Section 5.7 – Additional Trigonometric Graphs

Vertical Asymptote

A ***vertical asymptote*** is a vertical line that the graph approaches but does not intersect, while function values increase or decrease without bound as *x*-values get closer and closer to the line.

Graphing the *Tangent* Functions

The graphs of $y = A\tan(Bx + C) + D$ will have the following characteristics:

Domain: $\left\{x \mid x \neq (2n+1)\frac{\pi}{2},\ where\ n \in \mathbb{Z}\right\}$

Range: $(-\infty,\ \infty)$

- The graph is discontinuous at values of *x* of the form $x = (2n+1)\frac{\pi}{2}$ and has ***vertical asymptotes*** at these values.
- Its ***x-intercepts*** are of the form $x = n\pi$.
- Its period is π.
- Its graph has no amplitude, since there are no minimum or maximum values.
- The graph is symmetric with respect to the origin, so the function is an odd function. For all *x* in the domain, $\tan(-x) = -\tan(x)$.

No *Amplitude*

Period : $P = \frac{\pi}{|B|}$

Phase Shift : $\phi = -\frac{C}{B}$

Vertical translation: $y = D$

Vertical Asymptote (*VA*): $bx + c = (2n+1)\frac{\pi}{2}$

One cycle: $0 \le argument \le \pi$ $\boxed{or}$ $-\frac{\pi}{2} < argument \le \frac{\pi}{2}$

Example

Find the period, and the phase shift and sketch the graph of $y=\frac{1}{2}\tan\left(x+\frac{\pi}{4}\right)$

Solution

Period: $P=\frac{\pi}{|B|}=\pi$

Phase shift: $\phi=-\frac{C}{B}=-\frac{\frac{\pi}{4}}{1}=-\frac{\pi}{4}$

Vertical translation: $y = 0$

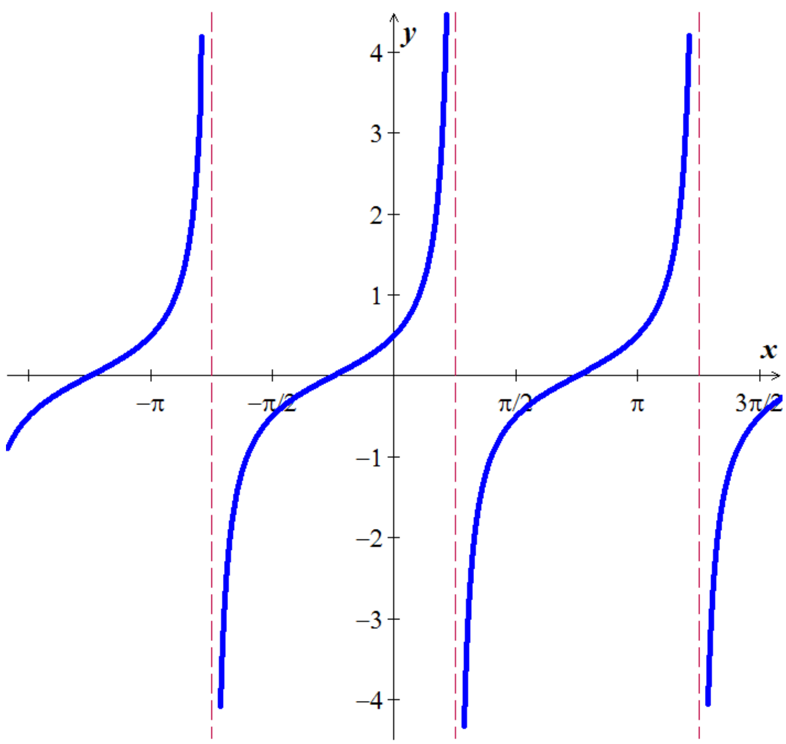
Vertical Asymptote: $x+\frac{\pi}{4}=(2n+1)\frac{\pi}{2}$

