SOLUTION *Section* 4.1 – Relations and Their Properties

Exercise

List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$ where $(a,\ b) \in R$ if and only if

a) $a = b$ **b)** $a + b = 4$ **c)** $a > b$

d) $a \mid b$ **e)** $gcd(a,b) = 1$ **f)** $lcm(a,b) = 2$

Solution

a) {(0, 0), (1, 1), (2, 2), (3, 3)}

b) {(4, 0), (1, 3), (3, 1), (2, 2)}

c) {(1, 0), (2, 0), (2,1), (3, 0), (3, 1), (3, 2), (4, 0), (4, 1), (4, 2), (4, 3)}

d) {(1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 0) , (3, 3) , 4, 0)} **(**means b is multiple of $a \neq 0$**)**

e) {(1, 0), (0, 1), (1,1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (4, 1), (4, 3)} **(**means *relatively prime***)**

f) {(1, 2), (2, 1), (2, 2)} **(**Mean *least common multiple* is 2**)**.

Exercise

a) List all the ordered pairs in the relation $R = \{(a,\ b) \mid a \text{ divides } b\}$ on the set {1, 2, 3, 4, 5, 6}

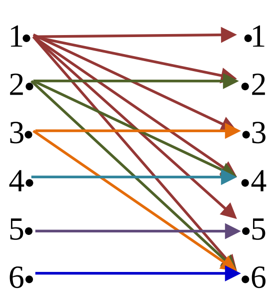
b) Display this relation graphically.

c) Display this relation in tabular form.

Solution

a) {(1, 1), (1, 2), (1,3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)}

b)



c)

R	1	2	3	4	5	6
1	×	×	×	×	×	×
2		×		×		×
3			×			×
4				×		
5					×	
6						×

Exercise

For each of these relations on the set {1, 2, 3, 4}, decide whether it is reflexive, symmetric, antisymmetric and transitive

a) {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}
b) {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}
c) {(2, 4), (4, 2)}
d) {(1, 2), (2, 3), (3, 4)}
e) {(1, 1), (2, 2), (3, 3), (4, 4)}
f) {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}

Solution

a) This relation is not reflexive, since (1, 1) is not included
It is not symmetric, since (2, 4) is included but not (4, 2)
It is not antisymmetric, since it includes (2, 3) and (3, 2) but $2 \neq 3$

$$\begin{array}{l} (2,\ 3)\ \&\ (3,\ 4)\ \rightarrow\ (2,\ 4) \\ (2,\ 3)\ \&\ (3,\ 2)\ \rightarrow\ (2,\ 2) \end{array}\ \&\ \begin{array}{l} (3,\ 2)\ \&\ (2,\ 4)\ \rightarrow\ (3,\ 4) \\ (3,\ 2)\ \&\ (2,\ 3)\ \rightarrow\ (3,\ 3) \end{array}$$

It is transitive.

b) This relation is reflexive, since (1, 1), (2, 2), (3, 3), and (4, 4)} are included
It is symmetric, since (2, 1) and (1, 2) are included
It is not antisymmetric, since it includes (2, 1) and (1, 2) but $2 \neq 1$

$$\begin{array}{l} (2,\ 1)\ \&\ (1,\ 2)\ \rightarrow\ (2,\ 2) \\ (1,\ 2)\ \&\ (2,\ 1)\ \rightarrow\ (1,\ 1) \end{array}$$

It is transitive.

c) This relation is not reflexive, since (1, 1) is not included
It is symmetric, since (2, 4) and (4, 2) are included
It is not antisymmetric, since it includes (2, 4) and (4, 2) but $2 \neq 4$
It is not transitive, since it includes (2, 4) and (4, 2) but not (2, 2)

d) This relation is not reflexive, since (1, 1) is not included
It is not symmetric, since (1, 2) is included but not (2, 1)
It is antisymmetric, since no cases of (*a*, *b*) and (*b*, *a*) both being in the relation
It is not transitive, since it includes (1, 2) and (2, 3) but not (1, 3)

