

Solution **Section 1.2 – Solutions to Separable Equations**

Exercise

Find the general solution of the differential equation $y' = xy$

Solution

$$\frac{dy}{dx} = xy$$

$$\frac{dy}{y} = x dx$$

$$\int \frac{dy}{y} = \int x dx$$

$$\ln|y| = \frac{1}{2}x^2 + C$$

$$|y| = e^{x^2/2+C}$$

$$y(x) = \pm e^{x^2/2} e^C$$

$$= \underline{A e^{x^2/2}}$$

Where $A = \pm e^C$

Exercise

Find the general solution of the differential equation $xy' = 2y$

Solution

$$x \frac{dy}{dx} = 2y$$

$$\frac{dy}{y} = 2 \frac{dx}{x}$$

$$\int \frac{dy}{y} = \int \frac{2}{x} dx$$

$$\ln|y| = 2 \ln|x| + C$$

$$= \ln x^2 + C$$

$$y(x) = \pm e^{\ln x^2 + C}$$

$$= \pm e^C x^2$$

$$= \underline{A x^2}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = e^{x-y}$

Solution

$$\frac{dy}{dx} = e^x e^{-y}$$

$$\frac{dy}{e^{-y}} = e^x dx$$

$$\int e^y dy = \int e^x dx$$

$$e^y = e^x + C$$

$$y(x) = \ln(e^x + C)$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = (1 + y^2)e^x$

Solution

$$\frac{dy}{dx} = (1 + y^2)e^x$$

$$\int \frac{dy}{1 + y^2} = \int e^x dx$$

$$\tan^{-1} y = e^x + C$$

$$y(x) = \tan(e^x + C)$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = xy + y$

Solution

$$\frac{dy}{dx} = (x + 1)y$$

$$\int \frac{dy}{y} = \int (x + 1) dx$$

$$\ln y = \frac{1}{2}x^2 + x + C$$

$$\rightarrow y(x) = e^{x^2/2 + x + C}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$$y' = ye^x - 2e^x + y - 2$$

Solution

$$\frac{dy}{dx} = (y - 2)e^x + y - 2$$

$$\frac{dy}{dx} = (y - 2)(e^x + 1)$$

$$\frac{dy}{y - 2} = (e^x + 1)dx$$

$$\int \frac{dy}{y - 2} = \int (e^x + 1)dx$$

$$\ln|y - 2| = e^x + x + C$$

$$y - 2 = \pm e^{e^x + x + C}$$

$$y - 2 = \pm e^C e^{e^x + x}$$

$$y(x) = \underline{De^{e^x + x} + 2} \quad D = \pm e^C$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \frac{x}{y + 2}$

Solution

$$\frac{dy}{dx} = \frac{x}{y + 2}$$

$$(y + 2)dy = xdx$$

$$\int (y + 2)dy = \int xdx$$

$$\frac{1}{2}y^2 + 2y = \frac{1}{2}x^2 + C$$

$$y^2 + 4y = x^2 + 2C$$

$$y^2 + 4y - x^2 - D = 0, \quad (D = 2C)$$

$$y = \frac{-4 \pm \sqrt{16 - 4(-x^2 - D)}}{2} = \frac{-4 \pm \sqrt{16 + 4x^2 + 4D}}{2} = \frac{-4 \pm 2\sqrt{x^2 + (4 + D)}}{2} = -2 \pm \sqrt{x^2 + E} \quad E = 4 + D$$

$$y(x) = \underline{-2 \pm \sqrt{x^2 + E}}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \frac{xy}{x-1}$

Solution

$$\frac{dy}{dx} = y \left(\frac{x}{x-1} \right)$$

$$\frac{dy}{y} = \left(\frac{x}{x-1} \right) dx$$

$$\int \frac{dy}{y} = \int \left(1 + \frac{1}{x-1} \right) dx$$

$$\ln|y| = x + \ln|x-1| + C$$

$$y(x) = \pm e^{x + \ln|x-1| + C}$$

$$= \pm e^C e^x e^{\ln|x-1|}$$

$$= \underline{De^x |x-1|}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$$y' = \frac{y^2 + ty + t^2}{t^2}$$

Solution

$$y' = \frac{y^2}{t^2} + \frac{y}{t} + 1$$

$$\text{Let } x = \frac{y}{t} \Rightarrow y = xt \rightarrow y' = x + tx'$$

$$y' = \frac{y^2}{t^2} + \frac{y}{t} + 1 = x^2 + x + 1$$

$$x + tx' = x^2 + x + 1$$

$$t \frac{dx}{dt} = x^2 + 1$$

$$\int \frac{dx}{x^2 + 1} = \int \frac{dt}{t}$$

$$\tan^{-1} x = \ln|t| + C$$

$$\tan^{-1} \frac{y}{t} = \ln|t| + C$$

$$\frac{y}{t} = \tan(\ln|t| + C)$$

$$y(t) = \underline{t \tan(\ln|t| + C)}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}$$

Solution

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}$$

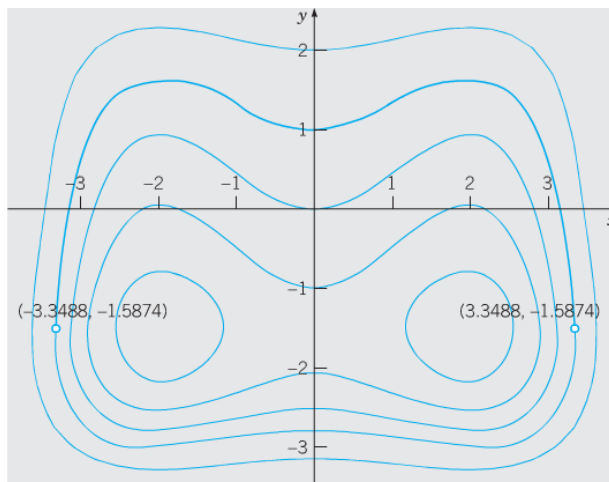
$$(4 + y^3)dy = (4x - x^3)dx$$

$$\int (4 + y^3)dy = \int (4x - x^3)dx$$

$$4y + \frac{1}{4}y^4 = 2x^2 - \frac{1}{4}x^4 + C_1$$

$$16y + y^4 = 8x^2 - x^4 + C$$

$$\underline{y^4 + 16y + x^4 - 8x^2 = +C}$$



Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$$y' = \frac{2xy + 2x}{x^2 - 1}$$

Solution

$$\frac{dy}{dx} = \frac{2x(y+1)}{x^2 - 1}$$

$$\frac{dy}{y+1} = \frac{2x}{x^2 - 1}dx$$

$$\int \frac{d(y+1)}{y+1} = \int \frac{d(x^2 - 1)}{x^2 - 1}$$

$$\ln|y+1| = \ln|x^2 - 1| + C$$

$$y+1 = e^{\ln|x^2 - 1| + C}$$

$$y = e^C e^{\ln|x^2 - 1|} - 1$$

$$\underline{y(x) = Ae^{\ln|x^2 - 1|} - 1}$$

$$d(x^2 - 1) = 2xdx$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} = \sin 5x$

Solution

$$\int dy = \int \sin 5x dx$$
$$\underline{y(x) = -\frac{1}{5} \cos 5x + C}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} = (x+1)^2$

Solution

$$\int dy = \int (x^2 + 2x + 1) dx$$
$$\underline{y(x) = \frac{1}{3}x^3 + x^2 + x + C}$$

Exercise

Find the general solution of the differential equation $dx + e^{3x} dy = 0$

Solution

$$\int dy = - \int e^{-3x} dx$$
$$\underline{y(x) = \frac{1}{3} e^{-3x} + C}$$

Exercise

Find the general solution of the differential equation $dy - (y-1)^2 dx = 0$

Solution

$$\int \frac{dy}{(y-1)^2} = \int dx$$
$$\int \frac{d(y-1)}{(y-1)^2} = \int dx$$
$$-\frac{1}{y-1} = x + C$$
$$\underline{y(x) = 1 - \frac{1}{x+C}}$$

Exercise

Find the general solution of the differential equation $x \frac{dy}{dx} = 4y$

Solution

$$\int \frac{dy}{y} = 4 \int \frac{dx}{x}$$

$$\ln y = 4 \ln x + \ln C$$

$$\ln y = \ln Cx^4$$

$$\underline{y(x) = Cx^4}$$

Exercise

Find the general solution of the differential equation $\frac{dx}{dy} = y^2 - 1$

Solution

$$\int dx = \int (y^2 - 1) dy$$

$$\underline{x = \frac{1}{3}y^3 - y + C}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} = e^{2y}$

