Solution   *Section* 1.6 – Exact Differential Equations

Exercise

Solve the differential equation $\quad (2x+y)dx+(x-6y)dy=0$

Solution

$$\frac{\partial\psi}{\partial x}=M=2x+y \quad\Rightarrow\quad M_y=1$$

$$\frac{\partial\psi}{\partial y}=N=x-6y \quad\Rightarrow\quad N_x=1$$

$$\Rightarrow M_y=N_x$$

$$\frac{\partial\psi}{\partial x}=2x+y \quad\Rightarrow\quad \psi=\int(2x+y)dx=x^2+xy+h(y)$$

$$\psi_y=x+h'(y)=x-6y \quad\Rightarrow\quad h'(y)=-6y$$

$$h(y)=\int -6y\,dy=-3y^2$$

$$\psi(x,y)=\boxed{x^2+xy-3y^2=C}$$

Exercise

Solve the differential equation $\quad (2x+3)dx+(2y-2)dy=0$

Solution

$$\frac{\partial\psi}{\partial x}=M=2x+3 \quad\Rightarrow\quad M_y=2$$

$$\frac{\partial\psi}{\partial y}=N=2y-2 \quad\Rightarrow\quad N_x=2$$

$$\Rightarrow M_y=N_x$$

$$\frac{\partial\psi}{\partial x}=2x+3 \quad\Rightarrow\quad \psi=\int(2x+3)dx=x^2+3x+h(y)$$

$$\psi_y=h'(y)=2y-2 \quad\Rightarrow\quad h(y)=\int(2y-2)dy$$

$$=y^2-2y+C$$

$$\psi(x,y)=\boxed{x^2+3x+y^2-2y=C}$$

Exercise

Solve the differential equation $\left(1-y\sin x\right)+\left(\cos x\right)y'=0$

Solution

$$\frac{\partial \psi}{\partial x}=M=1-y\sin x \quad \Rightarrow \quad M_y=-\sin x$$

$$\frac{\partial \psi}{\partial y}=N=\cos x \quad \Rightarrow \quad N_x=-\sin x$$

$$\Rightarrow M_y=N_x$$

$$\frac{\partial \psi}{\partial x}=1-y\sin x \quad \Rightarrow \quad \psi=\int\left(1-y\sin x\right)dx=x+y\cos x+h\left(y\right)$$

$$\psi_y=\cos x+h'\left(y\right)=\cos x \quad \Rightarrow \quad h'\left(y\right)=0$$

$$h\left(y\right)=C$$

$$\psi\left(x,\ y\right)=\underline{x+y\cos x=C}$$

Exercise

Solve the differential equation $\frac{dy}{dx}=-\frac{ax+by}{bx+cy}$

Solution

$$\left(bx+cy\right)dy=-\left(ax+by\right)dx$$

$$\left(ax+by\right)dx+\left(bx+cy\right)dy=0$$

$$\frac{\partial \psi}{\partial x}=M=ax+by \quad \Rightarrow \quad M_y=b$$

$$\frac{\partial \psi}{\partial y}=N=bx+cy \quad \Rightarrow \quad N_x=b$$

$$\Rightarrow M_y=N_x$$

$$\frac{\partial \psi}{\partial x}=ax+by \quad \Rightarrow \quad \psi=\int\left(ax+by\right)dx=\frac{1}{2}ax^2+bxy+h\left(y\right)$$

$$\psi_y=bx+h'\left(y\right)=bx+cy \quad \Rightarrow \quad h'\left(y\right)=cy$$

$$h\left(y\right)=\int cydy=\frac{1}{2}cy^2$$

$$\psi\left(x,\ y\right)=\frac{1}{2}ax^2+bxy+\frac{1}{2}cy^2=D$$

$$\underline{ax^2+2bxy+cy^2=E} \qquad \left(E=2D\right)$$

Exercise

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{3x^2 + y}{3y^2 - x}$

Solution

$$\left(3x^2 + y\right)dx - \left(3y^2 - x\right)dy = 0$$

$$\left.\begin{array}{l} \dfrac{\partial\psi}{\partial x} = M = 3x^2 + y \quad \Rightarrow \quad M_y = 1 \\ \dfrac{\partial\psi}{\partial y} = N = -3y^2 + x \quad \Rightarrow \quad N_x = 1 \end{array}\right. \Rightarrow M_y = N_x$$

$$\frac{\partial\psi}{\partial x} = 3x^2 + y \quad \Rightarrow \quad \psi = \int \left(3x^2 + y\right)dx = x^3 + xy + h(y)$$

$$\psi_y = x + h'(y) = -3y^2 + x \quad \Rightarrow \quad h'(y) = -3y^2$$

$$h(y) = \int -3y^2 dy = -y^3$$

$$\psi(x, y) = \boxed{x^3 + xy - y^2 = C}$$

Exercise

Solve the differential equation $2xydx + \left(x^2 - 1\right)dy = 0$

Solution

$$M(x, y) = 2xy \quad N(x, y) = x^2 - 1$$

$$\frac{\partial M}{\partial y} = 2x \quad \frac{\partial N}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int 2xy\ dx = x^2 y + h(y)$$

$$\psi_y = x^2 + h'(y) = x^2 - 1$$

$$h'(y) = -1 \quad \Rightarrow \quad h(y) = -y + C$$

$$\boxed{x^2 y - y = C}$$

Exercise

Find the general solution $y' = \dfrac{x^2 + y^2}{2xy}$

Solution

Let $y = xv \quad \Rightarrow \quad y' = v + xv'$

$$v + xv' = \frac{x^2 + x^2v^2}{2x^2v}$$

$$xv' = \frac{1+v^2}{2v} - v$$

$$x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\frac{2v}{1-v^2}dv = \frac{dx}{x}$$

$$-\int \frac{1}{1-v^2}d\left(1-v^2\right) = \int \frac{dx}{x}$$

$$-\ln\left|1-v^2\right| = \ln|x| + \ln C$$

$$\ln\frac{1}{\left|1-v^2\right|} = \ln|\mathrm{C}\,x|$$

$$\frac{1}{1-\left(\frac{y}{x}\right)^2} = \mathrm{C}\,x$$

$$\frac{x^2}{x^2-y^2} = \mathrm{C}\,x$$

$$\frac{x}{\mathrm{C}} = x^2 - y^2$$

$$\boxed{y^2 = x^2 - C_1x}$$

Exercise

Find the general solution $\quad 2xyy' = x^2 + 2y^2$

Solution

Let $y = xv \quad \Rightarrow \quad y' = v + xv'$

$2x(xv)(v + xv') = x^2 + 2x^2v^2$ ***Divide both side by*** x^2

$$2v^2 + 2xvv' = 1 + 2v^2$$

$$2xv\frac{dv}{dx} = 1$$

$$\int 2v\,dv = \int \frac{dx}{x}$$

$v^2 = \ln x + C \qquad \left(v = \frac{y}{x}\right)$

$$\frac{y^2}{x^2} = \ln x + C$$

$$y^2 = x^2\left(\ln x + C\right)$$

Exercise

Find the general solution $\quad xy' = y + 2\sqrt{xy}$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$$x\left(v + xv'\right) = vx + 2\sqrt{x^2 v}$$

$$x\left(v + xv'\right) = vx + 2x\sqrt{v} \qquad \textit{Divide both side by } x$$

$$v + xv' = v + 2\sqrt{v}$$

$$x\frac{dv}{dx} = 2\sqrt{v}$$

$$\int \frac{dv}{2\sqrt{v}} = \int \frac{dx}{x}$$

$$\sqrt{v} = \ln x + C \qquad \left(v = \frac{y}{x}\right)$$

$$\sqrt{\frac{y}{x}} = \ln x + C$$

$$\frac{y}{x} = \left(\ln x + C\right)^2$$

$$y = x\left(\ln x + C\right)^2$$

Exercise

Find the general solution $\quad xy^2y' = x^3 + y^3$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$$x^3v^2\left(v + xv'\right) = x^3 + x^3v^3 \qquad \textit{Divide both side by } x^3$$

$$v^2\left(v + xv'\right) = 1 + v^3$$

$$v^3 + xv^2v' = 1 + v^3$$

$$xv^2\frac{dv}{dx} = 1$$

$$\int v^2 dv = \int \frac{dx}{x}$$

$$\frac{1}{3}v^3 = \ln x + C \qquad \left(v = \frac{y}{x}\right)$$

$$\frac{y^3}{x^3} = 3(\ln x + C)$$

$$\boxed{y^3 = 3x^3(\ln x + C)}$$

Exercise

Find the general solution $\quad x^2 y' = xy + x^2 e^{y/x}$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$$x^2(v + xv') = x^2 v + x^2 e^{vx/x} \qquad \textit{Divide both side by } x^2$$

$$v + xv' = v + e^v$$

$$x\frac{dv}{dx} = e^v$$

$$\int e^{-v} dv = \int \frac{dx}{x}$$

$$e^{-v} = \ln x + \ln C \qquad \left(v = \frac{y}{x}\right)$$

$$e^{-\frac{y}{x}} = \ln\mathrm{C}\, x$$

$$-\frac{y}{x} = \ln(\ln\mathrm{C}\, x)$$

$$\boxed{y = -x\ln(\ln\mathrm{C}\, x)}$$

Exercise

Find the general solution $\quad x^2 y' = xy + y^2$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$$x^2(v + xv') = x^2 v + x^2 v^2 \qquad \textit{Divide both side by } x^2$$

$$v + xv' = v + v^2$$

$$x\frac{dv}{dx} = v^2$$

$$\int v^{-2} dv = \int \frac{dx}{x}$$

$$-v^{-1} = \ln x + \ln C \qquad \left(v = \frac{y}{x}\right)$$

$$\frac{x}{y} = -\ln Cx$$

$$\frac{x}{y} = \ln \frac{1}{Cx}$$

$$\boxed{y(x) = -\frac{x}{\ln Cx}}$$

Exercise

Find the general solution $\quad xyy' = x^2 + 3y^2$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$$vx^2(v + xv') = x^2 + 3x^2v^2 \qquad \textit{Divide both side by } x^2$$

$$v^2 + xvv' = 1 + 3v^2$$

$$xv\frac{dv}{dx} = 1 + 2v^2$$

$$\frac{v}{1+2v^2}dv = \frac{dx}{x}$$

$$\frac{1}{4}\int \frac{1}{1+2v^2}d\left(1+2v^2\right) = \int \frac{dx}{x}$$

$$\frac{1}{4}\ln\left(1+2v^2\right) = \ln x + \ln C$$

$$\ln\left(1+2v^2\right) = 4\ln \mathrm{C}\,x$$

$$\ln\left(1+2\frac{y^2}{x^2}\right) = \ln \mathrm{C}\,x^4$$

$$\frac{x^2+2y^2}{x^2} = \mathrm{C}\,x^4$$

$$\boxed{x^2 + 2y^2 = \mathrm{C}\,x^6}$$

Exercise

Find the general solution $\quad \left(x^2 - y^2\right)y' = 2xy$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$$\left(x^2 - v^2x^2\right)(v + xv') = 2x^2v \qquad \textit{Divide both side by } x^2$$

$$\left(1-v^2\right)\left(v+xv'\right)=2v$$

$$v+xv'=\frac{2v}{1-v^2}$$

$$x\frac{dv}{dx}=\frac{2v}{1-v^2}-v$$

$$x\frac{dv}{dx}=\frac{v^3+v}{1-v^2}$$

$$\int\frac{1-v^2}{v^3+v}dv=\int\frac{dx}{x}$$

$$\frac{1-v^2}{v\left(v^2+1\right)}=\frac{A}{v}+\frac{Bv+C}{v^2+1}=\frac{(A+B)v^2+Cv+A}{v\left(v^2+1\right)}$$

$$\begin{cases}A+B=-1\\ \quad A=1 \qquad \to B=-2\end{cases}$$

$$\int\left(\frac{1}{v}-\frac{2v}{v^2+1}\right)dv=\int\frac{dx}{x}$$

$$\int\frac{1}{v}dv-\int\frac{1}{v^2+1}d\left(v^2+1\right)=\int\frac{dx}{x}$$

$$\ln|v|-\ln\left(v^2+1\right)=\ln|x|+\ln C$$

$$\ln\frac{|v|}{v^2+1}=\ln \mathrm{C}\,x$$

$$\frac{y/x}{\frac{y^2}{x^2}+1}=\mathrm{C}\,x$$

$$\frac{y}{x}\frac{x^2}{y^2+x^2}=\mathrm{C}\,x$$

$$\boxed{y(x)=\mathrm{C}\left(y^2+x^2\right)}$$

Exercise

Find the general solution $\qquad xyy'=y^2+x\sqrt{4x^2+y^2}$

Solution

Let $y=vx \quad\Rightarrow\quad y'=v+xv'$

$$vx^2\left(v+xv'\right)=x^2v^2+x\sqrt{4x^2+v^2x^2}$$

$$vx^2\left(v+xv'\right)=x^2v^2+x^2\sqrt{4+v^2} \qquad \textit{Divide both side by } x^2$$

$$v^2 + xvv' = v^2 + \sqrt{4+v^2}$$

$$xv\frac{dv}{dx} = \sqrt{4+v^2}$$

$$\int \frac{v}{\sqrt{4+v^2}}dv = \int \frac{dx}{x}$$

$$\frac{1}{2}\int \frac{1}{\sqrt{4+v^2}}d\left(4+v^2\right) = \int \frac{dx}{x}$$

$$\sqrt{4+v^2} = \ln x + C$$

$$\sqrt{4+\frac{y^2}{x^2}} = \ln x + C$$

$$\frac{4x^2+y^2}{x^2} = (\ln x + C)^2$$

$$\boxed{4x^2 + y^2 = x^2(\ln x + C)^2}$$

Exercise

Find the general solution $\quad xy' = y + \sqrt{x^2+y^2}$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$$x(v+xv') = xv + \sqrt{x^2 + v^2x^2}$$

$$x(v+xv') = xv + x\sqrt{1+v^2} \qquad \textit{Divide both side by } x$$

$$v + xv' = v + \sqrt{1+v^2}$$

$$x\frac{dv}{dx} = \sqrt{1+v^2}$$

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$$

$$\ln\left|v + \sqrt{1+v^2}\right| = \ln x + \ln C$$

$$\ln\left|\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right| = \ln \mathrm{C}x$$

$$\frac{y}{x} + \frac{1}{x}\sqrt{x^2+y^2} = \mathrm{C}x$$

$$\boxed{y + \sqrt{x^2+y^2} = \mathrm{C}x^2}$$

Exercise

Find the general solution $\quad y^2 y' + 2xy^3 = 6x$

Solution

$$y' + 2xy = 6xy^{-2}$$

Let $\quad u = y^{1+2} = y^3 \quad \Rightarrow \quad y = u^{1/3}$

$$\frac{du}{dx} = 3y^2 \frac{dy}{dx} \quad \Rightarrow \quad y' = \frac{1}{3} y^{-2} u' = \frac{1}{3} u^{-2/3} u'$$

$$\frac{1}{3} u^{-2/3} u' + 2xu^{1/3} = 6xu^{-2/3} \qquad \textit{Multiply both sides by } 3u^{2/3}$$

$$u' + 6xu = 18x$$

$$e^{\int 6x dx} = e^{3x^2}$$

$$\int 18xe^{3x^2} dx = 3\int e^{3x^2} d\left(3x^2\right)$$

$$= 3e^{3x^2}$$

$$u = e^{-3x^2}\left(3e^{3x^2} + C\right)$$

$$\boxed{y^3 = 3 + Ce^{-3x^2}}$$

Exercise

Find the general solution $\quad x^2 y' + 2xy = 5y^4$

Solution

$$y' + 2\frac{1}{x} y = \frac{5}{x^2} y^4 \qquad \textit{Divide by } x^2$$

Let $\quad u = y^{1-4} = y^{-3} \quad \Rightarrow \quad y = u^{-1/3}$

$$\frac{du}{dx} = -3y^{-4} \frac{dy}{dx} \quad \Rightarrow \quad y' = -\frac{1}{3} y^4 u' = -\frac{1}{3} u^{-4/3} u'$$

$$-\frac{1}{3} u^{-4/3} u' + \frac{2}{x} u^{-1/3} = \frac{5}{x^2} u^{-4/3} \qquad \textit{Multiply both sides by } -3u^{4/3} \quad u = \frac{1}{x^{-6}}\left(\frac{15}{7} x^{-7} + C\right)$$

$$u' - \frac{6}{x} u = -\frac{15}{x^2}$$

$$e^{\int -\frac{6}{x} dx} = e^{-6\ln x}$$

$$= e^{\ln x^{-6}}$$

$$= x^{-6}$$

$$\int x^{-6}\left(-\frac{15}{x^2}\right)dx = -15\int x^{-8}dx$$

$$= \frac{15}{7}x^{-7}$$

$$y^{-3} = \frac{15+7Cx^7}{7x}$$

$$\boxed{y^3 = \frac{7x}{15+7Cx^7}}$$

Exercise

Find the general solution $\quad 2xy' + y^3e^{-2x} = 2xy$

Solution

$$2xy' - 2xy = -e^{-2x}y^3$$

$$y' - y = -\frac{e^{-2x}}{2x}y^3 \qquad \textit{Divide by } 2x$$

Let $\quad u = y^{1-3} = y^{-2} \quad \Rightarrow \quad y = u^{-1/2}$

$$\frac{du}{dx} = -2y^{-3}\frac{dy}{dx} \quad \Rightarrow \quad y' = -\frac{1}{2}y^3u' = -\frac{1}{2}u^{-3/2}u'$$

$$-\frac{1}{2}u^{-3/2}u' - u^{-1/2} = -\frac{e^{-2x}}{2x}u^{-3/2} \qquad \textit{Multiply both sides by } -2u^{3/2}$$

$$u' + 2u = \frac{e^{-2x}}{x}$$

$$e^{\int 2dx} = e^{2x}$$

$$\int \frac{e^{-2x}}{x}e^{2x}dx = \int \frac{dx}{x}$$

$$= \ln x$$

$$u = \frac{1}{e^{-2x}}(\ln x + C)$$

$$\frac{1}{y^2} = \frac{\ln x + C}{e^{2x}}$$

$$\boxed{y^2 = \frac{e^{2x}}{\ln x + C}}$$

Exercise

Find the general solution $\quad y^2\left(xy' + y\right)\left(1+x^4\right)^{1/2} = x$

Solution

$$y^2xy' + y^3 = x\left(1+x^4\right)^{-1/2}$$

$$y' + \frac{1}{x}y = \left(1+x^4\right)^{-1/2} y^{-2}$$ ***Divide both sides by*** xy^2

