Chapter 15 Section 1 - Oscillatory Motion

An oscillatory motion is a back and forth motion as a function of time. Often, an oscillatory motion is also a periodic motion. A periodic motion is a motion in which a certain pattern repeats itself again and again. The time taken by a periodic motion to make one complete pattern is called period (T). And the number of patterns executed per second is called frequency(f). The unit of frequency which is 1/second is called Hertz(Hz). Frequency and period are inverses of each other.

$$\boxed{f = \frac{1}{T}}$$

In this chapter, a special kind of oscillatory motion called simple harmonic motion where the displacement of a particle varies like a cosine or sine as a function of time will be considered.

<u>Simple Harmonic Motion</u>

A simple harmonic motion is a motion where the force acting on the particle is proportional and opposite in direction to the displacement of the particle. For one dimensional motion, this may be expressed mathematically as

$$\boxed{F_x = -cx}$$

Where F_x is the force acting on the particle, x is the displacement of the particle, and *c* is a constant of proportionality called <u>force constant.</u> The unit of *c* is Newton/meter (N/m). Since $F_x = ma_x$, the acceleration of the particle is also proportional to the displacement of the particle. $a_x = -\frac{c}{m}x$. And since $a_x = \frac{d^2x}{dt^2}$, it follows that for a simple harmonic motion the following differential equation applies

$$\frac{d^2x}{dt^2} + \frac{c}{m}x = 0 \quad \text{with} \quad \sqrt{\frac{c}{m}} = \omega_o$$

$$\boxed{\frac{d^2x}{dt^2} + \omega_o^2 x = 0} \quad \longrightarrow \text{Equation of a harmonic motion}$$

This is a 2nd order differential equation. The general solution of a 2nd order differential equation is two dimensional, which means all of the solutions of the equation can be expressed in terms of two linearly independent solutions of the equation (two functions are said to be linearly independent if they are not proportional to each other). By direct substitution it can be shown that $\cos \omega_o t$ & $\sin \omega_o t$ are solutions of this equation. Since $\cos \omega_o t$ & $\sin \omega_o t$ are linearly independent, the general solution of this equation can be written as

$$x(t) = c_1 \cos \omega_o t + c_2 \sin \omega_o t$$

Where c_1 & c_2 are arbitrary constants to be determined by two initial conditions. These two terms can be combined into one term by transforming the constants c_1 & c_2 into the constant A & ϕ using the transformation equations

$$c_1 = A\cos\phi \quad \text{and} \quad c_1 = A\sin\phi$$

$$\therefore \quad x(t) = A\cos\phi\cos\omega_o t + A\sin\phi\sin\omega_o t$$

These two terms can be combined into one term using the formula for the cosine of the difference of two angles ($\cos(a-b) = \cos a \cos b + \sin a \sin b$)

$$\boxed{x(t) = A\cos(\omega_o t - \phi)}$$

The constant A represents the maximum value of the displacement of the particle and is called the amplitude of the motion. The constant $\omega_o = \sqrt{c/m}$ determines how fast the particle oscillates back and forth and is called angular frequency of the motion. Its unit is rad/second. When $t = T$ (the period), $\omega_o T = 2\pi$ (which is the period of a cosine function).

$$\therefore \quad \boxed{\omega_o = \frac{2\pi}{T}} \quad or \quad \boxed{T = \frac{2\pi}{\omega_o} = 2\pi\sqrt{\frac{m}{c}}}$$

And since $T = 1/f$

$$\boxed{\omega_o = 2\pi f} \quad or \quad \boxed{f = \frac{\omega_o}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{c}{m}}}$$

The constant ϕ is called the phase angle of the motion, and its effect is to shift the graph of the cosine function to the right (if positive) or to the left (if negative).

The constants A & ϕ can be determined from initial conditions.

If $x|_{t=0} = x_o$ and $v|_{t=0} = v_o$,

$x_o = x|_{t=0} = A\cos(-\phi) = A\cos\phi$

$v_o = v|_{t=0} = \frac{dx}{dt}|_{t=0} = -A\omega_o \sin(\omega_o t - \phi)\,|_{t=0} = -A\omega_o \sin(-\phi)$

$= A\omega_o \sin\phi$

That is

$$A\cos\phi = x_o \;\ldots\ldots (1)$$

$$A\omega_o \sin\phi = v_o \;\ldots\ldots (2)$$

Squaring both equations and adding

$$A^2\cos^2\phi + A^2\sin^2\phi = x_o^2 + \left(\frac{v_o}{\omega_o}\right)^2$$

$$\Rightarrow \quad A^2(\cos^2\phi + \sin^2\phi) = x_o^2 + \left(\frac{v_o}{\omega_o}\right)^2$$

$$\boxed{A = \sqrt{x_o^2 + \left(\frac{v_o}{\omega_o}\right)^2}}$$

Dividing equation (2) by (1)

$$\frac{A\omega_o \sin\phi}{A\cos\phi} = \frac{v_0}{x_o}$$

$$\tan\phi = \frac{v_o}{w_o x_o}$$

$$\text{or} \quad \boxed{\phi = \tan^{-1}\left(\frac{v_o}{\omega_o x_o}\right)}$$

Example: By direct substitution show that $x(t) = A\cos(\omega_o t - \phi)$ is the solution of the equation of a harmonic motion $\frac{d^2x}{dt^2} + \omega_o x = 0$

<u>Solution</u>

$$\frac{d^2x}{dt^2} + \omega_o^2 x = \frac{d^2}{dt^2}[A\cos(\omega_o t - \phi)] + \omega_o^2 A\cos(\omega_o t - \phi)$$

$$= \frac{d}{dt}[-A\omega_o \sin(\omega_o t - \phi)] + \omega_o^2 A\cos(\omega_o t - \phi)$$

$$= -A\omega_o^2\cos(\omega_o t - \phi) + A\omega_o^2\cos(\omega_o t - \phi)$$

$$= \underline{0}$$

<u>PROVED</u>

Example: Give expressions for the velocity, acceleration and force acting as a function of time for a particle undergoing a simple harmonic motion.