Solution

$$\int e^{-2y} dy = \int dx$$

$$-\frac{1}{2}e^{-2y} = x + C$$

$$e^{-2y} = -2x + C_1$$

$$-2y = \ln(C_1 - 2x)$$

$$\underline{y(x) = -\frac{1}{2} \ln(C_1 - 2x)}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} + 2xy^2 = 0$

Solution

$$\frac{dy}{dx} = -2xy^2$$

$$-\int \frac{dy}{y^2} = \int 2x dx$$

$$\frac{1}{y} = x^2 + C$$

$$\underline{y(x) = \frac{1}{x^2 + C}}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} = e^{3x+2y}$

Solution

$$\frac{dy}{dx} = e^{3x} e^{2y}$$

$$\int e^{-2y} dy = \int e^{3x} dx$$

$$-\frac{1}{2} e^{-2y} = \frac{1}{3} e^{3x} + C$$

$$e^{-2y} = C_1 - \frac{2}{3} e^{3x}$$

$$-2y = \ln\left(C_1 - \frac{2}{3} e^{3x}\right)$$

$$\underline{y(x) = -\frac{1}{2} \ln\left(C_1 - \frac{2}{3} e^{3x}\right)}$$

Exercise

Find the general solution of the differential equation $e^x y \frac{dy}{dx} = e^{-y} + e^{-2x-y}$

Solution

$$e^x y \frac{dy}{dx} = e^{-y} (1 + e^{-2x})$$

$$y e^y dy = e^{-x} (1 + e^{-2x}) dx$$

$$\int y e^y dy = \int (e^{-x} + e^{-3x}) dx$$

$$\underline{(y-1)e^y = -e^{-x} - \frac{1}{3} e^{-3x} + C}$$

		$\int e^y$
+	y	e^y
-	1	e^y

Exercise

Find the general solution of the differential equation $y \ln x \frac{dx}{dy} = \left(\frac{y+1}{x}\right)^2$

Solution

$$x^2 \ln x dx = \frac{1}{y} (y^2 + 2y + 1) dy$$

$$\int x^2 \ln x dx = \int \left(y + 2 + \frac{1}{y} \right) dy$$

$$u = \ln x \quad dv = x^2 dx$$

$$du = \frac{dx}{x} \quad v = \frac{1}{3} x^3$$

$$\frac{1}{3} x^3 \ln x - \frac{1}{3} \int x^2 dx = \frac{1}{2} y^2 + 2y + \ln|y| + C$$

$$\underline{\frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 = \frac{1}{2} y^2 + 2y + \ln|y| + C}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} = \left(\frac{2y+3}{4x+5} \right)^2$

Solution

$$\int \frac{dy}{(2y+3)^2} = \int \frac{dx}{(4x+5)^2}$$

$$\frac{1}{2} \int \frac{d(2y+3)}{(2y+3)^2} = \frac{1}{4} \int \frac{d(4x+5)}{(4x+5)^2}$$

$$\frac{1}{2} \frac{-1}{2y+3} = \frac{1}{4} \frac{-1}{4x+5} + C$$

$$\underline{\frac{2}{2y+3} = \frac{1}{4x+5} + C}$$

Exercise

Find the general solution of the differential equation $\csc y dx + \sec^2 x dy = 0$

Solution

$$\csc y dx = -\sec^2 x dy$$

$$\frac{dy}{\csc y} = -\frac{dx}{\sec^2 x}$$

$$\sin y dy = -\cos^2 x dx$$

$$\int \sin y dy = -\frac{1}{2} \int (1 + \cos 2x) dx$$

$$-\cos y = -\frac{1}{2} \left(x + \frac{1}{2} \sin 2x \right) + C$$

$$\underline{\cos y = \frac{1}{2} x + \frac{1}{4} \sin 2x + C}$$

Exercise

Find the general solution of the differential equation $\sin 3x dx + 2y \cos^3 3x dy = 0$

Solution

$$\sin 3x dx = -2y \cos^3 3x dy$$

$$\int \frac{\sin 3x}{\cos^3 3x} dx = - \int 2y dy$$

$$-\frac{1}{3} \int \cos^{-3} 3x d(\cos 3x) = - \int 2y dy$$

$$-\frac{1}{6} \cos^{-2} 3x + C = y^2$$

$$\underline{y^2 = -\frac{1}{6} \sec^2 3x + C}$$

Exercise

Find the general solution of the differential equation $(e^y + 1)^2 e^{-y} dx + (e^x + 1)^3 e^{-x} dy = 0$

Solution

$$(e^y + 1)^2 e^{-y} dx = -(e^x + 1)^3 e^{-x} dy$$

$$\frac{e^x}{(e^x + 1)^3} dx = -\frac{e^y}{(e^y + 1)^2} dy$$

$$\int (e^x + 1)^{-3} d(e^x + 1) = - \int \frac{1}{(e^y + 1)^2} d(e^y + 1)$$

$$\underline{-\frac{1}{2} \frac{1}{(e^x + 1)^2} + C = \frac{1}{e^y + 1}}$$

Exercise

Find the general solution of the differential equation $x(1 + y^2)^{1/2} dx = y(1 + x^2)^{1/2} dy$

Solution

$$\int x(1 + x^2)^{-1/2} dx = \int y(1 + y^2)^{-1/2} dy$$

$$\frac{1}{2} \int (1 + x^2)^{-1/2} d(1 + x^2) = \frac{1}{2} \int (1 + y^2)^{-1/2} d(1 + y^2)$$

$$2(1+y^2)^{1/2} = 2(1+x^2)^{1/2} + C$$

$$\underline{(1+y^2)^{1/2} = (1+x^2)^{1/2} + C}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = y \sin x$

Solution

$$\int \frac{dy}{y} = \int \sin x \, dx$$

$$\ln|y| = -\cos x + C$$

$$y = e^{-\cos x + C}$$

$$\underline{= Ae^{-\cos x}}$$

Exercise

Find the general solution of the differential equation. $(1+x)\frac{dy}{dx} = 4y$

Solution

$$\int \frac{dy}{y} = \int \frac{4}{1+x} dx$$

$$\ln|y| = 4\ln|1+x| + \ln C$$

$$= \ln(1+x)^4 + \ln C$$

$$= \ln C(1+x)^4$$

$$\underline{y(x) = C(1+x)^4}$$

Exercise

Find the general solution of the differential equation. $2\sqrt{x}\frac{dy}{dx} = \sqrt{1-y^2}$

Solution

$$\int \frac{dy}{\sqrt{1-y^2}} = \frac{1}{2} \int x^{-1/2} dx$$

$$\underline{\arcsin y = \sqrt{x} + C}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = 3\sqrt{xy}$

Solution

$$\int y^{1/2} dy = 3 \int x^{1/2} dx$$

$$\frac{2}{3} y^{3/2} = 2x^{3/2} + C$$

$$\underline{y^{3/2} = 3x^{3/2} + C_1}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = (64xy)^{1/3}$

Solution

$$\int y^{-1/3} dy = \int 4x^{1/3} dx$$

$$\frac{3}{2} y^{2/3} = 3x^{4/3} + C_1$$

$$\underline{y^{2/3} = 2x^{4/3} + C}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = 2x \sec y$

Solution

$$\int \cos y \, dy = \int 2x \, dx$$

$$\underline{\sin y = x^2 + C}$$

Exercise

Find the general solution of the differential equation. $(1-x^2) \frac{dy}{dx} = 2y$

Solution

$$\int \frac{dy}{y} = 2 \int \frac{1}{1-x^2} dx$$

$$\int \frac{dy}{y} = \int \left(\frac{1}{1+x} + \frac{1}{1-x} \right) dx$$

$$\ln|y| = \ln|1+x| - \ln|1-x| + \ln C$$

$$\ln|y| = \ln C \left| \frac{1+x}{1-x} \right|$$

$$\underline{y(x) = C \frac{1+x}{1-x}}$$

Exercise

Find the general solution of the differential equation. $(1+x)^2 \frac{dy}{dx} = (1+y)^2$

Solution

$$\int \frac{dy}{(1+y)^2} = \int \frac{1}{(1+x)^2} dx$$

$$-\frac{1}{1+y} = -\frac{1}{1+x} + C$$

$$\frac{1}{1+y} = \frac{1+C+Cx}{1+x}$$

$$y+1 = \frac{1+x}{C_1+Cx}$$

$$y = \frac{1+x}{C_1+Cx} - 1$$

$$= \frac{1+x-C_1-Cx}{C_1+Cx} \quad A=1-C_1 \quad B=1-C$$

$$\underline{= \frac{A+Bx}{C_1+Cx}}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = xy^3$

