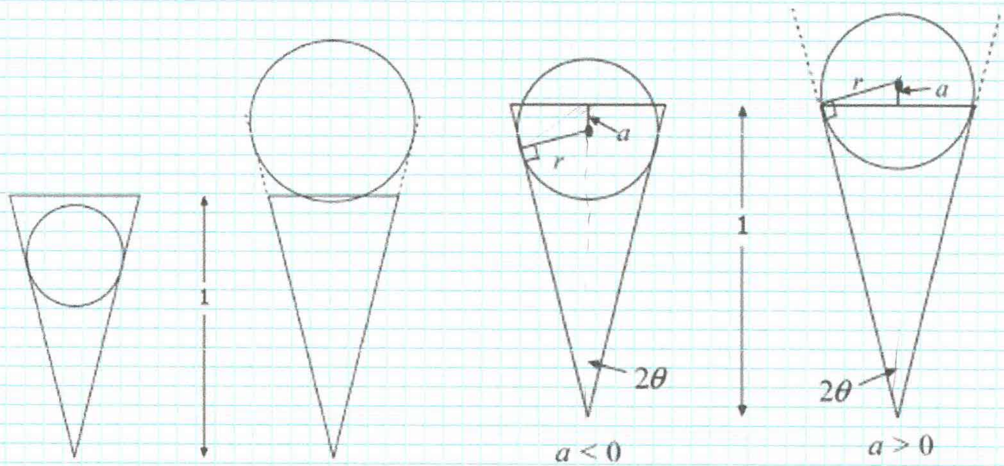
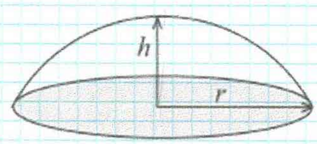
An ice cream cone with a height of one unit holds a sphere of ice cream. What is the radius of the ice cream sphere that maximizes the amount of the ice cream inside the cone?

You can see why this question results in an optimization problem: If the radius is small, much of the sphere is inside the cone, but the volume of the sphere is small. Alternatively, if the radius is large, the volume of the sphere is large, but only a small fraction of the sphere is inside the cone. Somewhere between these extremes, there should an optimal radius.



We assume that the cone has a base angle of 2θ and that the ice cream sphere is tangent to the sides of the cone. The solution requires the formula for the volume of a spherical cap. A cap of height h sliced from a sphere of radius r has volume of



$$V = \frac{\pi}{3} h^2 (3r - h)$$

To optimize the problem from the given information is to find a function. In this case, the objective function is the volume of the ice cream sphere that is inside the cone. Two cases must be considered. Let a be the distance between the center of the sphere and the top edge of the cone, where $a > 0$ means the center is below the top of the cone and $a < 0$ means the center is below the top of the cone.

a) In the case that $a > 0$, show that $r = (1+a)\sin\theta$. (The line from the center of the sphere to the point of tangency is perpendicular to the cone.)

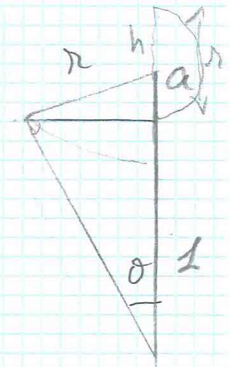
b) In the case that $a > 0$, show that the volume of ice cream inside the cone is $V(r) = \frac{\pi}{3}(r-a)^2(2r+a)$, where $a = \frac{r}{\sin\theta} - 1$

c) In the case that $a < 0$, show that $r = (1-|a|)\sin\theta = (1+a)\sin\theta$, which is the same relationship as in the case that $a > 0$.

d) For $a < 0$, show that the volume of the ice cream inside the cone is $V(r) = \frac{\pi}{3}(r-a)^2(2r+a)$, where $a = \frac{r}{\sin\theta} - 1$. Thus the volume function is the same for both $a > 0$ and $a < 0$.

e) Argue that the maximum value of r that needs to be considered occurs when the bottom of the sphere is at the top of the cone ($a > 0$). In this case $r_{\max} = a = \dfrac{\sin\theta}{1-\sin\theta}$.

f) Argue that the minimum value of r that needs to be considered occurs when the ~~bottom~~ top of the sphere is at the top of the cone ($a < 0$). In this case $r_{\min} = -a = \dfrac{\sin\theta}{1+\sin\theta}$.



