Solution *Section* 2.1 – Vectors in 2-Space, 3-Space, and *n*-Space

Exercise

Sketch the following vectors with initial points located at the origin

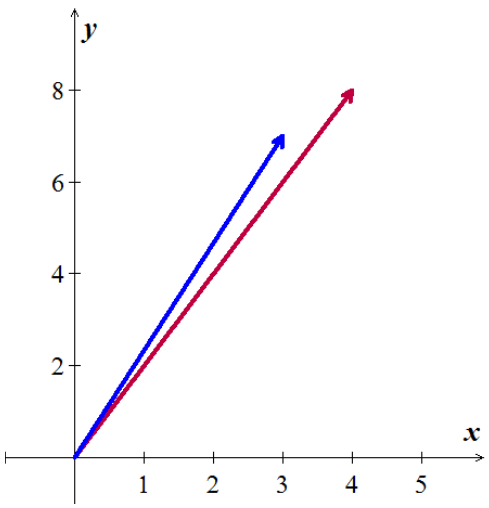
a) $P_1(4,8)$ $P_2(3,7)$

b) $P_1(-1,0,2)$ $P_2(0,-1,0)$

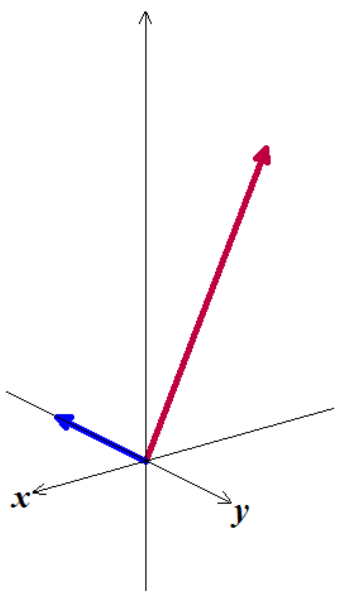
c) $P_1(3,-7,2)$ $P_2(-2,5,-4)$

Solution

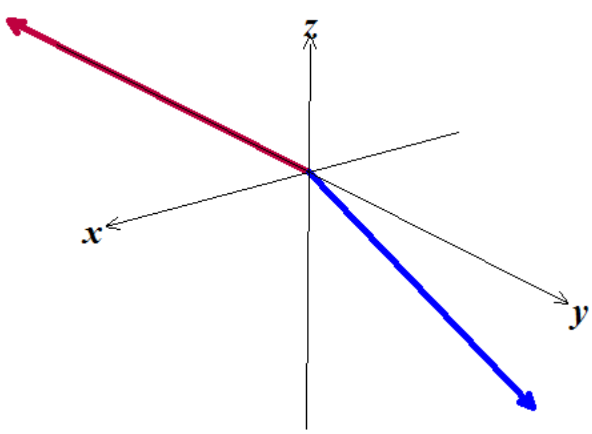
a)



b)



c)



Exercise

Find the components of the vector $\overrightarrow{P_1P_2}$

a) $P_1(3,5)\quad P_2(2,8)$

b) $P_1(5,-2,1)\quad P_2(2,4,2)$

c) $P_1(0,0,0)\quad P_2(-1,6,1)$

Solution

a) $\overrightarrow{P_1P_2} = (2-3,\ 8-5)\ \boxed{=(-1,\ 3)}$

b) $\overrightarrow{P_1P_2} = (2-5,\ 4-(-2),\ 2-1)\ \boxed{=(-3,\ 6,\ 1)}$

c) $\overrightarrow{P_1P_2} = (-1-0,\ 6-0,\ 1-0)\ \boxed{=(-1,\ 6,\ 1)}$

Exercise

Find the terminal point of the vector that is equivalent to ***u*** = (1, 2) and whose initial point is $A(1,1)$

Solution

The terminal point: $B(b_1,\ b_2)$

$$(b_1-1,\ b_2-1)=(1,2)$$

$$\begin{cases} b_1-1=1 & \Rightarrow b_1=2 \\ b_2-1=2 & \Rightarrow b_2=3 \end{cases}$$

The terminal point: $\boxed{B(2,\ 3)}$

Exercise

Find the initial point of the vector that is equivalent to ***u*** = (1, 1, 3) and whose terminal point is $B(-1,-1,2)$

Solution

The initial point: $A(x,y,z)$

$$(-1-x,-1-y,2-z)=(1,1,3)$$

$$\begin{cases} -1-x=1 & \Rightarrow x=-2 \\ -1-y=1 & \Rightarrow y=-2 \\ 2-z=3 & \Rightarrow z=-1 \end{cases}$$

The initial point: $\boxed{A(-2,-2,-1)}$

Exercise

Find a nonzero vector $\boldsymbol{u}$ with initial point $P(-1, 3, -5)$ such that

a) $\boldsymbol{u}$ has the same direction as $\boldsymbol{v} = (6, 7, -3)$

b) $\boldsymbol{u}$ is oppositely directed as $\boldsymbol{v} = (6, 7, -3)$

Solution

a) $\boldsymbol{u}$ has the same direction as $\boldsymbol{v} \Rightarrow \boldsymbol{u} = \boldsymbol{v} = (6, 7, -3)$

The initial point $P(-1, 3, -5)$ then the terminal point : $(-1+6, 3+7, -5-3) = \boxed{(5, 10, -8)}$

b) $\boldsymbol{u}$ is oppositely as $\boldsymbol{v} \Rightarrow \boldsymbol{u} = -\boldsymbol{v} = (-6, -7, 3)$

The initial point $P(-1, 3, -5)$ then the terminal point : $(-1-6, 3-7, -5+3) = \boxed{(-7, -4, -2)}$

Exercise

Let $\boldsymbol{u} = (-3, 1, 2)$, $\boldsymbol{v} = (4, 0, -8)$, and $\boldsymbol{w} = (6, -1, -4)$. Find the components

a) $v - w$

b) $6u + 2v$

c) $5(v - 4u)$

d) $-3(v - 8w)$

e) $(2u - 7w) - (8v + u)$

f) $-u + (v - 4w)$

Solution

a) $v - w = (4-6,\ 0-(-1),\ -8-(-4)) = \boxed{(-2,\ 1,\ -4)}$

b) $6u + 2v = (-18, 6, 12) + (8, 0, -16) = \boxed{(-10,\ 6,\ -4)}$

c) $5(v - 4u) = 5(4-(-12), 0-4, -8-8) = 5(16, -4, -16) = \boxed{(80,\ -20,\ -80)}$

d) $-3(v - 8w) = -3(4-48, 0-(-8), -8-(-32)) = -3(-44, 8, 24) = \boxed{(32,\ -24,\ -72)}$

e)
$$\begin{aligned}(2u - 7w) - (8v + u) &= [(-6, 2, 4) - (42, -7, -28)] - [(32, 0, -64) + (-3, 1, 2)] \\ &= (-48, 9, 32) - (29, 1, -62) \\ &\boxed{= (-77,\ 8,\ 94)}\end{aligned}$$

f)
$$\begin{aligned}-u + (v - 4w) &= (3, -1, -2) + [(4, 0, -8) - (24, -4, -16)] \\ &= (3, -1, -2) + (-20, 4, 8) \\ &\boxed{= (-17,\ 3,\ 6)}\end{aligned}$$

Exercise

Let $\boldsymbol{u}$ = (2, 1, 0, 1, –1) and $\boldsymbol{v}$ = (–2, 3, 1, 0, 2). Find scalars a and b so that $a\boldsymbol{u} + b\boldsymbol{v}$ = (–8, 8, 3, –1, 7)

Solution

$$\begin{aligned} au+bv &= a(2,1,0,1,-1)+b(-2,3,1,0,2) \\ &= (a-2b,\ a+3b,\ b,\ a,\ -a+2b) \\ &= (-8,8,3,-1,7) \end{aligned}$$

$$\begin{cases} a-2b=-8 \\ a+3b=8 \\ \quad\ b=3 \\ a\quad\ =-1 \\ -a+2b=7 \end{cases} \quad \rightarrow \quad a=-1 \quad b=3 \ \textit{Unique solution}$$

Exercise

Find all scalars $c_1,\ c_2,\ and\ c_3$ such that $c_1(1,2,0)+c_2(2,1,1,)+c_3(0,3,1)=(0,0,0)$

Solution

$$c_1(1,2,0)+c_2(2,1,1,)+c_3(0,3,1)=\left(c_1+2c_2, 2c_1+c_2+3c_3, c_2+c_3\right)=(0,0,0)$$

$$\begin{cases} c_1+2c_2 \qquad\quad =0 \\ 2c_1+c_2+3c_3=0 \\ \qquad\quad c_2+c_3=0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\boxed{c_1=c_2=c_3=0}$$

Exercise

Find the distance between the given points $\begin{bmatrix}5 & 1 & 8 & -1 & 2 & 9\end{bmatrix}$, $\begin{bmatrix}4 & 1 & 4 & 3 & 2 & 8\end{bmatrix}$

Solution

$$\begin{aligned} d &= \sqrt{(4-5)^2+(1-1)^2+(4-8)^2+(3+1)^2+(2-2)^2+(8-9)^2} \\ &= \sqrt{1+0+16+16+0+1} \\ &= \sqrt{34} \end{aligned}$$

Exercise

Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operation on $\boldsymbol{u}=(u_1,\ u_2)\quad \boldsymbol{v}=(v_1,\ v_2)$

$$\boldsymbol{u}+\boldsymbol{v}=(u_1+v_1+1,\ u_2+v_2+1) \qquad k\boldsymbol{u}=(ku_1,\ ku_2)$$

a) Compute $\boldsymbol{u}+\boldsymbol{v}$ and $k\boldsymbol{u}$ for $\boldsymbol{u}$ = (0, 4), $\boldsymbol{v}$ = (1, –3), and k = 2.

b) Show that (0, 0) ≠ **0**.

c) Show that (–1, –1) = 0.

d) Show that $\boldsymbol{u}+(-\boldsymbol{u})=0$ for $\boldsymbol{u}=(u_1,\ u_2)$

e) Find two vector space axioms that fail to hold.

Solution

a) $\boldsymbol{u}+\boldsymbol{v}=(0+1+1,\ 4-3+1)=\underline{(2,\ 2)|}$

$k\boldsymbol{u}=(ku_1,\ ku_2)=(2(0),\ 2(4))=\underline{(0,\ 8)|}$

b) $(0,0)+(u_1,u_2)=(0+u_1+1,\ 0+u_2+1)$

$=(u_1+1,\ u_2+1)$

$\neq(u_1,\ u_2)$

Therefore (0, 0) is not the zero vector **0** required (by Axiom).

c) $(-1,-1)+(u_1,u_2)=(-1+u_1+1,\ -1+u_2+1)$

$=(u_1,\ u_2)$

$(u_1,u_2)+(-1,-1)=(u_1-1+1,\ u_2-1+1)$

$=(u_1,\ u_2)$

Therefore (–1, –1) = **0** holds.

d) Let $\boldsymbol{u}=(u_1,\ u_2)\ -\boldsymbol{u}=(-2-u_1,\ -2-u_2)$

$\boldsymbol{u}+(-\boldsymbol{u})=(u_1+(-2-u_1)+1,\ u_2+(-2-u_2)+1)$

$=(-1,\ -1)$

$\underline{=\mathbf{0}|}$

$\boldsymbol{u}+(-\boldsymbol{u})=0$ holds

e) Axiom 7: $k(\boldsymbol{u}+\boldsymbol{v})\overset{?}{=}k\boldsymbol{u}+k\boldsymbol{v}$

$k(\boldsymbol{u}+\boldsymbol{v}) = k(u_1 + v_1 + 1,\ u_2 + v_2 + 1) = (ku_1 + kv_1 + k,\ ku_2 + kv_2 + k)$

$k\boldsymbol{u} + k\boldsymbol{v} = (ku_1,\ ku_2) + (kv_1,\ kv_2) = (ku_1 + kv_1 + 1,\ ku_2 + kv_2 + 1)$

Therefore, $k(\boldsymbol{u}+\boldsymbol{v}) \neq k\boldsymbol{u} + k\boldsymbol{v}$; Axiom 7 fails to hold

Axiom 8: $(k+m)\boldsymbol{u} \stackrel{?}{=} k\boldsymbol{u} + m\boldsymbol{u}$

$(k+m)\boldsymbol{u} = ((k+m)u_1,\ (k+m)u_2) = (ku_1 + mu_1,\ ku_2 + mu_2)$

$k\boldsymbol{u} + m\boldsymbol{u} = (ku_1,\ ku_2) + (mu_1,\ mu_2) = (ku_1 + mu_1 + 1,\ ku_2 + mu_2 + 1)$

Therefore, $(k+m)\boldsymbol{u} \neq k\boldsymbol{u} + m\boldsymbol{u}$; Axiom 8 fails to hold

Exercise

If $\|\vec{v}\|=5$ and $\|\vec{w}\|=3$, what are the smallest and largest possible values of $\|\vec{v}-\vec{w}\|$ and $\vec{v}.\vec{w}$?

Solution

$$\|\vec{v}-\vec{w}\| \le \|\vec{v}\|+\|\vec{w}\| = 5+3=8$$

$$\|\vec{v}-\vec{w}\| \ge \|\vec{v}\|-\|\vec{w}\| = 5-3=2$$

$$|\vec{v}.\vec{w}| = \|\vec{v}\|.\|\vec{w}\|.\cos\theta \le \|\vec{v}\|.\|\vec{w}\|$$

$$-\|\vec{v}\|.\|\vec{w}\| \le |\vec{v}.\vec{w}| \le \|\vec{v}\|.\|\vec{w}\|$$

$$-(3)(5) \le |\vec{v}.\vec{w}| \le (3)(5)$$

$$-15 \le |\vec{v}.\vec{w}| \le 15$$

The minimum value occurs when the dot product is a small as possible, v and w are parallel, but point in opposite directions. Thus the smallest value is -15.

The maximum value occurs when the dot product is a large as possible, v and w are parallel and point in same direction. Thus the largest value is 15.

Exercise

If $\|\vec{v}\|=7$ and $\|\vec{w}\|=3$, what are the smallest and largest possible values of $\|\vec{v}+\vec{w}\|$ and $\vec{v}.\vec{w}$?

Solution

$$\|\vec{v}+\vec{w}\| \le \|\vec{v}\|+\|\vec{w}\| = 7+3=10$$

$$\|\vec{v}+\vec{w}\| \ge \|\vec{v}\|-\|\vec{w}\| = 7-3=4$$

$$|\vec{v}.\vec{w}| \le \|\vec{v}\|.\|\vec{w}\|$$

$$-\|\vec{v}\|.\|\vec{w}\| \le |\vec{v}.\vec{w}| \le \|\vec{v}\|.\|\vec{w}\|$$

$$-(7)(3) \le |\vec{v}.\vec{w}| \le (7)(3)$$

$$-21 \le |\vec{v}.\vec{w}| \le 21$$

The minimum value occurs when the dot product is a small as possible, v and w are parallel, but point in opposite directions. Thus the smallest value is -21. $\vec{v}=(7,\ 0,\ 0,\ \cdots)$ and $\vec{w}=(-3,\ 0,\ 0,\ \cdots)$

The maximum value occurs when the dot product is a large as possible, v and w are parallel and point in same direction. Thus the largest value is 21. $\vec{v}=(7,\ 0,\ 0,\ \cdots)$ and $\vec{w}=(3,\ 0,\ 0,\ \cdots)$

Exercise

Given that $cos(\alpha)=\frac{v_1}{\|v\|}$ and $sin(\alpha)=\frac{v_2}{\|v\|}$. Similarly, $cos(\beta)=$ ______ and $sin(\beta)=$ ______. The angle θ is $\beta-\alpha$. Substitute into the trigonometry formula $cos(\alpha)cos(\beta)+sin(\alpha)sin(\beta)$ for $cos(\beta-\alpha)$ to find $cos(\theta)=\frac{v.w}{\|v\|.\|w\|}$

Solution

$$cos(\beta)=\frac{w_1}{\|w\|}$$

$$sin(\beta)=\frac{w_2}{\|w\|}$$

$$\begin{aligned}cos(\beta-\alpha)&=cos(\alpha)cos(\beta)+sin(\alpha)sin(\beta)\\ &=\frac{v_1}{\|v\|}\frac{w_1}{\|w\|}+\frac{v_2}{\|v\|}\frac{w_2}{\|w\|}\\ &=\frac{v_1w_1+v_2w_2}{\|v\|.\|w\|}\\ &=\frac{v.w}{\|v\|.\|w\|}\end{aligned}$$

Exercise

Can three vectors in the *xy* plane have $u.v<0$ and $v.w<0$ and $u.w<0$?

Solution

Let consider: $u=(1,\ 0)$, $v=\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$, $w=\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$

$$u.v=(1)\left(-\frac{1}{2}\right)+0\ \boxed{=-\frac{1}{2}}$$

$$\begin{aligned}v.w&=\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)+\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)\\ &=\frac{1}{4}-\frac{3}{4}\\ &\boxed{=-\frac{1}{2}}\end{aligned}$$

$$u.w=(1)\left(-\frac{1}{2}\right)+(0)\left(-\frac{\sqrt{3}}{2}\right)=\boxed{-\frac{1}{2}}$$

Yes, it is.

Exercise

Find the norm of v, a unit vector that has the same direction as v, and a unit vector that is oppositely directed.

a) $v = (4, -3)$

b) $v = (1, -1, 2)$

c) $v = (-2, 3, 3, -1)$

Solution

a) $\|v\| = \sqrt{4^2 + (-3)^2} = \underline{5}|$

Same direction unit vector: $u_1 = \frac{v}{\|v\|} = \frac{1}{5}(4, -3) = \underline{\left(\frac{4}{5}, -\frac{3}{5}\right)}|$

Opposite direction unit vector: $u_2 = -\frac{v}{\|v\|} = -\frac{1}{5}(4, -3) = \underline{\left(-\frac{4}{5}, \frac{3}{5}\right)}|$

b) $\|v\| = \sqrt{1^2 + (-1)^2 + 2^2} = \underline{\sqrt{6}}|$

Same direction unit vector:

$$u_1 = \frac{v}{\|v\|} = \frac{1}{\sqrt{6}}(1, -1, 2) = \underline{\left(\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)}|$$

Opposite direction unit vector:

$$u_2 = -\frac{v}{\|v\|} = -\frac{1}{\sqrt{6}}(1, -1, 2) = \underline{\left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right)}|$$

c) $\|v\| = \sqrt{(-2)^2 + (3)^2 + (3)^2 + (-1)^2} = \underline{\sqrt{23}}|$

Same direction unit vector:

$$u_1 = \frac{v}{\|v\|} = \frac{1}{\sqrt{23}}(-2, 3, 3, -1) = \underline{\left(\frac{-2}{\sqrt{23}},\ \frac{3}{\sqrt{23}},\ \frac{3}{\sqrt{23}},\ -\frac{1}{\sqrt{23}}\right)}|$$

Opposite direction unit vector:

$$u_2 = -\frac{v}{\|v\|} = -\frac{1}{\sqrt{23}}(-2, 3, 3, -1) = \underline{\left(\frac{2}{\sqrt{23}},\ -\frac{3}{\sqrt{23}},\ -\frac{3}{\sqrt{23}},\ \frac{1}{\sqrt{23}}\right)}|$$

Exercise

Evaluate the given expression with $\boldsymbol{u} = (2, -2, 3)$, $\boldsymbol{v} = (1, -3, 4)$, and $\boldsymbol{w} = (3, 6, -4)$

a) $\|u+v\|$ *b)* $\|-2u+2v\|$

c) $\|3u-5v+w\|$ *d)* $\|3v\|-3\|v\|$

e) $\|u\|+\|-2v\|+\|-3w\|$

Solution

a)
$$\begin{aligned}\|u+v\| &= \|(2,-2,3)+(1,-3,4)\| \\ &= \|(3,-5,7)\| \\ &= \sqrt{3^2+(-5)^2+7^2} \\ &\boxed{= \sqrt{83}}\end{aligned}$$

b)
$$\begin{aligned}\|-2u+2v\| &= \|(-4,4,-6)+(2,-6,8)\| \\ &= \|(-2,-2,2)\| \\ &= \sqrt{(-2)^2+(-2)^2+2^2} \\ &= \sqrt{12} \\ &\boxed{= 2\sqrt{3}}\end{aligned}$$

c)
$$\begin{aligned}\|3u-5v+w\| &= \|(6,-6,9)-(5,-15,20)+(3,6,-4)\| \\ &= \|(4,15,-15)\| \\ &= \sqrt{(4)^2+(15)^2+(-15)^2} \\ &\boxed{= \sqrt{466}}\end{aligned}$$

d) $\|3v\|-3\|v\| = 3\|v\|-3\|v\| = \boxed{0}$

$$\begin{aligned}&= \sqrt{3^2+(-9)^2+12^2}-3\sqrt{1^2+(-3)^2+4^2} \\ &= \sqrt{234}-3\sqrt{26} \\ &= 3\sqrt{26}-3\sqrt{26} \\ &\boxed{= 0}\end{aligned}$$

e)
$$\begin{aligned}\|u\|+\|-2v\|+\|-3w\| &= \|u\|-2\|v\|-3\|w\| \\ &= \sqrt{2^2+(-2)^2+3^2}-2\sqrt{1^2+(-3)^2+4^2}-3\sqrt{3^2+6^2+(-4)^2} \\ &\boxed{= \sqrt{17}-2\sqrt{26}-3\sqrt{61}}\end{aligned}$$

$$\|3v\| - 3\|v\| = \|(3,-9,12)\| - 3\|(1,-3,4)\|$$

Exercise

Let $\boldsymbol{v} = (1, 1, 2, -3, 1)$. Find all scalars k such that $\|kv\| = 5$

Solution

$$\begin{aligned} \|kv\| &= |k|\|v\| \\ &= |k| \ \|(1,1,2,-3,1)\| \\ &= |k| \ \sqrt{1^2+1^2+2^2+(-3)^2+1^2} \\ &= |k|\sqrt{49} \\ &= 7|k| \end{aligned}$$

$$7|k| = 5 \quad \rightarrow \quad |k| = \frac{5}{7} \Rightarrow \boxed{k = \pm\frac{5}{7}}$$

Exercise

Find $u \cdot v, \quad u \cdot u, \quad and \quad v \cdot v$

a) $u = (3, 1, 4)$, $v = (2, 2, -4)$

b) $u = (1, 1, 4, 6)$, $v = (2, -2, 3, -2)$

c) $u = (2, -1, 1, 0, -2)$, $v = (1, 2, 2, 2, 1)$

Solution

a) $u \cdot v = (3,1,4)\cdot(2,2,-4) = 3(2)+1(2)+4(-4) \ \underline{= -8}$

$u \cdot u = \|u\|^2 = 3^2+1^2+4^2 \ \underline{= 26}$

$v \cdot v = \|v\|^2 = 2^2+2^2+(-4)^2 \ \underline{= 24}$

b) $u \cdot v = (1,1,4,6)\cdot(2,-2,3,-2) = 1(2)+1(-2)+4(3)+6(-2) \ \underline{= 0}$

$u \cdot u = \|u\|^2 = 1^2+1^2+4^2+6^2 \ \underline{= 54}$

$v \cdot v = \|v\|^2 = 2^2+(-2)^2+3^2+(-2)^2 \ \underline{= 21}$

c) $u \cdot v = (2,-1,1,0,-2)\cdot(1,2,2,2,1) = 2(1)-1(2)+1(2)+0(2)-2(1) \ \underline{= 0}$

$u \cdot u = \|u\|^2 = 2^2+(-1)^2+1^2+0+(-2)^2 \ \underline{= 10}$

$v \cdot v = \|v\|^2 = 1^2+2^2+2^2+2^2+1^2 \ \underline{= 14}$

Exercise

Find the Euclidean distance between $\boldsymbol{u}$ and $\boldsymbol{v}$, then find the angle between them

a) $u = (3, 3, 3)$, $v = (1, 0, 4)$

b) $u = (1, 2, -3, 0)$, $v = (5, 1, 2, -2)$

c) $u = (0, 1, 1, 1, 2)$, $v = (2, 1, 0, -1, 3)$

Solution

a)
$$\begin{aligned} d = \|u - v\| &= \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2} \\ &= \sqrt{(-2)^2 + (-3)^2 + (1)^2} \\ &= \sqrt{14} \end{aligned}$$

$$\begin{aligned} \cos\theta &= \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|} \\ &= \frac{3(1) + 3(0) + 3(4)}{\sqrt{3^2 + 3^2 + 3^2}\sqrt{1^2 + 0^2 + 4^2}} \\ &= \frac{15}{\sqrt{27}\sqrt{17}} \end{aligned}$$

$$\theta = \cos^{-1}\left(\frac{15}{\sqrt{27}\sqrt{17}}\right) = 45.56^\circ$$

b)
$$\begin{aligned} d = \|u - v\| &= \sqrt{(1-5)^2 + (-2-1)^2 + (-3-2)^2 + (-2-0)^2} \\ &= \sqrt{(-4)^2 + (-3)^2 + (-5)^2 + (-2)^2} \\ &= \sqrt{46} \end{aligned}$$

$$\begin{aligned} \cos\theta &= \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|} \\ &= \frac{1(5) + 2(1) - 3(2) + 0(-2)}{\sqrt{1^2 + 2^2 + (-3)^2 + 0}\ \sqrt{5^2 + 1^2 + 2^2 + (-2)^2}} \\ &= \frac{1}{\sqrt{14}\sqrt{34}} \end{aligned}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{14}\sqrt{34}}\right) = 87.37^\circ$$

c) $d=\|u-v\|=\sqrt{(0-2)^2+(1-1)^2+(1-0)^2+(1-(-1))^2+(2-3)^2}$

$$=\sqrt{10}$$

$$\cos\theta=\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\ \|\boldsymbol{v}\|}$$

$$=\frac{0(2)+1(1)+1(0)+1(-1)+2(3)}{\sqrt{0+1^2+1^2+1^2+2^2}\ \sqrt{2^2+1^2+0+(-1)^2+(3)^2}}$$

$$=\frac{6}{\sqrt{7}\sqrt{15}}$$

$$\theta=\cos^{-1}\left(\frac{6}{\sqrt{7}\sqrt{15}}\right)=54.16^\circ$$

Exercise

Find a unit vector that has the same direction as the given vector

a) $(-4,-3)$ *b)* $\left(-3,\ 2,\ \sqrt{3}\right)$ *c)* $(1,\ 2,\ 3,\ 4,\ 5)$

Solution

a) $u=\frac{u}{\|u\|}=\frac{(-4,-3)}{\sqrt{(-4)^2+(-3)^2}}$

$$=\frac{(-4,-3)}{\sqrt{25}}$$

$$=\left(-\frac{4}{5},-\frac{3}{5}\right)$$

b) $u=\frac{1}{\sqrt{(-3)^2+(2)^2+\left(\sqrt{3}\right)^2}}\left(-3,2,\sqrt{3}\right)$

$$=\frac{1}{\sqrt{17}}\left(-3,\ 2,\ \sqrt{3}\right)$$

$$=\left(-\frac{3}{\sqrt{17}},\ \frac{2}{\sqrt{17}},\ \frac{\sqrt{3}}{\sqrt{17}}\right)$$

c) $u=\frac{1}{\sqrt{1^2+2^2+3^2+4^2+5^2}}(1,2,3,4,5)$

$$=\frac{1}{\sqrt{55}}(1,2,3,4,5)$$

$$=\left(\frac{1}{\sqrt{55}},\ \frac{2}{\sqrt{55}},\ \frac{3}{\sqrt{55}},\ \frac{4}{\sqrt{55}},\ \frac{5}{\sqrt{55}}\right)$$

Exercise

Find a unit vector that is oppositely to the given vector

a) $(-12, -5)$

b) $(3, -3, 3)$

c) $\left(-3,\ 1,\ \sqrt{6},\ 3\right)$

Solution

a) $$u = -\frac{1}{\sqrt{(-12)^2 + (-5)^2}}(-12, -5)$$

$$= -\frac{1}{\sqrt{169}}(-12, -5)$$

$$= \left(\frac{12}{13},\ \frac{5}{13}\right)$$

b) $$u = -\frac{1}{\sqrt{(3)^2 + (-3)^2 + (3)^2}}(3, -3, 3)$$

$$= -\frac{1}{\sqrt{27}}(3, -3, 3)$$

$$= -\frac{1}{3\sqrt{3}}(3, -3, 3)$$

$$= \left(-\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)$$

c) $$u = -\frac{1}{\sqrt{(-3)^2 + 1^2 + \left(\sqrt{6}\right)^2 + 3^2}}\left(-3,\ 1,\ \sqrt{6},\ 3\right)$$

$$= -\frac{1}{\sqrt{25}}\left(-3,\ 1,\ \sqrt{6},\ 3\right)$$

$$= \left(\frac{3}{5},\ -\frac{1}{5},\ -\frac{\sqrt{6}}{5},\ -\frac{3}{5}\right)$$

Exercise

Verify that the Cauchy-Schwarz inequality holds

a) $u = (-3, 1, 0)$, $v = (2, -1, 3)$

b) $u = (0, 2, 2, 1)$, $v = (1, 1, 1, 1)$

c) $u = (1, 3, 5, 2, 0, 1)$, $v = (0, 2, 4, 1, 3, 5)$

Solution

a)
$$\begin{aligned}|u \cdot v| &= |(-3,1,0)\cdot(2,-1,3)| \\ &= |-3(2)+1(-1)+0(3)| \\ &= |-7| \\ &= 7\end{aligned}$$

$$\begin{aligned}\|u\|\|v\| &= \sqrt{(-3)^2+1^2+0}\ \sqrt{(2)^2+(-1)^2+3^2} \\ &= \sqrt{10}\sqrt{14} \\ &\approx 11.83\end{aligned}$$

$|u \cdot v| \le \|u\|\|v\|$ Cauchy-Schwarz inequality holds

b)
$$\begin{aligned}|u \cdot v| &= |(0,2,2,1)\cdot(1,1,1,1)| \\ &= |0+2+2+1| \\ &= 5\end{aligned}$$

$$\begin{aligned}\|u\|\|v\| &= \sqrt{0+2^2+2^2+1^2}\ \sqrt{1^2+1^2+1^2+1^2} \\ &= \sqrt{9}\sqrt{4} \\ &= 6\end{aligned}$$

$|u \cdot v| \le \|u\|\|v\|$ Cauchy-Schwarz inequality holds

c)
$$\begin{aligned}|u \cdot v| &= |(1,3,5,2,0,1)\cdot(0,2,4,1,3,5)| \\ &= |0+6+20+2+0+5| \\ &= 23\end{aligned}$$

$$\begin{aligned}\|u\|\|v\| &= \sqrt{1^2+3^2+5^2+2^2+0+1^2}\ \sqrt{0+2^2+4^2+1^2+3^2+5^2} \\ &= \sqrt{40}\sqrt{55} \\ &\approx 46\end{aligned}$$

$|u \cdot v| \le \|u\|\|v\|$ Cauchy-Schwarz inequality holds

Exercise

Find $\boldsymbol{u} \cdot \boldsymbol{v}$ and then the angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$ $\quad \boldsymbol{u} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$

Solution

$$\boldsymbol{u} \cdot \boldsymbol{v} = 3 + 0 - 2 - 1 \underline{= 0|}$$

$$\theta = \cos^{-1} \frac{0}{\sqrt{15}\sqrt{3}} = \cos^{-1}(0) = \underline{90^\circ|}$$

Exercise

Find the norm: $\|\boldsymbol{u}\| + \|\boldsymbol{v}\|$, $\|\boldsymbol{u} + \boldsymbol{v}\|$ for $\boldsymbol{u} = (3,\ -1,\ -2,\ 1,\ 4)$ $\quad \boldsymbol{v} = (1,\ 1,\ 1,\ 1,\ 1)$

Solution

$$\|\boldsymbol{u}\| + \|\boldsymbol{v}\| = \sqrt{3^2 + (-1)^2 + (-2)^2 + 1^2 + 4^2} + \sqrt{1+1+1+1+1} \ \underline{= \sqrt{31} + \sqrt{5}|}$$

$$\|\boldsymbol{u} + \boldsymbol{v}\| = \|(4,\ 0,\ -1,\ 2,\ 5)\| = \sqrt{16 + 0 + 1 + 4 + 25} \ \underline{= \sqrt{46}|}$$

Exercise

Find all numbers r such that: $\|r(1,\ 0,\ -3,\ -1,\ 4,\ 1)\| = 1$

Solution

$$r\sqrt{1+9+1+16+1} = \pm 1$$

$$r\sqrt{28} = \pm 1$$

$$r = \pm \frac{1}{2\sqrt{7}} \underline{= \pm \frac{\sqrt{7}}{14}|}$$

Exercise

Find the distance between $P_1(7, -5, 1)$ and $P_2(-7, -2, -1)$

Solution

$$\begin{aligned} \|P_1P_2\| &= \sqrt{(-7-7)^2 + (-2+5)^2 + (-1-1)^2} \\ &= \sqrt{14^2 + 3^2 + (-2)^2} \\ &= \sqrt{196 + 9 + 4} \\ &\underline{= \sqrt{209}|} \end{aligned}$$

Exercise

Given $\boldsymbol{u} = (1, -5, 4)$, $\boldsymbol{v} = (3, 3, 3)$

a) Find $\boldsymbol{u} \cdot \boldsymbol{v}$

b) Find the cosine of the angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$.

