Lecture Three

Section 3.1 – Introduction to Linear Systems

Definition

A ***linear*** (algebraic) ***equation*** in n unknowns, $x_1, x_2, \ldots, x_n$, is an equation of the form

$$a_1x_1 + a_2x_2 + \ \ldots + a_nx_n = b$$

Where $a_1, a_2, \ldots, a_n$ and b are real numbers.

Matrices

$$\begin{array}{cc} & \text{Column} \\ & \begin{array}{ccc} C_1 & C_2 & C_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{c} Row\ 1 \to R_1 \\ Row\ 2 \to R_2 \\ Row\ 3 \to R_3 \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$

This is called Matrix (*Matrices*)

Each number in the array is an ***element*** or ***entry***

The matrix is said to be of order m x n

m: numbers of rows,

n: number of columns

When $m = n$, then matrix is said to be ***square***.

Given the system equations

$$\begin{aligned} 3x + \ \ y + 2z &= 31 \\ x + \ \ y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

Write into an ***augmented matrix*** form

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 31 \\ 1 & 1 & 2 & 19 \\ 1 & 3 & 2 & 25 \end{array}\right]$$

Example

Given the linear equations

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases}$$

The solution to this system is $(3,\ 1)$, which means that 2 lines meeting at a single point.

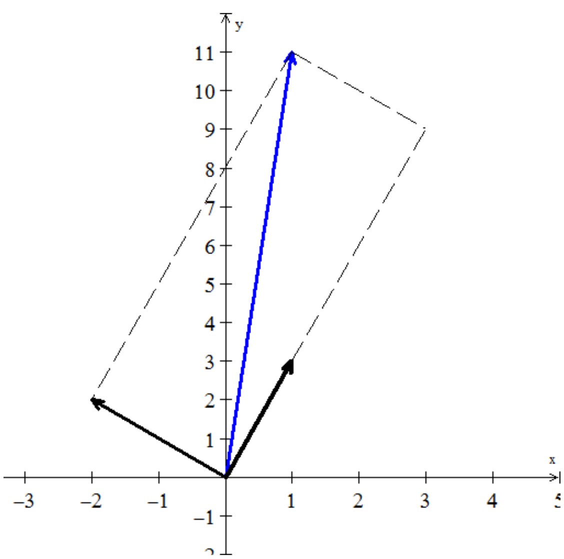
We can rewrite the system equation as linear combination:

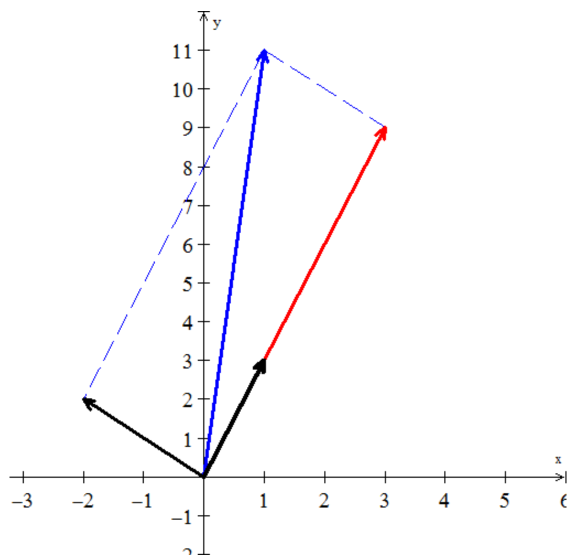
$$x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

$$x.v_1 + y.v_2 = v$$

$$\begin{bmatrix} 1+x \\ 3+y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} \Rightarrow \begin{bmatrix} x=3 \\ y=9 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 9 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$





Therefore, the side vectors are $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$ *and* $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$

The diagonal sum is $\begin{bmatrix} 3-2 \\ 9+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$

The linear combination is given by: $3\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$

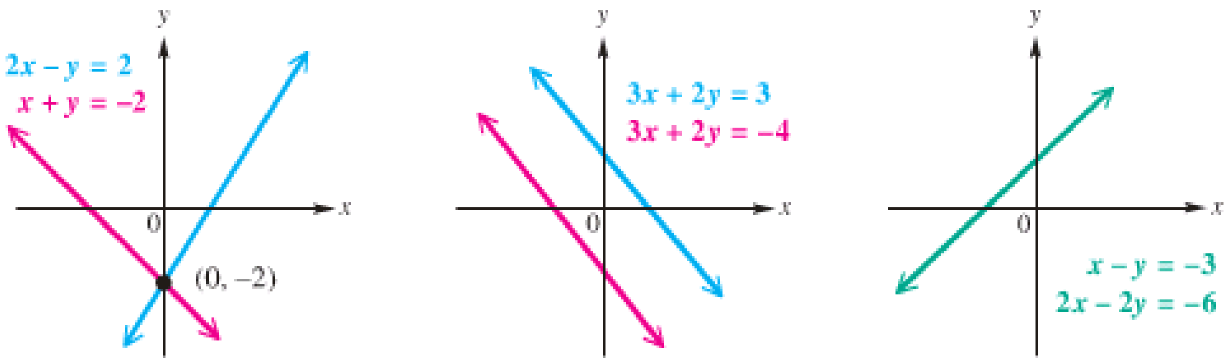
Thus, the solution is $x = 3 \quad y = 1$

