Lecture Four – Series

Section 4.1 – Introduction and Review of Power Series

4.1-1 Definition

A ***power series*** *about the point* x_0 is a series of the form

$$\sum_{n=0}^{\infty} a_n\left(x-x_0\right)^n = a_0 + a_1\left(x-x_0\right) + a_2\left(x-x_0\right)^2 + a_3\left(x-x_0\right)^3 + \cdots$$

The series is said to converge at x if the sequence of partial sums

$$\begin{aligned} S_N(x) &= \sum_{n=0}^{N} a_n\left(x-x_0\right)^n \\ &= a_0 + a_1\left(x-x_0\right) + a_2\left(x-x_0\right)^2 + \cdots + a_N\left(x-x_0\right)^N \end{aligned}$$

Converges as $N \to \infty$. The sum of the series at the point x is defined to be the limit at the partial sums,

$$\sum_{n=0}^{N} a_n\left(x-x_0\right)^n = \lim_{N\to\infty} S_N(x)$$

***Example* 1**

Show that the geometric series $\sum_{n=0}^{\infty} x^n$ converges for $|x|<1$ and that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ *for* $|x|<1$

Show that the series diverges for $|x| \geq 1$.

Solution

The partial sums $S_N(x) = \sum_{n=0}^{\infty} x^n$ can be evaluates as follows.

$$(1-x)S_N(x) = (1-x)\left(1+x+x^2+\cdots+x^N\right)$$
$$= \left(1+x+x^2+\cdots+x^N\right) - \left(x+x^2+\cdots+x^N+x^{N+1}\right)$$
$$= 1-x^{N+1}$$

$$S_N(x) = \sum_{n=0}^{N} x^n$$
$$= \frac{1-x^{N+1}}{1-x}, \qquad x \neq 1$$

If $|x|<1$, then $x^{N+1} \to 0 \quad$ *as* $N \to \infty$

$$\Rightarrow S_N(x) \to \frac{1}{1-x}$$

If $|x|>1$, then x^{N+1} diverges and therefore the $S_N(x)$ *diverges*

If $|x|=1$, then $S_N(1) = N+1$

4.1-2 Radius of Convergence of a Power Series

Corollary to Theorem

The convergence of the series $\sum c_n (x-a)^n$ is described by one of the following three cases:

1. There is a positive number R such the series diverges for x with $|x-a|>R$ but converges absolutely for x with $|x-a|<R$. The series may or may not converge at either of the endpoints $x=a-R$ and $x=a+R$.
2. The series converges absolutely for every x $(R=\infty)$.
3. The series converges at $x = a$ and diverges elsewhere ($R = 0$)

R is called the ***radius of convergence*** of the power series, and the interval of radius R centered at $x = a$ is called the ***interval of convergence***.

4.1-3 Interval of convergence

Theorem

For any power series $\sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n$ there is an ***R***, either a nonnegative number or ∞, such that the series converges if $\left|x - x_0\right| < R$ and diverges if $\left|x - x_0\right| > R$

4.1-4 The ratio Test

Theorem

Suppose the terms of the series $\sum_{n=0}^{\infty} A_n$ have the property that

$$\lim_{n\to\infty} \frac{\left|A_{n+1}\right|}{\left|A_n\right|} = L$$

exists. If $L < 1$ the series converges, while if $L > 1$ the series diverges

4.1-5 *Definition*

Suppose that $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists or is ∞. Then the power series $\sum c_n (x-a)^n$ has radius of convergence $R = \frac{1}{L}$.

If $L = 0$, then $R = \infty$

If $L = \infty$, then $R = 0$

And $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

4.1-6 How to Test a Power Series for Convergence

1. Use the Ratio Test (or Root Test) to find the interval where the series converges. Ordinarily, this is an open interval

$$|x-a|<R \quad or \quad a-R<x<a+R$$

2. If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint. Use the Comparison Test, the Integral Test, or the Alternating Series Test.

3. If the interval of absolute convergence is $a-R<x<a+R$, the series diverges for $|x-a|>R$ (it does not even converge conditionally) because the nth term does not approach zero for those values of x.

Example 2

Find the radius of convergence for the series. $\sum_{n=0}^{\infty} \frac{2^n x^{2n}}{2n(n+1)}$

Solution

$$\frac{|A_{n+1}|}{|A_n|} = \frac{2^{n+1}x^{2(n+1)}}{2(n+1)(n+2)} \cdot \frac{2n(n+1)}{2^n x^{2n}}$$

$$= \frac{2n}{(n+2)}x^2$$

$$\lim_{n\to\infty} \frac{|A_{n+1}|}{|A_n|} = \lim_{n\to\infty} \frac{2n}{n+2}x^2$$

$$\to 2x^2$$

By the ratio test, the series converges if $2x^2<1$, so the radius of convergence is $R=\frac{1}{\sqrt{2}}$

$$x^2<\frac{1}{2} \qquad -\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$$

Example 3

Determine the centre, radius, and interval of convergence of $\sum_{n=0}^{\infty} \frac{(2x+5)^n}{(n^2+1)3^n}$

Solution

$$\sum_{n=0}^{\infty} \frac{(2x+5)^n}{\left(n^2+1\right)3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \frac{1}{n^2+1}\left(x+\frac{5}{2}\right)^n$$

The centre of convergence is

$$x+\frac{5}{2}=0 \quad \Rightarrow \quad \boxed{x=-\frac{5}{2}}$$

$$L = \lim_{n\to\infty} \left| \frac{\left(\frac{2}{3}\right)^{n+1} \frac{1}{(n+1)^2+1}}{\left(\frac{2}{3}\right)^n \frac{1}{n^2+1}} \right|$$

$$= \lim_{n\to\infty} \frac{2}{3}\frac{n^2+1}{(n+1)^2+1}$$

$$\boxed{=\frac{2}{3}}$$

$$R = \frac{1}{L}$$

$$\boxed{=\frac{3}{2}}$$

The series converges absolutely on ***interval***

$$\left(-\frac{5}{2}-\frac{3}{2},\ -\frac{5}{2}+\frac{3}{2}\right) = \boxed{(-4,\ -1)} \qquad a-R<x<a+R$$

It diverges on $(-\infty,\ -4)\cup(-1,\ \infty)$

At $x=-4 \quad \Rightarrow \quad \displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{\left(n^2+1\right)3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$

1. $n<n+1$

$n^2<(n+1)^2$

$n^2+1<(n+1)^2+1$

$\dfrac{1}{n^2+1}>\dfrac{1}{(n+1)^2+1}$

2. $\displaystyle\lim_{x\to\infty} \frac{1}{n^2+1}=0$

Therefore, the series converges by Alternating Series Test.

At $x=-1 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \frac{3^n}{\left(n^2+1\right)3^n} = \sum_{n=0}^{\infty} \frac{1}{n^2+1}$

$$\int_0^{\infty} \frac{dx}{x^2+1} = \arctan x \Big|_0^{\infty}$$

$$= \arctan\infty - \arctan 0$$

$$= \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2}$$

Therefore, the series converges by Integral Test.

Both series converge (absolutely).

Therefore, the interval of convergence of the given power is $[-4,\ -1]$

4.1-7 Algebraic Operations on Series

The ***sum*** and ***difference*** of two series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$\sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + \cdots$$

$$\sum_{n=0}^{\infty} a_n x^n \pm \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} \left(a_n \pm b_n\right) x^n$$

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{m=0}^{\infty} b_m x^m\right) = \sum_{p=0}^{\infty} c_p x^p \qquad c_p = \sum_{k=0}^{p} a_{p-k} b_k$$

4.1-8 Differentiating Power Series

Theorem

The function: $f(x) = \sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n$

$$= a_0 + a_1\left(x - x_0\right) + a_2\left(x - x_0\right)^2 + a_3\left(x - x_0\right)^3 + \cdots$$

Can be differentiating the series by terms

$$f'(x) = \frac{d}{dx}\left[a_0 + a_1\left(x - x_0\right) + a_2\left(x - x_0\right)^2 + a_3\left(x - x_0\right)^3 + \cdots\right]$$

$$= a_1 + 2a_2\left(x - x_0\right) + 3a_3\left(x - x_0\right)^2 + \cdots$$

$$= \sum_{n=1}^{\infty} n a_n \left(x - x_0\right)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n \left(x - x_0\right)^{n-2}$$

In general: $\boxed{f^{(n)}(x) = n! a_n}$ $\Rightarrow$ $a_n = \dfrac{f^{(n)}(a)}{n!}$

4.1-9 Identity Theorem

Suppose that the series $f(x) = \sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n$ has a positive radius of convergence. Then

$$a_n = \frac{f^{(n)}\left(x_0\right)}{n!}$$

The coefficients of a power series are determined by the values of the sum $f(x)$.

If f has a series representation, then the series must be

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \cdots$$

4.1-10 Taylor and Maclaurin Series

Definitions

Let f be a function with derivatives of all orders throughout some interval containing ***a*** as an interior point. Then the ***Taylor series generated by*** f at $x=a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots$$

The ***Maclaurin series generated by*** f is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots,$$

The Taylor series generated by f at $x=0$.

Example 4

Find the Taylor series and the Taylor polynomials generated by $f(x)=\cos x$ at $x=0$

Solution

$$f(x)=\cos x, \qquad f'(x)=-\sin x,$$
$$f''(x)=-\cos x, \qquad f'''(x)=\sin x,$$
$$\vdots \qquad \vdots$$
$$f^{(2n)}(x)=(-1)^n\cos x, \qquad f^{(2n+1)}(x)=(-1)^{2n+1}\sin x$$

$$f^{(2n)}(0)=(-1)^n, \qquad f^{(2n+1)}(0)=0$$

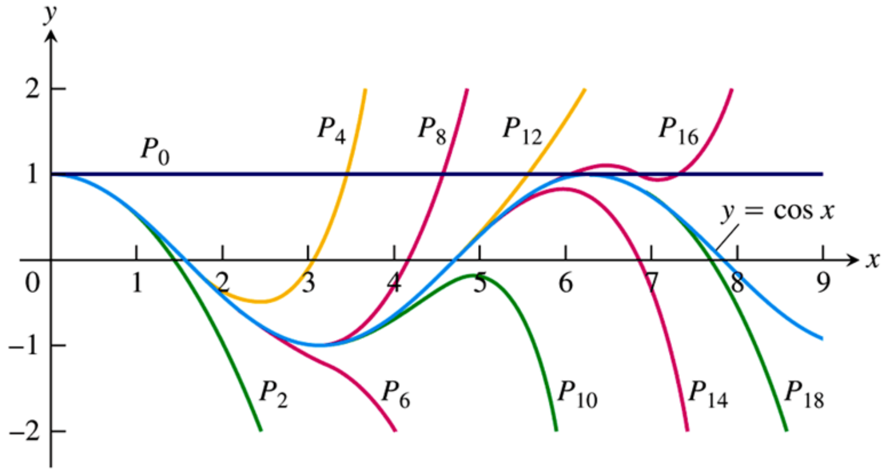
The Taylor series generated by f at $x=0$ is

$$\begin{aligned} f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots \\ &= 1+0\cdot x-\frac{x^2}{2!}+0\cdot x^3+\frac{x^4}{4!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots \\ &= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots \end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

$$P_{2n}(x) = P_{2n+1}(x)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}$$



Example 5

Find the Taylor series for $\ln x$ in powers of $x-2$. Where does the series converge to $\ln x$?

