SOLUTION *Section* 3.3 – Integral Test

Exercise

Use the ***Integral Test*** to determine if the series converge or diverge. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{0.2}}$

Solution

$$f(x)=\frac{1}{x^{0.2}}$$

$$\int_1^{\infty}\frac{dx}{x^{0.2}}=\int_1^{\infty}x^{-0.2}\,dx$$

$$=\frac{1}{0.8}x^{0.8}\Big|_1^{\infty}$$

$$=\frac{1}{0.8}(\infty-1)$$

$$=\infty$$

By the *Integral Test*, the given series ***diverges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+4}$

Solution

$$f(x)=\frac{1}{x^2+4}$$

$$\int_1^{\infty}\frac{dx}{x^2+4}=\frac{1}{2}\tan^{-1}\frac{x}{2}\Big|_1^{\infty}$$

$$=\frac{1}{2}\left(\tan^{-1}\infty-\tan^{-1}\frac{1}{2}\right)$$

$$=\frac{1}{2}\left(\frac{\pi}{2}-\tan^{-1}\frac{1}{2}\right)$$

By the *Integral Test*, the given series ***converges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\displaystyle\sum_{n=1}^{\infty} e^{-2n}$

Solution

$$f(x) = e^{-2x}$$

$$\int_1^{\infty} e^{-2x}\,dx = -\frac{1}{2}e^{-2x}\Big|_1^{\infty}$$

$$= -\frac{1}{2}\left(e^{-\infty} - e^{-2}\right)$$

$$= -\frac{1}{2}\left(\frac{1}{e^{\infty}} - \frac{1}{e^2}\right)$$

$$= -\frac{1}{2e^2}$$

By the *Integral Test*, the given series *converges*.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Solution

Function is positive, continuous, and decreasing for $x \ge 2$.

$$f(x) = \frac{1}{x(\ln x)^2}$$

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \int_2^{\infty} \frac{d(\ln x)}{(\ln x)^2} \qquad d(\ln x) = \frac{1}{x}dx$$

$$= -\frac{1}{\ln x}\Big|_2^{\infty}$$

$$= -\left(\frac{1}{\ln\infty} - \frac{1}{\ln 2}\right)$$

$$= \frac{1}{\ln 2}$$

By the *Integral Test*, the given series *converges*.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{e^{n/3}}$

Solution

$f(x)=\dfrac{x^2}{e^{x/3}}$ is positive, continuous for $x \ge 1$.

$$f'(x)=\frac{2xe^{x/3}-\frac{1}{3}x^2e^{x/3}}{\left(e^{x/3}\right)^2}$$

$$=\frac{6x-x^2}{3e^{x/3}}$$

$$=\frac{-x(x-6)}{3e^{x/3}}<0 \quad \textit{for} \quad x>6$$

		$\int e^{-x/3}\,dx$
+	x^2	$-3e^{-x/3}$
−	$2x$	$9e^{-x/3}$
+	2	$-27e^{-x/3}$

$$\int_7^{\infty}\frac{x^2}{e^{x/3}}dx=-\frac{3x^2}{e^{x/3}}-\frac{18x}{e^{x/3}}-\frac{54}{e^{x/3}}\Bigg|_7^{\infty}$$

$$=\frac{3}{e^{x/3}}(49+42+18)$$

$$=\frac{327}{e^{7/3}}$$

By the *Integral Test*, the given series *converges*.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\displaystyle\sum_{n=1}^{\infty}\frac{n-4}{n^2-2n+1}$

Solution

$$f(x)=\frac{x-4}{x^2-2x+1}$$

$$= \frac{x-4}{(x-1)^2} \quad \text{is continuous for } x \ge 2\text{, and positive } x > 4.$$

$$f'(x) = \frac{(x-1)^2 - 2(x-1)(x-4)}{(x-1)^4}$$

$$= \frac{(x-1)(x-1-2x+8)}{(x-1)^4}$$

$$= \frac{-x+7}{(x-1)^3} < 0 \quad \text{for} \quad x > 7$$

$$\int_8^\infty \frac{x-4}{(x-1)^2}dx = \lim_{b\to\infty}\left[\int_8^b \frac{x-1}{(x-1)^2}dx - \int_b^\infty \frac{3}{(x-1)^2}dx\right] \qquad d(x-1) = dx$$

$$= \lim_{b\to\infty}\left[\int_8^b \frac{1}{x-1}dx - \int_b^\infty \frac{3}{(x-1)^2}d(x-1)\right]$$

$$= \lim_{b\to\infty}\left(\ln|x-1|\right)\Big|_8^b - \lim_{b\to\infty}\left(\frac{3}{(x-1)}\right)\Big|_8^b$$

$$= \lim_{b\to\infty}\left[\ln|b-1| - \ln 7 - \frac{3}{b-1} + \frac{3}{7}\right]$$

$$\underline{= \infty}$$

By the *Integral Test*, the given series *diverges*.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

