

Solution **Section 4.5 – Partial Fraction Decomposition**

Exercise

Write the partial fraction decomposition of each rational expression $\frac{4}{x(x-1)}$

Solution

$$\frac{4}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$4 = A(x-1) + Bx$$

$$4 = Ax - A + Bx$$

$$4 = (A+B)x - A$$

$$\begin{cases} A+B=0 \\ -A=4 \end{cases} \rightarrow \begin{cases} B=-A=4 \\ A=-4 \end{cases}$$

$$\boxed{\frac{4}{x(x-1)} = -\frac{4}{x} + \frac{4}{x-1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{3x}{(x+2)(x-1)}$

Solution

$$\frac{3x}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$3x = A(x-1) + B(x+2)$$

$$3x = Ax - A + Bx + 2B$$

$$3x = (A+B)x - A + 2B$$

$$\begin{cases} A+B=3 \\ -A+2B=0 \end{cases}$$

$$3B=3$$

$$\boxed{B=1} \rightarrow \boxed{A=2}$$

$$\boxed{\frac{3x}{(x+2)(x-1)} = \frac{2}{x+2} + \frac{1}{x-1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{1}{x(x^2+1)}$

Solution

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$1 = A(x^2+1) + x(Bx+C)$$

$$1 = Ax^2 + A + Bx^2 + Cx$$

$$1 = (A+B)x^2 + Cx + A$$

$$\begin{cases} A+B=0 & \rightarrow \underline{B=-1} \\ \underline{C=0} \\ \underline{A=1} \end{cases}$$

$$\underline{\frac{1}{x(x^2+1)} = \frac{1}{x} - \frac{x}{x^2+1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{1}{(x+1)(x^2+4)}$

Solution

$$\frac{1}{(x+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4}$$

$$1 = A(x^2+4) + (x+1)(Bx+C)$$

$$1 = Ax^2 + 4A + Bx^2 + Cx + Bx + C$$

$$1 = (A+B)x^2 + (B+C)x + 4A + C$$

$$\begin{cases} A+B=0 \\ B+C=0 \\ 4A+C=1 \end{cases}$$

$$\rightarrow \begin{cases} A=-B & \rightarrow \underline{A=\frac{1}{5}} \\ C=-B & \rightarrow \underline{C=\frac{1}{5}} \\ -4B-B=1 & \rightarrow \underline{B=-\frac{1}{5}} \end{cases}$$

$$\frac{1}{(x+1)(x^2+4)} = \frac{\frac{1}{5}}{x+1} + \frac{\frac{1}{5}x + \frac{1}{5}}{x^2+4}$$

$$= \frac{1}{5} \frac{1}{x+1} + \frac{1}{5} \frac{x+1}{x^2+4} \quad |$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^2}{(x-1)^2(x+1)^2}$

Solution

$$\frac{x^2}{(x-1)^2(x+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

$$\begin{aligned} x^2 &= A(x-1)(x+1)^2 + B(x+1)^2 + C(x-1)^2(x+1) + D(x-1)^2 \\ &= A(x-1)(x^2+2x+1) + B(x^2+2x+1) + C(x^2-2x+1)(x+1) + D(x^2-2x+1) \\ &= Ax^3 + 2Ax^2 + Ax - Ax^2 - 2Ax - A + Bx^2 + 2Bx + B \\ &\quad + Cx^3 - 2Cx^2 + Cx + Cx^2 - 2Cx + C + Dx^2 - 2Dx + D \end{aligned}$$

$$x^2 = (A+C)x^3 + (A+B-C+D)x^2 + (-A+2B-C-2D)x - A+B+C+D$$

$$\begin{cases} A+C=0 & \rightarrow \underline{A=-C} \\ A+B-C+D=1 \\ -A+2B-C-2D=0 \\ -A+B+C+D=0 \end{cases}$$

$$\begin{cases} B-2C+D=1 \\ B-D=0 \\ B+2C+D=0 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 8$$

$$\Delta_B = \begin{vmatrix} 1 & -2 & 1 \\ 0 & 0 & -1 \\ 0 & 2 & 1 \end{vmatrix} = 2$$

$$\Delta_C = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix} = -2$$

$$\Delta_D = \begin{vmatrix} 1 & -2 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{vmatrix} = 2$$

$$\underline{B = \frac{1}{4} \quad C = -\frac{1}{4} \quad D = \frac{1}{4} \quad A = \frac{1}{4} \quad |}$$

$$\frac{x^2}{(x-1)^2(x+1)^2} = \frac{1}{4} \frac{1}{x-1} + \frac{1}{4} \frac{1}{(x-1)^2} - \frac{1}{4} \frac{1}{x+1} + \frac{1}{4} \frac{1}{(x+1)^2}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x+1}{x^2(x-2)^2}$

Solution

$$\frac{x+1}{x^2(x-2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2}$$

$$\begin{aligned} x+1 &= Ax(x-2)^2 + B(x-2)^2 + Cx^2(x-2) + Dx^2 \\ &= Ax(x^2 - 4x + 4) + B(x^2 - 4x + 4) + Cx^3 - 2Cx^2 + Dx^2 \\ &= Ax^3 - 4Ax^2 + 4Ax + Bx^2 - 4Bx + 4B + Cx^3 - 2Cx^2 + Dx^2 \\ &= (A+C)x^3 + (-4A-B-2C+D)x^2 + (4A-4B)x + 4B \end{aligned}$$

$$\begin{cases} A+C=0 \\ -4A-B-2C+D=0 \\ 4A-4B=1 \\ 4B=1 \end{cases}$$

$$\begin{cases} C = -\frac{1}{2} \\ D = 2 + \frac{1}{4} - 1 = \frac{5}{4} \\ A = \frac{1}{2} \\ B = \frac{1}{4} \end{cases}$$

$$\frac{x+1}{x^2(x-2)^2} = \frac{1}{2} \frac{1}{x} + \frac{1}{4} \frac{1}{x^2} - \frac{1}{2} \frac{1}{x-2} + \frac{5}{4} \frac{1}{(x-2)^2}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x-3}{(x+2)(x+1)^2}$

