

Exercises Section 4.5 – Working with Integrals

Exercise

If f is an odd function, why is $\int_{-a}^a f(x) dx = 0$?

Solution

If f is an odd function then it is symmetric about the origin, which the region between $-a$ and a , there is as much area above the axis and under f as there is below the axis and above f .
Therefore, the net area must be 0.

Exercise

If f is an even function, why is $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$?

Solution

If f is an even function then it is symmetric about the y -axis, which the region that between $-a$ and 0 has the same net area as the region between 0 and a .

$$\text{So } \int_{-a}^a f(x) dx = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx = 2 \int_0^a f(x) dx$$

Exercise

Is x^{12} an even or odd function? Is $\sin(x^2)$ an even or odd function?

Solution

$$f(x) = x^{12} \rightarrow f(-x) = (-x)^{12} = x^{12} = f(x); \text{ therefore } f(x) \text{ is an even function.}$$

$$g(x) = \sin(x^2) \rightarrow g(-x) = \sin((-x)^2) = \sin(x^2) = g(x) \text{ is also an even function.}$$

Exercise

Use symmetry to evaluate the following integrals $\int_{-2}^2 x^9 dx$

Solution

$$\text{Because } x^9 \text{ is an odd function, then } \int_{-2}^2 x^9 dx = 0$$

Exercise

Use symmetry to evaluate the following integrals $\int_{-200}^{200} 2x^5 dx$

Solution

Because $2x^5$ is an odd function, then $\int_{-200}^{200} 2x^5 dx = 0$

Exercise

Use symmetry to evaluate the following integrals $\int_{-\pi/4}^{\pi/4} \cos x dx$

Solution

Because $\cos x$ is an even function, then

$$\begin{aligned}\int_{-\pi/4}^{\pi/4} \cos x dx &= 2 \int_0^{\pi/4} \cos x dx \\ &= 2 \sin x \Big|_0^{\pi/4} \\ &= 2 \left(\frac{\sqrt{2}}{2} \right) \\ &= \sqrt{2}\end{aligned}$$

Exercise

Use symmetry to evaluate the following integrals $\int_{-2}^2 (x^9 - 3x^5 + 2x^2 - 10) dx$

Solution

$$\begin{aligned}\int_{-2}^2 (x^9 - 3x^5 + 2x^2 - 10) dx &= \overset{\text{Odd}}{\int_{-2}^2 (x^9 - 3x^5) dx} + \overset{\text{Even}}{\int_{-2}^2 (2x^2 - 10) dx} \\ &= 0 + 2 \int_0^2 (2x^2 - 10) dx \\ &= 2 \left[\frac{2}{3} x^3 - 10x \right]_0^2 \\ &= 2 \left(\frac{16}{3} - 20 \right) \\ &= -\frac{88}{3}\end{aligned}$$

Exercise

Use symmetry to evaluate the following integrals $\int_{-\pi/2}^{\pi/2} (\cos 2x + \cos x \sin x - 3 \sin x^5) dx$

Solution

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} (\cos 2x + \cos x \sin x - 3 \sin x^5) dx &= \overset{\text{Even}}{\int_{-\pi/2}^{\pi/2} \cos 2x dx} + \overset{\text{Odd}}{\int_{-\pi/2}^{\pi/2} (\cos x \sin x - 3 \sin x^5) dx} \\ &= 2 \int_0^{\pi/2} \cos 2x dx \\ &= \sin 2x \Big|_0^{\pi/2} \\ &= \underline{0} \end{aligned}$$

Exercise

Find the average value of the following functions on the given interval. $f(x) = x^3$ on $[-1, 1]$

Solution

$$\text{Average value} = \frac{1}{1 - (-1)} \int_{-1}^1 x^3 dx = \frac{1}{2} \frac{1}{4} x^4 \Big|_{-1}^1 = \underline{0}$$

Exercise

Find the average value of the following functions on the given interval. $f(x) = \frac{1}{x^2 + 1}$ on $[-1, 1]$

Solution

$$\begin{aligned} \text{Average value} &= \frac{1}{1 - (-1)} \int_{-1}^1 \frac{1}{x^2 + 1} dx \\ &= \frac{1}{2} \tan^{-1} x \Big|_{-1}^1 \\ &= \frac{1}{2} \left(\frac{\pi}{4} + \frac{\pi}{4} \right) \\ &= \underline{\frac{\pi}{4}} \end{aligned}$$

