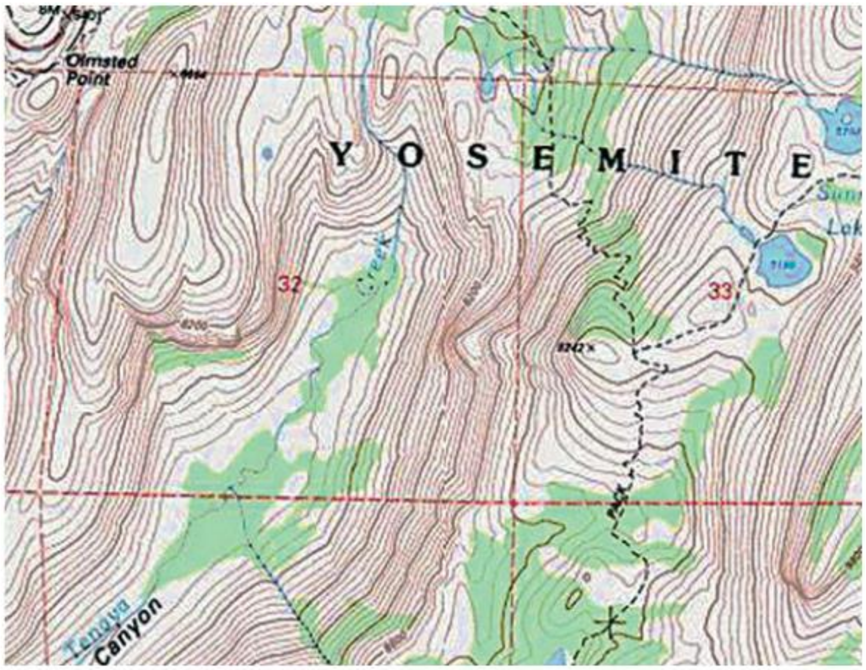
Section 2.5 – Directional Derivatives and the Gradient

You notice that the streams flow perpendicular to the contours. The streams are following paths of steepest descent so the waters reach lower elevation as quickly as possible. Therefore, the fastest instantaneous rate of change in a stream's elevation above the sea level has a particular direction.



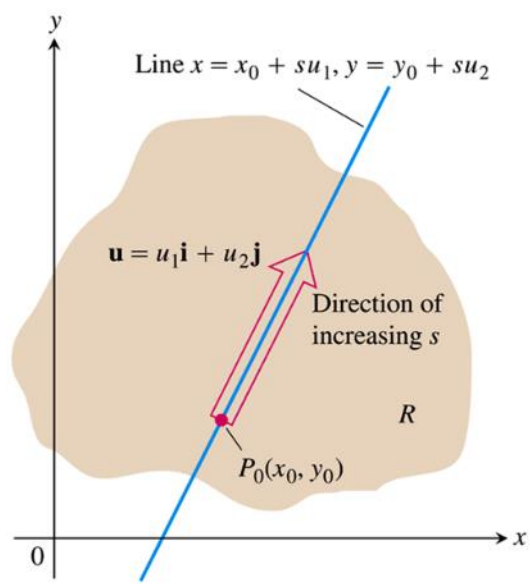
Directional Derivatives in the Plane

The rate at which *f* changes with respect to *t* along a differentiable curve $x = g(t),\ y = h(t)$ is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Suppose that the function $f(x, y)$ is defined throughout a region *R* in the *xy*-plane, that $P_0(x_0, y_0)$ is a point in *R*, and that $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j}$ is a unit vector. Then the equations

$x = x_0 + su_1, \quad y = y_0 + su_2$



Definition

The derivative of f at $P_0\left(x_0, y_0\right)$ in the direction of the unit vector $\boldsymbol{u}=u_1\boldsymbol{i}+u_2\boldsymbol{j}$ is the number

$$\left(\frac{df}{ds}\right)_{\boldsymbol{u},P_0}=\lim_{s\to 0}\frac{f\left(x_0+su_1, y_0+su_2\right)-f\left(x_0, y_0\right)}{s}$$

provided the limit exists.