Solution

$$\int y^{-3} dy = \int x dx$$

$$-\frac{1}{2y^2} = \frac{1}{2}x^2 + C_1$$

$$\frac{1}{y^2} = -x^2 + C$$

$$\underline{y^2 = \frac{1}{-x^2 + C}}$$

Exercise

Find the general solution of the differential equation. $y \frac{dy}{dx} = x(y^2 + 1)$

Solution

$$\int \frac{y}{y^2 + 1} dy = \int x dx$$

$$\frac{1}{2} \int \frac{1}{y^2 + 1} d(y^2 + 1) = \frac{1}{2} x^2 + C$$

$$\ln(y^2 + 1) = x^2 + C$$

$$y^2 + 1 = e^{x^2 + C}$$

$$\underline{y^2 = Ae^{x^2} - 1}$$

Exercise

Find the general solution of the differential equation. $y^3 \frac{dy}{dx} = (y^4 + 1) \cos x$

Solution

$$\int \frac{y^3}{y^4 + 1} dy = \int \cos x dx$$

$$\frac{1}{4} \ln(y^4 + 1) = \sin x + C$$

$$\ln(y^4 + 1) = 4 \sin x + C$$

$$y^4 + 1 = e^{4 \sin x + C}$$

$$\underline{y^4 = A e^{4 \sin x} - 1}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = \frac{1 + \sqrt{x}}{1 + \sqrt{y}}$

Solution

$$\int (1 + y^{1/2}) dy = \int (1 + x^{1/2}) dx$$

$$\underline{y + \frac{2}{3} y^{3/2} = x + \frac{2}{3} x^{3/2} + C}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = \frac{(x-1)y^5}{x^2(2y^3-y)}$

Solution

$$\begin{aligned}\left(\frac{2y^3-y}{y^5}\right)dy &= \left(\frac{x-1}{x^2}\right)dx \\ \int \left(2\frac{1}{y^2} - \frac{1}{y^4}\right)dy &= \int \left(\frac{1}{x} - \frac{1}{x^2}\right)dx \\ -\frac{2}{y} + \frac{1}{3y^3} &= \ln|x| + \frac{1}{x} + C \\ \frac{1-6y^2}{3y^3} &= \ln|x| + \frac{1}{x} + C\end{aligned}$$

Exercise

Find the general solution of the differential equation. $(x^2+1)(\tan y)y' = x$

Solution

$$\begin{aligned}\int \tan y \, dy &= \int \frac{x}{x^2+1} dx \\ \ln|\sec y| &= \frac{1}{2}\ln(x^2+1) + \ln C \\ &= \ln C\sqrt{x^2+1} \\ \sec y &= C\sqrt{x^2+1}\end{aligned}$$

Exercise

Find the general solution of the differential equation. $x^2y' = 1 - x^2 + y^2 - x^2y^2$

Solution

$$\begin{aligned}x^2y' &= 1 - x^2 + (1 - x^2)y^2 \\ x^2y' &= (1 - x^2)(1 + y^2) \\ \int \frac{1}{1+y^2} dy &= \int \frac{1-x^2}{x^2} dx\end{aligned}$$

$$\int \frac{1}{1+y^2} dy = \int \left(\frac{1}{x^2} - 1 \right) dx$$

$$\underline{\arctan y = -\frac{1}{x} - x + C}$$

Exercise

Find the general solution of the differential equation. $xy' + 4y = 0$

Solution

$$x \frac{dy}{dx} = -4y$$

$$\int \frac{dy}{y} = -4 \int \frac{dx}{x}$$

$$\ln|y| = -4 \ln|x| + C$$

$$\ln|y| = \ln x^{-4} + C$$

$$y(x) = e^{\ln x^{-4} + C}$$

$$= e^C e^{\ln x^{-4}}$$

$$\underline{= Ax^{-4}}$$

Exercise

Find the general solution of the differential equation. $(x^2 + 1)y' + 2xy = 0$

Solution

$$(x^2 + 1) \frac{dy}{dx} = -2xy$$

$$\int \frac{dy}{y} = - \int \frac{2x}{x^2 + 1} dx$$

$$\ln|y| = -\ln(x^2 + 1) + \ln C$$

$$\ln|y| = \ln \frac{C}{x^2 + 1}$$

$$\underline{y(x) = \frac{C}{x^2 + 1}}$$

Exercise

Find the general solution of the differential equation. $\frac{y'}{(x^2+1)y} = 3$

Solution

$$\int \frac{1}{y} dy = \int (3x^2 + 3) dx$$

$$\ln|y| = x^3 + 3x + C$$

$$\underline{y(x) = e^{x^3+3x+C}}$$

Exercise

Find the general solution of the differential equation. $y + e^x y' = 0$

Solution

$$e^x \frac{dy}{dx} = -y$$

$$\int \frac{dy}{y} = - \int e^{-x} dx$$

$$\ln|y| = e^{-x} + C$$

$$\underline{y(x) = e^{e^{-x}+C}}$$

Exercise

Find the general solution of the differential equation. $\frac{dx}{dt} = 3xt^2$

Solution

$$\int \frac{dx}{x} = \int 3t^2 dt$$

$$\ln|x| = t^3 + C$$

$$\underline{x(t) = e^{t^3+C} = Ae^{t^3}}$$

Exercise

Find the general solution of the differential equation. $x \frac{dy}{dx} = \frac{1}{y^3}$

Solution

$$\int y^3 dy = \int \frac{1}{x} dx$$

$$\frac{1}{4}y^4 = \ln|x| + C_1$$

$$y^4 = 4\ln|x| + C$$

$$\underline{y^4 = \ln x^4 + C}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = \frac{x}{y^2\sqrt{x+1}}$

Solution

$$\int y^2 dy = \int \frac{x}{\sqrt{x+1}} dx$$

$$\text{Let } u = x+1 \rightarrow x = u-1 \rightarrow du = dx$$

$$\frac{1}{3}y^3 = \int \frac{u-1}{u^{1/2}} du$$

$$\frac{1}{3}y^3 = \int (u^{1/2} - u^{-1/2}) du$$

$$\frac{1}{3}y^3 = \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C_1$$

$$\underline{y^3 = 2(x+1)^{3/2} - 6(x+1)^{1/2} + C}$$

Exercise

Find the general solution of the differential equation. $\frac{dx}{dt} - x^3 = x$

Solution

$$\frac{dx}{dt} = x^3 + x$$

$$\int \frac{dx}{x(x^2+1)} = \int dt$$

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$Ax^2 + A + Bx^2 + Cx = 1$$

$$\left\{ \begin{array}{l} x^2 \quad A+B=0 \\ x \quad \underline{C=0} \\ x^0 \quad \underline{A=1} \end{array} \right. \rightarrow \underline{B=-1}$$

$$\int \frac{dx}{x} - \int \frac{dx}{x^2+1} = t + K$$

$$\underline{\ln|x| - \arctan x = t + K}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = \frac{x}{ye^{x+2y}}$

Solution

$$\frac{dy}{dx} = \frac{x}{ye^{2y}e^x}$$

$$\int ye^{2y} dy = \int xe^{-x} dx$$

$$\frac{1}{2} ye^{2y} - \frac{1}{4} e^{2y} = -xe^{-x} - e^{-x} + C_1$$

$$\underline{(2y-1)e^{2y} = -4(x+1)e^{-x} + C}$$

		$\int e^{2y}$
+	y	$\frac{1}{2} e^{2y}$
-	1	$\frac{1}{4} e^{2y}$

		$\int e^{-x}$
+	x	$-e^{-x}$
-	1	e^{-x}

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = \frac{\sec^2 y}{1+x^2}$

Solution

$$\int \cos^2 y \, dy = \int \frac{dx}{1+x^2}$$

$$\frac{1}{2} \int (1 + \cos 2y) dy = \arctan x + C$$

$$\underline{\frac{1}{2} \left(y + \frac{1}{2} \sin 2y \right) = \arctan x + C}$$

Exercise

Find the general solution of the differential equation. $x \frac{dv}{dx} = \frac{1-4v^2}{3v}$

Solution

$$\int \frac{3v}{1-4v^2} dv = \int \frac{dx}{x}$$

$$-\frac{3}{8} \int \frac{1}{1-4v^2} d(1-4v^2) = \int \frac{dx}{x}$$

$$-\frac{3}{8} \ln|1-4v^2| = \ln|x| + \ln C$$

$$\ln\left(|1-4v^2|\right)^{-3/8} = \ln|Cx|$$

$$\underline{(1-4v^2)^{-3/8} = Cx}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = 3x^2(1+y^2)^{3/2}$