<u>Solution</u>

<u>Velocity</u>

$$v = \frac{dx}{dt} = \frac{d}{dt}[A\cos(\omega_o t - \phi)] = -A\omega_o \sin(\omega_o t - \phi)$$

Acceleration

$$a = \frac{dv}{dt} = \frac{d}{dt}[-A\omega_o \sin(\omega_o t - \phi)] = -A\omega_o^2 \cos(\omega_o t - \phi)$$

Force

$$F = ma = -m\omega_o^2 A\cos(\omega_o t - \phi)$$

Energy of a harmonic oscillator

The force acting on a harmonic oscillator is conservative. Therefore the mechanical energy of a harmonic oscillator is expected to be conserved. An expression for this conserved mechanical energy can be obtained by adding the kinetic and potential energy of a harmonic oscillator.

$$ME = KE + u$$

$$KE = \frac{1}{2}mv^2$$ And $$u = -\int F_x dx + C$$ (because for is conservative)

$$= -\int(-cx)dx + C$$

$$= \frac{1}{2}cx^2 + C_1$$

Assuming $u|_{x=0} = 0$, sets C_1 to zero

$$\therefore \quad u = \frac{1}{2}cx^2$$

$$ME = \frac{1}{2}mv^2 + \frac{1}{2}cx^2$$

But since $\omega_o = \sqrt{\frac{c}{m}}$, $c = m\omega_o^2$

$$\therefore \quad ME = \frac{1}{2}mv^2 + \frac{1}{2}m\omega_o^2 x^2$$

With $x(t) = A\cos(\omega_o t - \phi)$

& $v = \frac{dx}{dt} = -A\omega_o \sin(\omega_o t - \phi)$

$$ME = \frac{1}{2}mA^2\omega_o^2 \sin^2(\omega_o t - \phi) + \frac{1}{2}m\omega_o^2 A^2 \cos^2(\omega_o t - \phi)$$

$$= \frac{1}{2}m\omega_o^2 A^2[\sin^2(\omega_o t - \phi) + \cos^2(\omega_o t - \phi)]$$

$$= \frac{1}{2}m\omega_o^2 A^2$$

$$\therefore \quad \boxed{ME = \frac{1}{2}m\omega_o^2 A^2 = \frac{1}{2}cA^2}$$ $\longrightarrow$ mechanical energy of a harmonic oscillator

The mechanical energy of a harmonic oscillator is independent of time. It depends only on the force constant (c) and amplitude (A) only, which means mechanical energy is conserved as expected.

With this expression of the mechanical energy, an expression of the velocity of the particle as a function of displacement can be obtained.

$$ME = \frac{1}{2}mv^2 + \frac{1}{2}cx^2 = \frac{1}{2}cA^2$$

$$mv^2 = cA^2 - cx^2$$

$$v^2 = \frac{c}{m}(A^2 - x^2)$$

$$v = \pm\sqrt{\frac{c}{m}}(A^2 - x^2)^{1/2} \quad \text{with} \quad \omega_o = \sqrt{\frac{c}{m}}$$

$$\therefore \quad \boxed{v = \pm\omega_o\sqrt{A^2 - x^2}}$$

Examples of harmonic oscillators

1) An object attached to a spring

The dynamics of a spring is governed by Hook's law. Hook's law states that the force due to a spring is proportional but opposite to displacement. This may be mathematically expressed as

$$\boxed{F_s = -kx}$$

The constant of proportionality k is called Hook's constant and is constant for a given spring but may be different for different springs. Its unit of measurement is N/m.

Since the force acting on an object attached to a spring is proportional but opposite to displacement all the equations of a harmonic oscillator apply with $c = k$

$$\boxed{\begin{gathered} \omega_o = \sqrt{\frac{k}{m} = \frac{2\pi}{T}} \\ T = \frac{2\pi}{\omega_o} = 2\pi\sqrt{\frac{m}{k}} \\ f = \frac{\omega_o}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \\ x(t) = A\cos(\omega_o t - \phi) \\ ME = \frac{1}{2}kA^2 = \frac{1}{2}m\omega_o^2A^2 \\ v = \pm\omega_o\sqrt{A^2 - x^2} \end{gathered}}$$

Example: A certain spring extends by 2 cm when an object of mas 2 kg hands from it. By how much will it extend when an object of mass 7 kg hangs from it?

Solution

$$\begin{array}{l|l} m_1 = 2\,kg & m_2 = 7\,kg \\ x_1 = 2cm = 0.02m & x_2 = ?? \end{array}$$

$$|F_1| = k|x_1| \;\Rightarrow\; k = \frac{|F_1|}{|x_1|} \qquad F_1 = \text{weight of } m_1 = m_1|g|$$

$$\therefore \quad k = \frac{2(9.8)}{.02}\,{}^{N}\!/_{m} = 980\,{}^{N}\!/_{m}$$

$$|F_2| = k|x_2| \;\Rightarrow\; |x_1| = \frac{|F_1|}{k} \qquad F_2 = \text{weight of } m_2 = m_2|g|$$

$$= \frac{m_2|g|}{k} = \frac{7(9.8)}{980}\,{}^{N}\!/_{m} = .07\,m = \underline{7\,cm}$$

Example: An object of mass 2 kg is attached to a spring of Hook's constant 200 N/m on a horizontal frictionless surface. Then it is extended by 10 cm and then let free to oscillate.

a) How long will it take to make one complete oscillation?

