

2.7.2

2.7. C.D.E.

First Order D.E.

$$y' = \frac{dy}{dx} = f(x, y)$$

Ex: $y = \frac{c}{x} + 2 \rightarrow y' = \frac{1}{x} (2 - y)$ (0, 2)

$$y' = -\frac{c}{x^2}$$

$$y' \stackrel{?}{=} \frac{1}{x} (2 - y)$$

$$-\frac{c}{x^2} \stackrel{?}{=} \frac{1}{x} (2 - \frac{c}{x} - 2)$$

$$= \frac{1}{x} (-\frac{c}{x})$$

$$= -\frac{c}{x^2} \checkmark$$

Ex: $y = (x+1) - \frac{1}{3}e^x$ is soln! $y' = y - x$ $y(0) = \frac{2}{3}$

$$y' = 1 - \frac{1}{3}e^x$$

$$y' = y - x$$

$$1 - \frac{1}{3}e^x = x + 1 - \frac{1}{3}e^x - x$$
$$= 1 - \frac{1}{3}e^x \checkmark$$

$$y(0) = \frac{2}{3} \stackrel{?}{=} 0 + 1 - \frac{1}{3}e^0$$
$$= 1 - \frac{1}{3}$$
$$= \frac{2}{3} \checkmark$$

y' : slope

$$y' + P y = Q$$

Method: Separable Eqn.

$$\frac{dy}{dx} + P(x)y = 0 \quad \text{Homogeneous.}$$

$$\frac{dy}{dx} = -P y$$

$$\int \frac{dy}{y} = -\int P(x) dx$$

$$\ln|y| = -\int P(x) dx + C$$

$$y = e^{-\int P dx + C}$$

$$= e^{-\int P dx} e^C$$

e^C is constant

$$\Rightarrow y = e^{-\int P dx}$$

Ex $y' = t y^2$

$$\frac{dy}{dt} = t y^2$$

$$\int \frac{dy}{y^2} = \int t dt$$

$$-\frac{1}{y} = \frac{1}{2} t^2 + C$$

$$= \frac{t^2 + 2C}{2}$$

$$y = -\frac{2}{t^2 + 2C}$$

ex: $y' = -\frac{2xy}{x^2-1}$

$$\frac{dy}{dx} = \frac{2x(y+1)}{x^2-1}$$

$$\frac{dy}{y+1} = \frac{2x}{x^2-1} dx$$

$$\int \frac{d(y+1)}{y+1} = \int \frac{d(x^2-1)}{x^2-1}$$

$$\ln|y+1| = \ln|x^2-1| + \ln C$$

$$= \ln C|x^2-1|$$

$$y+1 = C|x^2-1|$$

$$y = C|x^2-1| - 1$$

$$y' + p y = f$$

ex: $x \frac{dy}{dx} = x^2 + 3y$

$$y' - \frac{3}{x}y = x$$

$$e^{\int \frac{3}{x} dx} = e^{-3 \ln x} = x^{-3}$$

$$\int x x^{-3} dx = \int x^{-2} dx$$

$$= -\frac{1}{x}$$

$$y(x) = \frac{1}{x^{-3}} \left(-\frac{1}{x} + C \right)$$

$$= -x^2 + C x^3 \quad x > 0$$

$$y(x) = \frac{1}{e^{\int p(x) dx}} \left(\int f(x) e^{\int p(x) dx} + C \right)$$

$$3. x^3 - y = \ln x + 1$$

$$y(1) = -2$$

$$y' - \frac{1}{3x} y = \frac{\ln x + 1}{3x}$$

$$\int \frac{1}{3x} dx = e^{-\frac{1}{3} \ln x}$$

$$= e^{\ln x^{-1/3}}$$

$$= x^{-1/3}$$

$$\int \frac{\ln x + 1}{3x} (x^{-1/3}) dx = \frac{1}{3} \int x^{-4/3} (\ln x + 1) dx$$

$$u = \ln x + 1$$

$$du = \frac{1}{x} dx$$

$$v = \int x^{-4/3} dx$$

$$= -3x^{-1/3}$$

$$= \frac{1}{3} \left[-3x^{-1/3} (\ln x + 1) + 3 \int \frac{x^{-1/3}}{x} dx \right]$$

$$= -x^{-1/3} (\ln x + 1) + \int x^{-4/3} dx$$

$$= -x^{-1/3} (\ln x + 1) - 3x^{-1/3}$$

$$y(x) = x^{1/3} \left(-\frac{\ln x + 1}{x^{1/3}} - \frac{3}{x^{1/3}} + C \right)$$

$$= -\ln x - 1 - 3 + C x^{1/3}$$

$$= -\ln x - 4 + C x^{1/3}$$

$$y(1) = -2$$

$$-2 = -4 + C$$

$$C = 2$$

$$y(x) = -\ln x - 4 + 2 \sqrt[3]{x}$$

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$$y' = xy = \frac{dy}{dx}$$

$$\int \frac{dy}{y} = \int x dx$$

$$\ln|y| = \frac{1}{2} x^2 + C$$

$$y = e^{\frac{1}{2} x^2 + C}$$

$$= e^{\frac{1}{2} x^2} e^C$$

$$= \underline{\underline{A e^{x^2/2}}}$$

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$$y' = e^{x-y}$$

$$\frac{dy}{dx} = e^x e^{-y}$$

$$\int e^y dy = \int e^x dx$$

$$e^y = e^x + C$$

$$\underline{\underline{y = \ln(e^x + C)}} \quad C \geq 0$$

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$$y' = y e^x - 2 e^x + y - 2$$

