Lecture Four

Section 4.1 – First-Order Systems

Consider a system of differential equations that can be solved for the highest-order derivatives of the dependent variables.

For instance, in the case of a system of two 2nd-order equations can be written in the form

$$\begin{cases} x_1' = f_1\left(t,\ x_1,\ x_2,\ x_1',\ x_2'\right) \\ x_2' = f_2\left(t,\ x_1,\ x_2,\ x_1',\ x_2'\right) \end{cases}$$

Any higher-order system can be transformed into an equivalent system of 1st-order equations. Consider a system consisting of the single nth-order equation.

$$x^{(n)} = f_2\left(t,\ x,\ x',\ \ldots,\ x^{(n-1)}\right)$$

We introduce the dependent variables $x_1,\ x_2,\ \ldots,\ x_n$ defined as follows:

$$x_1 = x, \quad x_2 = x', \quad x_3 = x'', \ \ldots \quad x_n = x^{(n-1)}$$

Note that $\quad x_1' = x', \quad x_2' = x'' = x_3, \quad \ldots$

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \quad \vdots \\ x_{n-1}' = x_n \end{cases}$$

$$x_n' = f_2\left(t,\ x_1,\ x_2,\ \ldots,\ x_n\right)$$

Example

The 3rd-order equation $x''' + 3x'' + 2x' - 5x = \sin 3t$ can be written in the form

$$x''' = f\left(t,\ x,\ x',\ x''\right) = 5x - 2x' - 3x'' + \sin 3t$$

Let $\quad x_1 = x, \quad x_2 = x' = x_1', \quad x_3 = x'' = x_2'$

Yield the system

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = 5x_1 - 2x_2 - 3x_3 + \sin 3t \end{cases}$$

Example

Transform this system into an equivalent 1st-order system

$$\begin{cases} x'' = -3x + y \\ y'' = 2x - 2y + 20\sin 2t \end{cases}$$

Solution

Let $\begin{matrix} x_1 = x & x_2 = x' = x_1' \\ y_1 = y & y_2 = y' = y_1' \end{matrix}$

$$\Rightarrow \quad \begin{cases} x_1' = x_2 \\ x_2' = -3x_1 + y_1 \end{cases} \qquad \begin{cases} y_1' = y_2 \\ y_2' = 2x_1 - 2y_1 + 20\sin 2t \end{cases}$$

Of 4 1st-order equations in the dependent variables x_1, x_2, y_1, y_2

Simple 2–Dimensional Systems

The linear 2nd-order differential equation $x'' + px' + qx = 0$

Let $x' = y \quad \Rightarrow \quad x'' = y'$

$$\begin{cases} x' = y \\ y' = -qx - py \end{cases}$$

Example

Solve the 2-dimensional system

$$\begin{cases} x' = -2y \\ y' = \frac{1}{2}x \end{cases}$$

Then solve using the initial values $x(0) = 2, \quad y(0) = 0$

Solution

$$x'' = -2y' = -2\left(\frac{1}{2}x\right) = -x$$

$$x'' + x = 0 \quad \Rightarrow \quad \lambda^2 + 1 = 0 \quad \to \lambda_{1,2} = \pm i$$

$\therefore$ Have a general solution: $x(t) = A\cos t + B\sin t$

$$\begin{aligned} y(t) &= -\frac{1}{2}x'(t) \\ &= -\frac{1}{2}(-A\sin t + B\cos t) \end{aligned}$$

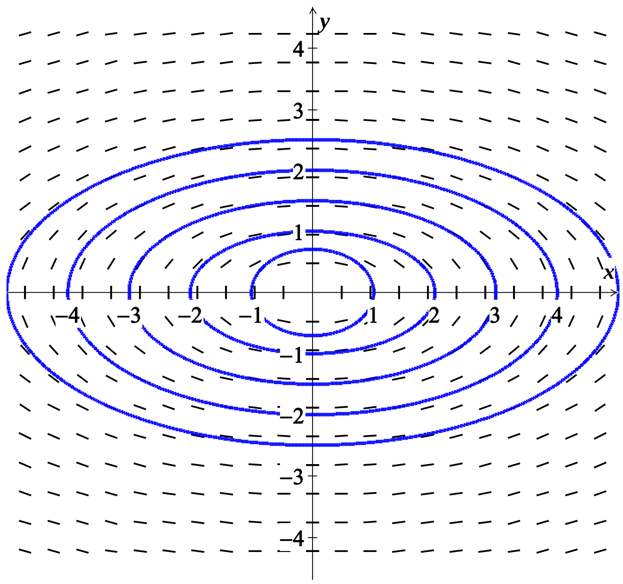
Let $A = C\cos\alpha \quad and \quad B = C\sin\alpha$

$$\begin{cases} x(t) = C\cos\alpha\cos t + C\sin\alpha\sin t = C\cos(t-\alpha) \\ y(t) = \frac{1}{2}(C\cos\alpha\sin t - C\sin\alpha\cos t) = \frac{1}{2}C\sin(t-\alpha) \end{cases}$$

$$\begin{cases} \cos(t-\alpha) = \dfrac{x(t)}{C} \\ \sin(t-\alpha) = \dfrac{2}{C}y(t) \end{cases}$$

$$\cos^2(t-\alpha) + \sin^2(t-\alpha) = 1$$

$$\frac{x^2}{C^2} + \frac{y^2}{(C/2)^2} = 1 \quad \therefore \quad \textit{Ellipse}$$

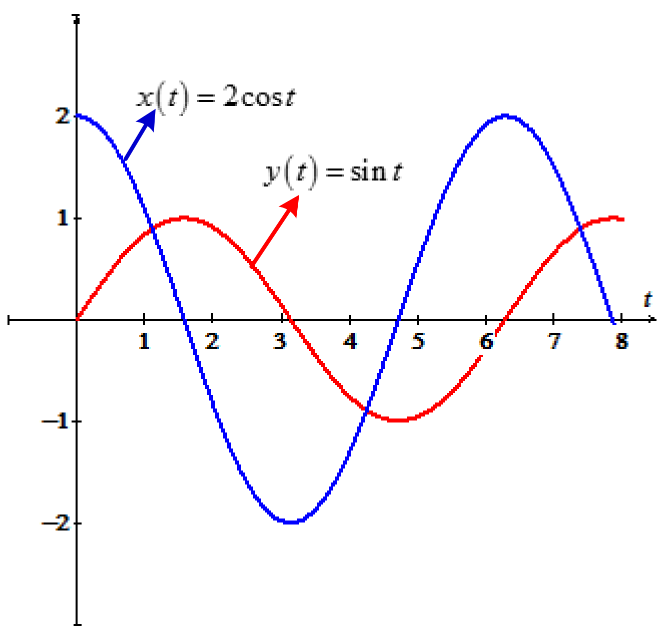


$$x(0) = 2, \quad y(0) = 0$$

$$x(0) = A = 2$$

$$y(0) = -\frac{1}{2}B = 0$$

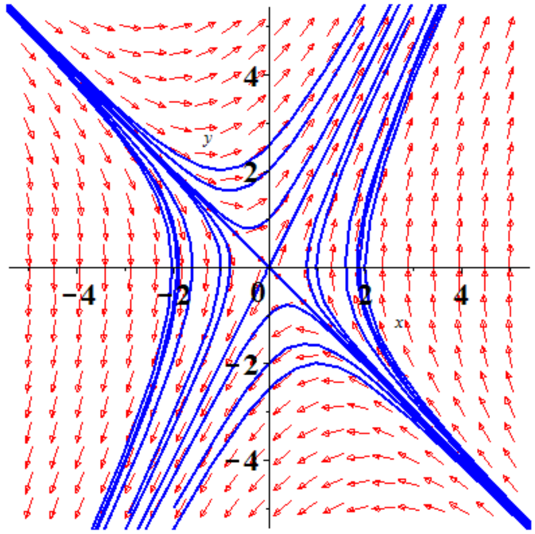
$$\begin{cases} x(t) = 2\cos t \\ y(t) = \sin t \end{cases}$$



Example

Find the general solution of the system

$$\begin{cases} x' = y \\ y' = 2x + y \end{cases}$$



Solution

$$x'' = y' = 2x + y$$

$$x'' = 2x + x'$$

$$x'' - x' - 2x = 0 \quad \Rightarrow \quad \lambda^2 - \lambda - 2 = 0$$

The eigenvalues are: $\lambda_1 = -1, \quad \lambda_2 = 2$

$\therefore$ General solution: $\begin{cases} x(t) = Ae^{-t} + Be^{2t} \\ y(t) = -Ae^{-t} + 2Be^{2t} \end{cases}$

Example

Solve the initial value problem

$$\begin{cases} x' = -y \\ y' = (1.01)x - (0.2)y \\ x(0) = 0, \quad y(0) = 1 \end{cases}$$

Solution

$$x'' = -y' = -1.01x + 0.2y$$

$$x'' = -y' = -1.01x - 0.2x'$$

$$x'' + 0.2x' + 1.01x = 0$$

$$\lambda^2 + 0.2\lambda + 1.01 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{-0.2 \pm \sqrt{0.04 - 4.04}}{2} = -0.1 \pm i$$

$$x(t) = e^{-0.1t}(A\cos t + B\sin t)$$

$$x(0) = 0 \quad \to A = 0 \quad \Rightarrow \quad x(t) = Be^{-0.1t}\sin t$$

$$y(t) = -x' = -0.1Be^{-0.1t}\sin t - Be^{-0.1t}\cos t$$

$$y(0) = 1 \quad \to -B = 1 \quad y(t) = -\frac{1}{10}e^{-t/10}\sin t + e^{-t/10}\cos t$$

$\therefore$ General solution: $\begin{cases} x(t) = -e^{-t/10}\sin t \\ y(t) = e^{-t/10}\left(\cos t - \frac{1}{10}\sin t\right) \end{cases}$

Exercises Section 4.1 – First-Order Systems

Transform the given differential equation or system into an equivalent system of 1st-order differential equation

1. $x'' + 3x' + 7x = t^2$

2. $x^{(4)} + 6x'' - 3x' + x = \cos 3t$

3. $t^2 x'' + tx' + \left(t^2 - 1\right)x = 0$

4. $t^3 x^{(3)} - 2t^2 x'' + 3tx' + 5x = \ln t$

5. $x'' - 5x + 4y = 0, \quad y'' + 4x - 5y = 0$

6. $x'' - 3x' + 4x - 2y = 0, \quad y'' + 2y' - 3x + y = \cos t$

7. $x'' = 3x - y + 2z, \quad y'' = x + y - 4z, \quad z'' = 5x - y - z$

8. $x'' = (1-y)x, \quad y'' = (1-x)y$

Find the general solution

9. $x' = y, \quad y' = -x$

10. $x' = y, \quad y' = -9x + 6y$

11. $x' = 8y, \quad y' = -2x$

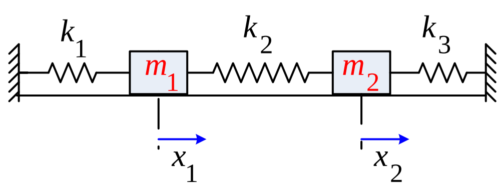
12. $x' = -2y, \quad y' = 2x; \quad x(0) = 1, \quad y(0) = 0$

13. $x' = y, \quad y' = 6x - y; \quad x(0) = 1, \quad y(0) = 2$

14. $x' = -y, \quad y' = 13x + 4y; \quad x(0) = 0, \quad y(0) = 3$

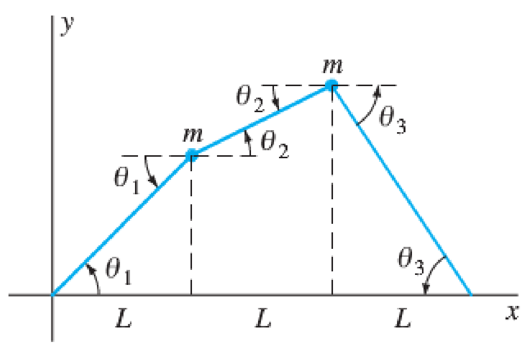
Derive the equations $\begin{cases} m_1 x_1'' = -\left(k_1 + k_2\right)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - \left(k_2 + k_3\right)x_2 \end{cases}$

For the displacements (from equilibrium) of the 2 masses.



15. Two particles each of mass m are attached to a string under (constant) tension T. Assume that the particles oscillate vertically (that is, parallel to the y-axis) with amplitudes so small that the sines of the angles shown are accurately approximated by their tangents. Show that the displacement y_1 and y_2 satisfy the equations

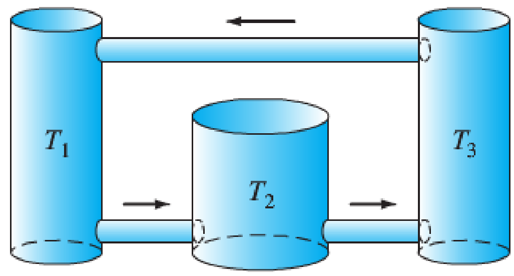
$$\begin{cases} ky_1'' = -2y_1 + y_2 \\ ky_2'' = y_1 - 2y_2 \end{cases} \qquad \text{where } k = \frac{mL}{T}$$



16. There 100-gal fermentation vats are connected, and the mixtures in each tank are kept uniform by stirring. Denote by $x_i(t)$ the amount (in pounds) of alcohol in tank T_i at time t (i = 1, 2, 3).

Suppose that the mixture circulates between the tanks at the rate of 10 gal/min. Derive the equations

$$\begin{cases} 10x_1' = -x_1 \qquad + x_3 \\ 10x_2' = x_1 - x_2 \\ 10x_3' = \qquad x_2 - x_3 \end{cases}$$



17. Suppose that a particle with mass m and electrical charge q moves in the xy-plane under the influence of the magnetic field $\vec{B} = B\hat{k}$ (thus a uniform field parallel to the z-axis), so the force on the particle is $\vec{F} = q\vec{v} \times \vec{B}$ if its velocity is $\vec{v}$. Show that the equations of motion of the particle are

$$mx'' = +qBy', \quad my'' = -qBx'$$

Section 4.2 – Matrices and Linear Systems

Let $a_{11}(t),\ a_{12}(t),\ \ldots,\ a_{nn}(t)$ and $b_1(t),\ b_2(t),\ \ldots,\ b_n(t)$ be continuous functions on the interval *I*.