Solution

Let $f(x) = \dfrac{1}{x\ln x}$

$$\int_2^\infty \frac{1}{x\ln x}dx = \int_2^\infty \frac{1}{\ln x}d(\ln x)$$

$$= \ln|\ln x|\Big|_2^\infty$$

$$= \infty - 0$$

$$\underline{= \infty}$$

Therefore; by the *Integral Test*, the given series ***diverges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{\sqrt{k^2+4}}$

Solution

Let $f(x)=\dfrac{x}{\sqrt{x^2+4}}$, $f(x)$ is continuous for $x \ge 1$.

$$f'(x)=\frac{\sqrt{x^2+4}-x^2\left(x^2+4\right)^{-1/2}}{\left(\sqrt{x^2+4}\right)^2}$$

$$=\frac{4}{\left(\sqrt{x^2+4}\right)^3} > 0$$

Thus $f(x)$ is increasing, and the conditions of the Integral Test are not satisfied.

Therefore; by the *Integral Test*, the given series ***diverges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\displaystyle\sum_{k=1}^{\infty} ke^{-2k^2}$

Solution

Let $f(x)=x\cdot e^{-2x^2}$, $f(x)$ is continuous for $x \ge 1$.

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} xe^{-2x^2}\,dx$$

$$=-\frac{1}{4}\int_1^{\infty} e^{-2x^2}\,d\left(-2x^2\right)$$

$$=-\frac{1}{4}e^{-2x^2}\Big|_1^{\infty}$$

$$=-\frac{1}{4}\left(0-e^{-2}\right)$$

$$\boxed{=\frac{1}{4e^2}}$$

Therefore; by the *Integral Test*, the given series ***converges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\sum_{k=0}^{\infty} \frac{1}{\sqrt{k+8}}$

Solution

Let $f(x) = \frac{1}{\sqrt{x+8}}$, $f(x)$ is continuous and decreasing for $x \ge 1$.

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{\sqrt{x+8}}\,dx$$

$$= 2\sqrt{x+8}\,\Big|_1^{\infty}$$

$$\underline{= \infty}$$

Therefore; by the *Integral Test*, the given series ***diverges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\sum_{k=2}^{\infty} \frac{1}{k \ln k \ln(\ln k)}$

Solution

Let $f(x) = \frac{1}{x \ln x \ln(\ln x)}$, $f(x)$ is ***not continuous*** at $x = e$.

Therefore; the *Integral Test* does not apply.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

Solution

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \int_1^{\infty} \frac{1}{x^2+1}\,d\left(x^2+1\right)$$

$$= \ln\left(x^2+1\right)\Big|_1^{\infty}$$

$$\underline{= \infty}$$

Therefore; by the *Integral Test*, the given series ***diverges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Solution

$$\int_1^{\infty} \frac{1}{x^2+1} dx = \arctan x \Big|_1^{\infty}$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Therefore; by the *Integral Test*, the given series ***converges***.

Exercise

Use the *Integral Test* st to determine if the series converge or diverge. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Solution

$$\int_1^{\infty} \frac{1}{x^3} dx = -\frac{1}{2}\frac{1}{x^2} \Big|_1^{\infty}$$

$$= -\frac{1}{2}(0-1)$$

$$= \frac{1}{2}$$

Therefore; by the *Integral Test*, the given series ***converges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

Solution

$$\int_1^{\infty} x^{-1/2} dx = \frac{1}{2}\sqrt{x} \Big|_1^{\infty}$$

$$= \infty$$

Therefore; by the *Integral Test*, the given series ***diverges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\sum_{n=1}^{\infty} \frac{1}{n^{1/4}}$

