Lecture Four – *Parametric Equations and Polar Coordinates*

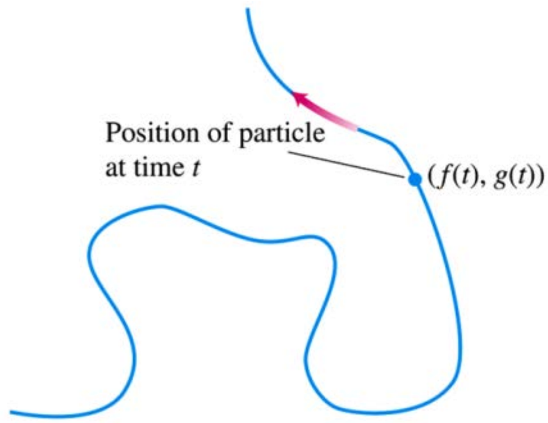
Section 4.1 – Parameterizations of Plane Curves

Definition

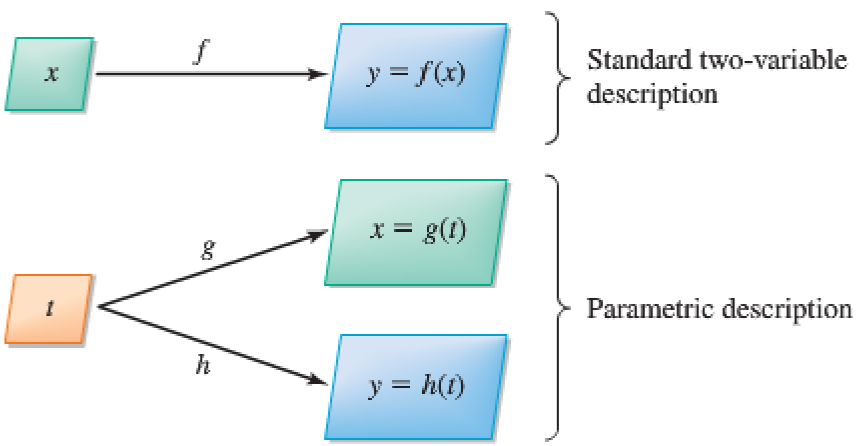
If x and y are given as functions

$$x = f(t), \quad y = g(t)$$

Over an interval I as t–values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a ***parametric curve***. The equations are ***parametric equations*** for the curve



The variable t is a parameter for the curve, and its domain I is the parameter interval.



Definition

The direction in which a parametric curve is generated as the parameter increases is called the forward or positive orientation of the curve.

Example

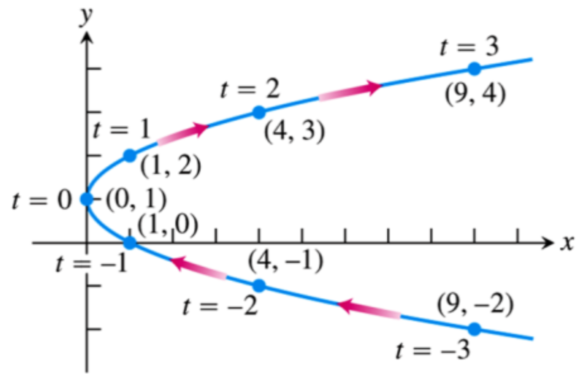
Sketch the curve defined by the parametric equations

$$x = t^2, \quad y = t + 1, \quad -\infty < t < \infty$$

Then obtain an algebraic equation in *x* and *y*.

Solution

t	$x = t^2$	$y = t + 1$
–3	9	–2
–2	4	–1
–1	1	0
0	0	1
1	1	2
2	4	3
3	9	4



$$y = t + 1 \Rightarrow t = y - 1$$

$$x = t^2 = (y-1)^2 = y^2 - 2y + 1 \qquad \textbf{\textit{represents a parabola}}$$

Example

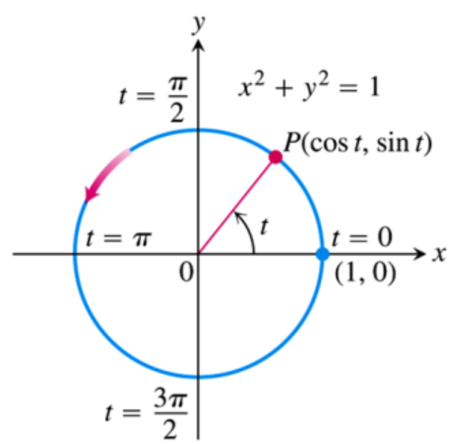
Graph the parametric curve $x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi$

Solution

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

This parametric curve lies along the unit circle $x^2 + y^2 = 1$.

As t increases from 0 to 2π, the point $(x, y) = (\cos t, \sin t)$ starts at (1, 0) and traces the entire circle once counterclockwise.



Example

Graph the parametric curve $x = a\cos t, \quad y = a\sin t, \quad 0 \le t \le 2\pi$

Solution

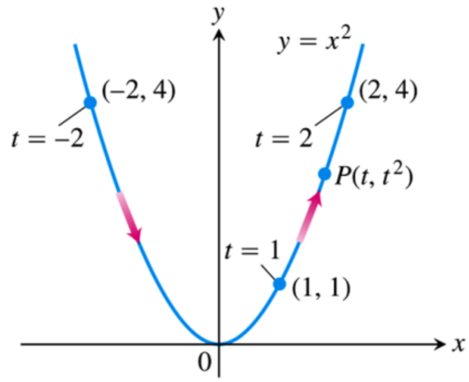
$$x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2\left(\cos^2 t + \sin^2 t\right) = a^2$$

This parametric curve lies along the unit circle $x^2 + y^2 = a^2$.

As *t* increases from 0 to 2π, the point $(x, y) = (a\cos t, a\sin t)$ starts at (*a*, 0) with a radius $r = a$ and traces the entire circle once counterclockwise.

Example

Graph the parametric curve $x = t, \quad y = t^2, \quad -\infty < t < \infty$



Solution

$y = x^2$

Example

Find a parameterization for the line through the point (a, b) having slope m.

Solution

A Cartesian equation of the line is $y - b = m(x - a)$

If $t = x - a \quad \Rightarrow \quad x = t + a$

$y - b = mt \quad \Rightarrow \quad y = mt + b$

That is, $x = t + a \quad y = mt + b, \quad -\infty < t < \infty$

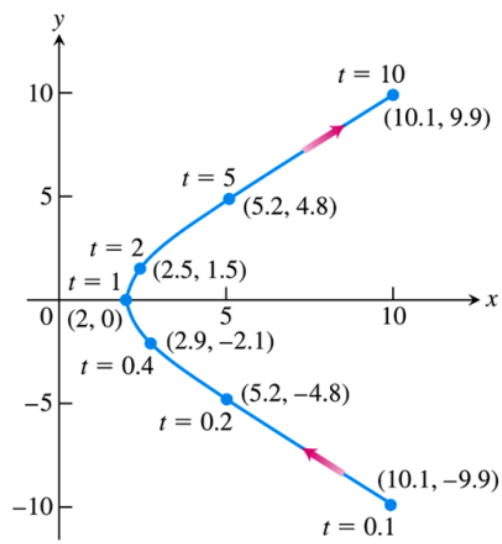
Example

Sketch the curve defined by the parametric equations $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t > 0$

Then obtain an algebraic equation in x and y.

Solution

t	$x = t + \frac{1}{t}$	$y = t - \frac{1}{t}$
.2	5.2	–4.8
.5	2.5	–1.5
1	2	0
2	2.5	1.5
5	5.2	4.8



$$x + y = t + \frac{1}{t} + t - \frac{1}{t} = 2t$$

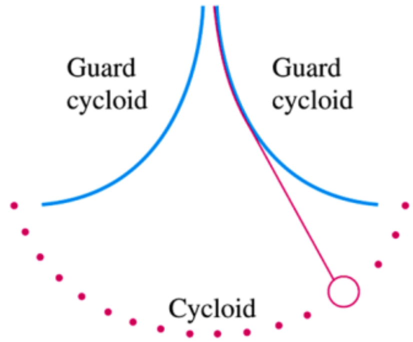
$$\underline{x - y} = t + \frac{1}{t} - t + \frac{1}{t} = \underline{\frac{2}{t}}$$

$$(x + y)(x - y) = 2t\left(\frac{2}{t}\right) = 4$$

$$\boxed{x^2 - y^2 = 4}$$

Cycloids

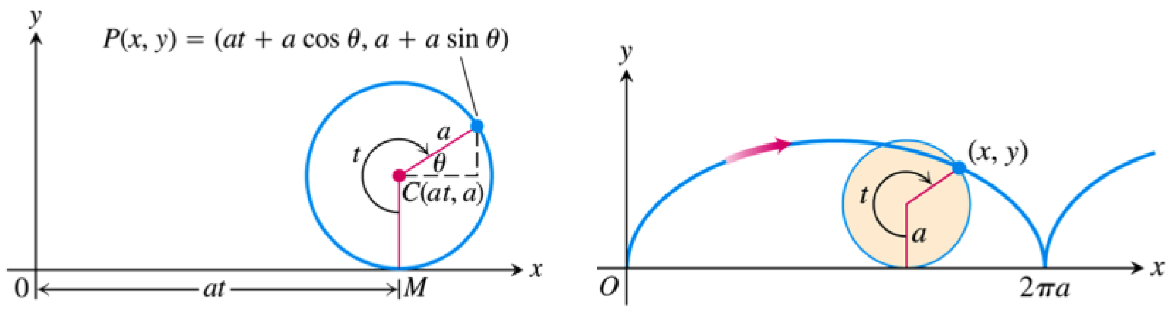
The problem with a pendulum clock whose bob swings in a circular arc is that the frequency of the swing depends on the amplitude of the swing.



