Solution *Section* 4.5 – Bessel's Equation and Bessel Functions

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$$x^2y'' + xy' + \left(x^2 - \upsilon^2\right)y = 0: \qquad x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0$$

Solution

$$\upsilon^2 = \frac{1}{9} \rightarrow \upsilon = \frac{1}{3}$$

The general solution is: $\boxed{y(x) = c_1 J_{1/3}(x) + c_2 J_{-1/3}(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$$x^2y'' + xy' + \left(x^2 - \upsilon^2\right)y = 0: \qquad x^2y'' + xy' + \left(x^2 - 1\right)y = 0$$

Solution

$$\upsilon^2 = 1 \rightarrow \upsilon = 1$$

The general solution is: $\boxed{y(x) = c_1 J_1(x) + c_2 Y_1(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$$x^2y'' + xy' + \left(x^2 - \upsilon^2\right)y = 0: \qquad 4x^2y'' + 4xy' + \left(4x^2 - 25\right)y = 0$$

Solution

$$\upsilon^2 = \frac{25}{4} \rightarrow \upsilon = \pm\frac{5}{2}$$

The general solution is: $\boxed{y(x) = c_1 J_{5/2}(x) + c_2 J_{-5/2}(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$$x^2y'' + xy' + \left(x^2 - \upsilon^2\right)y = 0: \qquad 16x^2y'' + 16xy' + \left(16x^2 - 1\right)y = 0$$

Solution

$$\upsilon^2 = \frac{1}{15} \rightarrow \upsilon = \pm\frac{1}{4}$$

The general solution is: $\underline{y(x) = c_1 J_{1/4}(x) + c_2 J_{-1/4}(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2 y'' + xy' + \left(x^2 - \upsilon^2\right) y = 0$: $\qquad xy'' + y' + xy = 0$

Solution

$\upsilon^2 = 0 \;\rightarrow\; \upsilon = 0$

The general solution is: $\underline{y(x) = c_1 J_0(x) + cY_0(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2 y'' + xy' + \left(x^2 - \upsilon^2\right) y = 0$: $\qquad xy'' + y' + \left(x - \frac{4}{x}\right) y = 0$

Solution

$\upsilon^2 = 4 \;\rightarrow\; \upsilon = 2$

The general solution is: $\underline{y(x) = c_1 J_2(x) + c_2 Y_2(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2 y'' + xy' + \left(\alpha^2 x^2 - \upsilon^2\right) y = 0$: $\qquad x^2 y'' + xy' + \left(9x^2 - 4\right) y = 0$

Solution

$$\begin{cases} \alpha^2 = 9 \;\rightarrow\; \alpha = 3 \\ \upsilon^2 = 4 \;\rightarrow\; \upsilon = 2 \end{cases}$$

The general solution is: $\underline{y(x) = c_1 J_2(3x) + c_2 Y_2(3x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2 y'' + xy' + \left(\alpha^2 x^2 - \upsilon^2\right) y = 0$: $\qquad x^2 y'' + xy' + \left(36x^2 - \frac{1}{4}\right) y = 0$

Solution

$$\begin{cases} \alpha^2 = 36 \;\to\; \alpha = 6 \\ \upsilon^2 = \frac{1}{4} \;\to\; \upsilon = \frac{1}{2} \end{cases}$$

The general solution is: $\underline{y(x) = c_1 J_{1/2}(6x) + c_2 J_{-1/2}(6x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2y'' + xy' + \left(\alpha^2x^2 - \upsilon^2\right)y = 0$: $\qquad x^2y'' + xy' + \left(25x^2 - \frac{4}{9}\right)y = 0$

Solution

$$\begin{cases} \alpha^2 = 25 \;\to\; \alpha = 5 \\ \upsilon^2 = \frac{4}{9} \;\to\; \upsilon = \frac{2}{3} \end{cases}$$

The general solution is: $\underline{y(x) = c_1 J_{2/3}(5x) + c_2 J_{-2/3}(5x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2y'' + xy' + \left(\alpha^2x^2 - \upsilon^2\right)y = 0$: $\qquad x^2y'' + xy' + \left(2x^2 - 64\right)y = 0$

Solution

$$\begin{cases} \alpha^2 = 2 \;\to\; \alpha = \sqrt{2} \\ \upsilon^2 = 64 \;\to\; \upsilon = 8 \end{cases}$$

The general solution is: $\underline{y(x) = c_1 J_8\left(\sqrt{2}x\right) + c_2 Y_8\left(\sqrt{2}x\right)}$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$4x^2y'' + 8xy' + \left(x^4 - 3\right)y = 0$$

Solution

$$\frac{1}{4}\times \quad 4x^2y'' + 8xy' + \left(x^4 - 3\right)y = 0$$

$$x^2y'' + 2xy' + \left(-\frac{3}{4} + \frac{1}{4}x^4\right)y = 0 \qquad\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$A = 2, \quad B = -\frac{3}{4}, \quad C = \frac{1}{4}, \quad p = 4$

$\alpha = \frac{1-2}{2} = -\frac{1}{2}, \quad \beta = \frac{4}{2} = 2, \quad k = \frac{2\sqrt{\frac{1}{4}}}{4} = \frac{1}{4}, \quad \upsilon = \frac{\sqrt{1+3}}{4} = \frac{1}{2}$

$y(x) = x^{-1/2}\left[c_1 J_{1/2}\left(\frac{1}{4}x^2\right) + c_2 J_{-1/2}\left(\frac{1}{4}x^2\right)\right]$ $\qquad y(x) = x^{\alpha}\left[c_1 J_{\upsilon}\left(kx^{\beta}\right) + c_2 J_{-\upsilon}\left(kx^{\beta}\right)\right]$

$= x^{-1/2}\left(c_1\sqrt{\frac{2}{\pi z}}\sin z + c_2\sqrt{\frac{2}{\pi z}}\cos z\right)$ $\qquad = c_1\left(\frac{2}{\pi x}\right)^{1/2}\sin x + c_2\left(\frac{2}{\pi x}\right)^{1/2}\cos x$

$= x^{-1/2}\left(c_1\frac{2}{x}\sqrt{\frac{2}{\pi}}\sin\frac{x^2}{4} + c_2\frac{2}{x}\sqrt{\frac{2}{\pi}}\cos\frac{x^2}{4}\right)$ $\qquad z = kx^{\beta} = \frac{x^2}{4}$

$$= x^{-3/2}\left(C_1\sqrt{\frac{2}{\pi}}\sin\frac{x^2}{4} + C_2\sqrt{\frac{2}{\pi}}\cos\frac{x^2}{4}\right)$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$y'' + 9xy = 0$$

Solution

$x^2 \times \quad y'' + 9xy = 0$

$x^2y'' + 9x^3y = 0$ $\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$

$A = 0, \quad B = 0, \quad C = 9, \quad p = 3$

$\alpha = \frac{1}{2}, \quad \beta = \frac{3}{2}, \quad k = 2, \quad \upsilon = \frac{1}{3}$ $\qquad \alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2-4B}}{p}$

$$y(x) = x^{1/2}\left[c_1 J_{1/3}\left(2x^{3/2}\right) + c_2 J_{-1/3}\left(2x^{3/2}\right)\right]$$ $\qquad y(x) = x^{\alpha}\left[c_1 J_{\upsilon}\left(kx^{\beta}\right) + c_2 J_{-\upsilon}\left(kx^{\beta}\right)\right]$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' + (x-3)y = 0$$

Solution

$x \times \quad xy'' - 3y + xy = 0$

$x^2y'' - 3xy + x^2y = 0$ $\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$

$A = -3, \quad B = 0, \quad C = 1, \quad p = 2$

$$\alpha = 2, \quad \beta = 1, \quad k = 1, \quad \upsilon = \frac{\sqrt{16}}{2} = 2$$

$$\alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\boxed{y(x) = x^2\left[c_1 Y_2(x) + c_2 J_2(x)\right]}$$

$$y(x) = x^\alpha\left[c_1 J_\upsilon\left(kx^\beta\right) + c_2 J_{-\upsilon}\left(kx^\beta\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' + \left(4x^3 - 1\right)y = 0$$