e) This relation is reflexive, since (1, 1), (2, 2), (3, 3), and (4, 4)} are included and it is *symmetric*
It is antisymmetric, since no cases of (*a*, *b*) and (*b*, *a*) both being in the relation
It is transitive, since the only time the hypothesis $(a,\ b) \in R \wedge (b,\ c) \in R$ is met is when $a \equiv b \equiv c$

f) This relation is not reflexive, since (1, 1) is not included
It is not symmetric, since (1, 4) is included but not (4, 1)
It is not antisymmetric, since it includes (1, 3) and (3, 1)
It is not transitive, since it includes (2, 3) and (3, 1) but not (2, 1)

Exercise

Determine whether the relation R on the set of all people is reflexive, symmetric, antisymmetric, and/or transitive, where $(a,\ b) \in R$ if and only if

a) *a* is taller than *b*.
b) *a* and *b* were born on the same day
c) *a* has the same first name as *b*.
d) *a* and *b* have a common grandparent.

Solution

a) I am *not* taller than myself, therefore *being taller* is not reflexive
It is *not* symmetric, since I am taller than my kid but my kid is not
It is antisymmetric since we never have *a* taller than *b* and *b* taller than *a* even if *a* = *b*
It is transitive since if *a* taller than *b* and *b* taller than *c* that implies that *A* taller then *c*

b) The relation is reflexive since *a* is born on the same day
It is symmetric, since *a* and *b* were born on the same day
It is *not* antisymmetric since *a* and *b* were born on the same day but $a \neq b$
It is transitive since if *a* and *b* were born on the same day and *b* and *c* were born on the same day that implies that *a* and *c* were born on the same day

c) The relation is reflexive since *a* has the same first name as *a*
It is symmetric, since *a* has the same first name as *b* than *b* has the same first name as *a*
It is *not* antisymmetric since *a* has the same first name as *b* but $a \neq b$
It is transitive since if *a* has the same first name as *b* and *c* has the same first name as *c* that implies that *a* has the same first name as *c*

d) The relation is reflexive since *a* and *a* have a common grandparent
It is symmetric, since *a* and *b* have a common grandparent than *b* and *a* have a common grandparent
It is *not* antisymmetric since *a* and *b* have a common grandparent but $a \neq b$
It is transitive since if *a* and *b* have a common grandparent and *b* and *c* have a common grandparent that implies that *a* and *c* have a common grandparent

Exercise

Determine whether the relation $\boldsymbol{R}$ on the set of all ***real numbers*** is reflexive, symmetric, antisymmetric, and/or transitive, where $(x,\ y) \in \boldsymbol{R}$ if and only if

a) $x+y=0$	***b)*** $x=\pm y$	***c)*** $x-y$ *is a rational number*	***d)*** $x=2y$
e) $xy \ge 0$	***f)*** $xy=0$	***g)*** $x=1$	***h)*** $x=1$ *or* $y=1$

Solution

a) The relation is *not* reflexive since $1+1 \neq 0$
It is symmetric, since $x+y=0$ then $y+x=0$ because $x+y=y+x$
It is *not* antisymmetric since $(1, -1) \in \boldsymbol{R}$ and $(-1, 1) \in \boldsymbol{R}$ but $1 \neq -1$

It is *not* transitive since (1, −1) and (−1, 1) $\in \boldsymbol{R}$ but $(1,\ 1)\notin \boldsymbol{R}$

b) The relation is reflexive since $x=\pm x$

It is symmetric, since $x=\pm y$ *iff* $y=\pm x$

It is *not* antisymmetric since (1, −1) $\in \boldsymbol{R}$ and (−1, 1) $\in \boldsymbol{R}$ but $1\neq -1$

It is transitive since the product 1's and −1's is ± 1

c) The relation is reflexive since $x-x=0$ is a rational number

It is symmetric, since $x-y$ is rational, then $-(x-y)=y-x$

It is *not* antisymmetric since (1, −1) $\in \boldsymbol{R}$ but (−1, 1) $\in \boldsymbol{R}$ but $1\neq -1$