Exercise

Find the average value of the following functions on the given interval. $f(x) = \frac{1}{x}$ on $[1, e]$

Solution

$$\begin{aligned}\text{Average value} &= \frac{1}{e-1} \int_1^e \frac{1}{x} dx \\ &= \frac{1}{e-1} [\ln|x|]_1^e \\ &= \frac{1}{e-1} (\ln e - 0) \\ &= \frac{1}{e-1}\end{aligned}$$

Exercise

Find the average value of the following functions on the given interval. $f(x) = e^{2x}$ on $[0, \ln 2]$

Solution

$$\begin{aligned}\text{Average value} &= \frac{1}{\ln 2} \int_0^{\ln 2} e^{2x} dx \\ &= \frac{1}{2 \ln 2} e^{2x} \Big|_0^{\ln 2} \\ &= \frac{1}{2 \ln 2} (e^{2 \ln 2} - 1) \\ &= \frac{1}{2 \ln 2} (4 - 1) \\ &= \frac{3}{2 \ln 2}\end{aligned}$$

Exercise

Suppose that $\int_0^4 f(x) dx = 10$ and $\int_0^4 g(x) dx = 20$. Furthermore, suppose that f is an even function and g is an odd function. Evaluate the integral

$$\int_{-4}^4 f(x) dx$$

Solution

f is an even function.

$$\begin{aligned}\int_{-4}^4 f(x) dx &= 2 \int_0^4 f(x) dx \\ &= 2(10) \\ &= 20\end{aligned}$$

Exercise

Suppose that $\int_0^4 f(x) dx = 10$ and $\int_0^4 g(x) dx = 20$. Furthermore, suppose that f is an even function and g is an odd function. Evaluate the integral

$$\int_{-4}^4 3g(x) dx$$

Solution

g is an odd function

$$\begin{aligned}\int_0^4 g(x) dx &= - \int_{-4}^0 g(x) dx \\ \int_{-4}^4 3g(x) dx &= 3 \int_{-4}^0 g(x) dx + 3 \int_0^4 g(x) dx \\ &= 3 \int_{-4}^0 g(x) dx - 3 \int_{-4}^0 g(x) dx \\ &= 0\end{aligned}$$

Exercise

Suppose that $\int_0^4 f(x) dx = 10$ and $\int_0^4 g(x) dx = 20$. Furthermore, suppose that f is an even function and g is an odd function. Evaluate the integral

$$\int_0^1 8xf(4x^2) dx$$

Solution

$$\int_0^1 8xf(4x^2) dx = \int_0^1 f(4x^2) d(4x^2) \qquad d(4x^2) = 8x dx$$

$$\begin{cases} x=1 & \rightarrow 4x^2=4 \\ x=0 & \rightarrow 4x^2=0 \end{cases}$$

$$\int_0^1 8xf(4x^2)dx = \int_0^4 f(4x^2)d(4x^2)$$

$$\underline{=10}$$

Exercise

Suppose that $\int_0^4 f(x)dx = 10$ and $\int_0^4 g(x)dx = 20$. Furthermore, suppose that f is an even function and g is an odd function. Evaluate the integral

$$\int_{-2}^2 3xf(x)dx$$

Solution

f is an even function, that implies $xf(x)$ is an odd function.

$$\int_{-2}^2 3xf(x)dx \underline{=0}$$

Exercise

Suppose that $\int_0^4 f(x)dx = 10$ and $\int_0^4 g(x)dx = 20$. Furthermore, suppose that f is an even function and g is an odd function. Evaluate the integral

$$\int_{-4}^4 (4f(x) - 3g(x))dx$$

Solution

f is an even function and g is an odd function.

$$\begin{aligned} \int_{-4}^4 (4f(x) - 3g(x))dx &= 4 \int_{-4}^4 f(x)dx - 3 \int_{-4}^4 g(x)dx \\ &= 8 \int_0^4 f(x)dx - 3(0) \\ &\underline{=80} \end{aligned}$$

Exercise

Suppose that f is an even function with $\int_0^8 f(x) dx = 9$. Evaluate the integral $\int_{-1}^1 x f(x^2) dx$

Solution

f is an even function, that implies $xf(x)$ is an odd function.

$$\int_{-1}^1 x f(x^2) dx = \underline{0}$$

Exercise

Suppose that f is an even function with $\int_0^8 f(x) dx = 9$. Evaluate the integral $\int_{-2}^2 x^2 f(x^3) dx$

