Solution Section 4.4 – Green's Theorem

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = (x-y)\,\hat{i} + (y-x)\,\hat{j}$ and curve C is the square bounded by $x=0,\ x=1,\ y=0,\ y=1$

Solution

$$M = x - y \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = -1$$

$$N = y - x \quad \Rightarrow \quad \frac{\partial N}{\partial x} = -1, \quad \frac{\partial N}{\partial y} = 1$$

$$\begin{aligned} Flux &= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy \\ &= \iint_R (1+1)\, dxdy \\ &= 2\int_0^1 \int_0^1 dxdy \\ &= 2\int_0^1 dy \\ &= 2 \end{aligned}$$

$$\begin{aligned} Circulation &= \iint_R \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dxdy \\ &= \int_0^1 \int_0^1 \big(-1-(-1)\big)\, dxdy \\ &= 0 \end{aligned}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \left(x^2 + 4y\right)\hat{i} + \left(x + y^2\right)\hat{j}$ and curve C is the square bounded by $x=0,\ x=1,\ y=0,\ y=1$

Solution

$$M = x^2 + 4y \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 2x, \quad \frac{\partial M}{\partial y} = 4$$

$$N = x + y^2 \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 1, \quad \frac{\partial N}{\partial y} = 2y$$

$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_0^1 (2x + 2y)\, dxdy$$

$$= \int_0^1 \left(x^2 + 2yx \Big|_0^1 \right. dy$$

$$= \int_0^1 (1 + 2y)\, dy$$

$$= y + y^2 \Big|_0^1$$

$$\underline{= 2}$$

$$Circulation = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_0^1 (1 - 4)\, dxdy$$

$$= -3 \int_0^1 dy$$

$$\underline{= -3}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = (x + y)\, \hat{i} - \left(x^2 + y^2\right) \hat{j}$ and curve C is the triangle bounded by $y = 0,\ x = 1,\ y = x$

Solution

$$M = x + y \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = 1$$

$$N = -\left(x^2 + y^2\right) \quad \Rightarrow \quad \frac{\partial N}{\partial x} = -2x, \quad \frac{\partial N}{\partial y} = -2y$$

$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_0^x (1-2y)\,dydx$$

$$= \int_0^1 \left(y - y^2 \,\Big|_0^x \right. dx$$

$$= \int_0^1 \left(x - x^2 \right) dx$$

$$= \frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$\boxed{= \frac{1}{6}}$$

$$Circulation = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_0^x (-2x-1)\,dydx$$

$$= \int_0^1 \left(-2xy - y \,\Big|_0^x \right. dx$$

$$= \int_0^1 \left(-2x^2 - x \right) dx$$

$$= -\frac{2}{3}x^3 - \frac{1}{2}x^2 \Big|_0^1$$

$$= -\frac{2}{3} - \frac{1}{2}$$

$$\boxed{= -\frac{7}{6}}$$

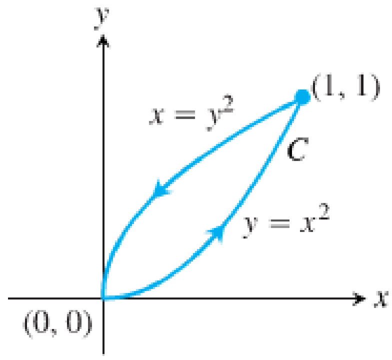
Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \left(xy + y^2 \right) \hat{i} + (x - y)\hat{j}$ and curve C

Solution

$$M = xy + y^2 \quad \Rightarrow \quad \frac{\partial M}{\partial x} = y, \quad \frac{\partial M}{\partial y} = x + 2y$$

$$N = x - y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 1, \quad \frac{\partial N}{\partial y} = -1$$



$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (y-1)\, dydx$$

$$= \int_0^1 \left(\frac{1}{2} y^2 - y \;\Big|_{x^2}^{\sqrt{x}} \right. dx$$

$$= \int_0^1 \left(\frac{1}{2} x - \sqrt{x} - \left(\frac{1}{2} x^4 - x^2 \right) \right) dx$$

$$= \int_0^1 \left(\frac{1}{2} x - x^{1/2} - \frac{1}{2} x^4 + x^2 \right) dx$$

$$= \frac{1}{4} x^2 - \frac{2}{3} x^{3/2} - \frac{1}{10} x^5 + \frac{1}{3} x^3 \Big|_0^1$$

$$= \frac{1}{4} - \frac{2}{3} - \frac{1}{10} + \frac{1}{3}$$

$$= -\frac{11}{60}$$

$$Circulation = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (1 - x - 2y)\, dydx$$

$$= \int_0^1 \left(y - xy - y^2 \;\Big|_{x^2}^{\sqrt{x}} \right. dx$$

$$= \int_0^1 \left(\sqrt{x} - x\sqrt{x} - x - x^2 + x^3 + x^4 \right) dx$$

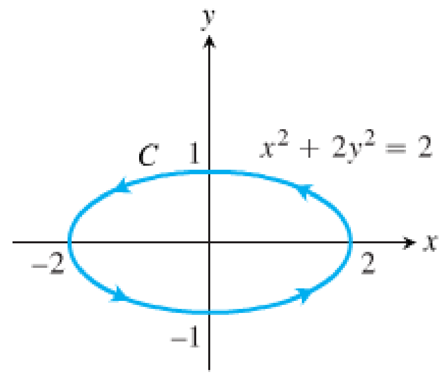
$$= \frac{2}{3} x^{3/2} - \frac{2}{5} x^{5/2} - \frac{1}{2} x^2 - \frac{1}{3} x^3 + \frac{1}{4} x^4 + \frac{1}{5} x^5 \Big|_0^1$$

$$= \frac{2}{3} - \frac{2}{5} - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

$$= -\frac{7}{60}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = (x+3y)\,\hat{i} + (2x-y)\,\hat{j}$ and curve C



Solution

$$M = x+3y \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = 3$$

$$N = 2x-y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 2, \quad \frac{\partial N}{\partial y} = -1$$

$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{(2-x^2)/2}}^{\sqrt{(2-x^2)/2}} (1-1)\, dydx$$

$$\underline{= 0}$$

$$Circulation = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{(2-x^2)/2}}^{\sqrt{(2-x^2)/2}} (2-3)\, dydx$$

$$= -\int_{-\sqrt{2}}^{\sqrt{2}} \left(\sqrt{\frac{2-x^2}{2}} + \sqrt{\frac{2-x^2}{2}}\right) dx$$

$$= -\frac{2}{\sqrt{2}} \int_{-\sqrt{2}}^{\sqrt{2}} \left(\sqrt{2-x^2}\right) dx$$

$$x = \sqrt{2}\,\sin\alpha \qquad \sqrt{2-x^2} = \sqrt{2}\,\cos\alpha$$

$$dx = \sqrt{2}\,\cos\alpha\; d\alpha$$

$$\int \sqrt{2-x^2}\; dx = \int \sqrt{2}\cos\alpha\left(\sqrt{2}\cos\alpha\right) d\alpha$$

$$= 2\int \cos^2\alpha\; d\alpha$$

$$= \int (1+\cos 2\alpha)\; d\alpha$$

$$= \alpha + \frac{1}{2}\sin 2\alpha$$

$$= \alpha + \sin\alpha\ \cos\alpha$$

$$= \sin^{-1}\frac{x}{\sqrt{2}} + \frac{x}{\sqrt{2}}\frac{\sqrt{2-x^2}}{\sqrt{2}}$$

$$= \sin^{-1}\frac{x}{\sqrt{2}} + \frac{1}{2}x\sqrt{2-x^2}$$

$$= -\frac{2}{\sqrt{2}}\left(\sin^{-1}\frac{x}{\sqrt{2}} + \frac{1}{2}x\sqrt{2-x^2}\right.\Bigg|_{-\sqrt{2}}^{\sqrt{2}}$$

$$= -\frac{2}{\sqrt{2}}\left[\sin^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) - \sin^{-1}\left(\frac{-\sqrt{2}}{\sqrt{2}}\right)\right]$$

$$= -\frac{2}{\sqrt{2}}\left(\sin^{-1}(1) + \sin^{-1}(1)\right)$$

$$= -\frac{2}{\sqrt{2}}\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$= -\frac{2\pi}{\sqrt{2}}$$

$$\underline{= -\pi\sqrt{2}}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \left(x + e^x \sin y\right)\hat{i} + \left(x + e^x \cos y\right)\hat{j}$ and curve C is the right-hand loop of the lemniscate $r^2 = \cos 2\theta$

Solution

$$M = x + e^x \sin y \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 1 + e^x \sin y, \quad \frac{\partial M}{\partial y} = e^x \cos y$$

$$N = x + e^x \cos y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 1 + e^x \cos y, \quad \frac{\partial N}{\partial y} = -e^x \sin y$$

$$Flux = \iint_R \left(1 + e^x \sin y - e^x \sin y\right) dxdy$$

$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$= \iint_R dxdy$$

$$= \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} r\ drd\theta$$

$$= \int_{-\pi/4}^{\pi/4} \left(\frac{1}{2}r^2 \Big|_0^{\sqrt{\cos 2\theta}}\right) d\theta$$

$$= \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos 2\theta \, d\theta$$

$$= \frac{1}{4} \sin 2\theta \Big|_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{4}\left(1-(-1)\right)$$

$$= \frac{1}{2}$$

$$Circulation = \iint_R \left(1+e^x \cos y - e^x \cos y\right) dxdy \qquad Circulation = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= \iint_R dxdy$$

$$= \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} r \, dr d\theta$$

$$= \frac{1}{2}$$

Exercise

Use Green's Theorem to find the counterclockwise circulation and outward flux for the field and curves

Square: $\vec{F} = (2xy + x)\,\hat{i} + (xy - y)\,\hat{j}$ $\quad C:$ *The square bounded by* $x = 0,\ x = 1,\ y = 0,\ y = 1$

Solution

$$M = 2xy + x \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 2y + 1, \quad \frac{\partial M}{\partial y} = 2x$$

$$N = xy - y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = y, \quad \frac{\partial N}{\partial y} = x - 1$$

$$Flux = \iint_R (2y + 1 + x - 1)\, dxdy \qquad Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$= \int_0^1 \int_0^1 (2y + x)\, dydx$$

$$= \int_0^1 \left(y^2 + xy \, \Big|_0^1 \right) dx$$

$$= \int_0^1 (1+x)\,dx$$

$$= x + \frac{1}{2}x^2 \,\Big|_0^1$$

$$= 1 + \frac{1}{2}$$

$$= \frac{3}{2}$$

$$Cir = \int_0^1 \int_0^1 (y-2x)\,dydx \qquad\qquad Circulation = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \int_0^1 \left(\frac{1}{2}y^2 - 2xy \, \Big|_0^1 \right) dx$$

$$= \int_0^1 \left(\frac{1}{2} - 2x \right) dx$$

$$= \frac{1}{2}x - x^2 \,\Big|_0^1$$

$$= \frac{1}{2} - 1$$

$$= -\frac{1}{2}$$

Exercise

Use Green's Theorem to find the counterclockwise circulation and outward flux for the field and curves

