Solution Section 1.1 – Rates of Change and Tangents to Curves

Exercise

Find the average rate of change of the function $f(x)=x^3+1$ over the interval $[2,\ 3]$

Solution

$$\begin{aligned}\frac{\Delta f}{\Delta x}&=\frac{f(3)-f(2)}{3-2}\\&=\frac{3^3+1-\left(2^3+1\right)}{1}\\&=27+1-(8+1)\\&=19\end{aligned}$$

Exercise

Find the average rate of change of the function $f(x)=x^2$ over the interval $[-1,\ 1]$

Solution

$$\begin{aligned}\frac{\Delta f}{\Delta x}&=\frac{f(1)-f(-1)}{1-(-1)}\\&=\frac{1^2-(-1)^2}{2}\\&=\frac{0}{2}\\&=0\end{aligned}$$

Exercise

Find the average rate of change of the function $f(t)=2+\cos t$ over the interval $[-\pi,\ \pi]$

Solution

$$\begin{aligned}\frac{\Delta f}{\Delta x}&=\frac{f(\pi)-f(-\pi)}{\pi-(-\pi)}\\&=\frac{2+\cos\pi-\left(2+\cos(-\pi)\right)}{2\pi}\\&=\frac{2-1-(2-1)}{2}\\&=0\end{aligned}$$

Exercise

Find the slope of $y = x^2 - 3$ at the point $P(2,\ 1)$ and an equation of the tangent line at this P.

Solution

$$\frac{\Delta y}{\Delta x} = \frac{f\left(x_1 + h\right) - f\left(x_1\right)}{h}$$

$$= \frac{f(2+h) - f(2)}{h}$$

$$= \frac{(2+h)^2 - 3 - \left(2^2 - 3\right)}{h}$$

$$= \frac{4 + 4h + h^2 - 3 - (4-3)}{h}$$

$$= \frac{4h + h^2}{h}$$

$$= \frac{4h}{h} + \frac{h^2}{h}$$

$$= 4 + h$$

As h approaches 0. Then the secant slope $h + 4 \to 4 = slope$

$$y - y_1 = m\left(x - x_1\right)$$

$$y - 1 = 4(x - 2)$$

$$y - 1 + 1 = 4x - 8 + 1$$

$$\boxed{y = 4x - 7}$$

Exercise

Find the slope of $y = x^2 - 2x - 3$ at the point $P(2,\ -3)$ and an equation of the tangent line at this P.

Solution

$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{f(2+h) - f(2)}{h} \\
&= \frac{(2+h)^2 - 2(2+h) - 3 - \left(2^2 - 2(2) - 3\right)}{h} \\
&= \frac{4 + 4h + h^2 - 4 - 2h - 3 - (-3)}{h} \\
&= \frac{2h + h^2}{h} \\
&= 2 + h
\end{aligned}$$

As h approaches 0. Then the secant slope $2 + h \to 2 = slope$

$$\begin{aligned}
& y - y_1 = m\left(x - x_1\right) \\
& y + 3 = 2(x - 2) \\
& y = 2x - 4 - 3 \\
& \boxed{y = 2x - 7}
\end{aligned}$$

Exercise

Find the slope of $y = x^3$ at the point $P(2,\ 8)$ and an equation of the tangent line at this P.

Solution

$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{f(2+h) - f(2)}{h} \\
&= \frac{(2+h)^3 - 2^3}{h} \\
&= \frac{8 + 12h + 6h^2 + h^3 - 8}{h} \\
&= 12 + 6h + h^2
\end{aligned}$$

As h approaches 0. Then $slope = 12$

$$\begin{aligned}
& y - y_1 = m\left(x - x_1\right) \\
& y - 8 = 12(x - 2) \\
& y = 12x - 24 + 8 \\
& \boxed{y = 12x - 16}
\end{aligned}$$

Exercise

Make a table of values for the function $f(x)=\dfrac{x+2}{x-2}$ at the points

$$x=1.2,\quad x=\frac{11}{10},\quad x=\frac{101}{100},\quad x=\frac{1001}{1000},\quad x=\frac{10001}{10000},\quad \textit{and } x=1$$

a) Find the average rate of change of $f(x)$ over the intervals $[1,\ x]$ for each $x\neq 1$ in the table

b) Extending the table if necessary, try to determine the rate of change of $f(x)$ at $x=1$.

Solution

a)

x	1.2	1.1	1.01	1.001	1.0001	1
$f(x)$	–4.0	$-3.\overline{4}$	$-3.\overline{04}$	$-3.\overline{004}$	$-3.\overline{004}$	–3

$$\frac{\Delta y}{\Delta x}=\frac{-4-(-3)}{1.2-1}=-5.0$$

$$\frac{\Delta y}{\Delta x}=\frac{-3.\overline{4}-(-3)}{1.1-1}=-4.\overline{4}$$

$$\frac{\Delta y}{\Delta x}=\frac{-3.\overline{04}-(-3)}{1.01-1}=-4.\overline{04}$$

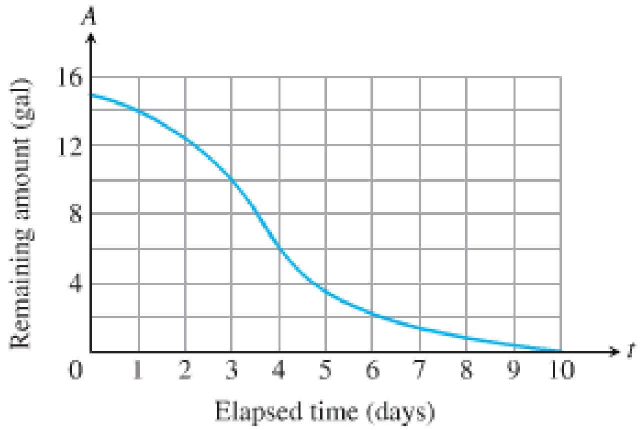
$$\frac{\Delta y}{\Delta x}=\frac{-3.\overline{004}-(-3)}{1.001-1}=-4.\overline{004}$$

$$\frac{\Delta y}{\Delta x}=\frac{-3.\overline{0004}-(-3)}{1.0001-1}=-4.\overline{0004}$$

b) The rate of change of $f(x)$ at $x=1$ is $\underline{-4}$

Exercise

The accompanying graph shows the total amount of gasoline A in the gas tank of an automobile after being driven for t days.



a) Estimate the average rate of gasoline consumption over the time intervals

$$[0,\ 3],\ [0,\ 5],\ and\ [7,\ 10]$$

b) Estimate the instantaneous rate of gasoline consumption over the time $t=1,\ t=4,\ and\ t=8$

c) Estimate the maximum rate of gasoline consumption and the specific time at which it occurs.

Solution

a) Average rate of gasoline consumption over the time intervals:

$$[0,\ 3] \quad \Rightarrow \quad Average\ Rate = \frac{10-15}{3-0} \approx \underline{= -1.67\ gal\ /\ day}|$$

$$[0,\ 5] \quad \Rightarrow \quad Average\ Rate = \frac{3.9-15}{3-0} \underline{\approx -2.2\ gal\ /\ day}|$$

$$[7,\ 10] \quad \Rightarrow \quad Average\ Rate = \frac{0-1.4}{10-7} \underline{\approx -0.5\ gal\ /\ day}|$$

b) At $t=1 \quad \to P(1,\ 14)$

Solution *Section* 1.2 – Limit of a Function and Limit Laws

Exercise

Find the limit: $\lim\limits_{x\to 1}(2x+4)$

Solution

$$\lim_{x\to 1}(2x+4) = 2*(1) + 4 = 6$$

Exercise

Find the limit: $\lim\limits_{x\to 1}\dfrac{x^2-4}{x-2}$

Solution

$$\lim_{x\to 1}\frac{x^2-4}{x-2} = \frac{1^2-4}{1-2}$$

$$= \frac{-3}{-1}$$

$$= 3$$

Exercise

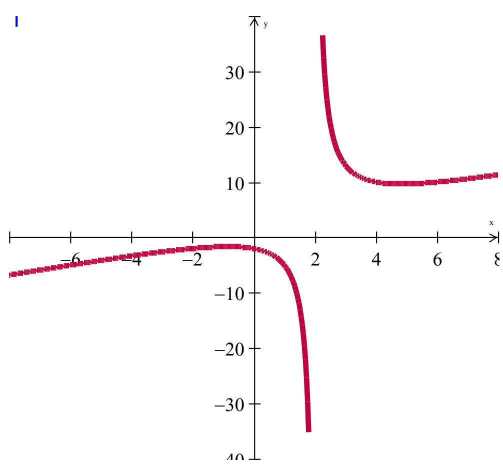
Find the limit: $\lim\limits_{x\to 2}\dfrac{x^2+4}{x-2}$

Solution

$$\lim_{x\to 2}\frac{x^2+4}{x-2} = \frac{2^2+4}{2-2}$$

$$= \frac{8}{0}$$

$= \infty$ (***Doesn't exist***)



Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{|x|}{x}$

Solution

$$\lim_{x\to 0} \frac{|x|}{x} = \frac{0}{0}$$

$$\lim_{x\to 0^-} \frac{|x|}{x} = \frac{x}{-x} = -1$$

$$\lim_{x\to 0^+} \frac{|x|}{x} = \frac{x}{x} = 1$$

Doesn't exist

Exercise

Find: $\lim\limits_{x\to 3} \dfrac{x^2 - x - 1}{\sqrt{x+1}}$

Solution

$$\lim_{x\to 3} \frac{x^2 - x - 1}{\sqrt{x+1}} = \frac{3^2 - 3 - 1}{\sqrt{3+1}}$$

$$= \frac{5}{2}$$

Exercise

Find: $\lim\limits_{x\to 2} \dfrac{x^2 + x - 6}{x - 2}$

Solution

$$\lim_{x\to 2} \frac{x^2 + x - 6}{x - 2} = \frac{2^2 + 2 - 6}{2 - 2}$$

$$= \frac{0}{0}$$

$$\lim_{x\to 2} \frac{(x+3)(x-2)}{x-2} = \lim_{x\to 2} (x+3)$$

$$= 5$$

Exercise

Find the limit: $\lim\limits_{x \to 0} (3x - 2)$

Solution

$$\lim_{x \to 0} (3x - 2) = 3(0) - 2 \underline{= -2}$$

Exercise

Find the limit: $\lim\limits_{x \to 1} (2x^2 - x + 4)$

Solution

$$\lim_{x \to 1} (2x^2 - x + 4) = 2(1)^2 - (1) + 4$$

$$\underline{= 5}$$

Exercise

Find the limit: $\lim\limits_{x \to -2} \left(x^3 - 2x^2 + 4x + 8\right)$

Solution

$$\lim_{x \to -2} \left(x^3 - 2x^2 + 4x + 8\right) = (-2)^3 - 2(-2)^2 + 4(-2) + 8$$

$$\underline{= -16}$$

Exercise

Find the limit: $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2}$

Solution

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \frac{2^2 - 4}{2 - 2} = \frac{0}{0}$$

$$\lim_{x \to 1.9} \frac{x^2 - 4}{x - 2} = \frac{1.9^2 - 4}{1.9 - 2} = 3.9$$

$$\lim_{x \to 2.1} \frac{x^2 - 4}{x - 2} = \frac{2.1^2 - 4}{2.1 - 2} = 4.1$$

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} \underline{= 4}$$

Exercise

Find the limit: $\lim\limits_{x\to 2} \frac{x^3-8}{x-2}$

Solution

$$\lim\limits_{x\to 2} \frac{x^3-8}{x-2} = \frac{0}{0}$$

$$\begin{aligned}\lim\limits_{x\to 2} \frac{x^3-8}{x-2} &= \lim\limits_{x\to 2} \frac{(x-2)(x^2+2x+4)}{x-2} \\ &= \lim\limits_{x\to 2} x^2+2x+4 \\ &= 2^2 + 2(2) + 4 \\ &= 12\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{x\to 3} \frac{x^2+x-12}{x-3}$

