Solution Section 6.1– Introduction

Exercise

Indicate the angle if it is an acute or obtuse. Then give the complement and the supplement of each angle.

a) $10°$ *b*) $52°$ *c*) $90°$ *d*) $120°$ *e*) $150°$

Solution

a) Acute;
Complement is $90° - 10° = 80°$;
Supplement is $180° - 10° = 170°$.

b) Acute;
Complement is $90° - 52° = 38°$;
Supplement is $180° - 52° = 128°$.

c) Neither (*right angle*);
Complement is $90° - 90° = 0°$;
Supplement is $180° - 90° = 90°$.

d) Obtuse;
Complement is $90° - 120° = -30°$;
Supplement is $180° - 120° = 60°$.

e) Obtuse;
Complement is $90° - 150° = -60°$;
Supplement is $180° - 150° = 30°$.

Exercise

Change to decimal degrees

a) $10°\ 45'$ *c)* $274°\ 18'\ 59''$ *e)* $98°\ 22'\ 45''$ *g)* $1°\ 2'\ 3''$
b) $34°\ 51'\ 35''$ *d)* $74°\ 8'\ 14''$ *f)* $9°\ 9'\ 9''$ *h)* $73°\ 40'\ 40''$

Solution

a) $10°\ 45' = 10° + 45'$

$$= 10° + 45'\frac{1°}{60'}$$
$$= 10° + 0.75°$$
$$\underline{= 10.75°}$$

b) $34°\ 51'\ 35'' = 34° + 51' + 35''$

$$= 34° + 51' \cdot \frac{1°}{60'} + 35'' \cdot \frac{1°}{3600''}$$
$$= 34° + 0.85° + 0.00972°$$

$$\underline{= 34.85972^\circ}$$

c) $274^\circ\ 18'\ 59'' = 274^\circ + 18' + 59''$

$$= 274^\circ + 18' \cdot \frac{1^\circ}{60'} + 59'' \cdot \frac{1^\circ}{3600''}$$

$$= 274^\circ + 0.3^\circ + 0.016389^\circ$$

$$\underline{= 274.316389^\circ}$$

d) $74^\circ\ 8'\ 14'' = 74^\circ + \frac{8^\circ}{60} + \frac{14^\circ}{3600}$

$$= 74^\circ + 0.1333^\circ + 0.0039^\circ$$

$$\underline{= 74.137^\circ}$$

e) $98^\circ\ 22'\ 45'' = 98^\circ + 22' + 45''$

$$= 98^\circ + 22' \cdot \frac{1^\circ}{60'} + 45'' \cdot \frac{1^\circ}{3600''}$$

$$= 98^\circ + 0.36667^\circ + 0.0125^\circ$$

$$\underline{= 98.37917^\circ}$$

f) $9^\circ\ 9'\ 9'' = 9^\circ + 9' + 9''$

$$= 9^\circ + 9' \cdot \frac{1^\circ}{60'} + 9'' \cdot \frac{1^\circ}{3600''}$$

$$= 9^\circ + 0.15^\circ + 0.0025^\circ$$

$$\underline{= 9.1525^\circ}$$

g) $1^\circ\ 2'\ 3'' = 1^\circ + 2' + 3''$

$$= 1^\circ + 2' \cdot \frac{1^\circ}{60'} + 3'' \cdot \frac{1^\circ}{3600''}$$

$$= 1^\circ + 0.03333^\circ + 0.000833^\circ$$

$$\underline{= 1.034163^\circ}$$

h) $73^\circ\ 40'\ 40'' = 73^\circ + 40' + 40''$

$$= 73^\circ + 40' \cdot \frac{1^\circ}{60'} + 40'' \cdot \frac{1^\circ}{3600''}$$

$$= 73^\circ + 0.6667^\circ + 0.0111^\circ$$

$$\underline{= 73.67778^\circ}$$

Exercise

Convert to degrees, minutes, and seconds.

a) $89.9004°$ *c)* $122.6853°$ *e)* $44.01°$ *g)* $29.411°$

b) $34.817°$ *d)* $178.5994°$ *f)* $19.99°$ *h)* $18.255°$

Solution

a) $89.9004° = 89° + 0.9004°$

$$= 89° + 0.9004° \cdot (60')$$

$$= 89° \quad 54.024'$$

$$= 89° \quad 54' + 0.024'$$

$$= 89° \quad 54' \quad 0.024' \cdot (60'')$$

$$\boxed{= 89° \quad 54' \quad 1.44''}$$

b) $34.817° = 34° + 0.817°$

$$= 34° + 0.817(60')$$

$$= 34° + 49.02'$$

$$= 34° + 49' + .02(60'')$$

$$= 34° + 49' + 1.2''$$

$$\boxed{= 34° \ 49' \ 1.2''}$$

c) $122.6853° = 122° + .6853°$

$$= 122° + \quad 0.6853 \cdot (60')$$

$$= 122° \quad 41.118'$$

$$= 122° \quad 41' + 0.118'$$

$$= 122° \quad 41' \quad 0.118 \cdot (60'')$$

$$\boxed{= 122° \quad 41' \quad 7.1''}$$

d) $178.5994° = 178° + .5994°$

$$= 178° + \quad .5994 \cdot (60')$$

$$= 178° \quad 35.964'$$

$$= 178° \quad 35' + .964'$$

$$= 178° \quad 35' \quad 0.964 \cdot (60'')$$

$$\boxed{= 178° \quad 35' \quad 57.84''}$$

e) $44.01° = 44° + .01°$

$$= 44° + \quad .01 \cdot (60')$$

$$= 44° \quad 0.6'$$

$= 44° \quad 0.6 \cdot (60'')$

$\underline{= 44° \quad 36''}|$

f) $19.99° = 19° + .99°$

$= 19° + \quad .99 \cdot (60')$

$= 19° \quad 59.4'$

$= 19° \quad 59' + 0.4'$

$= 19° \quad 59' \quad 0.4 \cdot (60'')$

$\underline{= 19° \quad 59' \quad 24''}|$

g) $29.411° = 29° + 0.411°$

$= 29° + \quad 0.411 \cdot (60')$

$= 29° \quad 24.66'$

$= 29° \quad 24' + 0.66'$

$= 29° \quad 24' \quad 0.66 \cdot (60'')$

$\underline{= 29° \quad 24' \quad 39.6''}|$

h) $18.255° = 18° + 0.255°$

$= 18° + \quad 0.255 \cdot (60')$

$= 18° \quad 15.3'$

$= 18° \quad 15' + 0.3'$

$= 18° \quad 15' \quad 0.3 \cdot (60'')$

$\underline{= 18° \quad 15' \quad 18''}|$

Exercise

Perform each calculation

a) $51° \ 29' + 32° \ 46'$ *b)* $90° - 73°12'$ *c)* $90° - 36° \ 18' \ 47''$ *d)* $75° \ 15' + 83° \ 32'$

Solution

a) $51°29' + 32°46'$

$$\begin{array}{r} 51° \quad 29' \\ +32° \quad 46' \\ \hline 83° \quad 75' \end{array}$$

$83° \quad 75' = 1°15'$ $\boxed{84° \quad 15'}$

b) $90° - 73°12'$

$$\begin{array}{r} 89^\circ \quad 60' \\ -73^\circ \quad 12' \\ \hline 16^\circ \quad 48' \end{array}$$

c) $90^\circ - 36^\circ\, 18'\, 47''$

$$\begin{array}{r} 90^\circ \qquad\qquad\quad \\ -36^\circ \quad 18' \quad 47'' \\ \hline \end{array} \Rightarrow \begin{array}{r} 89^\circ \quad 59' \quad 60'' \\ -36^\circ \quad 18' \quad 47'' \\ \hline 53^\circ \quad 41' \quad 13'' \end{array}$$

d) $75^\circ\, 15' + 83^\circ\, 32'$

$$\begin{array}{r} 75^\circ \quad 15' \\ 83^\circ \quad 32' \\ \hline 158^\circ \quad 47' \end{array}$$

Exercise

Find the angle of least possible positive measure coterminal with an angle of

a) -75° *b)* -800° *c)* 270°

Solution

a) $360^\circ - 75^\circ \underline{= 285^\circ}|$

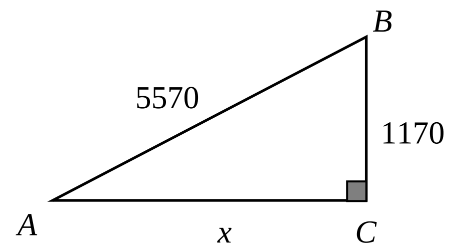
b) $3(360^\circ) - 800^\circ \underline{= 280^\circ}|$

c) $360^\circ + 270^\circ = \underline{630^\circ}|$

Exercise

A vertical rise of the Forest Double chair lift 1,170 *feet* and the length of the chair lift as 5,570 *feet*. To the nearest foot, find the horizontal distance covered by a person riding this lift.

Solution



$$x^2 + 1170^2 = 5570^2$$

$$x^2 = 5570^2 - 1170^2$$

$$x = \sqrt{5570^2 - 1170^2}$$

$$\boxed{x = 5,445.73 \ \ ft}$$

Exercise

A tire is rotating 600 times per minute. Through how many degrees does a point of the edge of the tire move in $\frac{1}{2}$ *second*?

Solution

$$\frac{1}{2} 600 \frac{rev}{\min} \cdot \frac{1\min}{60\sec} \cdot \frac{360°}{1rev} \ \boxed{= 1800 \ \ deg / sec}$$

Exercise

A windmill makes 90 *revolutions* per *minute*. How many revolutions does it make per *second*?

Solution

$$90 \frac{rev}{\min} \cdot \frac{1\min}{60\sec} \ \boxed{= 1.5 \ \ rev / \sec}$$

Exercise

Convert to radians

a) $256° \ 20'$ *b)* $-78.4°$ *c)* $330°$ *d)* $-60°$ *e)* $-225°$

Solution

a) $256° \ 20' = 256° + \frac{20°}{60}$

$$= 256° + \frac{2°}{6}$$

$$= \frac{1538°}{6} = \left(\frac{769}{3}\right)^{\circ}$$

$$\frac{769°}{3} \frac{\pi}{180°} \ \boxed{= \frac{769\pi}{540} \ rad} \ \boxed{\approx 4.47 \ rad}$$

b) $-78.4° = -78.4°\left(\frac{\pi}{180°}\right) rad$

$\boxed{\approx -1.37\ rad}$

c) $330° = 330°\left(\frac{\pi}{180°}\right) rad$

$\boxed{= \frac{11\pi}{6}\ rad}$

d) $-60° = -60°\left(\frac{\pi}{180°}\right) rad$

$\boxed{= -\frac{\pi}{3}\ rad}$

e) $-225° = -225°\left(\frac{\pi}{180°}\right) rad$

$\boxed{= -\frac{5\pi}{4}\ rad}$

Exercise

Convert to degrees

a) $\frac{11\pi}{6}$

b) $-\frac{5\pi}{3}$

c) $\frac{\pi}{6}$

d) 2.4

e) $\frac{\pi}{3}$

f) $-\frac{5\pi}{12}$

g) -4π

h) $\frac{7\pi}{13}$

Solution

a) $\frac{11\pi}{6}\ (rad) = \frac{11\pi}{6} \bullet \frac{180°}{\pi}$

$\boxed{= 330°}$

b) $-\frac{5\pi}{3}(rad) = -\frac{5\pi}{3} \bullet \frac{180°}{\pi}$

$\boxed{= -300°}$

c) $\frac{\pi}{6}(rad) = \frac{\pi}{6}\left(\frac{180}{\pi}\right)^{\circ}$

$\boxed{= 30^{\circ}}$

d) $2.4\ rad = 2.4 \bullet \frac{180°}{\pi}$

$= \frac{432°}{\pi}$

$\boxed{\approx 137.5°}$

e) $\frac{\pi}{3}(rad) = \frac{\pi}{3}\left(\frac{180}{\pi}\right)^{\circ}$

$$\underline{= 60^{\circ}}$$

f) $-\frac{5\pi}{12}(rad) = -\frac{5\pi}{12}\left(\frac{180}{\pi}\right)^{\circ}$

$$\underline{= -75^{\circ}}$$

g) $-4\pi(rad) = -4\pi\left(\frac{180}{\pi}\right)^{\circ}$

$$\underline{= -720^{\circ}}$$

h) $\frac{7\pi}{13}(rad) = \frac{7\pi}{13}\left(\frac{180}{\pi}\right)^{\circ}$

$$\underline{\approx 96.923^{\circ}}$$

Solution   Section 6.2 – Arc Length & Area – Velocity

Exercise

The minute hand of a clock is 1.2 *cm* long. How far does the tip of the minute hand travel in 40 *minutes*?

Solution

$$40\,\text{min} = 40\,\text{min}\,\frac{2\pi}{60}\,\frac{rad}{\text{min}}$$

$$= \frac{4\pi}{3}\,rad.$$

$$s = (1.2)\frac{4\pi}{3} \qquad\qquad s = r\theta$$

$$= \frac{(12)(4)\pi}{30}$$

$$\approx \frac{8\pi}{5}\ cm$$

Exercise

Find the radian measure if angle θ, if θ is a central angle in a circle of radius $r = 4$ *inches*, and θ cuts off an arc of length $s = 12\pi$ *inches*.

Solution

$$\theta = \frac{12\pi}{4} \qquad\qquad \theta = \frac{s}{r}$$

$$= 3\pi\ rad$$

Exercise

Give the length of the arc cut off by a central angle of 2 *radians* in a circle of radius 4.3 *inches*

Solution

Given: $\theta = 2\ rad, \quad r = 4.3\ in$

$$s = 4.3(2) \qquad\qquad s = r\theta$$

$$= 8.6\ in$$

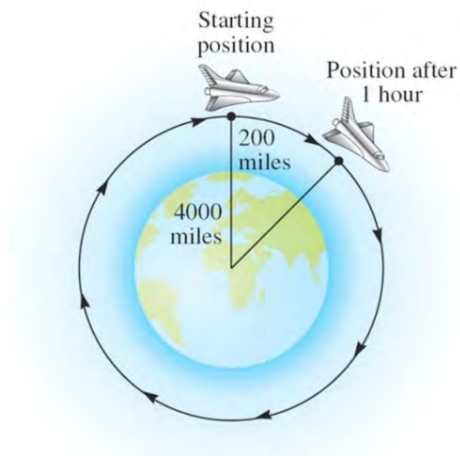
Exercise

A space shuttle 200 *miles* above the earth is orbiting the earth once every 6 *hours*. How long, in hours, does it take the space shuttle to travel 8,400 *miles*? (Assume the radius of the earth is 4,000 *miles*.) Give both the exact value and an approximate value for your answer.

Solution

$$\theta = \frac{8400}{4200} \qquad \theta = \frac{s}{r}$$

$$=2\ rad$$



$$\frac{2\ rad}{2\pi\ rad} = \frac{x\ hr}{6\ hr}$$

$$x = \frac{2\ (6)}{2\pi}$$

$$= \frac{6}{\pi}\ hrs$$

$$\approx 1.91\ hrs$$

Exercise

The pendulum on a grandfather clock swings from side to side once every second. If the length of the pendulum is 4 *feet* and the angle through which it swings is 20°. Find the total distance traveled in 1 *minute* by the tip of the pendulum on the grandfather clock.

Solution

Since $20° = 20 \cdot \frac{\pi}{180}$

$$= \frac{\pi}{9}\ rad$$

The length of the pendulum swings in 1 second:

$$s = r\theta = 4 \cdot \frac{\pi}{9}$$

$$= \frac{4\pi}{9}\ ft$$

In 60 seconds, the total distance traveled

$$d = 60 \cdot \frac{4\pi}{9}$$

$$= \frac{80\pi}{3}\ feet \qquad \approx 83.8\ feet$$

Exercise

Reno, Nevada is due north of Los Angeles. The latitude of Reno is 40°, while that of Los Angeles is 34° N. The radius of Earth is about 4000 *mi*. Find the north-south distance between the two cities.

Solution

The central angle between two cities: $40^\circ - 34^\circ = 6^\circ$

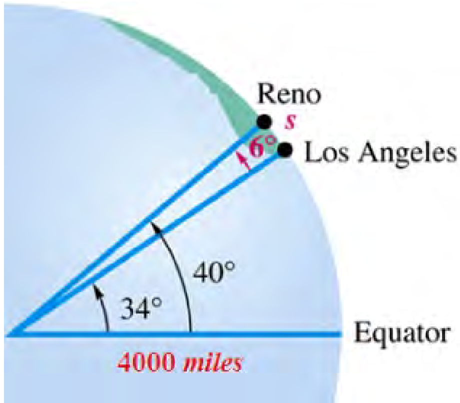
$$6^\circ = 6\left(\frac{\pi}{180}\right)$$

$$= \frac{\pi}{30}\ rad$$

$$s = 4000\frac{\pi}{30} \qquad\qquad s = r\theta$$

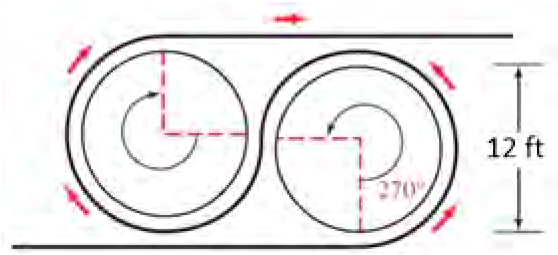
$$= \frac{400\pi}{3}\ \ miles$$

$$\approx 419\ miles$$



Exercise

The first cable railway to make use of the figure-eight drive system was a Sutter Street Railway. Each drive sheave was 12 *feet* in diameter.



Find the length of cable riding on one of the drive sheaves.

Solution

Since $270^\circ = 270 \cdot \frac{\pi}{180}$

$$= \frac{3\pi}{2}\ rad$$

The length of the cable riding on one of the drive sheaves is:

$$s = 6 \cdot \frac{3\pi}{2} \qquad\qquad s = r\theta$$

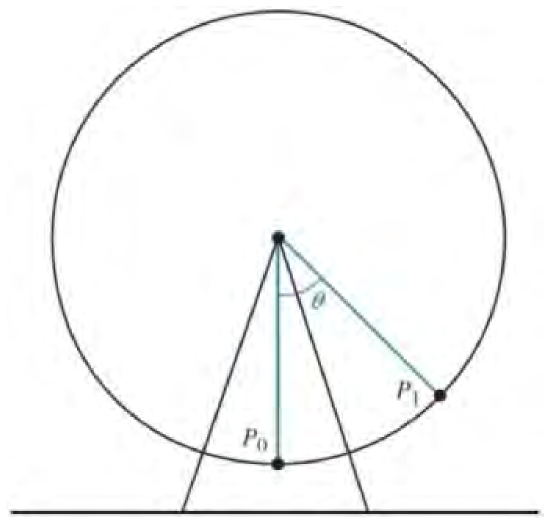
$$= 9\pi\ \ feet$$

$$\approx 28.3\ \ feet$$

Exercise

The diameter of a model of George Ferris's Ferris wheel is 250 *feet*, and θ is the central angle formed as a rider travels from his or her initial position P_0 to position P_1. Find the distance traveled by the rider if $\theta = 45°$ and if $\theta = 105°$.

Solution



$r = \frac{D}{2} = \frac{250}{2} = 125\, ft$

For $\theta = 45° = \frac{\pi}{4}$

$s = 125\left(\frac{\pi}{4}\right)$ $\quad s = r\theta$

$= \frac{125\pi}{4}$ *feet*

$\approx 98\ ft$

For $\theta = 105°$

$= 105\frac{\pi}{180}$

$= \frac{7\pi}{12}$

$s = 125\frac{7\pi}{12}$

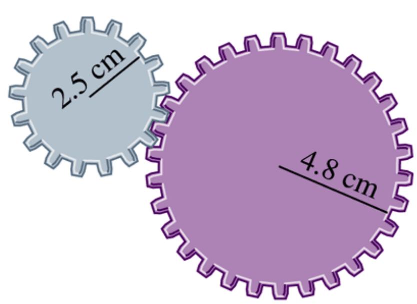
$= \frac{875\pi}{12}$ *feet*

$\approx 230\ ft$

Exercise

Two gears are adjusted so that the smaller gear drives the larger one. If the smaller gear rotates through an angle of 225°, through how many degrees will the larger gear rotate?

Solution



$s = r_1\theta_1 = r_2\theta_2$

$2.5(225°) = 4.8\theta_2$

$\theta_2 = \frac{2.5(225°)}{4.8}$

$= \frac{25(225°)}{48}$

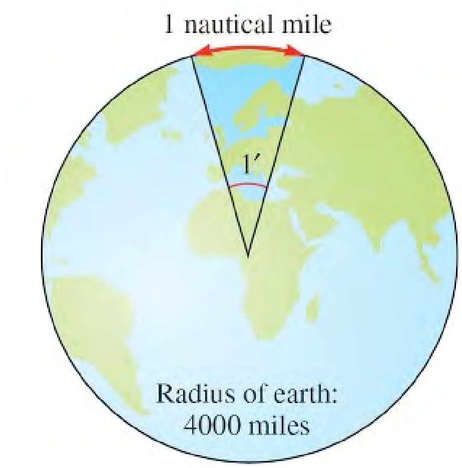
$= \left(\frac{5,625}{48}\right)^{\circ}$

$= 117.1875°$

Exercise

If a central angle with its vertex at the center of the earth has a measure of 1′, then the arc on the surface of the earth that is cut off by this angle (knows as the great circle distance) has a measure of 1 *nautical mile*.

Solution



$$\theta = 1'$$

$$= \frac{1}{60}^{\circ}$$

$$= \frac{1}{60} \cdot \frac{\pi}{180}$$

$$= \frac{\pi}{10800}\ rad$$

$$\frac{\pi}{10800} = \frac{s}{4000} \qquad \theta = \frac{s}{r}$$

$$s = \frac{4000}{10800}\pi$$

$$= \frac{10\pi}{27}\ miles \quad \approx 1.16\ mi$$

Exercise

If two ships are 20 *nautical miles* apart on the ocean, how many statute miles apart are they?

Solution

$$\theta = 20'$$

$$= \frac{20}{60}^{\circ}$$

$$= \frac{1}{3}^{\circ}$$

$$= \frac{1}{3} \cdot \frac{\pi}{180}$$

$$= \frac{\pi}{540}\ rad$$

$$\frac{\pi}{540} = \frac{s}{4000} \qquad \theta = \frac{s}{r}$$

$$s = \frac{4000\pi}{540}$$

$$= \frac{200\pi}{27}\ miles$$

$$s \approx 23.27\ mi$$

Exercise

Two gears are adjusted so that the smaller gear drives the larger one. If the smaller gear rotates through an angle of 300°, through how many degrees will the larger rotate?

Solution

Both gears travel the same arc distance (s), therefore:

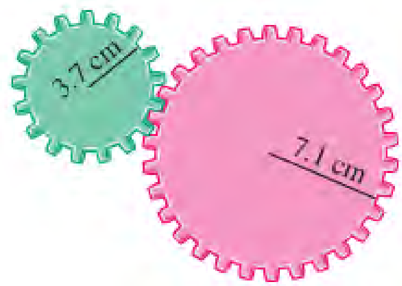
$$s = r_1\theta_1 = r_2\theta_2$$

$$3.7(300°) = 7.1\ \theta_2$$

$$\theta_2 = \frac{37}{71}(300°)$$

$$= \left(\frac{11,100}{71}\right)^{\circ}$$

$$\approx 156.34°$$



Exercise

The rotation of the smaller wheel causes the larger wheel to rotate. Through how many degrees will the larger wheel rotate if the smaller one rotates through 60.0°?

Solution

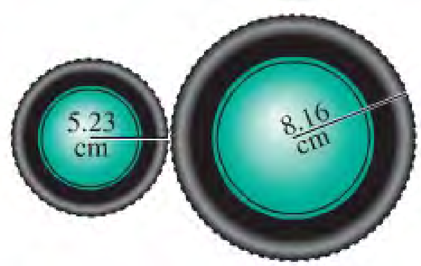
Both gears travel the same arc distance (s), therefore:

$$s = r_1\theta_1 = r_2\theta_2$$

$$5.23(60.0°) = 8.16\ \theta_2$$

$$\theta_2 = \frac{523}{816}(60°)$$

$$= \left(\frac{2,615}{68}\right)^{\circ} \approx 38.5°$$



Exercise

Find the radius of the larger wheel if the smaller wheel rotates 80° when the larger wheel rotates 50°.

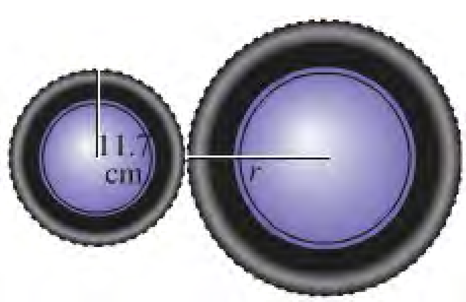
Solution

$$s = r_1\theta_1 = r_2\theta_2$$

$$11.7(80°) = r_2\ (50°)$$

$$r_2 = \frac{11.7(80°)}{50°}$$

$$= 18.72\ \ cm$$



Exercise

How many inches will the weight rise if the pulley is rotated through an angle of 71° 50′?
Through what angle, to the nearest minute, must the pulley be rotated to raise the weight 6 in?

Solution

$$\theta = \left(71^\circ + 50' \frac{1^\circ}{60'}\right)\frac{\pi}{180^\circ}$$

$$s = r\theta$$

$$= 9.27\left(71^\circ + 50' \frac{1^\circ}{60'}\right)\frac{\pi}{180^\circ}$$

$$\approx 11.622\ in$$

$$\theta = \frac{s}{r} = \frac{6}{9.27}\ rad$$

$$= \frac{6}{9.27}\frac{180^\circ}{\pi}$$

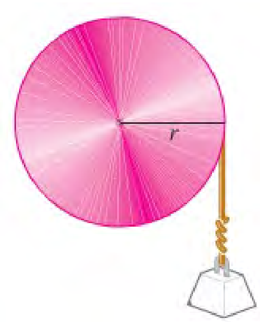
$$= 37.0846^\circ$$

$$= 37^\circ + .0846(60')$$

$$= 37^\circ\ 5'$$

Exercise

Find the radius of the pulley if a rotation of 51.6° raises the weight 11.4 *cm*.



Solution

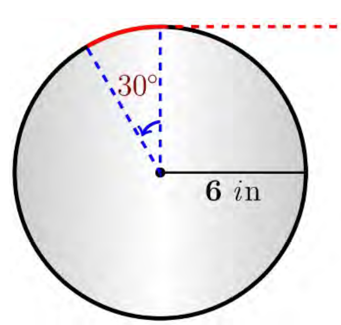
$$r = \frac{s}{\theta} = \frac{11.4}{51.6^\circ \frac{\pi}{180^\circ}}$$

$$= \frac{1,710}{43\pi}\ cm \qquad \approx 12.7\ cm$$

Exercise

A rope is being wound around a drum with radius 6 *inches*. How much rope will be wound around the drum if the drum is rotated through an angle of 30°?

Solution

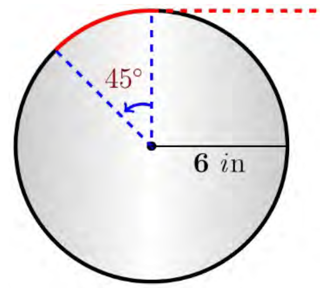


$$s = 6\left(30^\circ \frac{\pi}{180^\circ}\right) \qquad\qquad s = r\theta$$

$$= \pi\ \ in$$

Exercise

A rope is being wound around a drum with radius 6 *inches*. How much rope will be wound around the drum if the drum is rotated through an angle of 45°?



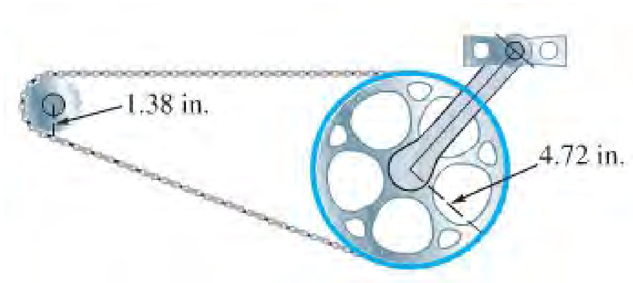
Solution

$$s = 6\left(45°\frac{\pi}{180°}\right) \qquad s = r\theta$$

$$= \frac{3\pi}{2}\ in$$

Exercise

The figure shows the chain drive of a bicycle. How far will the bicycle move if the pedals are rotated through 180°? Assume the radius of the bicycle wheel is 13.6 *in*.



Solution

$$\theta = 180° = \pi\ rad$$

The distance for the pedal gear:

$$s = r_2\theta = 4.72\pi$$

For the smaller gear:

$$\theta_2 = \frac{s}{r_2} = \frac{4.72\pi}{1.38}$$

$$= \frac{472\pi}{138}$$

$$= \frac{236\pi}{69}$$

The wheel distance:

$$s = r_3\theta_2 = \frac{136}{10}\left(\frac{236\pi}{69}\right)$$

$$= \frac{16,048\pi}{345}\ in \qquad = 146.12\ in$$

Exercise

The circular of a Medicine Wheel is 2500 *yrs* old. There are 27 aboriginal spokes in the wheel, all equally spaced.

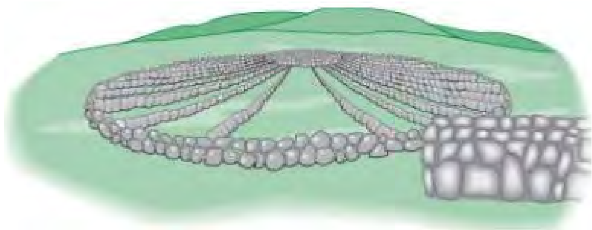
a) Find the measure of each central angle in degrees and in radians.
b) The radius measure of each of the wheel is 76.0 *ft*, find the circumference.
c) Find the length of each arc intercepted by consecutive pairs of spokes.
d) Find the area of each sector formed by consecutive spokes,

Solution

a) The central angle: $\theta = \frac{360°}{27} = \frac{40}{3}^{\circ}$

$$= \frac{40}{3}^{\circ} \frac{\pi\ rad}{180°}$$

$$= \boxed{\frac{2\pi}{27}\ rad}$$



b) $C = 2\pi r = 2\pi(76)$

$$\boxed{= 152\pi\ \ ft} \quad \boxed{\approx 477.5\ \ ft}$$

c) Since $r = 76$

$s = 76\frac{2\pi}{27}$ $\qquad s = r\theta$

$$\boxed{= \frac{152\pi}{27}\ ft} \quad \boxed{\approx 17.7\ ft}$$

d) $Area = \frac{1}{2}r^2\theta$

$$= \frac{1}{2}(76)^2\frac{2\pi}{27}$$

$$\boxed{= \frac{5,776\pi}{27}\ \ ft^2} \quad \boxed{\approx 672\ \ ft^2}$$

Exercise

The total arm and blade of a single windshield wiper was 10 *in*. long and rotated back and forth through an angle of 95°. The shaded region in the figure is the portion of the windshield cleaned by the 7-*in*. wiper blade. What is the area of the region cleaned?

Solution

The total angle:

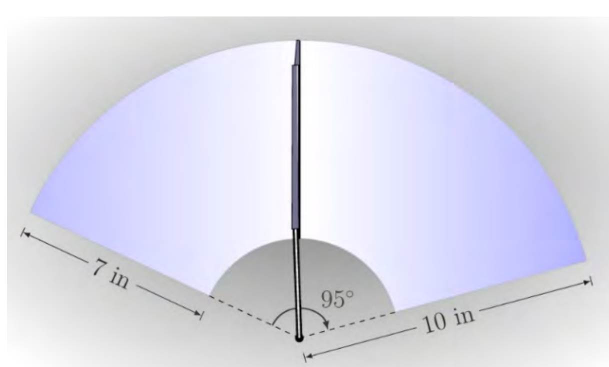
$$\theta = 95°\frac{\pi}{180°}$$

$$= \frac{19\pi}{36}\ rad$$

A_1 : The area of arm only (not cleaned by the blade).

$$A_1 = \frac{1}{2}(10-7)^2\frac{19\pi}{36}$$

$$= \frac{19\pi}{8}$$



A_2 : The area of arm and the blade.

$$A_2 = \frac{1}{2}(10)^2\frac{19\pi}{36}$$

$$\boxed{= \frac{475\pi}{18}}$$

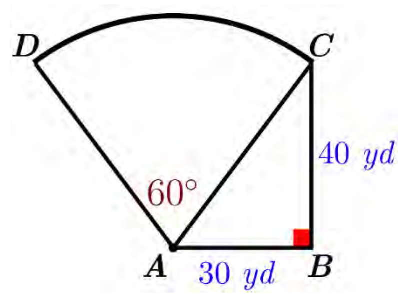
The total cleaned area:

$$A = A_2 - A_1$$
$$= \frac{475\pi}{18} - \frac{19\pi}{8}$$
$$= \frac{1900-171}{72}\pi$$
$$= \frac{1729\pi}{72}\ in^2 \qquad = 75.4\ in^2$$

Exercise

A frequent problem in surveying city lots and rural lands adjacent to curves of highways and railways is that of finding the area when one or more of the boundary lines is the arc of the circle. Find the area of the lot.

Solution



Using the Pythagorean theorem:

$$AC = \sqrt{30^2 + 40^2} = 50 = r$$

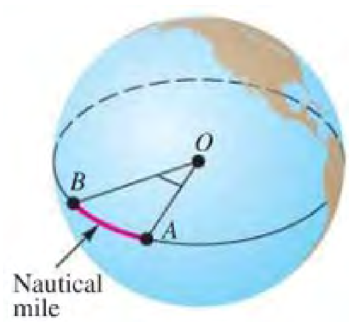
Total area = Area of the sector (ADC) + Area of the triangle (ABC)

$$\text{Total area } = \frac{1}{2}r^2(60°)\frac{\pi}{180°} + \frac{1}{2}(AB)(BC)$$
$$= \frac{1}{2}50^2(60°)\frac{\pi}{180°} + \frac{1}{2}(30)(40)$$
$$= \frac{1250}{3}\pi + 600\ \ yd^2$$
$$\approx 1909\ yd^2$$

Exercise

Nautical miles are used by ships and airplanes. They are different from statue miles, which equal 5280 *ft*. A nautical mile is defined to be the arc length along the equator intercepted by a central angle AOB of 1 *min*. If the equatorial radius is 3963 *mi*, use the arc length formula to approximate the number of statute miles in 1 *nautical mile*.

Solution



$$\theta = 1' \frac{1°}{60'}\frac{\pi}{180°}$$
$$= \frac{\pi}{10800}\ rad$$

The arc length:

$$s = 3963\frac{\pi}{10800} \qquad s = r\theta$$

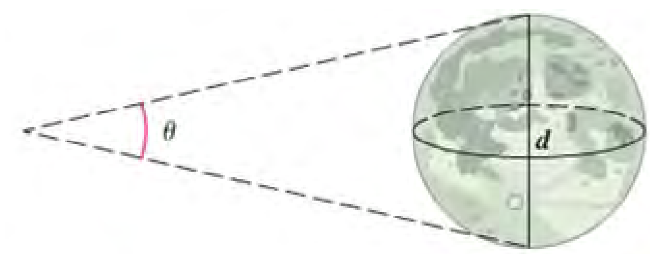
$= \frac{1321\pi}{3600}$

There are $\frac{1321\pi}{3600} \approx 1.15$ *statute miles* in 1 nautical mile.

Exercise

The distance to the moon is approximately 238,900 *mi*. Use the arc length formula to estimate the diameter d of the moon if angle θ is measured to be 0.5170°.

Solution



$s = 238900 \times 0.517° \frac{\pi}{180°}$ $\qquad s = r\theta$

$= \frac{1,235,113\pi}{1800}\ mi$

$\approx 2156\ mi$

Exercise

The minute hand of a clock is 1.2 *cm* long. To two significant digits, how far does the tip of the minute hand move in 20 *minutes*?

Solution

Given: $r = 1.2\ cm$

One complete rotation = 1 *hour* = 60 *minutes* = 2π

$\frac{\theta}{2\pi} = \frac{20}{60}$

$\theta = \frac{2\pi}{3}$

$s = 1.2 \frac{2\pi}{3}$ $\qquad s = r\theta$

$= 12 \frac{2\pi}{30}$

$= \frac{4\pi}{5}\ cm$

$\approx 2.5\ cm$

Exercise

If the sector formed by a central angle of 15° has an area of $\frac{\pi}{3}\text{cm}^2$, find the radius of a circle.

Solution

Given: $\theta = 15°\frac{\pi}{180} = \frac{\pi}{12};\qquad A = \frac{\pi}{3}$

$$A = \frac{1}{2}r^2\theta$$

$$\frac{\pi}{3} = \frac{1}{2}r^2\frac{\pi}{12}$$

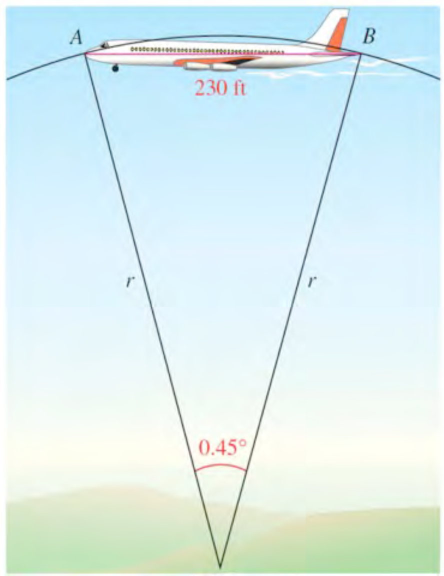
$$\frac{24}{\pi}\frac{\pi}{3} = \frac{1}{2}r^2\frac{\pi}{12}\frac{24}{\pi}$$

$$8 = r^2$$

$$\underline{r = 2\sqrt{2}\ cm}|$$

Exercise

A person standing on the earth notices that a 747 jet flying overhead subtends an angle 0.45°. If the length of the jet is 230 *ft*., find its altitude to the nearest thousand feet.



Solution

$$r = \frac{230}{0.45\left(\frac{\pi}{180}\right)} \qquad\qquad r = \frac{s}{\theta}$$

$$\underline{= \frac{92,000}{\pi}\ ft}|$$

$$\underline{\approx 29,285\ \ ft}|$$

Exercise

Suppose that P is on a circle with radius 10 *cm*, and ray OP is rotating with angular speed $\frac{\pi}{18}$ *rad* / sec.

a) Find the angle generated by P in 6 *seconds*

b) Find the distance traveled by P along the circle in 6 *seconds*.

c) Find the linear speed of P in *cm* per *sec*.

Solution

a) $\theta = \frac{\pi}{18}.6 \qquad\qquad \theta = \omega t$

$$\underline{= \frac{\pi}{3}\ rad}|$$

b) $s = 10\left(\frac{\pi}{3}\right)$ $\qquad s = r\theta$

$= \boxed{\frac{10\pi}{3}\ cm}$

c) $v = \frac{10\pi}{3}\frac{1}{6}$ $\qquad v = \frac{s}{t}$

$= \boxed{\frac{5\pi}{9}\ cm/\sec}$

Exercise

A belt runs a pulley of radius 6 *cm* at $80\ rev/\min$.

a) Find the angular speed of the pulley in *radians per sec*.

b) Find the linear speed of the belt in *cm per sec*.

Solution

a) $\omega = 80\frac{rev}{\min}\cdot\frac{1\min}{60\sec}\cdot\frac{2\pi}{1rev}$

$= \boxed{\frac{8\pi}{3}\ rad/\sec}$

b) $v = 6\left(\frac{8\pi}{3}\right)$ $\qquad v\ r\omega$

$= \boxed{16\pi\ \ cm/\sec}$ $\qquad \boxed{\approx 50\ \ cm/\sec}$

Exercise

Find the linear velocity of a point moving with uniform circular motion, if $s = 12\ cm$ and $t = 2\ sec$.

Solution

$v = \frac{12}{2}\frac{cm}{\sec}$ $\qquad v = \frac{s}{t}$

$= \boxed{6\ cm/\sec}$

Exercise

Find the distance ***s*** covered by a point moving with linear velocity $v = 55\ mi/hr$ and $t = 0.5\ hr$.

Solution

$s = 55\frac{mi}{hr}\times 0.5\ hr$ $\qquad s = vt$

$= \boxed{27.5\ \ miles}$

Exercise

Point *P* sweeps out central angle $\theta = 12\pi$ as it rotates on a circle of radius r with $t = 5\pi$ sec. Find the angular velocity of point *P*.

Solution

$$\omega = \frac{12\pi}{5\pi}\frac{rad}{\sec} \qquad\qquad \omega = \frac{\theta}{t}$$

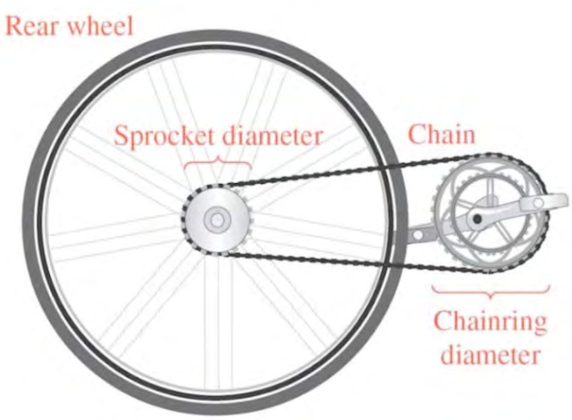
$$= \frac{12}{5}$$

$$\boxed{= 2.4 \ \ rad / \sec}$$

Exercise

When Lance Armstrong blazed up Mount Ventoux in the 2002 tour, he was equipped with a 150-*millimeter*-diameter chainring and a 95-*millimeter*-diameter sprocket. Lance is known for maintaining a very high cadence, or pedal rate. If he was pedaling at a rate of 90 *revolutions* per *minute*, find his speed in *kilometers* per *hour*. (1 *km* = 1,000,000 *mm* or 10^6 *mm*)

Solution



Chainring:

$$\omega = \frac{v}{r}$$

$$= 90\frac{rev}{\min} \times 2\pi\frac{radians}{rev} \times \frac{60}{1}\frac{\min}{hr}$$

$$\boxed{= 10,800\pi \ \ \frac{rad}{hr}}$$

$$v = r\omega$$

$$= \frac{150}{2}(mm) \times 10800\pi\frac{rad}{hr}$$

$$\boxed{= 810,000\pi \ \ \frac{mm}{hr}}$$

Sprocket:

$$\omega = \frac{v}{r}$$

$$= \frac{810000\pi\frac{mm}{hr}}{\frac{95}{2}mm}$$

$$\boxed{= 17,052.63\pi \ \ \frac{rad}{hr}}$$

$$v = r\omega$$

$$= 350(mm) \times \frac{1}{10^6}\frac{km}{mm} \times 17052.63\pi\frac{rad}{hr}$$

$$\boxed{= 18.8 \ \ \frac{km}{hr}}$$

Exercise

Find the angular velocity, in radians per minute, associated with given 7.2 *rpm*.

Solution

$$\omega = 7.2\frac{rev}{\min} \times 2\pi\frac{radians}{rev}$$

$$= \frac{144}{10}\pi$$

$$\boxed{= \frac{72\pi}{5}\ rad/min} \qquad \boxed{\approx 45.2\ \frac{rad}{\min}}$$

Exercise

Suppose that point P is on a circle with radius 60 *cm*, and ray OP is rotating with angular speed $\frac{\pi}{12}$ *radian per sec*.

a) Find the angle generated by P in 8 *sec*.

b) Find the distance traveled by P along the circle in 8 *sec*.

c) Find the linear speed of P.

Solution

a) $\theta = \omega t = \frac{\pi}{12}.8$

$$\boxed{= \frac{2\pi}{3}\ rad}$$

b) $s = r\theta = 60\left(\frac{2\pi}{3}\right)$

$$\boxed{= 40\pi\ cm}$$

c) $v = \frac{s}{t}\ \boxed{= \frac{40\pi}{6}\ cm/\sec}$

Exercise

Tires of a bicycle have radius 13 *in*. and are turning at the rate of 215 *revolutions per min*. How fast is the bicycle traveling in miles per hour? (*Hint*: 1 *mi* = 5280 *ft*.)

Solution

$$\omega = 215\ rev\ \frac{2\pi\ rad}{1\ rev}$$

$$= 430\pi\ rad/\min$$

$$v = r\omega = 13(430\pi)$$

$$= 5590\pi\ in/\min$$

$$v = 5590\pi \frac{in}{\min}\frac{60\min}{1hr}\frac{1\,ft}{12in}\frac{1mi}{5280\,ft}$$

$$= \frac{2,795\pi}{528}\ mph \qquad \approx 16.6\ mph$$

Exercise

Earth travels about the sun in an orbit that is almost circular. Assume that the orbit is a circle with radius 93,000,000 *mi*. Its angular and linear speeds are used in designing solar-power facilities.

a) Assume that a year is 365 *days*, and find the angle formed by Earth's movement in one day.

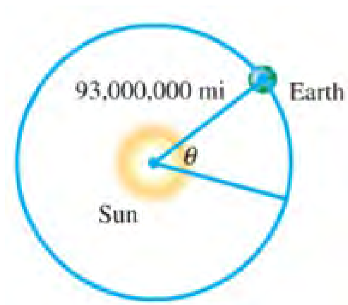
b) Give the angular speed in *radians per hour*.

c) Find the linear speed of Earth in *miles per hour*.

Solution

a) $\theta = \frac{1}{365}(2\pi)$

$$= \frac{2\pi}{365}\ rad$$



b) $\omega = \frac{2\pi\ rad}{365\ days}\ \frac{1\ day}{24\ hr}$

$$= \frac{\pi}{4380}\ rad\,/\,hr$$

c) $v = r\omega = (93,000,000)\frac{\pi}{4380}$

$$\approx 67,000\ \ mph$$

Exercise

Earth revolves on its axis once every 24 *hr*. Assuming that earth's radius is 6400 *km*, find the following.

a) Angular speed of Earth in radians per day and radians per hour.

b) Linear speed at the North Pole or South Pole

c) Linear speed ar a city on the equator

Solution

a) $\omega = \frac{2\pi}{1}\frac{rad}{day}$ $\qquad \omega = \frac{\theta}{t}$

$$= \frac{2\pi}{1}\frac{rad}{day}\frac{1\ day}{24\ hr}$$

$$= \frac{\pi}{12}\ \ rad\,/\,hr$$

b) At the poles, $r = 0$ so $\boldsymbol{v} = rw = 0$

c) At the equator, $r = 6400\ km$

$$v = 6400(2\pi) \qquad\qquad v = rw$$
$$= 12,800\pi \ km/day$$
$$= 12,800\pi \ \frac{km}{day}\ \frac{1\ day}{24\ hr}$$
$$\approx 533\pi \quad km/hr$$

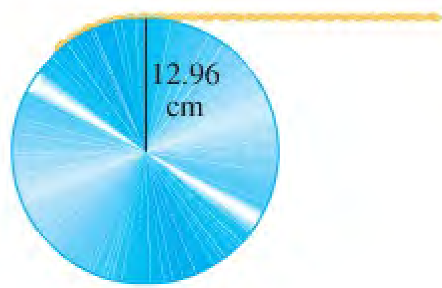
Exercise

The pulley has a radius of 12.96 *cm*. Suppose it takes 18 *sec* for 56 *cm* of belt to go around the pulley.

a) Find the linear speed of the belt in *cm per sec*.
b) Find the angular speed of the pulley in *rad per sec*.

Solution

Given: $s = 56\ cm$ in $t = 18$ sec $r = 12.96\ cm$



a) $v = \frac{s}{t} = \frac{56}{18}$

$$\approx 3.1\ cm/\sec$$

b) $\omega = \frac{v}{r} = \frac{3.1}{12.96}$

$$= \frac{310}{1296}$$
$$= \frac{155}{648} \quad rad/\sec \qquad \approx .24\ \ rad/\sec$$

Exercise

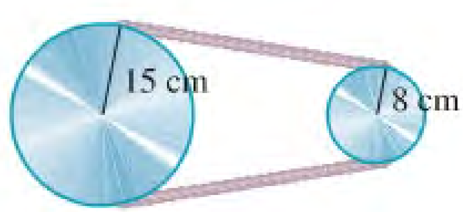
The two pulleys have radii of 15 *cm* and 8 *cm*, respectively. The larger pulley rotates 25 times in 36 *sec*. Find the angular speed of each pulley in *rad per sec*.

