Lecture Two – Functions

Section 2.1 – Functions and Graphs

Increasing and *Decreasing* Functions

- A function *rises from left to right (x-coordinate)*, the function *f* is said to be ***increasing*** on an open interval ***I*** (a, b) (*x*-coordinate)

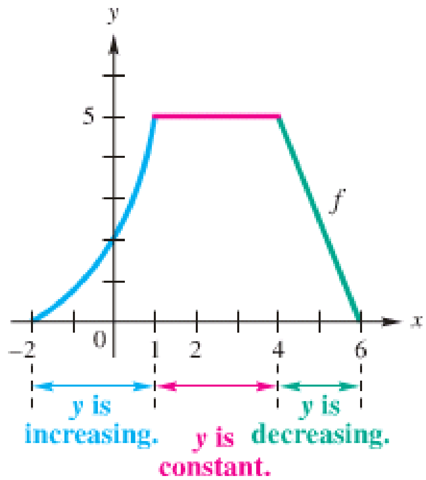
$$a < b \quad \Rightarrow \quad f(a) < f(b)$$

- A function *f* is said to be ***decreasing*** on an open interval ***I***

$$a < b \quad \Rightarrow \quad f(a) > f(b)$$

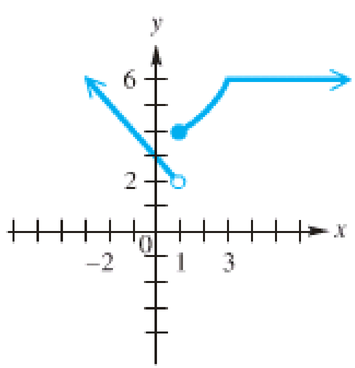
- A function *f* is said to be ***constant*** on an open interval ***I***

$$a < b \quad \Rightarrow \quad f(a) = f(b)$$



Example

Determine the intervals over which the function is increasing, decreasing, or constant



Increasing: [1, 3]

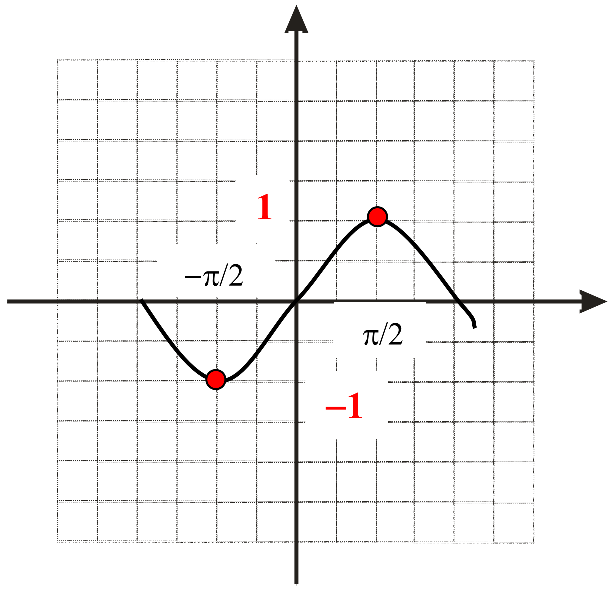
Decreasing: $(-\infty, 1)$

Constant: $[3, \infty)$

Relative *Maxima (um)* and *Minima (um)*

f(a) is a relative maximum if there exists an open interval I about *a* such that *f(a) > f(x),* for all *x* in I.

f(a) is a relative minimum if there exists an open interval I about *a* such that *f(a) < f(x),* for all *x* in I.

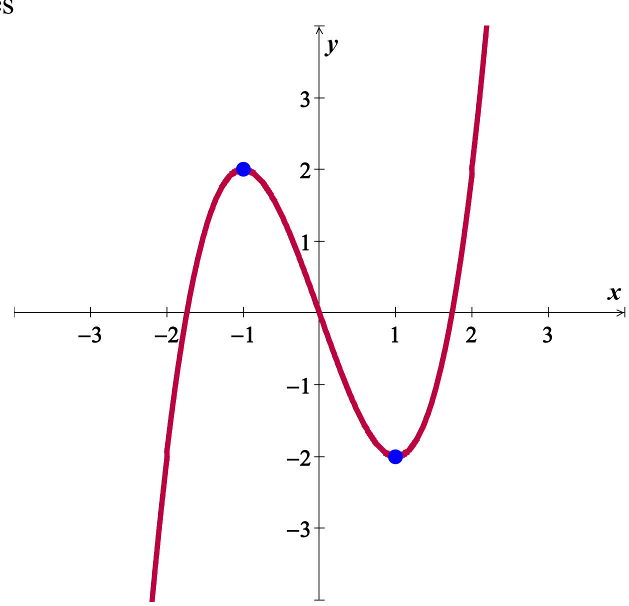


The relative minimum value of the function is −1 @ $x = -\pi/2$

The relative maximum value of the function is 1 @ $x = \pi/2$

Example

State the intervals on which the given function $f(x) = x^3 - 3x$ is increasing, decreasing, or constant, and determine the extreme values



Increasing $(-\infty,\ -1) \cup (1,\ \infty)$ ***RMIN*** $(1,\ -2)$

Decreasing $(-1,\ 1)$ ***RMAX*** $(-1,\ 2)$

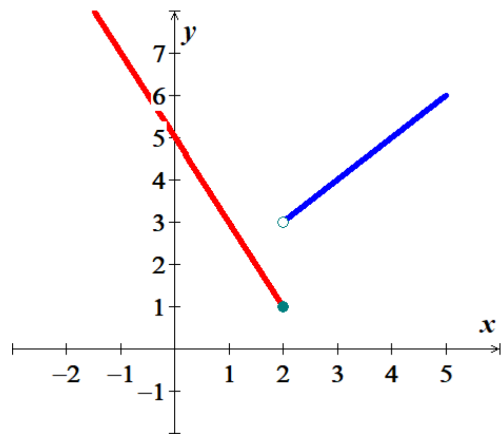
Piecewise-Defined Functions

Function are sometimes described by more than one expression, we call such functions ***piecewise-defined functions***.

Example

Graph function

$$f(x)=\begin{cases}-2x+5 & if \quad x\le 2\\ x+1 & if \quad x>2\end{cases}$$



Find: $f(2)=-2(2)+5=1$

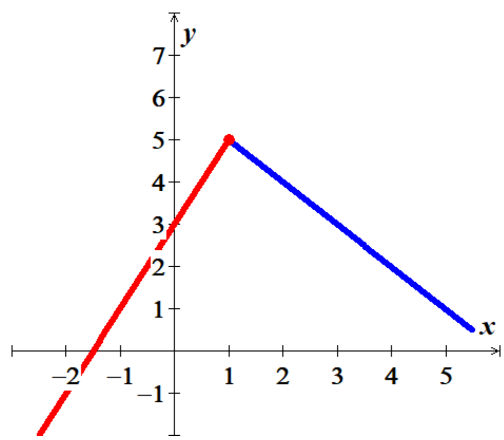
$f(0)=-2(0)+5=5$

$f(4)=4+1=5$

Example

Graph function

$$f(x)=\begin{cases}2x+3 & if \quad x\le 1\\ -x+6 & if \quad x>1\end{cases}$$



Example

$$C(t)=\begin{cases}20 & \text{if } 0\le t\le 60\\ 20+0.40(t-60) & \text{if } t>60\end{cases}$$

Find $C(40)$, $C(80)$, and $C(60)$

Solution

a) $C(40) = 20$

b) $C(80) = 20 + 0.40(80 - 60) = 28$

c) $C(60) = 20$

Exercise *Section* 2.1 – Functions and Graphs

1. $f(x)=\begin{cases} 2+x & if\ \ x<-4 \\ -x & if\ \ -4\le x\le 2 \\ 3x & if\ \ x>2 \end{cases}$ **Find**: $f(-5), f(-1), f(0),\ and\ f(3)$

2. $f(x)=\begin{cases} -2x & if\ \ x<-3 \\ 3x-1 & if\ \ -3\le x\le 2 \\ -4x & if\ \ x>2 \end{cases}$ **Find**: $f(-5), f(-1), f(0),\ and\ f(3)$

3. $f(x)=\begin{cases} x^3+3 & if\ \ -2\le x\le 0 \\ x+3 & if\ \ 0<x<1 \\ 4+x-x^2 & if\ \ 1\le x\le 3 \end{cases}$ **Find**: $f(-5), f(-1), f(0),\ and\ f(3)$

4. $h(x)=\begin{cases} \dfrac{x^2-9}{x-3} & if\ \ x\ne 3 \\ 6 & if\ \ x=3 \end{cases}$ **Find**: $h(5),\ h(0),\ and\ h(3)$

5. $f(x)=\begin{cases} 3x+5 & if\ \ x<0 \\ 4x+7 & if\ \ x\ge 0 \end{cases}$ **Find**

a) $f(0)$ *b)* $f(-2)$ *c)* $f(1)$ *d)* $f(3)+f(-3)$ *e)* Graph $f(x)$

6. $f(x)=\begin{cases} 6x-1 & if\ \ x<0 \\ 7x+3 & if\ \ x\ge 0 \end{cases}$ **Find**

a) $f(0)$ *b)* $f(-1)$ *c)* $f(4)$ *d)* $f(2)+f(-2)$ *e)* Graph $f(x)$

7. $f(x)=\begin{cases} 2x+1 & if\ \ x\le 1 \\ 3x-2 & if\ \ x>1 \end{cases}$ **Find**

a) $f(0)$ *b)* $f(2)$ *c)* $f(-2)$ *d)* $f(1)+f(-1)$ *e)* Graph $f(x)$

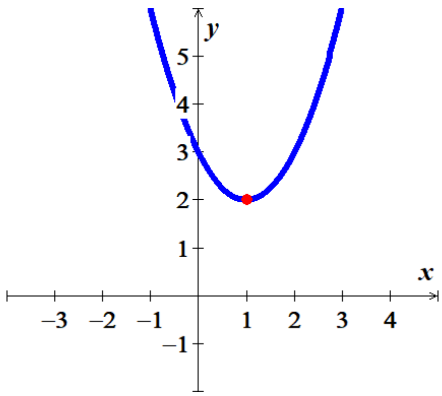
8. Graph the piecewise function defined by $f(x)=\begin{cases} 3 & if\ \ x\le -1 \\ x-2 & if\ \ x>-1 \end{cases}$

9. Sketch the graph $f(x)=\begin{cases} x+2 & if\ \ x\le -1 \\ x^3 & if\ \ -1<x<1 \\ -x+3 & if\ \ x\ge 1 \end{cases}$

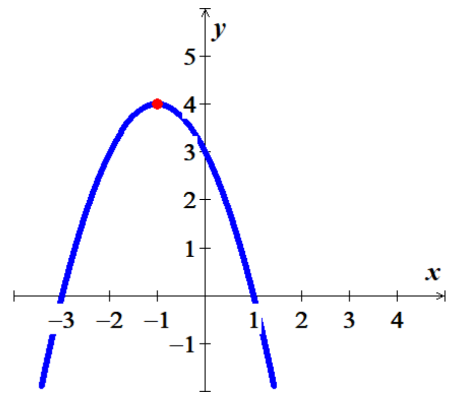
10. Sketch the graph $f(x)=\begin{cases} x-3 & if \quad x\le -2 \\ -x^2 & if \quad -2<x<1 \\ -x+4 & if \quad x\ge 1 \end{cases}$

(**37 – 42**) Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

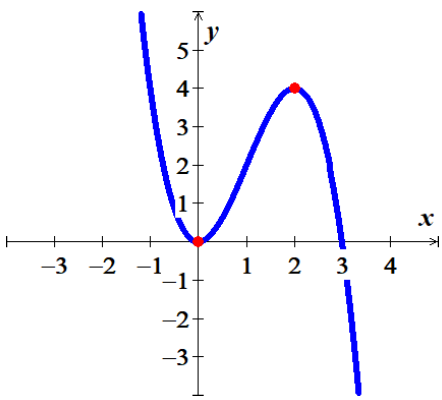
11. $f(x)=x^2-2x+3$



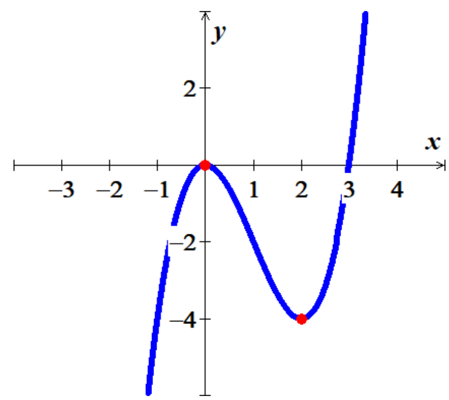
12. $f(x)=-x^2-2x+3$



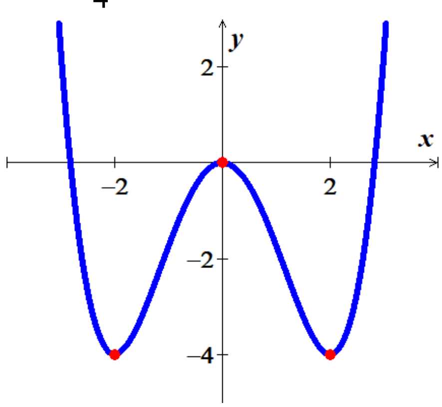
13. $f(x)=-x^3+3x^2$



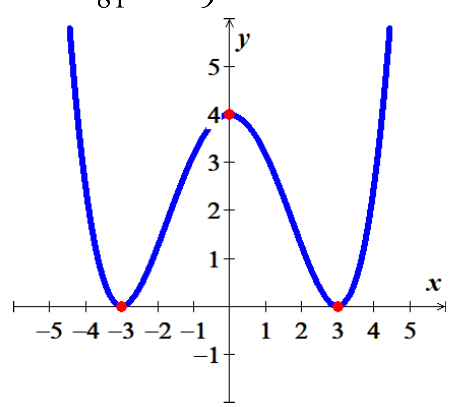
14. $f(x)=x^3-3x^2$



15. $f(x)=\frac{1}{4}x^4-2x^2$



16. $f(x)=\frac{4}{81}x^4-\frac{8}{9}x^2+4$

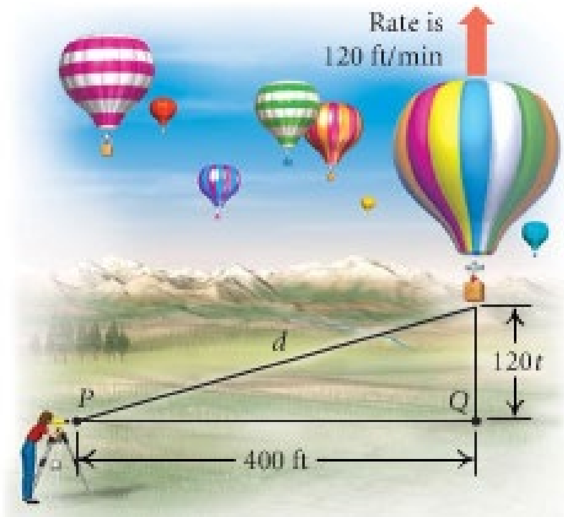


17. The elevation *H*, in *meters*, above sea level at which the boiling point of water is in ***t*** *degrees Celsius* is given by the function

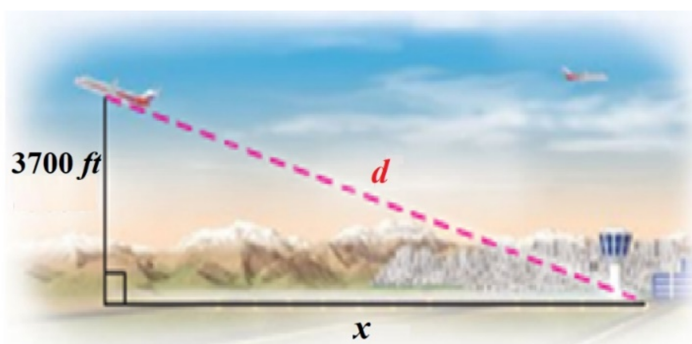
$$H(t) = 1000(100 - t) + 580(100 - t)^2$$

At what elevation is the boiling point 99.5°.

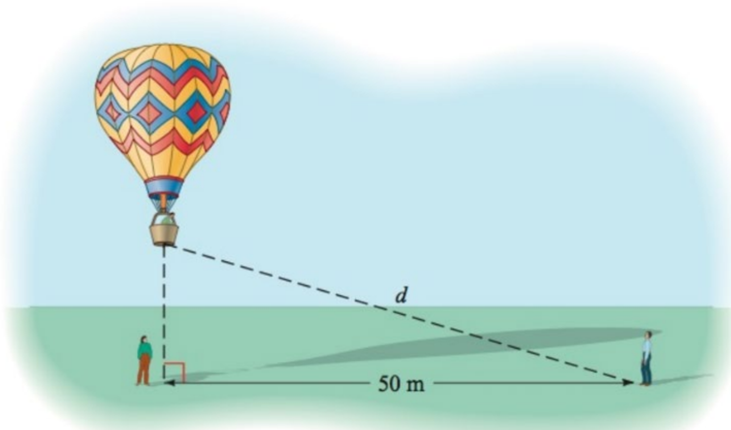
18. A hot-air balloon rises straight up from the ground at a rate of 120 *ft./min*. The balloon is tracked from a rangefinder on the ground at point *P*, which is 400 *feet*. from the release point *Q* of the balloon. Let *d* be the distance from the balloon to the rangefinder and *t* – the time, in *minutes*, since the balloon was released. Express *d* as a function of *t*.



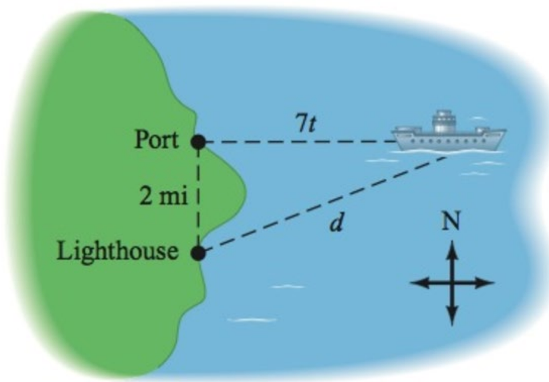
19. An airplane is flying at an altitude of 3700 *feet*. The slanted distance directly to the airport is *d feet*. Express the horizontal distance *x* as a function of *d*.



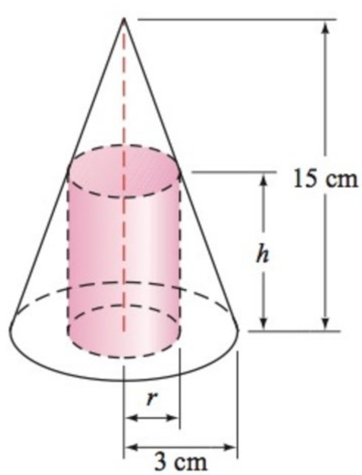
20. For the first minute of flight, a hot air balloon rises vertically at a rate of 3 *m/sec*. If t is the time in *seconds* that the balloon has been airborne, write the distance d between the balloon and a point on the ground 50 *meters* from the point to lift off as a function of t.



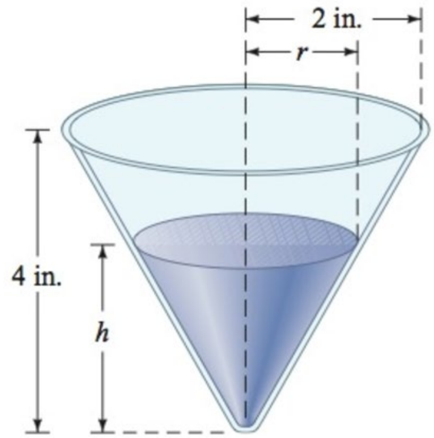
21. A light house is 2 *miles* south of a port. A ship leaves port and sails east at a rate of 7 *miles* per *hour*. Express the distance d between the ship and the lighthouse as a function of time, given that the ship has been sailing for t *hours*.



22. A cone has an altitude of 15 *cm* and a radius of 3 *cm*. A right circular cylinder of radius r and height h is inscribed in the cone. Use similar triangles to write h as a function of r.

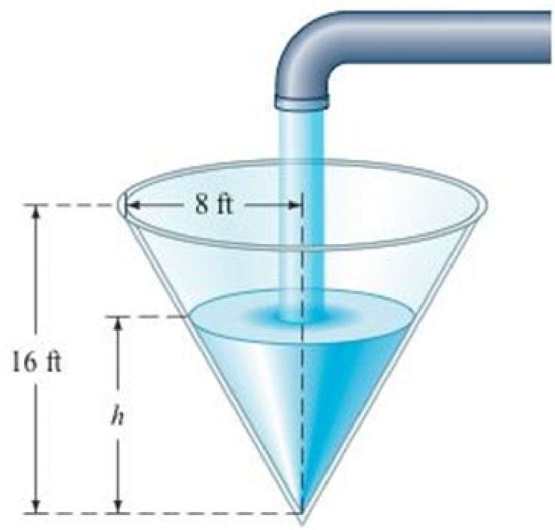


23. Water is flowing into a conical drinking cup with an altitude of 4 *inches* am a radius of 2 *inches*.



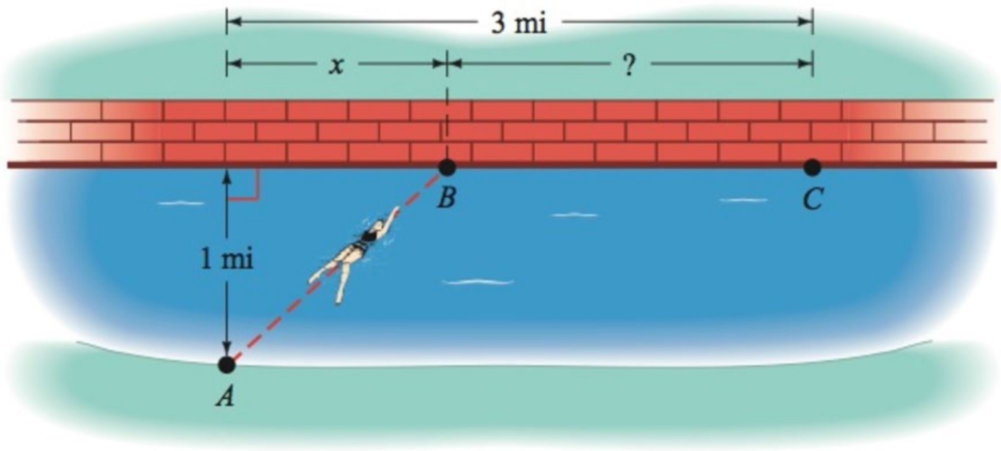
a) Write the radius r of the surface of the water as a function of its depth h.
b) Write the volume V of the water as a function of its depth h.

