Solution   Section 2.3 – Trigonometric Substitutions

Exercise

Evaluate the integral $\displaystyle\int \frac{3dx}{\sqrt{1+9x^2}}$

Solution

$$\int \frac{3dx}{\sqrt{1+9x^2}} = \frac{1}{3}\int \frac{\sec^2 t}{3\sec t}\,dt$$

$$= \int \sec t\ dt$$

$$= \ln\left|\sec t + \tan t\right| + C$$

$$= \ln\left|\sqrt{1+u^2} + u\right| + C$$

$$= \ln\left|\sqrt{1+9x^2} + 3x\right| + C$$

$3x = \tan t \quad \Rightarrow dx = \frac{1}{3}\sec^2 t\ dt$

$\sqrt{1+9x^2} = 3\sec^2 t$

Exercise

Evaluate the integral $\displaystyle\int \frac{5dx}{\sqrt{25x^2-9}}, \quad x > \frac{3}{5} = \sin^{-1}\frac{1}{2} - \sin^{-1} 0$

Solution

$$\int \frac{5dx}{\sqrt{25x^2-9}} = \int \frac{5\left(\frac{3}{5}\sec\theta\tan\theta d\theta\right)}{3\tan\theta}$$

$$= \int \sec\theta d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \ln\left|\frac{5}{3}x + \frac{1}{3}\frac{\sqrt{25x^2-9}}{3}\right| + C$$

$5x = 3\sec\theta \quad \rightarrow \quad dx = \frac{3}{5}\sec\theta\tan\theta d\theta$

$\sqrt{25x^2-9} = 3\tan\theta$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sqrt{y^2-49}}{y}dy, \quad y > 7$

Solution

$$\int \frac{\sqrt{y^2-49}}{y}dy = \int \frac{(7\tan\theta)}{7\sec\theta}(7\sec\theta\tan\theta)d\theta$$

$y = 7\sec\theta \quad \rightarrow \quad dy = 7\sec\theta\tan\theta d\theta$

$\sqrt{y^2-49} = 7\tan\theta$

$$= 7\int \tan^2 \theta d\theta$$

$$= 7\int \left(\sec^2 \theta - 1\right) d\theta$$

$$= 7\left(\tan\theta - \theta\right) + C$$

$$\boxed{= 7\left[\frac{\sqrt{y^2 - 49}}{7} - \sec^{-1}\left(\frac{y}{7}\right)\right] + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{2dx}{x^3\sqrt{x^2-1}}, \quad x > 1$

Solution

$$\int \frac{2dx}{x^3\sqrt{x^2-1}} = \int \frac{2\sec\theta\tan\theta d\theta}{\sec^3\theta\tan\theta}$$

$x = \sec\theta \quad dx = \sec\theta\tan\theta d\theta$

$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$

$$= 2\int \cos^2 \theta d\theta$$

$$= 2\int \frac{1+\cos 2\theta}{2} d\theta$$

$\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$$= \int \left(1 + \cos 2\theta\right) d\theta$$

$$= \theta + \frac{1}{2}\sin 2\theta + C$$

$$= \theta + \sin\theta\cos\theta + C$$

$$\boxed{= \sec^{-1} x + \frac{\sqrt{x^2-1}}{x^2} + C}$$

$x = \sec\theta = \dfrac{1}{\cos\theta} \Rightarrow \cos\theta = \dfrac{1}{x}$

$\sin\theta = \tan\theta\cos\theta = \sqrt{x^2-1}\left(\dfrac{1}{x}\right)$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^2}{4+x^2} dx$

Solution

$$\int \frac{x^2}{4+x^2} dx = \int \frac{4\tan^2\theta}{4\sec^2\theta} 2\sec^2\theta d\theta$$

$x = 2\tan\theta \quad dx = 2\sec^2\theta d\theta$

$4 + x^2 = 4 + 4\tan^2\theta = 4\sec^2\theta$

$$= 2\int \tan^2 \theta d\theta$$

$$= 2\int \left(\sec^2\theta - 1\right) d\theta \qquad\qquad \int \sec^2\theta d\theta = \tan\theta$$

$$= 2\left(\tan\theta - \theta\right) + C$$

$$= 2\left(\frac{x}{2} - \tan^{-1}\left(\frac{x}{2}\right)\right) + C$$

$$\boxed{= x - 2\tan^{-1}\left(\frac{x}{2}\right) + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+1}}$

Solution

$x = \tan\theta \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dx = \sec^2\theta d\theta$

$\sqrt{x^2+1} = \sqrt{\tan^2\theta + 1} = \sec\theta$

$$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec^2\theta d\theta}{\tan^2\theta\sec\theta}$$

$$= \int \frac{\sec\theta d\theta}{\tan^2\theta}$$

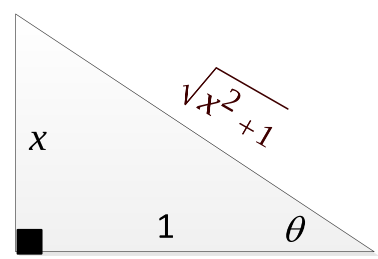
$$= \int \frac{\cos^2\theta d\theta}{\sin^2\theta\cos\theta}$$

$$= \int \frac{\cos\theta d\theta}{\sin^2\theta}$$

$$= \int \sin^{-2}\theta d\left(\sin\theta\right)$$

$$= -\frac{1}{\sin\theta} + C$$

$$\boxed{= -\frac{\sqrt{x^2+1}}{x} + C}$$



Exercise

Evaluate the integral $\displaystyle\int \frac{\left(1-x^2\right)^{1/2}}{x^4} dx$

Solution

$x = \sin\theta \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dx = \cos\theta d\theta$

$\left(1-x^2\right)^{1/2} = \left(1-\sin^2 x\right)^{1/2} = \cos\theta$

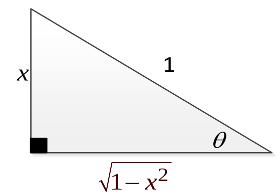
$$\int \frac{\left(1-x^2\right)^{1/2}}{x^4} dx = \int \frac{\cos\theta}{\sin^4\theta}\cos\theta d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\frac{1}{\sin^2\theta} d\theta$$

$$= \int \cot^2\theta \csc^2\theta\, d\theta$$

$$= -\frac{1}{3}\cot^3\theta + C$$

$$\boxed{= -\frac{1}{3}\left(\frac{\sqrt{1-x^2}}{x}\right)^3 + C}$$



Exercise

Evaluate the integral $\displaystyle\int \frac{x^3dx}{x^2-1}$

Solution

$$\int \frac{x^3dx}{x^2-1} = \int\left(x + \frac{x}{x^2-1}\right)dx$$

$$= \int xdx + \int \frac{x}{x^2-1}dx$$

$$= \int xdx + \frac{1}{2}\int \frac{d\left(x^2-1\right)}{x^2-1}$$

$$\boxed{= \frac{1}{2}x^2 + \frac{1}{2}\ln\left|x^2-1\right| + C}$$

$$\begin{array}{r|l} & \quad x \\ \hline x^2-1 & \quad x^3 \\ & \underline{\quad x^3 - x} \\ & \qquad\quad x \end{array}$$

$$d\left(x^2-1\right) = 2xdx \;\Rightarrow\; \frac{1}{2}d\left(x^2-1\right) = xdx$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}dx$

Solution

$$\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}dx = \int \frac{\cos\theta}{\sin\theta}\cos\theta\, d\theta$$

$$= \int \frac{\cos^2\theta}{\sin\theta}d\theta$$

$$= \int \frac{1-\sin^2\theta}{\sin\theta}d\theta$$

$$= \int \frac{1}{\sin\theta}d\theta - \int \sin\theta\, d\theta$$

$$= \int \csc\theta\, d\theta - \int \sin\theta\, d\theta$$

$$\ln x = \sin\theta \qquad 0 < \theta \le \frac{\pi}{2}$$

$$\frac{1}{x}dx = \cos\theta\, d\theta$$

$$\sqrt{1-(\ln x)^2} = \sqrt{1-\sin^2\theta} = \cos\theta$$

$$= -\ln\left|\csc\theta + \cot\theta\right| + \cos\theta + C$$

$$\boxed{= -\ln\left|\frac{1}{\ln x} + \frac{\sqrt{1-(\ln x)^2}}{\ln x}\right| + \sqrt{1-(\ln x)^2} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \sqrt{x}\,\sqrt{1-x}\,dx$

Solution

$$\int \sqrt{x}\,\sqrt{1-x}\,dx = \int u\sqrt{1-u^2}\,(2udu)$$

$u = \sqrt{x} \to u^2 = x \quad \Rightarrow dx = 2udu$

$u = \sin\theta \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$du = \cos\theta d\theta$

$$= 2\int u^2\sqrt{1-u^2}\,du$$

$$\int \sqrt{x}\,\sqrt{1-x}\,dx = 2\int u^2\sqrt{1-u^2}\,du = 2\int \sin^2\theta\cos\theta\cos\theta d\theta$$

$\sqrt{1-u^2} = \sqrt{1-\sin^2\theta} = \cos\theta$

$$= 2\int \sin^2\theta\cos^2\theta d\theta$$

$\sin 2\theta = 2\sin\theta\cos\theta \quad \to \sin^2 2\theta = 4\sin^2\theta\cos^2\theta$

$$= \frac{1}{2}\int \sin^2 2\theta d\theta$$

$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$

$$= \frac{1}{2}\int \frac{1-\cos 4\theta}{2}\,d\theta$$

$$= \frac{1}{4}\int d\theta - \frac{1}{4}\int \cos 4\theta d\theta$$

$$= \frac{1}{4}\theta - \frac{1}{16}\sin 4\theta + C$$

$$= \frac{1}{4}\theta - \frac{1}{16}2\sin 2\theta\cos 2\theta + C$$

$$= \frac{1}{4}\theta - \frac{1}{8}2\sin\theta\cos\theta\left(2\cos^2\theta - 1\right) + C$$

$$= \frac{1}{4}\theta - \frac{1}{2}\sin\theta\cos^3\theta + \frac{1}{4}\sin\theta\cos\theta + C$$

$$= \frac{1}{4}\sin^{-1}u - \frac{1}{2}u\left(1-u^2\right)^{3/2} + \frac{1}{4}u\sqrt{1-u^2} + C$$

$$\boxed{= \frac{1}{4}\sin^{-1}\sqrt{x} - \frac{1}{2}\sqrt{x}\,(1-x)^{3/2} + \frac{1}{4}\sqrt{x}\sqrt{1-x} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sqrt{x-2}}{\sqrt{x-1}}\,dx$