Solution

$$\lim\limits_{x\to 3} \frac{x^2+x-12}{x-3} = \frac{0}{0}$$

$$\begin{aligned}\lim\limits_{x\to 3} \frac{(x-3)(x+4)}{x-3} &= \lim\limits_{x\to 3} (x+4) \\ &= 3 + 4 \\ &\underline{= 7|}\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \frac{\sqrt{x+4}-2}{x}$

Solution

$$\lim\limits_{x\to 0} \frac{\sqrt{x+4}-2}{x} = \frac{\sqrt{4}-2}{0} = \frac{0}{0}$$

$$\begin{aligned}\lim\limits_{x\to 0} \frac{\sqrt{x+4}-2}{x} &= \lim\limits_{x\to 0} \frac{\sqrt{x+4}-2}{x} \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} \\ &= \lim\limits_{x\to 0} \frac{x+4-4}{x\left(\sqrt{x+4}+2\right)}\end{aligned}$$

$$= \lim_{x\to 0} \frac{x}{x\left(\sqrt{x+4}+2\right)}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{x+4}+2}$$

$$= \frac{1}{\sqrt{4}+2}$$

$$\boxed{= \frac{1}{4}}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{3}{\sqrt{3x+1}+1}$

Solution

$$\lim_{x\to 0} \frac{3}{\sqrt{3x+1}+1} = \frac{3}{\sqrt{3(0)+1}+1}$$

$$= \frac{3}{1+1}$$

$$\boxed{= \frac{3}{2}}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} f(x)$ $\quad f(x) = \begin{cases} x^2+1 & x<0 \\ 2x+1 & x>0 \end{cases}$

Solution

$$\lim_{x\to 0^-} x^2+1 = 1$$

$$\lim_{x\to 0+} 2x+1 = 1$$

$$\lim_{x\to 0} f(x) = 1$$

Exercise

Find the limit: $\lim\limits_{x\to -2} \dfrac{5}{x+2}$

Solution

$$\lim_{x\to -2} \frac{5}{x+2} = \frac{5}{0} = \infty \qquad (\textit{Doesn't exist})$$

Exercise

Find the limit: $\lim\limits_{x \to 3} \frac{\sqrt{x+1}-1}{x}$

Solution

$$\lim_{x \to 3} \frac{\sqrt{x+1}-1}{x} = \frac{\sqrt{3+1}-1}{3}$$

$$= \frac{2-1}{3}$$

$$= \frac{1}{3}$$

Exercise

Find the limit: $\lim\limits_{x \to 1} \frac{x^2-1}{x-1}$

Solution

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = \frac{0}{0}$$

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1}$$

$$= \lim_{x \to 1} (x+1)$$

$$= 1+1$$

$$= 2$$

Exercise

Find the limit: $\lim\limits_{x \to -2} \frac{|x+2|}{x+2}$

Solution

$$\lim_{x \to -2} \frac{|x+2|}{x+2} = \frac{|-2+2|}{-2+2} = \frac{0}{0}$$

$$\lim_{x \to -2^+} \frac{|x+2|}{x+2} = \frac{(x+2)}{(x+2)} = 1$$

$$\lim_{x \to -2^-} \frac{|x+2|}{x+2} = \frac{(x+2)}{-(x+2)} = -1$$

Doesn't exist

Exercise

Find the limit: $\lim\limits_{x\to 0}(2z-8)^{1/3}$

Solution

$$\lim_{x\to 0}(2z-8)^{1/3} = (2(0)-8)^{1/3}$$
$$= (-8)^{1/3}$$
$$\boxed{= -2}$$

Exercise

Find the limit: $\lim\limits_{x\to 2}\dfrac{x^2-7x+10}{x-2}$

Solution

$$\lim_{x\to 2}\frac{x^2-7x+10}{x-2} = \frac{2^2-7(2)+10}{2-2} = \frac{0}{0}$$

$$\lim_{x\to 2}\frac{x^2-7x+10}{x-2} = \lim_{x\to 2}\frac{(x-2)(x-5)}{x-2}$$
$$= \lim_{x\to 2}(x-5)$$
$$= 2-5$$
$$\boxed{= -3}$$

Exercise

Find the limit: $\lim\limits_{x\to 0}\dfrac{5x^3+8x^2}{3x^4-16x^2}$

Solution

$$\lim_{x\to 0}\frac{5x^3+8x^2}{3x^4-16x^2} = \frac{0}{0}$$

$$\lim_{x\to 0}\frac{5x^3+8x^2}{3x^4-16x^2} = \lim_{x\to 0}\frac{x^2(5x+8)}{x^2\left(3x^2-16\right)}$$
$$= \lim_{x\to 0}\frac{5x+8}{3x^2-16}$$
$$= \frac{8}{-16}$$
$$\boxed{= -\frac{1}{2}}$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{\frac{1}{x}-1}{x-1}$

Solution

$$\lim_{x\to 1} \frac{\frac{1}{x}-1}{x-1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$\begin{aligned}
\lim_{x\to 1} \frac{\frac{1}{x}-1}{x-1} &= \lim_{x\to 1} \frac{\frac{1-x}{x}}{x-1} \\
&= \lim_{x\to 1} \left(\frac{1-x}{x}\right)\left(\frac{1}{x-1}\right) \\
&= \lim_{x\to 1} \left(\frac{-(x-1)}{x}\right)\left(\frac{1}{x-1}\right) \\
&= \lim_{x\to 1} \frac{-1}{x} \\
&= -\frac{1}{1} \\
&= -1
\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{u\to 1} \dfrac{u^4-1}{u^3-1}$

Solution

$$\begin{aligned}
\lim_{u\to 1} \frac{u^4-1}{u^3-1} &= \lim_{u\to 1} \frac{\left(u^2-1\right)\left(u^2+1\right)}{(u-1)\left(u^2+u+1\right)} \\
&= \lim_{u\to 1} \frac{(u-1)(u+1)\left(u^2+1\right)}{(u-1)\left(u^2+u+1\right)} \\
&= \lim_{u\to 1} \frac{(u+1)\left(u^2+1\right)}{u^2+u+1} \\
&= \frac{(1+1)\left(1^2+1\right)}{1^2+1+1} \\
&= \frac{4}{3}
\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{x+3}-2}$

Solution

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2} = \frac{1-1}{\sqrt{1+3}-2} = \frac{0}{\sqrt{4}-2} = \frac{0}{0}$$

$$\begin{aligned}
\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2} &= \lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2} \bullet \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2} \\
&= \lim_{x\to 1} \frac{(x-1)\left(\sqrt{x+3}+2\right)}{x+3-4} \\
&= \lim_{x\to 1} \frac{(x-1)\left(\sqrt{x+3}+2\right)}{x-1} \\
&= \lim_{x\to 1} \left(\sqrt{x+3}+2\right) \\
&= \sqrt{1+3}+2 \\
&= \sqrt{4}+2 \\
&= 2+2 \\
&= 4
\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{x\to -1} \dfrac{\sqrt{x^2+8}-3}{x+1}$

Solution

$$\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} = \frac{\sqrt{(-1)^2+8}-3}{-1+1} = \frac{\sqrt{9}-3}{0} = \frac{0}{0}$$

$$\begin{aligned}
\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} &= \lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} \bullet \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3} \\
&= \lim_{x\to -1} \frac{x^2+8-9}{(x+1)\left(\sqrt{x^2+8}+3\right)} \\
&= \lim_{x\to -1} \frac{x^2-1}{(x+1)\left(\sqrt{x^2+8}+3\right)} \\
&= \lim_{x\to -1} \frac{(x-1)(x+1)}{(x+1)\left(\sqrt{x^2+8}+3\right)}
\end{aligned}$$

$$= \lim_{x\to -1} \frac{(x-1)}{\sqrt{x^2+8}+3}$$

$$= \frac{-1-1}{\sqrt{(-1)^2+8}+3}$$

$$= \frac{-2}{\sqrt{9}+3}$$

$$= \frac{-2}{3+3}$$

$$= \frac{-2}{6}$$

$$\boxed{= -\frac{1}{3}}$$

Exercise

Find the limit: $\lim\limits_{x\to -3} \dfrac{2-\sqrt{x^2-5}}{x+3}$

Solution

$$\lim_{x\to -3} \frac{2-\sqrt{x^2-5}}{x+3} = \frac{2-\sqrt{(-3)^2-5}}{-3+3} = \frac{2-\sqrt{9-5}}{0} = \frac{2-\sqrt{4}}{0} = \frac{0}{0}$$

$$\lim_{x\to -3} \frac{2-\sqrt{x^2-5}}{x+3} = \lim_{x\to -3} \frac{2-\sqrt{x^2-5}}{x+3} \bullet \frac{2+\sqrt{x^2-5}}{2+\sqrt{x^2-5}}$$

$$= \lim_{x\to -3} \frac{4-\left(x^2-5\right)}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{4-x^2+5}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{9-x^2}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{(x-3)(x+3)}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{(x-3)}{2+\sqrt{x^2-5}}$$

$$= \frac{-3-3}{2+\sqrt{(-3)^2-5}}$$

$$= \frac{-6}{2+\sqrt{9-5}}$$

$$= \frac{-6}{2+\sqrt{4}}$$

$$= -\frac{6}{4}$$

$$\underline{= -\frac{3}{2}}$$

Exercise

Find the limit: $\lim_{x\to 0} (2\sin x - 1)$

Solution

$$\lim_{x\to 0} (2\sin x - 1) = 2\sin(0) - 1$$

$$= 0 - 1$$

$$\underline{= -1}$$

Exercise

Find the limit: $\lim_{x\to 0} \sin^2 x$

Solution

$$\lim_{x\to 0} \sin^2 x = \sin^2(0) \;\underline{= 0}$$

Exercise

Find the limit: $\lim_{x\to 0} \sec x$

Solution

$$\lim_{x\to 0} \sec x = \sec(0)$$

$$= \frac{1}{\cos(0)}$$

$$= \frac{1}{1}$$

$$\underline{= 1}$$

Exercise

Find the limit: $\lim\limits_{x \to 0} \dfrac{1+x+\sin x}{3\cos x}$

Solution

$$\lim_{x \to 0} \frac{1+x+\sin x}{3\cos x} = \frac{1+0+\sin(0)}{3\cos(0)}$$

$$= \boxed{\frac{1}{3}}$$

Exercise

Find the limit: $\lim\limits_{x \to -\pi} \sqrt{x+4}\cos(x+\pi)$

Solution

$$\lim_{x \to -\pi} \sqrt{x+4}\cos(x+\pi) = \sqrt{-\pi+4}\cos(-\pi+\pi)$$

$$= \sqrt{-\pi+4}\cos(0)$$

$$= \boxed{.9265} \quad or \quad \sqrt{4-\pi}$$

Exercise

For the function $f(t)$ graphed, find the following limits or explain why they do not exist.