Solution

a) $\boldsymbol{u} \cdot \boldsymbol{v} = 3 - 15 + 12 = \underline{0}$

b) $\cos\theta = \dfrac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\| \; \|\boldsymbol{v}\|} \; \underline{= 0}$

Solution   *Section* 2.3 – Orthogonality

Exercise

Determine whether $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

a) $\boldsymbol{u}=(-6,\ -2),\quad \boldsymbol{v}=(5,\ -7)$

b) $\boldsymbol{u}=(6,\ 1,\ 4),\quad \boldsymbol{v}=(2,\ 0,\ -3)$

c) $\boldsymbol{u}=(1,\ -5,\ 4),\quad \boldsymbol{v}=(3,\ 3,\ 3)$

d) $\boldsymbol{u}=(-2,\ 2,\ 3),\quad \boldsymbol{v}=(1,\ 7,\ -4)$

Solution

a) $\boldsymbol{u}\cdot\boldsymbol{v}=(-6)(5)+(-2)(-7)$

$=-30+14$

$=-16\neq 0$

$\therefore$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are not orthogonal

b) $\boldsymbol{u}\cdot\boldsymbol{v}=6(2)+1(0)+4(-3)$

$=0$

$\therefore$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

c) $\boldsymbol{u}\cdot\boldsymbol{v}=1(3)-5(3)+4(3)$

$=0$

$\therefore$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

d) $\boldsymbol{u}\cdot\boldsymbol{v}=-2(1)+2(7)+3(-4)$

$=0$

$\therefore$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

Exercise

Determine whether the vectors form an orthogonal set

a) $\boldsymbol{v}_1=(2,\ 3),\quad \boldsymbol{v}_2=(3,\ 2)$

b) $\boldsymbol{v}_1=(1,\ -2),\quad \boldsymbol{v}_2=(-2,\ 1)$

c) $\boldsymbol{u}=(-4,\ 6,\ -10,\ 1)\quad \boldsymbol{v}=(2,\ 1,\ -2,\ 9)$

d) $\boldsymbol{u}=(a,\ b)\quad \boldsymbol{v}=(-b,\ a)$

e) $\boldsymbol{v}_1=(-2,\ 1,\ 1),\quad \boldsymbol{v}_2=(1,\ 0,\ 2),\quad \boldsymbol{v}_3=(-2,\ -5,\ 1)$

f) $\boldsymbol{v}_1=(1,\ 0,\ 1),\quad \boldsymbol{v}_2=(1,\ 1,\ 1),\quad \boldsymbol{v}_3=(-1,\ 0,\ 1)$

g) $\boldsymbol{v}_1=(2,-2,1),\quad \boldsymbol{v}_2=(2,1,-2),\quad \boldsymbol{v}_3=(1,2,2)$

Solution

a) $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 2(3) + 3(2)\ \underline{= 12 \neq 0}$

$\therefore$ Vectors don't form an orthogonal set

b) $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 1(-2) - 2(1)\ \underline{= -4 \neq 0}$

$\therefore$ Vectors don't form an orthogonal set

c) $\boldsymbol{u} \bullet \boldsymbol{v} = -8 + 6 + 20 + 9\ \underline{= 27 \neq 0}$; These vectors are not orthogonal

d) $\boldsymbol{u} \bullet \boldsymbol{v} = -ab + ab\ \underline{= 0}$; These vectors are orthogonal

e) $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = -2(1) + 1(0) + 1(2) \underline{= 0}$

$\boldsymbol{v}_1 \cdot \boldsymbol{v}_3 = -2(-2) + 1(-5) + 1(1) \underline{= 0}$

$\boldsymbol{v}_2 \cdot \boldsymbol{v}_3 = 1(-2) + 0(-5) + 2(1) \underline{= 0}$

$\therefore$ Vectors form an orthogonal set

f) $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 1(1) + 0(1) + 1(1)\ \underline{= 2 \neq 0}$

$\therefore$ Vectors don't form an orthogonal set

g) $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 2(2) - 2(1) + 1(-2) \underline{= 0}$

$\boldsymbol{v}_1 \cdot \boldsymbol{v}_3 = 2(1) - 2(2) + 1(2) \underline{= 0}$

$\boldsymbol{v}_2 \cdot \boldsymbol{v}_3 = 2(1) + 1(2) - 2(2) \underline{= 0}$

$\therefore$ Vectors form an orthogonal set

Exercise

Find a unit vector that is orthogonal to both $\boldsymbol{u}$ = (1, 0, 1) and $\boldsymbol{v}$ = (0, 1, 1)

Solution

Let $\boldsymbol{w} = (w_1, w_2, w_3)$ be the unit vector that is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$.

$\boldsymbol{u} \cdot \boldsymbol{w} = 1(w_1) + 0(w_2) + 1(w_3) = \underline{w_1 + w_3 = 0}$

$$\boxed{w_3 = -w_1}$$

$\boldsymbol{v} \cdot \boldsymbol{w} = 0(w_1) + 1(w_2) + 1(w_3) = \underline{w_2 + w_3 = 0}$

$$\boxed{w_3 = -w_2}$$

$w_1 = w_2 = -w_3$

The orthogonal vector to both $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\boldsymbol{w} = (1,\ 1,\ -1)$, therefore the unit vector is

$$\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = \frac{1}{\sqrt{1^2 + 1^2 + (-1)^2}}(1,\ 1,\ -1)$$

$$= \frac{1}{\sqrt{3}}(1,\ 1,\ -1)$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)$$

The possible vectors are: $\boxed{\pm\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)}$

Exercise

a) Show that $\boldsymbol{v} = (a, b)$ and $\boldsymbol{w} = (-b, a)$ are orthogonal vectors.

b) Use the result to find two vectors that are orthogonal to $\boldsymbol{v} = (2, -3)$.

c) Find two unit vectors that are orthogonal to (–3, 4)

Solution

a) $\boldsymbol{v} \cdot \boldsymbol{w} = a(-b) + b(a) = -ab + ab = 0$ are orthogonal vectors.

b) (2, 3) and (–2, 3).

c) $u_1 = \frac{1}{\sqrt{4^2 + 3^2}}(4,3) = \left(\frac{4}{5},\ \frac{3}{5}\right)$

$u_2 = -\frac{1}{\sqrt{4^2 + 3^2}}(4,3) = \left(-\frac{4}{5},\ -\frac{3}{5}\right)$

Exercise

Find the vector component of $\boldsymbol{u}$ along $\boldsymbol{a}$ and the vector component of $\boldsymbol{u}$ orthogonal to $\boldsymbol{a}$.

a) $\boldsymbol{u} = (6,\ 2),\quad \boldsymbol{a} = (3,\ -9)$

b) $\boldsymbol{u} = (3,\ 1,\ -7),\quad \boldsymbol{a} = (1,\ 0,\ 5)$

c) $\boldsymbol{u} = (1,\ 0,\ 0),\quad \boldsymbol{a} = (4,\ 3,\ 8)$

d) $\boldsymbol{u} = (1,\ 1,\ 1),\quad \boldsymbol{a} = (0,\ 2,\ -1)$

e) $\boldsymbol{u} = (2,\ 1,\ 1,\ 2),\quad \boldsymbol{a} = (4,\ -4,\ 2,\ -2)$

f) $\boldsymbol{u} = (5,0,-3,7),\quad \boldsymbol{a} = (2,1,-1,-1)$

Solution

a) $proj_{\boldsymbol{a}}\boldsymbol{u} = \frac{\boldsymbol{u} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$

$$= \frac{6(3)+2(-9)}{3^2+(-9)^2}(3,\ -9)$$

$$= \frac{0}{90}(3,\ -9)$$

$$\boxed{= (0,\ 0)}$$

$$\boldsymbol{u} - proj_{\boldsymbol{a}}\boldsymbol{u} = (6,\ 2)-(0,\ 0) = \boxed{(6,\ 2)}$$

b) $$proj_{\boldsymbol{a}}\boldsymbol{u} = \frac{\boldsymbol{u} \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} = \frac{3(1)+0-7(5)}{1^2+0+5^2}(1,0,5)$$

$$= \frac{-32}{26}(1,\ 0,\ 5)$$

$$\boxed{= \left(-\frac{16}{13}, 0, -\frac{80}{13}\right)}$$

$$\boldsymbol{u} - proj_{\boldsymbol{a}}\boldsymbol{u} = (1,0,5) - \left(-\frac{16}{13}, 0, -\frac{80}{13}\right) = \boxed{\left(\frac{55}{13}, 1, -\frac{11}{13}\right)}$$

c) $$proj_{\boldsymbol{a}}\boldsymbol{u} = \frac{\boldsymbol{u} \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

$$= \frac{1(4)+0+0}{4^2+3^2+8^2}(4,3,8)$$

$$= \frac{4}{89}(4,3,8)$$

$$\boxed{= \left(\frac{16}{89},\ \frac{12}{89},\ \frac{32}{89}\right)}$$

$$\boldsymbol{u} - proj_{\boldsymbol{a}}\boldsymbol{u} = (1,0,0) - \left(\frac{16}{89}, \frac{12}{89}, \frac{32}{89}\right) = \boxed{\left(\frac{73}{89},\ -\frac{12}{89},\ -\frac{32}{89}\right)}$$

d) $$proj_{\boldsymbol{a}}\boldsymbol{u} = \frac{\boldsymbol{u} \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

$$= \frac{1(0)+1(2)+1(-1)}{0^2+2^2+(-1)^2}(0,2,-1)$$

$$= \frac{1}{5}(0,2,-1)$$

$$\boxed{= \left(0,\ \frac{2}{5},\ -\frac{1}{5}\right)}$$

$$\boldsymbol{u} - proj_{\boldsymbol{a}}\boldsymbol{u} = (1,1,1) - \left(0, \frac{2}{5}, -\frac{2}{5}\right) = \boxed{\left(1,\ \frac{3}{5},\ \frac{6}{5}\right)}$$

e) $$proj_{\boldsymbol{a}}\boldsymbol{u} = \frac{\boldsymbol{u} \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

$$= \frac{2(4)+1(-4)+1(2)+2(-2)}{4^2+(-4)^2+2^2+(-2)^2}(4,-4,2,-2)$$

$$= \frac{2}{40}(4,-4,2,-2)$$

$$\boxed{= \left(\frac{1}{5},\ -\frac{1}{5},\ \frac{1}{10},\ -\frac{1}{10}\right)}$$

$$\boldsymbol{u} - proj_{\boldsymbol{a}}\boldsymbol{u} = (2,1,1,2) - \left(\frac{1}{5},-\frac{1}{5},\frac{1}{10},-\frac{1}{10}\right)$$

$$= \boxed{\left(\frac{9}{5},\ \frac{6}{5},\ \frac{9}{10},\ \frac{21}{10}\right)}$$

f) $$proj_{\boldsymbol{a}}\boldsymbol{u} = \frac{\boldsymbol{u} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

$$= \frac{5(2)+0(1)-3(-1)+7(-1)}{2^2+1^2+(-1)^2+(-1)^2}(2,1,-1,-1)$$

$$= \frac{6}{7}(2,1,-1,-1)$$

$$\boxed{= \left(\frac{12}{7},\ \frac{6}{7},\ -\frac{6}{7},\ -\frac{6}{7}\right)}$$

$$\boldsymbol{u} - proj_{\boldsymbol{a}}\boldsymbol{u} = (5,0,-3,7) - \left(\frac{12}{7},\frac{6}{7},-\frac{6}{7},-\frac{6}{7}\right)$$

$$= \boxed{\left(\frac{23}{7},\ -\frac{6}{7},\ -\frac{15}{7},\ \frac{55}{7}\right)}$$

Exercise

Project the vector $\boldsymbol{v}$ onto the line through $\boldsymbol{a}$, Check that $\boldsymbol{e}$ is perpendicular to $\boldsymbol{a}$:

$$a)\quad v=\begin{pmatrix}1\\2\\2\end{pmatrix}\quad and\quad a=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$b)\quad v=\begin{pmatrix}1\\3\\1\end{pmatrix}\quad and\quad a=\begin{pmatrix}-1\\-3\\-1\end{pmatrix}$$

$$c)\quad v=\begin{pmatrix}1\\1\\1\end{pmatrix}\quad and\quad a=\begin{pmatrix}1\\2\\2\end{pmatrix}$$

Solution

a) $$proj_{\boldsymbol{a}}\boldsymbol{v}=\frac{\boldsymbol{v}\bullet\boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

$$=\frac{1(1)+2(1)+2(1)}{1^2+1^2+1^2}(1,1,1)$$

$$=\frac{5}{3}(1,1,1)$$

$$=\left(\frac{5}{3},\ \frac{5}{3},\ \frac{5}{3}\right)$$

$$\boldsymbol{e}=\boldsymbol{v}-proj_{\boldsymbol{a}}\boldsymbol{v}=(1,2,2)-\left(\frac{5}{3},\frac{5}{3},\frac{5}{3}\right)$$

$$=\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$\boldsymbol{e}\cdot\boldsymbol{a}=\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)\cdot(1,1,1)$$

$$=-\frac{2}{3}+\frac{1}{3}+\frac{1}{3}$$

$$=0$$

$\boldsymbol{e}$ is perpendicular to $\boldsymbol{a}$

b) $$proj_{\boldsymbol{a}}\boldsymbol{v}=\frac{\boldsymbol{v}\bullet\boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

$$=\frac{1(-1)+3(-3)+1(-1)}{(-1)^2+(-3)^2+(-1)^2}(-1,-3,-1)$$

$$=\frac{-11}{11}(-1,-3,-1)$$

$$=(1,\ 3,\ 1)$$

$$\boldsymbol{e}=\boldsymbol{v}-proj_{\boldsymbol{a}}\boldsymbol{v}=(1,3,1)-(1,3,1)$$

$$= \boxed{(0,\ 0,\ 0)}$$

$$\boldsymbol{e} \cdot \boldsymbol{a} = (0,\ 0,\ 0) \cdot (-1, -3, -1)$$
$$= \boxed{0}$$

e is perpendicular to ***a***

c) $$proj_{\boldsymbol{a}} \boldsymbol{v} = \frac{\boldsymbol{v} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2} \boldsymbol{a}$$
$$= \frac{1(1)+1(2)+1(2)}{(1)^2+(2)^2+(2)^2}(1,2,2)$$
$$= \frac{5}{9}(1,2,2)$$
$$= \boxed{\left(\frac{5}{9},\ \frac{10}{9},\ \frac{10}{9}\right)}$$

$$\boldsymbol{e} = \boldsymbol{v} - proj_{\boldsymbol{a}} \boldsymbol{v} = (1,1,1) - \left(\frac{5}{9}, \frac{10}{9}, \frac{10}{9}\right)$$
$$= \boxed{\left(\frac{4}{9},\ -\frac{1}{9},\ -\frac{1}{9}\right)}$$

$$\boldsymbol{e} \cdot \boldsymbol{a} = \left(\frac{4}{9},\ -\frac{1}{9},\ -\frac{1}{9}\right) \cdot (1,2,2)$$
$$= \frac{4}{9} - \frac{2}{9} - \frac{2}{9}$$
$$= \boxed{0}$$

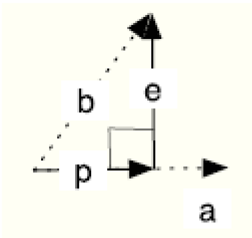
e is perpendicular to ***a***

Exercise

Draw the projection of ***b*** onto ***a*** and also compute it $\quad b = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad$ *and* $\quad a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Solution

$$proj_{\boldsymbol{a}} \boldsymbol{b} = \frac{\boldsymbol{b} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2} \boldsymbol{a}$$
$$= \frac{\cos\theta(1) + \sin\theta(0)}{(1)^2 + 0}(1,0)$$
$$= \cos\theta(1,\ 0)$$
$$= \boxed{(\cos\theta,\ 0)}$$

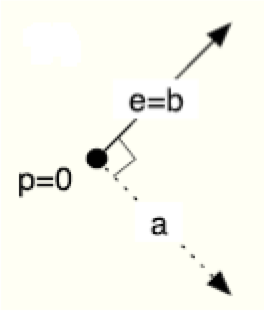


$$\boldsymbol{e} = \boldsymbol{b} - proj_{\boldsymbol{a}} \boldsymbol{b} = (\cos\theta, \sin\theta) - (\cos\theta, 0)$$
$$= \boxed{(0,\ \sin\theta)}$$

Exercise

Draw the projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ and also compute it $\qquad b=\begin{bmatrix}1\\1\end{bmatrix}$ *and* $a=\begin{bmatrix}1\\-1\end{bmatrix}$

Solution



$$proj_{\boldsymbol{a}}\boldsymbol{b}=\frac{\boldsymbol{b}\bullet\boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}=\frac{1(1)+1(-1)}{1^2+(-1)^2}(1,-1)$$

$$=\frac{0}{2}(1,-1)$$

$$\underline{=(0,\ 0)|}$$

$$\boldsymbol{e}=\boldsymbol{b}-proj_{\boldsymbol{a}}\boldsymbol{b}=(1,1)-(0,0)\ =\underline{(1,\ 1)|}$$

Exercise

Find the projection matrix $proj_{\boldsymbol{a}}\boldsymbol{u}=\frac{\boldsymbol{u}\bullet\boldsymbol{a}}{\|\boldsymbol{a}\|^2}$ onto the line through $a=\begin{bmatrix}1\\2\\2\end{bmatrix}$

Solution

$$a^Ta=\begin{bmatrix}1&2&2\end{bmatrix}\begin{bmatrix}1\\2\\2\end{bmatrix}=9$$

$$P=\frac{1}{a^Ta}a.a^T=\frac{1}{9}\begin{pmatrix}1\\2\\2\end{pmatrix}\begin{pmatrix}1&2&2\end{pmatrix}=\frac{1}{9}\begin{pmatrix}1&2&2\\2&4&4\\2&4&4\end{pmatrix}$$

Exercise

Show that if $\boldsymbol{v}$ is orthogonal to both $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$, then $\boldsymbol{v}$ is orthogonal to $k_1\boldsymbol{w}_1+k_2\boldsymbol{w}_2$ for all scalars k_1 and k_2.

Solution

$$\boldsymbol{v}\cdot\left(k_1\boldsymbol{w}_1+k_2\boldsymbol{w}_2\right)=\boldsymbol{v}\cdot\left(k_1\boldsymbol{w}_1\right)+\boldsymbol{v}\cdot\left(k_2\boldsymbol{w}_2\right)$$

$$=k_1\left(\boldsymbol{v}\cdot\boldsymbol{w}_1\right)+k_2\left(\boldsymbol{v}\cdot\boldsymbol{w}_2\right)$$

If v is orthogonal to $\boldsymbol{w}_1$ & $\boldsymbol{w}_2$

$\rightarrow\boldsymbol{v}\cdot\boldsymbol{w}_1=\boldsymbol{v}\cdot\boldsymbol{w}_2=0$

$$=k_1(0)+k_2(0)$$

$$\underline{=0|}$$

Exercise

a) Project the vector $\boldsymbol{v}$ = (3, 4, 4) onto the line through $\boldsymbol{a}$ = (2, 2, 1) and then onto the plane that also contains $\boldsymbol{a}^* = (1,\ 0,\ 0)$.

b) Check that the first error vector $\boldsymbol{v} - \boldsymbol{p}$ is perpendicular to $\boldsymbol{a}$, and the second error vector $\boldsymbol{v} - \boldsymbol{p}^*$ is also perpendicular to $\boldsymbol{a}^*$.

Solution

a) $$proj_{\boldsymbol{a}}\boldsymbol{v} = \frac{\boldsymbol{v} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

$$= \frac{3(2)+4(2)+4(1)}{(2)^2+(2)^2+(1)^2}(2,2,1)$$

$$= \frac{18}{9}(2,2,1)$$

$$= (4,\ 4,\ 2)$$

The plane contains the vectors $\boldsymbol{a}$ and $\boldsymbol{a}^*$ and is the column space of $\boldsymbol{A}$.

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 2 & 1 \end{bmatrix} \qquad \left(A^T A\right)^{-1} = \begin{bmatrix} 9 & 2 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ -2 & 9 \end{bmatrix}$$

$$P = A\left(A^T A\right)^{-1} A^T$$

$$= \frac{1}{5}\begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -2 & 9 \end{bmatrix}\begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & .8 & .4 \\ 0 & .4 & .2 \end{bmatrix}$$

b) The error vector: $e = v - p = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

$$ae = \begin{pmatrix} 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = 2(-1)+2(0)+1(2) = 0.$$

Therefore $\boldsymbol{e}$ is perpendicular to $\boldsymbol{a}$.

$$p^* = Pv = \begin{pmatrix} 1 & 0 & 0 \\ 0 & .8 & .4 \\ 0 & .4 & .2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4.8 \\ 2.4 \end{pmatrix}$$

The error vector: $e^* = v - p^* = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 4.8 \\ 2.4 \end{pmatrix} = \begin{pmatrix} 0 \\ -.8 \\ 1.6 \end{pmatrix}$

$a^* e^* = \begin{pmatrix} 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & -.8 & 1.6 \end{pmatrix} = 2(0) + 2(-.8) + 1(1.6) = 0$.

Therefore $\boldsymbol{e^*}$ is perpendicular to $\boldsymbol{a^*}$.

Exercise

Compute the projection matrices $\boldsymbol{aa}^T / \boldsymbol{a}^T\boldsymbol{a}$ onto the lines through $\boldsymbol{a}_1 = (-1,\ 2,\ 2)$ and $\boldsymbol{a}_2 = (2,\ 2,\ -1)$. Multiply those projection matrices and explain why their product P_1P_2 is what it is.

Project $\boldsymbol{v} = (1,\ 0,\ 0)$ onto the lines $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and also onto $\boldsymbol{a}_3 = (2,\ -1,\ 2)$. Add up the three projections $p_1 + p_2 + p_3$.

Solution

For $\boldsymbol{a}_1 = (-1,\ 2,\ 2)$

$$a_1a_1^T = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}\begin{pmatrix} -1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}$$

$$a_1^Ta_1 = \begin{pmatrix} -1 & 2 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = 9$$

$$P_1 = \frac{aa^T}{a^Ta} = \frac{1}{9}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}$$

For $\boldsymbol{a}_2 = (2,\ 2,\ -1)$

$$a_2a_2^T = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}\begin{pmatrix} 2 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

$$a_2^Ta_2 = \begin{pmatrix} 2 & 2 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = 9$$

$$P_2 = \frac{aa^T}{a^Ta} = \frac{1}{9}\begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

$$P_1P_2 = \frac{1}{9}\left(\frac{1}{9}\right)\begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}\begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

$$= \frac{1}{81}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{= 0}$$

This because a_1 *and* a_2 are perpendicular.

For $\boldsymbol{a}_3 = (2,\ -1,\ 2)$

$$a_3a_3^T = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}$$

$$a_3^Ta_3 = \begin{pmatrix} 2 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 9$$

$$P_3 = \frac{a_3a_3^T}{a_3^Ta_3} = \frac{1}{9}\begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}$$

$$p_3 = P_3v = \frac{1}{9}\begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ \frac{4}{9} \end{pmatrix}$$

$$p_1 = P_1v = \frac{1}{9}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{9} \\ -\frac{2}{9} \\ -\frac{2}{9} \end{pmatrix}$$

$$p_2 = P_2v = \frac{1}{9}\begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{4}{9} \\ \frac{4}{9} \\ -\frac{2}{9} \end{pmatrix}$$

$$p_1 + p_2 + p_3 = \begin{pmatrix} \frac{1}{9} \\ -\frac{2}{9} \\ -\frac{2}{9} \end{pmatrix} + \begin{pmatrix} \frac{4}{9} \\ \frac{4}{9} \\ -\frac{2}{9} \end{pmatrix} + \begin{pmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ \frac{4}{9} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \underline{= \boldsymbol{v}}$$

The reason is that a_3 is perpendicular to a_1 *and* a_2.

Hence, when you compute the three projections of a vector and add them up you get back to the vector you start with.

Exercise

If $P^2 = P$ show that $(I-P)^2 = I-P$. When P projects onto the column space of A, $I-P$ projects onto the ____.

Solution

$$\begin{aligned}(I-P)^2 v &= (I-P)(I-P)v \\ &= (I-P)(Iv-Pv) \\ &= I^2v - IPv - PIv + P^2v \\ &= v - Pv - Pv + P^2v \qquad\qquad P^2v = Pv \quad \textit{By definition} \\ &= v - Pv - Pv + Pv \\ &= v - Pv\end{aligned}$$

$$(I-P)^2 v = (I-P)v \Rightarrow \boxed{(I-P)^2 = (I-P)}$$

When P projects onto the column space of A, then $I-P$ projects onto the left nullspace.

Because $(I-P)^2 v = (I-P)v$; if $P\boldsymbol{v}$ is in the column space of A, then $v - Pv$ is a vector perpendicular to C(A).

Exercise

What linear combination of $(1,\ 2,\ -1)$ and $(1,\ 0,\ 1)$ is closest to $v = (2,\ 1,\ 1)$?

Solution

$$\tfrac{1}{2}(1,\ 2,\ -1) + \tfrac{3}{2}(1,\ 0,\ 1) = (2,\ 1,\ 1)$$

So this $\boldsymbol{v}$ is actually in the span of the two given vectors.

Exercise

Show that $\vec{u} - \vec{v}$ is orthogonal to $\vec{u} + \vec{v}$ if and only if $\|\vec{u}\| = \|\vec{v}\|$

Solution

Suppose that $\vec{u} - \vec{v}$ is orthogonal to $\vec{u} + \vec{v}$. Then

$$\begin{aligned}0 &= \langle \vec{u} - \vec{v},\ \vec{u} + \vec{v} \rangle \\ &= (\vec{u} - \vec{v})^T (\vec{u} + \vec{v})\end{aligned}$$

$$= \left(\vec{u}^T - \vec{v}^T\right)(\vec{u}+\vec{v})$$

$$= \vec{u}^T\vec{u} + \vec{u}^T\vec{v} - \vec{v}^T\vec{u} - \vec{v}^T\vec{v}$$

$$= \langle \vec{u}, \vec{u}\rangle + \langle \vec{u}, \vec{v}\rangle - \langle \vec{v}, \vec{u}\rangle - \langle \vec{v}, \vec{v}\rangle \qquad \langle \vec{u}, \vec{v}\rangle = \langle \vec{v}, \vec{u}\rangle$$

$$= \langle \vec{u}, \vec{u}\rangle - \langle \vec{v}, \vec{v}\rangle$$

So $\langle \vec{u}, \vec{u}\rangle = \langle \vec{v}, \vec{v}\rangle$. Therefore, $\|\vec{u}\|^2 = \|\vec{v}\|^2 \quad \Rightarrow \quad \|\vec{u}\| = \|\vec{v}\|$.

