Math 2318 – Linear Algebra **Exam 4** **Review**

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1. Find the standard matrix for the operator T defined by the formula

a) $T(x_1,x_2,x_3,x_4)=(7x_1+2x_2-x_3+x_4,\ \ x_2+x_3,\ \ -x_1)$

b) $T(x_1,x_2,x_3)=(2x_1+x_3,\ \ x_1+x_2-x_3,\ \ x_1-x_2+x_3)$

2. Find the characteristic equation, eigenvalues, and eigenvectors of $\begin{pmatrix}10 & -9\\ 4 & -2\end{pmatrix}$

3. Find the characteristic equation, eigenvalues, and eigenvectors of $\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}$

4. Find the eigenvalues, and eigenvectors of $\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$

5. Find the characteristic equation, eigenvalues, and eigenvectors of $\begin{pmatrix}3 & 0 & -5\\ \frac{1}{5} & -1 & 0\\ 1 & 1 & -2\end{pmatrix}$

6. Find the characteristic equation, eigenvalues, and eigenvectors of $\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 2 & -5 & 4\end{pmatrix}$

7. Find a matrix P that diagonalizes $A=\begin{bmatrix}-1 & 4 & -2\\ -3 & 4 & 0\\ -3 & 1 & 3\end{bmatrix}$

8. Let $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, determine when A is diagonalizable, not diagonalizable. (*Hint: discriminant of the characteristic equation*)

9. Show that $A=\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$ and $B=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$ are not similar matrices

10. Show that the matrix $A=\begin{bmatrix}2 & 0\\ 1 & 2\end{bmatrix}$ is not diagonalizable

11. Show that the function $T:\ \mathrm{R}^3\to R^2$ given the formula $T(x_1,\,x_2,\,x_3)=(2x_1-x_2+x_3\ ,\ x_2-4x_3)$ is linear transformation

12. Determine whether the function $T:\ M_{22}\to R$ is linear transformation

a) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 3a - 4b + c - d$

b) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a^2 + b^2$

13. Consider the basis $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ for R^3, where $\boldsymbol{v}_1 = (-2,\ 1)$ $\boldsymbol{v}_2 = (1,\ 3)$ and let $T : R^2 \to R^3$ be the linear transformation for which

$T(\boldsymbol{v}_1) = (-1,\ 2,\ 0)$, $T(\boldsymbol{v}_2) = (0,\ -3,\ 5)$

Find a formula for $T(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$, and then use that formula to compute $T(2, 4, -1)$

Solution

1. $a)\ \begin{pmatrix} 7 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$ $\qquad b)\ \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

1. $\lambda^2 - 8\lambda + 16$ $\quad Eigenvalue:\ \lambda = 4$ $\qquad Eigenvector:\ \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$

2. $\lambda^2 - 6\lambda + 8$ $\quad Eigenvalue:\ \lambda = 2, 4$ $\qquad Eigenvector:\ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

3. $\lambda^2 - 1 = 0$ $\quad Eigenvalue:\ \lambda = \pm 1$ $\qquad Eigenvector:\ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

4. $-\lambda^3 + 2\lambda = 0$ $\quad Eigenvalue:\ \lambda = 0,\ \pm\sqrt{2}$ $\qquad Eigenvector:\ \begin{pmatrix} \frac{5}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix} \begin{pmatrix} -\frac{5}{\sqrt{2}-3} \\ \frac{1}{7}\frac{3+\sqrt{2}}{1+\sqrt{2}} \\ 1 \end{pmatrix} \begin{pmatrix} \frac{5}{3+\sqrt{2}} \\ \frac{1}{7}\frac{3-\sqrt{2}}{1-\sqrt{2}} \\ 1 \end{pmatrix}$

5. $-\lambda^3 + 4\lambda^2 - 5\lambda + 2$ $\quad Eigenvalue:\ \lambda_{1,2} = 1,\ \lambda_3 = 2$ $\quad Eigenvector:\ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

6. $P = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

7. $diagonalizable:\ (a-d)^2 + 4bc > 0,\quad not\ diagonalizable:\ (a-d)^2 + 4bc < 0$

8. $\det(A) = -3 \ \neq\ \det(B) = 3$

9. $a)\ -4 \quad b)\ -16$

10. $(2-\lambda)^2 = 0 \ \rightarrow\ \lambda_{1,2} = 2$ repeated eigenvalues therefore is not diagonalizable

11. Let $\vec{u} = (u_1, u_2, u_3)$ $\ and\ $ $\vec{v} = (v_1, v_2, v_3)$

$$\begin{aligned} T(\boldsymbol{u}+\boldsymbol{v}) &= T(u_1+v_1,\ u_2+v_2,\ u_3+v_3) \\ &= (2u_1 + 2v_1 - u_2 - v_2 + u_3 + v_3,\ u_2 + v_2 - 4u_3 - 4v_3) \\ &= (2u_1 - u_2 + u_3,\ u_2 - 4u_3) + (2v_1 - v_2 + v_3,\ v_2 - 4v_3) \\ &= T(\boldsymbol{u}) + T(\boldsymbol{v}) \end{aligned}$$

$$
\begin{aligned}
T(r\boldsymbol{u}) &= T\left(ru_1,\ ru_2,\ ru_3\right) \\
&= \left(2ru_1 - ru_2 + ru_3,\ \ ru_2 - 4ru_3\right) \\
&= r\left(2u_1 - u_2 + u_3,\ \ u_2 - 4u_3\right) \\
&= rT(\boldsymbol{u})
\end{aligned}
$$

Since $T(\boldsymbol{u}+\boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$ and $T(r\boldsymbol{u}) = rT(\boldsymbol{u})$, then function T is a linear transformation.

12.

a)
$$
\begin{aligned}
T(A+B) &= T\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} \\
&= 3a_1 + 3a_2 - 4b_1 - 4b_2 + c_1 + c_2 - d_1 - d_2 \\
&= \left(3a_1 - 4b_1 + c_1 - d_1\right) + \left(3a_2 - 4b_2 + c_2 - d_2\right) \\
&= T(A) + T(B) \quad \surd
\end{aligned}
$$

$$
\begin{aligned}
T(kA) &= T\left(k\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} ka_1 & kb_1 \\ kc_1 & kd_1 \end{bmatrix}\right) \\
&= 3ka_1 - 4kb_1 + kc_1 - kd_1 \\
&= k\left(3a_1 - 4b_1 + c_1 - d_1\right) \\
&= kT(A) \quad \surd
\end{aligned}
$$

b)
$$
\begin{aligned}
T(A+B) &= T\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} \\
&= \left(a_1 + a_2\right)^2 + \left(b_1 - b_2\right)^2 \\
&= a_1^2 + a_2^2 + 2a_1a_2 + b_1^2 + b_2^2 + 2b_1a_2 \\
&= \left(a_1^2 + b_1^2\right) + \left(a_2^2 + b_2^2\right) + 2a_1a_2 + 2b_1a_2 \\
&= T(A) + T(B) \quad \#
\end{aligned}
$$

It is not a linear transformation

13. $T\left(x_1,\ x_2,\ x_3\right) = \left(-x_1 + 4x_2 - x_3,\ 5x_1 - 5x_2 - x_3,\ x_1 + 3x_3\right)$

$T(2,\ 4,\ -1) = (15,\ -9,\ -1)$