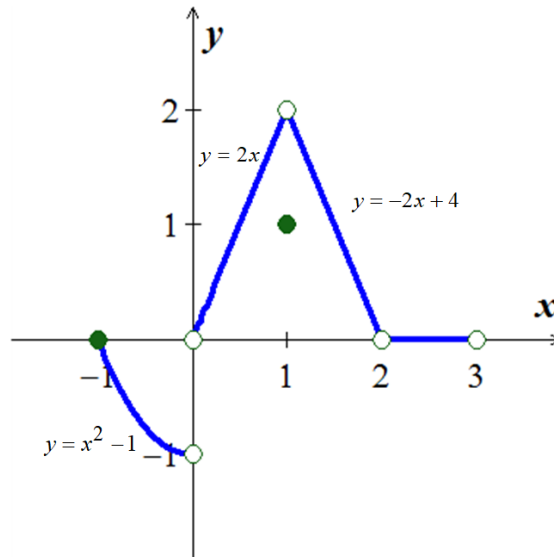# *Solution*   *Section* **1.5 – Continuity**

Exercise

Given the graphed function $f(x)$

  a) Does $f(-1)$ exist?

  b) Does $\lim\limits_{x \to -1^+} f(x)$ exist?

  c) Does $\lim\limits_{x \to -1^+} f(x) = f(-1)$?

  d) Is f continuous at $x = -1$?

  e) Does $f(1)$ exist?

  f) Does $\lim\limits_{x \to 1} f(x)$ exist?

  g) Does $\lim\limits_{x \to 1} f(x) = f(1)$?

  h) Is f continuous at $x = 1$?

Solution

  a) Yes $f(-1) = 0$

  b) Yes, $\lim\limits_{x \to -1^+} f(x) = 0$

  c) Yes

  d) Yes

  e) Yes, $f(1) = 1$

  f) Yes, $\lim\limits_{x \to 1} f(x) = 2$

  g) No

  h) No

Exercise

At what points is the function $y = \dfrac{1}{x-2} - 3x$ continuous?

Solution

  The function is continuous everywhere except when $x - 2 = 0 \Rightarrow x = 2$

Exercise

At what points is the function $y = \dfrac{x+3}{x^2 - 3x - 10}$ continuous?

Solution

The function is continuous everywhere except when $x^2 - 3x - 10 = 0 \Rightarrow x = -2,\ 5$

Exercise

At what points is the function $y = |x - 1| + \sin x$ continuous?

Solution

The function is continuous everywhere

Exercise

At what points is the function $y = \dfrac{x+2}{\cos x}$ continuous?

Solution

The function is continuous everywhere except when $\cos x = 0 \Rightarrow x = \dfrac{\pi}{2} + n\pi,\quad n \in \mathbb{Z}$

Exercise

At what points is the function $y = \tan \dfrac{\pi x}{2}$ continuous?

Solution

The function is continuous everywhere except when $x = 2n - 1,\quad n \in \mathbb{Z}$

Exercise

At what points is the function $y = \dfrac{x \tan x}{x^2 + 1}$ continuous?

Solution

The function is continuous everywhere except when $x = (2n - 1)\dfrac{\pi}{2},\quad n \in \mathbb{Z}$

Exercise

At what points is the function $y = \dfrac{\sqrt{x^4 + 1}}{1 + \sin^2 x}$ continuous?

Solution

The function is continuous everywhere

Exercise

At what points is the function $y = \sqrt{2x+3}$ continuous?

Solution

The function is continuous on the interval $2x + 3 \geq 0 \;\rightarrow\; x \geq -\frac{3}{2} \Rightarrow \left[-\frac{3}{2}, \infty\right)$, and discontinuous when $x < -\frac{3}{2}$

Exercise

At what points is the function $y = \sqrt[4]{3x-1}$ continuous?

Solution

The function is continuous on the interval $3x - 1 \geq 0 \rightarrow \left[\frac{1}{3}, \infty\right)$, and discontinuous when $x < \frac{1}{3}$

Exercise

At what points is the function $y = (2-x)^{1/5}$ continuous?

Solution

The function is continuous everywhere $\forall x$

Exercise

Find $\lim\limits_{x \to \pi} \sin(x - \sin x)$, then is the function continuous at the point being approached?

