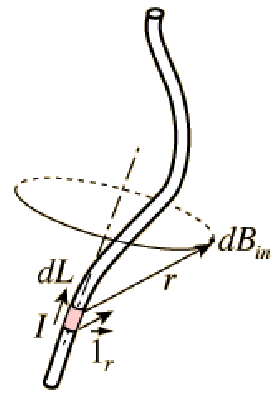
4.1 – Sources of the Magnetic Field

Magnetic field is produced by current (or a moving charge). The relationship between the current and the field is established by a law called the Biot-Savart law.

Biot-Savart Law. States that the magnetic field (dB) at a given point (P) due to a current (I) in a small path element (ds) is proportional to the current and to the sine of the angle between the path element and the position vector $\left(\vec{r}_p\right)$ of the point with respect to the path element, and inversely proportional to the square of the distance between the path element and the point.



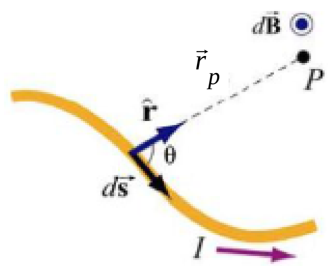
Mathematically:

$$dB = \frac{\mu_0 I ds \sin(\theta)}{4\pi r_p^2}$$

$$\mu_0 = 4\pi \times 10^{-7}\ Tm/A \ \rightarrow \text{Magnetic permeability in vacuum}$$

The direction of the field is perpendicular to the plane determined by $d\vec{s}$ and $\vec{r}_p$. In other words it is in the direction of the cross product between $d\vec{s}$ and $\vec{r}_p$ $\left(d\vec{s} \times \frac{\vec{r}_p}{r_p}\right)$ where $\frac{\vec{r}_p}{r_p}$ is a unit direction of $\vec{r}_p$. Thus the expression $ds\sin\theta$ can be represented as $d\vec{s} \times \frac{\vec{r}_p}{r_p}$ along with the direction.

Remember: $\left| d\vec{s} \times \frac{\vec{r}_p}{r_p} \right| = ds\sin\theta$.



Now Biot-Savart law can be written in vector form as

$$\boxed{d\vec{B} = \frac{\mu_0 I\, d\vec{s} \times \vec{r}_p}{4\pi r_p^3}}$$

$d\vec{B}$ → Magnetic field due to current I in a small path element $d\vec{s}$

$\vec{r}_p$ → Position vector of the point (P) with respect to the path element $d\vec{i}$

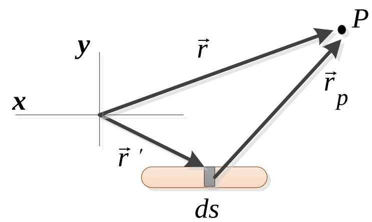
r_p → Distance between path element and the point.

μ_0 → A universal constant called magnetic permeability in vacuum.

Note. The direction of $d\vec{s}$ is the direction of the current.

If $\vec{r}$ and $\vec{r}\,'$ are position vectors of point P and the path element $d\vec{s}$ with respect to a certain coordinate system, then $\overrightarrow{r_p} = \vec{r} - \overrightarrow{r_1}$ and $r_p = |\vec{r} - \vec{r}\,'|$

Thus Biot-Savart law can alternately be written as



$$d\vec{B} = \frac{\mu_0 I\, d\vec{s} \times (\vec{r} - \vec{r}')}{4\pi \cdot |\vec{r} - \vec{r}'|^3}$$

The net magnetic field due to current in a wire is obtained by integrating $d\vec{B}$ over the whole wire.

$$\vec{B} = \int d\vec{B} = \int \frac{\mu_0}{4\pi} I \frac{d\vec{s} \times \overrightarrow{r_p}}{\overrightarrow{r_p}^3}$$

$$= \frac{\mu_0}{4\pi} I \int \frac{d\vec{s} \times \overrightarrow{r_p}}{\overrightarrow{r_p}^3}$$