Solution

$$x \times \quad xy'' - y + 4x^3 y = 0$$

$$x^2 y'' - xy + 4x^4 y = 0$$

$$x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = -1, \quad B = 0, \quad C = 4, \quad p = 4$$

$$\alpha = 1, \quad \beta = 2, \quad k = 1, \quad \upsilon = \frac{1}{2}$$

$$\alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\begin{aligned} y(x) &= x\left[c_1 J_{1/2}\left(x^2\right) + c_2 J_{-1/2}\left(x^2\right)\right] \\ &= x\left(c_1 \frac{1}{x}\sqrt{\frac{2}{\pi}}\sin x^2 + c_2 \frac{1}{x}\sqrt{\frac{2}{\pi}}\cos x^2\right) \\ &= c_1\sqrt{\frac{2}{\pi}}\sin x^2 + c_2\sqrt{\frac{2}{\pi}}\cos x^2 \\ &= \boxed{C_1 \sin x^2 + C_2 \cos x^2} \end{aligned}$$

$$y(x) = x^\alpha\left[c_1 J_\upsilon\left(kx^\beta\right) + c_2 J_{-\upsilon}\left(kx^\beta\right)\right]$$

$$y(z) = x^\alpha\left(c_1\left(\frac{2}{\pi z}\right)^{1/2}\sin z + c_2\left(\frac{2}{\pi z}\right)^{1/2}\cos z\right)$$

$$z = kx^\beta = x^2$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^2 y'' + xy' - \left(\frac{1}{4} + x^2\right)y = 0$$

Solution

$$x^2 y'' + xy' + \left(-\frac{1}{4} - x^2\right)y = 0$$

$$x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = 1, \quad B = -\frac{1}{4}, \quad C = -1, \quad p = 2$$

$$\alpha = 0, \quad \beta = 1, \quad k = i, \quad \upsilon = \frac{1}{2}$$

$$\alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$y(x) = c_1 I_{1/2}(x) + c_2 I_{-1/2}(x)$

$y(x) = x^{\alpha}\left[c_1 I_{\upsilon}\left(kx^{\beta}\right) + c_2 I_{-\upsilon}\left(kx^{\beta}\right)\right]$

$\boxed{y(x) = c_1\sqrt{\frac{2}{\pi x}}\sinh x + c_2\sqrt{\frac{2}{\pi x}}\cosh x}$

$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sinh x$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' + (2x+1)y' + (2x+1)y = 0$$

Solution

$x\times \quad xy'' + (2x+1)y' + (2x+1)y = 0$

$x^2y'' + x(2x+1)y' + \left(2x^2 + x\right)y = 0$

Let $Y = ye^{x} \quad \to \ y = Ye^{-x}$

$x^2(Y'' - 2Y' + Y)e^{-x} + x(2x+1)(Y' - Y)e^{-x} + \left(2x^2 + x\right)Ye^{-x} = 0$

$x^2Y'' - 2x^2Y' + x^2Y + \left(2x^2 + x\right)Y' - \left(2x^2 + x\right)Y + \left(2x^2 + x\right)Y = 0$

$x^2Y'' + xY' + x^2Y = 0$

$x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$

$A = 1, \quad B = 0, \quad C = 1, \quad p = 2$

$\alpha = 0, \quad \beta = 1, \quad k = 1, \quad \upsilon = 0$

$\alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$

$Y(x) = c_1 J_0(x) + c_2 Y_0(x)$

$y(x) = x^{\alpha}\left[c_1 J_0\left(kx^{\beta}\right) + c_2 Y_0\left(kx^{\beta}\right)\right]$

$\boxed{y(x) = \left(c_1 J_0(x) + c_2 Y_0(x)\right)e^{-x}}$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' - y' - xy = 0$$

Solution

$x\times \quad xy'' - y' - xy = 0$

$x^2y'' - xy' - x^2y = 0$

$x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$

$A = -1, \quad B = 0, \quad C = -1 = i, \quad p = 2$

Let $Y = \frac{y}{x}$ & $X = ix$

$$y = xY \quad \& \quad x = -iX$$

$$x^2\left(2Y' + xY''\right) - x\left(Y + xY'\right) - x^3Y = 0$$

$$x^3Y'' + x^2Y' - x\left(x^2 + 1\right)Y = 0$$

$$x^2Y'' + xY' - \left(x^2 + 1\right)Y = 0$$

$$-X^2Y'' - iXY' - \left(-X^2 + 1\right)Y = 0$$

$$X^2Y'' + XY' + \left(X^2 - 1\right)Y = 0$$

$$x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = 1, \quad B = -1, \quad C = 1, \quad p = 2$$

$$\alpha = 0, \quad \beta = 1, \quad k = 1, \quad \upsilon = 1$$

$$\alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$Y = Z_1(X)$$

$$y(x) = xZ_1(ix)$$

$$y(x) = x^\alpha\left[c_1J_0\left(kx^\beta\right) + c_2Y_0\left(kx^\beta\right)\right]$$

$$\boxed{= x\left(c_1I_1(x) + c_2K_1(x)\right)}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^4y'' + a^2y = 0$$

Solution

$$\frac{1}{x^2}\times \quad x^4y'' + a^2y = 0$$

$$x^2y'' + \frac{a^2}{x^2}y = 0$$

Let $Y = \frac{y}{\sqrt{x}} \quad \to y = \sqrt{x}\,Y$

$$X = \frac{a}{x} \quad \to \quad x = \frac{a}{X}$$

$$X^2Y'' + XY' + \left(X^2 - K^2\right)Y = 0$$

$$Y = x^{-1/2}y$$

$$Y' = -\frac{1}{2}x^{-3/2}y + x^{-1/2}y'$$

$$Y'' = \frac{3}{4}x^{-5/2}y - x^{-3/2}y' + x^{-1/2}y''$$

$$x^2\left(x^{-1/2}y'' - x^{-3/2}y' + \frac{3}{4}x^{-5/2}y\right) + x\left(-\frac{1}{2}x^{-3/2}y + x^{-1/2}y'\right) + \left(x^2 - K^2\right)x^{-1/2}y = 0$$

$$x^{3/2}y'' - x^{1/2}y' + \frac{3}{4}x^{-1/2}y - \frac{1}{2}x^{-1/2}y + x^{1/2}y' + \left(x^2 - K^2\right)x^{-1/2}y = 0$$

$$x^{3/2}y'' + \left(x^2 - K^2 + \frac{1}{4}\right)x^{-1/2}y = 0 \quad \times x^{1/2}$$

$$x^2y'' + \left(x^2 - K^2 + \frac{1}{4}\right)y = 0$$

$$x^2 - K^2 + \frac{1}{4} = x^2$$

$$-K^2 + \frac{1}{4} = 0 \quad \rightarrow \quad K^2 = \frac{1}{4}$$

$$X^2Y'' + XY' + \left(X^2 - \frac{1}{4}\right)Y = 0 \qquad\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = 1, \quad B = -\frac{1}{4}, \quad C = 1, \quad p = 2$$

$$\alpha = 0, \quad \beta = 1, \quad k = 1, \quad \upsilon = \frac{1}{2} \qquad\qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$Y = Z_{1/2}(X)$$

$$y(x) = \sqrt{x}Z_{1/2}\left(\frac{a}{x}\right) \qquad\qquad y(x) = x^{\alpha}\left[c_1J_{\upsilon}\left(kx^{\beta}\right) + c_2J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

$$= \sqrt{x}\left(c_1J_{1/2}\left(\frac{a}{x}\right) + c_2J_{-1/2}\left(\frac{a}{x}\right)\right) \qquad\qquad y(z) = x^{\alpha}\left(c_1\left(\frac{2}{\pi z}\right)^{1/2}\sin z + c_2\left(\frac{2}{\pi z}\right)^{1/2}\cos z\right)$$

$$= \sqrt{x}\left(c_1\sqrt{\frac{2x}{\pi a}}\sin\frac{a}{x} + c_2\sqrt{\frac{2x}{\pi a}}\cos\frac{a}{x}\right) \qquad\qquad z = kX^{\beta} = \frac{a}{x}$$

$$= x\left(c_1\sqrt{\frac{2}{\pi a}}\sin\frac{a}{x} + c_2\sqrt{\frac{2}{\pi a}}\cos\frac{a}{x}\right)$$

$$\boxed{= x\left(C_1\sin\frac{a}{x} + C_2\cos\frac{a}{x}\right)}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$y'' - x^2y = 0$$

Solution

$$x^2 \times \quad y'' - x^2y = 0$$

$$x^2y'' - x^4y = 0$$

$$A = 0, \quad B = 0, \quad C = -1, \quad p = 4$$

$$\alpha = \frac{1}{2}, \quad \beta = 1, \quad k = \frac{i}{2}, \quad \upsilon = 0 \qquad\qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

