

Section 4.4 – The Binomial Theorem

A binomial is a sum $a + b$, where a and b represent numbers. If n is a positive integer, then a general formula for expanding $(a + b)^n$ is given by the **binomial theorem**.

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

The expansions of $(a + b)^n$ for $n = 2, 3, 4$, and 5 have the following properties:

- ✓ There are $n + 1$ terms, the first being a^n and the last b^n
- ✓ The power of a decreases by 1 and the power of b increases by 1. For each term, the sum of the exponents of a and b is n .
- ✓ Each term has the form $(c)a^{n-k}b^k$, where the coefficient c is an integer and $k = 0, 1, 2, \dots, n$.
- ✓ The following formula is true for each of the first n terms of the expansion:

$$\frac{(\text{coefficient of term}) \cdot (\text{exponent of } a)}{\text{number of term}} = \text{coefficient of next term}$$

Coefficient of the $(k + 1)$ st Term in the Expansion of $(a + b)^n$

$$\frac{n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \dots \cdot (n-k+1)}{k \cdot (k-1) \cdot \dots \cdot 3 \cdot 2 \cdot 1}, \quad k = 1, 2, \dots, n$$

Factorial Notation

Definition of $n!$ (n factorial)

$$\begin{cases} n! = n(n-1)(n-2) \cdots 1 & \text{if } n > 0 \\ 0! = 1 \end{cases}$$

Calculators: Math \rightarrow Prob \rightarrow 4

Illustration

$$1! = 1$$

$$2! = 2 \cdot 1 = 2$$

$$3! = 3 \cdot 2 \cdot 1 = 6$$

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

Example

Simplify the quotient of factorial: $\frac{7!}{5!}$

Solution

$$\frac{7!}{5!} = \frac{7 \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 7 \cdot 6 = 42$$

Example

Simplify the quotient of factorial: $\frac{10!}{6!}$

Solution

$$\frac{10!}{6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot \cancel{6}!}{\cancel{6}!} = 10 \cdot 9 \cdot 8 \cdot 7 = 5040$$

Coefficient of the $(k+1)$ st Term in the Expansion of $(a+b)^n$ (Alternative Form)

$$\boxed{\binom{n}{k} = C(n, k) = \frac{n!}{k!(n-k)!}}, \quad k = 0, 1, 2, \dots, n$$

Example

Find $\binom{5}{2}, \binom{5}{0}$

Solution

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = \frac{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1}{(1 \cdot 2)(\cancel{1} \cdot \cancel{2} \cdot \cancel{3})} = \frac{20}{2} = \underline{10}$$

$$\binom{5}{0} = \frac{5!}{0!(5-0)!} = \frac{5!}{1(5!)} = \underline{1}$$

Binomial Theorem

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{k}a^{n-k}b^k + \dots + \binom{n}{n-1}ab^{n-1} + b^n$$

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \dots + \frac{n(n-1)(n-2)\dots(n-k+1)}{k!}a^{n-k}b^k + \dots + nab^{n-1} + b^n$$

