

Section 4.5 – Inverse Trigonometry Functions

Relationships Between f^{-1} and f

- $y = f^{-1}(x)$ if and only if $x = f(y)$, where x is in the domain of f^{-1} and y is in the domain of f
- Domain of f^{-1} = Range of f
- Range of f^{-1} = Domain of f
- $f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}
- $f^{-1}(f(y)) = y$ for every y in the domain of f
- The point (a, b) is on the graph of f *iff* the point (b, a) is on the graph of f^{-1} .
- The graphs of f^{-1} and f are reflections of each other through the line $y = x$.

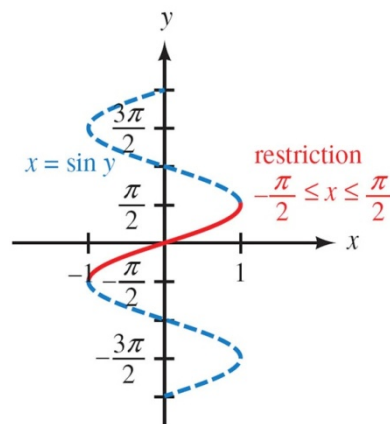
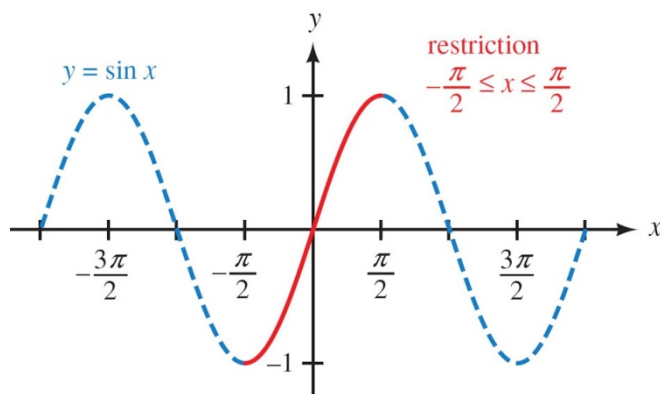
The Inverse *Sine* Function

$$y = \sin^{-1} x \quad \text{or} \quad y = \arcsin x \quad \text{iff} \quad x = \sin y \quad \text{for} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \quad \text{and} \quad -1 \leq x \leq 1$$

Properties of \sin^{-1}

$$\sin(\sin^{-1} x) = \sin(\arcsin x) = x \quad \text{if} \quad -1 \leq x \leq 1$$

$$\sin^{-1}(\sin y) = \arcsin(\sin y) = y \quad \text{if} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$



Example

Find the exact value: $\sin\left(\sin^{-1}\frac{1}{2}\right)$, $\sin^{-1}\left(\sin\frac{\pi}{4}\right)$

Solution

$$\sin\left(\sin^{-1}\frac{1}{2}\right) = \frac{1}{2} \quad \text{Since } -1 \leq \frac{1}{2} \leq 1$$

$$\sin^{-1}\left(\sin\frac{\pi}{4}\right) = \frac{\pi}{4} \quad \text{Since } -\frac{\pi}{2} \leq \frac{\pi}{4} \leq \frac{\pi}{2}$$