Let $Y = \frac{y}{\sqrt{x}} \quad \rightarrow y = \sqrt{x}\,Y$

$$X = \frac{1}{2}ix^2 \quad \rightarrow \quad x^2 = -2iX$$

$$X^2Y'' + XY' + \left(X^2 - K^2\right)Y = 0$$

$$Y = x^{-1/2}y$$

$$Y' = -\frac{1}{2}x^{-3/2}y + x^{-1/2}y'$$

$$Y'' = \frac{3}{4}x^{-5/2}y - x^{-3/2}y' + x^{-1/2}y''$$

$$x^2\left(x^{-1/2}y'' - x^{-3/2}y' + \frac{3}{4}x^{-5/2}y\right) + x\left(-\frac{1}{2}x^{-3/2}y + x^{-1/2}y'\right) + \left(x^2 - K^2\right)x^{-1/2}y = 0$$

$$x^{3/2}y'' - x^{1/2}y' + \frac{3}{4}x^{-1/2}y - \frac{1}{2}x^{-1/2}y + x^{1/2}y' + \left(x^2 - K^2\right)x^{-1/2}y = 0$$

$$x^{3/2}y'' + \left(x^2 - K^2 + \frac{1}{4}\right)x^{-1/2}y = 0 \qquad \times x^{1/2}$$

$$x^2y'' + \left(x^2 - K^2 + \frac{1}{4}\right)y = 0$$

$$x^2 - K^2 + \frac{1}{4} = -x^4$$

$$K = \frac{1}{4} \quad \rightarrow \quad K^2 = \frac{1}{16}$$

$$X^2Y'' + XY' + \left(X^2 - \frac{1}{16}\right)Y = 0$$

$$A = 1, \quad B = -\frac{1}{16}, \quad C = 1, \quad p = 2$$

$$\alpha = 0, \quad \beta = 1, \quad k = 1, \quad \upsilon = \frac{1}{4}$$

$$\alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$Y = Z_{1/4}(X)$$

$$y(x) = \sqrt{x}Z_{\frac{1}{4}}\left(\frac{i}{2}x^2\right)$$

$$= \sqrt{x}\left(c_1 I_{\frac{1}{4}}\left(\frac{x^2}{2}\right) + c_2 I_{-\frac{1}{4}}\left(\frac{x^2}{2}\right)\right)$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^2y'' - xy' + \left(1 + x^2\right)y = 0$$

Solution

$$x^2y'' - xy' + \left(1 + x^2\right)y = 0$$

$$x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = -1, \quad B = 1, \quad C = 1, \quad p = 2$$

$$\alpha = 1, \quad \beta = 1, \quad k = 1, \quad \upsilon = 0 \qquad\qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2-4B}}{p}$$

$$\boxed{y(x) = x\left[c_1 J_0(x) + c_2 Y_0(x)\right]}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' + 3y' + xy = 0$$

Solution

$$x \times \quad xy'' + 3y' + xy = 0$$

$$x^2 y'' + 3xy' + x^2 y = 0 \qquad\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = 3, \quad B = 0, \quad C = 1, \quad p = 2$$

$$\alpha = -1, \quad \beta = 1, \quad k = 1, \quad \upsilon = 1 \qquad\qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2-4B}}{p}$$

$$\boxed{y(x) = x^{-1}\left[c_1 J_1(x) + c_2 Y_1(x)\right]}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' - y' + 36x^3 y = 0$$

Solution

$$x \times \quad xy'' - y' + 36x^3 y = 0$$

$$x^2 y'' - xy' + 36x^4 y = 0 \qquad\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = -1, \quad B = 0, \quad C = 36, \quad p = 4$$

$$\alpha = 1, \quad \beta = 2, \quad k = 3, \quad \upsilon = \frac{1}{2} \qquad\qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2-4B}}{p}$$

$$\boxed{y(x) = x\left[c_1 J_{1/2}\left(3x^2\right) + c_2 J_{-1/2}\left(3x^2\right)\right]} \qquad\qquad y(x) = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^2y''-5xy'+(8+x)y=0$$

Solution

$$x^2y''-5xy'+(8+x)y=0 \qquad\qquad x^2\frac{d^2y}{dx^2}+Ax\frac{dy}{dx}+\left(B+Cx^p\right)y=0$$

$A=-5,\quad B=8,\quad C=1,\quad p=1$

$$\alpha=3,\quad \beta=\frac{1}{2},\quad k=2,\quad \upsilon=2 \qquad\qquad \alpha=\frac{1-A}{2},\ \beta=\frac{p}{2},\ k=\frac{2\sqrt{C}}{p},\ \upsilon=\frac{\sqrt{(1-A)^2-4B}}{p}$$

$$\boxed{y(x)=x^3\left[c_1J_2\left(2x^{1/2}\right)+c_2Y_2\left(2x^{1/2}\right)\right]} \qquad\qquad y(x)=x^{\alpha}\left[c_1J_{\upsilon}\left(kx^{\beta}\right)+c_2Y_{\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$36x^2y''+60xy'+\left(9x^3-5\right)y=0$$

Solution

$$x^2y''+\frac{5}{3}xy'+\left(\frac{1}{4}x^3-\frac{5}{36}\right)y=0 \qquad\qquad x^2\frac{d^2y}{dx^2}+Ax\frac{dy}{dx}+\left(B+Cx^p\right)y=0$$

$A=\frac{5}{3},\quad B=-\frac{5}{36},\quad C=\frac{1}{4},\quad p=3$

$$\alpha=-\frac{1}{3},\quad \beta=\frac{3}{2},\quad k=\frac{1}{3},\quad \upsilon=\frac{\sqrt{\left(-\frac{2}{3}\right)^2+\frac{5}{9}}}{3}=\frac{1}{3} \qquad\qquad \alpha=\frac{1-A}{2},\ \beta=\frac{p}{2},\ k=\frac{2\sqrt{C}}{p},\ \upsilon=\frac{\sqrt{(1-A)^2-4B}}{p}$$

$$\boxed{y(x)=x^{-1/3}\left[c_1J_{1/3}\left(\frac{1}{3}x^{3/2}\right)+c_2J_{-1/3}\left(\frac{1}{3}x^{3/2}\right)\right]} \qquad\qquad y(x)=x^{\alpha}\left[c_1J_{\upsilon}\left(kx^{\beta}\right)+c_2J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$16x^2y''+24xy'+\left(1+144x^3\right)y=0$$

Solution

$$x^2y''+\frac{3}{2}xy'+\left(\frac{1}{16}+9x^3\right)y=0 \qquad\qquad x^2\frac{d^2y}{dx^2}+Ax\frac{dy}{dx}+\left(B+Cx^p\right)y=0$$

$A=\frac{3}{2},\quad B=\frac{1}{16},\quad C=9,\quad p=3$

$$\alpha = -\frac{1}{4}, \quad \beta = \frac{3}{2}, \quad k = 2, \quad \upsilon = \frac{\sqrt{\left(-\frac{1}{2}\right)^2 - \frac{1}{4}}}{3} = 0 \qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\underline{y(x) = x^{-1/4}\left[c_1 J_0\left(2x^{3/2}\right) + c_2 Y_0\left(2x^{3/2}\right)\right]} \qquad y(x) = x^{\alpha}\left[c_1 J_\upsilon\left(kx^{\beta}\right) + c_2 Y_\upsilon\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^2 y'' + 3xy' + \left(1 + x^2\right)y = 0$$

Solution

$$x^2 y'' + 3xy' + \left(1 + x^2\right)y = 0 \qquad x^2 \frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = 3, \quad B = 1, \quad C = 1, \quad p = 2$$

$$\alpha = -1, \quad \beta = 1, \quad k = 1, \quad \upsilon = \frac{\sqrt{(-2)^2 - 4}}{3} = 0 \qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\underline{y(x) = x^{-1}\left[c_1 J_0(x) + c_2 Y_0(x)\right]} \qquad y(x) = x^{\alpha}\left[c_1 J_\upsilon\left(kx^{\beta}\right) + c_2 Y_\upsilon\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$4x^2 y'' - 12xy' + (15 + 16x)y = 0$$

Solution

$$x^2 y'' - 3xy' + \left(\frac{15}{4} + 4x\right)y = 0 \qquad x^2 \frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = -3, \quad B = \frac{15}{4}, \quad C = 4, \quad p = 1$$

$$\alpha = 2, \quad \beta = \frac{1}{2}, \quad k = 4, \quad \upsilon = \frac{\sqrt{(4)^2 - 15}}{1} = 1 \qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\underline{y(x) = x^{2}\left[c_1 J_1\left(4x^{1/2}\right) + c_2 Y_1\left(4x^{1/2}\right)\right]} \qquad y(x) = x^{\alpha}\left[c_1 J_\upsilon\left(kx^{\beta}\right) + c_2 Y_\upsilon\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$16x^2y'' - \left(5 - 144x^3\right)y = 0$$