Solution

$k = 200\, N/m$
$m = 2\, kg$
$T = ??$

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2}{200}} = \underline{\frac{\pi}{5}\ sec}$$

b) How many oscillations does it execute per second?

Solution

$T = \frac{\pi}{5}$ sec
$f = ??$

$$f = \frac{1}{T} = \frac{1}{\left({\pi}/{5}\right)} = \underline{\frac{5}{\pi}\ Hz}$$

c) Determine the amplitude and the phase angle of the motion and then give an expression for the displacement as a function of time.

$$x(t) = A\cos(\omega_o t - \phi)$$
$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{200}{2}} = 10\, rad/s$$
$$\therefore \quad x(t) = A\cos(10t - \phi)$$

$A\ \&\ \phi$ can be determined from initial condition

Extended by 0.1 m and then let go implied

$x|_{t=0} = 0.1$ and $v|_{t=0} = 0$ (released from rest)

$$x|_{t=0} = 0.1 = A\cos(-\phi) = A\cos\phi$$
$$v|_{t=0} = \frac{dx}{dt}|_{t=0} = [-A\omega_o\sin(10t - \phi)]|_{t=0} = -A\omega_o\sin(-\phi) = 0$$
$$\Rightarrow \sin\phi = 0$$
$$\Rightarrow \quad \phi = 0$$
$$\&\ A\cos(\phi) = 0.1$$
$$A\cos(0) = 0.1$$
$$A = 0.1$$
$$\therefore \quad A = 0.1 \ \ \&\ \ \phi = 0$$

Therefore the displacement as a function of time is given by

$$\underline{x(t) = 0.1m\cos(10t)}$$

d) Calculate its displacement, velocity, acceleration and force action on it after $\pi/10$ seconds.

Solution

Displacement

$$x(t) = 0.1m\cos(10t)$$
$$\Rightarrow\ x|_{t=\pi/10} = 0.1m\cos\left(10\left(\frac{\pi}{10}\right)\right) = 0.1m\cos(\pi)$$
$$= -0.1\ m$$

Velocity

$$v = \frac{dx}{dt} = -(0.1)(10)\sin(10t)$$
$$= -\sin(10t)$$
$$v|_{t=\pi/10} = -\sin\left(10\left(\frac{\pi}{10}\right)\right) = 0$$

Acceleration

$$a = \frac{dv}{dt} = -10\cos(10t)$$
$$a|_{t=\pi/10} = -10\cos\left(10\left(\frac{\pi}{10}\right)\right) = -10\cos\pi$$
$$= 10\, m/s^2$$

Force

$F = ma = (2)(10) = 20N$

e) Calculate its velocity by the time it is extended by 0.5 cm.

Solution

$x = 0.5cm = 0.05m$

$v = \pm\, \omega_o \sqrt{A^2 - x^2} = \pm 10\sqrt{0.1^2 - .05^2}$

$= \pm\, 0.866\, m/s$

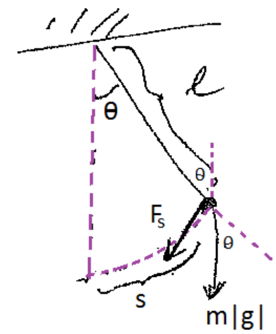
f) Calculate the mechanical energy of the object

$ME = \frac{1}{2}kA^2 = \frac{1}{2}(200)(0.1)^2 = 1\, J$

(Chapter 15 Lecture 2)

As will be shown shortly, the motion of a pendulum is harmonic only approximately for small angular displacements. Otherwise, generally, the motion of a pendulum is not harmonic.

The force responsible for the motion of a pendulum is the component of the weight of the object along its trajectory.



If F_s is the component of its weight along its trajectory, then $F_s = -m|g|\sin(\theta)$ (the negative indicates the fact that the motion is to the left at the point shown)

For small angles (less than 15°), $\sin\theta$ is approximately equal to 0.

$\therefore \quad F_s \approx -m|g|\theta$ for small angles

If the arc length subtended by the angle θ is s, then $\theta = s/l$

$$F_s \approx -m|g|\frac{s}{l} = -\left(\frac{m|g|}{l}\right)s$$

$$\boxed{F_s \cong -\left(\frac{m|g|}{l}\right)s} \quad \text{or since } F_s = \frac{md^2s}{dt^2} \quad \boxed{\frac{d^2s}{dt^2} + \frac{|g|}{l}s = 0}$$

That is, for small angles, the force acting on the pendulum is proportional but opposite to the displacement(s). Therefore for small angles, the pendulum is approximately harmonic and all the equations of a harmonic oscillator apply with $c = \frac{m|g|}{l}$.

$$\boxed{\begin{gathered}\omega_o = \sqrt{\frac{c}{m}} = \sqrt{\frac{m|g|}{l}} = \sqrt{\frac{|g|}{l}} \\ T = \frac{2\pi}{\omega_o} = 2\pi\sqrt{\frac{l}{|g|}} \\ f = 2\pi\omega_o = 2\pi\sqrt{\frac{|g|}{l}}\end{gathered}}$$

<u>Example:</u> A pendulum of length 0.098 m is displaced by a small angle and let free to oscillate.

a) How long will it take to make on complete oscillation

<u>Solution</u>

$l = 0.098\ m$

$T = ??$

$$T = 2\pi\sqrt{\frac{l}{|g|}} = 2\pi\sqrt{\frac{0.098}{9.8}}$$

$$T \approx 0.2\pi\ s$$

b) How many oscillations does it execute per second?

