Solution   *Section* 1.1 – Idea of Limits

Exercise

Find the average rate of change of the function $f(x)=x^3+1$ over the interval $[2,\ 3]$

Solution

$$\frac{\Delta f}{\Delta x}=\frac{f(3)-f(2)}{3-2}$$

$$=\frac{3^3+1-\left(2^3+1\right)}{1}$$

$$=27+1-(8+1)$$

$$=19$$

Exercise

Find the average rate of change of the function $f(x)=x^2$ over the interval $[-1,\ 1]$

Solution

$$\frac{\Delta f}{\Delta x}=\frac{f(1)-f(-1)}{1-(-1)}$$

$$=\frac{1^2-(-1)^2}{2}$$

$$=\frac{0}{2}$$

$$=0$$

Exercise

Find the average rate of change of the function $f(t)=2+\cos t$ over the interval $[-\pi,\ \pi]$

Solution

$$\frac{\Delta f}{\Delta x}=\frac{f(\pi)-f(-\pi)}{\pi-(-\pi)}$$

$$=\frac{2+\cos\pi-\left(2+\cos(-\pi)\right)}{2\pi}$$

$$=\frac{2-1-(2-1)}{2}$$

$$=0$$

Exercise

Find the slope of $y = x^2 - 3$ at the point $P(2,\ 1)$ and an equation of the tangent line at this *P*.

Solution

$$\frac{\Delta y}{\Delta x} = \frac{f(2+h) - f(2)}{h} \qquad\qquad \frac{\Delta y}{\Delta x} = \frac{f\left(x_1 + h\right) - f\left(x_1\right)}{h}$$

$$= \frac{(2+h)^2 - 3 - \left(2^2 - 3\right)}{h}$$

$$= \frac{4 + 4h + h^2 - 3 - (4-3)}{h}$$

$$= \frac{4h + h^2}{h}$$

$$\underline{= 4 + h|}$$

As h approaches 0. Then the secant slope $h + 4 \to 4 = slope$

$$y = 4(x-2) + 1 \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$y - 1 + 1 = 4x - 8 + 1$$

$$\underline{y = 4x - 7\ |}$$

Exercise

Find the slope of $y = x^2 - 2x - 3$ at the point $P(2,\ -3)$ and an equation of the tangent line at this *P*.

Solution

$$\frac{\Delta y}{\Delta x} = \frac{f(2+h) - f(2)}{h}$$

$$= \frac{(2+h)^2 - 2(2+h) - 3 - \left(2^2 - 2(2) - 3\right)}{h}$$

$$= \frac{4 + 4h + h^2 - 4 - 2h - 3 - (-3)}{h}$$

$$= \frac{2h + h^2}{h}$$

$= 2 + h$ As h approaches 0. Then the secant slope $2 + h \to 2 = slope$

$$y + 3 = 2(x-2) \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$y = 2x - 4 - 3$$

$$\underline{y = 2x - 7\ |}$$

Exercise

Find the slope of $y = x^3$ at the point $P(2,\ 8)$ and an equation of the tangent line at this P.

Solution

$$\frac{\Delta y}{\Delta x} = \frac{f(2+h) - f(2)}{h}$$

$$= \frac{(2+h)^3 - 2^3}{h}$$

$$= \frac{8 + 12h + 6h^2 + h^3 - 8}{h}$$

$$= 12 + 6h + h^2$$

As h approaches 0. Then $slope = 12$

$$y - 8 = 12(x - 2) \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$y = 12x - 24 + 8$$

$$y = 12x - 16$$

Exercise

Make a table of values for the function $f(x) = \dfrac{x+2}{x-2}$ at the points

$$x = 1.2, \quad x = \frac{11}{10}, \quad x = \frac{101}{100}, \quad x = \frac{1001}{1000}, \quad x = \frac{10001}{10000}, \quad and\ x = 1$$

a) Find the average rate of change of $f(x)$ over the intervals $[1,\ x]$ for each $x \ne 1$ in the table

b) Extending the table if necessary, try to determine the rate of change of $f(x)$ at $x = 1$.

Solution

a)

x	1.2	1.1	1.01	1.001	1.0001	1
$f(x)$	-4.0	$-3.\overline{4}$	$-3.\overline{04}$	$-3.\overline{004}$	$-3.\overline{004}$	-3

$$\frac{\Delta y}{\Delta x} = \frac{-4 - (-3)}{1.2 - 1} = -5.0$$

$$\frac{\Delta y}{\Delta x} = \frac{-3.\overline{4} - (-3)}{1.1 - 1} = -4.\overline{4}$$

$$\frac{\Delta y}{\Delta x} = \frac{-3.\overline{04} - (-3)}{1.01 - 1} = -4.\overline{04}$$

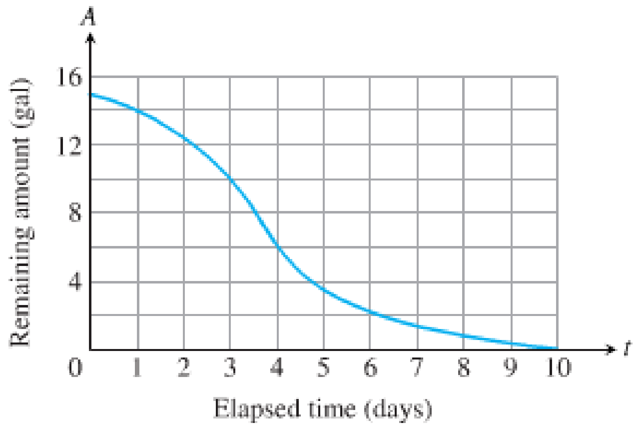
$$\frac{\Delta y}{\Delta x} = \frac{-3.\overline{004} - (-3)}{1.001 - 1} = -4.\overline{004}$$

$$\frac{\Delta y}{\Delta x} = \frac{-3.\overline{0004} - (-3)}{1.0001 - 1} = -4.\overline{0004}$$

b) The rate of change of $f(x)$ at $x = 1$ is -4

Exercise

The accompanying graph shows the total amount of gasoline A in the gas tank of an automobile after being driven for t days.



a) Estimate the average rate of gasoline consumption over the time intervals

$$[0,\ 3],\ [0,\ 5],\ and\ [7,\ 10]$$

b) Estimate the instantaneous rate of gasoline consumption over the time $t=1,\ t=4,\ and\ t=8$

Solution

a) Average rate of gasoline consumption over the time intervals:

$[0,\ 3]$

$$Average\ Rate = \frac{10-15}{3-0}$$

$$= -\frac{5}{3}$$

$$\approx -1.67\ gal\,/\,day$$

$[0,\ 5]$

$$Average\ Rate = \frac{3.9-15}{3-0}$$

$$\approx -2.2\ \ gal\,/\,day$$

$[7,\ 10]$

$$Average\ Rate = \frac{0-1.4}{10-7}$$

$$= -\frac{1.4}{3}$$

$$= -\frac{7}{15}$$

$$\approx -0.5\ \ gal\,/\,day$$

b) At $t=1 \quad \to P(1,\ 14)$

At $t=4 \quad \to P(4,\ 6)$

At $t=8 \quad \to P(8,\ 1)$

Solution   *Section* 1.2 – Definitions / Techniques of Limits

Exercise

Find the limit: $\lim\limits_{x\to 3}(-1)$

Solution

$$\lim_{x\to 3}(-1)\ \underline{=-1}$$

Exercise

Find the limit: $\lim\limits_{x\to -1}(3)$

Solution

$$\lim_{x\to -1}(3)\ \underline{=3}$$

Exercise

Find the limit: $\lim\limits_{x\to 1000} 18\pi^2$

Solution

$$\lim_{x\to 1000} 18\pi^2\ \underline{=18\pi^2}$$

Exercise

Find the limit: $\lim\limits_{x\to 1}\sqrt{5x+6}$

Solution

$$\lim_{x\to 1}\sqrt{5x+6}\ \underline{=\sqrt{11}}$$

Exercise

Find the limit: $\lim\limits_{x\to 9}\sqrt{x}$

Solution

$$\lim_{x\to 9}\sqrt{x}=\sqrt{9}$$

$$\underline{=3}$$

Exercise

Find the limit: $\lim\limits_{x\to -3}\left(x^2+3x\right)$

Solution

$$\lim_{x\to -3}\left(x^2+3x\right)=(-3)^2+3(-3)$$
$$=9-9$$
$$\underline{=0}$$

Exercise

Find the limit: $\lim\limits_{x\to -4}|x-4|$

Solution

$$\lim_{x\to -4}|x-4|=|-4-4|$$
$$=|-8|$$
$$\underline{=8}$$

Exercise

Find the limit: $\lim\limits_{x\to 4}(x+2)$

Solution

$$\lim_{x\to 4}(x+2)=4+2$$
$$\underline{=6}$$

Exercise

Find the limit: $\lim\limits_{x\to 4}(x-4)$

Solution

$$\lim_{x\to 4}(x-4)=4-4$$
$$\underline{=0}$$

Exercise

Find the limit: $\lim\limits_{x\to 2} (5x-6)^{3/2}$

Solution

$$\lim_{x\to 2} (5x-6)^{3/2} = (10-6)^{3/2}$$
$$= \sqrt{4^3}$$
$$\underline{= 8}$$

Exercise

Find the limit: $\lim\limits_{x\to 9} \dfrac{x-9}{\sqrt{x}-3}$

Solution

$$\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3} = \frac{9-9}{3-3} = \frac{0}{0}$$
$$= \lim_{x\to 9} \frac{\left(\sqrt{x}-3\right)\left(\sqrt{x}+3\right)}{\sqrt{x}-3}$$
$$= \lim_{x\to 9} \left(\sqrt{x}+3\right)$$
$$\underline{= 6}$$

Exercise

Find the limit: $\lim\limits_{x\to 1} (2x+4)$

Solution

$$\lim_{x\to 1} (2x+4) = 2(1)+4$$
$$\underline{= 6}$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{x^2-4}{x-2}$

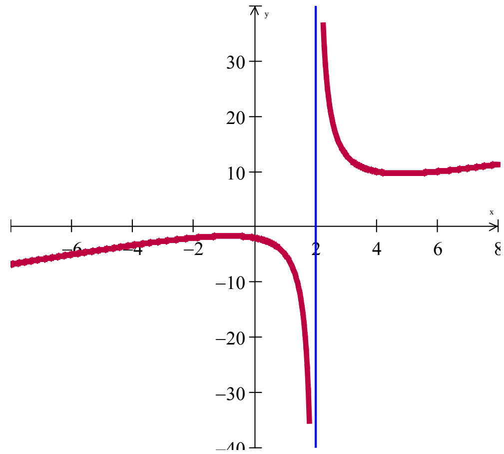
Solution

$$\lim_{x\to 1} \frac{x^2-4}{x-2} = \frac{1^2-4}{1-2}$$
$$= \frac{-3}{-1}$$

$\boxed{=3}$

Exercise

Find the limit: $\lim\limits_{x\to 2} \dfrac{x^2+4}{x-2}$



Solution

$$\lim_{x\to 2} \frac{x^2+4}{x-2} = \frac{2^2+4}{2-2}$$

$$= \frac{8}{0}$$

$$\boxed{=\infty} \qquad (\textbf{\textit{Doesn't exist}})$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{|x|}{x}$

Solution

$$\lim_{x\to 0} \frac{|x|}{x} = \frac{0}{0}$$

$$\lim_{x\to 0^-} \frac{|x|}{x} = \frac{x}{-x} = -1$$

$$\lim_{x\to 0^+} \frac{|x|}{x} = \frac{x}{x} = 1$$

Doesn't exist

Exercise

Find: $\lim\limits_{x\to 3} \dfrac{x^2-x-1}{\sqrt{x+1}}$

Solution

$$\lim_{x\to 3} \frac{x^2-x-1}{\sqrt{x+1}} = \frac{3^2-3-1}{\sqrt{3+1}}$$

$$\boxed{=\frac{5}{2}}$$

Exercise

Find: $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{x-2}$

Solution

$$\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \frac{2^2+2-6}{2-2} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{(x+3)(x-2)}{x-2}$$

$$= \lim_{x\to 2} (x+3)$$

$$= 5$$

Exercise

Find the limit: $\lim\limits_{x\to 0} (3x-2)$

Solution

$$\lim_{x\to 0} (3x-2) = 3(0)-2$$

$$= -2$$

Exercise

Find the limit: $\lim\limits_{x\to 1} (2x^2-x+4)$

Solution

$$\lim_{x\to 1} (2x^2-x+4) = 2(1)^2-(1)+4$$

$$= 5$$

Exercise

Find the limit: $\lim\limits_{x\to -2} \left(x^3-2x^2+4x+8\right)$

Solution

$$\lim_{x\to -2} \left(x^3-2x^2+4x+8\right) = (-2)^3-2(-2)^2+4(-2)+8$$

$$= -16$$

Exercise

Find the limit: $\lim\limits_{x\to 2} \frac{x^2-4}{x-2}$

Solution

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\frac{2^2-4}{2-2}=\frac{0}{0}$$

$$=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}$$

$$=\lim_{x\to 2}(x+2)$$

$$\underline{=4}$$

Exercise

Find the limit: $\lim\limits_{x\to 2}\frac{x^3-8}{x-2}$

Solution

$$\lim_{x\to 2}\frac{x^3-8}{x-2}=\frac{0}{0}$$

$$=\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{x-2}$$

$$=\lim_{x\to 2}x^2+2x+4$$

$$=2^2+2(2)+4$$

$$\underline{=12}$$

Exercise

Find the limit: $\lim\limits_{x\to 3}\frac{x^2+x-12}{x-3}$

Solution

$$\lim_{x\to 3}\frac{x^2+x-12}{x-3}=\frac{0}{0}$$

$$=\lim_{x\to 3}\frac{(x-3)(x+4)}{x-3}$$

$$=\lim_{x\to 3}(x+4)$$

$$\underline{=7}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{\sqrt{x+4}-2}{x}$

Solution

$$\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x} = \frac{\sqrt{4}-2}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sqrt{x+4}-2}{x} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$$

$$= \lim_{x\to 0} \frac{x+4-4}{x\left(\sqrt{x+4}+2\right)}$$

$$= \lim_{x\to 0} \frac{x}{x\left(\sqrt{x+4}+2\right)}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{x+4}+2}$$

$$= \frac{1}{\sqrt{4}+2}$$

$$= \frac{1}{4}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{3}{\sqrt{3x+1}+1}$

Solution

$$\lim_{x\to 0} \frac{3}{\sqrt{3x+1}+1} = \frac{3}{\sqrt{3(0)+1}+1}$$

$$= \frac{3}{1+1}$$

$$= \frac{3}{2}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} f(x) \qquad f(x) = \begin{cases} x^2+1 & x<0 \\ 2x+1 & x>0 \end{cases}$

Solution

$$\lim_{x\to 0^-} x^2+1 = 1$$

$$\lim_{x\to 0+} 2x+1 = 1$$

$$\lim_{x\to 0} f(x) \underline{=1}$$

Exercise

Find the limit: $\lim\limits_{x\to -2} \frac{5}{x+2}$

Solution

$$\lim_{x\to -2} \frac{5}{x+2} = \frac{5}{0}$$

$$\underline{=\infty}$$

Exercise

Find the limit: $\lim\limits_{x\to 3} \frac{\sqrt{x+1}-1}{x}$

Solution

$$\lim_{x\to 3} \frac{\sqrt{x+1}-1}{x} = \frac{\sqrt{3+1}-1}{3} = \frac{2-1}{3}$$

$$\underline{=\frac{1}{3}}$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \frac{x^2-1}{x-1}$

Solution

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1}$$

$$= \lim_{x\to 1} (x+1)$$

$$\underline{=2}$$

Exercise

Find the limit: $\lim\limits_{x\to -2} \dfrac{|x+2|}{x+2}$

Solution

$$\lim_{x\to -2} \frac{|x+2|}{x+2} = \frac{|-2+2|}{-2+2} = \frac{0}{0}$$

$$\lim_{x\to -2^+} \frac{|x+2|}{x+2} = \frac{(x+2)}{(x+2)}$$

$$\underline{=1}$$

$$\lim_{x\to -2^-} \frac{|x+2|}{x+2} = \frac{(x+2)}{-(x+2)}$$

$$\underline{=-1}$$

Doesn't exist

Exercise

Find the limit: $\lim\limits_{x\to 0} (2x-8)^{1/3}$

Solution

$$\lim_{x\to 0} (2x-8)^{1/3} = (2(0)-8)^{1/3}$$

$$=(-8)^{1/3}$$

$$\underline{=-2}$$

Exercise

Find the limit: $\lim\limits_{x\to 2} \dfrac{x^2-7x+10}{x-2}$

Solution

$$\lim_{x\to 2} \frac{x^2-7x+10}{x-2} = \frac{2^2-7(2)+10}{2-2} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{(x-2)(x-5)}{x-2}$$

$$= \lim_{x\to 2} (x-5)$$

$$= 2-5$$

$$\underline{=-3}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{5x^3+8x^2}{3x^4-16x^2}$

Solution

$$\lim_{x\to 0} \frac{5x^3+8x^2}{3x^4-16x^2} = \frac{0}{0}$$

$$\begin{aligned}\lim_{x\to 0} \frac{5x^3+8x^2}{3x^4-16x^2} &= \lim_{x\to 0} \frac{x^2(5x+8)}{x^2\left(3x^2-16\right)} \\ &= \lim_{x\to 0} \frac{5x+8}{3x^2-16} \\ &= \frac{8}{-16} \\ &= -\frac{1}{2}\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{\frac{1}{x}-1}{x-1}$

Solution

$$\lim_{x\to 1} \frac{\frac{1}{x}-1}{x-1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$\begin{aligned}\lim_{x\to 1} \frac{\frac{1}{x}-1}{x-1} &= \lim_{x\to 1} \frac{\frac{1-x}{x}}{x-1} \\ &= \lim_{x\to 1} \left(\frac{1-x}{x}\right)\left(\frac{1}{x-1}\right) \\ &= \lim_{x\to 1} \left(\frac{-(x-1)}{x}\right)\left(\frac{1}{x-1}\right) \\ &= \lim_{x\to 1} \frac{-1}{x} \\ &= -1\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{u\to 1} \dfrac{u^4-1}{u^3-1}$

Solution

$$\lim_{u\to 1}\frac{u^4-1}{u^3-1}=\lim_{u\to 1}\frac{\left(u^2-1\right)\left(u^2+1\right)}{(u-1)\left(u^2+u+1\right)}$$

$$=\lim_{u\to 1}\frac{(u-1)(u+1)\left(u^2+1\right)}{(u-1)\left(u^2+u+1\right)}$$

$$=\lim_{u\to 1}\frac{(u+1)\left(u^2+1\right)}{u^2+u+1}$$

$$=\frac{(1+1)\left(1^2+1\right)}{1^2+1+1}$$

$$=\frac{4}{3}$$

Exercise

Find the limit: $\lim_{x\to 1}\frac{x-1}{\sqrt{x+3}-2}$

Solution

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x+3}-2}=\frac{1-1}{\sqrt{1+3}-2}$$

$$=\frac{0}{\sqrt{4}-2}=\frac{0}{0}$$

$$=\lim_{x\to 1}\frac{x-1}{\sqrt{x+3}-2}\bullet\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}$$

$$=\lim_{x\to 1}\frac{(x-1)\left(\sqrt{x+3}+2\right)}{x+3-4}$$

$$=\lim_{x\to 1}\frac{(x-1)\left(\sqrt{x+3}+2\right)}{x-1}$$

$$=\lim_{x\to 1}\left(\sqrt{x+3}+2\right)$$

$$=\sqrt{1+3}+2$$

$$=4$$

Exercise

Find the limit: $\lim\limits_{x \to -1} \dfrac{\sqrt{x^2+8}-3}{x+1}$

Solution

$$\lim_{x \to -1} \frac{\sqrt{x^2+8}-3}{x+1} = \frac{\sqrt{(-1)^2+8}-3}{-1+1} = \frac{\sqrt{9}-3}{0} = \frac{0}{0}$$

$$= \lim_{x \to -1} \frac{\sqrt{x^2+8}-3}{x+1} \cdot \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}$$

$$= \lim_{x \to -1} \frac{x^2+8-9}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x \to -1} \frac{x^2-1}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x \to -1} \frac{(x-1)(x+1)}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x \to -1} \frac{(x-1)}{\sqrt{x^2+8}+3}$$

$$= \frac{-2}{\sqrt{9}+3}$$

$$= \frac{-2}{6}$$

$$\boxed{= -\frac{1}{3}}$$

Exercise

Find the limit: $\lim\limits_{x \to -3} \dfrac{2-\sqrt{x^2-5}}{x+3}$

Solution

$$\lim_{x \to -3} \frac{2-\sqrt{x^2-5}}{x+3} = \frac{2-\sqrt{(-3)^2-5}}{-3+3}$$

$$= \frac{2-\sqrt{9-5}}{0}$$

$$= \frac{2-\sqrt{4}}{0} = \frac{0}{0}$$

$$= \lim_{x\to -3} \frac{2-\sqrt{x^2-5}}{x+3} \cdot \frac{2+\sqrt{x^2-5}}{2+\sqrt{x^2-5}}$$

$$= \lim_{x\to -3} \frac{4-\left(x^2-5\right)}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{4-x^2+5}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{9-x^2}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{(x-3)(x+3)}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{(x-3)}{2+\sqrt{x^2-5}}$$

$$= \frac{-6}{2+\sqrt{9-5}}$$

$$= \frac{-6}{2+\sqrt{4}}$$

$$= -\frac{6}{4}$$

$$= -\frac{3}{2}$$

Exercise

Find the limit: $\lim_{x\to 0} (2\sin x - 1)$

Solution

$$\lim_{x\to 0} (2\sin x - 1) = 2\sin(0) - 1$$

$$= 0 - 1$$

$$= -1$$

Exercise

Find the limit: $\lim_{x\to 0} \sin^2 x$

Solution

$$\lim_{x\to 0} \sin^2 x = \sin^2(0)$$

$$\underline{= 0}$$

Exercise

Find the limit: $\lim_{x\to 0} \sec x$

Solution

$$\lim_{x\to 0} \sec x = \sec(0)$$

$$= \frac{1}{\cos(0)}$$

$$\underline{= 1}$$

Exercise

Find the limit: $\lim_{x\to 0} \frac{1+x+\sin x}{3\cos x}$

Solution

$$\lim_{x\to 0} \frac{1+x+\sin x}{3\cos x} = \frac{1+0+\sin(0)}{3\cos(0)}$$

$$\underline{= \frac{1}{3}}$$

Exercise

Find the limit: $\lim_{x\to -\pi} \sqrt{x+4}\, \cos(x+\pi)$

Solution

$$\lim_{x\to -\pi} \sqrt{x+4}\, \cos(x+\pi) = \sqrt{-\pi+4}\, \cos(-\pi+\pi)$$

$$= \sqrt{-\pi+4}\, \cos(0)$$

$$\underline{= \sqrt{4-\pi}}$$

Exercise

Find $\lim_{x\to -0.5^-} \sqrt{\frac{x+2}{x+1}}$

Solution

$$\lim_{x\to -0.5^-}\sqrt{\frac{x+2}{x+1}} = \sqrt{\frac{-0.5+2}{-0.5+1}}$$

$$= \sqrt{\frac{1.5}{0.5}}$$

$$= \sqrt{3}$$

Exercise

Find $\lim_{x\to 1^+}\sqrt{\frac{x-1}{x+2}}$

Solution

$$\lim_{x\to 1^+}\sqrt{\frac{x-1}{x+2}} = \sqrt{\frac{1-1}{1+2}}$$

$$= 0$$

Exercise

Find $\lim_{x\to -2^+}\left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right)$

Solution

$$\lim_{x\to -2^+}\left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right) = \left(\frac{-2}{-2+1}\right)\left(\frac{2(-2)+5}{(-2)^2+(-2)}\right)$$

$$= \left(\frac{-2}{-1}\right)\left(\frac{1}{2}\right)$$

$$= 1$$

Exercise

Find $\lim_{x\to 0^+}\frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}$

Solution

$$\lim_{x\to 0^+}\frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x} = \frac{\sqrt{5}-\sqrt{5}}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0^+}\frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}\frac{\sqrt{x^2+4x+5}+\sqrt{5}}{\sqrt{x^2+4x+5}+\sqrt{5}}$$

$$= \lim_{x\to 0^+} \frac{x^2+4x+5-5}{x\left(\sqrt{x^2+4x+5}+\sqrt{5}\right)}$$

$$= \lim_{x\to 0^+} \frac{x^2+4x}{x\left(\sqrt{x^2+4x+5}+\sqrt{5}\right)}$$

$$= \lim_{x\to 0^+} \frac{x(x+4)}{x\left(\sqrt{x^2+4x+5}+\sqrt{5}\right)}$$

$$= \lim_{x\to 0^+} \frac{x+4}{\sqrt{x^2+4x+5}+\sqrt{5}}$$

$$= \frac{0+4}{\sqrt{0^2+4(0)+5}+\sqrt{5}}$$

$$= \frac{4}{\sqrt{5}+\sqrt{5}}$$

$$= \frac{4}{2\sqrt{5}}$$

$$= \frac{2}{\sqrt{5}}$$

Exercise

Find $\displaystyle\lim_{x\to -2^+} (x+3)\frac{|x+2|}{x+2}$

Solution

$$\lim_{x\to -2^+} (x+3)\frac{|x+2|}{x+2} = (x+3)\frac{|-2+2|}{-2+2} = \frac{0}{0}$$

Since $x \to -2^+ \quad \Rightarrow \quad x > -2$

$\Rightarrow \quad |x+2| = (x+2)$

$$\lim_{x\to -2^+} (x+3)\frac{|x+2|}{x+2} = \lim_{x\to -2^+} (x+3)\frac{x+2}{x+2}$$

$$= \lim_{x\to -2^+} (x+3)$$

$$= -2+3$$

$$= 1$$

Exercise

Find $\lim\limits_{x\to 1^+} \dfrac{\sqrt{2x}(x-1)}{|x-1|}$

Solution

$$\lim_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|} = \frac{\sqrt{2(1)}(1-1)}{|1-1|} = \frac{0}{0}$$