The system of *n* 1st-order differential equations:

$$\begin{aligned}
x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + \ \ldots + a_{1n}(t)x_n + b_1(t)\\
x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + \ \ldots + a_{2n}(t)x_n + b_2(t)\\
&\ \vdots\\
x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ \ldots + a_{nn}(t)x_n + b_n(t)
\end{aligned}$$

Is called a 1st-order linear differential system.

The system is ***homogeneous*** if $b_1(t) \equiv b_2(t) \equiv \ \ldots \equiv b_m(t) \equiv 0$ on *I*, otherwise, the system is ***nonhomogeous*** if the functions $b_i(t)$ are not all identically zero on *I*.

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{pmatrix} \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad b(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}$$

The system can be written in the vector-matrix form $\quad X' = A(t)X + b(t) \qquad (S)$

$A(t)$: Coefficient matrix

$b(t)$: Constant matrix

A solution of the linear differential system (S) is a differentiable vector function

$$\vec{v} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

Satisfies (S) on the interval *I*.

The derivative of *A*: $\quad A'(t) = \dfrac{dA}{dt} = \left[\dfrac{da_{ij}}{dt}\right]$

Example

Find the derivative if $x(t)=\begin{pmatrix} t \\ t^2 \\ e^{-t} \end{pmatrix}$ $\qquad A(t)=\begin{pmatrix} \sin t & 1 \\ t & \cos t \end{pmatrix}$

Solution

$$x'(t)=\begin{pmatrix} 1 \\ 2t \\ -e^{-t} \end{pmatrix} \qquad A(t)=\begin{pmatrix} \cos t & 0 \\ 1 & -\sin t \end{pmatrix}$$

Example

The 1st-order system $\begin{cases} x_1' = 4x_1 - 3x_2 \\ x_2' = 6x_1 - 7x_2 \end{cases}$

$$\begin{aligned} X' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} &= \begin{bmatrix} 4x_1 - 3x_2 \\ 6x_1 - 7x_2 \end{bmatrix} \\ &= \begin{pmatrix} 4 & -3 \\ 6 & -7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} 4 & -3 \\ 6 & -7 \end{pmatrix} X \end{aligned}$$

$$\frac{dX}{dt} = P(t)X + f(t) \quad with \quad P(t)=\begin{pmatrix} 4 & -3 \\ 6 & -7 \end{pmatrix} \quad f(t)=\begin{pmatrix} 0 \\ 0 \end{pmatrix}=0$$

To verify that the vector functions:

$$x_1(t)=\begin{pmatrix} 3e^{2t} \\ 2e^{2t} \end{pmatrix} \quad x_2(t)=\begin{pmatrix} e^{-5t} \\ 3e^{-5t} \end{pmatrix}$$

Are both solutions of the matrix differential equations with coefficient matrix P.

$$Px_1 = \begin{pmatrix} 4 & -3 \\ 6 & -7 \end{pmatrix}\begin{pmatrix} 3e^{2t} \\ 2e^{2t} \end{pmatrix} = \begin{pmatrix} 6e^{2t} \\ 4e^{2t} \end{pmatrix} = x_1'$$

$$Px_2 = \begin{pmatrix} 4 & -3 \\ 6 & -7 \end{pmatrix}\begin{pmatrix} e^{-5t} \\ 3e^{-5t} \end{pmatrix} = \begin{pmatrix} -5e^{-5t} \\ -15e^{-5t} \end{pmatrix} = x_2'$$

When $f(t)=0 \;\Rightarrow\; \dfrac{dX}{dt} = P(t)X$ is a homogeneous equation

A ***homogeneous*** system

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0\\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0
\end{aligned}$$

Always has at least one solution namely $x_1 = x_2 = \ldots = x_n = 0$ called the ***trivial solution***

That is, homogeneous systems are always ***consistent***

Theorem

If $\vec{v}$ is a solution of (H) and α is any $\mathbb{R}$, then $\vec{u} = \alpha\vec{v}$ is also a solution of (H); any constant multiple of a solution of (H) is a solution of (H).

Theorem

If $\vec{v}_1$ and $\vec{v}_2$ are solutions of (H), then $\vec{u} = \vec{v}_1 + \vec{v}_2$ is also a solution of (H); the sum of any 2 solutions of (H) is a solution of (H).

$$\vec{v}_1' = A(t)\vec{v}_1 \qquad \vec{v}_1' + \vec{v}_2' = A(t)\vec{v}_1 + A(t)\vec{v}_2$$

$$\vec{v}_2' = A(t)\vec{v}_2 \qquad \left(\vec{v}_1 + \vec{v}_2\right)' = A(t)\left(\vec{v}_1 + \vec{v}_2\right)$$

$$\vec{u}' = A(t)\vec{u}' \qquad \text{Since } \vec{u} = \vec{v}_1 + \vec{v}_2$$

In general,

Theorem

If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are solutions of (H), and is $c_1, c_2, \ldots, c_n$ are $\mathbb{R}$ then

$$c_1\vec{v}_1,\ c_2\vec{v}_2,\ \ldots,\ c_n\vec{v}_n$$

Is a solution of (H); any linear combination of solutions of (H) is also a solution of (H).

$$\begin{aligned}
\vec{v}_1' &= A(t)\vec{v}_1 + c_1\\
\vec{v}_2' &= A(t)\vec{v}_2 + c_2\\
\vdots \quad & \quad \vdots \qquad \vdots\\
\vec{v}_n' &= A(t)\vec{v}_n + c_n
\end{aligned}$$

Linear Dependent and Independent

Let

$$\vec{x}_1(t)=\begin{pmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{pmatrix}, \quad \vec{x}_2(t)=\begin{pmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{pmatrix}, \quad ,\cdots \quad \vec{x}_m(t)=\begin{pmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{pmatrix}$$

Be vector functions defined on some interval *I*.

The vectors are linearly dependent on *I* if exist *n* real numbers $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1\vec{v}_1(t)+ c_2\vec{v}_2(t)+ \ldots + c_n\vec{v}_n(t)=0 \qquad on \;\; I$$

Otherwise the vectors are linearly independent on I.

Wronskian of solutions

Theorem

Let $x_1, x_2, \ldots, x_n$ are *n* solutions of the homogeneous linear equation $x'=P(t)x$ on an interval *I*.

Let $W=W\left(x_1, x_2, \ldots, x_n\right)$

$$W=\begin{vmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ v_{n1} & \cdots & \cdots & v_{nn} \end{vmatrix}=0 \qquad on \text{ I}$$

Called the Wronskian of the vector functions $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$

Special Case *n* solutions of (*H*)

Theorem

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be solution of (*H*). Exactly one of the following holds.

1. $W\left(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\right)(t)\equiv 0$ on *I* and the solutions are Linearly Dependent.
2. $W\left(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\right)(t)\neq 0$ for all $t\in I$ and the solutions are Linearly Independent.

Theorem

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be n L.I solutions of (H) $(W \neq 0)$

Let $\vec{u}$ be any solution of (H). Then there exists a unique set of constants $c_1, c_2, \ldots, c_n$ such that

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$$

That is, every solution of (H) can be written as a unique linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$

A set of n L.I solutions $(W \neq 0)$ of (H) is called a ***fundamental set of solutions***.

A fundamental set is also called a ***solution basis*** for (H).

Example

Determine if the solutions are linearly dependent or independent using Wronskian.

$$\vec{x}_1(t) = \begin{pmatrix} 2e^t \\ 2e^t \\ e^t \end{pmatrix}, \quad \vec{x}_2(t) = \begin{pmatrix} 2e^{3t} \\ 0 \\ -e^{3t} \end{pmatrix}, \quad \vec{x}_3(t) = \begin{pmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{pmatrix}$$

Solution

$$W = \begin{vmatrix} 2e^t & 2e^{3t} & 2e^{5t} \\ 2e^t & 0 & -2e^{5t} \\ e^t & -e^{3t} & e^{5t} \end{vmatrix} = -4e^{9t} - 4e^{9t} - 4e^{9t} - 4e^{9t} \underline{= -16e^{9t} \neq 0}$$

or $$W = \begin{vmatrix} 2e^t & 2e^{3t} & 2e^{5t} \\ 2e^t & 0 & -2e^{5t} \\ e^t & -e^{3t} & e^{5t} \end{vmatrix} = e^{9t}\begin{vmatrix} 2 & 2 & 2 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{vmatrix} \underline{= -16e^{9t} \neq 0}$$

The solutions $x_1, x_2, \text{ and } x_3$ are linearly independent.

Example

Find the general solution of: $y''' - 3y'' - 4y' + 12y = 6e^{t}$

Solution

$$\lambda^3 - 3\lambda^2 - 4\lambda + 12 = 0$$

$$\lambda^2(\lambda - 3) - 4(\lambda - 3) = 0$$

$$\left(\lambda^2 - 4\right)(\lambda - 3) = 0 \qquad \Rightarrow \lambda_1 = 3,\ \lambda_2 = 2,\ \lambda_3 = -2$$

The Fundamental set: $\left\{y_1 = e^{3t},\quad y_2 = e^{2t},\quad y_3 = e^{-2t}\right\}$

$$y_h = C_1 e^{3t} + C_2 e^{2t} + C_3 e^{-2t}$$

Particular solution: $z = e^{t} \quad \Rightarrow \quad z(t) = Ae^{t}$

$$z' = Ae^{t} \quad z'' = Ae^{t} \quad z''' = Ae^{t}$$

$$Ae^{t} - 3Ae^{t} - 4Ae^{t} + 12Ae^{t} = 6e^{t}$$

$$6Ae^{t} = 6e^{t} \quad \Rightarrow \quad \boxed{A = 1}$$

$$y_p = e^{t}$$

General solution: $y(t) = C_1 e^{3t} + C_2 e^{2t} + C_3 e^{-2t} + e^{t}$

$$y''' = 3y'' + 4y' - 12y + 6e^{t}$$

$$y = x_1 \quad y' = x_2 \quad y'' = x_3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} y' \\ y'' \\ y''' \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ 3x_3 + 4x_2 - 12x_1 + 6e^{t} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 6e^{t} \end{pmatrix}$$

$y = e^{3t} + e^{t}$ is a solution of the equation

Proof: $y' = 3e^{3t} + e^{t} \quad y'' = 9e^{3t} + e^{t} \quad y''' = 27e^{3t} + e^{t}$

$$y''' = 3\left(9e^{3t} + e^{t}\right) + 4\left(3e^{3t} + e^{t}\right) - 12\left(e^{3t} + e^{t}\right) + 6e^{t}$$

$$= 27e^{3t} + 3e^{t} + 12e^{3t} + 4e^{t} - 12e^{3t} - 12e^{t} + 6e^{t}$$

$$= 27e^{3t} + e^{t} \quad \checkmark$$

Therefore; $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} e^{3t} + e^{t} \\ 3e^{3t} + e^{t} \\ 9e^{3t} + e^{t} \end{pmatrix}$

For $y_1 = e^{3t}$ $\qquad x_1(t) = \begin{pmatrix} y_1 \\ y_1' \\ y_1'' \end{pmatrix} = \begin{pmatrix} e^{3t} \\ 3e^{3t} \\ 9e^{3t} \end{pmatrix}$

For $y_2 = e^{2t}$ $\qquad x_2(t) = \begin{pmatrix} y_2 \\ y_2' \\ y_2'' \end{pmatrix} = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix}$

For $y_1 = e^{-2t}$ $\qquad x_3(t) = \begin{pmatrix} y_3 \\ y_3' \\ y_3'' \end{pmatrix} = \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \\ 4e^{-2t} \end{pmatrix}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$W = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{vmatrix} = -12 \neq 0$$

Exercises *Section* 4.2 – Matrices and Linear Systems

Write the given system in the form $\boldsymbol{x}' = \boldsymbol{P}(t)x + \boldsymbol{f}(t)$

1. $x' = -3y, \quad y' = 3x$

2. $x' = 3x - 2y, \quad y' = 2x + y$

3. $x' = tx - e^t y + \cos t, \quad y' = e^{-t}x + t^2 y - \sin t$

4. $x' = y + z, \quad y' = z + x, \quad z' = x + y$

5. $x' = 2x - 3y, \quad y' = x + y + 2z, \quad z' = 5y - 7z$

6. $x' = 3x - 4y + z + t, \quad y' = x - 3z + t^2, \quad z' = 6y - 7z + t^3$

7. $x_1' = x_2, \quad x_2' = 2x_3, \quad x_3' = 3x_4, \quad x_4' = 4x_1$

8. $x_1' = x_2 + x_3 + 1, \quad x_2' = x_3 + x_4 + t, \quad x_3' = x_1 + x_4 + t^2, \quad x_4' = 4x_1 + x_2 + t^3$

For the systems below:

a) Verify that the given vectors are solutions of the given system.
b) Use the Wronskian to show that they are linearly independent.
c) Write the general solution of the system.
d) Find the particular solution that satisfies the given initial conditions

9. $\boldsymbol{x}' = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}\boldsymbol{x}; \quad \vec{x}_1 = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}$

10. $\boldsymbol{x}' = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}\boldsymbol{x}; \quad \vec{x}_1 = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 0 \\ x_2(0) = 5 \end{cases}$

11. $\boldsymbol{x}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\boldsymbol{x}; \quad \vec{x}_1 = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = e^{-2t}\begin{bmatrix} 1 \\ 5 \end{bmatrix}, \quad \begin{cases} x_1(0) = 5 \\ x_2(0) = -3 \end{cases}$

12. $\boldsymbol{x}' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\boldsymbol{x}; \quad \vec{x}_1 = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 8 \\ x_2(0) = 0 \end{cases}$

13. $\boldsymbol{x}' = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\boldsymbol{x}; \quad \vec{x}_1 = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} -2e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 0 \\ x_2(0) = 0 \\ x_3(0) = 4 \end{cases}$

14. $\boldsymbol{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\boldsymbol{x}; \quad \vec{x}_1 = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 10 \\ x_2(0) = 12 \\ x_3(0) = -1 \end{cases}$

15. $\boldsymbol{x}' = \begin{bmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{bmatrix} \boldsymbol{x}$; $\vec{x}_1 = \begin{bmatrix} e^{-t} \\ 0 \\ 0 \\ e^{-t} \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 0 \\ 0 \\ e^{-t} \\ 0 \end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{bmatrix}$, $\vec{x}_4 = \begin{bmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{bmatrix}$, $\begin{cases} x_1(0) = 1 \\ x_2(0) = 3 \\ x_3(0) = 4 \\ x_4(0) = 7 \end{cases}$

Section 4.3 – Eigenvalue Method for Linear System

A homogeneous first-order system with constant coefficients is given by

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n \end{aligned}$$

We can find *n* linear independent solution vectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ and the linear combination

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n$$

We apply the characteristics root method for solving a single homogeneous equation with constant coefficients.

$$\vec{x}(t) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \\ \vdots \\ v_n e^{\lambda t} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} e^{\lambda t} = \vec{v} e^{\lambda t}$$

Theorem

Let λ be an eigenvalue of the constant coefficient matrix *A* of the first-order linear system

$$\frac{dx}{dt} = Ax$$

If $\vec{v}$ is an eigenvector associated with λ, then

$$\vec{x}(t) = \vec{v}e^{\lambda t} \qquad \vec{v} \neq \vec{0}$$

is a nontrivial solution of the system

If $\lambda_1, \lambda_2, \dots, \lambda_n$ are distinct eigenvalues of *A* with corresponding $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$, then

$$\vec{x}_1 = e^{\lambda_1 t}\vec{v}_1, \quad \vec{x}_2 = e^{\lambda_2 t}\vec{v}_2, \quad \dots, \quad \vec{x}_n = e^{\lambda_k t}\vec{v}_n$$

form a fundamental set of solutions of $\vec{x}' = A\vec{x}$

And $\vec{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 + \dots + C_n\vec{x}_n$ is the general solution.