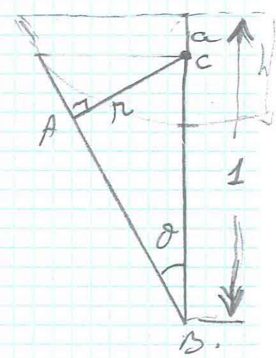
a) The tangency point is perpendicular to the cone from the Center of the sphere which makes a right triangle.

$\therefore \sin\theta = \dfrac{r}{a+1} \Rightarrow r = (a+1)\sin\theta.$

b) $V = \dfrac{\pi}{3} h^2 (3r - h)$, $h = r - a$

$= \dfrac{\pi}{3}(r-a)^2(3r - r + a)$

$= \dfrac{\pi}{3}(r-a)^2(2r+a)$



c) $|BC| = 1 - |a|$

$\triangle ABC$: $\sin\theta = \dfrac{r}{1-|a|} \Rightarrow r = (1-|a|)\sin\theta$

since $a<0 \Rightarrow 1 - a = 1 + a$; $(a>0)$

$\Rightarrow r = (1-|a|)\sin\theta = (1+a)\sin\theta.$

d) since $a<0 \Rightarrow h = r + a$

Cap is partially outside ~ The cap volume

$V = \dfrac{\pi}{3}(r+a)^2(3r - r - a)$

Circle $V = \dfrac{4}{3}\pi r^3$

Volume inside the cone:

$V = \dfrac{4\pi}{3} r^3 - \dfrac{\pi}{3}(r+a)^2(2r-a)$

$= \dfrac{\pi}{3}\left[4r^3 - (r^2 + 2ar + a^2)(2r-a)\right]$

$= \dfrac{\pi}{3}\left[4r^3 - 2r^3 + ar^2 - 4ar^2 + 2a^2 r - 2a^2 r + a^3\right]$

$= \dfrac{\pi}{3}(2r+a)(r^2 - 2ar + a^2)$

$= \dfrac{\pi}{3}(2r+a)(r-a)^2$; $a = \dfrac{r}{\sin\theta} - 1$

Cont

e) when the bottom of the sphere is @ the top of the cone $(a>0) \Rightarrow V(r)=0$

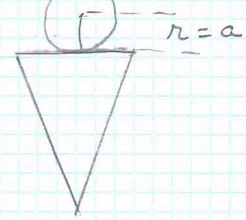
$r = a$

from (a) $\Rightarrow$ $r = (r+1)\sin\theta$

$r = r\sin\theta + \sin\theta$

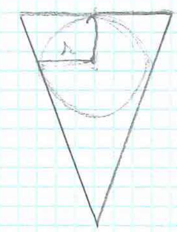
$r(1-\sin\theta) = \sin\theta$

$$r_{max} = \frac{\sin\theta}{1-\sin\theta} = a \checkmark$$



f) when the top of the sphere is at the top of the cone $(a<0) \Rightarrow$ all ice cream is inside the cone.

$\Rightarrow a = r$ + $V(r) = 0$.



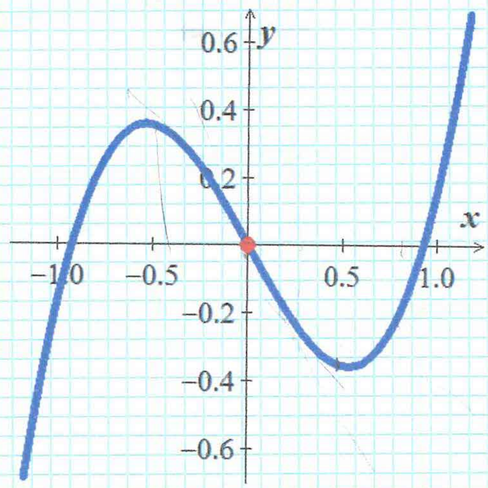
$$\sin\theta = \frac{r}{1+r}$$

$\sin\theta - r\sin\theta = r$

$r(1+\sin\theta) = \sin\theta$

$$r_{Min} = \frac{\sin\theta}{1+\sin\theta} = -a$$

A poorly chosen value of x_0 can lead to the unexpected results. The graph of $f(x) = x^3 - \sin x$ indicates that there are three roots of $f(x) = 0$: they are $x = 0$ and two roots near $x = 1$ *and* $x = -1$.



a) Verify by using Newton's method to approximates the known root $x = 0$ by using an initial value of $x = 0.49$. Calculate the approximation $x_1, x_2, x_3, \ldots$ until two consecutive values agree to 6 decimal places. What happens and why?

b) What happens if you use an initial value of $x = 0.4$?

c) What happens if you use an initial value of $x = 0.6$?

