Motion in Two Dimensions (Chapter 4 Lecture 1)

Motion in 2 dimensions is motion in a curved path in a plane.

4.1 Two Dimensional Motion Variables

Position Vector($\vec{r}$): of a particle is the vector whose tail is at the origin & whose head is at the location of a particle. The x & y components of a position vector are simply its x & y coordinates.

$$\boxed{r_x = x \quad \& \quad r_y = y}$$

In the $i - j$ notation a position vector of a particle may be written as

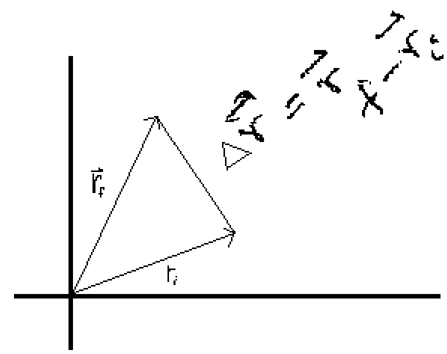
$$\boxed{\vec{r} = x\hat{\imath} + y\hat{\jmath}}$$

Displacement Vector ($\Delta\vec{r}$): of a particle is defined to be the change in its position vector.

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i$$

where $\vec{r}_i(\vec{r}_f)$ are initial(final) position vectors.

Graphically a displacement vector is the vector whose tail is at the initial location of the particle and whose head is at the final location of the particle.



In the $i - j$ notation:

$$\vec{r}_i = x_i\hat{\imath} + y_i\hat{\jmath}$$
$$\vec{r}_f = x_f\hat{\imath} + y_f\hat{\jmath}$$
$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i = \left(x_f\hat{\imath} + y_f\hat{\jmath}\right) - (x_i\hat{\imath} + y_i\hat{\jmath})$$
$$= \left(x_f - x_i\right)\hat{\imath} + \left(y_f - y_i\right)\hat{\jmath}$$
$$\boxed{\Delta\vec{r} = \left(x_f - x_i\right)\hat{\imath} + \left(y_f - y_i\right)\hat{\jmath} = \Delta x\hat{\imath} + \Delta y\hat{\jmath}}$$

Average Velocity($\bar{\vec{v}}$) is defined to be displacement per a unit time.

$$\bar{\vec{v}} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta x\hat{\imath} + \Delta y\hat{\jmath}}{\Delta t} = \left(\frac{\Delta x}{\Delta t}\right)\hat{\imath} + \left(\frac{\Delta y}{\Delta t}\right)\hat{\jmath}$$

$$\boxed{\bar{\vec{v}} = \left(\frac{\Delta x}{\Delta t}\right)\hat{\imath} + \left(\frac{\Delta y}{\Delta t}\right)\hat{\jmath}}$$

Also since $\bar{\vec{v}} = \bar{v}_x\hat{\imath} + \bar{v}_y\hat{\jmath}$

$$\boxed{\bar{v}_x = \frac{\Delta x}{\Delta t} \quad \& \quad \bar{v}_y = \frac{\Delta y}{\Delta t}}$$

Instantaneous Velocity ($\vec{v}$) is velocity at a given instant of time.

$$\vec{v} = \lim_{\Delta t \to 0}\frac{\Delta\vec{r}}{\Delta t} = \lim_{\Delta t \to 0}\left(\frac{\Delta x}{\Delta t}\hat{\imath} + \frac{\Delta y}{\Delta t}\hat{\jmath}\right)$$
$$= \lim_{\Delta t \to 0}\left(\frac{\Delta x}{\Delta t}\right)\hat{\imath} + \lim_{\Delta t \to 0}\left(\frac{\Delta y}{\Delta t}\right) = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath}$$
$$\boxed{\vec{v} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath}}$$

Also since $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$

$$v_X = \frac{dx}{dt} \quad \& \quad v_y = \frac{dy}{dt}$$

Average Acceleration ($\bar{\vec{a}}$): is defined to be change in velocity per a unit time.

$$\bar{\vec{a}} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\Delta v_x}{\Delta t}\hat{\imath} + \frac{\Delta v_y}{\Delta t}\hat{\jmath}$$

and since $\bar{\vec{a}} = \bar{a}_x\hat{\imath} + \bar{a}_y\hat{\jmath}$

$$\boxed{\bar{a}_x = \frac{\Delta v_x}{\Delta t} \quad \& \quad \bar{a}_y = \frac{\Delta v_y}{\Delta t}}$$

Instantaneous Acceleration ($\vec{a}$): is acceleration at a given instant of time.