Solution

Given: $\omega = \frac{25}{36}\ times/\sec \quad r_1 = 15\ cm \quad r_2 = 8\ cm$

The angular velocity of the larger pulley is:

$$\omega = \frac{25}{36}\frac{times}{\sec}\ \frac{2\pi\ rad}{1\ time}$$
$$= \frac{25\pi}{18}\ \ rad/\sec$$



The linear velocity of the larger pulley is:

$$v = r\omega = 15\left(\frac{25\pi}{18}\right)$$

$$= \boxed{\frac{125\pi}{6}\ cm/\sec}$$

The angular velocity of the smaller pulley is:

$$\omega = \frac{v}{r} = \frac{1}{8} \cdot \frac{125\pi}{6}$$

$$= \boxed{\frac{125\pi}{48}\ rad/\sec}$$

Exercise

A thread is being pulled off a spool at the rate of 59.4 *cm per sec*. Find the radius of the spool if it makes 152 *revolutions per min*.

Solution

Given: $\omega = 152\ rev/\min;\ \ v = 59.4\ cm/\sec$

$$r = \frac{1}{152\frac{rev}{\min}} 59.4 \frac{cm}{\sec} \qquad\qquad r = \frac{v}{\omega} = \frac{1}{\omega}v$$

$$= \left(\frac{1}{152}\frac{\min}{rev}\frac{60\ \sec}{1\ \min}\frac{1\ rev}{2\pi\ rad}\right)\left(\frac{594}{10}\frac{cm}{\sec}\right)$$

$$= \boxed{\frac{891}{76\pi}\ cm}$$

$$\boxed{\approx 3.7\ cm}$$

Exercise

A railroad track is laid along the arc of a circle of radius 1800 *feet*. The circular part of the track subtends a central angle of 40°. How long (in seconds) will it take a point on the front of a train traveling 30 *mph* to go around this portion of the track?

Solution

Given: $r = 1800\ ft.$

$$\theta = 40° = 40° \frac{\pi}{180°} = \frac{2\pi}{9} rad$$

$$v = 30\ mph$$

The arc length:

$$s = 1800\left(\frac{2\pi}{9}\right) \qquad\qquad s = r\theta$$

$$= \boxed{400\pi\ \ ft}$$

$$v = \frac{s}{t} \Rightarrow t = \frac{s}{v}$$

$$t = \frac{400\pi\ ft}{30\frac{mi}{hr}}$$

$$= \frac{40\pi}{3}\ ft \frac{hr}{mi} \frac{1mi}{5280\ ft} \frac{3600\sec}{1hr}$$

$$= \frac{1200\pi}{132} \sec$$

$$\approx 29 \sec$$

Exercise

A 90-*horsepower* outboard motor at full throttle will rotate it propeller at exactly 5,000 r*evolutions per min*. Find the angular speed of the propeller in radians per second.

Solution

$$\omega = 5000 \frac{rev}{\min} \frac{2\pi}{1} \frac{rad}{rev} \frac{1}{60} \frac{\min}{\sec}$$

$$= \frac{500\pi}{3}\ rad / \sec$$

$$\approx 523.6\ \ rad / \sec$$

Exercise

The shoulder joint can rotate at 25 *rad/min*. If a golfer's arm is straight and the distance from the shoulder to the club head is 5.00 *feet*., find the linear speed of the club head from the shoulder rotation.

Solution

Given: $\omega = 25\ rad / \min \quad r = 5\, ft$

$$v = r\omega = 5(25)$$

$$= 125\ \ ft / \min$$

Exercise

A vendor sells two sizes of pizza by the slice. The small slice is $\frac{1}{6}$ of a circular 18–*inch*–diameter pizza, and it sells for \$2.00. The large slice is $\frac{1}{8}$ of a circular 26–*inch*–diameter pizza, and it sells for \$3.00. Which slice provides more pizza per dollar?

Solution

Area of 18–*inch*–diameter: $A = \pi r^2 = \pi\left(\frac{18}{2}\right)^2 = 81\pi$

Area of 26–*inch*–diameter: $A = \pi r^2 = \pi\left(\frac{26}{2}\right)^2 = 169\pi$

For the small slice: $\frac{1}{6}81\pi\frac{1}{\$2} = 6.75\pi$

For the small slice: $\frac{1}{8}169\pi\frac{1}{\$3} \approx 7.04\pi$

$\therefore$ the Large size will provide more pizza per dollar

Exercise

A cone–shaped tent is made from a circular piece of canvas 24 *feet* in diameter by removing a sector with central angle $100°$ and connecting the ends. What is the surface area of the tent?

Solution

$$\begin{aligned}\theta &= 360° - 100° \\ &= 260°\frac{\pi}{180°} \\ &= \frac{13\pi}{9}\end{aligned}$$

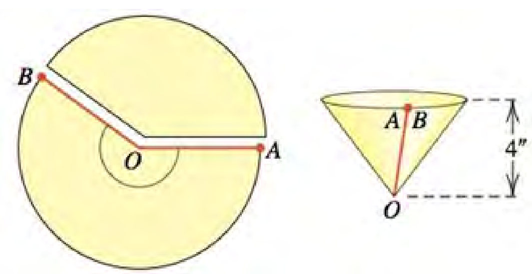
$$A_{sector} = \frac{1}{2}12^2\left(\frac{13\pi}{9}\right) \qquad\qquad A_{sector} = \frac{1}{2}r^2\theta$$

$$\underline{= 104\pi\ ft^2} \quad \approx 326.73\ ft^2$$

Exercise

A conical paper cup is constructed by removing a sector from a circle of radius 5 *inches* and attaching edge *OA* to *OB*. Find angle *AOB* so that the cap has a depth of 4 *inches*.

Solution



$r^2 + 4^2 = 5^2 \rightarrow \underline{r = 3\ in}$

The circumference of the rim of the cone is:

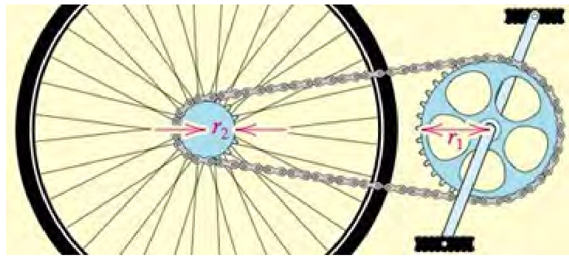
$2\pi r = 6\pi$

$$\begin{aligned}\theta &= \frac{s}{r} = \frac{6\pi}{5}\ rad \\ &= \frac{6(180)}{5} \\ &= \underline{216°}\end{aligned}$$

Exercise

The sprocket assembly for a bicycle is show in the figure. If the sprocket of radius r_1 rotates through an angle of θ_1 radians, find the corresponding angle of rotation for the sprocket of radius r_2

Solution



$$s_1 = s_2$$

$$r_1\theta_1 = r_2\theta_2$$

$$\boxed{\theta_2 = \frac{r_1\theta_1}{r_2}}$$

Exercise

A simple model of the core of a tornado is a right circular cylinder that rotates about its axis. If a tornado has a core diameter of 200 *feet* and maximum wind speed of 180 *mi/hr*. (or 264 *ft/sec*) at the perimeter of the core, approximate the number of revolutions the core makes each minute.

Solution

$$r = \frac{D}{2} = \frac{200}{2}$$

$$\boxed{= 100 \ ft}$$

$$264 = 100\theta \qquad v = r\omega$$

$$\theta = 2.64 \frac{rad}{\sec} \frac{60 \ \sec}{1 \ \min} \frac{1 \ rev}{2\pi \ rad}$$

$$\boxed{\approx 25.2 \ \ rev / \min}$$

Exercise

Earth rotates about its axis once every 23 *hours*, 56 *minutes*, and 4 *seconds*. Approximate the number of radians Earth rotates in one second.

Solution

$$23hr \frac{3600\sec}{1h} + 53\min \frac{60\sec}{1\min} + 4\sec \ \boxed{= 85{,}984 \ \sec}$$

Earth rotates in one second:

$$\frac{1 \ rev}{85{,}984 \ \sec} = 2\pi rad \left(\frac{1}{85{,}984 \ \sec} \right)$$

$$\boxed{\approx 7.31 \times 10^{-5} \ \ rad / \sec}$$

Exercise

A typical tire for a compact car is 22 *inches* in diameter. If the car is traveling at a speed of 60 *mi/hr*., find the number of revolutions the tire makes per minute.

Solution

$$r = \frac{D}{2} = \frac{22}{2}$$
$$\underline{= 11\ in|}$$

$$60 = 11\theta \qquad\qquad v = r\omega$$

$$\theta = \frac{1}{11\ in} 60 \frac{mi}{hr} \frac{12\ in}{1\ ft} \frac{5280\ ft}{1\ mil} \frac{1\ hr}{60\ \min} \frac{1\ rev}{2\pi\ rad}$$

$$\underline{\approx 916.73\ \ rev / \min|}$$

Exercise

A pendulum in a grandfather clock is 4 *feet* long and swings back and forth along a 6–*inch* arc. Approximate the angle (in *degrees*) through which the pendulum passes during one swing.

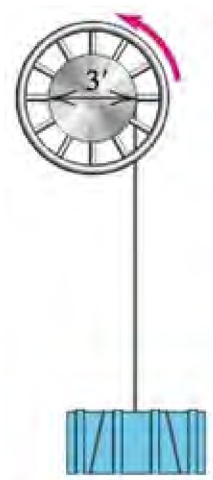
Solution

Given: $r = 4\ ft = 48\ in \quad s = 6\ in$

$$\theta = \frac{6}{48} = 0.125\ rad$$
$$= 0.125\ rad \frac{180^\circ}{\pi\ rad}$$
$$\underline{= 7.162^\circ|}$$

Exercise

A large winch of diameter 3 *feet* is used to hoist cargo.



a) Find the distance the cargo is lifted if the winch rotates through an angle measure $\frac{7\pi}{4}$.

b) Find the angle (in *radians*) through which the winch must rotate in order to lift the cargo *d feet*.

Solution

a) $s = \frac{3}{2} \frac{7\pi}{4}$

$$\underline{= \frac{21\pi}{8}\ ft \approx 8.25\ \ ft|} \qquad\qquad s = r\theta$$

b) $\theta = \dfrac{d}{\frac{3}{2}}$ $\qquad\qquad\qquad$ $\theta = \dfrac{s}{r}$

$$= \frac{2}{3}d$$

Solution   *Section* 6.3 – Trigonometric Functions

Exercise

Find the six trigonometry functions of θ if θ is in the standard position and the point $(-2, 3)$ is on the terminal side of θ.

Solution

$$r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$$

$$\sin\theta = \frac{y}{r} = \frac{3}{\sqrt{13}} \qquad \cos\theta = \frac{x}{r} = -\frac{2}{\sqrt{13}} \qquad \tan\theta = \frac{y}{x} = -\frac{3}{2}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x} = -\frac{\sqrt{13}}{2} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{r}{y} = \frac{\sqrt{13}}{3} \qquad \cot\theta = \frac{x}{y} = -\frac{2}{3}$$

Exercise

Find the six trigonometry functions of θ if θ is in the standard position and the point $(-3, -4)$ is on the terminal side of θ.

Solution

$3,\ 4 \ \rightarrow\ 5$

$$\sin\theta = -\frac{4}{5} \qquad \cos\theta = -\frac{3}{5} \qquad \tan\theta = \frac{-4}{-3} = \frac{4}{3}$$

$$\csc\theta = -\frac{5}{4} \qquad \sec\theta = -\frac{5}{3} \qquad \cot\theta = \frac{3}{4}$$

Exercise

Find the six trigonometry functions of θ in standard position with terminal side through the point $(-3, 0)$.

Solution

$$r = \sqrt{(-3)^2 + 0^2} = 3 \qquad r = \sqrt{x^2 + y^2}$$

$$\sin\theta = \frac{0}{3} = 0 \qquad \cos\theta = \frac{-3}{3} = -1 \qquad \tan\theta = \frac{0}{-3} = 0$$

$$\csc\theta = \frac{1}{0} \rightarrow \infty \qquad \sec\theta = \frac{1}{-1} = -1 \qquad \cot\theta = \frac{1}{0} = \infty$$

Exercise

Find the six trigonometry functions of θ if θ is in the standard position and the point $(12, -5)$ is on the terminal side of θ.

Solution

$$r = \sqrt{x^2 + y^2} = \sqrt{12^2 + (-5)^2} = \underline{13}$$

$\sin\theta = -\frac{5}{13}$ $\quad$ $\cos\theta = \frac{12}{13}$ $\quad$ $\tan\theta = -\frac{5}{12}$

$\csc\theta = -\frac{13}{5}$ $\quad$ $\sec\theta = \frac{13}{12}$ $\quad$ $\cot\theta = -\frac{12}{5}$

Exercise

Find the six trigonometry functions of θ if θ is in the standard position and the point $(5, \ -12)$ is on the terminal side of θ.

Solution

$5 \quad 12 \quad \to \quad 13$

$\sin\theta = -\frac{12}{13}$ $\quad$ $\cos\theta = \frac{5}{13}$ $\quad$ $\tan\theta = -\frac{12}{5}$

$\csc\theta = -\frac{13}{12}$ $\quad$ $\sec\theta = \frac{13}{5}$ $\quad$ $\cot\theta = -\frac{5}{12}$

Exercise

Find the six trigonometry functions of θ if θ is in the standard position and the point $(9, \ -12)$ is on the terminal side of θ.

Solution

$(9, \ -12) = 3(3, \ -4) \ \Rightarrow \ 3 \quad 4 \quad \to \quad 5$

$\sin\theta = -\frac{4}{5}$ $\quad$ $\cos\theta = \frac{3}{5}$ $\quad$ $\tan\theta = -\frac{4}{3}$

$\csc\theta = -\frac{5}{4}$ $\quad$ $\sec\theta = \frac{5}{3}$ $\quad$ $\cot\theta = -\frac{3}{4}$

Exercise

Find the six trigonometry functions of θ if θ is in the standard position and the point $(16,\ -12)$ is on the terminal side of θ.

Solution

$(16,\ -12) = 4(4,\ -3) \Rightarrow\ 4\quad 3\ \rightarrow\ 5$

$\sin\theta = -\frac{3}{5}$ $\qquad \cos\theta = \frac{4}{5}$ $\qquad \tan\theta = -\frac{3}{4}$

$\csc\theta = -\frac{5}{3}$ $\qquad \sec\theta = \frac{5}{4}$ $\qquad \cot\theta = -\frac{4}{3}$

Exercise

Find the six trigonometry functions of θ if θ is in the standard position and the point $(15,\ -8)$ is on the terminal side of θ.

Solution

$15\quad 8\ \rightarrow\ 17$

$\sin\theta = -\frac{8}{17}$ $\qquad \cos\theta = \frac{15}{17}$ $\qquad \tan\theta = -\frac{8}{15}$

$\csc\theta = -\frac{17}{8}$ $\qquad \sec\theta = \frac{17}{15}$ $\qquad \cot\theta = -\frac{15}{8}$

Exercise

Find the six trigonometry functions of θ if θ is in the standard position and the point $(-6,\ 8)$ is on the terminal side of θ.

Solution

$(-6,\ 8) = 2(-3,\ 4) \Rightarrow\ 3\quad 4\ \rightarrow\ 5$

$\sin\theta = \frac{4}{5}$ $\qquad \cos\theta = -\frac{3}{5}$ $\qquad \tan\theta = -\frac{4}{3}$

$\csc\theta = \frac{5}{4}$ $\qquad \sec\theta = -\frac{5}{3}$ $\qquad \cot\theta = -\frac{3}{4}$

Exercise

Find the six trigonometry functions of θ if θ is in the standard position and the point $(-15,\ 8)$ is on the terminal side of θ.

Solution

$15 \quad 8 \quad \to \quad 17$

$\sin\theta = \frac{8}{17}$ $\qquad \cos\theta = -\frac{15}{17}$ $\qquad \tan\theta = -\frac{8}{15}$

$\csc\theta = \frac{17}{8}$ $\qquad \sec\theta = -\frac{17}{15}$ $\qquad \cot\theta = -\frac{15}{8}$

Exercise

Find the six trigonometry functions of θ if θ is in the standard position and the point $(-7,\ 24)$ is on the terminal side of θ.

Solution

$7 \quad 24 \quad \to \quad 25$

$\sin\theta = \frac{24}{25}$ $\qquad \cos\theta = -\frac{7}{25}$ $\qquad \tan\theta = -\frac{24}{7}$

$\csc\theta = \frac{25}{24}$ $\qquad \sec\theta = -\frac{25}{7}$ $\qquad \cot\theta = -\frac{7}{24}$

Exercise

Find the six trigonometry functions of θ if θ is in the standard position and the point $(10,\ -24)$ is on the terminal side of θ.

Solution

$(10,\ -24) = 2(5,\ -12) \;\Rightarrow\; 5 \quad 12 \quad \to \quad 13$

$\sin\theta = -\frac{12}{13}$ $\qquad \cos\theta = \frac{5}{13}$ $\qquad \tan\theta = -\frac{12}{5}$

$\csc\theta = -\frac{13}{12}$ $\qquad \sec\theta = \frac{13}{5}$ $\qquad \cot\theta = -\frac{5}{12}$

Exercise

Find the six trigonometry functions of θ if θ is in the standard position and the point $(7,\ 24)$ is on the terminal side of θ.

Solution

$7 \quad 24 \quad \to \quad 25$

$\sin\theta = \frac{24}{25}$ $\qquad \cos\theta = \frac{7}{25}$ $\qquad \tan\theta = \frac{24}{7}$

$\csc\theta = \frac{25}{24}$ $\qquad \sec\theta = \frac{25}{7}$ $\qquad \cot\theta = \frac{7}{24}$

Exercise

Find the six trigonometry functions of θ if θ is in the standard position and the point $(-7,\ -24)$ is on the terminal side of θ.

Solution

$7 \quad 24 \quad \to \quad 25$

$\sin\theta = -\frac{24}{25}$ $\qquad \cos\theta = -\frac{7}{25}$ $\qquad \tan\theta = \frac{24}{7}$

$\csc\theta = -\frac{25}{24}$ $\qquad \sec\theta = -\frac{25}{7}$ $\qquad \cot\theta = \frac{7}{24}$

Exercise

Find the six trigonometry functions of θ if θ is in the standard position and the point $(-24,\ -7)$ is on the terminal side of θ.

Solution

$24 \quad 7 \quad \to \quad 25$

$\sin\theta = -\frac{7}{25}$ $\qquad \cos\theta = -\frac{24}{25}$ $\qquad \tan\theta = \frac{7}{24}$

$\csc\theta = -\frac{25}{7}$ $\qquad \sec\theta = -\frac{25}{24}$ $\qquad \cot\theta = \frac{24}{7}$

Exercise

Find the six trigonometry functions of θ if θ is in the standard position and the point $(24,\ -10)$ is on the terminal side of θ.

Solution

$$(24,\ -10) = 2(12,\ -5) \ \Rightarrow \ 12 \quad 5 \ \rightarrow \ 13$$

$$\sin\theta = -\frac{5}{13} \qquad \cos\theta = \frac{12}{13} \qquad \tan\theta = -\frac{5}{12}$$

$$\csc\theta = -\frac{13}{5} \qquad \sec\theta = \frac{13}{12} \qquad \cot\theta = -\frac{12}{5}$$

Exercise

Find the values of the six trigonometric functions for an angle of 90°.

Solution

$$\sin 90° = 1 \qquad \tan 90° = \infty \qquad \csc 90° = 1$$

$$\cos 90° = 0 \qquad \cot 90° = 0 \qquad \sec 90° = \infty$$

Exercise

Indicate the two quadrants θ could terminate in if $\cos\theta = \frac{1}{2}$

Solution

$$\cos\theta = \frac{1}{2} \qquad \rightarrow \boldsymbol{Q}\text{I} \ \& \ \boldsymbol{Q}\text{IV}$$

Exercise

Indicate the two quadrants θ could terminate in if $\csc\theta = -2.45$

Solution

$$\begin{aligned}\csc\theta &= -2.45 \\ &= \frac{1}{\sin\theta} \qquad \rightarrow \ \boldsymbol{Q}\text{III} \ \& \ \boldsymbol{Q}\text{IV}\end{aligned}$$

Exercise

Find the remaining trigonometric function of θ if $\sin\theta = \frac{12}{13}$ and θ terminates in ***Q*I**

Solution

$5 \quad 12 \quad \rightarrow \quad 13$

$\sin\theta = \frac{12}{13}$ $\qquad \cos\theta = \frac{5}{13}$ $\qquad \tan\theta = \frac{12}{5}$

$\csc\theta = \frac{13}{12}$ $\qquad \sec\theta = \frac{13}{5}$ $\qquad \cot\theta = \frac{5}{12}$

Exercise

Find the remaining trigonometric function of θ if $\cot\theta = -2$ and θ terminates in ***Q*II**.

Solution

$$\cot\theta = -2 = \frac{x}{y} \quad (\theta \in QII)$$

$$x = -2, \ \ y = 1$$

$$r = \sqrt{(-2)^2 + (1)^2} \qquad r = \sqrt{x^2 + y^2}$$

$$= \sqrt{5}$$

$\sin\theta = \frac{1}{\sqrt{5}}$ $\qquad \cos\theta = -\frac{2}{\sqrt{5}}$ $\qquad \tan\theta = -\frac{1}{2}$

$\csc\theta = \sqrt{5}$ $\qquad \sec\theta = -\frac{\sqrt{5}}{2}$

Exercise

Find the remaining trigonometric function of θ if $\tan\theta = \frac{3}{4}$ and θ terminates in ***Q*III**.

Solution

$$\tan\theta = \frac{3}{4} = \frac{y}{x} \quad (\theta \in QIII)$$

$$x = -4, \ \ y = -3$$

$4 \quad 3 \quad \rightarrow \quad 5$

$\sin\theta = -\frac{3}{5}$ $\qquad \cos\theta = -\frac{4}{5}$

$\csc\theta = -\frac{5}{3}$ $\qquad \sec\theta = -\frac{5}{4}$ $\qquad \cot\theta = \frac{4}{3}$

Exercise

Find the remaining trigonometric function of θ if $\cos\theta = \frac{24}{25}$ and θ terminates in **Q**IV.

Solution

$24 \quad 7 \quad \rightarrow \quad 25$

$\cos\theta = \frac{24}{25} \quad \theta \in QIV \quad \Rightarrow y = -7$

$\sin\theta = -\frac{7}{25}$ $\quad$ $\cos\theta = \frac{24}{25}$ $\quad$ $\tan\theta = -\frac{7}{24}$

$\csc\theta = -\frac{25}{7}$ $\quad$ $\sec\theta = \frac{25}{24}$ $\quad$ $\cot\theta = -\frac{24}{7}$

Exercise

Find the remaining trigonometric functions of θ if $\cos\theta = \frac{\sqrt{3}}{2}$ and θ is terminates in **Q**IV.

Solution

$\cos\theta = \frac{\sqrt{3}}{2} = \frac{x}{r} \qquad \Rightarrow x = \sqrt{3}, \ \ r = 2$

Since θ is **Q**IV

$$y = -\sqrt{2^2 - \sqrt{3}^2}$$

$$= -\sqrt{4-3}$$

$$\underline{= -1|}$$

$\sqrt{3} \quad 1 \quad \rightarrow \quad 2$

$\sin\theta = -\frac{1}{2}$ $\quad$ $\cos\theta = \frac{\sqrt{3}}{2}$ $\quad$ $\tan\theta = -\frac{1}{\sqrt{3}}$

$\csc\theta = -2$ $\quad$ $\sec\theta = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$ $\quad$ $\cot\theta = -\sqrt{3}$

Exercise

Find the remaining trigonometric functions of θ if $\tan\theta = -\frac{1}{2}$ and $\cos\theta > 0$.

Solution

$\tan\theta = \frac{\sin\theta}{\cos\theta} < 0 \quad \& \quad \cos\theta > 0$

$\sin\theta < 0 \quad \Rightarrow \theta$ in QIV

$\Rightarrow y = 1, \quad x = 2$

$r = \sqrt{1^2 + 2^2}$

$\quad = \sqrt{5}$

$2 \quad -1 \quad \to \quad \sqrt{5}$

$\sin\theta = -\frac{1}{\sqrt{5}}$ $\qquad \cos\theta = \frac{2}{\sqrt{5}}$ $\qquad \tan\theta = -\frac{1}{2}$

$\csc\theta = -\frac{\sqrt{5}}{5}$ $\qquad \sec\theta = \frac{\sqrt{5}}{2}$ $\qquad \cot\theta = -2$

Exercise

Find the remaining trigonometric functions of θ if $\cos\theta = \frac{3}{5}$ & $\theta \in QI$

Solution

$3 \quad 4 \quad \to \quad 5$

$\sin\theta = \frac{4}{5}$ $\qquad \cos\theta = \frac{3}{5}$ $\qquad \tan\theta = \frac{4}{3}$

$\csc\theta = \frac{5}{4}$ $\qquad \sec\theta = \frac{5}{3}$ $\qquad \cot\theta = \frac{3}{4}$

Exercise

Find the remaining trigonometric functions of θ if $\cos\theta = -\frac{4}{5}$ & $\theta \in QII$

Solution

$\theta \in QII$ & $\sin\theta > 0$

$-4 \quad 3 \quad \to \quad 5$

$\sin\theta = \frac{3}{5}$ $\qquad \cos\theta = -\frac{4}{5}$ $\qquad \tan\theta = -\frac{3}{4}$

$\csc\theta = \frac{5}{3}$ $\qquad \sec\theta = -\frac{5}{4}$ $\qquad \cot\theta = -\frac{4}{3}$

Exercise

Find the remaining trigonometric functions of θ if $\sin\theta = -\frac{3}{5}$ & $\theta \in \text{QIII}$

Solution

$\theta \in QIII$ & $\cos\theta < 0$

$-4 \quad -3 \quad \rightarrow \quad 5$

$\sin\theta = -\frac{3}{5}$ $\qquad \cos\theta = -\frac{4}{5}$ $\qquad \tan\theta = \frac{3}{4}$

$\csc\theta = -\frac{5}{3}$ $\qquad \sec\theta = -\frac{5}{4}$ $\qquad \cot\theta = \frac{4}{3}$

Exercise

Find the remaining trigonometric functions of θ if $\sin\theta = -\frac{3}{5}$ & $\theta \in QIV$

Solution

$\theta \in QIV$ & $\cos\theta > 0$

$4 \quad -3 \quad \rightarrow \quad 5$

$\sin\theta = -\frac{3}{5}$ $\qquad \cos\theta = \frac{4}{5}$ $\qquad \tan\theta = -\frac{3}{4}$

$\csc\theta = -\frac{5}{3}$ $\qquad \sec\theta = \frac{5}{4}$ $\qquad \cot\theta = -\frac{4}{3}$

Exercise

Find the remaining trigonometric functions of θ if $\cos\theta = -\frac{12}{13}$ & $\theta \in QIII$

Solution

$\theta \in QIII$ & $\sin\theta < 0$

$-12 \quad -5 \quad \rightarrow \quad 13$

$\sin\theta = -\frac{5}{13}$ $\qquad \cos\theta = -\frac{12}{13}$ $\qquad \tan\theta = \frac{5}{12}$

$\csc\theta = -\frac{13}{5}$ $\qquad \sec\theta = -\frac{13}{12}$ $\qquad \cot\theta = \frac{12}{5}$

Exercise

Find the remaining trigonometric functions of θ if $\cos\theta = -\frac{5}{13}$ & $\theta \in QII$

Solution

$\theta \in QII$ & $\sin\theta > 0$

$-5 \quad 12 \quad \rightarrow \quad 13$

$\sin\theta = \frac{12}{13}$ $\qquad \cos\theta = -\frac{5}{13}$ $\qquad \tan\theta = -\frac{12}{5}$

$\csc\theta = \frac{13}{12}$ $\qquad \sec\theta = -\frac{13}{5}$ $\qquad \cot\theta = -\frac{5}{12}$

Exercise

Find the remaining trigonometric functions of θ if $\cos\theta = \frac{12}{13}$ & $\theta \in QIV$

Solution

$\theta \in QIV$ & $\sin\theta < 0$

$12 \quad -5 \quad \rightarrow \quad 13$

$\sin\theta = -\frac{5}{13}$ $\qquad \cos\theta = \frac{12}{13}$ $\qquad \tan\theta = -\frac{5}{12}$

$\csc\theta = -\frac{13}{5}$ $\qquad \sec\theta = \frac{13}{12}$ $\qquad \cot\theta = -\frac{12}{5}$

Exercise

Find the remaining trigonometric functions of θ if $\sin\theta = -\frac{8}{17}$ & $\theta \in QIII$

Solution

$\theta \in QIII$ & $\cos\theta < 0$

$-15 \quad -8 \quad \rightarrow \quad 17$

$\sin\theta = -\frac{8}{17}$ $\qquad \cos\theta = -\frac{15}{17}$ $\qquad \tan\theta = \frac{8}{15}$

$\csc\theta = -\frac{17}{8}$ $\qquad \sec\theta = -\frac{17}{15}$ $\qquad \cot\theta = \frac{15}{8}$

Exercise

Find the remaining trigonometric functions of θ if $\cos\theta = -\frac{15}{17}$ & $\theta \in QII$

Solution

$\theta \in QII$ & $\sin\theta > 0$

$-15 \quad 8 \quad \rightarrow \quad 17$

$\sin\theta = \frac{8}{17}$ $\quad\cos\theta = -\frac{15}{17}$ $\quad\tan\theta = -\frac{8}{15}$

$\csc\theta = \frac{17}{8}$ $\quad\sec\theta = -\frac{17}{15}$ $\quad\cot\theta = -\frac{15}{8}$

Exercise

Find the remaining trigonometric functions of θ if $\cos\theta = -\frac{8}{17}$ & $\theta \in QII$

Solution

$\theta \in QII$ & $\sin\theta > 0$

$-8 \quad 15 \quad \rightarrow \quad 17$

$\sin\theta = \frac{15}{17}$ $\quad\cos\theta = -\frac{8}{17}$ $\quad\tan\theta = -\frac{15}{8}$

$\csc\theta = \frac{17}{15}$ $\quad\sec\theta = -\frac{17}{8}$ $\quad\cot\theta = -\frac{8}{15}$

Exercise

Find the remaining trigonometric functions of θ if $\cos\theta = -\frac{7}{25}$ & $\theta \in QII$

Solution

$\theta \in QII$ & $\sin\theta > 0$

$-7 \quad 24 \quad \rightarrow \quad 25$

$\sin\theta = \frac{24}{25}$ $\quad\cos\theta = -\frac{7}{25}$ $\quad\tan\theta = -\frac{24}{7}$

$\csc\theta = \frac{25}{24}$ $\quad\sec\theta = -\frac{25}{7}$ $\quad\cot\theta = -\frac{7}{24}$

Exercise

Find the remaining trigonometric functions of θ if $\sin\theta = -\frac{7}{25}$ & $\theta \in QIII$

Solution

$\theta \in QIII$ & $\cos\theta < 0$

$-24 \quad -7 \quad \rightarrow \quad 25$

$\sin\theta = -\frac{7}{25}$ $\qquad \cos\theta = -\frac{24}{25}$ $\qquad \tan\theta = \frac{7}{24}$

$\csc\theta = -\frac{25}{7}$ $\qquad \sec\theta = -\frac{25}{24}$ $\qquad \cot\theta = \frac{24}{7}$

Exercise

Find the remaining trigonometric functions of θ if $\sin\theta = -\frac{24}{25}$ & $\theta \in \text{QIV}$

Solution

$\theta \in QIV$ & $\cos\theta > 0$

$7 \quad -24 \quad \rightarrow \quad 25$

$\sin\theta = -\frac{24}{25}$ $\qquad \cos\theta = \frac{7}{25}$ $\qquad \tan\theta = -\frac{24}{7}$

$\csc\theta = -\frac{25}{24}$ $\qquad \sec\theta = \frac{25}{7}$ $\qquad \cot\theta = -\frac{7}{24}$

Exercise

If $\sin\theta = -\frac{5}{13}$, and θ is **Q**III, find $\cos\theta$ and $\tan\theta$.

Solution

$\sin\theta = -\frac{5}{13} = \frac{y}{r} \quad \rightarrow y = -5, \ r = 13$

$\Rightarrow x = \pm\sqrt{13^2 - 5^2} = \pm 12$ Since θ is Q III $\Rightarrow x = -12$ $\qquad x = \pm\sqrt{r^2 - y^2}$

$\cos\theta = -\frac{12}{13}$

$\tan\theta = \frac{5}{12}$

Exercise

If $\cos\theta = \frac{3}{5}$, and θ is **Q**IV, find $\sin\theta$ and $\tan\theta$.

Solution

$$\cos\theta = \frac{3}{5} = \frac{x}{r} \quad (\theta \in QIV) \quad \Rightarrow \boxed{x=3} \qquad y = -4$$

$$\sin\theta = -\frac{4}{5}, \quad \tan\theta = -\frac{4}{3}$$

Exercise

Use the reciprocal identities if $\cos\theta = \frac{\sqrt{3}}{2}$ find $\sec\theta$

Solution

$$\sec\theta = \frac{1}{\cos\theta}$$

$$= \frac{2}{\sqrt{3}}$$

$$= \frac{2\sqrt{3}}{3}$$

Exercise

Find $\cos\theta$, given that $\sec\theta = \frac{5}{3}$

Solution

$$\cos\theta = \frac{1}{\sec\theta}$$

$$= \frac{1}{\frac{5}{3}}$$

$$= \frac{3}{5}$$

Exercise

Find $\sin\theta$, given that $\csc\theta = -\frac{\sqrt{12}}{2}$

Solution

$$\sin\theta = \frac{1}{\csc\theta}$$

$$= -\frac{2}{\sqrt{12}}\frac{\sqrt{12}}{\sqrt{12}}$$

$$= -\frac{2\sqrt{12}}{12}$$

$$\boxed{= -\frac{\sqrt{12}}{6}}$$

Exercise

Use a ratio identity to find $\tan\theta$ if $\sin\theta = \frac{3}{5}$ and $\cos\theta = -\frac{4}{5}$

Solution

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad = \frac{\frac{3}{5}}{-\frac{4}{5}}$$

$$\boxed{= -\frac{3}{4}}$$

Exercise

If $\cos\theta = -\frac{1}{2}$ and θ terminates in ***Q*II**, find $\sin\theta$

Solution

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

$$= \sqrt{1-\frac{1}{4}}$$

$$= \sqrt{\frac{3}{4}}$$

$$\boxed{= \frac{\sqrt{3}}{2}}$$

Exercise

If $\sin\theta = \frac{3}{5}$ and θ terminated in ***Q*II**, find $\cos\theta$ and $\tan\theta$.

Solution

$$\boxed{\cos\theta = -\frac{4}{5}} \qquad (3,\ 4 \to 5)$$

$$\boxed{\tan\theta = -\frac{3}{4}}$$

Exercise

Find $\tan\theta$ if $\sin\theta = \frac{1}{3}$ and θ terminates in QI

Solution

$$\cos\theta = \sqrt{1-\sin^2\theta}$$

$$= \sqrt{1-\frac{1}{9}}$$

$$= \sqrt{\frac{8}{9}}$$

$$\boxed{= \frac{2\sqrt{2}}{3}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{3}}{\frac{2\sqrt{2}}{3}}$$

$$= \frac{1}{2\sqrt{2}}$$

$$\boxed{= \frac{\sqrt{2}}{4}}$$

Exercise

Find the remaining trigonometric ratios of θ, if $\sec\theta = -3$ and $\theta \in QIII$

Solution

$$\sec\theta = \frac{1}{\cos\theta} = -3$$

$$\boxed{\cos\theta = -\frac{1}{3}}$$

$$\sin\theta = -\sqrt{1-\cos^2\theta}$$

$$= -\sqrt{1-\frac{1}{9}}$$

$$= -\sqrt{\frac{8}{9}}$$

$$\boxed{= -\frac{2\sqrt{2}}{3}}$$

$$\boxed{\tan\theta = 2\sqrt{2}}$$

$$\cot\theta = \frac{1}{2\sqrt{2}}$$

$$= \boxed{\frac{\sqrt{2}}{4}}$$

$$\csc\theta = -\frac{3}{2\sqrt{2}} \qquad\qquad \csc\theta = \frac{1}{\sin\theta}$$

$$= \boxed{-\frac{3\sqrt{2}}{4}}$$

Exercise

Using the calculator and rounding your answer to the nearest hundredth, find the remaining trigonometric ratios of θ if $\csc\theta = -2.45$ and $\theta \in QIII$

Solution

$$\sin\theta = \frac{1}{-2.45} \qquad\qquad \sin\theta = \frac{1}{\csc\theta}$$

$$= -\frac{100}{245}$$

$$= \boxed{-\frac{20}{49}}$$

$$= \boxed{-0.41}$$

$$\cos\theta = -\sqrt{1-\sin^2\theta}$$

$$= -\sqrt{1-.41^2}$$

$$= \boxed{-0.91}$$

$$\tan\theta = \frac{-0.41}{-0.91}$$

$$= \boxed{\frac{41}{91}}$$

$$= \boxed{0.45}$$

$$\cot\theta = \frac{1}{0.45}$$

$$= \frac{100}{45}$$

$$= \boxed{\frac{20}{9}}$$

$$= \boxed{2.22}$$

$$\sec\theta = \frac{1}{-0.91}$$

$$= \boxed{-\frac{100}{91}}$$

$$= \boxed{-1.1}$$

Exercise

Write $\dfrac{\sec\theta}{\csc\theta}$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify if possible.

Solution

$$\frac{\sec\theta}{\csc\theta}=\frac{\frac{1}{\cos\theta}}{\frac{1}{\sin\theta}}$$

$$=\frac{1}{\cos\theta}\frac{\sin\theta}{1}$$

$$\boxed{=\frac{\sin\theta}{\cos\theta}}$$

Exercise

Write $\cot\theta-\csc\theta$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify if possible.

Solution

$$\cot\theta-\csc\theta=\frac{\cos\theta}{\sin\theta}-\frac{1}{\sin\theta}$$

$$\boxed{=\frac{\cos\theta-1}{\sin\theta}}$$

Exercise

Write $\dfrac{\sin\theta}{\cos\theta}+\dfrac{1}{\sin\theta}$ in terms of $\sin\theta$ and/or $\cos\theta$, and then simplify if possible.

Solution

$$\frac{\sin\theta}{\cos\theta}+\frac{1}{\sin\theta}\ \boxed{=\frac{\sin^2\theta+\cos\theta}{\cos\theta\sin\theta}}$$

Exercise

Write $\sin\theta\cot\theta+\cos\theta$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify if possible.

Solution

$$\sin\theta\cot\theta+\cos\theta=\sin\theta\frac{\cos\theta}{\sin\theta}+\cos\theta$$

$$=\cos\theta+\cos\theta$$

$$\boxed{=2\cos\theta}$$

Exercise

Multiply $(1-\cos\theta)(1+\cos\theta)$

Solution

$$(1-\cos\theta)(1+\cos\theta) = 1-\cos^2\theta$$
$$= \sin^2\theta$$

Exercise

Multiply $(\sin\theta+2)(\sin\theta-5)$

Solution

$$(\sin\theta+2)(\sin\theta-5) = \sin^2\theta - 3\sin\theta - 10$$

Exercise

Simplify the expression $\sqrt{25-x^2}$ as much as possible after substituting $5\sin\theta$ for x.

Solution

$$\sqrt{25-x^2} = \sqrt{25-(5\sin\theta)^2}$$
$$= \sqrt{25-25\sin^2\theta}$$
$$= \sqrt{25(1-\sin^2\theta)}$$
$$= \sqrt{25}\sqrt{\cos^2\theta}$$
$$= 5\cos\theta$$

Exercise

Simplify the expression $\sqrt{4x^2+16}$ as much as possible after substituting $2\tan\theta$ for x

Solution

$$\sqrt{4x^2+16} = \sqrt{4(2\tan\theta)^2+16}$$
$$= \sqrt{16\tan^2\theta+16}$$
$$= \sqrt{16(\tan^2\theta+1)}$$
$$= 4\sqrt{\tan^2\theta+1}$$

$$= 4\sqrt{\sec^2\theta}$$

$$\boxed{= 4\sec\theta}$$

Exercise

Simplify by using the table. $5\sin^2 30°$

Solution

$$5\sin^2 30° = 5\left(\frac{1}{2}\right)^2$$

$$\boxed{= \frac{5}{4}}$$

Exercise

Simplify by using the table. $\sin^2 60° + \cos^2 60°$

Solution

$$\sin^2 60° + \cos^2 60° = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2$$

$$= \frac{3}{4} + \frac{1}{4}$$

$$\boxed{= 1}$$

Exercise

Simplify by using the table. $(\tan 45° + \tan 60°)^2$

Solution

$$(\tan 45° + \tan 60°)^2 = \left(1 + \sqrt{3}\right)^2$$

$$= 1 + 3 + 2\sqrt{3}$$

$$\boxed{= 4 + 2\sqrt{3}}$$

Exercise

Find the exact value of $\csc 300°$

Solution

$$\hat{\theta} = 360° - 300° = 60° \quad \rightarrow 300° \in QIV$$

$$\csc 300^\circ = -\frac{1}{\sin 60^\circ}$$

$$= -\frac{1}{\frac{\sqrt{3}}{2}}$$

$$\boxed{= -\frac{2}{\sqrt{3}}}$$

Exercise

Find θ if $\sin\theta = -\frac{1}{2}$ and θ terminates in ***Q*III** with $0^\circ \le \theta \le 360^\circ$.

Solution

$$\hat{\theta} = \sin^{-1}\frac{1}{2} = 30^\circ$$

$$\theta \in \boldsymbol{Q}\text{III}$$

$$\Rightarrow \theta = 180^\circ + 30^\circ$$

$$\boxed{= 210^\circ}$$

Exercise

Find θ to the nearest degree if $\sec\theta = 3.8637$ and θ terminates in QIV with $0^\circ \le \theta \le 360^\circ$.

Solution

$$\sec\theta = 3.8637 = \frac{1}{\cos\theta}$$

$$\cos\theta = \frac{1}{3.8637}$$

$$\hat{\theta} = \cos^{-1}\frac{1}{3.8637} \qquad \textit{Calculator}: \ \cos^{-1}(1/3.8637)$$

$$\boxed{= 75^\circ}$$

$$\theta \in \text{QIV}$$

$$\Rightarrow \theta = 360^\circ - 75^\circ$$

$$\boxed{= 285^\circ}$$

Exercise

Find the exact value of $\cos 225°$

Solution

$\hat{\theta} = 225° - 180° = 45°$

$\rightarrow 225° \in QIII$

$\cos 225° = -\cos 45°$

$$\boxed{= -\frac{\sqrt{2}}{2}}$$

Exercise

Find the exact value of $\tan 315°$

Solution

$\hat{\theta} = 360° - 315° = 45° \qquad \rightarrow 315° \in QIV$

$\tan 315° = -\tan 45°$

$$\boxed{= -1}$$

Exercise

Find the exact value of $\cos 420°$

Solution

$\hat{\theta} = 420° - 360° = 60° \qquad \rightarrow 420° \in QI$

$\cos 420° = \cos 60°$

$$\boxed{= \frac{1}{2}}$$

Exercise

Find the exact value of $\cot 480°$

Solution

$\hat{\theta} = 480° - 360° = 120°$

$\hat{\theta} = 180° - 120° = 60° \qquad \rightarrow 480° \in QII$

$\cot 480° = -\frac{\cos 60°}{\sin 60°}$

$$=-\frac{1/2}{\sqrt{3}/2}$$

$$\boxed{=-\frac{1}{\sqrt{3}}}$$

Exercise

Use the calculator to find the value of $\csc 166.7^\circ$

Solution

$$\csc 166.7^\circ = \frac{1}{\sin 166.7^\circ}$$

$$\boxed{\approx 4.3469}$$

Exercise

Use the calculator to find the value of $\sec 590.9^\circ$

Solution

$$\sec 590.9^\circ = \frac{1}{\cos 590.9^\circ}$$

$$\boxed{\approx -1.5856}$$

Exercise

Use the calculator to find the value of $\tan 195^\circ\ 10'$

Solution

$$\tan\left(195^\circ\ 10'\right) = \tan\left(195^\circ + \frac{10}{60}\right)$$

$$= \tan 195.1667^\circ$$

$$\boxed{\approx 0.271}$$

Exercise

Use the calculator to find θ to the nearest degree if $\sin\theta = -0.3090$ with $\theta \in$ QIV with $0^\circ \le \theta \le 360^\circ$

Solution

$\hat{\theta} = \sin^{-1}(0.3090)$ Since $\theta \in$ QIV

$$\boxed{\approx 18.0^\circ}$$

$$\theta = 180^\circ + 40.0^\circ$$

$$\boxed{= 220.0^\circ}$$

Exercise

Use the calculator to find θ to the nearest degree if $\cos\theta = -0.7660$ with $\theta \in \boldsymbol{Q}\text{III}$ with $0° \le \theta \le 360°$

Solution

$$\hat{\theta} = \cos^{-1}(0.7660) \qquad \text{Since } \theta \in \boldsymbol{Q}\text{III}$$
$$\approx 40.0°$$
$$\theta = 180° + 40.0°$$
$$\underline{= 220.0°}$$

Exercise

Use the calculator to find θ to the nearest degree if $\sec\theta = -3.4159$ with $\theta \in \boldsymbol{Q}\text{II}$ with $0° \le \theta \le 360°$

Solution

$$\sec\theta = -3.4159$$
$$\cos\theta = -\frac{1}{3.4159}$$
$$\hat{\theta} = \cos^{-1}\left(\frac{1}{3.4159}\right) \qquad \text{Since } \theta \in \boldsymbol{Q}\text{II}$$
$$\underline{\approx 73.0°}$$
$$\theta \approx 180° - 73.0°$$
$$\underline{= 107.0°}$$

Exercise

Find θ to the nearest tenth of a degree if $\tan\theta = -0.8541$ and θ terminates in $\boldsymbol{Q}\text{IV}$ with $0° \le \theta \le 360°$.

Solution

$$\hat{\theta} = \tan^{-1} 0.8541 \qquad \theta \in \boldsymbol{Q}\text{IV}$$
$$\underline{\approx 40.5°}$$
$$\Rightarrow \theta = 360° - 40.5°$$
$$\underline{\approx 319.5°}$$

Exercise

Use the calculator to find θ to the nearest degree if $\sin\theta = 0.49368329$ with $\theta \in \boldsymbol{Q}\text{II}$ with $0° \le \theta < 360°$

Solution

$$\hat{\theta} = \sin^{-1} 0.49368329 \qquad \theta \in \boldsymbol{Q}\text{II}$$

$$= 29.6°$$

$$\Rightarrow \theta = 180° - 29.6°$$

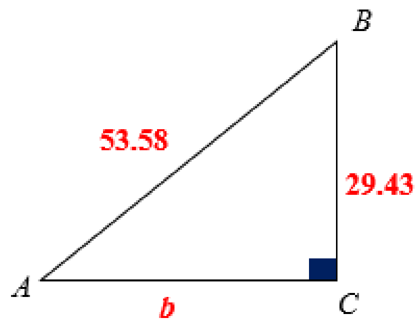
$$= 150.4°$$

Exercise

In the right triangle *ABC*, $a = 29.43$ and $c = 53.58$. Find the remaining side and angles.

Solution

$$c^2 = a^2 + b^2$$
$$b^2 = c^2 - a^2$$
$$b = \sqrt{53.58^2 - 29.43^2}$$
$$\approx 44.77$$



$$\sin A = \frac{29.43}{53.58}$$
$$A = \sin^{-1}\left(\frac{29.43}{53.58}\right)$$
$$\approx 33.32°$$

$$B = 90° - A$$
$$= 90° - 33.32°$$
$$\approx 56.68°$$

Exercise

In the right triangle *ABC*, $a = 2.73$ and $b = 3.41$. Find the remaining side and angles.

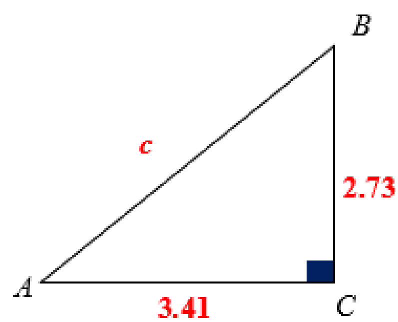
Solution

$$c^2 = a^2 + b^2$$
$$c = \sqrt{2.73^2 + 3.41^2} = 4.37$$

$$\tan A = \frac{a}{b} = \frac{2.73}{3.41}$$
$$A = \tan^{-1}\left(\frac{2.73}{3.41}\right) = 38.7°$$

and

$$\sin A = \frac{a}{c} = \frac{2.73}{4.37}$$
$$A = \sin^{-1}\left(\frac{2.73}{4.37}\right) \approx 38.7°$$

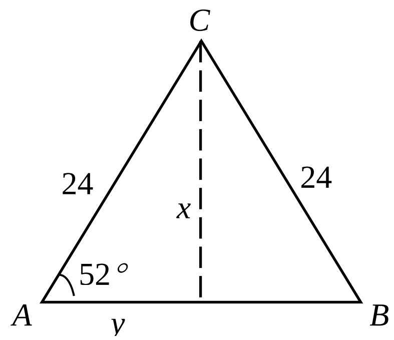


$$B = 90° - A$$
$$= 90° - 38.7°$$
$$\approx 51.3°$$

Exercise

The two equal sides of an isosceles triangle are each 24 *cm*. If each of the two equal angles measures 52°, find the length of the base and the altitude.

