Solution   *Section* 3.9 – Eigenvalues and Eigenvectors

Exercise

Find the eigenvalues and eigenvectors of $A,\ A^2,\ A^{-1},\ \text{and}\ \ A+4I$:

$$A=\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \textit{and} \quad A^2=\begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

Check the trace $\lambda_1+\lambda_2$ and the determinant $\lambda_1\lambda_2$ for A and also A^2.

Solution

For A:

$$\det(A-\lambda I)=\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix}$$

$$=(2-\lambda)^2-1$$

$$=\lambda^2-4\lambda+3=0$$

The eigenvalues of $\boldsymbol{A}$ are $\lambda_1=1$ *and* $\lambda_2=3$.

The trace of a square matrix A is the sum of the elements on the main diagonal: 2 + 2 agrees with 1+ 3. The det($\boldsymbol{A}$) = 3 agrees with the product $\lambda_1\lambda_2$.

The eigenvectors for $\boldsymbol{A}$ are:

$\lambda_1=1$: $(A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} 2-1 & -1 \\ -1 & 2-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}\to\begin{cases} x-y=0 \\ -x+y=0 \end{cases}\Rightarrow x=y$$

Therefore the eigenvector $V_1=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2=3$: $(A-\lambda_2 I)V_2=0$

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}\to\begin{cases} -x-y=0 \\ -x-y=0 \end{cases}\Rightarrow x=-y$$

Therefore the eigenvector $V_2=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

For A^2:

$$\det\left(A^2-\lambda I\right)=\begin{vmatrix}5-\lambda & -4\\ -4 & 5-\lambda\end{vmatrix}=(5-\lambda)^2-16=\lambda^2-10\lambda+9=0$$

The eigenvalues of A^2 are $\lambda_1=1 \quad and \quad \lambda_2=9$. $\qquad$ ***Or*** $\lambda_1=1^2=1 \quad and \quad \lambda_2=3^2=9$

$$\begin{cases} tr(A)=5+5=10 \\ \lambda_1+\lambda_2=1+9=10 \end{cases} \Rightarrow tr(A)=\lambda_1+\lambda_2$$

$$\begin{cases} \left|A^2\right|=\begin{vmatrix}5 & -4\\ -4 & 5\end{vmatrix}=9 \\ \lambda_1\lambda_2=1(9)=9 \end{cases} \Rightarrow \left|A^2\right|=\lambda_1\lambda_2$$

$\lambda_1=1:\left(A^2-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix}4 & -4\\ -4 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\rightarrow\begin{cases}4x-4y=0\\ -4x+4y=0\end{cases}\Rightarrow x=y$$

Therefore the eigenvector $V_1=\begin{pmatrix}1\\ 1\end{pmatrix}$

$\lambda_2=9:\left(A^2-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix}-4 & -4\\ -4 & -4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\rightarrow\begin{cases}-4x-4y=0\\ -4x-4y=0\end{cases}\Rightarrow x=y$$

Therefore the eigenvector $V_2=\begin{pmatrix}1\\ -1\end{pmatrix}$

For A^{-1}:

$$A^{-1}=\frac{1}{3}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}=\begin{pmatrix}\frac{2}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}\end{pmatrix}$$

$$\det\left(A^{-1}-\lambda I\right)=\begin{vmatrix}\frac{2}{3}-\lambda & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}-\lambda\end{vmatrix}=\left(\frac{2}{3}-\lambda\right)^2-\frac{1}{9}=\lambda^2-\frac{4}{3}\lambda+\frac{1}{3}=0$$

The eigenvalues of A^{-1} are $\lambda_1=1 \quad and \quad \lambda_2=\frac{1}{3}$.

$\lambda_1=1:\left(A^{-1}-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix} \frac{2}{3}-1 & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3}-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -\frac{1}{3}x + \frac{1}{3}y = 0 \\ \frac{1}{3}x - \frac{1}{3}y = 0 \end{cases} \to \boxed{x = y}$$

Therefore the eigenvector $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = \frac{1}{3}: \left(A^{-1} - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} \frac{2}{3}-\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3}-\frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \frac{1}{3}x + \frac{1}{3}y = 0 \\ \frac{1}{3}x + \frac{1}{3}y = 0 \end{cases} \to \boxed{x = -y}$$

Therefore the eigenvector $V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

For $A + 4I$:

$$A + 4I = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 1 & 6 \end{pmatrix}$$

$$\det\left(A^{-1} - \lambda I\right) = \begin{vmatrix} 6-\lambda & 1 \\ 1 & 6-\lambda \end{vmatrix} = (6-\lambda)^2 - 1 = \lambda^2 - 12\lambda + 35 = 0$$

The eigenvalues of A^{-1} are $\lambda_1 = 5$ *and* $\lambda_2 = 7$.

$\lambda_1 = 5: \left(A + 4I - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 6-5 & 1 \\ 1 & 6-5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x + y = 0 \\ x + y = 0 \end{cases} \to \boxed{x = -y}$$

Therefore the eigenvector $V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = 7: \left(A + 4I - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -x + \frac{1}{3}y = 0 \\ x - y = 0 \end{cases} \rightarrow \boxed{x = y}$$

Therefore the eigenvector $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The eigenvalues $(A) = \lambda$

The eigenvalues $\left(A^2\right) = \lambda^2$

The eigenvalues $\left(A^{-1}\right) = \frac{1}{\lambda}$

Exercise

Show directly that the given vectors are eigenvectors of the given matrix. What are the corresponding eigenvalues

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$$

Solution

$$A\boldsymbol{v}_1 = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 7 \\ -21 \end{bmatrix} = 7\begin{bmatrix} 1 \\ -3 \end{bmatrix} = 7\boldsymbol{v}_1$$

$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is an eigenvectors corresponding to the eigenvalue 7.

$$A\boldsymbol{v}_2 = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0\boldsymbol{v}_2$$

$\boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvectors corresponding to the eigenvalue 0.

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1-\lambda & -2 \\ -3 & 6-\lambda \end{vmatrix} \\ &= (1-\lambda)(6-\lambda) - 6 \\ &= 6 - 7\lambda + \lambda^2 - 6 \\ &= \lambda^2 - 7\lambda = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 = 0 \quad \textit{and} \quad \lambda_2 = 7$

Exercise

For which real numbers c does this matrix A have

$$A = \begin{pmatrix} 2 & -c \\ -1 & 2 \end{pmatrix}$$

a) Two real eigenvalues and eigenvectors.

b) A repeated eigenvalue with only one eigenvector

c) Two complex eigenvalues and eigenvectors.