$$x+\frac{\pi}{4}=\pi n+\frac{\pi}{2}$$

$$x+\frac{\pi}{4}-\frac{\pi}{4}=\pi n+\frac{\pi}{2}-\frac{\pi}{4}$$

$$\boxed{x=\pi n+\frac{\pi}{4}}$$

	x	$y=\frac{1}{2}\tan\left(x+\frac{\pi}{4}\right)$
$-\frac{\pi}{4}+0$	$-\frac{\pi}{4}$	0
$-\frac{\pi}{4}+\frac{1}{4}\pi$	0	0.5
$-\frac{\pi}{4}+\frac{1}{2}\pi$	$\frac{\pi}{4}$	∞
$-\frac{\pi}{4}+\frac{3}{4}\pi$	$\frac{\pi}{2}$	–0.5
$-\frac{\pi}{4}+\pi$	$\frac{3\pi}{4}$	0



One Complete cycle can be determined by:

$$-\frac{\pi}{2}\le x+\frac{\pi}{4}\le\frac{\pi}{2}$$

$$-\frac{\pi}{2}-\frac{\pi}{4}\le x+\frac{\pi}{4}-\frac{\pi}{4}\le\frac{\pi}{2}-\frac{\pi}{4}$$

$$-\frac{3\pi}{4}\le x\le\frac{\pi}{4}$$

Cotangent Functions

Domain: $\{x \mid x \neq n\pi,\ where\ n \in \mathbb{Z}\}$

Range: $(-\infty,\ \infty)$

- The graph is discontinuous at values of x of the form $x = n\pi$ and has ***vertical asymptotes*** at these values.
- Its ***x-intercepts*** are of the form $x = (2n+1)\frac{\pi}{2}$.
- Its period is π.
- Its graph has no amplitude, since there are no minimum or maximum values.
- The graph is symmetric with respect to the origin, so the function is an odd function. For all x in the domain, $\cot(-x) = -\cot(x)$.

Example

Find the period, and the phase shift and sketch the graph of $y = \cot\left(2x - \frac{\pi}{2}\right)$

Solution

Period: $P = \frac{\pi}{|B|} = \frac{\pi}{2}$

Phase shift: $\phi = -\frac{C}{B} = -\frac{-\frac{\pi}{2}}{2} = \frac{\pi}{4}$

One cycle: $0 \le 2x - \frac{\pi}{2} \le \pi$

$$\frac{\pi}{2} \le 2x \le \frac{3\pi}{2}$$

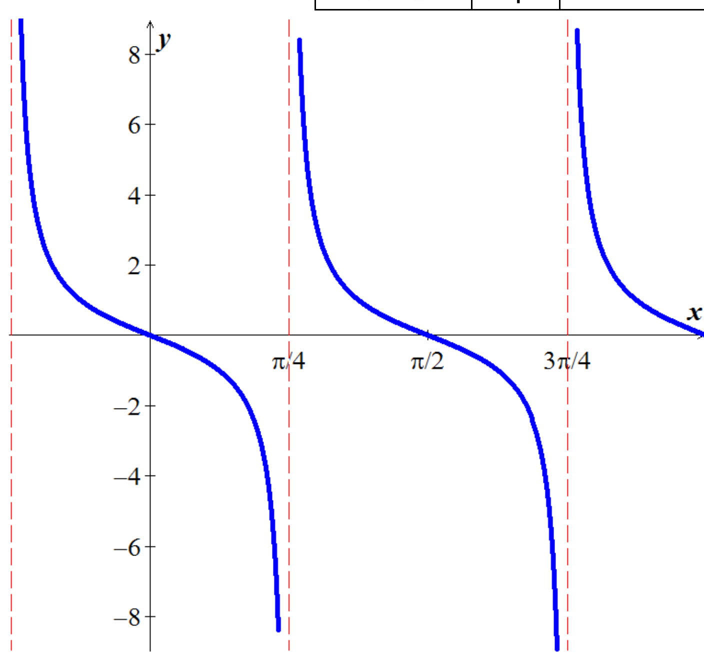
$$\frac{\pi}{4} \le x \le \frac{3\pi}{4}$$

	x	$y = \cot\left(2x - \frac{\pi}{2}\right)$
$\frac{\pi}{4} + 0$	$\frac{\pi}{4}$	∞
$\frac{\pi}{4} + \frac{\pi}{8}$	$\frac{3\pi}{8}$	1
$\frac{\pi}{4} + \frac{\pi}{4}$	$\frac{\pi}{2}$	0
$\frac{\pi}{4} + \frac{3\pi}{8}$	$\frac{5\pi}{8}$	-1
$\frac{\pi}{4} + \frac{\pi}{2}$	$\frac{3\pi}{4}$	∞

