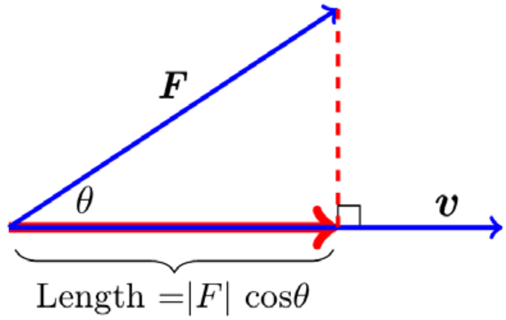
Section 1.2 – Dot Products

If a force ***F*** is applied to a particle moving along a path, we often need to know the magnitude of the force and the direction of motion.



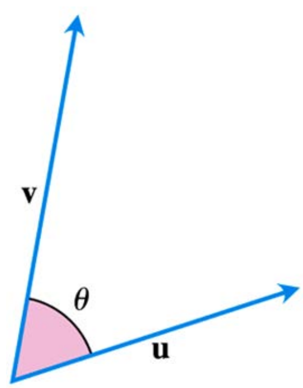
To calculate the angle between two vectors directly from their component, called the ***dot product***, also called *inner* or *scalar* products.

Angle between Vectors

Theorem

The angle θ between two nonzero vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ *and* $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\vec{u}|\,|\vec{v}|} \right)$$



Definition

The dot product $\vec{u} \cdot \vec{v}$ of vector $\vec{u} = \langle u_1, u_2, u_3 \rangle$ *and* $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Example

Find the dot product:

a) $\langle 1,-2,-1\rangle \bullet \langle -6,2,-3\rangle$ *b)* $\left(\frac{1}{2}\hat{i}+3\hat{j}+\hat{k}\right)\bullet\left(4\hat{i}-\hat{j}+2\hat{k}\right)$

Solution

a) $\langle 1,-2,-1\rangle \bullet \langle -6,2,-3\rangle = 1(-6)+(-2)(2)+(-1)(-3)$

$\underline{=-7}$

b) $\left(\frac{1}{2}\hat{i}+3\hat{j}+\hat{k}\right)\bullet\left(4\hat{i}-\hat{j}+2\hat{k}\right)=\frac{1}{2}(4)+3(-1)+1(2)$

$\underline{=1}$

Example

Find the angle between $\vec{u}=\hat{i}-2\hat{j}-2\hat{k}$ *and* $\vec{v}=6\hat{i}+3\hat{j}+2\hat{k}$

Solution

$\vec{u}\bullet\vec{v}=1(6)+(-2)(3)+(-2)(2)$

$\underline{=-4}$

$\vec{u}=\sqrt{1^2+(-2)^2+(-2)^2}$

$\underline{=3}$

$\vec{v}=\sqrt{6^2+3^2+2^2}$

$\underline{=7}$

$\theta=\cos^{-1}\frac{\vec{u}\bullet\vec{v}}{|\vec{u}|\;|\vec{v}|}$

$=\cos^{-1}\left(\frac{-4}{(3)(7)}\right)$

$\underline{=\cos^{-1}\left(-\frac{4}{21}\right)}$

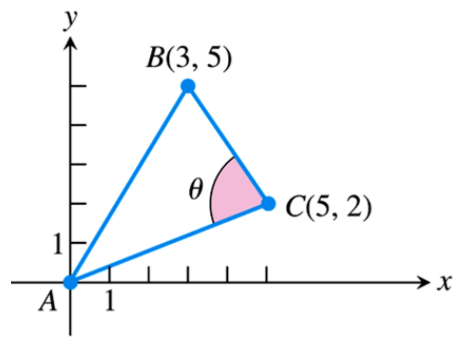
$\underline{\approx 1.76\ \ rad}$

Example

Find the angle θ of the triangle ABC determined by the vertices

$A=(0,\ 0),\quad B=(3,\ 5),\quad and\quad C=(5,\ 2)$

Solution



$\overrightarrow{CA}=\langle -5,\ -2\rangle \quad \overrightarrow{CB}=\langle -2,\ 3\rangle$