Solution

$$\int \frac{\sqrt{x-2}}{\sqrt{x-1}}\,dx = \int \frac{\sqrt{u^2-1}}{u} 2udu \qquad u=\sqrt{x-1} \quad \to u^2 = x-1 \quad \Rightarrow 2udu = dx$$

$$u = \sec\theta \qquad 0<\theta<\frac{\pi}{2}$$

$$du = \sec\theta\tan\theta d\theta$$

$$\sqrt{u^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$$

$$= 2\int \sqrt{u^2-1}\,du$$

$$= 2\int \tan\theta\sec\theta\tan\theta d\theta$$

$$w=\tan\theta \qquad dv = \sec\theta\tan\theta d\theta$$

$$dw = \sec^2\theta d\theta \qquad v=\sec\theta$$

$$2\int \tan\theta\sec\theta\tan\theta d\theta = 2\sec\theta\tan\theta - 2\int \sec^3\theta d\theta$$

$$= 2\sec\theta\tan\theta - 2\int \sec^2\theta\sec\theta d\theta$$

$$= 2\sec\theta\tan\theta - 2\int \left(\tan^2\theta+1\right)\sec\theta d\theta$$

$$= 2\sec\theta\tan\theta - 2\int \tan^2\theta\sec\theta d\theta - 2\int \sec\theta d\theta$$

$$2\int \tan^2\theta\sec\theta d\theta = 2\sec\theta\tan\theta - 2\int \tan^2\theta\sec\theta d\theta - 2\ln\left|\sec\theta+\tan\theta\right|$$

$$4\int \tan^2\theta\sec\theta d\theta = 2\sec\theta\tan\theta - 2\ln\left|\sec\theta+\tan\theta\right|$$

$$2\int \tan^2\theta\sec\theta d\theta = \sec\theta\tan\theta - \ln\left|\sec\theta+\tan\theta\right|$$

$$\int \frac{\sqrt{x-2}}{\sqrt{x-1}}\,dx = 2\int \tan\theta\sec\theta\tan\theta d\theta$$

$$= \sec\theta\tan\theta - \ln\left|\sec\theta+\tan\theta\right| + C$$

$$= u\sqrt{u^2-1} - \ln\left|u+\sqrt{u^2-1}\right| + C$$

$$= \sqrt{x-1}\sqrt{x-2} - \ln\left|\sqrt{x-1}+\sqrt{x-2}\right| + C$$

Exercise

Evaluate: $\displaystyle\int \frac{2dx}{\sqrt{1-4x^2}}$

Solution

$$\int \frac{2dx}{\sqrt{1-4x^2}} = \int \frac{du}{\sqrt{1-u^2}} \qquad\qquad u = 2x \quad \to \quad du = 2dx$$

$$= \sin^{-1} u + C$$

$$\boxed{= \sin^{-1} 2x + C}$$

Exercise

Evaluate: $\displaystyle\int \frac{dx}{\sqrt{4x^2-49}}$

Solution

$$2x = 7\sec\theta \quad \to dx = \frac{7}{2}\sec\theta\tan\theta\ d\theta$$

$$\sqrt{4x^2-49} = \frac{7}{2}\tan\theta$$

$$\int \frac{dx}{\sqrt{4x^2-49}} = \int \frac{dx}{2\sqrt{x^2-\left(\frac{7}{2}\right)^2}}$$

$$= \frac{1}{2}\int \frac{\frac{7}{2}\sec\theta\tan\theta d\theta}{\frac{7}{2}\tan\theta}$$

$$= \frac{1}{2}\int \sec\theta d\theta$$

$$= \frac{1}{2}\ln\left|\sec\theta+\tan\theta\right| + C$$

$$\boxed{= \frac{1}{2}\ln\left|\frac{2x}{7}+\frac{\sqrt{4x^2-49}}{7}\right| + C}$$

Exercise

Evaluate: $\displaystyle\int \frac{dx}{\sqrt{x^2+4}}$

Solution

Let: $x = 2\tan\theta \quad \to dx = 2\sec^2\theta d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\sqrt{x^2+4} = 2\left|\sec\theta\right|$$

$$\int \frac{dx}{\sqrt{x^2+4}} = \int \frac{2\sec^2\theta d\theta}{\sqrt{4\sec^2\theta}}$$

$$= \int \frac{2\sec^2\theta d\theta}{2|\sec\theta|}$$

$$= \int \sec\theta d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$\boxed{= \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\left(16-x^2\right)^{3/2}}$

Solution

$$\int \frac{dx}{\left(16-x^2\right)^{3/2}} = \int \frac{4\cos\theta}{(4\cos\theta)^3} d\theta$$

$x = 4\sin\theta \qquad \sqrt{16-x^2} = 4\cos\theta$

$dx = 4\cos\theta d\theta$

$$= \frac{1}{16}\int \frac{1}{\cos^2\theta} d\theta$$

$$= \frac{1}{16}\int \sec^2\theta \, d\theta$$

$$\boxed{= \frac{1}{16}\tan\theta + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\left(1+x^2\right)^2}$

Solution

$$\int \frac{dx}{\left(1+x^2\right)^2} = \int \frac{\sec^2\theta}{\sec^4\theta} d\theta$$

$x = \tan\theta \qquad 1+x^2 = \left(\sec^2\theta\right)^2$

$dx = \sec^2\theta \, d\theta$

$$= \int \frac{1}{\sec^2\theta} d\theta$$

$$= \int \cos^2\theta \, d\theta$$

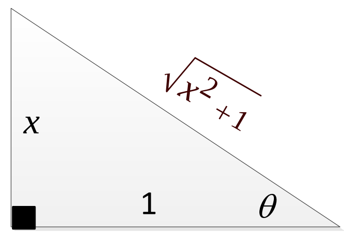
$$= \frac{1}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{2}\tan^{-1}x + \frac{1}{2}\sin\theta\cos\theta + C$$

$$= \frac{1}{2}\tan^{-1}x + \frac{1}{2}\frac{x}{\sqrt{1+x^2}}\,\frac{1}{\sqrt{1+x^2}} + C$$

$$= \frac{1}{2}\tan^{-1}x + \frac{x}{2\left(1+x^2\right)} + C$$



Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2+4}}$

Solution

$$\int \frac{dx}{\sqrt{x^2+4}} = \int \frac{2\sec^2\theta}{2\sec\theta}\, d\theta$$

$x = 2\tan\theta \qquad \sqrt{x^2+4} = 2\sec\theta$

$dx = 2\sec^2\theta \, d\theta$

$$= \int \sec\theta \, d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C$$

$$= \ln\left(\sqrt{x^2+4} + x\right) - \ln 2 + C$$

$$= \ln\left(\sqrt{x^2+4} + x\right) + C$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}}$

Solution

$$\int \frac{dx}{x^2\sqrt{9-x^2}} = \int \frac{3\cos\theta}{9\sin^2\theta(3\cos\theta)}\, d\theta$$

$x = 3\sin\theta \qquad \sqrt{9-x^2} = 3\cos\theta$

$dx = 3\cos\theta d\theta$

$$= \frac{1}{9}\int \csc^2\theta \, d\theta$$

$$= -\frac{1}{9}\cot\theta + C \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{9-x^2}}{3}\cdot\frac{3}{x}$$

$$\boxed{= -\frac{1}{9}\frac{\sqrt{9-x^2}}{x} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{4x^2+1}}$

Solution

$$\int \frac{dx}{\sqrt{4x^2+1}} = \frac{1}{2}\int \frac{\sec^2\theta}{\sec\theta}d\theta$$

$2x = \tan\theta \qquad \sqrt{4x^2+1} = \sec\theta$

$dx = \frac{1}{2}\sec^2\theta \, d\theta$

$$= \frac{1}{2}\int \sec\theta \, d\theta$$

$$= \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C$$

$$\boxed{= \frac{1}{2}\ln\left|\sqrt{4x^2+1} + 2x\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\left(x^2+1\right)^{3/2}}$

Solution

$$\int \frac{dx}{\left(x^2+1\right)^{3/2}} = \int \frac{\sec^2\theta}{(\sec\theta)^3}d\theta$$

$x = \tan\theta \qquad \sqrt{x^2+1} = \sec\theta$

$dx = \sec^2\theta \, d\theta$

$$= \int \frac{d\theta}{\sec\theta}$$

$$= \int \cos\theta \, d\theta$$

$$= \sin\theta + C \qquad \sin\theta = \frac{\tan\theta}{\sec\theta} = \frac{x}{\sqrt{x^2+1}}$$

$$\boxed{= \frac{x}{\sqrt{x^2+1}} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{4}{x^2\sqrt{16-x^2}}\,dx$

Solution

$$\int \frac{4}{x^2\sqrt{16-x^2}}\,dx = \int \frac{16\cos\theta}{16\sin^2\theta\,(4\cos\theta)}\,d\theta \qquad x = 4\sin\theta \quad \sqrt{16-x^2} = 4\cos\theta \quad dx = 4\cos\theta d\theta$$

$$= \frac{1}{4}\int \csc^2\theta\,d\theta$$

$$\boxed{= -\frac{1}{4}\cot\theta + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x^3}{\sqrt{9-x^2}}\,dx$

Solution

$$\int \frac{x^3}{\sqrt{9-x^2}}\,dx = \int \frac{27\sin^3\theta}{3\cos\theta}\,(3\cos\theta)\,d\theta \qquad x = 3\sin\theta \quad \sqrt{9-x^2} = 3\cos\theta \quad dx = 3\cos\theta d\theta$$

$$= 27\int \sin^3\theta\,d\theta$$

$$= 27\int \left(1-\cos^2\theta\right)\,d\left(\cos\theta\right)$$

$$= 27\left(\cos\theta - \frac{1}{3}\cos^3\theta\right) + C$$

$$\boxed{= 27\cos\theta - 9\cos^3\theta + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2-25}}$

Solution

$$\int \frac{dx}{\sqrt{x^2-25}} = \int \frac{5\sec\theta\tan\theta}{5\tan\theta}\,d\theta \qquad x = 5\sec\theta \quad \sqrt{x^2-25} = 5\tan\theta \quad dx = 5\sec\theta\tan\theta\,d\theta$$

$$= \int \sec\theta\,d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

$$\boxed{= \ln\left|\frac{x}{5} + \frac{1}{5}\sqrt{x^2-25}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{\sqrt{x^2-25}}{x}\,dx$

Solution

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \int \frac{5\tan\theta}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta$$

$$x = 5\sec\theta \qquad \sqrt{x^2-25} = 5\tan\theta$$

$$dx = 5\sec\theta\tan\theta\, d\theta$$

$$= 5\int \tan^2\theta\, d\theta$$

$$= 5\int \left(\sec^2\theta - 1\right)\, d\theta$$

$$= 5(\tan\theta - \theta) + C$$

$$\boxed{= \sqrt{x^2-25} - 5\,\text{arcsec}\,\frac{x}{5} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x^3}{\sqrt{x^2-25}}\,dx$

Solution

$$\int \frac{x^3}{\sqrt{x^2-25}}\,dx = \int \frac{5^3\sec^3\theta}{5\tan\theta}(5\sec\theta\tan\theta)\,d\theta$$

$$x = 5\sec\theta \qquad \sqrt{x^2-25} = 5\tan\theta$$

$$dx = 5\sec\theta\tan\theta\, d\theta$$

$$= 125\int \sec^4\theta\, d\theta$$

$$= 125\int \left(1+\tan^2\theta\right)\sec^2\theta\, d\theta$$

$$= 125\int \left(1+\tan^2\theta\right)\, d(\tan\theta)$$

$$= 125\left(\tan\theta + \frac{1}{3}\tan^3\theta\right) + C$$

$$= 125\left(\frac{\sqrt{x^2-25}}{5} + \frac{1}{3}\frac{\left(x^2-25\right)^{3/2}}{125}\right) + C$$

$$= \sqrt{x^2-25}\left(25 + \frac{x^2-25}{3}\right) + C$$

$$\boxed{= \frac{1}{3}\sqrt{x^2-25}\left(x^2+50\right) + C}$$

Exercise

Evaluate $\displaystyle\int x^3\sqrt{x^2-25}\,dx$

Solution

$$\int x^3\sqrt{x^2-25}\,dx = \int 5^3\sec^3\theta(5\tan\theta)(5\sec\theta\tan\theta)\,d\theta$$

$x = 5\sec\theta \qquad \sqrt{x^2-25} = 5\tan\theta$

$dx = 5\sec\theta\tan\theta\,d\theta$

$$= 5^5\int \sec^4\theta\tan^2\theta\,d\theta$$

$$= 5^5\int \sec^2\theta\left(1+\tan^2\theta\right)\tan^2\theta\,d\theta$$

$$= 5^5\int \left(\tan^2\theta+\tan^4\theta\right)\,d\left(\tan\theta\right)$$

$$= 5^5\left(\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta\right)+C$$

$$= 5^5\left(\frac{1}{3}\frac{1}{5^3}\left(x^2-25\right)^{3/2}+\frac{1}{5^6}\left(x^2-25\right)^{5/2}\right)+C$$

$$= \left(x^2-25\right)^{3/2}\left(\frac{25}{3}+\frac{1}{5}\left(x^2-25\right)\right)+C$$

$$= \frac{1}{15}\left(x^2-25\right)^{3/2}\left(125+3x^2-75\right)+C$$

$$= \frac{1}{15}\left(x^2-25\right)^{3/2}\left(3x^2+50\right)+C$$

Exercise

Evaluate $\displaystyle\int x\sqrt{x^2+1}\,dx$

Solution

$$\int x\sqrt{x^2+1}\,dx = \frac{1}{2}\int\left(x^2+1\right)^{1/2}\,d\left(x^2+1\right)$$

$$= \frac{1}{3}\left(x^2+1\right)^{3/2}+C$$

$$\int x\sqrt{x^2+1}\,dx = \int \tan\theta\sec^3\theta\,d\theta$$

$x=\tan\theta \qquad \sqrt{x^2+1}=\sec\theta$

$dx=\sec^2\theta\,d\theta$

$$= \int \sec^2\theta\,d\left(\sec\theta\right)$$

$$= \frac{1}{3}\sec^3\theta+C$$

$$= \frac{1}{3}\left(x^2+1\right)^{3/2}+C$$

Exercise

Evaluate $\displaystyle\int \frac{9x^3}{\sqrt{x^2+1}}\,dx$

Solution

$$\int \frac{9x^3}{\sqrt{x^2+1}}\,dx = \int \frac{9\tan^3\theta}{\sec\theta}\left(\sec^2\theta\right)d\theta$$

