

Solution **Section 2.3 – Product and Quotient Rules**

Exercise

Find the derivative of $y = (x+1)(\sqrt{x} + 2)$

Solution

$$\begin{aligned}y' &= (1)\left(x^{1/2} + 2\right) + (x+1)\left(\frac{1}{2}x^{-1/2}\right) \\&= x^{1/2} + 2 + \frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2} \\&= \underline{\underline{\frac{3}{2}x^{1/2} + \frac{1}{2}x^{-1/2} + 2}}\end{aligned}$$

Exercise

Find the derivative of $y = (4x + 3x^2)(6 - 3x)$

Solution

$$\begin{aligned}y' &= (4x + 3x^2)\frac{d}{dx}(6 - 3x) + (6 - 3x)\frac{d}{dx}(4x + 3x^2) & y = 24x + 6x^2 - 9x^3 \\&= (4x + 3x^2)(-3) + (6 - 3x)(4 + 6x) \\&= -12x - 9x^2 + 24 + 36x - 12x - 18x^2 \\&= \underline{\underline{-27x^2 + 12x + 24}}\end{aligned}$$

Exercise

Find the derivative of $y = \left(\frac{1}{x} + 1\right)(2x + 1)$

Solution

$$\begin{aligned}y' &= \left(x^{-1} + 1\right)\frac{d}{dx}(2x + 1) + (2x + 1)\frac{d}{dx}\left(x^{-1} + 1\right) \\&= \left(x^{-1} + 1\right)(2) + (2x + 1)\left(-x^{-2}\right) \\&= \frac{2}{x} + 2 + (2x + 1)\left(-\frac{1}{x^2}\right) \\&= \frac{2}{x} + 2 - \frac{2}{x} - \frac{1}{x^2} \\&= 2 - \frac{1}{x^2} \\&= \underline{\underline{\frac{2x^2 - 1}{x^2}}}\end{aligned}$$

Exercise

Find the derivative of $y = \frac{3 - \frac{2}{x}}{x + 4}$

Solution

$$\begin{aligned}y &= \frac{\frac{3x-2}{x}}{x+4} \\&= \frac{3x-2}{x} \cdot \frac{1}{x+4} \\&= \frac{3x-2}{x^2+4x}\end{aligned}$$

$$\begin{aligned}y' &= \frac{\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} x^2 + 2 \begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} x + \begin{vmatrix} 3 & -2 \\ 4 & 0 \end{vmatrix}}{(x^2+4x)^2} \\&= \frac{-3x^2+4x+8}{x^2(x+4)^2}\end{aligned}$$

OR

$$\begin{aligned}y' &= \frac{(x^2+4x)(3) - (3x-2)(2x+4)}{[x(x+4)]^2} \\&= \frac{3x^2+12x-6x^2-12x+4x+8}{x^2(x+4)^2} \\&= \frac{-3x^2+4x+8}{x^2(x+4)^2}\end{aligned}$$

Exercise

Find the derivative of $g(x) = \frac{x^2 - 4x + 2}{x^2 + 3}$

Solution

$$\begin{aligned}g'(x) &= \frac{\begin{vmatrix} 1 & -4 \\ 1 & 0 \end{vmatrix} x^2 + 2 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} x + \begin{vmatrix} -4 & 2 \\ 0 & 3 \end{vmatrix}}{(x^2+3)^2} \\&= \frac{4x^2+2x-12}{(x^2+3)^2}\end{aligned}$$

$$\frac{d}{dx} \left(\frac{ax^2+bx+c}{dx^2+ex+f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2+ex+f)^2}$$

$$\frac{d}{dx} \left(\frac{ax^2+bx+c}{dx^2+ex+f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2+ex+f)^2}$$

Or

$$\begin{aligned}g' &= \frac{(2x-4)(x^2+3) - (x^2-4x+2)(2x)}{(x^2+3)^2} \\&= \frac{2x^3+6x-4x^2-12-2x^3+8x^2-4x}{(x^2+3)^2} \\&= \frac{4x^2+2x-12}{(x^2+3)^2}\end{aligned}$$