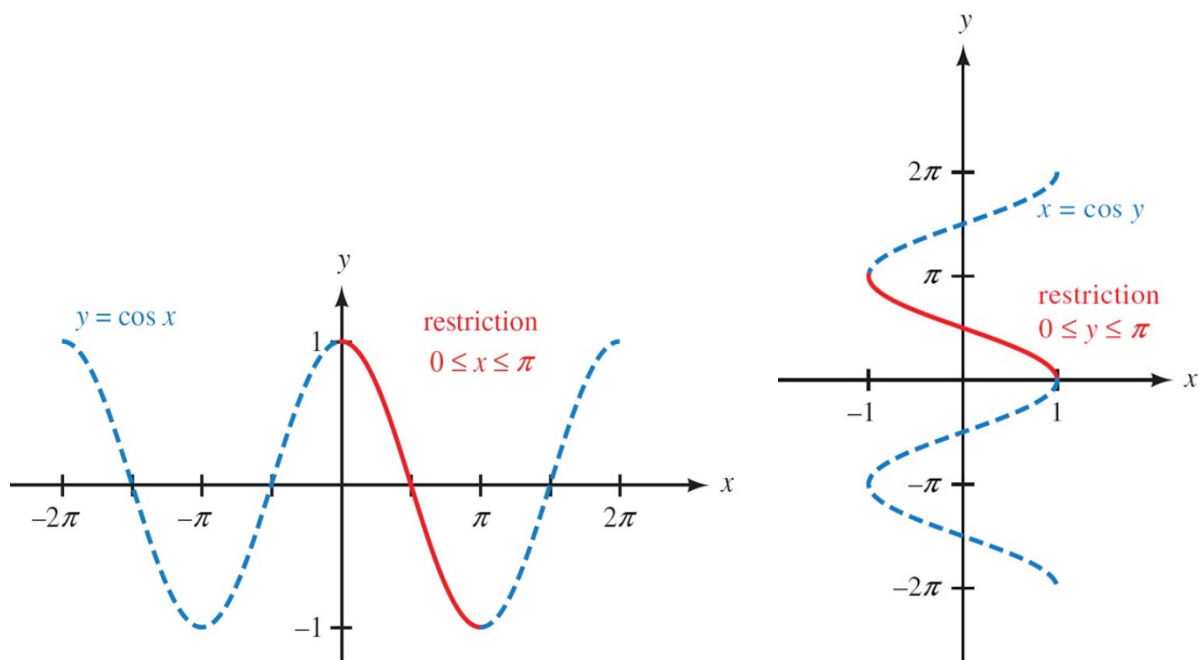
The Inverse *Cosine* Function

Definition

The inverse cosine function, denoted by \cos^{-1} , is defined by

$$y = \cos^{-1} x \text{ iff } x = \cos y \text{ for } 0 \leq y \leq \pi \text{ and } -1 \leq x \leq 1$$

Notation	Meaning
$y = \cos^{-1} x$ or $y = \arccos x$	$x = \cos y$ and $0 \leq y \leq \pi$



Properties of \cos^{-1}

$$\cos\left(\cos^{-1} x\right) = \cos(\arccos x) = x \quad \text{if } -1 \leq x \leq 1$$

$$\cos^{-1}(\cos y) = \arccos(\cos y) = y \quad \text{if } 0 \leq y \leq \pi$$

Example

Find the exact value: $\cos(\cos^{-1}(-0.5))$, $\cos^{-1}(\cos(3.14))$, $\cos^{-1}(\sin(-\frac{\pi}{6}))$

Solution

$$\cos(\cos^{-1}(-0.5)) = -0.5 \quad \text{Since } -1 \leq -0.5 \leq 1$$

$$\cos^{-1}(\cos(3.14)) = 3.14 \quad \text{Since } 0 \leq 3.14 \leq \pi$$

$$\cos^{-1}(\sin(-\frac{\pi}{6})) = \cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$$

Example

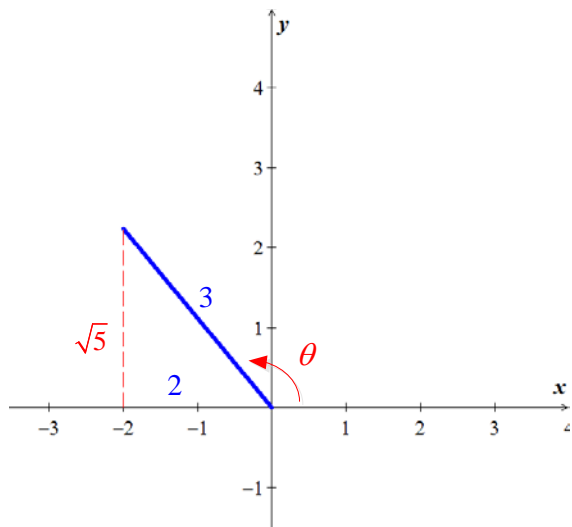
Find the exact value of $\sin\left[\arccos\left(-\frac{2}{3}\right)\right]$

Solution

$$\theta = \arccos\left(-\frac{2}{3}\right) \Rightarrow \cos \theta = -\frac{2}{3} \quad 0 \leq \theta \leq \pi$$

$$y = \sqrt{3^2 - 2^2} = \sqrt{5}$$

$$\sin\left[\arccos\left(-\frac{2}{3}\right)\right] = \sin \theta = \frac{\sqrt{5}}{3}$$



