -1$ $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -1 + \frac{1}{2^2} - \frac{1}{3^2} + \cdots$ *converges by alternating series*

$$u_{n+1} = \frac{1}{(n+1)^2} < \frac{1}{n^2} = u_n$$

$$\lim_{n\to\infty}\frac{1}{n^2} = 0$$

When $x = 1$ $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$ *converges by p-series*

Therefore; the interval of convergence $[-1,\ 1]$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{4^{n+1}}\cdot\frac{4^n}{x^n}\right|$$

$$= \frac{|x|}{4}$$

$\rightarrow \quad R = 4$

So, by the Ratio Test, the radius of convergence is $R = 4$.

The series centered at 0, it converges in the interval $(-4,\ 4)$

When $x = -4$ $\quad \sum_{n=1}^{\infty} (-1)^n = -1+1-1+\cdots$ *diverges by alternating series*

When $x = 4$ $\quad \sum_{n=1}^{\infty} 1$ *diverges*

Therefore; the interval of convergence $(-4,\ 4)$

Exercise

Find the interval of convergence of the power series $\quad \sum_{n=0}^{\infty} (2x)^n$

Solution

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{(2x)^{n+1}}{(2x)^n} \right|$$

$$\underline{= 2|x|\ }$$

$$2|x| = 1 \quad \to \quad R = \frac{1}{2}$$

So, by the Ratio Test, the radius of convergence is $R = \frac{1}{2}$.

The series converges in the interval $\left(-\frac{1}{2},\ \frac{1}{2}\right)$

When $x = -\frac{1}{2}$ $\quad \sum_{n=0}^{\infty} (-1)^n = -1+1-1+\cdots$ *diverges by alternating series*

When $x = \frac{1}{2}$ $\quad \sum_{n=0}^{\infty} 1$ *diverges*

Therefore; the interval of convergence $\left(-\frac{1}{2},\ \frac{1}{2}\right)$

Exercise

Find the interval of convergence of the power series $\quad \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right|$$

$$= |x| \lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$

$$= |x|$$

$|x| = 1 \quad \to \quad R = 1$

So, by the Ratio Test, the radius of convergence is $R = 1$.

The series converges in the interval $(-1,\ 1)$

When $x = -1$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges by p-series*

When $x = 1$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ *converges by Alternating Series*

Therefore; the interval of convergence $(-1,\ 1]$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} (-1)^n (n+1) x^n$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right|$$

$$= |x| \lim_{n\to\infty}\left|\frac{n+2}{n+1}\right|$$

$$= |x|$$

$|x| = 1 \quad \to \quad R = 1$

The series converges in the interval $(-1,\ 1)$

When $x = -1$ $\displaystyle\sum_{n=0}^{\infty} (n+1)$ *diverges*

When $x = 1$ $\displaystyle\sum_{n=0}^{\infty} (-1)^n (n+1)$ *diverges*

Therefore; the interval of convergence $(-1,\ 1)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{5n+5}}{(n+1)!}\cdot\frac{n!}{x^{5n}}\right|$$

$$=\left|x^5\right| \lim_{n\to\infty}\left|\frac{1}{n+1}\right|$$

$$\underline{=0\,}| \quad \to \quad R=\infty$$

The series converges for all x. Therefore; the interval of convergence $(-\infty,\ \infty)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(3x)^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{(3x)^n}\right|$$

$$=|3x| \lim_{n\to\infty}\left|\frac{1}{(2n+1)(2n+2)}\right|$$

$$\underline{=0\,}|$$

$\to \quad R=\infty$

The series converges for all x. Therefore; the interval of convergence $(-\infty,\ \infty)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} (2n)!\left(\frac{x}{3}\right)^n$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|(2n+2)!\left(\frac{x}{3}\right)^{n+1}\cdot(2n)!\left(\frac{x}{3}\right)^{-n}\right|$$

$$=\left|\frac{x}{3}\right| \lim_{n\to\infty}\left|(2n+1)(2n+2)\right|$$

$$\underline{=\infty\,}|$$

The series converges only for $x=0$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{(n+1)(n+2)}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+2)(n+3)}\cdot\frac{(n+1)(n+2)}{x^n}\right|$$

$$= |x|\lim_{n\to\infty}\left|\frac{n+1}{n+3}\right|$$

$$= |x|$$

$|x| = 1 \;\to\; R = 1$

The series converges in the interval $(-1,\ 1)$

When $x=-1$ $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}$ *converges by Alternating Series*

$$u_{n+1} = \frac{1}{(n+3)(n+2)} < \frac{1}{(n+1)(n+2)} = u_n$$

$$\lim_{n\to\infty}\frac{1}{(n+1)(n+2)} = 0$$

When $x=1$ $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)(n+2)}$ *converges by Limit Comparison Test to* $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$

Therefore; the interval of convergence $[-1,\ 1]$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{6^n}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{6^{n+1}}\cdot\frac{6^n}{x^n}\right|$$

$$= \frac{1}{6}|x|$$

$\frac{1}{6}|x| = 1 \;\to\; R = 6$

The series converges in the interval $(-6,\ 6)$

When $x=-6$ $\sum_{n=1}^{\infty}(-1)^n$ *diverges*

When $x=6$ $\sum_{n=1}^{\infty}(-1)^{n+1}$ *diverges*

Therefore; the interval of convergence $(-6,\ 6)$

Exercise

Find the interval of convergence of the power series $\sum_{n=0}^{\infty}(-1)^n\dfrac{n!(x-5)^n}{3^n}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)!(x-5)^{n+1}}{3^{n+1}}\cdot\frac{3^n}{n!(x-5)^n}\right|$$

$$=\left|\frac{x-5}{3}\right|\lim_{n\to\infty}(n+1)$$

$$=\infty$$

The series converges only for $x=5$

Exercise

Find the interval of convergence of the power series $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{(x-4)^n}{n\,9^n}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{(x-4)^{n+1}}{(n+1)9^{n+1}}\cdot\frac{n9^n}{(x-4)^n}\right|$$

$$=\frac{1}{9}|x-4|\lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$

$$=\frac{1}{9}|x-4|$$

$\frac{1}{9}|x-4|=1 \quad\to\quad R=9$

$$|x-4|=9 \ \Rightarrow\ \begin{cases} x-4=-9 & x=-5\\ x-4=9 & x=13\end{cases}$$

The series converges in the interval $(-5,\ 13)$ and center $x=4$

When $x=-5$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-9)^n}{n9^n}=\sum_{n=1}^{\infty}\frac{-1}{n} \quad \textit{diverges}$$

When $x=13$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(9)^n}{n\,9^n}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} \quad \textit{converges by Alternating Series}$$

$$u_{n+1}=\frac{1}{n+1}<\frac{1}{n}=u_n$$

$$\lim_{n\to\infty}\frac{1}{n}=0$$

Therefore; the interval of convergence $(-5,\ 13]$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty}\frac{(x-3)^{n+1}}{(n+1)4^{n+1}}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{(x-3)^{n+2}}{(n+2)4^{n+2}}\cdot\frac{(n+1)4^{n+1}}{(x-3)^{n+1}}\right|$$

$$=\frac{1}{4}|x-3|\lim_{n\to\infty}\left|\frac{n+1}{n+2}\right|$$

$$=\frac{1}{4}|x-3|$$

$$\frac{1}{4}|x-3|=1 \quad\rightarrow\quad R=4$$

$$|x-3|=4 \;\Rightarrow\; \begin{cases} x-3=-4 & x=-1 \\ x-3=4 & x=7 \end{cases}$$

The series converges in the interval $(-1,\ 7)$ and center $x=3$

When $x=-1$

$$\sum_{n=0}^{\infty}\frac{(-4)^{n+1}}{(n+1)4^{n+1}}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1} \quad \textit{converges by Alternating Series}$$

$$u_{n+1}=\frac{1}{n+2}<\frac{1}{n+1}=u_n$$

$$\lim_{n\to\infty}\frac{1}{n+1}=0$$

When $x=4$

$$\sum_{n=0}^{\infty} \frac{4^{n+1}}{(n+1)4^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{n+1} \quad \textit{diverges} \;\; \textit{by Integral Test}$$

$$\int_0^{\infty} \frac{dx}{x+1} = \ln(x+1) \Big|_0^{\infty}$$

$$\underline{= \infty \,|}$$

Therefore; the interval of convergence $[-1,\ 7)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-1)^{n+1}}{n+1}$

Solution

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{(x-1)^{n+2}}{n+2} \cdot \frac{n+1}{(x-1)^{n+1}} \right|$$

$$= |x-1| \lim_{n\to\infty} \left| \frac{n+1}{n+2} \right|$$

$$\underline{= |x-1| \,|}$$

$|x-1| = 1 \quad \to \quad R = 1$

$|x-1| = 1$

$$\Rightarrow \begin{cases} x-1=-1 & x=0 \\ x-1=1 & x=2 \end{cases}$$

The series converges in the interval $(0,\ 2)$ and center $x=1$

When $x=0$

$$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} \quad \textit{diverges} \;\; \textit{by Integral Test}$$

$$\int_0^{\infty} \frac{dx}{x+1} = \ln(x+1) \Big|_0^{\infty}$$

$$\underline{= \infty \,|}$$

When $x=1$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} \quad \textit{converges} \;\; \textit{by Alternative Test}$$

$$u_{n+1} = \frac{1}{n+2} < \frac{1}{n+1} = u_n$$

$$\lim_{n\to\infty} \frac{1}{n+1} = 0$$

Therefore; the interval of convergence $(0,\ 2]$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-2)^n}{n2^n}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n2^n}{(x-2)^n}\right|$$

$$= \frac{1}{2}|x-2| \lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$

$$= \frac{1}{2}|x-2|$$

$|x-2| = 2 \ \rightarrow \ R = 2$

$$|x-2| = 2 \ \Rightarrow \begin{cases} x-2=-2 & x=0 \\ x-2=2 & x=4 \end{cases}$$

The series converges in the interval $(0,\ 4)$ and center $x=2$

When $x=0$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-2)^n}{n2^n} = \sum_{n=0}^{\infty}\frac{-1}{n} \quad \textit{diverges by Integral Test}$$

$$\int_0^{\infty}\frac{-dx}{x} = -\ln x \Big|_0^{\infty}$$

$$= -\infty$$

When $x=4$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n2^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} \quad \textit{converges by Alternative Test}$$

$$u_{n+1} = \frac{1}{n+1} < \frac{1}{n} = u_n$$

$$\lim_{n\to\infty}\frac{1}{n} = 0$$

Therefore; the interval of convergence $(0,\ 4]$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^{n-1}}{3^{n-1}}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^n}{3^n}\cdot\frac{3^{n-1}}{(x-3)^{n-1}}\right|$$

$$=\frac{1}{3}|x-3|$$

$$\frac{1}{3}|x-3| = 1 \quad\to\quad R = 3$$

$$|x-3| = 3$$

$$\Rightarrow \begin{cases} x-3=-3 & x=0 \\ x-3=3 & x=6 \end{cases}$$

The series converges in the interval $(0,\ 6)$

When $x=0$

$$\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty}(-1) \quad \textit{diverges}$$

When $x=6$

$$\sum_{n=1}^{\infty}\frac{3^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty}1 \quad \textit{diverges}$$

Therefore; the interval of convergence $(0,\ 6)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{2n+3}\cdot\frac{2n+1}{x^{2n+1}}\right|$$

$$=x^2\lim_{n\to\infty}\left|\frac{2n+1}{2n+3}\right|$$

$$=x^2$$

$$\to\quad R=1$$

The series converges in the interval $(-1,\ 1)$

When $x=-1$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} \quad \textit{converges} \quad \textit{by Alternating Series}$$

When $x=1$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n}}{2n+1} \quad \textit{converges} \quad \textit{by Alternating Series}$$

Therefore; the interval of convergence $[-1,\ 1]$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}(-2x)^{n-1}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n+2}(-2x)^n \cdot \frac{n+1}{n(-2x)^{n-1}}\right|$$

$$=|-2x| \lim_{n\to\infty}\left|\frac{(n+1)^2}{n(n+2)}\right|$$

$$\underline{=2|x|}$$

$2|x|=1 \quad \to \quad R=\frac{1}{2}$

The series converges in the interval $\left(-\frac{1}{2},\ \frac{1}{2}\right)$

When $x=-\frac{1}{2}$

$$\sum_{n=1}^{\infty} \frac{n}{n+1} \quad \textit{diverges} \quad \textit{by nth Term Test}$$

$$\lim_{n\to\infty} \frac{n}{n+1}=1\neq 0$$

When $x=\frac{1}{2}$

$$\sum_{n=1}^{\infty} (-1)^{n-1}\frac{n}{n+1} \quad \textit{diverges} \quad \textit{by Alternating Series}$$

$$\lim_{n\to\infty} \frac{n}{n+1}=1\neq 0$$

Therefore; the interval of convergence $\left(-\frac{1}{2},\ \frac{1}{2}\right)$

Exercise

Find the interval of convergence of the power series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{(n+1)!}\cdot\frac{n!}{x^{2n}}\right|$$

$$= x^2 \lim_{n\to\infty}\left|\frac{1}{n+1}\right|$$

$$\underline{=0}$$

$\rightarrow \quad \underline{R=\infty}$

Therefore; the interval of convergence $(-\infty, \infty)$

Exercise

Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{3n+4}}{(3n+4)!}\cdot\frac{(3n+1)!}{x^{3n+1}}\right|$$

$$= \left|x^3\right| \lim_{n\to\infty}\left|\frac{1}{(3n+2)(3n+3)(3n+4)}\right|$$

$$\underline{=0}$$

$\rightarrow \quad \underline{R=\infty}$

Therefore; the interval of convergence $(-\infty, \infty)$

Exercise

Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{n!x^n}{(2n)!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{n!x^n}\right|$$

$$= |x| \lim_{n\to\infty}\left|\frac{n+1}{(2n+1)(2n+2)}\right|$$

$$\underline{=0}$$

$$\to \quad \underline{R=\infty}$$

Therefore; the interval of convergence $(-\infty,\ \infty)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{2\cdot 3\cdot 4\cdots(n+1)x^n}{n!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{2\cdot 3\cdot 4\cdots(n+1)(n+2)x^{n+1}}{(n+1)!}\cdot\frac{n!}{2\cdot 3\cdot 4\cdots(n+1)x^n}\right|$$

$$=|x|\lim_{n\to\infty}\left|\frac{n+2}{n+1}\right|$$

$$\underline{=|x|}$$

$|x|=1 \quad \to \quad R=1$

The series converges in the interval $(-1,\ 1)$

When $x=-1$

$$\sum_{n=1}^{\infty}(-1)^n\ \frac{2\cdot 3\cdot 4\cdots(n+1)}{n!}=\sum_{n=1}^{\infty}\ (-1)^n\,(n+1) \quad \textit{diverges}$$

When $x=1$

$$\sum_{n=1}^{\infty}\frac{2\cdot 3\cdot 4\cdots(n+1)}{n!}=\sum_{n=1}^{\infty}(n+1) \quad \textit{diverges}$$

Therefore; the interval of convergence $(-1,\ 1)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n}}{\sqrt{n+1}}$

Solution

$$R=\lim_{n\to\infty}\left|\frac{1}{\sqrt{n+2}}\cdot\sqrt{n+1}\right|$$

$$=\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}}$$

$$\underline{=1}$$

The *radius* of convergence is 1.

The *centre* of convergence is 0.

The *interval* of convergence is (−1, 1).

