

# Lecture Three – Infinite Sequences and Series

## Section 3.1 – Sequences

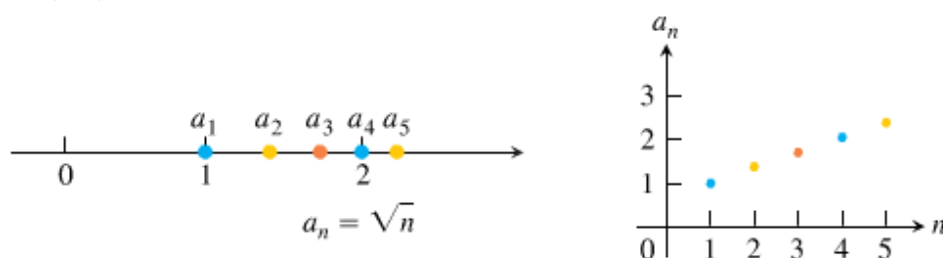
A sequence is a list of numbers

$$a_1, a_2, a_3, \dots, a_n, \dots$$

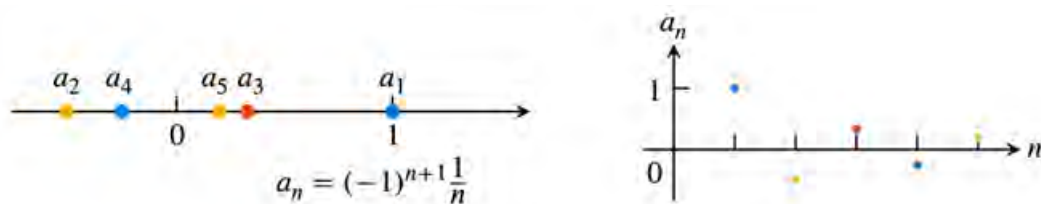
An **infinite sequence** of numbers is a function whose domain is the set of positive integers. These are the **terms** of the sequence. The integer ***n*** is called the **index** of  $a_n$ .

Sequences can be described by writing rules that specify their terms such as

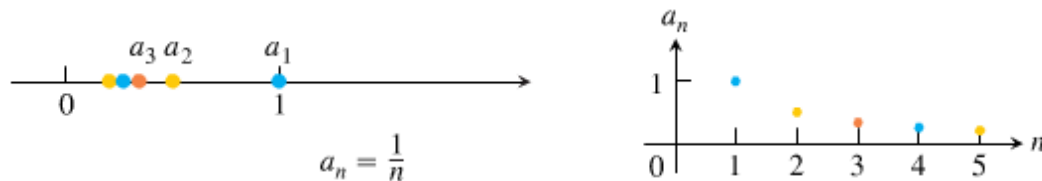
$$a_n = \sqrt{n} \Rightarrow \{a_n\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n}, \dots\}$$



$$a_n = (-1)^{n+1} \frac{1}{n} \Rightarrow \{a_n\} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots, (-1)^{n+1} \frac{1}{n}, \dots\right\}$$

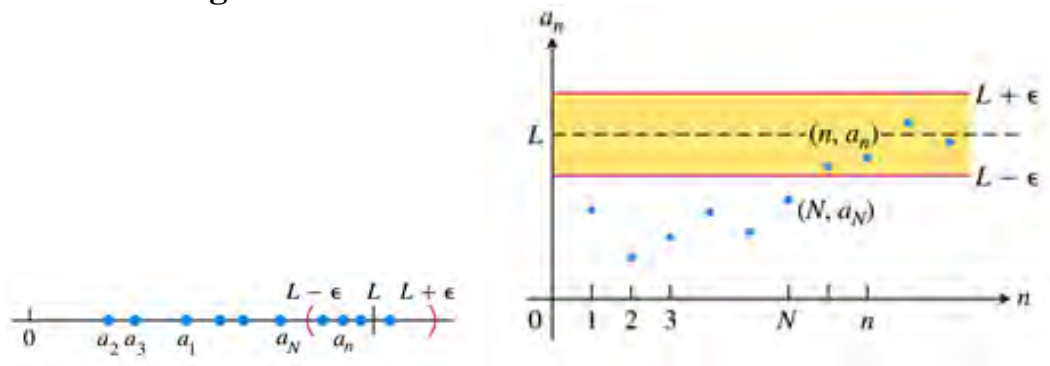


$$a_n = \frac{1}{n} \Rightarrow \{a_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots\right\}$$



