

Solution **Section 2.1 – Integration by Parts**

Exercise

Evaluate the integral $\int x \ln x \, dx$

Solution

$$\text{Let: } u = \ln x \Rightarrow du = \frac{1}{x} dx$$

$$dv = x dx \Rightarrow v = \int dv = \int x dx = \frac{1}{2} x^2$$

$$\begin{aligned} \int x \ln x dx &= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x^2 \frac{dx}{x} \\ &= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x dx \\ &= \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C \end{aligned}$$

Exercise

Evaluate the integral $\int \ln x^2 dx$

Solution

$$\begin{aligned} \int \ln x^2 dx &= 2 \int \ln x dx & u = \ln x \Rightarrow du = \frac{1}{x} dx & v = \int dx = x \\ \int \ln x^2 dx &= 2 \left[x \ln x - \int x \frac{1}{x} dx \right] \\ &= 2 \left[x \ln x - \int dx \right] \\ &= 2(x \ln x - x) + C \\ &= \underline{2x(\ln x - 1) + C} \end{aligned}$$

Exercise

Evaluate the integral $\int \ln(3x) dx$

Solution

$$u = \ln 3x \Rightarrow du = \frac{3}{3x} dx = \frac{1}{x} dx$$

$$dv = dx \Rightarrow v = x$$

$$\begin{aligned} \int \ln(3x) dx &= x \ln(3x) - \int x \frac{1}{x} dx \\ &= x \ln(3x) - \int dx \\ &= x \ln(3x) - x + C \\ &= \underline{x[\ln(3x) - 1] + C} \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{1}{x \ln x} dx$

Solution

$$\int \frac{1}{x \ln x} dx = \int \frac{1}{\ln x} \frac{1}{x} dx$$

$$u = \ln x \Rightarrow du = \frac{1}{x} dx$$

$$\begin{aligned} \int \frac{1}{x \ln x} dx &= \int \frac{1}{u} du \\ &= \ln u + C \\ &= \underline{\ln |\ln x| + C} \end{aligned}$$

Exercise

Evaluate the integrals $\int x(\ln x)^2 dx$

Solution

$$u = \ln x \rightarrow x = e^u$$

$$du = \frac{1}{x} dx \Rightarrow x du = dx \rightarrow dx = e^u du$$

$$\begin{aligned} \int x(\ln x)^2 dx &= \int e^u u^2 e^u du \\ &= \int u^2 e^{2u} du \\ &= \frac{1}{2} u^2 e^{2u} - \frac{1}{2} u e^{2u} + \frac{1}{4} e^{2u} + C \end{aligned}$$

		$\int e^{2u} du$
+	u^2	$\frac{1}{2} e^{2u}$
-	$2u$	$\frac{1}{4} e^{2u}$
+	2	$\frac{1}{8} e^{2u}$
-	0	

$$= \frac{1}{4} e^{2u} (2u^2 - 2u + 1) + C$$

$$\underline{= \frac{1}{4} x^2 (2(\ln x)^2 - 2 \ln x + 1) + C}$$

2nd Method

$$u = \ln x \quad dv = \int (x \ln x) dx$$

$$du = \frac{1}{x} dx \quad v = \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2$$

$$\int x (\ln x)^2 dx = (\ln x) \left(\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 \right) - \int \left(\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 \right) \frac{1}{x} dx$$

$$= \frac{1}{2} x^2 (\ln x)^2 - \frac{1}{4} x^2 \ln x - \int \left(\frac{1}{2} x \ln x - \frac{1}{4} x \right) dx$$

$$u = \ln x \Rightarrow du = \frac{1}{x} dx$$

$$dv = x dx \Rightarrow v = \frac{1}{2} x^2$$

$$\int x \ln x dx = \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x^2 \frac{dx}{x}$$

$$= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x dx$$

$$= \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2$$

$$= \frac{1}{2} x^2 (\ln x)^2 - \frac{1}{4} x^2 \ln x - \left(\frac{1}{2} \left(\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 \right) - \frac{1}{8} x^2 \right) + C$$

$$= \frac{1}{2} x^2 (\ln x)^2 - \frac{1}{4} x^2 \ln x - \frac{1}{4} x^2 \ln x + \frac{1}{8} x^2 + \frac{1}{8} x^2 + C$$

$$\underline{= \frac{1}{2} x^2 (\ln x)^2 - \frac{1}{2} x^2 \ln x + \frac{1}{4} x^2 + C}$$

3rd Method

$$u = (\ln x)^2 \quad dv = \int x dx$$

$$du = 2(\ln x) \frac{1}{x} dx \quad v = \frac{1}{2} x^2$$

$$\int x (\ln x)^2 dx = \frac{1}{2} x^2 (\ln x)^2 - \int \frac{1}{2} x^2 (2 \ln x) \frac{1}{x} dx$$

$$= \frac{1}{2} x^2 (\ln x)^2 - \int x \ln x dx$$

$$u = \ln x \Rightarrow du = \frac{1}{x} dx$$

$$\begin{aligned}
dv &= x dx \Rightarrow v = \frac{1}{2}x^2 \\
\int x \ln x dx &= \frac{1}{2}x^2 \ln x - \frac{1}{2} \int x^2 \frac{dx}{x} \\
&= \frac{1}{2}x^2 \ln x - \frac{1}{2} \int x dx \\
&= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 \\
&= \frac{1}{2}x^2 (\ln x)^2 - \frac{1}{4}x^2 \ln x - \left(\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 \right) + C \\
&= \frac{1}{2}x^2 (\ln x)^2 - \frac{1}{4}x^2 \ln x - \frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + C \\
&= \frac{1}{2}x^2 (\ln x)^2 - \frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + C
\end{aligned}$$

Exercise

Evaluate the integral $\int x^2 (\ln x)^2 dx$

Solution

$$\begin{aligned}
u &= (\ln x)^2 \quad v = \int x^2 dx \\
du &= 2 \frac{\ln x}{x} dx \quad v = \frac{1}{3}x^3 \\
\int x^2 (\ln x)^2 dx &= \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{3} \int x^2 \ln x dx \\
u &= \ln x \quad v = \int x^2 dx \\
du &= \frac{1}{x} dx \quad v = \frac{1}{3}x^3 \\
\int x^2 (\ln x)^2 dx &= \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{3} \left(\frac{1}{3}x^3 \ln x - \frac{1}{3} \int x^2 dx \right) \\
&= \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{9}x^3 \ln x + \frac{2}{27}x^3 + C \\
&= \frac{1}{27}x^3 (9 \ln^2 x - 6 \ln x + 2) + C
\end{aligned}$$

Or

$$\begin{aligned}
\text{Let } y &= \ln x \Rightarrow x = e^y \\
dx &= e^y dy
\end{aligned}$$

$$\int x^2 (\ln x)^2 dx = \int (e^y)^2 y^2 e^y dy$$

$$= \int y^2 e^{3y} dy$$

		$\int e^{3y} dy$
+	y^2	$\frac{1}{3} e^{3y}$
-	$2y$	$\frac{1}{9} e^{3y}$
+	2	$\frac{1}{27} e^{3y}$

$$\int x^2 (\ln x)^2 dx = e^{3y} \left(\frac{1}{3} y^2 - \frac{2}{9} y + \frac{2}{27} \right) + C$$

$$= x^3 \left(\frac{1}{3} \ln^2 x - \frac{2}{9} \ln x + \frac{2}{27} \right) + C$$

$$= \frac{1}{27} x^3 (9 \ln^2 x - 6 \ln x + 2) + C$$

Exercise

Evaluate the integral $\int \frac{(\ln x)^3}{x} dx$

Solution

$$\int \frac{(\ln x)^3}{x} dx = \int (\ln x)^3 d(\ln x)$$

$$= \frac{1}{4} (\ln x)^4 + C$$

$$d(\ln x) = \frac{dx}{x}$$

Exercise

Evaluate the integral $\int x^2 \ln x^3 dx$

Solution

$$u = \ln x \quad v = \int 3x^2 dx = x^3$$

$$du = \frac{1}{x} dx$$

$$\int u dv = uv - \int v du$$

$$\int x^2 \ln x^3 dx = \int 3x^2 \ln x dx$$

$$\begin{aligned}
 &= x^3 \ln x - \int x^2 dx \\
 &= x^3 \ln x - \frac{1}{3} x^3 + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \ln(x + x^2) dx$

Solution

$$u = \ln(x + x^2) \quad dv = dx$$

Let:

$$du = \frac{2x+1}{x+x^2} dx \quad v = x$$

$$\begin{aligned}
 \int \ln(x + x^2) dx &= x \ln(x + x^2) - \int x \frac{2x+1}{x+x^2} dx \\
 &= x \ln(x + x^2) - \int \frac{2x+1}{x(1+x)} x dx \\
 &= x \ln(x + x^2) - \int \frac{2x+2-1}{1+x} dx \\
 &= x \ln(x + x^2) - \int \frac{2(x+1)-1}{x+1} dx \\
 &= x \ln(x + x^2) - \int \left(2 - \frac{1}{x+1} \right) dx \\
 &= x \ln(x + x^2) - (2x - \ln|x+1|) + C \\
 &= x \ln(x + x^2) - 2x + \ln|x+1| + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int x \ln(x+1) dx$

Solution

$$u = \ln(x+1) \Rightarrow du = \frac{1}{x+1} dx$$

$$dv = x dx \Rightarrow v = \frac{1}{2} x^2$$

$$\int x \ln(x+1) dx = \frac{1}{2} x^2 \ln(x+1) - \frac{1}{2} \int \frac{x^2}{x+1} dx$$

$$\begin{aligned}
&= \frac{1}{2} x^2 \ln(x+1) - \frac{1}{2} \int \left(x - 1 + \frac{1}{x+1} \right) dx \\
&= \frac{1}{2} x^2 \ln(x+1) - \frac{1}{2} \left(\frac{1}{2} x^2 - x + \ln(x+1) \right) + C \\
&= \frac{1}{2} x^2 \ln(x+1) - \frac{1}{4} x^2 + \frac{1}{2} x - \frac{1}{2} \ln(x+1) + C \\
&= \underline{-\frac{1}{4} x^2 + \frac{1}{2} x + \frac{1}{2} (x^2 - 1) \ln(x+1) + C}
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{(\ln x)^2}{x} dx$

Solution

$$\begin{aligned}
\int \frac{(\ln x)^2}{x} dx &= \int (\ln x)^2 d(\ln x) \\
&= \underline{\frac{1}{3} (\ln x)^3 + C}
\end{aligned}$$

Exercise

Evaluate the integral $\int x^5 \ln 3x dx$

Solution

		$\int x^5 dx$
+	$\ln 3x$	$\frac{1}{6} x^6$
-	$\frac{1}{x}$	$\int \frac{1}{6} x^6$

$$\begin{aligned}
\int x^5 \ln 3x dx &= \frac{1}{6} x^6 \ln 3x - \frac{1}{6} \int x^6 \frac{1}{x} dx \\
&= \frac{1}{6} x^6 \ln 3x - \frac{1}{6} \int x^5 dx \\
&= \underline{\frac{1}{6} x^6 \ln 3x - \frac{1}{36} x^6 + C}
\end{aligned}$$

Exercise

Evaluate the integral $\int x^5 \ln x \, dx$

Solution

		$\int x^5 \, dx$
+	$\ln x$	$\frac{1}{6} x^6$
-	$\frac{1}{x}$	$\int \frac{1}{6} x^6$

$$\begin{aligned}\int x^5 \ln x \, dx &= \frac{1}{6} x^6 \ln x - \frac{1}{6} \int x^6 \frac{1}{x} \, dx \\ &= \frac{1}{6} x^6 \ln x - \frac{1}{6} \int x^5 \, dx \\ &= \frac{1}{6} x^6 \ln x - \frac{1}{36} x^6 + C\end{aligned}$$

Exercise

Evaluate the integral $\int \ln(x+1) \, dx$

Solution

		$\int dx$
+	$\ln(x+1)$	$\frac{1}{2} x$
-	$\frac{1}{x+1}$	$\frac{1}{2} \int x$

$$\begin{aligned}\int \ln(x+1) \, dx &= \frac{1}{2} x \ln(x+1) - \frac{1}{2} \int \frac{x}{x+1} \, dx \\ &= \frac{1}{2} x \ln(x+1) - \frac{1}{2} \int \left(1 - \frac{1}{x+1}\right) \, dx \\ &= \frac{1}{2} x \ln(x+1) - \frac{1}{2} (x - \ln(x+1)) + C \\ &= \frac{1}{2} x \ln(x+1) - \frac{1}{2} x + \frac{1}{2} \ln(x+1) + C \\ &= \frac{1}{2} (x+1) \ln(x+1) - \frac{1}{2} x + C\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\ln x}{x^{10}} dx$

Solution

		$\int x^{-10} dx$
+	$\ln x$	$-\frac{1}{9} x^{-9}$
-	$\frac{1}{x}$	$-\frac{1}{9} \int x^{-9}$

$$\begin{aligned}\int \frac{\ln x}{x^{10}} dx &= -\frac{1}{9x^9} \ln x + \frac{1}{9} \int \frac{1}{x} x^{-9} dx \\ &= -\frac{1}{9x^9} \ln x + \frac{1}{9} \int x^{-10} dx \\ &= -\frac{\ln x}{9x^9} - \frac{1}{81x^9} + C\end{aligned}$$

