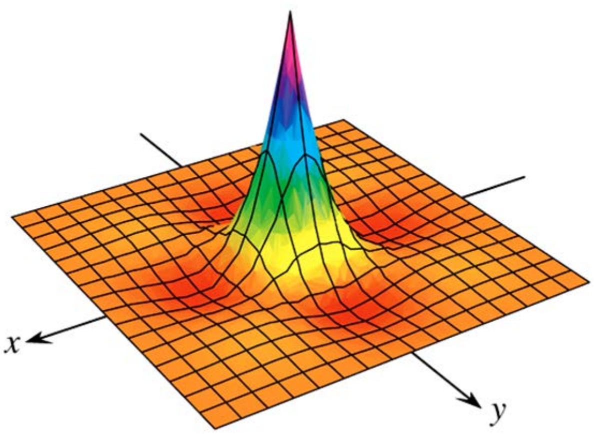
Section 2.7 – Maximum/Minimum Problems

Derivative Tests for Local Extreme Values

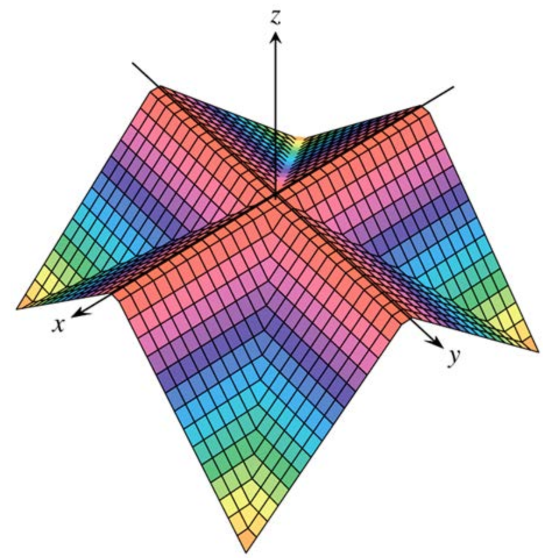
Definition

Let $f(x,y)$ be defined on a region R containing the point $(a,\ b)$. Then

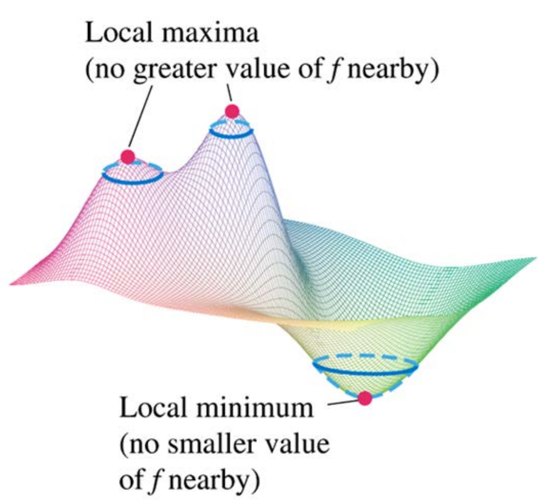
- $f(a,\ b)$ is a local maximum value of *f* if $f(a,\ b) \ge f(x,\ y)$ for all domain points $(x,\ y)$ in an open disk centered at $(a,\ b)$.
- $f(a,\ b)$ is a local minimum value of *f* if $f(a,\ b) \le f(x,\ y)$ for all domain points $(x,\ y)$ in an open disk centered at $(a,\ b)$.



