Solution *Section* 4.2 – Line Integrals

Exercise

Evaluate $\int_C (x+y)\,ds$ where C is the straight-line segment $x=t,\ y=(1-t),\ z=0$ from (0, 1, 0) to $(1,\ 0,\ 0)$.

Solution

$$\vec{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= t\hat{i} + (1-t)\hat{j}$$

$$\frac{dr}{dt} = \boldsymbol{i} - \boldsymbol{j} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right| = \sqrt{1+1} = \sqrt{2}$$

$$\begin{aligned} x &= t \\ y &= 1-t \end{aligned} \quad \rightarrow \quad x + y = t + 1 - t = 1$$

$$\int_C f(x,y,z) = \int_0^1 f(t, 1-t, 0)\left|\frac{d\vec{r}}{dt}\right| dt$$
$$= \int_0^1 (1)\sqrt{2}dt$$
$$= \sqrt{2}\,t \Big|_0^1$$
$$\boxed{= \sqrt{2}}$$

Exercise

Evaluate $\int_C (x-y+z-2)\,ds$ where C is the straight-line segment $x=t,\ y=(1-t),\ z=1$ from $(0,\ 1,\ 1)$ to $(1,\ 0,\ 1)$.

Solution

$$\vec{r}(t) = t\hat{i} + (1-t)\hat{j} + \hat{k} \quad 0 \le t \le 1$$

$$\frac{d\vec{r}}{dt} = \hat{i} - \hat{j} \quad \Rightarrow \quad \left|\frac{d\vec{r}}{dt}\right| = \sqrt{1+1} = \sqrt{2}$$

$$\begin{cases} x = t \\ y = 1 - t \\ z = 1 \end{cases}$$

$$x - y + z - 2 = t - 1 + t + 1 - 2$$
$$= 2t - 2$$

$$\int_C f(x, y, z) = \int_0^1 (2t - 2)\sqrt{2}\,dt$$

$$= \sqrt{2}\left(t^2 - 2t\right|_0^1$$

$$= \sqrt{2}(1 - 2)$$

$$= -\sqrt{2}$$

Exercise

Evaluate $\int_C (xy + y + z)\,ds$ along the curve $\vec{r}(t) = 2t\hat{i} + t\hat{j} + (2 - 2t)\hat{k}, \quad 0 \le t \le 1$

Solution

$$\vec{r}(t) = 2t\hat{i} + t\hat{j} + (2 - 2t)\hat{k}, \quad 0 \le t \le 1$$

$$\frac{d\vec{r}}{dt} = 2\hat{j} + \hat{j} - 2\hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{4 + 1 + 4} = 3$$

$$\begin{aligned} x &= 2t \\ y &= t \quad \rightarrow \quad xy + y + z = 2t^2 + t + 2 - 2t = 2t^2 - t + 2 \\ z &= 2 - 2t \end{aligned}$$

$$\int_C (xy + y + z)\,ds = \int_0^1 \left(2t^2 - t + 2\right)(3)\,dt$$

$$= 3\left(\frac{2}{3}t^3 - \frac{1}{2}t^2 + 2t\right|_0^1$$

$$= 3\left(\frac{2}{3} - \frac{1}{2} + 2\right)$$

$$= 3\left(\frac{13}{6}\right)$$

$$= \frac{13}{2}$$

Exercise

Evaluate $\int_C \left(xz - y^2\right) ds$ C: is the line segment from $(0,\ 1,\ 2)$ to $(-3,\ 7,\ -1)$.

Solution

Equation of the line is:

$$\begin{cases} x = 0 + (-3-0)t \\ y = 1 + (7-1)t \\ z = 2 + (-1-2)t \end{cases} \quad \rightarrow \quad \langle -3t,\ 1+6t,\ 2-3t \rangle$$

$$\vec{r}(t) = \langle -3t,\ 1+6t,\ 2-3t \rangle \quad 0 \le t \le 1$$

$$\vec{r}'(t) = \langle -3,\ 6,\ -3 \rangle$$

$$|\vec{r}'(t)| = \sqrt{9+36+9}$$

$$= 3\sqrt{6}$$

$$\int_C \left(xz - y^2\right) ds = 3\sqrt{6} \int_0^1 \left((-3t)(2-3t) - (1+6t)^2\right) dt$$

$$= 3\sqrt{6} \int_0^1 \left(-6t + 9t^2 - 1 - 12t - 36t^2\right) dt$$

$$= 3\sqrt{6} \int_0^1 \left(-27t^2 - 18t - 1\right) dt$$

$$= 3\sqrt{6} \left(-9t^3 - 9t^2 - t \right) \Big|_0^1$$

$$= 3\sqrt{6}(-9-9-1)$$

$$= -57\sqrt{6}$$

Exercise

Evaluate $\int_C xy\ ds$; C: is the unit circle $\vec{r}(t) = \langle \cos t,\ \sin t \rangle$; $0 \le t \le 2\pi$

Solution

$$\vec{r}'(t) = \langle -\sin t,\ \cos t \rangle$$

$$|\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t}$$

$$= 1$$

$$\int_C xy\,ds = \int_0^{2\pi} \cos t \sin t\ dt$$

$$= \int_0^{2\pi} \sin t\ d(\sin t)$$

$$= \frac{1}{2}\sin^2 t \Big|_0^{2\pi}$$

$$= 0$$

Exercise

Evaluate $\int_C (x+y)ds$ C: is the circle of radius 1 centered at $(0,\ 0)$

Solution

$$\vec{r}(t) = \langle \cos t,\ \sin t \rangle; \quad 0 \le t \le 2\pi$$

$$\vec{r}'(t) = \langle -\sin t,\ \cos t \rangle$$

$$|\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t}$$

$$= 1$$

$$\int_C (x+y)\ ds = \int_0^{2\pi} (\cos t + \sin t)\ dt$$

$$= \sin t - \cos t \Big|_0^{2\pi}$$

$$= -1 + 1$$

$$= 0$$

Exercise

Evaluate $\int_C \left(x^2 - 2y^2\right)ds$ C: is the line $\vec{r}(t) = \left\langle \frac{t}{\sqrt{2}},\ \frac{t}{\sqrt{2}} \right\rangle; \quad 0 \le t \le 4$