Example

A wheel of a radius *a* rolls along a horizontal straight line. Find parametric equations for the path traced by a point P on the wheel's circumference. The path is called a cycloid.

Solution



We take the line to be on the *x*-axis, mark a point *P* on the wheel, start the wheel with *P* at the origin, and roll the wheel. As parameter, we use the angle *t* through which the wheel turns, measured in radians. The wheel's center C lies at $(at,\ a)$ and the coordinates of *P* are

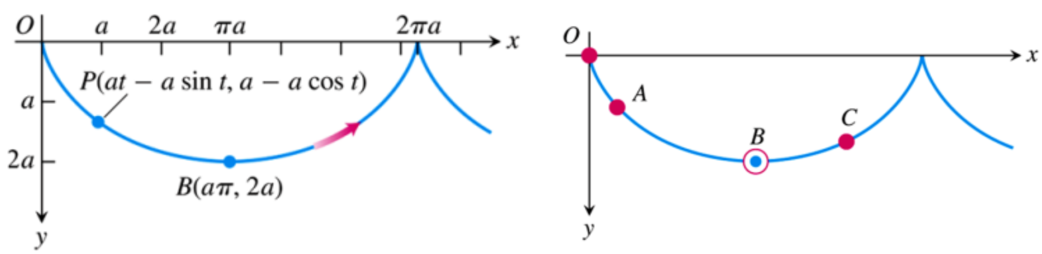
$$x = at + a\cos\theta, \quad y = a + a\sin\theta$$

To express θ in terms of *t*, we observe that $t + \theta = \frac{3\pi}{2}$, so that $\theta = \frac{3\pi}{2} - t$

That makes $\cos\theta = \cos\left(\frac{3\pi}{2} - t\right) = -\sin t, \quad \sin\theta = \sin\left(\frac{3\pi}{2} - t\right) = -\cos t$

$x = at - a\sin t, \quad y = a - a\cos t$

That implies to: $\boxed{x = a(t - \sin t), \quad y = a(1 - \cos t)}$



Exercises *Section* 4.1 – Parameterizations of Plane Curves

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

1. $x = 3t, \quad y = 9t^2, \quad -\infty < t < \infty$

2. $x = -\sqrt{t}, \quad y = t, \quad t \ge 0$

3. $x = 3 - 3t, \quad y = 2t, \quad 0 \le t \le 1$

4. $x = \cos 2t, \quad y = \sin 2t, \quad 0 \le t \le \pi$

5. $x = \cos(\pi - t), \quad y = \sin(\pi - t), \quad 0 \le t \le \pi$

6. $x = 4\sin t, \quad y = 5\cos t, \quad 0 \le t \le 2\pi$

7. $x = 1 + \sin t, \quad y = \cos t - 2, \quad 0 \le t \le 2\pi$

8. $x = t^2, \quad y = t^6 - 2t^4, \quad -\infty < t < \infty$

9. $x = \frac{t}{t-1}, \quad y = \frac{t-2}{t+1}, \quad -1 < t < 1$

10. $x = \sqrt{t+1}, \quad y = \sqrt{t}, \quad t \ge 0$

11. $x = 2\sinh t, \quad y = 2\cosh t, \quad -\infty < t < \infty$

12. $x = 4\cos 2\pi t, \quad y = 4\sin 2\pi t, \quad 0 \le t \le 1$

13. $x = \sqrt{t} + 4, \quad y = 3\sqrt{t}; \quad 0 \le t \le 16$

14. $x = (t+1)^2, \quad y = t + 2; \quad -10 \le t \le 10$

15. $x = t - 1, \quad y = t^3; \quad -4 \le t \le 4$

16. $x = e^{2t}, \quad y = e^t + 1; \quad 0 \le t \le 25$

17. $x = 3\cos t, \quad y = 3\sin t; \quad \pi \le t \le 2\pi$

18. $x = -7\cos 2t, \quad y = -7\sin 2t; \quad 0 \le t \le \pi$

19. $x = 1 - 3\sin 4\pi t, \quad y = 2 + 3\cos 4\pi t; \quad 0 \le t \le \frac{1}{2}$

20. Find parametric equation for the left half of the parabola $y = x^2 + 1$, originating at $(0,\ 1)$

21. Find parametric equation for the line that passes through the points $(1,\ 1)$ and $(3,\ 5)$, oriented in the direction of increasing *x*.

22. Find parametric equation for the lower half of the circle centered at $(-2,\ 2)$ with radius 6, oriented in the counterclockwise direction.

23. Find parametric equation for the upper half of the parabola $x = y^2$, originating at $(0,\ 0)$

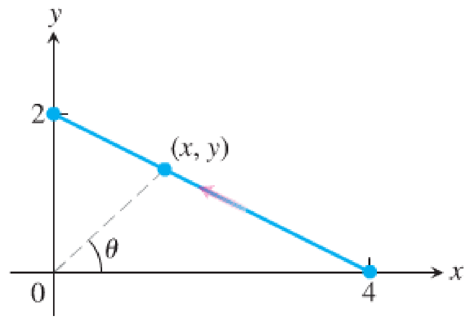
24. Find parametric equations (not unique) of an ellipse centered at the origin with major axis of length 6 on the *x-axis* and minor axis of length 3 on the *y-axis*, generated counterclockwise. Graph the ellipse and find a description in terms of *x* and *y*.

25. Find parametric equations (not unique) of an ellipse centered at the origin with major axis of length 12 on the *x-axis* and minor axis of length 2 on the *y-axis*, generated clockwise. Graph the ellipse and find a description in terms of *x* and *y*.

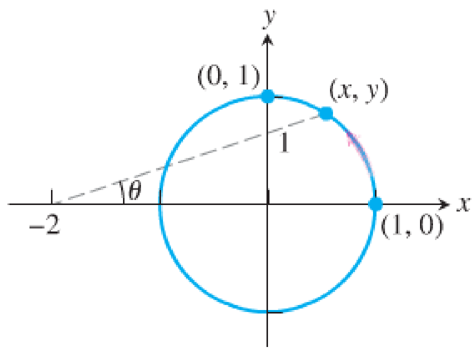
26. Find parametric equations (not unique) of an ellipse centered at $(-2,\ -3)$ with major and minor axes of lengths 30 and 20, parallel to the *x-axis* and *y-axis*, respectively. Graph the ellipse and find a description in terms of *x* and *y*.

27. Find a parametric equations and a parameter interval for the motion of a particle starting at the point (2, 0) and tracing the top half of the circle $x^2 + y^2 = 4$ four times.

28. Find a parametrization for the line segment joining points (0, 2) and (4, 0) using the angle θ in the accompanying figure as the parameter.



29. Find a parametrization for the circle $x^2 + y^2 = 1$ starting at (1, 0) and moving counterclockwise to the terminal point (0, 1), using the angle θ in the accompanying figure as the parameter.



30. A common task is to parameterize curves given either by either Cartesian equations or by graphs. Find a parametric representation of the following curves.

a) The segment of the parabola $y = 9 - x^2, \quad for \quad -1 \le x \le 3$

b) The complete curve $x = (y-5)^2 + \sqrt{y}$

c) The piecewise linear path that connects $P(-2,\ 0)$ to $Q(0,\ 3)$ to $R(4,\ 0)$ (in that order), where the parameter varies over the interval $0 \le t \le 2$

31. A projectile launched from the ground with an initial speed of 20 *m/s* and a launch angle θ follows a trajectory approximated by

$$x = (20\cos\theta)t, \quad y = -4.9t^2 + (20\sin\theta)t$$

Where *x* and *y* are the horizontal and vertical positions of the projectile relative to the launch point $(0,\ 0)$.

a) Graph the trajectory for various of θ in the range $0 < \theta < \frac{\pi}{2}$.

b) Based on your observations, what value of θ gives the greatest range (the horizontal distance between the launch and landing points)?

32. Many fascinating curves are generated by points on rolling wheels. The path of a light on the rim of a rolling when is a cycloid, which has the parametric equations

$$x = a(t - \sin t), \quad y = a(1 - \cos t), \quad \textit{for} \ \ t \geq 0$$

Where $a > 0$. Graph the cycloid with $a = 1$. On what interval does the parameter generate one arch of the cycloid?

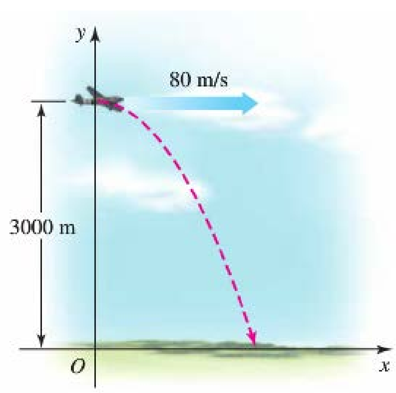
(18−20) Find parametric equations that describe the circular path of the objects. Assume (x, y) denotes the position of the object relative to the origin at the center of the circle.