Also, we can write:  $\{a_n\} = \{\sqrt{n}\}_{n=1}^{\infty}$

## Convergence and Divergence



$\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, 1 - \frac{1}{n}, \dots\right\}$  Terms approach 1.

$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots\right\}$  Terms approach 0.

### Definition

The sequence  $\{a_n\}$  **converges** to the number  $L$  if for every positive number  $\varepsilon$  there corresponds an integer  $N$  such that for all  $n$ ,

$$n > N \Rightarrow |a_n - L| < \varepsilon$$

If no such number  $L$  exists, we say  $\{a_n\}$  **diverges**.

The  $\{a_n\}$  **converges** to  $L$ , we write  $\lim_{n \rightarrow \infty} a_n = L$ , or simply  $a_n \rightarrow L$ , and call  $L$  the **limit** of the sequence.

### Example

Show that  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

### Solution

Let  $\varepsilon > 0$  be given. We must show that there exists an integer  $N$  such that for all  $n$ ,

$$n > N \Rightarrow \left| \frac{1}{n} - 0 \right| < \varepsilon$$

This implication will hold if  $\frac{1}{n} < \varepsilon$  or  $n > \frac{1}{\varepsilon}$ . If  $N$  is any integer greater than  $\frac{1}{\varepsilon}$ , the implication will hold for all  $n > N$ . This proves that  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

### Example

Show that  $\lim_{n \rightarrow \infty} k = k$  (any constant  $k$ )

### Solution

Let  $\varepsilon > 0$  be given. We must show that there exists an integer  $N$  such that for all  $n$ ,

$$n > N \Rightarrow |k - k| < \varepsilon$$

Since  $k - k = 0$ , we can use any positive integer for  $N$  and the implication will hold for all  $n > N$ . This proves that  $\lim_{n \rightarrow \infty} k = k$

### Definition

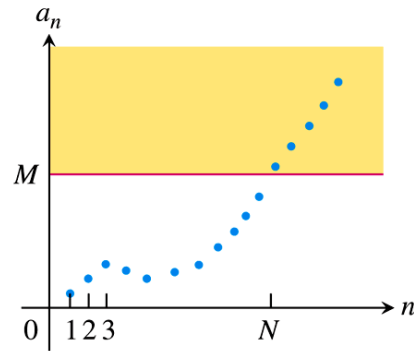
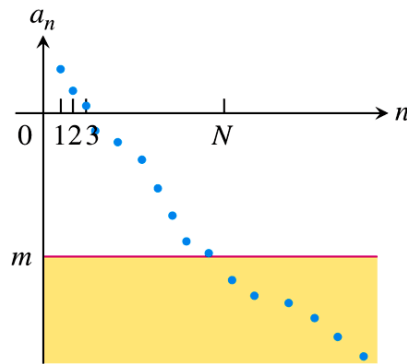
The sequence  $\{a_n\}$  **diverges** to infinity if for every number  $M$  there is an integer  $N$  such that for all  $n$  larger than  $N$ ,  $a_n > M$ . If this condition holds we write

$$\lim_{n \rightarrow \infty} a_n = \infty \quad \text{or} \quad a_n \rightarrow \infty$$

Similarly, if for every number  $m$  there is an integer  $N$  such that for all  $n > N$  we have  $a_n < m$ , then we say

$\{a_n\}$  **diverges to negative infinity** and write

$$\lim_{n \rightarrow \infty} a_n = -\infty \quad \text{or} \quad a_n \rightarrow -\infty$$