$x = \tan\theta \qquad \sqrt{x^2+1} = \sec\theta$

$dx = \sec^2\theta\, d\theta$

$$= 9\int \tan^2\theta \tan\theta \sec\theta\, d\theta$$

$$= 9\int \left(\sec^2\theta - 1\right)\, d\left(\sec\theta\right)$$

$$= 9\left(\frac{1}{3}\sec^3\theta - \sec\theta\right) + C$$

$$= 3\left(x^2+1\right)\sqrt{x^2+1} - 9\sqrt{x^2+1} + C$$

$$= 3\sqrt{x^2+1}\left(x^2+1-3\right) + C$$

$$\boxed{= 3\sqrt{x^2+1}\left(x^2-2\right) + C}$$

Exercise

Evaluate $\displaystyle\int \sqrt{16x^2+9}\,dx$

Solution

$$\int \sqrt{16x^2+9}\,dx = \int 3\sec\theta\left(\frac{3}{2}\sec^2\theta\right)d\theta$$

$2x = 3\tan\theta \qquad \sqrt{4x^2+9} = 3\sec\theta$

$2dx = 3\sec^2\theta\, d\theta$

$$= \frac{9}{2}\int \sec^3\theta\, d\theta$$

$$= \frac{9}{4}\sec\theta\tan\theta + \frac{9}{4}\ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \frac{9}{4}\frac{\sqrt{4x^2+9}}{3}\frac{2x}{3} + \frac{9}{4}\ln\left|\frac{\sqrt{4x^2+9}}{3} + \frac{2x}{3}\right| + C$$

$$\boxed{= \frac{1}{2}x\sqrt{4x^2+9} + \frac{9}{4}\ln\left|\frac{2x+\sqrt{4x^2+9}}{3}\right| + C}$$

$u = \sec x \qquad dv = \sec^2 x\, dx$

$du = \sec x\tan x\, dx \qquad v = \tan x$

$$\int \sec^3 x\, dx = \sec x\tan x - \int \tan x\left(\sec x\tan x\, dx\right)$$

$$= \sec x\tan x - \int \tan^2 x\sec x\, dx$$

$$= \sec x\tan x - \int\left(\sec^2 x - 1\right)\sec x\, dx$$

$$= \sec x\tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$2\int \sec^3 x\, dx = \sec x\tan x + \ln\left|\sec x + \tan x\right|$$

$$\int \sec^3 x\, dx = \boxed{\frac{1}{2}\sec x\tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right|}$$

Exercise

Evaluate $\displaystyle\int \sqrt{25-4x^2}\ dx$

Solution

$$\int \sqrt{25-4x^2}\ dx = \frac{25}{2}\int \cos^2\theta\ d\theta$$

$2x = 5\sin\theta \qquad \sqrt{25-4x^2} = 5\cos\theta$

$dx = \frac{5}{2}\cos\theta d\theta$

$$= \frac{25}{4}\int (1+\cos 2\theta)\ d\theta$$

$$= \frac{25}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{25}{4}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{25}{4}\left(\sin^{-1}\frac{2x}{5} + \frac{2x}{5}\frac{\sqrt{25-4x^2}}{5}\right) + C$$

$$= \frac{25}{4}\sin^{-1}\frac{2x}{5} + \frac{1}{2}x\sqrt{25-4x^2} + C$$

Exercise

Evaluate $\displaystyle\int \sqrt{5x^2-1}\ dx$

Solution

$$\int \sqrt{5x^2-1}\ dx = \frac{1}{\sqrt{5}}\int \sec\theta\ d\theta$$

$\sqrt{5}x = \sec\theta \qquad \sqrt{5x^2-1} = \tan\theta$

$dx = \frac{1}{\sqrt{5}}\sec\theta\tan\theta\ d\theta$

$$= \frac{1}{\sqrt{5}}\ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{x}{\sqrt{5}} + \sqrt{5x^2-1}\right| + C$$

Exercise

Evaluate $\displaystyle\int \frac{\sqrt{25x^2+4}}{x^4}dx$

Solution

$$\int \frac{\sqrt{25x^2+4}}{x^4}dx = \int \frac{2\sec\theta}{\left(\frac{2}{5}\right)^4\tan^4\theta}\frac{2}{5}\sec^2\theta\ d\theta$$

$5x = 2\tan\theta \qquad \sqrt{25x^2+4} = 2\sec\theta$

$5dx = 2\sec^2\theta\ d\theta$

$$=\frac{125}{4}\int \frac{\sec^3\theta}{\tan^4\theta}\,d\theta$$

$$=\frac{125}{4}\int \frac{\cos\theta}{\sin^4\theta}\,d\theta$$

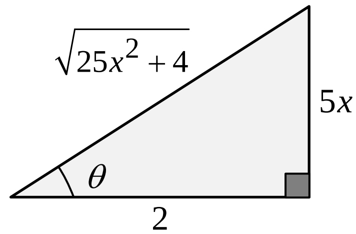
$$=\frac{125}{4}\int \sin^{-4}\theta\, d(\sin\theta)$$

$$=-\frac{125}{12}\frac{1}{\sin^3\theta}+C$$

$$=-\frac{125}{12}\csc^3\theta+C$$

$$=-\frac{125}{12}\left(\frac{\sqrt{25x^2+4}}{5x}\right)^3+C$$

$$\boxed{=-\frac{\left(25x^2+4\right)^{3/2}}{12x^3}+C}$$



Exercise

Evaluate $\displaystyle\int \frac{1}{x\sqrt{4x^2+9}}\,dx$

Solution

$$\int \frac{1}{x\sqrt{4x^2+9}}\,dx=\int \frac{1}{\frac{9}{2}\tan\theta\sec\theta}\left(\frac{3}{2}\sec^2\theta\right)d\theta$$

$2x=3\tan\theta \qquad \sqrt{4x^2+9}=3\sec\theta$

$dx=\frac{3}{2}\sec^2\theta\,d\theta$

$$=\int \frac{1}{\frac{9}{2}\tan\theta\sec\theta}\left(\frac{3}{2}\sec^2\theta\right)d\theta$$

$$=\frac{1}{3}\int \frac{\sec\theta}{\tan\theta}\,d\theta$$

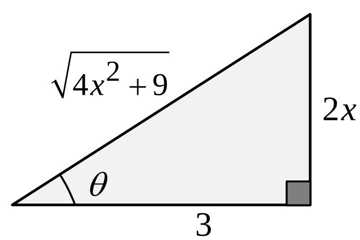
$$=\frac{1}{3}\int \frac{1}{\sin\theta}\,d\theta$$

$$=\frac{1}{3}\int \csc\theta\,d\theta$$

$$=-\frac{1}{3}\ln\left|\csc\theta+\cot\theta\right|+C$$

$$=-\frac{1}{3}\ln\left|\frac{\sqrt{4x^2+9}}{2x}+\frac{3}{2x}\right|+C$$

$$\boxed{=-\frac{1}{3}\ln\left|\frac{\sqrt{4x^2+9}+3}{2x}\right|+C}$$



Exercise

Evaluate $\displaystyle\int \frac{1}{\left(x^2+5\right)^{3/2}}dx$

Solution

$$\int \frac{1}{\left(x^2+5\right)^{3/2}}dx = \int \frac{1}{5\sqrt{5}\sec^3\theta}\left(\sqrt{5}\sec^2\theta\right)\, d\theta$$

$x = \sqrt{5}\tan\theta \qquad \sqrt{x^2+5} = \sqrt{5}\sec\theta$

$dx = \sqrt{5}\sec^2\theta\, d\theta$

$$= \frac{1}{5}\int \frac{1}{\sec\theta}\, d\theta$$

$$= \frac{1}{5}\int \cos\theta\, d\theta$$

$$\boxed{= \frac{1}{5}\sin\theta + C}$$

Exercise

Evaluate $\displaystyle\int e^x\sqrt{1-e^{2x}}\, dx$

Solution

$$\int e^x\sqrt{1-e^{2x}}\, dx = \int \cos^2\theta\, d\theta$$

$e^x = \sin\theta \qquad \sqrt{1-e^{2x}} = \cos\theta$

$e^x dx = \cos\theta d\theta$

$$= \frac{1}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{2}(\theta + \sin\theta\cos\theta) + C$$

$$\boxed{= \frac{1}{2}\left(\arcsin e^x + e^x\sqrt{1-e^{2x}}\right) + C}$$

Exercise

Evaluate $\displaystyle\int \frac{\sqrt{1-x}}{\sqrt{x}}\, dx$

Solution

$$\int \frac{\sqrt{1-x}}{\sqrt{x}}\, dx = \int \frac{\cos\theta}{\sin\theta}(2\sin\theta\cos\theta)\, d\theta$$

$\sqrt{x} = \sin\theta \to x = \sin^2\theta \qquad \sqrt{1-x} = \cos\theta$

$dx = 2\sin\theta\cos\theta d\theta$

$$= 2\int \cos^2\theta\, d\theta$$

$$= \int (1 + \cos 2\theta)\, d\theta$$

$$= \theta + \frac{1}{2}\sin 2\theta + C$$

$$= \theta + 2\sin\theta\cos\theta + C$$

$$= \arcsin\sqrt{x} + 2\sqrt{x}\sqrt{1-x} + C$$

Exercise

Evaluate $\displaystyle\int \frac{1}{x^4 + 4x^2 + 4}\, dx$

Solution

$$\int \frac{1}{x^4 + 4x^2 + 4}\, dx = \int \frac{dx}{\left(x^2+2\right)^2}$$

$x = \sqrt{2}\tan\theta \qquad x^2 + 2 = 2\sec^2\theta$

$dx = \sqrt{2}\sec^2\theta\, d\theta$

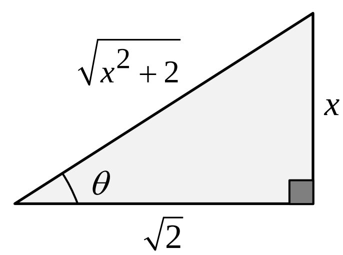
$$= \int \frac{\sqrt{2}\sec^2\theta}{4\sec^4\theta}\, d\theta$$

$$= \frac{\sqrt{2}}{4}\int \cos^2\theta\, d\theta$$

$$= \frac{\sqrt{2}}{8}\int (1 + \cos 2\theta)\, d\theta$$

$$= \frac{\sqrt{2}}{8}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{\sqrt{2}}{8}\left(\arctan\frac{x}{\sqrt{2}} + \frac{x\sqrt{2}}{x^2+2}\right) + C$$



Exercise

Evaluate $\displaystyle\int \frac{x^3 + x + 1}{x^4 + 2x^2 + 1}\, dx$

Solution

$$\int \frac{x^3 + x + 1}{x^4 + 2x^2 + 1}\, dx = \int \frac{x^3 + x}{x^4 + 2x^2 + 1}\, dx + \int \frac{1}{\left(x^2+1\right)^2}\, dx$$

$x = \tan\theta \qquad x^2 + 1 = \sec^2\theta$

$dx = \sec^2\theta\, d\theta$

$$= \frac{1}{4}\int \frac{1}{x^4 + 2x^2 + 1}\, d\left(x^4 + 2x^2 + 1\right) + \int \frac{1}{\sec^2\theta}\, d\theta$$

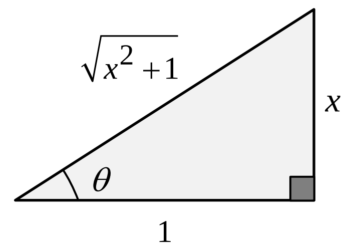
$$= \frac{1}{4}\ln\left(x^2+1\right)^2 + \int \cos^2\theta\, d\theta$$

$$= \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{2}\int (1 + \cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{2}\left(\theta + \sin\theta\cos\theta\right) + C$$

$$= \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right) + C$$



Exercise

Evaluate $\displaystyle\int \operatorname{arcsec} 2x \, dx \quad x > \frac{1}{2}$

