Section 1.2 – Polynomial Functions & Graphs

Polynomial Function

A *Polynomial function* $P(x)$ in x is a sum of the form is given by:

$$P(x)=a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$$

Where the coefficients $a_n,\ a_{n-1},\ \ldots, a_2,\ a_1, a_0$ are real numbers and the exponents are whole numbers.

$$a_n x^n$$

Degree

Leading Term

Leading Coefficient

Degree of f	***Form of f(x)***	***Graph of f(x)***
0	$f(x)=a_0$	A horizontal line
1	$f(x)=a_1 x + a_0$	A line with slope a_1
2	$f(x)=a_2 x^2 + a_1 x + a_0$	A parabola with a vertical axis

All polynomial functions are ***continuous functions***.

End Behavior $\left(a_n x^n\right)$

If n (degree) is ***even***:

If $a_n < 0 \quad \to \quad \begin{cases} x \to -\infty \ \Rightarrow \ f(x) \to -\infty \\ x \to \infty \ \Rightarrow \ f(x) \to -\infty \end{cases}$

If $a_n > 0 \quad \to \quad \begin{cases} x \to -\infty \ \Rightarrow \ f(x) \to \infty \\ x \to \infty \ \Rightarrow \ f(x) \to \infty \end{cases}$

If n (degree) is ***odd***:

If $a_n < 0 \quad \to \quad \begin{cases} x \to -\infty \ \Rightarrow \ f(x) \to \infty \\ x \to \infty \ \Rightarrow \ f(x) \to -\infty \end{cases}$

If $a_n > 0 \quad \to \quad \begin{cases} x \to -\infty \ \Rightarrow \ f(x) \to -\infty \\ x \to \infty \ \Rightarrow \ f(x) \to \infty \end{cases}$

The intermediate value *Theorem*

For any polynomial function $f(x)$ with real coefficients and $f(a) \neq f(b)$ for $a < b$, then f takes on every value between $f(a)$ and $f(b)$ in the interval $[a, b]$.

$\therefore$ $f(a)$ and $f(b)$ are the opposite signs. Then the function has a real zero between a and b.

Example

Using the intermediate value theorem, determine, if possible, whether the function has a real zero between *a* and *b*.

a) $f(x) = x^3 + x^2 - 6x; \quad a = -4, \ b = -2$

b) $f(x) = x^3 + x^2 - 6x; \quad a = -1, \ b = 3$

Solution

a) $f(x) = x^3 + x^2 - 6x; \quad a = -4, \ b = -2$

$$f(-4) = (-4)^3 + (-4)^2 - 6(-4)$$
$$= -24$$

$$f(-2) = (-2)^3 + (-2)^2 - 6(-2)$$
$$= 8$$

$\therefore$ $f(x)$ has a zero between -4 and -2.

b) $f(x) = x^3 + x^2 - 6x; \quad a = -1, \ b = 3$

$f(-1) = (-1)^3 + (-1)^2 - 6(-1) \ = 6$

$f(3) = (3)^3 + (3)^2 - 6(3) \ = 18$ *Can't be determined.*

The Rational Zeros *Theorem*

If the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ *has integer* coefficients, then

$$\text{possible rational zeros} = \frac{\text{possibilities for } a_0}{\text{possibilities for } a_n}$$

Example

Find all rational solutions of the equation: $3x^4 + 14x^3 + 14x^2 - 8x - 8 = 0$

Solution

Possibilities: $\pm\left\{\frac{8}{3}\right\} = \pm\left\{\frac{1,\ 2,\ 4,\ 8}{1,\ 3}\right\}$

$$= \pm\left\{1,\ 2,\ 4,\ 8,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3},\ \frac{8}{3}\right\}$$

The calculation will show that -2 is a zero.

$$\begin{array}{r|rrrrr}
-2 & 3 & 14 & 14 & -8 & -8 \\
 & & -6 & -16 & 4 & 8 \\
\hline
-\frac{2}{3} & 3 & 8 & -2 & -4 & 0 \\
 & & -2 & -4 & 4 & \\
\hline
 & 3 & 6 & -6 & 0 &
\end{array}$$

