3.4 Orthogonal Matrices

Defn A square matrix A is said to be orthogonal matrix:

$$A^{-1} = A^T \iff AA^T = A^TA = I$$

Ex

$$A = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I \checkmark$$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \underline{I} \checkmark$$

Theorem

- A is orthogonal
- row vectors of A form an orthonormal set
- column " " " "

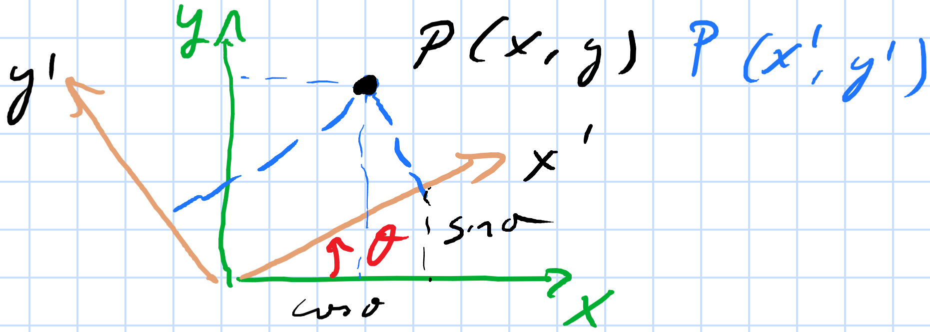
w/ Euclidean inner product.

✓ Inverse of an orthogonal matrix is orthogonal
✓ product " " matrices is "
✓ A is orthogonal det(A) = 1 or −1

if A is an orthogonal

$$\|A\vec{x}\| = \|\vec{x}\|$$

$$A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y}$$



$$\begin{bmatrix} \vec{x}' \\ \vec{y}' \end{bmatrix} = P^{-1} \begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix}$$

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad |P| = 1$$

$$P^{-1} = P^{T} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} \vec{x}' \\ \vec{y}' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}$$

$$\begin{cases} x' = x\cos\theta + y\sin\theta \\ y' = -x\sin\theta + y\cos\theta \end{cases}$$

Ex

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

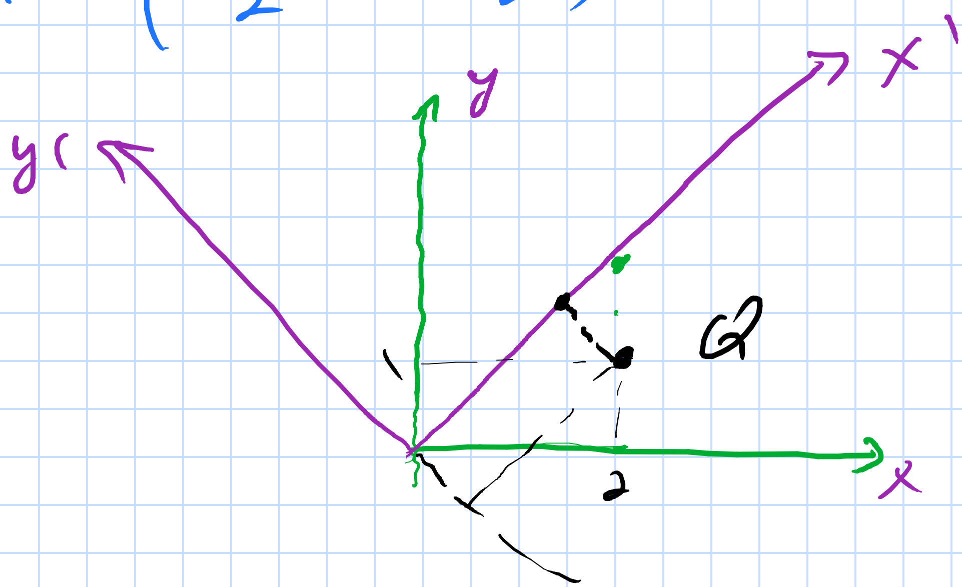
$Q\,(2,1) \qquad \theta = \frac{\pi}{4}$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} \end{pmatrix}$$

$$Q' = \left(\frac{3\sqrt{2}}{2} , -\frac{\sqrt{2}}{2} \right)$$



5/ $A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ $|A| = 1$

$$AA^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

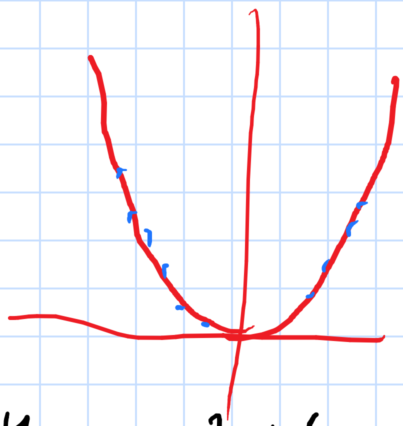
$$= I$$

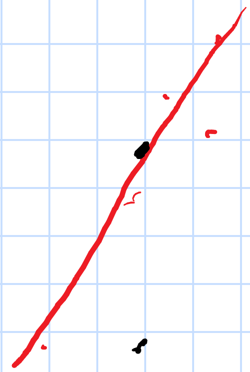
$$A^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