$$\vec{a} = \lim_{\Delta t\to 0}(\frac{\Delta\vec{v}}{\Delta t}) = \lim_{\Delta t\to 0}(\frac{\Delta v_x}{\Delta t}\hat{\imath} + \frac{\Delta v_y}{\Delta t}\hat{\jmath})$$

$$= \lim_{\Delta t\to 0}(\frac{\Delta v_x}{\Delta t}\hat{\imath}) + \lim_{\Delta t\to 0}(\frac{\Delta v_{y\hat{\jmath}}}{\Delta t}\hat{\jmath}) = \frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath}$$

$$\boxed{\vec{a} = \frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath}}$$

Also $\frac{dv_x}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$

$$\frac{dv_y}{dt} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$$

$$\boxed{\vec{a} = \frac{d^2x}{dt^2}\hat{\imath} + \frac{d^2y}{dt^2}\hat{\jmath}}$$

And since $\quad \vec{a} = a_x\hat{\imath} + a_y\hat{\jmath}$

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$$

$$a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}$$

Example: The position vector of a particle varies with time according to the equation

$$\vec{r} = (4t^2 + 2t - 1)\hat{\imath} + (8t - 5)\hat{\jmath}$$

a) Determine the position vector after 2 seconds

$\vec{r}|_{t=2} = ??$

$$\vec{r}|_{t=2} = (4t^2 + 2t - 1)|_{t=2}\,\hat{\imath} + (8t - 5)\,|_{t=2}\,\hat{\jmath}$$
$$= (4(2)^2 + 2(2) - 1)\hat{\imath} + (8(2) - 5)\hat{\jmath}$$
$$= 19\hat{\imath} + 11\hat{\jmath}$$

b) Calculate the displacement of the particle between t=1s & t=4s

$\Delta\vec{r} = \vec{r}_f - \vec{r}_i = \vec{r}|_{t=4} - \vec{r}|_{t=1}$

$\vec{r}|_{t=4} = (4(4)^2 + 2(4) - 1)\hat{\imath} + (8(4) - 5)\hat{\jmath} = 71\hat{\imath} + 27\hat{\jmath}$

$\vec{r}|_{t=1} = (4(1)^2 + 2(1) - 1)\hat{\imath} + (8(1) - 5)\hat{\jmath} = 5\hat{\imath} + 3\hat{\jmath}$

$\Delta\vec{r} = \vec{r}|_{t=4} - \vec{r}|_{t=1} = (71\hat{\imath} + 27\hat{\jmath}) - (5\hat{\imath} + 3\hat{\jmath}) = \underline{66\hat{\imath} + 24\hat{\jmath}}$

c) Determine its average velocity for the first 5 seconds

$\bar{\vec{v}} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}_f - \vec{r}_i}{t_f - t_i}$

$t_i = 0 \quad \& \quad t_f = 5$

$\vec{r}_f = \vec{r}|_{t=5} = (4 * 5^2 + 2 * 5 - 1)\hat{\imath} + (8 * 5 - 5)\hat{\jmath} = 109\hat{\imath} + 35\hat{\jmath}$

$\vec{r}_i = \vec{r}|_{t=0} = (4*0^2 + 2*0 - 1)\hat{\imath} + (8*0-5)\hat{\jmath} = -\hat{\imath} - 5\hat{\jmath}$

$$\bar{\vec{v}} = \frac{\vec{r}_f - \vec{r}_i}{t_f - t_i} = \frac{(109\hat{\imath}+35\hat{\jmath})-(-\hat{\imath}-5\hat{\jmath})}{5-0} = \frac{110\hat{\imath}+40\hat{\jmath}}{5} = (22\hat{\imath} + 8\hat{\jmath}) \text{ m/s}$$

d) Calculate its instantaneous velocity after 10 seconds.

$\vec{v}|_{t=10} = ?$ $\qquad \vec{v} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath}$

$\vec{r} = (4t^2 + 2t - 1)\hat{\imath} + (8t-5)\hat{\jmath}$

$x = 4t^2 + 2t - 1$

$y = 8t - 5$

$$\vec{v} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} = \frac{d}{dt}(4t^2 + 2t - 1)\hat{\imath} + \frac{d}{dt}(8t-5)\hat{\jmath}$$

$= (8t+2)\hat{\imath} + 8\hat{\jmath}$

$\vec{r}|_{t=10} = (8*10+2)\hat{\imath} + 8\hat{\jmath}$

$= (82\hat{\imath} + 8\hat{\jmath})$ m/s

e) Calculate its average acceleration between $t = 2s$ & $t = 4s$

$t_i = 2s \qquad t_f = 4s$

$$\bar{\vec{a}} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i}$$

$\vec{v}_f = \vec{v}|_{t=4}$ but $\vec{v} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} = (8t+2)\hat{\imath} + 8\hat{\jmath}$