It is transitive since $(x,y)\in \boldsymbol{R}$ then $x-y$ is a rational number $(y,z)\in \boldsymbol{R}$ then $x-y$ is a rational number, therefore $x-z$ is rational that means that $(x,z)\in \boldsymbol{R}$

d) The relation is *not* reflexive since $1\neq 2\cdot 1$

It is *not* symmetric, since $(2,\ 1)\in \boldsymbol{R}$ then $2=2\cdot 1$ but $1\neq 2\cdot 2$ therefore $(1,\ 2)\notin \boldsymbol{R}$

It is antisymmetric since $x=2y$ and $y=2x$ that implies to $y=2(2y)=4y$ which $y=0$

It is *not* transitive since $2=2\cdot 1$ and $4=2\cdot 2\ \Rightarrow\ 4\neq 2\cdot 1$ so $(4,\ 1)\notin \boldsymbol{R}$

e) The relation is reflexive since $x^2\geq 0$ always positive

It is symmetric, since $xy\geq 0\quad\Rightarrow\quad yx\geq 0$

It is *not* antisymmetric since (2, 3) $\in \boldsymbol{R}$ and (3, 2) $\in \boldsymbol{R}$ but $2\neq 3$

It is *not* transitive since $(1,\ 0)\in \boldsymbol{R}\Rightarrow 1\cdot 0\geq 0$ $(0,\ -1)\in \boldsymbol{R}\Rightarrow 0\cdot(-1)\geq 0$ but $1\cdot(-1)\not\geq 0\Rightarrow(1,\ -1)\notin \boldsymbol{R}$

f) $\boxed{xy=0}$ The relation is *not* reflexive since $(1,\ 1)\notin \boldsymbol{R}$

It is symmetric, since $xy=0\quad\rightarrow\quad yx=0$

It is antisymmetric since (2, 0) $\in \boldsymbol{R}$ and (0, 2) $\in \boldsymbol{R}$ but $2\neq 0$

It is *not* transitive since $2\cdot 0=0$ (2, 0) $\in \boldsymbol{R}$ and $0\cdot(-2)=0$ (0, −2) $\in \boldsymbol{R}\Rightarrow$ $2\cdot(-2)\neq 0$ so $(2,\ -2)\notin \boldsymbol{R}$

g) The relation is *not* reflexive since $(2,\ 2)\notin \boldsymbol{R}$

It is *not* symmetric, since $(1,\ 2)\in \boldsymbol{R}$ but $(2,\ 1)\notin \boldsymbol{R}$

It is antisymmetric since $(x, y)\in \boldsymbol{R}$ and $(y, x)\in \boldsymbol{R}$ then $x = 1$ and $y = 1$, so $x = y$

It is transitive since $(x, y)\in \boldsymbol{R}$ and $(y, z)\in \boldsymbol{R}$ then $x = 1$ and $y = 1$, so $(x, z)\in \boldsymbol{R}$

h) The relation is *not* reflexive since $(2,\ 2)\notin \boldsymbol{R}$

It is symmetric, since $(1,\ 2)\in \boldsymbol{R}$ and $(2,\ 1)\in \boldsymbol{R}$

It is *not* antisymmetric since $(1,\ 2)\in \boldsymbol{R}$ and $(2,\ 1)\in \boldsymbol{R}$ but $1\neq 2$

It is *not* transitive since $(2,\ 1)\in \boldsymbol{R}$ and $(1,\ 3)\in \boldsymbol{R}$ but $(2,\ 3)\notin \boldsymbol{R}$

Exercise

Determine whether the relation R on the set of all ***integers numbers*** is reflexive, symmetric, antisymmetric, and/or transitive, where $(x,\ y) \in R$ if and only if

a) $x \neq y$ **b)** $xy \geq 1$ **c)** $x = y+1$ *or* $x = y-1$ **d)** $x \equiv y(\textbf{mod } 7)$

e) x *is a multiple of* y **f)** $x = y^2$ **g)** $x \geq y^2$

Solution

a) This relation is not reflexive, since $1 \neq 1$ for instance

It is symmetric, if $x \neq y \Rightarrow y \neq x$

It is *not* antisymmetric since $1 \neq 2 \Rightarrow 2 \neq 1$

It is *not* transitive since $1 \neq 2$ *and* $2 \neq 1 \Rightarrow 1 \neq 1$

b) This relation is not reflexive, since (0, 0) is not included $(0 \not\geq 1)$

It is symmetric, because $xy = yx$ (commutative property of multiplication)

