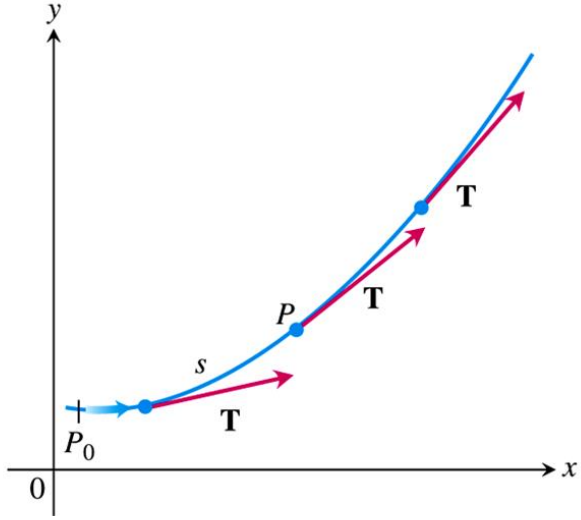
Section 1.8 – Curvature and Normal Vectors

Curvature of a Plane Curve

As a particle moves along a smooth curve in the plane, $\boldsymbol{T} = \frac{d\boldsymbol{r}}{ds}$ turns as the curve bends. Since $\boldsymbol{T}$ is a unit vector, its length remains constant and only its direction changes as particle moves along the curve. The rate at which $\boldsymbol{T}$ turns per unit of length along the curve is called the ***curvature***.



Definition

If $\boldsymbol{T}$ is the unit vector of a smooth curve, the ***curvature*** function of the curve is

$$\kappa = \left| \frac{d\boldsymbol{T}}{ds} \right|$$

Formula for Calculating Curvature

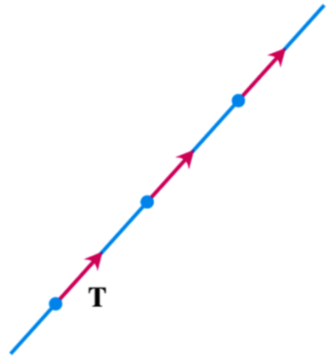
If $\boldsymbol{r}(t)$ is a smooth curve, then the curvature is

$$\kappa = \frac{1}{|v|} \left| \frac{d\boldsymbol{T}}{dt} \right|$$

Where $\boldsymbol{T} = \frac{v}{|v|}$ is the unit tangent vector.

Example

A straight line is parametrized by $\boldsymbol{r}(t)=\boldsymbol{C}+t\boldsymbol{v}$ for constant vectors $\boldsymbol{C}$ and $\boldsymbol{v}$. Thus $\boldsymbol{r}'(t)=\boldsymbol{v}$, and the unit tangent vector $\boldsymbol{T}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}$ is a constant vector that always points in the same direction and has derivative $\mathbf{0}$. It follows that, for any value of the parameter t, the curvature of the straight line is



$$\kappa=\frac{1}{|v|}\left|\frac{d\boldsymbol{T}}{dt}\right|=\frac{1}{|v|}|\mathbf{0}|=0$$

Example

Find the curvature of a circle $\boldsymbol{r}(t)=(a\cos t)\boldsymbol{i}+(a\sin t)\boldsymbol{j}$ of radius a.

Solution

$$\boldsymbol{v}(t)=\frac{d\boldsymbol{r}}{dt}=-(a\sin t)\boldsymbol{i}+(a\cos t)\boldsymbol{j}$$

$$\begin{aligned}|\boldsymbol{v}|&=\sqrt{(-a\sin t)^2+(a\cos t)^2}\\&=\sqrt{a^2\sin^2 t+a^2\cos^2 t}\\&=|a|\sqrt{\sin^2+\cos^2 t}\\&=a\end{aligned}$$

Since a > 0

$$\boldsymbol{T}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}=-(\sin t)\boldsymbol{i}+(\cos t)\boldsymbol{j}$$

$$\frac{d\boldsymbol{T}}{dt}=-(\cos t)\boldsymbol{i}-(\sin t)\boldsymbol{j}$$

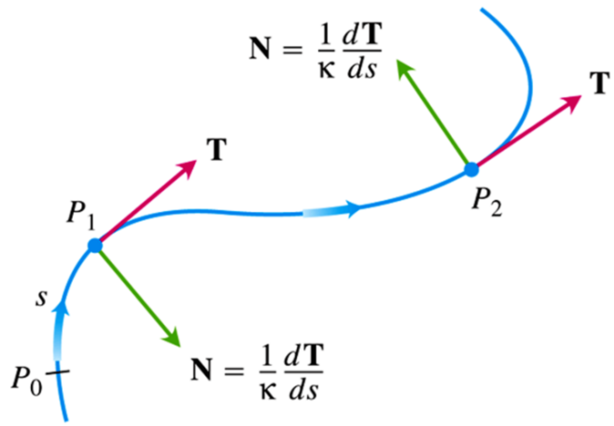
$$\left|\frac{d\boldsymbol{T}}{dt}\right|=\sqrt{\cos^2 t+\sin^2 t}=1$$

$$\begin{aligned}\kappa&=\frac{1}{|v|}\left|\frac{d\boldsymbol{T}}{dt}\right|\\&=\frac{1}{a}(1)\\&=\frac{1}{a}=\frac{1}{radius}\end{aligned}$$

Definition

At a point where $\kappa \neq 0$, the **principal unit normal** vector for a smooth curve in the plane is

$$\boxed{\boldsymbol{N} = \frac{1}{\kappa}\frac{d\boldsymbol{T}}{ds}}$$



Formula for Calculating *N*

If $\boldsymbol{r}(t)$ is a smooth curve, then the principal unit normal is

$$\boldsymbol{N} = \frac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|}$$

Where $\boldsymbol{T} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}$ is the unit tangent vector.