$\vec{v}_f = \vec{v}|_{t=4} = (8*4+2)\hat{\imath} + 8\hat{\jmath} = 34\hat{\imath} + 8\hat{\jmath}$

$\vec{v}_i = \vec{v}|_{t=2} = (8*2+2)\hat{\imath} + 8\hat{\jmath} = 18\hat{\imath} + 8\hat{\jmath}$

$$\bar{\vec{a}} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i} = \frac{(34\hat{\imath}+8\hat{\jmath}) - (18\hat{\imath}+8\hat{\jmath})}{4-2} = \frac{16\hat{\imath}}{2} = 8\hat{\imath}\, m/s^2$$

f) Calculate its instantaneous acceleration after 8 seconds

$\vec{a}|_{t=8} = ??$

$$\vec{a}(t) = \frac{d^2x}{dt^2}\hat{\imath} + \frac{d^2y}{dt^2}\hat{\jmath}$$

but

$x = 4t^2 + 2t - 1$ $\frac{dx}{dt} = 8t + 2$ $\frac{d^2x}{dt^2} = \frac{d}{dt}(8t+2) = 8$	$y = 8t - 5$ $\frac{dy}{dt} = 8$ $\frac{d^2t}{dt^2} = \frac{d}{dt}8 = 0$

$\vec{a}(t) = 8\hat{\imath} + 0\hat{\jmath}$ (independent of time)

$\vec{a}(t)|_{t=8} =$ $8\hat{\imath}\, m/s^2$

<u>4.2 Uniformly Accelerated Motion</u>

Motion with constant acceleration

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} = constant$$

$$a_x = constant$$

$$a_y = constant$$

<u>Equations of uniformly accelerated motion</u>

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} = \frac{d^2x}{dt^2}\hat{\imath} + \frac{d^2y}{dt^2}\hat{\jmath}$$

$$a_x = \frac{d^2x}{dt^2}$$
$$= \frac{d}{dt}\left(\frac{dx}{dt}\right)$$
$$= \frac{d}{dt}v_x$$
$$dv_x = a_x dt$$
$$\int dv_x = \int a_x\, dt = a_x \int dt$$
$$v_x = a_x t + C_1$$
$$If\ v_x|_{t=0} = v_{ix}$$
$$v_{ix} = a_x(0) + C \implies C_1 = v_{ix}$$
$$v_x = a_x t + v_i$$
$$v_{fx} = v_{ix} + a_x t$$

$$a_y = \frac{d^2y}{dt^2}$$
$$= \frac{d}{dt}\left(\frac{dy}{dt}\right)$$
$$= \frac{d}{dt}v_y$$
$$dv_y = a_y dt$$
$$\int dv_y = \int a_y\, dt = a_y \int dt$$
$$v_y = a_y t + C_1$$
$$If\ v_y|_{t=0} = v_{iy}$$
$$v_{iy} = a_y(0) + C \implies C_1 = v_{iy}$$
$$v_y = a_y t + v_{iy}$$
$$v_{fy} = v_{iy} + a_y t$$

$$v_x = \frac{dx}{dt}$$
$$\therefore v_x = \frac{dx}{dt} = v_{ix} + a_x t$$
$$\Rightarrow dx = (v_{ix} + a_x t)dt$$
$$\therefore \int dx = \int v_{ix}\, dt + \int a_x t\, dt$$
$$x = v_{ix}t + \frac{1}{2}a_x t^2 + C$$
$$If\ x(0) = x_i$$
$$x_i = v_{ix}(0) + \frac{1}{2}a_x(0)^2 + C$$
$$\Rightarrow C = x_i$$
$$\therefore\ \ x = x_i + v_{ix}t + \frac{1}{2}a_x t^2$$
$$\boxed{x - x_1 = \Delta x = v_{ix}t + \frac{1}{2}a_x t^2}$$

Applying the same procedure for the y component also

$$\boxed{y - y_1 = \Delta \mathrm{y} = v_{iy}t + \frac{1}{2}a_y t^2}$$

<u>Average Velocity</u>

$$\bar{v} = \frac{\Delta x}{t}\hat{\imath} + \frac{\Delta y}{t}\hat{\jmath} = \frac{\left(v_{ix}t + \frac{1}{2}a_x t^2\right)\hat{\imath}}{t} + \frac{\left(v_{iy}t + \frac{1}{2}a_y t^2\right)\hat{\jmath}}{t}$$

$$\implies \bar{v}_x = v_{ix} + \tfrac{1}{2}a_x t \quad \text{but} \quad a_{xt} = v_{fx} - v_{ix}$$

$$\therefore \quad \bar{v}_x = \tfrac{v_{ix}+v_{fx}}{2} \quad \text{similarly} \quad \bar{v}_y = \tfrac{v_{iy}+v_{fy}}{2}$$

$$\boxed{\begin{aligned}\Delta x &= \bar{v}_x t = \left(\frac{v_{ix}+v_{fx}}{2}\right)t \\ \Delta \mathrm{y} &= \bar{v}_y t = \left(\frac{v_{iy}+v_{fy}}{2}\right)t\end{aligned}}$$

