# Lecture Three

## *Section* 3.1 – Inner Products

### *Definition*

An ***inner product*** on a real vector space *V* is a function that associates a real number $\langle \vec{u},\ \vec{v} \rangle$ with each pair of vectors in *V* in such a way that the following axioms are satisfies for all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in *V* and all scalars *k*.

1. $\langle \vec{u},\ \vec{v} \rangle = \langle \vec{v},\ \vec{u} \rangle$ ***Symmetry axiom***
2. $\langle \vec{u}+\vec{v},\ \vec{w} \rangle = \langle \vec{u},\ \vec{w} \rangle + \langle \vec{v},\ \vec{w} \rangle$ ***Additivity axiom***
3. $\langle k\vec{u},\ \vec{v} \rangle = k\langle \vec{u},\ \vec{v} \rangle$ ***Homogeneity axiom***
4. $\langle \vec{v},\ \vec{v} \rangle \geq 0$ and $\langle \vec{v},\ \vec{v} \rangle = 0$ iff $\vec{v} = 0$ ***Positivity axiom***

A real vector space with an inner product is called a ***real inner product space***.

$$\boxed{\langle \vec{u},\ \vec{u} \rangle = \vec{u} \bullet \vec{v} = u_1 v_1 + u_2 v_2 + \cdots u_n v_n}$$

This is called the ***Euclidean inner product*** (or the ***standard inner product***)

### *Definition*

If *V* is a real inner product space, then the norm (or length) of a vector $\vec{v}$ in *V* is denoted by $\|\vec{v}\|$ and is defined by

$$\|\vec{v}\| = \sqrt{\langle \vec{v},\ \vec{v} \rangle}$$

And the ***distance*** between two vectors is denoted by $d(\vec{u},\ \vec{v})$ and is defined by

$$d(\vec{u},\ \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{\langle \vec{u} - \vec{v},\ \ \vec{u} - \vec{v} \rangle}$$

A vector of norm 1 is called a ***unit vector***.

### *Theorem*

If ***u*** and ***v*** are vectors in a real inner product space *V*, and if *k* is a scalar, then:

***a)*** $\|\vec{v}\| \geq 0$ with equality *iff* $\vec{v} = 0$

***b)*** $\|k\vec{v}\| = |k|\|\vec{v}\|$

***c)*** $d(\vec{u},\ \vec{v}) = d(\vec{v},\ \vec{u})$

***d)*** $d(\vec{u},\ \vec{v}) \geq 0$ with equality *iff* $\vec{u} = \vec{v}$

Although the Euclidean inner product is the most important inner product on $\mathbb{R}^n$, there are various applications in which is desirable to modify it by weighing each term differently. More precisely, if $w_1, w_2, \ldots, w_n$ are positive real numbers, which we will call weighs, and if $\vec{u} = (u_1, u_2, \ldots, u_n)$ and are vectors in $\mathbb{R}^n$, then it can be shown that the formula

$$\langle \vec{u}, \vec{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \ldots + w_n u_n v_n$$

Defines an inner product on $\mathbb{R}^n$ that we call the ***weighted Euclidean inner product*** with weights $\boldsymbol{w_1, w_2, \ldots, w_n}$

## *Example*

Let $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ be vectors in $\mathbb{R}^2$, verify that the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 3u_1 v_1 + 2u_2 v_2$ satisfies the four inner product axioms.

***Solution***

*Axiom* 1:
$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= 3u_1 v_1 + 2u_2 v_2 \\ &= 3v_1 u_1 + 2v_2 u_2 \\ &= \langle \vec{v}, \vec{u} \rangle \end{aligned}$$

*Axiom* 2:
$$\begin{aligned} \langle \vec{u} + \vec{v}, \vec{w} \rangle &= 3(u_1 + v_1) w_1 + 2(u_2 + v_2) w_2 \\ &= 3(u_1 w_1 + v_1 w_1) + 2(u_2 w_2 + v_2 w_2) \\ &= 3u_1 w_1 + 3v_1 w_1 + 2u_2 w_2 + 2v_2 w_2 \\ &= (3u_1 w_1 + 2u_2 w_2) + (3v_1 w_1 + 2v_2 w_2) \\ &= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \end{aligned}$$

*Axiom* 3:
$$\begin{aligned} \langle k\vec{u}, \vec{v} \rangle &= 3(ku_1) v_1 + 2(ku_2) v_2 \\ &= k(3u_1 v_1 + 2u_2 v_2) \\ &= k\langle \vec{u}, \vec{v} \rangle \end{aligned}$$

*Axiom* 4:
$$\begin{aligned} \langle \vec{v}, \vec{v} \rangle &= 3v_1 v_1 + 2v_2 v_2 \\ &= 3v_1^2 + 2v_2^2 \ge 0 \end{aligned}$$

$$v_1 = v_2 = 0 \quad iff \quad \vec{v} = \vec{0}$$

# *Exercises* *Section* 3.1 – Inner Products

**1.** Let $\langle \vec{u},\ \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (1,\ 1)$, $\vec{v} = (3,\ 2)$, $\vec{w} = (0,\ -1)$, and $k = 3$. Compute the following.

*a)* $\langle \vec{u},\ \vec{v} \rangle$ *c)* $\langle \vec{u} + \vec{v},\ \vec{w} \rangle$ *e)* $d(\vec{u},\ \vec{v})$

*b)* $\langle k\vec{v},\ \vec{w} \rangle$ *d)* $\|\vec{v}\|$ *f)* $\|\vec{u} - k\vec{v}\|$

**2.** Let $\langle \vec{u},\ \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (1,\ 1)$, $\vec{v} = (3,\ 2)$, $\vec{w} = (0,\ -1)$ and $k = 3$. Compute the following for the weighted Euclidean inner product $\langle \vec{u},\ \vec{v} \rangle = 2u_1v_1 + 3u_2v_2$ .

*a)* $\langle \vec{u},\ \vec{v} \rangle$ *c)* $\langle \vec{u} + \vec{v},\ \vec{w} \rangle$ *e)* $d(\vec{u},\ \vec{v})$

*b)* $\langle k\vec{v},\ \vec{w} \rangle$ *d)* $\|\vec{v}\|$ *f)* $\|\vec{u} - k\vec{v}\|$

**3.** Let $\langle \vec{u},\ \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (3,\ -2)$, $\vec{v} = (4,\ 5)$, $\vec{w} = (-1,\ 6)$, and $k = -4$. Verify the following.

*a)* $\langle \vec{u},\ \vec{v} \rangle = \langle \vec{v},\ \vec{u} \rangle$

*b)* $\langle \vec{u} + \vec{v},\ \vec{w} \rangle = \langle \vec{u},\ \vec{w} \rangle + \langle \vec{v},\ \vec{w} \rangle$

*c)* $\langle \vec{u},\ \vec{v} + \vec{w} \rangle = \langle \vec{u},\ \vec{v} \rangle + \langle \vec{u},\ \vec{w} \rangle$

*d)* $\langle k\vec{u},\ \vec{v} \rangle = k\langle \vec{u},\ \vec{v} \rangle$

*e)* $\langle \vec{0},\ \vec{v} \rangle = \langle \vec{v},\ \vec{0} \rangle = 0$

**4.** Let $\langle \vec{u},\ \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (3,\ -2)$, $\vec{v} = (4,\ 5)$, $\vec{w} = (-1,\ 6)$, and $k = -4$. Verify the following for the weighted Euclidean inner product $\langle \vec{u},\ \vec{v} \rangle = 4u_1v_1 + 5u_2v_2$

*a)* $\langle \vec{u},\ \vec{v} \rangle = \langle \vec{v},\ \vec{u} \rangle$

*b)* $\langle \vec{u} + \vec{v},\ \vec{w} \rangle = \langle \vec{u},\ \vec{w} \rangle + \langle \vec{v},\ \vec{w} \rangle$

*c)* $\langle \vec{u},\ \vec{v} + \vec{w} \rangle = \langle \vec{u},\ \vec{v} \rangle + \langle \vec{u},\ \vec{w} \rangle$

*d)* $\langle k\vec{u},\ \vec{v} \rangle = k\langle \vec{u},\ \vec{v} \rangle$

*e)* $\langle \vec{0},\ \vec{v} \rangle = \langle \vec{v},\ \vec{0} \rangle = 0$

**5.** Let $\vec{u} = (u_1,\ u_2)$ and $\vec{v} = (v_1,\ v_2)$. Show that the following are inner product on $\mathbb{R}^2$ by verifying that the inner product axioms hold. $\langle \vec{u},\ \vec{v} \rangle = 3u_1v_1 + 5u_2v_2$

**6.** Show that the following identity holds for the vectors in any inner product space

$$\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$$

**7.** Show that the following identity holds for the vectors in any inner product space

$$\langle \vec{u},\ \vec{v} \rangle = \frac{1}{4}\left( \|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2 \right)$$

**8.** Prove that $\|k\vec{v}\| = |k|\ \|\vec{v}\|$

# *Section* 3.2 – Angle and Orthogonality in Inner Product Spaces

### *Cosine Formula*

If $\vec{u}$ and $\vec{v}$ are nonzero vectors that implies

$$\cos\theta = \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$$

$$\theta = \cos^{-1}\left(\frac{\langle \vec{u},\ \vec{v}\rangle}{\|\vec{u}\|\ \|\vec{v}\|}\right)$$

$$-1 \le \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\ \|\vec{v}\|} \le 1$$

## *Example*

Let $\mathbb{R}^4$ have the Euclidean inner product. Find the cosine angle $\theta$ between the vectors $\vec{u} = (4,\ 3,\ 1,\ -2)$ and $\vec{v} = (-2,\ 1,\ 2,\ 3)$.