Solution

$$\lim_{x \to \pi} \sin(x - \sin x) = \sin(\pi - \sin \pi)$$
$$= \sin(\pi - 0)$$
$$= \sin(\pi)$$
$$= 0 \qquad\qquad \text{The function is continuous at } x = \pi$$

Exercise

Find $\lim\limits_{x \to 0} \tan\left(\frac{\pi}{4} \cos\left(\sin x^{1/3}\right)\right)$, then is the function continuous at the point being approached?

Solution

$$\lim_{x \to 0} \tan\left(\frac{\pi}{4} \cos\left(\sin x^{1/3}\right)\right) = \tan\left(\frac{\pi}{4} \cos\left(\sin (0)^{1/3}\right)\right)$$
$$= \tan\left(\frac{\pi}{4} \cos(0)\right)$$

$$= \tan\left(\frac{\pi}{4}\right)$$

$$\underline{= 1} \quad\quad\quad\quad \text{The function is continuous at } x = 0$$

Exercise

Find $\displaystyle \lim_{t \to 0} \cos\left(\frac{\pi}{\sqrt{19 - 3\sec 2t}}\right)$, then is the function continuous at the point being approached?

Solution

$$\lim_{t \to 0} \cos\left(\frac{\pi}{\sqrt{19 - 3\sec 2t}}\right) = \cos\left(\frac{\pi}{\sqrt{19 - 3\sec 2(0)}}\right)$$

$$= \cos\left(\frac{\pi}{\sqrt{19 - 3}}\right)$$

$$= \cos\left(\frac{\pi}{\sqrt{16}}\right)$$

$$= \cos\left(\frac{\pi}{4}\right)$$

$$\underline{= \frac{\sqrt{2}}{2}}$$

$\therefore$ The function is continuous at $t = 0$

Exercise

Explain why the equation $\cos x = x$ has at least one solution.

Solution

$$\cos x - x = 0$$

$$\begin{cases} \text{if } x = -\frac{\pi}{2} & \to \cos\left(-\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right) > 0 \\ \text{if } x = \frac{\pi}{2} & \to \cos\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\right) < 0 \end{cases}$$

$\Rightarrow \cos x - x = 0$

for some x between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

According to the Intermediate Value Theorem, and the function $\cos x = x$ is continuous and has at least one solution.

Exercise

Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$

Solution

$$f(-4) = (-4)^3 - 15(-4) + 1 = -3$$

$$f(-2) = (-2)^3 - 15(-2) + 1 = 23$$

$$f(-1) = (-1)^3 - 15(-1) + 1 = 15$$

$$f(1) = (1)^3 - 15(1) + 1 = -13$$

$$f(4) = (4)^3 - 15(4) + 1 = 5$$

By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-4 < x < -1$,

$-1 < x < 1$, and $1 < x < 4$. Thus, $x^3 - 15x + 1 = 0$ has three solutions in $[-4, 4]$. Since the polynomial of degree 3 can have at most 3 solutions, these are the solutions.
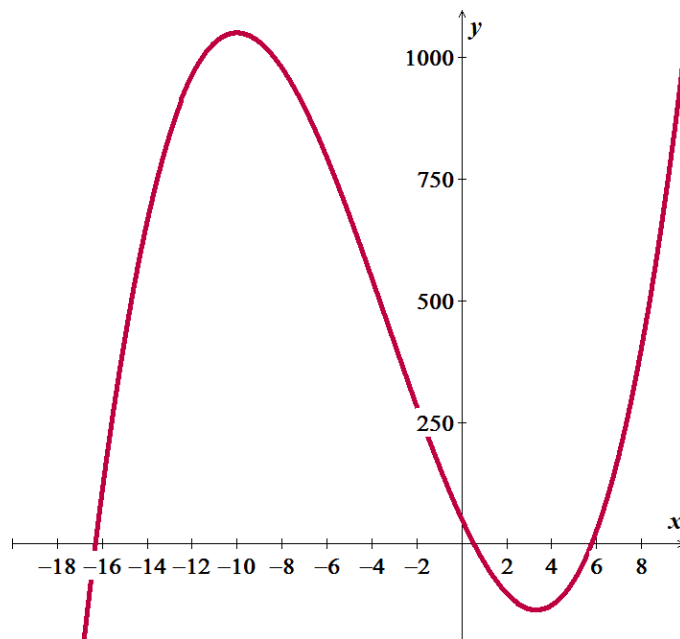
Exercise

Show that the equation has three solutions in the given interval $x^3 + 10x^2 - 100x + 50 = 0;$ $(-20, 10)$