Exercise

Find the derivative of $f(x) = \frac{(3-4x)(5x+1)}{7x-9}$

Solution

$$f(x) = \frac{-20x^2+11x+3}{7x-9}$$

$$f'(x) = \frac{\begin{vmatrix} -20 & 11 \\ 0 & 7 \end{vmatrix} x^2 + 2 \begin{vmatrix} -20 & 3 \\ 0 & -9 \end{vmatrix} x + \begin{vmatrix} 11 & 3 \\ 7 & -9 \end{vmatrix}}{(7x-9)^2}$$

$$= \frac{-140x^2+360x-120}{(7x-9)^2}$$

$$\frac{d}{dx} \left(\frac{ax^2+bx+c}{dx^2+ex+f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2+ex+f)^2}$$

Or

$$\begin{aligned}D_x \left[\frac{(3-4x)(5x+1)}{7x-9} \right] &= \frac{[(-4)(5x+1) + (3-4x)(5)](7x-9) - (3-4x)(5x+1)(7)}{(7x-9)^2} \\&= \frac{[-20x-4+15-20x](7x-9) - (15x+3-20x^2-4x)(7)}{(7x-9)^2} \\&= \frac{(-40x+11)(7x-9) - 7(-20x^2+11x+3)}{(7x-9)^2} \\&= \frac{-280x^2+360x+77x-99+140x^2-77x-21}{(7x-9)^2} \\&= \frac{-140x^2+360x-120}{(7x-9)^2}\end{aligned}$$

Exercise

Find the derivative of $f(x) = x\left(1 - \frac{2}{x+1}\right)$

Solution

$$f(x) = x - \frac{2x}{x+1}$$

$$\underline{f'(x) = 1 - \frac{2}{(x+1)^2}}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

$$f'(x) = 1 - \frac{2(x+1) - 2x}{(x+1)^2}$$

$$\left(\frac{2x}{x+1}\right)' \Rightarrow \begin{array}{ll} f = 2x & f' = 2 \\ g = x+1 & g' = 1 \end{array}$$

$$= 1 - \frac{2x+2-2x}{(x+1)^2}$$

$$\underline{= 1 - \frac{2}{(x+1)^2}}$$

Exercise

Find the derivative of $f(x) = (\sqrt{x} + 3)(x^2 - 5x)$

Solution

$$\begin{aligned} f' &= \left(\frac{1}{2}x^{-1/2}\right)(x^2 - 5x) + (\sqrt{x} + 3)(2x - 5) \\ &= \frac{1}{2}x^{3/2} - \frac{5}{2}x^{1/2} + 2x^{3/2} - 5x^{1/2} + 6x - 15 \\ &= \frac{5}{2}x^{3/2} - \frac{15}{2}x^{1/2} + 6x - 15 \\ &\underline{= \frac{5}{2}x^{3/2} + 6x - \frac{15}{2}x^{1/2} - 15} \end{aligned}$$

Exercise

Find the derivative of $y = (2x + 3)(5x^2 - 4x)$

Solution

$$\begin{aligned} y &= (2x + 3)(5x^2 - 4x) = 10x^3 - 8x^2 + 15x^2 - 12x \\ &= 10x^3 + 7x^2 - 12x \\ \underline{y' &= 30x^2 + 14x - 12} \end{aligned}$$

Exercise

Find the derivative of $y = \left(x^2 + 1\right)\left(x + 5 + \frac{1}{x}\right)$

Solution

$$y = x^3 + 5x^2 + x + x + 5 + \frac{1}{x}$$

$$= x^3 + 5x^2 + 2x + 5 + x^{-1}$$

$$y' = 3x^2 + 10x + 2 - x^{-2}$$

$$= 3x^2 + 10x + 2 - \frac{1}{x^2}$$

Exercise

Find the derivative of $y = \frac{x+4}{5x-2}$

Solution

$$y' = -\frac{22}{(5x-2)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

$$\begin{aligned} y' &= \frac{(5x-2) \frac{d}{dx}[(x+4)] - (x+4) \frac{d}{dx}[(5x-2)]}{(5x-2)^2} \\ &= \frac{(5x-2)(1) - (x+4)(5)}{(5x-2)^2} \\ &= \frac{5x-2-5x-20}{(5x-2)^2} \\ &= -\frac{22}{(5x-2)^2} \end{aligned}$$

Exercise

Find the derivative of $z = \frac{4-3x}{3x^2+x}$

Solution

$$z' = \frac{4-3x}{3x^2+x}$$

$$= \frac{9x^2 - 24x - 4}{(3x^2+x)^2}$$

$$\frac{d}{dx} \left(\frac{ax^2+bx+c}{dx^2+ex+f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2+ex+f)^2}$$