Solution

$$\vec{r}'(t) = \left\langle \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}} \right\rangle$$

$$|\vec{r}'(t)| = \sqrt{\frac{1}{2} + \frac{1}{2}}$$

$$\underline{=1}$$

$$\int_C \left(x^2 - 2y^2\right) ds = \int_0^4 \left(\frac{1}{2}t^2 - t^2\right) dt$$

$$= -\frac{1}{2}\int_0^4 t^2 dt$$

$$= -\frac{1}{6}t^3 \Big|_0^4$$

$$\boxed{= -\frac{32}{3}}$$

Exercise

Evaluate $\int_C x^2 y\, ds$ C: is the line $\vec{r}(t) = \left\langle \frac{t}{\sqrt{2}},\ 1 - \frac{t}{\sqrt{2}} \right\rangle;\quad 0 \le t \le 4$

Solution

$$\vec{r}'(t) = \left\langle \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}} \right\rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{\frac{1}{2} + \frac{1}{2}}$$

$$\underline{=1}$$

$$\int_C x^2 y\, ds = \int_0^4 \frac{1}{2}t^2\left(1 - \frac{1}{\sqrt{2}}t\right) dt$$

$$= \int_0^4 \left(\frac{1}{2}t^2 - \frac{1}{2\sqrt{2}}t^3\right) dt$$

$$= \frac{1}{6}t^3 - \frac{1}{8\sqrt{2}}t^4 \Big|_0^4$$

$$= \frac{32}{3} - \frac{32}{\sqrt{2}}$$

$$\boxed{= \frac{32 - 48\sqrt{2}}{3}}$$

Exercise

Evaluate $\int_C \left(x^2+y^2\right)ds$ C: is the circle of radius 4 centered at $(0,\ 0)$

Solution

$$\vec{r}(t)=\langle 4\cos t,\ 4\sin t\rangle;\quad 0\le t\le 2\pi$$

$$\vec{r}'(t)=\langle -4\sin t,\ 4\cos t\rangle$$

$$\left|\vec{r}'(t)\right|=\sqrt{16\sin^2 t+16\cos^2 t}$$

$$=4$$

$$\int_C \left(x^2+y^2\right)ds=4\int_0^{2\pi}\left(16\cos^2 t+16\sin^2 t\right)dt$$

$$=64\int_0^{2\pi} dt$$

$$=128\pi$$

Exercise

Evaluate $\int_C \left(x^2+y^2\right)ds$ C: is the line segment from $(0,\ 0)$ to $(5,\ 5)$

Solution

$$\vec{r}(t)=\langle 5t,\ 5t\rangle;\quad 0\le t\le 1$$

$$\vec{r}'(t)=\langle 5,\ 5\rangle$$

$$\left|\vec{r}'(t)\right|=\sqrt{25+25}$$

$$=5\sqrt{2}$$

$$\int_C \left(x^2+y^2\right)ds=5\sqrt{2}\int_0^1\left(25t^2+25t^2\right)dt$$

$$=250\sqrt{2}\int_0^1 t^2 dt$$

$$=\frac{250}{3}\sqrt{2}\ t^3\Big|_0^1$$

$$=\frac{250}{3}\sqrt{2}$$

Exercise

Evaluate $\displaystyle\int_C \frac{x}{x^2+y^2}\,ds$ C: is the line segment from $(1,\ 1)$ to $(10,\ 10)$

Solution

$$\vec{r}(t) = \langle t,\ t\rangle; \quad 1 \le t \le 10$$

$$\vec{r}\,'(t) = \langle 1,\ 1\rangle$$

$$\left|\vec{r}\,'(t)\right| = \sqrt{1+1}$$

$$= \sqrt{2}$$

$$\int_C \frac{x}{x^2+y^2}\,ds = \sqrt{2}\int_1^{10} \frac{t}{t^2+t^2}\,dt$$

$$= \frac{\sqrt{2}}{2}\int_1^{10} \frac{1}{t}\,dt$$

$$= \frac{\sqrt{2}}{2}\ \ln t\,\Big|_1^{10}$$

$$= \frac{\sqrt{2}}{2}\ln 10$$

Exercise

Evaluate $\displaystyle\int_C (xy)^{1/3}\,ds$ C: is the curve $y = x^2, \quad 0 \le x \le 1$

Solution

$$\vec{r}(t) = \left\langle t,\ t^2\right\rangle; \quad 0 \le t \le 1$$

$$\vec{r}\,'(t) = \langle 1,\ 2t\rangle$$

$$\left|\vec{r}\,'(t)\right| = \sqrt{1+4t^2}$$

$$\int_C (xy)^{1/3}\,ds = \int_0^1 \left(t^3\right)^{1/3}\ \sqrt{1+4t^2}\ dt$$

$$= \int_0^1 t\left(1+4t^2\right)^{1/2}\ dt$$

$$= \frac{1}{8}\int_0^1 \left(1+4t^2\right)^{1/2}\, d\left(1+4t^2\right)$$

$$= \frac{1}{12}\left(1+4t^2\right)^{3/2}\Big|_0^1$$

$$= \frac{1}{12}\left(5\sqrt{5}-1\right)$$

Exercise

Evaluate $\int_C xy\, ds$ C: is a portion of the ellipse $\frac{x^2}{4}+\frac{y^2}{16}=1$ in the first quadrant, oriented counterclockwise.