Solution



$$\sin 52° = \frac{x}{24}$$

$$x = 24 \sin 52°$$

$$\underline{x \approx 19\ cm}$$

$$\cos 52° = \frac{y}{24}$$

$$y = 24 \cos 52°$$

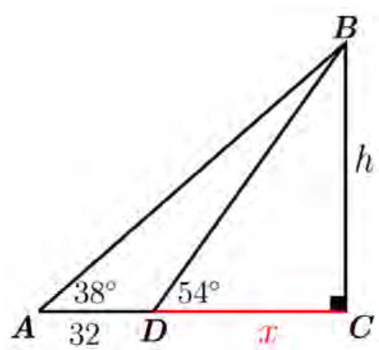
$$\underline{y \approx 15\ cm}$$

$$\Rightarrow \quad \underline{AB = 2y \approx 30\ cm}$$

Exercise

The distance from *A* to *D* is 32 *feet*. Use the figure to solve *x*, the distance between *D* and *C*.

Solution



Triangle ***DCB***

$$\tan 54° = \frac{h}{x}$$

$$\rightarrow \quad h = x \tan 54°$$

Triangle ***ACB***

$$\tan 38° = \frac{h}{x+32}$$

$$\rightarrow \quad h = (x+32) \tan 38°$$

$$h = x \tan 54° = (x+32) \tan 38°$$

$$x \tan 54° = x \tan 38° + 32 \tan 38°$$

$$x \tan 54° - x \tan 38° = 32 \tan 38°$$

$$x(\tan 54° - \tan 38°) = 32 \tan 38°$$

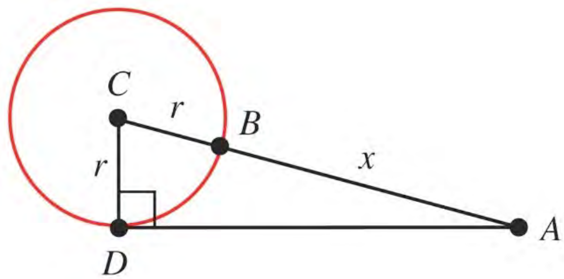
$$x = \frac{32 \tan 38°}{\tan 54° - \tan 38°}$$

$$\underline{= 42\ ft}$$

Exercise

If C = 26° and r = 19, find x.

Solution

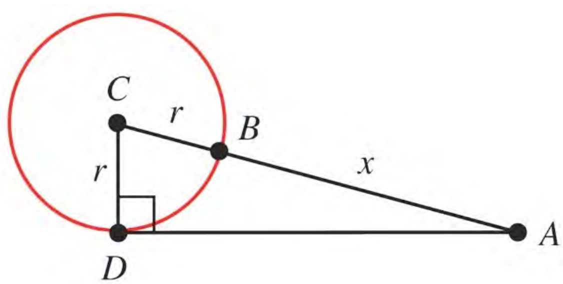


$$\cos 26° = \frac{r}{r+x}$$
$$= \frac{19}{19+x}$$
$$(19+x)\cos 26° = 19$$
$$19\cos 26° + x\cos 26° = 19$$
$$x\cos 26° = 19 - 19\cos 26°$$
$$x = \frac{19-19\cos 26°}{\cos 26°}$$
$$\approx 2.14$$

Exercise

If C = 30° and r = 15, find x.

Solution

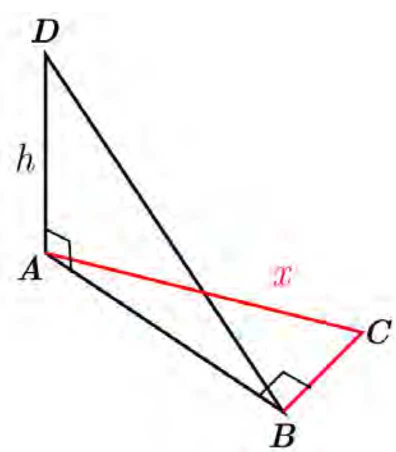


$$\cos 30° = \frac{r}{r+x}$$
$$= \frac{15}{15+x}$$
$$(15+x)\frac{\sqrt{3}}{2} = 15$$
$$15+x = \frac{30}{\sqrt{3}}$$
$$x = 10\sqrt{3} - 15$$

Exercise

If ∠ABD = 53°, C = 48°, and BC = 42, find x and then find h.

Solution

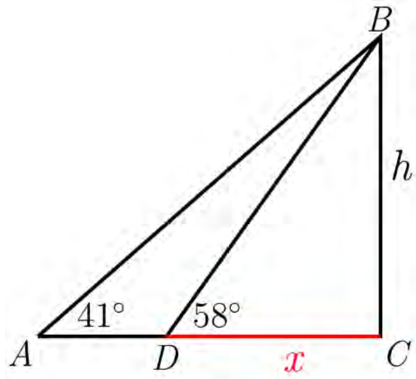


$$\tan 48° = \frac{x}{42}$$
$$x = 42\tan 48°$$
$$= 46.65 \approx 47$$
$$\tan 53° = \frac{h}{x}$$
$$h = 47\tan 53°$$
$$\approx 62$$

Exercise

If $A = 41°$, $\angle BDC = 58°$, and $AB = 28$, find $\boldsymbol{h}$, then $\boldsymbol{x}$.

Solution



$$\sin 41° = \frac{h}{AB}$$

$$h = 28\sin 41°$$

$$\approx 18$$

$$\tan 58° = \frac{h}{x}$$

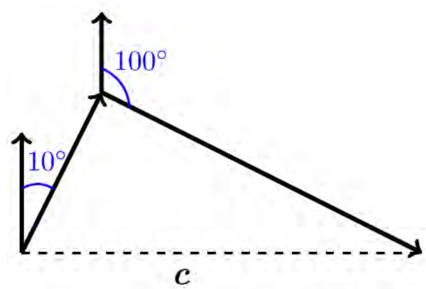
$$x = \frac{18}{\tan 58°}$$

$$\approx 11$$

Exercise

A plane flies 1.7 *hours* at 120 *mph* on a bearing of 10°. It then turns and flies 9.6 *hours* at the same speed on a bearing of 100°. How far is the plane from its starting point?

Solution



$$b = 120\frac{mi}{hr}1.7hrs$$

$$= 204\ mi$$

$$a = 120\frac{mi}{hr}9.6hrs$$

$$= 1152\ mi$$

The triangle is right triangle.

$$c = \sqrt{a^2 + b^2}$$

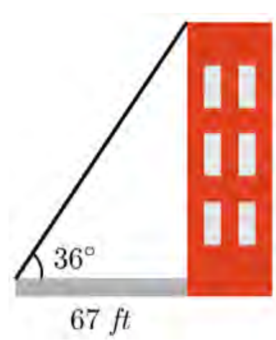
$$= \sqrt{1152^2 + 204^2}$$

$$\approx 1170\ mi$$

Exercise

The shadow of a vertical tower is 67.0 *feet* long when the angle of elevation of the sun is 36.0°. Find the height of the tower.

Solution



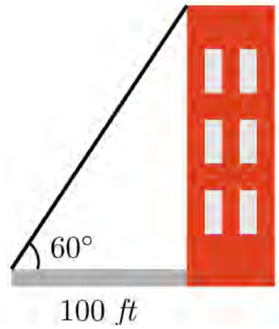
$$\tan 36° = \frac{h}{67}$$

$$h = 67\tan 36°$$

$$\approx 48.7\ ft$$

Exercise

The shadow of a vertical tower is 100 *feet* long when the angle of elevation of the sun is 60°. Find the height of the tower.



Solution

$$\tan 60^\circ = \frac{h}{100}$$

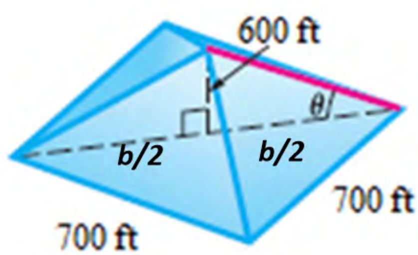
$$h = 100\tan 60^\circ$$

$$= 100\sqrt{3}\ \ ft$$

Exercise

The base of a pyramid is square with sides 700 *feet.* long, and the height of the pyramid is 600 *feet*. Find the angle of elevation of the edge indicated in the figure to two significant digits. (Hint: The base of the triangle in the figure is half the diagonal of the square base of the pyramid.)

Solution



$$b^2 = 700^2 + 700^2$$

$$b = \sqrt{2\left(700^2\right)}$$

$$= 700\sqrt{2}$$

$$\tan\theta = \frac{600}{b/2}$$

$$= \frac{600}{\frac{700\sqrt{2}}{2}}$$

$$= 600\frac{2}{700\sqrt{2}}$$

$$= \frac{6\sqrt{2}}{7}$$

$$\theta = \tan^{-1}\left(\frac{6\sqrt{2}}{7}\right)$$

$$\approx 50.48^\circ$$

Exercise

If a 73-*foot* flagpole casts a shadow 51 *feet* long, what is the angle of elevation of the sun (to the nearest tenth of a degree)?

Solution

$$\tan\theta = \frac{73}{51}$$

$$\theta = \tan^{-1}\left(\frac{73}{51}\right)$$

$$\approx 55.1^\circ$$

Exercise

If a 75-*foot* flagpole casts a shadow 43 *feet* long, to the nearest 10 *minutes* what is the angle of elevation of the sum from the tip of the shadow?

Solution

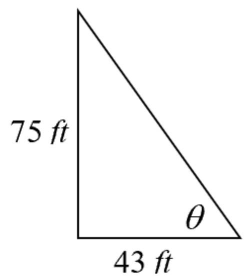
$$\tan\theta = \frac{75}{43}$$

$$\theta = \tan^{-1}\left(\frac{75}{43}\right)$$

$$= 60.17^\circ$$

$$= 60^\circ \;\; 0.17^\circ\left(\frac{60'}{1^\circ}\right)$$

$$\theta = 60^\circ \;\; 10'$$



Exercise

Suppose each edge of the cube is 3.00 *inches* long. Find the measure of the angle formed by diagonals DE and DG. *Round your answer to the nearest tenth of a degree.*

Solution

$$|DG| = \sqrt{3^2 + 3^2}$$

$$= 3\sqrt{2}$$

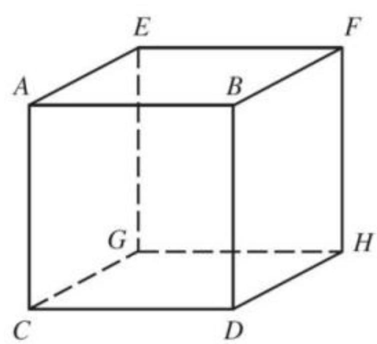
$$\tan(EDG) = \frac{EG}{GD}$$

$$= \frac{3}{3\sqrt{2}}$$

$$= \frac{\sqrt{2}}{2}$$

$$EDG = \tan^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

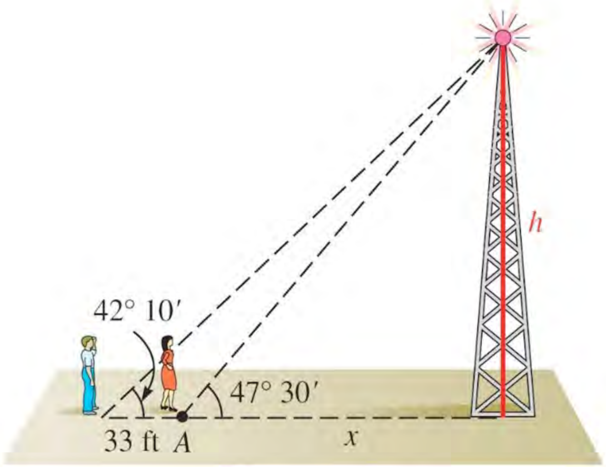
$$\angle EDG = 45^\circ$$



Exercise

A person standing at point ***A*** notices that the angle of elevation to the top of the antenna is 47° 30′. A second person standing 33.0 *feet* farther from the antenna than the person at ***A*** finds the angle of elevation to the top of the antenna to be 42° 10′. How far is the person at ***A*** from the base of the antenna?

Solution



$$47° \, 30' = 47 + 30\frac{1}{60}$$

$$= 47.5°$$

$$\tan 47.5° = \frac{h}{x}$$

$$\Rightarrow h = x \tan 47.5° \quad (1)$$

$$42° \, 10' = 42 + 10\frac{1}{60} = 42.167°$$

$$\tan 42.167° = \frac{h}{33+x}$$

$$\Rightarrow h = (33+x)\tan 42.167° \quad (2)$$

$$h = (33+x)\tan 42.167° = x\tan 47.5°$$

$$33\tan 42.167° + x\tan 42.167° = x\tan 47.5°$$

$$33\tan 42.167° = x\tan 47.5° - x\tan 42.167°$$

$$x = \frac{33\tan 42.167°}{\tan 47.5° - \tan 42.167°}$$

$$= 162 \ ft$$

Exercise

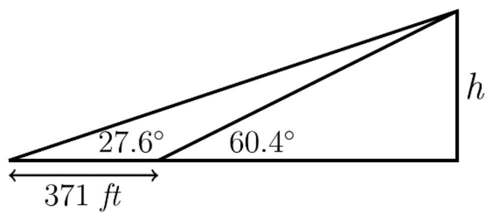
Find ***h*** as indicated in the figure.

Solution

$$h = \frac{371\tan 27.6° \tan 60.4°}{\tan 60.4° - \tan 27.6°}$$

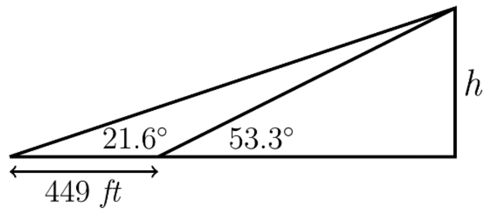
$$h = \frac{x\tan\alpha\tan\beta}{\tan\beta - \tan\alpha}$$

$$\approx 276 \ ft$$



Exercise

Find h as indicated in the figure.



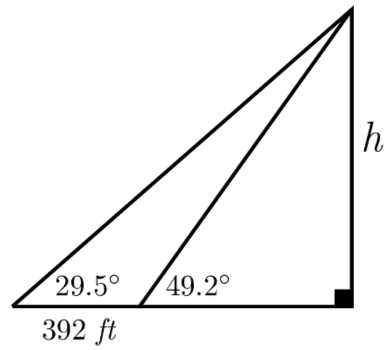
Solution

$$h = \frac{449 \tan 21.6° \tan 53.5°}{\tan 53.5° - \tan 21.6°} \qquad h = \frac{x \tan\alpha \tan\beta}{\tan\beta - \tan\alpha}$$

$$\underline{\approx 252 \quad ft}$$

Exercise

Find h as indicated in the figure.



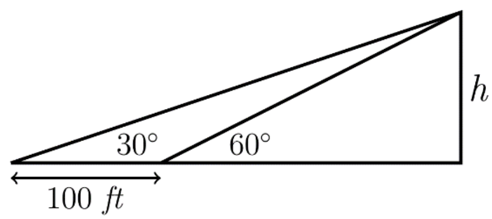
Solution

$$h = \frac{392 \tan 29.5° \ \tan 49.2°}{\tan 49.2° - \tan 29.5°} \qquad h = \frac{x \tan\alpha \tan\beta}{\tan\beta - \tan\alpha}$$

$$\underline{\approx 433.5 \quad ft}$$

Exercise

Find h as indicated in the figure.



Solution

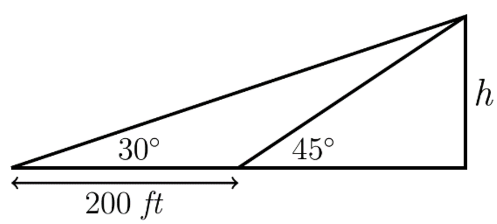
$$h = \frac{100 \tan 60° \ \tan 30°}{\tan 60° - \tan 30°} \qquad h = \frac{x \tan\alpha \tan\beta}{\tan\beta - \tan\alpha}$$

$$= \frac{100\sqrt{3}\left(\frac{1}{\sqrt{3}}\right)}{\sqrt{3} - \frac{1}{\sqrt{3}}}$$

$$\underline{= 50\sqrt{3} \ \ ft}$$

Exercise

Find h as indicated in the figure.



Solution

$$h = \frac{200 \tan 45° \ \tan 30°}{\tan 45° - \tan 30°} \qquad h = \frac{x \tan\alpha \tan\beta}{\tan\beta - \tan\alpha}$$

$$=\frac{100\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{3}}\right)}{\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}}$$

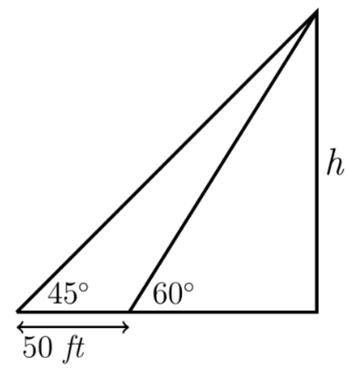
$$=\frac{100}{\sqrt{3}-\sqrt{2}}$$

$$=100\left(\sqrt{3}-\sqrt{2}\right)\ ft$$

Exercise

Find h as indicated in the figure.

Solution



$$h=\frac{50\tan 60^\circ\ \tan 45^\circ}{\tan 60^\circ-\tan 45^\circ} \qquad h=\frac{x\tan\alpha\tan\beta}{\tan\beta-\tan\alpha}$$

$$=\frac{50\sqrt{3}\left(\frac{\sqrt{2}}{2}\right)}{\sqrt{3}-\frac{\sqrt{2}}{2}}$$

$$=\frac{50\sqrt{6}}{2\sqrt{3}-\sqrt{2}}$$

$$=\frac{50\sqrt{6}}{12-2}\left(2\sqrt{3}+\sqrt{2}\right)$$

$$=25\left(2\sqrt{18}+\sqrt{12}\right)$$

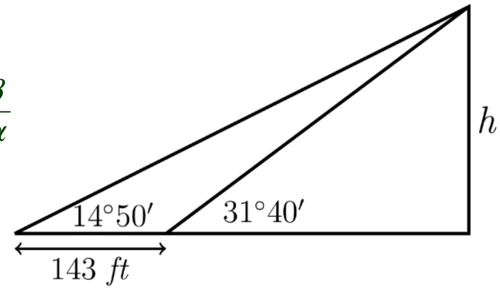
$$=25\left(6\sqrt{2}+2\sqrt{3}\right)$$

$$=50\left(3\sqrt{2}+\sqrt{3}\right)\ ft$$

Exercise

The angle of elevation from a point on the ground to the top of a pyramid is 31° 40′. The angle of elevation from a point 143 *ft* farther back to the top of the pyramid is 14° 50′. Find the height of the pyramid.

Solution



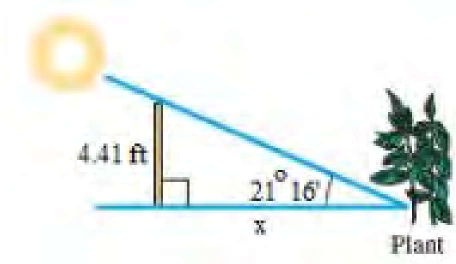
$$h=\frac{143\tan 14.833^\circ\tan 31.667^\circ}{\tan 31.667^\circ-\tan 14.833^\circ} \qquad h=\frac{x\tan\alpha\tan\beta}{\tan\beta-\tan\alpha}$$

$$\approx 66\ ft$$

Exercise

In one area, the lowest angle of elevation of the sun in winter is 21° 16′. Find the minimum distance, ***x***, that a plant needing full sun can be placed from a fence 4.41 *feet*. high.

Solution



$$\tan\left(21°16'\right) = \frac{4.41}{x}$$

$$x = \frac{4.41}{\tan\left(21° + \frac{16°}{60}\right)}$$

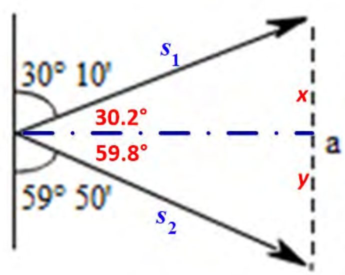
$$\boxed{\approx 11.33\, ft}$$

Exercise

A ship leaves its port and sails on a bearing of N 30° 10′ E, at speed 29.4 *mph*. Another ship leaves the same port at the same time and sails on a bearing of S 59° 50′ E, at speed 17.1 *mph*. Find the distance between the two ships after 2 *hrs*.

Solution

$$\begin{cases} 30°10' = 30° + \frac{10°}{60} \approx 30.16667° \\ 59°50' = 59° + \frac{50°}{60} \approx 59.8333° \end{cases}$$



After 2 *hours*:

$$\begin{cases} s_1 = 29.4 \frac{mi}{hr} . (2)\, hr = 58.8 \\ s_2 = 17.1 \frac{mi}{hr} . (2)\, hr = 34.2 \end{cases}$$

$$\begin{cases} \tan 30.2° = \frac{x}{s_1} \Rightarrow x = 58.8 \tan 30.2° \\ \tan 59.8° = \frac{y}{s_2} \Rightarrow y = 34.2 \tan 59.8° \end{cases}$$

$$a = x + y$$

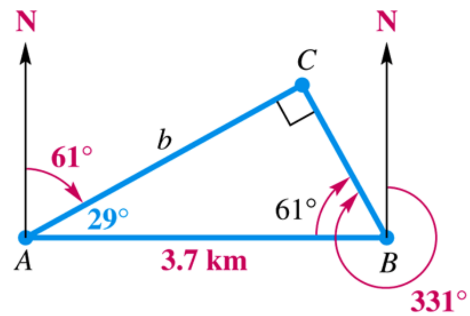
$$= 58.8 \tan 30.2° + 34.2 \tan 59.8°$$

$$\boxed{\approx 93\ miles}$$

Exercise

Radar stations A and B are on the east-west line, 3.7 *km* apart. Station A detects a place at C, on a bearing of 61°. Station B simultaneously detects the same plane, on a bearing of 331°. Find the distance from A to C.

Solution



$$A = 90° - 61° = 29°$$

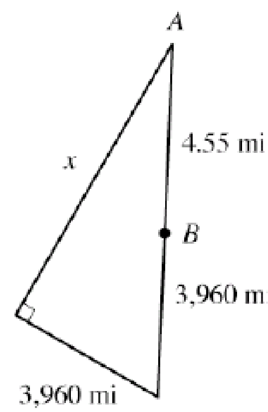
$$\cos 29° = \frac{b}{3.7}$$

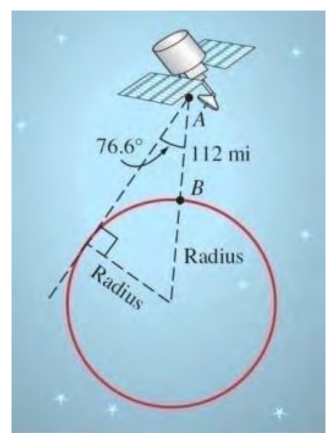
$$b = 3.7 \cos 29° \approx 3.2\ km$$

Exercise

Suppose the figure below is exaggerated diagram of a plane flying above the earth. If the plane is 4.55 *miles* above the earth and the radius of the earth is 3,960 *miles*, how far is it from the plane to the horizon? What is the measure of angle A?

Solution





$$x^2 + 3960^2 = 3964.55^2$$

$$x^2 = 3964.55^2 - 3960^2$$

$$x = \sqrt{3964.55^2 - 3960^2} \approx 190$$

The plane is 190 *miles* from the horizon.

$$\sin A = \frac{3960}{3964.55} \approx 0.9989$$

$$A = \sin^{-1}(0.9989) \approx 87.3°$$

Exercise

The Ferry wheel has a 250 *feet* diameter and 14 *feet* above the ground. If θ is the central angle formed as a rider moves from position P_0 to position P_1, find the rider's height above the ground h when θ is 45°.

Solution

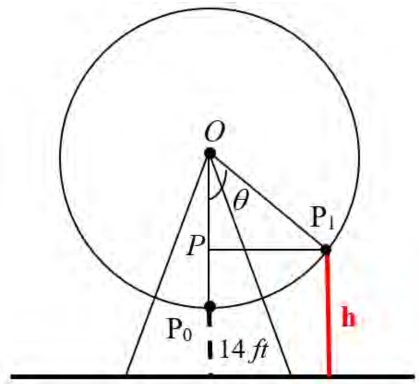
Distance between O *and* $P_0 = radius = \frac{250}{2} = 125\,ft$

$$\cos\theta = \frac{OP}{OP_1}$$

$$\cos 45° = \frac{OP}{125}$$

$$OP = 125\cos 45°$$

$$\begin{aligned} h &= PP_0 + 14 \\ &= OP_0 - OP + 14 \\ &= 125 - 125\cos 45° + 14 \\ &\approx 51\ ft \end{aligned}$$



Exercise

The length of the shadow of a building 34.09 *m* tall is 37.62 *m*. Find the angle of the elevation of the sun.

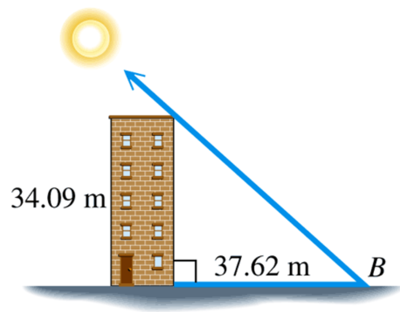
Solution

$$\tan B = \frac{34.09}{37.62}$$

$$B = \tan^{-1}\left(\frac{34.09}{37.62}\right)$$

$$\approx 42.18°$$

$\therefore$ The angle of elevation is $\approx 42.18°$



Exercise

The length of the shadow of a building 34.09 *m* tall is 37.62 *m*.
Find the angle of the elevation of the sun.

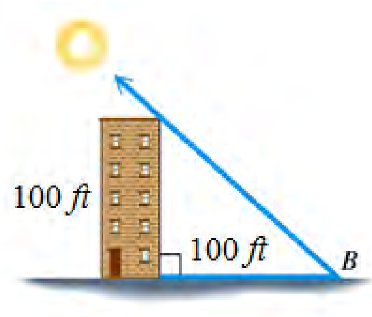
Solution

$$\tan B = \frac{100}{100}$$

$$B = \tan^{-1}(1)$$

$$= 45°$$

$\therefore$ The angle of elevation is $45°$



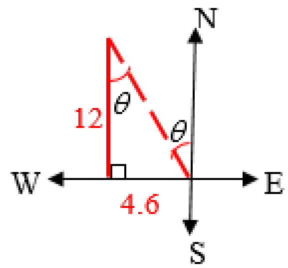
Exercise

San Luis Obispo, California is 12 *miles* due north of Grover Beach. If Arroyo Grande is 4.6 *miles* due east of Grover Beach, what is the bearing of San Luis Obispo from Arroyo Grande?

Solution

$$\tan\theta = \frac{4.6}{12}$$
$$= 0.3833$$

$$\theta = \tan^{-1} 0.3833$$
$$= 21^\circ$$



The bearing of San Luis Obispo from Arroyo Grande is N 21° W

Exercise

The bearing from A to C is S 52° E. The bearing from A to B is N 84° E. The bearing from B to C is S 38° W. A plane flying at 250 *mph* takes 2.4 hours to go from A to B. Find the distance from A to C.

Solution

$$\angle ABD = 180^\circ - 84^\circ$$
$$= 96^\circ$$

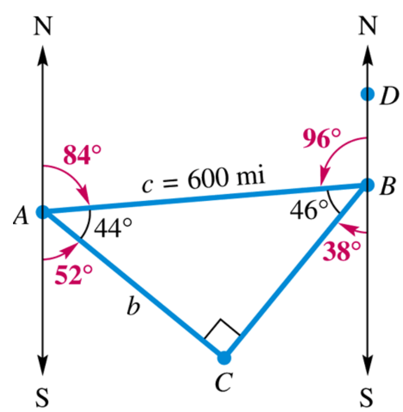
$$\angle ABC = 180^\circ - (96^\circ + 38^\circ)$$
$$= 46^\circ$$

$$\angle C = 180^\circ - (46^\circ + 44^\circ)$$
$$= 90^\circ$$

$$c = rate \times time$$
$$= 250(2.4)$$
$$= 600\ mi.$$

$$\sin 46^\circ = \frac{b}{c} = \frac{b}{600}$$

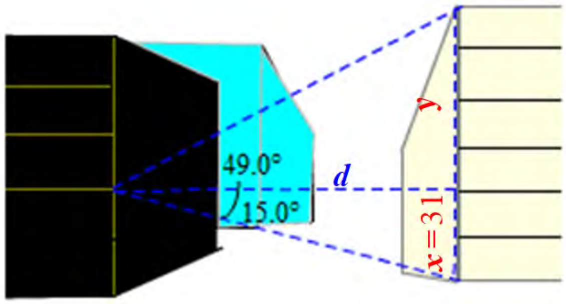
$$b = 600 \sin 46^\circ$$
$$\approx 430\ \ mi$$



Exercise

From a window 31.0 *feet*. above the street, the angle of elevation to the top of the building across the street is 49.0° and the angle of depression to the base of this building is 15.0°. Find the height of the building across the street.

Solution



$$\tan 15^\circ = \frac{31}{d}$$

$$d = \frac{31}{\tan 15^\circ}$$

$$\tan 49^\circ = \frac{y}{d}$$

$$y = \frac{31}{\tan 15^\circ}\tan 49^\circ$$

$$h = x + y$$

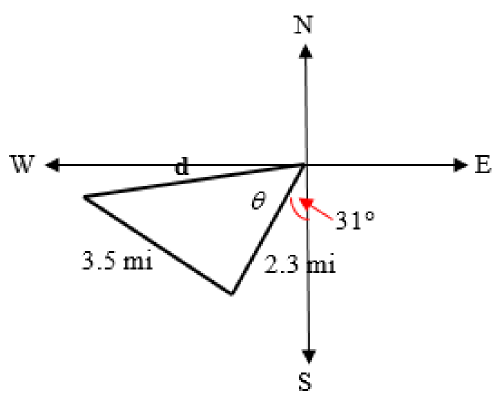
$$= 31 + \frac{31}{\tan 15^\circ}\tan 49^\circ$$

$$= 164\ ft$$

Exercise

A man wondering in the desert walks 2.3 *miles* in the direction *S* 31° *W*. He then turns 90° and walks 3.5 *miles* in the direction *N* 59° *W*. At that time, how far is he from his starting point, and what is his bearing from his starting point?

Solution



$$d = \sqrt{2.3^2 + 3.5^2} = 4.2$$

$$\cos\theta = \frac{2.3}{4.2} = .55$$

$$\theta = \cos^{-1} 0.55 \approx 57^\circ$$

S (57°+31°) W

$\rightarrow$ Bearing $S\ 88^\circ\ W$

Exercise

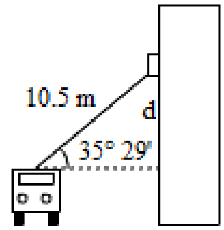
A 10.5-*m* fire truck ladder is leaning against a wall. Find the distance ***d*** the ladder goes up the wall (above the fire truck) if the ladder makes an angle of 35° 29′ with the horizontal.

Solution

$$\sin(35°29') = \frac{d}{10.5}$$

$$d = 10.5\sin\left(35° + \frac{29°}{60}\right)$$

$$= 6.1\ m$$



Exercise

A basic curve connecting two straight sections of road is often circular. In the figure, the points ***P*** and ***S*** mark the beginning and end of the curve. Let ***Q*** be the point of intersection where the two straight sections of highway leading into the curve would meet if extended. The radius of the curve is ***R***, and the central angle denotes how many degrees the curve turns.

a) If ***R*** = 965 ft. and ***θ*** = 37°, find the distance d between P and ***Q***.

b) Find an expression in terms of ***R*** and ***θ*** for the distance between points ***M*** and ***N***.

Solution

a) $\sin\frac{\theta}{2} = \frac{|PN|}{R}$

$$|PN| = 965\sin\left(\frac{37°}{2}\right)$$

$$\approx 306.2$$

$$\angle CPN = 90° - \frac{\theta}{2}$$

$$= 71.5°$$

$$\angle NPQ = 90° - \angle CPN$$

$$= 90° - 71.5°$$

$$= 18.5°$$

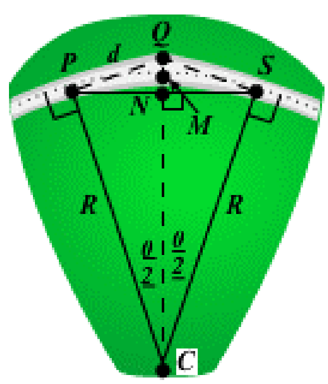
$$= \frac{\theta}{2}$$

$$\cos(NPQ) = \frac{|PN|}{d}$$

$$d = \frac{|PN|}{\cos 18.5°}$$

$$= \frac{306.2}{\cos 18.5°}$$

$$\approx 322.9$$



b) $\cos\frac{\theta}{2} = \frac{|CN|}{R}$

$$|CN| = R\cos\frac{\theta}{2}$$

$$R = |CQ| = |CM| + 2|NM|$$

$$2|NM| = R - |CM|$$

$$2|NM| = R - R\cos\frac{\theta}{2}$$

$$\boxed{|NM| = \frac{1}{2}R\left(1-\cos\frac{\theta}{2}\right)}$$

Exercise

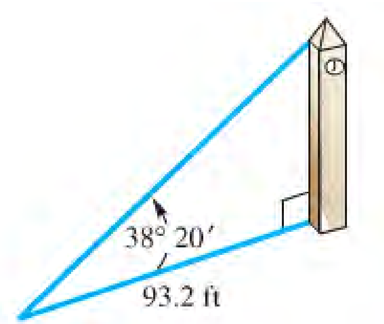
The angle of elevation from a point 93.2 *feet* from the base of a tower to the top of the tower is 38° 20′. Find the height of the tower.

Solution

$$\tan(38°\ 20') = \frac{h}{93.2}$$

$$h = 93.2\tan(38°\ 20')$$

$$\boxed{\approx 73.7\ \ ft}$$



Exercise

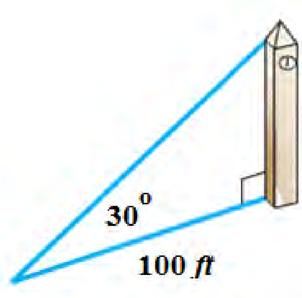
The angle of elevation from a point 100 *feet* from the base of a tower to the top of the tower is 30°. Find the height of the tower.

Solution

$$\tan(30°) = \frac{h}{100}$$

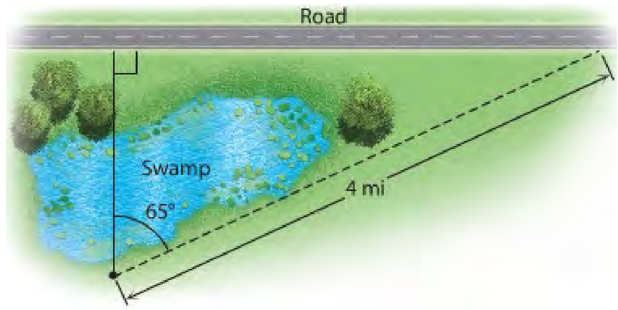
$$h = 100\tan(30°)$$

$$\boxed{= \frac{100}{\sqrt{3}}\ \ ft}$$



Exercise

Jane was hiking directly toward a long straight road when she encountered a swamp. She turned 65° to the right and hiked 4 *mi* in that direction to reach the road. How far was she forms the road when she encountered the swamp?



Solution

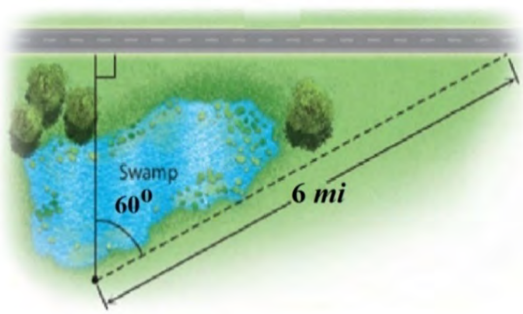
$$\cos 65° = \frac{d}{4}$$

$$d = 4\cos 65°$$

$$\approx 1.7 \ \ miles$$

Exercise

You were hiking directly toward a long straight road when you encountered a swamp. you turned 60° to the right and hiked 6 *mi* in that direction to reach the road. How far were you from the road when you encountered the swamp?



Solution

$$\cos 60° = \frac{d}{2}$$

$$d = 6\left(\frac{1}{2}\right)$$

$$= 3 \ \ miles$$

Exercise

From a highway overpass, 14.3 *m* above the road, the angle of depression of an oncoming car is measured at 18.3°. How far is the car from a point on the highway directly below the observer?

Solution

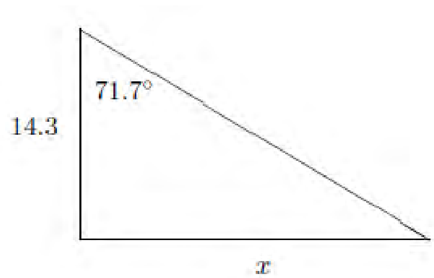
$$\alpha = 90° - 18.3°$$

$$= 71.7°$$

$$\tan(71.7°) = \frac{x}{14.3}$$

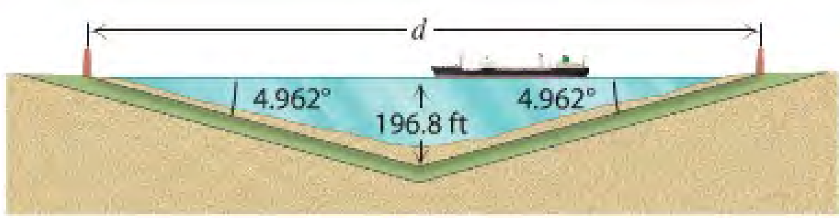
$$x = 14.3\tan(71.7°)$$

$$\approx 43.2 \ \ m$$



Exercise

A tunnel under a river is 196.8 *ft*. below the surface at its lowest point. If the angle of depression of the tunnel is 4.962° , then how far apart on the surface are the entrances to the tunnel? How long is the tunnel?



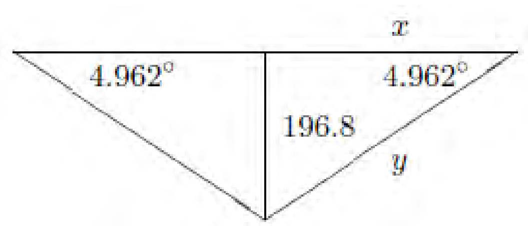
Solution

$$\tan 4.962° = \frac{196.8}{x}$$

$$x = \frac{196.8}{\tan 4.962°}$$

$$\approx 2266.75$$

$$\underline{|d} = 2x \ \underline{= 4533\ ft|}$$



$$\sin 4.962° = \frac{196.8}{y}$$

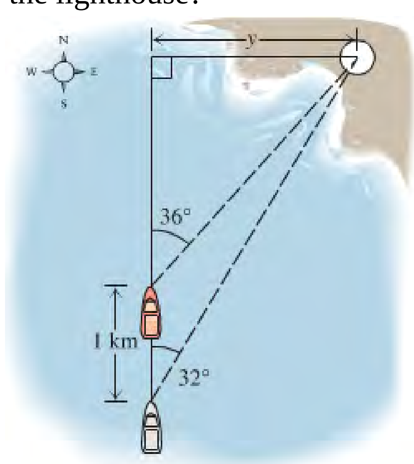
$$y = \frac{196.8}{\sin 4.962°}$$

$$\approx 2275.3$$

$\therefore$ The tunnel length: $2y = 4551$ *feet*

Exercise

A boat sailing north sights a lighthouse to the east at an angle of 32° from the north. After the boat travels one more *kilometer*, the angle of the lighthouse from the north is 36°. If the boat continues to sail north, then how close will the boat come to the lighthouse?



Solution

$\tan 36° = \dfrac{x}{y} \Rightarrow x = y\tan 36°$

$\tan 32° = \dfrac{x}{y+1} \Rightarrow x = (y+1)\tan 32°$

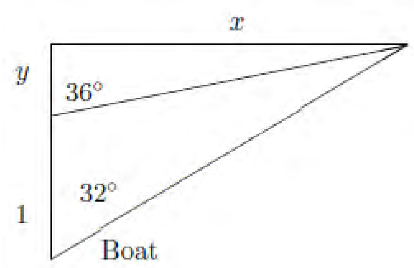
$x = y\tan 36° = (y+1)\tan 32°$

$y\tan 36° = y\tan 32° + \tan 32°$

$y\tan 36° - y\tan 32° = \tan 32°$

$y(\tan 36° - \tan 32°) = \tan 32°$

$y = \dfrac{\tan 32°}{\tan 36° - \tan 32°}$



$\Rightarrow x = y\tan 36°$

$= \dfrac{\tan 32°}{\tan 36° - \tan 32°}\tan 36°$

$\approx 4.5\ km$

∴ The closest will the boat come to the lighthouse is 4.5 *km*.

Exercise

The angle of elevation of a pedestrian crosswalk over a busy highway is 8.34°, as shown in the drawing. If the distance between the ends of the crosswalk measured on the ground is 342 *feet*., then what is the height *h* of the crosswalk at the center?

Solution

$\dfrac{342}{2} = 171$

$\tan(8.34°) = \dfrac{h}{171}$

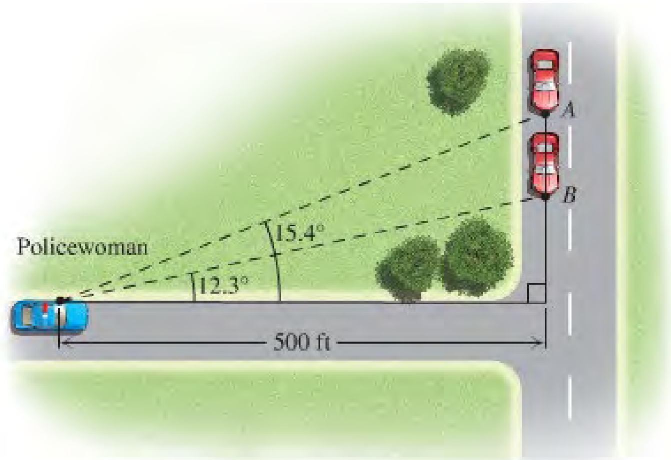
$h = 171\tan 8.34°$

$\approx 25.1\ ft$

Exercise

A policewoman has positioned herself 500 *feet*. from the intersection of two roads. She has carefully measured the angles of the lines of sight to points *A* and *B*. If a car passes from *A* to *B* is 1.75 *sec* and the speed limit is 55 *mph*, is the car speeding? (Hint: Find the distance from *B* to *A* and use *R* = *D/T*)

Solution



$$\tan 12.3° = \frac{b}{500}$$

$$b = 500\tan 12.3°$$

$$\tan 15.4° = \frac{b+a}{500}$$

$$b + a = 500\tan 15.4°$$

$$a = 500\tan 15.4° - b$$

$$= 500\tan 15.4° - 500\tan 12.3°$$

$$= 28.7\ ft\,\frac{1\ mi}{5280\ ft}$$

$$\approx 0.0054356\ mi$$

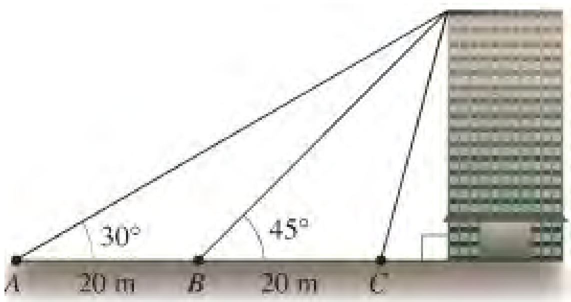
The speed is:

$$0.0054356\ mi\,\frac{1}{1.75\ \sec}\,\frac{3600\ \sec}{1\ hr} = 11.2\ \ mph$$

$\therefore$ The car is ***not*** speeding.

Exercise

From point *A* the angle of elevation to the top of the building is 30°. From point *B*, 20 meters closer to the building, the angle of elevation is 45°. Find the angle of elevation of the building from point *C*, which is another 20 *meters* closer to the building.



Solution

Let x be the distance between C and the building.

$$\tan 30° = \frac{h}{40+x}$$

$$h = (40+x)\tan 30°$$

$$= (40+x)\frac{1}{\sqrt{3}}$$

$$\tan 45° = \frac{h}{20+x}$$

$$h = (20+x)\tan 45°$$

$$= 20 + x$$

$$\Rightarrow h = \frac{1}{\sqrt{3}}(40+x) = 20+x$$

$$40 + x = 20\sqrt{3} + x\sqrt{3}$$

$$x - x\sqrt{3} = 20\sqrt{3} - 40$$

$$x\left(1-\sqrt{3}\right) = 20\sqrt{3} - 40$$

$$x = \frac{20\sqrt{3}-40}{1-\sqrt{3}} \approx 7.32$$

$$\Rightarrow h = (40+7.32)\frac{1}{\sqrt{3}} \approx 27.32$$

$$\tan C = \frac{h}{x} = \frac{27.32}{7.32}$$

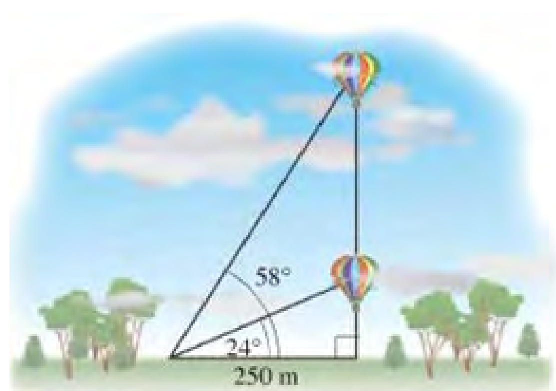
$$C = \tan^{-1}\left(\frac{27.32}{7.32}\right)$$

$$\approx 75°$$

Exercise

A hot air balloon is rising upward from the earth at a constant rate. An observer 250 *m* away spots the balloon at an angle of elevation of 24°. Two minutes later the angle of elevation of the balloon is 58°. At what rate is the balloon ascending?

Solution



$$\tan 24° = \frac{h_1}{250}$$

$$h_1 = 250\tan 24°$$

$$\tan 58° = \frac{h_2}{250}$$

$$h_2 = 250\tan 58°$$

It took 2 minutes to get from h_1 *to* h_2

$$rate = \frac{h_2 - h_1}{2}$$

$$= \frac{250\tan 58° - 250\tan 24°}{2}$$

$$\approx 144.4 \quad m/\min$$

Exercise

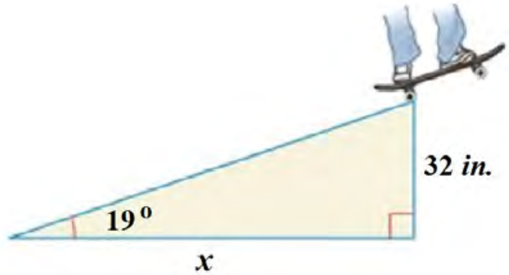
A skateboarder wishes to build a jump ramp that is inclined at a 19° angle and that has a maximum height of 32.0 *inches*. Find the horizontal width x of the ramp.

Solution

$$\tan 19^\circ = \frac{32}{x}$$

$$x = \frac{32}{\tan 19^\circ}$$

$$\approx 92.9 \ \ in$$



Exercise

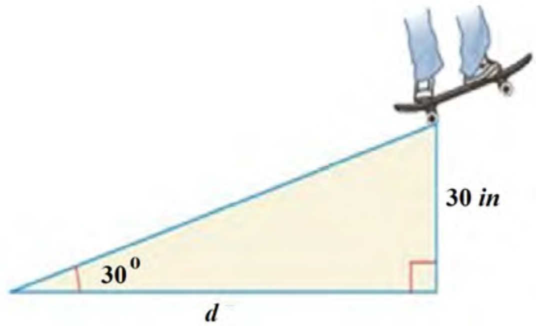
A skateboarder wishes to build a jump ramp that is inclined at a 30° angle and that has a maximum height of 30 *inches*. Find the horizontal width d of the ramp.