24. A water tank has the shape of a right circular cone with height 16 *feet* and radius 8 *feet*. Water is running into the tank so that the radius r (in *feet*) of the surface of the water is given by $r = 1.5t$, where t is the time (in *minutes*) that the water has been running.

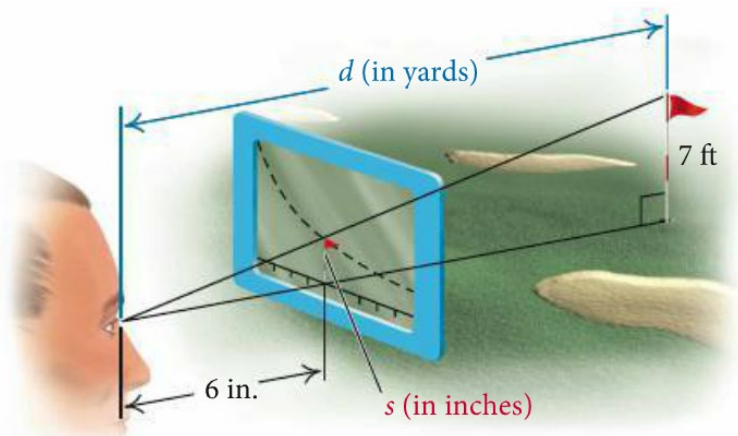


a) The area A of the surface of the water is $A = \pi r^2$. Find $A(t)$ and use it to determine the area of the surface of the water when $t = 2$ *minutes*.
b) The volume V of the water is given by $V = \frac{1}{3}\pi r^2 h$. Find $V(t)$ and use it to determine the volume of the water when $t = 3$ *minutes*

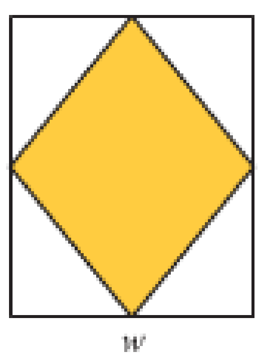
25. An athlete swims from point ***A*** to point ***B*** at a rate of 2 *miles* per *hour* and runs from point ***B*** to point ***C*** at a rate of 8 *miles* per *hour*. Use the dimensions in the figure to write the time t required to reach point ***C*** as a function of x.



26. A device used in golf to estimate the distance ***d***, in *yards*, to a hole measures the size ***s***, in *inches*, that the 7-*feet* pin appears to be in a viewfinder. Express the distance ***d*** as a function of ***s***.



27. A rhombus is inscribed in a rectangle that is ***w*** *meters* wide with a perimeter of 40 *m*. Each vertex of the rhombus is a midpoint of a side of the rectangle. Express the area of the rhombus as a function of the rectangle's width.

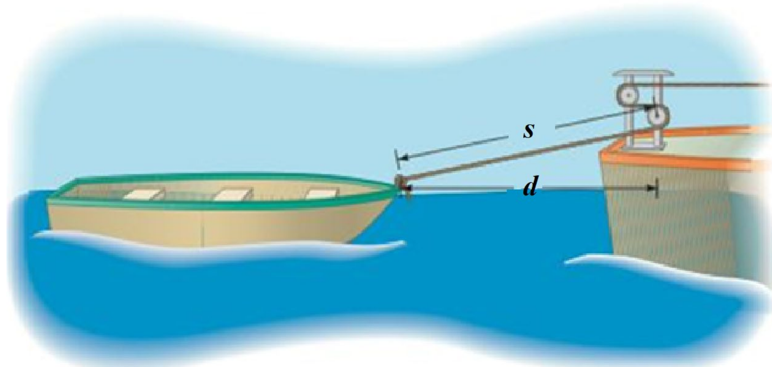


28. The surface area S of a right circular cylinder is given by the formula $S = 2\pi rh + 2\pi r^2$. if the height is twice the radius, find each of the following.

a) A function $S(r)$ for the surface area as a function of *r*.

b) A function $S(h)$ for the surface area as a function of *h*.

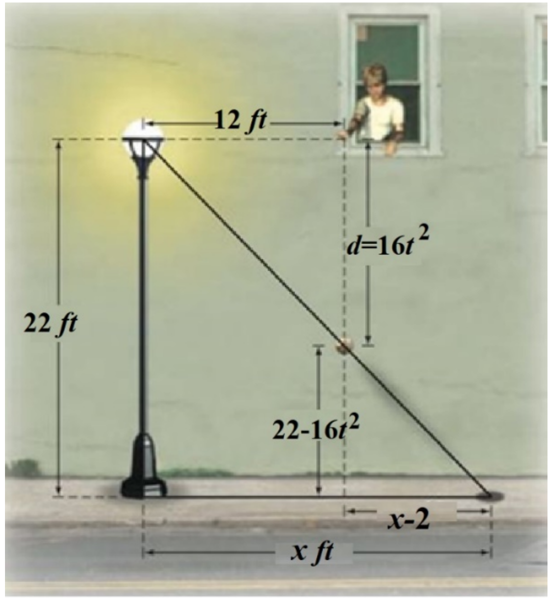
29. A boat is towed by a rope that runs through a pulley that is 4 *feet* above the point where the rope is tied to the boat. The length (in *feet*) of the rope from the boat to the pulley is given by $s = 48 - t$, where *t* is the time in *seconds* that the boat has been in tow. The horizontal distance from the pulley to the boat is *d*.



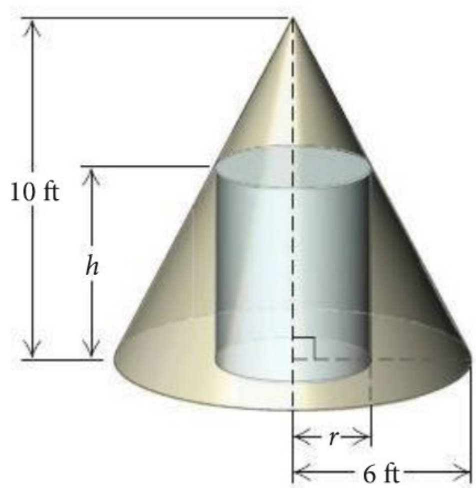
a) Find $d(t)$

b) Evaluate $s(35)$ and $d(35)$

30. The light from a lamppost casts a shadow from a ball that was dropped from a height of 22 *feet* above the ground. The distance *d*, in *feet*, the ball has dropped *t seconds* after it is released is given by $d(t) = 16t^2$. Find the distance *x*, in *feet*, of the shadow from the base of the lamppost as a function of time *t*.



31. *A right circular cylinder of height h and a radius r is inscribed in a right circular cone with a height of 10 *feet* and a base with radius 6 *feet*.



a) Express the height h of the cylinder as a function of r.
b) Express the volume V of the cylinder as a function of r.
c) Express the volume V of the cylinder as a function of h.

Section 2.2 – Function Operations

The *Domain* of a Function

1. ***Rational*** function: $\frac{f(x)}{h(x)}$ $\Rightarrow$ ***Domain***: $\boxed{h(x) \neq 0}$

Example: $f(x) = \frac{1}{x-3}$

Domain: $x \neq 3$ $\{x \mid x \neq 3\}$

Or $(-\infty, 3) \cup (3, \infty)$ *Interval Notation*

Or $\mathbb{R} - \{3\}$

2. ***Irrational*** function: $\sqrt{g(x)}$ $\Rightarrow$ ***Domain***: $\boxed{g(x) \geq 0}$

Example: $g(x) = \sqrt{3-x} + 5$

$$3 - x \geq 0$$
$$-x \geq -3$$

Domain: $x < 3$ $(-\infty, 3]$

3. ***Otherwise***: Domain all real numbers $(-\infty, \infty)$

Example: $f(x) = x^3 + |x|$

Domain: All real numbers $\mathbb{R}$

$(-\infty, \infty)$

(1) & (2) $\rightarrow$ Find the domain: $f(x) = \frac{x+1}{\sqrt{x-3}}$

$x > 3$

Domain: $(3, \infty)$

Example

Find the domain

a) $f(x)=x^2+3x-17$

Domain: $\underline{\mathbb{R}}$

b) $g(x)=\frac{5x}{x^2-49}$

$x^2 \neq 49$

$\underline{x \neq \pm 7}$

Domain: $\begin{cases} \{x \mid x \neq \pm 7\} & \text{or} \\ (-\infty,-7)\cup(-7,7)\cup(7,\infty) \end{cases}$

c) $h(x)=\sqrt{9x-27}$

$9x-27 \geq 0$

$9x \geq 27$

Domain: $\underline{x \geq 3}$ $\quad [3, \infty)$

The *Algebra* of Functions

$$(f+g)(x)=f(x)+g(x)$$

$$(f-g)(x)=f(x)-g(x)$$

$$(f\cdot g)(x)=f(x)\cdot g(x)$$

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$$

Example

Let $f(x)=x^2+1$ and $g(x)=3x+5$. Find each of the following $(f+g)(1)$, $(f-g)(-3)$, $(fg)(5)$, and $\left(\frac{f}{g}\right)(0)$

Solution

$$\begin{aligned}(f+g)(1)&=f(1)+g(1)\\&=1^2+1+3(1)+5\\&=1+1+3+5\\&\underline{=10}\end{aligned}$$

$$\begin{aligned}(f-g)(-3)&=f(-3)-g(-3)\\&=(-3)^2+1-(3(-3)+5)\\&\underline{=14}\end{aligned}$$

$$\begin{aligned}(fg)(5)&=f(5)\cdot g(5)\\&=\left(5^2+1\right)\cdot(3(5)+5)\\&=(26)\cdot(20)\\&\underline{=520}\end{aligned}$$

$$\begin{aligned}\left(\frac{f}{g}\right)(0)&=\frac{f(0)}{g(0)}\\&=\frac{0^2+1}{3(0)+5}\\&\underline{=\frac{1}{5}}\end{aligned}$$

Example

Let $f(x) = 8x - 9$ and $g(x) = \sqrt{2x-1}$. Find each of the following and give the domain

$(f+g)(x), \quad (f-g)(x), \quad (fg)(x), \left(\frac{f}{g}\right)(x)$

Solution

Domain of f: $\quad (-\infty, \infty)$

Domain of g: $\quad \left[\frac{1}{2}, \infty\right) \qquad \sqrt{2x-1 \geq 0} \ \to 2x \geq 1 \ \Rightarrow x \geq \frac{1}{2}$

a) $(f+g)(x) = 8x - 9 + \sqrt{2x-1}$

Domain: $x \geq \frac{1}{2} \qquad \left[\frac{1}{2}, \infty\right)$

b) $(f-g)(x) = 8x - 9 - \sqrt{2x-1}$

Domain: $x \geq \frac{1}{2} \qquad \left[\frac{1}{2}, \infty\right)$

c) $(fg)(x) = (8x-9)\sqrt{2x-1}$

Domain: $x \geq \frac{1}{2} \qquad \left[\frac{1}{2}, \infty\right)$

d) $\left(\frac{f}{g}\right)(x) = \dfrac{8x-9}{\sqrt{2x-1}}$

Domain: $x > \frac{1}{2} \qquad \left(\frac{1}{2}, \infty\right)$

Example

Let $f(x) = \sqrt{x-3}$ and $g(x) = \sqrt{x+1}$

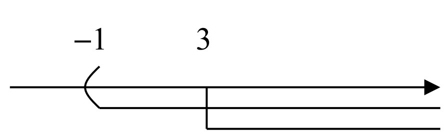
Find $(f+g)(x)$ and its domain, $\left(\frac{f}{g}\right)(x)$ and its domain

Solution

Domain $f(x)$: $x \geq 3 \quad$ *and* **Domain** $g(x)$: $x \geq -1$

a) $(f+g)(x) = \sqrt{x-3} + \sqrt{x+1}$

b) $x \geq 3$ *and* $x \geq -1 \ \Rightarrow$ **Domain**: $x \geq 3$



c) $\left(\frac{f}{g}\right)(x) = \dfrac{\sqrt{x-3}}{\sqrt{x+1}}$

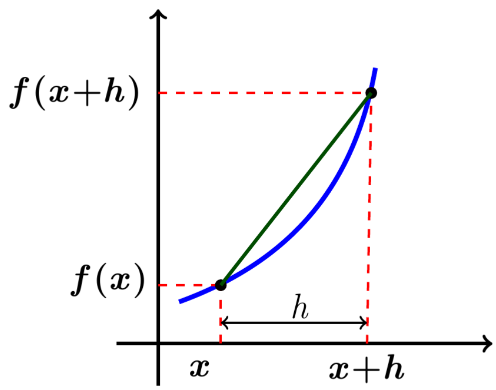
$$\rightarrow \begin{cases} x-3 \ge 0 & \Rightarrow \quad \underline{x \ge 3} \\ x+1 > 0 & \Rightarrow \quad \underline{x > -1} \end{cases}$$

Domain: $x \ge 3 \qquad [3, \infty)$

Difference Quotients

$$\frac{f(x+h)-f(x)}{(x+h)-x}$$

The difference quotient is given by: $\dfrac{f(x+h)-f(x)}{h}$



Example

For the function *f* given by $f(x)=2x-3$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$\begin{aligned} f(x+h) &= 2(--)-3 \\ &= 2(x+h)-3 \\ &= 2x+2h-3 \end{aligned}$$

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{\overbrace{2x+2h-3}^{f(x+h)} - \overbrace{(2x-3)}^{f(x)}}{h} \\ &= \frac{2x+2h-3-2x+3}{h} \\ &= \frac{2h}{h} \\ &= 2 \end{aligned}$$

Example

For the function *f* given by $f(x)=-2x^2+x+5$, find the difference quotient $\frac{f(x+h)-f(x)}{h}$

Solution

$$f(\boldsymbol{x+h})=-2(\boldsymbol{x+h})^2+(\boldsymbol{x+h})+5 \qquad (a+b)^2=a^2+2ab+b^2$$

$$f(x+h)=-2\left(x^2+2hx+h^2\right)+x+h+5$$

$$f(x+h)=-2x^2-4hx-2h^2+x+h+5$$

$$\frac{f(x+h)\ -f(x)}{h}=\frac{-2x^2-4hx-2h^2+x+h+5-(-2x^2+x+5)}{h}$$

$$=\frac{-2x^2-4hx-2h^2+x+h+5+2x^2-x-5}{h}$$

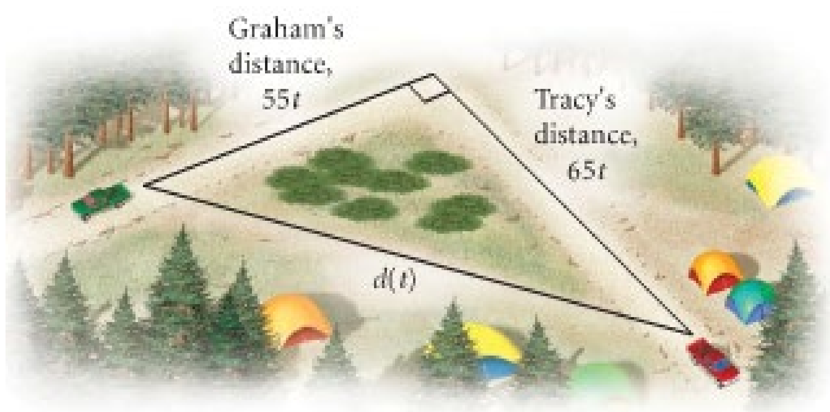
$$=\frac{-4hx-2h^2+h}{h}$$

$$=\frac{-4hx}{h}-\frac{2h^2}{h}+\frac{h}{h}$$

$$=-4x-2h+1$$

Example

Tracy and Graham drive away from a camp-ground at right angles to each other. Tracy's speed is 65 mph and Graham's is 55 *mph*.



a) Express the distance between the cars as a function of time.
b) Find the domain of the function.

Solution

a) *Distance* = *velocity* * *time*

Use Pythagorean Theorem:

$$d^2(t)=(65t)^2+(55t)^2$$

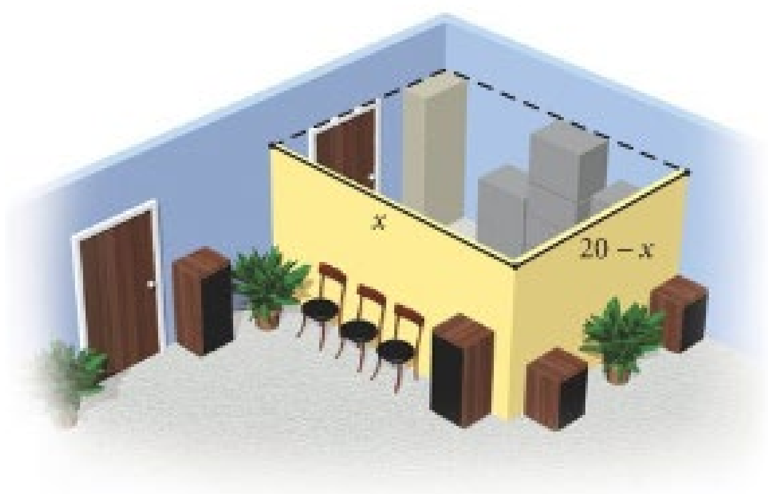
$$d^2 = 4225t^2 + 3025t^2$$
$$= 7250t^2$$

$$d(t) = \sqrt{7250t^2}$$
$$= \sqrt{7250}\sqrt{t^2}$$
$$\approx 85.15|t|$$
$$\underline{= 85.15\, t}$$

b) ***Domain***: $t \geq 0$

Example: (storage area)

The sound Shop has 20 *feet*. of dividers with which to set off a rectangular area for the storage of overstock. If a corner of the store is used for the storage area, the partition need only form two sides of a rectangle.



a) Express the floor area of the storage space as a function of the length of the partition.

b) Find the domain of the function.

