Section 2.4 – Chain Rule

Functions of Two Variables

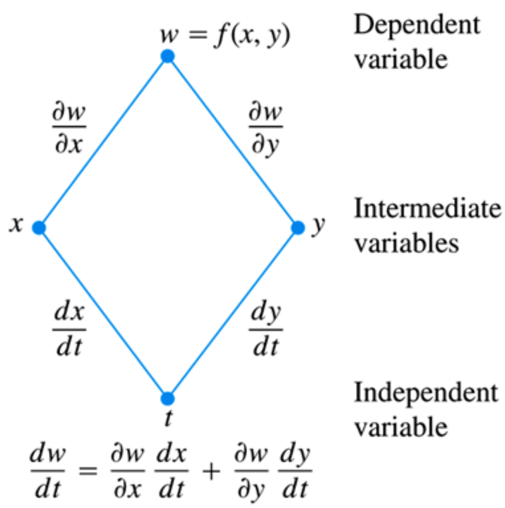
Theorem – Chain Rule for Functions of Two Independent Variables

If $w = f(x, y)$ is differentiable and if $x = x(t),\ y = y(t)$ are differentiable functions of *t*, then the composite $w = f(x(t), y(t))$ is a differentiable function of *t* and

$$\frac{dw}{dt} = f_x(x(t), y(t)) \bullet x'(t) + f_y(x(t), y(t)) \bullet y'(t)$$

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Chain Rule



Example

Use the Chain Rule to find the derivative of $w = xy$ with respect to *t* along the path $x = \cos t,\quad y = \sin t$.

What is the derivative's value at $t = \frac{\pi}{2}$?

Solution

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\
&= \frac{\partial(xy)}{\partial x}\frac{d}{dt}(\cos t) + \frac{\partial(xy)}{\partial y}\frac{d}{dt}(\sin t) \\
&= y(-\sin t) + x(\cos t) \\
&= (\sin t)(-\sin t) + (\cos t)(\cos t) \\
&= -\sin^2 t + \cos^2 t \\
&= \cos 2t
\end{aligned}$$

$w = xy$

$$= \cos t \sin t$$

$$= \frac{1}{2}\sin 2t$$

$$\frac{dw}{dt} = \frac{1}{2}(2\cos 2t)$$

$$\underline{= \cos 2t}$$

$$\left.\frac{dw}{dt}\right|_{t=\pi/2} = \cos 2\left(\frac{\pi}{2}\right)$$

$$= \cos(\pi)$$

$$\underline{= -1}$$

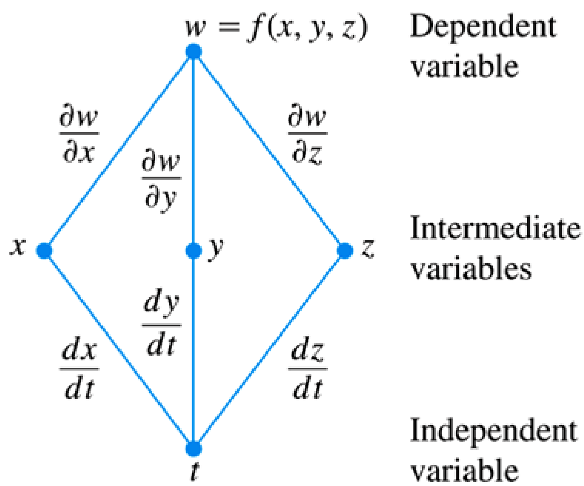
Functions of Three Variables

Theorem – Chain Rule for Functions of Three Independent Variables

If $w = f(x, y, z)$ is differentiable and if $x,\ y,\ and\ z$ are differentiable functions of *t*, then w is a differentiable function of *t* and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Chain Rule



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Example

Find $\frac{dw}{dt}$ if $w = xy + z, \quad x = \cos t, \quad y = \sin t, \quad z = t$

In this example the values of $w(t)$ are changing along the path of a helix as *t* changes. What is the derivative's value at *t* = 0?