Solution

Let $t = \frac{x-2}{2}$, then

$$\ln x = \ln(x - 2 + 2)$$

$$= \ln\left[2\left(\frac{x-2}{2} + 1\right)\right]$$

$$= \ln 2 + \ln(t+1)$$

$f(t) = \ln(t+1)$ $\qquad f(0) = \ln(1) = 0$

$f'(t) = \frac{1}{1+t}$ $\qquad f'(0) = 1$

$f''(t) = \frac{-1}{(1+t)^2}$ $\qquad f''(0) = -1$

$f'''(t) = \frac{2}{(1+t)^3}$ $\qquad f'''(0) = 2$

$$f^{(4)}(t) = \frac{-6}{(1+t)^4} \qquad\qquad f'''(0) = -6$$

$$f^{(n)}(t) = (-1)^{n+1}\frac{(n-1)!}{(1+t)^n}$$

$$\ln(1+t) = f(0) + f'(0)t + \frac{f''(0)}{2!}t^2 + \frac{f'''(0)}{3!}t^3 + \frac{f^{(IV)}(0)}{4!}t^4 + \cdots$$

$$= t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots$$

$$\ln x = \ln 2 + \ln(1+t)$$

$$= \ln 2 + t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots$$

$$= \ln 2 + \frac{x-2}{2} - \frac{(x-2)^2}{2\times 2^2} + \frac{(x-2)^3}{3\times 2^3} - \frac{(x-2)^4}{4\times 2^4} + \cdots$$

$$= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\times 2^n}(x-2)^n$$

Since the series for $\ln(1+t)$ is valid for $-1 < t \le 1$, this series for $\ln x$ is valid for $-1 < \frac{x-2}{2} \le 1$

$-2 < x-2 \le 2 \quad \rightarrow \quad \boxed{0 < x \le 4}$

4.1-11 Integrating Power Series

Theorem

Suppose the power series $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges for $|x-x_0| < R, \quad R > 0$

$$\int f(x)dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-x_0)^{n+1}}{n+1}$$

Exercises *Section* 4.1 – Introduction and Review of Power Series

(1 – 6) Determine the *centre*, *radius*, and *interval of convergence* of each of the power series

1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{\sqrt{n+1}}$

2. $\displaystyle\sum_{n=0}^{\infty} 3n(x+1)^n$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} x^n$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^3}(4-x)^n$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n}$

6. $\displaystyle\sum_{n=0}^{\infty} \frac{1+5^n}{n!} x^n$

(7 – 21) Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a

7. $f(x)=e^{2x}, \quad a=0$

8. $f(x)=\sin x, \quad a=0$

9. $f(x)=\ln(1+x), \quad a=0$

10. $f(x)=\frac{1}{x+2}, \quad a=0$

11. $f(x)=\sqrt{1-x}, \quad a=0$

12. $f(x)=x^3, \quad a=1$

13. $f(x)=8\sqrt{x}, \quad a=1$

14. $f(x)=\sin x, \quad a=\frac{\pi}{4}$

15. $f(x)=\cos x, \quad a=\frac{\pi}{6}$

16. $f(x)=\sqrt{x}, \quad a=9$

17. $f(x)=\sqrt[3]{x}, \quad a=8$

18. $f(x)=\ln x, \quad a=e$

19. $f(x)=\sqrt[4]{x}, \quad a=8$

20. $f(x)=\tan^{-1}x+x^2+1, \quad a=1$

21. $f(x)=e^x, \quad a=\ln 2$

(22 – 33) Find the nth Maclaurin polynomial for the function

22. $f(x)=e^{4x}, \quad n=4$

23. $f(x)=e^{-x}, \quad n=5$

24. $f(x)=e^{-x/2}, \quad n=4$

25. $f(x)=e^{x/3}, \quad n=4$

26. $f(x)=\sin x, \quad n=5$

27. $f(x)=\cos \pi x, \quad n=4$

28. $f(x)=xe^x, \quad n=4$

29. $f(x)=x^2e^{-x}, \quad n=4$

30. $f(x)=\frac{1}{x+1}, \quad n=5$

31. $f(x)=\frac{x}{x+1}, \quad n=4$

32. $f(x)=\sec x, \quad n=2$

33. $f(x)=\tan x, \quad n=3$

(**34 – 37**) Finding Taylor and Maclaurin Series generated by f at $x = a$

34. $f(x) = x^3 - 2x + 4, \quad a = 2$

35. $f(x) = 2x^3 + x^2 + 3x - 8, \quad a = 1$

36. $f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2, \quad a = -1$

37. $f(x) = \cos\left(2x + \frac{\pi}{2}\right), \quad a = \frac{\pi}{4}$

(**38 – 53**) Find the Maclaurin series for

38. xe^x

39. $5\cos \pi x$

40. $\dfrac{x^2}{x+1}$

41. e^{3x+1}

42. $\cos\left(2x^3\right)$

43. $\cos(2x - \pi)$

44. $x^2 \sin\left(\frac{x}{3}\right)$

45. $\cos^2\left(\frac{x}{2}\right)$

46. $\sin x \cos x$

47. $\tan^{-1}\left(5x^2\right)$

48. $\ln\left(2 + x^2\right)$

49. $\dfrac{1+x^3}{1+x^2}$

50. $\ln \dfrac{1+x}{1-x}$

51. $\dfrac{e^{2x^2} - 1}{x^2}$

52. $\cosh x - \cos x$

53. $\sinh x - \sin x$

Section 4.2 – Series Solutions near Ordinary Points

In this section, we consider methods of solving second order linear equations when the coefficients are function of the independent variable.

The second order linear homogeneous equation is given by:

$$P(x)\frac{d^2y}{dt^2}+Q(x)\frac{dy}{dt}+R(x)y=0$$

4.2-1 Example of a First-Order Equation

Find the series solution for the differential equation $y'-2xy=0$

Solution

We look for a solution of the form: $y(x)=\sum_{n=0}^{\infty}a_nx^n$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}$$

$$y'-2xy=0$$

$$\sum_{n=1}^{\infty}na_nx^{n-1}-2x\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=1}^{\infty}na_nx^{n-1}-\sum_{n=0}^{\infty}2a_nx^{n+1}=0$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-\sum_{n=0}^{\infty}2a_nx^{n+1}=0$$

$$a_1+\sum_{n=0}^{\infty}(n+2)a_{n+2}x^{n+1}-\sum_{n=0}^{\infty}2a_nx^{n+1}=0$$

$$a_1+\sum_{n=0}^{\infty}\left[(n+2)a_{n+2}-2a_n\right]x^{n+1}=0$$

$$\begin{cases} \boxed{a_1 = 0} \\ (n+2)a_{n+2} - 2a_n = 0 \quad \Rightarrow \quad \boxed{a_{n+2} = \frac{2a_n}{n+2}} \end{cases}$$

$\Rightarrow$ *Let* $a_0 = y(0)$ $\qquad\qquad\qquad a_1 = 0$

$$a_2 = \frac{2a_0}{2} = y(0) \qquad\qquad a_3 = \frac{2a_1}{3} = 0$$

$$a_4 = \frac{2a_2}{4} = \frac{1}{2}y(0) \qquad\qquad a_5 = \frac{2a_3}{5} = 0$$

$$a_6 = \frac{2a_4}{6} = \frac{1}{6}y(0)$$

$$a_8 = \frac{2a_6}{8} = \frac{1}{2.3.4}y(0)$$

$$y(x) = \sum_{k=0}^{\infty} a_{2k}x^{2k}$$

$$\boxed{= y(0)\sum_{k=0}^{\infty} \frac{x^{2k}}{k!}}$$

4.2-2 *Example 2*

Find the general series solution to the equation

$$y'' + xy' + y = 0$$

Find the particular solution with $y(0) = 0$ and $y'(0) = 2$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$y'' + xy' + y = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + x\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + na_n + a_n\right]x^n = 0$$

$$(n+2)(n+1)a_{n+2} + (n+1)a_n = 0$$

$$(n+2)(n+1)a_{n+2} = -(n+1)a_n$$

$$\boxed{a_{n+2} = -\frac{1}{n+2}a_n}$$

$a_0 = y(0) = 0$ | $a_1 = y'(0) = 2$

$a_2 = -\frac{1}{2}a_0$ | $a_3 = -\frac{1}{3}a_1$

$a_4 = -\frac{1}{4}a_2 = \frac{1}{2\cdot 4}a_0$ | $a_5 = -\frac{1}{5}a_3 = \frac{1}{3\cdot 5}a_1$

$a_6 = -\frac{1}{6}a_4 = -\frac{1}{2\cdot 4\cdot 6}a_0$ | $a_7 = -\frac{1}{7}a_7 = -\frac{1}{3\cdot 5\cdot 7}a_1$

The general solution can be written as:

$$y(x) = a_0\left[1 - \frac{1}{2}x^2 + \frac{1}{2\cdot 4}x^4 - \frac{1}{2\cdot 4\cdot 6}x^6 + \cdots\right] + a_1\left[x - \frac{1}{3}x^3 + \frac{1}{3\cdot 5}x^5 - \frac{1}{3\cdot 5\cdot 7}x^7 + \cdots\right]$$

For the given initial $y(0) = 0$ and $y'(0) = 2$, the solution is:

$$\boxed{y(x) = 2\left(x - \frac{1}{3}x^3 + \frac{1}{3\cdot 5}x^5 - \frac{1}{3\cdot 5\cdot 7}x^7 + \cdots\right)}$$

Summary

Front	x^0	x^1	x^2
$y(x)$	$\sum_{n=0}^{\infty} a_n x^n$	ka_*	$ka_* + qa_* x$
$y'(x)$	$\sum_{n=0}^{\infty} (n-1)a_{n-1} x^n$	$\sum_{n=1}^{\infty} na_n x^{n-1}$	$ka_* + qa_* x$
$y''(x)$	$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$	$\sum_{n=1}^{\infty} n(n+1)a_{n+1} x^{n-1}$	$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$

Use the appropriate *Sigma* corresponds to x's in front of $y^{(n)}(x)$ to eliminate changing the first value of n to combine the *Sigma*.

Example

For $n=0$ & $n=1$ always the product $n(n-1)=0$

That implies to
$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} x^n$$

For $ka_* + qa_* x$, is the result that we need to take out of *Sigma* to combine all *Sigma* into single one with the same first value of n.

Exercises Section 4.2 – Series Solutions near Ordinary Points

(**1 – 56**) Find the series solution.

1. $y' = 3y$
2. $y' = 4y$
3. $y' = x^2 y$
4. $y' + 2xy = 0$
5. $(x-2)y' + y = 0$
6. $(2x-1)y' + 2y = 0$
7. $2(x-1)y' = 3y$
8. $(1+x)y' - y = 0$
9. $(2-x)y' + 2y = 0$
10. $(x-4)y' + y = 0$
11. $x^2 y' = y - x - 1$
12. $(x-3)y' + 2y = 0$
13. $xy' + y = 0$
14. $x^3 y' - 2y = 0$
15. $y'' = 4y$
16. $y'' = 9y$
17. $y'' + y = 0$
18. $y'' - y = 0$
19. $y'' + y = x$
20. $y'' - xy = 0$
21. $y'' + xy = 0$
22. $y'' + xy' + y = 0$
23. $y'' - xy' - y = 0$
24. $y'' + x^2 y = 0$
25. $y'' + k^2 x^2 y = 0$
26. $y'' + 3xy' + 3y = 0$
27. $y'' - 2xy' + y = 0$
28. $y'' - xy' + 2y = 0$
29. $y'' - xy' - x^2 y = 0$
30. $y'' + x^2 y' + xy = 0$
31. $y'' + x^2 y' + 2xy = 0$
32. $y'' - x^2 y' - 3xy = 0$
33. $y'' + 2xy' + 2y = 0$
34. $2y'' + xy' + y = 0$
35. $3y'' + xy' - 4y = 0$
36. $5y'' - 2xy' + 10y = 0$
37. $(x-1)y'' + y' = 0$
38. $(x+2)y'' + xy' - y = 0$
39. $y'' - (x+1)y = 0$
40. $y'' - (x+1)y' - y = 0$
41. $(x^2+1)y'' - 6y = 0$
42. $(x^2+2)y'' + 3xy' - y = 0$
43. $(x^2-1)y'' + xy' - y = 0$
44. $(x^2+1)y'' + xy' - y = 0$
45. $(x^2+1)y'' - xy' + y = 0$
46. $(1-x^2)y'' - 6xy' - 4y = 0$
47. $y'' + (x-1)^2 y' - 4(x-1)y = 0$
48. $(2-x^2)y'' - xy' + 16y = 0$
49. $(x^2+1)y'' + 6xy' + 4y = 0$
50. $(x^2-1)y'' - 6xy' + 12y = 0$
51. $(x^2-1)y'' + 8xy' + 12y = 0$
52. $(x^2-1)y'' + 4xy' + 2y = 0$
53. $(x^2+1)y'' - 4xy' + 6y = 0$