## Theorem

Let  $\{a_n\}$  and  $\{b_n\}$  be sequences of real numbers, and let  $A$  and  $B$  real numbers. The following rules hold if

$$\lim_{n \rightarrow \infty} a_n = A \text{ and } \lim_{n \rightarrow \infty} b_n = B$$

**Sum Rule:**  $\lim_{n \rightarrow \infty} (a_n + b_n) = A + B$

**Difference Rule:**  $\lim_{n \rightarrow \infty} (a_n - b_n) = A - B$

**Constant Multiple Rule:**  $\lim_{n \rightarrow \infty} (ka_n) = kA$

**Product Rule:**  $\lim_{n \rightarrow \infty} (a_n \cdot b_n) = A \cdot B$

**Quotient Rule:**  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{A}{B}$  if  $B \neq 0$

## Example

a)  $\lim_{n \rightarrow \infty} \left(-\frac{1}{n}\right) = -1 \cdot \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right) = -1(0) = 0$

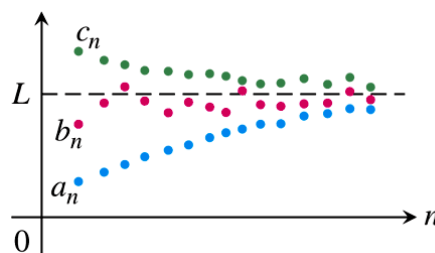
b)  $\lim_{n \rightarrow \infty} \left(\frac{n-1}{n}\right) = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) = \lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} \frac{1}{n} = 1 - 0 = 1$

c)  $\lim_{n \rightarrow \infty} \left(\frac{5}{n^2}\right) = 5 \cdot \lim_{n \rightarrow \infty} \left(\frac{1}{n^2}\right) = 5 \cdot \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right) \cdot \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right) = -1 \cdot 0 \cdot 0 = 0$

d)  $\lim_{n \rightarrow \infty} \left(\frac{4-7n^6}{n^6+3}\right) = \lim_{n \rightarrow \infty} \left(\frac{\frac{4}{n^6}-7}{1+\frac{3}{n^6}}\right) = \frac{0-7}{1+0} = -7$

## Theorem – The Sandwich Theorem for Sequences

Let  $\{a_n\}$ ,  $\{b_n\}$  and  $\{c_n\}$  be sequences of real numbers. If  $a_n \leq b_n \leq c_n$  holds for all  $n$  beyond some index  $N$ , and if  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$ , then  $\lim_{n \rightarrow \infty} b_n = L$  also.



### Example

Since  $\frac{1}{n} \rightarrow 0$ , we know that

- a)  $\frac{\cos n}{n} \rightarrow 0$  because  $-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$
- b)  $\frac{1}{2^n} \rightarrow 0$  because  $0 \leq \frac{1}{2^n} \leq \frac{1}{n}$
- c)  $(-1)^n \frac{1}{n} \rightarrow 0$  because  $-\frac{1}{n} \leq (-1)^n \frac{1}{n} \leq \frac{1}{n}$

### Theorem – The Continuous Function Theorem for Sequences

Let  $\{a_n\}$  be a sequence of real numbers. If  $a_n \rightarrow L$  and if  $f$  is a function that is continuous at  $L$  and defined at all  $a_n$ , then  $f(a_n) \rightarrow f(L)$ .

### Example

Show that  $\sqrt{\frac{n+1}{n}} \rightarrow 1$

#### Solution

We know that  $\frac{n+1}{n} \rightarrow 1$ . Taking  $f(x) = \sqrt{x}$  and  $L = 1$  that gives  $\sqrt{\frac{n+1}{n}} \rightarrow \sqrt{1} = 1$