Solution

$u = \operatorname{arcsec} 2x \qquad dv = dx$

$du = \dfrac{dx}{x\sqrt{4x^2-1}} \qquad v = x$

$$\int \operatorname{arcsec} 2x \, dx = x \operatorname{arcsec} 2x - \int \frac{dx}{\sqrt{4x^2-1}}$$

$2x = \sec\theta \qquad \sqrt{4x^2-1} = \tan\theta$

$dx = \frac{1}{2}\sec\theta\tan\theta \, d\theta$

$$= x \operatorname{arcsec} 2x - \frac{1}{2}\int \frac{\sec\theta\tan\theta}{\tan\theta} \, d\theta$$

$$= x \operatorname{arcsec} 2x - \frac{1}{2}\int \sec\theta \, d\theta$$

$$= x \operatorname{arcsec} 2x - \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C$$

$$= x \operatorname{arcsec} 2x - \frac{1}{2}\ln\left|2x + \sqrt{4x^2-1}\right| + C$$

Exercise

Evaluate $\displaystyle\int x \arcsin x \, dx$

Solution

$u = \arcsin x \qquad dv = x\,dx$

$du = \dfrac{dx}{\sqrt{1-x^2}} \qquad v = \frac{1}{2}x^2$

$$\int x \arcsin x \, dx = \frac{1}{2}x^2 \arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}} dx$$

$x = \sin\theta \qquad \sqrt{1-x^2} = \cos\theta$

$dx = \cos\theta \, d\theta$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\cos\theta}\cos\theta \, d\theta$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{2}\int \sin^2\theta \, d\theta$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{4}\int (1-\cos 2\theta) \, d\theta$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{4}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{4}(\theta - \sin\theta\cos\theta) + C$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{4}\left(\arcsin x - \frac{1}{x\sqrt{1-x^2}}\right) + C$$

$$\boxed{= \frac{1}{4}\left(2x^2 - 1\right)\arcsin x + \frac{1}{4}\frac{1}{x\sqrt{1-x^2}} + C}$$

Exercise

Evaluate $\displaystyle\int_0^{\sqrt{3}/2} \frac{x^2}{\left(1-x^2\right)^{3/2}} dx$

Solution

$$\int_0^{\sqrt{3}/2} \frac{x^2}{\left(1-x^2\right)^{3/2}} dx = \int_0^{\sqrt{3}/2} \frac{\sin^2\theta}{\cos^3\theta}(\cos\theta)\, d\theta$$

$x = \sin\theta \qquad \sqrt{1-x^2} = \cos\theta$

$dx = \cos\theta d\theta$

$$= \int_0^{\sqrt{3}/2} \tan^2\theta \, d\theta$$

$$= \int_0^{\sqrt{3}/2} \left(\sec^2\theta - 1\right) \, d\theta$$

$$= (\tan\theta - \theta)\Big|_0^{\sqrt{3}/2}$$

$$= \left(\frac{x}{\sqrt{1-x^2}} - \arcsin x\right)\Bigg|_0^{\sqrt{3}/2}$$

$$= \frac{\sqrt{3}}{2}\frac{1}{\sqrt{1-\frac{3}{4}}} - \frac{\pi}{3}$$

$$\boxed{= \sqrt{3} - \frac{\pi}{3}}$$

Exercise

Evaluate $\displaystyle\int_0^{\sqrt{3}/2} \frac{1}{\left(1-x^2\right)^{5/2}}\,dx$

Solution

$$\int_0^{\sqrt{3}/2} \frac{1}{\left(1-x^2\right)^{5/2}}\,dx = \int_0^{\sqrt{3}/2} \frac{1}{\cos^5\theta}\cos\theta\; d\theta$$

$x = \sin\theta \qquad \sqrt{1-x^2} = \cos\theta$

$dx = \cos\theta d\theta$

$$= \int_0^{\sqrt{3}/2} \sec^4\theta\; d\theta$$

$$= \int_0^{\sqrt{3}/2} \left(1+\tan^2\theta\right)\sec^2\theta\; d\theta$$

$$= \int_0^{\sqrt{3}/2} \left(1+\tan^2\theta\right)\, d\left(\tan\theta\right)$$

$$= \tan\theta + \frac{1}{3}\tan^3\theta \Big|_0^{\sqrt{3}/2}$$

$$= \frac{x}{\sqrt{1-x^2}} + \frac{1}{3}\frac{x^3}{\left(1-x^2\right)^{3/2}} \Big|_0^{\sqrt{3}/2}$$

$$= \frac{\sqrt{3}}{2}\frac{1}{\sqrt{1-\frac{3}{4}}} + \frac{\sqrt{3}}{8}\frac{1}{\left(\frac{1}{4}\right)^{3/2}}$$

$$= \sqrt{3} + \sqrt{3}$$

$$\boxed{= 2\sqrt{3}}$$

Exercise

Evaluate $\displaystyle\int_0^{3} \frac{x^3}{\sqrt{x^2+9}}\,dx$

Solution

$$\int_0^{3} \frac{x^3}{\sqrt{x^2+9}}\,dx = \int_0^{3} \frac{27\tan^3\theta}{3\sec\theta}3\sec^2\theta\; d\theta$$

$x = 3\tan\theta \qquad \sqrt{x^2+9} = 3\sec\theta$

$dx = 3\sec^2\theta\; d\theta$

$$= 27\int_0^{3} \tan^2\theta\tan\theta\sec\theta\; d\theta$$

$$= 27\int_0^3 \left(\sec^2\theta - 1\right)\, d\left(\sec\theta\right)$$

$$= 27\left(\frac{1}{3}\sec^3\theta - \sec\theta\right)\Big|_0^3$$

$$= 9\sqrt{x^2+9}\left(\frac{x^2+9}{27} - 1\right)\Big|_0^3$$

$$= \frac{1}{3}\sqrt{x^2+9}\left(x^2-18\right)\Big|_0^3$$

$$\boxed{= -9\sqrt{2} + 18}$$

Exercise

Evaluate $\displaystyle\int_0^{3/5} \sqrt{9-25x^2}\, dx$

Solution

$$\int_0^{3/5} \sqrt{9-25x^2}\, dx = \frac{9}{5}\int_0^{3/5} \cos^2\theta\, d\theta$$

$5x = 3\sin\theta \quad \sqrt{9-25x^2} = 3\cos\theta$

$dx = \frac{3}{5}\cos\theta d\theta$

$$= \frac{9}{10}\int_0^{3/5} \left(1+\cos 2\theta\right)\, d\theta$$

$$= \frac{9}{10}\left(\theta + \frac{1}{2}\sin 2\theta\right)_0^{3/5}$$

$\sin 2\theta = 2\sin\theta\cos\theta = 2\frac{5x}{3}\frac{5\sqrt{9-25x^2}}{3}$

$$= \frac{9}{10}\left(\arcsin\frac{5x}{3} + \frac{25}{9}x\sqrt{9-25x^2}\right)_0^{3/5}$$

$$\boxed{= \frac{9\pi}{20}}$$

Exercise

Evaluate $\displaystyle\int_4^6 \frac{x^2}{\sqrt{x^2-9}}\, dx$

Solution

$$\int_4^6 \frac{x^2}{\sqrt{x^2-9}}\, dx = \int_4^6 \frac{9\sec^2\theta}{3\tan\theta}\left(3\sec\theta\tan\theta\right)d\theta$$

$x = 3\sec\theta \quad \sqrt{x^2-9} = 3\tan\theta$

$dx = 3\sec\theta\tan\theta\, d\theta$

$$= 9\int_4^6 \sec^3\theta\, d\theta$$

$u = \sec x \quad dv = \sec^2 x dx$

$du = \sec x\tan x dx \quad v = \tan x$

$$= \frac{9}{2}\left[\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right]_4^6$$

$$= \frac{9}{2}\left[\frac{x}{3}\frac{\sqrt{x^2-9}}{3} + \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|\right]_4^6$$

$$= \frac{9}{2}\left(2\sqrt{3} + \ln\left(2+\sqrt{3}\right) - \frac{4\sqrt{7}}{9} - \ln\left(\frac{4+\sqrt{7}}{3}\right)\right)$$

$$\boxed{= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\ln\left(\frac{6+3\sqrt{3}}{4+\sqrt{7}}\right)}$$

$$\int \sec^3 x dx = \sec x \tan x - \int \tan x\left(\sec x \tan x dx\right)$$

$$= \sec x \tan x - \int \tan^2 x \sec x dx$$

$$= \sec x \tan x - \int \left(\sec^2 x - 1\right)\sec x \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$2\int \sec^3 x \, dx = \sec x \tan x + \ln\left|\sec x + \tan x\right|$$

$$\int \sec^3 x \, dx = \boxed{\frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right|}$$

Exercise

Evaluate $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x} dx$

Solution

$$\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x} dx = \int_{\sqrt{3}}^{2} \frac{\sqrt{3}\tan\theta}{\sqrt{3}\sec\theta}\left(\sqrt{3}\sec\theta\tan\theta\right) d\theta$$

$x = \sqrt{3}\sec\theta \qquad \sqrt{x^2-3} = \sqrt{3}\tan\theta$

$dx = \sqrt{3}\sec\theta\tan\theta \; d\theta$

$$= \sqrt{3}\int_{\sqrt{3}}^{2} \tan^2\theta \; d\theta$$

$$= \sqrt{3}\int_{\sqrt{3}}^{2} \left(\sec^2\theta - 1\right) \; d\theta$$

$$= \sqrt{3}\left(\tan\theta - \theta\right)\Big|_{\sqrt{3}}^{2}$$

$$= \sqrt{3}\left(\frac{\sqrt{x^2-3}}{\sqrt{3}} - \operatorname{arcsec}\frac{x}{\sqrt{3}}\right)\Big|_{\sqrt{3}}^{2}$$

$$= \sqrt{3}\left(\frac{1}{\sqrt{3}} - \frac{\pi}{6}\right)$$

$$\boxed{= 1 - \frac{\pi\sqrt{3}}{6}}$$

Exercise

Evaluate $\displaystyle\int_{1}^{4} \frac{\sqrt{x^2+4x-5}}{x+2} dx$

Solution

$$\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2}\,dx = \int_1^4 \frac{\sqrt{(x+2)^2-9}}{x+2}\,dx$$

$x+2=3\sec\theta \qquad \sqrt{(x+2)^2-9}=3\tan\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$

$$= \int_1^4 \frac{3\tan\theta}{3\sec\theta}(3\sec\theta\tan\theta)\ d\theta = 3\int_1^4 \tan^2\theta\ d\theta$$

$$= 3\int_1^4 \left(\sec^2\theta - 1\right)\ d\theta$$

$$= 3(\tan\theta - \theta)\Big|_1^4$$

$$= \sqrt{(x+2)^2-9} - 3\sec^{-1}\left(\frac{x+2}{3}\right)\Big|_1^4$$

$$= \sqrt{27} - 3\sec^{-1}(2) + 3\sec^{-1}(1)$$

$$\boxed{= 3\sqrt{3} - \pi}$$

$\theta = \sec^{-1}\left(\frac{x+2}{3}\right)$

$$\begin{cases} x=4 \ \to \ \theta = \sec^{-1}(2) = \frac{\pi}{3} \\ x=1 \ \to \ \theta = \sec^{-1}(1) = 0 \end{cases}$$

$$= 3(\tan\theta - \theta)\Big|_0^{\pi/3}$$

$$\boxed{= 3\sqrt{3} - \pi}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}}$

Solution

$$\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1}\frac{x}{3}\right]_0^{3/2}$$

$$\boxed{= \frac{\pi}{6}}$$

Exercise

Evaluate the integral $\displaystyle\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^x dx}{\left(1+e^{2x}\right)^{3/2}}$

Solution

$$\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^x dx}{\left(1+e^{2x}\right)^{3/2}} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{\tan\theta}{\left(\sec^2\theta\right)^{3/2}} \frac{\sec^2\theta}{\tan\theta}\,d\theta$$

$$= \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{\sec^2\theta}{\sec^3\theta}\,d\theta$$