Or

$$\begin{aligned} z' &= \frac{-3(3x^2 + x) - (6x + 1)(4 - 3x)}{(3x^2 + x)^2} \\ &= \frac{-9x^2 - 3x - (24x - 18x^2 + 4 - 3x)}{(3x^2 + x)^2} \\ &= \frac{-9x^2 - 3x - 24x + 18x^2 - 4}{(3x^2 + x)^2} \\ &= \frac{9x^2 - 24x - 4}{(3x^2 + x)^2} \end{aligned}$$

$$z' = \frac{u'v - v'u}{u^2}$$

$$\begin{aligned} u &= 4 - 3x & v &= 3x^2 + x \\ u' &= -3 & v' &= 6x + 1 \end{aligned}$$

Exercise

Find the derivative of $y = (2x - 7)^{-1}(x + 5)$

Solution

$$\begin{aligned} y' &= -(2x - 7)^{-2}(2)(x + 5) + (2x - 7)^{-1} \\ &= -(2x - 7)^{-2}(2x + 10) + (2x - 7)^{-1} \\ &= \left[-(2x - 7)^{-2}(2x + 10) + (2x - 7)^{-1} \right] \frac{(2x - 7)^2}{(2x - 7)^2} \\ &= \frac{-2x - 10 + 2x - 7}{(2x - 7)^2} \\ &= \frac{-17}{(2x - 7)^2} \end{aligned}$$

Exercise

Find the derivative of $f(x) = \frac{\sqrt{x} - 1}{\sqrt{x} + 1}$

Solution

$$\begin{aligned} f'(x) &= \frac{\frac{1}{2}(1 + 1)x^{-1/2}}{(\sqrt{x} + 1)^2} \\ &= \frac{1}{\sqrt{x}(\sqrt{x} + 1)^2} \end{aligned} \quad \left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx + d)^2}$$

Or

$$\begin{aligned}f'(x) &= \frac{\frac{1}{2}x^{-1/2}(x^{1/2}+1) - \frac{1}{2}x^{-1/2}(x^{1/2}-1)}{(\sqrt{x}+1)^2} \\&= \frac{\frac{1}{2}1+x^{-1/2}-1+x^{-1/2}}{(\sqrt{x}+1)^2} \\&= \frac{\frac{1}{2}2x^{-1/2}}{(\sqrt{x}+1)^2} \\&= \frac{1}{x^{1/2}(\sqrt{x}+1)^2} \\&= \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}\end{aligned}$$

$$\begin{aligned}u &= x^{1/2}-1 & v &= x^{1/2}+1 \\u' &= \frac{1}{2}x^{-1/2} & v' &= \frac{1}{2}x^{-1/2}\end{aligned}$$

Exercise

Find the derivative of $y = \frac{1}{(x^2-1)(x^2+x+1)}$

Solution

$$\begin{aligned}y &= \frac{1}{x^4+x^3+x^2-x^2-x-1} \\&= \frac{1}{x^4+x^3-x-1} \\y' &= \frac{-(4x^3+3x^2-1)}{(x^4+x^3-x-1)^2} \\&= \frac{-4x^3-3x^2+1}{(x^4+x^3-x-1)^2}\end{aligned}$$
$$\left(\frac{1}{v}\right)' = -\frac{v'}{v^2}$$

Exercise

Find the derivative of $f(x) = \frac{x^{3/2}(x^2+1)}{x+1}$

Solution

$$\begin{aligned}f(x) &= \frac{x^{7/2}+x^{3/2}}{x+1} \\u &= x^{7/2}+x^{3/2} & v &= x+1 \\u' &= \frac{7}{2}x^{5/2}+\frac{3}{2}x^{1/2} & v' &= 1\end{aligned}$$

$$f'(x) = \frac{\frac{7}{2}x^{7/2} + \frac{3}{2}x^{3/2} + \frac{7}{2}x^{5/2} + \frac{3}{2}x^{1/2} - x^{7/2} - x^{3/2}}{(x+1)^2}$$

$$= \frac{\frac{1}{2} \frac{5x^{7/2} + x^{3/2} + 7x^{5/2} + 3x^{1/2}}{(x+1)^2}}{\quad}$$