The series *does not converge* at $x=-1$ or $x=1$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} 3n(x+1)^n$

Solution

$$R=\lim_{n\to\infty}\left|\frac{3n}{3(n+1)}\right| \qquad\qquad R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$=\lim_{n\to\infty}\frac{3n}{3n}$$

$$=1$$

The radius of convergence is 1, and the centre of convergence is −1. $(x+1=0)$

$a-R<x<a+R \quad\Rightarrow\quad -1-1<x<-1+1$

Therefore; the given series convergences absolutely on $(-2,\ 0)$

At $x=-2$

The series is $\displaystyle\sum_{n=0}^{\infty} 3n(-1)^n$ which diverges.

At $x=0$

The series is $\displaystyle\sum_{n=0}^{\infty} 3n(1)^n=\sum_{n=0}^{\infty} 3n$ which diverges.

Hence, the interval of convergence is $(-2,\ 0)$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}}x^n$

Solution

$$R=\lim_{n\to\infty}\left|\frac{(n+1)^4 2^{2n+2}}{n^4 2^{2n}}\right| \qquad\qquad R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$= 4 \lim\limits_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^4\right|$

$\underline{=4}$

The radius of convergence is 4, and the centre of convergence is 0.

$a - R < x < a + R \quad \Rightarrow \quad -4 < x < 4$, the given series convergences absolutely on $(-4,\ 4)$

At $x = -4$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}}(-4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}}(-1)^n\, 2^{2n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^4} \text{ which converges } (p\text{-series}).$$

At $x = 4$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}}(4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} 2^{2n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \text{ which also converges.}$$

Hence, the interval of convergence is $[-4,\ 4]$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum\limits_{n=1}^{\infty} \frac{e^n}{n^3}(4-x)^n$

Solution

$$R = \lim_{n\to\infty}\left|\frac{e^n}{n^3} \cdot \frac{(n+1)^3}{e^{n+1}}\right| \qquad\qquad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= \frac{1}{e}\lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^3\right|$$

$$\underline{= \frac{1}{e}}$$

The *radius* of convergence is $\frac{1}{e}$.

The *centre* of convergence is 4. $(4 - x = 0 \ \Rightarrow \ x = 4)$

$a - R < x < a + R \quad \Rightarrow \quad 4 - \frac{1}{e} < x < 4 + \frac{1}{e}$, which the given series convergences absolutely

At $x=4-\frac{1}{e}$,

the series is $\sum_{n=1}^{\infty} \frac{e^n}{n^3}\left(\frac{1}{e}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ which converges (p-series).

At $x=4+\frac{1}{e}$,

the series is $\sum_{n=1}^{\infty} \frac{e^n}{n^3}\left(-\frac{1}{e}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ which also converges (p-series).

Hence, the interval of convergence is $\left[4-\frac{1}{e},\ 4+\frac{1}{e}\right]$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{1+5^n}{n!}x^n$

Solution

$$R = \lim_{n\to\infty}\left|\frac{1+5^n}{n!}\cdot\frac{(n+1)!}{1+5^{n+1}}\right| \qquad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1+5^n}{1+5^{n+1}}(n+1)\right|$$

$$=\infty$$

The *radius* of convergence is ∞ .

The *centre* of convergence is 0.

The *interval of convergence* is the real line $(-\infty,\ \infty)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n}$

Solution

$$\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n} = \sum_{n=1}^{\infty} \frac{4^n\left(x-\frac{1}{4}\right)^n}{n^n}$$

$$R = \lim_{n\to\infty}\left|\frac{4^n}{n^n}\cdot\frac{(n+1)^{n+1}}{4^{n+1}}\right| \qquad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$=\frac{1}{4}\lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^n(n+1)\right|$$

$$\boxed{=\infty}$$

The *radius* of convergence is ∞ .

$4x-1=0 \;\Rightarrow\; x=\frac{1}{4}$

The *centre* of convergence is $\boxed{x=\frac{1}{4}}$

The *interval of convergence* is the real line $(-\infty,\ \infty)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty}\frac{1+5^n}{n!}x^n$

Solution

$$a_n=\frac{1+5^n}{n!}$$

$$R=\lim_{n\to\infty}\left|\frac{\left(1+5^n\right)}{n!}\cdot\frac{(n+1)!}{\left(1+5^{n+1}\right)}\right| \qquad R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$=\lim_{n\to\infty}\left|(n+1)\frac{1+5^n}{1+5^{n+1}}\right|$$

$$=\lim_{n\to\infty}\left|(n+1)\frac{1}{5}\right|$$

$$\boxed{=\infty}$$

The *radius* of convergence is ∞ .

The *centre* of convergence is $\boxed{x=0}$.

The *interval of convergence* is the real line $(-\infty,\ \infty)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\displaystyle\sum\frac{n^2x^n}{n!}$

Solution

$$a_n=\frac{n^2}{n!}$$

$$R = \lim_{n\to\infty} \left| \frac{n^2 x^n}{n!} \cdot \frac{(n+1)!}{(n+1)^2 x^{n+1}} \right| \qquad\qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| (n+1)\left(\frac{n}{n+1}\right)^2 \right|$$

$$= \infty$$

The *radius* of convergence is ∞ .

The *centre* of convergence is $x = 0$.

The *interval of convergence* is the real line $(-\infty, \infty)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum \frac{x^{4n}}{n^2}$

Solution

$$a_n = \frac{1}{n^2} x^{4n}$$

$$R = \lim_{n\to\infty} \left(\frac{1}{n^2} \cdot \frac{(n+1)^2}{1} \right) \left| \frac{x^{4n}}{x^{4n+4}} \right| \qquad\qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^2 \left| \frac{1}{x^4} \right|$$

$$= 1$$

The *radius* of convergence is 1

The *centre* of convergence is $x = 0$

$$-1 < x < 1 \qquad\qquad a - R < x < a + R$$

which the given series convergences absolutely

At $x = -1$,

the series is $\sum \frac{(-1)^{4n}}{n^2} = \sum \frac{1}{n^2}$ which converges (*p*-series).

At $x = 1$,

the series is $\sum \frac{(1)^{4n}}{n^2} = \sum \frac{1}{n^2}$ which also converges (*p*-series).

The interval of convergence is the real line $[-1, 1]$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum (-1)^n \frac{(x+1)^{2n}}{n!}$

Solution

$$a_n = \frac{1}{n!}(x+1)^{2n}$$

$$R = \lim_{n\to\infty} \left(\frac{1}{n!}\cdot\frac{(n+1)!}{1}\right)\left|\frac{(x+1)^{2n}}{(x+1)^{2n+2}}\right| \qquad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= \lim_{n\to\infty} (n+1)\left|\frac{1}{(x+1)^2}\right|$$

$$= \infty$$

The *radius* of convergence is ∞

$x+1=0 \quad \to \quad x=-1$

The *centre* of convergence is $x=-1$

The *interval* of convergence is the real line $(-\infty,\ \infty)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum \frac{(x-1)^n}{n\cdot 5^n}$

Solution

$$a_n = \frac{1}{n\cdot 5^n}(x-1)^n$$

By *Ratio Test*:

$$R = \lim_{n\to\infty} \left(\frac{1}{n\cdot 5^n}\cdot\frac{(n+1)\cdot 5^{n+1}}{1}\right)\left|\frac{(x-1)^n}{(x-1)^{n+1}}\right| \qquad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= 5\lim_{n\to\infty} \frac{n+1}{n}\left|\frac{1}{x-1}\right|$$

$$= 5$$

The *radius* of convergence is 5

$x-1=0 \quad \to \quad x=1$

The *centre* of convergence is $x=1$

$-5+1 < x < 5+1 \qquad a-R < x < a+R$

$-4 < x < 6$

which the given series convergences absolutely

At $x=-4$,

the series is $\sum \frac{(-5)^n}{n\cdot 5^n} = \sum \frac{(-1)^n}{n}$

$\frac{1}{n} > \frac{1}{n+1}$

$\lim_{n\to\infty} \frac{1}{n} = 0$

which converges *Alternating Harmonic Series*.

At $x=6$,

the series is $\sum \frac{(5)^n}{n\cdot 5^n} = \sum \frac{1}{n}$ which *diverges* (p-series $p=1\le 1$)

The interval of convergence is the real line $[-4,\ 6)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum \left(\frac{x}{9}\right)^{3n}$

Solution

$a_n = \left(\frac{x}{9}\right)^{3n}$

By Root Test:

$R = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{x}{9}\right)^{3n}\right|}$ $\qquad R = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

$= \lim_{n\to\infty} \left(\frac{|x|}{9}\right)^3$

$= \frac{1}{729}\left|x^3\right| < 1$

$\left|\frac{x}{9}\right|^3 < 1$

$\left|\frac{x}{9}\right| < 1$

$-9 < x < 9$

The *radius* of convergence is 9

The *centre* of convergence is $x=0$

At $x=-9$,

the series is $\sum \left(\frac{-9}{9}\right)^{3n} = \sum (-1)$ which diverges by the *divergence Test*.

At $x = 9$,

the series is $\sum \left(\frac{9}{9}\right)^{3n} = \sum (1)$ which diverges by the *divergence Test*.

The interval of convergence is the real line $(-9,\ 9)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum \frac{(x+2)^n}{\sqrt{n}}$

Solution

$$a_n = \frac{(x+2)^n}{\sqrt{n}}$$

By *Ratio Test*:

$$R = \lim_{n\to\infty} \left(\frac{1}{\sqrt{n}} \cdot \frac{\sqrt{n+1}}{1}\right) \left|\frac{(x+2)^n}{(x+2)^{n+1}}\right| \qquad\qquad R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

$$= \lim_{n\to\infty} \sqrt{\frac{n+1}{n}} \left|\frac{1}{x+2}\right|$$

$$\underline{= 1}$$

The *radius* of convergence is $\underline{1}$

$x + 2 = 0 \quad \to \quad x = -2$

The *centre* of convergence is $\underline{x = -2}$

$-2 - 1 < x < -2 + 1 \qquad\qquad a - R < x < a + R$

$-3 < x < -1$

which the given series convergences absolutely

At $x = -3$,

the series is $\sum \frac{(-1)^n}{\sqrt{n}}$

$$\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

which converges *Alternating Series*.

At $x = -1$,

the series is $\sum \frac{(1)^n}{\sqrt{n}} = \sum \frac{1}{\sqrt{n}}$ which *diverges* (*p*-series $p = \frac{1}{2} \le 1$)

The interval of convergence is the real line $[-3, \ -1)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum \frac{(x+2)^k}{2^k \ln k}$

Solution

$$a_k = \frac{(x+2)^k}{2^k \ln k}$$

By *Ratio Test*:

$$R = \lim_{k\to\infty} \left(\frac{1}{2^k \ln k} \cdot \frac{2^{k+1} \ln(k+1)}{1} \right) \left| \frac{(x+2)^k}{(x+2)^{k+1}} \right| \qquad R = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right|$$

$$= 2 \lim_{n\to\infty} \frac{\ln(k+1)}{\ln k} \left| \frac{1}{x+2} \right|$$

$$= 2 \lim_{n\to\infty} \frac{\frac{1}{k+1}}{\frac{1}{k}}$$

$$= 2 \lim_{n\to\infty} \frac{k}{k+1}$$

$$= 2$$

The *radius* of convergence is 2

$x + 2 = 0 \quad \to \quad x = -2$

The *centre* of convergence is $x = -2$

$-2 - 2 < x < -2 + 2 \qquad\qquad a - R < x < a + R$

$-4 < x < 0$

which the given series convergences absolutely.

At $x = -4$,

the series is $\sum \frac{(-2)^k}{2^k \ln k} = \sum \frac{(-1)^k}{\ln k}$

$$\frac{1}{\ln k} > \frac{1}{\ln(k+1)}$$

$$\lim_{n\to\infty} \frac{1}{\ln k} = 0$$

which converges *Alternating Series*.

At $x = 0$,

the series is $\sum \frac{(2)^k}{2^k \ln k} = \sum \frac{1}{\ln k}$

$\ln k < k$

$\frac{1}{\ln k} > \frac{1}{k}$

$\frac{1}{k}$ *diverges* (*p*-series $p = 1 \le 1$)

$\therefore$ Which *diverges by Comprison Test.*

The *interval* of convergence is the real line $[-4,\ 0)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots$

Solution

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$$

$$a_k = \frac{x^{2k+1}}{2k+1}$$

By *Ratio Test*:

$$R = \lim_{k\to\infty} \left(\frac{1}{2k+1} \cdot \frac{2k+3}{1}\right) \left|\frac{x^{2k+1}}{x^{2k+3}}\right| \qquad\qquad R = \lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right|$$

$$= \lim_{n\to\infty} \frac{2k+3}{2k+1}\left(\frac{1}{x^2}\right)$$

$$\underline{= 1}$$

The *radius* of convergence is $\underline{1}$

The *centre* of convergence is $\underline{x = 0}$

$-1 < x < 1$

which the given series convergences absolutely

At $x = -1$,

the series is $\sum \frac{(-1)^{2k+1}}{2k+1} = \sum \frac{-1}{2k+1}$

$$\int_0^{\infty} \frac{-1}{2x+1} dx = -\frac{1}{2} \int_0^{\infty} \frac{1}{2x+1} d(2x+1)$$

$$= -\frac{1}{2}\ln(2x+1)\ \Big|_0^{\infty}$$

$$= -\frac{1}{2}(\ln\infty - \ln 1)$$

$$\underline{= -\infty}\ |$$

which diverges *Integral Test*.

At $x = 1$,

the series is $\sum \frac{(1)^{2k+1}}{2k+1} = \sum \frac{1}{2k+1}$

$$\int_0^{\infty} \frac{1}{2x+1}dx = \frac{1}{2}\int_0^{\infty} \frac{1}{2x+1}d(2x+1)$$

$$= \frac{1}{2}\ln(2x+1)\ \Big|_0^{\infty}$$

$$= \frac{1}{2}(\ln\infty - \ln 1)$$

$$\underline{= \infty}\ |$$

which diverges *Integral Test*.

The *interval* of convergence is the real line $(-1,\ 1)$

Exercise

For what value of *x* does the series $1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-3)^n + \cdots$ converges? What is its sum? What series do you get if you differentiate the given series term by term? For what value of *x* does the new series converge? What is its sum?

Solution

$$1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-3)^n + \cdots = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-3)^n$$

$$\lim_{b\to\infty}\left|\frac{(x-3)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(x-3)^n}\right| = \left|\frac{x-3}{2}\right| < 1$$

$$\Rightarrow\quad -1 < \frac{x-3}{2} < 1$$

$$-2 < x-3 < 2$$

$$\underline{1 < x < 5}\ |$$

When $x = 1$,

$$\sum_{n=1}^{\infty}(1)^n \text{ which is a divergent series}$$

When $x=5$,

$$\sum_{n=1}^{\infty}(-1)^n \text{ the series diverges}$$

The series is a geometric series, the sum is

$$\frac{1}{1+\frac{x-3}{2}}=\frac{2}{x-1}$$

If $f(x)=1-\frac{1}{2}(x-3)+\frac{1}{4}(x-3)^2+\cdots+\left(-\frac{1}{2}\right)^n(x-3)^n+\cdots$

$$=\frac{2}{x-1}$$

Then $f'(x)=-\frac{1}{2}+\frac{1}{2}(x-3)+\cdots+\left(-\frac{1}{2}\right)^n n(x-3)^{n-1}+\cdots$

$f'(x)$ is convergent when $1<x<5$ and divergent when $x=1$ *or* 5

The sum for $f'(x)$ is $\dfrac{-2}{(x-1)^2}$

Exercise

The series $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\frac{x^{11}}{11!}+\cdots$ converges to sin$x$ for all x.

a) Find the first six terms of a series for cos$x$. For what values of x should the series converge?

b) By replacing x by $2x$ in the series for sin$x$, find a series that converges to sin$2x$ for all x.

c) Using the result in part (a) and series multiplication, calculate the first six term of a series for $2\sin x\cos x$. Compare your answer with the answer in part (b).

