Lecture Four - Exponential and Logarithmic Functions

Section 4.1 – Inverse Functions

Inverse Relations

Interchanging the first and second coordinates of each ordered pair in a relation produces the inverse relation.

If a relation is defined by an equation, interchanging the variables produces an equation of the inverse relation

Given the relation: $\{(Zambia,4.2),(Columbia,4.5),(Poland,3.3),(Italy,3.3),(US,2.5)\}$

Inverse Relation: $\{(4.2,Zambia),(4.5,Columbia),(3.3,Poland),(3.3,Italy),(2.5,US)\}$

Example

Consider the relation g given by: $G=\{(2,4),(-1,3),(-2,0)\}$

Solution

The inverse relation: $G=\{(4,2),(3,-1),(0,-2)\}$

Example

Consider the relation given by: $F=\{(-2,2),(-1,1),(0,0),(1,3),(2,5)\}$

Solution

The inverse relation: $G=\{(2,-2),(1,-1),(0,0),(3,1),(5,2)\}$

One-to-One Functions

A function *f* is one-to-one (1 – 1) if different inputs have different outputs that is,

$$if\ \ a \neq b, \quad then \quad f(a) \neq f(b)$$

A function *f* is one-to-one (1 – 1) if different outputs the same, the inputs are the same – that is,

$$if\ \ f(a) = f(b), \quad then\ \ a = b$$

Example

Given the function *f* described by $f(x) = 2x - 3$, prove that *f* is one-to-one.

Solution

$f(a) = f(b)$

$2a - 3 = 2b - 3$ *Add 3 on both sides*

$2a = 2b$ *Divide by 2*

$a = b$ f is one-to-one

Example

Given the function *f* described by $f(x) = -4x + 12$, prove that *f* is one-to-one.

Solution

$f(a) = f(b)$

$-4a + 12 = -4b + 12$ *Subtract 12 from both sides*

$-4a = -4b$ *Divide by -4*

$a = b$

Example

Given the function *f* described by $f(x) = x^2$, prove that *f* is one-to-one.

Solution

$-1 \neq 1$

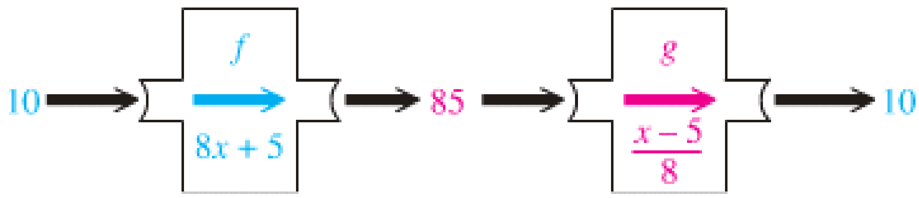
$\begin{cases} f(-1) = 1 \\ f(1) = 1 \end{cases} \Rightarrow f(-1) = f(1)$ f is *not* one-to-one

Definition of the Inverse of a Function

Let f and g be two functions such that

$$f(g(x)) = x \qquad and \qquad g(f(x)) = x$$

$$x \underset{g=f^{-1}}{\overset{f}{\rightleftarrows}} f(x) \qquad\qquad g(f(x)) = f^{-1}(f(x)) = x$$



If the inverse of a function f is also a function, it is named f^{-1} read "f - inverse"

The -1 in f^{-1} is not an exponent! And is not equal to $\cancel{\frac{1}{f(x)}}$

***Domain* and *Range* of** f *and* f^{-1}

$$domain\ of\ f^{-1} = range\ of\ f$$

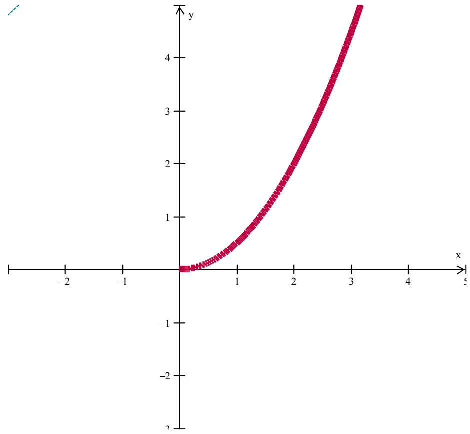
$$range\ of\ f^{-1} = domain\ of\ f$$

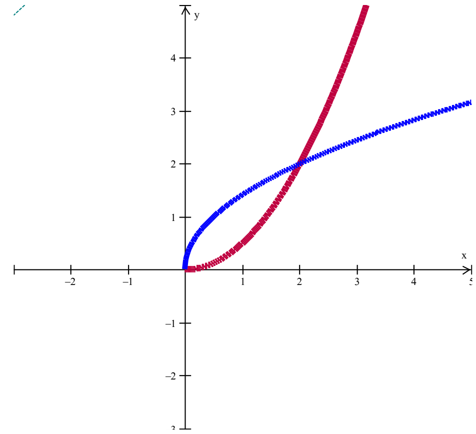
If a function f is one-to-one, then f^{-1} is the unique function such that each of the following holds.