$z = (\cos x)(\cos y)e^{-\sqrt{x^2+y^2}}$

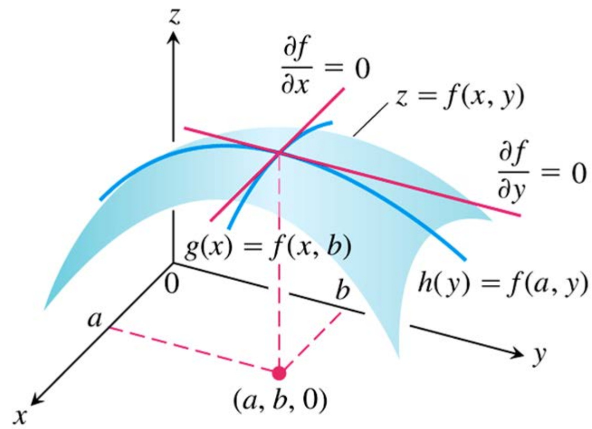


$z = \frac{1}{2}\left(\left||x|-|y|\right|-|x|-|y|\right)$



Theorem – **First derivative Test for Local Extreme Values**

If $f(x,y)$ has a local maximum or minimum value at an interior point $(a,\ b)$ of its domain and if the first partial derivatives exist there, then $f_x(a,\ b)=0$ *and* $f_y(a,\ b)=0$

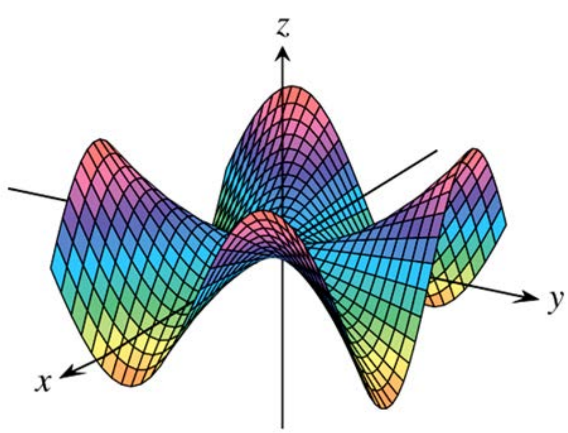


Definition

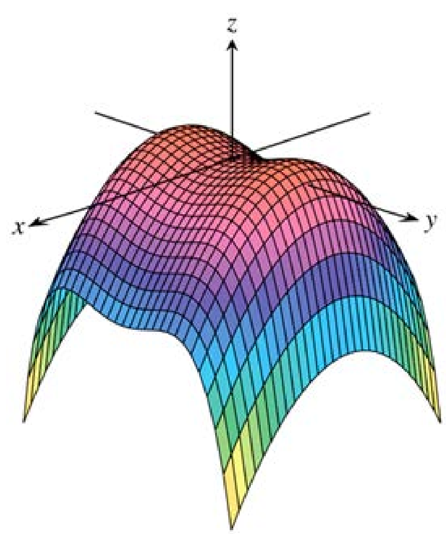
An interior point of the domain $f(x,y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a ***critical point*** of *f*.

Definition

A differentiable function $f(x,y)$ has a ***saddle point*** at a critical point $(a,\ b)$ if in every open disk centered at $(a,\ b)$ there are domain points (x,y) where $f(x,\ y)>f(a,\ b)$ and domain points (x,y) where $f(x,\ y)<f(a,\ b)$. The corresponding point $(a,\ b,\ f(a,\ b))$ on the surface $z=f(x,\ y)$ is called a saddle point of the surface.



$$z=\frac{xy\left(x^2-y^2\right)}{x^2+y^2}$$



$z=y^2-y^4-x^2$

Example

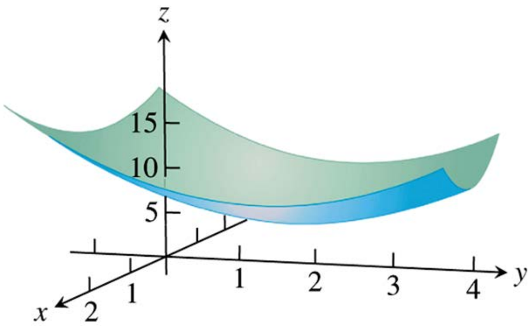
Find the local extreme values of $f(x,y)=x^2+y^2-4y+9$

Solution

The domain of f is the entire plane. The local extreme values occur:

$f_x=2x=0 \qquad f_y=2y-4=0$

Therefore, the critical point is $(0,\ 2)$ and the value $f(0,\ 2)=0+2^2-8+9=5$.



The critical point is a local minimum.