Triangle: $\vec{F} = \left(y - 6x^2 \right) \hat{i} + \left(x + y^2 \right) \hat{j}$

C: *The triangle made by the lines* $y = 0,\ y = x,$ *and* $x = 1$

Solution

$M = y - 6x^2 \qquad N = x + y^2$

$\frac{\partial M}{\partial x} = -12x \qquad \frac{\partial N}{\partial x} = 1$

$\frac{\partial M}{\partial x} = -12x \qquad \frac{\partial N}{\partial y} = 2y$

$$Flux = \int_0^1 \int_y^1 (-12x + 2y)\,dxdy \qquad\qquad Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$= \int_0^1 \left(-6x^2 + 2yx \,\Big|_y^1\right) dy$$

$$= \int_0^1 \left(-6 + 2y + 6y^2 - 2y^2 \,\Big|_y^1\right) dy$$

$$= \int_0^1 \left(4y^2 + 2y - 6\right) dy$$

$$= \frac{4}{3}y^3 + y^2 - 6y \,\Big|_0^1$$

$$= \frac{4}{3} + 1 - 6$$

$$\boxed{= -\frac{11}{3}}$$

$$Cir = \iint_R (1-1)\,dydx \qquad\qquad Circulation = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$\boxed{= 0}$$

Exercise

Find the circulation and the outward flux of the vector field $\vec{F} = \langle y - x,\ y\rangle$ for the curve $\vec{r}(t) = \langle 2\cos t,\ 2\sin t\rangle, \quad 0 \le t \le 2\pi$

Solution

$$\vec{F} = \langle 2\sin t - 2\cos t,\ 2\sin t\rangle$$

$$\vec{r}\,' = \langle -2\sin t,\ 2\cos t\rangle$$

$$\vec{F} \bullet \vec{r}\,' = \langle 2\sin t - 2\cos t,\ 2\sin t\rangle \bullet \langle -2\sin t,\ 2\cos t\rangle$$

$$= -4\sin^2 t + 4\cos t \sin t + 4\sin t \cos t$$

$$= -4\sin^2 t + 8\cos t \sin t$$

$$Cir = \int_0^{2\pi} (2\cos 2t - 2 + 4\sin 2t)\,dt \qquad\qquad Circulation = \int_C \vec{F} \bullet \vec{T}\ ds$$

$$= \sin 2t - 2t - 2\cos 2t \,\Big|_0^{2\pi}$$

$= -4\pi - 2 + 2$

$\underline{= -4\pi}\,\rfloor$

$dy = d\left(2\sin t\right) = 2\cos t\ dt$

$dx = d\left(2\cos t\right) = -2\sin 2t\ dt$

$$Flux = \int_0^{2\pi} \left(\left(2\sin t - 2\cos t\right)\left(2\cos t\right) - \left(2\sin t\right)\left(-2\sin t\right)\right) dt \qquad Flux = \int_C \left(Mdy - Ndx\right)\ dt$$

$$= \int_0^{2\pi} \left(4\sin t\cos t - 4\cos^2 t + 4\sin^2 t\right) dt$$

$$= \int_0^{2\pi} \left(2\sin 2t - 4\cos 2t\right) dt$$

$$= -\cos 2t - 2\sin 2t \Big|_0^{2\pi}$$

$= -1 + 1$

$\underline{= 0}\,\rfloor$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \langle x,\ y\rangle$; where R is the half-annulus $\left\{\left(r,\ \theta\right):\ \ 1 \le r \le 2,\quad 0 \le \theta \le \pi\right\}$

Solution

$M = y \quad \rightarrow \quad M_y = 1$

$N = x \quad \rightarrow \quad N_x = 1$

$$Cir = \iint_R \left(1 - 1\right) dA \qquad Cir = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$\underline{= 0}\,\rfloor$

$M = x \quad \rightarrow \quad M_x = 1$

$N = y \quad \rightarrow \quad N_y = 1$

$$Flux = \iint_R \left(1 + 1\right) dA \qquad Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$= 2\int_0^{\pi} d\theta\ \int_1^2 r\ dr$$

$$= 2\pi \left(\frac{1}{2}r^2\right|_1^2$$

$$\underline{= 3\pi}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \langle -y,\ x\rangle$; where R is the annulus $\{(r,\ \theta):\ \ 1 \le r \le 3,\ \ \ 0 \le \theta \le 2\pi\}$

Solution

$$Cir = \iint_R \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right) dA \qquad\qquad Cir = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= \iint_R (1+1)\, dA$$

$$= 2\int_0^{2\pi} d\theta \int_1^3 r\, dr$$

$$= 4\pi \left(\frac{1}{2}r^2\right|_1^3$$

$$\underline{= 16\pi}$$

$$Flux = \iint_R \left(\frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x)\right) dA \qquad\qquad Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$= \iint_R (0-0)\, dA$$

$$\underline{= 0}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \langle 2x+y,\ x-4y\rangle$; where R is the quarter-annulus $\left\{(r,\ \theta):\ \ 1 \le r \le 4,\ \ \ 0 \le \theta \le \frac{\pi}{2}\right\}$

Solution

$$Cir = \iint_R \left(\frac{\partial}{\partial x}(x-4y) - \frac{\partial}{\partial y}(2x+y)\right) dA \qquad\qquad Cir = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= \iint_R (1-1)\, dA$$

$$\underline{= 0}$$

$$Flux = \iint_R \left(\frac{\partial}{\partial x}(2x+y) + \frac{\partial}{\partial y}(x-4y) \right) dA \qquad Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \iint_R (2-4)\, dA$$

$$= -2 \int_0^{\frac{\pi}{2}} d\theta \int_1^4 r\, dr$$

$$= -\pi \left(\frac{1}{2} r^2 \right. \Big|_1^4$$

$$\underline{= -\frac{15}{2}\pi}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \langle x-y,\ 2y-x \rangle$; where R is the parallelogram $\{(x,\ y):\ \ 1-x \le y \le 3-x,\quad 0 \le x \le 1\}$

Solution

$$Cir = \iint_R \left(\frac{\partial}{\partial x}(2y-x) - \frac{\partial}{\partial y}(x-y) \right) dA \qquad Cir = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \iint_R (-1+1)\, dA$$

$$\underline{= 0}$$

$$Flux = \iint_R \left(\frac{\partial}{\partial x}(x-y) + \frac{\partial}{\partial y}(2y-x) \right) dA \qquad Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \iint_R (1+2)\, dA$$

$$= 3 \int_0^1 \int_{1-x}^{3-x} dydx$$

$$= 3 \int_0^1 y \Big|_{1-x}^{3-x} dx$$

$$= 3\int_0^1 2\, dx$$

$$\underline{= 6}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \left\langle \ln\left(x^2+y^2\right),\ \tan^{-1}\frac{y}{x} \right\rangle$; where R is the annulus $\left\{(r,\ \theta):\ \ 1 \le r \le 2,\ \ 0 \le \theta \le 2\pi\right\}$

Solution

$$Cir = \iint_R \left(\frac{\partial}{\partial x}\left(\tan^{-1}\frac{y}{x}\right) - \frac{\partial}{\partial y}\left(\ln\left(x^2+y^2\right)\right)\right) dA$$

$$Cir = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= \iint_R \left(-\frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2} - \frac{2y}{x^2+y^2}\right) dA$$

$$\left(\tan^{-1} u\right)' = \frac{u'}{1+u^2}$$

$$= \iint_R \left(-\frac{y}{x^2+y^2} - \frac{2y}{x^2+y^2}\right) dA$$

$$= -3\iint_R \left(\frac{y}{x^2+y^2}\right) dA$$

$$= -3\int_0^{2\pi}\int_1^2 \frac{r\sin\theta}{r^2} r\, drd\theta$$

$$= -3\int_0^{2\pi} \sin\theta\, d\theta \int_1^2 dr$$

$$= 3\left(\cos\theta \Big|_0^{2\pi}\right) \left(r \Big|_1^2\right)$$

$$= 3(1-1)(1)$$

$$\underline{= 0}$$

$$Flux = \iint_R \left(\frac{\partial}{\partial x}\left(\ln\left(x^2+y^2\right)\right) + \frac{\partial}{\partial y}\left(\tan^{-1}\frac{y}{x}\right)\right) dA$$

$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$= \iint_R \left(\frac{2x}{x^2+y^2} + \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2} \right) dA \qquad \left(\tan^{-1} u\right)' = \frac{u'}{1+u^2}$$

$$= \iint_R \left(\frac{2x}{x^2+y^2} + \frac{x}{x^2+y^2} \right) dA$$

$$= 3\iint_R \frac{x}{x^2+y^2}\, dA$$

$$= 3\int_0^{2\pi} \int_1^2 \frac{r\cos\theta}{r^2} r\, dr d\theta$$

$$= 3\int_0^{2\pi} \cos\theta\, d\theta \int_1^2 dr$$

$$= 3\left(\sin\theta \,\Big|_0^{2\pi}\right) \left(r \,\Big|_1^2\right)$$

$$= 3(0)(1)$$

$$\underline{= 0}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \nabla\left(\sqrt{x^2+y^2}\right)$; where R is the half-annulus $\{(r,\ \theta):\ \ 1 \le r \le 3,\ \ \ 0 \le \theta \le \pi\}$

Solution

$$\vec{F} = \nabla\left(\sqrt{x^2+y^2}\right)$$

$$= \left\langle \frac{x}{\sqrt{x^2+y^2}},\ \ \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

$$Cir = \iint_R \left(\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2}} \right) - \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}} \right) \right) dA \qquad Cir = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \iint_R \left(-\frac{xy}{\left(x^2+y^2\right)^{3/2}} + \frac{xy}{\left(x^2+y^2\right)^{3/2}} \right) dA$$

$$\underline{= 0}$$

$$Flux = \iint_R \left(\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}} \right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}} \right) \right) dA \qquad Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \iint_R \left(\frac{x^2+y^2-x^2}{\left(x^2+y^2\right)^{3/2}} + \frac{x^2+y^2-y^2}{\left(x^2+y^2\right)^{3/2}} \right) dA$$

$$= \iint_R \frac{x^2+y^2}{\left(x^2+y^2\right)^{3/2}} dA$$

$$= \iint_R \frac{1}{\left(x^2+y^2\right)^{1/2}} dA$$

$$= \int_0^{\pi} \int_1^3 \frac{1}{r} r \, drd\theta$$

$$= \int_0^{\pi} d\theta \int_1^3 dr$$

$$= 2\pi$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \langle y\cos x, \; -\sin x \rangle$; where R is the square $\left\{ (x, y): \quad 0 \le x \le \frac{\pi}{2}, \quad 0 \le y \le \frac{\pi}{2} \right\}$