Substituting for t from $v_{fx} = v_{ix} + a_x t$

$$\Delta x = \left(\frac{v_{ix}+v_{fx}}{2}\right)\left(\frac{v_{fx}-v_{ix}}{a_x}\right)$$

$$\Longrightarrow \boxed{v_{fx}^2 = v_{ix}^2 + 2a_x\Delta x} \text{ similarly } \boxed{v_{fy}^2 = v_{iy}^2 + 2a_y\Delta y}$$

Here is a list of equations of a uniformly accelerated two dimensional motion

<u>x-component</u>	<u>y-component</u>
$v_{fx} = v_{ix} + a_x t$ $\Delta x = v_{ix}t + \frac{1}{2}a_x t^2$ $\Delta x = \left(\frac{v_{ix}+v_{fx}}{2}\right)t$ $v_{fx}^2 = v_{ix}^2 + 2a_x\Delta x$	$v_{fy} = v_{iy} + a_y t$ $\Delta y = v_{iy}t + \frac{1}{2}a_y t^2$ $\Delta \mathrm{y} = \left(\frac{v_{iy}+v_{fy}}{2}\right)t$ $v_{fy}^2 = v_{iy}^2 + 2a_y\Delta \mathrm{y}$

Only the first two of each column are independent. The others are dependent equations included for simplicity.

<u>Example:</u> A particle is moving with a uniform acceleration of $(2\hat{\imath} + 4\hat{\jmath})m/s^2$. Its initial velocity is $(10\hat{\imath} + 5\hat{\jmath})m/s$

a) Determine its velocity after 4 seconds

$\vec{v}_i = v_{ix}\hat{\imath} + v_{iy}\hat{\jmath} = (10\hat{\imath} + 5\hat{\jmath})\frac{m}{s}$

$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} = (2\hat{\imath} + 4\hat{\jmath})\frac{m}{s^2}$

$t = 4s$

$\vec{v}_f = v_{fx}\hat{\imath} + v_{fy}\hat{\jmath} = ??$

$v_{fx} = v_{ix} + a_x t$ but $v_{ix} = 10\frac{m}{s}$ & $a_x = 2\frac{m}{s^2}$

$\therefore \quad v_{fx} = 10 + 2(4) = 18\frac{m}{s}$

$v_{fy} = v_{iy} + a_y t$ but $v_{iy} = 5\frac{m}{s}$ & $a_y = 4\frac{m}{s^2}$

$\therefore \quad v_{fy} = 5 + (4)(4) = 21\frac{m}{s}$

$\vec{v} = 18\frac{m}{s}\hat{\imath} + 21\frac{m}{s}\hat{\jmath}$

b) Determine its displacement after 10s

$$\Delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath} = ??$$

$$\Delta x = v_{ix}t + \frac{1}{2}a_x t^2 = (10)(10) + \frac{1}{2}(2)(10)^2 = 200\ m$$

$$\Delta y = v_{iy}t + \frac{1}{2}a_y t^2 = (5)(10) + \frac{1}{2}(4)(10)^2 = 250\ m$$

$$\therefore \quad \Delta\vec{r} = 200m\ \hat{\imath} + 250m\ \hat{\jmath}$$

4.3 Projectile Motion

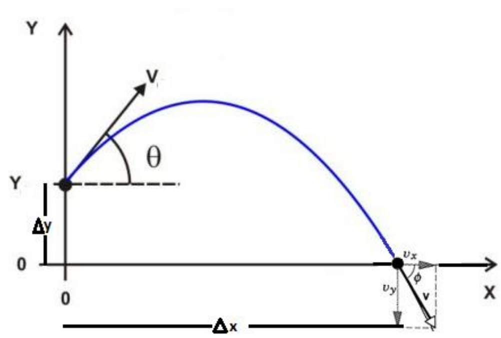
Projectile motion is motion under gravity in a plane. Motion under gravity is a uniformly accelerated motion; i.e., gravitational acceleration is a constant with a numerical value of $+9.8\frac{m}{s^2}$. Its direction is downward. Thus its x-component is zero & its y-component is $-9.8\frac{m}{s^2}$.
∴ for gravitational motion $a_x = 0$ & $a_y = -9.8\frac{m}{s^2} = g$

$$\boxed{\vec{a} = \boldsymbol{g}\,\hat{j} = \left(-9.8\frac{m}{s^2}\right)\hat{j}}$$