Solution

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 2-\lambda & -c \\ -1 & 2-\lambda \end{vmatrix} \\ &= (2-\lambda)^2 - c \\ &= \lambda^2 - 4\lambda + 4 - c = 0 \end{aligned}$$

$$\lambda = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-c)}}{2(1)}$$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16+4c}}{2}$$

a) Two real eigenvalues and eigenvectors, when $16 + 4c > 0 \ \rightarrow \ 4c > -16 \ \Rightarrow \ \boxed{c > -4}$

b) A repeated eigenvalue with only one eigenvector, when $16 + 4c = 0 \ \Rightarrow \ \boxed{c = -4}$

c) Two complex eigenvalues and eigenvectors, when $16 + 4c < 0 \ \Rightarrow \ \boxed{c < -4}$

Exercise

Find the eigenvalues of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{AB}$, and $\boldsymbol{BA}$:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad AB = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad BA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

a) The eigenvalues of $\boldsymbol{AB}$ (are equal to) (are not equal to) eigenvalues of $\boldsymbol{A}$ times eigenvalues of $\boldsymbol{B}$.

b) The eigenvalues of $\boldsymbol{AB}$ (are equal to) (are not equal to) eigenvalues of $\boldsymbol{BA}$.

Solution

Since $\boldsymbol{A}$ is a lower triangular, then $\lambda_1 = \lambda_2 = 1$

Since $\boldsymbol{B}$ is a upper triangular, then $\lambda_1 = \lambda_2 = 1$

$$\det(AB - I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 1 = \lambda^2 - 3\lambda + 1 = 0 \qquad \lambda_1 = \frac{3-\sqrt{5}}{2} \quad \lambda_2 = \frac{3+\sqrt{5}}{2}$$

$$\det(BA-I)=\begin{vmatrix}2-\lambda & 1\\ 1 & 1-\lambda\end{vmatrix}=(2-\lambda)(1-\lambda)-1=\lambda^2-3\lambda+1=0 \qquad \lambda_1=\frac{3-\sqrt{5}}{2} \quad \lambda_2=\frac{3+\sqrt{5}}{2}$$

a) The eigenvalues of $\boldsymbol{AB}$ are not equal to eigenvalues of $\boldsymbol{A}$ times eigenvalues of $\boldsymbol{B}$.

b) The eigenvalues of $\boldsymbol{AB}$ are equal to the eigenvalues of $\boldsymbol{BA}$.

Exercise

When $a+b=c+d$ show that (1, 1) is an eigenvector and find both eigenvalues of

$$A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

Solution

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}=\begin{pmatrix}a+b\\ c+d\end{pmatrix}=\begin{pmatrix}a+b\\ a+b\end{pmatrix}=(a+b)\begin{bmatrix}1\\ 1\end{bmatrix}$$

If $a+b=c+d=\lambda_1$

$tr(A)=a+d=\lambda_1+\lambda_2$

$$\begin{aligned}\lambda_2&=(a+d)-\lambda_1\\ &=a+d-(a+b)\\ &=a+d-a-b\\ &=d-b \qquad \text{or} \quad =a-c\end{aligned}$$

The eigenvalues for λ_2:

$$\begin{pmatrix}a-\lambda_2 & b\\ c & d-\lambda_2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}a-(a-c) & b\\ c & d-(d-b)\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}c & b\\ c & b\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\rightarrow\{cx+by=0 \quad \Rightarrow \boxed{cx=-by}$$

The eigenvector: $V_2=\begin{pmatrix}b\\ -c\end{pmatrix}$

Exercise

The eigenvalues of A equal to the eigenvalues of A^T. This is because $\det(A-\lambda I)$ equals $\det\left(A^T-\lambda I\right)$.

That is true because ______. Show by an example that the eigenvectors of A and A^T are not the same.

Solution

$$\det(A-\lambda I)=\det(A-\lambda I)^T=\det\left(A^T-(\lambda I)^T\right)=\det\left(A^T-\lambda I\right)$$

Therefore, A and A^T have the same eigenvalues.

Let consider the matrix: $A=\begin{pmatrix}0 & 1\\ 4 & 0\end{pmatrix} \Rightarrow A^T=\begin{pmatrix}0 & 4\\ 1 & 0\end{pmatrix}$

$$|A-\lambda I|=\begin{vmatrix}-\lambda & 1\\ 4 & -\lambda\end{vmatrix}=\lambda^2-4=0$$

The eigenvalues are: $\underline{\lambda=\pm 2}|$

For $\lambda=2$

$$(A-\lambda_1 I)V_1=\begin{pmatrix}-2 & 1\\ 4 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\to\begin{cases}-2x+y=0\\ 4x-2y=0\end{cases}\Rightarrow y=2x$$

$$V_1=\begin{pmatrix}1\\ 2\end{pmatrix}$$

$$\left(A^T-\lambda_1 I\right)V_1=\begin{pmatrix}-2 & 4\\ 1 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\to\begin{cases}-2x+4y=0\\ x-2y=0\end{cases}\Rightarrow x=2y$$

$$V_1=\begin{pmatrix}2\\ 1\end{pmatrix}$$

Exercise

Let $A=\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}$. Compute the eigenvalues and eigenvectors of A.

Solution

$$\det(A-\lambda I)=\begin{vmatrix}2-\lambda & -1\\ 1 & 2-\lambda\end{vmatrix}=(2-\lambda)^2+1=0$$

$$(2-\lambda)^2=-1$$

$$2-\lambda=\pm\sqrt{-1}=\pm i$$

The eigenvalues of $\boldsymbol{A}$ are: $\underline{\lambda=2\pm i}|$

For $\lambda_1=2-i\Rightarrow\left(A-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix} 2-(2-i) & -1 \\ 1 & 2-(2-i) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{cases} ix - y = 0 \\ x + iy = 0 \end{cases} \Rightarrow x = -iy$$

The eigenvector is: $V_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

For $\lambda_2 = 2 + i \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{cases} -ix - y = 0 \\ x - iy = 0 \end{cases} \Rightarrow x = iy$$

The eigenvector is: $V_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

Exercise

Let $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

a) What is the characteristic polynomial for A (i.e. compute $\det\left(A - \lambda I\right)$?

b) Verify that 1 is an eigenvalue of A. What is a corresponding eigenvector?

c) What are the other eigenvalues of A?