Let $u = y^{1+2} = y^3 \quad \Rightarrow \quad y = u^{1/3}$

$$\frac{du}{dx} = 3y^2\frac{dy}{dx} \quad \Rightarrow \quad y' = \frac{1}{3}y^{-2}u' = \frac{1}{3}u^{-2/3}u'$$

$$\frac{1}{3}u^{-2/3}u' + \frac{1}{x}u^{1/3} = \left(1+x^4\right)^{-1/2}u^{-2/3}$$ ***Multiply both sides by*** $3u^{2/3}$

$$u' + \frac{3}{x}u = 3\left(1+x^4\right)^{-1/2}$$

$$e^{\int \frac{3}{x}dx} = e^{3\ln x}$$
$$= e^{\ln x^3}$$
$$= x^3$$

$$\int 3\left(1+x^4\right)^{-1/2}x^3dx = \frac{3}{4}\int\left(1+x^4\right)^{-1/2}d\left(1+x^4\right)$$
$$= \frac{3}{2}\sqrt{1+x^4}$$

$$u = \frac{1}{x^3}\left(\frac{3}{2}\sqrt{1+x^4} + C\right)$$

$$\boxed{y^3 = \frac{1}{x^3}\left(\frac{3}{2}\sqrt{1+x^4} + C\right)}$$

Exercise

Find the general solution $\quad 3y^2y' + y^3 = e^{-x}$

Solution

$$3y' + y = e^{-x}y^{-2}$$ ***Divide both sides by*** y^2

Let $u = y^{1+2} = y^3 \quad \Rightarrow \quad y = u^{1/3}$

$$\frac{du}{dx} = 3y^2\frac{dy}{dx} \quad \Rightarrow \quad y' = \frac{1}{3}y^{-2}u' = \frac{1}{3}u^{-2/3}u'$$

$$u^{-2/3}u' + u^{1/3} = e^{-x}u^{-2/3}$$ ***Multiply both sides by*** $u^{2/3}$

$$u' + u = e^{-x}$$

$$e^{\int dx} = e^x$$

$$\int e^{-x}e^{x}dx = \int dx = x$$

$$u = \frac{1}{e^{x}}(x + C)$$

$$\boxed{y^3 = e^{-x}(x + C)}$$

Exercise

Find the general solution $\quad 3xy^2y' = 3x^4 + y^3$

Solution

$$3xy^2y' - y^3 = 3x^4 \qquad \textit{Divide both sides by } xy^2$$

$$3y' - \frac{1}{x}y = 3x^3y^{-2}$$

Let $\quad u = y^{1+2} = y^3 \quad \Rightarrow \quad y = u^{1/3} \quad \Rightarrow \quad y' = \frac{1}{3}u^{-2/3}u'$

$$u^{-2/3}u' - \frac{1}{x}u^{1/3} = 3x^3u^{-2/3} \qquad \textit{Multiply both sides by } u^{2/3}$$

$$u' - \frac{1}{x}u = 3x^3$$

$$e^{\int \frac{-1}{x}dx} = e^{-\ln x}$$

$$= x^{-1}$$

$$\int 3x^3x^{-1}dx = \int 3x^2dx$$

$$= x^3$$

$$u = x\left(x^3 + C\right)$$

$$y^3 = x^4 + Cx$$

$$\boxed{y = \sqrt[3]{x^4 + Cx}}$$

Exercise

Find the general solution $\quad xe^{y}y' = 2\left(e^{y} + x^3e^{2x}\right)$

Solution

Let $u = e^{y} \quad \Rightarrow \quad y = \ln u \quad \Rightarrow \quad y' = \frac{u'}{u}$

$$xu\frac{1}{u}u' = 2u + 2x^3e^{2x}$$

$$u' - \frac{2}{x}u = 2x^2e^{2x}$$

$$e^{\int \frac{-2}{x}dx} = e^{-2\ln x}$$

$$= x^{-2}$$

$$\int 2x^2 e^{2x} x^{-2} dx = 2\int e^{2x} dx$$

$$= e^{2x}$$

$$u = x^2\left(e^{2x} + C\right)$$

$$e^y = x^2 e^{2x} + Cx^2$$

$$\boxed{y = \ln\left(x^2 e^{2x} + Cx^2\right)}$$

Exercise

Find the general solution $\quad (2x\sin y\cos y)y' = 4x^2 + \sin^2 y$

Solution

Let $u = \sin y \quad \Rightarrow \quad u' = (\cos y)y'$

$$2xuu' = 4x^2 + u^2$$

$$u' = 2x\frac{1}{u} + \frac{1}{2x}u$$

$$u' - \frac{1}{2x}u = 2xu^{-1}$$

Let $\quad v = u^{1+1} = u^2 \quad \Rightarrow \quad u = v^{1/2}$

$$v' = 2uu' \quad \Rightarrow \quad u' = \frac{1}{2}u^{-1}v' = \frac{1}{2}v^{-1/2}v'$$

$$\frac{1}{2}v^{-1/2}v' - \frac{1}{2x}v^{1/2} = 2xv^{-1/2}$$ ***Multiply both sides by*** $2v^{1/2}$

$$v' - \frac{1}{x}v = 4x$$

$$e^{\int -\frac{1}{x}dx} = e^{-\ln x}$$

$$= e^{\ln x^{-1}}$$

$$= x^{-1}$$

$$\int x^{-1}(4x)dx = \int 4dx$$

$$= 4x$$

$$v = x(4x + C)$$

$$u^2 = 4x^2 + Cx$$

$$\boxed{\sin^2 y = 4x^2 + Cx}$$

Exercise

Find the general solution $\left(x+e^{y}\right)y'=xe^{-y}-1$

Solution

Let $u=e^{y} \quad \Rightarrow \quad y=\ln u \quad \Rightarrow y'=\dfrac{u'}{u}$

$$(x+u)\frac{u'}{u}=xu^{-1}-1$$

$$(x+u)u'=x-u$$

Let $u=vx \quad \Rightarrow \quad u'=v+xv'$

$$(x+vx)(v+xv')=x-vx$$

$$x(1+v)(v+xv')=x(1-v)$$

$$(1+v)(v+xv')=1-v$$

$$v+v^{2}+x(1+v)v'=1-v$$

$$x(1+v)\frac{dv}{dx}=1-2v-v^{2}$$

$$\int\frac{1+v}{1-2v-v^{2}}dv=\int\frac{dx}{x}$$

$$-\frac{1}{2}\ln\left|1-2v-v^{2}\right|=\ln x+\ln C$$

$$\ln\left|1-2v-v^{2}\right|=-2\ln Cx \qquad v=\frac{u}{x}=\frac{e^{y}}{x}$$

$$\ln\left|1-2\frac{e^{y}}{x}-\frac{e^{2y}}{x^{2}}\right|=\ln(Cx)^{-2}$$

$$\frac{x^{2}-2xe^{y}-e^{2y}}{x^{2}}=\frac{1}{(Cx)^{2}}$$

$$x^{2}-2xe^{y}-e^{2y}=C_{1}$$

Exercise

Find the general solution $\left(x^{2}+y^{2}\right)dx+\left(x^{2}-xy\right)dy=0$

Solution

$$M(x,\ y)=x^{2}+y^{2} \quad N(x,\ y)=x^{2}-xy$$

$$\frac{\partial M}{\partial y}=2y \quad \frac{\partial N}{\partial x}=2x-y$$

$$\Rightarrow \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

$$\frac{M_y - N_x}{N} = \frac{2y - 2x + y}{x^2 + xy}$$

Let $y = ux \;\Rightarrow\; dy = udx + xdu$

$$-\frac{x^2 + y^2}{x^2 - xy}dx = udx + xdu$$

$$\left(x^2 + y^2\right)dx + \left(x^2 - xy\right)(udx + xdu) = 0$$

$$\left(x^2 + y^2\right)dx + ux(x - y)dx + x^2(x - y)du = 0$$

$$\left(x^2 + u^2x^2\right)dx + ux(x - ux)dx + x^2(x - ux)du = 0$$

$$x^2(1 + u)dx + x^3(1 - u)du = 0$$

$$(1 + u)dx = x(u - 1)du$$

$$\frac{dx}{x} = \frac{u - 1}{u + 1}du$$

$$\int \frac{dx}{x} = \int \left(1 - \frac{2}{u + 1}\right)du$$

$$\ln|x| = u - 2\ln|u + 1| + \ln C$$

$$\ln|x| + \ln\left(\frac{y}{x} + 1\right)^2 - \ln C = \frac{y}{x}$$

$$\ln \frac{x}{C}\left(\frac{(x + y)^2}{x^2}\right) = \frac{y}{x}$$

$$\frac{(x + y)^2}{Cx} = e^{\frac{y}{x}}$$

$$\boxed{(x + y)^2 = Cxe^{\frac{y}{x}}}$$

Exercise

Find the general solution $\quad x\dfrac{dy}{dx} + y = x^2y^2$

Solution

$$y' + \frac{1}{x}y = xy^2$$

Let $u = y^{1-2} = y^{-1} \quad \Rightarrow \quad y = \dfrac{1}{u}$

$$\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx} \quad \Rightarrow \quad y' = -y^2u' = -\frac{1}{u^2}u'$$

$$-\frac{1}{u^2}u' + \frac{1}{x}\frac{1}{u} = x\frac{1}{u^2}$$

$$u' - \frac{1}{x}u = -x \qquad \left(\times -u^2\right)$$

$$e^{\int -\frac{1}{x}dx} = e^{-\ln x}$$

$$= e^{\ln x^{-1}}$$

$$= x^{-1}$$

$$\int -xx^{-1}dx = -x$$

$$u = x(-x + C)$$

$$\frac{1}{y} = -x^2 + Cx$$

$$\boxed{y(x) = \frac{1}{-x^2 + Cx}}$$

Exercise

Solve the differential equation $\left(3x^2 - 2xy + 2\right) + \left(6y^2 - x^2 + 3\right)y' = 0$

Solution

$$\frac{\partial \psi}{\partial x} = M = 3x^2 - 2xy + 2 \quad \Rightarrow \quad M_y = -2x$$

$$\frac{\partial \psi}{\partial y} = N = 6y^2 - x^2 + 3 \quad \Rightarrow \quad N_x = -2x$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = 3x^2 - 2xy + 2$$

$$\psi = \int \left(3x^2 - 2xy + 2\right)dx$$

$$= x^3 - x^2y + 2x + h(y)$$

$$\psi_y = -x^2 + h'(y) = 6y^2 - x^2 + 3 \quad \Rightarrow \quad h'(y) = 6y^2 + 3$$

$$h(y) = \int \left(6y^2 + 3\right)dy$$

$$= 2y^3 + 3y$$

$$\boxed{x^3 - x^2y + 2x + 2y^3 + 3y = C}$$

Exercise

Solve the differential equation $\left(e^x \sin y - 2y\sin x\right)dx + \left(e^x \cos y + 2\cos x\right)dy = 0$

Solution

$$\frac{\partial\psi}{\partial x} = M = e^x \sin y - 2y\sin x \quad\Rightarrow\quad M_y = e^x \cos y - 2\sin x$$

$$\frac{\partial\psi}{\partial y} = N = e^x \cos y + 2\cos x \quad\Rightarrow\quad N_x = e^x \cos y - 2\sin x$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial\psi}{\partial x} = e^x \sin y - 2y\sin x$$

$$\psi = \int\left(e^x \sin y - 2y\sin x\right)dx$$

$$= e^x \sin y + 2y\cos x + h(y)$$

$$\psi_y = e^x \cos y + 2\cos x + h'(y) = e^x \cos y + 2\cos x \quad\Rightarrow\quad h'(y) = 0$$

$$\rightarrow \underline{h(y) = C}$$

$$\boxed{e^x \sin y + 2y\cos x = C}$$

Exercise

Solve the differential equation $\left(\frac{y}{x} + 6x\right)dx + (\ln x - 2)dy = 0, \qquad x > 0$

Solution

$$\frac{\partial\psi}{\partial x} = M = \frac{y}{x} + 6x \quad\Rightarrow\quad M_y = \frac{1}{x}$$

$$\frac{\partial\psi}{\partial y} = N = \ln x - 2 \quad\Rightarrow\quad N_x = \frac{1}{x} \qquad\qquad \Rightarrow M_y = N_x$$

$$\frac{\partial\psi}{\partial x} = e^x \sin y - 2y\sin x$$

$$\psi = \int\left(\frac{y}{x} + 6x\right)dx$$

$$= y\ln x + 3x^2 + h(y)$$

$$\psi_y = \ln x + h'(y) = \ln x - 2 \quad\Rightarrow\quad h'(y) = -2$$

$$h(y) = \int -2dy = -2y$$

$$\boxed{y\ln x + 3x^2 - 2y = C}$$

Exercise

Solve the differential equation $\left(e^{2y} - y\cos xy\right)dx + \left(2xe^{2y} - x\cos xy + 2y\right)dy = 0$

Solution

$$M(x, y) = e^{2y} - y\cos xy \quad N(x, y) = 2xe^{2y} - x\cos xy + 2y$$

$$\frac{\partial M}{\partial y} = 2e^{2y} - \cos x + xy\sin xy \quad \frac{\partial N}{\partial x} = 2e^{2y} - \cos x + xy\sin xy$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(e^{2y} - y\cos xy\right)\, dx$$

$$= xe^{2y} - \sin xy + h(y)$$

$$\psi_y = 2xe^{2y} - x\cos xy + h'(y) = 2xe^{2y} - x\cos xy + 2y$$

$$h'(y) = 2y \quad \Rightarrow \quad h(y) = y^2 + C$$

$$\psi = \underline{xe^{2y} - \sin xy + y^2 = C}$$

Exercise

Solve the differential equation $\dfrac{xdx}{\left(x^2+y^2\right)^{3/2}} + \dfrac{ydy}{\left(x^2+y^2\right)^{3/2}} = 0$

Solution

Multiply both side by $\left(x^2+y^2\right)^{3/2}$ since $x^2 + y^2 \neq 0 \quad \Rightarrow \quad xdx + ydy = 0$

$$\frac{\partial \psi}{\partial x} = M = x \quad \Rightarrow \quad M_y = 0$$

$$\frac{\partial \psi}{\partial y} = N = y \quad \Rightarrow \quad N_x = 0 \qquad \Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = x$$

$$\psi = \int xdx$$

$$= \frac{1}{2}x^2 + h(y)$$

$$\psi_y = h'(y) = y$$

$$h(y) = \int ydy$$

$$= \frac{1}{2}y^2$$

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 = C_1$$

$$\boxed{x^2 + y^2 = C}$$

Exercise

Find the general solution $(2x-1)dx + (3y+7)dy = 0$

Solution

$$M(x, y) = 2x - 1 \quad N(x, y) = 3y + 7$$

$$\left.\begin{array}{l} \dfrac{\partial M}{\partial y} = 0 \\ \dfrac{\partial N}{\partial x} = 0 \end{array}\right. \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (2x-1)\, dx$$

$$= x^2 - x + h(y)$$

$$\psi_y = h'(y) = 3y + 7$$

$$h(y) = \int (3y+7)dy$$

$$= \frac{3}{2}y^2 + 7y$$

$$\boxed{x^2 - x + \frac{3}{2}y^2 + 7y = C}$$

Exercise

Find the general solution $(5x+4y)dx + \left(4x - 8y^3\right)dy = 0$

Solution

$$M(x, y) = 5x + 4y \quad N(x, y) = 4x - 8y^3$$

$$\left.\begin{array}{l} \dfrac{\partial M}{\partial y} = 4 \\ \dfrac{\partial N}{\partial x} = 4 \end{array}\right. \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (5x+4y)\, dx$$

$$= \frac{5}{2}x^2 + 4xy + h(y)$$

$$\psi_y = 4x + h'(y) = 4x - 8y^3$$

$$h'(y) = -8y^3$$

$$h(y) = -\int 8y^3\, dy$$

$$= -2y^4$$

$$\boxed{\frac{5}{2}x^2 + 4xy - 2y^4 = C}$$

Exercise

Find the general solution $\quad (\sin y - y\sin x)dx + (\cos x + x\cos y - y)dy = 0$

Solution

$$M(x,\ y) = \sin y - y\sin x \qquad N(x,\ y) = \cos x + x\cos y - y$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= \cos y - \sin x\\ \frac{\partial N}{\partial x} &= -\sin x + \cos y\end{aligned}\right. \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (\sin y - y\sin x)\ dx$$

$$= x\sin y + y\cos x + h(y)$$

$$\psi_y = x\cos y + \cos x + h'(y) = x\cos y + \cos x - y$$

$$h'(y) = -y$$

$$h(y) = -\int y\ dy$$

$$= -\frac{1}{2}y^2$$

$$\boxed{x\sin y + y\cos x - \frac{1}{2}y^2 = C}$$

Exercise

Find the general solution $\quad \left(2xy^2 - 3\right)dx + \left(2x^2y + 4\right)dy = 0$

Solution

$$M(x,\ y) = 2xy^2 - 3 \qquad N(x,\ y) = 2x^2 + 4$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 4xy \\ \frac{\partial N}{\partial x} &= 4xy\end{aligned}\right. \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(2xy^2 - 3\right)\, dx$$

$$= x^2 y^2 - 3x + h(y)$$

$$\psi_y = 2x^2 y + h'(y) = 2x^2 y + 4$$

$$\rightarrow h'(y) = 4$$

$$h(y) = \int 4\, dy$$

$$= 4y$$

$$\underline{x^2 y^2 - 3x + 4y = C}$$

Exercise

Find the general solution $\left(1 + \ln x + \frac{y}{x}\right)dx - (1 - \ln x)\,dy = 0$

Solution

$$M(x,\ y) = 1 + \ln x + \frac{y}{x} \quad N(x,\ y) = -1 + \ln x$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= \frac{1}{x} \\ \frac{\partial N}{\partial x} &= \frac{1}{x}\end{aligned}\right. \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(1 + \ln x + \frac{y}{x}\right)\, dx$$

$$= x + x\ln x - x + y\ln x + h(y)$$

$$= x\ln x + y\ln x + h(y)$$

$$\psi_y = \ln x + h'(y) = -1 + \ln x$$

$$\rightarrow h'(y) = -1$$

$$h(y) = -\int dy$$

$$= -y$$

$$\underline{x\ln x + y\ln x - y = C}$$

Exercise

Find the general solution $\left(x-y^3+y^2\sin x\right)dx-\left(3xy^2+2y\cos x\right)dy=0$

Solution

$$M(x,\ y)=x-y^3+y^2\sin x \quad N(x,\ y)=-3xy^2-2y\cos x$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y}&=-3y^2+2y\sin x\\ \frac{\partial N}{\partial x}&=-3y^2+2y\sin x\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\psi=\int\left(x-y^3+y^2\sin x\right)\,dx=\frac{1}{2}x^2-xy^3-y^2\cos x+h(y)$$

$$\psi_y=-3xy^2-2y\cos x+h'(y)=-3xy^2-2y\cos x$$

$$\rightarrow h'(y)=0 \quad\Rightarrow\quad h(y)=C$$

$$\boxed{\frac{1}{2}x^2-xy^3-y^2\cos x=C}$$

Exercise

Find the general solution $\left(x^3+y^3\right)dx+3xy^2dy=0$

Solution

$$M(x,\ y)=x^3+y^3 \quad N(x,\ y)=3xy^2$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y}&=3y^2\\ \frac{\partial N}{\partial x}&=3y^2\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\psi=\int\left(x^3+y^3\right)\,dx=\frac{1}{4}x^4+xy^3+h(y)$$

$$\psi_y=3xy^2+h'(y)=3xy^2$$

$$\rightarrow h'(y)=0 \quad\Rightarrow\quad h(y)=C$$

$$\boxed{\frac{1}{4}x^4+xy^3=C}$$

Exercise

Find the general solution $\left(3x^2y+e^y\right)dx+\left(x^3+xe^y-2y\right)dy=0$

Solution

$$M(x,\ y)=3x^2y+e^y \quad N(x,\ y)=x^3+xe^y-2y$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 3x^2 + e^y \\ \frac{\partial N}{\partial x} &= 3x^2 + e^y\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(3x^2 y + e^y\right) dx = x^3 y + xe^y + h(y)$$

$$\psi_y = x^3 + xe^y + h'(y) = x^3 + xe^y - 2y$$

$$\rightarrow h'(y) = -2y \quad \Rightarrow \quad h(y) = -y^2$$

$$x^3 y + xe^y - y^2 = C$$

Exercise

Find the general solution $\quad xdy + \left(y - 2xe^x - 6x^2\right)dx = 0$

Solution

$$M(x, y) = y - 2xe^x - 6x^2 \quad N(x, y) = x$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 1 \\ \frac{\partial N}{\partial x} &= 1\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(y - 2xe^x - 6x^2\right) dx$$

$$= xy - (2x - 2)e^x - 2x^3 + h(y)$$

$$\psi_y = x + h'(y) = x$$

$$\rightarrow h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$xy - 2xe^x + 2e^x - 2x^3 = C$$

Exercise

Find the general solution $\quad \left(1 - \frac{3}{y} + x\right)dy + \left(y - \frac{3}{x} + 1\right)dx = 0$

Solution

$$M(x, y) = y - \frac{3}{x} + 1 \quad N(x, y) = 1 - \frac{3}{y} + x$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 1 \\ \frac{\partial N}{\partial x} &= 1\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(y - \frac{3}{x} + 1\right) dx = xy - 3\ln|x| + x + h(y)$$

$$\psi_y = x + h'(y) = 1 - \frac{3}{y} + x$$

$$\rightarrow h'(y) = 1 - \frac{3}{y} \quad \Rightarrow \quad h(y) = y - 3\ln|y|$$

$$xy - 3\ln|x| + x + y - 3\ln|y| = C$$

$$\boxed{xy + x + y - 3\ln|xy| = C}$$

Exercise

Find the general solution $\left(x^2y^3 - \frac{1}{1+9x^2}\right)\frac{dx}{dy} + x^3y^2 = 0$