Example

Find $\boldsymbol{T}$ and $\boldsymbol{N}$ for the circular motion $\vec{r}(t) = (\cos 2t)\hat{i} + (\sin 2t)\hat{j}$

Solution

$$\vec{v}(t) = \vec{r}'(t) = -(2\sin 2t)\hat{i} + (2\cos 2t)\hat{j}$$

$$|\vec{v}| = \sqrt{4\sin^2 2t + 4\cos^2 2t} \underline{= 2}|$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = -(\sin 2t)\hat{i} + (\cos 2t)\hat{j}$$

$$\frac{d\vec{T}}{dt} = -(2\cos 2t)\hat{i} - (2\sin 2t)\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{4\cos^2 2t + 4\sin^2 2t} \underline{= 2}|$$

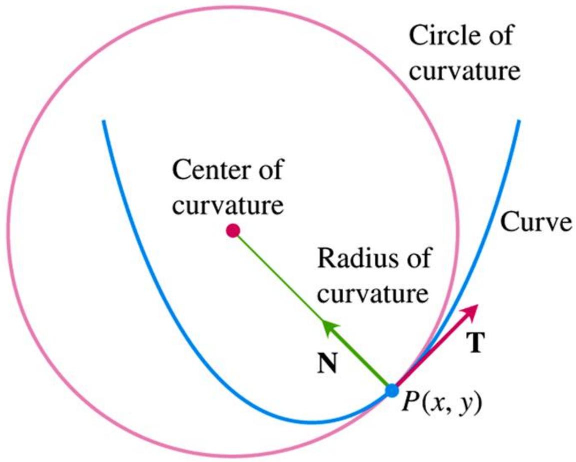
$$\vec{N} = \frac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|} = \frac{-(2\cos 2t)\hat{i} - (2\sin 2t)\hat{j}}{2}$$

$$\underline{= -(\cos 2t)\hat{i} - (\sin 2t)\hat{j}}|$$

Circle of Curvature for plane Curves

The ***circle of curvature*** or ***osculating circle*** at a point P on a plane where $\kappa \neq 0$ is the circle in the plane of the curve that

1. is tangent to the curve at P (has the same tangent line the curve has)
2. has the same curvature the curve has at P
3. lies toward the concave or inner side of the curve



The ***radius of curvature*** of the curve at P is the radius of the circle of curvature, which is

$$Radius\ of\ \ curvature = \rho = \frac{1}{\kappa}$$

To find ρ, we find κ and take the reciprocal. The ***center of curvature*** of the curve at P is the center of the circle of curvature.

Example

Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

Solution

Assume: $t = x$

$$\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j} = t\ \boldsymbol{i} + t^2\boldsymbol{j}$$

$$\boldsymbol{v} = \boldsymbol{r}' = \ \boldsymbol{i} + 2t\boldsymbol{j}$$

$$|\boldsymbol{v}| = \sqrt{1+4t^2}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{1+4t^2}}\hat{i} + \frac{2t}{\sqrt{1+4t^2}}\hat{j}$$

$$\frac{d\vec{T}}{dt} = -\frac{4t}{\left(1+4t^2\right)^{3/2}}\hat{i} + \frac{2\left(1+4t^2\right)^{1/2} - 8t^2\left(1+4t^2\right)^{-1/2}}{\left(1+4t^2\right)}\hat{j}$$

$$= -\frac{4t}{\left(1+4t^2\right)^{3/2}}\hat{i} + \frac{2\left(1+4t^2\right) - 8t^2}{\left(1+4t^2\right)^{3/2}}\hat{j}$$

$$= -\frac{4t}{\left(1+4t^2\right)^{3/2}}\hat{i} + \frac{2}{\left(1+4t^2\right)^{3/2}}\hat{j}$$

At the origin, $t = 0$, so the curvature is

$$\left.\frac{d\vec{T}}{dt}\right|_{t=0} = 0\hat{i} + 2\hat{j} \quad \underline{= 2\hat{j}\,\rfloor}$$

$$\kappa(0) = \frac{1}{|\vec{v}(0)|}\left|\frac{d\vec{T}}{dt}(0)\right|$$

$$= \frac{1}{\sqrt{1}}\left|2\hat{j}\right|$$

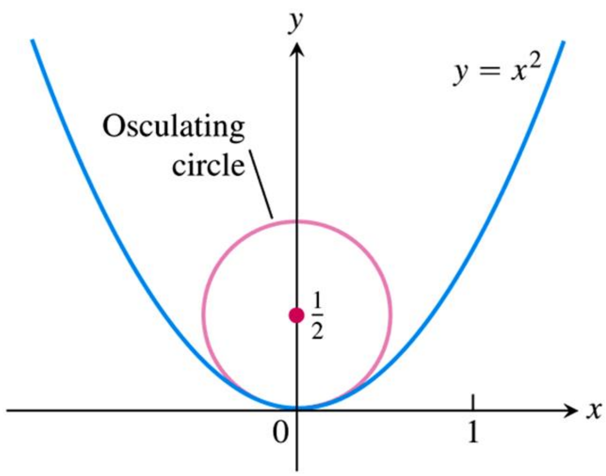
$$\underline{= 2\,\rfloor}$$

The radius of curvature is: $\rho = \frac{1}{\kappa} = \frac{1}{2}$

At the origin, t = 0, $\boldsymbol{T} = \boldsymbol{i} \quad \boldsymbol{N} = \boldsymbol{j}$