Solution

a) $\det\left(A - \lambda I\right) = \begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix}$

$$= (2-\lambda)(1-\lambda)(-1-\lambda) - 2 + 9 - 3(1-\lambda) - 3(2-\lambda) + 2(-1-\lambda)$$

$$= \left(2 - 3\lambda + \lambda^2\right)(-1-\lambda) + 7 - 3 + 3\lambda - 6 + 3\lambda - 2 - 2\lambda$$

$$= -2 + 3\lambda - \lambda^2 - 2\lambda + 3\lambda^2 - \lambda^3 + 4\lambda - 4$$

$$\underline{= -\lambda^3 + 2\lambda^2 + 5\lambda - 6}$$

b) *If* $\lambda = 1 \quad \to -\lambda^3 + 2\lambda^2 + 5\lambda - 6 \overset{?}{=} 0$

$$-1^3 + 2(1)^2 + 5(1) - 6 \overset{?}{=} 0$$

$$-1 + 2 + 5 - 6 \overset{?}{=} 0$$

$$\boxed{0 = 0}$$

1 is an eigenvalue of A.

$$\begin{pmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \rightarrow \begin{cases} x-2y+3z=0 \\ x \qquad\quad +z=0 \\ x+3y-2z=0 \end{cases}$$

$$\Rightarrow \begin{cases} \qquad\qquad \boxed{x=-z} \\ 3y = 2z - x = 2z + z = 3z \quad \Rightarrow \boxed{y=z} \end{cases}$$

The eigenvector for $\lambda = 1$ is $V = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

c) $-\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0 \;\Rightarrow\; \underline{\lambda_1 = 1 \quad \lambda_2 = -2 \quad \lambda_3 = 3}$

Exercise

For the matrix:

$a)\ \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$ $\quad b)\ \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$ $\quad c)\ \begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix}$

$d)\ \begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$ $\quad e)\ \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$ $\quad f)\ \begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$

$g)\ \begin{pmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{pmatrix}$ $\quad h)\ \begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\quad i)\ \begin{pmatrix} 10 & -9 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

a)

i. $\det(A-\lambda I)=\begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix}$

$$=(3-\lambda)(-1-\lambda)-0$$

$$=\lambda^2-2\lambda-3$$

The characteristic equation: $\lambda^2-2\lambda-3$

ii. $\lambda^2-2\lambda-3=0$

The eigenvalues are $\lambda_1=-1$ *and* $\lambda_2=3$

iii. $\lambda_1=-1 \quad \rightarrow \quad (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} 4 & 0 \\ 8 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}\rightarrow\begin{cases} 4x=0 \\ 8x=0 \end{cases}\Rightarrow x=0$$

Therefore the eigenvector $V_1=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda_2=3 \quad \rightarrow \quad (A-\lambda_2 I)V_2=0$

$$\begin{pmatrix} 0 & 0 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}\rightarrow\begin{cases} 0=0 \\ 8x-4y=0 \end{cases}\Rightarrow 8x=4y\rightarrow\boxed{2x=y}$$

Therefore the eigenvector $V_2 = \begin{pmatrix} x \\ 2x \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

b) For the matrix: $\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$

i. $\det(A-\lambda I) = \begin{vmatrix} 10-\lambda & -9 \\ 4 & -2-\lambda \end{vmatrix}$

$= (10-\lambda)(-2-\lambda)+36$

$= \lambda^2 - 8\lambda + 16$

$\Rightarrow$ The characteristic equation: $\underline{\lambda^2 - 8\lambda + 16|}$

ii. $\lambda^2 - 8\lambda + 16 = 0$

$\Rightarrow$ The eigenvalues are $\underline{\lambda_{1,2} = 4|}$

iii. $\lambda_1 = 4 \quad \rightarrow \quad (A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 6x-9y=0 \\ 4x-6y=0 \end{cases} \Rightarrow \boxed{2x=3y}$$

Therefore the eigenvector $V_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$

c) For the matrix: $\begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix}$

i. $\det(A-\lambda I) = \begin{vmatrix} -\lambda & 3 \\ 4 & -\lambda \end{vmatrix}$

$= \lambda^2 - 12$

$\Rightarrow$ The characteristic equation: $\underline{\lambda^2 - 12|}$

ii. $\lambda^2 - 12 = 0 \quad \Rightarrow \quad \lambda = \pm\sqrt{12}$

The eigenvalues are $\underline{\lambda_{1,2} = 4|}$

iii. For $\lambda_1 = \sqrt{12} \quad \rightarrow \quad \begin{pmatrix} -\sqrt{12} & 3 \\ 4 & -\sqrt{12} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -\sqrt{12} & 3 \\ 4 & -\sqrt{12} \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & -\frac{3}{\sqrt{12}} \\ 0 & 0 \end{pmatrix} \Rightarrow x - \frac{3}{\sqrt{12}}y = 0$$

Therefore the eigenvector $V_1 = \begin{pmatrix} \frac{3}{\sqrt{12}} \\ 1 \end{pmatrix}$

For $\lambda_2 = -\sqrt{12} \quad \rightarrow \quad \begin{pmatrix} \sqrt{12} & 3 \\ 4 & \sqrt{12} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} \sqrt{12} & 3 \\ 4 & \sqrt{12} \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & \frac{3}{\sqrt{12}} \\ 0 & 0 \end{pmatrix} \Rightarrow x + \frac{3}{\sqrt{12}} y = 0$$

Therefore the eigenvector $V_2 = \begin{pmatrix} -\frac{3}{\sqrt{12}} \\ 1 \end{pmatrix}$

d) For the matrix $\begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$

i. $\begin{vmatrix} -2-\lambda & -7 \\ 1 & 2-\lambda \end{vmatrix} = (-2-\lambda)(2-\lambda)+7$

$$= -4 + \lambda^2 + 7$$

$$= \lambda^2 + 3$$

The characteristic equation: $\boxed{\lambda^2 + 3 = 0}$

ii. $\lambda^2 = -3 \quad \rightarrow \quad$ *The eigenvalues* $\boxed{\lambda_{1,2} = \pm i\sqrt{3}}$

iii. For $\lambda_1 = -i\sqrt{3}$, we have: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -2+i\sqrt{3} & -7 \\ 1 & 2+i\sqrt{3} \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} \left(-2+i\sqrt{3}\right)x_1 - 7y_1 = 0 \\ x_1 + \left(2+i\sqrt{3}\right)y_1 = 0 \end{cases}$$

$$x_1 = -\left(2+i\sqrt{3}\right)y_1$$

Therefore the eigenvector $V_1 = \begin{pmatrix} 2+i\sqrt{3} \\ -1 \end{pmatrix}$