$f(x) = x^3 - \sin x \Rightarrow f'(x) = 3x^2 - \cos x$

a)

n	x_n	$f(x)$	$f'(x)$	$x_{n+1} = x_n - f/f'$
0	.49	−.352977	−.162033	−1.688428
	−1.688428	−3.820261	8.669726	−1.247784
	−1.247784	−.994472	4.353470	−1.019352
	−1.019352	−.207418	2.593317	−.939370
	−.939370	−.021729	2.056953	−.928806
	−.928806	−.000358	1.989254	−.928626
	−.928626	0		−.928626

$\boxed{X = -.928626}$

It's very close to the point of horizontal tangency.

b)

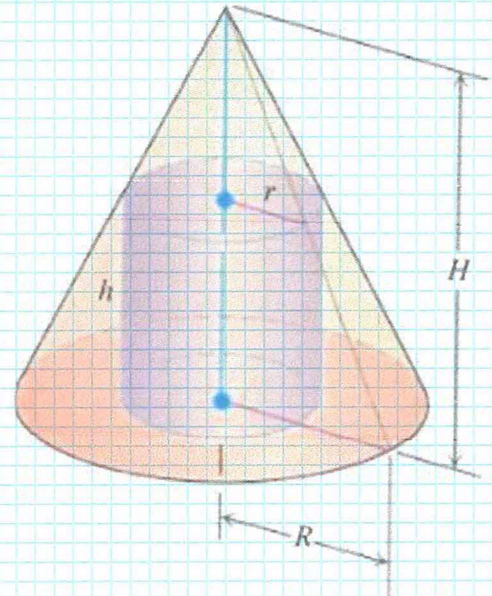
x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1}=x_n-f/f'$
0.4	-.325418	-.441061	-.337808
-0.337808	0.292871	-0.601141	0.149384
0.149384	-.145496	-.921916	-.008435
-.008435	.008434	-.999751	.00001
.00001	-.000001	-1.000000	0.000000
0.	0	-1	0.

is not close to horizontal tangent

c)

0.6	-0.348642	0.254664	1.969027
1.969027	6.712304	12.018993	1.410052
1.410552	1.819329	5.809416	1.097383
1.097383	.431507	3.156825	0.960693
.960693	.067065	2.195843	.930151
.930151	.003039	1.997832	.928630

Sometimes the solution of a max-min problem depends on the proportions of the shapes involved. As a case in point, suppose that a right circular cylinder of radius r and height h is inscribed in a right circular cone of radius R and height H. Find the value of r (in terms of R and H) that maximizes the total surface area of the cylinder (including top and bottom). As you will see, the solution depends on whether $H \le 2R$ or $H > 2R$



Surface area of the cylinder is

$$S = 2\pi r^2 + 2\pi r h$$

$$\frac{r}{R} = \frac{H-h}{H} \qquad \frac{\text{small } \Delta}{\text{Large } \Delta}$$

$$H - h = \frac{r}{R} H$$

$$h = H - \frac{rH}{R}$$

$$S = 2\pi\left(r^2 + r\left(H - \frac{H}{R} r\right)\right)$$

$$= 2\pi\left(r^2 + Hr - \frac{H}{R} r^2\right)$$

$$= 2\pi\left[\left(1 - \frac{H}{R}\right) r^2 + Hr\right] \quad ; \quad 0 \le r \le R.$$

If $H = R \Rightarrow S(r) = 2\pi H r \Rightarrow$ linear equation w/ a positive slope $\Rightarrow$ $S(r)$ is maximum at $r = R$.

