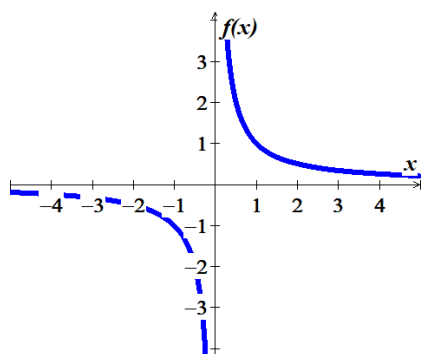# *Section* 2.6 – Graphing Rational Functions
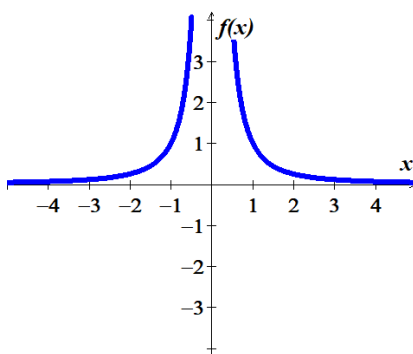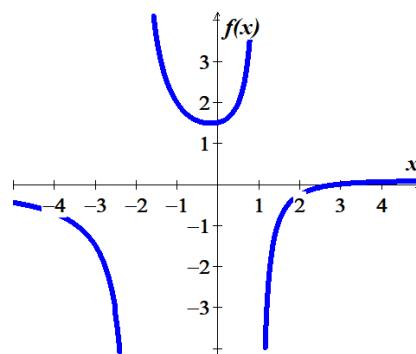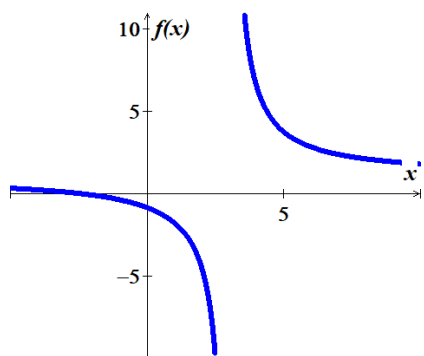
$$f(x) = \frac{1}{x}$$

$$f(x) = \frac{1}{x^2}$$

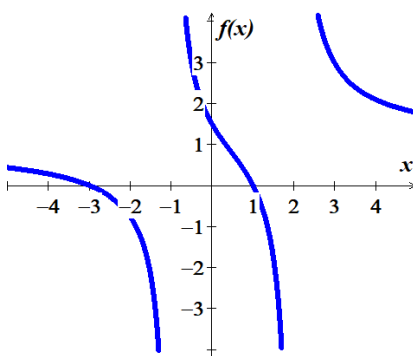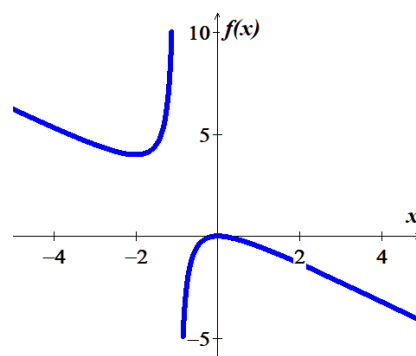$$f(x) = \frac{x-3}{x^2+x-2}$$

$$f(x) = \frac{2x+5}{2x-6}$$

$$f(x) = \frac{x^2+2x-3}{x^2-x-2}$$

$$f(x) = -\frac{x^2}{x+1}$$

Rational Function

A rational function is a function f that is a quotient of two polynomials, that is,

$$f(x) = \frac{g(x)}{h(x)}$$

Where $g(x)$ and $h(x)$ are polynomials. The domain of f consists of all real numbers *except* the zeros of the denominator $h(x)$.

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The Domain of a Rational Function

Example

Consider: $f(x) = \dfrac{1}{x-3}$

Find the domain and graph f.