Solution

$$\vec{r}(t) = \langle 2\cos t,\ 4\sin t\rangle;\quad 0 \le t \le \pi$$

$$\vec{r}\,'(t) = \langle -2\sin t,\ 4\cos t\rangle$$

$$\left|\vec{r}\,'(t)\right| = \sqrt{4\sin^2 t + 16\cos^2 t}$$

$$= 2\sqrt{\sin^2 t + 4\cos^2 t}$$

$$\int_C xy\, ds = 2\int_0^\pi (8\cos t\sin t)\sqrt{1-\cos^2 t + 4\cos^2 t}\, dt$$

$$= 16\int_0^\pi (\cos t \sin t)\left(1+3\cos^2 t\right)^{1/2}\, dt$$

$$= -\frac{8}{3}\int_0^\pi \left(1+3\cos^2 t\right)^{1/2}\, d\left(1+3\cos^2 t\right)$$

$$= -\frac{16}{9}\left(1+3\cos^2 t\right)^{3/2}\Big|_0^\pi$$

$$= -\frac{16}{9}(8-8)$$

$$= 0$$

Exercise

Evaluate $\int_C (2x-3y)\,ds$ C: is the line segment from $(-1,\ 0)$ to $(0,\ 1)$ followed by the line segment from $(0,\ 1)$ to $(1,\ 0)$

Solution

$(-1,\ 0)$ to $(0,\ 1)$

$\vec{r}_1(t) = \langle t-1,\ t\rangle \quad 0 \le t \le 1$

$\vec{r}_1'(t) = \langle 1,\ 1\rangle$

$\left|\vec{r}_1'(t)\right| = \sqrt{2}$

$(0,\ 1)$ to $(1,\ 0)$

$\vec{r}_2(t) = \langle t,\ 1-t\rangle$

$\vec{r}_2'(t) = \langle 1,\ -1\rangle$

$\left|\vec{r}_2'(t)\right| = \sqrt{2}$

$$\begin{aligned}
\int_C (2x-3y)\,ds &= \sqrt{2}\int_0^1 \big(2(t-1)-3t\big)\,dt + \sqrt{2}\int_0^1 (2t-3+3t)\,dt \\
&= \sqrt{2}\int_0^1 (-2-t)\,dt + \sqrt{2}\int_0^1 (5t-3)\,dt \\
&= \sqrt{2}\int_0^1 (-2-t+5t-3)\,dt \\
&= \sqrt{2}\int_0^1 (4t-5)\,dt \\
&= \sqrt{2}\left(2t^2-5t\right|_0^1 \\
&= \sqrt{2}(2-5) \\
&= -3\sqrt{2}
\end{aligned}$$

Exercise

Evaluate $\int_C (x+y+z)\,ds$; C is the circle $\vec{r}(t)=\langle 2\cos t,\ 0,\ 2\sin t\rangle \quad 0\le t\le 2\pi$

Solution

$$\vec{r}'(t)=\langle -2\sin t,\ 0,\ 2\cos t\rangle$$

$$\left|\vec{r}'(t)\right|=\sqrt{4\sin^2 t+4\cos^2 t}$$

$$=2$$

$$\int_C (x+y+z)\,ds=\int_0^{2\pi}(2\cos t+2\sin t)(2)\,dt$$

$$=4\left(-\sin t+\cos t\right.\Big|_0^{2\pi}$$

$$=0$$

Exercise

Evaluate $\int_C (x-y+2z)\ ds$; C is the circle $\vec{r}(t)=\langle 1,\ 3\cos t,\ 3\sin t\rangle \quad 0\le t\le 2\pi$

Solution

$$\vec{r}'(t)=\langle 0,\ -3\sin t,\ 3\cos t\rangle$$

$$\left|\vec{r}'(t)\right|=\sqrt{9\sin^2 t+9\cos^2 t}$$

$$=3$$

$$\int_C (x-y+2z)\ ds=3\int_0^{2\pi}(1-3\cos t+6\sin t)\,dt$$

$$=3\left(t-3\sin t-6\cos t\right.\Big|_0^{2\pi}$$

$$=3(2\pi-6+6)(t-3\sin t-6\cos t)$$

$$=6\pi$$

Exercise

Evaluate $\displaystyle\int_C xyz\ ds$; C is the circle $\vec{r}(t) = \langle 1,\ 3\cos t,\ 3\sin t\rangle \quad 0 \le t \le 2\pi$

Solution

$$\vec{r}'(t) = \langle 0,\ -3\sin t,\ 3\cos t\rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{9\sin^2 t + 9\cos^2 t}$$

$$= 3$$

$$\int_C xyz\ ds = 3\int_0^{2\pi} (9\cos t\sin t)\,dt$$

$$= 27\int_0^{2\pi} \sin t\ d(\sin t)$$

$$= \frac{27}{2}\sin^2 t\,\Big|_0^{2\pi}$$

$$= 0$$

Exercise

Evaluate $\displaystyle\int_C xyz\ ds$; C is the line segment from $(0,\ 0,\ 0)$ to $(1,\ 2,\ 3)$

Solution

$$\vec{r}(t) = \langle t,\ 2t,\ 3t\rangle \qquad 0 \le t \le 1$$

$$\vec{r}'(t) = \langle 1,\ 2,\ 3\rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{1+4+9}$$

$$= \sqrt{14}$$

$$\int_C xyz\ ds = \sqrt{14}\int_0^1 6t^3\ dt$$

$$= \frac{3}{2}\sqrt{14}\ t^4\,\Big|_0^1$$

$$= \frac{3}{2}\sqrt{14}$$

Exercise

Evaluate $\int_C \frac{xy}{z}\, ds$; C is the line segment from $(1,\ 4,\ 1)$ to $(3,\ 6,\ 3)$

Solution

$$\vec{r}(t) = \langle 2t+1,\ 2t+4,\ 2t+1 \rangle \qquad 0 \le t \le 1$$

$$\vec{r}(t) = \langle 2,\ 2,\ 2 \rangle$$

$$\boxed{|\vec{r}'(t)| = 2\sqrt{3}}$$

$$\begin{aligned}
\int_C \frac{xy}{z}\, ds &= 2\sqrt{3} \int_0^1 \frac{(2t+1)(2t+4)}{2t+1}\, dt \\
&= 2\sqrt{3} \int_0^1 (2t+4)\, dt \\
&= 2\sqrt{3} \left(t^2 + 4t \right|_0^1 \\
&= \boxed{10\sqrt{3}}
\end{aligned}$$