Example

Find the local extreme values of $f(x,y)=y^2-x^2$

Solution

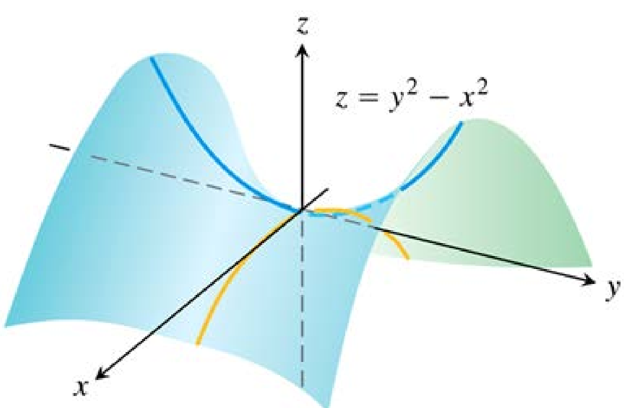
The domain of f is the entire plane.

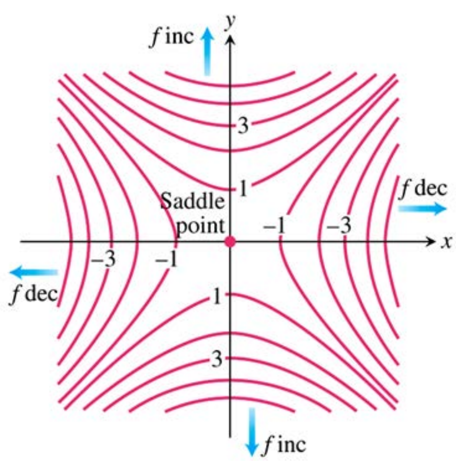
$f_x=-2x=0 \qquad f_y=2y=0$

Therefore, the local extreme is the origin $(0,\ 0)$ and the value $f(0,\ 0)=0$.

$f(0,y)=y^2\geq 0 \qquad f(x,0)=-x^2\leq 0$

The function has a saddle point at the origin and no local extreme values.





Theorem – Second Derivative Test for Local Extreme Values

Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

- f has a ***local maximum*** at (a, b) if $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a, b).
- f has a ***local minimum*** at (a, b) if $f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a, b).
- f has a ***saddle point*** at (a, b) if $f_{xx} f_{yy} - f_{xy}^2 < 0$ at (a, b).
- ***The test is inconclusive*** at (a, b) if $f_{xx} f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of *f* at (a, b).

Example

Find the local extreme values of $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

Solution

$f_x = y - 2x - 2 = 0 \qquad f_y = x - 2y - 2 = 0$

$$\begin{cases} -2x + y = 2 \\ x - 2y = 2 \end{cases} \rightarrow \boxed{x = y = -2}$$

Therefore, the critical point is $(-2, -2)$

$f_{xx} = -2 \quad f_{yy} = -2 \quad f_{xy} = 1$

$$f_{xx} f_{yy} - f_{xy}^2 = (-2)(-2) - 1^2 = 3 > 0$$

$$f_{xx} = -2 < 0$$

The function *f* has a local maximum at $(-2, -2)$ and the value is

$$f(-2, -2) = (-2)(-2) - (-2)^2 - (-2)^2 - 2(-2) - 2(-2) + 4$$

$$= 8$$

Example

Find the local extreme values of $f(x,y)=3y^2-2y^3-3x^2+6xy$

Solution

$$f_x=-6x+6y=0 \quad \textit{and} \quad f_y=6y-6y^2+6x=0$$

$$\begin{cases} -6x+6y=0 \\ 6y-6y^2+6x=0 \end{cases} \rightarrow \begin{array}{l} x=y \\ \qquad\qquad 6y-6y^2+6y=-6y(y-2)=0 \end{array}$$

$$\begin{cases} y=0=x & \boxed{(0,\ 0)} \\ y=2=x & \boxed{(2,\ 2)} \end{cases} \quad \text{are the critical points}$$

$$f_{xx}=-6 \quad f_y=6-12y \quad f_{xy}=6$$

$$\begin{aligned} f_{xx}f_{yy}-f_{xy}^2 &= (-6)(6-12y)-6^2 \\ &= -36+72y-36 \\ &= 72(y-1) \end{aligned}$$