$\to 3x^3 + 8x^2 - 2x - 4 \to \pm\left\{\frac{4}{3}\right\} = \pm\left\{1,\ 2,\ 4,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3}\right\}$

$\to 3x^2 + 6x - 6 = 0 \Rightarrow \underline{x = -1 \pm \sqrt{3}}$

Hence, the polynomial has roots $\underline{x = -2,\ -\frac{2}{3},\ -1 \pm \sqrt{3}}$

Sketching

Example

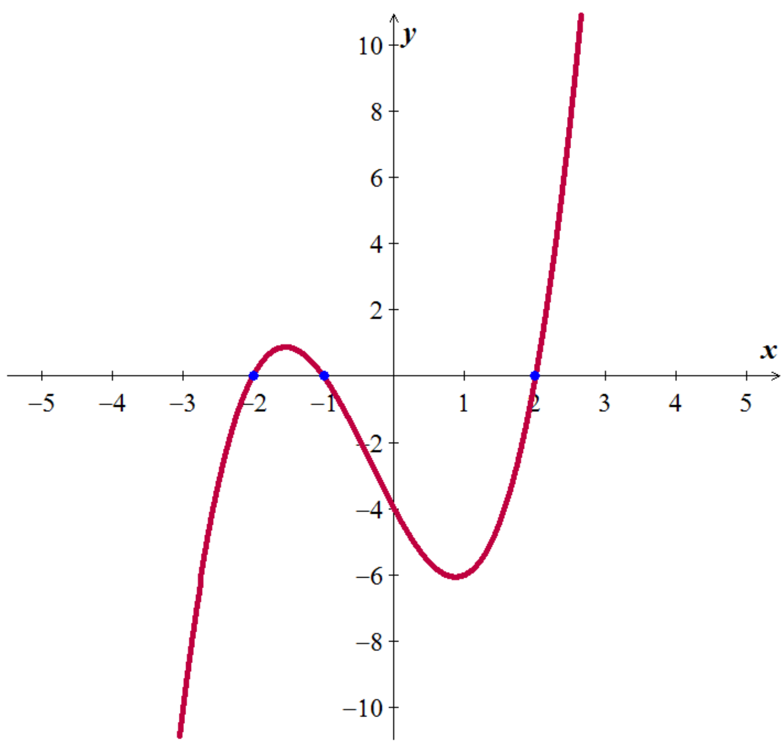
Let $f(x) = x^3 + x^2 - 4x - 4$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$$\begin{aligned} f(x) &= x^3 + x^2 - 4x - 4 \\ &= x^2(x+1) - 4(x+1) \\ &= (x+1)\left(x^2 - 4\right) \\ &= (x+1)(x+2)(x-2) \end{aligned}$$

The zeros of $f(x)$ (x-intercepts) are: $-2, \ -1, \ and \ 2$

Interval	$-\infty$ -2	-1	**0** 2	∞
Sign of $f(x)$	**–**	**+**	**–**	**+**
Position	**Below *x*-axis**	**Above *x*-axis**	**Below *x*-axis**	**Above *x*-axis**



We can conclude from the chart and the graph that:

$$f(x) > 0 \quad if \ \ x \ is \ in \ \ (-2, \ -1) \cup (2, \ \infty)$$

$$f(x) < 0 \quad if \ \ x \ is \ in \ \ (-\infty, \ -2) \cup (-1, \ 2)$$

Example

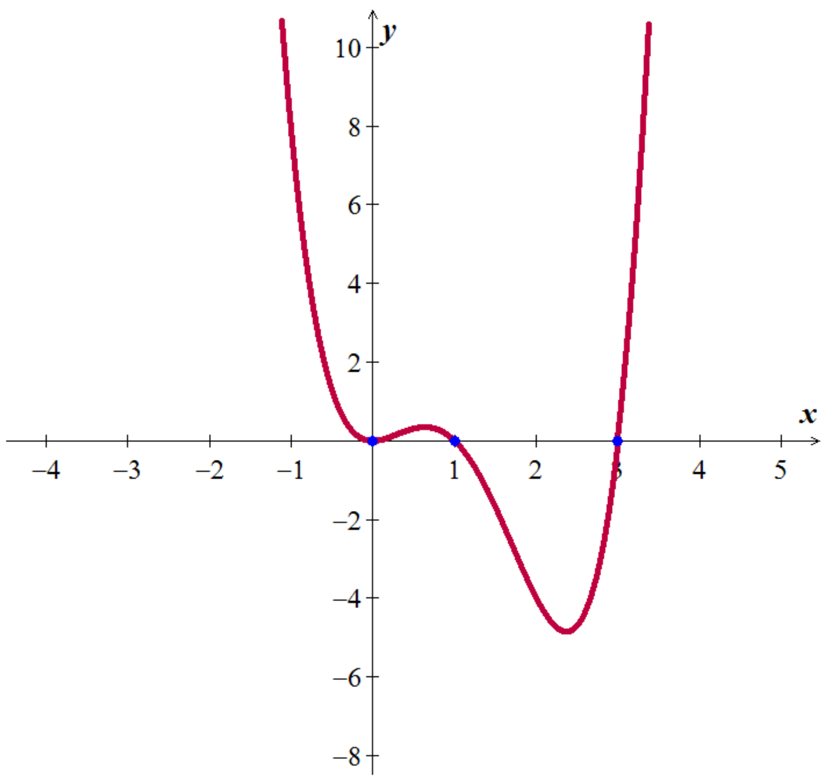
Let $f(x) = x^4 - 4x^3 + 3x^2$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$$f(x) = x^2\left(x^2 - 4x + 3\right)$$
$$= x^2(x-1)(x-3)$$

The zeros are: 0, 1, 3. Since the factor x^2 is always positive, it has no factor

$-\infty$	1	2	3	∞
+		–		+



$f(x) > 0 \quad if \;\; x \text{ is in } \;(-\infty,\ 0) \cup (0,\ 1) \cup (3,\ \infty)$

$f(x) < 0 \quad if \;\; x \text{ is in } \;(1,\ 3)$

Exercises *Section* 1.2 – Polynomial Functions & Graphs

Find the quotient and remainder if $f(x)$ is divided by $p(x)$

1. $f(x) = 2x^4 - x^3 + 7x - 12; \quad p(x) = x^2 - 3$

2. $f(x) = 3x^3 + 2x - 4; \quad p(x) = 2x^2 + 1$

3. $f(x) = 7x + 2; \quad p(x) = 2x^2 - x - 4$

4. $f(x) = 9x + 4; \quad p(x) = 2x - 5$

Use the remainder theorem to find $f(c)$

5. $f(x) = x^4 - 6x^2 + 4x - 8; \quad c = -3$

6. $f(x) = x^4 + 3x^2 - 12; \quad c = -2$

7. Use the factor theorem to show that $x - c$ is a factor of $f(x)$: $f(x) = x^3 + x^2 - 2x + 12; \quad c = -3$

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second

8. $2x^3 - 3x^2 + 4x - 5; \quad x - 2$

9. $5x^3 - 6x^2 + 15; \quad x - 4$

10. $9x^3 - 6x^2 + 3x - 4; \quad x - \frac{1}{3}$

Use the synthetic division to find $f(c)$

11. $f(x) = 2x^3 + 3x^2 - 4x + 4; \quad c = 3$

12. $f(x) = 8x^5 - 3x^2 + 7; \quad c = \frac{1}{2}$

13. $f(x) = x^3 - 3x^2 - 8; \quad c = 1 + \sqrt{2}$

14. Use the synthetic division to show that c is a zero of $f(x)$: $f(x) = 3x^4 + 8x^3 - 2x^2 - 10x + 4; \quad c = -2$

15. Use the synthetic division to show that c is a zero of $f(x)$: $f(x) = 27x^4 - 9x^3 + 3x^2 + 6x + 1; \quad c = -\frac{1}{3}$