The center of the circle is $\left(0,\ \frac{1}{2}\right)$

The equation of the osculating circle is: $\underline{x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}\,\rfloor}$



Curvature and Normal Vectors for Space Curves

If a smooth curve in space is specified by the position $\boldsymbol{r}(t)$ as a function of somw parameter t, and if s is the arc length parameter of the curve, then the unit tangernt vector $\boldsymbol{T}$ is $\frac{d\boldsymbol{r}}{ds}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}$. The curvature in space is then defiined to be

$$\kappa=\left|\frac{d\boldsymbol{T}}{ds}\right|=\frac{1}{|\boldsymbol{v}|}\left|\frac{d\boldsymbol{T}}{dt}\right|$$

Just as for plane curves. The vector $\frac{d\boldsymbol{T}}{ds}$ is orthogonal to $\boldsymbol{T}$, and we define the ***principal unit normal*** to be

$$\boldsymbol{N}=\frac{1}{\kappa}\frac{d\boldsymbol{T}}{ds}=\frac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|}$$

Example

Find the curvature for the helix $\vec{r}(t)=(a\cos t)\hat{i}+(a\sin t)\hat{j}+bt\,\hat{k}, \quad a,b\ge 0, \quad a^2+b^2\ne 0$

Solution

$$\vec{v}=-(a\sin t)\hat{i}+(a\cos t)\hat{j}+b\,\hat{k}$$

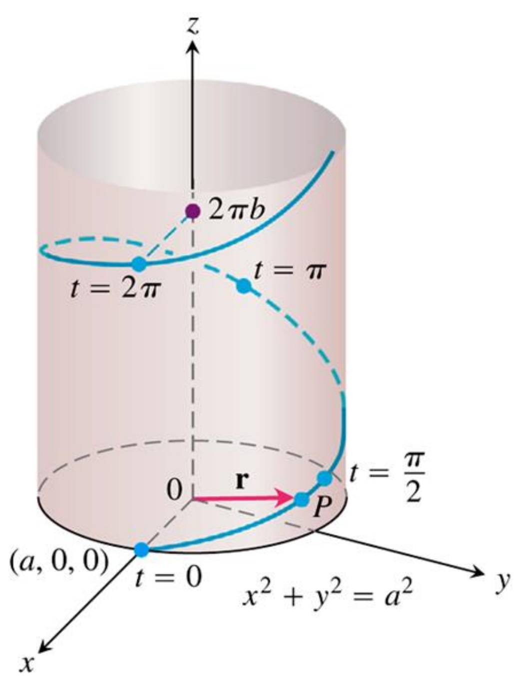
$$|\vec{v}|=\sqrt{a^2\sin^2 t+a^2\cos^2 t+b^2}$$

$$=\sqrt{a^2+b^2}$$

$$\vec{T}=\frac{\vec{v}}{|\vec{v}|}=\frac{1}{\sqrt{a^2+b^2}}\left(-(a\sin t)\hat{i}+(a\cos t)\hat{j}+b\,\hat{k}\right)$$

$$\frac{d\vec{T}}{dt}=\frac{1}{\sqrt{a^2+b^2}}\left(-(a\cos t)\hat{i}-(a\sin t)\hat{j}\right)$$

$$=\frac{-a}{\sqrt{a^2+b^2}}\left((\cos t)\hat{i}+(\sin t)\hat{j}\right)$$



$$\kappa=\frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

$$=\frac{1}{\sqrt{a^2+b^2}}\left|\frac{-a}{\sqrt{a^2+b^2}}\left((\cos t)\hat{i}+(\sin t)\hat{j}\right)\right|$$

$$=\frac{1}{\sqrt{a^2+b^2}}\frac{a}{\sqrt{a^2+b^2}}\sqrt{\sin^2 t+\cos^2 t}$$

$$\boxed{=\frac{a}{a^2+b^2}}$$

Example

Find $\boldsymbol{N}$ for the helix $\vec{r}(t) = (a\cos t)\hat{i} + (a\sin t)\hat{j} + bt\,\hat{k}$