The directional derivative is also noted by: $\left(D_{\boldsymbol{u}}f\right)_{P_0}$

Example

Find the derivative of $f(x, y)=x^2+xy$ at $P_0(1,\ 2)$ in the direction of the unit vector $\boldsymbol{u}=\frac{1}{\sqrt{2}}\boldsymbol{i}+\frac{1}{\sqrt{2}}\boldsymbol{j}$

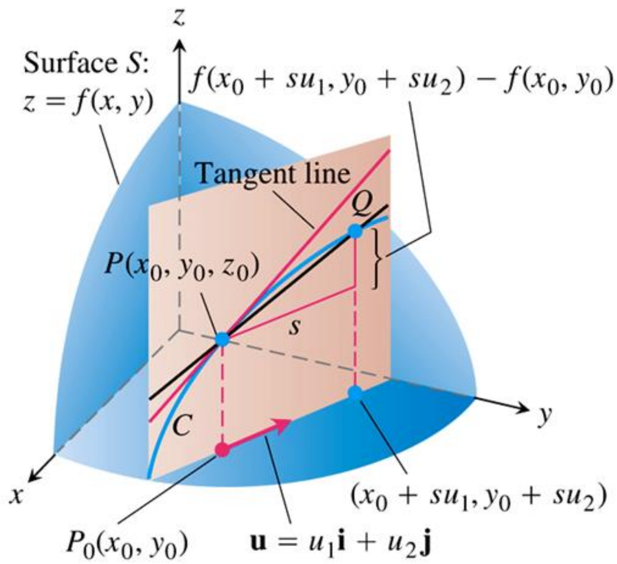
Solution

$$\begin{aligned}
\left(\frac{df}{ds}\right)_{\boldsymbol{u},P_0}&=\lim_{s\to 0}\frac{f\left(x_0+su_1, y_0+su_2\right)-f\left(x_0, y_0\right)}{s}\\
&=\lim_{s\to 0}\frac{f\left(1+s\frac{1}{\sqrt{2}},\ 2+s\frac{1}{\sqrt{2}}\right)-f(1,\ 2)}{s}\\
&=\lim_{s\to 0}\frac{\left(1+\frac{s}{\sqrt{2}}\right)^2+\left(1+\frac{s}{\sqrt{2}}\right)\left(2+\frac{s}{\sqrt{2}}\right)-\left(1^2+1\bullet 2\right)}{s}\\
&=\lim_{s\to 0}\frac{1+\frac{2}{\sqrt{2}}s+\frac{1}{2}s^2+2+\frac{3}{\sqrt{2}}s+\frac{1}{2}s^2-3}{s}\\
&=\lim_{s\to 0}\frac{s^2+\frac{5}{\sqrt{2}}s}{s}\\
&=\lim_{s\to 0}\left(s+\frac{5}{\sqrt{2}}\right)\\
&=\frac{5}{\sqrt{2}}
\end{aligned}$$

The rate of change of $f(x, y)=x^2+xy$ at $P_0(1,\ 2)$ in the direction $\boldsymbol{u}$ is $\frac{5}{\sqrt{2}}$

Interpretation of the Directional Derivative

The equation $z = f(x, y)$ represents a surface S in space. If $z_0 = f(x_0, y_0)$, then the point $P_0(x_0, y_0, z_0)$ lies on S. The vertical plane that passes through P and $P_0(x_0, y_0, z_0)$ parallel to $\boldsymbol{u}$ intersects S in a curve C.



When $\boldsymbol{u} = \boldsymbol{i}$, the directional derivative at P_0 is $\dfrac{\partial f}{\partial x}$ evaluated at (x_0, y_0).

When $\boldsymbol{u} = \boldsymbol{j}$, the directional derivative at P_0 is $\dfrac{\partial f}{\partial y}$ evaluated at (x_0, y_0).

The directional derivative generalizes the two partial derivatives.

