Section 2.7 – Derivatives of Exponential and Logarithmic Functions

Derivative of the Natural Exponential Function

$$\frac{d}{dx}\left[e^x\right]=e^x \qquad\qquad \frac{d}{dx}\left[e^U\right]=e^U\,\frac{dU}{dx}$$

Derivative of a^x

$$\frac{d}{dx}\left(a^x\right)=(\ln a)\,a^x \qquad\qquad \frac{d}{dx}\left[a^{g(x)}\right]=\ln(a)a^{g(x)}g'(x)$$

Example

Find the derivative of each function

a) $y=e^{5x}$

$$y'=(5x)'\,e^{5x}$$

$$=5e^{5x}$$

b) $s=3^t$

$$\frac{ds}{dt}=(\ln 3)3^t$$

c) $y=10e^{3x^2}$

$$y'=10e^{3x^2}\left(3x^2\right)'$$

$$=10e^{3x^2}(6x)$$

$$=60x\,e^{3x^2}$$

d) $s=8.10^{1/t}$

$$\frac{ds}{dt}=8(\ln 10)10^{1/t}\left(t^{-1}\right)'$$

$$=-\frac{8(\ln 10)10^{1/t}}{t^2}$$

Example

Find the derivative of $y = e^{x^2+1}\sqrt{5x+2}$

<u>*Solution*</u>

$$f = e^{x^2+1} \qquad\qquad g = (5x+2)^{1/2}$$

$$f' = 2xe^{x^2+1} \qquad g' = \frac{1}{2}5(5x+2)^{-1/2} = \frac{5}{2\sqrt{5x+2}}$$

$$y' = (2x)e^{x^2+1}\sqrt{5x+2} + e^{x^2+1}\frac{5}{2\sqrt{5x+2}}$$

$$= 2xe^{x^2+1}\sqrt{5x+2}\,\frac{2\sqrt{5x+2}}{2\sqrt{5x+2}} + \frac{5e^{x^2+1}}{2\sqrt{5x+2}}$$

$$= \frac{4xe^{x^2+1}(5x+2)}{2\sqrt{5x+2}} + \frac{5e^{x^2+1}}{2\sqrt{5x+2}}$$

$$= \frac{20x^2e^{x^2+1} + 8xe^{x^2+1} + 5e^{x^2+1}}{2\sqrt{5x+2}}$$

$$= \frac{e^{x^2+1}\left(20x^2+8x+5\right)}{2\sqrt{5x+2}}$$

Example

The demand function for the product is modeled by $p = 50e^{-0.0000125x}$ where p is the price per unit in dollars and x is the number of units. What price will yield maximum revenue?

<u>*Solution*</u>

$$R = xp = 50xe^{-0.0000125x}$$

$$R' = 50e^{-0.0000125x} + (-0.0000125)50xe^{-0.0000125x}$$

$$R' = 50e^{-0.0000125x} - 0.000625xe^{-0.0000125x}$$

$$R' = e^{-0.0000125x}(50 - 0.000625x) = 0$$

$$50 - 0.000625x = 0$$

$$-0.000625x = -50$$

$$x = \frac{-50}{-0.000625} = 80000$$

$$p(x = 80000) = 50e^{-0.0000125(80000)}$$

$$\approx \$18.39 \ / \ unit$$

Example

A company sells 990 units of a new product in the first year and 3213 units in the fourth year. They expect that sales can be approximated by a logistic function, leveling off at around 100,000 in the long run given by the formula

$$S(t)=\frac{100,000}{1+100e^{-100,000kt}}$$

a) Find ***k*** and rewrite the function

<u>*Solution*</u>

$$S(4)=3213$$

$$3213=\frac{100,000}{1+100e^{-100,000k4}}$$

$$3213=\frac{100,000}{1+100e^{-400,000k}}$$

$$3213+321300e^{-400,000k}=100,000$$ *Subtract 3213 from both sides*

$$321300e^{-400,000k}=96787$$ *Divide both sides by 321300*

$$e^{-400,000k}=0.3012$$ *ln both sides*

$$-400,000k=\ln 0.3012$$

$$k=\frac{\ln 0.3012}{-400,000}$$

$$\approx 3\times 10^{-6}$$

$$S(t)=\frac{100,000}{1+100e^{-0.3t}}$$

b) Find the rate of change of sales after 4 years

<u>*Solution*</u>

$$S'=\frac{3,000,000e^{-0.3t}}{\left(1+100e^{-0.3t}\right)^2}$$

$$S'(4)=\frac{3,000,000e^{-0.3(4)}}{\left(1+100e^{-0.3(4)}\right)^2}$$ $3000000e((-)0.3*4)/\left(1+100e((-)0.3*4)\right)\wedge 2$