Solution

$$d(x^3) = 3x^2 dx$$

$$\begin{cases} x = 2 & \rightarrow x^3 = 8 \\ x = -2 & \rightarrow x^3 = -8 \end{cases}$$

$$\begin{aligned} \int_{-2}^2 x^2 f(x^3) dx &= \frac{1}{3} \int_{-8}^8 f(x^3) d(x^3) \\ &= \frac{1}{3} \int_{-8}^0 f(x^3) d(x^3) + \frac{1}{3} \int_0^8 f(x^3) d(x^3) && f \text{ is an even function} \\ &= \frac{2}{3} \int_0^8 f(x^3) d(x^3) \\ &= \frac{2}{3} \cdot 9 \\ &= \underline{6} \end{aligned}$$

Exercise

Suppose that p is a nonzero real number and f is an odd integrable function with $\int_0^1 f(x) dx = \pi$.

Evaluate the integral $\int_0^{\frac{\pi}{2p}} (\cos px) f(\sin px) dx$

Solution

$$\begin{cases} x = \frac{\pi}{2p} & \rightarrow \sin px = 1 \\ x = 0 & \rightarrow \sin px = 0 \end{cases}$$

$$\int_0^{\frac{\pi}{2p}} (\cos px) f(\sin px) dx = \frac{1}{p} \int_0^1 f(\sin px) d(\sin px)$$

$$= \frac{\pi}{p}$$

$$d(\sin px) = p \cos px \, dx$$

Exercise

Suppose that p is a nonzero real number and f is an odd integrable function with $\int_0^1 f(x) dx = \pi$.

Evaluate the integral $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x) f(\sin x) dx$

Solution

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos px) f(\sin px) dx = \frac{1}{p} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\sin px) d(\sin px)$$

f is an odd function

$$= 0$$

Exercise

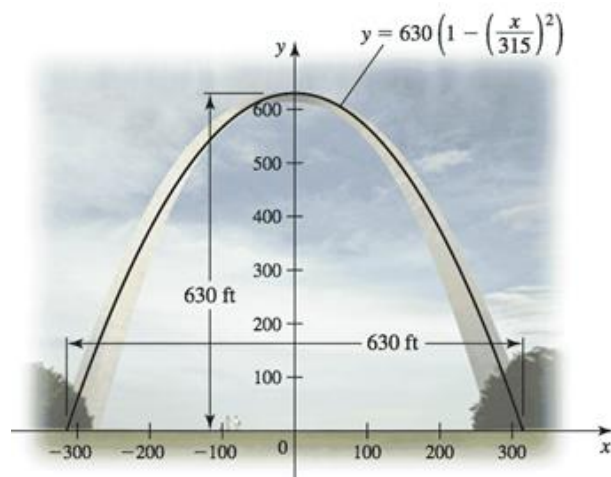
The Gateway Arch in St. Louis is 630 feet high and has a 630-ft base. Its shape can be modeled by the parabola

$$y = 630 \left(1 - \left(\frac{x}{315} \right)^2 \right)$$

Find the average height of the arch above the ground.

Solution

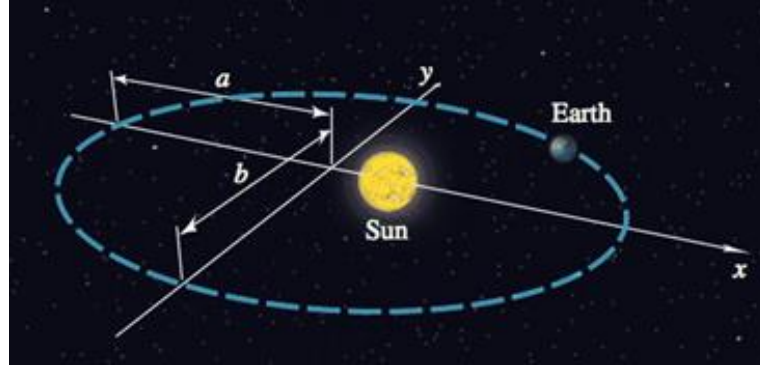
$$\begin{aligned} \text{Average height} &= \frac{1}{630} \int_{-315}^{315} 630 \left(1 - \frac{1}{315^2} x^2 \right) dx \\ &= x - \frac{1}{315^2} \frac{x^3}{3} \Big|_{-315}^{315} \\ &= 315 - 105 + 315 - 105 \\ &= 420 \text{ ft} \end{aligned}$$