Exercise

Evaluate the integral $\int x e^{2x} dx$

Solution

Let: $u = x \Rightarrow du = dx$

$$dv = e^{2x} dx \Rightarrow v = \int dv = \int e^{2x} dx = \frac{1}{2} e^{2x}$$

$$\int u dv = uv - \int v du$$

$$\begin{aligned}\int x e^{2x} dx &= \frac{1}{2} x e^{2x} - \int \frac{1}{2} e^{2x} dx \\ &= \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C\end{aligned}$$

Or

		$\int e^{2x} dx$
+	x	$\frac{1}{2} e^{2x}$
-	1	$\frac{1}{4} e^{2x}$

$$\int x e^{2x} dx = \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C$$

Exercise

Evaluate the integral $\int x^3 e^x dx$

Solution

		$\int e^x dx$
+	x^3	e^x
-	$3x^2$	e^x
+	$6x$	e^x
-	6	e^x

$$\int x^3 e^x dx = \underline{e^x (x^3 - 3x^2 + 6x - 6) + C}$$

Or

Let: $u = x^3 \Rightarrow du = 3x^2 dx$

$$dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$\begin{aligned} \int x^3 e^x dx &= x^3 e^x - \int e^x 3x^2 dx \\ &= x^3 e^x - 3 \int e^x x^2 dx \end{aligned}$$

Let: $u = x^2 \Rightarrow du = 2x dx$

$$dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$\int e^x x^2 dx = x^2 e^x - 2 \int x e^x dx$$

$$\begin{aligned} \int x^3 e^x dx &= x^3 e^x - 3 \left[x^2 e^x - 2 \int x e^x dx \right] \\ &= x^3 e^x - 3x^2 e^x + 6 \int x e^x dx \end{aligned}$$

Let: $u = x \Rightarrow du = dx$

$$dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x$$

$$\begin{aligned} \int x^3 e^x dx &= x^3 e^x - 3x^2 e^x + 6 \left[x e^x - e^x \right] + C \\ &= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C \end{aligned}$$

$$= e^x (x^3 - 3x^2 + 6x - 6) + C$$

Exercise

Evaluate the integral $\int \frac{2x}{e^x} dx$

Solution

		$\int e^{-x} dx$
+	$2x$	$-e^{-x}$
-	2	e^{-x}

$$\int \frac{2x}{e^x} dx = -e^{-x} (2x + 2) + C$$

Or

$$u = 2x \Rightarrow du = 2dx$$

$$dv = e^{-x} dx \Rightarrow v = -e^{-x}$$

$$\begin{aligned} \int \frac{2x}{e^x} dx &= 2x(-e^{-x}) - \int -e^{-x} 2dx \\ &= -2xe^{-x} + 2 \int e^{-x} dx \\ &= -2xe^{-x} - 2e^{-x} + C \\ &= -2e^{-x}(x+1) + C \\ &= -\frac{2(x+1)}{e^x} + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{x^3 e^{x^2}}{(x^2 + 1)^2} dx$

Solution

$$\begin{aligned} \text{Let: } u = x^2 e^{x^2} &\Rightarrow du = (2xe^{x^2} + 2xx^2 e^{x^2}) dx \\ &du = 2xe^{x^2} (1 + x^2) dx \end{aligned}$$

$$\begin{aligned}
 dv = x(x^2 + 1)^{-2} dx &\Rightarrow v = \int x(x^2 + 1)^{-2} dx \\
 &= \frac{1}{2} \int (x^2 + 1)^{-2} d(x^2 + 1) \\
 &= \frac{(x^2 + 1)^{-1}}{-1} \\
 &= -\frac{1}{2(x^2 + 1)}
 \end{aligned}$$

$$\begin{aligned}
 \int \frac{x^3 e^{x^2}}{(x^2 + 1)^2} dx &= x^2 e^{x^2} \left(-\frac{1}{2(x^2 + 1)} \right) - \int -\frac{1}{2(x^2 + 1)} 2x e^{x^2} (x^2 + 1) dx \\
 &= -\frac{x^2 e^{x^2}}{2(x^2 + 1)} + \int x e^{x^2} dx
 \end{aligned}$$

$$\text{Let: } u = x^2 \Rightarrow du = 2x dx$$

$$\begin{aligned}
 \int \frac{x^3 e^{x^2}}{(x^2 + 1)^2} dx &= -\frac{x^2 e^{x^2}}{2(x^2 + 1)} + \frac{1}{2} \int e^u du \\
 &= -\frac{x^2 e^{x^2}}{2(x^2 + 1)} + \frac{1}{2} e^u + C \\
 &= -\frac{x^2 e^{x^2}}{2(x^2 + 1)} + \frac{1}{2} e^{x^2} + C \\
 &= \frac{1}{2} e^{x^2} \left[-\frac{x^2}{(x^2 + 1)} + 1 \right] + C \\
 &= \frac{1}{2} e^{x^2} \left[\frac{-x^2 + x^2 + 1}{(x^2 + 1)} \right] + C \\
 &= \frac{e^{x^2}}{2(x^2 + 1)} + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int x^2 e^{-3x} dx$

Solution

$$u = x^2 \Rightarrow du = 2x dx$$

$$dv = e^{-3x} dx \Rightarrow v = -\frac{1}{3} e^{-3x}$$

$$\int x^2 e^{-3x} dx = -\frac{1}{3} x^2 e^{-3x} + \frac{2}{3} \int x e^{-3x} dx$$

$$u = x \Rightarrow du = dx$$

$$dv = e^{-3x} dx \Rightarrow v = -\frac{1}{3} e^{-3x}$$

$$\int x^2 e^{-3x} dx = -\frac{1}{3} x^2 e^{-3x} + \frac{2}{3} \left[-\frac{1}{3} x e^{-3x} + \frac{1}{3} \int e^{-3x} dx \right]$$

$$= -\frac{1}{3} x^2 e^{-3x} + \frac{2}{3} \left(-\frac{1}{3} x e^{-3x} - \frac{1}{9} e^{-3x} \right) + C$$

$$= -\frac{1}{3} x^2 e^{-3x} - \frac{2}{9} x e^{-3x} - \frac{2}{27} e^{-3x} + C$$

$$= \underline{-\frac{9x^2 + 6x + 2}{27} e^{-3x} + C}$$

Or

$\int e^{-3x}$		
+	x^2	$-\frac{1}{3} e^{-3x}$
-	$2x$	$\frac{1}{9} e^{-3x}$
+	2	$-\frac{1}{27} e^{-3x}$

$$\int x^2 e^{-3x} dx = \underline{-\frac{1}{3} x^2 e^{-3x} - \frac{2}{9} x e^{-3x} - \frac{2}{27} e^{-3x} + C}$$

Exercise

Evaluate the integral $\int (x^2 - 2x + 1)e^{2x} dx$

Solution

$\int e^{2x}$		
+	$x^2 - 2x + 1$	$\frac{1}{2}e^{2x}$
-	$2x - 2$	$\frac{1}{4}e^{2x}$
+	2	$\frac{1}{8}e^{2x}$

$$\begin{aligned}
 \int (x^2 - 2x + 1)e^{2x} dx &= \frac{1}{2}(x^2 - 2x + 1)e^{2x} - \frac{1}{4}(2x - 2)e^{2x} + \frac{1}{8}(2)e^{2x} + C \\
 &= \left(\frac{1}{2}x^2 - x + \frac{1}{2} - \frac{1}{2}x + \frac{1}{2} + \frac{1}{4} \right) e^{2x} + C \\
 &= \left(\frac{1}{2}x^2 - \frac{3}{2}x + \frac{5}{4} \right) e^{2x} + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int x^5 e^{x^3} dx$

Solution

Let: $u = x^3 \quad dv = x^2 e^{x^3} dx = \frac{1}{3} d(e^{x^3}) \quad d(e^{x^3}) = 3x^2 e^{x^3} dx$

$du = 3x^2 dx \quad v = \frac{1}{3} e^{x^3}$

$$\begin{aligned}
 \int x^5 e^{x^3} dx &= x^3 \frac{1}{3} e^{x^3} - \int \frac{1}{3} e^{x^3} 3x^2 dx & d(e^{x^3}) &= 3x^2 e^{x^3} dx & \int u dv &= uv - \int v du \\
 &= \frac{1}{3} x^3 e^{x^3} - \frac{1}{3} \int d(e^{x^3}) \\
 &= \frac{1}{3} x^3 e^{x^3} - \frac{1}{3} e^{x^3} + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int x e^{-4x} dx$

Solution

		$\int e^{-4x} dx$
+	x	$-\frac{1}{4}e^{-4x}$
-	1	$\frac{1}{16}e^{-4x}$

$$\int x e^{-4x} dx = \left(-\frac{x}{4} - \frac{1}{16} \right) e^{-4x} + C$$

Exercise

Evaluate the integral $\int \frac{x e^{2x}}{(2x+1)^2} dx$

Solution

$$u = x e^{2x} \rightarrow du = (2x+1) e^{2x} dx$$

$$dv = \frac{dx}{(2x+1)^2} = \frac{1}{2} \frac{d(2x+1)}{(2x+1)^2} \rightarrow v = -\frac{1}{2} \frac{1}{2x+1}$$

$$\begin{aligned} \int \frac{x e^{2x}}{(2x+1)^2} dx &= -\frac{x e^{2x}}{4x+2} + \frac{1}{2} \int e^{2x} dx \\ &= -\frac{x}{4x+2} e^{2x} + \frac{1}{4} e^{2x} + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{5x}{e^{2x}} dx$

Solution

		$\int e^{-2x} dx$
+	$5x$	$-\frac{1}{2}e^{-2x}$
-	5	$\frac{1}{4}e^{-2x}$

$$\begin{aligned} \int \frac{5x}{e^{2x}} dx &= \int 5x e^{-2x} dx \\ &= \left(-\frac{5}{2}x - \frac{5}{4} \right) e^{-2x} + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{e^{1/x}}{x^2} dx$

Solution

$$\int \frac{e^{1/x}}{x^2} dx = - \int e^{1/x} d\left(\frac{1}{x}\right) \\ = -e^{1/x} + C$$

Exercise

Evaluate the integral $\int x^2 e^{4x} dx$

Solution

$$\int x^2 e^{4x} dx = \left(\frac{1}{4} x^2 - \frac{1}{8} x + \frac{1}{32} \right) e^{4x} + C$$

$$\int x^n e^{ax} dx = e^{ax} \sum_{k=0}^n \frac{(-1)^k n!}{(n-k)! a^{k+1}} x^{n-k}$$

Exercise

Evaluate the integral $\int x^3 e^{-3x} dx$

Solution

$$\int x^3 e^{-3x} dx = \left(-\frac{1}{3} x^3 + \frac{1}{3} x^2 - \frac{2}{9} x + \frac{2}{27} \right) e^{-3x} + C$$

$$\int x^n e^{ax} dx = e^{ax} \sum_{k=0}^n \frac{(-1)^k n!}{(n-k)! a^{k+1}} x^{n-k}$$

Exercise

Evaluate the integral $\int x^4 e^x dx$

Solution

		$\int e^x dy$
+	x^4	e^x
-	$4x^3$	e^x
+	$12x^2$	e^x
-	$24x$	e^x
+	24	e^x

$$\int x^4 e^x dx = \left(x^4 + 4x^3 + 12x^2 + 24x + 24 \right) e^x + C$$

Exercise

Evaluate the integral $\int x^3 e^{4x} dx$

Solution

$$\int x^3 e^{4x} dx = e^{4x} \left(\frac{1}{4} x^3 - \frac{3}{16} x^2 + \frac{3}{32} x - \frac{3}{128} \right) + C$$

$$\int x^n e^{ax} dx = e^{ax} \sum_{k=0}^n \frac{(-1)^k n!}{(n-k)! a^{k+1}} x^{n-k}$$

Exercise

Evaluate the integral $\int (x+1)^2 e^x dx$

Solution

		$\int e^x dx$
+	$(x+1)^2$	e^x
-	$2(x+1)$	e^x
+	2	e^x

$$\begin{aligned} \int (x+1)^2 e^x dx &= e^x \left[(x+1)^2 - 2(x+1) + 2 \right] + C \\ &= e^x (x^2 + 2x + 1 - 2x - 2 + 2) + C \\ &= e^x (x^2 + 1) + C \end{aligned}$$

Exercise

Evaluate the integral $\int 2xe^{3x} dx$

Solution

		$\int e^{3x} dx$
+	$2x$	$\frac{1}{3} e^{3x}$
-	2	$\frac{1}{9} e^{3x}$

$$\begin{aligned} \int 2xe^{3x} dx &= e^{3x} \left(\frac{2}{3} x - \frac{2}{9} \right) + C \\ &= \frac{2}{9} e^{3x} (3x - 1) + C \end{aligned}$$

Exercise

Evaluate the integral $\int x^2 \sin x \, dx$

Solution

$\int \sin x$		
x^2	$(+)$	$-\cos x$
$2x$	$(-)$	$-\sin x$
2	$(+)$	$\cos x$
0		

$$\int x^2 \sin x \, dx = \underline{-x^2 \cos x + 2x \sin x + 2 \cos x + C}$$

Exercise

Evaluate the integral $\int \theta \cos \pi \theta \, d\theta$

Solution

Let: $u = \theta$ $dv = \cos \pi \theta \, d\theta$
 $du = d\theta$ $v = \int \cos \pi \theta \, d\theta = \frac{1}{\pi} \sin \pi \theta$

$$\begin{aligned} \int \theta \cos \pi \theta \, d\theta &= \frac{\theta}{\pi} \sin \pi \theta - \int \frac{1}{\pi} \sin \pi \theta \, d\theta \\ &= \frac{\theta}{\pi} \sin \pi \theta + \frac{1}{\pi} \frac{1}{\pi} \cos \pi \theta + C \\ &= \underline{\frac{\theta}{\pi} \sin \pi \theta + \frac{1}{\pi^2} \cos \pi \theta + C} \end{aligned}$$