Solution

$$\frac{x-3}{(x+2)(x+1)^2} = \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$\begin{aligned}
 x-3 &= A(x+1)^2 + B(x+1)(x+2) + C(x+2) \\
 &= Ax^2 + 2Ax + A + B(x^2 + 3x + 2) + Cx + 2C \\
 &= (A+B)x^2 + (2A+3B+C)x + A+2B+2C
 \end{aligned}$$

$$\begin{cases} A+B=0 \\ 2A+3B+C=1 \\ A+2B+2C=-3 \end{cases}$$

$$\begin{cases} A=-B \\ -2B+3B+C=1 \\ -B+2B+2C=-3 \end{cases}$$

$$\begin{cases} B+C=1 \\ B+2C=-3 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \quad \Delta_B = \begin{vmatrix} 1 & 1 \\ -3 & 2 \end{vmatrix} = 5 \quad \Delta_C = \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = -4$$

$$\underline{C = -4, \quad B = 5, \quad A = -5}$$

$$\underline{\frac{x-3}{(x+2)(x+1)^2} = -\frac{5}{x+2} + \frac{5}{x+1} - \frac{4}{(x+1)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^2+x}{(x+2)(x-1)^2}$

Solution

$$\frac{x^2+x}{(x+2)(x-1)^2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\begin{aligned}
 x^2+x &= A(x-1)^2 + B(x-1)(x+2) + C(x+2) \\
 &= Ax^2 - 2Ax + A + Bx^2 + Bx - 2B + Cx + 2C
 \end{aligned}$$

$$\begin{matrix} x^2 \\ x \\ x^0 \end{matrix} \begin{cases} A+B=1 \\ -2A+B+C=1 \\ A-2B+2C=0 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ -2 & 1 & 1 \\ 1 & -2 & 2 \end{vmatrix} = 9$$

$$\Delta_A = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix} = 2$$

$$\Delta_B = \begin{vmatrix} 1 & 1 & 0 \\ -2 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 7 \quad \Delta_C = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 1 \\ 1 & -2 & 0 \end{vmatrix} = 6$$

$$\left| \begin{array}{l} A = \frac{2}{9} \quad B = \frac{7}{9} \quad C = \frac{2}{3} \end{array} \right|$$

$$\left| \frac{x^2 + x}{(x+2)(x-1)^2} = \frac{\frac{2}{9}}{x+2} + \frac{\frac{7}{9}}{x-1} + \frac{\frac{2}{3}}{(x-1)^2} \right|$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{10x^2 + 2x}{(x-1)^2(x^2 + 2)}$

Solution

$$\frac{10x^2 + 2x}{(x-1)^2(x^2 + 2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx + D}{x^2 + 2}$$

$$\begin{aligned} 10x^2 + 2x &= A(x-1)(x^2 + 2) + B(x^2 + 2) + (Cx + D)(x-1)^2 \\ &= Ax^3 + 2Ax - Ax^2 - 2A + Bx^2 + 2B + (Cx + D)(x^2 - 2x + 1) \\ &= Ax^3 + 2Ax - Ax^2 - 2A + Bx^2 + 2B + Cx^3 - 2Cx^2 + Cx + Dx^2 - 2Dx + D \\ &= (A + C)x^3 + (B - 2A - 2C + D)x^2 + (2A + C - 2D)x - 2A + 2B + D \end{aligned}$$

$$\begin{cases} A + C = 0 & \rightarrow A = -C \\ B - 2A - 2C + D = 10 \\ 2A + C - 2D = 2 \\ -2A + 2B + D = 0 \end{cases}$$

$$\begin{cases} B + D = 10 \\ -C - 2D = 2 \\ 2B + 2C + D = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 2 & 2 & 1 \end{vmatrix} = 5$$

$$\Delta_B = \begin{vmatrix} 10 & 0 & 1 \\ 2 & -1 & -2 \\ 0 & 2 & 1 \end{vmatrix} = 34$$

$$\Delta_C = \begin{vmatrix} 1 & 10 & 1 \\ 0 & 2 & -2 \\ 2 & 0 & 1 \end{vmatrix} = -42$$

$$\Delta_D = \begin{vmatrix} 1 & 0 & 10 \\ 0 & -1 & 2 \\ 2 & 2 & 0 \end{vmatrix} = 16$$

$$\left| \begin{array}{l} B = \frac{34}{5} \quad C = -\frac{42}{5} \quad D = \frac{16}{5} \quad A = \frac{42}{5} \end{array} \right|$$

$$\left| \frac{10x^2 + 2x}{(x-1)^2(x^2+2)} = \frac{42}{5(x-1)} + \frac{34}{5(x-1)^2} + \frac{-42x+16}{5(x^2+2)} \right|$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^2 + 2x + 3}{(x+1)(x^2 + 2x + 4)}$