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k}a^{n-k}b^k$$

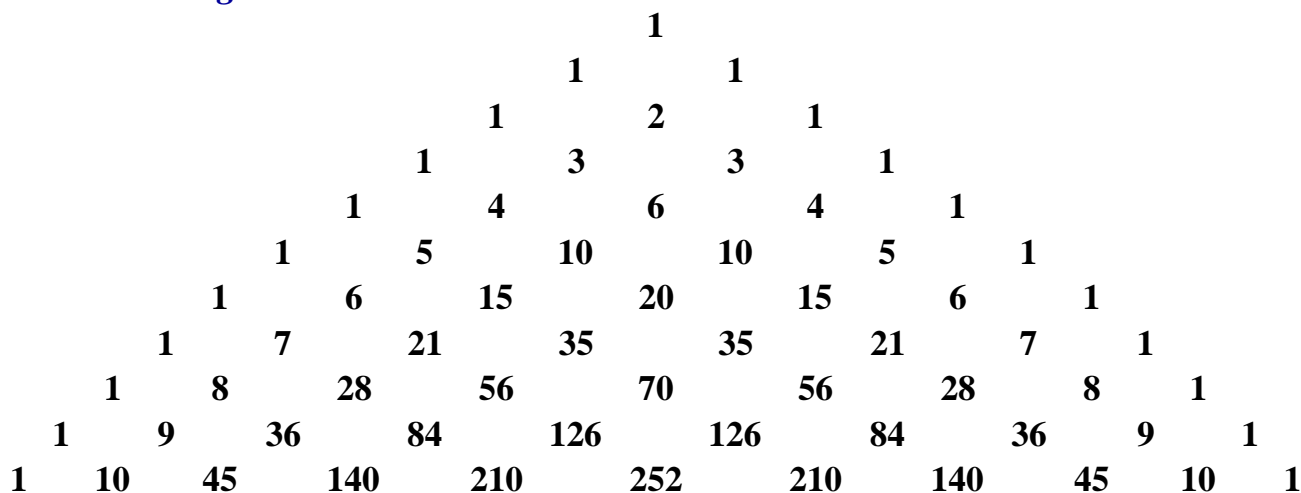
Example

Find the binomial expansion of $(2x + 3y^2)^4$

Solution

$$\begin{aligned}(2x + 3y^2)^4 &= (2x)^4 + \binom{4}{1}(2x)^3(3y^2)^1 + \binom{4}{2}(2x)^2(3y^2)^2 + \binom{4}{3}(2x)^1(3y^2)^3 + (3y^2)^4 \\&= 16x^4 + 4(8x^3)(3y^2) + 6(4x^2)(9y^4) + 4(2x)(27y^6) + 81y^8 \\&= 16x^4 + 96x^3y^2 + 216x^2y^4 + 216xy^6 + 81y^8\end{aligned}$$

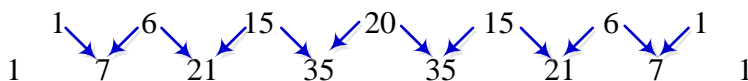
Pascal's Triangle



Example

Find the eighth row of the Pascal's triangle, and use it to expand $(a + b)^7$

Solution



$$(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

Example

Find the binomial expansion of $\left(\frac{1}{x} - 2\sqrt{x}\right)^5$

Solution

$$\begin{aligned} \left(\frac{1}{x} - 2\sqrt{x}\right)^5 &= \frac{1}{x^5} - 10\frac{1}{x^4}(\sqrt{x}) + 10\frac{1}{x^3}(4x) - 10\frac{1}{x^2}(8x\sqrt{x}) + 5\left(\frac{1}{x}\right)(16x^2) - 32x^{5/2} \\ &= \frac{1}{x^5} - 10\frac{1}{x^{7/2}} + 40\frac{1}{x^2} - 80\frac{1}{x^{1/2}} + 80x - 32x^{5/2} \end{aligned}$$

Exercises **Section 4.4 – The Binomial Theorem**

1. Find the *fifth* term in the expansion $(x^3 + \sqrt{y})^{13}$
2. Find the term involving q^{10} in the binomial expansion $(\frac{1}{3}p + q^2)^{12}$

Expand and simplify:

- | | | |
|-----------------------------------------|---------------------------------|-----------------------------------|
| 3. $(4x - y)^3$ | 16. $(ax + by)^5$ | 28. $(x^2 - 2y)^5$ |
| 4. $(x + y)^6$ | 17. $(\sqrt{x} - \sqrt{3})^4$ | 29. $(\frac{2}{x} + 3\sqrt{x})^4$ |
| 5. $(a - b)^6$ | 18. $(\sqrt{x} - \sqrt{2})^6$ | 30. $(2x + 5y)^7$ |
| 6. $(x - y)^7$ | 19. $(2x - 1)^{12}$ | 31. $(2x - 3)^{11}$ |
| 7. $(a + b)^8$ | 20. $(x - \frac{1}{x^2})^9$ | 32. $(2x - 3y)^6$ |
| 8. $(3t - 5x)^4$ | 21. $(\frac{2}{x} - 3y)^5$ | 33. $(2x + 3y)^5$ |
| 9. $(\frac{1}{3}x + y^2)^5$ | 22. $(3\sqrt{x} + 4\sqrt{x})^4$ | 34. $(3x - 2y)^4$ |
| 10. $(\frac{1}{x^2} + 3x)^6$ | 23. $(x + 1)^5$ | 35. $(x^2 + y^3)^3$ |
| 11. $(\sqrt{x} + \frac{1}{\sqrt{x}})^5$ | 24. $(x - 1)^5$ | 36. $(x^2 - y^2)^3$ |
| 12. $(2y - 3)^4$ | 25. $(x - 2)^6$ | 37. $(2 + i)^6$ |
| 13. $(x + 2)^5$ | 26. $(\frac{1}{x^3} - 2x)^5$ | 38. $(2 - i)^6$ |
| 14. $(x^2 - y^2)^6$ | 27. $(\frac{1}{x} - 2x)^6$ | 39. $(\sqrt{2} + i)^5$ |
| 15. $(ax - by)^4$ | | 40. $(3 - i)^4$ |