$a)\ \lim\limits_{t \to -2} f(t) \qquad b)\ \lim\limits_{t \to -1} f(t) \qquad c)\ \lim\limits_{t \to 0} f(t) \qquad d)\ \lim\limits_{t \to -0.5} f(t)$

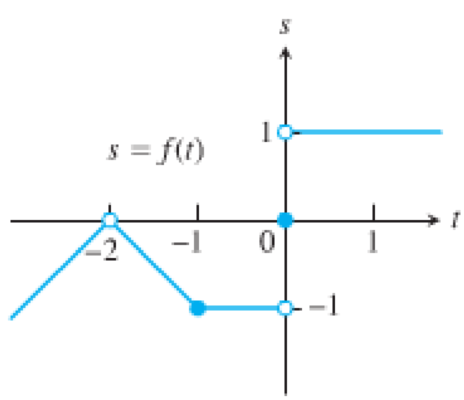
Solution

a) $\lim\limits_{t \to -2} f(t) = 0$

b) $\lim\limits_{t \to -1} f(t) = -1$

c) $\lim\limits_{t \to 0} f(t) =$ *doesn't exist*

d) $\lim\limits_{t \to -.5} f(t) = -1$



Exercise

Suppose $\lim\limits_{x\to c} f(x)=5$ *and* $\lim\limits_{x\to c} g(x)=-2$. Find

a) $\lim\limits_{x\to c} f(x)g(x)$

b) $\lim\limits_{x\to c} 2f(x)g(x)$

c) $\lim\limits_{x\to c} \left(f(x)+3g(x)\right)$

d) $\lim\limits_{x\to c} \dfrac{f(x)}{f(x)-g(x)}$

Solution

a) $\lim\limits_{x\to c} f(x)g(x)=\lim\limits_{x\to c} f(x)\bullet \lim\limits_{x\to c} g(x)=(5)(-2)\ \underline{=-10}$

b) $\lim\limits_{x\to c} 2f(x)g(x)=2\lim\limits_{x\to c} f(x)\bullet \lim\limits_{x\to c} g(x)=2(-10)\ \underline{=-20}$

c)
$$\begin{aligned}\lim_{x\to c}\left(f(x)+3g(x)\right)&=\lim_{x\to c} f(x)+3\lim_{x\to c} g(x)\\ &=5+3(-2)\\ &=5-6\\ &\underline{=-1}\end{aligned}$$

d)
$$\begin{aligned}\lim_{x\to c}\frac{f(x)}{f(x)-g(x)}&=\frac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} f(x)-\lim\limits_{x\to c} g(x)}\\ &=\frac{5}{5-(-2)}\\ &\underline{=\frac{5}{7}}\end{aligned}$$

Exercise

Explain why the limits do not exist for $\lim\limits_{x\to 0}\dfrac{x}{|x|}$

Solution

$$\lim_{x\to 0}\frac{x}{|x|}=\frac{0}{0}$$

$$\lim_{x\to 0^-}\frac{x}{|x|}=\frac{-x}{x}=-1$$

$$\lim_{x\to 0^+}\frac{x}{|x|}=\frac{x}{x}=1$$

Doesn't exist

Exercise

Evaluate the limit using the form $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ for $f(x)=x^2, \quad x=1$

Solution

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{(x+h)^2-x^2}{h}\\ &= \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}\\ &= \lim_{h\to 0}\left(\frac{2xh}{h}+\frac{h^2}{h}\right)\\ &= \lim_{h\to 0}(2x+h)\\ &= 2x\end{aligned}$$

Exercise

Evaluate the limit using the form $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ for $f(x)=\sqrt{3x+1}, \quad x=0$

Solution

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\sqrt{3(x+h)+1}-\sqrt{3x+1}}{h}\\ &= \lim_{h\to 0}\frac{\sqrt{3x+3h+1}-\sqrt{3x+1}}{h}\\ &= \lim_{h\to 0}\frac{\sqrt{3x+3h+1}-\sqrt{3x+1}}{h}\cdot\frac{\sqrt{3x+3h+1}+\sqrt{3x+1}}{\sqrt{3x+3h+1}+\sqrt{3x+1}}\\ &= \lim_{h\to 0}\frac{3x+3h+1-(3x+1)}{h\left(\sqrt{3x+3h+1}+\sqrt{3x+1}\right)}\\ &= \lim_{h\to 0}\frac{3x+3h+1-3x-1}{h\left(\sqrt{3x+3h+1}+\sqrt{3x+1}\right)}\\ &= \lim_{h\to 0}\frac{3h}{h\left(\sqrt{3x+3h+1}+\sqrt{3x+1}\right)}\\ &= \lim_{h\to 0}\frac{3}{\sqrt{3x+3h+1}+\sqrt{3x+1}}\\ &= \frac{3}{\sqrt{3(0)+1}+\sqrt{3(0)+1}} \qquad \textit{Given}: x=0\\ &= \frac{3}{2}\end{aligned}$$

Exercise

If $\lim\limits_{x\to 4}\dfrac{f(x)-5}{x-2}=1$, find $\lim\limits_{x\to 4} f(x)$

Solution

$$\lim_{x\to 4}\frac{f(x)-5}{x-2}=1$$

$$\frac{\lim\limits_{x\to 4} f(x)-5}{4-2}=1$$

$$\frac{\lim\limits_{x\to 4} f(x)-5}{2}=1$$ ***Multiply both sides by 2***

$$\lim_{x\to 4} f(x)-5=2$$ ***Add 5 on both sides***

$$\lim_{x\to 4} f(x)=7$$

Exercise

If $\lim\limits_{x\to 0}\dfrac{f(x)}{x^2}=1$, find $\lim\limits_{x\to 0} f(x)$ *and* $\lim\limits_{x\to 0}\dfrac{f(x)}{x}$

Solution

$$\lim_{x\to 0}\frac{f(x)}{x^2}=1$$

$$\frac{\lim\limits_{x\to 0} f(x)}{\lim\limits_{x\to 0} x^2}=1$$

$$\lim_{x\to 0} f(x)=\lim_{x\to 0} x^2 \underline{=0}$$

$$\lim_{x\to 0}\frac{f(x)}{x}=\lim_{x\to 0}\left(\frac{f(x)}{x^2}\bullet x\right)$$

$$=\lim_{x\to 0}\frac{f(x)}{x^2}\bullet\lim_{x\to 0} x$$

$$=1\bullet 0$$

$$\underline{=0}$$

Exercise

If $x^4 \le f(x) \le x^2; \quad -1 \le x \le 1$ and $x^2 \le f(x) \le x^4; \quad x < -1$ *and* $x > 1$. At what points c do you automatically know $\lim\limits_{x \to c} f(x)$? What can you say about the value of the limits at these points?

Solution

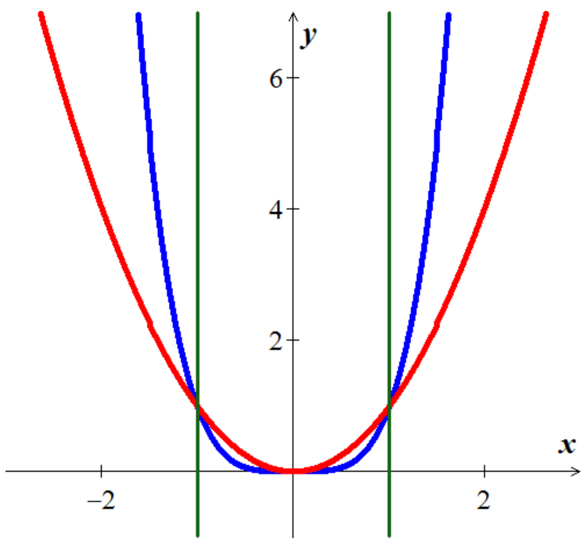
$$\lim_{x \to c} x^4 = \lim_{x \to c} x^2 \quad \Rightarrow \quad c^4 = c^2$$

$$c^4 - c^2 = 0$$

$$c^2\left(c^2 - 1\right) = 0$$

$$c^2 = 0 \qquad\qquad c^2 - 1 = 0$$

$$\boxed{c = 0} \qquad\qquad \boxed{c = \pm 1}$$



$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 = 0$$

$$\lim_{x \to -1} f(x) = \lim_{x \to 1} f(x) = 1$$

Solution *Section* 1.3 – Precise Definition of a Limit

Exercise

Sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \;\Rightarrow\; a < x < b$ for $a = 1, \quad b = 7, \quad x_0 = 5$

Solution

$$|x-5| < \delta \;\Rightarrow\; -\delta < x - 5 < \delta$$
$$-\delta + 5 < x < \delta + 5$$

$$-\delta + 5 = 1 \;\Rightarrow\; \delta = 4$$
$$\delta + 5 = 7 \;\Rightarrow\; \delta = 2$$

Exercise

Sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \;\Rightarrow\; a < x < b$ for $a = -\frac{7}{2}, \quad b = -\frac{1}{2}, \quad x_0 = -\frac{3}{2}$

Solution

$$\left|x + \frac{3}{2}\right| < \delta \;\Rightarrow\; -\delta < x + \frac{3}{2} < \delta$$
$$-\delta - \frac{3}{2} < x < \delta - \frac{3}{2}$$

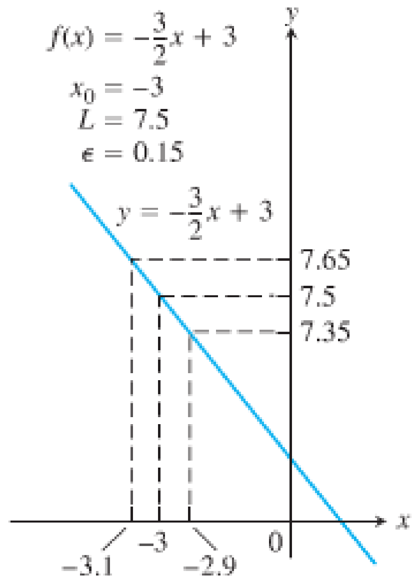
$$-\delta - \frac{3}{2} = -\frac{7}{2} \;\Rightarrow\; \boxed{\delta = \frac{7}{2} - \frac{3}{2} = 4}$$

$$\delta - \frac{3}{2} = -\frac{1}{2} \;\Rightarrow\; \boxed{\delta = \frac{1}{2} - \frac{3}{2} = -\frac{1}{2}}$$