Suppose $\|\vec{u}\| = \|\vec{v}\|$. Then

$$\langle \vec{u}-\vec{v}, \vec{u}+\vec{v}\rangle = (\vec{u}-\vec{v})^T(\vec{u}+\vec{v})$$

$$= \left(\vec{u}^T - \vec{v}^T\right)(\vec{u}+\vec{v})$$

$$= \vec{u}^T\vec{u} + \vec{u}^T\vec{v} - \vec{v}^T\vec{u} - \vec{v}^T\vec{v}$$

$$= \langle \vec{u}, \vec{u}\rangle + \langle \vec{u}, \vec{v}\rangle - \langle \vec{v}, \vec{u}\rangle - \langle \vec{v}, \vec{v}\rangle \qquad \langle \vec{u}, \vec{v}\rangle = \langle \vec{v}, \vec{u}\rangle$$

$$= \langle \vec{u}, \vec{u}\rangle - \langle \vec{v}, \vec{v}\rangle$$

$$= \|\vec{u}\|^2 - \|\vec{v}\|^2$$

$$= 0$$

So we can see that $\vec{u}-\vec{v}$ is orthogonal to $\vec{u}+\vec{v}$

We conclude that $\vec{u}-\vec{v}$ is orthogonal to $\vec{u}+\vec{v}$ if and only if $\|\vec{u}\| = \|\vec{v}\|$, as desired.

Exercise

Given $\boldsymbol{u} = (3, \ -1, \ 2) \quad \boldsymbol{v} = (4, \ -1, \ 5) \quad and \quad \boldsymbol{w} = (8, \ -7, \ -6)$

a) Find $3\boldsymbol{v} - 4(5\boldsymbol{u} - 6\boldsymbol{w})$

b) Find $\boldsymbol{u}\bullet\boldsymbol{v}$ and then the angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$.

Solution

a) $3\boldsymbol{v} - 4(5\boldsymbol{u} - 6\boldsymbol{w}) = 3(4, \ -1, \ 5) - 4\big(5(3, \ -1, \ 2) - 6(8, \ -7, \ -6)\big)$

$$= (12, \ -3, \ 15) - 4\big((15, \ -5, \ 10) - (48, \ -42, \ -36)\big)$$

$$= (12, \ -3, \ 15) - 4(-33, \ 37, \ 46)$$

$$= (12, \ -3, \ 15) - (-132, \ 148, \ 184)$$

$$= (144, \ -151, \ -169)$$

b) $\boldsymbol{u}\bullet\boldsymbol{v} = (3, \ -1, \ 2)\bullet(1, \ 1, \ -1)$

$$= 3 - 1 - 2$$

$$= 0$$

$$\theta = 90°$$

Exercise

Given: $\boldsymbol{u} = (3,\ 1,\ 3)$ $\quad \boldsymbol{v} = (4,\ 1,\ -2)$

a) Compute the projection $\boldsymbol{w}$ of $\boldsymbol{u}$ on $\boldsymbol{v}$

b) Find $\boldsymbol{p} = \boldsymbol{u} - \boldsymbol{w}$ and show that $\boldsymbol{p}$ is perpendicular to $\boldsymbol{v}$.

Solution

a) $\boldsymbol{w} = proj_{\boldsymbol{v}}\boldsymbol{u} = \dfrac{\boldsymbol{u} \bullet \boldsymbol{v}}{\|\boldsymbol{v}\|^2}\boldsymbol{v}$

$$= \frac{(3,\ 1,\ 3)\bullet(4,\ 1,\ -2)}{4^2 + 1^2 + (-2)^2}(4,\ 1,\ -2)$$

$$= \frac{12+1-6}{21}(4,\ 1,\ -2)$$

$$= \frac{7}{21}(4,\ 1,\ -2)$$

$$= \frac{1}{3}(4,\ 1,\ -2)$$

$$= \left(\frac{4}{3},\ \frac{1}{3},\ -\frac{2}{3}\right)$$

b) $\boldsymbol{p} = (3,\ 1,\ 3) - \left(\frac{4}{3},\ \frac{1}{3},\ -\frac{2}{3}\right) = \left(\frac{5}{3},\ \frac{2}{3},\ \frac{11}{3}\right)$

$\boldsymbol{p}\bullet\boldsymbol{u} = \left(\frac{5}{3},\ \frac{2}{3},\ \frac{11}{3}\right)\bullet(4,\ 1,\ -2) = \frac{20}{3} + \frac{2}{3} - \frac{22}{3}$ $= 0$; $\boldsymbol{p}$ is perpendicular to $\boldsymbol{v}$.

Exercise

a) Show that $\boldsymbol{v} = (a, b)$ and $\boldsymbol{w} = (-b, a)$ are orthogonal vectors

b) Use the result in part (*a*) to find two vectors that are orthogonal to $\boldsymbol{v} = (2, -3)$

c) Find two unit vectors that are orthogonal to $(-3, 4)$

Solution

a) $\boldsymbol{u}\cdot\boldsymbol{v} = -ab + ba = 0$; 2 vectors are orthogonal vectors.

b) $\boldsymbol{v} = (2, -3) \Rightarrow \boldsymbol{w} = (-3, -2)$ and $\boldsymbol{w} = (3, 2)$

c) $(-3, 4) \Rightarrow \boldsymbol{u} = \dfrac{(-3,\ 4)}{\sqrt{9+16}} = \left(-\frac{3}{5},\ \frac{4}{5}\right)$

$$\boldsymbol{u}_1 = \left(\frac{4}{5}, \frac{3}{5}\right) \quad and \quad \boldsymbol{u}_2 = \left(-\frac{4}{5}, -\frac{3}{5}\right)$$

Exercise

Show that $A(3, 0, 2)$, .$B(4, 3, 0)$, and $C(8, 1, -1)$ are vertices of a right triangle. At which vertex is the right angle?

Solution

$$AB = (4-3, 3-0, 0-2) = (1, 3, -2) \qquad AC = (5, 1, -3) \qquad BC = (4, -2, -1)$$

$$AB \bullet AC = 5+3+6 = 14$$

$$AB \bullet BC = 4-6+2 = 0$$

$$AC \bullet BC = 20-2+3 = 21$$

The right triangle at point B

Exercise

Establish the identity: $\boldsymbol{u} \bullet \boldsymbol{v} = \frac{1}{4}\|\boldsymbol{u}+\boldsymbol{v}\|^2 - \frac{1}{4}\|\boldsymbol{u}-\boldsymbol{v}\|^2$

Solution

Let $\boldsymbol{u}\left(u_1,\ u_2,\ \ldots,\ u_n\right)$ *and* $\boldsymbol{v} = \left(v_1,\ v_2,\ \ldots,\ v_n\right)$

$$\boldsymbol{u} \bullet \boldsymbol{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n$$

$$\boldsymbol{u}+\boldsymbol{v} = \left(u_1+v_1,\ u_2+v_2,\ \ldots, u_n+v_n\right)$$

$$\begin{aligned}
\|\boldsymbol{u}+\boldsymbol{v}\|^2 &= \left(u_1+v_1\right)^2 + \left(u_2+v_2\right)^2 + \cdots + \left(u_n+v_n\right)^2 \\
&= u_1^2 + v_1^2 + 2u_1v_1 + u_2^2 + v_2^2 + 2u_2v_2 + \cdots + u_2^2 + v_n^2 + 2u_nv_n
\end{aligned}$$

$$\boldsymbol{u}-\boldsymbol{v} = \left(u_1-v_1,\ u_2-v_2,\ \ldots, u_n-v_n\right)$$

$$\begin{aligned}
\|\boldsymbol{u}-\boldsymbol{v}\|^2 &= \left(u_1-v_1\right)^2 + \left(u_2-v_2\right)^2 + \cdots + \left(u_n-v_n\right)^2 \\
&= u_1^2 + v_1^2 - 2u_1v_1 + u_2^2 + v_2^2 - 2u_2v_2 + \cdots + u_2^2 + v_n^2 - 2u_nv_n
\end{aligned}$$

$$\begin{aligned}
\|\boldsymbol{u}+\boldsymbol{v}\|^2 - \|\boldsymbol{u}-\boldsymbol{v}\|^2 &= u_1^2 + v_1^2 + 2u_1v_1 + u_2^2 + v_2^2 + 2u_2v_2 + \cdots + u_2^2 + v_n^2 + 2u_nv_n \\
&\qquad -\left(u_1^2 + v_1^2 - 2u_1v_1 + u_2^2 + v_2^2 - 2u_2v_2 + \cdots + u_2^2 + v_n^2 - 2u_nv_n\right) \\
&= u_1^2 + v_1^2 + 2u_1v_1 + u_2^2 + v_2^2 + 2u_2v_2 + \cdots + u_2^2 + v_n^2 + 2u_nv_n \\
&\qquad - u_1^2 - v_1^2 + 2u_1v_1 - u_2^2 - v_2^2 + 2u_2v_2 - \cdots - u_2^2 - v_n^2 + 2u_nv_n \\
&= 4u_1v_1 + 4u_2v_2 + \cdots + 4u_nv_n
\end{aligned}$$

$$\frac{1}{4}\left(\|\boldsymbol{u}+\boldsymbol{v}\|^2 - \|\boldsymbol{u}-\boldsymbol{v}\|^2\right) = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

Therefore; $\boldsymbol{u} \bullet \boldsymbol{v} = \frac{1}{4}\|\boldsymbol{u}+\boldsymbol{v}\|^2 - \frac{1}{4}\|\boldsymbol{u}-\boldsymbol{v}\|^2$ is true.

2nd method:

$$\frac{1}{4}\|\boldsymbol{u}+\boldsymbol{v}\|^2-\frac{1}{4}\|\boldsymbol{u}-\boldsymbol{v}\|^2=\frac{1}{4}\left[(\boldsymbol{u}+\boldsymbol{v})(\boldsymbol{u}+\boldsymbol{v})-(\boldsymbol{u}-\boldsymbol{v})(\boldsymbol{u}-\boldsymbol{v})\right]$$

$$=\frac{1}{4}\left[\boldsymbol{uu}+2\boldsymbol{uv}+\boldsymbol{vv}-(\boldsymbol{uu}-2\boldsymbol{uv}+\boldsymbol{vv})\right]$$

$$=\frac{1}{4}\left[\boldsymbol{uu}+2\boldsymbol{uv}+\boldsymbol{vv}-\boldsymbol{uu}+2\boldsymbol{uv}-\boldsymbol{vv}\right]$$

$$=\frac{1}{4}(4\boldsymbol{uv})$$

$$=\boldsymbol{u}\bullet\boldsymbol{v}$$

Exercise

Find the Euclidean inner product $\boldsymbol{u}\bullet\boldsymbol{v}$: $\boldsymbol{u}=(-1,\ 1,\ 0,\ 4,\ -3)$ $\boldsymbol{v}=(-2,\ -2,\ 0,\ 2,\ -1)$

Solution

$$\boldsymbol{u}\bullet\boldsymbol{v}=2-2+0+8+3\ \underline{=11|}$$

Exercise

Find the Euclidean distance between $\boldsymbol{u}$ and $\boldsymbol{v}$: $\boldsymbol{u}=(3,\ -3,\ -2,\ 0,\ -3)$ $\boldsymbol{v}=(-4,\ 1,\ -1,\ 5,\ 0)$

Solution

$$d(\boldsymbol{u},\ \boldsymbol{v})=\|\boldsymbol{u}-\boldsymbol{v}\|=\sqrt{(u_1-v_1)^2+(u_2-v_2)^2+\ldots+(u_n-v_n)^2}$$

$$=\sqrt{(3+4)^2+(-3-1)^2+(-2+1)^2+(0-5)^2+(-3-0)^2}$$

$$=\sqrt{49+16+1+25+9}$$

$$=\sqrt{100}$$

$$\underline{=10|}$$

Solution   Section 2.4 – Cross Product

Exercise

Prove when the cross product $\boldsymbol{u}\times\boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$, then $\boldsymbol{u}.(\boldsymbol{u}\times\boldsymbol{v})=0$

Solution

Let $\boldsymbol{u}=(u_1,\ u_2,\ u_3)$ and $\boldsymbol{v}=(v_1,\ v_2,\ v_3)$

$$\begin{aligned}\boldsymbol{u}.(\boldsymbol{u}\times\boldsymbol{v}) &= (u_1,\ u_2,\ u_3).(u_2v_3-u_3v_2,\ u_3v_1-u_1v_3,\ u_1v_2-u_2v_1)\\ &= u_1(u_2v_3-u_3v_2)+u_2(u_3v_1-u_1v_3)+u_3(u_1v_2-u_2v_1)\\ &= u_1u_2v_3-u_1u_3v_2+u_2u_3v_1-u_2u_1v_3+u_3u_1v_2-u_3u_2v_1\\ &= 0\end{aligned}$$

Exercise

Find $\boldsymbol{u}\times\boldsymbol{v}$, where $\boldsymbol{u}=(1,\ 2,\ -2)$ *and* $\boldsymbol{v}=(3,\ 0,\ 1)$ and show that $\boldsymbol{u}\times\boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ and to $\boldsymbol{v}$.

Solution

$$u\times v=\left(\begin{vmatrix}2 & -2\\ 0 & 1\end{vmatrix},\ -\begin{vmatrix}1 & -2\\ 3 & 1\end{vmatrix},\ \begin{vmatrix}1 & 2\\ 3 & 0\end{vmatrix}\right)$$

$$=(2,\ -7,\ -6)$$

$$\begin{aligned}u\bullet(u\times v) &= (1,\ 2,\ -2)\bullet(2,\ -7,\ -6)\\ &= 2-14+12\\ &= 0\end{aligned}$$

$$\begin{aligned}v\bullet(u\times v) &= (3,\ 0,\ 1)\bullet(2,\ -7,\ -6)\\ &= 6-0-6\\ &= 0\end{aligned}$$

$\boldsymbol{u}\times\boldsymbol{v}$ is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$.

Exercise

Given $\boldsymbol{u}=(3,\ 2,\ -1),\quad \boldsymbol{v}=(0,\ 2,\ -3),\ \textit{and}\quad \boldsymbol{w}=(2,\ 6,\ 7)$ Compute the vectors

a) $\boldsymbol{u}\times\boldsymbol{v}$

b) $\boldsymbol{v}\times\boldsymbol{w}$

c) $\boldsymbol{u}\times(\boldsymbol{v}\times\boldsymbol{w})$

d) $(\boldsymbol{u}\times\boldsymbol{v})\times\boldsymbol{w}$

e) $\boldsymbol{u}\times(\boldsymbol{v}-2\boldsymbol{w})$

Solution

a) $$u\times v=\left(\begin{vmatrix}2&-1\\2&-3\end{vmatrix},\ -\begin{vmatrix}3&-1\\0&-3\end{vmatrix},\ \begin{vmatrix}3&2\\0&2\end{vmatrix}\right)$$
$$=(-4,\ 9,\ 6)$$

b) $$v\times w=\left(\begin{vmatrix}2&-3\\6&7\end{vmatrix},\ -\begin{vmatrix}0&-3\\2&7\end{vmatrix},\ \begin{vmatrix}0&2\\2&6\end{vmatrix}\right)$$
$$=(32,\ -6,\ -4)$$

c) $$u\times(v\times w)=(3,\ 2,\ -1)\times(32,\ -6,\ -4)$$
$$=\left(\begin{vmatrix}2&-1\\-6&-4\end{vmatrix},\ -\begin{vmatrix}3&-1\\32&-4\end{vmatrix},\ \begin{vmatrix}3&2\\32&-6\end{vmatrix}\right)$$
$$=(-14,\ -20,\ -82)$$

d) $$(u\times v)\times w=(-4,\ 9,\ 6)\times(2,\ 6,\ 7)$$
$$=\left(\begin{vmatrix}9&6\\6&7\end{vmatrix},\ -\begin{vmatrix}-4&6\\2&7\end{vmatrix},\ \begin{vmatrix}-4&9\\2&6\end{vmatrix}\right)$$
$$=(27,\ 40,\ -42)$$

e) $$u\times(v-2w)=(3,\ 2,\ -1)\times\left[(0,\ 2,\ -3)-2(2,\ 6,\ 7)\right]$$
$$=(3,\ 2,\ -1)\times(-4,\ -10,\ -17)$$
$$=\left(\begin{vmatrix}2&-1\\-10&-17\end{vmatrix},\ -\begin{vmatrix}3&-1\\-4&-17\end{vmatrix},\ \begin{vmatrix}3&2\\-4&-10\end{vmatrix}\right)$$
$$=(-44,\ 47,\ -22)$$

Exercise

Use the cross product to find a vector that is orthogonal to both

a) $\boldsymbol{u}=(-6,\ 4,\ 2),\quad \boldsymbol{v}=(3,\ 1,\ 5)$

b) $\boldsymbol{u}=(1,\ 1,\ -2),\quad \boldsymbol{v}=(2,\ -1,\ 2)$

c) $\boldsymbol{u}=(-2,\ 1,\ 5),\quad \boldsymbol{v}=(3,\ 0,\ -3)$

Solution

a) $\boldsymbol{u}\times\boldsymbol{v}=(-6,\ 4,\ 2)\times(3,\ 1,\ 5)$

$$=\left(\begin{vmatrix}4 & 2\\ 1 & 5\end{vmatrix},\ -\begin{vmatrix}-6 & 2\\ 3 & 5\end{vmatrix},\ \begin{vmatrix}-6 & 4\\ 3 & 1\end{vmatrix}\right)$$

$$=(18,\ 36,\ -18)$$

b) $\boldsymbol{u}\times\boldsymbol{v}=(1,\ 1,\ -2)\times(2,\ -1,\ 2)$

$$=\left(\begin{vmatrix}1 & -2\\ -1 & 2\end{vmatrix},\ -\begin{vmatrix}1 & -2\\ 2 & 2\end{vmatrix},\ \begin{vmatrix}1 & 1\\ 2 & -1\end{vmatrix}\right)$$

$$=(0,\ -6,\ -3)$$

c) $\boldsymbol{u}\times\boldsymbol{v}=(-2,\ 1,\ 5)\times(3,\ 0,\ -3)$

$$=\left(\begin{vmatrix}1 & 5\\ 0 & -3\end{vmatrix},\ -\begin{vmatrix}-2 & 5\\ 3 & -3\end{vmatrix},\ \begin{vmatrix}-2 & 1\\ 3 & 0\end{vmatrix}\right)$$

$$=(-3,\ 9,\ -3)$$

Exercise

Find the area of the parallelogram determined by the given vectors

a) $\boldsymbol{u}=(1,\ -1,\ 2)$ *and* $\boldsymbol{v}=(0,\ 3,\ 1)$

b) $\boldsymbol{u}=(3,\ -1,\ 4)$ *and* $\boldsymbol{v}=(6,\ -2,\ 8)$

c) $\boldsymbol{u}=(2,\ 3,\ 0)$ *and* $\boldsymbol{v}=(-1,\ 2,\ -2)$

Solution

a) $Area=\|u\times v\|$

$$=\left\|\left(\begin{vmatrix}-1 & 2\\ 3 & 1\end{vmatrix},\ -\begin{vmatrix}1 & 2\\ 0 & 1\end{vmatrix},\ \begin{vmatrix}1 & -1\\ 0 & 3\end{vmatrix}\right)\right\|$$

$$=\|(-7,\ -1,\ 3)\|$$

$$=\sqrt{7^2+1^2+3^2}$$

$$=\sqrt{59}\quad (Area)$$

b) $Area = \|u \times v\|$

$$= \left\| \left(\begin{vmatrix} -1 & 4 \\ -2 & 8 \end{vmatrix}, \ -\begin{vmatrix} 3 & 4 \\ 6 & 8 \end{vmatrix}, \ \begin{vmatrix} 3 & -1 \\ 6 & -2 \end{vmatrix} \right) \right\|$$

$$= \|(0,\ 0,\ 0)\|$$

$$= 0$$

c) $Area = \|u \times v\| = (2,\ 3,\ 0) \times (-1,\ 2,\ -2)$

$$= \left\| \left(\begin{vmatrix} 3 & 0 \\ 2 & -2 \end{vmatrix}, \ -\begin{vmatrix} 2 & 0 \\ -1 & -2 \end{vmatrix}, \ \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} \right) \right\|$$

$$= \|(-6,\ 4,\ 7)\|$$

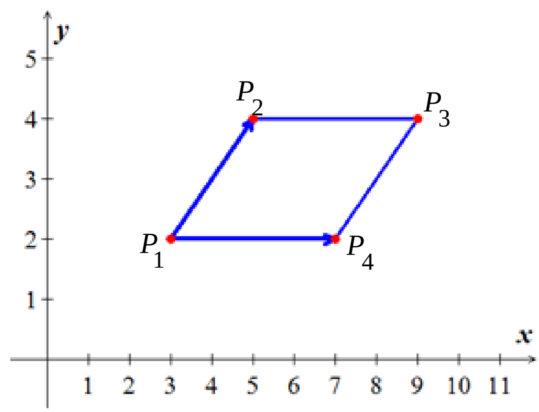
$$= \sqrt{(-6)^2 + 4^2 + 7^2}$$

$$= \sqrt{101} \quad (Area)$$

Exercise

Find the area of the parallelogram with the given vertices $P_1(3,2)$, $P_2(5,4)$, $P_3(9,4)$, $P_4(7,2)$

Solution



$$\overrightarrow{P_1P_2} = (5-3, 4-2) = (2,\ 2)$$

$$\overrightarrow{P_4P_3} = (9-7, 4-2) = (2,\ 2)$$

$$\overrightarrow{P_1P_4} = (7-3, 2-2) = (4,\ 0)$$

$$\overrightarrow{P_2P_3} = (9-5, 4-4) = (4,\ 0)$$

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_2} = (2,2) \times (4,0)$$

$$= \left(\begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix}, \ -\begin{vmatrix} 2 & 0 \\ 4 & 0 \end{vmatrix}, \ \begin{vmatrix} 2 & 2 \\ 4 & 0 \end{vmatrix} \right)$$

$$= (0,\ 0,\ -8)$$

The area of the parallelogram is

$$\left\| \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_2} \right\| = \sqrt{0 + 0 + (-8)^2} = 8$$

Exercise

Find the area of the triangle with the given vertices:

a) $A(2,0) \quad B(3,4) \quad C(-1,2)$

b) $A(1,1) \quad B(2,2) \quad C(3,-3)$

c) $P(2,\ 6,\ -1) \quad Q(1,\ 1,\ 1) \quad R=(4,\ 6,\ 2)$

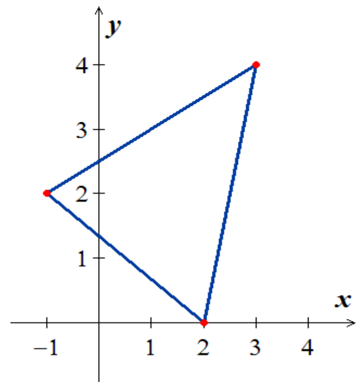
Solution

a) $\overrightarrow{AB}=(1,\ 4) \qquad \overrightarrow{AC}=(-3,\ 2)$

$$\overrightarrow{AB}\times\overrightarrow{AC}=(1,\ 4,\ 0)\times(-3,\ 2,\ 0)$$

$$=\left(\begin{vmatrix}4 & 0\\ 2 & 0\end{vmatrix},\ -\begin{vmatrix}1 & 0\\ -3 & 0\end{vmatrix},\ \begin{vmatrix}1 & 4\\ -3 & 2\end{vmatrix}\right)$$

$$=(0,\ 0,\ 14)$$



$$\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|=\sqrt{0+0+14^2}=14$$

The area of the triangle is $\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|=\frac{1}{2}14=7$

b) $\overrightarrow{AB}=(1,\ 1) \qquad \overrightarrow{AC}=(2,\ -4)$

$$\overrightarrow{AB}\times\overrightarrow{AC}=(1,\ 1,\ 0)\times(2,\ -4,\ 0)$$

$$=\left(\begin{vmatrix}1 & 0\\ -4 & 0\end{vmatrix},\ -\begin{vmatrix}1 & 0\\ 2 & 0\end{vmatrix},\ \begin{vmatrix}1 & 1\\ 2 & -4\end{vmatrix}\right)$$

$$=(0,\ 0,\ -6)$$

$$\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|=\sqrt{0+0+(-6)^2}=6$$

The area of the triangle is $\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|=\frac{1}{2}(6)=3$

c) $\overrightarrow{PQ}=(-1,\ -5,\ 2) \qquad \overrightarrow{PR}=(2,\ 0,\ 3)$

$$\overrightarrow{PQ}\times\overrightarrow{PR}=(-1,\ -5,\ 2)\times(2,\ 0,\ 3)$$

$$\begin{array}{ccccc}-1 & -5 & 2 & -1 & -5\\ 2 & 0 & 3 & 2 & 0\end{array}$$

$$=(-15,\ 7,\ 10)$$

$$\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\|=\sqrt{(-15)^2+7^2+10^2}=\sqrt{374}$$

The area of the triangle is $\frac{1}{2}\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\|=\frac{1}{2}\sqrt{374}$

Exercise

a) Find the area of the parallelogram with edges $v = (3,\ 2)$ and $w = (1,\ 4)$

b) Find the area of the triangle with sides ***v, w***, and ***v*** + ***w***. Draw it.

c) Find the area of the triangle with sides ***v, w***, and ***v*** – ***w***. Draw it.

Solution

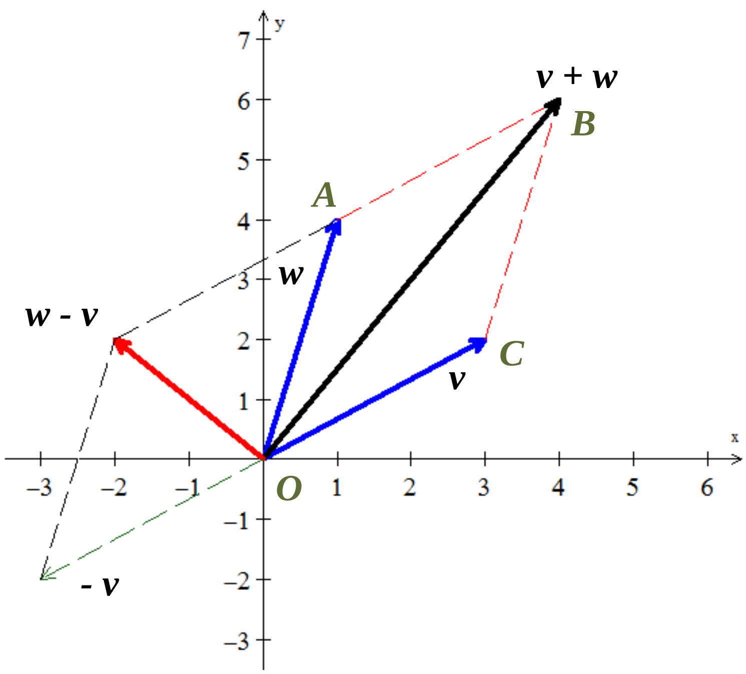
a) $Area = \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 10$ (which is the parallelogram *OABC*)

b) The area of the triangle with sides ***v, w,*** and ***v*** + ***w*** is the triangle OCB or OAB which it is half the parallelogram (by definition).

$Area = 5$

$v + w = (3,\ 2) + (1,\ 4) = (4,\ 6)$

$Area = \frac{1}{2}\begin{vmatrix} 3 & 4 \\ 2 & 6 \end{vmatrix} = \frac{1}{2}(10)\ \underline{= 5|}$



c) The area of the triangle with sides ***v, w,*** and ***v - w*** is equivalent to the triangle OAC which it is half the parallelogram (by definition).

