3.5 Taylor & Maclaurin Series

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n-1} + \cdots$$

$$= \sum_{n=1}^{\infty} nc_n (x-a)^{n-1}$$

$$f''(x) = \underbrace{\overset{1\cdot 2}{2}}_{2!} c_2 + \underbrace{\overset{1\cdot 2\cdot 3}{6}}_{3!} c_3(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2} + \cdots$$

$$f^{(n)}(x) = n!\, c_n$$

a_n

$$c_n = \frac{f^{(n)}(x)}{n!}$$

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Taylor series:

$$\sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$+ \boxed{\frac{f^{(n)}(a)}{n!}(x-a)^n} + \cdots$$

Maclaurin series: $\boxed{a = 0}$

$$\boxed{\frac{f^{(n)}}{n!}(x-a)^n}$$

Ex $f(x)=\frac{1}{x}$ @ $a=2$

$f(x)=\frac{1}{x}$ $\qquad$ $f(2)=\frac{1}{2}$

$f'(x)=-x^{-2}$ $\qquad$ $f'(2)=-\frac{1}{4}$

$f''(x)=2!\,x^{-3}$ $\qquad$ $f''(2)=\frac{2!}{2^3}$ $\qquad$ 2

$f'''(x)=-(3!)\,x^{-4}$ $\qquad$ $f'''(2)=-\frac{3!}{2^4}$

$f^{(n)}(x)=(-1)^n\,n!\,x^{-(n+1)}$ $\qquad$ $\boxed{f^{(n)}(2)=\frac{(-1)^n n!}{2^{n+1}}}$

$$\frac{1}{x}=\frac{1}{2}-\frac{1}{4}(x-2)+\frac{2!}{2^3\,2!}(x-2)^2+\cdots+$$

$$=\frac{1}{2}-\frac{1}{2^2}(x-2)+\frac{(x-2)^2}{2^3}+\cdots+\frac{(-1)^n(x-2)^n}{2^{n+1}}$$

$$\frac{f^{(n)}(2)}{n!}=\frac{(-1)^n}{2^{n+1}}$$

Ex $f(x) = e^x$ @ $x=0$

$f(x) = e^x$ $\quad$ $f(0) = 1$

$f'(x) = e^x$ $\quad$ $f'(0) = 1$

$\vdots$

$f^{(n)}(x) = e^x$

$$P(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$e = \underbrace{1+1}_{} + \frac{1}{2!} + \frac{1}{3!}$$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

$n=0 \Rightarrow$

fact $=1$

Ex $f(x) = \cos x$ @ $x=0$

$f(x) = \cos x$ $\quad f(0) = 1$ $\quad f'(x) = -\sin x$

$f''(x) = -\cos x$ $\quad f''(0) = -1$ $\quad f'''(x) = \sin x$

$f^{(4)}(x) = \cos x$ $\quad f^{(5)}(x) = -\sin x$

$\vdots$

$f^{(2n)}(x) = (-1)^n \cos x$ $\quad f^{(2n+1)}(0) = 0$

$$\boxed{f^{(2n)}(0) = (-1)^n}$$

$$T(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

---

about $\frac{\pi}{3}$

$$\cos x = \cos\left(x - \frac{\pi}{3} + \frac{\pi}{3}\right)$$

$$= \cos\left(x - \frac{\pi}{3}\right)\cos\frac{\pi}{3} - \sin\left(x - \frac{\pi}{3}\right)\sin\frac{\pi}{3}$$

$\underline{Ex}$ $\quad \ln x \to (x-2)$

$$t = \frac{x-2}{2}$$

$$\ln x = \ln\left(2+(x-2)\right)$$
$$= \ln 2\left(1+\frac{x-2}{2}\right)$$
$$= \ln 2(1+t)$$
$$= \underbrace{\ln 2}_{\text{constant}} + \ln(1+t)$$