Solution

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= (y)(-\sin t) + (x)(\cos t) + (1)(1)$$

$$= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1$$

$$= -\sin^2 t + \cos^2 t + 1$$

$$= \cos 2t + 1$$

$$\left.\frac{dw}{dt}\right|_{t=0} = \cos(0) + 1 = 2$$

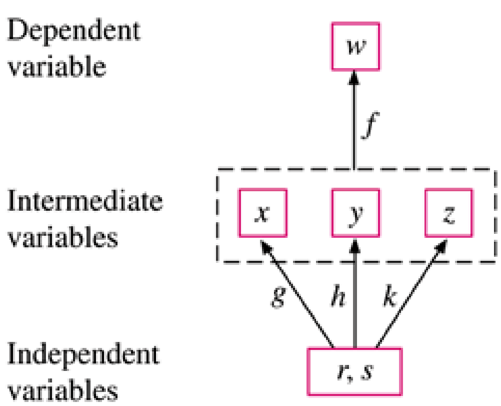
Functions Defined on Surfaces

Theorem – Chain Rule for Two Independent Variables and Three Intermediate Variables

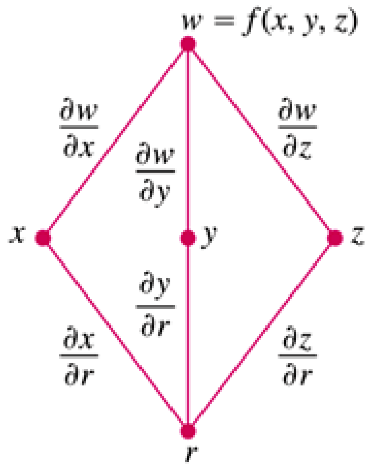
Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then *w* has partial derivatives with respect to *r* and *s*, given by the formulas

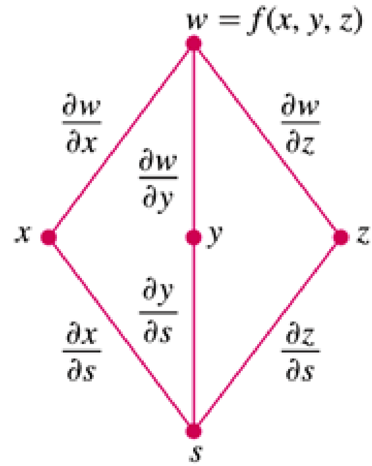
$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{dr} + \frac{\partial w}{\partial y}\frac{\partial y}{dr} + \frac{\partial w}{\partial z}\frac{\partial z}{dr}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$



$w = f(g(r, s), h(r, s), k(r, s))$





$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \qquad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Example

Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if $\quad w = x + 2y + z^2, \quad x = \frac{r}{s}, \quad y = r^2 + \ln s, \quad z = 2r$

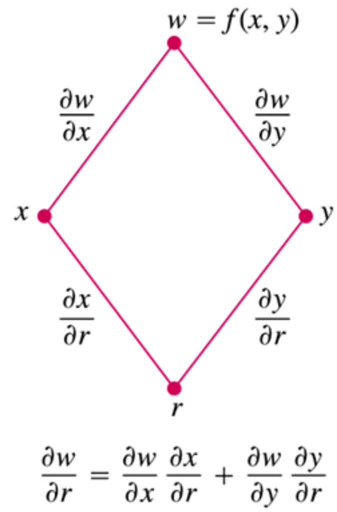
Solution

$$\begin{aligned}
\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{dr} + \frac{\partial w}{\partial y}\frac{\partial y}{dr} + \frac{\partial w}{\partial z}\frac{\partial z}{dr} \\
&= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2) \\
&= \frac{1}{s} + 4r + (4r)(2) \\
&\boxed{= \frac{1}{s} + 12r}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\
&= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) \\
&\boxed{= -\frac{r}{s^2} + \frac{2}{s}}
\end{aligned}$$

