Solution Section 1.5 – Applications and Models

Exercise

A rectangular park is 6 *miles* long and 2 *miles* wide. How long is a pedestrian route that runs diagonally across the park?

Solution

$$d^2 = 6^2 + 2^2$$

$$d^2 = 40$$

$$d = \sqrt{40}$$

$$\approx 6.32 \ \ miles$$

Exercise

What is the width of a 25-*inch* television set whose height is 15 *inches*?

Solution

$$w^2 + 15^2 = 25^2$$

$$w^2 = 25^2 - 15^2$$

$$w = \sqrt{625 - 225}$$

$$= 20 \ \ in$$

Exercise

The length of a rectangular sign is 3 *feet* longer than the width. If the sign's area is 54 square *feet*, find its length and width.

Solution

$$\ell = w + 3$$

$$Area = \ell w = 54$$

$$(w+3)w = 54$$

$$w^2 + 3w - 54 = 0$$

$$w = \frac{-3 \pm \sqrt{9 + 216}}{2} \qquad\qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-3 \pm 15}{2}$$

$$= \begin{cases} \frac{-3-15}{2} = -\times \\ \frac{-3+15}{2} = 6 \end{cases}$$

$$\ell = 6+3$$
$$= 9$$

$$w = \ell - 3$$
$$= 9 - 3$$
$$= 6$$

$\therefore$ the length of sign is **9** *feet* and width is **6** *feet*.

Exercise

A rectangular parking lot has a length that is 3 *yards* greater than the width. The area of the parking lot is 180 square *yards*, find the length and the width.

Solution

$$\ell = w + 3$$
$$Area = \ell w = 180$$
$$(w+3)w = 180$$
$$w^2 + 3w - 180 = 0$$
$$w = \frac{-3 \pm \sqrt{9+720}}{2} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-3 \pm 27}{2}$$
$$= \begin{cases} \frac{-3-27}{2} = -\times \\ \frac{-3+27}{2} = 12 \end{cases}$$

$$\ell = 12 + 3$$
$$= 15$$

$\therefore$ the length of sign is **15** *feet* and width is **12** *feet*.

Exercise

Each side of a square is lengthened by 3 *inches*. The area of this new, larger square is 64 square *inches*. Find the length of a side of the original square.

Solution

The new length of each side of a square is $= x + 3$

$A = (x+3)^2 = 64$

$x + 3 = \pm 8$

$x = -3 \pm 8$

$$= \begin{cases} -3-8 = -\cancel{} \\ -3+8 = 5 \end{cases}$$

$\therefore$ the length of the original square side is **5** *inches*.

Exercise

Each side of a square is lengthened by 2 *inches*. The area of this new, larger square is 36 square *inches*. Find the length of a side of the original square.

Solution

The new length of each side of a square is $= x + 2$

$A = (x+2)^2 = 36$

$x + 2 = \pm 6$

$x = -2 \pm 6$

$$= \begin{cases} -2-6 = -\cancel{} \\ -2+6 = 4 \end{cases}$$

$\therefore$ the length of the original square side is **4** *inches*.

Exercise

One number is 5 greater than another. The product of the numbers is 36. Find the numbers.

Solution

$n = m + 5$

$P = mn = 36$

$m(m+5) = 36$

$m^2 + 5m - 36 = 0$

$$m = \frac{-5 \pm \sqrt{25+144}}{2} \qquad m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-5 \pm 13}{2}$$

$$= \begin{cases} \frac{-5-13}{2} = -9 \\ \frac{-5+13}{2} = 4 \end{cases}$$

$n = -9 + 5 = -4$ $n = 4 + 5 = 9$

$\therefore$ The numbers are **4** & **9** **or** **−4** & **−9**

Exercise

One number is 6 less than another. The product of the numbers is 72. Find the numbers.

Solution

$n = m - 6$

$P = mn = 72$

$m(m-6) = 72$

$m^2 - 6m - 72 = 0$

$$m = \frac{6 \pm \sqrt{36+288}}{2} \qquad\qquad m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{6 \pm 18}{2}$$

$$= \begin{cases} \frac{6-18}{2} = -6 \\ \frac{6+18}{2} = 12 \end{cases}$$

$n = -6 - 6 = -12$

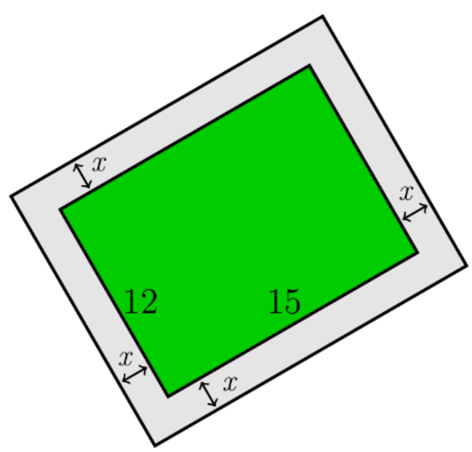
$n = 12 - 6 = 6$

$\therefore$ The numbers are **6** & **12** **or** **−6** & **−12**

Exercise

A vacant rectangular lot is being turned into a community vegetable garden measuring 15 *meters* by 12 *meters*. A path of uniform width is to surround the garden. If the area of the garden and path combined is 378 *square meters*, find the width of the path.

Solution



$Area = (15 + 2x)(12 + 2x)$

$378 = (15 + 2x)(12 + 2x)$

$378 = 180 + 30x + 24x + 4x^2$

$0 = 180 + 54x + 4x^2 - 378$

$0 = 4x^2 + 54x - 198$

$4x^2 + 54x - 198 = 0$

$$x = \frac{-(54) \pm \sqrt{(54)^2 - 4(4)(-198)}}{2(4)}$$

$$= \frac{-54 \pm \sqrt{6084}}{8}$$

$$= \frac{-54 \pm 78}{8}$$

$$= \begin{cases} \frac{-54+78}{8} = 3 \\ \frac{-54-78}{8} = -16.5 \end{cases}$$

∴ the width of the path is **3** *meters*.

Exercise

A pool measuring 10 *m* by 20 *m* is surrounded by a path of uniform width. If the area of the pool and the path combined is 600 m^2, what is the width of the path?

Solution

$A = lw$

$600 = (20 + 2x)(10 + 2x)$

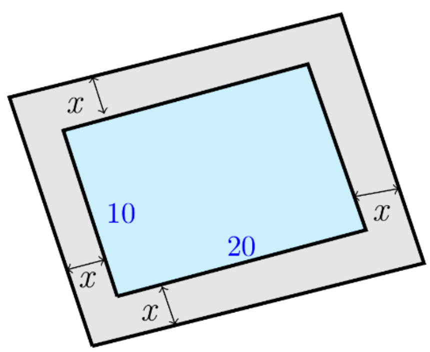
$600 = 200 + 40x + 20x + 4x^2$

$0 = -600 + 200 + 60x + 4x^2$

$0 = -400 + 60x + 4x^2$

$0 = -100 + 15x + x^2$

$x^2 + 15x - 100 = 0$



$$x = \frac{-15 \pm \sqrt{15^2 + 400}}{2(1)}$$

$$= \frac{-15 \pm \sqrt{625}}{2}$$

$$= \begin{cases} \frac{-15-25}{2} = -20 \\ \frac{-15+25}{2} = 5 \end{cases}$$

∴ The width of the path is **5** *m*

Exercise

You put in flower bed measuring 10 *feet* by 12 *feet*. You plan to surround the bed with uniform border of low-growing plants.

a) Write a polynomial that describes the area of the uniform border that surrounds your flowers.

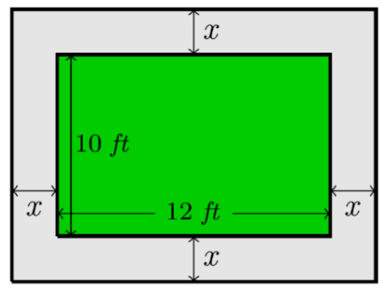
b) The low growing plants surrounding the flower bed require 1 square *foot* each when mature. If you have 168 of these plants, how wide a strip around the flower bed should you prepare for the border?

Solution

a) $Area = 4x^2 + 2(12x) + 2(10x)$

$$= 4x^2 + 44x$$

b) $A = 4x^2 + 44x = 168 \times 1$



$$4x^2 + 44x - 168 = 0$$

$$x^2 + 11x - 42 = 0$$

$$x = \frac{-11 \pm \sqrt{121 + 168}}{2} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-11 \pm 17}{2}$$

$$= \begin{cases} \frac{-11-17}{2} = -\times \\ \frac{-11+17}{2} = 3 \end{cases}$$

∴ The width of the path is **3** *feet*.

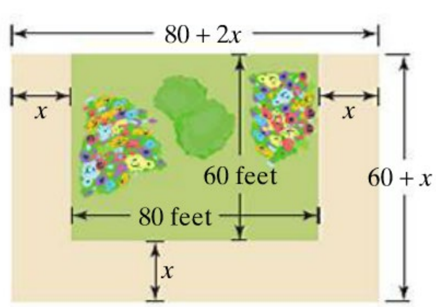
Exercise

A rectangular garden measures 80 *feet* by 60 *feet*. A large path of uniform width is to be added along both shorter sides and one longer side of the garden. The landscape designer doing the work wants to double the garden's area with the addition of this path. How wide should the path be?

Solution

Total Area $= 2 \times (\textit{area of the garden})$

$$(80 + 2x)(60 + x) = 2(60)(80)$$



$$4800 + 200x + 2x^2 = 9600$$

$$2x^2 + 200x - 4800 = 0$$

$$x^2 + 100x - 2400 = 0$$

$$x = \frac{-100 \pm \sqrt{10,000 + 9,600}}{2}$$

$$= \frac{-100 \pm 10\sqrt{196}}{2}$$

$$= \frac{-100 \pm 140}{2}$$

$$= \begin{cases} \frac{-100-140}{2} = -\cancel{\ } \\ \frac{-100+140}{2} = 20 \end{cases}$$

∴ the path should be **20** *feet*.

Exercise

The length of a rectangular poster is 1 *foot* more than the width, and a diagonal of the poster is 5 *feet*. Find the length and the width.

Solution

Given: $\ell = w + 1 \qquad d = 5$

$$\ell^2 + w^2 = d^2$$

$$(w+1)^2 + w^2 = 25$$

$$w^2 + 2w + 1 + w^2 = 25$$

$$2w^2 + 2w - 24 = 0$$

$$w^2 + w - 12 = 0$$

$$w = 3, \ \cancel{-4}$$

$$\ell = 3 + 1 = 4$$

∴ The length is **4** *feet* and the width is **3** *feet*.

Exercise

One leg of a right triangle is 7 *cm* less than the length of the other leg. The length of the hypotenuse is 13 *cm*. find the lengths of the legs.

Solution

Given: $x = y - 7 \qquad d = 13$

$$x^2 + y^2 = d^2$$

$$(y-7)^2 + y^2 = 169$$

$$y^2 - 14y + 49 + y^2 - 169 = 0$$

$$2y^2 - 14y - 120 = 0$$

$$y^2 - 7y - 60 = 0$$

$$y = \frac{7 \pm \sqrt{49+240}}{2}$$

$$= \frac{7 \pm \sqrt{289}}{2}$$

$$= \frac{7 \pm 17}{2}$$

$$= \begin{cases} \frac{7-17}{2} = -\cancel{\ } \\ \frac{7+17}{2} = 12 \end{cases}$$

$y = 12$

$x = 12 - 7 = 5$

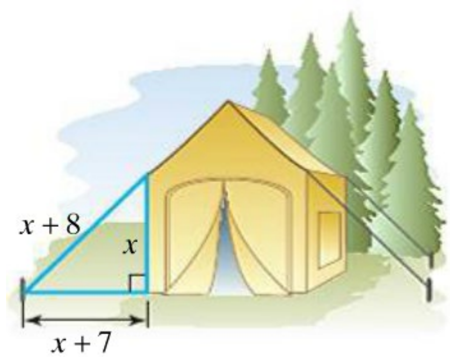
∴ The length of the leg: **5** & **12** *cm*.