It is *not* antisymmetric since (2, 3) and (3, 2) are both included

It is transitive holds between x and y if and only if either x and y are both positive or x and y are both negative

c) This relation is *not* reflexive, since (1, 1) is not included $(1 \neq 1+1)$

It is symmetric, because $x = y-1$ is equivalent to $y = x+1$

It is *not* antisymmetric since (1, 2) and (2, 1) are in the relation

It is *not* transitive since (1, 2) and (2, 1) are in the relation but (1, 1) is not

d) $x \equiv y(\textbf{mod } 7)$ means that $x - y = 7t$ for some t.

This relation is reflexive since $x - x = 7 \cdot 0$

It is symmetric since is $x \equiv y(\textbf{mod } 7)$ then $x - y = 7t$, therefore $y - x = 7(-t)$ so $y \equiv x(\textbf{mod } 7)$

It is *not* antisymmetric since $1 \equiv 8(\textbf{mod } 7)$ and $8 \equiv 1(\textbf{mod } 7)$

It is transitive since $x \equiv y(\textbf{mod } 7)$ means $x - y = 7t$ and $y \equiv z(\textbf{mod } 7)$ means $y - z = 7s$

$x - y = x - y + y - z = 7t + 7s = 7(t+s)$; therefore $x \equiv z(\textbf{mod } 7)$

e) x *is a multiple of* y means that $x = ty$ for some t.

This relation is reflexive since $x = x \cdot 1$

It is *not* symmetric since is $6 = 3 \cdot 2$ but $2 \neq 3 \cdot 6$

It is *not* antisymmetric since 2 is multiple of -2 but $2 \neq -2$

It is transitive since $x = ty$ and $y = sz \Rightarrow x = ty = tsz = (ts)z$ therefore x is a multiply of z.

f) This relation is *not* reflexive, since $3 \neq 3^2$

It is *not* symmetric since is $9 = 3^2$ but $3 \neq 9^2$

It is antisymmetric since $x = y^2$ and $y = x^2$

$$\Rightarrow x = y^2 = x^4$$

$$x - x^4 = 0$$

$$x(1 - x^3) = 0$$

$$x(1-x)\left(1+x+x^2\right)=0 \qquad \to x=0,\ 1$$

$x=y^2$ and $y=x^2$ when $x=y$

It is *not* transitive since $81=9^2$ and $9=3^2$ but $81\neq 3^2$

g) This relation is *not* reflexive, since $3\not\geq 3^2$

It is *not* symmetric since is $9\geq 3^2$ but $3\not\geq 9^2$

It is antisymmetric since $x\geq y^2$ and $y\geq x^2$, only when $x=0,\ 1$.

It is transitive since $x\geq y^2$ and $y\geq z^2$

$$\begin{aligned} x &\geq y^2 \\ &\geq \left(z^2\right)^2 \\ &= z^4 \\ &\geq z^2 \end{aligned}$$

Exercise

Show that the relation $R=\varnothing$ on nonempty set S is symmetric and transitive, but not reflexive.

Solution

If $R=\varnothing$, then the hypothesis of the conditional statements in the definitions of symmetric and transitive are never true, so those statements are always true by definition.

$S\neq\varnothing$ the statement $(a,a)\in R$ is false for an element of S, so $\forall a\ (a,a)\in R$ is not true, thus R is not reflexive.

Exercise

Show that the relation $R=\varnothing$ on nonempty set $S=\varnothing$ is reflexive, symmetric and transitive.