Solution

$$\int (1+y^2)^{-3/2} dy = \int 3x^2 dx \quad \begin{array}{l} y = \tan \theta \quad \sqrt{1+y^2} = \sec \theta \\ dy = \sec^2 \theta d\theta \end{array}$$

$$\int \sec^{-3} \theta \sec^2 \theta d\theta = x^3 + C$$

$$\int \sec \theta d\theta = x^3 + C$$

$$\ln |\sec \theta + \tan \theta| = x^3 + C$$

$$\boxed{\frac{1}{\sqrt{1+y^2}} + y = C_1 e^{x^3}}$$

Exercise

Find the general solution of the differential equation. $\frac{1}{y} dy + ye^{\cos x} \sin x dx = 0$

Solution

$$\int \frac{1}{y^2} dy = - \int e^{\cos x} \sin x dx$$

$$-\frac{1}{y} = e^{\cos x} + C$$

$$\boxed{y(x) = \frac{-1}{e^{\cos x} + C}}$$

Exercise

Find the general solution of the differential equation. $(x + xy^2)dx + e^{x^2} y dy = 0$

Solution

$$x(1+y^2)dx = -e^{x^2} y dy$$

$$\int x e^{-x^2} dx = - \int \frac{y}{1+y^2} dy$$

$$-\frac{1}{2} \int e^{-x^2} d(e^{-x^2}) = -\frac{1}{2} \int \frac{1}{1+y^2} d(1+y^2)$$

$$\boxed{e^{-x^2} = \ln(1+y^2) + C}$$

Exercise

Find the exact solution of the initial value problem. $y' = \frac{y}{x}$, $y(1) = -2$

Solution

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln|y| = \ln|x| + C$$

$$y = \pm e^{\ln|x| + C}$$

$$= \pm e^C e^{\ln|x|}$$

$$= D|x|$$

$$= Dx$$

$$y = Dx \Rightarrow D = \frac{y}{x} = \frac{-2}{1} = -2$$

$$\underline{y(x) = -2x}$$

Exercise

Find the exact solution of the initial value problem. $y' = -\frac{2t(1+y^2)}{y}$, $y(0) = 1$

Solution

$$\frac{dy}{dt} = -\frac{2t(1+y^2)}{y}$$

$$\int \frac{ydy}{1+y^2} = \int -2tdt$$

$$\frac{1}{2} \int \frac{1}{1+y^2} d(1+y^2) = -2 \int tdt$$

$$\frac{1}{2} \ln(1+y^2) = -t^2 + C$$

$$\ln(1+y^2) = -2t^2 + 2C$$

$$1+y^2 = e^{-2t^2+2C}$$

$$1+y^2 = e^{2C} e^{-2t^2}$$

$$1+y^2 = D e^{-2t^2}$$

$$1 + \textcolor{red}{1}^2 = De^{-2(\textcolor{red}{0})^2} \rightarrow \underline{2 = D}$$

$$y^2 = 2e^{-2t^2} - 1$$

$$y^2 = 2e^{-2t^2} - 1$$

$$y = \pm \sqrt{2e^{-2t^2} - 1}$$

$$\boxed{y(x) = \sqrt{2e^{-2t^2} - 1}}$$

$$2e^{-2t^2} - 1 > \textcolor{red}{0}$$

$$2e^{-2t^2} > 1$$

$$e^{-2t^2} > \frac{1}{2}$$

$$-2t^2 > \ln\left(\frac{1}{2}\right)$$

$$t^2 < -\frac{1}{2}\ln\left(\frac{1}{2}\right) = \frac{1}{2}\ln 2$$

$$t^2 < \ln \sqrt{2}$$

$$t < |\ln \sqrt{2}|$$

The interval of existence: $(-\ln \sqrt{2}, \ln \sqrt{2})$

Exercise

Find the exact solution of the initial value problem. Indicate the interval of existence.

$$y' = \frac{\sin x}{y}, \quad y\left(\frac{\pi}{2}\right) = 1$$

Solution

$$\frac{dy}{dx} = \frac{\sin x}{y}$$

$$ydy = \sin x dx$$

$$\int ydy = \int \sin x dx$$

$$\frac{1}{2}y^2 = -\cos x + C_1$$

$$y^2 = -2\cos x + C \quad (C = 2C_1)$$

$$y(x) = \pm \sqrt{-2\cos x + C}$$

$$y\left(\frac{\pi}{2}\right) = \sqrt{-2\cos \frac{\pi}{2} + C}$$

$$1 = \sqrt{C} \Rightarrow \boxed{C=1}$$

$$y(x) = \sqrt{1 - 2\cos x}$$

The interval of existence will be the interval containing $\frac{\pi}{2}$ and $1 - 2\cos x > 0$

$$\cos x < \frac{1}{2} \Rightarrow \boxed{\frac{\pi}{3} < x < \frac{5\pi}{3}}$$

Exercise

Find the exact solution of the initial value problem. $4tdy = (y^2 + ty^2)dt$, $y(1) = 1$

Solution

$$4tdy = y^2(1+t)dt$$

$$4 \int \frac{dy}{y^2} = \int \left(\frac{1}{t} + 1\right) dt \qquad \int \frac{dx}{x^2} = -\frac{1}{x}$$

$$-\frac{4}{y} = \ln|t| + t + C$$

$$y = \frac{-4}{\ln|t| + t + C}$$

$$1 = \frac{-4}{\ln|1| + 1 + C} \Rightarrow 1 = \frac{-4}{1+C} \rightarrow 1+C = -4 \Rightarrow \boxed{C=-5}$$

$$y = \frac{-4}{\ln|t| + t - 5}$$

Exercise

Find the exact solution of the initial value problem. $y' = \frac{1-2t}{y}$, $y(1) = -2$

Solution

$$y \frac{dy}{dt} = 1 - 2t$$

$$\int y dy = \int (1 - 2t) dt$$

$$\frac{1}{2} y^2 = t - t^2 + C_1$$

$$y^2 = 2t - 2t^2 + C$$

$$(-2)^2 = 2(1) - 2(1)^2 + C \Rightarrow \boxed{C=4}$$

$$y = -\sqrt{2t - 2t^2 + 4}$$

The negative value is taken to satisfy the initial condition.

Exercise

Find the exact solution of the initial value problem. $y' = y^2 - 4$, $y(0) = 0$

Solution

$$\frac{dy}{dt} = y^2 - 4$$

$$\frac{dy}{y^2 - 4} = dt \quad \frac{1}{y^2 - 4} = \frac{A}{y-2} + \frac{B}{y+2}$$

$$\frac{1}{y^2 - 4} = \frac{(A+B)y + 2A - 2B}{y-2} \quad \rightarrow \begin{cases} A+B=0 \\ 2A-2B=1 \end{cases} \Rightarrow A = \frac{1}{4} \quad B = -\frac{1}{4}$$

$$\left(\frac{1}{4(y-2)} - \frac{1}{4(y+2)} \right) dy = dt$$

$$\int \left(\frac{1}{4(y-2)} - \frac{1}{4(y+2)} \right) dy = \int dt$$

$$\frac{1}{4} (\ln|y-2| - \ln|y+2|) = t + C$$

$$\ln \left| \frac{y-2}{y+2} \right| = 4t + C$$

$$\frac{y-2}{y+2} = \pm e^{4t+C}$$

$$\frac{y-2}{y+2} = \pm e^C e^{4t} = k e^{4t}$$

$$y-2 = k e^{4t} y + 2k e^{4t}$$

$$y - k e^{4t} y = 2 + 2k e^{4t}$$

$$y(1 - k e^{4t}) = 2 + 2k e^{4t}$$

$$y = \frac{2 + 2k e^{4t}}{1 - k e^{4t}}$$

$$0 = \frac{2 + 2k e^{4(0)}}{1 - k e^{4(0)}}$$

$$0 = 2 + 2k \Rightarrow \boxed{k = -1}$$

$$y = \frac{2 - 2e^{4t}}{1 + e^{4t}}$$

Exercise

Find the exact solution of the initial value problem. Indicate the interval of existence.