Note

$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$ is called the coefficient matrix

The matrix form of the system is written as $\quad Ax = b$

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$



One solution (lines intersect)	***No Solution (lines //)***	***Infinite solution***
Consistent	***Inconsistent***	***Consistent***
Independent	***Independent***	***Dependent***

***Three* Equations in 3 Unknowns**

A linear equation in three unknowns *x, y, z*:

$$ax + by + cz = \alpha$$

A solution of the equation is an ordered triple of numbers $(x,\ y,\ z)$

If $a = b = c = 0,\ and\ \alpha = 0$, all ordered triples satisfy the equation $0x + 0y + 0z = 0$ (***infinitely many***)

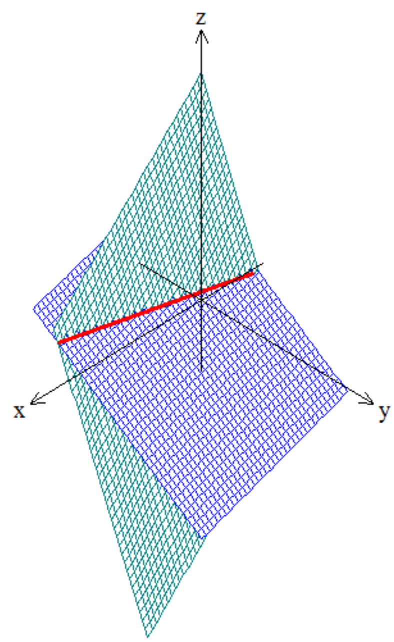
If $a = b = c = 0,\ and\ \alpha \neq 0$, no ordered triples satisfies the equation $0x + 0y + 0z \neq 0$ (***no solution***)

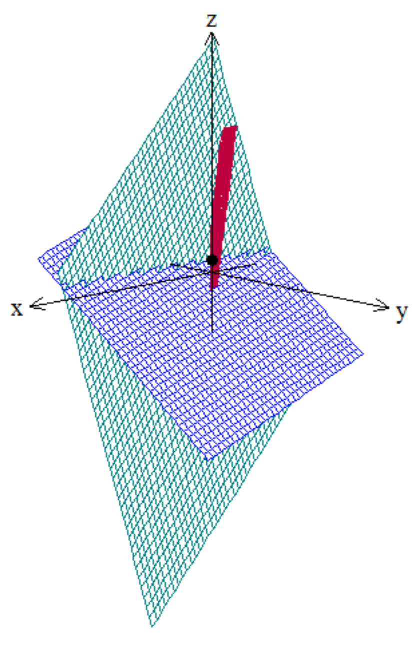
If *a, b, c*, not all 0, then the set of all ordered triples that satisfy the equation is a plane (in 3-space)

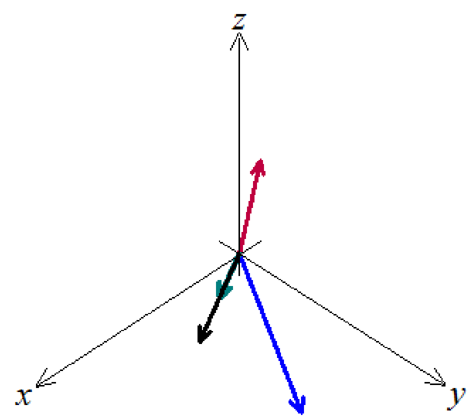
Example

Given the system equations

$$x + 2y + 3z = 6$$
$$2x + 5y + 2z = 4$$
$$6x - 3y + z = 2$$







This system can be written as linear combination:

$$x\begin{bmatrix}1\\2\\6\end{bmatrix}+y\begin{bmatrix}2\\5\\-3\end{bmatrix}+z\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}6\\4\\2\end{bmatrix}\qquad \text{Let } b=\begin{bmatrix}6\\4\\2\end{bmatrix}$$

We want to multiply the three column vectors by $x,\ y,\ z$ to produce $\boldsymbol{b}$.
The combination of the three vectors that produces
vector b is 2 times the third vector. $2(3,\ 2,\ 1)=(6,\ 4,\ 2)=b$
Therefore the coefficients that we need are $x=0,\quad y=0,\quad and\ z=2$.