Solution

a) $(\sin x)'=\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}+\cdots$

$$\cos x=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n}}{(2n)!}$$

$$\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right|=x^2\lim_{n\to\infty}\left|\frac{1}{(2n+1)(2n+2)}\right|=0<1 \quad (\forall x)$$

The series converges for all values of x.

b) $\sin 2x = 2x - \frac{2^3 x^3}{3!} + \frac{2^5 x^5}{5!} - \frac{2^7 x^7}{7!} + \frac{2^9 x^9}{9!} - \frac{2^{11} x^{11}}{11!} + \cdots$

$$= 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \frac{512x^9}{9!} - \frac{2048x^{11}}{11!} + \cdots$$

c) $2\sin x \cos x = 2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right)$

$$= 2x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right) - 2\frac{x^3}{3!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right)$$

$$+ 2\frac{x^5}{5!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right) - 2\frac{x^7}{7!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right)$$

$$+ 2\frac{x^9}{9!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right) - 2\frac{x^{11}}{11!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right)$$

$$= 2x - \frac{2x^3}{2!} + \frac{2x^5}{4!} - \frac{2x^7}{6!} + \frac{2x^9}{8!} - \frac{2x^{11}}{10!} - \frac{2x^3}{3!} + \frac{2x^5}{2!3!} - \frac{2x^7}{4!3!} + \frac{2x^9}{6!3!} - \frac{2x^{11}}{8!3!}$$

$$+ \frac{2x^5}{5!} - \frac{2x^7}{5!2!} + \frac{2x^9}{5!4!} - \frac{2x^{11}}{5!6!} - \frac{2x^7}{7!} + \frac{2x^9}{7!2!} - \frac{2x^{11}}{7!4!} + \frac{2x^9}{9!} - \frac{2x^{11}}{9!2!} - \frac{2x^{11}}{11!} + \cdots$$

$$= 2x - \frac{2^3 x^3}{3!} + \frac{2^5 x^5}{5!} - \frac{2^7 x^7}{7!} + \frac{2^9 x^9}{9!} - \frac{2^{11} x^{11}}{11!} + \cdots$$

Exercise

Find the sum of the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ by the first finding the sum of the power series

$$\sum_{n=1}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + 16x^4 + \cdots$$

Solution

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}$$

$$\sum_{n=1}^{\infty} nx^n = x + x^2 + 3x^3 + 4x^4 + \cdots$$

$$= x\left(1 + x + 3x^2 + 4x^3 + \cdots\right)$$

$$= x\frac{d}{dx}\left(1 + x + x^2 + \cdots + x^n + \cdots\right)$$

$$= x\left(\frac{1}{1-x}\right)'$$

$$= \frac{x}{(1-x)^2}$$

$$\frac{d}{dx}\sum_{n=1}^{\infty} nx^n = \left(\frac{x}{(1-x)^2}\right)'$$

$$= \frac{(1-x)^2 + 2x(1-x)}{(1-x)^4}$$

$$= \frac{1+x}{(1-x)^3}$$

$$\frac{d}{dx}\sum_{n=1}^{\infty} nx^n = \sum_{n=1}^{\infty} n^2 x^{n-1}$$

$$= \frac{1+x}{(1-x)^3}$$

Multiply by x both sides

$$x\sum_{n=1}^{\infty} n^2 x^{n-1} = x\frac{1+x}{(1-x)^3}$$

$$\sum_{n=1}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + 16x^4 + \cdots$$

$$\boxed{= \frac{x(1+x)}{(1-x)^3}}$$

Let $x = \frac{1}{2}$

$$\sum_{n=1}^{\infty} n^2 \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}\frac{3}{2}}{\left(\frac{1}{2}\right)^3}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} \boxed{= 6}$$

Exercise

Find a series representation of $f(x)=\dfrac{1}{2+x}$ in powers of $x-1$. What is the interval of convergence of this series?

Solution

Let $t=x-1 \quad \Rightarrow \quad x=t+1$, we have

$$\frac{1}{2+x}=\frac{1}{3+t}$$

$$=\frac{1}{3}\frac{1}{1+\frac{t}{3}}$$

$$=\frac{1}{3}\left(1-\frac{t}{3}+\frac{t^2}{3^2}-\frac{t^3}{3^3}+\cdots\right) \qquad \left(-1<\frac{t}{3}<1\right)$$

$$=\sum_{n-0}^{\infty}(-1)^n\frac{t^n}{3^{n+1}} \qquad (-3<t<3)$$

$$=\sum_{n-0}^{\infty}(-1)^n\frac{(x-1)^n}{3^{n+1}} \qquad (-2<x<4)$$

$$R=\lim_{n\to\infty}\frac{3^{n+2}}{3^{n+1}}$$

$$\underline{=3}$$

The *radius* of convergence of this series is 3.

The distance from the centre of convergence $x-1=0 \Rightarrow \underline{x=1}$, to the point -2 where the denominator is 0.

Exercise

Determine the Cauchy product of the series $1+x+x^2+x^3+\cdots$ *and* $-x+x^2-x^3+\cdots$. On what interval and to what function does the product series converge?

Solution

$$1+x+x^2+x^3+\cdots=\frac{1}{1-x}$$

$$=\sum_{n=0}^{\infty}x^n$$

$$-x+x^2-x^3+\cdots=\frac{1}{1+x}$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n$$

Let $a_n = 1$ *and* $b_n = (-1)^n$, then the series holds for $-1 < x < 1$

We have

$$c_n = \sum_{j=0}^{n} (-1)^{n-j}$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

Then the Cauchy product is

$$1 + x^2 + x^4 + \cdots = \sum_{n=0}^{\infty} x^{2n}$$

$$= \frac{1}{1-x} \cdot \frac{1}{1+x}$$

$$= \boxed{\frac{1}{1-x^2}} \quad \text{for} \quad -1 < x < 1$$

Exercise

Determine the power series expansion of $\dfrac{1}{(1-x)^2}$ by formally dividing $1 - 2x + x^2$ into 1.

Use the power series $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad -1 < x < 1$

Solution

$$\begin{array}{r|l} & 1 + 2x + 3x^2 + 4x^3 + \cdots \\ \hline 1 - 2x + x^2 & 1 \\ & \underline{1 - 2x + x^2} \\ & \quad 2x - x^2 \\ & \quad \underline{2x - 4x^2 + 2x^3} \\ & \qquad 3x^2 - 2x^3 \\ & \qquad \underline{3x^2 - 6x^3 + 3x^4} \\ & \qquad\quad 4x^3 + 3x^4 \\ & \qquad\quad \underline{4x^3 - 8x^4 + 4x^5} \\ & \qquad\qquad 11x^4 - \cdots \end{array}$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^n \quad \text{for} \quad -1 < x < 1$$

Exercise

Determine the interval of convergence and the sum of the series

$$1-4x+16x^2-64x^3+\cdots=\sum_{n=0}^{\infty}(-1)^n(4x)^n$$

Solution

$$1-4x+16x^2-64x^3+\cdots=1+(-4x)+(-4x)^2+(-4x)^3+\cdots \qquad \frac{1}{1-u}=1+u+u^2+u^3+\cdots$$

$$=\frac{1}{1-(-4x)}$$

$$=\frac{1}{1+4x}$$

Therefore; the interval of convergence is $-\frac{1}{4}<x<\frac{1}{4}$

Exercise

Determine the interval of convergence and the sum of the series

$$3+4x+5x^2+6x^3+\cdots=\sum_{n=0}^{\infty}(n+3)x^n$$

Solution

$$\sum_{n=0}^{\infty}(n+3)x^n=\sum_{n=0}^{\infty}n\,x^n+\sum_{n=0}^{\infty}3x^n$$

$$\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\cdots=\frac{1}{1-x}$$

$$\left(\sum_{n=0}^{\infty}x^n\right)'=1+2x+3x^2+4x^3+\cdots \qquad \textit{Differentiate}$$

$$=\frac{1}{(1-x)^2}$$

$$x\left(1+2x+3x^2+4x^3+\cdots\right)=\frac{x}{(1-x)^2} \qquad \textit{Multiply by } x$$

$$\sum_{n=0}^{\infty}nx^n=x+2x^2+3x^3+\cdots$$

$$=\frac{x}{(1-x)^2}$$

Then,

$$\sum_{n=0}^{\infty}(n+3)x^n = \sum_{n=0}^{\infty} nx^n + \sum_{n=0}^{\infty} 3x^n$$

$$= \frac{x}{(1-x)^2} + 3\frac{1}{1-x}$$

$$= \boxed{\frac{3-2x}{(1-x)^2}} \qquad (-1 < x < 1)$$

Exercise

Determine the interval of convergence and the sum of the series

$$\frac{1}{3} + \frac{x}{4} + \frac{x^2}{5} + \frac{x^3}{6} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n+3}$$

Solution

$$\frac{1}{3} + \frac{x}{4} + \frac{x^2}{5} + \frac{x^3}{6} + \cdots = \frac{1}{x^3}\left(\frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \cdots\right)$$

$$= \frac{1}{x^3}\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \cdots - x - \frac{x^2}{2}\right) \qquad \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$

$$= \frac{1}{x^3}\left(-\ln(1-x) - x - \frac{x^2}{2}\right)$$

$$= \boxed{-\frac{1}{x^3}\ln(1-x) - \frac{1}{x^2} - \frac{1}{2x}} \qquad (-1 \le x < 1,\ \ x \ne 0)$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=e^{2x}, \quad a=0$

Solution

$$f(x)=e^{2x} \quad \to \quad f(0)=e^{2(0)}=1$$
$$f'(x)=2e^{2x} \quad \to \quad f'(0)=2$$
$$f''(x)=4e^{2x} \quad \to \quad f''(0)=4$$
$$f'''(x)=8e^{2x} \quad \to \quad f'''(0)=8$$

$$P_0(x)=f(0)=1$$
$$P_1(x)=f(0)+f'(0)(x-0)$$
$$=1+2x$$
$$P_2(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2$$
$$=1+2x+2x^2$$
$$P_3(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3$$
$$=1+2x+2x^2+\frac{4}{3}x^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\sin x, \quad a=0$

Solution

$$f(x)=\sin x \quad \to \quad f(0)=0$$
$$f'(x)=\cos x \quad \to \quad f'(0)=1$$
$$f''(x)=-\sin x \quad \to \quad f''(0)=0$$
$$f'''(x)=-\cos x \quad \to \quad f'''(0)=-1$$

$$P_0(x)=f(0)=0$$
$$P_1(x)=f(0)+f'(0)(x-0)$$
$$=x$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$$
$$= x$$
$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$
$$= x - \frac{1}{6}x^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = \ln(1+x), \quad a = 0$

Solution

$$f(x) = \ln(1+x) \quad \to \quad f(0) = 0$$
$$f'(x) = \frac{1}{1+x} \quad \to \quad f'(0) = 1$$
$$f''(x) = -\frac{1}{(1+x)^2} \quad \to \quad f''(0) = -1$$
$$f'''(x) = \frac{2}{(1+x)^3} \quad \to \quad f'''(0) = 2$$
$$P_0(x) = f(0)$$
$$= 0$$
$$P_1(x) = f(0) + f'(0)(x-0)$$
$$= x$$
$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$$
$$= x - \frac{1}{2}x^2$$
$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$
$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = \frac{1}{x+2}, \quad a = 0$

Solution

$$f(x) = (x+2)^{-1} \quad \to \quad f(0) = \frac{1}{2}$$

$$f'(x) = -(x+2)^{-2} \quad \to \quad f'(0) = -\tfrac{1}{4}$$

$$f''(x) = 2(x+2)^{-3} \quad \to \quad f''(0) = \tfrac{1}{4}$$

$$f'''(x) = -6(x+2)^{-4} \quad \to \quad f'''(0) = -\tfrac{3}{8}$$

$$P_0(x) = f(0)$$

$$\boxed{= \tfrac{1}{2}}$$

$$P_1(x) = f(0) + f'(0)(x-0)$$

$$\boxed{= \tfrac{1}{2} - \tfrac{1}{4}x}$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$\boxed{= \tfrac{1}{2} - \tfrac{1}{4}x + \tfrac{1}{8}x^2}$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$$\boxed{= \tfrac{1}{2} - \tfrac{1}{4}x + \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = \sqrt{1-x}, \quad a = 0$

Solution

$$f(x) = (1-x)^{1/2} \quad \to \quad f(0) = 1$$

$$f'(x) = -\tfrac{1}{2}(1-x)^{-1/2} \quad \to \quad f(0) = -\tfrac{1}{2}$$

$$f''(x) = -\tfrac{1}{4}(1-x)^{-3/2} \quad \to \quad f(0) = -\tfrac{1}{4}$$

$$f'''(x) = -\tfrac{3}{8}(1-x)^{-5/2} \quad \to \quad f(0) = -\tfrac{3}{8}$$

$$P_0(x) = f(0)$$

$$\boxed{= 1}$$

$$P_1(x) = f(0) + f'(0)(x-0)$$

$$\boxed{= 1 - \tfrac{1}{2}x}$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$=1-\frac{1}{2}x-\frac{1}{8}x^2$$

$$P_3(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3$$

$$=1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=x^3, \quad a=1$

Solution

$f(x)=x^3 \quad \to \quad f(1)=1$

$f'(x)=3x^2 \quad \to \quad f'(1)=3$

$f''(x)=6x \quad \to \quad f''(1)=6$

$f'''(x)=6 \quad \to \quad f'''(1)=6$

$$P_0(x)=f(a)$$

$$=1$$

$$P_1(x)=f(a)+f'(a)(x-a)$$

$$=1+3(x-1)$$

$$P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$=1+3(x-1)+3(x-1)^2$$

$$P_3(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3$$

$$=1+3(x-1)+3(x-1)^2+(x-1)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=8\sqrt{x}, \quad a=1$

Solution

$f(x)=8x^{1/2} \quad \to \quad f(1)=8$

$f'(x)=4x^{-1/2} \quad \to \quad f'(1)=4$

$f''(x)=-2x^{-3/2} \quad \to \quad f''(1)=-2$

$f'''(x)=3x^{-5/2} \quad \to \quad f'''(1)=3$

$P_0(x) = f(a)$

$\boxed{= 8}$

$P_1(x) = f(a) + f'(a)(x-a)$

$\boxed{= 8 + 4(x-1)}$

$P_2(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2$

$\boxed{= 8 + 4(x-1) - (x-1)^2}$

$P_3(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \dfrac{f'''(a)}{3!}(x-a)^3$

$\boxed{= 8 + 4(x-1) - (x-1)^2 + 3(x-1)^3}$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = \sin x, \quad a = \frac{\pi}{4}$

Solution

$f(x) = \sin x \quad \to \quad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(x) = \cos x \quad \to \quad f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f''(x) = -\sin x \quad \to \quad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f'''(x) = -\cos x \quad \to \quad f'''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$P_0(x) = f(a)$

$\boxed{= \frac{\sqrt{2}}{2}}$

$P_1(x) = f(a) + f'(a)(x-a)$

$\boxed{= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)}$

$P_2(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2$

$\boxed{= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2}$

$P_3(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \dfrac{f'''(a)}{3!}(x-a)^3$

$\boxed{= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3}$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\cos x, \quad a=\frac{\pi}{6}$