$e^x = \tan\theta \to x = \ln(\tan\theta)$

$dx = \frac{\sec^2\theta}{\tan\theta}\,d\theta$

$\tan^{-1}\left(\frac{3}{4}\right) < \theta < \tan^{-1}\left(\frac{4}{3}\right)$

$1+e^{2x} = 1+\tan^2\theta = \sec^2\theta$

$$= \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{1}{\sec\theta} d\theta$$

$$= \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \cos\theta d\theta$$

$$= \sin\theta \Big|_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)}$$

$$= \sin\left(\tan^{-1}(3/4)\right) - \sin\left(\tan^{-1}(4/3)\right)$$

$$= \frac{4}{5} - \frac{3}{5}$$

$$= \frac{1}{5}$$

Exercise

Evaluate the integral $\displaystyle\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}}$

Solution

$$\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}} = \int_0^{\pi/4} \frac{e^{\tan\theta}\sec^2\theta}{e^{\tan\theta}\sec\theta} d\theta$$

$y = e^{\tan\theta}$ $\quad 1 \le y \le e \to \quad 0 \le \theta = \tan^{-1}(\ln y) \le \frac{\pi}{4}$

$dy = e^{\tan\theta}\sec^2\theta d\theta$

$$= \int_0^{\pi/4} \sec\theta d\theta$$

$\sqrt{1+(\ln y)^2} = \sqrt{1+\tan^2\theta} = \sec\theta$

$$= \left[\ln\left|\sec\theta + \tan\theta\right|\right]_0^{\pi/4}$$

$$= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln\left|\sec 0 + \tan 0\right|$$

$$= \ln\left(1+\sqrt{2}\right)$$

Exercise

Evaluate $\displaystyle\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2-1}}$

Solution

Let: $\quad u = 3y \quad \Rightarrow \quad du = 3dy \quad \to \quad \frac{du}{3} = dy$

$$\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2-1}} = \int_{-2/3}^{-\sqrt{2}/3} \frac{\frac{du}{3}}{\frac{u}{3}\sqrt{u^2-1}}$$

$$= \int_{-2/3}^{-\sqrt{2}/3} \frac{du}{u\sqrt{u^2-1}} \qquad \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right|$$

$$= \sec^{-1}|3y|\Big|_{-2/3}^{-\sqrt{2}/3}$$

$$= \sec^{-1}\left|-\sqrt{2}\right| - \sec^{-1}|-2|$$

$$= \frac{\pi}{4} - \frac{\pi}{3}$$

$$= -\frac{\pi}{12}$$

Exercise

Evaluate $\displaystyle\int_0^2 \sqrt{1+4x^2}\,dx$

Solution

$$\int_0^2 \sqrt{1+4x^2}\,dx = \frac{1}{2}\int_0^2 \sec^3\theta\, d\theta$$

$2x = \tan\theta \qquad \sqrt{1+4x^2} = \sec\theta$

$dx = \frac{1}{2}\sec^2\theta d\theta$

$$= \frac{1}{4}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_0^2$$

$$= \frac{1}{4}\left(2x\sqrt{1+4x^2} + \ln\left|2x + \sqrt{1+4x^2}\right|\right)\Big|_0^2$$

$$= \frac{1}{4}\left(4\sqrt{17} + \ln\left|4+\sqrt{17}\right|\right)$$

$$= \sqrt{17} + \frac{1}{4}\ln\left(4+\sqrt{17}\right)$$

$u = \sec x \qquad dv = \sec^2 x dx$

$du = \sec x \tan x dx \qquad v = \tan x$

$$\int \sec^3 x dx = \sec x \tan x - \int \tan x\left(\sec x \tan x dx\right)$$

$$= \sec x \tan x - \int \tan^2 x \sec x dx$$

$$= \sec x \tan x - \int\left(\sec^2 x - 1\right)\sec x\, dx$$

$$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$2\int \sec^3 x\, dx = \sec x \tan x + \ln|\sec x + \tan x|$$

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x|$$

Exercise

Consider the region bounded by the graph $y=\sqrt{x\tan^{-1}x}$ and $y=0$ for $0\le x\le 1$. Find the volume of the solid formed by revolving this region about the x-axis.

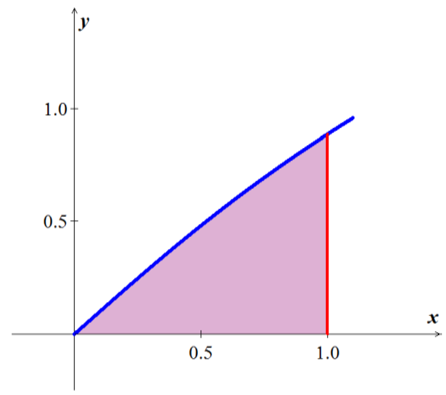
Solution

$$V=\pi\int_0^1\left(\sqrt{x\tan^{-1}x}\right)^2dx$$

$$=\pi\int_0^1 x\tan^{-1}x\,dx$$

$u=\tan^{-1}x \qquad dv=xdx$

$du=\frac{1}{x^2+1}dx \qquad v=\frac{1}{2}x^2$



$$V=\pi\left(\frac{1}{2}\left[x^2\tan^{-1}x\right]_0^1-\frac{1}{2}\int_0^1\frac{x^2}{1+x^2}dx\right)$$

$$=\frac{\pi}{2}\left(\left(1\tan^{-1}1-0\right)-\int_0^1\left(1-\frac{1}{1+x^2}\right)dx\right)$$

$$=\frac{\pi}{2}\left(\frac{\pi}{4}-\int_0^1 dx+\int_0^1\frac{1}{1+x^2}dx\right)$$

$$=\frac{\pi}{2}\left(\frac{\pi}{4}-\left[x\right]_0^1+\left[\tan^{-1}x\right]_0^1\right)$$

$$=\frac{\pi}{2}\left(\frac{\pi}{4}-1+\tan^{-1}1\right)$$

$$=\frac{\pi}{2}\left(\frac{\pi}{4}-1+\frac{\pi}{4}\right)$$

$$=\frac{\pi}{2}\left(\frac{\pi}{2}-1\right)$$

$$=\frac{\pi^2}{4}-\frac{\pi}{2}$$

Exercise

Use two approach to show that the area of a cap (or segment) of a circle of radius r subtended by an angle θ is given by

$$A_{seg}=\frac{1}{2}r^2(\theta-\sin\theta)$$

a) Find the area using geometry (no calculus).

b) Find the area using calculus

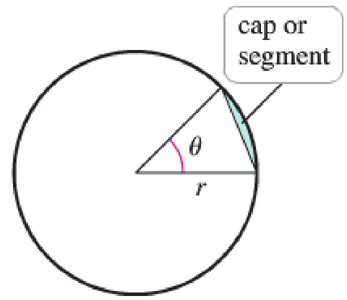
Solution

a) Area of a segment (*cap*) = Area of a sector ***minus*** Area of the isosceles triangle

The area of a sector: $A = \frac{1}{2}\theta r^2$

Area of the isosceles triangle: $A = \frac{1}{2}r^2 \sin\theta$

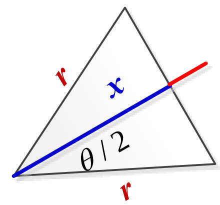
$$A_{seg} = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta = \frac{1}{2}r^2(\theta - \sin\theta)$$



b) $0 \le \theta \le \pi \quad \to \quad 0 \le \frac{\theta}{2} \le \frac{\pi}{2}$

$x = r\cos\frac{\alpha}{2} \quad \to \quad dx = -\frac{1}{2}r\sin\frac{\alpha}{2}\, d\alpha$

$\sqrt{r^2 - x^2} = r\sin\frac{\alpha}{2}$



$$\begin{aligned}
A_{cap} &= 2\int_{r\cos\theta/2}^{r} \sqrt{r^2 - x^2}\, dx \\
&= 2\int_{\theta}^{0} \left(r\sin\frac{\alpha}{2}\right)\left(-\frac{1}{2}r\sin\frac{\alpha}{2}\right) d\alpha \\
&= r^2\int_0^{\theta} \left(\sin^2\frac{\alpha}{2}\right) d\alpha \\
&= \frac{1}{2}r^2\int_0^{\theta} (1 - \cos\alpha)\, d\alpha \\
&= \frac{1}{2}r^2(\alpha - \sin\alpha)\Big|_0^{\theta} \\
&= \boxed{\frac{1}{2}r^2(\theta - \sin\theta)}
\end{aligned}$$

Exercise

A lune is a crescent-shaped region bounded by the arcs of two circles. Let C_1 be a circle of radius 4 centered at the origin. Let C_2 be a circle of radius 3 centered at the point $(2,\ 0)$. Find the area of the lune that lies inside C_1 and outside C_2.

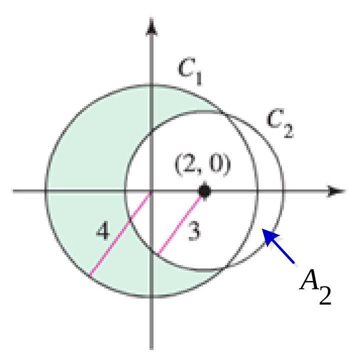
Solution

$C_1 \quad \to \quad x^2 + y^2 = 16 \ \Rightarrow\ y^2 = 16 - x^2$

$C_2 \quad \to \quad (x-2)^2 + y^2 = 9 \ \Rightarrow\ y^2 = 9 - (x-2)^2$

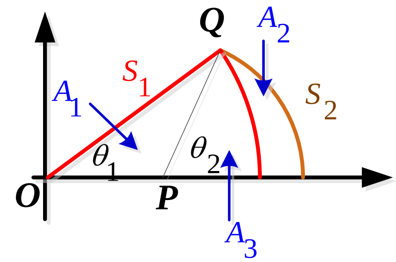
$16 - x^2 = 9 - x^2 + 4x - 4$

$11 = 4x \quad \to \quad \boxed{x = \frac{11}{4}} \ \Rightarrow \boxed{y = \pm\frac{\sqrt{135}}{4} = \pm\frac{3\sqrt{15}}{4}}$



For ***sector*** C_1: $\theta_1 = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{3\sqrt{15}}{11}$

$$\text{Area: } S_1 = \frac{1}{2}r^2\theta_1 = 8\tan^{-1}\left(\frac{3\sqrt{15}}{11}\right)$$



For ***sector*** C_2: $x_2 = \frac{11}{4} - 2 = \frac{3}{4}$

$$\theta_2 = \tan^{-1}\frac{y}{x_2} = \tan^{-1}\sqrt{15}$$

$$\text{Area: } S_2 = \frac{1}{2}r_2^2\theta_2 = \frac{9}{2}\tan^{-1}\left(\sqrt{15}\right)$$

$OQ = 4, \quad PQ = 3, \quad OP = 2$

$$Area(\Delta APQ) = A_1 = \frac{1}{2}(4)(2)\sin\theta_1 = 4\frac{y}{4} = \boxed{\frac{3\sqrt{15}}{4}}$$

$$A_2 = S_2 - S_1 + A_1$$

$$= \boxed{\frac{9}{2}\tan^{-1}\left(\sqrt{15}\right) - 8\tan^{-1}\left(\frac{3\sqrt{15}}{11}\right) + \frac{3\sqrt{15}}{4}}$$

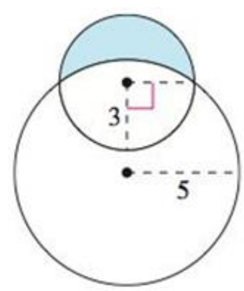
$$A_{lune} = A_{C_1} - A_{C_2} + 2A_2$$

$$= 16\pi - 9\pi + 9\tan^{-1}\left(\sqrt{15}\right) - 16\tan^{-1}\left(\frac{3\sqrt{15}}{11}\right) + \frac{3\sqrt{15}}{2}$$

$$= \boxed{7\pi + 9\tan^{-1}\left(\sqrt{15}\right) - 16\tan^{-1}\left(\frac{3\sqrt{15}}{11}\right) + \frac{3\sqrt{15}}{2}} \quad \boxed{\approx 26.66} \quad unit^2$$

Exercise

The crescent-shaped region bounded by two circles forms a lune. Find the area of the lune given that the radius of the smaller circle is 3 and the radius of the larger circle is 5.