$$S'(r) = 2\pi\left[2\left(1 - \frac{H}{R}\right) r + H\right] = 0$$

$$2\left(\frac{R-H}{R}\right) r = -H$$

$$r = \frac{-HR}{2(R-H)} = \boxed{\frac{HR}{2(H-R)}}$$

① If $H = 2R \Rightarrow r = \dfrac{2R^2}{2R} = R. \Rightarrow$

$S(r)$ is maximum at $r = R$

Cont

② If $H < R \Rightarrow H - R < 0 \Rightarrow r < 0$ #

$R < H < 2R \Rightarrow H - 2R < 0 \Rightarrow H > 2(H-R)$

$$\Rightarrow r = \frac{RH}{2(H-R)} > \frac{RH}{H} = R$$

∴ The maximum occurs at the right endpoint R

$0 \le r \le R$ & $S(r)$ is an increasing fcts of r

③ If $H > 2R \Rightarrow 2R + H < 2H$

$H < 2H - 2R = 2(H-R)$

$$\frac{H}{2(H-R)} < 1$$

$$\frac{HR}{2(H-R)} < R$$

$\Rightarrow r < R$, ∴ $S(r)$ is a max. @ $r = \frac{RH}{2(H-R)}$

∴ If $H \in (0, 2R] \Rightarrow$ Max. Surface area is @ $r = R$

If $H \in (2R, \infty) \Rightarrow$ " " " " $r = \frac{RH}{2(H-R)}$

Let $f(x) = \frac{x}{x^2+1}$

a) Show that Newton's method takes the form $x_{n+1} = \frac{2x_n^3}{x_n^2 - 1}$

b) Let $x_0 = \frac{1}{\sqrt{3}}$ and then find the exact values of $x_1, x_2, x_3, x_4, x_5, \ldots$

c) Graph $f(x) = \frac{x}{x^2+1}$; $-1 \le x \le 1$; $-0.5 \le f(x) \le 0.5$ and illustrate how $x_1, x_2, x_3,$ *and* x_4 were found. Does Newton's method lead to an approximate solution to $\frac{x}{x^2+1} = 0$ if $x_0 = \frac{1}{\sqrt{3}}$? Why or why not?

a) $f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)} = x_n - \frac{x_n}{x_n^2+1} \cdot \frac{(x_n^2+1)^2}{1-x_n^2}$$

$$= x_n - \frac{x_n^3 + x_n}{1-x_n^2}$$

$$= \frac{x_n - x_n^3 - x_n^3 - x_n}{1-x_n^2}$$

$$= \boxed{\frac{-2x_n^3}{1-x_n^2}} = \boxed{\frac{2x_n^3}{x_n^2-1}} \checkmark$$

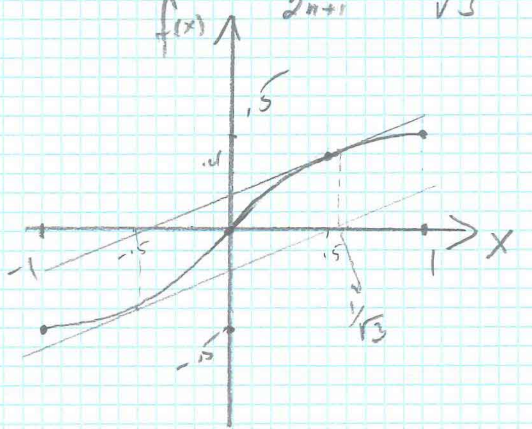
b) $x_0 = \frac{1}{\sqrt{3}} \Rightarrow x_1 = \frac{2x_0^3}{x_0^2-1} = \frac{2\,\frac{1}{3\sqrt{3}}}{\frac{1}{3}-1} = -\frac{1}{\sqrt{3}}$

$$x_2 = \frac{2\,\frac{-1}{3\sqrt{3}}}{\frac{1}{3}-1} = \frac{1}{\sqrt{3}}$$

$$x_3 = \frac{2\,\frac{1}{3\sqrt{3}}}{\frac{1}{3}-1} = \frac{-1}{\sqrt{3}}$$

$x_{2n+1} = -\frac{1}{\sqrt{3}}$, $x_{2k} = \frac{1}{\sqrt{3}}$ $k \in \mathbb{N}$.

c)



x	f(x)
-1	-.5
-.5	-2/5
0	0
.5	2/5
1	.5

Tangent line to the curve $f(x)$ @ $x = \frac{1}{\sqrt{3}}$ intersects x-axis @ $x = -\frac{1}{\sqrt{3}}$ and vice versa.

$\therefore$ Newton's method will never lead to an approximation soln w/ initial estimate @ $x_0 = 1/\sqrt{3}$