33. A go-cart moves counterclockwise with constant speed around a circular track of radius 400 *m*, completing in 1.5 *min*.

34. The tip of the 15-*in* second hand of a clock completes one revolution in 60 *sec*.

35. A Ferris wheel has a radius of 20 *m* and completes a revolution in the clockwise direction at constant speed in 3 *min*. Assume that *x* and *y* measure the horizontal and vertical positions of a seat on the Ferris wheel relative to a coordinate system whose origin is at the low point of the wheel. Assume the seat begins moving at the origin.

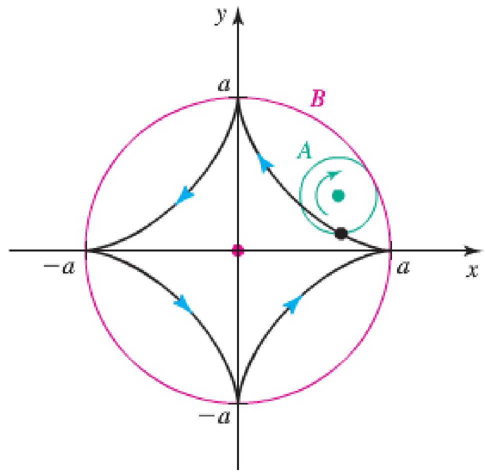
36. A plane traveling horizontally at 80 *m/s* over flat ground at an elevation of 3000 *m* releases an emergency packet. The trajectory of the packet is given by



$$x = 80t, \quad y = -4.9t^2 + 3000, \quad \textit{for} \ \ t \geq 0$$

Where the origin is the point on the ground directly beneath the plane at the moment of the release. Graph the trajectory of the packet and find the coordinates of the point where the packet lands.

37. The path of a point on circle A with radius $\frac{a}{4}$ that rolls on the inside of circle B with a radius a is an asteroid or hypocycloid. Its parametric equations are



$$x = a\cos^3 t, \quad y = a\sin^3 t, \quad 0 \le t \le 2\pi$$

Graph the asteroid with $a = 1$ and find its equation in terms of x and y.

Section 4.2 – Calculus with Parametric Curves

Tangents and Areas

A parametrized curve $x = f(t)$ and $y = g(t)$ is differentiable at *t* if *f* and *g* are differentiable at *t*.

Parametric Formula for dy/dx

If all three derivatives exist and $\frac{dx}{dt} \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

The derivatives $\frac{dy}{dt}, \frac{dx}{dt}$, and $\frac{dy}{dx}$ are related by the Chain Rule $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

Parametric Formula for d^2y/dx^2

If the equations $x = f(t)$, $y = g(t)$ define *y* as a twice–differentiable function of *x*, then at any point where $\frac{dx}{dt} \neq 0$ and $y' = \frac{dy}{dx}$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

Example

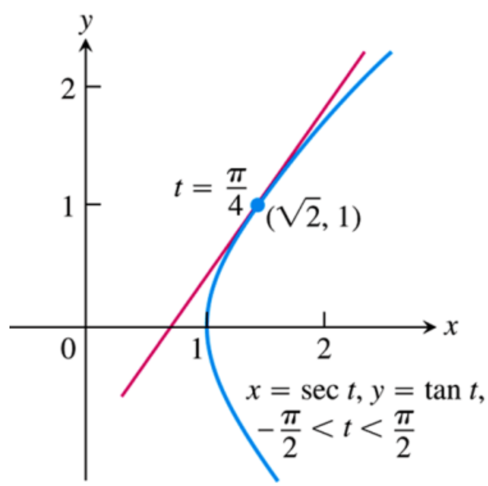
Find the tangent to the curve $x = \sec t$, $y = \tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$, at the point $(\sqrt{2}, 1)$, where $t = \frac{\pi}{4}$

Solution

The slope of the curve at *t* is:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t}$$

$$t = \frac{\pi}{4} \quad \Rightarrow \quad \boxed{m = \frac{dy}{dx}\bigg|_{t=\frac{\pi}{4}}} = \frac{\sec\frac{\pi}{4}}{\tan\frac{\pi}{4}} = \frac{\sqrt{2}}{1} = \boxed{\sqrt{2}}$$



The tangent line is

$$y = m(x - x_1) + y_1$$

$$y = \sqrt{2}(x - \sqrt{2}) + 1 = \sqrt{2}x - 2 + 1 = \sqrt{2}\,x - 1$$

$$\boxed{= \sqrt{2}\,x - 1}$$

Example

Find $\dfrac{d^2y}{dx^2}$ as a function of t if $x=t-t^2, \quad y=t-t^3$

Solution

$$y'=\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{1-3t^2}{1-2t}$$

$$\frac{dy'}{dt}=\frac{d}{dt}\left(\frac{1-3t^2}{1-2t}\right) \qquad\qquad \begin{array}{ll} u=1-3t^2 & v=1-2t \\ u'=-6t & v'=-2 \end{array}$$

$$=\frac{-6t(1-2t)-(-2)\left(1-3t^2\right)}{(1-2t)^2}$$

$$=\frac{-6t+12t^2+2-6t^2}{(1-2t)^2}$$

$$=\frac{6t^2-6t+2}{(1-2t)^2}$$

$$\frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}$$

$$=\frac{6t^2-6t+2}{(1-2t)^2}\div(1-2t)$$

$$=\frac{6t^2-6t+2}{(1-2t)^3}$$

Example

Find the area enclosed by the asteroid: $x=\cos^3 t, \quad y=\sin^3 t, \quad 0\le t\le 2\pi$

Solution

By symmetry, the enclosed area is 4 times the area beneath the curve in the first quadrant where $0\le t\le \frac{\pi}{2}$.

$$A=4\int_0^1 y\,dx \qquad\qquad dx=d\left(\cos^3 t\right)=3\cos^2 t\sin t\,dt$$

$$=4\int_0^{\pi/2}\sin^3 t\cdot 3\cos^2 t\sin t\ dt$$

$$= 12\int_0^{\pi/2} \sin^4 t \cdot \cos^2 t \; dt$$

$$= 12\int_0^{\pi/2} \left(\frac{1-\cos 2t}{2}\right)^2 \left(\frac{1+\cos 2t}{2}\right) \; dt$$

$$= \frac{3}{2}\int_0^{\pi/2} \left(1-2\cos 2t+\cos^2 2t\right)\left(1+\cos 2t\right) \; dt$$

$$= \frac{3}{2}\int_0^{\pi/2} \left(1-2\cos 2t+\cos^2 2t+\cos 2t-2\cos^2 2t+\cos^3 2t\right) \; dt$$

$$= \frac{3}{2}\int_0^{\pi/2} \left(1-\cos 2t-\cos^2 2t+\cos^3 2t\right) \; dt$$

$$= \frac{3}{2}\left[t-\frac{1}{2}\sin 2t\right]_0^{\pi/2} - \frac{3}{2}\int_0^{\pi/2} \cos^2 2t \; dt + \frac{3}{2}\int_0^{\pi/2} \cos^3 2t \; dt \qquad \cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

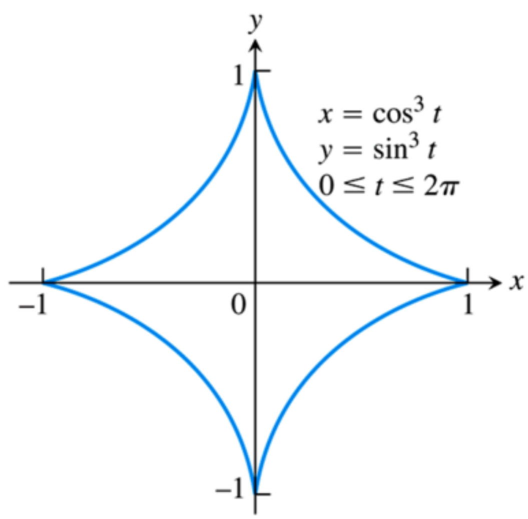
$$= \frac{3}{2}\left(\frac{\pi}{2}-0\right) - \frac{3}{2}\int_0^{\pi/2} \frac{1+\cos 2t}{2} \; dt + \frac{3}{2}\int_0^{\pi/2} \left(1-\sin^2 2t\right)\cos 2t \; dt$$

$$= \frac{3\pi}{4} - \frac{3}{4}\left[t+\frac{1}{2}\sin 2t\right]_0^{\pi/2} + \frac{3}{4}\int_0^{\pi/2} \left(1-\sin^2 2t\right) d\left(\sin 2t\right)$$

$$= \frac{3\pi}{4} - \frac{3}{4}\left(\frac{\pi}{2}-0\right) + \frac{3}{4}\left[\sin 2t-\frac{1}{3}\sin^3 2t\right]_0^{\pi/2}$$

$$= \frac{3\pi}{4} - \frac{3\pi}{8} + \frac{3}{4}(0-0)$$

$$= \boxed{\frac{3\pi}{8}}$$

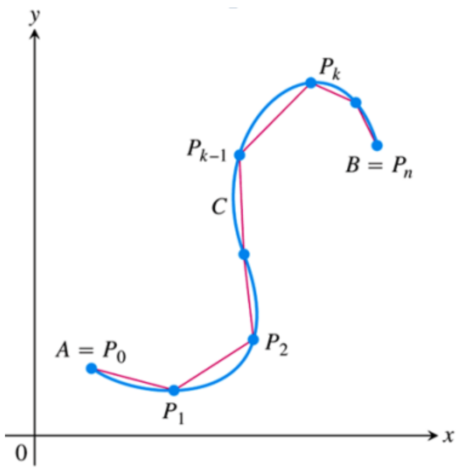


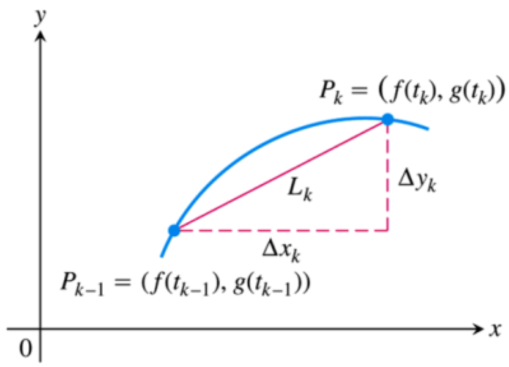
Length of a Parametrically Defined Curve