***Solution***

$$\|\vec{u}\| = \sqrt{4^2 + 3^2 + 1^2 + (-2)^2}$$

$$= \sqrt{30}$$

$$\|\vec{v}\| = \sqrt{4+1+4+9}$$

$$= \sqrt{18}$$

$$= 3\sqrt{2}$$

$$\langle \vec{u},\ \vec{v}\rangle = 4(-2)+3(1)+1(2)-2(3)$$

$$= -9$$

$$\cos\theta = -\frac{9}{3\sqrt{30}\sqrt{2}} \qquad \cos\theta = \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$$

$$= -\frac{3}{\sqrt{60}}$$

$$= -\frac{3}{2\sqrt{15}}$$

## *Theorem* – Cauchy-Schwarz Inequality

If $\vec{v}$ and $\vec{w}$ are vectors in a real inner product space $V$, then

$$\left\|\langle \vec{u},\ \vec{v}\rangle\right\| \le \|\vec{u}\|\ \|\vec{v}\|$$

## *Proof*

If either $\vec{u}$ or $\vec{v}$ is equal to zero, then both sides equal to zero

Inequality holds.

Suppose that $\vec{u},\ \vec{v} \ne 0$ and if $\vec{w}$ any vector

$\|\vec{w}\| = \vec{w}\ \vec{w} \ge 0$

Let $\vec{w} = \vec{u} - t\vec{v}$ , then:

$$\begin{aligned}
0 &\le \vec{w}\vec{w} \\
&= (\vec{u} - t\vec{v})(\vec{u} - t\vec{v}) \\
&= \vec{u}\bullet\vec{u} - t(\vec{u}\bullet\vec{v}) - t(\vec{v}\bullet\vec{u}) + t^2(\vec{v}\bullet\vec{v}) \\
&= \vec{u}\bullet\vec{u} - 2t\,(\vec{u}\bullet\vec{v}) + t^2(\vec{v}\bullet\vec{v}) \qquad \text{Let } t = \frac{\vec{u}\bullet\vec{v}}{\vec{v}\bullet\vec{v}} \\
&= \vec{u}\bullet\vec{u} - 2\left(\frac{\vec{u}\bullet\vec{v}}{\vec{v}\bullet\vec{v}}\right)(\vec{u}\bullet\vec{v}) + \left(\frac{\vec{u}\bullet\vec{v}}{\vec{v}\bullet\vec{v}}\right)^2(\vec{v}\bullet\vec{v}) \\
&= \vec{u}\bullet\vec{u} - 2\frac{(\vec{u}\bullet\vec{v})^2}{\vec{v}\bullet\vec{v}} + \frac{(\vec{u}\bullet\vec{v})^2}{\vec{v}\bullet\vec{v}} \\
&= \vec{u}\bullet\vec{u} - \frac{(\vec{u}\bullet\vec{v})^2}{\vec{v}\bullet\vec{v}} \\
&= \frac{(\vec{u}\bullet\vec{u})(\vec{v}\bullet\vec{v}) - (\vec{u}\bullet\vec{v})^2}{\vec{v}\bullet\vec{v}} \qquad \text{Since } \vec{v}\bullet\vec{v} > 0 \\
&\le (\vec{u}\bullet\vec{u})(\vec{v}\bullet\vec{v}) - (\vec{u}\bullet\vec{v})^2
\end{aligned}$$

$$(\vec{u}\bullet\vec{v})^2 \le (\vec{u}\bullet\vec{u})(\vec{v}\bullet\vec{v})$$

$$\left\|\langle \vec{u},\ \vec{v}\rangle\right\| \le \|\vec{u}\|\ \|\vec{v}\|$$

The following two alternative forms of the Cauchy-Schwarz inequality are useful to know:

$$\langle \vec{u},\ \vec{v}\rangle^2 \le \langle \vec{u},\ \vec{u}\rangle\langle \vec{v},\ \vec{v}\rangle$$

$$\langle \vec{u},\ \vec{v}\rangle^2 \le \|\vec{u}\|^2\ \|\vec{v}\|^2$$

### *Theorem*

If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors in a real inner product space *V*, and if *k* is any scalar, then

**a)** $\|\vec{u}+\vec{v}\| \le \|\vec{u}\|+\|\vec{v}\|$ (***Triangle inequality for vectors***)

**b)** $d(\vec{u},\ \vec{v}) \le d(\vec{u},\ \vec{w}) + d(\vec{w},\ \vec{v})$ (***Triangle inequality for distances***)

***Proof*** **(*a*)**

$$\begin{aligned}\|\vec{u}+\vec{v}\|^2 &= \langle \vec{u}+\vec{v},\ \vec{u}+\vec{v}\rangle \\ &= \langle \vec{u},\ \vec{u}\rangle + 2\langle \vec{u},\ \vec{v}\rangle + \langle \vec{v},\ \vec{v}\rangle \\ &\le \langle \vec{u},\ \vec{u}\rangle + 2\left|\langle \vec{u},\ \vec{v}\rangle\right| + \langle \vec{v},\ \vec{v}\rangle \\ &\le \langle \vec{u},\ \vec{u}\rangle + 2\|\vec{u}\|\ \|\vec{v}\| + \langle \vec{v},\ \vec{v}\rangle \\ &= \|\vec{u}\|^2 + 2\|\vec{u}\|\ \|\vec{v}\| + \|\vec{v}\|^2 \\ &= \left(\|\vec{u}\|+\|\vec{v}\|\right)^2\end{aligned}$$

$$\|\vec{u}+\vec{v}\|^2 \le \left(\|\vec{u}\|+\|\vec{v}\|\right)^2$$

$$\|\vec{u}+\vec{v}\| \le \|\vec{u}\|+\|\vec{v}\|$$

### *Definition*

Two vectors $\vec{u}$ and $\vec{v}$ in an inner product space are called orthogonal if $\langle \vec{u},\ \vec{v}\rangle = 0$

### *Example*

The vectors $\vec{u}=(1,\ 1)$ and $\vec{v}=(1,\ -1)$ are orthogonal with respect to the Euclidean inner product on $\mathbb{R}^2$, since

$$\begin{aligned}\vec{u}\bullet\vec{v} &= 1(1)+1(-1) \\ &= 0\ \rfloor\end{aligned}$$

They are not orthogonal with the respect to the weighted Euclidean inner product $\langle \vec{u},\ \vec{v}\rangle = 3u_1v_1 + 2u_2v_2$, since

$$\begin{aligned}\langle \vec{u},\ \vec{v}\rangle &= 3(1)(1) + 2(1)(-1) \\ &= 1 \ne 0\ \rfloor\end{aligned}$$

### *Example*

$U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ *and* $V = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ are orthogonal, since

$$\begin{aligned} U \bullet V &= 1(0) + 0(2) + 1(0) + 1(0) \\ &\underline{= 0\,|} \end{aligned}$$

### *Definition*

If $W$ is a subspace of an inner product space $V$, then the set of all vectors are orthogonal to every vector in $W$ is called the ***orthogonal complement*** of $W$ and is denoted by the symbol $W^{\perp}$

### *Theorem*

If $W$ is a subspace of an inner product space $V$, then:

***a)*** $W^{\perp}$ is a subspace of $V$.

***b)*** $W \cap W^{\perp} = \{0\}$

***Proof***

***a)*** Let set $W^{\perp}$ contains at least the zero vector, since $\langle \vec{0}, \vec{w} \rangle = 0$ for every vector $\vec{w}$ in $W$. We need to show that $W^{\perp}$ is closed under addition and scalar multiplication.

Suppose that $\vec{u}$ and $\vec{v}$ are vectors in $W^{\perp}$, so every vector $\vec{w}$ in $W$ we have $\langle \vec{u}, \vec{w} \rangle = 0$ and $\langle \vec{v}, \vec{w} \rangle = 0$

$$\begin{aligned} \langle \vec{u} + \vec{v}, \vec{w} \rangle &= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \\ &= 0 + 0 \\ &\underline{= 0\,|} \end{aligned}$$

***Closed under addition***

$$\begin{aligned} \langle k\vec{u}, \vec{w} \rangle &= k \langle \vec{u}, \vec{w} \rangle \\ &= k(0) \\ &\underline{= 0\,|} \end{aligned}$$

***Closed under scalar multiplication***

Which proves that $\vec{u} + \vec{w}$ and $k\vec{u}$ are in $W^{\perp}$

***b)*** If $\vec{v}$ is any vector in both $W$ and $W^{\perp}$, then $\vec{v}$ is orthogonal to itself; that is, $\langle \vec{v}, \vec{v} \rangle = 0$. It follows from the positivity axiom for inner products that $\vec{v} = \vec{0}$

## *Theorem*

If $W$ is a subspace of a finite-dimensional inner product space $V$, then the orthogonal complement of $W^{\perp}$ is $W$; that is

$$\left(W^{\perp}\right)^{\perp}=W$$

## *Example*

Let $W$ be the subspace of $\mathbb{R}^6$ spanned by the vectors

$$\vec{w}_1=(1,\ 3,\ -2,\ 0,\ 2,\ 0), \qquad \vec{w}_2=(2,\ 6,\ -5,\ -2,\ 4,\ -3)$$
$$\vec{w}_3=(0,\ 0,\ 5,\ 10,\ 0,\ 15), \qquad \vec{w}_4=(2,\ 6,\ 0,\ 8,\ 4,\ 18)$$

Find a basis for the orthogonal complement of $W$.

***Solution***

The Space $W$ is the same as the row space of the matrix

$$A=\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{pmatrix} \begin{array}{l} \\ R_2-2R_1 \\ \\ R_4-2R_1 \end{array}$$

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 0 & 0 & 4 & 8 & 0 & 18 \end{pmatrix} \begin{array}{l} R_1-2R_2 \\ \\ R_3+5R_2 \\ R_4+4R_2 \end{array}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 6 \\ 0 & 0 & -1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix} \begin{array}{l} \\ -R_2 \\ \\ \frac{1}{6}R_3 \end{array}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 6 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} R_1-6R_3 \\ R_2-3R_3 \\ \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The solution

$$\left(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6\right) = \left(-3x_2 - 4x_4 - 2x_5,\ x_2,\ -2x_4,\ x_4,\ x_5,\ 0\right)$$

$$= x_2\left(-3, 1, 0, 0, 0, 0\right) + x_4\left(-4, 0, -2, 1, 0, 0\right) + x_5\left(-2, 0, 0, 0, 1, 0\right)$$

$\vec{v}_1 = (-3, 1, 0, 0, 0, 0)$, $\vec{v}_2 = (-4, 0, -2, 1, 0, 0)$, $\vec{v}_3 = (-2, 0, 0, 0, 1, 0)$

## *Definition*

A collection of vectors in $\mathbb{R}^n$ (or inner space) is called orthogonal if any 2 are perpendicular.

$$\vec{v}_i . \vec{v}_j = \vec{v}^T \vec{v} = \begin{cases} 0 & for\ i \neq j \quad (orthogonal\ vectors) \\ 1 & for\ i = j \quad (unit\ vectors) \end{cases}$$

## *Theorem*

If $\vec{v}_1, \ \ldots, \ \vec{v}_m$ are nonzero orthogonal vectors, then they are linearly independent.