Solution

$$Cir = \iint_R \left(\frac{\partial}{\partial x}(-\sin x) - \frac{\partial}{\partial y}(y\cos x) \right) dA \qquad Cir = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \iint_R (-\cos x - \cos x) \, dA$$

$$= -2\int_0^{\frac{\pi}{2}} dy \int_0^{\frac{\pi}{2}} \cos x \, dx$$

$$= -\pi \sin x \Big|_0^{\frac{\pi}{2}}$$

$$= -\pi$$

$$Flux = \iint_R \left(\frac{\partial}{\partial x}(y\cos x) + \frac{\partial}{\partial y}(-\sin x) \right) dA \qquad Flux = \iint\limits_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \iint_R (-y\sin x + 0)\, dA$$

$$= -\int_0^{\frac{\pi}{2}} y\ dy \quad \int_0^{\frac{\pi}{2}} \sin x\ dx$$

$$= \frac{1}{2} y^2 \Big|_0^{\frac{\pi}{2}} \quad \cos x \Big|_0^{\frac{\pi}{2}}$$

$$\boxed{= -\frac{\pi^2}{8}}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \left\langle x + y^2,\ x^2 - y \right\rangle$; where $R = \left\{ (x,\ y):\ \ 3y^2 \le x \le 36 - y^2 \right\}$

Solution

$$x = 36 - y^2 = 3y^2$$

$$4y^2 = 36 \ \rightarrow\ \underline{y = \pm 3}$$

$$Cir = \iint_R \left(\frac{\partial}{\partial x}\left(x^2 - y\right) - \frac{\partial}{\partial y}\left(x + y^2\right) \right) dA \qquad Cir = \iint\limits_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \iint_R (2x - 2y)\, dA$$

$$= 2\int_{-3}^{3} \int_{3y^2}^{36-y^2} (x - y)\, dxdy$$

$$= 2\int_{-3}^{3} \left(\frac{1}{2}x^2 - yx \Big|_{3y^2}^{36-y^2} \right) dy$$

$$= 2\int_{-3}^{3} \left(648 - 36y^2 + \frac{1}{2}y^4 - 36y + y^3 - \frac{9}{2}y^4 + 3y^3 \right) dy$$

$$= 2\int_{-3}^{3} \left(648 - 36y - 36y^2 + 4y^3 - 4y^4 \right) dy$$

$$= 8\left(162y - \frac{9}{2}y^2 - 3y^3 + \frac{1}{4}y^4 - \frac{1}{5}y^5\right|_{-3}^{3}$$

$$= 8\left(486 - \frac{81}{2} - 81 + \frac{81}{4} - \frac{243}{5} + 486 + \frac{81}{2} - 81 - \frac{81}{4} - \frac{243}{5}\right)$$

$$= 8\left(810 - \frac{486}{5}\right)$$

$$\boxed{= \frac{28,512}{5}}$$

$$Flux = \iint_R \left(\frac{\partial}{\partial x}\left(x + y^2\right) + \frac{\partial}{\partial y}\left(x^2 - y\right)\right) dA \qquad Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$= \iint_R (1 - 1)\, dA$$

$$\boxed{= 0}$$

Exercise

Find the outward flux for the field $\boldsymbol{F} = \left(3xy - \frac{x}{1+y^2}\right)\boldsymbol{i} + \left(e^x + \tan^{-1} y\right)\boldsymbol{j}$ across the cardioid $r = a(1 + \cos\theta), \;\; a > 0$

Solution

$$M = 3xy - \frac{x}{1+y^2} \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 3y - \frac{1}{1+y^2}$$

$$N = e^x + \tan^{-1} y \quad \Rightarrow \quad \frac{\partial N}{\partial y} = \frac{1}{1+y^2}$$

$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$= \iint_R \left(3y - \frac{1}{1+y^2} + \frac{1}{1+y^2}\right) dxdy$$

$$= \iint_R 3y\; dxdy$$

$$= 3\int_0^{2\pi} \int_0^{a(1+\cos\theta)} (r\sin\theta)\; rdrd\theta$$

$$= 3\int_0^{2\pi} \frac{1}{3}\sin\theta \left(r^3 \right|_0^{a(1+\cos\theta)} d\theta$$

$$= a^3 \int_0^{2\pi} \sin\theta\left(1+\cos\theta\right)^3 \, d\theta \qquad\qquad d\left(1+\cos\theta\right) = -\sin\theta d\theta$$

$$= -a^3 \int_0^{2\pi} \left(1+\cos\theta\right)^3 \, d\left(1+\cos\theta\right)$$

$$= -\frac{1}{4}a^3\left(1+\cos\theta\right)^4 \Big|_0^{2\pi}$$

$$= -\frac{1}{4}a^3\left(2^4 - 2^4\right)$$

$$\underline{= 0}$$

Exercise

Find the work done by $\boldsymbol{F} = 2xy^3\boldsymbol{i} + 4x^2y^2\boldsymbol{j}$ in moving a particle once counterclockwise around the curve C: The boundary of the triangular region in the first quadrant enclosed by the x-axis, the line $x = 1$ and the curve $y = x^3$

Solution

$$M = 2xy^3 \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 6xy^2$$

$$N = 4x^2y^2 \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 8xy^2$$

$$Work = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy \qquad\qquad Work = \oint_C Mdx + Ndy$$

$$= \int_0^1 \int_0^{x^3} \left(8xy^2 - 6xy^2\right)dydx$$

$$= \int_0^1 \int_0^{x^3} \left(2xy^2\right)dydx$$

$$= \int_0^1 \left(\frac{2}{3}xy^3 \Big|_0^{x^3}\right) dx$$

$$= \frac{2}{3}\int_0^1 x^{10}\, dx$$

$$= \frac{2}{33}x^{11}\Big|_0^1$$

$$\underline{= \frac{2}{33}}$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint_C \left(y^2dx + x^2dy\right)$ C: The triangle bounded by $x = 0,\ x + y = 1,\ y = 0$

Solution

$$M = y^2 \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 2y$$

$$N = x^2 \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 2x$$

$$\begin{aligned}
\oint_C \left(y^2dx + x^2dy\right) &= \int_0^1 \int_0^{1-x} (2x - 2y)\, dydx \\
&= \int_0^1 \left(2xy - y^2 \Big|_0^{1-x}\right) dx \\
&= \int_0^1 \left(2x(1-x) - (1-x)^2\right) dx \\
&= \int_0^1 \left(2x - 2x^2 - 1 + 2x - x^2\right) dx \\
&= \int_0^1 \left(-3x^2 + 4x - 1\right) dx \\
&= -x^3 + 2x^2 - x \Big|_0^1 \\
&= -1 + 2 - 1 \\
&= 0
\end{aligned}$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint_C (3ydx + 2xdy)$ C: The boundary of $0 \le x \le \pi, \quad 0 \le y \le \sin x$

Solution

$$M = 3y \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 3$$

$$N = 2x \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 2$$

$$\oint_C (3ydx + 2xdy) = \int_0^{\pi} \int_0^{\sin x} (2-3)\,dydx$$

$$= -\int_0^{\pi} \left(y \Big|_0^{\sin x} \right) dx$$

$$= -\int_0^{\pi} \sin x \; dx$$

$$= \cos x \Big|_0^{\pi}$$

$$\underline{= -2}$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint_C \left(3y - e^{\sin x}\right)dx + \left(7x + \sqrt{y^4 + 1}\right)dy$: where C is the circle $x^2 + y^2 = 9$

Solution

$$\oint_C \left(3y - e^{\sin x}\right)dx + \left(7x + \sqrt{y^4+1}\right)dy = \iint_R \left(\frac{\partial}{\partial x}\left(7x + \sqrt{y^4+1}\right) - \frac{\partial}{\partial y}\left(3y - e^{\sin x}\right) \right) dA$$

$$= \iint_R (7-3)\,dA$$

$$= 4\iint_R dA$$

$$= 4\int_0^{2\pi} d\theta \; \int_0^3 r\; dr$$

$$= 8\pi \left(\frac{1}{2} r^2 \Big|_0^3 \right.$$

$$\underline{= 36\pi}$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint_C (3x-5y)\,dx+(x-6y)\,dy$: where C is the ellipse $\frac{x^2}{4}+y^2=1$

Solution

$$\oint_C (3x-5y)\,dx+(x-6y)\,dy = \iint_R \left(\frac{\partial}{\partial x}(x-6y)-\frac{\partial}{\partial y}(3x-5y)\right)dA$$

$$= \iint_R (1-(-5))\,dA$$

$$= 6\iint_R dA$$

$$\underline{= 6\times Area\ of\ ellipse}$$

$$\frac{x^2}{4}+y^2=1$$

$x=2\cos t \quad \to \quad dx=-2\sin t\ dt$

$y=\sin t \quad \to \quad dy=\cos t\ dt$

$$A=\frac{1}{2}\int_0^{2\pi}\left(2\cos t(\cos t)-\sin t(-2\sin t)\right)dt \qquad\qquad A=\frac{1}{2}\oint_C xdy-ydx$$

$$=\int_0^{2\pi}\left(\cos^2 t+\sin^2 t\right)dt$$

$$=\int_0^{2\pi} dt$$

$$\underline{=2\pi}$$

$$\oint_C (3x-5y)\,dx+(x-6y)\,dy \ \underline{=12\pi}$$

Exercise

Use either form of Green's Theorem to evaluate the line integral $\oint_C \left(x^3+xy\right)dy+\left(2y^2-2x^2y\right)dx$; C is the square with vertices $(\pm1,\ \pm1)$ with *counterclockwise* orientation

Solution

$$N = x^3+xy \quad \rightarrow \quad N_x = 3x^2+y$$

$$M = 2y^2-2x^2y \quad \rightarrow \quad M_y = 4y-2x^2$$

$$\begin{aligned}
\oint_C \left(x^3+xy\right)dy+\left(2y^2-2x^2y\right)dx &= \int_{-1}^{1}\int_{-1}^{1}\left(3x^2+y-4y+2x^2\right)dydx \\
&= \int_{-1}^{1}\int_{-1}^{1}\left(5x^2-3y\right)dydx \\
&= \int_{-1}^{1}\left(5x^2y-\frac{3}{2}y^2\ \Big|_{-1}^{1}\right)dx \\
&= \int_{-1}^{1}\left(5x^2-\frac{3}{2}+5x^2+\frac{3}{2}\right)dx \\
&= \int_{-1}^{1}10x^2\ dx \\
&= \frac{10}{3}x^3\ \Big|_{-1}^{1} \\
&= \underline{\frac{20}{3}}
\end{aligned}$$

Exercise

Use either form of Green's Theorem to evaluate the line integral $\oint_C 3x^3dy-3y^3dx$; C is the circle of radius 4 centered at the origin with *clockwise* orientation.