Since $x \to 1^+ \quad \Rightarrow \quad x > 1$

$\Rightarrow \quad |x-1| = x-1$

$$\lim_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|} = \lim_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{x-1}$$

$$= \lim_{x\to 1^+} \sqrt{2x}$$

$$= \sqrt{2}$$

Exercise

Find $\lim\limits_{\theta\to 0} \dfrac{\sin\sqrt{2}.\theta}{\sqrt{2}.\theta}$

Solution

Let: $\sqrt{2}\theta = x \to 0$

$$\lim_{\theta\to 0} \frac{\sin\sqrt{2}.\theta}{\sqrt{2}.\theta} = \lim_{x\to 0} \frac{\sin x}{x}$$

$$= 1$$

Exercise

Find $\lim\limits_{x\to 0} \dfrac{\sin 3x}{4x}$

Solution

$$\lim_{x\to 0} \frac{\sin 3x}{4x} = \lim_{x\to 0} \frac{\sin 3x}{4x}\frac{3}{3}$$

$$= \frac{3}{4}\lim_{x\to 0} \frac{\sin 3x}{3x}$$ Let: $3x = u$

$$= \frac{3}{4}\lim_{u\to 0} \frac{\sin u}{u}$$ ***By definition:*** $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

$$= \frac{3}{4}$$

Exercise

Find $\lim\limits_{x\to 0^-} \dfrac{x}{\sin 3x}$

Solution

$$\lim_{x\to 0^-} \frac{x}{\sin 3x} = \lim_{x\to 0^-} \frac{x}{\sin 3x}\left(\frac{3}{3}\right)$$

$$= \frac{1}{3}\lim_{x\to 0^-} \frac{3x}{\sin 3x}$$

$$= \frac{1}{3}\lim_{x\to 0^-} \frac{1}{\frac{\sin 3x}{3x}}$$

By definition: $\lim\limits_{u\to 0} \dfrac{\sin u}{u} = 1$

$$= \frac{1}{3}$$

Exercise

Find $\lim\limits_{x\to 0} \dfrac{\tan 2x}{x}$

Solution

$$\lim_{x\to 0} \frac{\tan 2x}{x} = \lim_{x\to 0} \frac{\frac{\sin 2x}{\cos 2x}}{x}$$

$$= \lim_{x\to 0} \left(\frac{\sin 2x}{x} \bullet \frac{1}{\cos 2x}\right)$$

$$= \lim_{x\to 0} \left(2\frac{\sin 2x}{2x}\right) \lim_{x\to 0} \left(\frac{1}{\cos 2x}\right)$$

$$= 2\,\frac{1}{\cos 0}$$

$$= 2$$

Exercise

Find $\lim\limits_{x\to 0} 6x^2(\cot x)(\csc 2x)$

Solution

$$\lim_{x\to 0} 6x^2(\cot x)(\csc 2x) = \lim_{x\to 0} 6x^2\left(\frac{\cos x}{\sin x}\right)\left(\frac{1}{\sin 2x}\right)$$

$$= \lim_{x\to 0} 3\cos x\left(\frac{x}{\sin x}\right)\left(\frac{2x}{\sin 2x}\right)$$

$$= 3\lim_{x\to 0} (\cos x) \bullet \lim_{x\to 0} \left(\frac{x}{\sin x}\right) \bullet \lim_{2x\to 0} \left(\frac{2x}{\sin 2x}\right)$$

$$= (3)(1)(1)(1)$$

$$= 3$$

Exercise

Find $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\sin 2\theta}$

Solution

$$\lim_{\theta \to 0} \frac{\sin\theta}{\sin 2\theta} = \lim_{\theta \to 0} \frac{\sin\theta}{\sin 2\theta}\frac{2\theta}{2\theta}$$

$$= \frac{1}{2}\lim_{\theta \to 0}\left(\frac{2\theta}{\sin 2\theta} \cdot \frac{\sin\theta}{\theta}\right)$$

$$= \frac{1}{2}(1)(1)$$

$$= \frac{1}{2}$$

Exercise

Find $\lim\limits_{h \to 0} \dfrac{\sin(\sin h)}{\sin h}$

Solution

Let: $\sin h = \theta$ $\qquad \theta = \sin h \xrightarrow[h \to 0]{} 0$

$$\lim_{h \to 0} \frac{\sin(\sin h)}{\sin h} = \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}$$

$$= 1$$

Exercise

Find $\lim\limits_{\theta \to 0} \dfrac{\theta \cot 4\theta}{\sin^2\theta \cot^2 2\theta}$

Solution

$$\lim_{\theta \to 0} \frac{\theta \cot 4\theta}{\sin^2\theta \cot^2 2\theta} = \lim_{\theta \to 0} \frac{\theta \frac{\cos 4\theta}{\sin 4\theta}}{\sin^2\theta \frac{\cos^2 2\theta}{\sin^2 2\theta}}$$

$$= \lim_{\theta \to 0} \theta \frac{\cos 4\theta}{2\sin 2\theta \cos 2\theta} \frac{\sin^2 2\theta}{\sin^2\theta \cos^2 2\theta}$$

$$= \lim_{\theta \to 0} \left(\frac{1}{2} \cdot \theta \cdot \cos 4\theta \bullet \frac{2\sin\theta\cos\theta}{\sin^2\theta} \cdot \frac{1}{\cos^3 2\theta}\right)$$

$$= \lim_{\theta \to 0} \left(\cos 4\theta \bullet \frac{\theta}{\sin\theta} \cdot \cos\theta \cdot \frac{1}{\cos^3 2\theta}\right)$$

$$= \lim_{\theta\to 0}(\cos 4\theta) \quad \lim_{\theta\to 0}\left(\frac{\theta}{\sin\theta}\right) \quad \lim_{\theta\to 0}\left(\frac{\cos\theta}{\cos^3 2\theta}\right)$$

$$=(1)(1)(1)$$

$$\underline{=1\,|}$$

Exercise

Find $\displaystyle \lim_{\theta\to\pi/4}\frac{\sin^2\theta-\cos^2\theta}{\sin\theta-\cos\theta}$

Solution

$$\lim_{\theta\to\pi/4}\frac{\sin^2\theta-\cos^2\theta}{\sin\theta-\cos\theta}=\frac{\frac{1}{2}-\frac{1}{2}}{\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}}=\frac{0}{0}$$

$$=\lim_{\theta\to\pi/4}\frac{(\sin\theta-\cos\theta)(\sin\theta+\cos\theta)}{\sin\theta-\cos\theta}$$

$$=\lim_{\theta\to\pi/4}(\sin\theta+\cos\theta)$$

$$=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}$$

$$\underline{=\sqrt{2}\,|}$$

Exercise

Find $\displaystyle \lim_{x\to\pi/2}\frac{\frac{1}{\sqrt{\sin x}}-1}{x+\frac{\pi}{2}}$

Solution

$$\lim_{x\to\pi/2}\frac{\frac{1}{\sqrt{\sin x}}-1}{x+\frac{\pi}{2}}=\frac{1-1}{\frac{\pi}{2}+\frac{\pi}{2}}$$

$$\underline{=0\,|}$$

Exercise

Find $\displaystyle \lim_{x\to 1}\frac{x^3-7x^2+12x}{4-x}$

Solution

$$\lim_{x\to 1}\frac{x^3-7x^2+12x}{4-x}=\frac{1-7+12}{4-1}$$

$$\underline{=2\,|}$$

Exercise

Find $\lim\limits_{x\to 4} \dfrac{x^3-7x^2+12x}{4-x}$

Solution

$$\lim_{x\to 4} \frac{x^3-7x^2+12x}{4-x} = \frac{64-112+48}{4-4} = \frac{0}{0}$$

$$= \lim_{x\to 4} \frac{x(x-3)(x-4)}{4-x}$$

$$= \lim_{x\to 4} -x(x-3)$$

$$=-4$$

Exercise

Find $\lim\limits_{x\to 1} \dfrac{1-x^2}{x^2-8x+7}$

Solution

$$\lim_{x\to 1} \frac{1-x^2}{x^2-8x+7} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{(1-x)(1+x)}{(x-1)(x-7)}$$

$$= -\lim_{x\to 1} \frac{1+x}{x-7}$$

$$= \frac{1}{3}$$

Exercise

Find $\lim\limits_{x\to 3} \dfrac{\sqrt{3x+16}-5}{x-3}$

Solution

$$\lim_{x\to 3} \frac{\sqrt{3x+16}-5}{x-3} = \frac{\sqrt{9+16}-5}{3-3} = \frac{5-5}{0} = \frac{0}{0}$$

$$= \lim_{x\to 3} \frac{\sqrt{3x+16}-5}{x-3}\frac{\sqrt{3x+16}+5}{\sqrt{3x+16}+5}$$

$$= \lim_{x\to 3} \frac{3x+16-25}{(x-3)\left(\sqrt{3x+16}+5\right)}$$

$$= \lim_{x\to 3} \frac{3(x-3)}{(x-3)\left(\sqrt{3x+16}+5\right)}$$

$$= \lim_{x \to 3} \frac{3}{\sqrt{3x+16}+5}$$

$$= \frac{3}{5+5}$$

$$= \frac{3}{10}$$

Exercise

Find $\displaystyle \lim_{x \to 3} \frac{1}{x-3}\left(\frac{1}{\sqrt{x+1}} - \frac{1}{2}\right)$

Solution

$$\lim_{x \to 3} \frac{1}{x-3}\left(\frac{1}{\sqrt{x+1}} - \frac{1}{2}\right) = \frac{1}{0}\left(\frac{1}{2} - \frac{1}{2}\right) = \frac{0}{0}$$

$$= \lim_{x \to 3} \frac{1}{x-3}\left(\frac{2-\sqrt{x+1}}{\sqrt{x+1}}\right)\left(\frac{2+\sqrt{x+1}}{2+\sqrt{x+1}}\right)$$

$$= \lim_{x \to 3} \frac{1}{x-3}\left(\frac{4-x-1}{2\sqrt{x+1}+x+1}\right)$$

$$= \lim_{x \to 3} \frac{x-3}{x-3}\left(\frac{-1}{2\sqrt{x+1}+x+1}\right)$$

$$= \lim_{x \to 3} \frac{-1}{2\sqrt{x+1}+x+1}$$

$$= -\frac{1}{8}$$

Exercise

Find $\displaystyle \lim_{x \to 1/3} \frac{x-\frac{1}{3}}{(3x-1)^2}$

Solution

$$\lim_{x \to 1/3} \frac{x-\frac{1}{3}}{(3x-1)^2} = \frac{\frac{1}{3}-\frac{1}{3}}{\left(3\frac{1}{3}-1\right)^2} = \frac{0}{0}$$

$$= \lim_{x \to 1/3} \frac{x-\frac{1}{3}}{9\left(x-\frac{1}{3}\right)^2}$$

$$= \lim_{x \to 1/3} \frac{1}{9\left(x-\frac{1}{3}\right)} = \frac{1}{0}$$

$$= \infty$$

Exercise

Find $\lim\limits_{x\to 3} \dfrac{x^4-81}{x-3}$

Solution

$$\lim_{x\to 3} \frac{x^4-81}{x-3} = \frac{81-81}{3-3} = \frac{0}{0}$$

$$= \lim_{x\to 3} \frac{(x-3)(x+3)\left(x^2+9\right)}{x-3} \qquad a^4-b^4 = \left(a^2-b^2\right)\left(a^2+b^2\right) = (a-b)(a+b)\left(a^2+b^2\right)$$

$$= \lim_{x\to 3} (x+3)\left(x^2+9\right) = 6(18)$$

$$\underline{=108}$$

Exercise

Find $\lim\limits_{x\to 1} \dfrac{x^5-1}{x-1}$

Solution

$$\lim_{x\to 1} \frac{x^5-1}{x-1} = \frac{1-1}{1-1} = \frac{0}{0} \qquad \left(a^5-b^5\right) = (a-b)\left(a^4+a^3b+a^2b^2+ab^3+b^4\right)$$

$$= \lim_{x\to 1} \frac{(x-1)\left(x^4+x^3+x^2+x+1\right)}{x-1}$$

$$= \lim_{x\to 1} \left(x^4+x^3+x^2+x+1\right)$$

$$\underline{=5}$$

Exercise

Find $\lim\limits_{x\to 81} \dfrac{\sqrt[4]{x}-3}{x-81}$

Solution

$$\lim_{x\to 81} \frac{\sqrt[4]{x}-3}{x-81} = \frac{3-3}{81-81} = \frac{0}{0}$$

$$= \lim_{x\to 81} \frac{\sqrt[4]{x}-3}{\left(\sqrt{x}+9\right)\left(\sqrt{x}-9\right)}$$

$$= \lim_{x\to 81} \frac{\sqrt[4]{x}-3}{\left(\sqrt{x}+9\right)\left(\sqrt[4]{x}+3\right)\left(\sqrt[4]{x}-3\right)}$$

$$= \lim_{x\to 81} \frac{1}{\left(\sqrt{x}+9\right)\left(\sqrt[4]{x}+3\right)}$$

$$= \frac{1}{(18)(6)}$$

$$= \frac{1}{108}$$

Exercise

Find the limit: $\displaystyle \lim_{x\to 1} \frac{\sqrt[3]{x}-1}{x-1}$

Solution

$$\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{x-1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\left(\sqrt[3]{x}\right)^3 - 1^3}$$

$$= \lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\left(\sqrt[3]{x}-1\right)\left(x^{2/3}+\sqrt[3]{x}+1\right)}$$

$$= \lim_{x\to 1} \frac{1}{x^{2/3}+\sqrt[3]{x}+1}$$

$$= \frac{1}{3}$$

Exercise

Find the limit: $\displaystyle \lim_{x\to 2} \frac{x^5-32}{x-2}$

Solution

$$\begin{array}{r|rrrrrr} 2 & 1 & 0 & 0 & 0 & 0 & -32 \\ & & 2 & 4 & 8 & 16 & 32 \\ \hline & 1 & 2 & 4 & 8 & 16 & 0 \end{array}$$

$$\lim_{x\to 2} \frac{x^5-32}{x-2} = \frac{2^5-32}{2-2} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{(x-2)\left(x^4+2x^3+4x^2+8x+16\right)}{x-2}$$

$$= \lim_{x\to 2} \left(x^4+2x^3+4x^2+8x+16\right)$$

$$= 16+16+16+16+16$$

$$= 80$$

Exercise

Find the limit: $\lim\limits_{x \to 1} \frac{x^6 - 1}{x - 1}$

Solution

$$\begin{array}{c|ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ & & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}$$

$$\lim\limits_{x \to 1} \frac{x^6 - 1}{x - 1} = \frac{0}{0}$$

$$= \lim\limits_{x \to 1} \frac{(x-1)\left(x^5 + x^4 + x^3 + x^2 + x + 1\right)}{x - 1}$$

$$= \lim\limits_{x \to 1} \left(x^5 + x^4 + x^3 + x^2 + x + 1\right)$$

$$= 6$$

Exercise

Find the limit: $\lim\limits_{x \to -1} \frac{x^7 + 1}{x + 1}$

Solution

$$\begin{array}{c|cccccccc} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ & & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 0 \end{array}$$

$$\lim\limits_{x \to -1} \frac{x^7 + 1}{x + 1} = \frac{-1+1}{-1+1} = \frac{0}{0}$$

$$= \lim\limits_{x \to 1} \frac{(x+1)\left(x^6 - x^5 + x^4 - x^3 + x^2 - x + 1\right)}{x + 1}$$

$$= \lim\limits_{x \to 1} \left(x^6 - x^5 + x^4 - x^3 + x^2 - x + 1\right)$$

$$= 1$$

Exercise

Find the limit: $\lim\limits_{x \to a} \frac{x^5 - a^5}{x - a}$

Solution

$$\begin{array}{c|cccccc} a & 1 & 0 & 0 & 0 & 0 & -a^5 \\ & & a & a^2 & a^3 & a^4 & a^5 \\ \hline & 1 & a & a^2 & a^3 & a^4 & \color{red}{0} \end{array}$$

$$\lim_{x \to a} \frac{x^5 - a^5}{x - a} = \frac{a^5 - a^5}{a - a} \color{red}{= \frac{0}{0}}$$

$$= \lim_{x \to a} \frac{(x-a)\left(x^4 + ax^3 + a^2x^2 + a^3x + a^4\right)}{x - a}$$

$$= \lim_{x \to a} \left(x^4 + ax^3 + a^2x^2 + a^3x + a^4\right)$$

$$= a^4 + a^4 + a^4 + a^4 + a^4$$

$$\color{blue}{= 5a^4}$$

Exercise

Find the limit: $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} \quad n \in \mathbb{Z}^+$

Solution

$$\begin{array}{c|ccccccc} a & 1 & 0 & 0 & 0 & \cdots & 0 & -a^n \\ & & a & a^2 & a^3 & \cdots & a^{n-1} & a^n \\ \hline & 1 & a & a^2 & a^3 & \cdots & a^{n-1} & \color{red}{0} \end{array}$$

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \frac{a^n - a^n}{a - a} \color{red}{= \frac{0}{0}}$$

$$= \lim_{x \to a} \frac{(x-a)\left(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-2}x + a^{n-1}\right)}{x - a}$$

$$= \lim_{x \to a} \left(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-2}x + a^{n-1}\right)$$

$$= a^{n-1} + a^{n-1} + a^{n-1} + \ldots + a^{n-1} + a^{n-1}$$

$$\color{blue}{= na^{n-1}}$$

Exercise

Find the limit: $\lim\limits_{h \to 0} \dfrac{100}{(10h - 1)^{11} + 2}$

Solution

$$\lim_{h\to 0} \frac{100}{(10h-1)^{11}+2} = \frac{100}{(-1)^{11}+2}$$

$$= \frac{100}{-1+2}$$

$$\underline{= 100}$$

Exercise

Find the limit: $\displaystyle \lim_{h\to 0} \frac{(5+h)^2 - 25}{h}$

Solution

$$\lim_{h\to 0} \frac{(5+h)^2 - 25}{h} = \frac{5^2 - 25}{0} = \frac{0}{0}$$

$$= \lim_{h\to 0} \frac{((5+h)-5)((5+h)+5)}{h}$$

$$= \lim_{h\to 0} \frac{h(h+10)}{h}$$

$$= \lim_{h\to 0} (h+10)$$

$$\underline{= 10}$$

Exercise

Find the limit: $\displaystyle \lim_{x\to 3} \frac{\frac{1}{x^2+2x} - \frac{1}{15}}{x-3}$

Solution

$$\lim_{x\to 3} \frac{\frac{1}{x^2+2x} - \frac{1}{15}}{x-3} = \frac{\frac{1}{15} - \frac{1}{15}}{0} = \frac{0}{0}$$

$$= \lim_{x\to 3} \frac{1}{x-3}\left(\frac{1}{x(x+2)} - \frac{1}{15}\right)$$

$$= \lim_{x\to 3} \frac{1}{x-3}\left(\frac{15-x^2-2x}{15x(x+2)}\right)$$

$$= \lim_{x\to 3} \frac{-(x-3)(x+5)}{15x(x+2)(x-3)}$$

$$= \lim_{x\to 3} \frac{-(x+5)}{15x(x+2)}$$

$$= -\frac{8}{15(3)(5)}$$

$$\boxed{= -\frac{8}{225}}$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \frac{\sqrt{10x-9}-1}{x-1}$

Solution

$$\lim_{x\to 1} \frac{\sqrt{10x-9}-1}{x-1} = \frac{1-1}{0} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{\sqrt{10x-9}-1}{x-1} \cdot \frac{\sqrt{10x-9}+1}{\sqrt{10x-9}+1}$$

$$= \lim_{x\to 1} \frac{10x-9-1}{(x-1)\left(\sqrt{10x-9}+1\right)}$$

$$= \lim_{x\to 1} \frac{10(x-1)}{(x-1)\left(\sqrt{10x-9}+1\right)}$$

$$= \lim_{x\to 1} \frac{10}{\sqrt{10x-9}+1}$$

$$= \frac{10}{2}$$

$$\boxed{= 5}$$

Exercise

Find the limit: $\lim\limits_{x\to 2} \left(\frac{1}{x-2} - \frac{2}{x^2-2x}\right)$

Solution

$$\lim_{x\to 2} \left(\frac{1}{x-2} - \frac{2}{x^2-2x}\right) = \frac{1}{0} - \frac{2}{0} = \infty - \infty$$

$$= \lim_{x\to 2} \left(\frac{1}{x-2} - \frac{2}{x(x-2)}\right)$$

$$= \lim_{x\to 2} \frac{x-2}{x(x-2)}$$

$$= \lim_{x\to 2} \frac{1}{x}$$

$$\boxed{= \frac{1}{2}}$$

Exercise

Find the limit: $\lim\limits_{x \to c} \dfrac{x^2 - 2cx + c^2}{x - c}$

Solution

$$\lim_{x \to c} \frac{x^2 - 2cx + c^2}{x - c} = \frac{c^2 - 2c^2 + c^2}{0} = \frac{0}{0}$$

$$= \lim_{x \to c} \frac{(x-c)^2}{x-c}$$

$$= \lim_{x \to c} (x-c)$$

$$= 0$$

Exercise

Find the limit: $\lim\limits_{x \to -c} \dfrac{x^2 + 5cx + 4c^2}{x^2 + cx}$

Solution

$$\lim_{x \to -c} \frac{x^2 + 5cx + 4c^2}{x^2 + cx} = \frac{c^2 - 5c^2 + 4c^2}{c^2 - c^2} = \frac{0}{0}$$

$$= \lim_{x \to -c} \frac{(x+c)(x+4c)}{x(x+c)}$$

$$= \lim_{x \to -c} \frac{x+4c}{x}$$

$$= \frac{-c+4c}{-c}$$

$$= \frac{3c}{-c}$$

$$= -3$$

Exercise

Find the limit: $\lim\limits_{x \to 16} \dfrac{\sqrt[4]{x} - 2}{x - 16}$

Solution

$$\lim_{x \to 16} \frac{\sqrt[4]{x} - 2}{x - 16} = \frac{\sqrt[4]{16} - 2}{16 - 16} = \frac{2-2}{0} = \frac{0}{0}$$

$$\lim_{x \to 16} \frac{\sqrt[4]{x} - 2}{\left(\sqrt[4]{x}\right)^4 - 2^4}$$

$$a^4 - b^4 = \left(a^2 + b^2\right)(a-b)(a+b)$$

$$= \lim_{x\to 16} \frac{\sqrt[4]{x}-2}{\left(\sqrt{x}+2^2\right)\left(\sqrt[4]{x}+2\right)\left(\sqrt[4]{x}-2\right)}$$

$$= \lim_{x\to 16} \frac{1}{\left(\sqrt{x}+4\right)\left(\sqrt[4]{x}+2\right)}$$

$$= \frac{1}{\left(\sqrt{16}+4\right)\left(\sqrt[4]{16}+2\right)}$$

$$= \frac{1}{(4+4)(2+2)}$$

$$= \frac{1}{(8)(4)}$$

$$= \frac{1}{32}$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{x}-1}$

Solution

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}{\sqrt{x}-1}$$

$$= \lim_{x\to 1} \left(\sqrt{x}+1\right)$$

$$= 2$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{4x+5}-3}$

Solution

$$\lim_{x\to 1} \frac{x-1}{\sqrt{4x+5}-3} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{x-1}{\sqrt{4x+5}-3} \bullet \frac{\sqrt{4x+5}+3}{\sqrt{4x+5}+3}$$

$$= \lim_{x\to 1} \frac{(x-1)\left(\sqrt{4x+5}+3\right)}{4x+5-9}$$

$$= \lim_{x\to 1} \frac{(x-1)\left(\sqrt{4x+5}+3\right)}{4(x-1)}$$

$$= \frac{1}{5}\lim_{x\to 1} \left(\sqrt{4x+5}+3\right)$$

$$= \frac{6}{5}$$

Exercise

Find the limit: $\displaystyle \lim_{x\to 4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}}$

Solution

$$\lim_{x\to 4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}} = \frac{0}{3-3} = \frac{0}{0}$$

$$= \lim_{x\to 4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}} \bullet \frac{3+\sqrt{x+5}}{3+\sqrt{x+5}}$$

$$= 3\lim_{x\to 4} \frac{(x-4)\left(3+\sqrt{x+5}\right)\sqrt{x+5}}{9-(x+5)}$$

$$= 3\lim_{x\to 4} \frac{(x-4)\left(3+\sqrt{x+5}\right)\sqrt{x+5}}{4-x}$$

$$= -3\lim_{x\to 4} \left(3+\sqrt{x+5}\right)\sqrt{x+5}$$

$$= -3\,(6)(3)$$

$$= -54$$

Exercise

Find the limit: $\displaystyle \lim_{x\to 0} \frac{x}{\sqrt{ax+1}-1} \quad (a \neq 0)$

Solution

$$\lim_{x\to 0} \frac{x}{\sqrt{ax+1}-1} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{x}{\sqrt{ax+1}-1} \bullet \frac{\sqrt{ax+1}+1}{\sqrt{ax+1}+1}$$

$$= \lim_{x\to 0} \frac{x\left(\sqrt{ax+1}+1\right)}{ax+1-1}$$

$$= \lim_{x\to 0} \frac{x\left(\sqrt{ax+1}+1\right)}{ax}$$

$$= \frac{1}{a} \lim_{x \to 0} \left(\sqrt{ax+1} + 1 \right)$$

$$\underline{= \frac{2}{a}}$$

Exercise

Find the limit: $\displaystyle \lim_{x \to \pi} \frac{\cos^2 x + 3\cos x + 2}{\cos x + 1}$