$$\approx 933$$

Derivatives of Logarithmic

Derivative **of** $\log_a x$

$$\frac{d}{dx}\left[\log_a |x|\right] = \frac{1}{(\ln a)x} \qquad \frac{d}{dx}\left[\log_a |g(x)|\right] = \frac{1}{(\ln a)} \bullet \frac{g'(x)}{g(x)}$$

Derivative of ln

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx} = \frac{u'}{u}$$

Other Bases

$$\frac{d}{dx}\left[a^x\right] = a^x \ln a \qquad \frac{d}{dx}\left[a^u\right] = a^u (\ln a) \frac{du}{dx}$$

Example

Find the derivative of each function

a) $f(x) = \ln 6x$

$$f'(x) = \frac{(6x)'}{6x}$$

$$= \frac{6}{6x}$$

$$= \frac{1}{x}$$

b) $f(x) = \log x$

$$f' = \frac{1}{(\ln 10)x}$$

c) $f(x) = \ln\left(x^2+1\right)$ $\qquad \frac{d}{dx}(\ln u) = \frac{u'}{u}$

$$f' = \frac{2x}{x^2+1}$$

Example

Find the derivative of function $f(x)=\log_2\left(3x^2-4x\right)$

<u>*Solution*</u>

$$f'=\frac{1}{\ln 2}\frac{6x-4}{3x^2-4x}$$

$$\frac{d}{dx}\left[\log_a|g(x)|\right]=\frac{1}{(\ln a)}\cdot\frac{g'(x)}{g(x)}$$

$$=\frac{6x-4}{\ln 2\left(3x^2-4x\right)}$$

Example

Find the derivative of function $y=\ln|5x|$

<u>*Solution*</u>

$$y'=\frac{5}{5x}$$

$$=\frac{1}{x}$$

Example

Find the derivative of function $y=3x\ln x^2$

<u>*Solution*</u>

$$y'=3\ln x^2+3x\frac{2x}{x^2}$$

$$=3\ln x^2+6$$

Example

Find the derivative of function $y=x\ 3^{x+1}$

<u>*Solution*</u>

$$f=x \qquad g=3^{x+1}$$

$$f'=1 \quad g'=3^{x+1}\ln(3)$$

$$y'=3^{x+1}+x\ 3^{x+1}\ln 3$$

$$=3^{x+1}\left[1+x\ \ln 3\right]$$

Example

Find the derivative of function $s(t)=\dfrac{\log_8\left(t^{3/2}+1\right)}{t}$

<u>*Solution*</u>

$$f=\log_8\left(t^{3/2}+1\right) \qquad g=t$$

$$f'=\frac{1}{\ln 8}\frac{\frac{3}{2}t^{1/2}}{t^{3/2}+1} \qquad g=1$$

$$s'=\frac{\frac{1}{\ln 8}\frac{\frac{3}{2}t^{1/2}}{t^{3/2}+1}\bullet t-\log_8\left(t^{3/2}+1\right)}{t^2}$$

$$=\frac{\frac{1}{\ln 8}\frac{\frac{3}{2}t^{3/2}}{t^{3/2}+1}-\log_8\left(t^{3/2}+1\right)}{t^2}\bullet\frac{2\ln 8\left(t^{3/2}+1\right)}{2\ln 8\left(t^{3/2}+1\right)}$$

$$\boxed{=\frac{3t^{3/2}-2\left(t^{3/2}+1\right)(\ln 8)\log_8\left(t^{3/2}+1\right)}{t^2\left(t^{3/2}+1\right)\ln 8}}$$