Exercise

Evaluate $\int_C (y-z)\, ds$; C is the helix $\vec{r}(t) = \langle 3\cos t,\ 3\sin t,\ t \rangle \quad 0 \le t \le 2\pi$

Solution

$$\vec{r}'(t) = \langle -3\sin t,\ 3\cos t,\ 1 \rangle$$

$$\begin{aligned}
|\vec{r}'(t)| &= \sqrt{9\sin^2 t + 9\cos^2 t + 1} \\
&= \boxed{\sqrt{10}}
\end{aligned}$$

$$\begin{aligned}
\int_C (y-z)\, ds &= \sqrt{10} \int_0^{2\pi} (3\sin t - t)\, dt \\
&= \sqrt{10} \left(-3\cos t - \frac{1}{2}t^2 \right|_0^{2\pi} \\
&= \sqrt{10}\left(-3 - 2\pi^2 + 3 \right) \\
&= \boxed{-2\pi\sqrt{10}}
\end{aligned}$$

Exercise

Evaluate $\int_C xe^{yz}\,ds$; C is $\vec{r}(t)=\langle t,\ 2t,\ -4t\rangle \quad 1\le t\le 2$

Solution

$$\vec{r}'(t)=\langle 1,\ 2,\ -4\rangle$$

$$\boxed{|\vec{r}'(t)|=\sqrt{21}}$$

$$\int_C xe^{yz}\,ds=\sqrt{21}\int_1^2 te^{-8t^2}\,dt$$

$$=-\frac{\sqrt{21}}{16}\int_1^2 e^{-8t^2}\,d\left(-8t^2\right)$$

$$=-\frac{\sqrt{21}}{16}\,e^{-8t^2}\Big|_1^2$$

$$=-\frac{\sqrt{21}}{16}\left(e^{-32}-e^{-8}\right)$$

$$=-\frac{\sqrt{21}}{16e^8}\left(\frac{1}{e^{24}}-1\right)$$

$$\boxed{=\frac{\sqrt{21}}{16e^{32}}\left(e^{24}-1\right)}$$

Exercise

Find the integral of $f(x,y,z)=x+y+z$ over the straight–line segment from $(1,\ 2,\ 3)$ to $(0,\ -1,\ 1)$

Solution

$$\vec{r}(t)=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+t\left((0-1)\,\hat{i}+(-1-2)\hat{j}+(1-3)\,\hat{k}\right)$$

$$=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+t\left(-\hat{i}-3\hat{j}-2\hat{k}\right)$$

$$=(1-t)\,\hat{i}+(2-3t)\hat{j}+(3-2t)\,\hat{k},\quad 0\le t\le 1$$

$$\frac{d\vec{r}}{dt}=-\hat{i}-3\hat{j}-2\hat{k}$$

$$\left|\frac{dr}{dt}\right|=\sqrt{1+9+4}$$

$$\boxed{=\sqrt{14}}$$

$$x = 1 - t$$
$$y = 2 - 3t \quad \rightarrow \quad x + y + z = 1 - t + 2 - 3t + 3 - 2t$$
$$z = 3 - 2t$$

$$\underline{x + y + z = 6 - 6t}$$

$$\int_C (x + y + z)\, ds = \int_0^1 (6 - 6t)\left(\sqrt{14}\right) dt$$
$$= \sqrt{14}\left(6t - 3t^2 \right|_0^1$$
$$\underline{= 3\sqrt{14}}$$

Exercise

Find the integral of $f(x, y, z) = \dfrac{\sqrt{3}}{x^2 + y^2 + z^2}$ over the curve $\vec{r}(t) = t\,\hat{i} + t\hat{j} + t\,\hat{k}, \quad 1 \le t \le \infty$

Solution

$$\vec{r}(t) = t\,\hat{i} + t\hat{j} + t\,\hat{k}, \quad 1 \le t \le \infty$$
$$\frac{d\vec{r}}{dt} = \hat{i} + \hat{j} + \hat{k}$$
$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{1 + 1 + 1}$$
$$\underline{= \sqrt{3}}$$

$$x^2 + y^2 + z^2 = t^2 + t^2 + t^2$$
$$\underline{= 3t^2}$$

$$\int_C \frac{\sqrt{3}}{x^2 + y^2 + z^2}\, ds = \int_1^\infty \frac{\sqrt{3}}{3t^2}\left(\sqrt{3}\right) dt$$
$$= -\frac{1}{t}\Big|_1^\infty$$
$$= -\left(\frac{1}{\infty} - 1\right)$$
$$\underline{= 1}$$

Exercise

Evaluate $\int_C x\, ds$ where C is

a) The straight-line segment $x=t,\ \ y=\frac{t}{2}$, from (0, 0) to (4, 2).

b) The parabolic curve $x=t,\ \ y=t^2$, from (0, 0) to (2, 4).

Solution

a) $x=t \quad \to \quad \begin{cases} x=0 & t=0 \\ x=4 & t=4 \end{cases}$

$t=2y \quad \to \quad \begin{cases} y=0 & t=0 \\ y=2 & t=4 \end{cases}$

$\vec{r}(t)=t\,\hat{i}\ +\frac{t}{2}\hat{j}, \quad 0\le t\le 4$

$\frac{d\vec{r}}{dt}=\hat{i}\ +\frac{1}{2}\hat{j}$

$$\left|\frac{d\vec{r}}{dt}\right|=\sqrt{1+\frac{1}{4}}$$

$$=\frac{\sqrt{5}}{2}$$

$$\int_C x\, ds=\int_0^4 t\frac{\sqrt{5}}{2}dt$$

$$=\frac{\sqrt{5}}{2}\left(\frac{1}{2}t^2\right|_0^4$$

$$=4\sqrt{5}$$

b) $x=t \quad \to \quad \begin{cases} x=0 & t=0 \\ x=2 & t=2 \end{cases}$

$t=\sqrt{y} \quad \to \quad \begin{cases} y=0 & t=0 \\ y=4 & t=2 \end{cases}$

$\vec{r}(t)=t\,\hat{i}\ +t^2\hat{j}, \quad 0\le t\le 2$

$\frac{d\vec{r}}{dt}=\hat{i}\ +2t\hat{j}$

$$\left|\frac{d\vec{r}}{dt}\right|=\sqrt{1+4t^2}$$

$$\int_C x\, ds=\int_0^2 t\sqrt{1+4t^2}\ dt \qquad\qquad d\left(1+4t^2\right)=8tdt$$

