Section R.2 – Integration

Definition of Antiderivative

A Function F is an Antiderivative of a function f if for every x in the domain of f, it follows that $F'(x) = f(x)$

Notation for Antiderivatives and indefinite integrals

The notation

$$\int f(x)dx = F(x) + C$$

where C is an arbitrary constant, means that F is an Antiderivative of f.
That is $F'(x) = f(x)$ for all x in the domain of f.

$\int f(x)dx$ Indefinite integral

Integrand

Integral sign

$$\int f(x)dx = \overbrace{F(x) + C}^{\text{Antiderivative}}$$

Differential

Basic Integration Rules

$$\int kdx = kx + C$$

$$\int kf(x)dx = k\int f(x)dx$$

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

The General Power Rule

The Simple Power Rule is given by:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

$$\int \overbrace{\left(x^2+1\right)^3}^{u^3} \underbrace{2x dx}_{du} = \int u^3 du$$

$$= \frac{u^4}{4} + C$$

General Power Rule for Integration

If *u* is a differentiable function of *x*, then

$$\int u^n \frac{du}{dx} dx = \int u^n du = \frac{u^{n+1}}{n+1} + C \qquad n \neq -1$$

Example

Find each indefinite integral.

$$\int 5x dx = \int 5x^1 dx$$

$$= 5\frac{x^{1+1}}{1+1} + C$$

$$\boxed{= \frac{5}{2}x^2 + C}$$

$$\int \sqrt[3]{x} dx = \int x^{1/3} dx$$

$$= \frac{x^{1/3+1}}{1/3+1} + C$$

$$= \frac{x^{4/3}}{4/3} + C$$

$$\boxed{= \frac{3}{4}x^{4/3} + C} \quad or \quad = \frac{3}{4}x\sqrt[3]{x} + C$$

Example

Find the integral $\int x^2 \sin\left(x^3\right) dx$

Solution

$$\int x^2 \sin\left(x^3\right) dx = \frac{1}{3}\int \sin x^3 \cdot d\left(x^3\right) \qquad d\left(x^3\right) = 3x^2 dx$$

$$\underline{= -\frac{1}{3}\cos\left(x^3\right) + C}$$

Example

Find the integral $\int x\sqrt{2x+1}\, dx$

Solution

Let: $u = 2x+1 \quad \Rightarrow \quad du = 2dx$

$$dx = \frac{1}{2}du$$

$u = 2x+1 \rightarrow 2x = u-1 \quad \Rightarrow \quad x = \frac{u-1}{2}$

$$\int x\sqrt{2x+1}\, dx = \int \frac{1}{2}(u-1)\sqrt{u}\ \frac{1}{2}du$$

$$= \frac{1}{4}\int (u-1)u^{1/2}\, du$$

$$= \frac{1}{4}\int \left(u^{3/2} - u^{1/2}\right) du$$

$$= \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C$$

$$\underline{= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C}$$

Theorem – The Fundamental Theorem of Calculus, P-2

If f is continuous at every point in $[a, b]$, then F is any antiderivative of f on $[a, b]$, then

$$\boxed{\int_a^b f(x)dx = F(b) - F(a)}$$

Example

a) $\displaystyle\int_0^{\pi} \cos x \, dx = \sin x \Big|_0^{\pi}$

$$= \sin \pi - \sin 0$$

$$= \underline{0}$$

b) $\displaystyle\int_{-\frac{\pi}{4}}^{0} \sec x \tan x \, dx = \sec x \Big|_{-\frac{\pi}{4}}^{0}$

$$= \sec 0 - \sec\left(-\frac{\pi}{4}\right)$$

$$= \underline{1 - \sqrt{2}}$$

c) $\displaystyle\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx = \left[x^{3/2} + \frac{4}{x}\right]_1^4$

$$= \left((4)^{3/2} + \frac{4}{4}\right) - \left((1)^{3/2} + \frac{4}{1}\right)$$