Solution

$$\lim_{x \to \pi} \frac{\cos^2 x + 3\cos x + 2}{\cos x + 1} = \frac{1 - 3 + 2}{-1 + 1} = \frac{0}{0}$$

$$= \lim_{x \to \pi} \frac{(\cos x + 1)(\cos x + 2)}{\cos x + 1}$$

$$= \lim_{x \to \pi} (\cos x + 2)$$

$$= -1 + 2$$

$$\underline{= 1}$$

Exercise

Find the limit: $\displaystyle \lim_{x \to \frac{3\pi}{2}} \frac{\sin^2 x + 6\sin x + 5}{\sin^2 x - 1}$

Solution

$$\lim_{x \to \frac{3\pi}{2}} \frac{\sin^2 x + 6\sin x + 5}{\sin^2 x - 1} = \frac{1 - 6 + 5}{1 - 1} = \frac{0}{0}$$

$$= \lim_{x \to \frac{3\pi}{2}} \frac{(\sin x + 1)(\sin x + 5)}{(\sin x - 1)(\sin x + 1)}$$

$$= \lim_{x \to \frac{3\pi}{2}} \frac{\sin x + 5}{\sin x - 1}$$

$$= \frac{-1 + 5}{-1 - 1}$$

$$\underline{= -2}$$

Exercise

Find the limit: $\displaystyle \lim_{x \to \frac{\pi}{2}} \frac{\sin x - 1}{\sqrt{\sin x} - 1}$

Solution

$$\lim_{x\to\frac{\pi}{2}} \frac{\sin x - 1}{\sqrt{\sin x} - 1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\left(\sqrt{\sin x} - 1\right)\left(\sqrt{\sin x} + 1\right)}{\sqrt{\sin x} - 1}$$

$$= \lim_{x\to\frac{\pi}{2}} \left(\sqrt{\sin x} + 1\right)$$

$$= 2$$

Exercise

Find the limit: $\lim_{x\to 0} \dfrac{\frac{1}{2+\sin x} - \frac{1}{2}}{\sin x}$

Solution

$$\lim_{x\to 0} \frac{\frac{1}{2+\sin x} - \frac{1}{2}}{\sin x} = \frac{\frac{1}{2} - \frac{1}{2}}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{1}{\sin x} \cdot \frac{2 - \sin x - 2}{2(2+\sin x)}$$

$$= \frac{1}{2} \lim_{x\to 0} \frac{1}{\sin x} \frac{-\sin x}{(2+\sin x)}$$

$$= -\frac{1}{2} \lim_{x\to 0} \frac{1}{2+\sin x}$$

$$= -\frac{1}{2}\left(\frac{1}{2}\right)$$

$$= -\frac{1}{4}$$

Exercise

Find the limit: $\lim_{x\to 0} \dfrac{e^{2x} - 1}{e^{x} - 1}$

Solution

$$\lim_{x\to 0} \frac{e^{2x} - 1}{e^{x} - 1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\left(e^{x} - 1\right)\left(e^{x} + 1\right)}{e^{x} - 1}$$

$$= \lim_{x\to 0} \left(e^{x} + 1\right)$$

$$= 2$$

Exercise

Find the limit: $\lim\limits_{x \to \frac{\pi}{4}} \csc x$

Solution

$$\lim_{x \to \frac{\pi}{4}} \csc x = \csc \frac{\pi}{4}$$

$$= \frac{1}{\cos \frac{\pi}{4}}$$

$$= \sqrt{2}$$

Exercise

Find the limit: $\lim\limits_{x \to 4} \frac{x-5}{\left(x^2 - 10x + 24\right)^2}$

Solution

$$\lim_{x \to 4} \frac{x-5}{\left(x^2 - 10x + 24\right)^2} = \frac{-1}{(16 - 41 + 24)^2}$$

$$= -1$$

Exercise

Find the limit: $\lim\limits_{x \to 0} \frac{\cos x - 1}{\sin^2 x}$

Solution

$$\lim_{x \to 0} \frac{\cos x - 1}{\sin^2 x} = \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\cos x - 1}{(1 - \cos x)(1 + \cos x)}$$

$$= -\lim_{x \to 0} \frac{1}{1 + \cos x}$$

$$= -\frac{1}{2}$$

Exercise

Find the limit: $\lim\limits_{x \to 0} \frac{1 - \cos^2 x}{\sin x}$

Solution

$$\lim_{x\to 0}\frac{1-\cos^2 x}{\sin x}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{\sin x}$$

$$=\lim_{x\to 0}\sin x$$

$$\underline{=0}\,\rfloor$$

Exercise

Find $\lim_{x\to 0}\frac{x^3-5x^2}{x^2}$

Solution

$$\lim_{x\to 0}\frac{x^3-5x^2}{x^2}=\frac{0}{0}$$

$$=\lim_{x\to 0}(x-5)$$

$$\underline{=-5}\,\rfloor$$

Exercise

Find $\lim_{x\to 5}\frac{4x^2-100}{x-5}$

Solution

$$\lim_{x\to 5}\frac{4x^2-100}{x-5}=\frac{0}{0}$$

$$=\lim_{x\to 5}\frac{4(x-5)(x+5)}{x-5}$$

$$=\lim_{x\to 5}4(x+5)$$

$$\underline{=40}\,\rfloor$$

Exercise

Find $\lim_{x\to 3}\frac{\sqrt{9-6x+x^2}}{x-3}$

Solution

$$\lim_{x\to 3}\frac{\sqrt{9-6x+x^2}}{x-3}=\frac{\sqrt{9-18+9}}{3-3}=\frac{0}{0}$$

$$= \lim_{x\to 3} \frac{\sqrt{(x-3)^2}}{x-3}$$

$$= \lim_{x\to 3} \frac{x-3}{x-3}$$

$$\underline{= 1}$$

Exercise

Find $\displaystyle \lim_{x\to 3} \frac{\sqrt{9+6x+x^2}}{x-3}$

Solution

$$\lim_{x\to 3} \frac{\sqrt{9+6x+x^2}}{x-3} = \frac{\sqrt{9+18+9}}{3-3}$$

$$= \frac{\sqrt{36}}{0}$$

$$\underline{= \infty}$$

Exercise

Find $\displaystyle \lim_{x\to 3} \frac{\sqrt{x^2-9}}{x-3}$

Solution

$$\lim_{x\to 3} \frac{\sqrt{x^2-9}}{x-3} = \frac{\sqrt{9-9}}{3-3} = \frac{0}{0}$$

$$= \lim_{x\to 3} \frac{\sqrt{(x-3)(x+3)}}{x-3}$$

$$= \lim_{x\to 3} \sqrt{\frac{x+3}{x-3}}$$

$$= \sqrt{\frac{6}{0}}$$

$$\underline{= \infty}$$

Exercise

Find $\displaystyle \lim_{x\to \frac{4\pi}{3}} \sin x$

Solution

$$\lim_{x \to \frac{4\pi}{3}} \sin x = \sin \frac{4\pi}{3}$$

$$\boxed{= -\frac{\sqrt{3}}{2}}$$

Exercise

Find $\displaystyle \lim_{x \to \frac{2\pi}{3}} \cos x$

Solution

$$\lim_{x \to \frac{2\pi}{3}} \cos x = \cos \frac{2\pi}{3}$$

$$\boxed{= -\frac{1}{2}}$$

Exercise

Find $\displaystyle \lim_{x \to \frac{7\pi}{4}} \sin x$

Solution

$$\lim_{x \to \frac{7\pi}{4}} \sin x = \sin \frac{7\pi}{4}$$

$$\boxed{= -\frac{\sqrt{2}}{2}}$$

Exercise

Find $\displaystyle \lim_{x \to 1} \frac{\sin \sqrt{1-x}}{\sqrt{1-x^2}}$

Solution

$$\lim_{x \to 1} \frac{\sin \sqrt{1-x}}{\sqrt{1-x^2}} = \frac{\sin 0}{0} = \frac{0}{0}$$

$$= \lim_{x \to 1} \frac{\sin \sqrt{1-x}}{\sqrt{1-x}} \frac{1}{\sqrt{1+x}}$$

$$= \lim_{(1-x) \to 0} \frac{\sin \sqrt{1-x}}{\sqrt{1-x}} \lim_{x \to 1} \frac{1}{\sqrt{1+x}}$$

$$= 1\left(\frac{1}{\sqrt{2}}\right)$$

$$\boxed{= \frac{1}{\sqrt{2}}}$$

Exercise

Find $\displaystyle\lim_{x\to 2} \frac{\sin\sqrt{2-x}}{\sqrt{4-x^2}}$

Solution

$$\lim_{x\to 2} \frac{\sin\sqrt{2-x}}{\sqrt{4-x^2}} = \frac{\sin 0}{0} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{\sin\sqrt{2-x}}{\sqrt{2-x}} \frac{1}{\sqrt{2+x}}$$

$$= \lim_{\sqrt{2-x}\to 0} \frac{\sin\sqrt{2-x}}{\sqrt{2-x}} \lim_{x\to 2} \frac{1}{\sqrt{2+x}}$$

$$= 1\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}$$

Exercise

Find $\displaystyle\lim_{x\to 0} \frac{\sin\left(\sqrt{5}\,x\right)}{\sin\left(\sqrt{3}\,x\right)}$

Solution

$$\lim_{x\to 0} \frac{\sin\left(\sqrt{5}\,x\right)}{\sin\left(\sqrt{3}\,x\right)} = \frac{\sin 0}{\sin 0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sqrt{5}\,x}{\sqrt{3}\,x} \frac{\sin\left(\sqrt{5}\,x\right)}{\sqrt{5}\,x} \cdot \frac{1}{\frac{\sin\left(\sqrt{3}\,x\right)}{\sqrt{3}\,x}}$$

$$= \frac{\sqrt{5}}{\sqrt{3}} \lim_{\sqrt{5}\,x\to 0} \frac{\sin\left(\sqrt{5}\,x\right)}{\sqrt{5}\,x} \cdot \frac{1}{\lim_{\sqrt{3}\,x\to 0} \frac{\sin\left(\sqrt{3}\,x\right)}{\sqrt{3}\,x}}$$

$$= \frac{\sqrt{5}}{\sqrt{3}}$$

Exercise

Find $\displaystyle\lim_{x\to 0} \frac{\sin\left(\sqrt{15}\,x\right)}{\sin\left(\sqrt{3}\,x\right)}$

Solution

$$\lim_{x\to 0} \frac{\sin\left(\sqrt{15}\,x\right)}{\sin\left(\sqrt{3}\,x\right)} = \frac{\sin 0}{\sin 0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sqrt{15}\,x}{\sqrt{3}\,x}\;\frac{\sin\left(\sqrt{5}\,x\right)}{\sqrt{15}\,x}\cdot\frac{1}{\dfrac{\sin\left(\sqrt{3}\,x\right)}{\sqrt{3}\,x}}$$

$$= \sqrt{\frac{15}{3}}\;\lim_{\sqrt{15}\,x\to 0}\frac{\sin\left(\sqrt{15}\,x\right)}{\sqrt{15}\,x}\cdot\frac{1}{\displaystyle\lim_{\sqrt{3}\,x\to 0}\frac{\sin\left(\sqrt{3}\,x\right)}{\sqrt{3}\,x}}$$

$$= \sqrt{3}$$

Exercise

Find $\displaystyle\lim_{x\to 0^+} \frac{x-\sqrt{x}}{\sqrt{\sin x}}$

Solution

$$\lim_{x\to 0^+} \frac{x-\sqrt{x}}{\sqrt{\sin x}} = \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{x-\sqrt{x}}{\dfrac{\sqrt{\sin x}}{\sqrt{x}}}\cdot\frac{1}{\sqrt{x}}$$

$$= \lim_{x\to 0^+} \frac{1}{\sqrt{\dfrac{\sin x}{x}}}\;\lim_{x\to 0^+}\frac{x-\sqrt{x}}{\sqrt{x}}$$

$$= (1)\;\lim_{x\to 0^+}\left(\frac{x}{\sqrt{x}}-\frac{\sqrt{x}}{\sqrt{x}}\right)$$

$$= \lim_{x\to 0^+}\left(\sqrt{x}-1\right)$$

$$= -1$$

Exercise

Find $\lim\limits_{x \to 1} \dfrac{x - \sqrt{x}}{\sqrt{\sin x}}$

Solution

$$\lim_{x \to 1} \frac{x - \sqrt{x}}{\sqrt{\sin x}} = \frac{0}{\sqrt{\sin 1}}$$

$$\underline{= 0}$$

Exercise

Find $\lim\limits_{x \to \pi} \dfrac{x - \sqrt{x}}{\sqrt{\sin x}}$

Solution

$$\lim_{x \to \pi} \frac{x - \sqrt{x}}{\sqrt{\sin x}} = \frac{\pi - \sqrt{\pi}}{\sqrt{\sin \pi}}$$

$$= \frac{\pi - \sqrt{\pi}}{0}$$

$$\underline{= \infty}$$

Exercise

Find $\lim\limits_{x \to 0} e^{x^3}$

Solution

$$\lim_{x \to 0} e^{x^3} = e^0$$

$$\underline{= 1}$$

Exercise

Find $\lim\limits_{x \to 1} e^{x^2}$

Solution

$$\lim_{x \to 1} e^{x^2} = e^1$$

$$\underline{= e}$$

Exercise

Find $\lim\limits_{x\to 1} e^{x^3-1}$

Solution

$$\lim_{x\to 1} e^{x^3-1} = e^{1-1}$$

$$= e^0$$

$$\boxed{= 1}$$

Exercise

Find $\lim\limits_{x\to -1} e^{x^3-1}$

Solution

$$\lim_{x\to -1} e^{x^3-1} = e^{-1-1}$$

$$= e^{-2}$$

$$\boxed{= \frac{1}{e^2}}$$

Exercise

Find $\lim\limits_{x\to 2} \left(e^{x^2} - \ln x\right)$

Solution

$$\boxed{\lim_{x\to 2} \left(e^{x^2} - \ln x\right) = e^4 - \ln 2}$$

Exercise

Find $\lim\limits_{x\to 1} \left(e^{x^2} - \ln x\right)$

Solution

$$\lim_{x\to 1} \left(e^{x^2} - \ln x\right) = e - \ln 1$$

$$\boxed{= e}$$

Exercise

Find $\quad \lim\limits_{x\to e} \ln x$

Solution

$$\lim_{x\to e} \ln x = \ln e$$

$$= 1$$

Exercise

Find $\quad \lim\limits_{x\to e} \ln x^2$

Solution

$$\lim_{x\to e} \ln x^2 = \ln e^2$$

$$= 2\ln e$$

$$= 2$$

Exercise

Find $\quad \lim\limits_{x\to 0^+} \ln x$

Solution

$$\lim_{x\to 0^+} \ln x = \ln 0^+$$

$$= -\infty$$

Exercise

Find $\quad \lim\limits_{x\to 1} \frac{1}{\ln x}$

Solution

$$\lim_{x\to 1} \frac{1}{\ln x} = \frac{1}{\ln 1}$$

$$= \frac{1}{0}$$

$$= \infty$$

Exercise

Find $\lim\limits_{x \to e} \ln e^{2x}$

Solution

$$\lim_{x \to e} \ln e^{2x} = \ln e^{2e}$$

$$= 2e \ln e$$

$$\boxed{= 2e}$$

Exercise

Find $\lim\limits_{x \to 1} \ln e^{x^2}$

Solution

$$\lim_{x \to 1} \ln e^{x^2} = \ln e$$

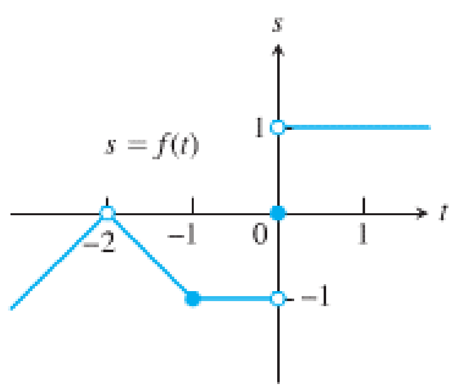
$$\boxed{= 1}$$

Exercise

For the function $f(t)$ graphed, find the following limits or explain why they do not exist.

$a)\ \lim\limits_{t \to -2} f(t) \quad b)\ \lim\limits_{t \to -1} f(t) \quad c)\ \lim\limits_{t \to 0} f(t) \quad d)\ \lim\limits_{t \to -0.5} f(t)$

Solution



a) $\boxed{\lim\limits_{t \to -2} f(t) = 0}$

b) $\boxed{\lim\limits_{t \to -1} f(t) = -1}$

c) $\lim\limits_{t \to 0} f(t) =$ *doesn't exist*

d) $\boxed{\lim\limits_{t \to -.5} f(t) = -1}$

Exercise

Suppose $\lim\limits_{x\to c} f(x)=5$ *and* $\lim\limits_{x\to c} g(x)=-2$. Find

a) $\lim\limits_{x\to c} f(x)g(x)$

b) $\lim\limits_{x\to c} 2f(x)g(x)$

c) $\lim\limits_{x\to c} \left(f(x)+3g(x)\right)$

d) $\lim\limits_{x\to c} \dfrac{f(x)}{f(x)-g(x)}$

Solution

a) $\lim\limits_{x\to c} f(x)g(x)=\lim\limits_{x\to c} f(x)\bullet\lim\limits_{x\to c} g(x)$

$$=(5)(-2)$$

$$=-10$$

b) $\lim\limits_{x\to c} 2f(x)g(x)=2\lim\limits_{x\to c} f(x)\bullet\lim\limits_{x\to c} g(x)$

$$=2(-10)$$

$$=-20$$

c) $\lim\limits_{x\to c} \left(f(x)+3g(x)\right)=\lim\limits_{x\to c} f(x)+3\lim\limits_{x\to c} g(x)$

$$=5+3(-2)$$

$$=-1$$

d) $\lim\limits_{x\to c} \dfrac{f(x)}{f(x)-g(x)}=\dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} f(x)-\lim\limits_{x\to c} g(x)}$

$$=\frac{5}{5-(-2)}$$

$$=\frac{5}{7}$$

Exercise

Explain why the limits do not exist for $\lim\limits_{x\to 0}\dfrac{x}{|x|}$

Solution

$$\lim_{x\to 0}\frac{x}{|x|}=\frac{0}{0}$$

$$\lim_{x\to 0^-}\frac{x}{|x|}=\frac{-x}{x}=-1$$

Doesn't exist

$$\lim_{x\to 0^+}\frac{x}{|x|}=\frac{x}{x}=1$$

Exercise

Evaluate the limit using the form $\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$ for $f(x)=x^2, \quad x=1$

Solution

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$

$$=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$

$$=\lim_{h\to 0}\left(\frac{2xh}{h}+\frac{h^2}{h}\right)$$

$$=\lim_{h\to 0}(2x+h)$$

$$=2x$$

Exercise

Evaluate the limit using the form $\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$ for $f(x)=\sqrt{3x+1}, \quad x=0$

Solution

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{3(x+h)+1}-\sqrt{3x+1}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{3x+3h+1}-\sqrt{3x+1}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{3x+3h+1}-\sqrt{3x+1}}{h}\cdot\frac{\sqrt{3x+3h+1}+\sqrt{3x+1}}{\sqrt{3x+3h+1}+\sqrt{3x+1}}$$

$$=\lim_{h\to 0}\frac{3x+3h+1-(3x+1)}{h\left(\sqrt{3x+3h+1}+\sqrt{3x+1}\right)}$$

$$=\lim_{h\to 0}\frac{3x+3h+1-3x-1}{h\left(\sqrt{3x+3h+1}+\sqrt{3x+1}\right)}$$

$$=\lim_{h\to 0}\frac{3h}{h\left(\sqrt{3x+3h+1}+\sqrt{3x+1}\right)}$$

$$=\lim_{h\to 0}\frac{3}{\sqrt{3x+3h+1}+\sqrt{3x+1}}$$

$$=\frac{3}{\sqrt{3(0)+1}+\sqrt{3(0)+1}} \qquad \textit{Given}: x=0$$

$$=\frac{3}{2}$$

Exercise

If $\lim\limits_{x\to 4}\dfrac{f(x)-5}{x-2}=1$, find $\lim\limits_{x\to 4} f(x)$

Solution

$$\lim_{x\to 4}\frac{f(x)-5}{x-2}=1$$

$$\frac{\lim\limits_{x\to 4} f(x)-5}{4-2}=1$$

$$\frac{\lim\limits_{x\to 4} f(x)-5}{2}=1$$ ***Multiply both sides by 2***

$$\lim_{x\to 4} f(x)-5=2$$ ***Add 5 on both sides***

$$\lim_{x\to 4} f(x)=7$$

Exercise

If $\lim\limits_{x\to 0}\dfrac{f(x)}{x^2}=1$, find $\lim\limits_{x\to 0} f(x)$ *and* $\lim\limits_{x\to 0}\dfrac{f(x)}{x}$

Solution

$$\lim_{x\to 0}\frac{f(x)}{x^2}=1$$

$$\frac{\lim\limits_{x\to 0} f(x)}{\lim\limits_{x\to 0} x^2}=1$$

$$\lim_{x\to 0} f(x)=\lim_{x\to 0} x^2$$

$$=0$$

$$\lim_{x\to 0}\frac{f(x)}{x}=\lim_{x\to 0}\left(\frac{f(x)}{x^2}\bullet x\right)$$

$$=\lim_{x\to 0}\frac{f(x)}{x^2}\bullet\lim_{x\to 0} x$$

$$=1\bullet 0$$

$$=0$$

Exercise

If $x^4 \le f(x) \le x^2;\quad -1 \le x \le 1$ and $x^2 \le f(x) \le x^4;\quad x < -1$ *and* $x > 1$. At what points c do you automatically know $\lim\limits_{x \to c} f(x)$? What can you say about the value of the limits at these points?

Solution

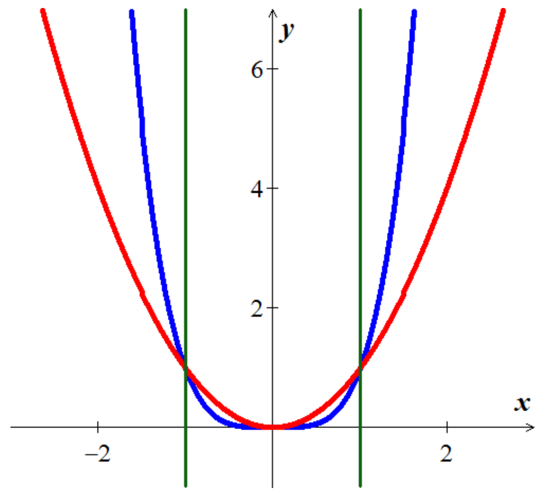
$$\lim_{x \to c} x^4 = \lim_{x \to c} x^2 \quad \Rightarrow \quad c^4 = c^2$$

$$c^4 - c^2 = 0$$

$$c^2\left(c^2 - 1\right) = 0$$

$$c^2 = 0 \qquad\qquad c^2 - 1 = 0$$

$$\boxed{c = 0} \qquad\qquad \boxed{c = \pm 1}$$



$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2$$

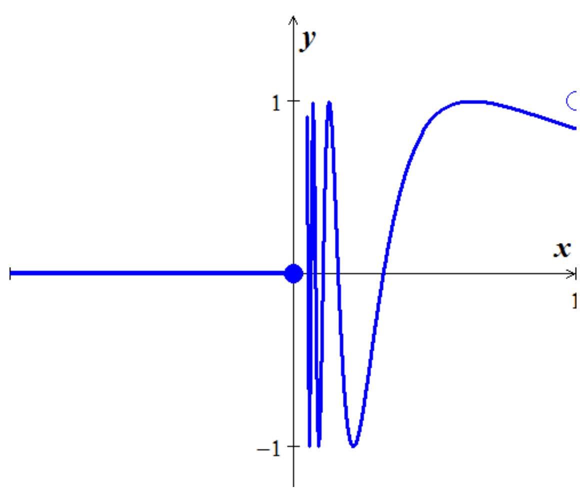
$$= 0$$

$$\lim_{x \to -1} f(x) = \lim_{x \to 1} f(x)$$

$$= 1$$

Exercise

Let $f(x) = \begin{cases} 0, & x \le 0 \\ \sin\frac{1}{x}, & x > 0 \end{cases}$



a) Does $\lim\limits_{x \to 0^+} f(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim\limits_{x\to 0^-} f(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim\limits_{x\to 0} f(x)$ exist? If so, what is it? If not, why not?

Solution

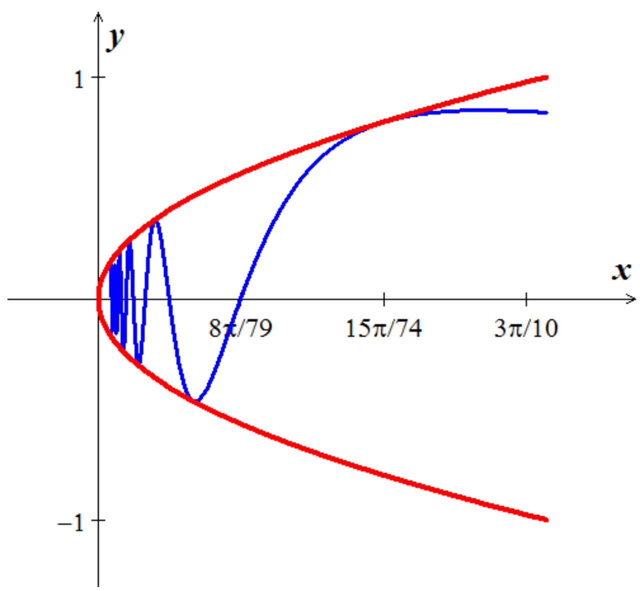
a) $\lim\limits_{x\to 0^+} f(x)$ doesn't exist, since $\sin\left(\frac{1}{x}\right)$ doesn't approach any single value as $x \to 0$

b) $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} 0 = 0$

c) $\lim\limits_{x\to 0} f(x)$ doesn't exist, since $\lim\limits_{x\to 0^+} f(x)$ doesn't exist

Exercise

Let $g(x) = \sqrt{x}\ \sin\frac{1}{x}$



a) Does $\lim\limits_{x\to 0^+} g(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim\limits_{x\to 0^-} g(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim\limits_{x\to 0} g(x)$ exist? If so, what is it? If not, why not?