Note

- Recall that an eigenvalue λ of the matrix *A* is a solution of the characteristic equation $|A - \lambda I| = 0$
- An eigenvector $\vec{v}$ associated with λ is then a solution of the eigenvector equation $(A - \lambda I)\vec{v} = 0$

Distinct Real *Eigenvalues*

Examples

Find a general solution of the system

$$\begin{cases} x_1' = 4x_1 + 2x_2 \\ x_2' = 3x_1 - x_2 \end{cases}$$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = (4-\lambda)(-1-\lambda) - 6$$

$$= \lambda^2 - 3\lambda - 10 = 0$$

The distinct real eigenvalues: $\lambda_1 = -2, \quad \lambda_2 = 5$

For $\lambda_1 = -2 \quad \Rightarrow \quad (A + 2I)V_1 = 0$

$$\begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad 3x_1 + y_1 = 0 \;\rightarrow\; y_1 = -3x_1$$

$$\rightarrow V_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad \Rightarrow \quad x_1(t) = \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-2t}$$

For $\lambda_2 = 5 \quad \Rightarrow \quad (A - 5I)V_2 = 0$

$$\begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad -x_2 + 2y_2 = 0 \;\rightarrow\; x_2 = 2y_2$$

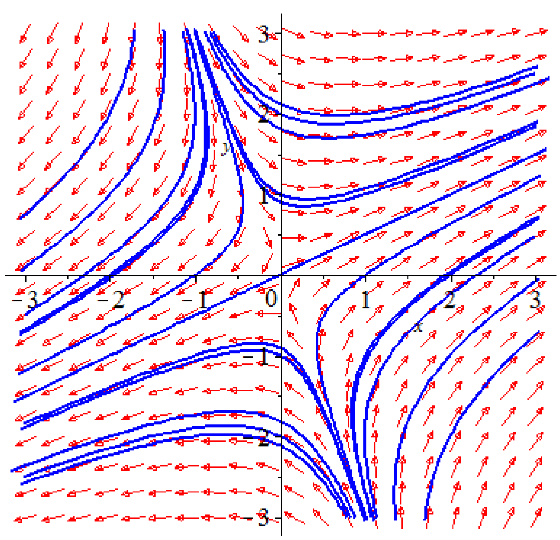
$$\rightarrow V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \Rightarrow \quad x_2(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t}$$

$$x_1(t) = \begin{pmatrix} e^{-2t} \\ -3e^{-2t} \end{pmatrix} \qquad x_2(t) = \begin{pmatrix} 2e^{5t} \\ e^{5t} \end{pmatrix}$$

Using Wronskian: $\begin{vmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{vmatrix} = 7e^{3t} \neq 0$

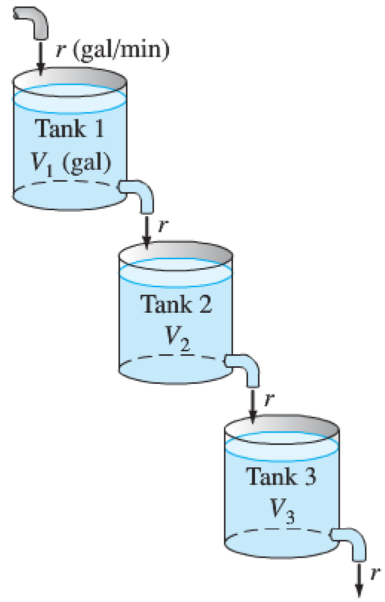
The general solution: $x(t) = C_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t}$

OR $\begin{cases} x_1(t) = C_1 e^{-2t} + 2C_2 e^{5t} \\ x_2(t) = -3C_1 e^{-2t} + C_2 e^{5t} \end{cases}$



Examples

If $V_1 = 20\ gal$, $V_2 = 40\ gal$, $V_3 = 50\ gal$, $r = 10\ gal/\min$ and the initial amounts of salt in 3 brine tanks, in lbs, are $x_1(0) = 15 \quad x_2(0) = x_3(0) = 0$. Find the amount of salt in each tank at time $t \ge 0$.



Solution

$$\begin{cases} x_1' = -k_1 x_1 \\ x_2' = k_1 x_1 - k_2 x_2 \\ x_3' = \qquad k_2 x_2 - k_3 x_3 \end{cases} \qquad \text{where } k_i = \frac{r}{v_i} \quad i = 1,2,3$$

$$k_1 = \frac{10}{20} = .5 \quad k_2 = \frac{10}{40} = .25 \quad k_3 = \frac{10}{50} = .2$$

$$\begin{cases} x_1' = -.5x_1 \\ x_2' = .5x_1 - .25x_2 \\ x_3' = \qquad .25x_2 - .2x_3 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -.5 & 0 & 0 \\ .5 & -.25 & 0 \\ 0 & .25 & -.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad with\ x(0) = \begin{pmatrix} 15 \\ 0 \\ 0 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -.5-\lambda & 0 & 0 \\ .5 & -.25-\lambda & 0 \\ 0 & .25 & -.2-\lambda \end{vmatrix}$$

$$= (-.5-\lambda)(-.25-\lambda)(-.2-\lambda) = 0$$

The eigenvalues are: $\lambda_1 = -.5 \quad \lambda_2 = -.25 \quad \lambda_3 = -.2$

For $\lambda_1 = -.5 \Rightarrow (A + .5I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ .5 & .25 & 0 \\ 0 & .25 & .3 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} .5a_1 + .25b_1 = 0 \rightarrow 2a_1 = -b_1 \\ .25b_1 + .3c_1 = 0 \rightarrow 6c_1 = -5b_1 \end{cases}$$

$$\rightarrow V_1 = \begin{pmatrix} 3 \\ -6 \\ 5 \end{pmatrix} \Rightarrow x_1(t) = \begin{pmatrix} 3 \\ -6 \\ 5 \end{pmatrix} e^{-.5t}$$

For $\lambda_2 = -.25 \Rightarrow (A + .25I)V_2 = 0$

$$\begin{pmatrix} -.25 & 0 & 0 \\ .5 & 0 & 0 \\ 0 & .25 & .05 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a_2 = 0 \\ .25b_2 + .05c_2 = 0 \rightarrow c_2 = -5b_2 \end{cases}$$

$$\to V_2=\begin{pmatrix}0\\1\\-5\end{pmatrix} \quad\Rightarrow\quad x_2(t)=\begin{pmatrix}0\\1\\-5\end{pmatrix}e^{-.25t}$$

For $\lambda_3=-.2 \quad\Rightarrow\quad (A+.2I)V_3=0$

$$\begin{pmatrix}-.3 & 0 & 0\\ .5 & -.05 & 0\\ 0 & .25 & 0\end{pmatrix}\begin{pmatrix}a_3\\b_3\\c_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \quad\Rightarrow\quad \begin{cases} a_3=0\\ b_3=0\\ 0c_3=0 \ \to\ c_3=1\end{cases}$$

$$\to V_3=\begin{pmatrix}0\\0\\1\end{pmatrix} \quad\Rightarrow\quad x_3(t)=\begin{pmatrix}0\\0\\1\end{pmatrix}e^{-.2t}$$

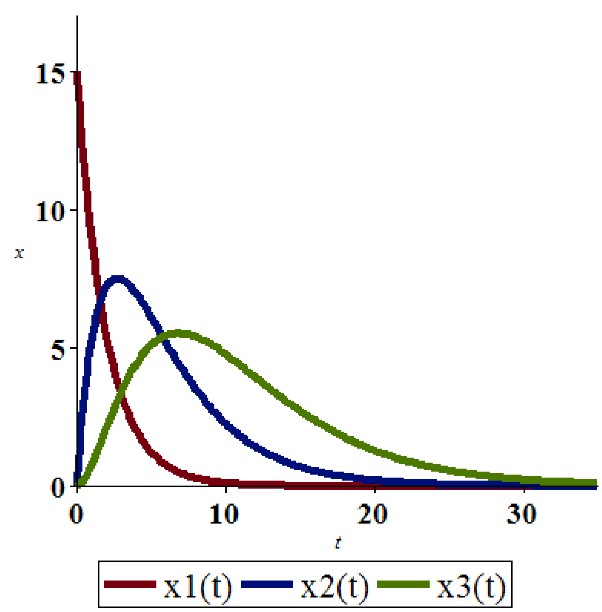
$$\Rightarrow\ x(t)=C_1\begin{pmatrix}3\\-6\\5\end{pmatrix}e^{-.5t}+C_2\begin{pmatrix}0\\1\\-5\end{pmatrix}e^{-.25t}+C_3\begin{pmatrix}0\\0\\1\end{pmatrix}e^{-.2t}$$

$$\begin{cases} x_1(t)=3C_1e^{-.5t}\\ x_2(t)=-6C_1e^{-.5t}+C_2e^{-.25t}\\ x_3(t)=5C_1e^{-.5t}-5C_2e^{-.25t}+C_3e^{-.2t}\end{cases}$$

With *initial* values

$$\begin{cases} 15=3C_1\\ 0=-6C_1+C_2\\ 0=5C_1-5C_2+C_3\end{cases} \quad\to\quad \begin{cases} \boxed{5=C_1}\\ \boxed{C_2=30}\\ \boxed{C_3=-5(5)+5(30)=125}\end{cases}$$

$$\begin{cases} x_1(t)=15e^{-.5t}\\ x_2(t)=-30e^{-.5t}+30e^{-.25t}\\ x_3(t)=25e^{-.5t}-150e^{-.25t}+125e^{-.2t}\end{cases}$$



Complex *Eigenvalues*

Examples

Find a general solution of the system

$$\begin{cases} x_1' = 4x_1 - 3x_2 \\ x_2' = 3x_1 + 4x_2 \end{cases}$$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 + 9 = 0$$

$$(4-\lambda)^2 = -9 \quad \Rightarrow \quad 4-\lambda = \pm 3i$$

The distinct real eigenvalues: $\lambda_{1,2} = 4 \pm 3i$

For $\lambda_1 = 4 - 3i \quad \Rightarrow \quad \big(A - (4-3i)I\big)V = 0$

$$\begin{pmatrix} 3i & -3 \\ 3 & 3i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad 3ia - 3b = 0 \ \rightarrow \ b = ia \quad \rightarrow V = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

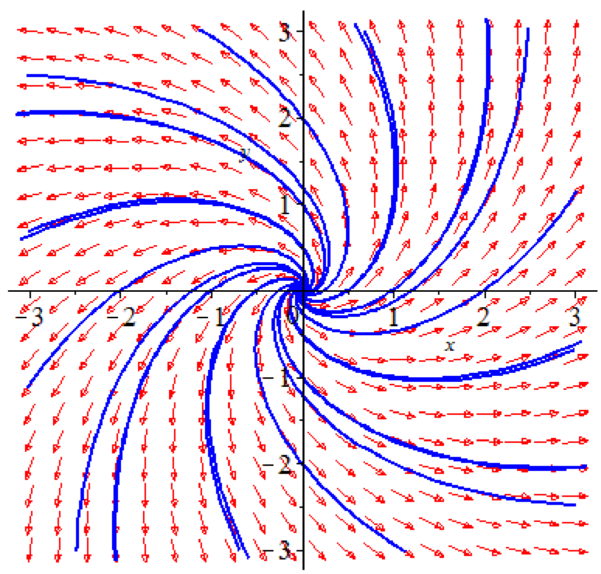
$$x(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(4-3i)t}$$

$$= \begin{pmatrix} 1 \\ i \end{pmatrix} e^{4t} e^{-3it}$$

$$e^{ait} = \cos at + i \sin at$$

$$= \begin{pmatrix} 1 \\ i \end{pmatrix} e^{4t} (\cos 3t - i \sin 3t)$$

$$= \begin{pmatrix} \cos 3t - i \sin 3t \\ i \cos 3t + \sin 3t \end{pmatrix} e^{4t}$$



$$\vec{x}_1(t) = \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} e^{4t} \qquad \vec{x}_2(t) = \begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix} e^{4t}$$

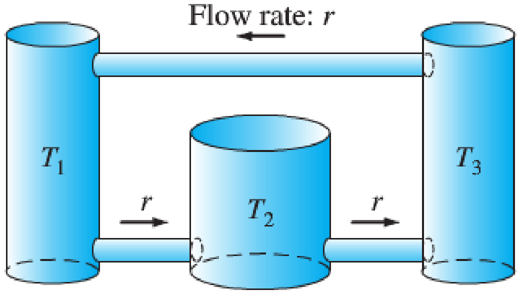
$$x(t) = C_1 \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} e^{4t} + C_2 \begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix} e^{4t}$$

$$\begin{cases} x_1(t) = \big(C_1 \cos 3t - C_2 \sin 3t\big) e^{4t} \\ x_2(t) = \big(-C_1 \sin 3t + C_2 \cos 3t\big) e^{4t} \end{cases}$$

Examples

If $V_1 = 50\ gal, \quad V_2 = 25\ gal, \quad V_3 = 50\ gal, \quad r = 10\ gal/\min$, find the amount $x_1(t),\ x_2(t),\ x_3(t)$ of salt in each tank at time $t \ge 0$