Solution

Let x = the length

$width + length = 20$

$width = 20 - length$

a) ***Area*** = $length * width$

$$= x(20 - x)$$
$$\underline{= 20x - x^2}$$

b) ***Domain***: x value varies from 0 to 20 $\Rightarrow$ (0, 20)

Exercises *Section* 2.2 – Function Operations

(**1 – 80**) Find the Domain

1. $f(x)=7x+4$

2. $f(x)=|3x-2|$

3. $f(x)=3x+\pi$

4. $f(x)=\sqrt{7}x+\frac{1}{2}$

5. $f(x)=-2x^2+3x-5$

6. $f(x)=x^3-2x^2+x-3$

7. $f(x)=x^2-2x-15$

8. $f(x)=4-\frac{2}{x}$

9. $f(x)=\frac{1}{x^4}$

10. $g(x)=\frac{3}{x-4}$

11. $y=\frac{2}{x-3}$

12. $y=\frac{-7}{x-5}$

13. $f(x)=\frac{x+5}{2-x}$

14. $f(x)=\frac{8}{x+4}$

15. $f(x)=\frac{1}{x+4}$

16. $f(x)=\frac{1}{x-4}$

17. $f(x)=\frac{3x}{x+2}$

18. $f(x)=x-\frac{2}{x-3}$

19. $f(x)=x+\frac{3}{x-5}$

20. $f(x)=\frac{1}{2}x-\frac{8}{x+7}$

21. $f(x)=\frac{1}{x-3}-\frac{8}{x+7}$

22. $f(x)=\frac{1}{x+4}-\frac{2x}{x-4}$

23. $f(x)=\frac{3x^2}{x+3}-\frac{4x}{x-2}$

24. $f(x)=\frac{1}{x^2-2x+1}$

25. $f(x)=\frac{x}{x^2+3x+2}$

26. $f(x)=\frac{x^2}{x^2-5x+4}$

27. $f(x)=\frac{1}{x^2-4x-5}$

28. $g(x)=\frac{2}{x^2+x-12}$

29. $h(x)=\frac{5}{\frac{4}{x}-1}$

30. $y=\sqrt{x}$

31. $f(x)=\sqrt{8-3x}$

32. $y=\sqrt{4x+1}$

33. $y=\sqrt{7-2x}$

34. $f(x)=\sqrt{8-x}$

35. $f(x)=\sqrt{3-2x}$

36. $f(x)=\sqrt{3+2x}$

37. $f(x)=\sqrt{5-x}$

38. $f(x)=\sqrt{x-5}$

39. $f(x)=\sqrt{6-3x}$

40. $f(x)=\sqrt{3x-6}$

41. $f(x)=\sqrt{2x+7}$

42. $f(x)=\sqrt{x^2-16}$

43. $f(x)=\sqrt{16-x^2}$

44. $f(x)=\sqrt{9-x^2}$

45. $f(x)=\sqrt{x^2-25}$

46. $f(x)=\sqrt{x^2-5x+4}$

47. $f(x)=\sqrt{x^2+5x+4}$

48. $f(x)=\sqrt{x^2+3x+2}$

49. $f(x)=\sqrt{x^2-3x+2}$

50. $f(x)=\sqrt{x-4}+\sqrt{x+1}$

51. $f(x)=\sqrt{3-x}+\sqrt{x-2}$

52. $f(x)=\sqrt{1-x}+\sqrt{4-x}$

53. $f(x)=\sqrt{1-x}-\sqrt{x-3}$

54. $f(x)=\sqrt{x+4}-\sqrt{x-1}$

55. $f(x)=\frac{\sqrt{x+1}}{x}$

56. $g(x)=\frac{\sqrt{x-3}}{x-6}$

57. $f(x)=\frac{\sqrt{x+4}}{\sqrt{x-1}}$

58. $f(x)=\frac{\sqrt{5-x}}{x}$

59. $f(x)=\frac{x}{\sqrt{5-x}}$

60. $f(x)=\dfrac{1}{x\sqrt{5-x}}$

61. $f(x)=\dfrac{x+1}{x^3-4x}$

62. $f(x)=\dfrac{\sqrt{x+5}}{x}$

63. $f(x)=\dfrac{x}{\sqrt{x+5}}$

64. $f(x)=\dfrac{1}{x\sqrt{x+5}}$

65. $f(x)=\dfrac{x+3}{\sqrt{x-3}}$

66. $f(x)=\dfrac{\sqrt{x+3}}{\sqrt{x-3}}$

67. $f(x)=\dfrac{\sqrt{x-2}}{\sqrt{x+2}}$

68. $f(x)=\dfrac{\sqrt{2-x}}{\sqrt{x+2}}$

69. $f(x)=\dfrac{x-4}{\sqrt{x-2}}$

70. $f(x)=\dfrac{1}{(x-3)\sqrt{x+3}}$

71. $f(x)=\sqrt{x+2}+\sqrt{2-x}$

72. $f(x)=\sqrt{(x-2)(x-6)}$

73. $f(x)=\sqrt{x+3}-\sqrt{4-x}$

74. $f(x)=\dfrac{\sqrt{4x-3}}{x^2-4}$

75. $f(x)=\dfrac{4x}{6x^2+13x-5}$

76. $f(x)=\dfrac{\sqrt{2x-3}}{x^2-5x+4}$

77. $f(x)=\dfrac{x^2}{\sqrt{x^2-5x+4}}$

78. $f(x)=\dfrac{x+2}{\sqrt{x^2+5x+4}}$

79. $f(x)=\dfrac{\sqrt{x+2}}{\sqrt{x^2+3x+2}}$

80. $f(x)=\dfrac{\sqrt{2x+3}}{x^2-6x+5}$

81. Let $f(x)=4x-3$ and $g(x)=5x+7$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\dfrac{f}{g}\right)(x)$

82. Let $f(x)=2x^2+3$ and $g(x)=3x-4$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\dfrac{f}{g}\right)(x)$

83. Let $f(x)=x^2-2x-3$ and $g(x)=x^2+3x-2$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\dfrac{f}{g}\right)(x)$

84. Let $f(x)=\sqrt{4x-1}$ and $g(x)=\dfrac{1}{x}$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\dfrac{f}{g}\right)(x)$

85. Given that $f(x)=x+1$ and $g(x)=\sqrt{x+3}$

a) Find $(f+g)(x)$

b) Find the domain of $(f+g)(x)$

c) Find: $(f+g)(6)$

86. Given that $f(x) = x^2 - 4$ and $g(x) = x + 2$

a) Find $(f+g)(x)$ and its domain

b) Find $(f/g)(x)$ and its domain

87. Let $f(x) = x^2 + 1$ and $g(x) = 3x + 5$. Find $(f+g)(1)$, $(f-g)(-3)$, $(fg)(5)$, and $\left(\frac{f}{g}\right)(0)$

88. Find $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, *and* $(f/g)(x)$ and the domain of

$f(x) = \sqrt{3-2x}$, $\quad g(x) = \sqrt{x+4}$

89. Find $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, *and* $(f/g)(x)$ and the domain of

$f(x) = \dfrac{2x}{x-4}$, $\quad g(x) = \dfrac{x}{x+5}$

90. Find $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, *and* $(f/g)(x)$ of $f(x) = x - 5$ and $g(x) = x^2 - 1$

(**88 – 103**) Find and simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$ for the given function

91. $f(x) = 9x + 5$

92. $f(x) = 6x + 2$

93. $f(x) = 4x + 11$

94. $f(x) = 3x - 5$

95. $f(x) = -2x - 3$

96. $f(x) = -4x + 3$

97. $f(x) = 3x - 6$

98. $f(x) = -5x - 7$

99. $f(x) = 2x^2$

100. $f(x) = 5x^2$

101. $f(x) = 3x^2 - 4x$

102. $f(x) = 2x^2 - 3x$

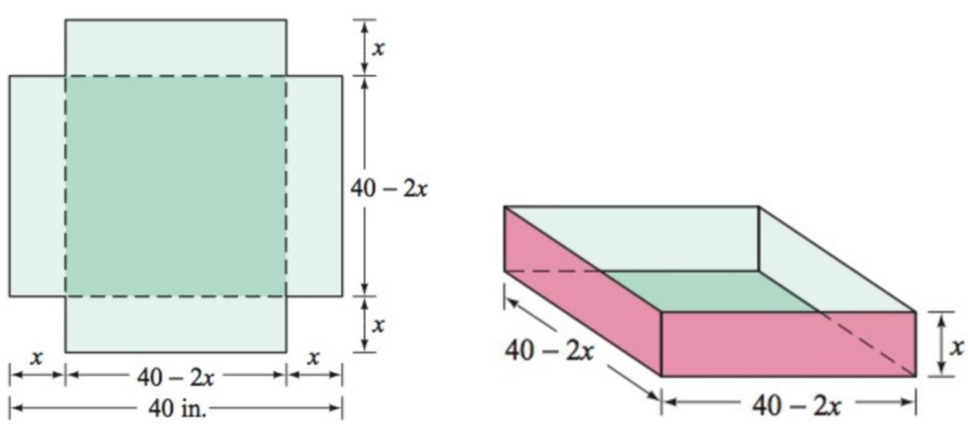
103. $f(x) = 2x^2 - x - 3$

104. $f(x) = x^2 - 2x + 5$

105. $f(x) = 3x^2 - 2x + 5$

106. $f(x) = -2x^2 - 3x + 7$

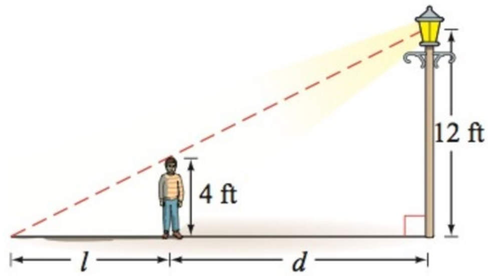
107. An open box is to be made from a square piece of cardboard that measures 40 *inches* on each side, to construct the box, squares that measure x *inches* on each side are cut from each corner of the cardboard.



a) Express the volume V of the box as a function of x.

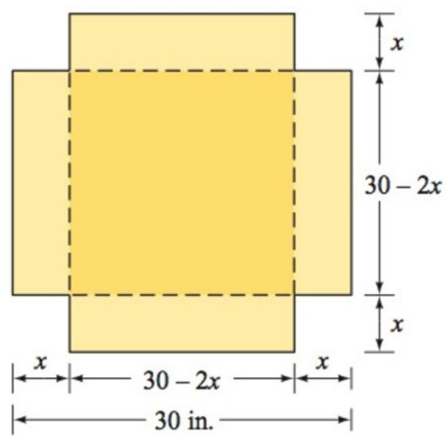
b) Determine the domain of V.

108. A child 4 *feet* tall is standing near a street lamp that is 12 *feet* high. The light from the lamp casts a shadow.



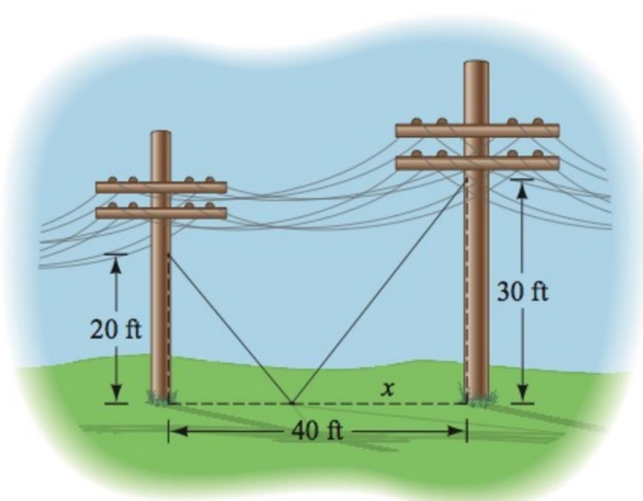
a) Find the length l of the shadow as a function of the distance d of the child from the lamppost.
b) What is the domain of the function?
c) What is the length of the shadow when the child is 8 *feet* from the base of the lamppost?

109. An open box is to be made from a square piece of cardboard with the dimensions 30 *inches* by 30 *inches* by cutting out squares of area x^2 from each corner.



a) Express the volume V of the box as a function of x.
b) Determine the domain of V.

110. Two guy wires are attached to utility poles that are 40 *feet* apart.

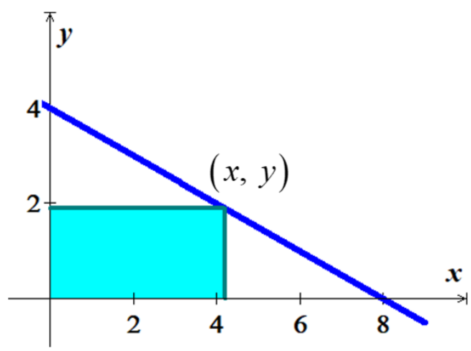


a) Find the total length of the two guy wires as a function of x.
b) What is the domain of this function?

111. A rancher has 360 *yard*s. of fencing with which to enclose two adjacent rectangular corrals, one for sheep and one for cattle. A river forms one side of the corrals. Suppose the width of each corral is x *yards*.

a) Express the total area of the two corrals as a function of x.
b) Find the domain of the function.

112. A rectangle is bounded by the x- and y-*axis* of $y=-\frac{1}{2}x+4$



a) Find the area of the rectangle as a function of x.
b) What is the domain of this function.

Section 2.3 – Composition Functions

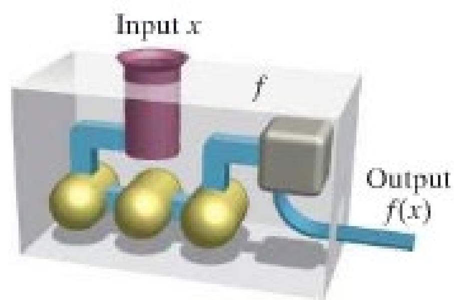
Composition of Functions

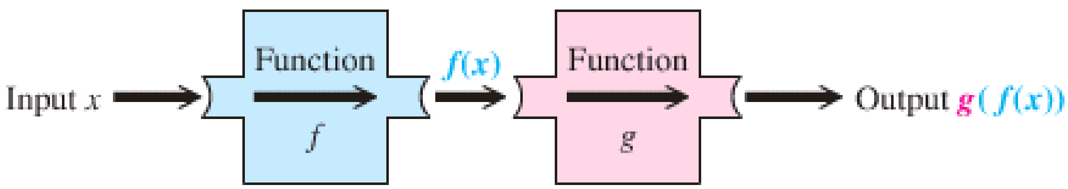
The composite function $g \circ f$, the composite of f and g, is defined as

$$(g \circ f)(x) = g\big(f(x)\big)$$

Where x is in the domain of f

and $g(x)$ is in the domain of f





Example

Given that $f(x) = 5x + 6$ and $g(x) = 2x^2 - x - 1$, find $(f \circ g)(x)$ and $(g \circ f)(x)$

Solution

$(f \circ g)(x) = f\big(g(x)\big)$ $\qquad = f(2x^2 - x - 1)$ ***Domain:*** All real numbers

$= 5(\text{---------}) + 6$

$= 5(2x^2 - x - 1) + 6$

$= 10x^2 - 5x - 5 + 6$

$\underline{= 10x^2 - 5x + 1}$ ***Domain:*** All real numbers

$(g \circ f)(x) = g\big(f(x)\big)$

$= g(5x + 6)$ ***Domain:*** All real numbers

$= 2(\qquad)^2 - (\qquad) - 1$

$= 2(5x + 6)^2 - (5x + 6) - 1$

$= 2(25x^2 + 60x + 36) - 5x - 6 - 1$

$= 50x^2 + 120x + 72 - 5x - 7$

$\underline{= 50x^2 + 115x + 65}$ ***Domain:*** All real numbers

Example

Let $f(x)=\sqrt{x}$ and $g(x)=4x+2$, find each of the following and its domain.

a) $(f\circ g)(x)$

b) $(g\circ f)(x)$

Solution

a) $(f\circ g)(x)=f(g(x))$

$$=f(4x+2) \qquad (-\infty,\infty)$$

$$=\sqrt{4x+2}$$

$$4x+2\ge 0$$
$$4x\ge -2$$
$$x\ge -\frac{2}{4}$$

Domain: $\boxed{x\ge -\frac{1}{2}}$ $\quad \left[-\frac{1}{2},\infty\right)$

b) $(g\circ f)(x)=g(f(x))$

$$=g\left(\sqrt{x}\right) \qquad x\ge 0$$

$$=4\sqrt{x}+2 \qquad x\ge 0$$

Domain: $\boxed{x\ge 0}$ $\quad [0,\infty)$

Example

Let $f(x)=2x-1$ and $g(x)=\dfrac{4}{x-1}$ Find:

a) $(f\circ g)(2)$

b) $(g\circ f)(-3)$

Solution

a) $(f\circ g)(2)=f(g(2))$

$$=f\left(\frac{4}{2-1}\right)$$

$$=f(4)$$

$$=2(4)-1$$

$$\boxed{=7}$$

b) $(g\circ f)(-3)=g(f(-3))$

$$=g(2(-3)-1)$$

$$= g(-7)$$

$$= \frac{4}{-7-1}$$

$$= \frac{4}{-8}$$

$$\boxed{= -\frac{1}{2}}$$

Example

Given that $f(x) = \frac{4}{x+2}$ and $g(x) = \frac{1}{x}$, find

a) $(f \circ g)(x)$

b) Domain of $(f \circ g)(x)$

Solution

a) $(f \circ g)(x) = f(g(x))$

$$= f\left(\frac{1}{x}\right) \qquad \textbf{\textit{Domain:}}: x \ne 0$$

$$= \frac{4}{\frac{1}{x}+2}$$

$$= \frac{4}{\frac{1+2x}{x}}$$

$$= 4 \div \frac{1+2x}{x}$$

$$= 4\frac{x}{1+2x}$$

$$\boxed{= \frac{4x}{1+2x}} \qquad \textbf{\textit{Domain:}}: x \ne -\frac{1}{2}$$

b) Domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0\right) \cup (0, \infty)$

Exercises Section 2.3 – Composition Functions

1. Given that $f(x)=2x-5$ and $g(x)=x^2-3x+8$, find $(f\circ g)(x)$, $(g\circ f)(x)$ and their domain then find $(f\circ g)(7)$

2. Given that $f(x)=\sqrt{x}$ and $g(x)=x-1$, find

a) $(f\circ g)(x)=f(g(x))$

b) $(g\circ f)(x)=g(f(x))$

c) $(f\circ g)(2)=f(g(2))$

3. Given that $f(x)=\frac{x}{x+5}$ and $g(x)=\frac{6}{x}$, find

a) $(f\circ g)(x)=f(g(x))$

b) $(g\circ f)(x)=g(f(x))$

c) $(f\circ g)(2)=f(g(2))$

4. Find $(f\circ g)(x)$, $(g\circ f)(x)$, $f\big(g(-2)\big)$ *and* $g\big(f(3)\big)$: $f(x)=2x^2+3x-4$, $g(x)=2x-1$

5. Find $(f\circ g)(x)$, $(g\circ f)(x)$, $f\big(g(-2)\big)$ *and* $g\big(f(3)\big)$: $f(x)=x^3+2x^2$, $g(x)=3x$

6. Find $(f\circ g)(x)$, $(g\circ f)(x)$, $f\big(g(-2)\big)$ *and* $g\big(f(3)\big)$: $f(x)=|x|$, $g(x)=-7$

(**7 – 36**) For the given function; find:

a) Find $(f\circ g)(x)$ and the ***domain*** of $f\circ g$

b) Find $(g\circ f)(x)$ and the ***domain*** of $g\circ f$

7. $f(x)=x-3$ *and* $g(x)=x+3$

8. $f(x)=\frac{2}{3}x$ *and* $g(x)=\frac{3}{2}x$

9. $f(x)=x-1$ *and* $g(x)=3x^2-2x-1$

10. $f(x)=3x-2$ *and* $g(x)=x^2-5$

11. $f(x)=x^2-2$ *and* $g(x)=4x-3$

12. $f(x)=4x^2-x+10$ *and* $g(x)=2x-7$

13. $f(x)=\sqrt{x}$ *and* $g(x)=x+3$

14. $f(x)=\sqrt{x}$ *and* $g(x)=2-3x$

15. $f(x)=3x+2$ *and* $g(x)=\sqrt{x}$

16. $f(x)=x^4$ *and* $g(x)=\sqrt[4]{x}$

17. $f(x)=x^n$ *and* $g(x)=\sqrt[n]{x}$

18. $f(x)=x^2-3x$ *and* $g(x)=\sqrt{x+2}$

19. $f(x)=\sqrt{x-2}$ *and* $g(x)=\sqrt{x+5}$

20. $f(x)=x^2+2$ *and* $g(x)=\sqrt{3-x}$

21. $f(x)=x^5-2$ *and* $g(x)=\sqrt[5]{x+2}$

22. $f(x)=1-x^2$ *and* $g(x)=\sqrt{x^2-25}$

23. $f(x) = 2x + 3$ *and* $g(x) = \dfrac{x-3}{2}$

24. $f(x) = 4x - 5$ *and* $g(x) = \dfrac{x+5}{4}$

25. $f(x) = \dfrac{4}{1-5x}$ *and* $g(x) = \dfrac{1}{x}$

26. $f(x) = \dfrac{1}{x-2}$ *and* $g(x) = \dfrac{x+2}{x}$

27. $f(x) = \dfrac{1}{1+x}$ *and* $g(x) = \dfrac{1-x}{x}$

28. $f(x) = \dfrac{3x+5}{2}$ *and* $g(x) = \dfrac{2x-5}{3}$

29. $f(x) = \dfrac{x-1}{x-2}$ *and* $g(x) = \dfrac{x-3}{x-4}$

30. $f(x) = \dfrac{6}{x-3}$ *and* $g(x) = \dfrac{1}{x}$

31. $f(x) = \dfrac{6}{x}$ *and* $g(x) = \dfrac{1}{2x+1}$

32. $f(x) = 3x - 7$ *and* $g(x) = \dfrac{x+7}{3}$

33. $f(x) = \dfrac{2x+3}{x-4}$ *and* $g(x) = \dfrac{4x+3}{x-2}$

34. $f(x) = \dfrac{2x+3}{x+4}$ *and* $g(x) = \dfrac{-4x+3}{x-2}$

35. $f(x) = x + 1$ *and* $g(x) = x^3 - 5x^2 + 3x + 7$

36. $f(x) = x - 1$ *and* $g(x) = x^3 + 2x^2 - 3x - 9$

(37 – 48) Evaluate each composite function, where $f(x) = 2x - 3$ *and* $g(x) = x^2 - 5x$

37. $(f \circ g)(4)$

38. $(g \circ f)(4)$

39. $(f \circ g)(-2)$

40. $(g \circ f)(-2)$

41. $(f \circ f)(-3)$

42. $(g \circ g)(7)$

43. $(f \circ g)(\sqrt{2})$

44. $(g \circ f)(\sqrt{3})$

45. $(f \circ g)(2a)$

46. $(g \circ f)(3b)$

47. $(f \circ g)(k+1)$

48. $(g \circ f)(k-1)$

Section 2.4 – Properties of Division

Long Division

Divide $\left(x^3+2x^2-5x-6\right) \div (x+1)$

$$
\begin{array}{rl}
 & \overbrace{x^2+x-6}^{\textit{Quotient}} \\
\cline{2-2}
\underset{\textit{Divisor}}{x+1} & \big) \, x^3+2x^2-5x-6 \longleftarrow \textit{Dividend} \\
 & \underline{x^3+x^2} \\
 & \quad x^2-5x \\
 & \quad \underline{x^2-x} \\
 & \qquad -6x-6 \\
 & \qquad \underline{-6x-6} \\
 & \qquad\qquad 0 \longleftarrow \textit{Remainder}
\end{array}
$$

$\underline{Q(x)=x^2+x-6}$

$\underline{R(x)=0}$

Example

Use the long division to find the quotient and the remainder: $\left(x^4-16\right) \div \left(x^2+3x+1\right)$

Solution

$$
\begin{array}{rl}
 & x^2-3x+8 \\
\cline{2-2}
x^2+3x+1 & \big) \, x^4+0x^3+0x^2+0x-16 \\
 & \underline{x^4+3x^3+x^2} \\
 & \quad -3x^3-x^2 \\
 & \quad \underline{-3x^3-9x^2-3x} \\
 & \qquad\qquad 8x^2+\ \ 3x-16 \\
 & \qquad\qquad \underline{8x^2+24x+\ 8} \\
 & \qquad\qquad\quad -21x-24
\end{array}
$$

$$\frac{x^4-16}{x^2+3x+1}=x^2-3x+8+\frac{-21x-24}{x^2+3x+1}$$

$$\underline{x^4-16=\left(x^2+3x+1\right)\left(x^2-3x+8\right)+(-21x-24)}$$

Remainder *Theorem*

If a number c is substituted for x in the polynomial $f(x)$, then the result $f(c)$ is the remainder that would be obtained by dividing $f(x)$ by $x-c$.