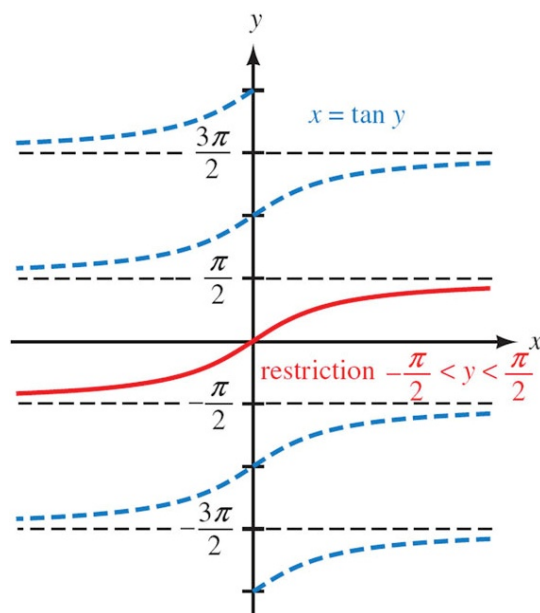
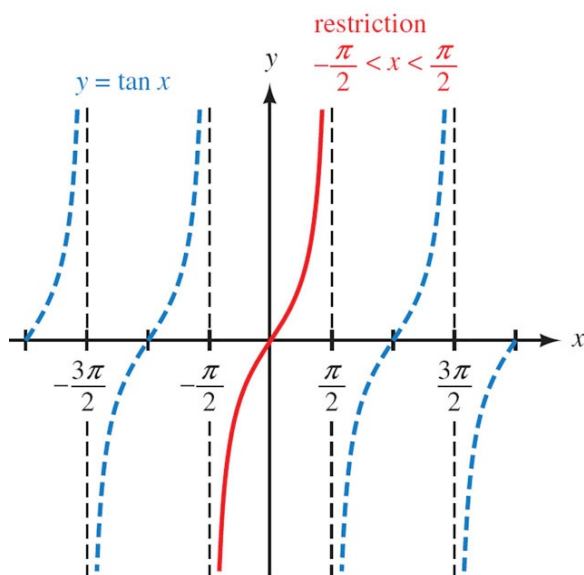
The Inverse *Tangent* Function

Definition

The inverse cosine function, denoted by \tan^{-1} , is defined by

$$y = \tan^{-1} x \quad \text{iff} \quad x = \tan y \quad \text{for any real number } x \text{ and for } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

$$y = \tan^{-1} x \quad \text{or} \quad y = \arctan x$$



Properties of \tan^{-1}

$$\tan(\tan^{-1} x) = \tan(\arctan x) = x \quad \text{for every } x$$

$$\tan^{-1}(\tan y) = \arctan(\tan y) = y \quad \text{if } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

Example

Find the exact value: $\tan(\tan^{-1}(1000))$, $\tan^{-1}(\tan \frac{\pi}{4})$, $\arctan(\tan \pi)$

Solution

$$\tan(\tan^{-1} 1000) = 1000$$

$$\tan^{-1}(\tan \frac{\pi}{4}) = \frac{\pi}{4} \quad \text{Since } -\frac{\pi}{2} \leq \frac{\pi}{4} \leq \frac{\pi}{2}$$

$$\arctan(\tan \pi) = \arctan(0) = 0 \quad \because \pi > \frac{\pi}{2}$$

Example

Evaluate in radians without using a calculator or tables.

a. $\sin^{-1} \frac{1}{2}$

$$-\frac{\pi}{2} \leq \text{angle} \leq \frac{\pi}{2} \Rightarrow \sin \frac{\pi}{6} = \frac{1}{2}$$
$$\sin^{-1} \frac{1}{2} = \frac{\pi}{6}$$

b. $\arccos\left(-\frac{\sqrt{3}}{2}\right)$

$$0 < \text{angle} < \pi \Rightarrow \cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$$
$$\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$$

c. $\tan^{-1}(-1)$

$$-\frac{\pi}{2} < \text{angle} < \frac{\pi}{2} \Rightarrow \tan\left(-\frac{\pi}{4}\right) = -1$$
$$\tan^{-1}(-1) = -\frac{\pi}{4}$$