Equations for a projectile motion can be obtained from the equations of a uniformly accelerated motion with $a_x = 0$ & $a_y = g = -9.8\,m/s^2$

x-component	y-component
$v_{fx} = v_{ix}$	$v_{fy} = v_{iy} + gt$
$\Delta x = v_{ix}t$	$\Delta y = v_{iy}t + \frac{1}{2}gt^2$
	$\Delta y = \left(\frac{v_{iy} + v_{fy}}{2}\right)t$
	$v_{fy}^2 = v_{iy}^2 + 2g\Delta y$

$$v_{ix} = v_i \cos\theta_i$$
$$v_{iy} = v_i \sin\theta_i$$
$$v_{fx} = v_f \cos\theta_f$$
$$v_{fy} = v_f \sin\theta_f$$



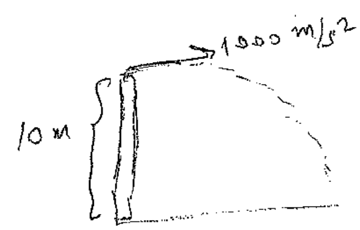
Example: A bullet is fired from a 10 m building horizontally with a speed of 1000 m/s

a) Calculate the time taken to reach the ground

$v_i = 1000\frac{m}{s}$ $\theta_i = 0$ (because it is horizontal)

$$v_{ix} = 1000\cos 0 = 1000\frac{m}{s}; v_{iy} = 1000\sin 0 = 0$$

$\Delta y = -10m$ (negative because final location is below initial location)



$$\Delta y = v_{iy}t + \frac{1}{2}gt^2$$
$$-10 = 0(t) + \frac{1}{2}(-10)t^2 \left(with\ g \approx -10\frac{m}{s^2}\right)$$

$t^2 = 2 \quad \Longrightarrow t = \sqrt{2} \approx 1.4\ seconds$

b) Calculate the x & y components of its velocity by the time it hits the ground.

$$v_{fx} = ?? \qquad v_{fy} = ??$$

$$v_{fx} = v_{ix} = 1000\frac{m}{s}$$

$$v_{fy} = v_{iy} + gt = 0(t) - 10(1.4) = -14\ m/s$$

c) Calculate the magnitude and direction of its velocity by the time it hits the ground.

$v_f = ?? \qquad \theta_f = ??$

$$v_f = \sqrt{v_{fx}^2 + v_{fy}^2} = \sqrt{1000^2 + (-14)^2} \approx 1000.1$$

d) How far did it fall horizontally? $\Delta x = ??$

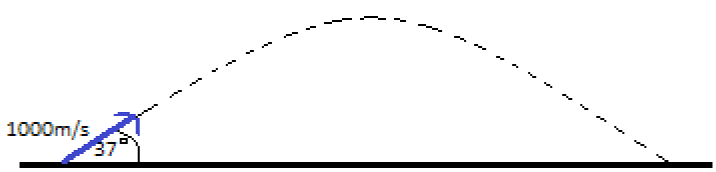
$$\Delta x = v_{ix}t = 1000(1.4) = 1400\ m$$

e) Express its displacement and velocity by the time it hits the ground in the $\hat{\imath} - \hat{\jmath}$ notation

$$\Delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath}$$
$$1400\ m\ \hat{\imath} - 10\ m\ \hat{\jmath}$$

$$\vec{v}_f = v_{fx}\hat{\imath} + v_{fy}\hat{\jmath}$$
$$= (1000\hat{\imath} - 14\hat{\jmath})\ m/s$$

Example: A bullet is fired from the ground making an angle of 37° with the horizontal and a speed of 1000 m/s.



a) How high will it rise?