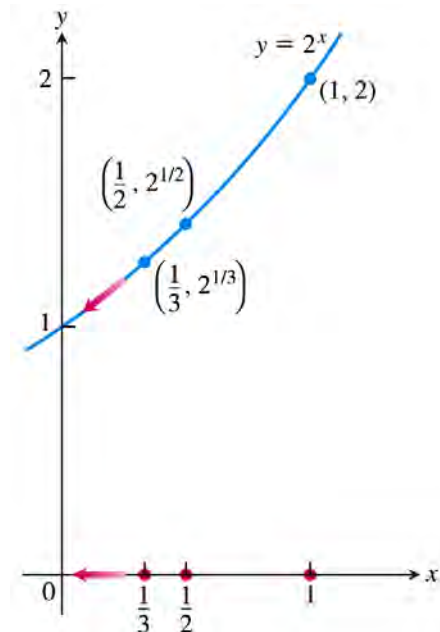
### Example

The sequence  $\left\{\frac{1}{n}\right\}$  converges to 0.

By taking  $a_n = \frac{1}{n}$ ,  $f(x) = 2^x$ , and  $L = 0$ .

We see that  $2^{1/n} = f\left(\frac{1}{n}\right) \rightarrow f(L) = 2^0 = 1$ .

The sequence  $\left\{2^{1/n}\right\}$  converges to 1.



## Using L'Hôpital's Rule

### Theorem

Suppose that  $f(x)$  is a function for all  $x \geq n_0$  and that  $\{a_n\}$  is a sequence of real numbers such that  $a_n = f(n)$  for  $n \geq n_0$ . Then

$$\lim_{x \rightarrow \infty} f(x) = L \Rightarrow \lim_{n \rightarrow \infty} a_n = L$$

### Example

Show that  $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$

### Solution

The function  $\frac{\ln x}{x}$  is defined for all  $x \geq 1$  and agrees with the given sequence at positive integers.

Therefore;

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\ln n}{n} &= \lim_{x \rightarrow \infty} \frac{\ln x}{x} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} \\ &= 0 \end{aligned}$$

### Example

Does the sequence whose  $n$ th term is  $a_n = \left(\frac{n+1}{n-1}\right)^n$  converge? If so, find  $\lim_{n \rightarrow \infty} a_n$

### Solution

The limit leads to the indeterminate form  $1^\infty$ .

$$\begin{aligned} \ln a_n &= \ln \left( \frac{n+1}{n-1} \right)^n \\ &= n \ln \left( \frac{n+1}{n-1} \right) && \infty \cdot 0 \text{ form} \\ &= \frac{\ln \left( \frac{n+1}{n-1} \right)}{\frac{1}{n}} && 0/0 \text{ form} \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \ln a_n &= \lim_{n \rightarrow \infty} \frac{\ln \left( \frac{n+1}{n-1} \right)}{\frac{1}{n}} && \left( \ln \frac{n+1}{n-1} \right)' = \frac{\frac{n-1-(n+1)}{(n-1)^2}}{\frac{n+1}{n-1}} = \frac{-2}{(n+1)(n-1)} \end{aligned}$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \frac{\frac{-2}{n^2-1}}{-\frac{1}{n^2}} \\
&= \lim_{n \rightarrow \infty} \frac{2n^2}{n^2-1} \\
&= 2
\end{aligned}$$

$\lim_{n \rightarrow \infty} a_n = e^2$

### ***Theorem***

The following six sequences converge to the limits listed below:

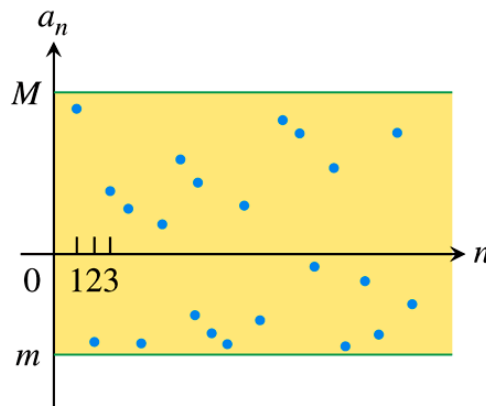
1.  $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$
2.  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$
3.  $\lim_{n \rightarrow \infty} x^{1/n} = 1 \quad x > 0$
4.  $\lim_{n \rightarrow \infty} x^n = 1 \quad |x| < 1$
5.  $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$
6.  $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$

## Bounded Monotonic Sequences

### Definitions

A sequence  $\{a_n\}$  is **bounded from above** if there exists a number  $M$  such that  $a_n \leq M$  for all  $n$ . The number  $M$  is an **upper bound** for  $\{a_n\}$  but no number less than  $M$  is an upper bound for  $\{a_n\}$ , then  $M$  is the **least upper bound** for  $\{a_n\}$ .