Solution

a) $\lim\limits_{x\to 0^+} g(x)$ exists, by the sandwich theorem $-\sqrt{x} \le g(x) \le \sqrt{x}.$ *for* $x > 0$

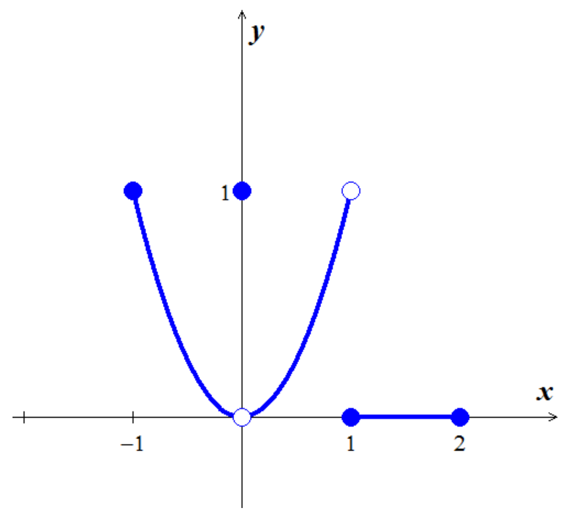
b) $\lim\limits_{x\to 0^-} g(x)$ doesn't exist, since $\sqrt{x}$ is not defined for $x < 0$

c) $\lim_{x \to 0} g(x)$ doesn't exist, since $\lim_{x \to 0^-} g(x)$ doesn't exist.

Exercise

Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

Solution



a) $\lim_{x \to -1^+} f(x) = 1$ ***True***

b) $\lim_{x \to 0^-} f(x) = 0$ ***True***

c) $\lim_{x \to 0^-} f(x) = 1$ ***False***

d) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$ ***True***

e) $\lim_{x \to 0} f(x)$ *exists* ***True***

f) $\lim_{x \to 0} f(x) = 0$ ***True***

g) $\lim_{x \to 0} f(x) = 1$ ***False***

h) $\lim_{x \to 1} f(x) = 1$ ***False***

i) $\lim_{x \to 1} f(x) = 0$ ***False***

j) $\lim_{x \to 2^-} f(x) = 2$ ***False***

k) $\lim_{x \to -1^-} f(x) = 0$ *does not exist* ***True***

l) $\lim_{x \to 2^+} f(x) = 0$ ***False***

Solution   Section 1.3 – Infinite Limits

Exercise

Find $\displaystyle\lim_{x\to 5} \frac{x-7}{x(x-5)^2}$

Solution

$$\lim_{x\to 5} \frac{x-7}{x(x-5)^2} = \frac{-2}{0}$$

$$\underline{= \infty}$$

Exercise

Find $\displaystyle\lim_{x\to -5^+} \frac{x-5}{x+5}$

Solution

$$\lim_{x\to -5^+} \frac{x-5}{x+5} = \frac{-10}{0^+}$$

$$\underline{= -\infty}$$

Exercise

Find $\displaystyle\lim_{x\to 3^-} \frac{x-4}{x^2-3x}$

Solution

$$\lim_{x\to 3^-} \frac{x-4}{x^2-3x} = \frac{-1}{0^-}$$

$$\underline{= \infty}$$

Exercise

Find $\displaystyle\lim_{x\to 0^+} \frac{1}{3x}$

Solution

$$\lim_{x\to 0^+} \frac{1}{3x} = \frac{1}{0^+}$$

$$\underline{= \infty}$$

Exercise

Find $\lim\limits_{x\to -5^-} \dfrac{3x}{2x+10}$

Solution

$$\lim_{x\to -5^-} \frac{3x}{2x+10} = \lim_{x\to -5^-} \frac{3}{2+\frac{10}{x}}$$

$$\underline{= \infty}$$

Exercise

Find $\lim\limits_{x\to 0} \dfrac{1}{x^{2/3}}$

Solution

$$\lim_{x\to 0} \frac{1}{x^{2/3}} = \lim_{x\to 0} \frac{1}{\left(x^{1/3}\right)^2}$$

$$\underline{= \infty}$$

Exercise

Find $\lim\limits_{x\to 0^-} \dfrac{1}{3x^{1/3}}$

Solution

$$\lim_{x\to 0^-} \frac{1}{3x^{1/3}} = \frac{1}{0^-}$$

$$\underline{= -\infty}$$

Exercise

Find $\lim\limits_{x\to\left(-\frac{\pi}{2}\right)^+} \sec x$

Solution

$$\lim_{x\to\left(-\frac{\pi}{2}\right)^+} \sec x \underline{= \infty}$$

Exercise

Find $\lim\limits_{\theta\to 0^-} (1+\csc\theta)$

Solution

$$\lim_{\theta\to 0^-}(1+\csc\theta)=\lim_{\theta\to 0^-}\left(1+\frac{1}{\sin\theta}\right)$$

$$\underline{=-\infty}$$

Exercise

Find $\lim_{\theta\to 0^+}\csc\theta$

Solution

$$\lim_{\theta\to 0^+}\csc\theta=\lim_{\theta\to 0^+}\frac{1}{\sin\theta} \qquad \text{As } \theta\to 0^+ \ \sin\theta>0$$

$$\underline{=+\infty}$$

Exercise

Find $\lim_{x\to 0+}(-10\cot x)$

Solution

$$\lim_{x\to 0^+}(-10\cot x)=-10\lim_{x\to 0^+}\frac{\cos\theta}{\sin\theta}=-10\left(\frac{1}{0}\right) \qquad \text{As } x\to 0^+ \ \cos\theta>0; \quad \sin\theta>0$$

$$\underline{=-\infty}$$

Exercise

Find $\lim_{\theta\to\frac{\pi}{2}^+}\frac{1}{3}\tan\theta$

Solution

$$\lim_{\theta\to\frac{\pi}{2}^+}\frac{1}{3}\tan\theta=\frac{1}{3}\lim_{\theta\to\frac{\pi}{2}^+}\frac{\sin\theta}{\cos\theta}=\frac{1}{3}\left(-\frac{1}{0}\right) \qquad \text{As } \theta\to\frac{\pi}{2}^+ \ \cos\theta<0; \quad \sin\theta>0$$

$$\underline{=-\infty}$$

Exercise

Find $\lim_{x\to 2^+}\frac{1}{x-2}$

Solution

$$\lim_{x\to 2^+}\frac{1}{x-2}=\frac{1}{2^+-2}=\frac{1}{0^+}$$

$$\underline{=\infty}$$

Exercise

Find $\lim\limits_{x \to 2^-} \dfrac{1}{x-2}$

Solution

$$\lim_{x \to 2^-} \frac{1}{x-2} = \frac{1}{2^- - 2} = \frac{1}{0^-}$$

$$\underline{= -\infty}$$

Exercise

Find $\lim\limits_{x \to 2} \dfrac{1}{x-2}$

Solution

$$\lim_{x \to 2} \frac{1}{x-2} = \frac{1}{0}$$

$$\underline{= \infty}$$

Exercise

Find $\lim\limits_{x \to 3^+} \dfrac{2}{(x-3)^3}$

Solution

$$\lim_{x \to 3^+} \frac{2}{(x-3)^3} = \frac{2}{0^+}$$

$$\underline{= \infty}$$

Exercise

Find $\lim\limits_{x \to 3^-} \dfrac{2}{(x-3)^3}$

Solution

$$\lim_{x \to 3^-} \frac{2}{(x-3)^3} = \frac{2}{0^-}$$

$$\underline{= -\infty}$$

Exercise

Find $\lim\limits_{x \to 3} \dfrac{2}{(x-3)^3}$

Solution

$$\lim_{x \to 3} \frac{2}{(x-3)^3} = \frac{2}{0}$$

$$\boxed{= \infty}$$

Exercise

Find $\displaystyle\lim_{x \to 4^+} \frac{x-5}{(x-4)^2}$

Solution

$$\lim_{x \to 4^+} \frac{x-5}{(x-4)^2} = \frac{-1}{0}$$

$$\boxed{= -\infty}$$

Exercise

Find $\displaystyle\lim_{x \to 4^-} \frac{x-5}{(x-4)^2}$

Solution

$$\lim_{x \to 4^-} \frac{x-5}{(x-4)^2} = \frac{-1}{0}$$

$$\boxed{= -\infty}$$

Exercise

Find $\displaystyle\lim_{x \to 4} \frac{x-5}{(x-4)^2}$

Solution

$$\lim_{x \to 4^-} \frac{x-5}{(x-4)^2} = \frac{-1}{0}$$

$$\boxed{= -\infty}$$

Exercise

Find $\displaystyle\lim_{x \to 1^+} \frac{x-2}{(x-1)^3}$

Solution

$$\lim_{x \to 1^+} \frac{x-2}{(x-1)^3} = \frac{-1}{0^+}$$

$$\boxed{= -\infty}$$

Exercise

Find $\displaystyle\lim_{x\to 1^-} \frac{x-2}{(x-1)^3}$

Solution

$$\lim_{x\to 1^-} \frac{x-2}{(x-1)^3} = \frac{-1}{0^-}$$

$$= \infty$$

Exercise

Find $\displaystyle\lim_{x\to 1} \frac{x-2}{(x-1)^3}$

Solution

$$\lim_{x\to 1} \frac{x-2}{(x-1)^3} = \frac{-1}{0^+}$$

$$= \nexists$$

Exercise

Find $\displaystyle\lim_{x\to 3^+} \frac{(x-1)(x-2)}{x-3}$

Solution

$$\lim_{x\to 3^+} \frac{(x-1)(x-2)}{x-3} = \frac{2}{0}$$

$$= \infty$$

Exercise

Find $\displaystyle\lim_{x\to 3^-} \frac{(x-1)(x-2)}{x-3}$

Solution

$$\lim_{x\to 3^-} \frac{(x-1)(x-2)}{x-3} = \frac{2}{0^-}$$

$$= -\infty$$

Exercise

Find $\lim\limits_{x\to 3} \frac{(x-1)(x-2)}{x-3}$

Solution

$$\lim_{x\to 3^-} \frac{(x-1)(x-2)}{x-3} = \frac{2}{0^-}$$

$$= -\infty$$

$$\lim_{x\to 3^+} \frac{(x-1)(x-2)}{x-3} = \infty$$

$$\lim_{x\to 3} \frac{(x-1)(x-2)}{x-3} = \nexists$$

Exercise

Find $\lim\limits_{x\to -2^+} \frac{x-4}{x(x+2)}$

Solution

$$\lim_{x\to -2^+} \frac{x-4}{x(x+2)} = \frac{-6}{-0^+}$$

$$= \infty$$

Exercise

Find $\lim\limits_{x\to -2^-} \frac{x-4}{x(x+2)}$

Solution

$$\lim_{x\to -2^-} \frac{x-4}{x(x+2)} = \frac{-6}{0^+}$$

$$= -\infty$$

Exercise

Find $\lim\limits_{x\to -2} \frac{x-4}{x(x+2)}$

Solution

$$\lim_{x\to -2^+} \frac{x-4}{x(x+2)} = \infty$$

$$\lim_{x \to -2^-} \frac{x-4}{x(x+2)} = -\infty$$

$$\lim_{x \to -2} \frac{x-4}{x(x+2)} = \boxed{\nexists}$$

Exercise

Find $\lim_{x \to 2^+} \frac{x^2-4x+3}{(x-2)^2}$

Solution

$$\lim_{x \to 2^+} \frac{x^2-4x+3}{(x-2)^2} = \frac{-1}{0^+}$$

$$= \boxed{-\infty}$$

Exercise

Find $\lim_{x \to 2^-} \frac{x^2-4x+3}{(x-2)^2}$

Solution

$$\lim_{x \to 2^-} \frac{x^2-4x+3}{(x-2)^2} = \frac{-1}{0^+}$$

$$= \boxed{-\infty}$$

Exercise

Find $\lim_{x \to 2} \frac{x^2-4x+3}{(x-2)^2}$

Solution

$$\lim_{x \to 2} \frac{x^2-4x+3}{(x-2)^2} = \frac{-1}{0}$$

$$= \boxed{-\infty}$$

Exercise

Find $\lim_{x \to -2^+} \frac{x^3-5x^2+6x}{x^4-4x^2}$

Solution

$$\lim_{x \to -2^+} \frac{x^3-5x^2+6x}{x^4-4x^2} = \lim_{x \to -2^+} \frac{x(x-2)(x-3)}{x^2(x-2)(x+2)}$$

$$= \lim_{x \to -2^+} \frac{x-3}{x(x+2)} \quad \frac{-}{-(+)}$$

$$\underline{= \infty}$$

Exercise

Find $\displaystyle \lim_{x \to -2^-} \frac{x^3 - 5x^2 + 6x}{x^4 - 4x^2}$

Solution

$$\lim_{x \to -2^-} \frac{x^3 - 5x^2 + 6x}{x^4 - 4x^2} = \lim_{x \to -2^-} \frac{x(x-2)(x-3)}{x^2(x-2)(x+2)}$$

$$= \lim_{x \to -2^-} \frac{x-3}{x(x+2)} \quad \frac{-}{-(-)}$$

$$\underline{= -\infty}$$

Exercise

Find $\displaystyle \lim_{x \to -2} \frac{x^3 - 5x^2 + 6x}{x^4 - 4x^2}$

Solution

$$\lim_{x \to -2} \frac{x^3 - 5x^2 + 6x}{x^4 - 4x^2} = \frac{-8 - 20 - 12}{16 - 16}$$

$$= \frac{-40}{0}$$

$$\underline{= -\infty}$$

Exercise

Find $\displaystyle \lim_{u \to 0^+} \frac{u-1}{\sin u}$

Solution

$$\lim_{u \to 0^+} \frac{u-1}{\sin u} = \frac{-1}{0^+}$$

$$\underline{= -\infty}$$

Exercise

Find $\displaystyle \lim_{x \to 0^-} \frac{2}{\tan x}$

Solution

$$\lim_{x \to 0^-} \frac{2}{\tan x} = \frac{2}{0^-}$$

$$= -\infty$$

Exercise

Find $\lim\limits_{x\to 1^+} \dfrac{x^2-5x+6}{x-1}$

Solution

$$\lim_{x\to 1^+} \frac{x^2-5x+6}{x-1} = \frac{2}{0^+}$$

$$= \infty$$

Exercise

Find $\lim\limits_{x\to 2\pi^-} \csc x$

Solution

$$\lim_{x\to 2\pi^-} \csc x = \frac{1}{\sin\left(2\pi^-\right)} = \frac{1}{0^-}$$

$$= -\infty$$

Exercise

Find $\lim\limits_{x\to 0^+} e^{\sqrt{x}}$

Solution

$$\lim_{x\to 0^+} e^{\sqrt{x}} = 1$$

Exercise

Find $\lim\limits_{x\to \frac{\pi}{2}^-} \dfrac{1+\sin x}{\cos x}$

Solution

$$\lim_{x\to \frac{\pi}{2}^-} \frac{1+\sin x}{\cos x} = \frac{2}{0^+}$$

$$= \infty$$

Exercise

Find $\displaystyle\lim_{x\to\frac{\pi}{2}^{+}} \frac{1+\sin x}{\cos x}$

Solution

$$\lim_{x\to\frac{\pi}{2}^{+}} \frac{1+\sin x}{\cos x} = \frac{2}{0^{-}}$$

$$\boxed{= -\infty}$$

Exercise

Find $\displaystyle\lim_{x\to 0^{-}} \frac{e^{x}}{1+e^{x}}$

Solution

$$\lim_{x\to 0^{-}} \frac{e^{x}}{1-e^{x}} = \frac{1}{0^{+}}$$

$$\boxed{= \infty}$$

Exercise

Find $\displaystyle\lim_{x\to 0^{+}} \frac{e^{x}}{1-e^{x}}$

Solution

$$\lim_{x\to 0^{+}} \frac{e^{x}}{1-e^{x}} = \frac{1}{0^{-}}$$

$$\boxed{= -\infty}$$

Exercise

Find $\displaystyle\lim_{x\to 1^{-}} \frac{x}{\ln x}$

Solution

$$\lim_{x\to 1^{-}} \frac{x}{\ln x} = \frac{1}{0^{-}}$$

$$\boxed{= -\infty}$$

Exercise

Find $\lim\limits_{x\to 0^+} \dfrac{x}{\ln x}$

Solution

$$\lim_{x\to 0^+} \frac{x}{\ln x} = \frac{0}{-\infty}$$

$$\underline{= 0}$$

Exercise

Find $\lim\limits_{x\to 0^-} \dfrac{2e^x + 5e^{3x}}{e^{2x} - e^{3x}}$

Solution

$$\lim_{x\to 0^-} \frac{2e^x + 5e^{3x}}{e^{2x} - e^{3x}} = \lim_{x\to 0^-} \frac{2e^x + 5e^{3x}}{e^{2x}\left(1 - e^x\right)}$$

$$= \frac{7}{0}$$

$$\underline{= \infty}$$

Exercise

Find $\lim\limits_{x\to 0^+} \dfrac{2e^x + 5e^{3x}}{e^{2x} - e^{3x}}$

Solution

$$\lim_{x\to 0^+} \frac{2e^x + 5e^{3x}}{e^{2x} - e^{3x}} = \lim_{x\to 0^+} \frac{2e^x + 5e^{3x}}{e^{2x}\left(1 - e^x\right)}$$

$$= \frac{7}{0^-}$$

$$\underline{= -\infty}$$

Exercise

Find $\lim\limits_{x\to 1^-} \dfrac{\ln x}{\sin^{-1} x}$

Solution

$$\lim_{x\to 1^-} \frac{\ln x}{\sin^{-1} x} = \frac{\ln 1}{\sin^{-1} 1}$$

$$= \frac{0}{\frac{\pi}{2}}$$

$$\underline{= 0}$$

Exercise

Let $f(x) = \dfrac{x^2 - 7x + 12}{x - a}$

a) For what values of a, if any, does $\lim\limits_{x \to a^+} f(x)$ equal a finite number?

b) For what values of a, if any, does $\lim\limits_{x \to a^+} f(x) = \infty$?

c) For what values of a, if any, does $\lim\limits_{x \to a^+} f(x) = -\infty$?

Solution

$$f(x) = \frac{x^2 - 7x + 12}{x - a} = \frac{(x-3)(x-4)}{x-a}$$

a) If $a = 3$, then

$$\lim_{x \to 3} \frac{(x-3)(x-4)}{x-3} = \lim_{x \to 3} (x-4)$$

$$\underline{= -1}$$

If $a = 4$, then

$$\lim_{x \to 4} \frac{(x-3)(x-4)}{x-4} = \lim_{x \to 4} (x-1)$$

$$\underline{= 1}$$

b) $\lim\limits_{x \to a^+} f(x) = \infty$ for any number other than 3 or 4.

As $x \to a^+$, then $(x-a)$ is always positive.

$(x-3)(x-4) > 0 \quad \Rightarrow \quad (-\infty, 3) \cup (4, \infty)$

c) $\lim\limits_{x \to a^+} f(x) = -\infty$ for any number other than 3 or 4.

As $x \to a^+$, then $(x-a)$ is always positive, and $(3, 4)$

Exercise

Analyze $\lim\limits_{x\to 1^+}\sqrt{\dfrac{x-1}{x-3}}$ and $\lim\limits_{x\to 1^-}\sqrt{\dfrac{x-1}{x-3}}$

Solution

$$\lim_{x\to 1^+}\sqrt{\frac{x-1}{x-3}}=\sqrt{\frac{0^+}{-2}} \quad \nexists$$

$$\lim_{x\to 1^-}\sqrt{\frac{x-1}{x-3}}=\sqrt{\frac{0^-}{-2}}$$

$$\underline{=0\,|}$$

Solution Section 1.4 – Limits at Infinity

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $h(x) = \dfrac{-5+\frac{7}{x}}{3-\frac{1}{x^2}}$

Solution

$$\lim_{x\to\infty} \frac{-5+\frac{7}{x}}{3-\frac{1}{x^2}} = -\frac{5}{3}$$

$$\lim_{x\to-\infty} \frac{-5+\frac{7}{x}}{3-\frac{1}{x^2}} = -\frac{5}{3}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{2x+3}{5x+7}$

Solution

$$\lim_{x\to\infty} \frac{2x+3}{5x+7} = \lim_{x\to\infty} \frac{2+\frac{3}{x}}{5+\frac{7}{x}}$$

$$= \frac{2}{5}$$

$$\lim_{x\to-\infty} \frac{2x+3}{5x+7} = \lim_{x\to-\infty} \frac{2+\frac{3}{x}}{5+\frac{7}{x}}$$

$$= \frac{2}{5}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{2x^3+7}{x^3-x^2+x+7}$

Solution

$$\lim_{x\to\infty} \frac{2x^3+7}{x^3-x^2+x+7} = \lim_{x\to\infty} \frac{2+\frac{7}{x^3}}{1-\frac{1}{x}+\frac{1}{x^2}+\frac{7}{x^3}}$$

$$= 2$$

$$\lim_{x\to-\infty} \frac{2x^3+7}{x^3-x^2+x+7} = \lim_{x\to-\infty} \frac{2+\frac{7}{x^3}}{1-\frac{1}{x}+\frac{1}{x^2}+\frac{7}{x^3}}$$

$$= 2$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \frac{x+1}{x^2+3}$

Solution

$$\lim_{x\to\infty} \frac{x+1}{x^2+3} = \lim_{x\to\infty} \frac{\frac{x}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{3}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x^2}}$$

$$= 0$$

$$\lim_{x\to-\infty} \frac{x+1}{x^2+3} = \lim_{x\to-\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x^2}}$$

$$= 0$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \frac{7x^3}{x^3-3x^2+6x}$

Solution

$$\lim_{x\to\infty} \frac{7x^3}{x^3-3x^2+6x} = \lim_{x\to\infty} \frac{7}{1-\frac{3}{x}+\frac{6}{x^2}}$$

$$= 7$$

$$\lim_{x\to-\infty} \frac{7x^3}{x^3-3x^2+6x} = \lim_{x\to-\infty} \frac{7}{1-\frac{3}{x}+\frac{6}{x^2}}$$

$$= 7$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{9x^4 + x}{2x^4 + 5x^2 - x + 6}$

Solution

$$\lim_{x\to\infty} \frac{9x^4 + x}{2x^4 + 5x^2 - x + 6} = \lim_{x\to\infty} \frac{\frac{9x^4}{x^4} + \frac{x}{x^4}}{\frac{2x^4}{x^4} + \frac{5x^2}{x^4} - \frac{x}{x^4} + \frac{6}{x^4}}$$

$$= \lim_{x\to\infty} \frac{9 + \frac{1}{x^3}}{2 + \frac{5}{x^2} - \frac{1}{x^3} + \frac{6}{x^4}}$$

$$\boxed{= \frac{9}{2}}$$

$$\lim_{x\to-\infty} \frac{9x^4 + x}{2x^4 + 5x^2 - x + 6} = \lim_{x\to-\infty} \frac{9 + \frac{1}{x^3}}{2 + \frac{5}{x^2} - \frac{1}{x^3} + \frac{6}{x^4}}$$

$$\boxed{= \frac{9}{2}}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x}$

Solution

$$\lim_{x\to\infty} \frac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x} = \lim_{x\to\infty} \frac{-2 - \frac{2}{x^2} + \frac{3}{x^3}}{3 + \frac{3}{x} - \frac{5}{x^2}}$$

$$\boxed{= -\frac{2}{3}}$$

$$\lim_{x\to-\infty} \frac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x} = \lim_{x\to-\infty} \frac{-2 - \frac{2}{x^2} + \frac{3}{x^3}}{3 + \frac{3}{x} - \frac{5}{x^2}}$$

$$\boxed{= -\frac{2}{3}}$$

Exercise

Find $\lim\limits_{x\to\infty} x^{12}$

Solution

$$\lim_{x\to\infty} x^{12} \underline{= \infty}$$

Exercise

Find $\lim\limits_{x\to-\infty} 3x^{9}$

Solution

$$\lim_{x\to-\infty} 3x^{9} \underline{= -\infty}$$

Exercise

Find $\lim\limits_{x\to-\infty} x^{-8}$

Solution

$$\lim_{x\to-\infty} x^{-8} = \frac{1}{(-\infty)^8}$$

$$\underline{= 0}$$

Exercise

Find $\lim\limits_{x\to-\infty} x^{-9}$

Solution

$$\lim_{x\to-\infty} x^{-9} = \frac{1}{(-\infty)^9}$$

$$\underline{= 0}$$

Exercise

Find $\lim\limits_{x\to-\infty} 2x^{-6}$

Solution

$$\lim_{x\to-\infty} 2x^{-6} = \frac{2}{\infty}$$

$$\underline{= 0}$$

Exercise

Find $\lim\limits_{x\to\infty}\left(3x^{12}-9x^{7}\right)$

Solution

$$\lim_{x\to\infty}\left(3x^{12}-9x^{7}\right)\underline{=\infty}$$

Exercise

Find $\lim\limits_{x\to-\infty}\left(3x^{7}+x^{2}\right)$

Solution

$$\lim_{x\to-\infty}\left(3x^{7}+x^{2}\right)=\lim_{x\to-\infty}x^{2}\left(3x^{5}+1\right)$$

$$\underline{=-\infty}$$

Exercise

Find $\lim\limits_{x\to-\infty}\left(-2x^{16}+2\right)$

Solution

$$\lim_{x\to-\infty}\left(-2x^{16}+2\right)\underline{=-\infty}$$

Exercise

Find $\lim\limits_{x\to-\infty}\left(2x^{-6}+4x^{5}\right)$

Solution

$$\lim_{x\to-\infty}\left(2x^{-6}+4x^{5}\right)=\lim_{x\to-\infty}x^{-6}\left(2+4x^{11}\right) \qquad +\infty(-\infty)$$

$$\underline{=-\infty}$$

Exercise

Find $\lim\limits_{x\to-\infty}\dfrac{\cos x}{3x}$

Solution

$$-\frac{1}{3x} \le \frac{\cos x}{3x} \le \frac{1}{3x}, \quad -1 \le \cos x \le 1$$

$$\lim_{x\to-\infty} \frac{\cos x}{3x} \underline{= 0|} \qquad \textit{By the Sandwich Theorem}$$