Exercise

Find the derivative of $f(x) = \frac{x^3 - 4x^2 + x}{x - 2}$

Solution

$$f'(x) = \frac{3x^3 - 8x^2 + x - 6x^2 + 16x - 2 - x^3 + 4x^2 - x}{(x-2)^2}$$

$$= \frac{2x^3 - 10x^2 + 16x - 2}{(x-2)^2}$$

$$u = x^3 - 4x^2 + x \quad v = x - 2$$

$$u' = 3x^2 - 8x + 1 \quad v' = 1$$

Exercise

Find the derivative of $g(x) = \frac{x(3-x)}{2x^2}$

Solution

$$g(x) = \frac{1}{2} \frac{3-x}{x}$$

$$g'(x) = \frac{1}{2} \frac{-x - 3 + x}{x^2}$$

$$= -\frac{3}{2x^2}$$

$$u = 3 - x \quad v = x$$

$$u' = -1 \quad v' = 1$$

Exercise

Find the derivative of $y = \frac{2x^2}{3x+1}$

Solution

$$y' = 2 \frac{6x^2 + 2x - 3x^2}{(3x+1)^2}$$

$$= \frac{6x^2 + 4x}{(3x+1)^2}$$

$$u = x^2 \quad v = 3x + 1$$

$$u' = 2x \quad v' = 3$$

Exercise

Find the derivative of $f(x) = \frac{x^9 + x^8 + 4x^5 - 7x}{x^4 - 3x^2 + 2x + 1}$

Solution

$$u = x^9 + x^8 + 4x^5 - 7x \quad v = x^4 - 3x^2 + 2x + 1$$

$$u' = 9x^8 + 8x^7 + 20x^4 - 7 \quad v' = 4x^3 - 6x + 2$$

x^{12}	x^{11}	x^{10}	x^9	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x	x^0
			9									
		18										
9	8	-27	-24	16			20			21	-14	
-12	-4	6	6	20	8	-60	40	-7		42	14	-7
				-16		24	-8	-28				
			-2									
				-2								

$$f'(x) = \frac{-3x^{12} + 4x^{11} - 21x^{10} - 2x^9 + 27x^8 + 8x^7 - 36x^6 + 32x^5 - 15x^4 + 63x^2 - 7}{(x^4 - 3x^2 + 2x + 1)^2}$$