$$= y(e^x + 1) - 2(e^x + 1)$$

$$\frac{dy}{dx} = (y-2)(e^x + 1)$$

$$\int \frac{d(y-2)}{y-2} = \int (e^x + 1) dx$$

$$d(y-2) = dy$$

$$\underline{\underline{\ln|y-2| = e^x + x + C}}$$

130. $y' + (\tan x) y = \cos^2 x$ $(\tan x) y \quad \frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$

$$e^{\int \tan x dx} = e^{\ln |\sec x|} = \sec x$$

$$\int \cos^2 x \frac{1}{\cos x} dx = \int \cos x dx = \sin x$$

$$y(x) = \cos x (\sin x + C)$$

136. $y' = y + 2xe^{2x}$ $y(0) = 3$

$$y' - y = 2xe^{2x}$$

$$e^{\int -1 dx} = e^{-x}$$

$$\int 2xe^{2x} e^{-x} dx = 2 \int xe^x dx$$

$$= 2e^x(x-1)$$

$$y(x) = e^x (2e^x(x-1) + C)$$

$$= 2(x-1)e^{2x} + Ce^x$$

$$y(0) = 3 = -2 + C$$

$$C = 5$$

$$y(x) = 2(x-1)e^{2x} + 5e^x$$

Mixture Problems

$$\text{Rate of change} = \text{rate in} - \text{rate out}$$

$$\frac{dy}{dt}$$



$$\text{Rate} = \text{Volume} \frac{\text{gal}}{\text{min}} \times \text{concentration} \frac{\text{lb}}{\text{gal}}$$

$$\text{Rate}_{\text{out}} = \frac{y(t)}{V(t)} \cdot \text{outflow rate}$$

gal, lb, min

$$V(0) = 3000 \text{ gal} \quad y(0) = 100 \text{ lb}$$

$$\text{rate in: } 2 \text{ lb/gal} \quad 40 \text{ gal/min}$$

$$\text{out: } ? \quad 45 \text{ gal/min}$$

$$V(t) = 3000 + (40 - 45)t$$

$$= 3000 - 5t$$

$$\text{* Tank to be empty: } 3000 - 5t = 0$$

$$t = 600 \text{ min}$$

$$\text{Rate out} = \frac{y(t)}{3000 - 5t} \cdot 45$$

$$\text{Rate in} = (2)(40) = 80$$

$$\frac{dy}{dt} = 80 - \frac{45}{3000 - 5t} y$$

$$y' + \frac{45}{3000 - 5t} y = 80$$

$$e^{\int \frac{45}{3000 - 5t} dt} = e^{-9 \int \frac{d(3000 - 5t)}{3000 - 5t}}$$

$$= e^{-9 \ln |3000 - 5t|}$$

$$= (3000 - 5t)^{-9}$$

$$\int 80 (3000 - 5t)^{-9} dt = -16 \int (3000 - 5t)^{-9} d(2000 - 5t)$$

$$= 2 (3000 - 5t)^{-8}$$

$$y(t) = (3000 - 5t)^9 \left[2 (3000 - 5t)^{-8} + C \right]$$

$$= 3000 - 5t + C (3000 - 5t)^9$$

$$y(0) = 100$$

$$100 = 3000 + C (3000)^9$$

$$C = -\frac{2900}{(3000)^9}$$

$$y(t) = 3000 - 5t - \frac{2900}{(3000)^9} (3000 - 5t)^9$$

$$y(20) = 2900 - \frac{2900}{3000^9} 2900^9$$

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$$y'(t) = 2y + 4 \quad y(0) = 8$$

$$y' - 2y = 4$$

$$e^{\int -2dt} = e^{-2t}$$

$$\int 4e^{-2t} dt = -2e^{-2t}$$

$$y(t) = e^{2t} (-2e^{-2t} + C)$$

$$= -2 + Ce^{2t}$$

$$y(0) = 8 = -2 + C$$

$$C = 10$$

$$y(t) = -2 + 10e^{2t}$$

198 $x dy + (y - \cos x) dx = 0 \quad y\left(\frac{\pi}{2}\right) = 0$

$$x \frac{dy}{dx} + y - \cos x = 0$$

$$y' + \frac{1}{x}y = \frac{\cos x}{x}$$

$$e^{\int \frac{dx}{x}} = e^{\ln x} = x$$

$$\int \frac{\cos x}{x} x dx = \int \cos x dx = \sin x$$

$$y(x) = \frac{1}{x} (\sin x + C)$$

$$y\left(\frac{\pi}{2}\right) = 0 = \frac{2}{\pi} (1 + C)$$

$$1 + C = 0 \Rightarrow C = -1$$

$$y(x) = \frac{1}{x} (\sin x - 1)$$

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$$(x^2+1)y' + 3xy = 6x$$

$$y(0) = -1$$

$$y' + \frac{3x}{x^2+1} y = \frac{6x}{x^2+1}$$

$$e^{\int \frac{3x}{x^2+1} dx} = e^{\frac{3}{2} \int \frac{d(x^2+1)}{x^2+1}} = e^{\frac{3}{2} \ln(x^2+1)} \\ = (x^2+1)^{3/2}$$

$$\int \frac{6x}{x^2+1} (x^2+1)^{3/2} dx = 6 \int x (x^2+1)^{1/2} dx \\ = 3 \int (x^2+1)^{1/2} d(x^2+1) \\ = 2 (x^2+1)^{3/2}$$

$$J(x) = \frac{1}{(x^2+1)^{3/2}} (2(x^2+1)^{3/2} + C) \\ = 2 + \frac{C}{(x^2+1)^{3/2}}$$

$$y(0) = -1 = 2 + C \\ \underline{C = -3}$$

$$\underline{y(x) = 2 - \frac{3}{(x^2+1)^{3/2}}}$$