For $\lambda_2 = i\sqrt{3}$, we have: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -2-i\sqrt{3} & -7 \\ 1 & 2-i\sqrt{3} \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} \left(-2-i\sqrt{3}\right)x_2 - 7y_2 = 0 \\ x_2 + \left(2-i\sqrt{3}\right)y_2 = 0 \end{cases}$$

$$x_2 = -\left(2-i\sqrt{3}\right)y_2$$

Therefore the eigenvector $V_2 = \begin{pmatrix} 2-i\sqrt{3} \\ -1 \end{pmatrix}$

e) For the matrix: $\begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

i. $\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda)(1-\lambda) + 2(1-\lambda)$

$$= (1-\lambda)\left[(4-\lambda)(1-\lambda)+2\right]$$

$$= (1-\lambda)\left(\lambda^2 - 5\lambda + 6\right) \quad \Rightarrow \text{The characteristic equation: } \underline{-\lambda^3 + 6\lambda^2 - 11\lambda + 6}$$

ii. $-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0 \quad \Rightarrow \quad$ The eigenvalues are $\boxed{\lambda = 1,\ 2,\ 3}$

iii. $\lambda_1 = 1 \;\rightarrow\; \begin{pmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 3x_1 + x_3 = 0 \\ x_1 = 0 \end{cases} \Rightarrow x_1 = x_3 = 0$

Therefore the eigenvector $V_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 2 \;\rightarrow\; \begin{pmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \;\rightarrow \begin{cases} 2x_1 + x_3 = 0 \\ -2x_1 - x_2 = 0 \\ -2x_1 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = -2x_1 \\ x_2 = -2x_1 \end{cases}$

Therefore the eigenvector $V_2 = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$

$\lambda_3 = 3 \;\rightarrow\; \begin{pmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \;\rightarrow \begin{cases} x_1 + x_3 = 0 \\ -2x_1 - 2x_2 = 0 \\ -2x_1 - 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = -x_1 \\ x_2 = -x_1 \end{cases}$

Therefore the eigenvector $V_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

f) For the matrix: $\begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$

i. $\det(A-\lambda I)=\begin{vmatrix} 3-\lambda & 0 & -5 \\ \frac{1}{5} & -1-\lambda & 0 \\ 1 & 1 & -2-\lambda \end{vmatrix}$

$$
\begin{aligned}
&=(3-\lambda)(-1-\lambda)(-2-\lambda)-1+5(-1-\lambda)\\
&=(3-\lambda)\left(\lambda^2+3\lambda+2\right)-1-5-5\lambda\\
&=3\lambda^2+9\lambda+6-\lambda^3-3\lambda^2-2\lambda-6-5\lambda\\
&=-\lambda^3+2\lambda
\end{aligned}
$$

$\Rightarrow$ The characteristic equation: $-\lambda^3+2\lambda$

ii. $-\lambda^3+2\lambda=0 \quad \Rightarrow$ The eigenvalues are $\boxed{\lambda=0,\ \pm\sqrt{2}}$

iii. $\lambda_1=-\sqrt{2} \quad \rightarrow \begin{pmatrix} 3+\sqrt{2} & 0 & -5 \\ \frac{1}{5} & -1+\sqrt{2} & 0 \\ 1 & 1 & -2+\sqrt{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \left(3+\sqrt{2}\right)x_1-5x_3=0 \\ \frac{1}{5}x_1+\left(-1+\sqrt{2}\right)x_2=0 \\ x_1+x_2+\left(-2+\sqrt{2}\right)x_3=0 \end{cases}$

$\rightarrow \begin{cases} x_3=\dfrac{3+\sqrt{2}}{5}x_1 \\ \left(-1+\sqrt{2}\right)x_2=-\frac{1}{5}x_1 \end{cases} \quad \Rightarrow x_2=-\dfrac{1}{5\left(-1+\sqrt{2}\right)}x_1$

Therefore the eigenvector $V_1=\begin{pmatrix} 1 \\ \dfrac{1}{5\left(1-\sqrt{2}\right)} \\ \dfrac{3+\sqrt{2}}{5} \end{pmatrix}$

$\lambda_2=0 \quad \rightarrow \begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 3x_1-5x_3=0 \\ \frac{1}{5}x_1-x_2=0 \\ x_1+x_2-2x_3=0 \end{cases} \Rightarrow \begin{cases} x_3=\frac{3}{5}x_1 \\ x_2=\frac{1}{5}x_1 \end{cases}$

Therefore the eigenvector $V_2=\begin{pmatrix} 5 \\ \frac{1}{5} \\ \frac{3}{5} \end{pmatrix}$

$$\lambda_3 = \sqrt{2} \quad \to \quad \begin{pmatrix} 3-\sqrt{2} & 0 & -5 \\ \frac{1}{5} & -1-\sqrt{2} & 0 \\ 1 & 1 & -2-\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{cases} \left(3-\sqrt{2}\right)x_1 - 5x_3 = 0 \\ \frac{1}{5}x_1 + \left(-1-\sqrt{2}\right)x_2 = 0 \\ x_1 + x_2 + \left(-2-\sqrt{2}\right)x_3 = 0 \end{cases}$$

$$\to \begin{cases} x_3 = \dfrac{3-\sqrt{2}}{5}x_1 0 \\ \left(-1-\sqrt{2}\right)x_2 = -\frac{1}{5}x_1 \end{cases} \Rightarrow x_2 = \frac{1}{5\left(1+\sqrt{2}\right)}x_1$$