Calculation and Gradients

$$\left(\frac{df}{ds}\right)\bigg|_{\boldsymbol{u},P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0}\frac{dx}{ds} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\frac{dy}{ds}$$

$$= \left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} u_2$$

$$= \underbrace{\left[\left(\frac{\partial f}{\partial x}\right)_{P_0}\boldsymbol{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\boldsymbol{j}\right]}_{\text{Gradient of at } P_0} \bullet \underbrace{(u_1\boldsymbol{i} + u_2\boldsymbol{j})}_{\text{Directional } u}$$

Definition

The ***gradient vector*** (*gradient*) of $f(x,y)$ at a point $P_0(x_0,y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}$$

obtained by evaluating the partial derivatives of *f* at P_0

Theorem – The directional Derivative is a Dot Product

If $f(x,y)$ is differentiable in an open region containing $P_0(x_0,y_0)$, then

$$\left(\frac{df}{ds}\right)_{\boldsymbol{u},P_0} = (\nabla f)_{P_0} \bullet \boldsymbol{u}$$

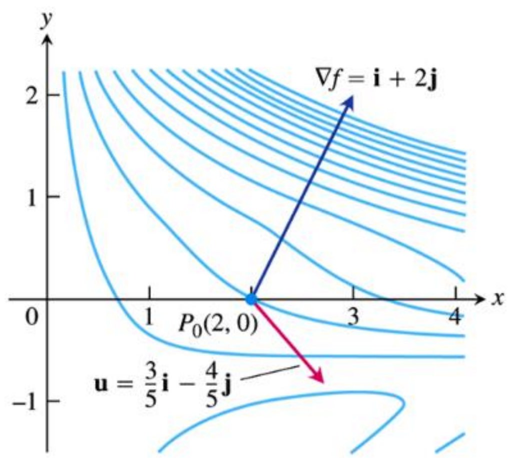
The dot product of the gradient ∇f at P_0 and $\boldsymbol{u}$.

Example

Find the derivative of $f(x,y) = xe^y + \cos(xy)$ at the point (2, 0) in the direction of $\boldsymbol{v} = 3\boldsymbol{i} - 4\boldsymbol{j}$

Solution

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{3\boldsymbol{i} - 4\boldsymbol{j}}{\sqrt{3^2 + 4^2}} = \frac{3}{5}\boldsymbol{i} - \frac{4}{5}\boldsymbol{j}$$



The partial derivatives of *f* are continuous and at (2, 0)

$$f_x(2,0) = \left(e^y - y\sin(xy)\right)_{(2,0)} = e^0 - 0 = 1$$

$$f_y(2,0) = \left(xe^y - x\sin(xy)\right)_{(2,0)} = 2e^0 - 0 = 2$$

$$\begin{aligned}\nabla f &= f_x(2,0)\boldsymbol{i} + f_y(2,0)\boldsymbol{j}\\ &= \boldsymbol{i} + 2\boldsymbol{j}\end{aligned}$$

Therefore, the derivative of *f* at (2, 0) in the direction of $\boldsymbol{v}$ is

$$\begin{aligned}\left(D_{\boldsymbol{u}} f\right)\Big|_{(2,0)} &= (\nabla f)_{(2,0)} \bullet \boldsymbol{u}\\ &= (\boldsymbol{i} + 2\boldsymbol{j})\bullet\left(\frac{3}{5}\boldsymbol{i} - \frac{4}{5}\boldsymbol{j}\right)\\ &= \frac{3}{5} - \frac{8}{5}\\ &= -1\end{aligned}$$

Properties of the Directional Derivative $D_{\boldsymbol{u}} f = \nabla f \cdot \boldsymbol{u} = |\nabla f| \cos\theta$

1. The function *f* increases most rapidly when $\cos\theta = 1$ or when $\theta = 0$ and $\boldsymbol{u}$ is the direction of ∇f. That is, at each point *P* in its domain, *f* increases most rapidly in the direction of the gradient vector ∇f at *P*. The derivative in this direction is $D_{\boldsymbol{u}} f = |\nabla f| \cos 0 = |\nabla f|$
2. Similarly, *f* decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\boldsymbol{u}} f = |\nabla f| \cos(\pi) = -|\nabla f|$
3. Any direction $\boldsymbol{u}$ orthogonal to a gradient ∇f is a direction of zero change in *f* because θ then equals $\frac{\pi}{2}$ and $D_{\boldsymbol{u}} f = |\nabla f| \cos\left(\frac{\pi}{2}\right) = |\nabla f| \cdot (0) = 0$

Example

Find the directions in which $f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$

a) Increases most rapidly at the point (1, 1)
b) decreases most rapidly at the point (1, 1)
c) What are the directions of zero change in *f* at (1, 1)