Solution

$$f(x)=\cos x \quad \rightarrow \quad f\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$$

$$f'(x)=-\sin x \quad \rightarrow \quad f'\left(\frac{\pi}{6}\right)=-\frac{1}{2}$$

$$f''(x)=-\cos x \quad \rightarrow \quad f''\left(\frac{\pi}{6}\right)=-\frac{\sqrt{3}}{2}$$

$$f'''(x)=\text{sinx} \quad \rightarrow \quad f'''\left(\frac{\pi}{6}\right)=\frac{1}{2}$$

$$P_0(x)=f(a)$$

$$=\frac{\sqrt{3}}{2}$$

$$P_1(x)=f(a)+f'(a)(x-a)$$

$$=\frac{\sqrt{2}}{2}-\frac{1}{2}\left(x-\frac{\pi}{6}\right)$$

$$P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$=\frac{\sqrt{2}}{2}-\frac{1}{2}\left(x-\frac{\pi}{6}\right)-\frac{\sqrt{3}}{4}\left(x-\frac{\pi}{6}\right)^2$$

$$P_3(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3$$

$$=\frac{\sqrt{3}}{2}-\frac{1}{2}\left(x-\frac{\pi}{6}\right)-\frac{\sqrt{3}}{4}\left(x-\frac{\pi}{6}\right)^2+\frac{1}{12}\left(x-\frac{\pi}{6}\right)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\sqrt{x}, \quad a=9$

Solution

$$f(x)=x^{1/2} \quad \rightarrow \quad f(9)=3$$

$$f'(x)=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}} \quad \rightarrow \quad f'(9)=\frac{1}{6}$$

$$f''(x)=-\frac{1}{4}x^{-3/2}=-\frac{1}{4x\sqrt{x}} \quad \rightarrow \quad f''(9)=-\frac{1}{4\times 3^3}$$

$$f'''(x)=\frac{3}{8}x^{-5/2}=\frac{3}{8x^2\sqrt{x}} \quad \rightarrow \quad f'''(9)=\frac{1}{2^3\times 3^4}$$

$$P_0(x) = f(a)$$

$$\boxed{= 3}$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$\boxed{= 3 + \frac{1}{6}(x-9)}$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{= 3 + \frac{1}{2 \cdot 3}(x-9) - \frac{1}{2^3 \cdot 3^3}(x-9)^2}$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$\boxed{= 3 + \frac{1}{2 \cdot 3}(x-9) - \frac{1}{2^2 \cdot 3^3}(x-9)^2 + \frac{1}{2^4 \cdot 3^5}(x-9)^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = \sqrt[3]{x}, \quad a = 8$

Solution

$$f(x) = x^{1/3} \quad \rightarrow \quad f(8) = 2$$

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} \quad \rightarrow \quad f'(8) = \frac{1}{2^2 \times 3}$$

$$f''(x) = -\frac{2}{9}x^{-5/3} = -\frac{2}{3^2 x^{5/3}} \quad \rightarrow \quad f''(8) = -\frac{1}{3^2 \times 2^4}$$

$$f'''(x) = \frac{10}{3^3}x^{-8/3} = \frac{2 \cdot 5}{3^3 x^{8/3}} \quad \rightarrow \quad f'''(8) = \frac{5}{2^7 \times 3^3}$$

$$P_0(x) = f(a)$$

$$\boxed{= 2}$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$\boxed{= 2 + \frac{1}{2^2 \cdot 3}(x-8)}$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{= 2 + \frac{1}{2^2 \cdot 3}(x-8) - \frac{1}{2^5 \cdot 3^2}(x-8)^2}$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$\boxed{= 2 + \frac{1}{2^2 \cdot 3}(x-8) - \frac{1}{2^5 \cdot 3^2}(x-8)^2 + \frac{1}{2^8 \cdot 3^4}(x-8)^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by *f at a*: $f(x)=\ln x, \quad a=e$

Solution

$$f(x)=\ln x \quad \rightarrow \quad f(e)=1$$

$$f'(x)=\frac{1}{x} \quad \rightarrow \quad f'(e)=\frac{1}{e}$$

$$f''(x)=-\frac{1}{x^2} \quad \rightarrow \quad f''(e)=-\frac{1}{e^2}$$

$$f'''(x)=\frac{2}{x^3} \quad \rightarrow \quad f'''(e)=\frac{2}{e^3}$$

$$P_0(x)=f(a)$$

$$=1$$

$$P_1(x)=f(a)+f'(a)(x-a)$$

$$=1+\frac{1}{e}(x-e)$$

$$P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$=1+\frac{1}{e}(x-e)-\frac{1}{2e^2}(x-e)^2$$

$$P_3(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3$$

$$=1+\frac{1}{e}(x-e)-\frac{1}{2e^2}(x-e)^2+\frac{1}{3e^3}(x-e)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by *f at a*: $f(x)=\sqrt[4]{x}, \quad a=8$

Solution

$$f(x)=x^{1/4} \quad \rightarrow \quad f(8)=\sqrt[4]{8}$$

$$f'(x)=\frac{1}{4}x^{-3/4}=\frac{1}{4x^{3/4}} \quad \rightarrow \quad f'\left(8=2^3\right)=\frac{1}{2^2\times 2^{9/4}}=\frac{1}{2^4\sqrt[4]{2}}$$

$$f''(x)=-\frac{3}{16}x^{-7/4}=-\frac{3}{2^4x^{7/4}} \quad \rightarrow \quad f''(8)=-\frac{3}{2^4\,2^{21/4}}=-\frac{3}{2^9\sqrt[4]{2}}$$

$$f'''(x)=\frac{21}{2^6}x^{-11/4}=\frac{21}{2^6x^{11/4}} \quad \rightarrow \quad f'''(8)=\frac{21}{2^6\times 2^{33/4}}=\frac{21}{2^{14}\sqrt[4]{2}}$$

$$P_0(x)=f(a)$$

$$=\sqrt[4]{8}$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$= \sqrt[4]{8} + \frac{1}{2^4 \cdot \sqrt[4]{2}}(x-8)$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$= \sqrt[4]{8} + \frac{1}{2^4 \cdot \sqrt[4]{2}}(x-8) - \frac{3}{2^{10} \cdot \sqrt[4]{2}}(x-8)^2$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= \sqrt[4]{8} + \frac{1}{2^4 \cdot \sqrt[4]{2}}(x-8) - \frac{3}{2^{10} \cdot \sqrt[4]{2}}(x-8)^2 + \frac{7}{2^{15} \cdot \sqrt[4]{2}}(x-8)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = \tan^{-1} x + x^2 + 1, \quad a = 1$

Solution

$$f(x) = \tan^{-1} x + x^2 + 1 \quad \to \quad f(1) = \frac{\pi}{4} + 2$$

$$f'(x) = \frac{1}{x^2+1} + 2x \quad \to \quad f'(1) = \frac{5}{2}$$

$$f''(x) = -\frac{2x}{\left(x^2+1\right)^2} + 2 \quad \to \quad f''(1) = -\frac{1}{2} + 2 = \frac{3}{2}$$

$$f'''(x) = -\frac{2x^2+2-8x^2}{\left(x^2+1\right)^3} = -\frac{2-2x^2}{\left(x^2+1\right)^3} \quad \to \quad f'''(1) = 0 \qquad \left(U^n V^m\right)' = U^{n-1} V^{m-1}\left(nU'V + mUV'\right)$$

$$P_0(x) = f(a)$$

$$= \frac{\pi}{4} + 2$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$= \frac{\pi}{4} + 2 + \frac{5}{2}(x-1)$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$= \frac{\pi}{4} + 2 + \frac{5}{2}(x-1) - \frac{3}{4}(x-1)^2$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= \frac{\pi}{4} + 2 + \frac{5}{2}(x-1) - \frac{3}{4}(x-1)^2$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a: $f(x)=e^x, \quad a=\ln 2$

Solution

$f(x)=e^x \quad \rightarrow \quad f(\ln 2)=2$

$f'(x)=e^x \quad \rightarrow \quad f'(\ln 2)=2$

$f''(x)=e^x \quad \rightarrow \quad f''(\ln 2)=2$

$f'''(x)=e^x \quad \rightarrow \quad f'''(\ln 2)=2$

$P_0(x)=f(a)$

$\quad =2$

$P_1(x)=f(a)+f'(a)(x-a)$

$\quad =2+2(x-\ln 2)$

$P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$

$\quad =2+2(x-\ln 2)+(x-\ln 2)^2$

$P_3(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3$

$\quad =2+2(x-\ln 2)+(x-\ln 2)^2+\frac{1}{3}(x-\ln 2)^3$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a: $\quad f(x)=e^{3x}; \quad a=0$

Solution

$f(x)=e^{3x}$	$\rightarrow f(0)=1$
$f'(x)=3e^{3x}$	$\rightarrow f'(0)=3$
$f''(x)=9e^{3x}$	$\rightarrow f''(0)=9$
$f'''(x)=27e^{3x}$	$\rightarrow f'''(0)=27$

$P_0(x)=f(a)$

$\quad =1$

$P_1(x)=f(a)+f'(a)(x-a)$

$$=1+3x$$

$$P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$=1+3x+\frac{9}{2}x^2$$

$$P_3(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3$$

$$=1+3x+\frac{9}{2}x^2+\frac{9}{2}x^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a: $\quad f(x)=\frac{1}{x};\quad a=1$

Solution

$f(x)=\frac{1}{x}$	$\to f(1)=1$
$f'(x)=-\frac{1}{x^2}$	$\to f'(1)=-1$
$f''(x)=\frac{2}{x^3}$	$\to f''(1)=2$
$f'''(x)=-\frac{6}{x^4}$	$\to f'''(1)=-6$

$$P_0(x)=f(a)$$

$$=1-(x-1)$$

$$=1$$

$$P_1(x)=f(a)+f'(a)(x-a)$$

$$=2-x$$

$$P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_2(x)=1-(x-1)+(x-1)^2$$

$$P_3(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3$$

$$=1-(x-1)+(x-1)^2-(x-1)^3$$

$$f(x)=\sum_{k=0}^{\infty}(-1)^k(x-1)^k$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a: $\quad f(x)=\cos x; \quad a=\frac{\pi}{2}$

Solution

$f(x)=\cos x \qquad \rightarrow f\left(\frac{\pi}{2}\right)=0$

$f'(x)=-\sin x \qquad \rightarrow f'\left(\frac{\pi}{2}\right)=-1$

$f''(x)=-\cos x \qquad \rightarrow f''\left(\frac{\pi}{2}\right)=0$

$f'''(x)=\sin x \qquad \rightarrow f'''\left(\frac{\pi}{2}\right)=1$

$$P_0(x)=f(a)$$
$$=0$$

$$P_1(x)=f(a)+f'(a)(x-a)$$
$$=-\left(x-\frac{\pi}{2}\right)$$

$$P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$
$$=-\left(x-\frac{\pi}{2}\right)$$

$$P_3(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3$$
$$=-\left(x-\frac{\pi}{2}\right)+\frac{1}{6}\left(x-\frac{\pi}{2}\right)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a: $\quad f(x)=\frac{1}{x+1}; \quad a=0$

Solution

$f(x)=\frac{1}{x+1} \qquad \rightarrow f(0)=1$

$f'(x)=-\frac{1}{(x+1)^2} \qquad \rightarrow f'(0)=-1$

$f''(x)=\frac{2}{(x+1)^3} \qquad \rightarrow f''(0)=2$

$$f'''(x) = -\frac{6}{(x+1)^4} \quad \to f'''(0) = -6$$

$$P_0(x) = f(a)$$

$$\underline{= 1}$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$\underline{= 1 - x}$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\underline{= 1 - x + x^2}$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$\underline{= 1 - x + x^2 - x^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $\quad f(x) = \tan^{-1} 4x; \quad a = 0$

Solution

$$f(x) = \tan^{-1} 4x \quad \to f(0) = 0$$

$$f'(x) = \frac{4}{1+16x^2} \quad \to f'(0) = 4$$

$$f''(x) = -\frac{128x}{\left(1+16x^2\right)^2} \quad \to f''(0) = 0$$

$$f'''(x) = -\frac{128\left(1+16x^2\right) - 2(32x)(128x)}{\left(1+16x^2\right)^3} = \frac{6144x^2 - 128}{\left(1+16x^2\right)^3} \quad \to f'''(0) = -128$$

$$P_0(x) = f(a)$$

$$\underline{= 0}$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$\underline{= 4x}$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\underline{= 4x}$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$\underline{= x - \frac{64}{3}x^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $\quad f(x) = \sin 2x; \quad a = -\frac{\pi}{2}$

Solution

$f(x) = \sin 2x$	$\rightarrow f\left(-\frac{\pi}{2}\right) = 0$
$f'(x) = 2\cos 2x$	$\rightarrow f'\left(-\frac{\pi}{2}\right) = -2$
$f''(x) = -4\sin 2x$	$\rightarrow f''\left(-\frac{\pi}{2}\right) = 0$
$f'''(x) = -8\cos 2x$	$\rightarrow f'''\left(-\frac{\pi}{2}\right) = 8$

$$P_0(x) = f(a)$$

$$\underline{= 0}$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$\underline{= -2\left(x + \frac{\pi}{2}\right)}$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\underline{= -2\left(x + \frac{\pi}{2}\right)}$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$\underline{= -2\left(x + \frac{\pi}{2}\right) + \frac{4}{3}\left(x + \frac{\pi}{2}\right)^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $\quad f(x) = \cosh 3x; \quad a = 0$

Solution

$$f(x)=\cosh 3x \qquad \to f(0)=1$$

$$f'(x)=3\sinh 3x \qquad \to f'(0)=0$$

$$f''(x)=9\cosh 3x \qquad \to f''(0)=9$$

$$f'''(x)=27\cosh 3x \qquad \to f'''(0)=0$$

$$P_0(x)=f(a)$$

$$\underline{=1\,}|$$

$$P_1(x)=f(a)+f'(a)(x-a)$$

$$\underline{=1\,}|$$

$$P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$\underline{=1+\frac{9}{2}x^2\,}|$$

$$P_3(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3$$

$$\underline{=1+\frac{9}{2}x^2\,}|$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $\qquad f(x)=\frac{1}{4+x^2}; \quad a=0$

Solution

$$f(x)=\frac{1}{4+x^2} \qquad \to f(0)=\frac{1}{4}$$

$$f'(x)=-\frac{2x}{\left(4+x^2\right)^2} \qquad \to f'(0)=0$$

$$f''(x)=-\frac{8+2x^2-2(2x)^2}{\left(4+x^2\right)^3}$$

$$=-\frac{8-6x^2}{\left(4+x^2\right)^3} \qquad \to f''(0)=-\frac{1}{8}$$

$$f'''(x) = -\frac{-12x\left(4+x^2\right) - 6x\left(8-6x^2\right)}{\left(4+x^2\right)^4} \qquad \to f'''(0) = 0$$

$$= \frac{96x - 24x^2}{\left(4+x^2\right)^4}$$

$$P_0(x) = f(a)$$

$$\boxed{= \frac{1}{4}}$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$\boxed{= \frac{1}{4}}$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{= \frac{1}{4} - \frac{1}{16}x^2}$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$\boxed{= \frac{1}{4} - \frac{1}{16}x^2}$$

Exercise

Find out the third term of the Maclaurin series for the following function. $f(x) = (1+x)^{1/3}$

Solution

$f(x) = (1+x)^{1/3}$	$\to f(0) = 1$
$f'(x) = \frac{1}{3}(1+x)^{-2/3}$	$\to f'(0) = \frac{1}{3}$
$f''(x) = -\frac{2}{9}(1+x)^{-5/3}$	$\to f''(0) = -\frac{2}{9}$
$f'''(x) = \frac{10}{27}(1+x)^{-8/3}$	$\to f'''(0) = \frac{10}{27}$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3$$

$$\boxed{= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3}$$

Exercise

Find out the third term of the Maclaurin series for the following function. $f(x)=(1+x)^{-1/2}$