That is, if $f(x)=(x-c)Q(x)+R(x)$ *then* $f(c)=R$

Example

If $f(x)=x^3-3x^2+x+5$, use the remainder theorem to find $f(2)$

Solution

$$\begin{array}{r|l} & x^2-x-1 \\ \hline x-2 & x^3-3x^2+x+5 \\ & \underline{x^3-2x^2} \\ & \quad -x^2+x \\ & \quad \underline{-x^2+2x} \\ & \qquad\quad -x+5 \\ & \qquad\quad \underline{-x+2} \\ & \qquad\qquad\quad \boxed{3} \end{array}$$

$f(2)=3$

Factor *Theorem*

A polynomial $f(x)$ has a factor $x-c$ if and only if $f(c)=0$

Example

Show that $x-2$ is a factor of $f(x)=x^3-4x^2+3x+2$.

Solution

Since $f(2)=(2)^3-4(2)^2+3(2)$

$=0$

From the factor theorem; $x-2$ is a factor of $f(x)$.

Synthetic Division

Use synthetic division to find the quotient and the remainder of $\left(4x^3 - 3x^2 + x + 7\right) \div (x-2)$

$$\begin{array}{c|cccc} & x^3 & x^2 & x^1 & x^0 \\ & 4 & -3 & 1 & 7 \\ 2 & & 8 & 10 & 22 \\ \hline & 4 & 5 & 11 & \boxed{29} \\ & x^2 & x^1 & x^0 & \end{array}$$

Quotient : $Q(x) = 4x^2 + 5x + 11$

Remainder : $R(x) = 29$

Example

If $f(x) = 3x^5 - 38x^3 + 5x^2 - 1$, use the synthetic division to find $f(4)$.

Solution

$$\begin{array}{r|rrrrrr} 4 & 3 & 0 & -38 & 5 & 0 & -1 \\ & & 12 & 48 & 40 & 180 & 720 \\ \hline & 3 & 12 & 10 & 45 & 180 & \boxed{719} \end{array}$$

$f(4) = 719$

Example

Show that -11 is a zero of the polynomial $f(x) = x^3 + 8x^2 - 29x + 44$

Solution

$$\begin{array}{r|rrrr} -11 & 1 & 8 & -29 & 44 \\ & & -11 & 33 & -44 \\ \hline & 1 & -3 & 4 & \boxed{0} \end{array}$$

Thus, $f(-11) = 0$, and -11 is a zero of f.

The Rational Zeros *Theorem*

If the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ *has integer* coefficients and if $\frac{c}{d}$ is a rational zero of $f(x)$ such that ***c*** and ***d*** have no common prime factor, then

1. The numerator c of the zero is a factor of the constant term a_0
2. The denominator d of the zero is a factor of the leading coefficient a_n

$$\textit{possible rational zeros} = \frac{\textit{factors of the constant term } a_0}{\textit{factors of the leading coefficient } a_n} = \frac{\textit{possibilities for } a_0}{\textit{possibilities for } a_n}$$

Example

Find all rational solutions of the equation: $3x^4 + 14x^3 + 14x^2 - 8x - 8 = 0$

Solution

possibilities for a_0	$\pm 1,\ \pm 2,\ \pm 4,\ \pm 8$
possibilities for a_n	$\pm 1,\ \pm 3$
possibilities for c/d	$\pm 1,\ \pm 2,\ \pm 4,\ \pm 8,\ \pm\frac{1}{3},\ \pm\frac{2}{3},\ \pm\frac{4}{3},\ \pm\frac{8}{3}$

Using the calculator, the result will show that -2 is a zero.

$$\begin{array}{r|rrrrr} -2 & 3 & 14 & 14 & -8 & -8 \\ & & -6 & -16 & 4 & 8 \\ \hline & 3 & 8 & -2 & -4 & \boxed{0} \end{array}$$

We have the factorization of: $(x+2)\left(3x^3 + 8x^2 - 2x - 4\right) = 0$

For $3x^3 + 8x^2 - 2x - 4 \quad \Rightarrow \quad \frac{c}{d} = \frac{\pm 1,\ \pm 2,\ \pm 4}{\pm 1,\ \pm 3}$

$x = -\frac{2}{3}$ is another solution.

$$\begin{array}{r|rrrr} -\frac{2}{3} & 3 & 8 & -2 & -4 \\ & & -2 & -4 & 4 \\ \hline & 3 & 6 & -6 & \boxed{0} \end{array}$$

We have the factorization of: $(x+2)\left(x + \frac{2}{3}\right)\left(3x^2 + 6x - 6\right) = 0$

By applying quadratic formula to solve: $3x^2 + 6x - 6 = 0 \Rightarrow x = -1 \pm \sqrt{3}$

Hence, the polynomial has two rational roots $x = -2$ *and* $-\frac{2}{3}$ and two irrational roots $x = -1 \pm \sqrt{3}$.

Exercises Section 2.4 – Properties of Division

1. Find the quotient and remainder if $f(x)$ is divided by $p(x)$:
$f(x) = 2x^4 - x^3 + 7x - 12; \quad p(x) = x^2 - 3$

(2 – 4) Find the quotient and remainder if $f(x)$ is divided by $p(x)$

2. $f(x) = 3x^3 + 2x - 4; \quad p(x) = 2x^2 + 1$

3. $f(x) = 7x + 2; \quad p(x) = 2x^2 - x - 4$

4. $f(x) = 9x + 4; \quad p(x) = 2x - 5$

5. Use the remainder theorem to find $f(c)$: $f(x) = x^4 - 6x^2 + 4x - 8; \quad c = -3$

6. Use the remainder theorem to find $f(c)$: $f(x) = x^4 + 3x^2 - 12; \quad c = -2$

7. Use the factor theorem to show that $x - c$ is a factor of $f(x)$: $f(x) = x^3 + x^2 - 2x + 12; \quad c = -3$

8. Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $2x^3 - 3x^2 + 4x - 5; \quad x - 2$

9. Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $5x^3 - 6x^2 + 15; \quad x - 4$

10. Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $9x^3 - 6x^2 + 3x - 4; \quad x - \frac{1}{3}$

(11 – 13) Use the synthetic division to find $f(c)$:

11. $f(x) = 2x^3 + 3x^2 - 4x + 4; \quad c = 3$

12. $f(x) = 8x^5 - 3x^2 + 7; \quad c = \frac{1}{2}$

13. $f(x) = x^3 - 3x^2 - 8; \quad c = 1 + \sqrt{2}$

14. Use the synthetic division to show that c is a zero of $f(x)$:
$f(x) = 3x^4 + 8x^3 - 2x^2 - 10x + 4; \quad c = -2$

15. Use the synthetic division to show that c is a zero of $f(x)$:
$f(x) = 27x^4 - 9x^3 + 3x^2 + 6x + 1; \quad c = -\frac{1}{3}$

16. Find all values of k such that $f(x)$ is divisible by the given linear polynomial:
$f(x) = kx^3 + x^2 + k^2x + 3k^2 + 11; \quad x + 2$

(**17 – 62**) Find all solutions of the equation

17. $x^3 - x^2 - 10x - 8 = 0$

18. $x^3 + x^2 - 14x - 24 = 0$

19. $2x^3 - 3x^2 - 17x + 30 = 0$

20. $12x^3 + 8x^2 - 3x - 2 = 0$

21. $x^3 + x^2 - 6x - 8 = 0$

22. $x^3 - 19x - 30 = 0$

23. $2x^3 + x^2 - 25x + 12 = 0$

24. $3x^3 + 11x^2 - 6x - 8 = 0$

25. $2x^3 + 9x^2 - 2x - 9 = 0$

26. $x^3 + 3x^2 - 6x - 8 = 0$

27. $3x^3 - x^2 - 6x + 2 = 0$

28. $x^3 - 8x^2 + 8x + 24 = 0$

29. $x^3 - 7x^2 - 7x + 69 = 0$

30. $x^3 - 3x - 2 = 0$

31. $x^3 - 2x + 1 = 0$

32. $x^3 - 2x^2 - 11x + 12 = 0$

33. $x^3 - 2x^2 - 7x - 4 = 0$

34. $x^3 - 10x - 12 = 0$

35. $x^3 - 5x^2 + 17x - 13 = 0$

36. $6x^3 + 25x^2 - 24x + 5 = 0$

37. $8x^3 + 18x^2 + 45x + 27 = 0$

38. $3x^3 - x^2 + 11x - 20 = 0$

39. $x^4 - x^3 - 9x^2 + 3x + 18 = 0$

40. $2x^4 - 9x^3 + 9x^2 + x - 3 = 0$

41. $6x^4 + 5x^3 - 17x^2 - 6x = 0$

42. $x^4 - 2x^2 - 16x - 15 = 0$

43. $x^4 - 2x^3 - 5x^2 + 8x + 4 = 0$

44. $2x^4 - 17x^3 + 4x^2 + 35x - 24 = 0$

45. $x^4 + x^3 - 3x^2 - 5x - 2 = 0$

46. $6x^4 - 17x^3 - 11x^2 + 42x = 0$

47. $x^4 - 5x^2 - 2x = 0$

48. $3x^4 - 4x^3 - 11x^2 + 16x - 4 = 0$

49. $6x^4 + 23x^3 + 19x^2 - 8x - 4 = 0$

50. $4x^4 - 12x^3 + 3x^2 + 12x - 7 = 0$

51. $2x^4 - 9x^3 - 2x^2 + 27x - 12 = 0$

52. $2x^4 - 19x^3 + 51x^2 - 31x + 5 = 0$

53. $4x^4 - 35x^3 + 71x^2 - 4x - 6 = 0$

54. $2x^4 + 3x^3 - 4x^2 - 3x + 2 = 0$

55. $x^4 + 3x^3 - 30x^2 - 6x + 56 = 0$

56. $6x^5 + 19x^4 + x^3 - 6x^2 = 0$

57. $3x^5 - 10x^4 - 6x^3 + 24x^2 + 11x - 6 = 0$

58. $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 = 0$

59. $x^5 - x^4 - 7x^3 + 7x^2 + 12x - 12 = 0$

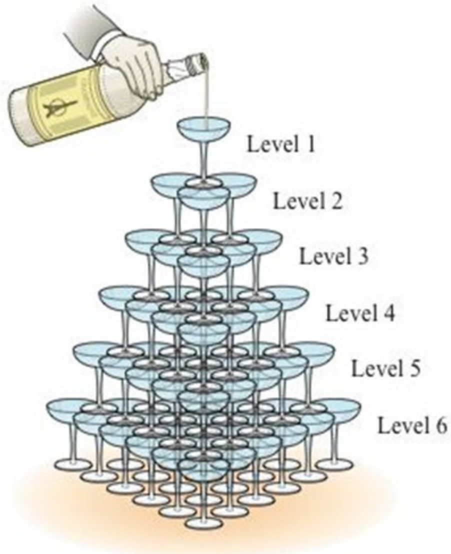
60. $x^5 - 2x^3 - 8x = 0$

61. $x^5 - 32 = 0$

62. $3x^6 - 10x^5 - 29x^4 + 34x^3 + 50x^2 - 24x - 24 = 0$

63. Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

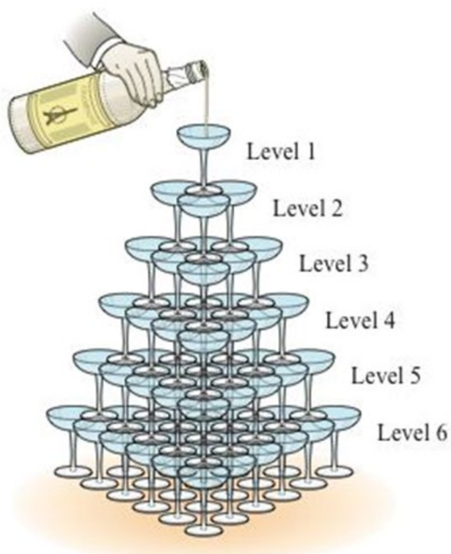
$$T(k) = \frac{1}{6}\left(k^3 + 3k^2 + 2k\right)$$



Where k is the number of levels in the pyramid. If 220 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

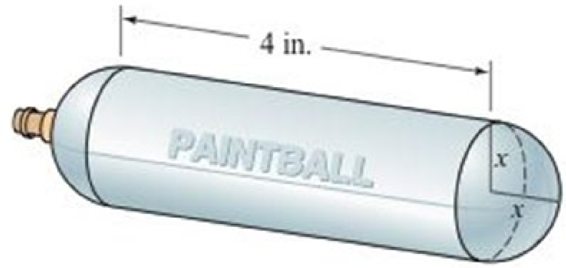
64. Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T(k) = \frac{1}{6}\left(2k^3 + 3k^2 + k\right)$$



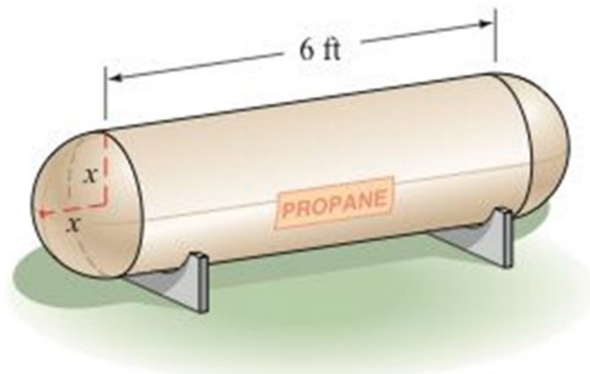
Where k is the number of levels in the pyramid. If 140 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

65. A carbon dioxide cartridge for a paintball rifle has the shape of a right circular cylinder with a hemisphere at each end. The cylinder is 4 *inches* long, and the volume of the cartridge is $2\pi\ in^3$.

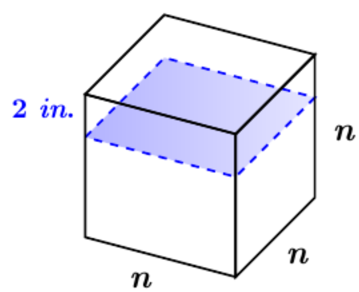


The common interior radius of the cylinder and the hemispheres is denoted by x. Estimate the length of the radius x.

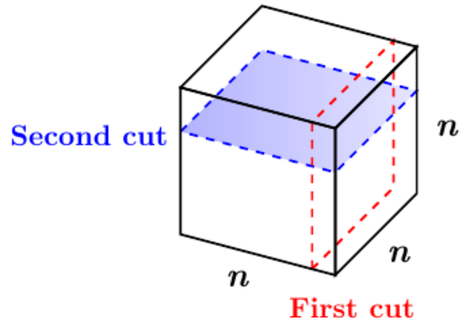
66. A propane tank has the shape of a circular cylinder with a hemisphere at each end. The cylinder is 6 *feet* long and the volume of the tank is $9\pi\ ft^3$. Find the length of the radius x.



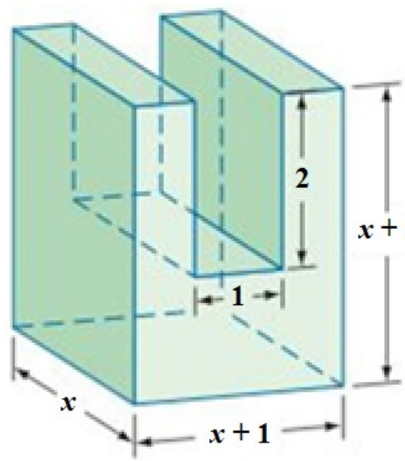
67. A cube measures n inches on each edge. If a slice 2 *inches* thick is cut from one face of the cube, the resulting solid has a volume of $567\ in^3$. Find n.



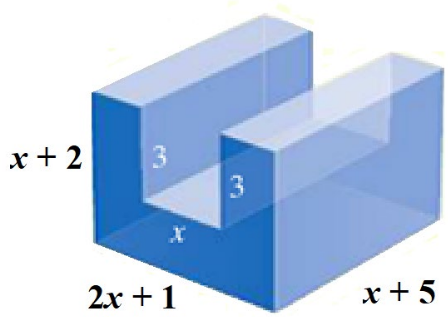
68. A cube measures n inches on each edge. If a slice 1 *inch* thick is cut from one face of the cube and then a slice 3 inches thick is cut from another face of the cube, the resulting solid has a volume of $1560\ in^3$. Find the dimensions of the original cube.



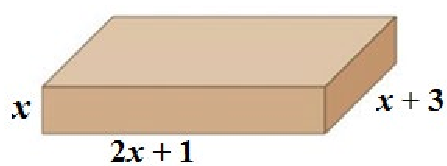
69. For what value of x will the volume of the following solid be $112\ in^3$



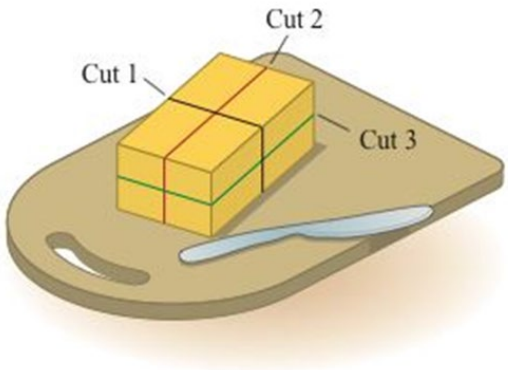
70. For what value of x will the volume of the following solid be $208\ in^3$



71. The length of rectangular box is 1 *inch* more than twice the height of the box, and the width is 3 *inches* more than the height. If the volume of the box is $126\ in^3$, find the dimensions of the box.



72. One straight cut through a thick piece of cheese produces two pieces. Two straight cuts can produce a maximum of four pieces. Two straight cuts can produce a maximum of four pieces. Three straight cuts can produce a maximum of eight pieces.



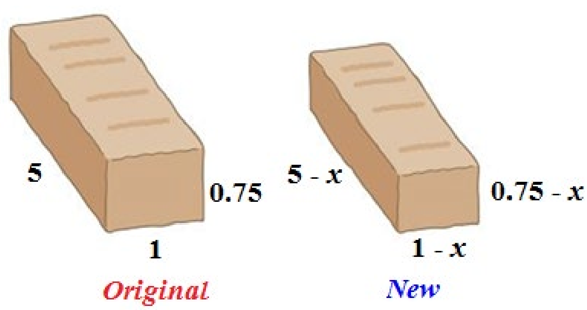
You might be inclined to think that every additional cut double numbers of pieces. However, for four straight cuts, you get a maximum of 15 pieces. The maximum number of pieces P that can be produced by n straight cuts is given by

$$P(n) = \frac{n^3 + 5n + 6}{6}$$

a) Determine number of pieces that can be produces by five straight cuts.
b) What is the fewest number of straight cuts that are needed to produce 64 pieces?

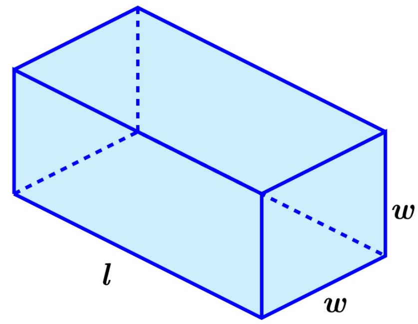
73. The number of ways one can select three cards from a group of n cards (the order of the selection matters), where $n \geq 3$, is given by $P(n) = n^3 - 3n^2 + 2n$. For a certain card trick, a magician has determined that there are exactly 504 *ways* to choose three cards from a given group. How many cards are in the group?

74. A nutrition bar in the shape of a rectangular solid measure 0.75 *in*. by 1 *in*. by 5 *inches*.



To reduce costs, the manufacturer has decided to decrease each of the dimensions of the nutrition bar by x *inches*, what value of x will produce a new bar with a volume that is $0.75\ in^3$ less than the present bar's volume.

75. A rectangular box is square on two ends and has length plus girth of 81 *inches*. (Girth: distance *around* the box). Determine the possible lengths $l\ (l > w)$ of the box if its volume is $4900\ in^3$.



Section 2.5 – Polynomial Functions

Polynomial Function

A *Polynomial function* $P(x)$ in x is a sum of the form is given by:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x + a_0$$

Where the coefficients $a_n,\ a_{n-1},\ \dots, a_2,\ a_1, a_0$ are real numbers and the exponents are whole numbers.

Degree

$$\underline{a_n} x^n$$

Leading Term

Leading Coefficient

Non-polynomial Functions: $\frac{1}{x} + 2x$; $\sqrt{x^2 - 3} + x$; $\frac{x-5}{x^2+2}$

Degree of f	***Form of f(x)***	***Graph of f(x)***
0	$f(x) = a_0$	A horizontal line
1	$f(x) = a_1 x + a_0$	A line with slope a_1
2	$f(x) = a_2 x^2 + a_1 x + a_0$	A parabola with a vertical axis

All polynomial functions are ***continuous functions***.

End Behavior $\left(a_n x^n\right)$

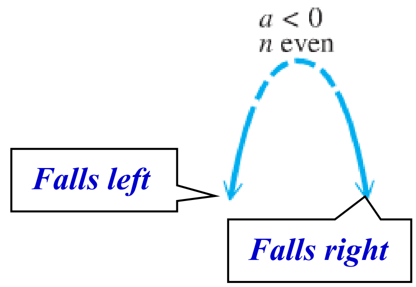
If n (degree) is ***even***:

If $a_n < 0$ (in front x^n is negative).

Then the function falls from the left and right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to -\infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to -\infty$$

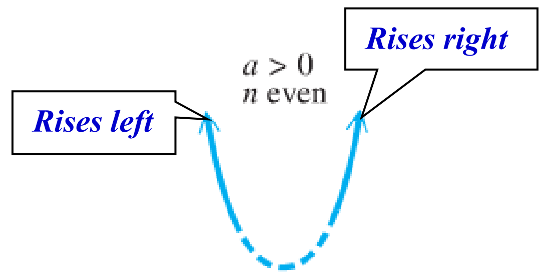


If $a_n > 0$ (in front x^n is positive).