Solution

$$\tan 30^\circ = \frac{30}{d}$$

$$d = \frac{30}{\frac{1}{\sqrt{3}}}$$

$$= 30\sqrt{3} \ \ in.$$



Exercise

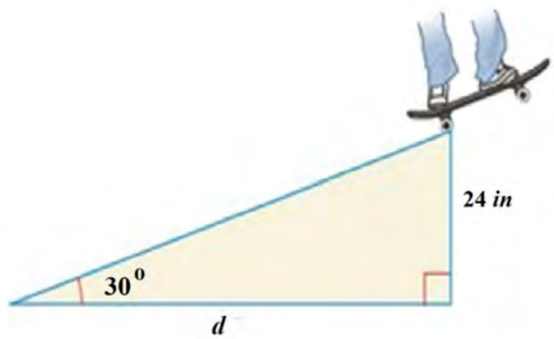
A skateboarder wishes to build a jump ramp that is inclined at a 30° angle and that has a maximum height of 24 *inches*. Find the horizontal width d of the ramp.

Solution

$$\tan 30^\circ = \frac{24}{d}$$

$$d = \frac{24}{\frac{1}{\sqrt{3}}}$$

$$= 24\sqrt{3} \ \ in.$$



Exercise

For best illumination of a piece of art, a lighting specialist for an art gallery recommends that a ceiling-mounted light be 6 *feet* from the piece of art and that the angle of depression of the light be 38°. How far from a wall should the light be placed so that the recommendations of the specialist are met? Notice that the art extends outward 4 *inches* from the wall.

Solution

$$\cos 38° = \frac{x}{6}$$

$$x = 6\cos 38°$$

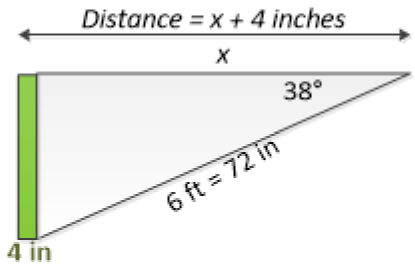
$$\approx 4.7 \ \ feet$$

$$distance = 4.7\,ft\,\frac{12\ in}{1\ ft} + 4in$$

$$= 60.7\ in$$

$$distance = \frac{60.7}{12}$$

$$\approx 5.1\ \ ft$$



Exercise

For best illumination of a piece of art, a lighting specialist for an art gallery recommends that a ceiling-mounted light be 6 *feet* from the piece of art and that the angle of depression of the light be 38°. How far from a wall should the light be placed so that the recommendations of the specialist are met? Notice that the art extends outward 4 *inches* from the wall.

Solution

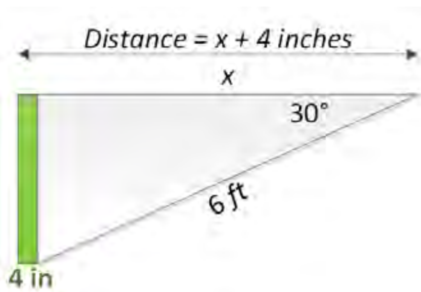
$$\cos 30° = \frac{x}{6}$$

$$x = 6\left(\frac{\sqrt{3}}{2}\right)$$

$$= 3\sqrt{3}\ \ ft$$

$$distance = 3\sqrt{3}\ ft\,\frac{12\ in}{1\ ft} + 4\ in$$

$$= 36\sqrt{3} + 4 \ \ in.$$



Exercise

A surveyor determines that the angle of elevation from a transit to the top of a building is 27.8°. The transit is positioned 5.5 *feet* above ground level and 131 *feet* from the building. Find the height of the building to the nearest tenth of a foot.

Solution

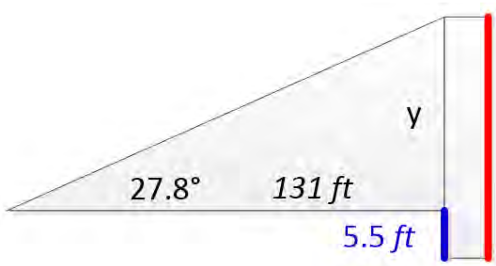
$$\tan 27.8^\circ = \frac{y}{131}$$

$$y = 131\tan 27.8^\circ$$

$$h = y + 5.5$$

$$= 131\tan 27.8^\circ + 5.5$$

$$\approx 74.6\ ft$$



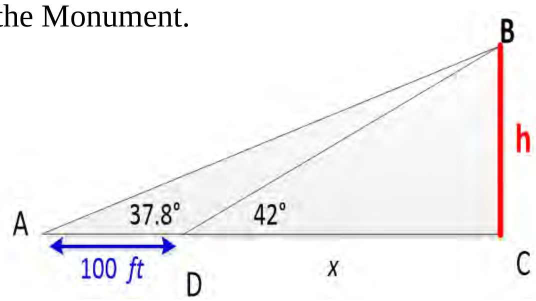
Exercise

From a point *A* on a line from the base of the Washington Monument, the angle of elevation to the top of the monument is 42.0°. From a point 100 *feet* away from *A* and on the same line, the angle to the top is 37.8°. Find the height, to the nearest foot, of the Monument.

Solution

$$\boldsymbol{h} = \frac{100\tan 37.8^\circ \tan 42^\circ}{\tan 42^\circ - \tan 37.8^\circ}$$

$$\approx 560\ ft$$



Exercise

A method that surveyors use to determine a small distance *d* between two points *P* and *Q* is called the ***subtense bar method***. The subtense bar with length *b* is centered at *Q* and situated perpendicular to the line of sight between *P* and *Q*. Angle θ is measured, then the distance d can be determined.

a) Find *d* with $\theta = 1^\circ\ 23'\ 12''$ and $b = 2.000\ cm$

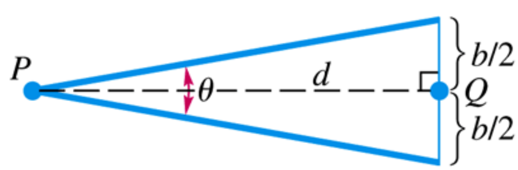
b) Angle θ usually cannot be measured more accurately than to the nearest 1″. How much change would there be in the value of *d* if θ were measured 1″ larger?

Solution

a) $\cot\frac{\theta}{2} = \frac{d}{b/2}$

$$d = \frac{b}{2}\cot\frac{\theta}{2}$$

$$\theta = 1^\circ\ 23'\ 12''$$



$$=1^\circ + \frac{23^\circ}{60} + \frac{12^\circ}{3600}$$

$$\underline{\approx 1.38667^\circ}|$$

$$d = \frac{2}{2}\cot\frac{1.38667^\circ}{2}$$

$$\underline{\approx 82.6341\ cm}|$$

b) $\theta = 1^\circ\ 23'\ 12'' + 1''$

$$= 1^\circ\ 23'\ 13''$$

$$\approx 1.386944^\circ$$

$$d = \frac{2}{2}\cot\frac{1.386944^\circ}{2}$$

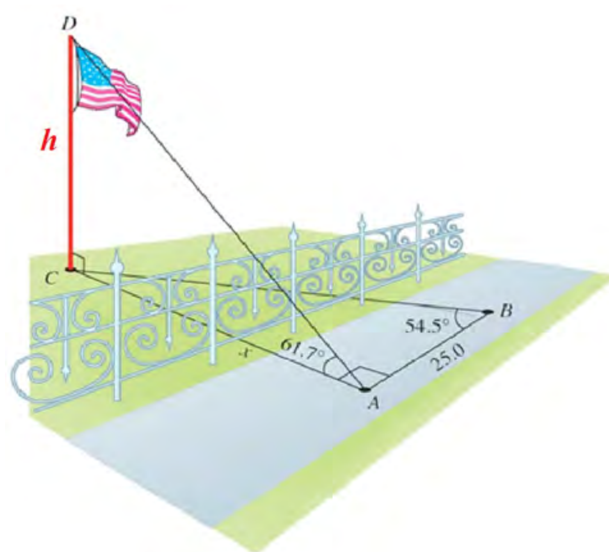
$$\approx 82.617558\ cm$$

$\therefore$ The change is: $82.6341 - 82.6175$ $\underline{\approx 0.0166\ cm}|$

Exercise

A diagram that shows how Diane estimates the height of a flagpole. She can't measure the distance between herself and the flagpole directly because there is a fence in the way. So she stands at point *A* facing the pole and finds the angle of elevation from point *A* to the top of the pole to be 61.7°. Then she turns 90° and walks 25.0 *ft* to point *B*, where she measures the angle between her path and a line from *B* to the base of the pole. She finds that angle is 54.5°. Use this information to find the height of the pole.

Solution



$$\tan 54.5^\circ = \frac{x}{25.0}$$

$$x = 25.0\tan 54.5^\circ$$

$$\underline{\approx 35.0487\ ft}|$$

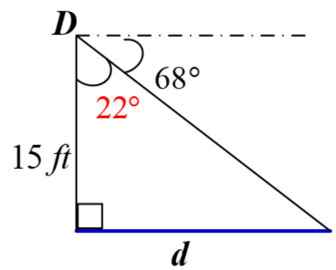
$$\tan 61.7^\circ = \frac{h}{35.0487}$$

$$h = 35.0487\tan 61.7^\circ$$

$$\underline{\approx 65.1\ ft}|$$

Exercise

From a point 15 *feet* above level ground, a surveyor measures the angle of depression of an object on the ground at 68°. Approximate the distance from the object to the point on the ground directly beneath the surveyor.



Solution

$$D = 90^\circ - 68^\circ = 22^\circ$$

$$\tan 22^\circ = \frac{d}{15}$$

$$d = 15 \tan 22^\circ$$

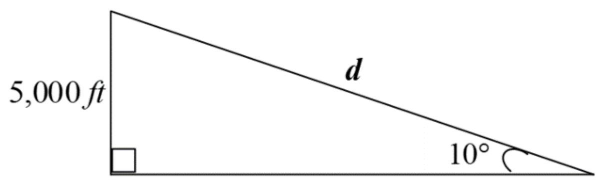
$$= 6.1 \ ft$$

Distance from the object to the point is about 6.1 *feet*.

Exercise

A pilot, flying at an altitude of 5,000 *feet* wishes to approach the numbers on a runway at an angle of 10°. Approximate, to the nearest 100 *feet*, the distance from the airplane to the numbers at the beginning of the descent.

Solution



$$\sin 10^\circ = \frac{5,000}{d}$$

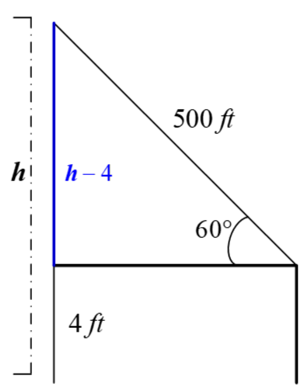
$$d = \frac{5,000}{\sin 10^\circ}$$

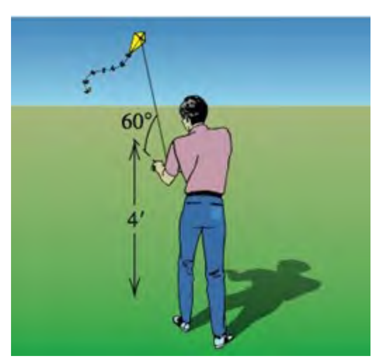
$$\approx 28,793.85$$

$$\approx 28,800 \ ft$$

Exercise

A person flying a kite holds the string 4 *feet* above ground level. The string of the kite is taut and make an angle of 60° with the horizontal. Approximate the height of the kite above level ground if 500 *feet* of sting is paved out.





Solution

$$\sin 60^\circ = \frac{h-4}{500}$$

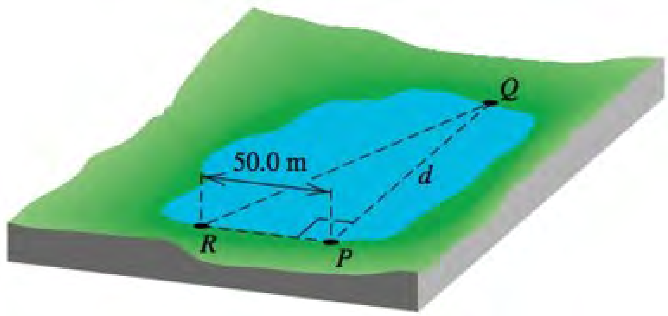
$$h - 4 = 500\frac{\sqrt{3}}{2}$$

$$h = 250\sqrt{3} + 4$$

$$\approx 437 \ ft$$

Exercise

To find the distance d between two points P and Q on opposite shores of a lake, a surveyor locates a point R that is 50.0 meters from P such that RP is perpendicular to PQ. Nest, using a transit, the surveyor measures angle PRQ as $72°\ 40'$. Find d.



Solution

Given: $\angle PRQ = 72°\ 40'$

$$\tan\left(72°\ 40'\right) = \frac{d}{50}$$

$$d = 50\tan\left(72°\ 40'\right)$$

$$\approx 160\ m$$

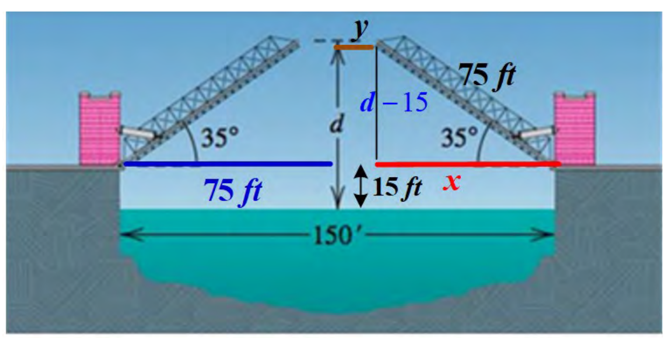
Exercise

A drawbridge is 150 *feet* long when stretched across a river. The two sections of the bridge can be rotated upward through an angle of 35°.

a) If the water level is 15 *feet* below the closed bridge, find the distance d between the end of a section and the water level when the bridge is fully open.

b) Approximately how far apart are the ends of the two sections when the bridge is fully opened?

Solution



a) $\sin 35° = \dfrac{d-15}{75}$

$$d - 15 = 75\sin 35°$$

$$d = 75\sin 35° + 15$$

$$\approx 58\ ft$$

b) $\cos 35° = \dfrac{x}{75}$

$$x = 75\cos 35°$$

$$y = 75 - 75\cos 35°$$

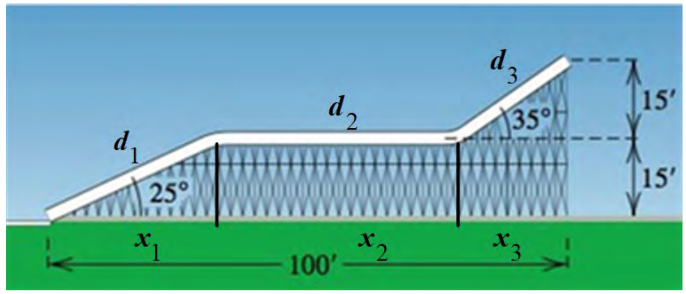
$$\approx 13.56$$

The two sections are apart: $2 \times 13.56 \approx 27\ ft$

Exercise

Find the total length of a design for a water slide to the nearest *foot*.

Solution



$$\sin 25° = \frac{15}{d_1}$$

$$d_1 = \frac{15}{\sin 25°}$$

$$\approx 35.49\ ft$$

$$\sin 35° = \frac{15}{d_3}$$

$$d_3 = \frac{15}{\sin 35°}$$

$$\approx 26.15\ ft$$

$$\tan 25° = \frac{15}{x_1} \quad \Rightarrow \quad x_1 = \frac{15}{\tan 25°} \approx 32.17\ ft$$

$$\tan 35° = \frac{15}{x_2} \quad \Rightarrow \quad x_2 = \frac{15}{\tan 35°} \approx 21.42\ ft$$

$$d_2 = 100 - x_1 - x_2$$

$$= 100 - 32.17 - 21.42$$

$$\approx 46.41\ ft$$

$$\text{Total length } = 35.49 + 26.15 + 46.41 \approx 108.05\ ft$$

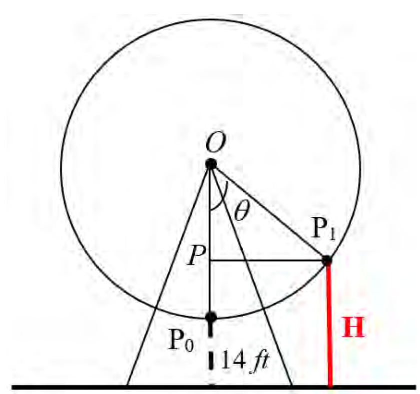
Exercise

The diameter of the Ferris wheel is 250 *feet*, the distance from the ground to the bottom of the wheel is 14 *feet*, and one complete revolution takes 20 *minutes*, find

a) The linear velocity, in miles per hour, of a person riding on the wheel.

b) The height of the rider in terms of the time *t*, where *t* is measured in minutes.

Solution



Given: $\theta = 1\ rev = 2\pi\ rad;\ t = 20\ min.;$

$$r = \frac{D}{2} = \frac{250}{2} = 125\,ft$$

a) $\omega = \dfrac{\theta}{t}$ *or* $\boxed{v = \dfrac{r\theta}{t}}$

$$= \frac{2\pi}{20}$$

$$= \frac{\pi}{10}\ rad / min$$

$$v = r\omega$$
$$= (125\,ft)\left(\frac{\pi}{10}\,rad\,/\,\min\right)$$
$$\approx 39.27\ ft\,/\,\min$$

$$v \approx 39.27\ \frac{ft}{\min}\ \frac{60\min}{1hr}\ \frac{1mile}{5{,}280\,ft}$$
$$\approx 0.45\ mi\,/\,hr$$

b) $\cos\theta = \dfrac{OP}{OP_1} = OP_0 - OP + 14$

$$= \frac{OP}{125}$$

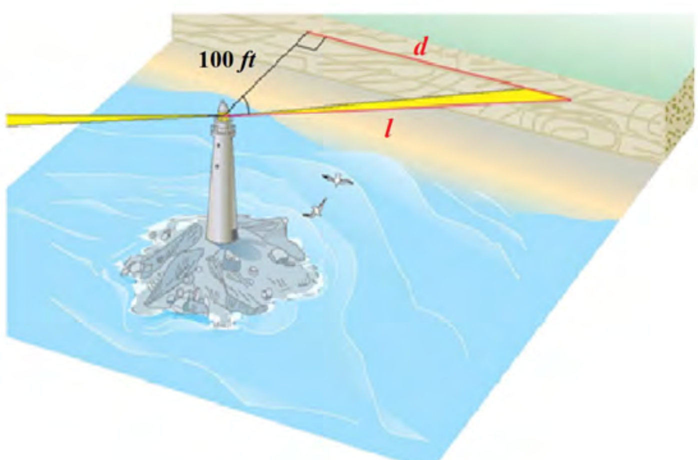
$$OP = 125\cos\theta$$
$$H = PP_0 + 14$$
$$= 125 - 125\cos\theta + 14$$
$$= 139 - 125\cos\theta$$

$$\omega = \frac{\theta}{t}$$
$$\theta = \omega\, t$$
$$\theta = \frac{\pi}{10}\ t$$
$$H(t) = 139 - 125\cos\left(\frac{\pi}{10}t\right)$$

Exercise

Find an equation that expresses l in terms of time t. Find l when t is 0.5 *sec*, 1.0 *sec*, and 1.5 *sec*. (assume the light goes through one rotation every 4 *seconds*.)

Solution



$$\omega = \frac{\theta}{t} = \frac{2\pi}{4}\frac{rad}{\sec}$$
$$= \frac{\pi}{2}\ rad\,/\,\sec$$
$$\frac{\theta}{t} = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{2}\ t$$
$$\cos\left(\frac{\pi}{2}\ t\right) = \frac{100}{l}$$
$$l\cos\left(\frac{\pi}{2}\ t\right) = 100$$
$$l = \frac{100}{\cos\left(\frac{\pi}{2}\ t\right)}$$

$$= 100\sec\left(\frac{\pi}{2}\ t\right)$$

For $t = 0.5$ *sec*

$$\rightarrow \ \underline{|l} = \frac{100}{\cos\left(\frac{\pi}{2}\frac{1}{2}\right)} = \frac{100}{\cos\left(\frac{\pi}{4}\right)}$$

$$= \frac{100}{\frac{1}{\sqrt{2}}}$$

$$\underline{= 100\sqrt{2}\ \ ft|} \qquad \underline{\approx 141\ \ ft|}$$

For $t = 1.0$ *sec*

$$l = \frac{100}{\cos\left(\frac{\pi}{2}\right)}$$

$$= \frac{100}{0}$$

$$\underline{= \textit{Undefined}\ |}$$

For $t = 1.5$ sec

$$l = \frac{100}{\cos\left(\frac{\pi}{2}\frac{3}{2}\right)}$$

$$= \frac{100}{\cos\left(\frac{3\pi}{4}\right)}$$

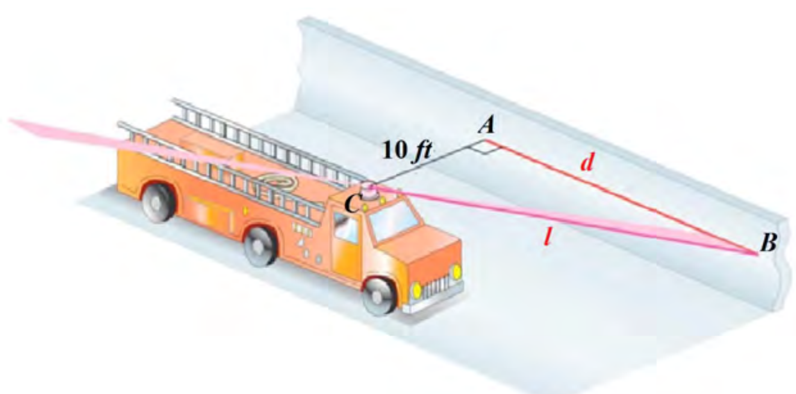
$$= -\frac{100}{\frac{1}{\sqrt{2}}}$$

$$\underline{= -100\sqrt{2}\ \ ft|} \qquad \underline{\approx -141\ \ ft|}$$

Exercise

A fire truck parked on the shoulder of a freeway next to a long block wall. The red light on the top of the truck is 10 *feet* from the wall and rotates through a complete revolution every 2 *seconds*. Find the equations that give the lengths d and ℓ in terms of time.

Solution



$$\omega = \frac{\theta}{t} = \frac{2\pi}{2} = \pi\ rad\ /\ sec$$

$$\tan\theta = \frac{d}{10}$$

$$d = 10\tan\theta$$

$$= 10\tan\pi t$$

$$\sec\theta = \frac{l}{10}$$

$$l = 10\sec\theta$$

$$\boxed{= 10\sec \pi t \quad ft}$$

Exercise

A Ferris wheel has radius 50.0 *feet*. A person takes a seat and then the wheel turns $\frac{2\pi}{3}$ *rad*.

a) How far is the person above the ground?

b) If it takes 30 *sec* for the wheel to turn $\frac{2\pi}{3}$ *rad*, what is the angular speed of the wheel?

Solution

a) $\alpha = \frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$

$$\cos\alpha = \frac{h_1}{r}$$

$$h_1 = r\cos\alpha$$

$$= 50\cos\frac{\pi}{6}$$

$$\boxed{= 25\sqrt{3} \quad ft} \quad = 43.3\ ft$$

Person is $50 + 25\sqrt{3} = 93.3\ ft$ above the ground

b) $\omega = \frac{\theta}{t} = \frac{\frac{2\pi}{3}\ rad}{30\ sec}$

$$\boxed{= \frac{\pi}{45} \quad rad\ /\ sec}$$

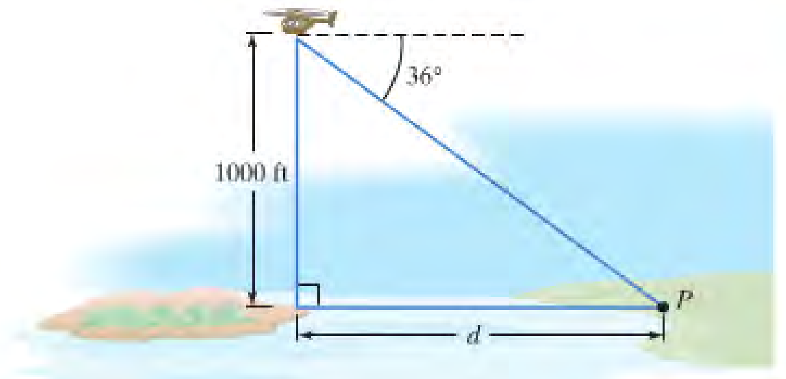
$$t = \frac{400\pi\ ft}{30\ \frac{mi}{hr}}$$

$$= \frac{40\pi}{3}\ ft\ \frac{hr}{mi}\ \frac{1mi}{5280\ ft}\ \frac{3600\sec}{1hr}$$

$$\boxed{\approx 29 \quad \sec}$$

Exercise

A helicopter hovers 1,000 *feet* above a small island. The angle of depression from the helicopter to point *P* on the coast is 36°. How far off the coast is the island?



Solution

$$\tan 36° = \frac{1,000}{d}$$

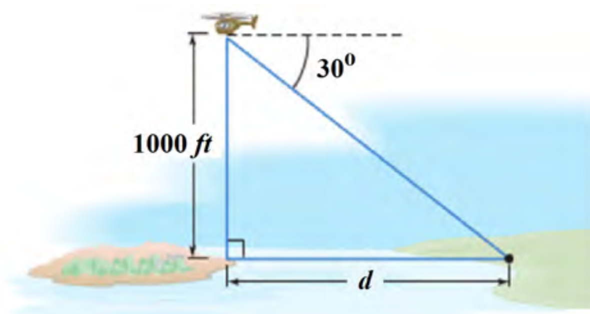
$$d = \frac{1,000}{\tan 36°}$$

$$\approx 1,376 \ \ feet$$

∴The island is approximately 1,376 *feet* off the coast.

Exercise

A helicopter hovers 1,000 *feet* above a small island. The angle of depression from the helicopter to point *P* on the coast is 30°. How far off the coast is the island?



Solution

$$\tan 30° = \frac{1,000}{d}$$

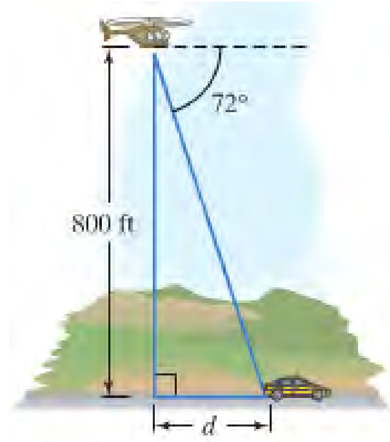
$$d = \frac{1,000}{\frac{1}{\sqrt{3}}}$$

$$= 1,000\sqrt{3} \ \ feet$$

∴The island is approximately 1,376 *feet* off the coast.

Exercise

A police helicopter is flying at 800 *feet*. A stolen car is sighted at an angle of depression of 72°. Find the distance of the stolen car from a point directly below the helicopter.



Solution

$$\tan 72^\circ = \frac{800}{d}$$

$$d = \frac{800}{\tan 72^\circ}$$

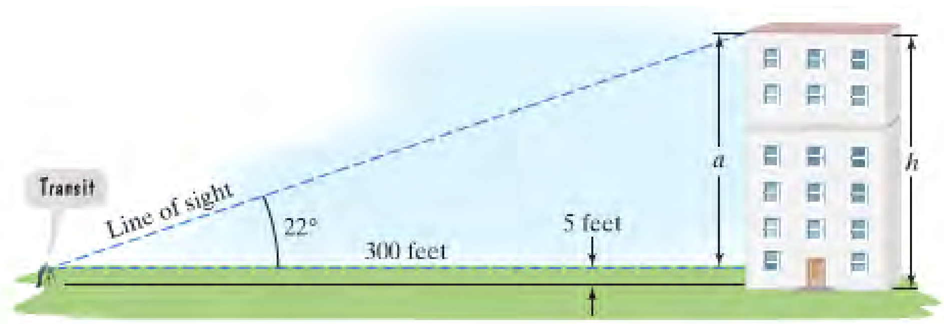
$$\approx 260 \ ft$$

∴ The stolen car is approximately 260 *feet* from a point directly below the helicopter.

Exercise

Sighting the top of a building a surveyor measured the angle of elevation to be 22°. The transit is 5 *feet* above the ground and 300 *feet* from the building. Find the building's height.

Solution



$$\tan 22^\circ = \frac{a}{300}$$

$$a = 300 \tan 22^\circ$$

$$\approx 121$$

$$h = 5 + 121$$

$$\approx 126 \ ft$$

Exercise

Sighting the top of a building a surveyor measured the angle of elevation to be 30°. The transit is 5 *feet* above the ground and 250 *feet* from the building. Find the building's height.

Solution

$$\tan 30^\circ = \frac{a}{250}$$

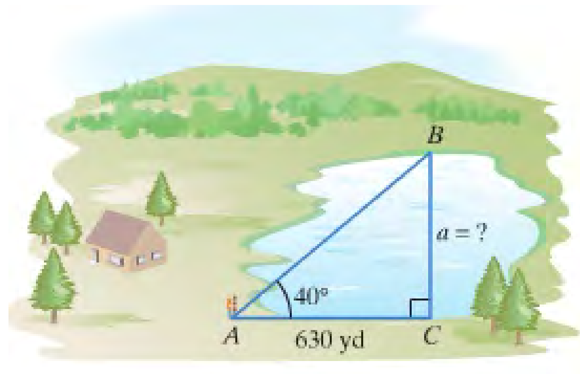
$$a = \frac{250}{\sqrt{3}}$$

$$h = 5 + \frac{250\sqrt{3}}{3}$$

$$= \frac{15 + 250\sqrt{3}}{3} \ ft$$

Exercise

Determine how far it is across the lake.



Solution

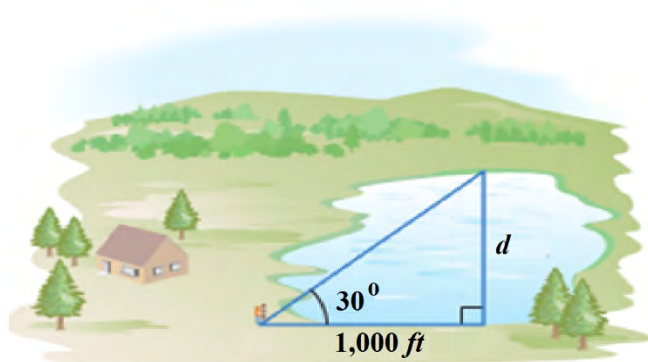
$$\tan 40^\circ = \frac{a}{630}$$

$$a = 630 \tan 40^\circ$$

$$\approx 529\ yd$$

Exercise

Determine how far it is across the lake.



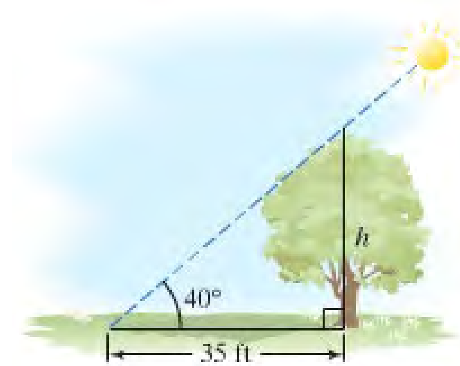
Solution

$$\tan 30^\circ = \frac{d}{1,000}$$

$$d = \frac{1,000}{\sqrt{3}}\ yd.$$

Exercise

At a certain time of day, the angle of elevation of the sun is 40°. Find the height of a tree whose shadow is 35 *feet* long.



Solution

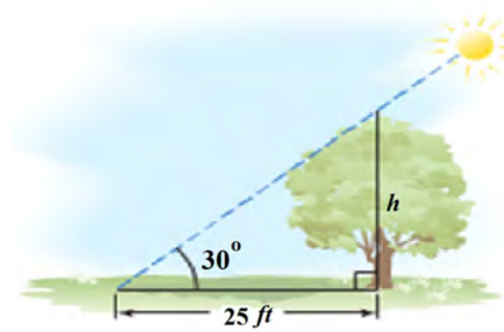
$$\tan 40^\circ = \frac{h}{35}$$

$$h = 35 \tan 40^\circ$$

$$\approx 29.4\ ft$$

Exercise

At a certain time of day, the angle of elevation of the sun is 30°. Find the height of a tree whose shadow is 25 *feet* long.



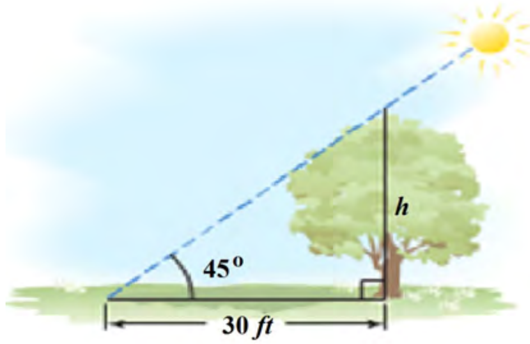
Solution

$$\tan 30^\circ = \frac{h}{25}$$

$$h = \frac{25}{\sqrt{3}}\ ft$$

Exercise

At a certain time of day, the angle of elevation of the sun is 45°.
Find the height of a tree whose shadow is 30 *feet* long.



Solution

$$\tan 45° = \frac{h}{25}$$

$$\boxed{h = \frac{25\sqrt{2}}{2}\ \ ft}$$

Exercise

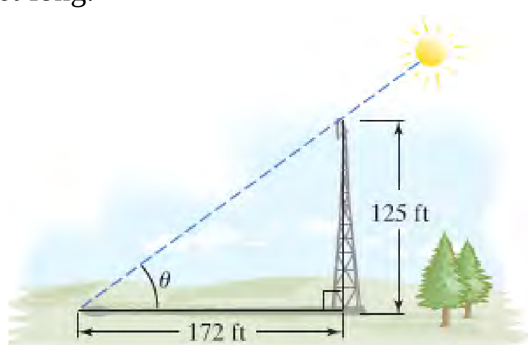
A tower that is 125 *feet* casts a shadow 172 *feet* long.
Find the angle of elevation of the sun.

Solution

$$\tan\theta = \frac{125}{172}$$

$$\theta = \tan^{-1}\left(\frac{125}{172}\right)$$

$$\boxed{\approx 36°}$$



Exercise

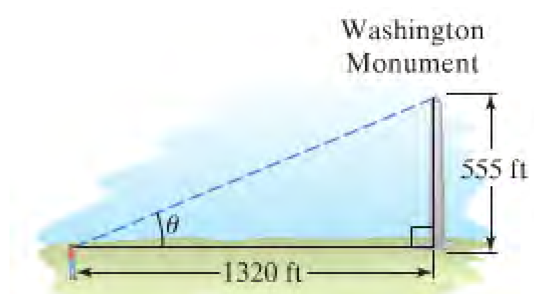
The Washington Monument is 555 *feet* high. If you are standing one quarter of a mile, or 1,320 *feet*, from the base of the monument and looking to the top, find the angle of elevation.

Solution

$$\tan\theta = \frac{555}{1320}$$

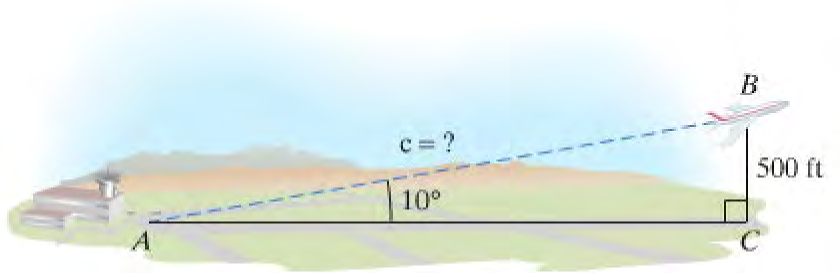
$$\theta = \tan^{-1}\left(\frac{555}{1320}\right)$$

$$\boxed{\approx 22.8°}$$



Exercise

A plane rises from take-off and flies at an angle of $10°$ with the horizontal runway. When it has gained 500 *feet*, find the distance the plane has flown.



Solution

$$\sin 10° = \frac{500}{c}$$

$$c = \frac{500}{\sin 10°}$$

$$\approx 2,879.4\ ft$$

Exercise

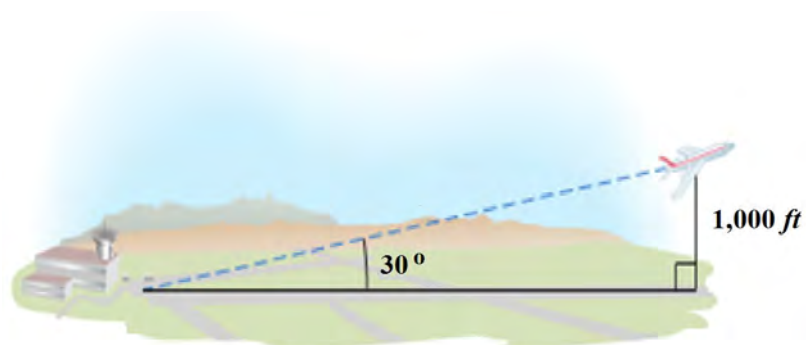
A plane rises from take-off and flies at an angle of $30°$ with the horizontal runway. When it has gained 1,000 *feet*, find the distance the plane has flown.

Solution

$$\sin 30° = \frac{1,000}{d}$$

$$d = \frac{1,000}{\frac{1}{2}}$$

$$= 2,000\ \ ft$$



Exercise

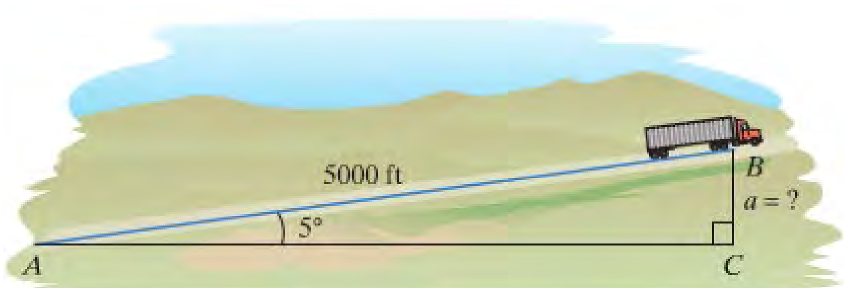
A road is inclined at an angle of 5°. After driving 5,000 *feet* along this road, find the driver's increase in altitude.

Solution

$$\sin 5° = \frac{a}{5,000}$$

$$a = 5,000 \sin 5°$$

$$\approx 435.8\ \ ft$$



Exercise

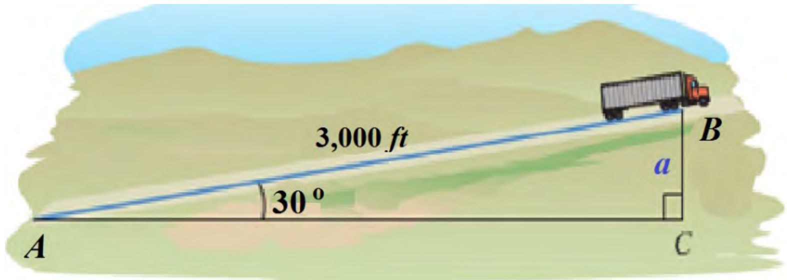
A road is inclined at an angle of 30°. After driving 3,000 *feet* along this road, find the driver's increase in altitude.

Solution

$$\sin 30^\circ = \frac{a}{3,000}$$

$$a = 3,000\left(\frac{1}{2}\right)$$

$$= 1,500 \ ft$$



Exercise

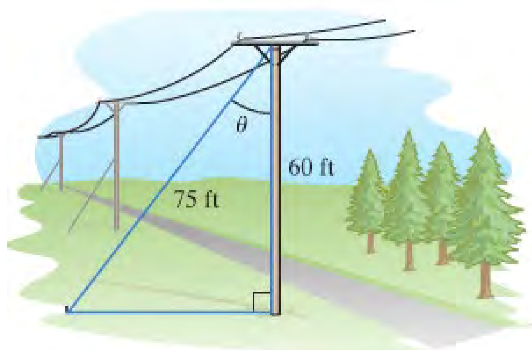
A telephone pole is 60 *feet* tall. A guy wire 75 *feet* long is attached from the ground to the top of the pole. Find the angle between the wire and the pole.

Solution

$$\cos\theta = \frac{60}{75}$$

$$\theta = \cos^{-1}\left(\frac{60}{75}\right)$$

$$\approx 37^\circ$$



Exercise

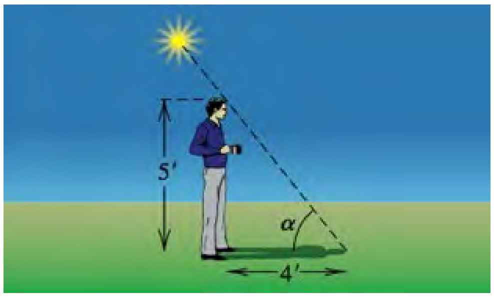
Approximate the angle of elevation α of the sun if a person 5.0 *feet* tall casts a shadow 4.0 *feet* long on level ground.

Solution

$$\tan\alpha = \frac{5}{4}$$

$$\alpha = \tan^{-1}\frac{5}{4}$$

$$\approx 51.34^\circ$$



Exercise

A spotlight with intensity 5000 candles is located 15 *feet* above a stage. If the spotlight is rotated through an angle θ, the illuminance E (in foot-candles) in the lighted area of the stage is given by

$$E = \frac{5,000\cos\theta}{s^2}$$

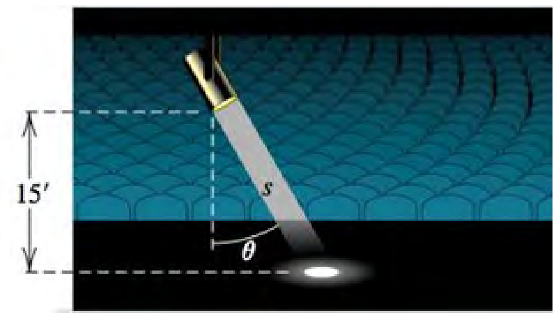
Where s is the distance (in *feet*) that the light must travel.

a) Find the illuminance if the spotlight is rotated through an angle of 30°.

b) The maximum illuminance occurs when $\theta = 0°$. For what value of θ is the illuminance one-half the maximum value.

Solution

a) $\cos\theta = \frac{15}{s} \quad \Rightarrow \quad s = \frac{15}{\cos\theta}$



$$E = \frac{5,000\cos\theta}{s^2} = 5,000\cos\theta\,\frac{\cos^2\theta}{15^2}$$

$$= \frac{200}{9}\cos^3\theta$$

$$= \frac{200}{9}\cos^3(30°)$$

$$= \frac{200}{9}\left(\frac{\sqrt{3}}{2}\right)^3$$

$$\underline{= \frac{25\sqrt{3}}{3}\ \ ft - candles} \quad \approx 14.43\ \ ft - candles$$

b) $E = \frac{1}{2}E_{\max}$

$$\frac{200}{9}\cos^3\theta = \frac{1}{2}\frac{200}{9}\cos^3 0°$$

$$\cos^3\theta = \frac{1}{2}$$

$$\cos\theta = \sqrt[3]{\frac{1}{2}}$$

$$\theta = \cos^{-1}\sqrt[3]{\frac{1}{2}}$$

$$\underline{\approx 37.47°}$$

Exercise

A conveyor belt 9 *meters* long can be hydraulically rotated up to an angle of 40° to unload cargo from airplanes.

a) Find, to the nearest degree, the angle through which the conveyor belt should be rotated up to reach a door that is 4 *meters* above the platform supporting the belt.

b) Approximate the maximum height above the platform that the belt can reach.

Solution

$$\sin\alpha = \frac{4}{9}$$

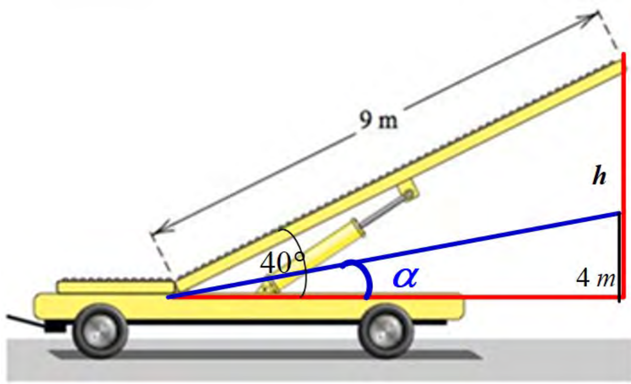
$$\alpha = \sin^{-1}\frac{4}{9}$$

$$\underline{\approx 26.4^\circ}$$

$$\sin 40^\circ = \frac{h}{9}$$

$$h = 9\sin 40^\circ$$

$$\underline{\approx 5.785\ m}$$



Exercise

A rectangular box has dimensions $8'' \times 6'' \times 4''$. Approximate, to the nearest tenth of a degree, the angle θ formed by a diagonal of the base and the diagonal of the box.

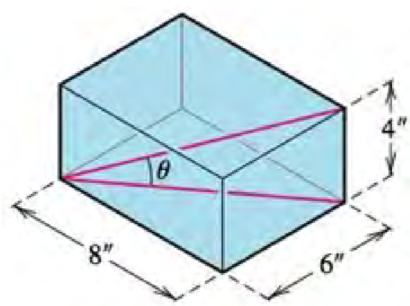
Solution

$$d = \sqrt{8^2 + 6^2}$$

$$\underline{= 10}$$

$$\theta = \tan^{-1}\frac{4}{10}$$

$$\underline{\approx 21.8^\circ}$$



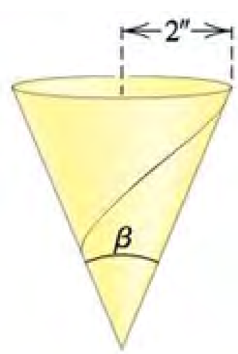
Exercise

A conical paper cup has a radius of 2 *inches*, approximate, to the nearest degree, the angle β so that the cone will have a volume of $20\ in^3$.

Solution

$$V = \frac{1}{3}\pi r^2 h$$

$$= 20\ in^3$$



$$h = \frac{60}{\pi\left(2^2\right)}$$

$$= \frac{15}{\pi} \underline{\approx 4.77\ in}$$

$$\tan\frac{\beta}{2} = \frac{2}{4.77}$$

$$\frac{\beta}{2} = \tan^{-1}\frac{2}{4.77}$$

$$\approx 22.75°$$

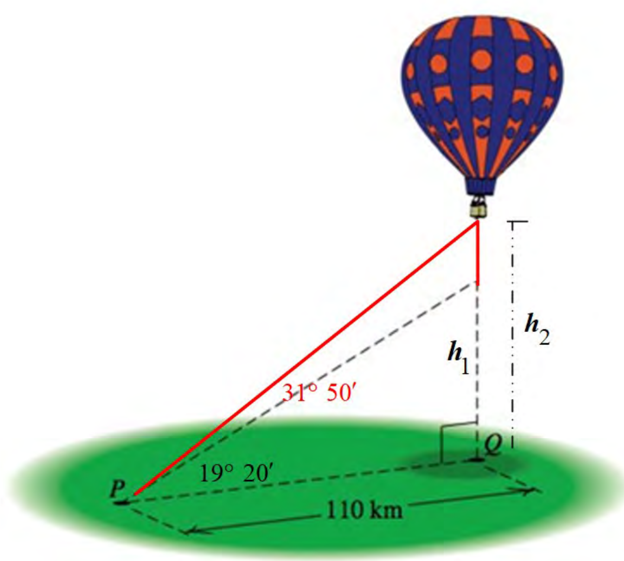
$$\beta = 2\left(22.75°\right)$$

$$\underline{\approx 45.5°}$$

Exercise

As a hot-air balloon rises vertically, its angle of elevation from a point P on level ground 100 *km* from the point Q directly underneath the balloon changes from $19°\ 20'$ to $31°\ 50'$.
Approximately how far does the balloon rise during this period?