$V.A$: $2x - \frac{\pi}{2} = n\pi$

$$2x = n\pi + \frac{\pi}{2}$$

$$\boxed{x = \frac{\pi}{2}n + \frac{\pi}{4}}$$



Graphing the *Secant* Function

Domain: $\left\{x \mid x \neq (2n+1)\frac{\pi}{2},\ where\ n \in \mathbb{Z}\right\}$

Range: $(-\infty,\ -1] \cup [1,\ \infty)$

- The graph is discontinuous at values of x of the form $x = (2n+1)\frac{\pi}{2}$ and has ***vertical asymptotes*** at these values.
- There are ***no x-intercepts***.
- Its period is 2π.
- Its graph has no amplitude, since there are no minimum or maximum values.
- The graph is symmetric with respect to the y-axis, so the function is an even function. For all x in the domain, $\sec(-x) = \sec(x)$.

Example

Sketch the graph of $y = 2\sec\left(x - \frac{\pi}{4}\right)$

Solution

Period $= \frac{2\pi}{1} = 2\pi$

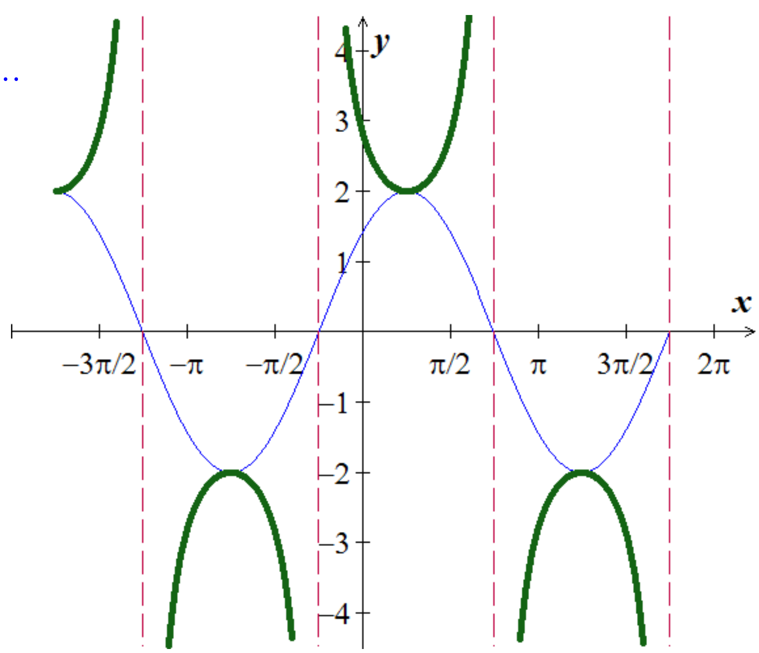
First, graph $y = 2\cos\left(x - \frac{\pi}{4}\right)$

Phase shift: $\phi = -\frac{C}{B} = -\frac{-\frac{\pi}{4}}{1} = \frac{\pi}{4}$

Vertical Asymptote: $x = \frac{\pi}{4} + \frac{\pi}{2}$

$= \frac{3\pi}{4},\ \frac{7\pi}{4},\ \frac{11\pi}{4},\ \ldots$

x	$y = 2\cos\left(x - \frac{\pi}{4}\right)$
$0 + \frac{\pi}{4} = \frac{\pi}{4}$	2
$\frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$	0
$\pi + \frac{\pi}{4} = \frac{5\pi}{4}$	–2
$\frac{3\pi}{2} + \frac{\pi}{4} = \frac{7\pi}{4}$	0
$2\pi + \frac{\pi}{4} = \frac{9\pi}{4}$	2