Exercises *Section* 2.7 – Derivatives of Exponential and Logarithmic Functions

Find the derivative:

1. $f(x)=e^{3x}$

2. $f(x)=e^{-2x^3}$

3. $f(x)=4e^{x^2}$

4. $f(x)=e^{-2x}$

5. $f(x)=x^2e^x$

6. $f(x)=\frac{e^x+e^{-x}}{2}$

7. $f(x)=\frac{e^x}{x^2}$

8. $f(x)=x^2e^x-e^x$

9. $f(x)=(1+2x)e^{4x}$

10. $y=x^2e^{5x}$

11. $f(x)=\frac{100,000}{1+100e^{-0.3x}}$

12. $y=x^2e^{-2x}$

13. $y=\frac{e^x+e^{-x}}{x}$

14. $y=\sqrt{e^{2x^2}+e^{-2x^2}}$

15. $y=\frac{x}{e^{2x}}$

16. $y=\ln\sqrt{x+5}$

17. $y=(3x+7)\ln(2x-1)$

18. $y=e^{x^2}\ln x$

19. $y=\log_7\sqrt{4x-3}$

20. $f(x)=\ln\sqrt[3]{x+1}$

21. $f(x)=\ln\left[x^2\sqrt{x^2+1}\right]$

22. $y=\ln\dfrac{1+e^{x}}{1-e^{x}}$

23. $y=\ln\dfrac{x^{2}}{x^{2}+1}$

24. $y=\ln\left[\dfrac{x^{2}(x+1)^{3}}{(x+3)^{1/2}}\right]$

25. $y=\ln(x^{2}+1)$

26. $y=\dfrac{\ln x}{e^{2x}}$

27. $f(x)=\ln\left(x^{2}-4\right)$

28. $f(x)=x^{2}\ln x$

29. $f(x)=-\dfrac{\ln x}{x^{2}}$

30. $f(x)=\dfrac{e^{\sqrt{x}}}{\ln\left(\sqrt{x}+1\right)}$

31. $f(x)=e^{2x}\ln\left(xe^{x}+1\right)$

32. $f(x)=\dfrac{xe^{x}}{\ln\left(x^{2}+1\right)}$

33. $f(x)=2\ln\left(x^{2}-3x+4\right)$

34. $f(x)=e^{x^{2}+3x+1}$

35. $f(x)=3\ln\left(1+x^{2}\right)$

36. $f(x)=(1+\ln x)^{3}$

37. $f(x)=(x-2\ln x)^{4}$

38. $f(x)=\dfrac{e^{x}}{x^{2}+1}$

39. $f(x)=\dfrac{1-e^{x}}{1+e^{x}}$

40. $f(x)=\dfrac{\ln x}{1+x}$

41. $f(x)=\dfrac{2x}{1+\ln x}$

42. $f(x) = x^2 e^x$

43. $f(x) = x^3 \ln x$

44. $f(x) = 6x^4 \ln x$

45. $f(x) = 2x^3 e^x$

46. $f(x) = \dfrac{3e^x}{1+e^x}$

47. $f(x) = 5e^x + 3x + 1$

48. $f(x) = \dfrac{\ln x}{2x+5}$

49. $f(x) = -2\ln x + x^2 - 4$

50. $f(x) = e^x + x - \ln x$

51. $f(x) = \ln x + 2e^x - 3x^2$

52. $f(x) = \ln x^8$

53. $f(x) = 5x - \ln x^5$

54. $f(x) = \ln x^2 + 4e^x$

55. $f(x) = \ln x^{10} + 2\ln x$

56. Find the second derivative of $y = 3e^{5x^3+1}$

57. Find the equation of the tangent line to $f(x) = e^x$ at the point (0, 1)

58. Find the equation of the tangent line to $f(x) = e^x$ at the point (1, e)

59. Find the equation of the tangent lines to $f(x) = 4e^{-8x}$ at the points (0, 4)

60. Find the equation of the tangent line to $y = 4xe^{-x} + 5$ at $x = 1$

61. The percentage of people of any particular age group that will die in a given year may be approximated by the formula

$$P(t) = 0.00239e^{0.0957t}$$

where t is the age of the person in years

a) Find $P(25)$

b) Find $P'(25)$

62. Assume the cost of a gallon of milk is \$2.90. With continuous compounding, find the time it would take the cost to be 5 times as much (to the nearest tenth of a year), at an annual inflation rate of 6 %.