Solution

$$N = 3x^3 \quad \rightarrow \quad N_x = 9x^2$$

$$M = -3y^3 \quad \rightarrow \quad M_y = -9y^2$$

$$\oint_C 3x^3dy-3y^3dx = \iint_R \left(9x^2+9y^2\right)dA$$

$$= 9\int_0^{2\pi}\int_0^4 r^2\ r\, dr d\theta$$

$$= 9\int_0^{2\pi} d\theta \quad \int_0^4 r^3\ dr$$

$$= 9(2\pi)\ \left(\frac{1}{4}r^4\ \right|_0^4$$

$$= 18\pi(64)$$

$$\underline{= 1152\pi\ }|$$

Since the orientation is *cw*: -1152π

Exercise

Evaluate $\int_C (x-y)dx + (x+y)dy$ counterclockwise around the triangle with vertices (0, 0), $(1,\ 0)$ and $(0,\ 1)$

Solution

Along $(0,0) \to (1,0)$: $\vec{r}(t) = t\,\hat{i}\,, \quad 0 \le t \le 1$

$$\vec{F} = (x-y)\,\hat{i}\, + (x+y)\,\hat{j}$$

$$= t\,\hat{i}\, + t\,\hat{j}$$

$$\frac{d\vec{r}}{dt} = \hat{i}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left(t\,\hat{i}\, + t\,\hat{j}\right) \bullet \left(\hat{i}\right)$$

$$\underline{= t\ }|$$

Along $(1,0) \to (0,1)$: $\vec{r}(t) = (1-t)\,\hat{i}\, + t\,\hat{j}\,, \quad 0 \le t \le 1$

$$\vec{F} = (x-y)\,\hat{i}\, + (x+y)\,\hat{j}$$

$$= (1-2t)\,\hat{i}\, + \hat{j}$$

$$\frac{d\vec{r}}{dt} = -\hat{i}\, + \hat{j}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left((1-2t)\,\hat{i}\, + \hat{j}\right) \bullet \left(-\hat{i}\, + \hat{j}\right)$$

$$= -1 + 2t + 1$$

$$\underline{= 2t\ }|$$

Along $(0,1) \to (0,0)$: $\vec{r}(t) = (1-t)\,\hat{j}\,, \quad 0 \le t \le 1$

$$\vec{F} = (x-y)\,\hat{i}\, + (x+y)\,\hat{j}$$

$$= (t-1)\,\hat{i} + (1-t)\,\hat{j}$$

$$\frac{d\vec{r}}{dt} = -\hat{j}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left((t-1)\,\hat{i} + (1-t)\,\hat{j}\right) \bullet \left(-\hat{j}\right)$$

$$\underline{= t-1 \,\big|}$$

$$\int_C (x-y)\,dx + (x+y)\,dy = \int_0^1 t\,dt + \int_0^1 2t\,dt + \int_0^1 (t-1)\,dt$$

$$= \int_0^1 (t+2t+t-1)\,dt$$

$$= \int_0^1 (4t-1)\,dt$$

$$= 2t^2 - t \Big|_0^1$$

$$= 2-1$$

$$\underline{= 1 \,\big|}$$

Exercise

Use Green's theorem to evaluate the line integral $\oint xy^2dx + x^2ydy$; C is the triangle with vertices $(0,\ 0)$, $(2,\ 0)$, $(0,\ 2)$ with counterclockwise orientation.

Solution

$$\oint xy^2dx + x^2ydy = \iint_R \left(\frac{\partial}{\partial x}\left(x^2y\right) - \frac{\partial}{\partial x}\left(x^2y\right)\right)dxdy$$

$$= \iint_R (2xy - 2xy)\,dxdy$$

$$\underline{= 0 \,\big|}$$

Exercise

Use Green's theorem to evaluate the line integral $\oint \left(-3y + x^{3/2}\right)dx + \left(x - y^{2/3}\right)dy$; C is the boundary of the half disk $\left\{(x, y): \ x^2 + y^2 \le 2, \ y \ge 0\right\}$ with counterclockwise orientation.

Solution

$$\oint \left(-3y + x^{3/2}\right)dx + \left(x - y^{2/3}\right)dy = \iint_C \left(\frac{\partial}{\partial x}\left(x - y^{2/3}\right) - \frac{\partial}{\partial y}\left(-3y + x^{3/2}\right)\right)dA$$

$$= \iint_C (1+3)\,dA$$

$$= \iint_C 4\,dA \qquad\qquad \textit{Semicircle } A = \pi$$

$$= 4\pi$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint_{(0,\,1)} \left(2x + e^{y^2}\right)dy - \left(4y^2 + e^{x^2}\right)dx$: C is the boundary of the square with vertices $(0,\ 0)$, $(1,\ 0)$, $(1,\ 1)$ with counterclockwise orientation.

Solution

$$\oint_{(0,\,1)} \left(2x + e^{y^2}\right)dy - \left(4y^2 + e^{x^2}\right)dx = \iint_C \left(\frac{\partial}{\partial x}\left(2x + e^{y^2}\right) + \frac{\partial}{\partial y}\left(4y^2 + e^{x^2}\right)\right)dA$$

$$= \iint_C (2 + 8y)\,dA$$

$$= \int_0^1 \int_0^1 (2 + 8y)\,dxdy$$

$$= \int_0^1 (2 + 8y)\,dy$$

$$= 2y + 4y^2 \Big|_0^1$$

$$= 6$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint_C (2x-3y)\,dy-(3x+4y)\,dx$: C is the unit circle

Solution

$$\oint_C (2x-3y)\,dy-(3x+4y)\,dx = \iint_C \left(\frac{\partial}{\partial x}(2x-3y)+\frac{\partial}{\partial y}(3x+4y)\right)dA$$

$$= \iint_C (2+4)\,dA$$

$$= 6\times(\text{Area of the unit circle})$$

$$\underline{= 6\pi}$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint fdy-gdx$; where $\langle f,\ g\rangle = \langle 0,\ xy\rangle$ and C is the triangle with vertices $(0,\ 0)$, $(2,\ 0)$, $(0,\ 4)$ with counterclockwise orientation.

Solution

$(2,\ 0)-(0,\ 4)$: $\rightarrow\ y=\frac{4}{-2}x+4\ \underline{=4-2x}$

$$\oint fdy-gdx = \iint_R \left(\frac{\partial}{\partial x}(f)+\frac{\partial}{\partial y}(g)\right)dA$$

$$= \iint_R \left(\frac{\partial}{\partial x}(0)+\frac{\partial}{\partial y}(xy)\right)dA$$

$$= \int_0^2\int_0^{4-2x} x\ dydx$$

$$= \int_0^2 xy\Big|_0^{4-2x}\,dx$$

$$= \int_0^2 \left(4x-2x^2\right)dx$$

$$= 2x^2-\frac{2}{3}x^3\Big|_0^2$$

$$= 8-\frac{16}{3}$$

$$\underline{=\frac{8}{3}}$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint fdy - gdx;$ where $\langle f,\ g\rangle = \langle x^2,\ 2y^2\rangle$ and C is the upper half of the unit circle and the line segment $-1 \le x \le 1$ with clockwise orientation.

Solution

$$x^2 + y^2 = 1 \quad \to \quad y = \sqrt{1-x^2} \qquad \textit{upper half of the unit circle}$$

$$\oint fdy - gdx = -\iint_R \left(\frac{\partial}{\partial x}(f) + \frac{\partial}{\partial y}(g)\right) dA \qquad \textit{clockwise orientation}$$

$$= -\iint_R \left(\frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}\left(2y^2\right)\right) dA$$

$$= -\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (2x+4y)\,dydx$$

$$= -\int_{-1}^{1} \left(2xy + 2y^2 \Big|_0^{\sqrt{1-x^2}}\right. dx$$

$$= -\int_{-1}^{1} \left(2x\sqrt{1-x^2} + 2\left(1-x^2\right)\right) dx$$

$$= \int_{-1}^{1} \left(1-x^2\right)^{1/2} d\left(1-x^2\right) - 2\int_{-1}^{1} \left(1-x^2\right) dx$$

$$= \frac{2}{3}\left(1-x^2\right)^{3/2} - 2x + \frac{2}{3}x^3 \Big|_{-1}^{1}$$

$$= -2 + \frac{2}{3} - 2 + \frac{2}{3}$$

$$= -\frac{8}{3}$$

Exercise

Apply Green's Theorem to evaluate the integral, the circulation line integral of $\vec{F} = \langle x^2 + y^2,\ 4x + y^3\rangle$, where C is the boundary of $\{(x,\ y):\ 0 \le y \le \sin x,\quad 0 \le x \le \pi\}$

Solution

Using Circulation form

$$\iint_C \left(\frac{\partial}{\partial x}\left(4x+y^3\right) - \frac{\partial}{\partial y}\left(x^2+y^2\right)\right) dA = \iint_C (4-2y)\,dA$$

$$= \int_0^{\pi} \int_0^{\sin x} (4 - 2y)\, dydx$$

$$= \int_0^{\pi} \left(4y - y^2 \Big|_0^{\sin x} dx\right.$$

$$= \int_0^{\pi} \left(4\sin x - \sin^2 x\right) dx$$

$$= \int_0^{\pi} \left(4\sin x - \frac{1}{2} + \frac{1}{2}\cos 2x\right) dx$$

$$= -4\cos x - \frac{1}{2}x + \frac{1}{4}\sin 2x \Big|_0^{\pi}$$

$$= 4 - \frac{\pi}{2} + 4$$

$$= 8 - \frac{\pi}{2}$$

Exercise

Apply Green's Theorem to evaluate the integral, the circulation line integral of $\vec{F} = \left\langle 2xy^2 + x,\ 4x^3 + y \right\rangle$, where C is the boundary of $\{(x, y):\ 0 \le y \le \sin x, \quad 0 \le x \le \pi\}$

Solution

Using Circulation form

$$\iint_C \left(\frac{\partial}{\partial x}\left(4x^3 + y\right) - \frac{\partial}{\partial y}\left(2xy^2 + x\right)\right) dA = \iint_C \left(12x^2 - 4xy\right) dA$$

$$= \int_0^{\pi} \int_0^{\sin x} \left(12x^2 - 4xy\right) dydx$$

$$= \int_0^{\pi} \left(12x^2 y - 2xy^2 \Big|_0^{\sin x} dx\right.$$

$$= \int_0^{\pi} \left(12x^2 \sin x - 2x\sin^2 x\right) dx$$

$$= \int_0^{\pi} \left(12x^2 \sin x - 2x\left(\frac{1 - \cos 2x}{2}\right)\right) dx$$

$$= \int_0^{\pi} \left(12x^2 \sin x - x + x\cos 2x\right) dx$$

		$\int \sin x$				$\int \cos 2x$
+	$12x^2$	$-\cos x$		+	x	$\frac{1}{2}\sin 2x$
−	$24x$	$-\sin x$		−	1	$-\frac{1}{4}\cos 2x$
+	24	$\cos x$				

$$= \left(-12x^2\cos x + 24x\sin x + 24\cos x - \frac{1}{2}x^2 + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x \right|_0^{\pi}$$

$$= 12\pi^2 - 24 - \frac{\pi^2}{2} + \frac{1}{4} + 12\pi^2 - 24 - \frac{1}{4}$$

$$= \frac{23\pi^2}{2} - 48$$

Exercise

Apply Green's Theorem to evaluate the integral, the flus line integral of $\vec{F} = \left\langle e^{x-y},\ e^{y-x} \right\rangle$, where C is the boundary of $\{(x,\ y):\ 0 \le y \le x,\quad 0 \le x \le 1\}$