Exercise

The planets orbit the Sun in elliptical orbits with the Sun at one focus. The equation of an ellipse whose dimensions are $2a$ in the x -direction and $2b$ in the y -direction is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



- a) Let d^2 denote the square of the distance from a planet to the center of the ellipse at $(0, 0)$. Integrate over the interval $[-a, a]$ to show that the average value of d^2 is $\frac{a^2 + 2b^2}{3}$.
- b) Show that in the case of a circle ($a = b = R$), the average value in part (a) is R^2 .
- c) Assuming $0 < b < a$, the coordinates of the Sun are $(\sqrt{a^2 - b^2}, 0)$. Let D^2 denote the square of the distance from the planet to the Sun. Integrate over the interval $[-a, a]$ to show that the average value of D^2 is $\frac{4a^2 - b^2}{3}$.

Solution

$$a) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \rightarrow y^2 = b^2 - \frac{b^2}{a^2}x^2$$

$$d^2 = x^2 + y^2 = x^2 + b^2 - \frac{b^2}{a^2}x^2$$

$$= b^2 + \left(1 - \frac{b^2}{a^2}\right)x^2$$

$$\text{The average value of } d^2 = \frac{1}{2a} \int_{-a}^a \left(b^2 + \left(1 - \frac{b^2}{a^2}\right)x^2 \right) dx$$

$$= \frac{1}{2a} \left[b^2 x + \frac{1}{3} \left(1 - \frac{b^2}{a^2}\right) x^3 \right]_{-a}^a$$

$$= \frac{1}{2a} \left(ab^2 + \frac{1}{3} \left(1 - \frac{b^2}{a^2}\right) a^3 + ab^2 + \frac{1}{3} \left(1 - \frac{b^2}{a^2}\right) a^3 \right)$$

$$= \frac{1}{2a} \left(2ab^2 + \frac{2}{3}a^3 - \frac{2}{3}b^2a \right)$$

$$= \frac{\frac{2}{3}b^2 + \frac{a^2}{3}}{\quad}$$

b) If $a = b = R$

The average value of $d^2 = \frac{2R^2}{3} + \frac{R^2}{3} = R^2$

c) $D^2 = \left(x - \sqrt{a^2 - b^2} \right)^2 + y^2$

$$= x^2 - 2x\sqrt{a^2 - b^2} + a^2 - b^2 + b^2 - \frac{b^2}{a^2}x^2$$

$$= \left(1 - \frac{b^2}{a^2} \right) x^2 - 2x\sqrt{a^2 - b^2} + a^2$$

The average value of $D^2 = \frac{1}{2a} \int_{-a}^a \left(\left(1 - \frac{b^2}{a^2} \right) x^2 - 2x\sqrt{a^2 - b^2} + a^2 \right) dx$

$$= \frac{1}{2a} \left(\frac{1}{3} \left(1 - \frac{b^2}{a^2} \right) x^3 - x^2 \sqrt{a^2 - b^2} + a^2 x \right)_{-a}^a$$

$$= \frac{1}{2a} \left(\frac{1}{3} \left(1 - \frac{b^2}{a^2} \right) a^3 - a^2 \sqrt{a^2 - b^2} + a^3 + \frac{1}{3} \left(1 - \frac{b^2}{a^2} \right) a^3 + a^2 \sqrt{a^2 - b^2} + a^3 \right)$$

$$= \frac{1}{3} \left(1 - \frac{b^2}{a^2} \right) a^2 + a^2$$

$$= \frac{1}{3} a^2 - \frac{1}{3} b^2 + a^2$$

$$= \frac{\frac{4}{3}a^2 - \frac{1}{3}b^2}{\quad}$$

Exercise

A particle moves along a line with a velocity given by $v(t) = 5 \sin \pi t$ starting with an initial position $s(0) = 0$. Find the displacement of the particle between $t = 0$ and $t = 2$, which is given by

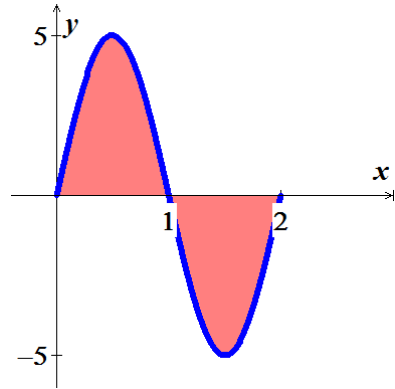
$s(t) = \int_0^2 v(t) dt$. Find the distance traveled by the particle during this interval, which is $\int_0^2 |v(t)| dt$.