Exercise

Find $\displaystyle \lim_{x\to\infty} \frac{x+\sin x}{2x+7-5\sin x}$

Solution

$$\lim_{x\to\infty} \frac{x+\sin x}{2x+7-5\sin x} = \lim_{x\to\infty} \frac{1+\frac{\sin x}{x}}{2+\frac{7}{x}-\frac{5\sin x}{x}}$$

$$= \frac{1+0}{2+0-0}$$

$$\underline{= \frac{1}{2}|}$$

Exercise

Find $\displaystyle \lim_{x\to\infty} \sqrt{\frac{8x^2-3}{2x^2+x}}$

Solution

$$\lim_{x\to\infty} \sqrt{\frac{8x^2-3}{2x^2+x}} = \lim_{x\to\infty} \sqrt{\frac{8-\frac{3}{x^2}}{2+\frac{1}{x}}}$$

$$= \sqrt{\frac{8}{2}}$$

$$\underline{= 2|}$$

Exercise

Find $\displaystyle \lim_{x\to-\infty} \left(\frac{x^2+x-1}{8x^2-3}\right)^{1/3}$

Solution

$$\lim_{x\to-\infty} \left(\frac{x^2+x-1}{8x^2-3}\right)^{1/3} = \lim_{x\to-\infty} \left(\frac{1+\frac{1}{x}-\frac{1}{x^2}}{8-\frac{3}{x^2}}\right)^{1/3}$$

$$= \left(\frac{1}{8}\right)^{1/3}$$

$$= \frac{1}{2}$$

Exercise

Find $\displaystyle\lim_{x\to\infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7}$

Solution

$$\lim_{x\to\infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7} = \lim_{x\to\infty} \frac{\frac{2\sqrt{x}}{x} + \frac{x^{-1}}{x}}{3 - \frac{7}{x}}$$

$$= \lim_{x\to\infty} \frac{\frac{2}{x^{1/2}} + \frac{1}{x^2}}{3 - \frac{7}{x}}$$

$$= 0$$

Exercise

Find $\displaystyle\lim_{x\to\infty} \frac{x^{-1} + x^{-4}}{x^{-2} + x^{-3}}$

Solution

$$\lim_{x\to\infty} \frac{x^{-1} + x^{-4}}{x^{-2} + x^{-3}} = \lim_{x\to\infty} \frac{\frac{x^{-1}}{x^{-2}} + \frac{x^{-4}}{x^{-2}}}{\frac{x^{-2}}{x^{-2}} + \frac{x^{-3}}{x^{-2}}}$$

$$= \lim_{x\to\infty} \frac{x + \frac{1}{x^2}}{1 + \frac{1}{x}}$$

$$= \infty$$

Exercise

Find $\displaystyle\lim_{x\to-\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}}$

Solution

$$\lim_{x\to-\infty} \frac{4-3x^3}{\sqrt{x^6+9}} = \lim_{x\to-\infty} \frac{\frac{4-3x^3}{\sqrt{x^6}}}{\frac{\sqrt{x^6+9}}{\sqrt{x^6}}}$$

$$= \lim_{x\to-\infty} \frac{\frac{4-3x^3}{x^3}}{\sqrt{\frac{x^6+9}{x^6}}}$$

$$= \lim_{x\to-\infty} \frac{\frac{4}{x3}-3}{\sqrt{1+\frac{9}{x^6}}}$$

$$= \frac{-3}{\sqrt{1}}$$

$$\underline{= -3 \,|}$$

Exercise

Find $\lim_{x\to-\infty} \left(\sqrt{x^2+3}+x\right)$

Solution

$$\lim_{x\to-\infty} \left(\sqrt{x^2+3}+x\right) = \lim_{x\to-\infty} \left(\sqrt{x^2+3}+x\right)\frac{\sqrt{x^2+3}-x}{\sqrt{x^2+3}-x}$$

$$= \lim_{x\to-\infty} \frac{x^2+3-x^2}{\sqrt{x^2+3}-x}$$

$$= \lim_{x\to-\infty} \frac{3}{\sqrt{x^2+3}-x}$$

$$= \lim_{x\to-\infty} \frac{\frac{3}{x}}{\sqrt{\frac{x^2}{x^2}+\frac{3}{x^2}}-\frac{x}{x}}$$

$$= \lim_{x\to-\infty} \frac{\frac{3}{x}}{\sqrt{1+\frac{3}{x^2}}+1}$$

$$= \frac{0}{\sqrt{1}+1}$$

$$\underline{= 0 \,|}$$

Exercise

Find $\lim\limits_{x\to\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-2x}\right)$

Solution

$$\lim_{x\to\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-2x}\right)=\lim_{x\to\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-2x}\right)\frac{\sqrt{x^2+3x}+\sqrt{x^2-2x}}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$$

$$=\lim_{x\to\infty}\frac{\left(x^2+3x\right)-\left(x^2-2x\right)}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$$

$$=\lim_{x\to\infty}\frac{x^2+3x-x^2+2x}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$$

$$=\lim_{x\to\infty}\frac{5x}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$$

$$=\lim_{x\to\infty}\frac{\frac{5x}{\sqrt{x^2}}}{\sqrt{\frac{x^2}{x^2}+\frac{3x}{x^2}}+\sqrt{\frac{x^2}{x^2}-\frac{2x}{x^2}}}$$

$$=\lim_{x\to\infty}\frac{5}{\sqrt{1+\frac{3}{x}}+\sqrt{1-\frac{2}{x}}}$$

$$=\frac{5}{\sqrt{1}+\sqrt{1}}$$

$$\boxed{=\frac{5}{2}}$$

Exercise

Find $\lim\limits_{x\to\infty}\frac{2x-3}{4x+10}$

Solution

$$\lim_{x\to\infty}\frac{2x-3}{4x+10}\boxed{=\frac{1}{2}}$$

Exercise

Find $\lim\limits_{x\to\infty}\frac{x^4-1}{x^5+2}$

Solution

$$\lim_{x\to\infty} \frac{x^4-1}{x^5+2} \underline{=0}$$

Exercise

Find $\lim_{x\to-\infty} \left(-3x^3+5\right)$

Solution

$$\lim_{x\to-\infty} \left(-3x^3+5\right) \underline{=\infty}$$

Exercise

Find $\lim_{x\to\infty} \left(e^{-2x}+\frac{2}{x}\right)$

Solution

$$\lim_{x\to\infty} \left(e^{-2x}+\frac{2}{x}\right) = e^{-\infty}+0$$

$$\underline{=0}$$

Exercise

Find $\lim_{x\to\infty} \frac{1}{\ln x+1}$

Solution

$$\lim_{x\to\infty} \frac{1}{\ln x+1} = \frac{1}{\infty}$$

$$\underline{=0}$$

Exercise

Find $\lim_{x\to\infty} \left(3+\frac{10}{x^2}\right)$

Solution

$$\lim_{x\to\infty} \left(3+\frac{10}{x^2}\right) = 3+0$$

$$\underline{=3}$$

Exercise

Find $\lim\limits_{x\to\infty}\left(5+\frac{1}{x}+\frac{10}{x^2}\right)$

Solution

$$\lim\limits_{x\to\infty}\left(5+\frac{1}{x}+\frac{10}{x^2}\right)=5+0+0$$

$$\underline{=5}$$

Exercise

Find $\lim\limits_{x\to\infty}\frac{4x^2+2x+3}{x^2}$

Solution

$$\lim\limits_{x\to\infty}\frac{4x^2+2x+3}{x^2}=\lim\limits_{x\to\infty}\frac{4x^2}{x^2}$$

$$\underline{=4}$$

Exercise

Find $\lim\limits_{x\to\infty}\left(5+\frac{100}{x}+\frac{\sin^4 x^3}{x^2}\right)$

Solution

$$-1\le\sin\theta\le1$$

$$0\le\sin^4\theta\le1$$

$$0\le\frac{\sin^4\theta}{x^2}\le\frac{1}{x^2}\quad\to0$$

$$\lim\limits_{x\to\infty}\left(5+\frac{100}{x}+\frac{\sin^4 x^3}{x^2}\right)\underline{=5}$$

Exercise

Find $\lim\limits_{\theta\to\infty}\frac{\cos\theta}{\theta^2}$

Solution

$$-1\le\cos\theta\le1$$

$$-\frac{1}{\theta^2}\le\frac{\cos\theta}{\theta^2}\le\frac{1}{\theta^2}\quad\to0$$

$$\lim_{\theta\to\infty} \frac{\cos\theta}{\theta^2} = 0$$

Exercise

Find $\lim_{\theta\to\infty} \frac{\cos\theta^5}{\sqrt{\theta}}$

Solution

$$-1 \le \cos\theta^5 \le 1$$

$$-\frac{1}{\sqrt{\theta}} \le \frac{\cos\theta^5}{\sqrt{\theta}} \le \frac{1}{\sqrt{\theta}} \quad \to 0$$

$$\lim_{\theta\to\infty} \frac{\cos\theta^5}{\sqrt{\theta}} = 0$$

Exercise

Find $\lim_{x\to\infty} \frac{4x}{20x+1}$

Solution

$$\lim_{x\to\infty} \frac{4x}{20x+1} = \frac{4}{20}$$

$$= \frac{1}{5}$$

Exercise

Find $\lim_{x\to-\infty} \frac{4x}{20x+1}$

Solution

$$\lim_{x\to-\infty} \frac{4x}{20x+1} = \lim_{x\to-\infty} \frac{4x}{20x}$$

$$= \frac{1}{5}$$

Exercise

Find $\lim_{x\to\infty} \frac{3x^2-7}{x^2+5x}$

Solution

$$\lim_{x\to\infty} \frac{3x^2-7}{x^2+5x} \underline{= 3}$$

Exercise

Find $\lim_{x\to-\infty} \frac{3x^2-7}{x^2+5x}$

Solution

$$\lim_{x\to-\infty} \frac{3x^2-7}{x^2+5x} = \lim_{x\to-\infty} \frac{3x^2}{x^2}$$

$$\underline{= 3}$$

Exercise

Find $\lim_{x\to\infty} \frac{6x^2-9x+8}{3x^2+2}$

Solution

$$\lim_{x\to\infty} \frac{6x^2-9x+8}{3x^2+2} = \lim_{x\to\infty} \frac{6x^2}{3x^2}$$

$$= \frac{6}{3}$$

$$\underline{= 2}$$

Exercise

Find $\lim_{x\to-\infty} \frac{6x^2-9x+8}{3x^2+2}$

Solution

$$\lim_{x\to-\infty} \frac{6x^2-9x+8}{3x^2+2} = \lim_{x\to-\infty} \frac{6x^2}{3x^2}$$

$$= \frac{6}{3}$$

$$\underline{= 2}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{4x^2-7}{8x^2+5x+2}$

Solution

$$\lim_{x\to\infty} \frac{4x^2-7}{8x^2+5x+2} = \lim_{x\to\infty} \frac{4x^2}{8x^2}$$

$$= \frac{4}{8}$$

$$= \frac{1}{2}$$

Exercise

Find $\lim\limits_{x\to-\infty} \dfrac{4x^2-7}{8x^2+5x+2}$

Solution

$$\lim_{x\to-\infty} \frac{4x^2-7}{8x^2+5x+2} = \lim_{x\to-\infty} \frac{4x^2}{8x^2}$$

$$= \frac{4}{8}$$

$$= \frac{1}{2}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4}$

Solution

$$\lim_{x\to\infty} \frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4} = \lim_{x\to\infty} \frac{\sqrt{16x^4}+x^2}{2x^2}$$

$$= \lim_{x\to\infty} \frac{4x^2+x^2}{2x^2}$$

$$= \lim_{x\to\infty} \frac{5x^2}{2x^2}$$

$$= \frac{5}{2}$$

Exercise

Find $\lim\limits_{x\to-\infty} \dfrac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4}$

Solution

$$\lim_{x\to-\infty} \frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4} = \lim_{x\to-\infty} \frac{\sqrt{16x^4}+x^2}{2x^2}$$

$$= \lim_{x\to-\infty} \frac{4x^2+x^2}{2x^2}$$

$$= \lim_{x\to-\infty} \frac{5x^2}{2x^2}$$

$$= \frac{5}{2}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{3x^4+3x^3-36x^2}{x^4-25x^2+144}$

Solution

$$\lim_{x\to\infty} \frac{3x^4+3x^3-36x^2}{x^4-25x^2+144} = \lim_{x\to\infty} \frac{3x^4}{x^4}$$

$$= 3$$

Exercise

Find $\lim\limits_{x\to-\infty} \dfrac{3x^4+3x^3-36x^2}{x^4-25x^2+144}$

Solution

$$\lim_{x\to-\infty} \frac{3x^4+3x^3-36x^2}{x^4-25x^2+144} = \lim_{x\to-\infty} \frac{3x^4}{x^4}$$

$$= 3$$

Exercise

Find $\lim\limits_{x\to\infty} 16x^2\left(4x^2-\sqrt{16x^4+1}\right)$

Solution

$$\lim_{x\to\infty} 16x^2\left(4x^2-\sqrt{16x^4+1}\right)=\infty-\infty$$

$$=\lim_{x\to\infty} 16x^2\left(4x^2-\sqrt{16x^4+1}\right)\cdot\frac{4x^2+\sqrt{16x^4+1}}{4x^2+\sqrt{16x^4+1}}$$

$$=\lim_{x\to\infty} 16x^2\,\frac{16x^4-16x^4-1}{4x^2+\sqrt{16x^4+1}}$$

$$=\lim_{x\to\infty} 16x^2\,\frac{-1}{4x^2+\sqrt{16x^4+1}}$$

$$=\lim_{x\to\infty} \frac{-16x^2}{4x^2+4x^2}$$

$$=\lim_{x\to\infty} \frac{-16x^2}{8x^2}$$

$$\underline{=-2}$$

Exercise

Find $\lim_{x\to-\infty} 16x^2\left(4x^2-\sqrt{16x^4+1}\right)$

Solution

$$\lim_{x\to-\infty} 16x^2\left(4x^2-\sqrt{16x^4+1}\right)=\infty-\infty$$

$$=\lim_{x\to-\infty} 16x^2\left(4x^2-\sqrt{16x^4+1}\right)\cdot\frac{4x^2+\sqrt{16x^4+1}}{4x^2+\sqrt{16x^4+1}}$$

$$=\lim_{x\to-\infty} 16x^2\,\frac{16x^4-16x^4-1}{4x^2+\sqrt{16x^4+1}}$$

$$=\lim_{x\to-\infty} 16x^2\,\frac{-1}{4x^2+\sqrt{16x^4+1}}$$

$$=\lim_{x\to-\infty} \frac{-16x^2}{4x^2+4x^2}$$

$$=\lim_{x\to-\infty} \frac{-16x^2}{8x^2}$$

$$\underline{=-2}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{x-1}{x^{2/3}-1}$

Solution

$$\lim\limits_{x\to\infty} \frac{x-1}{x^{2/3}-1} = \lim\limits_{x\to\infty} \frac{x}{x^{2/3}}$$

$$= \lim\limits_{x\to\infty} x^{1/3}$$

$$\boxed{=\infty}$$

Exercise

Find $\lim\limits_{x\to-\infty} \dfrac{x-1}{x^{2/3}-1}$

Solution

$$\lim\limits_{x\to-\infty} \frac{x-1}{x^{2/3}-1} = \lim\limits_{x\to-\infty} \frac{x}{x^{2/3}}$$

$$= \lim\limits_{x\to-\infty} x^{1/3}$$

$$\boxed{=-\infty}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+2x+6}-3}{x-1}$

Solution

$$\lim\limits_{x\to\infty} \frac{\sqrt{x^2+2x+6}-3}{x-1} = \lim\limits_{x\to\infty} \frac{\sqrt{x^2}}{x}$$

$$= \lim\limits_{x\to\infty} \frac{x}{x}$$

$$\boxed{=1}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{\left|1-x^2\right|}{x(x+1)}$

Solution

$$\lim_{x\to\infty} \frac{\left|1-x^2\right|}{x(x+1)} = \lim_{x\to\infty} \frac{x^2-1}{x^2+1}$$

$$= \lim_{x\to\infty} \frac{x^2}{x^2}$$

$$\underline{=1}$$

Exercise

Find $\lim_{x\to\infty} \left(\sqrt{|x|} - \sqrt{|x-1|}\right)$

Solution

$$\lim_{x\to\infty} \left(\sqrt{|x|} - \sqrt{|x-1|}\right) = \infty - \infty \qquad x\to\infty \Rightarrow |x| = x \ \ \& \ \ |x-1| = x-1$$

$$= \lim_{x\to\infty} \left(\sqrt{x} - \sqrt{x-1}\right)\cdot\frac{\sqrt{x}+\sqrt{x-1}}{\sqrt{x}+\sqrt{x-1}}$$

$$= \lim_{x\to\infty} \frac{x-x+1}{\sqrt{x}+\sqrt{x-1}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x}+\sqrt{x-1}}$$

$$= \frac{1}{\infty}$$

$$\underline{=0}$$

Exercise

Find $\lim_{x\to\infty} \frac{\tan^{-1}x}{x}$

Solution

$$-\frac{\pi}{2} \le \tan^{-1}x \le \frac{\pi}{2}$$

$$-\frac{\pi}{2x} \le \frac{\tan^{-1}x}{x} \le \frac{\pi}{2x} \quad \to \ 0$$

$$\lim_{x\to\infty} \frac{\tan^{-1}x}{x} \underline{=0}$$

Exercise

Find $\displaystyle\lim_{x\to\infty} \frac{\cos x}{e^{3x}}$

Solution

$$-1 \le \cos x \le 1$$

$$-\frac{1}{e^{3x}} \le \frac{\cos x}{e^{3x}} \le \frac{1}{e^{3x}} \to 0$$

$$\lim_{x\to\infty} \frac{\cos x}{e^{3x}} = 0$$

Exercise

Find $\displaystyle\lim_{x\to 0} \frac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}}$

Solution

$$\lim_{x\to 0} \frac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}} = \frac{2+10}{1+1}$$

$$= 6$$

Exercise

Find $\displaystyle\lim_{x\to\infty} \frac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}}$

Solution

$$\lim_{x\to\infty} \frac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}} = \lim_{x\to\infty} \frac{2e^{x}}{e^{x}} \qquad \lim_{x\to\infty} e^{-x} = 0$$

$$= 2$$

Exercise

Find $\displaystyle\lim_{x\to -\infty} \frac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}}$

Solution

$$\lim_{x\to\infty} \frac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}} = \lim_{x\to\infty} \frac{10e^{-x}}{e^{-x}} \qquad \lim_{x\to -\infty} e^{x} = 0$$

$$= 10$$

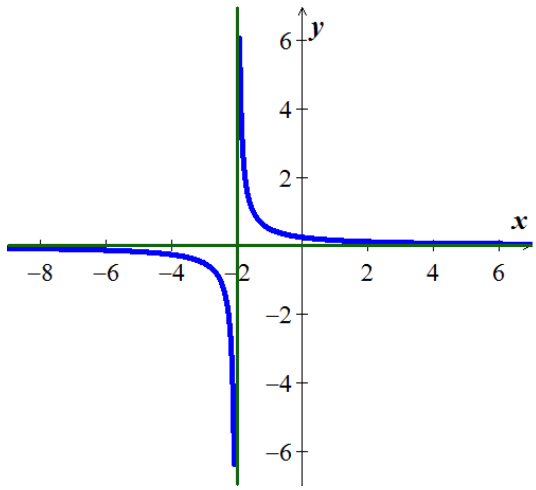
Exercise

Graph the rational function $y = \dfrac{1}{2x+4}$. Include the equations of the asymptotes.

Solution

$\boldsymbol{VA}$: $2x = 4 = 0 \quad \Rightarrow \quad \boxed{x = -2}$

$\boldsymbol{HA}$: $\boxed{y = 0}$



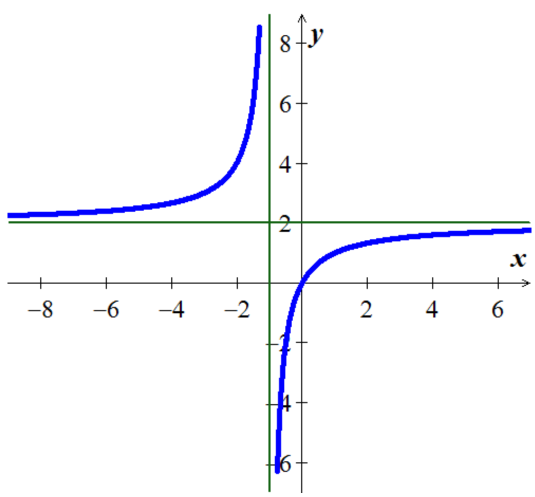
Exercise

Graph the rational function $y = \dfrac{2x}{x+1}$. Include the equations of the asymptotes.

Solution

$\boldsymbol{VA}$: $\boxed{x = -1}$

$\boldsymbol{HA}$: $\boxed{y = 2}$



Exercise

Graph the rational function $y=\dfrac{x^2}{x-1}$. Include the equations of the asymptotes.

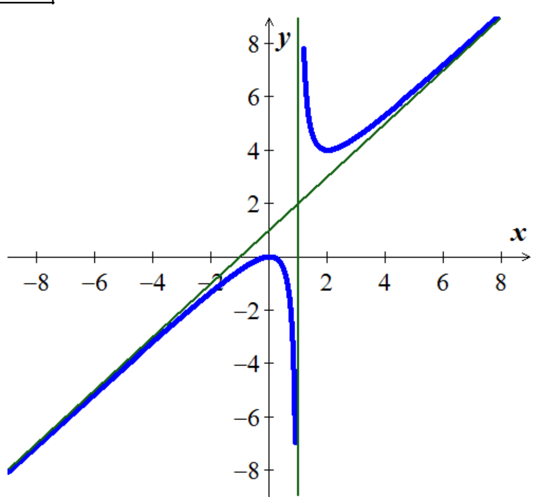
Solution

$$\begin{array}{r|l} x-1 & \overline{\color{blue}{x+1}} \\ & x^2 \\ & \underline{x^2-x} \\ & \quad x \\ & \quad \underline{x-1} \\ & \qquad \boxed{1} \end{array}$$

$$y=\frac{x^2}{x-1}$$
$$=x+1+\frac{1}{x-1}$$

VA: $\underline{x=1}$

Oblique Asymptote: $\underline{y=x+1}$



Exercise

Graph the rational function $y=\dfrac{x^3+1}{x^2}$. Include the equations of the asymptotes.

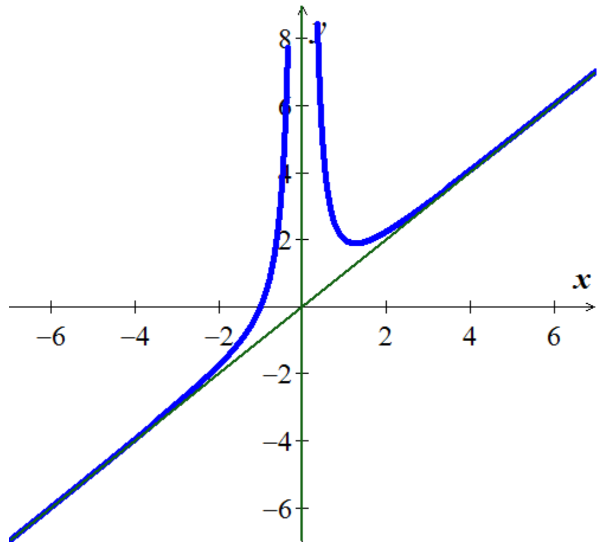
Solution

$$\begin{array}{r|l} x^2 & \overline{\color{blue}{x}} \\ & x^3+1 \\ & \underline{x^3} \\ & \quad \boxed{1} \end{array}$$

$$y=\frac{x^3+1}{x^2}=x+\frac{1}{x^2}$$

VA: $\boxed{x=0}$

Oblique Asymptote: $\boxed{y=x}$



Exercise

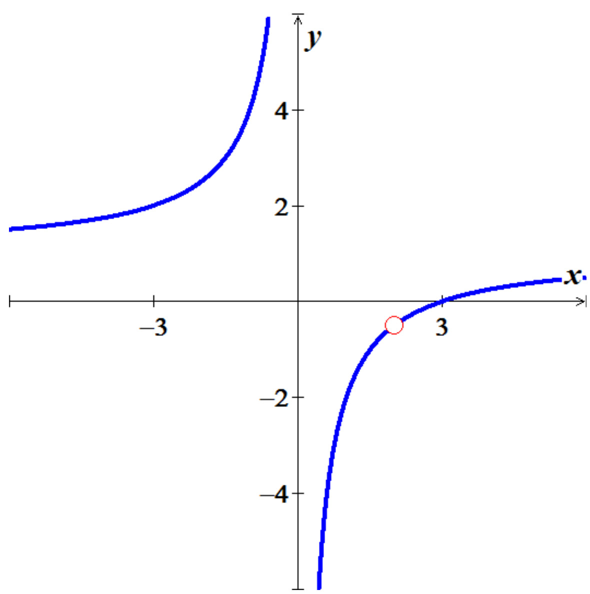
Let $\qquad f(x)=\dfrac{x^2-5x+6}{x^2-2x}$

a) Analyze $\lim\limits_{x\to 0^-} f(x)$, $\lim\limits_{x\to 0^+} f(x)$, $\lim\limits_{x\to 2^-} f(x)$, and $\lim\limits_{x\to 2^+} f(x)$

b) Does the graph of f have any vertical asymptotes? Explain?