## *Definition*

A vector $\vec{v}$ is called normal if $\|\vec{v}\| = 1$

A collection of vectors $\vec{v}_1, \ \ldots, \ \vec{v}_m$ is called orthonormal if they are orthogonal and each $\|\vec{v}_i\| = 1$.

An orthonormal basis is a basis made up of orthonormal vectors.

### *Example*

***Q*** rotates every vector in the plane through the angle $\theta$.

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$Q^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad \cos^2\theta + \sin^2\theta = 1$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\underline{= Q^T}$$

The dot product $(\cos\theta \sin\theta - \sin\theta \cos\theta = 0)$, the columns are orthogonal.

They are unit vectors because $\cos^2\theta + \sin^2\theta = 1$. Those columns give an orthonormal basis for the plane $\mathbb{R}^2$.

We have: $QQ^T = I = Q^T Q$ (This type is called ***rotation***)

# *Exercises* Section 3.2 – Angle and Orthogonality in Inner Product Spaces

**1.** Which of the following form orthonormal sets?

*a)* (1, 0), (0, 2) in $\mathbb{R}^2$

*b)* $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in $\mathbb{R}^2$

*c)* $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in $\mathbb{R}^2$

*d)* $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ in $\mathbb{R}^3$

*e)* $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ in $\mathbb{R}^3$

*f)* $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ in $\mathbb{R}^3$

**2.** Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.

*a)* $\vec{u}=(1, -3), \quad \vec{v}=(2, 4)$

*b)* $\vec{u}=(-1, 0), \quad \vec{v}=(3, 8)$

*c)* $\vec{u}=(-1, 5, 2), \quad \vec{v}=(2, 4, -9)$

*d)* $\vec{u}=(4, 1, 8), \quad \vec{v}=(1, 0, -3)$

*e)* $\vec{u}=(1, 0, 1, 0), \quad \vec{v}=(-3, -3, -3, -3)$

*f)* $\vec{u}=(2, 1, 7, -1), \quad \vec{v}=(4, 0, 0, 0)$

*g)* $\vec{u}=(1, 3, -5, 4), \quad \vec{v}=(2, -4, 4, 1)$

*h)* $\vec{u}=(1, 2, 3, 4), \quad \vec{v}=(-1, -2, -3, -4)$

**3.** Find the cosine of the angle between ***A*** and ***B***.

*a)* $A=\begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix} \quad B=\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$

*b)* $A=\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} \quad B=\begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$

*c)* $A=\begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix} \quad B=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

*d)* $A=\begin{pmatrix} 1 & -2 & 7 \\ 6 & -3 & 4 \end{pmatrix} \quad B=\begin{pmatrix} 1 & -2 & 3 \\ 6 & 5 & -4 \end{pmatrix}$

**4.** Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

*a)* $\vec{u}=(-1, 3, 2), \quad \vec{v}=(4, 2, -1)$

*b)* $\vec{u}=(a, b), \quad \vec{v}=(-b, a)$

*c)* $\vec{u}=(-2, -2, -2), \quad \vec{v}=(1, 1, 1)$

*d)* $\vec{u}=(-4, 6, -10, 1), \quad \vec{v}=(2, 1, -2, 9)$

*e)* $\vec{u}=(-4, 6, -10, 1), \quad \vec{v}=(2, 1, -2, 9)$

**5.** Do there exist scalars $k$ and $l$ such that the vectors $\vec{u}=(2, k, 6), \quad \vec{v}=(l, 5, 3), \quad and \quad \vec{w}=(1, 2, 3)$ are mutually orthogonal with respect to the Euclidean inner product?

**6.** Let $\mathbb{R}^3$ have the Euclidean inner product. For which values of *k* are $\vec{u}$ and $\vec{v}$ orthogonal?

*a)* $\vec{u} = (2, 1, 3), \quad \vec{v} = (1, 7, k)$ *b)* $\vec{u} = (k, k, 1), \quad \vec{v} = (k, 5, 6)$

**7.** Let *V* be an inner product space. Show that if $\vec{u}$ and $\vec{v}$ are orthogonal unit vectors in *V*, then $\|\vec{u} - \vec{v}\| = \sqrt{2}$

**8.** Let $\boldsymbol{S}$ be a subspace of $\mathbb{R}^n$. Explain what $\left(\boldsymbol{S}^\perp\right)^\perp = \boldsymbol{S}$ means and why it is true.

**9.** The methane molecule $CH_4$ is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$ – (***note*** that all six edges have length $\sqrt{2}$, so the tetrahedron is regular). What is the cosine of the angle between the rays going from the center $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ to the vertices?

**10.** Determine if the given vectors are orthogonal.

$\vec{x}_1 = (1, 0, 1, 0), \quad \vec{x}_2 = (0, 1, 0, 1), \quad \vec{x}_3 = (1, 0, -1, 0), \quad \vec{x}_4 = (1, 1, -1, -1)$

**11.** Which of the following sets of vectors are orthogonal with respect to the Euclidean inner

*a)* $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \ \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \ \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

*b)* $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \ \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) \ \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

**12.** Consider vectors $\vec{u} = (2, 3, 5) \quad \vec{v} = (1, -4, 3) \quad in \quad \mathbb{R}^3$

*a)* $\langle \vec{u}, \vec{v} \rangle$ *b)* $\|\vec{u}\|$ *c)* $\|\vec{v}\|$ *d)* Cosine between $\vec{u}$ *and* $\vec{v}$

**13.** Consider vectors $\vec{u} = (1, 1, 1) \quad \vec{v} = (1, 2, -3) \quad in \quad \mathbb{R}^3$

*a)* $\langle \vec{u}, \vec{v} \rangle$ *b)* $\|\vec{u}\|$ *c)* $\|\vec{v}\|$ *d)* Cosine $\theta$ between $\vec{u}$ *and* $\vec{v}$

**14.** Consider vectors $\vec{u} = (1, 2, 5) \quad \vec{v} = (2, -3, 5) \quad \vec{w} = (4, 2, -3) \quad in \quad \mathbb{R}^3$

*a)* $\langle \vec{u}, \vec{v} \rangle$
*b)* $\langle \vec{u}, \vec{w} \rangle$
*c)* $\langle \vec{v}, \vec{w} \rangle$
*d)* $\|\vec{u}\|$
*e)* $\|\vec{v}\|$
*f)* $\|\vec{w}\|$
*g)* Cosine $\alpha$ between $\vec{u}$ *and* $\vec{v}$
*h)* Cosine $\beta$ between $\vec{u}$ *and* $\vec{w}$
*i)* Cosine $\theta$ between $\vec{v}$ *and* $\vec{w}$
*j)* $(\vec{u} + \vec{v}) \bullet \vec{w}$

**15.** Consider polynomial $f(t)=3t-5;\quad g(t)=t^2 \quad in \quad \mathbb{P}(t)$

*a)* $\langle f, g\rangle$ *b)* $\|f\|$ *c)* $\|g\|$ *d)* Cosine between $f$ *and* $g$

**16.** Consider polynomial $f(t)=t+2\ ;\quad g(t)=3t-2\ ;\quad h(t)=t^2-2t-3 \quad in \quad \mathbb{P}(t)$

*a)* $\langle f, g\rangle$ *d)* $\|f\|$ *g)* Cosine $\alpha$ between $f$ *and* $g$

*b)* $\langle f, h\rangle$ *e)* $\|g\|$ *h)* Cosine $\beta$ between $f$ *and* $h$

*c)* $\langle g, h\rangle$ *f)* $\|h\|$ *i)* Cosine $\theta$ between $g$ *and* $h$

**17.** Suppose $\langle \vec{u}, \vec{v}\rangle = 3+2\,i$ in a complex inner product space *V*. Find:

*a)* $\langle (2-4\,i)\vec{u}, \vec{v}\rangle$ *b)* $\langle \vec{u}, (4+3\,i)\vec{v}\rangle$ *c)* $\langle (3-6\,i)\vec{u}, (5-2\,i)\vec{v}\rangle$ *d)* $\|\vec{u}, \vec{v}\|$

**18.** Find the Fourier coefficient *c* and the projection $c\vec{v}$ of $\vec{u}=(3+4\,i,\ 2-3\,i)$ along $\vec{v}=(5+i,\ 2i)$ in $\mathbb{C}^2$

**19.** Suppose $\vec{v}=(1,\ 3,\ 5,\ 7)$. Find the projection of $\vec{v}$ onto *W* or find $\vec{w}\in W$ that minimizes $\|\vec{v}-\vec{w}\|$, where *W* is the subspace of $\mathbb{R}^4$ spanned by:

*a)* $\vec{u}_1=(1,\ 1,\ 1,\ 1)$ *and* $\vec{u}_2=(1,\ -3,\ 4,\ -2)$

*b)* $\vec{v}_1=(1,\ 1,\ 1,\ 1)$ *and* $\vec{v}_2=(1,\ 2,\ 3,\ 2)$

**20.** Suppose $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ is an orthogonal set of vectors. Prove that (*Pythagoras*)

$$\|\vec{u}_1+\vec{u}_2+\ \ldots+\vec{u}_n\|^2=\|\vec{u}_1\|^2+\|\vec{u}_2\|^2+\ \ldots+\|\vec{u}_n\|^2$$

**21.** Suppose *A* is an orthogonal matrix. Show that $\langle \vec{u}A, \vec{v}A\rangle=\langle \vec{u}, \vec{v}\rangle$ for any $\vec{u}, \vec{v}\ \in V$

**22.** Suppose *A* is an orthogonal matrix. Show that $\|\vec{u}A\|=\|\vec{u}\|$ for every $\vec{u}\ \in V$

**23.** Let *V* be an inner product space over $\mathbb{R}$ or $\mathbb{C}$. Show that

$$\|\vec{u}+\vec{v}\|=\|\vec{u}\|+\|\vec{v}\|$$

If and only if

$$\|s\vec{u}+t\vec{v}\|=s\|\vec{u}\|+t\|\vec{v}\| \quad for\ all \quad s,\ t\ge 0$$

**24.** Let *V* be an inner product vector space over $\mathbb{R}$.

*a)* If $e_1, e_2, e_3$ are three vectors in *V* with pairwise product negative, that is,

$$\langle e_1, e_2\rangle<0,\quad i,\ j=1,\ 2,\ 3,\quad i\ne j$$

Show that $e_1, e_2, e_3$ are linearly independent.

