$$T(e_1) = \begin{bmatrix} \cos^2\theta \\ \sin\theta\cos\theta \end{bmatrix} \qquad T(e_2) = \begin{bmatrix} \sin\theta\cos\theta \\ \sin^2\theta \end{bmatrix}$$

$$P_0 = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

$$\theta = \frac{\pi}{6} \qquad \vec{x} = (1, 5)^{\rightarrow}$$

$$P_0 = \begin{bmatrix} \cos^2\frac{\pi}{6} & \sin\frac{\pi}{2}\cos\frac{\pi}{6} \\ \sin\frac{\pi}{6}\cos\frac{\pi}{6} & \sin^2\frac{\pi}{6} \end{bmatrix}$$

$$= \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}$$

$$P_0\vec{x} = \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3+5\sqrt{3}}{4} \\ \frac{\sqrt{3}+5}{4} \end{pmatrix}$$

Ex $\quad T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$T(\vec{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

Find image of $\vec{u} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$$T(\vec{u}) = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \checkmark$$

$$\underline{\vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}} \Rightarrow T(\vec{v}) = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \checkmark$$

$$\vec{u} + \vec{v} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

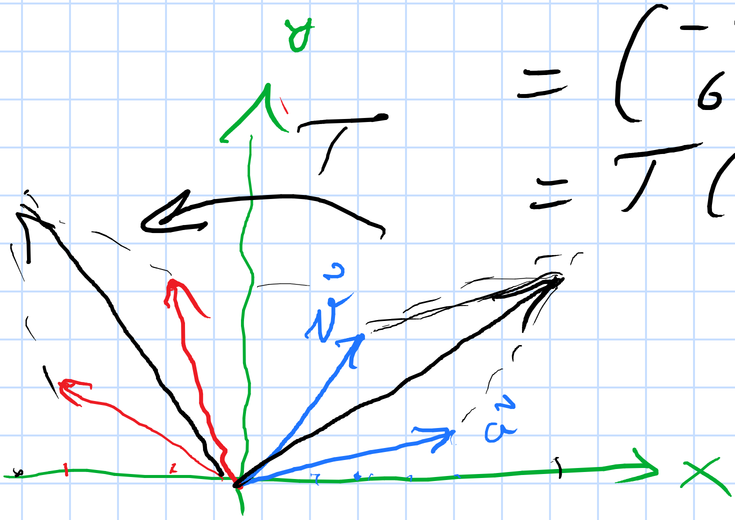
$$= \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$T(\vec{u} + \vec{v}) = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$$

$$T(\vec{u}) + T(\vec{v}) = \begin{pmatrix} -1 \\ 4 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ 6 \end{pmatrix}$$

$$= T(\vec{u} + \vec{v})$$



4 Subspaces:

1. Row space
2. Column "
3. nullspace
4. Left nullspace

$$\begin{bmatrix} 1 & 3 & 5 & 0 & 9 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{matrix} x_1 = -3x_2 - 5x_3 - 9x_5 \\ x_4 = -8x_5 \end{matrix}$$

rank = 2

dimension = 3 $\qquad (5-2) = 3$

row space: $\qquad$ rank = 2

free var: x_2, x_3, x_5

pivots variables: x_1, x_4

Nullspace $\begin{pmatrix} -3 & -5 & -9 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -8 \\ 0 & 0 & 1 \end{pmatrix}$

left nullspace

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 0 & 0 \\ 0 & 1 & 0 \\ 9 & 8 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$y_1 = 0$

$y_2 = 0$

$y_3 \ \forall$.

$(0, 0, y_3)$

$$\begin{cases} w_1 = 2x_1 - 3x_2 + 4x_4 \\ w_2 = 3x_1 + 5x_2 - x_4 \end{cases}$$

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & -3 & 0 & 4 \\ 3 & 5 & 0 & -1 \end{pmatrix}}_{\text{standard matrix}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$T(x_1, x_2) = (2x_1 - x_2,\ x_1 + x_2)$$

$$= \begin{pmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}}_{\text{std matrix}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

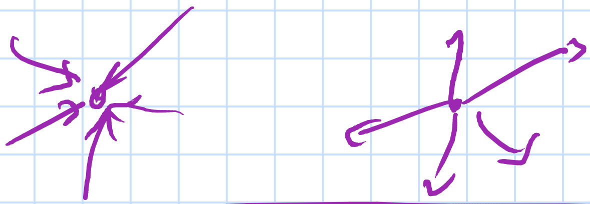
4.2 Linear Transformation

combine

$$T(c\vec{v}) = cT(\vec{v})$$

$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$$

$$T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$$



dialation

Ex $T(x,y,z) = (z-x,\ z-y)$ ←

linear Transformation?