Solution

$$f(x)=\frac{x^2-5x+6}{x^2-2x}$$
$$=\frac{(x-2)(x-3)}{x(x-2)}$$
$$=\frac{x-3}{x}$$



a) $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}\dfrac{x-3}{x}$

$$=\frac{-3}{0^-}$$

$$\boxed{=\infty}$$

$\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+}\dfrac{x-3}{x}$

$$=\frac{-3}{0^+}$$

$$\boxed{= -\infty}$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x-3}{x}$$

$$= \frac{2-3}{2}$$

$$\boxed{= -\frac{1}{2}}$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x-3}{x}$$

$$= \frac{2-3}{2}$$

$$\boxed{= -\frac{1}{2}}$$

b) ***VA***$: x = 0$ ***Hole***$: \ x = 2 \to f(2) = -\frac{1}{2}$

HA$: y = 1$ ***OA***$: \ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y = \dfrac{3x}{1-x}$

Solution

VA$: x = 1,$ ***Hole***$: \ n/a,$ ***HA***$: y = -3,$ ***OA***$: \ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y = \dfrac{x^2}{x^2+9}$

Solution

VA$: \ n/a;$ ***Hole***$: \ n/a;$ ***HA***$: y = 1;$ ***OA***$: \ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y = \dfrac{x-2}{x^2-4x+3}$

Solution

VA$: x = 1, \ 3;$ ***Hole***$: \ n/a;$ ***HA***$: y = 0;$ ***OA***$: \ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y = \dfrac{5x-1}{1-3x}$

Solution

$$\textbf{VA}: \ x = \frac{1}{3}; \quad \textbf{Hole}: \ n/a; \quad \textbf{HA}: \ y = -\frac{5}{3}; \quad \textbf{OA}: \ n/a$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y = \dfrac{3}{x-5}$

Solution

$$\textbf{VA}: x = 5, \quad \textbf{Hole}: \ n/a, \quad \textbf{HA}: y = 0, \quad \textbf{OA}: \ n/a$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y = \dfrac{x^3-1}{x^2+1}$

Solution

$$\begin{array}{r|l} & x \\ \hline x^2+1 & x^3 - 1 \\ & x^3 + x \\ \hline & \boxed{-x-1} \end{array}$$

$$\begin{aligned} y &= \frac{x^3-1}{x^2+1} \\ &= x + \frac{-x-1}{x^2+1} \\ &= x - \frac{x+1}{x^2+1} \end{aligned}$$

$$\textbf{VA}: \ n/a, \quad \textbf{Hole}: \ n/a, \quad \textbf{HA}: \ n/a, \quad \textbf{OA}: \ y = x$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y = \dfrac{3x^2-27}{(x+3)(2x+1)}$

Solution

$$\textbf{VA}: x = -3, \ -\frac{1}{2}; \quad \textbf{Hole}: \ n/a; \quad \textbf{HA}: y = \frac{3}{2}; \quad \textbf{OA}: \ n/a$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{x^3+3x^2-2}{x^2-4}$

Solution

$$\begin{array}{r|l} & \quad x+3 \\ x^2-4 & x^3+3x^2-2 \\ & \underline{x^3 \qquad -4x} \\ & \quad 3x^2+4x-2 \end{array}$$

$$\begin{aligned} y &= \frac{x^3+3x^2-2}{x^2-4} \\ &= x+3+\frac{4x+10}{x^2-4} \end{aligned}$$

$\boldsymbol{VA}:\ x=\pm 2,\quad \boldsymbol{Hole}:\ n/a,\quad \boldsymbol{HA}:\ n/a,\quad \boldsymbol{OA}:\ y=x+3$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{x-3}{x^2-9}$

Solution

$\boldsymbol{VA}:\ x=-3;\quad \boldsymbol{Hole}:\ x=3;\quad \boldsymbol{HA}:\ y=0;\quad \boldsymbol{OA}:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{6}{\sqrt{x^2-4x}}$

Solution

$\boldsymbol{VA}:\ x=0,\ 4;\quad \boldsymbol{Hole}:\ n/a;\quad \boldsymbol{HA}:\ y=0;\quad \boldsymbol{OA}:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{4x^3+1}{1-x^3}$

Solution

$\boldsymbol{VA}:\ x=1;\quad \boldsymbol{Hole}:\ n/a;\quad \boldsymbol{HA}:\ y=-4;\quad \boldsymbol{OA}:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x) = \dfrac{x+1}{\sqrt{9x^2+x}}$

Solution

$$\textbf{\textit{VA}}:\ x = 0,\ -\frac{1}{9};\quad \textbf{\textit{Hole}}:\ n/a;\quad \textbf{\textit{HA}}:\ y = \frac{1}{3};\quad \textbf{\textit{OA}}:\ n/a$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x) = 1 - e^{-2x}$

Solution

$$\textbf{\textit{VA}}:\ n/a;\quad \textbf{\textit{Hole}}:\ n/a;\quad \textbf{\textit{HA}}:\ y = 1;\quad \textbf{\textit{OA}}:\ n/a$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x) = \dfrac{1}{\ln x^2}$

Solution

$$\textbf{\textit{VA}}:\ x = 0;\quad \textbf{\textit{Hole}}:\ n/a;\quad \textbf{\textit{HA}}:\ y = 0;\quad \textbf{\textit{OA}}:\ n/a$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x) = \dfrac{1}{\tan^{-1} x}$

Solution

$$\textbf{\textit{VA}}:\ x = 0;\quad \textbf{\textit{Hole}}:\ n/a;\quad \textbf{\textit{HA}}:\ y = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi};\quad \textbf{\textit{OA}}:\ n/a$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x) = \dfrac{2x^2+6}{2x^2+3x-2}$

Solution

$$\textbf{\textit{VA}}:\ x = -2,\ \frac{1}{2};\quad \textbf{\textit{Hole}}:\ n/a;\quad \textbf{\textit{HA}}:\ y = 1;\quad \textbf{\textit{OA}}:\ n/a$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{3x^2+2x-1}{4x+1}$

Solution

$$\begin{array}{r|l} & \frac{3}{4}x+\frac{5}{16} \\ \hline 4x+1 & 3x^2+2x-1 \\ & \underline{3x^2+\frac{3}{4}x} \\ & \qquad \frac{5}{4}x-1 \end{array}$$

$\boldsymbol{VA}:\ x=-\frac{1}{4};\quad \boldsymbol{Hole}:\ n/a;\quad \boldsymbol{HA}:\ n/a;\quad \boldsymbol{OA}:\ y=\frac{3}{4}x+\frac{5}{16}$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{9x^2+4}{(2x-1)^2}$

Solution

$\boldsymbol{VA}:\ x=\frac{1}{2};\quad \boldsymbol{Hole}:\ n/a;\quad \boldsymbol{HA}:\ y=\frac{9}{4};\quad \boldsymbol{OA}:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{1+x-2x^2-x^3}{x^2+1}$

Solution

$$\begin{array}{r|l} & \qquad -x-2 \\ \hline x^2+1 & -x^3-2x^2+x+1 \\ & \underline{-x^3 \qquad\quad -x} \\ & \quad -2x^2+2x \end{array}$$

$\boldsymbol{VA}:\ n/a;\quad \boldsymbol{Hole}:\ n/a;\quad \boldsymbol{HA}:\ n/a;\quad \boldsymbol{OA}:\ y=-x-2$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{x(x+2)^3}{3x^2-4x}$

Solution

$$f(x)=\frac{x\left(x^3+6x^2+12x+8\right)}{x(3x-4)}$$

$$=\frac{x^3+6x^2+12x+8}{3x-4}$$

$$\begin{array}{r|l} & \frac{1}{3}x^2+\frac{22}{9}x+\frac{196}{27} \\ \hline 3x-4 & x^3+6x^2+12x+8 \\ & x^3-\frac{4}{3}x^2 \\ \hline & \frac{22}{3}x^2+12x \\ & \frac{22}{3}x^2-\frac{88}{9}x \\ \hline & \frac{196}{9}x \end{array}$$

$\boldsymbol{VA}:\ x=\frac{4}{3};\quad \boldsymbol{Hole}:\ (0,\ -2);\quad \boldsymbol{HA}:\ n/a;\quad \boldsymbol{OA}:\ y=\frac{1}{3}x^2+\frac{22}{9}x+\frac{196}{27}$

Exercise

Find the limit $\displaystyle\lim_{x\to 0}\frac{x^2-4x+4}{x^3+5x^2-14x}$

Solution

$$\lim_{x\to 0}\frac{x^2-4x+4}{x^3+5x^2-14x}=\frac{4}{0}$$

$$=\infty$$

Exercise

Find the limit $\displaystyle\lim_{x\to 2}\frac{x^2-4x+4}{x^3+5x^2-14x}$

Solution

$$\lim_{x\to 2}\frac{x^2-4x+4}{x^3+5x^2-14x}=\frac{4-8+4}{8+20-28}=\frac{0}{0}$$

$$=\lim_{x\to 2}\frac{(x-2)(x-2)}{x(x-2)(x+7)}$$

$$=\lim_{x\to 2}\frac{x-2}{x(x+7)}$$

$$=\frac{0}{18}$$

$$=0$$

Exercise

Find the limit $\lim\limits_{x\to a} \dfrac{x^2-a^2}{x^4-a^4}$

Solution

$$\lim_{x\to a} \frac{x^2-a^2}{x^4-a^4} = \frac{a^2-a^2}{a^4-a^4} = \frac{0}{0}$$

$$= \lim_{x\to a} \frac{x^2-a^2}{\left(x^2-a^2\right)\left(x^2+a^2\right)}$$

$$= \lim_{x\to a} \frac{1}{x^2+a^2}$$

$$= \frac{1}{a^2+a^2}$$

$$= \frac{1}{2a^2}$$

Exercise

Find the limit $\lim\limits_{x\to 0} \dfrac{(x+h)^2-x^2}{h}$

Solution

$$\lim_{x\to 0} \frac{(x+h)^2-x^2}{h} = \frac{h^2}{h}$$

$$= h$$

Exercise

Find the limit $\lim\limits_{h\to 0} \dfrac{(x+h)^2-x^2}{h}$

Solution

$$\lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \frac{x^2-x^2}{0} = \frac{0}{0}$$

$$= \lim_{h\to 0} \frac{x^2+2hx+h^2-x^2}{h}$$

$$= \lim_{h\to 0} \frac{2hx+h^2}{h}$$

$$= \lim_{h \to 0} (2x + h)$$

$$\underline{= 2x}$$

Exercise

Find the limit $\displaystyle \lim_{x \to 1} \frac{1-\sqrt{x}}{1-x}$

Solution

$$\lim_{x \to 1} \frac{1-\sqrt{x}}{1-x} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$= \lim_{x \to 1} \frac{1-\sqrt{x}}{\left(1-\sqrt{x}\right)\left(1+\sqrt{x}\right)}$$

$$= \lim_{x \to 1} \frac{1}{1+\sqrt{x}}$$

$$\underline{= \frac{1}{2}}$$

$$\lim_{x \to 1} \frac{1-\sqrt{x}}{1-x} = \lim_{x \to 1} \frac{1-\sqrt{x}}{1-x} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}}$$

$$= \lim_{x \to 1} \frac{1-x}{(1-x)\left(1+\sqrt{x}\right)}$$

$$= \lim_{x \to 1} \frac{1}{1+\sqrt{x}}$$

$$\underline{= \frac{1}{2}}$$

Exercise

Find the limit $\displaystyle \lim_{x \to 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x}$

Solution

$$\lim_{x \to 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = \frac{\frac{1}{2} - \frac{1}{2}}{0} = \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{1}{x}\left(\frac{2-2-x}{2(2+x)}\right)$$

$$= \frac{1}{2} \lim_{x \to 0} \frac{1}{x}\left(\frac{-x}{2+x}\right)$$

$$= -\frac{1}{2}\lim_{x\to 0}\ \frac{1}{2+x}$$

$$= -\frac{1}{2}\left(\frac{1}{2}\right)$$

$$= -\frac{1}{4}$$

Exercise

Find the limit $\lim\limits_{x\to 1}\ \dfrac{x^{1/3}-1}{\sqrt{x}-1}$

Solution

$$\lim_{x\to 1}\ \frac{x^{1/3}-1}{\sqrt{x}-1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$= \lim_{x\to 1}\ \frac{x^{1/3}-1}{\sqrt{x}-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}$$

$$= \lim_{x\to 1}\ \frac{\left(x^{1/3}-1\right)\left(\sqrt{x}+1\right)}{x-1}$$

$$= \lim_{x\to 1}\ \frac{\left(x^{1/3}-1\right)\left(\sqrt{x}+1\right)}{\left(x^{1/3}\right)^3-1^3} \qquad a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$$

$$= \lim_{x\to 1}\ \frac{\left(x^{1/3}-1\right)\left(\sqrt{x}+1\right)}{\left(x^{1/3}-1\right)\left(x^{2/3}+x^{1/3}+1\right)}$$

$$= \lim_{x\to 1}\ \frac{\sqrt{x}+1}{x^{2/3}+x^{1/3}+1}$$

$$= \frac{2}{3}$$

Exercise

Find the limit $\lim\limits_{x\to 64}\ \dfrac{x^{2/3}-16}{\sqrt{x}-8}$

Solution

$$\lim_{x\to 64}\ \frac{x^{2/3}-16}{\sqrt{x}-8} = \frac{\left(4^3\right)^{2/3}-16}{8-8}$$

$$= \frac{16-16}{0} = \frac{0}{0}$$

$$= \lim_{x\to 64} \frac{\left(x^{1/3}\right)^2 - 16}{\sqrt{x}-8} \cdot \frac{\sqrt{x}+8}{\sqrt{x}+8}$$

$$= \lim_{x\to 64} \frac{\left(x^{1/3}-4\right)\left(x^{1/3}+4\right)\left(\sqrt{x}+8\right)}{x-64}$$

$$= \lim_{x\to 64} \frac{\left(x^{1/3}-4\right)\left(x^{1/3}+4\right)\left(\sqrt{x}+8\right)}{\left(x^{1/3}\right)^3 - 4^3}$$

$$a^3 - b^3 = (a-b)\left(a^2+ab+b^2\right)$$

$$= \lim_{x\to 64} \frac{\left(x^{1/3}-4\right)\left(x^{1/3}+4\right)\left(\sqrt{x}+8\right)}{\left(x^{1/3}-4\right)\left(x^{2/3}+4x^{1/3}+16\right)}$$

$$= \lim_{x\to 64} \frac{\left(x^{1/3}+4\right)\left(\sqrt{x}+8\right)}{x^{2/3}+4x^{1/3}+16}$$

$$= \frac{(4+4)(8+8)}{16+16+16}$$

$$\boxed{= \frac{8}{3}}$$

Exercise

Find the limit $\displaystyle \lim_{x\to 0} \frac{\tan(2x)}{\tan(\pi x)}$

Solution

$$\lim_{x\to 0} \frac{\tan(2x)}{\tan(\pi x)} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin 2x}{\cos 2x} \cdot \frac{\cos(\pi x)}{\sin(\pi x)}$$

$$= \lim_{x\to 0} \frac{\cos(\pi x)}{\cos 2x} \cdot \frac{\sin 2x}{2x} \cdot \frac{2x}{\pi x} \cdot \frac{\pi x}{\sin(\pi x)}$$

$$= \frac{2}{\pi} \frac{\cos 0}{\cos 0} \cdot \lim_{2x\to 0} \frac{\sin 2x}{2x} \lim_{\pi x\to 0} \frac{1}{\frac{\sin \pi x}{\pi x}}$$

$$\boxed{= \frac{2}{\pi}}$$

Exercise

Find the limit $\lim\limits_{x\to\pi^-} \csc x$

Solution

$$\lim_{x\to\pi^-} \csc x = \frac{1}{\sin \pi^-}$$
$$= \frac{1}{0^-}$$
$$\underline{= -\infty}$$

Exercise

Find the limit $\lim\limits_{x\to\pi} \sin\left(\frac{x}{2} + \sin x\right)$

Solution

$$\lim_{x\to\pi} \sin\left(\frac{x}{2} + \sin x\right) = \sin\left(\frac{\pi}{2} + \sin \pi\right)$$
$$= \sin\frac{\pi}{2}$$
$$\underline{= 1}$$

Exercise

Find the limit $\lim\limits_{x\to\pi} \cos^2(x - \tan x)$

Solution

$$\lim_{x\to\pi} \cos^2(x - \tan x) = \cos^2(\pi - \tan\pi)$$
$$= \cos^2(\pi)$$
$$= (-1)^2$$
$$\underline{= 1}$$

Exercise

Find the limit $\lim\limits_{x\to 0} \frac{8x}{3\sin x - x}$

Solution

$$\lim_{x\to 0}\frac{8x}{3\sin x - x} = \frac{0}{0}$$

$$= \lim_{x\to 0}\frac{8\frac{x}{x}}{3\frac{\sin x}{x} - \frac{x}{x}}$$

$$= \frac{8}{3\lim_{x\to 0}\frac{\sin x}{x} - 1} \qquad \lim_{x\to 0}\frac{\sin x}{x} = 1$$

$$= \frac{8}{3-1}$$

$$\underline{= 4}$$

Exercise

Find the limit $\lim_{x\to 0}\frac{\cos 2x - 1}{\sin x}$

Solution

$$\lim_{x\to 0}\frac{\cos 2x - 1}{\sin x} = \frac{0}{0}$$

$$= \lim_{x\to 0}\frac{1 - 2\sin^2 x - 1}{\sin x}$$

$$= \lim_{x\to 0}\frac{-2\sin^2 x}{\sin x}$$

$$= -2\lim_{x\to 0}\sin x$$

$$\underline{= 0}$$

Exercise

Find the limit $\lim_{x\to -\infty}\frac{4 - 3x^3}{\sqrt{x^6 + 9}}$

Solution

$$\lim_{x\to -\infty}\frac{4 - 3x^3}{\sqrt{x^6 + 9}} = \lim_{x\to -\infty}\frac{3x^3}{\sqrt{x^6}}$$

$$= \lim_{x\to -\infty}\frac{3x^3}{x^3}$$

$$\underline{= 3}$$

Exercise

Find the limit $\displaystyle \lim_{x \to -\infty} \frac{x^2 - 4x + 8}{3x^3}$

Solution

$$\lim_{x \to -\infty} \frac{x^2 - 4x + 8}{3x^3} = \lim_{x \to -\infty} \frac{x^2}{3x^3}$$

$$= \lim_{x \to -\infty} \frac{1}{3x}$$

$$\underline{= 0}$$

Exercise

Find the limit $\displaystyle \lim_{x \to -\infty} \frac{2x^2 + 3}{5x^2 + 7}$

Solution

$$\lim_{x \to -\infty} \frac{2x^2 + 3}{5x^2 + 7} = \frac{2}{5}$$

Exercise

Find the limit $\displaystyle \lim_{x \to \infty} \frac{x^4 + x^3}{12x^3 + 128}$

Solution

$$\lim_{x \to \infty} \frac{x^4 + x^3}{12x^3 + 128} = \infty$$

Exercise

Find the limit $\displaystyle \lim_{x \to -\infty} \frac{2 + \sqrt{x}}{2 - \sqrt{x}}$

Solution

Since $x \to -\infty$ and inside the square root can't be negative

$$\lim_{x \to -\infty} \frac{2 + \sqrt{x}}{2 - \sqrt{x}} = \nexists$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{2+\sqrt{x}}{2-\sqrt{x}}$

Solution

$$\lim_{x\to\infty} \frac{2+\sqrt{x}}{2-\sqrt{x}} = \lim_{x\to\infty} \frac{\sqrt{x}}{-\sqrt{x}}$$

$$\underline{=-1}$$

Exercise

Find the limit $\lim\limits_{x\to-\infty} \dfrac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}}$

Solution

$$\lim_{x\to-\infty} \frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}} = \lim_{x\to-\infty} \frac{\sqrt[3]{x}}{\sqrt[3]{x}}$$

$$\underline{=1}$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{\frac{1}{x}+\frac{1}{x^4}}{\frac{1}{x^2}-\frac{1}{x^3}}$

Solution

$$\lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^4}}{\frac{1}{x^2}-\frac{1}{x^3}} = \frac{0}{0}$$

$$= \lim_{x\to\infty} \frac{\frac{x^3+1}{x^4}}{\frac{x-1}{x^3}}$$

$$= \lim_{x\to\infty} \frac{x^3+1}{x-1}\cdot\frac{x^3}{x^4}$$

$$= \lim_{x\to\infty} \frac{x^3+1}{x(x-1)}$$

$$= \lim_{x\to\infty} \frac{x^3}{x^2}$$

$$\underline{=\infty}$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}}$

Solution

$$\lim_{x\to\infty} \frac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}} = \lim_{x\to\infty} \frac{2x^{5/3}}{x^{8/5}}$$

$$= \lim_{x\to\infty} 2x^{\left(\frac{5}{3}-\frac{8}{5}\right)}$$

$$= \lim_{x\to\infty} 2x^{\frac{1}{15}}$$

$$\underline{= \infty}$$

Exercise

Find the limit $\lim\limits_{x\to 2^+} \ln(x-2)$

Solution

$$\lim_{x\to 2^+} \ln(x-2) = \ln\left(0^+\right)$$

$$\underline{= -\infty}$$

Exercise

Find the limit $\lim\limits_{x\to 1} x^2 \ln\left(2 - \sqrt{x}\right)$

Solution

$$\lim_{x\to 1} x^2 \ln\left(2 - \sqrt{x}\right) = \ln(2-1)$$

$$= \ln 1$$

$$\underline{= 0}$$

Exercise

Find the limit $\lim\limits_{\theta\to 0^+} \sqrt{\theta}\, e^{\cos\frac{\pi}{\theta}}$

Solution

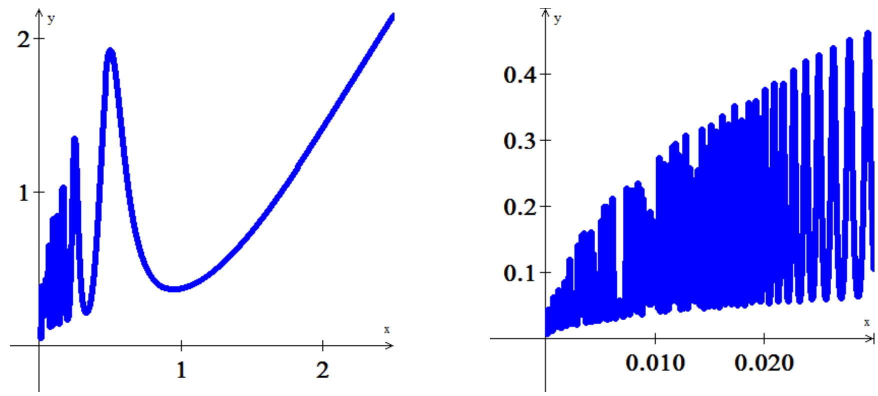
$$\lim_{\theta\to 0^+} \sqrt{\theta}\, e^{\cos\frac{\pi}{\theta}} = 0\cdot e^{\cos\infty}$$

$$-1 \le \cos\frac{\pi}{\theta} \le 1$$

$$e^{-1} \le e^{\cos\frac{\pi}{\theta}} \le e$$

$$0\cdot\frac{1}{e} \le 0\cdot e^{\cos\frac{\pi}{\theta}} \le 0\cdot e$$

$$\boxed{\lim_{\theta\to 0^+} \sqrt{\theta}\, e^{\cos\frac{\pi}{\theta}} = 0}$$



Exercise

Find the limit $\displaystyle\lim_{x\to\infty} \frac{2x-3}{5x+6}$

Solution

$$\lim_{x\to\infty} \frac{2x-3}{5x+6} = \lim_{x\to\infty} \frac{2x}{5x}$$

$$\boxed{=\frac{2}{5}}$$

Exercise

Find the limit $\displaystyle\lim_{x\to\infty} \frac{2x^2-3}{5x^2+6}$

Solution

$$\lim_{x\to\infty} \frac{2x^2-3}{5x^2+6} = \lim_{x\to\infty} \frac{2x^2}{5x^2}$$

$$\boxed{=\frac{2}{5}}$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{2x-3}{5x^3+6}$

Solution

$$\lim_{x\to\infty} \frac{2x-3}{5x^3+6} = \lim_{x\to\infty} \frac{2x}{5x^3}$$

$$\underline{=0}$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{1}{5x^2-3x+6}$

Solution

$$\lim_{x\to\infty} \frac{1}{5x^2-3x+6} = \lim_{x\to\infty} \frac{1}{5x^2}$$

$$\underline{=0}$$

Exercise

Find the limit $\lim\limits_{\theta\to 0} \dfrac{\theta\ \cot 4\theta}{\sin^2\theta\ \cot^2 2\theta}$

Solution

$$\lim_{\theta\to 0} \frac{\theta\ \cot 4\theta}{\sin^2\theta\ \cot^2 2\theta} = \frac{0}{0}$$

$$= \lim_{\theta\to 0} \frac{1}{\frac{\sin\theta}{\theta}} \cdot \frac{1}{\sin\theta} \cdot \frac{\cos 4\theta}{\sin 4\theta} \cdot \frac{\sin^2 2\theta}{\cos^2 2\theta}$$

$$= \lim_{\theta\to 0} \frac{1}{\frac{\sin\theta}{\theta}} \lim_{\theta\to 0} \frac{\cos 4\theta}{\cos^2 2\theta} \lim_{\theta\to 0} \frac{1}{\sin\theta} \cdot \frac{\sin 2\theta\ \sin 2\theta}{2\sin 2\theta\ \cos 2\theta}$$

$$= (1)(1) \lim_{\theta\to 0} \frac{1}{\sin\theta} \cdot \frac{2\sin\theta\cos\theta}{2\cos 2\theta}$$

$$= \lim_{\theta\to 0} \frac{\cos\theta}{\cos 2\theta}$$

$$\underline{=1}$$

Exercise

Find the limit $\lim\limits_{x\to 0^+} \frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}$