Solution



$$\tan\left(19°\ 20'\right) = \frac{h_1}{100}$$

$$h_1 = 100\tan\left(19°\ 20'\right)$$

$$\underline{\approx 38.59\ \ km}$$

$$\tan\left(31°\ 50'\right) = \frac{h_2}{100}$$

$$h_2 = 100\tan\left(31°\ 50'\right)$$

$$\underline{\approx 68.29\ \ km}$$

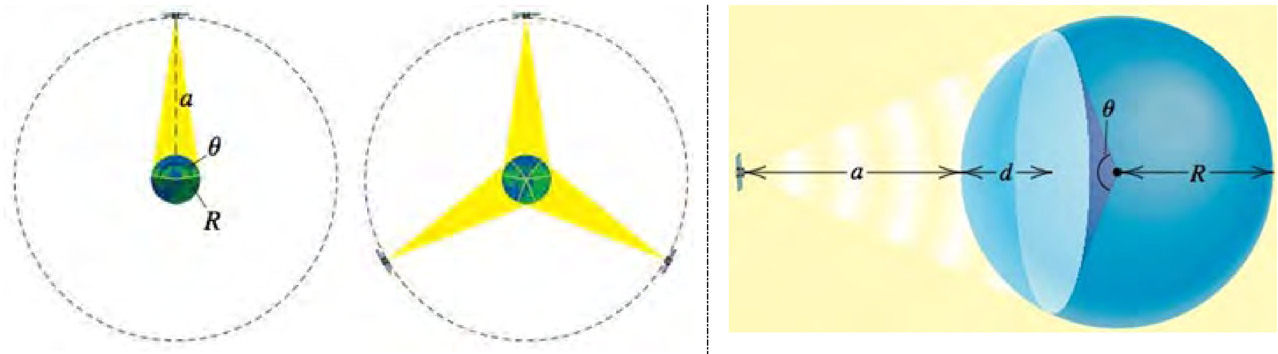
The change in elevation is:

$$h_2 - h_1 \approx 68.29 - 38.59$$

$$\underline{= 29.7\ \ km}$$

Exercise

Shown in the left part of the figure is a communications satellite with an equatorial orbit–that is, a nearly circular orbit in the plane determined by Earth's equator. If the satellite circles Earth at an altitude of $a = 22{,}300\ mi$, its speed is the same as the rotational speed of Earth; to an observer on the equator, the satellite appears to be stationary–that is, its orbit is synchronous.



a ***b***

a) Using $R = 4{,}000\ mi$ for the radius of Earth, determine the percentage of the equator that is within signal range of such a satellite.

b) As shown in the right part of the figure (a), three satellites are equally spaced in equatorial synchronous orbits. Use the value of θ obtained in part (a) to explain why all points on the equator are within signal range of at least one of the three satellites.

c) The figure shows the area served by a communication satellite circling a planet of radius R at an altitude a. The portion of the planet's surface within range of the satellite is a spherical cap of depth d and surface area $A = 2\pi Rd$. Express d in terms of R and θ.

d) Estimate the percentage of the planet's surface that is within signal range of a single satellite in equatorial synchronous orbit.

Solution

a) $\cos\frac{\theta}{2} = \frac{R}{R+a}$

$$= \frac{4{,}000}{26{,}300}$$

$$\frac{\theta}{2} = \cos^{-1}\left(\frac{4{,}000}{26{,}300}\right)$$

$$\approx 81.25^\circ$$

$$\theta \approx 162.5^\circ$$

The percentage of the equator that is within signal rage is:

$$\frac{162.5^\circ}{360^\circ}\times 100 \ \approx 45\%$$

b) Each satellite has a signal range of more than 120°, and this all 3 will cover all points on the equator.

c) $\cos\frac{\theta}{2} = \frac{R-d}{R}$

$R\cos\frac{\theta}{2} = R - d$

$\boxed{d = R\left(1-\cos\frac{\theta}{2}\right)}$

d) $d = R\left(1-\cos 81.25°\right)$

$\approx 0.8479R$

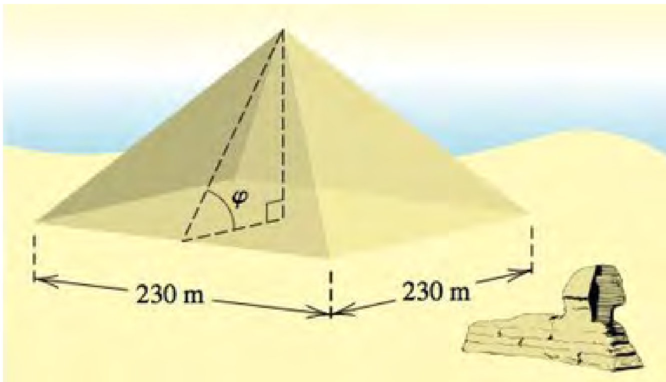
$\frac{d}{2R} \approx \frac{.8479R}{2R}$

$= 0.4239$

The percentage of the planet's surface that is within signal range of a single satellite is: 42.39%

Exercise

The great Pyramid of Egypt is 147 *meters* high, with a square base of side 230 *meters*. Approximate, to the nearest degree, the angle φ formed when an observer stands at the midpoint of one the sides and views the apex of the pyramid.



Solution

$\tan\varphi = \frac{147}{\frac{1}{2}230}$

$\varphi = \tan^{-1}\frac{147}{115}$

$\approx 52°$

Exercise

A tunnel for a new highway is to be cut through a mountain that is 260 *feet* high. At a distance of 200 *feet* from the base of the mountain, the angle of elevation is 36°. From a distance of 150 *feet* on the other side, the angle of elevation is 47°. Approximate the length of the tunnel to the nearest foot.

Solution

Left triangle:

$\tan 36° = \frac{260}{200+d_1}$

$d_1 = \frac{260}{\tan 36°} - 200$

$\approx 157.86\ \ ft$

Right triangle:

$$\tan 47° = \frac{260}{150 + d_2}$$

$$d_2 = \frac{260}{\tan 47°} - 150$$

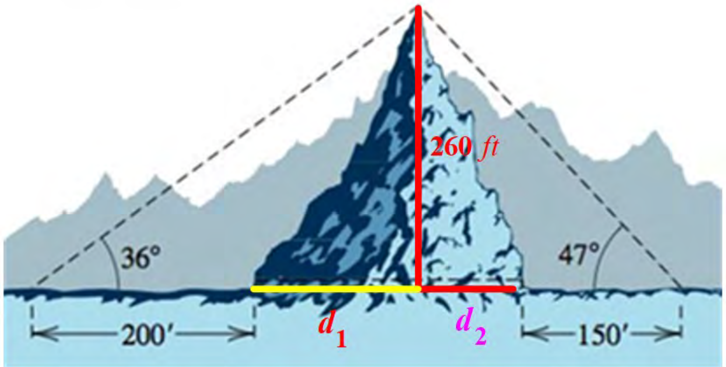
$$\boxed{\approx 92.45\ ft}$$

Length of the tunnel:

$$d = d_1 + d_2$$

$$\approx 157.86 + 92.45$$

$$\boxed{= 250\ ft}$$



Exercise

When a certain skyscraper is viewed from the top of a building 50 *feet* tall, the angle of elevation is 59°. When viewed from the street next to the shorter building, the angle of elevation is 62°.

a) Approximately how far apart are the two structures?

b) Approximate the height of the skyscraper to the nearest tenth of a *foot*.

Solution

a) $\tan 59° = \frac{h-50}{x}$

$\rightarrow \quad h = x\tan 59° + 50$

$\tan 62° = \frac{h}{x}$

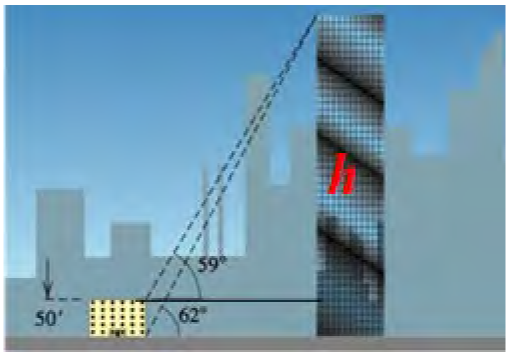
$\rightarrow \quad h = x\tan 62°$

$x\tan 59° + 50 = x\tan 62°$

$x\left(\tan 62° - \tan 59°\right) = 50$

$x = \frac{50}{\tan 62° - \tan 59°}$

$\boxed{\approx 231.0\ ft}$



b) $h = x\tan 62°$

$= 231\tan 62°$

$\boxed{\approx 434.5\ ft}$

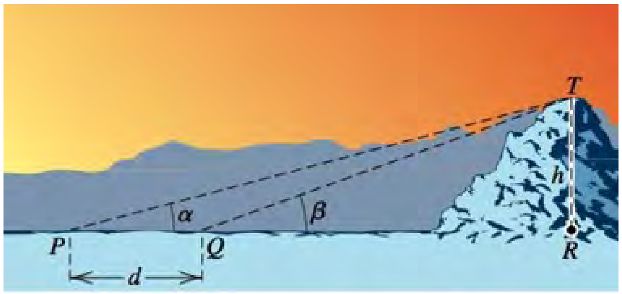
Exercise

When a mountaintop is viewed from the point P, the angle of elevation is α. From a point Q, which is d miles closer to the mountain, the angle of elevation increases to β.

a) Show that the height h of the mountain is given by: $h = \dfrac{d}{\cot\alpha - \cot\beta}$.

b) If $d = 2mi, \quad \alpha = 15°, \quad and \quad \beta = 20°$, approximate the height of the mountain.

Solution



a) $h = \dfrac{d\tan\alpha\tan\beta}{\tan\beta - \tan\alpha}$

$$= d\frac{\frac{1}{\cot\alpha}\frac{1}{\cot\beta}}{\frac{1}{\cot\beta} - \frac{1}{\cot\alpha}}$$

$$= d\frac{\frac{1}{\cot\alpha\cot\beta}}{\frac{\cot\alpha - \cot\beta}{\cot\alpha\cot\beta}}$$

$$= \boxed{\frac{d}{\cot\alpha - \cot\beta}} \quad \checkmark$$

b) **Given**: $d = 2mi, \quad \alpha = 15°, \quad and \quad \beta = 20°$

$$h = \frac{2\tan 15°\tan 20°}{\tan 20° - \tan 15°}$$

$$\approx 2.03\ mi$$

Exercise

An observer of height h stands on an incline at a distance d from the base of a building of height T. The angle of elevation from the observer to the top of the building is θ, and the incline makes an angle of α with the horizontal.

a) Express T in terms of h, d, α, and θ.

b) If $d = 50\ ft, \quad h = 6\ ft, \quad \alpha = 15°, \quad and \quad \theta = 31.4°$, estimate the height of the building.

Solution

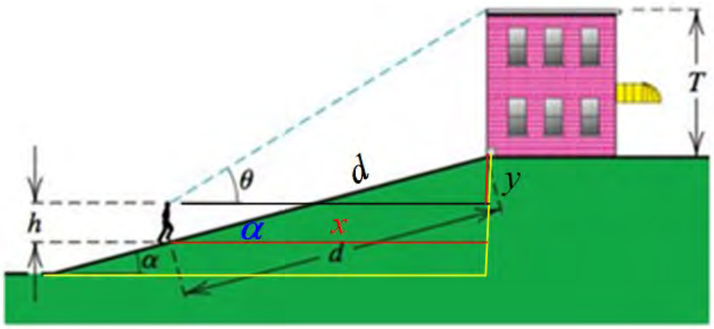
a) From ΔABD:

$\cos\alpha = \dfrac{x}{d}$

$\rightarrow \quad x = d\cos\alpha$

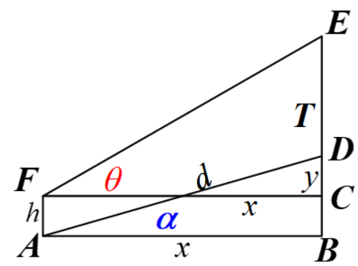
$\sin\alpha = \dfrac{y+h}{d}$

$\rightarrow \quad y = d\sin\alpha - h$



From ΔFCE :

$$\tan\theta = \frac{T+y}{x} \quad \rightarrow \quad x\tan\theta = T+y$$



$$x\tan\theta = T+y$$

$$d\cos\alpha\tan\theta = T + d\sin\alpha - h$$

$$\underline{T = d\left(\cos\alpha\tan\theta - \sin\alpha\right) + h \,\big|}$$

b) ***Given***: $d = 50\ ft, \quad h = 6\ ft, \quad \alpha = 15°, \quad and \quad \theta = 31.4°$

$$T = 50\left(\cos 15°\tan 31.4° - \sin 15°\right) + 6$$

$$\underline{\approx 22.54 \quad ft \,\big|}$$

Exercise

In triangle *ABC*, $B = 110°$, $C = 40°$ and $b = 18\ in$. Find the length of side *c*.

Solution

$$A = 180° - (B + C)$$
$$= 180° - 110° - 40°$$
$$\boxed{= 30°}$$

$$\frac{a}{\sin 30°} = \frac{18}{\sin 110°} \qquad \frac{a}{\sin A} = \frac{b}{\sin B}$$

$$a = \frac{18\sin 30°}{\sin 110°}$$
$$\boxed{\approx 9.6\ \ in}$$

$$\frac{c}{\sin 40°} = \frac{18}{\sin 110°} \qquad \frac{c}{\sin C} = \frac{b}{\sin B}$$

$$c = \frac{18\sin 40°}{\sin 110°}$$
$$\boxed{\approx 12.3\ \ in}$$

Exercise

In triangle *ABC*, $A = 110.4°$, $C = 21.8°$ and $c = 246\ in$. Find all the missing parts.

Solution

$$B = 180° - A - C$$
$$= 180° - 110.4° - 21.8°$$
$$\boxed{= 47.8°}$$

$$\frac{a}{\sin 110.4°} = \frac{246}{\sin 21.8°} \qquad \frac{a}{\sin A} = \frac{c}{\sin C}$$

$$a = \frac{246\sin 110.4°}{\sin 21.8°}$$
$$\boxed{\approx 621\ in}$$

$$\frac{b}{47.8} = \frac{246}{\sin 21.8°} \qquad \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$b = \frac{246\sin 47.8°}{\sin 21.8°}$$
$$\boxed{\approx 491\ in}$$

Exercise

Find the missing parts of triangle *ABC* if $B = 34°, C = 82°$, and $a = 5.6\ cm$.

Solution

$$A = 180° - (B + C)$$
$$= 180° - (34° + 82°)$$
$$= 180° - 116°$$
$$= 64°$$

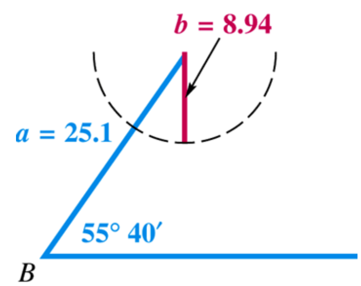
$$b = \frac{a \sin B}{\sin A} \qquad \frac{b}{\sin B} = \frac{a}{\sin A}$$
$$= \frac{5.6 \sin 34°}{\sin 64°}$$
$$\approx 3.5\ cm$$

$$c = \frac{a \sin C}{\sin A} \qquad \frac{c}{\sin C} = \frac{a}{\sin A}$$
$$= \frac{5.6 \sin 82°}{\sin 64°}$$
$$\approx 6.2\ \ cm$$

Exercise

Solve triangle *ABC* if $B = 55°40'$, $b = 8.94\ m$, and $a = 25.1\ m$.

Solution



$$\frac{\sin A}{25.1} = \frac{\sin\left(55° + \frac{40°}{60}\right)}{8.94} \qquad \frac{\sin A}{a} = \frac{\sin B}{b}$$
$$\sin A = \frac{25.1 \sin(55.667°)}{8.94}$$
$$\approx 2.3184 > 1$$

Since sin *A* > 1 is impossible, *no such triangle* ***exists***.

Exercise

Solve triangle *ABC* if $A = 55.3°$, $a = 22.8\ ft$, and $b = 24.9$ ft

Solution

$$\sin B = \frac{24.9 \sin 55.3°}{22.8} \approx 0.89787 \qquad \frac{\sin B}{b} = \frac{\sin A}{a}$$
$$B = \sin^{-1}(0.89787)$$
$$B = 63.9° \quad \text{and} \quad B = 180° - 63.9° = 116.1°$$

$C = 180° - A - B$

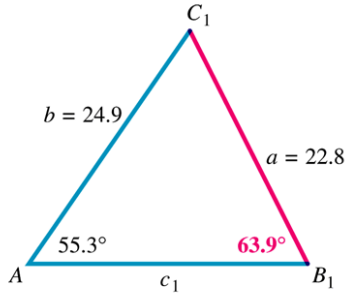
$C = 180° - 55.3° - 63.9°$

$\boxed{C = 60.8°}$

$c = \dfrac{a \sin C}{\sin A}$

$= \dfrac{22.8 \sin 60.8°}{\sin 55.3°}$

$\boxed{= 24.2\ ft}$



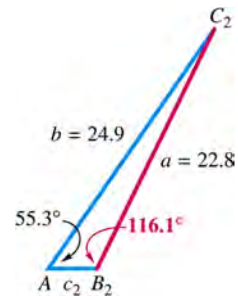
$C = 180° - 55.3° - 116.1°$

$\boxed{C = 8.6°}$

$c = \dfrac{a \sin C}{\sin A}$

$= \dfrac{22.8 \sin 8.6°}{\sin 55.3°}$

$\boxed{= 4.15\ ft}$



Exercise

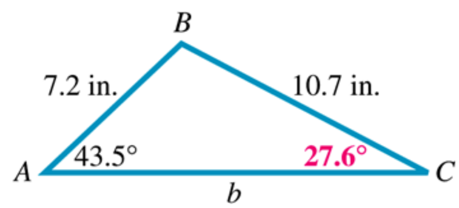
Solve triangle *ABC* given $A = 43.5°$, $a = 10.7$ in., and $c = 7.2$ in

Solution

$\sin C = \dfrac{7.2 \sin 43.5°}{10.7} \approx 0.4632$ $\qquad \dfrac{\sin C}{c} = \dfrac{\sin A}{a}$

$C = \sin^{-1}(0.4632)$

$C = 27.6° \quad and \quad C = 180° - 27.6° = 152.4°$



$B = 180° - A - C$

$B = 180° - 43.5° - 27.6°$

$\boxed{B = 108.9°}$

$b = \dfrac{a \sin B}{\sin A}$

$= \dfrac{10.7 \sin 108.9°}{\sin 43.5°}$

$\boxed{= 14.7\ in}$

$B = 180° - 43.5° - 152.4°$

$B = -15.9°$

Is not possible

Exercise

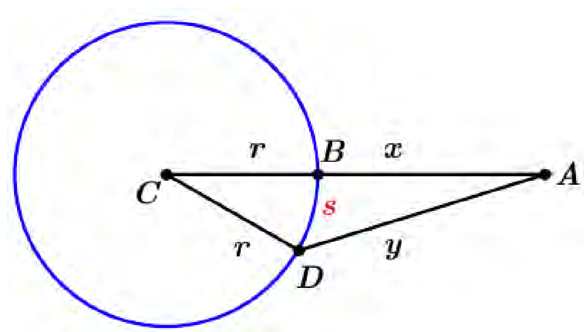
If $A = 26°,\ s = 22,\ and\ \ r = 19$ find x

Solution

$C = \theta = \frac{s}{r}\ rad$

$= \frac{22}{19} \frac{180°}{\pi}$

$\boxed{\approx 66°}$



$$\begin{aligned} D &= 180 - A - C \\ &= 180° - 26° - 66° \\ &\underline{= 88°} \end{aligned}$$

$$\frac{r+x}{\sin D} = \frac{r}{\sin A}$$

$$19 + x = \frac{19 \sin 88°}{\sin 26°}$$

$$\begin{aligned} x &= \frac{19 \sin 88°}{\sin 26°} - 19 \\ &\underline{\approx 24} \end{aligned}$$

Exercise

If $a = 13\ yd$, $b = 14\ yd$, and $c = 15\ yd$, find the largest angle.

Solution

$$\begin{aligned} C &= \cos^{-1}\left(\frac{13^2 + 14^2 - 15^2}{2(13)(14)}\right) \\ &\underline{\approx 67°} \end{aligned} \qquad C = \cos^{-1} \frac{a^2 + b^2 - c^2}{2ab}$$

Exercise

Solve triangle ABC if $b = 63.4\ km$, and $c = 75.2\ km$, $A = 124°\ \ 40'$

Solution

$$\begin{aligned} a &= \sqrt{b^2 + c^2 - 2bc \cos A} \\ &= \sqrt{(63.4)^2 + (75.2)^2 - 2(63.4)(75.2)\cos\left(124° + \frac{40°}{60}\right)} \\ &\underline{\approx 122.9\ km} \end{aligned}$$

$$\sin B = \frac{b \sin A}{a}$$

$$\begin{aligned} \underline{B} &= \sin^{-1}\left(\frac{63.4 \sin 124.67°}{122.9}\right) \\ &\underline{\approx 25.1°} \end{aligned}$$

$$\begin{aligned} \underline{C} &= 180° - A - B \\ &= 180° - 124.67° - 25.1° \\ &\underline{\approx 30.23°} \end{aligned}$$

Exercise

Solve triangle *ABC* if $a = 832\ ft$, $b = 623\ ft$, and $c = 345\ ft$

Solution

$$C = \cos^{-1}\left(\frac{832^2 + 623^2 - 345^2}{2(832)(623)}\right) \qquad C = \cos^{-1}\frac{a^2 + b^2 - c^2}{2ab}$$

$$\approx 22°$$

$$B = \sin^{-1}\left(\frac{623\sin 22°}{345}\right) \qquad \sin B = \frac{b\sin C}{c}$$

$$\approx 43°$$

$$A = 180° - 22° - 43°$$

$$= 115°$$

Exercise

Solve triangle *ABC* if $A = 42.3°$, $b = 12.9m$, $and\ c = 15.4m$

Solution

$$a = \sqrt{12.9^2 + 15.4^2 - 2(12.9)(15.4)\cos 42.3°} \qquad a = \sqrt{b^2 + c^2 - 2bc\cos A}$$

$$\approx 10.47\ m$$

$$\sin B = \frac{12.9\sin 42.3°}{10.47} \qquad \sin B = \frac{b\sin A}{a}$$

$$B = \sin^{-1}\left(\frac{12.9\sin 42.3°}{10.47}\right)$$

$$\approx 56.0°$$

$$C = 180° - 42.3° - 56°$$

$$= 81.7°$$

Exercise

Solve triangle *ABC* if $a = 9.47\ ft$, $b = 15.9\ ft$, $and\ c = 21.1\ ft$

Solution

$$C = \cos^{-1}\left(\frac{9.47^2 + 15.9^2 - 21.1^2}{2(9.47)(15.9)}\right) \qquad C = \cos^{-1}\frac{a^2 + b^2 - c^2}{2ab}$$

$$\approx 109.9°$$

$$\sin B = \frac{15.9\sin 109.9°}{21.1} \qquad \sin B = \frac{b\sin C}{c}$$

$$\underline{|B} = \sin^{-1}\left(\frac{15.9\sin 109.9^\circ}{21.1}\right)$$

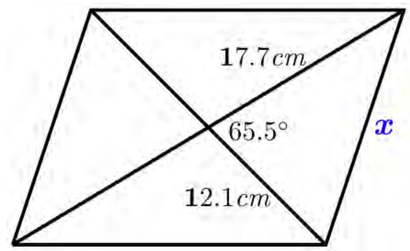
$$\underline{\approx 25.0^\circ}$$

$$\underline{|A} = 180^\circ - 25^\circ - 109.9^\circ$$

$$\underline{= 45.1^\circ}$$

Exercise

The diagonals of a parallelogram are 24.2 *cm* and 35.4 *cm* and intersect at an angle of 65.5°. Find the length of the shorter side of the parallelogram



Solution

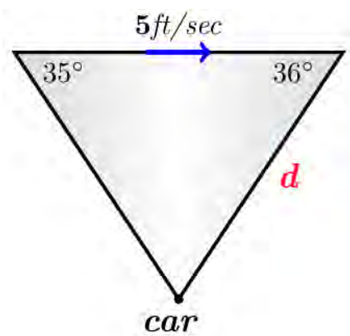
$$x = \sqrt{17.7^2 + 12.1^2 - 2(17.7)(12.1)\cos 65.5^\circ}$$

$$\underline{= 16.8\ \ cm}$$

Exercise

A man flying in a hot-air balloon in a straight line at a constant rate of 5 *feet* per *second*, while keeping it at a constant altitude. As he approaches the parking lot of a market, he notices that the angle of depression from his balloon to a friend's car in the parking lot is 35°. A minute and a half later, after flying directly over this friend's car, he looks back to see his friend getting into the car and observes the angle of depression to be 36°. At that time, what is the distance between him and his friend?

Solution



$$\angle car = 180^\circ - 35^\circ - 36^\circ = 109^\circ$$

$$\frac{d}{\sin 35^\circ} = \frac{450}{\sin 109^\circ}$$

$$d = \frac{450\sin 35^\circ}{\sin 109^\circ}$$

$$\underline{\approx 273\ \ ft}$$

Exercise

A satellite is circling above the earth. When the satellite is directly above point *B*, angle *A* is 75.4°. If the distance between points *B* and *D* on the circumference of the earth is 910 *miles* and the radius of the earth is 3,960 *miles*, how far above the earth is the satellite?

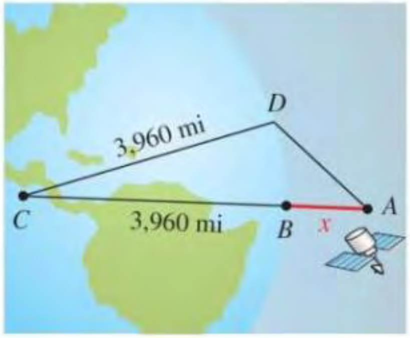
Solution

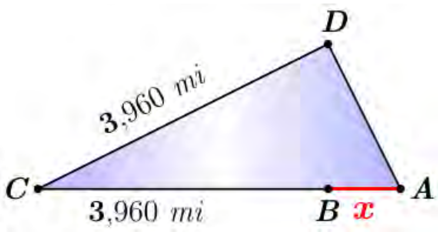
$$\theta = \frac{s}{r}$$

$C = arc\ length\ BD$ divides by *radius*

$$C = \frac{910}{3960} rad$$
$$= \frac{910}{3960} \frac{180°}{\pi}$$
$$= 13.2°$$

$$D = 180° - (A + C)$$
$$= 180° - (75.4° + 13.2°)$$
$$\underline{= 91.4°}$$





$$\frac{CA}{\sin D} = \frac{3960}{\sin A}$$

$$\frac{x + 3960}{\sin 91.4°} = \frac{3960}{\sin 75.4°}$$

$$x + 3960 = \frac{3960 \sin 91.4°}{\sin 75.4°}$$

$$x = \frac{3960 \sin 91.4°}{\sin 75.4°} - 3960$$

$$\underline{x = 130 \ \ mi}$$

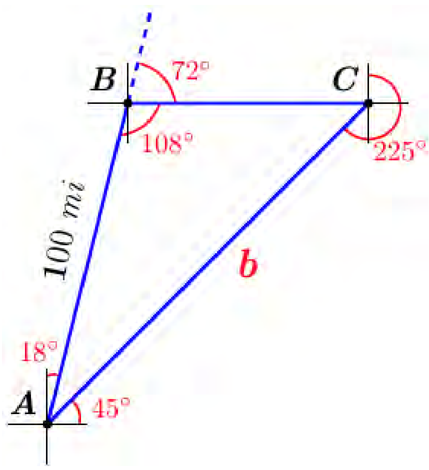
Exercise

A pilot left Fairbanks in a light plane and flew 100 *miles* toward Fort in still air on a course with bearing of 18°. She then flew due east (bearing 90°) for some time drop supplies to a snowbound family. After the drop, her course to return to Fairbanks had bearing of 225°. What was her maximum distance from Fairbanks?

Solution

From the triangle *ABC*:

$$\angle ABC = 90° + 18° = 108°$$
$$\angle ACB = 360° - 225° - 90° = 45°$$
$$\angle BAC = 90° - 18° - 45° = 27°$$



The length *AC* is the maximum distance from Fairbanks:

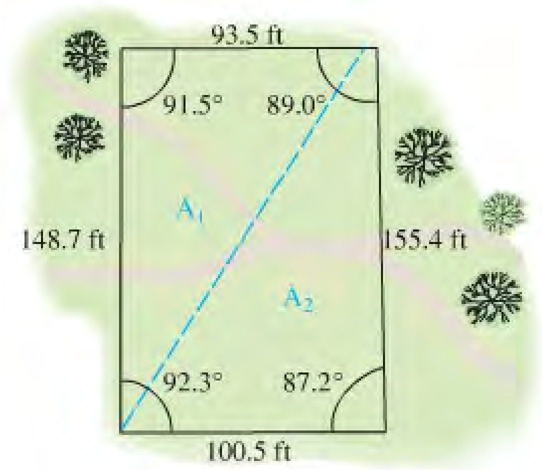
$$\frac{b}{\sin 108°} = \frac{100}{\sin 45°}$$

$$b = \frac{100 \sin 108°}{\sin 45°}$$
$$\underline{\approx 134.5 \ \ miles}$$

Exercise

The dimensions of a land are given in the figure. Find the area of the property in square feet.

Solution



$$A_1 = \frac{1}{2}(148.7)(93.5)\sin 91.5°$$
$$\approx 6949.3\ ft^2$$
$$A_2 = \frac{1}{2}(100.5)(155.4)\sin 87.2°$$
$$\approx 7799.5\ ft^2$$

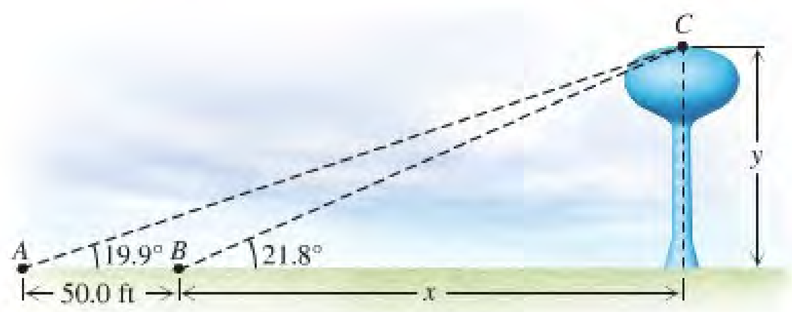
The total area $= A_1 + A_2 = 6949.3 + 7799.5$

$$= 14,748.8\ ft^2$$

Exercise

The angle of elevation of the top of a water tower from point *A* on the ground is 19.9°. From point *B*, 50.0 feet closer to the tower, the angle of elevation is 21.8°. What is the height of the tower?

Solution



$$\angle ABC = 180° - 21.8° = 158.2°$$
$$\angle ACB = 180° - 19.9° - 158.2° = 1.9°$$

Apply the law of sines in triangle *ABC*:

$$\frac{BC}{\sin 19.9°} = \frac{50}{\sin 1.9°}$$
$$\Rightarrow BC = \frac{50\sin 19.9°}{\sin 1.9°} \approx 513.3$$

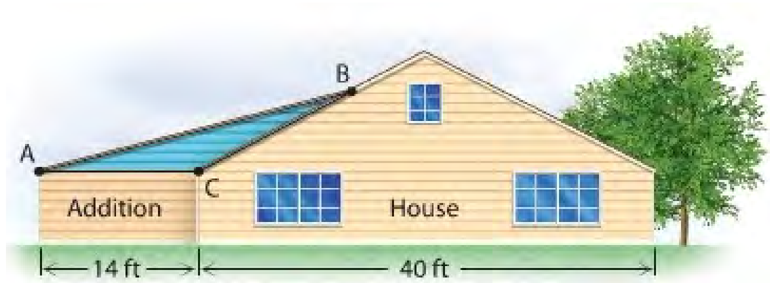
Using the right triangle: $\sin 21.8° = \frac{y}{BC}$

$$y = 513.3\sin 21.8°$$
$$\approx 191\ ft$$

Or $\quad y = \dfrac{50\tan 19.9°\tan 21.8°}{\tan 21.8° - \tan 19.9°} \approx 190.6\ ft$

Exercise

A 40-*feet* wide house has a roof with a 6-12 pitch (the roof rises 6 *feet* for a run of 12 *feet*). The owner plans a 14-*feet* wide addition that will have a 3-12 pitch to its roof. Find the lengths of $\overline{AB}$ and $\overline{BC}$.



Solution

$$\tan\gamma = \frac{6}{12}$$

$$\gamma = \tan^{-1}\left(\frac{6}{12}\right)$$

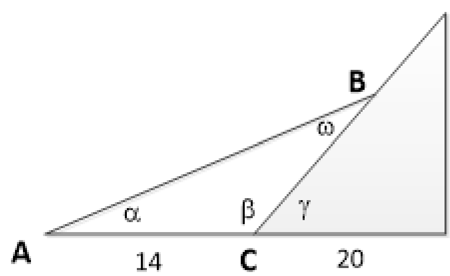
$$= 26.565^\circ$$

$$\tan\alpha = \frac{3}{12}$$

$$\alpha = \tan^{-1}\left(\frac{3}{12}\right)$$

$$= 14.036^\circ$$

$$\beta = 180^\circ - \gamma$$
$$= 180^\circ - 26.565^\circ$$
$$= 153.435^\circ$$



$$\omega = 180^\circ - 14.036^\circ - 153.435^\circ$$

$$= 12.529^\circ$$

$$\frac{AB}{\sin 153.435^\circ} = \frac{14}{\sin 12.529^\circ}$$

$$|AB| = \frac{14\sin 153.435^\circ}{\sin 12.529^\circ}$$

$$\approx 28.9 \ ft$$

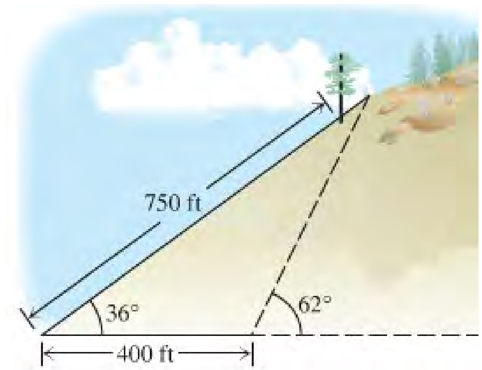
$$\frac{BC}{\sin 14.036^\circ} = \frac{14}{\sin 12.529^\circ}$$

$$|BC| = \frac{14\sin 14.036^\circ}{\sin 12.529^\circ}$$

$$\approx 15.7 \ ft$$

Exercise

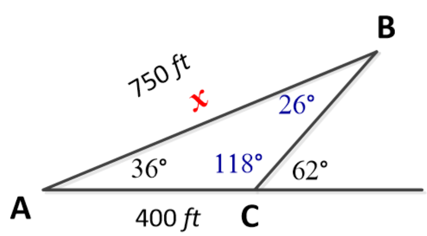
A hill has an angle of inclination of 36°. A study completed by a state's highway commission showed that the placement of a highway requires that 400 *feet* of the hill, measured horizontally, be removed. The engineers plan to leave a slope alongside the highway with an angle of inclination of 62°. Located 750 *feet* up the hill measured from the base is a tree containing the nest of an endangered hawk. Will this tree be removed in the excavation?



Solution

$\angle ACB = 180° - 62° = 118°$

$\angle ABC = 180° - 118° - 36° = 26°$



Using the law of sines:

$$\frac{x}{\sin 118°} = \frac{400}{\sin 26°}$$

$$x = \frac{400 \sin 118°}{\sin 26°}$$

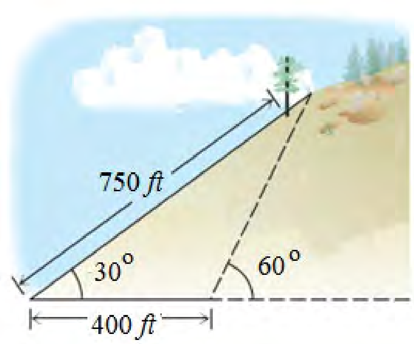
$$\approx 805.7 \ ft \quad > 750$$

Yes, the tree will have to be excavated.

Exercise

A hill has an angle of inclination of 30°. A study completed by a state's highway commission showed that the placement of a highway requires that 400 *feet* of the hill, measured horizontally, be removed. The engineers plan to leave a slope alongside the highway with an angle of inclination of 60°. Located 750 *feet* up the hill measured from the base is a tree containing the nest of an endangered hawk. Will this tree be removed in the excavation?

Solution



$\angle ACB = 180° - 60° = 120°$

$\angle ABC = 180° - 120° - 30° = 30°$

That implies: $|CA| = |CB|$

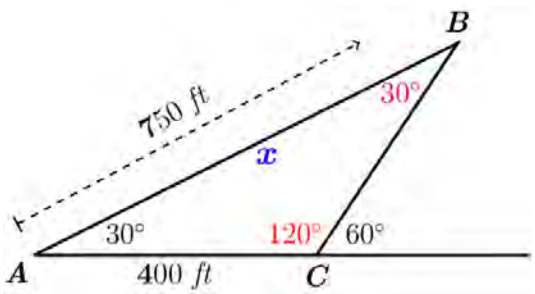
Using the law of sines:

$$\frac{x}{\sin 120^\circ} = \frac{400}{\sin 30^\circ}$$

$$x = \frac{400 \sin 120^\circ}{\sin 30^\circ}$$

$$x = \frac{400\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

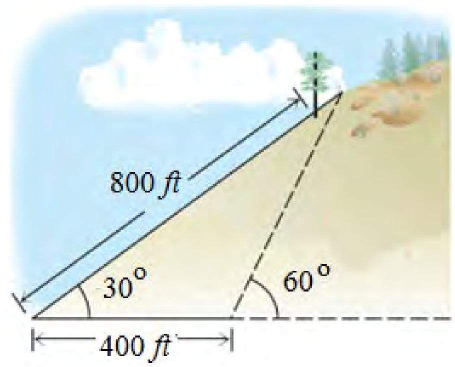
$$= 400\sqrt{3}\ ft \quad < 750$$



∴ the tree will ***not*** have to be excavated.

Exercise

A hill has an angle of inclination of 30°. A study completed by a state's highway commission showed that the placement of a highway requires that 400 *feet* of the hill, measured horizontally, be removed. The engineers plan to leave a slope alongside the highway with an angle of inclination of 60°. Located 800 *feet* up the hill measured from the base is a tree containing the nest of an endangered hawk. Will this tree be removed in the excavation?



Solution

$\angle ACB = 180^\circ - 60^\circ = 120^\circ$

$\angle ABC = 180^\circ - 120^\circ - 30^\circ = 30^\circ$

That implies: $|CA| = |CB|$

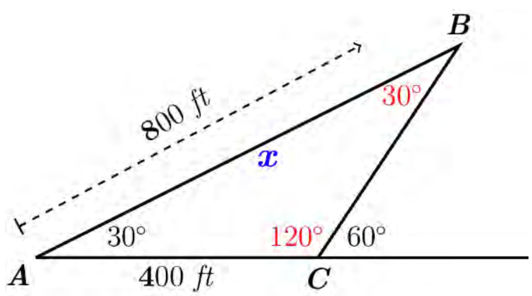
Using the law of sines:

$$\frac{x}{\sin 120^\circ} = \frac{400}{\sin 30^\circ}$$

$$x = \frac{400 \sin 120^\circ}{\sin 30^\circ}$$

$$x = \frac{400\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

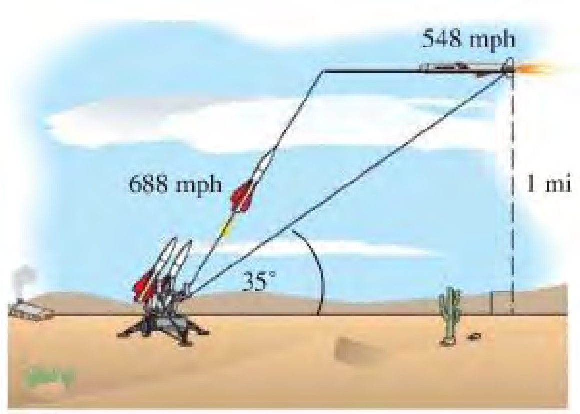
$$= 400\sqrt{3}\ ft \quad < 800$$



∴ the tree will ***not*** have to be excavated.

Exercise

A cruise missile is traveling straight across the desert at 548 *mph* at an altitude of 1 *mile*. A gunner spots the missile coming in his direction and fires a projectile at the missile when the angle of elevation of the missile is 35°. If the speed of the projectile is 688 *mph*, then for what angle of elevation of the gun will the projectile hit the missile?



Solution

$$\angle ACB = 35°$$
$$\angle BAC = 180° - 35° - \beta$$

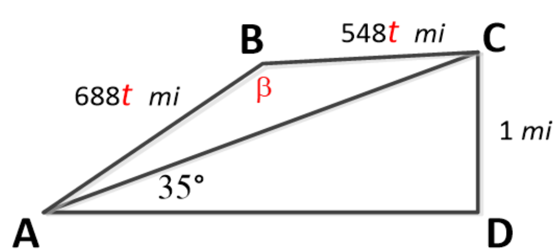
After ***t*** seconds;

The cruise missile distance: $548\frac{t}{3600}\,miles$

The Projectile distance: $688\frac{t}{3600}\,miles$

Using the law of sines:

$$\frac{\frac{548t}{3600}}{\sin(145° - \beta)} = \frac{\frac{688t}{3600}}{\sin 35°}$$



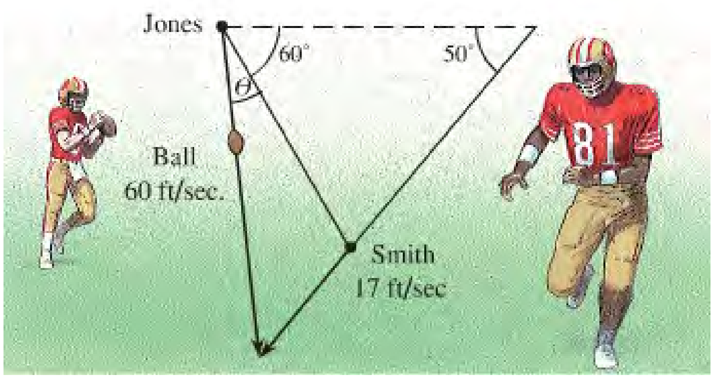
$$\frac{548t}{3600}\sin 35° = \frac{688t}{3600}\sin(145° - \beta)$$
$$548\sin 35° = 688\sin(145° - \beta)$$
$$\sin(145° - \beta) = \frac{548}{688}\sin 35°$$
$$145° - \beta = \sin^{-1}\left(\frac{548}{688}\sin 35°\right)$$
$$\beta = 145° - \sin^{-1}\left(\frac{548}{688}\sin 35°\right)$$
$$\approx 117.8°$$

$$\Rightarrow \angle BAC = 180° - 35° - 117.8°$$
$$\approx 27.2°$$

The angle of elevation of the projectile must be $(= 35° + 27.2°)$ $62.2°$

Exercise

When the ball is snapped, Smith starts running at a 50° angle to the line of scrimmage. At the moment when Smith is at a 60° angle from Jones, Smith is running at 17 *feet/sec* and Jones passes the ball at 60 *feet/sec* to Smith. However, to complete the pass, Jones must lead Smith by the angle θ. Find θ (find θ in his head. Note that θ can be found without knowing any distances.)

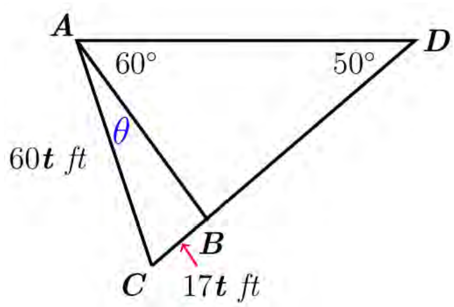


Solution

$$\angle ABD = 180° - 60° - 50°$$
$$= 70°$$

$$\angle ABC = 180° - 70°$$
$$= 110°$$

Using the law of sines:



$$\frac{17t}{\sin\theta} = \frac{60t}{\sin 110°}$$

$$\frac{17}{\sin\theta} = \frac{60}{\sin 110°}$$

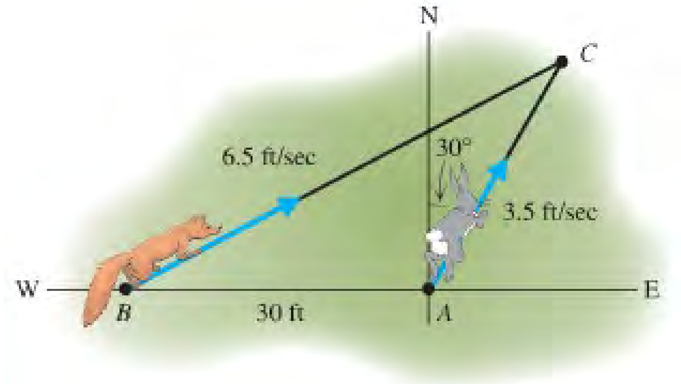
$$\sin\theta = \frac{17\sin 110°}{60}$$

$$\theta = \sin^{-1}\left(\frac{17\sin 110°}{60}\right)$$

$$= 15.4°$$

Exercise

A rabbit starts running from point *A* in a straight line in the direction 30° from the north at 3.5 *ft/sec*. At the same time a fox starts running in a straight line from a position 30 *feet* to the west of the rabbit 6.5 *ft/sec*. The fox chooses his path so that he will catch the rabbit at point *C*. In how many seconds will the fox catch the rabbit?

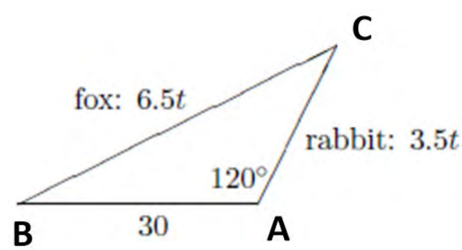


Solution

$$\angle BAC = 90° + 30°$$
$$= 120°$$

$$\frac{6.5t}{\sin 120°} = \frac{3.5t}{\sin B}$$

$$\frac{6.5}{\sin 120°} = \frac{3.5}{\sin B}$$



$$\sin B = \frac{3.5\sin 120°}{6.5}$$

$$B = \sin^{-1}\left(\frac{3.5\sin 120°}{6.5}\right)$$
$$\approx 28°$$

$$C = 180° - 120° - 28°$$
$$= 32°$$

$$\frac{3.5t}{\sin 28} = \frac{30}{\sin 32°}$$

$$t = \frac{30\sin 28}{3.5\sin 32°}$$
$$\approx 7.6 \text{ sec}$$

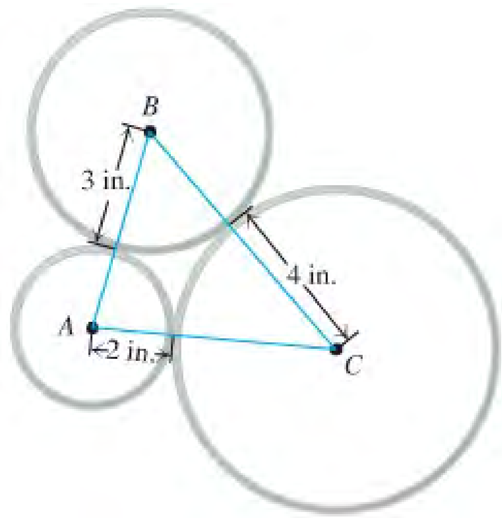
It will take 7.6 *sec*. to catch the rabbit.

Exercise

An engineer wants to position three pipes at the vertices of a triangle. If the pipes *A*, *B*, and *C* have radii 2 *in*, 3 *in*, and 4 *in*, respectively, then what are the measures of the angles of the triangle *ABC*?

Solution

$AC = 6 \quad AB = 5 \quad BC = 7$



$$A = \cos^{-1}\left(\frac{5^2 + 6^2 - 7^2}{2(5)(6)}\right)$$

$$\approx 78.5^\circ$$

$$\frac{6}{\sin B} = \frac{7}{\sin 78.5^\circ}$$

$$\sin B = \frac{6\sin 78.5^\circ}{7}$$

$$B = \sin^{-1}\left(\frac{6\sin 78.5^\circ}{7}\right)$$

$$\approx 57.1^\circ$$

$$C = 180^\circ - 78.5^\circ - 57.1^\circ$$

$$\approx 44.4^\circ$$

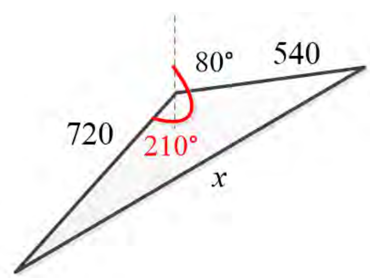
Exercise

Andrea and Steve left the airport at the same time. Andrea flew at 180 *mph* on a course with bearing 80°, and Steve flew at 240 *mph* on a course with bearing 210°. How far apart were they after 3 *hr*.?

Solution

After 3 *hrs*. Steve flew: 3(240) = 720 *mph*

Andrea flew: 3(180) = 540 *mph*



$$x = \sqrt{720^2 + 540^2 - 2(720)(540)\cos 210^\circ}$$

$$= 10\sqrt{72^2 + 54^2 + 2(72)(54)\frac{\sqrt{3}}{2}}$$

$$= 10\sqrt{(9\times 8)^2 + (9\times 6)^2 + (9\times 8)(9\times 6)\sqrt{3}}$$

$$= 90\sqrt{64 + 36 + 48\sqrt{3}}$$

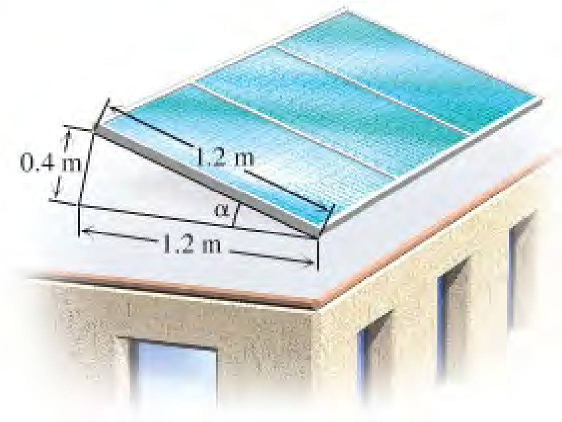
$$= 90\sqrt{100 + 48\sqrt{3}}$$

$$= 180\sqrt{25 + 12\sqrt{3}}$$

$$\approx 1218 \text{ miles}$$

Exercise

A solar panel with a width of 1.2 *m* is positioned on a flat roof.
What is the angle of elevation α of the solar panel?



