Section 4.4 – Properties of Logarithms

Product Rule

$$\log_b MN = \log_b M + \log_b N \qquad \textbf{\textit{For M > 0 and N > 0}}$$

$$\begin{cases} \log_b M = x \;\Rightarrow\; M = b^x \\ \log_b N = y \;\Rightarrow\; N = b^y \end{cases} \Rightarrow \text{MN} = b^x\, b^y = b^{x+y}$$

Convert back to logarithmic form: $\log_b MN = x + y$

$$\log_b MN = \log_b M + \log_b N$$

Example

Use the product rule to expand the logarithmic expression

$$\log(100x) = \log 100 + \log x$$

Power Rule

$$\log_b M^{\,p} = p \log_b M$$

Example

Use the power rule to expand each logarithmic expression

$$\ln \sqrt[3]{x} = \ln(x)^{1/3} = \frac{1}{3}\ln x$$

Quotient Rule

$$\log_b \frac{M}{N} = \log_b M - \log_b N$$

Example

Use the quotient rule to expand the logarithmic expression

$$\ln\left(\frac{e^5}{11}\right) = \ln e^5 - \ln 11 = 5 - \ln 11$$

Express each of the following in terms of sums and differences of logarithms

a) $\log_6(7.9)$

$$\log_6(7.9) = \log_6 7 + \log_6 9 \qquad \textit{Product Rule}$$

b) $\log_9\left(\frac{15}{7}\right)$

$$\log_9\left(\frac{15}{7}\right) = \log_9 15 - \log_9 7 \qquad \textit{Quotient Rule}$$

c) $\log_5\sqrt{8}$

$$\log_5\sqrt{8} = \log_5 8^{1/2}$$

$$= \frac{1}{2}\log_5 8 \qquad \textit{Power Rule}$$

d) $\log_b\left(x^4\sqrt[3]{y}\right) = \log_b\left(x^4\right) + \log_b\left(\sqrt[3]{y}\right)$ *Product Rule*

$$= \log_b\left(x^4\right) + \log_b\left(y^{1/3}\right) \qquad \textit{Power Rule}$$

$$= 4\log_b(x) + \frac{1}{3}\log_b(y)$$

e) $\log_a\left(\frac{mnq}{p^2r^4}\right) = \log_a(mnq) - \log_a\left(p^2r^4\right)$ *Quotient Rule*

$$= \log_a m + \log_a n + \log_a q - \left(\log_a p^2 + \log_a r^4\right) \qquad \textit{Product Rule}$$

$$= \log_a m + \log_a n + \log_a q - \log_a p^2 - \log_a r^4$$

$$= \log_a m + \log_a n + \log_a q - 2\log_a p - 4\log_a r \qquad \textit{Power Rule}$$

f) $\log_5\left(\frac{\sqrt{x}}{25y^3}\right) = \log_5\left(x^{1/2}\right) - \log_5\left(25y^3\right)$ *Quotient Rule*

$$= \log_5\left(x^{1/2}\right) - \left[\log_5\left(5^2\right) + \log_5\left(y^3\right)\right] \qquad \textit{Product Rule}$$

$$= \log_5\left(x^{1/2}\right) - \log_5\left(5^2\right) - \log_5\left(y^3\right) \qquad \textit{log}_5\left(5^2\right) = 2$$

$$= \frac{1}{2}\log_5(x) - 2 - 3\log_5(y)$$

Example

Write as a single logarithmic

a) $\log(7x+6) - \log x = \log \dfrac{7x+6}{x}$

b) $\log_3 (x+2) + \log_3 x - \log_3 2 = \log_3 x(x+2) - \log_3 2$ *Product Rule*

$= \log_3 \dfrac{x(x+2)}{2}$ *Quotient Rule*

c) $2\ln x + \frac{1}{3}\ln(x+5) = \ln x^2 + \ln(x+5)^{1/3}$ *Power Rule*

$= \ln x^2 \ (x+5)^{1/3}$ *Product Rule*

$= \ln x^2 \sqrt[3]{x+5}$

d) $2\log(x-3) - \log x = \log(x-3)^2 - \log x$ *Power Rule*

$= \log \dfrac{(x-3)^2}{x}$ *Quotient Rule*

e) $\frac{1}{4}\log_b x - 2\log_b 5 - 10\log_b y = \log_b x^{1/4} - \log_b 5^2 - \log_b y^{10}$ *Power Rule*