Exercise

Evaluate the integral $\int 4x \sec^2 2x \, dx$

Solution

Let: $u = 4x \rightarrow du = 4$ $dv = \sec^2 2x \, dx \rightarrow v = \frac{1}{2} \tan 2x$

$$\int 4x \sec^2 2x \, dx = 2x \tan 2x - \int 4 \left(\frac{1}{2} \tan 2x \right) dx$$

$$= 2x \tan 2x - 2 \frac{1}{2} \ln |\sec 2x| + C$$

$$= 2x \tan 2x - \ln |\sec 2x| + C$$

Exercise

Evaluate the integral $\int x^3 \sin x \, dx$

Solution

		$\int \sin x$
+	x^3	$-\cos x$
-	$3x^2$	$-\sin x$
+	$6x$	$\cos x$
-	6	$\sin x$

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$$

Exercise

Evaluate the integral $\int (x^3 - 2x) \sin 2x \, dx$

Solution

		$\int \sin 2x dx$
+	$x^3 - 2x$	$-\frac{1}{2} \cos 2x$
-	$3x^2 - 2$	$-\frac{1}{4} \sin 2x$
+	$6x$	$\frac{1}{8} \cos 2x$
-	6	$\frac{1}{16} \sin 2x$

$$\int (x^3 - 2x) \sin 2x \, dx = -\frac{1}{2} (x^3 - 2x) \cos 2x + \frac{1}{4} (3x^2 - 2) \sin 2x + \frac{3}{4} x \cos 2x - \frac{3}{8} \sin 2x + C$$

$$= \left(-\frac{1}{2} x^3 + x + \frac{3}{4} x \right) \cos 2x + \left(\frac{3}{4} x^2 - \frac{1}{2} - \frac{3}{8} \right) \sin 2x + C$$

$$= \left(-\frac{1}{2} x^3 + \frac{7}{4} x \right) \cos 2x + \left(\frac{3}{4} x^2 - \frac{7}{8} \right) \sin 2x + C$$

Exercise

Evaluate the integral $\int x^2 \sin 2x \, dx$

Solution

		$\int \sin 2x dx$
+	x^2	$-\frac{1}{2} \cos 2x$
-	$2x$	$-\frac{1}{4} \sin 2x$
+	2	$\frac{1}{8} \cos 2x$

$$\int x^2 \sin 2x \, dx = -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + C$$

$$= -\frac{1}{4} (2x^2 - 1) \cos 2x + \frac{1}{2} x \sin 2x + C$$

Exercise

Evaluate the integral $\int x^2 \sin(1-x) \, dx$

Solution

		$\int \sin(1-x) dx$
+	x^2	$\cos(1-x)$
-	$2x$	$-\sin(1-x)$
+	2	$\cos(1-x)$

$$\int x^2 \sin(1-x) \, dx = x^2 \cos(1-x) + 2x \sin(1-x) + 2 \cos(1-x) + C$$

$$= (x^2 + 2) \cos(1-x) + 2x \sin(1-x) + C$$

Exercise

Evaluate the integral $\int x \sin x \cos x \, dx$

Solution

		$\int \sin 2x dx$
+	x	$-\frac{1}{2} \cos 2x$
-	1	$-\frac{1}{4} \sin 2x$

$$\begin{aligned}
 \int x \sin x \cos x \, dx &= \frac{1}{2} \int x \sin 2x \, dx \\
 &= \frac{1}{2} \left(-\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x \right) \\
 &= \underline{-\frac{1}{4} x \cos 2x + \frac{1}{8} \sin 2x + C}
 \end{aligned}$$

Exercise

Evaluate the integral $\int x \cos x \, dx$

Solution

		$\int \cos x$
+	x	$\sin x$
-	1	$-\cos x$

$$\int x \cos x \, dx = \underline{x \sin x + \cos x + C}$$

Exercise

Evaluate the integral $\int x \csc x \cot x \, dx$

Solution

$$u = x \rightarrow du = dx$$

$$dv = \csc x \cot x \, dx \rightarrow v = -\csc x$$

$$\begin{aligned}
 \int x \csc x \cot x \, dx &= -x \csc x + \int \csc x \, dx \\
 &= \underline{-x \csc x - \ln |\csc x + \cot x| + C}
 \end{aligned}$$

Exercise

Evaluate the integral $\int x^2 \cos x \, dx$

Solution

		$\int \cos x$
+	x^2	$\sin x$
-	$2x$	$-\cos x$
+	2	$-\sin x$

$$\int x^2 \cos x \, dx = \underline{x^2 \sin x + 2x \cos x - 2 \sin x + C}$$

Exercise

Evaluate the integral $\int x^3 \cos 2x \, dx$

Solution

		$\int \cos 2x$
+	x^3	$\frac{1}{2} \sin 2x$
-	$3x^2$	$-\frac{1}{4} \cos 2x$
+	$6x$	$-\frac{1}{8} \sin 2x$
-	6	$\frac{1}{16} \cos 2x$

$$\begin{aligned} \int x^3 \cos 2x \, dx &= \frac{1}{2} x^3 \sin 2x + \frac{3}{4} x^2 \cos 2x - \frac{3}{4} x \sin 2x - \frac{3}{8} \cos 2x + C \\ &= \underline{\left(\frac{1}{2} x^3 - \frac{3}{4} x \right) \sin 2x + \left(\frac{3}{4} x^2 - \frac{3}{8} \right) \cos 2x + C} \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx$

Solution

$$\text{Let: } u = \sqrt{x} \rightarrow du = \frac{1}{2\sqrt{x}} \, dx \Rightarrow 2du = \frac{1}{\sqrt{x}} \, dx$$

$$\begin{aligned} \int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx &= \int (\cos u)(2du) \\ &= 2 \int \cos u \, du \end{aligned}$$

$$= 2 \sin u + C$$

$$= \underline{2 \sin \sqrt{x} + C}$$

Exercise

Evaluate the integral $\int x \sinh x \, dx$

Solution

		$\int \sinh x \, dx$
+	x	$\cosh x$
-	1	$\sinh x$

$$\int x \sinh x \, dx = \underline{x \cosh x - \sinh x + C}$$

Exercise

Evaluate the integral $\int x^2 \cosh x \, dx$

Solution

		$\int \cosh x$
+	x^2	$\sinh x$
-	$2x$	$\cosh x$
+	2	$\sinh x$

$$\int x^2 \cosh x \, dx = x^2 \sinh x - 2x \cosh x + 2 \sinh x + C$$

$$= \underline{(x^2 + 2) \sinh x - 2x \cosh x + C}$$

Exercise

Evaluate the integral $\int e^{2x} \cos 3x \, dx$

Solution

		$\int \cos 3x \, dx$
+	e^{2x}	$\frac{1}{3} \sin 3x$

-	$2e^{2x}$	$-\frac{1}{9}\cos 3x$
+	$4e^{2x}$	$-\frac{1}{9}\int \cos 3x \, dx$

$$\int e^{2x} \cos 3x dx = \frac{1}{3}e^{2x} \sin 3x + \frac{2}{9}e^{2x} \cos 3x - \frac{4}{9} \int e^{2x} \cos 3x dx$$

$$\int e^{2x} \cos 3x dx + \frac{4}{9} \int e^{2x} \cos 3x dx = \frac{1}{3}e^{2x} \sin 3x + \frac{2}{9}e^{2x} \cos 3x$$

$$\frac{13}{9} \int e^{2x} \cos 3x dx = \frac{1}{3}e^{2x} \sin 3x + \frac{2}{9}e^{2x} \cos 3x$$

$$\int e^{2x} \cos 3x dx = \underline{\underline{\frac{e^{2x}}{13}(3 \sin 3x + 2 \cos 3x) + C}}$$

Exercise

Evaluate the integral $\int e^{-3x} \sin 5x \, dx$

Solution

		$\int \sin 5x$
+	e^{-3x}	$-\frac{1}{5}\cos 5x$
-	$-3e^{-3x}$	$-\frac{1}{25}\sin 5x$
+	$9e^{-3x}$	$-\int \frac{1}{25}\sin 5x$

$$\int e^{-3x} \sin 5x \, dx = -\frac{1}{5}e^{-3x} \cos 5x - \frac{3}{25}e^{-3x} \sin 5x - \frac{9}{25} \int e^{-3x} \sin 5x \, dx$$

$$\left(1 + \frac{9}{25}\right) \int e^{-3x} \sin 5x \, dx = -\frac{1}{25}(5 \cos 5x + 3 \sin 5x)e^{-3x}$$

$$\frac{34}{25} \int e^{-3x} \sin 5x \, dx = -\frac{1}{25}(5 \cos 5x + 3 \sin 5x)e^{-3x}$$

$$\int e^{-3x} \sin 5x \, dx = \underline{\underline{-\frac{1}{34}(5 \cos 5x + 3 \sin 5x)e^{-3x} + C}}$$

Exercise

Evaluate the integral $\int e^{-x} \sin 4x \, dx$

Solution

		$\int \sin 4x \, dx$
+	e^{-x}	$-\frac{1}{4} \cos 4x$
-	$-e^{-x}$	$-\frac{1}{16} \sin 4x$
+	e^{-x}	$-\frac{1}{16} \int \sin 4x \, dx$

$$\int e^{-x} \sin 4x \, dx = -\frac{1}{4} e^{-x} \cos 4x - \frac{1}{16} e^{-x} \sin 4x - \frac{1}{16} \int e^{-x} \sin 4x \, dx$$

$$\left(1 + \frac{1}{16}\right) \int e^{-x} \sin 4x \, dx = -\frac{1}{4} e^{-x} \cos 4x - \frac{1}{16} e^{-x} \sin 4x$$

$$\frac{17}{16} \int e^{-x} \sin 4x \, dx = -\frac{1}{16} e^{-x} (4 \cos 4x + \sin 4x)$$

$$\int e^{-x} \sin 4x \, dx = \underline{-\frac{e^{-x}}{17} (4 \cos 4x + \sin 4x) + C}$$

Exercise

Evaluate the integral $\int e^{-2\theta} \sin 6\theta \, d\theta$

Solution

		$\int \sin 6\theta \, d\theta$
+	$e^{-2\theta}$	$-\frac{1}{6} \cos 6\theta$
-	$-2e^{-2\theta}$	$-\frac{1}{36} \sin 6\theta$
+	$4e^{-2\theta}$	$-\frac{1}{36} \int \sin 6\theta \, d\theta$

$$\int e^{-2\theta} \sin 6\theta \, d\theta = -\frac{1}{6} e^{-2\theta} \cos 6\theta - \frac{1}{18} e^{-2\theta} \sin 6\theta - \frac{1}{9} \int e^{-2\theta} \sin 6\theta \, d\theta$$

$$\left(1 + \frac{1}{9}\right) \int e^{-2\theta} \sin 6\theta \, d\theta = -\frac{1}{18} e^{-2\theta} (3 \cos 6\theta + \sin 6\theta)$$

$$\frac{10}{9} \int e^{-2\theta} \sin 6\theta \, d\theta = -\frac{1}{18} e^{-2\theta} (3 \cos 6\theta + \sin 6\theta)$$

$$\int e^{-2\theta} \sin 6\theta \, d\theta = \underline{-\frac{e^{-2\theta}}{20} (3 \cos 6\theta + \sin 6\theta) + C}$$

Exercise

Evaluate the integral $\int e^{-3x} \sin 4x \, dx$

Solution

		$\int \sin 4x$
+	e^{-3x}	$-\frac{1}{4} \cos 4x$
-	$-3e^{-3x}$	$-\frac{1}{16} \sin 4x$
+	$9e^{-3x}$	$-\frac{1}{16} \int \sin 4x$

$$\int e^{-3x} \sin 4x \, dx = -\frac{1}{4} e^{-3x} \cos 4x - \frac{3}{16} e^{-3x} \sin 4x - \frac{9}{16} \int e^{-3x} \sin 4x \, dx$$

$$\left(1 + \frac{9}{16}\right) \int e^{-3x} \sin 4x \, dx = -\frac{1}{16} (4 \cos 4x + 3 \sin 4x) e^{-3x}$$

$$\frac{25}{16} \int e^{-3x} \sin 4x \, dx = -\frac{1}{16} (4 \cos 4x + 3 \sin 4x) e^{-3x}$$

$$\int e^{-3x} \sin 4x \, dx = \underline{-\frac{1}{25} (4 \cos 4x + 3 \sin 4x) e^{-3x} + C}$$

Exercise

Evaluate the integral $\int e^{4x} \cos 2x \, dx$

Solution

		$\int \cos 2x$
+	e^{4x}	$\frac{1}{2} \sin 2x$
-	$4e^{4x}$	$-\frac{1}{4} \cos 2x$
+	$16e^{4x}$	$-\frac{1}{4} \int \cos 2x$

$$\int e^{4x} \cos 2x \, dx = \frac{1}{2} e^{4x} \sin 2x + e^{4x} \cos 2x - 4 \int e^{4x} \cos 2x \, dx$$

$$5 \int e^{4x} \cos 2x \, dx = \frac{1}{2} (\sin 2x + 2 \cos 2x) e^{4x}$$

$$\int e^{4x} \cos 2x \, dx = \underline{\frac{1}{10} (\sin 2x + 2 \cos 2x) e^{4x} + C}$$