- If $w = f(x, y)$, $x = g(r, s)$ and $y = h(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{dr} + \frac{\partial w}{\partial y}\frac{\partial y}{dr} \quad \textit{and} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$



Example

Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if $w = x^2 + y^2, \quad x = r - s, \quad y = r + s$

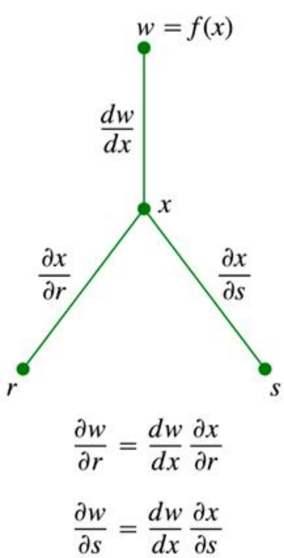
Solution

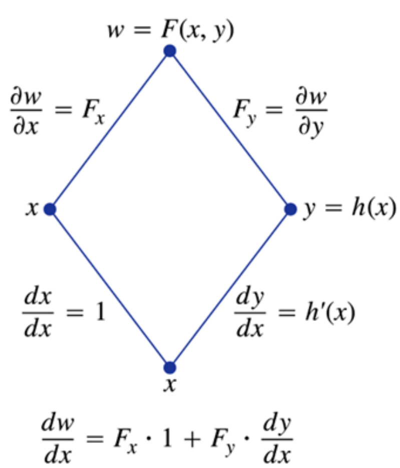
$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{dr} + \frac{\partial w}{\partial y}\frac{\partial y}{dr}\\ &= (2x)(1) + (2y)(1)\\ &= 2(r - s) + 2(r + s)\\ &= 2r - 2s + 2r + 2s\\ &= 4r\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}\\ &= (2x)(-1) + (2y)(1)\\ &= -2(r - s) + 2(r + s)\\ &= -2r + 2s + 2r + 2s\\ &= 4s\end{aligned}$$

- If $w = f(x)$, $x = g(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{dr} \quad \textit{and} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s}$$





Implicit Differentiation Revisited

Theorem – A Formula for Implicit Differentiation

Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y)=0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx}=-\frac{F_x}{F_y}$$

$$\frac{dz}{dx}=-\frac{F_x}{F_z} \qquad \frac{dz}{dy}=-\frac{F_y}{F_z}$$

Example

Find $\frac{dy}{dx}$ if $y^2-x^2-\sin xy=0$

Solution

$$F(x, y)=y^2-x^2-\sin xy$$

$$\begin{aligned}\frac{dy}{dx}&=-\frac{F_x}{F_y}\\&=-\frac{-2x-y\cos xy}{2y-x\cos xy}\\&=\boxed{\frac{2x+y\cos xy}{2y-x\cos xy}}\end{aligned}$$

Example

Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ at (0, 0, 0) if $x^3+z^2+ye^{xz}+z\cos y=0$

Solution

$$F(x, y, z)=x^3+z^2+ye^{xz}+z\cos y$$

$$F_x=3x^2+yze^{xz}, \quad F_y=e^{xz}-z\sin y, \quad \textit{and} \quad F_z=2z+xye^{xz}+\cos y$$

$$F(0,\ 0,\ 0)=0 \quad F_z=1\neq 0$$

$$\frac{dz}{dx}=-\frac{F_x}{F_z}=-\frac{3x^2+yze^{xz}}{2z+xye^{xz}+\cos y} \qquad \frac{dz}{dy}=-\frac{F_y}{F_z}=-\frac{e^{xz}-z\sin y}{2z+xye^{xz}+\cos y}$$

$$\left.\frac{dz}{dx}\right|_{(0,0,0)}=-\frac{0}{1}\ \boxed{=0} \qquad \frac{dz}{dy}=-\frac{F_y}{F_z}=-\frac{1}{1}\ \boxed{=-1}$$

Exercises — *Section* 2.4 – Chain Rule

Express $\frac{dw}{dt}$ as a function of t, then evaluate $\frac{dw}{dt}$ at the given value of t.