Exercise

Use the graph to find a $\delta > 0$ such that for all x

$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$



Solution

Given: $a = -3.1, \quad b = -2.9, \quad x_0 = -3$

$$|x+3| < \delta \quad \Rightarrow \quad -\delta < x + 3 < \delta$$
$$-\delta - 3 < x < \delta - 3$$

$$-\delta - 3 = -3.1 \;\Rightarrow\; \underline{\delta = 3.1 - 3 = 0.1}$$

$$\delta - 3 = -2.9 \;\Rightarrow\; \underline{\delta = 3 - 2.9 = 0.1}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

$$f(x) = x + 1, \quad L = 5, \quad x_0 = 4, \quad \varepsilon = 0.01$$

Solution

$$|(x+1) - 5| < .01 \quad \Rightarrow \quad |x - 4| < .01$$
$$-.01 < x - 4 < .01$$
$$-.01 + 4 < x - 4 + 4 < .01 + 4$$
$$3.99 < x < 4.01$$

$$|x - 4| < \delta \quad \Rightarrow \quad -\delta < x - 4 < \delta$$
$$-\delta + 4 < x < \delta + 4$$

$$-\delta + 4 = 3.99 \;\Rightarrow\; \underline{\delta = 4 - 3.99 = 0.01}$$

$$\delta + 4 = 4.01 \;\Rightarrow\; \underline{\delta = 4.01 - 4 = 0.01}$$

$$\Rightarrow \underline{\delta = .01}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

$$f(x)=\sqrt{x+1}, \quad L=1, \quad x_0=0, \quad \varepsilon=0.1$$

Solution

$$\left|\sqrt{x+1}-1\right|<0.1 \quad \Rightarrow \quad -0.1<\sqrt{x+1}-1<0.1$$

$$-0.1+1<\sqrt{x+1}-1+1<0.1+1$$

$$.9<\sqrt{x+1}<1.1$$

$$(.9)^2<\left(\sqrt{x+1}\right)^2<(1.1)^2$$

$$.81<x+1<1.21$$

$$.81-1<x+1-1<1.21-1$$

$$-0.19<x<0.21$$

$$|x-0|<\delta \quad \Rightarrow \quad -\delta<x<\delta$$

$$\left.\begin{array}{l} -\delta=-0.19 \Rightarrow \delta=0.19 \\ \delta=0.21 \end{array}\right\} \rightarrow \boxed{\delta=0.19}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

$$f(x)=\sqrt{x-7}, \quad L=4, \quad x_0=23, \quad \varepsilon=1$$

Solution

$$\left|\sqrt{x-7}-4\right|<1 \quad \Rightarrow \quad -1<\sqrt{x-7}-4<1$$

$$3<\sqrt{x-7}<5$$

$$(3)^2<\left(\sqrt{x-7}\right)^2<(5)^2$$

$$9<x-7<25$$

$$9+7<x-7+7<25+7$$

$$16<x<32$$

$$|x-23|<\delta \quad \Rightarrow \quad -\delta<x-23<\delta$$

$$-\delta+23<x<\delta+23$$

$$\left.\begin{array}{l} -\delta+23=16 \Rightarrow \delta=23-16=7 \\ \delta+23=32 \Rightarrow \delta=32-23=9 \end{array}\right\} \rightarrow \boxed{\delta=7}$$

Exercise

Find an open interval about x_0 on which the inequality $\left|f(x)-L\right|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<\left|x-x_0\right|<\delta$ the inequality $\left|f(x)-L\right|<\varepsilon$ holds.

$$f(x)=x^2, \quad L=3, \quad x_0=\sqrt{3}, \quad \varepsilon=0.1$$

Solution

$$\left|x^2-3\right|<0.1 \quad \Rightarrow \quad -0.1<x^2-3<0.1$$

$$2.9<x^2<3.1$$

$$\sqrt{2.9}<x<\sqrt{3.1}$$

$$\left|x-\sqrt{3}\right|<\delta \quad \Rightarrow \quad -\delta<x-\sqrt{3}<\delta$$

$$-\delta+\sqrt{3}<x<\delta+\sqrt{3}$$

$$\left.\begin{aligned} -\delta+\sqrt{3}=\sqrt{2.9} \ &\Rightarrow \ \delta=\sqrt{3}-\sqrt{2.9}=.029 \\ \delta+\sqrt{3}=\sqrt{3.1} \ &\Rightarrow \ \delta=\sqrt{3.1}-\sqrt{3}=.029 \end{aligned}\right\} \to \boxed{\delta=.029}$$

Exercise

Find an open interval about x_0 on which the inequality $\left|f(x)-L\right|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<\left|x-x_0\right|<\delta$ the inequality $\left|f(x)-L\right|<\varepsilon$ holds.

$$f(x)=\frac{120}{x}, \quad L=5, \quad x_0=24, \quad \varepsilon=1$$

Solution

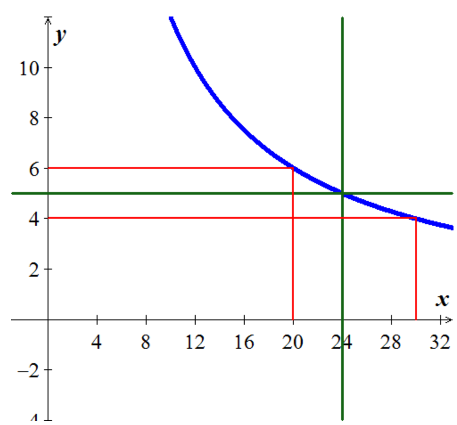
$$\left|\frac{120}{x}-5\right|<0.1 \quad \Rightarrow \quad -1<\frac{120}{x}-5<1$$

$$4<\frac{120}{x}<6$$

$$\frac{1}{6}<\frac{x}{120}<\frac{1}{4}$$

$$\frac{1}{6}(120)<x<\frac{1}{4}(120)$$

$$20<x<30$$



$$|x-24|<\delta \quad \Rightarrow \quad -\delta<x-24<\delta$$

$$-\delta+24<x<\delta+24$$

$$\left.\begin{aligned} -\delta+24=20 \ &\Rightarrow \ \delta=24-20=4 \\ \delta+24=30 \ &\Rightarrow \ \delta=30-24=6 \end{aligned}\right\} \to \boxed{\delta=4}$$

Exercise

Prove that $\lim\limits_{x \to 4} (9-x) = 5$

Solution

$$|(9-x)-5| < \varepsilon \quad \Rightarrow \quad -\varepsilon < 4 - x < \varepsilon$$

$$-\varepsilon - 4 < -x < \varepsilon - 4 \qquad \textit{divide by } (-).$$

$$\varepsilon + 4 > x > 4 - \varepsilon$$

$$4 - \varepsilon < x < \varepsilon + 4$$

$$|x-4| < \delta \quad \Rightarrow \quad -\delta < x - 4 < \delta$$

$$-\delta + 4 < x < \delta + 4$$

$$\left.\begin{array}{l} -\delta + 4 = 4 - \varepsilon \ \Rightarrow \ -\delta = -\varepsilon \Rightarrow \delta = \varepsilon \\ \delta + 4 = \varepsilon + 4 \ \Rightarrow \ \delta = \varepsilon \end{array}\right. \to \boxed{\delta = \varepsilon}$$

Exercise

Prove that $\lim\limits_{x \to 1} \frac{1}{x} = 1$

Solution

$$\left|\frac{1}{x} - 1\right| < \varepsilon \quad \Rightarrow \quad -\varepsilon < \frac{1}{x} - 1 < \varepsilon$$

$$-\varepsilon + 1 < \frac{1}{x} < \varepsilon + 1$$

$$\frac{1}{\varepsilon + 1} > x > \frac{1}{-\varepsilon + 1}$$

$$\frac{1}{1+\varepsilon} < x < \frac{1}{1-\varepsilon}$$

$$|x-1| < \delta \quad \Rightarrow \quad -\delta < x - 1 < \delta$$

$$1 - \delta < x < 1 + \delta$$

$$\left.\begin{array}{l} 1 - \delta = \dfrac{1}{1+\varepsilon} \ \Rightarrow \ \delta = 1 + \dfrac{1}{1+\varepsilon} = \dfrac{2+\varepsilon}{1+\varepsilon} \\ 1 + \delta = \dfrac{1}{1-\varepsilon} \ \Rightarrow \ \delta = \dfrac{1}{1-\varepsilon} - 1 = \dfrac{\varepsilon}{1-\varepsilon} \end{array}\right. \to \textit{the smallest}: \boxed{\delta = \frac{\varepsilon}{1-\varepsilon}}$$

Exercise

Prove that $\lim\limits_{x\to 0} f(x) = 0 \quad if \quad f(x) = \begin{cases} 2x, & x < 0 \\ \frac{x}{2}, & x \ge 0 \end{cases}$

Solution

For $x < 0$: $|2x - 0| < \varepsilon \quad \Rightarrow \quad -\varepsilon < 2x < 0$

$$-\frac{\varepsilon}{2} < x < 0$$

For $x \ge 0$: $\left|\frac{x}{2} - 0\right| < \varepsilon \quad \Rightarrow \quad 0 \le \frac{x}{2} < \varepsilon$

$$0 \le x < 2\varepsilon$$

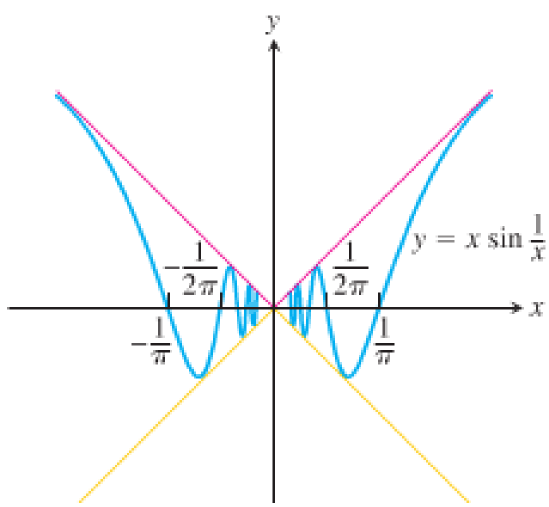
$|x - 0| < \delta \quad \Rightarrow \quad -\delta < x < \delta$

$-\delta = -\frac{\varepsilon}{2} \quad \Rightarrow \quad \delta = \frac{\varepsilon}{2} \quad \to \textit{the smallest}: \boxed{\delta = \frac{\varepsilon}{2}}$

$\delta = 2\varepsilon$

Exercise

Prove that $\lim\limits_{x\to 0} x\frac{1}{\sin x} = 0$



Solution

$$\left.\begin{array}{ll} -x \le x\sin\frac{1}{x} \le x, & \forall x > 0 \\ -x \ge x\sin\frac{1}{x} \ge x, & \forall x < 0 \end{array}\right\} \to \lim_{x\to 0}(-x) = \lim_{x\to 0}(x) = 0$$