Solution

$$\frac{x^2 + 2x + 3}{(x+1)(x^2 + 2x + 4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2 + 2x + 4}$$

$$x^2 + 2x + 3 = A(x^2 + 2x + 4) + (Bx + C)(x+1)$$

$$= Ax^2 + 2Ax + 4A + Bx^2 + Bx + Cx + C$$

$$= (A+B)x^2 + (2A+B+C)x + 4A+C$$

$$\begin{cases} A+B=1 \\ 2A+B+C=2 \\ 4A+C=3 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 4 & 0 & 1 \end{vmatrix} = 3$$

$$\Delta_A = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix} = 2$$

$$\Delta_B = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = 1$$

$$\Delta_C = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 4 & 0 & 3 \end{vmatrix} = 1$$

$$\left| \begin{array}{l} A = \frac{2}{3} \quad B = \frac{1}{3} \quad C = \frac{1}{3} \end{array} \right|$$

$$\left| \frac{x^2 + 2x + 3}{(x+1)(x^2 + 2x + 4)} = \frac{\frac{2}{3}}{x+1} + \frac{\frac{1}{3}x + \frac{1}{3}}{x^2 + 2x + 4} \right|$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^2 - 11x - 18}{x(x^2 + 3x + 3)}$

Solution

$$\frac{x^2 - 11x - 18}{x(x^2 + 3x + 3)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 3x + 3}$$

$$x^2 - 11x - 18 = Ax^2 + 3Ax + 3A + Bx^2 + Cx$$

$$= (A + B)x^2 + (3A + C)x + 3A$$

$$\begin{cases} A + B = 1 & \rightarrow \underline{B = 7} \\ 3A + C = -11 & \rightarrow \underline{C = 7} \\ 3A = -18 & \rightarrow \underline{A = -6} \end{cases}$$

$$\frac{x^2 - 11x - 18}{x(x^2 + 3x + 3)} = -\frac{6}{x} + \frac{7x + 7}{x^2 + 3x + 3}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{1}{(2x + 3)(4x - 1)}$

Solution

$$\frac{1}{(2x + 3)(4x - 1)} = \frac{A}{2x + 3} + \frac{B}{4x - 1}$$

$$1 = 4Ax - A + 2Bx + 3B$$

$$1 = (4A + 2B)x - A + 3B$$

$$\begin{cases} 4A + 2B = 0 \\ -A + 3B = 1 \end{cases}$$

$$\Delta = \begin{vmatrix} 4 & 2 \\ -1 & 3 \end{vmatrix} = 14 \quad \Delta_A = \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = -2 \quad \Delta_B = \begin{vmatrix} 4 & 0 \\ -1 & 1 \end{vmatrix} = 4$$

$$\underline{A = -\frac{1}{7}} \quad \underline{B = \frac{2}{7}}$$

$$\frac{1}{(2x + 3)(4x - 1)} = -\frac{\frac{1}{7}}{2x + 3} + \frac{\frac{2}{7}}{4x - 1}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^2 + 2x + 3}{(x^2 + 4)^2}$

Solution

$$\frac{x^2 + 2x + 3}{(x^2 + 4)^2} = \frac{Ax + B}{x^2 + 4} + \frac{Cx + D}{(x^2 + 4)^2}$$

$$\begin{aligned}x^2 + 2x + 3 &= (Ax + B)(x^2 + 4) + Cx + D \\&= Ax^3 + 4Ax + Bx^2 + 4B + Cx + D \\&= Ax^3 + Bx^2 + (4A + C)x + 4B + D\end{aligned}$$

$$\left\{ \begin{array}{l} \underline{A=0} \\ \underline{B=1} \\ 4A + C = 2 \\ 4B + D = 3 \end{array} \right. \rightarrow \left\{ \begin{array}{l} \underline{C=2} \\ \underline{D=3-4B=-1} \end{array} \right.$$

$$\frac{x^2 + 2x + 3}{(x^2 + 4)^2} = \frac{1}{x^2 + 4} + \frac{2x - 1}{(x^2 + 4)^2}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^3 + 1}{(x^2 + 16)^2}$

Solution

$$\frac{x^3 + 1}{(x^2 + 16)^2} = \frac{Ax + B}{x^2 + 16} + \frac{Cx + D}{(x^2 + 16)^2}$$

$$\begin{aligned}x^3 + 1 &= (Ax + B)(x^2 + 16) + Cx + D \\&= Ax^3 + 16Ax + Bx^2 + 16B + Cx + D\end{aligned}$$

$$\left\{ \begin{array}{l} \overset{x^3}{A=1} \\ \overset{x^2}{B=0} \\ \overset{x}{16A + C = 0} \\ \overset{x^0}{16B + D = 1} \end{array} \right. \rightarrow \left\{ \begin{array}{l} \underline{C=-16} \\ \underline{D=1} \end{array} \right.$$

$$\frac{x^3+1}{(x^2+16)^2} = \frac{x}{x^2+16} + \frac{-16x+1}{(x^2+16)^2}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{7x+3}{x^3-2x^2-3x}$

Solution

$$\begin{aligned}\frac{7x+3}{x^3-2x^2-3x} &= \frac{7x+3}{x(x+1)(x-3)} \\ &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-3}\end{aligned}$$

$$\begin{aligned}7x+3 &= A(x+1)(x-3) + Bx(x-3) + Cx(x+1) \\ &= Ax^2 - 2Ax - 3A + Bx^2 - 3B + Cx^2 + Cx \\ &= (A+B+C)x^2 + (C-2A)x - 3A - 3B\end{aligned}$$

$$\begin{cases} A+B+C=0 \\ C-2A=7 \\ -3A-3B=3 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 0 & 1 \\ -3 & -3 & 0 \end{vmatrix} = 6 \quad \Delta_A = \begin{vmatrix} 0 & 1 & 1 \\ 7 & 0 & 1 \\ 3 & -3 & 0 \end{vmatrix} = -18$$

$$\Delta_B = \begin{vmatrix} 1 & 0 & 1 \\ -2 & 7 & 1 \\ -3 & 3 & 0 \end{vmatrix} = 12 \quad \Delta_C = \begin{vmatrix} 1 & 1 & 0 \\ -2 & 0 & 7 \\ -3 & -3 & 3 \end{vmatrix} = 6$$

$$A = -3 \quad B = 2 \quad C = 1$$

$$\frac{7x+3}{x^3-2x^2-3x} = \frac{-3}{x} + \frac{2}{x+1} + \frac{1}{x-3}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^2}{x^3-4x^2+5x-2}$