$Area = 5$

$Area = \frac{1}{2}\begin{vmatrix} 2 & -2 \\ -3 & -2 \end{vmatrix} = \frac{1}{2}|-10|\ \underline{= 5|}$

Exercise

Find the volume of the parallelepiped with sides $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$.

a) $\boldsymbol{u}=(2,-6,2),\quad \boldsymbol{v}=(0,4,-2),\quad \boldsymbol{w}=(2,2,-4)$

b) $\boldsymbol{u}=(3,1,2),\quad \boldsymbol{v}=(4,5,1),\quad \boldsymbol{w}=(1,2,4)$

Solution

a) $\boldsymbol{u}.(\boldsymbol{v}\times\boldsymbol{w})=\begin{vmatrix}2 & -6 & 2\\ 0 & 4 & -2\\ 2 & 2 & -4\end{vmatrix}=-16$

The volume of the parallelepiped is $|-16|=\underline{16}|$

b) $\boldsymbol{u}.(\boldsymbol{v}\times\boldsymbol{w})=\begin{vmatrix}3 & 1 & 2\\ 4 & 5 & 1\\ 1 & 2 & 4\end{vmatrix}=45$

The volume of the parallelepiped is $\boxed{45}$

Exercise

Compute the scalar triple product $\boldsymbol{u}.(\boldsymbol{v}\times\boldsymbol{w})$

a) $\boldsymbol{u}=(-2,0,6),\quad \boldsymbol{v}=(1,-3,1),\quad \boldsymbol{w}=(-5,-1,1)$

b) $\boldsymbol{u}=(-1,2,4),\quad \boldsymbol{v}=(3,4,-2),\quad \boldsymbol{w}=(-1,2,5)$

c) $\boldsymbol{u}=(a,0,0),\quad \boldsymbol{v}=(0,b,0),\quad \boldsymbol{w}=(0,0,c)$

d) $\boldsymbol{u}=3\boldsymbol{i}-2\boldsymbol{j}-5\boldsymbol{k},\quad \boldsymbol{v}=\boldsymbol{i}+4\boldsymbol{j}-4\boldsymbol{k},\quad \boldsymbol{w}=3\boldsymbol{j}+2\boldsymbol{k}$

e) $\boldsymbol{u}=(3,\ -1,\ 6)\quad \boldsymbol{v}=(2,\ 4,\ 3)\quad \boldsymbol{w}=(5,\ -1,\ 2)$

Solution

a) $\boldsymbol{u}.(\boldsymbol{v}\times\boldsymbol{w})=\begin{vmatrix}-2 & 0 & 6\\ 1 & -3 & 1\\ -5 & -1 & 1\end{vmatrix}=\underline{-92}|$

b) $\boldsymbol{u}.(\boldsymbol{v}\times\boldsymbol{w})=\begin{vmatrix}-1 & 2 & 4\\ 3 & 4 & -2\\ -1 & 2 & 5\end{vmatrix}=\underline{-10}|$

c) $\boldsymbol{u}.(\boldsymbol{v}\times\boldsymbol{w})=\begin{vmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{vmatrix}=\underline{abc}|$

d) $\boldsymbol{u}.(\boldsymbol{v}\times\boldsymbol{w})=\begin{vmatrix}3 & -2 & -5\\ 1 & 4 & -4\\ 0 & 3 & 2\end{vmatrix}\ \underline{=49}|$

$$e)\quad \boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w}) = \begin{vmatrix} 3 & -1 & 6 \\ 2 & 4 & 3 \\ 5 & -1 & 2 \end{vmatrix} \underline{= -110|}$$

Exercise

Use the cross product to find the sine of the angle between the vectors $\boldsymbol{u} = (2, 3, -6), \quad \boldsymbol{v} = (2, 3, 6)$

Solution

$$\boldsymbol{u}\times\boldsymbol{v} = (2,3,-6)\times(2,3,6)$$

$$= \left(\begin{vmatrix} 3 & -6 \\ 3 & 6 \end{vmatrix}, -\begin{vmatrix} 2 & -6 \\ 2 & 6 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} \right)$$

$$(36, \ -24, \ 0)$$

$$\|\boldsymbol{u}\times\boldsymbol{v}\| = \sqrt{36^2 + (-24)^2 + 0} = \sqrt{1872} = 12\sqrt{13}$$

$$\sin\theta = \left(\frac{\|\boldsymbol{u}\times\boldsymbol{v}\|}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} \right)$$

$$= \frac{12\sqrt{13}}{\sqrt{2^2+3^2+(-6)^2}\sqrt{2^2+3^2+6^2}}$$

$$= \frac{12\sqrt{13}}{(7)(7)}$$

$$\underline{= \frac{12}{49}\sqrt{13}|}$$

Exercise

Simplify $(\boldsymbol{u}+\boldsymbol{v})\times(\boldsymbol{u}-\boldsymbol{v})$

Solution

$$(\boldsymbol{u}+\boldsymbol{v})\times(\boldsymbol{u}-\boldsymbol{v}) = (\boldsymbol{u}+\boldsymbol{v})\times\boldsymbol{u} - (\boldsymbol{u}+\boldsymbol{v})\times\boldsymbol{v}$$

$$= (\boldsymbol{u}\times\boldsymbol{u}) + (\boldsymbol{v}\times\boldsymbol{u}) - [(\boldsymbol{u}\times\boldsymbol{v}) + (\boldsymbol{v}\times\boldsymbol{v})]$$

$$= 0 + (\boldsymbol{v}\times\boldsymbol{u}) - [(\boldsymbol{u}\times\boldsymbol{v}) + 0]$$

$$= (\boldsymbol{v}\times\boldsymbol{u}) - (\boldsymbol{u}\times\boldsymbol{v})$$

$$= (\boldsymbol{v}\times\boldsymbol{u}) - (-(\boldsymbol{v}\times\boldsymbol{u}))$$

$$= (\boldsymbol{v}\times\boldsymbol{u}) + (\boldsymbol{v}\times\boldsymbol{u})$$

$$\underline{= 2(\boldsymbol{v}\times\boldsymbol{u})|}$$

Exercise

Prove Lagrange's identity: $\|\boldsymbol{u}\times\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2\ \|\boldsymbol{v}\|^2 - (\boldsymbol{u}.\boldsymbol{v})^2$

Solution

Let $\boldsymbol{u} = (u_1,\ u_2,\ u_3)$ and $\boldsymbol{v} = (v_1,\ v_2,\ v_3)$

$$\|\boldsymbol{u}\|^2 = u_1^2 + u_2^2 + u_3^2$$

$$\|\boldsymbol{v}\|^2 = v_1^2 + v_2^2 + v_3^2$$

$$(\boldsymbol{u}.\boldsymbol{v})^2 = (u_1v_1 + u_2v_2 + u_3v_3)^2$$

$$\begin{aligned}\|\boldsymbol{u}\times\boldsymbol{v}\|^2 &= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\ &= u_2^2v_3^2 - 2u_2v_3u_3v_2 + u_3^2v_2^2 + u_3^2v_1^2 - 2u_3v_1u_1v_3 + u_1^2v_3^2 + u_1^2v_2^2 - 2u_2v_1u_2v_1 + u_2^2v_1^2\end{aligned}$$

$$\begin{aligned}\|\boldsymbol{u}\|^2\ \|\boldsymbol{v}\|^2 - (\boldsymbol{u}.\boldsymbol{v})^2 &= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \\ &= u_1^2v_1^2 + u_1^2v_2^2 + u_1^2v_3^2 + u_2^2v_1^2 + u_2^2v_2^2 + u_2^2v_3^2 + u_3^2v_1^2 + u_3^2v_2^2 + u_3^2v_3^2 \\ &\qquad - u_1^2v_1^2 - u_1v_1u_2v_2 - u_1v_1u_3v_3 \\ &\qquad - u_2v_2u_1v_1 - u_2^2v_2^2 - u_2v_2u_3v_3 \\ &\qquad - u_1v_1u_3v_3 - u_2v_2u_3v_3 - u_3^2v_3^2 \\ &= u_2^2v_3^2 - 2u_2v_2u_3v_3 + u_3^2v_2^2 \\ &\qquad + u_3^2v_1^2 - 2u_1v_1u_3v_3 + u_1^2v_3^2 \\ &\qquad + u_1^2v_2^2 - 2u_1v_1u_2v_2 + u_2^2v_1^2\end{aligned}$$

$$\Rightarrow \boxed{\|\boldsymbol{u}\times\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2\ \|\boldsymbol{v}\|^2 - (\boldsymbol{u}.\boldsymbol{v})^2}$$

Exercise

Polar coordinates satisfy $x = r\cos\theta$ *and* $y = \sin\theta$. Polar area $J\ dr\ d\theta$ includes J:

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

The two columns are orthogonal. Their lengths are ______. Thus J = ______.

Solution

The length of the first column is: $= \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$

The length of the second column is: $= \sqrt{r^2\sin^2\theta + r^2\cos^2\theta}$

$$= \sqrt{r^2\left(\sin^2\theta + \cos^2\theta\right)}$$

$$= \sqrt{r^2}$$

$$= r$$

So J is the product $1.\ r = r$.

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$

$$= r\left(\cos^2\theta + \sin^2\theta\right)$$

$$= r$$

Solution   *Section* 2.5 – Subspaces, Span and Null Space

Exercise

Suppose S and T are two subspaces of a vector space **V**.

a) The sum $S+T$ contains all sums $s+t$ of a vector s in S and a vector t in T. Show that $S+T$ satisfies the requirements (addition and scalar multiplication) for a vector space.

b) If S and T are lines in $\boldsymbol{R}^m$, what is the difference between $S+T$ and $S \cup T$? That union contains all vectors from S and T or both. Explain this statement: The span of $S \cup T$ is $S+T$.

Solution

a) Let $s,\ s'$ be vectors in S, Let $t,\ t'$ be vectors in T, and let c be a scalar. Then

$$(s+t)+(s'+t')=(s+s')+(t+t') \text{ and}$$

$$c(s+t)=cs+ct$$

Thus $S+T$ is closed under addition and scalar multiplication, it satisfies the two requirements for a vector space.

b) If S and T are distinct lines, then S and T is a plane, whereas $S \cup T$ is not even closed under addition. The span of $S \cup T$ is the set of all combinations of vectors in this union. In particular, it contains all sums $s+t$ of a vector $\boldsymbol{s}$ in S and a vector $\boldsymbol{t}$ in T, and these sums form $S+T$. $S+T$ contains both S and T; so it contains $S \cup T$. $S+T$ is a vector space.

c) So it contains all combinations of vectors in itself; in particular, it contains the span of $S \cup T$. Thus the span of $S \cup T$ is $S+T$.

Exercise

Determine which of the following are subspaces of $\boldsymbol{R}^3$?

a) All vectors of the form $(a,\ 0,\ 0)$

b) All vectors of the form $(a,\ 1,\ 1)$

c) All vectors of the form $(a,\ b,\ c)$, where $b = a + c$

d) All vectors of the form $(a,\ b,\ c)$, where $b = a + c + 1$

e) All vectors of the form $(a,\ b,\ 0)$

Solution

a) $(a_1,\ 0,\ 0)+(a_2,\ 0,\ 0)=(a_1+a_2,\ 0,\ 0)$

$k(a,\ 0,\ 0)=(ka,\ 0,\ 0)$

This is a subspace of $\boldsymbol{R}^3$

b) $(a_1,\ 1,\ 1)+(a_2,\ 1,\ 1)=(a_1+a_2,\ 2,\ 2)$ which is not in the set.

Therefore, this is not a subspace of $\boldsymbol{R}^3$

c) $(a_1, b_1, c_1)+(a_2, b_2, c_2)=(a_1+a_2, b_1+b_2, c_1+c_2)$

$$=(a_1+a_2, a_1+c_1+a_2+c_2, c_1+c_2)$$

$$=(a_1+a_2, (a_1+a_2)+(c_1+c_2), c_1+c_2)$$

$$=(a_1, a_1+c_1, c_1)+(a_2, a_2+c_2, c_2)$$

$k(a, b, c)=(ka, kb, kc)$

$$=(ka, k(a+c), kc)$$

$$=k(a, (a+c), c)$$

This is a subspace of $\boldsymbol{R}^3$

d) $k(a+c+1)\neq ka+kc+1$ so $k(a,b,c)$ is not in the set.

Therefore, this is not a subspace of $\boldsymbol{R}^3$

e) $(a_1, b_1, 0)+(a_2, b_2, 0)=(a_1+a_2, b_1+b_2, 0)$

$k(a, b, 0)=(ka, kb, 0)$

This is a subspace of $\boldsymbol{R}^3$

Exercise

Determine which of the following are subspaces of $\boldsymbol{R}^\infty$?

a) All sequences $\boldsymbol{v}$ in $\boldsymbol{R}^\infty$ of the form $v = (v, 0, v, 0, \ldots)=(kv, k, kv, k, \cdots)$

b) All sequences $\boldsymbol{v}$ in $\boldsymbol{R}^\infty$ of the form $v = (v, 1, v, 1, \ldots)$

c) All sequences $\boldsymbol{v}$ in $\boldsymbol{R}^\infty$ of the form $v = (v, 2v, 4v, 8v, 16v, \ldots)$

Solution

a) $(v_1,0,v_1,0,\ldots)+(v_2,0,v_2,0,\ldots)=(v_1+v_2, 0, v_1+v_2, 0, \ldots)$

$kv=k(v,0,v,0,\ldots)=(kv,0,kv,0,\ldots)$

This is a subspace of $\boldsymbol{R}^\infty$

b) $kv=k(v,1,v,1,\ldots)$

kv is not in the set

Since $k \neq 1$, then is not a subspace of $\boldsymbol{R}^\infty$

c) $\left(v_1,2v_1,4v_1,8v_1,\ldots\right)+\left(v_2,2v_2,4v_2,8v_2,\ldots\right)=\left(v_1+v_2,2v_1+2v_2,4v_1+4v_2,8v_1+8v_2,\ldots\right)$

$$=\left(v_1+v_2,2\left(v_1+v_2\right),4\left(v_1+v_2\right),8\left(v_1+v_2\right),\ldots\right)$$

$k\left(v,2v,4v,8v,\ldots\right)=\left(kv,\ 2kv,\ 4kv,\ 8kv,\ \ldots\right)$

This is a subspace of $\boldsymbol{R}^{\infty}$

Exercise

Which of the following are linear combinations of $\boldsymbol{u}$ = (0, –2, 2) and $\boldsymbol{v}$ = (1, 3, –1)?

a) (2, 2, 2) *b)* (3, 1, 5) *c)* (0, 4, 5) *d)* (0, 0 ,0)

Solution

$$A=\begin{bmatrix}0&1\\-2&3\\2&-1\end{bmatrix}\quad x=\begin{bmatrix}x_1\\x_2\end{bmatrix}\quad b=\begin{bmatrix}a\\b\\c\end{bmatrix}$$

a) $b=\begin{bmatrix}2\\2\\2\end{bmatrix}\rightarrow\left[\begin{array}{cc|c}0&1&2\\-2&3&2\\2&-1&2\end{array}\right]\xrightarrow{rref}\left[\begin{array}{cc|c}1&0&2\\0&1&2\\0&0&0\end{array}\right]$

(2, 2, 2) = 2$\boldsymbol{u}$ + 2$\boldsymbol{v}$ is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.

b) $b=\begin{bmatrix}3\\1\\5\end{bmatrix}\rightarrow\left[\begin{array}{cc|c}0&1&3\\-2&3&1\\2&-1&5\end{array}\right]\xrightarrow{rref}\left[\begin{array}{cc|c}1&0&4\\0&1&3\\0&0&0\end{array}\right]$

(3, 1, 5) = 4$\boldsymbol{u}$ + 3$\boldsymbol{v}$ is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.

c) $b=\begin{bmatrix}0\\4\\5\end{bmatrix}\rightarrow\left[\begin{array}{cc|c}0&1&0\\-2&3&4\\2&-1&5\end{array}\right]\xrightarrow{rref}\left[\begin{array}{cc|c}1&0&0\\0&1&0\\0&0&1\end{array}\right]$

(0, 4, 5) is not a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.

d) $b=\begin{bmatrix}0\\0\\0\end{bmatrix}\rightarrow\left[\begin{array}{cc|c}0&1&0\\-2&3&0\\2&-1&0\end{array}\right]\xrightarrow{rref}\left[\begin{array}{cc|c}1&0&0\\0&1&0\\0&0&0\end{array}\right]$

(0, 0, 0) = 0$\boldsymbol{u}$ + 0$\boldsymbol{v}$ is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.

Exercise

Which of the following are linear combinations of $\boldsymbol{u} = (2, 1, 4)$, $\boldsymbol{v} = (1, -1, 3)$ and $\boldsymbol{w} = (3, 2, 5)$?

a) (–9, –7, –15)

b) (6, 11, 6)

c) (0, 0 ,0)

Solution

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 3 & 5 \end{pmatrix}$$

a) $$\left[\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Therefore, $(-9,\ -7,\ -15) = -2\boldsymbol{u} + 1\boldsymbol{v} - 2\boldsymbol{w}$

b) $$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 6 \\ 1 & -1 & 2 & 11 \\ 4 & 3 & 5 & 6 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Therefore, $(6,\ 11,\ 6) = 4\boldsymbol{u} - 5\boldsymbol{v} + 1\boldsymbol{w}$

c) $$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore, $(0,\ 0,\ 0) = 0\boldsymbol{u} + 0\boldsymbol{v} + 0\boldsymbol{w}$

Exercise

Determine whether the given vectors span $\boldsymbol{R}^3$

a) $v_1 = (2,2,2),\quad v_2 = (0,0,3),\quad v_3 = (0,1,1)$

b) $v_1 = (2,-1,3),\quad v_2 = (4,1,2),\quad v_3 = (8,-1,8)$

c) $v_1 = (3,1,4),\quad v_2 = (2,-3,5),\quad v_3 = (5,-2,9),\quad v_4 = (1,4,-1)$

Solution

a) $det\begin{pmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{pmatrix} = -6 \neq 0$

The system is consistent for all values so the given vectors span $\boldsymbol{R}^3$.

b) $det\begin{pmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{pmatrix} = 0$

The system is not consistent for all values so the given vectors do not span $\boldsymbol{R}^3$.

c) $\left[\begin{array}{cccc|c} 3 & 2 & 5 & 1 & b_1 \\ 1 & -3 & -2 & 4 & b_2 \\ 4 & 5 & 9 & -1 & b_3 \end{array}\right] \xrightarrow{\text{leads to}} \left[\begin{array}{cccc|c} 1 & -3 & -2 & 4 & b_2 \\ 0 & 1 & 1 & -1 & \frac{1}{11}b_1 - \frac{3}{11}b_2 \\ 0 & 0 & 0 & 0 & -\frac{17}{11}b_1 + \frac{7}{11}b_2 + b_3 \end{array}\right]$

The system has a solution only if $-\frac{17}{11}b_1 + \frac{7}{11}b_2 + b_3 = 0$. But since this is a restriction that the given vectors don't span on all of $\boldsymbol{R}^3$. So the given vectors do not span $\boldsymbol{R}^3$.

Exercise

Which of the following are linear combinations of $A = \begin{pmatrix} 4 & 0 \\ -2 & -2 \end{pmatrix},\quad B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix},\quad C = \begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}$

a) $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ *b)* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *c)* $\begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix}$

Solution

$$\begin{pmatrix} 4 & 1 & 0 \\ 0 & -1 & 2 \\ -2 & 2 & 1 \\ -2 & 3 & 4 \end{pmatrix}$$

a) $\left[\begin{array}{ccc|c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & -5 \\ -2 & 2 & 1 & -1 \\ -2 & 3 & 4 & -8 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = 1A + 2B - 3C$ is a linear combinations of *A*, *B*, and *C*.

b) $\left[\begin{array}{ccc|c} 4 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ -2 & 2 & 1 & 0 \\ -2 & 3 & 4 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0A + 0B + 0C$ is a linear combinations of *A*, *B*, and *C*.

c) $\left[\begin{array}{ccc|c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & 0 \\ -2 & 2 & 1 & 3 \\ -2 & 3 & 4 & 8 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix} = 1A + 2B + 1C$ is a linear combinations of *A*, *B*, and *C*.

Exercise

Suppose that $v_1 = (2,1,0,3)$, $v_2 = (3,-1,5,2)$, $v_3 = (-1,0,2,1)$. Which of the following vectors are in span $\{v_1, v_2, v_3\}$

a) (2, 3, –7, 3) *b)* (0, 0, 0, 0) *c)* (1, 1, 1, 1) *d)* (–4, 6, –13, 4)

Solution

In order to be span $\{v_1, v_2, v_3\}$, there must exists scalars *a*, *b*, *c* that $av_1 + bv_2 + cv_3 = \boldsymbol{w}$

$$A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

a) $\left[\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 1 & -1 & 0 & 3 \\ 0 & 5 & 2 & -7 \\ 3 & 2 & 1 & 3 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

This system is consistent, it has only solution is $a = 2$, $b = -1$, $c = -1$

$2v_1 - 1v_2 - 1v_3 = (2,3,-7,3)$, therefore (2, 3, –7, 3) is in span $\{v_1, v_2, v_3\}$

b) The vector (0, 0 , 0, 0) is obviously in span $\{v_1, v_2, v_3\}$ since $0v_1 + 0v_2 + 0v_3 = (0,0,0,0)$

c)
$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 5 & 2 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

This system is inconsistent, therefore (1, 1, 1, 1) is not in span $\{v_1, v_2, v_3\}$

d)
$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & -4 \\ 1 & -1 & 0 & 6 \\ 0 & 5 & 2 & -13 \\ 3 & 2 & 1 & 4 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This system is consistent, it has only solution is $a = 3$, $b = -3$, $c = 1$

$3v_1 - 3v_2 + 1v_3 = (-4,6,-13,4)$, therefore (–4, 6, –13, 4) is in span $\{v_1, v_2, v_3\}$

Exercise

Let $\boldsymbol{f} = \cos^2 x$ and $\boldsymbol{g} = \sin^2 x$. Which of the following lie in the space spanned by $\boldsymbol{f}$ and $\boldsymbol{g}$

a) $\cos 2x$ *b)* $3 + x^2$ *c)* $\sin x$ *d)* 0

Solution

a) $\cos 2x = \cos^2 x - \sin^2 x$, therefore $\cos 2x$ is in span $\{\boldsymbol{f}, \boldsymbol{g}\}$

b) In order for $3 + x^2$ to be in span $\{\boldsymbol{f}, \boldsymbol{g}\}$, there must exist scalars *a* and *b* such that

$$a\cos^2 x + b\sin^2 x = 3 + x^2$$

When $\left.\begin{array}{l} x = 0 \quad \Rightarrow \quad a = 3 \\ x = \pi \quad \Rightarrow \quad a = 3 + \pi^2 \end{array}\right\} \rightarrow \textit{contradiction}$

Therefore $3 + x^2$ is not in span $\{\boldsymbol{f}, \boldsymbol{g}\}$

c) In order for $\sin x$ to be in span $\{\boldsymbol{f}, \boldsymbol{g}\}$, there must exist scalars *a* and *b* such that

$$a\cos^2 x + b\sin^2 x = \sin x$$

When $\left.\begin{array}{l} x = \frac{\pi}{2} \quad \Rightarrow \quad b = 1 \\ x = -\frac{\pi}{2} \quad \Rightarrow \quad b = -1 \end{array}\right\} \rightarrow \textit{contradiction}$

Therefore $\sin x$ is not in span $\{\boldsymbol{f}, \boldsymbol{g}\}$

d) In order for 0 to be in span $\{\boldsymbol{f}, \boldsymbol{g}\}$, there must exist scalars a and b such that

$$0\cos^2 x + 0\sin^2 x = 0$$

Therefore $\mathbf{0}$ is in span $\{\boldsymbol{f}, \boldsymbol{g}\}$

Exercise

$V = \mathbb{R}^3, \quad S = \{(0,\ s,\ t) \mid s,\ t \text{ are real numbers}\}$ where V is a vector space and S is subset of V

a) Is S closed under addition?

b) Is S closed under scalar multiplication?

c) Is S a subspace of V?

Solution

a) Let $\boldsymbol{u} = (0,\ s_1,\ t_1)$ *and* $\boldsymbol{v} = (0,\ s_2,\ t_2)$

$$\boldsymbol{u} + \boldsymbol{v} = (0,\ s_1 + s_2,\ t_1 + t_2) = (0,\ s,\ t)$$

Yes, S is closed under addition

b) $k\boldsymbol{u} = (0,\ ks_1,\ kt_1) = (0,\ s,\ t)$

Yes, S is closed under scalar multiplication

c) Since S is closed under addition and scalar multiplication, then S is a subspace of V.

Exercise

$V = \mathbb{R}^3, \quad S = \{(x,\ y,\ z) \mid x,\ y\ ,\ z\ \geq 0\}$ where V is a vector space and S is subset of V

a) Is S closed under addition?

b) Is S closed under scalar multiplication?

c) Is S a subspace of V?

Solution

a) Let $\boldsymbol{u} = (x_1, y_1, z_1)$ *and* $\boldsymbol{v} = (x_2, y_2, z_2)$

$$\boldsymbol{u} + \boldsymbol{v} = (x_1 + x_2, y_1 + y_2, z_1 + z_2) = (x,\ y,\ z) \qquad \textit{where } x = x_1 + x_2,\ y = y_1 + y_2,\ z = z_1 + z_2$$

Yes, S is closed under addition

b) $(-1)\boldsymbol{u} = (-x_1,\ -y_1,\ -z_1)$

S is ***not*** closed under scalar multiplication since $x_1 \geq 0 \ \Rightarrow\ -x_1 \leq 0$

c) S is ***not*** a subspace of V.

Exercise

$V=\mathbb{R}^3, \quad S=\{(x,\ y,\ z)|\ z=x+y+1\}$ where V is a vector space and S is subset of V

a) Is S closed under addition?

b) Is S closed under scalar multiplication?

c) Is S a subspace of V?

Solution

a) Let $\boldsymbol{u}=(0,\ 1,\ 2) \quad \textit{and} \quad \boldsymbol{v}=(1,\ 2,\ 4)$

$$\boldsymbol{u}+\boldsymbol{v}=(1,\ 3,\ 6)$$
$$\neq(1,\ 3,\ 1+3+1)$$

No, S is *not* closed under addition

b)
$$\begin{aligned} 2\boldsymbol{u} &= \left(2x_1,\ 2y_1,\ 2z_1\right) \\ &= \left(2x_1,\ 2y_1,\ 2\left(x_1+y_1+1\right)\right) \\ &= \left(2x_1,\ 2y_1,\ 2x_1+2y_1+2\right) \qquad \textit{Where } x=2x_1,\ y=2y_1, 2z=2\left(x_1+y_1+1\right) \\ &= (x,\ y,\ z) \end{aligned}$$

Yes, S is closed under scalar multiplication

c) S is ***not*** a subspace of V.

Exercise

$V=M_{33}, \quad S=\{A|\ A \textit{ is invertible}\}$ where V is a vector space and S is subset of V

a) Is S closed under addition?

b) Is S closed under scalar multiplication?

c) Is S a subspace of V?

Solution

a) Let assume: $A=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ *and* $B=\begin{pmatrix}1&1\\0&-1\end{pmatrix}$ are invertible but $A+B=\begin{pmatrix}2&1\\0&0\end{pmatrix}$ is not invertible.

S is not closed under addition

b) S is not closed under scalar multiplication if $k=0$

c) S is ***not*** a subspace of V.

Exercise

Given: $A=\begin{bmatrix}1&3&2\\2&6&4\end{bmatrix}$

a) Find $NS(A)$

b) For which n is $NS(A)$ a subspace of $\mathbb{R}^n$

c) Sketch $NS(A)$ in $\mathbb{R}^2$ or $\mathbb{R}^3$

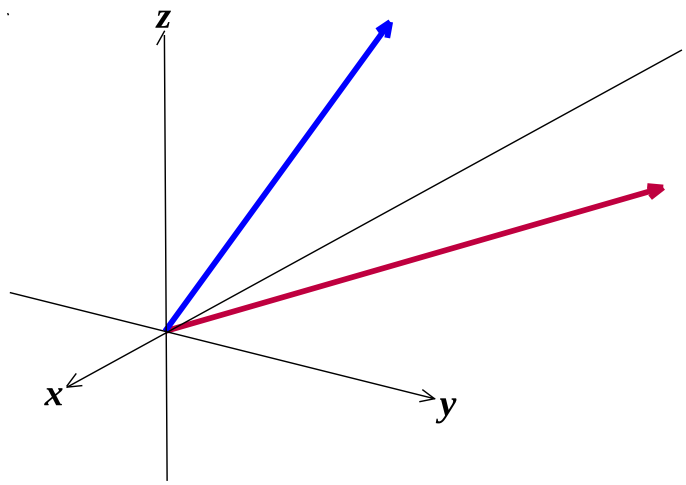
Solution

a) $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix} \xrightarrow[R_2 - 2R_1]{rref} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x = -3y - 2z$

$$\left\{ y\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \middle| y, z \in \mathbb{R} \right\}$$

b) $n = 3$

c)



Exercise

Determine which of the following are subspaces of M_{22}

a) All 2×2 matrices with integer entries

b) All matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a + b + c + d = 0$

Solution

a) Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ where $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are integers

$A + B = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix}$ where $a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3,\ a_4 + b_4$ are integers too.

Then, it is closed under addition.

$$\frac{1}{2}A = \frac{1}{2}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix}$$

It is not closed under multiplication if the scalar is a real number.

Therefore; it is ***not*** a subspace of M_{22}

b) Let $A=\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ $a_1+a_2+a_3+a_4=0$ and $B=\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ $b_1+b_2+b_3+b_4=0$

$$A+B=\begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{bmatrix}$$

$$a_1+a_2+a_3+a_4+b_1+b_2+b_3+b_4=0$$

$$\left(a_1+b_1\right)+\left(a_2+b_2\right)+\left(a_3+b_3\right)+\left(a_4+b_4\right)=0$$

Then, it is closed under addition.

$$kA=\begin{bmatrix} ka_1 & ka_2 \\ ka_3 & ka_4 \end{bmatrix} \quad ka_1+ka_2+ka_3+ka_4=k\left(a_1+a_2+a_3+a_4\right)=k(0)=0$$

It is closed under multiplication

Therefore; it is a subspace of M_{22}

Solution Section 2.6 – Linear Independence

Exercise

Given three independent vectors w_1, w_2, w_3. Take combinations of those vectors to produce v_1, v_2, v_3. Write the combinations in a matrix form as $V = WM$.

$$\begin{aligned} v_1 &= w_1 + w_2 \\ v_2 &= w_1 + 2w_2 + w_3 \\ v_1 &= w_2 + cw_3 \end{aligned} \quad \text{which is } \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & c \end{bmatrix}$$

What is the test on a matrix **V** to see if its columns are linearly independent?

If $c \neq 1$ show that v_1, v_2, v_3 are linearly independent.

If $c = 1$ show that $v's$ are linearly *dependent*.

Solution

The nullspace of **V** must contain only the *zero* vector. Then $x = (0,\ 0,\ 0)$ is the only combination of the columns that gives **V**x = zero vector.

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & c \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & c \end{bmatrix} \xrightarrow[R_3 - R_2]{R_1 - R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & \boxed{c-1} \end{bmatrix}$$

If $c \neq 1$, then the matrix M is invertible. So if x is any nonzero vector we know that Mx is nonzero. Since $\boldsymbol{w}$'s are given as independent and WMx is nonzero. Since $V = WM$, this says that x is not in the nullspace of **V**. therefore; v_1, v_2, v_3 are independent.

$$\text{If } c = 1\text{, that implies } \begin{aligned} v_1 &= w_1 + w_2 \\ v_2 &= w_1 + w_2 + w_2 + w_3 \\ v_3 &= w_2 + w_3 \end{aligned} \Rightarrow \begin{aligned} v_1 &= w_1 + w_2 \\ &\boxed{v_2 = v_1 + v_3} \\ v_3 &= w_2 + w_3 \end{aligned}$$

$-v_1 + v_2 - v_3 = 0$, which means that $v's$ are linearly *dependent*.

The other way, the vector $x = (1,\ -1,\ 1)$ is in that nullspace, and $Mx = 0$. Then certainly $WMx = 0$ which is the same as $Vx = 0$. So the $v's$ are dependent.

Exercise

Find the largest possible number of independent vectors among

$$v_1=\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}\quad v_2=\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}\quad v_3=\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\quad v_4=\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\quad v_5=\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}\quad v_6=\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$$

Solution

Since $v_4=v_2-v_1,\ v_5=v_3-v_1,\ \textit{and}\ \ v_6=v_3-v_2$, there are at most three independent vectors among these: furthermore, applying row reduction to the matrix $\begin{bmatrix}v_1 & v_2 & v_3\end{bmatrix}$ gives three pivots, showing that v_1, v_2, v_3 are independent.

Exercise

Show that $v_1,\ v_2,\ v_3$ are independent but $v_1,\ v_2,\ v_3,\ v_4$ are dependent:

$$v_1=\begin{pmatrix}1\\0\\0\end{pmatrix}\quad v_2=\begin{pmatrix}1\\1\\0\end{pmatrix}\quad v_3=\begin{pmatrix}1\\1\\1\end{pmatrix}\quad v_4=\begin{pmatrix}2\\3\\4\end{pmatrix}$$

Solve either $c_1v_1+c_2v_2+c_3v_3=0$ *or* $Ax=0$. The v's go in the columns of ***A***.

Solution

$$\begin{pmatrix}v_1 & v_2 & v_3\end{pmatrix}=\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$$

This matrix has 3 pivots with rank of 3 equals to rows that implies the $v_1,\ v_2,\ v_3$ are independent.