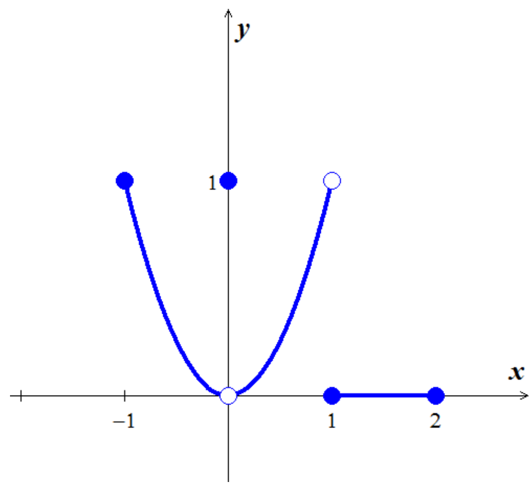
Then by the sandwich theorem, $\lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$

Solution   *Section* 1.4 – One-Sided Limits

Exercise

Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

Solution



a) $\lim\limits_{x\to -1^+} f(x) = 1$ ***True***

b) $\lim\limits_{x\to 0^-} f(x) = 0$ ***True***

c) $\lim\limits_{x\to 0^-} f(x) = 1$ ***False***

d) $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x)$ ***True***

e) $\lim\limits_{x\to 0} f(x)$ *exists* ***True***

f) $\lim\limits_{x\to 0} f(x) = 0$ ***True***

g) $\lim\limits_{x\to 0} f(x) = 1$ ***False***

h) $\lim\limits_{x\to 1} f(x) = 1$ ***False***

i) $\lim\limits_{x\to 1} f(x) = 0$ ***False***

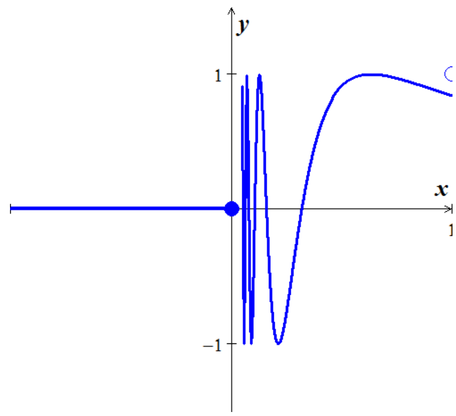
j) $\lim\limits_{x\to 2^-} f(x) = 2$ ***False***

k) $\lim\limits_{x\to -1^-} f(x) = 0$ *does not exist* ***True***

l) $\lim\limits_{x\to 2^+} f(x) = 0$ ***False***

Exercise

Let $f(x)=\begin{cases} 0, & x\le 0 \\ \sin\frac{1}{x}, & x>0 \end{cases}$



a) Does $\lim\limits_{x\to 0^+} f(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim\limits_{x\to 0^-} f(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim\limits_{x\to 0} f(x)$ exist? If so, what is it? If not, why not?

Solution

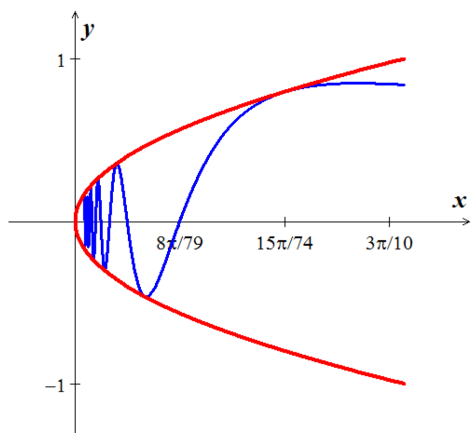
a) $\lim\limits_{x\to 0^+} f(x)$ doesn't exist, since $\sin\left(\frac{1}{x}\right)$ doesn't approach any single value as $x\to 0$

b) $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-} 0=0$

c) $\lim\limits_{x\to 0} f(x)$ doesn't exist, since $\lim\limits_{x\to 0^+} f(x)$ doesn't exist

Exercise

Let $g(x)=\sqrt{x}\ \sin\frac{1}{x}$



a) Does $\lim\limits_{x\to 0^+} g(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim\limits_{x\to 0^-} g(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim\limits_{x\to 0} g(x)$ exist? If so, what is it? If not, why not?

Solution

a) $\lim\limits_{x\to 0^+} g(x)$ exists, by the sandwich theorem $-\sqrt{x}\le g(x)\le\sqrt{x}.$ *for* $x>0$

b) $\lim\limits_{x\to 0^-} g(x)$ doesn't exist, since $\sqrt{x}$ is not defined for $x<0$

c) $\lim\limits_{x\to 0} g(x)$ doesn't exist, since $\lim\limits_{x\to 0^-} g(x)$ doesn't exist

Exercise

Find $\lim\limits_{x\to -0.5^-} \sqrt{\dfrac{x+2}{x+1}}$

Solution

$$\lim_{x\to -0.5^-} \sqrt{\frac{x+2}{x+1}} = \sqrt{\frac{-0.5+2}{-0.5+1}}$$

$$= \sqrt{\frac{1.5}{0.5}}$$

$$= \sqrt{3}$$

Exercise

Find $\lim\limits_{x\to 1^+} \sqrt{\dfrac{x-1}{x+2}}$

Solution

$$\lim_{x\to 1^+} \sqrt{\frac{x-1}{x+2}} = \sqrt{\frac{1-1}{1+2}}$$

$$= \sqrt{0}$$

$$= 0$$

Exercise

Find $\lim\limits_{x\to -2^+} \left(\dfrac{x}{x+1}\right)\left(\dfrac{2x+5}{x^2+x}\right)$

Solution

$$\lim_{x\to -2^+} \left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right) = \left(\frac{-2}{-2+1}\right)\left(\frac{2(-2)+5}{(-2)^2+(-2)}\right)$$

$$= \left(\frac{-2}{-1}\right)\left(\frac{1}{2}\right)$$

$$= 1$$

Exercise

Find $\lim\limits_{x\to 0^+} \dfrac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}$

Solution

$$\lim_{x\to 0^+} \frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x} = \frac{\sqrt{5}-\sqrt{5}}{0} = \frac{0}{0}$$

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x} &= \lim_{x\to 0^+} \frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}\,\frac{\sqrt{x^2+4x+5}+\sqrt{5}}{\sqrt{x^2+4x+5}+\sqrt{5}} \\
&= \lim_{x\to 0^+} \frac{x^2+4x+5-5}{x\left(\sqrt{x^2+4x+5}+\sqrt{5}\right)} \\
&= \lim_{x\to 0^+} \frac{x^2+4x}{x\left(\sqrt{x^2+4x+5}+\sqrt{5}\right)} \\
&= \lim_{x\to 0^+} \frac{x(x+4)}{x\left(\sqrt{x^2+4x+5}+\sqrt{5}\right)} \\
&= \lim_{x\to 0^+} \frac{x+4}{\sqrt{x^2+4x+5}+\sqrt{5}} \\
&= \frac{0+4}{\sqrt{0^2+4(0)+5}+\sqrt{5}} \\
&= \frac{4}{\sqrt{5}+\sqrt{5}} \\
&= \frac{4}{2\sqrt{5}} \\
&= \frac{2}{\sqrt{5}}
\end{aligned}$$

Exercise

Find $\lim\limits_{x\to -2^+} (x+3)\frac{|x+2|}{x+2}$

Solution

$$\lim\limits_{x\to -2^+} (x+3)\frac{|x+2|}{x+2} = (x+3)\frac{|-2+2|}{-2+2} = \frac{0}{0}$$

Since $x \to -2^+ \quad \Rightarrow \quad x < -2$

$$|x+2| = -(x+2)$$

$$\begin{aligned}\lim\limits_{x\to -2^+} (x+3)\frac{|x+2|}{x+2} &= \lim\limits_{x\to -2^+} (x+3)\frac{-(x+2)}{x+2} \\ &= \lim\limits_{x\to -2^+} \left[-(x+3)\right] \\ &= -(-2+3) \\ &= -1\end{aligned}$$

Exercise

Find $\lim\limits_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|}$

Solution

$$\lim\limits_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|} = \frac{\sqrt{2(1)}(1-1)}{|1-1|} = \frac{0}{0}$$

Since $x \to 1^+ \quad \Rightarrow \quad x > 1$

$$|x-1| = x-1$$

$$\begin{aligned}\lim\limits_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|} &= \lim\limits_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{x-1} \\ &= \lim\limits_{x\to 1^+} \sqrt{2x} \\ &= \sqrt{2}\end{aligned}$$

Exercise

Find $\lim\limits_{\theta\to 0}\dfrac{\sin\sqrt{2}.\theta}{\sqrt{2}.\theta}$

Solution

Let: $\sqrt{2}\theta = x$

$$\lim_{\theta\to 0}\frac{\sin\sqrt{2}.\theta}{\sqrt{2}.\theta} = \lim_{x\to 0}\frac{\sin x}{x} \, \underline{=1|}$$

Exercise

Find $\lim\limits_{x\to 0}\dfrac{\sin 3x}{4x}$

Solution

$$\lim_{x\to 0}\frac{\sin 3x}{4x} = \lim_{x\to 0}\frac{\sin 3x}{4x}\frac{3}{3}$$

$$= \frac{3}{4}\lim_{x\to 0}\frac{\sin 3x}{3x}$$ Let: $3x = u$

$$= \frac{3}{4}\lim_{u\to 0}\frac{\sin u}{u}$$ ***By definition:*** $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$

$$\underline{= \frac{3}{4}|}$$

Exercise

Find $\lim\limits_{x\to 0^-}\dfrac{x}{\sin 3x}$

Solution

$$\lim_{x\to 0^-}\frac{x}{\sin 3x} = \lim_{x\to 0^-}\frac{x}{\sin 3x}\left(\frac{3}{3}\right)$$

$$= \frac{1}{3}\lim_{x\to 0^-}\frac{3x}{\sin 3x}$$

$$= \frac{1}{3}\lim_{x\to 0^-}\frac{1}{\frac{\sin 3x}{3x}}$$

$$= \frac{1}{3}\frac{1}{\lim\limits_{x\to 0^-}\frac{\sin 3x}{3x}}$$ ***By definition:*** $\lim\limits_{u\to 0}\dfrac{\sin u}{u} = 1$

$$\underline{= \frac{1}{3}|}$$

Exercise

Find $\lim\limits_{x\to 0}\dfrac{\tan 2x}{x}$

Solution

$$\lim_{x\to 0}\frac{\tan 2x}{x}=\lim_{x\to 0}\frac{\frac{\sin 2x}{\cos 2x}}{x}$$