$$\overrightarrow{CA}\cdot\overrightarrow{CB}=(-5)(-2)+(-2)(3)$$

$$=4$$

$$\left|\overrightarrow{CA}\right|=\sqrt{(-5)^2+(-2)^2}$$

$$=\sqrt{29}$$

$$\left|\overrightarrow{CB}\right|=\sqrt{(-2)^2+(3)^2}$$

$$=\sqrt{13}$$

$$\theta=\cos^{-1}\frac{\overrightarrow{CA}\cdot\overrightarrow{CB}}{\left|\overrightarrow{CA}\right|\left|\overrightarrow{CB}\right|}$$

$$=\cos^{-1}\left(\frac{4}{\sqrt{29}\sqrt{13}}\right)$$

$$=\cos^{-1}\left(\frac{4}{\sqrt{377}}\right)$$

$\approx 1.36\ rad$ *or* 78.1°

Perpendicular (*Orthogonal*) Vectors

Definition

Vectors $\vec{u}$ and $\vec{v}$ are orthogonal (*or* perpendicular) *iff* $\vec{u} \bullet \vec{v} = 0$

Example

Determine if the two vectors are orthogonal

a) $\vec{u} = \langle 3, \ -2 \rangle$ *and* $\vec{v} = \langle 4, \ 6 \rangle$

b) $\vec{u} = 3\hat{i} - 2\hat{j} + \hat{k}$ *and* $\vec{v} = 2\hat{j} + 4\hat{k}$

Solution

a) $\vec{u} \bullet \vec{v} = 3(4) + (-2)(6)$

$= 0$

$\therefore$ The two vectors are orthogonal

b) $\vec{u} \bullet \vec{v} = 3(0) + (-2)(2) + 1(4)$

$= 0$

$\therefore$ The two vectors are orthogonal

Dot Product Properties and Vector Projection

Properties of the Dot Product

If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are any vectors and c is a scalar, then

a) $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$

b) $\vec{u} \bullet \vec{u} = |\vec{u}|^2$

c) $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$

d) $\vec{u} \bullet (\vec{v} - \vec{w}) = \vec{u} \bullet \vec{v} - \vec{u} \bullet \vec{w}$

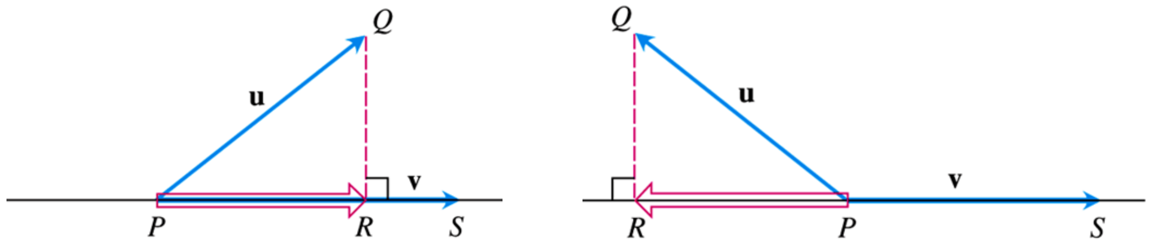
e) $(\vec{u} + \vec{v}) \bullet \vec{w} = \vec{u} \bullet \vec{w} + \vec{v} \bullet \vec{w}$

f) $(\vec{u} - \vec{v}) \bullet \vec{w} = \vec{u} \bullet \vec{w} - \vec{v} \bullet \vec{w}$

g) $c(\vec{u} \bullet \vec{v}) = (c\vec{u}) \bullet \vec{v} = \vec{u} \bullet (c\vec{v})$