Therefore the eigenvector $V_3 = \begin{pmatrix} 1 \\ \dfrac{1}{5\left(1+\sqrt{2}\right)} \\ \dfrac{3-\sqrt{2}}{5} \end{pmatrix}$

g) For the matrix: $\begin{pmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{pmatrix}$

i. $\det\left(A-\lambda I\right) = \begin{vmatrix} -2-\lambda & 0 & 1 \\ -6 & -2-\lambda & 0 \\ 19 & 5 & -4-\lambda \end{vmatrix}$

$$= \left(-2-\lambda\right)^2\left(-4-\lambda\right) - 30 - 19\left(-2-\lambda\right)$$

$$= \left(4+4\lambda+\lambda^2\right)\left(-4-\lambda\right) - 30 + 38 + 19\lambda$$

$$= -16 - 16\lambda - 4\lambda^2 - 4\lambda - 4\lambda^2 - \lambda^3 + 8 + 19\lambda$$

$$= -\lambda^3 - 8\lambda^2 - \lambda - 8$$

$\Rightarrow$ The characteristic equation: $\underline{-\lambda^3 - 8\lambda^2 - \lambda - 8}$

ii. $\lambda^3 + 8\lambda^2 + \lambda + 8 = 0 \quad \Rightarrow \quad (\lambda+8)\left(\lambda^2+1\right) = 0$

$\lambda^3 + 8\lambda^2 + \lambda + 8 = 0 \quad \Rightarrow$ The eigenvalues are $\boxed{\lambda_{1,2,3} = -8, \ \pm i}$

iii. $\lambda_1 = -8 \quad \to \quad \begin{pmatrix} 6 & 0 & 1 \\ -6 & 6 & 0 \\ 19 & 5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 6 & 0 & 1 \\ -6 & 6 & 0 \\ 19 & 5 & 4 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & \frac{1}{6} \\ 0 & 1 & \frac{1}{6} \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 + \frac{1}{6}z_1 = 0 \\ y_1 + \frac{1}{6}z_1 = 0 \end{cases}$$

Therefore the eigenvector $V_1 = \begin{pmatrix} -\frac{1}{6} \\ -\frac{1}{6} \\ 1 \end{pmatrix}$

For $\lambda_2 = -i \quad \rightarrow \quad \begin{pmatrix} -2+i & 0 & 1 \\ -6 & -2+i & 0 \\ 19 & 5 & -4+i \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\rightarrow \begin{cases} (-2+i)x_2 + z_2 = 0 \\ -6x_2 + (-2+i)y_2 = 0 \\ 19x_2 + 5y_2 + (-4+i)z_2 = 0 \end{cases}$$

Therefore the eigenvector $V_2 = \begin{pmatrix} 1 \\ -\frac{12}{5} - i\frac{6}{5} \\ 2-i \end{pmatrix}$

For $\lambda_3 = i \quad \rightarrow \quad \begin{pmatrix} -2-i & 0 & 1 \\ -6 & -2-i & 0 \\ 19 & 5 & -4-i \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{cases} (-2-i)x_2 + z_2 = 0 \\ -6x_2 + (-2-i)y_2 = 0 \\ 19x_2 + 5y_2 + (-4-i)z_2 = 0 \end{cases}$$

Therefore the eigenvector $V_2 = \begin{pmatrix} 1 \\ -\frac{12}{5} + i\frac{6}{5} \\ 2+i \end{pmatrix}$

h) For the matrix: $\begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

i. $\det(A-\lambda I)=\begin{pmatrix}-\lambda & 0 & 2 & 0\\ 1 & -\lambda & 1 & 0\\ 0 & 1 & -2-\lambda & 0\\ 0 & 0 & 0 & 1-\lambda\end{pmatrix}=(1-\lambda)\begin{vmatrix}-\lambda & 0 & 2\\ 1 & -\lambda & 1\\ 0 & 1 & -2-\lambda\end{vmatrix}$

$$=(1-\lambda)\left(\lambda^2(-2-\lambda)+2+\lambda\right)$$

$$=(1-\lambda)\left(-\lambda^3-2\lambda^2+\lambda+2\right)$$

$$=\lambda^4+\lambda^3-3\lambda^2-\lambda+2$$

$\Rightarrow$ The characteristic equation: $\lambda^4+\lambda^3-3\lambda^2-\lambda+2$

ii. $\lambda^4+\lambda^3-3\lambda^2-\lambda+2=0 \quad\Rightarrow\quad$ The eigenvalues are $\boxed{\lambda=-2,-1,1,1}$

iii. $\lambda_1=-2 \quad\to\quad \begin{pmatrix}2 & 0 & 2 & 0\\ 1 & 2 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \to \begin{cases}2x_1+2x_3=0\\ x_1+2x_2+x_3=0\\ x_2=0\\ x_4=0\end{cases}$

$$\to\begin{cases}x_1=-x_3\\ x_1=-x_3\\ x_2=0\\ x_4=0\end{cases}$$

Therefore the eigenvector $V_1=\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}$

$\lambda_2=-1 \quad\to\quad \begin{pmatrix}1 & 0 & 2 & 0\\ 1 & 1 & 1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \to \begin{cases}x_1+2x_3=0\\ x_1+x_2+x_3=0\\ x_2-x_3=0\\ x_4=0\end{cases}$

$$\to\begin{cases}x_1=-2x_3\\ x_1=-x_2-x_3\\ x_2=x_3\end{cases}$$

Therefore the eigenvector $V_2=\begin{pmatrix}-2\\1\\1\\0\end{pmatrix}$

$$\lambda_3 = 1 \quad \rightarrow \quad \begin{pmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -x_1 + 2x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_2 - 3x_3 = 0 \end{cases}$$

$$\rightarrow \begin{cases} x_1 = 2x_3 \\ x_1 = x_2 - x_3 \\ x_2 = 3x_3 \end{cases}$$

Therefore the eigenvector $V_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_4 = 1 \quad \rightarrow$ Therefore the eigenvector $V_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

i) For the matrix: $\begin{pmatrix} 10 & -9 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

i. $\det(A - \lambda I) = \begin{pmatrix} 10-\lambda & -9 & 0 & 0 \\ 4 & -2-\lambda & 0 & 0 \\ 0 & 0 & -2-\lambda & -7 \\ 0 & 0 & 1 & 2-\lambda \end{pmatrix}$

$$= (10-\lambda)\begin{vmatrix} -2-\lambda & 0 & 0 \\ 0 & -2-\lambda & -7 \\ 0 & 1 & 2-\lambda \end{vmatrix} + 9\begin{vmatrix} 4 & 0 & 0 \\ 0 & -2-\lambda & -7 \\ 0 & 1 & 2-\lambda \end{vmatrix}$$

$$= (10-\lambda)\left[(-2-\lambda)^2(2-\lambda) + 7(-2-\lambda)\right] + 9\left[(4)(-2-\lambda)(2-\lambda) + 28\right]$$

