Section 3.5 – The Ratio and Root Tests

Theorem – The Ratio Test

Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$$

Then

a) the series ***converges*** if $\rho < 1$,

b) the series ***diverges*** if $\rho > 1$, or ρ is infinite

c) the test is ***inconclusive*** if $\rho = 1$,

The value ρ doesn't mean the sum of the series.

Example

Investigate the convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$

Solution

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}+5}{3^{n+1}}}{\frac{2^n+5}{3^n}}$$

$$= \frac{1}{3}\frac{2^{n+1}+5}{2^n+5}$$

$$= \frac{1}{3}\cdot\frac{2+5\cdot 2^{-n}}{1+5\cdot 2^{-n}}$$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

$$= \lim_{n\to\infty} \frac{1}{3}\cdot\frac{2+5\cdot 2^{-n}}{1+5\cdot 2^{-n}}$$

$$= \frac{1}{3}\cdot\frac{2}{1}$$

$$= \frac{2}{3} < 1$$

The series ***converges*** since $\rho < 1$.

$$\sum_{n=0}^{\infty}\frac{2^n+5}{3^n}=\sum_{n=0}^{\infty}\frac{2^n}{3^n}+\sum_{n=0}^{\infty}\frac{5}{3^n}$$

$$=\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n+\sum_{n=0}^{\infty}\frac{5}{3^n}$$

$$=\frac{1}{1-\frac{2}{3}}+\frac{5}{1-\frac{1}{3}}$$

$$=\frac{21}{2}$$

Example

Investigate the convergence of the series $\sum_{n=1}^{\infty}\frac{(2n)!}{n!n!}$

Solution

$$\frac{a_{n+1}}{a_n}=\frac{\frac{(2(n+1))!}{(n+1)!(n+1)!}}{\frac{(2n)!}{n!n!}}$$

$$=\frac{1}{(n+1)(n+1)}\frac{(2n+2)!}{(2n)!}$$

$$=\frac{(2n+2)(2n+1)}{(n+1)(n+1)}$$

$$=\frac{2(n+1)(2n+1)}{(n+1)(n+1)}$$

$$=\frac{4n+1}{n+1}$$

$$\rho=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$$

$$=\lim_{n\to\infty}\frac{4n+1}{n+1}$$

$$=4>1$$

The series ***diverges*** since $\rho > 1$.

Example

Investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{4^n n!n!}{(2n)!}$

Solution

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}(n+1)!(n+1)!}{(2(n+1))!} \cdot \frac{(2n)!}{4^n n!n!}$$

$$= \frac{4(n+1)(n+1)}{(2n+2)(2n+1)}$$

$$= \frac{4(n+1)}{2(2n+1)}$$

$$= \frac{2(n+1)}{2n+1}$$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

$$= \lim_{n\to\infty} \frac{2n+2}{2n+1}$$

$$=1$$

Because the limit is $\rho = 1$, we can't decide from the *Ratio Test* whether the series converges.

However, since a_{n+1} *always* $>$ a_n, then the series ***diverges***.

$$a_{n+1} \ ? \ a_n$$

$$\frac{4^{n+1}(n+1)!(n+1)!}{(2(n+1))!} \ ? \ \frac{4^n n!n!}{(2n)!}$$

$$\frac{4^{n+1}(n+1)!(n+1)!}{4^n n!n!} \ ? \ \frac{(2(n+1))!}{(2n)!}$$

$$4(n+1)(n+1) \ ? \ (2n+1)(2n+2)$$

$$4(n+1)(n+1) \ ? \ 2(n+1)(2n+1)$$

$$2n+2 > 2n+1$$

$$a_{n+1} > a_n$$

Theorem – The Root Test

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$$

Then

- ***a)*** the series ***converges*** if $\rho < 1$,
- ***b)*** the series ***diverges*** if $\rho > 1$, or ρ is infinite
- ***c)*** the test is ***inconclusive*** if $\rho = 1$,

Example

Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges or diverges using the Root Test

Solution

$$\sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}}$$

$$= \frac{\left(\sqrt[n]{n}\right)^2}{2}$$

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n}$$

$$= \lim_{n\to\infty} \frac{n^{2/n}}{2}$$

$$= \frac{\infty^0}{2}$$

$$= \frac{1}{2} < 1$$

The series ***converges*** by the *Root Test*.