Exercise

A tent with wires attached to help stabilize it, as shown below. The length of each wire is 8 *feet* greater than the distance from the ground to where it is attached to the tent.

The distance from the base of the tent to where the wire is anchored exceeds this height by 7 *feet*, Find the length of each wire used to stabilize the tent.

Solution



$$x^2 + (x+7)^2 = (x+8)^2$$

$$x^2 + x^2 + 14x + 49 = x^2 + 16x + 64$$

$$x^2 - 2x - 15 = 0$$

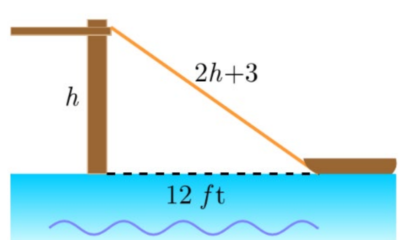
$x = 5,\ \cancel{-3}$

∴ The length of each wire: **5** *feet*, **12** *feet*, and **13** *feet*.

Exercise

A boat is being pulled into a dock with a rope attached to the boat at water level. Where the boat is 12 *feet* from the dock, the length of the rope from the boat to the dock is 3 *feet* longer than twice the height of the dock above the water. Find the height of the dock.

Solution



$$(2h+3)^2 = h^2 + 12^2$$

$$4h^2 + 12h + 9 = h^2 + 144$$

$$4h^2 + 12h + 9 - h^2 - 144 = 0$$

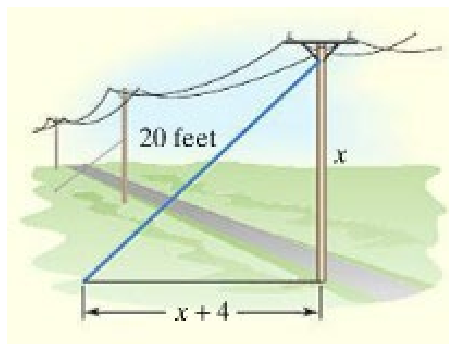
$$3h^2 + 12h - 135 = 0$$
$$h^2 + 4h - 45 = 0$$
$$(h+9)(h-5) = 0$$
$$h = -9, \quad 5$$

Height = 5 *feet*.

Exercise

A piece of wire measuring 20 *feet* is attached to a telephone pole as a guy wire. The distance along the ground from the bottom of the pole to the end of the wire is 4 *feet* greater than the height where the wire is attached to the pole. How far up the pole does the guy wire reach?

Solution



$$(x+4)^2 + x^2 = 20^2$$
$$x^2 + 8x + 16 + x^2 = 400$$
$$2x^2 + 8x - 384 = 0$$
$$x^2 + 4x - 192 = 0$$
$$x = \frac{-4 \pm \sqrt{16 + 768}}{2}$$
$$= \frac{-4 \pm \sqrt{784}}{2}$$
$$= \frac{-4 \pm 28}{2}$$
$$= \begin{cases} \frac{-4-28}{2} = -\cancel{\ } \\ \frac{-4+28}{2} = 12 \end{cases}$$

∴ the guy wire reaches the pole at **12** *feet* high.

Exercise

Logan and Cassidy leave a campsite, Logan biking due north and Cassidy biking due east. Logan bikes 7 *km/h* slower than Cassidy. After 4 *hr*, they are 68 *km* apart. Find the speed of each bicyclist.

Solution

$$4r^2 + [4(r-7)]^2 = 68^2$$
$$16r^2 + 16(r^2 - 14r + 49) = 4624$$
$$16r^2 + 16r^2 - 224r + 784 = 4624$$
$$32r^2 - 224r + 784 - 4624 = 0$$

$$32r^2 - 224r - 3840 = 0$$

$$r^2 - 7r - 120 = 0$$

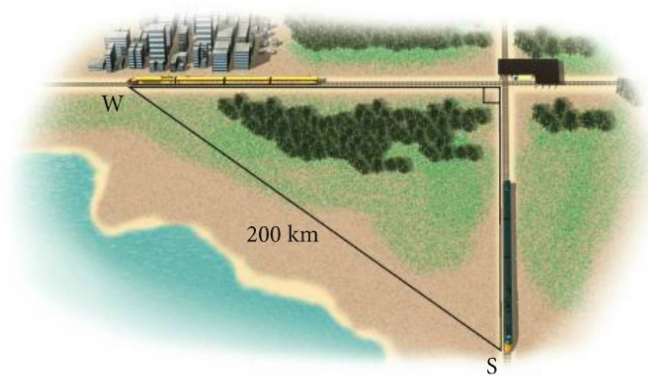
$$\to r = -8, \ 15$$

$\Rightarrow Cassidy's = 15\ km/h$ $\qquad\qquad$ $\Rightarrow Logan's = 8\ km/h$

Exercise

Two trains leave a station at the same time. One train travels due west, and the other travels due south. The train traveling west travels 20 *km/hr* faster than the train traveling south. After 2 *hr*., the trains are 200 *km* apart. Find the speed of each train.

Solution



Given: $w = s + 20 \quad \& \quad t = 2$

$$\left[2(s+20)\right]^2 + (2s)^2 = 200^2$$

$$4\left(s^2 + 40s + 400\right) + 4s^2 = 40,000$$

$$s^2 + 40s + 400 + s^2 = 10,000$$

$$2s^2 + 40s + 9,600 = 0$$

$$s^2 + 20s + 4,800 = 0$$

$$\to \underline{s = \cancel{-80}, \ 60}$$

$\therefore$ Speed of south train: $60 \ \ km/hr$

Speed of west train: $60 + 20 = 80 \ \ km/hr$

Exercise

Towers are 1482 *feet* tall. How long would it take an object dropped from the top to reach the ground? Given $s = t^2$

Solution

$$1482 = 16t^2$$

$$\frac{1482}{16} = t^2$$

$$t = \sqrt{\frac{1482}{16}}$$

$$= \frac{\sqrt{1482}}{4}$$

$$\approx 9.624 \ \ sec$$

Exercise

The formula $P = 0.01A^2 + .05A + 107$ models a woman's normal Point systolic blood pressure, *P*, an age *A*. Use this formula to find the age, to the nearest year, of a woman whose normal systolic blood pressure is 115 *mm* Hg.

Solution

$$0.01A^2 + 0.05A + 107 = 115 \quad \Rightarrow \quad 0.01A^2 + 0.05A - 8 = 0$$

$$A = \frac{-.05 \pm \sqrt{.05^2 - 4(.01)(-8)}}{2(.01)}$$

$$= \frac{-.05 \pm \sqrt{.0025 + .32}}{.02}$$

$$= \frac{-.05 \pm .567}{.02}$$

$$= \begin{cases} \dfrac{-.05 - .567}{.02} = \cancel{-31} \ (\textit{Not a Solution}) \\ \dfrac{-.05 + .567}{.02} = 25.89 \approx 26 \end{cases}$$

Exercise

A rectangular piece of metal is 10 *in*. longer than it is wide. Squares with sides 2 ***in***. long are cut from the four corners, and the flaps folded upward to form an open box. If the volume of the box is 832 in^3, what were the original dimensions of the piece of metal?

Solution

$l = w + 10$

Bottom width: $w - 4$

Bottom length: $l - 4 = w + 10 - 4 = w + 6$

$$V = lwh = (w+6)(w-4)2$$

$$= 2(w^2 - 4w + 6w - 24)$$

$$= 2w^2 + 4w - 48$$

$$2w^2 + 4w - 48 = 832$$

$$2w^2 + 4w - 880 = 0$$

$$w^2 + 2w - 440 = 0$$

$$(w + 22)(w - 20) = 0$$

$$w + 22 = 0 \qquad w - 20 = 0$$

$$w = -22 \qquad w = 20$$

Width of the metal is 20 *in* by the length (20+10) 30 *in*.

Exercise

An astronaut on the moon throws a baseball upward. The astronaut is 6 *ft*., 6 *in*., tall, and the initial velocity of the ball is 30 *ft/sec*. The height s of the ball in feet is given by the equation

$$s = -2.7t^2 + 30t + 6.5$$

Where ***t*** is the number of seconds after the ball was thrown.

a) After how many seconds is the ball 12 *feet* above the moon's surface?
b) How many seconds will it take for the ball to return to the surface?

Solution

a) After how many seconds is the ball 12 *feet* above the moon's surface?

$$12 = -2.7t^2 + 30t + 6.5$$

$$0 = -2.7t^2 + 30t + 6.5 - 12$$

$$0 = -2.7t^2 + 30t - 5.5$$

$$t = \frac{-30 \pm \sqrt{(30)^2 - 4(-2.7)(-5.5)}}{2(-2.7)}$$

$$\approx \frac{-30 \pm 29}{-5.4}$$

$$t \approx \frac{-30-29}{-5.4} \qquad t \approx \frac{-30+29}{-5.4}$$

$$\underline{t \approx 10.9\ sec} \qquad \underline{t \approx 0.12\ sec}$$

b) How many seconds will it take for the ball to return to the surface?

$$0 = -2.7t^2 + 30t + 6.5$$

$$t = \frac{-30 \pm \sqrt{(30)^2 - 4(-2.7)(6.5)}}{2(-2.7)} \approx \frac{-30 \pm 31.15}{-5.4}$$

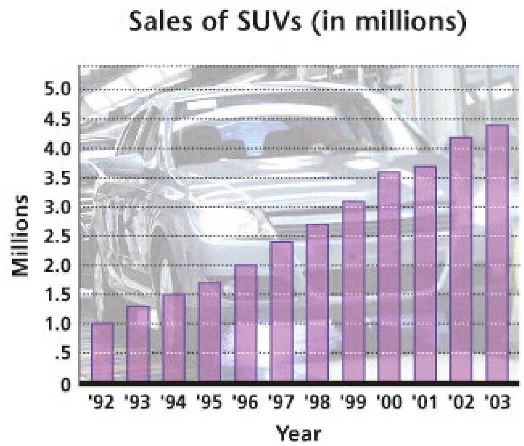
$$t \approx \frac{-30-31.15}{-5.4} \qquad t \approx \frac{-30+31.15}{-5.4}$$

$$t \approx 11.32 \qquad t \approx -0.212$$

It will take 11.32 *sec*.

Exercise

The bar graph shows of SUVs (sport utility vehicles in the US, in *millions*. The quadratic equation $S = .00579x^2 + .2579x + .9703$ models sales of SUVs from 1992 to 2003, where *S* represents sales in *millions*, and $x = 0$ represents 1992, $x = 1$ represents 1993 and so on.



a) Use the model to determine sales in 2002 and 2003. Compare the results to the actual figures of 4.2 million and 4.4 million from the graph.

b) According to the model, in what year do sales reach 3.5 million? Is the result accurate?

Solution

a) For 2002 $\Rightarrow x = 10$

$$S = .00579(10)^2 + .2579(10) + .9703$$
$$\approx 4.1 \ \ million$$

For 2003 $\Rightarrow x = 11$

$$S = .00579(11)^2 + .2579(11) + .9703$$
$$\approx 4.5 \ \ million$$

b) $3.5 = .00579x^2 + .2579x + .9703$

$$0 = .00579x^2 + .2579x + .9703 - 3.5$$
$$0 = .00579x^2 + .2579x - 2.5297$$

$$x = \frac{-.2579 \pm \sqrt{(.2579)^2 - 4(.00579)(-2.5297)}}{2(.00579)}$$

$$= \frac{-.2579 \pm \sqrt{.1251}}{.01158}$$

$$x = \frac{-.2579 - .3537}{.01158} \qquad x = \frac{-.2579 + .3537}{.01158}$$

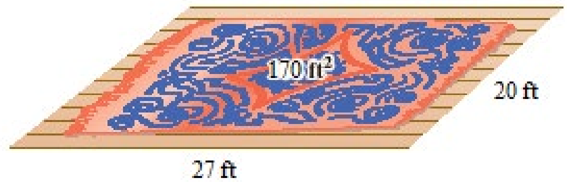
$$x \approx -52.8 \qquad x \approx 8.3$$

According to the model, the number reached 3.5 *million* in the year 2000. The model closely matches the graph, so it is accurate

Exercise

Cynthia wants to buy a rug for a room that is 20 *feet.* wide and 27 *feet*. long. She wants to leave a uniform strip of floor around the rug. She can afford to buy 170 *square feet* of carpeting. What dimension should the rug have?