Solution

$$\frac{x^2}{x^3-4x^2+5x-2} = \frac{x^2}{(x-2)(x-1)^2}$$

$$= \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\begin{aligned} x^2 &= A(x-1)^2 + B(x-2)(x-1) + C(x-2) \\ &= Ax^2 - 2Ax + A + Bx^2 - 3Bx + 2B + Cx - 2C \\ &= (A+B)x^2 + (-2A-3B+C)x + A+2B-2C \end{aligned}$$

$$\begin{cases} A+B=1 \\ -2A-3B+C=0 \\ A+2B-2C=0 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 0 \\ -2 & -3 & 1 \\ 1 & 2 & -2 \end{vmatrix} = 1$$

$$D_A = \begin{vmatrix} 1 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 2 & -2 \end{vmatrix} = 4$$

$$D_B = \begin{vmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & 0 & -2 \end{vmatrix} = -3$$

$$D_C = \begin{vmatrix} 1 & 1 & 1 \\ -2 & -3 & 0 \\ 1 & 2 & 0 \end{vmatrix} = -1$$

$$\underline{A=4 \quad B=-3 \quad C=-1}$$

$$\underline{\frac{x^2}{x^3-4x^2+5x-2} = \frac{4}{x-2} - \frac{3}{x-1} - \frac{1}{(x-1)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^3}{(x^2+16)^3}$

Solution

$$\frac{x^3}{(x^2+16)^3} = \frac{Ax+B}{x^2+16} + \frac{Cx+D}{(x^2+16)^2} + \frac{Ex+F}{(x^2+16)^3}$$

$$x^3 = (Ax+B)(x^2+16)^2 + (Cx+D)(x^2+16) + Ex + F$$

$$= (Ax+B)(x^4+32x^2+256) + Cx^3+16Cx+Dx^2+16D+Ex+F$$

$$= Ax^5+32Ax^3+256Ax+Bx^4+32Bx^2+256B+Cx^3+Dx^2+(16C+E)x+16D+F$$

$$= Ax^5+Bx^4+(32A+C)x^3+(32B+D)x^2+(256A+16C+E)x+256B+16D+F$$

$$\begin{array}{ll}
 x^5 & \underline{A=0} \\
 x^4 & \underline{B=0} \\
 x^3 & 32A+C=1 \rightarrow \underline{C=1} \\
 x^2 & 32B+D=0 \rightarrow \underline{D=0} \\
 x^1 & 256A+16C+E=0 \rightarrow \underline{E=-16} \\
 x^0 & 256B+16D+F=0 \rightarrow \underline{F=0}
 \end{array}$$

$$\frac{x^3}{(x^2+16)^3} = \frac{x}{(x^2+16)^2} + \frac{-16x}{(x^2+16)^3}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{4}{2x^2-5x-3}$

Solution

$$\begin{aligned}
 \frac{4}{2x^2-5x-3} &= \frac{4}{(2x+1)(x-3)} \\
 &= \frac{A}{2x+1} + \frac{B}{x-3}
 \end{aligned}$$

$$\begin{aligned}
 4 &= Ax - 3A + 2Bx + B \\
 &= (A+2B)x - 3A + B
 \end{aligned}$$

$$\begin{cases} A+2B=0 \\ -3A+B=4 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix} = 7 \quad \Delta_A = \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = -8 \quad \Delta_B = \begin{vmatrix} 1 & 0 \\ -3 & 4 \end{vmatrix} = 4$$

$$\underline{A = -\frac{8}{7} \quad B = \frac{4}{7}}$$

$$\frac{4}{2x^2-5x-3} = \frac{-\frac{8}{7}}{2x+1} + \frac{\frac{4}{7}}{x-3}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{2x+3}{x^4-9x^2}$

Solution

$$\frac{2x+3}{x^4-9x^2} = \frac{2x+3}{x^2(x-3)(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3} + \frac{D}{x+3}$$

$$\begin{aligned} 2x+3 &= Ax(x^2-9) + B(x^2-9) + Cx^2(x+3) + Dx^2(x-3) \\ &= Ax^3 - 9Ax + Bx^2 - 9B + Cx^3 + 3Cx^2 + Dx^3 - 3Dx^2 \\ &= (A+C+D)x^3 + (B+3C-3D)x^2 - 9Ax - 9B \end{aligned}$$

$$\left\{ \begin{array}{l} A+C+D=0 \\ B+3C-3D=0 \\ -9A=2 \rightarrow A=-\frac{2}{9} \\ -9B=3 \rightarrow B=-\frac{1}{3} \end{array} \right. \quad \begin{array}{l} C+D=\frac{2}{9} \\ 3C-3D=\frac{1}{3} \end{array}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 3 & -3 \end{vmatrix} = -6 \quad \Delta_C = \begin{vmatrix} \frac{2}{9} & 1 \\ \frac{1}{3} & -3 \end{vmatrix} = -1 \quad \Delta_D = \begin{vmatrix} 1 & \frac{2}{9} \\ 3 & \frac{1}{3} \end{vmatrix} = -\frac{1}{3}$$

$$\underline{C = \frac{1}{6} \quad D = \frac{1}{18}}$$

$$\underline{\frac{2x+3}{x^4-9x^2} = -\frac{2}{9} \frac{1}{x} - \frac{1}{3} \frac{1}{x^2} + \frac{1}{6} \frac{1}{x-3} + \frac{1}{18} \frac{1}{x+3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^2+9}{x^4-2x^2-8}$