54. $\left(x^2+2\right)y''+4xy'+2y=0$

55. $\left(x^2-3\right)y''+2xy'=0$

56. $\left(x^2+3\right)y''-7xy'+16y=0$

(57 – 80) Find the series solution to the initial value problems

57. $y''+4y=0$; $y(0)=0,\quad y'(0)=3$

58. $y''+x^2y=0$; $y(0)=1,\quad y'(0)=0$

59. $y''-2xy'+8y=0$; $y(0)=3,\quad y'(0)=0$

60. $y''+y'-2y=0$; $y(0)=1,\quad y'(0)=-2$

61. $y''-2y'+y=0$; $y(0)=0,\quad y'(0)=1$

62. $y''+xy'+y=0\quad y(0)=1\quad y'(0)=0$

63. $y''-xy'-y=0\quad y(0)=2\quad y'(0)=1$

64. $y''-xy'-y=0\quad y(0)=1\quad y'(0)=0$

65. $y''+xy'-2y=0\quad y(0)=1\quad y'(0)=0$

66. $y''+(x-1)y'+y=0\quad y(1)=2\quad y'(1)=0$

67. $(x-1)y''-xy'+y=0$; $y(0)=-2,\quad y'(0)=6$

68. $(x+1)y''-(2-x)y'+y=0$; $y(0)=2,\quad y'(0)=-1$

69. $(1-x)y''+xy'-2y=0$; $y(0)=0,\quad y'(0)=1$

70. $\left(x^2+1\right)y''+2xy'=0$; $y(0)=0,\quad y'(0)=1$

71. $\left(2+x^2\right)y''-xy'+4y=0\quad y(0)=-1\quad y'(0)=3$

72. $\left(2-x^2\right)y''-xy'+4y=0\quad y(0)=1\quad y'(0)=0$

73. $\left(4-x^2\right)y''+2y=0\quad y(0)=0\quad y'(0)=1$

74. $\left(x^2-4\right)y''+3xy'+y=0$; $y(0)=4,\quad y'(0)=1$

75. $\left(x^2+1\right)y''+2xy'-2y=0$; $y(0)=0,\quad y'(0)=1$

76. $\left(x^2-1\right)y''+3xy'+xy=0$; $y(0)=4,\quad y'(0)=6$

77. $\left(2x-x^2\right)y''-6(x-1)y'-4y=0$; $y(1)=0,\quad y'(1)=1$

78. $\left(x^2-6x+10\right)y''-4(x-3)y'+6y=0$; $y(3)=2,\quad y'(3)=0$

79. $\left(4x^2+16x+17\right)y''-8y=0\ ;\quad y(-2)=1,\quad y'(-2)=0$

80. $\left(x^2+6x\right)y''+(3x+9)y'-3y=0\ ;\quad y(-3)=0,\quad y'(-3)=2$

(81 – 84) Find the series solution near the given value

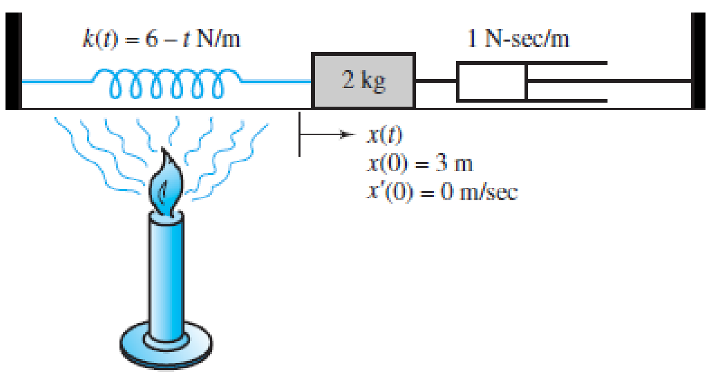
81. $y''-(x-2)y'+2y=0\ ;\quad near\ x=2$

82. $y''+(x-1)^2y'-4(x-1)y=0\ ;\quad near\ x=1$

83. $y''+(x-1)y=e^x\ ;\quad near\ x=1$

84. $y''+xy'+(2x-1)y=0\ ;\quad near\ x=-1\quad y(-1)=2,\quad y'(-1)=-2$

85. As a spring is heated, its spring "constant" decreases. Suppose the spring is heated so that the spring "constant" at time *t* is $k(t)=6-t\ N/m$.



If the unforced mass-spring system has mass $m=2\ kg$ and a damping constant $b=1\ N\text{-}sec/m$ with initial conditions $x(0)=3\ m$ and $x'(0)=0\ m/sec$, then the displacement $x(t)$ is governed by the initial value problem

$$2x''(t)+x'(t)+(6-t)x(t)=0\ ;\quad x(0)=3,\quad x'(0)=0$$

Find at least the first four nonzero terms in a power series expansion about $t=0$ for the displacement.

Section 4.3 – Legendre's Equation

Solving a homogeneous linear differential equation with constant coefficients can be reduced to the algebraic problem of finding the roots of its characteristic equation.

4.3-1 Legendre's Equation of order n

The Legendre's equation of order *n* is important in many applications. It has the form

$$\left(1-x^2\right)y''-2xy'+n(n+1)y=0$$

$$y''-\frac{2x}{1-x^2}y'+\frac{n(n+1)}{1-x^2}y=0$$

$$y''+P(x)y'+Q(x)y=0$$

Any solution of that equation is called a ***Legendre*** function.

Note that: $P(x)=-\frac{2x}{1-x^2}$ and $Q(x)=\frac{n(n+1)}{1-x^2}$ are analytic at $x=0$. *P* at $x=\pm 1$.

Hence Legendre's equation has power series solutions of the form

$$y=\sum_{m=0}^{\infty}a_m x^m$$

$$y'=\sum_{m=1}^{\infty}ma_m x^{m-1}$$

$$y''=\sum_{m=2}^{\infty}m(m-1)a_m x^{m-2}$$

$$\left(1-x^2\right)y''-2xy'+n(n+1)y=0$$

$$\left(1-x^2\right)\sum_{m=2}^{\infty}m(m-1)a_m x^{m-2}-2x\sum_{m=1}^{\infty}ma_m x^{m-1}+n(n+1)\sum_{m=0}^{\infty}a_m x^m=0$$

$$\sum_{m=2}^{\infty}m(m-1)a_m x^{m-2}-\sum_{m=2}^{\infty}m(m-1)a_m x^m-\sum_{m=1}^{\infty}2ma_m x^m+\sum_{m=0}^{\infty}n(n+1)a_m x^m=0$$

To obtain the same general power x^k, then we must set $m-2=k \quad \Rightarrow \quad m=k+2$

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k-\sum_{k=2}^{\infty}k(k-1)a_kx^k-\sum_{k=1}^{\infty}2ka_kx^k+\sum_{k=0}^{\infty}n(n+1)a_kx^k=0$$

$k=0$	$2\cdot 1\cdot a_2+n(n+1)a_0$
$k=1$	$3\cdot 2\cdot a_3+\left[-2+n(n+1)\right]a_1$
$k=2$	$4\cdot 3\cdot a_4+\left[-2-4+n(n+1)\right]a_2$
k	$(k+2)(k+1)a_{k+2}+\left[-k(k-1)-2k+n(n+1)\right]a_k$

$$(k+2)(k+1)a_{k+2}+\left[-k^2-k+n(n+1)\right]a_k=0$$

$$\begin{aligned}a_{k+2}&=-\frac{-k^2-k+n^2+n}{(k+2)(k+1)}a_k\\&=-\frac{(n-k)(n+k+1)}{(k+2)(k+1)}a_k\end{aligned}$$

This is called a ***recurrence relation*** or ***recursion formula***.

$a_2=-\frac{n(n+1)}{2!}a_0$	$a_3=-\frac{(n-1)(n+2)}{3!}a_1$
$a_4=-\frac{(n-2)(n+3)}{4\cdot 3}a_2$	$a_5=-\frac{(n-3)(n+4)}{5\cdot 4}a_3$
$=\frac{(n-2)n(n+1)(n+3)}{4!}a_0$	$=\frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_1$
$\vdots \quad \vdots$	$\vdots \quad \vdots$

The general Legendre equation solution is: $y(x)=a_0y_1(x)+a_1y_2(x)$

Where

$$\begin{cases}y_1(x)=1-\frac{n(n+1)}{2!}x^2+\frac{(n-2)n(n+1)(n+3)}{4!}x^4-\cdots\\ y_2(x)=x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^4-\cdots\end{cases}$$

4.3-2 Legendre Polynomials $P_n(x)$

For Legendre's equation $\left(1-x^2\right)y''-2xy'+n(n+1)y=0$ will happen when the parameter $\boldsymbol{n}$ is nonnegative integer. Otherwise, when n is even, $y_1(x)$ reduces to a polynomial of degree n. If n is odd, $y_2(x)$ reduces (the same) to a polynomial of degree n.

$$a_n=\frac{(2n)!}{2^n(n!)^2}=\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!} \qquad n\in\mathbb{Z}^+$$

If $n=0 \quad\Rightarrow\quad a_n=1$

From previous Proof (4.3-1):

$$a_{k+2}=-\frac{(n-k)(n+k+1)}{(k+2)(k+1)}a_k$$

Then, $a_k=-\dfrac{(k+2)(k+1)}{(n-k)(n+k+1)}a_{k+2} \qquad (k\le n-2)$

If $k=n-2$

$$\begin{aligned}
a_{n-2}&=-\frac{n(n-1)}{2(2n-1)}a_n\\
&=-\frac{n(n-1)(2n)!}{2(2n-1)2^n(n!)^2}\\
&=-\frac{n(n-1)(2n)(2n-1)(2n-2)!}{2(2n-1)2^n\left[n(n-1)!\right]\left[n(n-1)(n-2)!\right]}\\
&=\boxed{-\frac{(2n-2)!}{2^n(n-1)!(n-2)!}}
\end{aligned}$$

$$\begin{aligned}
a_{n-4}&=-\frac{(n-2)(n-3)}{4(2n-3)}a_{n-2}\\
&=\frac{(n-2)(n-3)}{4(2n-3)}\cdot\frac{(2n-2)!}{2^n(n-1)!(n-2)!}\\
&=\frac{(n-2)(n-3)}{4(2n-3)}\cdot\frac{(2n-2)(2n-3)(2n-4)!}{2^n(n-1)(n-2)(n-3)(n-4)!(n-2)!}\\
&=\frac{2(n-1)(2n-4)!}{4\cdot 2^n(n-1)(n-4)!(n-2)!}
\end{aligned}$$

$$= \frac{(2n-4)!}{2^n 2!(n-2)!(n-4)!}$$

In general; $a_{n-2k} = (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!}$

The resulting solution of Legendre's differential equation is called the Legendre polynomial of degree ***n*** and is denoted by $P_n(x)$.

$$P_n(x) = \sum_{k=0}^{K} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}$$

$$= \frac{(2n)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n 1!(n-1)!(n-2)!} x^{n-2} + \cdots$$

$P_0(x) = 1$

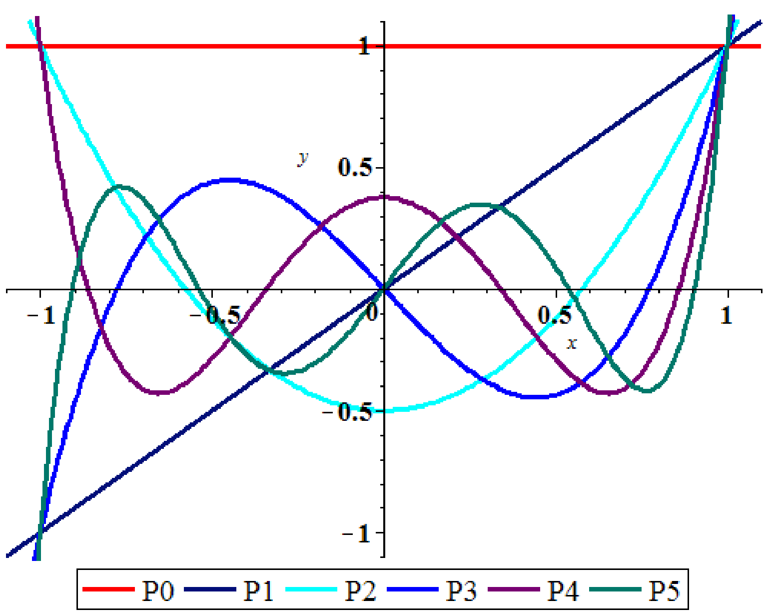
$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$P_5(x) = \frac{1}{8}(65x^5 - 70x^3 + 15x)$



Exercise Section 4.3 – Legendre's Equation

1. Establish the recursion formula using the following two steps

a) Differentiate both sides of equation

$$g(t,\ x)=\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n \quad \text{with respect to } t \text{ to show that}$$

$$(x-t)\sum_{n=0}^{\infty}P_n(x)\,t^n=\left(1-2xt+t^2\right)\sum_{n=1}^{\infty}nP_n(x)\,t^{n-1}$$

b) Equate the coefficients of t^n in this equation to show that

$$P_1(x)=xP_0(x)$$

and

$$(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x) \quad \text{for } n\ge 1$$

2. Show that $P_{2n+1}(0)=0$ *and* $P_{2n}(0)=(-1)^n\frac{2n!}{2^{2n}(n!)^2}$

3. Show that $P_n'(0)=(-1)^{n+1}P_n'(-1)$

$$=\frac{n(n+1)}{2}$$

Hint: Use Legendre's equation $\left(1-x^2\right)y''-2xy'+n(n+1)y=0$

4. The differential equation $y''+xy=0$ is called ***Airy's equation***, and its solutions are called ***Airy functions***. Find the series for the solutions y_1 and y_2 where $y_1(0)=1$ and $y_1'(0)=0$, while $y_2(0)=0$ and $y_2'(0)=1$. What is the radius of convergence for these two series?