$v_i = 1000$ m/s $\qquad \theta_i = 37°$

$v_{ix} = 1000\cos 37 = 800$ m/s

$v_{iy} = 1000\sin 37 = 600$ m/s

$v_{fy} = 0$ (the vertical component of its velocity at the maximum height is zero)

$$v_{fy}^2 = v_{iy}^2 + 2\,g\,\Delta y$$
$$0^2 = 600^2 + 2(-10)\Delta y$$
$$-36000 = -20\Delta y$$
$$\Delta y = 18000\ m$$

b) How long does it take to reach its maximum height?

$$v_{fy} = v_{iy} + g\ t$$
$$0 = 600 - 10t$$
$$t = 60\ sec$$

c) How far did it fall? (horizontally)

t = twice the time for maximum height

$= 2(60) = 120\ sec$

$$\Delta x = ??$$
$$\Delta x = v_{ix}t = 800 * 120 = 96000$$

Example: A bullet is fired from a 100 m tall building with a speed of 1000 m/s making an angle of 53° with the horizontal right.

a) Determine the time taken to hit the ground

$v_i = 1000$ m/s $\quad \theta_i = 53°$

$v_{ix} = v_i \cos\theta_i = 1000 \cos 53 = 600$ m/s

$v_{iy} = v_i \sin\theta_i = 1000 \sin 53 = 800$ m/s

$\Delta y = -100m$ $v_{fy}^2 = v_i^2 + 2g\Delta y$ $v_{fy}^2 = 800^2 + 2(-10)(-100)$ $v_{fy}^2 = 642000$ $v_{fy} = \sqrt{642000} \approx -800$	$v_{fy} = v_{iy} + gt$ $-800 = 800 - 10t$ $t = 160\ sec$

b) How far (horizontally did it fall?

$\Delta x = ??$

$\Delta x = v_{ix} t$

$\Delta x = 600 * 160 = 96000$ meters

4.4 Circular Motion

Circular motion is motion in a circular path.

Uniform Circular Motion: is motion in a circular path with a constant speed. The time taken for one complete revolution is called period (T). The number of cycles executed per second is called frequency (f). Unit of measurement for frequency is 1/sec which is defined to be the Hertz (Hz). Period and frequency are inverses of each other.

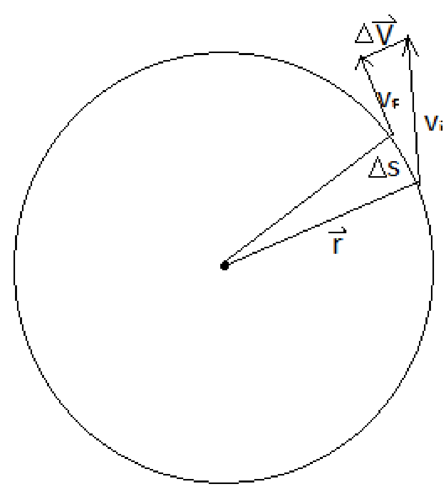
$$\boxed{f = \frac{1}{T}}$$

The number of radians executed per second is called angular speed (ω). Since there are 2π radians in a revolution

$$\boxed{\omega = 2\pi f = \frac{2\pi}{T}}$$

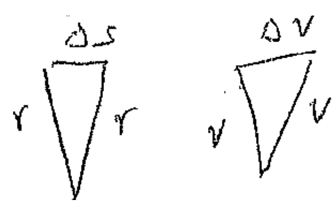
The period of a circular motion can be obtained as a ratio between circumference and speed (v)

$\boxed{T = \frac{2\pi r}{v}}$ r stands for the radius of a circle



Uniform Circular motion is an accelerated motion even though the speed is constant because direction is changing constantly. Acceleration due to change in direction is called centripetal or radial acceleration.

The diagram shows the change in the direction of the velocity in a small interval of time, Δt. The direction of centripetal acceleration is always towards the center. Since the velocity and the



position vector (radius) turn by the same angle in a time Δt, we have the following two similar triangles.

Since corresponding sides of two similar triangles are proportional.

$$\frac{\Delta v}{v} = \frac{\Delta s}{r} \Rightarrow \Delta v = \Delta s * \frac{v}{r}$$

$$\frac{\Delta v}{\Delta t} = \frac{\Delta s}{\Delta t}\frac{v}{r} \quad but \frac{\Delta s}{\Delta t} = v \quad \& \ \frac{\Delta v}{\Delta t} = a_c$$

$$\boxed{a_c = \frac{v^2}{r}}$$

$a_c \rightarrow$ centripetal acceleration
$v \rightarrow$ speed
$r \rightarrow$ radius

Example: A particle is revolving in a circular path of radius 4m with a constant speed. If it makes 10 revolutions in 10 seconds, calculate its centripetal acceleration.