Solution

Let $f(x) = \frac{1}{x^{1/4}}$

$$\int_1^\infty x^{-1/4} dx = \frac{4}{3} x^{3/4} \Big|_1^\infty$$

$$\boxed{= \infty}$$

Therefore; by the *Integral Test*, the given series ***diverges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\sum_{n=1}^{\infty} \frac{1}{n^5}$

Solution

Let $f(x) = \frac{1}{x^5}$

$$\int_1^\infty x^{-5} dx = -\frac{1}{4} \frac{1}{x^4} \Big|_1^\infty$$

$$= -\frac{1}{4}(0 - 1)$$

$$\boxed{= \frac{1}{4}}$$

Therefore; by the *Integral Test*, the given series ***converges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\sum_{n=2}^{\infty} \frac{1}{e^n}$

Solution

Let $f(x) = \frac{1}{e^x}$

$$\int_2^\infty e^{-x} dx = -e^{-x} \Big|_2^\infty$$

$$= -e^{-\infty} + e^{-2}$$

$$=\boxed{\frac{1}{e^2}}$$

Therefore; by the *Integral Test*, the given series ***converges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$

Solution

Let $f(x)=\dfrac{x}{e^x}$

		$\int e^{-x}$
+	x	$-e^{-x}$
−	1	e^{-x}

$$\int_1^{\infty} xe^{-x}dx = e^{-x}(-x-1)\Big|_1^{\infty}$$

$$=0-e^{-1}(-2)$$

$$=\boxed{\frac{2}{e}}$$

Therefore; by the *Integral Test*, the given series ***converges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\displaystyle\sum_{k=1}^{\infty} \frac{|\sin k|}{k^2}$

Solution

$-1 \le \sin k \le 1$

$0 \le |\sin k| \le 1$

The *Integral Test* does not apply, because the series is not decreasing.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{\left(k^2+1\right)^3}$

Solution

Let $f(x)=\dfrac{x}{\left(x^2+1\right)^3}$

$$\int_1^{\infty}\frac{x}{\left(x^2+1\right)^3}\,dx=\frac{1}{2}\int_1^{\infty}\left(x^2+1\right)^{-3}\,d\left(x^2+1\right)$$

$$=-\frac{1}{4}\left(x^2+1\right)^{-2}\Big|_1^{\infty}$$

$$=-\frac{1}{4}\left(0-\frac{1}{8}\right)$$

$$=\frac{1}{16}$$

Therefore; by the *Integral Test*, the given series ***converges***.

Exercise

Use the *Integral Test* to determine if the series converge or diverge. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{\sqrt[3]{k+10}}$

Solution

Let $f(x)=\dfrac{1}{\sqrt[3]{x+10}}$

$$\int_1^{\infty}\frac{1}{\sqrt[3]{x+10}}\,dx=\int_1^{\infty}(x+10)^{-1/3}\,d(x+10)$$

$$=\frac{3}{2}(x+10)^{2/3}\Big|_1^{\infty}$$

$$=\infty$$

Therefore; by the *Integral Test*, the given series ***diverges***.

Exercise

Use the ***p-series*** *Test* to determine if the series converge or diverge. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^9}$

Solution

Which is *p-series* with $p=9>1$

Therefore; by the ***p-series*** *Test*, the given series ***converges***.

Exercise

Use the ***p-series*** *Test* to determine if the series converge or diverge. $\sum_{k=1}^{\infty} \frac{1}{k^6}$

Solution

Which is *p-series* with $p = 6 > 1$

Therefore; by the ***p-series*** *Test*, the given series ***converges***.

Exercise

Use the ***p-series*** *Test* to determine if the series converge or diverge. $\sum_{k=1}^{\infty} \frac{1}{k^{1/9}}$

Solution

v $p = \frac{1}{9} \le 1$

Therefore; by the ***p-series*** *Test*, the given series ***diverges***.

Exercise

Use the ***p-series*** *Test* to determine if the series converge or diverge. $\sum_{k=1}^{\infty} k^{-2}$

Solution

$$\sum_{k=1}^{\infty} k^{-2} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Which is *p-series* with $p = 2 > 1$

Therefore; by the ***p-series*** *Test*, the given series ***converges***.

Exercise

Use the ***p-series*** *Test* to determine if the series converge or diverge. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k}}$

Solution

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/4}}$$

Which is *p-series* with $p = \frac{1}{4} \le 1$

Therefore; by the ***p-series*** *Test*, the given series ***diverges***.