Solution

$f = ??$

$$f = \frac{1}{T} = \frac{1}{0.2\pi} = \underline{\frac{5}{\pi}\ Hz}$$

Example: How long does a pendulum have to be if it is to make one complete oscillation in one second?

Solution

$T = 1s$

$l = ??$

$$T = 2\pi\sqrt{\frac{l}{|g|}}$$

$$T^2 = 4\pi^2\frac{l}{|g|}$$

$$l = \frac{|g|T^2}{4\pi^2} = \frac{(9.8)(1)^2}{4\pi^2}$$

$$l \cong \underline{0.25\ m}$$

Example: If a pendulum makes one complete revolution in 2 seconds on earth, how long will it take to make one complete oscillation on the moon where the gravitational acceleration is 1/6th of that on earth?

Solution

$T_{earth} = 2\ s$

$T_{moon} = ??$

$|g|_{moon} = \frac{1}{6}|g|$

$$T_{earth} = 2\pi\sqrt{\frac{l}{|g|}}$$

$$T_{moon} = 2\pi\sqrt{\frac{l}{|g|/6}}$$

$$\frac{T_{moon}}{T_{earth}} = \sqrt{\frac{\cancel{l}}{\cancel{|g|}/6} * \frac{\cancel{|g|}}{\cancel{l}}} = \sqrt{6}$$

$$T_{moon} = \sqrt{6}\,T_{earth} = \underline{2\sqrt{6}\ sec}$$

Example: If the length of a pendulum is doubled, by what factor will its period change?

Solution

$\frac{l'}{l} = 2$

$\frac{T'}{T} = ??$

$$T' = 2\pi\sqrt{\frac{l'}{|g|}}$$

$$T = 2\pi\sqrt{\frac{l}{|g|}}$$

$$\frac{T'}{T} = \sqrt{\frac{l'}{|g|}} \cdot \sqrt{\frac{|g|}{l}} = \sqrt{\frac{l'}{l}} = \sqrt{2}$$

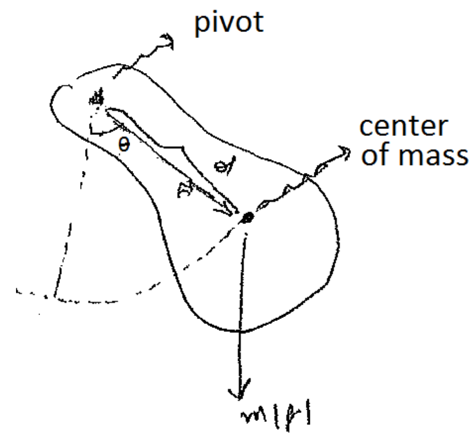
$$\therefore\quad T' = \sqrt{2} \cdot T$$

It will be multiplied by a factor of $\sqrt{2}$

Physical Pendulum

A physical pendulum is an object pivoted at one its points, displaced about the pivot by a certain angle and the let free to oscillate.

The force responsible for the rotation about the pivot is the weight of the object. For purposes of calculating torque, the whole weight can be assumed to act at the center of mass.



$$\therefore \quad \vec{\tau} = \vec{r} \times \vec{F}$$

$$\vec{F} = -m|g|\hat{\jmath}$$

$\vec{r}$ is the position vector of the center of mass with respect to the pivot

$$\vec{r} = d\cos\left(\theta + \frac{3\pi}{2}\right)\hat{\imath} + d\sin\left(\theta + \frac{3\pi}{2}\right)\hat{\jmath}$$
$$= d\sin\theta\,\hat{\imath} - d\cos\theta\,\hat{\jmath}$$

$$\therefore \quad \vec{\tau} = \vec{r} \times \vec{F} \quad = [d\sin\theta\,\hat{\imath} - d\cos\theta\,\hat{\jmath}] \times [-m|g|\hat{\jmath}]$$

$$\vec{\tau} = -m|g|\sin\theta\,\hat{k}$$

$$\tau = -m|g|d\sin\theta \ \ldots\ldots (1)$$

For small angles $\sin\theta = \theta$

$$\therefore \quad \tau = -m|g|d\,\theta$$

But also from the equations of motion

$$\tau = I\frac{d^2\theta}{dt^2}$$

Where I is the moment of inertia about the axis of rotation. θ is angular displacement about the axis of rotation, which is θ in this case.

$$\tau = I\frac{d^2\theta}{dt^2} \ \ldots\ldots (2)$$

Equating (1) & (2)

$$I\frac{d^2\theta}{dt^2} = -m|g|d\,\theta$$

$$I\frac{d^2\theta}{dt^2} + m|g|d\,\theta = 0 \quad or \quad \frac{d^2\theta}{dt^2} + \frac{m|g|d}{I}\theta = 0$$

This is an equation of a harmonic motion with $\omega_o = \frac{m|g|d}{I}$ and all the conclusions of a harmonic motion apply

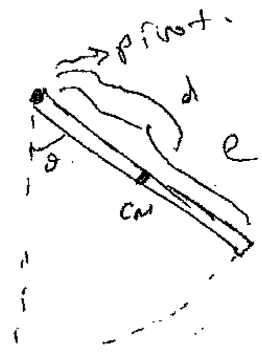
$$\omega_o = \sqrt{\frac{m|g|d}{I}}$$

$$T = 2\pi\sqrt{\frac{I}{m|g|d}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{m|g|d}{I}}$$

$$\theta = A\cos(\omega_o t - \theta)$$

Example: A uniform rod of length 2m is pivoted at one of its endpoints, displaced by a small angular displacement and let free to oscillate. Its mass is 4 kg.



a) Calculate its moment of inertia about the axis of rotation.

$$I = I_{cm} + md^2$$

For a uniform rod

	$I_{cm} = \frac{ml^2}{12}$	and since the center of mass is located at the center $d = \frac{l}{2}$
$m = 4\,kg$ $l = 2m$	$I = \frac{ml^2}{12} + m\left(\frac{l}{2}\right)^2 = \frac{ml^2}{12} + \frac{ml^2}{4} = \frac{ml^2 + 3ml^2}{12} = \frac{4}{12}ml^2 = \frac{ml^2}{3}$	

$$\therefore \quad I = \frac{ml^2}{3} = \frac{(4)(2)^2}{3} = \frac{16}{3}kg\cdot m$$

b) How long does it take to make one complete oscillation?