Graphing the *Cosecant* Function

Domain: $\{x \mid x \neq n\pi,\ where\ n \in \mathbb{Z}\}$

Range: $(-\infty,\ -1] \cup [1,\ \infty)$

- The graph is discontinuous at values of x of the form $x = n\pi$ and has ***vertical asymptotes*** at these values.
- There are no x-intercepts.
- Its period is 2π.
- Its graph has no amplitude, since there are no minimum or maximum values.
- The graph is symmetric with respect to the *origin*, so the function is an odd function. For all x in the domain $\csc(-x) = -\csc(x)$.

Example

Find the period and sketch the graph of $y = \csc(2x+\pi)$

Solution

$$y = \csc(2x+\pi) = \frac{1}{\sin(2x+\pi)}$$

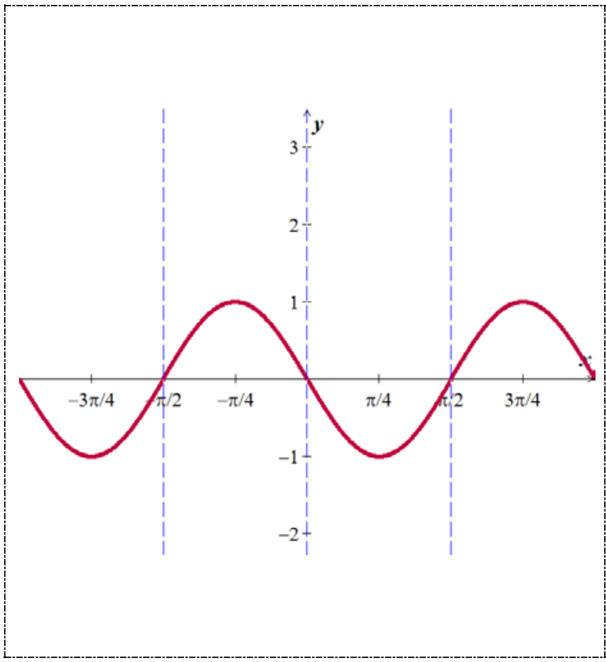
Period $= \frac{2\pi}{2} = \pi$

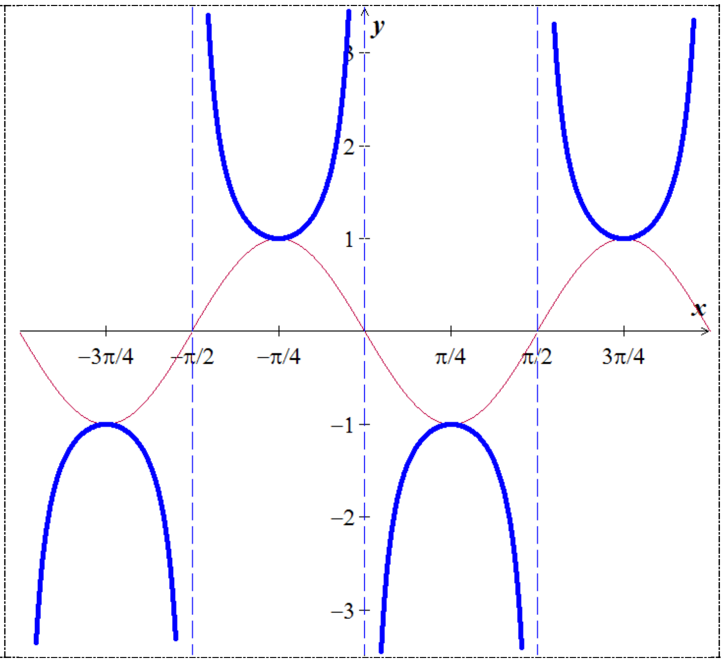
First, graph $y = \sin(2x+\pi)$

Phase shift: $\phi = -\frac{C}{B} = -\frac{\pi}{2}$

Vertical Asymptote: $x = 0, \pm\frac{\pi}{2}, \pm\pi, \dots$

x	$y = \sin(2x+\pi)$
$0 - \frac{\pi}{2} = -\frac{\pi}{2}$	0
$\frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$	1
$\frac{\pi}{2} - \frac{\pi}{2} = 0$	0
$\frac{3\pi}{4} - \frac{\pi}{2} = \frac{\pi}{4}$	–1
$\pi - \frac{\pi}{2} = \frac{\pi}{2}$	0