5. The Hermite equation of order α is $\quad y''-2xy'+2\alpha y=0$

a) Find the general solution is $y(x)=a_0y_1(x)+a_1y_2(x)$

Show that $y_1(x)$ is a polynomial if α is an even integer, whereas $y_2(x)$ is a polynomial if α is an odd integer.

b) When $\alpha = n$, use $y_1(x)$ to find polynomial solutions for $n=0$, $n=2$, and $n=4$, then use $y_2(x)$ to find polynomial solutions for $n=1$, $n=3$, and $n=5$.

c) The Hermite polynomial of degree *n* is denoted by $H_n(x)$. It is the *n*th-degree polynomial solution of Hermite's equation, multiplied by a suitable constant so that the coefficient of x^n is 2^n. Use part (b) to show the first six Hermite polynomials are

$$H_0(x) = 1$$
$$H_1(x) = 2x$$
$$H_2(x) = 4x^2 - 2$$
$$H_3(x) = 8x^3 - 12x$$
$$H_4(x) = 16x^4 - 48x^2 + 12$$
$$H_5(x) = 32x^5 - 160x^3 + 120x$$

d) A general formula for the Hermite polynomials is

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

Verify that this formula does in fact give an *n*th-degree polynomial.

6. Rodrigues's Formula is given by: $\quad P_n(x) = \dfrac{1}{n!\,2^n}\dfrac{d^n}{dx^n}\left(x^2-1\right)^n$

For the *n*th-degree Legendre polynomial.

a) Show that $v = \left(x^2-1\right)^n$ satisfies the differential equation $\left(1-x^2\right)v' + 2nxv = 0$

Differentiate each side of this equation to obtain

$$\left(1-x^2\right)v'' + 2(n-1)xv' + 2nv = 0$$

b) Differentiate each side of the last equation *n* times in succession to obtain

$$\left(1-x^2\right)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} = 0$$

which satisfies Legendre's equation of order *n*.

c) Show that the coefficient of x^n in u is $\dfrac{(2n)!}{n!}$; then state why this proves Rodrigues. Formula.

Note: that the coefficient of x^n in $P_n(x)$ is $\dfrac{(2n)!}{2^n(n!)^2}$

Section 4.4 – Solution about Singular Points

4.4-1 Solution about Singular Points

The Standard form $y'' + P(x)y' + Q(x)y = 0$

In the neighborhood of a singular point, as the behavior of the solutions.

Definition

A point x_0 is an ***ordinary point*** if both $P(x)$ and $Q(x)$ are analytic at x_0. If a point in not ordinary it is a ***singular point***.

4.4-2 *Definition* (*Regular and Irregular Singular Points*)

A singular point $x = x_0$ is said to be a ***regular singular*** point of a differential equation if the functions $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2 Q(x)$ are both analytic at x_0.

A singular point is not regular is said to be an ***irregular singular point*** of the equation.

The singular points are those points where $p(x)$ or $q(x)$ fails to be analytic, when the denominators are zero.

- If $x - x_0$ appears at most to the first power in the denominator of $P(x)$ and at most to the second power in the denominator of $Q(x)$, then $x = x_0$ is a ***regular singular point***.

Example 1

Determine the singular points for $\left(x^2 - 4\right)^2 y'' + 3(x-2)y' + 5y = 0$

Solution

$$(x-2)^2 (x+2)^2 y'' + 3(x-2)y' + 5y = 0$$

$$y'' + 3\frac{x-2}{(x-2)^2(x+2)^2} y' + \frac{5}{(x-2)^2(x+2)^2} y = 0$$

$$P(x) = \frac{3}{(x-2)(x+2)^2} \qquad Q(x) = \frac{5}{(x-2)^2(x+2)^2}$$

The singular points are: $\underline{x_0 = -2,\ 2}|$

At $x_0 = -2$

$$p(x) = (x+2)\frac{3}{(x-2)(x+2)^2}$$

$$= \frac{3}{(x-2)(x+2)}$$

Denominator: $x = \pm 2$

$\therefore$ It is *not* an analytic at $x_0 = -2$, it fails because $(x+2)$ in the denominator.

$$q(x) = (x+2)^2 \frac{5}{(x-2)^2(x+2)^2}$$

$$= \frac{5}{(x-2)^2}$$

Denominator: $x = 2$

$\therefore$ It is an analytic at $x_0 = -2$

At $x_0 = 2$

$$p(x) = (x-2)\frac{3}{(x-2)(x+2)^2}$$

$$= \frac{3}{(x+2)^2}$$

Denominator: $x = -2$

$\therefore$ It is an analytic at $x_0 = 2$

$$q(x) = (x-2)^2 \frac{5}{(x-2)^2(x+2)^2}$$

$$= \frac{5}{(x+2)^2}$$

Denominator: $x = -2$

$\therefore$ It is an analytic at $x_0 = 2$

4.4-3 Frobenius *Theorem*

If $x = x_0$ is a regular singular point of the differential equation. There exists at least one solution of the form

$$y(x) = \left(x - x_0\right)^r \sum_{n=0}^{\infty} c_n \left(x - x_0\right)^n$$

$$= \sum_{n=0}^{\infty} c_n \left(x - x_0\right)^{n+r}$$

r: constant to be determined.

The series will converge at least on some interval $0 < x - x_0 < R$

4.4-4 The model of Frobenius

The simplest equation, of a second-order linear differential equation near the regular singular point $x = 0$, is the constant-coefficient ***equidimensional*** equation

$$x^2 y'' + p_0 x y' + q_0 y = 0$$

If r is a root of the quadratic equation: $\quad r(r-1) + p_0 r + q_0 = 0$

The two possible *Frobenius* series solutions are then of the forms

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n \qquad \textit{and} \qquad y_2(x) = x^{r_2} \sum_{n=0}^{\infty} b_n x^n$$

Example 2

Find the exponents in the possible Frobenius series solutions of the equation

$$2x^2(1+x)y'' + 3x(1+x)^3 y' - \left(1 - x^2\right)y = 0$$

Solution

$$y'' + \frac{3x(1+x)^3}{2x^2(1+x)} y' - \frac{(1-x)(1+x)}{2x^2(1+x)} y = 0$$

$$y'' + \frac{3}{2}\frac{(1+x)^2}{x} y' - \frac{1}{2}\frac{1-x}{x^2} y = 0$$

Therefore; $p_0 = \frac{3}{2}, \quad q_0 = -\frac{1}{2}$

The indicial equation is:

$$r(r-1) + \frac{3}{2}r - \frac{1}{2} = r^2 + \frac{1}{2}r - \frac{1}{2}$$
$$= (r+1)\left(r - \frac{1}{2}\right) = 0$$

With roots $r_1 = \frac{1}{2}$ *and* $r_2 = -1$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/2}\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{-1}\sum_{n=0}^{\infty} b_n x^n$$

4.4-5 *Theorem* – Frobenius Series Solutions

Suppose that $x = 0$ is a regular point of the equation $x^2 y'' + p_0 x y' + q_0 y = 0$

Let $\rho > 0$ denote the minimum of the radii of convergence of the power series

$$p(x) = \sum_{n=0}^{\infty} p_n x^n \quad \textit{and} \quad q(x) = \sum_{n=0}^{\infty} q_n x^n$$

Let r_1 and r_2 be the (real) roots, with $r_1 \ge r_2$, of the ***indicial equation***

$I(x) = r(r-1) + p_0 r + q_0 = 0$. Then

✓ For $x > 0$, there exist a solution of the form

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n \quad (a_0 \neq 0) \text{ corresponding to the larger root } r_1.$$

$$y_2(x) = y_1(x)\ln x + x^{r_1+1}\sum_{n=0}^{\infty} b_n x^n$$

✓ If $r_1 - r_2 = N$, a positive integer, then the equation has two solutions y_1 and y_2 of the form

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n$$

and

$$y_2(x) = Cy_1(x) + x^{r_2} \sum_{n=0}^{\infty} b_n x^n$$

Where $a_0,\ b_0 \neq 0$

The radii of convergence of the power series of this theorem are all at least ρ. The coefficients in these series (and the constant C) may be determined by direct substitution of the series.

Example **3**

Find the general solution to the equation $2xy'' + y' - 4y = 0$ near the point $x_0 = 0$

Solution

$$\left(\frac{x}{2}\right)2xy'' + \left(\frac{x}{2}\right)y' - \left(\frac{x}{2}\right)4y = 0$$

$$x^2 y'' + \frac{1}{2}xy' - 2xy = 0$$

That implies to $p(x) = \frac{1}{2}$ and $q(x) = -2x$, both are analytic.

Hence, $x_0 = 0$ is a regular point

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$2x^2 y'' + xy' - 4xy = 0$$

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} + x\sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} - 4x\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n\, x^{n+r} + \sum_{n=0}^{\infty} (n+r)a_n\, x^{n+r} - 4x\sum_{n=0}^{\infty} a_n\, x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[2(n+r)(n+r-1)+(n+r)\right]a_n\, x^{n+r} - 4x\sum_{n=0}^{\infty} a_n\, x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r-2)+(n+r)\right]a_n\, x^{n+r} - 4\sum_{n=0}^{\infty} a_n\, x^{n+r+1} = 0$$

$$x^r\left(\sum_{n=0}^{\infty} \left[(n+r)(2n+2r-1)\right]a_n\, x^{n} - 4\sum_{n=0}^{\infty} a_n\, x^{n+1}\right) = 0$$

$$x^r\left(r(2r-1)a_0 + \underbrace{\sum_{n=1}^{\infty} (n+r)(2n+2r-1)a_n\, x^{n}}_{k=n} - 4\underbrace{\sum_{n=0}^{\infty} a_n\, x^{n+1}}_{k=n+1}\right) = 0$$

$$x^r\left(r(2r-1)a_0 + \sum_{k=1}^{\infty} (k+r)(2k+2r-1)a_k\, x^{k} - 4\sum_{k=1}^{\infty} a_{k-1}\, x^{k}\right) = 0$$

$$x^r\left(r(2r-1)a_0 + \sum_{k=1}^{\infty} \left[(k+r)(2k+2r-1)a_k - 4a_{k-1}\right]x^{k}\right) = 0$$

$$\begin{cases} r(2r-1)a_0 = 0 & \Rightarrow\ \underline{r = 0,\ \tfrac{1}{2}} \\ (k+r)(2k+2r-1)a_k - 4a_{k-1} = 0 & \Rightarrow\ \underline{a_k = \dfrac{4}{(k+r)(2k+2r-1)}a_{k-1}} \end{cases}$$

$r = 0$

$$a_k = \frac{4}{k(2k-1)}a_{k-1}$$

$$a_1 = \frac{4}{1}a_0$$

$$a_2 = \frac{4}{2\cdot 3}a_1 = \frac{4^2}{1\cdot 2\cdot 3}a_0$$

$r = \frac{1}{2}$

$$a_k = \frac{4}{\left(k+\frac{1}{2}\right)\left(2k+2\frac{1}{2}-1\right)}a_{k-1} = \frac{4}{k(2k+1)}a_{k-1}$$

$$a_1 = \frac{4}{1\cdot 3}a_0$$

$$a_2 = \frac{4}{2\cdot 5}a_1 = \frac{4^2}{1\cdot 2\cdot 3\cdot 5}a_0$$

$$a_3 = \frac{4}{3\cdot 5}a_2 = \frac{4^3}{1\cdot 2\cdot 3\cdot 3\cdot 5}a_0 \qquad a_3 = \frac{4}{3\cdot 7}a_2 = \frac{4^3}{3!(3\cdot 5\cdot 7)}a_0$$

$$a_4 = \frac{4}{4\cdot 7}a_3 = \frac{4^3}{4!(1\cdot 3\cdot 5\cdot 7)}a_0 \qquad a_4 = \frac{4}{4\cdot 9}a_3 = \frac{4^3}{4!(3\cdot 5\cdot 7\cdot 9)}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_n = \frac{4^n}{n!\ 1\cdot 3\cdot 5\cdots(2n-1)}a_0 \qquad a_n = \frac{4^n}{n!\ 3\cdot 5\cdots(2n+1)}a_0$$

$$y_1(x) = x^0\left(a_0 + \sum_{n=0}^{\infty}\frac{4}{n!\ 1\cdot 3\cdot 5\cdots(2n-1)}a_0 x^n\right)$$

$$= a_0\left(1 + \sum_{n=0}^{\infty}\frac{4}{n!\ 1\cdot 3\cdot 5\cdots(2n-1)}x^n\right)$$

$$y_2(x) = x^{1/2}\left(a_0 + \sum_{n=0}^{\infty}\frac{4^n}{n!\ 3\cdot 5\cdots(2n+1)}a_0 x^n\right)$$

$$= a_0 x^{1/2}\left(1 + \sum_{n=0}^{\infty}\frac{4^n}{n!\ 3\cdot 5\cdots(2n+1)}x^n\right)$$