Definition

If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = a$, then the length of C is the definite integral

$$L = \int_a^b \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}\, dt$$





Example

Find the length of the circle of radius r defined parametrically by $x = r\cos t,\quad y = r\sin t,\quad 0 \le t \le 2\pi$

Solution

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_0^{2\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt \\
&= \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt \\
&= \int_0^{2\pi} \sqrt{r^2\left(\sin^2 t + \cos^2 t\right)}\, dt \\
&= \int_0^{2\pi} r\, dt \\
&= rt\Big|_0^{2\pi} \\
&= 2\pi r \ \text{ unit}
\end{aligned}$$

Example

Find the length of the asteroid: $x=\cos^3 t,\quad y=\sin^3 t,\quad 0\le t\le 2\pi$

Solution

Because of the curve's symmetry with respect to the coordinate axes, its length is 4 times the length of the first quadrant.

$$\left(\frac{dx}{dt}\right)^2=\left[3\cos^2 t(-\sin t)\right]^2=9\cos^4 t\sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2=\left[3\sin^2 t(\cos t)\right]^2=9\sin^4 t\cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t}$$

$$=3\left|\cos t\sin t\right|\sqrt{\cos^2 t+\sin^2 t} \qquad \cos^2 t+\sin^2 t=1$$

$$=3\cos t\sin t \qquad \cos t\sin t\ge 0,\quad 0\le t\le \tfrac{\pi}{2}$$

$$L=\int_0^{2\pi}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$=4\int_0^{\pi/2}3\cos t\sin t\,dt \qquad \sin 2t=2\cos t\sin t$$

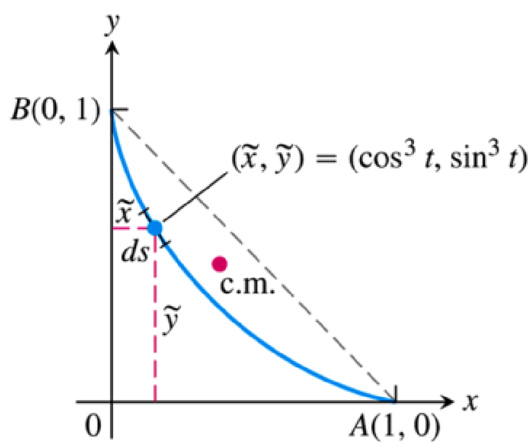
$$=\frac{12}{2}\int_0^{\pi/2}\sin 2t\,dt$$

$$=-\frac{6}{2}\cos 2t\Big|_0^{\pi/2}$$

$$=-3(-1-1)$$

$$=-3(-2)$$

$$=6\ \ unit$$



Area of Surface of Revolution for Parametrized Curves

If a smooth curve $x = f(t)$ and $y = g(t)$, $a \le t \le b$, is traversed exactly once as *t* increases from *a* to *b*, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. **Revolution about the *x*–axis ($y \ge 0$):**

$$S = 2\pi \int_a^b y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

2. **Revolution about the *y*–axis ($x \ge 0$):**

$$S = 2\pi \int_a^b x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Example

The standard parametrization of the circle of radius 1 centered at the point (0, 1) in the *xy*–plane is

$$x = \cos t, \quad y = 1 + \sin t, \quad 0 \le t \le 2\pi$$

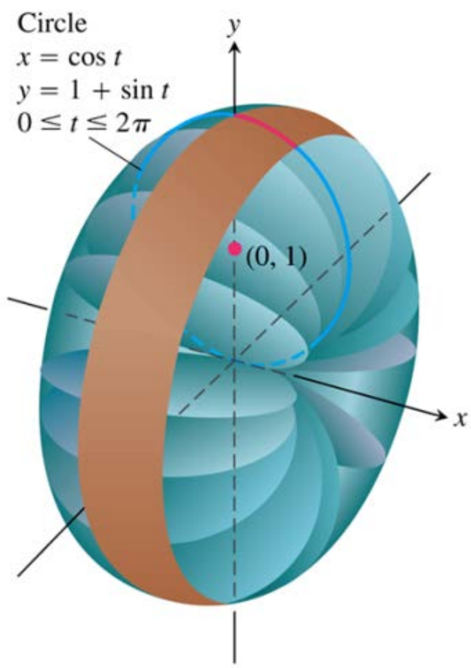
Use the parametrization to find the area of the surface swept out by revolving the circle about the x–axis.

Solution

$$x = \cos t \quad \Rightarrow \quad \left(\frac{dx}{dt}\right)^2 = (-\sin t)^2$$

$$y = 1 + \sin t \quad \Rightarrow \quad \left(\frac{dy}{dt}\right)^2 = (\cos t)^2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\sin^2 t + \cos^2 t} = 1$$



$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= \int_0^{2\pi} 2\pi(1 + \sin t)\ dt$$

$$= 2\pi \left[t - \cos t\right]_0^{2\pi}$$

$$= 2\pi\left(2\pi - 1 - (0 - 1)\right)$$

$$= 4\pi^2 \ \ unit^2$$

Exercises *Section* 4.2 – Calculus with Parametric Curves

Find all the points at which the curve has the given slope.

1. $x = 4\cos t, \quad y = 4\sin t; \quad slope = \frac{1}{2}$

2. $x = 2\cos t, \quad y = 8\sin t; \quad slope = -1$

3. $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}; \quad slope = 1$

4. $x = 2 + \sqrt{t}, \quad y = 2 - 4t; \quad slope = -8$

Find an equation of the line tangent to the curve at the point corresponding to the given value of t.

5. $x = \sin t, \quad y = \cos t, \quad t = \frac{\pi}{4}$

6. $x = t^2 - 1, \quad y = t^3 + t, \quad t = 2$

7. $x = e^t, \quad y = \ln(t+1), \quad t = 0$

8. $x = \cos t + t\sin t, \quad y = \sin t - t\cos t, \quad t = \frac{\pi}{4}$

9. $x = 6t, \quad y = t^2 + 4, \quad t = 1$

10. $x = t - 2, \quad y = \frac{1}{t} + 3, \quad t = 1$

11. $x = t^2 - t + 2, \quad y = t^3 - 3t, \quad t = -1$

12. $x = -t^2 + 3t, \quad y = 2t^{3/2}, \quad t = \frac{1}{4}$

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point

13. $x = \sin 2\pi t, \quad y = \cos 2\pi t, \quad t = -\frac{1}{6}$

14. $x = \cos t, \quad y = \sqrt{3}\cos t, \quad t = \frac{2\pi}{3}$

15. $x = t, \quad y = \sqrt{t}, \quad t = \frac{1}{4}$

16. $x = \sec^2 t - 1, \quad y = \tan t, \quad t = -\frac{\pi}{4}$

17. $x = \frac{1}{t+1}, \quad y = \frac{t}{t-1}, \quad t = 2$

18. $x = t + e^t, \quad y = 1 - e^t, \quad t = 0$

19. $x = 4t, \quad y = 3t - 2, \quad t = 3$

20. $x = \sqrt{t}, \quad y = 3t - 1, \quad t = 1$

21. $x = t + 1, \quad y = t^2 + 3t, \quad t = -1$

22. $x = t^2 + 5t + 4, \quad y = 4t, \quad t = 0$

23. $x = 4\cos\theta, \quad y = 4\sin\theta, \quad \theta = \frac{\pi}{4}$

24. $x = \cos\theta, \quad y = 3\sin\theta, \quad \theta = 0$

25. $x = 2 + \sec\theta, \quad y = 1 + 2\tan\theta, \quad \theta = \frac{\pi}{6}$

26. $x = \sqrt{t}, \quad y = \sqrt{t-1}, \quad t = 2$

27. $x = \cos^3\theta, \quad y = \sin^3\theta, \quad \theta = \frac{\pi}{4}$

28. $x = \theta - \sin\theta, \quad y = 1 - \cos\theta, \quad \theta = \pi$

Find the equations of the tangent lines at the point where the curve crosses itself

29. $x = 2\sin 2t, \quad y = 3\sin t$

30. $x = 2 - \pi\cos t, \quad y = 2t - \pi\sin t$

31. $x = t^2 - t, \quad y = t^3 - 3t - 1$

32. $x = t^3 - 6t, \quad y = t^2$

Find the slope of the curve $x = f(t),\ y = g(t)$ at the given value of t. Define x and y as differentiable functions.