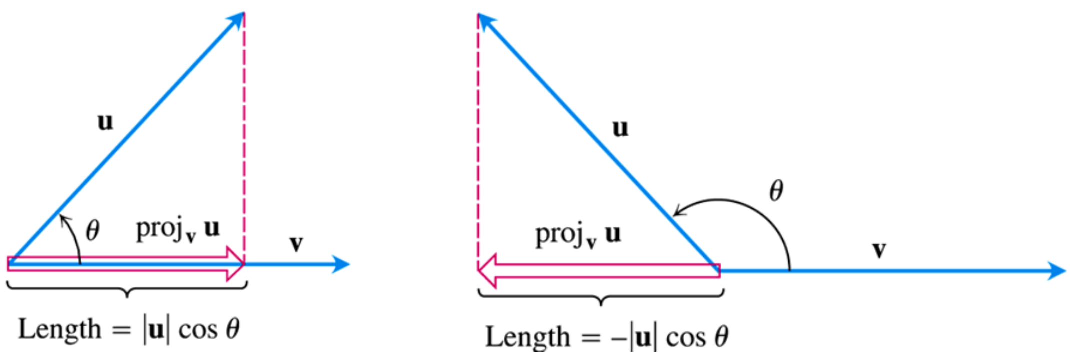
h) $\vec{0} \bullet \vec{v} = \vec{v} \bullet \vec{0} = 0$

The vector projection of $\boldsymbol{u} = \overrightarrow{PQ}$ onto a nonzero vector $\boldsymbol{v} = \overrightarrow{PS}$ is the vector $\overrightarrow{PR}$ determined by dropping a perpendicular from Q to the line PS.



The notation for this vector is

$$proj_{\boldsymbol{v}}\boldsymbol{u} \qquad \text{(The vector projection of } \boldsymbol{u} \text{ onto } \boldsymbol{v}\text{)}$$



$$
\begin{aligned}
proj_{\vec{v}}\,\vec{u} &= \left(|\vec{u}|\cos\theta\right)\frac{\vec{u}}{|\vec{v}|} \\
&= \left(\frac{\vec{u}\bullet\vec{v}}{|\vec{v}|^2}\right)\vec{v}
\end{aligned}
$$

The scalar component of $\vec{u}$ in the direction of $\vec{v}$ is the scalar: $|\vec{u}|\cos\theta = \dfrac{\vec{u}\bullet\vec{v}}{|\vec{v}|} = \vec{u}\bullet\dfrac{\vec{v}}{|\vec{v}|}$

Example

Find the vector projection of $\vec{u} = 6\hat{i} + 3\hat{j} + 2\hat{k}$ *onto* $\vec{v} = \hat{i} - 2\hat{j} - 2\hat{k}$ and the scalar component of $\vec{u}$ in the direction of $\vec{v}$.

Solution

$$
\begin{aligned}
proj_{\vec{v}}\,\vec{u} &= \frac{\vec{u}\bullet\vec{v}}{|\vec{v}|^2}\vec{v} \\
&= \frac{6(1)+3(-2)+2(-2)}{1^2+(-2)^2+(-2)^2}\left(\hat{i}-2\hat{j}-2\hat{k}\right) \\
&= \frac{-4}{9}\left(\hat{i}-2\hat{j}-2\hat{k}\right)
\end{aligned}
$$

$$\boxed{= -\frac{4}{9}\hat{i} + \frac{8}{9}\hat{j} + \frac{8}{9}\hat{k}}$$

$$\vec{u}\cos\theta = \vec{u} \bullet \frac{\vec{v}}{|\vec{v}|}$$

$$= \left(6\hat{i} + 3\hat{j} + 2\hat{k}\right) \bullet \frac{\hat{i} - 2\hat{j} - 2\hat{k}}{\sqrt{1^2 + (-2)^2 + (-2)^2}}$$

$$= \left(6\hat{i} + 3\hat{j} + 2\hat{k}\right) \bullet \left(\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k}\right)$$

$$= 6\left(\frac{1}{3}\right) + 3\left(-\frac{2}{3}\right) + 2\left(-\frac{2}{3}\right)$$

$$= 2 - 2 - \frac{4}{3}$$

$$\boxed{= -\frac{4}{3}}$$

Example

Find the vector projection of a force $\vec{F} = 5\hat{i} + 2\hat{j}$ *onto* $\vec{v} = \hat{i} - 3\hat{j}$ and the scalar component of $\vec{F}$ in the direction of $\vec{v}$.