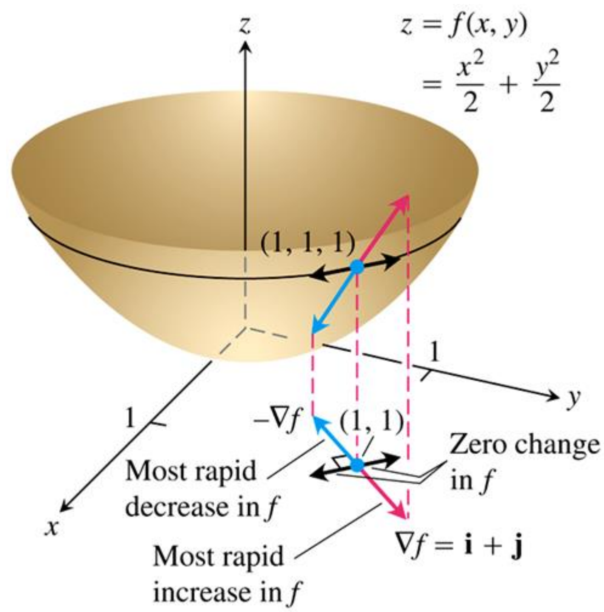
Solution

a) The function increases most rapidly at the point (1, 1).
The gradient is

$$\nabla f = f_x \boldsymbol{i} + f_y \boldsymbol{j} = x\boldsymbol{i} + y\boldsymbol{j}$$

$$(\nabla f)_{(1,1)} = (x\boldsymbol{i} + y\boldsymbol{j})_{(1,1)} = \boldsymbol{i} + \boldsymbol{j}$$

Its direction is: $\boldsymbol{u} = \frac{\boldsymbol{i} + \boldsymbol{j}}{|\boldsymbol{i} + \boldsymbol{j}|} = \frac{1}{\sqrt{2}}\boldsymbol{i} + \frac{1}{\sqrt{2}}\boldsymbol{j}$



b) The function decreases most rapidly at the point (1, 1).

The gradient is $-(\nabla f)_{(1,1)} = -\boldsymbol{i} - \boldsymbol{j}$

Its direction is: $-\boldsymbol{u} = -\frac{1}{\sqrt{2}}\boldsymbol{i} - \frac{1}{\sqrt{2}}\boldsymbol{j}$

c) The directions of zero change at (1, 1) are the direction orthogonal to ∇f:

$$\boldsymbol{n} = -\frac{1}{\sqrt{2}}\boldsymbol{i} + \frac{1}{\sqrt{2}}\boldsymbol{j} \quad \textit{and} \quad -\boldsymbol{n} = \frac{1}{\sqrt{2}}\boldsymbol{i} - \frac{1}{\sqrt{2}}\boldsymbol{j}$$

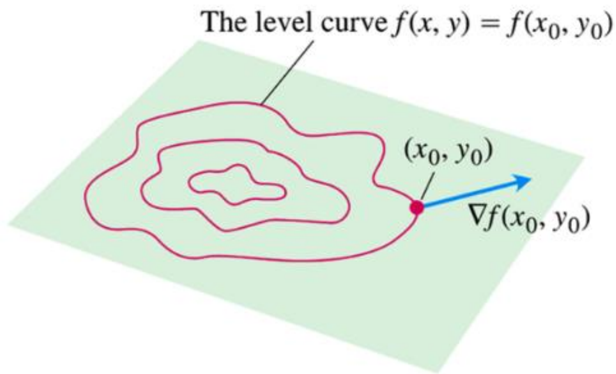
Gradients and Tangents to Level Curves

If a differentiable function $f(x,y)$ has a constant value c along a smooth curve $r = g(t)\boldsymbol{i} + h(t)\boldsymbol{j}$, then $f(g(t),h(t)) = c$. Differentiating both sides of this equation with respect to t leads to the equations

$$\frac{d}{dt} f(g(t),h(t)) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} = 0$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt}\boldsymbol{i} + \frac{dh}{dt}\boldsymbol{j}\right)}_{\frac{d\boldsymbol{r}}{dt}} = 0$$



- At every point (x_0, y_0) in the domain of a differentiable function $f(x,y)$, the gradient of f is normal to the level curve through (x_0, y_0).