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1$$

Solution

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2y - 2}$$

$$(2y - 2)dy = (3x^2 + 4x + 2)dx$$

$$\int (2y - 2)dy = \int (3x^2 + 4x + 2)dx$$

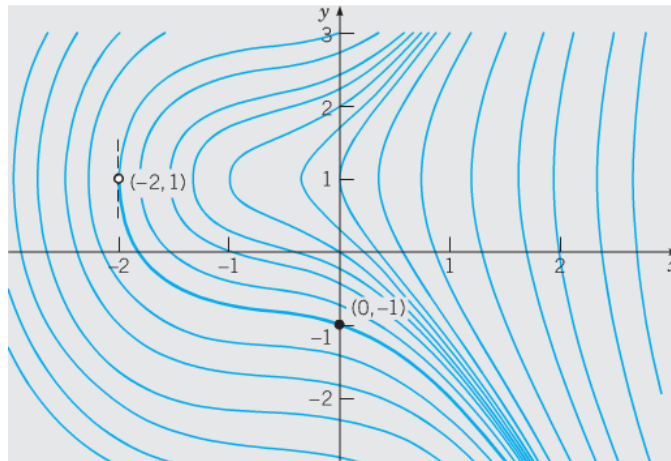
$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$

$$y(0) = -1$$

$$(-1)^2 - 2(-1) = (0)^3 + 2(0)^2 + 2(0) + C$$

$$\Rightarrow \boxed{C = 3}$$

$$\underline{y^2 - 2y = x^3 + 2x^2 + 2x + 3}$$



Exercise

Find the exact solution of the initial value problem. $y' = \frac{x}{1+2y}$, $y(-1) = 0$

Solution

$$\frac{dy}{dx} = \frac{x}{1+2y}$$

$$\int (1+2y)dy = \int xdx$$

$$y + y^2 = \frac{1}{2}x^2 + C \quad y(-1) = 0$$

$$0 = \frac{1}{2}(-1)^2 + C \Rightarrow C = -\frac{1}{2}$$

$$\underline{y + y^2 = \frac{1}{2}x^2 - \frac{1}{2}}$$

Exercise

Find the exact solution of the initial value problem $(e^{2y} - y)\cos x \frac{dy}{dx} = e^y \sin 2x$, $y(0) = 0$

Solution

$$\frac{e^{2y} - y}{e^y} dy = \frac{2 \sin x \cos x}{\cos x} dx$$

$$\int (e^y - ye^{-y}) dy = \int 2 \sin x dx$$

$$e^y + ye^{-y} + e^{-y} = -2 \cos x + C$$

$$y(0) = 0 \quad 1 + 1 = -2 + C \quad \rightarrow \underline{C = 4}$$

$$\underline{e^y + ye^{-y} + e^{-y} = 4 - 2 \cos x}$$

		$\int e^{-y} dy$
+	y	$-e^{-y}$
-	1	e^{-y}

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = e^{-x^2}, \quad y(3) = 5$

Solution

$$\int_3^x \frac{dy}{dt} dt = \int_3^x e^{-t^2} dt$$

$$y(x) - y(3) = \int_3^x e^{-t^2} dt$$

$$\underline{y(x) = 5 + \int_3^x e^{-t^2} dt}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dx} + 2y = 1, \quad y(0) = \frac{5}{2}$

Solution

$$\frac{dy}{dx} = 1 - 2y$$

$$\frac{dy}{1 - 2y} = dx$$

$$-\frac{1}{2} \int \frac{d(1 - 2y)}{1 - 2y} = \int dx$$

$$-\frac{1}{2} \ln|1 - 2y| = x + C$$

$$\ln|1 - 2y| = -2x + C \quad y(0) = \frac{5}{2}$$

$$\ln|1 - 5| = C \quad \rightarrow \quad \underline{C = \ln 4}$$

$$1 - 2y = e^{-2x + \ln 4}$$

$$1 - 2y = e^{-2x} e^{\ln 4}$$

$$\underline{y = \frac{1}{2} - 2e^{-2x}}$$

Exercise

Find the exact solution of the initial value problem. $\sqrt{1-y^2}dx - \sqrt{1-x^2}dy = 0, \quad y(0) = \frac{\sqrt{3}}{2}$

Solution

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{dy}{\sqrt{1-y^2}}$$

$$\sin^{-1} x + C = \sin^{-1} y \quad y(0) = \frac{\sqrt{3}}{2}$$

$$\sin^{-1} 0 + C = \sin^{-1} \frac{\sqrt{3}}{2} \Rightarrow \underline{C = \frac{\pi}{3}}$$

$$\sin^{-1} y = \sin^{-1} x + \frac{\pi}{3}$$

$$y = \sin\left(\sin^{-1} x + \frac{\pi}{3}\right)$$

$$= \sin\left(\sin^{-1} x\right)\cos\frac{\pi}{3} + \cos\left(\sin^{-1} x\right)\sin\frac{\pi}{3}$$

$$\underline{y(x) = \frac{x}{2} + \frac{\sqrt{3}}{2}\sqrt{1-x^2}}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a}$$

$$\alpha = \sin^{-1} x \rightarrow \sin \alpha = x \quad \cos \alpha = \sqrt{1-\sin^2 \alpha} = \sqrt{1-x^2}$$

Exercise

Find the exact solution of the initial value problem. $(1+x^4)dy + x(1+4y^2)dx = 0, \quad y(1) = 0$

Solution

$$\int \frac{1}{1+(2y)^2} dy = - \int \frac{x}{1+(x^2)^2} dx$$

$$\int \frac{1}{1+(2y)^2} dy = -\frac{1}{2} \int \frac{1}{1+(x^2)^2} d(x^2)$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$\frac{1}{2} \tan^{-1} 2y = -\frac{1}{2} \tan^{-1} x^2 + C$$

$$\tan^{-1} 2y + \tan^{-1} x^2 = C_1 \quad y(1) = 0 \quad \tan^{-1} 0 + \tan^{-1} 1 = C_1 \Rightarrow \underline{C_1 = \frac{\pi}{4}}$$

$$\underline{\tan^{-1} 2y + \tan^{-1} x^2 = \frac{\pi}{4}}$$

$$2y = \tan\left(\frac{\pi}{4} - \tan^{-1} x^2\right)$$

$$= \frac{\tan \frac{\pi}{4} - \tan\left(\tan^{-1} x^2\right)}{1 + \tan\left(\frac{\pi}{4}\right)\tan\left(\tan^{-1} x^2\right)}$$

$$\underline{y(x) = \frac{1}{2} \frac{1-x^2}{1+x^2}}$$

Exercise

Find the exact solution of the initial value problem. $e^{-2t} \frac{dy}{dt} = \frac{1+e^{-2t}}{y}, \quad y(0) = 0$

Solution

$$y dy = (1 + e^{-2t}) e^{2t} dt$$

$$\int y dy = \int (e^{2t} + 1) dt$$

$$\frac{1}{2} y^2 = \frac{1}{2} e^{2t} + t + C_1$$

$$y^2 = e^{2t} + 2t + C \quad y(0) = 0$$

$$0 = 1 + C \rightarrow \underline{C = -1}$$

$$\underline{y^2 = e^{2t} + 2t - 1}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt} = \frac{t+2}{y}, \quad y(0) = 2$

Solution

$$\int y dy = \int (t + 2) dt$$

$$\frac{1}{2} y^2 = \frac{1}{2} t^2 + 2t + C_1$$

$$y^2 = t^2 + 4t + C \quad y(0) = 2 \quad \underline{4 = C}$$

$$\underline{y = \sqrt{t^2 + 4t + 4}}$$

Exercise

Find the exact solution of the initial value problem. $\frac{1}{t^2} \frac{dy}{dt} = y, \quad y(0) = 1$

Solution

$$\int \frac{1}{y} dy = \int t^2 dt$$

$$\ln|y| = \frac{1}{3} t^3 + C \quad y(0) = 1$$

$$\ln|1| = C \rightarrow \underline{C = 0}$$

$$\ln|y| = \frac{1}{3} t^3$$

$$\underline{y(t) = e^{t^3/3}}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt} = -y^2 e^{2t}; \quad y(0) = 1$

Solution

$$-\int \frac{1}{y^2} dy = \int e^{2t} dt$$

$$\frac{1}{y} = \frac{1}{2} e^{2t} + C \quad y(0) = 1 \quad 1 = \frac{1}{2} + C \rightarrow C = \frac{1}{2}$$

$$\frac{1}{y} = \frac{1}{2} (e^{2t} + 1)$$

$$y(t) = \frac{2}{e^{2t} + 1}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt} - (2t + 1)y = 0; \quad y(0) = 2$