Solution

$f(x)=(1+x)^{-1/2}$	$\to f(0)=1$
$f'(x)=-\frac{1}{2}(1+x)^{-3/2}$	$\to f'(0)=-\frac{1}{2}$
$f''(x)=\frac{3}{4}(1+x)^{-5/2}$	$\to f''(0)=\frac{3}{4}$
$f'''(x)=-\frac{15}{8}(1+x)^{-7/2}$	$\to f'''(0)=-\frac{15}{8}$

$$P_3(x)=f(a)+f'(a)(x-a)+\frac{1}{2!}f''(a)(x-a)^2+\frac{1}{3!}f'''(a)(x-a)^3$$

$$=1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3$$

Exercise

Find out the third term of the Maclaurin series for the following function. $f(x)=\left(1+\frac{x}{2}\right)^{-3}$

Solution

$f(x)=\left(1+\frac{x}{2}\right)^{-3}$	$\to f(0)=1$
$f'(x)=-\frac{3}{2}\left(1+\frac{x}{2}\right)^{-4}$	$\to f'(0)=-\frac{3}{2}$
$f''(x)=3\left(1+\frac{x}{2}\right)^{-5}$	$\to f''(0)=3$
$f'''(x)=-\frac{15}{2}\left(1+\frac{x}{2}\right)^{-6}$	$\to f'''(0)=-\frac{15}{2}$

$$P_3(x)=f(a)+f'(a)(x-a)+\frac{1}{2!}f''(a)(x-a)^2+\frac{1}{3!}f'''(a)(x-a)^3$$

$$=1-\frac{3}{2}x+\frac{3}{2}x^2-\frac{5}{4}x^3$$

Exercise

Find out the third term of the Maclaurin series for the following function. $f(x)=(1+2x)^{-5}$

Solution

$f(x)=(1+2x)^{-5} \quad \rightarrow f(0)=1$

$f'(x)=-10(1+2x)^{-6} \quad \rightarrow f'(0)=-10$

$f''(x)=120(1+2x)^{-7} \quad \rightarrow f''(0)=120$

$f'''(x)=-1680(1+2x)^{-8} \quad \rightarrow f'''(0)=-1680$

$$P_3(x)=f(a)+f'(a)(x-a)+\frac{1}{2!}f''(a)(x-a)^2+\frac{1}{3!}f'''(a)(x-a)^3$$

$$=1-10x+60x^2-280x^3$$

Exercise

Find the nth Maclaurin polynomial for the function $f(x)=e^{4x}, \quad n=4$

Solution

$f(x)=e^{4x} \quad \rightarrow \quad f(0)=1$

$f'(x)=4e^{4x} \quad \rightarrow \quad f'(0)=4$

$f''(x)=16e^{4x} \quad \rightarrow \quad f''(0)=16$

$f'''(x)=64e^{4x} \quad \rightarrow \quad f'''(0)=64$

$f^{(4)}(x)=256e^{4x} \quad \rightarrow \quad f^{(4)}(0)=256$

$$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$

$$P_4(x)=1+4x+8x^2+\frac{32}{3}x^3+\frac{32}{3}x^4$$

Exercise

Find the nth Maclaurin polynomial for the function $f(x)=e^{-x}, \quad n=5$

Solution

$f(x)=e^{-x} \quad \rightarrow \quad f(0)=1$

$f'(x)=-e^{-x} \quad \rightarrow \quad f'(0)=-1$

$f''(x)=e^{-x} \quad \rightarrow \quad f''(0)=1$

$$f'''(x) = -e^{-x} \quad \to \quad f'''(0) = -1$$

$$f^{(4)}(x) = e^{-x} \quad \to \quad f^{(4)}(0) = 1$$

$$f^{(5)}(x) = -e^{-x} \quad \to \quad f^{(5)}(0) = -1$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$\boxed{P_5(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5}$$

Exercise

Find the *n*th Maclaurin polynomial for the function $f(x) = e^{-x/2}$, $n = 4$

Solution

$$f(x) = e^{-x/2} \quad \to \quad f(0) = 1$$

$$f'(x) = -\frac{1}{2}e^{-x/2} \quad \to \quad f'(0) = -\frac{1}{2}$$

$$f''(x) = \frac{1}{4}e^{-x/2} \quad \to \quad f''(0) = \frac{1}{4}$$

$$f'''(x) = -\frac{1}{8}e^{-x/2} \quad \to \quad f'''(0) = -\frac{1}{8}$$

$$f^{(4)}(x) = \frac{1}{16}e^{-x/2} \quad \to \quad f^{(4)}(0) = \frac{1}{16}$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\boxed{P_4(x) = 1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4}$$

Exercise

Find the *n*th Maclaurin polynomial for the function $f(x) = e^{x/3}$, $n = 4$

Solution

$$f(x) = e^{x/3} \quad \to \quad f(0) = 1$$

$$f'(x) = \frac{1}{3}e^{x/3} \quad \to \quad f'(0) = \frac{1}{3}$$

$$f''(x) = \frac{1}{9}e^{x/3} \quad \to \quad f''(0) = \frac{1}{9}$$

$$f'''(x)=\frac{1}{27}e^{x/3} \quad\to\quad f'''(0)=\frac{1}{27}$$

$$f^{(4)}(x)=\frac{1}{81}e^{x/3} \quad\to\quad f^{(4)}(0)=\frac{1}{81}$$

$$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$

$$\boxed{P_4(x)=1+\frac{1}{3}x+\frac{1}{18}x^2+\frac{1}{162}x^3+\frac{1}{1944}x^4}$$

Exercise

Find the *n*th Maclaurin polynomial for the function $\quad f(x)=\sin x, \quad n=5$

Solution

$$f(x)=\sin x \quad\to\quad f(0)=0$$

$$f'(x)=\cos x \quad\to\quad f'(0)=1$$

$$f''(x)=-\sin x \quad\to\quad f''(0)=0$$

$$f'''(x)=-\cos x \quad\to\quad f'''(0)=-1$$

$$f^{(4)}(x)=\sin x \quad\to\quad f^{(4)}(0)=0$$

$$f^{(5)}(x)=\cos x \quad\to\quad f^{(5)}(0)=1$$

$$P_5(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\frac{f^{(5)}(0)}{5!}x^5$$

$$\boxed{P_5(x)=x-\frac{1}{6}x^3+\frac{1}{120}x^5}$$

Exercise

Find the *n*th Maclaurin polynomial for the function $\quad f(x)=\cos \pi x, \quad n=4$

Solution

$$f(x)=\cos \pi x \quad\to\quad f(0)=1$$

$$f'(x)=-\pi \sin \pi x \quad\to\quad f'(0)=0$$

$$f''(x)=-\pi^2 \cos \pi x \quad\to\quad f''(0)=-\pi^2$$

$$f'''(x)=\pi^3 \sin \pi x \quad\to\quad f'''(0)=0$$

$$f^{(4)}(x) = \pi^4 \cos \pi x \quad \rightarrow \quad f^{(4)}(0) = \pi^4$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$P_4(x) = 1 - \frac{\pi^2}{2}x^2 + \frac{\pi^4}{24}x^4$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x) = xe^x, \quad n = 4$

Solution

$$f(x) = xe^x \quad \rightarrow \quad f(0) = 0$$

$$f'(x) = e^x + xe^x \quad \rightarrow \quad f'(0) = 1$$

$$f''(x) = 2e^x + xe^x \quad \rightarrow \quad f''(0) = 2$$

$$f'''(x) = 3e^x + xe^x \quad \rightarrow \quad f'''(0) = 3$$

$$f^{(4)}(x) = 4e^x + xe^x \quad \rightarrow \quad f^{(4)}(0) = 4$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$P_4(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x) = x^2e^{-x}, \quad n = 4$

Solution

$$f(x) = x^2e^{-x} \quad \rightarrow \quad f(0) = 0$$

$$f'(x) = 2xe^{-x} - x^2e^{-x} \quad \rightarrow \quad f'(0) = 0$$

$$f''(x) = 2e^{-x} - 4xe^{x} + x^2e^{-x} \quad \rightarrow \quad f''(0) = 2$$

$$f'''(x) = -6e^{-x} + 6xe^{-x} - x^2e^{-x} \quad \rightarrow \quad f'''(0) = -6$$

$$f^{(4)}(x) = 12e^{-x} - 8xe^{-x} + x^2e^{-x} \quad \rightarrow \quad f^{(4)}(0) = 12$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\boxed{P_4(x) = x^2 - x^3 + \frac{1}{2}x^4}$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x) = \frac{1}{x+1}, \quad n = 5$

Solution

$$f(x) = \frac{1}{x+1} \quad \rightarrow \quad f(0) = 1$$

$$f'(x) = -(x+1)^{-2} \quad \rightarrow \quad f'(0) = -1$$

$$f''(x) = 2(x+1)^{-3} \quad \rightarrow \quad f''(0) = 2$$

$$f'''(x) = -6(x+1)^{-4} \quad \rightarrow \quad f'''(0) = -6$$

$$f^{(4)}(x) = 24(x+1)^{-5} \quad \rightarrow \quad f^{(4)}(0) = 24$$

$$f^{(5)}(x) = -120(x+1)^{-6} \quad \rightarrow \quad f^{(5)}(0) = -120$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$\boxed{P_5(x) = 1 - x + x^2 - x^3 + x^4 - x^5}$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x) = \frac{x}{x+1}, \quad n = 4$

Solution

$$f(x) = \frac{x}{x+1} = 1 - \frac{1}{x+1} \quad \rightarrow \quad f(0) = 0$$

$$f'(x) = (x+1)^{-2} \quad \rightarrow \quad f'(0) = 1$$

$$f''(x) = -2(x+1)^{-3} \quad \rightarrow \quad f''(0) = -2$$

$$f'''(x) = 6(x+1)^{-4} \quad \rightarrow \quad f'''(0) = 6$$

$$f^{(4)}(x) = -24(x+1)^{-5} \quad \rightarrow \quad f^{(4)}(0) = -24$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\boxed{P_4(x) = x - x^2 + x^3 - x^4}$$

Exercise

Find the *n*th Maclaurin polynomial for the function $f(x) = \sec x, \quad n = 2$

Solution

$$f(x) = \sec x \quad \rightarrow \quad f(0) = 1$$

$$f'(x) = \sec x \tan x \quad \rightarrow \quad f'(0) = 0$$

$$f''(x) = \sec x \tan^2 x + \sec^3 x \quad \rightarrow \quad f''(0) = 1$$

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$

$$\boxed{P_2(x) = 1 + \frac{1}{2}x^2}$$

Exercise

Find the *n*th Maclaurin polynomial for the function $f(x) = \tan x, \quad n = 3$

Solution

$$f(x) = \tan x \quad \rightarrow \quad f(0) = 0$$

$$f'(x) = \sec^2 x \quad \rightarrow \quad f'(0) = 1$$

$$f''(x) = 2\sec^2 x \tan x \quad \rightarrow \quad f''(0) = 0$$

$$f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x \quad \rightarrow \quad f'''(0) = 2$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3$$

$$\boxed{P_4(x) = x + \frac{1}{3}x^3}$$

Exercise

Find the Maclaurin series for: xe^x

Solution

$$f(x) = xe^x \quad \to \quad f(0) = 0$$

$$f'(x) = e^x + xe^x \quad \to \quad f'(0) = 1$$

$$f''(x) = 2e^x + xe^x \quad \to \quad f''(0) = 2$$

$$f'''(x) = 3e^x + xe^x \quad \to \quad f'''(0) = 3$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$f^{(n)}(x) = ne^x + xe^x \quad \to \quad f^{(n)}(0) = n$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$xe^x = x + x^2 + \frac{1}{2}x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n-1)!}x^n$$

Exercise

Find the Maclaurin series for: $5\cos \pi x$

Solution

$$f(x) = 5\cos \pi x \quad \to \quad f(0) = 5$$

$$f'(x) = -5\pi \sin \pi x \quad \to \quad f'(0) = 0$$

$$f''(x) = -5\pi^2 \cos \pi x \quad \to \quad f''(0) = -5\pi^2$$

$$f'''(x) = 5\pi^3 \sin \pi x \quad \to \quad f'''(0) = 0$$

$$5\cos \pi x = 5 - \frac{5\pi^2 x^2}{2!} + \frac{5\pi^4 x^4}{4!} - \frac{5\pi^6 x^6}{6!} + \cdots$$

$$= 5\sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!}$$

Exercise

Find the Maclaurin series for: $\dfrac{x^2}{x+1}$

Solution

$$f(x) = \frac{x^2}{x+1} \quad \to \quad f(0) = 0$$

$$f'(x)=\frac{2x(x+1)-x^2}{(x+1)^2}=\frac{x^2+2x}{(x+1)^2}\quad\to\quad f'(0)=0$$

$$f''(x)=\frac{(2x+2)(x+1)^2-2(x+1)\left(x^2+2x\right)}{(x+1)^4}$$

$$=\frac{2x^2+4x+2-2x^2-4x}{(x+1)^3}$$

$$=\frac{2}{(x+1)^3}\quad\to\quad f''(0)=2$$

$$f'''(x)=\frac{-6}{(x+1)^4}\quad\to\quad f'''(0)=-6$$

$$\frac{x^2}{x+1}=\frac{2}{2!}x^2-\frac{6x^3}{3!}+\frac{24x^4}{4!}-\cdots$$

$$=x^2-x^3+x^4-\cdots$$

$$=\sum_{n=2}^{\infty}(-1)^n x^n$$

Exercise

Find the Maclaurin series for: e^{3x+1}

Solution

$$e^{3x+1}=e\cdot e^{3x}$$

$$=e\left(\sum_{n=0}^{\infty}\frac{(3x)^n}{n!}\right)$$

$$=e\sum_{n=0}^{\infty}\frac{3^n x^n}{n!}\quad(for\ all\ x)$$

Exercise

Find the Maclaurin series for: $\cos\left(2x^3\right)$

Solution

$$\cos\left(2x^3\right)=1-\frac{\left(2x^3\right)^2}{2!}+\frac{\left(2x^3\right)^4}{4!}-\frac{\left(2x^3\right)^6}{6!}+\cdots$$

$$= 1 - \frac{2^2 x^3}{2!} + \frac{2^4 x^{12}}{4!} - \frac{2^6 x^{18}}{6!} + \cdots$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n)!} x^{6n}} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\cos(2x - \pi)$

Solution

$$\cos(2x - \pi) = \cos(2x)\cos\pi + \sin(2x)\sin\pi$$

$$= -\cos(2x)$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x)^{2n}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^{n+1} 4^n}{(2n)!} x^{2n}} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $x^2 \sin\left(\frac{x}{3}\right)$

Solution

$$x^2 \sin\left(\frac{x}{3}\right) = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{x}{3}\right)^{2n+1}}{(2n+1)!}$$

$$= x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{3^{2n+1}(2n+1)!}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{3^{2n+1}(2n+1)!}} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\cos^2\left(\frac{x}{2}\right)$

Solution

$$\cos^2\left(\frac{x}{2}\right)=\frac{1}{2}(1+\cos x)$$

$$=\frac{1}{2}\left(1+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}\right)$$

$$=\frac{1}{2}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$

$$=1+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} \quad (for\ all\ x)$$

Exercise

Find the Maclaurin series for: $\sin x\cos x$

Solution

$$\sin x\cos x=\frac{1}{2}\sin(2x)$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2x)^{2n+1}$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}2^{2n+1}x^{2n+1}$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}}{(2n+1)!}x^{2n+1} \quad (for\ all\ x)$$

Exercise

Find the Maclaurin series for: $\tan^{-1}\left(5x^2\right)$

Solution

$$\tan^{-1}\left(5x^2\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(5x^2\right)^{2n+1}$$

$$5x^2\le 1 \to x^2\le\frac{1}{5} \Rightarrow -\frac{1}{\sqrt{5}}\le x\le\frac{1}{\sqrt{5}}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n 5^{2n+1}}{2n+1}x^{4n+2} \quad \left(for\ -\frac{1}{\sqrt{5}}\le x\le\frac{1}{\sqrt{5}}\right)$$