Solution

$$proj_{\vec{v}}\,\vec{F} = \frac{\vec{F} \bullet \vec{v}}{|\vec{v}|^2}\vec{v}$$

$$= \frac{5(1) + 2(-3)}{1^2 + (-3)^2}\left(\hat{i} - 3\hat{j}\right)$$

$$= -\frac{1}{10}\left(\hat{i} - 3\hat{j}\right)$$

$$\boxed{= -\frac{1}{10}\hat{i} + \frac{3}{10}\hat{j}}$$

$$\vec{F}\cos\theta = \vec{F} \bullet \frac{\vec{v}}{|\vec{v}|}$$

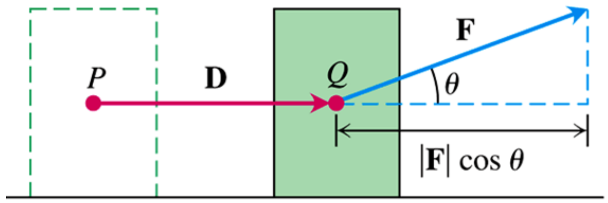
$$= \frac{\left(5\hat{i} + 2\hat{j}\right) \bullet \left(\hat{i} - 3\hat{j}\right)}{\sqrt{1^2 + (-3)^2}}$$

$$= \frac{5 - 6}{\sqrt{10}}$$

$$\boxed{= -\frac{1}{\sqrt{10}}}$$

Work

The work is done by a constant force of magnitude $\boldsymbol{F}$ in moving an object through a distance $\boldsymbol{d}$ as $W = Fd$.



$$\begin{aligned} Work &= \begin{pmatrix} scalar\ component\ of\ \boldsymbol{F} \\ \text{in}\ the\ direction\ of\ \boldsymbol{D} \end{pmatrix} (length\ of\ \boldsymbol{D}) \\ &= \left(|\boldsymbol{F}|\cos\theta\right)|\boldsymbol{D}| \\ &= \boldsymbol{F} \bullet \boldsymbol{D} \end{aligned}$$

Definition

The work done by a constant force $\vec{F}$ acting through a displacement $\vec{D} = \overrightarrow{PQ}$ is

$$W = \vec{F} \bullet \vec{D}$$

Example

If $\left|\vec{F}\right| = 40\ N,\quad \left|\vec{D}\right| = 3\ m,\quad and\quad \theta = 60^\circ$ find the work done by $\vec{F}$ in acting from P to Q.

Solution

$$\begin{aligned} Work &= \vec{F} \bullet \vec{D} \\ &= \left|\vec{F}\right| \bullet \left|\vec{D}\right| \cos\theta \\ &= (40)(3)\cos 60^\circ \\ &= 60\ \ J\ \ (joules) \end{aligned}$$

Exercises *Section* 1.2 – Dot Products

(**1 – 5**) Find

a) $\vec{v}\cdot\vec{u},\ |\vec{v}|,\ |\vec{u}|$

b) The cosine of the angle between $\vec{v}$ and $\vec{u}$

c) The scalar component of $\vec{u}$ in the direction of $\vec{v}$

d) The vector $proj_{\vec{v}}\vec{u}$

1. $\vec{v}=2\hat{i}-4\hat{j}+\sqrt{5}\hat{k}$, $\vec{u}=-2\hat{i}+4\hat{j}-\sqrt{5}\hat{k}$

2. $\vec{v}=\frac{3}{5}\hat{i}+\frac{4}{5}\hat{k}$, $\vec{u}=5\hat{i}+12\hat{j}$

3. $\vec{v}=2\hat{i}+10\hat{j}-11\hat{k}$, $\vec{u}=2\hat{i}+2\hat{j}+\hat{k}$

4. $\vec{v}=-\hat{i}+\hat{j}$, $\vec{u}=2\hat{i}+\sqrt{17}\hat{j}$

5. $\vec{v}=\left\langle \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}}\right\rangle$, $\vec{u}=\left\langle \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{3}}\right\rangle$