Exercise

Use the ***p-series*** *Test* to determine if the series converge or diverge. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{16k^2}}$

Solution

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{16k^2}} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$$

Which is *p-series* with $p = \frac{1}{2} \le 1$

Therefore; by the ***p-series*** *Test*, the given series ***diverges***.

Exercise

Determine if the series converge or diverge $\sum_{k=1}^{\infty} \frac{1}{k^8}$

Solution

Which is *p-series* with $p = 8 > 1$

Therefore; by the ***p-series*** *Test*, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\sum_{k=1}^{\infty} \frac{1}{3^k}$

Solution

Let $f(x) = \frac{1}{3^x}$

$$\int_1^{\infty} \frac{1}{3^x}\, dx = \int_1^{\infty} 3^{-x}\, dx$$

$$= -(\ln 3)\ 3^{-x} \Big|_1^{\infty}$$

$$= -(\ln 3)\left(0 - \frac{1}{3}\right)$$

$$= \boxed{\frac{1}{3}\ln 3}$$

Therefore; by the *Integral Test*, the given series ***converges***.

$$\sum_{k=1}^{\infty} \frac{1}{3^k} = \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k$$

By Geometric series $r = \frac{1}{3} < 1$

Therefore; by the *Geometric Test*, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\sum_{k=1}^{\infty} \frac{1}{k^{5/2}}$

Solution

Which is *p-series* with $p = \frac{5}{2} > 1$

Therefore; by the ***p-series*** *Test*, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\sum_{n=1}^{\infty} ne^{-n^2}$

Solution

Let $f(x) = xe^{-x^2}$

$$\int_1^{\infty} xe^{-x^2}\, dx = -\frac{1}{2}\int_1^{\infty} e^{-x^2}\, d\left(-x^2\right)$$

$$= -\frac{1}{2}e^{-x^2} \Big|_1^{\infty}$$

$$= -\frac{1}{2}(0-1)$$

$$= \boxed{\frac{1}{2}}$$

Therefore; by the *Integral Test*, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=0}^{\infty} \frac{10}{n^2+9}$

Solution

Let $f(x) = \dfrac{10}{x^2+9}$

$$\int_0^{\infty} \frac{10}{x^2+9}\,dx = \frac{10}{3}\tan^{-1}\frac{x}{3}\Big|_0^{\infty}$$

$$= \frac{10}{3}\left(\tan^{-1}\infty - \tan^{-1}0\right)$$

$$= \frac{10}{3}\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{5\pi}{3}$$

Therefore; by the *Integral Test*, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n+1)(3n+4)}$

Solution

Let $f(x) = \dfrac{1}{(3x+1)(3x+4)}$

$$= \frac{A}{3x+1} + \frac{B}{3x+4}$$

$$3Ax + 4A + 3Bx + B = 1$$

$$\begin{cases} x & 3A + 3B = 0 \\ x^0 & 4A + B = 1 \end{cases}$$

$$\Delta = \begin{vmatrix} 3 & 3 \\ 4 & 1 \end{vmatrix} = -9 \qquad \Delta_A = \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} = -3 \qquad \Delta_B = \begin{vmatrix} 3 & 0 \\ 4 & 1 \end{vmatrix} = 3$$

$$A = \frac{1}{3} \quad B = -\frac{1}{3}$$

$$\int_1^{\infty} \frac{1}{(3x+1)(3x+4)}\,dx = \int_1^{\infty} \left(\frac{1}{3}\frac{1}{3x+1} - \frac{1}{3}\frac{1}{3x+4}\right)dx$$

$$= \frac{1}{9}\int_1^{\infty} \frac{1}{3x+1}\, d(3x+1) - \frac{1}{9}\int_1^{\infty} \frac{1}{3x+4}\, d(3x+4)$$

$$= \frac{1}{9}\left(\ln(3x+1) - \ln(3x+4)\right) \Big|_1^{\infty}$$

$$= \frac{1}{9}\left(\ln\left(\frac{3x+1}{3x+4}\right)\right) \Big|_1^{\infty}$$

$$= \frac{1}{9}\left(\ln 1 - \ln \frac{4}{7}\right)$$

$$\boxed{= -\frac{1}{9}\ln \frac{4}{7}}$$

Therefore; by the *Integral Test*, the given series ***converges***.