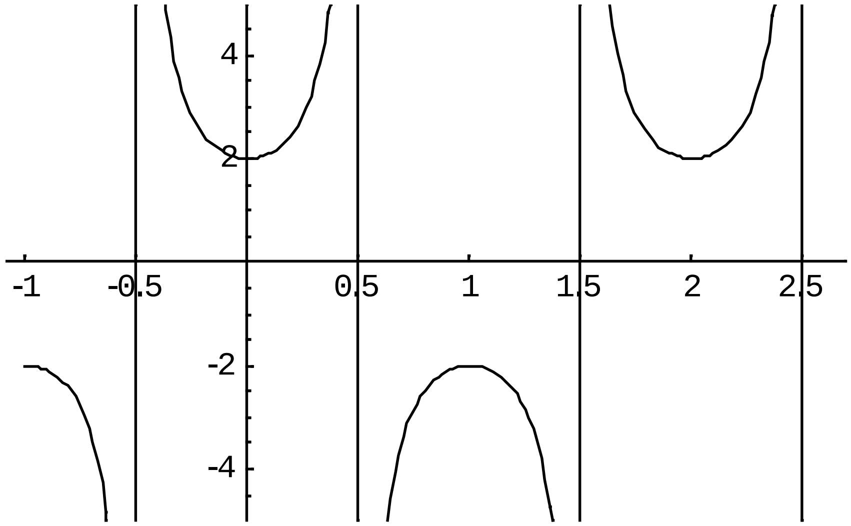




Finding the *Secant* and *Cosecant* Functions from the Graph

Example

Find an equation $y = k + A\sec(Bx + C)$ or $y = k + A\csc(Bx + C)$ to match the graph



Solution

For cosine:

$\boldsymbol{A} = 2$

$P = 2 = \frac{2\pi}{B} \Rightarrow \boxed{B = \frac{2\pi}{2} = \pi}$

$Phase\ shift = -\frac{C}{B} = 0 \Rightarrow \boxed{C = 0}$

$y = 2\sec(\pi x)$ from –1 to 2.5.