$$= (10-\lambda)(-2-\lambda)\left(3+\lambda^2\right) + 9\left(4\lambda^2 + 12\right)$$

$$= \left(3+\lambda^2\right)\left(-8\lambda + \lambda^2 + 16\right)$$

$$= \left(3+\lambda^2\right)(\lambda - 4)^2$$

$\Rightarrow$ The characteristic equation: $\left(3+\lambda^2\right)(\lambda-4)^2$

ii. $\left(3+\lambda^2\right)(\lambda-4)^2 = 0 \quad \Rightarrow$ The eigenvalues are $\boxed{\lambda = 4,\ 4,\ \pm i\sqrt{3}}$

iii. $\lambda_1 = 4 \rightarrow \begin{pmatrix} 6 & -9 & 0 & 0 \\ 4 & -6 & 0 & 0 \\ 0 & 0 & -6 & -7 \\ 0 & 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 6x_1 - 9x_2 = 0 \\ 4x_1 - 6x_2 = 0 \\ -6x_3 - 7x_4 = 0 \\ x_3 - 2x_4 = 0 \end{cases}$

$$\begin{cases} 6x_1 = 9x_2 \\ 6x_3 = -7x_4 \\ x_3 = 2x_4 \end{cases} \Rightarrow \begin{matrix} x_1 = \frac{3}{2}x_2 \\ x_3 = x_4 = 0 \end{matrix}$$

Therefore the eigenvector $V_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda_2 = 4 \rightarrow$ Therefore the eigenvector $V_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\lambda_3 = -i\sqrt{3} \rightarrow \begin{pmatrix} 10+i\sqrt{3} & -9 & 0 & 0 \\ 4 & -2+i\sqrt{3} & 0 & 0 \\ 0 & 0 & -2+i\sqrt{3} & -7 \\ 0 & 0 & 1 & 2+i\sqrt{3} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} \left(10+i\sqrt{3}\right)x_1 - 9x_2 = 0 \\ 4x_1 + \left(-2+i\sqrt{3}\right)x_2 = 0 \\ \left(-2+i\sqrt{3}\right)x_3 - 7x_4 = 0 \\ x_3 + \left(2+i\sqrt{3}\right)x_4 = 0 \end{cases}$$

$$\rightarrow \begin{cases} \left(10+i\sqrt{3}\right)x_1 = 9x_2 \\ 4x_1 = -\left(-2+i\sqrt{3}\right)x_2 \\ \left(-2+i\sqrt{3}\right)x_3 = 7x_4 \\ x_3 = -\left(2+i\sqrt{3}\right)x_4 \end{cases} \Rightarrow \begin{cases} x_1 = x_2 = 0 \\ x_3 = \dfrac{7}{-2+i\sqrt{3}}x_4\left(\dfrac{-2-i\sqrt{3}}{-2-i\sqrt{3}}\right) = -\left(2+i\sqrt{3}\right)x_4 \\ x_3 = -\left(2+i\sqrt{3}\right)x_4 \end{cases}$$

Therefore the eigenvector $V_3 = \begin{pmatrix} 0 \\ 0 \\ -\left(2+i\sqrt{3}\right) \\ 1 \end{pmatrix}$

$$\lambda_4 = i\sqrt{3} \rightarrow \begin{pmatrix} 10-i\sqrt{3} & -9 & 0 & 0 \\ 4 & -2-i\sqrt{3} & 0 & 0 \\ 0 & 0 & -2-i\sqrt{3} & -7 \\ 0 & 0 & 1 & 2-i\sqrt{3} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} \left(10-i\sqrt{3}\right)x_1 - 9x_2 = 0 \\ 4x_1 + \left(-2-i\sqrt{3}\right)x_2 = 0 \\ \left(-2-i\sqrt{3}\right)x_3 - 7x_4 = 0 \\ x_3 + \left(2-i\sqrt{3}\right)x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = x_2 = 0 \\ x_3 = \dfrac{7}{-2-i\sqrt{3}}x_4\left(\dfrac{-2+i\sqrt{3}}{-2+i\sqrt{3}}\right) = \left(-2+i\sqrt{3}\right)x_4 \\ x_3 = -\left(2-i\sqrt{3}\right)x_4 \end{cases}$$

Therefore the eigenvector $V_4 = \begin{pmatrix} 0 \\ 0 \\ -2+i\sqrt{3} \\ 1 \end{pmatrix}$

j) For the matrix $\begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$

i. $\begin{vmatrix} -1-\lambda & 0 & 1 \\ -1 & 3-\lambda & 0 \\ -4 & 13 & -1-\lambda \end{vmatrix} = (-1-\lambda)^2(3-\lambda) - 13 + 4(3-\lambda)$

$$= \left(\lambda^2 + 2\lambda + 1\right)(3-\lambda) - 13 + 12 - 4\lambda$$

$$= 3\lambda^2 + 6\lambda + 3 - \lambda^3 - 2\lambda^2 - \lambda - 1 - 4\lambda$$

$$= -\lambda^3 + \lambda^2 + \lambda + 2$$

The characteristic equation: $\underline{-\lambda^3 + \lambda^2 + \lambda + 2 = 0}$

ii. $\rightarrow$ *The eigenvalues* $\underline{\lambda_{1,2,3} = 2,\ -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}}$

iii. For $\lambda_1 = 2$, we have: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -3 & 0 & 1 \\ -1 & 1 & 0 \\ -4 & 13 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -3x_1 + z_1 = 0 \\ -x_1 + y_1 = 0 \\ -4x_1 + 13y_1 - 3z_1 = 0 \end{cases} \Rightarrow \begin{cases} z_1 = 3x_1 \\ y_1 = x_1 \end{cases}$$

If we let $x_1 = 1$; therefore the eigenvector $V_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

For $\lambda_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, we have: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -\frac{1}{2} + i\frac{\sqrt{3}}{2} & 0 & 1 \\ -1 & \frac{7}{2} + i\frac{\sqrt{3}}{2} & 0 \\ -4 & 13 & -\frac{1}{2} + i\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)x_2 + z_2 = 0 \\ -x_2 + \left(\frac{7}{2} + i\frac{\sqrt{3}}{2}\right)y_2 = 0 \\ -4x_2 + 13y_2 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)z_2 = 0 \end{cases}$$

$$\rightarrow \begin{cases} z_2 = -\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)x_2 \\ y_2 = \left(\frac{2}{7 + i\sqrt{3}}\right)x_2 \end{cases}$$