Solution

$$\left(x^2y^3 - \frac{1}{1+9x^2}\right)dx + \left(x^3y^2\right)dy = 0$$

$$M(x, y) = x^2y^3 - \frac{1}{1+9x^2} \qquad N(x, y) = x^3y^2$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= 3x^2y^2 \\ \frac{\partial N}{\partial x} &= 3x^2y^2 \end{aligned} \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(x^2y^3 - \frac{1}{1+9x^2}\right) dx$$

$$= \frac{1}{3}x^3y^3 - \frac{1}{3}\arctan(3x) + h(y)$$

$$\psi_y = x^3y^2 + h'(y) = x^3y^2$$

$$\rightarrow h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$\frac{1}{3}x^3y^3 - \frac{1}{3}\arctan(3x) = C_1$$

$$\boxed{x^3y^3 - \arctan(3x) = C}$$

Exercise

Find the general solution $(5y - 2x)y' - 2y = 0$

Solution

$$M(x, y) = -2y \qquad N(x, y) = 5y - 2x$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= -2\\ \frac{\partial N}{\partial x} &= -2\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (-2y)\ dx = -2xy + h(y)$$

$$\psi_y = -2x + h'(y) = 5y - 2x$$

$$\to h'(y) = 5y \quad \Rightarrow \quad h(y) = \frac{5}{2}y^2$$

$$-2xy + \frac{5}{2}y^2 = C$$

Exercise

Find the general solution $\quad (x-y)dx - xdy = 0$

Solution

$$M(x,\ y) = x - y \quad N(x,\ y) = -x$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= -1\\ \frac{\partial N}{\partial x} &= -1\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (x-y)\ dx$$

$$= \frac{1}{2}x^2 - xy + h(y)$$

$$\psi_y = -x + h'(y) = -x$$

$$\to h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$\frac{1}{2}x^2 - xy = C$$

Exercise

Find the general solution $\quad (x+y)dx + xdy = 0$

Solution

$$M(x,\ y) = x + y \quad N(x,\ y) = x$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 1\\ \frac{\partial N}{\partial x} &= 1\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (x+y)\ dx$$
$$= \frac{1}{2}x^2 + xy + h(y)$$
$$\psi_y = x + h'(y) = x$$
$$\to h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$
$$\frac{1}{2}x^2 + xy = C$$

Exercise

Find the general solution $\quad \frac{dy}{dx} = -\frac{2xy^2+1}{2x^2y}$

Solution

$$2x^2ydy = -\left(2xy^2+1\right)dx$$
$$\left(2xy^2+1\right)dx + 2x^2ydy = 0$$
$$M(x,\ y) = 2xy^2+1 \quad N(x,\ y) = 2x^2y$$
$$\left.\begin{array}{l}\frac{\partial M}{\partial y} = 4xy \\ \frac{\partial N}{\partial x} = 4xy\end{array}\right. \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
$$\psi = \int \left(2xy^2+1\right)\ dx$$
$$= x^2y^2 + x + h(y)$$
$$\psi_y = 2x^2y + h'(y) = 2x^2y$$
$$\to h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$
$$x^2y^2 + x = C$$

Exercise

Find the general solution $\quad \left(1+e^xy+xe^xy\right)dx + \left(xe^x+2\right)dy = 0$

Solution

$$M(x,\ y) = 1+e^xy+xe^xy \quad N(x,\ y) = xe^x+2$$
$$\left.\begin{array}{l}\frac{\partial M}{\partial y} = e^x + xe^x \\ \frac{\partial N}{\partial x} = e^x + xe^x\end{array}\right. \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(1 + e^x y + xe^x y\right)\, dx$$

$$= x + e^x y + (x-1)e^x y + h(y)$$

$$= x + xe^x y + h(y)$$

$$\psi_y = xe^x + h'(y) = xe^x + 2$$

$$\rightarrow h'(y) = 2 \quad \Rightarrow \quad h(y) = 2y + C$$

$$\boxed{x + xe^x y + 2y + C = 0}$$

Exercise

Find the general solution $\left(2xy^3 + 1\right)dx + \left(3x^2y^2 - \frac{1}{y}\right)dy = 0$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(2xy^3 + 1\right) = 6xy^2 \\ N_x &= \frac{\partial}{\partial x}\left(3x^2y^2 - \frac{1}{y}\right) = 6xy^2 \end{aligned}\right\} \Rightarrow M_y = N_x$$

$$\psi = \int \left(2xy^3 + 1\right)\, dx$$

$$= x^2y^3 + x + h(y)$$

$$\psi_y = 3x^2y^2 + h'(y)$$

$$= 3x^2y^2 - \frac{1}{y}$$

$$\rightarrow h'(y) = -\frac{1}{y} \quad \Rightarrow \quad h(y) = -\ln|y| + C$$

$$\boxed{x^2y^3 + x - \ln|y| = C}$$

Exercise

Find the general solution $(2x + y)dx + (x - 2y)dy = 0$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}(2x + y) = 1 \\ N_x &= \frac{\partial}{\partial x}(x - 2y) = 1 \end{aligned}\right\} \Rightarrow M_y = N_x$$

$$\psi = \int (2x + y)\, dx$$

$$= x^2 + xy + h(y)$$

$$\psi_y = x + h'(y) = x - 2y$$

$$\to h'(y) = -2y \quad \Rightarrow \quad h(y) = -y^2$$

$$\boxed{x^2 + xy - y^2 = C}$$

Exercise

Find the general solution $\quad e^x (y - x)dx + \left(1 + e^x\right)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}\left(e^x (y - x)\right) = e^x$$

$$N_x = \frac{\partial}{\partial x}\left(1 + e^x\right) = e^x \qquad \Rightarrow M_y = N_x$$

$$\psi = \int \left(ye^x - xe^x\right) dx$$

$$= ye^x - (x-1)e^x + h(y)$$

$$\psi_y = e^x + h'(y) = 1 + e^x$$

$$\to h'(y) = 1 \quad \Rightarrow \quad h(y) = y$$

$$\boxed{ye^x - (x-1)e^x + y = C}$$

$$\boxed{y(x) = \frac{(x-1)e^x + C}{1 + e^x}}$$

Exercise

Find the general solution $\quad \left(ye^{xy} - \frac{1}{y}\right)dx + \left(xe^{xy} + \frac{x}{y^2}\right)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}\left(ye^{xy} - \frac{1}{y}\right) = e^{xy} + xye^{xy} + \frac{1}{y^2}$$

$$N_x = \frac{\partial}{\partial x}\left(xe^{xy} + \frac{x}{y^2}\right) = e^{xy} + xye^{xy} + \frac{1}{y^2} \qquad \Rightarrow M_y = N_x$$

$$\psi = \int \left(ye^{xy} - \frac{1}{y} \right) dx$$

$$= e^{xy} - \frac{x}{y} + h(y)$$

$$\psi_y = xe^{xy} + \frac{x}{y^2} + h'(y) = xe^{xy} + \frac{x}{y^2}$$

$$\rightarrow h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$\boxed{e^{xy} - \frac{x}{y} = C}$$

Exercise

Find the general solution $\quad (\tan x - \sin x \sin y)dx + (\cos x \cos y)dy = 0$

Solution

$$M(x, y) = \tan x - \sin x \sin y \quad N(x, y) = \cos x \cos y$$

$$\left. \begin{aligned} \frac{\partial M}{\partial y} &= -\sin x \cos y \\ \frac{\partial N}{\partial x} &= -\sin x \cos y \end{aligned} \right. \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (\tan x - \sin x \sin y) \, dx$$

$$= \ln|\sec x| + \cos x \sin y + h(y)$$

$$\psi_y = \cos x \cos y + h'(y) = \cos x \cos y$$

$$\rightarrow h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$\boxed{\ln|\sec x| + \cos x \sin y = C}$$

Exercise

Find the general solution $\quad \left(2x^3 - xy^2 - 2y + 3\right)dx - \left(x^2 y + 2x\right)dy = 0$

Solution

$$M(x, y) = 2x^3 - xy^2 - 2y + 3 \quad N(x, y) = -x^2 y - 2x$$

$$\left. \begin{aligned} \frac{\partial M}{\partial y} &= -2xy - 2 \\ \frac{\partial N}{\partial x} &= -2xy - 2 \end{aligned} \right. \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(2x^3 - xy^2 - 2y + 3\right)\, dx$$

$$= \frac{1}{2}x^4 - \frac{1}{2}x^2y^2 - 2xy + 3x + h(y)$$

$$\psi_y = -x^2y - 2x + h'(y) = -x^2y - 2x$$

$$\rightarrow h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$\boxed{\frac{1}{2}x^4 - \frac{1}{2}x^2y^2 - 2xy + 3x = C}$$

Exercise

Find the general solution $\quad (x + \sin y)dx + (x\cos y - 2y)dy = 0$

Solution

$$M(x, y) = x + \sin y \quad N(x, y) = x\cos y - 2y$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= \cos y \\ \frac{\partial N}{\partial x} &= \cos y \end{aligned}\right. \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (x + \sin y)\, dx$$

$$= \frac{1}{2}x^2 + x\sin y + h(y)$$

$$\psi_y = x\cos y + h'(y) = x\cos y - 2y$$

$$\rightarrow h'(y) = -2y \quad \Rightarrow \quad h(y) = -y^2$$

$$\boxed{\frac{1}{2}x^2 + x\sin y - y^2 = C}$$

Exercise

Find the general solution $\quad \left(x + \dfrac{1}{\sqrt{y^2 - x^2}}\right)dx + \left(1 - \dfrac{x}{y\sqrt{y^2 - x^2}}\right)dy = 0$

Solution

$$M(x, y) = x + \frac{1}{\sqrt{y^2 - x^2}} \quad N(x, y) = 1 - \frac{x}{y\sqrt{y^2 - x^2}}$$

$$\frac{\partial M}{\partial y} = -\frac{1}{2}(2y)\frac{1}{\sqrt{y^2-x^2}} = -\frac{y}{\sqrt{y^2-x^2}}$$

$$\frac{\partial N}{\partial x} = -\frac{1}{y}\frac{1}{\sqrt{y^2-x^2}}\left(y^2-x^2+x^2\right) = -\frac{y}{\sqrt{y^2-x^2}}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int\left(x + \frac{1}{\sqrt{y^2-x^2}}\right)dx \qquad \int\frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$= \frac{1}{2}x^2 + \sin^{-1}\frac{x}{y} + h(y)$$

$$\psi_y = -\frac{1}{y^2\sqrt{1-\left(\frac{x}{y}\right)^2}} + h'(y) \qquad \frac{d}{dx}(\arcsin u) = \frac{u'}{\sqrt{1-u^2}}$$

$$= -\frac{1}{y\sqrt{y^2-x^2}} + h'(y) = 1 - \frac{1}{y\sqrt{y^2-x^2}}$$

$$\to h'(y) = 1 \quad \Rightarrow \quad h(y) = y$$

$$\boxed{\frac{1}{2}x^2 + \sin^{-1}\frac{x}{y} + y = C}$$

Exercise

Find the general solution $\left(2x+y^2-\cos(x+y)\right)dx + \left(2xy-\cos(x+y)-e^y\right)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}\left(2x+y^2-\cos(x+y)\right) = 2y+\sin(x+y)$$

$$N_x = \frac{\partial}{\partial x}\left(2xy-\cos(x+y)-e^y\right) = 2y+\sin(x+y)$$

$$\Rightarrow M_y = N_x$$

$$\psi = \int\left(2x+y^2-\cos(x+y)\right)dx$$

$$= x^2 + xy^2 - \sin(x+y) + h(y)$$

$$\psi_y = 2xy - \cos(x+y) + h'(y) = 2xy - \cos(x+y) - e^y$$

$$\to h'(y) = -e^y \quad \Rightarrow \quad h(y) = -e^y$$

$$\boxed{x^2 + xy^2 - \sin(x+y) - e^y = C}$$

Exercise

Find the general solution $\left(\frac{2}{\sqrt{1-x^2}}+y\cos(xy)\right)dx+\left(x\cos(xy)-y^{-1/3}\right)dy=0$

Solution

$$M_y=\frac{\partial}{\partial y}\left(\frac{2}{\sqrt{1-x^2}}+y\cos(xy)\right)=\cos(xy)-xy\sin(xy)$$

$$N_x=\frac{\partial}{\partial x}\left(x\cos(xy)-y^{-1/3}\right)=\cos(xy)-xy\sin(xy)$$

$$\Rightarrow M_y=N_x$$

$$\psi=\int\left(x\cos(xy)-y^{-1/3}\right)\,dy$$

$$=\sin(xy)-\frac{3}{2}y^{2/3}+h(x)$$

$$\psi_x=y\cos(xy)+h'(x)=\frac{2}{\sqrt{1-x^2}}+y\cos(xy)$$

$$\to h'(x)=\frac{2}{\sqrt{1-x^2}} \quad\Rightarrow\quad h(x)=2\arcsin x$$

$$\sin(xy)-\frac{3}{2}y^{2/3}+2\arcsin x=C$$

Exercise

Find the general solution $\left(2x+y\cos(xy)\right)dx+\left(x\cos(xy)-2y\right)dy=0$

Solution

$$M_y=\frac{\partial}{\partial y}\left(2x+y\cos(xy)\right)=\cos(xy)-xy\sin(xy)$$

$$N_x=\frac{\partial}{\partial x}\left(x\cos(xy)-2y\right)=\cos(xy)-xy\sin(xy)$$

$$\Rightarrow M_y=N_x$$

$$\psi=\int\left(2x+y\cos(xy)\right)\,dy$$

$$=x^2+\sin(xy)+h(y)$$

$$\psi_y=x\cos(xy)+h'(y)=x\cos(xy)-2y$$

$$\to h'(y)=-2y \quad\Rightarrow\quad h(y)=-y^2$$

$$x^2+\sin(xy)-y^2=C$$

Exercise

Find the general solution $\left(e^{x}\sin y-3x^{2}\right)dx+\left(e^{x}\cos y+\frac{1}{3}y^{-2/3}\right)dy=0$

Solution

$$M_{y}=\frac{\partial}{\partial y}\left(e^{x}\sin y-3x^{2}\right)=e^{x}\cos y$$

$$N_{x}=\frac{\partial}{\partial x}\left(e^{x}\cos y+\frac{1}{3}y^{-2/3}\right)=e^{x}\cos y \qquad \Rightarrow M_{y}=N_{x}$$

$$\psi=\int\left(e^{x}\sin y-3x^{2}\right)\,dy$$

$$=e^{x}\sin y-x^{3}+h(y)$$

$$\psi_{y}=e^{x}\cos y+h'(y)=e^{x}\cos y+\frac{1}{3}y^{-2/3}$$

$$\to h'(y)=\frac{1}{3}y^{-2/3} \quad\Rightarrow\quad h(y)=y^{1/3}$$

$$\boxed{e^{x}\sin y-x^{3}+y^{1/3}=C}$$

Exercise

Find the general solution $\left(2y\sin x\cos x-y+2y^{2}e^{xy^{2}}\right)dx=\left(x-\sin^{2}x-4xye^{xy^{2}}\right)dy$

Solution

$$\left(2y\sin x\cos x-y+2y^{2}e^{xy^{2}}\right)dx-\left(x-\sin^{2}x-4xye^{xy^{2}}\right)dy=0$$

$$M(x,\ y)=2y\sin x\cos x-y+2y^{2}e^{xy^{2}} \qquad N(x,\ y)=-x+\sin^{2}x+4xye^{xy^{2}}$$

$$\frac{\partial M}{\partial y}=2\sin x\cos x-1+4ye^{xy^{2}}+4xy^{3}e^{xy^{2}}$$

$$\frac{\partial N}{\partial x}=-1+2\sin x\cos x+4ye^{xy^{2}}+4xy^{3}e^{xy^{2}} \qquad \Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\psi=\int\left(2y\sin x\cos x-y+2y^{2}e^{xy^{2}}\right)\,dx$$

$$=\int(2y\sin x)\,d(\sin x)-xy+2e^{xy^{2}}$$

$$=y\sin^{2}x-xy+2e^{xy^{2}}$$

$$\psi_{y}=\sin^{2}x-x+4xye^{xy^{2}}+h'(y)=-x+\sin^{2}x+4xye^{xy^{2}}$$

$$\to h'(y)=0 \quad\Rightarrow\quad h(y)=C$$

$$\boxed{y\sin^{2}x-xy+2e^{xy^{2}}=C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$x^2y^3 + x\left(1+y^2\right)y' = 0, \qquad \mu(x,\ y) = \frac{1}{xy^3}$$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(x^2y^3\right) = 3x^2y^2 \\ N_x &= \frac{\partial}{\partial x}\left(x + xy^2\right) = 1 + y^2 \end{aligned}\right. \quad \Rightarrow M_y \neq N_x$$

$$x^2y^3\left(\frac{1}{xy^3}\right) + x\left(1+y^2\right)\left(\frac{1}{xy^3}\right)y' = 0$$

$$x + \left(\frac{1+y^2}{y^3}\right)\frac{dy}{dx} = 0 \quad \Rightarrow \quad \left(\frac{1+y^2}{y^3}\right)dy = -xdx$$

$$\int\left(y^{-3} + \frac{1}{y}\right)dy = -\int xdx$$

$$-\frac{1}{2}y^{-2} + \ln|y| = -\frac{1}{2}x^2 + C_0$$

$$\boxed{x^2 - y^{-2} + \ln|y| = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$y^2 - xy + \left(x^2\right)y' = 0, \qquad \mu(x,\ y) = \frac{1}{xy^2}$$

Solution

$$M_y = \frac{\partial}{\partial y}\left(y^2 - xy\right) = 2y - x \qquad N_x = \frac{\partial}{\partial x}\left(x^2\right) = 2x \quad \Rightarrow M_y \neq N_x$$

$$\left(y^2 - xy\right)\left(\frac{1}{xy^2}\right) + \left(x^2\right)\left(\frac{1}{xy^2}\right)y' = 0$$

$$\left(\frac{1}{x} - \frac{1}{y}\right) + \left(\frac{x}{y^2}\right)y' = 0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(\frac{1}{x} - \frac{1}{y}\right) = \frac{1}{y^2} \\ N_x &= \frac{\partial}{\partial x}\left(\frac{x}{y^2}\right) = \frac{1}{y^2} \end{aligned}\right. \quad \Rightarrow M_y = N_x$$

$$\frac{\partial\psi}{\partial x} = \frac{1}{x} - \frac{1}{y}$$

$$\psi = \int\left(\frac{1}{x} - \frac{1}{y}\right)dx$$

$$= \ln|x| - \frac{x}{y} + h(y)$$

$$\psi_y = \frac{x}{y^2} + h'(y) = \frac{x}{y^2}$$

$$\Rightarrow \quad h'(y) = 0 \quad \to h(y) = C$$

$$\boxed{\ln|x| - \frac{x}{y} = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$x^2y^3 - y + x\left(1 + x^2y^2\right)y' = 0, \qquad \mu(x, y) = \frac{1}{xy}$$

Solution

$$M_y = \frac{\partial}{\partial y}\left(x^2y^3 - y\right) = 3y^2 - 1 \qquad N_x = \frac{\partial}{\partial x}\left(x + x^3y^2\right) = 1 + 3x^2y^2 \quad \Rightarrow M_y \neq N_x$$

$$\left(x^2y^3 - y\right)\left(\frac{1}{xy}\right) + x\left(1 + x^2y^2\right)\left(\frac{1}{xy}\right)y' = 0$$

$$\left(xy^2 - \frac{1}{x}\right) + \left(\frac{1}{y} + x^2y\right)y' = 0$$

$$M_y = \frac{\partial}{\partial y}\left(xy^2 - \frac{1}{x}\right) = 2xy$$

$$N_x = \frac{\partial}{\partial x}\left(\frac{1}{y} + x^2y\right) = 2xy$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial\psi}{\partial x} = xy^2 - \frac{1}{x}$$

$$\psi = \int\left(xy^2 - \frac{1}{x}\right)dx$$

$$= \frac{1}{2}x^2y^2 - \ln|x| + h(y)$$

$$\psi_y = x^2y + h'(y) = \frac{1}{y} + x^2y$$

$$\Rightarrow \quad h'(y) = \frac{1}{y} \quad \to h(y) = \ln|y|$$

$$\boxed{\frac{1}{2}x^2y^2 - \ln x + \ln y = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$\left(\frac{\sin y}{y} - 2e^{-x}\sin x\right)dx + \left(\frac{\cos y + 2e^{-x}\cos x}{y}\right)dy = 0, \qquad \mu(x, y) = ye^{x}$$