Solution

$$s(t) = \int_0^2 5 \sin \pi t \, dt$$

$$\begin{aligned}
 &= -\frac{5}{\pi} \cos \pi t \Big|_0^2 \\
 &= -\frac{5}{\pi} (1-1) \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 s(t) &= \int_0^2 |5 \sin \pi t| \, dt \\
 &= 5 \int_0^1 \sin \pi t \, dt + 5 \int_1^2 (-\sin \pi t) \, dt \\
 &= -\frac{5}{\pi} \cos \pi t \Big|_0^1 + \frac{5}{\pi} \cos \pi t \Big|_1^2 \\
 &= -\frac{5}{\pi} (-1-1) + \frac{5}{\pi} (1+1) \\
 &= \frac{10}{\pi} + \frac{10}{\pi} \\
 &= \frac{20}{\pi}
 \end{aligned}$$



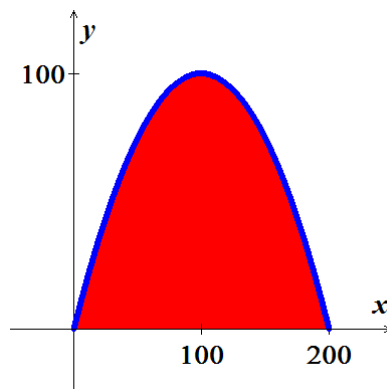
Exercise

A baseball is launched into the outfield on a parabolic trajectory given by $y = 0.01x(200 - x)$. Find the average height of the baseball over the horizontal extent of its flight.

Solution

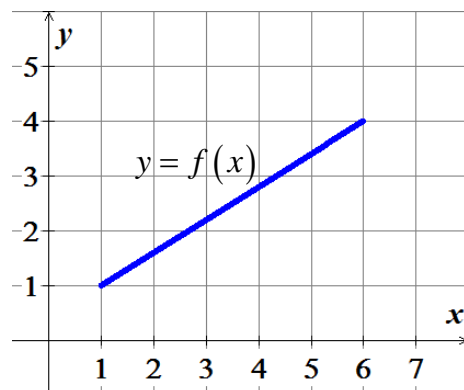
$$y = 0.01x(200 - x) = 0 \rightarrow x = 0, 200$$

$$\begin{aligned}
 Avg &= \frac{1}{200} \int_0^{200} (2x - 0.01x^2) \, dx \\
 &= \frac{1}{200} \left(x^2 - \frac{0.01}{3} x^3 \right) \Big|_0^{200} \\
 &= \frac{1}{200} \left(4 \times 10^4 - \frac{1}{300} 8 \times 10^6 \right) \\
 &= \frac{4 \times 10^4}{200} \left(1 - \frac{2}{3} \right) \\
 &= \frac{200}{3}
 \end{aligned}$$



Exercise

Find the average value of f shown in the figure on the interval $[1, 6]$ and then find the point(s) c in $(1, 6)$ guaranteed to exist by the Mean Value Theorem for Integrals



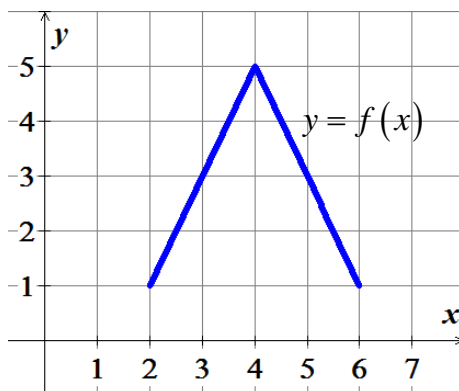
Solution

Since it is a straight line, then the average value is 2.5

The average value occurs at the midpoint of the interval which is $(3.5, 2.5)$

Exercise

Find the average value of f shown in the figure on the interval $[2, 6]$ and then find the point(s) c in $(2, 6)$ guaranteed to exist by the Mean Value Theorem for Integrals



Solution

Over interval $[2, 4]$; it is a straight line, then the average value is 3

Over interval $[4, 6]$; it is a straight line, then the average value is 3 .

Therefore; the overage is 3 over $[2, 6]$