Exercises *Section* 5.7 – Additional Trigonometric Graphs

(1 – 12) Find the period, show the asymptotes, and sketch the graph of

1. $y=\tan\left(x-\frac{\pi}{4}\right)$

2. $y=2\tan\left(2x+\frac{\pi}{2}\right)$

3. $y=-\frac{1}{4}\tan\left(\frac{1}{2}x+\frac{\pi}{3}\right)$

4. $y=\cot\left(x+\frac{\pi}{4}\right)$

5. $y=2\cot\left(2x+\frac{\pi}{2}\right)$

6. $y=-\frac{1}{2}\cot\left(\frac{1}{2}x+\frac{\pi}{4}\right)$

7. $y=\sec\left(x-\frac{\pi}{2}\right)$

8. $y=2\sec\left(2x-\frac{\pi}{2}\right)$

9. $y=-3\sec\left(\frac{1}{3}x+\frac{\pi}{3}\right)$

10. $y=\csc\left(x-\frac{\pi}{2}\right)$

11. $y=2\csc\left(2x+\frac{\pi}{2}\right)$

12. $y=4\csc\left(\frac{1}{2}x-\frac{\pi}{4}\right)$

(13 – 15) Graph over a **2**-*period* interval

13. $y=1-2\cot 2\left(x+\frac{\pi}{2}\right)$

14. $y=\frac{2}{3}\tan\left(\frac{3}{4}x-\pi\right)-2$

15. $y=-2-\cot\left(x-\frac{\pi}{4}\right)$

(16 – 17) Graph over a one-period interval

16. $y=1-\frac{1}{2}\csc\left(x-\frac{3\pi}{4}\right)$

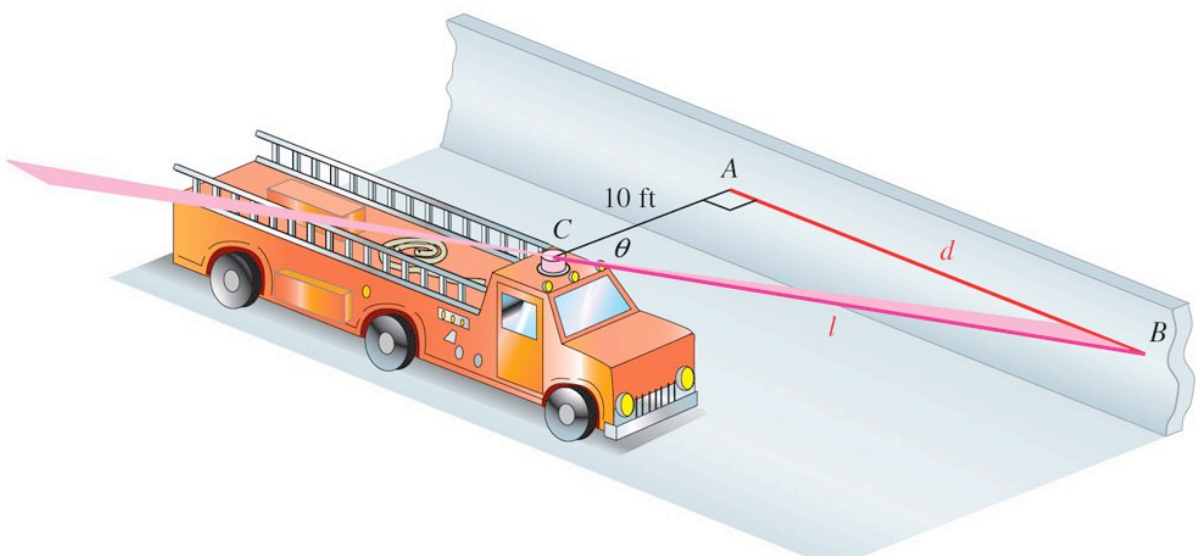
17. $y=2+\frac{1}{4}\sec\left(\frac{1}{2}x-\pi\right)$

18. $y=3+2\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)$

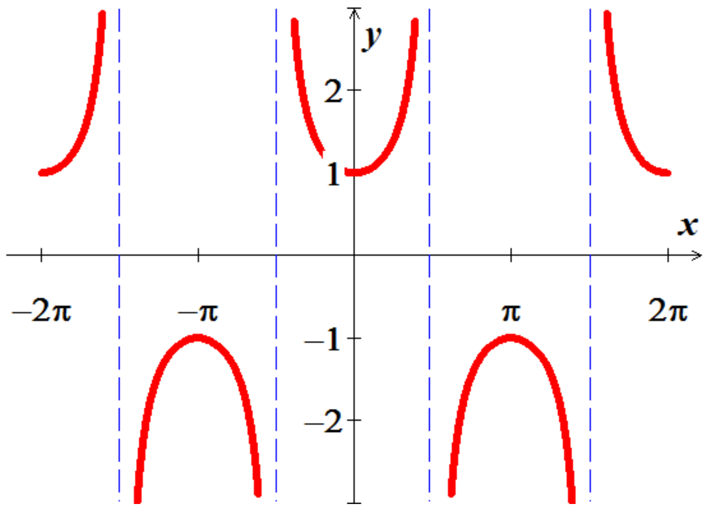
19. $y=-1-3\csc\left(\frac{\pi x}{2}+\frac{3\pi}{4}\right)$