Find all values of k such that $f(x)$ is divisible by the given linear polynomial:

16. $f(x) = kx^3 + x^2 + k^2x + 3k^2 + 11; \quad x + 2$

17. $f(x) = x^3 + k^3x^2 + +2kx - 2k^4; \quad x - 1.6$

18. $f(x) = k^2x^3 - 4kx + 3; \quad x - 1$

Find all solutions of the equation

19. $x^3 - x^2 - 10x - 8 = 0$

20. $x^3 + x^2 - 14x - 24 = 0$

21. $2x^3 - 3x^2 - 17x + 30 = 0$

22. $12x^3 + 8x^2 - 3x - 2 = 0$

23. $x^4 + 3x^3 - 30x^2 - 6x + 56 = 0$

24. $3x^5 - 10x^4 - 6x^3 + 24x^2 + 11x - 6 = 0$

27. $2x^4 - 9x^3 + 9x^2 + x - 3 = 0$

28. $8x^3 + 18x^2 + 45x + 27 = 0$

25. $6x^5 + 19x^4 + x^3 - 6x^2 = 0$

26. $x^4 - x^3 - 9x^2 + 3x + 18 = 0$

29. $3x^3 - x^2 + 11x - 20 = 0$

30. $6x^4 + 5x^3 - 17x^2 - 6x = 0$

31. If $f(x) = 3x^3 - kx^2 + x - 5k$, find a number k such that the graph of f contains the point $(-1, 4)$.

32. If $f(x) = kx^3 + x^2 - kx + 2$, find a number k such that the graph of f contains the point $(2, 12)$.

33. If one zero of $f(x) = x^3 - 2x^2 - 16x + 16k$ is 2, find two other zeros.

34. If one zero of $f(x) = x^3 - 3x^2 - kx + 12$ is -2, find two other zeros.

35. Find a polynomial $f(x)$ of degree 3 that has the zeros -1, 2, 3; and satisfies the given condition: $f(-2) = 80$

36. Find a polynomial $f(x)$ of degree 3 that has the zeros $-2i$, $2i$, 3; and satisfies the given condition: $f(1) = 20$

37. Find a polynomial $f(x)$ of degree 4 with leading coefficient 1 such that both -4 and 3 are zeros of multiplicity 2, and sketch the graph of f.

Find the zeros of the following functions and state the multiplicity of each zero