$$=\lim_{x\to 0}\left(\frac{\sin 2x}{x}\bullet\frac{1}{\cos 2x}\right)$$

$$=\lim_{x\to 0}\left(\frac{\sin 2x}{x}\right)\lim_{x\to 0}\left(\frac{1}{\cos 2x}\right)$$

$$=1\bullet\frac{1}{\cos 0}$$

$$=1\bullet\frac{1}{1}$$

$$=1$$

Exercise

Find $\lim\limits_{x\to 0} 6x^2(\cot x)(\csc 2x)$

Solution

$$\lim_{x\to 0} 6x^2(\cot x)(\csc 2x)=\lim_{x\to 0} 6x^2\left(\frac{\cos x}{\sin x}\right)\left(\frac{1}{\sin 2x}\right)$$

$$=\lim_{x\to 0} 3\cos x\left(\frac{x}{\sin x}\right)\left(\frac{2x}{\sin 2x}\right)$$

$$=3\lim_{x\to 0}(\cos x)\bullet\lim_{x\to 0}\left(\frac{x}{\sin x}\right)\bullet\lim_{x\to 0}\left(\frac{2x}{\sin 2x}\right)$$

$$=3\bullet 1\bullet 1\bullet 1$$

$$=3$$

Exercise

Find $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\sin 2\theta}$

Solution

$$\lim_{\theta\to 0}\frac{\sin\theta}{\sin 2\theta}=\lim_{\theta\to 0}\frac{\sin\theta}{\sin 2\theta}\frac{2\theta}{2\theta}$$

$$=\frac{1}{2}\lim_{\theta\to 0}\left(\frac{2\theta}{\sin 2\theta}\bullet\frac{\sin\theta}{\theta}\right)$$

$$=\frac{1}{2}\bullet 1\bullet 1$$

$$=\frac{1}{2}$$

Exercise

Find $\displaystyle\lim_{h\to 0}\frac{\sin(\sin h)}{\sin h}$

Solution

Let: $\sin h=\theta$

$$\theta=\sin h\xrightarrow[h\to 0]{}0$$

$$\lim_{h\to 0}\frac{\sin(\sin h)}{\sin h}=\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\ =1$$

Exercise

Find $\displaystyle\lim_{\theta\to 0}\frac{\theta\cot 4\theta}{\sin^2\theta\cot^2 2\theta}$

Solution

$$\lim_{\theta\to 0}\frac{\theta\cot 4\theta}{\sin^2\theta\cot^2 2\theta}=\lim_{\theta\to 0}\frac{\theta\frac{\cos 4\theta}{\sin 4\theta}}{\sin^2\theta\frac{\cos^2 2\theta}{\sin^2 2\theta}}$$

$$=\lim_{\theta\to 0}\theta\frac{\cos 4\theta}{2\sin 2\theta\cos 2\theta}\frac{\sin^2 2\theta}{\sin^2\theta\cos^2 2\theta}$$

$$=\lim_{\theta\to 0}\left(\frac{1}{2}\cdot\theta\cdot\cos 4\theta\bullet\frac{2\sin\theta\cos\theta}{\sin^2\theta}\cdot\frac{1}{\cos^3 2\theta}\right)$$

$$=\lim_{\theta\to 0}\left(\cos 4\theta\bullet\frac{\theta}{\sin\theta}\cdot\cos\theta\cdot\frac{1}{\cos^3 2\theta}\right)$$

$$=\lim_{\theta\to 0}(\cos 4\theta)\bullet\lim_{\theta\to 0}\left(\frac{\theta}{\sin\theta}\right)\bullet\lim_{\theta\to 0}\left(\frac{\cos\theta}{\cos^3 2\theta}\right)$$

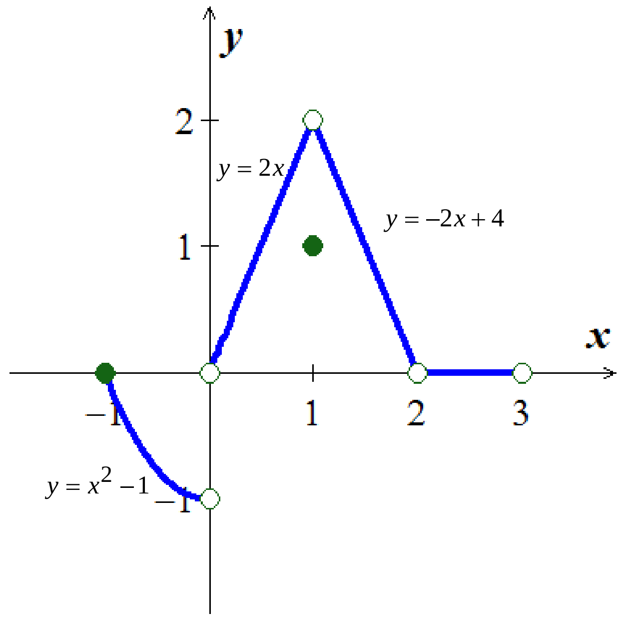
$$=1\bullet 1\bullet 1$$

$$=1$$

Solution   *Section* 1.5 – Continuity

Exercise

Given the graphed function $f(x)$



a) Does $f(-1)$ exist?

b) Does $\lim\limits_{x\to -1^+} f(x)$ exist?

c) Does $\lim\limits_{x\to -1^+} f(x) = f(-1)$?

d) Is f continuous at $x=-1$?

e) Does $f(1)$ exist?

f) Does $\lim\limits_{x\to 1} f(x)$ exist?

g) Does $\lim\limits_{x\to 1} f(x) = f(1)$?

h) Is f continuous at $x=1$?

Solution

a) Yes $f(-1)=0$

b) Yes, $\lim\limits_{x\to -1^+} f(x)=0$

c) Yes

d) Yes

e) Yes, $f(1)=1$

f) Yes, $\lim\limits_{x\to 1} f(x)=2$

g) No

h) No

Exercise

At what points is the function $y=\dfrac{1}{x-2}-3x$ continuous?

Solution

The function is continuous everywhere except when $x-2=0 \Rightarrow x=2$

Exercise

At what points is the function $y=\dfrac{x+3}{x^2-3x-10}$ continuous?

Solution

The function is continuous everywhere except when $x^2-3x-10=0 \Rightarrow x=-2,\ 5$

Exercise

At what points is the function $y=|x-1|+\sin x$ continuous?

Solution

The function is continuous everywhere

Exercise

At what points is the function $y=\dfrac{x+2}{\cos x}$ continuous?

Solution

The function is continuous everywhere except when $\cos x=0 \Rightarrow x=\frac{\pi}{2}+n\pi, \quad n\in\mathbb{Z}$

Exercise

At what points is the function $y=\tan\frac{\pi x}{2}$ continuous?

Solution

The function is continuous everywhere except when $x=2n-1, \quad n\in\mathbb{Z}$

Exercise

At what points is the function $y=\dfrac{x\tan x}{x^2+1}$ continuous?

Solution

The function is continuous everywhere except when $x=(2n-1)\frac{\pi}{2}, \quad n\in\mathbb{Z}$

Exercise

At what points is the function $y=\frac{\sqrt{x^4+1}}{1+\sin^2 x}$ continuous?

Solution

The function is continuous everywhere

Exercise

At what points is the function $y=\sqrt{2x+3}$ continuous?

Solution

The function is continuous on the interval $2x+3\geq 0 \quad \rightarrow \quad x\geq -\frac{3}{2} \Rightarrow \left[-\frac{3}{2},\ \infty\right)$, and discontinuous when $x<-\frac{3}{2}$

Exercise

At what points is the function $y=\sqrt[4]{3x-1}$ continuous?

Solution

The function is continuous on the interval $3x-1\geq 0 \quad \rightarrow \quad x\geq \frac{1}{3} \Rightarrow \left[\frac{1}{3},\ \infty\right)$, and discontinuous when $x<\frac{1}{3}$

Exercise

At what points is the function $y=(2-x)^{1/5}$ continuous?

Solution

The function is continuous everywhere $\forall x$

Exercise

Find $\lim\limits_{x\to\pi} \sin(x-\sin x)$, then is the function continuous at the point being approached?

Solution

$$
\begin{aligned}
\lim_{x\to\pi} \sin(x-\sin x) &= \sin(\pi-\sin\pi) \\
&= \sin(\pi-0) \\
&= \sin(\pi) \\
&= 0
\end{aligned}
$$

The functions is continuous at $x=\pi$

Exercise

Find $\lim\limits_{x\to 0} \tan\left(\frac{\pi}{4}\cos\left(\sin x^{1/3}\right)\right)$, then is the function continuous at the point being approached?

Solution

$$\lim_{x\to 0} \tan\left(\frac{\pi}{4}\cos\left(\sin x^{1/3}\right)\right) = \tan\left(\frac{\pi}{4}\cos\left(\sin(0)^{1/3}\right)\right)$$

$$= \tan\left(\frac{\pi}{4}\cos(0)\right)$$

$$= \tan\left(\frac{\pi}{4}\right)$$

$$\underline{=1|} \qquad \text{The functions is continuous at } x=0$$

Exercise

Find $\lim\limits_{t\to 0} \cos\left(\frac{\pi}{\sqrt{19-3\sec 2t}}\right)$, then is the function continuous at the point being approached?

Solution

$$\lim_{t\to 0} \cos\left(\frac{\pi}{\sqrt{19-3\sec 2t}}\right) = \cos\left(\frac{\pi}{\sqrt{19-3\sec 2(0)}}\right)$$

$$= \cos\left(\frac{\pi}{\sqrt{19-3}}\right)$$

$$= \cos\left(\frac{\pi}{\sqrt{16}}\right)$$

$$= \cos\left(\frac{\pi}{4}\right)$$

$$\underline{=\frac{\sqrt{2}}{2}|} \qquad \text{The functions is continuous at } t=0$$

Exercise

Explain why the equation $\cos x = x$ has at least one solution.

Solution

$\cos x - x = 0$

$$\begin{cases} \text{if } \ x=-\frac{\pi}{2} & \to \cos\left(-\frac{\pi}{2}\right)-\left(-\frac{\pi}{2}\right) > 0 \\ \text{if } \ x=\frac{\pi}{2} & \to \cos\left(\frac{\pi}{2}\right)-\left(\frac{\pi}{2}\right) < 0 \end{cases} \Rightarrow \cos x - x = 0 \ \text{ for some } x \text{ between } -\frac{\pi}{2} \text{ and } \frac{\pi}{2}$$

According to the Intermediate Value Theorem, and the function $\cos x = x$ is continuous and has at least one solution.

Exercise

Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval [–4, 4]

Solution

$$f(-4) = (-4)^3 - 15(-4) + 1 = -3$$

$$f(-2) = (-2)^3 - 15(-2) + 1 = 23$$

$$f(-1) = (-1)^3 - 15(-1) + 1 = 15$$

$$f(1) = (1)^3 - 15(1) + 1 = -13$$

$$f(4) = (4)^3 - 15(4) + 1 = 5$$

By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-4 < x < -1$, $-1 < x < 1$, and $1 < x < 4$.