20. Graph $y=\frac{1}{3}\sec 2x$ *for* $-\frac{3\pi}{2}\le x\le\frac{3\pi}{2}$

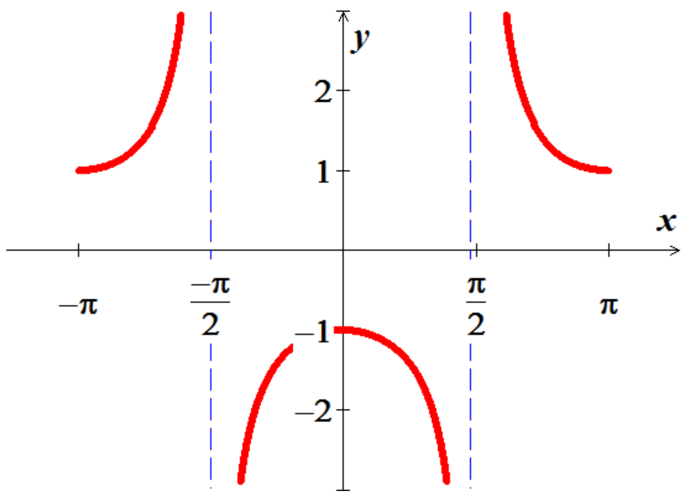
21. A fire truck parked on the shoulder of a freeway next to a long block wall. The red light on the top is 10 *feet* from the wall and rotates through one complete revolution every 2 *seconds*. Graph the function that gives the length ***d*** in terms of time *t* from ***t*** = 0 to ***t*** = 2.