Solution

Using flux form

$$\iint_C \left(\frac{\partial}{\partial x}\left(e^{x-y}\right) + \frac{\partial}{\partial y}\left(e^{y-x}\right)\right) dA = \iint_C \left(e^{x-y} + e^{y-x}\right) dA$$

$$= \int_0^1 \int_0^x \left(e^{x-y} + e^{y-x}\right) dy dx$$

$$= \int_0^1 \left(-e^{x-y} + e^{y-x} \right|_0^x dx$$

$$= \int_0^1 \left(-1 + 1 + e^x - e^{-x}\right) dx$$

$$= \int_0^1 \left(e^x - e^{-x}\right) dx$$

$$= e^x + e^{-x} \Big|_0^1$$

$$= e + e^{-1} - 2$$

Exercise

Evaluate $\displaystyle\int_C y^2dx + x^2dy$ $\quad C$ is the circle $x^2 + y^2 = 4$

Solution

$M = y^2 \quad \to \quad M_y = 2y$

$N = x^2 \quad \to \quad N_x = 2x$

$$\int_C y^2dx + x^2dy = \int_C (2x - 2y)\,dxdy$$

$$= 2\int_0^{2\pi}\int_0^2 (r\cos\theta - r\sin\theta)\; rdrd\theta$$

$$= 2\int_0^{2\pi} (\cos\theta - \sin\theta)\,d\theta \int_0^2 r^2\,dr$$

$$= 2\,\left(\sin\theta + \cos\theta\right|_0^{2\pi}\;\left(\tfrac{1}{3}r^3\right|_0^2$$

$$= 2(1-1)\left(\frac{8}{3}\right)$$

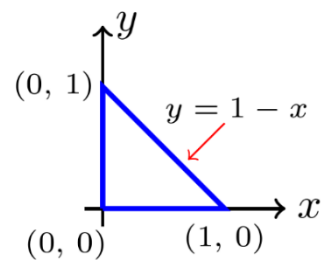
$$\underline{= 0}$$

Exercise

Use the flux form to Green's Theorem to evaluate $\displaystyle\iint_R \left(2xy + 4y^3\right)dA$, where R is the triangle with vertices $(0,\ 0)$, $(1,\ 0)$, and $(0,\ 1)$.

Solution

$(1,\ 0) - (0,\ 1)$: $\quad y = -x + 1$



$$\iint_R \left(2xy + 4y^3\right)dA = \int_0^1\int_0^{1-x} \left(2xy + 4y^3\right)dydx$$

$$= \int_0^1 \left(xy^2 + y^4\right|_0^{1-x}\,dx$$

$$= \int_0^1 \left(x - 2x^2 + x^3 + 1 - 4x + 6x^2 - 4x^3 + x^4\right)dx$$

$$= \int_0^1 \left(1 - 3x + 4x^2 - 3x^3 + x^4\right) dx$$

$$= x - \frac{3}{2}x^2 + \frac{4}{3}x^3 - \frac{3}{4}x^4 + \frac{1}{5}x^5 \Big|_0^1$$

$$= 1 - \frac{3}{2} + \frac{4}{3} - \frac{3}{4} + \frac{1}{5}$$

$$= \frac{-30 + 80 - 45 + 12}{60}$$

$$= \frac{17}{60}$$

Exercise

Show that $\oint_C \ln x \sin y\, dy - \frac{\cos y}{x} dx = 0$ for any closed curve C to which Green's Theorem applies.

Solution

$$M = -\frac{\cos y}{x} \quad \to \quad M_y = \frac{\sin y}{x}$$

$$N = \ln x \sin y \quad \to \quad N_x = \frac{\ln y}{x}$$

$$\oint_C \ln x \sin y dy - \frac{\cos y}{x} dx = \iint_R \left(\frac{\sin y}{x} - \frac{\sin y}{x}\right) dxdy$$

$$= 0 \quad \checkmark$$

Exercise

Prove that the radial field $\vec{F} = \frac{\vec{r}}{|\vec{r}|^p}$ where $\vec{r} = \langle x, y\rangle$ and p is a real number, is conservative on $\mathbb{R}^2$ with the origin removed. For what value of p is $\vec{F}$ conservative on $\mathbb{R}^2$ (including the origin)?

Solution

$$\vec{F} = \frac{\vec{r}}{|\vec{r}|^p}$$

$$= \frac{\langle x, y\rangle}{\left(\sqrt{x^2 + y^2}\right)^p}$$

$$\varphi_x = \frac{x}{\left(x^2+y^2\right)^{p/2}}; \quad \varphi_y = \frac{y}{\left(x^2+y^2\right)^{p/2}}$$

$$\varphi = \int \frac{x}{\left(x^2+y^2\right)^{p/2}}\,dx$$

$$= \frac{1}{2}\int \left(x^2+y^2\right)^{-p/2}\, d\left(x^2+y^2\right)$$

$$= \frac{1}{2}\frac{1}{\frac{2-p}{2}}\left(x^2+y^2\right)^{1-p/2} + C$$

$$= \frac{1}{2-p}\left(x^2+y^2\right)^{1-p/2} + C(x,y) \qquad \textit{for } p \neq 2$$

For $p \neq 2$

$$\varphi = \frac{1}{2-p}\left(x^2+y^2\right)^{1-p/2} + C(x,y)$$

$$\varphi_y = \frac{1}{2-p}\frac{2-p}{2}(2y)\left(x^2+y^2\right)^{1-\frac{p}{2}-1} + C_y$$

$$= y\left(x^2+y^2\right)^{-\frac{p}{2}} + C_y = \frac{y}{\left(x^2+y^2\right)^{p/2}}$$

$$\Rightarrow \quad \boxed{C_y = 0}$$

$$\therefore \quad \varphi = \frac{1}{(2-p)\left(x^2+y^2\right)^{\frac{p-2}{2}}}$$

$$= \frac{-1}{(p-2)\left(r^2\right)^{\frac{p-2}{2}}}$$

$$= \frac{-1}{(p-2)\ |r|^{p-2}}$$

For $p = 2$

$$\vec{F} = \frac{\langle x,\ y\rangle}{\left(\sqrt{x^2+y^2}\right)^2}$$

$$= \frac{\langle x,\ y\rangle}{x^2+y^2}$$

$$\varphi_x = \frac{x}{x^2+y^2}; \quad \varphi_y = \frac{y}{x^2+y^2}$$

$$\varphi = \int \frac{x}{x^2+y^2}\,dx$$

$$= \frac{1}{2}\int \frac{1}{x^2+y^2}\,d\left(x^2+y^2\right)$$

$$= \frac{1}{2}\ln\left(x^2+y^2\right) + C(x,y)$$

$$\varphi_y = \frac{y}{x^2+y^2} + C_y = \frac{y}{x^2+y^2}$$

$$\Rightarrow \; C_y = 0$$

$$\varphi = \frac{1}{2}\ln\left(|r|^2\right)$$

Thus $\vec{F}$ is conservative on all $\mathbb{R}^2$ for $p < 0$

Exercise

Find the area of the elliptical region cut from the plane $x+y+z=1$ by the cylinder $x^2+y^2=1$

Solution

$$f(x,y,z) = x+y+z-1$$

$$\nabla f = \langle 1,\ 1,\ 1\rangle$$

$$|\nabla f| = \sqrt{3}$$

$$Area = \sqrt{3}\int_0^{2\pi}\int_0^1 r\,drd\theta \qquad\qquad Area = \iint_R |\nabla f|\,dA$$

$$= \sqrt{3}\int_0^{2\pi} d\theta \quad \left(\frac{1}{2}r^2\right|_0^1$$

$$= \sqrt{3}(2\pi)\frac{1}{2}$$

$$= \pi\sqrt{3}\ \ unit^2$$

Exercise

Find the area of the cap cut from the paraboloid $x^2 + y^2 + z^2 = 1$ by the plane $z = \frac{\sqrt{2}}{2}$

Solution

$$f(x, y, z) = x^2 + y^2 + z^2 - 1$$

$$\nabla f = \langle 2x,\ 2y,\ 2z \rangle$$

$$\begin{aligned} |\nabla f| &= \sqrt{4x^2 + 4y^2 + 4z^2} \\ &= 2\sqrt{x^2 + y^2 + z^2} \\ &= 2 \end{aligned}$$

$$\begin{aligned} |\nabla f \bullet \vec{p}| &= \langle 2x,\ 2y,\ 2z \rangle \bullet \hat{k} \\ &= 2|z| \\ &= 2z \end{aligned}$$

$$Area = \iint_R \frac{|\nabla f|}{|\nabla f \bullet \vec{p}|} dA$$

$$\begin{aligned} Area &= \iint_R \frac{2}{2z} dA \\ &= \iint_R \frac{1}{\sqrt{1 - x^2 - y^2}} dxdy \\ &= \int_0^{2\pi} d\theta \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1 - r^2}} r\, dr \\ &= -\pi \int_0^{\frac{1}{\sqrt{2}}} \left(1 - r^2\right)^{-1/2} d\left(1 - r^2\right) \\ &= -2\pi \left(1 - r^2\right)^{1/2} \Big|_0^{\frac{1}{\sqrt{2}}} \\ &= -2\pi\left(\frac{1}{\sqrt{2}} - 1\right) \\ &= 2\pi\left(1 - \frac{\sqrt{2}}{2}\right) \\ &= \pi\left(2 - \sqrt{2}\right)\ unit^2 \end{aligned}$$

Exercise

Evaluate both integrals in Green's theorem of the vector field. Is the vector field conservative?

$\vec{F} = \langle x,\ y\rangle;\quad R = \left\{(x,\ y):\quad x^2 + y^2 \le 2\right\}$

Solution

$$M = x \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 0$$

$$N = y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 0$$

$$Curl = 0 - 0 \qquad\qquad Curl = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$\underline{= 0}\,\rfloor$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R (0 - 0)\, dA$$

$$\underline{= 0}\,\rfloor$$

$$\vec{r}(t) = \left\langle \sqrt{2}\cos t,\ \sqrt{2}\sin t\right\rangle$$

$$\vec{r}\,' = \left\langle -\sqrt{2}\sin t,\ \sqrt{2}\cos t\right\rangle$$

$$\vec{F} = \left\langle \sqrt{2}\cos t,\ \sqrt{2}\sin t\right\rangle$$

$$\vec{F} \cdot \frac{d\vec{r}}{dt} = \left\langle \sqrt{2}\cos t,\ \sqrt{2}\sin t\right\rangle \cdot \left\langle -\sqrt{2}\sin t,\ \sqrt{2}\cos t\right\rangle$$

$$= -2\cos t \sin t + 2\sin t \cos t$$

$$\underline{= 0}\,\rfloor$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} 0\ dt$$

$$\underline{= 0}\,\rfloor$$

$\therefore$ The vector field is conservative since its curl is zero.