Solution

$x - 3 = 0$

$x = 3$

Thus, the domain is: $\{x \mid x \neq 3\}$ *or* $(-\infty,\ 3) \cup (3,\ \infty)$
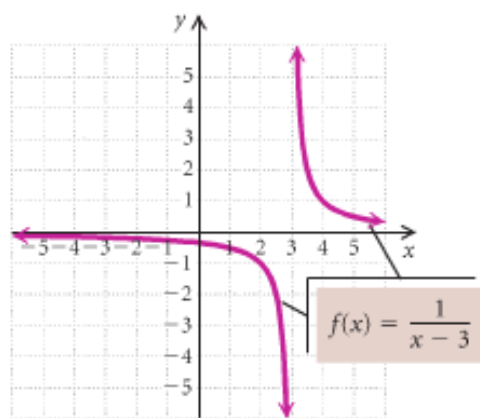
Function		Domain	
$f(x) = \dfrac{1}{x}$	$\{x \mid x \neq 0\}$	$(-\infty,\ 0) \cup (0,\ \infty)$	
$f(x) = \dfrac{1}{x^2}$	$\{x \mid x \neq 0\}$	$(-\infty,\ 0) \cup (0,\ \infty)$	
$f(x) = \dfrac{x-3}{x^2 + x - 2}$	$\{x \mid x \neq -2 \text{ and } x \neq 1\}$	$(-\infty,\ -2) \cup (-2,\ 1) \cup (1,\ \infty)$	
$f(x) = \dfrac{2x+5}{2x-6} = \dfrac{2x+5}{2(x-3)}$	$\{x \mid x \neq 3\}$	$(-\infty,\ 3) \cup (3,\ \infty)$	

Asymptotes

Vertical Asymptote (VA) - *Think Domain*

The line $x = a$ is a *vertical asymptote* for the graph of a function f if

$$f(x) \to \infty \quad or \quad f(x) \to -\infty$$

As x approaches a from either the left or the right

Example

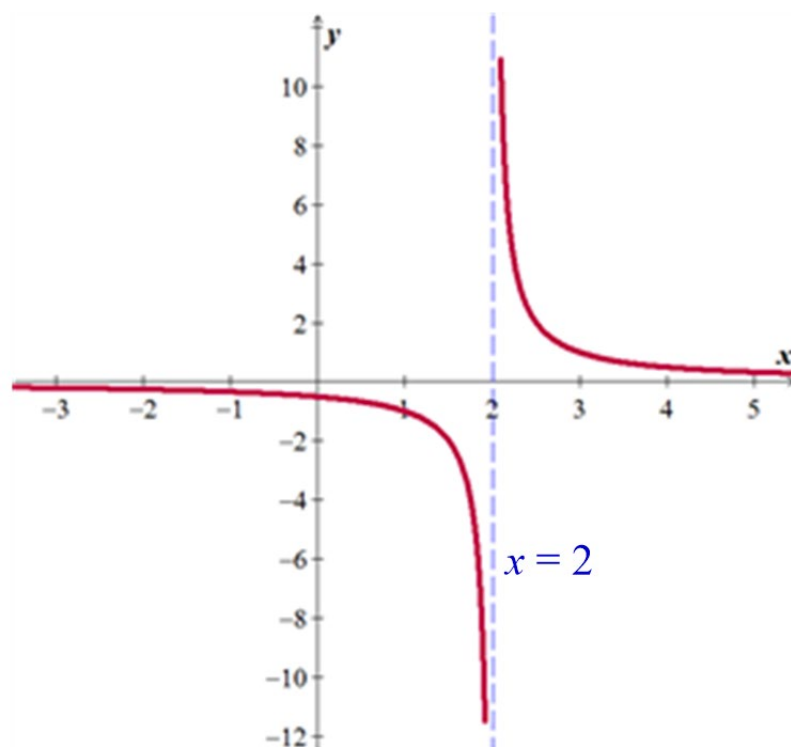Find the vertical asymptote of $f(x) = \dfrac{1}{x-2}$, and sketch the graph.