Solution

$$x^2y'' + \left(9x^3 - \frac{5}{16}\right)y = 0 \qquad\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = 0, \quad B = -\frac{5}{16}, \quad C = 9, \quad p = 3$$

$$\alpha = \frac{1}{2}, \quad \beta = \frac{3}{2}, \quad k = 2, \quad \upsilon = \frac{\sqrt{1+\frac{5}{4}}}{3} = \frac{1}{2} \qquad\qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\underline{y(x) = x^{1/2}\left[c_1J_{1/2}\left(2x^{3/2}\right) + c_2J_{-1/2}\left(2x^{3/2}\right)\right]} \qquad\qquad y(x) = x^{\alpha}\left[c_1J_{\upsilon}\left(kx^{\beta}\right) + c_2J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$2x^2y'' + 3xy' - \left(28 - 2x^5\right)y = 0$$

Solution

$$x^2y'' + \frac{3}{2}xy' + \left(x^5 - 14\right)y = 0 \qquad\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = \frac{3}{2}, \quad B = -14, \quad C = 1, \quad p = 5$$

$$\alpha = -\frac{1}{4}, \quad \beta = \frac{5}{2}, \quad k = \frac{2}{5}, \quad \upsilon = \frac{\sqrt{\left(-\frac{1}{2}\right)^2 + 56}}{5} = \frac{\frac{15}{2}}{5} = \frac{3}{2} \qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\underline{y(x) = x^{-1/4}\left[c_1J_{3/2}\left(\frac{2}{5}x^{5/2}\right) + c_2J_{-3/2}\left(\frac{2}{5}x^{5/2}\right)\right]} \qquad\qquad y(x) = x^{\alpha}\left[c_1J_{\upsilon}\left(kx^{\beta}\right) + c_2J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$y'' + x^4y = 0$$

Solution

$$x^2 \times \quad y'' + x^4y = 0$$

$$x^2y'' + x^6y = 0 \qquad\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$A=0,\quad B=0,\quad C=1,\quad p=6$

$\alpha=\frac{1}{2},\quad \beta=3,\quad k=\frac{1}{3},\quad \upsilon=\frac{1}{6}$ $\qquad \alpha=\frac{1-A}{2},\ \beta=\frac{p}{2},\ k=\frac{2\sqrt{C}}{p},\ \upsilon=\frac{\sqrt{(1-A)^2-4B}}{p}$

$$\underline{y(x)=x^{1/2}\left[c_1J_{1/6}\left(\frac{1}{3}x^3\right)+c_2J_{-1/6}\left(\frac{1}{3}x^3\right)\right]} \qquad y(x)=x^{\alpha}\left[c_1J_{\upsilon}\left(kx^{\beta}\right)+c_2J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$y''+4x^3y=0$$

Solution

$x^2\times\quad y''+4x^3y=0$

$x^2y''+4x^5y=0$ $\qquad x^2\frac{d^2y}{dx^2}+Ax\frac{dy}{dx}+\left(B+Cx^p\right)y=0$

$A=0,\quad B=0,\quad C=4,\quad p=5$

$\alpha=\frac{1}{2},\quad \beta=\frac{5}{2},\quad k=\frac{4}{5},\quad \upsilon=\frac{1}{5}$ $\qquad \alpha=\frac{1-A}{2},\ \beta=\frac{p}{2},\ k=\frac{2\sqrt{C}}{p},\ \upsilon=\frac{\sqrt{(1-A)^2-4B}}{p}$

$$\underline{y(x)=x^{1/2}\left[c_1J_{1/5}\left(\frac{4}{5}x^{5/2}\right)+c_2J_{-1/5}\left(\frac{4}{5}x^{5/2}\right)\right]} \qquad y(x)=x^{\alpha}\left[c_1J_{\upsilon}\left(kx^{\beta}\right)+c_2J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Find a Frobenius solution of Bessel's equation of order zero $x^2y''+xy'+x^2y=0$

Solution

$$y''+\frac{1}{x}y'+y=0$$

Therefore, $x=0$ is a regular singular point, and that $p_0=1,\quad q_0=0$ and $p(x)\equiv 1,\quad q(x)=x^2$.

The indicial equation is: $r(r-1)+r=r^2=0 \quad\to\quad \boxed{r=0}$

There is only one Frobenius series solution: $y(x)=\sum_{n=0}^{\infty}a_nx^n$

$$y=\sum_{n=0}^{\infty}a_nx^n$$

$$y'=\sum_{n=0}^{\infty}na_nx^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$$

$$x^2 y'' + xy' + x^2 y = 0$$

$$x^2 \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=0}^{\infty} na_n x^{n-1} + x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} [n(n-1)+n]a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\sum_{n=0}^{\infty} n^2 a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0 \quad a_4 = -\frac{a_2}{4^2} = \frac{a_0}{2^2 \cdot 4^2}$$

$$0 + a_1 x + \sum_{n=2}^{\infty} n^2 a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$a_1 x + \sum_{n=2}^{\infty} \left(n^2 a_n + a_{n-2}\right) x^n = 0$$

$$a_1 = 0 \quad \to \quad a_{n(odd)} = 0$$

$$n^2 a_n + a_{n-2} = 0 \quad \Rightarrow \quad a_n = -\frac{a_{n-2}}{n^2} \quad (n \ge 2)$$

$$a_2 = -\frac{a_0}{4}$$

$$a_6 = -\frac{a_4}{6^2} = -\frac{a_0}{2^2 \cdot 4^2 \cdot 6^2}$$

$$a_{2n} = \frac{(-1)^n}{2^2 \cdot 4^2 \cdots (2n)^2} a_0 = \frac{(-1)^n}{2^{2n} \cdot (n!)^2} a_0$$

The choice $a_0 = 1$ gives us the Bessel function of order zero of the first kind.

$$J_0(x) = \frac{(-1)^n x^{2n}}{2^{2n} \cdot (n!)^2} \; \boxed{= 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots}$$

Exercise

Derive the formula $\quad x\,J'_{\upsilon}(x) = \upsilon\,J_{\upsilon}(x) - x\,J_{\upsilon+1}(x)$

Solution

$$x\,J_{\upsilon}(x) = x\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$x\,J'_{\upsilon}(x) = x\sum_{n=0}^{\infty}\frac{(-1)^n(2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= 2\sum_{n=0}^{\infty}\frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \upsilon\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \upsilon\,J_v(x)$$

$$= \upsilon\,J_v(x) + x\sum_{n=1}^{\infty}\frac{(-1)^n}{(n-1)!\,\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= \upsilon\,J_v(x) + x\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n!\,\Gamma(2+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon+1}$$

$$\underline{= \upsilon\,J_{\upsilon}(x) - x\,J_{\upsilon+1}(x)} \quad \checkmark$$

Exercise

Derive the formula $\quad x\,J'_{\upsilon}(x) = -\upsilon\,J_{\upsilon}(x) + x\,J_{\upsilon-1}(x)$

Solution

$$x\,J'_{\upsilon}(x) = x\sum_{n=0}^{\infty}\frac{(-1)^n(2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= 2\sum_{n=0}^{\infty}\frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \upsilon\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$-\upsilon J_\upsilon(x)+xJ_{\upsilon-1}(x)=-\upsilon\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}+x\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$=-\sum_{n=0}^{\infty}\frac{(-1)^n\upsilon}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$=-\sum_{n=0}^{\infty}\frac{(-1)^n\upsilon}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}+\sum_{n=0}^{\infty}\frac{(-1)^n(\upsilon+n)}{n!\Gamma(1+\upsilon+n)}2\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(-\upsilon+2n+2\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$=x\sum_{n=0}^{\infty}\frac{(-1)^n(2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$=xJ'_\upsilon(x) \quad \checkmark$$

Exercise

Derive the formula $2\upsilon J'_\upsilon(x)=xJ_{\upsilon+1}(x)+xJ_{\upsilon-1}(x)$

Solution

From previous proofs:

$$\begin{array}{rl} & xJ'_\upsilon(x)=\upsilon J_\upsilon(x)-xJ_{\upsilon+1}(x) \\ - & xJ'_\upsilon(x)=-\upsilon J_\upsilon(x)+xJ_{\upsilon-1}(x) \\ \hline & 0=2\upsilon J_\upsilon(x)-xJ_{\upsilon+1}(x)-xJ_{\upsilon-1}(x) \end{array}$$

$$2\upsilon J'_\upsilon(x)=xJ_{\upsilon+1}(x)+xJ_{\upsilon-1}(x) \quad \checkmark$$