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{s} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}}$$

Example

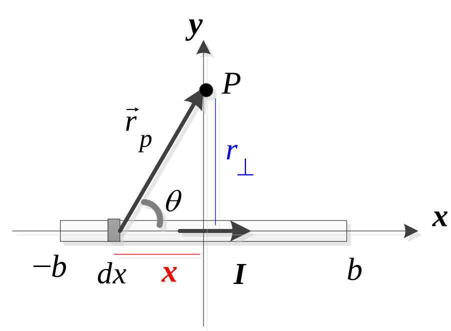
The wire shown extends between $x = -b$ and $x = b$ and carries a current I. The perpendicular distance between the point P shown and the wire is $r_\perp$. Find an expression for the net magnetic field at point P due to the current in the wire.

Solution

Consider a small element dx in the wire whose Cordiant with respect to the coordinate shown is x.

$$d\vec{s} = dx\hat{i}$$

$$\overrightarrow{r_p} = r_p \cos(\theta)\hat{i} + r_p \sin(\theta)\hat{j} = x\hat{i} + r_\perp \hat{j}$$



$$\therefore d\vec{s} \times \overrightarrow{r_p} = dx\hat{i} \times (x\hat{i} \times r_\perp \hat{j})$$

$$= xdx(\hat{i} \times \hat{i}) + r_\perp dx(\hat{i} \times \hat{j}) \qquad \hat{i} \times \hat{i} = 0, \quad \hat{i} \times \hat{j} = \hat{k}$$

$$= r_\perp dx\, \hat{k}$$

(That is the direction is ⊥'y out of the paper at point P)

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{s} \times \vec{r}_p}{r_p^3} \qquad \text{Also } r_p = \left(x^2 + r_\perp^2\right)^{1/2}$$

$$= \frac{\mu_0 I}{4\pi} \int_{-b}^{b} \frac{r_\perp \hat{k}}{\left(x^2 + r_\perp^2\right)^{3/2}} dx$$

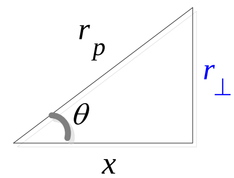
$$= \frac{\mu_0}{4\pi} I r_\perp \hat{k} \int_{-b}^{b} \frac{dx}{\left(x^2 + r_\perp^2\right)^{3/2}}$$

$$\text{Let } x = r_\perp \cot\theta \quad \Rightarrow \quad dx = -r_\perp \csc^2\theta d\theta$$

$$\left(x^2 + r_\perp^2\right)^{3/2} = \left(r_\perp^2 \cot^2\theta + r_\perp^2\right)^{3/2} = r_\perp^3 \left(\cot^2\theta + 1\right)^{3/2}$$

$$= r_\perp^3 \left(\csc^2\theta\right)^{3/2}$$

$$\boxed{= r_\perp^3 \csc^3\theta}$$



$$\vec{B} = \frac{\mu_0}{4\pi} I r_\perp \hat{k} \int_{-b}^{b} \frac{1}{r_\perp^3 \csc^3\theta} \left(-r_\perp \csc^2\theta d\theta\right)$$

$$= -\frac{\mu_0}{4\pi r_\perp} I\hat{k} \int_{-b}^{b} \frac{1}{\csc\theta} d\theta$$

$$= -\frac{\mu_0}{4\pi r_\perp} I\hat{k} \int_{-b}^{b} \sin\theta d\theta \qquad \cos\theta = \frac{x}{r_p} = \frac{x}{\left(x^2 + r_\perp^2\right)^{1/2}}$$

$$= \frac{\mu_0}{4\pi r_\perp} I\hat{k} \left[\cos\theta\right]_{-b}^{b}$$

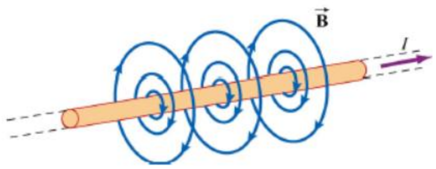
$$= \frac{\mu_0}{4\pi r_\perp} I\hat{k} \left.\frac{x}{\left(x^2 + r_\perp^2\right)^{1/2}}\right|_{-b}^{b}$$