Exercise

Find the derivative of $f(x) = \frac{x}{1+x^2}$

Solution

$$f'(x) = \frac{1+x^2 - 2x^2}{(1+x^2)^2}$$

$$u = x \quad v = 1+x^2$$

$$u' = 1 \quad v' = 2x$$

$$= \frac{1-x^2}{(1+x^2)^2}$$

Exercise

Find the derivative of $y = \frac{x^2 - 2ax + a^2}{x - a}$

Solution

$$y = \frac{(x-a)^2}{x-a} = x-a$$

$$y' = 1$$

Exercise

Find the derivative of $f(x) = \frac{x^2 + 4x^{1/2}}{x^2}$

Solution

$$f(x) = 1 + 4x^{-3/2}$$

$$\underline{f'(x) = -6x^{-5/2}}$$

Exercise

Find the derivative of $f(x) = (2x+1)(3x^2+2)$

Solution

$$f'(x) = 2(3x^2+2) + (6x)(2x+1)$$

$$= 6x^2 + 4 + 12x^2 + 6x$$

$$\underline{= 18x^2 + 6x + 4}$$

Exercise

Find the derivative of $f(x) = \frac{x^2-1}{x^2+1}$

Solution

$$f'(x) = \frac{2x^2 + 2x - 2x^3 + 2x}{(x^2+1)^2}$$

$$\underline{= \frac{-2x^3 + 2x^2 + 4x}{(x^2+1)^2}}$$

$$u = x^2 - 1 \quad v = x^2 + 1$$

$$u' = 2x \quad v' = 2x$$

Exercise

Find the derivative of $y = \frac{4x^3 + 3x + 1}{2x^5}$

Solution

$$y = 2x^{-2} + \frac{3}{2}x^{-4} + \frac{1}{2}x^{-5}$$

$$y' = -4x^{-3} - 6x^{-5} - \frac{5}{2}x^{-6}$$

$$= -\frac{1}{2}x^{-6}(8x^3 + 12x + 5)$$

$$\underline{= -\frac{8x^3 + 12x + 5}{2x^6}}$$

Exercise

Find the derivative of $y = \frac{4}{3-x}$

Solution

$$\underline{y' = \frac{4}{(3-x)^2}}$$

$$\left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$$

Exercise

Find the derivative of $y = \frac{2}{1-x^2}$

Solution

$$\underline{y' = \frac{4x}{(1-x^2)^2}}$$

$$\left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$$

Exercise

Find the derivative of $f(x) = \frac{\pi}{2-\pi x}$

Solution

$$\underline{f'(x) = \frac{\pi^2}{(2-\pi x)^2}}$$

$$\left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$$

Exercise

Find the derivative of $y = \frac{x-4}{5x-2}$

Solution

$$\underline{y' = \frac{1(-2) - (-4)(5)}{(5x-2)^2} = \frac{18}{(5x-2)^2}}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \frac{3x-4}{2x-1}$

Solution

$$y' = \frac{-3+8}{(2x-1)^2}$$
$$= \frac{5}{(2x-1)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \frac{3x+4}{2x+1}$

Solution

$$y' = \frac{3-8}{(2x+1)^2}$$
$$= \frac{-5}{(2x+1)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \frac{-3x+4}{2x+1}$

Solution

$$y' = \frac{-3-8}{(2x+1)^2}$$
$$= -\frac{11}{(2x+1)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \frac{-3x-4}{2x-1}$

Solution

$$y' = \frac{3+8}{(2x-1)^2}$$
$$= \frac{11}{(2x-1)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \frac{2x-3}{x+1}$

Solution

$$y' = \frac{2+3}{(x+1)^2}$$
$$= \frac{5}{(x+1)^2}$$

$$\left(\frac{ax+b}{cx+d} \right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \frac{3x}{3x-2}$

Solution

$$y' = \frac{-6}{(3x-2)^2}$$

$$\left(\frac{ax+b}{cx+d} \right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \frac{x-3}{2x+5}$

Solution

$$y' = \frac{11}{(2x+5)^2}$$

$$\left(\frac{ax+b}{cx+d} \right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \frac{5x-3}{2x+5}$

Solution

$$y' = \frac{31}{(2x+5)^2}$$

$$\left(\frac{ax+b}{cx+d} \right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \frac{6x-8}{2x-3}$

Solution

$$y' = -\frac{2}{(2x-3)^2}$$

$$\left(\frac{ax+b}{cx+d} \right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \frac{x^2 - 4}{5x^2 - 2}$

Solution

$$y' = \frac{2(-2 + 20)x}{(5x^2 - 2)^2}$$
$$= \frac{36x}{(5x^2 - 2)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

Exercise

Find the derivative of $y = \frac{3x^2 - 4}{2x^2 - 1}$

Solution

$$y' = \frac{2(-3 + 8)x}{(2x^2 - 1)^2}$$
$$= \frac{10x}{(2x^2 - 1)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

Exercise

Find the derivative of $y = \frac{3x^2 + 4}{2x^2 + 1}$

Solution

$$y' = \frac{2(3 - 8)x}{(2x^2 + 1)^2}$$
$$= -\frac{10x}{(2x^2 + 1)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

Exercise

Find the derivative of $y = \frac{2x^2 - 3}{x^2 + 1}$

Solution

$$y' = \frac{2(2+3)x}{(x^2+1)^2}$$

$$= \frac{10x}{(x^2+1)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

Exercise

Find the derivative of $y = \frac{3x^2}{3x^2 - 2}$

Solution

$$y' = -\frac{12x}{(3x^2 - 2)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

Exercise

Find the derivative of $y = \frac{5x^2 - 3}{2x^2 + 5}$

Solution

$$y' = \frac{2(25+6)x}{(2x^2+5)^2}$$

$$= \frac{62x}{(2x^2+5)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

Exercise

Find the derivative of $y = \frac{6x^2 - 8}{2x^2 + 1}$

Solution

$$y' = \frac{2(6+16)x}{(2x^2+1)^2}$$

$$= \frac{44x}{(2x^2+1)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

Exercise

Find the derivative of $y = \frac{6x^3 + 8}{2x^3 + 1}$

Solution

$$y' = \frac{3(6-16)x^2}{(2x^3+1)^2}$$
$$= -\frac{30x^2}{(2x^3+1)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