Then the function rises from the left and right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to \infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to \infty$$



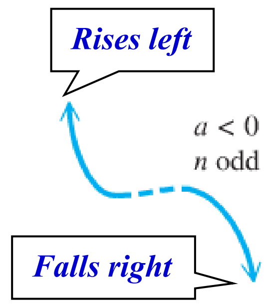
If n (degree) is ***odd***:

If $a_n < 0$ (negative).

Then the function rises from the left side and falls from the right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to \infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to -\infty$$

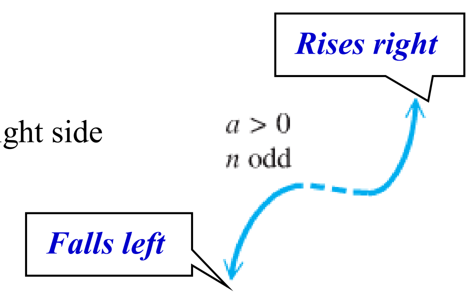


If $a_n > 0$ (positive).

Then the function falls from the left side and rises from the right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to -\infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to \infty$$



Example

Determine the end behavior of the graph of the polynomial function $f(x) = -4x^5 + 7x^2 - x + 9$

Solution

Leading term: $-4x^5$ with 5th degree (n is odd)

$x \to -\infty \quad \Rightarrow \quad f(x) = -(-)^5 = (-)(-) = + \to \infty \qquad f(x)$ rises left

$x \to \infty \quad \Rightarrow \quad f(x) \to -\infty \qquad f(x)$ falls right

The Intermediate Value *Theorem*

For any polynomial function $f(x)$ with real coefficients and $f(a) \neq f(b)$ for $a < b$, then f takes on every value between $f(a)$ and $f(b)$ in the interval $[a,\ b]$.

$\therefore$ $f(a)$ and $f(b)$ are the ***opposite signs***. Then the function has a real zero between *a* and *b*.

Example

Using the intermediate value theorem, determine, if possible, whether the function has a real zero between *a* and *b*.

a) $f(x) = x^3 + x^2 - 6x; \quad a = -4,\ b = -2$

b) $f(x) = x^3 + x^2 - 6x; \quad a = -1,\ b = 3$

Solution

a) $f(x) = x^3 + x^2 - 6x; \quad a = -4,\ b = -2$

$f(-4) = (-4)^3 + (-4)^2 - 6(-4)$

$= -24$

$f(-2) = (-2)^3 + (-2)^2 - 6(-2)$

$= 8$

$\therefore$ $f(x)$ has a zero between -4 and -2

b) $f(x) = x^3 + x^2 - 6x; \quad a = -1,\ b = 3$

$f(-1) = (-1)^3 + (-1)^2 - 6(-1)$

$= 6$

$f(3) = (3)^3 + (3)^2 - 6(3) = 18$

$= 18$

$\therefore$ $f(x)$ *zeros **can't be determined***

Example

Show that $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3$ has a zero between 1 and 2.

Solution

$f(1) = 1 + 2 - 6 + 2 - 3$

$= -4$

$f(2) = (2)^5 + 2(2)^4 - 6(2)^3 + 2(2) - 3$

$\underline{= 17}$

Since $f(1)$ *and* $f(2)$ have opposite signs.

Therefore, $f(c) = 0$ for at least one real number c between 1 and 2.

Exercises *Section* 2.5 – Polynomial Functions

(**1 – 12**) Determine the end behavior of the graph of the polynomial function

1. $f(x)=5x^3+7x^2-x+9$

2. $f(x)=11x^3-6x^2+x+3$

3. $f(x)=-11x^3-6x^2+x+3$

4. $f(x)=2x^3+3x^2-23x-42$

5. $f(x)=5x^4+7x^2-x+9$

6. $f(x)=11x^4-6x^2+x+3$

7. $f(x)=-5x^4+7x^2-x+9$

8. $f(x)=-11x^4-6x^2+x+3$

9. $f(x)=5x^5-16x^2-20x+64$

10. $f(x)=-5x^5-16x^2-20x+64$

11. $f(x)=-3x^6-16x^3+64$

12. $f(x)=3x^6-16x^3+4$

(**13 – 32**) Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers.

13. $f(x)=x^3-x-1;\quad between\ \ 1\ and\ \ 2$

14. $f(x)=x^3-4x^2+2;\quad between\ \ 0\ and\ 1$

15. $f(x)=2x^4-4x^2+1;\quad between\ \ -1\ and\ \ 0$

16. $f(x)=x^4+6x^3-18x^2;\quad between\ \ 2\ and\ \ 3$

17. $f(x)=x^3+x^2-2x+1;\quad between\ \ -3\ and\ \ -2$

18. $f(x)=x^5-x^3-1;\quad between\ \ 1\ and\ \ 2$

19. $f(x)=3x^3-10x+9;\quad between\ \ -3\ and\ \ -2$

20. $f(x)=3x^3-8x^2+x+2;\quad between\ \ 2\ and\ \ 3$

21. $f(x)=3x^3-8x^2+x+2;\quad between\ \ 1\ and\ \ 2$

22. $f(x)=x^5+2x^4-6x^3+2x-3;\quad between\ \ 0\ and\ 1$

23. $P(x)=2x^3+3x^2-23x-42,\quad a=3,\quad b=4$

24. $P(x)=4x^3-x^2-6x+1,\quad a=0,\quad b=1$

25. $P(x)=3x^3+7x^2+3x+7,\quad a=-3,\quad b=-2$

26. $P(x)=2x^3-21x^2-2x+25,\quad a=1,\quad b=2$

27. $P(x)=4x^4+7x^3-11x^2+7x-15,\quad a=1,\quad b=\frac{3}{2}$

28. $P(x)=5x^3-16x^2-20x+64,\quad a=3,\quad b=\frac{7}{2}$

29. $P(x)=x^4-x^2-x-4,\quad a=1,\quad b=2$

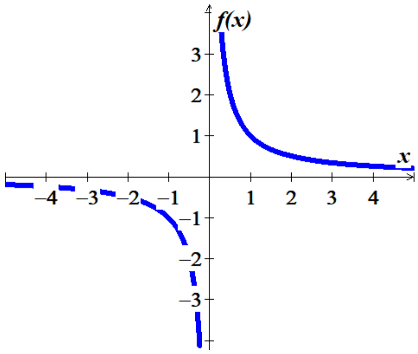
30. $P(x) = x^3 - x - 8, \quad a = 2, \quad b = 3$

31. $P(x) = x^3 - x - 8, \quad a = 0, \quad b = 1$

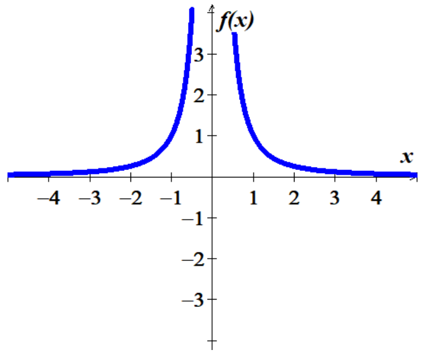
32. $P(x) = x^3 - x - 8, \quad a = 2.1, \quad b = 2.2$

Section 2.6 – Rational Functions

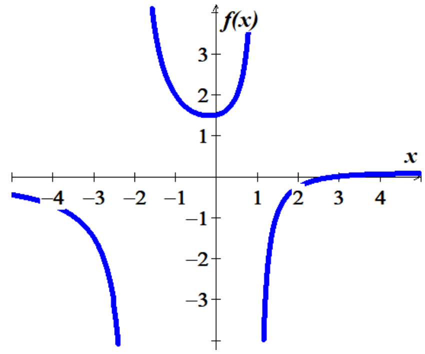
$$f(x)=\frac{1}{x}$$



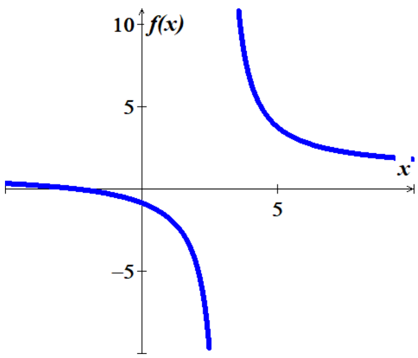
$$f(x)=\frac{1}{x^2}$$



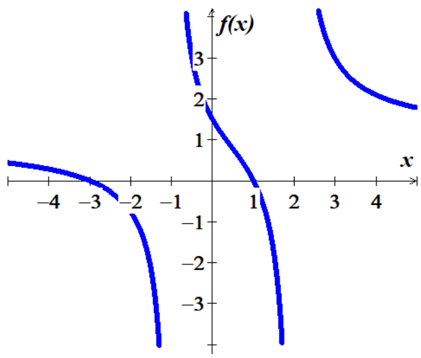
$$f(x)=\frac{x-3}{x^2+x-2}$$



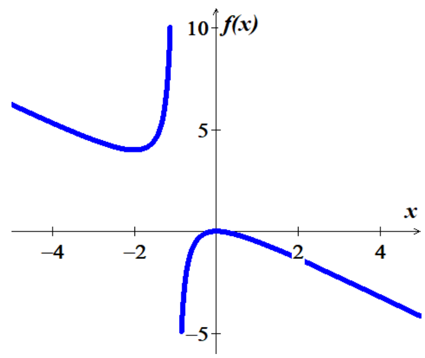
$$f(x)=\frac{2x+5}{2x-6}$$



$$f(x)=\frac{x^2+2x-3}{x^2-x-2}$$



$$f(x)=-\frac{x^2}{x+1}$$



Rational Function

A rational function is a function f that is a quotient of two polynomials, that is,

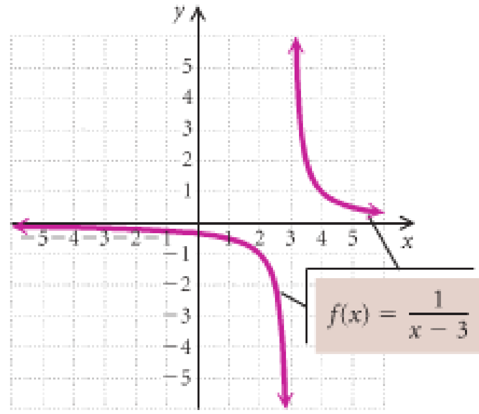
$$f(x)=\frac{g(x)}{h(x)}$$

Where $g(x)$ and $h(x)$ are polynomials. The domain of f consists of all real numbers ***except*** the zeros of the denominator $h(x)$.

The Domain of a Rational Function

Example

Consider: $f(x)=\dfrac{1}{x-3}$



Find the domain and graph *f*.

Solution

$x-3=0$

$x=3$

Thus, the domain is: $\{x \mid x \neq 3\}$ *or* $(-\infty,\ 3)\cup(3,\ \infty)$

Function	**Domain**	
$f(x)=\dfrac{1}{x}$	$\{x \mid x\neq 0\}$	$(-\infty,\ 0)\cup(0,\ \infty)$
$f(x)=\dfrac{1}{x^2}$	$\{x \mid x\neq 0\}$	$(-\infty,\ 0)\cup(0,\ \infty)$
$f(x)=\dfrac{x-3}{x^2+x-2}$	$\{x \mid x\neq -2 \text{ and } x\neq 1\}$	$(-\infty,\ -2)\cup(-2,\ 1)\cup(1,\ \infty)$
$f(x)=\dfrac{2x+5}{2x-6}=\dfrac{2x+5}{2(x-3)}$	$\{x \mid x\neq 3\}$	$(-\infty,\ 3)\cup(3,\ \infty)$

Asymptotes

Vertical Asymptote (*VA*) - *Think Domain*

The line $x = a$ is a ***vertical asymptote*** for the graph of a function f if

$$f(x) \to \infty \quad or \quad f(x) \to -\infty$$

As ***x*** approaches ***a*** from either the left or the right

Example

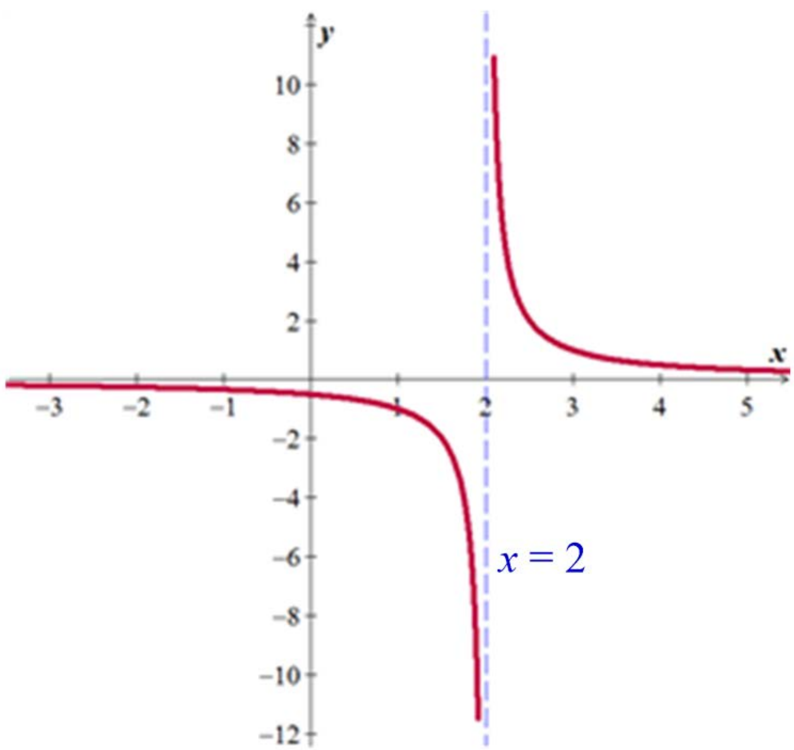
Find the vertical asymptote of $f(x) = \dfrac{1}{x-2}$, and sketch the graph.

Solution

VA: $x = 2$

$f(x) \to \infty \quad as \quad x \to 2^+$

$f(x) \to -\infty \quad as \quad x \to 2^-$



Horizontal Asymptote (*HA*)

The line $y = c$ is a ***horizontal asymptote*** for the graph of a function f if

$$f(x) \to c \quad as \quad x \to -\infty \quad or \quad x \to -\infty$$

Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0} = \dfrac{a_n x^n}{b_m x^m}$ be a rational function.

1. **If the degree of numerator is less than of denominator** $(n < m) \Rightarrow y = 0$

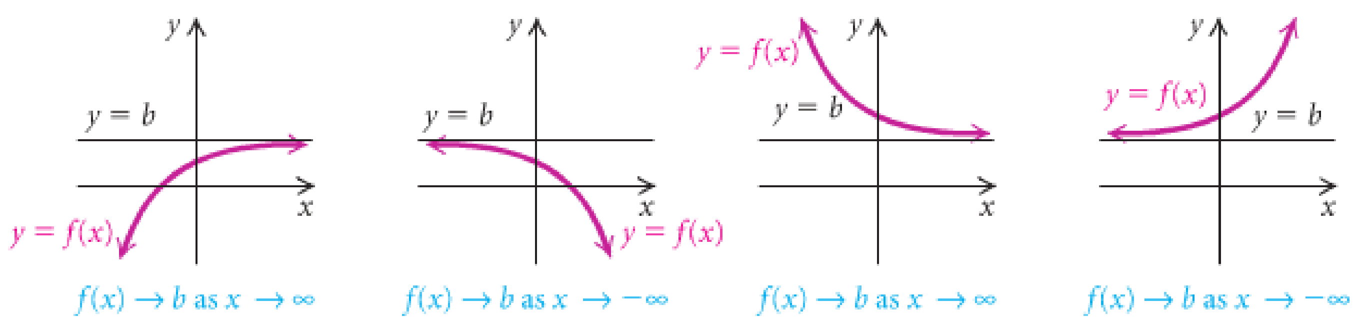
$$y = \frac{2x+1}{4x^2+5} \quad \Rightarrow \boxed{y = 0}$$

2. **If the degree of numerator is equal of denominator** $(n = m) \Rightarrow y = \dfrac{a_n}{b_m}$

$$y = \frac{2x^2+1}{4x^2+5} \quad \Rightarrow \boxed{y = \frac{2}{4} = \frac{1}{2}}$$

3. **If the degree of numerator is greater than of denominator** $(n > m) \Rightarrow$ No horizontal asymptote

$$y = \frac{2x^3+1}{4x^2+5} \quad \Rightarrow No \quad HA$$



Example

Determine the horizontal asymptote of $f(x) = \dfrac{-7x^4 - 10x^2 + 1}{11x^4 + x - 2}$

Solution

$$f(x) = \frac{-7x^4}{11x^4} = -\frac{7}{11}$$

Therefore, the horizontal asymptote (***HA***) is: $\boxed{y = -\frac{7}{11}}$

Example

Find the vertical and the horizontal asymptote for the graph of f, if it exists

a) $f(x)=\frac{3x-1}{x^2-x-6}$

b) $f(x)=\frac{5x^2+1}{3x^2-4}$

c) $f(x)=\frac{2x^4-3x^2+5}{x^2+1}$

Solution

a) $f(x)=\frac{3x-1}{x^2-x-6}$

$$x^2-x-6=0 \quad \to \quad x=-2,\ 3$$

VA: $x=-2, \quad x=3$

HA: $y=0$

b) $f(x)=\frac{5x^2+1}{3x^2-4}$

$$3x^2-4=0$$

$$3x^2=4$$

$$x^2=\frac{4}{3}$$

$$x=\pm\frac{2}{\sqrt{3}}$$

VA: $x=-\frac{2}{\sqrt{3}}, \quad x=\frac{2}{\sqrt{3}}$

HA: $y=\frac{5}{3}$

c) $f(x)=\frac{2x^4-3x^2+5}{x^2+1}$

$$x^2+1=0 \quad \to \quad x^2=-1$$

VA: *n/a*

HA: *n/a*

Slant or *Oblique* Asymptotes

When the degree of the numerator is one greater than the degree of the numerator, the graph has a slant or oblique asymptote and it is a line $y = ax + b, \quad a \neq 0$. To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

$$y = \frac{3x^2 - 1}{x + 2}$$

$$\begin{array}{r|l} & \quad 3x - 6 \\ \hline x+2 & 3x^2 + 0x - 1 \\ & \underline{3x^2 + 6x} \\ & \qquad -6x - 1 \\ & \qquad \underline{-6x - 12} \\ & \qquad\quad R = 11 \end{array}$$

$$y = \frac{3x^2 - 1}{x + 2} = 3x - 6 + \frac{11}{x + 2}$$

The ***oblique asymptote*** is the line ***y* = 3*x* – 6**

Example

Find all the asymptotes of $f(x) = \dfrac{2x^2 - 3x - 1}{x - 2}$

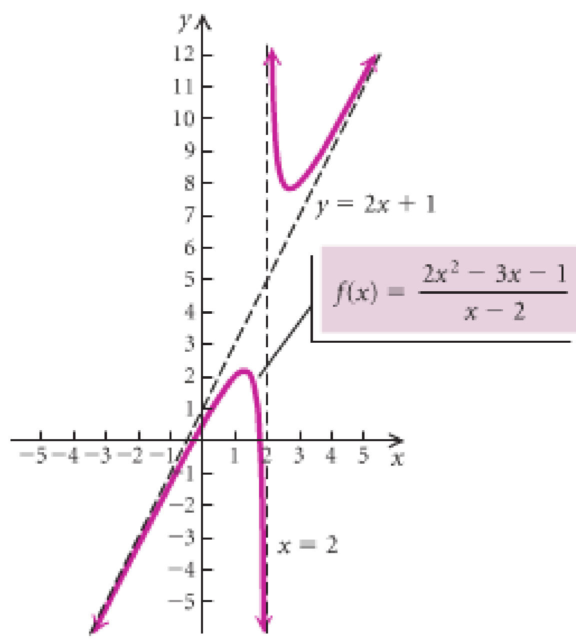
Solution

$$\begin{array}{r|l} & \quad 2x + 1 \\ \hline x-2 & 2x^2 - 3x - 1 \\ & \underline{-2x^2 + 4x} \\ & \qquad x - 1 \\ & \qquad \underline{-x + 2} \\ & \qquad\quad 1 \end{array}$$

$$f(x) = \frac{2x^2 - 3x - 1}{x - 2} = (2x + 1) + \frac{1}{x - 2}$$

The ***oblique asymptote*** is the line $y = 2x + 1$

VA:: $x = 2$



Graph That Has a *Hole*

Example

Sketch the graph of g if $g(x)=\dfrac{3x^2+x-4}{2x^2-7x+5}$

Solution

$$g(x)=\frac{(3x+4)(x-1)}{(2x-5)(x-1)}$$

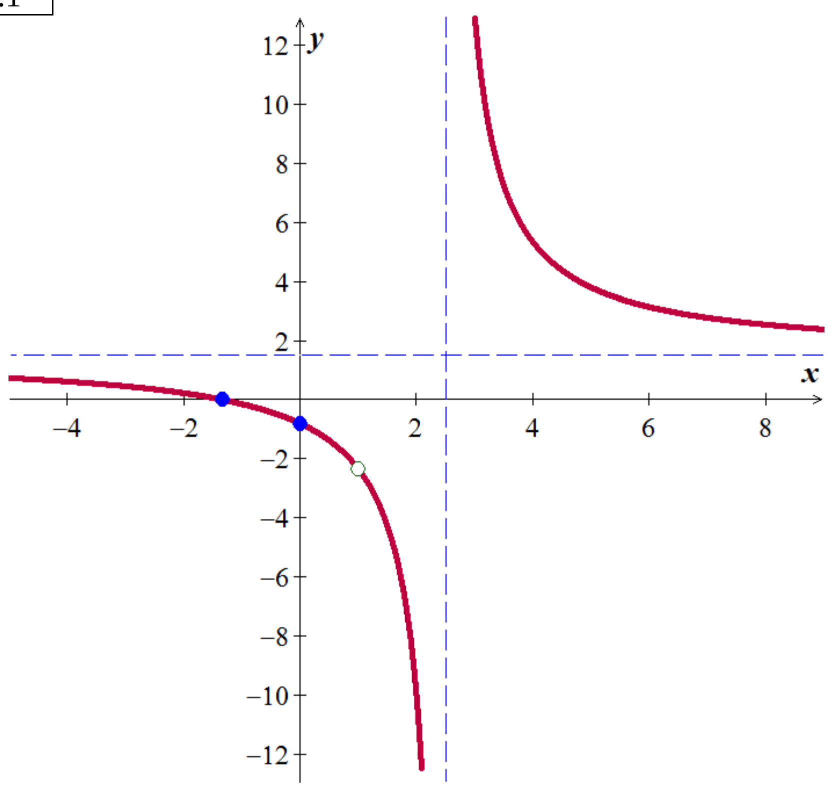
$$=\frac{3x+4}{2x-5}=f(x)$$

VA: $x=\frac{5}{2}$

HA: $y=\frac{3}{2}$

The only different between the graphs that g has a ***hole*** at $x=1\to f(1)=-\frac{7}{3}$