Solution

Since the domain is empty, then the relation is vacuously reflexive, symmetric and transitive s

Exercise

Give an example of a relation on a set that is

a) both symmetric and antisymmetric

b) neither symmetric nor antisymmetric

Solution

a) The empty set on $\{a\}$ – vacuously symmetric and antisymmetric

b) $\{(a, b), (b, a), (a, c)\}$ on $\{a, b, c\}$

Exercise

A relation R is called ***asymmetric*** if $(a,\ b) \in R$ implies that $(b,\ a) \notin R$. Explore the notion of an asymmetric relation to the following

a) {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}
b) {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}
c) {(2, 4), (4, 2)}
d) {(1, 2), (2, 3), (3, 4)}
e) {(1, 1), (2, 2), (3, 3), (4, 4)}
f) {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}
g) *a* is taller than *b*.
h) *a* and *b* were born on the same day
i) *a* has the same first name as *b*.
j) *a* and *b* have a common grandparent.

Solution

The relations (***a***), (***b***), and (***c***) are not *asymmetric* since they contain pairs of the form (*x, x*)

The relation (***f***) is not *asymmetric* since both (1, 3) and (3, 1) are in the relation

The relation (***d***) is not *asymmetric*

The relation (***g***) is *asymmetric* since if *a* taller than *b*, then *b* can't be taller than *a*.

The relation (***h***) is not *asymmetric* since *a* and *b* were born on the same day but $a \neq b$

The relation (***i***) is not *asymmetric* since *a* has the same first name as *b* but $a \neq b$

The relation (***j***) is not *asymmetric* since *a* and *b* have a common grandparent but $a \neq b$

Exercise

Let R be the relation $R = \{(a,\ b) \mid a < b\}$ on the set of integers. Find

a) R^{-1} ***b)*** $\overline{R}$

Solution

a) $R^{-1} = \{(b,a) \mid (a,b) \in R\} = \{(b,a) \mid a < b\} = \{(a,b) \mid a > b\}$

b) $\overline{R} = \{(b,a) \mid (a,b) \notin R\} = \{(b,a) \mid a \not< b\} = \{(a,b) \mid a \geq b\}$

Exercise

Let R be the relation $R = \{(a,\ b) \mid a \text{ divides } b\}$ on the set of positive integers. Find

a) R^{-1} ***b)*** $\overline{R}$

Solution

a) $R^{-1} = \{(a,\ b) \mid b \text{ divides } a\}$

b) $\overline{R} = \{(a,\ b) \mid a \text{ does not divide } b\}$

Exercise

Let R be the relation on the set of all states in the U.S. consisting of pairs (a, b) where state a borders state b. Find

a) R^{-1} ***b)*** $\overline{R}$

Solution

a) Since this relation is symmetric, $R^{-1} = R$

b) This relation consists of all pairs (a, b) in which state a does not border state b.

Exercise

Let $R_1 = \{(1,\ 2),\ (2,\ 3),\ (3,\ 4)\}$ and

$R_2 = \{(1,\ 1),\ (1,\ 2),\ (2,\ 1),\ (2,\ 2),\ (2,\ 3),\ (3,\ 1),\ (3,\ 2),\ (3,\ 3),\ (3,\ 4)\}$ be relation from {1, 2, 3} to {1, 2, 3, 4}.

Find

a) $R_1 \cup R_2$ ***b***) $R_1 \cap R_2$ ***c***) $R_1 - R_2$ ***d***) $R_2 - R_1$

Solution

a) $R_1 \cup R_2 = \{(1,\ 1),\ (1,\ 2),\ (2,\ 1),\ (2,\ 2),\ (2,\ 3),\ (3,\ 1),\ (3,\ 2),\ (3,\ 3),\ (3,\ 4)\} = R_2$

b) $R_1 \cap R_2 = \{(1,\ 2),\ (2,\ 3),\ (3,\ 4)\} = R_1$

c) $R_1 - R_2 = \varnothing$

d) $R_2 - R_1 = \{(1,\ 1),\ (2,\ 1),\ (2,\ 2),\ (3,\ 1),\ (3,\ 2),\ (3,\ 3)\}$

Exercise

Let the relation R = {(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)} and the relation S = {(2, 1), (3, 1), (3, 2), (4, 2)}
Find $S \circ R$