Solution

$$\alpha = \cos^{-1}\left(\frac{1.2^2 + 1.2^2 - 0.4^2}{2(1.2)(1.2)}\right)$$

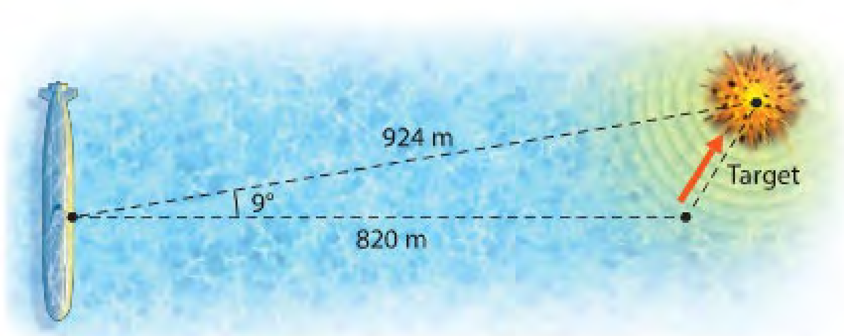
$$\approx 19.2^\circ$$

or $\alpha = \dfrac{0.4}{1.2}$ $\qquad \alpha = \dfrac{s}{r}$

$$= \frac{1}{3} \ rad$$

Exercise

A submarine sights a moving target at a distance of 820 *m*. A torpedo is fired 9° ahead of the target, and travels 924 *m* in a straight line to hit the target. How far has the target moved from the time the torpedo is fired to the time of the hit?



Solution

$$x = \sqrt{820^2 + 924^2 - 2(820)(924)\cos 9^\circ}$$

$$\approx 171.7 \ m$$

Exercise

A tunnel is planned through a mountain to connect points *A* and *B* on two existing roads. If the angle between the roads at point *C* is 28°, what is the distance from point *A* to *B*? Find $\angle CBA$ and $\angle CAB$ to the nearest tenth of a degree.

Solution

$$AB = \sqrt{5.3^2 + 7.6^2 - 2(5.3)(7.6)\cos 28^\circ}$$

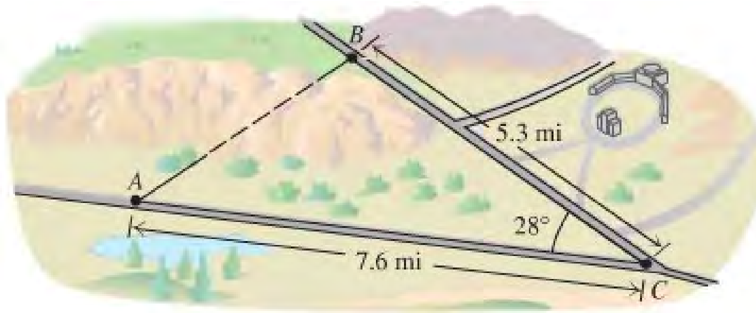
≈ 3.8

$\angle CBA = \cos^{-1} \frac{3.8^2 + 5.3^2 - 7.6^2}{2(3.8)(5.3)}$

$\approx 112°$

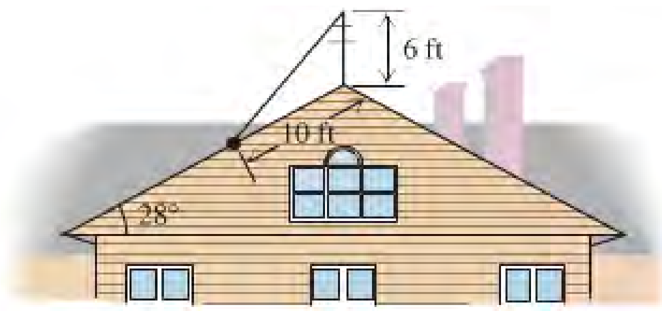
$\angle BAC = 180° - 112° - 28°$

$\approx 40°$



Exercise

A 6-*feet* antenna is installed at the top of a roof. A guy wire is to be attached to the top of the antenna and to a point 10 *feet* down the roof. If the angle of elevation of the roof is 28°, then what length guy wire is needed?



Solution

$\alpha = 90° - 28°$

$= 62°$

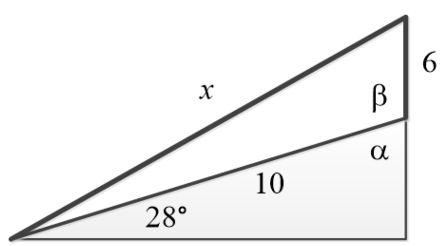
$\beta = 180° - 62°$

$= 118°$

By the cosine law,

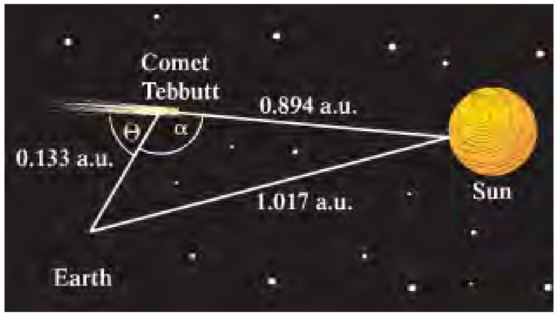
$x = \sqrt{6^2 + 100^2 - 2(6)(10)\cos 118°}$

$\approx 13.9 \ \ ft$



Exercise

On June 30, 1861, Comet Tebutt, one of the greatest comets, was visible even before sunset. One of the factors that causes a comet to be extra bright is a small scattering angle θ. When Comet Tebutt was at its brightest, it was 0.133 *a.u.* from the earth, 0.894 *a.u.* from the sun, and the earth was 1.017 *a.u.* from the sun. Find the phase angle α and the scattering angle θ for Comet Tebutt on June 30, 1861. (One astronomical unit (*a.u*) is the average distance between the earth and the sub.)



Solution

By the cosine law:

$$\alpha = \cos^{-1}\left(\frac{0.133^2 + 0.894^2 - 1.017^2}{2(0.133)(0.891)}\right)$$

$$\approx 156^\circ$$

$$\theta = 180^\circ - \alpha$$

$$= 180^\circ - 156^\circ$$

$$\approx 24^\circ$$

Exercise

A human arm consists of an upper arm of 30 *cm* and a lower arm of 30 *cm*. To move the hand to the point (36, 8), the human brain chooses angle θ_1 *and* θ_2 to the nearest tenth of a degree.

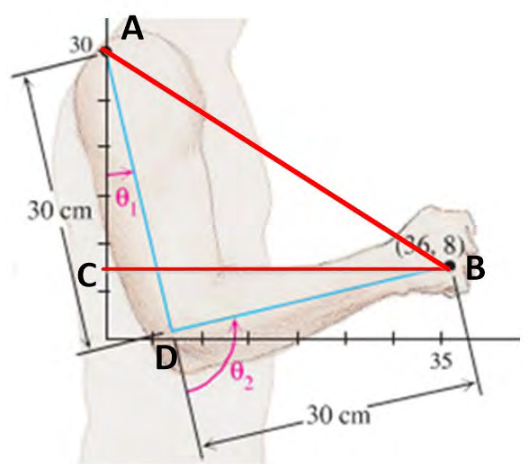
Solution

$$AC = 30 - 8 = 22 \qquad BC = 36$$

$$AB = \sqrt{AC^2 + CB^2}$$

$$= \sqrt{22^2 + 36^2}$$

$$\approx 42.19$$



$$\angle ADB = \cos^{-1}\left(\frac{AD^2 + DB^2 - AB^2}{2(AD)(DB)}\right)$$

$$= \cos^{-1}\left(\frac{30^2 + 30^2 - 42.19^2}{2(30)(30)}\right)$$

$$\approx 89.4^\circ$$

$$\theta_2 \approx 180^\circ - 89.4^\circ$$

$$\approx 90.6^\circ$$

$$\tan(\angle CAB) = \frac{BC}{AC} = \frac{36}{22}$$

$$\angle CAB = \tan^{-1}\frac{36}{22}$$

$$\approx 58.57^\circ$$

$$\frac{\sin DAB}{30} = \frac{\sin 89.4^\circ}{42.19}$$

$$\sin DAB = \frac{30\sin 89.4^\circ}{42.19}$$

$$\angle DAB = \sin^{-1}\frac{30\sin 89.4^\circ}{42.19}$$

$$\approx 45.32^\circ$$

$$\theta_1 = \angle CAB - \angle DAB$$

$$= 58.57^\circ - 45.32^\circ$$

$$\approx 13.25^\circ$$

Exercise

A forest ranger is 150 *feet* above the ground in a fire tower when she spots an angry grizzly bear east of the tower with an angle of depression of 10°. Southeast of the tower she spots a hiker with an angle of depression of 15°. Find the distance between the hiker and the angry bear.

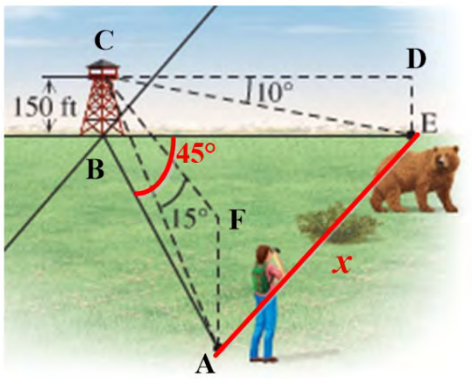
Solution

$$\angle BEC = \angle ECD = 10^\circ$$

From triangle EBC: $\tan 10^\circ = \frac{150}{BE}$

$$\Rightarrow BE = \frac{150}{\tan 10^\circ} \approx 850.692$$

$$\angle BAC = \angle ACF = 15^\circ$$



From triangle ABC:

$$\tan 15^\circ = \frac{150}{AB}$$

$$AB = \frac{150}{\tan 15^\circ}$$

$$\approx 559.808$$

$$x = \sqrt{AB^2 + BE^2 - 2(AB)(BE)\cos 45^\circ}$$

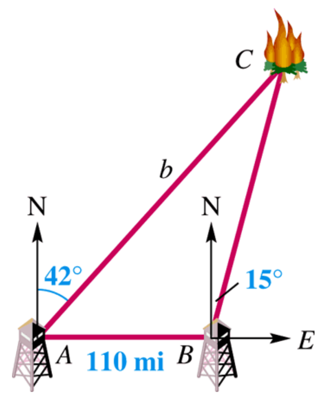
$$= \sqrt{559.808^2 + 850.692^2 - 2(559.808)(850.692)\cos 45°}$$

$$\approx 603 \ \ ft$$

Exercise

Two ranger stations are on an east-west line 110 *mi* apart. A forest fire is located on a bearing *N* 42° *E* from the western station at *A* and a bearing of *N* 15° *E* from the eastern station at *B*. How far is the fire from the western station?

Solution



$$\angle BAC = 90° - 42°$$
$$= 48°$$

$$\angle ABC = 90° + 15°$$
$$= 105°$$

$$\angle C = 180° - 105° - 48°$$
$$= 27°$$

$$\frac{b}{\sin 105°} = \frac{110}{\sin 27°} \qquad \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$b = \frac{110 \sin 105°}{\sin 27°}$$

$$b \approx 234 \ \ mi$$

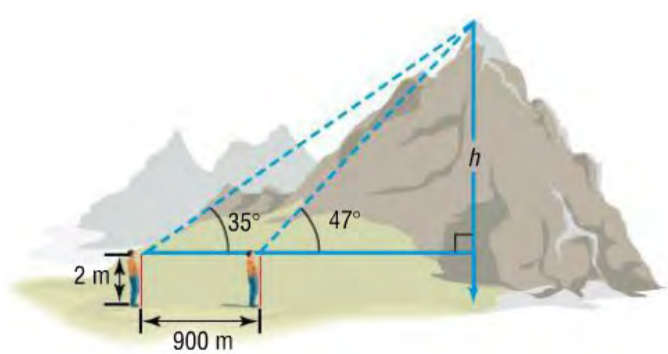
The fire is about **234** *miles* from the western station.

Exercise

To measure the height of a mountain, a surveyor takes two sightings of the peak at a distance 900 meters apart on a direct line to the mountain. The first observation results in an angle of elevation of 47°, and the second results in an angle of elevation of 35°. If the transit is 2 *meters* high, what is the height *h* of the mountain?

Solution

$$h = 2 + \frac{900 \tan 35° \tan 47°}{\tan 47° - \tan 35°}$$
$$\approx 2 + 1815.86$$
$$\approx 1817.86 \text{ m}$$



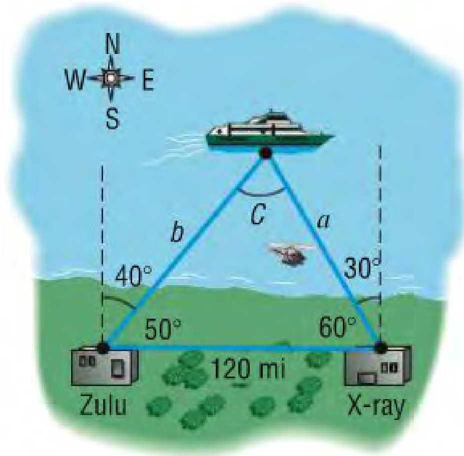
Exercise

A Station Zulu is located 120 *miles* due west of Station X-ray. A ship at sea sends an *SOS* call that is received by each station. The call to Station Zulu indicates that the bearing of the ship from Zulu is $N\ 40°\ E$. The call to Station *X*-ray indicates that the bearing of the ship from *X*-ray is $N\ 30°\ W$.

a) How far is each station from the ship?

b) If a helicopter capable of flying 200 *miles* per *hour* is dispatched from the nearest station to the ship, how long will it take to reach the ship?

Solution



a) $C = 180° - 50° - 60° = 70°$

$$b = \frac{120\sin 60°}{\sin 70°}$$

$$\approx 110.59\ mi$$

$$a = \frac{120\sin 50°}{\sin 70°}$$

$$\approx 97.82\ mi$$

Station Zulu is about 111 *miles* from the ship

and Station *X*-ray is about 98 *miles* from the ship.

b) ***Given***: $v = 200\ mi/hrs$

$$t = \frac{d}{v} = \frac{97.82}{200}$$

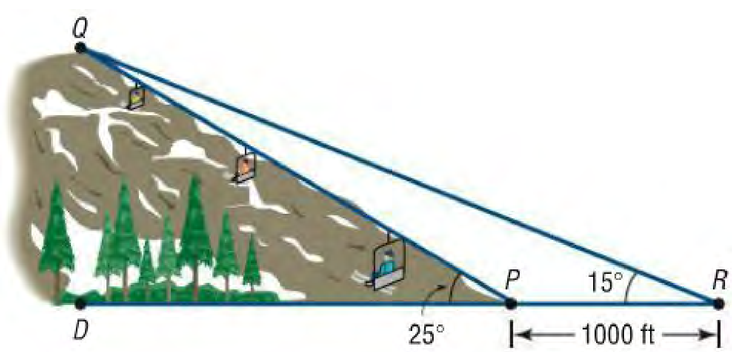
$$\approx 0.49\ hrs \qquad \approx 29\ \min$$

It will takes 29 *minutes* to reach Station *X*-ray.

Exercise

To find the length of the span of a proposed ski lift from *P* to *Q*, a surveyor measures $\angle DPQ$ to be $25°$ and then walks back a distance of 1000 *feet* to *R* and measures $\angle DRQ$ to be $15°$. What is the distance from *P* to *Q*.

Solution



$$\angle RPQ = 180° - 25° = 155°$$

$$\angle PQR = 180° - 155° - 15° = 10°$$

$$\frac{|PQ|}{\sin 15°} = \frac{|PR|}{\sin 10°}$$

$$|PQ| = \frac{10^3 \sin 15°}{\sin 10°}$$

$$\approx 1,490.5\ ft$$

Exercise

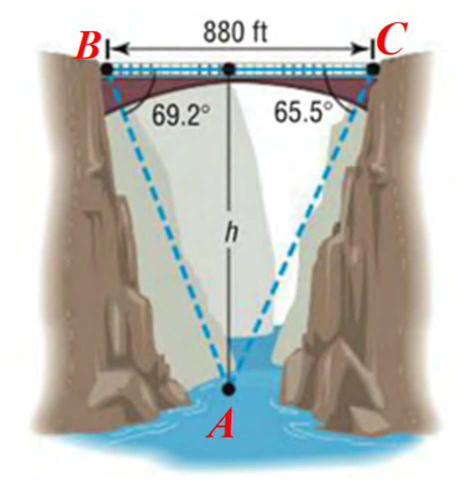
The highest bridge in the world is the bridge over the Royal Gorge of the Arkansas River in Colorado, sightings to the same point at water level directly under the bridge are taken from each side of the 880–foot–long bridge. How high is the bridge?

Solution

$$A = 180° - 69.2° - 65.5°$$
$$\underline{= 45.3°}$$

$$\frac{b}{\sin 65.5°} = \frac{880}{\sin 45.3°}$$

$$c = \frac{880 \sin 65.5°}{\sin 45.3°}$$
$$\underline{\approx 1126.57 \ \ ft}$$



$$\sin 69.2° = \frac{h}{c}$$

$$\boldsymbol{h} = 1126.57 \sin 69.2°$$
$$\underline{\approx 1,053.15 \ \ ft}$$

Exercise

Find the area of the triangle $b = 1, \quad c = 3, \quad A = 80°$

Solution

$$K = \frac{1}{2} bc \sin A$$
$$= \frac{1}{2}(1)(3)\sin 80°$$
$$\underline{\approx 1.48 \ \ unit^2}$$

Exercise

Find the area of the triangle $b = 4, \quad c = 1, \quad A = 120°$

Solution

$$K = \frac{1}{2} bc \sin A$$
$$= \frac{1}{2}(4)(1)\sin 120°$$
$$\underline{\approx 1.732 \ \ unit^2}$$

Exercise

Find the area of the triangle $a=2, \quad c=1, \quad B=10^\circ$

Solution

$$K=\frac{1}{2}ac\sin B$$

$$=\frac{1}{2}(2)(1)\sin 10^\circ$$

$$\approx 0.174\ \ unit^2$$

Exercise

Find the area of the triangle $a=3, \quad c=2, \quad B=110^\circ$

Solution

$$K=\frac{1}{2}ac\sin B$$

$$=\frac{1}{2}(3)(2)\sin 110^\circ$$

$$\approx 2.819\ \ unit^2$$

Exercise

Find the area of the triangle $a=8, \quad b=6, \quad C=30^\circ$

Solution

$$K=\frac{1}{2}ab\sin C$$

$$=\frac{1}{2}(8)(6)\sin 30^\circ$$

$$=12\ \ unit^2$$

Exercise

Find the area of the triangle $a=3, \quad b=4, \quad C=60^\circ$

Solution

$$K=\frac{1}{2}(3)(4)\sin 60^\circ \qquad\qquad K=\frac{1}{2}ab\sin C$$

$$\approx 5.196\ \ unit^2$$

Exercise

Find the area of the triangle $a = 6, \quad b = 4, \quad C = 60°$

Solution

$$K = \frac{1}{2}(6)(4)\sin 60° \qquad\qquad K = \frac{1}{2}ab\sin C$$

$$\approx 10.392 \;\; unit^2$$

Exercise

Find the area of the triangle $a = 4, \quad b = 5, \quad c = 7$

Solution

$$s = \frac{1}{2}(4+5+7) \qquad\qquad s = \frac{1}{2}(a+b+c)$$

$$= 8 \;\; unit$$

$$K = \sqrt{8(8-4)(8-5)(8-7)} \qquad\qquad K = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{8(4)(3)(1)}$$

$$= \sqrt{96}$$

$$\approx 9.8 \;\; unit^2$$

Exercise

Find the area of the triangle $a = 12, \quad b = 13, \quad c = 5$

Solution

$$s = \frac{1}{2}(12+13+5) \qquad\qquad s = \frac{1}{2}(a+b+c)$$

$$= 15 \;\; unit$$

$$K = \sqrt{15(15-12)(15-13)(15-5)} \qquad\qquad K = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{15(3)(2)(10)}$$

$$= 30 \;\; unit^2$$

Exercise

Find the area of the triangle $a = 3, \quad b = 3, \quad c = 2$

Solution

$$s = \frac{1}{2}(3+3+2) \qquad\qquad s = \frac{1}{2}(a+b+c)$$

$$\underline{= 4 \ \ unit}$$

$$K = \sqrt{4(4-3)(4-3)(4-2)} \qquad\qquad K = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\underline{\approx 2.83 \ \ unit^2}$$

Exercise

Find the area of the triangle $a = 4, \quad b = 5, \quad c = 3$

Solution

$$s = \frac{1}{2}(4+5+3) \qquad\qquad s = \frac{1}{2}(a+b+c)$$

$$\underline{= 6 \ \ unit}$$

$$K = \sqrt{6(6-4)(6-5)(6-3)} \qquad\qquad K = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{6(2)(1)(3)}$$

$$\underline{= 6 \ \ unit^2}$$

Exercise

Find the area of the triangle $a = 5, \quad b = 8, \quad c = 9$

Solution

$$s = \frac{1}{2}(5+8+9) \qquad\qquad s = \frac{1}{2}(a+b+c)$$

$$\underline{= 11 \ \ unit}$$

$$K = \sqrt{11(11-5)(11-8)(11-9)} \qquad\qquad K = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{11(6)(3)(2)}$$

$$= \sqrt{96}$$

$$\underline{\approx 19.9 \ \ unit^2}$$

Exercise

Find the area of the triangle $a = 2, \quad b = 2, \quad c = 2$

Solution

$$s = \frac{1}{2}(2+2+2) \qquad\qquad s = \frac{1}{2}(a+b+c)$$

$=3 \;\; unit$

$$K=\sqrt{3(3-2)(3-2)(3-2)} \qquad K=\sqrt{s(s-a)(s-b)(s-c)}$$

$=\sqrt{3}$

$\approx 1.732 \;\; unit^2$

Exercise

Find the area of the triangle $a=4, \quad b=3, \quad c=6$

Solution

$$s=\frac{1}{2}(4+3+6) \qquad s=\frac{1}{2}(a+b+c)$$

$=6.5 \;\; unit$

$$K=\sqrt{6.5(6.5-4)(6.5-3)(6.5-6)} \qquad K=\sqrt{s(s-a)(s-b)(s-c)}$$

$=\sqrt{6.5(2.5)(3.5)(0.5)}$

$\approx 5.33 \;\; unit^2$

Exercise

The dimensions of a triangular lot are 100 *feet* by 50 *feet* by 75 *feet*. If the price of such land is $3 per square foot, how much does the lot cost?

Solution

$$s=\frac{1}{2}(100+50+75) \qquad s=\frac{1}{2}(a+b+c)$$

$=112.5 \;\; unit$

$$K=\sqrt{112.5(112.5-100)(112.5-50)(112.5-75)} \qquad K=\sqrt{s(s-a)(s-b)(s-c)}$$

$=\sqrt{112.5(12.5)(62.5)(37.5)}$

$\approx 1{,}815.46 \;\; ft^2$

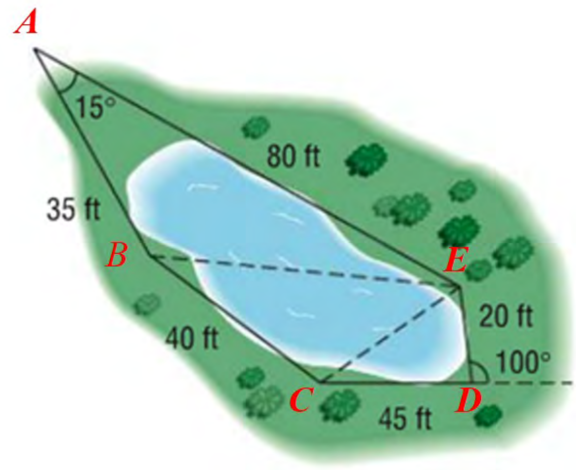
$Cost=(1{,}815.46)(\$3)$

$=\$5,446.38$

Exercise

To approximate the area of a lake, a surveyor walks around the perimeter of the lake. What is the approximate area of the lake?

Solution



Triangle ***ABE***:

$$|BE| = \sqrt{35^2 + 80^2 - 2(35)(80)\cos 15°}$$

$$\approx 47.072 \ ft$$

$$A_{\Delta ABE} = \frac{1}{2}(|AB|)(|BE|)\sin A$$

$$= \frac{1}{2}(80)(35)\sin 15°$$

$$\approx 362.3 \ ft^2$$

Triangle ***CDE***:

$$D = 180° - 100° = 80°$$

$$|CE| = \sqrt{45^2 + 20^2 - 2(45)(20)\cos 80°}$$

$$a = \sqrt{e^2 + b^2 - 2eb\cos A}$$

$$\approx 45.961 \ ft$$

$$A_{\Delta CDE} = \frac{1}{2}(|CD|)(|DE|)\sin D$$

$$= \frac{1}{2}(45)(20)\sin 80°$$

$$\approx 443.2 \ ft^2$$

Triangle ***BCE***:

$$s = \frac{1}{2}(40 + 45.961 + 47.072)$$

$$\approx 66.5$$

$$A_{\Delta BCE} = \sqrt{66.5(66.5 - 40)(66.5 - 45.961)(66.5 - 47.072)}$$

$$\approx 839.6 \ ft^2$$

$$A_{Total} = A_{\Delta ABE} + A_{\Delta CDE} A_{\Delta BCE}$$

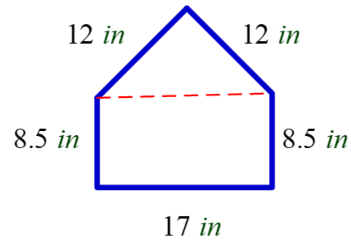
$$= 362.3 + 443.2 + 839.6$$

$$\approx 1,645.1 \ ft^2$$

Exercise

The dimensions of home plate at any major league baseball stadium are shown. Find the area of home plate

Solution



Divide the home plate into a rectangle and triangle.

$$A_{rectangle} = 17 \times 8.5$$

$$\underline{= 144.5 \ ft^2|}$$

$$s = \frac{1}{2}(12+12+17) \qquad\qquad s = \frac{1}{2}(a+b+c)$$

$$\underline{= 20.5 \ unit|}$$

$$A_{triangle} = \sqrt{20.5(20.5-12)(20.5-12)(20.5-17)}$$

$$= \sqrt{20.5(8.5)(8.5)(3.5)}$$

$$= \sqrt{5{,}183.9375}$$

$$\underline{\approx 72 \ in^2|}$$

$$A_{total} = A_{\text{rectangle}} + A_{triangle}$$

$$\approx 144.5 + 72$$

$$\underline{\approx 216.5 \ in^2|}$$

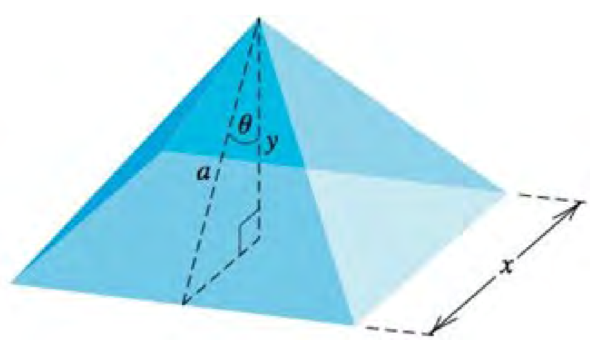
Exercise

A pyramid has a square base and congruent triangular faces. Let θ be the angle that the altitude a of a triangular face makes with the altitude y of the pyramid, and let x be the length of a side.

a) Express the total surface area S of the four faces in terms of a and θ.

b) The volume V of the pyramid equals one-third the area of the base times the altitude. Express V in terms of a and θ.

Solution



a) $\sin\theta = \dfrac{\frac{1}{2}x}{a}$

$$x = 2a\sin\theta$$

Area of one face is:

$$= \frac{1}{2}(base)(height) = \frac{1}{2}xa$$

$$= \frac{1}{2}(2a\sin\theta)(a)$$

$$= a\sin^2\theta$$

Total surface area: $\boxed{S = 4a\sin^2\theta}$

b) $V = \frac{1}{3}(base\ area)(height) = \frac{1}{3}x^2 y$

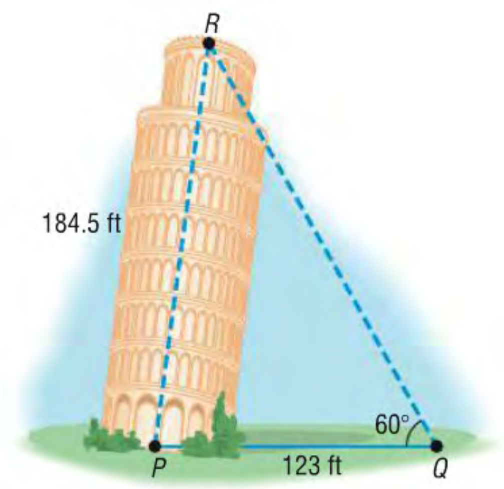
$\cos\theta = \frac{y}{a} \quad \rightarrow \quad y = a\cos\theta$

$V = \frac{1}{3}(2a\sin\theta)^2(a\cos\theta)$

$\boxed{= \frac{4}{3}a^3\sin^2\theta\cos\theta}$

Exercise

The famous Leaning Tower of Pisa was originally 184.5 *feet* high. At a distance of 123 *feet* from the base if the tower, the angle of elevation to the top of the tower is found to be 60°. Find the $\angle RPQ$ indicated in the figure. Also, find the perpendicular distance from R to PQ.



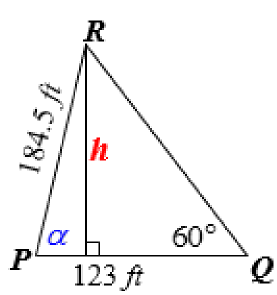
Solution

Let $h: height$

$\frac{\sin R}{123} = \frac{\sin 60°}{184.5}$

$R = \sin^{-1}\left(\frac{123\sin 60°}{184.5}\right)$

$\boxed{\approx 35.3°}$



$\angle RPQ = 180° - 60° - 35.3°$

$\boxed{\approx 84.7°}$

$\sin 84.7° = \frac{h}{184.5}$

$h = 184.5\sin 84.7°$

$\approx 183.71 \ ft$

Exercise

If a mountaintop is viewed from a point P due south of the mountain, the angle of elevation is α. If viewed from a point Q that is d miles cast of P, the angle of elevation is β.

a) Show that the height h of the mountain is given by $h = \dfrac{d\sin\alpha\sin\beta}{\sqrt{\sin^2\alpha - \sin^2\beta}}$

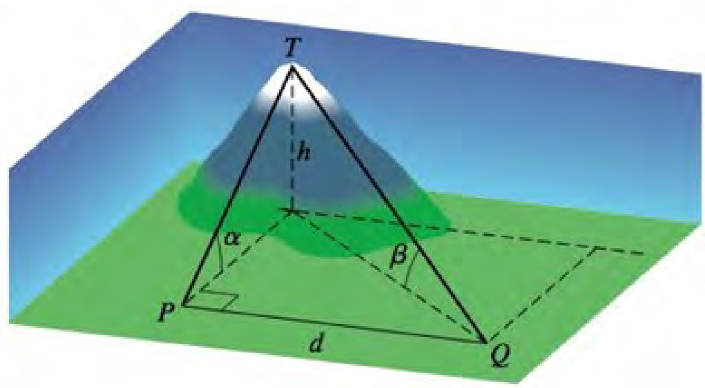
b) If $\alpha = 30°,\quad \beta = 20°,\quad and\quad d = 10\ mi$, approximate h.

Solution

a) Let $\overline{PT} = q \quad \& \quad \overline{QT} = p$

$\sin\alpha = \dfrac{h}{q} \ \rightarrow\ q = \dfrac{h}{\sin\alpha}$

$\sin\beta = \dfrac{h}{p} \ \rightarrow\ p = \dfrac{h}{\sin\beta}$



ΔTPQ: is right triangle at P:

$$d^2 + q^2 = p^2$$

$$d^2 + \left(\frac{h}{\sin\alpha}\right)^2 = \left(\frac{h}{\sin\beta}\right)^2$$

$$h^2\left(\frac{1}{\sin^2\beta} - \frac{1}{\sin^2\alpha}\right) = d^2$$

$$h^2\left(\frac{\sin^2\alpha - \sin^2\beta}{\sin^2\beta\sin^2\alpha}\right) = d^2$$

$$h^2 = \frac{d^2\sin^2\beta\sin^2\alpha}{\sin^2\alpha - \sin^2\beta}$$

$$h = \frac{d\ \sin\beta\sin\alpha}{\sqrt{\sin^2\alpha - \sin^2\beta}} \qquad \checkmark$$

b) **Given**: $\alpha = 30°,\quad \beta = 20°,\quad and\quad d = 10\ mi$

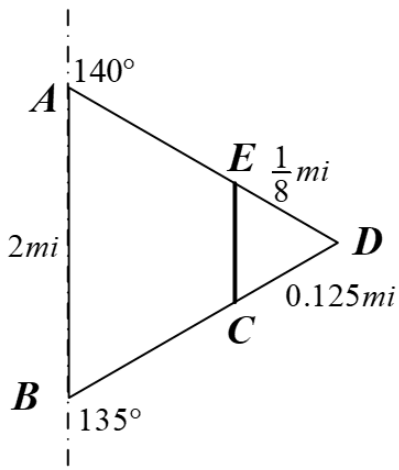
$$h = \frac{10\ \sin 20°\sin 30°}{\sqrt{\sin^2 30° - \sin^2 20°}}$$

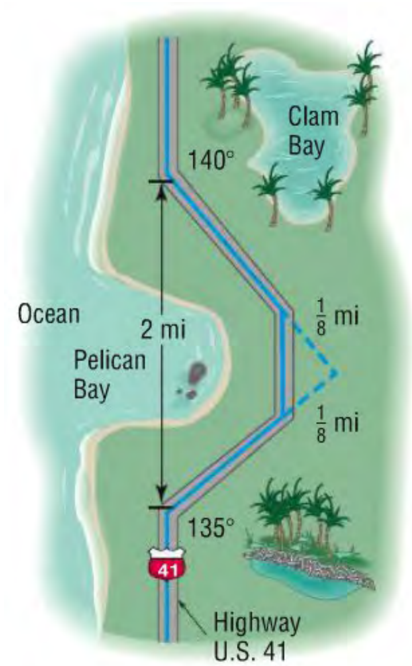
$\approx 4.69\ \ mi$

Exercise

A highway whose primary directions are north–south, is being constructed along the west coast of Florida. Near Naples, a bay obstructs the straight path of the road. Since the cost of a bridge is prohibitive, engineers decide to go around the bay. The path that they decide on and the measurements taken as shown in the picture. What is the length of highway needed to go around the bay?

Solution





$$A = 180° - 140°$$
$$\underline{= 40°|}$$

$$B = 180° - 135°$$
$$\underline{= 45°|}$$

$$D = 180° - 40° - 45°$$
$$\underline{= 95°|}$$

$$\frac{BD}{\sin 40°} = \frac{2}{\sin 95°}$$

$$BD = \frac{2\sin 40°}{\sin 95°}$$
$$\underline{\approx 1.29\ mi|}$$

$$BC = 1.29 - .125$$
$$\underline{\approx 1.165\ mi|}$$

$$\frac{AD}{\sin 45°} = \frac{2}{\sin 95°}$$

$$AD = \frac{2\sin 45°}{\sin 95°}$$
$$\underline{\approx 1.42\ mi|}$$

$$BC = 1.42 - .125$$
$$\underline{\approx 1.295\ mi|}$$

$$CE = \sqrt{(0.125)^2 + (0.125)^2 - 2(0.125)^2 \cos 95°}$$
$$\underline{\approx 0.184\ mi|}$$

The approximation length of highway needed to go around the bay:

$$1.165 + 1.295 + 0.184 \ \underline{\approx 2.64\ \ mi\ |}$$

Exercise

Derive the Mollweide's formula: $\frac{a-b}{c}=\frac{\sin\left[\frac{1}{2}(A-B)\right]}{\cos\left(\frac{1}{2}C\right)}$

Solution

$$\frac{a-b}{c}=\frac{a}{c}-\frac{b}{c}$$

$$=\frac{\sin A}{\sin C}-\frac{\sin B}{\sin C}$$

$$=\frac{\sin A-\sin B}{\sin C}$$

$$=\frac{2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)}{\sin\left(2\cdot\frac{C}{2}\right)} \qquad \sin A-\sin B=2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

$$=\frac{2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)}{2\sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)} \qquad \sin 2\alpha=2\sin\alpha\cos\alpha$$

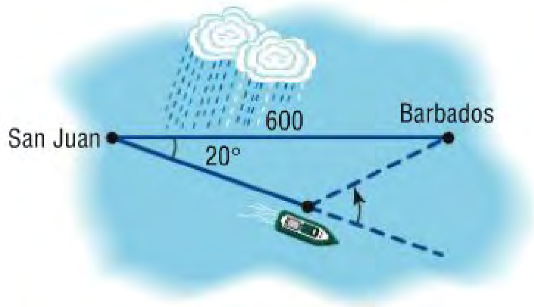
$$=\frac{\sin\left(\frac{A-B}{2}\right)}{\cos\left(\frac{C}{2}\right)}\frac{\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)}{\sin\left(\frac{C}{2}\right)} \qquad \cos A=\sin\left(\frac{\pi}{2}-A\right)$$

$$=\frac{\sin\left(\frac{A-B}{2}\right)}{\cos\left(\frac{C}{2}\right)}\frac{\sin\left(\frac{C}{2}\right)}{\sin\left(\frac{C}{2}\right)}$$

$$=\frac{\sin\left(\frac{A-B}{2}\right)}{\cos\left(\frac{C}{2}\right)} \quad \checkmark$$

Exercise

A cruise ship maintains an average speed of 15 *knots* in going from San Juan, Puerto Rico, to Barbados, West Indies, a distance of 600 *nautical miles*. To avoid a tropical storm, the captain heads out to San Juan in a direction of 20° off a direct heading to Barbados. The captain maintains the 15–knots speed for 10 *hours*, after which time the path to Barbados becomes clear of storms.



a) Through what angle should the captain turn to head directly to Barbados?

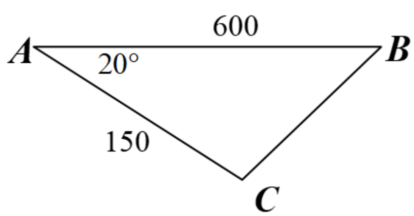
b) Once the turn is made, how long will it be before the ship reaches Barbados if the same 15–*knot* speed is maintained?

Solution

a) After 10 *hrs*., the ship travels $15\times 10=150$ *nautical miles*

$$c=\sqrt{600^2+150^2-2(600)(150)\cos 20^\circ}$$

$$\approx 461.9 \text{ nautical miles}$$



$$C=\cos^{-1}\frac{150^2+461.9^2-600^2}{2(150)(461.9)}$$

$$\approx 153.6^\circ$$

The captain needs to turn the ship through an angle of:

$$180^\circ-153.6^\circ=26.4^\circ$$

b) $t=\dfrac{461.9}{15}$

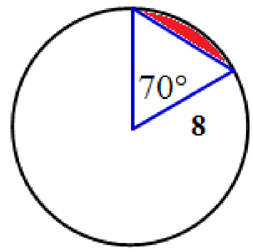
$$\approx 30.8 \ \ hrs$$

The total time for the trip will be about $10+30.8\approx 40.8\ \ hrs$

Exercise

Find the area of the segment (shaded in blue in the figure) of a circle whose radius is 8 *feet*, formed by a cebtral angle of $70°$

Solution



$$\theta = 70° \frac{\pi}{180°}$$
$$= \frac{7\pi}{18}$$

$$A_{\sec tor} = \frac{1}{2}8^2\left(\frac{7\pi}{18}\right)$$
$$\boxed{= \frac{112\pi}{9}\ ft^2}$$

$$A_{\sec tor} = \frac{1}{2}r^2\theta$$

$$A_{triangle} = \frac{1}{2}8^2 \sin 70°$$
$$\boxed{\approx 30.07\ ft^2}$$

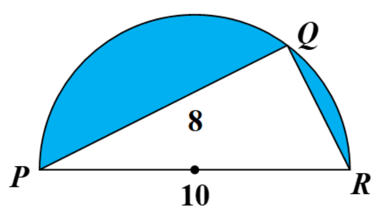
$$A_{triangle} = \frac{1}{2}ab\sin\theta$$

$$A_{segment} = \frac{112\pi}{9} - 30.07$$
$$\boxed{\approx 9.03\ ft^2}$$

Exercise

Find the area of the shaded region enclosed in a semicircle of diameter 10 *inches*. The length of the chord *PQ* is 8 *inches*.

Solution



$$A_{semicircle} = \frac{1}{2}\pi r^2$$
$$\boxed{= \frac{25\pi}{2}\ in^2}$$

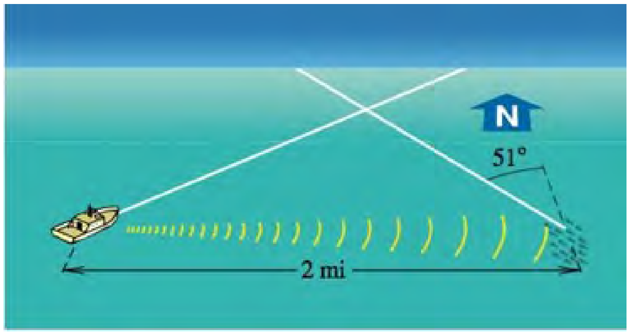
$$\overline{QR} = \sqrt{10^2 - 8^2} = 6$$

$$A_{triangle} = \frac{1}{2}6(8)$$
$$\boxed{= 24\ in^2}$$

$$A_{shaded} = \frac{25\pi}{2} - 24$$
$$\boxed{= 15.27\ in^2}$$

Exercise

A commercial fishing boat uses sonar equipment to detect a school of fish 2 *miles* east of the boat and traveling in the direction of $N\ 51°\ W$ at a rate of 8 mi/hr



a) The boat travels at 20 mi/hr, approximate the direction it should head to intercept the school of fish.

b) Find, to the nearest minute, the time it will take the boat to reach the fish.

Solution

a) $B = 90° - 51°$

$\underline{= 39°}$

Boat distance: $a = 20t$

School fish distance: $b = 8t$

$$\frac{\sin A}{a} = \frac{\sin 39°}{b}$$

$$\frac{\sin A}{20t} = \frac{\sin 39°}{8t}$$

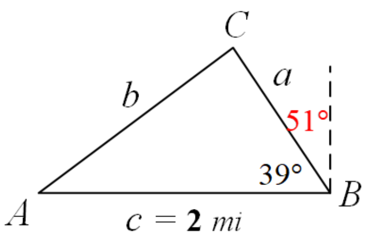
$$A = \sin^{-1}\left(\frac{5}{2}\sin 39°\right)$$

$$\underline{\approx 14.6°}$$

$90° - 14.6° = 75.4°$; the boat should travel in the (approximate) direction $N\ 75.4°\ E$

b) $C = 180° - 14.6° - 39°$

$\underline{\approx 126.4°}$



$$\frac{a}{\sin 14.6} = \frac{2}{\sin 126.4°}$$

$$a = \frac{2}{\sin 126.4°}\sin 14.6$$

$$\underline{\approx 1.56\ mi}$$

$$t = \frac{a}{20}$$

$$\approx \frac{1.56}{20}$$

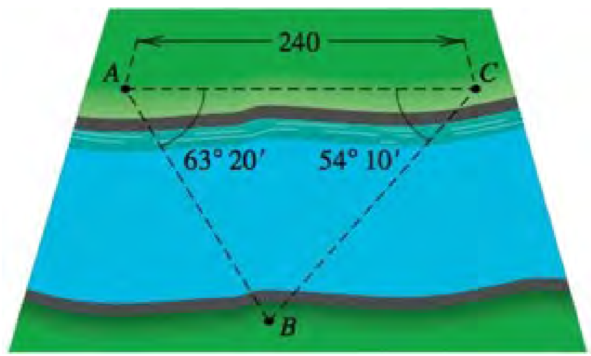
$$\approx 0.08hr$$

$$\underline{\approx 5\ \text{min}}$$

Exercise

To find the distance between two points *A* and *B* that lie on opposite banks of a river, a surveyor lays off a line segment *AC* of length 240 *yards* along one bank and determines that the measures of $\angle BAC$ and $\angle ACB$ are $63^\circ\ 20'$ and $54^\circ\ 10'$, respectively.
Approximate the distance between *A* and *B*.

Solution



$$\angle B = 180^\circ - 63^\circ\ 20' - 54^\circ\ 10'$$
$$\underline{= 62^\circ\ 30'}$$

$$\frac{|AB|}{\sin(54^\circ\ 10')} = \frac{240}{\sin(62^\circ\ 30')}$$

$$|AB| = \frac{240\sin(54^\circ\ 10')}{\sin(62^\circ\ 30')}$$

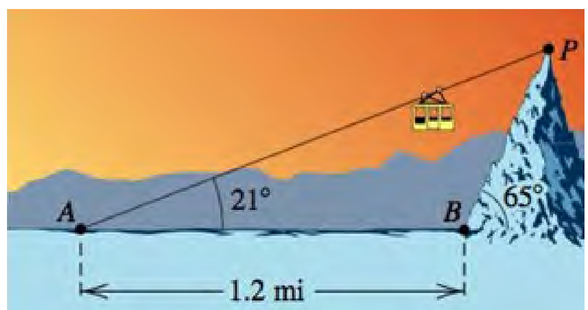
$$\underline{\approx 219.4\ \ yds}$$

Exercise

A cable car carries passengers from a point *A*, which is 1.2 *miles* from a point *B* at the base of a mountain, to a point *P* at the top of the mountain. The angle of elevation of *P* from *A* and *B* are 21° and 65°, respectively.

a) Approximate the distance between *A* and *P*.
b) Approximate the height of the mountain.

Solution



$$\angle ABP = 180^\circ - 65^\circ \ \underline{= 115^\circ}$$

$$\angle APB = 180^\circ - 115^\circ - 21^\circ \ \underline{= 44^\circ}$$

a) $$\frac{|AP|}{\sin 115^\circ} = \frac{1.2}{\sin 44^\circ}$$

$$|AP| = \frac{1.2\sin 115^\circ}{\sin 44^\circ}$$

$$\underline{\approx 1.57\ \ mi}$$

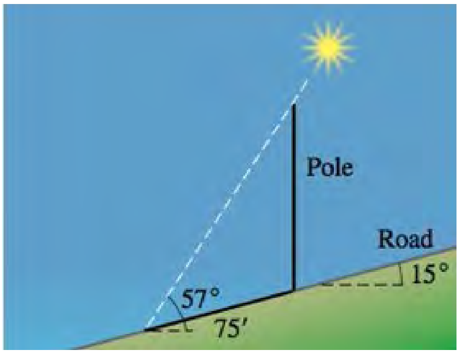
b) $$\sin 21^\circ = \frac{h}{|AP|}$$

$$h \approx (1.57)\sin 21^\circ$$

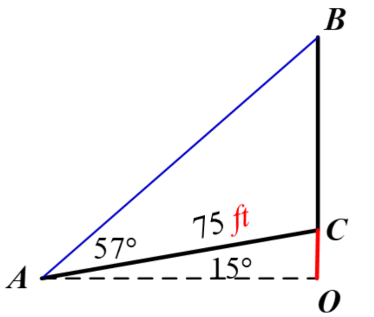
$$\underline{\approx 0.56\ \ mi}$$

Exercise

A straight road makes an angle of $15°$ with the horizontal. When the angle of elevation of the sun is $57°$, a vertical pole at the side of the road casts a shadow 75 *feet* long directly down the road. Approximate the length of the pole.



Solution



$$B = 90° - 57°$$

$$\underline{= 33°|}$$

$$\frac{|BC|}{\sin(57° - 15°)} = \frac{75}{\sin 33°}$$

$$|BC| = \frac{75 \sin 42°}{\sin 33°}$$

$$\underline{\approx 92.14 \ \ ft \ |}$$

Exercise

The angles of elevation of a balloon from two points *A* and *B* on level ground are $24° \ 10'$ and $47° \ 40'$, respectively. Points *A* and *B* are 8.4 *miles* apart, and the balloon is between the points, in the same vertical plane. Approximate the height of the balloon above the ground.