$$\left(\frac{1}{u^n}\right)' = \frac{-n u'}{u^{n+1}}$$

$f(t) = \ln(1+t)$ $\qquad$ $f(0) = \ln 1 = 0$

$f'(t) = \dfrac{1}{1+t}$ $\qquad$ $f'(0) = 1$

$f''(t) = -\dfrac{1}{(1+t)^2}$ $\qquad$ $f''(0) = -1$

$f'''(t) = \dfrac{2}{(1+t)^3}$ $\qquad$ $f'''(0) = 2$

$f^{(4)}(t) = -\dfrac{6}{(1+t)^4}$ $\qquad$ $f^{(4)}(0) = -6$

$$f^{(n)}(t) = (-1)^{n+1}\frac{(n-1)!}{(1+t)^n}$$

$$\ln(t+1) = t - \frac{1}{2!}t^2 + \frac{2}{3!}t^3 - \frac{6}{4!}t^4 + \dots$$

$$\ln x = \ln 2 + t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4$$

$$\ln x = \ln 2 + \frac{(x-2)}{2} - \frac{1}{2}\left(\frac{x-2}{2}\right)^2 + \frac{1}{3}\left(\frac{x-2}{2}\right)^3 + \cdots$$

$$= \ln 2 + \frac{1}{1\cdot 2}(x-2) - \frac{1}{2\cdot 2^2}(x-2)^2 + \frac{1}{3\cdot 2^3}(x-2)^3 + \cdots$$

$$= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n\cdot 2^n}$$

$\ln(>0)$

$\boxed{\ln x = ?}$ $\qquad \sum$

condition for x's ?

$$\begin{cases} x = 1 \Rightarrow \ln 1 = 0 \\ 0 < x < 1 \Rightarrow \ln x < 0 \\ x > 1 \Rightarrow \ln x > 0 \end{cases}$$

1) $f(x) = e^{2x} \qquad a=0 \qquad 0,1,2,3$

$f(x) = e^{2x} \qquad f(0) = 1 \qquad 2^0$

$f'(x) = 2e^{2x} \qquad f'(0) = 2 \qquad 2^1$

$f''(x) = 4e^{2x} \qquad f''(0) = 4 \qquad 2^2$

$f'''(x) = 8e^{2x} \qquad f'''(0) = 8 \qquad 2^3$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

---

2) $f(x) = \sin x \qquad a=0 \qquad 0,1,2,3$

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$\sin x = x - \frac{x^3}{3!}$$

30/ $f(x) = xe^x \quad n=4 \quad$ MacLaurin $\boxed{a=0}$

$f(x) = xe^x \qquad f(0) = 0$

$f'(x) = (x+1)e^x \qquad f'(0) = 1$

$f''(x) = (x+2)e^x \qquad f''(0) = 2$

$f'''(x) = (x+3)e^x \qquad f'''(0) = 3$

$f^{(4)}(x) = (x+4)e^x \qquad f^{(4)}(0) = 4$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$xe^x = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$$

~~31~~ $f(x) = x^2e^x \quad n=4 \quad a=0$

$f(x) = x^2e^x \qquad f(0) = 0$

$f'(x) = 2xe^x + x^2e^x \qquad f'(0) = 0$

$f''(x) = 2e^x + 2xe^x + 2xe^x + x^2e^x \qquad f''(0) = 2$

$= (2+4x+x^2)e^x$

$f'''(x) = (4+2x)e^x + (2+4x+x^2)e^x \qquad f'''(0) = 6$

$= (6+6x+x^2)e^x$

$f^{(4)}(x) = (6+2x)e^x + (6+6x+x^2)e^x \qquad f^{(4)}(0) = 12$

$= (12+8x+x^2)e^x$

$$P_4(x) = x^2 + x^3 + \frac{1}{2}x^4$$

3) $f(x) = \ln(x+1)$ $\quad a = 0 \qquad 0, 1, 2, 3$

$f(x) = \ln(x+1) \rightarrow f(0) = \ln 1 = 0$

$f'(x) = \frac{1}{x+1} \qquad f'(0) = 1$

$f''(x) = \frac{-1}{(x+1)^2} \qquad f''(0) = -1$

$f'''(x) = \frac{2}{(x+1)^3} \qquad f'''(x) = 2$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3$$

12) $f(x) = \ln x$ $\qquad a = e$

$f(x) = \ln x \qquad f(e) = \ln e = 1$

$f'(x) = \frac{1}{x} \qquad f'(e) = \frac{1}{e}$

$f''(x) = -\frac{1}{x^2} \qquad f''(e) = -\frac{1}{e^2}$

$f'''(x) = \frac{2}{x^3} \qquad f'''(e) = \frac{2}{e^3}$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3$$

$$= 1 + \sum_{n=1} \frac{(-1)^{(2n+1)}(x-e)^n}{n e^n}$$

$$\frac{f^{(n)}(x)}{n!}(x-a)^n$$