Exercise

Find the Maclaurin series for: $\ln\left(2+x^2\right)$

Solution

$$\ln\left(2+x^2\right)=\ln 2\left(1+\frac{x^2}{2}\right)$$

$$=\ln 2+\ln\left(1+\frac{x^2}{2}\right)$$

$$=\ln 2+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{x^2}{2}\right)^n \qquad \frac{x^2}{2}\le 1 \to x^2\le 2 \quad\Rightarrow\quad -\sqrt{2}\le x\le\sqrt{2}$$

$$=\ln 2+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}\frac{x^{2n}}{2^n} \qquad \left(for\ \ -\sqrt{2}\le x\le\sqrt{2}\right)$$

Exercise

Find the Maclaurin series for: $\dfrac{1+x^3}{1+x^2}$

Solution

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n=1-x+x^2-x^3+\cdots$$

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+\cdots$$

$$\frac{1+x^3}{1+x^2}=\left(1+x^3\right)\left(1-x^2+x^4-x^6+\cdots\right)$$

$$=1-x^2+x^4-x^6+\cdots+x^3-x^5+x^7-x^9+\cdots$$

$$=1-x^2+x^3+x^4-x^5-x^6+x^7+x^8-x^9-\cdots$$

$$=1-x^2+\sum_{n=2}^{\infty}(-1)^n\left(x^{2n-1}+x^{2n}\right) \qquad \left(for\ \ |x|<1\right)$$

Exercise

Find the Maclaurin series for: $\ln\dfrac{1+x}{1-x}$

Solution

$$\ln\frac{1+x}{1-x} = \ln(1+x) - \ln(1-x) \qquad\qquad = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots\right) = 2x + 2\frac{x^3}{3} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$= \sum_{n=0}^{\infty} \left((-1)^n + 1\right) \frac{x^{n+1}}{n+1}$$

$$= 2\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1} \qquad (-1 < x < 1)$$

Exercise

Find the Maclaurin series for: $\dfrac{e^{2x^2} - 1}{x^2}$

Solution

$$\frac{e^{2x^2} - 1}{x^2} = \frac{1}{x^2}\left(e^{2x^2} - 1\right)$$

$$= \frac{1}{x^2}\left(1 + 2x^2 + \frac{\left(2x^2\right)^2}{2!} + \frac{\left(2x^2\right)^3}{3!} + \cdots - 1\right)$$

$$= \frac{1}{x^2}\left(2x^2 + \frac{2^2x^4}{2!} + \frac{2^3x^6}{3!} + \cdots\right)$$

$$= 2 + \frac{2^2x^2}{2!} + \frac{2^3x^4}{3!} + \frac{2^4x^6}{4!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{2^{n+1}}{(n+1)!} x^{2n} \qquad (\text{for all } \ x \neq 0)$$

Exercise

Find the Maclaurin series for: $\cosh x - \cos x$

Solution

$$\cosh x - \cos x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)=2\frac{x^2}{2!}+2\frac{x^6}{6!}+\cdots$$

$$=\sum_{n=0}^{\infty}\left(1-(-1)^n\right)\frac{x^{2n}}{(2n)!}$$

$$=2\sum_{n=0}^{\infty}\frac{x^{4n+2}}{(4n+2)!} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\sinh x-\sin x$

Solution

$$\sinh x-\sin x=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

$$=\sum_{n=0}^{\infty}\left(1-(-1)^n\right)\frac{x^{2n+1}}{(2n+1)!}$$

$$=2\sum_{n=0}^{\infty}\frac{x^{4n+3}}{(4n+3)!} \quad (\text{for all } x)$$

Exercise

Finding Taylor and Maclaurin Series generated by f at $x = a$: $f(x)=x^3-2x+4, \quad a=2$

Solution

$$f(x)=x^3-2x+4 \quad \to \quad f(2)=8$$

$$f'(x)=3x^2-2 \quad \to \quad f'(2)=10$$

$$f''(x)=6x \quad \to \quad f''(2)=12$$

$$f'''(x)=6 \quad \to \quad f'''(2)=6$$

$$f^{(n)}(x)=0 \quad \to \quad f^{(n)}(2)=0 \quad (n>3)$$

$$P_n(x)=f(2)+f'(2)(x-2)+\frac{f''(2)}{2!}(x-2)^2+\frac{f'''(2)}{3!}(x-2)^3+\cdots$$

$$x^3-2x+4=8+10(x-2)+6(x-2)^2+(x-2)^3$$

Exercise

Finding Taylor and Maclaurin Series generated by f *at* $x = a$: $\quad f(x) = 2x^3 + x^2 + 3x - 8, \quad a = 1$

Solution

$$f(x) = 2x^3 + x^2 + 3x - 8 \quad \to \quad f(1) = -2$$

$$f'(x) = 6x^2 + 2x + 3 \quad \to \quad f'(1) = 11$$

$$f''(x) = 12x + 2 \quad \to \quad f''(1) = 14$$

$$f'''(x) = 12 \quad \to \quad f'''(1) = 12$$

$$f^{(n)}(x) = 0 \quad \to \quad f^{(n)}(1) = 0 \quad (n \ge 4)$$

$$P_n(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots$$

$$\underline{2x^3 + x^2 + 3x - 8 = -2 + 11(x-1) + 7(x-1)^2 + 2(x-1)^3}$$

Exercise

Finding Taylor and Maclaurin Series generated by f *at* $x = a$:

$$f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2, \quad a = -1$$

Solution

$$f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2 \quad \to \quad f(-1) = -7$$

$$f'(x) = 15x^4 - 4x^3 + 6x^2 + 2x \quad \to \quad f'(-1) = 23$$

$$f''(x) = 60x^3 - 12x^2 + 12x + 2 \quad \to \quad f''(-1) = -82$$

$$f'''(x) = 180x^2 - 24x + 12 \quad \to \quad f'''(-1) = 216$$

$$f^{(4)}(x) = 360x - 24 \quad \to \quad f^{(4)}(-1) = -384$$

$$f^{(5)}(x) = 360 \quad \to \quad f^{(5)}(-1) = 360$$

$$f^{(n)}(x) = 0 \quad \to \quad f^{(n)}(-1) = 0 \quad (n \ge 6)$$

$$P_n(x) = f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \frac{f'''(-1)}{3!}(x+1)^3 + \frac{f^{(4)}(-1)}{4!}(x+1)^2 + \frac{f^{(4)}(-1)}{5!}(x+1)^3$$

$$3x^5 - x^4 + 2x^3 + x^2 - 2 = -7 + 23(x+1) - \frac{82}{2!}(x+1)^2 + \frac{216}{3!}(x+1)^3 - \frac{384}{4!}(x+1)^4 + \frac{360}{5!}(x+1)^3$$

$$\underline{= -7 + 23(x+1) - 41(x+1)^2 + 36(x+1)^3 - 16(x+1)^4 + 3(x+1)^3}$$

Exercise

Finding Taylor and Maclaurin Series generated by f at $x = a$: $\quad f(x)=\cos\left(2x+\frac{\pi}{2}\right), \quad a=\frac{\pi}{4}$

Solution

$$f(x)=\cos\left(2x+\tfrac{\pi}{2}\right) \rightarrow f\left(\tfrac{\pi}{4}\right)=-1$$

$$f'(x)=-2\sin\left(2x+\tfrac{\pi}{2}\right) \rightarrow f'\left(\tfrac{\pi}{4}\right)=0$$

$$f''(x)=-4\cos\left(2x+\tfrac{\pi}{2}\right) \rightarrow f''\left(\tfrac{\pi}{4}\right)=4$$

$$f'''(x)=8\sin\left(2x+\tfrac{\pi}{2}\right) \rightarrow f'''\left(\tfrac{\pi}{4}\right)=0$$

$$f^{(4)}(x)=16\cos\left(2x+\tfrac{\pi}{2}\right) \rightarrow f^{(4)}\left(\tfrac{\pi}{4}\right)=-16$$

$$f^{(5)}(x)=-32\sin\left(2x+\tfrac{\pi}{2}\right) \rightarrow f^{(5)}\left(\tfrac{\pi}{4}\right)=0$$

$$\rightarrow f^{(2n)}\left(\tfrac{\pi}{4}\right)=(-1)^n 2^{2n}$$

$$\begin{aligned}\cos\left(2x+\tfrac{\pi}{2}\right)&=-1+\frac{4}{2!}\left(x-\tfrac{\pi}{4}\right)^2-\frac{16}{4!}\left(x-\tfrac{\pi}{4}\right)^4+\cdots\\&=-1+2\left(x-\tfrac{\pi}{4}\right)^2-\frac{2}{3}\left(x-\tfrac{\pi}{4}\right)^4+\cdots\\&=\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}}{(2n)!}\left(x-\tfrac{\pi}{4}\right)^{2n}\end{aligned}$$

Exercise

Find the Taylor series of the functions, where is the series representation valid? $f(x)=e^{-2x}$ *about* -1

Solution

Let $t=x+1 \Rightarrow x=t-1$

$$\begin{aligned}f(x)=e^{-2x}&=e^{-2x-2+2}=e^{-2(x+1)+2}\\&=e^2\sum_{n=0}^{\infty}\frac{(-2(x+1))^n}{n!}\\&=e^2\sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{n!}(x+1)^n \qquad (\textit{for all } x)\end{aligned}$$

Exercise

Find the Taylor series of the functions, where is the series representation valid? $f(x)=\sin x \quad about \ \frac{\pi}{2}$

Solution

Let $y=x-\frac{\pi}{2} \quad \Rightarrow \quad x=y+\frac{\pi}{2}$

$$\sin x=\sin\left(y+\frac{\pi}{2}\right)=\cos y$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}y^{2n}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{2}\right)^{2n}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{2}\right)^{2n} \qquad (for\ all\ x)$$

Exercise

Find the Taylor series of the functions, where is the series representation valid?

$f(x)=\ln x \quad in\ powers\ of\ x-3$

Solution

Let $y=x-3 \quad \Rightarrow \quad x=y+3$

$$\ln x=\ln(y+3)=\ln 3\left(1+\frac{y}{3}\right)$$

$$=\ln 3+\ln\left(1+\frac{y}{3}\right)$$

$$=\ln 3+\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\left(\frac{y}{3}\right)^{n+1}$$

$$=\ln 3+\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\frac{(x-3)^{n+1}}{3^{n+1}}$$

$$=\ln 3+\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)3^{n+1}}(x-3)^{n+1} \qquad (0<x\le 6)$$

Exercise

Find the Taylor series of the functions, where is the series representation valid?

$$f(x) = \ln(2+x) \quad \textit{in powers of } x-2$$

Solution

$$\ln(2+x) = \ln(2+x-2+2) = \ln(4+x-2)$$

$$= \ln 4\left(1+\frac{x-2}{4}\right)$$

$$= \ln 4 + \ln\left(1+\frac{x-2}{4}\right)$$

$$= \ln 4 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\left(\frac{x-2}{4}\right)^{n+1} \qquad \frac{|x-2|}{4} < 1 \Rightarrow |x-2| < 4 \qquad -4 < x-2 < 4 \qquad -2 < x < 6$$

$$\boxed{= \ln 4 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)4^{n+1}}(x-2)^{n+1}} \qquad (-2 < x \le 6)$$

Exercise

Find the Taylor series of the functions, where is the series representation valid?

$$f(x) = e^{2x+3} \quad \textit{in powers of } x+1$$

Solution

$$e^{2x+3} = e^{2x+2-2+3} = e^{2(x+1)+1}$$

$$= e \cdot e^{2(x+1)}$$

$$= e\sum_{n=0}^{\infty} \frac{(2(x+1))^n}{n!}$$

$$\boxed{= e\sum_{n=0}^{\infty} \frac{2^n (x+1)^n}{n!}} \qquad (\textit{for all } x)$$

Exercise

Find the Taylor series of the functions, where is the series representation valid?

$$f(x) = \sin x - \cos x \quad \textit{about } \frac{\pi}{4}$$

Solution

$$\sin x - \cos x = \sin\left(x - \frac{\pi}{4} + \frac{\pi}{4}\right) - \cos\left(x - \frac{\pi}{4} + \frac{\pi}{4}\right)$$

$$= \cos\left(x - \frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(x - \frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)$$

$$= \cos\left(x - \frac{\pi}{4}\right)\frac{\sqrt{2}}{2} + \sin\left(x - \frac{\pi}{4}\right)\frac{\sqrt{2}}{2} - \cos\left(x - \frac{\pi}{4}\right)\frac{\sqrt{2}}{2} + \sin\left(x - \frac{\pi}{4}\right)\frac{\sqrt{2}}{2}$$

$$= \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$$

$$= \sqrt{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(x - \frac{\pi}{4}\right)^{2n+1} \qquad (\textit{for all } x)$$

Exercise

Find the Taylor series of the functions, where is the series representation valid?

$$f(x) = \cos^2 x \quad \textit{about } \frac{\pi}{8}$$

Solution

$$\cos^2(x) = \frac{1}{2}(1 + \cos 2x) = \frac{1}{2}\left(1 + \cos\left(2x - \frac{\pi}{4} + \frac{\pi}{4}\right)\right)$$

$$= \frac{1}{2}\left(1 + \cos\left(2x - \frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(2x - \frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)\right)$$

$$= \frac{1}{2}\left(1 + \frac{\sqrt{2}}{2}\cos\left(2\left(x - \frac{\pi}{8}\right)\right) - \frac{\sqrt{2}}{2}\sin\left(2\left(x - \frac{\pi}{8}\right)\right)\right)$$

$$= \frac{1}{2} + \frac{\sqrt{2}}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(2\left(x - \frac{\pi}{8}\right)\right)^{2n} - \frac{\sqrt{2}}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(2\left(x - \frac{\pi}{8}\right)\right)^{2n+1}$$

$$= \frac{1}{2} + \frac{1}{2\sqrt{2}}\sum_{n=0}^{\infty}(-1)^n\left[\frac{2^{2n}}{(2n)!}\left(x - \frac{\pi}{8}\right)^{2n} - \frac{2^{2n+1}}{(2n+1)!}\left(x - \frac{\pi}{8}\right)^{2n+1}\right] \qquad (\textit{for all } x)$$

Exercise

Find the Taylor series of the functions, where is the series representation valid?

$$f(x) = \frac{x}{1+x} \quad \textit{in powers of } x - 1$$

Solution

$$\frac{x}{1+x} = \frac{x - 1 + 1}{1 + x - 1 + 1} = \frac{(x-1)+1}{(x-1)+2}$$

$$= 1 - \frac{1}{(x-1)+2}$$

$$= 1 - \frac{1}{2\left(1 + \frac{x-1}{2}\right)}$$

$$= 1 - \frac{1}{2}\left(1 - \frac{x-1}{2} + \left(\frac{x-1}{2}\right)^2 - \left(\frac{x-1}{2}\right)^3 + \cdots\right)$$

$$= \frac{1}{2} - \frac{1}{2^2}(x-1) + \frac{1}{2^3}(x-1)^2 - \frac{1}{2^4}(x-1)^3 + \cdots \qquad |x-1| < 1 \quad -1 < x-1 < 1$$

$$\boxed{= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n+1}}(x-1)^n} \qquad (0 < x < 2)$$

Exercise

Find the Taylor series of the functions, where is the series representation valid?