$$0\begin{bmatrix}1\\2\\6\end{bmatrix}+0\begin{bmatrix}2\\5\\-3\end{bmatrix}+2\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}6\\4\\2\end{bmatrix}$$

Homogeneous Systems

The system of linear equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ \dots + a_{2n}x_n &= b_2 \\
&\ \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ \dots + a_{mn}x_n &= b_m
\end{aligned}
$$

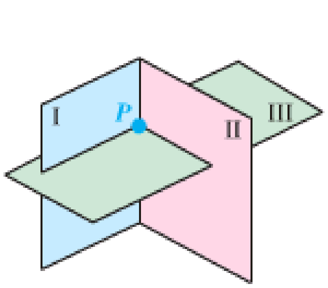
Is ***homogeneous*** if $b_1 = b_2 = \ \dots = b_m = 0$ otherwise, the system is ***nonhomogeous***.

A ***homogeneous*** system

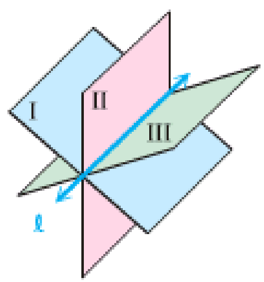
$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ \dots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \ \dots + a_{2n}x_n &= 0 \\
&\ \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ \dots + a_{mn}x_n &= 0
\end{aligned}
$$

Always has at least one solution namely $x_1 = x_2 = \ \dots = x_n = 0$ called the ***trivial solution***

That is, homogeneous systems are always ***consistent***

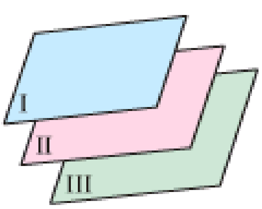


A single solution

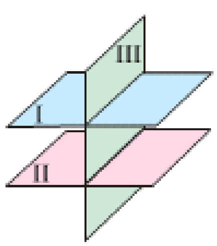


Points of a line in common

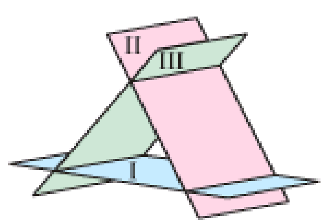
All points in common



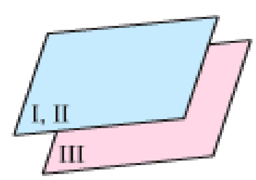
No points in common



No points in common



No points in common



No points in common

Exercises   *Section* 3.1 – Introduction to Linear Systems

1. Find a solution for x, y, z to the system of equations

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3e + 2\sqrt{2} + \pi \\ 6e + 5\sqrt{2} + 4\pi \\ 9e + 8\sqrt{2} + 7\pi \end{pmatrix}$$

2. Draw the two pictures in two planes for the equations: $x - 2y = 0, \quad x + y = 6$

3. Normally 4 planes in 4-dimensional space meet at a ________. Normally 4 column vectors in 4-deimensional space can combine to produce b. what combinations of $(1,\ 0,\ 0,\ 0)$, $(1,\ 1,\ 0,\ 0)$, $(1,\ 1,\ 1,\ 0)$, $(1,\ 1,\ 1,\ 1)$ produces $b = (3,\ 3,\ 3,\ 2)$?
What 4 equations for $x,\ y,\ z,\ w$ are you solving?

4. What 2 by 2 matrix A rotates every vector through 45° ?
The vector (1, 0) goes to $\left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$. The vector (0, 1) goes to $\left(-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$.
Those determine the matrix. Draw these particular vectors is the xy-plane and find A.

5. What two vectors are obtained by rotating the plane vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by 30° (cw) ?
Write a matrix A such that for every vector v in the plane, Av is the vector obtained by rotating v clockwise by 30°.
Find a matrix B such that for every 3-dimensional vector v, the vector Bv is the reflection of v through the plane $x + y + z = 0$. *Hint* $: v = (1,\ 0,\ 0)$

6. In each part, find a system of linear equation corresponding to the given augmented matrix

a) $\begin{bmatrix} 0 & 3 & -1 & -1 & -1 \\ 5 & 2 & 0 & -3 & -6 \end{bmatrix}$ *b)* $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \\ 5 & -6 & 1 & 1 \\ -8 & 0 & 0 & 3 \end{bmatrix}$

7. Find the augmented matrix for the given system of linear equations.

a) $\begin{cases} -2x_1 = 6 \\ 3x_1 = 8 \\ 9x_1 = -3 \end{cases}$ *b)* $\begin{cases} 3x_1 - 2x_2 = -1 \\ 4x_1 + 5x_2 = 3 \\ 7x_1 + 3x_2 = 2 \end{cases}$ *c)* $\begin{cases} 2x_1 + 2x_3 = 1 \\ 3x_1 - x_2 + 4x_3 = 7 \\ 6x_1 + x_2 - x_3 = 0 \end{cases}$