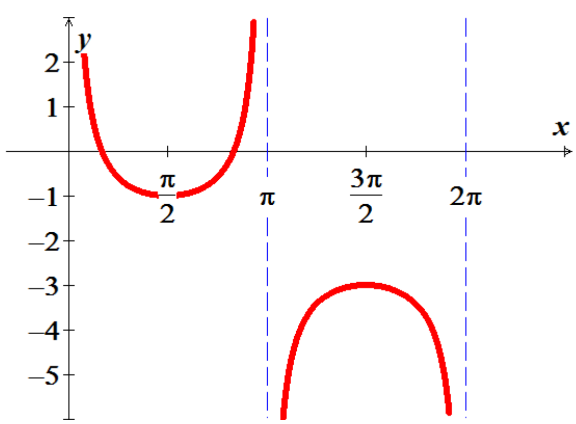
22. Find an equation to match the graph



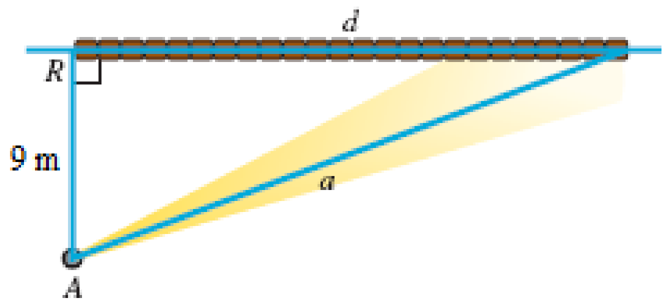
23. Find an equation to match the graph



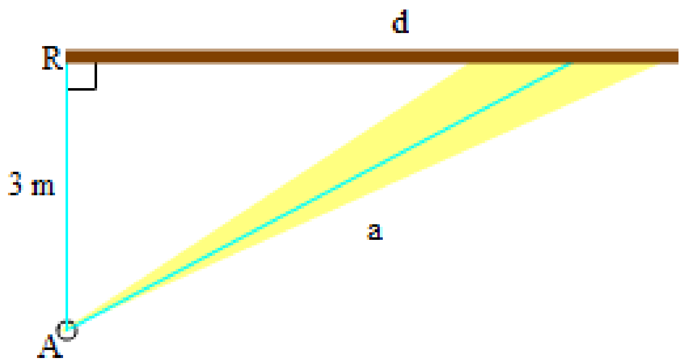
24. Find an equation to match the graph



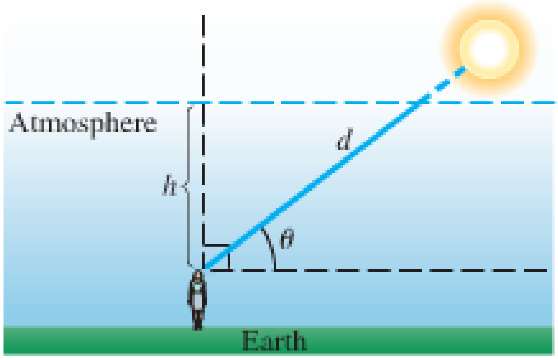
25. A rotating beacon is located at point A next to a long wall. The beacon is 9 m from the wall. The distance $\boldsymbol{a}$ is given by $a = 9|\sec 2\pi t|$, where t is time measured in seconds since the beacon started rotating. (When t = 0, the beacon is aimed at point R.) Find $\boldsymbol{a}$ for t = 0.45



26. A rotating beacon is located 3 m south of point R on an east-west wall. d, the length of the light display along the wall from R, is given by $d = 3\tan 2\pi t$, where t is time measured in seconds since the beacon started rotating. (When t = 0, the beacon is aimed at point R. When the beacon is aimed to the right of R, the value of d is positive; d is negative if the beacon is aimed to the left of R.) Find a for t = 0.8



27. The shortest path for the sun's rays through Earth's atmosphere occurs when the sun is directly overhead. Disregarding the curvature of Earth, as the sun moves lower on the horizon, the distance that sunlight passes through the atmosphere increases by a factor of $\csc\theta$, where θ is the angle of elevation of the sun. This increased distance reduces both the intensity of the sun and the amount of ultraviolet light that reached Earth's surface.

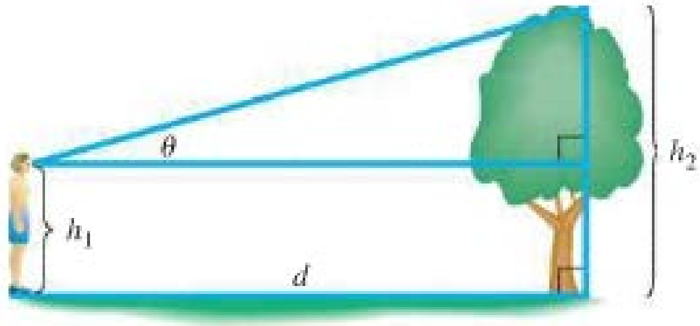


a) Verify that $d = h\csc\theta$

b) Determine θ when $d = 2h$

c) The atmosphere filters out the ultraviolet light that causes skin to burn, Compare the difference between sunbathing when $\theta = \frac{\pi}{2}$ and when $\theta = \frac{\pi}{3}$. Which measure gives less ultraviolet light?

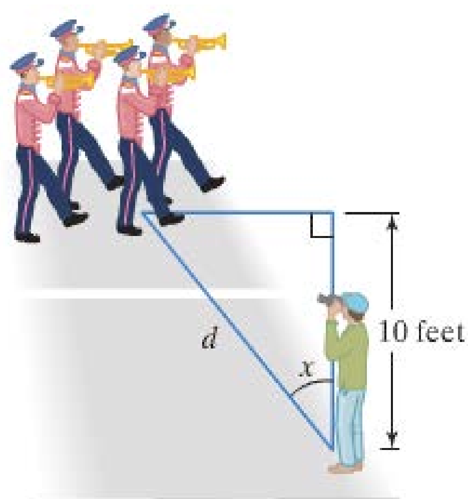
28. Let a person whose eyes are h_1 feet from the ground stand ***d*** feet from an object h_1 feet tall, where $h_2 > h_1$ feet. Let θ be the angle of elevation to the top of the object.



a) Show that $d = \left(h_2 - h_1\right)\cot\theta$

b) Let $h_2 = 55$ and $h_1 = 5$. Graph ***d*** for the interval $0 < \theta \le \frac{\pi}{2}$

29. Your friend is marching with a band and has asked you to film him. You have set yourself up 10 *feet* from the street where your friend will be passing from left to right. If *d* represents your distance, in feet, from your friend and *x* is the radian measure of the angle.



a) Express *d* in terms of a trigonometric function of *x*.

b) Graph the function for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$