33. $x^3 + 2t^2 = 9, \quad 2y^3 - 3t^2 = 4, \quad t = 2$

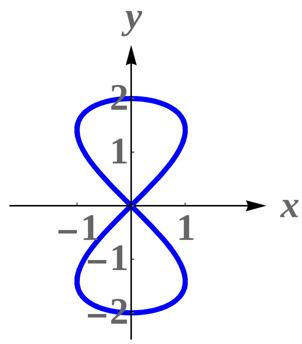
34. $x + 2x^{3/2} = t^2 + t, \quad y\sqrt{t+1} + 2t\sqrt{y} = 4, \quad t = 0$

35. $t = \ln(x - t), \quad y = te^t, \quad t = 0$

36. Consider Lissajous curve, estimate the coordinates of the points on the curve at which there is

$$x = \sin 2t, \quad y = 2\sin t; \quad 0 \le t \le 2\pi$$

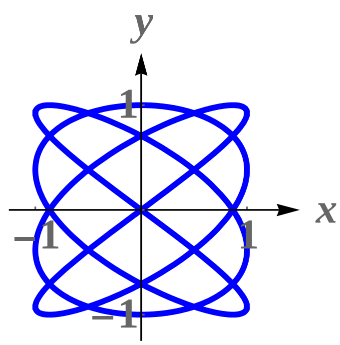
a) A horizontal tangent line
b) A vertical tangent line.



37. Consider Lissajous curve, estimate the coordinates of the points on the curve at which there is

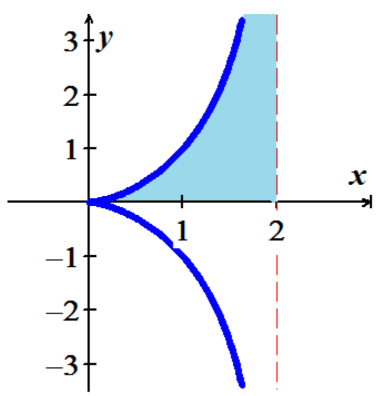
$$x = \sin 4t, \quad y = \sin 3t; \quad 0 \le t \le 2\pi$$

a) A horizontal tangent line
b) A vertical tangent line.

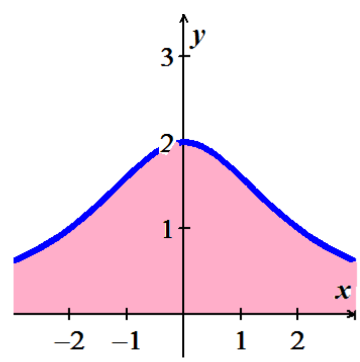


Find the area of the region

38. $x = 2\sin^2\theta, \quad y = 2\sin^2\theta\tan\theta, \quad 0 \le \theta < \frac{\pi}{2}$



39. $x = 2\cot\theta, \quad y = 2\sin^2\theta, \quad 0 \le \theta < \pi$



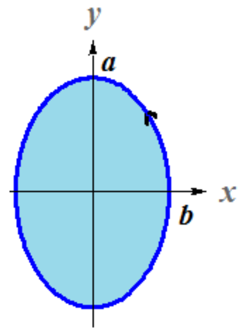
40. Find the area under one arch of the cycloid $x = a(t - \sin t), \quad y = a(1 - \cos t)$

41. Find the area enclosed by the y–axis and the curve $x = t - t^2, \quad y = 1 + e^{-t}$

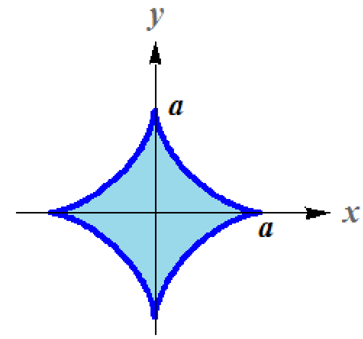
42. Find the area enclosed by the ellipse $x = a\cos t, \quad y = b\sin t, \quad 0 \le t \le 2\pi$

Find the area of the closed curve

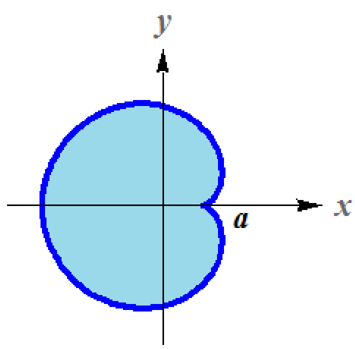
43. *Ellipse* $\begin{cases} x = b\cos t \\ y = a\sin t \end{cases}$ $0 \le t \le 2\pi$



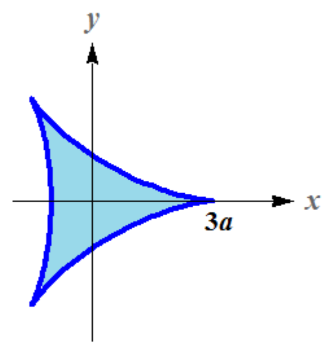
44. *Astroid* $\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases}$ $0 \le t \le 2\pi$



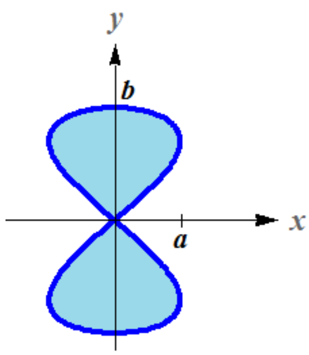
45. *Cardioid* $\begin{cases} x = 2a\cos t - a\cos 2t \\ y = 2a\sin t - a\sin 2t \end{cases}$ $0 \le t \le 2\pi$



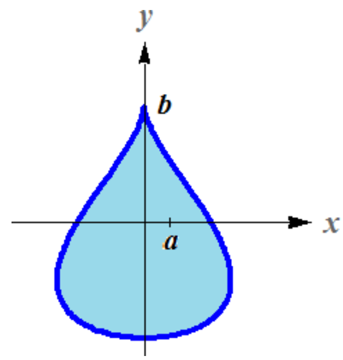
46. *Deltoid* $\begin{cases} x = 2a\cos t + a\cos 2t \\ y = 2a\sin t - a\sin 2t \end{cases}$ $0 \le t \le 2\pi$



47. *Hourglass* $\begin{cases} x = a\sin 2t \\ y = b\sin t \end{cases}$ $0 \le t \le 2\pi$



48. *Teardrop* $\begin{cases} x = 2a\cos t - a\sin 2t \\ y = b\sin t \end{cases}$ $0 \le t \le 2\pi$



Find the lengths of the curves

49. $x = \cos t, \quad y = t + \sin t, \quad 0 \le t \le \pi$

50. $x = t^3, \quad y = \frac{3}{2}t^2, \quad 0 \le t \le \sqrt{3}$

51. $x = 8\cos t + 8t\sin t, \quad y = 8\sin t - 8t\cos t, \quad 0 \le t \le \frac{\pi}{2}$

52. $x = \ln(\sec t + \tan t) - \sin t, \quad y = \cos t, \quad 0 \le t \le \frac{\pi}{3}$

53. Hypocycloid perimeter curve: $x = a\cos\theta, \quad y = a\sin\theta$

54. Circle circumference: $x = a\cos^3\theta, \quad y = a\sin^3\theta$

55. Cycloid arch: $x = a(\theta - \sin\theta), \quad y = a(1-\cos\theta)$

56. Involute of a circle: $x = \cos\theta + \theta\sin\theta, \quad y = \sin\theta - \theta\cos\theta$

Find the areas of the surfaces generated by revolving the curves

57. $x = \frac{1}{3}t^3, \quad y = t+1, \quad 1 \le t \le 2, \quad y\text{-}axis$

58. $x = \frac{2}{3}t^{3/2}, \quad y = 2\sqrt{t}, \quad 0 \le t \le \sqrt{3}; \quad x - axis$

59. $x = t + \sqrt{2}, \quad y = \frac{t^2}{2} + \sqrt{2}t, \quad -\sqrt{2} \le t \le \sqrt{2}; \quad y - axis$

60. $x = 2t, \quad y = 3t; \quad 0 \le t \le 3 \quad x\text{-}axis$

61. $x = 2t, \quad y = 3t; \quad 0 \le t \le 3 \quad y\text{-}axis$

62. $x = t, \quad y = 4 - 2t; \quad 0 \le t \le 2 \quad x\text{-}axis$

63. $x = t, \quad y = 4 - 2t; \quad 0 \le t \le 2 \quad y\text{ - }axis$

64. $x = 5\cos\theta, \quad y = 5\sin\theta, \quad 0 \le \theta \le \frac{\pi}{2}, \quad y\text{ - }axis$

65. $x = a\cos^3\theta, \quad y = a\sin^3\theta, \quad 0 \le \theta \le \pi, \quad x\text{-}axis$

66. $x = a\cos\theta, \quad y = b\sin\theta, \quad 0 \le \theta \le 2\pi$

a) *x-axis* *b)* *y-axis*

67. $x = 2t, \quad y = 3t, \quad 0 \le t \le 3$

a) *x-axis* *b)* *y-axis*

68. $x = t, \quad y = 4 - 2t, \quad 0 \le t \le 2$

a) *x-axis* *b)* *y-axis*

69. Use the parametric equations $x = t^2\sqrt{3} \quad and \quad y = 3t - \frac{1}{3}t^3$ to

a) Graph the curve on the interval $-3 \le t \le 3$.

b) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

c) Find the equation of the tangent line at the point $\left(\sqrt{3}, \ \frac{8}{3}\right)$

d) Find the length of the curve

e) Find the surface area generated by revolving the curve about the *x-axis*

70. Use the parametric equations $x = a(\theta - \sin\theta) \quad and \quad y = a(1-\cos\theta) \quad a > 0$

a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

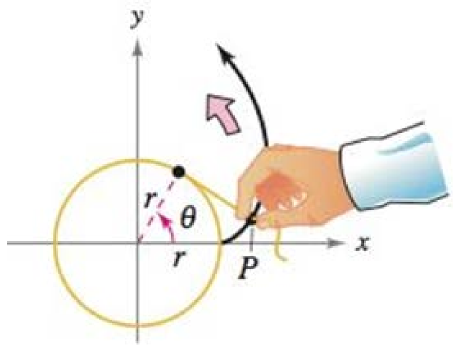
b) Find the equation of the tangent line at the point where $\theta = \frac{\pi}{6}$

c) Find all points (if any) of horizontal tangency.

d) Determine where the curve is concave upward or concave downward.

e) Find the length of one arc of the curve

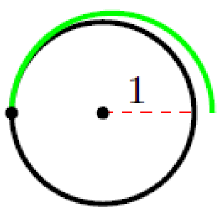
71. The involute of a circle is described by the endpoint *P* of a string that is held taut as it is unwound from a spool that does not turn.



Show that a parametric representation of the involute is

$$x = r(\cos\theta + \theta\sin\theta) \quad \textit{and} \quad y = r(\sin\theta - \theta\cos\theta)$$

72. The figure shows a piece of string tied to a circle with a radius of one unit. The string is just long enough to reach the opposite side if the circle.

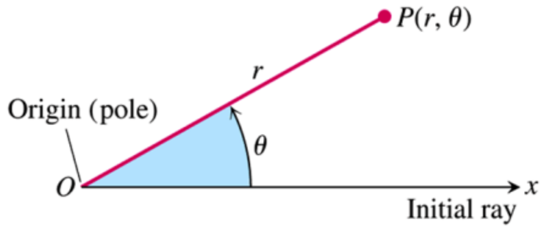


Find the area that is covered when the string is unwounded counterclockwise.

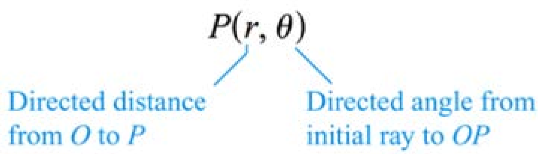
Section 4.3 – Polar Coordinates and Graphs

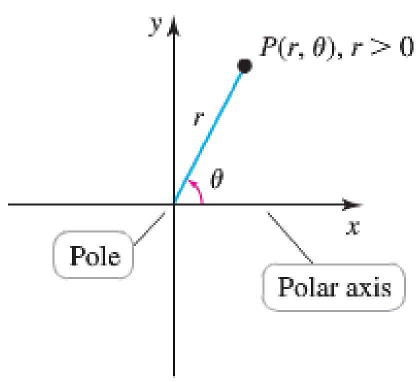
Definition of Polar Coordinates

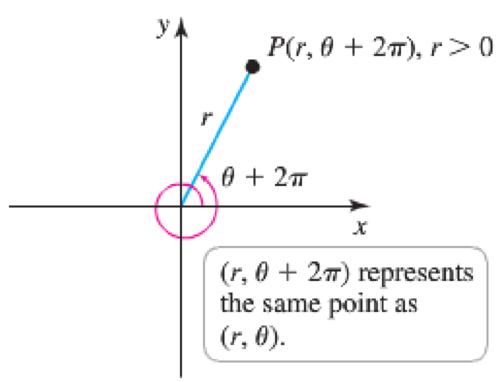
To define polar coordinates, let an ***origin*** O (called the ***pole***) and an ***initial ray*** from O. Then each point P can be located by assigning to it a ***polar coordinate pair*** $(r,\ \theta)$ in which r gives the directed from O to P and θ gives the directed angle from the initial ray to yay OP.

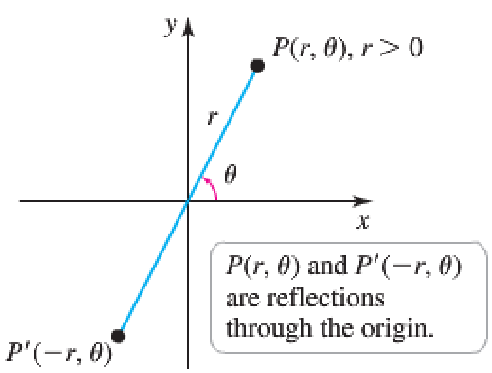


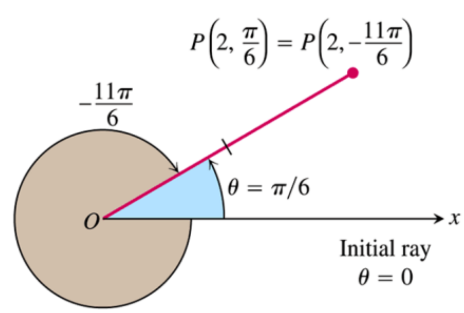
Polar Coordinates





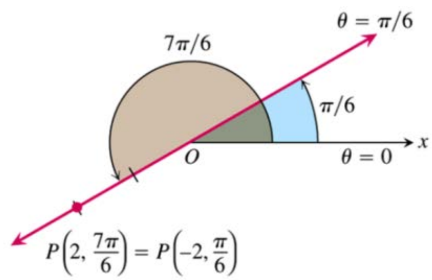






Example

Find all the polar coordinates of the point $P\left(2,\ \frac{\pi}{6}\right)$



Solution

For $r=2 \quad \Rightarrow \quad \theta = \frac{\pi}{6},\ \frac{\pi}{6} \pm 2\pi,\ \frac{\pi}{6} \pm 4\pi,\ \ldots$

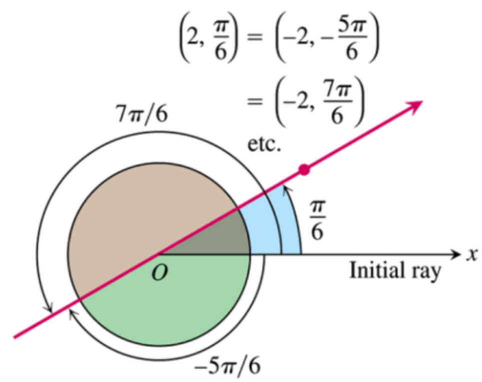
For $r=-2 \quad \Rightarrow \quad \theta = -\frac{5\pi}{6},\ -\frac{5\pi}{6} \pm 2\pi,\ -\frac{5\pi}{6} \pm 4\pi,\ \ldots$

The corresponding coordinate pairs of P are

$$\left(2,\ \frac{\pi}{6} + 2n\pi\right), \quad n = 0,\ \pm 1,\ \pm 2,\ \ldots$$

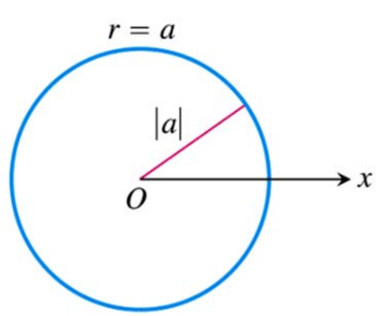
And

$$\left(-2,\ -\frac{5\pi}{6} + 2n\pi\right), \quad n = 0,\ \pm 1,\ \pm 2,\ \ldots$$



Polar Equations and Graphs

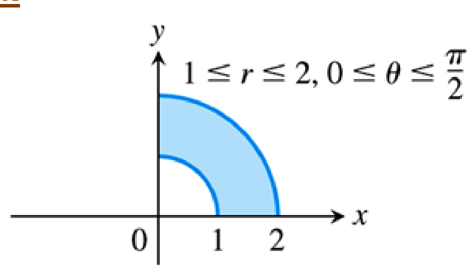
Equation	***Graph***
$r = a$	Circle of radius $\lvert a \rvert$ centered at O
$\theta = \theta_0$	Line through O making an angle θ_0 with the initial ray



Example

Graph the polar coordinate $1 \le r \le 2 \quad and \quad 0 \le \theta \le \frac{\pi}{2}$

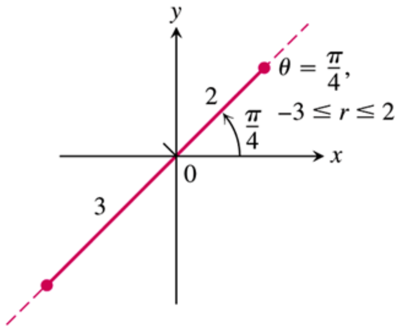
Solution



Example

Graph the polar coordinate $-3 \le r \le 2 \quad and \quad \theta = \frac{\pi}{4}$