Solution

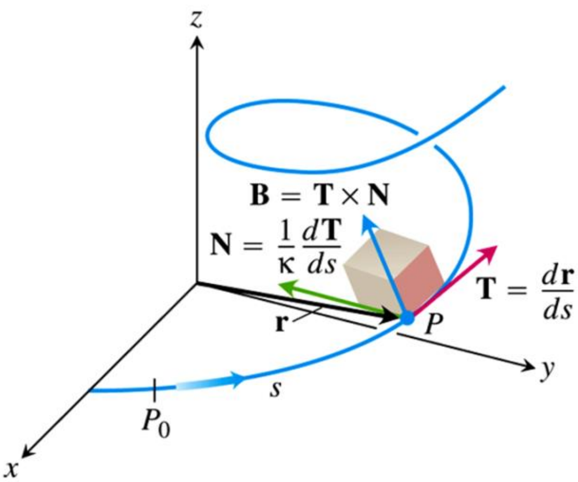
$$\frac{d\vec{T}}{dt} = \frac{-a}{\sqrt{a^2+b^2}}\left((\cos t)\hat{i} + (\sin t)\hat{j}\right)$$

$$\left|\frac{d\boldsymbol{T}}{dt}\right| = \frac{a}{\sqrt{a^2+b^2}}$$

$$\begin{aligned}
\vec{N} &= \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|} \\
&= \frac{-a}{\sqrt{a^2+b^2}}\left((\cos t)\hat{i} + (\sin t)\hat{j}\right) \bullet \frac{\sqrt{a^2+b^2}}{a} \\
&= -\left((\cos t)\hat{i} + (\sin t)\hat{j}\right) \\
&= -(\cos t)\hat{i} - (\sin t)\hat{j}
\end{aligned}$$

TNB Frame

The ***binormal vector*** of a curve in space $\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N}$, a unit vector orthogonal to both $\boldsymbol{T}$ and $\boldsymbol{N}$. Together ***T***, ***N***, and ***B*** define a moving right-handed vector frame that play a significant role in calculating the paths of particles moving through space. It is called the ***Frenet frame*** or ***TNB frame***.

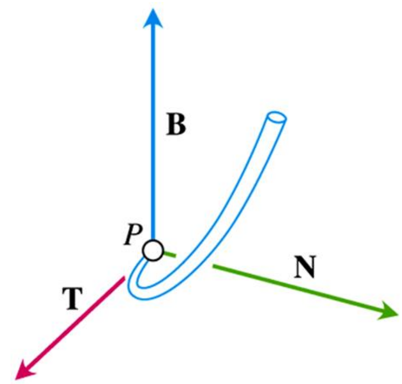


Tangential and Normal Components of Acceleration

When an object is accelerated by gravity, brakes, or a combination of rocket motors, how much of the acceleration acts in the direction of motion, in the tangential direction ***T***.

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}}{ds}\frac{ds}{dt} = \boldsymbol{T}\frac{ds}{dt}$$

$$\begin{aligned}
\vec{a} &= \frac{d\vec{v}}{dt} \\
&= \frac{d}{dt}\left(\boldsymbol{T}\frac{ds}{dt}\right) \\
&= \frac{d^2s}{dt^2}\boldsymbol{T} + \frac{ds}{dt}\frac{d\boldsymbol{T}}{dt} \\
&= \frac{d^2s}{dt^2}\boldsymbol{T} + \frac{ds}{dt}\left(\frac{d\boldsymbol{T}}{ds}\frac{ds}{dt}\right) \\
&= \frac{d^2s}{dt^2}\boldsymbol{T} + \frac{ds}{dt}\left(\kappa\boldsymbol{N}\frac{ds}{dt}\right) \\
&= \frac{d^2s}{dt^2}\boldsymbol{T} + \kappa\left(\frac{ds}{dt}\right)^2\boldsymbol{N}
\end{aligned}$$



Definition

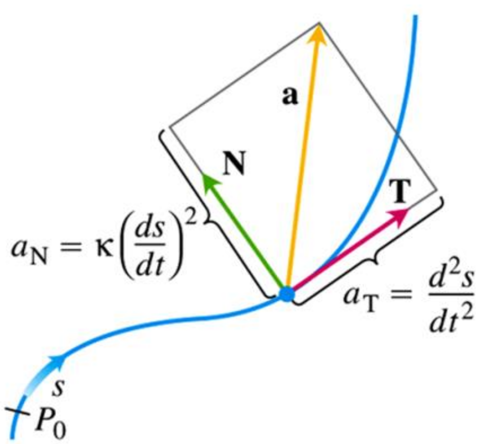
If the acceleration vector is written as

$$\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$$

then

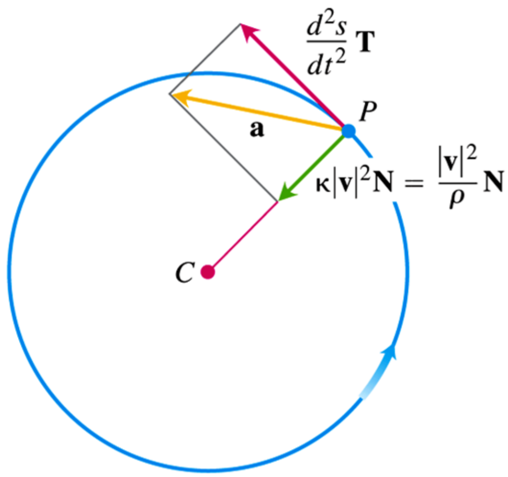
$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}|\boldsymbol{v}| \quad \textit{and} \quad a_N = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\boldsymbol{v}|^2$$

are the ***tangential*** and ***normal*** scalar components of acceleration.



