Section 4.3 – Definite Integral

Definition

Let $f(x)$ be a function defined on a closed interval [*a*, *b*]. We say that a number *J* is the ***definite integral* of *f* over [*a, b*]** and that *J* is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\Delta x_k$ if the following condition is satisfied:

Given any number ε > 0 there is a corresponding number δ > 0 such that for every partition $P = \{x_0, x_1, \cdots, x_n\}$ of [a, b] with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\Delta x_k - J \right| < \varepsilon$$

Leibniz introduced a notation for the definite integral that captures its construction as a limit of Riemann sums.

Upper limit of integration

Function is integrand

Integral sign

$$\int_a^b f(x)\,dx$$

x is the variable of integration

Lower limit of integration

Integral of *f* from *a* to *b*.

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = J = \int_a^b f(x)\,dx$$

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k)\Delta x_k = J = \int_a^b f(x)\,dx$$

Theorem – Integrability of Continuous Functions

If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)\,dx$ exists and f in integrable over $[a, b]$

Properties of Definite Integrals

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \qquad\qquad \int_a^a f(x)\,dx = 0$$

Theorem

When f and g are integrable over the interval $[a, b]$, the definite integral satisfies the rules:

Order of Integration: $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

Zero Width Interval: $\int_a^a f(x)\,dx = 0$

Constant Multiple: $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

Sum and Difference: $\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

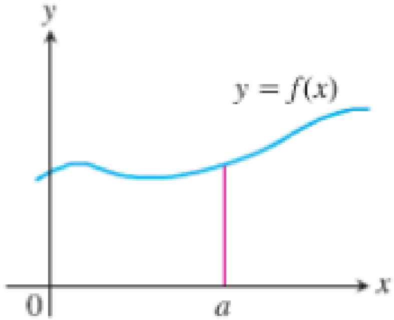
Additivity: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

Max-Min Inequality: If f has ***maximum*** value *max* f and ***minimum*** value *min* f on $[a, b]$, then

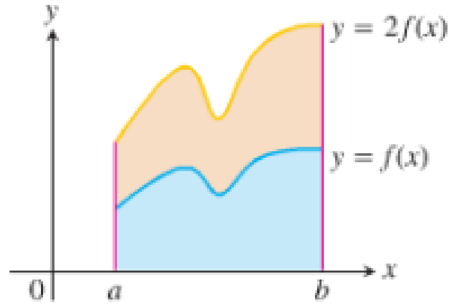
$$(\min f)\cdot(b-a) \le \int_a^b f(x)\,dx \le (\max f)\cdot(b-a)$$

Domination: $f(x) \ge g(x)$ *on* $[a, b] \;\Rightarrow\; \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

$$f(x) \ge 0 \text{ on } [a, b] \;\Rightarrow\; \int_a^b f(x)\,dx \ge 0$$

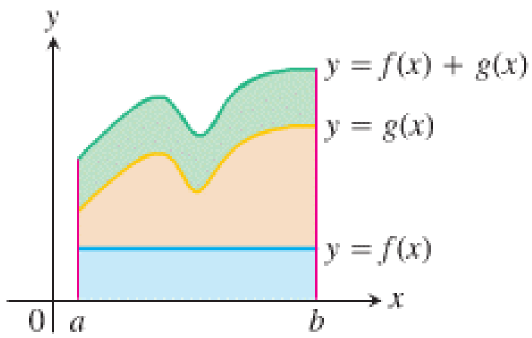


Zero Width Interval: $\int_a^a f(x)\,dx = 0$



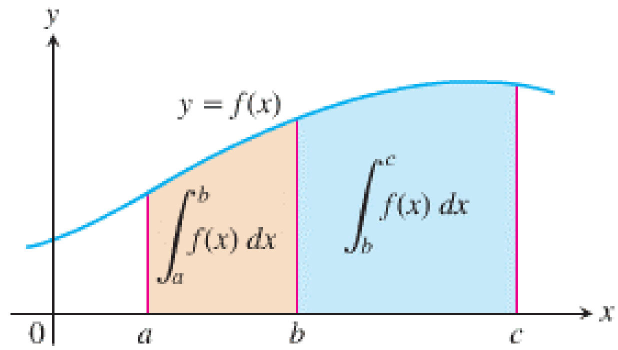
Constant Multiple: ($k = 2$)

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$



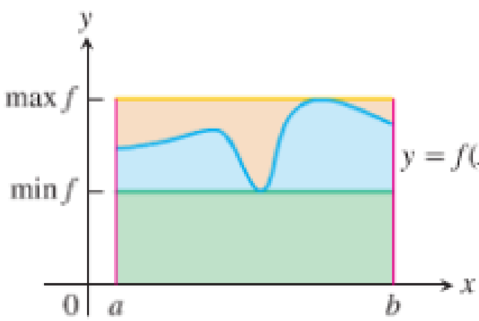
Sum: (*areas add*)

$$\int_a^b \left(f(x)+g(x)\right)dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