This is a telescoping series with

$$S_n = \frac{1}{3}\left(\frac{1}{4} - \frac{1}{3n+4}\right)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{3}\left(\frac{1}{4} - \frac{1}{3n+4}\right)$$

$$\boxed{= \frac{1}{12}}$$

Exercise

Determine if the series converge or diverge $\sum_{n=1}^{\infty} e^{-n}$

Solution

The series is a geometric series with $r = \frac{1}{e} < 1$

Therefore; it ***converges***.

Exercise

Determine if the series converge or diverge $\sum_{n=1}^{\infty} \frac{n}{n+1}$

Solution

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$$

Therefore; the given series ***diverges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$

Solution

$$\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}} = 3\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

Which is a divergent p-series $\left(p = \frac{1}{2} \leq 1\right)$.

Therefore; the given series ***diverges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} \frac{-8}{n}$

Solution

$$\sum_{n=1}^{\infty} \frac{-8}{n} = -8\sum_{n=1}^{\infty} \frac{1}{n},$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

Therefore; the given series ***diverges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=2}^{\infty} \frac{\ln}{n}$

Solution

By the Integral Test:

$$\int_2^{\infty} \frac{\ln x}{x}\,dx = \int_2^{\infty} \ln x \; d(\ln x)$$

$$= \frac{1}{2}\ln^2 x \Big|_2^{\infty}$$

$$=\frac{1}{2}\left(\ln^2 \infty - \ln^2 2\right)$$

$$\underline{= \infty}\,\big|$$

Therefore; the given series ***diverges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{4^n + 3}$

Solution

Using L'Hôpital rule:

$$\sum_{n=1}^{\infty} \frac{5^n}{4^n + 3} = \sum_{n=1}^{\infty} \frac{5^n \ln 5}{4^n \ln 4}$$

$$= \frac{\ln 5}{\ln 4} \sum_{n=1}^{\infty} \left(\frac{5}{4}\right)^n$$

Using the Geometric series: $|r| = \frac{5}{4} \geq 1$ which diverges.

Therefore; by Geoemtric test, the given series ***diverges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} n \sin\frac{1}{n}$

Solution

$$\lim_{n \to \infty} n \sin\frac{1}{n} = \lim_{n \to \infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{x \to 0} \frac{\sin x}{x}$$

$$\underline{= 1 \neq 0}\,\big|$$

Therefore; the given series ***diverges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{10}}$

Solution

This is a *p*-series with $p = 10$.

Therefore; by the *p*-series, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=2}^{\infty}\frac{n^{e}}{n^{\pi}}$

Solution

$$\sum_{n=2}^{\infty}\frac{n^{e}}{n^{\pi}}=\sum_{n=2}^{\infty}\frac{1}{n^{\pi-e}}$$

$$=\sum_{n=2}^{\infty}\frac{1}{n^{3.1416-2.71828}}$$

$$=\sum_{n=2}^{\infty}\frac{1}{n^{0.42331}}$$

$0.42331<1$

Therefore; the given series ***diverges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=3}^{\infty}\frac{1}{(n-2)^{4}}$

Solution

$$\sum_{n=3}^{\infty}\frac{1}{(n-2)^{4}}=\sum_{n=1}^{\infty}\frac{1}{n^{4}}$$

This is a *p*-series with $p = 4$.

Therefore; by the *p*-series, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} 2n^{-3/2}$

Solution

$$\sum_{n=1}^{\infty} 2n^{-3/2} = 2\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

This is a *p*-series with $p = \frac{3}{2}$

Therefore; by the *p*-series, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{k=0}^{\infty} \frac{k}{\sqrt{k^2+4}}$

Solution

$$\lim_{k\to\infty} \frac{k}{\sqrt{k^2+4}} = \lim_{k\to\infty} \frac{k}{\sqrt{k^2}}$$

$$= \lim_{k\to\infty} \frac{k}{k}$$

$$\underline{=1} \neq 0$$

Therefore; the given series ***diverges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{k=1}^{\infty} \frac{2^k+3^k}{4^k}$

Solution

$$\int_1^{\infty} \left(\frac{2^x}{4^x} + \frac{3^x}{4^x}\right) dx = \int_1^{\infty} \left(\left(\frac{1}{2}\right)^x + \left(\frac{3}{4}\right)^x\right) dx$$

$$= -\frac{1}{\ln(2)}\left(\frac{1}{2}\right)^x - \frac{1}{\ln(3/4)}\left(\frac{3}{4}\right)^x \Bigg|_1^{\infty}$$

$$= 0 - \left(-\frac{1}{2\ln(2)} - \frac{3}{4\ln(3/4)}\right)$$

$$\underline{\approx 3.3284 \ < \infty}$$

Therefore; by the integral test, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{k=2}^{\infty} \frac{4}{k\ln^2 k}$