$$= (9) - (5)$$

$$= \underline{4}$$

Other Indefinite Integrals

$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax} \quad \to \quad \int e^{ax}dx = \frac{1}{a}e^{ax} + C$$

$$\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x} \quad \to \quad \int \frac{dx}{x} = \ln|x| + C$$

$$\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{a^2 - x^2}} \quad \to \quad \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{a}{x^2 + a^2} \quad \to \quad \int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\frac{d}{dx}\left(\sec^{-1}\left|\frac{x}{a}\right|\right) = \frac{a}{x\sqrt{x^2 - a^2}} \quad \to \quad \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$$

Example

Evaluate $\int e^{-10x}dx$

Solution

$$\int e^{-10x}dx = -\frac{1}{10}e^{-10x} + C$$

Example

Evaluate $\int \frac{5}{x}dx$

Solution

$$\int \frac{5}{x}dx = 5\ln|x| + C$$

Example

Evaluate $\int \frac{4}{\sqrt{9 - x^2}}dx$

Solution

$$\int \frac{4}{\sqrt{9 - x^2}}dx = 4\sin^{-1}\left(\frac{x}{3}\right) + C \qquad a^2 = 9 \quad \to \quad a = 3$$

Exercises *Section R.2* – Integration

Find each indefinite integral.

1. $\displaystyle\int \frac{x+2}{\sqrt{x}}dx$
2. $\displaystyle\int 4y^{-3}dy$
3. $\displaystyle\int (x^3-4x+2)dx$
4. $\displaystyle\int \left(\sqrt[4]{x^3}+1\right)dx$
5. $\displaystyle\int \sqrt{x}(x+1)dx$
6. $\displaystyle\int (1+3t)t^2dt$
7. $\displaystyle\int \frac{x^2-5}{x^2}dx$
8. $\displaystyle\int (-40x+250)dx$
9. $\displaystyle\int \left(7-3x-3x^2\right)(2x+1)\,dx$
10. $\displaystyle\int (1+\cos 3\theta)\,d\theta$
11. $\displaystyle\int 2\sec^2\theta\,d\theta$
12. $\displaystyle\int \sec 2x\tan 2x\,dx$
13. $\displaystyle\int 2e^{2x}dx$
14. $\displaystyle\int \frac{12}{x}dx$
15. $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}$
16. $\displaystyle\int \frac{dx}{x^2+1}$
17. $\displaystyle\int \frac{1+\tan\theta}{\sec\theta}d\theta$

Find the general solution of the differential equation

18. $y' = 2t+3$
19. $y' = 3t^2+2t+3$
20. $y' = \sin 2t + 2\cos 3t$
21. $y' = x^3(3x^4+1)^2$
22. $y' = 5x\sqrt{x^2-1}$
23. $y' = x\sqrt{x^2+4}$

Evaluate the integrals

24. $\displaystyle\int_{-2}^{2}\left(x^3-2x+3\right)dx$
25. $\displaystyle\int_{0}^{1}\left(x^2+\sqrt{x}\right)dx$
26. $\displaystyle\int_{0}^{\pi/3} 4\sec u\tan u\,du$
27. $\displaystyle\int_{\pi/4}^{3\pi/4} \csc\theta\cot\theta\,d\theta$
28. $\displaystyle\int_{-\pi/3}^{-\pi/4}\left(4\sec^2 t+\frac{\pi}{t^2}\right)dt$
29. $\displaystyle\int_{-3}^{-1}\frac{y^5-2y}{y^3}dy$
30. $\displaystyle\int_{1}^{8}\frac{\left(x^{1/3}+1\right)\left(2-x^{2/3}\right)}{x^{1/3}}dx$
31. $\displaystyle\int_{0}^{1}(2t+3)^3dt$
32. $\displaystyle\int_{-1}^{1} r\sqrt{1-r^2}\,dr$