Solution

$$\frac{x^2+9}{x^4-2x^2-8} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+2}$$

$$\begin{aligned} x^2+9 &= A(x+2)(x^2+2) + B(x-2)(x^2+2) + (Cx+D)(x^2-4) \\ &= Ax^3 + 2Ax + 2Ax^2 + 4A + Bx^3 + 2Bx - 2Bx^2 - 4B + Cx^3 - 4Cx + Dx^2 - 4D \\ &= (A+B+C)x^3 + (2A-2B+D)x^2 + (2A+2B-4C)x + 4A-4B-4D \end{aligned}$$

$$\left\{ \begin{array}{l} A+B+C=0 \\ 2A-2B+D=1 \\ 2A+2B-4C=0 \\ 4A-4B-4D=9 \end{array} \right.$$

$$\boxed{A = \frac{13}{24} \quad B = -\frac{13}{24} \quad C = 0 \quad D = -\frac{7}{6}}$$

$$\boxed{\frac{x^2+9}{x^4-2x^2-8} = \frac{\frac{13}{24}}{x-2} - \frac{\frac{13}{24}}{x+2} - \frac{\frac{7}{6}}{x^2+2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{y}{y^2-2y-3}$

Solution

$$\frac{y}{y^2-2y-3} = \frac{A}{y-3} + \frac{B}{y+1}$$

$$y = (A+B)y + A - 3B$$

$$\begin{cases} A+B=1 \\ A-3B=0 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = -4 \quad \Delta_A = \begin{vmatrix} 1 & 1 \\ 0 & -3 \end{vmatrix} = -3 \quad \Delta_B = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$\boxed{A = \frac{3}{4} \quad B = \frac{1}{4}}$$

$$\boxed{\frac{y}{y^2-2y-3} = \frac{\frac{3}{4}}{y-3} + \frac{\frac{1}{4}}{y+1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x+3}{2x^3-8x}$

Solution

$$\frac{x+3}{2x^3-8x} = \frac{1}{2} \frac{x+3}{x(x^2-4)}$$

$$= \frac{1}{2} \left(\frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2} \right)$$

$$= \frac{1}{2} \frac{A(x+2)(x-2) + Bx(x-2) + Cx(x+2)}{x(x+2)(x-2)}$$

$$(A+B+C)x^2 + (2C-2B)x - 4A = x+3$$

$$\left\{ \begin{array}{l} A+B+C=0 \\ 2C-2B=1 \\ -4A=3 \end{array} \right. \rightarrow \left\{ \begin{array}{l} B+C=\frac{3}{4} \\ -B+C=\frac{1}{2} \\ A=-\frac{3}{4} \end{array} \right.$$

$$\left| \begin{array}{l} B=\frac{1}{8} \quad C=\frac{5}{8} \end{array} \right|$$

$$\left| \frac{x+3}{2x^3-8x} = \frac{1}{2} \left(-\frac{\frac{3}{4}}{x} + \frac{\frac{1}{8}}{x+2} + \frac{\frac{5}{8}}{x-2} \right) \right|$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^2}{(x-1)(x^2+2x+1)}$

Solution

$$\frac{x^2}{(x-1)(x^2+2x+1)} = \frac{x^2}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

$$= (A+B)x^2 + (2A+C)x + A-B-C$$

$$\left\{ \begin{array}{l} A+B=1 \\ 2A+C=0 \\ A-B-C=0 \end{array} \right.$$

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & -1 & -1 \end{vmatrix} = 4$$

$$\Delta_A = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{vmatrix} = 1$$

$$\Delta_B = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & -1 \end{vmatrix} = 3$$

$$\Delta_C = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 1 & -1 & 0 \end{vmatrix} = -2$$

$$\left| \begin{array}{l} A=\frac{1}{4} \quad B=\frac{3}{4} \quad C=-\frac{1}{2} \end{array} \right|$$

$$\left| \frac{x^2}{(x-1)(x^2+2x+1)} = \frac{\frac{1}{4}}{x-1} + \frac{\frac{3}{4}}{x+1} - \frac{\frac{1}{2}}{(x+1)^2} \right|$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{3x^2 + x + 4}{x^3 + x}$

Solution

$$\begin{aligned}\frac{3x^2 + x + 4}{x^3 + x} &= \frac{A}{x} + \frac{Bx + C}{x^2 + 1} \\ &= \frac{(A + B)x^2 + Cx + A}{x(x^2 + 1)}\end{aligned}$$

$$3x^2 + x + 4 = (A + B)x^2 + Cx + A$$

$$\begin{cases} A + B = 3 & \rightarrow \underline{B = -1} \\ \underline{C = 1} \\ \underline{A = 4} \end{cases}$$

$$\underline{\frac{3x^2 + x + 4}{x^3 + x} = \frac{4}{x} + \frac{-x + 1}{x^2 + 1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{8x^2 + 8x + 2}{(4x^2 + 1)^2}$

Solution

$$\begin{aligned}\frac{8x^2 + 8x + 2}{(4x^2 + 1)^2} &= \frac{Ax + B}{4x^2 + 1} + \frac{Cx + D}{(4x^2 + 1)^2} \\ &= \frac{(Ax + B)(4x^2 + 1) + Cx + D}{(4x^2 + 1)^2}\end{aligned}$$

$$\begin{aligned}8x^2 + 8x + 2 &= (Ax + B)(4x^2 + 1) + Cx + D \\ &= 4Ax^3 + 4Bx^2 + (A + C)x + B + D\end{aligned}$$

$$\begin{cases} \underline{A = 0} \\ 4B = 8 & \rightarrow \underline{B = 2} \\ A + C = 8 & \rightarrow \underline{C = 8} \\ B + D = 2 & \rightarrow \underline{D = 0} \end{cases}$$

$$\frac{8x^2 + 8x + 2}{(4x^2 + 1)^2} = \frac{2}{4x^2 + 1} + \frac{8x}{(4x^2 + 1)^2}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{1}{x^2 + 2x}$