Solution

$$\int_2^{\infty} \frac{4}{x\ln^2 x}dx = 4\int_2^{\infty} \frac{1}{\ln^2 x}d(\ln x)$$

$$= -4\frac{1}{\ln x}\Big|_2^{\infty}$$

$$\boxed{= 4\frac{1}{\ln 2} < \infty}$$

Therefore; by the integral test, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$

Solution

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/5}}$$

This is a p-series with $p = \frac{1}{5} < 1$

Therefore; by the p-series, the given series ***diverges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$

Solution

This is a p-series with $p = \frac{5}{3} > 1$

Therefore; by the p-series, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$

Solution

This is a p-series with $p = 1.04 > 1$

Therefore; by the p-series, the given series ***converges***.

Exercise

Determine if the series converge or diverge $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$

Solution

This is a p-series with $p = \pi > 1$

Therefore; by the p-series, the given series ***converges***.

Exercise

Determine if the series converge or diverge $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots$

Solution

$$1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots = \sum_{1}^{\infty} \frac{1}{n^{3/2}}$$

This is a p-series with $p = \frac{3}{2} > 1$

Therefore; by the p-series, the given series ***converges***.

Exercise

Determine if the series converge or diverge $1 + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{9}} + \frac{1}{\sqrt[3]{16}} + \frac{1}{\sqrt[3]{25}} + \cdots$

Solution

$$1 + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{9}} + \frac{1}{\sqrt[3]{16}} + \frac{1}{\sqrt[3]{25}} + \cdots = \sum_{1}^{\infty} \frac{1}{n^{2/3}}$$

This is a p-series with $p = \frac{2}{3} < 1$

Therefore; by the p-series, the given series ***diverges***.

Exercise

Consider the series $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$, where p is a real number.

a) Use the Integral Test to determine the values of p for which this series converges.

b) Does this series converge faster for $p = 2$ or $p = 3$? Explain.

Solution

a) Let $f(x) = \frac{1}{x(\ln x)^p}$

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\, dx = \int_2^{\infty} (\ln x)^{-p}\, d(\ln x)$$

$$= \frac{1}{1-p}(\ln x)^{1-p} \Big|_2^{\infty}$$

In order for the integral to exist doe the given series to converge, then the value(s) of p:

$1 - p < 0 \quad \rightarrow \quad \underline{p > 1}$

b) Since series converges for $p > 1$

For $p = 2$

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx = \int_2^{\infty} \frac{1}{(\ln x)^2}\, d(\ln x)$$

$$= -\frac{1}{\ln x} \Big|_2^{\infty}$$

$$= -\left(0 - \frac{1}{\ln 2}\right)$$

$$= \frac{1}{\ln 2}$$

For $p = 3$

$$\int_2^{\infty} \frac{1}{x(\ln x)^3}\, dx = \int_2^{\infty} (\ln x)^{-3}\, d(\ln x)$$

$$= -\frac{1}{2}\frac{1}{(\ln x)^2} \Big|_2^{\infty}$$

$$= -\frac{1}{2}\left(0 - \frac{1}{(\ln 2)^2}\right)$$

$$= \frac{1}{2}\frac{1}{\ln^2 2}$$

$p=2$	$p=3$
$\frac{1}{\ln 2}$	$\frac{1}{2}\frac{1}{\ln^2 2}$
1	$\frac{1}{2}\frac{1}{\ln 2}$

From the table, the value of $p=3$ is smaller than $p=2$

Therefore; the series converges faster for $p=3$.