Thus, $x^3 - 15x + 1 = 0$ has three solutions in [–4, 4]. Since the polynomial of degree 3 can have at most 3 solutions, these are the solutions.

Exercise

If functions $f(x)$ and $g(x)$ are continuous for $0 \le x \le 1$, could $\dfrac{f(x)}{g(x)}$ possibly be discontinuous at a point of [0, 1]? Give reason for your answer.

Solution

Yes, if we can get a value of $g(x)$ is between [0, 1], $x = \frac{1}{2} \Rightarrow g(x) = 2x - 1$ and $f(x) = x$.

Then $\dfrac{f(x)}{g(x)} = \dfrac{x}{2x-1} \Rightarrow \dfrac{f(x)}{g(x)}$ is discontinuous at $x = \frac{1}{2}$

Exercise

Suppose that a function f is continuous on the closed interval [0, 1] and that $0 \le f(x) \le 1$ for every x in [0, 1]. Show that there must exist a number c in [0, 1] such that $f(c) = c$ (***c*** is called a ***fixed point*** of f).

Solution

Let $f(x) = x \Rightarrow f(0) = 0 \quad or \quad f(1) = 1$. In these cases, $c = 0$ or $c = 1$.

Let $f(0) = a > 0 \quad and \quad f(1) = b < 1$ because $0 \le f(x) \le 1$.

Define $g(x) = f(x) - x \Rightarrow g$ is continuous on [0, 1].

$$\Rightarrow \begin{cases} g(0) = f(0) - 0 = a > 0 \\ g(1) = f(1) - 1 = b - 1 < 0 \end{cases}$$

By the Intermediate Value Theorem there is a number c in [0, 1] such that

$g(c) = 0 \Rightarrow f(c) - c = 0 \Rightarrow f(c) = c$

Solution Section 1.6 – Limits Involving Infinity; Asymptotes

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $h(x) = \dfrac{-5+\frac{7}{x}}{3-\frac{1}{x^2}}$

Solution

$$\lim_{x\to\infty} \frac{-5+\frac{7}{x}}{3-\frac{1}{x^2}} = -\frac{5}{3}$$

$$\lim_{x\to-\infty} \frac{-5+\frac{7}{x}}{3-\frac{1}{x^2}} = -\frac{5}{3}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{2x+3}{5x+7}$

Solution

$$\lim_{x\to\infty} \frac{2x+3}{5x+7} = \lim_{x\to\infty} \frac{2+\frac{3}{x}}{5+\frac{7}{x}} = \frac{2}{5}$$

$$\lim_{x\to-\infty} \frac{2x+3}{5x+7} = \lim_{x\to-\infty} \frac{2+\frac{3}{x}}{5+\frac{7}{x}} = \frac{2}{5}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{2x^3+7}{x^3-x^2+x+7}$

Solution

$$\lim_{x\to\infty} \frac{2x^3+7}{x^3-x^2+x+7} = \lim_{x\to\infty} \frac{2+\frac{7}{x^3}}{1-\frac{1}{x}+\frac{1}{x^2}+\frac{7}{x^3}} = 2$$

$$\lim_{x\to-\infty} \frac{2x^3+7}{x^3-x^2+x+7} = \lim_{x\to-\infty} \frac{2+\frac{7}{x^3}}{1-\frac{1}{x}+\frac{1}{x^2}+\frac{7}{x^3}} = 2$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{x+1}{x^2+3}$

Solution

$$\lim_{x\to\infty} \frac{x+1}{x^2+3} = \lim_{x\to\infty} \frac{\frac{x}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{3}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x^2}} \, \boxed{=0}$$

$$\lim_{x\to-\infty} \frac{x+1}{x^2+3} = \lim_{x\to-\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x^2}} \, \boxed{=0}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{7x^3}{x^3-3x^2+6x}$

Solution

$$\lim_{x\to\infty} \frac{7x^3}{x^3-3x^2+6x} = \lim_{x\to\infty} \frac{7}{1-\frac{3}{x}+\frac{6}{x^2}} \, \boxed{=7}$$

$$\lim_{x\to-\infty} \frac{7x^3}{x^3-3x^2+6x} = \lim_{x\to-\infty} \frac{7}{1-\frac{3}{x}+\frac{6}{x^2}} \, \boxed{=7}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{9x^4+x}{2x^4+5x^2-x+6}$

Solution

$$\lim_{x\to\infty} \frac{9x^4+x}{2x^4+5x^2-x+6} = \lim_{x\to\infty} \frac{\frac{9x^4}{x^4}+\frac{x}{x^4}}{\frac{2x^4}{x^4}+\frac{5x^2}{x^4}-\frac{x}{x^4}+\frac{6}{x^4}} = \lim_{x\to\infty} \frac{9+\frac{1}{x^3}}{2+\frac{5}{x^2}-\frac{1}{x^3}+\frac{6}{x^4}} \, \boxed{=\frac{9}{2}}$$

$$\lim_{x\to-\infty} \frac{9x^4+x}{2x^4+5x^2-x+6} = \lim_{x\to-\infty} \frac{9+\frac{1}{x^3}}{2+\frac{5}{x^2}-\frac{1}{x^3}+\frac{6}{x^4}} \, \boxed{=\frac{9}{2}}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f\left(x\right) = \dfrac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x}$

Solution

$$\lim_{x\to\infty} \frac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x} = \lim_{x\to\infty} \frac{-2 - \frac{2}{x^2} + \frac{3}{x^3}}{3 + \frac{3}{x} - \frac{5}{x^2}} = -\frac{2}{3}$$

$$\lim_{x\to-\infty} \frac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x} = \lim_{x\to-\infty} \frac{-2 - \frac{2}{x^2} + \frac{3}{x^3}}{3 + \frac{3}{x} - \frac{5}{x^2}} = -\frac{2}{3}$$

Exercise

Find $\lim\limits_{x\to-\infty} \dfrac{\cos x}{3x}$

Solution

$$-\frac{1}{3x} \le \frac{\cos x}{3x} \le \frac{1}{3x}, \quad -1 \le \cos x \le 1$$

$$\lim_{x\to-\infty} \frac{\cos x}{3x} = 0 \qquad \textit{By the Sandwich Theorem}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{x + \sin x}{2x + 7 - 5\sin x}$

Solution

$$\lim_{x\to\infty} \frac{x + \sin x}{2x + 7 - 5\sin x} = \lim_{x\to\infty} \frac{1 + \frac{\sin x}{x}}{2 + \frac{7}{x} - \frac{5\sin x}{x}}$$

$$= \frac{1 + 0}{2 + 0 - 0}$$

$$= \frac{1}{2}$$

Exercise

Find $\lim\limits_{x\to\infty}\sqrt{\dfrac{8x^2-3}{2x^2+x}}$

Solution

$$\lim_{x\to\infty}\sqrt{\frac{8x^2-3}{2x^2+x}} = \lim_{x\to\infty}\sqrt{\frac{8-\frac{3}{x^2}}{2+\frac{1}{x}}}$$

$$=\sqrt{\frac{8}{2}}$$

$$=\sqrt{4}$$

$$\underline{=2}$$

Exercise

Find $\lim\limits_{x\to-\infty}\left(\dfrac{x^2+x-1}{8x^2-3}\right)^{1/3}$

Solution

$$\lim_{x\to-\infty}\left(\frac{x^2+x-1}{8x^2-3}\right)^{1/3} = \lim_{x\to-\infty}\left(\frac{1+\frac{1}{x}-\frac{1}{x^2}}{8-\frac{3}{x^2}}\right)^{1/3}$$

$$=\left(\frac{1}{8}\right)^{1/3}$$

$$\underline{=\frac{1}{2}}$$

Exercise

Find $\lim\limits_{x\to\infty}\dfrac{2\sqrt{x}+x^{-1}}{3x-7}$

Solution

$$\lim_{x\to\infty}\frac{2\sqrt{x}+x^{-1}}{3x-7} = \lim_{x\to\infty}\frac{\frac{2\sqrt{x}}{x}+\frac{x^{-1}}{x}}{3-\frac{7}{x}} = \lim_{x\to\infty}\frac{\frac{2}{x^{1/2}}+\frac{1}{x^2}}{3-\frac{7}{x}}\ \underline{=0}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{x^{-1}+x^{-4}}{x^{-2}+x^{-3}}$

Solution

$$\lim_{x\to\infty} \frac{x^{-1}+x^{-4}}{x^{-2}+x^{-3}} = \lim_{x\to\infty} \frac{\frac{x^{-1}}{x^{-2}}+\frac{x^{-4}}{x^{-2}}}{\frac{x^{-2}}{x^{-2}}+\frac{x^{-3}}{x^{-2}}}$$

$$= \lim_{x\to\infty} \frac{x+\frac{1}{x^2}}{1+\frac{1}{x}}$$

$$\underline{= \infty}$$

Exercise

Find $\lim\limits_{x\to-\infty} \dfrac{4-3x^3}{\sqrt{x^6+9}}$

Solution

$$\lim_{x\to-\infty} \frac{4-3x^3}{\sqrt{x^6+9}} = \lim_{x\to-\infty} \frac{\frac{4-3x^3}{\sqrt{x^6}}}{\frac{\sqrt{x^6+9}}{\sqrt{x^6}}}$$

$$= \lim_{x\to-\infty} \frac{\frac{4-3x^3}{x3}}{\sqrt{\frac{x^6+9}{x^6}}}$$

$$= \lim_{x\to-\infty} \frac{\frac{4}{x3}-3}{\sqrt{1+\frac{9}{x^6}}}$$

$$= \frac{4}{\sqrt{1}}$$

$$\underline{= 4}$$

Exercise

Find $\lim\limits_{x\to 0^+} \frac{1}{3x}$

Solution

$$\lim_{x\to 0^+} \frac{1}{3x} \ \boxed{=\infty}$$

Exercise

Find $\lim\limits_{x\to -5^-} \frac{3x}{2x+10}$

Solution

$$\lim_{x\to -5^-} \frac{3x}{2x+10} = \lim_{x\to -5^-} \frac{3}{2+\frac{10}{x}} \ \boxed{=\infty}$$

Exercise

Find $\lim\limits_{x\to 0} \frac{1}{x^{2/3}}$

Solution

$$\lim_{x\to 0} \frac{1}{x^{2/3}} = \lim_{x\to 0} \frac{1}{\left(x^{1/3}\right)^2} \ \boxed{=\infty}$$

Exercise

Find $\lim\limits_{x\to 0^-} \frac{1}{3x^{1/3}}$

Solution

$$\lim_{x\to 0^-} \frac{1}{3x^{1/3}} \ \boxed{=-\infty}$$

Exercise

Find $\displaystyle\lim_{x\to\left(-\frac{\pi}{2}\right)^+} \sec x$

Solution

$$\lim_{x\to\left(-\frac{\pi}{2}\right)^+} \sec x \underline{\underline{=\infty}}$$