38. $f(x) = x^2(3x+2)(2x-5)^3$

39. $f(x) = 4x^5 + 12x^4 + 9x^3$

40. $f(x) = \left(x^2 + x - 12\right)^3 \left(x^2 - 9\right)^2$

41. $f(x) = \left(6x^2 + 7x - 5\right)^4 \left(4x^2 - 1\right)^2$

42. $f(x) = x^4 + 7x^2 - 144$

43. $f(x) = x^4 + 21x^2 - 100$

(**44 – 102**) Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f

44. $f(x) = x^4 - 4x^2$

45. $f(x) = x^4 + 3x^3 - 4x^2$

46. $f(x) = x^3 + 2x^2 - 4x - 8$

47. $f(x) = x^3 - 3x^2 - 9x + 27$

48. $f(x) = -x^4 + 12x^2 - 27$

49. $f(x) = x^2(x+2)(x-1)^2(x-2)$

50. $f(x) = 2x^3 + 11x^2 - 7x - 6$

51. $f(x) = x^3 + 2x^2 - 5x - 6$

52. $f(x) = x^3 + 8x^2 + 11x - 20$

53. $f(x) = x^4 + x^2 - 2$

54. $f(x) = x^4 - x^3 - 6x^2 + 4x + 8$

55. $f(x) = 4x^5 - 8x^4 - x + 2$

56. $f(x) = 2x^4 - x^3 - 5x^2 + 2x + 2$

57. $f(x) = x^3 - x^2 - 10x - 8$

58. $f(x) = x^3 + x^2 - 14x - 24$

59. $f(x) = 2x^3 - 3x^2 - 17x + 30$

60. $f(x) = 12x^3 + 8x^2 - 3x - 2$

61. $f(x) = x^3 + x^2 - 6x - 8$

62. $f(x) = x^3 - 19x - 30$

63. $f(x) = 2x^3 + x^2 - 25x + 12$

64. $f(x) = 3x^3 + 11x^2 - 6x - 8$

65. $f(x) = 2x^3 + 9x^2 - 2x - 9$

66. $f(x) = x^3 + 3x^2 - 6x - 8$

67. $f(x) = 3x^3 - x^2 - 6x + 2$

68. $f(x) = x^3 - 8x^2 + 8x + 24$

69. $f(x) = x^3 - 7x^2 - 7x + 69$

70. $f(x) = x^3 - 3x - 2$

71. $f(x) = x^3 - 2x + 1$

72. $f(x) = x^3 - 2x^2 - 11x + 12$

73. $f(x) = x^3 - 2x^2 - 7x - 4$

74. $f(x) = x^3 - 10x - 12$

75. $f(x) = x^3 - 5x^2 + 17x - 13$

76. $f(x) = 6x^3 + 25x^2 - 24x + 5$

77. $f(x) = 8x^3 + 18x^2 + 45x + 27$

78. $f(x) = 3x^3 - x^2 + 11x - 20$

79. $f(x) = x^4 - x^3 - 9x^2 + 3x + 18$

80. $f(x) = 2x^4 - 9x^3 + 9x^2 + x - 3$

81. $f(x) = 6x^4 + 5x^3 - 17x^2 - 6x$

82. $f(x) = x^4 - 2x^2 - 16x - 15$

83. $f(x) = x^4 - 2x^3 - 5x^2 + 8x + 4$

84. $f(x) = 2x^4 - 17x^3 + 4x^2 + 35x - 24$

85. $f(x) = x^4 + x^3 - 3x^2 - 5x - 2$

86. $f(x) = 6x^4 - 17x^3 - 11x^2 + 42x$

87. $f(x) = x^4 - 5x^2 - 2x$

88. $f(x) = 3x^4 - 4x^3 - 11x^2 + 16x - 4$

89. $f(x) = 6x^4 + 23x^3 + 19x^2 - 8x - 4$

90. $f(x) = 4x^4 - 12x^3 + 3x^2 + 12x - 7$

91. $f(x) = 2x^4 - 9x^3 - 2x^2 + 27x - 12$

92. $f(x) = 2x^4 - 19x^3 + 51x^2 - 31x + 5$

93. $f(x) = 4x^4 - 35x^3 + 71x^2 - 4x - 6$

94. $f(x) = 2x^4 + 3x^3 - 4x^2 - 3x + 2$

95. $f(x) = x^4 + 3x^3 - 30x^2 - 6x + 56$

96. $f(x) = 3x^5 - 10x^4 - 6x^3 + 24x^2 + 11x - 6$

97. $f(x) = 6x^5 + 19x^4 + x^3 - 6x^2$

98. $f(x) = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$

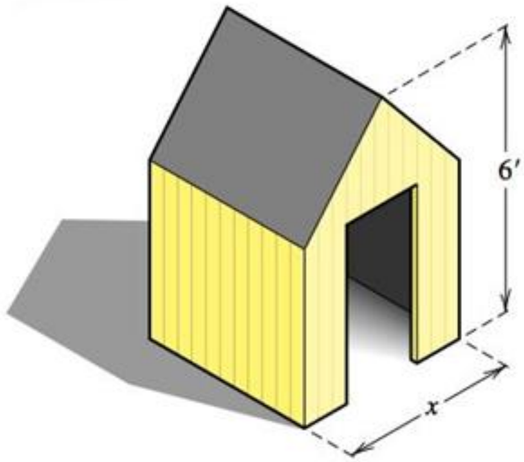
99. $f(x) = x^5 - x^4 - 7x^3 + 7x^2 + 12x - 12$