Example

Find an equation for the tangent to the ellipse $\frac{x^2}{4} + y^2 = 2$ at the point $(-2,\ 1)$

Solution

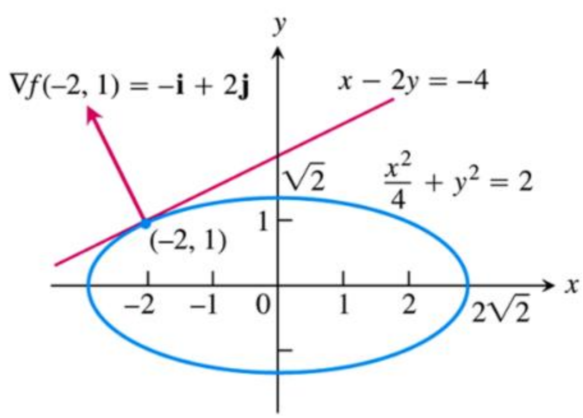
$$f(x,y) = \frac{x^2}{4} + y^2$$

The gradient of f at $(-2,\ 1)$ is

$$\nabla f = f_x \boldsymbol{i} + f_y \boldsymbol{j} = \frac{x}{2}\boldsymbol{i} + 2y\boldsymbol{j}$$

$$\nabla f\Big|_{(-2,1)} = \left(\frac{x}{2}\boldsymbol{i} + 2y\boldsymbol{j}\right)_{(-2,1)}$$

$$= -\boldsymbol{i} + 2\boldsymbol{j}$$



The equation of the line is given by: $f_x(x - x_0) + f_y(y - y_0) = 0$

$$(-1)(x - (-2)) + (2)(y - 1) = 0$$

$$-x - 2 + 2y - 2 = 0$$

$$-x + 2y = 4 \quad \to \quad \boxed{x - 2y = -4}$$

Algebra Rules for Gradients

Sum Rule: $\nabla(f+g) = \nabla f + \nabla g$

Difference Rule: $\nabla(f-g) = \nabla f - \nabla g$

Constant Multiple Rule: $\nabla(kf) = k\nabla f \qquad \forall k$

Product Rule: $\nabla(fg) = f\nabla g + g\nabla f$

Quotient Rule: $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Functions of Three Variables

For a differentiable function $f(x,y,z)$ and a unit $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}$ in space, we have

$$\nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}$$

$$D_{\boldsymbol{u}} f = \nabla f \bullet \boldsymbol{u} = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 + \frac{\partial f}{\partial z}u_3$$

The directional derivative can be written in the form

$$D_{\boldsymbol{u}} f = \nabla f \bullet \boldsymbol{u} = |\nabla f||\boldsymbol{u}|\cos\theta = |\nabla f|\cos\theta$$

Example

a) Find the derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $P_0(1,\ 1,\ 0)$ in the direction of $\vec{v} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

b) In what directions does *f* change most rapidly at P_0, and what are the rates of change in these directions?

Solution

a) The direction of $\boldsymbol{v}$ is obtained by dividing $\boldsymbol{v}$ by its length:

$$|\vec{v}| = \sqrt{2^2 + (-3)^2 + 6^2}$$

$$\underline{= 7\,}|$$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$$

The partial derivatives of *f* at P_0 are

$$f_x \Big|_{(1,1,0)} = \left(3x^2 - y^2\right)\Big|_{(1,1,0)} = 2$$

$$f_y \Big|_{(1,1,0)} = -2y\Big|_{(1,1,0)} = -2$$

$$f_z \Big|_{(1,1,0)} = -1$$

The gradient of *f* at P_0 is

$$\nabla f \Big|_{(1,1,0)} = \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)\Big|_{(1,1,0)}$$

$$= 2\hat{i} - 2\hat{j} - \hat{k}$$

Therefore, the derivative of *f* at P_0 in the direction of ***v*** is

$$D_{\boldsymbol{u}} f \Big|_{(1,1,0)} = \nabla f \Big|_{(1,1,0)} \bullet \boldsymbol{u}$$

$$= \left(2\hat{i} - 2\hat{j} - \hat{k}\right) \bullet \left(\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right)$$

$$= (2)\left(\frac{2}{7}\right) + (-2)\left(-\frac{3}{7}\right) + (-1)\left(\frac{6}{7}\right)$$

$$= \frac{4}{7}$$

b) The function increases most rapidly in the direction of $\nabla f = 2\hat{i} - 2\hat{j} - \hat{k}$ and decreases most rapidly in the direction of $-\nabla f$.