*b)* Is it possible for three vectors on the *xy*–plane to have pairwise negative products?

*c)* Does part (a) remain valid when the word “negative: is replaced with positive?

*d)* Suppose $\vec{u},\ \vec{v},\ \textit{and}\ \vec{w}$ are three unit vectors in the *xy*–plane. What are the maximum and minimum values that

$$\langle \vec{u},\ \vec{v}\rangle + \langle \vec{v},\ \vec{w}\rangle + \langle \vec{w},\ \vec{u}\rangle$$

Can attain? And when?

# *Section* 3.3 – Gram-Schmidt Process

## *Definition*

A set of two or more vectors in a real inner product space is said to be ***orthogonal*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be ***orthonormal***.

## *Theorem*

**1.** If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is an orthogonal basis for an inner product space $V$, and if $\vec{u}$ is any vector in $V$, then

$$\vec{u} = \frac{\langle \vec{u}, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 + \frac{\langle \vec{u}, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 + \cdots + \frac{\langle \vec{u}, \vec{v}_n \rangle}{\|\vec{v}_n\|^2}\vec{v}_n$$

**2.** If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is an orthonormal basis for an inner product space $V$, and if $\vec{u}$ is any vector in $V$, then

$$\vec{u} = \langle \vec{u}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{u}, \vec{v}_2 \rangle \vec{v}_2 + \cdots + \langle \vec{u}, \vec{v}_n \rangle \vec{v}_n$$

## *Proof*

**1.** Since $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for $V$, every vector $\vec{u}$ in $V$ can be expressed in the form

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

Let show that $c_i = \dfrac{\langle \vec{u}, \vec{v}_i \rangle}{\|\vec{v}_i\|^2}$ *for* $i = 1, 2, \ldots, n$

$$\begin{aligned}\langle \vec{u}, \vec{v}_i \rangle &= \langle c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n, \ \vec{v}_i \rangle \\ &= c_1 \langle \vec{v}_1, \vec{v}_i \rangle + c_2 \langle \vec{v}_2, \vec{v}_i \rangle + \cdots + c_n \langle \vec{v}_n, \vec{v}_i \rangle\end{aligned}$$

Since $S$ is an orthogonal set, all of the inner products in the last equality are zero except the $i^{th}$, so we have

$$\begin{aligned}\langle \vec{u}, \vec{v}_i \rangle &= c_i \langle \vec{v}_i, \vec{v}_i \rangle \\ &= c_i \|\vec{v}_i\|^2\end{aligned}$$

## *The Gram-Schmidt Process*

To convert a basis $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r\}$ into an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$, perform the following computations:

**Step 1**: $\vec{v}_1 = \vec{u}_1$

**Step 2**: $\vec{v}_2 = \vec{u}_2 - \dfrac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$

**Step 3**: $\vec{v}_3 = \vec{u}_3 - \dfrac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \dfrac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$

**Step 4**: $\vec{v}_4 = \vec{u}_4 - \dfrac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \dfrac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \dfrac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$

To convert the orthogonal basis into an orthonormal basis $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$, normalize the orthogonal basis vectors. $\boxed{\vec{q}_i = \dfrac{\vec{v}_i}{\|\vec{v}_i\|}}$

## *Example*

Assume that the vector space $\mathbb{R}$ has the Euclidean inner product. Apply the Gram-Schmidt process to transform the basis vectors

$$\vec{u}_1 = (1, 1, 1) \quad \vec{u}_2 = (0, 1, 1) \quad \vec{u}_3 = (0, 0, 1)$$

Into the orthogonal basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, and then normalize the ***orthogonal*** basis vectors to obtain an orthonormal basis $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$

***Solution***

$$\vec{v}_1 = \vec{u}_1$$
$$= (1, 1, 1)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$=(0,\ 1,\ 1)-\frac{0+1+1}{1^2+1^2+1^2}(1,\ 1,\ 1)$$

$$=(0,\ 1,\ 1)-\frac{2}{3}(1,\ 1,\ 1)$$

$$=\boxed{\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}$$

$$\vec{v}_3=\vec{u}_3-proj_{\vec{v}_2}\vec{u}_3$$

$$=\vec{u}_3-\frac{\langle \vec{u}_3,\vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1-\frac{\langle \vec{u}_3,\vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$=(0,\ 0,\ 1)-\frac{0+0+1}{1^2+1^2+1^2}(1,\ 1,\ 1)-\frac{0+0+\frac{1}{3}}{\left(\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$=(0,\ 0,\ 1)-\frac{1}{3}(1,\ 1,\ 1)-\frac{\frac{1}{3}}{\frac{2}{3}}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$=(0,\ 0,\ 1)-\frac{1}{3}(1,\ 1,\ 1)-\frac{1}{2}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$=\boxed{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}$$

$$\vec{q}_1=\frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$=\frac{(1,\ 1,\ 1)}{\sqrt{3}}$$

$$=\boxed{\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)}$$

$$\vec{q}_2=\frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$=\frac{\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}{\sqrt{\left(\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2}}$$

$$=\frac{\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}{\frac{\sqrt{6}}{3}}$$

$$=\boxed{\left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)}$$

$$
\begin{aligned}
\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
&= \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\sqrt{0^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}} \\
&= \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\frac{\sqrt{2}}{2}} \\
&= \left(0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)
\end{aligned}
$$

## *Gram-Schmidt* Process (*Orthonormal*)

Suppose $\vec{v}_1, \ldots, \vec{v}_n$ linearly independent in $\mathbb{R}^n$, construct $n$ ***orthonormal*** $\vec{u}_1, \ldots, \vec{u}_n$ that span the same space: span $\{\vec{u}_1, \ldots, \vec{u}_k\}$ = span $\{\vec{v}_1, \ldots, \vec{v}_k\}$

***Step* 1**: Since $\vec{v}_i$ are linearly independent ($\neq 0$), so $\|\vec{v}_1\| \neq 0$ (to create a normal vector)

Let $\vec{u}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \vec{q}_1$, then $\|\vec{u}_1\| = 1$ since $\vec{u}_1$ is orthonormal and span $\{\vec{u}_1\} = span\{\vec{v}_1\}$

$\vec{w}_1 = \vec{v}_1 \Rightarrow \vec{v}_1 = \|\vec{w}_1\|\vec{u}_1$

***Step* 2**: $\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \bullet \vec{q}_1)\vec{q}_1$

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \bullet \vec{u}_1}{\|\vec{v}_1\|}\vec{v}_1 \qquad (\vec{w}_2 \perp \vec{u}_1)$$

$$\vec{v}_2 = \|\vec{w}_2\|\vec{u}_2 + (\vec{v}_2 \bullet \vec{u}_1)\vec{u}_1 \qquad \vec{w}_2 = \|\vec{w}_2\|\vec{u}_2$$

$$\boxed{\vec{q}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}}$$

***Step* 3**: $\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \bullet \vec{q}_1)\vec{q}_1 - (\vec{v}_3 \bullet \vec{q}_2)\vec{q}_2$

$$\boxed{\vec{q}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}}$$

| | |
|---|---|
| | $\vec{q}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$ |
| $\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \bullet \vec{q}_1)\vec{q}_1$ | $\vec{q}_2 = \dfrac{\vec{w}_2}{\|\vec{w}_2\|}$ |
| $\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \bullet \vec{q}_1)\vec{q}_1 - (\vec{v} \bullet \vec{q}_2)\vec{q}_2$ | $\vec{q}_3 = \dfrac{\vec{w}_3}{\|\vec{w}_3\|}$ |
| $\vec{w}_n = \vec{v}_n - (\vec{v}_n \bullet \vec{q}_1)\vec{q}_1 - (\vec{v}_n \bullet \vec{q}_2)\vec{q}_2 - \ \ldots\ - (\vec{v}_n \bullet \vec{q}_{n-1})\vec{q}_{n-1}$ | $\vec{q}_n = \dfrac{\vec{w}_n}{\|\vec{w}_n\|}$ |

### *Example*

Use the Gram-Schmidt process to find an ***orthonormal*** basis for the subspaces of

$$\vec{v}_1 = (1,\ 1,\ 0,\ 0) \quad \vec{v}_2 = (0,\ 1,\ 1,\ 0) \quad \vec{v}_3 = (1,\ 0,\ 1,\ 1)$$

***Solution***

**Step 1**: $\vec{q}_1 = \dfrac{(1,\ 1,\ 0,\ 0)}{\sqrt{1^2+1^2+0+0}}$ $\qquad \vec{q}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$

$$= \frac{(1,\ 1,\ 0,\ 0)}{\sqrt{2}}$$

$$\boxed{= \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)}$$

**Step 2**: $\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \bullet \vec{q}_1)\vec{q}_1$

$$= (0,\ 1,\ 1,\ 0) - \left[(0,\ 1,\ 1,\ 0) \bullet \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)\right]\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)$$

$$= (0,\ 1,\ 1,\ 0) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)$$

$$= (0,\ 1,\ 1,\ 0) - \left(\frac{1}{2},\ \frac{1}{2},\ 0,\ 0\right)$$

$$\boxed{= \left(-\frac{1}{2},\ \frac{1}{2},\ 1,\ 0\right)}$$

$$\|\vec{w}_2\| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 1}$$

$$= \frac{\sqrt{3}}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}}$$

$$\boxed{= \frac{\sqrt{6}}{2}}$$

$$\vec{q}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{\left(-\frac{1}{2},\ \frac{1}{2},\ 1,\ 0\right)}{\frac{\sqrt{6}}{2}}$$

$$= \frac{2}{\sqrt{6}}\left(-\frac{1}{2},\ \frac{1}{2},\ 1,\ 0\right)$$

$$\boxed{= \left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ 0\right)}$$