$$= \frac{1}{8}\int_0^2 \left(1+4t^2\right)^{1/2} d\left(1+4t^2\right)$$

$$= \frac{1}{8}\left(\frac{2}{3}\left(1+4t^2\right)^{3/2}\right|_0^2$$

$$= \frac{1}{12}\left[(17)^{3/2}-1\right]$$

$$\underline{= \frac{1}{12}\left(17\sqrt{17}-1\right)}$$

Exercise

Evaluate $\int_C \sqrt{x+2y}\ ds$ where C is

a) The straight-line segment $x=t,\ \ y=4t$, from (0, 0) to (1, 4).

b) $C_1 \cup C_2$: C_1 is the line segment (0, 0) to (1, 0) and C_2 is the line segment (1, 0) to (1, 2).

Solution

a) $x = t \quad \rightarrow \begin{cases} x=0 & t=0 \\ x=1 & t=1 \end{cases}$

$t = \frac{y}{4} \quad \rightarrow \begin{cases} y=0 & t=0 \\ y=4 & t=1 \end{cases}$

$\vec{r}(t) = t\,\hat{i} + 4t\hat{j}, \quad 0 \le t \le 1$

$\frac{d\vec{r}}{dt} = \hat{i} + 4\hat{j}$

$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{1+16}$

$\underline{= \sqrt{17}}$

$$\int_C \sqrt{x+2y}\ ds = \int_0^1 \sqrt{t+8t}\left(\sqrt{17}\right)dt$$

$$= \sqrt{17}\int_0^1 \sqrt{9t}\ dt$$

$$= 3\sqrt{17}\left(\frac{2}{3}t^{3/2}\right|_0^1$$

$$\underline{= 2\sqrt{17}}$$

b) C_1 : $\vec{r}(t) = \left(0\,\hat{i} + 0\hat{j}\right) + t\left(\hat{i} + 0\hat{j}\right) \underline{= t\,\hat{i}} \quad 0 \le t \le 1$

$\frac{d\vec{r}}{dt} = \hat{i} \quad \Rightarrow \quad \left|\frac{d\vec{r}}{dt}\right| = 1$

$$C_2: \quad \vec{r}(t) = \left(1\hat{i} + 0\hat{j}\right) + t\left((1-1)\hat{i} + (2-0)\hat{j}\right)$$

$$= \hat{i} + 2t\hat{j} \qquad\qquad 0 \le t \le 2$$

$$\frac{d\vec{r}}{dt} = 2\hat{j} \quad \Rightarrow \quad \left|\frac{d\vec{r}}{dt}\right| = 2$$

$$\int_C \sqrt{x+2y}\ ds = \int_0^1 \sqrt{t}\,(1)\,dt + \int_0^2 \sqrt{1+4t}\,(2)\,dt$$

$$= \frac{2}{3}t^{3/2}\Big|_0^1 + \frac{1}{2}\int_0^2 (1+4t)^{1/2}\ d(1+4t)$$

$$= \frac{2}{3} + \frac{1}{3}\left((1+4t)^{3/2}\ \Big|_0^2\right.$$

$$= \frac{2}{3} + \frac{1}{3}\left[(9)^{3/2} - 1\right]$$

$$= \frac{2}{3} + \frac{1}{3}(26)$$

$$\boxed{= \frac{28}{3}}$$

Exercise

Find the line integral of $f(x,y) = \dfrac{\sqrt{y}}{x}$ along the curve $\vec{r}(t) = t^3\,\hat{i} + t^4\hat{j}, \quad \frac{1}{2} \le t \le 1$

Solution

$$\vec{r}(t) = t^3\,\hat{i} + t^4\hat{j}, \quad \frac{1}{2} \le t \le 1$$

$$\frac{d\vec{r}}{dt} = 3t^2\,\hat{i} + 4t^3\hat{j}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{9t^4 + 16t^6}$$

$$\boxed{= t^2\sqrt{9+16t^2}}$$

$$\int_C \frac{\sqrt{y}}{x}\ ds = \int_{1/2}^1 \frac{\sqrt{t^4}}{t^3}\left(t^2\sqrt{9+16t^2}\right)dt$$

$$= \int_{1/2}^1 t\left(9+16t^2\right)^{1/2}\ dt \qquad\qquad d\left(9+16t^2\right) = 32t\,dt$$

$$= \frac{1}{32}\int_{1/2}^1 \left(9+16t^2\right)^{1/2} d\left(9+16t^2\right)$$

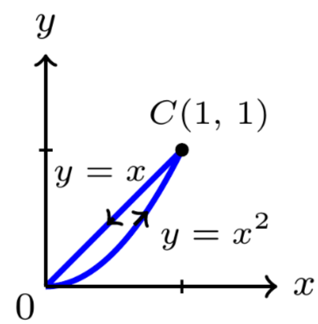
$$=\frac{1}{32}\left(\frac{2}{3}\right)\left(\left(9+16t^2\right)^{3/2}\ \Big|_{1/2}^{1}\right.$$

$$=\frac{1}{48}\left[(25)^{3/2}-(13)^{3/2}\right]$$

$$\boxed{=\frac{1}{48}\left(125-13\sqrt{13}\right)}$$

Exercise

Evaluate $\int_C \left(x+\sqrt{y}\right)\,ds$ where C is



Solution

$C_1:\quad \vec{r}(t)=t\,\hat{i}\,+t^2\hat{j}\quad 0\le t\le 1$

$$\frac{d\vec{r}}{dt}=\hat{i}\,+2t\hat{j}\quad\Rightarrow\quad\left|\frac{d\vec{r}}{dt}\right|=\sqrt{1+4t^2}$$