A sequence  $\{a_n\}$  is **bounded from below** if there exists a number  $m$  such that  $a_n \geq m$  for all  $n$ . The number  $m$  is a **lower bound** for  $\{a_n\}$ . If  $m$  is a lower bound for  $\{a_n\}$  but no number greater than  $m$  is a lower bound for  $\{a_n\}$ , then  $m$  is the **greatest lower bound** for  $\{a_n\}$ .



If  $\{a_n\}$  is bounded from above and below, the  $\{a_n\}$  is **bounded**.

If  $\{a_n\}$  is not bounded, then  $\{a_n\}$  is an **unbounded** sequence.



## Definition

A sequence  $\{a_n\}$  is **nondecreasing** if  $a_n \leq a_{n+1}$  for all  $n$ . That is  $a_1 \leq a_2 \leq a_3 \leq \dots$

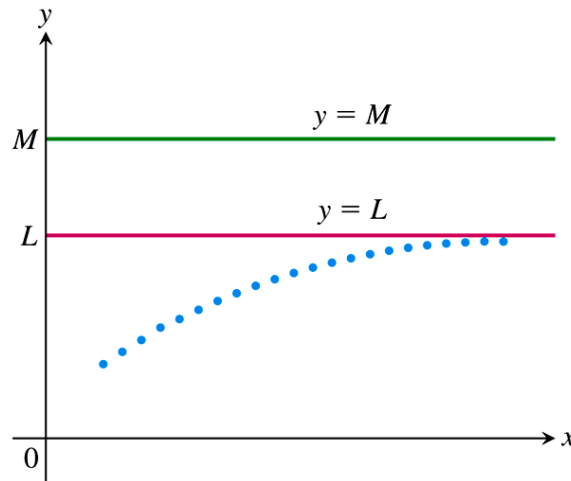
Which each term is greater than or equal to its predecessor  $(a_{n+1} \geq a_n)$

**Example:**  $\left\{1 - \frac{1}{n}\right\}_{n=1}^{\infty} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}$

A sequence  $\{a_n\}$  is **nonincreasing** if  $a_n \geq a_{n+1}$  for all  $n$ , which each term is less than or equal to its predecessor  $(a_{n+1} \leq a_n)$

**Example:**  $\left\{1 + \frac{1}{n}\right\}_{n=1}^{\infty} = \left\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots\right\}$

The sequence  $\{a_n\}$  is **monotonic** if it is either nondecreasing or nonincreasing.



## Theorem

If a sequence  $\{a_n\}$  is both *bounded* and *monotonic*, then the sequence converges.

## Example

The sequence  $\{1, 2, 3, \dots, n, \dots\}$  is nondecreasing

The sequence  $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots\right\}$  is nondecreasing

The sequence  $\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, \frac{1}{2^n}, \dots\right\}$  is nonincreasing