Solution



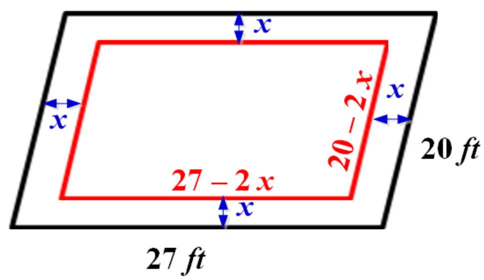
The area of the rug is:

$(27-2x)(20-2x)=170$

$540-54x-40x+4x^2=170$

$540-94x+4x^2-170=0$

$4x^2-94x+370=0$ ***Solve for x.***



$\cancel{x=18.5}$ *or* $x=5$

$20-2x=20-2(5)=10$

and $27-2x=27-2(5)=17$

Therefore, the dimensions are: **10**, **20** *feet*.

Exercise

Erik finds a piece of property in the shape of a right triangle. He finds that the longer leg is 20 *m* longer than twice the length of the shorter leg. The hypotenuse is 10 *m* longer than the length of the longer leg. Find the lengths of the sides of the triangular lot.

Solution

$l:$ *longer leg*
$s:$ *shorter leg*

Longer leg is 20 *m* longer than twice the length of the shorter leg

$$l=2s+20$$

The hypotenuse is 10 *m* longer than the length of the longer leg

$$\begin{aligned} h &= l+10 \\ &= 2s+20+10 \\ &= 2s+30 \end{aligned}$$

$l^2+s^2=h^2$

$(2s+20)^2+s^2=(2s+30)^2$

$4s^2+80s+400+s^2=4s^2+120s+900$

$4s^2+80s+400+s^2-4s^2-120s-900=0$

$s^2-40s-500=0$

$(s+10)(s-50)=0$

$$s+10=0 \qquad\qquad s-50=0$$
$$s=-10 \qquad\qquad s=50$$

The shorter length is 50 m.

The longer length is $l=2s+20=2(50)+20=120$

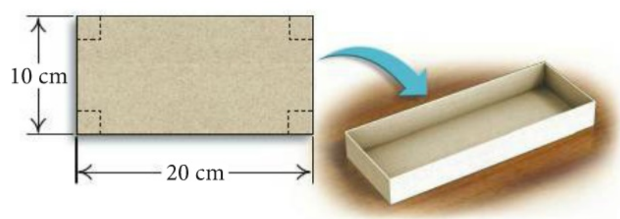
$$h=l+10=120+10=130\ m$$

Exercise

An open box is made from a 10-*cm* by 20-*cm* piece of tin by cutting a square from each corner and folding up the edges. The area of the resulting base is 96 cm^2. What is the length of the sides of the squares?

Solution

$$\begin{aligned}\text{Area of the base} &= (10-2x)(20-2x)\\ &= 200-20x-40x+4x^2\\ &= 4x^2-60x+200\end{aligned}$$



$4x^2-60x+200=96$

$4x^2-60x+104=0$

$x^2-15x+26=0$

$(x-13)(x-2)=0$

$$\begin{cases} x-13=0 \rightarrow x=13 \\ x-2=0 \rightarrow x=2 \end{cases}$$

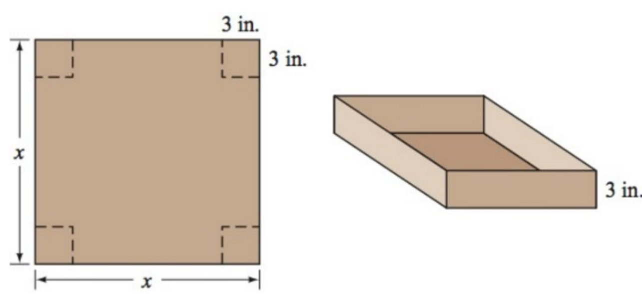
$\Rightarrow x=2 \quad (only)$

Therefore, the length of the sides are 2 cm.

Exercise

A square piece of cardboard is formed into a box by cutting out 3-*inch* squares from each of the corners and folding up the sides. If the volume of the box needs to be 126.75 cubic *inches*, what size square piece of cardboard is needed?

Solution



$V=3(x-6)^2=126.75$

$(x-6)^2=42.25$

$x-6=\sqrt{\dfrac{4225}{100}}$

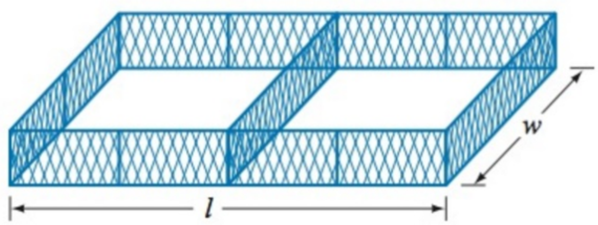
$$x = 6 + \frac{65}{10}$$
$$= 6 + \frac{13}{2}$$
$$= \frac{25}{2}$$
$$\underline{= 12.5 \ \ in.}$$

Exercise

You want to use 132 *feet* of chain-link fencing to enclose a rectangular region and subdivide the region into two smaller rectangular regions. If the total enclosed area is 576 *square feet*, find the dimensions of the enclosed region.

Solution

$$P = 2l + 3w = 132$$
$$\underline{l = \frac{1}{2}(132 - 3w)}$$
$$A = lw = 576$$
$$\frac{w}{2}(132 - 3w) = 576$$
$$132w - 3w^2 = 1{,}152$$
$$3w^2 - 132w + 1{,}152 = 0$$
$$w^2 - 44w + 384 = 0$$
$$w = \frac{44 \pm \sqrt{1936 - 1536}}{2}$$
$$= \frac{44 \pm \sqrt{400}}{2}$$
$$= \frac{44 \pm 20}{2}$$
$$= \begin{cases} \frac{44 - 20}{2} = 12 \\ \frac{44 + 20}{2} = 32 \end{cases}$$



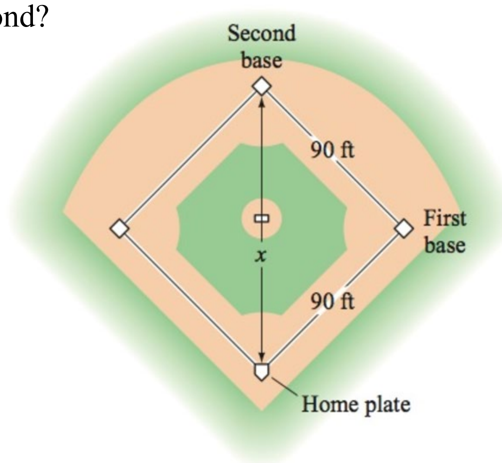
$$w = 12 \quad \to \quad l = \frac{1}{2}(132 - 36) \ \underline{= 48}$$
$$w = 32 \quad \to \quad l = \frac{1}{2}(132 - 96) \ \underline{= 18}$$

∴ the dimensions: Length 48 *feet*, width 12 *feet*.

Or Length 18 *feet*, width 32 *feet*.

Exercise

How far is it from home plate to second base on a baseball diamond?



Solution

$$x^2 = 90^2 + 90^2$$

$$= 2\left(90^2\right)$$

$$\underline{x = 90\sqrt{2}\,\big|}$$

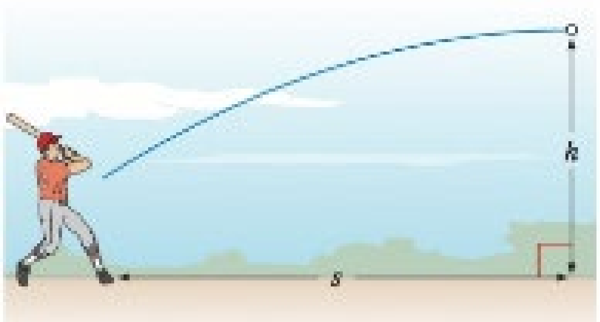
∴ The distance between home plate and second base is

$90\sqrt{2}$ *feet*

Exercise

Two equations can be used to track the position of a baseball *t* seconds after it is hit.

For instance, suppose $h = -16t^2 + 50t + 4.5$ gives the height, in *feet*, of a baseball *t* seconds after it is hit and $s = 103.9t$ gives the horizontal distance, in *feet*, of the ball from home plate *t* seconds after it is hit.



Use these equations to determine whether this particular baseball will clear a 10-*foot* fence positioned 360 *feet* from home plate.

Solution

$$s = 103.9t = 360$$

$$t = \frac{360}{103.9}$$

$$= \frac{3600}{1039}\ sec \qquad \approx 3.46\ sec$$

$$h(3.46) = -16(3.46)^2 + 50(3.46) + 4.5$$

$$\approx -14.05$$

Since the height is negative, then the ball hit the ground before the fence.

∴ The baseball will ***not*** clear the 10-*foot* fence.

Exercise

A ball is thrown downward with an initial velocity of 5 *feet* per *second* from the Golden Gate Bridge, which is 220 *feet* above the water. How long will it take for the ball to hit the water?

Solution

$s(t) = -16t^2 - 5t + 220$ $\qquad\qquad$ $s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$

$$-16t^2 - 5t + 220 = 0$$

$$t = \frac{5 \pm \sqrt{25 + 4(16)(220)}}{-32}$$

$$= \frac{5 \pm \sqrt{25 + 14,080}}{-32}$$

$$= \frac{-5 + \sqrt{14,105}}{32}$$

∴ It will take for the ball to hit the water $\boxed{\frac{-5 + \sqrt{14,105}}{32} \approx 3.56 \ \ sec}$

Exercise

A television screen measures 60 *inches* diagonally, and its aspect ratio is 16 to 9. This means that the ratio of the width of the screen to the height of the screen is 16 to 9.

Find the width and height of the screen.

Solution

$$(16x)^2 + (9x)^2 = 60^2$$

$$256x^2 + 81x^2 = 3600$$

$$337x^2 = 3600$$

$$x^2 = \frac{3600}{337}$$

$$x = \sqrt{\frac{3600}{337}}$$

$$\boxed{= \frac{60}{\sqrt{337}} \ \ in.} \quad \approx 3.268 \ \ in.$$

∴ The width of TV is $16 \times \frac{60}{\sqrt{337}}$ $\boxed{= \frac{960}{\sqrt{337}} \ \ in.}$ $\approx 52 \ \ in.$

The height of TV is $9 \times \frac{60}{\sqrt{337}}$ $\boxed{= \frac{540}{\sqrt{337}} \ \ in.}$ $\approx 29.4 \ \ in.$

Exercise

A company makes rectangular solid candy bars that measures 5 *inches* by 2 *inches* by 0.5 *inch*. Due to difficult financial times, the company has decided to keep the price of the candy bar fixed and reduce the volume of the bar by 20%. What should the dimensions be for the new candy bar if the company keeps the height at 0.5 *inch* and makes length of the candy bar 3 *inches* longer than the width?