Exercise

Find the derivative of $y = \frac{5x^3 - 3}{2x^3 + 5}$

Solution

$$y' = \frac{3(25+6)x^2}{(2x^3+5)^2}$$
$$= \frac{93x^2}{(2x^3+5)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

Exercise

Find the derivative of $y = \frac{x^3}{3x^3 - 2}$

Solution

$$y' = -\frac{6x^2}{(3x^3-2)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

Exercise

Find the derivative of $y = \frac{2x^3 - 3}{2x^3 + 1}$

Solution

$$y' = \frac{24x^2}{(2x^3+1)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

Exercise

Find the derivative of $y = \frac{2x^4 - 3}{2x^4 + 1}$

Solution

$$y' = \frac{4(2+6)x^3}{(2x^4 + 1)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

$$= \frac{32x^3}{(2x^4 + 1)^2}$$

Exercise

Find the derivative of $y = \frac{x^2 - 4x + 1}{5x^2 - 2x - 1}$

Solution

$$y' = \frac{\begin{vmatrix} 1 & -4 \\ 5 & -2 \end{vmatrix} x^2 + 2 \begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} x + \begin{vmatrix} -4 & 1 \\ -2 & -1 \end{vmatrix}}{(5x^2 - 2x - 1)^2}$$

$$\frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

$$= \frac{18x^2 - 12x + 6}{(5x^2 - 2x - 1)^2}$$

Exercise

Find the derivative of $y = \frac{3x^2 - 4x + 2}{2x^2 + x - 1}$

Solution

$$y' = \frac{\begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} x^2 + 2 \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} x + \begin{vmatrix} -4 & 2 \\ 1 & -1 \end{vmatrix}}{(2x^2 + x - 1)^2}$$

$$\frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

$$= \frac{11x^2 - 14x + 6}{(2x^2 + x - 1)^2}$$

Exercise

Find the derivative of $y = \frac{3x^2 + x - 4}{2x^2 + 1}$

Solution

$$y' = \frac{\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} x^2 + 2 \begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} x + \begin{vmatrix} 1 & -4 \\ 0 & 1 \end{vmatrix}}{(2x^2 + 1)^2}$$
$$= \frac{2x^2 + 22x + 1}{(2x^2 + 1)^2}$$

$$\frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

Exercise

Find the derivative of $y = \frac{2x^2 - 3}{x^2 + 5x + 1}$

Solution

$$y' = \frac{\begin{vmatrix} 2 & 0 \\ 1 & 5 \end{vmatrix} x^2 + 2 \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} x + \begin{vmatrix} 0 & -3 \\ 5 & 1 \end{vmatrix}}{(x^2 + 5x + 1)^2}$$
$$= \frac{10x^2 + 10x + 15}{(x^2 + 5x + 1)^2}$$

$$\frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

Exercise

Find the derivative of $y = \frac{3x^2}{3x^2 + 6x - 8}$

Solution

$$y' = \frac{\begin{vmatrix} 3 & 0 \\ 3 & 6 \end{vmatrix} x^2 + 2 \begin{vmatrix} 3 & 0 \\ 3 & -8 \end{vmatrix} x + \begin{vmatrix} 0 & 0 \\ 6 & -8 \end{vmatrix}}{(3x^2 + 6x - 8)^2}$$
$$= \frac{18x^2 - 48x}{(3x^2 + 6x - 8)^2}$$

$$\frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

Exercise

Find the derivative of $y = \frac{x^2 + 2x}{2x^2 + x - 5}$

Solution

$$y' = \frac{\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} x^2 + 2 \begin{vmatrix} 1 & 0 \\ 2 & -5 \end{vmatrix} x + \begin{vmatrix} 2 & 0 \\ 1 & -5 \end{vmatrix}}{(2x^2 + x - 5)^2}$$
$$= \frac{-3x^2 - 10x - 10}{(2x^2 + x - 5)^2}$$

$$\frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

Exercise

Find the derivative of $y = \frac{x^2 + 5x + 1}{x^2}$

Solution

$$y' = \frac{\begin{vmatrix} 1 & 5 \\ 1 & 0 \end{vmatrix} x^2 + 2 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} x + \begin{vmatrix} 5 & 1 \\ 0 & 0 \end{vmatrix}}{x^4}$$
$$= \frac{-5x^2 - 4x}{x^4}$$
$$= \frac{-5x - 4}{x^3}$$