Solution

$$\lim_{x\to 0^+} \frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x} = \frac{\sqrt{5}-\sqrt{5}}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x} \cdot \frac{\sqrt{x^2+4x+5}+\sqrt{5}}{\sqrt{x^2+4x+5}+\sqrt{5}}$$

$$= \lim_{x\to 0^+} \frac{x^2+4x+5-5}{x\left(\sqrt{x^2+4x+5}+\sqrt{5}\right)}$$

$$= \lim_{x\to 0^+} \frac{x(x+4)}{x\left(\sqrt{x^2+4x+5}+\sqrt{5}\right)}$$

$$= \lim_{x\to 0^+} \frac{x+4}{\sqrt{x^2+4x+5}+\sqrt{5}}$$

$$= \frac{4}{\sqrt{5}+\sqrt{5}}$$

$$= \frac{4}{2\sqrt{5}}$$

$$= \frac{2}{\sqrt{5}}$$

Exercise

Find the limit $\lim\limits_{x\to 2} \frac{x^4-16}{x-2}$

Solution

$$\lim_{x\to 2} \frac{x^4-16}{x-2} = \frac{16-16}{2-2} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{(x-2)(x+2)\left(x^2+4\right)}{x-2} \qquad a^4-b^4 = (a-b)(a+b)\left(a^2+b^2\right)$$

$$= \lim_{x\to 2} (x+2)\left(x^2+4\right)$$

$$= (4)(8)$$

$$= 32$$

Exercise

Find the limit $\lim\limits_{x\to 2} \dfrac{x^3-8}{x-2}$

Solution

$$\lim_{x\to 2} \frac{x^3-8}{x-2} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{(x-2)\left(x^2+2x+4\right)}{x-2} \qquad a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$$

$$= \lim_{x\to 2} \left(x^2+2x+4\right)$$

$$= 4+4+4$$

$$= 12$$

Exercise

Find the limit $\lim\limits_{x\to -\infty} \dfrac{\sqrt[3]{x}-5x+3}{2x+x^{2/3}-4}$

Solution

$$\lim_{x\to -\infty} \frac{\sqrt[3]{x}-5x+3}{2x+x^{2/3}-4} = \lim_{x\to -\infty} \frac{-5x}{2x}$$

$$= -\frac{5}{2}$$

Exercise

Find the limit $\lim\limits_{x\to -\infty} \dfrac{\sqrt{x^2+1}}{x+1}$

Solution

$$\lim_{x\to -\infty} \frac{\sqrt{x^2+1}}{x+1} = \lim_{x\to -\infty} \frac{\sqrt{x^2}}{x}$$

$$= \lim_{x\to -\infty} \frac{|x|}{x}$$

$$= -1$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+1}}{x+1}$

Solution

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x+1} = \lim_{x\to\infty} \frac{\sqrt{x^2}}{x}$$

$$= \lim_{x\to\infty} \frac{|x|}{x}$$

$$\underline{=1}$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{x-3}{\sqrt{4x^2+25}}$

Solution

$$\lim_{x\to\infty} \frac{x-3}{\sqrt{4x^2+25}} = \lim_{x\to\infty} \frac{x}{2|x|}$$

$$\underline{=\frac{1}{2}}$$

Exercise

Find the limit $\lim\limits_{x\to-\infty} \dfrac{4-3x^3}{\sqrt{x^6+9}}$

Solution

$$\lim_{x\to-\infty} \frac{4-3x^3}{\sqrt{x^6+9}} = \lim_{x\to-\infty} \frac{3x^3}{x^3}$$

$$\underline{=3}$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{x^4-x}{15x^3+4}$

Solution

$$\boxed{\lim_{x\to\infty} \frac{x^4 - x}{15x^3 + 4} = \infty}$$

Exercise

Find the limit $\displaystyle \lim_{x\to\infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x}$

Solution

$$-1 \le \sin x \le 1$$

$$\lim_{x\to\infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x} = \lim_{x\to\infty} \frac{x}{x}$$

$$\boxed{=1}$$

Exercise

Find the limit $\displaystyle \lim_{x\to\infty} \frac{x^{2/3} - x^{-1}}{x^{2/3} + \cos^2 x}$

Solution

$$-1 \le \cos x \le 1$$

$$0 \le \cos^2 x \le 1$$

$$\lim_{x\to\infty} \frac{x^{2/3} - \frac{1}{x}}{x^{2/3} + \cos^2 x} = \lim_{x\to\infty} \frac{x^{2/3}}{x^{2/3}}$$

$$\boxed{=1}$$

Exercise

Find the limit $\displaystyle \lim_{x\to\infty} \frac{\sin 2x}{x}$

Solution

$$-1 \le \sin 2x \le 1$$

$$-\lim_{x\to\infty} \frac{1}{x} \le \lim_{x\to\infty} \frac{\sin 2x}{x} \le \lim_{x\to\infty} \frac{1}{x}$$

$$0 \le \lim_{x\to\infty} \frac{\sin 2x}{x} \le 0$$

$$\boxed{\lim_{x\to\infty} \frac{\sin 2x}{x} = 0}$$

Exercise

Find the limit $\lim\limits_{x\to 0} \dfrac{\sin 5x}{3x}$

Solution

$$\lim_{x\to 0} \frac{\sin 5x}{3x} = \lim_{5x\to 0} \frac{5}{3} \cdot \frac{\sin 5x}{5x}$$

$$\boxed{= \frac{5}{3}}$$

Exercise

Find the limit $\lim\limits_{x\to -\infty} \dfrac{\cos x}{2x}$

Solution

$$-1 \le \cos x \le 1$$

$$-\lim_{x\to\infty} \frac{1}{2x} \le \lim_{x\to\infty} \frac{\cos x}{2x} \le \lim_{x\to\infty} \frac{1}{2x}$$

$$0 \le \lim_{x\to\infty} \frac{\cos x}{2x} \le 0$$

$$\boxed{\lim_{x\to\infty} \frac{\cos x}{2x} = 0}$$

Exercise

Find the limit $\lim\limits_{x\to -\infty} \left(\dfrac{x^2+x-1}{8x^2-3}\right)^{1/3}$

Solution

$$\lim_{x\to -\infty} \left(\frac{x^2+x-1}{8x^2-3}\right)^{1/3} = \lim_{x\to -\infty} \left(\frac{x^2}{8x^2}\right)^{1/3}$$

$$= \left(\frac{1}{2^3}\right)^{1/3}$$

$$\boxed{= \frac{1}{2}}$$

Exercise

Find the limit $\lim\limits_{x\to -1} \dfrac{\sqrt{x^2+8}-3}{x+1}$

Solution

$$\lim\limits_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} = \frac{3-3}{-1+1} = \frac{0}{0}$$

$$= \lim\limits_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} \cdot \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}$$

$$= \lim\limits_{x\to -1} \frac{x^2+8-9}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim\limits_{x\to -1} \frac{x^2-1}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim\limits_{x\to -1} \frac{(x-1)(x+1)}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim\limits_{x\to -1} \frac{x+1}{\sqrt{x^2+8}+3}$$

$$= \frac{0}{6}$$

$$= 0$$

Exercise

Find the limit $\lim\limits_{x\to -\infty} \left(\dfrac{1-x^3}{x^2+7x}\right)^5$

Solution

$$\lim\limits_{x\to -\infty} \left(\frac{1-x^3}{x^2+7x}\right)^5 = \lim\limits_{x\to -\infty} \left(\frac{-x^3}{x^2}\right)^5$$

$$= \lim\limits_{x\to -\infty} \left(-x^5\right)$$

$$= \infty$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \sqrt{\dfrac{x^2-5x}{x^3+x-2}}$

Solution

$$\lim_{x\to\infty} \sqrt{\frac{x^2-5x}{x^3+x-2}} = \lim_{x\to\infty} \sqrt{\frac{x^2}{x^3}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x}}$$

$$\underline{= 0}$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{2\sqrt{x}+x^{-1}}{3x-7}$

Solution

$$\lim_{x\to\infty} \frac{2\sqrt{x}+x^{-1}}{3x-7} = \lim_{x\to\infty} \frac{2\sqrt{x}}{3x}$$

$$= \lim_{x\to\infty} \frac{2}{3\sqrt{x}}$$

$$\underline{= 0}$$

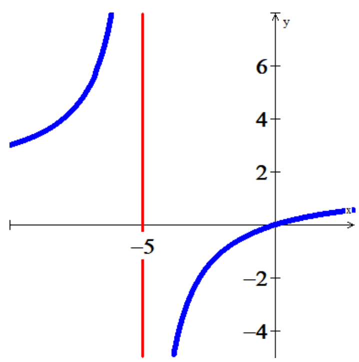
Exercise

Find the limit $\lim\limits_{x\to-5^-} \dfrac{3x}{2x+10}$

Solution

$$\lim_{x\to-5^-} \frac{3x}{2x+10} = \frac{-15}{0^-}$$

$$\underline{= \infty}$$



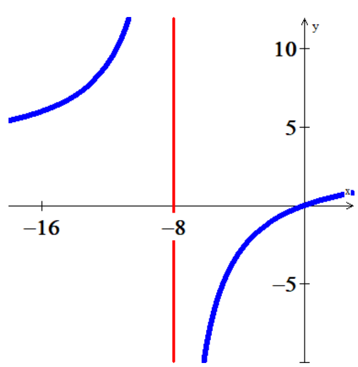
Exercise

Find the limit $\lim\limits_{x\to-8^+} \dfrac{3x}{x+8}$

Solution

$$\lim_{x\to-8^+} \frac{3x}{x+8} = \frac{-24}{0^+}$$

$$\underline{= -\infty}$$



Exercise

Find the limit $\lim\limits_{x \to 0} \dfrac{-1}{x^2(x+1)}$

Solution

$$\lim_{x \to 0} \frac{-1}{x^2(x+1)} = -\frac{1}{0}$$

$$= -\infty$$

Exercise

Find the limit $\lim\limits_{x \to 7} \dfrac{4}{(x-7)^2}$

Solution

$$\lim_{x \to 7} \frac{4}{(x-7)^2} = \frac{4}{0}$$

$$= \infty$$

Exercise

Find the limit $\lim\limits_{x \to 0} \dfrac{1}{x^{2/3}}$

Solution

$$\lim_{x \to 0} \frac{1}{x^{2/3}} = \infty$$

Exercise

Find the limit $\lim\limits_{x \to -\infty} \left(x + \sqrt{x^2 - 4x + 2} \right)$

Solution

$$\lim_{x \to -\infty} \left(x + \sqrt{x^2 - 4x + 2} \right) = -\infty + \infty$$

$$= \lim_{x \to -\infty} \left(x + \sqrt{x^2 - 4x + 2} \right) \cdot \frac{x - \sqrt{x^2 - 4x + 2}}{x - \sqrt{x^2 - 4x + 2}}$$

$$= \lim_{x \to -\infty} \frac{x^2 - x^2 + 4x - 2}{x - \sqrt{x^2 - 4x + 2}} \ .$$

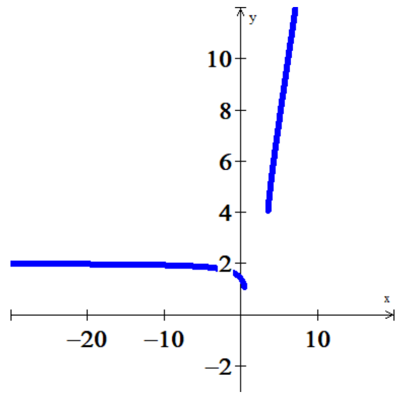
$$= \lim_{x \to -\infty} \frac{4x-2}{x-\sqrt{x^2-4x+2}}$$

$$= \lim_{x \to -\infty} \frac{4x-2}{x-|x|} \qquad x \to -\infty \quad (x<0) \to \quad |x| = -x$$

$$= \lim_{x \to -\infty} \frac{4x}{x+x}$$

$$= \lim_{x \to -\infty} \frac{4x}{2x}$$

$\underline{= 2}$



Solution   *Section* 1.5 – Continuity

Exercise

Given the graphed function $f(x)$

a) Does $f(-1)$ exist?

b) Does $\lim\limits_{x\to -1^+} f(x)$ exist?

c) Does $\lim\limits_{x\to -1^+} f(x) = f(-1)$?

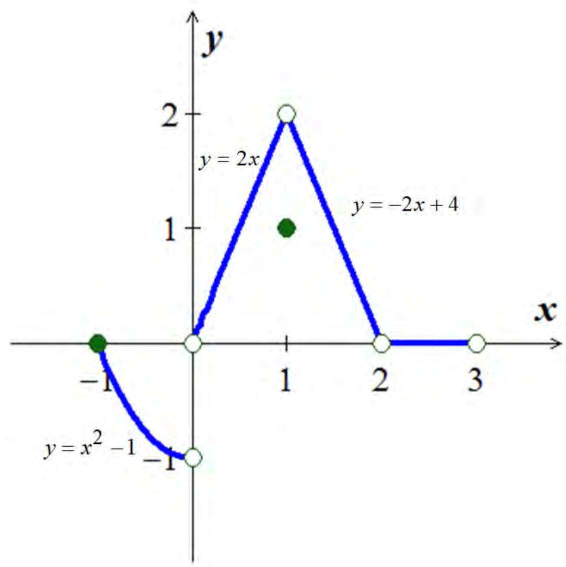
d) Is f continuous at $x=-1$?

e) Does $f(1)$ exist?

f) Does $\lim\limits_{x\to 1} f(x)$ exist?

g) Does $\lim\limits_{x\to 1} f(x) = f(1)$?

h) Is f continuous at $x=1$?



Solution

a) Yes $f(-1) = 0$

b) Yes, $\lim\limits_{x\to -1^+} f(x) = 0$

c) Yes

d) Yes

e) Yes, $f(1) = 1$

f) Yes, $\lim\limits_{x\to 1} f(x) = 2$

g) No

h) No

Exercise

At what points is the function $y = \dfrac{1}{x-2} - 3x$ continuous?

Solution

The function is continuous everywhere except when $x-2=0 \Rightarrow x=2$

Exercise

At what points is the function $y = \dfrac{x+3}{x^2 - 3x - 10}$ continuous?

Solution

The function is continuous everywhere except when $x^2 - 3x - 10 = 0 \Rightarrow x = -2,\ 5$

Exercise

At what points is the function $y = |x-1| + \sin x$ continuous?

Solution

The function is continuous everywhere

Exercise

At what points is the function $y = \dfrac{x+2}{\cos x}$ continuous?

Solution

The function is continuous everywhere except when $\cos x = 0 \Rightarrow x = \dfrac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}$

Exercise

At what points is the function $y = \tan \dfrac{\pi x}{2}$ continuous?

Solution

The function is continuous everywhere except when $x = 2n - 1, \quad n \in \mathbb{Z}$

Exercise

At what points is the function $y = \dfrac{x \tan x}{x^2 + 1}$ continuous?

Solution

The function is continuous everywhere except when $x = (2n-1)\dfrac{\pi}{2}, \quad n \in \mathbb{Z}$

Exercise

At what points is the function $y = \dfrac{\sqrt{x^4 + 1}}{1 + \sin^2 x}$ continuous?

Solution

The function is continuous everywhere

Exercise

At what points is the function $y = \sqrt{2x+3}$ continuous?

Solution

The function is continuous on the interval $2x+3 \ge 0 \quad \rightarrow \quad x \ge -\frac{3}{2} \Rightarrow \left[-\frac{3}{2},\ \infty\right)$, and discontinuous when $x < -\frac{3}{2}$

Exercise

At what points is the function $y = \sqrt[4]{3x-1}$ continuous?

Solution

The function is continuous on the interval $3x-1 \ge 0 \rightarrow \left[\frac{1}{3},\ \infty\right)$, and discontinuous when $x < \frac{1}{3}$

Exercise

At what points is the function $y = (2-x)^{1/5}$ continuous?

Solution

The function is continuous everywhere $\forall x$

Exercise

Find $\lim\limits_{x\to\pi} \sin(x - \sin x)$, then is the function continuous at the point being approached?

Solution

$$\begin{aligned}\lim_{x\to\pi} \sin(x-\sin x) &= \sin(\pi - \sin\pi) \\ &= \sin(\pi - 0) \\ &= \sin(\pi) \\ &= 0\end{aligned}$$

The function is continuous at $x = \pi$

Exercise

Find $\lim\limits_{x\to 0} \tan\left(\frac{\pi}{4}\cos\left(\sin x^{1/3}\right)\right)$, then is the function continuous at the point being approached?

Solution

$$\begin{aligned}\lim_{x\to 0} \tan\left(\frac{\pi}{4}\cos\left(\sin x^{1/3}\right)\right) &= \tan\left(\frac{\pi}{4}\cos\left(\sin(0)^{1/3}\right)\right) \\ &= \tan\left(\frac{\pi}{4}\cos(0)\right)\end{aligned}$$

$$= \tan\left(\frac{\pi}{4}\right)$$

$$\underline{=1}$$ The function is continuous at $x = 0$

Exercise

Find $\lim\limits_{t\to 0} \cos\left(\dfrac{\pi}{\sqrt{19-3\sec 2t}}\right)$, then is the function continuous at the point being approached?

Solution

$$\lim_{t\to 0} \cos\left(\frac{\pi}{\sqrt{19-3\sec 2t}}\right) = \cos\left(\frac{\pi}{\sqrt{19-3\sec 2(0)}}\right)$$

$$= \cos\left(\frac{\pi}{\sqrt{19-3}}\right)$$

$$= \cos\left(\frac{\pi}{\sqrt{16}}\right)$$

$$= \cos\left(\frac{\pi}{4}\right)$$

$$\underline{= \frac{\sqrt{2}}{2}}$$

$\therefore$ The function is continuous at $t = 0$

Exercise

Explain why the equation $\cos x = x$ has at least one solution.

Solution

$\cos x - x = 0$

$$\begin{cases} if \quad x = -\frac{\pi}{2} & \rightarrow \cos\left(-\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right) > 0 \\ if \quad x = \frac{\pi}{2} & \rightarrow \cos\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\right) < 0 \end{cases}$$

$\Rightarrow \cos x - x = 0$

for some x between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

According to the Intermediate Value Theorem, and the function $\cos x = x$ is continuous and has at least one solution.

Exercise

Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval [–4, 4]

Solution

$$f(-4) = (-4)^3 - 15(-4) + 1 = -3$$

$$f(-2) = (-2)^3 - 15(-2) + 1 = 23$$

$$f(-1) = (-1)^3 - 15(-1) + 1 = 15$$

$$f(1) = (1)^3 - 15(1) + 1 = -13$$

$$f(4) = (4)^3 - 15(4) + 1 = 5$$

By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-4 < x < -1$, $-1 < x < 1$, and $1 < x < 4$. Thus, $x^3 - 15x + 1 = 0$ has three solutions in [–4, 4]. Since the polynomial of degree 3 can have at most 3 solutions, these are the solutions.

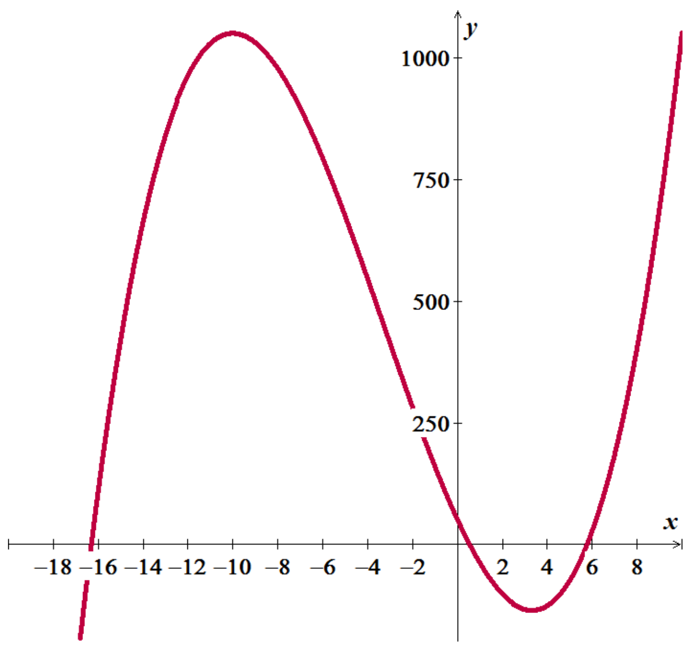
Exercise

Show that the equation has three solutions in the given interval $x^3 + 10x^2 - 100x + 50 = 0; \quad (-20,\ 10)$

Solution

x	**y**
–19	–1299
–18	–742
–17	–273
–16	114
–15	425
–14	666
–13	962
–12	1029
–10	1050
–9	1031
–8	978
–7	897
–6	794
–5	675
–4	546

-3	413
-2	282
-1	159
0	50
1	-39
2	-102
3	-133
4	-126
5	-75
6	26



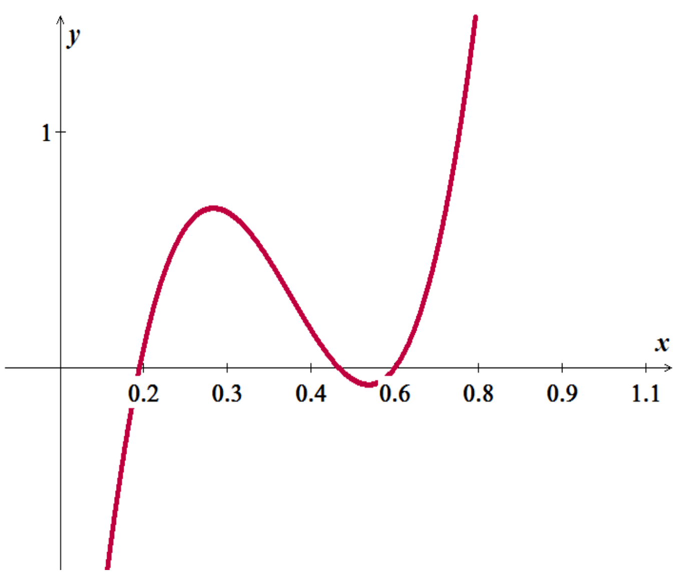
By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-17 < x < -16$, $0 < x < 1$, and $5 < x < 6$.

Exercise

Show that the equation has three solutions in the given interval $70x^3 - 87x^2 + 32x - 3 = 0; \quad (0,\ 1)$

Solution

x	y
.05	–1.6
.1	–0.6
.15	0.08
.2	.48
.25	.656
.3	.66
.35	.543
.4	.36
.45	.161
.5	0
.55	–.07
.6	0
.65	.266
.7	.78
.75	1.6
.8	2.76
.85	4.33
.9	6.36
.95	8.9



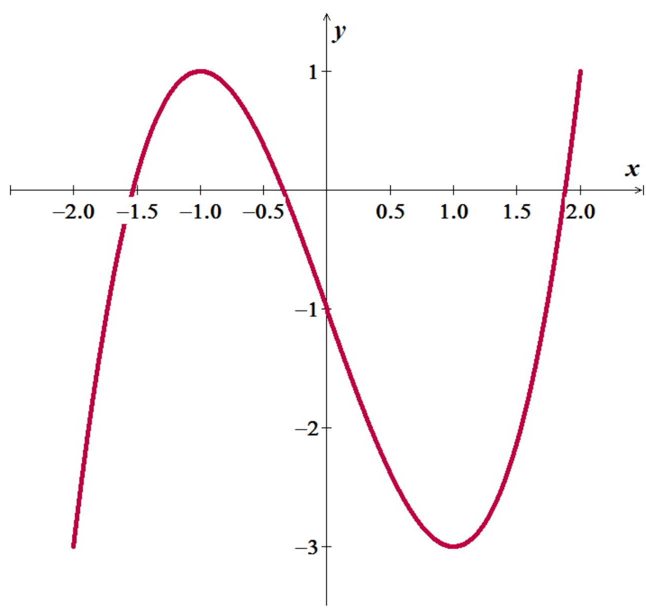
By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $0.1 < x < 0.15$, $0.5 < x < 0.55$, and $0.55 < x < 0.6$.

Exercise

Show that the equation has three solutions in the given interval $x^3 - 3x - 1 = 0;\quad [-2,\ 2]$

Solution

x	y
–2	–3.0
–1.75	–1.109
–1.5	0.125
–1.25	0.797
–1.0	1
–0.75	0.828
–0.5	0.375
–0.25	–0.266
0	–1.0
0.25	–1.73
0.5	–2.375
0.75	–2.828
1.0	–3.0
1.25	–2.797
1.5	–2.12
1.75	–0.89
2.	1.0



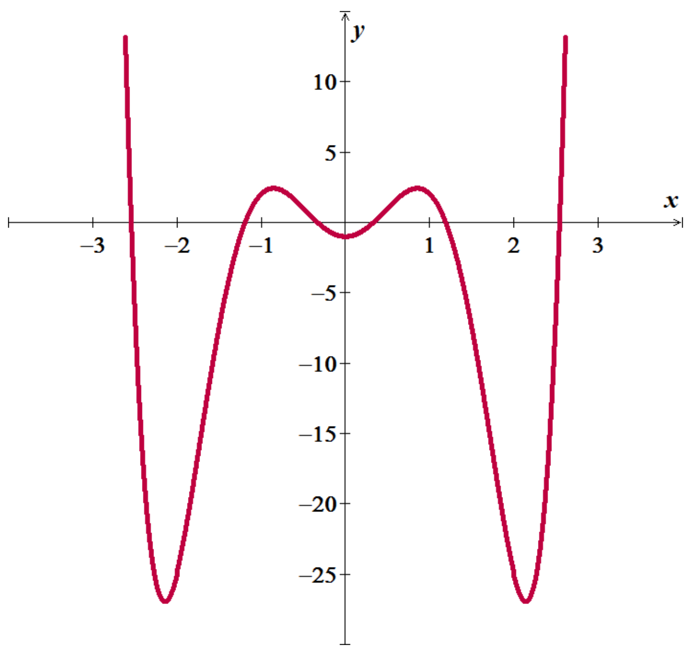
By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-1.75 < x < -1.5$, $-0.5 < x < -0.25$, and $1.75 < x < 2$.

