2.4

$$y'' + py' + qy = f(x) \quad (N) \text{ non-homogeneous}$$

inhomogeneous

Homogeneous: $y'' + py' + qy = 0$

$$\lambda^2 + p\lambda + q = 0$$

$2\ \mathbb{R} \Rightarrow y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$

$2\ \mathbb{C} \Rightarrow y(t) = e^{at}(C_1 \cos bt + C_2 \sin bt)$

2 repeated $\Rightarrow y(t) = (C_1 + C_2 t)e^{\lambda t}$

$$y(x) = \underbrace{y_p(x)}_{\text{Particular}} + \underbrace{C_1 y_1(x) + C_2 y_2(x)}_{\text{homogeneous}}$$

Ex $\quad y'' - y' - 2y = \underbrace{2e^{-2t}}_{\text{Forcing Term}} \qquad y_p(t)$

Let $y = ae^{-2t}$

$$y' = -2ae^{-2t}$$

$$y'' = 4ae^{-2t}$$

$$4ae^{-2t} + 2ae^{-2t} - 2ae^{-2t} = 2e^{-2t}$$

$$4a = 2$$

$$a = \frac{1}{2}$$

$$\boxed{y_p(t) = \frac{1}{2}e^{-2t}}$$

$f(t) = A\cos\omega t + B\sin\omega t$ (either/both)

$y_p(t) = a\cos\omega t + b\sin\omega t$.

Ex $y'' + 2y' - 3y = 5\sin 3t \quad y_p(t) = ?$

let $-3y_p = a\sin 3t + b\cos 3t$

$2\times y' = 3a\cos 3t - 3b\sin 3t$

$y'' = -9a\sin 3t - 9b\cos 3t$

$\cos 3t$ $\quad$ $\sin 3t$

$-3b + 6a - 9b = 0 \qquad -3a - 6b - 9a = 5$

$$\begin{cases} 6a - 12b = 0 \\ -12a - 6b = 5 \end{cases} \rightarrow \begin{cases} a - 2b = 0 \\ 12a + 6b = -5 \end{cases}$$