Solution

$$M_y = \frac{\partial}{\partial y}\left(\frac{\sin y}{y} - 2e^{-x}\sin x\right) = \frac{y\cos y - \sin y}{y^2}$$

$$N_x = \frac{\partial}{\partial x}\left(\frac{\cos y + 2e^{-x}\cos x}{y}\right) = \frac{1}{y}\left(-2e^{-x}\cos x - 2e^{-x}\sin x\right)$$

$$\Rightarrow M_y \neq N_x$$

$$\left(ye^{x}\right)\left(\frac{\sin y}{y} - 2e^{-x}\sin x\right)dx + \left(ye^{x}\right)\left(\frac{\cos y + 2e^{-x}\cos x}{y}\right)dy = 0$$

$$\left(e^{x}\sin y - 2y\sin x\right)dx + \left(e^{x}\cos y + 2\cos x\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(e^{x}\sin y - 2y\sin x\right) = e^{x}\cos y - 2\sin x$$

$$N_x = \frac{\partial}{\partial x}\left(e^{x}\cos y + 2\cos x\right) = e^{x}\cos y - 2\sin x$$

$$\Rightarrow M_y = N_x$$

$$\psi = \int \left(e^{x}\sin y - 2y\sin x\right)dx$$

$$= e^{x}\sin y + 2y\cos x + h(y)$$

$$\psi_y = e^{x}\cos y + 2\cos x + h'(y)$$

$$= e^{x}\cos y + 2\cos x$$

$$\Rightarrow \quad h'(y) = 0 \quad \rightarrow h(y) = C$$

$$e^{x}\sin y + 2y\cos x = C$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$(x+2)\sin y dx + x\cos y dy = 0, \qquad \mu(x, y) = xe^x$$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left((x+2)\sin y\right) = (x+2)\cos y \\ N_x &= \frac{\partial}{\partial x}\left(x\cos y\right) = -x\sin y \end{aligned}\right. \Rightarrow M_y \neq N_x$$

$$\left(xe^x\right)(x+2)\sin y dx + \left(xe^x\right)x\cos y dy = 0$$

$$\left(x^2+2x\right)e^x \sin y dx + x^2 e^x \cos y dy = 0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(\left(x^2+2x\right)e^x \sin y\right) = \left(x^2+2x\right)e^x \cos y \\ N_x &= \frac{\partial}{\partial x}\left(x^2 e^x \cos y\right) = \left(2xe^x + x^2\right)e^x \cos y \end{aligned}\right. \Rightarrow M_y = N_x$$

$$\psi = \int \left(x^2 e^x \cos y\right) dy = x^2 e^x \sin y + h(x)$$

$$\psi_x = \left(x^2+2x\right)e^x \sin y + h'(x) = \left(x^2+2x\right)e^x \sin y$$

$$\Rightarrow \quad h'(x) = 0 \;\rightarrow\; h(x) = C$$

$$\boxed{x^2 e^x \sin y = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$\left(x^2+y^2-x\right)dx - y dy = 0, \qquad \mu(x, y) = \frac{1}{x^2+y^2}$$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(x^2+y^2-x\right) = 2y \\ N_x &= \frac{\partial}{\partial x}(-y) = 0 \end{aligned}\right. \Rightarrow M_y \neq N_x$$

$$\frac{1}{x^2+y^2}\left(x^2+y^2-x\right)dx - \frac{y}{x^2+y^2}dy = 0$$

$$\left(1 - \frac{x}{x^2+y^2}\right)dx - \frac{y}{x^2+y^2}dy = 0$$

$$M_y = \left(1 - \frac{x}{x^2+y^2}\right) = \frac{2xy}{\left(x^2+y^2\right)^2}$$
$$N_x = \left(\frac{-y}{x^2+y^2}\right) = \frac{2xy}{\left(x^2+y^2\right)^2}$$
$$\Rightarrow M_y = N_x$$

$$\frac{d\psi}{dx} = 1 - \frac{x}{x^2+y^2}$$

$$\psi = \int\left(1 - \frac{x}{x^2+y^2}\right)dx$$
$$= \int dx - \frac{1}{2}\int \frac{1}{x^2+y^2}\, d\left(x^2+y^2\right)$$
$$= x - \frac{1}{2}\ln\left(x^2+y^2\right) + h(y)$$

$$\psi_y = -\frac{y}{x^2+y^2} + h'(y)$$
$$= -\frac{y}{x^2+y^2}$$

$$h'(y) = 0 \quad \to \quad h(y) = C$$

$$\boxed{x - \frac{1}{2}\ln\left(x^2+y^2\right) = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation $(2y-6x)dx + \left(3x - 4x^2y^{-1}\right)dy = 0, \qquad \mu(x, y) = xy^2$

Solution

$$M_y = \frac{\partial}{\partial y}(2y-6x) = 2$$
$$N_x = \frac{\partial}{\partial x}\left(3x - 4x^2y^{-1}\right) = 3 - \frac{8x}{y}$$
$$\Rightarrow M_y \neq N_x$$

$$xy^2(2y-6x)dx + xy^2\left(3x-4x^2y^{-1}\right)dy = 0$$

$$\left(2xy^3 - 6x^2y^2\right)dx + \left(3x^2y^2 - 4x^3y\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(2xy^3 - 6x^2y^2\right) = 6xy^2 - 12x^2y$$
$$N_x = \frac{\partial}{\partial x}\left(3x^2y^2 - 4x^3y\right) = 6xy^2 - 12x^2y$$
$$\Rightarrow M_y = N_x$$

$$\psi = \int \left(2xy^3 - 6x^2y^2\right) dx$$

$$= x^2y^3 - 2x^3y^2 + h(y)$$

$$\psi_y = 3x^2y^2 - 4x^3y + h'(y)$$

$$= 3x^2y^2 - 4x^3y$$

$$h'(y) = 0 \quad \to \quad \boxed{h(y) = C}$$

$$\boxed{x^2y^3 - 2x^3y^2 = C}$$

Exercise

Find the general solution of the homogenous equation $\left(x^2 + y^2\right)dx - 2xydy = 0$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(x^2 + y^2\right) = 2y \\ N_x &= \frac{\partial}{\partial x}(-2xy) = -2y \end{aligned}\right\} \Rightarrow M_y \neq N_x$$

$$\frac{M_y - N_x}{N} = \frac{2y + 2y}{-2xy} = -\frac{4y}{2xy} = -\frac{2}{x}$$

$$\frac{d\mu}{dx} = -\mu\frac{2}{x} \quad \Rightarrow \quad \int \frac{d\mu}{\mu} = -2\int \frac{dx}{x}$$

$$\ln\mu = -2\ln x$$

$$\ln\mu = \ln x^{-2} \quad \Rightarrow \quad \boxed{\mu = \frac{1}{x^2}}$$

$$\frac{1}{x^2}\left(x^2 + y^2\right)dx - \frac{1}{x^2}2xydy = 0 \qquad \Rightarrow \quad \left(1 + \frac{y^2}{x^2}\right)dx - \frac{2y}{x}dy = 0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(1 + \frac{y^2}{x^2}\right) = \frac{2y}{x^2} \\ N_x &= \frac{\partial}{\partial x}\left(-\frac{2y}{x}\right) = \frac{2y}{x^2} \end{aligned}\right\} \Rightarrow M_y = N_x$$

$$\psi = \int \left(1 + \frac{y^2}{x^2}\right)dx$$

$$= x - \frac{y^2}{x} + h(y)$$

$$\psi_y = -\frac{2y}{x} + h'(y) = -\frac{2y}{x}$$

$$h'(y)=0 \quad \Rightarrow \quad h(y)=C$$

$$x-\frac{y^2}{x}=C \qquad\qquad \textit{multiply by } x$$

$$\boxed{x^2-y^2=Cx}$$

Exercise

Find the general solution of the homogenous equation $(x+y)dx+(y-x)dy=0$

Solution

$$(x+y)dx=-(y-x)dy$$

$$\frac{dy}{dx}=\frac{x+y}{x-y}$$

$$=\frac{\frac{x+y}{x}}{\frac{x-y}{x}}$$

$$=\frac{1+\frac{y}{x}}{1-\frac{y}{x}}$$

$$\frac{dy}{dx}=\frac{1+v}{1-v}=x\frac{dv}{dx}+v$$

$$x\frac{dv}{dx}=\frac{1+v}{1-v}-v=\frac{1+v^2}{1-v}$$

$$\frac{1-v}{1+v^2}dv=\frac{dx}{x}$$

$$\int\frac{dx}{x}=\int\frac{1}{1+v^2}dv-\int\frac{v}{1+v^2}dv \qquad\qquad \int\frac{1}{a^2+x^2}dx=\arctan\frac{x}{a}$$

$$\ln x=\arctan v-\frac{1}{2}\int\frac{1}{1+v^2}d\left(1+v^2\right)$$

$$\ln x+C=\arctan v-\frac{1}{2}\ln\left(1+v^2\right)$$

$$\ln x+C=\arctan\frac{y}{x}-\frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right)$$

$$\boxed{\arctan\frac{y}{x}-\frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right)-\ln x=C}$$

Exercise

Find the general solution of the homogenous equation $\frac{dy}{dx}=\frac{y\left(x^2+y^2\right)}{xy^2-2x^3}$

Solution

$$\frac{dy}{dx}=\frac{y}{x}\frac{\frac{x^2+y^2}{x^2}}{\frac{y^2-2x^2}{x^2}}$$

$$=\frac{y}{x}\frac{1+\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)^2-2}$$

$$=v\frac{1+v^2}{v^2-2}=x\frac{dv}{dx}+v$$

$$\frac{v+v^3}{v^2-2}-v=x\frac{dv}{dx}$$

$$x\frac{dv}{dx}=\frac{v+v^3-v^3+2v}{v^2-2}=\frac{3v}{v^2-2}$$

$$\int\frac{dx}{x}=\int\frac{v^2-2}{3v}dv=\frac{1}{3}\int\left(v-\frac{2}{v}\right)dv$$

$$3\ln x+C=\frac{1}{2}v^2-2\ln v$$

$$3\ln x+C=\frac{1}{2}\frac{y^2}{x^2}-2\ln\frac{y}{x}$$

$$3\ln x+C=\frac{1}{2}\frac{y^2}{x^2}-2(\ln y-\ln x)$$

$$6\ln x+C=\frac{y^2}{x^2}-4\ln y+4\ln x$$

$$\boxed{\frac{y^2}{x^2}-4\ln y-2\ln x=C}$$

Exercise

Find an integrating factor and solve the given equation $\left(3x^2y+2xy+y^3\right)dx+\left(x^2+y^2\right)dy=0$

Solution

$$M_y=\frac{\partial}{\partial y}\left(3x^2y+2xy+y^3\right)=3x^2+2x+3y^2$$

$$N_x=\frac{\partial}{\partial x}\left(x^2+y^2\right)=2x$$

$$\Rightarrow M_y\neq N_x$$

$$\frac{M_y - N_x}{N} = \frac{3x^2 + 2x + 3y^2 - 2x}{x^2 + y^2} = 3$$

$$\frac{d\mu}{dx} = 3\mu \quad \Rightarrow \quad \int \frac{d\mu}{\mu} = 3\int dx$$

$$\ln \mu = 3x \quad \Rightarrow \quad \mu = e^{3x}$$

$$e^{3x}\left(3x^2 y + 2xy + y^3\right)dx + e^{3x}\left(x^2 + y^2\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left[e^{3x}\left(3x^2 y + 2xy + y^3\right)\right] = e^{3x}\left(3x^2 + 2x + 3y^2\right)$$

$$N_x = \frac{\partial}{\partial x} e^{3x}\left(x^2 + y^2\right) = 3e^{3x}\left(x^2 + y^2\right) + 2xe^{3x} = e^{3x}\left(3x^2 + 3y^2 + 2x\right)$$

$$\Rightarrow M_y = N_x$$

$$\psi = \int \left(e^{3x}\left(3x^2 y + 2xy + y^3\right)\right)dx = +h(y)$$

$$= e^{3x}\left(x^2 y + \frac{2}{3}xy + \frac{1}{3}y^3 - \frac{2}{3}xy - \frac{2}{9}y + \frac{2}{9}y\right) + h(y)$$

$$= e^{3x}\left(x^2 y + \frac{1}{3}y^3\right) + h(y)$$

	$\int e^{3x}$
$3x^2 y + 2xy + y^3$	$\frac{1}{3}e^{3x}$
$6xy + 2y$	$\frac{1}{9}e^{3x}$
$6y$	$\frac{1}{27}e^{3x}$

$$\psi_y = e^{3x}\left(x^2 + y^2\right) + h'(y) = e^{3x}\left(x^2 + y^2\right)$$

$$h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$\boxed{e^{3x}\left(x^2 y + \frac{1}{3}y^3\right) = C}$$

Exercise

Find an integrating factor and solve the given equation $\quad dx + \left(\frac{x}{y} - \sin y\right)dy = 0$

Solution

$$ydx + \left(x - y\sin y\right)dy = 0 \qquad \textit{Multiply by } \boldsymbol{y} \textit{ both sides}$$

$$M_y = \frac{\partial}{\partial y}(y) = 1; \quad N_x = \frac{\partial}{\partial x}(x - y\sin y) = 1; \quad M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = 2x + y^2 \quad \Rightarrow \quad \psi = \int ydx = xy + h(y)$$

$$\psi_y = x + h'(y) = x - y\sin y \quad \Rightarrow \quad h'(y) = -y\sin y$$

$$h(y) = -\int y\sin ydy = y\cos y - \sin y$$

	$\int -\sin y$
y	$\cos y$
1	$\sin y$

$$\boxed{xy + y\cos y - \sin y = C}$$

Exercise

Find an integrating factor and solve the given equation $\quad e^x dx + \left(e^x \cot y + 2y\csc y\right)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}\left(e^x\right) = 0$$
$$N_x = \frac{\partial}{\partial x}\left(e^x \cot y + 2y\csc y\right) = e^x \quad \Rightarrow M_y \neq N_x$$

$$e^x dx + \left(e^x \frac{\cos y}{\sin y} + 2y\frac{1}{\sin y}\right)dy = 0 \qquad \textit{Multiply by } \boldsymbol{\sin y} \textit{ both sides}$$

$$(\sin y)e^x dx + (\sin y)\left(e^x \frac{\cos y}{\sin y} + 2y\frac{1}{\sin y}\right)dy = 0$$

$$e^x \sin y dx + \left(e^x \cos y + 2y\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(e^x \sin y\right) = e^x \cos y$$
$$N_x = \frac{\partial}{\partial x}\left(e^x \cos y + 2y\right) = e^x \cos y \quad \Rightarrow M_y = N_x$$

$$\psi = \int \left(e^x \sin y\right)dx = e^x \sin y + h(y)$$

$$\psi_y = e^x \cos y + h'(y) = e^x \cos y + 2y \quad \to h'(y) = 2y \;\Rightarrow\; h(y) = y^2$$

$$\psi(x, y) = e^x \sin y + y^2 = C$$

$$e^x \sin y + y^2 = C$$

Exercise

Find an integrating factor and solve the given equation $\quad \left(3x + \frac{6}{y}\right)dx + \left(\frac{x^2}{y} + 3\frac{y}{x}\right)dy = 0$

Solution

$$xy\left(3x + \frac{6}{y}\right)dx + xy\left(\frac{x^2}{y} + 3\frac{y}{x}\right)dy = 0$$

$$\left(3x^2 y + 6x\right)dx + \left(x^3 + 3y^2\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(3x^2 y + 6x\right) = 3x^2$$
$$N_x = \frac{\partial}{\partial x}\left(x^3 + 3y^2\right) = 3x^2 \quad \Rightarrow M_y = N_x$$

$$\psi = \int \left(3x^2 y + 6x\right)dx = x^3 y + 3x^2 + h(y)$$

$$\psi_y = x^3 + h'(y) = x^3 + 3y^2$$

$$h'(y) = 3y^2 \quad \Rightarrow \quad h(y) = y^3$$

$$\underline{x^3 y + 3x^2 + y^3 = C}$$

Exercise

Find an integrating factor and solve the given equation $\quad \left(x + 3x^3 \sin y\right)dx + \left(x^4 \cos y\right)dy = 0$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(x + 3x^3 \sin y\right) = 3x^3 \cos y \\ N_x &= \frac{\partial}{\partial x}\left(x^4 \cos y\right) = 4x^3 \cos y \end{aligned}\right\} \Rightarrow M_y \neq N_x$$

$$\frac{M_y - N_x}{N} = \frac{3x^3 \cos y - 4x^3 \cos y}{x^4 \cos y} = -\frac{1}{x}$$

$$\mu = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x} \qquad\qquad \mu = e^{\int \frac{M_y - N_x}{N}dx}$$

$$\frac{1}{x}\left(x + 3x^3 \sin y\right)dx + \frac{1}{x}\left(x^4 \cos y\right)dy = 0$$

$$\left(1 + 3x^2 \sin y\right)dx + \left(x^3 \cos y\right)dy = 0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(1 + 3x^2 \sin y\right) = 3x^2 \cos y \\ N_x &= \frac{\partial}{\partial x}\left(x^3 \cos y\right) = 3x^2 \cos y \end{aligned}\right\} \Rightarrow M_y = N_x$$

$$\psi = \int \left(1 + 3x^2 \sin y\right)dx = x + x^3 \sin y + h(y)$$

$$\psi_y = -x^3 \cos y + h'(y) = -x^3 \cos y$$

$$h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$\underline{x + x^3 \sin y = C}$$

Exercise

Find an integrating factor and solve the given equation $\quad \left(2x^2 + y\right)dx + \left(x^2 y - x\right)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}\left(2x^2 + y\right) = 1$$
$$N_x = \frac{\partial}{\partial x}\left(x^2 y - x\right) = 2xy - 1 \qquad \Rightarrow M_y \neq N_x$$

$$\frac{M_y - N_x}{N} = \frac{1 - 2xy + 1}{x^2 y - x} = \frac{2(1 - xy)}{x(xy - 1)} = -\frac{2}{x}$$

$$\mu = e^{-\int \frac{2}{x} dx} = e^{-2\ln x} \underline{= x^{-2}|} \qquad \mu = e^{\int \frac{M_y - N_x}{N} dx}$$

$$x^{-2}\left(2x^2 + y\right)dx + x^{-2}\left(x^2 y - x\right)dy = 0$$

$$\left(2 + \frac{y}{x^2}\right)dx + \left(y - \frac{1}{x}\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(2 + \frac{y}{x^2}\right) = \frac{1}{x^2}$$
$$N_x = \frac{\partial}{\partial x}\left(y - \frac{1}{x}\right) = \frac{1}{x^2} \qquad \Rightarrow M_y = N_x$$

$$\psi = \int \left(2 + \frac{y}{x^2}\right)dx$$
$$= 2x - \frac{y}{x} + h(y)$$

$$\psi_y = -\frac{1}{x} + h'(y) = y - \frac{1}{x}$$

$$h'(y) = y \Rightarrow h(y) = \frac{1}{2}y^2$$

$$\underline{2x - \frac{y}{x} + \frac{1}{2}y^2 = C|}$$

Exercise

Find an integrating factor and solve the given equation $\left(3x^2 + y\right)dx + \left(x^2 y - x\right)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}\left(3x^2 + y\right) = 1$$
$$N_x = \frac{\partial}{\partial x}\left(x^2 y - x\right) = 2xy - 1 \qquad \Rightarrow M_y \neq N_x$$

$$\frac{M_y - N_x}{N} = \frac{1 - 2xy + 1}{x^2 y - x}$$

$$= \frac{2(1-xy)}{x(xy-1)}$$

$$= -\frac{2}{x}$$

$$e^{-\int \frac{2}{x}dx} = e^{-2\ln x} \underline{= x^{-2}}|$$

$$x^{-2}\left(3x^2+y\right)dx + x^{-2}\left(x^2y-x\right)dy = 0$$

$$\left(3+\frac{y}{x^2}\right)dx + \left(y-\frac{1}{x}\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(3+\frac{y}{x^2}\right) = \frac{1}{x^2}$$

$$N_x = \frac{\partial}{\partial x}\left(y-\frac{1}{x}\right) = \frac{1}{x^2}$$

$$\Rightarrow M_y = N_x$$

$$\psi = \int\left(3+\frac{y}{x^2}\right)dx$$

$$= 3x - \frac{y}{x} + h(y)$$

$$\psi_y = -\frac{1}{x} + h'(y)$$

$$= y - \frac{1}{x}$$

$$h'(y) = y \;\Rightarrow\; h(y) = \frac{1}{2}y^2$$

$$\boxed{3x - \frac{y}{x} + \frac{1}{2}y^2 = C}$$

Exercise

Find an integrating factor and solve the given equation $\left(2y^2+2y+4x^2\right)dx + (2xy+x)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}\left(2y^2+2y+4x^2\right) = 4y+2$$

$$N_x = \frac{\partial}{\partial x}(2xy+x) = 2y+1$$

$$\Rightarrow M_y \neq N_x$$

$$\frac{M_y - N_x}{N} = \frac{4y+2-2y-1}{x(2y+1)} = \frac{2y+1}{x(2y+1)} = \frac{1}{x}$$

$$\mu = e^{\int \frac{1}{x}dx} = e^{\ln x} \underline{= x}|$$

$$\mu = e^{\int \frac{M_y - N_x}{N}dx}$$

$$x\left(2y^2+2y+4x^2\right)dx+x(2xy+x)dy=0$$

$$\left(2xy^2+2xy+4x^3\right)dx+\left(2x^2y+x^2\right)dy=0$$

$$\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(2xy^2+2xy+4x^3\right)=4xy+2x \\ N_x &= \frac{\partial}{\partial x}\left(2x^2y+x^2\right)=4xy+2x \end{aligned} \quad \Rightarrow M_y=N_x$$

$$\psi=\int\left(2xy^2+2xy+4x^3\right)dx$$

$$=x^2y^2+x^2y+x^4+h(y)$$

$$\psi_y=2x^2y+x^2+h'(y)=2x^2y+x^2$$

$$h'(y)=0 \;\Rightarrow\; h(y)=C$$

$$\boxed{x^2y^2+x^2y+x^4=C}$$

Exercise

Find an integrating factor and solve the given equation $\left(x^4-x+y\right)dx-xdy=0$