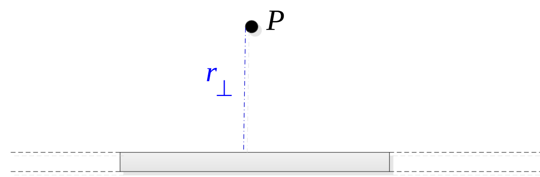
$$= \frac{\mu_0}{4\pi r_\perp} I\hat{k} \frac{2b}{\left(b^2 + r_\perp^2\right)^{1/2}}$$

$$\boxed{= \frac{bI\mu_0}{2\pi r_\perp \sqrt{b^2 + r_\perp^2}} \hat{k}}$$

Magnetic field due to a current carrying infinitely long straight wire

The magnetic field due to a current carrying long straight wire can be obtained from the expression of the preceding example by letting b approach to infinity





As shown in the example, the magnetic field at a point a perpendicular distance of $r_\perp$ from the wire that extend from $x = -b$ to $x = b$ is given by

$$\vec{B} = \frac{I\mu_0 \, b}{2\pi r_\perp \sqrt{b^2 + r_\perp^2}} \hat{k}$$

The field due to a wire that extends from $-\infty$ to ∞ can be obtained from the limit as b approaches ∞

$$\lim_{b\to\infty}\left\{\frac{I\mu_0 \, b}{2\pi r_\perp \sqrt{b^2 + r_\perp^2}} \hat{k}\right\} = \frac{I\mu_0}{2\pi r_\perp}\hat{k} \lim_{b\to\infty}\left\{\frac{b}{\sqrt{b^2 + r_\perp^2}}\right\}$$

$$= \frac{I\mu_0}{2\pi r_\perp}\hat{k} \lim_{b\to\infty}\frac{b}{b\sqrt{1+\left(\frac{r_\perp}{b}\right)^2}}$$

$$= \frac{I\mu_0}{2\pi r_\perp}\hat{k} \lim_{b\to\infty}\frac{1}{\sqrt{1+\left(\frac{r_\perp}{b}\right)^2}} \qquad \frac{r_\perp}{b} \xrightarrow[b\to\infty]{} 0$$

$$= \frac{I\mu_0}{2\pi r_\perp}\hat{k}$$

$$\boxed{B = \frac{I\mu_0}{2\pi r_\perp}}$$

B $\rightarrow$ Magnitude of the magnetic field due to an infinitely long current carrying straight wire

I $\rightarrow$ Current in the wire

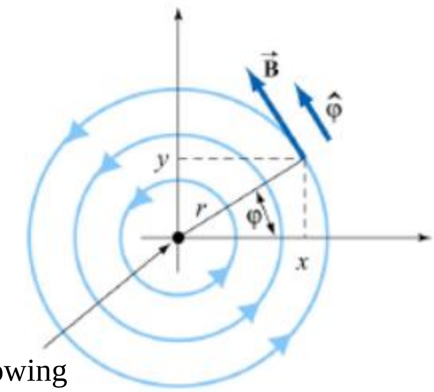
$r_\perp$ $\rightarrow \perp$ distance between the wire and the point

$$\mu_0 = 4\pi \times 10^{-7} \ \frac{Tm}{A}$$

The magnetic field lines due to a current carrying infinitely long straight wire are circles concentric with the wire. The line of action of the field at any point should be tangent to the circular field line that passes through the point.

To distinguish between the two possible directions of the tangent line (related to clockwise or counterclockwise along the circle), the right hand rule is used. If thumb is aligned along the direction of the current and fingers are wrapped around the wire, the direction of the fingers represents the direction of the arrow along the circular field lines.