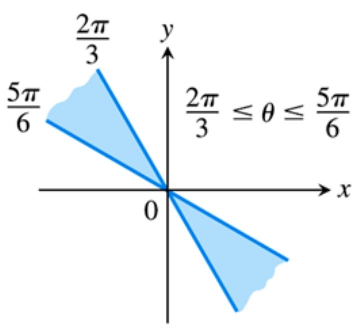
Solution



Example

Graph the polar coordinate $\frac{2\pi}{3} \le \theta \le \frac{5\pi}{6} \qquad (no\ restriction\ on\ \ r)$

Solution

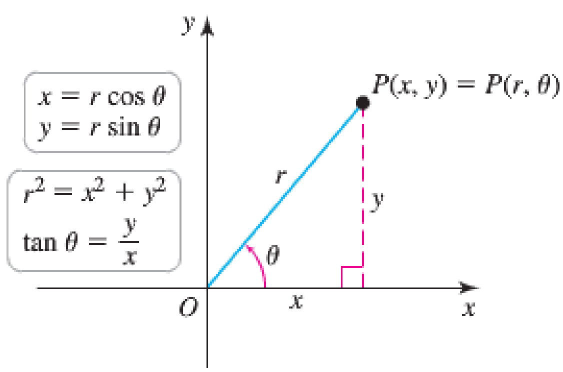


Relating Polar and Cartesian Coordinates

When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial polar ray as the positive *x*-axis. The ray $\theta = \frac{\pi}{2}, \quad r > 0$ becomes the positive *y*-axis. The two coordinate systems are then related by the following equations

Equations Relating Polar and Cartesian Coordinates

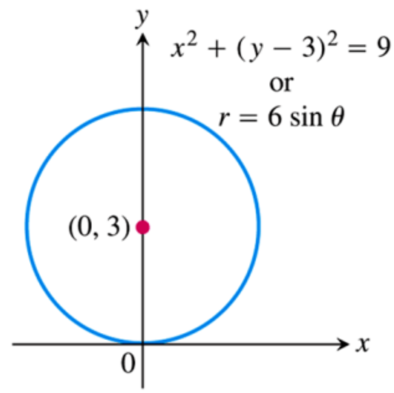
$$\begin{cases} x = r\cos\theta, & y = r\sin\theta \\ r^2 = x^2 + y^2, & \tan\theta = \frac{y}{x} \end{cases}$$



Polar equation	*Cartesian equation*
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

Example

Find a polar equation for the circle $x^2 + (y-3)^2 = 9$



Solution

$x^2 + (y-3)^2 = 9$

$x^2 + y^2 - 6y + 9 = 9$

$x^2 + y^2 - 6y = 0$ $\quad\quad x^2 + y^2 = r^2$

$r^2 - 6r\sin\theta = 0$

$r(r - 6\sin\theta) = 0 \quad \Rightarrow \quad \boxed{r = 0} \quad \boxed{r = 6\sin\theta}$

Example

Replace the polar equation by equivalent Cartesian equation and identify the graph: $r\cos\theta = -4$

Solution

$r\cos\theta = -4 \quad \Rightarrow \quad x = -4$

The graph: Vertical line through $x = -4$

Example

Replace the polar equation by equivalent Cartesian equation and identify the graph: $r^2 = 4r\cos\theta$

Solution

$r^2 = 4r\cos\theta$

$x^2 + y^2 = 4x$

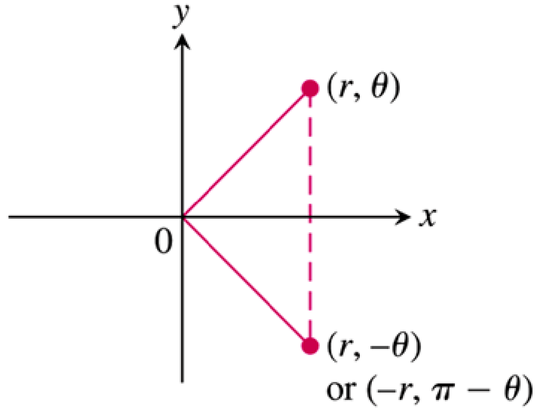
$x^2 - 4x + y^2 = 0$

$x^2 - 4x + 4 + y^2 = 4$

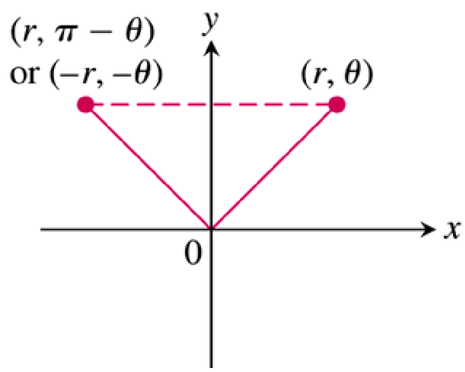
$(x-2)^2 + y^2 = 4$ The ***graph***: Circle with center (2, 0) and radius 2.

Symmetry Test for Polar Graphs

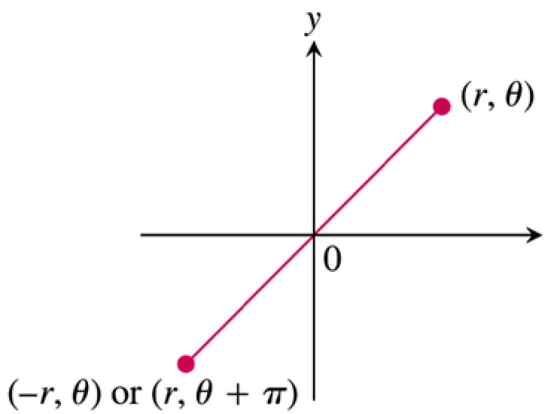
1. ***Symmetry about the x-axis***: If the point (r,θ) lies on the graph, then the point $(r,-\theta)$ or $(-r,\pi-\theta)$ lies on the graph.



2. ***Symmetry about the y-axis***: If the point (r,θ) lies on the graph, then the point $(r,\pi-\theta)$ or $(-r,-\theta)$ lies on the graph.



3. ***Symmetry about the origin***: If the point (r,θ) lies on the graph, then the point $(-r,\theta)$ or $(r,\theta+\pi)$ lies on the graph.



Example

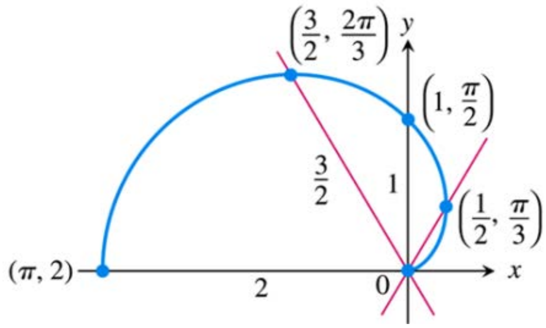
Graph the curve $r = 1 - \cos\theta$

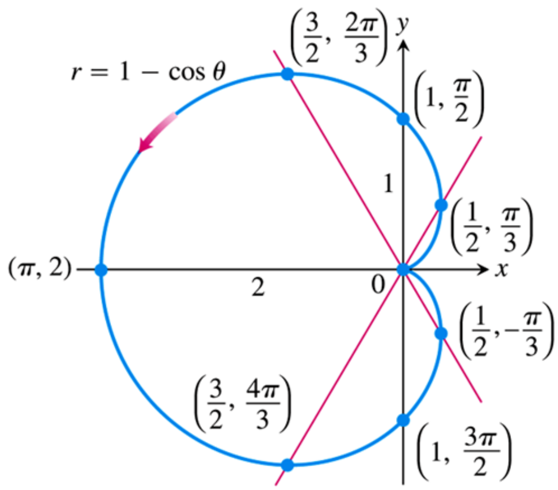
Solution

The curve is symmetric about the *x*-axis:

$1 - \cos(-\theta) = 1 - \cos\theta = r$

θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2





Example

Graph the curve $r^2 = 4\cos\theta$

Solution

The curve is symmetric about the *x*-axis:

$r^2 = 4\cos\theta$

$r^2 = 4\cos(-\theta)$

$(r, -\theta)$

The curve is symmetric about the *origin*:

$r^2 = 4\cos\theta$

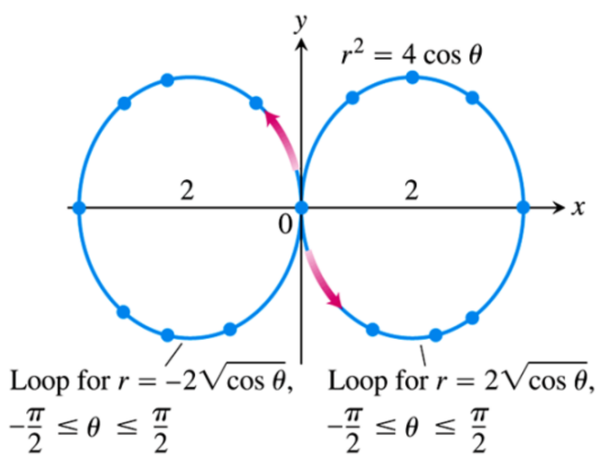
$(-r)^2 = 4\cos\theta$

$(-r, \theta)$

Therefore, the curve is also symmetric about the *y*-axis.

$$r^2 = 4\cos\theta \quad \Rightarrow \quad r = \pm 2\sqrt{\cos\theta}$$

θ	$r = \pm 2\sqrt{\cos\theta}$
0	± 2
$\pm\frac{\pi}{6}$	$\approx \pm 1.9$
$\pm\frac{\pi}{4}$	$\approx \pm 1.7$
$\pm\frac{\pi}{3}$	$\approx \pm 1.4$
$\pm\frac{\pi}{2}$	0



A Technique for Graphing

One way to graph a polar equation $r = f(\theta)$ is to make a table of (r, θ) values, plot the corresponding points, and connect them in order of increasing.