Solution

$$\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(x^4-x+y\right)=1 \\ N_x &= \frac{\partial}{\partial x}(-x)=-1 \end{aligned} \quad \Rightarrow M_y \neq N_x$$

$$\frac{M_y-N_x}{N}=-\frac{2}{x}$$

$$\mu=e^{-\int\frac{2}{x}dx}=e^{-2\ln x}\boxed{=x^{-2}} \qquad \mu=e^{\int\frac{M_y-N_x}{N}dx}$$

$$x^{-2}\left(x^4-x+y\right)dx-x^{-2}xdy=0$$

$$\left(x^2-x^{-1}+yx^{-2}\right)dx-x^{-1}dy=0$$

$$\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(x^2-x^{-1}+yx^{-2}\right)=x^{-2} \\ N_x &= \frac{\partial}{\partial x}\left(-x^{-1}\right)=x^{-2} \end{aligned} \quad \Rightarrow M_y=N_x$$

$$\psi=\int\left(x^2-x^{-1}+yx^{-2}\right)dx$$

$$=\frac{1}{3}x^3-\ln|x|-\frac{y}{x}+h(y)$$

$$\psi_y=-\frac{1}{x}+h'(y)$$

$$=-\frac{1}{x}$$

$$h'(y)=0 \;\Rightarrow\; h(y)=C$$

$$\boxed{\frac{1}{3}x^3-\ln|x|-\frac{y}{x}=C}$$

Exercise

Find an integrating factor and solve the given equation $(2xy)dx+\left(y^2-3x^2\right)dy=0$

Solution

$$M_y=\frac{\partial}{\partial y}(2xy)=2x$$

$$N_x=\frac{\partial}{\partial x}\left(y^2-3x^2\right)=-6x$$

$$\Rightarrow M_y\neq N_x$$

$$\frac{N_x-M_y}{M}=\frac{-8x}{2xy}=-\frac{4}{y}$$

$$\mu=e^{-\int\frac{4}{y}dy}=e^{-4\ln y}\underline{=y^{-4}}$$

$$\mu=e^{\int\frac{N_x-M_y}{M}dy}$$

$$y^{-4}(2xy)dx+y^{-4}\left(y^2-3x^2\right)dy=0$$

$$\left(2xy^{-3}\right)dx+\left(y^{-2}-3x^2y^{-4}\right)dy=0$$

$$M_y=\frac{\partial}{\partial y}\left(2xy^{-3}\right)=-6xy^{-4}$$

$$N_x=\frac{\partial}{\partial x}\left(y^{-2}-3x^2y^{-4}\right)=-6xy^{-4}$$

$$\Rightarrow M_y=N_x$$

$$\psi=\int\left(2xy^{-3}\right)dx=x^2y^{-3}+h(y)$$

$$\psi_y=-3x^2y^{-4}+h'(y)=y^{-2}-3x^2y^{-4}$$

$$h'(y)=\frac{1}{y^2}\;\Rightarrow\; h(y)=-\frac{1}{y}$$

$$\boxed{x^2y^{-3}-\frac{1}{y}=C}$$

Exercise

Solve the given initial-value problem $\frac{dy}{dx}=\frac{xy^2-\cos x\sin x}{y\left(1-x^2\right)}, \quad y(0)=2$

Solution

$$\left(xy^2-\cos x\sin x\right)dx-y\left(1-x^2\right)dy=0$$

$$M_y=\frac{\partial}{\partial y}\left(xy^2-\cos x\sin x\right)=2xy \qquad N_x=\frac{\partial}{\partial x}\left(-y+yx^2\right)=2xy$$

$$\Rightarrow \underline{M_y=N_x|}$$

$$\begin{aligned}\psi&=\int\left(xy^2-\cos x\sin x\right)dx\\&=\int\left(xy^2-\tfrac{1}{2}\sin 2x\right)dx\\&=\tfrac{1}{2}x^2y^2+\tfrac{1}{4}\cos 2x+h(y)\end{aligned}$$

$$\psi_y=x^2y+h'(y)=-y+yx^2$$

$$h'(y)=-y \quad\Rightarrow\quad h(y)=-\tfrac{1}{2}y^2$$

$$\psi=\tfrac{1}{2}x^2y^2+\tfrac{1}{4}\cos 2x-\tfrac{1}{2}y^2=C$$

$$y(0)=2 \quad\Rightarrow\quad \tfrac{1}{4}-\tfrac{1}{2}(4)=C\to\boxed{C=-\tfrac{7}{4}}$$

$$\tfrac{1}{2}x^2y^2+\tfrac{1}{4}\cos 2x-\tfrac{1}{2}y^2=-\tfrac{7}{4}$$

$$\boxed{2x^2y^2+\cos 2x-2y^2=-7} \qquad 2x^2y^2+2\cos^2 x-1-2y^2=-7$$

$$\underline{x^2y^2+\cos^2 x-y^2=-3|}$$

Exercise

Solve the given initial-value problem $(x+y)^2\,dx+\left(2xy+x^2-1\right)dy, \quad y(1)=1$

Solution

$$M_y=\frac{\partial}{\partial y}(x+y)^2=2(x+y)$$

$$N_x=\frac{\partial}{\partial x}\left(2xy+x^2-1\right)=2y+2x \qquad \Rightarrow \underline{M_y=N_x|}$$

$$\psi=\int\left(x^2+2xy+y^2\right)dx$$

$$= \frac{1}{3}x^3 + x^2 y + xy^2 + h(y)$$

$$\psi_y = x^2 + 2xy + h'(y)$$

$$= 2xy + x^2 - 1$$

$$h'(y) = -1 \quad \Rightarrow \quad h(y) = -y$$

$$\psi = \frac{1}{3}x^3 + x^2 y + xy^2 - y = C$$

$$y(1) = 1 \quad \Rightarrow \quad \frac{1}{3} + 1 + 1 - 1 = C \quad \to \quad \underline{C = \frac{4}{3}}$$

$$\underline{\frac{1}{3}x^3 + x^2 y + xy^2 - y = \frac{4}{3}}$$

Exercise

Solve the given initial-value problem $\left(e^x + y\right)dx + \left(2 + x + ye^y\right)dy, \quad y(0) = 1$

Solution

$$M_y = \frac{\partial}{\partial y}\left(e^x + y\right) = 1$$

$$N_x = \frac{\partial}{\partial x}\left(2 + x + ye^y\right) = 1 \quad \Rightarrow \quad \underline{M_y = N_x}$$

$$\psi = \int \left(e^x + y\right)dx$$

$$= e^x + xy + h(y)$$

$$\psi_y = x + h'(y)$$

$$= 2 + x + ye^y$$

$$h'(y) = 2 + ye^y$$

$$h(y) = 2y + e^y (y - 1)$$

$$e^x + xy + 2y + e^y (y - 1) = C$$

$$y(0) = 1 \quad \Rightarrow \quad 1 + 2 = C \quad \to \quad \underline{C = 3}$$

$$\underline{e^x + xy + 2y + e^y (y - 1) = 3}$$

Exercise

Solve the given initial-value problem $(2x - y)dx + (2y - x)dy, \quad y(1) = 3$

Solution

$$\begin{cases} M_y = \frac{\partial}{\partial y}(2x - y) = -1 \\ N_x = \frac{\partial}{\partial x}(2y - x) = -1 \end{cases} \Rightarrow \underline{M_y = N_x}$$

$$\psi = \int (2x - y)\,dx$$

$$= x^2 - xy + h(y)$$

$$\psi_y = -x + h'(y) = 2y - x$$

$$h'(y) = 2y \quad \rightarrow h(y) = y^2$$

$$x^2 - xy + y^2 = C$$

$$y(1) = 3 \quad \Rightarrow \quad 1 - 3 + 9 = C \quad \rightarrow \quad \underline{C = 7}$$

$$\boxed{x^2 - xy + y^2 = 7}$$

$$y^2 - xy + x^2 - 7 = 0 \quad \rightarrow \quad y = \frac{x \pm \sqrt{x^2 - 4x^2 + 28}}{2} = \frac{x \pm \sqrt{-3x^2 + 28}}{2}$$

$$since \;\; y(1) = 3 \quad \rightarrow \quad y = \frac{1}{2}\left(1 \pm \sqrt{-3(1)^2 + 28}\right) = \frac{1}{2}(1 \pm 5) \begin{cases} \frac{1+5}{2} = 3 \\ \cancel{\frac{1-5}{2} = -2} \end{cases}$$

$$\underline{y(x) = \frac{x + \sqrt{-3x^2 + 28}}{2}} \qquad |x| < \sqrt{\frac{28}{3}}$$

Exercise

Solve the given initial-value problem $\left(9x^2 + y - 1\right)dx - (4y - x)\,dy, \quad y(1) = 0$

Solution

$$\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(9x^2 + y - 1\right) = 1 \\ N_x &= \frac{\partial}{\partial x}(x - 4y) = 1 \end{aligned} \Rightarrow \underline{M_y = N_x}$$

$$\psi = \int \left(9x^2 + y - 1\right)dx$$

$$= 3x^3 + xy - x + h(y)$$

$$\psi_y = x + h'(y) = x - 4y$$

$$h'(y) = -4y \quad \rightarrow h(y) = -2y^2$$

$$3x^3 + xy - x - 2y^2 = C$$

$$y(1) = 0 \quad \Rightarrow \quad 3 - 1 = C \quad \rightarrow \quad \underline{C = 2}$$

$$\boxed{-2y^2 + xy + 3x^3 - x = 2}$$

$$-2y^2 + xy + 3x^3 - x - 2 = 0 \quad \to \quad y = \frac{-x \pm \sqrt{x^2 + 24x^3 - 8x - 16}}{-4}$$

$$\text{since } y(1) = 0 \quad \to \quad y = -\frac{1}{4}\left(-1 \pm \sqrt{1 + 24 - 8 - 16}\right) = -\frac{1}{4}(-1 \pm 1) \begin{cases} -\frac{-1+1}{4} = 0 \\ \cancel{\frac{-1-1}{-4} = \frac{1}{2}} \end{cases}$$

$$\boxed{y(x) = \frac{x - \sqrt{x^2 + 24x^3 - 8x - 16}}{4}}$$

Exercise

Solve the given initial-value problem $\left(x + y^3\right)y' + y + x^3 = 0, \quad y(0) = -2$

Solution

$$M_y = \frac{\partial}{\partial y}\left(y + x^3\right) = 1$$

$$N_x = \frac{\partial}{\partial x}\left(x + y^3\right) = 1$$

$$\Rightarrow \boxed{M_y = N_x}$$

$$\psi = \int \left(y + x^3\right) dx = xy + \frac{1}{4}x^4 + h(y)$$

$$\psi_y = x + h'(y) = x + y^3$$

$$h'(y) = y^3 \quad \to h(y) = \frac{1}{4}y^4$$

$$xy + \frac{1}{4}x^4 + \frac{1}{4}y^4 = C$$

$$y(0) = -2 \quad \to \quad \boxed{4 = C}$$

$$\boxed{xy + \frac{1}{4}x^4 + \frac{1}{4}y^4 = 4}$$

Exercise

Solve the given initial-value problem $y' = \left(3x^2 + 1\right)\left(y^2 + 1\right), \quad y(0) = 1$

Solution

$$\frac{1}{y^2 + 1}y' - \left(3x^2 + 1\right) = 0$$

$$M_y = \frac{\partial}{\partial y}\left(-3x^2 - 1\right) = 0$$

$$N_x = \frac{\partial}{\partial x}\left(\frac{1}{y^2 + 1}\right) = 0$$

$$\Rightarrow M_y = N_x$$

$$\psi = \int \left(-3x^2 - 1\right)dx = -x^3 - x + h(y)$$

$$\psi_y = h'(y) = \frac{1}{y^2+1} \quad \rightarrow h(y) = \tan^{-1} y$$

$$-x^3 - x + \tan^{-1} y = C$$

$$y(0) = 1 \quad \rightarrow \tan^{-1} 1 = C \;\Rightarrow\; \boxed{C = \frac{\pi}{4}}$$

$$-x^3 - x + \tan^{-1} y = \frac{\pi}{4}$$

$$\tan^{-1} y = x^3 + x + \frac{\pi}{4}$$

$$\boxed{y(x) = \tan\left(x^3 + x + \frac{\pi}{4}\right)}$$

Exercise

Solve the given initial-value problem $\left(y^3 + \cos t\right)y' = 2 + y\sin t, \quad y(0) = -1$

Solution

$$\left(y^3 + \cos t\right)y' - (2 + y\sin t) = 0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}(-2 - y\sin t) = -\sin t \\ N_t &= \frac{\partial}{\partial t}\left(y^3 + \cos t\right) = -\sin t \end{aligned}\right\} \Rightarrow \boxed{M_y = N_t}$$

$$\psi = \int (-2 - y\sin t)\,dt$$

$$= -2t + y\cos t + h(y)$$

$$\psi_y = \cos t + h'(y) = y^3 + \cos t$$

$$h'(y) = y^3 \quad \rightarrow h(y) = \frac{1}{4}y^4$$

$$-2t + y\cos t + \frac{1}{4}y^4 = C$$

$$y(0) = -1 \quad \rightarrow -1 + \frac{1}{4} = C \;\Rightarrow\; \boxed{C = -\frac{3}{4}}$$

$$\boxed{-2t + y\cos t + \frac{1}{4}y^4 = -\frac{3}{4}}$$

Exercise

Solve the given initial-value problem $\left(y^3 - t^3\right)y' = 3t^2 y + 1, \quad y(-2) = -1$

Solution

$$\left(y^3-t^3\right)y'-\left(3t^2y+1\right)=0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(-3t^2y-1\right)=-3t^2 \\ N_t &= \frac{\partial}{\partial t}\left(y^3-t^3\right)=-3t^2 \end{aligned}\right. \quad \Rightarrow \boxed{M_y=N_t}$$

$$\psi=\int\left(-3t^2y-1\right)dt$$

$$=-t^3y-t+h(y)$$

$$\psi_y=-t^3+h'(y)$$

$$=y^3-t^3$$

$$h'(y)=y^3 \quad \to h(y)=\frac{1}{4}y^4$$

$$-t^3y-t+\frac{1}{4}y^4=C$$

$$y(-2)=-1 \quad -8+2+\frac{1}{4}=C \quad \Rightarrow \boxed{C=-\frac{23}{4}}$$

$$\boxed{-t^3y-t+\frac{1}{4}y^4=-\frac{23}{4}}$$

Exercise

Solve the given initial-value problem $\frac{dy}{dx}=(-2x+y)^2-7, \quad y(0)=0$

Solution

Let $u=-2x+y \quad \Rightarrow \quad \frac{du}{dx}=-2+\frac{dy}{dx}$

$$\frac{du}{dx}+2=u^2-7$$

$$\frac{du}{dx}=u^2-9$$

$$\int\frac{du}{u^2-9}=\int dx$$

$$\frac{1}{u^2-9}=\frac{A}{u-3}+\frac{B}{u+3}$$

$$Au+3A+Bu-3B=0 \quad \begin{cases} A+B=0 \\ 3A-3B=1 \end{cases} \quad \to A=\frac{1}{6}, \quad B=-\frac{1}{6}$$

$$\frac{1}{6}\int\left(\frac{1}{u-3}-\frac{1}{u+3}\right)du=\int dx$$

$$\frac{1}{6}\left(\ln|u-3|-\ln|u+3|\right)=x+C$$

$$\ln\left|\frac{u-3}{u+3}\right|=6x+C$$

$$\frac{u-3}{u+3}=e^{6x+C}=Ae^{6x}$$

$$u-3=Aue^{6x}+3Ae^{6x}$$

$$u=\frac{3+3Ae^{6x}}{1-Ae^{6x}}=-2x+y$$

$$y=2x+\frac{3+3Ae^{6x}}{1-Ae^{6x}} \qquad y(0)=0$$

$$0=\frac{3+3A}{1-A} \to \boxed{A=-1}$$

$$\boxed{y(x)=2x+\frac{3\left(1-e^{6x}\right)}{1+e^{6x}}}$$

Exercise

Solve the given initial-value problem $\quad (2y-x)y'-y+2x=0, \quad y(1)=0$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}(2y-x)=2 \\ N_x &= \frac{\partial}{\partial x}(-y+2x)=2 \end{aligned}\right. \Rightarrow \boxed{M_y=N_x}$$

$$\psi=\int(-y+2x)\,dx=x^2-xy+h(y)$$

$$\psi_y=-x+h'(y)=2y-x \quad \Rightarrow h'(y)=2y \quad \to h(y)=y^2$$

$$x^2-xy+y^2=C$$

$$y(1)=0 \quad \to 1=C$$

$$x^2-xy+y^2=1$$

$$y^2-xy+x^2-1=0$$

$$y=\frac{x\pm\sqrt{x^2-4x^2+4}}{2}=\frac{x\pm\sqrt{4-3x^2}}{2}$$

Since $y(1)=0 \quad \to \boxed{y(x)=\frac{x-\sqrt{4-3x^2}}{2}}$

Exercise

Solve the given initial-value problem $\quad \left(e^{2y}+t^2y\right)y'+ty^2+\cos t=0, \quad y\left(\frac{\pi}{2}\right)=0$

Solution

$$M_y = \frac{\partial}{\partial y}\left(ty^2 + \cos t\right) = 2yt$$
$$N_t = \frac{\partial}{\partial t}\left(e^{2y} + t^2 y\right) = 2ty \quad \Rightarrow \boxed{M_y = N_t}$$

$$\psi = \int \left(ty^2 + \cos t\right) dt$$
$$= \frac{1}{2}t^2y^2 + \sin t + h(y)$$

$$\psi_y = t^2y + h'(y)$$
$$= t^2y + e^{2y}$$

$$h'(y) = e^{2y} \quad \to h(y) = \frac{1}{2}e^{2y}$$

$$\frac{1}{2}t^2y^2 + \sin t + \frac{1}{2}e^{2y} = C$$

$$y\left(\frac{\pi}{2}\right) = 0 \quad 1 + \frac{1}{2} = C \Rightarrow \boxed{C = \frac{3}{2}}$$

$$\boxed{\frac{1}{2}t^2y^2 + \sin t + \frac{1}{2}e^{2y} = \frac{3}{2}}$$

Exercise

Solve the given initial-value problem $\quad y' = -\dfrac{y\cos(ty) + 1}{t\cos(ty) + 2ye^{y^2}}, \quad y(\pi) = 0$

Solution

$$\left(t\cos(ty) + 2ye^{y^2}\right)y' + \left(y\cos(ty) + 1\right) = 0$$

$$M_y = \frac{\partial}{\partial y}\left(y\cos ty + 1\right) = \cos ty - ty\sin ty$$
$$N_t = \frac{\partial}{\partial t}\left(t\cos ty + 2ye^{y^2}\right) = \cos ty - ty\sin ty \quad \Rightarrow \boxed{M_y = N_t}$$

$$\psi = \int \left(y\cos ty + 1\right) dt$$
$$= \sin ty + t + h(y)$$

$$\psi_y = t\cos ty + h'(y)$$
$$= t\cos ty + 2ye^{y^2}$$

$$h'(y) = 2ye^{y^2} \quad \to h(y) = e^{y^2}$$

$$\sin ty + t + e^{y^2} = C$$

$$y(\pi)=0 \quad \Rightarrow \underline{C=-\pi-1}$$

$$\underline{\sin ty+t+e^{y^2}=\pi+1}$$

Exercise

Solve the given initial-value problem $\left(2ty+\frac{1}{y}\right)y'+y^2=1, \quad y(1)=1$

