Solution   *Section* 1.8 – Set Operations

Exercise

Let A be the set of students who live within one mile of school and let B be the set of students who walk to classes. Describe the students in each of these sets

a) $A \cap B$
b) $A \cup B$
c) $A - B$
d) $B - A$

Solution

a) The set of students who live one mile of school and walk to classes.

b) The set of students who live one mile of school or walk to classes.

c) The set of students who live one mile of school but not walk to class.

d) The set of students who live more than one mile from school but nevertheless walk to class.

Exercise

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$

a) $A \cup B$
b) $A \cap B$
c) $A - B$
d) $B - A$

Solution

a) $\{0, 1, 2, 3, 4, 5, 6\}$

b) $\{3\}$

c) $(1, 2, 4, 5\}$

d) $\{0, 6\}$

Exercise

Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$

a) $A \cup B$
b) $A \cap B$
c) $A - B$
d) $B - A$

Solution

a) $\{a, b, c, d, e, f, g, h\} = B$

b) $\{a, b, c, d, e\} = A$

c) $\varnothing$, since there are no elements in A that are not in B.

d) $\{f, g, h\}$

Exercise

Prove the domination laws by showing that

a) $A \bigcup U = U$

b) $A \bigcap U = A$

c) $A \bigcup \varnothing = A$

d) $A \bigcap \varnothing = \varnothing$

Solution

a)
$$\begin{aligned} A \bigcup U &= \{x \mid x \in A \vee x \in U\} \\ &= \{x \mid x \in A \vee \boldsymbol{T}\} \\ &= \{x \mid \boldsymbol{T}\} \\ &= U \end{aligned}$$

b)
$$\begin{aligned} A \bigcap U &= \{x \mid x \in A \wedge x \in U\} \\ &= \{x \mid x \in A \wedge \boldsymbol{T}\} \\ &= \{x \mid x \in A\} \\ &= A \end{aligned}$$

c)
$$\begin{aligned} A \bigcup \varnothing &= \{x \mid x \in A \vee x \in \varnothing\} \\ &= \{x \mid x \in A \vee \boldsymbol{F}\} \\ &= \{x \mid x \in A\} \\ &= A \end{aligned}$$

d)
$$\begin{aligned} A \bigcap \varnothing &= \{x \mid x \in A \wedge x \in \varnothing\} \\ &= \{x \mid x \in A \wedge \boldsymbol{F}\} \\ &= \{x \mid \boldsymbol{F}\} \\ &= \varnothing \end{aligned}$$

Exercise

Prove the complement laws by showing that

a) $A \bigcup \overline{A} = U$

b) $A \bigcap \overline{A} = \varnothing$

Solution

a) $A\bigcup\overline{A}=\left\{x\middle|x\in A\vee x\in\overline{A}\right\}$

$$=\left\{x\middle|x\in A\vee x\notin A\right\}$$

$$=\left\{x\middle|\boldsymbol{T}\right\}$$

$$=U$$

b) $A\bigcap\overline{A}=\left\{x\middle|x\in A\wedge x\in\overline{A}\right\}$

$$=\left\{x\middle|x\in A\wedge x\notin A\right\}$$

$$=\left\{x\middle|\boldsymbol{F}\right\}$$

$$=\varnothing$$

Exercise

Show that

a) $A-\varnothing=A$

b) $\varnothing-A=\varnothing$

Solution

a) $A-\varnothing=\left\{x\middle|x\in A\wedge x\notin\varnothing\right\}$

$$=\left\{x\middle|x\in A\wedge\boldsymbol{T}\right\}$$

$$=\left\{x\middle|x\in A\right\}$$

$$=A$$

b) $\varnothing-A=\left\{x\middle|x\in\varnothing\wedge x\notin A\right\}$

$$=\left\{x\middle|\boldsymbol{F}\wedge x\notin A\right\}$$

$$=\left\{x\middle|\boldsymbol{F}\right\}$$

$$=\varnothing$$

Exercise

Prove the absorption law by showing that if *A* and *B* are sets, then

a) $A\bigcap\left(A\bigcup B\right)=A$

b) $A\bigcup\left(A\bigcap B\right)=A$

Solution

a) Suppose $x\in A\bigcap\left(A\bigcup B\right)$, then $x\in A$ and $x\in A\bigcup B$ by the definition of intersection. We have $x\in A$ and in the latter case $x\in A\ \ or\ x\in B$ by the definition of union. Since both of these are true, $x\in A\bigcup B$ by the definition of intersection, and we have shown that the right-hand side is a subset of the left-hand side.

b) Suppose $x \in A \bigcup (A \bigcap B) \quad \Rightarrow \quad x \in A$ *or* $x \in (A \bigcap B)$ by definition of union.

$x \in A$ *or* $(x \in A$ *and* $x \in B)$

By the definition of the intersection, in any event, $x \in A$. Therefore, $x \in A \bigcup (A \bigcap B)$ as well.

That proves that the right-hand side is a subset of the left-hand side.

Exercise

Show that if *A*, *B*, and *C* are sets, then $\overline{A \bigcap B \bigcap C} = \overline{A} \bigcup \overline{B} \bigcup \overline{C}$

Solution

Suppose $x \in \overline{A \bigcap B \bigcap C}$, then $x \notin A \bigcap B \bigcap C$, which means that *x* fails to be in at least one of these three sets. In other words, $x \notin A$ *or* $x \notin B$ *or* $x \notin C$. This is equivalent to saying that $x \in \overline{A}$ *or* $x \in \overline{B}$ *or* $x \in \overline{C}$. Therefore $x \in \overline{A} \bigcup \overline{B} \bigcup \overline{C}$.