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

$$= C_1\left(1 + \sum_{n=0}^{\infty}\frac{4^n}{n!\ 1\cdot 3\cdot 5\cdots(2n-1)}x^n\right) + C_2 x^{1/2}\left(1 + \sum_{n=0}^{\infty}\frac{4^n}{n!\ 3\cdot 5\cdots(2n+1)}x^n\right)$$

Example 4

Find the general solution to the equation $3xy'' + y' - y = 0$

Solution

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)c_n\, x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n\, x^{n+r-2}$$

$$3xy'' + y' - y = 0$$

$$3x\sum_{n=0}^{\infty} (n+r+2)(n+r+1)c_{n+2}\, x^{n+r} + \sum_{n=0}^{\infty} (n+r+1)c_{n+1}\, x^{n+r} - \sum_{n=0}^{\infty} c_n\, x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 3(n+r+2)(n+r+1)c_{n+2}\, x^{n+r+1} + \sum_{n=0}^{\infty} \left[(n+r+1)c_{n+1} - c_n\right] x^{n+r} = 0$$

$$3\sum_{n=0}^{\infty} (n+r)(n+r-1)c_n\, x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)c_n\, x^{n+r-1} - \sum_{n=0}^{\infty} c_n\, x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} c_n(n+r)(3n+3r-3+1)\, x^{n-1}x^r - \sum_{n=0}^{\infty} c_n\, x^n x^r = 0$$

$$x^r\left(\sum_{n=0}^{\infty} c_n(n+r)(3n+3r-2)\, x^{n-1} - \sum_{n=0}^{\infty} c_n\, x^n\right) = 0$$

$$x^r\left(c_0 r(3r-2)x^{-1} + \underbrace{\sum_{n=1}^{\infty} c_n(n+r)(3n+3r-2)\, x^{n-1}}_{k=n-1} - \underbrace{\sum_{n=0}^{\infty} c_n\, x^n}_{k=n}\right) = 0$$

$$x^r\left(c_0 r(3r-2)x^{-1} + \sum_{k=0}^{\infty} c_{k+1}(k+r+1)(3k+3r+1)\, x^k - \sum_{k=0}^{\infty} c_k\, x^k\right) = 0$$

$$x^r\left(c_0 r(3r-2)x^{-1} + \sum_{k=0}^{\infty} \left[c_{k+1}(k+r+1)(3k+3r+1) - c_k\right] x^k\right) = 0$$

$$\begin{cases} c_0 r(3r-2)=0 & \Rightarrow \quad \boxed{r=0,\ \frac{2}{3}} \\ c_{k+1}(k+r+1)(3k+3r+1)-c_k=0 & \Rightarrow \quad \boxed{c_{k+1}=\dfrac{c_k}{(k+r+1)(3k+3r+1)}} \end{cases}$$

$r=0$

$$c_{k+1}=\frac{c_k}{(k+1)(3k+1)}$$

$$c_1=c_0$$

$$c_2=\frac{c_1}{2\cdot 4}=\frac{c_0}{(2)(4)}$$

$$c_3=\frac{c_2}{3\cdot 7}=\frac{c_0}{2\cdot 3(4\cdot 7)}$$

$$c_4=\frac{c_3}{4\cdot 10}=\frac{c_0}{2\cdot 3\cdot 4(4\cdot 7\cdot 10)}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$c_n=\frac{c_0}{n!\ 4\cdot 7\cdot 10\cdots(3n-2)}$$

$r=\frac{2}{3}$

$$c_{k+1}=\frac{c_k}{\left(k+\frac{5}{3}\right)(3k+3)}=\frac{c_k}{(3k+5)(k+1)}$$

$$c_1=\frac{c_0}{5\cdot 1}$$

$$c_2=\frac{c_1}{8\cdot 2}=\frac{c_0}{5\cdot 8\cdot 1\cdot 2}$$

$$c_3=\frac{c_2}{11\cdot 3}=\frac{c_0}{(5\cdot 8\cdot 11)(1\cdot 2\cdot 3)}$$

$$c_4=\frac{c_3}{14\cdot 4}=\frac{c_0}{(5\cdot 8\cdot 11\cdot 14)(1\cdot 2\cdot 3\cdot 4)}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$c_n=\frac{c_0}{n!\ 5\cdot 8\cdot 11\cdots(3n+2)}$$

$$y_1(x)=x^0\left(c_0+\sum_{n=0}^{\infty}\frac{c_0}{n!\ 4\cdot 7\cdot 10\cdots(3n-2)}\,x^n\right)$$

$$=c_0\left(1+\sum_{n=0}^{\infty}\frac{1}{n!\ 4\cdot 7\cdot 10\cdots(3n-2)}\,x^n\right)$$

$$y_2(x)=x^{2/3}\left(c_0+\sum_{n=0}^{\infty}\frac{c_0}{n!\ 5\cdot 8\cdots(3n+2)}\,x^n\right)$$

$$=c_0x^{2/3}\left(1+\sum_{n=0}^{\infty}\frac{1}{n!\ 5\cdot 8\cdots(3n+2)}\,x^n\right)$$

$$y(x)=C_1y_1(x)+C_2y_2(x)$$

$$= C_1\left(1+\sum_{n=0}^{\infty}\frac{1}{n!\;4\cdot 7\cdot 10\cdots(3n-2)}x^n\right)+C_2x^{2/3}\left(1+\sum_{n=0}^{\infty}\frac{1}{n!\;5\cdot 8\cdots(3n+2)}x^n\right)$$

OR

$$y''+\frac{1}{3x}y'-\frac{1}{3x}y=0$$

$$p(x)=\left(x-x_0\right)P(x)=x\frac{1}{3x}=\frac{1}{3} \qquad p(x)=a_0+a_1x+\cdots$$

$$q(x)=\left(x-x_0\right)^2Q(x)=x^2\left(-\frac{1}{3x}\right)=-\frac{1}{3}x \qquad q(x)=b_0+b_1x+\cdots$$

$$r(r-1)+a_0r+b_0=0$$

$$r(r-1)+\frac{1}{3}r+0=0$$

$$r^2-r+\frac{1}{3}r=0$$

$$3r^2-2r=0$$

$$r(3r-2)=0$$

4.4-6 *Theorem* The Extended Theorem and Procedure of ***Frobenius***

The *ODE* is given by: $x^2y''+xp(x)y'+q(x)y=0$

Has a regular singular point at $x=0$. The extended Method of ***Frobenius*** produce *two* independent solutions of the *ODE* if the indicial roots are real.

- Find the indicial roots r_1 and r_2 of the indicial polynomial $f(r)=r^2+\left(p_0-1\right)r+q_0$

  Verify that they are real; index them such that $r_2\le r_1$

- Construct the solution $y_1(x)=x^{r_1}\sum_{n=0}^{\infty}a_n x^n \quad \left(a_0=1\right)$ by the method of Frobenius. The

  recursion formula is $f\left(r_1+n\right)a_n=\sum_{k=0}^{n-1}\left[\left(k+r_1\right)p_{n-k}+q_{n-k}\right]a_k$

- If $r_1=r_2 \;\Rightarrow\; y_2(x)=y_1(x)\ln x+x^{r_1}\sum_{n=0}^{\infty}c_n x^n \qquad (x>0)$

- If $r_1 - r_2$ is a positive integer, then a second independent solution has the form

$$y_2(x) = \alpha y_1(x) \ln x + x^{r_2}\left(1 + \sum_{n=1}^{\infty} d_n x^n\right)$$

Exercises *Section* 4.4 – Solution about Singular Points

(**1 – 19**) Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

1. $x^2y'' + 3y' - xy = 0$

2. $\left(x^2 + x\right)y'' + 3y' - 6xy = 0$

3. $\left(x^2 - 1\right)y'' + (1 - x)y' + \left(x^2 - 2x + 1\right)y = 0$

4. $e^x y'' - \left(x^2 - 1\right)y' + 2xy = 0$

5. $\ln(x - 1)y'' + (\sin 2x)y' - e^x y = 0$

6. $xy'' + x(1 - x)^{-1}y' + (\sin x)y = 0$

7. $x^3y'' + 4x^2y' + 3y = 0$

8. $x(x + 3)^2 y'' - y = 0$

9. $\left(x^2 - 9\right)^2 y'' + (x + 3)y' + 2y = 0$

10. $y'' - \frac{1}{x}y' + \frac{1}{(x - 1)^3}y = 0$

11. $\left(x^3 + 4x\right)y'' - 2xy' + 6y = 0$

12. $x^2(x - 5)^2 y'' + 4xy' + \left(x^2 - 25\right)y = 0$

13. $\left(x^2 + x - 6\right)y'' + (x + 3)y' + (x - 2)y = 0$

14. $x\left(x^2 + 1\right)^2 y'' + y = 0$

15. $x^3\left(x^2 - 25\right)(x - 2)^2 y'' + 3x(x - 2)y' + 7(x + 5)y = 0$

16. $\left(x^3 - 2x^2 - 3x\right)^2 y'' + x(x - 3)^2 y' - (x + 1)y = 0$

17. $\left(1 - x^2\right)y'' + (\tan x)y' + x^{5/3}y = 0$

18. $x(x - 1)^2(x + 2)y'' + x^2y' - \left(x^3 + 2x - 1\right)y = 0$

19. $x^4\left(x^2 + 1\right)(x - 1)^2 y'' + 4x^3(x - 1)y' + (x + 1)y = 0$

(20 – 21) Determine whether $x=0$ is an ordinary point, singular point, or irregular singular point of the given differential equation

20. $xy''+(1-\cos x)y'+x^2y=0$

21. $\left(e^x-1-x\right)y''+xy=0$

(22 – 53) Find the Frobenius series solutions near the point $x=0$

22. $2x^2y''+3xy'-\left(1+x^2\right)y=0$

23. $2x^2y''-xy'+\left(1+x^2\right)y=0$

24. $2xy''+(1+x)y'+y=0$

25. $xy''+2y'+xy=0$

26. $2xy''-y'+2y=0$

27. $2xy''+5y'+xy=0$

28. $4xy''+\frac{1}{2}y'+y=0$

29. $2x^2y''-xy'+\left(x^2+1\right)y=0$

30. $2xy''-(3+2x)y'+y=0$

31. $3xy''+(2-x)y'-y=0$

32. $xy''+(x-6)y'-3y=0$

33. $x(x-1)y''+3y'-2y=0$

34. $x^2y''-\left(x-\frac{2}{9}\right)y=0$

35. $x^2y''+x(3+x)y'-3y=0$

36. $x^2y''+\left(x^2-2x\right)xy'+2y=0$

37. $x^2y''+\left(x^2+2x\right)y'-2y=0$

38. $2xy''+3y'-y=0$

39. $2xy''-y'-y=0$

40. $2xy''+(1+x)y'+y=0$

41. $2xy''+\left(1-2x^2\right)y'-4xy=0$

42. $2x^2y''+xy'-\left(1+2x^2\right)y=0$

43. $2x^2y''+xy'-\left(3-2x^2\right)y=0$

44. $3xy''+2y'+2y=0$

45. $3x^2y''+2xy'+x^2y=0$

46. $3x^2y''-xy'+y=0$

47. $4xy''+2y'+y=0$

48. $6x^2y''+7xy'-\left(x^2+2\right)y=0$

49. $xy''+y'+2y=0$

50. $2x(1-x)y''+(1+x)y'-y=0$

51. $x^2y''+\left(x^2+\frac{1}{2}x\right)y'+xy=0$

52. $18x^2y''+3x(x+5)y'-(10x+1)y=0$

53. $2x^2y''+7x(x+1)y'-3y=0$

54. Find the Frobenius series solutions:

$$x(1-x)y''+\left[c-(a+b+1)x\right]y'-aby=0 \quad (\textit{Gauss' Hypergeometric})$$

Section 4.5 – Bessel's Equation and Bessel Functions

4.5-1 Bessel's Equation

In this section we consider three special cases of ***Bessel's equation***

$$x^2 y'' + xy' + \left(x^2 - \upsilon^2\right) y = 0$$

Where υ is a constant, and the solutions are called ***Bessel functions***.