Solution

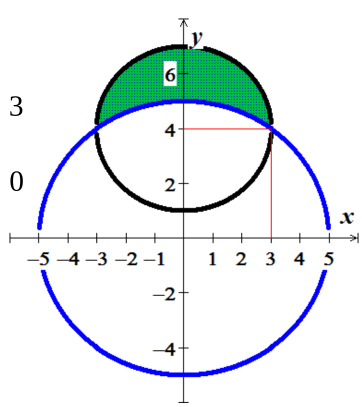
Large Circle: $x^2 + y^2 = 25 \;\rightarrow\; y = \sqrt{25 - x^2}$

Small Circle: $r = 3 \;\rightarrow\; y = \sqrt{25 - 9} = \boxed{4}$

$$x^2 + (y-4)^2 = 9 \;\rightarrow\; y = 4 + \sqrt{9 - x^2}$$

$$A = 2\int_0^3 \left(4 + \sqrt{9 - x^2} - \sqrt{25 - x^2}\right)dx$$

$$= 2\left[4x + \frac{1}{2}\left(9\arcsin\left(\frac{x}{3}\right) + x\sqrt{9 - x^2}\right) - \frac{1}{2}\left(25\arcsin\left(\frac{x}{5}\right) + x\sqrt{25 - x^2}\right)\right]_0^3$$



$$= 2\left[12 + \frac{1}{2}\left(9\frac{\pi}{2}\right) - \frac{1}{2}\left(25\arcsin\left(\frac{3}{5}\right) + 12\right)\right]$$

$$= \boxed{12 + \frac{9\pi}{2} - 25\arcsin\left(\frac{3}{5}\right)}$$

Exercise

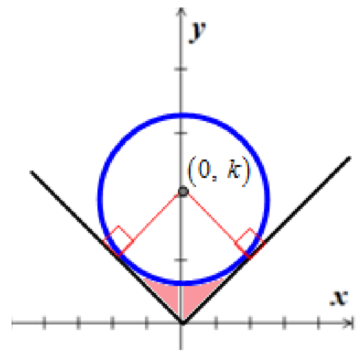
The surface of a machine part is the region between the graphs of $y = |x|$ *and* $x^2 + (y-k)^2 = 25$

a) Find k when the circle is tangent to the graph of $y = |x|$

b) Find the area of the surface of the machine part.

c) Find the area of the surface of the machine part as a function of the radius r of the circle.

Solution



a) $x^2 + (y-k)^2 = 25 \quad \rightarrow \quad \underline{r = 5}$

$k^2 = 5^2 + 5^2 = 50 \quad \rightarrow \quad \underline{k = 5\sqrt{2}}$

b) $Area = area\ square - \frac{1}{4}(area\ circle)$

$$= 5^2 - \frac{1}{4}\pi 5^2$$

$$= 25\left(1 - \frac{\pi}{4}\right)\ unit^2$$

c) $Area = area\ square - \frac{1}{4}(area\ circle)$

$$= r^2 - \frac{1}{4}\pi r^2$$

$$= r^2\left(1 - \frac{\pi}{4}\right)\ unit^2$$

Exercise

Consider the function $f(x) = \left(9 + x^2\right)^{-1/2}$ and the region $\boldsymbol{R}$ on the interval [0, 4].

a) Find the area of R.

b) Find the volume of the solid generated when R is revolved about the *x-axis*.

c) Find the volume of the solid generated when R is revolved about the *y-axis*.

Solution

$x = 3\tan\theta \quad \rightarrow \quad dx = 3\sec^2\theta\, d\theta$

$\sqrt{9 + x^2} = 3\sec\theta$

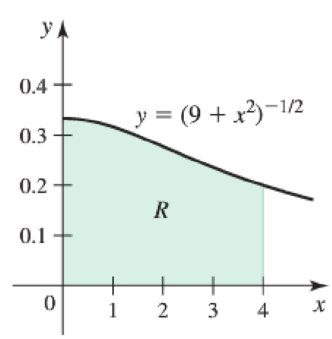
a) $A = \int_0^4 \frac{dx}{\sqrt{9 + x^2}}$

$$= \int_0^4 \frac{3\sec^2\theta\, d\theta}{3\sec\theta}$$

$$= \int_0^4 \sec\theta\, d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right|\Big|_0^4$$

$$= \ln\left|\frac{\sqrt{9 + x^2}}{3} + \frac{x}{3}\right|\Big|_0^4$$



$$= \ln\left(\frac{5}{3} + \frac{4}{3}\right) - 0$$

$$\underline{= \ln 3} \quad unit^2$$

b) $V = \pi \int_0^4 \frac{dx}{9 + x^2}$ $\qquad x = 3\tan\theta \rightarrow dx = 3\sec^2\theta\, d\theta$

$9 + x^2 = 9\sec^2\theta$

$$= \pi \int_0^4 \frac{3\sec^2\theta\, d\theta}{9\sec^2\theta}$$

$$= \frac{\pi}{3} \int_0^4 d\theta$$

$$= \frac{\pi}{3}\theta \Big|_0^4$$

$$= \frac{\pi}{3}\tan^{-1}\frac{x}{3}\Big|_0^4$$

$$\underline{= \frac{\pi}{3}\tan^{-1}\frac{4}{3}} \quad unit^3$$

c) $V = 2\pi \int_0^4 \frac{x}{\sqrt{9 + x^2}} dx$ $\qquad d\left(9 + x^2\right) = 2x dx$

$$= \pi \int_0^4 \left(9 + x^2\right)^{-1/2} d\left(9 + x^2\right)$$

$$= 2\pi\left(9 + x^2\right)^{1/2}\Big|_0^4$$

$$= 2\pi(5 - 3)$$

$$\underline{= 4\pi} \quad unit^3$$

Exercise

A total of $\boldsymbol{Q}$ is distributed uniformly on a line segment of length $2L$ along the *y-axis*. The x-component of the electric field at a point $(a,\ 0)$ is given by

$$E_x = \frac{kQa}{2L} \int_{-L}^{L} \frac{dy}{\left(a^2 + y^2\right)^{3/2}}$$

Where k is a physical constant and $a > 0$

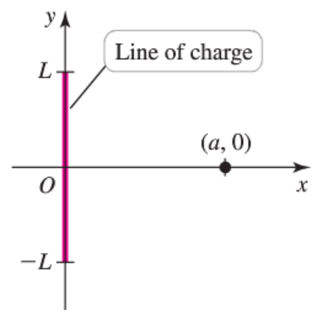
a) Confirm that $E_x(a) = \dfrac{kQ}{a\sqrt{a^2 + L^2}}$

b) Letting $\rho = \dfrac{Q}{2L}$ be the charge density on the line segment, show that if $L \to \infty$, *then* $E_x = \dfrac{2k\rho}{a}$

Solution

$y = a\tan\theta \;\to dy = a\sec^2\theta\, d\theta$

$\sqrt{a^2+y^2} = a\sec\theta$



a) $$E_x = \frac{kQa}{2L}\int_{-L}^{L} \frac{dy}{\left(a^2+y^2\right)^{3/2}}$$

$$= \frac{kQa}{2L}\int_{-L}^{L} \frac{a\sec^2\theta\, d\theta}{a^3\sec^3\theta}$$

$$= \frac{kQ}{2aL}\int_{-L}^{L} \frac{d\theta}{\sec\theta}$$

$$= \frac{kQ}{2aL}\int_{-L}^{L} \cos\theta d\theta$$

$$= \frac{kQ}{2aL}\sin\theta \Big|_{-L}^{L}$$

$$= \frac{kQ}{2aL}\left(\frac{y}{\sqrt{a^2+y^2}}\right)\Big|_{-L}^{L}$$

$$= \frac{kQ}{2aL}\left(\frac{2L}{\sqrt{a^2+L^2}}\right)$$

$$\boxed{= \frac{kQ}{a\sqrt{a^2+L^2}}}$$

b) Let $\rho = \dfrac{Q}{2L} \quad \to \quad Q = 2\rho L$

$$E_x(a) = \frac{kQa}{2L}\lim_{L\to\infty}\int_{-L}^{L} \frac{dy}{\left(a^2+y^2\right)^{3/2}}$$

$$= \frac{kQa}{2L}\lim_{L\to\infty}\left(\frac{2L}{a^2\sqrt{a^2+L^2}}\right)$$

$$= k\rho a\ \frac{2}{a^2}$$

$$\boxed{= \frac{2k\rho}{a}}$$

Exercise

A long, straight wire of length $2L$ on the *y-axis* carries a current I. according to the Biot-Savart Law, the magnitude of the field due to the current at a point $(a,\ 0)$ is given by

$$B(a) = \frac{\mu_0 I}{4\pi}\int_{-L}^{L} \frac{\sin\theta}{r^2}dy$$

Where μ_0 is a physical constant, $a > 0$, and θ, r, and y are related to the figure *a)* Show that the magnitude of the magnetic field at $(a,\ 0)$ is $B(a)=\dfrac{\mu_0 IL}{2\pi a\sqrt{a^2+L^2}}$

b) What is the magnitude of the magnetic field at $(a,\ 0)$ due to an infinitely long wire $(L\to\infty)$?

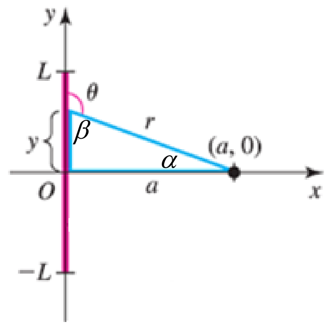
Solution

a) $\beta=\pi-\theta \quad \& \quad \alpha+\beta=\frac{\pi}{2}$

$$\sin\theta=\sin(\pi-\beta)=\sin\left(\frac{\pi}{2}+\alpha\right)=\cos\alpha=\frac{a}{r}$$

$$r^2=y^2+a^2$$

$$\frac{\sin\theta}{r^2}=\frac{a}{r^3}=\frac{a}{\left(a^2+y^2\right)^{3/2}}$$



$$B(a)=\frac{\mu_0 I}{4\pi}\int_{-L}^{L}\frac{\sin\theta}{r^2}dy$$

$$=\frac{\mu_0 I}{4\pi}\int_{-L}^{L}\frac{a}{\left(a^2+y^2\right)^{3/2}}dy$$

$y=a\tan u \qquad \sqrt{a^2+y^2}=a\sec u$

$dy=a\sec^2 u\ du$

$$=\frac{\mu_0 I}{2\pi}\int_0^L\frac{a^2\sec^2 u\ du}{a^3\sec^3 u}$$

$$=\frac{\mu_0 I}{2a\pi}\int_0^L\frac{1}{\sec u}du$$

$$=\frac{\mu_0 I}{2a\pi}\int_0^L\cos u\ du$$

$$=\frac{\mu_0 I}{2a\pi}\sin u\Big|_0^L$$

$$=\frac{\mu_0 I}{2a\pi}\frac{y}{\sqrt{a^2+y^2}}\Big|_0^L$$

$$\boxed{=\frac{\mu_0 IL}{2a\pi\sqrt{a^2+L^2}}}$$

b) $$\lim_{L\to\infty}B(a)=\lim_{L\to\infty}\frac{\mu_0 IL}{2a\pi\sqrt{a^2+L^2}}$$

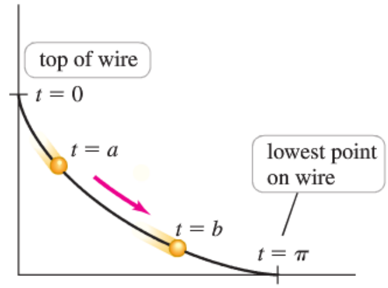
$$=\frac{\mu_0 I}{2a\pi}\lim_{L\to\infty}\frac{L}{\sqrt{a^2+L^2}}$$

$$\lim_{L\to\infty}\frac{L}{\sqrt{a^2+L^2}}=\lim_{L\to\infty}\frac{L}{\sqrt{L^2}}=1$$

$$\boxed{=\frac{\mu_0 I}{2a\pi}}$$

Exercise

The cycloid is the curve traced by a point on the rim of a rolling wheel. Imagine a wire shaped like an inverted cycloid.