$v_4=v_1+v_2-4v_3$ *or* $v_1+v_2-4v_3-v_4=0$ that shows that $v_1,\ v_2,\ v_3,\ v_4$ are dependent.

Exercise

Decide the dependence or independence of

a) The vectors (1, 3, 2) and (2, 1, 3) and (3, 2, 1).

b) The vectors $(1,\ -3,\ 2)$ and $(2,\ 1,\ -3)$ and $(-3,\ 2,\ 1)$.

Solution

a) These are linearly independent. $x_1(1,\ 3,\ 2)+x_2(2,\ 1,\ 3)+x_3(3,\ 2,\ 1)=(0,\ 0,\ 0)$ only if $x_1=x_2=x_3=0$

b) These are linearly dependent: $1(1,\ -3,\ 2)+1(2,\ 1,\ -3)+1(-3,\ 2,\ 1)=(0,\ 0,\ 0)$

Exercise

Find two independent vectors on the plane $x+2y-3z-t=0$ in $\boldsymbol{R}^4$. Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

Solution $(3,\ 0,\ 1,\ 0)$

This plane is the nullspace of the matrix

$$A=\begin{pmatrix}1 & 2 & -3 & -1\end{pmatrix}$$

$x_1+2x_2-3x_3-x_4=0$

The pivot is 1st column, and the rest are 3 variables.

If $x_2=-1 \quad x_3=x_4=0 \Rightarrow x_1=2$. The vector is $(2,\ -1,\ 0,\ 0)$

If $x_3=1 \quad x_1=x_4=0 \Rightarrow x_1=3$. The vector is

If $x_4=1 \quad x_1=x_3=0 \Rightarrow x_1=1$. The vector is $(1,\ 0,\ 0,\ 1)$

The 3 vectors $(2,\ -1,\ 0,\ 0)$, $(3,\ 0,\ 1,\ 0)$, $(1,\ 0,\ 0,\ 1)$ are linearly independent.

We can't find 4 independent vectors because the nullspace only has dimension 3 (have 3 variables).

Exercise

Determine whether the vectors are linearly dependent or linearly independent in $\boldsymbol{R}^3$

a) (4, –1, 2), (–4, 10, 2)

b) (8, –1, 3), (4, 0, 1)

c) (–3, 0, 4), (5, –1, 2), (1, 1, 3)

d) (–2, 0, 1), (3, 2, 5), (6, –1, 1), (7, 0, –2)

Solution

a) The vector equation $a(4, -1, 2) + b(-4, 10, 2) = (0, 0, 0)$

$$\left[\begin{array}{cc|c} 4 & -4 & 0 \\ -1 & 10 & 0 \\ 2 & 2 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system has only the trivial solution $a = b = 0$.
We conclude that the given set of vectors is linearly independent.

b) $a(8, -1, 3) + b(4, 0, 1) = (0, 0, 0)$

$$\left[\begin{array}{cc|c} 8 & 4 & 0 \\ -1 & 0 & 0 \\ 3 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore, the system has only one trivial solution $a = b = 0$.
We conclude that the given set of vectors is linearly independent

c) The vector equation $a(-3, 0, 4) + b(5, -1, 2) + c(1, 1, 3) = (0, 0, 0)$

$$\left[\begin{array}{ccc|c} -3 & 5 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 4 & 2 & 3 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore the system has only the trivial solution $a = b = c = 0$.
We conclude that the given set of vectors is linearly independent.

d) The vector equation $a(-2, 0, 1) + b(3, 2, 5) + c(6, -1, 1) + d(7, 0, -2) = (0, 0, 0)$

$$\left[\begin{array}{cccc|c} -2 & 3 & 6 & 7 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 1 & 5 & 1 & -2 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{79}{29} & 0 \\ 0 & 1 & 0 & \frac{3}{29} & 0 \\ 0 & 0 & 1 & \frac{6}{29} & 0 \end{array}\right]$$

Therefore the system has nontrivial solutions $a = \frac{79}{29}t, \quad b = -\frac{3}{29}t, \quad c = -\frac{6}{29}t, \quad d = t$

We conclude that the given set of vectors is linearly dependent.

Exercise

Determine whether the vectors are linearly dependent or linearly independent in $\boldsymbol{R}^4$

a) (3, 8, 7, –3), (1, 5, 3, –1), (2, –1, 2, 6), (1, 4, 0, 3)

b) (0, 0, 2, 2), (3, 3, 0, 0), (1, 1, 0, –1)

c) (0, 3, –3, –6), (–2, 0, 0, –6), (0, –4, –2, –2), (0, –8, 4, –4)

d) (3, 0, –3, 6), (0, 2, 3, 1), (0, –2, –2, 0), (–2, 1, 2, 1)

Solution

a) $\det\begin{pmatrix} 3 & 1 & 2 & 1 \\ 8 & 5 & -1 & 4 \\ 7 & 3 & 2 & 0 \\ -3 & -1 & 6 & 3 \end{pmatrix} = 128 \neq 0$

The system has only the trivial solution and the vectors are linearly independent.

b) $k_1(0,0,2,2) + k_2(3,3,0,0) + k_3(1,1,0,-1) = (0,0,0,0)$

$$\left[\begin{array}{ccc|c} 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 = k_2 = k_3 = 0$

The system has only the trivial solution and the vectors are linearly independent.

c) $\det\begin{pmatrix} 0 & -2 & 0 & 0 \\ 3 & 0 & -4 & -8 \\ -3 & 0 & -2 & 4 \\ -6 & -6 & -2 & -4 \end{pmatrix} = 480 \neq 0$

The system has only the trivial solution and the vectors are linearly independent.

d) $a(3,\ 0,\ -3,\ 6) + b(0,\ 2,\ 3,\ 1) + c(0,\ -2,\ -2,\ 0) + d(-2,\ 1,\ 2,\ 1) = (0,\ 0,\ 0,\ 0)$

$$\left[\begin{array}{cccc|c} 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & 1 & 0 \\ -3 & 3 & -2 & 2 & 0 \\ 6 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore, the system has only one trivial solution $a = b = c = d = 0$.

The given set of vectors is linearly independent

Exercise

a) Show that the three vectors $v_1 = (1,2,3,4)$ $\quad v_2 = (0,1,0,-1)$ $\quad v_3 = (1,3,3,3)$ form a linearly dependent set in $\boldsymbol{R}^4$.

b) Express each vector in part (*a*) as a linear combination of the other two.

Solution

a) The vector equation $k_1(1,2,3,4) + k_2 = (0,1,0,-1) + k_3(1,3,3,3) = (0,0,0,0)$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 \\ 3 & 0 & 3 & 0 \\ 4 & -1 & 3 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The solution: $k_1 = -t, \ k_2 = -t, \ k_3 = t$

Since the system has nontrivial solutions, the given set of vectors is linearly dependent.

b) Since $k_1 = -t, \ k_2 = -t, \ k_3 = t$ and if we let $t = 1$, then $-v_1 - v_2 + v_3 = 0$

$$v_1 = -v_2 + v_3, \quad v_2 = -v_1 + v_3, \quad v_3 = v_1 + v_2$$

Exercise

For which real values of λ do the following vectors form a linearly dependent set in $\boldsymbol{R}^3$

$$v_1 = \left(\lambda, -\tfrac{1}{2}, -\tfrac{1}{2}\right) \quad v_2 = \left(-\tfrac{1}{2}, \lambda, -\tfrac{1}{2}\right) \quad v_3 = \left(-\tfrac{1}{2}, -\tfrac{1}{2}, \lambda\right)$$

Solution

$$k_1\left(\lambda, -\tfrac{1}{2}, -\tfrac{1}{2}\right) + k_2 = \left(-\tfrac{1}{2}, \lambda, -\tfrac{1}{2}\right) + k_3\left(-\tfrac{1}{2}, -\tfrac{1}{2}, \lambda\right) = (0,0,0,0)$$

$$\det\begin{pmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{pmatrix} = \frac{1}{4}\left(4\lambda^3 - 3\lambda - 1\right)$$

For $\lambda = 1 \quad \lambda = -\dfrac{1}{2}$, the determinant is zero and the vectors form a linearly dependent set.

Exercise

Show that if $S=\{v_1, v_2, \ldots, v_n\}$ is a linearly independent set of vectors, then so is every nonempty subset of S.

Solution

Let $\{v_a, v_b, \ldots, v_r\}$ be a nonempty subset of *S*.

If this set is linearly dependent, then there would be a nonzero solution $(k_a, k_b, \ldots, k_r)$ to $k_a v_a + k_b v_b + \ldots + k_r v_r = 0$. This can be expanded to a nonzero solution of $k_1 v_1 + k_2 v_2 + \ldots + k_n v_n = 0$ by taking all other coefficients as 0. This contradicts the linear independence of S, so the subset must be linearly independent.

Exercise

Show that if $S=\{v_1, v_2, \ldots, v_r\}$ is a linearly dependent set of vectors in a vector space *V*, and if $v_{r+1}, \ldots, v_n$ are vectors in *V* that are not in *S*, then $\{v_1, v_2, \ldots, v_r, v_{r+1}, \ldots, v_n\}$ is also linearly dependent.

Solution

If *S* is linearly dependent, then there is a nonzero solution $(k_1, k_2, \ldots, k_r)$ to $k_1 v_1 + k_2 v_2 + \ldots + k_r v_r = 0$. Thus $(k_1, k_2, \ldots, k_r, 0, 0, \ldots, 0)$ is a nonzero solution to $k_1 v_1 + k_2 v_2 + \ldots + k_r v_r + k_{r+1} v_{r+1} \ldots + k_n v_n = 0$ so the set $\{v_1, v_2, \ldots, v_r, v_{r+1}, \ldots, v_n\}$ is linearly dependent.

Exercise

Show that $\{v_1, v_2\}$ is linearly independent and v_3 does not lie in span $\{v_1, v_2\}$, then $\{v_1, v_2, v_3\}$ is a linearly independent.

Solution

If $\{v_1, v_2, v_3\}$ are linearly dependent, there exist a nonzero solution to $k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$ with $k_3 \neq 0$ (since v_1 and v_2 are linearly independent).

$k_3 v_3 = -k_1 v_1 - k_2 v_2 \quad \Rightarrow \quad v_3 = -\dfrac{k_1}{k_3} v_1 - \dfrac{k_2}{k_3} v_2$ which contradicts that v_3 is not in span $\{v_1, v_2\}$.

Thus $\{v_1, v_2, v_3\}$ is a linearly independent.

Exercise

By using the appropriate identities, where required, determine $F(-\infty, \infty)$ are linearly dependent.

a) $6,\ 3\sin^2 x,\ 2\cos^2 x$

b) $x,\ \cos x$

c) $1,\ \sin x,\ \sin 2x$

d) $(3-x)^2,\ x^2-6x,\ 5$

e) $\cos 2x,\ \sin^2 x,\ \cos^2 x$

Solution

a) From the identity $\sin^2 x + \cos^2 x = 1$

$$(-1)(6)+(2)\left(3\sin^2 x\right)+(3)\left(2\cos^2 x\right) = -6+6\left(\sin^2 x+\cos^2 x\right) \underline{= 0}$$

Therefore, the set is linearly dependent.

b) $ax + b\cos x = 0$

$x = 0 \quad \Rightarrow \quad b = 0$

$x = \frac{\pi}{2} \quad \Rightarrow \quad a = 0$

Therefore, the set is linearly independent.

c) $a(1) + b\sin x + c\sin 2x = 0$

$x = 0 \quad \Rightarrow \quad a = 0$

$x = \frac{\pi}{2} \quad \Rightarrow \quad b = 0$

$x = \frac{\pi}{4} \quad \Rightarrow \quad c = 0$

Therefore, the set is linearly independent.

d) $(3-x)^2 = 9 - 6x + x^2$

$(3-x)^2 - \left(9-6x+x^2\right) = 0$

$(3-x)^2 - \left(x^2-6x\right) - 9 = 0$

$(1)(3-x)^2 + (-1)\left(x^2-6x\right) + \left(-\frac{9}{5}\right)5 = 0$

Therefore, the set is linearly dependent.

e) By using the double angle:

$\cos 2x = \cos^2 x - \sin^2 x$ are linearly dependent.

Exercise

$f_1(x) = \sin x, \quad f_2(x) = \cos x$ are linearly independent in $F(-\infty, \infty)$ because neither function is a scalar multiple of the other. Confirm the linear independence using Wroński's test.

Solution

The Wronskian: $W(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$

$$= -\sin^2 x - \cos^2 x$$

$$= -\left(\sin^2 x + \cos^2 x\right)$$

$$\underline{= -1 \neq 0|}$$

$\sin x \quad$ *and* $\quad \cos x$ are linearly independent

Exercise

Use the Wronskian to show that $f_1(x) = \sin x, \quad f_2(x) = \cos x, \quad f_3(x) = x\cos x$ span a three-dimensional subspace of $F(-\infty, \infty)$

Solution

The Wronskian: $W(x) = \begin{vmatrix} \sin x & \cos x & x\cos x \\ \cos x & -\sin x & \cos x - x\sin x \\ -\sin x & -\cos x & -2\sin x - x\cos x \end{vmatrix}$

$$= 2\sin^3 x + x\sin^2 x\cos x - \sin x\cos^2 x + x\sin^2 x\cos x - x\cos^3 x$$
$$\quad - x\sin^2 x\cos x + \sin x\cos^2 x - x\sin^2 x\cos x + 2\sin x\cos^2 x + x\cos^3 x$$

$$= 2\sin^3 x + 2\sin x\cos^2 x$$

$$= 2\sin x\left(\sin^2 x + \cos^2 x\right)$$

$$\underline{= 2\sin x|}$$

Since $\sin x \neq 0$ for all real x values, the vectors are linearly independent.

Exercise

Show by inspection that the vectors are linearly dependent.

$\boldsymbol{v}_1(4,\ -1,\ 3), \quad \boldsymbol{v}_2(2,\ 3,\ -1), \quad \boldsymbol{v}_3(-1,\ 2,\ -1), \quad \boldsymbol{v}_4(5,\ 2,\ 3), \quad$ *in* $\mathbb{R}^3$

Solution

$$\begin{bmatrix} 4 & 2 & -1 & 5 \\ -1 & 3 & 2 & 2 \\ 3 & -1 & -1 & 3 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & \frac{11}{7} \\ 0 & 1 & 0 & \frac{1}{7} \\ 0 & 0 & 1 & \frac{11}{7} \end{bmatrix}$$

$7\boldsymbol{v}_4 = 11\boldsymbol{v}_1 + \boldsymbol{v}_2 + 11\boldsymbol{v}_2$

Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

a) $(2,\ -1,\ 3),\ \ (3,\ 4,\ 1),\ \ (2,\ -3,\ 4),\ \ in\ \mathbb{R}^3.$

b) $(1,\ 0,\ 0,\ 0),\ \ (1,\ 1,\ 0,\ 0),\ \ (1,\ 1,\ 1,\ 0),\ \ (1,\ 1,\ 1,\ 1),\ \ in\ \mathbb{R}^4.$

c) $A_1\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix},\ \ A_2\begin{bmatrix} -2 & 4 \\ 1 & 0 \end{bmatrix},\ \ A_3\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix},\ \ in\ M_{22}$

Solution

a) $a(2,\ -1,\ 3) + b(3,\ 4,\ 1) + c(2,\ -3,\ 4) = (0,\ 0,\ 0)$

$$\begin{bmatrix} 2 & 3 & 2 & 0 \\ -1 & 4 & -3 & 0 \\ 3 & 1 & 4 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The system has only he trivial solution $a = b = c = 0$.

$$\begin{vmatrix} 2 & 3 & 2 \\ -1 & 4 & -3 \\ 3 & 1 & 4 \end{vmatrix} = 32 - 27 - 2 - 24 + 6 + 12 \neq 0$$

The system has only the trivial solution and the vectors are linearly independent

b) $$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

The system has only the trivial solution and the vectors are linearly independent

c) $$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 2 & 4 & -1 & 0 \\ 3 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ The vectors are linearly independent

Exercise

Suppose that the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \text{ and } \boldsymbol{u}_3$ are linearly dependent. Are the vectors $\boldsymbol{v}_1 = \boldsymbol{u}_1 + \boldsymbol{u}_2$, $\boldsymbol{v}_2 = \boldsymbol{u}_1 + \boldsymbol{u}_3$, and $\boldsymbol{v}_3 = \boldsymbol{u}_2 + \boldsymbol{u}_3$ also linearly dependent?

(***Hint***: Assume that $a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3 = 0$, and see what the $a_i\,'s$ can be.)

Solution

Given: $\boldsymbol{u}_1, \boldsymbol{u}_2, \text{ and } \boldsymbol{u}_3$ are linearly dependent, then there are scalar $b_1, b_2, \text{ and } b_3$ such that $b_1\boldsymbol{u}_1 + b_2\boldsymbol{u}_2 + b_3\boldsymbol{u}_3 = 0$.

Assume that $a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3 = 0$

$$a_1\left(\boldsymbol{u}_1 + \boldsymbol{u}_2\right) + a_2\left(\boldsymbol{u}_1 + \boldsymbol{u}_3\right) + a_3\left(\boldsymbol{u}_2 + \boldsymbol{u}_3\right) = 0$$

$$a_1\boldsymbol{u}_1 + a_1\boldsymbol{u}_2 + a_2\boldsymbol{u}_1 + a_2\boldsymbol{u}_3 + a_3\boldsymbol{u}_2 + a_3\boldsymbol{u}_3 = 0$$

$$\left(a_1 + a_2\right)\boldsymbol{u}_1 + \left(a_1 + a_3\right)\boldsymbol{u}_2 + \left(a_2 + a_3\right)\boldsymbol{u}_3 = 0$$

If $a_1 + a_2 = b_1 \quad a_1 + a_3 = b_2 \quad a_2 + a_3 = b_3$ and since $\boldsymbol{u}_1, \boldsymbol{u}_2, \text{ and } \boldsymbol{u}_3$ are linearly dependent, therefore, $\boldsymbol{v}_1, \boldsymbol{v}_2, \text{ and } \boldsymbol{v}_3$ are linearly dependent

Solution Section 2.7 – Coordinates, Basis and Dimension

Exercise

Suppose $v_1, \ldots, v_n$ is a basis for R^n and the n by n matrix A is invertible. Show that $Av_1, \ldots, Av_n$ is also a basis for R^n.

Solution

Put the basis vectors $v_1, \ldots, v_n$ in the columns of an invertible matrix **V**. then $Av_1, \ldots, Av_n$ are the columns of $\boldsymbol{AV}$. Since $\boldsymbol{A}$ is invertible, so is $\boldsymbol{AV}$ and its column give a basis.

Suppose $c_1Av_1 + \ldots + c_nAv_n = 0$. This is $Av = 0$ with $v = c_1v_1 + \ldots + c_nv_n$. Multiply by A^{-1} to get $v = 0$. By linear independence of $v's$, all $c_i = 0$. So the $Av's$ are independent.

Exercise

Consider the matrix $A = \begin{pmatrix} 1 & 0 & a \\ 2 & -1 & b \\ 1 & 1 & c \\ -2 & 1 & d \end{pmatrix}$

a) Which vectors $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ will make the columns of $\boldsymbol{A}$ linearly dependent?

b) Which vectors $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ will make the columns of $\boldsymbol{A}$ a basis for $\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : y + w = 0 \right\}$?

c) For $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \\ 2 \end{pmatrix}$, compute a basis for the four subspaces.

Solution

a) All linear combination of $\begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

b) To satisfy $b + d = 0$. For example $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$$A\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + B\begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \end{pmatrix} + C\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} \quad ; \quad A \neq 0$$

c) $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \\ 2 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \end{pmatrix} + 4\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \\ 2 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & -2 \\ 1 & 1 & 5 \\ -2 & 1 & 2 \end{pmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 + 2R_1 \end{matrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -4 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{pmatrix} \begin{matrix} \\ \\ R_3 + R_2 \\ R_4 + R_2 \end{matrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rightarrow \begin{cases} x_1 + x_3 = 0 \\ -x_2 - 4x_3 = 0 \end{cases}$$

The first 2 columns span the column space $C(\boldsymbol{A})$.

If $x_3 = 1$ that implies that the nullspace $N(\boldsymbol{A})$: $\left\{ \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix} \right\}$

Rank($\boldsymbol{A}$) = 2 and $\begin{bmatrix} -1 & -4 & 1 \end{bmatrix}^T$ is a basis for the one-dimensional $N(\boldsymbol{A})$.

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Exercise

Find a basis for $x-2y+3z=0$ in $\boldsymbol{R}^3$.

Find a basis for the intersection of that plane with *xy* plane. Then find a basis for all vectors perpendicular to the plane.

Solution

This plane is the nullspace of the matrix

$$A=\begin{pmatrix}1 & -2 & 3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

The special solutions: $s_1=\begin{pmatrix}2\\1\\0\end{pmatrix}$ $s_2=\begin{pmatrix}-3\\0\\1\end{pmatrix}$ give a basis for the nullspace, and for the plane.

The intersection of this plane with the *xy*-plane is a line $(x,\ -2x,\ 3x)$ and the vector $(1,\ -2,\ 3)^T$ lies in the *xy*-plane.

The vector $v_3=\begin{pmatrix}1\\-2\\3\end{pmatrix}$ is perpendicular to both vectors s_1 *and* s_2 : the space vectors perpendicular to a plane $\mathbb{R}^3$ is one-dimensional, it gives a basis.

Exercise

U comes from ***A*** by subtracting row 1 from row 3:

$$A=\begin{bmatrix}1 & 3 & 2\\ 0 & 1 & 1\\ 1 & 3 & 2\end{bmatrix} \quad \textit{and} \quad U=\begin{bmatrix}1 & 3 & 2\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$$

Find the bases for the two column spaces. Find the bases for the two row spaces. Find bases for the two nullspaces.

Solution

$$\begin{bmatrix}1 & 3 & 2\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}\xrightarrow{R_1-3R_2}\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}\rightarrow\begin{cases}x_1-x_3=0\\ x_2+x_3=0\end{cases}$$

a) The pivots are in the first two columns, so one possible basis for $C(\boldsymbol{A})$ is $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}3\\1\\3\end{bmatrix}\right\}$ and for $C(\boldsymbol{U})$ is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}3\\1\\0\end{bmatrix}\right\}$.

b) Both $\boldsymbol{A}$ and $\boldsymbol{U}$ have the same nullspace $N(\boldsymbol{A}) = N(\boldsymbol{U})$, with basis $\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$

c) Both $\boldsymbol{A}$ and $\boldsymbol{U}$ have the same row space $C\left(A^T\right)=C\left(U^T\right)$, *with basis* $\left\{\begin{bmatrix}1\\3\\2\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$

Exercise

Write a 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly independent. (Assume a combination gives $c_1P_1+\ldots+c_5P_5=0$, and check entries to prove c_i is zero.) The five permutation matrices are a basis for the subspace of 3 by 3 matrices with row and column sums all equal.

Solution

$$P_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\quad P_2=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\quad P_3=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\quad P_4=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\quad P_5=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

$$P_1+P_2+P_3=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\text{ and }P_4+P_5=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$$

$$P_1+P_2+P_3-P_4-P_5=I$$

$$c_1P_1+c_2P_2+c_3P_3+c_4P_4+c_5P_5=\begin{pmatrix}c_3&c_1+c_4&c_2+c_5\\c_1+c_5&c_2&c_3+c_4\\c_2+c_4&c_3+c_5&c_1\end{pmatrix}=0$$

$$c_1=c_2=c_3=0\ (diagonal)\Rightarrow\begin{pmatrix}0&0+c_4&0+c_5\\0+c_5&0&0+c_4\\0+c_4&0+c_5&0\end{pmatrix}=0\Rightarrow c_4=c_5=0$$

Exercise

Choose three independent columns of $A = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 12 & 15 & 2 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix}$. Then choose a different three independent columns. Explain whether either of these choices forms a basis for $C(A)$.

Solution

$$\begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 12 & 15 & 2 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix} R_2 - 2R_1 \begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix} R_4 - R_2 \begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & \frac{1}{2} & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \frac{1}{2}R_1 \\ \frac{1}{6}R_2 \\ \frac{1}{9}R_3 \end{matrix} \begin{pmatrix} 1 & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 1 & \frac{7}{6} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} R_1 - \frac{1}{2}R_3 \begin{pmatrix} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{7}{6} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank(***A***) = 3, the columns space is 3 which form a basis of $C(A)$. The variable is x_3

If $x_3 = 1 \Rightarrow \begin{pmatrix} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{7}{6} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} x_1 + \frac{1}{4}x_3 = 0 \\ x_2 + \frac{7}{6}x_3 = 0 \\ x_4 = 0 \end{cases} \rightarrow x_1 = -\frac{1}{4} \quad x_2 = -\frac{7}{6}$

$N(A)$ is spanned by $x_n = \begin{pmatrix} -\frac{1}{4} \\ -\frac{7}{6} \\ 1 \\ 0 \end{pmatrix}$, which gives the relation of the columns. The special solution x_n gives a relation $-\frac{1}{4}v_1 - \frac{7}{6}v_2 + v_3 = 0$. If we take any two columns from the first three columns and the column 4, they will span a three dimensional space since there will be no relation among them. Hence they form a basis of $C(A)$.

Exercise

Which of the following sets of vectors are bases for $\boldsymbol{R}^2$?

a) $\{(2,1),\ (3,0)\}$

b) $\{(0,0),\ (1,3)\}$

Solution

a) $k_1(2,1)+k_2(3,0)=(0,0)$

$k_1(2,1)+k_2(3,0)=(b_1,b_2)$

$\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} = -3 \neq 0$

Therefore the vectors $\{(2,1),\ (3,0)\}$ are linearly independent and span $\boldsymbol{R}^2$, so they form a basis for $\boldsymbol{R}^2$.

b) $k_1(0,0)+k_2(1,3)=(0,0)$

$k_1(0,0)+k_2(1,3)=(b_1,b_2)$

$\begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix} = 0$

Therefore the vectors $\{(0,0),\ (1,3)\}$ are linearly dependent, so they don't form a basis for $\boldsymbol{R}^2$.

Exercise

Which of the following sets of vectors are bases for $\boldsymbol{R}^3$?

a) $\{(1,0,0),\ (2,2,0),\ (3,3,3)\}$

b) $\{(3,1,-4),\ (2,5,6),\ (1,4,8)\}$

c) $\{(2,-3,1),\ (4,1,1),\ (0,-7,1)\}$

d) $\{(1,\ 6,\ 4),\ (2,\ 4,-1),\ (-1,\ 2,\ 5)\}$

Solution

a) $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{vmatrix} = 6 \neq 0$ Therefore the set of vectors are linearly independent.

The set form a basis for $\boldsymbol{R}^3$.

b) $\begin{vmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{vmatrix} = 26 \neq 0$ Therefore the set of vectors are linearly independent.

The set form a basis for $\boldsymbol{R}^3$.

c) $\begin{vmatrix} 2 & 4 & 0 \\ -3 & 1 & -7 \\ 1 & 1 & 1 \end{vmatrix} = 0$ Therefore the set of vectors are linearly dependent.

The set don't form a basis for $\boldsymbol{R}^3$.

d) $\begin{vmatrix} 1 & 2 & -1 \\ 6 & 4 & 2 \\ 4 & -1 & 5 \end{vmatrix} = 0$

Therefore the set of vectors are linearly dependent.

The set don't form a basis for $\boldsymbol{R}^3$.

Exercise

Let V be the space spanned by $v_1 = \cos^2 x, \quad v_2 = \sin^2 x, \quad v_3 = \cos 2x$

a) Show that $S = \{v_1, v_2, v_3\}$ is not a basis for V.

b) Find a basis for V.

Solution

a) $\cos 2x = \cos^2 x - \sin^2 x$

$k_1 \cos^2 x + k_2 \sin^2 x + k_3 \cos 2x = 0$

$k_1 \cos^2 x + k_2 \sin^2 x + k_3 \left(\cos^2 x - \sin^2 x\right) = 0$

$\left(k_1 + k_3\right)\cos^2 x + \left(k_2 - k_3\right)\sin^2 x = 0 \quad \Rightarrow \quad \begin{cases} k_1 + k_3 = 0 & \rightarrow k_1 = -k_3 \\ k_2 - k_3 = 0 & \rightarrow k_2 = k_3 \end{cases}$

If $k_3 = -1 \quad \Rightarrow k_1 = 1, \quad k_2 = -1$

$(1)\cos^2 x + (-1)\sin^2 x + (-1)\cos 2x = 0$

This shows that $\{v_1, v_2, v_3\}$ is linearly dependent, therefore it is not a basis for V.

b) For $c_1 \cos^2 x + c_2 \sin^2 x = 0$ to hold for all real x values, we must have $c_1 = 0 \quad (x = 0)$ and $c_2 = 0 \quad \left(x = \frac{\pi}{2}\right)$. Therefore the vectors $v_1 = \cos^2 x \quad v_2 = \sin^2 x$ are linearly independent.

$\mathbf{v} = k_1 \cos^2 x + k_2 \sin^2 x + k_3 \cos 2x$

$\quad = \left(k_1 + k_3\right)\cos^2 x + \left(k_2 - k_3\right)\sin^2 x$

This proves that the vectors $v_1 = \cos^2 x \quad \textit{and} \quad v_2 = \sin^2 x$ span V. We can conclude that $v_1 = \cos^2 x \quad \textit{and} \quad v_2 = \sin^2 x$ can form a basis for V.