$C_2:\quad \vec{r}(t)=\left(1\hat{i}\,+1\hat{j}\right)+t\left(-\hat{i}\,-\hat{j}\right)$

$$=(1-t)\,\hat{i}\,+(1-t)\hat{j}\qquad 0\le t\le 1$$

$$\frac{dr}{dt}=-\hat{i}\,-\hat{j}\quad\Rightarrow\quad\left|\frac{dr}{dt}\right|=\sqrt{2}$$

$$\int_C\left(x+\sqrt{y}\right)\,ds=\int_0^1\left(t+\sqrt{t^2}\right)\left(\sqrt{1+4t^2}\right)dt\;+\;\int_0^1\left(1-t+\sqrt{1-t}\right)\left(\sqrt{2}\right)dt$$

$$=\int_0^1 2t\left(\sqrt{1+4t^2}\right)dt\;-\;\sqrt{2}\int_0^1\left((1-t)+\sqrt{1-t}\right)d(1-t)$$

$$=\frac{1}{4}\int_0^1\left(1+4t^2\right)^{1/2}d\left(1+4t^2\right)\;-\;\sqrt{2}\left(\frac{1}{2}(1-t)^2+\frac{2}{3}(1-t)^{3/2}\ \Big|_0^1\right.$$

$$=\frac{1}{6}\left(1+4t^2\right)^{3/2}\ \Big|_0^1\;-\;\sqrt{2}\left(-\frac{1}{2}-\frac{2}{3}\right)$$

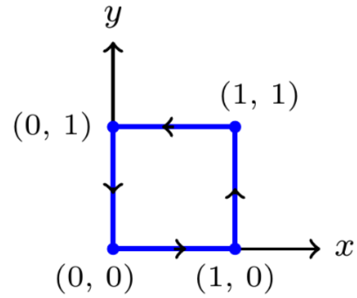
$$=\frac{1}{6}\left((5)^{3/2}-1\right)+\frac{7\sqrt{2}}{6}$$

$$\boxed{=\frac{5\sqrt{5}-1+7\sqrt{2}}{6}}$$

Exercise

Evaluate $\displaystyle\int_C \frac{1}{x^2+y^2+1}\,ds$ where C is

Solution



$C_1:\ \vec{r}(t)=t\,\hat{i} \qquad 0\le t\le 1$

$$\frac{d\vec{r}}{dt}=\hat{i} \quad\Rightarrow\quad \left|\frac{d\vec{r}}{dt}\right|=1$$

$C_2:\ \vec{r}(t)=\hat{i}+t\hat{j} \qquad 0\le t\le 1$

$$\frac{d\vec{r}}{dt}=\hat{j} \quad\Rightarrow\quad \left|\frac{d\vec{r}}{dt}\right|=1$$

$C_3:\ \vec{r}(t)=(1-t)\,\hat{i}+\hat{j} \qquad 0\le t\le 1$

$$\frac{d\vec{r}}{dt}=-\hat{i} \quad\Rightarrow\quad \left|\frac{d\vec{r}}{dt}\right|=1$$

$C_4:\ \vec{r}(t)=(1-t)\hat{j} \qquad 0\le t\le 1$

$$\frac{d\vec{r}}{dt}=-\hat{j} \quad\Rightarrow\quad \left|\frac{d\vec{r}}{dt}\right|=1$$

$$\int_C \frac{1}{x^2+y^2+1}\,ds=\int_0^1 \frac{1}{t^2+1}(1)\,dt+\int_0^1 \frac{1}{1+t^2+1}(1)\,dt$$

$$+\int_0^1 \frac{1}{(1-t)^2+1+1}(1)\,dt+\int_0^1 \frac{1}{(1-t)^2+1}(1)\,dt$$

$$=\int_0^1 \frac{1}{t^2+1}\,dt+\int_0^1 \frac{1}{t^2+2}\,dt-\int_0^1 \frac{1}{(1-t)^2+2}\,d(1-t)$$

$$-\int_0^1 \frac{1}{(1-t)^2+1}\,d(1-t)$$

$$=\tan^{-1}t+\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1-t}{\sqrt{2}}\right)-\tan^{-1}(1-t)\,\Big|_0^1$$

$$=\frac{\pi}{4}+\frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}+\frac{\pi}{4}$$

$$=\frac{\pi}{2}+\frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

Exercise

Find the line integral of $f(x,y) = \frac{x^3}{y}$ over the curve $C: \quad y = \frac{x^2}{2}, \quad 0 \le x \le 2$

Solution

$$\vec{r}(t) = x\,\hat{i} + y\hat{j}$$

$$= x\,\hat{i} + \frac{1}{2}x^2\hat{j} \qquad\qquad 0 \le x \le 2$$

$$\frac{d\vec{r}}{dt} = \hat{i} + x\hat{j}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{1+x^2}$$

$$\int_C f(x,y)\,ds = \int_C \frac{x^3}{y^2}\,ds$$

$$= \int_0^2 2x\sqrt{1+x^2}\,dx \qquad\qquad d\left(1+x^2\right) = 2x\,dx$$

$$= \int_0^2 \left(1+x^2\right)^{1/2} d\left(1+x^2\right)$$

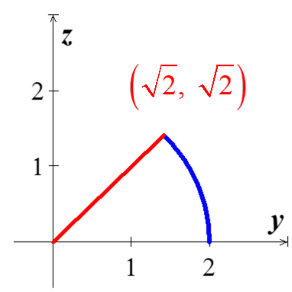
$$= \frac{2}{3}\left(1+x^2\right)^{3/2}\Big|_0^2$$

$$= \frac{2}{3}\left((5)^{3/2} - 1\right)$$

$$\boxed{= \frac{10\sqrt{5}-2}{3}}$$

Exercise

Find the line integral of $f(x,y) = x^2 - y$ over the curve $C: \quad x^2 + y^2 = 4$ in the first quadrant from (0, 2) to $\left(\sqrt{2},\ \sqrt{2}\right)$