## Exercises      Section 3.1 – Sequences

1. Find the values of  $a_1$ ,  $a_2$ ,  $a_3$ , and  $a_4$  for  $a_n = \frac{1-n}{n^2}$
  2. Find the values of  $a_1$ ,  $a_2$ ,  $a_3$ , and  $a_4$  for  $a_n = \frac{1}{n!}$
  3. Find the values of  $a_1$ ,  $a_2$ ,  $a_3$ , and  $a_4$  for  $a_n = \frac{(-1)^{n+1}}{2n-1}$
  4. Find the values of  $a_1$ ,  $a_2$ ,  $a_3$ , and  $a_4$  for  $a_n = 2 + (-1)^n$
  5. Find the values of  $a_1$ ,  $a_2$ ,  $a_3$ , and  $a_4$  for  $a_n = \frac{2^n - 1}{2^n}$
  6. Write the first ten terms of the sequence  $a_1 = 1$ ,  $a_{n+1} = a_n + \frac{1}{2^n}$
  7. Write the first ten terms of the sequence  $a_1 = 1$ ,  $a_{n+1} = \frac{a_n}{n+1}$
  8. Write the first ten terms of the sequence  $a_1 = 2$ ,  $a_2 = -1$ ,  $a_{n+2} = \frac{a_{n+1}}{a_n}$
  9. Find a formula for the  $n$ th term of the sequence  $-1, 1, -1, 1, -1, \dots$
  10. Find a formula for the  $n$ th term of the sequence  $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \dots$
  11. Find a formula for the  $n$ th term of the sequence  $\frac{1}{9}, \frac{2}{12}, \frac{2^2}{15}, \frac{2^3}{18}, \frac{2^4}{21}, \dots$
  12. Find a formula for the  $n$ th term of the sequence  $-3, -2, -1, 0, 1, \dots$
  13. Find a formula for the  $n$ th term of the sequence  $\frac{1}{25}, \frac{8}{125}, \frac{27}{625}, \frac{64}{3125}, \frac{125}{15625}, \dots$
  14. Find a formula for the  $n$ th term of the sequence  $0, 1, 1, 2, 2, 3, 3, 4, \dots$
- (15 – 43) Determine if the sequence converge or diverge? Then find the limit of each convergent sequence.

- |                                   |   |   |
|-----------------------------------|---|---|
| 15. $a_n = \frac{n + (-1)^n}{n}$  | 18. $a_n = \left(2 - \frac{1}{2^n}\right) \left(3 + \frac{1}{2^n}\right)$ | 22. $a_n = \frac{\sin^2 n}{2^n}$                  |
| 16. $a_n = \frac{1-2n}{1+2n}$     | 19. $a_n = n\pi \cos(n\pi)$   | 23. $a_n = \frac{\ln n}{\ln 2n}$                  |
| 17. $a_n = \frac{1-n^3}{70-4n^2}$ | 20. $a_n = n - \sqrt{n^2 - n}$  | 24. $a_n = \frac{3^n \cdot 6^n}{2^{-n} \cdot n!}$ |
|                                   | 21. $a_n = \sqrt{\frac{2n}{n+1}}$   | 25. $a_n = \sqrt{n} \sin \frac{1}{\sqrt{n}}$      |

$$26. \quad a_n = \frac{n^2}{2^n - 1}$$

$$27. \quad \{c_n\} = \left\{(-1)^n \frac{1}{n!}\right\}$$

$$28. \quad a_n = \frac{5}{n+2}$$

$$29. \quad a_n = 8 + \frac{5}{n}$$

$$30. \quad a_n = (-1)^n \left(\frac{n}{n+1}\right)$$

$$31. \quad a_n = \frac{1 + (-1)^n}{n^2}$$

$$32. \quad a_n = \frac{10n^2 + 3n + 7}{2n^2 - 6}$$

$$33. \quad a_n = \frac{\sqrt[3]{n}}{\sqrt[3]{n} + 1}$$

$$34. \quad a_n = \frac{\ln(n^3)}{2n}$$

$$35. \quad a_n = \frac{5^n}{3^n}$$

$$36. \quad a_n = \frac{(n+1)!}{n!}$$

$$37. \quad a_n = \frac{(n-2)!}{n!}$$

$$38. \quad a_n = \frac{n^p}{e^n}, \quad p > 0$$

$$39. \quad a_n = n \sin \frac{1}{n}$$

$$40. \quad a_n = 2^{1/n}$$

$$41. \quad a_n = -3^{-n}$$

$$42. \quad a_n = \frac{\sin n}{n}$$

$$43. \quad a_n = \frac{\cos \pi n}{n^2}$$