***Step* 3**: $\vec{v}_3 \bullet \vec{q}_1 = (1,\ 0,\ 1,\ 1) \bullet \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)$

$$= \frac{1}{\sqrt{2}}$$

$$\vec{v}_3 \bullet \vec{q}_2 = (1,\ 0,\ 1,\ 1) \bullet \left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ 0\right)$$

$$= -\frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}}$$

$$\vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \bullet \vec{q}_1\right)\vec{q}_1 - \left(\vec{v} \bullet \vec{q}_2\right)\vec{q}_2$$

$$= (1,\ 0,\ 1,\ 1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right) - \frac{1}{\sqrt{6}}\left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ 0\right)$$

$$= (1,\ 0,\ 1,\ 1) - \left(\frac{1}{2},\ \frac{1}{2},\ 0,\ 0\right) - \left(-\frac{1}{6},\ \frac{1}{6},\ \frac{2}{6},\ 0\right)$$

$$= \left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$\vec{q}_3 = \frac{\vec{w}_3}{\left\|\vec{w}_3\right\|}$$

$$= \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + 1^2}}\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$= \frac{1}{\sqrt{\frac{21}{9}}}\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$= \frac{3}{\sqrt{21}}\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$= \left(\frac{2}{\sqrt{21}},\ -\frac{2}{\sqrt{21}},\ \frac{2}{\sqrt{21}},\ \frac{3}{\sqrt{21}}\right)$$

The ***orthonormal*** basis:

$$\left\{ \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right),\ \left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ 0\right),\ \left(\frac{2}{\sqrt{21}},\ -\frac{2}{\sqrt{21}},\ \frac{2}{\sqrt{21}},\ \frac{3}{\sqrt{21}}\right) \right\}$$

## *QR*–Decomposition

### Problem

If *A* is an $m \times n$ matrix with linearly independent column vectors, and if *Q* is the matrix that results by applying the Gram-Schmidt process to the column vectors of *A*, what relationship, if any, exists between *A* and *Q*?

To solve this problem, suppose that the column vectors of *A* are $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ and the orthonormal column vectors of *Q* are $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n$.

$$\begin{array}{lll}
\vec{u}_1 = \left\langle \vec{u}_1, \vec{q}_1 \right\rangle \vec{q}_1 + \left\langle \vec{u}_1, \vec{q}_2 \right\rangle \vec{q}_2 + \cdots + \left\langle \vec{u}_1, \vec{q}_n \right\rangle \vec{q}_n \\
\vec{u}_2 = \left\langle \vec{u}_2, \vec{q}_1 \right\rangle \vec{q}_1 + \left\langle \vec{u}_2, \vec{q}_2 \right\rangle \vec{q}_2 + \cdots + \left\langle \vec{u}_2, \vec{q}_n \right\rangle \vec{q}_n \\
\quad\vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
\vec{u}_n = \left\langle \vec{u}_n, \vec{q}_1 \right\rangle \vec{q}_1 + \left\langle \vec{u}_n, \vec{q}_2 \right\rangle \vec{q}_2 + \cdots + \left\langle \vec{u}_n, \vec{q}_n \right\rangle \vec{q}_n
\end{array}$$

$$R = \begin{bmatrix} \left\langle \vec{u}_1, \vec{q}_1 \right\rangle & \left\langle \vec{u}_2, \vec{q}_1 \right\rangle & \cdots & \left\langle \vec{u}_n, \vec{q}_1 \right\rangle \\ 0 & \left\langle \vec{u}_2, \vec{q}_2 \right\rangle & \cdots & \left\langle \vec{u}_n, \vec{q}_2 \right\rangle \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \left\langle \vec{u}_n, \vec{q}_n \right\rangle \end{bmatrix}$$

The equation $A = QR$ is a factorization of *A* into the product of a matrix *Q* with orthonormal column vectors and an invertible upper triangular matrix *R*. We call it the ***QR-decomposition of A***.

### Theorem

If *A* is an $m \times n$ matrix with linearly independent column vectors, then *A* can be factored as

$$A = QR$$

Where *Q* is an $m \times n$ matrix with orthonormal column vectors, and *R* is an $n \times n$ invertible upper triangular matrix.

### *Example*

Find the $QR$-decomposition of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

***Solution***

The column vectors of are

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

From the previous example

$$\vec{q}_1 = \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \qquad \vec{q}_2 = \left( -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right) \qquad \vec{q}_3 = \left( 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle & \langle \vec{u}_3, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle & \langle \vec{u}_3, \vec{q}_2 \rangle \\ 0 & 0 & \langle \vec{u}_3, \vec{q}_3 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} (1)\frac{1}{\sqrt{3}} + (1)\frac{1}{\sqrt{3}} + (1)\frac{1}{\sqrt{3}} & 0 + (1)\frac{1}{\sqrt{3}} + (1)\frac{1}{\sqrt{3}} & 0 + 0 + (1)\frac{1}{\sqrt{3}} \\ 0 & 0\left(\frac{-2}{\sqrt{6}}\right) + (1)\frac{1}{\sqrt{6}} + (1)\frac{1}{\sqrt{6}} & 0\left(\frac{-2}{\sqrt{6}}\right) + 0\frac{1}{\sqrt{6}} + (1)\frac{1}{\sqrt{6}} \\ 0 & 0 & 0 + (0)\frac{-1}{\sqrt{2}} + (1)\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

***A*** **=** ***Q*** ***R***

## ***Calculus***: Applying the Gram-Schmidt Process

We can apply the Gram-Schmidt orthogonalization procedure to generate some polynomials that are orthonormal on the interval $x \in [-1,\ 1]$ with inner product

$$\langle p,\ q\rangle = \int_{-1}^{1} p(x)q(x)dx$$

### ***Example***

Apply the Gram-Schmidt orthonormalization process to the basis $B = \{1,\ x,\ x^2\}$ in $\mathbb{P}_2$ using the inner product

***Solution***

$B = \{1,\ x,\ x^2\}$

Let $\vec{u}_1 = 1, \quad \vec{u}_2 = x, \quad \vec{u}_3 = x^2$

$\underline{\vec{v}_1 = \vec{u}_1 = 1}$

$$\begin{aligned}\langle \vec{v}_1, \vec{v}_1\rangle &= \int_{-1}^{1} dx \\ &= x \Big|_{-1}^{1} \\ &= 2\end{aligned}$$

$$\begin{aligned}\langle \vec{u}_2, \vec{v}_1\rangle &= \int_{-1}^{1} x\ dx \\ &= \frac{1}{2}x^2 \Big|_{-1}^{1} \\ &= 0\end{aligned}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 \\ &= x - \frac{0}{2}(1) \\ &= x\end{aligned}$$

$$\langle \vec{v}_2, \vec{v}_2\rangle = \int_{-1}^{1} x^2\ dx$$

$$= \frac{1}{3}x^3 \Big|_{-1}^{1}$$

$$\boxed{= \frac{2}{3}}$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_{-1}^{1} x^2 \, dx$$

$$= \frac{1}{3}x^3 \Big|_{-1}^{1}$$

$$\boxed{= \frac{2}{3}}$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^{1} x^3 \, dx$$

$$= \frac{1}{4}x^4 \Big|_{-1}^{1}$$

$$\boxed{= 0}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= x^2 - \frac{3}{2}(x)(0) - \frac{1}{2}\frac{2}{3}$$

$$\boxed{= x^2 - \frac{1}{3}}$$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 \, dx$$

$$= \int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) dx$$

$$= \left(\frac{1}{5}x^5 - \frac{2}{9}x^3 + \frac{1}{9}x\right) \Big|_{-1}^{1}$$

$$= \frac{1}{5} - \frac{2}{9} + \frac{1}{9} + \frac{1}{5} - \frac{2}{9} + \frac{1}{9}$$

$$\boxed{= \frac{8}{45}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\boxed{= \frac{1}{\sqrt{2}}}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{x}{\sqrt{2/3}}$$

$$\boxed{= \frac{\sqrt{3}}{\sqrt{2}} x}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3}\right)$$

$$\boxed{= \frac{3\sqrt{5}}{2\sqrt{2}} \left(x^2 - \frac{1}{3}\right)}$$

The ***orthonormal*** basis is $\left\{ \frac{1}{\sqrt{2}},\ \sqrt{\frac{3}{2}}\, x,\ \frac{1}{2}\sqrt{\frac{5}{2}}\left(3x^2 - 1\right) \right\}$