Exercise

Consider the series $\sum\limits_{k=2}^{\infty}\dfrac{1}{k(\ln k)\ (\ln(\ln k))^p}$, where p is a real number.

a) For what values of p does this series converge?

b) Which he following series converge faster? Explain.

$$\sum_{k=2}^{\infty}\frac{1}{k(\ln k)^2} \quad \text{or} \quad \sum_{k=2}^{\infty}\frac{1}{k(\ln k)\ (\ln(\ln k))^2}$$

Solution

a) Let $f(x)=\dfrac{1}{x\ln x\ (\ln(\ln x))^p}$

$$\int_2^{\infty}\frac{1}{x\ln x\ (\ln(\ln x))^p}=\int_2^{\infty}(\ln(\ln x))^{-p}\ d(\ln(\ln x)) \qquad d(\ln(\ln x))=\frac{1}{\ln x}\frac{1}{x}dx$$

$$=\frac{1}{1-p}(\ln(\ln x))^{1-p}\ \Big|_2^{\infty}$$

In order for the integral to exist doe the given series to converge, then the value(s) of p:

$1-p<0 \quad \to \quad \boxed{p>1}$

b) $$\int_2^{\infty}\frac{1}{x\ln x\ (\ln(\ln x))^2}=-\frac{1}{\ln\ln x}\Big|_2^{\infty}$$

$$=-\left(0-\frac{1}{\ln\ln 2}\right)$$

$$=\boxed{\frac{1}{\ln\ln 2}}$$

$$\int_2^{\infty}\frac{1}{x(\ln x)^2}\ dx=\int_2^{\infty}\frac{1}{(\ln x)^2}\ d(\ln x)$$

$$= -\frac{1}{\ln x}\Big|_2^{\infty}$$

$$= -\left(0 - \frac{1}{\ln 2}\right)$$

$$= \frac{1}{\ln 2}$$

$\ln 2 > \ln \ln 2$

$\frac{1}{\ln 2} < \frac{1}{\ln \ln 2}$

Therefore; the first series converges faster because the terms get smaller faster.

Exercise

Consider a wedding cake of infinite height, each layer of which is a right circular cylinder of height 1. The bottom layer of the cake has a radius of 1, the second layer has a radius of $\frac{1}{2}$, the third layer has a radius of $\frac{1}{3}$, and the n^{th} layer has a radius of $\frac{1}{n}$.

a) To determine how much frosting is needed to cover the cake, find the area of the lateral (vertical sides of the wedding cake. What is the area of the horizontal surfaces of the cake?

b) Determine the volume of the cake.

c) Comment on your answer to parts (a) and (b)

Solution

a) The circumference of the k^{th} layer is: $2\pi\frac{1}{k}$, so its area $\frac{2\pi}{k}$

The total vertical surface area:

$$\sum_{k=1}^{\infty}\frac{2\pi}{k} = 2\pi\sum_{k=1}^{\infty}\frac{1}{k}$$

$$= \infty$$

Looking at the cake from above, the horizontal area

$$Area = \pi r^2 = \pi \cdot 1^2$$

$$= \pi$$

b) The volume of a cylinder $= \pi r^2 h$

Volume of the k^{th} layer $= \pi\frac{1}{k^2}\cdot 1 = \frac{\pi}{k^2}$

Thus the volume of the cake is:

$$\sum_{k=1}^{\infty}\frac{\pi}{k^2} = \pi\sum_{k=1}^{\infty}\frac{1}{k^2}$$

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6} \quad (Leonhard\ \ Euler)$$

$$= \frac{\pi^3}{6}$$

$$\approx 5.168$$

c) This cake has infinite area, it has finite volume.

Exercise

The Riemann zeta function is the subject of extensive research and is associated with several renowned unsolved problems. Is its defined by $\zeta(x) = \sum_{k=1}^{\neq} \frac{1}{k^x}$, when x is a real number, the zeta function becomes a p-series. For even positive integers ρ, the value of $\zeta(\rho)$ is known exactly. For example,

$$\sum_{k=1}^{\neq} \frac{1}{k^2} = \frac{\pi^2}{6}, \quad \sum_{k=1}^{\neq} \frac{1}{k^4} = \frac{\pi^4}{90}, \quad \text{and} \quad \sum_{k=1}^{\neq} \frac{1}{k^6} = \frac{\pi^6}{945}, \quad \ldots$$

a) Use the estimation techniques to approximate $\zeta(3)$ and $\zeta(5)$ (whose values are not known exactly) with a remainder less than 10^{-3}.

b) Determine the sum of the reciprocals of the squares of the odd positive integers by rearranging the terms of the series $(x = 2)$ without changing the value of the series.