Formula for Calculating the Normal Component of Acceleration

$$a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2}$$



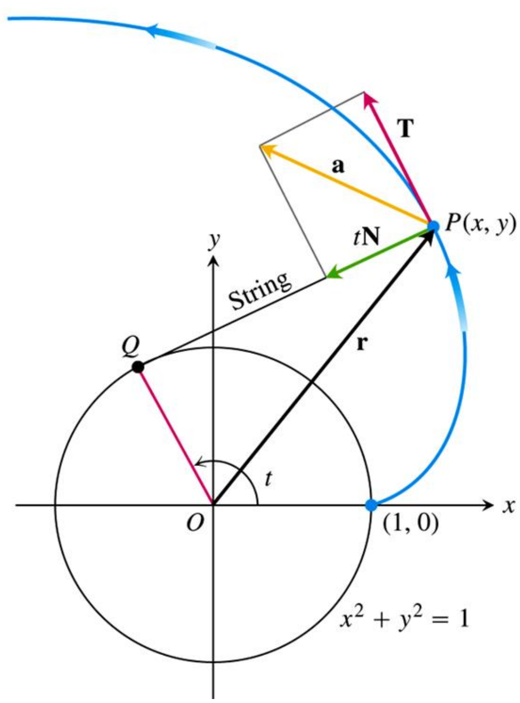
Example

Without finding $\boldsymbol{T}$ and $\boldsymbol{N}$, write the acceleration of the motion

$$\boldsymbol{r}(t) = (\cos t + t\sin t)\boldsymbol{i} + (\sin t - t\cos t)\boldsymbol{j}, \quad t > 0$$

In the form $\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$

Solution



$$\boldsymbol{v} = \boldsymbol{r}' = (-\sin t + \sin t + t\cos t)\boldsymbol{i} + (\cos t - \cos t + t\sin t)\boldsymbol{j}$$

$$= (t\cos t)\boldsymbol{i} + (t\sin t)\boldsymbol{j}$$

$$|\boldsymbol{v}| = \sqrt{t^2\cos^2 t + t^2\sin^2 t}$$

$$= \sqrt{t^2(\cos^2 t + \sin^2 t)}$$

$$= |t| \qquad \boldsymbol{t > 0}$$

$$= t$$

$$a_T = \frac{d}{dt}|\boldsymbol{v}|$$

$$= \frac{d}{dt}(t)$$

$$\underline{= 1|}$$

$$\boldsymbol{a} = \boldsymbol{v}' = (\cos t - t\sin t)\boldsymbol{i} + (\sin t + t\cos t)\boldsymbol{j}$$

$$|\boldsymbol{a}|^2 = (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2$$

$$= \cos^2 t - 2t\cos t\sin t + t^2\sin^2 t + \sin^2 t + 2t\cos t\sin t + t^2\cos^2 t$$

$$= 1 + t^2\left(\sin^2 t + \cos^2 t\right)$$

$$= 1 + t^2$$

$$a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2}$$

$$= \sqrt{1 + t^2 - 1}$$

$$\boxed{= t}$$

$$\boldsymbol{a} = a_T \boldsymbol{T} + a_N \boldsymbol{N}$$

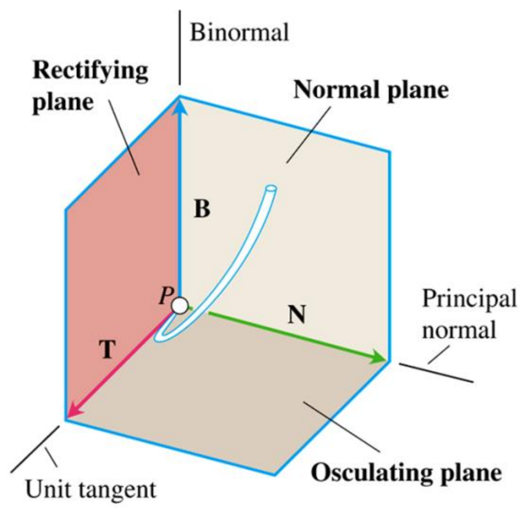
$$\boxed{= \boldsymbol{T} + t\boldsymbol{N}}$$

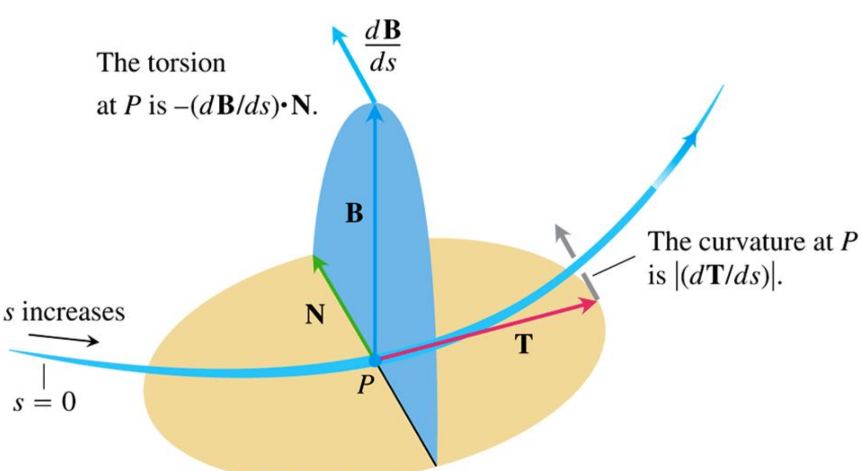
Torsion

Definition

Let $\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N}$. The torsion function of a smooth curve is

$$\tau = -\frac{d\boldsymbol{B}}{ds} \bullet \boldsymbol{N}$$





Computation Formulas for Curves in Space

Unit tangent vector: $\boldsymbol{T} = \dfrac{\boldsymbol{v}}{|\boldsymbol{v}|}$

Principal unit normal vector: $\boldsymbol{N} = \dfrac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|} = \dfrac{1}{\kappa}\dfrac{d\boldsymbol{T}}{ds}$