Exercise

Prove that $\frac{d}{dx}\left[x^{\upsilon+1}J_{\upsilon+1}(x)\right]=x^{\upsilon+1}J_\upsilon(x)$

Solution

$$\frac{d}{dx}\left[x^{\upsilon+1}J_{\upsilon+1}(x)\right]=\frac{d}{dx}\left[x^{\upsilon+1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+1+n)}\left(\frac{x}{2}\right)^{2n+\upsilon+1}\right]$$

$$=\frac{d}{dx}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\upsilon+n+2)}\left(\frac{x}{2}\right)^{2n+2\upsilon+2}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(2n+2\upsilon+2)}{n!\Gamma(\upsilon+n+2)}\left(\frac{x}{2}\right)^{2n+2\upsilon+1}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n 2(n+\upsilon+1)}{n!2\Gamma(\upsilon+n+2)}\left(\frac{x}{2}\right)^{2n+2\upsilon+1} \qquad 2\Gamma(\upsilon+n+2)=2(\upsilon+n+1)\Gamma(\upsilon+n+1)$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n 2(n+\upsilon+1)}{n!\,2(\upsilon+n+1)\Gamma(\upsilon+n+1)}\left(\frac{x}{2}\right)^{2n+2\upsilon+1}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(\upsilon+n+1)}\left(\frac{x}{2}\right)^{2n+2\upsilon+1}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(\upsilon+n+1)}\left(\frac{x}{2}\right)^{2n+\upsilon}\left(\frac{x}{2}\right)^{\upsilon+1}$$

$$\underline{=x^{\upsilon+1}J_{\upsilon}(x)} \quad \checkmark$$

Exercise

Show that $y=\sqrt{x}\,J_{3/2}(x)$ is a solution of $x^2y''+\left(x^2-2\right)y=0$

Solution

$$x^2y''+\left(x^2-2\right)y=0$$

$J_{3/2}(x)$ is the solution of Bessel's equation of order $\frac{3}{2}$:

$$x^2J''_{3/2}(x)+xJ'_{3/2}(x)+\left(x^2-\frac{9}{4}\right)J_{3/2}(x)=0$$

$$x^2\left(\sqrt{x}\,J_{3/2}(x)\right)''+\left(x^2-2\right)\sqrt{x}\,J_{3/2}(x)=$$

$$=x^2\left[-\frac{1}{4}x^{-3/2}J_{3/2}(x)+x^{-1/2}J'_{3/2}(x)+x^{1/2}J''_{3/2}(x)\right]+\left(x^2-2\right)\sqrt{x}\,J_{3/2}(x)$$

$$=-\frac{1}{4}x^{1/2}J_{3/2}(x)+x^{3/2}J'_{3/2}(x)+x^{5/2}J''_{3/2}(x)+x^{5/2}J_{3/2}(x)-2\sqrt{x}\,J_{3/2}(x)$$

$$= \sqrt{x}\left[x^2 J''_{3/2}(x) + x J'_{3/2}(x) + \left(x^2 - \frac{9}{4}\right) J_{3/2}(x)\right]$$

$$\underline{= 0}$$

Exercise

Show that $\quad 4J''_{\upsilon}(x) = J_{\upsilon-2}(x) - 2J_{\upsilon}(x) + J_{\upsilon+2}(x)$

Solution

$$J_{\upsilon}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$x J'_{\upsilon}(x) = x \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= 2\sum_{n=0}^{\infty} \frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \upsilon \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$-\upsilon J_{\upsilon}(x) + x J_{\upsilon-1}(x) = -\upsilon \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + x \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n \upsilon}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n \upsilon}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \sum_{n=0}^{\infty} \frac{(-1)^n (\upsilon+n)}{n!\Gamma(1+\upsilon+n)} 2\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (-\upsilon+2n+2\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= x\sum_{n=0}^{\infty} \frac{(-1)^n (2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$\underline{= x J'_{\upsilon}(x)}$$

$$x J'_\upsilon(x) = x\sum_{n=0}^{\infty}\frac{(-1)^n(2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= 2\sum_{n=0}^{\infty}\frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \upsilon\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \upsilon J_v(x)$$

$$= \upsilon J_v(x) + x\sum_{n=1}^{\infty}\frac{(-1)^n}{(n-1)!\,\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= \upsilon J_v(x) + x\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n!\,\Gamma(2+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon+1}$$

$$\underline{= \upsilon J_\upsilon(x) - x J_{\upsilon+1}(x)\,\Big|}$$

$$x J'_\upsilon(x) = -\upsilon J_\upsilon(x) + x J_{\upsilon-1}(x)$$

$$\begin{array}{rl} & x J'_\upsilon(x) = \upsilon J_\upsilon(x) - x J_{\upsilon+1}(x) \\ + & x J'_\upsilon(x) = -\upsilon J_\upsilon(x) + x J_{\upsilon-1}(x) \\ \hline & 2x J'_\upsilon(x) = -x J_{\upsilon+1}(x) + x J_{\upsilon-1}(x) \end{array}$$

$$J'_\upsilon(x) = \frac{1}{2}\left(J_{\upsilon-1}(x) - J_{\upsilon+1}(x)\right)$$

$$J''_\upsilon(x) = \frac{1}{2}\left(J'_{\upsilon-1}(x) - J'_{\upsilon+1}(x)\right)$$

$$J'_\upsilon(x) = \frac{1}{2}\left(J_{\upsilon-1}(x) - J_{\upsilon+1}(x)\right) \quad \to (\upsilon = \upsilon - 1) \quad J'_{\upsilon-1}(x) = \frac{1}{2}\left(J_{\upsilon-2}(x) - J_\upsilon(x)\right)$$

$$J'_\upsilon(x) = \frac{1}{2}\left(J_{\upsilon-1}(x) - J_{\upsilon+1}(x)\right) \quad \to (\upsilon = \upsilon + 1) \quad J'_{\upsilon+1}(x) = \frac{1}{2}\left(J_\upsilon(x) - J_{\upsilon+2}(x)\right)$$

$$J''_\upsilon(x) = \frac{1}{2}\left(J'_{\upsilon-1}(x) - J'_{\upsilon+1}(x)\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}J_{\upsilon-2}(x) - \frac{1}{2}J_\upsilon(x) - \frac{1}{2}J_\upsilon(x) + \frac{1}{2}J_{\upsilon+2}(x)\right)$$

$$= \frac{1}{4}\left(J_{\upsilon-2}(x) - 2J_\upsilon(x) + J_{\upsilon+2}(x)\right)$$

$$\underline{4J''_\upsilon(x) = J_{\upsilon-2}(x) - 2J_\upsilon(x) + J_{\upsilon+2}(x)\,\Big|} \quad \checkmark$$

Exercise

Show that $y = x^{1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right)$ is a solution of Airy's differential equation $y'' + \alpha^2 xy = 0$, $x > 0$, whenever w is a solution of Bessel's equation of order $\frac{2}{3}$, that is, $t^2w'' + tw' + \left(t^2 - \frac{1}{9}\right)w = 0$, $t > 0$.

[*Hint*: After differentiating, substituting, and simplifying, then let $t = \frac{2}{3}\alpha x^{3/2}$].

Solution

$$y = x^{1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right)$$

$$y' = \frac{1}{2}x^{-1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right) + x^{1/2}\left(\alpha x^{1/2}\right)w'\left(\frac{2}{3}\alpha x^{3/2}\right)$$

$$= \alpha x w'\left(\frac{2}{3}\alpha x^{3/2}\right) + \frac{1}{2}x^{-1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right)$$

$$y'' = \alpha x\left(\alpha x^{1/2}\right)w''\left(\frac{2}{3}\alpha x^{3/2}\right) + \alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right) + \frac{1}{2}x^{-1/2}\left(\alpha x^{1/2}\right)w'\left(\frac{2}{3}\alpha x^{3/2}\right) - \frac{1}{4}x^{-3/2}w\left(\frac{2}{3}\alpha x^{3/2}\right)$$

$$= \alpha^2 x^{3/2}w''\left(\frac{2}{3}\alpha x^{3/2}\right) + \frac{3}{2}\alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right) - \frac{1}{4}x^{-3/2}w\left(\frac{2}{3}\alpha x^{3/2}\right)$$

$$y'' + \alpha^2 xy = 0$$

$$\alpha^2 x^{3/2}w''\left(\frac{2}{3}\alpha x^{3/2}\right) + \frac{3}{2}\alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right) - \frac{1}{4}x^{-3/2}w\left(\frac{2}{3}\alpha x^{3/2}\right) + \alpha^2 x^{3/2}w\left(\frac{2}{3}\alpha x^{3/2}\right) = 0$$