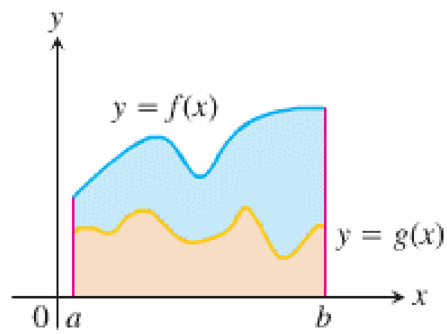
Additive for definite integrals:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



Max-Min Inequality:

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a)$$



Domination

$$f(x) \ge g(x) \text{ on } [a,b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

Example

Suppose that $\int_{-1}^{1} f(x)\,dx = 5,\quad \int_{1}^{4} f(x)\,dx = -2,\quad \int_{-1}^{1} h(x)\,dx = 7$. Find:

a) $\int_{4}^{1} f(x)\,dx$

b) $\int_{-1}^{1} \left[2f(x)+3h(x)\right]dx$

Solution

a) $\int_{4}^{1} f(x)\,dx = -\int_{1}^{4} f(x)\,dx = -(-2) = \underline{2}$

b) $\int_{-1}^{1} \left[2f(x)+3h(x)\right]dx = 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx$

$$= 2(5) + 3(7)$$

$$\underline{= 31}$$

Example

Show that the value of $\int_{0}^{1} \sqrt{1+\cos x}\,dx$ is less than or equal to $\sqrt{2}$

Solution

min $f \bullet (b-a)$: is the lower bound

max $f \bullet (b-a)$: is the upper bound

The maximum value of $\sqrt{1+\cos x}$ on [0, 1] is $\sqrt{1+1} = \sqrt{2}$

So, $\int_{0}^{1} \sqrt{1+\cos x}\,dx \le \sqrt{2}\cdot(1-0) = \sqrt{2}$

Area Under the Graph of a Nonnegative Function

Definition

If $y = f(x)$ is nonnegative and integrable over a closed interval [a, b], then the area under the curve $y = f(x)$ over [a, b] is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx$$

Example

Compute $\int_0^b x\,dx$ and find the area A under $y = x$ over the interval [0, b], $b > 0$.

Solution

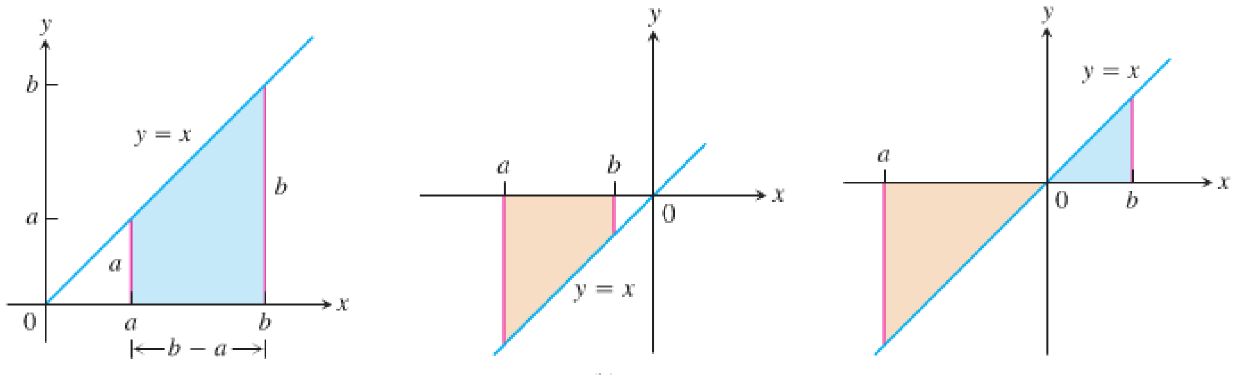
To Compute the definite integral, we consider the partition P subdivides the interval [0, b] into n subintervals of equal width $\Delta x = \frac{b-0}{n} = \frac{b}{n}$.

$$P = \left\{0,\ \frac{b}{n},\ \frac{2b}{n},\ \frac{3b}{n},\ \ldots,\ \frac{nb}{n}\right\} \quad \textit{and} \quad c_k = \frac{kb}{n}$$

$$\begin{aligned}\sum_{k=1}^{n} f(c_k)\Delta x &= \sum_{k=1}^{n} \frac{kb}{n}\cdot\frac{b}{n}\\ &= \sum_{k=1}^{n} \frac{kb^2}{n^2}\\ &= \frac{b^2}{n^2}\sum_{k=1}^{n} k\\ &= \frac{b^2}{n^2}\cdot\frac{n(n+1)}{2}\\ &= \frac{b^2}{2}\left(1+\frac{1}{n}\right)\end{aligned}$$

$$\lim_{n\to\infty}\left[\frac{b^2}{2}\left(1+\frac{1}{n}\right)\right] = \frac{b^2}{2}$$

$$\int_0^b x\,dx = \boxed{\frac{b^2}{2}}$$



$$A = \int_0^b x\,dx = \frac{b^2}{2}$$

$$\int_a^b x\,dx = \int_a^0 x\,dx + \int_0^b x\,dx$$

$$= -\int_0^a x\,dx + \int_0^b x\,dx$$

$$= -\frac{a^2}{2} + \frac{b^2}{2}$$

$$\boxed{\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2} \qquad a < b}$$

$$\boxed{\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3} \qquad a < b}$$