Solution

$$x = r\cos t \quad y = r\sin t$$

$$\vec{r}(t) = (2\sin t)\,\hat{i} + (2\cos t)\,\hat{j} \qquad 0 \le t \le \frac{\pi}{4}$$

$$\frac{d\vec{r}}{dt} = (2\cos t)\,\hat{i} - (2\sin t)\,\hat{j}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{4\cos^2 t + 4\sin^2 t}$$

$$\boxed{= 2}$$

$$f(x,y) = x^2 - y = 4\sin^2 t - 2\cos t$$

$$\int_C f(x,y)\,ds = \int_0^{\pi/4} \left(4\sin^2 t - 2\cos t\right)(2)\,dt \qquad \sin^2 t = \frac{1-\cos 2t}{2}$$

$$= 4\int_0^{\pi/4} (1 - \cos 2t - \cos t)\,dt$$

$$= 4\left(t - \frac{1}{2}\sin 2t - \sin t\right|_0^{\pi/4}$$

$$= 4\left(\frac{\pi}{4} - \frac{1}{2} - \frac{\sqrt{2}}{2}\right)$$

$$= 4\left(\frac{\pi}{4} - \frac{1+\sqrt{2}}{2}\right)$$

$$= \pi - 2\left(1+\sqrt{2}\right)$$

Exercise

Evaluate the line integral $\int_C \left(x^2 - 2xy + y^2\right)ds$; C is the upper half of a circle

$\vec{r}(t) = \langle 5\cos t,\ 5\sin t\rangle, \quad 0 \le t \le \pi \quad (ccw)$

Solution

$$\vec{r}\,' = \langle -5\sin t,\ 5\cos t\rangle$$

$$|\vec{r}\,'| = \sqrt{25\sin^2 t + 25\cos^2 t}$$

$$= 5$$

$$\int_C \left(x^2 - 2xy + y^2\right)ds = 5\int_0^{\pi} \left(25\cos^2 t - 50\cos t\sin t + 25\sin^2 t\right)dt$$

$$= 125\int_0^{\pi} (1 - 2\cos t\sin t)\,dt$$

$$= 125\int_0^{\pi} (1 - \sin 2t)\,dt$$

$$= 125\left(t + \frac{1}{2}\cos 2t\right|_0^{\pi}$$

$$= 125\left(\pi + \frac{1}{2} - \frac{1}{2}\right)$$

$$\underline{= 125\pi}$$

Exercise

Evaluate the line integral $\int_C y\,e^{-xz}\,ds$; C is the path $\vec{r}(t) = \langle t,\ 3t,\ -6t\rangle$, $0 \le t \le \ln 8$

Solution

$$\vec{r}' = \langle 1,\ 3,\ 6\rangle$$

$$|\vec{r}'| = \sqrt{1+9+36}$$

$$\underline{= \sqrt{46}}$$

$$\int_C ye^{-xz}ds = \sqrt{46}\int_0^{\ln 8} 3t\,e^{6t^2}\,dt$$

$$= \frac{\sqrt{46}}{4}\int_0^{\ln 8} e^{6t^2}\,d\left(6t^2\right)$$

$$= \frac{\sqrt{46}}{4}\left(e^{6t^2}\right|_0^{3\ln 2}$$

$$\underline{= \frac{\sqrt{46}}{4}\left(e^{54\ln 2} - 1\right)}$$

Exercise

Integrate $f(x,y,z) = \sqrt{x^2+z^2}$ over the circle $\vec{r}(t) = (a\cos t)\,\hat{j} + (a\sin t)\,\hat{k}$, $0 \le t \le 2\pi$

Solution

$$\vec{r}' = \langle 0,\ -a\sin t,\ a\cos t\rangle$$

$$|\vec{r}'| = \sqrt{a^2\sin^2 t + a^2\cos^2 t}$$

$$\underline{= a}$$

$$f(t) = \sqrt{0 + a^2\sin^2 t}$$

$$\underline{= a|\sin t|}$$

$$\int_C f\ |r'|\,dt = a^2\int_0^{2\pi} |\sin t|\,dt$$

$$= a^2 \int_0^{\pi} \sin t \, dt + a^2 \int_{\pi}^{2\pi} \sin t \, dt$$

$$= -a^2 \left(\cos t \,\right|_0^{\pi} - a^2 \left(\cos t \,\right|_{\pi}^{2\pi}$$

$$= -a^2(-1-1) - a^2(1+1)$$

$$= 2a^2 + 2a^2$$

$$= 4a^2$$

Exercise

Integrate $f(x,y,z) = \sqrt{x^2+y^2}$ over the involute curve

$\vec{r}(t) = \langle \cos t + t\sin t, \quad \sin t - t\cos t \rangle, \quad 0 \le t \le \sqrt{3}$

Solution

$$\vec{r}' = \langle -\sin t + \sin t + t\cos t, \quad \cos t - \cos t + t\sin t \rangle$$

$$= \langle t\cos t, \quad t\sin t \rangle$$

$$|v| = \sqrt{t^2\cos^2 t + t^2\sin^2 t}$$

$$= t$$

$$f(t) = \sqrt{(\cos t + t\sin t)^2 + (\sin t - t\cos t)^2}$$

$$= \sqrt{\cos^2 t + 2t\cos t\sin t + t^2\sin^2 t + \sin^2 t - 2t\cos t\sin t + t^2\cos^2 t}$$

$$= \sqrt{1+t^2}$$

$$\int_C f\,|\vec{v}|\,dt = \int_0^{\sqrt{3}} t\sqrt{1+t^2}\,dt$$

$$= \frac{1}{2}\int_0^{\sqrt{3}} \left(1+t^2\right)^{1/2} dt$$

$$= \frac{1}{3}\left(1+t^2\right)^{3/2} \Big|_0^{\sqrt{3}}$$

$$= \frac{1}{3}(8-1)$$

$$= \frac{7}{3}$$

Exercise

Find the average of the function on the given curves

$f(x, y) = x + 2y$ on the line segment from $(1, 1)$ to $(2, 5)$

Solution

$$\vec{r}(t) = \langle (2-1)t + 1,\ (5-1)t + 1 \rangle$$
$$= \langle t + 1,\ 4t + 1 \rangle$$

$$|\vec{r}'(t)| = \sqrt{1+16}$$
$$= \sqrt{17}$$

$$\int_C (x+2y)\,ds = \int_0^1 (t+1+2(4t+1)) \cdot \sqrt{17}\ dt$$
$$= \sqrt{17} \int_0^1 (9t+3)\ dt$$
$$= \sqrt{17} \left(\frac{9}{2}t^2 + 3t \right) \Big|_0^1$$
$$= \sqrt{17} \left(\frac{9}{2} + 3 \right)$$
$$= \frac{15}{2}\sqrt{17}$$

The length of the line segment is $\sqrt{17}$

$\therefore$ The average value is $\frac{15}{2}$

Exercise

Find the average of the function on the given curves

$f(x, y) = x^2 + 4y^2$ on the circle of radius 9 centered at the origin.