$$\alpha^2 x^{3/2}w''\left(\frac{2}{3}\alpha x^{3/2}\right) + \frac{3}{2}\alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right) + \left(\alpha^2 x^{3/2} - \frac{1}{4x^{3/2}}\right)w\left(\frac{2}{3}\alpha x^{3/2}\right) = 0$$

$$t = \frac{2}{3}\alpha x^{3/2} \quad \to \quad \alpha x^{3/2} = \frac{3}{2}t$$

$$\frac{3}{2}\frac{\alpha}{t}\left[t^2w''(t) + tw'(t) + \left(t^2 - \frac{1}{9}\right)w(t)\right] = 0$$

$$\boxed{t^2w'' + tw' + \left(t^2 - \frac{1}{9}\right)w = 0} \quad \checkmark$$

Exercise

Use the relation $\Gamma(x+1) = x\Gamma(x)$ and if p is nonnegative integer, then show that

$$J_\upsilon(x) = \frac{1}{\Gamma(\upsilon+1)}\left(\frac{x}{2}\right)^\upsilon\left[1 + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(\upsilon+1)(\upsilon+2)\cdots(\upsilon+n)}\left(\frac{x}{2}\right)^{2n}\right]$$

Solution

$$J_\upsilon(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(\upsilon+n+1)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

Given: $\Gamma(x+1) = x\Gamma(x)$

$$\Gamma(\upsilon+n+1)=(\upsilon+1)(\upsilon+2)\cdots(\upsilon+n)\Gamma(\upsilon+n)$$

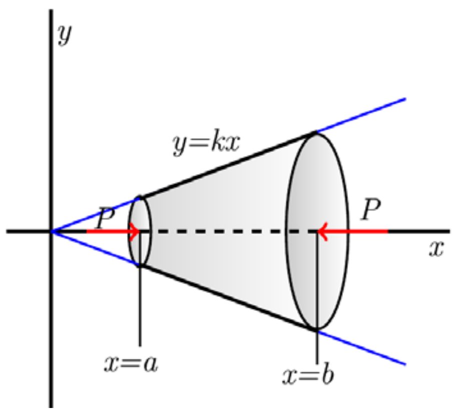
$$J_\upsilon(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(\upsilon+1)(\upsilon+2)\cdots(\upsilon+n)\Gamma(\upsilon+n)}\left(\frac{x}{2}\right)^{2n}\left(\frac{x}{2}\right)^{\upsilon}$$

$$=\frac{1}{\Gamma(\upsilon+1)}\left(\frac{x}{2}\right)^{\upsilon}\left[1+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(\upsilon+1)(\upsilon+2)\cdots(\upsilon+n)}\left(\frac{x}{2}\right)^{2n}\right] \quad \checkmark$$

Exercise

A linearly tapered rod with circular cross section, subject to an axial force P of compression. Its deflection curve $y=y(x)$ satisfies the endpoint value problem

$$EIy''+Py=0\ ; \quad y(a)=y(b)=0 \quad (1)$$



Here, however, the moment of inertia $I=I(x)$ of the cross section at x is given by

$$I(x)=\frac{1}{4}\pi(kx)^4=I_0\left(\frac{x}{b}\right)^4 \quad (2)$$

Where $I_0=I(b)$, the value of I at $x=b$. Substitution of $I(x)$ in the differential equation (1) yields to the eigenvalue problem

$$x^4y''+\lambda y=0\ ; \quad y(a)=y(b)=0 \quad (3)$$

Where $\lambda=\mu^2=\frac{Pb^4}{EI_0}$

a) Show that the general solution of $x^4y''+\mu^2y=0$ is $\quad y(x)=x\left(A\cos\frac{\mu}{x}+B\sin\frac{\mu}{x}\right)$

b) Conclude that the nth eigenvalue is given by $\mu_n=n\pi\frac{ab}{L}$, where $L=b-a$ is the length of the rod, and hence that the nth buckling force is

$$P_n=\frac{n^2\pi^2}{L^2}\left(\frac{a}{b}\right)^2EI_0$$

Solution

a) $x^{-2}\times \quad x^4y'' + \mu^2 y = 0$

$$x^2y'' + \mu^2 x^{-2} y = 0$$

$$A = 0, \quad B = 0, \quad C = \mu^2, \quad p = -2$$

$$\alpha = \frac{1}{2}, \quad \beta = -1, \quad k = \mu, \quad \upsilon = \frac{1}{2}$$

$$\alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$y(x) = x^{1/2}\left[c_1 J_{1/2}\left(\mu x^{-1}\right) + c_2 J_{-1/2}\left(\mu x^{-1}\right)\right]$$

$$y(x) = x^{\alpha}\left(c_1 J_{1/2}\left(kx^{\beta}\right) + c_2 J_{-1/2}\left(kx^{\beta}\right)\right)$$

$$= \sqrt{x}\left(c_1\sqrt{\frac{2x}{\pi\mu}}\cos\left(\frac{\mu}{x}\right) + c_2\sqrt{\frac{2x}{\pi\mu}}\sin\left(\frac{\mu}{x}\right)\right)$$

$$= x^{\alpha}\left(c_1\left(\frac{2}{\pi k x^{\beta}}\right)^{1/2}\sin\left(kx^{\beta}\right) + c_2\left(\frac{2}{\pi k x^{\beta}}\right)^{1/2}\cos\left(kx^{\beta}\right)\right)$$

$$= x\left(c_1\sqrt{\frac{2}{\pi\mu}}\cos\left(\frac{\mu}{x}\right) + c_2\sqrt{\frac{2}{\pi\mu}}\sin\left(\frac{\mu}{x}\right)\right)$$

$$A = c_1\sqrt{\frac{2}{\pi\mu}}, \quad B = c_2\sqrt{\frac{2}{\pi\mu}}$$

$$\boxed{= x\left(A\cos\left(\frac{\mu}{x}\right) + B\sin\left(\frac{\mu}{x}\right)\right)}$$

b) ***Given***: $\mu_n = n\pi\frac{ab}{L}; \quad y(a) = y(b) = 0, \quad L = b - a$

$$\begin{cases} y(a) = a\left(A\cos\left(\frac{\mu}{a}\right) + B\sin\left(\frac{\mu}{a}\right)\right) = 0 \\ y(b) = b\left(A\cos\left(\frac{\mu}{b}\right) + B\sin\left(\frac{\mu}{b}\right)\right) = 0 \end{cases}$$

$$\begin{cases} A\cos\left(\frac{\mu}{a}\right) + B\sin\left(\frac{\mu}{a}\right) = 0 \\ A\cos\left(\frac{\mu}{b}\right) + B\sin\left(\frac{\mu}{b}\right) = 0 \end{cases} \qquad (a, \ b \neq 0)$$

$$\Delta = \begin{vmatrix} \cos\frac{\mu}{a} & \sin\frac{\mu}{a} \\ \cos\frac{\mu}{b} & \sin\frac{\mu}{b} \end{vmatrix}$$

$$= \cos\frac{\mu}{a}\sin\frac{\mu}{b} - \sin\frac{\mu}{a}\cos\frac{\mu}{b}$$

$$= \sin\left(\frac{\mu}{b} - \frac{\mu}{a}\right)$$

$$= \sin\left(\frac{b-a}{ab}\mu\right)$$

$$\boxed{= \sin\left(\frac{L}{ab}\mu\right)}$$

$$\lambda = \mu^2 = \frac{Pb^4}{EI_0}$$

$$P = \frac{EI_0}{b^4}\mu^2$$

$$= \frac{EI_0}{b^4}\left(n\pi\frac{ab}{L}\right)^2$$

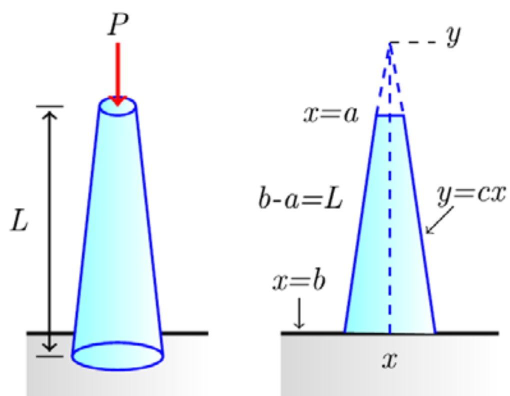
$$\boxed{= \frac{n^2\pi^2}{L^2}\left(EI_0\right)\left(\frac{a}{b}\right)^2}$$