Solution

The original volume is given:

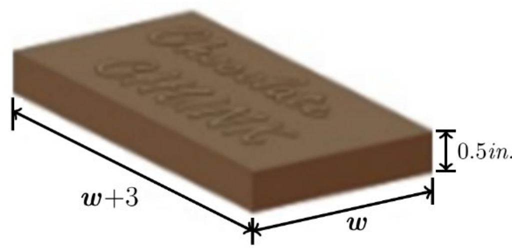
$$V = 5 \times 2 \times \frac{1}{2}$$

$$= 5 \ \ in^3$$

Reduction the volume of the bar by 20% which leave 80% of the new candy.

$$V_{new} = (.8)(5)$$

$$= 4 \ \ in^3$$



$$V = lwh$$

$$4 = (w+3)(w)\left(\frac{1}{2}\right)$$

$$w^2 + 3w = 8$$

$$w^2 + 3w - 8 = 0$$

$$w = \frac{-3 \pm \sqrt{9+32}}{2}$$

$$= \frac{-3 \pm \sqrt{41}}{2}$$

$$w = \frac{-3+\sqrt{41}}{2} \qquad \cancel{w = \frac{-3-\sqrt{41}}{2}} < 0$$

$\therefore$ The new width of the chocolate bar is $= \dfrac{-3+\sqrt{41}}{2} \ \ in.$

$\approx 1.7 \ \ in.$

The new length of the chocolate bar is $\dfrac{-3+\sqrt{41}}{2} + 3 = \dfrac{3+\sqrt{41}}{2} \ \ in.$

$\approx 4.7 \ in.$

Exercise

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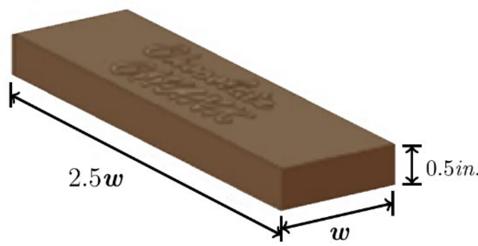


$$V = lwh$$

$$4 = \left(\frac{3}{2}w\right)(w)\left(\frac{1}{2}\right)$$

$$3w^2 = 16$$

$$w^2 = \frac{16}{3}$$

$$w = \frac{4}{\sqrt{3}}$$

$\therefore$ The new width of the chocolate bar is $\frac{4\sqrt{3}}{3}$ *in.*

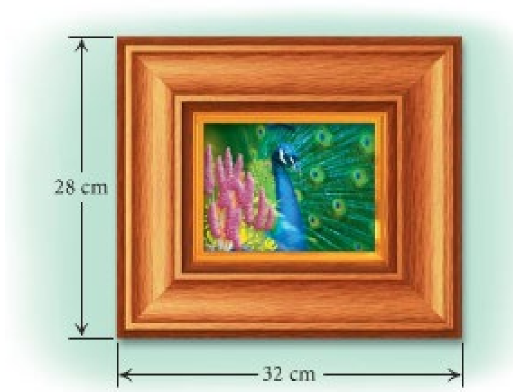
The new length of the chocolate bar is $3\frac{4\sqrt{3}}{3} = 4\sqrt{3}$ *in.*

Exercise

A picture frame measures 28 *cm* by 32 *cm* and is of uniform width. What is the width of the frame if 192 cm^2 of the picture shows?

Solution

Area of the picture $= (32-2x)(28-2x) = 192$



$$896 - 64x - 56x + 4x^2 = 192$$

$$896 - 120x + 4x^2 - 192 = 0$$

$$4x^2 - 120x + 704 = 0$$

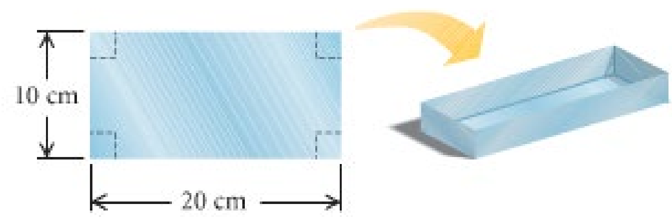
$$x^2 - 30x + 176 = 0$$

$$\begin{cases} x-8=0 \to \underline{x=8|} \\ x-22=0 \to \underline{x=22|} \end{cases}$$

$\therefore$ The width of the frame is **8** *cm*.

Exercise

An open box is made from a 10-*cm* by 20-*cm* of tin by cutting a square from each corner and folding up the edges. The area of the resulting base is 96 cm^2. What is the length of the sides of the squares?



Solution

Area of the base $=(20-2x)(10-2x)=96$

$$200-40x-20x+4x^2=96$$

$$4x^2-60x+200-96=0$$

$$4x^2-60x+104=0 \qquad \textit{Solve for } x$$

$$\boxed{x=2,\ \cancel{13}}$$

The length of the sides of the squares is 3-*cm*

Exercise

You have 600 *feet* of fencing to enclose a rectangular plot that borders on a river. If you do not fence the side along the river.

a) Find the length and width of the plot that will maximize the area.

b) What is the largest area that can be enclosed?

Solution

a) $P=l+2w$

$$600=l+2w \quad \to l=600-2w$$

$$A=(600-2w)w \qquad A=lw$$

$$=600w-2w^2$$

$$=-2w^2+600w$$

$$w=-\frac{600}{2(-2)} \qquad x_{vertex}=-\frac{b}{2a}$$

$\boxed{= 150 \ \ feet}$

$l = 600 - 2w$

$\boxed{= 300 \ \ feet}$

b) $A = lw = (300)(150)$

$\boxed{= 45000 \ ft^2}$

Exercise

You have 60 *yards* of fencing to enclosed a rectangular region.

a) Find the dimensions of the rectangle that maximize the enclosed area.
b) What is the maximum area?

Solution

a) $P = 2(\ell + w)$

$60 = 2(\ell + w)$

$\ell + w = 30$

$\underline{\ell = 30 - w}$

$A = (30 - w)w$

$= -w^2 + 30w$

$w = \frac{30}{2}$ $\qquad x_{vertex} = -\frac{b}{2a}$

$\boxed{= 15 \ \ yards}$

$\ell = 30 - 15$

$\boxed{= 15 \ \ yards}$

The dimensions of the rectangle 15×15

b) $Area = 15 \times 15$

$\boxed{= 225 \ \ yard^2}$

Exercise

You have 80 *yards* of fencing to enclosed a rectangular region.

a) Find the dimensions of the rectangle that maximize the enclosed area.
b) What is the maximum area?

Solution

a) $P = 2(\ell + w)$

$80 = 2(\ell + w)$

$\ell + w = 40$

$\underline{\ell = 40 - w}$

$A = (40 - w)w$

$= -w^2 + 40w$

$w = \frac{40}{2}$ $\qquad x_{vertex} = -\frac{b}{2a}$

$\underline{= 20 \;\; yards}$

$\ell = 40 - 20$

$\underline{= 20 \;\; yards}$

The dimensions of the rectangle 20×20

b) $Area = 20 \times 20$

$\underline{= 400 \;\; yard^2}$

Exercise

The sum of the length *l* and the width *w* of a rectangle tangular area is 240 *meters*.

a) Write *w* as a function of *l*.

b) Write the area *A* as a function of *l*.

c) Find the dimensions that produce the greatest area.

Solution

a) $P = 2(\ell + w)$

$240 = 2(\ell + w)$

$\ell + w = 120$

$\underline{w = 120 - \ell}$

b) $A = \ell(120 - \ell)$

$\underline{= -\ell^2 + 120\ell}$

c) $\ell = \frac{120}{2}$ $\qquad x_{vertex} = -\frac{b}{2a}$

$\underline{= 60 \;\; m}$

$w = 120 - 60$

$\underline{= 60 \;\; m}$

The dimensions of the rectangle 60×60

Exercise

You use 600 *feet* of chainlink fencing to enclose a rectangular region and to subdivide the region into two smallerrectangular regions by placing a fence parallel to one of the sides.

a) Write w as a function of l.

b) Write the area A as a function of l.

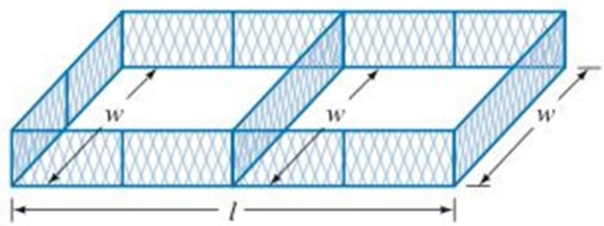
c) Find the dimensions that produce the greatest area.

Solution

a) $P = 2\ell + 3w$

$600 = 2\ell + 3w$

$w = \frac{1}{3}(600 - 2\ell)$



b) $A = \ell\frac{1}{3}(600 - 2\ell)$

$= -\frac{2}{3}\ell^2 + 200\ell$

c) $\ell = 200\frac{3}{4}$ $\qquad x_{vertex} = -\frac{b}{2a}$

$= 150\ ft$

$w = \frac{1}{3}(600 - 300)$

$= 100\ ft$

Exercise

You use 1,200 *feet* of chainlink fencing to enclose a rectangular region and to subdivide the region into three smallerrectangular regions by placing a fence parallel to one of the sides.

a) Write w as a function of l.

b) Write the area A as a function of l.

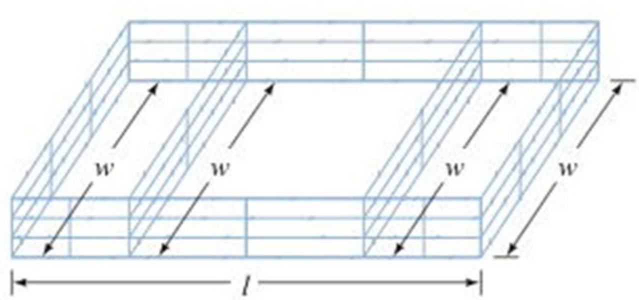
c) Find the dimensions that produce the greatest area.

Solution

a) $P = 2\ell + 4w$

$1,200 = 2\ell + 4w$

$w = 300 - \frac{1}{2}\ell$



b) $A = \ell\left(300 - \frac{1}{2}\ell\right)$

$= -\frac{1}{2}\ell^2 + 300\ell$

c) $\underline{\ell = 300\ \ ft}|$ $\qquad\qquad x_{vertex} = -\frac{b}{2a}$

$w = 300 - 150$

$\quad \underline{= 150\ \ ft}|$

Exercise

A landscaper has enough stone to enclose a rectangular pond next to exiting garden wall of the house with 24 *feet* of stone wall. If the garden wall forms one side of the rectangle.

a) What is the maximum area that the landscaper can enclose?

b) What dimensions of the pond will yield this area?

Solution

a) $P = \ell + 2w$

$24 = \ell + 2w$

$\underline{\ell = 24 - 2w}|$

$A = (24 - 2w)w$

$\quad = -2w^2 + 24w$

$w = \frac{24}{4}$ $\qquad\qquad x_{vertex} = -\frac{b}{2a}$

$\quad \underline{= 6\ \ ft}|$

$\ell = 24 - 12$

$\quad \underline{= 12\ \ ft}|$

$Area = 12 \times 6$

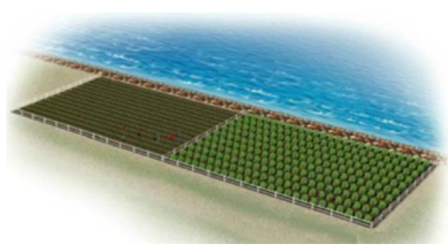
$\underline{= 72\ \ ft^2}|$

b) The dimensions of the rectangle $6 \times 12\ \ feet$

Exercise

A berry former needs to separate and enclose two adjacent rectangular fields, one for strawberries and one for blueberries. If a lake forms one side of the fields and 1,000 *feet* of fencing is available, what is the largest total area that can be enclosed?