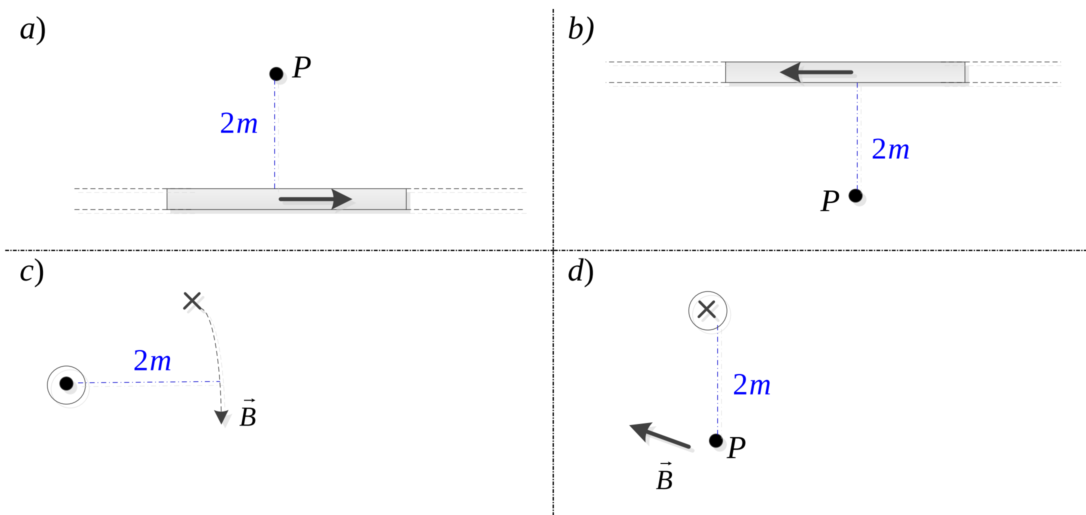
The following diagram shows the field lines due to an infinitely long straight wire penetrating the plane of the paper perpendicularly & carrying current perpendicularly out of the paper.



Current flowing
out the paper

Example

For each of the following find the magnitude and direction of the magnetic field due to the infinitely long current carrying wire at point P. In each case assume the current is $2A$.



Solution

Given: $I = 2A, \quad r_{\perp} = 2m$

a) Direction: Perpendicularly out $(\bullet)$

$$B = \frac{I\mu_0}{2\pi r_{\perp}} = \frac{(2)\left(4\pi \times 10^{-7}\right)}{2\pi(2)} \underline{= 2\times 10^{-7}\ T}$$

b) Direction: Perpendicularly out $(\bullet)$

$B = 2\times10^{-7}\ T$

c) Wire $\perp$ to paper & carrying current perpendicularly out

$B = 2\times10^{-7}\ T$

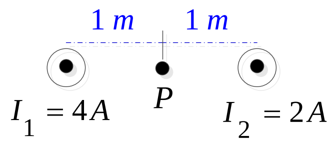
Direction: South from the right hand rule

d) $B = 2\times10^{-7}\ T$

Direction: West

Example

Point P is located midway between two wires penetrating the plane of the paper perpendicularly as shown



The two wires are separated by a distance of $2m$. The left wire carries a current of $4A$ and the right wire carries of a current of $2A$.

Calculate the magnitude & direction of the net magnetic field due to both wires at point P.

Solution

From the right hand rule the direction of the magnetic field at point P is south. $\left(i.e\ \vec{B}_1 = B_1\left(-\hat{j}\right)\right)$

And that due to wire 2 is norm $\left(i.e\ \vec{B}_2 = B_2\,\hat{j}\right)$.

$$\vec{B}_{net} = \vec{B}_1 + \vec{B}_2$$
$$= B_1\left(-\hat{j}\right) + B_2\,\hat{j}$$
$$= \left(B_2 - B_1\right)\hat{j}$$