Solution

a) $\zeta(m) = \int_n^{\infty} \frac{1}{x^m} dx = \frac{1}{m-1} x^{1-m} \Big|_n^{\infty} \quad = \frac{1}{m-1} n^{1-m}$

For $\zeta(3) = \frac{1}{2} n^{-2} < 10^{-3}$

$$\frac{1}{2n^2} < \frac{1}{10^3}$$

$$2n^2 > 10^3$$

$$n > \sqrt{500} \approx 23$$

$$\sum_{k=1}^{23} \frac{1}{k^3} \approx 1.201151955$$

```
for i = 1:n
    kk = 1 / (i^x);
    k = k + kk;
end
```

The true value is ≈ 1.202056903

For $\zeta(5) = \frac{1}{4} n^{-4} < 10^{-3}$

$$\frac{1}{4n^4} < \frac{1}{10^3}$$

$$4n^4 > 10^3$$

$$n > (250)^{1/4} \approx 4$$

$$\sum_{k=1}^{4} \frac{1}{k^5} \approx 1.0363417888$$

The true value is ≈ 1.036927755

b) $$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

$$= \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

$$\frac{3}{4}\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{3}{4}\frac{\pi^2}{6}$$

$$= \frac{\pi^2}{8}$$

Exercise

Consider a set of identical dominoes that are 2 inches long. The dominoes are stacked on top of each other with their long edges aligned so that each domino overhangs the one beneath is as far as possible

a) If there are n dominoes in the stack, what is the greatest distance that the top domino can be made to overhang the bottom domino? (*Hint*: Put the n^{th} domino beneath the previous $n-1$ dominoes.)

b) If we allow for infinitely many dominoes in the stack, what is the greatest distance that the top domino can be made to overhang the bottom domino?

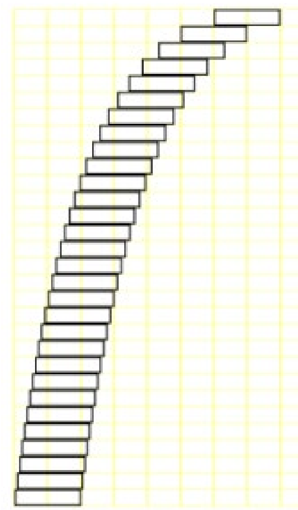
Solution

a) The center of gravity of any stack of dominoes is the average of the locations of their centers.

Define the midpoint of the zeroth (top) domino to be $x=0$, and stack additional dominoes down and to its right (to increasingly positive x-coordinates).

Let $m(n)$ be the x-coordinate of the midpoint of the n^{th} domino. Then in order for the stack not to fall over, the left edge of the n^{th} domino must be placed directly under the center of gravity of dominos 0 through $n-1$, which is $\frac{1}{n}\sum_{i=0}^{n-1} m(i)$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$$



So $m(n)=1+\frac{1}{n}\sum_{i=0}^{n-1} m(i)=\sum_{k=1}^{n}\frac{1}{k}$

Proof by induction;

For $n=1 \Rightarrow m(1)=1$ ✓ P_1 is true

Let P_j is true $m(j)=\sum_{k=1}^{j}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{j}$;

we need to prove it is also true for P_{j+1}

$$\begin{aligned} m(j+1) &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{j}+\frac{1}{j+1} \\ &= \sum_{k=1}^{j}\frac{1}{k}+\frac{1}{j+1} \\ &= \sum_{k=1}^{j+1}\frac{1}{k} \end{aligned}$$

Therefore; the formula is clearly true by mathematical induction.

b) For infinite number of dominos, because the overhang is the harmonic series, the distance is potentially infinite. This series diverges so with enough dominoes.

Exercise

A theorem states that the sequence of prime numbers $\{p_k\}$ satisfies $\lim_{k\to\infty}\frac{p_k}{k\ln k}$.

Show that $\sum_{k=2}^{\infty}\frac{1}{k\ln k}$ diverges, which implies that the series $\sum_{k=1}^{\infty}\frac{1}{p_k}$

(A prime number is a positive integer number that is divisible only by 1 and itself).

Solution

Let $f(x) = \frac{1}{x\ln x}$

$$\int_2^{\infty} \frac{1}{x\ln x}dx = \int_2^{\infty} \frac{1}{\ln x}d(\ln x)$$

$$= \ln|\ln x| \Big|_2^{\infty}$$

$$= \infty - \ln\ln 2$$

$$= \infty$$

Therefore; by the *Integral Test*, the given series ***diverges***.