6. Find the angles between the vectors $\vec{u}=2\hat{i}+\hat{j}$, $\vec{v}=\hat{i}+2\hat{j}-\hat{k}$

7. Find the angles between the vectors $\vec{u}=\sqrt{3}\hat{i}-7\hat{j}$, $\vec{v}=\sqrt{3}\hat{i}+\hat{j}+\hat{k}$

8. Find the angles between the vectors $\vec{u}=\hat{i}+\sqrt{2}\hat{j}-\sqrt{2}\hat{k}$, $\vec{v}=-\hat{i}+\hat{j}+\hat{k}$

9. Consider $\vec{u}=-3\hat{j}+4\hat{k}$, $\vec{v}=-4\hat{i}+\hat{j}+5\hat{k}$

a) Find the angle between $\vec{u}$ and $\vec{v}$.

b) Compute $proj_{\vec{v}}\vec{u}$ and $scal_{\vec{v}}\vec{u}$

c) Compute $proj_{\vec{u}}\vec{v}$ and $scal_{\vec{u}}\vec{v}$

10. Consider $\vec{u}=-\hat{i}+2\hat{j}+2\hat{k}$, $\vec{v}=3\hat{i}+6\hat{j}+6\hat{k}$

a) Find the angle between $\vec{u}$ and $\vec{v}$.

b) Compute $proj_{\vec{v}}\vec{u}$ and $scal_{\vec{v}}\vec{u}$

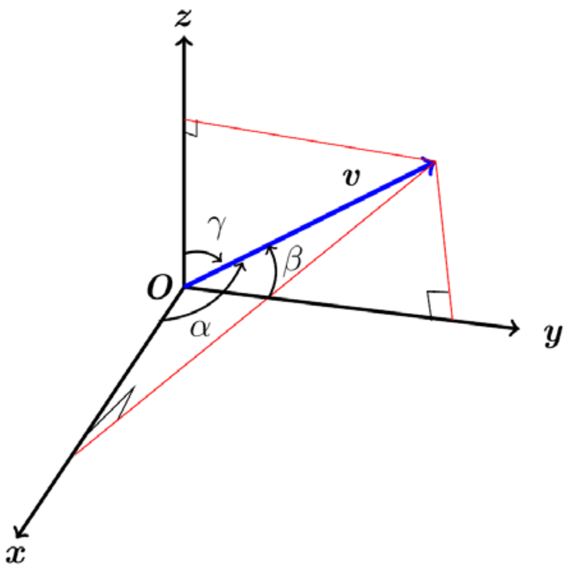
c) Compute $proj_{\vec{u}}\vec{v}$ and $scal_{\vec{u}}\vec{v}$

11. The direction angles α, β, and γ of a vector $\vec{v}=a\hat{i}+b\hat{j}+c\hat{k}$ are defined as follows:

α is the angle between ***v*** and the positive *x*-axis $(0\le\alpha\le\pi)$

β is the angle between ***v*** and the positive *y*-axis $(0\le\beta\le\pi)$

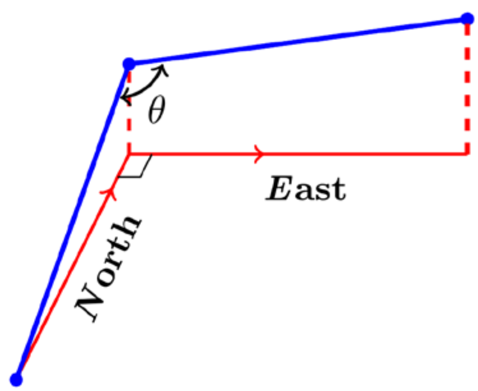
γ is the angle between ***v*** and the positive *z*-axis $(0\le\gamma\le\pi)$



a) Show that $\cos\alpha = \frac{a}{|\vec{v}|}$, $\cos\beta = \frac{b}{|\vec{v}|}$, $\cos\gamma = \frac{c}{|\vec{v}|}$, and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. These cosines are called the direction cosines of $\vec{v}$.