Exercise

Evaluate the integral $\int e^{3x} \cos 3x \, dx$

Solution

		$\int \cos 3x$
+	e^{3x}	$\frac{1}{3} \sin 3x$
-	$3e^{3x}$	$-\frac{1}{9} \cos 3x$
+	$9e^{3x}$	$-\frac{1}{9} \int \cos 3x$

$$\int e^{3x} \cos 3x \, dx = \frac{1}{3} e^{3x} \sin 3x + \frac{1}{3} e^{3x} \cos 3x - \int e^{3x} \cos 3x \, dx$$

$$2 \int e^{3x} \cos 3x \, dx = \frac{1}{3} (\sin 3x + \cos 3x) e^{3x}$$

$$\int e^{3x} \cos 3x \, dx = \underline{\frac{1}{6} (\sin 3x + \cos 3x) e^{3x} + C}$$

Exercise

Evaluate the integral $\int e^{3x} \cos 2x \, dx$

Solution

		$\int \cos 2x$
+	e^{3x}	$\frac{1}{2} \sin 2x$
-	$3e^{3x}$	$-\frac{1}{4} \cos 2x$
+	$9e^{3x}$	$-\frac{1}{4} \int \cos 2x$

$$\int e^{3x} \cos 2x \, dx = e^{3x} \left(\frac{1}{2} \sin 2x + \frac{3}{4} \cos 2x \right) - \frac{9}{4} \int e^{3x} \cos 2x \, dx$$

$$\left(1 + \frac{9}{4}\right) \int e^{3x} \cos 2x \, dx = \frac{1}{4} e^{3x} (2 \sin 2x + 3 \cos 2x)$$

$$\frac{13}{4} \int e^{3x} \cos 2x \, dx = \frac{1}{4} e^{3x} (2 \sin 2x + 3 \cos 2x)$$

$$\int e^{3x} \cos 2x \, dx = \frac{1}{13} e^{3x} (2 \sin 2x + 3 \cos 2x) + C$$

Exercise

Evaluate the integral $\int e^x \sin x \, dx$

Solution

		$\int \sin x$
+	e^x	$-\cos x$
-	e^x	$-\sin x$
+	e^x	$-\int \sin x$

$$\int e^x \sin x \, dx = e^x (-\cos x + \sin x) - \int e^x \sin x \, dx$$

$$2 \int e^x \sin x \, dx = e^x (\sin x - \cos x)$$

$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

Exercise

Evaluate the integral $\int e^{-2x} \sin 3x \, dx$

Solution

		$\int \sin 3x$
+	e^{-2x}	$-\frac{1}{3} \cos 3x$
-	$-2e^{-2x}$	$-\frac{1}{9} \sin 3x$
+	$4e^{-2x}$	$-\frac{1}{9} \int \sin 3x$

$$\begin{aligned}
\int e^{-2x} \sin 3x \, dx &= e^{-2x} \left(-\frac{1}{3} \cos 3x - \frac{2}{9} \sin 3x \right) - \frac{4}{9} \int e^{-2x} \sin 3x \, dx \\
\left(1 + \frac{4}{9} \right) \int e^{-2x} \sin 3x \, dx &= -\frac{1}{9} e^{-2x} (3 \cos 3x + 2 \sin 3x) \\
\frac{13}{9} \int e^{-2x} \sin 3x \, dx &= -\frac{1}{9} e^{-2x} (3 \cos 3x + 2 \sin 3x) \\
\int e^{-2x} \sin 3x \, dx &= \underline{-\frac{1}{13} e^{-2x} (3 \cos 3x + 2 \sin 3x) + C}
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{x}{\sqrt{x-1}} \, dx$

Solution

Let: $u = x \Rightarrow du = dx$

$$\begin{aligned}
dv = \frac{dx}{\sqrt{x-1}} \Rightarrow v &= \int (x-1)^{-1/2} d(x-1) \\
&= \frac{(x-1)^{1/2}}{1/2} \\
&= \underline{2(x-1)^{1/2}}
\end{aligned}$$

$$\begin{aligned}
\int \frac{x}{\sqrt{x-1}} \, dx &= 2x\sqrt{x-1} - 2 \int (x-1)^{1/2} \, dx \\
&= 2x\sqrt{x-1} - 2 \frac{(x-1)^{3/2}}{3/2} + C \\
&= 2x\sqrt{x-1} - \frac{4}{3} (x-1)\sqrt{x-1} + C \\
&= \sqrt{x-1} \left[2x - \frac{4}{3}x + \frac{4}{3} \right] + C \\
&= \sqrt{x-1} \left[\frac{6x - 4x + 4}{3} \right] + C \\
&= \sqrt{x-1} \left[\frac{2x + 4}{3} \right] + C \\
&= \underline{\frac{2}{3} \sqrt{x-1} (x+2) + C}
\end{aligned}$$

Or —————

Let: $u = x-1 \Rightarrow x = u+1$
 $du = dx$

$$\begin{aligned}
\int \frac{x}{\sqrt{x-1}} dx &= \int (u+1)u^{-1/2} du \\
&= \int \left(u^{1/2} + u^{-1/2}\right) du \\
&= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + C \\
&= (x-1)^{1/2} \left(\frac{2}{3}x - \frac{2}{3} + 2\right) + C \\
&= \sqrt{x-1} \left[\frac{2x+4}{3}\right] + C \\
&= \underline{\underline{\frac{2}{3}\sqrt{x-1}(x+2) + C}}
\end{aligned}$$

Exercise

Evaluate the integral $\int x\sqrt{x-5} \, dx$

Solution

$$\begin{aligned}
\text{Let } u &= \sqrt{x-5} \rightarrow u^2 = x-5 \Rightarrow x = u^2 + 5 \\
2udu &= dx
\end{aligned}$$

$$\begin{aligned}
\int x\sqrt{x-5} \, dx &= \int (u^2 + 5)u(2udu) \\
&= \int (2u^4 + 10u^2) du \\
&= \underline{\underline{\frac{2}{5}u^5 + \frac{10}{3}u^3 + C}}
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{x}{\sqrt{6x+1}} \, dx$

Solution

$$u = x \rightarrow du = dx$$

$$dv = (6x+1)^{-1/2} \, dx$$

$$= \frac{1}{6}(6x+1)^{-1/2} d(6x+1)$$

$$v = \frac{1}{3}(6x+1)^{1/2}$$

$$\begin{aligned}
 \int \frac{x}{\sqrt{6x+1}} dx &= \frac{1}{3} x \sqrt{6x+1} - \frac{1}{3} \int (6x+1)^{1/2} dx \\
 &= \frac{1}{3} x \sqrt{6x+1} - \frac{1}{18} \int (6x+1)^{1/2} d(6x+1) \\
 &= \frac{1}{3} x \sqrt{6x+1} - \frac{1}{27} (6x+1)^{3/2} + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{x}{2\sqrt{x+2}} dx$

Solution

		$\int (x+2)^{-1/2} d(x+2)$
+	x	$2(x+2)^{1/2}$
-	1	$\frac{4}{3}(x+2)^{3/2}$

$$\begin{aligned}
 \int \frac{x}{2\sqrt{x+2}} dx &= \frac{1}{2} \int x(x+2)^{-1/2} dx \\
 &= \frac{1}{2} \left[2x\sqrt{x+2} - \frac{4}{3}(x+2)^{3/2} \right] + C \\
 &= \frac{1}{3} \sqrt{x+2} (3x - 2(x+2)) + C \\
 &= \frac{1}{3} \sqrt{x+2} (3x - 2x - 4) + C \\
 &= \frac{1}{3} \sqrt{x+2} (x - 4) + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{2x^2 - 3x}{(x-1)^3} dx$

Solution

		$\int (x-1)^{-3} d(x-1)$
+	$2x^2 - 3x$	$-\frac{1}{2}(x-1)^{-2}$
-	$4x - 3$	$\frac{1}{2}(x-1)^{-1}$
+	4	$\frac{1}{2} \ln x-1 $

$$\int \frac{2x^2 - 3x}{(x-1)^3} dx = -\frac{1}{2} \frac{2x^2 - 3x}{(x-1)^2} - \frac{1}{2} \frac{4x-3}{x-1} + 2 \ln|x-1| + C$$

Exercise

Evaluate the integral $\int \frac{x^2 + 3x + 4}{\sqrt[3]{2x+1}} dx$

Solution

		$\frac{1}{2} \int (2x+1)^{-1/3} d(2x+1)$
+	$x^2 + 3x + 4$	$\frac{3}{4} (2x+1)^{2/3}$
-	$2x + 3$	$\frac{1}{2} \frac{9}{20} (2x+1)^{5/3}$
+	2	$\frac{1}{2} \frac{27}{320} (2x+1)^{8/3}$

$$\int \frac{x^2 + 3x + 4}{\sqrt[3]{2x+1}} dx = \frac{3}{4} (x^2 + 3x + 4) (2x+1)^{2/3} - \frac{9}{40} (2x+3) (2x+1)^{5/3} + \frac{27}{320} (2x+1)^{8/3} + C$$

Exercise

Evaluate the integral $\int \frac{x}{\sqrt{x+1}} dx$

Solution

		$\int (x+1)^{-1/2} dx$
+	x	$2(x+1)^{1/2}$
-	1	$\frac{4}{3} (x+1)^{3/2}$

$$\int \frac{x}{\sqrt{x+1}} dx = 2x(x+1)^{1/2} - \frac{4}{3} (x+1)^{3/2} + C$$

Exercise

Evaluate the integral $\int \frac{x^5}{\sqrt{1-2x^3}} dx$

Solution

		$\int x^2 (1-2x^3)^{-1/2} dx = -\frac{1}{6} \int (1-2x^3)^{-1/2} d(1-2x^3)$
+	x^3	$-\frac{1}{3} (1-2x^3)^{1/2}$
-	$3x^2$	$\int -\frac{1}{3} (1-2x^3)^{1/2}$

$$\begin{aligned}
 \int \frac{x^5}{\sqrt{1-2x^3}} dx &= -\frac{1}{3} x^3 \sqrt{1-2x^3} + \int x^2 (1-2x^3)^{1/2} dx \\
 &= -\frac{1}{3} x^3 \sqrt{1-2x^3} - \frac{1}{6} \int (1-2x^3)^{1/2} d(1-2x^3) \\
 &= -\frac{1}{3} x^3 \sqrt{1-2x^3} - \frac{1}{9} (1-2x^3)^{3/2} + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int x\sqrt{1-3x} dx$

Solution

		$\int (1-3x)^{1/2} dx = -\frac{1}{3} \int (1-3x)^{1/2} d(1-3x)$
+	x	$-\frac{2}{9} (1-3x)^{3/2}$
-	1	$-\left(-\frac{1}{3}\right) \frac{2}{9} \frac{2}{5} (1-3x)^{5/2}$

$$\int x\sqrt{1-3x} dx = -\frac{2x}{9} (1-3x)^{3/2} - \frac{4}{135} (1-3x)^{5/2} + C$$

Exercise

Evaluate the integral $\int \sin(\ln x) dx$

Solution

		$\int dx$
+	$\sin(\ln x)$	x
-	$\frac{\cos(\ln x)}{x}$	$\int x dx$

$$\begin{aligned}\int \sin(\ln x) \, dx &= x \sin(\ln x) - \int x \frac{\cos(\ln x)}{x} dx \\ &= x \sin(\ln x) - \int \cos(\ln x) \, dx\end{aligned}$$

		$\int dx$
+	$\cos(\ln x)$	x
-	$-\frac{\sin(\ln x)}{x}$	$\int x dx$

$$= x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x) \, dx$$

$$2 \int \sin(\ln x) \, dx = x \sin(\ln x) - x \cos(\ln x)$$

$$\int \sin(\ln x) \, dx = \underline{\underline{\frac{1}{2} x \sin(\ln x) - \frac{1}{2} x \cos(\ln x) + C}}$$

Exercise

Evaluate the integral $\int \tan^{-1} y \, dy$

Solution

$$u = \tan^{-1} y \quad dv = dy$$

Let:

$$du = \frac{dy}{1+y^2} \quad v = y$$

$$\int \tan^{-1} y \, dy = y \tan^{-1} y - \int \frac{y dy}{1+y^2}$$

$$d(1+y^2) = 2y dy \quad \rightarrow \quad \frac{1}{2} d(1+y^2) = y dy$$

$$= y \tan^{-1} y - \int \frac{\frac{1}{2} d(1+y^2)}{1+y^2}$$

$$= y \tan^{-1} y - \frac{1}{2} \ln(1+y^2) + C$$

$$= \underline{\underline{y \tan^{-1} y - \ln \sqrt{1+y^2} + C}}$$

Exercise

Evaluate the integral $\int \sin^{-1} y \, dy$

Solution

Let: $u = \sin^{-1} y \quad dv = dy$
 $du = \frac{dy}{\sqrt{1-y^2}} \quad v = y$

$$\begin{aligned} \int \sin^{-1} y \, dy &= y \sin^{-1} y - \int \frac{y dy}{\sqrt{1-y^2}} & d(1-y^2) &= -2y dy \rightarrow -\frac{1}{2} d(1-y^2) = y dy \\ &= y \sin^{-1} y + \frac{1}{2} \int (1-y^2)^{-1/2} d(1-y^2) \\ &= y \sin^{-1} y + \frac{1}{2} (2) (1-y^2)^{1/2} + C \\ &= y \sin^{-1} y + \sqrt{1-y^2} + C \end{aligned}$$

Or

		$\int dy$
+	$\sin^{-1} y$	y
-	$\frac{1}{\sqrt{1-y^2}}$	$\int y$

$$\begin{aligned} \int \sin^{-1} y \, dy &= y \sin^{-1} y - \int \frac{y dy}{\sqrt{1-y^2}} \\ &= y \sin^{-1} y + \frac{1}{2} \int (1-y^2)^{-1/2} d(1-y^2) \\ &= y \sin^{-1} y + \sqrt{1-y^2} + C \end{aligned}$$