Solution



$P = \ell + 3w$

$1,000 = \ell + 3w$

$\underline{\ell = 1,000 - 3w}|$

$A = (1,000 - 3w)w$

$= -3w^2 + 1,000w$

$w = \frac{1,000}{6}$ $\qquad x_{vertex} = -\frac{b}{2a}$

$\boxed{= \frac{500}{3}\ ft}$

$\ell = 1,000 - 500$

$\boxed{= 500\ ft}$

$Area = 500 \times \frac{500}{3}$

$\boxed{= \frac{250,000}{3}\ ft^2}$

Exercise

A fourth-grade class decides to enclose a rectangular garden, using the side of the school as one side of the rectangle. What is the maximum area that the class can enclose with 32 *feet* of fence? What should the dimensions of the garden be in order to yield this area?

Solution

Perimeter: $P = l + 2w = 32$

$l = 32 - 2w$

Area: $A = lw$

$A = (32 - 2w)w$

$= 32w - 2w^2$

$= -2w^2 + 32w$

$w = -\frac{32}{2(-2)}$ $\qquad x_{vertex} = -\frac{b}{2a}$

$\boxed{= 8}$

$l = 32 - 2(8)$

$\boxed{= 16}$

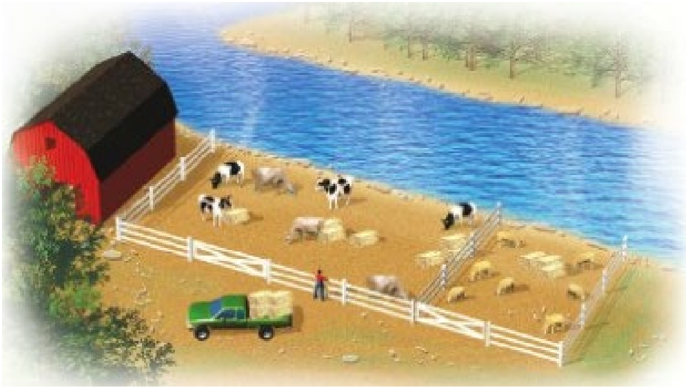
$A = lw = (16)(8)$

$\boxed{= 128\ ft^2}$

Exercise

A rancher needs to enclose two adjacent rectangular corrals, one for cattle and one for sheep. If a river forms one side of the corrals and 240 *yards* of fencing is available, what is the largest total area that can be enclosed?

Solution



Perimeter: $P = l + 3w = 240$

$$l = 240 - 3w$$

Area: $A = lw$

$$A = (240 - 3w)w$$

$$= 240w - 3w^2$$

$$= -3w^2 + 240w$$

$$w = -\frac{240}{2(-3)} \qquad x_{vertex} = -\frac{b}{2a}$$

$$\boxed{= 40}$$

$$l = 240 - 3(40)$$

$$\boxed{= 120}$$

$$A = lw = (120)(40)$$

$$\boxed{= 4800\ yd^2}$$

Exercise

A Norman window is a rectangle with a semicircle on top. Sky Blue Windows is designing a Norman window that will require 24 *feet* of trim on the outer edges. What dimensions will allow the maximum amount of light to enter a house?

Solution

Perimeter of the semi-circle $= \frac{1}{2}(2\pi x)$

Perimeter of the rectangle $= 2x + 2y$

Total perimeter: $\pi x + 2x + 2y = 24$

$$2y = 24 - \pi x - 2x$$

$$y = 12 - \frac{\pi}{2}x - x$$

A Norman window

$$\text{Area } = \frac{1}{2}\left(\pi x^2\right) + (2x)y$$

$$= \frac{\pi}{2}x^2 + 2x\left(12 - \frac{\pi}{2}x - x\right)$$

$$= \frac{\pi}{2}x^2 + 24x - \pi x^2 - 2x^2$$

$$= 24x - \left(\frac{\pi}{2} + 2\right)x^2$$

$$= -\left(\frac{\pi}{2} + 2\right)x^2 + 24x$$

$$x = -\frac{24}{2\left(-\frac{\pi}{2} - 2\right)} \qquad x_{vertex} = -\frac{b}{2a}$$

$$= -\frac{24}{-2\left(\frac{\pi + 4}{2}\right)}$$

$$= \frac{24}{\pi + 4}$$

$$y = 12 - \frac{\pi}{2}\frac{24}{\pi + 4} - \frac{24}{\pi + 4}$$

$$= \frac{24\pi + 96 - 24\pi - 48}{2(\pi + 4)}$$

$$= \frac{24}{\pi + 4}$$

Exercise

A Norman window has the shape of a rectangle surmounted by a semicircle. The exterior perimeter of the window is 48 *feet*.

Find the height h and the radius r that will allow the maximum amount of light to enter the window?

Solution

Perimeter of the semi-circle $= \frac{1}{2}(2\pi r)$

$$= \pi r$$

Perimeter of the rectangle $= 2r + 2h$

Total perimeter:

$$\pi r + 2r + 2h = 48$$

$$2h = 48 - \pi r - 2r$$

$$h = 24 - \frac{1}{2}\pi r - r$$

$$Area = \frac{1}{2}\pi r^2 + (2r)h$$

$$= \frac{1}{2}\pi r^2 + 2r\left(24 - \frac{1}{2}\pi r - r\right)$$

$$= \frac{1}{2}\pi r^2 + 48r - \pi r^2 - 2r^2$$

$$=-\left(\frac{1}{2}\pi+2\right)r^2+48r$$

$$r=-\frac{48}{2\left(-\frac{\pi}{2}-2\right)} \qquad x_{vertex}=-\frac{b}{2a}$$

$$=\frac{48}{\pi+4}$$

$$h=24-\left(\frac{\pi}{2}+1\right)r$$

$$=24-\left(\frac{\pi+2}{2}\right)\frac{48}{\pi+4}$$

$$=24-24\frac{\pi+2}{\pi+4}$$

$$=24\left(1-\frac{\pi+2}{\pi+4}\right)$$

$$=\frac{48}{\pi+4}$$

Exercise

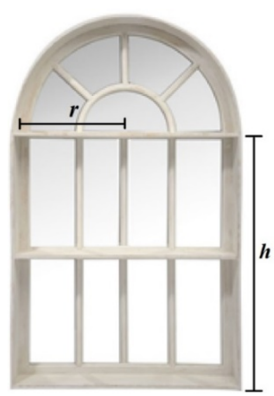
A Norman window has the shape of a rectangle surmounted by a semicircle. It requires 36 *feet* of trim on the outer edges. What dimensions will allow the maximum amount of light to enter a house?

Solution

Perimeter of the semi-circle $=\frac{1}{2}(2\pi r)$

$$=\pi r$$

Perimeter of the rectangle $=2r+2h$



Total perimeter:

$$\pi r+2r+2h=36$$

$$2h=36-\pi r-2r$$

$$h=18-\frac{1}{2}\pi r-r$$

$$Area=\frac{1}{2}\pi r^2+(2r)h$$

$$=\frac{1}{2}\pi r^2+2r\left(18-\frac{1}{2}\pi r-r\right)$$

$$=\frac{1}{2}\pi r^2+36r-\pi r^2-2r^2$$

$$=-\left(\frac{1}{2}\pi+2\right)r^2+36r$$

$$r=-\frac{36}{2\left(-\frac{\pi}{2}-2\right)} \qquad r_{vertex}=-\frac{b}{2a}$$

$$=\boxed{\dfrac{36}{\pi+4}}$$

$$h=18-\left(\frac{\pi}{2}+1\right)r$$

$$=18-\left(\frac{\pi+2}{2}\right)\frac{36}{\pi+4}$$

$$=18-18\frac{\pi+2}{\pi+4}$$

$$=18\left(1-\frac{\pi+2}{\pi+4}\right)$$

$$=\boxed{\dfrac{36}{\pi+4}}$$

Exercise

The temperature $T(t)$, in degrees Fahrenheit, during the day can be modeled by the equation $T(t)=-0.7t^2+9.4t+59.3$, where t is the number of hours after 6:00 AM.

a) At what time the temperature a maximum?

b) What is the maximum temperature?

Solution

a) $t=-\dfrac{9.4}{2(-0.7)}$

$$=\frac{94}{14}$$

$$=\frac{47}{7}\ hrs$$

$$=\left(6+\frac{5}{7}\right)hrs$$

$$=6hrs\quad \frac{5}{7}hr\frac{60\ \text{min}}{hr}$$

$$=6\ hrs\quad \frac{300}{7}\ min$$

$$=6\ hrs\quad 42\ min\quad \frac{6}{7}\ min$$

$$=6\ hrs\quad 42\ min\quad \frac{6}{7}\ min\frac{60\ sec}{min}$$

$$=6\ hrs\quad 42\ min\quad \frac{360}{7}\ sec$$

$$\approx \boxed{6\ hrs\quad 42\ min\quad 51\ sec}$$

The maximum temperature is around $\boxed{12:43\ \text{PM}}$

b) $T\left(\frac{47}{7}\right)=-\frac{7}{10}\left(\frac{2209}{49}\right)+\frac{94}{10}\left(\frac{47}{7}\right)+\frac{593}{10}$

$$= -\frac{2209}{70} + \frac{2209}{35} + \frac{593}{10}$$

$$= \frac{2209}{70} + \frac{593}{10}$$

$$= \frac{6360}{70}$$

$$\boxed{= \frac{636}{7} \ {}^\circ F}$$

$$\boxed{\approx 90.86 \ {}^\circ F}$$

Exercise

When a softball player swings a bat, the amount of energy $E(t)$, in *joules*, that is transferred to the bat can be approximated by the function

$$E(t) = -279.67t^2 + 82.86t$$

Where $0 \le t \le 0.3$ and t is measured in *seconds*. According to this model, what is the maximum energy of the bat?

Solution

$$t = -\frac{82.86}{2(-279.67)} \qquad\qquad t_{vertex} = -\frac{b}{2a}$$

$$= \frac{8286}{2(27967)}$$

$$= \frac{4243}{27967}$$

$$\boxed{\approx 0.15 \ sec}$$

The maximum energy is

$$E(0.15) = -279.67(0.15)^2 + 82.86(0.15)$$

$$\boxed{\approx 6.136 \ joules}$$

Exercise

Some softball fields are built in a parabolic mound shape so that water will drain off the field. A model for the shape of a certain field is given by

$$h(x) = -0.0002348x^2 + 0.0375x$$

Where $h(x)$ is the height, in *feet*, of the field at a distance of *x feet* from one sideline. Find the maximum height of the field.

Solution

$$x = -\frac{0.0375}{2(-0.0002348)} \qquad x_{vertex} = -\frac{b}{2a}$$

$$\approx 79.86 \ ft$$

The maximum height of the field is

$$h(79.86) = -0.0002348(79.86)^2 + 0.0375(79.86)$$

$$\approx 4.5 \ feet$$

Exercise

The fuel efficiency for a certain midsize car is given by

$$E(v) = -0.018v^2 + 1.476v + 3.4$$

Where $E(v)$ is the fuel efficiency in *miles* per *gallon* for a car traveling v in *miles* per *hour*.

a) What speed will yield the maximum fuel efficiency?

b) What is the maximum fuel efficiency for this car?

Solution

a) $v = -\frac{1.476}{2(-0.018)} \qquad v_{vertex} = -\frac{b}{2a}$

$$= 41 \ mi/hr$$

b) $E(41) = -0.018(41)^2 + 1.476(41) + 3.4$

$$\approx 33.658 \ mi/gal$$

Exercise

If the initial velocity of a projectile is 128 *feet* per *second*, then the height h, in *feet*, is a function of time t, in *seconds*, given by the equation

$$h(t) = -16t^2 + 128t$$

a) Find the time t when the projectile achieves its maximum height.

b) Find the maximum height of the projectile.

c) Find the time t when the projectile hits the ground.