Solution

$$\frac{1}{x^2 + 2x} = \frac{A}{x} + \frac{B}{x + 2}$$

$$1 = Ax + 2A + Bx$$

$$x \quad 2A = 1 \quad \rightarrow A = \frac{1}{2}$$

$$x^0 \quad A + B = 0 \quad \rightarrow B = -\frac{1}{2}$$

$$\frac{1}{x^2 + 2x} = \frac{1}{2} \frac{1}{x} - \frac{1}{2} \frac{1}{x + 2}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{2x + 1}{x^2 - 7x + 12}$

Solution

$$\frac{2x + 1}{x^2 - 7x + 12} = \frac{A}{x - 4} + \frac{B}{x - 3}$$

$$2x + 1 = Ax - 3A + Bx - 4B$$

$$x \quad A + B = 2$$

$$x^0 \quad -3A - 4B = 1$$

$$A = \frac{\begin{vmatrix} 2 & 1 \\ 1 & -4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -3 & -4 \end{vmatrix}} = \frac{-9}{-1} = 9 \quad B = \frac{\begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix}}{-1} = \frac{7}{-1} = -7$$

$$\frac{2x + 1}{x^2 - 7x + 12} = \frac{9}{x - 4} - \frac{7}{x - 3}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^2 + x}{x^4 - 3x^2 - 4}$

Solution

$$\begin{aligned}\frac{x^2 + x}{x^4 - 3x^2 - 4} &= \frac{x^2 + x}{(x^2 - 4)(x^2 + 1)} \\ &= \frac{A}{x - 2} + \frac{B}{x + 2} + \frac{Cx + D}{x^2 + 1}\end{aligned}$$

$$\begin{aligned}x^2 + x &= A(x + 2)(x^2 + 1) + B(x - 2)(x^2 + 1) + (Cx + D)(x^2 - 4) \\ &= Ax^3 + Ax + 2Ax^2 + 2A + Bx^3 + Bx - 2Bx^2 - 2B + Cx^3 - 4Cx + Dx^2 - 4D \\ &= (A + B + C)x^3 + (2A - 2B + D)x^2 + (A + B - 4C)x + 2A - 2B - 4D\end{aligned}$$

$$\begin{cases} x^3 & A + B + C = 0 & (1) \\ x^2 & 2A - 2B + D = 1 & (2) \\ x & A + B - 4C = 1 & (3) \\ x^0 & 2A - 2B - 4D = 0 & (4) \end{cases}$$

$$(1) - (3) \rightarrow 5C = -1 \quad \underline{C = -\frac{1}{5}}$$

$$(2) - (4) \rightarrow 5D = 1 \quad \underline{D = \frac{1}{5}}$$

$$\begin{cases} A + B = \frac{1}{5} \\ 2A - 2B = \frac{4}{5} \end{cases} \rightarrow \begin{cases} 2A + 2B = \frac{2}{5} \\ 2A - 2B = \frac{4}{5} \end{cases}$$

$$4A = \frac{6}{5} \rightarrow \underline{A = \frac{3}{10}}$$

$$B = \frac{1}{5} - \frac{3}{10} \rightarrow \underline{B = -\frac{1}{10}}$$

$$\underline{\frac{x^2 + x}{x^4 - 3x^2 - 4} = \frac{3}{10} \frac{1}{x - 2} - \frac{1}{10} \frac{1}{x + 2} + \frac{1}{5} \frac{-x + 1}{x^2 + 1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{(\theta^2 + 1)^3}$

Solution

$$\frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{(\theta^2 + 1)^3} = \frac{A\theta + B}{\theta^2 + 1} + \frac{C\theta + D}{(\theta^2 + 1)^2} + \frac{E\theta + F}{(\theta^2 + 1)^3}$$

$$\begin{aligned}\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1 &= (A\theta + B)(\theta^2 + 1)^2 + (C\theta + D)(\theta^2 + 1) + E\theta + F \\ &= (A\theta + B)(\theta^4 + 2\theta^2 + 1) + C\theta^3 + C\theta + D\theta^2 + D + E\theta + F \\ &= A\theta^5 + B\theta^4 + (2A + C)\theta^3 + (2B + D)\theta^2 + (A + C + E)\theta + B + D + F\end{aligned}$$

$$\left\{ \begin{array}{l} \boxed{A=0} \\ \boxed{B=1} \\ 2A + C = -4 \\ 2B + D = 2 \\ A + C + E = -3 \\ B + D + F = 1 \end{array} \right. \rightarrow \boxed{C=-4} \quad \boxed{D=0} \quad \boxed{E=1} \quad \boxed{F=0}$$

$$\frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{(\theta^2 + 1)^3} = \frac{1}{\theta^2 + 1} - 4\frac{\theta}{(\theta^2 + 1)^2} + \frac{\theta}{(\theta^2 + 1)^3}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{3x^2 + 7x - 2}{x^3 - x^2 - 2x}$

Solution

$$\frac{3x^2 + 7x - 2}{x^3 - x^2 - 2x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}$$

$$\begin{aligned}3x^2 + 7x - 2 &= A(x+1)(x-2) + Bx(x-2) + Cx(x+1) \\ &= Ax^2 - Ax - 2A \\ &\quad Bx^2 - 2Bx \\ &\quad Cx^2 + Cx\end{aligned}$$

$$\begin{cases} A + B + C = 3 \\ -A - 2B + C = 7 \\ -2A = -2 \end{cases} \rightarrow \underline{A = 1}$$

$$\begin{cases} B + C = 2 \\ -2B + C = 8 \end{cases} \rightarrow \underline{B = -2} \quad \underline{C = 4}$$

$$\underline{\frac{3x^2 + 7x - 2}{x^3 - x^2 - 2x} = \frac{1}{x} - \frac{2}{x+1} + \frac{4}{x-2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{3x^2 + 2x + 5}{(x-1)(x^2 - x - 20)}$