Solution

$(1,\ 2) \in R \text{ and } (2,\ 1) \in S \Rightarrow (1,\ 1) \in S \circ R$

$(1,\ 3) \in R \text{ and } (3,\ 2) \in S \Rightarrow (1,\ 2) \in S \circ R$

$(2,\ 3) \in R \text{ and } (3,\ 1) \in S \Rightarrow (2,\ 1) \in S \circ R$

$(2,\ 4) \in R \text{ and } (4,\ 2) \in S \Rightarrow (2,\ 2) \in S \circ R$

$S \circ R = \{(1, 1),\ (1,\ 2),\ (2,\ 1),\ (2,\ 2)\}$

Exercise

$$R_1 = \{(a,b) \in \boldsymbol{R}^2 | a > b\} \qquad R_3 = \{(a,b) \in \boldsymbol{R}^2 | a < b\} \qquad R_5 = \{(a,b) \in \boldsymbol{R}^2 | a = b\}$$

$$R_2 = \{(a,b) \in \boldsymbol{R}^2 | a \ge b\} \qquad R_4 = \{(a,b) \in \boldsymbol{R}^2 | a \le b\} \qquad R_6 = \{(a,b) \in \boldsymbol{R}^2 | a \ne b\}$$

Find the following:

a) $R_1 \cup R_3$ ***b***) $R_1 \cup R_5$ ***c***) $R_2 \cap R_4$ ***d***) $R_3 \cap R_5$ ***e***) $R_1 - R_2$

f) $R_2 - R_1$ ***g***) $R_1 \oplus R_3$ ***h***) $R_2 \oplus R_4$ ***i***) $R_1 \circ R_1$ ***j***) $R_1 \circ R_2$

k) $R_1 \circ R_3$ ***l***) $R_1 \circ R_4$ ***m***) $R_1 \circ R_5$ ***n***) $R_1 \circ R_6$ ***o***) $R_2 \circ R_3$

Solution

a) $R_1 \cup R_3 = \{(a,b) \in \boldsymbol{R}^2 \mid a > b \text{ or } a < b\}$

$= \{(a,b) \in \boldsymbol{R}^2 \mid a \ne b\}$

$= R_6$

b) $R_1 \cup R_5 = \{(a,b) \in \boldsymbol{R}^2 \mid a > b \text{ or } a = b\}$

$= \{(a,b) \in \boldsymbol{R}^2 \mid a \le b\}$

$= R_2$

c) $R_2 \cap R_4 = \{(a,b) \in \boldsymbol{R}^2 \mid a \ge b \text{ and } a \le b\}$

$= \{(a,b) \in \boldsymbol{R}^2 \mid a = b\}$

$= R_5$

d) $R_3 \cap R_5 = \{(a,b) \in \boldsymbol{R}^2 \mid a < b \text{ and } a = b\}$

$= \varnothing$

e) $R_1 - R_2 = R_1 \cap \overline{R}_2$

$= \{(a,b) \in \boldsymbol{R}^2 \mid a > b \text{ and } a < b\}$

$= \varnothing$

f) $R_2 - R_1 = R_2 \cap \overline{R}_1$

$= \{(a,b) \in \boldsymbol{R}^2 \mid a \ge b \text{ and } a \le b\}$

$= \{(a,b) \in \boldsymbol{R}^2 \mid a = b\}$

$= R_5$

g) $R_1 \oplus R_3 = \left(R_1 \cap \bar{R}_3\right) \cup \left(R_3 \cap \bar{R}_1\right)$

$= \left\{(a,b) \in \boldsymbol{R}^2 \mid a > b \ and \ a \geq b\right\} \cup \left\{(a,b) \in \boldsymbol{R}^2 \mid a < b \ and \ a \leq b\right\}$

$= R_1 \cup R_3$ **(From part *a*)**

$= R_6$

h) $R_2 \oplus R_4 = \left(R_2 \cap \bar{R}_4\right) \cup \left(R_4 \cap \bar{R}_2\right)$