A bead sliding down this wire without friction has some remarkable properties. Among all wire shapes, the cycloid is the shape that produces the fastest descent time. It can be shown that the descent time between any two points $0 \le a < b \le \pi$ on the curve is

$$descent\ time = \int_a^b \sqrt{\frac{1-\cos t}{g(\cos a - \cos t)}}\,dt$$

Where g is the acceleration due to gravity, $t = 0$ corresponds to the top of the wire, and $t = \pi$ corresponds to the lowest point on the wire.

a) Find the descent time on the interval $[a,\ b]$.

b) Show that when $b = \pi$, the descent time is the same for all values of a; that is, the descent time to the bottom of the wire is the same for all starting points.

Solution

a) $$\int_a^b \sqrt{\frac{1-\cos t}{g(\cos a - \cos t)}}\,dt = \int_a^b \sqrt{\frac{(1-\cos t)(1+\cos t)}{g(\cos a - \cos t)(1+\cos t)}}\,dt$$

$$= \frac{1}{\sqrt{g}}\int_a^b \sqrt{\frac{(1-\cos^2 t)}{\cos a + (\cos a - 1)\cos t - \cos^2 t}}\,dt$$

$$= \frac{1}{\sqrt{g}}\int_a^b \frac{\sin t}{\sqrt{\cos a + \left(\frac{\cos a - 1}{2}\right)^2 - \left(\frac{\cos a - 1}{2}\right)^2 + (\cos a - 1)\cos t - \cos^2 t}}\,dt$$

$$= \frac{1}{\sqrt{g}}\int_a^b \frac{\sin t}{\sqrt{\cos a + \left(\frac{\cos a - 1}{2}\right)^2 - \left(\left(\frac{\cos a - 1}{2}\right) - \cos t\right)^2}}\,dt$$

Let: $v = \sqrt{\cos a + \left(\frac{\cos a - 1}{2}\right)^2}$

$$= \frac{1}{2}\sqrt{4\cos a + \cos^2 a - 2\cos a + 1}$$

$$= \frac{1}{2}(\cos a + 1)$$

$$\frac{\cos a - 1}{2} - \cos t = v\sin\theta \quad \to \quad \sin t\ dt = v\cos\theta\ d\theta$$

$$\sqrt{v - \left(\left(\frac{\cos a - 1}{2}\right) - \cos t\right)^2} = v\cos\theta$$

$$= \frac{1}{\sqrt{g}}\int_a^b \frac{v\cos\theta}{v\cos\theta}\ d\theta$$

$$= \frac{1}{\sqrt{g}}\theta\ \Big|_a^b$$

$$\theta = \sin^{-1}\left(\frac{\cos a - 1 - 2\cos t}{2}\frac{2}{1+\cos a}\right)$$

$$= \frac{1}{\sqrt{g}}\sin^{-1}\left(\frac{\cos a - 1 - 2\cos t}{1+\cos a}\right)\ \Big|_a^b$$

$$= \frac{1}{\sqrt{g}}\left(\sin^{-1}\left(\frac{\cos a - 1 - 2\cos b}{1+\cos a}\right) - \sin^{-1}(-1)\right)$$

$$= \frac{1}{\sqrt{g}}\left(\sin^{-1}\left(\frac{\cos a - 1 - 2\cos b}{1+\cos a}\right) + \frac{\pi}{2}\right)$$

b) $\frac{1}{\sqrt{g}}\left(\sin^{-1}\left(\frac{\cos a - 1 - 2\cos b}{1+\cos a}\right) + \frac{\pi}{2}\right)\Big|_{b=\pi} = \frac{1}{\sqrt{g}}\left(\sin^{-1}\left(\frac{\cos a - 1 + 2}{1+\cos a}\right) + \frac{\pi}{2}\right)$

$$= \frac{1}{\sqrt{g}}\left(\sin^{-1}(1) + \frac{\pi}{2}\right)$$

$$= \frac{\pi}{\sqrt{g}}$$

Exercise

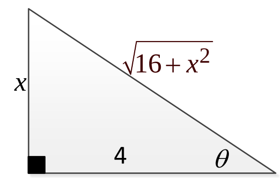
Find the area of the region bounded by the curve $f(x) = \left(16 + x^2\right)^{-3/2}$ and the x-*axis* on the interval $[0,\ 3]$

Solution

$$A = \int_0^3 \frac{dx}{\left(16 + x^2\right)^{3/2}} \qquad x = 4\tan\theta \quad \to dx = 4\sec^2\theta\ d\theta \qquad 16 + x^2 = 16\sec^2\theta$$

$$= \int_0^3 \frac{4\sec^2\theta d\theta}{\left(16\sec^2\theta\right)^{3/2}}$$

$$= \int_0^3 \frac{4\sec^2\theta}{4^3\sec^3\theta}d\theta$$



$$= \frac{1}{16}\int_0^3 \cos\theta \, d\theta$$

$$= \frac{1}{16}\sin\theta \Big|_0^3$$

$$= \frac{1}{16}\frac{x}{\sqrt{16+x^2}}\Big|_0^3$$

$$= \frac{1}{16}\left(\frac{3}{5}-0\right)$$

$$= \frac{3}{80}\ unit^2$$

Exercise

Find the length of the curve $y = ax^2$ from $x = 0$ to $x = 10$, where $a > 0$ is a real number.

Solution

$$1+(y')^2 = 1+(2ax)^2$$

$$L = \int_0^{10} \sqrt{1+4a^2x^2}\ dx$$

$$= \int_0^{10} 2a\sqrt{\frac{1}{4a^2}+x^2}\ dx \qquad x=\frac{1}{2a}\tan\theta \quad \frac{1}{4a^2}+x^2=\frac{1}{4a^2}\sec^2\theta$$

$$= \int_0^{10} 2a\frac{1}{2a}\sec\theta\frac{1}{4a^2}\sec^2\theta\ d\theta \qquad dx = \frac{1}{4a^2}\sec^2\theta\ d\theta$$

$$= \frac{1}{2a}\int_0^{10} \sec^3\theta\ d\theta$$

$$u=\sec x \qquad dv=\sec^2 xdx$$

$$du=\sec x\tan xdx \qquad v=\tan x$$

$$\int \sec^3 xdx = \sec x\tan x - \int \tan x(\sec x\tan xdx)$$

$$= \sec x\tan x - \int \tan^2 x\sec xdx$$

$$= \sec x\tan x - \int \left(\sec^2 x - 1\right)\sec x\ dx$$

$$= \sec x\tan x - \int \sec^3 x\ dx + \int \sec x\ dx$$

$$\int \sec^3 x\ dx + \int \sec^3 x\ dx = \sec x\tan x - \int \sec^3 x\ dx + \int \sec^3 x\ dx + \int \sec x\ dx$$

$$2\int \sec^3 x\ dx = \sec x\tan x + \int \sec x\ dx$$

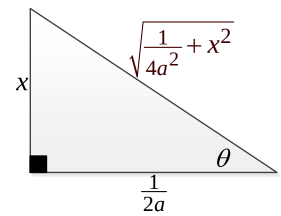
$$= \sec x\tan x + \ln|\sec x+\tan x|$$

$$= \frac{1}{4a}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right)\Big|_0^{10}$$

$$= \frac{1}{4a}\left(2a\sqrt{\frac{1}{4a^2} + x^2}\,(2ax) + \ln\left|\sqrt{1+4a^2x^2} + 2ax\right|\right)\Big|_0^{10}$$

$$= \frac{1}{4a}\left((2ax)\sqrt{1+4a^2x^2} + \ln\left|\sqrt{1+4a^2x^2} + 2ax\right|\right)\Big|_0^{10}$$

$$\boxed{= \frac{1}{4a}\left((20a)\sqrt{1+400a^2} + \ln\left|\sqrt{1+400a^2} + 20a\right|\right)}$$



Exercise

Find the arc length of the graph of $f(x) = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$

Solution

$$1 + (f')^2 = 1 + x^2$$

$$L = \int_0^1 \sqrt{1+x^2}\, dx$$

$x = \tan\theta$ $\qquad \sqrt{x^2+1} = \sec\theta$

$dx = \sec^2\theta\, d\theta$

$$= \int_0^1 \sec^3\theta\, d\theta$$

$$= \frac{1}{2}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right)\Big|_0^1$$

$$= \frac{1}{2}\left(x\sqrt{x^2+1} + \ln\left|x + \sqrt{x^2+1}\right|\right)\Big|_0^1$$

$$\boxed{= \frac{1}{2}\left(\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right)}$$

Exercise

A projectile is launched from the ground with an initial speed V at an angle θ from the horizontal. Assume that the x-*axis* is the horizontal ground and y is the height above the ground. Neglecting air resistance and letting g be the acceleration due to gravity, it can be shown that the trajectory of the projectile is given by

$$y = -\frac{1}{2}kx^2 + y_{max} \quad \textit{where} \quad k = \frac{g}{(V\cos\theta)^2}$$

$$\textit{and} \qquad y_{max} = \frac{(V\sin\theta)^2}{2g}$$

a) Note that the high point of the trajectory occurs at $\left(0,\ y_{max}\right)$. If the projectile is on the ground at $(-a,\ 0)$ and $(a,\ 0)$, what is a?

b) Show that the length of the trajectory (arc length) is $2\int_0^a \sqrt{1+k^2x^2}\,dx$

c) Evaluate the arc length integral and express your result in the terms of V, g, and θ.

d) For fixed value of V and g, show that the launch angle θ that maximizes the length of the trajectory satisfies $(\sin\theta)\ln(\sec\theta+\tan\theta)=1$

Solution

a) At $(\pm a,\ 0) \to\ y=0=-\frac{1}{2}ka^2+y_{max}$

$$a^2=\frac{2}{k}y_{max} \quad\Rightarrow\quad \boxed{a=\sqrt{\frac{2y_{max}}{k}}}$$

b) $y'=-kx \quad\Rightarrow\ 1+(y')^2=1+k^2x^2$

$$L=\int_{-a}^{a}\sqrt{1+k^2x^2}\,dx \qquad \text{since } y(x) \text{ is an even function}$$

$$=2\int_0^a\sqrt{1+k^2x^2}\,dx$$

c) $L=2\int_0^a\sqrt{1+k^2x^2}\,dx$ $\qquad x=\frac{1}{k}\tan\theta \ \Rightarrow\ dx=\frac{1}{k}\sec^2\theta\,d\theta;\quad 1+k^2x^2=\sec^2\theta$

$$=2\int_0^a\frac{1}{k}\sec\theta\sec^2\theta\,d\theta$$

$$=\frac{2}{k}\int_0^a\sec^3\theta\,d\theta$$

$$=\frac{1}{k}\left(\sec\theta\tan\theta+\ln\left|\sec\theta+\tan\theta\right|\right)\Big|_0^a$$

$$=\frac{1}{k}\left(\sqrt{1+k^2x^2}\,(kx)+\ln\left|\sqrt{1+k^2x^2}+kx\right|\right)\Big|_0^a$$

$$\boxed{=\frac{1}{k}\left(ak\sqrt{1+k^2a^2}+\ln\left|\sqrt{1+k^2a^2}+ka\right|\right)}$$

$$a=\sqrt{\frac{2}{k}\frac{(V\sin\theta)^2}{2g}} =\frac{V\sin\theta}{\sqrt{g\frac{g}{(V\cos\theta)^2}}} =\frac{V^2}{g}\sin\theta\cos\theta$$

$$k=\frac{g}{(V\cos\theta)^2}$$

$$ak=\tan\theta$$

$$L(\theta)=\frac{(V\cos\theta)^2}{g}\left(\tan\theta\sqrt{1+\tan^2\theta}+\ln\left|\sqrt{1+\tan^2\theta}+\tan\theta\right|\right)$$

$$=\frac{V^2\cos^2\theta}{g}\left(\tan\theta\sec\theta+\ln\left|\sec\theta+\tan\theta\right|\right)$$

$$=\frac{V^2}{g}\sin\theta+\frac{V^2}{g}\cos^2\theta\ln\left|\sec\theta+\tan\theta\right|$$

$$\boxed{=\frac{V^2}{g}\left(\sin\theta+\cos^2\theta\sinh^{-1}(\tan\theta)\right)}$$

d) $L'(\theta)=\dfrac{V^2}{g}\left(\cos\theta-2\cos\theta\sin\theta\sinh^{-1}(\tan\theta)+\cos^2\theta\dfrac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\right)$

$$=\frac{V^2}{g}\left(\cos\theta-2\cos\theta\sin\theta\sinh^{-1}(\tan\theta)+\cos^2\theta\sec\theta\right)$$

$$=\frac{2V^2\cos\theta}{g}\left(1-\sin\theta\sinh^{-1}(\tan\theta)\right)=0$$

$$\sin\theta\sinh^{-1}(\tan\theta)=1$$

$$\sin\theta\ln(\sec\theta+\tan\theta)=1 \quad \checkmark$$

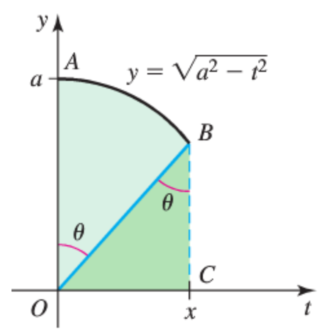
Exercise

Let $F(x)=\int_0^x\sqrt{a^2-t^2}\,dt$. The figure shows that $F(x)=$ *area of sector OAB* + *area of triangle OBC*

a) Use the figure to prove that $F(x)=\dfrac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\dfrac{x\sqrt{a^2-x^2}}{2}$

b) Conclude that $\int\sqrt{a^2-x^2}\,dx=\dfrac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\dfrac{x\sqrt{a^2-x^2}}{2}+C$