Solution

$$\frac{3x^2 + 2x + 5}{(x-1)(x^2 - x - 20)} = \frac{A}{x-1} + \frac{B}{x-5} + \frac{C}{x+4}$$

$$3x^2 + 2x + 5 = (A + B + C)x^2 + (-A + 3B - 6C)x - 20A - 4B + 5C$$

$$\begin{cases} x^2 & A + B + C = 3 \\ x & -A + 3B - 6C = 2 \\ x^0 & -20A - 4B + 5C = 5 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 3 & -6 \\ -20 & -4 & 5 \end{vmatrix} = 180$$

$$D_A = \begin{vmatrix} 3 & 1 & 1 \\ 2 & 3 & -6 \\ 5 & -4 & 5 \end{vmatrix} = -90$$

$$D_B = \begin{vmatrix} 1 & 3 & 1 \\ -1 & 2 & -6 \\ -20 & 5 & 5 \end{vmatrix} = 450$$

$$D_C = \begin{vmatrix} 1 & 1 & 3 \\ -1 & 3 & 2 \\ -20 & -4 & 5 \end{vmatrix} = 180$$

$$\underline{A = \frac{1}{2}, \quad B = \frac{5}{2}, \quad C = 1}$$

$$\underline{\frac{3x^2 + 2x + 5}{(x-1)(x^2 - x - 20)} = \frac{1}{2} \frac{1}{x-1} + \frac{5}{2} \frac{1}{x-5} + \frac{1}{x+4}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{5x^2 - 3x + 2}{x^3 - 2x^2}$

Solution

$$\frac{5x^2 - 3x + 2}{x^3 - 2x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$5x^2 - 3x + 2 = Ax^2 - 2Ax + Bx - 2B + Cx^2$$

$$\begin{cases} x^2 & A + C = 5 & C = 4 \\ x & -2A + B = -3 & A = 1 \\ x^0 & -2B = 2 & \rightarrow B = -1 \end{cases}$$

$$\frac{5x^2 - 3x + 2}{x^3 - 2x^2} = \frac{1}{x} - \frac{1}{x^2} + \frac{4}{x-2}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{7x^2 - 13x + 13}{(x-2)(x^2 - 2x + 3)}$

Solution

$$\frac{7x^2 - 13x + 13}{(x-2)(x^2 - 2x + 3)} = \frac{A}{x-2} + \frac{Bx + C}{x^2 - 2x + 3}$$

$$7x^2 - 13x + 13 = Ax^2 - 2Ax + 3A + Bx^2 - 2Bx + Cx - 2C$$

$$\begin{cases} x^2 & A + B = 7 \\ x^1 & -2A - 2B + C = -13 \\ x^0 & 3A - 2C = 13 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 0 \\ -2 & -2 & 1 \\ 3 & 0 & -2 \end{vmatrix} = 3$$

$$D_A = \begin{vmatrix} 7 & 1 & 0 \\ -13 & -2 & 1 \\ 13 & 0 & -2 \end{vmatrix} = 15$$

$$D_B = \begin{vmatrix} 1 & 7 & 0 \\ -2 & -13 & 1 \\ 3 & 13 & -2 \end{vmatrix} = 6$$

$$D_C = \begin{vmatrix} 1 & 1 & 7 \\ -2 & -2 & -13 \\ 3 & 0 & 13 \end{vmatrix} = 3$$

$$\underline{A = 5; B = 2; C = 1}$$

$$\frac{7x^2 - 13x + 13}{(x-2)(x^2 - 2x + 3)} = \frac{5}{x-2} + \frac{2x+1}{x^2 - 2x + 3}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{1}{x^2 - 5x + 6}$

Solution

$$\frac{1}{x^2 - 5x + 6} = \frac{A}{x-2} + \frac{B}{x-3}$$

$$Ax - 3A + Bx - 2B = 1$$

$$\rightarrow \begin{cases} A + B = 0 \\ -3A - 2B = 1 \end{cases} \rightarrow A = -1 \quad B = 1$$

$$\frac{1}{x^2 - 5x + 6} = \frac{-1}{x-2} + \frac{1}{x-3}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{1}{x^2 - 5x + 5}$

Solution

$$\frac{1}{x^2 - 5x + 5} = \frac{A}{x - \frac{5+\sqrt{5}}{2}} + \frac{B}{x - \frac{5-\sqrt{5}}{2}}$$

$$Ax - \left(\frac{5-\sqrt{5}}{2}\right)A + Bx - \left(\frac{5+\sqrt{5}}{2}\right)B = 1$$

$$\begin{cases} A + B = 0 \\ -\frac{5-\sqrt{5}}{2}A - \frac{5+\sqrt{5}}{2}B = 1 \end{cases} \rightarrow \begin{cases} \frac{5-\sqrt{5}}{2}A + \frac{5-\sqrt{5}}{2}B = 0 \\ -\frac{5-\sqrt{5}}{2}A - \frac{5+\sqrt{5}}{2}B = 1 \end{cases}$$

$$-\sqrt{5}B = 1 \rightarrow B = -\frac{1}{\sqrt{5}} \Rightarrow A = \frac{1}{\sqrt{5}}$$

$$\frac{1}{x^2 - 5x + 5} = \frac{\sqrt{5}}{5} \frac{2}{2x-5-\sqrt{5}} - \frac{\sqrt{5}}{5} \frac{2}{2x-5+\sqrt{5}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}$