Binormal vector: $\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N} = \dfrac{1}{v}\left|\dfrac{d\boldsymbol{T}}{dt}\right|$

Curvature: $\kappa = \left|\dfrac{d\boldsymbol{T}}{ds}\right| = \dfrac{|\boldsymbol{v} \times \boldsymbol{a}|}{|\boldsymbol{v}|^3} = \dfrac{1}{|v|}\left|\dfrac{d\boldsymbol{T}}{dt}\right|$

Torsion: $\tau = -\dfrac{d\boldsymbol{B}}{ds} \bullet \boldsymbol{N} = \dfrac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\boldsymbol{v} \times \boldsymbol{a}|^2}$

Tangential and normal scalar components of acceleration:

$$\boldsymbol{a} = a_T \boldsymbol{T} + a_N \boldsymbol{N}$$

$$a_T = \frac{d}{dt}|\boldsymbol{v}|$$

$$a_N = \kappa|\boldsymbol{v}|^2 = \sqrt{|\boldsymbol{a}|^2 - a_T^2}$$

$$= |\boldsymbol{v}|\left|\frac{dT}{dt}\right|$$

Exercises Section 1.8 – Curvature and Normal Vectors

Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the plane curves:

1. $\boldsymbol{r}(t) = t\boldsymbol{i} + (\ln\cos t)\boldsymbol{j}, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$
2. $\boldsymbol{r}(t) = (\ln\sec t)\boldsymbol{i} + t\boldsymbol{j}, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$
3. $\boldsymbol{r}(t) = (2t+3)\boldsymbol{i} + \left(5 - t^2\right)\boldsymbol{j}$
4. $\boldsymbol{r}(t) = (\cos t + t\sin t)\boldsymbol{i} + (\sin t - t\cos t)\boldsymbol{j}, \quad t > 0$
5. $\boldsymbol{r}(t) = (3\sin t)\boldsymbol{i} + (3\cos t)\boldsymbol{j} + 4t\boldsymbol{k}$
6. $\boldsymbol{r}(t) = \left(e^t\cos t\right)\boldsymbol{i} + \left(e^t\sin t\right)\boldsymbol{j} + 2\boldsymbol{k}$
7. $\boldsymbol{r}(t) = \frac{t^3}{3}\boldsymbol{i} + \frac{t^2}{2}\boldsymbol{j}, \quad t > 0$
8. $\boldsymbol{r}(t) = \left(\cos^3 t\right)\boldsymbol{i} + \left(\sin^3 t\right)\boldsymbol{j}, \quad 0 < t < \frac{\pi}{2}$
9. $\boldsymbol{r}(t) = (\cosh t)\boldsymbol{i} - (\sinh t)\boldsymbol{j} + t\boldsymbol{k}$

10. Find an equation for the circle of curvature of the curve $\boldsymbol{r}(t) = t\boldsymbol{i} + (\sin t)\boldsymbol{j}$, at the point $\left(\frac{\pi}{2}, 1\right)$. (The curve parametrizes the graph $y = \sin x$ in the xy-plane.)

Write $\boldsymbol{a}$ of the motion $\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$ without finding $\boldsymbol{T}$ and $\boldsymbol{N}$.

11. $\vec{r}(t) = (a\cos t)\hat{i} + (a\sin t)\hat{j} + bt\hat{k}$
12. $\vec{r}(t) = (1+3t)\hat{i} + (t-2)\hat{j} - 3t\hat{k}$
13. $\vec{r}(t) = (t+1)\hat{i} + 2t\hat{j} + t^2\hat{k}, \quad t = 1$
14. $\vec{r}(t) = (t\cos t)\hat{i} + (t\sin t)\hat{j} + t^2\hat{k}, \quad t = 0$
15. $\vec{r}(t) = \left(e^t\cos t\right)\hat{i} + \left(e^t\sin t\right)\hat{j} + \sqrt{2}e^t\hat{k}, \quad t = 0$
16. $\vec{r}(t) = \left(2 + 3t + 3t^2\right)\hat{i} + \left(4t + 4t^2\right)\hat{j} - (6\cos t)\hat{k} \quad t = 0$
17. $\vec{r}(t) = (2+t)\hat{i} + \left(t + 2t^2\right)\hat{j} + \left(1 + t^2\right)\hat{k} \quad t = 0$

Graph the curves and sketch their velocity and acceleration vectors at the given values of t. Then write $\boldsymbol{a}$ of the motion $\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$ without finding $\boldsymbol{T}$ and $\boldsymbol{N}$, and find the value of κ at the given values of t.