The rates of change in the directions are

$$|\nabla f| = \sqrt{4+4+1} = 3 \quad \textit{and} \quad -|\nabla f| = -3$$

Exercises *Section* 2.5 – Directional Derivatives and the Gradient

Find the gradient of the function at the given point. Then sketch the gradient together with the level curve that passes through the point

1. $f(x,y) = y - x, \quad (2,\ 1)$

2. $f(x,y) = \ln\left(x^2 + y^2\right), \quad (1,\ 1)$

3. $f(x,y) = \sqrt{2x + 3y}, \quad (-1,\ 2)$

Find ∇f at the given point

4. $f(x,y,z) = x^2 + y^2 - 2z^2 + z\ln x, \quad (1,\ 1,\ 1)$

5. $f(x,y,z) = 2x^3 - 3\left(x^2 + y^2\right)z + \tan^{-1} xz, \quad (1,\ 1,\ 1)$

6. $f(x,y,z) = e^{x+y}\cos z + (y+1)\sin^{-1} x, \quad \left(0,\ 0,\ \frac{\pi}{6}\right)$

7. Find the derivative of the function $f(x,y) = 2xy - 3y^2$ at $P_0(5,\ 5)$ in the direction of $\boldsymbol{v} = 4\boldsymbol{i} + 3\boldsymbol{j}$

8. Find the derivative of the function $f(x,y) = \dfrac{x - y}{xy + 2}$ at $P_0(1,\ -1)$ in the direction of $\boldsymbol{v} = 12\boldsymbol{i} + 5\boldsymbol{j}$

9. Find the derivative of the function $h(x,y) = \tan^{-1}\left(\dfrac{y}{x}\right) + \sqrt{3}\sin^{-1}\left(\dfrac{xy}{2}\right)$ at $P_0(1,\ 1)$ in the direction of $\boldsymbol{v} = 3\boldsymbol{i} - 2\boldsymbol{j}$

10. Find the derivative of the function $f(x,y,z) = xy + yz + zx$ at $P_0(1,\ -1,\ 2)$ in the direction of $\boldsymbol{v} = 3\boldsymbol{i} + 6\boldsymbol{j} - 2\boldsymbol{k}$

11. Find the derivative of the function $g(x,y,z) = 3e^x\cos yz$ at $P_0(0,\ 0,\ 0)$ in the direction of $\boldsymbol{v} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$

12. Find the derivative of the function $h(x,y,z) = \cos xy + e^{yz} + \ln zx$ at $P_0\left(1,\ 0,\ \frac{1}{2}\right)$ in the direction of $\boldsymbol{v} = \boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$

13. Find the directions in which the function $f(x,y) = x^2 + xy + y^2$ increase and decrease most rapidly at $P_0(-1,\ 1)$. Then find the derivatives of the function in these directions.

14. Find the directions in which the function $f(x,y) = x^2y + e^{xy}\sin y$ increase and decrease most rapidly at $P_0(1,\ 0)$. Then find the derivatives of the function in these directions.

15. Find the directions in which the function $g(x,y,z)=xe^{y}+z^{2}$ increase and decrease most rapidly at $P_0\left(1,\ \ln 2,\ \frac{1}{2}\right)$. Then find the derivatives of the function in these directions.

16. Find the directions in which the function $h(x,y,z)=\ln\left(x^2+y^2-1\right)+y+6z$ increase and decrease most rapidly at $P_0\,(1,\ 1,\ 0)$. Then find the derivatives of the function in these directions.

17. Sketch the curve $x^2+y^2=4;\quad (f(x,y)=c)$ together with ∇f and the tangent line at the point $\left(\sqrt{2},\ \sqrt{2}\right)$. Then write an equation for the tangent line.

18. Sketch the curve $x^2-y=1;\quad (f(x,y)=c)$ together with ∇f and the tangent line at the point $\left(\sqrt{2},\ 1\right)$. Then write an equation for the tangent line.