$a = \frac{-10}{30} = \frac{-1}{3} \qquad b = \frac{-1}{6}$

$y_p(t) = -\frac{1}{3}\sin 3t - \frac{1}{6}\cos 3t$

$$f(t) = a_0 t^n + a_1 t^{n-1} + \text{---} + a_n$$

Ex $y'' + 2y' - 3y = 3t + 4$

$$y_p = at + b$$

$$y' = a$$

$$y'' = 0$$

$$2a - \underline{3at} - 3b = \underline{3t} + 4$$

$$-3a = 3 \Rightarrow \boxed{a = -1}$$

$$2(-1) - 3b = 4 \Rightarrow b = -2$$

$$\boxed{y_p(t) = -t - 2}$$

Ex $y'' - y' - 2y = 3e^{-t}$

$y = ae^{-t}$

$y' = -ae^{-t}$

$y'' = ae^{-t}$

$ae^{-t} + ae^{-t} - 2ae^{-t} = 3e^{-t}$

$0 = 3e^{-t}$ ##

$y = ate^{-t} \to y = at^2e^{-t}$

(H): $\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda = -1, 2$

$y = ate^{-t}$

$y' = (a - at)e^{-t} = ae^{-t} - ate^{-t}$

$y'' = (-a - a + at)e^{-t}$

$= (at - 2a)e^{-t}$

$ate^{-t} - 2ae^{-t} - ae^{-t} + ate^{-t} - 2ae^{-t} = 3e^{-t}$

$at - 2a - a + at - 2at = 3 \qquad e^{-t} \neq 0$

$-3a = 3 \Rightarrow \boxed{a = -1}$

$\boxed{y(t) = -te^{-t}}$

Ex $2y'' - 5y' + 3y = 4e^{3t}$

$y(0) = 1, \; y'(0) = 0$

soln $2\lambda^2 - 5\lambda + 3 = 0 \Rightarrow \lambda_{1,2} = 1, \frac{3}{2}$

$y_h = C_1 e^t + C_2 e^{3t/2}$

$y_p = a e^{3t}$

$y' = 3a e^{3t}$

$y'' = 9a e^{3t}$

$(18a - 15a + 3a) e^{3t} = 4e^{3t}$

$6a = 4 \Rightarrow a = \frac{2}{3}$

$y(t) = C_1 e^t + C_2 e^{3t/2} + \frac{2}{3} e^{3t}$

$y(0) = C_1 + C_2 + \frac{2}{3} = 1$

$C_1 + C_2 = \frac{1}{3}$ ①

$y'(t) = C_1 e^t + \frac{3}{2} C_2 e^{3t/2} + 2e^{3t}$

$y'(0) = C_1 + \frac{3}{2} C_2 + 2 = 0$

$2C_1 + 3C_2 = -4$ ②

-2 ① $\Rightarrow -2C_1 - 2C_2 = -\frac{2}{3}$

② $\Rightarrow 2C_1 + 3C_2 = -4$

$C_2 = -\frac{14}{3}$

① $\boxed{C_1 = \frac{1}{3} + \frac{14}{3} = 5}$

$y(t) = 5e^t - \frac{14}{3} e^{3t/2} + \frac{2}{3} e^{3t}$

\# 73 $y'' - 3y' - 10y = 2x - 3$

$\lambda^2 - 3\lambda - 10 = 0$

$\lambda_{1,2} = -2, 5$

$y_h = C_1 e^{-2t} + C_2 e^{5t}$

$y_p = ax + b$

$y' = a$

$y'' = 0$

$-3a - 10ax - 10b = 2x - 3$

$-10a = 2$ $\qquad$ $+3a + 10b = +3$

$a = \frac{1}{5}$ $\qquad$ $10b = 3 - \frac{3}{5} = \frac{12}{5}$

$b = \frac{6}{25}$

$y(t) = C_1 e^{-2t} + C_2 e^{5t} + \frac{1}{5}x + \frac{6}{25}$

\#153 $y'' + 4y' + 8y = \sin t$

$y(0) = 1 \quad y'(0) = 0$ $\qquad \frac{-b}{2a} \quad \sqrt{16-32}$

$\lambda^2 + 4\lambda + 8 = 0 \Rightarrow \lambda = -2 \pm 2i$

$y_h(t) = e^{-2t}(C_1 \cos 2t + C_2 \sin 2t)$

$8)\ y_p = a\cos t + b \sin t$

$4)\ y' = -a \sin t + b \cos t$

$y'' = -a\cos t - b \sin t$

$\cos t$

$8a + 4b - a = 0$

$\sin t$

$8b - 4a - b = 1$

$$\begin{cases} 7a + 4b = 0 \\ -4a + 7b = 1 \end{cases}$$

$a = \frac{-4}{65}$ $\qquad b = \frac{-7}{4}\left(\frac{-4}{65}\right) = \frac{7}{65}$

$y(t) = (C_1 \cos 2t + C_2 \sin 2t)e^{-2t} - \frac{4}{65}\cos t + \frac{7}{65}\sin t$

$y(0) = C_1 - \frac{4}{65} = 1 \Rightarrow C_1 = \frac{69}{65}$

$y' = (-2C_1 \sin 2t + 2C_2 \cos 2t - 2C_1 \cos 2t - 2C_2 \sin 2t)e^{-2t}$
$+ \frac{4}{65}\sin t + \frac{7}{65}\cos t$

$y'(0) = 2C_2 - 2\left(\frac{69}{65}\right) + \frac{7}{65} = 0$

$2C_2 = \frac{-7 + 138}{65} \Rightarrow C_2 = \frac{131}{130}$

$y(t) = \left(\frac{69}{65}\cos 2t + \frac{131}{130}\sin 2t\right)e^{-2t}$
$- \frac{4}{65}\cos t + \frac{7}{65}\sin t$

2.5 Variation of Parameters.

$$y = C_1 y_1 + C_2 y_2 \quad \Big\} \; y'' + p y' + q y = \underline{g(t)}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

$$\begin{cases} u_1 = -\int \dfrac{y_2 \, g(t)}{W} \, dt \\ u_2 = \int \dfrac{y_1 \, g(t)}{W} \, dt \end{cases}$$

$$y_p = u_1 y_1 + u_2 y_2$$

Ex1 $\quad y_1(x) = x^4 \qquad y_2(x) = x^2$

$$y'' - \frac{5}{x}y' + \frac{8}{x^2}y = 4x^3$$

$$W = \begin{vmatrix} x^4 & x^2 \\ 4x^3 & 2x \end{vmatrix}$$

$$= 2x^5 - 4x^5$$

$$= -2x^5 \neq 0$$

$$v_1 = -\int \frac{x^2(4x^3)}{-2x^5}\,dx$$

$$= \int 2\,dx$$

$$= 2x$$

$$v_2 = \int \frac{x^4(4x^3)}{-2x^5}\,dx$$

$$= -2\int x^2\,dx$$

$$= -\frac{2}{3}x^3$$

$$y_p = (2x)(x^4) + \left(-\frac{2}{3}x^3\right)(x^2)$$

$$= 2x^5 - \frac{2}{3}x^5$$

$$= \frac{4}{3}x^5$$

$$y(x) = C_1x^4 + C_2x^2 + \frac{4}{3}x^5$$

$$\left\{ y_1(x) = e^{2x},\ y_2(x) = xe^{2x} \right\}$$

$$y'' - 4y' + 4y = \frac{e^{2x}}{x}$$

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix}$$

$$= e^{4x} + 2xe^{4x} - 2xe^{4x}$$

$$= e^{4x}$$

$$v_1 = -\int \frac{xe^{2x}}{e^{4x}} \frac{e^{2x}}{x} dx$$

$$= \boxed{-x}$$

$$v_2 = \int \frac{e^{2x}}{e^{4x}} \frac{e^{2x}}{x} dx$$

$$= \int \frac{dx}{x}$$

$$= \ln|x|$$

$$y_p = -xe^{2x} + x(\ln x)e^{2x}$$

$$y(x) = C_1 e^{2x} + \overbrace{C_2 xe^{2x} - xe^{2x}} + x(\ln x)e^{2x}$$

$$C_1 e^{2x} + C_3 xe^{2x} + x(\ln x)e^{2x}$$

$$\boxed{C_3 = C_2 - 1}$$

Ex $y'' + y = \tan t$

$$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \qquad a=0$$

$$y_h = C_1 \underbrace{\cos t}_{y_1} + C_2 \underbrace{\sin t}_{y_2}$$

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}$$
$$= 1$$

$$v_1 = -\int \sin t \tan t \, dt$$
$$= -\int \frac{\sin^2 t}{\cos t} dt$$
$$= -\int \frac{1-\cos^2 t}{\cos t} dt$$
$$= -\int (\sec t - \cos t) dt$$
$$= -\left(\ln|\sec t + \tan t| - \sin t\right)$$
$$= \sin t - \ln|\sec t + \tan t|$$

$$v_2 = \int \cos t \tan t \, dt$$
$$= \int \sin t \, dt$$
$$= -\cos t \qquad (\overset{C_4}{\overset{\shortparallel}{C_2 + 1}}) \sin t$$

$$y(t) = C_1 \cos t + C_2 \sin t + \sin t - \ln|\sec t + \tan t| \cos t - \sin t \cos t$$
$$= C_3 \cos t + C_4 \sin t - \ln|\sec t + \tan t|$$