Solution



$$\begin{cases} x_1' = -k_1x_1 \qquad + k_3x_3 \\ x_2' = k_1x_1 - k_2x_2 \\ x_3' = \qquad k_2x_2 - k_3x_3 \end{cases} \qquad \text{where } k_i = \frac{r}{v_i} \quad i = 1, 2, 3$$

$$k_1 = \frac{10}{50} = .2 \quad k_1 = \frac{10}{25} = .4 \quad k_1 = \frac{10}{50} = .2$$

$$\begin{cases} x_1' = -.2x_1 \qquad + .2x_3 \\ x_2' = .2x_1 - .4x_2 \\ x_3' = \qquad .4x_2 - .2x_3 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -.2 & 0 & .2 \\ .2 & -.4 & 0 \\ 0 & .4 & -.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -.2-\lambda & 0 & .2 \\ .2 & -.4-\lambda & 0 \\ 0 & .4 & -.2-\lambda \end{vmatrix}$$

$$= (-.2-\lambda)(-.4-\lambda)(-.2-\lambda) + (.2)(.2)(.4)$$

$$= -\lambda^3 - .8\lambda^2 - .2\lambda$$

$$= -\lambda\left(\lambda^2 + .8\lambda + .2\right) = 0 \qquad \lambda^2 + .8\lambda + .2 = 0 \quad \lambda = \frac{-.8 \pm \sqrt{.64 - .8}}{2} = -.4 \pm .2i$$

The eigenvalues are: $\lambda_1 = 0 \quad \lambda_{2,3} = -.4 \pm .2i$

For $\lambda_1 = 0 \;\Rightarrow\; (A - 0I)V_1 = 0$

$$\begin{pmatrix} -.2 & 0 & .2 \\ .2 & -.4 & 0 \\ 0 & .4 & -.2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -.2a + .2c = 0 \to a = c \\ .2a - .4b = 0 \to a = 2b \end{cases}$$

$$\to V_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \Rightarrow x_1(t) = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

For $\lambda = -.4 - .2i \;\Rightarrow\; \left(A + (.4 + .2i)\right)V_2 = 0$

$$\begin{pmatrix} .2+.2i & 0 & .2 \\ .2 & .2i & 0 \\ 0 & .4 & .2+.2i \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} (.2+.2i)a=-.2c \\ .2a=-.2ib \end{cases}$$

$$\text{Let } b=i \quad \Rightarrow a=1 \quad c=-1-i$$

$$\to V_2=\begin{pmatrix} 1 \\ i \\ -1-i \end{pmatrix} \quad \Rightarrow \quad x_{2,3}(t)=\begin{pmatrix} 1 \\ i \\ -1-i \end{pmatrix} e^{-.4t}e^{-.2ti}$$

$$x_{2,3}(t)=\begin{pmatrix} 1 \\ i \\ -1-i \end{pmatrix} e^{-.4t}\left(\cos(.2t)-i\sin(.2t)\right)$$

$$=\begin{pmatrix} \cos .2t-i\sin .2t \\ \sin .2t+i\cos .2t \\ -\cos .2t-\sin .2t-i(\cos .2t-\sin .2t) \end{pmatrix} e^{-.4t}$$

$$x_1(t)=\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad x_2(t)=\begin{pmatrix} \cos .2t \\ \sin .2t \\ -\cos .2t-\sin .2t \end{pmatrix} e^{-.4t} \quad x_3(t)=\begin{pmatrix} -\sin .2t \\ \cos .2t \\ \sin .2t-\cos .2t \end{pmatrix} e^{-.4t}$$

$$\begin{cases} x_1(t)=2C_1+\left(C_2\cos 0.2t-C_3\sin 0.2t\right)e^{-.4t} \\ x_2(t)=C_1+\left(C_2\sin 0.2t+C_3\cos 0.2t\right)e^{-.4t} \\ x_3(t)=2C_1+\left(\left(-C_2-C_3\right)\cos 0.2t+\left(C_3-C_2\right)\sin 0.2t\right)e^{-.4t} \end{cases}$$

Exercises Section 4.3 – Eigenvalue Method for Linear System

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system.

1. $x_1' = x_1 + 2x_2, \quad x_2' = 2x_1 + x_2$

2. $x_1' = 2x_1 + 3x_2, \quad x_2' = 2x_1 + x_2$

3. $x_1' = 6x_1 - 7x_2, \quad x_2' = x_1 - 2x_2$

4. $x_1' = -3x_1 + 4x_2, \quad x_2' = 6x_1 - 5x_2$

5. $x_1' = x_1 - 5x_2, \quad x_2' = x_1 - x_2$

6. $x_1' = -3x_1 - 2x_2, \quad x_2' = 9x_1 + 3x_2$

7. $x_1' = x_1 - 5x_2, \quad x_2' = x_1 + 3x_2$

8. $x_1' = 5x_1 - 9x_2, \quad x_2' = 2x_1 - x_2$

9. $x_1' = 3x_1 + 4x_2, \quad x_2' = 3x_1 + 2x_2; \quad x_1(0) = x_2(0) = 1$

10. $x_1' = 9x_1 + 5x_2, \quad x_2' = -6x_1 - 2x_2; \quad x_1(0) = 1,\ x_2(0) = 0$

11. $x_1' = 2x_1 - 5x_2, \quad x_2' = 4x_1 - 2x_2; \quad x_1(0) = 2,\ x_2(0) = 3$

12. $x_1' = x_1 - 2x_2, \quad x_2' = 2x_1 + x_2; \quad x_1(0) = 0,\ x_2(0) = 4$

Find the general solution of the given system.

13. $x_1' = 4x_1 + x_2 + 4x_3, \quad x_2' = x_1 + 7x_2 + x_3, \quad x_3' = 4x_1 + x_2 + 4x_3$

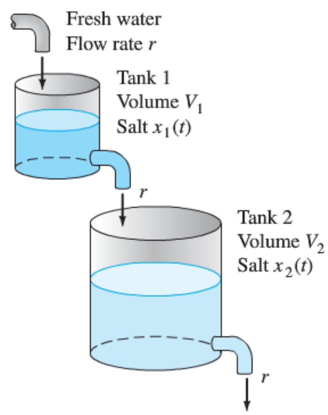
14. $x_1' = x_1 + 2x_2 + 2x_3, \quad x_2' = 2x_1 + 7x_2 + x_3, \quad x_3' = 2x_1 + x_2 + 7x_3$

15. $x_1' = 4x_1 + x_2 + x_3, \quad x_2' = x_1 + 4x_2 + x_3, \quad x_3' = x_1 + x_2 + 4x_3$

16. $x_1' = 5x_1 + x_2 + 3x_3, \quad x_2' = x_1 + 7x_2 + x_3, \quad x_3' = 3x_1 + x_2 + 5x_3$

17. $x_1' = 5x_1 - 6x_3, \quad x_2' = 2x_1 - x_2 - 2x_3, \quad x_3' = 4x_1 - 2x_2 - 4x_3$

18. $x_1' = 3x_1 + 2x_2 + 2x_3, \quad x_2' = -5x_1 - 4x_2 - 2x_3, \quad x_3' = 5x_1 + 5x_2 + 3x_3$

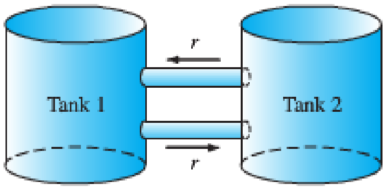


Find the amount $x_1(t), x_2(t)$ of salt in each tank at time $t \ge 0$, with $x_1(0) = 15\ lb \quad x_2(0) = 0$. If

19. $V_1 = 50\ gal, \quad V_2 = 25\ gal, \quad r = 10\ gal/\min$

20. $V_1 = 25\ gal, \quad V_2 = 40\ gal, \quad r = 10\ gal/\min$

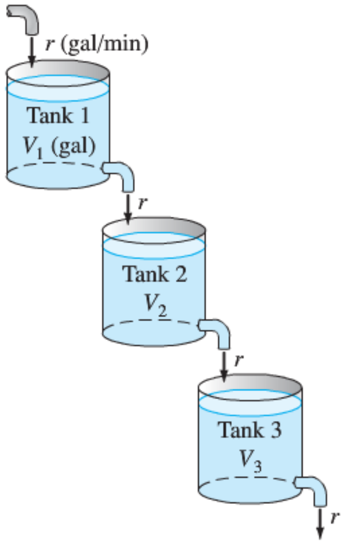
Find the amount $x_1(t),\ x_2(t)$ of salt in each tank at time $t \geq 0$, with $x_1(0) = 15\ lb \quad x_2(0) = 0$. If



21. $V_1 = 50\ gal, \quad V_2 = 25\ gal, \quad r = 10\ gal/\min$

22. $V_1 = 25\ gal, \quad V_2 = 40\ gal, \quad r = 10\ gal/\min$

Find the amount $x_1(t),\ x_2(t),\ x_3(t)$ of salt in each tank at time $t \geq 0$, if



23. $V_1 = 30\ gal, \quad V_2 = 15\ gal, \quad V_3 = 10\ gal, \quad r = 30\ gal/\min$

$x_1(0) = 27\ lb \quad x_2(0) = x_3(0) = 0$

24. $V_1 = 20\ gal, \quad V_2 = 30\ gal, \quad V_3 = 60\ gal, \quad r = 60\ gal/\min$

$x_1(0) = 45\ lb \quad x_2(0) = x_3(0) = 0$

25. $V_1 = 15\ gal, \quad V_2 = 10\ gal, \quad V_3 = 30\ gal, \quad r = 60\ gal/\min$

$x_1(0) = 45\ lb \quad x_2(0) = x_3(0) = 0$

Section 4.4 – Second-Order System & Mechanical Applications

Second-Order Homogeneous Linear systems

Theorem

Let matrix $A\ (n \times n)$, If A has distinct negative eigenvalues $-\omega_1^2,\ -\omega_2^2,\ \ldots,\ -\omega_n^2$ with associated real eigenvalues $\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_n$, then a general solution of

$$\vec{x}'' = A\vec{x}$$

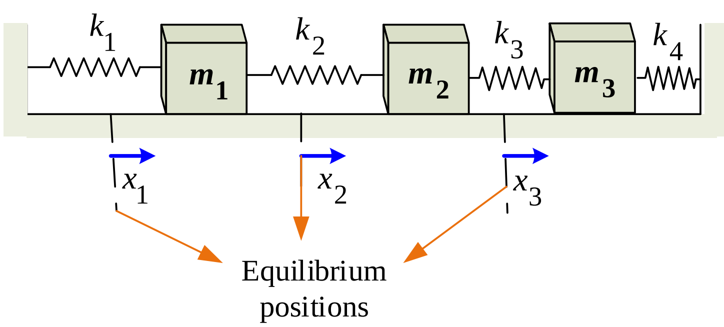
Is given by
$$\vec{x}(t) = \sum_{i=1}^{n} \left(a_i \cos \omega_i t + b_i \sin \omega_i t\right)\vec{v}_i$$

With a_i and b_i arbitrary constants.

In the special case of a nonrepeated zero eigenvalue λ_0 with associated eigenvector $\vec{v}_0$

$$\vec{x}_0(t) = \left(a_0 + b_0 t\right)\vec{v}_0$$

Example



Consider the mass-and–spring systems, as shown above. Three masses connected to each other and to two walls by 4 indicated springs. Assume the masses slide without friction and each spring obeys Hooke's law $(F = -kx)$.

By applying Newton's law $F = ma$ to the 3-masses:

$$m_1 x_1'' = -k_1 x_1 \qquad\qquad + k_2\left(x_2 - x_1\right)$$
$$m_2 x_2'' = -k_2\left(x_2 - x_1\right) + k_3\left(x_3 - x_2\right)$$
$$m_3 x_3'' = -k_3\left(x_3 - x_2\right) - k_4 x_3$$

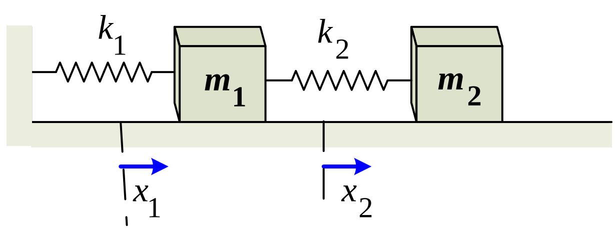
The displacement vector: $\quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

The mass matrix $$M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}$$

The stiffness matrix $$K = \begin{pmatrix} -k_1 - k_2 & k_2 & 0 \\ k_2 & -k_2 - k_3 & k_3 \\ 0 & k_3 & -k_3 - k_4 \end{pmatrix}$$

Example

Consider the mass-and-spring system.