Solution

VA: $x = 2$

$f(x) \to \infty \quad as \quad x \to 2^+$

$f(x) \to -\infty \quad as \quad x \to 2^-$

Horizontal Asymptote (HA)

The line $y = c$ is a **horizontal asymptote** for the graph of a function f if

$$f(x) \to c \quad as \quad x \to -\infty \quad or \quad x \to -\infty$$

Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0} = \dfrac{a_n x^n}{b_m x^m}$ be a rational function.

1. If the degree of numerator is less than of denominator $(n < m) \Rightarrow y = 0$

$$y = \frac{2x + 1}{4x^2 + 5} \quad \Rightarrow \boxed{y = 0}$$

2. If the degree of numerator is equal of denominator $(n = m) \Rightarrow y = \dfrac{a_n}{b_m}$

$$y = \frac{2x^2 + 1}{4x^2 + 5} \quad \Rightarrow \boxed{y = \frac{2}{4} = \frac{1}{2}}$$

3. If the degree of numerator is greater than of denominator $(n > m) \Rightarrow$ No horizontal asymptote

$$y = \frac{2x^3 + 1}{4x^2 + 5} \quad \Rightarrow No \ HA$$

$f(x) \to b$ as $x \to \infty$ $f(x) \to b$ as $x \to -\infty$ $f(x) \to b$ as $x \to \infty$ $f(x) \to b$ as $x \to -\infty$
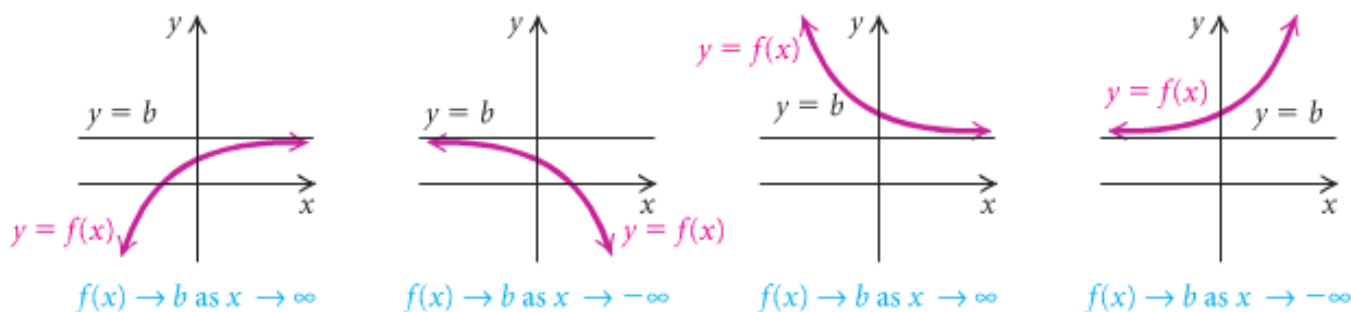
Example

Determine the horizontal asymptote of $f(x) = \dfrac{-7x^4 - 10x^2 + 1}{11x^4 + x - 2}$

Solution

$$f(x) = \frac{-7x^4}{11x^4} = -\frac{7}{11}$$

Therefore, the horizontal asymptote (HA) is: $\boxed{y = -\frac{7}{11}}$

Example

Find the vertical and the horizontal asymptote for the graph of f, if it exists

a) $f(x) = \dfrac{3x-1}{x^2-x-6}$

b) $f(x) = \dfrac{5x^2+1}{3x^2-4}$

c) $f(x) = \dfrac{2x^4-3x^2+5}{x^2+1}$

Solution

a) $f(x) = \dfrac{3x-1}{x^2-x-6}$

$$x^2 - x - 6 = 0 \quad \rightarrow \quad x = -2,\ 3$$

VA: $x = -2, \quad x = 3$

HA: $y = 0$

b) $f(x) = \dfrac{5x^2+1}{3x^2-4}$

$$3x^2 - 4 = 0$$
$$3x^2 = 4$$
$$x^2 = \frac{4}{3}$$
$$x = \pm\frac{2}{\sqrt{3}}$$