Exercise

Evaluate both integrals in Green's theorem of the vector field. Is the vector field conservative?

$\vec{F} = \langle y,\ x\rangle$; R is the square with vertices $(0,\ 0),\ (1,\ 0),\ (1,\ 1),\ (0,\ 1)$

Solution

$$M = y \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 1$$

$$N = x \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 1$$

$Curl = 1 - 1$ $\qquad Curl = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

$\underline{= 0}$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \iint_R (1-1)\, dA$$

$$\underline{= 0}$$

$(0,\ 0) - (1,\ 0)$

$\vec{r}_1(t) = \langle t,\ 0 \rangle$

$\vec{r}_1{}' = \langle 1,\ 0 \rangle$

$\vec{F}_1 = \langle 0,\ t \rangle$

$\vec{F}_1 \cdot \vec{r}_1{}' = \langle 0,\ t \rangle \cdot \langle 1,\ 0 \rangle$

$\underline{= 0}$

$(1,\ 0) - (1,\ 1)$

$\vec{r}_2(t) = \langle 1,\ t \rangle$

$\vec{r}_2{}' = \langle 0,\ 1 \rangle$

$\vec{F}_2 = \langle t,\ 1 \rangle$

$\vec{F}_2 \cdot \vec{r}_2{}' = \langle t,\ 1 \rangle \cdot \langle 0,\ 1 \rangle$

$\underline{= 1}$

$(1,\ 1) - (0,\ 1)$

$\vec{r}_3(t) = \langle 1-t,\ 1 \rangle$

$\vec{r}_3{}' = \langle -1,\ 0 \rangle$

$\vec{F}_3 = \langle 1,\ 1-t \rangle$

$\vec{F}_3 \cdot \vec{r}_3{}' = \langle 1,\ 1-t \rangle \cdot \langle -1,\ 0 \rangle$

$\underline{= -1}$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}_1 \cdot \vec{r}_1{}'\, dt + \int_0^1 \vec{F}_2 \cdot \vec{r}_2{}'\, dt + \int_0^1 \vec{F}_3 \cdot \vec{r}_3{}'\, dt$$

$$= 0 + 1 - 1$$

$$\underline{= 0}$$

∴ The vector field is conservative since its curl is zero.

Exercise

Evaluate both integrals in Green's theorem of the vector field. Is the vector field conservative?

$\vec{F} = \langle 2y,\ -2x\rangle$; R is the region bounded by $y = \sin x$ and $y = 0$ for $0 \le x \le \pi$

Solution

$$M = 2y \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 2$$

$$N = -2x \quad \Rightarrow \quad \frac{\partial N}{\partial x} = -2$$

$$Curl = -2 - 2 \qquad\qquad Curl = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$\underline{= -4}$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R (-4)\, dA$$

$$= -4\int_0^{\pi} \sin x\, dx$$

$$= 4\cos x \Big|_0^{\pi}$$

$$= 4(-1-1)$$

$$\underline{= -8}$$

$y = 0$

$$\vec{r}_1(t) = \langle t,\ 0\rangle$$

$$\vec{r}_1{}' = \langle 1,\ 0\rangle$$

$$\vec{F} = \langle 0,\ -2t\rangle$$

$$\vec{F}_1 \cdot \vec{r}_1{}' = \langle 0,\ -2t\rangle \cdot \langle 1,\ 0\rangle$$

$$\underline{= 0}$$

$y = \sin x$

$$\vec{r}_2(t) = \langle t,\ \sin t\rangle$$

$$\vec{r}_2{}' = \langle 1,\ \cos t\rangle$$

$$\vec{F}_2 = \langle 2\sin t,\ -2t\rangle$$

$$\vec{F}_2 \cdot \vec{r}_2{}' = \langle 2\sin t,\ -2t\rangle \cdot \langle 1,\ \cos t\rangle$$

$$\underline{= 2\sin t - 2t\cos t}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi} \vec{F}_1 \cdot \vec{r}_1{}'\ dt + \int_{\pi}^{0} \vec{F}_2 \cdot \vec{r}_2{}'\ dt$$

$$= 0 + \int_{\pi}^{0} (2\sin t - 2t\cos t)\,dt$$

$$= -2\cos t - 2t\sin t - 2\cos t \Big|_{\pi}^{0}$$

$$= -4\cos t - 2t\sin t \Big|_{\pi}^{0}$$

$$= -4 - 4$$

$$\underline{= -8\,|}$$

$\therefore$ The vector field is ***not*** conservative since its curl is nonzero.

Exercise

Evaluate both integrals in Green's theorem of the vector field. Is the vector field conservative?

$\vec{F} = \langle -3y,\ 3x\rangle$; R is the triangle with vertices $(0,\ 0)$, $(1,\ 0)$, $(0,\ 2)$

Solution

$$M = -3y \quad \Rightarrow \quad \frac{\partial M}{\partial y} = -3$$

$$N = 3x \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 3$$

$$Curl = 3 + 3 \qquad\qquad Curl = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$\underline{= 6\,|}$$

$$y = \frac{2-0}{0-1}(x-1)$$

$$\underline{= -2x + 2\,|}$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R 6\ dA$$

$$= 6\int_0^1 (2-2x)\,dx$$

$$= 6\left(2x - x^2\right.\Big|_0^1$$

$$\underline{= 6\,|}$$

$(0,\ 0) - (1,\ 0)$

$$\vec{r}_1(t) = \langle t,\ 0 \rangle$$

$$\vec{r}_1{}' = \langle 1,\ 0 \rangle$$

$$\vec{F}_1 = \langle 0,\ 3t \rangle$$

$$\vec{F}_1 \cdot \vec{r}_1{}' = \langle 0,\ 3t \rangle \cdot \langle 1,\ 0 \rangle$$

$$= 0$$

$(1,\ 0) - (0,\ 2)$

$$\vec{r}_2(t) = \langle 1-t,\ 2t \rangle$$

$$\vec{r}_2{}' = \langle -1,\ 2 \rangle$$

$$\vec{F}_2 = \langle -6t,\ 3-3t \rangle$$

$$\vec{F}_2 \cdot \vec{r}_2{}' = \langle -6t,\ 3-3t \rangle \cdot \langle -1,\ 2 \rangle$$

$$= 6t + 6 - 6t$$

$$= 6$$

$(0,\ 2) - (0,\ 0)$

$$\vec{r}_3(t) = \langle 0,\ 2-2t \rangle$$

$$\vec{r}_3{}' = \langle 0,\ -2 \rangle$$

$$\vec{F}_3 = \langle 6t-6,\ 0 \rangle$$

$$\vec{F}_3 \cdot \vec{r}_3{}' = \langle 6t-6,\ 0 \rangle \cdot \langle 0,\ -2 \rangle$$

$$= 0$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}_1 \cdot \vec{r}_1{}'\ dt + \int_0^1 \vec{F}_2 \cdot \vec{r}_2{}'\ dt + \int_0^1 \vec{F}_3 \cdot \vec{r}_3{}'\ dt$$

$$= 0 + \int_0^1 6\ dt + 0$$

$$= 6$$

$\therefore$ The vector field is conservative since its curl is zero.

Exercise

Evaluate both integrals in Green's theorem of the vector field. Is the vector field conservative?

$\vec{F} = \left\langle 2xy,\ x^2 - y^2 \right\rangle$; R is the region bounded by $y = x(2-x)$ and $y = 0$

Solution

$M = 2xy \quad \Rightarrow \quad \dfrac{\partial M}{\partial y} = 2x$

$N = x^2 - y^2 \quad \Rightarrow \quad \dfrac{\partial N}{\partial x} = 2x$

$Curl = 2x - 2x$ $\qquad\qquad$ $Curl = \dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$

$\qquad \underline{= 0}$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \iint_R (2x - 2x)\, dA$$

$$\underline{= 0}$$

$y = 0$

$\vec{r}_1(t) = \langle t,\ 0 \rangle$

$\vec{r}_1{}' = \langle 1,\ 0 \rangle$

$\vec{F}_1 = \left\langle 0,\ t^2 \right\rangle$

$\vec{F}_1 \cdot \vec{r}_1{}' = \left\langle 0,\ t^2 \right\rangle \cdot \langle 1,\ 0 \rangle$

$\qquad \underline{= 0}$

$y = 2x - x^2$

$\vec{r}_2(t) = \left\langle t,\ 2t - t^2 \right\rangle$

$\vec{r}_2{}' = \langle 1,\ 2 - 2t \rangle$

$\vec{F}_2 = \left\langle 4t^2 - 2t^3,\ \ -3t^2 + 4t^3 - t^4 \right\rangle$

$$\begin{aligned}
\vec{F}_2 \cdot \vec{r}_2{}' &= \left\langle 4t^2 - 2t^3,\ \ -3t^2 + 4t^3 - t^4 \right\rangle \cdot \langle 1,\ 2 - 2t \rangle \\
&= 4t^2 - 2t^3 + \left(-3t^2 + 4t^3 - t^4 \right)(2 - 2t) \\
&= 4t^2 - 2t^3 - 6t^2 + 8t^3 - 2t^4 + 6t^3 - 8t^4 + 2t^5 \\
&= 2t^5 - 10t^4 + 12t^3 - 2t^2
\end{aligned}$$

$y = 2t - t^2 = 0 \quad \to \quad \underline{t = 0,\ 2}$

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^2 \vec{F}_1 \cdot \vec{r}_1{}'\ dt + \int_2^0 \vec{F}_2 \cdot \vec{r}_2{}'\ dt \\
&= 0 + \int_2^0 \left(2t^5 - 10t^4 + 12t^3 - 2t^2 \right) dt
\end{aligned}$$

$$= \frac{1}{3}t^6 - 2t^5 + 3t^4 - \frac{2}{3}t^3 \Big|_2^0$$

$$= -\frac{64}{3} + 64 - 48 + \frac{16}{3}$$

$$= -\frac{48}{3} + 16$$

$$\underline{= 0}$$

$\therefore$ The vector field is conservative since its curl is zero.

Exercise

Evaluate both integrals in Green's theorem of the vector field. Is the vector field conservative?