When solving partial differential equations involving the Laplacian in polar and cylindrical coordinates. The Bessel's equation is a whole family of differential equations, one for each value of υ.

From the *Frobenius* model: $x^2 y'' + p_0 xy' + q_0 y = 0$

$$\rightarrow \ p_0 = 1 \quad \& \quad q_0 = -\upsilon^2$$

The indicial equation is given by:

$$I(r) = r(r-1) + p_0 r + q_0$$

$$= r(r-1) + r - \upsilon^2 = 0$$

$$r^2 - \upsilon^2 = 0 \quad \rightarrow \quad \underline{r = \pm\upsilon}$$

We will consider the three cases $\upsilon = 0$, $\upsilon = \frac{1}{2}$, and $\upsilon = 1$ for the interval $x > 0$.

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + x \sum_{n=1}^{\infty} (n+r) a_n x^{n+r-1} + \left(x^2 - \upsilon^2\right) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}+\sum_{n=1}^{\infty}(n+r)a_n x^{n+r}+\sum_{n=0}^{\infty}a_n x^{n+r+2}-\upsilon^2\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$x^r\left(\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)+(n+r)-\upsilon^2\right]a_n x^n+\sum_{n=0}^{\infty}a_n x^{n+2}\right)=0$$

$$(n+r)(n+r-1)+(n+r)=(n+r)(n+r-1+1)=(n+r)^2$$

$$\left(r^2-\upsilon^2\right)a_0+\left((1+r)^2-\upsilon^2\right)a_1+\underbrace{\sum_{n=2}^{\infty}\left[(n+r)^2-\upsilon^2\right]a_n x^n}_{k=n-2}+\underbrace{\sum_{n=0}^{\infty}a_n x^{n+2}}_{k=n}=0$$

$$\sum_{k=0}^{\infty}\left[(k+2+r)^2-\upsilon^2\right]a_{k+2}x^{k+2}+\sum_{k=0}^{\infty}a_k x^{k+2}=0$$

$$\sum_{k=0}^{\infty}\left[\left((k+2+r)^2-\upsilon^2\right)a_{k+2}+a_k\right]x^{k+2}=0$$

$$\left((k+2+r)^2-\upsilon^2\right)a_{k+2}+a_k=0$$

$$a_{k+2}=\frac{-a_k}{(k+2+r)^2-\upsilon^2}\qquad (k+2+r)^2-\upsilon^2=(k+2)^2+2r(k+2)+r^2-\upsilon^2$$

$$=(k+2)(k+2+2r)+r^2-\upsilon^2\qquad r^2-\upsilon^2=0$$

$$a_{k+2}=\frac{-a_k}{(k+2)(k+2+2\upsilon)}$$

We must choose $a_1=0 \ \rightarrow\ a_3=a_5=\cdots=0$

$$a_{2n}=-\frac{1}{2n(2n+2\upsilon)}a_{n-2}$$

$$=-\frac{1}{2^2 n(n+\upsilon)}a_{n-2}\qquad (2n=k+2)$$

$$a_2=-\frac{1}{2^2\cdot 1\cdot(1+\nu)}a_0$$

$$a_4(0)=-\frac{1}{2^2\cdot 2(2+\nu)}a_2\ =\frac{1}{2^4\cdot 1\cdot 2(1+\nu)(2+\nu)}a_0$$

$$a_6(0) = -\frac{1}{2^2 \cdot 3(3+\nu)} a_4 \quad = -\frac{1}{2^6 \cdot 1 \cdot 2 \cdot 3(1+\nu)(2+\nu)(3+\nu)} a_0$$

$$\vdots \qquad \vdots \;\; \vdots$$

$$\boxed{a_{2n} = \frac{(-1)^n}{2^{2n} n!(1+\upsilon)(2+\upsilon)\cdots(n+\upsilon)} a_0, \quad n = 1,\ 2,\ 3,\ \ldots}$$

4.5-2 Gamma Function

Many important functions in applied sciences are defined via improper integrals. Maybe the most famous among them is the ***Gamma Function***.

$$(\upsilon+1)\cdot(\upsilon+2)\cdot\ldots\cdot(\upsilon+n) = \frac{(\upsilon+n)!}{\upsilon!}$$

The gamma function is defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt \quad \textit{for } x > 0$$

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}$$

$$= \frac{\Gamma(x+n)}{x\cdot(x+1)\cdot\ldots\cdot(x+n-1)}$$

$$x! = \Gamma(x+1)$$

$$(\upsilon+n)! = \Gamma(\upsilon+n+1)$$

$$a_{2n} = \frac{(-1)^n}{2^{2n+\upsilon} n!\Gamma(1+\upsilon+n)}, \quad n = 0,1,2,3,\ldots$$

The series solution is denoted by $J_\upsilon(x)$:

$$J_\upsilon(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon}$$

For $r_2 = -\upsilon$, then

$$J_{-\upsilon}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1-\upsilon+n)}\left(\frac{x}{2}\right)^{2n-\upsilon}$$

The functions $J_{\upsilon}(x)$ and $J_{-\upsilon}(x)$ are called the ***Bessel function of the first kind*** of order υ and $-\upsilon$.

4.5-3 Bessel Equation of Order *Zero*

In this case $\upsilon=0$, that implies to Bessel's equation: $x^2y''+xy'+x^2y=0$

The roots of the indicial equation are equal: $r_1=r_2=0$

Hence, $y_1(x)=a_0\left[1+\sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right]$

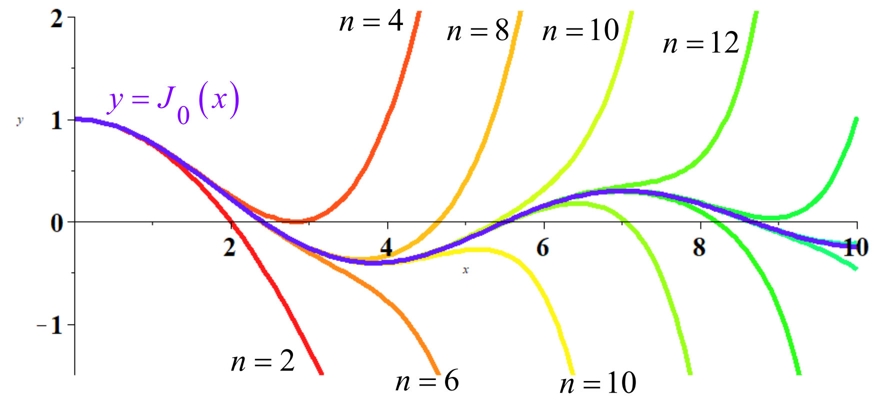
$$J_0(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+n)}\left(\frac{x}{2}\right)^{2n}$$

$$y_2(x)=J_0(x)\ln x+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H(n)}{(n!)^2}\left(\frac{x}{2}\right)^{2n} \qquad H(n)=1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{n}$$

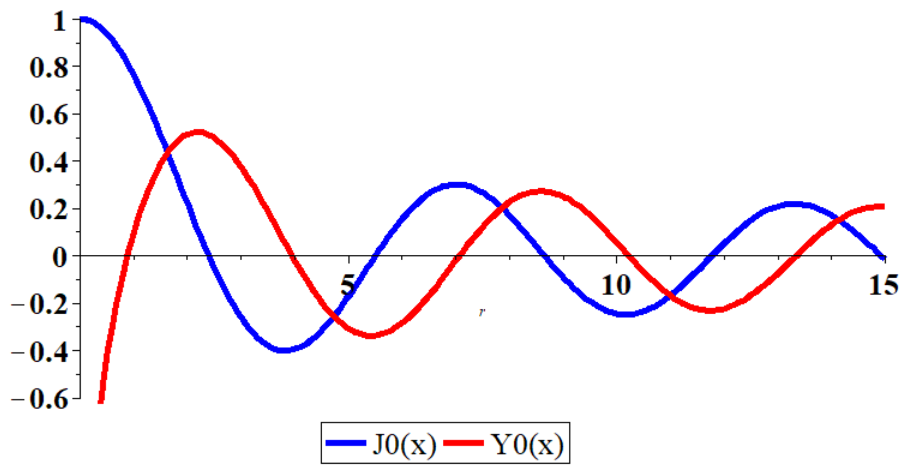
$$\begin{aligned}Y_0(x)&=\frac{2}{\pi}\left[y_2(x)+(\gamma-\ln 2)J_0(x)\right]\\&=\frac{2}{\pi}\left[\left(\ln\frac{x}{2}+\gamma\right)J_0(x)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H(n)}{(n!)^2}\left(\frac{x}{2}\right)^{2n}\right]\end{aligned}$$

Where γ is ***Euler's*** constant, defined by

$$\begin{aligned}\gamma&=\lim_{n\to\infty}\left(H(n)-\ln n\right)\\&=\lim_{x\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right)\\&=0.5772156\cdots\end{aligned}$$



n = 4
n = 8
n = 10
n = 12
y = J0(x)
n = 2
n = 6
n = 10



J0(x)
Y0(x)

4.5-4 Bessel Equation of Order *One-Half*

In this case $\upsilon = \frac{1}{2}$, that implies to Bessel's equation:

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$$

The roots of the indicial equation are equals: $r_1 = \frac{1}{2}, \quad r_2 = -\frac{1}{2}$

$$\begin{aligned} a_{2n} &= -\frac{1}{2^2 n(n+\upsilon)} a_{n-2} \\ &= -\frac{1}{2^2 n\left(n+\frac{1}{2}\right)} a_{n-2} \\ &= -\frac{1}{2n(2n+1)} a_{n-2} \end{aligned}$$

$$\boxed{a_{2n} = \frac{(-1)^n}{2^{2n} n!(1+\upsilon)(2+\upsilon)\cdots(n+\upsilon)} a_0}\ ; \qquad n = 1,\ 2,\ 3,\ \ldots$$

Taking $a_0 = 1$, we obtain

$$\begin{aligned} y_1(x) &= x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}\right] \\ &= x^{-1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \end{aligned}$$

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x, \quad x > 0$$

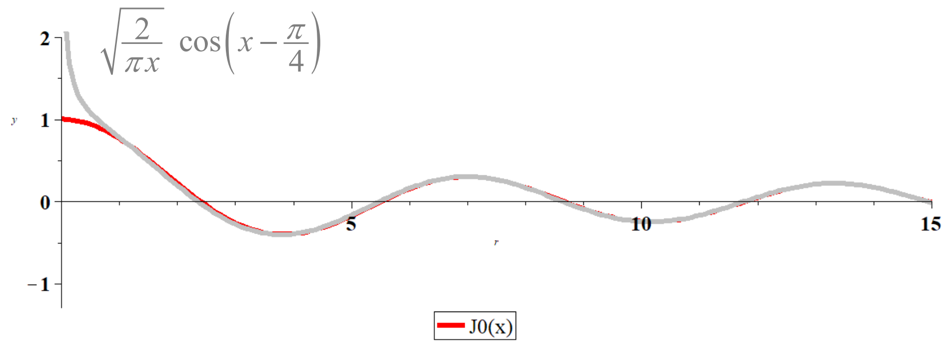
For $r_2 = -\frac{1}{2}$,

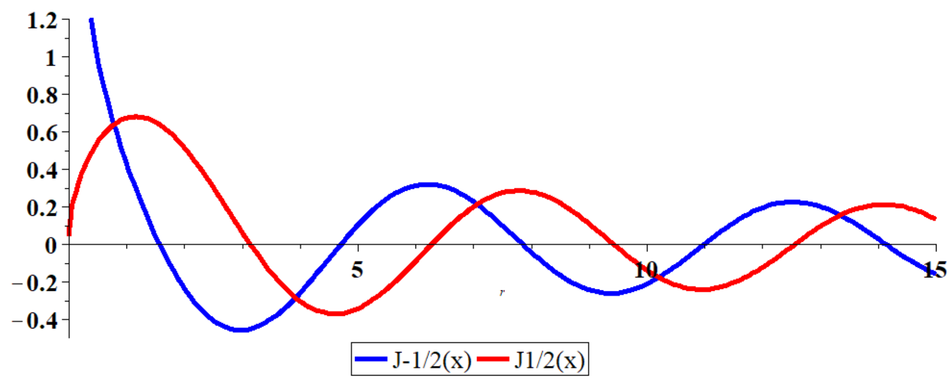
$$a_{2n} = \frac{(-1)^n}{(2n)!} a_0, \quad a_{2n+1} = \frac{(-1)^n}{(2n+1)!} a_1, \quad n = 1, 2, \ldots$$

$$\begin{aligned} y_2(x) &= x^{-1/2}\left[a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right] \\ &= a_0 \frac{\cos x}{x^{1/2}} + a_1 \frac{\sin x}{x^{1/2}} \end{aligned}$$