Where $m_1 = 2$, $m_2 = 1$, $k_1 = 100$, $k_2 = 50$ and $M\vec{x}'' = K\vec{x}$

Solution

$$\begin{cases} m_1 x_1'' = -k_1 x_1 + k_2 \left(x_2 - x_1\right) \\ m_2 x_2'' = -k_2 \left(x_2 - x_1\right) \end{cases} \to \begin{cases} m_1 x_1'' = \left(-k_1 - k_2\right) x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - k_2 x_2 \end{cases}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} x'' = \begin{pmatrix} -150 & 50 \\ 50 & -50 \end{pmatrix} \vec{x} \qquad M = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow M^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$$

$$x'' = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -150 & 50 \\ 50 & -50 \end{pmatrix} \vec{x} \qquad M^{-1} M\vec{x}'' = M^{-1} K\vec{x}$$

$$= \begin{pmatrix} -75 & 25 \\ 50 & -50 \end{pmatrix} \vec{x} \qquad \vec{x}'' = \boldsymbol{A}\vec{x}$$

$$A = \begin{pmatrix} -75 & 25 \\ 50 & -50 \end{pmatrix}$$

$$\begin{aligned} \left|A - \lambda I\right| &= \begin{vmatrix} -75 - \lambda & 25 \\ 50 & -50 - \lambda \end{vmatrix} \\ &= \left(-75 - \lambda\right)\left(-50 - \lambda\right) - 1250 \\ &= \lambda^2 + 125\lambda + 2500 = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 = -100$, $\lambda_2 = -25$

By the theorem, the natural frequencies: $\omega_1 = 10 \quad and \quad \omega_2 = 5$

For $\lambda_1 = -100 \quad \Rightarrow \quad (A + 100I)V_1 = 0$

$$\begin{pmatrix} 25 & 25 \\ 50 & 50 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad a = -b \qquad \rightarrow V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\lambda_2 = -25 \quad \Rightarrow \quad (A + 25I)V_2 = 0$

$$\begin{pmatrix} -50 & 25 \\ 50 & -25 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad 2a = b \qquad \rightarrow V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The free oscillation of the mass-and-spring system, follows by:

$$\vec{x}(t) = (a_1 \cos 10t + b_1 \sin 10t)V_1 + (a_2 \cos 5t + b_2 \sin 5t)V_2$$

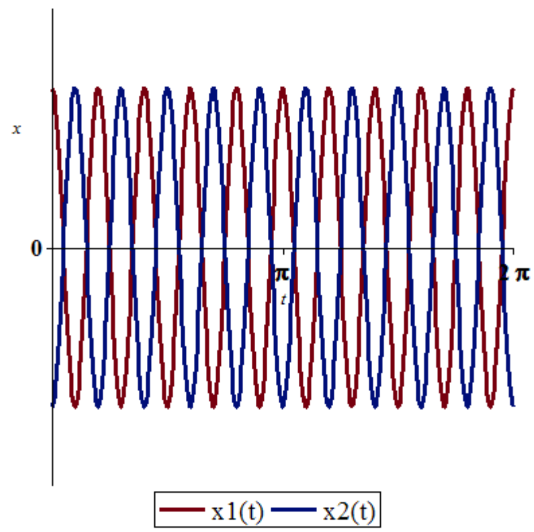
The natural mode:

$$\vec{x}_1(t) = (a_1 \cos 10t + b_1 \sin 10t)V_1$$
$$= c_1 \cos(10t - \alpha_1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Where $c_1 = \sqrt{a_1^2 + b_1^2}$; $\cos\alpha_1 = \frac{a_1}{c_1} \quad \sin\alpha_1 = \frac{b_1}{c_1}$

Which has the scalar equations:

$$\begin{cases} x_1(t) = c_1 \cos(10t - \alpha_1) \\ x_2(t) = -c_1 \cos(10t - \alpha_1) \end{cases}$$



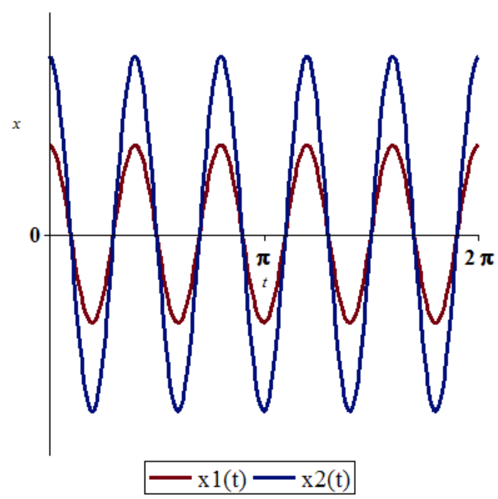
The second part:

$$\vec{x}_2(t) = (a_2 \cos 5t + b_2 \sin 5t)V_2$$
$$= c_2 \cos(5t - \alpha_2)\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Where $c_2 = \sqrt{a_2^2 + b_2^2}$; $\cos\alpha_2 = \frac{a_2}{c_2} \quad \sin\alpha_2 = \frac{b_2}{c_2}$

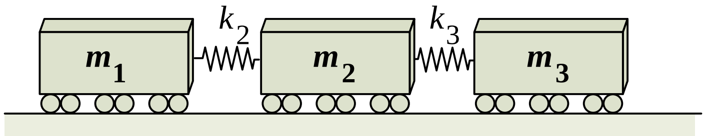
Which has the scalar equations:

$$\begin{cases} x_1(t) = c_2 \cos(5t - \alpha_2) \\ x_2(t) = 2c_2 \cos(5t - \alpha_2) \end{cases}$$



Example

Three railway cars are connected by buffer springs that react when compressed, but disengage instead of stretching.



Given that $k_2 = k_3 = k = 3000\ lb/ft$ and $m_1 = m_3 = 750\ lbs$ *and* $m_2 = 500\ lbs$

Suppose that the leftmost car is moving to the right with velocity v_0 and at time $t=0$ strikes the other 2 cars. The corresponding initial conditions are:

$$x_1(0) = x_2(0) = x_3(0) = 0$$
$$x_1'(0) = v_0 \quad x_2'(0) = x_3'(0) = 0$$

Solution

$$m_1 x_1'' = k_2(x_2 - x_1)$$
$$m_2 x_2'' = -k_2(x_2 - x_1) + k_3(x_3 - x_2)$$
$$m_3 x_3'' = -k_3(x_3 - x_2)$$

$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \vec{x}'' = \begin{pmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{pmatrix} \vec{x}$$

$$\begin{pmatrix} 750 & 0 & 0 \\ 0 & 500 & 0 \\ 0 & 0 & 750 \end{pmatrix} \vec{x}'' = \begin{pmatrix} -3000 & 3000 & 0 \\ 3000 & -6000 & 3000 \\ 0 & 3000 & -3000 \end{pmatrix} \vec{x} \qquad \begin{pmatrix} 750 & 0 & 0 \\ 0 & 500 & 0 \\ 0 & 0 & 750 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{750} & 0 & 0 \\ 0 & \frac{1}{500} & 0 \\ 0 & 0 & \frac{1}{750} \end{pmatrix}$$

$$\vec{x}'' = \begin{pmatrix} \frac{1}{750} & 0 & 0 \\ 0 & \frac{1}{500} & 0 \\ 0 & 0 & \frac{1}{750} \end{pmatrix} \begin{pmatrix} -3000 & 3000 & 0 \\ 3000 & -6000 & 3000 \\ 0 & 3000 & -3000 \end{pmatrix} \vec{x}$$

$$= \begin{pmatrix} -4 & 4 & 0 \\ 6 & -12 & 6 \\ 0 & 4 & -4 \end{pmatrix} \vec{x} \qquad A = \begin{pmatrix} -4 & 4 & 0 \\ 6 & -12 & 6 \\ 0 & 4 & -4 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -4-\lambda & 4 & 0 \\ 6 & -12-\lambda & 6 \\ 0 & 4 & -4-\lambda \end{vmatrix}$$

$$= (-4-\lambda)^2(-12-\lambda) - 24(-4-\lambda) - 24(-4-\lambda)$$

$$=(-4-\lambda)\left[48+16\lambda+\lambda^2-48\right]$$
$$=\lambda(-4-\lambda)(\lambda+16)=0$$

The eigenvalues are: $\lambda_1=0\to\omega_1=0,\quad \lambda_2=-4\to\omega_2=2,\quad \lambda_3=-16\to\omega_3=4$

For $\lambda_1=0\ (\omega_1=0)\ \Rightarrow\ (A-0I)V_1=0$

$$\begin{pmatrix}-4&4&0\\6&-12&6\\0&4&-4\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\ \Rightarrow\ \begin{matrix}a=b\\ \\ b=c\end{matrix}\ \to V_1=\begin{pmatrix}1\\1\\1\end{pmatrix}\ \Rightarrow\vec{x}_1(t)=\left(a_1+b_1t\right)\begin{pmatrix}1\\1\\1\end{pmatrix}$$

For $\lambda_2=-4\ (\omega_2=2)\ \Rightarrow\ (A+4I)V_2=0$

$$\begin{pmatrix}0&4&0\\6&-8&6\\0&4&0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\ \Rightarrow\ \begin{matrix}b=0\\ a=-c\end{matrix}\ \to V_2=\begin{pmatrix}1\\0\\-1\end{pmatrix}\ \Rightarrow\vec{x}_2(t)=\left(a_2\cos 2t+b_2\sin 2t\right)\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

For $\lambda_3=-16\ (\omega_3=4)\ \Rightarrow\ (A+16I)V_3=0$

$$\begin{pmatrix}12&4&0\\6&4&6\\0&4&12\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\ \Rightarrow\ \begin{matrix}3a=-b\\ \\ b=-3c\end{matrix}\ \to V_3=\begin{pmatrix}1\\-3\\1\end{pmatrix}\ \Rightarrow\vec{x}_3(t)=\left(a_3\cos 4t+b_3\sin 4t\right)\begin{pmatrix}1\\-3\\1\end{pmatrix}$$

$$\vec{x}(t)=a_1\begin{pmatrix}1\\1\\1\end{pmatrix}+b_1t\begin{pmatrix}1\\1\\1\end{pmatrix}+a_2\begin{pmatrix}1\\0\\-1\end{pmatrix}\cos 2t+b_2\begin{pmatrix}1\\0\\-1\end{pmatrix}\sin 2t+a_3\begin{pmatrix}1\\-3\\1\end{pmatrix}\cos 4t+b_3\begin{pmatrix}1\\-3\\1\end{pmatrix}\sin 4t$$

$$\begin{cases}\vec{x}_1(t)=a_1+b_1t+a_2\cos 2t+b_2\sin 2t+a_3\cos 4t+b_3\sin 4t\\ \vec{x}_2(t)=a_1+b_1t-3a_3\cos 4t-3b_3\sin 4t\\ \vec{x}_3(t)=a_1+b_1t-a_2\cos 2t-b_2\sin 2t+a_3\cos 4t+b_3\sin 4t\end{cases}$$

Applying the initial values

$$\vec{x}_1(0)=a_1+a_2+a_3=0$$
$$\vec{x}_2(0)=a_1-3a_3=0 \qquad a_1=3a_3 \qquad \Rightarrow\ \underline{a_1=a_2=a_3=0}$$
$$\vec{x}_3(0)=a_1-a_2+a_3=0\quad (1)\,\&\,(3)\to 2a_1+2a_3=0$$

$$\begin{cases}\vec{x}_1(t)=b_1t+b_2\sin 2t+b_3\sin 4t\\ \vec{x}_2(t)=b_1t-3b_3\sin 4t\\ \vec{x}_3(t)=b_1t-b_2\sin 2t+b_3\sin 4t\end{cases}$$

$$\begin{cases}\vec{x}_1'(t)=b_1+2b_2\cos 2t+4b_3\cos 4t\\ \vec{x}_2'(t)=b_1-12b_3\cos 4t\\ \vec{x}_3'(t)=b_1-2b_2\cos 2t+4b_3\cos 4t\end{cases}$$

$$\begin{cases} \vec{x}_1'(0) = b_1 + 2b_2 + 4b_3 = v_0 \\ \vec{x}_2'(0) = b_1 \qquad -12b_3 = 0 \\ \vec{x}_3'(0) = b_1 - 2b_2 + 4b_3 = 0 \end{cases} \to \begin{matrix} b_1 = 12b_3 \\ 2b_2 = 16b_3 \end{matrix} \to 12b_3 + 16b_3 + 4b_3 = v_0 \Rightarrow \begin{matrix} b_3 = \frac{1}{32}v_0 \\ b_1 = \frac{3}{8}v_0 \\ b_2 = \frac{1}{4}v_0 \end{matrix}$$

$$\begin{cases} \vec{x}_1(t) = \frac{1}{32}v_0(12t + 8\sin 2t + \sin 4t) \\ \vec{x}_2(t) = \frac{1}{32}v_0(12t \qquad -3\sin 4t) \\ \vec{x}_3(t) = \frac{1}{32}v_0(12t - 8\sin 2t + \sin 4t) \end{cases} \qquad \begin{cases} \vec{x}_1'(t) = \frac{1}{32}v_0(12 + 16\cos 2t + 4\cos 4t) \\ \vec{x}_2'(t) = \frac{1}{32}v_0(12 - 12\cos 4t) \\ \vec{x}_3'(t) = \frac{1}{32}v_0(12 - 16\cos 2t + 4\cos 4t) \end{cases}$$

For these equations to hold, only when the 2 buffer springs remain compressed; that is, while both

$$x_2 - x_1 < 0 \quad and \quad x_3 - x_2 < 0$$

$$\begin{aligned} x_2(t) - x_1(t) &= \tfrac{1}{32}v_0(12t - 3\sin 4t) - \tfrac{1}{32}v_0(12t + 8\sin 2t + \sin 4t) \\ &= \tfrac{1}{32}v_0(-8\sin 2t - 4\sin 4t) \\ &= -\tfrac{1}{8}v_0(2\sin 2t + 2\sin 2t\cos 2t) \\ &= -\tfrac{1}{4}v_0\sin 2t(1 + \cos 2t) < 0 \end{aligned}$$

$$\sin 2t = 0 \Rightarrow (2t = 0, \pi) \to \boxed{t = 0, \ \tfrac{\pi}{2}} \quad \cos 2t = -1 \to (2t = \pi) \to \boxed{t = \tfrac{\pi}{2}}$$

$$x_2 - x_1 < 0 \ \Rightarrow \ t \in \left(0, \ \tfrac{\pi}{2}\right)$$

$$\begin{aligned} x_3(t) - x_2(t) &= \tfrac{1}{32}v_0(12t - 8\sin 2t + \sin 4t) - \tfrac{1}{32}v_0(12t - 3\sin 4t) \\ &= \tfrac{1}{32}v_0(-8\sin 2t + 4\sin 4t) \\ &= -\tfrac{1}{8}v_0(2\sin 2t - 2\sin 2t\cos 2t) \\ &= -\tfrac{1}{4}v_0(\sin 2t)(1 - \cos 2t) < 0 \end{aligned}$$

$$\sin 2t = 0 \Rightarrow (2t = 0, \pi) \to \boxed{t = 0, \ \tfrac{\pi}{2}} \quad \cos 2t = 1 \to (2t = 0) \to \boxed{t = 0}$$

$$x_3 - x_2 < 0 \ \Rightarrow \ t \in \left(0, \ \tfrac{\pi}{2}\right)$$

$$x_2 - x_1 < 0 \quad and \quad x_3 - x_2 < 0 \text{ until } \boxed{t = \tfrac{\pi}{2} \approx 1.57 \ sec}$$

$$x_1\left(\tfrac{\pi}{2}\right) = x_2\left(\tfrac{\pi}{2}\right) = x_3\left(\tfrac{\pi}{2}\right) = \tfrac{1}{32}v_0\left(12\tfrac{\pi}{2}\right) \boxed{= \tfrac{3\pi}{16}v_0}$$

$$x_1'\left(\tfrac{\pi}{2}\right) = x_2'\left(\tfrac{\pi}{2}\right) = 0 \quad x_3'\left(\tfrac{\pi}{2}\right) = \tfrac{1}{32}v_0(32) \boxed{= v_0}$$

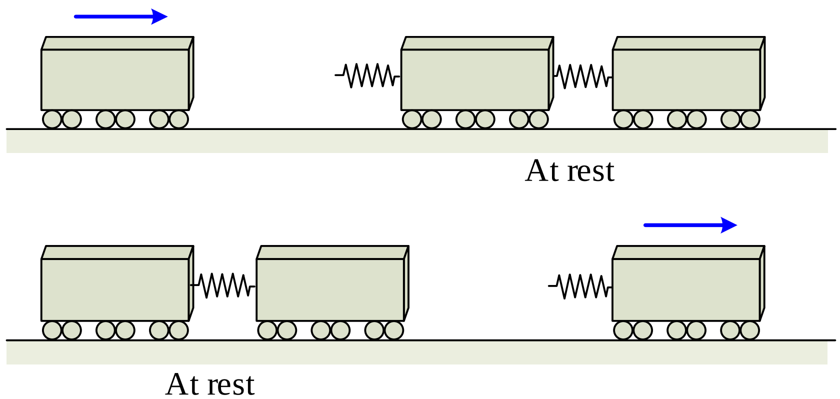
We conclude that the 3 railway cars remain engaged and moving to the right until disengagement occurs at time $t = \frac{\pi}{2}$.