Solution

a) $t = -\frac{128}{-32} \qquad t_{vertex} = -\frac{b}{2a}$

$$= 4 \ sec$$

b) $h(4) = -16(16) + 128(4)$

$$\boxed{= 256\ ft}$$

c) $h(t) = -16t^2 + 128t = 0$

$-16t(t-8) = 0$

$t = 0 \quad t = 8$

The projectile hits the ground in $\boxed{t = 8\ sec}$

Exercise

If the initial velocity of a projectile is 64 *feet* per *second* and an initial height of 80 *feet*, then the height *h*, in *feet*, is a function of time *t*, in *seconds*, given by the equation

$$h(t) = -16t^2 + 64t + 80$$

a) Find the time *t* when the projectile achieves its maximum height.

b) Find the maximum height of the projectile.

c) Find the time *t* when the projectile hits the ground.

Solution

a) $t = -\dfrac{64}{-32}$ $\qquad\qquad t_{vertex} = -\dfrac{b}{2a}$

$\boxed{= 2\ sec}$

b) $h(2) = -16(4) + 64(2) + 80$

$\boxed{= 144\ ft}$

c) $h(t) = -16t^2 + 64t + 80 = 0$

$$t = \frac{-64 \pm \sqrt{4,096 + 5,120}}{-32}$$

$$= \frac{64 \pm \sqrt{9,216}}{32}$$

$$= \frac{64 \pm 96}{32}$$

$$= \begin{cases} \dfrac{64-96}{32} = - \\ \dfrac{64+96}{32} = 5 \end{cases}$$

The projectile hits the ground in $\boxed{t = 5\ sec}$

Exercise

If the initial velocity of a projectile is 100 *feet* per *second* and an initial height of 20 *feet*, then the height h, in *feet*, is a function of time t, in *seconds*, given by the equation

$$h(t) = -16t^2 + 100t + 20$$

a) Find the time t when the projectile achieves its maximum height.
b) Find the maximum height of the projectile.
c) Find the time t when the projectile hits the ground.

Solution

a) $t = -\dfrac{100}{-32}$ $\qquad t_{vertex} = -\dfrac{b}{2a}$

$= \dfrac{25}{8}\ sec$

$= 3.125\ sec$

b) $h(3.125) = -16(3.125)^2 + 100(3.125) + 20$

$= 176.25\ ft$

c) $h(t) = -16t^2 + 100t + 20 = 0$

$$t = \frac{-100 \pm \sqrt{10,000 + 1,280}}{-32}$$

$$= \frac{64 \pm \sqrt{11,280}}{32}$$

$$= \begin{cases} \dfrac{64 - 106.2}{32} = - \\ \dfrac{64 + 106.2}{32} = 5.3 \end{cases}$$

The projectile hits the ground in $t = 5.3\ sec$

Exercise

A frog leaps from a stump 3.5-*foot*-high and lands 3.5 *feet* from the base of the stump.
It is determined that the height of the frog as a function of its distance, x, from the base of the stump is given by the function $h(x) = -0.5x^2 + 0.75x + 3.5$ where h is in feet.

a) How high is the frog when its horizontal distance from the base of the stump is 2 *feet*?
b) At what two distances from the base of the stump after is jumped was the frog 3.6 *feet* above the ground?
c) At what distance from the base did the frog reach its highest point?
d) What was the maximum height reached by the frog?

Solution

a) At $x = 2\,ft$. $\quad Find \quad h(x=2)$

$$h(2) = -0.5\left(2^2\right) + 0.75(2) + 3.5$$

$$\underline{= 3\,ft}$$

b) $h(x) = -0.5x^2 + 0.75x + 3.5 = 3.6$

$$-0.5x^2 + 0.75x + 3.5 - 3.6 = 0$$

$$-0.5x^2 + 0.75x - .1 = 0$$

Solve for x: $\underline{x = 0.1,\ \ 1.4\,ft}$

c) The distance from the base for the frog to reach the highest point is

$$x = -\frac{b}{2a} = -\frac{.75}{2(-.5)} \underline{= .75\,ft}$$

d) Maximum height:

$$h(x = .75) = -0.5(.75)^2 + 0.75(.75) + 3.5 \underline{= 3.78\,ft}$$

Exercise

The height of an arch is given by

$$h(x) = -\frac{3}{64}x^2 + 27, \quad -24 \le x \le 24$$

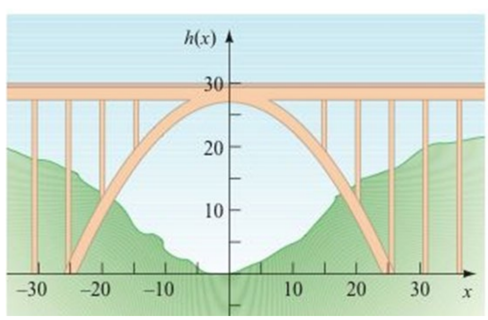
Where $|x|$ is the horizontal distance in *feet* from the center of the arch to the ground

a) What is the maximum height of the arch?
b) What is the height of the arch 10 *feet* to the right of center?
c) How far from the center is the arch 8 *feet* tall?

Solution

a) $\underline{x = 0\ \ ft}$ $\qquad\qquad x_{vertex} = -\dfrac{b}{2a}$

$\underline{h(0) = 27\ \ ft}$



b) $h(10) = -\dfrac{3}{64}(100) + 27$

$$= -\frac{75}{16} + 27$$

$$= \frac{357}{16}$$

$$\underline{= 22.3125\ \ ft}$$

c) $h(x) = -\dfrac{3}{64}x^2 + 27 = 8$

$$-\frac{3}{64}x^2 = -19$$

$$x^2 = \frac{1,216}{3}$$

$$x = \pm\sqrt{\frac{1,216}{3}}$$

$$\boxed{= \pm 8\sqrt{\frac{19}{3}}}$$

$$\boxed{\approx \pm 20.13 \ ft}$$

Exercise

A weightless environment can be created in an airplane by flying in a series of parabolic paths. This is one method that NASA uses to train astronauts for the experience of weightlessness. Suppose the height h, in *feet*, of NASA's airplane is modeled by

$$h(t) = -6.6t^2 + 430t + 28,000$$

Where t is the time, in *seconds,* after the plane enters its parabolic path.
Find the maximum height of the plane.

Solution

$$t = \frac{430}{13.2} \qquad\qquad t_{vertex} = -\frac{b}{2a}$$

$$= \frac{4300}{132}$$

$$= \frac{1,075}{33}$$

$$\boxed{\approx 32.58 \ sec}$$

$$h(32.58) = -6.6(32.58)^2 + 430(32.58) + 28,000$$

$$\boxed{\approx 35,000 \ ft}$$

Exercise

You drop a screwdriver from the top of an elevator shaft. Exactly 5 *seconds* later, you hear the sound of the screwdriver hitting the bottom of the shaft. The speed of sound is $1,100 \ ft/sec$. How tall is the elevator shaft?

Solution

$$t_1 + t_2 = 5$$

$$s(t) = 16t^2$$

$$t^2 = \frac{s}{16}$$

$$\boxed{t_1 = \frac{\sqrt{s}}{4}}$$

$$s = 1{,}100\, t_2$$

$$\boxed{t_2 = \frac{s}{1{,}100}}$$

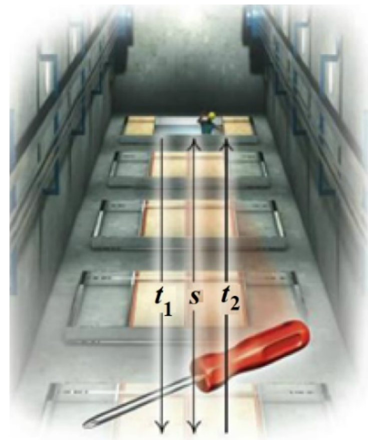
$$t_1 + t_2 = 5$$

$$\frac{\sqrt{s}}{4} + \frac{s}{1{,}100} = 5$$

$$s + 275\sqrt{s} - 5{,}500 = 0$$

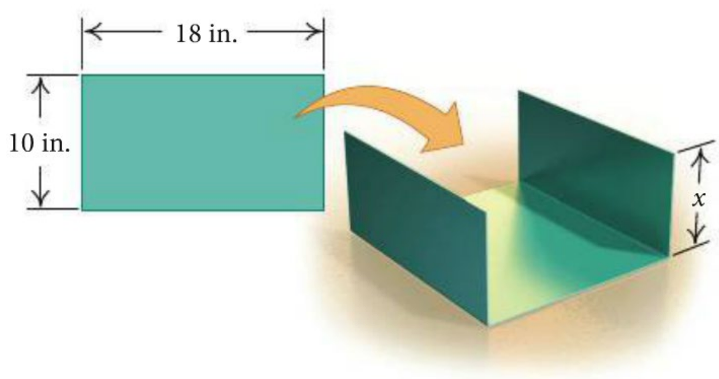
$$\sqrt{s} = \begin{cases} \dfrac{-275 - 312.5}{2} = - \\ \dfrac{-275 + 312.5}{2} = 18.725 \end{cases}$$

$$\boxed{s = 350.6 \ \ feet}$$



Exercise

A company plans to produce a one- compartment vertical file by bending the long side of a 10-*in*. by 18-*in*. sheet of metal along two lines to form a $\lfloor\_\rfloor$− shape. How tall should the file be in order to maximize the volume that it can hold?



Solution

Height $= x$

If the length is 18 *in*.

Width of the base $= 10 - 2x$

$$Volume = 18x(10 - 2x)$$

$$= -36x^2 + 180x$$

$$x = \frac{180}{72} \qquad\qquad x_{vertex} = -\frac{b}{2a}$$

$\boxed{= \frac{5}{2} \ \ in.}$

$\boxed{= 2.5 \ \ in.}$

$$Max.\ Area = 18\frac{5}{2}(10-5)$$

$$\boxed{= 225 \ \ in^3}$$

If the length is 10 *in*.

Width of the base $= 18 - 2x$

$Volume = 10x(18-2x)$

$= -20x^2 + 180x$

$x = \frac{180}{40}$ $\qquad x_{vertex} = -\frac{b}{2a}$

$\boxed{= \frac{9}{2} \ \ in.}$

$\boxed{= 4.5 \ \ in.}$

$$Max.\ Area = 10\frac{9}{2}(18-9)$$

$$\boxed{= 405 \ \ in^3}$$

To maximize the volume, the length should be 10 *in*. and bent on 18 *in*. side with 4.5 *in*. height to give a volume of $405 \ \ in^3$

Exercise

The sum of the base and the height of a triangle is 20 *cm*. Find the dimensions for which the area is a maximum.

Solution

$b + h = 20$

$b = 20 - h$

$A = \frac{1}{2}bh$

$= \frac{1}{2}(20-h)h$

$= -\frac{1}{2}h^2 + 10h$

$\boxed{h = 10 \ \ cm}$ $\qquad h_{vertex} = -\frac{b}{2a}$

$b = 20 - 10$

$\boxed{= 10 \ \ cm}$

The triangle dimensions for the maximum area is $10 \times 10 \ \ cm$

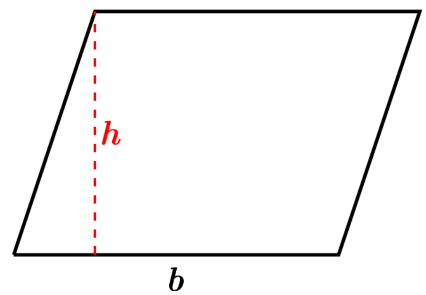
Exercise

The sum of the base and the height of a parallelogram is 14 *in*. Find the dimensions for which the area is a maximum.

Solution

$b + h = 14$

$b = 14 - h$



$$Area = bh$$
$$= (14 - h)h$$
$$= -h^2 + 14h$$

$\boxed{h = 7 \ \ in.}$ $\qquad h_{vertex} = -\frac{b}{2a}$

$b = 14 - 7$

$\boxed{= 7 \ \ in.}$

The parallelogram dimensions for the maximum area is $7 \times 7 \ \ cm$

Exercise

An arch in the shape of a parabola has the dimensions shown in the figure. How wide is the arch 9 *feet* up?