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x, \quad x > 0$$

The general solution is: $\qquad y(x) = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$





4.5-5 Bessel Equation of Order *One*

In this case $\upsilon = 1$, that implies to Bessel's equation: $x^2 y'' + xy' + \left(x^2 - 1\right) y = 0$

The roots of the indicial equation are equal: $r_1 = 1, \quad r_2 = -1$

$$a_{2n} = -\frac{1}{2^2 n(n+1)} a_{n-2}$$

$$a_{2n} = \frac{(-1)^n}{2^{2n} n!(n+1)!} a_0, \quad n = 1, 2, 3, \ldots$$

Taking $a_0 = \frac{1}{2}$, we obtain

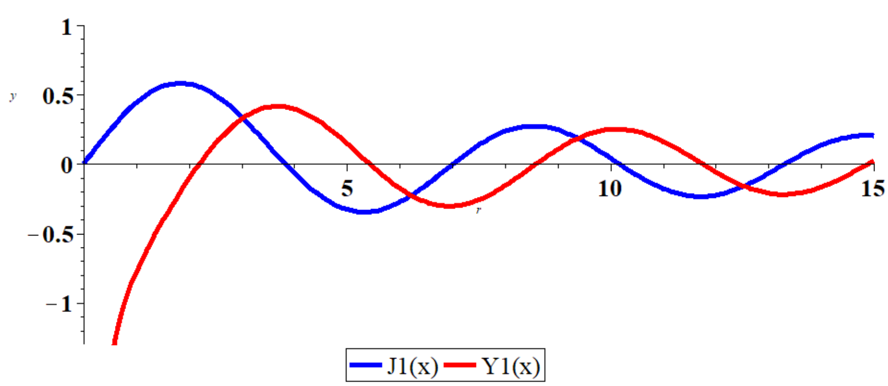
$$J_1(x) = \frac{x}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n+1)!n!}$$

$$y_2(x) = -J_1(x) \ln x + \frac{1}{x}\left[1 - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(H_n + H_{n-1}\right)}{2^{2n} n!(n-1)!} x^{2n}\right]$$

$$Y_1(x) = \frac{2}{\pi}\left(-y_2(x) + (\gamma - \ln 2) J_1(x)\right)$$

The general solution is:

$$y(x) = c_1 J_1(x) + c_2 Y_1(x)$$



4.5-6 Applications of Bessel Functions

The importance of Bessel functions stems not only from the frequent appearance of Bessel's equation in applications, but also from the fact that the solutions of many other second-order linear differential equations can be expressed in terms of Bessel functions.

The Bessel's equation is given by:

$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}+\left(z^2-\upsilon^2\right)w=0$$

Let $\quad w=x^{-\alpha}y, \quad z=kx^{\beta}$

$$z=kx^{\beta}\quad\rightarrow x=\left(\frac{z}{k}\right)^{1/\beta}$$

$$\begin{aligned}
\frac{dw}{dz}&=\frac{dw}{dx}\frac{dx}{dz}\\
&=\frac{d}{dx}\left(x^{-\alpha}y\right)\frac{d}{dz}\left(\left(\frac{z}{k}\right)^{1/\beta}\right)\\
&=\left(-\alpha x^{-\alpha-1}y+x^{-\alpha}\frac{dy}{dx}\right)\left(\frac{1}{k\beta}\left(\frac{z}{k}\right)^{1/\beta-1}\right)\\
&=\frac{1}{k\beta}\left(-\alpha x^{-\alpha-1}y+x^{-\alpha}\frac{dy}{dx}\right)\left(x^{\beta}\right)^{1/\beta-1}\\
&=\frac{1}{k\beta}\left(-\alpha x^{-\alpha-1}y+x^{-\alpha}\frac{dy}{dx}\right)x^{1-\beta}\\
&=\boxed{\frac{1}{k\beta}\left(-\alpha x^{-\alpha-\beta}y+x^{1-\alpha-\beta}\frac{dy}{dx}\right)}
\end{aligned}$$

$$\begin{aligned}
\frac{d^2w}{dz^2}&=\frac{d}{dx}\left(\frac{dw}{dz}\right)\frac{dx}{dz}\\
&=\frac{1}{k\beta}\frac{d}{dx}\left(-\alpha x^{-\alpha-\beta}y+x^{1-\alpha-\beta}\frac{dy}{dx}\right)\frac{d}{dz}\left(\left(\frac{z}{k}\right)^{1/\beta}\right)\\
&=\frac{1}{k\beta}\left(\left(\alpha^2+\alpha\beta\right)x^{-\alpha-\beta-1}y-\alpha x^{-\alpha-\beta}\frac{dy}{dx}+(1-\alpha-\beta)x^{-\alpha-\beta}\frac{dy}{dx}+x^{1-\alpha-\beta}\frac{d^2y}{dx^2}\right)\left(\frac{1}{k\beta}x^{1-\beta}\right)\\
&=\frac{1}{k^2\beta^2}\left(\left(\alpha^2+\alpha\beta\right)x^{-\alpha-2\beta}y+\left((1-\alpha-\beta)x^{1-\alpha-2\beta}-\alpha x^{1-\alpha-2\beta}\right)\frac{dy}{dx}+x^{2-\alpha-2\beta}\frac{d^2y}{dx^2}\right)\\
&=\boxed{\frac{1}{k^2\beta^2}\left(\left(\alpha^2+\alpha\beta\right)x^{-\alpha-2\beta}y+(1-2\alpha-\beta)x^{1-\alpha-2\beta}\frac{dy}{dx}+x^{2-\alpha-2\beta}\frac{d^2y}{dx^2}\right)}
\end{aligned}$$

$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}+\left(z^2-\upsilon^2\right)w=0$$

$$k^2x^{2\beta}\frac{1}{k^2\beta^2}\left(\left(\alpha^2+\alpha\beta\right)x^{-\alpha-2\beta}y+(1-2\alpha-\beta)x^{1-\alpha-2\beta}\frac{dy}{dx}+x^{2-\alpha-2\beta}\frac{d^2y}{dx^2}\right)$$
$$+kx^{\beta}\frac{1}{k\beta}\left(-\alpha x^{-\alpha-\beta}y+x^{1-\alpha-\beta}\frac{dy}{dx}\right)+\left(k^2x^{2\beta}-\upsilon^2\right)x^{-\alpha}y=0$$

$$\frac{1}{\beta^2}\left(\left(\alpha^2+\alpha\beta\right)x^{-\alpha}y+(1-2\alpha-\beta)x^{1-\alpha}\frac{dy}{dx}+x^{2-\alpha}\frac{d^2y}{dx^2}\right)+\frac{1}{\beta}\left(-\alpha x^{-\alpha}y+x^{1-\alpha}\frac{dy}{dx}\right)$$
$$+\left(k^2x^{2\beta}-\upsilon^2\right)x^{-\alpha}y=0$$

$$\left(\alpha^2+\alpha\beta\right)x^{-\alpha}y+(1-2\alpha-\beta)x^{1-\alpha}\frac{dy}{dx}+x^{2-\alpha}\frac{d^2y}{dx^2}-\alpha\beta x^{-\alpha}y+\beta x^{1-\alpha}\frac{dy}{dx}$$
$$+\left(k^2\beta^2x^{2\beta}-\beta^2\upsilon^2\right)x^{-\alpha}y=0$$

$$x^2x^{-\alpha}\frac{d^2y}{dx^2}+(1-2\alpha-\beta+\beta)xx^{-\alpha}\frac{dy}{dx}+\left(\alpha^2+\alpha\beta+k^2\beta^2x^{2\beta}-\beta^2\upsilon^2-\alpha\beta\right)x^{-\alpha}y=0$$

Then substitute into the Bessel's equation:

$$x^2\frac{d^2y}{dx^2}+(1-2\alpha)x\frac{dy}{dx}+\left(\alpha^2-\beta^2\upsilon^2+k^2\beta^2x^{2\beta}\right)y=0$$

That is equivalent to:

$$x^2\frac{d^2y}{dx^2}+Ax\frac{dy}{dx}+\left(B+Cx^p\right)y=0$$

Where $\quad A=1-2\alpha,\quad B=\alpha^2-\beta^2\upsilon^2,\quad C=\beta^2k^2,\quad p=2\beta$

$$\to \upsilon^2=\frac{\alpha^2-B}{\beta^2}$$

$$\alpha=\frac{1-A}{2},\quad \beta=\frac{p}{2},\quad k=\frac{2\sqrt{C}}{p},\quad \upsilon=\frac{\sqrt{(1-A)^2-4B}}{p}\quad \left((1-A)^2-4B>0\right)$$

Which follows that the general solution is:

$$y(x)=x^{\alpha}w(z)$$
$$=x^{\alpha}w\left(kx^{\beta}\right)$$

Where

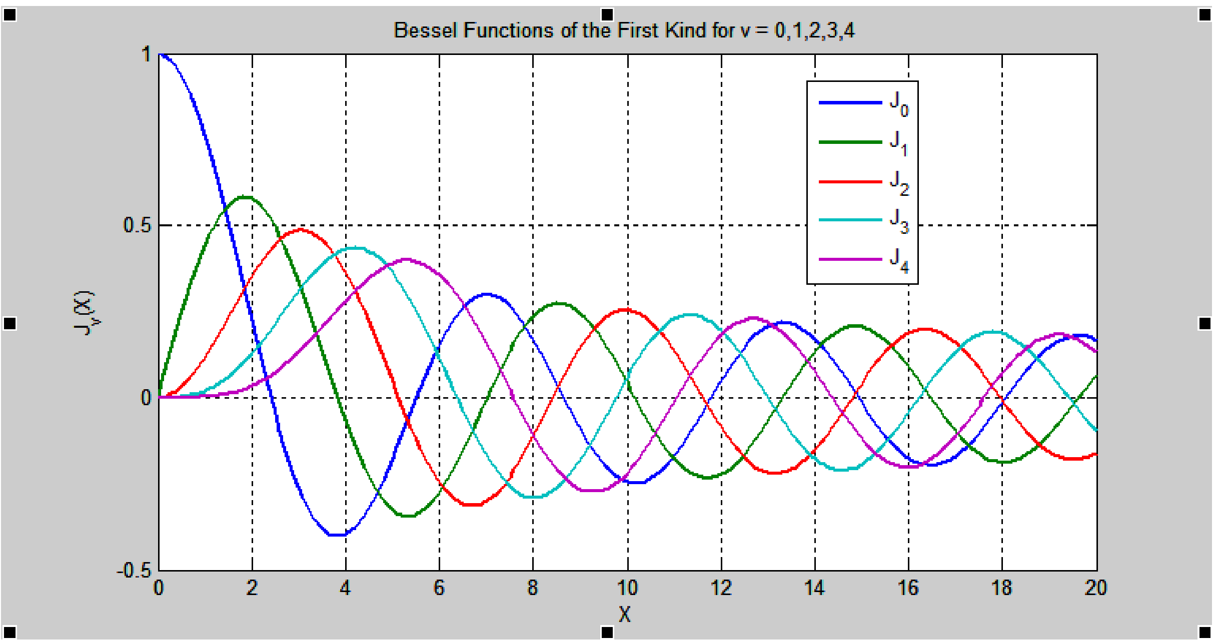
$$w(z)=c_1J_{\upsilon}(z)+c_2Y_{-\upsilon}(z)$$

4.5-7 ***Theorem***: Solutions in Bessel Functions

If $C > 0$, $p \neq 0$, and $(1-A)^2 \geq 4B$, then the general solution (for $x > 0$)

$$y(x) = x^{\alpha}\left[c_1 J_{\upsilon}\left(kx^{\beta}\right) + c_2 J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

Where α, β, k, and υ are given. If υ is an integer, then $J_{-\upsilon}$ is to be replaced by Y_{υ}.