At $(0,\ 0)$

$$f_{xx}f_{yy}-f_{xy}^2=-72<0$$

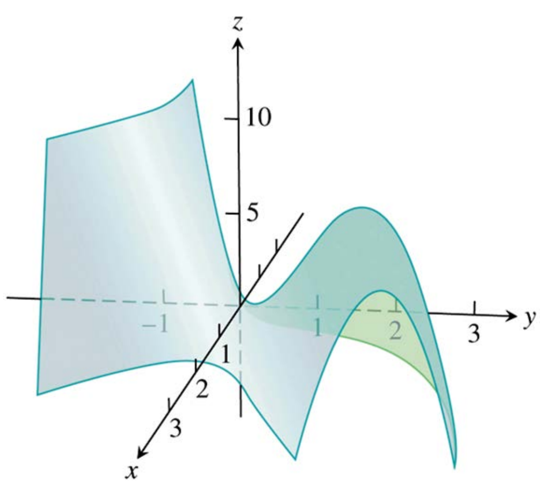
So, the function has a saddle point at the origin.

At $(2,\ 2)$

$$f_{xx}f_{yy}-f_{xy}^2=72>0 \ \text{ and } \ f_{xx}=-6<0$$

So, the function has a local maximum at $(2,\ 2)$ with a value of

$$\begin{aligned} f(2,\ 2) &= 12-16-12+24 \\ &= 8 \end{aligned}$$



Absolute Maxima and Minima on Closed Bounded Regions

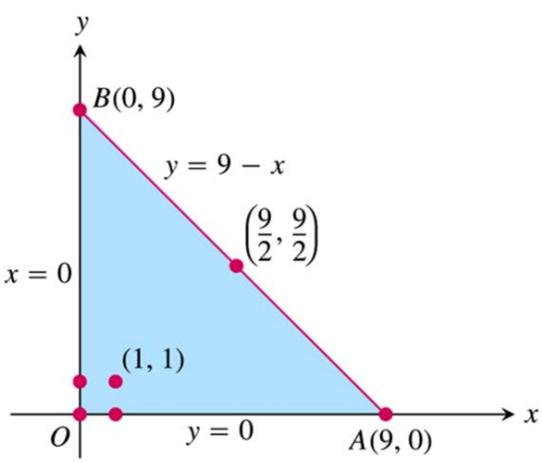
The absolute extrema of a continuous function $f(x,y)$ on a closed and bounded region *R* into three steps

1. *List the interior points of R* where *f* may have local maxima and minima and evaluate *f* at these points. These are the critical points of *f*.
2. *List the boundary points* of *R* where *f* may have local maxima and minima and evaluate *f* at these points.
3. *Look through the lists* for the maximum and minimum values of *f*. These will be the absolute maximum and minimum values of *f* on *R*. Since absolute maxima and minima are also local maxima and minima, the absolute maximum and minimum values of appear somewhere in the lists made in Steps 1 and 2

Example

Find the absolute maximum and minimum values of $f(x,\ y)=2+2x+2y-x^2-y^2$ on the triangular region in the first quadrant bounded by the lines $x=0,\quad y=0,\quad y=9-x$

Solution



$f_x=2-2x=0 \quad f_y=2-2y=0$

$x=1 \quad y=1$

The critical point is $(1,\ 1)$. The value of *f* is

$f(1,\ 1)=2+2+2-1-1\underline{=4|}$

Boundary points:

i. On the segment *OA*, *y* = 0. The function

$f(x,\ 0)=2+2x-x^2$

This function is defined on the closed interval $0\le x\le 9$.

$$\begin{cases} x=0 & \to f(0,\ 0)=\underline{2|} \\ x=9 & \to f(9,\ 0)=2+18-81=\underline{-61|} \end{cases}$$