If we let $x_2 = 1$; therefore the eigenvector $V_2 = \begin{pmatrix} 1 \\ \frac{1 - i\sqrt{3}}{2} \\ \frac{2}{7 + i\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1 - i\sqrt{3}}{2} \\ \frac{7 - i\sqrt{3}}{26} \end{pmatrix}$

For $\lambda_3 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, we have: $\left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -\frac{1}{2} - i\frac{\sqrt{3}}{2} & 0 & 1 \\ -1 & \frac{7}{2} - i\frac{\sqrt{3}}{2} & 0 \\ -4 & 13 & -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)x_3 + z_3 = 0 \\ -x_3 + \left(\frac{7}{2} - i\frac{\sqrt{3}}{2}\right)y_3 = 0 \\ -4x_3 + 13y_3 + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)z_3 = 0 \end{cases}$$

$$\rightarrow \begin{cases} z_3 = -\left(\frac{1 + i\sqrt{3}}{2}\right)x_3 \\ y_3 = \left(\frac{2}{7 - i\sqrt{3}}\right)x_3 \end{cases}$$

If we let $x_3 = 1$; therefore the eigenvector $V_3 = \begin{pmatrix} 1 \\ \frac{1 + i\sqrt{3}}{2} \\ \frac{2}{7 - i\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1 + i\sqrt{3}}{2} \\ \frac{7 + i\sqrt{3}}{26} \end{pmatrix}$

Exercise

Find the eigenvalues of A^9 for $A = \begin{pmatrix} 1 & 3 & 7 & 11 \\ 0 & \frac{1}{2} & 3 & 8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

Solution

The eigenvalues are: $\lambda = 1,\ \frac{1}{2},\ 0,\ 2$

The eigenvalues of A^9 are: $1^9 \ \underline{=1|}$ $\left(\frac{1}{2}\right)^9 \ \underline{=\frac{1}{512}|}$ $0^9 \ \underline{=0|}$ $2^9 \ \underline{=512|}$

Exercise

Find the eigenvalues of the matrices

$$A = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0.61 & 0.52 \\ 0.39 & 0.48 \end{pmatrix}, \quad A^\infty = \begin{pmatrix} 0.57143 & 0.57143 \\ 0.42857 & 0.42857 \end{pmatrix}, \quad \textit{and} \quad B = \begin{pmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{pmatrix}$$

Solution

The eigenvalues for A:

$$\begin{vmatrix} 0.7-\lambda & 0.4 \\ 0.3 & 0.6-\lambda \end{vmatrix} = (0.7-\lambda)(0.6-\lambda) - .12$$

$$= \lambda^2 - 1.3\lambda + .3 = 0 \qquad \lambda_{1,2} = 0.65 \pm 0.35$$

The eigenvalues are: $\lambda_1 = 1 \quad \lambda_2 = 0.3$

The eigenvalues for A^2: $\lambda_1 = 1^2 = 1 \quad \lambda_2 = 0.3^2 = 0.09$

The eigenvalues for A^∞: $\lambda^2 - \lambda = 0 \quad \lambda_1 = 1 \quad \lambda_2 = 0.3^\infty = 0$

The eigenvalues for B:

$$\begin{vmatrix} 0.3-\lambda & 0.6 \\ 0.7 & 0.4-\lambda \end{vmatrix} = \lambda^2 - .7\lambda - .3 = 0 \qquad \lambda_{1,2} = 0.35 \pm 0.65$$

The eigenvalues are: $\lambda_1 = 1 \quad \lambda_2 = -0.3$

Exercise

Given the matrix $\begin{bmatrix} -1 & -3 \\ -3 & 7 \end{bmatrix}$

a) Find the characteristic polynomial.

b) Find the eigenvalues

c) Find the bases for its eigenspaces

d) Graph the eigenspaces

e) Verify directly that $A\mathbf{v} = \lambda\mathbf{v}$, for all associated eigenvectors and eigenvalues.

Solution

a) $$\begin{vmatrix} -1-\lambda & -3 \\ -3 & 7-\lambda \end{vmatrix} = (-1-\lambda)(7-\lambda)-9$$
$$= -7-6\lambda+\lambda^2-9$$
$$= \lambda^2-6\lambda-16$$

The characteristic polynomial is $\underline{\lambda^2-6\lambda-16=0}$

b) $\lambda^2-6\lambda-16=0 \quad \Rightarrow \quad \lambda_1=-2 \quad \textit{and} \quad \lambda_2=8$

c) For $\lambda_1=-2$, we have: $(A+2I)V_1=0$

$$\begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} x_1-3y_1=0 \\ -3x_1+9y_1=0 \end{cases} \quad \Rightarrow x_1=3y_1$$

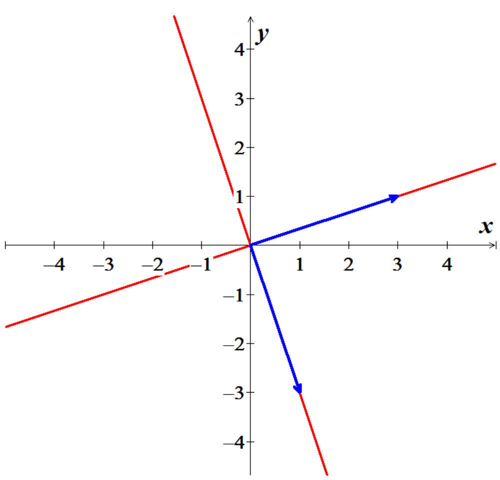
Therefore the eigenvector $V_1=\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

For $\lambda_2=8$, we have: $(A+8I)V_2=0$

$$\begin{pmatrix} -9 & -3 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} -9x_2-3y_2=0 \\ -3x_2-y_2=0 \end{cases} \quad \Rightarrow y_2=-3x_2$$

Therefore the eigenvector $V_2=\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

d)



e) $AV_1 = \lambda V_1 \quad \to \quad \begin{pmatrix} -1 & -3 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = -2\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\begin{pmatrix} -6 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \end{pmatrix}$ √

$AV_2 = \lambda V_2 \quad \to \quad \begin{pmatrix} -1 & -3 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = 8\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$\begin{pmatrix} 8 \\ -24 \end{pmatrix} = \begin{pmatrix} -6 \\ -24 \end{pmatrix}$ √

Exercise

Given the matrix $\begin{bmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{bmatrix}$

a) Find the characteristic polynomial.
b) Find the eigenvalues
c) Find the bases for its eigenspaces
d) Graph the eigenspaces
e) Verify directly that $A\mathbf{v} = \lambda\mathbf{v}$, for all associated eigenvectors and eigenvalues.