Exercise

Find $\displaystyle\lim_{\theta\to 0^-} (1+\csc\theta)$

Solution

$$\lim_{\theta\to 0^-} (1+\csc\theta) \lim_{\theta\to 0^-}\left(1+\frac{1}{\sin\theta}\right) \underline{\underline{=-\infty}}$$

Exercise

Find $\displaystyle\lim_{x\to-\infty}\left(\sqrt{x^2+3}+x\right)$

Solution

$$\begin{aligned}
\lim_{x\to-\infty}\left(\sqrt{x^2+3}+x\right) &= \lim_{x\to-\infty}\left(\sqrt{x^2+3}+x\right)\frac{\sqrt{x^2+3}-x}{\sqrt{x^2+3}-x} \\
&= \lim_{x\to-\infty}\frac{x^2+3-x^2}{\sqrt{x^2+3}-x} \\
&= \lim_{x\to-\infty}\frac{x^2+3-x^2}{\sqrt{x^2+3}-x} \\
&= \lim_{x\to-\infty}\frac{3}{\sqrt{x^2+3}-x} \\
&= \lim_{x\to-\infty}\frac{\frac{3}{x}}{\sqrt{\frac{x^2}{x^2}+\frac{3}{x^2}}-\frac{x}{x}} \\
&= \lim_{x\to-\infty}\frac{\frac{3}{x}}{\sqrt{1+\frac{3}{x^2}}+1} \\
&= \frac{0}{\sqrt{1}+1} \\
&\underline{\underline{=0}}
\end{aligned}$$

Exercise

Find $\lim\limits_{x\to\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-2x}\right)$

Solution

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-2x}\right) &= \lim_{x\to\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-2x}\right)\frac{\sqrt{x^2+3x}+\sqrt{x^2-2x}}{\sqrt{x^2+3x}+\sqrt{x^2-2x}} \\
&= \lim_{x\to\infty}\frac{\left(x^2+3x\right)-\left(x^2-2x\right)}{\sqrt{x^2+3x}+\sqrt{x^2-2x}} \\
&= \lim_{x\to\infty}\frac{x^2+3x-x^2+2x}{\sqrt{x^2+3x}+\sqrt{x^2-2x}} \\
&= \lim_{x\to\infty}\frac{5x}{\sqrt{x^2+3x}+\sqrt{x^2-2x}} \\
&= \lim_{x\to\infty}\frac{\frac{5x}{\sqrt{x^2}}}{\sqrt{\frac{x^2}{x^2}+\frac{3x}{x^2}}+\sqrt{\frac{x^2}{x^2}-\frac{2x}{x^2}}} \\
&= \lim_{x\to\infty}\frac{5}{\sqrt{1+\frac{3}{x}}+\sqrt{1-\frac{2}{x}}} \\
&= \frac{5}{\sqrt{1}+\sqrt{1}} \\
&= \boxed{\frac{5}{2}}
\end{aligned}$$

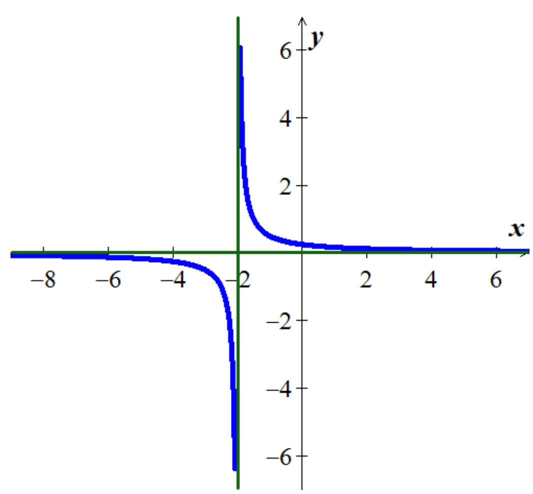
Exercise

Graph the rational function $y=\dfrac{1}{2x+4}$. Include the equations of the asymptotes.

Solution

VA: $2x=4=0 \Rightarrow \boxed{x=-2}$

HA: $\boxed{y=0}$



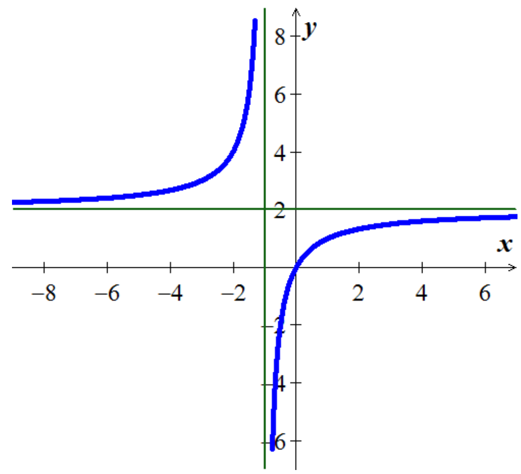
Exercise

Graph the rational function $y=\dfrac{2x}{x+1}$. Include the equations of the asymptotes.

Solution

VA: $\boxed{x=-1}$

HA: $\boxed{y=2}$



Exercise

Graph the rational function $y=\dfrac{x^2}{x-1}$. Include the equations of the asymptotes.

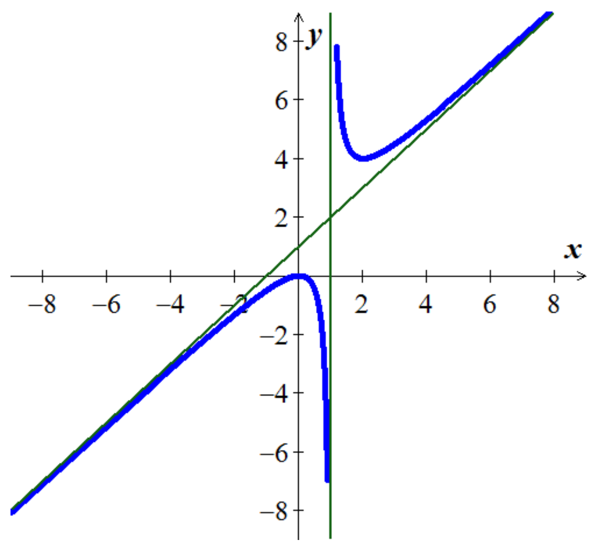
Solution

$$\begin{array}{r} x+1 \\ x-1\overline{)\;x^2\quad} \\ \underline{x^2-x} \\ x \\ \underline{x-1} \\ \boxed{1} \end{array}$$

$$y=\frac{x^2}{x-1}=x+1+\frac{1}{x-1}$$

VA: $\boxed{x=1}$

Oblique Asymptote: $\boxed{y=x+1}$



Exercise

Graph the rational function $y = \dfrac{x^3 + 1}{x^2}$. Include the equations of the asymptotes.

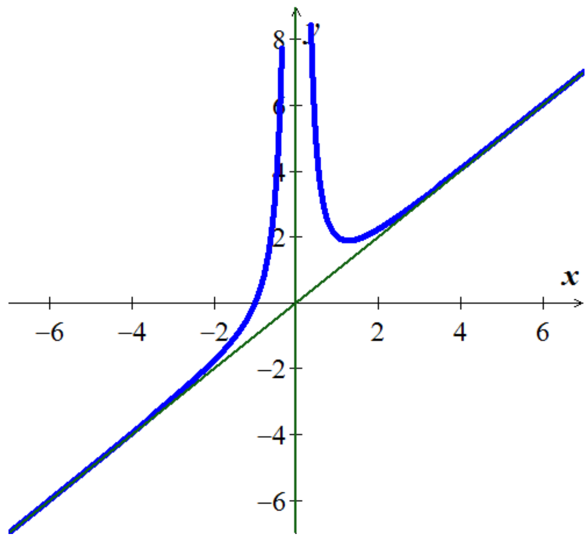
Solution

$$\begin{array}{r} x \\ x^2 \overline{\smash{\big)}\, x^3 + 1} \\ \underline{x^3 \quad\;\;} \\ \boxed{1} \end{array}$$

$$y = \frac{x^3 + 1}{x^2} = x + \frac{1}{x^2}$$

VA: $\underline{x = 0|}$

Oblique Asymptote: $\underline{y = x|}$



Exercise

Find the vertical, horizontal and oblique asymptotes (if any) of $y = \dfrac{3x}{1 - x}$

Solution

$\boldsymbol{VA}: x = 1$

$\boldsymbol{HA}: y = -3$

Exercise

Find the vertical, horizontal and oblique asymptotes (if any) of $y = \dfrac{x^2}{x^2 + 9}$

Solution

$\boldsymbol{HA}: y = 1$

Exercise

Find the vertical, horizontal and oblique asymptotes (if any) of $y = \dfrac{x - 2}{x^2 - 4x + 3}$

Solution

$\boldsymbol{VA}: x = 1, 3$

$\boldsymbol{HA}: y = 0$

Exercise

Find the vertical, horizontal and oblique asymptotes (if any) of $y=\dfrac{3}{x-5}$

Solution

VA: $x = 5$

HA: $y = 0$

Exercise

Find the vertical, horizontal and oblique asymptotes (if any) of $y=\dfrac{x^3-1}{x^2+1}$

Solution

$$
\begin{array}{r|l}
 & x \\
x^2+1 & x^3-1 \\
 & \underline{x^3+x} \\
 & \boxed{-x-1}
\end{array}
$$

$$y=\frac{x^3-1}{x^2+1}=x+\frac{-x-1}{x^2+1}=x-\frac{x+1}{x^2+1}$$

Oblique Asymptote: $y=x$

Exercise

Find the vertical, horizontal and oblique asymptotes (if any) of $y=\dfrac{3x^2-27}{(x+3)(2x+1)}$

Solution

VA: $x = -3,\ -\frac{1}{2}$

HA: $y = \frac{3}{2}$

Exercise

Find the vertical, horizontal and oblique asymptotes (if any) of $y = \dfrac{x^3 + 3x^2 - 2}{x^2 - 4}$

Solution

$$\begin{array}{r|l} & x+3 \\ \hline x^2-4 & x^3+3x^2-2 \\ & \underline{x^3 \qquad -4x} \\ & \quad 3x^2+4x-2 \\ & \quad \underline{3x^2 \qquad -12} \\ & \qquad \boxed{4x+10} \end{array}$$

$$y = \frac{x^3 + 3x^2 - 2}{x^2 - 4} = x + 3 + \frac{4x+10}{x^2 - 4}$$

VA: $x = \pm 2$

Oblique Asymptote: $y = x + 3$

Exercise

Find the vertical, horizontal and oblique asymptotes (if any) of $y = \dfrac{x-3}{x^2 - 9}$

Solution

VA: $x = -3$

Hole: $x = 3$

Exercise

Find the vertical, horizontal and oblique asymptotes (if any) of $y = \dfrac{6}{\sqrt{x^2 - 4x}}$

Solution

VA: $x = 0, 4$

Exercise

Find the vertical, horizontal and oblique asymptotes (if any) of $y = \dfrac{5x-1}{1-3x}$

Solution

VA: $x = \frac{1}{3}$

HA: $y = \frac{5}{3}$