$T = ??$

$$T = 2\pi\sqrt{\frac{I}{m|g|d}}$$

$$= 2\pi\sqrt{\frac{16/3}{4(9.8)(1)}} \cong \underline{2.32\,sec}$$

$$d = \frac{l}{2} = \frac{2}{2} = 1m$$

Torsional Pendulum

A torsional pendulum is a pendulum where a wire is twisted and let free to oscillate back and forth. In such a case the restoring torque is proportional and opposite to the angular displacement.

$$\boxed{\tau = -k\theta}$$ k is a constant called torsional constant

But also from the equation of motion

$$\tau = I\frac{d^2\theta}{dt^2} \qquad \text{Where } I \text{ is the moment of inertia}$$

$$\therefore \quad I\frac{d^2\theta}{dt^2} = -k\theta$$

$$\frac{d^2\theta}{dt^2} + \frac{k}{I}\theta = 0$$

This is an equation of harmonic motion with $\omega_o^2 = \frac{k}{I}$.

Therefore all properties of a harmonic motion apply

$$\boxed{\begin{gathered}\omega_o = \sqrt{\frac{k}{I}} \\ T = 2\pi\sqrt{\frac{I}{k}} \\ f = \frac{1}{2\pi}\sqrt{\frac{k}{I}} \\ \theta = A\cos(\omega_o t - \theta)\end{gathered}}$$

Example: A uniform disc of mass 2 kg and radius 0.1 m is hanging from a wire of torsional constant 0.1 N m. The wire is twisted and let free to oscillate. How many oscillations will it execute for second?

Solution

$M = 2\ kg$
$R = 0.1\ m$
$k = 0.1\ Nm$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{I}}$$

$$I = I_{cm}^{disk} = \frac{MR^2}{2}$$

$$= \frac{(2)(0.1)^2}{2}$$

$$= .01\ kg\ m^2$$

$$\therefore \quad f = \frac{1}{2\pi}\sqrt{\frac{0.1}{.01}}$$

$$= 0.5\ Hz$$

Damped Oscillations

Brief review of homogenous 2nd order differential equations with constant coefficients

A 2nd order homogenous differential equation with constant coefficients has the form

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

Where a,b, and c are constants. The solution of a 2nd order differential equation is two dimensional. That is, the general solution is the linear combination of any two linearly independent (not proportional to each other) solutions of the differential equation. If $x_1(t)$ & $x_2(t)$ are any two linearly independent solutions, then the general solution of the equation can be expresses as $x(t) = c_1x_1(t) + c_2x_2(t)$ where c_1 & c_2 are arbitrary constants. c_1 & c_2 can be determined from two initial (or other) conditions. Therefore the general strategy of solving the equation is to find two linearly independent functions that satisfy the equation.

Since the derivatives of e^{rt} are proportional to the functions itself, it makes sense to try solutions of this form. By substituting e^{rt}(where r is a constant) directly into the equation, we can find values of r for which e^{rt} is a solution

$$\therefore \text{ with } x(t) = e^{rt}$$

$$a\left(\frac{d^2}{dt^2}(e^{rt})\right) + b\frac{d}{dt}(e^{rt}) + c\,e^{rt} = 0$$

$$(ar^2 + br + c)e^{rt} = 0$$

$$\Rightarrow \quad \boxed{ar^2 + br + c = 0}$$

⟶ This is called the characteristic equation of the differential equation.

$$\Rightarrow \quad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r_1 = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \quad \& \quad r_2 = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

Now let's look at different scenarios

1) $\underline{b^2 - 4ac > 0}$

If $b^2 - 4ac > 0$, then bot r_1 & r_2 are real numbers & different from each other and the two linearly independent solutions are $e^{r_1 t}$ & $e^{r_2 t}$. Therefore the general solution is

$$\boxed{x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$$

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2) $\underline{b^2 - 4ac = 0}$

If $b^2 - 4ac = 0$, $r_1 = r_2$ and only one solution is obtained. But a second linearly independent solution is needed. It can be shown by direct substitution that $t\,e^{rt}$ is also a solution.

<u>Proof:</u> with $x(t) = t\,e^{r_1 t}$ where $r_1 = -\frac{b}{2a}$

$$a\frac{d^2}{dt^2}[t\,e^{r_1 t}] + b\frac{d}{dt}[t\,e^{r_1 t}] + c\,t\,e^{r_1 t} \overset{?}{=} 0$$

$$\Rightarrow \quad a\frac{d}{dt}[e^{r_1 t} + r_1 t\,e^{r_1 t}] + b\{e^{r_1 t} + r_1 t\,e^{r_1 t}] + c\,t\,e^{r_1 t} = 0$$

$$\Rightarrow a[r_1 e^{r_1 t} + r_1^2\,t\,e^{r_1 t} + r_1\,e^{r_1 t}] + b[e^{r_1 t} + r_1 t\,e^{r_1 t}] + c\,t\,e^{r_1 t} \overset{?}{=} 0$$

$$\Rightarrow a[2_{r1} + r_1^2 t] + b[1 + r_1 t] + c\,t \overset{?}{=} 0$$

$$\Rightarrow \quad 2ar_1 + b + t[ar_1^2 + br_1 + c] \overset{?}{=} 0$$

$$ar_1^2 + br_1 + c = 0$$

Because r_1 is a solution of $ar_1^2 + br_1 + c = 0$

$$2ar_1 + b \stackrel{?}{=} 0 \qquad \text{But } r_1 = -\frac{b}{2a}$$

$$2a\left(-\frac{b}{2a}\right) + b \stackrel{?}{=} 0$$

$$-b + b = 0$$

Therefore the general solution for the case $b^2 - 4ac$ can be given as

$$\boxed{x(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t}} \quad \text{where } r_1 = -b/2a$$