Solution

$$\left(2ty+\frac{1}{y}\right)y'+y^2-1=0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(y^2-1\right)=2y \\ N_t &= \frac{\partial}{\partial t}\left(2ty+\frac{1}{y}\right)=2y \end{aligned}\right\} \Rightarrow \underline{M_y=N_t}$$

$$\begin{aligned} \psi &= \int\left(y^2-1\right)dt \\ &= ty^2-t+h(y) \end{aligned}$$

$$\begin{aligned} \psi_y &= t\cos ty+h'(y) \\ &= 2ty+h'(y) \\ &= 2ty+\frac{1}{y} \end{aligned}$$

$$h'(y)=\frac{1}{y} \quad \rightarrow h(y)=\ln y$$

$$ty^2-t+\ln|y|=C$$

$$y(1)=1 \quad \Rightarrow \underline{C=0}$$

$$\underline{ty^2-t+\ln|y|=0}$$

Exercise

Solve the given initial-value problem $\left(ye^x+1\right)dx+\left(e^x-1\right)dy=0 \quad y(1)=1$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(ye^x+1\right)=e^x \\ N_x &= \frac{\partial}{\partial x}\left(e^x-1\right)=e^x \end{aligned}\right\} \Rightarrow M_y=N_x$$

$$\psi=\int\left(ye^x+1\right)dx$$

$$= ye^x + x + h(y)$$

$$\psi_y = e^x + h'(y) = e^x - 1$$

$$\to h'(y) = -1 \Rightarrow h(y) = -y$$

$$\underline{ye^x + x - y = C}$$

$$y(1) = 1 \quad \to \quad \underline{C = e}$$

$$\underline{ye^x + x - y = e}$$

Exercise

Solve the given initial-value problem $\quad 2xy^2 + 4 = 2\left(3 - x^2y\right)y' \quad y(-1) = 8$

Solution

$$2xy^2 + 4 - 2\left(3 - x^2y\right)y' = 0$$

$$\left.\begin{aligned} M &= 2xy^2 + 4 \;\Rightarrow\; M_y = 4xy \\ N &= -6 + 2x^2y \;\Rightarrow\; N_x = 4xy \end{aligned}\right\} \to \; M_y = N_x$$

$$\psi = \int \left(2xy^2 + 4\right) dx \qquad\qquad \psi = \int M dx$$

$$= x^2y^2 + 4x + h(y)$$

$$\psi_y = 2x^2y + h'(y)$$

$$= 2x^2y - 6$$

$$h'(y) = -6 \quad \to \quad h(y) = -6y$$

$$\underline{x^2y^2 + 4x - 6y = C}$$

$$y(-1) = 8 \quad \to 64 - 4 - 48 = C \quad \underline{C = 12}$$

$$\underline{x^2y^2 + 4x - 6y = 12}$$

Exercise

Solve the given initial-value problem $\quad y' + \frac{4}{x}y = x^3y^2 \quad y(2) = -1$

Solution

Let $u = y^{1-2} = y^{-1} \quad \Rightarrow \quad y = \frac{1}{u}$

$$\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx} \quad \Rightarrow \quad u' = -\frac{1}{y^2}y'$$

$$y' = -y^2 u' = -\frac{1}{u^2}u'$$

$$y' + \frac{4}{x}y = x^3 y^2$$

$$-\frac{1}{u^2}u' + \frac{4}{x}\frac{1}{u} = x^3\frac{1}{u^2}$$

$$u' - \frac{4}{x}u = -x^3$$

$$e^{\int -\frac{4}{x}dx} = e^{-4\ln x}$$
$$= x^{-4}$$

$$\int -x^3 x^{-4}dx = -\int \frac{1}{x}dx$$
$$= -\ln x$$

$$u = x^4\left(-\ln x + C\right)$$

$$y = \frac{1}{x^4\left(C - \ln x\right)} \qquad\qquad y = \frac{1}{u}$$

$$y(2) = -1 \quad \rightarrow -1 = \frac{1}{16\left(C - \ln 2\right)}$$

$$\boxed{C = \ln 2 - \frac{1}{16}}$$

$$\boxed{y(x) = \frac{1}{x^4\left(\ln 2 - \frac{1}{16} - \ln x\right)}}$$

Exercise

Solve the given initial-value problem $\quad y' = 5y + e^{-2x}y^{-2} \quad y(0) = 2$

Solution

$$y^2 y' - 5y^3 = e^{-2x}$$

Let $u = y^3 \quad \Rightarrow \quad y = u^{1/3}$

$$u' = 3y^2 y' \quad \Rightarrow \quad y' = \frac{1}{3}u^{-2/3}u'$$

$$\frac{1}{3}u' - 5u = e^{-2x}$$

$$u' - 15u = 3e^{-2x}$$

$$e^{\int -15dx} = e^{-15x}$$

$$\int 3e^{-2x}e^{-15x}dx = 3\int e^{-17x}dx$$

$$= -\frac{3}{17}e^{-17x}$$

$$u = e^{15x}\left(-\frac{3}{17}e^{-17x} + C\right) \qquad u = y^3$$

$$y^3 = e^{15x}C - \frac{3}{17}e^{-2x}$$

$$y(0) = 2 \quad \to \ 8 = C - \frac{3}{17}$$

$$\Rightarrow \boxed{C = \frac{139}{17}}$$

$$\boxed{y(x) = \left(\frac{139e^{15x} - 3e^{-2x}}{17}\right)^{1/3}}$$

Exercise

Solve the given initial-value problem $\quad 6y' - 2y = xy^4 \quad y(0) = -2$

Solution

$$6y^{-4}y' - 2y^{-3} = x$$

Let $u = y^{-3} \quad \Rightarrow \quad y = u^{-1/3}$

$$y' = -\frac{1}{3}u^{-4/3}u'$$

$$6u^{4/3}\left(-\frac{1}{3}u^{-4/3}\right)u' - 2u = x$$

$$-2u' - 2u = x$$

$$u' + u = -\frac{1}{2}x$$

$$e^{\int dx} = e^x$$

$$-\int \frac{1}{2}xe^x dx = -\frac{1}{2}(x-1)e^x$$

$$u = \frac{1}{e^x}\left(-\frac{1}{2}(x-1)e^x + C\right) \qquad u = y^{-3}$$

$$y^{-3} = \frac{C}{e^x} - \frac{1}{2}(x-1)$$

$$y(0) = -2 \quad \to \ -\frac{1}{8} = C + \frac{1}{2}$$

$$\Rightarrow \boxed{C = -\frac{5}{8}}$$

$$\frac{1}{y^3} = -\frac{5}{8}e^{-x} - \frac{1}{2}(x-1)$$

$$= -\frac{5e^{-x} - 4x + 4}{8}$$

$$\boxed{y(x) = -\frac{2}{\left(5e^{-x} - 4x + 4\right)^{1/3}}}$$

Exercise

Solve the given initial-value problem $\quad y' + \frac{y}{x} - \sqrt{y} = 0 \quad y(1) = 0$

Solution

$$y' + \frac{1}{x}y = y^{1/2}$$

$$y^{-1/2}y' + \frac{1}{x}y^{1/2} = 1$$

Let $u = y^{1/2} \quad \Rightarrow \quad y = u^2$

$$y' = 2u\,u'$$

$$u^{-1}(2uu') + \frac{1}{x}u = 1$$

$$2u' + \frac{1}{x}u = 1$$

$$u' + \frac{1}{2x}u = \frac{1}{2}$$

$$e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\ln x}$$

$$= x^{1/2}$$

$$\int \frac{1}{2}x^{1/2}dx = \frac{1}{3}x^{3/2}$$

$$u = \frac{1}{x^{1/2}}\left(\frac{1}{3}x^{3/2} + C\right)$$

$$y^{1/2} = \frac{1}{3}x + Cx^{-1/2}$$

$$y(1) = 0 \quad \rightarrow \quad 0 = C + \frac{1}{3}$$

$$\Rightarrow \quad \boxed{C = -\frac{1}{3}}$$

$$\boxed{y(x) = \left(\frac{1}{3}x + \frac{1}{3}x^{-1/2}\right)^2}$$

Exercise

Solve the given initial-value problem $\quad xyy' + 4x^2 + y^2 = 0 \quad y(2) = -7$

Solution

$$xyy' = -4x^2 - y^2$$

$$\frac{y}{x}y' = -4 - \left(\frac{y}{x}\right)^2$$

Let $\frac{y}{x} = v \quad \to y = xv$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v\left(v + x\frac{dv}{dx}\right) = -4 - v^2$$

$$vx\frac{dv}{dx} = -4 - 2v^2$$

$$-\frac{v}{4+2v^2}dv = \frac{dx}{x}$$

$$-\frac{1}{4}\int \frac{1}{4+2v^2}d\left(4+2v^2\right) = \int \frac{1}{x}dx$$

$$-\frac{1}{4}\ln\left(4+2v^2\right) = \ln x + \ln C_1$$

$$\ln\left(4+2v^2\right)^{-\frac{1}{4}} = \ln C_1 x$$

$$\left(4+2v^2\right)^{-\frac{1}{4}} = C_1 x$$

$$4 + 2\left(\frac{y}{x}\right)^2 = Cx^{-4}$$

$$\frac{2y^2}{x^2} = \frac{C}{x^4} - 4$$

$$y^2 = \frac{1}{2}\frac{C - 4x^4}{x^2}$$

$$y(2) = -7 \quad \to 49 = \frac{1}{2}\frac{C-64}{4}$$

$$\to \underline{C = 456|}$$

$$y^2 = \frac{1}{2}\frac{456 - 4x^4}{x^2}$$

$$y^2 = \frac{228 - 2x^4}{x^2}$$

$$y = \pm\frac{\sqrt{228 - 2x^4}}{x}$$

Since the given initial $y(2)=-7$

$$\boxed{y=-\frac{\sqrt{228-2x^4}}{x}}$$

Exercise

Solve the given initial-value problem $\quad xy'=y(\ln x-\ln y) \quad y(1)=4$

Solution

$$y'=\frac{y}{x}\ln\frac{x}{y}$$

Let $\quad \frac{y}{x}=v \quad \rightarrow y=xv$

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$v+xv'=v\ln\frac{1}{v}$$

$$x\frac{dv}{dx}=-v(1+\ln v)$$

$$\int\frac{dv}{v(1+\ln v)}=-\int\frac{dx}{x}$$

$$\int\frac{d(1+\ln v)}{1+\ln v}=-\int\frac{dx}{x}$$

$$\ln(1+\ln v)=-\ln x+\ln C$$

$$\ln\left(1+\ln\frac{y}{x}\right)=\ln\frac{C}{x}$$

$$1+\ln\frac{y}{x}=\frac{C}{x}$$

$$y(1)=4 \quad \rightarrow \quad \boxed{C=1+\ln 4}$$

$$\ln\frac{y}{x}=\frac{1+\ln 4}{x}-1$$

$$\boxed{y(x)=xe^{\frac{1+\ln 4}{x}-1}}$$

Exercise

Solve the given initial-value problem $\quad y'-(4x-y+1)^2=0 \quad y(0)=2$

Solution

Let $\quad v=4x-y \quad \Rightarrow \quad v'=4-y'$

$$y'=4-v'$$

$$4 - v' - (v+1)^2 = 0$$

$$v' = 4 - (v+1)^2$$

$$= -v^2 - 2v + 3$$

$$\int \frac{dv}{v^2 + 2v - 3} = -\int dx$$

$$\int \frac{dv}{(v-1)(v+3)} = -\int dx$$

$$\frac{1}{(v-1)(v+3)} = \frac{A}{v-1} + \frac{B}{v+3}$$

$$Av + 3A + Bv - B = 1$$

$$\begin{cases} A + B = 0 \\ 3A - B = 1 \end{cases} \rightarrow \underline{A = \frac{1}{4}, \ B = -\frac{1}{4}}$$

$$\frac{1}{4}\int \left(\frac{1}{v-1} - \frac{1}{v+3}\right) = -\int dx$$

$$\frac{1}{4}\left(\ln(v-1) - \ln(v+3)\right) = -x + C_1$$

$$\ln \frac{v-1}{v+3} = C - 4x$$

$$\ln \left|\frac{4x - y - 1}{4x - y + 3}\right| = C - 4x$$

$$y(0) = 2 \quad \rightarrow \quad \underline{C = \ln 3}$$

$$\frac{4x - y - 1}{4x - y + 3} = e^{\ln 3 - 4x}$$

$$4x - y - 1 = 3(4x - y + 3)e^{-4x}$$

$$(4x - y - 1)e^{4x} = 12x - 3y + 9$$

$$3y - ye^{4x} = 12x + 9 + (1 - 4x)e^{4x}$$

$$\underline{y(x) = \frac{12x + 9 + (1 - 4x)e^{4x}}{3 - e^{4x}}}$$

Exercise

Solve the given initial-value problem $\left(e^{t+y} + 2y\right)y' + \left(e^{t+y} + 3t^2\right) = 0, \quad y(0) = 0$

Solution

$$M_y = \frac{\partial}{\partial y}\left(e^{t+y} + 3t^2\right) = e^{t+y}$$

$$N_t = \frac{\partial}{\partial t}\left(e^{t+y} + 2y\right) = e^{t+y}$$

$$\Rightarrow \underline{M_y = N_t}$$

$$\psi = \int \left(e^{t+y} + 3t^2\right)dt$$

$$= e^{t+y} + t^3 + h(y)$$

$$\psi_y = e^{t+y} + h'(y) = e^{t+y} + 2y$$

$$h'(y) = 2y \quad \to h(y) = y^2$$

$$e^{t+y} + t^3 + y^2 = C$$

$$y(0) = 0 \quad \Rightarrow \quad \underline{C = 1}$$

$$\underline{e^{t+y} + t^3 + y^2 = 1}$$

Exercise

Solve the given initial-value problem $\quad (4y + 2x - 5)dx + (6y + 4x - 1)dy, \quad y(-1) = 2$

Solution

$$M_y = \frac{\partial}{\partial y}(4y + 2x - 5) = 4$$

$$N_x = \frac{\partial}{\partial x}(6y + 4x - 1) = 4 \quad \Rightarrow \underline{M_y = N_x}$$

$$\psi = \int (4y + 2x - 5)dx$$

$$= 4xy + x^2 - 5x + h(y)$$

$$\psi_y = 4x + h'(y)$$

$$= 6y + 4x - 1$$

$$\Rightarrow h'(y) = 6y - 1 \quad \to h(y) = 3y^2 - y$$

$$\psi = 4xy + x^2 - 5x + 3y^2 - y = C$$

$$y(-1) = 2 \quad \Rightarrow \quad 4(-1)(2) + 1 + 5 + 12 - 2 = C$$

$$\to \underline{C = 8}$$

$$\underline{4xy + x^2 - 5x + 3y^2 - y = 8}$$

Exercise

Solve the given initial-value problem $\quad \left(ye^{xy} - \frac{1}{y}\right)dx + \left(xe^{xy} + \frac{x}{y^2}\right)dy = 0 \quad y(1) = 1$

Solution

$$M_y = \frac{\partial}{\partial y}\left(ye^{xy} - \frac{1}{y}\right) = (1 + xy)e^{xy} + \frac{1}{y^2}$$

$$N_x = \frac{\partial}{\partial x}\left(xe^{xy} + \frac{x}{y^2}\right) = (1 + xy)e^{xy} + \frac{1}{y^2} \quad \Rightarrow M_y = N_x$$

$$\psi = \int \left(ye^{xy} - \frac{1}{y} \right) dx = e^{xy} - \frac{x}{y} + h(y)$$

$$\psi_y = xe^{xy} + \frac{x}{y^2} + h'(y) = xe^{xy} + \frac{x}{y^2}$$

$$\to h'(y) = 0 \ \Rightarrow \ h(y) = C$$

$$\underline{e^{xy} - \frac{x}{y} = C}$$

$$y(1) = 1 \quad \to \quad \underline{C = e - 1}$$

$$\underline{e^{xy} - \frac{x}{y} = e - 1}$$

Exercise

Solve the given initial-value problem $\quad (2y\ln t - t\sin y)y' + \frac{1}{t}y^2 + \cos y = 0, \quad y(2) = 0$

Solution

$$M_y = \frac{\partial}{\partial y}\left(\frac{1}{t}y^2 + \cos y \right) = \frac{2y}{t} - \sin y$$

$$N_t = \frac{\partial}{\partial t}(2y\ln t - t\sin y) = \frac{2y}{t} - \sin y$$

$$\Rightarrow \underline{M_y = N_t}$$

$$\psi = \int \left(\frac{1}{t}y^2 + \cos y \right) dt$$

$$= y^2 \ln|t| + t\cos y + h(y)$$

$$\psi_y = 2y\ln t - t\sin y + h'(y) = 2y\ln t - t\sin y$$

$$\Rightarrow h'(y) = 0 \quad \to h(y) = C$$

$$y^2 \ln|t| + t\cos y + C = 0$$

$$y(2) = 0 \quad \Rightarrow \quad \underline{C = -2}$$

$$\underline{y^2 \ln|t| + t\cos y - 2 = 0}$$

Exercise

Solve the given initial-value problem $\quad (\tan y - 2)dx + \left(x\sec^2 y + \frac{1}{y} \right) dy = 0 \quad y(0) = 1$

Solution

$$M_y = \frac{\partial}{\partial y}(\tan y - 2) = \sec^2 y$$

$$N_x = \frac{\partial}{\partial x}\left(x\sec^2 y + \frac{1}{y} \right) = \sec^2 y$$

$$\Rightarrow M_y = N_x$$

$$\psi = \int (\tan y - 2)\,dx$$
$$= x\tan y - 2x + h(y)$$

$$\psi_y = x\sec^2 y + h'(y)$$
$$= x\sec^2 y + \frac{1}{y}$$

$$\to\ h'(y) = \frac{1}{y}\ \Rightarrow\ h(y) = \ln|y|$$

$$x\tan y - 2x + \ln|y| = C$$

$$y(0) = 1 \quad \to\ C = 0$$

$$x\tan y - 2x + \ln y = 0$$

Exercise

Solve the given initial-value problem $\quad 2xy - 9x^2 + \left(2y + x^2 + 1\right)\frac{dy}{dx} = 0 \quad y(0) = -3$

Solution

$$\left.\begin{aligned} M &= 2xy - 9x^2 \ \Rightarrow\ M_y = 2x \\ N &= 2y + x^2 + 1 \ \Rightarrow\ N_x = 2x \end{aligned}\right\} \to\ M_y = N_x$$

$$\psi = \int \left(2xy - 9x^2\right) dx \qquad\qquad \psi = \int M dx$$
$$= x^2 y - 3x^3 + h(y)$$

$$\psi_y = x^2 + h'(y)$$
$$= 2y + x^2 + 1$$

$$h'(y) = 2y + 1 \ \to\ h(y) = y^2 + y$$

$$x^2 y - 3x^3 + y^2 + y = C$$

$$y(0) = -3 \ \to\ 9 - 3 = C \quad C = 6$$

$$x^2 y - 3x^3 + y^2 + y = 6$$

Exercise

Solve the given initial-value problem $\quad \frac{2t}{t^2+1}y - 2t + \left(2 - \ln\left(t^2 + 1\right)\right)\frac{dy}{dt} = 0 \quad y(5) = 0$

Solution

$$M=\frac{2t}{t^2+1}y-2t \quad \Rightarrow \quad M_y=\frac{2t}{t^2+1}$$
$$N=2-\ln\left(t^2+1\right) \quad \Rightarrow \quad N_t=\frac{2t}{t^2+1} \quad \rightarrow \quad M_y=N_t$$

$$\psi=\int\left(\frac{2t}{t^2+1}y-2t\right)dt \qquad\qquad \psi=\int M dt$$
$$=y\int\frac{1}{t^2+1}d\left(t^2+1\right)-t^2$$
$$=\ln\left(t^2+1\right)y-t^2+h(y)$$

$$\psi_y=\ln\left(t^2+1\right)+h'(y)$$
$$=2-\ln\left(t^2+1\right)$$

$$h'(y)=-2 \quad \rightarrow \quad h(y)=-2y$$

$$\ln\left(t^2+1\right)y-t^2-2y=C$$

$$y(5)=0 \quad \rightarrow \quad \underline{C=-25}$$

$$\left(\ln\left(t^2+1\right)-2\right)y-t^2=-25$$

$$\boxed{y(t)=\frac{t^2-25}{\ln\left(t^2+1\right)-2}}$$

Exercise

Solve the given initial-value problem $\quad 3y^3e^{3xy}-1+\left(2ye^{3xy}+3xy^2e^{3xy}\right)y'=0 \quad y(0)=1$