$x^2 y'' + xy' + \left(x^2 - \upsilon^2\right) y = 0$	
$\upsilon = 0$	$Y_0(x) = \frac{2}{\pi}\left[y_2(x) + (\gamma - \ln 2) J_0(x)\right]$ $= \frac{2}{\pi}\left[\left(\ln\frac{x}{2} + \gamma\right) J_0(x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H(n)}{(n!)^2}\left(\frac{x}{2}\right)^{2n}\right]$
$\upsilon = \frac{1}{2}$	$y(x) = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x) \quad = c_1\left(\frac{2}{\pi x}\right)^{1/2} \sin x + c_2\left(\frac{2}{\pi x}\right)^{1/2} \cos x$
$\upsilon = 1$	$y(x) = c_1 J_1(x) + c_2 Y_1(x)$ $= c_1 \frac{x}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n+1)!n!} + c_2 \frac{2}{\pi}\left[-y_2(x) + (\gamma - \ln 2) J_1(x)\right]$

Zeros of J_0, J_1, Y_0 *and* Y_1			
$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$
2.4048	0.0000	0.8936	2.1971
5.5201	3.8317	3.9577	5.4297
8.6537	7.156	7.0861	8.5960
11.7915	10.1735	10.2223	11.7492
14.9309	13.3237	13.3611	14.8974

Numerical Values J_0, J_1, Y_0 *and* Y_1				
x	$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$
0	1.0000	0.00	–	–
1	0.7652	0.4401	0.0883	–0.7812
2	0.2239	0.5767	0.5104	–0.1070
3	–0.2601	0.3391	0.3769	0.3247
4	–0.3971	–0.0660	–0.0169	0.3979
5	–0.1776	–0.3276	–0.3085	0.1479
6	0.1506	–0.2767	–0.2882	–0.1750
7	0.3001	–0.0047	–0.0259	–0.3027
8	0.1717	0.2346	0.2235	–0.1581
9	–0.0903	0.2453	0.2499	0.143
10	–0.2459	0.0435	0.0557	0.2490
11	–0.1712	–0.1768	–0.1688	0.1637
12	0.0477	–0.2234	–0.2252	–0.0571
13	0.2069	–0.0703	–0.0782	–0.2101
14	0.1711	0.1334	0.1272	–0.1666
15	–0.0142	0.2051	0.2055	0.0211

Exercises Section 4.5 – Bessel's Equation and Bessel Functions

(1 – 6) Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$$x^2y'' + xy' + \left(x^2 - \upsilon^2\right)y = 0$$

1. $x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0$

2. $x^2y'' + xy' + \left(x^2 - 1\right)y = 0$

3. $4x^2y'' + 4xy' + \left(4x^2 - 25\right)y = 0$

4. $16x^2y'' + 16xy' + \left(16x^2 - 1\right)y = 0$

5. $xy'' + y' + xy = 0$

6. $xy'' + y' + \left(x - \frac{4}{x}\right)y = 0$

(7 – 10) Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$$x^2y'' + xy' + \left(\alpha^2x^2 - \upsilon^2\right)y = 0$$

7. $x^2y'' + xy' + \left(9x^2 - 4\right)y = 0$

8. $x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right)y = 0$

9. $x^2y'' + xy' + \left(25x^2 - \frac{4}{9}\right)y = 0$

10. $x^2y'' + xy' + \left(2x^2 - 64\right)y = 0$

(11 – 31) Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

11. $4x^2y'' + 8xy' + \left(x^4 - 3\right)y = 0$

12. $y'' + 9xy = 0$

13. $xy'' + (x - 3)y = 0$

14. $xy'' + \left(4x^3 - 1\right)y = 0$

15. $x^2y'' + xy' - \left(\frac{1}{4} + x^2\right)y = 0$

16. $xy'' + (2x + 1)y' + (2x + 1)y = 0$

17. $xy'' - y' - xy = 0$

18. $x^4y'' + a^2y = 0$

19. $y'' - x^2y = 0$

20. $x^2y'' - xy' + \left(1 + x^2\right)y = 0$

21. $xy'' + 3y' + xy = 0$

22. $xy'' - y' + 36x^3y = 0$

23. $x^2y'' - 5xy' + (8 + x)y = 0$

24. $36x^2y'' + 60xy' + \left(9x^3 - 5\right)y = 0$

25. $16x^2y'' + 24xy' + \left(1 + 144x^3\right)y = 0$

26. $x^2y'' + 3xy' + \left(1 + x^2\right)y = 0$

27. $4x^2y'' - 12xy' + (15 + 16x)y = 0$

28. $16x^2y'' - \left(5 - 144x^3\right)y = 0$

29. $2x^2y'' + 3xy' - \left(28 - 2x^5\right)y = 0$

30. $y'' + x^4y = 0$

31. $y'' + 4x^3y = 0$

32. Find a Frobenius solution of Bessel's equation of order ***zero*** $x^2y'' + xy' + x^2y = 0$

33. Derive the formula $\quad xJ'_\upsilon(x) = \upsilon J_\upsilon(x) - xJ_{\upsilon+1}(x)$

34. Derive the formula $\quad xJ'_\upsilon(x) = -\upsilon J_\upsilon(x) + xJ_{\upsilon-1}(x)$

35. Derive the formula $\quad 2\upsilon J'_\upsilon(x) = xJ_{\upsilon+1}(x) + xJ_{\upsilon-1}(x)$

36. Prove that $\quad \frac{d}{dx}\left[x^{\upsilon+1}J_{\upsilon+1}(x)\right] = x^{\upsilon+1}J_\upsilon(x)$

37. Show that $y = \sqrt{x}\,J_{3/2}(x)$ is a solution of $x^2y'' + \left(x^2 - 2\right)y = 0$

38. Show that $\quad 4J''_\upsilon(x) = J_{\upsilon-2}(x) - 2J_\upsilon(x) + J_{\upsilon+2}(x)$

39. Show that $y = x^{1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right)$ is a solution of Airy's differential equation $y'' + \alpha^2 xy = 0$, $x > 0$

, whenever w is a solution of Bessel's equation of order $\frac{2}{3}$, that is, $t^2w'' + tw' + \left(t^2 - \frac{1}{9}\right)w = 0$, $t > 0$

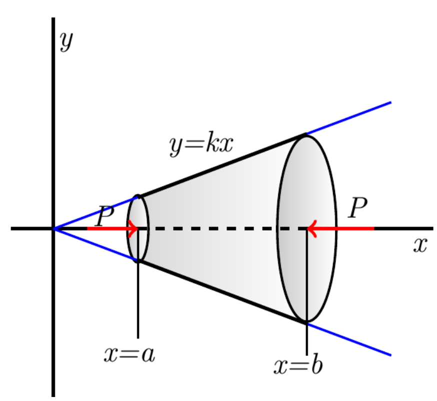
[*Hint*: After differentiating, substituting, and simplifying, then let $t = \frac{2}{3}\alpha x^{3/2}$].

40. Use the relation $\Gamma(x+1) = x\Gamma(x)$ and if υ is nonnegative integer, then show that

$$J_\upsilon(x) = \frac{1}{\Gamma(\upsilon+1)}\left(\frac{x}{2}\right)^\upsilon \left[1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(\upsilon+1)(\upsilon+2)\cdots(\upsilon+n)}\left(\frac{x}{2}\right)^{2n}\right]$$

41. A linearly tapered rod with circular cross section, subject to an axial force P of compression. Its deflection curve $y = y(x)$ satisfies the endpoint value problem

$$EIy'' + Py = 0\,; \quad y(a) = y(b) = 0 \quad (1)$$



Here, however, the moment of inertia $I = I(x)$ of the cross section at x is given by

$$I(x) = \frac{1}{4}\pi(kx)^4 = I_0\left(\frac{x}{b}\right)^4 \quad (2)$$

Where $I_0 = I(b)$, the value of I at $x = b$. Substitution of $I(x)$ in the differential equation (1) yields to the eigenvalue problem

$$x^4 y'' + \lambda y = 0 \; ; \quad y(a) = y(b) = 0 \quad (3)$$

Where $\lambda = \mu^2 = \dfrac{Pb^4}{EI_0}$

a) Show that the general solution of $x^4 y'' + \mu^2 y = 0$ is $y(x) = x\left(A\cos\dfrac{\mu}{x} + B\sin\dfrac{\mu}{x}\right)$

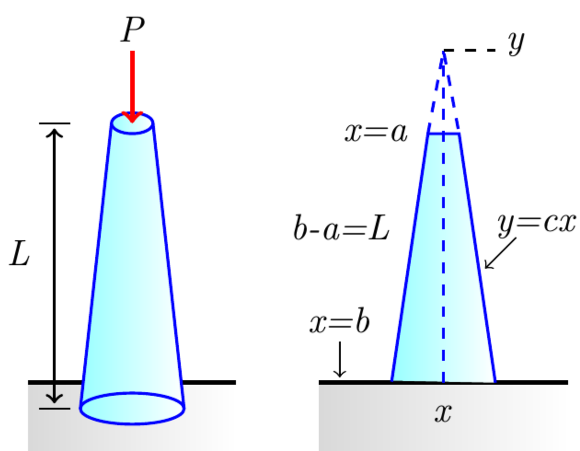
b) Conclude that the nth eigenvalue is given by $\mu_n = n\pi\dfrac{ab}{L}$, where $L = b - a$ is the length of the rod, and hence that the nth buckling force is

$$P_n = \frac{n^2\pi^2}{L^2}\left(\frac{a}{b}\right)^2 EI_0$$

42. When a constant vertical compressive force or load P was applied to a thin column of uniform cross section, the deflection $y(x)$ was a solution of the boundary-value problem

$$EI\frac{d^2y}{dy^2} + Py = 0 \; ; \quad y(0) = 0, \quad y(L) = 0$$

The assumption here is that the column is hinged at both ends. The column will buckle or deflect only when the compression force is a critical load P_n



a) Let assume that the column is of length L, is hinged at both ends, has circular cross sections, and is tapered. If the column, a truncated cone, has a linear taper $y = cx$ in cross section, the moment of inertia of a cross section with respect to an axis perpendicular to the $xy-plane$ is $I = \frac{1}{4}\pi r^4$, where $r = y$ and $y = cx$. Hence, we can write $I(x) = I_0(x\ b)^4$

, where $I_0 = I(b) = \frac{1}{4}\pi(cb)^4$. Substituting $I(x)$ into the differential equation, we see that the deflection in this case is determine from the *BVP*?

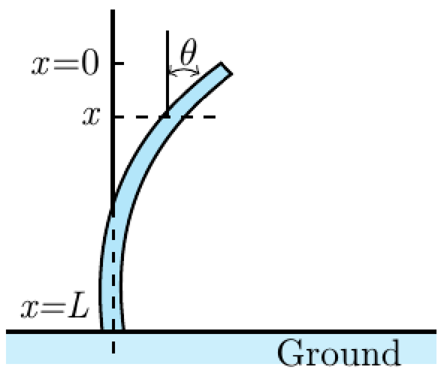
$$x^4 \frac{d^2y}{dx^2} + \lambda y = 0\ ; \quad y(a) = 0, \quad y(b) = 0$$

Where $\lambda = Pb^4 EI_0$

Find the critical loads P_n for the tapered column. Use an appropriate identity to express the buckling modes $y_n(x)$ as a single function.

b) Plot the graph of the first buckling mode $y_1(x)$ corresponding to the Euler load P_1 when $b = 11$ and $a = 1$

43. For a practical application, we now consider the problem of determining when a uniform vertical column will buckle under its own weight (after, perhaps, being nudged laterally just a bit by a passing breeze). We take $x = 0$ at the free top end of the column and $x = L > 0$ at its bottom; we assume that the bottom is rigidly imbedded in the ground, perhaps in concrete.



Denote the angular deflection of the column at the point x by $\theta(x)$. From the theory of elasticity, it follows that

$$EI\frac{d^2\theta}{dx^2} + g\rho x\theta = 0$$

Where E is the Young's modulus of the material of the column,

I is the cross-sectional moment of inertia

ρ is the linear density of the column

g is gravitational acceleration.

For physical reasons – no bending at the free top of the column and no deflection at its imbedded bottom – the boundary conditions are $\theta'(0) = 0, \quad \theta(L) = 0$

Determine the general equation of the length L.