---

3) $\underline{b^2 - 4ac < 0}$
4) If $b^2 - 4ac < 0$, then the two solutions are complex. Let the two complex solutions be represented as $r_1 = -\beta + i\omega$ & $\beta - i\omega$ where $\beta = +b/2a$ & $\omega = \sqrt{4ac - b^2}/2a$. The general solution can be written as

$$x(t) = c_1 e^{(-\beta + i\omega)t} + c_2 e^{(\beta - i\omega)t}$$

$$= e^{-\beta t}\left[c_1 e^{i\omega t} + c_2 e^{-i\omega t}\right]$$

$$\text{But} \quad \begin{aligned} e^{i\omega t} &= \cos\omega t + i\sin\omega t \\ e^{-i\omega t} &= \cos\omega t - i\sin\omega t \end{aligned}$$

$$\therefore \quad x(t) = e^{-\beta t}[c_1(\cos\omega t + i\sin\omega t) + c_2(\cos\omega t - i\sin\omega t)]$$

$$= e^{-\beta t}[(c_1 + c_2)\cos\omega t + ([c_1 - c_2]i)\sin\omega t]$$

The sum or difference of constants is also a constant.

Let $c_1' = c_1 + c_2$ & $c_2' = i(c_1 - c_2)$

$$\therefore \quad \boxed{x(t) = e^{-\beta t}[c_1'\cos\omega t + c_2'\sin\omega t]} \quad \beta = -\frac{b}{2a} \;\; \& \;\; \omega = \frac{\sqrt{4ac - b^2}}{2a}$$

If $\beta = 0$, this turns out to be the solution of a harmonic motion as expected

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Damped Harmonic Motion

A damped harmonic motion is a motion where a particle is subjected to a resistive force proportional to the velocity of the particle in addition to the force which is proportional and opposite to the displacement. Typical resistance forces are air resistance and water resistance.

$$\boxed{F_x = -cx - b\frac{dx}{dt}}$$

b is the constant of proportionality between the resistive force and the velocity $v = dx/dt$.

$$m\frac{d^2x}{dt^2} = -cx - b\frac{dx}{dt}$$

$$m\frac{d^2x}{dt^2} + cx + b\frac{dx}{dt} = 0$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{c}{m}x = 0$$

Let $\omega_o^2 = \frac{c}{m}$ where ω_0 is the frequency of the non-damped harmonic motion. Let $\frac{b}{m} = \beta$. β is called the damping constant.

$$\Rightarrow \quad \boxed{\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_o^2 x = 0}$$

Where $\beta = \frac{b}{2m}$ & $\omega_o = \sqrt{\frac{c}{m}}$

The characteristic equation of this differential equation which can be obtained by direct substitution of $x(t) = e^{rt}$ is

$$r^2 + 2\beta + \omega_o^2 = 0$$

$$\boxed{r_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_o^2}}$$

Or

$$\boxed{\begin{aligned} r_1 &= -\beta + \sqrt{\beta^2 - \omega_o^2} \\ r_2 &= -\beta - \sqrt{\beta^2 - \omega_o^2} \end{aligned}}$$

Or

$$\boxed{\begin{aligned} r_1 &= -\frac{b}{2m} + \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{c}{m}} \\ r_2 &= -\frac{b}{2m} - \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{c}{m}} \end{aligned}}$$

Under Damped Oscillations

Damped oscillations occur when $\beta^2 < \omega_o^2$ and the roots are complex. With $\omega = \sqrt{\omega_o^2 - \beta^2}$, the general solution can be written as

$$\boxed{x(t) = e^{-\beta t}(\mathrm{c}_1\cos\omega t + c_2\sin\omega t)}$$

Combining the cosine and sine into a single cosine by transforming the variables c_1 & c_2 into A & ϕ through the equations $c_1 = A\cos\phi$ & $c_2 = A\sin\phi$

$$\boxed{x(t) = A\,e^{-\beta t}\cos(\omega t - \phi)}$$

Where $\beta = b/2m$ is the damping constant and $\omega = \sqrt{\omega_o^2 - \left(\frac{b}{2m}\right)^2}$ is the frequency of the oscillation. Remember ω_o is the frequency of the harmonic motion without resistive force $(b = 0)$