Conversely, if $x \in \overline{A} \bigcup \overline{B} \bigcup \overline{C}$, then $x \in \overline{A}$ *or* $x \in \overline{B}$ *or* $x \in \overline{C}$. This means $x \notin A$ *or* $x \notin B$ *or* $x \notin C$, so *x* cannot be in the intersection of *A, B*, and *C*. Since $x \notin A \bigcap B \bigcap C$, we conclude that $x \in \overline{A \bigcap B \bigcap C}$, as desired.

Or

A	B	C	$A \bigcap B \bigcap C$	$\overline{A \bigcap B \bigcap C}$	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{A} \bigcup \overline{B} \bigcup \overline{C}$
1	1	1	**1**	**0**	**0**	**0**	**0**	**0**
1	1	0	**0**	**1**	**0**	**0**	**1**	**1**
1	0	1	**0**	**1**	**0**	**1**	**0**	**1**
1	0	0	**0**	**1**	**0**	**1**	**1**	**1**
0	1	1	**0**	**1**	**1**	**0**	**0**	**1**
0	1	0	**0**	**1**	**1**	**0**	**1**	**1**
0	0	1	**0**	**1**	**1**	**1**	**0**	**1**
0	0	0	**0**	**1**	**1**	**1**	**1**	**1**

Exercise

Let A and B be sets. Show that

a) $(A \cap B) \subseteq A$

b) $A \subseteq (A \cup B)$

c) $(A - B) \subseteq A$

d) $A \cap (B - A) = \varnothing$

e) $A \cup (B - A) = A \cup B$

Solution

a) If x is in $A \cap B$, then, by definition of intersection, it is in A.

b) If x is in A, then perforce, by definition of union, it is in $A \cup B$.

c) If x is in $A - B$, then perforce, by definition of difference, it is in A.

d) Is $x \in A$ then $x \notin B - A$. Therefore there can be no elements in $A \cap (B - A)$, so $A \cap (B - A) = \varnothing$.

e) The left-hand side consists of elements of either A or B or both. This is precisely the definition of the right-hand side.

Exercise

Draw the Venn diagrams for each of these combinations of the sets A, B, and C.

a) $A \cap (B - C)$

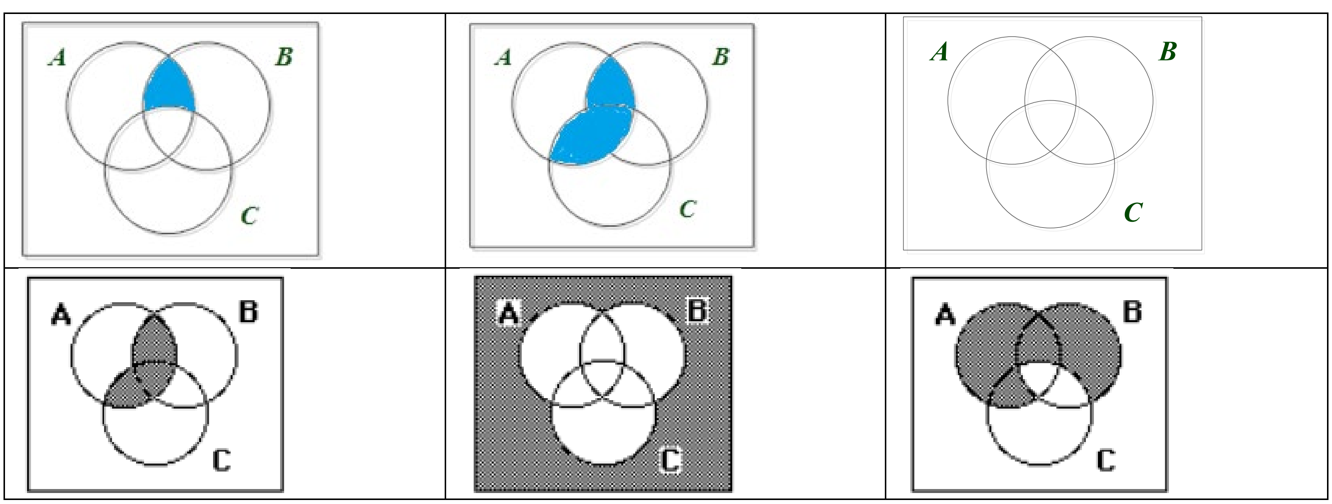
b) $(A \cap B) \cup (A \cap C)$

c) $(A \cap \overline{B}) \cup (A \cap \overline{C})$

d) $\overline{A} \cap \overline{B} \cap \overline{C}$

e) $(A - B) \cup (A - C) \cup (B - C)$

Solution



Exercise

Show that $A \oplus B = (A \cup B) - (A \cap B)$

Solution

This is just a restatement of the definition. An element is in $(A \cup B) - (A \cap B)$ if it in the union that in either A or B, but not in the intersection (i.e., not in both A and B)

Exercise

Show that $A \oplus B = (A - B) \cup (B - A)$

Solution

There are two ways that an item can be in either A or B but not both. It can be in A but not B (which is equivalent to saying that it is in $A - B$), or it can be in B but not A (which is equivalent to saying that it is in $B - A$).

Thus an element is in $A \oplus B$ if and only if it is in $(A - B) \cup (B - A)$.