Solution

x	y
-19	-1299
-18	-742
-17	-273
-16	114
-15	425
-14	666
-13	962
-12	1029
-10	1050
-9	1031
-8	978
-7	897
-6	794
-5	675
-4	546

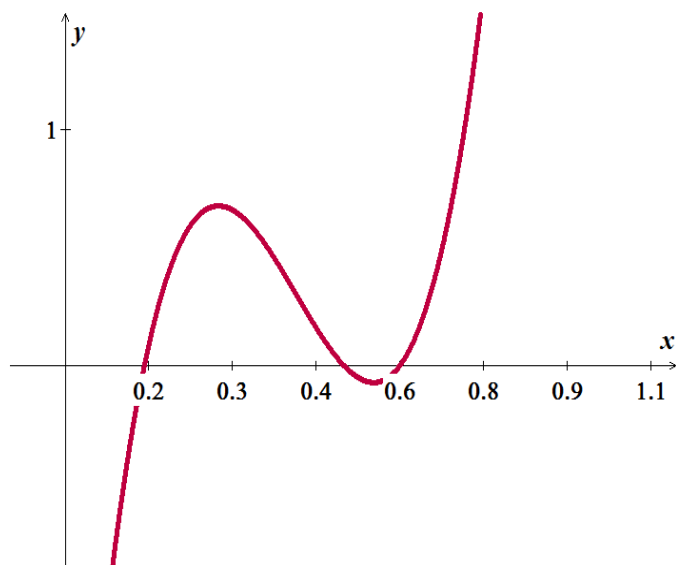|	
−3	413
−2	282
−1	159
0	50
1	−39
2	−102
3	−133
4	−126
5	−75
6	26

By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-17 < x < -16$, $0 < x < 1$, and $5 < x < 6$.

Exercise

Show that the equation has three solutions in the given interval $70x^3 - 87x^2 + 32x - 3 = 0$; $(0, 1)$

Solution

| x	y
.05	−1.6
.1	−0.6
.15	0.08
.2	.48
.25	.656
.3	.66
.35	.543
.4	.36
.45	.161
.5	0
.55	−.07
.6	0
.65	.266
.7	.78
.75	1.6
.8	2.76
.85	4.33
.9	6.36
.95	8.9

By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $0.1 < x < 0.15$, $0.5 < x < 0.55$, and $0.55 < x < 0.6$.

Exercise

Show that the equation has three solutions in the given interval $x^3 - 3x - 1 = 0$; $[-2, 2]$

Solution

| x	y
−2	−3.0
−1.75	−1.109
−1.5	0.125
−1.25	0.797
−1.0	1
−0.75	0.828
−0.5	0.375
−0.25	−0.266
0	−1.0
0.25	−1.73
0.5	−2.375
0.75	−2.828
1.0	−3.0
1.25	−2.797
1.5	−2.12
1.75	−0.89
2.	1.0
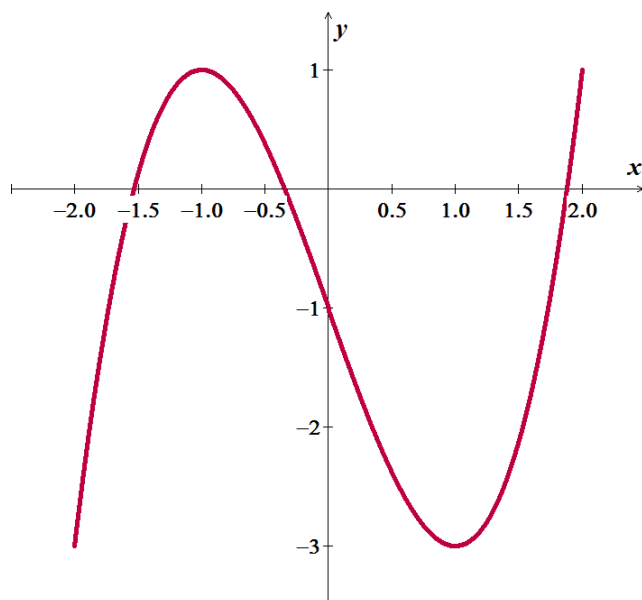
By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-1.75 < x < -1.5$, $-0.5 < x < -0.25$, and $1.75 < x < 2$.