1. $w = x^2 + y^2, \quad x = \cos t, \quad y = \sin t, \quad t = \pi$

2. $w = x^2 + y^2, \quad x = \cos t + \sin t, \quad y = \cos t - \sin t, \quad t = 0$

3. $w = \ln\left(x^2 + y^2 + z^2\right), \quad x = \cos t, \quad y = \sin t, \quad z = 4\sqrt{t}, \quad t = 3$

4. $w = z - \sin xy, \quad x = t, \quad y = \ln t, \quad z = e^{t-1}, \quad t = 1$

5. $w = \sin(xy + \pi), \quad x = e^t, \quad y = \ln(t+1) \quad t = 0$

6. $w = xe^y + y\sin z - \cos z, \quad x = 2\sqrt{t}, \quad y = t - 1 + \ln t, \quad z = \pi t, \quad t = 1$

7. Express $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ as functions of u and v if $z = 4e^x \ln y, \quad x = \ln(u\cos v), \quad y = u\sin v$, then evaluate $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ at the point $(u,v) = \left(2, \frac{\pi}{4}\right)$.

8. Express $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ as functions of u and v if $w = xy + yz + xz, \quad x = u + v, \quad y = u - v, \quad z = uv$, then evaluate $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ at the point $(u,v) = \left(\frac{1}{2}, 1\right)$.

9. Express $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ as functions of x, y and z if

$u = e^{qr}\sin^{-1} p, \quad p = \sin x, \quad q = z^2 \ln y, \quad r = \frac{1}{z}$, then evaluate $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ at the point

$(x,y,z) = \left(\frac{\pi}{4}, \frac{1}{2}, -\frac{1}{2}\right)$.

10. Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z^3 - xy + yz + y^3 - 2 = 0$ at the point $(1, 1, 1)$

11. Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $\sin(x+y) + \sin(y+z) + \sin(x+z) = 0$ at the point (π, π, π)

12. Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0$ at the point $(1, \ln 2, \ln 3)$

13. Find $\frac{\partial w}{\partial r}$ when $r = 1, s = -1$ if $w = (x+y+z)^2, \quad x = r - s, \quad y = \cos(r+s), \quad z = \sin(r+s)$

14. Find $\frac{\partial z}{\partial u}$ when $u = 0, v = 1$ if $z = \sin xy + x\sin y, \quad x = u^2 + v^2, \quad y = uv$

15. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ when $u = \ln 2, v = 1$ if $z = 5\tan^{-1} x, \quad x = e^u + \ln v$

16. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ when $u=1,\ v=-2$ if $z=\ln q, \quad q=\sqrt{v+3}\ \tan^{-1}u$

17. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ when $r=\pi$ and $s=0$ if $w=\sin(2x-y), \quad x=r+\sin s, \quad y=rs$

18. Assume that $w=f\left(s^3+t^2\right)$ and $f'(x)=e^x$. Find $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$

Evaluate the derivatives

19. $w'(t)$, where $w=xy\sin z, \quad x=t^2, \quad y=4t^3, \quad and \quad z=t+1$

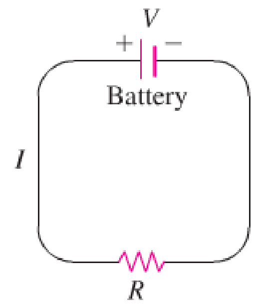
20. $w'(t)$, where $w=\sqrt{x^2+y^2+z^2}, \quad x=\sin t, \quad y=\cos t, \quad and \quad z=\cos t$

21. w_s and w_t , where $w=xyz, \quad x=2st, \quad y=st^2, \quad and \quad z=s^2t$