At the interior points where $f_x = 0$. The only point is $x=1$ where $f(1,\ 0)=3$

ii. On the segment *OB*, *x* = 0. The function

$$f(0,\ y)=2+2y-y^2$$

$$f(0,\ 0)=2, \quad f(0,\ 9)=-61, \quad f(1,\ 0)=3$$

iii. Left the interior points of the segment *AB*. With $y=9-x$, then

$$\begin{aligned} f(x,\ y) &= 2+2x+2(9-x)-x^2-(9-x)^2 \\ &= 2+2x+18-2x-x^2-81+18x-x^2 \\ &= -2x^2+18x-61 \end{aligned}$$

$$f'(x, 9-x)=-4x+18=0 \quad \Rightarrow \quad \underline{x=\frac{9}{2}\Big|}$$

At $x=\frac{9}{2} \quad \Rightarrow \quad y=9-x=\frac{9}{2}$

$$\begin{aligned} f\left(\frac{9}{2},\ \frac{9}{2}\right) &= 2+2\left(\frac{9}{2}\right)+2\left(9-\frac{9}{2}\right)-\left(\frac{9}{2}\right)^2-\left(9-\frac{9}{2}\right)^2 \\ &= \underline{-\frac{41}{2}\Big|} \end{aligned}$$

$\therefore\ 4,\ 2,\ -61,\ 3,\ -\frac{41}{2}.$

The maximum is 4, which *f* assumes at (1, 1). The minimum is –61, which *f* assumes at (0, 9) and (9, 0).

Example

A delivery company accepts only rectangular boxes the sum of whose length and girth (perimeter of a cross-section) does not exceed 108 *in*. Find the dimensions of an acceptable box of largest volume.

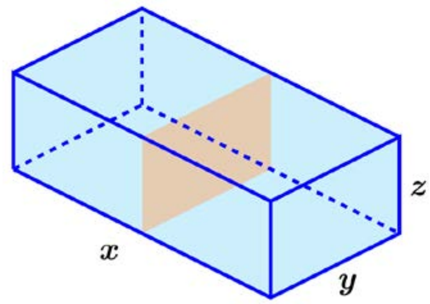
Solution

Let *x*, *y*, and *z* represent the length, width, and height.

The girth is: $= 2y + 2z\left(= P\right)$

Volume: $V = xyz$

We want to maximize the volume of the box satisfying:



$$x + 2y + 2z = 108$$
$$x = 108 - 2y - 2z$$

$$V\left(y,z\right) = \left(108 - 2y - 2z\right)yz$$
$$= 108yz - 2y^2z - 2yz^2$$

$$V_y\left(y,z\right) = 108z - 4yz - 2z^2$$
$$= 2z\left(54 - 2y - z\right) = 0$$

$$V_z\left(y,z\right) = 108y - 2y^2 - 4yz$$
$$= 2y\left(54 - y - 2z\right) = 0$$

$$\begin{cases} 2z\left(54 - 2y - z\right) = 0 & \rightarrow \boxed{z = 0} \quad 54 - 2y - z = 0 \\ 2y\left(54 - y - 2z\right) = 0 & \rightarrow \boxed{y = 0} \quad 54 - y - 2z = 0 \end{cases}$$

$$\begin{cases} 2y + z = 54 \\ y + 2z = 54 \end{cases} \rightarrow \boxed{y = z = 18}$$

$$\begin{cases} if \ \ y = 0 \quad 54 - 2y - z = 0 \quad \Rightarrow z = 54 \rightarrow \boxed{(0,\ 54)} \\ if \ \ z = 0 \quad 54 - y - 2z = 0 \quad \Rightarrow y = 54 \rightarrow \boxed{(54,\ 0)} \end{cases}$$

$\therefore$ The critical points are: $(0,\ 0),\ (0,\ 54),\ (54,\ 0),\ (18,\ 18)$