Exercise

Evaluate the integral $\int x \tan^{-1} x \, dx$

Solution

$u = \tan^{-1} x \quad v = \int x dx$
 $du = \frac{dx}{x^2+1} \quad v = \frac{1}{2} x^2$

$$\begin{aligned}
\int x \tan^{-1} x \, dx &= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{x^2 + 1} dx \\
&= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{x^2 + 1} \right) dx \\
&= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \left(x - \tan^{-1} x \right) + C \\
&= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + C \\
&= \underline{\frac{1}{2} (x^2 + 1) \tan^{-1} x - \frac{1}{2} x + C}
\end{aligned}$$

Exercise

Evaluate the integral $\int \sinh^{-1} x \, dx$

Solution

		$\int dx$
+	$\sinh^{-1} x$	x
-	$\frac{1}{\sqrt{x^2 + 1}}$	$\int x \, dx$

$$\begin{aligned}
\int \sinh^{-1} x \, dx &= x \sinh^{-1} x - \int \frac{x}{\sqrt{x^2 + 1}} dx \\
&= x \sinh^{-1} x - \frac{1}{2} \int (x^2 + 1)^{-1/2} d(x^2 + 1) \\
&= \underline{x \sinh^{-1} x - \sqrt{x^2 + 1} + C}
\end{aligned}$$

Exercise

Evaluate the integral $\int \tan^{-1} 3x \, dx$

Solution

		$\int dx$
+	$\tan^{-1} 3x$	x
-	$\frac{3}{9x^2 + 1}$	$\int x \, dx$

$$\begin{aligned}
 \int \tan^{-1} 3x \, dx &= x \tan^{-1} 3x - \int \frac{3x}{9x^2 + 1} dx \\
 &= x \tan^{-1} 3x - \frac{1}{6} \int \frac{1}{9x^2 + 1} d(9x^2 + 1) \\
 &= x \tan^{-1} 3x - \frac{1}{6} \ln(9x^2 + 1) + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \cos^{-1}\left(\frac{x}{2}\right) dx$

Solution

		$\int dx$
+	$\cos^{-1}\left(\frac{x}{2}\right)$	x
-	$\frac{\frac{1}{2}}{\sqrt{1 - \frac{1}{4}x^2}}$	$\int x \, dx$

$$\begin{aligned}
 \frac{\frac{1}{2}}{\sqrt{1 - \frac{1}{4}x^2}} &= \frac{\frac{1}{2}}{\sqrt{\frac{4 - x^2}{4}}} \\
 &= \frac{1}{\sqrt{4 - x^2}}
 \end{aligned}$$

$$\begin{aligned}
 \int \cos^{-1}\left(\frac{x}{2}\right) dx &= x \cos^{-1}\left(\frac{x}{2}\right) - \int \frac{x}{\sqrt{4 - x^2}} dx \\
 &= x \cos^{-1}\left(\frac{x}{2}\right) + \frac{1}{2} \int (4 - x^2)^{-1/2} d(4 - x^2) \\
 &= x \cos^{-1}\left(\frac{x}{2}\right) + \sqrt{4 - x^2} + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int x \sec^{-1} x \, dx \quad x \geq 1$

Solution

		$\int x dx$
+	$\sec^{-1} x$	$\frac{1}{2} x^2$
-	$\frac{1}{ x \sqrt{x^2-1}}$	$\int \frac{1}{2} x^2$

$$\begin{aligned}
 \int x \sec^{-1} x \, dx &= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \int \frac{1}{x\sqrt{x^2-1}} x^2 dx \\
 &= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \int x (x^2-1)^{-1/2} dx \\
 &= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{4} \int (x^2-1)^{-1/2} d(x^2-1) \\
 &= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \sqrt{x^2-1} + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int_{-1}^0 2x^2 \sqrt{x+1} \, dx$

Solution

		$\int (x+1)^{1/2} d(x+1)$
+	$2x^2$	$\frac{2}{3} (x+1)^{3/2}$
-	$4x$	$\frac{4}{15} (x+1)^{5/2}$
+	4	$\frac{8}{105} (x+1)^{7/2}$

$$\begin{aligned}
 \int_{-1}^0 2x^2 \sqrt{x+1} \, dx &= \frac{4}{3} x^2 (x+1)^{3/2} - \frac{16}{15} x (x+1)^{5/2} + \frac{32}{105} (x+1)^{7/2} \Big|_{-1}^0 \\
 &= \frac{32}{105}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_0^{1/\sqrt{2}} x \tan^{-1} x^2 dx$

Solution

$$\begin{aligned} \int_0^{1/\sqrt{2}} x \tan^{-1} x^2 dx &= \frac{1}{2} \int_0^{1/\sqrt{2}} \tan^{-1} x^2 d(x^2) \\ &= \frac{1}{2} \int_0^{1/\sqrt{2}} \tan^{-1} y dy \quad \left(\text{Let } y = x^2 \right) \end{aligned}$$

		$\int dy$
+	$\tan^{-1} y$	y
-	$\frac{1}{1+y^2}$	$\int y dy$

$$\begin{aligned} \int_0^{1/\sqrt{2}} x \tan^{-1} x^2 dx &= \frac{1}{2} \left[y \tan^{-1} y \Big|_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} \frac{y}{1+y^2} dy \right] \\ &= \frac{1}{2} x^2 \tan^{-1} x^2 \Big|_0^{1/\sqrt{2}} - \frac{1}{4} \int_0^{1/\sqrt{2}} \frac{1}{1+y^2} d(1+y^2) \\ &= \frac{1}{4} \tan^{-1} \frac{1}{2} - \frac{1}{4} \ln(1+x^4) \Big|_0^{1/\sqrt{2}} \\ &= \frac{1}{4} \tan^{-1} \frac{1}{2} - \frac{1}{4} \ln\left(1 + \frac{1}{4}\right) \\ &= \frac{1}{4} \tan^{-1} \frac{1}{2} - \frac{1}{4} \ln \frac{5}{4} \end{aligned}$$

Exercise

Evaluate the integral $\int_1^e x^2 \ln x dx$

Solution

		$\int x^2$
+	$\ln x$	$\frac{1}{3} x^3$
-	$\frac{1}{x}$	$\int \frac{1}{3} x^3$

$$\begin{aligned}
 \int_1^e x^2 \ln x \, dx &= \frac{1}{3} x^3 \ln x \Big|_1^e - \frac{1}{3} \int_1^e \frac{1}{x} x^3 \, dx \\
 &= \frac{1}{3} (e^3 - 0) - \frac{1}{3} \int_1^e x^2 \, dx \\
 &= \frac{1}{3} e^3 - \frac{1}{9} \left(x^3 \Big|_1^e \right) \\
 &= \frac{1}{3} e^3 - \frac{1}{9} (e^3 - 1) \\
 &= \frac{1}{3} e^3 - \frac{1}{9} e^3 + \frac{1}{9} \\
 &= \frac{1}{9} (2e^3 + 1)
 \end{aligned}$$

Exercise

Evaluate the integral $\int_{-1}^{\ln 2} \frac{3t}{e^t} dt$

Solution

		$\int e^{-t}$
+	t	$-e^{-t}$
-	1	e^{-t}

$$\begin{aligned}
 \int_{-1}^{\ln 2} \frac{3t}{e^t} dt &= 3e^{-t} (-t - 1) \Big|_{-1}^{\ln 2} \\
 &= -3 \left(e^{-\ln 2} (\ln 2 + 1) - e(0) \right) \\
 &= -\frac{3}{2} (\ln 2 + 1)
 \end{aligned}$$

Exercise

Evaluate the integral $\int_{\pi}^{2\pi} \cot \frac{x}{3} \, dx$

Solution

$$\int_{\pi}^{2\pi} \cot \frac{x}{3} \, dx = \int_{\pi}^{2\pi} \frac{\cos \frac{x}{3}}{\sin \frac{x}{3}} \, dx$$

$$\begin{aligned}
&= 3 \int_{\pi}^{2\pi} \frac{1}{\sin \frac{x}{3}} d\left(\sin \frac{x}{3}\right) \\
&= 3 \ln \left| \sin \frac{x}{3} \right| \Big|_{\pi}^{2\pi} \\
&= 3 \left(\ln \left| \sin \frac{2\pi}{3} \right| - \ln \left| \sin \frac{\pi}{3} \right| \right) \\
&= 3 \left(\ln \frac{\sqrt{3}}{2} - \ln \frac{\sqrt{3}}{2} \right) \\
&= 0
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2) dx$

Solution

$$u = \sin^{-1}(x^2) \quad dv = 2x dx$$

$$du = \frac{2x}{\sqrt{1-x^4}} dx \quad v = x^2$$

$$\int u dv = uv - \int v du$$

$$\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2) dx = \left[x^2 \sin^{-1}(x^2) \right]_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} x^2 \frac{2x}{\sqrt{1-x^4}} dx \quad d(1-x^4) = -4x^3 dx$$

$$= \left(\left(\frac{1}{\sqrt{2}} \right)^2 \sin^{-1} \left(\left(\frac{1}{\sqrt{2}} \right)^2 \right) - 0 \right) + \int_0^{1/\sqrt{2}} \frac{d(1-x^4)}{2\sqrt{1-x^4}}$$

$$= \frac{1}{2} \sin^{-1} \left(\frac{1}{2} \right) + \left[\sqrt{1-x^4} \right]_0^{1/\sqrt{2}}$$

$$= \frac{1}{2} \frac{\pi}{6} + \left(\sqrt{1-\frac{1}{4}} - 1 \right)$$

$$= \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

$$= \frac{1}{12} (\pi + 6\sqrt{3} - 12)$$

Exercise

Evaluate the integral $\int_1^e x^3 \ln x dx$

Solution

$$u = \ln x \quad v = \int x^3 dx = \frac{1}{4}x^4$$

$$du = \frac{1}{x} dx$$

$$\begin{aligned} \int_1^e x^3 \ln x dx &= \left[\frac{1}{4} x^4 \ln x \right]_1^e - \frac{1}{4} \int_1^e x^4 \frac{dx}{x} \\ &= \frac{1}{4} (e^4 \ln e - 1^4 \ln 1) - \frac{1}{4} \int_1^e x^3 dx \\ &= \frac{e^4}{4} - \frac{1}{16} [x^4]_1^e \\ &= \frac{e^4}{4} - \frac{1}{16} (e^4 - 1) \\ &= \frac{4}{4} \frac{e^4}{4} - \frac{1}{16} e^4 + \frac{1}{16} \\ &= \frac{3e^4 + 1}{16} \end{aligned}$$

Exercise

Evaluate the integral $\int_0^1 x\sqrt{1-x} dx$

Solution

Let: $u = x \quad dv = \sqrt{1-x} dx = (1-x)^{1/2} dx \quad d(1-x) = -dx$

$$du = dx \quad v = -\int (1-x)^{1/2} d(1-x) = -\frac{2}{3}(1-x)^{2/3}$$

$$\begin{aligned} \int_0^1 x\sqrt{1-x} dx &= \left[x \left(-\frac{2}{3}(1-x)^{2/3} \right) \right]_0^1 - \int_0^1 -\frac{2}{3}(1-x)^{2/3} dx \\ &= \left[-\frac{2}{3}x(1-x)^{2/3} \right]_0^1 + \frac{2}{3} \int_0^1 (1-x)^{2/3} (-d(1-x)) \\ &= -\frac{2}{3} \left[(1)(0)^{2/3} - 0 \right] - \left[\frac{2}{3} \left(\frac{3}{5} \right) (1-x)^{5/3} \right]_0^1 \\ &= -\frac{4}{15} \left[0 - (1)^{5/3} \right] \end{aligned}$$

$$\int u dv = uv - \int v du$$

$$= \frac{4}{15} \Big|$$

Or

		$-\int (1-x)^{1/2} d(1-x)$
+	x	$-\frac{2}{3}(1-x)^{3/2}$
-	1	$\frac{4}{15}(1-x)^{5/2}$

$$\int_0^1 x\sqrt{1-x} dx = -\frac{2}{3}x(1-x)^{3/2} - \frac{4}{15}(1-x)^{5/2} \Big|_0^1$$

$$= \frac{4}{15} \Big|$$

Exercise

Evaluate the integral $\int_0^{\pi/3} x \tan^2 x dx$

Solution

$$u = x \rightarrow dv = \tan^2 x dx = \frac{\sin^2 x}{\cos^2 x} dx = \frac{1 - \cos^2 x}{\cos^2 x} dx$$

$$du = dx \rightarrow v = \int \left(\frac{1}{\cos^2 x} - 1 \right) dx = \tan x - x$$

$$\int_0^{\pi/3} x \tan^2 x dx = \left[x(\tan x - x) \right]_0^{\pi/3} - \int_0^{\pi/3} (\tan x - x) dx$$

$$= \left[\frac{\pi}{3} \left(\tan \frac{\pi}{3} - \frac{\pi}{3} \right) - 0 \right] - \left[-\ln |\cos x| - \frac{x^2}{2} \right]_0^{\pi/3}$$

$$= \frac{\pi}{3} \left(\sqrt{3} - \frac{\pi}{3} \right) + \left[\ln \left| \cos \frac{\pi}{3} \right| + \frac{1}{2} \left(\frac{\pi}{3} \right)^2 - \ln |1| - 0 \right]$$

$$= \frac{\pi}{3} \sqrt{3} - \frac{\pi^2}{9} + \ln \left| \frac{1}{2} \right| + \frac{\pi^2}{18}$$

$$= \frac{\pi}{3} \sqrt{3} - \ln 2 - \frac{\pi^2}{18} \Big|$$

$$\int u dv = uv - \int v du$$

Exercise

Evaluate the integral $\int_0^{\pi} x \sin x \, dx$

Solution

		$\int \sin x \, dx$
+	x	$-\cos x$
-	1	$-\sin x$

$$\int_0^{\pi} x \sin x \, dx = -x \cos x + \sin x \Big|_0^{\pi} \\ = \pi$$

Exercise

Evaluate the integral $\int_1^e \ln 2x \, dx$

Solution

$$\begin{aligned} \int_1^e \ln 2x \, dx &= \frac{1}{2} \int_1^e \ln 2x \, d(2x) \\ &= x \ln 2x - x \Big|_1^e \\ &= e \ln 2e - e - \ln 2 + 1 \\ &= e(\ln 2 + \ln e) - e - \ln 2 + 1 \\ &= e \ln 2 - \ln 2 + 1 \\ &= (e-1) \ln 2 + 1 \end{aligned}$$

$$\int \ln x \, dx = x \ln x - x$$

Exercise

Evaluate the integral $\int_0^{\pi/2} x \cos 2x \, dx$

Solution

		$\int \cos 2x \, dx$
+	x	$\frac{1}{2} \sin 2x$
-	1	$-\frac{1}{4} \cos 2x$

$$\begin{aligned}\int_0^{\pi/2} x \cos 2x \, dx &= \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x \Big|_0^{\pi/2} \\ &= -\frac{1}{4} - \frac{1}{4} \\ &= \underline{-\frac{1}{2}}\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\ln 2} x e^x \, dx$