Critically Damped Motion

Critically damped motion is the transition between oscillatory and non-oscillatory damped motion. It occurs when $\beta^2 = \omega_o^2$ or when there is only one root of the characteristic equation. The general solution is given by

$$x(t) = c_1 e^{-\beta t} + c_2 t e^{-\beta t}$$

$$\Rightarrow \boxed{x(t) = (c_1 + c_2 t)e^{-\beta t}}$$

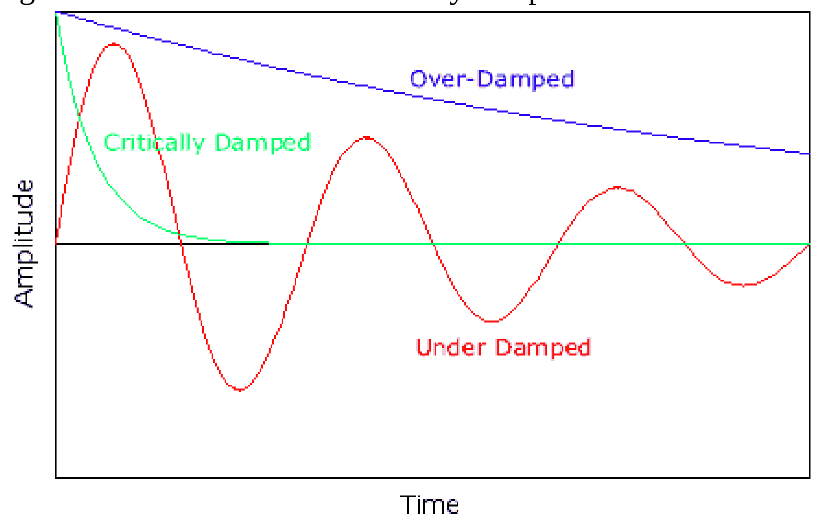
Where $\beta = b/2m$ is the damping constant since $e^{-\beta t}$ decreases faster (at least eventually) than $(c_1 + c_2 t)$, the particle goes to rest without oscillations.

Over Damped Motion

Over damped motion occurs when $\beta^2 > \omega_o^2$ or when the characteristic equation has two real roots. When $\beta^2 > \omega_o^2$, both roots $r_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_o^2}$ are negative. As a result both of the exponentials will decrease with time. The general solution is

$$\boxed{x(t) = c_1 e^{\left(-\beta+\sqrt{\beta^2-\omega_o^2}\right)t} + c_2 e^{\left(-\beta-\sqrt{\beta^2-\omega_o^2}\right)t}}$$

Since both exponentials decrease with time, the particle will eventually go to rest without any oscillations. The difference between an over damped motion and a critically damped motion is that an over damped motion goes slower to rest than a critically damped motion does.



Example: A spring has a Hook's constant if 100 N/m. A particle is attached to the spring on a frictionless horizontal surface, extended and then let free. The resistive force, F_r, due to air resistance varies with the velocity of the particle according to the equation $F_r = -2v$ (where v is velocity.

a) Determine the range of masses of the particle for which the motion would be under damped oscillation.

Solution

$k = 100\, N/m$

$b = 2 \left(\begin{array}{l} F_r = -bv \\ \quad = -2v \end{array} \right)$

Under damped oscillation

$\beta^2 - \omega_o^2 < 0$

$$but\ \beta = \frac{b}{2m} \quad \& \quad \omega_o = \sqrt{\frac{k}{m}}$$

$$\Rightarrow \left(\frac{b}{2m}\right)^2 - \frac{k}{m} < 0$$

$$\frac{b^2}{4m^2} < \frac{k}{m}$$

$$4mk > b^2$$

$$m > \frac{b^2}{4k} > \frac{2^2}{4(100)} > 0.01\, kg$$

$\therefore$ under damped oscillation will occur when the mass of the particle is greater than 0.01 kg.

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b) Determine the mass of the particle that will result in a critically damped oscillation.
<u>Solution</u>

Critically damped $\Rightarrow$ $\beta^2 - \omega_o^2 = 0$

$$\left(\frac{b}{2m}\right)^2 = \frac{k}{m}$$

$$m = \frac{b^2}{4k} = .01kg$$

$\therefore$ The motion of the particle will be critically damped when the mass is 0.01 kg

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c) Find the range of masses of the particle for which over damped oscillation will take place.
<u>Solution</u>

Over damped $\Rightarrow$ $\beta^2 - \omega_o^2 > 0$

$$\left(\frac{b}{2m}\right)^2 > \frac{k}{m}$$

$$m < \frac{b^2}{4k} < .01kg$$

$\therefore$ Over damped oscillation will take place when the mass of the particle is less than 0.01 kg.

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d) If the mass of the particle is 1 kg & the spring is extended by 0.01 m and then let go
i) Calculate the damping constant
<u>Solution</u>

$m = 1\ kg$
$x|_{t=0} = x_o = 0.01\ m$
$v|_{t=0} = v_o = \frac{dx}{dt}|_{t=0} = 0$
(released from rest)
$\beta = ??$

$$\beta = \frac{b}{2m} = \frac{2}{2(1)} = 1\frac{1}{s}$$

ii) Since the mass is greater than the .01 kg, the motion should be under damped oscillation. Calculate the angular frequency of the oscillation.
<u>Solution</u>

$\omega = ??$

$$\omega = \sqrt{\omega_o^2 - \beta^2}$$
$$= \sqrt{100 - 1^1}$$
$$= \sqrt{99}\,rad/s$$

$\omega_o^2 = \frac{k}{m} = \frac{100}{1}$
$\omega_o^2 = 100$
$\beta = 1$ (as calculated above)

iii) Express the displacement of the particle as a function of time.
<u>Solution</u>

The general solution of an under damped oscillation is given by

$$x(t) = A\,e^{-\beta t}\cos(\omega t - \phi)$$

Where A & ϕ have to be determined from initial conditions.