63. The sales in thousands of a new type of product are given by $S(t)=30-90e^{-0.5t}$, where t represents time in years. Find the rate of change of sales at the time when $t=3$

64. A company's total cost, in millions of dollars, is given by $C(t)=300-70e^{-t}$ where t = time in years. Find the marginal cost when $t=3$.

65. A company's total cost, in millions of dollars, is given by $C(t)=280-30e^{-t}$ where t = time in years. Find the marginal cost when $t=2$.

66. The demand function for a certain book is given by the function $x=D(p)=70e^{-0.006p}$. Find the marginal demand $D'(p)$

67. Suppose that the amount in grams of a radioactive substance present at time t (in years) is given by $A(t)=840e^{-0.63t}$. Find the rate of change of the quantity present at the time when $t=2$.

68. Researchers have found that the maximum number of successful trials that a laboratory rat can complete in a week is given by

$$P(t)=53\left(1-e^{-0.4t}\right)$$

where t is the number of weeks the rat has been trained. Find the rate of change $P'(t)$.

69. When a telephone wire is hung between two poles, the wire forms a U-shape curve called a Catenary. For instance, the function $y=30\left(e^{x/60}+e^{-x/60}\right)\quad -30\le x\le 30$ models the shape of the telephone wire strung between two poles that are 60 ft. apart (x & y are measured in ft.). Show that the lowest point on the wire is midway between two poles. How much does the wire sag between the two poles?

70. Find $f''(x)$ for $f(x)=\dfrac{\ln x}{7x}$, then find $f''(0)$ and $f''(2)$

71. Suppose the average test score p and was modeled by $p=92.3-16.9\ln(t+1)$, where t is the time in months. How would the rate at which the average test score changed after 1 year?

72. Suppose that the population of a certain collection of rare ants is given by

$$P(t)=(t+100)\ln(t+2)$$

Where t represents the time in days. Find the rates of change of the population on the second day and on the eighth day.

73. Suppose that the demand function for x units of a certain item is $P(x)=100+\dfrac{180\ln(x+5)}{x}$ where P is the price per unit, in dollars. Find the marginal revenue.

74. The population of coyotes in the northwestern portion of Alabama is given by the formula $P(t) = \left(t^2 + 100\right)\ln\left(t+2\right)$, where *t* represents the time in years since 2000 (the year 2000 corresponds to $\left(t = 0\right)$ Find the rate of change of the coyote population in 2013 $\left(t = 13\right)$.

75. Students in a math class took a final exam. They took equivalent forms of the exam in monthly intervals thereafter. The average score $S(t)$, in percent, after t months was found to be given by

$$S(t) = 73 - 17\ln\left(t+1\right), \qquad t \geq 0$$

Find $S'(t)$.

76. Suppose that the population of a town is given by $P(t) = 8\ln\sqrt{8t+7}$ where *t* is the time in years after 1980 and *P* is the population of the town in thousands. Find $P'(t)$.

77. The following formula accurately models the relationship between the size of a certain type of tumor and the amount of time that it has been growing:

$$V(t) = 450\left(1 - e - 0.0022t\right)^3$$

where *t* is in months and $V(t)$ is measured in cubic centimeters. Calculate the rate of change of tumor volume at 80 months.

78. A yeast culture at room temperature (68° F) is placed in a refrigerator set at a constant temperature of 38° F. After *t* hours, the temperature *T* of the culture is given approximately by

$$T = 30e^{-0.58t} + 38 \quad t \geq 0$$

What is the rate of change of temperature of the culture at the end of 1 hour? At the end of 4 hours?

79. A mathematical model for the average age of a group of people learning to type is given by

$$N\left(t\right) = 10 + 6\ln t \quad t \geq 1$$

Where $N\left(t\right)$ is the number of words per minute typed after *t* hours of instruction and practice (2 hours per day, 5 days per week). What is the rate of learning after 10 hours of instruction and practice? After 100 hours?