Exercise

When a constant vertical compressive force or load *P* was applied to a thin column of uniform cross section, the deflection $y(x)$ was a solution of the boundary-value problem

$$EI\frac{d^2y}{dy^2} + Py = 0\ ; \quad y(0) = 0, \quad y(L) = 0$$

The assumption here is that the column is hinged at both ends. The column will buckle or deflect only when the compression force is a critical load P_n



a) Let assume that the column is of length *L*, is hinged at both ends, has circular cross sections, and is tapered. If the column, a truncated cone, has a linear taper $y = cx$ in cross section, the moment of inertia of a cross section with respect to an axis perpendicular to the $xy-plane$ is $I = \frac{1}{4}\pi r^4$, where $r = y$ and $y = cx$. Hence we can write $I(x) = I_0\,(x\ b)^4$, where $I_0 = I(b) = \frac{1}{4}\pi(cb)^4$. Substituting $I(x)$ into the differential equation, we see that the deflection in this case is determine from the *BVP*?

$$x^4\frac{d^2y}{dx^2} + \lambda y = 0\ ; \quad y(a) = 0, \quad y(b) = 0$$

Where $\lambda = Pb^4 EI_0$

Find the critical loads P_n for the tapered column. Use an appropriate identity to express the buckling modes $y_n(x)$ as a single function.

b) Plot the graph of the first buckling mode $y_1(x)$ corresponding to the Euler load P_1 when $b=11$ and $a=1$

Solution

c) $x^{-2}\times \quad x^4y''+\lambda y=0$

$$x^2y''+\lambda x^{-2}y=0$$

$A=0, \quad B=0, \quad C=\lambda, \quad p=-2$

$$\alpha=\frac{1}{2}, \quad \beta=-1, \quad k=\sqrt{\lambda}, \quad \upsilon=\frac{1}{2} \qquad\qquad \alpha=\frac{1-A}{2},\ \beta=\frac{p}{2},\ k=\frac{2\sqrt{C}}{p},\ \upsilon=\frac{\sqrt{(1-A)^2-4B}}{p}$$

$$y(x)=x^{1/2}\left[c_1J_{1/2}\left(\sqrt{\lambda}x^{-1}\right)+c_2J_{-1/2}\left(\sqrt{\lambda}x^{-1}\right)\right] \qquad y(x)=x^{\alpha}\left(c_1J_{1/2}\left(kx^{\beta}\right)+c_2J_{-1/2}\left(kx^{\beta}\right)\right)$$

$$=\sqrt{x}\left(c_1\sqrt{\frac{2x}{\pi\sqrt{\lambda}}}\cos\left(\frac{\sqrt{\lambda}}{x}\right)+c_2\sqrt{\frac{2x}{\pi\sqrt{\lambda}}}\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right)$$

$$=x^{\alpha}\left(c_1\sqrt{\frac{2}{\pi kx^{\beta}}}\sin\left(kx^{\beta}\right)+c_2\sqrt{\frac{2}{\pi kx^{\beta}}}\cos\left(kx^{\beta}\right)\right)$$

$$=x\left(c_1\sqrt{\frac{2x}{\pi\sqrt{\lambda}}}\cos\left(\frac{\sqrt{\lambda}}{x}\right)+c_2\sqrt{\frac{2x}{\pi\sqrt{\lambda}}}\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right) \qquad A=c_1\sqrt{\frac{2}{\pi\sqrt{\lambda}}},\quad B=c_2\sqrt{\frac{2}{\pi\sqrt{\lambda}}}$$

$$\boxed{=x\left(A\cos\left(\frac{\sqrt{\lambda}}{x}\right)+B\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right)}$$

Given: $\lambda=Pb^4EI_0$; $y(a)=y(b)=0, \quad L=b-a$

$$\begin{cases} y(a)=a\left(A\cos\left(\frac{\sqrt{\lambda}}{a}\right)+B\sin\left(\frac{\sqrt{\lambda}}{a}\right)\right)=0 \\ y(b)=b\left(A\cos\left(\frac{\sqrt{\lambda}}{b}\right)+B\sin\left(\frac{\sqrt{\lambda}}{b}\right)\right)=0 \end{cases}$$

$$\begin{cases} A\cos\left(\frac{\sqrt{\lambda}}{a}\right)+B\sin\left(\frac{\sqrt{\lambda}}{a}\right)=0 \\ A\cos\left(\frac{\sqrt{\lambda}}{b}\right)+B\sin\left(\frac{\sqrt{\lambda}}{b}\right)=0 \end{cases} \qquad (a,\ b\neq 0)$$

$$\Delta=\begin{vmatrix} \cos\frac{\sqrt{\lambda}}{a} & \sin\frac{\sqrt{\lambda}}{a} \\ \cos\frac{\sqrt{\lambda}}{b} & \sin\frac{\sqrt{\lambda}}{b} \end{vmatrix}$$

$$=\cos\frac{\sqrt{\lambda}}{a}\sin\frac{\sqrt{\lambda}}{b}-\sin\frac{\sqrt{\lambda}}{a}\cos\frac{\sqrt{\lambda}}{b}$$

$$=\sin\left(\frac{\sqrt{\lambda}}{b}-\frac{\sqrt{\lambda}}{a}\right)$$

$$= \sin\left(\frac{b-a}{ab}\sqrt{\lambda}\right)$$

$$= \sin\left(\frac{L}{ab}\sqrt{\lambda}\right) = 0$$

$$\frac{L}{ab}\sqrt{\lambda} = n\pi \quad \to \quad \sqrt{\lambda} = \frac{n\pi ab}{L} \qquad (n \in \mathbb{N})$$

$$\lambda = \frac{n^2\pi^2a^2b^2}{L^2} = Pb^4EI_0$$

$$P_n = \frac{n^2\pi^2}{L^2}\left(EI_0\right)\left(\frac{a}{b}\right)^2$$

If we let $B = -A\dfrac{\sin\frac{\sqrt{\lambda}}{a}}{\cos\frac{\sqrt{\lambda}}{a}}$

$$y(x) = x\left(A\cos\left(\frac{\sqrt{\lambda}}{x}\right) + B\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right)$$

$$= x\left(A\cos\left(\frac{\sqrt{\lambda}}{x}\right) - A\frac{\sin\frac{\sqrt{\lambda}}{a}}{\cos\frac{\sqrt{\lambda}}{a}}\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right)$$

$$= \frac{A}{\cos\frac{\sqrt{\lambda}}{a}}x\left(\cos\frac{\sqrt{\lambda}}{a}\cos\left(\frac{\sqrt{\lambda}}{x}\right) - \sin\frac{\sqrt{\lambda}}{a}\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right) \qquad \frac{A}{\cos\frac{\sqrt{\lambda}}{a}} = C$$

$$= Cx\sin\left(\frac{\sqrt{\lambda}}{x} - \frac{\sqrt{\lambda}}{a}\right)$$

$$= Cx\sin\sqrt{\lambda}\left(\frac{1}{x} - \frac{1}{a}\right)$$

$$y_n(x) = Cx\sin\sqrt{\lambda}\left(\frac{1}{x} - \frac{1}{a}\right) \qquad \left(\sqrt{\lambda} = \frac{n\pi ab}{L}\right)$$

$$= Cx\sin\frac{n\pi ab}{L}\left(\frac{1}{x} - \frac{1}{a}\right)$$

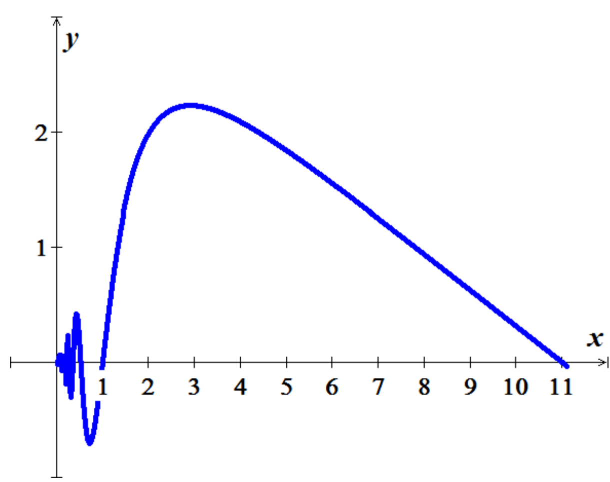
$$= Cx\sin\frac{n\pi b}{L}\left(\frac{a}{x} - 1\right)$$

$$= C_1x\sin\frac{n\pi b}{L}\left(1 - \frac{a}{x}\right)$$

d) ***Given***: $n = 1, \quad a = 1, \quad b = 11$

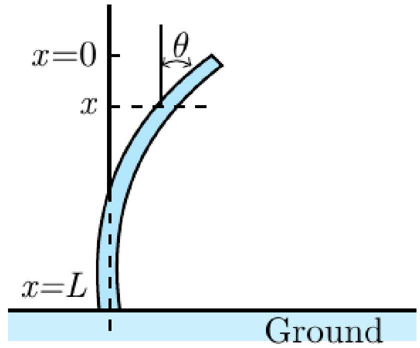
Let $C_1 = 1$

$$y_1(x) = x\sin\frac{11\pi}{10}\left(1 - \frac{1}{x}\right)$$



Exercise

For a practical application, we now consider the problem of determining when a uniform vertical column will buckle under its own weight (after, perhaps, being nudged laterally just a bit by a passing breeze). We take $x = 0$ at the free top end of the column and $x = L > 0$ at its bottom; we assume that the bottom is rigidly imbedded in the ground, perhaps in concrete.