$$f(x) = xe^x \quad in\ powers\ of\ x+2$$

Solution

$$xe^x = (x+2-2)e^{x+2-2}$$

$$= (x+2-2)e^{-2}e^{x+2}$$

$$= (x+2)e^{-2}e^{x+2} - 2e^{-2}e^{x+2}$$

$$= \frac{1}{e^2}(x+2)\sum_{n=0}^{\infty}\frac{(x+2)^n}{n!} - \frac{2}{e^2}\sum_{n=0}^{\infty}\frac{(x+2)^n}{n!}$$

$$= \frac{1}{e^2}\sum_{n=0}^{\infty}\frac{(x+2)^{n+1}}{n!} - \frac{2}{e^2}\sum_{n=0}^{\infty}\frac{(x+2)^n}{n!}$$

$$= \frac{1}{e^2}\sum_{n=1}^{\infty}\frac{(x+2)^n}{(n-1)!} - \frac{2}{e^2}\left(1 + \sum_{n=1}^{\infty}\frac{(x+2)^n}{n!}\right)$$

$$= -\frac{2}{e^2} + \frac{1}{e^2}\sum_{n=1}^{\infty}\frac{(x+2)^n}{(n-1)!} - \frac{2}{e^2}\sum_{n=1}^{\infty}\frac{(x+2)^n}{n!}$$

$$\boxed{= -\frac{2}{e^2} + \frac{1}{e^2}\sum_{n=1}^{\infty}\left(\frac{1}{(n-1)!} - \frac{2}{n!}\right)(x+2)^n} \qquad (for\ all\ x)$$

Exercise

Find the nth Taylor polynomial centered at c for the function $\quad f(x)=\frac{2}{x}, \quad n=3, \quad c=1$

Solution

$$f(x)=\frac{2}{x} \quad \rightarrow \quad f(1)=2$$

$$f'(x)=-\frac{2}{x^2} \quad \rightarrow \quad f'(1)=-2$$

$$f''(x)=\frac{4}{x^3} \quad \rightarrow \quad f''(1)=4$$

$$f'''(x)=-\frac{12}{x^4} \quad \rightarrow \quad f'''(0)=-12$$

$$P_3(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f^{(3)}(c)}{3!}(x-c)^3$$

$$\boxed{P_3(x)=2-2(x-1)+2(x-1)^2-2(x-1)^3}$$

Exercise

Find the nth Taylor polynomial centered at c for the function $\quad f(x)=\frac{1}{x^2}, \quad n=4, \quad c=2$

Solution

$$f(x)=\frac{1}{x^2} \quad \rightarrow \quad f(2)=\frac{1}{4}$$

$$f'(x)=-\frac{2}{x^3} \quad \rightarrow \quad f'(2)=-\frac{1}{4}$$

$$f''(x)=\frac{6}{x^4} \quad \rightarrow \quad f''(2)=\frac{3}{8}$$

$$f'''(x)=-\frac{24}{x^5} \quad \rightarrow \quad f'''(2)=-\frac{3}{4}$$

$$f^{(4)}(x)=\frac{120}{x^4} \quad \rightarrow \quad f^{(4)}(2)=\frac{15}{8}$$

$$P_4(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f^{(3)}(c)}{3!}(x-c)^3+\frac{f^{(4)}(c)}{4!}(x-c)^4$$

$$\boxed{P_4(x)=\frac{1}{4}-\frac{1}{4}(x-2)+\frac{3}{16}(x-2)^2-\frac{1}{8}(x-2)^3+\frac{5}{64}(x-2)^4}$$

Exercise

Find the nth Taylor polynomial centered at c for the function $\quad f(x)=\sqrt{x}, \quad n=3, \quad c=4$

Solution

$$f(x) = x^{1/2} \quad \to \quad f(4) = 2$$

$$f'(x) = \frac{1}{2}x^{-1/2} \quad \to \quad f'(4) = \frac{1}{4}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} \quad \to \quad f''(4) = -\frac{1}{4}\frac{1}{\left(2^2\right)^{3/2}} = -\frac{1}{32}$$

$$f'''(x) = \frac{3}{8}x^{-5/2} \quad \to \quad f'''(4) = \frac{3}{256}$$

$$P_3(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3$$

$$\boxed{P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3}$$

Exercise

Find the nth Taylor polynomial centered at c for the function $f(x) = \sqrt[3]{x}, \quad n = 3, \quad c = 8$

Solution

$$f(x) = x^{1/3} \quad \to \quad f(8) = 2$$

$$f'(x) = \frac{1}{3}x^{-2/3} \quad \to \quad f'(8) = \frac{1}{12}$$

$$f''(x) = -\frac{2}{9}x^{-5/3} \quad \to \quad f''(8) = -\frac{1}{144}$$

$$f'''(x) = \frac{10}{27}x^{-8/3} \quad \to \quad f'''(8) = \frac{5}{3456}$$

$$P_3(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3$$

$$\boxed{P_3(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20,736}(x-8)^3}$$

Exercise

Find the nth Taylor polynomial centered at c for the function $f(x) = \ln x, \quad n = 4, \quad c = 2$

Solution

$$f(x) = \ln x \quad \to \quad f(2) = \ln 2$$

$$f'(x) = \frac{1}{x} \quad \to \quad f'(2) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{x^2} \quad \to \quad f''(2) = -\frac{1}{4}$$

$$f'''(x) = \frac{2}{x^3} \quad \to \quad f'''(2) = \frac{1}{4}$$

$$f^{(4)}(x) = -\frac{6}{x^4} \quad \to \quad f^{(4)}(2) = -\frac{3}{8}$$

$$P_4(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + \frac{f^{(4)}(c)}{4!}(x-c)^4$$

$$\boxed{P_4(x) = \ln 2 + \frac{1}{2}\frac{1}{4}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \frac{1}{64}(x-2)^4}$$

Exercise

Find the nth Taylor polynomial centered at c for the function $f(x) = x^2 \cos x, \quad n = 2, \quad c = \pi$

Solution

$$f(x) = x^2 \cos x \quad \to \quad f(\pi) = -\pi^2$$

$$f'(x) = 2x\cos x - x^2 \sin x \quad \to \quad f'(\pi) = -2\pi$$

$$f''(x) = 2\cos x - 4x\sin x - x^2 \cos x \quad \to \quad f''(\pi) = -2 + \pi^2$$

$$P_2(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2$$

$$\boxed{= -\pi^2 - 2\pi(x-\pi) + \frac{\pi^2 - 2}{2}(x-\pi)^2}$$

Exercise

Find the nth−order Taylor polynomial centered at c for the function $f(x) = \sin 2x; \quad n = 3, \quad c = 0$

Solution

$f(x) = \sin 2x$	$\to f(0) = 0$
$f'(x) = 2\cos 2x$	$\to f'(0) = 2$
$f''(x) = -4\sin 2x$	$\to f''(0) = 0$
$f'''(x) = -8\cos 2x$	$\to f'''(0) = -8$

$$P(x) = 2x - \frac{8}{3!}x^3$$

$$\boxed{= 2x - \frac{1}{3!}(2x)^3}$$

Exercise

Find the nth–order Taylor polynomial centered at c for the function $f(x)=\cos x^2;\quad n=2,\quad c=0$

Solution

$f(x)=\cos x^2 \quad \to f(0)=1$

$f'(x)=-2x\sin x^2 \quad \to f'(0)=0$

$f''(x)=-2\sin x^2-4x^2\cos x^2 \quad \to f''(0)=0$

$\boxed{P(x)=1}$

Exercise

Find the nth–order Taylor polynomial centered at c for the function $f(x)=e^{-x};\quad n=2,\quad c=0$

Solution

$f(x)=e^{-x} \quad \to f(0)=1$

$f'(x)=-e^{-x} \quad \to f'(0)=-1$

$f''(x)=e^{-x} \quad \to f''(0)=1$

$\boxed{P(x)=1-x-\frac{1}{2}x^2}$

Exercise

Find the nth–order Taylor polynomial centered at c for the function $f(x)=\cos x;\quad n=2,\quad c=\frac{\pi}{4}$

Solution

$f(x)=\cos x \quad \to f\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$

$f'(x)=-\sin x \quad \to f'\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$

$f''(x)=-\cos x \quad \to f''\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$

$\boxed{P(x)=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2}$

Exercise

Find the nth–order Taylor polynomial centered at c for the function $f(x)=\ln x; \quad n=2, \quad c=1$

Solution

$f(x)=\ln x$	$\to f(1)=0$
$f'(x)=\frac{1}{x}$	$\to f'(1)=1$
$f''(x)=-\frac{1}{x^2}$	$\to f''(1)=-1$

$$P(x)=x-1-\frac{1}{2}(x-1)^2$$

Exercise

Find the nth–order Taylor polynomial centered at c for the function $f(x)=\sinh 2x; \quad n=4, \quad c=0$

Solution

$f(x)=\sinh 2x$	$\to f(0)=0$
$f'(x)=2\cosh 2x$	$\to f'(0)=2$
$f''(x)=4\sinh 2x$	$\to f''(0)=0$
$f'''(x)=8\cosh 2x$	$\to f'''(0)=8$
$f^{(iv)}(x)=16\sinh 2x$	$\to f^{(iv)}(0)=0$

$$P(x)=2x-\frac{8}{3!}x^3$$

$$=2x-\frac{1}{6}(2x)^3$$

Exercise

Find the nth–order Taylor polynomial centered at c for the function $f(x)=\cosh x; \quad n=3, \quad c=\ln 2$

Solution

$$f(x)=\cosh x \qquad \to f(\ln 2)=\frac{e^{\ln 2}+e^{-\ln 2}}{2}$$

$$=\frac{1}{2}\left(2+\frac{1}{2}\right)$$

$$=\boxed{\frac{5}{4}}$$

$$f'(x)=\sinh x \quad \to f'(\ln 2)=\frac{e^{\ln 2}-e^{-\ln 2}}{2}$$

$$=\frac{1}{2}\left(2-\frac{1}{2}\right)$$

$$=\boxed{\frac{3}{4}}$$

$$f''(x)=\cosh x \quad \to f''(\ln 2)=\frac{e^{\ln 2}+e^{-\ln 2}}{2}$$

$$=\frac{1}{2}\left(2+\frac{1}{2}\right)$$

$$=\boxed{\frac{5}{4}}$$

$$f'''(x)=\sinh x \quad \to f'''(\ln 2)=\frac{e^{\ln 2}-e^{-\ln 2}}{2}$$

$$=\frac{1}{2}\left(2-\frac{1}{2}\right)$$

$$=\boxed{\frac{3}{4}}$$

$$P(x)=\frac{5}{4}+\frac{3}{4}(x-\ln 2)+\frac{5}{4}\frac{1}{2!}(x-\ln 2)^2+\frac{3}{4}\frac{1}{3!}(x-\ln 2)^3$$

$$=\boxed{\frac{5}{4}+\frac{3}{4}(x-\ln 2)+\frac{5}{8}(x-\ln 2)^2+\frac{1}{8}(x-\ln 2)^3}$$

Exercise

Find the sums of the series $1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\frac{x^8}{4!}+\cdots$

Solution

$$1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\frac{x^8}{4!}+\cdots=1+\left(x^2\right)^1+\frac{\left(x^2\right)^2}{2!}+\frac{\left(x^2\right)^3}{3!}+\frac{\left(x^2\right)^4}{4!}+\cdots$$

$$=\boxed{e^{x^2}}$$

Exercise

Find the sums of the series $1+\frac{x^2}{3!}+\frac{x^4}{5!}+\frac{x^6}{7!}+\frac{x^8}{9!}+\cdots$

Solution

$$1+\frac{x^2}{3!}+\frac{x^4}{5!}+\frac{x^6}{7!}+\frac{x^8}{9!}+\cdots=\frac{1}{x}\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\frac{x^9}{9!}+\cdots\right)$$

$$= \frac{1}{x}\sinh x$$

$$= \begin{cases} \dfrac{e^x - e^{-x}}{2x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Exercise

Find the sums of the series $x^3 - \frac{x^9}{3!\times 4} + \frac{x^{15}}{5!\times 16} - \frac{x^{21}}{7!\times 64} + \frac{x^{27}}{9!\times 256} - \cdots$

Solution

$$x^3 - \frac{x^9}{3!\times 4} + \frac{x^{15}}{5!\times 16} - \frac{x^{21}}{7!\times 64} + \frac{x^{27}}{9!\times 256} - \cdots = 2\left(\frac{x^3}{2} - \frac{1}{3!}\left(\frac{x^3}{2}\right)^3 + \frac{1}{5!}\left(\frac{x^3}{2}\right)^5 - \frac{1}{7!}\left(\frac{x^3}{2}\right)^7 + \cdots\right)$$

$$= 2\sin\left(\frac{x^3}{2}\right) \qquad \textit{for all } x$$

Exercise

Use the geometric series $\sum\limits_{k=0}^{\infty} x^k = \frac{1}{1-x}$, for $|x| < 1$, to determine the Maclaurin series and the interval of convergence for the following function

$$f(x) = \frac{1}{1-x^2}$$

Solution

The Maclaurin series for $f(x) = \frac{1}{1-x^2}$ is $\sum\limits_{k=0}^{\infty} x^{2k}$

By the *Root test*:

$$\sqrt[k]{x^{2k}} = x^2 < 1$$

$$-1 < x < 1$$

At $x = -1$,

the series is $\sum (-1)^{2k} = \sum 1$ which diverges

At $x = 1$,

the series is $\sum (1)^{2k} = \sum 1$ which diverges

The *interval* of convergence is the real line $(-1,\ 1)$

Exercise

Use the geometric series $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, for $|x|<1$, to determine the Maclaurin series and the interval of convergence for the following function

$$f(x) = \frac{1}{1+x^3}$$

Solution

The Maclaurin series for $f(x) = \frac{1}{1+x^3}$ is

$$\sum_{k=0}^{\infty} (-x)^{3k} = \sum_{k=0}^{\infty} (-1)^k x^{3k}$$

By the *Root test*:

$$\sqrt[k]{\left|(-x)^{3k}\right|} = x^3 < 1$$

$$-1 < x < 1$$

At $x = -1$,

the series is $\sum (1)^{3k} = \sum 1$ which diverges

At $x = 1$,

the series is $\sum (-1)^{3k} = \sum (-1)^k$ which diverges absolutely (*harmonic*)

The *interval* of convergence is the real line $(-1,\ 1)$

Exercise

Use the geometric series $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, for $|x|<1$, to determine the Maclaurin series and the interval of convergence for the following function

$$f(x) = \frac{1}{1+5x}$$

Solution

The Maclaurin series for $f(x) = \frac{1}{1+5x}$ is

$$\sum_{k=0}^{\infty} (-5x)^k = \sum_{k=0}^{\infty} (-5)^k x^k$$

By the *Root test*:

$$\sqrt[k]{\left|(-5x)^k\right|} = |5x| < 1$$

$$-\frac{1}{5} < x < \frac{1}{5}$$

At $x = -\frac{1}{5}$,

the series is $\sum (1)^k = \sum 1$ which diverges

At $x = \frac{1}{5}$,

the series is $\sum (-1)^k$ which diverges absolutely (*harmonic*)

The *interval* of convergence is the real line $\left(-\frac{1}{5},\ \frac{1}{5}\right)$

Exercise

Use the geometric series $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, for $|x| < 1$, to determine the Maclaurin series and the interval of convergence for the following function

$$f(x) = \frac{10}{1+x}$$

Solution

The Maclaurin series for $f(x) = \frac{10}{1+x}$ is

$$\sum_{k=0}^{\infty} 10(-x)^k$$

By the *Root test*:

$$\sqrt[k]{\left|(-x)^k\right|} = |x| < 1$$

$$-1 < x < 1$$

At $x = -1$,

the series is $\sum 10(1)^k = \sum 10$ which diverges

At $x = 1$,

the series is $\sum 10(-1)^k$ which diverges absolutely (*harmonic*)