$= \log_b x^{1/4} - \left[\log_b 5^2 + \log_b y^{10}\right]$ *Factor the minus*

$= \log_b x^{1/4} - \left[\log_b \left(5^2 y^{10}\right)\right]$ *Product Rule*

$= \log_b \dfrac{\sqrt[4]{x}}{5^2 y^{10}}$ *Quotient Rule*

Exercises *Section* 4.4 – Properties of Logarithms

1. Express as a sum of logarithms: $\log_3(ab)$

2. Express as a sum of logarithms: $\log_7(7x)$

Express the following in terms of sums and differences of logarithms

3. $\log \frac{x}{1000}$

4. $\log_5\left(\frac{125}{y}\right)$

5. $\log_b x^7$

6. $\ln\sqrt[7]{x}$

7. $\log_a \frac{x^2 y}{z^4}$

8. $\log_b \frac{x^2 y}{b^3}$

9. $\log_b\left(\frac{x^3 y}{z^2}\right)$

10. $\log_b\left(\frac{\sqrt[3]{x}y^4}{z^5}\right)$

11. $\log_a \sqrt[4]{\frac{m^8 n^{12}}{a^3 b^5}}$

12. $\log_p \sqrt[3]{\frac{m^5 n^4}{t^2}}$

13. $\log\left(\frac{100x^3\sqrt[3]{5-x}}{3(x+7)^2}\right)$

14. $\log_b \sqrt[n]{\frac{x^3 y^5}{z^m}}$

15. $\log_a \sqrt[3]{\frac{a^2 b}{c^5}}$

16. $\log_b\left(x^4\sqrt[3]{y}\right)$

17. $\log_5\left(\frac{\sqrt{x}}{25y^3}\right)$

18. $\log_a \frac{x^3 w}{y^2 z^4}$

19. $\log_a \frac{\sqrt{y}}{x^4\sqrt[3]{z}}$

20. $\ln\sqrt[4]{\frac{x^7}{y^5 z}}$

21. $\ln x\sqrt[3]{\frac{y^4}{z^5}}$

Write the expression as a single logarithm

22. $2\log_a x+\frac{1}{3}\log_a(x-2)-5\log_a(2x+3)$

23. $5\log_a x-\frac{1}{2}\log_a(3x-4)-3\log_a(5x+1)$

24. $\log\left(x^3y^2\right)-2\log\left(x\sqrt[3]{y}\right)-3\log\left(\frac{x}{y}\right)$

25. $\ln y^3+\frac{1}{3}\ln\left(x^3y^6\right)-5\ln y$

26. $2\ln x-4\ln\left(\frac{1}{y}\right)-3\ln(xy)$

27. $\frac{1}{2}\log_a x+4\log_a y-3\log_a x$

28. $\frac{2}{3}\left[\ln\left(x^2-9\right)-\ln(x+3)\right]+\ln(x+y)$

29. $4\ln x+7\ln y-3\ln z$

30. $\frac{1}{3}\left[5\ln(x+6)-\ln x-\ln(x^2-25)\right]$

31. $\frac{2}{3}\left[\ln\left(x^2-4\right)-\ln(x+2)\right]+\ln(x+y)$

32. $\frac{1}{2}\log_b m+\frac{3}{2}\log_b 2n-\log_b m^2 n$

33. $\frac{1}{2}\log_y p^3q^4-\frac{2}{3}\log_y p^4q^3$

34. $\frac{1}{4}\log_b x-2\log_b 5-10\log_b y$

35. Assume that $\log_{10} 2=.3010$. Find each logarithm $\log_{10} 4$, $\log_{10} 5$

36. Given that: $\log_a 2\approx 0.301$, $\log_a 7\approx 0.845$, and $\log_a 11\approx 1.041$ find each of the following:

$\log_a \frac{2}{11}$, $\log_a 14$, $\log_a 98$, $\log_a \frac{1}{7}$, $\log_a 9$