x	y
–4	.6
1.3	0
0	–.8
4	5.3
6	3.1



Exercises *Section* 2.6 – Rational Functions

(**1 – 21**) Determine all asymptotes of the function

1. $y=\dfrac{3x}{1-x}$

2. $y=\dfrac{x^2}{x^2+9}$

3. $y=\dfrac{x-2}{x^2-4x+3}$

4. $y=\dfrac{3}{x-5}$

5. $y=\dfrac{x^3-1}{x^2+1}$

6. $y=\dfrac{3x^2-27}{(x+3)(2x+1)}$

7. $y=\dfrac{x^3+3x^2-2}{x^2-4}$

8. $y=\dfrac{x-3}{x^2-9}$

9. $y=\dfrac{6}{\sqrt{x^2-4x}}$

10. $y=\dfrac{5x-1}{1-3x}$

11. $f(x)=\dfrac{2x-11}{x^2+2x-8}$

12. $f(x)=\dfrac{x^2-4x}{x^3-x}$

13. $f(x)=\dfrac{x-2}{x^3-5x}$

14. $f(x)=\dfrac{4x}{x^2+10x}$

15. $f(x)=\dfrac{3-x}{(x-4)(x+6)}$

16. $f(x)=\dfrac{x^3}{2x^3-x^2-3x}$

17. $f(x)=\dfrac{3x^2+5}{4x^2-3}$

18. $f(x)=\dfrac{x+6}{x^3+2x^2}$

19. $f(x)=\dfrac{x^2+4x-1}{x+3}$

20. $f(x)=\dfrac{x^2-6x}{x-5}$

21. $f(x)=\dfrac{x^3-x^2+x-4}{x^2+2x-1}$

(**22 – 53**) Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

22. $f(x)=\dfrac{-3x}{x+2}$

23. $f(x)=\dfrac{x+1}{x^2+2x-3}$

24. $f(x)=\dfrac{2x^2-2x-4}{x^2+x-12}$

25. $f(x)=\dfrac{-2x^2+10x-12}{x^2+x}$

26. $f(x)=\dfrac{x^2-x-6}{x+1}$

27. $f(x)=\dfrac{x^3+1}{x-2}$

28. $f(x)=\dfrac{2x^2+x-6}{x^2+3x+2}$

29. $f(x)=\dfrac{x-1}{1-x^2}$

30. $f(x)=\dfrac{x^2+x-2}{x+2}$

31. $f(x)=\dfrac{x^3-2x^2-4x+8}{x-2}$

32. $f(x)=\dfrac{2x^2-3x-1}{x-2}$

33. $f(x)=\dfrac{2x+3}{3x^2+7x-6}$

34. $f(x)=\dfrac{x^2-1}{x^2+x-6}$

35. $f(x)=\dfrac{-2x^2-x+15}{x^2-x-12}$

36. $f(x)=\dfrac{1}{x-3}$

37. $f(x)=\dfrac{-2}{x+3}$

38. $f(x)=\dfrac{x}{x+2}$

39. $f(x)=\dfrac{x-5}{x+4}$

40. $f(x)=\dfrac{2x^2-2}{x^2-9}$

41. $f(x)=\dfrac{x^2-3}{x^2+4}$

42. $f(x)=\dfrac{x^2+4}{x^2-3}$

43. $f(x) = \dfrac{x^2}{x^2 - 6x + 9}$

44. $f(x) = \dfrac{x^2 + x + 4}{x^2 + 2x - 1}$

45. $f(x) = \dfrac{2x^2 + 14}{x^2 - 6x + 5}$

46. $f(x) = \dfrac{x^2 - 4x - 5}{2x + 5}$

47. $f(x) = \dfrac{x - 3}{x^2 - 3x + 2}$

48. $f(x) = \dfrac{x^2 + 2}{x^2 + 3x + 2}$

49. $f(x) = \dfrac{x - 2}{x^2 - 3x + 2}$

50. $f(x) = \dfrac{x^2 + x}{x + 1}$

51. $f(x) = \dfrac{x^2 - 2x}{x - 2}$

52. $f(x) = \dfrac{x^2 - 3x}{x + 3}$

53. $f(x) = \dfrac{x^3 + 3x^2 - 4x + 6}{x + 2}$

(**54 – 59**) Find an equation of a rational function f that satisfies the given conditions

54. $\begin{cases} \textit{vertical asymptote}: \ x = 4 \\ \textit{horizontal asymptote}: \ y = -1 \\ x - \textit{intercept}: \ 3 \end{cases}$

55. $\begin{cases} \textit{vertical asymptote}: \ x = -4, x = 5 \\ \textit{horizontal asymptote}: \ y = \frac{3}{2} \\ x - \textit{intercept}: \ -2 \end{cases}$

56. $\begin{cases} \textit{vertical asymptote}: \ x = 5 \\ \textit{horizontal asymptote}: \ y = -1 \\ x - \textit{intercept}: \ 2 \end{cases}$

57. $\begin{cases} \textit{vertical asymptote}: \ x = -2, \ x = 0 \\ \textit{horizontal asymptote}: \ y = 0 \\ x - \textit{intercept}: \ 2, \quad f(3) = 1 \end{cases}$

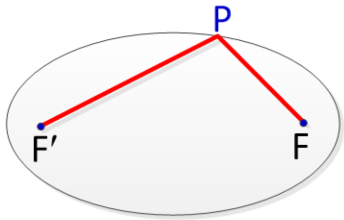
58. $\begin{cases} \textit{vertical asymptote}: \ x = -3, \ x = 1 \\ \textit{horizontal asymptote}: \ y = 0 \\ x - \textit{intercept}: \ -1, \ f(0) = -2 \\ \textit{hole at} \ x = 2 \end{cases}$

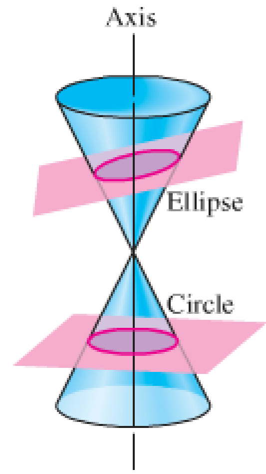
59. $\begin{cases} \textit{vertical asymptote}: \ x = -1, \ x = 3 \\ \textit{horizontal asymptote}: \ y = 2 \\ x - \textit{intercept}: \ -2, \ 1 \\ \textit{hole}: \ x = 0 \end{cases}$

Section 4.7 – Ellipses

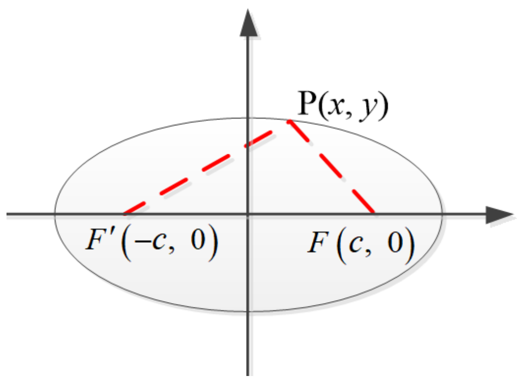
Definition of an *Ellipse*

An ***ellipse*** is the set of all points in a plane, the sum of whose distances from two fixed points (the ***foci***) in the plane is a positive constant.





An ellipse is a curve that is the locus of all points in the plane the sum of the distances $d(P,F)$ and $d(P,F')$ from two fixed points $F'(-c,\ 0)$ and $F(c,\ 0)$ (the ***foci***) separated by a distance of $2c$, with the center of the ellipse at the origin, is the distance length of the string and hence is constant. The constant of the distances of P from F and F' will be denoted by $2a$.



$$d(P,F)+d(P,F')=2a$$

$$\sqrt{(x-c)^2+(y-0)^2}+\sqrt{(x+c)^2+(y-0)^2}=2a$$

$$\sqrt{(x-c)^2+y^2}=2a-\sqrt{(x+c)^2+y^2}$$

$$\left(\sqrt{(x-c)^2+y^2}\right)^2=\left(2a-\sqrt{(x+c)^2+y^2}\right)^2$$

$$(x-c)^2+y^2=4a^2-4a\sqrt{(x+c)^2+y^2}+(x+c)^2+y^2$$

$$x^2-2cx+c^2+y^2=4a^2-4a\sqrt{(x+c)^2+y^2}+x^2+2cx+c^2+y^2$$

$$-2cx=4a^2-4a\sqrt{(x+c)^2+y^2}+2cx$$

$$4a\sqrt{(x+c)^2+y^2}=4a^2+2cx+2cx$$

$$\left(a\sqrt{(x+c)^2+y^2}\right)^2=\left(a^2+cx\right)^2$$

$$a^2\left(x^2+2cx+c^2+y^2\right)=a^4+2a^2cx+c^2x^2$$

$$a^2x^2+2a^2cx+a^2c^2+a^2y^2=a^4+2a^2cx+c^2x^2$$

$$a^2x^2-c^2x^2+a^2y^2=a^4-a^2c^2$$

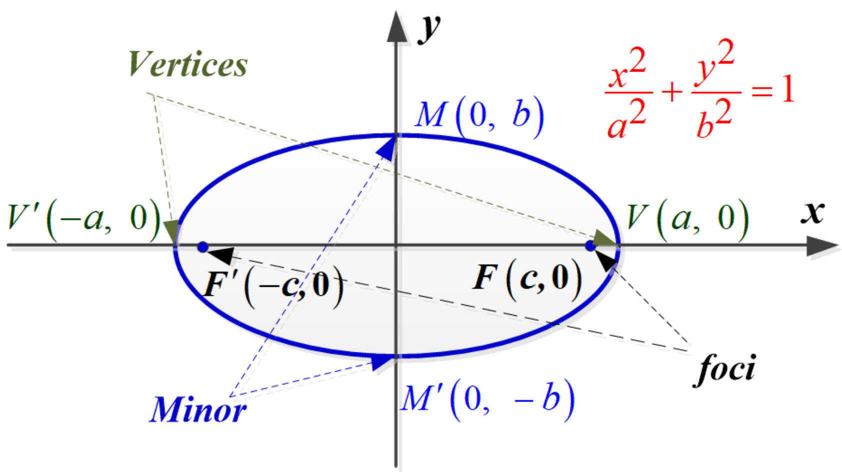
$$x^2\left(a^2-c^2\right)+a^2y^2=a^2\left(a^2-c^2\right)$$

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1$$

Since $a>c \Rightarrow a^2-c^2>0$, we let $b=\sqrt{a^2-c^2}\Rightarrow b^2=a^2-c^2$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

The coordinates of every point (x, y) on the ellipse satisfy the equation.



The ***x**-intercepts* are a and $-a$. The corresponding points $V(a,\ 0)$ and $V'(-a,\ 0)$ are called the ***vertices*** of the ellipse. The line segment $V'V$ is called the ***major axis***.

The ***y**-intercepts* are b and $-b$. The corresponding points $M(0,\ b)$ and $M'(0,\ -b)$ are called the ***minor axis*** of the ellipse.

Standard Equations of an *Ellipse* with Center at the Origin

The graph of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *or* $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$

Where $a > b > 0$, is an ellipse with center at the origin. The length of the major axis is $2a$, and the length of the minor axis is $2b$. The foci are the distance $\boldsymbol{c}$ from the origin where $c^2=a^2-b^2$

Example

Sketch the graph of $2x^2+9y^2=18$, and find the foci.

Solution

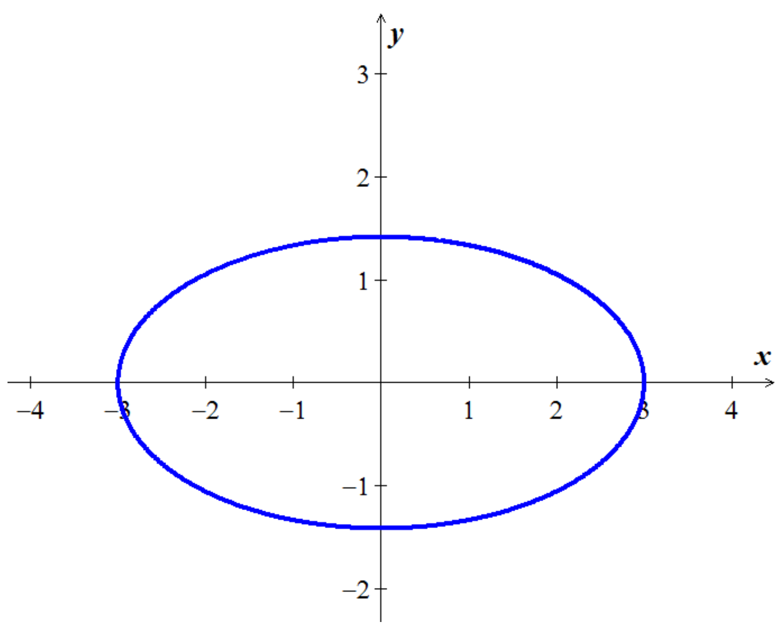
$$\frac{2x^2}{18}+\frac{9y^2}{18}=\frac{18}{18}$$ ***Divide each term by 18***

$$\frac{x^2}{9}+\frac{y^2}{2}=1$$

$$\begin{cases} a^2=9 \to a=3 \\ b^2=2 \to b=\sqrt{2} \end{cases}$$

The ***vertices*** are: $V'(-3,0)$ and $V(3,0)$

The ***minors*** are: $M'(0,-\sqrt{2})$ and $M(0,\sqrt{2})$



$$c=\sqrt{a^2-b^2}=\sqrt{9-2}=\sqrt{7}$$

The ***foci*** are $F'\left(-\sqrt{7},0\right)$ and $F\left(\sqrt{7},0\right)$

Example

Sketch the graph of $9x^2 + 4y^2 = 25$, and find the foci.

Solution

$$\frac{9x^2}{25} + \frac{4y^2}{25} = \frac{25}{25} \qquad \textit{Divide each term by } 25$$

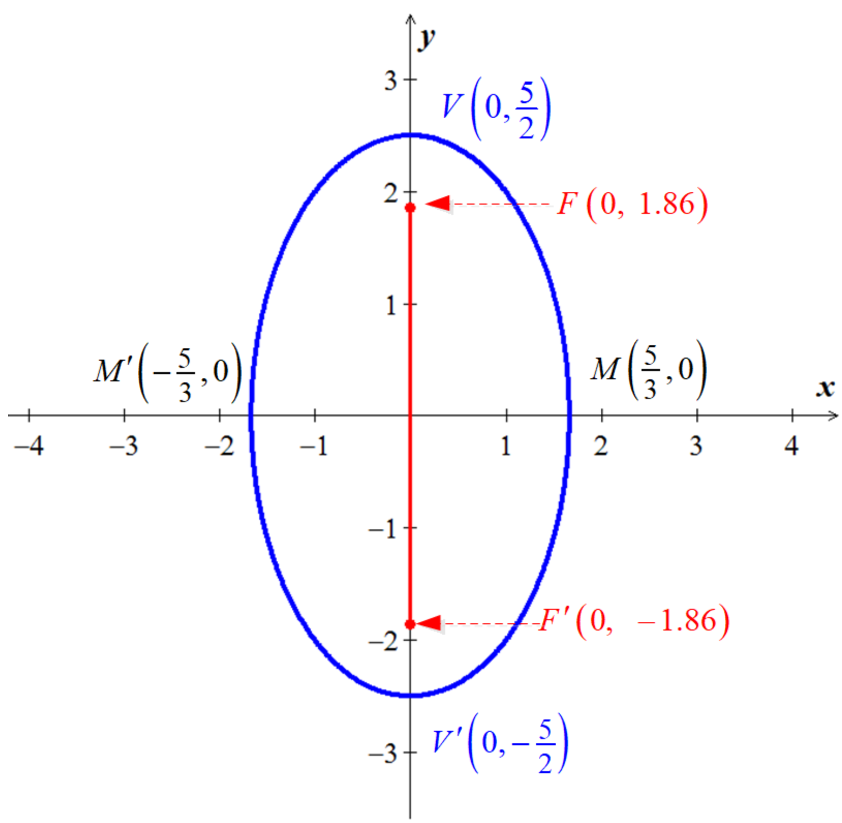
$$\frac{x^2}{\frac{25}{9}} + \frac{y^2}{\frac{25}{4}} = 1$$

Since $\frac{25}{4} > \frac{25}{9}$, the major axis and the foci are on the y-axis.

$$\begin{cases} a^2 = \frac{25}{4} & \rightarrow\ a = \frac{5}{2} \\ b^2 = \frac{25}{9} & \rightarrow\ b = \frac{5}{3} \end{cases}$$

The ***vertices*** are: $V'\left(0,\ -\frac{5}{2}\right)$ and $V\left(0,\ \frac{5}{2}\right)$

The ***minors*** are: $M'\left(-\frac{5}{3},\ 0\right)$ and $M\left(\frac{5}{3},\ 0\right)$



$$c = \pm\sqrt{\frac{25}{4} - \frac{25}{9}} \qquad\qquad c = \pm\sqrt{a^2 - b^2}$$

$$= \pm 5\sqrt{\frac{1}{4} - \frac{1}{9}}$$

$$= \pm 5\sqrt{\frac{5}{36}}$$

$$\boxed{= \pm\frac{5\sqrt{5}}{6}}$$

The ***foci*** are $F'\left(0,\ -\frac{5\sqrt{5}}{6}\right)$ and $F\left(0,\ \frac{5\sqrt{5}}{6}\right)$

Example

Find an equation of the ellipse with vertices $(\pm 4,\ 0)$ and foci $(\pm 2,\ 0)$

Solution

Given: $a = 4, \quad c = 2$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$c^2 = a^2 - b^2$$

$$b^2 = a^2 - c^2$$

$$= 4^2 - 2^2$$

$$\boxed{= 12}$$

$$\boxed{\frac{x^2}{16} + \frac{y^2}{12} = 1}$$

Ellipse with _center_ (_h_, _k_) $\qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \;\rightarrow\; \frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

Example

Sketch the graph of the equation $16x^2+9y^2+64x-18y-71=0$

Solution

$$\left(16x^2+64x\right)+\left(9y^2-18y\right)=71$$

$$16\left(x^2+4x+\_\_\right)+9\left(y^2-2y+\_\_\right)=71$$

$$16\left(x^2+4x+4\right)+9\left(y^2-2y+1\right)=71+(16)4+(9)1$$

$$16(x+2)^2+9(y-1)^2=144$$

$$\frac{16(x+2)^2}{144}+\frac{9(y-1)^2}{144}=\frac{144}{144}$$

$$\frac{(x+2)^2}{9}+\frac{(y-1)^2}{16}=1$$

The center of the ellipse is $C(-2, 1)$ and major axis on the vertical line $x = -2$.

$a=4, \quad b=3$

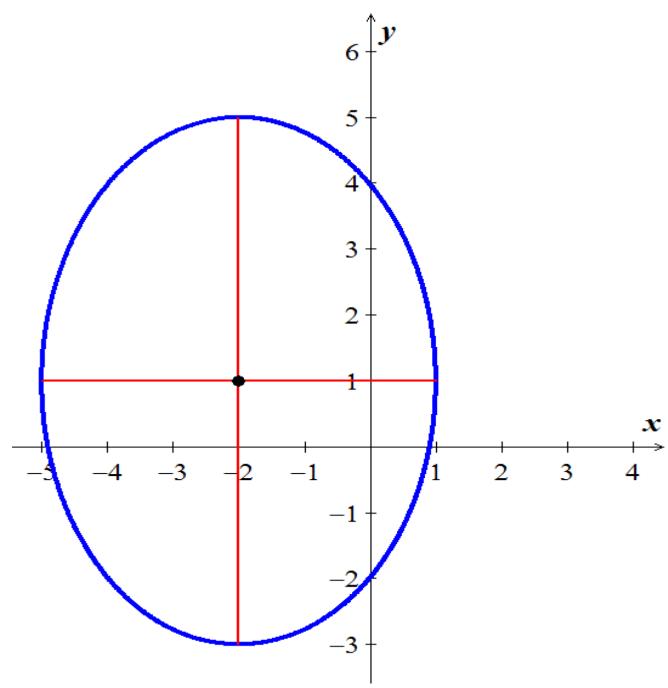
The ***vertices*** are: $V'(-2,\ -3)$ and $V(-2,\ 5)$

The ***minors*** are: $M'(-5,\ -1)$ and $M(1,\ 1)$

$$c=\sqrt{16-9} \qquad\qquad c=\sqrt{a^2-b^2}$$

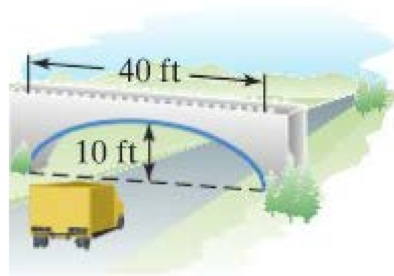
$$=\boxed{\sqrt{7}}$$

The ***foci*** are $\boxed{F=\left(-2,\ 1\pm\sqrt{7}\right)}$



Example

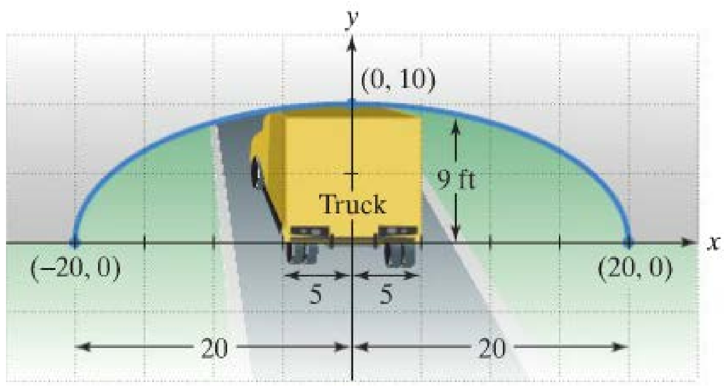
A semielliptical archway over a one-way road has a height of 10 *feet* and a width of 40 *feet*.