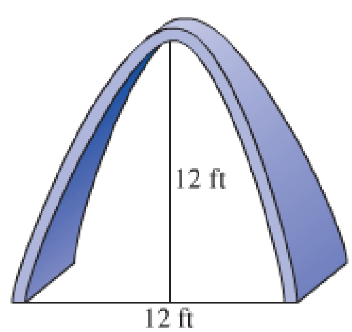
Solution

$Vertex: \ V(0, \ 12)$

$(x-h)^2 = 4p(y-k)$

$(x-0)^2 = 4p(y-12)$

$x^2 = 4p(y-12)$



The parabola passes through the point $(6, \ 0)$

$6^2 = 4p(0-12)$

$-48p = 36$

$p = -\frac{36}{48} = \boxed{-\frac{3}{4}}$

The equation is: $x^2 = -3(y-12)$

The arch is 9 feet up that is the y-coordinate,

$x^2 = -3(9-12) = 9$

$\boxed{x = 3}$

The width is $2(3) \ \boxed{= 6 \ \ feet}$

Exercise

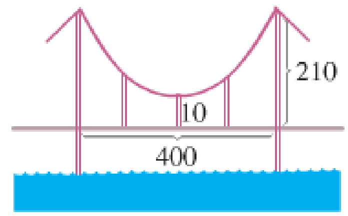
The cable in the center portion of a bridge is supported as shown in the figure to form a parabola. The center support is 10 *feet* high, the tallest supports are 210 *feet* high, and the distance between the two tallest supports is 400 *feet*. Find the height of the remaining supports if the supports are evenly spaced.

Solution

$Vertex: \ V(0,\ 10)$

$(x-h)^2 = 4p(y-k)$

$(x-0)^2 = 4p(y-10) \quad \Rightarrow \quad x^2 = 4p(y-10)$



The parabola passes through the point $(200,\ 210)$

$200^2 = 4p(210-10)$

$800p = 200^2$

$p = \dfrac{40000}{800}$

$\underline{= 50 \ |}$

The equation is: $x^2 = 200(y-10)$

The x-coordinate of one of the supports is 100.

$100^2 = 200(y-10)$

$y - 10 = \dfrac{10000}{200} = 50$

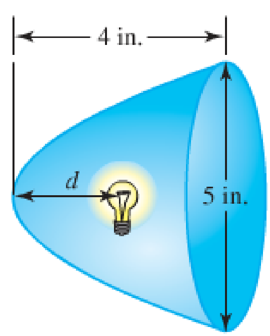
$y = 50 + 10$

$\underline{= 60 \ \ feet \ |}$

$\therefore$ The height is $60 \ \ feet$

Exercise

A headlight is being constructed in the shape of a paraboloid with depth 4 *inches* and diameter 5 *inches*. Determine the distance *d* that the bulb should be form the vertex in order to have the beam of light shine straight ahead.



Solution

Let the vertex be at the origin $V(0,\ 0)$

The equation is: $y^2 = 4px$

Which it passes through the point $V(4,\ 2.5)$

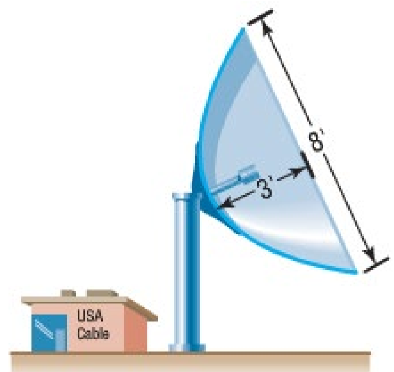
$(2.5)^2 = 4p(4)$

$$p = \frac{(2.5)^2}{16} = \frac{25}{64}$$

The bulb should be $\frac{25}{64} \approx 0.39$ inch from the vertex

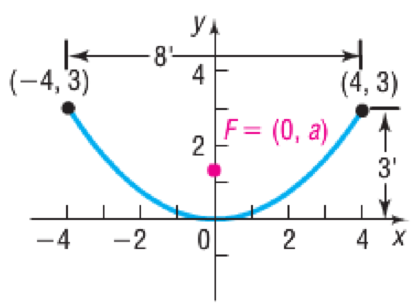
Exercise

A satellite dish is shaped like a paraboloid of revolution. The signals that emanate from a satellite strike the surface of the dish and are reflected to a single point, where the receiver is located. If the dish is 8 *feet* across at its opening and 3 *feet* deep at its center, at what position should the receiver be placed? That is, where is the focus?



Solution

From the figure, we can draw the parabola used to form the dish on a rectangular coordinate system so that the vertex of the parabola is at the origin and its focus on the positive *y*-axis.



The equation from of the parabola is: $x^2 = 4py$

Since $(4,\ 3)$ is a point on the graph

$$4^2 = 4p(3)$$

$$p = \frac{16}{12} = \frac{4}{3}$$

Therefore, the receiver should be located $\frac{4}{3}\,ft$ from the base of the dish, along its axis of symmetry.

Exercise

A cable TV receiving dish is in the shape of a paraboloid of revolution. Find the location of the receiver, which is placed at the focus, if the dish is 6 *feet* across at its opening and 2 feet deep.

Solution

Given: Parabola is 6 *feet* across and 2 *feet* deep.

Let the vertex of the parabola is at (0, 0) and it opens up, then the equation of the parabola has the form $x^2 = 4ay$

Therefore, the point (3, 2) and (−3, 2) are on the parabola.

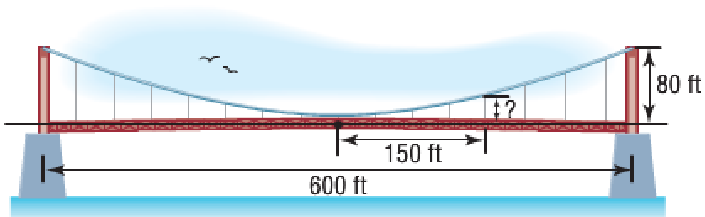
$$3^2 = 4a(2) \quad \to \quad a = \frac{9}{8} = 1.125$$

Where a is the distance from the vertex to the focus.

Thus, the receiver (located at the focus) is 1.125 *feet* or 13.5 *inches* from the base of the dish, along the axis of the parabola.

Exercise

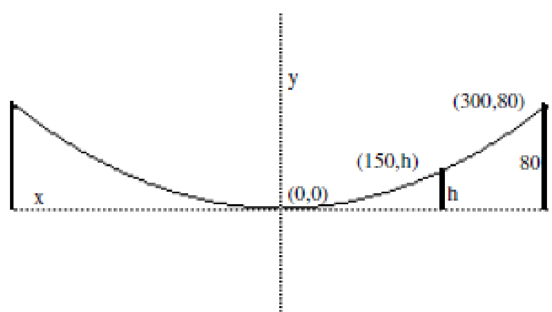
The cables of a suspension bridge are in the shape of a parabola, as shown below. The towers supporting the cable are 600 *feet* apart and 80 *feet* high. If the cables touch the road surface midway between the towers, what is the height of the cable from the road at a point 150 *feet* from the center of the bridge?



Solution

Let the vertex of the parabola is at (0, 0) and it opens up, then the equation of the parabola has the form $x^2 = cy$

The point (300, 80) is a point on the parabola.



$$300^2 = c(80) \quad \to \quad c = \frac{300^2}{80} = 1125$$

$$x^2 = 1125y$$

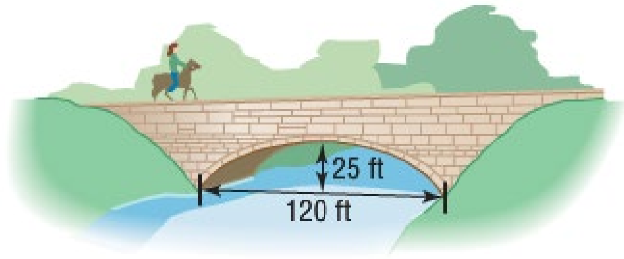
The height of the cable 150 feet from the center is:

$$150^2 = 1125h \quad \to \quad h = \frac{150^2}{1125} = 20$$

The height of the cable 150 feet from the center is 20 *feet*.

Exercise

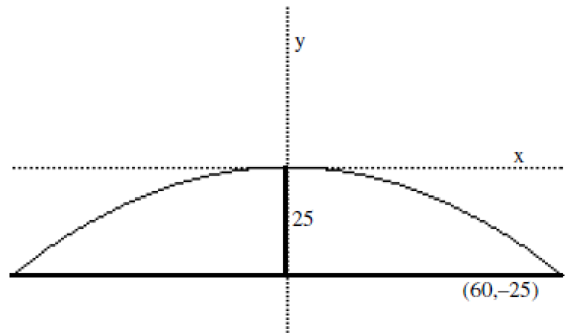
A bridge is built in the shape of a parabolic arch. The bridge has a span of 120 *feet* and a maximum height of 25 *feet*. Choose a suitable rectangular coordinate system and find the height of the arch at distances of 10, 30, and 50 *feet* from the center.



Solution

Let the vertex of the parabola is at (0, 0) and it opens down, then the equation of the parabola has the form $x^2 = cy$

The point (60, –25) is a point on the parabola.



$$60^2 = c(-25) \quad \rightarrow \quad c = \frac{60^2}{-25} = -144$$

$$x^2 = -144y$$

The height of the arch at

Distance 10:

$$10^2 = -144y$$

$$y = \frac{100}{-144} \approx -0.69$$

The height of the bridge 10 feet from the center is about $25 - 0.69 = 24.31\ ft$

Distance 30:

$$30^2 = -144y$$

$$y = \frac{900}{-144} \approx -6.25$$

The height of the bridge 30 feet from the center is about $25 - 6.25 = 18.75\ ft$

Distance 50:

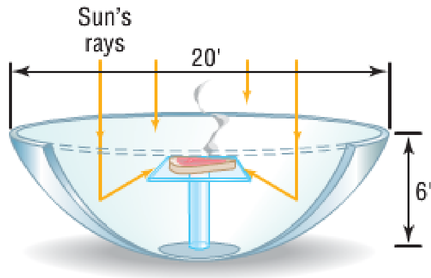
$$50^2 = -144y$$

$$y = \frac{2500}{-144} \approx -17.36$$

The height of the bridge 10 feet from the center is about $25 - 17.36 = 7.64\ ft$

Exercise

A mirror is shaped like a paraboloid of revolution and will be used to concentrate the rays of the sun at its focus, creating a heat source. If the mirror is 20 *feet* across at its opening and is 6 *feet* deep, where will the heat source be concentrated?



Solution

Let the vertex of the parabola is at (0, 0) and it opens up, then the equation of the parabola has the form $x^2 = 4ay$

Since the parabola is 20 *feet* across and 6 feet deep.

The points (10, 6) and (–10, 6) are on the parabola.

$$10^2 = 4a(6) \quad \rightarrow \quad a = \frac{100}{24} \approx 4.17 \ ft$$

The heat will be concentrated about 4.17 *feet* from the base, along the axis of symmetry.

Exercise

A reflecting telescope contains a mirror shaped a paraboloid of revolution. If the mirror is 4 *inches* across at its opening and is 3 *inches* deep, where will the collected light be concentrated?

Solution

Let the vertex of the parabola is at (0, 0) and it opens up.

Then the equation of the parabola has the form $x^2 = 4ay$

Since the parabola is 4 *inches* across and 3 *inches* deep.

The points (2, 3) and (–2, 3) are on the parabola.

$$2^2 = 4a(3) \quad \rightarrow \quad a = \frac{4}{12} \approx \frac{1}{2} \ in$$

The collected light will be concentrated 1/3 inch from the base of the mirror along the axis of symmetry.