Solution

$$\frac{dy}{dt} = (2t + 1)y$$

$$\int \frac{dy}{y} = \int (2t + 1) dt$$

$$\ln|y| = t^2 + t + C \quad y(0) = 2$$

$$\ln 2 = C$$

$$\ln|y| = t^2 + t + \ln 2$$

$$y(t) = e^{\ln 2} e^{t^2 + t}$$

$$= 2e^{t^2 + t}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt} + 4ty^2 = 0; \quad y(0) = 1$

Solution

$$-\int \frac{dy}{y^2} = \int 4t dt$$

$$\frac{1}{y} = 2t^2 + C \quad y(0) = 1 \quad 1 = C$$

$$\frac{1}{y} = 2t^2 + 1$$

$$\underline{y(t) = \frac{1}{2t^2 + 1}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = ye^x; \quad y(0) = 2e$

Solution

$$\int \frac{dy}{y} = \int e^x dx$$

$$\ln|y| = e^x + \ln C$$

$$y(0) = 2e \rightarrow \ln|2e| = 1 + \ln C$$

$$\ln 2 + 1 = 1 + \ln C \Rightarrow \underline{C = 2}$$

$$\ln|y| = e^x + \ln 2$$

$$y(x) = e^{e^x + \ln 2}$$

$$= e^{e^x} e^{\ln 2}$$

$$\underline{= 2e^{e^x}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 3x^2(y^2 + 1); \quad y(0) = 1$

Solution

$$\int \frac{1}{y^2 + 1} dy = \int 3x^2 dx$$

$$\arctan y = x^3 + C$$

$$y(0) = 1 \rightarrow \arctan 1 = C \Rightarrow \underline{C = \frac{\pi}{4}}$$

$$\underline{y(x) = \tan\left(x^3 + \frac{\pi}{4}\right)}$$

Exercise

Find the exact solution of the initial value problem $2y \frac{dy}{dx} = \frac{x}{\sqrt{x^2 - 16}}; \quad y(5) = 2$

Solution

$$\int 2y dy = \int \frac{x}{\sqrt{x^2 - 16}} dx$$

$$y^2 = \frac{1}{2} \int (x^2 - 16)^{-1/2} d(x^2 - 16)$$

$$y^2 = (x^2 - 16)^{1/2} + C$$

$$y(5) = 2 \rightarrow 4 = (9)^{1/2} + C \Rightarrow C = 4 - 3 = \underline{1}$$

$$\underline{y^2 = 1 + \sqrt{x^2 - 16}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 4x^3y - y; \quad y(1) = -3$

Solution

$$\frac{dy}{dx} = (4x^3 - 1)y$$

$$\int \frac{dy}{y} = \int (4x^3 - 1) dx$$

$$\ln|y| = x^4 - x + C$$

$$y = Ce^{x^4 - x}$$

$$y(1) = -3 \quad \underline{-3 = C}$$

$$\underline{y(x) = -3e^{x^4 - x}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} + 1 = 2y; \quad y(1) = 1$

Solution

$$\int \frac{dy}{2y - 1} = \int dx$$

$$\frac{1}{2} \ln(2y - 1) = x + C$$

$$\ln(2y - 1) = 2x + C$$

$$2y - 1 = e^{2x + C}$$

$$y(x) = Ae^{2x} + \frac{1}{2}$$

$$y(1) = 1 \quad 1 = Ae^2 + \frac{1}{2} \rightarrow \underline{A = e^{-2}}$$

$$\underline{y(x) = e^{2x-2} + \frac{1}{2}}$$

Exercise

Find the exact solution of the initial value problem $(\tan x) \frac{dy}{dx} = y; \quad y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

Solution

$$\int \frac{dy}{y} = \int \frac{dx}{\tan x} = \int \frac{\cos x dx}{\sin x}$$

$$\ln y = \ln(\sin x) + \ln C$$

$$y(x) = C \sin x$$

$$y\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \rightarrow \frac{\pi}{2} = C$$

$$y(x) = \frac{\pi}{2} \sin x$$

Exercise

Find the exact solution of the initial value problem $x \frac{dy}{dx} - y = 2x^2 y; \quad y(1) = 1$

Solution

$$x \frac{dy}{dx} = 2x^2 y + y$$

$$x \frac{dy}{dx} = (2x^2 + 1)y$$

$$\int \frac{dy}{y} = \int \left(2x + \frac{1}{x}\right) dx$$

$$\ln y = x^2 + \ln x + \ln C$$

$$y(x) = e^{x^2 + \ln x + \ln C}$$

$$= Cxe^{x^2}$$

$$y(1) = 1 \rightarrow 1 = Ce \Rightarrow C = e^{-1}$$

$$y(x) = xe^{x^2-1}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 2xy^2 + 3x^2 y^2; \quad y(1) = -1$

Solution

$$\frac{dy}{dx} = (2x + 3x^2)y^2$$

$$\int \frac{dy}{y^2} = \int (2x + 3x^2) dx$$

$$-\frac{1}{y} = x^2 + x^3 + C$$

$$y(x) = \frac{-1}{x^2 + x^3 + C}$$

$$y(1) = -1 \rightarrow -1 = \frac{-1}{C} \Rightarrow \underline{C=1}$$

$$\underline{y(x) = \frac{-1}{x^2 + x^3 + 1}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 6e^{2x-y}; \quad y(0) = 0$

Solution

$$\int e^y dy = \int 6e^{2x} dx$$

$$e^y = 3e^{2x} + C$$

$$y(x) = \ln(3e^{2x} + C)$$

$$y(0) = 0 \rightarrow 0 = \ln(3 + C) \Rightarrow 3 + C = 1 \rightarrow \underline{C = -2}$$

$$\underline{y(x) = \ln(3e^{2x} - 2)}$$

Exercise

Find the exact solution of the initial value problem $2\sqrt{x} \frac{dy}{dx} = \cos^2 y; \quad y(4) = \frac{\pi}{4}$

Solution

$$\frac{dy}{\cos^2 y} = \frac{1}{2} x^{-1/2} dx$$

$$\int \sec^2 y dy = \int \frac{1}{2} x^{-1/2} dx$$

$$\tan y = \sqrt{x} + C$$

$$y(x) = \tan^{-1}(\sqrt{x} + C)$$

$$y(4) = \frac{\pi}{4} \rightarrow \frac{\pi}{4} = \arctan(2 + C) \rightarrow 2 + C = 1 \Rightarrow \underline{C = -1}$$

$$\underline{y(x) = \tan^{-1}(\sqrt{x} - 1)}$$

Exercise

Find the exact solution of the initial value problem $y' + 3y = 0$; $y(0) = -3$

Solution

$$\frac{dy}{dx} = -3y$$

$$\int \frac{dy}{y} = -3 \int dx$$

$$\ln|y| = -3x + C$$

$$y(x) = e^{-3x+C}$$

$$= Ae^{-3x}$$

$$y(0) = -3 \rightarrow -3 = A$$

$$y(x) = -3e^{-3x}$$

Exercise

Find the exact solution of the initial value problem $2y' - y = 0$; $y(-1) = 2$

Solution

$$2 \frac{dy}{dx} = y$$

$$\int \frac{dy}{y} = \frac{1}{2} \int dx$$

$$\ln|y| = \frac{1}{2}x + C$$

$$y(x) = e^{x/2+C}$$

$$= Ae^{x/2}$$

$$y(-1) = 2 \rightarrow 2 = Ae^{-1/2} \Rightarrow A = 2e^{1/2}$$

$$y(x) = 2e^{1/2}e^{x/2} = 2e^{(x+1)/2}$$

Exercise

Find the exact solution of the initial value problem $2xy - y' = 0$; $y(1) = 3$

Solution

$$\frac{dy}{dx} = 2xy$$

$$\int \frac{dy}{y} = \int 2x dx$$

$$\ln|y| = x^2 + C$$

$$y(x) = e^{x^2 + C}$$

$$= Ae^{x^2}$$

$$y(1) = 3 \rightarrow 3 = Ae \Rightarrow \underline{A = 3e^{-1}}$$

$$\underline{y(x) = \frac{3}{e} e^{x^2}}$$

Exercise

Find the exact solution of the initial value problem $y \frac{dy}{dx} - \sin x = 0; \quad y\left(\frac{\pi}{2}\right) = -2$

Solution

$$\int y \, dy = \int \sin x \, dx$$

$$\frac{1}{2} y^2 = -\cos x + C$$

$$y\left(\frac{\pi}{2}\right) = -2 \rightarrow \underline{2 = C}$$

$$\frac{1}{2} y^2 = -\cos x + 2$$

$$y^2 = 4 - 2\cos x$$

$$\underline{y(x) = -\sqrt{4 - 2\cos x}} \quad \text{Initial value is negative}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dt} = \frac{1}{y^2}; \quad y(1) = 2$