Solution

		$\int e^x \, dx$
+	x	e^x
-	1	e^x

$$\begin{aligned}\int_0^{\ln 2} x e^x \, dx &= e^x (x - 1) \Big|_0^{\ln 2} \\ &= 2(\ln 2 - 1) + 1 \\ &= \underline{2 \ln 2 - 1}\end{aligned}$$

Exercise

Evaluate the integral $\int_1^{e^2} x^2 \ln x \, dx$

Solution

$$\begin{aligned}\int x^2 \ln x \, dx &= \frac{1}{3} x^3 \ln x - \frac{1}{3} \int x^2 \, dx \\ \int_1^{e^2} x^2 \ln x \, dx &= \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 \Big|_1^{e^2} \\ &= \frac{2}{3} e^6 - \frac{1}{9} e^6 + \frac{1}{9} \\ &= \underline{\frac{5}{9} e^6 + \frac{1}{9}}\end{aligned}$$

$$u = \ln x$$

$$du = \frac{1}{x} dx \quad v = \int x^2 dx = \frac{1}{3} x^3$$

Exercise

Evaluate the integral $\int_0^3 x e^{x/2} dx$

Solution

$$\int x^n e^{ax} dx = e^{ax} \sum_{k=0}^n \frac{(-1)^k n!}{(n-k)! a^{k+1}} x^{n-k}$$

$$\int_0^3 x e^{x/2} dx = (2x - 4) e^{x/2} \Big|_0^3$$

$$= 2e^{3/2} + 4$$

Exercise

Evaluate the integral $\int_0^2 x^2 e^{-2x} dx$

Solution

$$\int_0^2 x^2 e^{-2x} dx = \left(-\frac{1}{2} x^2 + \frac{1}{2} x - \frac{1}{4} \right) e^{-2x} \Big|_0^2$$

$$= \left(-2 + 1 - \frac{1}{4} \right) e^{-4} + \frac{1}{4}$$

$$= \frac{1}{4} - \frac{5}{4} e^{-4}$$

$$\int x^n e^{ax} dx = e^{ax} \sum_{k=0}^n \frac{(-1)^k n!}{(n-k)! a^{k+1}} x^{n-k}$$

Exercise

Evaluate the integral $\int_0^{\pi/4} x \cos 2x dx$

Solution

		$\int \cos 2x dx$
+	x	$\frac{1}{2} \sin 2x$
-	1	$-\frac{1}{4} \cos 2x$

$$\int_0^{\pi/4} x \cos 2x dx = \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x \Big|_0^{\pi/4}$$

$$= \frac{\pi}{8} - \frac{1}{4}$$

Exercise

Evaluate the integral $\int_0^{\pi} x \sin 2x \, dx$

Solution

		$\int \sin 2x \, dx$
+	x	$-\frac{1}{2} \cos 2x$
-	1	$-\frac{1}{4} \sin 2x$

$$\int_0^{\pi} x \sin 2x \, dx = -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x \Big|_0^{\pi}$$
$$= -\frac{\pi}{2} \Big|$$

Exercise

Evaluate the integral $\int_1^4 e^{\sqrt{x}} \, dx$

Solution

$$u = \sqrt{x} \rightarrow u^2 = x$$
$$2u \, du = dx$$

$$\int_1^4 e^{\sqrt{x}} \, dx = 2 \int_1^4 u e^u \, du$$

		$\int e^u \, du$
+	u	e^u
-	1	e^u

$$\int_1^4 e^{\sqrt{x}} \, dx = 2e^u (u-1) \Big|_1^4$$
$$= 2e^{\sqrt{x}} (\sqrt{x}-1) \Big|_1^4$$
$$= 2 \left[e^2 (2-1) - e(1-1) \right]$$
$$= 2e^2 \Big|$$

Exercise

Use integration by parts to establish the reduction formula

$$\int x^n \sin x \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx$$

Solution

$$u = x^n \quad dv = \sin x \, dx$$

$$du = nx^{n-1} \, dx \quad v = -\cos x$$

$$\int x^n \sin x \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx \quad \checkmark$$

Exercise

Use integration by parts to establish the reduction formula

$$\int x^n e^{ax} \, dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx, \quad a \neq 0$$

Solution

$$u = x^n \quad dv = e^{ax} \, dx$$

$$du = nx^{n-1} \, dx \quad v = \frac{1}{a} e^{ax}$$

$$\int x^n e^{ax} \, dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx, \quad a \neq 0 \quad \checkmark$$

Exercise

Use integration by parts to establish the reduction formula

$$\int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx$$

Solution

$$u = (\ln x)^n \quad dv = \int dx$$

$$du = n(\ln x)^{n-1} \frac{1}{x} \, dx \quad v = x$$

$$\int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx \quad \checkmark$$

Exercise

Use integration by parts to establish the reduction formula

$$\int_a^b \left(\int_x^b f(t) dt \right) dx = \int_a^b (x-a) f(x) dx$$

Solution

$$u = x - a \quad dv = f(x) dx$$

$$du = dx \quad v = \int_b^x f(t) dt$$

$$\begin{aligned} \int_a^b (x-a) f(x) dx &= \left[(x-a) \int_b^x f(t) dt \right]_a^b - \int_a^b \left(\int_b^x f(t) dt \right) dx \\ &= (b-a) \int_b^b f(t) dt - (a-a) \int_b^a f(t) dt - \int_a^b \left(- \int_x^b f(t) dt \right) dx \\ &= \int_a^b \left(\int_x^b f(t) dt \right) dx \quad \checkmark \end{aligned} \quad \int_b^b f(t) dt = 0, \quad a-a=0$$

Exercise

Use integration by parts to establish the reduction formula

$$\int \sqrt{1-x^2} dx = \frac{1}{2} x \sqrt{1-x^2} + \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} dx$$

Solution

$$u = \sqrt{1-x^2} \quad dv = dx$$

$$du = \frac{-x}{\sqrt{1-x^2}} dx \quad v = x$$

$$\begin{aligned} \int \sqrt{1-x^2} dx &= x \sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}} dx \\ &= x \sqrt{1-x^2} - \int \left(\frac{1-x^2-1}{\sqrt{1-x^2}} \right) dx \\ &= x \sqrt{1-x^2} - \int \left(\frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \right) dx \end{aligned}$$

$$\begin{aligned}
&= x\sqrt{1-x^2} - \int \sqrt{1-x^2} dx + \int \frac{1}{\sqrt{1-x^2}} dx \\
&\int \sqrt{1-x^2} dx + \int \sqrt{1-x^2} dx = x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}} dx \\
&2 \int \sqrt{1-x^2} dx = x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}} dx \\
&\int \sqrt{1-x^2} dx = \frac{1}{2} x\sqrt{1-x^2} + \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} dx \quad \checkmark
\end{aligned}$$

Exercise

Find the indefinite integral: $\int 5x^n \ln ax \, dx \quad a \neq 0, n \neq -1$

Solution

$$\begin{aligned}
u &= \ln ax & dv &= x^n dx \\
du &= \frac{a}{ax} dx = \frac{dx}{x} & v &= \frac{x^{n+1}}{n+1} \\
\int 5x^n \ln ax \, dx &= 5 \left[\frac{x^{n+1}}{n+1} \ln ax - \frac{1}{n+1} \int \frac{x^{n+1}}{x} dx \right] \\
&= 5 \left[\frac{x^{n+1}}{n+1} \ln ax - \frac{1}{n+1} \int x^n dx \right] \\
&= 5 \left[\frac{x^{n+1}}{n+1} \ln ax - \frac{1}{n+1} \frac{x^{n+1}}{n+1} \right] + C \\
&= \frac{5x^{n+1}}{n+1} \left(\ln ax - \frac{1}{n+1} \right) + C
\end{aligned}$$

Exercise

Find the volume of the solid generated by the region bounded by $f(x) = x \ln x$, and the x -axis on $[1, e^2]$ is revolved about the y -axis.

Solution

Using **Disk** Method:

$$\begin{aligned}
 V &= \pi \int_1^{e^2} (x \ln x)^2 dx \\
 &= \pi \int_1^{e^2} x^2 \ln^2 x dx
 \end{aligned}$$

$$\begin{aligned}
 \text{Let } y = \ln x &\Rightarrow x = e^y \\
 dx &= e^y dy
 \end{aligned}$$

$$\begin{aligned}
 \int x^2 (\ln x)^2 dx &= \int (e^y)^2 y^2 e^y dy \\
 &= \int y^2 e^{3y} dy
 \end{aligned}$$

		$\int e^{3y} dy$
+	y^2	$\frac{1}{3} e^{3y}$
-	$2y$	$\frac{1}{9} e^{3y}$
+	2	$\frac{1}{27} e^{3y}$

$$\begin{aligned}
 V &= \pi e^{3y} \left(\frac{1}{3} y^2 - \frac{2}{9} y + \frac{2}{27} \right) \Bigg|_1^{e^2} \\
 &= \pi x^3 \left(\frac{1}{3} \ln^2 x - \frac{2}{9} \ln x + \frac{2}{27} \right) \Bigg|_1^{e^2} \\
 &= \pi (e^2)^3 \left(\frac{1}{3} (\ln e^2)^2 - \frac{2}{9} \ln e^2 + \frac{2}{27} \right) - \pi \left(\frac{2}{27} \right) \\
 &= \pi e^6 \left(\frac{4}{3} - \frac{4}{9} + \frac{2}{27} \right) - \frac{2}{27} \pi \\
 &= \pi e^6 \left(\frac{36 - 12 + 2}{27} \right) - \frac{2\pi}{27} \\
 &= \pi e^6 \left(\frac{26}{27} \right) - \frac{2\pi}{27} \\
 &= \frac{2\pi}{27} (13e^2 - 1) \text{ unit}^3
 \end{aligned}$$

Exercise

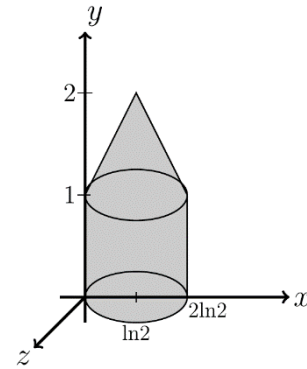
Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$

Solution

$$\begin{aligned} V &= 2\pi \int_0^{\ln 2} (\ln 2 - x) e^x dx \\ &= 2\pi \int_0^{\ln 2} (\ln 2 e^x - x e^x) dx \\ &= 2\pi \ln 2 \left(e^x \right) \Big|_0^{\ln 2} - 2\pi \int_0^{\ln 2} x e^x dx \end{aligned}$$

	e^x	
+	x	e^x
-	1	e^x

$$\begin{aligned} &= 2\pi \ln 2 (e^{\ln 2} - e^0) - 2\pi \left[x e^x - e^x \right]_0^{\ln 2} \\ &= 2\pi \ln 2 (2 - 1) - 2\pi \left[\ln 2 e^{\ln 2} - e^{\ln 2} - (0 - 1) \right] \\ &= 2\pi \ln 2 - 2\pi [2 \ln 2 - 2 + 1] \\ &= 2\pi \ln 2 - 4\pi \ln 2 + 2\pi \\ &= -2\pi \ln 2 + 2\pi \\ &= \underline{2\pi(1 - \ln 2) \text{ unit}^3} \end{aligned}$$



Exercise

Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $x = 1$, about

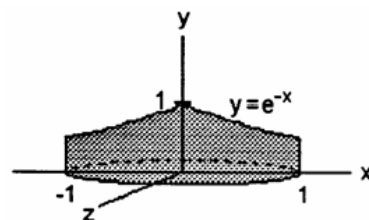
- the line y -axis
- the line $x = 1$

Solution

$$a) \quad V = 2\pi \int_0^1 x e^{-x} dx$$

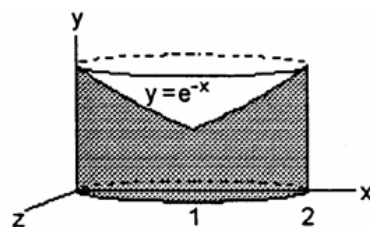
$\int e^{-x}$		
(+)	x	$-e^{-x}$
(-)	1	e^{-x}

$$\begin{aligned}
 &= 2\pi \left(\left[-xe^{-x} - e^{-x} \right]_0^1 \right) \\
 &= 2\pi \left(-e^{-1} - e^{-1} + 0 + 1 \right) \\
 &= 2\pi \left(-\frac{1}{e} - \frac{1}{e} + 1 \right) \\
 &= 2\pi \left(-\frac{2}{e} + 1 \right) \\
 &= \underline{2\pi - \frac{4\pi}{e}} \text{ unit}^3
 \end{aligned}$$



b) $V = 2\pi \int_0^1 (1-x)e^{-x} dx$

$$\begin{aligned}
 &= 2\pi \left(\int_0^1 e^{-x} dx - \int_0^1 xe^{-x} dx \right) \\
 &= 2\pi \left(\left[-e^{-x} - (-xe^{-x} - e^{-x}) \right]_0^1 \right) \\
 &= 2\pi \left[e^{-x} + xe^{-x} - e^{-x} \right]_0^1 \\
 &= 2\pi \left[xe^{-x} \right]_0^1 \\
 &= 2\pi \left(e^{-1} \right) \\
 &= \underline{\frac{2\pi}{e}} \text{ unit}^3
 \end{aligned}$$



Exercise

Find the volume of the solid that is generated by the region bounded by $f(x) = e^{-x}$, $x = \ln 2$, and the coordinate axes is revolved about the y -axis.