# *Exercises* *Section* 3.3 – Gram-Schmidt Process

**(1 – 14)** Use the Gram-Schmidt process to find an ***orthonormal*** basis for the subspaces of $\mathbb{R}^m$.

**1.** $\vec{u}_1 = (1,\ -3),\quad \vec{u}_2 = (2,\ 2)$

**2.** $\vec{u}_1 = (1,\ 0),\quad \vec{u}_2 = (3,\ -5)$

**3.** $\{(1,\ 1,\ 1),\ (-1,\ 1,\ 0),\ (1,\ 2,\ 1)\}$

**4.** $\{(1,\ 1,\ 1),\ (0,\ 1,\ 1),\ (0,\ 0,\ 1)\}$

**5.** $\{(1,\ 1,\ 1),\ (0,\ 2,\ 1),\ (1,\ 0,\ 3)\}$

**6.** $\{(2,\ 2,\ 2),\ (1,\ 0,\ -1),\ (0,\ 3,\ 1)\}$

**7.** $\{(1,\ -1,\ 0),\ (0,\ 1,\ 0),\ (2,\ 3,\ 1)\}$

**8.** $\{(3,\ 0,\ 4),\ (-1,\ 0,\ 7),\ (2,\ 9,\ 11)\}$

**9.** $\{(1,\ 1,\ 1,\ 1),\ (1,\ 2,\ 1,\ 0),\ (1,\ 3,\ 0,\ 0)\}$

**10.** $\{(0,\ 2,\ -1,\ 1),\ (0,\ 0,\ 1,\ 1),\ (-2,\ 1,\ 1,\ -1)\}$

**11.** $\vec{u}_1 = (1,\ 0,\ 0),\quad \vec{u}_2 = (3,\ 7,\ -2),\quad \vec{u}_3 = (0,\ 4,\ 1)$

**12.** $\vec{u}_1 = (1, 1, 1, 1),\quad \vec{u}_2 = (1,\ 2,\ 4,\ 5),\quad \vec{u}_3 = (1,\ -3,\ -4,\ -2)$

**13.** $\vec{u}_1 = (1, 1, 1, 1),\quad \vec{u}_2 = (1,\ 1,\ 2,\ 4),\quad \vec{u}_3 = (1,\ 2,\ -4,\ -3)$

**14.** $\vec{u}_1 = (0,\ 2,\ 1,\ 0),\quad \vec{u}_2 = (1,\ -1,\ 0,\ 0),\quad \vec{u}_3 = (1,\ 2,\ 0,\ -1),\quad \vec{u}_4 = (1,\ 0,\ 0,\ 1)$

**(15 – 26)** Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

**15.** $\{(1,\ 1,\ 0),\ (1,\ 2,\ 0),\ (0,\ 1,\ 2)\}$

**16.** $\{(1,\ -2,\ 2),\ (2,\ 2,\ 1),\ (2,\ -1,\ -2)\}$

**17.** $\{(1,\ 0,\ 0),\ (1,\ 1,\ 1),\ (1,\ 1,\ -1)\}$

**18.** $\{(4,\ -3,\ 0),\ (1,\ 2,\ 0),\ (0,\ 0,\ 4)\}$

**19.** $\{(0,\ 1,\ 2),\ (2,\ 0,\ 0),\ (1,\ 1,\ 1)\}$

**20.** $\{(0,\ 1,\ 1),\ (1,\ 1,\ 0),\ (1,\ 0,\ 1)\}$

**21.** $\{(1,\ 2,\ -2),\ (1,\ 0,\ -4),\ (5,\ 2,\ 0),\ (1,\ 1,\ -1)\}$

**22.** $\{(-3,\ 1,\ 2),\ (1,\ 1,\ 1),\ (2,\ 0,\ -1),\ (1,\ -3,\ 2)\}$

**23.** $\{(2,\ 1,\ 1),\ (0,\ 3,\ -1),\ (3,\ -4,\ -2),\ (-1,\ -1,\ 3)\}$

**24.** $\vec{u}_1 = (1, 1, 0,\ -1),\quad \vec{u}_2 = (1, 3, 0, 1),\quad \vec{u}_3 = (4,\ 2,\ 2,\ 0)$

**25.** $\vec{u}_1 = (1, 1, 1, 1),\quad \vec{u}_2 = (1,\ 1,\ 2,\ 4),\quad \vec{u}_3 = (1,\ 2,\ -4,\ -3)$

**26.** $\{(3,\ 4,\ 0,\ 0),\ (-1,\ 1,\ 0,\ 0),\ (2,\ 1,\ 0,\ -1),\ (0,\ 1,\ 1,\ 0)\}$

**27.** Find the ***QR***-decomposition of

*a)* $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

*b)* $\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix}$

*c)* $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$

*d)* $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

*e)* $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

**28.** Verify that the Cauchy-Schwarz inequality holds for the given vectors using the Euclidean inner product

$\vec{u} = (0,\ -2,\ 2,\ 1), \quad \vec{v} = (-1,\ -1,\ 1,\ 1)$

(**29 – 33**) Apply the Gram-Schmidt ***orthonormalization*** process in $\mathbb{C}^0\ [-1,\ 1]$ spanned by the functions, using the inner product

**29.** $f_1(x) = x + 2, \quad f_2(x) = x^2 - 3x + 4$

**30.** $f_1(x) = x, \quad f_2(x) = x^3, \quad f_3(x) = x^5$

**31.** $f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x) = \frac{1}{2}\left(3x^2 - 1\right)$

**32.** $f_1(x) = 1, \quad f_2(x) = \sin \pi x, \quad f_3(x) = \cos \pi x$

**33.** $f_1(x) = \sin \pi x, \quad f_2(x) = \sin 2\pi x, \quad f_3(x) = \sin 3\pi x$

**34.** For $\mathbb{P}_3[x]$, define the inner product over $\mathbb{R}$ as

$$\langle f,\ g \rangle = \int_{-1}^{1} f(x) g(x)\ dx$$

*a)* If $f(x) = 1$ is a unit vector in $\mathbb{P}_3[x]$?

*b)* Find an orthonormal basis for the subspace spanned by $x\ and\ x^2$.

*c)* Complete the basis in part (*b*) to an orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product.

*d)* Is

$$[f,\ g] = \int_{0}^{1} f(x) g(x)\ dx$$

Also, an inner product for $\mathbb{P}_3[x]$

*e)* Find a pair of vectors $\vec{v}$ and $\vec{w}$ such that

$$\langle \vec{v},\ \vec{w} \rangle = 0 \quad \textit{but} \quad [\vec{v},\ \vec{w}] \neq 0$$

*f)* Is the basis found in part ($c$) are orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product in part ($d$)?

# *Section* 3.4 – Orthogonal Matrices

## *Definition*

A square matrix $A$ is said to be orthogonal if its transpose is the same as its inverse, that is, if

$$A^{-1} = A^T$$

or, equivalently, if

$$AA^T = A^T A = I$$

### *Example*

The matrix $A = \begin{pmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{pmatrix}$

***Solution***

$$A^T A = \begin{pmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{pmatrix}\begin{pmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

### *Example*

The matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

***Solution***

$$A^T A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

### *Theorem*

The following are equivalent for $n \times n$ matrix $A$.

***a)*** $A$ is orthogonal.

***b)*** The row vectors of $A$ form an orthonormal set in $R^n$ with the Euclidean inner product.

***c)*** The column vectors of $A$ form an orthonormal set in $R^n$ with the Euclidean inner product.

### *Theorem*

***a)*** The inverse of an orthogonal matrix is orthogonal

***b)*** A product of orthogonal matrices is orthogonal

***c)*** If $A$ is orthogonal, then $\det(A) = 1$ or $\det(A) = -1$

### *Theorem*

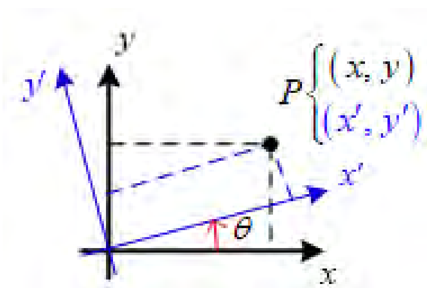
If $A$ is an $n \times n$ matrix, then the following are equivalent

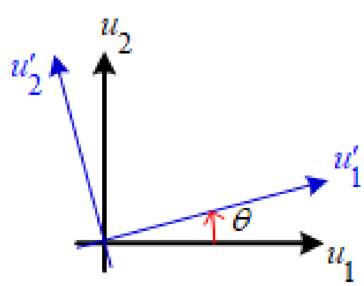
***a)*** $A$ is orthogonal.

***b)*** $\|A\vec{x}\| = \|\vec{x}\|$ for all $\boldsymbol{x}$ in $R^n$.

***c)*** $A\vec{x} \bullet A\vec{y} = \vec{x} \bullet \vec{y}$ for all $\vec{x}$ and $\vec{y}$ in $R^n$.

Let $\vec{u}_1$ and $\vec{u}_2$ be the unit vectors along the $x$- and $y$-axes and unit vectors $\vec{u}'_1$ and $\vec{u}'_2$ along the $x'$- and $y'$-axes.



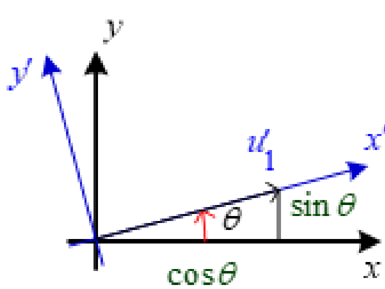




The new coordinates $(x', y')$ and the old coordinates $(x, y)$ of a point $P$ will be related by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$$

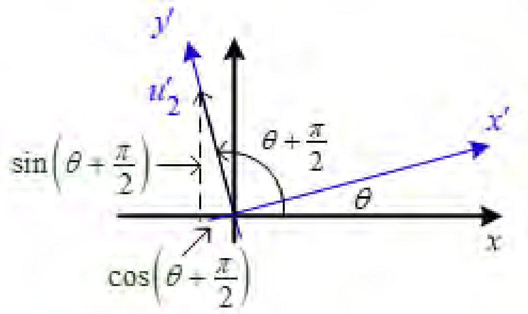
$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$P^{-1} = P^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$



$$\begin{bmatrix} \vec{x}' \\ \vec{y}' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix}$$

$$\to \begin{cases} x' = x\cos\theta + y\sin\theta \\ y' = -x\sin\theta + y\cos\theta \end{cases}$$




These are sometimes called the ***rotation equations***.

## *Example*

Use the form $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ to find the new coordinates of the point $Q(2,\ 1)$ if the coordinate axes of a rectangular coordinate system are rotated through an angle of $\theta = \frac{\pi}{4}$.

***Solution***

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

The new coordinates of $Q$ are $(x',\ y') = \left( \frac{3}{\sqrt{2}},\ -\frac{1}{\sqrt{2}} \right)$

# *Exercises* *Section* 3.4 – Orthogonal Matrices

(**1 – 2**) Show that the matrix is orthogonal

1. $A=\begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$

2. $A=\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

(**3 – 12**) Determine if the matrix is orthogonal. For those that is orthogonal find the inverse.

3. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

5. $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

6. $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$

7. $\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$

8. $\begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

9. $\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

10. $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$

11. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$

12. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{bmatrix}$

13. Find a last column so that the resulting matrix is orthogonal

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & \cdots \end{bmatrix}$$

14. Determine if the given matrix is orthogonal. If it is, find its inverse

$$\begin{bmatrix} \frac{1}{9} & \frac{4}{5} & \frac{3}{7} \\ \frac{4}{9} & \frac{3}{5} & -\frac{2}{7} \\ \frac{8}{9} & -\frac{2}{5} & \frac{3}{7} \end{bmatrix}$$

15. Prove that if $A$ is orthogonal, then $A^T$ is orthogonal.

16. Prove that if $A$ is orthogonal, then $A^{-1}$ is orthogonal.

17. Prove that if $A$ and $B$ are orthogonal, then $AB$ is orthogonal.

18. Let $Q$ be an $n \times n$ orthogonal matrix, and let $A$ be an $n \times n$ matrix. Show that $\det\left(QAQ^T\right) = \det(A)$

19. Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$

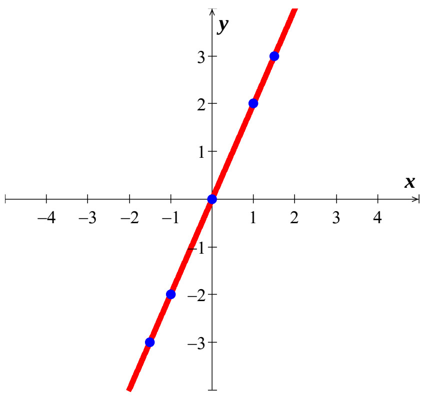
*a)* Is matrix $A$ an orthogonal matrix?
*b)* Let $B$ be the matrix obtained by normalizing each row of $A$, find $B$.
*c)* Is $B$ an orthogonal matrix?
*d)* Are the columns of $B$ orthogonal?

# *Section* 3.5 – Least Squares Analysis

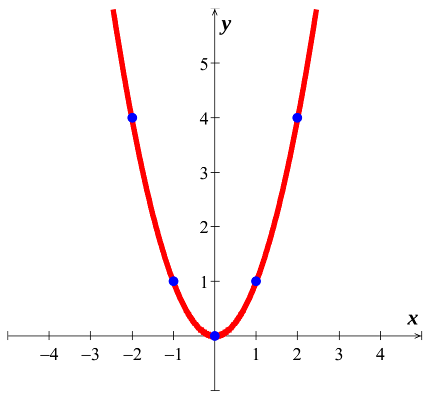
The use to ***best*** fit data, we will use results about orthogonal projections in inner product spaces to obtain a technique for fitting a line or other polynomial.

## Fitting a Curve to Data

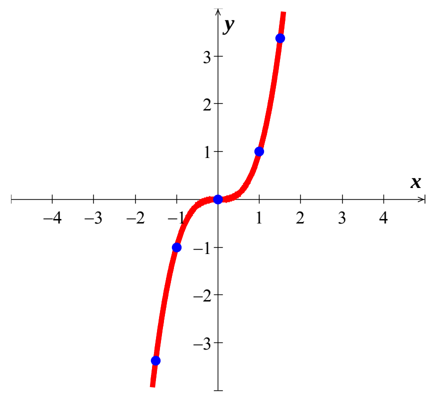
The common problem is to obtain a mathematical relationship between 2 variables $x$ and $y$ by ***fitting*** a curve to points in the $xy$-plane.
Some possibility of fitting the data




$y = ax + b$




$y = ax^2 + bx + c$




$y = ax^3 + bx^2 + cx + d$

## Least Squares Fit of a Straight Line

Recall that a system of equations $A\vec{x} = \vec{y}$ is called inconsistent if it does not have a solution. Suppose we want to fit a straight line $y = mx + b$ to the determined points $\left(x_1, y_1\right), \ \ldots, \ \left(x_n, y_n\right)$

If the data points were collinear, the line would pass through all $n$ points and the unknown coefficients $m$ and $b$ would satisfy the equations

$$\begin{array}{l} y_1 = mx_1 + b \\ y_2 = mx_2 + b \\ \vdots \quad\quad \vdots \quad\quad \vdots \\ y_n = mx_n + b \end{array} \quad \Rightarrow \quad \underbrace{\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ b \end{bmatrix}}_{\vec{x}} \underbrace{=}_{=} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\vec{y}}$$

The problem is to find $m$ and $b$ that minimize the errors is some sense.

## Least Square Problem

Given a linear system $A\vec{x} = \vec{y}$ of $m$ equations in $n$ unknowns, find a vector $\vec{x}$ that minimizes $\|\vec{y} - A\vec{x}\|$ with respect to the Euclidean inner product on $\mathbb{R}^m$. We call such as $\vec{x}$ a least squares solution of the system, we call $\vec{y} - A\vec{x}$ the least squares error vectors, and we call $\|\vec{y} - A\vec{x}\|$ the least squares error.

$$A\boldsymbol{x} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{pmatrix}$$

The term "***least square solution***" results from the fact the minimizing $\|\vec{y} - A\vec{x}\| = e_1^2 + e_2^2 + \ldots + e_m^2$

### *Example*

Find the sums of squares of the errors of (2, 4), (4, 8), (6, 6)

***Solution***

$$4 = 2m + b \quad \Rightarrow \quad 4 - 2m - b = e_1$$

$$8 = 4m + b \quad \Rightarrow \quad 8 - 4m - b = e_2$$

$$6 = 6m + b \quad \Rightarrow \quad 6 - 6m - b = e_3$$

$$e_1^2 + e_2^2 + \ldots + e_m^2 = (4 - 2m - b)^2 + (8 - 4m - b)^2 + (6 - 6m - b)^2$$

The least squares problem for this example to find the values $m$ and $b$ for which $e_1^2 + e_2^2 + \ldots + e_m^2$ is a minimum.

### *Theorem*

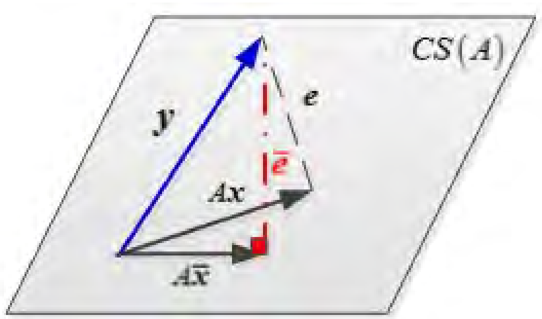
If $A$ is an $m \times n$ matrix, the equation $A\vec{x} = \vec{y}$ has a solution if and only if $\vec{y}$ is in the column space of $A$.

$$\vec{y} - A\vec{x} = \vec{e}$$

$A\vec{x}$ is a vector that is in the column space of $A$. For this $A$ the column space is a plane is $\mathbb{R}^m$

$\vec{y}$ is a vector, not in the column space of $A$ (otherwise $A\vec{x} = \vec{y}$ has an exact solution)

$\vec{e}$ is the error vector, the difference between $\vec{y}$ and $A\vec{x}$




The length $\|\vec{e}\|$ is a *minimum* exactly when $\vec{e} \perp CS(A)$

### Best Approximation *Theorem*

If $CS(A)$ is a finite dimensional subspace of an inner product space, and if $\vec{y}$ is a vector in ***V***, then $proj_{CS(A)}\vec{y}$ is the best approximation to ***y*** from $CS(A)$ is the sense that

$$\left\| \vec{y} - proj_{CS(A)}\vec{y} \right\| < \| \vec{y} - CS(A) \|$$

For every vector $\vec{w}$ in $CS(A)$ that is different from $proj_{CS(A)}\vec{y}$

### *Theorem*

For every linear system $A\vec{x} = \vec{y}$, the associated normal system

$$A^T A\vec{x} = A^T \vec{y}$$

Is consistent, and all solutions are least squares solutions of $A\vec{x} = \vec{y}$

If the columns of $A$ are linearly independent, then $A^T A$ is invertible so has a unique solution $\overline{x}$.
This solution is often expressed theoretically as

$$\left(A^T A\right)^{-1} A^T A\overline{x} = \left(A^T A\right)^{-1} A^T \vec{y}$$

$$\bar{x} = \left(A^T A\right)^{-1} A^T \vec{y}$$

## *Proof*

Let the vector $\overline{\boldsymbol{x}}$ is a least squares solution to $A\vec{x} = \vec{y} \quad \Leftrightarrow \quad (\vec{y} - A\bar{x}) \perp CS(A)$

$$(\vec{y} - A\bar{x}) \cdot \vec{\boldsymbol{z}} = 0 \qquad \vec{\boldsymbol{z}} \ in \ CS(A) \quad \& \quad \vec{\boldsymbol{z}} = A\vec{w}$$

$$(\vec{y} - A\bar{x}) \cdot A\vec{w} = 0 \qquad \vec{w} \ in \ \mathbb{R}^n$$

$$A^T (\vec{y} - A\bar{x}) \cdot \vec{w} = 0$$

$$A^T (\vec{y} - A\overline{\boldsymbol{x}}) = 0$$

$$A^T \vec{y} - A^T A\overline{\boldsymbol{x}} = 0$$

$$A^T \vec{y} = A^T A\bar{x}$$

## *Theorem*

If $A$ is an $m \times n$ matrix, then the following are equivalent

***a)*** $A$ has linearly independent column vectors.

***b)*** $A^T A$ is invertible.

## *Example*

Find the equation of the line that best fits the given points in the least-squares sense.