Exercise

Show that the graph of an equation of the form $Ax^2 + Dx + Ey + F = 0 \qquad A \neq 0$

a) Is a parabola if $E \neq 0$

b) Is a vertical line if $E = 0$ *and* $D^2 - 4AF = 0$

c) Is two vertical lines if $E = 0$ *and* $D^2 - 4AF > 0$

d) Contains no points if $E = 0$ *and* $D^2 - 4AF < 0$

Solution

a) If $E \neq 0 \quad \to \quad Ax^2 + Dx + Ey + F = 0$

The x-vertex: $x = -\frac{b}{2a} = -\frac{D}{2A}$

$$A\left(-\frac{D}{2A}\right)^2 + D\left(-\frac{D}{2A}\right) + Ey + F = 0$$

$$\frac{D^2}{4A} - \frac{D^2}{2A} + Ey + F = 0$$

$$Ey = \frac{D^2}{4A} - F$$

$$y = \frac{D^2 - 4AF}{4AE}$$

This is the equation of a parabola whose vertex is: $\left(-\frac{D}{2A}, \frac{D^2 - 4AF}{4AE}\right)$ and whose axis of symmetry is parallel to the y-axis.

b) If $E = 0 \quad \to \quad Ax^2 + Dx + F = 0$

$$x = \frac{-D \pm \sqrt{D^2 - 4AF}}{2A}$$

$$= -\frac{D}{2A} \qquad \textit{Since } D^2 - 4AF = 0$$

This is a single vertical line.

c) If $E = 0 \quad \to \quad Ax^2 + Dx + F = 0$

$$x = \frac{-D \pm \sqrt{D^2 - 4AF}}{2A}$$

If $D^2 - 4AF > 0$, then

$x = \frac{-D - \sqrt{D^2 - 4AF}}{2A}$ *and* $x = \frac{-D + \sqrt{D^2 - 4AF}}{2A}$ are two vertical lines.

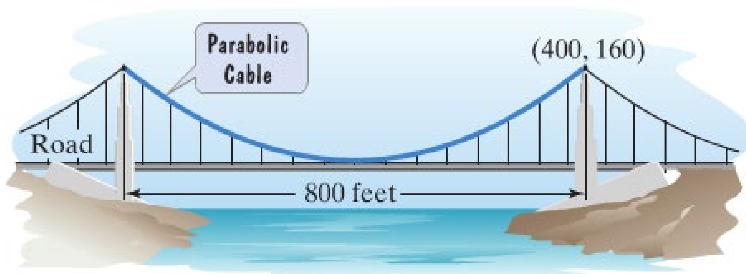
d) If $E = 0 \quad \to \quad Ax^2 + Dx + F = 0$

$$x = \frac{-D \pm \sqrt{D^2 - 4AF}}{2A}$$

If $D^2 - 4AF < 0$, then there is no real solution. The graph contains no points.

Exercise

The towers of a suspension bridge are 800 *feet* apart and rise 160 *feet* above the road. The cable between the towers has the shape of a parabola and the cable just touches the sides of the road midway between the towers. What is the height of the cable 100 *feet* from a tower?



Solution

Given the point: $(400,\ 160)$

$(400)^2 = 4p(160)$ $\qquad\qquad x^2 = 4py$

$p = \frac{400^2}{640} = 250$

$x^2 = 1{,}000y$

$x = 400 - 100 = 300$

$(300)^2 = 1{,}000y$ $\qquad\qquad x^2 = 4py$

$y = \frac{300^2}{1{,}000} = 90$

The height is 90 *feet*.

Exercise

The cables of a suspension bridge are in the shape of a parabola. The towers supporting the cable are 400 *feet* apart and 100 *feet* high. If the cables are at a height of 10 *feet* midway between the towers, what is the height of the cable at a point 50 *feet* from the center of the bridge?

Solution

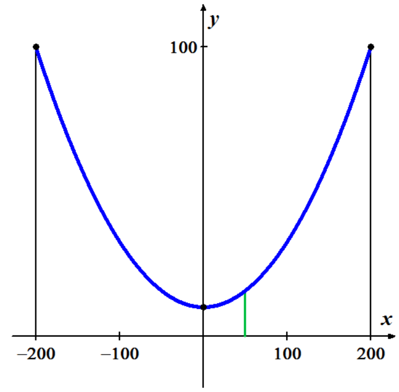
Vertex point: $(0,\ 10)$ and the parabola is open up

A point on parabola: $(200,\ 100)$

$200^2 = c(100 - 10)$ $\qquad\qquad (x-h)^2 = c(y-k)$

$$c=\frac{40{,}000}{90}=\frac{4000}{9}$$

$$\boxed{x^2=\frac{4000}{9}(y-10)}$$



The height of the cable 50 feet from the center – $(50,\ h)$

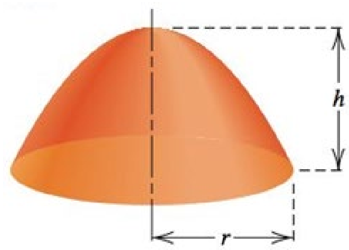
$$y=\frac{9}{4000}x^2+10$$

$$h=\frac{9}{4000}(50)^2+10\ \boxed{\approx 15.625\ ft}$$

The height of the cable 50 *feet* from the center is about 15.625 *feet*.

Exercise

The focal length of the (finite) paraboloid is the distance p between its vertex and focus



a) Express p in terms of r and h.

b) A reflector is to be constructed with a focal length of 10 *feet* and a depth of 5 *feet*. Find the radius of the reflector.

Solution

a) The point $(r,\ h)$ is on the parabola.

$$r^2=4p(h) \qquad\qquad x^2=4py$$

$$\boxed{p=\frac{r^2}{4h}}$$

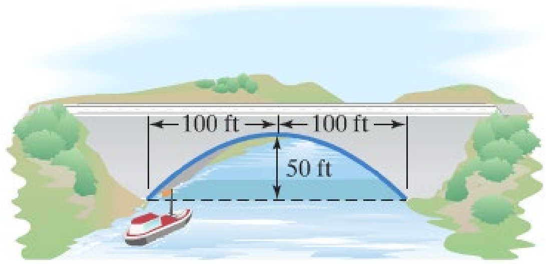
b) Given: $p=10;\quad h=5$

$$r=\sqrt{4(10)(5)}$$

$$\boxed{=10\sqrt{2}}$$

Exercise

The parabolic arch is 50 *feet* above the water at the center and 200 *feet* wide at the base. Will a boat that is 30 *feet* tall clear the arch 30 *feet* from the center?



Solution

$$\left(\frac{200}{2}\right)^2 = 4p(-50) \qquad\qquad x^2 = 4py$$

$$p = \frac{200^2}{-200}$$

$$= -200$$

$$x^2 = -200y$$

Given the boat tall: $x = 30$

$$(30)^2 = -200y \qquad\qquad x^2 = 4py$$

$$y = \frac{900}{-200}$$

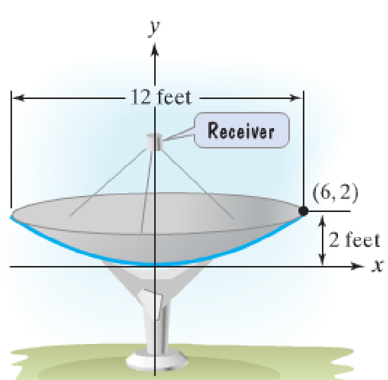
$$= -4.5$$

Height of bridge $= 50 - 4.5 = 45.5\ ft$

Yes, the boat will clear the arch.

Exercise

A satellite dish, as shown below, is in the shape of a parabolic surface. Signals coming from a satellite strike the surface of the dish and are reflected to the focus, where the receiver if located. The satellite dish shown has a diameter of 12 *feet* and a depth of 2 *feet*. How far from the base of the dish should the receiver be placed?



Solution

$$6^2 = 4p(2) \qquad\qquad x^2 = 4py$$

$$p = \frac{36}{8}$$

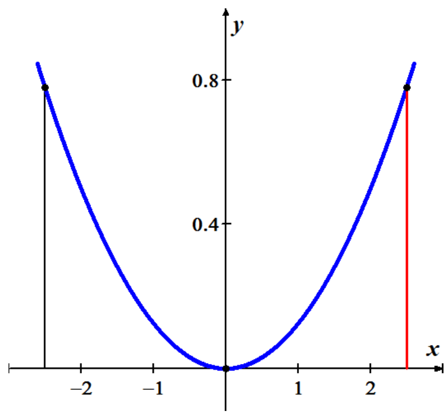
$$= 4.5$$

The receiver should be located 4.5 *feet* from the base of the dish.

Exercise

A searchlight is shaped like a paraboloid of revolution. If the light source is located 2 *feet* from the base along the axis of symmetry and the opening is 5 *feet* across, how deep should the searchlight be?

Solution



Vertex point: $(0,\ 0)$ and the parabola is open up.

Given: $p = 2$

$$x^2 = 8y \qquad\qquad x^2 = 4py$$

The opening is 5 *feet* across – $(2.5,\ y)$

$$y = \frac{x^2}{8}$$

$$= \frac{2.5^2}{8}$$

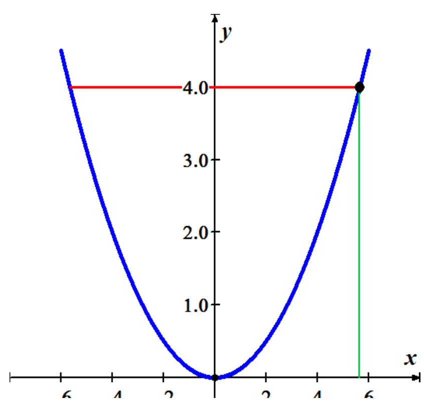
$$= 0.78125\ ft$$

The depth of the searchlight should be 0.78125 *feet*.

Exercise

A searchlight is shaped like a paraboloid of revolution. If the light source is located 2 *feet* from the base along the axis of symmetry and the depth of the searchlight is 4 *feet* across, how deep should the opening be?

Solution



Vertex point: $(0,\ 0)$ and the parabola is open up.

Given: $p = 2$

$$x^2 = 8y \qquad\qquad x^2 = 4py$$

The depth is 4 *feet* – $(x,\ 4)$

$$x^2 = 8(4)$$

$$= 32$$

$x = \pm 4\sqrt{2}\ ft$

The width of the opening of the searchlight should be $2\left(4\sqrt{2}\right) = 11.31\ \ feet$.

Exercise

A searchlight is shaped like a paraboloid, with the light source at the focus. If the reflector is 3 *feet* across at the opening and 1 *foot* deep, where is the focus?

Solution

Vertex point: $(0,\ 0)$ and the parabola is open up.

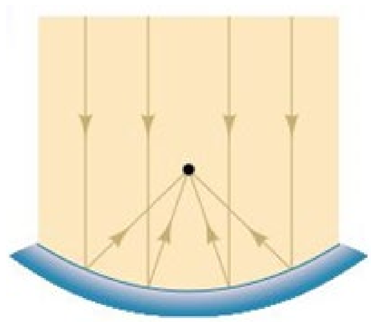
$2x = 3 \ \rightarrow \ x = \frac{3}{2}$

$1 = \frac{1}{4p}\left(\frac{3}{2}\right)^2 \qquad\qquad y = \frac{1}{4p}x^2$

$p = \frac{9}{16}\ \ ft$

Exercise

A mirror for a reflecting telescope has the shape of a (finite) paraboloid of diameter 8 *inches* and depth 1 *inch*. How far from the center the mirror will the incoming light collect?



Solution

Vertex point: $(0,\ 0)$ and passing through $P\left(\frac{8}{2},\ 1\right) = (4,\ 1)$

$1 = \frac{1}{4p}(4)^2 \qquad\qquad y = \frac{1}{4p}x^2$

$p = \frac{16}{4} = 4$

The light will collect 4 *inches* from the center of the mirror.