Exercise

Find the coordinate vector of $\boldsymbol{w}$ relative to the basis $S=\{u_1, u_2\}$ for $\boldsymbol{R}^2$

a) $u_1=(1,\ 0),\quad u_2=(0,\ 1),\quad w=(3,\ -7)$

b) $u_1=(2,\ -4),\quad u_2=(3,\ 8),\quad w=(1,\ 1)$

c) $u_1=(1,\ 1),\quad u_2=(0,\ 2),\quad w=(a,\ b)$

d) $u_1=(1,\ -1),\quad u_2=(1,\ 1),\quad w=(0,\ 1)$

e) $u_1=(1,\ -1),\quad u_2=(1,\ 1),\quad w=(1,\ 1)$

Solution

a) We must first express $\boldsymbol{w}$ as a linear combination of the vectors in *S*; $\boldsymbol{w}=c_1u_1+c_2u_2$

$$(3,-7)=3(1,0)-7(0,1)$$

$$=3u_1-7u_2$$

Therefore, $(w)_S=\boxed{(3,-7)}$

b) Solve $c_1u_1+c_2u_2=\boldsymbol{w} \Rightarrow c_1(2,-4)+c_2(3,8)=(1,1)$

$$\rightarrow \begin{cases} 2c_1+3c_2=1 \\ -4c_1+8c_2=1 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ -4 & 8 & 1 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & \frac{5}{28} \\ 0 & 1 & \frac{3}{14} \end{array}\right]$$

Therefore, $(w)_S=\boxed{\left(\frac{5}{28},\ \frac{3}{14}\right)}$

c) Solve $c_1u_1+c_2u_2=\boldsymbol{w} \Rightarrow c_1(1,1)+c_2(0,2)=(a,b)$

$$\rightarrow \begin{cases} \boxed{c_1=a} \\ c_1+2c_2=b \end{cases} \Rightarrow \boxed{c_2=\frac{b-a}{2}}$$

Therefore, $(w)_S=\boxed{\left(a,\ \frac{b-a}{2}\right)}$

d) Solve $c_1u_1+c_2u_2=\boldsymbol{w} \Rightarrow c_1(1,-1)+c_2(1,1)=(0,1)$

$$\rightarrow \begin{cases} c_1+c_2=0 \\ -c_1+c_2=1 \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & 1 & 1 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{array}\right] \rightarrow \boxed{c_1=-\frac{1}{2}} \quad \boxed{c_2=\frac{1}{2}}$$

Therefore, $(w)_S=\boxed{\left(-\frac{1}{2},\ \frac{1}{2}\right)}$

e) Solve $c_1u_1+c_2u_2=\boldsymbol{w} \Rightarrow c_1(1,-1)+c_2(1,1)=(1,1)$

$$\rightarrow \begin{cases} c_1 + c_2 = 1 \\ -c_1 + c_2 = 1 \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right] \rightarrow \boxed{c_1 = 0} \quad \boxed{c_2 = 1}$$

Therefore, $(w)_S = \underline{(0,\ 1)}|$

Exercise

Find the coordinate vector of v relative to the basis $S = \{v_1, v_2, v_3\}$

a) $v = (2,-1,3), \quad v_1 = (1,0,0), \quad v_2 = (2,2,0), \quad v_3 = (3,3,3)$

b) $v = (5,-12,3), \quad v_1 = (1,2,3), \quad v_2 = (-4,5,6), \quad v_3 = (7,-8,9)$

Solution

a) Solve $c_1v_1 + c_2v_2 + c_3v_3 = \mathbf{v} \quad \Rightarrow \quad c_1(1,0,0) + c_2(2,2,0) + c_2(3,3,3) = (2,-1,3)$

$$\rightarrow \begin{cases} c_1 + 2c_2 + 3c_3 = 2 \\ \quad 2c_2 + 3c_3 = -1 \\ \quad\quad 3c_3 = 3 \end{cases} \qquad \begin{array}{l} \rightarrow c_1 = 2 - 2c_2 - 3c_3 = 3 \\ \rightarrow c_2 = \dfrac{-3c_3 - 1}{2} = -2 \\ \rightarrow c_3 = 1 \end{array}$$

Therefore, $(v)_S = \underline{(3,\ -2,\ 1)}|$

b) Solve $c_1v_1 + c_2v_2 + c_3v_3 = \mathbf{v} \quad \Rightarrow \quad c_1(1,2,3) + c_2(-4,5,6) + c_2(7,-8,9) = (5,-12,3)$

$$\rightarrow \begin{cases} c_1 - 4c_2 + 7c_3 = 5 \\ 2c_1 + 5c_2 - 8c_3 = -12 \\ 3c_1 + 6c_2 + 9c_3 = 3 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -4 & 7 & 5 \\ 2 & 5 & -8 & -12 \\ 3 & 6 & 9 & 3 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \begin{cases} c_1 = -2 \\ c_2 = 0 \\ c_3 = 1 \end{cases}$$

Therefore, $(v)_S = \underline{(-2,\ 0,\ 1)}|$

Exercise

Show that $\{A_1, A_2, A_3, A_4\}$ is a basis for M_{22}, and express A as a linear combination of the basis vectors

a) $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $A_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $A = \begin{pmatrix} 6 & 2 \\ 5 & 3 \end{pmatrix}$

b) $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ $A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

c) $A = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$, $A_1 = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Solution

a) Matrices $\{A_1, A_2, A_3, A_4\}$ are linearly independent if the equation

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = \mathbf{0}$$

$$k_1\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + k_2\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + k_3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + k_4\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \mathbf{0}$$

Has only the trivial solution.

For these matrices to span M_{22}, it must be expressed every matrix $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ as

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = \mathbf{A}$$

$$k_1\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + k_2\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + k_3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + k_4\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 3 \end{pmatrix}$$

The 2 equations can be written as linear systems

$$\begin{cases} k_1 + k_2 + k_3 = 0 \\ k_2 = 0 \\ k_1 + k_4 = 0 \\ k_3 = 0 \end{cases} \quad \textit{and} \quad \begin{cases} k_1 + k_2 + k_3 = a_1 \\ k_2 = a_2 \\ k_1 + k_4 = a_3 \\ k_3 = a_4 \end{cases}$$

$\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -1 \neq 0$, that the homogeneous system has only the trivial solution.

$\{A_1, A_2, A_3, A_4\}$ span M_{22}

$$\rightarrow\begin{cases} k_1 + k_2 + k_3 & = 6 \\ k_2 & = 2 \\ k_1 + k_4 & = 5 \\ k_3 & = 3 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 0 & 3 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \quad \begin{matrix} k_1 = 1 \\ k_2 = 2 \\ k_3 = 3 \\ k_4 = 4 \end{matrix}$$

$$\mathbf{A} = \underline{A_1 + 2A_2 + 3A_3 + 4A_4}$$

b) Matrices $\{A_1, A_2, A_3, A_4\}$ are linearly independent if the equation

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = \mathbf{0}$$

$$k_1\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + k_2\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + k_3\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + k_4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{0}$$

Has only the trivial solution.

For these matrices to span M_{22}, it must be expressed every matrix $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ as

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = \mathbf{A}$$

$$k_1\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + k_2\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + k_3\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + k_4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

The 2 equations can be written as linear systems

$$\begin{aligned} k_1 &= 0 \\ k_1 + k_2 &= 0 \\ k_1 + k_2 + k_3 &= 0 \\ k_1 + k_2 + k_3 + k_4 &= 0 \end{aligned} \quad \textit{and} \quad \begin{aligned} k_1 &= a_1 \\ k_1 + k_2 &= a_2 \\ k_1 + k_2 + k_3 &= a_3 \\ k_1 + k_2 + k_3 + k_4 &= a_4 \end{aligned}$$

$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$, that the homogeneous system has only the trivial solution.

$\{A_1, A_2, A_3, A_4\}$ span M_{22}

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \quad \begin{array}{l} k_1 = 1 \\ k_2 = -1 \\ k_3 = 1 \\ k_4 = -1 \end{array}$$

$$\mathbf{A} = \underline{A_1 - A_2 + A_3 - A_4|}$$

c) $k_1\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + k_2\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + k_3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + k_4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \quad \begin{array}{l} k_1 = 1 \\ k_2 = 1 \\ k_3 = -1 \\ k_4 = 3 \end{array}$$

$$\mathbf{A} = \underline{A_1 + A_2 - A_3 + 3A_4|}$$

Exercise

Consider the eight vectors

$$\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}$$

a) List all of the one-element. Linearly dependent sets formed from these.

b) What are the two-element, linearly dependent sets?

c) Find a three-element set spanning a subspace of dimension three, and dimension of two? One? Zero?

d) Which four-element sets are linearly dependent? Explain why.

Solution

a) $\begin{pmatrix}0\\0\\0\end{pmatrix}$ zero vector is the only linearly dependent.

b) The set that contains zero vector and any other vector.

c) 2-dimension:

$$\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \quad \textit{or} \quad \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \quad \textit{or} \quad \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}$$

1-dimensional subspace if we allow duplicates (zero vector) $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$

d) All four-element sets are linearly dependent in three-dimensional space.

Exercise

Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space

a) $\begin{cases} x_1 + x_2 - x_3 = 0 \\ -2x_1 - x_2 + 2x_3 = 0 \\ -x_1 \qquad + x_3 = 0 \end{cases}$

b) $\begin{cases} 3x_1 + x_2 + x_3 + x_4 = 0 \\ 5x_1 - x_2 + x_3 - x_4 = 0 \end{cases}$

c) $\begin{cases} x_1 - 3x_2 + x_3 = 0 \\ 2x_1 - 6x_2 + 2x_3 = 0 \\ 3x_1 - 9x_2 + 3x_3 = 0 \end{cases}$

d) $\begin{cases} x + y + z = 0 \\ 3x + 2y - 2z = 0 \\ 4x + 3y - z = 0 \\ 6x + 5y + z = 0 \end{cases}$

e) $\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 \qquad + 5x_3 = 0 \\ \qquad x_2 + x_3 = 0 \end{cases}$

Solution

a) $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -2 & -1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad x_1 - x_3 = 0 \to x_1 = x_3$, $\quad x_2 = 0$

The solution: $(x_1,\ 0,\ x_1) = x_1(1,\ 0,\ 1)$

The solution space has dimension 1 and a basis $\underline{(1,\ 0,\ 1)|}$

b) $\left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{array}\right]$ $\quad x_3 = s \qquad x_1 = -\frac{1}{4}x_3 = s$

$x_4 = t \quad x_2 = -\frac{1}{4}x_3 - x_4 = -\frac{1}{4}s - t$

The solution: $(x_1,\ x_2,\ x_3,\ x_4) = \left(-\frac{1}{4}s,\ -\frac{1}{4}s - t,\ s,\ t\right)$

$$= s\left(-\frac{1}{4},\ -\frac{1}{4},\ 1,\ 0\right) + t(0,\ -1,\ 0,\ 1)$$

The solution space has dimension 2 and a basis $\underline{\left(-\frac{1}{4},\ -\frac{1}{4},\ 1,\ 0\right),\ \ (0,\ -1,\ 0,\ 1)|}$

c) $\left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 2 & -6 & 2 & 0 \\ 3 & -9 & 3 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad x_1 - 3x_2 + x_3 = 0 \to x_1 = 3x_2 - x_3$

The solution: $(x_1,\ x_2,\ x_3) = (3x_2 - x_3,\ x_2,\ x_3)$

$$= x_2(3,\ 1,\ 0) + x_3(-1,\ 0,\ 1)$$

The solution space has dimension 2 and a basis $\underline{(3,\ 1,\ 0)\ \ \textit{and}\ \ (-1,\ 0,\ 1)|}$

d) $\begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 3 & 2 & -2 & | & 0 \\ 4 & 3 & -1 & | & 0 \\ 6 & 5 & 1 & | & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -4 & | & 0 \\ 0 & 1 & 5 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$ $\begin{aligned} x &= 4z \\ y &= -5z \end{aligned}$

The solution: $(x,\ y,\ z) = (4z,\ -5z,\ z) = z(4,\ -5,\ 1)$

The solution space has dimension 1 and a basis $(4,\ -5,\ 1)$

e) $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 5 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ No basis and dimension = 0

Exercise

If $AS = SA$ for the shift matrix S. Show that ***A*** must have this special form:

$$If \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$then\ \ A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$$

"The subspace of matrices that commute with the shift S has dimension _______."

Solution

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a & b \\ 0 & d & e \\ 0 & g & h \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & a & b \\ 0 & d & e \\ 0 & g & h \end{bmatrix} = \begin{bmatrix} d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow d = g = h = 0 \quad b = f \quad a = e = i$$

$$A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$$

The subspace of matrices that commute with the shift S has dimension 3, because the matrix has only three variables

Exercise

Find bases for the following subspaces of $\boldsymbol{R}^3$

a) All vectors of the form $(a, b, c, 0)$

b) All vectors of the form (a, b, c, d), where $d = a + b$ and $c = a - b$.

c) All vectors of the form (a, b, c, d), where $a = b = c = d$.

Solution

a) The subspace can be expressed as span $S = \{(1,0,0,0),(0,1,0,0),(0,0,1,0)\}$ is a set of linearly independent vectors. Therefore S forms a basis for the subspace, so its dimension is 3.

b) The subspace contains all vectors $(a,\ b,\ a+b,\ a-b) = a(1,0,1,1) + b(0,1,1,-1)$, the set $S = \{(1,0,1,1),(0,1,1,-1)\}$ is linearly independent vectors. Therefore S forms a basis for the subspace, so its dimension is 2.

c) The subspace contains all vectors $(a,\ a,\ a,\ a) = a(1,1,1,1)$, we can express the set $S = \{(1,1,1,1)\}$ as span S and it is linearly independent. Therefore S forms a basis for the subspace, so its dimension is 1.

Exercise

Find a basis for the null space of A.

a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

Solution

a) $$A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix} \xrightarrow{\ rref\ } \begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \to x_1 = 16x_3 = 16t \\ \to x_2 = 19x_3 = 19t \\ \ \end{matrix}$$

The general form of the solution of A$\boldsymbol{x}$ = $\mathbf{0}$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$, therefore the vector $\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$ forms a basis for the null space of A.

b) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{matrix} \to x_1 = -x_3 + \frac{2}{7}x_4 = -t + \frac{2}{7}s \\ \to x_2 = -x_3 - \frac{4}{7}x_4 = -t - \frac{4}{7}s \end{matrix}$

The general form of the solution of A$\boldsymbol{x}$ = $\mathbf{0}$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$, therefore the vectors

$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of A.

c) $\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\to \begin{cases} x_1 = -x_3 - 2x_4 - x_5 = -r - 2s - t \\ x_2 = -x_3 - x_4 - 2x_5 = -r - s - 2t \end{cases}$$

The general form of the solution of A$\boldsymbol{x}$ = $\mathbf{0}$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, therefore the vectors

$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, *and* $\begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of A.

Exercise

Find a basis for the subspace of $\boldsymbol{R}^4$ spanned by the given vectors

a) $(1,1,-4,-3),\quad (2,0,2,-2),\quad (2,-1,3,2)$

b) $(-1,1,-2,0),\quad (3,3,6,0),\quad (9,0,0,3)$

Solution

a) $\begin{pmatrix} 1 & 1 & -4 & -3 \\ 2 & 0 & 2 & -2 \\ 2 & -1 & 3 & 2 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 1 & -4 & -3 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & 1 & -\frac{1}{2} \end{pmatrix}$

A basis for the subspace is $(1,1,-4,-3),\quad (0,1,-5,-2),\quad \left(0,0,1,-\frac{1}{2}\right)$

b) $\begin{pmatrix} -1 & 1 & -2 & 0 \\ 3 & 3 & 6 & 0 \\ 9 & 0 & 0 & 3 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{6} \end{pmatrix}$

A basis for the subspace is $(1,-1,2,0),\quad (0,1,0,0),\quad \left(0,0,1,-\frac{1}{6}\right)$

Exercise

Determine whether the given vectors form a basis for the given vector space

a) $\boldsymbol{v}_1(3,\ -2,\ 1),\quad \boldsymbol{v}_2(2,\ 3,\ 1),\quad \boldsymbol{v}_3(2,\ 1,\ -3),\quad in\ \mathbb{R}^3$

b) $\boldsymbol{v}_1=(1,\ 1,\ 0,\ 0),\quad \boldsymbol{v}_2=(0,\ 1,\ 1,\ 0),\quad \boldsymbol{v}_3=(0,\ 0,\ 1,\ 1),\quad \boldsymbol{v}_4=(1,\ 0,\ 0,\ 1),\quad for\ \mathbb{R}^4$

c) $M_1=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\quad M_2=\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\quad M_3=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},\quad M_4=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\quad M_{22}$

Solution

a) $\begin{vmatrix} 3 & 2 & 2 \\ -2 & 3 & 1 \\ 1 & 1 & -3 \end{vmatrix} = 14 \neq 0$

The given vectors are linearly independent and span $\boldsymbol{R}^3$, so they form a basis for $\boldsymbol{R}^3$.

b) $\begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = 2 \neq 0$

The given vectors are linearly independent and span $\boldsymbol{R}^4$, so they form a basis for $\boldsymbol{R}^4$.

$$M_1=\begin{bmatrix}1&0\\0&0\end{bmatrix},\quad M_2=\begin{bmatrix}1&1\\0&0\end{bmatrix},\quad M_3=\begin{bmatrix}1&1\\1&0\end{bmatrix},\quad M_4=\begin{bmatrix}1&1\\1&1\end{bmatrix}\quad M_{22}$$

c) $\begin{vmatrix}1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{vmatrix}=1\neq 0$

They form a basis for M_{22}.

Exercise

Find a basis for, and the dimension of, the null space of the given matrix $\begin{bmatrix}2&1&-1&1\\4&-2&-2&1\end{bmatrix}$

Solution

$$\begin{bmatrix}2&1&-1&1\\4&-2&-2&1\end{bmatrix}\xrightarrow{rref}\begin{bmatrix}1&0&-\frac{1}{2}&\frac{3}{8}\\0&1&0&\frac{1}{4}\end{bmatrix}\quad \begin{aligned}x_1-\tfrac{1}{2}x_3+\tfrac{3}{8}x_4&=0\\ x_2+\tfrac{1}{4}x_4&=0\end{aligned}$$

$$x_1=\frac{1}{2}x_3-\frac{3}{8}x_4$$

$$x_2=-\frac{1}{4}x_4$$

$$\boldsymbol{x}=x_3\begin{bmatrix}\frac{1}{2}\\0\\1\\0\end{bmatrix}+x_4\begin{bmatrix}-\frac{3}{8}\\\frac{1}{4}\\0\\1\end{bmatrix}$$

The *bases* are: $\begin{bmatrix}\frac{1}{2}\\0\\1\\0\end{bmatrix}$ *and* $\begin{bmatrix}-\frac{3}{8}\\\frac{1}{4}\\0\\1\end{bmatrix}$

Dimension: 2

Solution   *Section* 2.8 – Row and Column Spaces

Exercise

List the row vectors and column vectors of the matrix

$$\begin{bmatrix} 2 & -1 & 0 & 1 \\ 3 & 5 & 7 & -1 \\ 1 & 4 & 2 & 7 \end{bmatrix}$$

Solution

Row vectors: $r_1 = \begin{bmatrix} 2 & -1 & 0 & 1 \end{bmatrix}$, $r_2 = \begin{bmatrix} 3 & 5 & 7 & -1 \end{bmatrix}$, $r_3 = \begin{bmatrix} 1 & 4 & 2 & 7 \end{bmatrix}$

Column vectors: $c_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $c_2 = \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix}$, $c_3 = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$, $c_4 = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$

Exercise

Express the product A$\boldsymbol{x}$ as a linear combination of the column vectors of *A*.

a) $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

b) $\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$

c) $\begin{bmatrix} -3 & 6 & 2 \\ 5 & -4 & 0 \\ 2 & 3 & -1 \\ 1 & 8 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$

Solution

a) $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

b) $\begin{bmatrix} -3 & 6 & 2 \\ 5 & -4 & 0 \\ 2 & 3 & -1 \\ 1 & 8 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = -1\begin{bmatrix} -3 \\ 5 \\ 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 6 \\ -4 \\ 3 \\ 8 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix}$

c) $\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} = -2\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 6 \\ -1 \end{bmatrix} + 5\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$

Exercise

Determine whether $\boldsymbol{b}$ is in the column space of A, and if so, express $\boldsymbol{b}$ as a linear combination of the column vectors of A.

a) $A=\begin{bmatrix}1 & 3\\ 4 & -6\end{bmatrix},\quad b=\begin{bmatrix}-2\\ 10\end{bmatrix}$

b) $A=\begin{bmatrix}1 & -1 & 1\\ 9 & 3 & 1\\ 1 & 1 & 1\end{bmatrix},\quad b=\begin{bmatrix}5\\ 1\\ -1\end{bmatrix}$

c) $A=\begin{bmatrix}1 & 1 & 2\\ 1 & 0 & 1\\ 2 & 1 & 3\end{bmatrix}\quad \boldsymbol{b}=\begin{bmatrix}-1\\ 0\\ 2\end{bmatrix}$

d) $A=\begin{bmatrix}1 & 2 & 0 & 1\\ 0 & 1 & 2 & 1\\ 1 & 2 & 1 & 3\\ 0 & 1 & 2 & 2\end{bmatrix},\quad b=\begin{bmatrix}4\\ 3\\ 5\\ 7\end{bmatrix}$

Solution

a) $\left[\begin{array}{cc|c}1 & 3 & -2\\ 4 & -6 & 10\end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c}1 & 0 & 1\\ 0 & 1 & -1\end{array}\right]$

$$\begin{bmatrix}-2\\ 10\end{bmatrix}=\begin{bmatrix}1\\ 4\end{bmatrix}-\begin{bmatrix}3\\ -6\end{bmatrix}$$

b) $\left[\begin{array}{ccc|c}1 & -1 & 1 & 5\\ 9 & 3 & 1 & 1\\ 1 & 1 & 1 & -1\end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right]$

The system $A\boldsymbol{x}=\boldsymbol{b}$ is inconsistent and $\boldsymbol{b}$ is not in the column space of A.

c) $\left[\begin{array}{ccc|c}1 & 1 & 2 & -1\\ 1 & 0 & 1 & 0\\ 2 & 1 & 3 & 2\end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right]$

The system $A\boldsymbol{x}=\boldsymbol{b}$ is inconsistent and $\boldsymbol{b}$ is not in the column space of A.

d) $\left[\begin{array}{cccc|c}1 & 2 & 0 & 1 & 4\\ 0 & 1 & 2 & 1 & 3\\ 1 & 2 & 1 & 3 & 5\\ 0 & 1 & 2 & 2 & 7\end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c}1 & 0 & 0 & 0 & -26\\ 0 & 1 & 0 & 0 & 13\\ 0 & 0 & 1 & 0 & -7\\ 0 & 0 & 0 & 1 & 4\end{array}\right]$

$$\begin{bmatrix}4\\ 3\\ 5\\ 7\end{bmatrix}=-26\begin{bmatrix}1\\ 0\\ 1\\ 0\end{bmatrix}+13\begin{bmatrix}2\\ 1\\ 2\\ 1\end{bmatrix}-7\begin{bmatrix}0\\ 2\\ 1\\ 2\end{bmatrix}+4\begin{bmatrix}1\\ 1\\ 3\\ 2\end{bmatrix}$$

Exercise

Suppose that $x_1 = -1,\quad x_2 = 2,\quad x_3 = 4,\quad x_4 = -3$ is a solution of a nonhomogeneous linear system A$\boldsymbol{x}$ = $\boldsymbol{b}$ and that the solution set of the homogeneous system A$\boldsymbol{x}$ = **0** is given by the formulas

$$x_1 = -3r + 4s,\quad x_2 = r - s,\quad x_3 = r,\quad x_4 = s$$

a) Find a vector form of the general solution of A$\boldsymbol{x}$ = **0**

b) Find a vector form of the general solution of A$\boldsymbol{x}$ = $\boldsymbol{b}$

Solution

a) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = r\begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 4 \\ -3 \end{bmatrix} + r\begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

Exercise

Find the vector form of the general solution of the given linear system A$\boldsymbol{x}$ = $\boldsymbol{b}$; then use that result to find the vector form of the general solution of A$\boldsymbol{x}$ = **0**.

a) $\begin{cases} x_1 - 3x_2 = 1 \\ 2x_1 - 6x_2 = 2 \end{cases}$

b) $\begin{cases} x_1 + x_2 + 2x_3 = 5 \\ x_1 \qquad + x_3 = -2 \\ 2x_1 + x_2 + 3x_3 = 3 \end{cases}$

c) $\begin{cases} x_1 + 2x_2 - 3x_3 + x_4 = 4 \\ -2x_1 + x_2 + 2x_3 + x_4 = -1 \\ -x_1 + 3x_2 - x_3 + 2x_4 = 3 \\ 4x_1 - 7x_2 \qquad - 5x_4 = -5 \end{cases}$

d) $\begin{cases} x_1 - 2x_2 + x_3 + 2x_4 = -1 \\ 2x_1 - 4x_2 + 2x_3 + 4x_4 = -2 \\ -x_1 + 2x_2 - x_3 - 2x_4 = 1 \\ 3x_1 - 6x_2 + 3x_3 + 6x_4 = -3 \end{cases}$

Solution

a) $\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & -6 & 2 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 = 1 + 3x_2$

The solution of A$\boldsymbol{x}$ = $\boldsymbol{b}$ is $x_1 = 1 + 3t,\quad x_2 = t$ or $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

The general form of the solution of A$\boldsymbol{x}$ = $\mathbf{0}$ is $\boldsymbol{x}=t\begin{bmatrix}3\\1\end{bmatrix}$

b) $\left[\begin{array}{ccc|c}1&1&2&5\\1&0&1&-2\\2&1&3&3\end{array}\right]\xrightarrow{rref}\left[\begin{array}{ccc|c}1&0&1&-2\\0&1&1&7\\0&0&0&0\end{array}\right]\begin{array}{l}\Rightarrow x_1=-2-x_3\\\Rightarrow x_2=7-x_3\\\end{array}$

The solution of A$\boldsymbol{x}$ = $\boldsymbol{b}$ is $x_1=-2-t,\quad x_2=7-t,\quad x_3=t$ or $\boldsymbol{x}=\begin{bmatrix}-2\\7\\0\end{bmatrix}+t\begin{bmatrix}-1\\-1\\1\end{bmatrix}$

The general form of the solution of A$\boldsymbol{x}$ = $\mathbf{0}$ is $\boldsymbol{x}=t\begin{bmatrix}-1\\-1\\1\end{bmatrix}$

c) $\left[\begin{array}{cccc|c}1&2&-3&1&4\\-2&1&2&1&-1\\-1&3&-1&2&3\\4&-7&0&-5&-5\end{array}\right]\xrightarrow{rref}\left[\begin{array}{cccc|c}1&0&-\frac{7}{5}&-\frac{1}{5}&\frac{6}{5}\\0&1&-\frac{4}{5}&\frac{3}{5}&\frac{7}{5}\\0&0&0&0&0\\0&0&0&0&0\end{array}\right]\begin{array}{l}\Rightarrow x_1=\frac{6}{5}+\frac{7}{5}x_3+\frac{1}{5}x_4\\\Rightarrow x_2=\frac{7}{5}+\frac{4}{5}x_3-\frac{3}{5}x_4\\\\\end{array}$

The solution of A$\boldsymbol{x}$ = $\boldsymbol{b}$ is $\boldsymbol{x}=\begin{bmatrix}\frac{6}{5}\\\frac{7}{5}\\0\\0\end{bmatrix}+s\begin{bmatrix}\frac{7}{5}\\\frac{4}{5}\\1\\0\end{bmatrix}+t\begin{bmatrix}\frac{1}{5}\\-\frac{3}{5}\\0\\1\end{bmatrix}$

The general form of the solution of A$\boldsymbol{x}$ = $\mathbf{0}$ is $\boldsymbol{x}=s\begin{bmatrix}\frac{7}{5}\\\frac{4}{5}\\1\\0\end{bmatrix}+t\begin{bmatrix}\frac{1}{5}\\-\frac{3}{5}\\0\\1\end{bmatrix}$

d) $\left[\begin{array}{cccc|c}1&-2&1&2&-1\\2&-4&2&4&-2\\-1&2&-1&-2&1\\3&-6&3&6&-3\end{array}\right]\xrightarrow{rref}\left[\begin{array}{cccc|c}1&-2&1&2&-1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{array}\right]\quad x_1=-1+2x_2-x_3-2x_4$

Let $x_2=s\quad x_3=t\quad x_4=r$

The solution of A$\boldsymbol{x}$ = $\boldsymbol{b}$ is $\boldsymbol{x}=\begin{bmatrix}-1\\0\\0\\0\end{bmatrix}+s\begin{bmatrix}2\\1\\0\\0\end{bmatrix}+t\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}+r\begin{bmatrix}-2\\0\\0\\1\end{bmatrix}$

The general form of the solution of A$\boldsymbol{x}$ = **0** is $\boldsymbol{x} = s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Exercise

Given the vectors $v_1 = (1,\ 2,\ 0)$ and $v_2 = (2,\ 3,\ 0)$

a) Are they linearly independent?

b) Are they a basis for any space?

c) What space **V** do they span?

d) What is the dimension of that space?

e) What matrices ***A*** have **V** as their column space?

f) Which matrices have **V** as their nullspace?

g) Describe all vectors v_3 that complete a basis v_1, v_2, v_3 for $\boldsymbol{R}^3$.

Solution

a) v_1, v_2 are independent – the only combination to give **0** is $0.v_1 + 0.v_2$.

b) Yes, they are a basis for whatever space **V** they span.

c) That space **V** contains all vectors $(x,\ y,\ 0)$. It is the *xy* plane in $\boldsymbol{R}^3$.

d) The dimension of **V** is 2 since the basis contains 2 vectors.

e) This **V** is the column space of any 3 by *n* matrix ***A*** of rank 2, if every column is a combination of v_1 *and* v_2. In particular ***A*** could just have columns v_1 *and* v_2.

f) This **V** is the nullspace of any *m* by 3 matrix ***B*** of rank 1, if every row is a multiple of $(0,\ 0,\ 1)$. In particular take $B = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. Then $Bv_1 = 0$ *and* $Bv_2 = 0$.

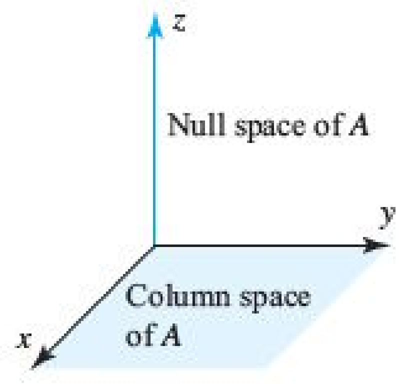
g) Any third vector $v_3 = (a,\ b,\ c)$ will complete a basis for $\boldsymbol{R}^3$ provided $c \ne 0$.

Exercise

a) Let $A=\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}$

Show that relative to an *xyz*-coordinate system in 3-space the null space of *A* consists of all points on the *z*-axis and that the column space consists of all points in the *xy*-plane.

b) Find a 3 x 3 matrix whose null space is the *x*-axis and whose column space is the *yz*-plane.