Solution

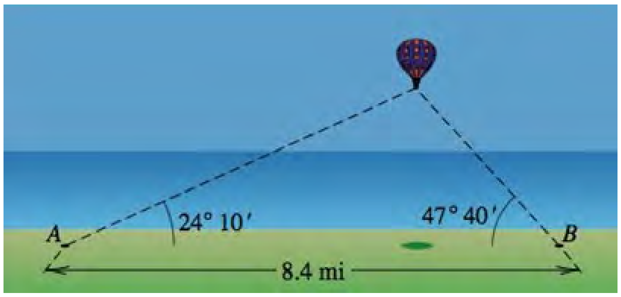
$$\angle C = 180° - 24° \ 10' - 47° \ 40'$$

$$\underline{= 108° \ 10'|}$$

$$\frac{|AC|}{\sin(47° \ 40')} = \frac{84}{\sin(108° \ 10')}$$

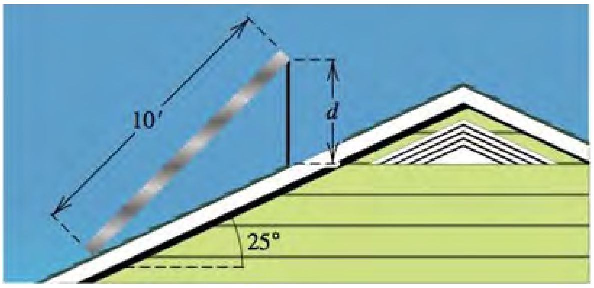
$$|AB| = \frac{84 \sin(47° \ 40')}{\sin(108° \ 10')}$$

$$\underline{\approx 65.4 \ \ miles \ |}$$



Exercise

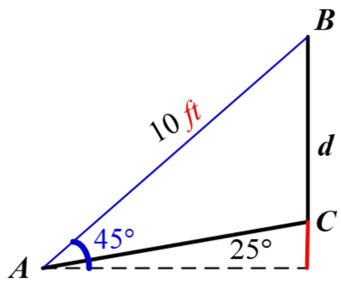
A solar panel 10 *feet* in width, which is to be attached to a roof that makes an angle of $25°$ with the horizontal. Approximate the length *d* of the brace that is needed for the panel to make an angle of $45°$ with the horizontal.



Solution

$$\angle B = 90° - 45°$$
$$= 45°$$

$$\angle C = 180° - (45° - 25°) - 45°$$
$$= 115°$$



$$\frac{d}{\sin(45° - 25°)} = \frac{10}{\sin(115°)}$$

$$d = \frac{10\ \sin(20°)}{\sin(115°)}$$

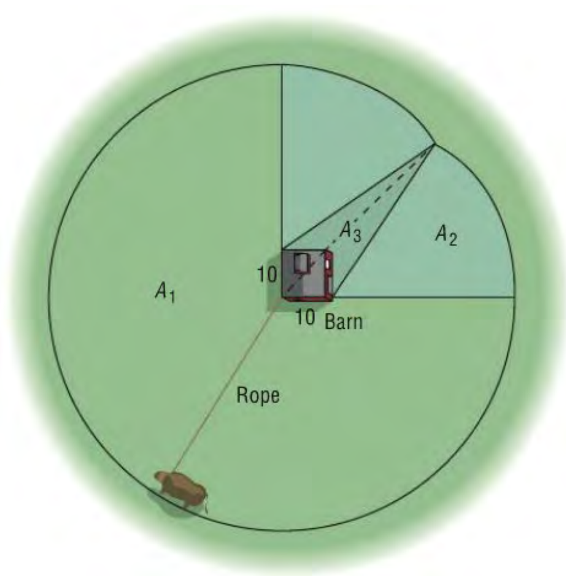
$$\approx 3.8\ ft$$

Exercise

A cow is tethered to one corner of a square barn, 10 *feet* by 10 *feet*, with a rope 100 feet long.

a) What is the maximum grazing area for the cow?

b) If the barn is rectangular, 10 *feet* by 20 *feet*, what is the maximum grazing area for the cow?



Solution

a) $A_1 = \frac{3}{4}(Area\ of\ \ the\ circle)$

$$= \frac{3}{4}\left(\pi r^2\right)$$

$$= \frac{3}{4}\left(\pi 10^4\right)$$

$$= 7,500\pi\ \ ft^2 \qquad \approx 23,561.94\ ft^2$$

$\angle ABC = 45° \quad (square) \quad AB = 10, \quad AC = 90$

$$\frac{\sin C}{10} = \frac{\sin 45°}{90}$$

$$\sin C = \frac{1}{9}\frac{\sqrt{2}}{2}$$

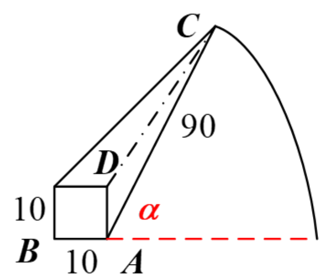
$$\underline{|C} = \sin^{-1}\left(\frac{\sqrt{2}}{18}\right) \ \underline{\approx 4.51^\circ|}$$

$$\angle CAB \approx 180^\circ - 45^\circ - 4.51^\circ$$

$$\underline{\approx 130.49^\circ|}$$

$$\angle DAC = 130.49^\circ - 90^\circ$$

$$\underline{\approx 40.49^\circ|}$$



$$A_3 = 2Area(ADC)$$

$$= 2 \cdot \frac{1}{2}(AD)(AC)\cos A$$

$$= (10)(90)\cos 40.49^\circ$$

$$\underline{\approx 584.38 \ ft^2|}$$

$$\alpha = 180^\circ - 130.49^\circ \ \underline{\approx 49.51^\circ|}$$

$$A_2 = \frac{1}{2}90^2\left(49.51^\circ \frac{\pi}{180^\circ}\right) \qquad A = \frac{1}{2}r^2\theta$$

$$\underline{\approx 3,499.66 \ ft^2|}$$

Total grazing area $\approx 23,561.94 + 584.38 + 2(3,499.66)$

$$\underline{\approx 31,146 \ ft^2|}$$

b) Let BC be the radius 90 center at B, where $B(0,\ 10)$ & $C(x,\ y)$

$$x^2 + (y-10)^2 = 90^2 \quad (\mathbf{1})$$

Let AC be radius 80 center at A, where $A(20,\ 0)$:

$$(x-20)^2 + y^2 = 80^2 \quad (\mathbf{2})$$

So the point C is the intersection of the 2 circles (**1**) & (**2**) using graphing tool:

$$C(77.705,\ 55.409)$$

$$A_1 = \frac{3}{4}(Area\ of\ the\ circle)$$

$$\underline{\approx 23,561.94 \ ft^2|}$$

Let $D(20,\ 10)$

$$CD = \sqrt{(77.7-20)^2 + (55.4-10)^2}$$

$$\underline{\approx 73.4|}$$

$$\angle CAD = \cos^{-1}\frac{80^2 + 10^2 - 73.4^2}{2(80)(10)}$$

$$\approx 45.95^\circ$$

$$A_3 = \frac{1}{2}(80)(10)\sin 45.95^\circ$$

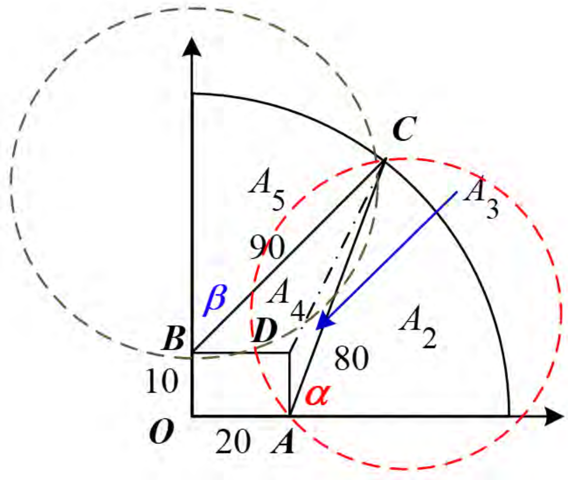
$$\approx 287.49 \ ft^2$$

$$\alpha = 90^\circ - 45.95^\circ$$

$$\approx 44.05^\circ$$

$$A_2 = \frac{1}{2}r^2\theta = \frac{1}{2}80^2\left(44.05^\circ \frac{\pi}{180^\circ}\right)$$

$$\approx 2,460.22 \ ft^2$$



$$\angle CBD = \cos^{-1}\frac{90^2 + 20^2 - 73.4^2}{2(90)(20)}$$

$$\approx 30.17^\circ$$

$$A_4 = \frac{1}{2}(90)(20)\sin 30.17^\circ$$

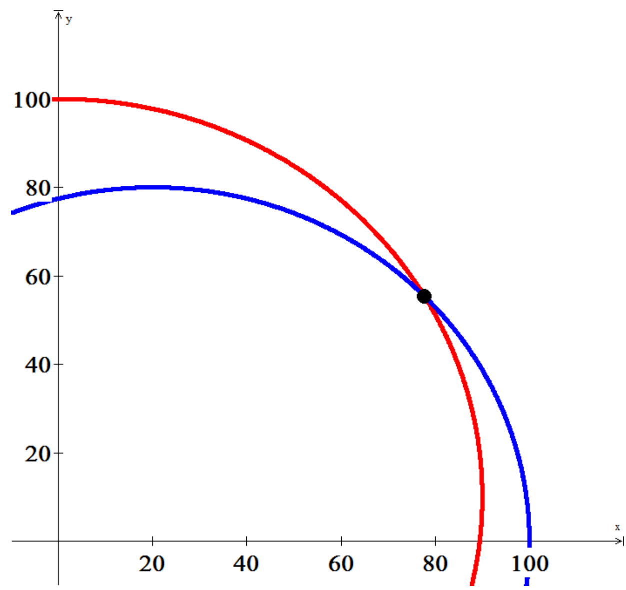
$$\approx 542.31 \ ft^2$$

$$\alpha = 90^\circ - 30.17^\circ$$

$$\approx 59.83^\circ$$

$$A_5 = \frac{1}{2}r^2\theta = \frac{1}{2}90^2\left(59.83^\circ \frac{\pi}{180^\circ}\right)$$

$$\approx 4,229.13 \ ft^2$$



Total grazing area

$$\approx 23,561.94 + 2,460.22 + 287.49 + 542.31 + 4,229.13$$

$$\approx 31,081 \ ft^2$$

Exercise

For any triangle, show that $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$ *where* $s = \frac{1}{2}(a+b+c)$

Solution

$$\cos\frac{C}{2} = \sqrt{\frac{1+\cos C}{2}}$$

$$= \sqrt{\frac{1}{2}\left(1+\frac{a^2+b^2-c^2}{2ab}\right)}$$

$$= \sqrt{\frac{2ab+a^2+b^2-c^2}{4ab}}$$

$$= \sqrt{\frac{(a+b)^2-c^2}{4ab}} \qquad x^2-y^2=(x-y)(x+y)$$

$$= \sqrt{\frac{(a+b+c)(a+b-c)}{4ab}} \qquad 2s=a+b+c \quad 2s-c=a+b$$

$$= \sqrt{\frac{2s(2s-c-c)}{4ab}}$$

$$= \sqrt{\frac{2s(2s-2c)}{4ab}}$$

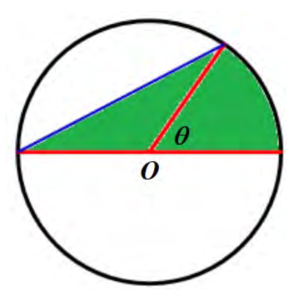
$$= \sqrt{\frac{4s(s-c)}{4ab}}$$

$$= \sqrt{\frac{s(s-c)}{ab}} \quad \checkmark$$

Exercise

The figure shows a circle of radius r with center at O. find the area K of the shaded region as a function of the central angle θ.

Solution



$$A_{sector} = \frac{1}{2}r^2\theta$$

$$A_{triangle} = \frac{1}{2}r^2\sin(\pi-\theta)$$

$$K = \frac{1}{2}r^2\theta + \frac{1}{2}r^2\sin(\pi-\theta)$$

$$= \frac{1}{2}r^2(\theta+\sin(\pi-\theta)) \qquad \sin(\pi-\theta)=\sin\theta$$

$$\boxed{= \frac{1}{2}r^2(\theta+\sin\theta)}$$

Exercise

Refer to the figure, in which a unit circle is drawn. The line segment *DB* is tangent to the circle and θ is acute.

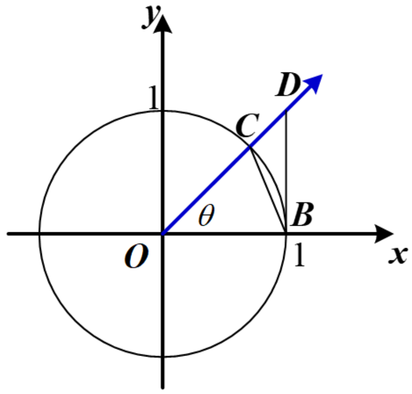
a) Express the area of ΔOBC in terms of $\sin\theta$ and $\cos\theta$.

b) Express the area of ΔOBD in terms of $\sin\theta$ and $\cos\theta$.

c) The area of the sector $\overset{\frown}{OBC}$ if the circle is $\frac{1}{2}\theta$, where θ is measured in *radians*. Use the results of part (*a*) and (*b*) and the fact that

$$Area\ \Delta OBC < Area\ \overset{\frown}{OBC} < Area\ \Delta OBD$$

To show that $1 < \dfrac{\theta}{\sin\theta} < \dfrac{1}{\cos\theta}$



Solution

a) Area of $\Delta OBC = \frac{1}{2}(1)(1)\sin\theta$

$$= \boxed{\frac{1}{2}\sin\theta}$$

b) $\tan\theta = \dfrac{\overline{BD}}{1} \quad \to \quad \overline{BD} = \tan\theta$

Area of $\Delta OBD = \frac{1}{2}\overline{OB} \times \overline{BD}$

$$= \frac{1}{2}(1)\tan\theta$$

$$= \boxed{\frac{\sin\theta}{2\cos\theta}}$$

c) $Area\ \Delta OBC < Area\ \overset{\frown}{OBC} < Area\ \Delta OBD$

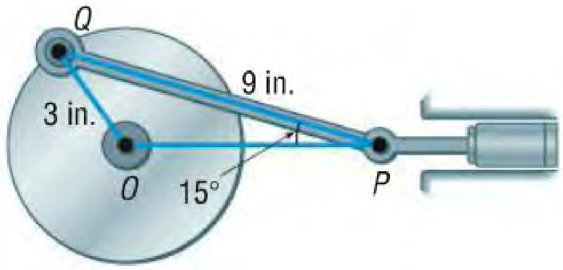
$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\frac{\sin\theta}{\cos\theta}$$

$$\sin\theta < \theta < \frac{\sin\theta}{\cos\theta} \qquad \times\frac{1}{\sin\theta}$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

Exercise

On a certain automobile, the crankshaft is 3 *inches* long and the connecting rod is 9 *inches* long. At the time when $\angle OPQ$ is 15°, how far is the piston *P* from the center *O* of the crankshaft?



Solution

$$\frac{\sin O}{9} = \frac{\sin 15°}{3}$$

$$\hat{O} = \sin^{-1}(3\sin 15°)$$
$$\approx 50.94°$$

$$O = 50.94°$$
$$Q = 180° - 50.94° - 15°$$
$$= 114.06°$$

$$\frac{q}{\sin 114.06°} = \frac{3}{\sin 15°}$$

$$q = \frac{3\sin 114.06°}{\sin 15°}$$
$$\approx 10.58\ in$$

$$O = 180° - 50.94°$$
$$= 129.06°$$

$$Q = 180° - 129.06° - 15°$$
$$= 35.94°$$

$$\frac{q}{\sin 35.94°} = \frac{3}{\sin 15°}$$

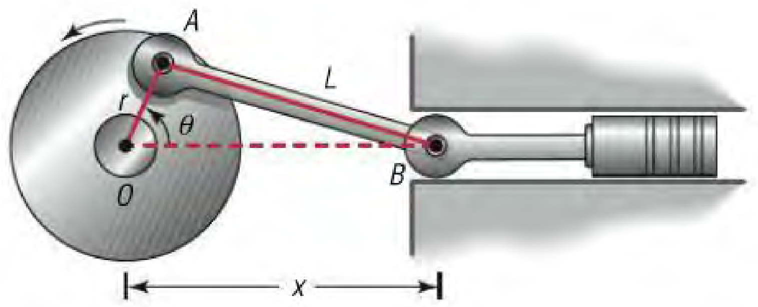
$$q = \frac{3\sin 35.94°}{\sin 15°}$$
$$\approx 6.80\ in$$

The distance from the piston P to the center O of the crankshaft is approximately either 10.58 *inches* or 6.8 *inches*.

Exercise

Rod OA rotates about the fixed point O so that point A travels on a circle of radius r. Connected to point A is another rod AB of length $L > 2r$, and point B is connected to a piston. Show that the distance x between point O and point B is given by

$$x = r\cos\theta + \sqrt{r^2\cos^2\theta + L^2 - r^2}$$



Where θ is the angle of rotation of rod OA.

Solution

$$L^2 = r^2 + x^2 - 2rx\cos\theta$$ *Law of cosine*

$$x^2 - 2rx\cos\theta + r^2 - L^2 = 0$$

$$x = \frac{2r\cos\theta \pm \sqrt{4r^2\cos^2\theta - 4\left(r^2 - L^2\right)}}{2}$$ $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2r\cos\theta \pm 2\sqrt{r^2\cos^2\theta - r^2 + L^2}}{2}$$

$$= r\cos\theta + \sqrt{r^2\cos^2\theta - r^2 + L^2} \quad \checkmark$$

Exercise

Find the area of the segment of a circle whose radius is 5 *inches*, formed by a central angle of $40°$.

Solution

$$\theta = 40° \frac{\pi}{180°}$$

$$= \frac{2\pi}{9}$$

$$A_{sector} = \frac{1}{2}5^2\left(\frac{2\pi}{9}\right) \qquad A_{sector} = \frac{1}{2}r^2\theta$$

$$= \frac{25\pi}{9}\ in^2$$

$$A_{triangle} = \frac{1}{2}5^2 \sin 40° \qquad A_{triangle} = \frac{1}{2}ab\sin\theta$$

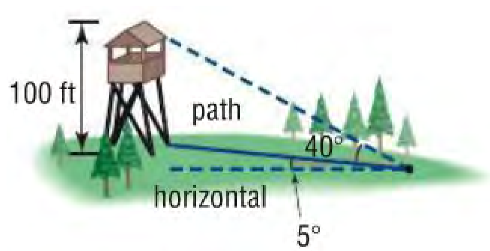
$$= \frac{25}{2}\sin 40°\ \ in^2$$

$$A_{segment} = \frac{25\pi}{9} - \frac{25}{2}\sin 40°$$

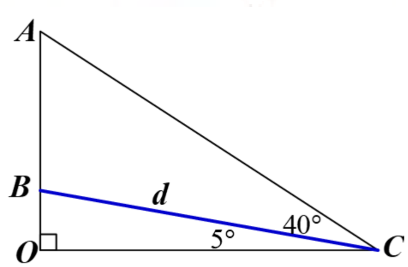
$$\approx 0.69\ \ in^2$$

Exercise

A forest ranger is walking on a path inclined at 5° to the horizontal directly toward a 100–*foot*–tall fire observation tower. The angle of elevation from the path to the top of the tower is 40°. How far is the ranger from the tower at this time?



Solution



$$\angle OCA = 40° + 5°$$

$$= 45°$$

$$A = 90° - 45°$$

$$= 45°$$

$$\angle ABC = B = 180° - 45° - 40°$$

$$\underline{= 95°}$$

$$\frac{d}{\sin 45°} = \frac{100}{\sin 40°}$$

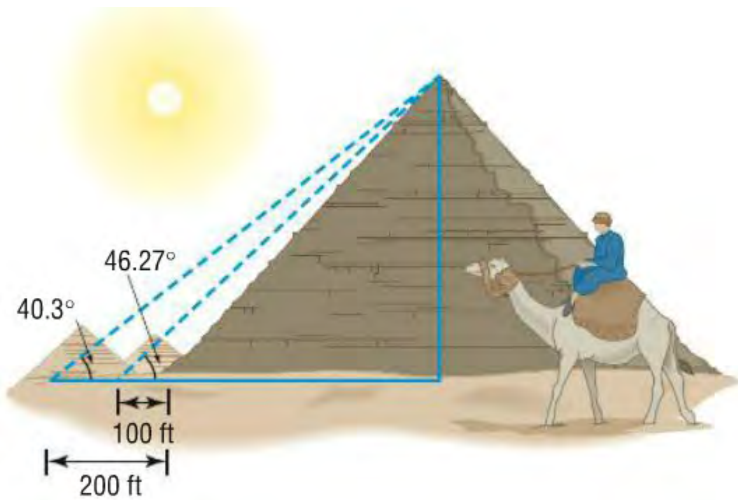
$$d = \frac{100 \sin 45°}{\sin 40°}$$

$$\underline{\approx 110.01 \ \ feet}$$

The ranger distance is about **110.01** *feet* from the tower.

Exercise

One of the original Seven Wonders of the world, the Great Pyramid of Cheops was built about 2580 BC. Its original height was 480 *feet* 11 *inches*, but owing to the loss of its topmost stones, it is now shorter. Find the current height of the Great Pyramid using the information shown in the picture.



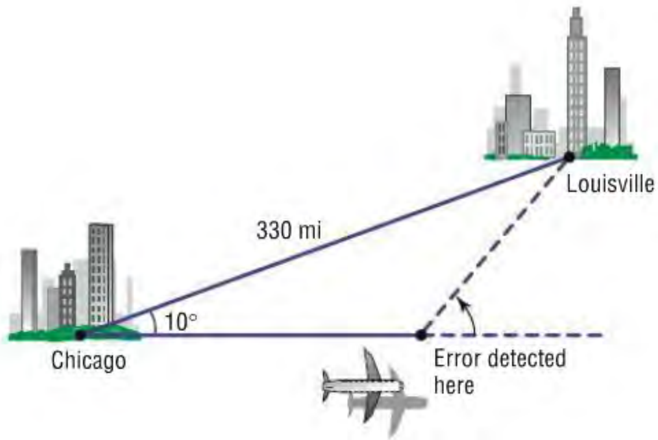
Solution

$$h = \frac{100 \tan 43.27° \tan 40.3°}{\tan 43.27° - \tan 40.3°} \qquad\qquad h = \frac{x \tan\alpha \tan\beta}{\tan\alpha - \tan\beta}$$

$$\underline{\approx 449.36 \ \ ft}$$

Exercise

In attempting to fly from Chicago to Louisville, a distance of 330 *miles*, a pilot inadvertently took a course that was 10° in error.



a) If the aircraft maintains an average speed of 220 *miles* per *hours*, and if the error in direction is discovered after 15 *minutes*, through what angle should the pilot turn to head toward Louisville?

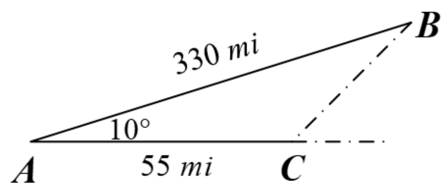
b) What new average speed should the pilot maintain so that the total time of the trip is 90 *minutes*?

Solution

a) After 15 *min*., the ship travels

$$15 \times 220 = 3,300 \times \frac{1\ hr}{60\ min}$$

$$= 55\ \ miles$$



$$c = \sqrt{330^2 + 55^2 - 2(330)(55)\cos 10^\circ}$$

$$\approx 276\ miles$$

$$C = \cos^{-1} \frac{55^2 + 276^2 - 330^2}{2(55)(276)}$$

$$\approx 168^\circ$$

The pilot needs to turn through an angle of:

$$180^\circ - 1168^\circ \ = 12^\circ$$

b) The total time of the trip is 90 *min*
Since the pilot found that 25 *min* were used (*error*).
Then, there are $90 - 15 = 75\,min$ to complete the trip.

The plane must travel 276 miles in $75min \times \frac{1hr}{60min} = 1.25\ hrs$.

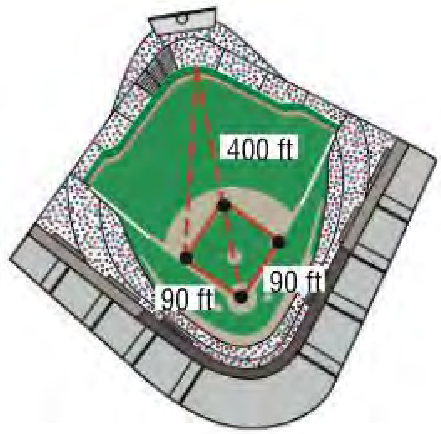
$$r = \frac{276}{1.25}$$

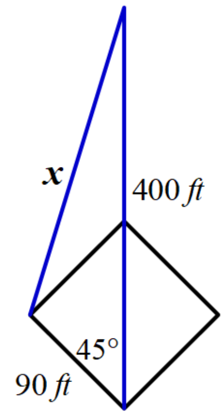
$$\approx 220.8\ \ \ mi\ /\ hr$$

The pilot must maintain a speed of 200.8 *mi/hr.* to complete the trip in 90 *min*.

Exercise

The distance from home plate to the fence in dead center is 400 *feet*. How far is it for the fence in dead center to third base?





Solution

$$x = \sqrt{400^2 + 90^2 - 2(400)(90)\cos 45^\circ}$$

$$= \sqrt{160000 + 8100 - 36000\sqrt{2}}$$

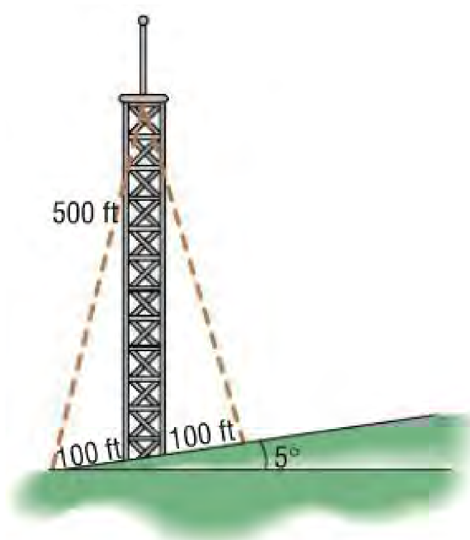
$$= 10\sqrt{1681 - 360\sqrt{2}}$$

$$\approx 342.33 \ ft$$

It is approximately 342.33 *feet* from dead center to third base.

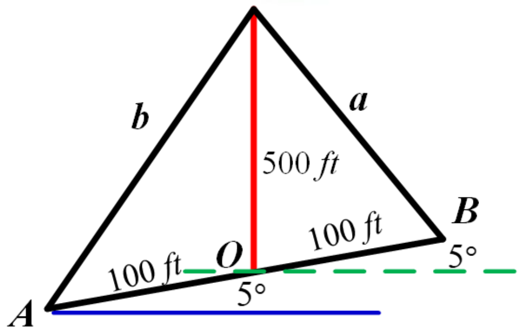
Exercise

A radio tower 500 *feet* high is located on the side of a hill with an inclination to the horizontal of 5°. How long should two guy wires be if they are to connect to the top of the tower and be secured at two points 100 *feet* directly above and directly below the base of the tower?



Solution

$$\angle BOC = 90° - 5°$$
$$\boxed{= 85°}$$
$$\angle AOC = 90° + 5°$$
$$\boxed{= 95°}$$



$$a = \sqrt{500^2 + 100^2 - 2(500)(100)\cos 85°}$$
$$\boxed{\approx 501.28\ ft}$$

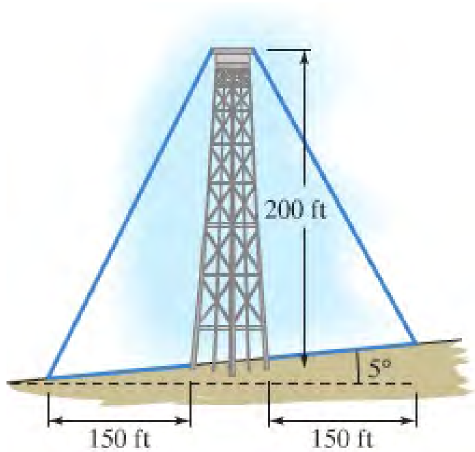
Right of the tower, it needs about **501.28** *feet* of wires

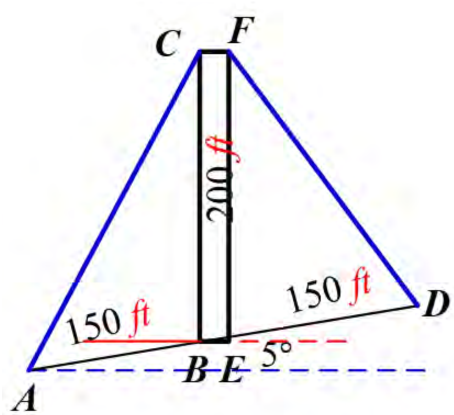
$$b = \sqrt{500^2 + 100^2 - 2(500)(100)\cos 95°}$$
$$\boxed{\approx 518.38\ ft}$$

Left of the tower, it needs about **518.38** *feet* of wires

Exercise

A 200-*foot* tower on the side of a hill that forms a 5° angle with the horizontal. Find the length of each of the two guy wires that are anchored 150 *feet* uphill and downhill from the tower's base and extend to the top of the tower.





Solution

$$B = 90° + 5° = 95°$$

$$b = \sqrt{200^2 + 150^2 - 2(200)(150)\cos 95°} \qquad b^2 = a^2 + c^2 - 2ac\cos B$$
$$\boxed{\approx 260.2\ ft}$$

$$E = 90° - 5° = 85°$$

$$e = \sqrt{200^2 + 150^2 - 2(200)(150)\cos 85°} \qquad e^2 = d^2 + f^2 - 2df\cos E$$
$$\boxed{\approx 239.3\ ft}$$

Exercise

When the angle of elevation of the sun is 62°, a telephone pole that is tilted at an angle of 8° directly away from the sun casts a shadow 20 *feet* long. Determine the length of the pole.

Solution

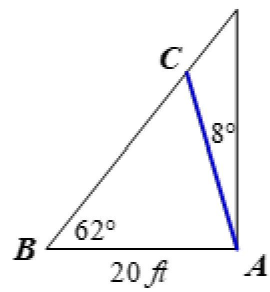
$A = 90° - 8° \ \underline{= 82°}$

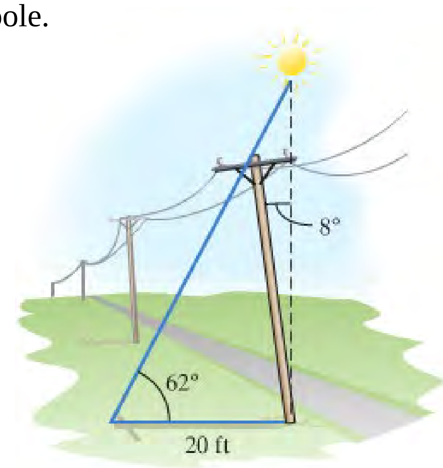
$C = 180° - 82° - 62° \ \underline{= 36°}$

$\dfrac{|AC|}{\sin 62°} = \dfrac{20}{\sin 82°}$

$|AC| = \dfrac{20\sin 62°}{\sin 82°}$

$\underline{\approx 30.0 \ ft}$



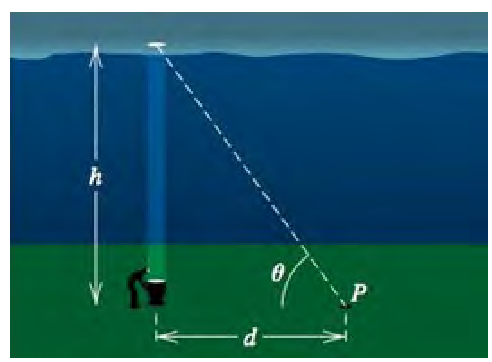


Exercise

To measure the height h of a cloud cover, a meteorology student directs a spotlight vertically upward from the ground. From a point P on level ground that is d meters from the spotlight, the angle of elevation θ of the light image on the clouds is then measured.

a) Express h in terms of d and θ

b) Approximate h if $d = 1000 \ m$ and $\theta = 59°$



Solution

a) $\tan\theta = \dfrac{h}{d}$

$\underline{h = d\tan\theta}$

b) **Given**: $d = 1000 \ m$ and $\theta = 59°$

$h = 1,000 \tan 59°$

$\underline{\approx 1,664.3 \ m}$

Exercise

A hot–air balloon is rising vertically. From a point on level ground 125 *feet* from the point directly under the passenger compartment, the angle of elevation to the balloon changes from $19.2°$ to $31.7°$. How far does the balloon rise during this period?

Solution

$$\tan 19.2^\circ = \frac{y}{125}$$

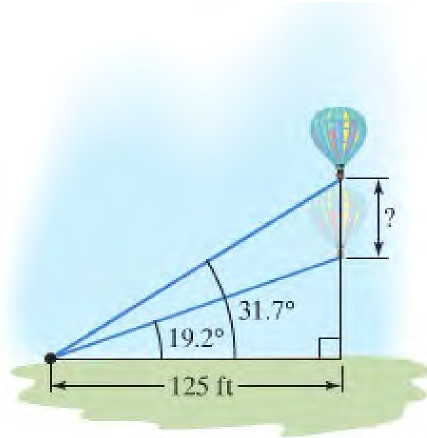
$$\underline{y = 125 \tan 19.2^\circ}|$$

$$\tan 31.7^\circ = \frac{x+y}{125}$$

$$x = 125 \tan 31.7^\circ - y$$

$$= 125 \tan 31.7^\circ - 125 \tan 19.2^\circ$$

$$\approx 33.7 \ ft$$



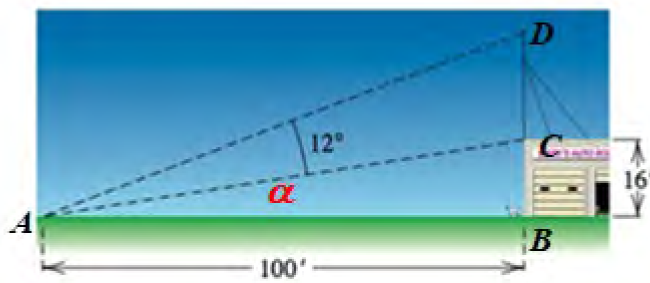
Exercise

A *CB* antenna is located on the top of a garage that is 16 *feet* tall. From a point on level ground that is 100 *feet* from a point directly below the antenna, the antenna subtends an angle of 12°. Approximate the length of the antenna.

Solution

$$\alpha = \tan^{-1} \frac{16}{100}$$

$$\approx 9.09^\circ$$



$$\tan(12^\circ + 9.09^\circ) = \frac{16 + |CD|}{100}$$

$$|CD| = 100 \tan(12^\circ + 9.09^\circ) - 26$$

$$\approx 38.567 \ ft$$

The length of the antenna is ≈ 39 *feet*

Exercise

A tsunami is a tidal wave caused by an earthquake beneath the sea. These waves can be more than 100 *feet* in height and can travel at great speeds. Engineers sometimes represent such waves by trigonometric expressions of the form $y = a \cos bt$ and use these representations to estimate the effectiveness of sea walls. Suppose that a wave has height $h = 50\ ft$ and period time 30 *minutes* and is traveling at the rate of $180\ ft / \sec$

a) Let (x, y) be a point on the wave represented in the figure. Express *y* as a function of *t* if $y = 25\ ft$ when $t = 0$.

b) The wave length *L* is the distance between two successive crests of the wave. Approximate *L* in *feet*.

Solution

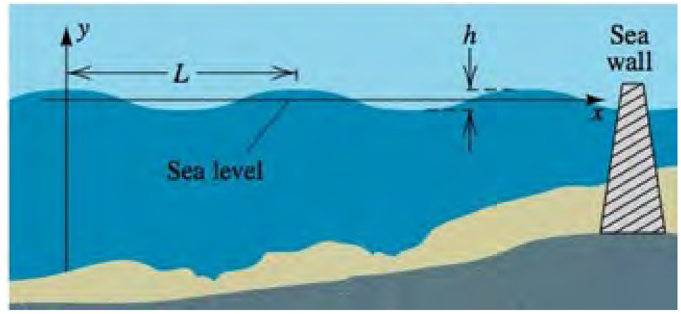
a) **Given**: $y = a\cos bt \quad |A| = h, \quad P = L$

$a = |A| \; \underline{= h|}$

$P = \dfrac{2\pi}{b} = L$

$\underline{b = \dfrac{2\pi}{L}|}$

$\underline{y = h\cos\dfrac{2\pi}{L}t|}$



Given: $y = 25\ ft$ when $t = 0$

$25 = h\cos 0$

$\underline{h = 25\ \ ft|}$

$\underline{y(t) = 25\cos\dfrac{2\pi}{L}t\ |}$

Exercise

Two fire–lookout stations are 20 *miles* apart, with station *B* directly east of station *A*. Both stations spot fire on a mountain to the north. The bearing from station *A* to the fire is N50°E. The bearing from station *B* to the fire is N36°W. How far is the fire from station *A*?

Solution

$C = 180° - 40° - 54°$

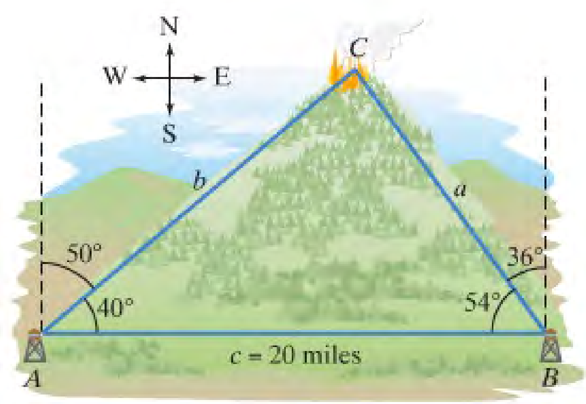
$\underline{= 86°|}$

$\dfrac{b}{\sin 54°} = \dfrac{20}{\sin 86°}$

$b = \dfrac{20\sin 54°}{\sin 86°}$

$\underline{\approx 16.2\ \ miles\ |}$

The fire is approximately 16.2 *miles* from station *A*.



Exercise

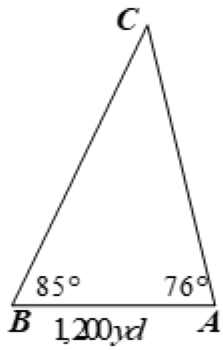
A 1200–yard–long sand beach and an oils platform in the ocean. The angle made with the platform from one end of the beach is 85° and from the other end is 76°. Find the distance of the oil platform from each end of the beach.

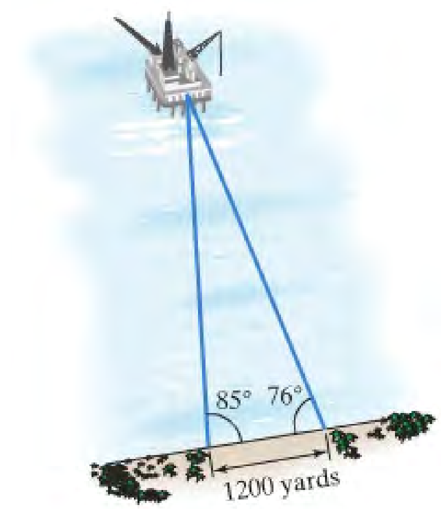
Solution

$$C = 180° - 85° - 76°$$
$$\underline{= 19°|}$$

$$\frac{b}{\sin 85°} = \frac{1200}{\sin 19°}$$
$$b = \frac{1200 \sin 85°}{\sin 19°}$$
$$\underline{\approx 3671.8 \ yds|}$$

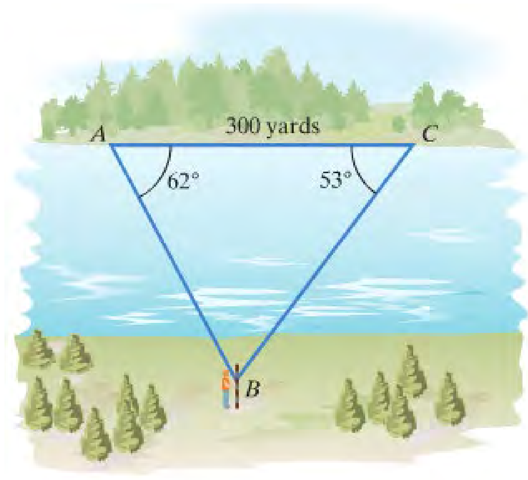
$$\frac{a}{\sin 76°} = \frac{1200}{\sin 19°}$$
$$a = \frac{1200 \sin 75°}{\sin 19°}$$
$$\underline{\approx 3576.4 \ yds|}$$





Exercise

A surveyor needs to determine the distance between two points that lie on opposite banks of a river. 300 *yards* are measured along one bank. The angle from each end of this line segment to a point on the opposite bank are 62° and 53°. Find the distance between A and B.



Solution

$$B = 180^\circ - 62^\circ - 53^\circ$$
$$= 65^\circ$$

$$\frac{|AB|}{\sin 53^\circ} = \frac{300}{\sin 65^\circ}$$

$$|AB| = \frac{300 \sin 53^\circ}{\sin 65^\circ}$$
$$\approx 264.4 \ yds$$

Exercise

A pine tree growing on a hillside makes a 75° angle with the hill. From a point 80 *feet* up the hill, the angle of elevation to the top of the tree is 62° and the angle of depression to the bottom is 23°. Find the height of the tree.

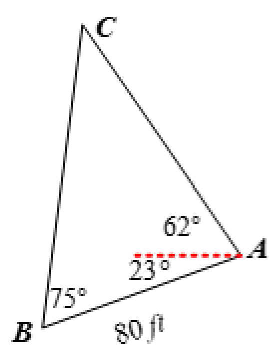
Solution

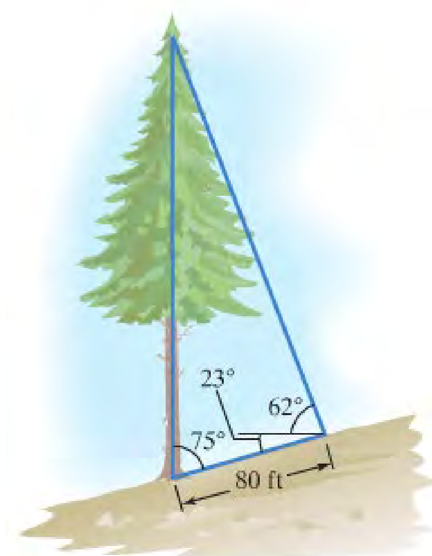
$$A = 62^\circ + 23^\circ$$
$$= 85^\circ$$

$$C = 180^\circ - 75^\circ - 85^\circ$$
$$= 20^\circ$$

$$\frac{|BC|}{\sin 85^\circ} = \frac{80}{\sin 20^\circ}$$

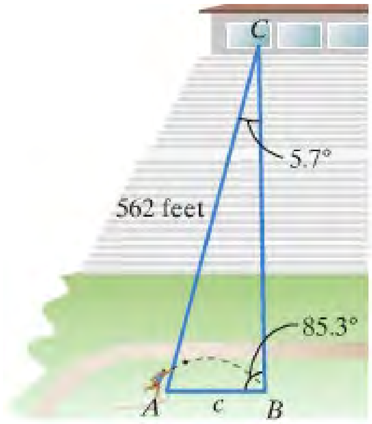
$$|BC| = \frac{80 \sin 85^\circ}{\sin 20^\circ}$$
$$\approx 233 \ ft$$





Exercise

The shot of a hot-put ring is tossed from A lands at B. Using modern electronic equipment, the distance of the toss can be measured without the use of measuring tapes. When the shot lands at B, an electronic transmitter placed at B sends a signal to a device in the official's booth above the track. The device determines the angles B and C. At a track meet, the distance from the official' booth to the shot-ring is 562 *feet*. If $B = 85.3°$ and $C = 5.7°$, determine the length of the toss.



Solution

$$\frac{c}{\sin 5.7°} = \frac{562}{\sin 85.3°}$$

$$c = \frac{562 \sin 5.7°}{\sin 85.3°}$$

$$\approx 56 \;\; ft$$

Exercise

A pier forms an 85° angle with a straight shore. At a distance of 100 *feet* from a pier, the line of sight to the tip forms a 37° angle. Find the length of the pier.

Solution

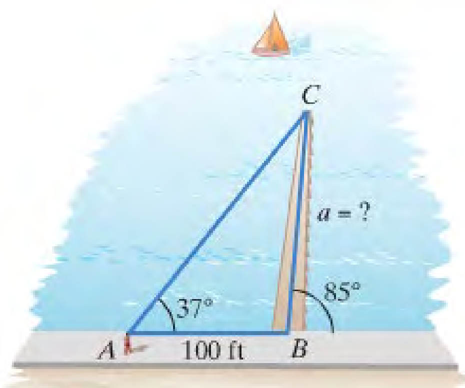
$$B = 180° - 85°$$
$$= 95°$$

$$C = 180° - 37° - 95°$$
$$= 48°$$

$$\frac{a}{\sin 37°} = \frac{100}{\sin 48°}$$

$$a = \frac{100 \sin 37°}{\sin 48°}$$

$$\approx 81.0 \; ft$$



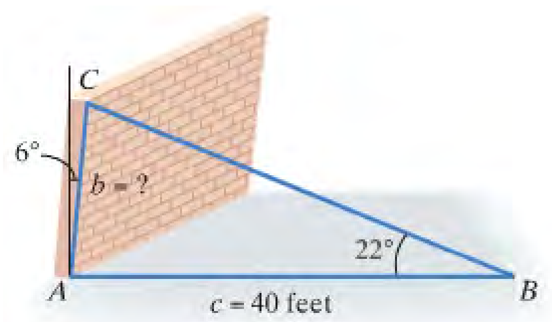
Exercise

A leaning wall is inclined 6° from the vertical. At a distance of 40 *feet* from the wall, the angle of elevation to the top is 22°.Find the height of the wall.

Solution

$$A = 90° - 6°$$
$$= 84°$$

$$C = 180° - 84° - 22°$$
$$= 74°$$



$$\frac{b}{\sin 22°} = \frac{40}{\sin 74°}$$

$$b = \frac{40 \sin 22°}{\sin 74°}$$

$$\approx 15.6 \ ft$$

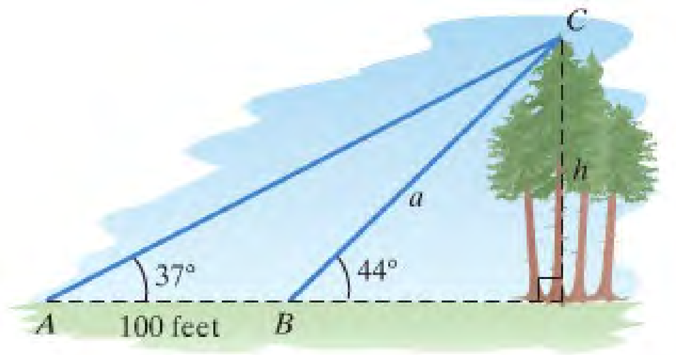
Exercise

Redwood trees are hundreds of feet tall. The height of one of these is represented by *h*.

a) Find the height of the tree.

b) Find *a*.

Solution



a) $h = \frac{100 \tan 44° \tan 37°}{\tan 44° - \tan 37°}$

$$\approx 343.0 \ ft$$

b) $\sin 44° = \frac{h}{a}$

$$a = \frac{343}{\sin 44°}$$

$$\approx 493.8 \ ft$$

Exercise

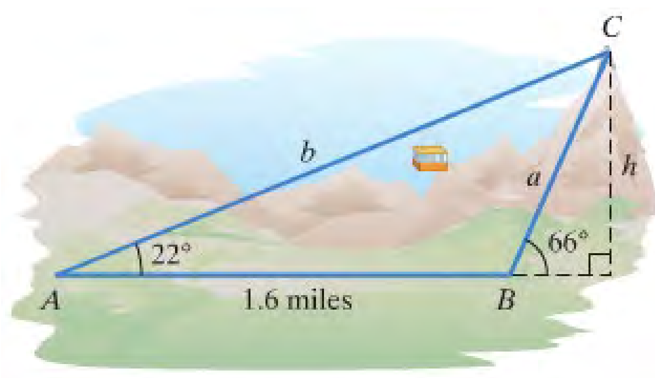
A carry cable car that carries passengers from *A* to *C*. Point *A* is 1.6 *miles* from the base of the mountain. The angles of elevation from *A* and *B* to the mountain's peak are 22° and 66°, respectively.

a) Find the height of the mountain.

b) Determine the distance covered by the cable car.

c) Find *a*.