Exercise

Show that the equation has six solutions in the given interval $x^6 - 8x^4 + 10x^2 - 1 = 0;\quad [-3,\ 3]$

Solution

x	y
−3.0	170.0
−2.5	−6.86
−2.0	−25.0
−1.5	−7.61
−1.0	2.0
−0.5	1.02
0.0	−1.0
0.5	1.01
1.0	2.0
1.5	−7.6
2.0	−25.0
2.5	−6.86
3.0	170.0
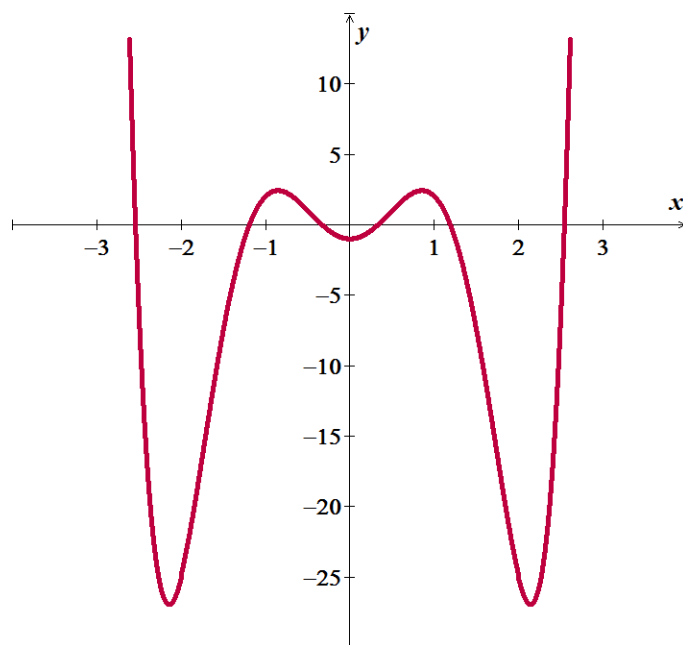
By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-3.0 < x < -2.5$, $-1.5 < x < -1.0$, $-0.5 \le x \le 0$, $-0.0 \le x \le 0.5$, $1.0 \le x \le 1.5$ and $2.5 < x < 3.0$.

Exercise

If functions $f(x)$ and $g(x)$ are continuous for $0 \le x \le 1$, could $\dfrac{f(x)}{g(x)}$ possibly be discontinuous at a point of [0, 1]? Give reason for your answer.

Solution

Yes, if we can get a value of $g(x)$ is between [0, 1], $x = \dfrac{1}{2} \Rightarrow g(x) = 2x - 1$ and $f(x) = x$.

Then $\dfrac{f(x)}{g(x)} = \dfrac{x}{2x-1} \Rightarrow \dfrac{f(x)}{g(x)}$ is discontinuous at $x = \dfrac{1}{2}$

Exercise

Suppose that a function f is continuous on the closed interval [0, 1] and that $0 \le f(x) \le 1$ for every x in [0, 1]. Show that there must exist a number c in [0, 1] such that $f(c) = c$ (c is called a **fixed point** of f).

Solution

Let $f(x) = x \Rightarrow f(0) = 0 \quad or \quad f(1) = 1$. In these cases, $c = 0$ or $c = 1$.

Let $f(0) = a > 0 \quad and \quad f(1) = b < 1$ because $0 \le f(x) \le 1$.

Define $g(x) = f(x) - x \Rightarrow g$ is continuous on [0, 1].

$$\Rightarrow \begin{cases} g(0) = f(0) - 0 = a > 0 \\ g(1) = f(1) - 1 = b - 1 < 0 \end{cases}$$

By the Intermediate Value Theorem there is a number c in [0, 1] such that

$$g(c) = 0 \Rightarrow f(c) - c = 0 \Rightarrow f(c) = c$$

Exercise

Use the Intermediate Value Theorem to show that the equation $x^5 + 7x + 5 = 0$ has a solution in the interval $(-1, \ 0)$.

Solution

$f(-1) = -1 - 7 + 5 = -3 < 0$

$f(0) = 5 > 0$

By Intermediate value theorem, the function has a solution in $(-1, \ 0)$

Exercise

The amount of an antibiotic (in *mg*) in the blood t hours after an intravenous line is opened is given by

$$m(t) = 100\left(e^{-0.1t} - e^{-0.3t}\right)$$

a) Use the Intermediate Value Theorem to show that the amount of drug is 30 *mg* at some time in the interval $[0,\ 5]$ and again at some time in the interval $[5,\ 15]$