100. $f(x) = x^5 - 2x^3 - 8x$

101. $f(x) = x^5 - 7x^4 + 19x^3 - 37x^2 + 60x - 36$

102. $f(x) = 3x^6 - 10x^5 - 29x^4 + 34x^3 + 50x^2 - 24x - 24$

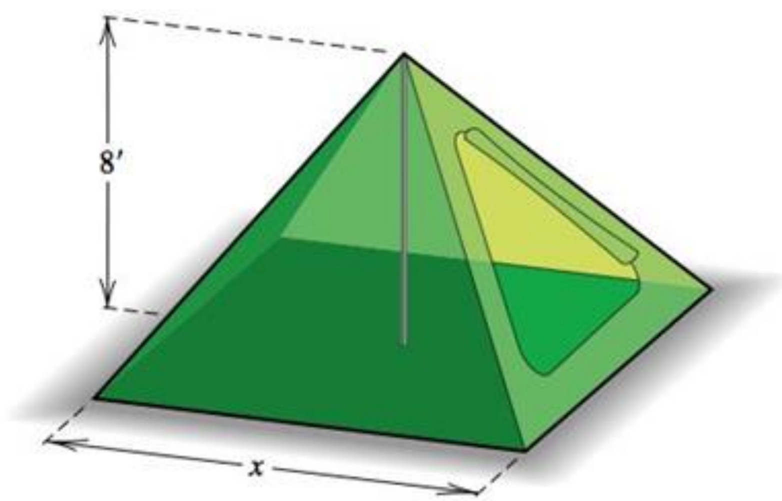
103. A storage shelter is to be constructed in the shape of a cube with a triangular prism forming the roof. The length x of a side of the cube is yet to be determined.



a) If the total height of the structure is 6 *feet*, show that its volume V is given by
$V = x^3 + \frac{1}{2}x^2(6-x)$

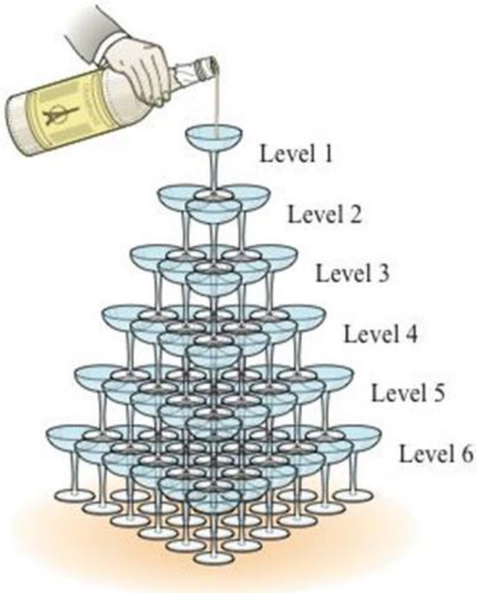
b) Determine x so that the volume is $80\ ft^3$

104. A canvas camping tent is to be constructed in the shape of a pyramid with a square base. An 8–*foot* pole will form the center support. Find the length x of a side of the base so that the total amount of canvas needed for the sides and bottom is $384\ ft^2$



105. Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

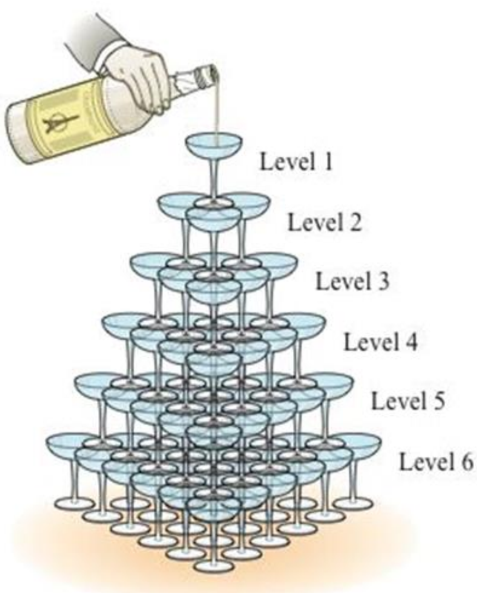
$$T(k) = \frac{1}{6}\left(k^3 + 3k^2 + 2k\right)$$