Your truck has a width of 10 *feet* and a height of 9 *feet*. Will your truck clear the opening of the archway?

Solution

Given: $a = \frac{40}{2} = 20, \quad b = 10$



$$\frac{x^2}{40^2} + \frac{y^2}{10^2} = 1 \qquad\qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The edge of the 10-foot-truck corresponds to $x = 5$

$$\frac{5^2}{40^2} + \frac{y^2}{10^2} = 1$$

$$400\frac{25}{400} + 400\frac{y^2}{100} = 400$$

$$25 + 4y^2 = 400$$

$$y^2 = \frac{375}{4}$$

$$\boxed{y = \frac{5\sqrt{15}}{2} \ \ ft}$$

$$\boxed{\approx 9.68 \ \ ft}$$

The truck will clear about 0.68 *feet* (8.16 *inches*)

Exercises Section 2.7 – Ellipses

(**1 –17**) Find the ***center***, ***vertices, minors*** and ***foci*** of the ellipse, and then sketch the graph of

1. $\frac{x^2}{9}+\frac{y^2}{4}=1$

2. $\frac{x^2}{16}+\frac{y^2}{36}=1$

3. $\frac{x^2}{15}+\frac{y^2}{16}=1$

4. $\frac{25x^2}{36}+\frac{64y^2}{9}=1$

5. $12x^2+8y^2=96$

6. $4x^2+y^2=16$

7. $4x^2+25y^2=1$

8. $\frac{(x-3)^2}{16}+\frac{(y+4)^2}{9}=1$

9. $9x^2+4y^2-18x+16y-11=0$

10. $\frac{(x+3)^2}{16}+\frac{(y-2)^2}{36}=1$

11. $\frac{(x+1)^2}{64}+\frac{(y-2)^2}{49}=1$

12. $4x^2+9y^2-32x-36y+64=0$

13. $x^2+2y^2+2x-20y+43=0$

14. $25x^2+4y^2-250x-16y+541=0$

15. $4x^2+y^2=2y$

16. $2x^2+3y^2-8x+6y+5=0$

17. $4x^2+3y^2+8x-6y-5=0$

18. Find an equation for an ellipse with: *$x-$intercepts:* ± 4; *foci* $(-2,\ 0)$ *and* $(2,\ 0)$

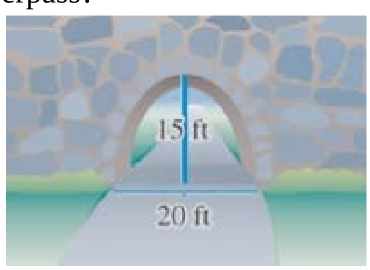
19. Find an equation for an ellipse with: *Endpoints of major axis at* $(6,\ 0)$ *and* $(-6,\ 0)$; $c=4$

20. Find an equation for an ellipse with: Center $(3,-2)$; $a=5$; $c=3$; major axis vertical

21. Find an equation for an ellipse with: *major axis of length* 6; *foci* $(0,\ 2)$ *and* $(0,\ -2)$

22. A patient's kidney stone is placed 12 *units* away from the source of the shock waves of a lithotripter. The lithotripter is based on an ellipse with a minor axis that measures 16 *units*. Find an equation of an ellipse that would satisfy this situation.

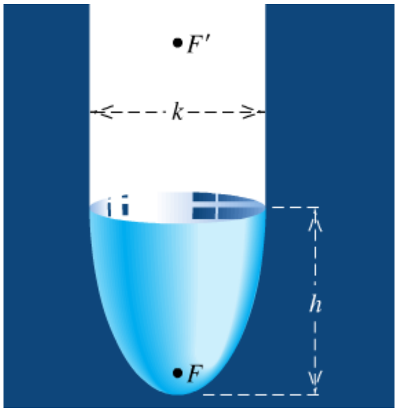
23. A one-way road passes under an overpass in the form of half of an ellipse 15 *feet* high at the center and 20 *feet* wide. Assuming that a truck is 12 *feet* wide, what is the height of the tallest truck that can pass under the overpass?



24. The basic shape of an elliptical reflector is a hemi-ellipsoid of height *h* and diameter *k*. Waves emitted from focus *F* will reflect off the surface into focus F'.

a) Express the distance $d(V,\ F)$ and $d(V,\ F')$ in terms of *h* and *k*.

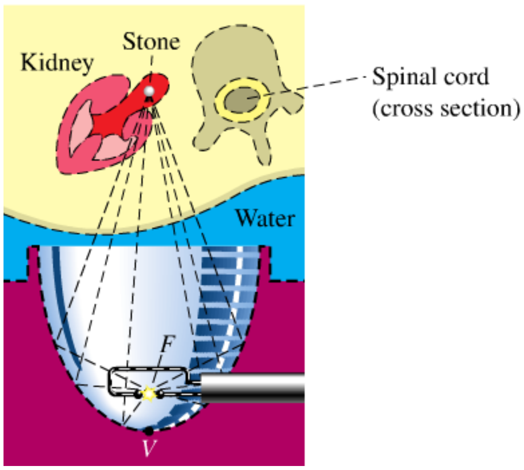
b) An elliptical reflector of height 17 *cm* is to be constructed so that waves emitted from *F* are reflected to a point F' that is 32 *cm* from *V*. Find the diameter of the reflector and the location of *F*.



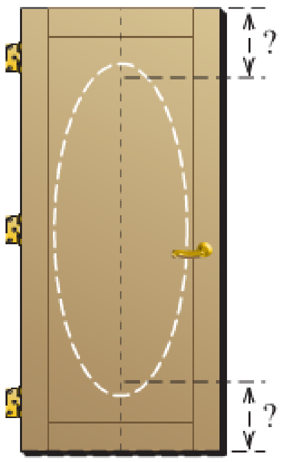
25. A lithotripter of height 15 *cm* and diameter 18 *cm* is to be constructed. High-energy underwater shock waves will be emitted from the focus *F* that is closest to the vertex *V*.

a) Find the distance from *V* to *F*.

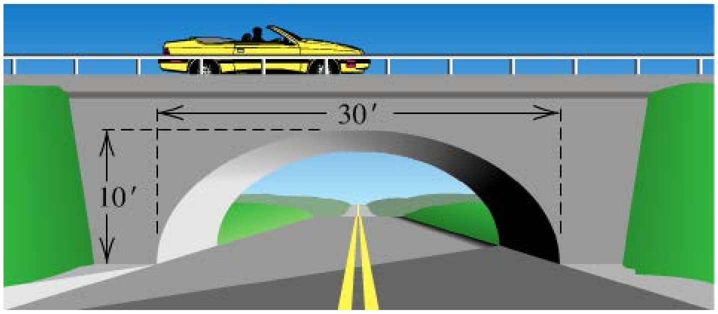
b) How far from *V* (in the vertical direction) should a kidney stone located?



26. An Artist plans to create an elliptical design with major axis 60″ and minor axis 24″, centered on a door that measures 80″ by 36″. On a vertical line that dissects the door, approximately how far from each end of the door should the push-pins be inserted? How long should the string be?

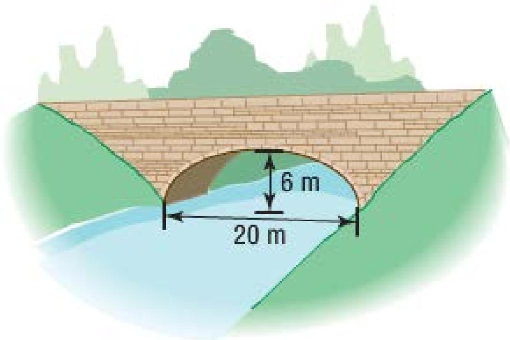


27. An arch of a bridge is semi-elliptical, with major axis horizontal. The base of the arch is 30 *feet*. across, and the highest part of the arch is 10 *feet*. above the horizontal roadway. Find the height of the arch 6 *feet*. from the center of the base.



28. The whispering gallery in the Museum of Science and Industry in Chicago is 47.3 *feet* long. The distance from the center of the room to the foci is 20.3 *feet*. Find an equation that describes the shape of the room. How high is the room at its center?

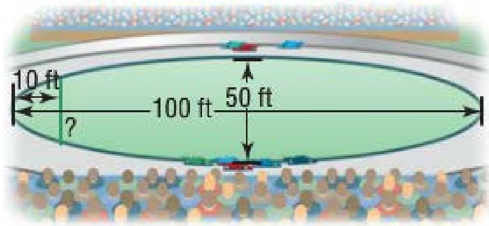
29. An arch in the shape of the upper half of an ellipse is used to support a bridge that is to span a river 20 *meters* wide. The center of the arch is 6 *meters* above the center of the river. Write an equation for the ellipse in which the *x*-axis coincides with the water level and the *y*-axis passes through the center of the arch.



30. A bridge is built in the shape of a semielliptical arch. The bridge has a span of 120 *feet* and a maximum height of 25 *feet*. Choose a rectangular coordinate system and find the height of the arch at distances of 10, 30, and 50 feet from the center.

31. A bridge is built in the shape of a semielliptical arch. The bridge has a span of 100 *feet*. The height of the arch is 10 *feet*. Find the height of the arch at its center.

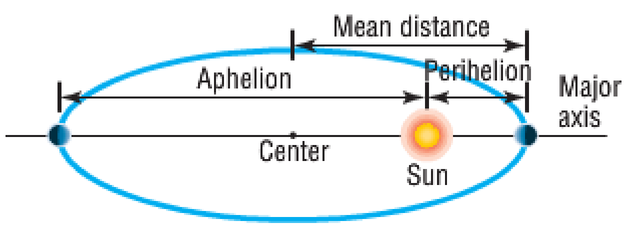
33. A racetrack is in the shape of an ellipse, 100 *feet* long and 50 feet wide. What is the width 10 *feet* from a vertex?



34. A homeowner is putting in a fireplace that has a 4-*inch* radius vent pipe. He needs to cut an elliptical hole in his roof to accommodate the pipe. If the pitch of his roof is $\frac{5}{4}$ (a rise of 5, run of 4) what are the dimensions of the hole?

35. A football is in the shape of a ***prolate spheroid***, which is simply a solid obtained by rotating an ellipse about its major axis. An inflated NFL football averages 11.125 inches in length and 28.25 inches in center circumference. If the volume of a prolate spheroid is $\frac{4}{3}\pi ab^2$, how much air does the football contain? (Neglect material thickness)

36. The fact that the orbit of a planet about the Sun is an ellipse with the Sun at one focus. The ***aphelion*** of a planet is its greatest distance from the Sun, and the ***perihelion*** is its shortest distance. The ***mean distance*** of a planet from the Sun is the length of the semi-major axis of the elliptical orbit.



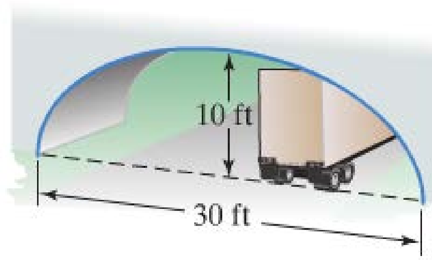
a) The mean distance of Earth from the Sun is 93 million *miles*. If the aphelion of Earth is 94.5 million miles, what is the perihelion? Write an equation for the orbit of Earth around the Sun.

b) The mean distance of Mars from the Sun is 142 million *miles*. If the perihelion of Mars is 128.5 million miles, what is the aphelion? Write an equation for the orbit of Mars about the Sun.

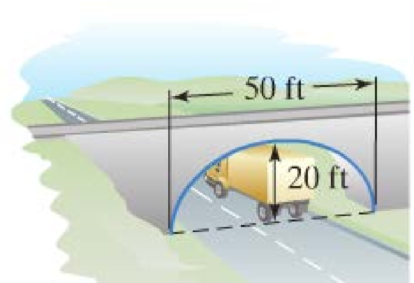
c) The aphelion of Jupiter is 507 million *miles*. If the distance from the center of it elliptical orbit to the Sun is 23.2 million *miles*, what is the perihelion? What is the mean distance? Write an equation for the orbit of Jupiter around the Sun.

d) The perihelion of Pluto is 4551 million *miles*, and the distance from the center of its elliptical orbit to the Sun is 897.5 million *miles*. Find the aphelion of Pluto. What is the mean distance of Pluto from the Sun? Write an equation for the orbit of Pluto about the Sun.

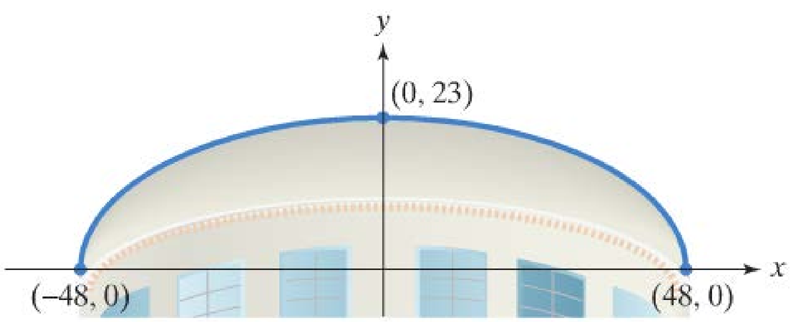
37. Will a truck that is 8 *feet* wide carrying a load that reaches 7 *feet* above the ground the semielliptical arch on the one-way road that passes under the bridge?



38. A semielliptic archway has a height of 20 *feet* and a width of 50 *feet* and a width of 50 *feet*. Can a truck 14 *feet* high and 10 *feet* wide drive under the archway without going into the other lane?



39. The elliptical ceiling in Statuary Hall is 96 *feet* long and 23 *feet* tall.



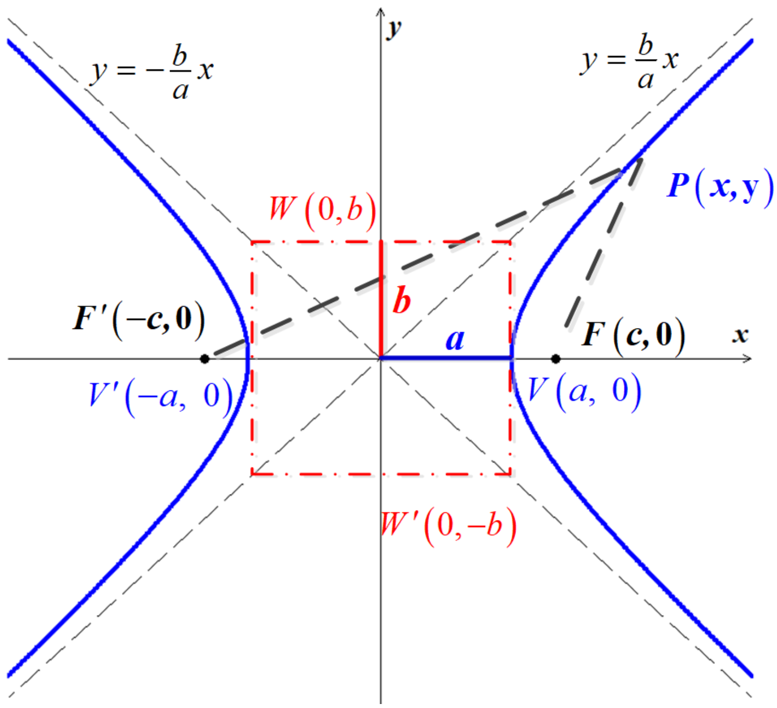
a) Using the rectangular coordinate system in the figure shown, write the standard form of the equation of the elliptical ceiling.

b) John Quincy Adams discovered that he could overhear the conversations of opposing party leaders near the left side of the chamber if he situated his desk at the focus at the right side of the chamber. How far from the center of the ellipse along the major axis did Adams situate his desk?

Section 4.8 – Hyperbolas

Definition of a Hyperbola

A ***hyperbola*** is the set of all points in a plane, the difference of whose distances from two fixed points (the ***foci***) in the plane is a positive constant.



Let $P(x,y)$ be a point on the hyperbola and $F'(-c,\ 0)$ and $F(c,\ 0)$ (the ***foci***), where the midpoint of $F'F$ (the origin) is called the *center*. The following is true:

$$d(P,F') - d(P,F) = 2a \quad \textit{or} \quad d(P,F) - d(P,F') = 2a$$

That implies to:

$$\left| d(P,F) - d(P,F') \right| = 2a$$

$$\left| \sqrt{(x-c)^2 + (y-0)^2} - \sqrt{(x+c)^2 + (y-0)^2} \right| = 2a$$

$$\left| \sqrt{x^2 - 2cx + c^2 + y^2} - \sqrt{x^2 + 2cx + c^2 + y^2} \right| = 2a$$

$$\sqrt{x^2 - 2cx + c^2 + y^2} = 2a + \sqrt{x^2 + 2cx + c^2 + y^2}$$

$$\left(\sqrt{x^2 - 2cx + c^2 + y^2} \right)^2 = \left(2a + \sqrt{x^2 + 2cx + c^2 + y^2} \right)^2$$

$$x^2 - 2cx + c^2 + y^2 = 4a^2 + 4a\sqrt{x^2 + 2cx + c^2 + y^2} + x^2 + 2cx + c^2 + y^2$$

$$-2cx = 4a^2 + 4a\sqrt{x^2 + 2cx + c^2 + y^2} + 2cx$$

$$-4cx-4a^2=4a\sqrt{x^2+2cx+c^2+y^2}$$

$$\left(-cx-a^2\right)^2=\left(a\sqrt{x^2+2cx+c^2+y^2}\right)^2$$

$$c^2x^2+2a^2cx+a^4=a^2\left(x^2+2cx+c^2+y^2\right)$$

$$c^2x^2+2a^2cx+a^4=a^2x^2+2a^2cx+a^2c^2+a^2y^2$$

$$c^2x^2+a^4=a^2x^2+a^2c^2+a^2y^2$$

$$c^2x^2-a^2x^2-a^2y^2=a^2c^2-a^4$$

$$\left(c^2-a^2\right)x^2-a^2y^2=a^2\left(c^2-a^2\right)$$

$$\frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=1$$

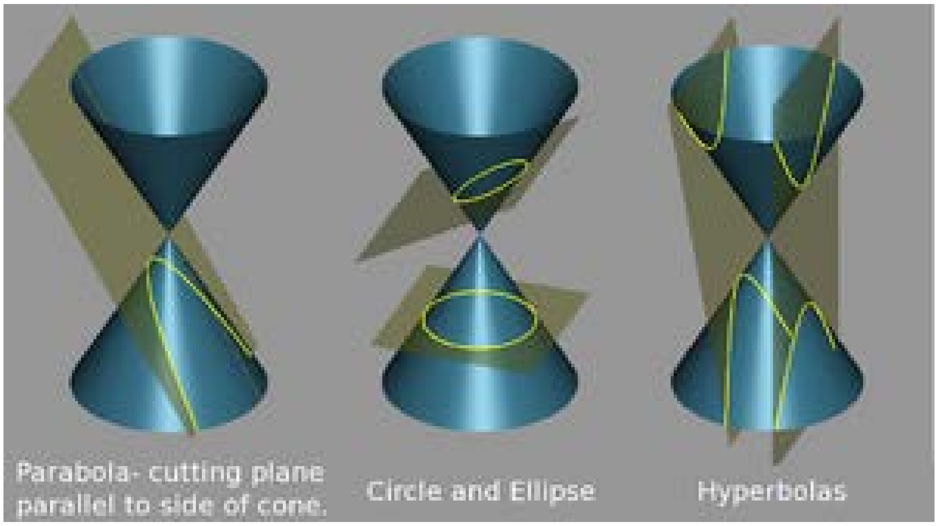
Finally, if we let $b^2=c^2-a^2;\quad b>0$

$$\boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1}$$

Applying the tests for symmetry, we see that the hyperbola is symmetric with the respect to both axes and the origin.

The ***x**-intercepts* are a and $-a$. The corresponding points $V(a,\ 0)$ and $V'(-a,\ 0)$ are called the ***vertices*** of the ellipse. The line segment $V'V$ is called the ***transverse axis***.

The graph has no ***y***-intercept, since $-\frac{y^2}{b^2}=1$ has the *complex* solutions $y=\pm bi$. The points $W(0,b)$ and $W'(0,-b)$ are endpoints of the ***conjugate axis*** $W'W$ (there are not on the hyperbola)



Example

Sketch the graph of $9x^2 - 4y^2 = 36$. Find the foci and equations of the asymptotes.