Example

Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ converges or diverges using the Root Test

Solution

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{2^n}{n^3}}$$

$$= \frac{\sqrt[n]{2^n}}{\sqrt[n]{n^3}}$$

$$= \frac{2}{\left(\sqrt[n]{n}\right)^3}$$

$$\rho = \lim_{n\to\infty} \frac{2}{\left(\sqrt[n]{n}\right)^3}$$

$$= \frac{2}{1}$$

$$\underline{= 2 > 1}$$

The series ***diverges*** by the *Root Test*.

Example

Determine if the series $\sum_{n=1}^{\infty}\left(\frac{1}{1+n}\right)^n$ converges or diverges using the Root Test

Solution

$$\sqrt[n]{\left(\frac{1}{1+n}\right)^n} = \frac{1}{1+n}$$

$$\lim_{n\to\infty} \frac{1}{1+n} = 0 < 1$$

The series ***converges*** by the *Root Test*.

Exercises Section 3.5 – The Ratio and Root Tests

Use the ***Ratio Test*** to determine if the series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}}{n3^{n-1}}$

3. $\displaystyle\sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 (n+2)!}{n!3^{2n}}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n5^n}{(2n+3)\ln(n+1)}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{99^n}{n!}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{2^n}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

12. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{3^n}$

13. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

14. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{6}{5}\right)^n$

15. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{7}{8}\right)^n$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4^n}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n^4}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$

20. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(\frac{3}{2}\right)^n}{n^2}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n3^n}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

Use the ***Root Test*** to determine if the series converges or diverges.

24. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$

25. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5}\right)^n$

26. $\displaystyle\sum_{n=1}^{\infty} \left(\ln\left(e^2+\frac{1}{n}\right)\right)^{n+1}$

27. $\displaystyle\sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$

28. $\displaystyle\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

32. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

33. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n}{n+1}\right)^{n}$

34. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3n+2}{n+3}\right)^{n}$

35. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n-2}{5n+1}\right)^{n}$

36. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n}}{(\ln n)^{n}}$

37. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{-3n}{2n+1}\right)^{3n}$

38. $\displaystyle\sum_{n=1}^{\infty}\left(2\sqrt[n]{n}+1\right)^{n}$

39. $\displaystyle\sum_{n=0}^{\infty}e^{-3n}$

40. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{3^{n}}$

41. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{500}\right)^{n}$

42. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n^{2}}\right)^{n}$

43. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\ln n}{n}\right)^{n}$

44. $\displaystyle\sum_{n=2}^{\infty}\frac{n}{(\ln n)^{n}}$

Use any method to determine if the series converges or diverges.

45. $\displaystyle\sum_{n=1}^{\infty}\frac{n^{\sqrt{2}}}{2^{n}}$

46. $\displaystyle\sum_{n=1}^{\infty}n^{2}e^{-n}$

47. $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{10^{n}}$

48. $\displaystyle\sum_{n=1}^{\infty}\frac{(\ln n)^{n}}{n^{n}}$

49. $\displaystyle\sum_{n=1}^{\infty}\frac{n2^{n}(n+1)!}{3^{n}n!}$

50. $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^{2}}{(2n)!}$

51. $\displaystyle\sum_{n=1}^{\infty}\frac{n^{2}+1}{n^{3}+1}$

52. $\displaystyle\sum_{n=1}^{\infty}\left|\sin\frac{1}{n^{2}}\right|$

53. $\displaystyle\sum_{n=8}^{\infty}\frac{1}{\pi^{n}+5}$

54. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{(\ln n)^{3}}$

55. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\pi^{n}-n^{\pi}}$

56. $\displaystyle\sum_{n=0}^{\infty}\frac{1+n}{2+n}$

57. $\displaystyle\sum_{n=1}^{\infty}\frac{1+n^{4/3}}{2+n^{5/3}}$

58. $\displaystyle\sum_{n=1}^{\infty}\frac{n^{2}}{1+n\sqrt{n}}$

59. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)(\ln\ln n)^{2}}$

60. $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n(\ln n)\sqrt{\ln\ln n}}$