Where k is the number of levels in the pyramid. If 220 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

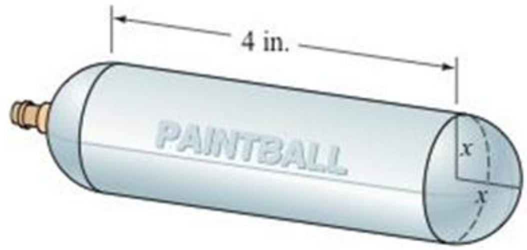
106. Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T(k) = \frac{1}{6}\left(2k^3 + 3k^2 + k\right)$$



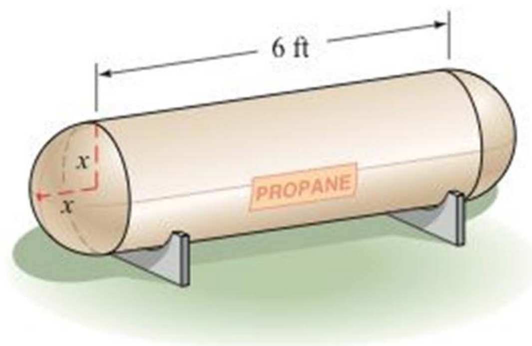
Where k is the number of levels in the pyramid. If 140 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

107. A carbon dioxide cartridge for a paintball rifle has the shape of a right circular cylinder with a hemisphere at each end. The cylinder is 4 *inches* long, and the volume of the cartridge is $2\pi \ \ in^3$.

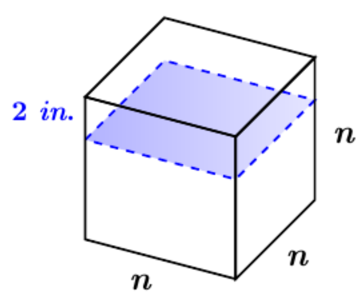


The common interior radius of the cylinder and the hemispheres is denoted by *x*. Estimate the length of the radius *x*.

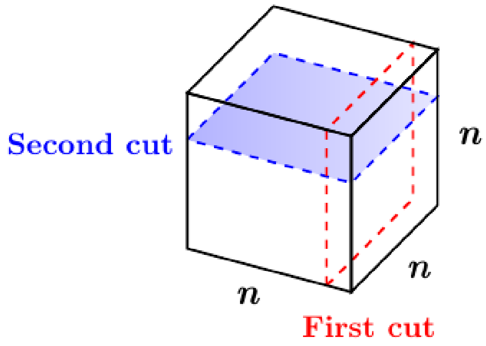
108. A propane tank has the shape of a circular cylinder with a hemisphere at each end. The cylinder is 6 *feet* long and the volume of the tank is $9\pi \ \ ft^3$. Find the length of the radius *x*.



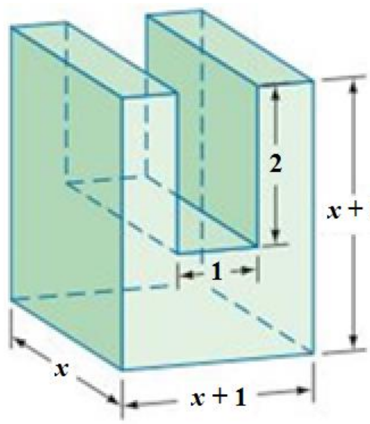
109. A cube measures *n* inches on each edge. If a slice 2 *inches* thick is cut from one face of the cube, the resulting solid has a volume of $567 \ \ in^3$. Find *n*.