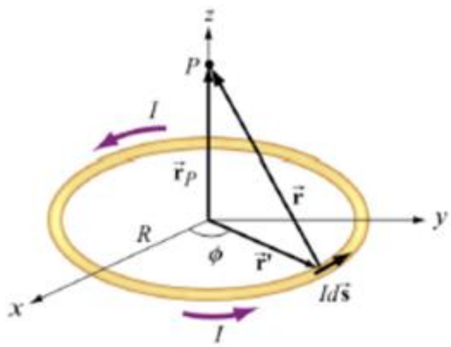
$$B_1 = \frac{I_1\mu_0}{2\pi r_{\perp 1}}$$
$$= \frac{(4)\left(4\pi\times10^{-7}\right)}{2\pi(1)}$$
$$= 8\times10^{-7}\ T$$

$$B_2 = \frac{I_2\mu_0}{2\pi r_{\perp 2}} = \frac{(2)\left(4\pi\times10^{-7}\right)}{2\pi(1)} = 4\times10^{-7}\ T$$

$$\vec{B}_{net} = \left(B_2 - B_1\right)\hat{j} = \left(4\times10^{-7} - 8\times10^{-7}\right)\hat{j} = -4\times10^{-7}\,\hat{j}\ T \quad (South)$$

Example

Consider a circular loop of radius R carrying current I as shown.

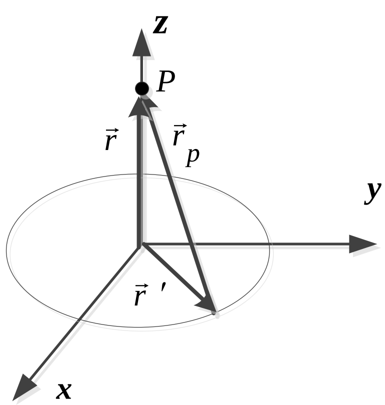


Obtain an expression for the magnetic field at point P located on its axis at a distance ***a*** from its center.

<u>Solution</u>

If the position vectors of $d\vec{s}$ & point P with respect to the region are respectively $\vec{r}\ '$ & $\vec{r}$, then $\vec{r}_p$ (position vector of point P with respect to the path element $d\vec{s}$) is given by

$$\vec{r}_p = \vec{r} - \vec{r}\ '$$



But $\vec{r} = a\hat{k}$ *and* $\vec{r}\ ' = R\cos\phi\hat{i} + R\sin\phi\hat{j}$

(ϕ is angle between x-axis & $\vec{r}\ '$)

$$\vec{r}_p = \vec{r} - \vec{r}\ '$$

$$= a\hat{k} - \left(R\cos\phi\hat{i} + R\sin\phi\hat{j}\right)$$

$$= a\hat{k} - R\cos\phi\hat{i} - R\sin\phi\hat{j}$$

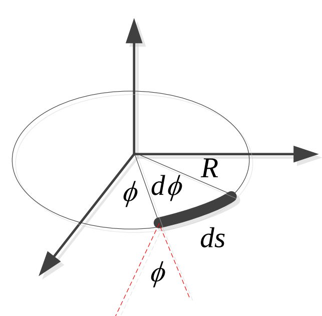
And $r_p^2 = a^2 + R^2\cos^2\phi + R^2\sin^2\phi = a^2 + R^2$

$$d\vec{s} = ds\cos\left(\phi + 90°\right)\ \hat{i} + ds\sin\left(\phi + 90°\right)\ \hat{j}$$

(Angle formed between $d\vec{s}$ which is tangent to the circle & the x-axis is $\phi + 90°$)

$$d\vec{s} = -\sin\phi ds\ \hat{i} + \cos\phi ds\ \hat{j}$$

Also ds can be expressed in terms of R & $d\phi$: $ds = Rd\phi$



$$d\vec{s} = -\sin\phi Rd\phi\ \hat{i} + \cos\phi Rd\phi\ \hat{j}$$

$$d\vec{s} \times \vec{r}_p = \left(-\sin\phi Rd\phi\ \hat{i} + \cos\phi Rd\phi\ \hat{j}\right) \times \left(-R\cos\phi\hat{i} - R\sin\phi\hat{j} + a\hat{k}\right)$$