Solution

$$\int y^2 \, dy = \int dt$$

$$\frac{1}{3} y^3 = t + C$$

$$y(1) = 2 \rightarrow \frac{8}{3} = 1 + C \Rightarrow \underline{C = \frac{5}{3}}$$

$$\frac{1}{3} y^3 = t + \frac{5}{3}$$

$$y^3 = 3t + 5$$

$$\underline{y(t) = (3t + 5)^{1/3}}$$

Exercise

Find the exact solution of the initial value problem $y' + \frac{1}{y+1} = 0$; $y(1) = 0$

Solution

$$\frac{dy}{dx} = -\frac{1}{y+1}$$

$$\int (y+1) dy = - \int dx$$

$$\frac{1}{2} y^2 + y = -x + C$$

$$y(1) = 0 \rightarrow C = 1$$

$$\frac{1}{2} y^2 + y = -x + 1$$

$$y^2 + 2y + 2(x-1) = 0 \rightarrow y = \frac{-2 \pm 2\sqrt{1-2x+2}}{2}$$

$$\underline{y(x) = -1 + \sqrt{3-2x}} \quad (\text{initial condition + sign})$$

Exercise

Find the exact solution of the initial value problem $y' + e^y t = e^y \sin t$; $y(0) = 0$

Solution

$$\frac{dy}{dt} = (-t + \sin t) e^y$$

$$\int e^{-y} dy = \int (-t + \sin t) dt$$

$$-e^{-y} = -\frac{1}{2} t^2 - \cos t + C$$

$$e^{-y} = \frac{1}{2} t^2 + \cos t + C$$

$$y(0) = 0 \rightarrow 1 = 1 + C \Rightarrow C = 0$$

$$e^{-y} = \frac{1}{2} t^2 + \cos t$$

$$-y = \ln\left(\frac{1}{2} t^2 + \cos t\right)$$

$$\underline{y(x) = -\ln\left(\frac{1}{2} t^2 + \cos t\right)}$$

Exercise

Find the exact solution of the initial value problem $y' - 2ty^2 = 0$; $y(0) = -1$

Solution

$$\frac{dy}{dt} = 2ty^2$$

$$\int \frac{dy}{y^2} = \int 2t \, dt$$

$$-\frac{1}{y} = t^2 + C$$

$$y(0) = -1 \Rightarrow \underline{C = 1}$$

$$-\frac{1}{y} = t^2 + 1$$

$$\underline{y(x) = \frac{-1}{t+1}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 1 + y^2$; $y\left(\frac{\pi}{4}\right) = -1$

Solution

$$\int \frac{dy}{1+y^2} = \int dx$$

$$\tan^{-1} y = x + C$$

$$y\left(\frac{\pi}{4}\right) = -1 \rightarrow -\frac{\pi}{4} = \frac{\pi}{4} + C \Rightarrow \underline{C = -\frac{\pi}{2}}$$

$$\tan^{-1} y = x - \frac{\pi}{2}$$

$$\underline{y(x) = \tan\left(x - \frac{\pi}{2}\right)}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dt} = t - ty^2$; $y(0) = \frac{1}{2}$

Solution

$$\frac{dy}{dt} = t(1 - y^2)$$

$$\int \frac{dy}{1-y^2} = \int t \, dt$$

$$\frac{1}{1-y^2} = \frac{A}{1-y} + \frac{B}{1+y}$$

$$1 = A + Ay + B - By$$

$$\begin{cases} A - B = 0 \\ A + B = 1 \end{cases} \rightarrow \underline{A = \frac{1}{2} = B}$$

$$\int \left(\frac{1}{2} \frac{1}{1-y} + \frac{1}{2} \frac{1}{1+y} \right) dy = \int t \, dt$$

$$-\frac{1}{2}\ln|1-y| + \frac{1}{2}\ln|1+y| = \frac{1}{2}t + C$$

$$\ln|1+y| - \ln|1-y| = t + C$$

$$\ln\left|\frac{1+y}{1-y}\right| = t + C$$

$$\frac{1+y}{1-y} = Ae^t$$

$$y(0) = \frac{1}{2} \rightarrow \frac{\frac{3}{2}}{\frac{1}{2}} = A \Rightarrow \underline{A=3}$$

$$\frac{1+y}{1-y} = 3e^t$$

$$1+y = 3e^t - 3ye^t$$

$$y(1+3e^t) = 3e^t - 1$$

$$\underline{y(x) = \frac{3e^t - 1}{1 + 3e^t}}$$

Exercise

Find the exact solution of the initial value problem $3y^2 \frac{dy}{dt} + 2t = 1; \quad y(-1) = -1$

Solution

$$\int 3y^2 dy = \int (1-2t) dt$$

$$y^3 = t - t^2 + C$$

$$y(-1) = -1 \rightarrow -1 = -1 - 1 + C \Rightarrow \underline{C=1}$$

$$y^3 = t - t^2 + 1$$

$$\underline{y(t) = (t - t^2 + 1)^{1/3}}$$

Exercise

Find the exact solution of the initial value problem $e^x y' + (\cos y)^2 = 0; \quad y(0) = \frac{\pi}{4}$

Solution

$$e^x \frac{dy}{dx} = -(\cos y)^2$$

$$\int \sec^2 y \, dy = -\int e^{-x} dx$$

$$\tan y = e^{-x} + C$$

$$y(0) = \frac{\pi}{4} \rightarrow 1 = 1 + C \Rightarrow \underline{C = 0}$$

$$\tan y = e^{-x}$$

$$\underline{y(x) = \arctan(e^{-x})}$$

Exercise

Find the exact solution of the initial value problem $(2y - \sin y)y' + x = \sin x; \quad y(0) = 0$

Solution

$$(2y - \sin y) \frac{dy}{dx} = -x + \sin x$$

$$\int (2y - \sin y) dy = \int (-x + \sin x) dx$$

$$y^2 + \cos y = -\frac{1}{2}x^2 - \cos x + C$$

$$y(0) = 0 \rightarrow 1 = -1 + C \Rightarrow \underline{C = 2}$$

$$\underline{y^2 + \cos y = -\frac{1}{2}x^2 - \cos x + 2}$$

Exercise

Find the exact solution of the initial value problem $e^y y' + \frac{x}{y+1} = \frac{2}{y+1}; \quad y(1) = 2$

Solution

$$e^y \frac{dy}{dx} = \frac{2-2x}{y+1}$$

$$\int (y+1)e^y dy = \int (2-2x) dx$$

$$ye^y = 2x - x^2 + C$$

$$y(1) = 2 \rightarrow 2e^2 = 2 - 1 + C \Rightarrow \underline{C = 2e^2 - 1}$$

$$\underline{ye^y = 2x - x^2 + 2e^2 - 1}$$

Exercise

Find the exact solution of the initial value problem $(\ln y)y' + x = 1; \quad y(3) = e$

Solution

$$(\ln y) \frac{dy}{dx} = 1 - x$$

$$\int (\ln y) dy = \int (1-x) dx \quad \begin{array}{ll} u = \ln y & dv = dy \\ du = \frac{1}{y} dy & v = y \end{array}$$

$$y \ln y - \int dy = x - \frac{1}{2} x^2 + C$$

$$y \ln y - y = x - \frac{1}{2} x^2 + C$$

$$y(3) = e \rightarrow e \ln e = 3 - \frac{9}{2} + C \Rightarrow C = e + \frac{3}{2}$$

$$\underline{y \ln y - y = x - \frac{1}{2} x^2 + e + \frac{3}{2}}$$

Exercise

Find the exact solution of the initial value problem $y' = x^3(1-y); \quad y(0) = 3$

Solution

$$\int \frac{dy}{1-y} = \int x^3 dx$$

$$-\ln|1-y| = \frac{1}{4} x^4 + C_1$$

$$\ln|1-y| = -\frac{1}{4} x^4 + C$$

$$y(0) = 3 \rightarrow \ln 2 = C$$

$$1-y = e^{-\frac{1}{4} x^4 + C}$$

$$y = 1 - e^{\ln 2 - \frac{1}{4} x^4}$$

$$\underline{y(x) = 1 - 2e^{-x^4/4}}$$

Exercise

Find the exact solution of the initial value problem $y' = (1+y^2) \tan x; \quad y(0) = \sqrt{3}$