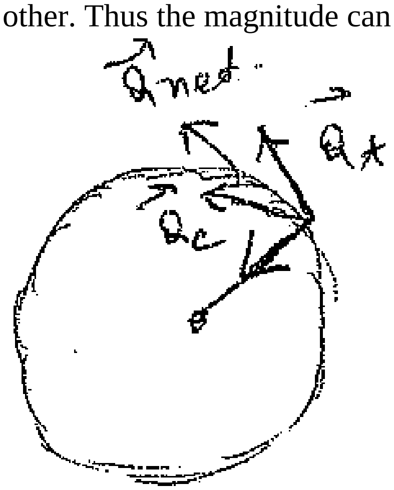
$r = 4m$	$T = \frac{10s}{10rev} = 1\ sec$	$a_c = ??$
$a_c = \frac{v^2}{r}$	$v = \frac{2\pi r}{T} = \frac{2\pi(4)}{1} = 8\pi$ m/s	

$$a_c = \frac{(8\pi)^2}{4} \approx 157.914 \text{ m}/s^2$$

<u>Non Uniform Circular Motion</u>: is a circular motion where the speed is not constant. In this case, there are two kinds of acceleration. The acceleration due to change of directions which is called centripetal acceleration, and the acceleration due to change of speed which is called tangential acceleration. The direction of tangential acceleration is always tangent to the circular path, while the direction of centripetal acceleration is radial or towards the center. Therefor centripetal and tangential acceleration are perpendicular to each other. Thus the magnitude can be obtained from the Pythagorean Theorem.

$$\boxed{a_{net} = \sqrt{a_c^2 + a_t^2}}$$

$a_{net} \longrightarrow$ net acceleration
$a_c \rightarrow$ centripetal acceleration
$a_t \rightarrow$ tangential acceleration



Example: The speed of a particle travelling in a circular path of radius 2 m changed uniformly from 5 m/s to 10m/s in 10 seconds

a) Calculate its tangential acceleration

$v_i = 5$ m/s $\qquad v_f = 10$ m/s $\qquad t = 10\ seconds$

$a_t = ??$

$$a_t = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t} = \frac{10 - 5}{10} = .5\ m/s^2$$

b) Calculate its initial centripetal acceleration

a_c =?? $v_i = 5$ m/s $r = 2m$

$$a_c = \frac{v_i^2}{r} = \frac{5^2}{2} = 12.5 \text{ m/}s^2$$

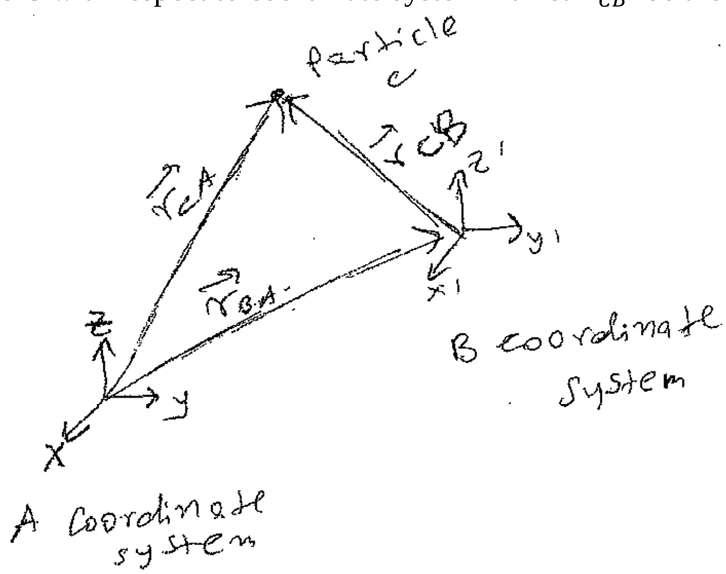
c) Calculate its initial net acceleration

$a_{net} = \sqrt{a_c^2 + a_t^2} = \sqrt{12.5^2 + .5^2} \approx 12.5\, m/s^2$

4.5 Relative Velocity

Let $\vec{r}_{CA}$ be the position vector of particle C with respect to coordinate system A. Let $\vec{r}_{CB}$ be the position vector of particle C with respect to coordinate system B. And let $\vec{r}_{BA}$ be the position vector of coordinate system B with respect to coordinate system A. Then from vector addition

$$\boxed{\vec{r}_{CA} = \vec{r}_{BA} + \vec{r}_{CB}}$$



Taking the change of both sides

$$\Delta \vec{r}_{CA} = \Delta \vec{r}_{BA} + \Delta \vec{r}_{CB}$$

Diving both sides by the time interval for the change

$$\frac{\Delta\vec{r}_{CA}}{\Delta t} = \frac{\Delta\vec{r}_{BA}}{\Delta t} + \frac{\Delta\vec{r}_{CB}}{\Delta t}$$

But $\frac{\Delta\vec{r}_{CA}}{\Delta t} = \vec{v}_{CA}; \frac{\Delta\vec{r}_{BA}}{\Delta t} = \vec{v}_{BA}; \frac{\Delta\vec{r}_{CB}}{\Delta t} = \vec{v}_{CB}$

$$\boxed{\vec{v}_{CA} = \vec{v}_{BA} + \vec{v}_{CB}}$$

$\vec{v}_{CA} \longrightarrow$ velocity of particle C wrt A
$\vec{v}_{BA} \longrightarrow$ velocity of B wrt A
$\vec{v}_{CB} \longrightarrow$ velocity of C wrt B

Example: Car A is moving to the right with a speed of 20 m/s. Car B is traveling to the right with a speed of 50 m/s.