Exercise

Show that the equation has six solutions in the given interval $x^6 - 8x^4 + 10x^2 - 1 = 0;\quad [-3,\ 3]$

Solution

x	**y**
–3.0	170.0
–2.5	–6.86
–2.0	–25.0
–1.5	–7.61
–1.0	2.0
–0.5	1.02
0.0	–1.0
0.5	1.01
1.0	2.0
1.5	–7.6
2.0	–25.0
2.5	–6.86
3.0	170.0



By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-3.0 < x < -2.5$, $-1.5 < x < -1.0$, $-0.5 \le x \le 0$, $-0.0 \le x \le 0.5$, $1.0 \le x \le 1.5$ and $2.5 < x < 3.0$.

Exercise

If functions $f(x)$ and $g(x)$ are continuous for $0 \le x \le 1$, could $\dfrac{f(x)}{g(x)}$ possibly be discontinuous at a point of [0, 1]? Give reason for your answer.

Solution

Yes, if we can get a value of $g(x)$ is between [0, 1], $x = \frac{1}{2} \;\Rightarrow g(x) = 2x - 1$ and $f(x) = x$.

Then $\dfrac{f(x)}{g(x)} = \dfrac{x}{2x-1} \;\Rightarrow \dfrac{f(x)}{g(x)}$ is discontinuous at $x = \frac{1}{2}$

Exercise

Suppose that a function f is continuous on the closed interval [0, 1] and that $0 \le f(x) \le 1$ for every x in [0, 1]. Show that there must exist a number c in [0, 1] such that $f(c) = c$ (***c*** is called a ***fixed point*** of f).

Solution

Let $f(x) = x \Rightarrow f(0) = 0 \quad or \quad f(1) = 1$. In these cases, $c = 0$ or $c = 1$.

Let $f(0) = a > 0 \quad and \quad f(1) = b < 1$ because $0 \le f(x) \le 1$.

Define $g(x) = f(x) - x \Rightarrow g$ is continuous on [0, 1].

$$\Rightarrow \begin{cases} g(0) = f(0) - 0 = a > 0 \\ g(1) = f(1) - 1 = b - 1 < 0 \end{cases}$$

By the Intermediate Value Theorem there is a number c in [0, 1] such that

$g(c) = 0 \Rightarrow f(c) - c = 0 \Rightarrow f(c) = c$

Exercise

Use the Intermediate Value Theorem to show that the equation $x^5 + 7x + 5 = 0$ has a solution in the interval $(-1,\ 0)$.

Solution

$f(-1) = -1 - 7 + 5 = -3 < 0$

$f(0) = 5 > 0$

By Intermediate value theorem, the function has a solution in $(-1,\ 0)$

Exercise

The amount of an antibiotic (in *mg*) in the blood *t* hours after an intravenous line is opened is given by

$$m(t)=100\left(e^{-0.1t}-e^{-0.3t}\right)$$

a) Use the Intermediate Value Theorem to show that the amount of drug is 30 *mg* at some time in the interval $[0,\ 5]$ and again at some time in the interval $[5,\ 15]$

b) Estimate the times at which $m=30\ mg$

c) Is the amount of drug in the blood ever 50 *mg*?

Solution

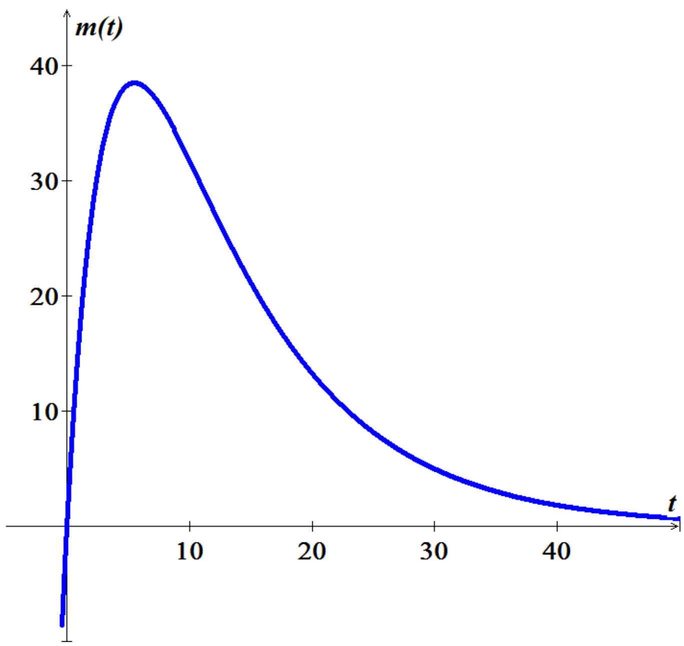
a) $m(0)=100(1-1)=0$

$m(5)\approx 38.34>30$

$m(15)\approx 21.2<30$

30 is an intermediate value between for both $[0,\ 5]$ and $[5,\ 15]$.

b) $m(t)=100\left(e^{-0.1t}-e^{-0.3t}\right)=30$



$$e^{-0.1t}-e^{-0.3t}=0.3\ \xrightarrow{software}\ \begin{cases}\boxed{t_1\approx 2.4}\\ \boxed{t_2\approx 10.8}\end{cases}$$

c) No, peak is 38.5 (using the graph)

Exercise

Determine whether the following functions are continuous at a. $f(x)=\frac{1}{x-5};\quad a=5$

Solution

$f(5)\ \nexists$

The function is continuous everywhere except @ $x=5$

Exercise

Determine whether the following functions are continuous at a. $h(x)=\sqrt{x^2-9};\quad a=3$

Solution

$\lim\limits_{x\to 3^-} h(x)\ \nexists \quad \therefore\ h$ is discontinuous @ 3

Exercise

Determine whether the following functions are continuous at a. $g(x)=\begin{cases}\frac{x^2-16}{x-4} & if \quad x\neq 4\\ 9 & if \quad x=4\end{cases};\quad a=4$

Solution

$$\lim_{x\to 4} g(x)=\lim_{x\to 4}\frac{(x-4)(x+4)}{x-4}=\lim_{x\to 4}(x+4)=8\neq 9=g(4)$$

$\therefore\ g$ is discontinuous @ 4

Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $\quad f(x)=\sqrt{x^2-5}$

Solution

$\sqrt{x^2-5\geq 0} \quad\Rightarrow\quad x\leq -5\ \ \&\ \ x\geq 5$

The function is continuous at –5 to the left and right of $x=5$

Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $\quad f(x)=e^{\sqrt{x-2}}$

Solution

The function is continuous at and to the right of $x=2$

Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $f(x) = \dfrac{2x}{x^3 - 25x}$

Solution

The function is continuous everywhere except at $x = 0, \ \pm 5$

The function is continuous to the left of –5, then to the right of –5 to the left of 0, then to the right of 0 thru the left of 5 then to the tight of 5.

Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $f(x) = \cos e^x$

Solution

The function is continuous everywhere.

Exercise

Let $g(x) = \begin{cases} 5x - 2 & if \quad x < 1 \\ a & if \quad x = 1 \\ ax^2 + bx & if \quad x > 1 \end{cases}$

Determine values of the constants a and b for which $g(x)$ is continuous at $x = 1$

Solution

$$\lim_{x \to 1^-} g(x) = g(1)$$

$$= 5 - 2$$

$$\underline{= 3 = a}$$

$$\lim_{x \to 1^-} g(x) = g(1)$$

$$= a + b$$

$$= 3 + b = 3$$

$\to \quad \underline{b = 0}$

Solution Section 1.6 – Precise Definition of Limits

Exercise

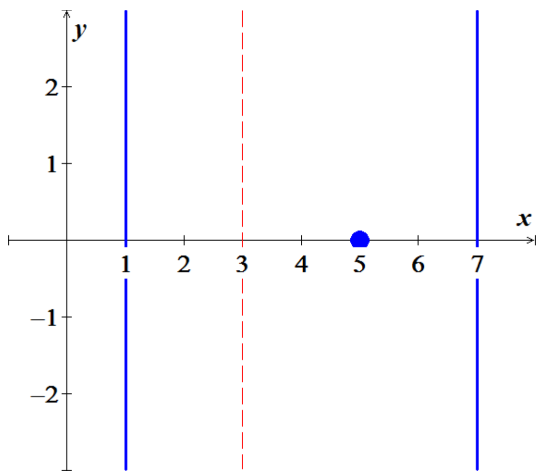
Sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \Rightarrow a < x < b$ for $a = 1, \quad b = 7, \quad x_0 = 5$

Solution

$$|x-5| < \delta \Rightarrow -\delta < x-5 < \delta$$
$$-\delta + 5 < x < \delta + 5$$

$$-\delta + 5 = 1 \Rightarrow \underline{\delta = 4}|$$

$$\delta + 5 = 7 \Rightarrow \underline{\delta = 2}|$$



Exercise

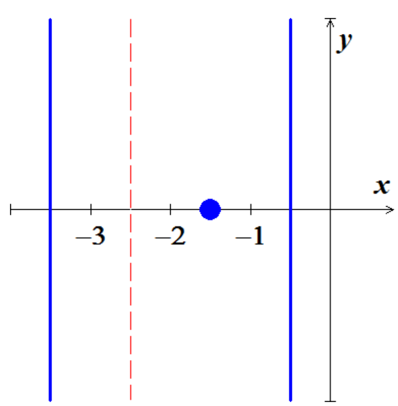
Sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \Rightarrow a < x < b$ for $a = -\frac{7}{2}, \quad b = -\frac{1}{2}, \quad x_0 = -\frac{3}{2}$

Solution

$$\left|x + \frac{3}{2}\right| < \delta$$
$$-\delta < x + \frac{3}{2} < \delta$$
$$-\delta - \frac{3}{2} < x < \delta - \frac{3}{2}$$

$$-\delta - \frac{3}{2} = -\frac{7}{2} \Rightarrow \underline{|\delta = \frac{7}{2} - \frac{3}{2} = 2|}$$

$$\delta - \frac{3}{2} = -\frac{1}{2} \Rightarrow \underline{|\delta = \frac{1}{2} - \frac{3}{2} = -1|}$$

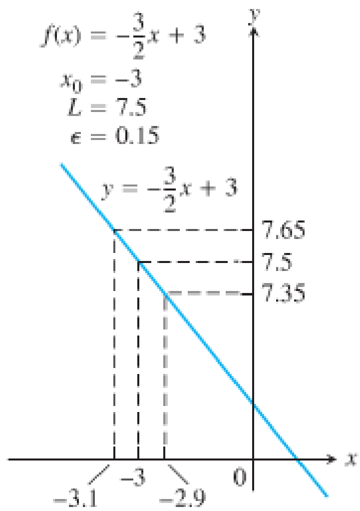


Exercise

Use the graph to find a $\delta > 0$ such that for all x $\quad 0 < |x - x_0| < \delta \;\Rightarrow\; |f(x) - L| < \varepsilon$

Solution

Given: $\quad a = -3.1, \quad b = -2.9, \quad x_0 = -3$



$|x + 3| < \delta$

$-\delta < x + 3 < \delta$

$-\delta - 3 < x < \delta - 3$

$-\delta - 3 = -3.1$

$\Rightarrow \; \underline{\delta = 3.1 - 3 = 0.1}|$

$\delta - 3 = -2.9$

$\Rightarrow \; \underline{\delta = 3 - 2.9 = 0.1}|$

Exercise

Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

$$f(x) = x + 1, \quad L = 5, \quad x_0 = 4, \quad \varepsilon = 0.01$$

Solution

$|(x+1) - 5| < .01$

$|x - 4| < .01$

$-.01 < x - 4 < .01$

$-.01 + 4 < x - 4 + 4 < .01 + 4$

$3.99 < x < 4.01$

$|x - 4| < \delta$

$-\delta < x - 4 < \delta$

$-\delta + 4 < x < \delta + 4$

$-\delta + 4 = 3.99$

$\underline{\delta = 4 - 3.99 = 0.01}|$

$\delta + 4 = 4.01$

$\underline{\delta = 4.01 - 4 = 0.01}|$

$\Rightarrow \; \underline{\delta = .01}\,|$

Exercise

Find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

$$f(x)=2x-1, \quad L=3, \quad x_0=2, \quad \varepsilon=0.1$$

Solution

$|2x-1-3|<.1$

$$\begin{aligned}
&|2x-4|<.1\\
&-.1<2x-4<.1\\
&-.1+4<2x-4+4<.1+4\\
&3.9<2x<4.1\\
&\frac{3.9}{2}<x<\frac{4.1}{2}\\
&1.95<x<2.05
\end{aligned}$$

$|x-2|<\delta$

$$\begin{aligned}
&-\delta<x-2<\delta\\
&-\delta+2<x<\delta+2
\end{aligned}$$

$-\delta+2=1.95$

$$\delta=2-1.95 \underline{=0.05|}$$

$\delta+2=2.05$

$$\delta=2.05-2 \underline{=0.05|}$$

$\Rightarrow \underline{\delta=.05|}$

Exercise

Find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

$$f(x)=x+2, \quad L=3, \quad x_0=1, \quad \varepsilon=0.001$$

Solution

$|x+2-3|<.001$

$$\begin{aligned}
&|x-1|<.001\\
&-.001<x-1<.001\\
&-.001+1<x-1+1<.001+1\\
&0.999<x<1.001
\end{aligned}$$

$$|x-1| < \delta$$

$$-\delta < x-1 < \delta$$

$$-\delta+1 < x < \delta+1$$

$$-\delta+1 = .999$$

$$\delta = 1-.999 \ \underline{= 0.001|}$$

$$\delta+1 = 1.001$$

$$\delta = 1.001-1 \ \underline{= 0.001|}$$

$$\Rightarrow \ \underline{\delta = .001 \ |}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x)-L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x-x_0| < \delta$ the inequality $|f(x)-L| < \varepsilon$ holds.

$$f(x) = 3x+2, \quad L = 2, \quad x_0 = 0, \quad \varepsilon = 0.1$$

Solution

$$|3x+2-2| < .1$$

$$|3x| < .1$$

$$-.1 < 3x < .1$$

$$-\frac{.1}{3} < x < \frac{.1}{3}$$

$$-\frac{1}{30} < x < \frac{1}{30}$$

$$|x-0| < \delta$$

$$-\delta < x < \delta$$

$$-\delta = -\frac{1}{30}$$

$$\delta \ \underline{= \frac{1}{30}|}$$

$$\Rightarrow \ \underline{\delta = \frac{1}{30} \ |}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

$$f(x)=\sqrt{x+1}, \quad L=1, \quad x_0=0, \quad \varepsilon=0.1$$

Solution

$$\left|\sqrt{x+1}-1\right|<0.1$$

$$-0.1<\sqrt{x+1}-1<0.1$$

$$-0.1+1<\sqrt{x+1}-1+1<0.1+1$$

$$.9<\sqrt{x+1}<1.1$$

$$(.9)^2<\left(\sqrt{x+1}\right)^2<(1.1)^2$$

$$.81<x+1<1.21$$

$$.81-1<x+1-1<1.21-1$$

$$-0.19<x<0.21$$

$$|x-0|<\delta \quad\Rightarrow\quad -\delta<x<\delta$$

$$-\delta=-0.19 \;\Rightarrow\; \underline{\delta=0.19} \qquad \to \quad \underline{\delta=0.19}$$

$$\underline{\delta=0.21}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

$$f(x)=\sqrt{x-7}, \quad L=4, \quad x_0=23, \quad \varepsilon=1$$

Solution

$$\left|\sqrt{x-7}-4\right|<1$$

$$-1<\sqrt{x-7}-4<1$$

$$3<\sqrt{x-7}<5$$

$$(3)^2<\left(\sqrt{x-7}\right)^2<(5)^2$$

$$9<x-7<25$$

$$9+7<x-7+7<25+7$$

$$16<x<32$$

$$|x-23|<\delta$$

$$-\delta < x - 23 < \delta$$

$$-\delta + 23 < x < \delta + 23$$

$$-\delta + 23 = 16 \;\Rightarrow\; \delta = 23 - 16 = 7$$

$$\delta + 23 = 32 \;\Rightarrow\; \delta = 32 - 23 = 9$$

$$\Rightarrow \; \delta = 7$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

$$f(x) = x^2, \quad L = 3, \quad x_0 = \sqrt{3}, \quad \varepsilon = 0.1$$

Solution

$$|x^2 - 3| < 0.1$$

$$-0.1 < x^2 - 3 < 0.1$$

$$2.9 < x^2 < 3.1$$

$$\sqrt{2.9} < x < \sqrt{3.1}$$

$$|x - \sqrt{3}| < \delta$$

$$-\delta < x - \sqrt{3} < \delta$$

$$-\delta + \sqrt{3} < x < \delta + \sqrt{3}$$

$$-\delta + \sqrt{3} = \sqrt{2.9} \;\Rightarrow\; \delta = \sqrt{3} - \sqrt{2.9} = .029$$

$$\delta + \sqrt{3} = \sqrt{3.1} \;\Rightarrow\; \delta = \sqrt{3.1} - \sqrt{3} = .029$$

$$\Rightarrow \; \delta = .029$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

$$f(x) = \frac{120}{x}, \quad L = 5, \quad x_0 = 24, \quad \varepsilon = 1$$

Solution

$$\left|\frac{120}{x} - 5\right| < 0.1$$

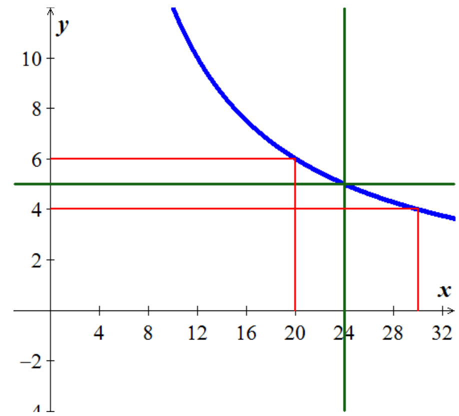
$$-1 < \frac{120}{x} - 5 < 1$$

$$4 < \frac{120}{x} < 6$$

$$\frac{1}{6} < \frac{x}{120} < \frac{1}{4}$$

$$\frac{1}{6}(120) < x < \frac{1}{4}(120)$$

$$20 < x < 30$$



$$|x - 24| < \delta$$

$$-\delta < x - 24 < \delta$$

$$-\delta + 24 < x < \delta + 24$$

$$-\delta + 24 = 20 \quad \Rightarrow \quad \delta = 24 - 20 = \underline{4}$$

$$\delta + 24 = 30 \quad \Rightarrow \quad \delta = 30 - 24 = \underline{6}$$

$$\Rightarrow \quad \underline{\delta = 4}$$

Exercise

Prove that $\displaystyle\lim_{x \to 4} (9 - x) = 5$

Solution

$$|(9 - x) - 5| < \varepsilon$$

$$-\varepsilon < 4 - x < \varepsilon$$

$$-\varepsilon - 4 < -x < \varepsilon - 4 \qquad \textit{divide by } (-).$$

$$\varepsilon + 4 > x > 4 - \varepsilon$$

$$4 - \varepsilon < x < \varepsilon + 4$$

$$|x - 4| < \delta$$

$$-\delta < x - 4 < \delta$$

$$-\delta + 4 < x < \delta + 4$$

$$-\delta + 4 = 4 - \varepsilon \quad \Rightarrow \quad -\delta = -\varepsilon \Rightarrow \quad \underline{\delta = \varepsilon}$$

$$\delta + 4 = \varepsilon + 4 \quad \Rightarrow \quad \underline{\delta = \varepsilon}$$

$$\Rightarrow \quad \underline{\delta = \varepsilon}$$

Exercise

Prove that $\lim\limits_{x\to 1} \frac{1}{x} = 1$

Solution

$$\left|\frac{1}{x}-1\right| < \varepsilon$$

$$-\varepsilon < \frac{1}{x}-1 < \varepsilon$$

$$-\varepsilon + 1 < \frac{1}{x} < \varepsilon + 1$$

$$\frac{1}{\varepsilon+1} > x > \frac{1}{-\varepsilon+1}$$

$$\frac{1}{1+\varepsilon} < x < \frac{1}{1-\varepsilon}$$

$$|x-1| < \delta$$

$$-\delta < x-1 < \delta$$

$$1-\delta < x < 1+\delta$$

$$1-\delta = \frac{1}{1+\varepsilon} \quad \Rightarrow \quad \delta = 1+\frac{1}{1+\varepsilon} = \frac{2+\varepsilon}{1+\varepsilon}$$

$$1+\delta = \frac{1}{1-\varepsilon} \quad \Rightarrow \quad \delta = \frac{1}{1-\varepsilon} - 1 = \frac{\varepsilon}{1-\varepsilon}$$

The smallest : $\boxed{\delta = \frac{\varepsilon}{1-\varepsilon}}$

Exercise

Prove that $\lim\limits_{x\to 5} \frac{x^2-25}{x-5} = 10$

Solution

$$\left|\frac{x^2-25}{x-5} - 10\right| < \varepsilon$$

$$-\varepsilon < \frac{(x-5)(x+5)}{x-5} - 10 < \varepsilon$$

$$-\varepsilon + 10 < x+5 < \varepsilon + 10$$

$$-\varepsilon + 5 < x < \varepsilon + 15$$

$$|x-10| < \delta$$

$$-\delta < x-10 < \delta$$

$$10-\delta < x < 10+\delta$$

$$10-\delta = 5-\varepsilon \quad \Rightarrow \quad \boxed{\delta = 5+\varepsilon}$$

$10+\delta=\varepsilon+15 \quad \Rightarrow \quad \underline{\delta=\varepsilon+5}$

The smallest: $\delta=\varepsilon+5$

Exercise

Prove that $\quad \lim\limits_{x\to 0} f(x)=0 \quad if \quad f(x)=\begin{cases} 2x, & x<0 \\ \frac{x}{2}, & x\ge 0 \end{cases}$

Solution

For $x<0$: $|2x-0|<\varepsilon$

$$-\varepsilon<2x<0$$

$$-\frac{\varepsilon}{2}<x<0$$

For $x\ge 0$: $\left|\frac{x}{2}-0\right|<\varepsilon$

$$0\le \frac{x}{2}<\varepsilon$$

$$0\le x<2\varepsilon$$

$|x-0|<\delta \quad \Rightarrow \quad -\delta<x<\delta$

$$-\delta=-\frac{\varepsilon}{2} \;\Rightarrow\; \delta=\frac{\varepsilon}{2} \quad \to \text{ the smallest}: \ \delta=\frac{\varepsilon}{2}$$

$$\delta=2\varepsilon$$

Exercise

Prove that $\quad \lim\limits_{x\to 1} (5x-2)=3$

Solution

$|(5x-2)-3|<\varepsilon$

$$-\varepsilon<5x-5<\varepsilon$$

$$5-\varepsilon<5x<\varepsilon+5$$

$$1-\frac{1}{5}\varepsilon<x<1+\frac{1}{5}\varepsilon$$

$|x-3|<\delta$

$$-\delta<x-3<\delta$$

$$3-\delta<x<3+\delta$$

$$3-\delta=1-\frac{1}{5}\varepsilon \quad \Rightarrow \quad \delta=\frac{1}{5}\varepsilon+2$$

$$3+\delta=1+\frac{1}{5}\varepsilon \quad \Rightarrow \quad \delta=\frac{1}{5}\varepsilon-2 \quad \to \text{ the smallest}: \ \delta=\frac{1}{5}\varepsilon-2$$

Exercise

Prove that $\lim\limits_{x\to 2} \dfrac{1}{(x-2)^4} = \infty$

Solution

Let $N > 0$ and let $\delta = \dfrac{1}{\sqrt[4]{N}}$

Suppose that $0 < |x-2| < \delta$

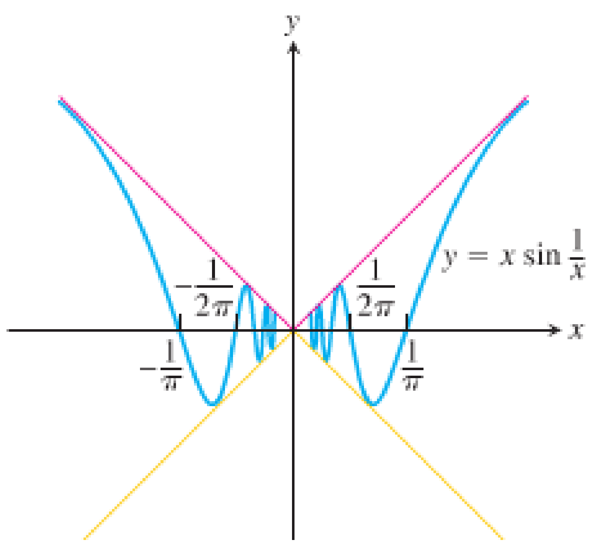
$|x-2| < \delta = \dfrac{1}{\sqrt[4]{N}}$

$\dfrac{1}{|x-2|} > \sqrt[4]{N}$

$\boxed{\dfrac{1}{(x-2)^4} > N}$ ✓

Exercise

Prove that $\lim\limits_{x\to 0} x\dfrac{1}{\sin x} = 0$



Solution

$$\left.\begin{array}{ll} -x \le x\sin\dfrac{1}{x} \le x, & \forall x > 0 \\ -x \ge x\sin\dfrac{1}{x} \ge x, & \forall x < 0 \end{array}\right\} \to \lim_{x\to 0}(-x) = \lim_{x\to 0}(x) = 0$$

Then by the sandwich theorem, $\boxed{\lim\limits_{x\to 0} x\sin\left(\dfrac{1}{x}\right) = 0}$