VA: $x = -\dfrac{2}{\sqrt{3}}, \quad x = \dfrac{2}{\sqrt{3}}$

HA: $y = \dfrac{5}{3}$

c) $f(x) = \dfrac{2x^4-3x^2+5}{x^2+1}$

$$x^2 + 1 = 0 \quad \rightarrow \quad x^2 = -1$$

VA: n/a

HA: n/a

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Slant or *Oblique* **Asymptotes**

When the degree of the numerator is one greater than the degree of the numerator, the graph has a slant or oblique asymptote and it is a line $y = ax + b, \quad a \neq 0$. To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

$$y = \frac{3x^2 - 1}{x + 2}$$

$$\begin{array}{r} 3x - 6 \\ x + 2 \overline{\smash{)}3x^2 + 0x - 1} \\ \underline{3x^2 + 6x} \\ -6x - 1 \\ \underline{-6x - 12} \\ R = 11 \end{array}$$

$$y = \frac{3x^2 - 1}{x + 2} = 3x - 6 + \frac{11}{x + 2}$$

The *oblique asymptote* is the line **y = 3x − 6**

Example

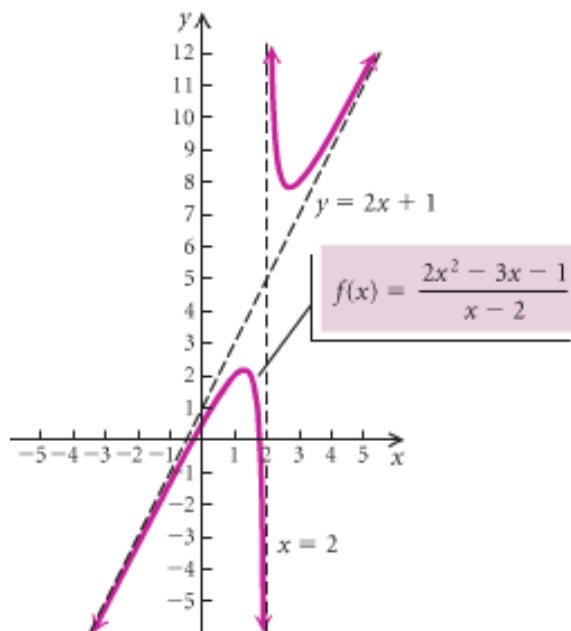Find all the asymptotes of $\quad f(x) = \frac{2x^2 - 3x - 1}{x - 2}$

Solution

$$\begin{array}{r} 2x + 1 \\ x - 2 \overline{\smash{)}2x^2 - 3x - 1} \\ \underline{-2x^2 + 4x} \\ x - 1 \\ \underline{-x + 2} \\ 1 \end{array}$$

$$f(x) = \frac{2x^2 - 3x - 1}{x - 2} = (2x + 1) + \frac{1}{x - 2}$$

The *oblique asymptote* is the line $y = 2x + 1$

VA:: $x = 2$

$y = 2x + 1$

$f(x) = \dfrac{2x^2 - 3x - 1}{x - 2}$

$x = 2$

Graph That Has a *Hole*

Example

Sketch the graph of g if $g(x) = \dfrac{3x^2 + x - 4}{2x^2 - 7x + 5}$

Solution

$$g(x) = \dfrac{(3x+4)(x-1)}{(2x-5)(x-1)}$$

$$= \dfrac{3x+4}{2x-5} = f(x)$$

VA: $x = \dfrac{5}{2}$

HA: $y = \dfrac{3}{2}$

The only different between the graphs that g has a **hole** at $x = 1 \rightarrow f(1) = -\dfrac{7}{3}$