Solution

Using *Shells Method*:

$$V = 2\pi \int_0^{\ln 2} x e^{-x} dx$$

$$V = \int_a^b 2\pi (\text{radius})(\text{height}) dx$$

		$\int e^{-x} dx$
+	x	$-e^{-x}$
-	1	e^{-x}

$$\begin{aligned}
 &= 2\pi \left[e^{-x}(-x-1) \right]_0^{\ln 2} \\
 &= 2\pi \left(e^{-\ln 2}(-\ln 2 - 1) + 1 \right) \\
 &= 2\pi \left(\frac{1}{2}(-\ln 2 - 1) + 1 \right) \\
 &= 2\pi \left(-\frac{1}{2}\ln 2 + \frac{1}{2} \right) \\
 &= \pi(1 - \ln 2) \text{ unit}^3
 \end{aligned}$$

Exercise

Find the volume of the solid that is generated by the region bounded by $f(x) = e^{-x}$, and the x -axis on $[1, \ln 2]$ is revolved about the line $x = \ln 2$.

Solution

Using *Shells Method*:

$$V = 2\pi \int_0^{\ln 2} (\ln 2 - x) e^{-x} dx$$

$$V = \int_a^b 2\pi (\text{radius})(\text{height}) dx$$

$$= 2\pi \ln 2 \int_0^{\ln 2} e^{-x} dx - 2\pi \int_0^{\ln 2} x e^{-x} dx$$

		$\int e^{-x}$
+	x	$-e^{-x}$
-	1	e^{-x}

$$\begin{aligned}
&= 2\pi \left(-(\ln 2)e^{-x} - (-x-1)e^{-x} \right) \Big|_0^{\ln 2} \\
&= 2\pi \left(-(\ln 2)e^{-\ln 2} + (\ln 2 + 1)e^{-\ln 2} - \ln 2 - 1 \right) \\
&= 2\pi \left(-\frac{1}{2}\ln 2 + \frac{1}{2}(\ln 2 + 1) + \ln 2 - 1 \right) \\
&= 2\pi \left(-\frac{1}{2}\ln 2 + \frac{1}{2}\ln 2 + \frac{1}{2} + \ln 2 - 1 \right) \\
&= 2\pi \left(\ln 2 - \frac{1}{2} \right) \\
&= \pi (2\ln 2 - 1) \\
&= \pi (\ln 4 - 1) \text{ unit}^3
\end{aligned}$$

Exercise

Find the volume of the solid that is generated by the region bounded by $f(x) = \sin x$, and the x -axis on $[0, \pi]$ is revolved about the y -axis.

Solution

Using **Shells Method**:

$$V = 2\pi \int_0^{\pi} x \sin x \, dx$$

		$\int \sin x$
+	x	$-\cos x$
-	1	$-\sin x$

$$= 2\pi \left[-x \cos x + \sin x \right] \Big|_0^{\pi}$$

$$= 2\pi^2 \text{ unit}^3$$

$$V = \int_a^b 2\pi (\text{radius})(\text{height}) \, dx$$

Exercise

Find the area of the region generated when the region bounded by $y = \sin x$ and $y = \sin^{-1} x$ on the interval $\left[0, \frac{1}{2}\right]$.

Solution

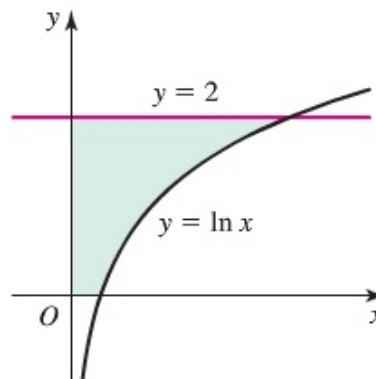
$$\begin{aligned}
 A &= \int_0^{1/2} (\sin^{-1} x - \sin x) dx & u &= \sin^{-1} x \\
 & & du &= \frac{dx}{\sqrt{1-x^2}} & v &= \int dx = x \\
 &= x \sin^{-1} x \Big|_0^{1/2} - \int_0^{1/2} \frac{x dx}{\sqrt{1-x^2}} + \cos x \Big|_0^{1/2} \\
 &= x \sin^{-1} x + \cos x \Big|_0^{1/2} + \frac{1}{2} \int_0^{1/2} (1-x^2)^{-1/2} d(1-x^2) \\
 &= x \sin^{-1} x + \cos x + (1-x^2)^{1/2} \Big|_0^{1/2} \\
 &= \frac{1}{2} \sin^{-1} \frac{1}{2} + \cos \frac{1}{2} + \left(1 - \frac{1}{4}\right)^{1/2} - 1 - 1 \\
 &= \frac{\pi}{12} + \cos \frac{1}{2} + \frac{\sqrt{3}}{2} - 2 \text{ unit}^2
 \end{aligned}$$

Exercise

Determine the area of the shaded region bounded by $y = \ln x$, $y = 2$, $y = 0$, and $x = 0$

Solution

$$\begin{aligned}
 y = \ln x = 0 &\rightarrow x = 1 \\
 y = \ln x = 2 &\rightarrow x = e^2 \\
 A &= 1 \times 2 + \int_1^2 (2 - \ln x) dx \\
 &= 2 + (2x - x \ln x + x) \Big|_1^2 \\
 &= 2 + 4 - 2 \ln 2 + 2 - 2 - 1 \\
 &= 5 - 2 \ln 2 \text{ unit}^2
 \end{aligned}$$



Exercise

Find the area between the curves $y = \ln x^2$, $y = \ln x$, and $x = e^2$

Solution

$$y = \ln x^2 = \ln x \quad \text{with } x > 0$$

$$x^2 = x \Rightarrow x = 1$$

$$A = \int_1^{e^2} (\ln x^2 - \ln x) dx$$

$$= \int_1^{e^2} (2 \ln x - \ln x) dx$$

$$= \int_1^{e^2} \ln x dx$$

$$u = \ln x \quad dv = dx$$

$$du = \frac{1}{x} dx \quad v = \int dx = x$$

$$\int \ln x dx = x \ln x - \int x \frac{1}{x} dx$$

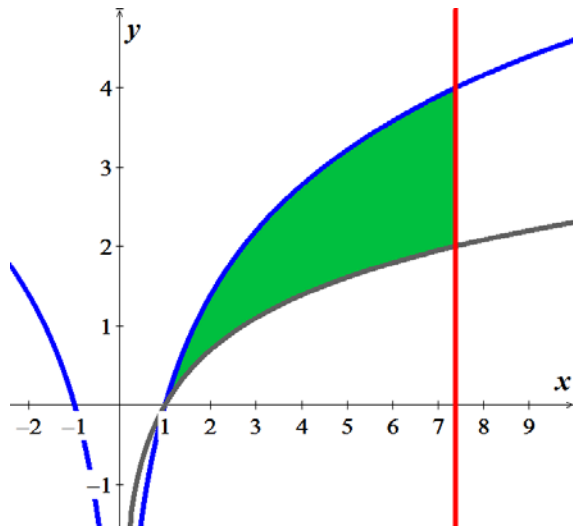
$$= x \ln x - \int dx$$

$$= x \ln x - x \Big|$$

$$= (x \ln x - x) \Big|_1^{e^2}$$

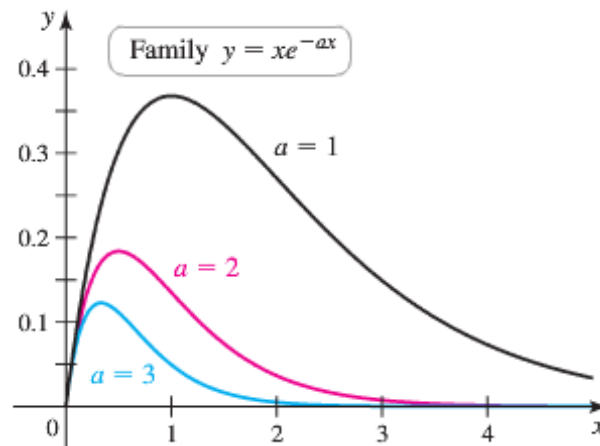
$$= e^2 \ln e^2 - e^2 + 1$$

$$= e^2 + 1 \text{ unit}^2 \Big|$$



Exercise

The curves $y = xe^{-ax}$ are shown in the figure for $a = 1, 2$, and 3 .



- Find the area of the region bounded by $y = xe^{-x}$ and the x -axis on the interval $[0, 4]$.
- Find the area of the region bounded by $y = xe^{-ax}$ and the x -axis on the interval $[0, 4]$ where $a > 0$.
- Find the area of the region bounded by $y = xe^{-ax}$ and the x -axis on the interval $[0, b]$. Because this area depends on a and b , we call it $A(a, b)$ where $a > 0$ and $b > 0$.
- Use part (c) to show that $A(1, \ln b) = 4A(2, \frac{1}{2}\ln b)$.
- Does this pattern continue? Is it true that $A(1, \ln b) = a^2 A(a, \frac{1}{a}\ln b)$?

Solution

$$a) \int_0^4 xe^{-x} dx = e^{-x}(-x-1) \Big|_0^4$$

		$\int e^{-x} dx$
+	x	$-e^{-x}$
-	1	e^{-x}

$$= e^{-4}(-5) - (-1)$$

$$= 1 - \frac{5}{e^4} \text{ unit}^2$$

$$b) \int_0^4 xe^{-ax} dx = e^{-ax} \left(-\frac{1}{a}x - \frac{1}{a^2} \right) \Big|_0^4$$

$$= e^{-4a} \left(-\frac{4}{a} - \frac{1}{a^2} \right) - \left(-\frac{1}{a^2} \right)$$

$$= \frac{1}{a^2} - e^{-4a} \left(\frac{4a+1}{a^2} \right)$$

		$\int e^{-ax} dx$
+	x	$-\frac{1}{a}e^{-ax}$
-	1	$\frac{1}{a^2}e^{-ax}$

$$\left. = \frac{1}{a^2} \left(1 - \frac{4a+1}{e^{-4a}} \right) unit^2 \right|$$

$$\begin{aligned} c) \quad \int_0^b x e^{-ax} dx &= e^{-ax} \left(-\frac{1}{a} x - \frac{1}{a^2} \right) \Big|_0^b \\ &= e^{-ab} \left(-\frac{b}{a} - \frac{1}{a^2} \right) - \left(-\frac{1}{a^2} \right) \\ &= \frac{1}{a^2} - e^{-ab} \left(\frac{ab+1}{a^2} \right) \\ &= \frac{1}{a^2} \left(1 - \frac{ab+1}{e^{ab}} \right) unit^2 \end{aligned} \left| \right.$$

$$d) \quad A(a, b) = \frac{1}{a^2} \left(1 - \frac{ab+1}{e^{ab}} \right)$$

$$\begin{aligned} A(1, \ln b) &= 1 - \frac{\ln b + 1}{e^{\ln b}} \\ &= 1 - \frac{\ln b + 1}{b} \end{aligned} \left| \right.$$

$$\begin{aligned} A\left(2, \frac{1}{2} \ln b\right) &= \frac{1}{4} \left(1 - \frac{\ln b + 1}{e^{\ln b}} \right) \\ &= \frac{1}{4} \left(1 - \frac{\ln b + 1}{b} \right) \\ &= \frac{1}{4} A(1, \ln b) \end{aligned}$$

$$\therefore \left. A(1, \ln b) = 4A\left(2, \frac{1}{2} \ln b\right) \right|$$

$$\begin{aligned} e) \quad A\left(a, \frac{1}{a} \ln b\right) &= \frac{1}{a^2} \left(1 - \frac{\ln b + 1}{e^{\ln b}} \right) \\ &= \frac{1}{a^2} \left(1 - \frac{\ln b + 1}{b} \right) \\ &= \frac{1}{a^2} A(1, \ln b) \end{aligned}$$

$$\text{Yes, there is a pattern: } \left. A(1, \ln b) = a^2 A\left(a, \frac{1}{a} \ln b\right) \right|$$