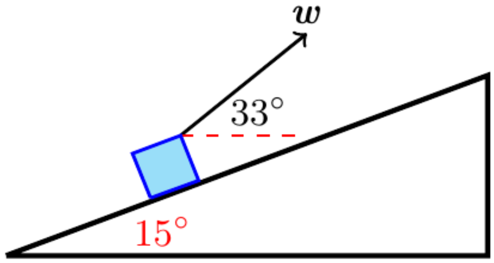
b) Show that if $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$ is a unit vector, then *a*, *b*, and *c* are the direction cosines of $\vec{v}$.

12. A water main is to be constructed with 20% grade in the north direction and a 10% grade in the east direction. Determine the angle θ required in the water main for the turn from north to east.



13. A gun with muzzle velocity of 1200 *ft/sec* is fired at an angle of 8° above the horizontal. Find the horizontal and vertical components of the velocity.

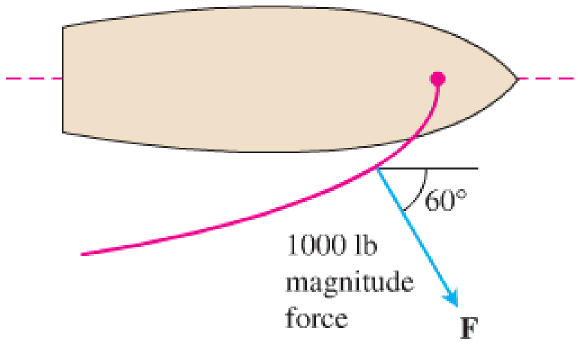
14. Suppose that a box is being towed up an inclined plane. Find the force ***w*** needed to make the component of the force parallel to the indicated plane equal to 2.5 *lb*.



15. Find the work done by a force $\vec{F} = 5\hat{i}$ (magnitude 5 *N*) in moving an object along the line from the origin to the point (1, 1) (distance in meters)

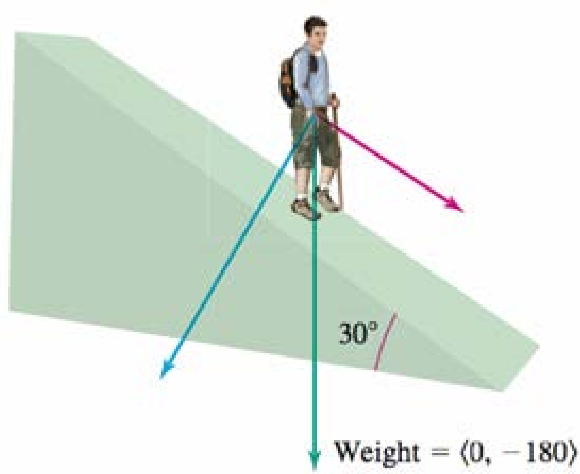
16. How much work does it take to slide a crate 20 *m* along a loading dock by pulling on it with a 200 *N* force at an angle of 30° from the horizontal?

17. The wind passing over a boat's sail exerted a 100-*lb* magnitude force *F*. How much work did the wind perform in moving the boat forward 1 *mile* ? Answer in foot-pounds.



18. Use a dot product to find an equation of the line in the *xy*-plane passing through the point $\left(x_0, y_0\right)$ perpendicular to the vector $\langle a, b\rangle$.

19. A 180-*lb* man stands on a hillside that makes an angle of 30° with the horizontal, producing a force of $W = \langle 0, \ -180\rangle$ *lb*s.



a) Find the component of his weight in the downward direction perpendicular to the hillside and in the downward parallel to the hillside.
b) How much work is done when the man moves 10 *ft* up the hillside?