Solution

a) $\begin{vmatrix} 5-\lambda & 0 & -4 \\ 0 & -3-\lambda & 0 \\ -4 & 0 & -1-\lambda \end{vmatrix} = (5-\lambda)(-3-\lambda)(-1-\lambda) - 16(-3-\lambda)$

$$= (5-\lambda)\left(3+4\lambda+\lambda^2\right) + 48 + 16\lambda$$
$$= 15 + 20\lambda + 5\lambda^2 - 3\lambda - 4\lambda^2 - \lambda^3 + 48 + 16\lambda$$
$$= -\lambda^3 + \lambda^2 + 33\lambda + 63$$

The characteristic polynomial is $\underline{-\lambda^3 + \lambda^2 + 33\lambda + 63 = 0}$

b) $-\lambda^3 + \lambda^2 + 33\lambda + 63 = 0 \quad \Rightarrow \quad \lambda = -3, \ -3, \ 7$

c) For $\lambda_{1,2} = -3$, we have: $(A + 3I)V_1 = 0$

$$\begin{pmatrix} 8 & 0 & -4 \\ 0 & 0 & 0 \\ -4 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} 8x_1 - 4z_1 = 0 \\ \\ -4x_1 + 2z_1 = 0 \end{cases} \quad \Rightarrow z_1 = 2x_1$$

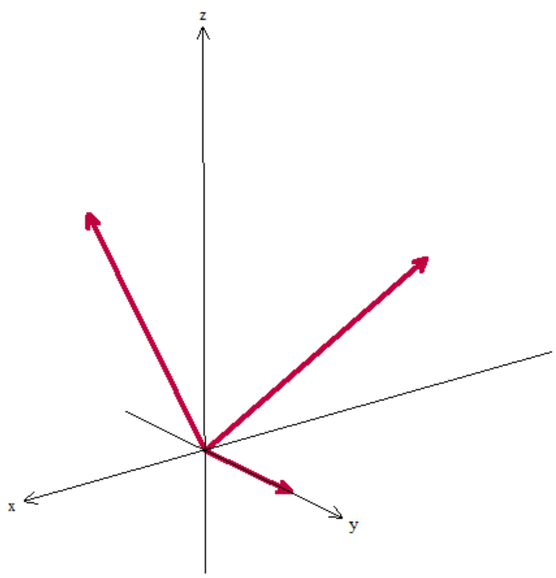
Therefore the eigenvector $V_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ *and* $V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda_3 = 7$, we have: $(A - 7I)V_3 = 0$

$$\begin{pmatrix} -2 & 0 & -4 \\ 0 & -10 & 0 \\ -4 & 0 & -8 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -2x_1 - 4z_1 = 0 \\ -10y_1 = 0 \\ -4x_1 - 8z_1 = 0 \end{cases} \Rightarrow x_1 = -2z_1 \text{ and } y_1 = 0$$

Therefore the eigenvector $V_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

d)



e) $AV_1 = \lambda V_1 \rightarrow \begin{pmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} -3 \\ 0 \\ -6 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ -6 \end{pmatrix} \checkmark$$

$AV_2 = \lambda V_2 \rightarrow \begin{pmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} \checkmark$$

$$AV_3 = \lambda V_3 \quad \rightarrow \quad \begin{pmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = 7 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -14 \\ 0 \\ 7 \end{pmatrix} = \begin{pmatrix} -14 \\ 0 \\ 7 \end{pmatrix} \checkmark$$

Exercise

Given: $A = \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}$. Compute A^{11}

Solution

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 7 & -1 \\ 0 & 1-\lambda & 0 \\ 0 & 15 & -2-\lambda \end{vmatrix}$$

$$= (-1-\lambda)(1-\lambda)(-2-\lambda)$$

The eigenvalues are: $-1,\ 1,\ -2$

For $\lambda_1 = -1$, we have: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & 7 & -1 \\ 0 & 2 & 0 \\ 0 & 15 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 7y_1 - z_1 = 0 \\ 2y_1 = 0 \\ 15y_1 - z_1 = 0 \end{cases} \quad \Rightarrow \begin{cases} z_1 = 7y_1 \\ y_1 = 0 \\ \end{cases}$$

The eigenvector $V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

For $\lambda_2 = 1$, we have: $(A - I)V_2 = 0$

$$\begin{pmatrix} -2 & 7 & -1 \\ 0 & 0 & 0 \\ 0 & 15 & -3 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} -2x_2 + 7y_2 - z_2 = 0 \\ \\ 15y_2 - 3z_2 = 0 \end{cases} \quad \Rightarrow \begin{cases} 2x_2 = 7y_2 - z_2 \\ \\ 5y_2 = z_2 \end{cases}$$

If we let $y_2 = 1 \ \rightarrow z_2 = 5$ and $x_2 = \frac{7-5}{2} = 1$;

The eigenvector $V_2 = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$

For $\lambda_3 = -2$, we have: $(A+2I)V_3 = 0$

$$\begin{pmatrix} 1 & 7 & -1 \\ 0 & 3 & 0 \\ 0 & 15 & 0 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} x_3 + 7y_3 - z_3 = 0 \\ 3y_3 = 0 \\ 15y_3 = 0 \end{cases} \quad \Rightarrow \begin{cases} x_3 = -7y_3 + z_3 \\ y_3 = 0 \end{cases}$$

The eigenvector $V_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \Rightarrow D^{11} = \begin{pmatrix} (-1)^{11} & 0 & 0 \\ 0 & 1^{11} & 0 \\ 0 & 0 & (-2)^{11} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2048 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix} \Rightarrow P^{-1} = \begin{pmatrix} 1 & 4 & -1 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

$$A^{11} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2048 \end{pmatrix}\begin{pmatrix} 1 & 4 & -1 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & -2048 \\ 0 & 1 & 0 \\ 0 & 5 & -2048 \end{pmatrix}\begin{pmatrix} 1 & 4 & -1 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 10237 & 2047 \\ 0 & 1 & 0 \\ 0 & 10245 & -2048 \end{pmatrix}$$