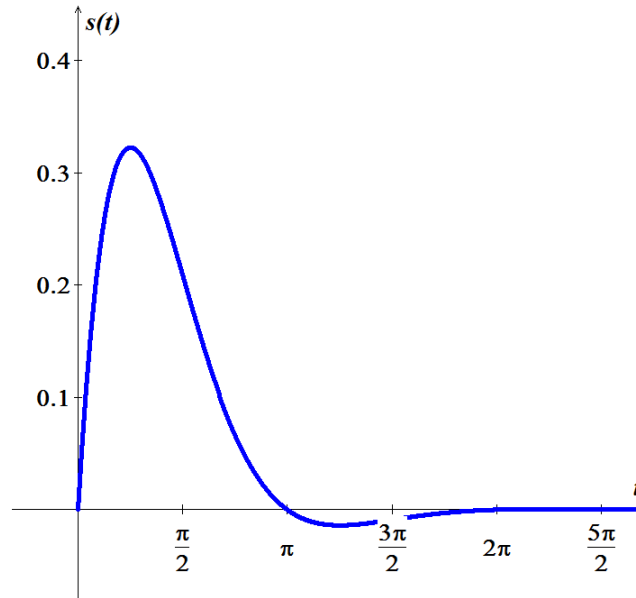
Exercise

Suppose a mass on a spring that is slowed by friction has the position function $s(t) = e^{-t} \sin t$

- Graph the position function. At what times does the oscillator pass through the position $s = 0$?
- Find the average value of the position on the interval $[0, \pi]$.
- Generalize part (b) and find the average value of the position on the interval $[n\pi, (n+1)\pi]$, for $n = 0, 1, 2, \dots$

Solution

a) $s(t) = e^{-t} \sin t = 0 \quad \sin t = 0 \rightarrow \underline{t = n\pi}$



b) $\int e^{-t} \sin t \, dt = -e^{-t} (\cos t + \sin t) - \int e^{-t} \sin t \, dt$

$$2 \int e^{-t} \sin t \, dt = -e^{-t} (\cos t + \sin t)$$

$$\text{Average} = \frac{1}{\pi} \int_0^{\pi} e^{-t} \sin t \, dt$$

$$= -\frac{1}{2\pi} e^{-t} (\cos t - \sin t) \Big|_0^{\pi}$$

$$= -\frac{1}{2\pi} (-e^{-\pi} - 1)$$

$$= \underline{\underline{\frac{1}{2\pi} (e^{-\pi} + 1)}}$$

		$\int \sin t$
+	e^{-t}	$-\cos t$
-	$-e^{-t}$	$-\sin t$
+	e^{-t}	$-\int \sin t \, dt$

c) $\text{Average} = \frac{1}{\pi} \int_{n\pi}^{(n+1)\pi} e^{-t} \sin t \, dt$

$$\begin{aligned}
&= -\frac{1}{2\pi} e^{-t} (\cos t - \sin t) \Big|_{n\pi}^{(n+1)\pi} \\
&= -\frac{1}{2\pi} \left(e^{-(n+1)\pi} (\cos((n+1)\pi) - \sin((n+1)\pi)) - e^{-n\pi} (\cos n\pi - \sin n\pi) \right) \\
&= -\frac{1}{2\pi} \left(e^{-(n+1)\pi} \cos((n+1)\pi) - e^{-n\pi} \cos n\pi \right) \\
&= \frac{e^{-n\pi}}{2\pi} (\cos n\pi - e^{-\pi} \cos(n+1)\pi) \\
&= \frac{e^{-n\pi}}{2\pi} ((-1)^n - e^{-\pi} (-1)^{n+1}) \\
&= \underline{(-1)^n \frac{e^{-n\pi}}{2\pi} (1 + e^{-\pi})}
\end{aligned}$$

Exercise

Given the region bounded by the graphs of $y = x \sin x$, $y = 0$, $x = 0$, $x = \pi$, find

- The area of the region.
- The volume of the solid generated by revolving the region about the x -axis
- The volume of the solid generated by revolving the region about the y -axis
- The centroid of the region

Solution

$$a) \quad A = \int_0^{\pi} x \sin x \, dx$$

		$\int \sin x$
+	x	$-\cos x$
-	1	$-\sin x$

$$= -x \cos x + \sin x \Big|_0^{\pi}$$

$$= \underline{\pi \text{ unit}^2}$$

$$b) \quad V = \pi \int_0^{\pi} (x \sin x)^2 \, dx$$

$$= \pi \int_0^{\pi} x^2 \sin^2 x \, dx$$

$$= \frac{\pi}{2} \int_0^{\pi} x^2 (1 - \cos 2x) \, dx$$

		$\int \cos 2x$
+	x^2	$\frac{1}{2} \sin 2x$
-	$2x$	$-\frac{1}{4} \cos 2x$
+	2	$-\frac{1}{8} \sin 2x$

$$\begin{aligned}
&= \frac{\pi}{2} \int_0^{\pi} (x^2 - x^2 \cos 2x) dx \\
&= \frac{\pi}{2} \left(\frac{1}{3} x^3 - \frac{1}{2} x^2 \sin 2x - \frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x \right) \Big|_0^{\pi} \\
&= \frac{\pi}{2} \left(\frac{1}{3} \pi^3 - \frac{\pi}{2} \right) \\
&= \frac{\pi^4}{6} - \frac{\pi^2}{4} \text{ unit}^3 \Big|
\end{aligned}$$

c) $V = 2\pi \int_0^{\pi} x(x \sin x) dx$

$$\begin{aligned}
&= 2\pi \int_0^{\pi} (x^2 \sin x) dx \\
&= 2\pi \left(-x^2 \cos x + 2x \sin x + 2 \cos x \right) \Big|_0^{\pi} \\
&= 2\pi (\pi^2 - 2 - 2) \\
&= 2\pi^3 - 8\pi \text{ unit}^3 \Big|
\end{aligned}$$

		$\int \sin x$
+	x^2	$-\cos x$
-	$2x$	$-\sin x$
+	2	$\cos x$

d) $m = \int_0^{\pi} x \sin x dx$

$$\begin{aligned}
&= -x \cos x + \sin x \Big|_0^{\pi} \\
&= \pi \Big|
\end{aligned}$$

From (a)

$$\begin{aligned}
M_x &= \frac{1}{2} \int_0^{\pi} (x \sin x)^2 dx \\
&= \frac{1}{2} \left(\frac{\pi^3}{6} - \frac{\pi}{4} \right) \Big|
\end{aligned}$$

From (b)

$$\begin{aligned}
M_y &= \int_0^{\pi} x(x \sin x) dx \\
&= \frac{2\pi^3 - 8\pi}{2\pi} \\
&= \pi^2 - 4 \Big|
\end{aligned}$$

From (c)

$$\bar{x} = \frac{M_y}{m} = \frac{\pi^2 - 4}{\pi} \approx 1.8684 \Big|$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{\pi} \left(\frac{\pi^3}{12} - \frac{\pi}{8} \right) = \frac{\pi^2}{12} - \frac{1}{8} \approx 0.6975 \Big|$$

Exercise

The region R is bounded by the curve $y = \ln x$ and the x -axis on the interval $[1, e]$. Find the volume of the solid that is generated when R is revolved in the following ways

- a) About the x -axis
- b) About the y -axis
- c) About the line $x = 1$
- d) About the line $y = 1$

Solution

- a) About the x -axis

$$V = \pi \int_1^e (\ln x)^2 dx$$

$$\text{Let } y = \ln x \Rightarrow x = e^y$$

$$dx = e^y dy$$

$$V = \pi \int_1^e y^2 e^y dy$$

		$\int e^y dy$
+	y^2	e^y
-	$2y$	e^y
+	2	e^y

$$= \pi (y^2 - 2y + 2) e^y \Big|_1^e$$

$$= \pi x ((\ln x)^2 - 2 \ln x + 2) \Big|_1^e$$

$$= \pi (e(1 - 2 + 2) - 2)$$

$$= \pi(e - 2) \text{ unit}^3$$

- b) About the y -axis

$$V = 2\pi \int_1^e x \ln x dx$$

		$\int x dx$
+	$\ln x$	$\frac{1}{2} x^2$
-	$\frac{1}{x}$	$\int \frac{1}{2} x^2$

$$\begin{aligned}
&= 2\pi \left[\frac{1}{2} x^2 \ln x - \int_1^e \frac{1}{2} x^2 \frac{1}{x} dx \right] \\
&= 2\pi \left[\frac{1}{2} x^2 \ln x - \frac{1}{2} \int_1^e x dx \right] \\
&= \pi \left(x^2 \ln x - \frac{1}{2} x^2 \right) \Big|_1^e \\
&= \pi \left(e^2 \ln e - \frac{1}{2} e^2 + \frac{1}{2} \right) \\
&= \frac{\pi}{2} (e^2 + 1) \text{ unit}^3
\end{aligned}$$

c) About the line $x = 1$

$$V = 2\pi \int_1^e (x-1) \ln x dx$$

		$\int (x-1) dx$
+	$\ln x$	$\frac{1}{2} x^2 - x$
-	$\frac{1}{x}$	$\int \left(\frac{1}{2} x^2 - x \right)$

$$\begin{aligned}
&= 2\pi \left[\left(\frac{1}{2} x^2 - x \right) \ln x - \int_1^e \left(\frac{1}{2} x^2 - x \right) \frac{1}{x} dx \right] \\
&= 2\pi \left[\left(\frac{1}{2} x^2 - x \right) \ln x - \int_1^e \left(\frac{1}{2} x - 1 \right) dx \right] \\
&= 2\pi \left(\left(\frac{1}{2} x^2 - x \right) \ln x - \left(\frac{1}{4} x^2 - x \right) \right) \Big|_1^e \\
&= 2\pi \left(\frac{1}{2} e^2 - e - \frac{1}{4} e^2 + e + \frac{1}{4} - 1 \right) \\
&= 2\pi \left(\frac{1}{4} e^2 - \frac{3}{4} \right) \\
&= \frac{\pi}{2} (e^2 - 3) \text{ unit}^3
\end{aligned}$$

d) About the line $y = 1$

$$\begin{aligned}
V &= \pi \int_1^e \left(1 - (1 - \ln x)^2 \right) dx \\
&= \pi \int_1^e \left(1 - 1 + 2 \ln x - (\ln x)^2 \right) dx
\end{aligned}$$

$$= \pi \int_1^e \left(2 \ln x - (\ln x)^2 \right) dx$$

$$\text{Let } y = \ln x \Rightarrow x = e^y$$

$$dx = e^y dy$$

		$\int e^y dy$
+	y^2	e^y
-	$2y$	e^y
+	2	e^y

$$\int (\ln x)^2 dx = \left(y^2 - 2y + 2 \right) e^y$$

$$= x \left((\ln x)^2 - 2 \ln x + 2 \right)$$

		$\int 2dx$
+	$\ln x$	$2x$
-	$\frac{1}{x}$	$\int 2x$

$$\int 2 \ln x dx = 2x \ln x - \int 2x \frac{1}{x} dx$$

$$= 2x \ln x - 2 \int dx$$

$$= 2x \ln x - 2x$$

$$V = \pi \int_1^e \left(2 \ln x - (\ln x)^2 \right) dx$$

$$= \pi \left(2x \ln x - 2x - x \left((\ln x)^2 - 2 \ln x + 2 \right) \right) \Big|_1^e$$

$$= \pi \left(2x \ln x - 2x - x (\ln x)^2 + 2x \ln x - 2x \right) \Big|_1^e$$

$$= \pi \left(4x \ln x - 4x - x (\ln x)^2 \right) \Big|_1^e$$

$$= \pi (4e - 4e - e + 4)$$

$$= \pi (4 - e) \text{ unit}^3$$

Exercise

A string stretched between the two points $(0, 0)$ and $(2, 0)$ is plucked by displacing the string h units at its midpoint. The motion of the string is modeled by a **Fourier Sine series** whose coefficients are given by

$$b_n = h \int_0^1 x \sin \frac{n\pi x}{2} dx + h \int_1^2 (-x+2) \sin \frac{n\pi x}{2} dx$$

Find b_n

Solution

$$b_n = h \int_0^1 x \sin \frac{n\pi x}{2} dx + h \int_1^2 (-x+2) \sin \frac{n\pi x}{2} dx$$

		$\int \sin \frac{n\pi x}{2} dx$	
+	x	$-\frac{2}{n\pi} \cos \frac{n\pi x}{2}$	$-x+2$
-	1	$-\frac{4}{n^2\pi^2} \sin \frac{n\pi x}{2}$	-1

$$= h \left(-\frac{2}{n\pi} x \cos \frac{n\pi x}{2} + \frac{4}{n^2\pi^2} \sin \frac{n\pi x}{2} \right) \Big|_0^1 + h \left(-\frac{2}{n\pi} (2-x) \cos \frac{n\pi x}{2} - \frac{4}{n^2\pi^2} \sin \frac{n\pi x}{2} \right) \Big|_1^2$$

$$= h \left(-\frac{2}{n\pi} \cos \frac{n\pi}{2} + \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} \right) + h \left(-\frac{4}{n^2\pi^2} \sin n\pi + \frac{2}{n\pi} \cos \frac{n\pi}{2} + \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} \right)$$

$$= h \left(\frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} - \frac{4}{n^2\pi^2} \sin n\pi + \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} \right) \quad \left(\cos \frac{n\pi}{2} = 0 \quad \sin n\pi = 0 \right)$$

$$= \frac{8h}{n^2\pi^2} \sin \frac{n\pi}{2}$$

$$= (-1)^n \frac{8h}{n^2\pi^2}$$