b) Estimate the times at which $m = 30$ *mg*

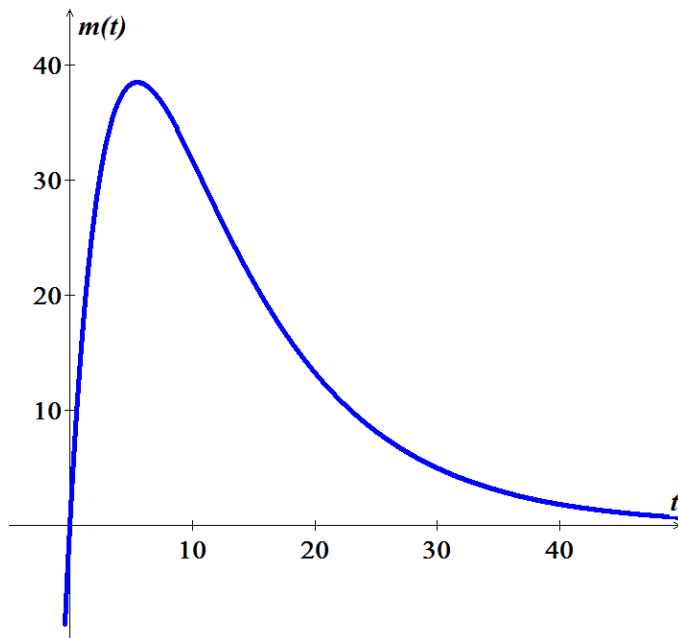c) Is the amount of drug in the blood ever 50 *mg*?

Solution

a) $m(0) = 100(1-1) = 0$

$m(5) \approx 38.34 > 30$

$m(15) \approx 21.2 < 30$

30 is an intermediate value between for both $[0,\ 5]$ and $[5,\ 15]$.

b) $m(t) = 100\left(e^{-0.1t} - e^{-0.3t}\right) = 30$

$e^{-0.1t} - e^{-0.3t} = 0.3 \quad \xrightarrow{\ software\ } \quad \begin{cases} t_1 \approx 2.4 \\ t_2 \approx 10.8 \end{cases}$

c) No, peak is 38.5 (using the graph)

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Exercise

Determine whether the following functions are continuous at a. $f(x) = \dfrac{1}{x-5}$; $a = 5$

Solution

    $f(5)$ $\not\exists$

The function is continuous everywhere except @ $x = 5$

Exercise

Determine whether the following functions are continuous at a. $h(x) = \sqrt{x^2 - 9}$; $a = 3$

Solution

    $\lim\limits_{x \to 3^-} h(x)$ $\not\exists$ $\therefore$ h is discontinuous @ 3

Exercise

Determine whether the following functions are continuous at a. $g(x) = \begin{cases} \dfrac{x^2 - 16}{x - 4} & if \quad x \neq 4 \\ 9 & if \quad x = 4 \end{cases}$; $a = 4$

Solution

    $\lim\limits_{x \to 4} g(x) = \lim\limits_{x \to 4} \dfrac{(x-4)(x+4)}{x-4} = \lim\limits_{x \to 4} (x+4) = 8 \neq 9 = g(4)$

    $\therefore$ g is discontinuous @ 4

Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $f(x) = \sqrt{x^2 - 5}$

Solution

    $\sqrt{x^2 - 5} \geq 0$ $\Rightarrow$ $x \leq -5$ & $x \geq 5$

The function is continuous at -5 to the left and right of $x = 5$

Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $f(x) = e^{\sqrt{x-2}}$

Solution

    The function is continuous at and to the right of $x = 2$

Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $\quad f(x) = \dfrac{2x}{x^3 - 25x}$

Solution

The function is continuous everywhere except at $x = 0,\ \pm 5$

The function is continuous to the left of -5, then to the right of -5 to the left of 0, then to the right of 0 thru the left of 5 then to the tight of 5.

Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $\quad f(x) = \cos e^x$

Solution

The function is continuous everywhere.

Exercise

Let $g(x) = \begin{cases} 5x - 2 & if \quad x < 1 \\ a & if \quad x = 1 \\ ax^2 + bx & if \quad x > 1 \end{cases}$

Determine values of the constants a and b for which $g(x)$ is continuous at $x = 1$

Solution

$$\lim_{x \to 1^-} g(x) = g(1)$$
$$= 5 - 2$$
$$\underline{= 3 = a}|$$

$$\lim_{x \to 1^-} g(x) = g(1)$$
$$= a + b$$
$$= 3 + b = 3$$

$\rightarrow \quad \underline{b = 0}|$