At (0, 0):

$$V\left(0,0\right) = 108yz - 2y^2z - 2yz^2 \Big|_{(0,0)} = 0$$

At (0, 54):

$$V\left(0,54\right) = 108yz - 2y^2z - 2yz^2 \Big|_{(0,54)} = 0$$

At (54, 0):

$$V\left(54,0\right) = 108yz - 2y^2z - 2yz^2 \Big|_{(54,0)} = 0$$

At (18, 18):

$$V(18,18) = 108yz - 2y^2z - 2yz^2\Big|_{(18,18)} = 11,664$$

$$V_{yy} = -4z, \quad V_{zz} = -4y, \quad V_{yz} = 108 - 4y - 4z$$

$$\begin{aligned} V_{xx}V_{yy} - V_{xy}^2 &= (-4z)(-4y) - (108 - 4y - 4z)^2 \\ &= \left[16yz - 16(27 - y - z)^2\right]_{(18,18)} \\ &= 16(18)(18) - 16(27 - 18 - 18)^2 \\ &= 3888 > 0 \end{aligned}$$

$$V_{yy}(18,18) = -4(18) < 0$$

That implies (18, 18) give a maximum volume.

$$x = 108 - 2(18) - 2(18) = 36$$

$$V = xyz = 36(18)(18)$$

$$= 11,664$$

The dimensions of the package are: $\boldsymbol{x} = 36$ *in*., $\boldsymbol{y} = 18$ *in*, $\boldsymbol{z} = 18$ *in*.

The maximum volume is $11,664\ in^3$

Summary of *Max–Min* Tests

The extreme values of $f(x, y)$ can occur only at

- ***i.*** ***Boundary points*** of the domain of *f*.
- ***ii.*** ***Critical points*** (interior points where $f_x = f_y = 0$ or points where f_x *or* f_y fail to exist)

If the first- and second-order partial derivatives of *f* are continuous throughout a disk centered at a point (a, b) and $f_x(a, b) = f_y(a, b) = 0$, the nature of $f(a, b)$ can be tested with the ***Second Derivative Test***:

- ***i.*** $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at $(a, b) \Rightarrow$ ***local maximum***
- ***ii.*** $f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at $(a, b) \Rightarrow$ ***local minimum***
- ***iii.*** $f_{xx} f_{yy} - f_{xy}^2 < 0$ at $(a, b) \Rightarrow$ ***saddle point***
- ***iv.*** $f_{xx} f_{yy} - f_{xy}^2 = 0$ at $(a, b) \Rightarrow$ ***test is inconclusive.***

Exercises 2.7 – Maximum/Minimum Problems

(**1 – 30**) Find all the local maxima, local minima, and saddle points of the function

1. $f(x,y)=x^2+xy+y^2+3x-3y+4$

2. $f(x,y)=x^2-xy+y^2+2x+2y-4$

3. $f(x,y)=x^3+y^3-3xy+15$

4. $f(x,y)=x^4-8x^2+3y^2-6y$

5. $f(x,y)=2xy-5x^2-2y^2+4x+4y-4$

6. $f(x,y)=x^2-4xy+y^2+6y+2$

7. $f(x,y)=2x^2+3xy+4y^2-5x+2y$

8. $f(x,y)=x^2-y^2-2x+4y+6$

9. $f(x,y)=\sqrt{56x^2-8y^2-16x-31}+1-8x$

10. $f(x,y)=1-\sqrt[3]{x^2+y^2}$

11. $f(x,y)=x^3+y^3+3x^2-3y^2-8$

12. $f(x,y)=4xy-x^4-y^4$

13. $f(x,y)=\dfrac{1}{x^2+y^2-1}$

14. $f(x,y)=\dfrac{1}{x}+xy+\dfrac{1}{y}$

15. $f(x,y)=y\sin x$

16. $f(x,y)=e^{2x}\cos y$

17. $f(x,y)=e^y-ye^x$

18. $f(x,y)=e^{-y}\left(x^2+y^2\right)$

19. $f(x,y)=2\ln x+\ln y-4x-y$

20. $f(x,y)=\ln(x+y)+x^2-y$

21. $f(x,\ y)=1+x^2+y^2$

22. $f(x,\ y)=x^2-6x+y^2+8y$

23. $f(x,\ y)=(3x-2)^2+(y-4)^2$

24. $f(x,\ y)=3x^2-4y^2$

25. $f(x,\ y)=x^4+y^4-16xy$

26. $f(x,\ y)=\frac{1}{3}x^3-\frac{1}{3}y^3+3xy$

27. $f(x,\ y)=x^4-2x^2+y^2-4y+5$

28. $f(x,\ y)=x^2+xy-2x-y+1$

29. $f(x,\ y)=x^2+6x+y^2+8$

30. $f(x,\ y)=e^{x^2y^2-2xy^2+y^2}$

(**31 – 34**) Identify the critical points of the functions. Then determine whether each critical point corresponds to a local maximum, local minimum, or saddle point. State when your analysis is inconclusive.