19. Sketch the curve $x^2-xy+y^2=7;\quad (f(x,y)=c)$ together with ∇f and the tangent line at the point $(-1,\ 2)$. Then write an equation for the tangent line.

20. In what direction is the derivative of $f(x,y)=xy+y^2$ at $P(3,\ 2)$ equal to zero?

Compute the gradient of the function, evaluate it at the given point *P*, and evaluate the directional derivative at that point in the given direction

21. $f(x,\ y)=x^2;\quad P=(1,\ 2);\quad \vec{u}=\left\langle \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right\rangle$

22. $f(x,\ y)=x^2y^3;\quad P=(-1,\ 1);\quad \vec{u}=\left\langle \frac{5}{13},\ \frac{12}{13}\right\rangle$

23. $f(x,\ y)=\frac{x}{y^2};\quad P=(0,\ 3);\quad \vec{u}=\left\langle \frac{\sqrt{3}}{2},\ \frac{1}{2}\right\rangle$

24. $f(x,\ y)=\sqrt{2+x^2+2y^2};\quad P=(2,\ 1);\quad \vec{u}=\left\langle \frac{3}{5},\ \frac{4}{5}\right\rangle$

25. $f(x,\ y,\ z)=xy+yz+xz+4;\quad P=(2,\ -2,\ 1);\quad \vec{u}=\left\langle 0,\ -\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right\rangle$

26. $f(x,\ y,\ z)=1+\sin(x+2y-z);\quad P=\left(\frac{\pi}{6},\ \frac{\pi}{6},\ -\frac{\pi}{6}\right);\quad \vec{u}=\left\langle \frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right\rangle$

Find the direction in which f increases and decreases most rapidly at P_0 and find the derivative of f in each direction. Also, find the derivative of f at P_0 in the direction of the vector $\boldsymbol{v}$.

27. $f(x,y)=\cos x\cos y,\quad P_0\left(\frac{\pi}{4},\ \frac{\pi}{4}\right),\quad \vec{v}=3\hat{i}+4\hat{j}$

28. $f(x, y) = x^2 e^{-2y}, \quad P_0(1,\ 0), \quad \vec{v} = \hat{i} + \hat{j}$

29. $f(x, y, z) = \ln(2x + 3y + 6z), \quad P_0(-1,\ -1,\ 1), \quad \vec{v} = 2\hat{i} + 3\hat{j} + 6\hat{k}$

30. $f(x, y, z) = x^2 + 3xy - z^2 + 2y + z + 4, \quad P_0(0,\ 0,\ 0), \quad \vec{v} = \hat{i} + \hat{j} + \hat{k}$

31. Let $f(x,\ y) = \ln(1 + xy); \quad P = (2,\ 3)$

a) Find the unit vectors that give the direction of steepest ascent and steepest descent at *P*.
b) Find a unit vector that points in ta direction of no change.

32. Let $f(x,\ y) = \sqrt{4 - x^2 - y^2}; \quad P = (-1,\ 1)$

a) Find the unit vectors that give the direction of steepest ascent and steepest descent at *P*.
b) Find a unit vector that points in ta direction of no change.

Let $f(x,\ y) = 8 - 2x^2 - y^2$, for the level curves $f(x,\ y) = C$ and points $(a,\ b)$, compute the slope of the line tangent to the level curve at $(a,\ b)$ and verify that the tangent line is orthogonal to the gradient at that point.

33. $f(x,\ y) = 5; \quad (a,\ b) = (1,\ 1)$

34. $f(x,\ y) = 0; \quad (a,\ b) = (2,\ 0)$

35. Find the direction in which the function $f(x,\ y) = 4x^2 - y^2$ has zero change at the point $(1,\ 1,\ 3)$. Express the directions in terms of unit vectors.

36. An infinitely long charged cylinder of radius *R* with its axis along *z*-axis has an electric potential $V = k \ln\left(\frac{R}{r}\right)$, where *r* is the distance between a variable point $P(x,\ y)$ and the axis of the cylinder $\left(r^2 = x^2 + y^2\right)$ and *k* is a physical constant. The electric field at a point $(x,\ y)$ in the *xy*-plane is given by $\boldsymbol{E} = -\nabla V$, where ∇V is the two-dimensional gradient. Compute the electric field at a point $(x,\ y)$ with $r > R$.