18. $\vec{r}(t) = (4\cos t)\hat{i} + \left(\sqrt{2}\sin t\right)\hat{j}, \quad t = 0$ *and* $\frac{\pi}{4}$
19. $\vec{r}(t) = \left(\sqrt{3}\sec t\right)\hat{i} + \left(\sqrt{3}\tan t\right)\hat{j}, \quad t = 0$

Find $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{B}$, τ, and κ at the given value of t for the plane curves

20. $\vec{r}(t) = \frac{4}{9}(1+t)^{3/2}\hat{i} + \frac{4}{9}(1-t)^{3/2}\hat{j} + \frac{1}{3}t\hat{k}; \quad t = 0$
21. $\vec{r}(t) = \left(e^t\sin 2t\right)\hat{i} + \left(e^t\cos 2t\right)\hat{j} + 2e^t\hat{k}; \quad t = 0$

22. $\vec{r}(t) = t\,\hat{i} + \left(\frac{1}{2}e^{2t}\right)\hat{j}; \quad t = \ln 2$

Find $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{B}$, τ and κ at the given value of *t*. Then find equations for the osculating, normal, and rectifying planes at that value of *t*.

23. $\boldsymbol{r}(t) = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} - \boldsymbol{k}, \quad t = \frac{\pi}{4}$

24. $\boldsymbol{r}(t) = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} + t\boldsymbol{k}, \quad t = 0$

Find $\boldsymbol{B}$ and τ for:

25. $\boldsymbol{r}(t) = (3\sin t)\boldsymbol{i} + (3\cos t)\boldsymbol{j} + 4t\boldsymbol{k}$

26. $\boldsymbol{r}(t) = (\cos t + t\sin t)\boldsymbol{i} + (\sin t - t\cos t)\boldsymbol{j} + 3\boldsymbol{k}$

27. $\boldsymbol{r}(t) = (6\sin 2t)\boldsymbol{i} + (6\cos 2t)\boldsymbol{j} + 5t\boldsymbol{k}$

28. The speedometer on your car reads a steady 35 *mph*, could you be accelerating? Explain.

29. Can anything be said about the acceleration of a particle that is moving at a constant speed? Give reasons for your answer.

30. Find $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{B}$, τ and κ as functions of $\boldsymbol{t}$ for the plane curves: $\boldsymbol{r}(t) = (\sin t)\boldsymbol{i} + \left(\sqrt{2}\cos t\right)\boldsymbol{j} + (\sin t)\boldsymbol{k}$, then write $\boldsymbol{a}$ of the motion $\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$

31. Consider the ellipse $\vec{r}(t) = \langle 3\cos t,\ 4\sin t\rangle \quad for \quad 0 \le t \le 2\pi$

a) Find the tangent vector $\vec{r}'$, the unit vector $\vec{T}$, and the principal unit normal vector $\vec{N}$ at all points on the curve.

b) At what points does $|\vec{r}'|$ have maximum and minimum values?

c) At what points does the curvature have maximum and minimum values? Interpret this result in light of part (*b*).

d) Find the points (if any) at which $\vec{r}$ and $\vec{N}$ are parallel.

32. Find the following for all values of *t* for which the given curve is defined by $\vec{r}(t) = \langle 6\cos t,\ 3\sin t\rangle, \quad 0 \le t \le 2\pi$

a) Find the tangent vector and the unit tangent vector

b) Find the curvature.

c) Find the principal unit normal vector.

d) Verify that $|\vec{N}| = 1 \quad and \quad \vec{T} \bullet \vec{N} = 0$

e) Graph the curve and sketch $\vec{T}$ and $\vec{N}$ at two points.

33. Find the following for all values of t for which the given curve is defined by

$$\vec{r}(t) = (\cos t)\hat{i} + (2\sin t)\hat{j} + \hat{k}, \quad 0 \le t \le 2\pi$$

a) Find the tangent vector and the unit tangent vector
b) Find the curvature.
c) Find the principal unit normal vector.
d) Verify that $\left|\vec{N}\right| = 1 \quad and \quad \vec{T} \bullet \vec{N} = 0$
e) Graph the curve and sketch $\vec{T}$ and $\vec{N}$ at two points.

34. Find the following for all values of t for which the given curve is defined by

$$\vec{r}(t) = t\hat{i} + (2\cos t)\hat{j} + (2\sin t)\hat{k}, \quad 0 \le t \le 2\pi$$

a) Find the tangent vector and the unit tangent vector
b) Find the curvature.
c) Find the principal unit normal vector.
d) Verify that $\left|\vec{N}\right| = 1 \quad and \quad \vec{T} \bullet \vec{N} = 0$
e) Graph the curve and sketch $\vec{T}$ and $\vec{N}$ at two points.

35. Find the following for all values of t for which the given curve is defined by

$$\vec{r}(t) = (\cos t)\hat{i} + (2\cos t)\hat{j} + \left(\sqrt{5}\sin t\right)\hat{k}, \quad 0 \le t \le 2\pi$$

a) Find the tangent vector and the unit tangent vector
b) Find the curvature.
c) Find the principal unit normal vector.
d) Verify that $\left|\vec{N}\right| = 1 \quad and \quad \vec{T} \bullet \vec{N} = 0$
e) Graph the curve and sketch $\vec{T}$ and $\vec{N}$ at two points.