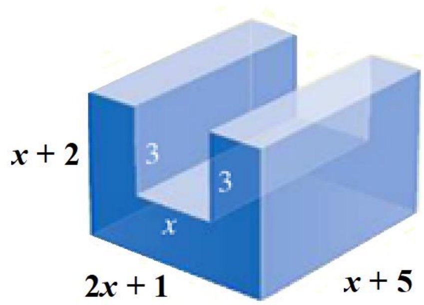
110. A cube measures *n* inches on each edge. If a slice 1 *inch* thick is cut from one face of the cube and then a slice 3 inches thick is cut from another face of the cube, the resulting solid has a volume of $1560 \ \ in^3$. Find the dimensions of the original cube.



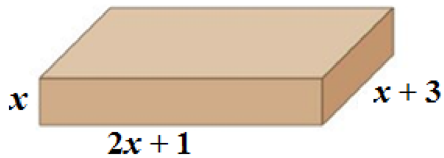
111. For what value of x will the volume of the following solid be $112\ in^3$



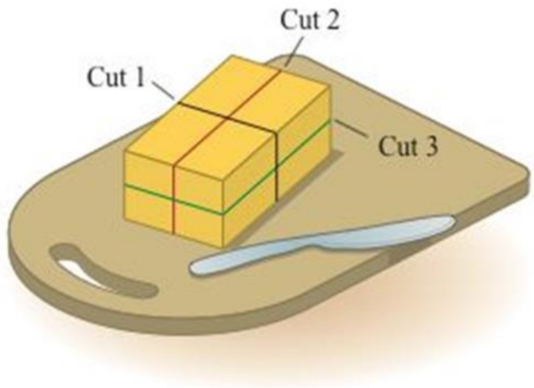
112. For what value of x will the volume of the following solid be $208\ in^3$



113. The length of rectangular box is 1 *inch* more than twice the height of the box, and the width is 3 *inches* more than the height. If the volume of the box is $126\ in^3$, find the dimensions of the box.



114. One straight cut through a thick piece of cheese produces two pieces. Two straight cuts can produce a maximum of four pieces. Two straight cuts can produce a maximum of four pieces. Three straight cuts can produce a maximum of eight pieces.



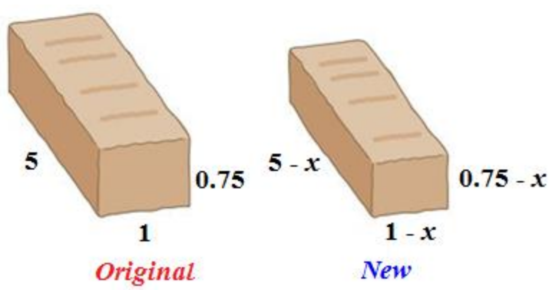
You might be inclined to think that every additional cut doubles number of pieces. However, for four straight cuts, you get a maximum of 15 pieces. The maximum number of pieces P that can be produced by n straight cuts is given by

$$P(n) = \frac{n^3 + 5n + 6}{6}$$

a) Determine number of pieces that can be produces by five straight cuts.
b) What is the fewest number of straight cuts that are needed to produce 64 pieces?

115. The number of ways one can select three cards from a group of n cards (the order of the selection matters), where $n \geq 3$, is given by $P(n) = n^3 - 3n^2 + 2n$. For a certain card trick, a magician has determined that there are exactly 504 *ways* to choose three cards from a given group. How many cards are in the group?

116. A nutrition bar in the shape of a rectangular solid measure 0.75 *in*. by 1 *in*. by 5 *inches*.



To reduce costs, the manufacturer has decided to decrease each of the dimensions of the nutrition bar by x *inches*, what value of x will produce a new bar with a volume that is $0.75\ in^3$ less than the present bar's volume.

117. A rectangular box is square on two ends and has length plus girth of 81 *inches*. (Girth: distance *around* the box). Determine the possible lengths $l\ (l > w)$ of the box if its volume is $4900\ in^3$.