Solution

$$M=3y^3e^{3xy}-1 \quad \Rightarrow \quad M_y=9y^2e^{3xy}+9xy^3e^{3xy}$$
$$N=2ye^{3xy}+3xy^2e^{3xy} \quad \Rightarrow \quad N_x=6y^2e^{3xy}+3y^2e^{3xy}+9xy^3e^{3xy}$$

$$\rightarrow \quad M_y=N_x$$

$$\psi=\int\left(3y^3e^{3xy}-1\right)dx \qquad\qquad \psi=\int M dx$$
$$=y^2e^{3xy}-x+h(y)$$

$$\psi_y=2ye^{3xy}+3xy^2e^{3xy}+h'(y)$$
$$=2ye^{3xy}+3xy^2e^{3xy}$$

$$\Rightarrow h'(y) = 0 \quad \to \quad h(y) = C$$

$$y^2 e^{3xy} - x = C$$

$$y(0) = 1 \quad \to \quad \underline{C = 1}$$

$$\boxed{y^2 e^{3xy} - x = 1}$$

Exercise

Solve the given initial-value problem $\quad 2xydx + \left(1 + x^2\right)dy = 0\ ; \quad y(2) = -5$

Solution

$$\left.\begin{array}{l} M = 2xy \quad \Rightarrow \quad M_y = 2x \\ N = 1 + x^2 \quad \Rightarrow \quad N_x = 2x \end{array}\right. \quad \to \quad M_y = N_x$$

$$\psi = \int (2xy)\ dx$$

$$= x^2 y + h(y)$$

$$\psi_y = x^2 + h'(y) = 1 + x^2$$

$$\Rightarrow h'(y) = 1 \quad \to \quad h(y) = y$$

$$x^2 y + y = C$$

$$y\left(x^2 + 1\right) = C$$

$$y(2) = -5 \quad \to -20 - 5 = C$$

$$\Rightarrow \underline{C = -25}$$

$$\boxed{y(x) = -\frac{25}{1 + x^2}}$$

Exercise

Solve the given initial-value problem $\quad \dfrac{dy}{dx} = -\dfrac{2x\cos y + 3x^2 y}{x^3 - x^2 \sin y - y}\ ; \quad y(0) = 2$

Solution

$$\left(x^3 - x^2 \sin y - y\right)dy = -\left(2x\cos y + 3x^2 y\right)dx$$

$$\left(2x\cos y + 3x^2 y\right)dx + \left(x^3 - x^2 \sin y - y\right)dy = 0$$

$$M = 2x\cos y + 3x^2 y \quad \Rightarrow \quad M_y = -2x\sin y + 3x^2$$

$$N = x^3 - x^2\sin y - y \quad \Rightarrow \quad N_x = 3x^2 - 2x\sin y \qquad \rightarrow \quad M_y = N_x$$

$$\psi = \int \left(2x\cos y + 3x^2 y\right) dx$$

$$= x^2 \cos y + x^3 y + h(y)$$

$$\psi_y = -x^2 \sin y + x^3 + h'(y) = x^3 - x^2 \sin y - y$$

$$\Rightarrow h'(y) = -y \quad \rightarrow \quad h(y) = -\frac{1}{2}y^2$$

$$x^2 \cos y + x^3 y - \frac{1}{2}y^2 = C$$

$$y(0) = 2 \quad \rightarrow \quad \underline{-2 = C}$$

$$\boxed{x^2 \cos y + x^3 y - \frac{1}{2}y^2 = -2}$$

Exercise

Find an integrating factor of the form $x^n y^m$ and solve the equation

$$\left(2y^2 - 6xy\right)dx + \left(3xy - 4x^2\right)dy = 0$$

Solution

$$x^n y^m \left(2y^2 - 6xy\right)dx + x^n y^m \left(3xy - 4x^2\right)dy = 0$$

$$\left(2x^n y^{m+2} - 6x^{n+1}y^{m+1}\right)dx + \left(3x^{n+1}y^{m+1} - 4x^{n+2}y^m\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(2x^n y^{m+2} - 6x^{n+1}y^{m+1}\right) = 2(m+2)x^n y^{m+1} - 6(m+1)x^{n+1}y^m$$

$$N_x = \frac{\partial}{\partial x}\left(3x^{n+1}y^{m+1} - 4x^{n+2}y^m\right) = 3(n+1)x^n y^{m+1} - 4(n+2)x^{n+1}y^m$$

For the equation to be exact, then

$$2(m+2)x^n y^{m+1} - 6(m+1)x^{n+1}y^m = 3(n+1)x^n y^{m+1} - 4(n+2)x^{n+1}y^m$$

$$2(m+2)y - 6(m+1)x = 3(n+1)y - 4(n+2)x$$

$$\begin{cases} 2m+4 = 3n+3 \\ 3m+3 = 2n+4 \end{cases} \rightarrow \begin{cases} 2m-3n = -1 \\ 3m-2n = 1 \end{cases} \Rightarrow \underline{m=1, \quad n=1}$$

$$xy\left(2y^2 - 6xy\right)dx + xy\left(3xy - 4x^2\right)dy = 0$$

$$\left(2xy^3 - 6x^2y^2\right)dx + \left(3x^2y^2 - 4x^3y\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(2xy^3 - 6x^2y^2\right) = 6xy^2 - 12x^2y$$
$$N_x = \frac{\partial}{\partial x}\left(3x^2y^2 - 4x^3y\right) = 6xy^2 - 12x^2y \quad \Rightarrow \quad M_y = N_x$$

$$\psi = \int\left(2xy^3 - 6x^2y^2\right)dx$$
$$= x^2y^3 - 2x^3y^2 + h(y)$$

$$\psi_y = 3x^2y^2 - 4x^3y + h'(y)$$
$$= 3x^2y^2 - 4x^3y$$

$$h'(y) = 0 \;\Rightarrow\; h(y) = C$$

$$\boxed{x^2y^3 - 2x^3y^2 = C}$$

Exercise

Find an integrating factor of the form x^ny^m and solve the equation

$$(12 + 5xy)dx + \left(6xy^{-1} + 3x^2\right)dy = 0$$

Solution

$$x^ny^m(12 + 5xy)dx + x^ny^m\left(6xy^{-1} + 3x^2\right)dy = 0$$

$$\left(12x^ny^m + 5x^{n+1}y^{m+1}\right)dx + \left(6x^{n+1}y^{m-1} + 3x^{n+2}y^m\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(12x^ny^m + 5x^{n+1}y^{m+1}\right) = 12mx^ny^{m-1} + 5(m+1)x^{n+1}y^m$$
$$N_x = \frac{\partial}{\partial x}\left(6x^{n+1}y^{m-1} + 3x^{n+2}y^m\right) = 6(n+1)x^ny^{m-1} + 3(n+2)x^{n+1}y^m$$

For the equation to be exact, then

$$12mx^ny^{m-1} + 5(m+1)x^{n+1}y^m = 6(n+1)x^ny^{m-1} + 3(n+2)x^{n+1}y^m$$
$$12m + 5(m+1)xy = 6(n+1) + 3(n+2)xy$$

$$\begin{cases} 12m = 6n + 6 \\ 5m + 5 = 3n + 6 \end{cases} \rightarrow \begin{cases} 2m - n = 1 \\ 5m - 6n = 1 \end{cases} \Rightarrow \boxed{n = 3, \;\; m = 2}$$

$$x^3y^2(12 + 5xy)dx + x^3y^2\left(6xy^{-1} + 3x^2\right)dy = 0$$

$$\left(12x^3y^2 + 5x^4y^3\right)dx + \left(6x^4y + 3x^5y^2\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(12x^3y^2 + 5x^4y^3\right) = 24x^3y + 15x^4y^2$$
$$N_x = \frac{\partial}{\partial x}\left(6x^4y + 3x^5y^2\right) = 24x^3y + 15x^4y^2 \qquad \Rightarrow \quad M_y = N_x$$

$$\psi = \int\left(12x^3y^2 + 5x^4y^3\right)dx$$
$$= 3x^4y^2 + x^5y^3 + h(y)$$

$$\psi_y = 6x^4y + 3x^5y^2 + h'(y)$$
$$= 6x^4y + 3x^5y^2$$

$$h'(y) = 0 \;\Rightarrow\; h(y) = C$$

$$\boxed{3x^4y^2 + x^5y^3 = C}$$

Exercise

Find an integrating factor of the form $x^n y^m$ and solve the equation

$$\left(3y + 4xy^2\right)dx + \left(2x + 3x^2y\right)dy = 0$$

Solution

$$x^n y^m\left(3y + 4xy^2\right)dx + x^n y^m\left(2x + 3x^2y\right)dy = 0$$

$$\left(3x^n y^{m+1} + 4x^{n+1}y^{m+2}\right)dx + \left(2x^{n+1}y^m + 3x^{n+2}y^{m+1}\right)dy = 0$$

$$M = 3x^n y^{m+1} + 4x^{n+1}y^{m+2} \;; \quad N = 2x^{n+1}y^m + 3x^{n+2}y^{m+1}$$

$$M_y = \frac{\partial}{\partial y}\left(3x^n y^{m+1} + 4x^{n+1}y^{m+2}\right) = 3(m+1)x^n y^m + 4(m+2)x^{n+1}y^{m+1}$$
$$N_x = \frac{\partial}{\partial x}\left(2x^{n+1}y^m + 3x^{n+2}y^{m+1}\right) = 2(n+1)x^n y^m + 3(n+2)x^{n+1}y^{m+1}$$

For the equation to be exact, then

$$3(m+1)x^n y^m + 4(m+2)x^{n+1}y^{m+1} = 2(n+1)x^n y^m + 3(n+2)x^{n+1}y^{m+1}$$

$$3(m+1) + 4(m+2)xy = 2(n+1) + 3(n+2)xy$$

$$\begin{cases} 3m + 3 = 2n + 2 \\ 4m + 8 = 3n + 6 \end{cases} \rightarrow \begin{cases} 3m - 2n = -1 \\ 4m - 3n = -2 \end{cases}$$

$$\underline{m} = \frac{\begin{vmatrix} -1 & -2 \\ -2 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 4 & -3 \end{vmatrix}} = \frac{-1}{-1} \;\underline{=1} \qquad \underline{n} = \frac{\begin{vmatrix} 3 & -1 \\ 4 & -2 \end{vmatrix}}{-1} \;\underline{=2}$$

$$x^2y\left(3y+4xy^2\right)dx+x^2y\left(2x+3x^2y\right)dy=0$$

$$\left(3x^2y^2+4x^3y^3\right)dx+\left(2x^3y+3x^4y^2\right)dy=0$$

$$\left.\begin{aligned}M_y&=\frac{\partial}{\partial y}\left(3x^2y^2+4x^3y^3\right)=6x^2y+12x^3y^2\\N_x&=\frac{\partial}{\partial x}\left(2x^3y+3x^4y^2\right)=6x^2y+12x^3y^2\end{aligned}\right.\quad\Rightarrow\quad M_y=N_x$$

$$\psi=\int\left(3x^2y^2+4x^3y^3\right)dx$$

$$=x^3y^2+x^4y^3+h(y)$$

$$\psi_y=2x^3y+3x^4y^2+h'(y)=2x^3y+3x^4y^2$$

$$h'(y)=0\ \Rightarrow\ h(y)=C$$

$$\boxed{x^3y^2+x^4y^3=C}$$

Exercise

Find the general solution by using either Bernoulli $\quad\frac{dy}{dx}-5y=-\frac{5}{2}xy^3$

Solution

$$y^3\ \Rightarrow\ n=3$$

Let $\quad u=y^{1-3}=y^{-2}=\frac{1}{y^2}\quad\Rightarrow\quad y=\frac{1}{\sqrt{u}}$

$$\frac{du}{dx}=-2y^{-3}\frac{dy}{dx}\quad\Rightarrow\quad-\frac{1}{2}y^3\frac{du}{dx}=\frac{dy}{dx}$$

$$\frac{dy}{dx}-5y=-\frac{5}{2}xy^3$$

$$-\frac{1}{2}u^{-3/2}\frac{du}{dx}-5u^{-1/2}=-\frac{5}{2}xu^{-3/2}\qquad\times\ -2u^{3/2}$$

$$\frac{du}{dx}+10u=5x$$

$$e^{\int 10dx}=e^{10x}$$

$$\int 5xe^{10x}dx=\left(\frac{1}{2}x-\frac{1}{20}\right)e^{10x}$$

		$\int e^{10x}$
+	$5x$	$\frac{1}{10}e^{10x}$
−	5	$\frac{1}{100}e^{10x}$

$$u=\frac{1}{e^{10x}}\left(\left(\frac{1}{2}x-\frac{1}{20}\right)e^{10x}+C\right)$$

$$\boxed{\frac{1}{y^2}=\frac{1}{2}x-\frac{1}{20}+Ce^{-10x}}$$

Exercise

Find the general solution by using either Bernoulli $\frac{dy}{dx}+\frac{y}{x}=x^2y^2$

Solution

$$y^2 \Rightarrow n=2$$

$$\text{Let } \quad u=y^{1-2}=\frac{1}{y} \quad \Rightarrow \quad y=\frac{1}{u}$$

$$\frac{du}{dx}=-\frac{1}{y^2}\frac{dy}{dx} \quad \Rightarrow \quad -\frac{1}{u^2}\frac{du}{dx}=\frac{dy}{dx}$$

$$\frac{dy}{dx}+\frac{y}{x}=x^2y^2$$

$$-\frac{1}{u^2}\frac{du}{dx}+\frac{1}{xu}=x^2\frac{1}{u^2} \qquad \times\ -u^2$$

$$\frac{du}{dx}-\frac{1}{x}u=-x^2$$

$$e^{\int -\frac{1}{x}dx}=e^{-\ln|x|}$$

$$=\frac{1}{x}$$

$$\int x^2\frac{1}{x}dx=\frac{1}{2}x^2$$

$$u=x\left(\frac{1}{2}x^2+C_1\right)$$

$$\frac{1}{y}=\frac{1}{2}x^3+C_1x$$

$$\boxed{y(x)=\frac{2}{x^3+Cx}}$$

Exercise

Find the general solution by using either Bernoulli $\frac{dy}{dx}-y=e^{2x}y^3$

Solution

$$y^3 \Rightarrow n=3$$

$$\text{Let } \quad u=y^{1-3}=y^{-2}=\frac{1}{y^2} \quad \Rightarrow \quad y=\frac{1}{\sqrt{u}}=u^{-1/2}$$

$$\frac{du}{dx}=-2y^{-3}\frac{dy}{dx} \quad \Rightarrow \quad -\frac{1}{2}y^3\frac{du}{dx}=\frac{dy}{dx}$$

$$\frac{dy}{dx}-y=e^{2x}y^3$$

$$-\frac{1}{2}u^{-3/2}\frac{du}{dx}-u^{-1/2}=e^{2x}u^{-3/2} \qquad \times\ -2u^{3/2}$$

$$\frac{du}{dx} + 2u = -2e^{2x}$$

$$e^{\int 2dx} = e^{2x}$$

$$\int -2e^{2x}e^{2x}dx = -\frac{1}{2}e^{4x}$$

$$u = e^{-2x}\left(-\frac{1}{2}e^{4x} + C\right)$$

$$\boxed{y^{-2} = -\frac{1}{2}e^{2x} + Ce^{-2x}} \qquad y = \pm\sqrt{\frac{-2}{e^{2x} + Ce^{-2x}}}$$

Exercise

Find the general solution by using either Bernoulli $\frac{dy}{dx} + \frac{y}{x-2} = 5(x-2)y^{1/2}$

Solution

$$y^{1/2} \Rightarrow n = \frac{1}{2}$$

Let $u = y^{1-\frac{1}{2}} = y^{1/2} \Rightarrow y = u^2 \qquad \to \frac{dy}{dx} = 2u\frac{du}{dx}$

$$\frac{dy}{dx} + \frac{1}{x-2}y = 5(x-2)y^{1/2}$$

$$2u\frac{du}{dx} + \frac{1}{x-2}u^2 = 5(x-2)u \qquad \times \frac{1}{2u}$$

$$\frac{du}{dx} + \frac{1}{2}\frac{1}{x-2}u = \frac{5}{2}(x-2)$$

$$e^{\frac{1}{2}\int \frac{1}{x-2}dx} = e^{\frac{1}{2}\ln|x-2|}$$

$$= \sqrt{x-2}$$

$$\int \frac{1}{2}(x-2)\sqrt{x-2}\ dx = \int \frac{1}{2}(x-2)^{3/2}\ dx$$

$$= \frac{1}{5}(x-2)^{5/2}$$

$$u = \frac{1}{\sqrt{x-2}}\left(\frac{1}{5}(x-2)^{5/2} + C\right)$$

$$y^{1/2} = \frac{1}{5}(x-2)^{3/2} + \frac{C}{\sqrt{x-2}}$$

$$\boxed{y(x) = \left(\frac{1}{5}(x-2)^{3/2} + \frac{C}{\sqrt{x-2}}\right)^2}$$

Exercise

Find the general solution by using either Bernoulli $\frac{dy}{dx}+y=e^{x}y^{-2}$

Solution

$y^{-2} \Rightarrow n=-2$

Let $u=y^{1+2}=y^{3} \Rightarrow y=u^{1/3}$

$\rightarrow \frac{dy}{dx}=\frac{1}{3}u^{-2/3}\frac{du}{dx}$

$\frac{dy}{dx}+y=e^{x}y^{-2}$

$\frac{1}{3}u^{-2/3}\frac{du}{dx}+u^{1/3}=e^{x}u^{-2/3}$ $\times\ 3u^{2/3}$

$\frac{du}{dx}+3u=3e^{x}$

$e^{\int 3dx}=e^{3x}$

$\int 3e^{x}e^{3x}\,dx=\frac{3}{4}e^{4x}$

$u=e^{-3x}\left(\frac{3}{4}e^{4x}+C\right)$

$\boxed{y^{3}=\frac{3}{4}e^{x}+Ce^{-3x}}$

Exercise

Find the general solution by using either Bernoulli $\frac{dy}{dx}+y^{3}x+y=0$

Solution

$\frac{dy}{dx}+y=-xy^{3}$

$y^{3} \Rightarrow n=3$

Let $u=y^{1-3}=y^{-2}=\frac{1}{y^{2}} \Rightarrow y=\frac{1}{\sqrt{u}}=u^{-1/2}$

$\frac{dy}{dx}=-\frac{1}{2}u^{-3/2}\frac{du}{dx}$

$-\frac{1}{2}u^{-3/2}\frac{du}{dx}+u^{-1/2}=-xu^{-3/2}$ $\times\ -2u^{3/2}$

$\frac{du}{dx}-2u=2x$

$e^{\int -2dx}=e^{-2x}$

		$\int e^{-2x}$
+	$2x$	$-\frac{1}{2}e^{-2x}$
−	2	$\frac{1}{4}e^{-2x}$

$\int 2xe^{-2x}dx=\left(-x-\frac{1}{2}\right)e^{-2x}$

$$u = e^{2x}\left(\left(-x-\frac{1}{2}\right)e^{-2x}+C\right)$$

$$\boxed{\frac{1}{y^2} = -x-\frac{1}{2}+Ce^{2x}}$$

$$y(x) = \pm\frac{1}{\sqrt{Ce^{2x}-x-\frac{1}{2}}}$$

Exercise

Find the general solution by using homogeneous equations. $\left(xy+y^2\right)dx - x^2dy = 0$

Solution

$$\frac{dy}{dx} = \frac{xy+y^2}{x^2} \quad = \frac{y}{x}+\left(\frac{y}{x}\right)^2$$

Let $v = \frac{y}{x} \to y = xv \quad \Rightarrow \quad y' = v + xv'$

$$v + xv' = v + v^2$$

$$x\frac{dv}{dx} = v^2$$

$$\int \frac{dv}{v^2} = \int \frac{dx}{x}$$

$$-\frac{1}{v} = \ln|x| + C$$

$$-\frac{x}{y} = \ln|x| + C$$

$$\boxed{y(x) = -\frac{x}{\ln|x|+C}}$$

Exercise

Find the general solution by using homogeneous equations. $\left(x^2+y^2\right)dx + 2xydy = 0$