$\boxed{\left(f^{-1} \circ f\right)(x) = f^{-1}(f(x)) = x}$ for each x in the domain of f, and

$\boxed{\left(f \circ f^{-1}\right)(x) = f(f^{-1}(x)) = x}$ for each x in the domain of f^{-1}

The condition that f is one-to-one in the definition of inverse function is important; otherwise, g will not define a function





Example

Let $f(x) = x^3 - 1$ and $g(x) = \sqrt[3]{x+1}$, is g the inverse function of f?

Solution

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\sqrt[3]{x+1}\right) \\ &= \left(\sqrt[3]{x+1}\right)^3 - 1 \\ &= x+1-1 \\ &= x\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g\left(x^3 - 1\right) \\ &= \sqrt[3]{x^3 - 1 + 1} \\ &= \sqrt[3]{x^3} \\ &= x\end{aligned}$$

g is the inverse function of f

Example

Show that each function is the inverse of the other: $f(x) = 4x - 7$ *and* $g(x) = \dfrac{x+7}{4}$

Solution

$$\begin{aligned}f(g(x)) &= f\left(\frac{x+7}{4}\right) \\ &= 4\left(\frac{x+7}{4}\right) - 7 \\ &= x + 7 - 7 \\ &= x\end{aligned}$$

$$\begin{aligned}g(f(x)) &= g(4x-7) \\ &= \frac{4x-7+7}{4} \\ &= \frac{4x}{4} \\ &= x\end{aligned}$$

Finding the *Inverse Function*

	Example
Finding an Inverse Function	$f(x) = 2x + 7$
1. Replace $f(x)$ with y	$y = 2x + 7$
2. Interchange x and y	$x = 2y + 7$
3. Solve for y	$x - 7 = 2y$
	$\frac{x-7}{2} = y$
4. Replace y with $f^{-1}(x)$	$f^{-1}(x) = \frac{x-7}{2}$

Example

Find the inverse of $f(x) = 4x^3 - 1$

Solution

$$y = 4x^3 - 1$$

$$x = 4y^3 - 1$$

$$x + 1 = 4y^3$$

$$\frac{x+1}{4} = y^3$$

$$y = \left(\frac{x+1}{4}\right)^{1/3}$$

$$= \boxed{\sqrt[3]{\frac{x+1}{4}} = f^{-1}(x)}$$

Example

Find a formula for the inverse $f(x)=\frac{5x-3}{2x+1}$

Solution

$$y=\frac{5x-3}{2x+1}$$

$$x=\frac{5y-3}{2y+1}$$

$$x(2y+1)=5y-3$$

$$2xy+x=5y-3$$

$$2xy-5y=-x-3$$

$$y(2x-5)=-x-3$$

$$y=\frac{-x-3}{2x-5}$$

$$\boxed{f^{-1}(x)=-\frac{x+3}{2x-5}}$$

Exercise   *Section* 4.1 – Inverse Functions

Determine whether the function is one-to-one

1. $f(x)=3x-7$

2. $f(x)=x^2-9$

3. $f(x)=\sqrt{x}$

4. $f(x)=\sqrt[3]{x}$

5. $f(x)=|x|$

Prove that the given function *f* is one-to-one

6. $f(x)=\dfrac{2}{x+3}$

7. $f(x)=(x-2)^3$

8. $y=x^2+2$

9. $f(x)=\dfrac{x+1}{x-3}$

10. Let $f(x)=x^3-1$ and $g(x)=\sqrt[3]{x+1}$, is g the inverse function of *f*?

11. Given that $f(x)=5x+8$, use composition of functions to show that $f^{-1}(x)=\dfrac{x-8}{5}$

12. Given the function $f(x)=(x+8)^3$

a) Find $f^{-1}(x)$

b) Graph f and f^{-1} in the same rectangular coordinate system

c) Find the domain and the range of f and f^{-1}

Prove the f *and* g are inverse functions of each other, and sketch the graphs of f *and* g

13. $f(x)=3x-2 \quad g(x)=\dfrac{x+2}{3}$

14. $f(x)=x^2+5, x\le 0 \quad g(x)=-\sqrt{x-5}, \ \ x\ge 5$

15. $f(x)=x^3-4; \quad g(x)=\sqrt[3]{x+4}$

Determine the domain and range of f^{-1}(Hint: first find the domain and range of *f*)

16. $f(x)=-\dfrac{2}{x-1}$

17. $f(x)=\dfrac{5}{x+3}$

18. $f(x)=\dfrac{4x+5}{3x-8}$

Find the inverse function of

19. $f(x)=3x+5$

20. $f(x)=\dfrac{1}{3x-2}$

21. $f(x)=\dfrac{3x+2}{2x-5}$

22. $f(x)=\dfrac{4x}{x-2}$

23. $f(x)=2-3x^2; \quad x\le 0$

24. $f(x)=2x^3-5$

25. $f(x)=\sqrt{3-x}$

26. $f(x)=\sqrt[3]{x}+1$

27. $f(x)=\left(x^3+1\right)^5$

28. $f(x)=x^2-6x; \quad x\ge 3$

29. $f(x)=(x-2)^3$

30. $f(x)=\dfrac{x+1}{x-3}$

31. $f(x)=\dfrac{2x+1}{x-3}$