x	y
−4	.6
1.3	0
0	−.8
4	5.3
6	3.1
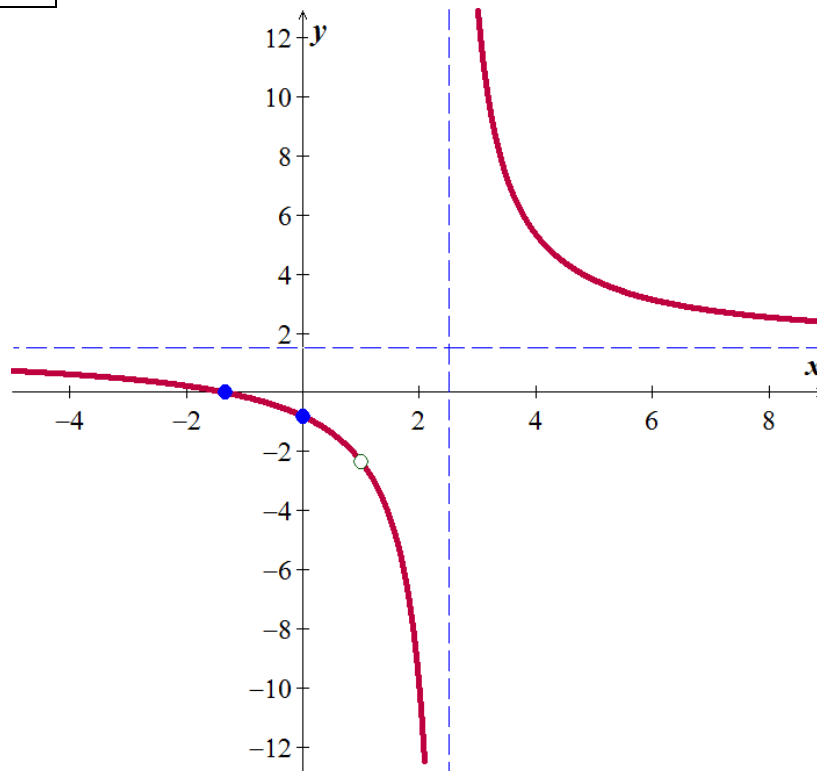
Exercises   Section 2.6 – Rational Functions