31. $f(x,\ y)=x^4+y^4-16xy$

32. $f(x,\ y)=\frac{1}{3}x^3-\frac{1}{3}y^3+2xy$

33. $f(x,\ y)=xy(2+x)(y-3)$

34. $f(x,\ y)=10-x^3-y^3-3x^2+3y^2$

(**35 – 53**) Find the absolute maximum and minimum values of the function on the specified region *R*.

35. $f(x,\ y)=\frac{1}{3}x^3-\frac{1}{3}y^3+2xy$ on the rectangle $R=\{(x,\ y):\ 0\le x\le 3,\ -1\le y\le 1\}$

36. $f(x,\ y)=x^4+y^4-4xy+1$ on the square $R=\{(x,\ y):\ -2\le x\le 2,\ -2\le y\le 2\}$

37. $f(x,\ y)=x^2y-y^3$ on the triangle $R=\{(x,\ y):\ 0\le x\le 2,\ 0\le y\le 2-x\}$

38. $f(x, y) = xy$ on the semicircular disk $R = \left\{(x, y):\ -1 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2}\right\}$

39. $f(x, y) = x^2 + y^2 - 2y + 1;\quad R = \left\{(x, y):\ x^2 + y^2 \le 4\right\}$

40. $f(x, y) = 2x^2 + y^2;\quad R = \left\{(x, y):\ x^2 + y^2 \le 16\right\}$

41. $f(x, y) = 4 + 2x^2 + y^2;\quad R = \left\{(x, y):\ -1 \le x \le 1,\ -1 \le y \le 1\right\}$

42. $f(x, y) = 6 - x^2 - 4y^2;\quad R = \left\{(x, y):\ -2 \le x \le 2,\ -1 \le y \le 1\right\}$

43. $f(x, y) = 2x^2 - 4x + 3y^2 + 2;\quad R = \left\{(x, y):\ (x-1)^2 + y^2 \le 1\right\}$

44. $f(x, y) = -2x^2 + 4x - 3y^2 - 6y - 1;\quad R = \left\{(x, y):\ (x-1)^2 + (y+1)^2 \le 1\right\}$

45. $f(x, y) = \sqrt{x^2 + y^2 - 2x + 2};\quad R = \left\{(x, y):\ x^2 + y^2 \le 4,\ y \ge 0\right\}$

46. $f(x, y) = \dfrac{-x^2 + 2y^2}{2 + 2x^2y^2}$; R is the closed region bounded by the lines $y = x$, $y = 2x$, and $y = 2$

47. $f(x, y) = \sqrt{x^2 + y^2}$; R is the closed region bounded by the ellipse $\dfrac{x^2}{4} + y^2 = 1$

48. $f(x, y) = x^2 + y^2 - 4;\quad R = \left\{(x, y):\ x^2 + y^2 < 4\right\}$

49. $f(x, y) = x + 3y;\quad R = \left\{(x, y):\ |x| < 1,\ |y| < 2\right\}$

50. $f(x, y) = 2e^{-x-y};\quad R = \left\{(x, y):\ x \ge 0,\ y \ge 0\right\}$

51. $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0,\quad y = 2,\quad y = 2x$ in the first quadrant.

52. $D(x,y) = x^2 - xy + y^2 + 1$ on the closed triangular plate bounded by the lines $x = 0,\quad y = 4,\quad y = x$ in the first quadrant.