Solution

a) $A=\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix} \rightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$ $\quad \begin{matrix}x=0\\y=0\\z=t\end{matrix}$

The general form of the solution of A$\boldsymbol{x}$ = **0** is, $t\begin{bmatrix}0\\0\\1\end{bmatrix}$ therefore the null space of A is the *z*-axis, and

the column space is the span of $c_1=\begin{bmatrix}0\\1\\0\end{bmatrix}$, $c_2=\begin{bmatrix}1\\0\\0\end{bmatrix}$ which is all linear combinations of *y* and *x*

(*xy*-plane)

b) $\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}$

Exercise

If we add an extra column b to a matrix *A*, then the column space gets larger unless _____. Give an example where the column space gets larger and an example where it doesn't. Why is $Ax=b$ solvable exactly when the column space doesn't get larger – it is the same for *A* and $\begin{bmatrix}A & b\end{bmatrix}$?

Solution

If we add an extra column ***b*** to a matrix ***A***, then the column space gets larger unless ***it contains b*** that is a linear combination of the columns of ***A***.

Let $A=\begin{pmatrix}1&0\\0&0\end{pmatrix}$; then the column space gets larger if $b=\begin{pmatrix}0\\1\end{pmatrix}$ and it doesn't if $b=\begin{pmatrix}1\\0\end{pmatrix}$.

The equation $Ax=b$ is solvable exactly when ***b*** is a (nontrivial) linear combination of the column of ***A***.

The equation $Ax=b$ is solvable exactly when ***b*** lies in the column space, when the column space doesn't get larger.

Exercise

For which right sides (find a condition on b_1, b_2, b_3) are these solvable. (Use the column space $C(A)$ and the equation $Ax = b$)

a) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Solution

a) The column space consists of the vectors for $\begin{pmatrix} x_1 + 4x_2 + 2x_3 \\ 2x_1 + 8x_2 + 4x_3 \\ -x_1 - 4x_2 - 2x_3 \end{pmatrix}$ is $\begin{pmatrix} z \\ 2z \\ -z \end{pmatrix} = z\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

They are scalar multiples of $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

b) By substituting $x_1 + 4x_2$ with new variable *z*, then the column space consists of the vectors

$$\begin{pmatrix} x_1 + 4x_2 \\ 2x_1 + 9x_2 \\ -x_1 - 4x_2 \end{pmatrix} = z\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

They are linear combinations of $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Exercise

Show that the matrices *A* and $\begin{bmatrix} A & AB \end{bmatrix}$ (with extra columns) have the same column space. But find a square matrix with $C\left(A^2\right)$ smaller than $C(A)$. Important point: An *n* by *n* matrix has $C(A) = \mathbf{R}^n$ exactly when *A* is an ______ matrix.

Solution

Each column of $\boldsymbol{AB}$ is a combination of the columns of $\boldsymbol{A}$ (the combining coefficients are the entries in the corresponding column of B). So any combination of the columns of $\begin{bmatrix} A & AB \end{bmatrix}$ is a combination of the columns of $\boldsymbol{A}$ alone. Thus $\boldsymbol{A}$ and $\begin{bmatrix} A & AB \end{bmatrix}$ have the same column space.

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; then $A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, so $C\left(A^2\right) = Z$.

$C(A)$ is the line through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Any n by n matrix has $C(A) = \boldsymbol{R}^n$ exactly when $\boldsymbol{A}$ is an ***invertible*** matrix, because $Ax = b$ is solvable for any given $\boldsymbol{b}$ when $\boldsymbol{A}$ is invertible.

Exercise

The column of AB are combinations of the columns of A. This means: The column space of AB is contained in (possibly equal to) to the column space of A. Give an example where the column spaces A and AB are not equal.

Solution

The column space of $\boldsymbol{AB}$ is contained in (possibly equal to) to the column space of $\boldsymbol{A}$.
$B = 0$ *and* $A \neq 0$ is a case when $AB = 0$ has a smaller column space than $\boldsymbol{A}$.

Exercise

Find a square matrix A where $C\left(A^2\right)$ (the column space of A^2 is smaller than $C(A)$.

Solution

For example, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; then $A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Thus $C(A)$ is generated by vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which is of one dimensional, but $C\left(A^2\right)$ is a zero space.

Hence $C\left(A^2\right)$ is strictly smaller than $C(A)$.

Exercise

Suppose $Ax = b$ and $Cx = b$ have the same (complete) solutions for every $\boldsymbol{b}$. Is true that $A = C$?

Solution

Yes, if $A = C$, let $\boldsymbol{y}$ be any vector of the correct size, and set $b = Ay$. Then $\boldsymbol{y}$ is a solution to $Ax = b$ and it is also a solution to $Cx = b$; $b = Ay = Cy$

Exercise

Apply Gauss-Jordan elimination to $Ux=0$ and $Ux=c$. Reach $Rx=0$ and $Rx=d$:

$$\begin{bmatrix} U & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \qquad \begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

Solve $Rx=0$ to find x_n (its free variable is $x_2=1$).

Solve $Rx=d$ to find x_p (its free variable is $x_2=0$).

Solution

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \xrightarrow[\frac{1}{4}R_2]{} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1-3R_2} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The free variable is x_2, since it is the only one. We have to let $x_2=1$

$$\begin{cases} x_1+2x_2=0 \\ x_3=0 \end{cases} \rightarrow x_1=-2x_2$$

The special solution is $s_1(-2,\ 1,\ 0) \Rightarrow x_n = x_2\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix} \xrightarrow[\frac{1}{4}R_2]{} \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1-3R_2} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The free variable is x_2 that implies to $x_2=0$

$$\begin{cases} x_1+2x_2=-1 \\ x_3=2 \end{cases} \rightarrow x_1=-1$$

The particular solution is $x_p = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

Exercise

Which of the following subsets of $\boldsymbol{R}^3$ are actually subspaces?

a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$

b) The plane of vectors with $b_1 = 1$.

c) The vectors with $b_1 b_2 b_3 = 0$.

d) All linear combinations of $v = (1, 4, 0)$ and $w = (2, 2, 2)$.

e) All vectors that satisfies $b_1 + b_2 + b_3 = 0$

f) All vectors with $b_1 \le b_2 \le b_3$.

Solution

a) This is subspace

- For $v = (b_1, b_2, b_3)$ with $b_1 = b_2$ and $w = (c_1, c_2, c_3)$ with $c_1 = c_2$ the sum $v + w = (b_1 + c_1, b_2 + c_2, b_3 + c_3)$ is in the same set as $b_1 + c_1 = b_2 + c_2$
- For an element $v = (b_1, b_2, b_3)$ with $b_1 = b_2$, $cv = (cb_1, cb_2, cb_3)$ and $cb_1 = cb_2$, thus it is in the same set.

b) This is not a subspace. For example for $v = (1, 0, 0)$ and $cv = -v = (-1, 0, 0)$ is not in the set.

c) This is not a subspace. For example for $v = (1, 1, 0)$ and $w = (1, 0, 1)$ are in the set, but their sum $v + w = (2, 1, 1)$ is not in the set.

d) This is subspace, by definition of linear combination.

- For 2 vectors $v_1 = \alpha_1 v + \beta_1 w$ and $v_2 = \alpha_2 v + \beta_2 w$ the sum

$$\begin{aligned} v_1 + v_2 &= \alpha_1 v + \beta_1 w + \alpha_2 v + \beta_2 w \\ &= (\alpha_1 + \alpha_2) v + (\beta_1 + \beta_2) w \end{aligned}$$

  is still the linear combination of v and w.

- For an element $v_1 = \alpha_1 v + \beta_1 w$, $cv_1 = c\alpha_1 v + c\beta_1 w$ is still the linear combination of v and w, thus it is the same set

e) This is subspace, these are the vectors orthogonal to $(1, 1, 1)$

- For $v = (b_1, b_2, b_3)$ with $b_1 + b_2 + b_3 = 0$ and $w = (c_1, c_2, c_3)$ with $c_1 + c_2 + c_3 = 0$ the sum $v + w = (b_1 + c_1, b_2 + c_2, b_3 + c_3)$ is in the same set as $b_1 + c_1 + b_2 + c_2 + b_3 + c_3 = 0$
- For an element $v = (b_1, b_2, b_3)$ with $b_1 + b_2 + b_3 = 0$, $cv = (cb_1, cb_2, cb_3)$ and $cb_1 + cb_2 + cb_3 = 0$, thus it is in the same set.

f) This is not a subspace. For example for $v = (1, 2, 3)$ and $-v = (-1, -2, -3)$ is not in the set.

Exercise

We are given three different vectors b_1, b_2, b_3. Construct a matrix so that the equations $Ax = b_1$ and $Ax = b_2$ are solvable, but $Ax = b_3$ is not solvable.

a) How can you decide if this possible?

b) How could you construct A?

Solution

The equations $Ax = b_1$ and $Ax = b_2$ will be solvable.

$$Ax = \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b_3 \text{ (solvable?)}$$

If $Ax = b_3$ is not solvable, we have the desired matrix A.

If $Ax = b_3$ is solvable, then it is not possible to construct A.

When the column space contains b_1 and b_2, it will have to contain their linear combinations.

So b_3 would necessarily be in that column space and $Ax = b_3$ would necessarily be solvable.

Exercise

For which vectors (b_1, b_2, b_3) do these systems have a solution?

a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Solution

a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \begin{array}{l} \to x_1 + x_2 + x_3 = b_1 \\ \to x_2 + x_3 = b_2 \\ \to x_3 = b_3 \end{array}$

$$\begin{cases} x_1 = b_1 - b_2 - b_3 \\ x_2 = b_2 - b_3 \\ x_3 = b_3 \end{cases} \Rightarrow \left(b_1 - b_2 - b_3,\ b_2 - b_3,\ b_3\right)$$

Solution for every b.

b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \begin{array}{l} \to x_1 + x_2 + x_3 = b_1 \\ \to x_2 + x_3 = b_2 \\ \to 0x_3 = b_3 \end{array}$

$$\begin{cases} x_1 = b_1 - b_2 \\ x_2 = b_2 \\ 0 = b_3 \end{cases} \Rightarrow \left(b_1 - b_2,\ b_2,\ 0\right)$$

Solvable only if $b_3 = 0$

c) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \xrightarrow{R_2 - R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 0 & b_3 - b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right]$

$$\Rightarrow b_3 - b_2 = 0 \Rightarrow \boxed{b_3 = b_2}$$

$$\begin{cases} x_1 = b_1 - 2b_3 \\ 0 = 0 \\ x_3 = b_3 \end{cases} \quad \Rightarrow \quad \left(b_1 - 2b_3,\ 0,\ b_3\right)$$

Solvable only if $b_3 = b_2$

Exercise

Find a basis for the null space of A. $\quad A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & 2 & \frac{4}{3} \\ 0 & 1 & 0 & 0 & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & -\frac{5}{12} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $x_4 = s \quad x_5 = t \quad \rightarrow \begin{cases} x_1 = -2x_4 - \frac{4}{3}x_5 = 2s - \frac{4}{3}t \\ x_2 = \frac{1}{6}x_5 = \frac{1}{6}t \\ x_3 = \frac{5}{12}x_5 = \frac{5}{12}t \end{cases}$

The general form of the solution of A$\boldsymbol{x}$ = $\mathbf{0}$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{4}{3} \\ \frac{1}{6} \\ \frac{5}{12} \\ 0 \\ 1 \end{bmatrix}$

Therefore the vectors $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -\frac{4}{3} \\ \frac{1}{6} \\ \frac{5}{12} \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of A.

Exercise

Verify that $rank(A) = rank\left(A^T\right)$

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -\frac{6}{7} & -\frac{4}{7} \\ 0 & 1 & \frac{17}{7} & \frac{2}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad rank(A) = 2$$

$$A^T = \begin{bmatrix} 1 & -3 & -2 \\ 2 & 1 & 3 \\ 4 & 5 & 6 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad rank\left(A^T\right) = 2$$

$$rank(A) = rank\left(A^T\right) = \underline{2|}$$

Exercise

Find the rank and nullity of the matrix; then verify that the values obtained satisfy $rank(A) + N(A) = n$

a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

Solution

a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$

$rank(A)=2;\quad nullity(A)=1;\quad rank(A)+nullity(A)=2+1=3=n \quad \leftarrow number\ of\ columns$

b) $A=\begin{bmatrix}1&4&5&2\\2&1&3&0\\-1&3&2&2\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1&0&1&-\frac{2}{7}\\0&1&1&\frac{4}{7}\\0&0&0&0\end{bmatrix}$

$rank(A)=2;\quad nullity(A)=1;\quad rank(A)+nullity(A)=2+1=3=n$

c) $\begin{bmatrix}1&4&5&6&9\\3&-2&1&4&-1\\-1&0&-1&-2&-1\\2&3&5&7&8\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1&0&1&2&1\\0&1&1&1&2\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$

$rank(A)=2;\quad nullity(A)=2;\quad rank(A)+nullity(A)=2+2=4=n$

d) $A=\begin{bmatrix}1&-3&2&2&1\\0&3&6&0&-3\\2&-3&-2&4&4\\3&-6&0&6&5\\-2&9&2&-4&-5\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1&0&0&2&\frac{4}{3}\\0&1&0&0&-\frac{1}{6}\\0&0&1&0&-\frac{5}{12}\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$

$rank(A)=3;\ NS(A)=2;$ Number of column = 5; $rank(A)+NS(A)=3+2=5=n$

Exercise

If A is an $m\times n$ matrix, what is the largest possible value for its rank and the smallest possible value of the nullity of A.

Solution

The largest possible value for the rank of an $m\times n$ matrix:

- $n \quad if \quad m\ge n$ (when every column of the *rref* (A) contains a leading 1)
- $m \quad if \quad m<n$ (when every row of the *rref* (A) contains a leading 1)

The smallest possible value for the nullity of an $m\times n$ matrix:

- $0 \quad if \quad m\ge n$ (when every column of the *rref* (A) contains a leading 1)
- $n-m \quad if \quad m<n$ (when every row of the *rref* (A) contains a leading 1)

Exercise

Discuss how the rank of *A* varies with *t*.

$$a)\ A=\begin{bmatrix}1 & 1 & t\\ 1 & t & 1\\ t & 1 & 1\end{bmatrix} \qquad b)\ A=\begin{bmatrix}t & 3 & -1\\ 3 & 6 & -2\\ -1 & -3 & t\end{bmatrix}$$

Solution

a) $$\begin{vmatrix}1 & 1 & t\\ 1 & t & 1\\ t & 1 & 1\end{vmatrix}=t+t+t-t^3-1-1$$

$$=-t^3+3t-2=0$$

Solve for t: $\boxed{t=1,-2,-2}$

Therefore, $rank(A)=3$ for $\forall t-\{1,-2,-2\}$

If $t=1$, $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ $\quad rank(A)=1$

If $t=-2$, $A=\begin{bmatrix}1 & 1 & -2\\ 1 & -2 & 1\\ -2 & 1 & 1\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$ $\quad rank(A)=2$

b) $$\begin{vmatrix}t & 3 & -1\\ 3 & 6 & -2\\ -1 & -3 & t\end{vmatrix}=6t^2+6+9-6-6t-9t$$

$$=6t^2-15t+9=0$$

Solve for t: $\boxed{t=1,\ \frac{3}{2}}$

Therefore, $rank(A)=3$ for $\forall t-\left\{1,\ \frac{3}{2}\right\}$

If $t=1$, $A=\begin{bmatrix}1 & 3 & -1\\ 3 & 6 & -2\\ -1 & -3 & 1\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -\frac{1}{3}\\ 0 & 0 & 0\end{bmatrix}$ $\quad rank(A)=2$

If $t=\frac{3}{2}$, $A=\begin{bmatrix}\frac{3}{2} & 3 & -1\\ 3 & 6 & -2\\ -1 & -3 & \frac{3}{2}\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -\frac{5}{6}\\ 0 & 0 & 0\end{bmatrix}$ $\quad rank(A)=2$

Exercise

Are there values of *r* and *s* for which

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

Has rank 1? Has rank 2? If so, find those values.

Solution

Since the third column will always have a nonzero entry, the *rank* will never be 1. (row 1 and row 4 never have a nonzero entry).

If $r = 2$ and $s = 1$, that implies to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow rank = 2$$

Exercise

Find the row reduced form ***R*** and the rank *r* of ***A*** (those depend on *c*).
Which are the pivot columns of ***A***? Which variables are free? What are the special solutions and the nullspace matrix ***N*** (always depending on *c*)?

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & c \end{bmatrix} \quad and \quad A = \begin{bmatrix} c & c \\ c & c \end{bmatrix}$$

Solution

a) $c \neq 4 \quad R = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$

$rank(A) = 2$, the pivot columns are 1 and 3, the second variable x_2 is free.

The special solution: $x_2 = 1$ which yields to $N = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$c = 4 \quad R = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$

$rank(A) = 1$, the pivot column is column 1, the second and third variables x_2, x_3 are free.

The special solution goes into $N = \begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

b) $c \neq 0 \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$,

$rank(A) = 1$, the pivot column is the first column, the second variable x_2 is free.

The special solution: $x_2 = 1$ which yields to $N = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$c = 0 \quad R = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$rank(A) = 0$, the matrix has no pivot column, and both variable are free.

The special solution is: $N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Exercise

Find the row reduced form R and the rank r of A (those depend on c).
Which are the pivot columns of A? Which variables are free? What are the special solutions and the nullspace matrix N (always depending on c)?

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \quad \textit{and} \quad A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & c-1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a) If $c = 1$, then

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This has only one pivot (first column) and 3 free variables x_2, x_3, x_4.

The nullspace matrix: $\begin{pmatrix} -1 & -2 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b) If $c \neq 1$, then

$$\begin{pmatrix}1 & 1 & 2 & 2\\ 0 & c-1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \xrightarrow{\frac{1}{c-1}R_2} \begin{pmatrix}1 & 1 & 2 & 2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \xrightarrow{R_2-R_1} \begin{pmatrix}1 & 0 & 2 & 2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

There are two pivots $\left(C_1,\ C_2\right)$ and 2 free variables $x_3,\ x_4$

The nullspace matrix: $\begin{pmatrix}-2 & -2\\ 0 & 0\\ 1 & 0\\ 0 & 1\end{pmatrix}$

$$A = \begin{bmatrix}1-c & 2\\ 0 & 2-c\end{bmatrix}$$

a) If $c = 1 \Rightarrow\ A = \begin{bmatrix}0 & 2\\ 0 & 1\end{bmatrix} \xrightarrow{R_1-2R_2} \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$

$$\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \to 0a + b = 0 \Rightarrow b = 0$$

This has a single pivot in the second column and one free variable with the nullspace matrix $\begin{pmatrix}1\\ 0\end{pmatrix}$

b) If $c = 2 \Rightarrow\ A = \begin{bmatrix}-1 & 2\\ 0 & 0\end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix}1 & -2\\ 0 & 0\end{bmatrix}$

$$\begin{pmatrix}1 & -2\\ 0 & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \to a - 2b = 0 \Rightarrow if\ \ b = 1 \qquad a = 2$$

This has a single pivot in the first column with the nullspace matrix $\begin{pmatrix}2\\ 1\end{pmatrix}$

c) Otherwise $c \ne 1,\ 2\ \Rightarrow\ A = \begin{bmatrix}1-c & 2\\ 0 & 2-c\end{bmatrix} \xrightarrow{\frac{1}{1-c}R_1} \begin{bmatrix}1 & \frac{2}{1-c}\\ 0 & 2-c\end{bmatrix}$

$$\begin{bmatrix}1 & \frac{2}{1-c}\\ 0 & 2-c\end{bmatrix} \xrightarrow[\frac{1}{2-c}R_2]{} \begin{bmatrix}1 & \frac{2}{1-c}\\ 0 & 1\end{bmatrix} \xrightarrow{R_1-\frac{2}{1-c}R_2} \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

The result is the identity matrix with 2 pivots, which has (2 – 2) 0 null space.

Exercise

If A has a rank r, then it has an r by r sub-matrix S that is invertible. Remove $m-r$ rows and $n-r$ columns to find an invertible sub-matrix S inside each A (you could keep the pivot rows and pivot columns of A).

$$A=\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 4\end{pmatrix} \quad A=\begin{pmatrix}1 & 2 & 3\\ 2 & 4 & 6\end{pmatrix} \quad A=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Solution

If a matrix $\boldsymbol{A}$ has rank r, then the

(*dimension of the column space*) = (*dimension of the row space*) = *r*

For the invertible sub-matrix S, we need to find r linearly independent rows and r linearly independent columns.

For matrix $\boldsymbol{A}$:

$$\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 4\end{pmatrix} \xrightarrow[R_2-R_1]{} \begin{pmatrix}1 & 2 & 3\\ 0 & 0 & 1\end{pmatrix} \xrightarrow{R_1-3R_2} \begin{pmatrix}1 & 2 & 0\\ 0 & 0 & 1\end{pmatrix}$$

The 1st and 3rd columns are linearly independent, and the 1st and 2nd rows are also linearly independent.

Rank ($\boldsymbol{A}$) = 2.

The sub matrices are: $S_A=\begin{pmatrix}1 & 3\\ 1 & 4\end{pmatrix}$ $S_A=\begin{pmatrix}2 & 3\\ 2 & 4\end{pmatrix}$ $S_A=\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}$

For matrix $\boldsymbol{B}$:

$$\begin{pmatrix}1 & 2 & 3\\ 2 & 4 & 6\end{pmatrix} \xrightarrow[R_2-2R_1]{} \begin{pmatrix}1 & 2 & 3\\ 0 & 0 & 0\end{pmatrix}$$

Rank ($\boldsymbol{B}$) = 1.

The submatrix is: $S_A=(1)$

For matrix $\boldsymbol{C}$:

$$C=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Rank ($\boldsymbol{C}$) = 2.

The submatrix is by disregarding (deleting) 1st column and 2nd row: $S_A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

Exercise

Suppose that column 3 of 4 x 6 matrix is all zero. Then x_3 must be a ______ variable. Give one special solution for this matrix.

Solution

The x_3 must be a ***free variable***.

A special solution for this variable can be taken to be.

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Exercise

Fill in the missing numbers to make A rank1, rank 2, rank 3.(your solution should be 3 matrices)

$$A = \begin{pmatrix} & -3 & \\ 1 & 3 & -1 \\ & 9 & -3 \end{pmatrix}$$

Solution

$$A = \begin{pmatrix} a & -3 & b \\ 1 & 3 & -1 \\ c & 9 & -3 \end{pmatrix}$$

If rank (A) = 1, then we need the 1st and 3rd to be multiple of the 2nd row to get zero in these rows.

$$A = \begin{pmatrix} a & -3 & b \\ 1 & 3 & -1 \\ c & 9 & -3 \end{pmatrix} \begin{matrix} R_1 + R_2 \\ \\ R_3 - 3R_2 \end{matrix} \begin{pmatrix} a+1 & 0 & b-1 \\ 1 & 3 & -1 \\ c-3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} a+1=0 \\ b-1=0 \\ c-3=0 \end{cases} \to \begin{cases} a=-1 \\ b=1 \\ c=3 \end{cases}$$

$$A = \begin{pmatrix} -1 & -3 & 1 \\ 1 & 3 & -1 \\ 3 & 9 & -3 \end{pmatrix}$$

If rank (A) = 2, then we need the 1st ***or*** 3rd to be multiple of the 2nd row to get zero row.

$$A=\begin{pmatrix} a & -3 & b \\ 1 & 3 & -1 \\ c & 9 & -3 \end{pmatrix} \begin{matrix} R_1+R_2 \\ \\ R_3-3R_2 \end{matrix} \begin{pmatrix} a+1 & 0 & b-1 \\ 1 & 3 & -1 \\ c-3 & 0 & 0 \end{pmatrix} \quad \boxed{c \neq 3}$$

$$A=\begin{pmatrix} -1 & -3 & 1 \\ 1 & 3 & -1 \\ 2 & 9 & -3 \end{pmatrix}$$

If rank (A) = 3 (full rank), then the appropriate to start using 0's or 1's to fill the blank.

$$A=\begin{pmatrix} 0 & -3 & 0 \\ 1 & 3 & -1 \\ 1 & 9 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -1 \\ 0 & -3 & 0 \\ 1 & 9 & -3 \end{pmatrix} \xrightarrow[R_3-R_1]{} \begin{pmatrix} 1 & 3 & -1 \\ 0 & -3 & 0 \\ 0 & 6 & -2 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{3}R_2} \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 6 & -2 \end{pmatrix} \xrightarrow[R_1-6R_2]{R_1-3R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\xrightarrow[-\frac{1}{2}R_3]{} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1+R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, it has rank 3.

Exercise

Fill out these matrices so that they have rank 1:

$$A=\begin{pmatrix} 1 & 2 & 4 \\ 2 & & \\ 4 & & \end{pmatrix} \qquad B=\begin{pmatrix} 2 & & \\ 1 & & \\ 2 & 6 & -3 \end{pmatrix} \qquad M=\begin{pmatrix} a & b \\ c & \end{pmatrix}$$

Solution

Rank = 1 means that all the rows are multiples of each other.

$$A=\begin{pmatrix} 1 & 2 & 4 \\ 2 & a & b \\ 4 & c & d \end{pmatrix} \xrightarrow[R_3=4R_1]{R_2=2R_1} \begin{matrix} a=2(2) & b=2(4) \\ c=4(2) & d=4(4) \end{matrix}$$

$$A=\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{pmatrix}$$

$$B=\begin{pmatrix} 2 & a & b \\ 1 & c & d \\ 2 & 6 & -3 \end{pmatrix} \xrightarrow[R_2=\frac{1}{2}R_3]{R_1=R_3} \begin{matrix} a=6 & b=-3 \\ c=3 & d=-\frac{3}{2} \end{matrix}$$

$$B=\begin{pmatrix} 2 & 6 & -3 \\ 1 & 3 & -\frac{3}{2} \\ 2 & 6 & -3 \end{pmatrix}$$

$$M=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow[R_2=\frac{c}{a}R_1]{} d=\frac{c}{a}b \to M=\begin{pmatrix} a & b \\ c & \frac{bc}{a} \end{pmatrix}$$

Exercise

Suppose A and B are n by n matrices, and $AB = I$. Prove from $rank(AB) \le rank(A)$ that the $rank(A) = n$. So A is invertible and B must be its two-sided inverse. Therefore $BA = I$ (which is not so obvious!).

Solution

Since A is n by n $\Rightarrow rank(A) \le n$

$$n = rank\left(I_n\right) = rank(AB) \le rank(A)$$

Exercise

Every m by n matrix of rank r reduces to (m by r) times (r by n):

$$A = \text{(pivot columns of } A\text{) (first } r \text{ rows of } R\text{)} = (COL)(ROW)^T$$

Write the 3 by 4 matrix $A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{pmatrix}$ as the product of the 3 by 2 from the pivot columns and the 2 by 4 matrix from R.

Solution

$$A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{pmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \end{pmatrix}$$

$$\xrightarrow[R_3 - 2R_2]{} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The pivots columns are the 1st and 2nd column.

$$A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Exercise

Suppose R is m by n matrix of rank r, with pivot columns first: $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

a) What are the shapes of those 4 blocks?
b) Find the right-inverse B with $RB = I$ if $r = m$.
c) Find the right-inverse C with $CR = I$ if $r = n$.
d) What is the reduced row echelon form of R^T (with shapes)?
e) What is the reduced row echelon form of $R^T R$ (with shapes)?
Prove that $R^T R$ has the same nullspace as R. Then show that $A^T A$ always has the same nullspace as A (a value fact).
f) Suppose you allow elementary column operations on $\boldsymbol{A}$ as well as elementary row operations (which get to $\boldsymbol{R}$). What is the "row-and-column reduced form" for an m by n matrix of rank $\boldsymbol{r}$?

Solution

a) $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} r \times r & r \times (n-r) \\ (m-r) \times r & (m-r) \times (n-r) \end{bmatrix}$

b) $R = \begin{bmatrix} I & F \end{bmatrix}$

$RB = I \Rightarrow \begin{bmatrix} I & F \end{bmatrix} B = I$

$$\begin{bmatrix} I & F \end{bmatrix} \begin{pmatrix} M \\ N \end{pmatrix} = I$$

$$IM + FN = I \Rightarrow \begin{cases} M = I \\ N = 0 \end{cases} \rightarrow \quad F : r \times (n-r)$$

$$B = \begin{bmatrix} I_{r \times r} \\ 0_{(n-r) \times r} \end{bmatrix}$$

c) $R = \begin{bmatrix} I & 0 \end{bmatrix}$

$CR = I \Rightarrow C \begin{bmatrix} I & 0 \end{bmatrix} = I$

$C = \begin{bmatrix} I_{r \times r} & 0_{r \times (m-r)} \end{bmatrix}$

d) $R^T = \begin{bmatrix} I_{r \times r} & 0_{(m-r) \times r} \\ F_{r \times (n-r)} & 0_{(m-r) \times (n-r)} \end{bmatrix} \Rightarrow rref\left(R^T\right) = \begin{bmatrix} I_{r \times r} & 0_{(m-r) \times r} \\ 0_{(n-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix}$

e) $R^T R = \begin{bmatrix} I & 0 \\ F & 0 \end{bmatrix} \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & F \\ FI & 0 \end{bmatrix}$

$FI : r \times (n-r) \quad r \times r$, the inner is not equal but to make work, we can use the F transpose.

$(n-r) \times r \quad r \times r \Rightarrow F^T I = F^T$

$$R^T R = \begin{bmatrix} I & 0 \\ F & 0 \end{bmatrix} \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} I & F \\ F^T & 0 \end{bmatrix}$$

$$rref\left(R^T R\right) = \begin{bmatrix} I_{r\times r} & F_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix} = R$$

So that $N(A) = N\left(rref(A)\right)$ for any matrix $\boldsymbol{A}$. So, $N(A) = N\left(R^T R\right)$

f) After getting to R we can use the column operations to get rid of F.

$$\begin{bmatrix} I_{r\times r} & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}$$

Exercise

True or False (check addition or give a counterexample)

a) The symmetric matrices in M $\left(with\ A^T = A\right)$ from a subspace.

b) The skew-symmetric matrices in M $\left(with\ A^T = -A\right)$ from a subspace.

c) The un-symmetric matrices in M $\left(with\ A^T \neq A\right)$ from a subspace.

d) Invertible matrices

e) Singular matrices

Solution

a) True: $A^T = A$ and $B^T = B$ lead to $(A+B)^T = A^T + B^T = A + B$

b) True: $A^T = -A$ and $B^T = -B$ lead to $(A+B)^T = A^T + B^T = -A - B = -(A+B)$

c) False: $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

d) False: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ are invertible matrices but $A + B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ is not invertible. $\therefore$ The zero matrix is not invertible but any linear subspace should contain the zero matrix. So invertible matrices do not form a linear subspace.

e) False: $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ are singular matrices but $A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ is not singular.