$= \left\{(a,b) \in \boldsymbol{R}^2 \mid a \geq b \ and \ a > b\right\} \cup \left\{(a,b) \in \boldsymbol{R}^2 \mid a \leq b \ and \ a < b\right\}$

$= R_1 \cup R_3$ **(From part *a*)**

$= R_6$

i) $R_1 \circ R_1 = \left\{(a,b) \in R_1 \ and \ (b,c) \in R_1\right\}$

$a > b \ and \ b > c \Rightarrow a > c$ (*clearly*) that means $(a,c) \in R_1$ (*Transitive*).

Therefore, $R_1 \circ R_1 = R_1$

j) $R_1 \circ R_2 = \left\{(a,b) \in R_2 \ and \ (b,c) \in R_1\right\}$

$a \geq b \ and \ b > c \Rightarrow a > c$ (*clearly*) that means $(a,c) \in R_1$. Therefore, $R_1 \circ R_2 = R_1$

k) $R_1 \circ R_3 = \left\{(a,b) \in R_3 \ and \ (b,c) \in R_1\right\}$

$a < b \ and \ b > c$. Therefore, $R_1 \circ R_3 = \boldsymbol{R}^2$

l) $R_1 \circ R_4 = \left\{(a,b) \in R_4 \ and \ (b,c) \in R_1\right\}$

$a \leq b \ and \ b > c$. Clearly this can always be done simply by choosing b to be large enough.

Therefore, $R_1 \circ R_4 = \boldsymbol{R}^2$

m) $R_1 \circ R_5 = \left\{(a,b) \in R_5 \ and \ (b,c) \in R_1\right\}$

$a = b \ and \ b > c$ *iff* $a > c$. Therefore, $R_1 \circ R_5 = R_1$

n) $R_1 \circ R_6 = \left\{(a,b) \in R_6 \ and \ (b,c) \in R_1\right\}$

$a \neq b \ and \ b > c$. Clearly this can always be done simply by choosing b to be large enough.

Therefore, $R_1 \circ R_6 = \boldsymbol{R}^2$

o) $R_2 \circ R_3 = \left\{(a,b) \in R_3 \ and \ (b,c) \in R_2\right\}$

$a < b \ and \ b \geq c$. Clearly this can always be done simply by choosing b to be large enough.

Therefore, $R_2 \circ R_3 = \boldsymbol{R}^2$

Exercise

Let R_1 *and* R_2 be the "*divides*" and "*is a multiple of*" relations on the set of all positive integers, respectively.

That is $R_1 = \{(a,b) \mid a \text{ divides } b\}$ and $R_2 = \{(a,b) \mid a \text{ is a multiple of } b\}$

Find the following:

a) $R_1 \cup R_2$ ***b***) $R_1 \cap R_2$ ***c***) $R_1 - R_2$ ***d***) $R_2 - R_1$ ***e***) $R_1 \oplus R_2$

Solution

a) $(a,\ b) \in R_1 \cup R_2$ if and only if $a|b$ or $b|a$

b) $(a,\ b) \in R_1 \cup R_2$ if and only if $a|b$ and $b|a$ with $a = \pm b$ and $a \neq 0$

c) $R_1 - R_2 = R_1 \cap \overline{R}_2$ this relation holds between 2 integers if R_1 holds and R_2 does not hold.

$(a,\ b) \in R_1 \cap R_2$ if and only if $a|b$ and $b \nmid a$ $(a \neq \pm b)$

d) $R_2 - R_1 = R_2 \cap \overline{R}_1$ this relation holds between 2 integers if R_2 holds and R_1 does not hold.

$(a,\ b) \in R_1 \cap R_2$ if and only if $b|a$ and $a \nmid b$ $(a \neq \pm b)$

e) $R_1 \oplus R_2 = (R_1 - R_2) \cup (R_2 - R_1)$ this relation holds between 2 integers if R_2 holds and R_1 does not hold and R_2 holds and R_1 does not hold. if and only if $a|b$ or $b|a$ $(a \neq \pm b)$