$$= \cos\phi Rad\phi\ \hat{i} + \sin\phi Rad\phi\ \hat{j} + R^2 d\phi\ \hat{k}$$

$$\begin{array}{ccccc} \hat{i} & \hat{j} & \hat{k} & & \\ -\sin\phi Rd\phi & \cos\phi Rd\phi & 0 & -\sin\phi Rd\phi & \cos\phi Rd\phi \\ -R\cos\phi & -R\sin\phi & a & -R\cos\phi & -R\sin\phi \end{array} = \cos\phi Rad\phi\hat{i} + \sin\phi Rad\phi\ \hat{j} + \left(\sin^2\phi R^2 d\phi + \cos^2\phi R^2 d\phi\right)\hat{k}$$

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{s}\times\vec{r}_p}{r_p^3} \qquad r_p^2 = a^2 + R^2 \;\Rightarrow\; r_p = \sqrt{a^2+R^2} \;\Rightarrow\; r_p^3 = \left(a^2+R^2\right)^{3/2}$$

$$= \frac{\mu_0 I}{4\pi\left(a^2+R^2\right)^{3/2}}\int_0^{2\pi}\left(\cos\phi R a d\phi\,\hat{i} + \sin\phi R a d\phi\,\hat{j} + R^2 d\phi\,\hat{k}\right)$$

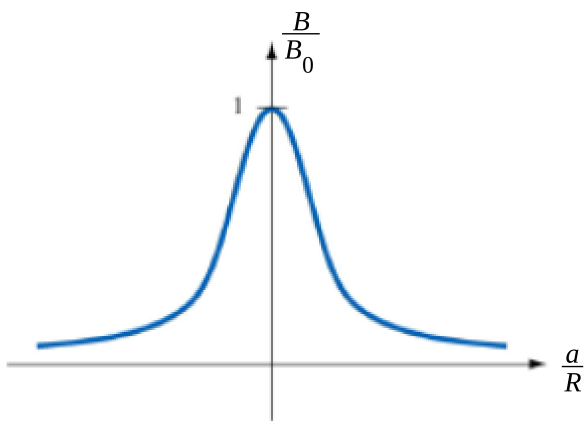
$$= \frac{\mu_0 I}{4\pi\left(a^2+R^2\right)^{3/2}}\left(Ra\int_0^{2\pi}\cos\phi d\phi\,\hat{i} + Ra\int_0^{2\pi}\sin\phi d\phi\,\hat{j} + R^2\int_0^{2\pi} d\phi\,\hat{k}\right)$$

$$= \frac{\mu_0 I}{4\pi\left(a^2+R^2\right)^{3/2}}\left(Ra\left[\sin\phi\right]_0^{2\pi}\,\hat{i} + Ra\left[\cos\phi\right]_0^{2\pi}\,\hat{j} + R^2\left[\phi\right]_0^{2\pi}\,\hat{k}\right)$$

$$= \frac{\mu_0 I}{4\pi\left(a^2+R^2\right)^{3/2}}\left(Ra\left(\sin 2\pi - \sin 0\right)\,\hat{i} + Ra\left(\cos 2\pi - \cos 0\right)\,\hat{j} + R^2\left(2\pi\right)\,\hat{k}\right)$$

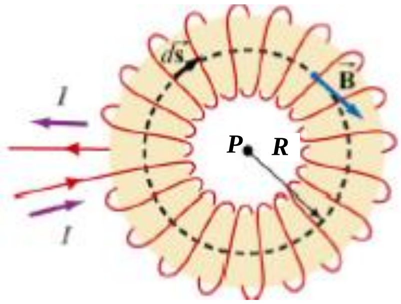
$$= \frac{\mu_0 I}{4\pi\left(a^2+R^2\right)^{3/2}}\left(2\pi R^2\,\hat{k}\right)$$

$$\boxed{\vec{B} = \frac{\mu_0 I\, R^2}{2\left(a^2+R^2\right)^{3/2}}\,\hat{k}}$$



Magnetic field due to a circular coil of *N* turns at the center of the coil

If there are *N* turns then the net field is *N* times the field due to a single turn



$$\vec{B} = NB_{one\ turn}$$

The expression for $B_{one\ turn}$ can be obtained from the result of the preceding example with $a = 0$.