61. $\displaystyle\sum_{n=1}^{\infty}\frac{1+(-1)^{n}}{\sqrt{n}}$

62. $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{n^{2}e^{n}}$

63. $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)!6^{n}}{(3n)!}$

64. $\displaystyle\sum_{n=2}^{\infty}\frac{\sqrt{n}}{3^{n}\ln n}$

65. $\displaystyle\sum_{n=0}^{\infty}\frac{n^{100}2^{n}}{\sqrt{n!}}$

66. $\displaystyle\sum_{n=1}^{\infty}\frac{1+n!}{(1+n)!}$

67. $\displaystyle\sum_{n=1}^{\infty}\frac{2^{n}}{3^{n}-n^{3}}$

68. $\displaystyle\sum_{n=1}^{\infty}\frac{n^{n}}{\pi^{n}n!}$

69. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

70. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$

71. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

72. $\displaystyle\sum_{n=1}^{\infty} \frac{100}{n}$

73. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$

74. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2\pi}{3}\right)^n$

75. $\displaystyle\sum_{n=1}^{\infty} \frac{5n}{2n-1}$

76. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$

77. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3^{n-2}}{2^n}$

78. $\displaystyle\sum_{n=1}^{\infty} \frac{10}{3\sqrt{n^3}}$

79. $\displaystyle\sum_{n=1}^{\infty} \frac{10n+3}{n2^n}$

80. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{4n^2-1}$

81. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{3^n}$

82. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n\,7^n}$

83. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

84. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$

85. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{\ln(k+1)}\right)^k$

86. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2(\ln k)^2}$

87. $\displaystyle\sum_{k=3}^{\infty} \frac{1}{\ln k}$

88. $\displaystyle\sum_{k=2}^{\infty} \frac{5\ln k}{k}$

89. $\displaystyle\sum_{k=1}^{\infty} \ln\left(\frac{k+2}{k+1}\right)$

90. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2 \ln k}$

91. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^{\ln k}}$

92. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

93. $\dfrac{1+\sqrt{2}}{2} + \dfrac{1+\sqrt{3}}{4} + \dfrac{1+\sqrt{4}}{8} + \cdots$

94. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{3\cdot 5\cdot 7\cdots(2n+1)}$

95. $\displaystyle\sum_{n=1}^{\infty} \frac{3\cdot 5\cdot 7\cdots(2n+1)}{18^n(2n-1)n!}$

96. Use the integral test to show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges. Show that the sum s of the series is less than $\dfrac{\pi}{2}$

97. Use the root test to show that $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}}{n^n}$ converges

98. Use the root test to test that $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ converges

99. Try to use the ratio test to determine whether $\sum_{n=1}^{\infty} \frac{2^{2n}(n!)^2}{(2n)!}$ converges. What happen?

Now observe that $\frac{2^{2n}(n!)^2}{(2n)!} = \frac{\left[2n(2n-2)(2n-4)\ \cdots\ 6\times 4\times 2\right]^2}{2n(2n-1)(2n-2)\ \cdots\ 3\times 2\times 1}$

$$= \frac{2n}{2n-1} \times \frac{2n-2}{2n-3} \times \frac{4}{3} \times \frac{2}{1}$$

Does the given series converge? Why or why not?

100. Suppose $a_n > 0$ and $\frac{a_{n+1}}{a_n} \ge \frac{n}{n+1}$ for all n. Show that $\sum_{n=1}^{\infty} a_n$ diverges.

$\left(a_n \ge \frac{K}{n}\ \textit{for some constant } K\right)$

101. Working in the early 1600s, the mathematicians Wallis, Pascal, and Fermat were calculating the area of the region under the curve $y = x^p$ between $x = 0$ and $x = 1$, where p is the positive integer. Using arguments that predated the Fundamental Theorem of Calculus, they were able to prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{k}{n}\right)^p = \frac{1}{p+1}$$

Use Riemann sums and integrals to verify this limit.

102. Complete the following steps to find the values of $p > 0$ for which the series $\sum_{k=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2k-1)}{p^k k!}$ converges

a) Use the Ratio Test to show that $\sum_{k=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2k-1)}{p^k k!}$ converges for $p > 2$.

b) Use Stirling's formula, $k! = \sqrt{2\pi k}\, k^k e^{-k}$ for large k, to determine whether the series converges when $p = 2$.

$$\left(\textit{Hint}:\ 1\cdot 3\cdot 5\cdots(2k-1) = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2k-1)2k}{2\cdot 4\cdot 6\cdots 2k}\right)$$