Denote the angular deflection of the column at the point *x* by $\theta(x)$. From the theory of elasticity it follows that

$$EI\frac{d^2\theta}{dx^2} + g\rho x\theta = 0$$

Where *E* is the Young's modulus of the material of the column,

I is its cross-sectional moment of inertia

ρ is the linear density of the column

g is gravitational acceleration.

For physical reasons – no bending at the free top of the column and no deflection at its imbedded bottom – the boundary conditions are $\theta'(0) = 0, \quad \theta(L) = 0$

Determine the general equation of the length *L*.

Solution

$$EI\theta'' + g\rho x\theta = 0$$

$$\theta'' + \frac{g\rho}{EI}x\theta = 0$$

Let $\lambda = \frac{g\rho}{EI} = \gamma^2$

$x^2 \times \quad \theta'' + \gamma^2 x\theta = 0$

$x^2\theta'' + \gamma^2 x^3\theta = 0$; $\theta'(0) = 0, \quad \theta(L) = 0$

$A = 0, \quad B = 0, \quad C = \gamma^2, \quad p = 3$

$\alpha = \frac{1}{2}, \quad \beta = \frac{3}{2}, \quad k = \frac{2\gamma}{3}, \quad \upsilon = \frac{1}{3}$ $\qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$

$\theta(x) = x^{1/2}\left[c_1 J_{1/3}\left(\frac{2}{3}\gamma x^{3/2}\right) + c_2 J_{-1/3}\left(\frac{2}{3}\gamma x^{3/2}\right)\right]$ $\qquad y(x) = x^{\alpha}\left(c_1 J_{\nu}\left(kx^{\beta}\right) + c_2 J_{-\nu}\left(kx^{\beta}\right)\right)$

$$J_{1/3}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma\left(1+\frac{1}{3}+n\right)}\left(\frac{x}{2}\right)^{2n+\frac{1}{3}}$$

$$J_{\upsilon}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma\left(n+\frac{4}{3}\right)}\left(\frac{x}{2}\right)^{2n+\frac{1}{3}}$$

$$J_{\upsilon}(x)=\frac{x^{\upsilon}}{2^{\upsilon}\Gamma(\upsilon+1)}\left\{1-\frac{x^2}{2(2\upsilon+2)}+\frac{x^4}{2\cdot4\cdot(2\upsilon+2)(2\upsilon+4)}-\cdots\right\}$$

$$=\frac{x^{1/3}}{2^{1/3}\Gamma\left(\frac{4}{3}\right)}\left\{1-\frac{x^2}{2\left(\frac{2}{3}+2\right)}+\frac{x^4}{2\cdot4\cdot\left(\frac{2}{3}+2\right)\left(\frac{2}{3}+4\right)}-\cdots\right\}$$

$$=\frac{x^{1/3}}{2^{1/3}\Gamma\left(\frac{4}{3}\right)}\left\{1-\frac{3x^2}{2^3}+\frac{3^2x^4}{112\times2^3}-\cdots\right\}$$

$$J_{1/3}\left(\frac{2}{3}\gamma x^{3/2}\right)=\frac{1}{2^{1/3}\Gamma\left(\frac{4}{3}\right)}\left(\frac{2}{3}\gamma x^{3/2}\right)^{1/3}\left\{1-\frac{3}{2^3}\left(\frac{2}{3}\gamma x^{3/2}\right)^2+\frac{3^2}{896}\left(\frac{2}{3}\gamma x^{3/2}\right)^4-\cdots\right\}$$

$$=\frac{\gamma^{1/3}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)}x^{1/2}\left\{1-\frac{1}{12}\gamma^2x^3+\frac{1}{504}\gamma^4x^6-\cdots\right\}$$

$$J_{-1/3}\left(\frac{2}{3}\gamma x^{3/2}\right)=\frac{1}{2^{-1/3}\Gamma\left(\frac{4}{3}\right)}\left(\frac{2}{3}\gamma x^{3/2}\right)^{-1/3}\left\{1-\frac{1}{2\left(2-\frac{1}{3}\right)}\left(\frac{2}{3}\gamma x^{3/2}\right)^2+\frac{1}{8\left(2-\frac{1}{3}\right)\left(4-\frac{1}{3}\right)}\left(\frac{2}{3}\gamma x^{3/2}\right)^4-\cdots\right\}$$

$$=\frac{3^{1/3}}{\gamma^{1/3}\Gamma\left(\frac{2}{3}\right)}x^{-1/2}\left\{1-\frac{1}{6}\gamma^2x^3+\frac{1}{180}\gamma^4x^6-\cdots\right\}$$

$$\theta(x)=x^{1/2}\left[c_1J_{1/3}\left(\frac{2}{3}\gamma x^{3/2}\right)+c_2J_{-1/3}\left(\frac{2}{3}\gamma x^{3/2}\right)\right]$$

$$=x^{1/2}\left[c_1\frac{\gamma^{1/3}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)}x^{1/2}\left\{1-\frac{1}{12}\gamma^2x^3+\frac{1}{504}\gamma^4x^6-\cdots\right\}+c_2\frac{3^{1/3}}{\gamma^{1/3}\Gamma\left(\frac{2}{3}\right)}x^{-1/2}\left\{1-\frac{1}{6}\gamma^2x^3+\frac{1}{180}\gamma^4x^6-\cdots\right\}\right]$$

$$=c_1\frac{\gamma^{1/3}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)}\left\{x-\frac{1}{12}\gamma^2x^4+\frac{1}{504}\gamma^4x^7-\cdots\right\}+c_2\frac{3^{1/3}}{\gamma^{1/3}\Gamma\left(\frac{2}{3}\right)}\left\{1-\frac{1}{6}\gamma^2x^3+\frac{1}{180}\gamma^4x^6-\cdots\right\}$$

Given: $\theta(L)=0,\quad \theta'(0)=0$

$$\theta'(x)=c_1\frac{\gamma^{1/3}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)}\left\{1-\frac{1}{3}\gamma^2x^3+\frac{1}{72}\gamma^4x^6-\cdots\right\}+\frac{3^{1/3}}{\gamma^{1/3}\Gamma\left(\frac{2}{3}\right)}\left\{\frac{1}{2}\gamma^2x^2+\frac{1}{30}\gamma^4x^5-\cdots\right\}$$

$$\theta'(0)=c_1\frac{\gamma^{1/3}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)}=0\quad\rightarrow\quad c_1=0$$

$$\frac{3^{1/3}c_2}{\gamma^{1/3}\Gamma\left(\frac{2}{3}\right)}\left\{1-\frac{1}{6}\gamma^2L^3+\frac{1}{180}\gamma^4L^6-\cdots\right\}=0$$

$$c_2J_{-1/3}\left(\frac{2}{3}\gamma L^{3/2}\right)=0 \quad\rightarrow\quad J_{-1/3}\left(\frac{2}{3}\gamma L^{3/2}\right)=0$$

$$J_{-1/3}\left(z=\frac{2}{3}\gamma L^{3/2}\right)=0$$

Using MatLab:

```
fzero('besselj(-1/3,z)',2)
syms z y
fplot(besselj(-1/3, z))
axis([0 10 -0.5 1.1])
grid on
ylabel('J_-1/3(z)')
legend('J_-1/3')
```

$$z=1.8664$$

$$z=\frac{2}{3}\gamma L^{3/2} \quad\rightarrow\quad L=\left(\frac{3z}{2\gamma}\right)^{2/3}$$

$$L=\left(\frac{3(1.86635)}{2\sqrt{\frac{g\rho}{EI}}}\right)^{2/3}$$

$$\approx 1.986352\left(\frac{EI}{g\rho}\right)^{1/3}$$