$$B_{one\ turn}(a) = \frac{\mu_0 I\ R^2}{2\left(R^2 + a^2\right)^{3/2}}\ \hat{k}$$

$$B_{one\ turn}(a = 0) = \frac{\mu_0 I\ R^2}{2\left(R^2\right)^{3/2}}\ \hat{k}$$

$$= \frac{\mu_0 I\ R^2}{2R^3}\ \hat{k}$$

$$= \boxed{\frac{\mu_0 I}{2R}\ \hat{k}}$$

The field at the center of the coil is

$$\vec{B} = NB_{one\ turn}$$

$$= \boxed{\frac{\mu_0 NI}{2R}\ \hat{k}}$$

Field due to a coil of *N* turns at the center of the coil
$R \rightarrow$ Radius
$I \rightarrow$ Current

Example

A coil has 10 turns & has a radius of 2 *cm*. If it is carrying a current of 3*A*. Calculate the magnetic field

a) At its center

b) On a point on its axis a distance of 10 *cm* from the center

Solution

a) ***Given***: $N = 10, \quad I = 3A, \quad R = 0.02m$

$$\vec{B} = \frac{\mu_0 NI}{2R}\ \hat{k}$$

$$= \frac{4\pi \times 10^{-7}(10)(3)}{2(.02)}\ \hat{k}$$

$$\approx 9.3\times 10^{-7}\ \hat{k}\ \ T$$

b) ***Given***: $a = 0.1\ m$

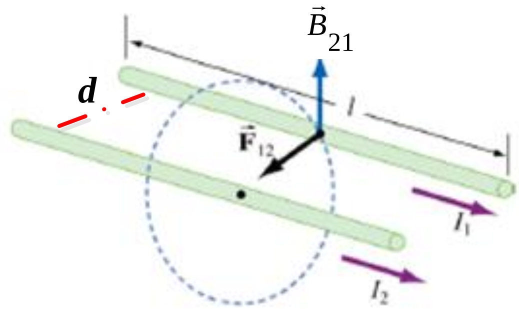
$$\vec{B} = NB_{one\ turn}$$

$$= N\frac{\mu_0 I\ R^2}{2\left(R^2 + a^2\right)^{3/2}}\ \hat{k}$$

$$= (10)\frac{4\pi \times 10^{-7}(3)(0.02)^2}{2\left(0.02^2 + 0.1^2\right)^{3/2}}\ \hat{k}$$

$$\approx 7.1\times 10^{-6}\ \hat{k}\ \ T$$

Magnetic field between two parallel infinitely long current carrying wires



Let $\vec{B}_{21}$ be the magnetic field at the location of wire 2 due to wire 1.

This field will exert magnetic force on wire 1 by wire 2 is given by

$$F_{12} = I_1 \vec{l} \times \vec{B}_{21}$$

And $\vec{B}_{21} = -\frac{\mu_0 I_2}{2\pi d}\,\hat{j}$

$$\vec{F}_{12} = I_1 \vec{l} \times \vec{B}_{21}$$

$$= I_1 \left(\ell\,\hat{i}\right) \times \left(-\frac{\mu_0 I_2}{2\pi d}\,\hat{j}\right)$$

$$= -\frac{\mu_0 I_1 I_2 \ell}{2\pi d}\,\hat{k}$$

$$\boxed{F_{12} = \frac{\mu_0 I_1 I_2 \ell}{2\pi d}} \quad or \quad \boxed{\frac{F_{12}}{\ell} = \frac{\mu_0 I_1 I_2}{2\pi d}}$$