$\vec{u} = (x_1, y_1, z_1)$ $\quad\vec{v} = (x_2, y_2, z_2)$

$$\begin{aligned}
T(\vec{u}+\vec{v}) &= T(x_1+x_2,\ y_1+y_2,\ z_1+z_2) \\
&= (z_1+z_2-(x_1+x_2),\ z_1+z_2-(y_1+y_2)) \\
&= (z_1+z_2-x_1-x_2,\ z_1+z_2-y_1-y_2) \\
&= ((z_1-x_1)+(z_2-x_2),\ z_1-y_1+(z_2-y_2)) \\
&= (z_1-x_1,\ z_1-y_1) + (z_2-x_2,\ z_2-y_2) \\
&= T(x_1,y_1,z_1) + T(x_2,y_2,z_2) \\
&= T(\vec{u}) + T(\vec{v}) \checkmark
\end{aligned}$$

$$T(r\vec{u}) = T(rx_1, ry_1, rz_1)$$
$$= (rz_1 - rx_1, rz_1 - ry_1)$$
$$= r(z_1 - x_1, z_1 - y_1)$$
$$= r\,T(x_1, y_1, z_1)$$
$$= r\,T(\vec{u}) \checkmark$$

Since $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

& $T(r\vec{u}) = r\,T(\vec{u})$

Then, the fctn T is a linear transformation

Domain: $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$T(x,y,z) = \begin{pmatrix} z-x \\ z-y \end{pmatrix}$$

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

Theorem Let $T: V \to W$ be linear transformation

$V \to S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a basis for V

$$T(\vec{v}) = c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \dots + c_n T(\vec{v}_n)$$

Ex $\vec{v}_1 = (1,1,1)$ $v_2 = (1,1,0)$ $v_3 = (1,0,0)$

$T: \mathbb{R}^3 \to \mathbb{R}^2$

$T(\vec{v}_1) = (1,0)$ $T(\vec{v}_2) = (2,-1)$ $T(\vec{v}_3) = (4,3)$

$T(\vec{x}_1, \vec{x}_2, \vec{x}_3) = ?$

$$(x_1, x_2, x_3) = c_1 (1,1,1) + c_2 (1,1,0) + c_3 (1,0,0)$$

$$\begin{aligned} c_1 + c_2 + c_3 &= x_1 \Rightarrow c_3 = x_1 - x_2 \\ c_1 + c_2 &= x_2 \Rightarrow c_2 = x_2 - x_3 \\ c_1 &= x_3 \end{aligned}$$

$$(x_1, x_2, x_3) = x_3 (1,1,1) + (x_2 - x_3)(1,1,0) + (x_1 - x_2)(1,0,0)$$

$$= x_3 v_1 + (x_2 - x_3) \vec{v}_2 + (x_1 - x_2) \vec{v}_3$$

$$T(x_1, x_2, x_3) = x_3 T(\vec{v}_1) + (x_2 - x_3) T(\vec{v}_2) + (x_1 - x_2) T(v_3)$$

$$T(x_1, x_2, x_3) = x_3(1,0) + (x_2 - x_3)(2,-1) + (x_1 - x_2)(4,3)$$

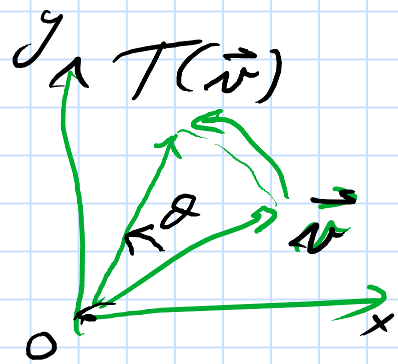
$$= (x_3 + 2x_2 - 2x_3 + 4x_1 - 4x_2, \; -x_2 + x_3 + 3x_1 - 3x_2$$

$$= (4x_1 - 2x_2 - x_3, \; 3x_1 - 4x_2 + x_3)$$

$$T(2,-3,5) = (8+6-5, \; 6+12+5) = \boxed{(9, 23)}$$

Kernel & Range of a Rotation

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$



$\vec{0}$ rotate itself $(\vec{0})$

$\ker(T) = \{0\}$

$T: V \longrightarrow W$

Kernel of T is a subspace of V

Range of T " W

$$
\begin{tikzcd}
\vec{x} \arrow[r, "T"] \arrow[d] & T(\vec{x}) \arrow[d] \\
{[\vec{x}]_B} \arrow[r, "A"'] & {[T(\vec{x})]_{B'}}
\end{tikzcd}
$$

direct

multiply

$$A[\vec{x}]_B = [T(\vec{x})]_{B'}$$

Ex $T: P_1 \to P_2$ $\quad T(p(x)) = x\,p(x)$

$\vec{B} = \{\vec{u}_1, \vec{u}_2\}$ $\quad B' = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$\vec{u}_1 = 1 \quad \vec{u}_2 = x \quad \vec{v}_1 = 1, \ \vec{v}_2 = x, \ \vec{v}_3 = x^2$

$$
\begin{aligned}
T(\vec{u}_1) &= T(1) \\
&= x(1) \\
&= x
\end{aligned}
$$

$$
\begin{aligned}
T(\vec{u}_2) &= T(x) \\
&= x(x) \\
&= x^2
\end{aligned}
$$

$$[T(\vec{u}_1)]_{B'} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad [T(\vec{u}_2)]_{B'} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$[T]_{BB'} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$T(p(x)) = x\,p(x)$$

$$T(\underbrace{a+bx}) = x(a+bx)$$

$$= ax + bx^2$$

$$B = \{1, x\} \qquad B' = \{1, x, x^2\}$$

$$\vec{x} = a + bx$$

$$[\vec{x}_B] = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{bmatrix} 0 \\ a \\ b \end{bmatrix} = [T(\vec{x})]_{B'}$$