Exercise

Let $A=\begin{pmatrix}1 & 2 & -2 & 3 & 0\\ 2 & 4 & -3 & 7 & 0\\ 3 & 6 & -5 & 10 & -2\\ 5 & 10 & -9 & 16 & 0\end{pmatrix}$

a) Reduce A to row-reduced echelon from.

b) What is the rank of A?

c) What are the pivots?

d) What are the free variables?

e) Find the special solutions. What is the nullspace $N(A)$?

f) Exhibit an r x r submatrix of A which is invertible, where $r=rank(A)$. (An r x r submatrix of A is obtained by keeping r rows and r columns of A)

Solution

a) $A=\begin{pmatrix}1 & 2 & -2 & 3 & 0\\ 2 & 4 & -3 & 7 & 0\\ 3 & 6 & -5 & 10 & -2\\ 5 & 10 & -9 & 16 & 0\end{pmatrix}\begin{matrix}\\ R_2-2R_1\\ R_3-3R_1\\ R_4-5R_1\end{matrix}\begin{pmatrix}1 & 2 & -2 & 3 & 0\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1 & -2\\ 0 & 0 & 1 & 1 & 0\end{pmatrix}$

$$\begin{matrix}\\ \\ R_3-R_2\\ R_4-R_2\end{matrix}\begin{pmatrix}1 & 2 & -2 & 3 & 0\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & -2\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

b) Rank(***A***) = 3

c) The pivots are x_1, x_3, x_5

d) The free variables are x_2, x_4

e) $\begin{pmatrix}1 & 2 & -2 & 3 & 0\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & -2\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}\begin{matrix}\\ \\ -\frac{1}{2}R3\\ \\ \end{matrix}\begin{pmatrix}1 & 2 & -2 & 3 & 0\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}\begin{matrix}R_1+2R_2\\ \\ \\ \\ \end{matrix}\begin{pmatrix}1 & 2 & 0 & 5 & 0\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}$

Let $x=x_2s_2+x_4s_4$

$$Rx=\begin{pmatrix}1 & 2 & 0 & 5 & 0\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\\ 0\end{pmatrix}\qquad\to\begin{cases}x_1+2x_2+5x_4=0\\ x_3+x_4=0\\ x_5=0\end{cases}$$

1. Set $x_2 = 1, \quad x_4 = 0 \rightarrow \begin{cases} x_1 + 2 = 0 \Rightarrow x_1 = -2 \\ x_3 = 0 \\ x_5 = 0 \end{cases}$

The special solution: $s_2 = (-2,\ 1,\ 0,\ 0,\ 0)$

2. Set $x_2 = 0, \quad x_4 = 1 \rightarrow \begin{cases} x_1 + 5 = 0 \Rightarrow x_1 = -5 \\ x_3 + 1 = 0 \Rightarrow x_3 = -1 \\ x_5 = 0 \end{cases}$

The special solution: $s_3 = (-5,\ 0,\ -1,\ 1,\ 0)$

The nullspace is the set $\left\{ x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -5 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

f) The pivot rows and columns must be included in a submatrix. To do that, just take the rows and columns of $\boldsymbol{A}$ containing pivots, which are columns 1, 3, 5 and rows 1, 2, 3. That will give us a 3 by 3 submatrix. Therefore, this submatrix of $\boldsymbol{A}$ will be invertible.

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & -3 & 0 \\ 3 & -5 & -2 \end{pmatrix}$$

Exercise

Let $A=\begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 2 & 1 & 0 & 0 & -15 \\ 6 & -1 & -8 & -1 & -47 \\ 0 & 2 & 4 & 3 & 16 \end{pmatrix}$

a) Reduce ***A*** to (ordinary) echelon from.

b) What the pivots?

c) What are the free variables?

d) Reduce ***A*** to row-reduced echelon form.

e) Find the special solutions. What is the nullspace $N(A)$?

f) What is the rank of ***A***?

g) Give the complete solution to $Ax=b,\ where\ \ b=A\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}$

Solution

a) $A=\begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 2 & 1 & 0 & 0 & -15 \\ 6 & -1 & -8 & -1 & -47 \\ 0 & 2 & 4 & 3 & 16 \end{pmatrix} \xrightarrow[R_3+6R_1]{R_2+2R_1} \begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 0 & 5 & 10 & 0 & -5 \\ 0 & 11 & 22 & -1 & -17 \\ 0 & 2 & 4 & 3 & 16 \end{pmatrix}$

$\begin{array}{c} \\ \\ 5R_3-11R_2 \\ 5R_4-2R_2 \end{array}\begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 0 & 5 & 10 & 0 & -5 \\ 0 & 0 & 0 & -5 & -30 \\ 0 & 0 & 0 & 15 & 90 \end{pmatrix} \xrightarrow{R_4+3R_3} \begin{pmatrix} \boxed{-1} & 2 & 5 & 0 & 5 \\ 0 & \boxed{5} & 10 & 0 & -5 \\ 0 & 0 & 0 & \boxed{-5} & -30 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

b) The pivots are -1, 5, and -5 (Columns 1, 2, 4)

c) The free variables are 3rd and 5th $(x_3,\ x_5)$

d) $\begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 0 & 5 & 10 & 0 & -5 \\ 0 & 0 & 0 & -5 & -30 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{c} -R_1 \\ \frac{1}{5}R_2 \\ -\frac{1}{5}R_3 \\ \\ \end{array} \begin{pmatrix} 1 & -2 & -5 & 0 & -5 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$\xrightarrow{R_1+2R_2} \begin{pmatrix} 1 & 0 & -1 & 0 & -7 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $\qquad R=\begin{pmatrix} 1 & 0 & -1 & 0 & -7 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

e) Let $x=x_3s_1+x_5s_2$

$$Rx=\begin{pmatrix}1&0&-1&0&-7\\0&1&2&0&-1\\0&0&0&1&6\\0&0&0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\quad\rightarrow\begin{cases}x_1-x_3-7x_5=0\\x_2+2x_3-x_5=0\\x_4+6x_5=0\end{cases}$$

1. Set $x_3=1,\quad x_5=0\rightarrow\begin{cases}x_1-1=0\\x_2+2=0\\x_4=0\end{cases}\Rightarrow\begin{cases}x_1=1\\x_2=-2\\x_4=0\end{cases}$

The special solution: $s_1=(1,\ -2,\ 1,\ 0,\ 0)$

2. Set $x_3=0,\quad x_5=1\rightarrow\begin{cases}x_1-7=0\\x_2-1=0\\x_4+6=0\end{cases}\Rightarrow\begin{cases}x_1=7\\x_2=1\\x_4=-6\end{cases}$

The special solution: $s_2=(7,\ 1,\ 0,\ -6,\ 1)$

The nullspace is the set $\left\{x_3\begin{pmatrix}1\\-2\\1\\0\\0\end{pmatrix}+x_5\begin{pmatrix}7\\1\\0\\-6\\1\end{pmatrix}\right\}$

f) Rank($\boldsymbol{A}$) = 3

g) $Ax=b,\ where\ \ b=A\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}$

The complete solution = (the particular solution) + (special solution)

$$x=x_p+x_n$$

$$x=\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}+x_3\begin{pmatrix}1\\-2\\1\\0\\0\end{pmatrix}+x_5\begin{pmatrix}7\\1\\0\\-6\\1\end{pmatrix}$$

Exercise

The 3 by 3 matrix A has rank 2.

$$Ax=b \quad is \quad \begin{matrix} x_1+2x_2+3x_3+5x_4=b_1 \\ 2x_1+4x_2+8x_3+12x_4=b_2 \\ 3x_1+6x_2+7x_3+13x_4=b_3 \end{matrix}$$

a) Reduce $\begin{bmatrix} A & b \end{bmatrix}$ to $\begin{bmatrix} U & c \end{bmatrix}$, so that $Ax=b$ becomes triangular system $Ux=c$.

b) Find the condition on (b_1, b_2, b_3) for $Ax=b$ to have a solution

c) Describe the column space of A. Which plane in $\boldsymbol{R}^3$?

d) Describe the nullspace of A. Which special solutions in $\boldsymbol{R}^4$?

e) Find a particular solution to $Ax=(0,\ 6,\ -6)$ and then complete solution.

Solution

a) $$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2-2b_1 \\ 0 & 0 & -2 & -2 & b_3-3b_1 \end{array}\right]$$

$$\xrightarrow[R_3+R_2]{} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & b_3+b_2-5b_1 \end{array}\right]$$

b) The last equation $b_3+b_2-5b_1=0$ shows the solvability condition.

c) (***i***) The column space is the plane containing all combinations of the pivot columns: 1st (1, 2, 3) and 3rd (3, 8, 7).

(***ii***) The column space contains all vectors with $b_3+b_2-5b_1=0$. That makes $Ax=b$ solvable, so $\boldsymbol{b}$ is in the column space. All columns of A pass this test $b_3+b_2-5b_1=0$. This is the equation for the plane in (***i***).

d) The special solutions have free variables:

$$\begin{cases} x_1+2x_2+3x_3+5x_4=0 \\ \qquad 2x_3+2x_4=0 \end{cases} \Rightarrow \begin{cases} x_1=-2x_2-2x_4 \\ \quad x_3=-x_4 \end{cases}$$

$$x_2=1,\ x_4=0 \Rightarrow \begin{cases} x_1=-2 \\ x_3=0 \end{cases} \to s_1=\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$x_2 = 0,\ x_4 = 1 \Rightarrow \begin{cases} x_1 = -2 \\ x_3 = -1 \end{cases} \to s_2 = \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

The nullspace $N(A)$ in $\boldsymbol{R}^4$ contains all $x_n = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

e) One particular solution x_p has free variables = zero.

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 5x_4 = 0 \\ \quad 2x_3 + 2x_4 = 6 \end{cases} \Rightarrow \begin{cases} x_1 = -2x_2 - 3x_3 - 5x_4 \\ x_3 = 3 - x_4 \end{cases} \to\Rightarrow \begin{cases} x_1 = -2x_2 - 9 - 2x_4 \\ x_3 = 3 - x_4 \end{cases}$$

$$x_2 = x_4 = 0 \Rightarrow \begin{cases} x_1 = -9 \\ x_3 = 3 \end{cases}$$

$$x_p = \begin{pmatrix} -9 \\ 0 \\ -3 \\ 0 \end{pmatrix}$$

The complete solution to $Ax = (0,\ 6,\ -6)$ is $x = x_p + all\ x_n$

Exercise

Find the special solutions and describe the complete solution to $Ax = 0$ for

$$A_1 = 3 \text{ by } 4 \text{ zero matrix} \qquad A_2 = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \qquad A_3 = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$$

Which are the pivot columns? Which are the free variables? What is the R (Reduced Row Echelon matrix) in each case?

Solution

$A_1 x = 0$ has 4 solutions. They are the columns s_1, s_2, s_3, s_4 of the identity matrix (4 by 4).

The Nullspace is of $\boldsymbol{R}^4$.

The complete solution: $x = c_1 s_1 + c_2 s_2 + c_3 s_3 + c_4 s_4$ in $\boldsymbol{R}^4$.

There are no pivot columns; all variables are free, the reduced R is the same zero matrix as A_1.

$A_2 x = 0$

$$\Rightarrow A_2 x = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 + 2x_2 = 0$$

The vector solution: $s = (-2, 1)$, The first column of A_2 is its pivot column, and x_2 is the free variable.

$$\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \xrightarrow[R_2 - R_1]{} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

$$R_3 = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

All variables are free. There are three special solutions to $A_3 x = 0$

$$s_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad s_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad s_3 = \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The complete solution: $x = c_1 s_1 + c_2 s_2 + c_3 s_3$.

Exercise

Create a 3 by 4 matrix whose special solutions to $Ax=0$ are s_1 *and* s_2:

$$s_1 = \begin{pmatrix} -3 \\ 2 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad s_2 = \begin{pmatrix} -2 \\ 0 \\ -6 \\ 1 \end{pmatrix}$$

You could create the matrix A in row reduced form R. Then describe all possible matrices A with the required Nullspace $N(A) =$ all combinations of s_1 *and* s_2.

Solution

We can write the solution:

$x = x_2 s_1 + x_4 s_2$

$$x_2 \begin{pmatrix} -3 \\ 2 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -6 \\ 1 \end{pmatrix} = \begin{pmatrix} -3x_2 - 2x_4 \\ 2x_2 \\ -6x_4 \\ x_4 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3x_2 - 2x_4 \\ 2x_2 \\ -6x_4 \\ x_4 \end{pmatrix} \to \begin{cases} x_1 = -3x_2 - 2x_4 \\ \\ x_3 = -6x_4 \end{cases}$$

$$\to \begin{cases} x_1 + 3x_2 + 2x_4 = 0 \\ \\ x_3 + 6x_4 = 0 \end{cases} \Rightarrow \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The entries 3, 2, 6 are the negatives of -3, -2, -6 in the special solutions.

Every 3 by 4 matrix has at least one special solution. These A's have two.

Exercise

The plane $x-3y-z=12$ is parallel to the plane $x-3y-z=0$. One particular point on this plane is $(12,\ 0,\ 0)$. All points on the plane have the form (fill the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}$$

Solution

$$x-3y-z=12 \Rightarrow x=3y+z+12$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} 12+3y+z \\ y \\ z \end{pmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} z \\ 0 \\ 1 \end{bmatrix}$$

Exercise

Construct a matrix whose column space contains (1, 1, 5) and (0, 3, 1) and whose Nullspace contains $(1,\ 1,\ 2)$.

Solution

$$A = \begin{pmatrix} 1 & 0 & a \\ 1 & 3 & b \\ 5 & 1 & c \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & a \\ 1 & 3 & b \\ 5 & 1 & c \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+2a \\ 1+3+2b \\ 5+1+2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 1+2a=0 \\ 4+2b=0 \\ 6+2c=0 \end{cases} \rightarrow \begin{cases} 2a=-1 \\ 2b=-4 \\ 2c=-6 \end{cases} \rightarrow \begin{cases} a=-\frac{1}{2} \\ b=-2 \\ c=-3 \end{cases}$$

$$A = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{pmatrix}$$

Exercise

Construct a matrix whose column space contains (1, 1, 0) and (0, 1, 1) and whose Nullspace contains (1, 0, 1) and (0, 0, 1).

Solution

It is impossible. Matrix $\boldsymbol{A}$ must be 3 by 3. Since the nullspace is supposed to contain two independent vectors, $\boldsymbol{A}$ can have at most $3-2=1$ pivots. Since the column space supposes to contain two independent vectors. A must have at least 2 pivots. These conditions can't both be met.

Exercise

Construct a matrix whose column space contains (1, 1, 1) and whose Nullspace contains $(1,\ 1,\ 1,\ 1)$.

Solution

The matrix needs to be 3 by 4 matrix.

$$\begin{pmatrix} 1 & a & b & c \\ 1 & d & e & f \\ 1 & g & h & i \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 1+a+b+c=0 \\ 1+d+e+f=0 \\ 1+g+h+i=0 \end{cases} \Rightarrow \begin{cases} a+b+c=-1 \\ d+e+f=-1 \\ g+h+i=-1 \end{cases} \rightarrow \begin{cases} if \quad b=c=0 \quad a=-1 \\ if \quad d=f=0 \quad e=-1 \\ if \quad g=h=0 \quad i=-1 \end{cases}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

Exercise

How is the Nullspace N(C) related to the spaces N(A) and N(B), if $C=\begin{bmatrix} A \\ B \end{bmatrix}$?

Solution

$$Cx=\begin{bmatrix} Ax \\ Bx \end{bmatrix}=0=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If and only if Ax =0 and Bx = 0.

$N(C)=N(A)\bigcap N(B)$

Exercise

Why does no 3 by 3 matrix have a nullspace that equals its column space?

Solution

If nullspace = column space then $n - r = r$ (there are r pivots). For $n = 3 \Rightarrow 3 = 2\boldsymbol{r}$ is impossible.

Exercise

If $AB = 0$ then the column space B is contained in the _______ of A. Give an example of A and B.

Solution

If $AB = 0$ then the column space B is contained in the ***nullspace*** of A.

Example: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

Exercise

True or false (with reason if true or example to show it is false)

a) A square matrix has no free variables.

b) An invertible matrix has no free variables.

c) An m by n matrix has no more than $\boldsymbol{n}$ pivot variables.

d) An m by n matrix has no more than $\boldsymbol{m}$ pivot variables.

Solution

a) False. Any matrix with fewer than full number of pivots will. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

b) True. Since it is invertible, we will get the full number of pivots. The nullspace has dimension, so we have 0 free variables.

c) True, the number of pivot variables is the dimension of the nullspace, which is at most the number of columns. The nullspace dimension + column space dimension = number of columns.

d) True, in reduced echelon matrix the pivot columns are all 0 except for a single 1, and there are only up to m vectors of this type.

Exercise

Suppose an m by n matrix has r pivots. The number of special solutions is ______.

The Nullspace contains only $x = 0$ when $r =$ _______.

The column space is all of $\mathbf{R}^m$ when $r =$ ______.

Solution

Suppose an m by n matrix has r pivots. The number of special solutions is _ $\boldsymbol{n-r}$_.

The Nullspace contains only $x = 0$ when $r =$ _ $\boldsymbol{n}$ _.

The column space is all of $\mathbf{R}^m$ when $r =$ _$\boldsymbol{m}$ _.

Exercise

Find the complete solution in the form $x_p + x_n$ to these full rank system:

$a)\ \ x+y+z=4 \qquad\qquad b)\ \begin{matrix} x+y+z=4 \\ x-y+z=4 \end{matrix}$

Solution

a) $x+y+z=4$

The equivalent matrix is given by: $\begin{cases} Ax=4 \\ A=\begin{pmatrix}1 & 1 & 1\end{pmatrix} \end{cases}$

The complete solution in the form $x = x_p + x_n$

x_n is the homogeneous solution to $Ax_n = 0$

Size of $\boldsymbol{A}$ is $m = 1$ and $n = 3$, rank($\boldsymbol{A}$) = $r = 1$

$$Ax_n = 0 \Rightarrow \begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}x_1 \\ x_2 \\ x_3\end{pmatrix} = \begin{pmatrix}0 \\ 0 \\ 0\end{pmatrix} \Rightarrow x_1 + x_2 + x_3 = 0 \Rightarrow \boxed{x_1 = -x_2 - x_3}$$

Set $x_2 = 1, \quad x_3 = 0$ The special solution: $s_1 = (-1,\ 1,\ 0)$

Set $x_2 = 0, \quad x_3 = 1$ The special solution: $s_2 = (-1,\ 0,\ 1)$

The nullspace is the set $\left\{ x_2\begin{pmatrix}-1 \\ 1 \\ 0\end{pmatrix} + x_3\begin{pmatrix}-1 \\ 0 \\ 1\end{pmatrix} \right\}$

$x = 4 - y - z \ \Rightarrow x_1 = 4 - x_2 - x_3$

Set $x_2 = 0, \quad x_3 = 0$ that implies to the particular solution: $x_p = \begin{pmatrix}4 \\ 0 \\ 0\end{pmatrix}$

The complete solution in the form $x=\begin{pmatrix}4\\0\\0\end{pmatrix}+x_2\begin{pmatrix}-1\\1\\0\end{pmatrix}+x_3\begin{pmatrix}-1\\0\\1\end{pmatrix}$

Note: that the null space of A is spanned by the two linearly independent vectors $(-1,\ 1,\ 0)^T$ *and* $(-1,\ 0,\ 1)^T$

b) $\begin{matrix} x+y+z=4 \\ x-y+z=4 \end{matrix}$

The equivalent matrix is given by: $A=\begin{pmatrix}1&1&1\\1&-1&1\end{pmatrix}$ and $A\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}4\\4\end{pmatrix}$

$$\left[\begin{array}{ccc|c}1&1&1&4\\1&-1&1&4\end{array}\right]\xrightarrow[R_2-R_1]{}\left[\begin{array}{ccc|c}1&1&1&4\\0&-2&0&0\end{array}\right]\xrightarrow[-\frac{1}{2}R_2]{}\left[\begin{array}{ccc|c}1&1&1&4\\0&1&0&0\end{array}\right]\xrightarrow{R_1-R_2}$$

$$\left[\begin{array}{ccc|c}1&0&1&4\\0&1&0&0\end{array}\right]$$

The pivots are $x_1,\ x_2$; The free variable is x_3

Rank r = 2, n = 2, m = 3. The nullspace has dimension $m - r$ = 1.

$$\begin{pmatrix}1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow\begin{cases}x_1+x_3=0\to x_1=-x_3\\ \quad x_2=0\end{cases}$$

If $x_3=1\Rightarrow x_1=-1$ The special solution: $s_1=(-1,\ 0,\ 1)$

The nullspace is the set $\left\{x_3\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$

Set $x_3=0$ that implies $\begin{pmatrix}1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}4\\0\end{pmatrix}\Rightarrow\begin{cases}x_1=4\\x_2=0\end{cases}$

Then the particular solution: $x_p=\begin{pmatrix}4\\0\\0\end{pmatrix}$

The complete solution in the form $x=\begin{pmatrix}4\\0\\0\end{pmatrix}+x_3\begin{pmatrix}-1\\0\\1\end{pmatrix}$

Exercise

Find the complete solution in the form $x_p + x_n$ to the system:

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

Solution

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{cccc|c} 1 & 3 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 2 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The pivots are $x_1,\ x_3$; The free variables are $x_2,\ x_4$

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 + 3x_2 = 0 \\ x_3 + 2x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -3x_2 \\ x_3 = -2x_4 \end{cases}$$

1. Set $x_2 = 1,\quad x_4 = 0$ The special solution: $s_1 = (-3,\ 1,\ 0,\ 0)$

2. Set $x_2 = 0,\quad x_4 = 1$ The special solution: $s_2 = (0,\ 0,\ -2,\ 1)$

The special solution: $x_n = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 + 3(0) = \frac{1}{2} \\ x_3 + 2(0) = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} x_1 = \frac{1}{2} \\ x_3 = \frac{1}{2} \end{cases}$$

Then the particular solution: $x_p = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$

The complete solution in the form $x = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}$

Exercise

If $\boldsymbol{A}$ is 3 x 7 matrix, its largest possible rank is ________. In this case, there is a pivot in every ________ of $\boldsymbol{U}$ and $\boldsymbol{R}$, the solution to $Ax = b$ _________ (always exists or is unique), and the column space of $\boldsymbol{A}$ is _________. Construct an example of such a matrix $\boldsymbol{A}$.

Solution

If $\boldsymbol{A}$ is 3 x 7 matrix, its largest possible rank is **3**. In this case, there is a pivot in every ***row*** of $\boldsymbol{U}$ and R, the solution to $Ax = b$ ***always exists***, and the column space of $\boldsymbol{A}$ is $\boldsymbol{R}^3$.

$$A = \begin{pmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{pmatrix}$$

$rank(A) \le 3$, that implies that you have 3 pivots (1 each row)

$$A = \begin{pmatrix} 1 & 0 & 0 & * & * & * & * \\ 0 & 1 & 0 & * & * & * & * \\ 0 & 0 & 1 & * & * & * & * \end{pmatrix} \to A = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 5 & 6 & 7 & 8 \\ 0 & 0 & 1 & 9 & 10 & 11 & 12 \end{pmatrix}$$

Exercise

If $\boldsymbol{A}$ is 6 x 3 matrix, its largest possible rank is ________. In this case, there is a pivot in every ________ of $\boldsymbol{U}$ and R, the solution to $Ax = b$ _________ (always exists or is unique), and the nullspace of $\boldsymbol{A}$ is _________. Construct an example of such a matrix $\boldsymbol{A}$.

Solution

If $\boldsymbol{A}$ is 6 x 3 matrix, its largest possible rank is **3**. In this case, there is a pivot in every ***column*** of $\boldsymbol{U}$ and R, the solution to $Ax = b$ ***is unique***, and the column space of $\boldsymbol{A}$ is $\{\mathbf{0}\}$.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Exercise

Find the rank of $A, A^T A$ and AA^T for $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix}$

Solution

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix} \underset{R_3 + R_1}{\rightarrow} \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 3 \end{pmatrix} \tfrac{1}{2}R_2 \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 3 \end{pmatrix} R_3 - 3R_2 \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \boxed{rank(A) = 2}$$

$$A^T A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 9 \end{pmatrix} \xrightarrow[2R_2 + R_1]{} \begin{pmatrix} 2 & -1 \\ 0 & 17 \end{pmatrix} \Rightarrow \boxed{rank\left(A^T A\right) = 2}$$

$$AA^T = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 4 \\ 1 & 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 4 \\ 1 & 4 & 5 \end{pmatrix} \xrightarrow[2R_3 - R_1]{R_2 - R_1} \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 6 & 9 \end{pmatrix} \xrightarrow[R_3 - 3R_2]{} \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \boxed{rank\left(AA^T\right) = 2}$$

$\therefore rank(A) = rank\left(A^T A\right) = rank\left(AA^T\right)$ for any matrix $\boldsymbol{A}$.

Exercise

Explain why these are all false:

a) The complete solution is any linear combination of x_p *and* x_n.

b) A system $Ax = b$ has at most one particular solution.

c) The solution x_p with all free variables zero is the shortest solution (minimum length $\|x\|$). Find a 2 by 2 counterexample.

d) If A is invertible there is no solution x_n in the null space.

Solution

a) The coefficient of x_p must be one.

b) If $x_n \in N(A)$ is the nullspace of $\boldsymbol{A}$ and x_p is one particular solution, then x_p *and* x_n is also a particular solution.

c) If $\boldsymbol{A}$ is a 2 by 2 matrix of rank 1, then the solution to $Ax = b$ form a line parallel to the line that the nullspace. The line $x + y = 1$ gives such an example.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad x_p = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Then $\left\| x_p \right\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{2\frac{1}{4}} = \frac{1}{\sqrt{2}} < 1$ while the particular solutions having some coordinate equal to zero are (1, 0) and (0, 1) and they both have $\left\| x_p \right\| = 1$

d) There is always $x_n = 0$

Exercise

Write down all known relation between *r* and *m* and *n* if $Ax = b$ has

a) No solution for some ***b***.

b) Infinitely many solutions for every ***b***.

c) Exactly one solution for some ***b***, no solution for other ***b***.

d) Exactly one solution for every ***b***.

Solution

a) The system has less than full row rank: $r < m$.

b) The system has full row rank and less than full column rank: $m = r < n$.

c) The system has full column rank and less than full row rank: $n = r < m$.

d) The system has full row and column rank (it is invertible): $m = r = n$.

Exercise

Find a basis for its row space, find a basis for its column space, and determine its rank

$$a)\ \begin{bmatrix} 0 & 2 & -3 & 4 & 1 & 2 & 1 & 7 \\ 0 & 0 & 3 & -2 & 0 & 4 & -5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 6 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad b)\ \begin{bmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{bmatrix}$$

Solution

a) Row Space: every row

Column Space: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ -2 \\ 2 \\ 0 \end{bmatrix}$

Rank = 4

b) $\begin{bmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & \frac{13}{3} \\ 0 & 1 & -7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Row Space: $\begin{bmatrix} 3 & 2 & -1 \end{bmatrix}, \begin{bmatrix} 6 & 3 & 5 \end{bmatrix}$

Column Space: $\begin{bmatrix} 3 \\ 6 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \\ -1 \end{bmatrix}$

Rank = 2

Exercise

Find a basis for the row space, find a basis for the null space, find $\dim RS$, find $\dim NS$, and verify $\dim RS + \dim NS = n$

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ 3 & 1 & -3 & -1 \\ 5 & -3 & 5 & 1 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ 3 & 1 & -3 & -1 \\ 5 & -3 & 5 & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -\frac{2}{7} & -\frac{1}{7} \\ 0 & 1 & -\frac{15}{7} & -\frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Space: $\begin{bmatrix} 1 & -2 & 4 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 & -3 & -1 \end{bmatrix}$

Column Space: $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$\dim RS = 2$

$\dim NS = 2$

$2 + 2 = 2 \quad \Rightarrow \quad \dim RS + \dim NS = n$

Exercise

Determine if ***b*** lies in the column space of the given matrix. If it does, express ***b*** as linear combination of the column.

$$\begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

Solution

$$\left[\begin{array}{cc|c} 2 & -3 & 4 \\ -4 & 6 & -6 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 1 \end{array}\right]$$

b does not lies in the column space

Exercise

Find the transition matrix from B to C and find $[\boldsymbol{x}]_c$

a) $B = \{(3, 1), (-1, -2)\}, \quad C = \{(1, -3), (5, 0)\}, \quad [x]_B = [-1 \quad -2]^T$

b) $B = \{(1, 1, 1),\ (-2, -1, 0),\ (2, 1, 2)\}, \quad C = \{(-6, -2, 1),\ (-1, 1, 5),\ (-1, -1, 1)\},$

$[\boldsymbol{x}]_B = [-3 \quad 2 \quad 4]^T$

Solution

a) $$\left[\begin{array}{cc|cc} 1 & 5 & 3 & -1 \\ -3 & 0 & 1 & -2 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{array}\right]$$

$$[\boldsymbol{x}]_c = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

b) $$\left[\begin{array}{ccc|ccc} -6 & -1 & -1 & 1 & -2 & 2 \\ -2 & 1 & -1 & 1 & -1 & 1 \\ 1 & 5 & 1 & 1 & 0 & 2 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{13} & \frac{4}{13} & -\frac{6}{13} \\ 0 & 1 & 0 & \frac{4}{13} & -\frac{3}{26} & \frac{11}{26} \\ 0 & 0 & 1 & -\frac{5}{13} & \frac{7}{26} & \frac{9}{26} \end{array}\right]$$

$$[\boldsymbol{x}]_c = \begin{bmatrix} -\frac{2}{13} & \frac{4}{13} & -\frac{6}{13} \\ \frac{4}{13} & -\frac{3}{26} & \frac{11}{26} \\ -\frac{5}{13} & \frac{7}{26} & \frac{9}{26} \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -\frac{10}{13} \\ \frac{17}{13} \\ \frac{35}{13} \end{bmatrix}$$