36. Find equations for the osculating, normal and rectifying planes of the curve $\vec{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$ at the point (1, 1, 1).

37. Consider the position vector $\vec{r}(t) = \left(t^2 + 1\right)\hat{i} + (2t)\hat{j}, \quad t \ge 0$ of the moving objects

a) Find the normal and tangential components of the acceleration.
b) Graph the trajectory and sketch the normal and tangential components of the acceleration at two points on the trajectory. Show that their sum gives the total acceleration.

38. Consider the position vector $\vec{r}(t) = (2\cos t)\hat{i} + (2\sin t)\hat{j}, \quad 0 \le t \le 2\pi$ of the moving objects

c) Find the normal and tangential components of the acceleration.
d) Graph the trajectory and sketch the normal and tangential components of the acceleration at two points on the trajectory. Show that their sum gives the total acceleration.

39. Consider the position vector $\vec{r}(t) = 3t\,\hat{i} + (4-t)\hat{j} + t\,\hat{k}, \quad t \ge 0$ of the moving objects

Find the normal and tangential components of the acceleration.

40. Consider the position vector $\vec{r}(t) = (2\cos t)\hat{i} + (2\sin t)\hat{j} + (10t)\hat{k}, \quad 0 \le t \le 2\pi$ of the moving objects

a) Find the normal and tangential components of the acceleration.

b) Graph the trajectory and sketch the normal and tangential components of the acceleration at two points on the trajectory. Show that their sum gives the total acceleration.

41. Compute the unit binormal vector $\boldsymbol{B}$ and the torsion of the curve $\boldsymbol{r}(t) = \left\langle t,\ t^2,\ t^3 \right\rangle, \quad at\ t = 1$

42. At point *P*, the velocity and acceleration of a particle moving in the plane are $\vec{v} = 3\hat{i} + 4\hat{j}$ and $\vec{a} = 5\hat{i} + 15\hat{j}$. Find the curvature of the particle's path at *P*.

43. Consider the curve $C:\ \boldsymbol{r}(t) = \langle 3\sin t,\ 4\sin t,\ 5\cos t \rangle, \quad for\ 0 \le t \le 2\pi$

a) Find $\boldsymbol{T}(t)$ at all points of *C*.

b) Find $\boldsymbol{N}(t)$ and the curvature at all points of *C*.

c) Sketch the curve and show $\boldsymbol{T}(t)$ and $\boldsymbol{N}(t)$ at the points of *C* corresponding to $t = 0$ and $t = \frac{\pi}{2}$.

d) Are the results of parts (*a*) and (*b*) consistent with the graph?

e) Find $\boldsymbol{B}(t)$ at all points of *C*.

f) Describe three calculations that serve to check the accuracy of your results in part (*a*) – (*f*).

g) Compute the torsion at all points of *C*. Interpret this result.

44. Consider the curve $C:\ \boldsymbol{r}(t) = \langle 3\sin t,\ 3\cos t,\ 4t \rangle, \quad for\ 0 \le t \le 2\pi$

a) Find $\boldsymbol{T}(t)$ at all points of *C*.

b) Find $\boldsymbol{N}(t)$ and the curvature at all points of *C*.

c) Sketch the curve and show $\boldsymbol{T}(t)$ and $\boldsymbol{N}(t)$ at the points of *C* corresponding to $t = 0$ and $t = \frac{\pi}{2}$.

d) Are the results of parts (*a*) and (*b*) consistent with the graph?

e) Find $\boldsymbol{B}(t)$ at all points of *C*.

f) Describe three calculations that serve to check the accuracy of your results in part (*a*) – (*f*).

g) Compute the torsion at all points of *C*. Interpret this result.

45. Suppose $\vec{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$, where *f*, *g*, and *h* are the quadratic functions $f(t) = a_1 t^2 + b_1 t + c_1$, $g(t) = a_2 t^2 + b_2 t + c_2$, and $h(t) = a_3 t^2 + b_3 t + c_3$, and where at least one of the leading coefficients $a_1,\ a_2,\ or\ a_3$ is nonzero. Apart from a set of degenerate cases (for example $\vec{r}(t)=\left\langle t^2,\ t^2,\ t^2\right\rangle$, whose graph is a line), it can be shown that the graph of $\vec{r}(t)$ is a parabola that lies in a plane

a) Show by direct computation that $\vec{v}\times\vec{a}$ is constant. Then explain why the unit binormal vector is constant at all points on the curve. What does this result say about the torsion of the curve?

b) Compute $\boldsymbol{a}'(t)$ and explain why the torsion is zero at all points on the curve for which the torsion is defined.

46. Let *f* and *g* be continuous on an interval *I*. consider the curve

$$C:\ \vec{r}(t)=\left\langle a_1 f(t)+a_2 g(t)+a_3,\ \ b_1 f(t)+b_2 g(t)+b_3,\ \ c_1 f(t)+c_2 g(t)+c_3\right\rangle$$

For *t* in *I*, and where $a_i,\ b_i,\ and\ c_i$, for $i=1,\ 2,\ and\ 3$, are real numbers

a) Show that, in general, *C* lies in a plane.

b) Explain why the torsion is zero at all points of *C* for which the torsion is defined.