Solution

$$\frac{dy}{dx} = -\frac{x^2+y^2}{2xy}$$

$$= -\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right)$$

Let $v = \frac{y}{x} \to y = xv \quad \Rightarrow \quad y' = v + xv'$

$$v + xv' = -\frac{1}{2}v - \frac{1}{2v}$$

$$xv' = -\frac{3}{2}v - \frac{1}{2v}$$

$$x\frac{dv}{dx} = -\left(\frac{3v^2+1}{2v}\right)$$

$$\int \frac{2v}{3v^2+1}dv = -\int \frac{1}{x}dx$$

$$\frac{1}{3}\ln\left(3v^2+1\right) = -\ln|x| + \ln C_1$$

$$\ln\left(3v^2+1\right) = -3\ln\left|C_1 x\right|$$

$$3v^2+1 = \frac{C}{|x|^3}$$

$$3\frac{y^2}{x^2} = \frac{C}{|x|^3} - 1$$

$$3xy^2 = C - x^3$$

$$\boxed{3xy^2 + x^3 = C}$$

Exercise

Find the general solution by using homogeneous equations. $\left(y^2 - xy\right)dx + x^2 dy = 0$

Solution

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2}$$

$$= \frac{y}{x} - \left(\frac{y}{x}\right)^2$$

Let $v = \frac{y}{x} \to y = xv \quad \Rightarrow \quad y' = v + xv'$

$$v + xv' = v - v^2$$

$$x\frac{dv}{dx} = -v^2$$

$$-\int \frac{1}{v^2}dv = \int \frac{1}{x}dx$$

$$\frac{1}{v} = \ln|x| + C$$

$$\frac{x}{y} = \ln|x| + C$$

$$\boxed{y(x) = \frac{x}{\ln|x| + C}}$$

Exercise

Find the general solution by using homogeneous equations. $\frac{dy}{d\theta} = \frac{1}{\theta}\left(\theta \sec\left(\frac{y}{\theta}\right) + y\right)$

Solution

$$\frac{dy}{d\theta} = \sec\left(\frac{y}{\theta}\right) + \frac{y}{\theta}$$

Let $v = \frac{y}{\theta} \to y = \theta v \quad \Rightarrow \quad y' = v + \theta v'$

$$v + \theta v' = \sec v + v$$

$$\int \cos v \, dv = \int \frac{d\theta}{\theta}$$

$$\sin v = \ln|\theta| + C$$

$$\frac{y}{\theta} = \arcsin\left(\ln|\theta| + C\right)$$

$$\boxed{y(\theta) = \theta \arcsin\left(\ln|\theta| + C\right)}$$

Exercise

Find the general solution by using homogeneous equations. $\frac{dy}{dx} = \frac{y(\ln y - \ln x + 1)}{x}$

Solution

$$\frac{dy}{dx} = \frac{y}{x}\left(\ln\frac{y}{x} + 1\right)$$

Let $v = \frac{y}{x} \to y = xv \quad \Rightarrow \quad y' = v + xv'$

$$v + xv' = v\ln|v| + v$$

$$x\frac{dv}{dx} = v\ln|v|$$

$$\frac{dv}{v\ln|v|} = \frac{dx}{x}$$

$$\int \frac{1}{\ln|v|}\, d(\ln v) = \int \frac{dx}{x}$$

$$\ln|\ln v| = \ln|x| + \ln C$$

$$\ln|\ln v| = \ln|Cx|$$

$$\ln v = Cx$$

$$v = e^{Cx} = \frac{y}{x}$$

$$\boxed{y(x) = xe^{Cx}}$$

Exercise

Find the general solution by using Equation with Linear Coefficients

$$(-3x+y-1)dx+(x+y+3)dy=0$$

Solution

$$\begin{cases} a_1b_2=(-3)(1)=-3 \\ a_2b_1=(1)(1)=1 \end{cases} \rightarrow \quad a_1b_2 \neq a_2b_1$$

$$x=u+h \quad y=v+k$$

$$\begin{cases} -3h+k-1=0 \\ h+k+3=0 \end{cases} \rightarrow \begin{cases} -3h+k=1 \\ h+k=-3 \end{cases} \rightarrow \quad \underline{h=-1,\ k=-2|}$$

$$\begin{cases} x=u+h=u-1 \\ y=v+k=v-2 \end{cases}$$

$$(-3u+3+v-2-1)du+(u-1+v-2+3)dv=0$$

$$(-3u+v)du+(u+v)dv=0 \quad \rightarrow \quad \frac{dv}{du}=\frac{3-\frac{v}{u}}{1+\frac{v}{u}}$$

Let $w=\frac{v}{u} \quad \rightarrow v=uw \quad \rightarrow \frac{dv}{du}=w+u\frac{dw}{du}$

$$w+u\frac{dw}{du}=\frac{3-w}{1+w}$$

$$u\frac{dw}{du}=\frac{3-w}{1+w}-w$$

$$u\frac{dw}{du}=\frac{3-2w-w^2}{1+w}$$

$$\frac{w+1}{w^2+2w-3}dw=-\frac{du}{u}$$

$$\frac{1}{2}\int\frac{1}{w^2+2w-3}d\left(w^2+2w-3\right)=-\int\frac{du}{u}$$

$$\frac{1}{2}\ln\left|w^2+2w-3\right|=-\ln u+\ln C_1$$

$$\ln\sqrt{w^2+2w-3}=\ln C_1\frac{1}{u}$$

$$\sqrt{w^2+2w-3}=C_1\frac{1}{u}$$

$$w^2+2w-3=C\frac{1}{u^2}$$

$$\frac{v^2}{u^2}+2\left(\frac{v}{u}\right)-3=C\frac{1}{u^2}$$

$$v^2+2uv-3u^2=C \qquad x=u-1 \quad y=v-2$$

$$\underline{(y+2)^2+2(x+1)(y+2)-3(x+1)^2=C\,|}$$

Exercise

Find the general solution by using Equation with Linear Coefficients

$$(x+y-1)dx+(y-x-5)dy=0$$

Solution

$$\begin{cases} a_1b_2=(1)(-1)=-1 \\ a_2b_1=(1)(1)=1 \end{cases} \quad \rightarrow \quad a_1b_2 \neq a_2b_1$$

$x=u+h \quad y=v+k$

$$\begin{cases} h+k-1=0 \\ -h+k-5=0 \end{cases} \rightarrow \begin{cases} h+k=1 \\ -h+k=5 \end{cases} \quad \rightarrow \quad \underline{h=-2,\ k=3}$$

$$\begin{cases} x=u+h=u-2 \\ y=v+k=v+3 \end{cases}$$

$$(u-2+v+3-1)du+(v+3-u+2-5)dv=0$$

$$(u+v)du+(v-u)dv=0 \quad \rightarrow \quad \frac{dv}{du}=\frac{u+v}{u-v}=\frac{1+\frac{v}{u}}{1-\frac{v}{u}}$$

Let $w=\frac{v}{u} \quad \rightarrow \quad v=uw \quad \rightarrow \frac{dv}{du}=w+u\frac{dw}{du}$

$$w+u\frac{dw}{du}=\frac{1+w}{1-w}$$

$$u\frac{dw}{du}=\frac{1+w}{1-w}-w$$

$$u\frac{dw}{du}=\frac{1+w^2}{1-w}$$

$$\frac{1-w}{w^2+1}dw=\frac{du}{u}$$

$$\int\frac{1}{w^2+1}dw-\int\frac{w}{w^2+1}dw=\int\frac{du}{u}$$

$$\arctan\frac{v}{u}-\frac{1}{2}\ln\left(\frac{v^2}{u^2}+1\right)=\ln u+C_1$$

$$2\arctan\frac{v}{u}-\ln\left(\frac{v^2+u^2}{u^2}\right)=2\ln u+2C_1$$

$$2\arctan\frac{v}{u}-\ln\left(\frac{v^2+u^2}{u^2}\right)-\ln u^2=C$$

$$2\arctan\frac{v}{u}-\ln\left(v^2+u^2\right)=C \qquad u=x+2 \quad v=y-3$$

$$\underline{2\arctan\frac{y-3}{x+2}-\ln\left((x+2)^2+(y-3)^2\right)=C}$$

Exercise

Find the general solution by using Equation with Linear Coefficients

$$(2x+y+4)dx+(x-2y-2)dy=0$$

Solution

$$\begin{cases} a_1b_2=(2)(-2)=-4 \\ a_2b_1=(1)(1)=1 \end{cases} \rightarrow \quad a_1b_2 \neq a_2b_1$$

$$x=u+h \quad y=v+k$$

$$\begin{cases} 2h+k+4=0 \\ h-2k-2=0 \end{cases} \rightarrow \begin{cases} 2h+k=-4 \\ h-2k=2 \end{cases} \rightarrow \quad \boxed{h=-\frac{6}{5},\ k=-\frac{8}{5}}$$

$$\begin{cases} x=u+h=u-\frac{6}{5} \\ y=v+k=v-\frac{8}{5} \end{cases}$$

$$\left(2u-\frac{12}{5}+v-\frac{8}{5}+4\right)du+\left(u-\frac{6}{5}-2v+\frac{16}{5}-2\right)dv=0$$

$$(2u+v)du+(u-2v)dv=0 \quad \rightarrow \quad \frac{dv}{du}=\frac{2u+v}{2v-u}=\frac{2+\frac{v}{u}}{2\frac{v}{u}-1}$$

Let $w=\frac{v}{u} \quad \rightarrow \ v=uw \quad \rightarrow \frac{dv}{du}=w+u\frac{dw}{du}$

$$w+u\frac{dw}{du}=\frac{2+w}{2w-1}$$

$$u\frac{dw}{du}=\frac{2+w}{2w-1}-w$$

$$u\frac{dw}{du}=\frac{2+2w-2w^2}{2w-1}$$

$$\frac{2w-1}{2+2w-2w^2}dw=\frac{du}{u}$$

$$-\int \frac{1}{w^2-w-1}d\left(w^2-w-1\right)=2\int \frac{du}{u}$$

$$-\ln\left|w^2-w-1\right|=2\ln|u|+C_1$$

$$\ln\left|w^2-w-1\right|+\ln u^2=C_1$$

$$\ln\left|u^2\left(\frac{v^2}{u^2}-\frac{v}{u}-1\right)\right|=\ln C$$

$$v^2-uv-u^2=C \qquad u=x+\frac{6}{5} \quad v=y+\frac{8}{5}$$

$$\boxed{\left(x+\frac{8}{5}\right)^2-\left(x+\frac{8}{5}\right)\left(x+\frac{6}{5}\right)-\left(x+\frac{6}{5}\right)^2=C} \qquad (5x+8)^2-(5x+8)(5x+6)-(5x+6)^2=C_2$$

Exercise

Find the general solution by using Equation with Linear Coefficients

$$(2x-y)dx+(4x+y-3)dy=0$$

Solution

$$\begin{cases} a_1b_2 = (2)(1) = 2 \\ a_2b_1 = (-1)(4) = -4 \end{cases} \rightarrow \quad a_1b_2 \neq a_2b_1$$

$$x = u + h \quad y = v + k$$

$$\begin{cases} 2h - k = 0 \\ 4h + k = 3 \end{cases} \rightarrow \quad \underline{h = \tfrac{1}{2},\ k = 1}|$$

$$\begin{cases} x = u + h = u + \frac{1}{2} \\ y = v + k = v + 1 \end{cases}$$

$$(2u+1-v-1)du + (4u+2+v+1-3)dv = 0$$

$$(2u-v)du + (4u+v)dv = 0 \quad \rightarrow \quad \frac{dv}{du} = \frac{v-2u}{4u+v} = \frac{\frac{v}{u}-2}{4+\frac{v}{u}}$$

Let $w = \frac{v}{u} \quad \rightarrow \ v = uw \quad \rightarrow \frac{dv}{du} = w + u\frac{dw}{du}$

$$w + u\frac{dw}{du} = \frac{w-2}{4+w}$$

$$u\frac{dw}{du} = \frac{w-2}{4+w} - w$$

$$u\frac{dw}{du} = \frac{-w^2-3w-2}{4+w}$$

$$\frac{w+4}{w^2+3w+2}dw = -\frac{du}{u}$$

$$\frac{w+4}{w^2+3w+2} = \frac{A}{w+1} + \frac{B}{w+2} \quad \rightarrow (A+B)w + 2A + B = w + 4$$

$$\begin{cases} A + B = 1 \\ 2A + B = 4 \end{cases} \rightarrow \quad A = 3, \ \ B = -2$$

$$\int \frac{3}{w+1}dw - \int \frac{2}{w+2}dw = -\int \frac{du}{u}$$

$$3\ln|w+1| - 2\ln|w+2| = -\ln|u| + \ln C$$

$$\ln|w+1|^3 - \ln(w+2)^2 + \ln|u| = \ln C$$

$$\ln\frac{u(w+1)^3}{(w+2)^2} = \ln C$$

$$\frac{u(w+1)^3}{(w+2)^2} = C$$

$$u\left(\frac{v}{u}+1\right)^3 = C\left(\frac{v}{u}+2\right)^2 \qquad u = x - \frac{1}{2} \quad v = y - 1$$

$$\frac{1}{u^2}(v+u)^3 = C\frac{1}{u^2}(v+2u)^2$$

$$\left(x - \frac{1}{2} + y - 1\right)^3 = C(y - 1 + 2x - 1)^2$$

$$\left(x - \frac{3}{2} + y\right)^3 = C(y + 2x - 2)^2$$

$$\frac{1}{8}(2x - 3 + 2y)^3 = C(y + 2x - 2)^2 \qquad C_1 = 8C$$

$$\boxed{(2x - 3 + 2y)^3 = C_1(y + 2x - 2)^2}$$

Exercise

Prove that $Mdx + Ndy = 0$ has an integrating factor that depends only on the sum $x + y$ if and only if the expression

$$\frac{N_x - M_y}{M - N} \text{ depends only on } x + y$$

Use the prove to solve the equation $(3 + y + xy)dx + (3 + x + xy)dy = 0$

Solution

An equation $Mdx + Ndy = 0$ has an integrating factor $\mu(x+y)$ iff $\mu(x+y)Mdx + \mu(x+y)Ndy = 0$

For the equation to be exact, then

$$\frac{\partial}{\partial y}\left[\mu(x+y)M(x,\, y)\right] = \frac{\partial}{\partial x}\left[\mu(x+y)N(x,\, y)\right]$$

$$\mu'(x+y)M + \mu(x+y)M_y = \mu'(x+y)N + \mu(x+y)N_x$$

$$\mu'(x+y)(M - N) = \mu(x+y)\left(N_x - M_y\right)$$

$$\frac{\mu'(x+y)}{\mu(x+y)} = \frac{N_x - M_y}{M - N}$$

$$\int \frac{\mu'(x+y)}{\mu(x+y)} = \int \frac{N_x - M_y}{M - N} d(x+y)$$

$$\ln\left|\mu(x+y)\right| = \int \frac{N_x - M_y}{M - N} d(x+y)$$

$$\mu(x+y) = \pm e^{\int \frac{N_x - M_y}{M - N} d(x+y)}$$

$$(3+y+xy)dx+(3+x+xy)dy=0$$

$$M_y=\frac{\partial}{\partial y}(3+y+xy)=1+x$$

$$N_x=\frac{\partial}{\partial x}(3+x+xy)=1+y$$

$$\frac{N_x-M_y}{M-N}=\frac{1+y-(1+x)}{3+y+xy-(3+x+xy)}$$

$$=\frac{y-x}{y-x}$$

$$\underline{=1}$$

$$\mu(x+y)=e^{\int d(x+y)}$$

$$=e^{x+y}$$

$$e^{x+y}(3+y+xy)dx+e^{x+y}(3+x+xy)dy=0$$

$$M_y=\frac{\partial}{\partial y}(3+y+xy)e^{x+y}=(1+x)e^{x+y}+(3+y+xy)e^{x+y}=(4+x+y+xy)e^{x+y}$$

$$N_x=\frac{\partial}{\partial x}(3+x+xy)e^{x+y}=(1+y)e^{x+y}+(3+x+xy)e^{x+y}=(4+x+y+xy)e^{x+y}$$

$$\Rightarrow\ M_y=N_x$$

$$\psi=\int(3+y+xy)e^{x+y}dx$$

$$=(3+y+y(x-1))e^{x+y}+h(y)$$

$$=(3+xy)e^{x+y}+h(y)$$

$$\psi_y=(3+xy+x)e^{x+y}+h'(y)=(3+xy+x)e^{x+y}$$

$$h'(y)=0\ \Rightarrow\ h(y)=C$$

$$\boxed{(3+xy)e^{x+y}=C}$$

Exercise

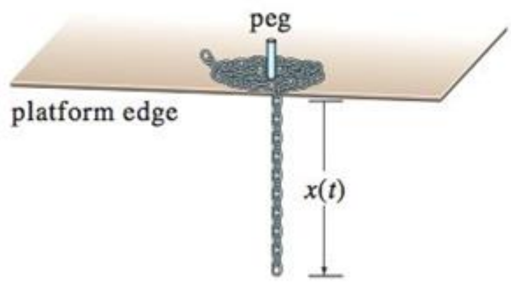
A portion of a uniform chain of length 8 *feet* is loosely coiled around a peg at the edge of a high horizontal platform, and the remaining portion of the chain hangs at rest over the edge of the platform.

Suppose the length of the overhanging chain is 3 *feet*, that the chain weighs 2 2 *lb/ft* , and that the positive direction is downward. Starting at $t=0$ seconds, the weight of the overhanging portion causes the chain on the table to uncoil smoothly and to fall to the floor. If $x(t)$ denotes the length of the chain overhanging the table at time $t > 0$, then $v = \frac{dx}{dt}$ is its velocity. When all resistive forces are ignored, it can be shown that a mathematical model relating v to x is given by

$$xv\frac{dv}{dx} + v^2 = 32x$$

a) Rewrite this model in differential form and solve the *DE* for v in terms of x by finding am appropriate integrating factor. Find an explicit solution $v(x)$.

b) Determine the velocity with which the chain leaves the platform.



Solution

a) $\left(v^2 - 32x\right)dx + xvdv = 0$

$M(x,\ v) = v^2 - 32x \quad N(x,\ v) = xv$

$\frac{\partial M}{\partial v} = 2v \quad \frac{\partial N}{\partial x} = v \quad \Rightarrow \frac{\partial M}{\partial v} \neq \frac{\partial N}{\partial x}$

$\frac{M_v - N_x}{N} = \frac{2v - v}{xv} = \frac{1}{x}$

$\frac{d\mu}{dx} = \frac{\mu}{x}$

$\int \frac{d\mu}{\mu} = \int \frac{dx}{x}$

$\ln \mu = \ln x$

$\rightarrow \boxed{\mu(x) = x}$

$x\left(v^2 - 32x\right)dx + x^2vdv = 0$

$$\begin{cases} M_v = \frac{\partial}{\partial v}\left(v^2x - 32x^2\right) = 2vx \\ N_x = \frac{\partial}{\partial x}\left(vx^2\right) = 2vx \end{cases} \quad \Rightarrow M_y = N_x$$

$\psi = \int \left(v^2x - 32x^2\right)dx$

$\quad = \frac{1}{2}v^2x^2 - \frac{32}{3}x^3 + h(v)$

$\psi_v = vx^2 + h'(v) = vx^2 \quad \Rightarrow \quad h'(v) = 0 \qquad \Rightarrow h(y) = C$

$\frac{1}{2}v^2x^2 - \frac{32}{3}x^3 = C$

Given: $v = 0 \ \ x = 3 \quad \Rightarrow \boxed{-288 = C}$

$\frac{1}{2}v^2x^2 - \frac{32}{3}x^3 = -288$

$3v^2x^2 - 64x^3 = -1728$

$3v^2x^2 = 64\left(x^3 - 27\right)$

$$v^2 = 64\frac{x^3-27}{3x^2}$$

$$\boxed{v(x) = \frac{8}{x}\sqrt{\frac{x^3}{3}-9}}$$

b) The chain leaves the platform when $x=8$

$$v(8) = \sqrt{\frac{512}{3}-9} = \boxed{\sqrt{\frac{485}{3}}\ ft/s} \quad \boxed{\approx 12.7\ ft/s}$$