(1 – 21)   Determine all asymptotes of the function

1. $y = \dfrac{3x}{1-x}$

2. $y = \dfrac{x^2}{x^2+9}$

3. $y = \dfrac{x-2}{x^2-4x+3}$

4. $y = \dfrac{3}{x-5}$

5. $y = \dfrac{x^3-1}{x^2+1}$

6. $y = \dfrac{3x^2-27}{(x+3)(2x+1)}$

7. $y = \dfrac{x^3+3x^2-2}{x^2-4}$

8. $y = \dfrac{x-3}{x^2-9}$

9. $y = \dfrac{6}{\sqrt{x^2-4x}}$

10. $y = \dfrac{5x-1}{1-3x}$

11. $f(x) = \dfrac{2x-11}{x^2+2x-8}$

12. $f(x) = \dfrac{x^2-4x}{x^3-x}$

13. $f(x) = \dfrac{x-2}{x^3-5x}$

14. $f(x) = \dfrac{4x}{x^2+10x}$

15. $f(x) = \dfrac{3-x}{(x-4)(x+6)}$

16. $f(x) = \dfrac{x^3}{2x^3-x^2-3x}$

17. $f(x) = \dfrac{3x^2+5}{4x^2-3}$

18. $f(x) = \dfrac{x+6}{x^3+2x^2}$

19. $f(x) = \dfrac{x^2+4x-1}{x+3}$

20. $f(x) = \dfrac{x^2-6x}{x-5}$

21. $f(x) = \dfrac{x^3-x^2+x-4}{x^2+2x-1}$

(22 – 53)   Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

22. $f(x) = \dfrac{-3x}{x+2}$

23. $f(x) = \dfrac{x+1}{x^2+2x-3}$

24. $f(x) = \dfrac{2x^2-2x-4}{x^2+x-12}$

25. $f(x) = \dfrac{-2x^2+10x-12}{x^2+x}$

26. $f(x) = \dfrac{x^2-x-6}{x+1}$

27. $f(x) = \dfrac{x^3+1}{x-2}$

28. $f(x) = \dfrac{2x^2+x-6}{x^2+3x+2}$

29. $f(x) = \dfrac{x-1}{1-x^2}$

30. $f(x) = \dfrac{x^2+x-2}{x+2}$

31. $f(x) = \dfrac{x^3-2x^2-4x+8}{x-2}$

32. $f(x) = \dfrac{2x^2-3x-1}{x-2}$

33. $f(x) = \dfrac{2x+3}{3x^2+7x-6}$

34. $f(x) = \dfrac{x^2-1}{x^2+x-6}$

35. $f(x) = \dfrac{-2x^2-x+15}{x^2-x-12}$

36. $f(x) = \dfrac{1}{x-3}$

37. $f(x) = \dfrac{-2}{x+3}$

38. $f(x) = \dfrac{x}{x+2}$

39. $f(x) = \dfrac{x-5}{x+4}$

40. $f(x) = \dfrac{2x^2-2}{x^2-9}$

41. $f(x) = \dfrac{x^2-3}{x^2+4}$

42. $f(x) = \dfrac{x^2+4}{x^2-3}$

43. $f(x) = \dfrac{x^2}{x^2 - 6x + 9}$

44. $f(x) = \dfrac{x^2 + x + 4}{x^2 + 2x - 1}$

45. $f(x) = \dfrac{2x^2 + 14}{x^2 - 6x + 5}$

46. $f(x) = \dfrac{x^2 - 4x - 5}{2x + 5}$

47. $f(x) = \dfrac{x - 3}{x^2 - 3x + 2}$

48. $f(x) = \dfrac{x^2 + 2}{x^2 + 3x + 2}$

49. $f(x) = \dfrac{x - 2}{x^2 - 3x + 2}$

50. $f(x) = \dfrac{x^2 + x}{x + 1}$

51. $f(x) = \dfrac{x^2 - 2x}{x - 2}$

52. $f(x) = \dfrac{x^2 - 3x}{x + 3}$

53. $f(x) = \dfrac{x^3 + 3x^2 - 4x + 6}{x + 2}$

(54 − 59) Find an equation of a rational function f that satisfies the given conditions

54.
$$\begin{cases} vertical\ asymptote:\ x = 4 \\ horizontal\ asymptote:\ y = -1 \\ x-intercept:\ 3 \end{cases}$$

55.
$$\begin{cases} vertical\ asymptote:\ x = -4,\, x = 5 \\ horizontal\ asymptote:\ y = \dfrac{3}{2} \\ x-intercept:\ -2 \end{cases}$$

56.
$$\begin{cases} vertical\ asymptote:\ x = 5 \\ horizontal\ asymptote:\ y = -1 \\ x-intercept:\ 2 \end{cases}$$

57.
$$\begin{cases} vertical\ asymptote:\ x = -2,\, x = 0 \\ horizontal\ asymptote:\ y = 0 \\ x-intercept:\ 2,\quad f(3) = 1 \end{cases}$$

58.
$$\begin{cases} vertical\ asymptote:\ x = -3,\ x = 1 \\ horizontal\ asymptote:\ y = 0 \\ x-intercept:\ -1,\ f(0) = -2 \\ hole\ at\ x = 2 \end{cases}$$

59.
$$\begin{cases} vertical\ asymptote:\ x = -1,\ x = 3 \\ horizontal\ asymptote:\ y = 2 \\ x-intercept:\ -2,\ 1 \\ hole:\ x = 0 \end{cases}$$