Example

Use a calculator to evaluate each expression to the nearest tenth of a degree

a. $\arcsin(0.5075)$

$$\arcsin(0.5075) = 30.5^\circ$$

b. $\arcsin(-0.5075)$

$$\arcsin(-0.5075) = -30.5^\circ$$

c. $\cos^{-1}(0.6428)$

$$\cos^{-1}(0.6428) = 50.0^\circ$$

d. $\cos^{-1}(-0.6428)$

$$\cos^{-1}(-0.6428) = 130.0^\circ$$

e. $\arctan(4.474)$

$$\arctan(4.474) = 77.4^\circ$$

f. $\arctan(-4.474)$

$$\arctan(-4.474) = -77.4^\circ$$

Example

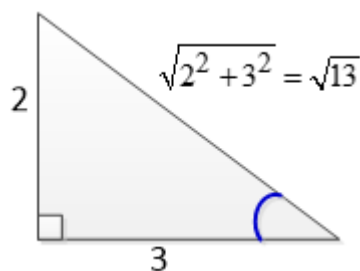
Find the exact value: $\sec\left(\arctan\frac{2}{3}\right)$

Solution

$$\alpha = \arctan\frac{2}{3} \rightarrow \tan \alpha = \frac{2}{3}$$

$$\sec\left(\arctan\frac{2}{3}\right) = \sec \alpha$$

$$= \frac{\sqrt{13}}{3}$$



Example

Find the exact value: $\sin\left(\arctan\frac{1}{2} - \arccos\frac{4}{5}\right)$

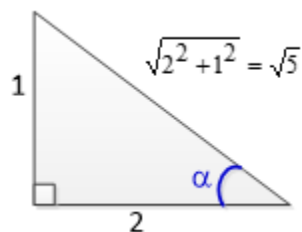
Solution

$$\alpha = \arctan\frac{1}{2} \quad \beta = \arccos\frac{4}{5}$$

$$\tan \alpha = \frac{1}{2} \quad \cos \beta = \frac{4}{5}$$

$$\sin \alpha = \frac{1}{\sqrt{5}} \quad \sin \beta = \frac{3}{5}$$

$$\cos \alpha = \frac{2}{\sqrt{5}}$$



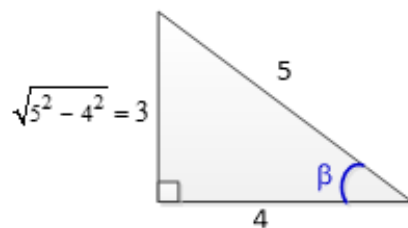
$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$= \frac{1}{\sqrt{5}} \cdot \frac{4}{5} - \frac{2}{\sqrt{5}} \cdot \frac{3}{5}$$

$$= \frac{4}{5\sqrt{5}} - \frac{6}{5\sqrt{5}}$$

$$= -\frac{2}{5\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$$

$$= -\frac{2\sqrt{5}}{25}$$



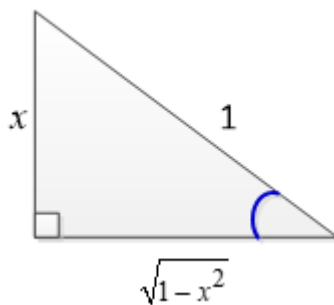
Example

If $-1 \leq x \leq 1$, rewrite $\cos(\sin^{-1} x)$ as an algebraic expression in x .

Solution

$$\alpha = \sin^{-1} x \rightarrow \sin \alpha = x = \frac{x}{1}$$

$$\begin{aligned}\cos(\sin^{-1} x) &= \cos \alpha \\ &= \frac{\sqrt{1-x^2}}{1} \\ &= \sqrt{1-x^2}\end{aligned}$$