$$\beta = 1\ \ \&\ \ \omega = \sqrt{99}\ \Rightarrow\ x(t) = A\,e^{-t}\cos\left(\sqrt{99}t - \phi\right)$$

$$x|_{t=0} = 0.01\ m$$

$$\Rightarrow \quad A\, e^0 \cos(-\phi) = .01$$
$$A \cos(\phi) = .01 \ldots (1)$$

$$\frac{dx}{dt}|_{t=0} \Rightarrow \frac{d}{dt}\left[Ae^{-t}\cos\left(\sqrt{99}t - \phi\right)\right]|_{t=0} = 0$$
$$\Rightarrow \quad \left[-Ae^{-t}\cos\left(\sqrt{99}t - \phi\right) + \left(-Ae^{-t}\sqrt{99}\sin\left(\sqrt{99}t - \phi\right)\right]|_{t=0} = 0$$

$$-A\cos(-\phi) - A\sqrt{99}\sin(-\phi) = 0$$
$$-A\cos\phi + A\sqrt{99}\sin\phi = 0 \ldots (2)$$

$$A\cos\phi = A\sqrt{99}\sin\phi$$

$$\Rightarrow \frac{\sin\phi}{\cos\phi} = \frac{1}{\sqrt{99}} = \tan\phi$$
$$\phi = \tan^{-1}\left(\frac{1}{\sqrt{99}}\right) = 0.1\, rad$$

$$(1) \quad \Rightarrow \quad A\cos\phi = .01$$
$$A\cos(.1) = .01$$
$$A = \frac{.01}{\cos(.1)} = 0.001\, m$$

$\therefore$ the displacement as a function of time is given by

$$\underline{x(t) = .001\, e^{-t}\cos\left(\sqrt{99}t - 0.1\right)}$$

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e) If a mass of 0.01 kg is attached to the spring, extended by 0.01 m and then let go, obtain an expression for the displacement of the particle as a function of time.

Solution

$m = 0.01\, kg$

As shown earlier a mass of 0.01 kg will result in a critically damped oscillation. The general solution of a critically damped oscillation is given by

$x(t) = (c_1 + c_2 t)e^{-\beta t}$ where c_1 & c_2 are to be determined from initial conditions

$$\beta = \frac{b}{2m} = \frac{2}{2(.01)} = 100\, {}^{1}/_{s}$$

$$\therefore \quad x(t) = (c_1 + c_2 t)e^{-100t}$$

$$x|_{t=0} \quad \Rightarrow \quad [c_1 + c_2(0)]e^{-100(0)} = 0.01$$
$$c_1 = 0.01\, m$$

$$\frac{dx}{dt} = \frac{d}{dt}[(c_1 + c_2 t)e^{-100t}]$$
$$= c_2 e^{-100\, t} + (-100)(c_1 + c_2 t)e^{-100t}$$

$$\frac{dx}{dt}|_{t=0} = 0$$
$$\Rightarrow \quad c_2 - 100c_1 = 0$$
$$c_2 = 100c_1 = 100(.01) = 1\,m$$
$$c_2 = 1\,m$$

$\therefore$ the displacement caries with time according to the formula

$$x(t) = (0.01 + t)e^{-100t}$$

---

f) If a mass of 0.005 kg is attached to the spring, extended by 0.01 m and then let go, find an expression for the displacement of the particle as a function of time.

Solution

$m = .005\,kg$

As shown earlier, since 0.005 kg < .01 kg the motion will be an over damped oscillation. The general solution of an over damped motion is given by

$$x(t) = c_1 e^{\left(-\beta+\sqrt{\beta^2-\omega_o^2}\right)t} + c_2 e^{\left(-\beta-\sqrt{\beta^2-\omega_o^2}\right)t}$$

Where c_1 & c_2 are to be determined from initial conditions

$$\beta = \frac{b}{2m} = \frac{2}{2(.005)} = 200\,{}^{1}/_{s}$$
$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{.005}} = 100\sqrt{2}\,rad/s$$
$$\therefore \quad \text{with} \quad \beta = 200 \ \& \sqrt{\beta^2 - \omega_o^2} = 100\sqrt{2}$$
$$\beta + \sqrt{\beta^2 - \omega_o^2} = -200 + 100\sqrt{2} = 100\left(-2 + \sqrt{2}\right)$$
$$\beta - \sqrt{\beta^2 - \omega_o^2} = -200 - 100\sqrt{2} = 100\left(-2 - \sqrt{2}\right)$$
$$\& \quad x(t) = c_1 e^{(-2+\sqrt{2})100t} + c_2 e^{(-2-\sqrt{2})100t}$$

$$x|_{t=0} = 0.01 \quad \Rightarrow \quad c_1 + c_2 = 0.01 \ldots (1)$$

$$\frac{dx}{dt} = \left(-2 + \sqrt{2}\right)c_1 e^{(-2+\sqrt{2})100t} + \left(-2 - \sqrt{2}\right)c_2 e^{(-2-\sqrt{2})100t}$$

$$\frac{dx}{dt}|_{t=0} \quad \Rightarrow \quad \left(-2 + \sqrt{2}\right)c_1 + \left(-2 - \sqrt{2}\right)c_2 = 0$$
$$\left(-2 + \sqrt{2}\right)c_1 = \left(2 + \sqrt{2}\right)c_2$$

$$c_1 = \left(\frac{2 + \sqrt{2}}{-2 + \sqrt{2}}\right)c_2 = -5.83\,c_2 \ldots (2)$$

Substituting (2) into (1)

$$c_1 + c_2 = 0.01$$
$$-5.83\,c_2 + c_2 = 0.01$$
$$-4.83c_2 = 0.01$$

$$c_2 = \frac{.01}{-4.83} = -.002$$

$$c_1 = 0.01 - c_2 = 0.01 - (-.002)$$
$$c_1 = .012$$

$$\therefore \quad c_1 = 0.012 \ \ \& \ \ c_2 = -.002$$

The displacement varies with time according to the equation

$$\underline{x(t) = .012\, e^{(-2+\sqrt{2})100t} - .002\, e^{(-2-\sqrt{2})100t}}$$