(40, 482), (45, 467), (50, 452), (55, 432), (60, 421)

***Solution***

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 40 & 1 \\ 45 & 1 \\ 50 & 1 \\ 55 & 1 \\ 60 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 482 \\ 467 \\ 452 \\ 432 \\ 421 \end{pmatrix}$$

Where $A = \begin{pmatrix} 40 & 1 \\ 45 & 1 \\ 50 & 1 \\ 55 & 1 \\ 60 & 1 \end{pmatrix} \quad \boldsymbol{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \boldsymbol{y} = \begin{pmatrix} 482 \\ 467 \\ 452 \\ 432 \\ 421 \end{pmatrix}$

Using the normal equation formula: $A^T Ax = A^T y$

$$\begin{pmatrix} 40 & 45 & 50 & 55 & 60 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 40 & 1 \\ 45 & 1 \\ 50 & 1 \\ 55 & 1 \\ 60 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 40 & 45 & 50 & 55 & 60 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 482 \\ 467 \\ 452 \\ 432 \\ 421 \end{pmatrix}$$

$$\begin{pmatrix} 12,750 & 250 \\ 250 & 5 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 111,970 \\ 2,255 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{1250}\begin{pmatrix} 5 & -250 \\ -250 & 12,750 \end{pmatrix}\begin{pmatrix} 111,970 \\ 2,255 \end{pmatrix}$$

$$= \begin{pmatrix} -3.12 \\ 607 \end{pmatrix}$$

***Or***

$$m = \frac{\begin{vmatrix} 111,970 & 250 \\ 2,255 & 5 \end{vmatrix}}{\begin{vmatrix} 12,750 & 250 \\ 250 & 5 \end{vmatrix}}$$

$$= \frac{-3,900}{1,250}$$

$$\boxed{= -\frac{78}{25}}$$

$$b = \frac{\begin{vmatrix} 12,750 & 111,970 \\ 250 & 2,255 \end{vmatrix}}{1,250}$$

$$= \frac{758,750}{1,250}$$

$$\boxed{= 607}$$

Thus, $\boxed{y = -\frac{78}{25}x + 607}$ *or* $\boxed{y = -3.12x + 607}$

### *Example*

Given the system equation: $\begin{cases} x_1 - x_2 = 4 \\ 3x_1 + 2x_2 = 1 \\ -2x_1 + 4x_2 = 3 \end{cases}$

*a)* Find the least-squares solution of the linear system $A\vec{x} = \vec{y}$

*b)* Find the orthogonal projection of $\vec{y}$ on the column space of $A$

*c)* Find the ***error vector*** and the ***error***

***Solution***

***a)*** $A = \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$

$$A^T A\vec{x} = A^T \vec{y}$$

$$\begin{pmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 14 & -3 \\ -3 & 21 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{285}\begin{pmatrix} 21 & 3 \\ 3 & 14 \end{pmatrix}\begin{pmatrix} 1 \\ 10 \end{pmatrix} \qquad X = A^{-1}B$$

$$= \begin{pmatrix} \frac{51}{285} \\ \frac{143}{285} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{17}{95} \\ \frac{143}{285} \end{pmatrix}$$

Thus $\underline{y = \frac{17}{95}x + \frac{143}{285}}$ *or* $\underline{y = 0.1789x + 0.5018}$

***b)*** The orthogonal projection of $\vec{y}$ on the column space of $A$

$$A\vec{x} = \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} \frac{17}{95} \\ \frac{143}{285} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{pmatrix}$$

***c)*** $$\vec{y} - A\vec{x} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1232}{285} \\ -\frac{154}{285} \\ \frac{4}{3} \end{pmatrix}$$

The ***error***: $\|\vec{y} - A\vec{x}\| = \sqrt{\left(\frac{1232}{285}\right)^2 + \left(-\frac{154}{285}\right)^2 + \left(\frac{4}{3}\right)^2}$

$$\underline{\approx 4.556 \,|}$$

# *Exercises* ***Section* 3.5 – Least Squares Analysis**

**(1 – 7)** Find the equation of the line that best fits the given points in the least-squares sense and find the error.

**1.** $\{(0,\ 2),\ (1,\ 2),\ (2,\ 0)\}$

**2.** $\{(1,\ 5),\ (2,\ 4),\ (3,\ 1),\ (4,\ 1),\ (5,\ -1)\}$

**3.** $\{(0,\ 1),\ (1,\ 3),\ (2,\ 4),\ (3,\ 4)\}$

**4.** $\{(-2,\ 0),\ (-1,\ 0),\ (0,\ 1),\ (1,\ 3),\ (2,\ 5)\}$

**5.** $\{(2,\ 3),\ (3,\ 2),\ (5,\ 1),\ (6,\ 0)\}$

**6.** $\{(-1,\ 0),\ (0,\ 1),\ (1,\ 2),\ (2,\ 4)\}$

**7.** $\{(1,\ 0),\ (2,\ 1),\ (4,\ 2),\ (5,\ 3)\}$

**(8 – 10)** Find the orthogonal projection of the vector $\vec{u}$ on the subspace of $\mathbb{R}^4$ spanned by the vectors

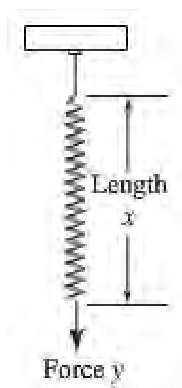
**8.** $\vec{u}=(-3,\ -3,\ 8,\ 9);\ \ \vec{v}_1=(3,\ 1,\ 0,\ 1),\ \ \vec{v}_2=(1,\ 2,\ 1,\ 1),\ \ \vec{v}_3=(-1,\ 0,\ 2,\ -1)$

**9.** $\vec{u}=(6,\ 3,\ 9,\ 6);\ \ \vec{v}_1=(2,\ 1,\ 1,\ 1),\ \ \vec{v}_2=(1,\ 0,\ 1,\ 1),\ \ \vec{v}_3=(-2,\ -1,\ 0,\ -1)$

**10.** $\vec{u}=(-2,\ 0,\ 2,\ 4);\ \ v_1=(1,\ 1,\ 3,\ 0),\ \ \vec{v}_2=(-2,\ -1,\ -2,\ 1),\ \ \vec{v}_3=(-3,\ -1,\ 1,\ 3)$

**11.** Find the standard matrix for the orthogonal projection *P* of $\mathbb{R}^2$ on the line passes through the origin and makes an angle $\theta$ with the positive *x*-axis.

**12.** Hooke's law in physics states that the length *x* of a uniform spring is a linear function of the force *y* applied to it. If we express the relationship as $y = mx + b$, then the coefficient *m* is called the spring constant.




Suppose a particular unstretched spring has a measured length of 6.1 *inches*.(i.e., *x* = 6.1 when $y = 0$). Forces of 2 pounds, 4 pounds, and 6 pounds are then applied to the spring, and the corresponding lengths are found to be 7.6 inches, 8.7 inches, and 10.4 inches. Find the spring constant.

**13.** Prove: If *A* has a linearly independent column vectors, and if $\boldsymbol{b}$ is orthogonal to the column space of *A*, then the least squares solution of $A\vec{x} = \vec{b}$ is $\vec{x} = \vec{0}$.

**14.** Let *A* be an $m \times n$ matrix with linearly independent row vectors. Find a standard matrix for the orthogonal projection of $\mathbb{R}^n$ onto the row space of *A*.

**15.** Let *W* be the line with parametric equations $x = 2t, \quad y = -t, \quad z = 4t$

*a)* Find a basis for *W*.
*b)* Find the standard matrix for the orthogonal projection on *W*.
*c)* Use the matrix in part (*b*) to find the orthogonal projection of a point $P_0\left(x_0, y_0, z_0\right)$ on *W*.
*d)* Find the distance between the point $P_0\left(2, 1, -3\right)$ and the line *W*.

**16.** In $\mathbb{R}^3$, consider the line *l* given by the equations $x = t, \quad y = t, \quad z = t$
And the line *m* given by the equations $x = s, \quad y = 2s - 1, \quad z = 1$

Let *P* be the point on *l*, and let *Q* be a point on *m*.

Find the values of $t$ and $s$ that minimize the distance between the lines by minimizing the squared distance $\|P - Q\|^2$

**17.** Determine whether the statement is true or false,

*a)* If *A* is an $m \times n$ matrix, then $A^T A$ is a square matrix.
*b)* If $A^T A$ is invertible, then *A* is invertible.
*c)* If *A* is invertible, then $A^T A$ is invertible.
*d)* If $A\vec{x} = \vec{b}$ is a consistent linear system, then $A^T A\vec{x} = A^T\vec{b}$ is also consistent.
*e)* If $A\vec{x} = \vec{b}$ is an inconsistent linear system, then $A^T A\vec{x} = A^T\vec{b}$ is also inconsistent.
*f)* Every linear system has a least squares solution.
*g)* Every linear system has a unique least squares solution.
*h)* If *A* is an $m \times n$ matrix with linearly independent columns and $\vec{b}$ is in $R^m$, then $A\vec{x} = \vec{b}$ has a unique least squares solution.

**18.** A certain experiment produces the data $\{(1, 1.8), (2, 2.7), (3, 3.4), (4, 3.8), (5, 3.9)\}$.

Find the function that it will fit these data in the form of $y = \beta_1 x + \beta_2 x^2$

**19.** According to Kepler's first law, a comet should have an ellipse, parabolic, or hyperbolic orbit (with gravitational attractions from the planets ignored). In suitable polar coordinates, the position $(r, \upsilon)$ of a comet satisfies an equation of the form

$$r = \beta + e(r \cdot \cos\upsilon)$$

Where $\beta$ is a constant and $e$ is the eccentricity of the orbit, with $0 \le e < 1$ for an ellipse, $e = 1$ for a parabolic, and $e > 1$ for a hyperbola.

Suppose observations of a newly discovered comet provide the data below.

| $\upsilon$ | .88 | 1.10 | 1.42 | 1.77 | 2.14 |
|---|---|---|---|---|---|
| $r$ | 3.00 | 2.30 | 1.65 | 1.25 | 1.01 |

Determine the type of orbit, and predict where the orbit will be when $\upsilon = 4.6$ $(radians)$?

**20.** To measure the takeoff performance of an airplane, the horizontal position of the plane was measured every second, from $t = 0$ to $t = 12$

The position (in *feet*) were:

$0,\ 8.8,\ 29.9,\ 62.0,\ 104.7,\ 159.1,\ 222.0,\ 294.5,\ 380.4,\ 471.1,\ 571.7,\ 686.8,$ *and* $809.2$

*a)* Find the least square cubic curve $y = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$ for these data.

*b)* Estimate the velocity of the plane when $t = 4.5$ *sec*, using the result from part (*a*).