$\vec{F} = \left\langle 0,\ x^2 + y^2 \right\rangle; \quad R = \left\{ (x,\ y):\ \ x^2 + y^2 \le 1 \right\}$

Solution

$$M = 0 \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 0$$

$$N = x^2 + y^2 \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 2x$$

$$Curl = 2x - 0 \qquad\qquad Curl = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$\underline{= 2x}$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \iint_R 2x\ dA$$

$$= \int_0^{2\pi} \int_0^1 2r\cos\theta\ r\ drd\theta$$

$$= \int_0^{2\pi} \cos\theta\ d\theta \quad \int_0^1 2r^2\ dr$$

$$= \sin\theta \Big|_0^{2\pi} \quad \left(\frac{2}{3} r^3 \Big|_0^1 \right.$$

$$\underline{= 0}$$

$$\vec{r}(t) = \langle \cos t,\ \sin t \rangle$$

$$\frac{d\vec{r}}{dt} = \langle -\sin t,\ \cos t \rangle$$

$$\vec{F} = \left\langle 0,\ \cos^2 t + \sin^2 t \right\rangle$$

$$= \langle 0,\ 1 \rangle$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \langle 0,\ 1 \rangle \bullet \langle -\sin t,\ \cos t \rangle$$

$$= \cos t$$

$$\int_C \vec{F} \bullet d\vec{r} = \int_0^{2\pi} \cos t \ dt$$

$$= \sin t \Big|_0^{2\pi}$$

$$= 0$$

∴ The vector field is ***not*** conservative since its curl is nonzero.

Exercise

Find the area of the region using line integral of the region enclosed by the ellipse $x^2 + 4y^2 = 16$

Solution

$$x^2 + 4y^2 = 16$$

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

$$\begin{cases} x = 4\cos t \\ y = 2\sin t \end{cases} \qquad 0 \le t \le 2\pi$$

$$A = \frac{1}{2}\oint_C \left(4\cos t \frac{d}{dt}(2\sin t) - 2\sin t \frac{d}{dt}(4\cos t)\right) dt$$

$$= 4\int_0^{2\pi} \left(\cos^2 t + \sin^2 t\right) dt$$

$$= 4\int_0^{2\pi} dt$$

$$= 8\pi \ \ unit^2$$

Exercise

Find the area of the region using line integral of the region bounded by the hypocycloid $\vec{r}(t) = \left\langle \cos^3 t,\ \sin^3 t \right\rangle$ for $0 \le t \le 2\pi$.

Solution

$$A = \frac{1}{2}\int_0^{2\pi} \left(\cos^3 t\frac{d}{dt}\left(\sin^3 t\right) - \sin^3 t\frac{d}{dt}\left(\cos^3 t\right)\right)dt \qquad A = \frac{1}{2}\oint_C xdy - ydx$$

$$= \frac{1}{2}\int_0^{2\pi} \left(\cos^3 t\left(3\sin^2 t\cos t\right) - \sin^3 t\left(-3\cos^2 t\sin t\right)\right)dt$$

$$= \frac{3}{2}\int_0^{2\pi} \left(\cos^4 t\sin^2 t + \sin^4 t\cos^2 t\right)dt$$

$$= \frac{3}{2}\int_0^{2\pi} \sin^2 t\cos^2 t\left(\cos^2 t + \sin^2 t\right)dt$$

$$= \frac{3}{2}\int_0^{2\pi} \sin^2 t\cos^2 t\ dt$$

$$= \frac{3}{2}\int_0^{2\pi} \left(\frac{1-\cos 2t}{2}\right)\left(\frac{1+\cos 2t}{2}\right)\ dt$$

$$= \frac{3}{8}\int_0^{2\pi} \left(1-\cos^2 2t\right)\ dt$$

$$= \frac{3}{8}\int_0^{2\pi} \left(\frac{1}{2}-\frac{1}{2}\cos 4t\right)\ dt$$

$$= \frac{3}{8}\left(\frac{t}{2}-\frac{1}{8}\sin 4t\right)\Big|_0^{2\pi}$$

$$= \frac{3\pi}{8}\ unit^2$$

Exercise

Find the area of the region using line integral of the region enclosed by a disk of radius 5

Solution

$$x = 5\cos t \quad \rightarrow \quad dx = -5\sin t\ dt$$

$$y = 5\sin t \quad \rightarrow \quad dy = 5\cos t\ dt$$

$$A = \frac{1}{2}\int_0^{2\pi} \left(5\cos t\left(5\cos t\right) - 5\sin t\left(-5\sin t\right)\right)dt \qquad A = \frac{1}{2}\oint_C xdy - ydx$$

$$= \frac{25}{2}\int_0^{2\pi} \left(\cos^2 t + \sin^2 t\right)dt$$

$$= \frac{25}{2} \int_0^{2\pi} dt$$

$$= 25\pi$$

Exercise

Find the area of the region using line integral of the region bounded by an ellipse with semi-major and semi-minor axes of length 12 and 8, respectively.

Solution

$$\frac{x^2}{6^2} + \frac{y^2}{4^2} = 1$$

$x = 6\cos t \quad \rightarrow \quad dx = -6\sin t \ dt$

$y = 4\sin t \quad \rightarrow \quad dy = 4\cos t \ dt$

$$A = \frac{1}{2} \int_0^{2\pi} \left(6\cos t(4\cos t) - 4\sin t(-6\sin t)\right) dt \qquad A = \frac{1}{2}\oint_C xdy - ydx$$

$$= 12 \int_0^{2\pi} \left(\cos^2 t + \sin^2 t\right) dt$$

$$= 12 \int_0^{2\pi} dt$$

$$= 24\pi$$

Exercise

Find the area of the region using line integral of the region bounded by an ellipse $9x^2 + 25y^2 = 225$.

Solution

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

$x = 5\cos t \quad \rightarrow \quad dx = -5\sin t \ dt$

$y = 3\sin t \quad \rightarrow \quad dy = 3\cos t \ dt$

$$A = \frac{1}{2} \int_0^{2\pi} \left(5\cos t(3\cos t) - 3\sin t(-5\sin t)\right) dt \qquad A = \frac{1}{2}\oint_C xdy - ydx$$

$$=\frac{15}{2}\int_0^{2\pi}\left(\cos^2 t+\sin^2 t\right)dt$$

$$=\frac{15}{2}\int_0^{2\pi} dt$$

$$\underline{=15\pi}$$

Exercise

Find the area of the region using line integral of the region $\left\{(x,\ y):\ \ x^2+y^2\leq 16\right\}$

Solution

$x=4\cos t \quad\rightarrow\quad dx=-4\sin t\ dt$

$y=4\sin t \quad\rightarrow\quad dy=4\cos t\ dt$

$$A=\frac{1}{2}\int_0^{2\pi}\left(4\cos t\,(4\cos t)-4\sin t\,(-4\sin t)\right)dt \qquad\qquad A=\frac{1}{2}\oint_C xdy-ydx$$

$$=8\int_0^{2\pi}\left(\cos^2 t+\sin^2 t\right)dt$$

$$=8\int_0^{2\pi} dt$$

$$\underline{=16\pi}$$

Exercise

Find the area of the region using line integral of the region bounded by the parabolas $\vec{r}(t)=\left\langle t,\ 2t^2\right\rangle$ and $\vec{r}(t)=\left\langle t,\ 12-t^2\right\rangle$ for $-2\leq t\leq 2$

Solution

$$A=\frac{1}{2}\oint_C xdy-ydx$$

$$A=\frac{1}{2}\int_{-2}^{2}\left(t\frac{d}{dt}\left(2t^2\right)-2t^2\frac{d}{dt}(t)\right)dt\ -\ \frac{1}{2}\int_{-2}^{2}\left(t\frac{d}{dt}\left(12-t^2\right)-\left(12-t^2\right)\frac{d}{dt}(t)\right)dt$$

$$=\frac{1}{2}\int_{-2}^{2}\left(t(4t)-2t^2\right)dt\ -\ \frac{1}{2}\int_{-2}^{2}\left(t(-2t)-12+t^2\right)dt$$

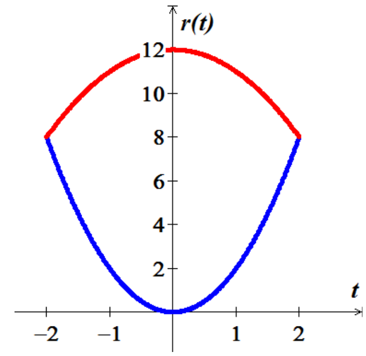
$$= \frac{1}{2} \int_{-2}^{2} \left(4t^2 - 2t^2 + 2t^2 + 12 - t^2 \right) dt$$

$$= \frac{1}{2} \int_{-2}^{2} \left(3t^2 + 12 \right) dt$$

$$= \frac{1}{2} \left(t^3 + 12t \right|_{-2}^{2}$$

$$= \frac{1}{2}(8 + 24 + 8 + 24)$$

$$= 32$$



Exercise

Find the area of the region using line integral of the region bounded by the curve

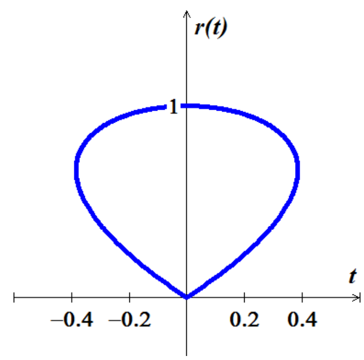
$\vec{r}(t) = \left\langle t\left(1 - t^2\right),\ 1 - t^2 \right\rangle$ for $-1 \le t \le 1$

Solution

$\vec{r}(-1) = \langle 0,\ 0 \rangle$

$\vec{r}\left(-\frac{1}{2}\right) = \left\langle \frac{1}{8},\ \frac{3}{4} \right\rangle$

$\vec{r}(0) = \langle 0,\ 1 \rangle$



The curve travels in counterclockwise, therefore;

$$A = \frac{1}{2} \int_{1}^{-1} \left(\left(t - t^3\right)(-2t) - \left(1 - t^2\right)\left(1 - 3t^2\right) \right) dt \qquad A = \frac{1}{2} \oint_C x\,dy - y\,dx$$

$$= \frac{1}{2} \int_{1}^{-1} \left(-2t^2 + 2t^4 - 1 + 3t^2 + t^2 - 3t^4 \right) dt$$

$$= \frac{1}{2} \int_{1}^{-1} \left(2t^2 - t^4 - 1 \right) dt$$

$$= \frac{1}{2} \left(\frac{2}{3}t^3 - \frac{1}{5}t^5 - t \right|_{1}^{-1}$$

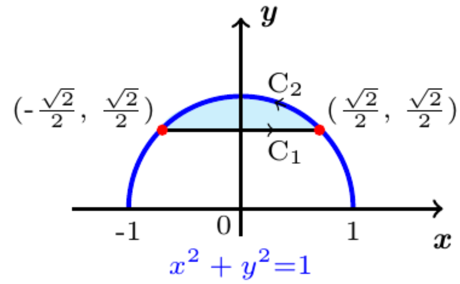
$$= -\frac{2}{3} + \frac{1}{5} + 1$$

$$= \frac{8}{3}$$

Exercise

Find the area of the region using line integral of the shaded region

Solution



For the path C_1 :

$$\begin{cases} t=0 & \to x=-\frac{\sqrt{2}}{2} \\ t=1 & \to x=\frac{\sqrt{2}}{2} \end{cases}$$

$$x=\frac{\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}}{1-0}t-\frac{\sqrt{2}}{2}$$

$$=\sqrt{2}t-\frac{\sqrt{2}}{2}$$

$$y=\frac{\sqrt{2}}{2}$$

$$C_1:\ \vec{r}_1(t)=\left\langle \sqrt{2}t-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right\rangle \quad 0\le t\le 1$$

$$\frac{d\vec{r}_1}{dt}=\left\langle \sqrt{2},\ 0\right\rangle$$

For the path C_2 :

$$C_2:\ \vec{r}_2(t)=\langle \cos t,\ \sin t\rangle \quad -\frac{\pi}{4}\le t\le\frac{\pi}{4}$$

$$\frac{d\vec{r}_2}{dt}=\langle -\sin t,\ \cos t\rangle$$

$$A=\frac{1}{2}\int_0^1\left(\left(\sqrt{2}t-\frac{\sqrt{2}}{2}\right)(0)-\left(\frac{\sqrt{2}}{2}\right)\left(\sqrt{2}\right)\right)dt\ +\ \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(\cos^2 t+\sin^2 t\right)dt \qquad A=\frac{1}{2}\oint_C xdy-ydx$$

$$=-\frac{1}{2}\int_0^1 dt+\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} dt$$

$$=-\frac{1}{2}+\frac{1}{2}t\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$=\frac{\pi}{4}-\frac{1}{2}$$