Solution

$$\frac{9x^2}{36} - \frac{4y^2}{36} = \frac{36}{36}$$

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

$$\to \begin{cases} a^2 = 4 \to a = \pm 2 \\ b^2 = 9 \to b = \pm 3 \end{cases} \qquad \Rightarrow c = \pm\sqrt{a^2 + b^2} = \pm\sqrt{4+9} = \pm\sqrt{13}$$

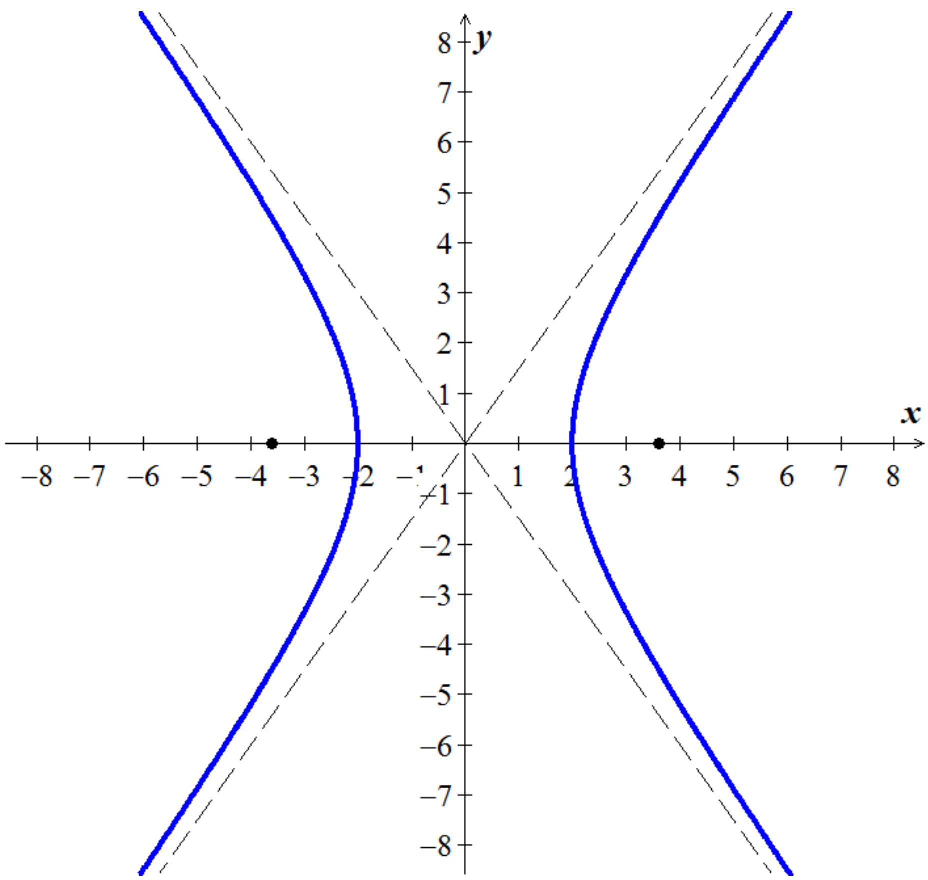
There are no ***y***–intercepts.

The ***endpoints***: $(0, \ \pm 3)$

The ***vertices***: $(\pm 2, \ 0)$

The ***foci*** are $F\left(\sqrt{13}, \ 0\right)$ *and* $F'\left(-\sqrt{13}, \ 0\right)$

The equations of the ***asymptotes*** are: $\boxed{y = \pm\frac{3}{2}x}$ $\qquad y = \pm\frac{b}{a}x$



Example

Sketch the graph of $4y^2 - 2x^2 = 1$. Find the foci and equations of the asymptotes.

Solution

$$\frac{y^2}{\frac{1}{4}} - \frac{x^2}{\frac{1}{2}} = 1$$

$$\rightarrow \begin{cases} a^2 = \frac{1}{4} \rightarrow a = \pm\frac{1}{2} \\ b^2 = \frac{1}{2} \rightarrow b = \pm\frac{1}{\sqrt{2}} \end{cases}$$

$$\Rightarrow c = \pm\sqrt{a^2 + b^2} = \pm\sqrt{\frac{1}{4} + \frac{1}{2}} = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$$

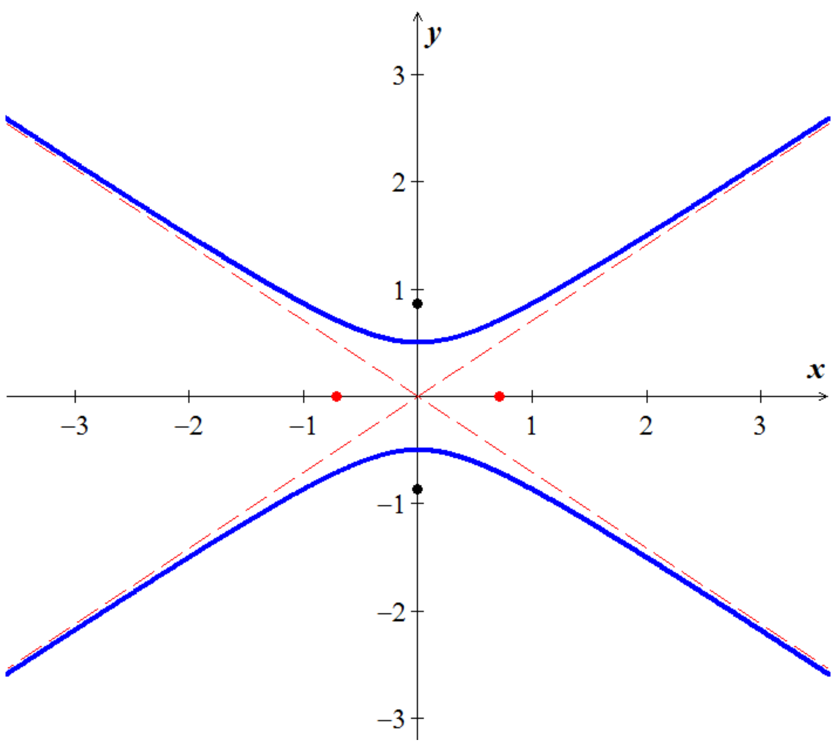
There are no x-intercepts.

The ***endpoints***: $\left(\pm\frac{1}{\sqrt{2}},\ 0\right)$

The ***vertices***: $\left(0,\ \pm\frac{1}{2}\right)$

The ***foci*** are $\left(0,\ \pm\frac{\sqrt{3}}{2}\right)$

The equations of the ***asymptotes*** are: $y = \pm\frac{\frac{1}{2}}{\frac{1}{\sqrt{2}}}x = \boxed{\pm\frac{\sqrt{2}}{2}x}$ $\qquad y = \pm\frac{a}{b}x$



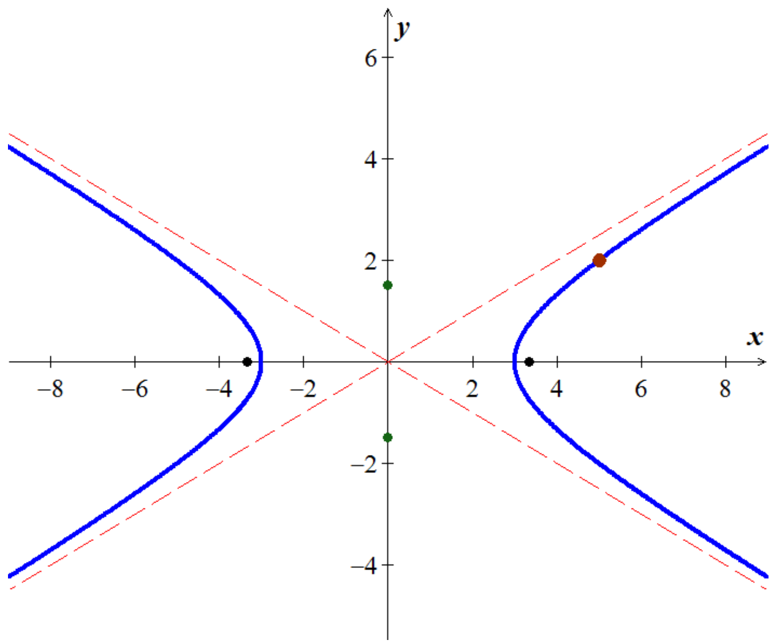
Example

A hyperbola has vertices $(\pm 3,\ 0)$ and passes through the point $P(5,\ 2)$. Find its equation, foci and asymptotes.

Solution

$$\frac{x^2}{3^2} - \frac{y^2}{b^2} = 1$$

Since $P(5,\ 2)$ in on the hyperbola $\Rightarrow \frac{5^2}{3^2} - \frac{2^2}{b^2} = 1$



$$-\frac{4}{b^2} = 1 - \frac{25}{9}$$

$$-\frac{4}{b^2} = -\frac{16}{9}$$

$$\frac{b^2}{4} = \frac{9}{16}$$

$$b^2 = \frac{9}{4}$$

$$\frac{x^2}{9} - \frac{y^2}{\frac{9}{4}} = 1$$

$$\frac{x^2}{9} - \frac{4y^2}{9} = 1$$

$$x^2 - 4y^2 = 9$$

$$c = \sqrt{9 + \frac{9}{4}} \qquad\qquad c = \sqrt{a^2 + b^2}$$

$$= \sqrt{\frac{45}{4}}$$

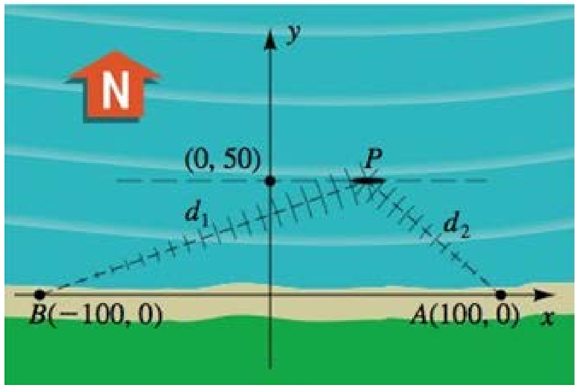
$$= \boxed{\frac{3\sqrt{5}}{2}}$$

The ***foci***: $\left(\pm\frac{3\sqrt{5}}{2},\ 0\right)$

The equations of the ***asymptotes*** are: $y = \pm\frac{\frac{3}{2}}{3}x = \boxed{\pm\frac{1}{2}x}$ $\qquad\qquad y = \pm\frac{b}{a}x$

Example

Coast Guard station *A* is 200 *miles* directly east of another station *B*. A ship is sailing on a line parallel to and 50 *miles* north of the line through *A* and *B*. Radio signals are sent out from *A* and *B* at the rate of 980 $ft/\mu\text{sec}$ (microsecond). If, at 1:00 PM, the signal from *B* reaches the ship 400 microseconds after the signal from *A*, locate the position of the ship at that time.



Solution

Given: $v = 980 \ \ ft/\mu\text{sec} \qquad t = 400 \ \ \mu\text{sec}$

$$d_2 - d_1 = (980)(400) = 392{,}000 \ ft$$

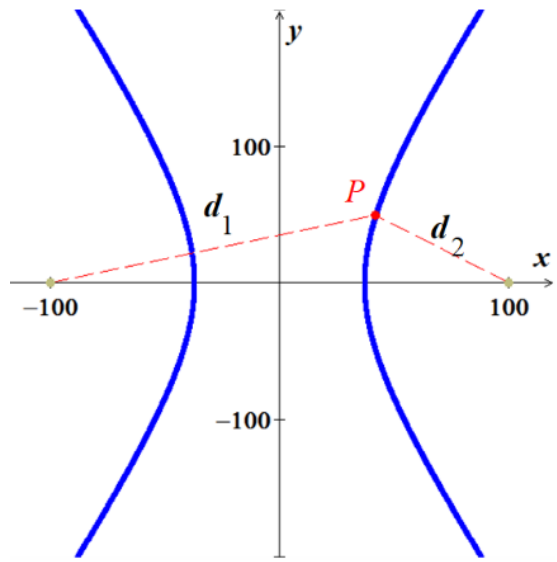
$$= 392{,}000 \ ft \frac{1 \ mi}{5280 \ ft}$$

$$= 74.\overline{24} \ \ mi$$

Since $d_2 - d_1 = 2a$

$$a = \frac{74.\overline{24}}{2} = 37.\overline{12}$$

$$a^2 = \left(37.\overline{12}\right)^2 \approx 1378$$



Distance from the origin to either focus is $c = 100$

Then, $b^2 = c^2 - a^2 \approx 10{,}000 - 1378 \approx 8622$

$$\frac{x^2}{1378} - \frac{y^2}{8622} = 1 \qquad\qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Since $y_P = 50$

$$\frac{x^2}{1378} - \frac{50^2}{8622} = 1$$

$$x^2 = 1{,}378\left(1 + \frac{2{,}500}{8{,}622}\right)$$

$$x = \sqrt{1{,}378\left(\frac{11{,}122}{8{,}622}\right)} \approx 42.16$$

$\therefore \ P(42, \ 50)$

Exercises *Section* 4.8 – Hyperbolas

(**1 – 15**) Find the ***center, vertices***, the ***foci***, ***endpoints*** and the equations of the ***asymptotes*** of the hyperbola. Sketch its graph, showing the asymptotes and the foci.

1. $\frac{x^2}{9} - \frac{y^2}{4} = 1$

2. $\frac{y^2}{9} - \frac{x^2}{4} = 1$

3. $x^2 - \frac{y^2}{24} = 1$

4. $y^2 - 4x^2 = 16$

5. $16x^2 - 36y^2 = 1$

6. $\frac{(y+2)^2}{9} - \frac{(x+2)^2}{4} = 1$

7. $\frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} = 1$

8. $(y-2)^2 - 4(x+2)^2 = 4$

9. $(x+4)^2 - 9(y-3)^2 = 9$

10. $144x^2 - 25y^2 + 864x - 100y - 2404 = 0$

11. $4y^2 - x^2 + 40y - 4x + 60 = 0$

12. $4x^2 - 16x - 9y^2 + 36y = -16$

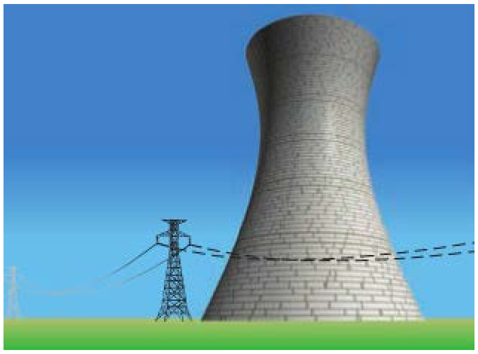
13. $2x^2 - y^2 + 4x + 4y = 4$

14. $2y^2 - x^2 + 2x + 8y + 3 = 0$

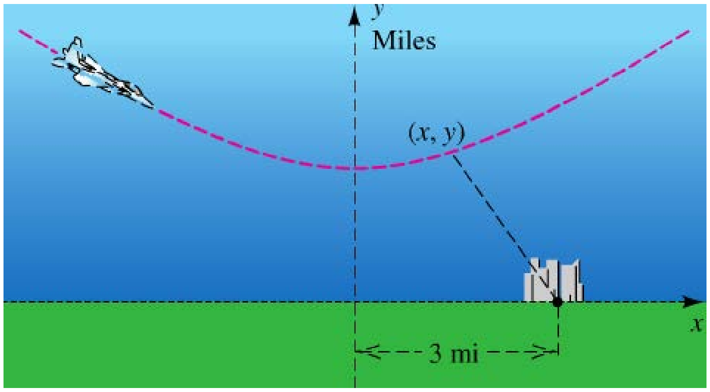
15. $2y^2 - 4x^2 - 16x - 2y - 19 = 0$

16. Suppose a hyperbola has center at the origin, foci at $F'(-c,\ 0)$ and $F(c,\ 0)$, and equation $d(P,\ F') - d(P,\ F) = 2a$. Let $b^2 = c^2 - a^2$, and show that an equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

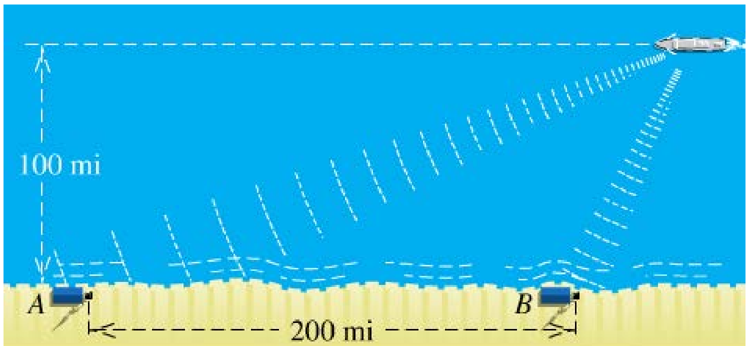
17. A cooling tower is a hydraulic structure. Suppose its base diameter is 100 meters and its smallest diameter of 48 meters occurs 84 meters from the base. If the tower is 120 meters high approximate its diameter at the top.



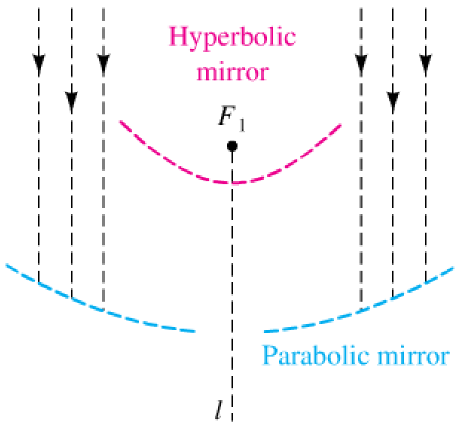
18. An airplane is flyting along the hyperbolic path. If an equation of the path is $2y^2 - x^2 = 8$, determine how close the airplane comes to town located at (3, 0). (Hunt: Let *S* denote the square of the distance from a point (*x*, *y*) on the path to (3, 0), and find the minimum value of *S*.)



19. A ship is traveling a course that is 100 miles from, and parallel tom a straight shoreline. The ship sends out a distress signal that is received by two Coast Guard stations *A* and *B*, located 200 miles apart. By measuring the difference in signal reception times, it is determined that the ship is 160 miles closer to *B* than to *A*. Where is the ship?

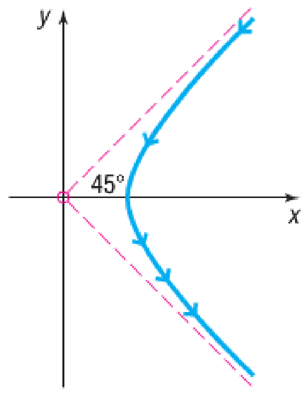


20. The Cassegrain telescope design (dating back to 1672) makes use of the reflective properties of both the parabola and the hyperbola. The figure shows a (split) parabolic mirror, with one focus at F_1 and axis along the line *l*, and a hyperbolic mirror, with one focus also at F_1 and transverse axis along *l*. Where do incoming light waves parallel to the common axis finally collect?



21. Suppose that two people standing 1 *mile* apart both see a flash of lightning. After a period of time, the person standing at point *A* hears the thunder. One second later, the person standing at point *B* hears the thunder. If the person at *B* is due west of the person at *A* and the lightning strike is known to occur due north of the person standing at point *A*, where did the lightning strike occur? (Sound travels at 1100 ft / sec and 1 $mile = 5280\ ft$)

22. Ernest Rutherford published a paper that he described the motion of alpha particles as they are shot at a piece of gold foil 0.00004 *cm* thick. Before conducting this experiment, Rutherford expected that the alpha particles would shoot through the foil just as a bullet would shoot through the foil just as a bullet would shoot through snow. Instead, a small fraction of the alpha particles bounced off the foil. This led to the conclusion that the nucleus of an atom is dense, while the remainder of the atom is sparse. Only the density of the nucleus could cause the alpha particles to deviate from their path. The figure shows a diagram from Rutherford's paper that indicates that the deflected alpha particles follow the path of one branch of a hyperbola.



a) Find an equation of the asymptotes under this scenario.

b) If the vertex of the path of the alpha particles is 10 *cm* from the center of the hyperbola, find a model that describes the path of the particle.

23. Hyperbolas have interesting reflective properties that make them useful for lenses and mirrors. For example, if a ray of light strikes a convex hyperbolic mirror on a line that would (theoretically) pass through its rear focus, it is reflected through its rear focus, it is reflected through the front focus. This property and that of the parabola were used to develop the *Cassegrain* telescope in 1672. The focus of the parabolic mirror and the rear focus of the hyperbolic mirror are the same point. The rays are collected by the parabolic mirror, reflected toward the common focus, and thus are reflected by the hyperbolic mirror through the opening to its front focus, where the eyepiece is located. If the equation of the hyperbola is $\frac{y^2}{9}-\frac{x^2}{16}=1$ and the focal length (distance from the vertex to the focus) of the parabola is 6, find the equation of the parabola.

24. The ***eccentricity e*** of a hyperbola is defined as the number $\frac{c}{a}$, where a is the distance of a vertex from the center and c is the distance of a focus from the center. Because $c > a$, it follows that $e > 1$. Describe the general shape of a hyperbola whose eccentricity is close to 1. What is the shape if e is very large?

25. An explosive is recorded by two microphone that are 1 *mile* apart. Microphone M_1 received the sound 2 *seconds* before microphone M_2 . Assuming sound travels at 1,100 *feet* per *second*, determine the possible locations of the explosion relative to the location of the microphones.

26. Radio towers *A* and *B*, 200 *km* apart, are situated along the coast, with *A* located due west of *B*. Simultaneous radio signals are sent from each tower to a ship, with the signal from *B* received $500\ \mu\text{sec}$ before the signal from *A*.

a) Assuming that the radio signals travel $300\ m/\mu\text{sec}$, determine the equation of the hyperbola on which the ship is located.

b) If the ship lies due north of tower *B*, how far out at sea is it?

27. An architect designs two houses that are shaped and positioned like a part of the branches of the hyperbola whose equation is $625y^2 - 400x^2 = 250{,}000$, where *x* and *y* are in yards. How far apart are the houses at their closest point?