Calculate the speed of car B with respect to car A.

Let the speed of car A wrt to the ground be $\vec{v}_{A\,g}$

Let the speed of car B wrt to the ground be $\vec{v}_{B\,g}$

Let the speed of car B wrt to the car A be $\vec{v}_{BA}$

$\vec{v}_{A\,g} = 20$ m/s $\hat{\imath}$ $\vec{v}_{B\,g} = 50$ m/s $\hat{\imath}$ $\vec{v}_{BA}$= ??

$$\vec{v}_{Bg} = \vec{v}_{BA} + \vec{v}_{Ag}$$

$$\vec{v}_{Bg} - \vec{v}_{Ag} = \vec{v}_{BA}$$

$$(50\text{m/s } \hat{\imath}) - (20 \text{ m/s } \hat{\imath}) = \vec{v}_{BA}$$

$$30 \text{ m/s } \hat{\imath} = \vec{v}_{BA}$$

Example: Car A is travelling with a speed of 50 m/s in a direction of 37° north of east. Car B is travelling with a speed of 40 m/s in a direction of west.

a) Determine the velocity of car B with respect to car A

Solution

$\vec{v}_{Ag}$ = 50 m/s 37° north of east
$= (50 \cos 37\,\hat{\imath} + 50 \sin 37\,\hat{\jmath})$
$\vec{v}_{Ag}$ = 40 m/s $\hat{\imath}$ + 30 m/s $\hat{\jmath}$

$\vec{v}_{Bg}$ = 40 m/s west
= -40 m/s $\hat{\imath}$

$$\vec{v}_{Bg} = \vec{v}_{BA} + \vec{v}_{Ag}$$
$$\vec{v}_{BA} = \vec{v}_{Bg} - \vec{v}_{Ag}$$
$$= (-40 \text{ m/s } \hat{\imath}) - (40 \text{ m/s } \hat{\imath} + 30 \text{ m/s } \hat{\jmath})$$
$$= (-80\,\hat{\imath} - 30\,\hat{\jmath}) \text{ m/s}$$

b) Determine the velocity of car A with respect to car B

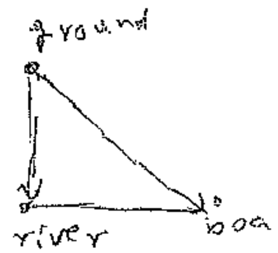
Solution

$\vec{v}_{AB}$ = ??

$$\vec{v}_{AB} = -\vec{v}_{BA}$$
$$= -(-80\,\hat{\imath} - 30\,\hat{\jmath}) \text{ m/s}$$
$$= (80\,\hat{\imath} + 30\,\hat{\jmath}) \text{ m/s}$$

Example: A river is flowing towards south with a speed of 5 m/s. A boat is travelling east with a speed of 8 m/s with respect to the river.

a) Determine the velocity of the boat with respect to the ground.



$$\vec{v}_{bg} = \vec{v}_{rg} + \vec{v}_{br}$$

$\vec{v}_{bg} \longrightarrow$ velocity of boat wrt ground
$\vec{v}_{rg} \longrightarrow$ velocity of river wrt ground
$\vec{v}_{br} \longrightarrow$ velocity of boat wrt river

$\vec{v}_{rg}$ = 5 m/s south
= -5 m/s $\hat{\jmath}$

$\vec{v}_{br}$ = 8 m/s $\hat{\imath}$
$\vec{v}_{bg}$ = ??

$$\vec{v}_{bg} = \vec{v}_{rg} + \vec{v}_{br}$$
$$= (-5 \text{ m/s}\hat{\jmath}) + (8 \text{ m/s } \hat{\imath})$$
$$= (8\hat{\imath} - 5\hat{\jmath}) \text{ m/s}$$

b) How long with the boat take to cross the river if the width of the river is 40 m.

Solution

width = $|\vec{v}_{br}|$(time)

time = $\dfrac{40m}{8\,\frac{m}{s}} = 5$ seconds

Note: $\vec{v}_{br}$ is selected for the velocity because it is velocity across the river.