Solution

$$\begin{aligned}\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} &= \frac{5x^2 + 20x + 6}{x(x+1)^2} \\ &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}\end{aligned}$$

$$Ax^2 + 2Ax + A + Bx^2 + Bx + Cx = 5x^2 + 20x + 6$$

$$\begin{cases} A + B = 5 \\ 2A + B + C = 20 \\ A = 6 \end{cases} \rightarrow \underline{B = -1} \quad \underline{C = 9}$$

$$\underline{\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} = \frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)}$

Solution

$$\frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)} = \frac{2x^3 - 4x - 8}{x(x-1)(x^2 + 4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx + D}{x^2 + 4}$$

$$Ax^3 - Ax^2 + 4Ax - 4A + Bx^3 + 4Bx + Cx^3 - Cx^2 + Dx^2 - Dx = 2x^3 - 4x - 8$$

$$\begin{cases} x^3 & A + B + C = 2 \\ x^2 & -A - C + D = 0 \\ x^1 & 4A + 4B - D = -4 \\ x^0 & -4A = -8 \end{cases} \rightarrow \begin{cases} B + C = 0 \\ -C + D = 2 \\ 4B - D = -12 \\ \underline{A = 2} \end{cases}$$

$$\Rightarrow \begin{cases} B + D = 2 \\ 4B - D = -12 \end{cases}$$

$$\underline{A = 2 \quad B = -2 \quad C = 2 \quad D = 4}$$

$$\frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)} = \frac{2}{x} - \frac{2}{x-1} + \frac{2x}{x^2 + 4} + \frac{4}{x^2 + 4}$$

Exercise

Write the partial fraction decomposition of each rational expression

$$\frac{8x^3 + 13x}{(x^2 + 2)^2}$$

Solution

$$\frac{8x^3 + 13x}{(x^2 + 2)^2} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{(x^2 + 2)^2}$$

$$Ax^3 + 2Ax + Bx^2 + 2B + Cx + D = 8x^3 + 13x$$

$$\begin{cases} x^3 & A = 8 \\ x^2 & B = 0 \\ x^1 & 2A + C = 13 \\ x^0 & D = 0 \end{cases} \rightarrow \underline{C = -3}$$

$$\frac{8x^3 + 13x}{(x^2 + 2)^2} = \frac{8x}{x^2 + 2} - \frac{3x}{(x^2 + 2)^2}$$

Exercise

Write the partial fraction decomposition of each rational expression

$$\frac{1}{x^2 - 9}$$

Solution

$$\frac{1}{x^2 - 9} = \frac{A}{x-3} + \frac{B}{x+3}$$

$$Ax + 3A + Bx - 3B = 1$$

$$\Rightarrow \begin{cases} A + B = 0 \\ 3A - 3B = 1 \end{cases} \rightarrow \underline{A = \frac{1}{6} \quad B = -\frac{1}{6}}$$

$$\frac{1}{x^2 - 9} = \frac{1}{6} \frac{1}{x-3} - \frac{1}{6} \frac{1}{x+3}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{2}{9x^2 - 1}$

Solution

$$\frac{2}{9x^2 - 1} = \frac{A}{3x - 1} + \frac{B}{3x + 1}$$

$$3Ax + A + 3Bx - B = 2$$

$$\Rightarrow \begin{cases} 3A + 3B = 0 \\ A - B = 2 \end{cases} \rightarrow \underline{A = 1 \quad B = -1}$$

$$\frac{2}{9x^2 - 1} = \frac{1}{3x - 1} - \frac{1}{3x + 1}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{5}{x^2 + 3x - 4}$

Solution

$$\frac{5}{x^2 + 3x - 4} = \frac{A}{x - 1} + \frac{B}{x + 4}$$

$$Ax + 4A + Bx - B = 5$$

$$\Rightarrow \begin{cases} A + B = 0 \\ 4A - B = 5 \end{cases} \rightarrow \underline{A = 1 \quad B = -1}$$

$$\frac{5}{x^2 + 3x - 4} = \frac{1}{x - 1} - \frac{1}{x + 4}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{3 - x}{3x^2 - 2x - 1}$

Solution

$$\frac{3 - x}{3x^2 - 2x - 1} = \frac{A}{x - 1} + \frac{B}{3x + 1}$$

$$3Ax + A + Bx - B = 3 - x$$

$$\Rightarrow \begin{cases} 3A + B = -1 \\ A - B = 3 \end{cases} \rightarrow \underline{A = \frac{1}{2} \quad B = -\frac{5}{2}}$$

$$\frac{3 - x}{3x^2 - 2x - 1} = \frac{1}{2} \frac{1}{x - 1} - \frac{5}{2} \frac{1}{3x + 1}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^2 + 12x + 12}{x^3 - 4x}$

Solution

$$\frac{x^2 + 12x + 12}{x^3 - 4x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$$

$$Ax^2 - 4A + Bx^2 + 2Bx + Cx^2 - 2Cx = x^2 + 12x + 12$$

$$\begin{cases} x^2 & A + B + C = 1 \\ x^1 & 2B - 2C = 12 \\ x^0 & -4A = 12 \end{cases} \rightarrow A = -3 \quad B = 5 \quad C = -1$$

$$\frac{x^2 + 12x + 12}{x^3 - 4x} = -\frac{3}{x} + \frac{5}{x-2} - \frac{1}{x+2}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{5x-2}{(x-2)^2}$

Solution

$$\frac{5x-2}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$$

$$Ax - 2A + B = 5x - 2$$

$$\Rightarrow \begin{cases} A = 5 \\ -2A + B = -2 \end{cases} \rightarrow B = 8$$

$$\frac{5x-2}{(x-2)^2} = \frac{5}{x-2} + \frac{8}{(x-2)^2}$$