Solution

$$\vec{r}(t) = \langle 9\cos t,\ 9\sin t \rangle \quad 0 \le t \le 2\pi$$
$$\vec{r}'(t) = \langle -9\sin t,\ 9\cos t \rangle$$
$$|\vec{r}'(t)| = \sqrt{81\sin^2 t + 81\cos^2 t}$$
$$= 9$$

$$\int_C (x^2 + 4y^2)\,ds = 9 \int_0^{2\pi} (81\cos^2 t + 324\sin^2 t)\ dt$$

$$= 729\int_0^{2\pi}\left(\frac{1}{2}+\frac{1}{2}\cos 2t + 2 - 2\cos 2t\right)\, dt$$

$$= 729\int_0^{2\pi}\left(\frac{5}{2}-\frac{3}{2}\cos 2t\right)\, dt$$

$$= 729\left(\frac{5}{2}t-\frac{3}{4}\sin 2t\right)\Big|_0^{2\pi}$$

$$= 3,645\pi$$

The circumference of the circle is $9(2\pi) = 18\pi$

$\therefore$ The average value is $\frac{3645\pi}{18\pi} = \frac{405}{2}$

Exercise

Find the average of the function on the given curves

$f(x, y) = xe^y$ on the circle of radius 1 centered at the origin.

Solution

$$\vec{r}(t) = \langle \cos t,\ \sin t\rangle \quad 0 \le t \le 2\pi$$

$$\vec{r}'(t) = \langle -\sin t,\ \cos t\rangle$$

$$|\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t}$$

$$= 1$$

$$\int_C xe^y ds = \int_0^{2\pi} \cos t\ e^{\sin t}\ dt$$

$$= \int_0^{2\pi} e^{\sin t}\ d(\sin t)$$

$$= e^{\sin t}\Big|_0^{2\pi}$$

$$= 1 - 1$$

$$= 0$$

$\therefore$ The average value is 0

Exercise

Find the average of the function on the given curves

$$f(x, y) = \sqrt{4 + 9y^{2/3}} \quad \text{on the curve } y = x^{3/2}, \text{ for } 0 \le x \le 5$$

Solution

$$\vec{r}(t) = \left\langle t, \ t^{3/2} \right\rangle$$

$$\vec{r}'(t) = \left\langle 1, \ \frac{3}{2}t^{1/2} \right\rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{1 + \frac{9}{4}t}$$

$$= \frac{1}{2}\sqrt{4 + 9t}$$

$$\int_C \sqrt{4 + 9y^{2/3}}\, ds = \frac{1}{2}\int_0^5 \sqrt{4 + 9\left(t^{3/2}\right)^{2/3}} \ \sqrt{4 + 9t} \ dt$$

$$= \frac{1}{2}\int_0^5 \sqrt{4 + 9t} \ \sqrt{4 + 9t} \ dt$$

$$= \frac{1}{2}\int_0^5 (4 + 9t) \ dt$$

$$= \frac{1}{2} \left(4t + \frac{9}{2}t^2 \right) \Big|_0^5$$

$$= \frac{1}{2}\left(20 + \frac{225}{2}\right)$$

$$= \frac{265}{4}$$

The length of the curve is

$$\int_0^5 \sqrt{4 + 9\left(x^{3/2}\right)^{2/3}} \ dx = \frac{1}{2}\int_0^5 \sqrt{4 + 9x} \ dx$$

$$= \frac{1}{18}\int_0^5 (4 + 9x)^{1/2} \ d(4 + 9x)$$

$$= \frac{1}{27} \ (4 + 9x)^{3/2} \Big|_0^5$$

$$= \frac{1}{27}(343 - 8)$$

$$= \frac{335}{27} \quad unit$$

$\therefore$ The average value is $= \frac{265}{4} \times \frac{27}{335} = \frac{1431}{268}$

Exercise

Find the length of the curve $\vec{r}(t) = \left\langle 20\sin\frac{t}{4},\ 20\cos\frac{t}{4},\ \frac{t}{2} \right\rangle \quad 0 \le t \le 2$

Solution

$$\vec{r}'(t) = \left\langle 5\cos\frac{t}{4},\ -5\sin\frac{t}{4},\ \frac{1}{2} \right\rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{25\cos^2\frac{t}{4} + 25\sin^2\frac{t}{4} + \frac{1}{4}}$$

$$= \sqrt{25 + \frac{1}{4}}$$

$$= \frac{1}{2}\sqrt{101}$$

$$L = \int_0^2 \frac{1}{2}\sqrt{101}\ dt$$

$$= \frac{1}{2}\sqrt{101}(2)$$

$$= \sqrt{101} \quad unit$$

Exercise

Find the length of the curve $\vec{r}(t) = \langle 30\sin t,\ 40\sin t,\ 50\cos t \rangle \quad 0 \le t \le 2\pi$

Solution

$$\left|\vec{r}'(t)\right| = \sqrt{900\cos^2 t + 1600\cos^2 t + 2500\sin^2 t}$$

$$= \sqrt{2500\cos^2 t + 2500\sin^2 t}$$

$$= 50$$

$$L = \int_0^{2\pi} 50\ dt$$

$$= 100\pi \quad unit$$