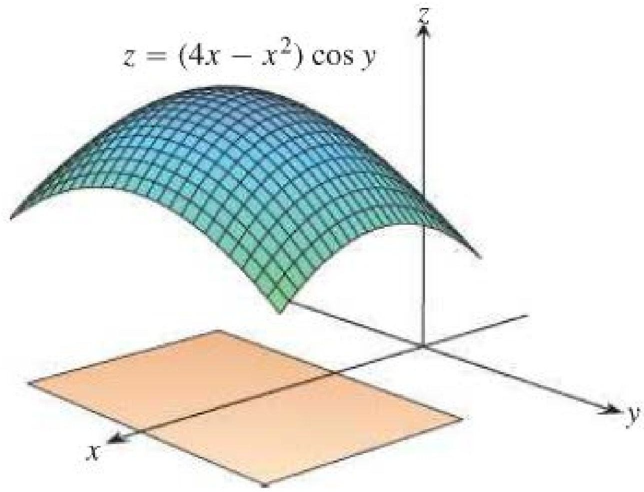
53. $T(x,y) = x^2 + xy + y^2 - 6x + 2$ on the triangular plate $0 \le x \le 5,\quad -3 \le y \le 0$.

54. Find the point on the graph of $z = x^2 + y^2 + 10$ nearest the plane $x + 2y - z = 0$

55. Find the minimum distance from the point $(2,\ -1,\ 1)$ to the plane $x + y - z = 2$

56. Find the maximum value of $s = xy + yz + xz$ where $x + y + z = 6$

57. Find the absolute maxima and minima of the function $f(x,y)=\left(4x-x^2\right)\cos y$ on the triangular plate $1\le x\le 3,\quad -\frac{\pi}{4}\le y\le \frac{\pi}{4}$.



58. Among all triangles with a perimeter of 9 *units*, find the dimensions of the triangle with the maximum area. It may be easiest to use Heron's formula, which states that the area of a triangle with side length *a*, *b*, and *c* is $A=\sqrt{s(s-a)(s-b)(s-c)}$, where 2*s* is the perimeter of the triangle.

59. Let *P* be a plane tangent to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ at a point in the first octane. Let *T* be the tetrahedron in the first octant bounded by *P* and the coordinate planes $x=0$, $y=0$, and $z=0$. Find the minimum volume *T*. (the volume of a tetrahedron in one-third the area of the base times the height.)

60. Given three distinct noncollinear points *A*, *B*, and *C* in the plane, find the point *P* in the plane such the sum of the distances $|AP|+|BP|+|CP|$ is a minimum. Here is how to procees with three points, assuming that the triangle formed by the three points has no angle greater than $\left(120°=\frac{2\pi}{3}\right)$

a) Assume the coordinates of the threee given points are $A\left(x_1, y_1\right)$, $B\left(x_2, y_2\right)$, and $C\left(x_3, y_3\right)$. Let $d_1(x, y)$ be the distance between $A\left(x_1, y_1\right)$ and a variable point $P(x, y)$. Compute the gradient of d_1 and show that it is a unit vector pointing along the line between the two points.

b) Define d_2 and d_3 in a similar way and show that ∇d_2 and ∇d_3 are also unit vectors in the direction of line between the two points.

c) The goal is to minimize $f(x,y,z)=d_1+d_2+d_3$. Show that the condition $f_x=f_y=0$ implies that $\nabla d_1+\nabla d_2+\nabla d_3=0$.

d) Explain why part (c) implies that the optimal point P has the property the the three line segments AP, BP, and CP all intersect symmetrically in angles of $\frac{2\pi}{3}$.

e) What is the optimal solution if one of the angles in the triangle is greater than $\frac{2\pi}{3}$ (draw a picture)?

f) Estimate the Steiner point for the three points $(0,\ 0)$, $(0,\ 1)$, $(2,\ 0)$

(**61 – 62**) Show that the following two functions have two local maxima but no other extreme points (thus no saddle or basin between the mountains).

61. $f(x,\ y) = -\left(x^2 - 1\right)^2 - \left(x^2 - e^y\right)^2$

62. $f(x,\ y) = 4x^2e^y - 2x^4 - e^{4y}$