$$\frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

Exercise

Find the derivative of $y = \frac{x^2 - 3x + 1}{x^2 - 8x + 5}$

Solution

$$y' = \frac{\begin{vmatrix} 1 & -3 \\ 1 & -8 \end{vmatrix} x^2 + 2 \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} x + \begin{vmatrix} -3 & 1 \\ -8 & 5 \end{vmatrix}}{(x^2 - 8x + 5)^2}$$
$$= \frac{-5x^2 + 8x - 7}{(x^2 - 8x + 5)^2}$$

$$\frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

Exercise

Find the **first** and **second** derivative $y = \frac{x^2 + 5x - 1}{x^2}$

Solution

$$\begin{aligned} y' &= \frac{(2x+5)x^2 - 2x(x^2 + 5x - 1)}{x^4} & \left(\frac{u}{v}\right)' &= \frac{u'v - v'u}{v^2} & u &= x^2 + 5x - 1 & v &= x^2 \\ &= \frac{(2x+5)x^2 - 2x(x^2 + 5x - 1)}{x^4} & u' &= 2x + 5 & v' &= 2x \\ &= x \frac{(2x+5)x - 2(x^2 + 5x - 1)}{x^4} \\ &= \frac{2x^2 + 5x - 2x^2 - 10x + 2}{x^3} \\ &= \frac{-5x + 2}{x^3} \end{aligned}$$

$$\begin{aligned} y'' &= \frac{(-5)x^3 - 3x^2(-5x + 2)}{x^6} & u &= -5x + 2 & v &= x^3 \\ &= x^2 \frac{-5x^3 + 15x - 6}{x^6} & u' &= -5 & v' &= 3x^2 \\ &= \frac{-5x^3 + 15x - 6}{x^4} \end{aligned}$$

Exercise

Find y' , y'' , y''' : $y = (x-3)\sqrt{x+2}$

Solution

$$\begin{aligned} y' &= \sqrt{x+2} + \frac{1}{2}(x-3)(x+2)^{-1/2} \\ y'' &= \frac{1}{2}(x+2)^{-1/2} + \frac{1}{2}(x+2)^{-1/2} - \frac{1}{4}(x-3)(x+2)^{-3/2} \\ &= (x+2)^{-1/2} - \frac{1}{4}(x-3)(x+2)^{-3/2} \\ y''' &= -\frac{1}{2}(x+2)^{-3/2} - \frac{1}{4}(x+2)^{-3/2} + \frac{3}{8}(x-3)(x+2)^{-5/2} \\ &= -\frac{3}{4}(x+2)^{-3/2} + \frac{3}{8}(x-3)(x+2)^{-5/2} \end{aligned}$$

Exercise

For what value(s) of x is the line tangent to the curve $y = x\sqrt{6-x}$ horizontal? Vertical?

Solution

$$y' = \sqrt{6-x} - \frac{x}{2\sqrt{6-x}}$$
$$= \frac{12-3x}{2\sqrt{6-x}} = 0$$

$$12-3x=0 \rightarrow \underline{x=4, y=4\sqrt{2}}$$

\therefore Point $(4, 4\sqrt{2})$ is a horizontal tangent line.

\therefore The vertical tangent line inside the square root of y . $\Rightarrow \underline{x=6}$

$$\lim_{x \rightarrow 6} y' = \lim_{x \rightarrow 6} \frac{12-3x}{2\sqrt{6-x}}$$
$$= \frac{-6}{0}$$
$$\underline{= -\infty}$$

Exercise

Find an equation of the tangent line to the graph of $y = \frac{x^2-4}{2x+5}$ when $x=0$

Solution

$$y' = \frac{(2x+5)(2x) - (x^2-4)(2)}{(2x+5)^2}$$
$$= \frac{4x^2+10x-2x^2+8}{(2x+5)^2}$$
$$= \frac{2x^2+10x+8}{(2x+5)^2}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$\Rightarrow x=0 \rightarrow y' = \frac{8}{25} = m$$

$$x=0 \rightarrow y = \frac{x^2-4}{2x+5} = -\frac{4}{5}$$

$$y = \frac{8}{25}(x-0) - \frac{4}{5}$$

$$\underline{y = \frac{8}{25}x - \frac{4}{5}}$$

$$y = m(x - x_1) + y_1$$