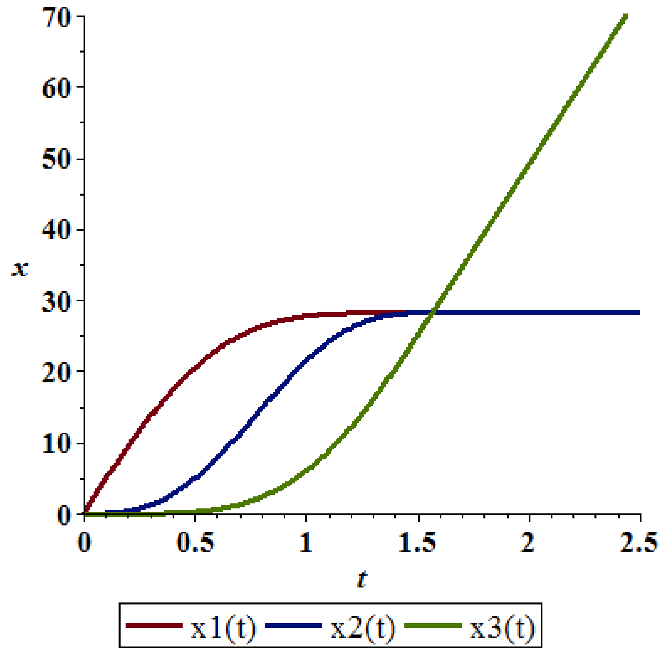
At $t > \frac{\pi}{2}$

$$x_1(t) = x_2(t) = \frac{3\pi}{16} v_0$$

$$\frac{3\pi}{16} v_0 = v_0 \left(\frac{\pi}{2} - \beta\right) \quad \rightarrow \quad \beta = \frac{\pi}{2} - \frac{3\pi}{16} = \frac{5\pi}{16}$$

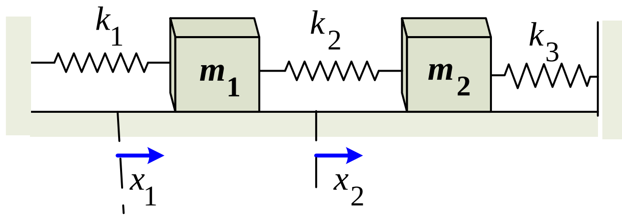
$$x_3(t) = v_0 \left(t - \frac{5\pi}{16}\right) = v_0 t - \frac{5\pi}{16} v_0$$





Exercises Section 4.4 – Second-Order System & Mechanical Applications

Consider the mass-and-spring system shown below and with the given masses and spring constants values.



Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

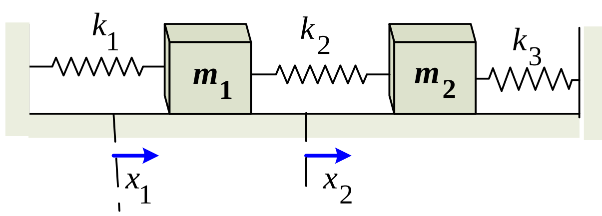
1. $m_1 = m_2 = 1; \quad k_1 = 0,\ k_2 = 2,\ k_3 = 0 \quad (no\ walls)$

2. $m_1 = m_2 = 1; \quad k_1 = 1,\ k_2 = 2,\ k_3 = 1$

3. $m_1 = m_2 = 1; \quad k_1 = 2,\ k_2 = 1,\ k_3 = 2$

4. $m_1 = 1,\ m_2 = 2; \quad k_1 = 2,\ k_2 = k_3 = 4$

Consider the mass-and-spring system shown below and with the given masses and spring constants values.



The mass-and-spring system is set in motion from rest $x_1'(0) = x_2'(0) = 0$ in its equilibrium position $x_1(0) = x_2(0) = 0$.

Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

For the given external forces $F_1(t)$ and $F_2(t)$ acting on the masses m_1 and m_2, respectively.

Find the resulting motion of the system and describe it as a superposition of oscillations at three different frequencies.

5. $m_1 = m_2 = 1; \quad k_1 = 1,\ k_2 = 4,\ k_3 = 1 \quad F_1(t) = 96\cos 5t,\ F_2(t) = 0$

6. $m_1 = 1,\ m_2 = 2; \quad k_1 = 1,\ k_2 = k_3 = 2; \quad F_1(t) = 0,\ F_2(t) = 120\cos 3t$

7. $m_1 = m_2 = 1; \quad k_1 = 4,\ k_2 = 6,\ k_3 = 4; \quad F_1(t) = 30\cos t,\ F_2(t) = 60\cos t$

8. Consider a mass-and-spring system containing two masses $m_1 = m_2 = 1$ whose displacement functions $x(t)$ and $y(t)$ satisfy the differential equations

$$\begin{aligned} x'' &= -40x + 8y \\ y'' &= 12x - 60y \end{aligned}$$

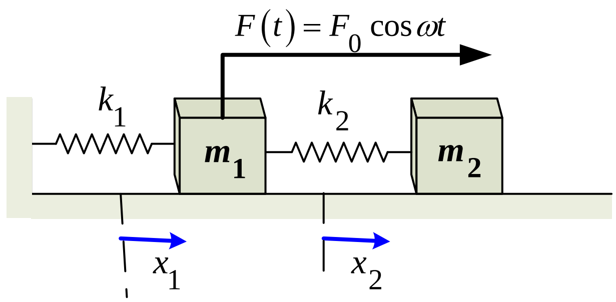
a) Describe the two fundamental modes of free oscillation of the system.

b) Assume that the two masses start in motion with the initial conditions

$$x(0)=19,\ x'(0)=12 \quad \textit{and} \quad y(0)=3,\ y'(0)=6$$

And are acted on by the same force, $F_1(t)=F_2(t)=-195\cos 7t$. Describe the resulting motion as a superposition of oscillations at three different frequencies.

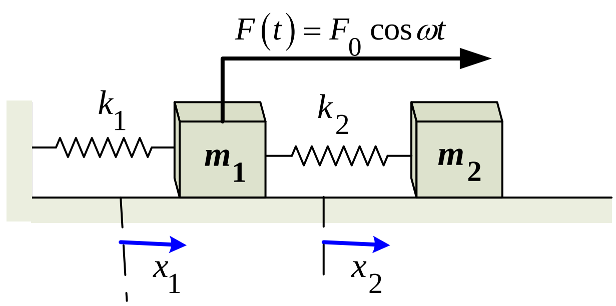
9. Consider a mass-and-spring system shown below. Assume that $m_1=1;\ \ k_1=50;\ \ F_0=5$ in mks units, and that $\omega=10$. Then find m_2 so that in the resulting steady periodic oscillations, the mass m_1 will remain at rest (!).



Thus the effect of the second mass-and-spring pair will be to neutralize the effect of the force on the first mass. This is an example of a dynamic damper. It has an electrical analogy that some cable companies use to prevent your reception of certain cable channels.

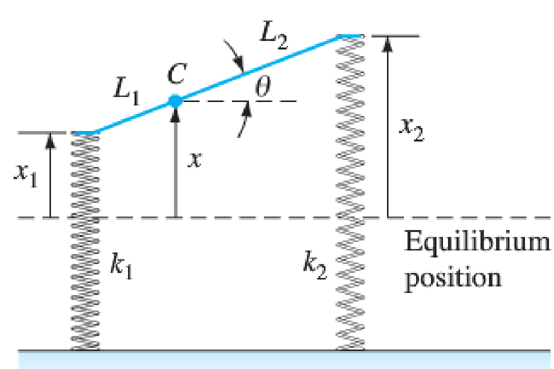
10. Consider a mass-and-spring system shown below. Assume that

$$m_1=2,\ m_2=\frac{1}{2};\quad k_1=75,\ k_2=25;\quad F_0=100 \quad \textit{and} \quad \omega=10 \text{ (in mks units).}$$



Find the solution of the system $M\vec{x}''=K\vec{x}+F$ that satisfies the initial conditions $\vec{x}(0)=\vec{x}'(0)=\mathbf{0}$

11. A car with two axles and with separate front and rear suspension systems.



We assume that the car body acts as would a solid bar of mass m and length $L = L_1 + L_2$. It has moment of inertia I about its center of mass C, which is at distance L_1 from the front of the car. The car has front and back suspension springs with Hooke's constants k_1 and k_2, respectively. When the car is in motion, let $x(t)$ denote the vertical displacement of the center of mass of the car from equilibrium; let $\theta(t)$ denote its angular displacement (in radians) from the horizontal. Then Newton's laws of motion for linear and angular acceleration can be used to derive the equations.

$$mx'' = -(k_1 + k_2)x + (k_1L_1 - k_2L_2)\theta$$
$$I\theta'' = (k_1L_1 - k_2L_2)x - \left(k_1L_1^2 + k_2L_2^2\right)\theta$$

Suppose that $m = 75$ *slugs* (the car weighs 2400 *lb*), $L_1 = 7\,ft$, $L_2 = 3\,ft$ (it's a rear engine car), $k_1 = k_2 = 2000\ lb/ft$, and $I = 1000\ ft.lb.s^2$.

a) Find the two natural frequencies ω_1 and ω_2 of the car.

b) Now suppose that the car is driven at a speed of v $ft/\sec$ along a washboard surface shaped like a sine curve with a wavelength of 40 *ft*. The result is a periodic force on the car with frequency $\omega = \frac{2\pi}{40}v = \frac{\pi}{20}v$. Resonance occurs when $\omega = \omega_1$ *or* $\omega = \omega_2$. Find the corresponding two critical speeds of the car (in *ft/sec*)

The system is taken as a model for an undamped car with the given parameters in *fps* units.

a) Find the two natural frequencies ω_1 and ω_2 of the car (in hertz).

b) Assume that his car is driven along a sinusoidal washboard surface with a wavelength of 40 *ft*. The result is a periodic force on the car with frequency $\omega = \frac{2\pi}{40}v = \frac{\pi}{20}v$. Resonance occurs when $\omega = \omega_1$ *or* $\omega = \omega_2$. Find the corresponding two critical speeds of the car (in *ft/sec*)

12. $m = 100;\quad I = 800;\quad L_1 = L_2 = 5;\quad k_1 = k_2 = 2000$

13. $m = 100;\quad I = 1000;\quad L_1 = 6,\ L_2 = 4;\quad k_1 = k_2 = 2000$

14. $m = 100;\quad I = 800;\quad L_1 = L_2 = 5;\quad k_1 = 1000,\ k_2 = 2000$

Section 4.5 – Multiple Eigenvalues Solutions

Matrix $A\ (n\times n)$ has n distinct (real or complex) eigenvalues $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$ with respective eigenvectors $\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_n$, then a general solution of the system is given by

$$\vec{x}(t)=c_1\vec{v}_1e^{\lambda_1 t}+c_2\vec{v}_2e^{\lambda_2 t}+\ \ldots+c_n\vec{v}_ne^{\lambda_n t}$$

When the characteristic equation $|A-\lambda I|=0$ doesn't have n distinct roots, and thus has at least one repeated root.

An eigenvalue is of multiplicity $k>1$ if it is a k-fold root. For each eigenvalue λ, the eigenvector equation

$$(A-\lambda I)V=0$$

has at least one nonzero solution V, so there is at least one eigenvector with λ.

Example

Find a general solution of the system

$$x'=\begin{bmatrix}9 & 4 & 0\\ -6 & -1 & 0\\ 6 & 4 & 3\end{bmatrix}x$$

Solution

The characteristic equation:

$$\begin{aligned}|A-\lambda I|&=\begin{vmatrix}9-\lambda & 4 & 0\\ -6 & -1-\lambda & 0\\ 6 & 4 & 3-\lambda\end{vmatrix}\\
&=(9-\lambda)(-1-\lambda)(3-\lambda)+24(3-\lambda)\\
&=(3-\lambda)\left[-9-8\lambda+\lambda^2+24\right]\\
&=(3-\lambda)\left(\lambda^2-8\lambda+15\right)\\
&=(3-\lambda)^2(5-\lambda)=0\end{aligned}$$

The distinct eigenvalues are: $\lambda_1=5,\quad \lambda_{2,3}=3\ (\textit{repeated})$ of multiplicity $k=2$.