Solution

$$\int \frac{1}{1+y^2} dy = \int \tan x dx$$

$$\tan^{-1} y = \ln|\sec x| + C$$

$$y(0) = \sqrt{3} \rightarrow \frac{\pi}{3} = \ln 1 + C \Rightarrow C = \frac{\pi}{3}$$

$$\underline{y(x) = \tan\left(\ln|\sec x| + \frac{\pi}{3}\right)}$$

Exercise

Find the exact solution of the initial value problem $\frac{1}{2} \frac{dy}{dx} = \sqrt{1+y} \cos x; \quad y(\pi) = 0$

Solution

$$\frac{1}{2} \int (1+y)^{-1/2} dy = \int \cos x dx$$

$$\sqrt{1+y} = \sin x + C$$

$$y(\pi) = 0 \quad \underline{1 = C}$$

$$\sqrt{1+y} = \sin x + 1$$

$$\underline{y(x) = (\sin x + 1)^2 - 1}$$

Exercise

Find the exact solution of the initial value problem $x^2 \frac{dy}{dx} = \frac{4x^2 - x - 2}{(x+1)(y+1)}; \quad y(1) = 1$

Solution

$$(y+1)dy = \frac{4x^2 - x - 2}{x^2(x+1)} dx$$

$$\frac{4x^2 - x - 2}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$4x^2 - x - 2 = Ax^2 + Ax + Bx + B + Cx^2$$

$$\begin{cases} x^2 & A + C = 4 \quad \underline{C = 3} \\ x & A + B = -1 \quad \underline{A = 1} \\ x^0 & B = -2 \end{cases}$$

$$\int (y+1)dy = \int \frac{dx}{x} - \int \frac{2}{x^2} dx + 3 \int \frac{dx}{x+1}$$

$$\frac{1}{2} y^2 + y = \ln|x| + \frac{2}{x} + 3 \ln|x+1| + C$$

$$y(1) = 1 \rightarrow \frac{1}{2} + 1 = 2 + 3 \ln 2 + C \Rightarrow \underline{C = -\frac{1}{2} - \ln 8}$$

$$\underline{\frac{1}{2} y^2 + y = \ln|x| + \frac{2}{x} + 3 \ln|x+1| - \frac{1}{2} - \ln 8}$$

Exercise

Find the exact solution of the initial value problem $\frac{1}{\theta} \frac{dy}{d\theta} = \frac{y \sin \theta}{y^2 + 1} \quad y(\pi) = 1$

Solution

$$\int \frac{y^2+1}{y} dy = \int \theta \sin \theta d\theta$$

$$\int \left(y + \frac{1}{y} \right) dy = -\theta \cos \theta + \sin \theta + C$$

$$\frac{1}{2} y^2 + \ln|y| = -\theta \cos \theta + \sin \theta + C$$

$$y(\pi) = 1 \quad \frac{1}{2} = \pi + C \Rightarrow C = \frac{1}{2} - \pi$$

$$\frac{1}{2} y^2 + \ln|y| = -\theta \cos \theta + \sin \theta + \frac{1}{2} - \pi$$

Exercise

Find the exact solution of the initial value problem $x^2 dx + 2y dy = 0$; $y(0) = 2$

Solution

$$\int 2y dy = - \int x^2 dx$$

$$y^2 = -\frac{1}{3} x^3 + C$$

$$y(0) = 2 \rightarrow 4 = C$$

$$y^2 = -\frac{1}{3} x^3 + 4$$

Exercise

Find the exact solution of the initial value problem $\frac{1}{t} \frac{dy}{dt} = 2 \cos^2 y$; $y(0) = \frac{\pi}{4}$

Solution

$$\int \sec^2 y dy = \int 2t dt$$

$$\tan y = t^2 + C$$

$$y(0) = \frac{\pi}{4} \rightarrow 1 = C$$

$$y(t) = \tan^{-1}(t^2 + 1)$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 8x^3 e^{-2y}$; $y(1) = 0$

Solution

$$\int e^{2y} dy = \int 8x^3 dx$$

$$\frac{1}{2}e^{2y} = 2x^4 + C$$

$$y(1) = 0 \rightarrow \underline{\frac{1}{2} = C}$$

$$\frac{1}{2}e^{2y} = 2x^4 + \frac{1}{2}$$

$$e^{2y} = 4x^4 + 1$$

$$\underline{y(x) = \frac{1}{2} \ln(4x^4 + 1)}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = x^2(1+y); \quad y(0) = 3$

Solution

$$\int \frac{1}{1+y} dy = \int x^2 dx$$

$$\ln|1+y| = \frac{1}{3}x^3 + C$$

$$y(0) = 3 \rightarrow \underline{\ln 4 = C}$$

$$1+y = e^{\frac{1}{3}x^3 + \ln 4}$$

$$\underline{y(x) = 4e^{x^3/3} - 1}$$

Exercise

Find the exact solution of the initial value problem $\sqrt{y}dx + (1+x)dy = 0; \quad y(0) = 1$

Solution

$$\int y^{-1/2} dy = - \int \frac{1}{x+1} dx$$

$$2\sqrt{y} = -\ln|x+1| + C$$

$$y(0) = 1 \rightarrow \underline{2 = C}$$

$$\underline{2\sqrt{y} = -\ln|x+1| + 2}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 6y^2x$, $y(1) = \frac{1}{25}$

Solution

$$\int \frac{dy}{y^2} = \int 6x dx$$

$$-\frac{1}{y} = 3x^2 + C \quad y(1) = \frac{1}{25}$$

$$-25 = 3 + C \rightarrow \underline{C = -28}$$

$$-\frac{1}{y} = 3x^2 - 28$$

$$\underline{y(x) = \frac{1}{28 - 3x^2}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = \frac{3x^2 + 4x - 4}{2y - 4}$, $y(1) = 3$

Solution

$$\int (2y - 4) dy = \int (3x^2 + 4x - 4) dx$$

$$y^2 - 4y = x^3 + 2x^2 - 4x + C \quad y(1) = 3$$

$$9 - 12 = 1 + 2 - 4 + C \rightarrow \underline{C = -2}$$

$$\underline{y^2 - 4y = x^3 + 2x^2 - 4x - 2}$$

Exercise

Find the exact solution of the initial value problem $y' = e^{-y}(2x - 4)$ $y(5) = 0$

Solution

$$\int e^y dy = \int (2x - 4) dx$$

$$e^y = x^2 - 4x + C \quad y(5) = 0$$

$$e^0 = 25 - 20 + C \rightarrow \underline{C = -4}$$

$$e^y = x^2 - 4x - 4$$

$$\underline{y(x) = \ln|x^2 - 4x - 4|}$$

Exercise

Find the exact solution of the initial value problem $\frac{dr}{d\theta} = \frac{r^2}{\theta}, \quad r(1) = 2$

Solution

$$\begin{aligned} \int \frac{dr}{r^2} &= \int \frac{d\theta}{\theta} \\ -\frac{1}{r} &= \ln|\theta| + C & r(1) = 2 \\ -\frac{1}{2} &= C \\ -\frac{1}{r} &= \ln|\theta| - \frac{1}{2} \\ \frac{1}{r} &= \frac{1 - 2\ln|\theta|}{2} \\ r(\theta) &= \frac{2}{1 - 2\ln|\theta|} \end{aligned}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dt} = e^{y-t} (1+t^2) \sec y, \quad y(0) = 0$

Solution

$$\begin{aligned} \frac{dy}{dt} &= e^{-t} (1+t^2) e^y \sec y \\ \int (e^{-y} \cos y) dy &= \int (1+t^2) e^{-t} dt \\ \int (e^{-y} \cos y) dy &= e^{-y} (\sin y - \cos y) - \int (e^{-y} \cos y) dy \\ 2 \int (e^{-y} \cos y) dy &= e^{-y} (\sin y - \cos y) \\ \int (e^{-y} \cos y) dy &= \frac{1}{2} e^{-y} (\sin y - \cos y) \\ \int (1+t^2) e^{-t} dt &= e^{-t} (-1 - t^2 - 2t - 2) \\ \frac{1}{2} e^{-y} (\sin y - \cos y) &= -e^{-t} (t^2 + 2t + 3) + C & y(0) = 0 \\ -\frac{1}{2} &= -3 + C \rightarrow C = \frac{5}{2} \end{aligned}$$

		$\int \cos y$		$\int e^{-t}$
+	e^{-y}	$\sin y$	$1+t^2$	$-e^{-t}$
-	$-e^{-y}$	$-\cos y$	$2t$	e^{-t}
+	e^{-y}		2	$-e^{-t}$

$$\frac{1}{2} e^{-y} (\sin y - \cos y) = -e^{-t} (t^2 + 2t + 3) + \frac{5}{2}$$