Exercise

Prove the identity $\oint_C dx = \oint_C dy = 0$, where C is a simple closed smooth oriented curve

Solution

$$\oint_C dx = \oint_C dy$$

$$\oint_C dx - \oint_C dy = \oint_C (1dx - 1dy)$$

This is an outward flux of the constant vector field $\vec{F} = \langle 1,\ 1 \rangle$

$$\oint_C dx - \oint_C dy = \underbrace{\iint_R \left(\frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(1) \right) dA}_{=0}$$

$$\oint_C dx = \oint_C dy = 0 \quad \checkmark$$

Exercise

Prove the identity $\oint_C f(x)\,dx + g(y)\,dy = 0$, where f and g have continuous derivatives on the region enclosed by C (is a simple closed smooth oriented curve)

Solution

By Green's Theorem:

$$\oint_C f(x)\,dx + g(y)\,dy = \underbrace{\iint \left(\frac{\partial}{\partial x}(g(y)) - \frac{\partial}{\partial y}(f(x)) \right) dA}_{=0} \quad \checkmark$$

Exercise

Show that the value of $\oint_C xy^2 dx + \left(x^2 y + 2x\right) dy$ depends only on the area of the region enclosed by C.

Solution

$$\oint_C xy^2 dx + \left(x^2 y + 2x\right) dy = \iint_R \left(\frac{\partial}{\partial x}\left(x^2 y + 2x\right) - \frac{\partial}{\partial y}\left(xy^2\right) \right) dA$$

$$= \iint_R (2xy + 2 - 2xy)\, dA$$

$$= 2\iint_R dA$$

$$\boxed{= 2 \times Area\ of\ A}$$

$\therefore \oint_C xy^2 dx + \left(x^2 y + 2x\right) dy$ depends only on the area of the region

Exercise

In terms of the parameters a and b, how is the value of $\oint_C aydx + bxdy$ related to the area of the region enclosed by C, assuming counterclockwise orientation of C?

Solution

$$\oint_C aydx + bxdy = \iint_R \left(\frac{\partial}{\partial x}(bx) - \frac{\partial}{\partial y}(ay)\right) dA$$

$$= \iint_R (b-a)\, dA$$

$$\boxed{= (b-a) \times Area\ of\ A}$$

Exercise

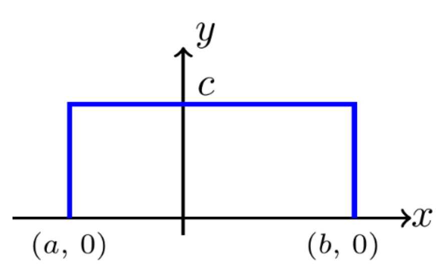
Show that if the circulation form of Green's Theorem is applied to the vector field $\left\langle 0,\ \frac{f(x)}{c} \right\rangle$ and $R = \left\{(x, y):\ \ a \le x \le b,\ \ 0 \le y \le c\right\}$, then the result is the Fundamental Theorem of Calculus,

$$\int_a^b \frac{df}{dx} dx = f(b) - f(a)$$

Solution

If $f(x)$ is continuous, then the circulation form of Green's Theorem is given

$$\oint_C \frac{f(x)}{c} dy = \frac{1}{c}\iint_R \frac{df}{dx} dA$$



$$\frac{1}{c}\iint_R \frac{df}{dx}\,dA = \frac{1}{c}\int_a^b \int_0^c \frac{df}{dx}\,dydx$$

$$= \frac{1}{c}\int_a^b \frac{df}{dx}\left(y \Big|_0^c\right) dx$$

$$= \int_a^b \frac{df}{dx}\,dx$$

$(a,\ 0)\ -\ (b,\ 0)$:

$$\vec{r}_1(t) = \langle (b-a)t + a,\ 0\rangle$$

$$\frac{d}{dt}\left(\vec{r}_1\right) = \langle b-a,\ 0\rangle$$

$$\vec{F}_1 = \left\langle 0,\ \frac{f\left((b-a)t+a\right)}{c}\right\rangle$$

$$\vec{F}_1 \bullet \vec{r}_1{}' = \left\langle 0,\ \frac{f\left((b-a)t+a\right)}{c}\right\rangle \bullet \langle b-a,\ 0\rangle$$

$$\underline{= 0\,|}$$

$(b,\ 0)\ -\ (b,\ c)$:

$$\vec{r}_2(t) = \langle b,\ ct\rangle$$

$$\vec{r}_2{}' = \langle 0,\ c\rangle$$

$$\vec{F}_2 = \left\langle 0,\ \frac{f(b)}{c}\right\rangle$$

$$\vec{F}_2 \bullet \vec{r}_2{}' = \left\langle 0,\ \frac{f(b)}{c}\right\rangle \bullet \langle 0,\ c\rangle$$

$$\underline{= f(b)\,|}$$

$(b,\ c)\ -\ (a,\ c)$:

$$\vec{r}_3(t) = \langle (a-b)t + b,\ c\rangle$$

$$\vec{r}_3{}' = \langle a-b,\ 0\rangle$$

$$\vec{F}_3 = \left\langle 0,\ \frac{f\left((a-b)t+b\right)}{c}\right\rangle$$

$$\vec{F}_3 \bullet \vec{r}_3{}' = \left\langle 0,\ \frac{f\left((a-b)t+b\right)}{c}\right\rangle \bullet \langle a-b,\ 0\rangle$$

$$\underline{= 0\,|}$$

$(a,\ c)\ -\ (a,\ 0)$:

$$\vec{r}_4(t) = \langle a,\ -ct + c \rangle$$

$$\vec{r}_4\,' = \langle 0,\ -c \rangle$$

$$\vec{F}_3 = \left\langle 0,\ \frac{f(a)}{c} \right\rangle$$

$$\vec{F}_3 \cdot \vec{r}_3\,' = \left\langle 0,\ \frac{f(a)}{c} \right\rangle \cdot \langle 0,\ -c \rangle$$

$$= -f(a)\ \rfloor$$

$$\oint_C \frac{f(x)}{c} dy = \int_0^1 \left(0 + f(b) + 0 - f(a)\right) dt$$

$$= \int_0^1 \left(f(b) - f(a)\right) dt$$

$$= \left(f(b) - f(a)\right)\, t \Big|_0^1$$

$$= f(b) - f(a)\ \rfloor$$

$$\int_a^b \frac{df}{dx} dx = f(b) - f(a)\ \rfloor$$

Exercise

Show that if the flux form of Green's Theorem is applied to the vector field $\left\langle \frac{f(x)}{c},\ 0 \right\rangle$ and $R = \{(x,\ y):\ \ a \le x \le b,\ \ \ 0 \le y \le c\}$, then the result is the Fundamental Theorem of Calculus,

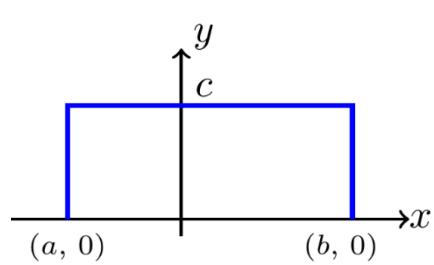
$$\int_a^b \frac{df}{dx} dx = f(b) - f(a)$$

Solution

If $f(x)$ is continuous, then the circulation form of Green's Theorem is given

$$\oint_C \frac{f(x)}{c} dy = \frac{1}{c} \iint_R \frac{df}{dx} dA$$

$$\frac{1}{c} \iint_R \frac{df}{dx} dA = \frac{1}{c} \int_a^b \int_0^c \frac{df}{dx} dy dx$$



$$= \frac{1}{c}\int_a^b \frac{df}{dx}\ \left(y\ \Big|_0^c\right. dx$$

$$= \int_a^b \frac{df}{dx} dx$$

$(a,\ 0)\ -\ (b,\ 0)$:

$$\vec{r}_1(t) = \langle (b-a)t + a,\ 0\rangle$$

$$\vec{r}_1{}' = \langle b-a,\ 0\rangle$$

$$\vec{F}_1 = \left\langle \frac{f((b-a)t+a)}{c},\ 0\right\rangle$$

$$\frac{f((b-a)t+a)}{c}(0) + 0(b-a) = 0 \quad (1)$$

$(b,\ 0)\ -\ (b,\ c)$:

$$\vec{r}_2(t) = \langle b,\ ct\rangle$$

$$\vec{r}_2{}' = \langle 0,\ c\rangle$$

$$\vec{F}_2 = \left\langle \frac{f(b)}{c},\ 0\right\rangle$$

$$\frac{f(b)}{c}(c) + 0 = f(b) \quad (2)$$

$(b,\ c)\ -\ (a,\ c)$:

$$\vec{r}_3(t) = \langle (a-b)t + b,\ c\rangle$$

$$\vec{r}_3{}' = \langle a-b,\ 0\rangle$$

$$\vec{F}_3 = \left\langle \frac{f((a-b)t+b)}{c},\ 0\right\rangle$$

$$\frac{f((a-b)t+b)}{c}(0) + 0(a-b) = 0 \quad (3)$$

$(a,\ c)\ -\ (a,\ 0)$:

$$\vec{r}_4(t) = \langle a,\ -ct + c\rangle$$

$$\vec{r}_4{}' = \langle 0,\ -c\rangle$$

$$\vec{F}_3 = \left\langle \frac{f(a)}{c},\ 0\right\rangle$$

$$\frac{f(a)}{c}(c) + 0 = f(a) \quad (2)$$

$$\oint_C \frac{f(x)}{c}\,dy = \int_0^1 \left(0 + f(b) + 0 - f(a)\right)dt$$

$$= \int_0^1 \left(f(b) - f(a)\right)dt$$

$$= \left(f(b) - f(a)\right)\, t\,\Big|_0^1$$

$$= f(b) - f(a)$$

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$