For $\lambda_1=5 \;\Rightarrow\; (A-5I)V_1=0$

$$\begin{pmatrix}4 & 4 & 0\\ -6 & -6 & 0\\ 6 & 4 & -2\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \;\Rightarrow\; \begin{matrix}a=-b\\ \\ 6a+4b-2c=0\to c=a\end{matrix} \quad\to V_1=\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}$$

For $\lambda_2=3 \;\Rightarrow\; (A-3I)V_2=0$

$$\begin{pmatrix} 6 & 4 & 0 \\ -6 & -4 & 0 \\ 6 & 4 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \quad 3a = -2b \qquad \to V_2 = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$$

$$\text{If } a = b = 0 \quad then \quad c = 1 \qquad \to V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

V_2 and V_2 are linearly independent eigenvectors.

$$\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} + c_3 \vec{v}_3 e^{\lambda_3 t}$$

$$= c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} e^{3t} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{3t}$$

$$\begin{cases} x_1(t) = c_1 e^{5t} + 2c_2 e^{3t} \\ x_2(t) = -c_1 e^{5t} - 3c_2 e^{3t} \\ x_3(t) = c_1 e^{5t} \qquad\qquad + c_3 e^{3t} \end{cases}$$

Defective Eigenvalues

Example

Find a general solution of the system $A = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix}$

Solution

The characteristic equation:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -3 \\ 3 & 7-\lambda \end{vmatrix}$$

$$= (1-\lambda)(7-\lambda) + 9$$

$$= \lambda^2 - 8\lambda + 16 = 0$$

The eigenvalues are: $\lambda_{1,2} = 4 \quad (multiplicity\ 2)$

For $\lambda = 4 \quad \Rightarrow \quad (A - 4I)V_1 = 0$

$$\begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \quad a = -b \qquad \to V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Since the eigenvalue $\lambda_{1,2} = 4 \ (multiplicity\ 2)$ has only one independent eigenvector, and hence is incomplete.

An eigenvalue λ of multiplicity $k>1$ is called ***defective*** if it is not complete.
If λ has only $p<k$ linearly independent eigenvectors, then the number

$$d=k-p$$

of ***missing*** eigenvectors is called the defect of the defective eigenvalue λ.

Defective Multiplicity 2 Eigenvalues

1. First find a nonzero solution $\vec{v}_2$ of the equation

$$(A-\lambda I)^2\vec{v}_2=\vec{0} \quad \textit{such that} \quad (A-\lambda I)\vec{v}_2=\vec{v}_1$$

is nonzero, and therefore is an eigenvectors $\vec{v}_1$ associated with λ.

2. Then from the two independent solutions

$$\vec{x}_1(t)=\vec{v}_1e^{\lambda t} \quad \textit{and} \quad \vec{x}_2(t)=\left(\vec{v}_1t+\vec{v}_2\right)e^{\lambda t}$$

Example

Find a general solution of the system $A=\begin{bmatrix}1 & -3\\ 3 & 7\end{bmatrix}$

Solution

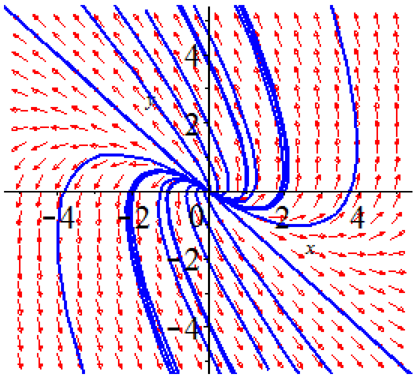
$$|A-\lambda I|=\begin{vmatrix}1-\lambda & -3\\ 3 & 7-\lambda\end{vmatrix}=(1-\lambda)(7-\lambda)+9$$

$$=\lambda^2-8\lambda+16=0$$ The eigenvalues are: $\lambda_{1,2}=4$ $\left(\textit{multiplicity } 2\right)$

$$(A-4I)^2=\begin{pmatrix}-3 & -3\\ 3 & 3\end{pmatrix}\begin{pmatrix}-3 & -3\\ 3 & 3\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

Since $(A-\lambda I)^2\vec{v}_2=\vec{0} \Rightarrow \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\vec{v}_2=\vec{0}$ and $\vec{v}_2$ is a nonzero vector, we can let $\vec{v}_2=\begin{pmatrix}1\\ 0\end{pmatrix}$

$$(A-4I)\vec{v}_2=\vec{v}_1 \quad \Rightarrow \quad \begin{pmatrix}-3 & -3\\ 3 & 3\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}-3\\ 3\end{pmatrix}=\vec{v}_1$$



$$\begin{cases}\vec{x}_1(t)=\begin{pmatrix}-3\\ 3\end{pmatrix}e^{4t}\\ \vec{x}_2(t)=\left(\begin{pmatrix}-3\\ 3\end{pmatrix}t+\begin{pmatrix}1\\ 0\end{pmatrix}\right)e^{4t}\end{cases} \rightarrow \begin{cases}\vec{x}_1(t)=\begin{pmatrix}-3\\ 3\end{pmatrix}e^{4t}\\ \vec{x}_2(t)=\begin{pmatrix}-3t+1\\ 3t\end{pmatrix}e^{4t}\end{cases}$$

The general solution: $\vec{x}(t)=c_1\vec{x}_1(t)+c_2\vec{x}_2(t)$

$$\begin{cases}x_1(t)=\left(-3c_2t+c_2-3c_1\right)e^{4t}\\ x_2(t)=\left(3c_2t+3c_1\right)e^{4t}\end{cases}$$

Generalized Eigenvectors

If λ is an eigenvalue of the matrix $\boldsymbol{A}$, then a rank r generalized eigenvector $\vec{v}$ such that

$$(A-\lambda I)^{r}\,\vec{v}=\vec{0} \quad but \quad (A-\lambda I)^{r-1}\vec{v}\neq\vec{0}$$

$$\begin{cases} \vec{x}_1(t)=\vec{v}_1 e^{\lambda t} \\ \vec{x}_2(t)=\left(\vec{v}_1 t+\vec{v}_2\right)e^{\lambda t} \\ \vec{x}_3(t)=\left(\frac{1}{2}\vec{v}_1 t^2+\vec{v}_2 t+\vec{v}_3\right)e^{\lambda t} \\ \quad\vdots \quad\quad \vdots \quad\quad\quad\quad \vdots \\ \vec{x}_k(t)=\left(\frac{\vec{v}_1}{(k-1)!}t^{k-1}+\cdots+\frac{\vec{v}_{k-2}}{2!}t^2+\vec{v}_{k-1}t+\vec{v}_k\right)e^{\lambda t} \end{cases}$$

Example

Find three linearly independent solutions of the system $\boldsymbol{x}'=\begin{bmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{bmatrix}\boldsymbol{x}$

Solution

$$|A-\lambda I|=\begin{vmatrix} -\lambda & 1 & 2 \\ -5 & -3-\lambda & -7 \\ 1 & 0 & -\lambda \end{vmatrix}$$

$$=\lambda^2(-3-\lambda)-7-2(-3-\lambda)-5\lambda$$

$$=-\lambda^3-3\lambda^2-3\lambda-1$$

$$=-(\lambda+1)^3=0$$

The eigenvalues are $\lambda_{1,2,3}=-1 \quad of \;\; multiplicity\; 3$

For $\lambda=-1 \;\Rightarrow\; (A+I)V=0$

$$\begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} a+b+2c=0\to b=a \\ \\ a=-c \end{array} \to V=a\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad (a\neq 0)$$

The defect of $\lambda=-1$ is 2.

To apply the method for triple eigenvalues, then

$$(A+I)^2=\begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix}=\begin{pmatrix} -2 & -1 & -3 \\ -2 & -1 & -3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$(A+I)^3=\begin{pmatrix}1&1&2\\-5&-2&-7\\1&0&1\end{pmatrix}\begin{pmatrix}-2&-1&-3\\-2&-1&-3\\2&1&3\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

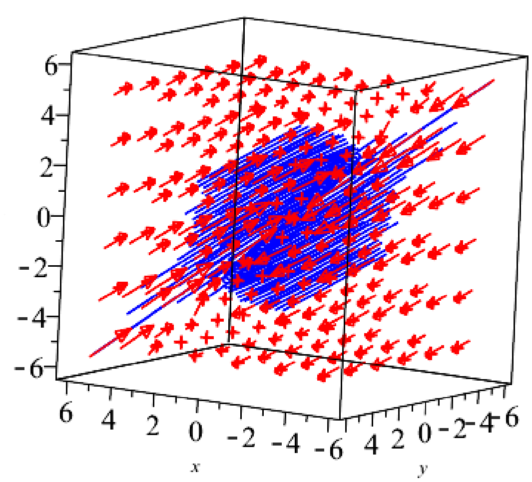
Since $(A+I)^3\vec{v}_3=0$, therefore any nonzero vector $\vec{v}_3=\begin{bmatrix}1&0&0\end{bmatrix}^T$ will be a solution.

$$\vec{v}_2=(A+I)\vec{v}_3=\begin{pmatrix}1&1&2\\-5&-2&-7\\1&0&1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\-5\\1\end{pmatrix}$$

$$\vec{v}_1=(A+I)\vec{v}_2=\begin{pmatrix}1&1&2\\-5&-2&-7\\1&0&1\end{pmatrix}\begin{pmatrix}1\\-5\\1\end{pmatrix}=\begin{pmatrix}-2\\-2\\2\end{pmatrix}$$

$$\begin{cases}\vec{x}_1(t)=\vec{v}_1e^{-t}\\ \vec{x}_2(t)=\left(\vec{v}_1t+\vec{v}_2\right)e^{-t}\\ \vec{x}_3(t)=\left(\frac{1}{2}\vec{v}_1t^2+\vec{v}_2t+\vec{v}_3\right)e^{-t}\end{cases}\rightarrow\begin{cases}\vec{x}_1(t)=\begin{pmatrix}-2\\-2\\2\end{pmatrix}e^{-t}\\ \vec{x}_2(t)=\left(\begin{pmatrix}-2\\-2\\2\end{pmatrix}t+\begin{pmatrix}1\\-5\\1\end{pmatrix}\right)e^{-t}\\ \vec{x}_3(t)=\left(\frac{1}{2}\begin{pmatrix}-2\\-2\\2\end{pmatrix}t^2+\begin{pmatrix}1\\-5\\1\end{pmatrix}t+\begin{pmatrix}1\\0\\0\end{pmatrix}\right)e^{-t}\end{cases}$$

$$\begin{cases}\vec{x}_1(t)=\begin{pmatrix}-2\\-2\\2\end{pmatrix}e^{-t}\\ \vec{x}_2(t)=\begin{pmatrix}-2t+1\\-2t-5\\2t+1\end{pmatrix}e^{-t}\\ \vec{x}_3(t)=\begin{pmatrix}-2t^2+t+1\\-t^2-5t\\t^2+t\end{pmatrix}e^{-t}\end{cases}$$



The general solution:

$$\vec{x}(t)=c_1\vec{x}_1(t)+c_2\vec{x}_2(t)+c_3\vec{x}_3(t)$$

$$\begin{cases}x_1(t)=\left(-c_3t^2+\left(c_3-2c_2\right)t+c_3+c_2-2c_1\right)e^{-t}\\ x_2(t)=\left(-c_3t^2-\left(5c_3+2c_2\right)t-5c_2-2c_1\right)e^{-t}\\ x_3(t)=\left(c_3t^2+\left(c_3+2c_2\right)t+c_2+2c_1\right)e^{-t}\end{cases}$$

Example

Suppose that the matrix $\boldsymbol{A}$ (6×6) has two multiplicity 3 eigenvalues $\lambda_1 = -2$ *and* $\lambda_2 = 3$ with defects 1 and 2, respectively.

Then λ_1 must have an eigenvector $\vec{u}_1$ and a length 2 chain $\{\vec{v}_1, \vec{v}_2\}$ of generalized eigenvectors.

$\left(\vec{u}_1 \text{ and } \vec{v}_1 \text{ are L.I}\right)$

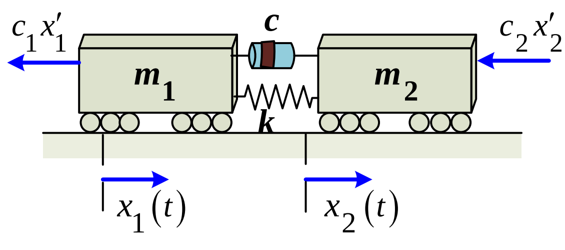
And λ_2 must have a length 3 chain $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ of generalized eigenvectors.

The six eigenvectors $\vec{u}_1, \vec{v}_1, \vec{v}_2, \vec{w}_1, \vec{w}_2, \vec{w}_3$ are then L.I and yield the following 6 independent solutions.

$$\begin{cases} \vec{x}_1(t) = \vec{u}_1 e^{-2t} \\ \vec{x}_2(t) = \vec{v}_1 e^{-2t} \\ \vec{x}_3(t) = \left(\vec{v}_1 t + \vec{v}_2\right) e^{-2t} \\ \vec{x}_4(t) = \vec{w}_1 e^{3t} \\ \vec{x}_5(t) = \left(\vec{w}_1 t + \vec{w}_2\right) e^{3t} \\ \vec{x}_6(t) = \left(\frac{1}{2}\vec{w}_1 t^2 + \vec{w}_2 t + \vec{w}_3\right) e^{3t} \end{cases}$$

Example

Two railway cars that are connected with a spring (permanently attached to both cars) and with a damper that exerts opposite forces on the two cars, of magnitude $c\left(x_1' - x_2'\right)$ proportional to their relative velocity. The two cars are also subject to frictional resistance forces $c_1 x_1'$ and $c_2 x_2'$ proportional to their respective velocities.



Let $m_1 = m_2 = c = 1$ *and* $c_1 = c_2 = k = 2$

Solution

The equations of motion:

$$\begin{cases} m_1x_1'' = k\left(x_2 - x_1\right) - c_1x_1' - c\left(x_1' - x_2'\right) \\ m_2x_2'' = k\left(x_1 - x_2\right) - c_2x_2' - c\left(x_2' - x_1'\right) \end{cases}$$

The equations can be written in the form: $Mx'' = Kx + Rx'$

where $R = \begin{bmatrix} -\left(c+c_1\right) & c \\ c & -\left(c+c_2\right) \end{bmatrix}$ is the ***resistance*** matrix.