Solution



a) Area of sector OAB is $\frac{1}{2}\theta a^2$

From the triangle OBC: $\sin\theta=\frac{x}{a} \ \rightarrow\ \theta=\sin^{-1}\frac{x}{a}$

$$|BC|=\sqrt{a^2-x^2}$$

Area of sector OAB is $\frac{1}{2}a^2\sin^{-1}\frac{x}{a}$

Area of triangle OBC: $\frac{1}{2}x\sqrt{a^2-x^2}$

$F(x)=$ *area of sector OAB* + *area of triangle OBC*

$$=\frac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\frac{x\sqrt{a^2-x^2}}{2}$$

b) $\dfrac{d}{dx}\left(\dfrac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\dfrac{x\sqrt{a^2-x^2}}{2}+C\right)=\dfrac{a^2}{2}\dfrac{\frac{1}{a}}{\sqrt{1-\left(\frac{x}{a}\right)^2}}+\dfrac{1}{2}\sqrt{a^2-x^2}-\dfrac{1}{2}\dfrac{x^2}{\sqrt{a^2-x^2}}$

$$=\frac{1}{2}\frac{a^2}{\sqrt{a^2-x^2}}+\frac{1}{2}\frac{a^2-2x^2}{\sqrt{a^2-x^2}}$$

$$= \frac{1}{2}\frac{2a^2 - 2x^2}{\sqrt{a^2 - x^2}}$$

$$= \frac{a^2 - x^2}{\sqrt{a^2 - x^2}}$$

$$= \sqrt{a^2 - x^2}$$

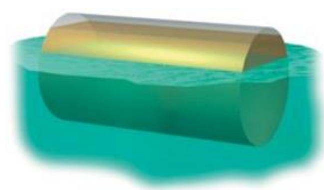
By the antiderivative:

$$\int \sqrt{a^2 - x^2}\,dx = \frac{a^2 \sin^{-1}\left(\frac{x}{a}\right)}{2} + \frac{x\sqrt{a^2 - x^2}}{2} + C \quad \checkmark$$

Exercise

A sealed barrel of oil (weighing 48 pounds per cubic foot) is floating in seawater (weighing 64 pounds per cubic foot). The barrel is not completely full of oil. With the barrel lying on its side, the top 0.2 *foot* of the barrel is empty.

Compare the fluid forces against one end of the barrel from the inside and from the outside.



Solution

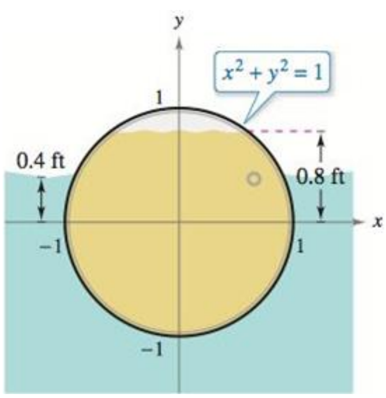
$$x^2 + y^2 = 1 \quad \to \quad 2x = 2\sqrt{1 - y^2}$$

$$F_{inside} = 48\int_{-1}^{0.8} (0.8 - y)(2)\sqrt{1 - y^2}\ dy \qquad\qquad F = w\int_c^d h(y)L(y)dy$$

$$= 76.8\int_{-1}^{0.8} \sqrt{1 - y^2}\ dy - 96\int_{-1}^{0.8} y\sqrt{1 - y^2}\ dy$$

$$= 76.8\int_{-1}^{0.8} \sqrt{1 - y^2}\ dy + 48\int_{-1}^{0.8} \left(1 - y^2\right)^{1/2}\ d\left(1 - y^2\right) \qquad \begin{matrix} y = \sin\theta & \sqrt{1 - y^2} = \cos\theta \\ dy = \cos\theta\, d\theta & \end{matrix}$$

$$= 76.8\int_{-1}^{0.8} \cos^2\theta\ d\theta + 32\left(1 - y^2\right)^{3/2}\Big|_{-1}^{0.8}$$



$$= 38.4\int_{-1}^{0.8} (1 + \cos 2\theta)\ d\theta + 32(0.16)^{3/2}$$

$$= 38.4\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_{-1}^{0.8} + 32(0.4)^3$$

$$= 38.4\left(\arcsin y + y\sqrt{1 - y^2}\right)\Big|_{-1}^{0.8} + 2.048$$

$$= 38.4\left(\arcsin 0.8 + 0.32 + \frac{\pi}{2}\right) + 2.048$$

$$\approx 121.3\ \ lbs$$

$$F_{outside} = 64\int_{-1}^{0.4}(0.4-y)(2)\sqrt{1-y^2}\ dy \qquad\qquad F = w\int_c^d h(y)L(y)dy$$

$$= 51.2\int_{-1}^{0.4}\sqrt{1-y^2}\ dy - 128\int_{-1}^{0.4} y\sqrt{1-y^2}\ dy$$

$$= 25.6\left(\arcsin y + y\sqrt{1-y^2}\right)\Big|_{-1}^{0.4} + \frac{128}{3}\left(1-y^2\right)^{3/2}\Big|_{-1}^{0.4}$$

$$\boxed{\approx 93.0\ \ lbs}$$

Exercise

The axis of a storage tank in the form of a right circular cylinder is horizontal. The radius and length of the tank are 1 *meter* and 3 *meters*, respectively.

a) Determine the volume of fluid in the tank as a function of its depth *d*.
b) Graph the function in part (a).
c) Design a dip stick for the tank with markings of $\frac{1}{4},\ \frac{1}{2},\ and\ \frac{3}{4}$
d) Fluid is entering the tank at a rate of $\frac{1}{4}\ m^3/min$. Determine the rate of change of the depth of the fluid as a function of its depth *d*.
e) Graph the function in part (*d*).\When will the rate of change of the depth be minimum?

Solution

a) Consider the center at $(0,\ 1)$: $x^2+(y-1)^2=1\ \rightarrow\ x=\sqrt{1-(y-1)^2}$

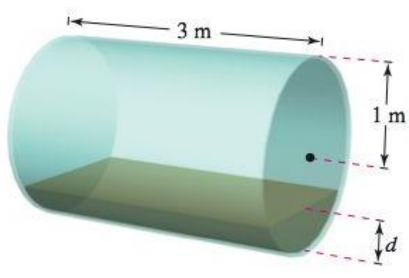
The depth: $0 \le d \le 2$

$$V = \int_0^d (3)\left(2\sqrt{1-(y-1)^2}\right)dy$$

$$= 6\int_0^d \sqrt{1-(y-1)^2}\ d(y-1) \qquad\qquad y-1=\sin\theta \qquad \sqrt{1-(y-1)^2}=\cos\theta$$

$$d(y-1)=\cos\theta d\theta$$

$$= 6\int_0^d \cos^2\theta\ d\theta$$



$$= 3\int_0^d (1+\cos 2\theta)\ d\theta$$

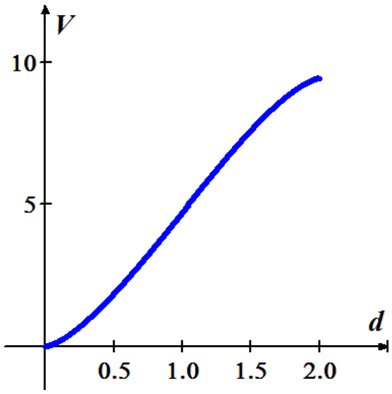
$$= 3\left(\theta+\frac{1}{2}\sin 2\theta\right)_0^d$$

$$= 3(\theta+\sin\theta\cos\theta)\Big|_0^d$$

$$= 3\left(\arcsin(y-1)+(y-1)\sqrt{1-(y-1)^2}\right)_0^d$$

$$= 3\arcsin(d-1) + 3(d-1)\sqrt{2d-d^2} + \frac{3\pi}{2}$$

b)



c) The full tank holds $3\pi\ m^3$

A dip stick for the tank with markings of $\frac{1}{4},\ \frac{1}{2},\ and\ \frac{3}{4}$

The horizontal lines are: $y = \frac{3\pi}{4},\quad y = \frac{3\pi}{2},\quad y = \frac{9\pi}{4}$

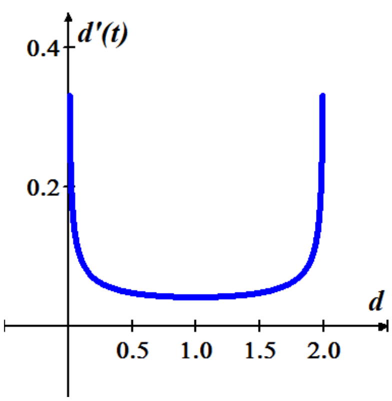
Intersect the curve at $d = 0.596,\quad d = 1.0,\quad d = 1.404$

d) $V = 6\int_0^d \sqrt{1-(y-1)^2}\ dy \quad \rightarrow \quad \frac{dV}{dt} = \frac{dV}{dd}\frac{dd}{dt}$

$$\frac{dV}{dt} = 6\sqrt{1-(d-1)^2}\cdot d'(t) = \frac{1}{4}$$

$$d'(t) = \frac{1}{24\sqrt{1-(d-1)^2}}$$

e)



From the graph, the minimum occurs at $d = 1$, which is the widest part of the tank.

Exercise

The field strength H of a magnet of length $2L$ on a particle r units from the center of the magnet is

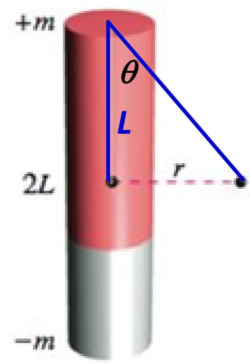
$$H = \frac{2mL}{\left(r^2+L^2\right)^{3/2}}$$

Where $\pm m$ are the poles of the magnet.

Find the average field strength as the particle moves from 0 to R units from the center by evaluating the integral

$$\frac{1}{R}\int_0^R \frac{2mL}{\left(r^2+L^2\right)^{3/2}}dr$$

Solution



$$r = L\tan\theta \quad \to \quad dr = L\sec^2\theta d\theta$$

$$r^2 + L^2 = L^2\tan^2\theta + L^2 \; \underline{= L^2\sec^2\theta}|$$

$$\frac{1}{R}\int_0^R \frac{2mL}{\left(r^2+L^2\right)^{3/2}}dr = \frac{1}{R}\int_0^R \frac{2mL}{(L\sec\theta)^3}L\sec^2\theta\; d\theta$$

$$= \frac{2m}{RL}\int_0^R \frac{1}{\sec\theta}\; d\theta$$

$$= \frac{2m}{RL}\int_0^R \cos\theta\; d\theta$$

$$= \frac{2m}{RL}\sin\theta\Big|_0^R$$

$$= \frac{2m}{RL}\frac{r}{\sqrt{r^2+L^2}}\Bigg|_0^R$$

$$= \frac{2m}{L\sqrt{R^2+L^2}}$$