$\frac{F_{12}}{\ell}$ → Force (magnetic) per unit length exerted by wire 2

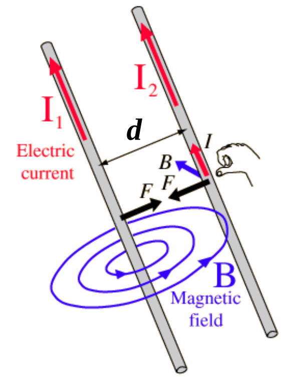
I_1, I_2 → Currents of wire 1, 2

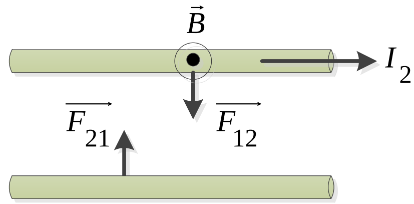
d → Distance between the parallel wires

The fore exerted by wire 1 on wire 2 also will have the same magnitude but opposite direction because they are action reaction forces.

$$F_{12} = F_{21} \quad or \quad \vec{F}_{12} = -\vec{F}_{21}$$

Since the direction of $\vec{B}_{12}$ is perpendicularly out & direction of I_2 is to the right, from the right hand rule or screw rule the direction of the magnetic force should be downward; which also implies the direction of the force exerted by wire 2 on wire 1 is upwards. In other words, the two wire will attract. This always true if the wires carry current in the same direction.

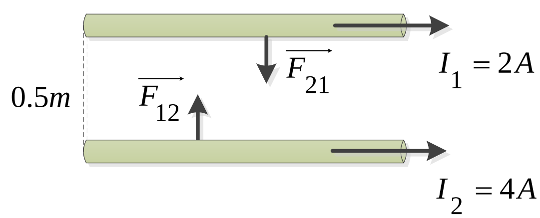




Generally wire carrying current in the same direction attract while wires current in opposite directions repel each other.

Example

For the wires shown



a) Calculate the magnitude & direction of the magnetic force exerted by wire 1 on wire 2

b) Calculate the magnitude & direction of the magnetic force exerted by wire 2 on wire 1

Solution

a) Given: $I_1 = 2A, \quad I_2 = 4A, \quad d = 0.5m$

$$\frac{F_{12}}{\ell} = \frac{\mu_0 I_1 I_2}{2\pi d}$$

$$= \frac{4\pi \times (2)(4)}{2\pi (0.5)}$$

$$\approx 32 \times 10^{-7} \ N/m$$

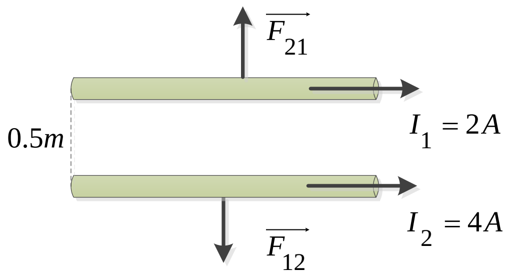
Since the currents have the same direction. Thus the force is attractive; which means the force exerted by wire 1 on wire 2 is upwards or north.

b) The two forces are action reaction forces $\Rightarrow \quad \vec{F}_{12} = -\vec{F}_{21}$

$F_{21} \approx 32 \times 10^{-7} \ N/m$ (South)

Example

For the wires shown



a) Calculate the magnitude & direction of the magnetic force exerted by wire 1 on wire 2

b) Calculate the magnitude & direction of the magnetic force exerted by wire 2 on wire 1

Solution

a) Given: $I_1 = 2A, \quad I_2 = 4A, \quad d = 0.5m$

$$\frac{F_{12}}{\ell} = \frac{\mu_0 I_1 I_2}{2\pi d}$$

$$= \frac{4\pi \times (2)(4)}{2\pi(0.5)}$$

$$\approx 32 \times 10^{-7} \ N/m \quad \text{(North)}$$

b) The two forces are action reaction forces $\Rightarrow \quad \vec{F}_{12} = -\vec{F}_{21}$

$$F_{21} \approx 32 \times 10^{-7} \ N/m \quad \text{(South)}$$