To use the equations as a 1[st]-order system, let assume $x_3(t) = x_1'(t)$ *and* $x_4(t) = x_2'(t)$

$$\begin{cases} x_1' = x_3 \\ x_2' = x_4 \\ x_3' = -kx_1 + kx_2 - \left(c_1 + c\right)x_3 + cx_4 \\ x_4' = kx_1 - kx_2 + cx_3 - \left(c_2 + c\right)x_4 \end{cases} \rightarrow \begin{cases} x_1' = x_3 \\ x_2' = x_4 \\ x_3' = -2x_1 + 2x_2 - 3x_3 + x_4 \\ x_4' = 2x_1 - 2x_2 + x_3 - 3x_4 \end{cases}$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{pmatrix}$$

$$\begin{aligned} \left|A - \lambda I\right| &= \begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -2 & 2 & -3-\lambda & 1 \\ 2 & -2 & 1 & -3-\lambda \end{vmatrix} \\ &= -\lambda\left[-\lambda(-3-\lambda)^2 + 2 + 2(-3-\lambda) + \lambda\right] - 2\lambda - 2\lambda(-3-\lambda) \\ &= -\lambda\left(-9\lambda - 6\lambda^2 - \lambda^3 - 4 - \lambda\right) + 4\lambda + 2\lambda^2 \\ &= \lambda^4 + 6\lambda^3 + 12\lambda^2 + 8\lambda \\ &= \lambda\left(\lambda^3 + 6\lambda^2 + 12\lambda + 8\right) \\ &= \lambda(\lambda + 2)^3 = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 = 0$ and $\lambda_{2,3,4} = -2$ (***triple***)

For $\lambda_1 = 0 \quad \Rightarrow \quad \left(A - 0I\right)V_1 = 0$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} & c = 0 \\ & d = 0 \\ & -2a + 2b = 0 \rightarrow a = b \end{aligned} \quad \rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda_2 = -2 \quad \Rightarrow \quad \left(A + 2I\right)V_2 = 0$

$$\begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} 2a = -c \\ 2b = -d \\ -2a+2b-c+d=0 \\ 2a-2b+c-3d=0 \end{array}$$

$$\text{Let } a=1 \Rightarrow c=-2 \quad b=0 \Rightarrow d=0 \quad \to \vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

$$\text{Let } a=0 \Rightarrow c=0 \quad b=1 \Rightarrow d=-2 \quad \to \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \end{pmatrix}$$

$$\vec{w}_1 = \vec{u}_1 + \vec{u}_2 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \\ -2 \end{pmatrix}$$

$$(A+2I)^2 \vec{v}_2 = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -3 \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -3 \end{pmatrix}\vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \to 2a_2 + 2b_2 + c_2 + d_2 = 0 \quad \to \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\underline{|\vec{v}_1} = (A+2I)\vec{v}_2 = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \\ 2 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = \vec{v}_1 e^{0t} \\ \vec{x}_2(t) = \vec{w}_1 e^{-2t} \\ \vec{x}_3(t) = \vec{v}_2 e^{-2t} \\ \vec{x}_4(t) = \left(\vec{v}_2 t + \vec{v}_3\right) e^{-2t} \end{cases} \to \begin{cases} \vec{x}_1(t) = [1 \quad 1 \quad 0 \quad 0]^T \\ \vec{x}_2(t) = [1 \quad 1 \quad -2 \quad -2]^T \; e^{-2t} \\ \vec{x}_3(t) = [1 \quad -1 \quad -2 \quad 2]^T \; e^{-2t} \\ \vec{x}_4(t) = [t \quad -t \quad -2t+1 \quad 2t-1]^T \; e^{-2t} \end{cases}$$

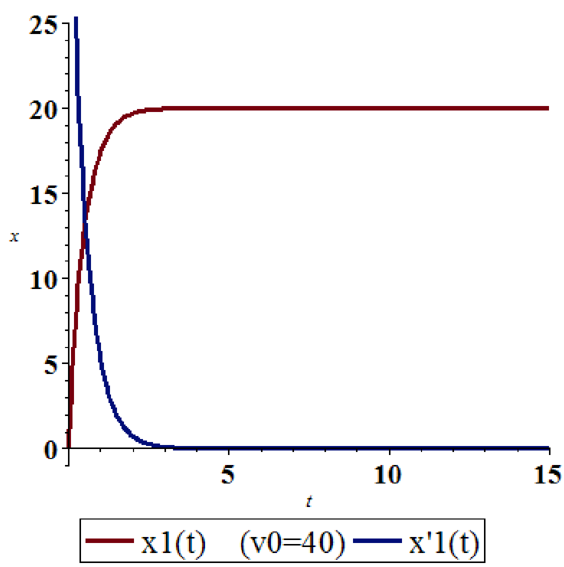
The general solution: $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + c_4\vec{x}_4(t)$

$$\begin{cases} x_1(t) = c_1 + \left(c_2 + c_3 + c_4 t\right)e^{-2t} \\ x_2(t) = c_1 + \left(c_2 - c_3 - c_4 t\right)e^{-2t} \\ x_3(t) = \left(-2c_2 - 2c_3 + c_4 - 2c_4 t\right)e^{-2t} \\ x_4(t) = \left(-2c_2 + 2c_3 - c_4 + 2c_4 t\right)e^{-2t} \end{cases}$$

Recall that $x_3(t) = x_1'(t)$, $x_4(t) = x_2'(t)$ and since the position of the 2 cars in initial positon at rest, so $x_1(0) = x_2(0) = 0$ with initial velocity of $x_1'(0) = x_2'(0) = v_0$

$$\begin{cases} x_1(0) = c_1 + c_2 + c_3 = 0 \\ x_2(0) = c_1 + c_2 - c_3 = 0 \\ x_3(0) = x_1'(0) = -2c_2 - 2c_3 + c_4 = v_0 \\ x_4(0) = x_2'(0) = -2c_2 + 2c_3 - c_4 = v_0 \end{cases} \qquad \begin{aligned} c_1 &= -c_2 \\ c_3 &= 0 \\ c_2 &= -\tfrac{1}{2}v_0 \\ c_4 &= 0 \end{aligned}$$

$$\begin{cases} x_1(t) = x_2(t) = \frac{1}{2}v_0\left(1 - e^{-2t}\right) \\ x_1'(t) = x_2'(t) = v_0 e^{-2t} \end{cases}$$



Diagonalization

Suppose the n by n matrix $\boldsymbol{A}$ has n linearly independent eigenvectors $x_1, \ldots, x_n$. Put them into the column of an ***eigenvector matrix P***. Then $P^{-1}AP$ is the eigenvalue matrix $\boldsymbol{A}$:

$$P^{-1}AP = \Lambda = D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Definition

A square matrix A is called ***diagonalizable*** if there is an invertible matrix P such that $P^{-1}AP$ is diagonal; the matrix P is said to ***diagonalize*** A.

Theorem

Independent x from different λ - Eigenvectors $x_1, \ldots, x_n$ that correspond to distinct (all different) eigenvalues are linearly independent. An n by n matrix that has n different eigenvalues (no repeated λ's) must be diagonalizable.

The *Jordan* Form

For every A, we want to choose M so that $M^{-1}AM$ is as nearly diagonal as possible. When A has a full set of n eigenvectors, they go into the columns of M. Then $M = P$. The matrix $P^{-1}AP$ is diagonal.

If A has $\boldsymbol{s}$ independent eigenvectors, it is similar to a matrix J that has $\boldsymbol{s}$ Jordan blocks on its diagonal. There is a matrix M such that

$$M^{-1}AM = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix} = J$$

Each block in J has one eigenvalue λ_i, one eigenvector, and 1's above the diagonal:

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

A is similar to B if they share the same Jordan form J – not otherwise.

Similar Matrices

Definition

If A and B are square matrices, then we say that ***B* is similar to *A*** if there exists an invertible matrix P such that $B = P^{-1}AP \quad or \quad A = PBP^{-1}$

Example

Jordan matrix J has triple eigenvalues 5, 5, 5. Its only eigenvectors are multiples of (1, 0, 0). Algebraic multiplicity 3, geometric multiplicity 1:

$$\text{If } J = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \quad then \quad J - 5I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ has rank 2.}$$

Every similar matrix $B = M^{-1}JM$ has the same triple eigenvalues 5, 5, 5. Also $B - 5I$ must have the same rank 2. Its nullspace has dimension 3 – 2 = 1. So each similar matrix B also has only one independent eigenvector.

The transpose matrix J^T has the same eigenvalues 5, 5, 5, and $J^T - 5I$ has the same rank 2. ***Jordan's theory says that J^T is similar to J***. The matrix that produces the similarity happens to be the reserve identity M:

$$J^T = M^{-1}JM \quad is \quad \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

There is one line of eigenvectors $(x_1,\ 0,\ 0)$ for J and another line $(0,\ 0,\ x_3)$ for J^T.

Example

Find Jordan form of the matrix $A = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -3 \\ 3 & 7-\lambda \end{vmatrix} = (1-\lambda)(7-\lambda) + 9$$

$$= \lambda^2 - 8\lambda + 16 = 0 \quad \text{The eigenvalues are: } \lambda_{1,2} = 4 \quad (multiplicity\ 2)$$

$$(A - 4I)^2 = \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Since $(A - \lambda I)^2 \vec{v}_2 = \vec{0} \quad \Rightarrow \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \vec{v}_2 = \vec{0}$ and $\vec{v}_2$ is a nonzero vector, we can let $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$(A-4I)\vec{v}_2=\vec{v}_1 \quad\Rightarrow\quad \begin{pmatrix}-3 & -3\\ 3 & 3\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}-3\\3\end{pmatrix}=\vec{v}_1$$

$$Q=\left[\vec{v}_1 \quad \vec{v}_2\right]=\begin{pmatrix}-3 & 1\\ 3 & 0\end{pmatrix} \quad\rightarrow\quad Q^{-1}=\begin{pmatrix}0 & \frac{1}{3}\\ 1 & 1\end{pmatrix}$$

$$\begin{aligned}\boldsymbol{J}=Q^{-1}AQ&=\begin{pmatrix}0 & \frac{1}{3}\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & -3\\ 3 & 7\end{pmatrix}\begin{pmatrix}-3 & 1\\ 3 & 0\end{pmatrix}\\ &=\begin{pmatrix}0 & \frac{1}{3}\\ 1 & 1\end{pmatrix}\begin{pmatrix}-12 & 1\\ 12 & 3\end{pmatrix}\\ &=\begin{pmatrix}4 & 1\\ 0 & 4\end{pmatrix}\end{aligned}$$

$\boldsymbol{J}=\boldsymbol{J}_1$ is a single 2×2 Jordan block corresponding to the single eigenvalue $\lambda=4$ of *A*.

The General Cayley-Hamilton Theorem

Every diagonalizable matrix A satisfies its characteristic equation $p(\lambda)=|A-\lambda I|=0$ $\left(p(A)=0\right)$. Using Jordan normal form to show that this is true whether or not *A* is diagonalizable.

If $\boldsymbol{J}=Q^{-1}AQ \quad\Rightarrow\quad p(A)=Q^{-1}p(\boldsymbol{J})Q$

If the Jordan blocks $\boldsymbol{J}_1,\ \boldsymbol{J}_2,\ \ldots,\ \boldsymbol{J}_s$ have sizes $k_1,\ k_2,\ \ldots,\ k_s$ that is $\boldsymbol{J}_i$ $\left(k_i\times k_i\right)$ matrix and the corresponding eigenvalues $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_s$ *respectively*, then

$$p(\lambda)=\left(\lambda_1-\lambda\right)^{k_1}\left(\lambda_2-\lambda\right)^{k_2}\ \ldots\left(\lambda_s-\lambda\right)^{k_s}$$

$$\rightarrow\qquad p(\boldsymbol{J})=\left(\lambda_1 I-\boldsymbol{J}\right)^{k_1}\left(\lambda_2 I-\boldsymbol{J}\right)^{k_2}\ \ldots\left(\lambda_s I-\boldsymbol{J}\right)^{k_s}$$

$p(\boldsymbol{J})$ has the same block-diagonal structure as $\boldsymbol{J}$ itself

$$\left(\lambda_i I-\boldsymbol{J}_i\right)^{k_i}=\begin{bmatrix}0 & -1 & 0 & \cdots & 0\\ 0 & 0 & -1 & \cdots & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & \cdots & \cdots & 0 & -1\\ 0 & \cdots & \cdots & \cdots & 0\end{bmatrix}^{k_i}$$

Exercises *Section* 4.5 – Multiple Eigenvalues Solutions

Find the general solutions

1. $\boldsymbol{x}' = \begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix}\boldsymbol{x}$

2. $\boldsymbol{x}' = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\boldsymbol{x}$

3. $\boldsymbol{x}' = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}\boldsymbol{x}$

4. $\boldsymbol{x}' = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}\boldsymbol{x}$

5. $\boldsymbol{x}' = \begin{bmatrix} 2 & 0 & 0 \\ -7 & 9 & 7 \\ 0 & 0 & 2 \end{bmatrix}\boldsymbol{x}$

6. $\boldsymbol{x}' = \begin{bmatrix} 25 & 12 & 0 \\ -18 & -5 & 0 \\ 6 & 6 & 13 \end{bmatrix}\boldsymbol{x}$

7. $\boldsymbol{x}' = \begin{bmatrix} -3 & 0 & -4 \\ -1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}\boldsymbol{x}$

8. $\boldsymbol{x}' = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}\boldsymbol{x}$

9. $\boldsymbol{x}' = \begin{bmatrix} 0 & 0 & 1 \\ -5 & -1 & -5 \\ 4 & 1 & -2 \end{bmatrix}\boldsymbol{x}$

10. $\boldsymbol{x}' = \begin{bmatrix} 39 & 8 & -16 \\ -36 & -5 & 16 \\ 72 & 16 & -29 \end{bmatrix}\boldsymbol{x}$

11. $\boldsymbol{x}' = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}\boldsymbol{x}$

12. $\boldsymbol{x}' = \begin{bmatrix} -1 & -4 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}\boldsymbol{x}$

13. The characteristic equation of the coefficient matrix $\boldsymbol{A}$ of the system

$$\boldsymbol{x}' = \begin{bmatrix} 3 & -4 & 1 & 0 \\ 4 & 3 & 0 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{bmatrix}\boldsymbol{x} \qquad \text{is} \quad p(\lambda) = \left(\lambda^2 - 6\lambda + 25\right)^2 = 0$$

Therefore, $\boldsymbol{A}$ has the repeated complex pair $3 \pm 4i$ of eigenvalues. First show that the complex vectors $\vec{v}_1 = \begin{bmatrix} 1 & i & 0 & 0 \end{bmatrix}^T$ *and* $\vec{v}_2 = \begin{bmatrix} 0 & 0 & 1 & i \end{bmatrix}^T$ form a length 2 chain $\{\vec{v}_1, \vec{v}_2\}$ associated with the eigenvalue $\lambda = 3 - 4i$. Then calculate the real and imaginary parts of the complex-valued solutions

$$\vec{v}_1 e^{\lambda t} \quad \textit{and} \quad \left(\vec{v}_1 t + \vec{v}_2\right)e^{\lambda t}$$

To find four independent real-valued solutions of $\boldsymbol{x}' = A\boldsymbol{x}$