The interval of convergence is the real line $(-1,\ 1)$

Exercise

Use the geometric series $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, for $|x|<1$, to determine the Maclaurin series and the interval of convergence for the following function

$$f(x)=\frac{1}{(1-10x)^2}$$

Solution

$$\frac{1}{1-10x}=\sum_{k=0}^{\infty}(10x)^k$$

$$\frac{1}{10}\cdot\frac{1}{1-10x}=\frac{1}{10}\sum_{k=0}^{\infty}(10x)^k$$

$$\left(\frac{1}{10}\cdot\frac{1}{1-10x}\right)'=\frac{1}{(1-10x)^2}$$

Thus, the Maclaurin series for $f(x)$

$$\left(\frac{1}{10}\sum_{k=0}^{\infty}(10x)^k\right)'=\frac{1}{10}\sum_{k=0}^{\infty}10k(10x)^{k-1}$$

$$=\sum_{k=0}^{\infty}k(10x)^{k-1}$$

$$L=\lim_{k\to\infty}\left|\frac{k(10x)^{k-1}}{(k+1)(10x)^k}\right| \qquad L=\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$$

$$=\left|(10x)^{-1}\right|<1$$

$$|x|<\frac{1}{10} \quad\to\quad -\frac{1}{10}<x<\frac{1}{10}$$

At $x=-\frac{1}{10}$,

the series is $\sum k(-1)^k$ which diverges absolutely

At $x=\frac{1}{10}$,

the series is $\sum k$ which diverges

The *interval* of convergence is the real line $\left(-\frac{1}{10},\ \frac{1}{10}\right)$

Exercise

Use the geometric series $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, for $|x|<1$, to determine the Maclaurin series and the interval of convergence for the following function

$$f(x) = \ln(1-4x)$$

Solution

$$-4\int \frac{dx}{1-4x} = \int \frac{d(1-4x)}{1-4x}$$

$$= \ln(1-4x)$$

$$\int \frac{dx}{1-4x} = -\frac{1}{4}\ln(1-4x)$$

$$= -\frac{1}{4} f(x)$$

$$\int \left(-\frac{1}{4}\sum_{k=0}^{\infty} (4x)^k \right) = -\sum_{k=0}^{\infty} \frac{1}{k+1}(4x)^{k+1}$$

$$f(x) = -\sum_{k=0}^{\infty} \frac{1}{k+1}(4x)^{k+1}$$

$$L = \lim_{k\to\infty} \left| \frac{(4x)^{k+2}}{k+2} \cdot \frac{k+1}{(4x)^{k+1}} \right| \qquad\qquad L = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$$

$$= \lim_{k\to\infty} \frac{k+1}{k+2}|4x|$$

$$= |4x| < 1$$

$$|x| < \frac{1}{4} \quad \rightarrow \quad -\frac{1}{4} < x < \frac{1}{4}$$

At $x = -\frac{1}{4}$,

the series is $f(x) = \ln 2$ which converges

At $x = \frac{1}{4}$,

the series is $f(x) = \ln(0) = -\infty$ which diverges

The *interval* of convergence is the real line $\left[-\frac{1}{4}, \frac{1}{4}\right)$

Exercise

The limit $\lim\limits_{n\to\infty} \dfrac{n!}{\sqrt{2\pi}\, n^{n+1/2}e^{-n}} = 1$ that is the relative error in the approximation $n! \approx \sqrt{2\pi}\, n^{n+1/2}e^{-n}$

Approaches zero as n increases. That is $n!$ grows at a rate comparable to $\sqrt{2\pi}\, n^{n+1/2}e^{-n}$. This result, known as Stirling's Formula, is often very useful in applied mathemmatics and statistics. Prove it by carrying out the following steps.

a) Use the identity $\ln(n!) = \sum\limits_{j=1}^{n} \ln j$ and the increasing nature of ln to show that if $n \ge 1$,

$$\int_0^n \ln x\, dx < \ln(n!) < \int_1^{n+1} \ln x\, dx$$

And hence that $n\ln n - n < \ln(n!) < (n+1)\ln(n+1) - n$

b) If $c_n = \ln(n!) - \left(n+\frac{1}{2}\right)\ln n + n$, show that

$$c_n - c_{n+1} = \left(n+\frac{1}{2}\right)\ln\left(\frac{n+1}{n}\right) - 1$$

$$= \left(n+\frac{1}{2}\right)\ln\left(\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}}\right) - 1$$

c) Use the Maclaurin series for $\ln\dfrac{1+t}{1-t}$ to show that

$$0 < c_n - c_{n+1} < \frac{1}{3}\left(\frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^4} + \frac{1}{(2n+1)^6} + \cdots\right)$$

$$= \frac{1}{12}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

and therefore that $\{c_n\}$ is decreasing and $\left\{c_n - \frac{1}{12n}\right\}$ is increasing. Hence conclude that $\lim\limits_{n\to\infty} c_n = c$ exists, and that

$$\lim_{n\to\infty} \frac{n!}{n^{n+1/2}e^{-n}} = \lim_{n\to\infty} e^{c_n} = e^c$$

Solution

a) $\ln(k-1) < \displaystyle\int_{k-1}^{k} \ln x\, dx < \ln k, \quad k = 1,\ 2,\ 3,\ \cdots$

$$n\ln n - n = \int_0^n \ln x\, dx < \ln(n!) < \int_1^{n+1} \ln x\, dx$$

$$= (n+1)\ln(n+1) - n - 1$$

$$< (n+1)\ln(n+1)$$

b) If $c_n = \ln(n!) - \left(n+\frac{1}{2}\right)\ln n + n$, then

$$c_n - c_{n+1} = \ln(n!) - \left(n+\frac{1}{2}\right)\ln n + n - \left[\ln((n+1)!) - \left(n+1+\frac{1}{2}\right)\ln(n+1) + n+1\right]$$

$$= \ln(n!) - \left(n+\frac{1}{2}\right)\ln n - \ln((n+1)!) + \left(n+\frac{1}{2}+1\right)\ln(n+1) - 1$$

$$= \ln(n!) - \ln((n+1)!) - \left(n+\frac{1}{2}\right)\ln n^{n+1/2} + \left(n+\frac{1}{2}\right)\ln(n+1) + \ln(n+1) - 1$$

$$= \ln\left(\frac{n!}{(n+1)!}\right) - \left(n+\frac{1}{2}\right)\ln\frac{n}{n+1} + \ln(n+1) - 1$$

$$= \ln\left(\frac{1}{n+1}\right) - \left(n+\frac{1}{2}\right)\ln\frac{n}{n+1} + \ln(n+1) - 1$$

$$= -\ln(n+1) + \left(n+\frac{1}{2}\right)\ln\left(\frac{n}{n+1}\right)^{-1} + \ln(n+1) - 1$$

$$= \left(n+\frac{1}{2}\right)\ln\left(\frac{n+1}{n}\right) - 1$$

$$= \left(n+\frac{1}{2}\right)\ln\left(\frac{2n+2}{2n}\right) - 1$$

$$= \left(n+\frac{1}{2}\right)\ln\left(\frac{\frac{2n+2}{2n+1}}{\frac{2n}{2n+1}}\right) - 1$$

$$= \left(n+\frac{1}{2}\right)\ln\left(\frac{\frac{2n+1+1}{2n+1}}{\frac{2n+1-1}{2n+1}}\right) - 1$$

$$= \left(n+\frac{1}{2}\right)\ln\left(\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}}\right) - 1$$

c) $\ln\frac{1+t}{1-t} = \ln(1+t) - \ln(1-t)$

$$= t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots + t + \frac{t^2}{2} + \frac{t^3}{3} - \cdots$$

$$= 2\left(t + \frac{t^3}{3} + \frac{t^5}{5} + \cdots\right) \quad \textit{for} \quad -1 < t < 1$$

$$0 < c_n - c_{n+1} = \left(\frac{2n+1}{2}\right)\ln\left(\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}}\right) - 1$$

$$\ln\frac{1+t}{1-t} = 2\left(t + \frac{t^3}{3} + \frac{t^5}{5} + \cdots\right)$$

$$=\frac{1}{2}(2n+1)(2)\left(\frac{1}{2n+1}+\frac{1}{3}\left(\frac{1}{2n+1}\right)^3+\frac{1}{5}\left(\frac{1}{2n+1}\right)^5+\cdots\right)-1$$

$$=1+\frac{1}{3(2n+1)^2}+\frac{1}{5(2n+1)^4}+\cdots-1$$

$$=\frac{1}{3(2n+1)^2}+\frac{1}{5(2n+1)^4}+\frac{1}{7(2n+1)^6}+\cdots$$

$$<\frac{1}{3}\left(\frac{1}{(2n+1)^2}+\frac{1}{(2n+1)^4}+\cdots\right)$$

Geometric series $S_n = a_1\frac{1}{1-r}$

$$=\frac{1}{3}\cdot\frac{1}{(2n+1)^2}\cdot\frac{1}{1-\frac{1}{(2n+1)^2}}$$

$$=\frac{1}{3}\cdot\frac{1}{(2n+1)^2}\cdot\frac{1}{1-\frac{1}{(2n+1)^2}}$$

$$=\frac{1}{3}\cdot\frac{1}{(2n+1)^2-1}$$

$$=\frac{1}{3}\cdot\frac{1}{4n^2+4n}$$

$$=\frac{1}{12}\cdot\frac{1}{n(n+1)}$$

$$=\frac{1}{12}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

These inequalities imply that $\{c_n\}$ is decreasing and $\left\{c_n-\frac{1}{12n}\right\}$ is increasing.

Thus $\{c_n\}$ is bounded below by $c_1-\frac{1}{12}=1-\frac{1}{12}=\frac{11}{12}$ $\quad\left(c_1=\ln(1!)-\left(1+\frac{1}{2}\right)\ln 1+1\right)$

So $\lim\limits_{n\to\infty} c_n = c$ exists.

Since $e^{c_n}=\dfrac{n!}{n^{(n+1/2)}e^{-n}}$, we have

$$\lim_{n\to\infty}\frac{n!}{n^{(n+1/2)}e^{-n}}=\lim_{n\to\infty}e^{c_n}$$

$$=e^c \quad \text{exists.}$$

Exercise

Suppose you want to approximate $\sqrt[3]{128}$ to within 10^{-4} of the exact value.

a) Use a Taylor polynomial for $f(x)=(125+x)^{1/3}$ centered at 0.

b) Use a Taylor polynomial for $f(x)=x^{1/3}$ centered at 125.

c) Compare the two approaches. Are they equivalent?

Solution

$\underline{\sqrt[3]{128} \approx 5.03968419958}$

a) $a=0$

$f(x)=(125+x)^{1/3} \qquad \to f(0)=(125)^{1/3}=5$

$f'(x)=\frac{1}{3}(125+x)^{-2/3} \qquad \to f'(0)=\frac{1}{3}(125)^{-2/3}=\frac{1}{3}\left(5^3\right)^{-2/3}=\frac{1}{3}(5)^{-2}=\frac{1}{75}$

$f''(x)=-\frac{2}{9}(125+x)^{-5/3} \qquad \to f''(1)=-\frac{2}{9}\left(5^3\right)^{-5/3}=-\frac{2}{9}\frac{1}{5^5}=-\frac{2}{28,125}$

$f(x)=5+\frac{1}{75}x-\frac{1}{28,125}x^2$

$125+x=128 \quad \Rightarrow \quad \underline{x=3}$

$$\begin{aligned} f(3)&=5+\frac{1}{75}(3)-\frac{1}{28,125}(9) \\ &=5+\frac{1}{25}-\frac{1}{3,125} \\ &=5+.04-.00032 \\ &\approx 5.03968 \end{aligned}$$

b) $a=125=5^3$

$f(x)=x^{1/3} \qquad \to f(125)=5$

$f'(x)=\frac{1}{3}x^{-2/3} \qquad \to f'(0)=\frac{1}{3}\left(5^3\right)^{-2/3}=\frac{1}{75}$

$$f''(x) = -\frac{2}{9}x^{-5/3} \qquad \to f''(1) = -\frac{2}{9}\left(5^3\right)^{-5/3}$$

$$= -\frac{2}{28,125}$$

$$f(x) = 5 + \frac{1}{75}(x-125) - \frac{1}{28,125}(x-125)^2$$

$$f(128) = 5 + \frac{1}{75}(3) - \frac{1}{28,125}(3)^2$$

$$= 5 + \frac{1}{25} - \frac{1}{3,125}$$

$$= 5 + .04 - .00032$$

$$\approx 5.03968$$

c) Both the results from part (a) and (b) are the same since they are just shifting.

Exercise

Consider the function

$$f(x) = \begin{cases} e^{-1/x^2} & if \;\; x \neq 0 \\ 0 & if \;\; x = 0 \end{cases}$$

a) Use the definition of the derivative to show that $f'(0) = 0$

b) Assume the fact that $f^k(0) = 0$ for $k = 1, 2, 3, \ldots$. Write the Taylor series for f centered at 0.

c) Explain why the Taylor series for f does not converge to f for $x \neq 0$

Solution

a) $f'(0) = \lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x - 0}$

$$= \lim_{x\to 0} \frac{e^{-1/x^2} - 0}{x}$$

$$= \lim_{x\to 0} \frac{e^{-1/x^2}}{x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\frac{2}{x^3}e^{-1/x^2}}{1}$$

$$= \lim_{x\to 0} \frac{2e^{-1/x^2}}{x^3} = \frac{0}{0}$$

Let $y = \frac{1}{x^2} \quad \Rightarrow \quad x = \frac{1}{\sqrt{y}}$

$x \to 0 \quad \Rightarrow \quad y \to \infty$

$$f'(0) = \lim_{y\to\infty} \frac{e^{-y}}{\frac{1}{\sqrt{y}}}$$

$$= \lim_{y\to\infty} \frac{\sqrt{y}}{e^{y}}$$

$$\underline{= 0|} \quad \checkmark$$

b) ***Given***: $f^{k}(0) = 0$

Since the Taylor series centered at 0 has only one term $f(x) = f(0) = 0$ and $f^{k}(0) = 0$ (derivaties are equal to 0).
Therefore; the Taylor series is zero.

c) It does not converge to $f(x)$ because when $x \neq 0$, $f(x) \neq 0$

Exercise

Teams A and B go into sudden death overtime after playing to a tie. The teams alternate possession of the ball and the first team to score wins. Each team has a $\frac{1}{6}$ chance of scoring when it has the ball, with Team A having the ball first.

a) The probability that Team A ultimately wins is $\sum\limits_{k=0}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^{2k}$. Evaluate this series.

b) The expected number of rounds (possessions by either team) required for the overtime to end is $\frac{1}{6}\sum\limits_{k=1}^{\infty} k\left(\frac{5}{6}\right)^{k-1}$. Evaluate this series.

Solution

a) $$\sum_{k=0}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^{2k} = \frac{1}{6}\sum_{k=0}^{\infty} \left(\frac{25}{36}\right)^{k}$$

It is a *Geometric* series with $r = \frac{25}{36} < 1$, then

$$\sum_{k=0}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^{2k} = \frac{1}{6}\cdot\frac{1}{1-\frac{25}{36}}$$

$$= \frac{1}{6}\cdot\frac{36}{11}$$

$$=\boxed{\frac{6}{11}}$$

b) Using the series $\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$

$$\left(\sum_{k=1}^{\infty} x^k\right)' = \left(\frac{x}{1-x}\right)'$$

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

Let $x = \frac{5}{6}$

$$\frac{1}{6}\sum_{k=1}^{\infty} k\left(\frac{5}{6}\right)^{k-1} = \frac{1}{6}\frac{1}{\left(1-\frac{5}{6}\right)^2}$$

$$= \frac{1}{6}\frac{1}{\left(\frac{1}{6}\right)^2}$$

$$\boxed{=6}$$