22. w_r, w_s, and w_t , where $w=\ln\left(xy^2\right), \quad x=rst, \quad and \quad y=r+s$

23. The voltage V in a circuit that satisfies the law $V=IR$ is slowly dropping as the battery wears out. At the same time, the resistance R is increasing as the resistor heats up. Use the equation

$$\frac{dV}{dt}=\frac{\partial V}{\partial I}\frac{dI}{dt}+\frac{\partial V}{\partial R}\frac{dR}{dt}$$



To find how the current is changing at the instant when $R=600\ \Omega$, $I=0.04A$, $\frac{dR}{dt}=0.5\ ohm/\sec$, and $\frac{dV}{dt}=-0.01\ volt/\sec$

24. The lengths *a*, *b*, and *c* of the edges of a rectangular box are changing with time. At the instant in question, $a=1\ m, \quad b=2\ m, \quad c=3\ m$, $\frac{da}{dt}=\frac{db}{dt}=1\ m/\sec$, and $\frac{dc}{dt}=-3\ m/\sec$. At what rates the box's volume V and surface area S changing at that instant? Are the box's interior diagonals increasing in length or decreasing?

25. Let $T=f(x,y)$ be the temperature at the point (x,y) on the circle $x=\cos t, \quad y=\sin t, \ 0\le t\le 2\pi$ and suppose that

$$\frac{\partial T}{\partial x}=8x-4y, \quad \frac{\partial T}{\partial y}=8y-4x$$

a) Find where the maximum and minimum temperatures on the circle occur by examining the derivatives $\frac{dT}{dt}$ and $\frac{d^2T}{dt^2}$.

b) Suppose that $T = 4x^2 - 4xy + 4y^2$. Find the maximum and minimum values of T on the circle.

Evaluate $\frac{dy}{dx}$

26. $x^2 - 2y^2 - 1 = 0$

27. $x^3 + 3xy^2 - y^5 = 0$

28. $2\sin xy = 1$

29. $ye^{xy} - 2 = 0$

30. $\sqrt{x^2 + 2xy + y^4} = 3$

31. $y\ln\left(x^2 + y^2 + 4\right) = 3$

32. $y\ln\left(x^2 + y^2\right) = 4$

33. $2x^2 + 3xy - 3y^4 = 2$

Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ at the given point.

34. $z^3 - xy + yz + y^3 - 2 = 0; \quad (1,\ 1,\ 1)$

35. $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0; \quad (2,\ 3,\ 6)$

36. $\sin(x+y) + \sin(y+z) + \sin(x+z) = 0; \quad (\pi,\ \pi,\ \pi)$

37. $xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0; \quad (1,\ \ln 2,\ \ln 3)$

38. Consider the surface and parameterized curves C in the xy-plane

$$z = 4x^2 + y^2 - 2; \quad C: \ x = \cos t, \quad y = \sin t, \quad for \ \ 0 \le t \le 2\pi$$

a) Find $z'(t)$ on C.

b) Imagine that you are walking on the surface directly above C consistent with the positive orientation of C. Find the values of $\boldsymbol{t}$ for which you are walking uphill.

39. Consider the surface and parameterized curves C in the xy-plane

$$z = 4x^2 - 2y^2 + 4; \quad C: \ x = 2\cos t, \quad y = 2\sin t, \quad for \ \ 0 \le t \le 2\pi$$

a) Find $z'(t)$ on C.

b) Imagine that you are walking on the surface directly above C consistent with the positive orientation of C. Find the values of $\boldsymbol{t}$ for which you are walking uphill.

40. Find the value of the derivative of $f(x,y,z) = xy + yz + xz$ with respect to t on the curve $x = \cos t, \quad y = \sin t, \quad z = \cos 2t$ at $t = 1$

41. Define y as a differentiable function of x for $2xy + e^{x+y} - 2 = 0$, find the values of $\frac{dy}{dx}$ at point $P(0,\ \ln 2)$