Exercises **Section 4.5 – Inverse Trigonometric Functions**

(1 – 18) Find the exact value of the expression whenever it is defined

- | | | |
|---|--|---|
| 1. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ | 7. $\cos^{-1}\left[\cos\left(\frac{5\pi}{6}\right)\right]$ | 13. $\cos\left[\arctan\left(-\frac{3}{4}\right) - \arcsin\frac{4}{5}\right]$ |
| 2. $\arccos\left(\frac{\sqrt{2}}{2}\right)$ | 8. $\tan^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right]$ | 14. $\tan\left[\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)\right]$ |
| 3. $\arctan\left(-\frac{\sqrt{3}}{3}\right)$ | 9. $\arcsin\left[\sin\left(-\frac{\pi}{2}\right)\right]$ | 15. $\sin\left[2\arccos\left(-\frac{3}{5}\right)\right]$ |
| 4. $\sin\left[\arcsin\left(-\frac{3}{10}\right)\right]$ | 10. $\arccos[\cos(0)]$ | 16. $\cos\left[2\sin^{-1}\left(\frac{15}{17}\right)\right]$ |
| 5. $\tan[\arctan(14)]$ | 11. $\arctan\left[\tan\left(-\frac{\pi}{4}\right)\right]$ | 17. $\tan\left[2\tan^{-1}\left(\frac{3}{4}\right)\right]$ |
| 6. $\sin\left[\sin^{-1}\left(\frac{2}{3}\right)\right]$ | 12. $\sin\left[\arcsin\left(\frac{1}{2}\right) + \arccos 0\right]$ | 18. $\cos\left[\frac{1}{2}\tan^{-1}\left(\frac{8}{15}\right)\right]$ |

(19 – 28) Evaluate without using a calculator

- | | | |
|--|---|--|
| 19. $\cos\left(\cos^{-1}\frac{3}{5}\right)$ | 23. $\cos\left(\sin^{-1}\frac{1}{2}\right)$ | 26. $\tan\left(\sin^{-1}\frac{3}{5}\right)$ |
| 20. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ | 24. $\sin\left(\sin^{-1}\frac{3}{5}\right)$ | 27. $\sec\left(\cos^{-1}\frac{1}{\sqrt{5}}\right)$ |
| 21. $\tan\left(\cos^{-1}\frac{3}{5}\right)$ | 25. $\cos\left(\tan^{-1}\frac{3}{4}\right)$ | 28. $\cot\left(\tan^{-1}\frac{1}{2}\right)$ |
| 22. $\sin\left(\cos^{-1}\frac{1}{\sqrt{5}}\right)$ | | |

(29 – 41) Write an equivalent expression that involves x only for

- | | |
|--|---|
| 29. $\cos(\cos^{-1}x)$ | 34. $\cot\left(\sin^{-1}\frac{\sqrt{x^2-9}}{x}\right) \quad x > 0$ |
| 30. $\tan(\cos^{-1}x)$ | 35. $\sin(2\sin^{-1}x) \quad x > 0$ |
| 31. $\csc\left(\sin^{-1}\frac{1}{x}\right)$ | 36. $\cos(2\tan^{-1}x), \quad x > 0$ |
| 32. $\sin(\tan^{-1}x); \quad x > 0$ | 37. $\cos\left(\frac{1}{2}\arccos x\right), \quad x > 0$ |
| 33. $\sec\left(\sin^{-1}\frac{x}{\sqrt{x^2+4}}\right) \quad x > 0$ | 38. $\tan\left(\frac{1}{2}\cos^{-1}\frac{1}{x}\right), \quad x > 0$ |

$$39. \sec\left(\tan^{-1}\frac{2}{\sqrt{x^2-4}}\right) \quad x > 0$$

$$41. \sin\left(\cos^{-1}\frac{x}{\sqrt{x^2+4}}\right) \quad x > 0$$

$$40. \sec\left(\sin^{-1}\frac{\sqrt{x^2-25}}{x}\right) \quad x > 0$$

(42 – 44) Sketch the graph of the equation:

$$42. \quad y = \sin^{-1} 2x$$

$$43. \quad y = \sin^{-1}(x-2) + \frac{\pi}{2}$$

$$44. \quad y = \cos^{-1}\frac{1}{2}x$$

$$45. \text{ Evaluate } \sin\left(\tan^{-1}\frac{3}{4}\right) \text{ without using a calculator}$$

$$46. \text{ Evaluate } \sin(\cos^{-1} x) \text{ as an equivalent expression in } x \text{ only}$$