Solution



a) $h = \frac{1.6 \ \tan 66° \tan 22°}{\tan 66° - \tan 22°}$

$$\approx 0.788 \ mi$$

b) $\sin 22° = \frac{h}{b}$

$$b = \frac{0.788}{\sin 22°}$$

$$\approx 2.1042 \ mi$$

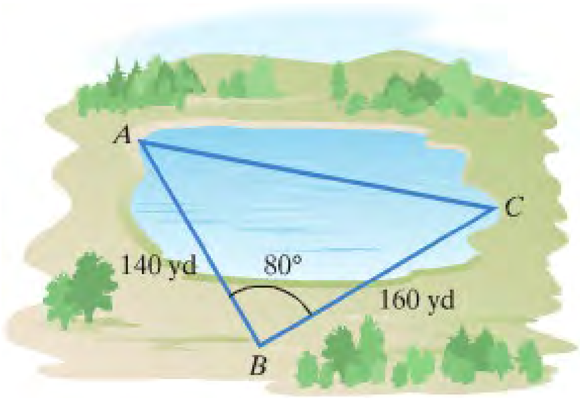
c) $\sin 66° = \frac{h}{a}$

$$b = \frac{1.2687}{\sin 66°}$$

$$\approx 1.389 \ \ mi$$

Exercise

Find the distance across the lake from *A* to *C*.



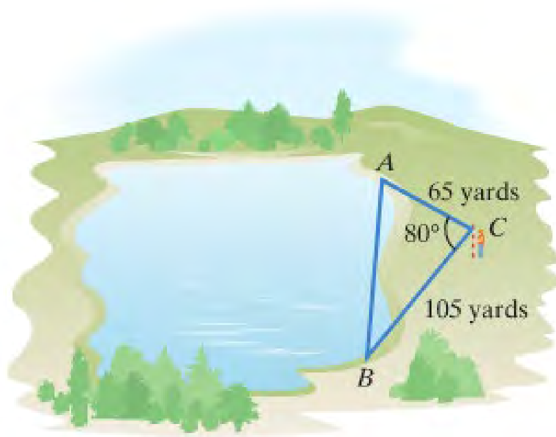
Solution

$$b = \sqrt{160^2 + 140^2 - 2(160)(140)\cos 80°} \qquad b^2 = a^2 + c^2 - 2ac\cos B$$

$$\approx 193 \ \ yd$$

Exercise

To find the distance across a protected cove at a lake, a surveyor makes the measurements. Find the distance from *A* to *B*.



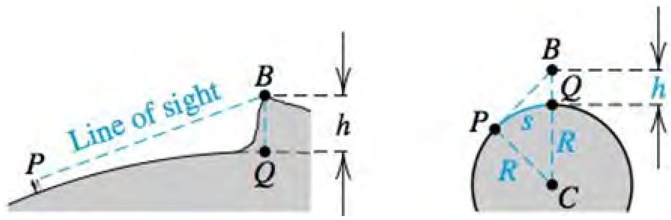
Solution

$$c = \sqrt{105^2 + 65^2 - 2(105)(65)\cos 80°} \qquad c^2 = a^2 + b^2 - 2ab\cos C$$

$$\approx 113 \ \ yd$$

Exercise

A surveyor using a transit, sights the edge B of a bluff, as shown in the left of the figure. Because of the curvature of Earth, the true elevation $\boldsymbol{h}$ of the bluff is larger than that measured by the surveyor. A cross-sectional schematic view of Earth is shown in the right part of the figure.



a) If s is the length of arc PQ and R is the distance from P to the center C of Earth, express h in terms of R and s.

b) If $R = 4{,}000\ mi$ and $s = 50\ mi$, estimate the elevation of the bluff in feet.

Solution

a) Right triangle CPB: $\cos\theta = \dfrac{R}{R+h}$

$$R + h = \frac{R}{\cos\theta} = R\sec\theta$$

$$h = R\sec\theta - R$$

From the arc: $\theta = \dfrac{s}{R}$

$$h = R\sec\frac{s}{R} - R$$

b) ***Given***: $R = 4{,}000\ mi$, $s = 50\ mi$

$$h = 4{,}000\sec\frac{50}{4{,}000} - 4{,}000$$

$$\approx 1{,}650\ \ ft$$

Exercise

Shown in the figure is a design for a rain gutter.

a) Express the volume V as a function of θ.

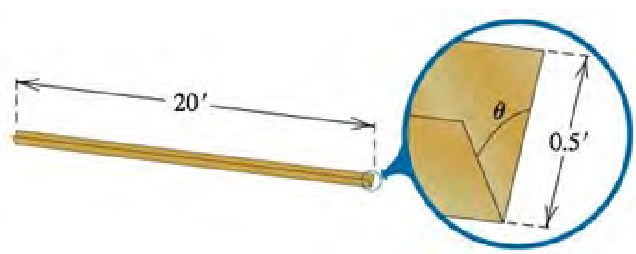
b) Approximate the acute angle θ that results in a volume of $2\ ft^3$

Solution

a) $Volume = 20 \times (Area\ of\ \ the\ sector\text{-}triangle)$

$$= 20 \times \left(\tfrac{1}{2}(0.5)^2 \sin\theta\right)$$

$$= 2.5\ \sin\theta$$



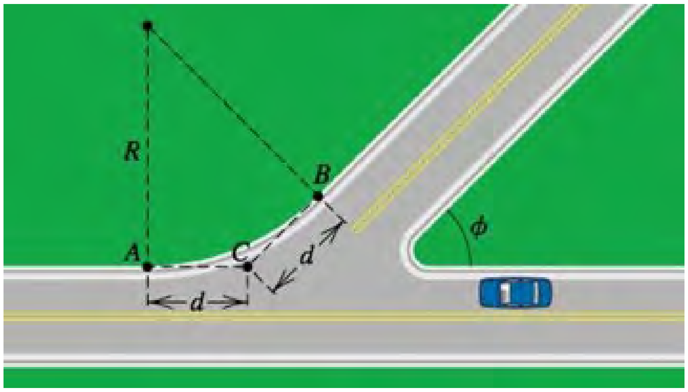
b) $\dfrac{5}{2}\sin\theta = 2$

$$\theta = \sin^{-1}\left(\frac{4}{5}\right)$$

$$\approx 53.13°$$

Exercise

A highway engineer is designing curbing for a street at an intersection where two highways meet at an angle ϕ, as shown in the figure, the curbing between points A and B is to be constructed using a circle that is tangent to the highway at these two points.



a) Show that the relationship between the radius R of the circle and the distance d in the figure is given by the equation $d = R\tan\frac{\phi}{2}$.

b) If $\phi = 45°$ and $d = 20\ ft$, approximate R and the length of the curbing.

Solution

a) $\tan\frac{\phi}{2} = \frac{d}{R}$

$$\boxed{d = R\tan\frac{\phi}{2}}$$

b) Given: $\phi = 45°$, $d = 20\ ft$

$$R = \frac{d}{\tan\frac{\phi}{2}}$$

$$= \frac{20}{\tan\frac{45°}{2}}$$

$$\boxed{\approx 48.28\ \ ft}$$

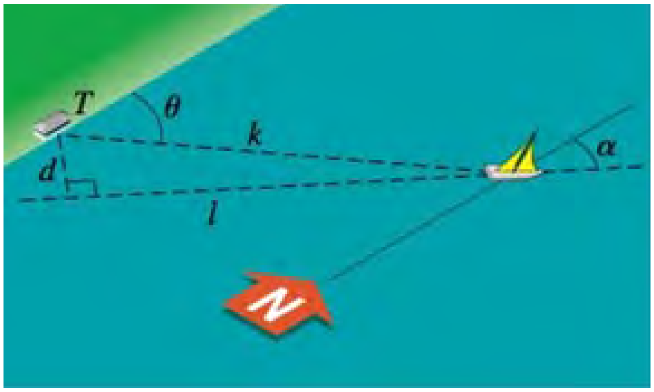
Length of the curbing (s) $= R\phi$

$$= 48.28 \times \frac{\pi}{4}$$

$$\boxed{\approx 37.92\ \ ft}$$

Exercise

A sailboat is following a straight line course l. (Assume that the shoreline is parallel to the north-south line.) The shortest distance from a tracking station T to the course is d miles. As the boat sails, the tracking station records its distance k from T and its direction θ with respect to T. Angle α specifies the direction of the sailboat.



a) Express α in terms of d, k, and θ.

b) Estimate α to the nearest degree if $d = 50\ mi,\ k = 210\ mi,$ and $\theta = 53.4°$

Solution

a) $\beta = \theta - \alpha$

$\sin\beta = \frac{d}{k} \quad \to \quad \beta = \sin^{-1}\frac{d}{k}$

$\beta = \theta - \alpha = \sin^{-1}\frac{d}{k}$

$\alpha = \theta - \sin^{-1}\frac{d}{k}$

b) **Given**: $d = 50\ mi,\ k = 210\ mi,\ \theta = 53.4°$

$\alpha = 53.4° - \sin^{-1}\frac{50}{210}$

$\approx 39.6°$

Exercise

An art critic whose eye level is 6 *feet* above the floor views a painting that is 10 *feet* in height and is mounted 4 *feet* above the floor.

a) If the critic is standing x *feet* from the wall, express the viewing angle θ in terms of x.

b) Use the addition formula for the tangent to show that $\theta = \tan^{-1}\left(\frac{10x}{x^2 - 16}\right)$

c) For what value of x is $\theta = 45°$?

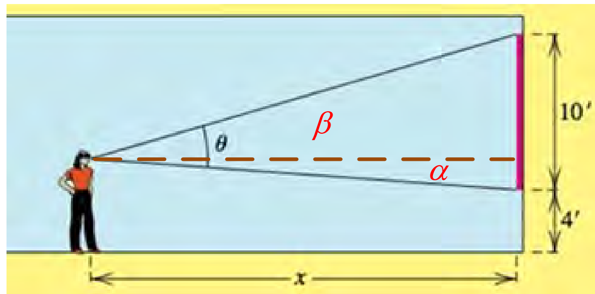
Solution

a) $\tan\alpha = \frac{2}{x} \quad \to \quad \alpha = \tan^{-1}\frac{2}{x}$

$$\tan\beta = \frac{8}{x} \quad \to \quad \boxed{\beta = \tan^{-1}\frac{8}{x}}$$

$$\theta = \alpha + \beta$$

$$\boxed{= \tan^{-1}\frac{8}{x} + \tan^{-1}\frac{2}{x}}$$



b) $\tan\theta = \tan(\alpha + \beta)$

$$\tan\theta = \frac{\frac{2}{x} + \frac{8}{x}}{1 - \frac{16}{x^2}}$$

$$= \frac{10x}{x^2 - 16}$$

$$\boxed{\theta = \tan^{-1}\left(\frac{10x}{x^2 - 16}\right)}$$

c) $\tan 45° = \frac{10x}{x^2 - 16} = 1$

$$x^2 - 16 = 10x$$

$$x^2 - 10x - 16 = 0 \qquad x = \frac{10 \pm \sqrt{100 + 64}}{2} = \frac{10 \pm \sqrt{164}}{2} = \frac{10 \pm 2\sqrt{41}}{2}$$

$$\boxed{x = 5 + \sqrt{41} \ \ ft}$$

Exercise

When an individual is walking, the magnitude *F* of the vertical force of one foot on the ground can be described by

$$F = A(\cos bt - a\cos 3bt), \qquad \text{where } t \text{ is time in seconds, } A > 0,\ b > 0 \text{ and } 0 < a < 1$$

a) Show that $F = 0$, when $t = -\frac{\pi}{2b}$ and $t = \frac{\pi}{2b}$. (the time $t = -\frac{\pi}{2b}$ corresponds to the moment when the foot first touches the ground and the weight of the body is being supported by the other foot.)

b) The maximum force occurs when $3a\sin 3bt = \sin bt$.

If $a = \frac{1}{3}$, find the solutions of this equation for the interval $-\frac{\pi}{2b} < t < \frac{\pi}{2b}$.

c) If $a = \frac{1}{3}$, express the maximum force in terms of *A*.

Solution

a) $F = A(\cos bt - a\cos 3bt) = 0 \quad \Rightarrow \quad \cos bt = a\cos 3bt$

Since $a \neq 0$, $\cos bt = \cos 3bt = 0$

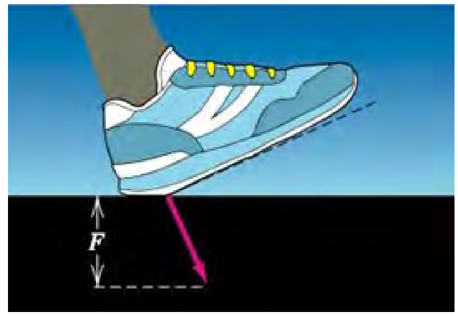
$$\cos bt = 0$$

$$bt = \pm\frac{\pi}{2}$$

$$\boxed{t = \pm\frac{\pi}{2b}}$$

$$\cos 3bt = 0$$

$$\to\ 3bt = \pm\frac{\pi}{2} + 2k\pi$$



b) **Given**: $a = \frac{1}{3}$

$$3a\sin 3bt = 3\frac{1}{3}\sin 3bt$$

$$= \boxed{\sin 3bt = \sin bt}$$

$$\sin 3bt = \sin bt \to \begin{cases} \boxed{t = 0} \\ bt = \pm\frac{\pi}{4} & \boxed{t = \pm\frac{\pi}{4b}} \\ 3bt = \pm\frac{\pi}{4} \end{cases}$$

c) $F = A\left(\cos bt - \frac{1}{3}\cos 3bt\right)$

$$= A\left(\cos b\frac{\pi}{4b} - \frac{1}{3}\cos 3b\frac{\pi}{4b}\right)$$

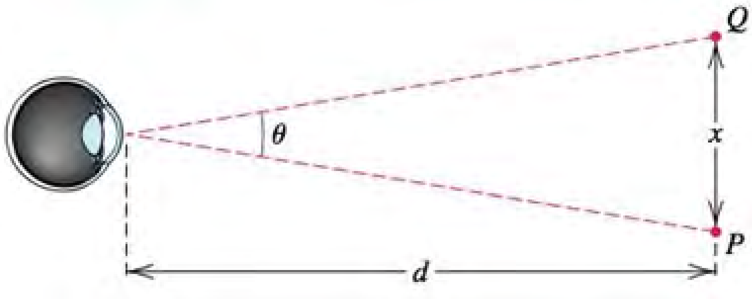
$$= A\left(\cos\frac{\pi}{4} - \frac{1}{3}\cos\frac{3\pi}{4}\right)$$

$$= A\left(\frac{\sqrt{2}}{2} + \frac{1}{3}\frac{\sqrt{2}}{2}\right)$$

$$\boxed{= \frac{2\sqrt{2}}{3}A}$$

Exercise

The human eye can distinguish between two distant points P and Q provided the angle of resolution θ is not too small. Suppose P and Q are x units apart and are d units from the eye.



a) Express x in terms of d and θ.

b) For a person with normal vision, the smallest distinguishable angle of resolution is about 0.0005 *radian*. If a pen 6 *inches* long is viewed by such an individual at a distance of d *feet*, for what values of d will be the end points of the pen be distinguishable?

Solution

a) $\tan\frac{\theta}{2} = \frac{\frac{x}{2}}{d}$

$$\boxed{x = 2d\tan\frac{\theta}{2}}$$

b) $d = \frac{x}{2}\frac{1}{\tan\frac{\theta}{2}}$

$$= 3\frac{1}{\tan\frac{.0005}{2}}\frac{1\ ft}{12\ in}$$

$$\boxed{\approx 999.99\ ft}$$

$$\boxed{d \le 1{,}000\ ft}$$

Exercise

A satellite S circles a planet at a distance d *miles* from the planet's surface. The portion of the planet's surface that is visible from the satellite is determined by the angle θ.

a) Assuming that the planet is spherical in shape, express d in terms of θ and the radius r of the planet.

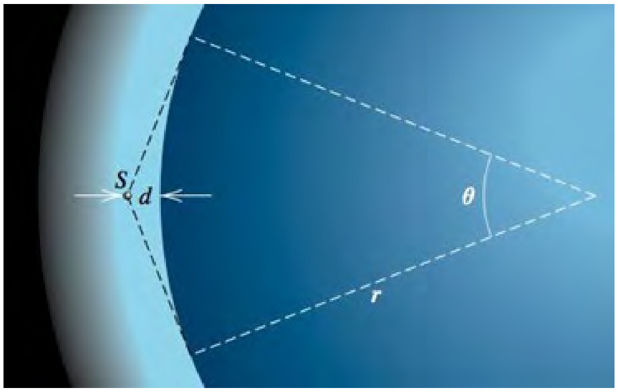
b) Approximate θ for a satellite 300 *miles* from the surface of Earth, using $r = 4{,}000\ mi$.

Solution

a) $\cos\frac{\theta}{2} = \frac{r}{r+d}$

$$r + d = \frac{r}{\cos\frac{\theta}{2}}$$

$$\boxed{d = r\left(\sec\frac{\theta}{2} - 1\right)}$$



b) ***Given***: $r = 4{,}000\ mi$ & $d = 300\ mi$

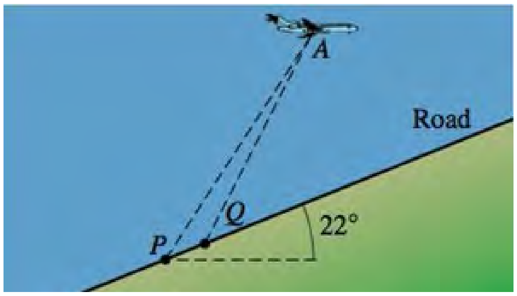
$$\frac{\theta}{2} = \cos^{-1}\frac{r}{r+d}$$

$$\theta = 2\cos^{-1}\frac{4{,}000}{4{,}300}$$

$$\approx 43.06^\circ$$

Exercise

A straight road makes an angle of 22° with the horizontal. From a certain point *P* on the road, the angle of elevation of an airplane at point *A* is 57°. At the same instant, form another point *Q*, 100 *meters* farther up the road, the angle of elevation is 63°. The points *P*, *Q*, and *A* lie in the same vertical plane.



Approximate the distance from *P* to the airplane.

Solution

$$\angle APQ = 57^\circ - 22^\circ$$

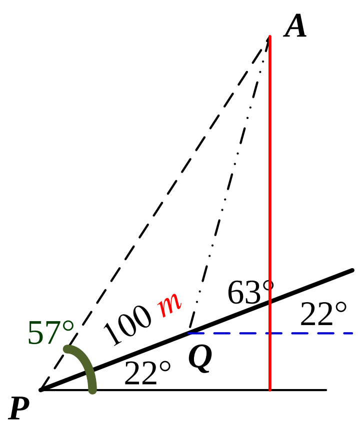
$$\underline{= 35^\circ}$$

$$\angle AQP = 180^\circ - \left(63^\circ - 22^\circ\right)$$

$$\underline{= 139^\circ}$$

$$\angle QAP = 180^\circ - 139^\circ - 35^\circ$$

$$\underline{= 6^\circ}$$



$$\frac{|PA|}{\sin\angle AQP} = \frac{100}{\sin\angle QAP}$$

$$|PA| = \frac{100\ \sin 139^\circ}{\sin 6^\circ}$$

$$\approx 627.64\ m$$

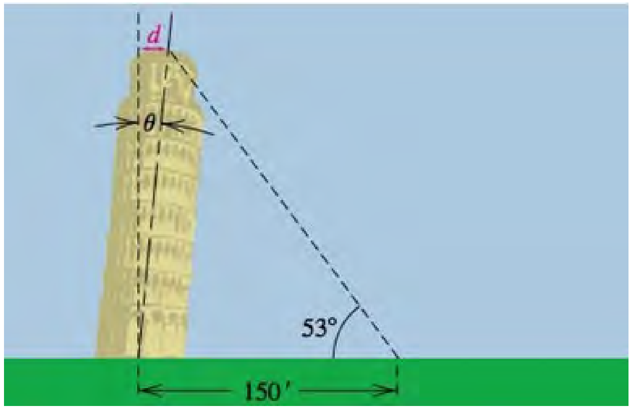
Exercise

The leaning tower of Pisa was originally perpendicular to the ground and 179 *feet* tall. Because of sinking into the earth, it now leans at a certain angle θ from the perpendicular. When the top of the tower is viewed from a point 150 *feet* from the center of its base, the angle of elevation is 53°.

a) Approximate the angle θ.

b) Approximate the distance *d* that the center of the top the tower has moved from the perpendicular.

Solution



a) $\sin 53^\circ = \dfrac{179}{x}$

$$x = \frac{179}{\sin 53^\circ}$$

$$\approx 224.13$$

$$y = \sqrt{224.13^2 + 150^2 - 2(150)(224.13)\cos 53^\circ}$$

$$\approx 179.635$$

$$\cos\theta = \frac{179}{179.635}$$

$$\theta = \cos^{-1}\frac{179}{179.635}$$

$$\approx 4.8^\circ$$

b) $\tan\theta = \dfrac{d}{179}$

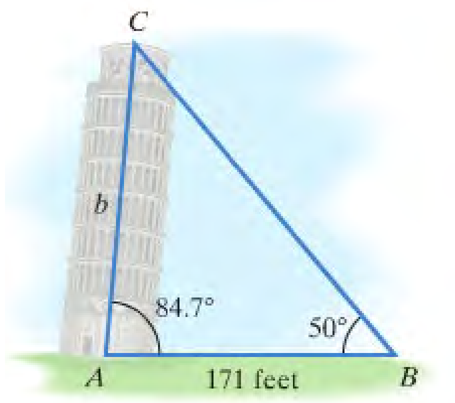
$$d = 179\tan 5^\circ$$

$$\approx 15.6\ ft$$

Exercise

The leaning Tower of Pisa in Italy leans at an angle of about 84.7°, 171 *feet* from the base of the tower, the angle of elevation to the top is 50°. Find the distance from the base to the top of the tower.

Solution



$$C = 180^\circ - 84.7^\circ - 50^\circ$$

$$= 45.3^\circ$$

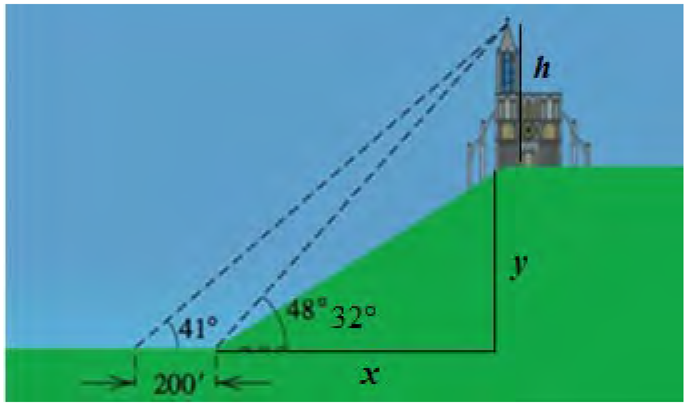
$$\frac{b}{\sin 50^\circ} = \frac{171}{\sin 45.3^\circ}$$

$$b = \frac{171\sin 50^\circ}{\sin 45.3^\circ}$$

$$\approx 184\ ft$$

Exercise

A cathedral is located on a hill. When the top of the spire is viewed from the base of the hill, the angle of elevation is $48°$. When it is viewed at a distance of 200 *feet* from the base of the hill, the angle is $41°$. The hill rises at an angle of $32°$.



Approximate the height of the cathedral.

Solution

$$h + y = \frac{200\ \tan 48° \tan 41°}{\tan 48° - \tan 41°}$$

$$\approx 800.114$$

$$\tan 32° = \frac{y}{x} \quad \rightarrow \quad x = \frac{y}{\tan 32°}$$

$$\tan 48° = \frac{h+y}{x} \quad \rightarrow \quad x = \frac{h+y}{\tan 48°}$$

$$\frac{y}{\tan 32°} = \frac{h+y}{\tan 48°}$$

$$y = \frac{800.114 \tan 32°}{\tan 48°}$$

$$\approx 450.172$$

$$h \approx 800.114 - 450.172$$

$$\approx 350\ \ ft$$

Exercise

A helicopter hovers at an altitude that is 1,000 *feet* above a mountain peak of altitude 5,210 *feet*. A second, taller peak is viewed from both the mountaintop and the helicopter. From the helicopter, the angle of depression is $43°$, and from the mountaintop, the angle of elevation is $18°$.

a) Approximate the distance from peak to peak.

b) Approximate the altitude of the taller peak.

Solution

a) $\angle BAC = 90° - 43° = 47°$

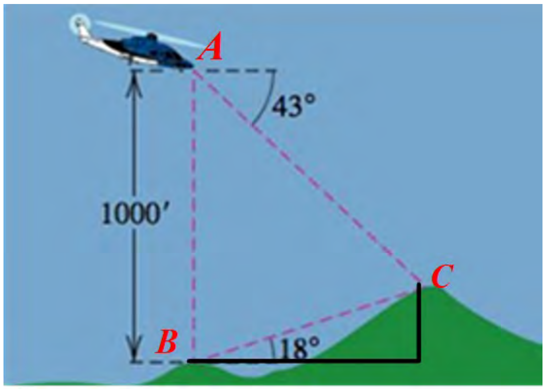
$\angle ABC = 90° - 18° = 72°$

$\angle ACB = 180° - 72° - 47° = 61°$

$$\frac{|BC|}{\sin 47°} = \frac{1000}{\sin 61°}$$

$$|BC| = \frac{1000 \sin 47°}{\sin 61°}$$

$$\approx 836.2 \ ft$$



b) $\sin 18° = \dfrac{h}{|BC|}$

$h = |BC| \sin 18°$

$\approx 258.4 \ ft$

Exercise

The volume V of the right triangular prism shown in the figure is $\frac{1}{3}Bh$, where B is the area of the base and h is the height of the prism.

a) Approximate h.

b) Approximate V.

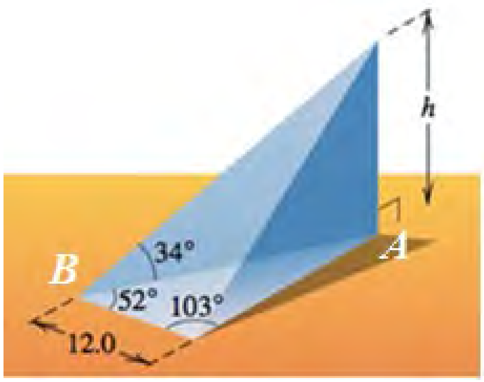
Solution

a) $\angle A = 180° - 103° - 52° = 25°$

$$\frac{|AB|}{\sin 103°} = \frac{12}{\sin 25°}$$

$$|AB| = \frac{12 \sin 103°}{\sin 25°}$$

$$\approx 27.7$$



$\tan 34° = \dfrac{h}{|AB|}$

$h = 27.7 \tan 34°$

≈ 18.7

b) $B = \frac{1}{2}(27.7)(12)\sin 52° \approx 131$

$V = \frac{1}{3}Bh$

$= \frac{1}{3}(131)(18.7)$

$\approx 816\ \ unit^3$

Exercise

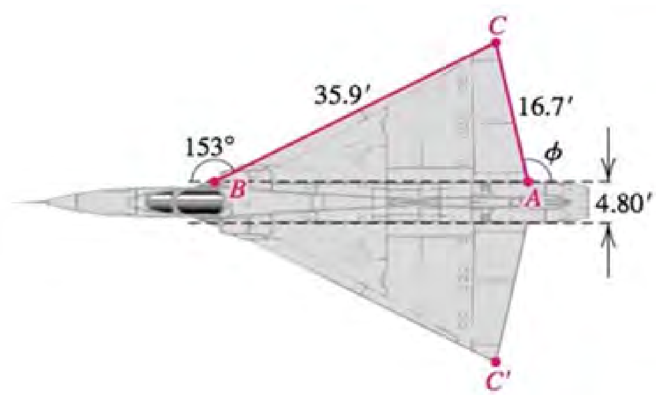
Shown in the figure is a plan for the top of a wing of a jet fighter.

a) Approximate angle ϕ.

b) If the fuselage is 4.80 *feet* wide, approximate the wing span CC'.

c) Approximate the area of the triangle *ABC.*

Solution



a) $\angle ABC = 180° - 153°$

$= 27°$

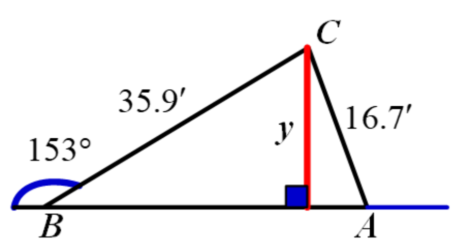
$\frac{\sin A}{35.9} = \frac{\sin 27°}{16.7}$

$A = \sin^{-1}\left(\frac{35.9}{16.7}\sin 27°\right)$

$\approx 77.4°$

$\phi = 180° - 77.4°$

$\approx 102.6°$



b) $\sin A = \frac{y}{16.7}$

$y = 16.7\sin 77.4°$

≈ 16.3

$CC' = 2(16.3) + 4.8$

$\approx 37.4\ \ ft$

c) $\angle ACB = 180° - 27° - 77.4°$

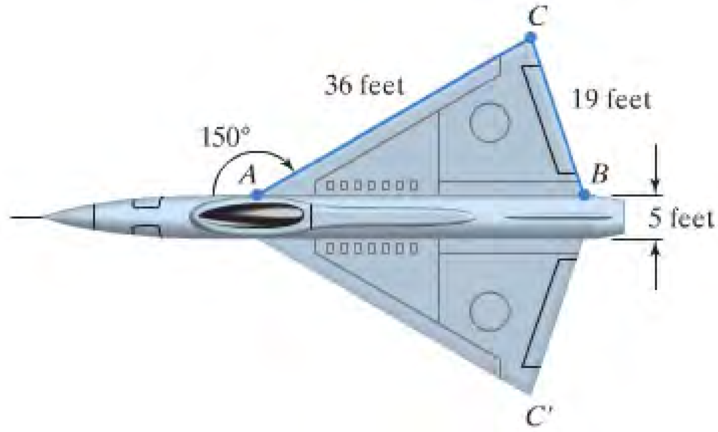
$= 75.6°$

$Area = \frac{1}{2}(35.9)(16.7)\sin 75.6°$

$\approx 290.35\ \ ft^2$

Exercise

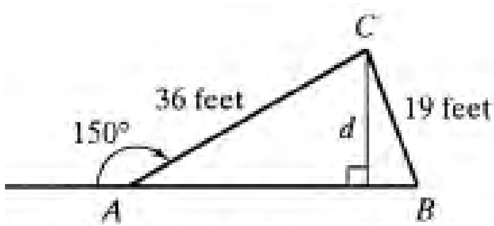
Shown in the figure is a plan for the top of a wing of a jet fighter. The fuselage is 5 *feet* wide. Find the wing span CC'



Solution

$$\angle BAC = 180° - 150°$$
$$= 30°$$

$$\sin A = \frac{d}{36}$$
$$d = 36\sin 30°$$
$$= 36\left(\frac{1}{2}\right)$$
$$= 18\ ft$$

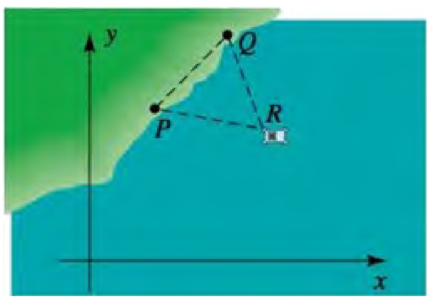


$$CC' = 2(18) + 5$$
$$= 41\ \ ft$$

Exercise

Computer software for surveyors makes use of coordinate systems to locate geographic positions. An offshore oil well at point *R* is viewed from points *P* and *A* and $\angle QPR$ and $\angle RQP$ are found to be $55°\ 50'$ and $65°\ 22'$, respectively. If points *P* and *Q* have coordinates $(1487.7,\ 3452.8)$ and $(3145.8,\ 5127.5)$, respectively. Approximate the coordinates of *R*.

Solution



Find: $R(x,\ y)$

$$|PQ| = \sqrt{(3,145.8 - 1,487.7)^2 + (5,127.5 - 3,452.8)^2}$$
$$\approx 2,356.6$$

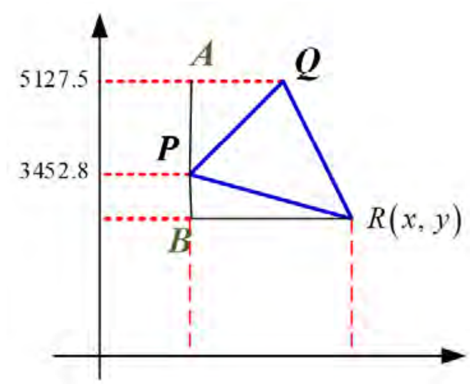
$$\angle PRQ = 180° - 65°\ 22' - 55°\ 50'$$
$$= 180° - 121°\ 12'$$

$$= 58.8^\circ$$

$$\frac{|PR|}{\sin(65.37^\circ)} = \frac{2,356.6}{\sin(58.8^\circ)}$$

$$|PR| = \frac{2,356.6\sin(65.4^\circ)}{\sin(58.8^\circ)}$$

$$\approx 2,505.0$$



$$|AP| = 5127.5 - 3452.8$$

$$= 1,647.7$$

$$\angle APQ = \cos^{-1}\frac{|AP|}{|PQ|}$$

$$= \cos^{-1}\frac{1,647.7}{2,356.6}$$

$$\approx 45.6^\circ$$

$$\angle BPR \approx 180^\circ - 45.6^\circ - 55.83^\circ$$

$$\approx 78.57^\circ$$

$$\sin 78.57^\circ = \frac{|BR|}{|PR|}$$

$$|BR| = 2,505\sin 78.57^\circ$$

$$\approx 2,455.3$$

$$x = x_P + |BR|$$

$$= 1487.7 + 2455.3$$

$$\approx 3,943$$

$$\cos 78.57^\circ = \frac{|PB|}{|PR|}$$

$$|PB| = 2,505\cos 78.57^\circ$$

$$\approx 496.4$$

$$y = y_P - |PB|$$

$$= 3452.8 - 496.4$$

$$\approx 2,956.4$$

$$R(3,943,\ \ 2,946.4)$$

Exercise

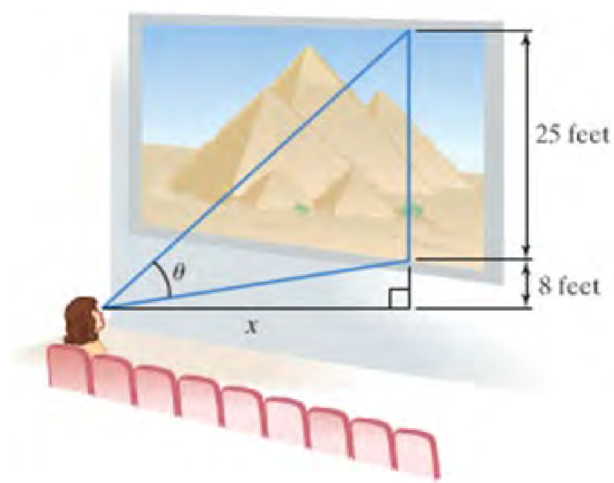
Your movie theater has a 25–*foot*–high screen located 8 *feet* above your eye level. If you sit too close to the screen, your viewing angle is too small resulting in a distorted picture. By contrast, if you sit too far back, the image is quite small, diminishing the movie's visual impact. If you sit *x feet* back from the screen, your viewing angle θ, is giving by

$$\theta = \tan^{-1}\frac{33}{x} - \tan^{-1}\frac{8}{x}$$

Find the viewing angle, in radians, at distances of 5 *feet*, 10 *feet*, 15 *feet*, 25 *feet*, and 25 *feet*.

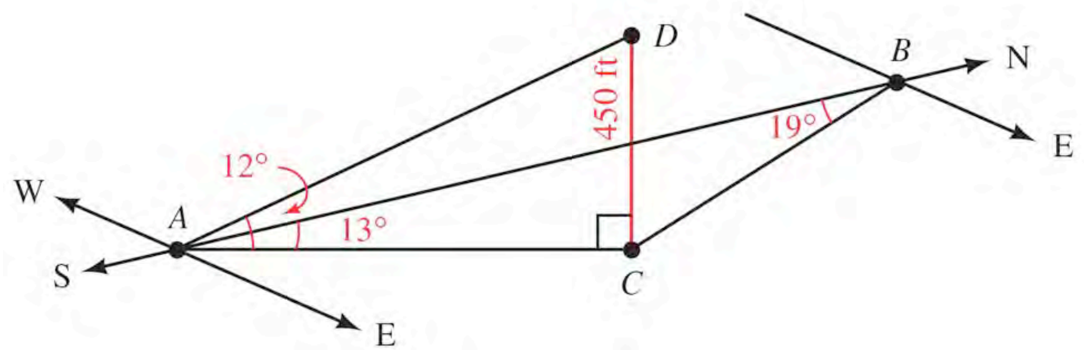
Solution

x	$\theta = \tan^{-1}\frac{33}{x} - \tan^{-1}\frac{8}{x}$
5	$\theta = \tan^{-1}\frac{33}{5} - \tan^{-1}\frac{8}{5}$ $\approx 0.408\ rad$
10	$\theta = \tan^{-1}\frac{33}{10} - \tan^{-1}\frac{8}{10}$ $\approx 0.602\ rad$
15	$\theta = \tan^{-1}\frac{33}{15} - \tan^{-1}\frac{8}{15}$ $\approx 0.654\ rad$
20	$\theta = \tan^{-1}\frac{33}{20} - \tan^{-1}\frac{8}{20}$ $\approx 0.645\ rad$
25	$\theta = \tan^{-1}\frac{33}{25} - \tan^{-1}\frac{8}{25}$ $\approx 0.613\ rad$



Exercise

A hot-air balloon is flying over a dry lake when the wind stops blowing. The balloon comes to a stop 450 *feet* above the ground at point *D*. A jeep following the balloon runs out of gas at point *A*. The nearest service station is due north of the jeep at point *B*. The bearing of the balloon from the jeep at *A* is N 13° E, while the bearing of the balloon from the service station at *B* is S 19° E. If the angle of elevation of the balloon from *A* is 12°, how far will the people in the jeep have to walk to reach the service station at point *B*?



Solution

$$\tan 12° = \frac{DC}{AC}$$

$$AC = \frac{DC}{\tan 12°}$$

$$= \frac{450}{\tan 12°}$$

$$\approx 2{,}117 \ ft$$

$$\angle ACB = 180° - (13° + 19°)$$

$$= 148°$$

Using triangle *ABC*

$$\frac{AB}{\sin 148°} = \frac{2117}{\sin 19°}$$

$$AB = \frac{2117 \sin 148°}{\sin 19°}$$

$$\approx 3{,}446 \ ft$$

Exercise

Derive the formula: $\dfrac{a-b}{a+b}=\dfrac{\tan\left[\frac{1}{2}(A-B)\right]}{\tan\left[\frac{1}{2}(A+B)\right]}$

Solution

$$\frac{a-b}{a+b}=\frac{\frac{a-b}{c}}{\frac{a+b}{c}}$$

$$=\frac{a-b}{c}\div\frac{a+b}{c}$$

$$=\left(\frac{a}{c}-\frac{b}{c}\right)\div\left(\frac{a}{c}+\frac{b}{c}\right) \qquad \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}\ \rightarrow \frac{a}{c}=\frac{\sin A}{\sin C}\quad \frac{b}{c}=\frac{\sin B}{\sin C}$$

$$=\left(\frac{\sin A}{\sin C}-\frac{\sin B}{\sin C}\right)\div\left(\frac{\sin A}{\sin C}+\frac{\sin B}{\sin C}\right)$$

$$=\left(\frac{\sin A-\sin B}{\sin C}\right)\div\left(\frac{\sin A+\sin B}{\sin C}\right)$$

$$=\frac{\sin A-\sin B}{\sin A+\sin B} \qquad \sin x-\sin y=2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) \qquad \sin x+\sin y=2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$=\frac{2\cos\left(\frac{1}{2}(A+B)\right)\sin\left(\frac{1}{2}(A-B)\right)}{2\sin\left(\frac{1}{2}(A+B)\right)\cos\left(\frac{1}{2}(A-B)\right)}$$

$$=\frac{\sin\left(\frac{1}{2}(A-B)\right)}{\cos\left(\frac{1}{2}(A-B)\right)}\cdot\frac{\cos\left(\frac{1}{2}(A+B)\right)}{\sin\left(\frac{1}{2}(A+B)\right)}$$

$$=\frac{\tan\left[\frac{1}{2}(A-B)\right]}{\tan\left[\frac{1}{2}(A+B)\right]}$$

Exercise

For any triangle, show that $\sin\dfrac{C}{2}=\sqrt{\dfrac{(s-a)(s-b)}{ab}}$ *where* $s=\dfrac{1}{2}(a+b+c)$

Solution

$$\sin\frac{C}{2}=\sqrt{\frac{1}{2}(1-\cos C)}$$

$$=\sqrt{\frac{1}{2}\left(1-\frac{a^2+b^2-c^2}{2ab}\right)}$$

$$=\sqrt{\frac{2ab-a^2-b^2+c^2}{4ab}}$$

$$= \sqrt{\frac{-\left(a^2 + b^2 - 2ab - c^2\right)}{4ab}}$$

$$= \sqrt{\frac{-\left((a-b)^2 - c^2\right)}{4ab}} \qquad x^2 - y^2 = (x-y)(x+y)$$

$$= \sqrt{\frac{-(a-b+c)(a-b-c)}{4ab}} \qquad 2s-b=a+c \quad 2s-a=b+c$$

$$= \sqrt{\frac{(2s-b-b)(2s-a-a)}{4ab}}$$

$$= \sqrt{\frac{4(s-b)(s-a)}{4ab}}$$

$$= \sqrt{\frac{(s-b)(s-a)}{ab}} \quad \checkmark$$

Exercise

Prove the identity $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$

Solution

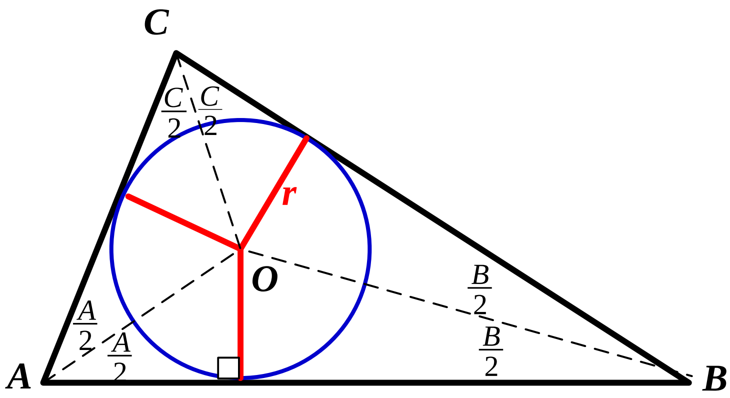
$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{1}{a}\frac{b^2 + c^2 - a^2}{2bc} + \frac{1}{b}\frac{a^2 + c^2 - b^2}{2ac} + \frac{1}{c}\frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{b^2 + c^2 - a^2 + a^2 + c^2 - b^2 + a^2 + b^2 - c^2}{2abc}$$

$$= \frac{a^2 + b^2 + c^2}{2abc} \quad \checkmark$$

Exercise

The lines that bisect each angle of a triangle meet in a single point O, and perpendicular distance r from O to each side of the triangle is the same. The circle with center at O and radius r is called the inscribed circle of the triangle.



a) Show that $r = \dfrac{c\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}}$

b) Show that $\cot\frac{C}{2} = \frac{s-c}{r}$ where $s = \frac{1}{2}(a+b+c)$

c) Show that $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \frac{s}{r}$

d) Show that the area K of triangle ABC is $K = rs$, where $s = \frac{1}{2}(a+b+c)$.

e) Show that $r = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$

Solution

a) $\Delta AOB \quad \to \quad \angle AOB = \pi - \left(\frac{A}{2} + \frac{B}{2}\right)$

$\sin\frac{A}{2} = \frac{r}{\overline{OA}} \quad \Rightarrow \quad \overline{OA} = \dfrac{r}{\sin\frac{A}{2}}$

$\dfrac{\overline{OA}}{\sin\frac{B}{2}} = \dfrac{c}{\sin(\angle AOB)}$

$$\begin{aligned} \sin(\angle AOB) &= \sin\left(\pi - \left(\frac{A}{2} + \frac{B}{2}\right)\right) && \sin(\pi - \alpha) = \sin\alpha \\ &= \sin\left(\frac{A}{2} + \frac{B}{2}\right) \\ &= \sin\left(\frac{A+B}{2}\right) && \sin\alpha = \cos\left(\tfrac{1}{2} - \alpha\right) \\ &= \cos\left(\frac{\pi}{2} - \frac{A+B}{2}\right) \\ &= \cos\left(\frac{\pi - (A+B)}{2}\right) && C = \pi - (A+B) \end{aligned}$$

$$= \cos\left(\frac{C}{2}\right)$$

$$\frac{r}{\sin\frac{A}{2}}\frac{1}{\sin\frac{B}{2}} = \frac{c}{\cos\left(\frac{C}{2}\right)}$$

$$\boxed{r = \frac{c\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}}} \quad \checkmark$$

b) $\cot\frac{C}{2} = \dfrac{\cos\frac{C}{2}}{\sin\frac{C}{2}}$

From part (a): $\cos\frac{C}{2} = \dfrac{c\sin\frac{A}{2}\sin\frac{B}{2}}{r}$

$$\begin{aligned}
\sin\frac{C}{2} &= \sqrt{\frac{1}{2}(1-\cos C)} \\
&= \sqrt{\frac{1}{2}\left(1-\frac{a^2+b^2-c^2}{2ab}\right)} \\
&= \sqrt{\frac{2ab-a^2-b^2+c^2}{4ab}} \\
&= \sqrt{\frac{-\left(a^2+b^2-2ab-c^2\right)}{4ab}} \\
&= \sqrt{\frac{-\left((a-b)^2-c^2\right)}{4ab}} && x^2-y^2=(x-y)(x+y) \\
&= \sqrt{\frac{-(a-b+c)(a-b-c)}{4ab}} && 2s-b=a+c \quad 2s-a=b+c \\
&= \sqrt{\frac{(2s-b-b)(2s-a-a)}{4ab}} \\
&= \sqrt{\frac{4(s-b)(s-a)}{4ab}} \\
&= \sqrt{\frac{(s-b)(s-a)}{ab}}
\end{aligned}$$

$$\cot\frac{C}{2} = \frac{\dfrac{c\sin\frac{A}{2}\sin\frac{B}{2}}{r}}{\sqrt{\dfrac{(s-b)(s-a)}{ab}}}$$

$$= \frac{c}{r}\frac{\sqrt{ab}}{\sqrt{(s-b)(s-a)}}\sin\frac{A}{2}\sin\frac{B}{2} \qquad \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \quad \& \quad \sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}$$

$$= \frac{c}{r}\frac{\sqrt{ab}}{\sqrt{(s-b)(s-a)}}\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-c)}{ac}}$$

$$= \frac{c}{r}\sqrt{\frac{ab}{abc^2}}\sqrt{\frac{(s-a)(s-b)(s-c)^2}{(s-b)(s-a)}}$$

$$= \frac{c}{r}\frac{1}{c}(s-c)$$

$$\boxed{= \frac{s-c}{r}} \quad \checkmark$$

c) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \frac{s-a}{r} + \frac{s-b}{r} + \frac{s-c}{r}$

$$= \frac{3s-a-b-c}{r}$$

$$= \frac{3s-(a+b+c)}{r} \qquad 2s = a+b+c$$

$$= \frac{3s-2s}{r}$$

$$\boxed{= \frac{s}{r}} \quad \checkmark$$

d) $K = Area(\Delta AOB) + Area(\Delta AOC) + Area(\Delta BOC)$

$$Area(\Delta AOB) = \frac{1}{2}(height)(base) = \frac{1}{2}rc$$

$$Area(\Delta AOC) = \frac{1}{2}(height)(base) = \frac{1}{2}rb$$

$$Area(\Delta BOC) = \frac{1}{2}(height)(base) = \frac{1}{2}ra$$

$$K = \frac{1}{2}rc + \frac{1}{2}rb + \frac{1}{2}ra$$

$$= \frac{1}{2}r(a+b+c) \qquad 2s = a+b+c$$

$$= \frac{1}{2}r(2s)$$

$$\boxed{= rs} \quad \checkmark$$

e) $K = \sqrt{s(s-a)(s-b)(s-c)}$

$$rs = \sqrt{s(s-a)(s-b)(s-c)}$$

$$r = \frac{1}{s}\sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2}}$$

$$\boxed{= \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}} \quad \checkmark$$