orthogonal & find Inverse

3.5 Least Square Analysis

Fitting a curve





$y = ax + b$

$y = ax^2 + bx + c$

$y = ax^3 + bx^2 + cx + d$

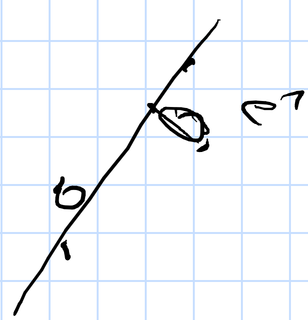
$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$

a_i ---

Linear Eqn : $y = ax + b$

$y = mx + b$

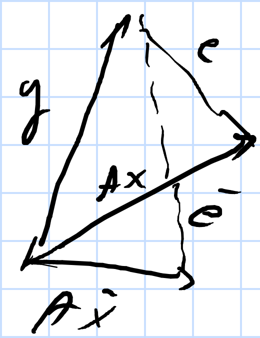
$$\left.\begin{array}{l} y_1 = m x_1 + b \\ y_2 = m x_2 + b \\ \vdots \\ y_n = m x_n + b \end{array}\right\} \Rightarrow \overset{y}{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}} = \overset{A}{\begin{pmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{pmatrix}} \overset{x}{\begin{pmatrix} m \\ b \end{pmatrix}}$$



$$\| \hat{y} - A\hat{x} \|$$

$$A\vec{x} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{pmatrix}$$

$$\| \hat{y} - A\vec{x} \| = e_1^2 + e_2^2 + \cdots + e_m^2$$



$$A\vec{x} = \vec{y}$$

$$\boxed{A^T A\vec{x} = A^T\vec{y}}$$

$$(A^TA)^{-1} A^TA\vec{x} = (A^TA)^{-1}A^T\vec{y}$$

$$I\vec{x} = (A^TA)^{-1}A^T\vec{y}$$

$$\boxed{\vec{x} = (A^TA)^{-1}A^T\vec{y}}$$

Ex $(40, 482)$ $(45, 467)$ $(50, 452)$ $(55, 432)$ $(60, 421)$

$$A = \begin{pmatrix} 40 & 1 \\ 45 & 1 \\ 50 & 1 \\ 55 & 1 \\ 60 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 482 \\ 467 \\ 452 \\ 432 \\ 421 \end{pmatrix}$$

$$\boxed{\vec{x} = (A^TA)^{-1}A^T\vec{y}}$$

$$A^TA = \begin{pmatrix} 40 & 45 & 50 & 55 & 60 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 40 & 1 \\ 45 & 1 \\ 50 & 1 \\ 55 & 1 \\ 60 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 12,750 & 250 \\ 250 & 5 \end{pmatrix}$$

$$(A^TA)^{-1} = \frac{1}{1250} \begin{pmatrix} 5 & -250 \\ -250 & 12,750 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{1250} \begin{pmatrix} 5 & -250 \\ -250 & 12750 \end{pmatrix} \begin{pmatrix} 40 & 45 & 50 & 55 & 60 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 482 \\ 467 \\ 452 \\ 432 \\ 421 \end{pmatrix}$$

$$= \frac{1}{1250} \begin{pmatrix} 5 & -250 \\ -250 & 12750 \end{pmatrix} \begin{pmatrix} 111,970 \\ 2,255 \end{pmatrix}$$

$$= \begin{pmatrix} -3.12 \\ 607 \end{pmatrix}$$

$$\underline{y = -3.12x + 607}$$

Ex

$$\begin{cases} x_1 - x_2 = 4 \\ 3x_1 + 2x_2 = 1 \\ -2x_1 + 4x_2 = 3 \end{cases}$$

$$A = \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & -3 \\ -3 & 21 \end{pmatrix}$$

$$(A^T A)^{-1} = \frac{1}{285} \begin{pmatrix} 21 & 3 \\ 3 & 14 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{285} \begin{pmatrix} 21 & 3 \\ 3 & 14 \end{pmatrix} \begin{pmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

$$= \frac{1}{285} \begin{pmatrix} 21 & 3 \\ 3 & 14 \end{pmatrix} \begin{pmatrix} 1 \\ 10 \end{pmatrix}$$

$$= \frac{1}{285} \begin{pmatrix} 51 \\ 143 \end{pmatrix}$$

$$m = \frac{51}{285} \quad \frac{17}{95} \qquad b = \frac{143}{285}$$

$$\boxed{y = \frac{17}{95} x + \frac{143}{285}}$$