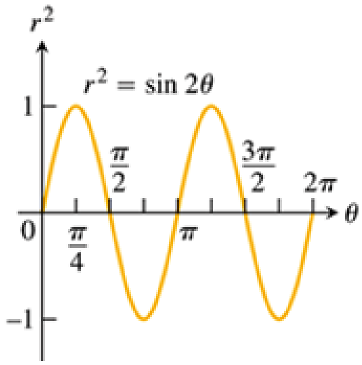
Another method of graphing more reliable is

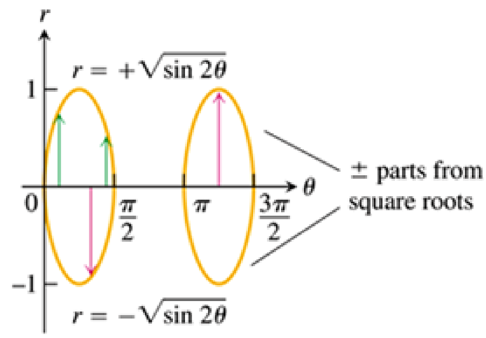
1. First graph $r = f(\theta)$ in the *Cartesian* $r\theta -$ *plane*,
2. Then use the *Cartesian* graph as a table and guide to sketch the *polar coordinate* graph.

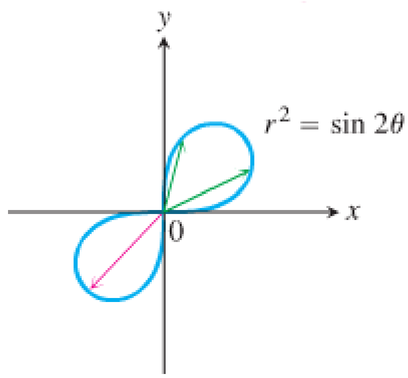
Example

Graph the *lemniscate* curve $\quad r^2 = \sin 2\theta$

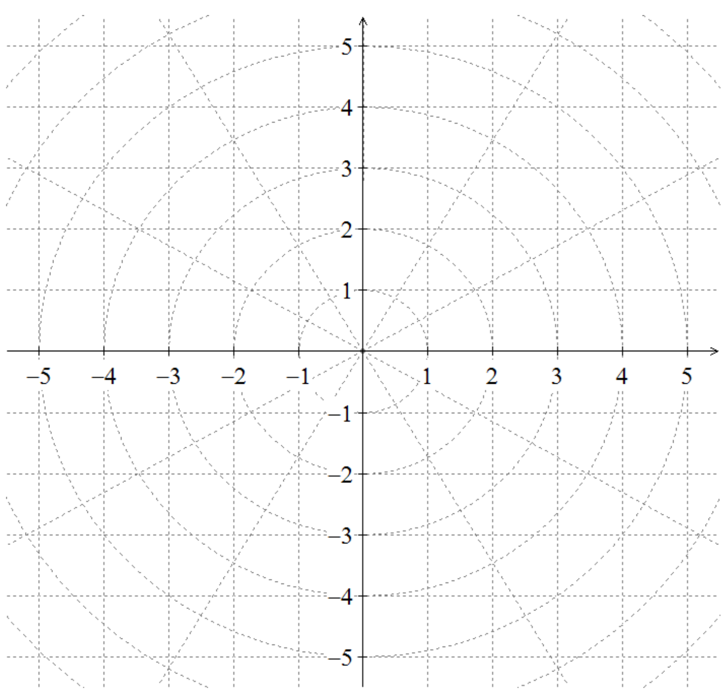
Solution







θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{7\pi}{6}$
$r = \pm\sqrt{\sin 2\theta}$	0	$\pm\sqrt{\frac{\sqrt{3}}{2}} \approx \pm.93$	± 1	$\pm\sqrt{\frac{\sqrt{3}}{2}} \approx \pm.93$	0	0	$\pm\sqrt{\frac{\sqrt{3}}{2}} \approx \pm.93$



Exercises Section 4.3 – Polar Coordinates

1. Find the Cartesian coordinates of the following points (given in polar coordinates)

$a)\ \left(\sqrt{2},\ \frac{\pi}{4}\right)\quad b)\ (1,\ 0)\quad c)\ \left(0,\ \frac{\pi}{2}\right)\quad d)\ \left(-\sqrt{2},\ \frac{\pi}{4}\right)$

2. Find the polar coordinates, $0 \le \theta < 2\pi$ *and* $r \ge 0$, of the following points given in Cartesian coordinates

$a)\ (1,\ 1)\quad b)\ (-3,\ 0)\quad c)\ \left(\sqrt{3},\ -1\right)\quad d)\ (-3,\ 4)$

3. Find the polar coordinates, $-\pi \le \theta < \pi$ *and* $r \ge 0$, of the following points given in Cartesian coordinates

$a)\ (-2,\ -2)\quad b)\ (0,\ 3)\quad c)\ \left(-\sqrt{3},\ 1\right)\quad d)\ (5,\ -12)$

Graph

4. $1 \le r \le 2$

5. $0 \le \theta \le \frac{\pi}{6},\quad r \ge 0$

6. $\theta = \frac{\pi}{2},\quad r \le 0$

7. $-\frac{\pi}{4} \le \theta \le \frac{3\pi}{4},\quad 0 \le r \le 1$

8. $0 \le \theta \le \frac{\pi}{2},\quad 1 \le |r| \le 2$

Replace the polar equation with equivalent Cartesian equation and identify the graph

9. $r\cos\theta = 2$

10. $r\sin\theta = -1$

11. $r = -3\sec\theta$

12. $r\cos\theta + r\sin\theta = 1$

13. $r^2 = 4r\sin\theta$

14. $r = \dfrac{5}{\sin\theta - 2\cos\theta}$

15. $r = 4\tan\theta\sec\theta$

16. $r\sin\theta = \ln r + \ln\cos\theta$

17. $\cos^2\theta = \sin^2\theta$

18. $r = 2\cos\theta + 2\sin\theta$

19. $r\sin\left(\frac{2\pi}{3} - \theta\right) = 5$

20. $r = \dfrac{4}{2\cos\theta - \sin\theta}$

Replace the Cartesian equation with equivalent polar equation

21. $x = y$

22. $x^2 - y^2 = 1$

23. $\frac{x^2}{9} + \frac{y^2}{4} = 1$

24. $xy = 1$

25. $x^2 + xy + y^2 = 1$

26. $x^2 + (y-2)^2 = 4$

27. $(x+2)^2 + (y-5)^2 = 16$

28. ***a)*** Show that every vertical line in the *xy*–plane has a polar equation of the form $r = a\sec\theta$
b) Find the analogous polar equation for horizontal lines in the *xy*–plane.

Identify the symmetries of the curves. Then sketch the curves.

29. $r = 2 - 2\cos\theta$

31. $r = 2 + \sin\theta$

33. $r^2 = -\sin\theta$

30. $r = 1 + \sin\theta$

32. $r^2 = \sin\theta$

34. $r^2 = -\cos\theta$

Graph the lemniscate. What symmetries do these curves have?

35. $r^2 = 4\cos 2\theta$

36. $r^2 = 4\sin 2\theta$

37. $r^2 = -\cos 2\theta$

Graph the limaçons is Old French for "snail". Equations for limaçons have the form
$r = a \pm b\cos\theta$ *or* $r = a \pm b\sin\theta$

38. $r = \frac{1}{2} + \cos\theta$

39. $r = \frac{1}{2} + \sin\theta$

40. $r = 1 - \cos\theta$

41. $r = \frac{3}{2} - \sin\theta$

42. $r = 2 + \cos\theta$

Graph the equation

43. $r = 1 - 2\sin 3\theta$

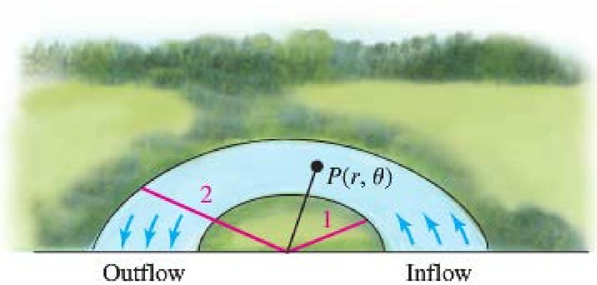
44. $r = \sin^2 \frac{\theta}{2}$

45. $r = 1 - \sin\theta$

46. $r^2 = 4\sin\theta$

47. Graph the ***nephroid*** of ***Freeth*** equation $r = 1 + 2\sin\frac{\theta}{2}$

48. Water flows in a shallow semicircular channel with inner and outer radii of 1 *m* and 2 *m*. At a point $P(r,\ \theta)$ in the channel, the flow is in the tangential direction (counterclockwise along circles), and it depends only on *r*, the distance from the center of the semicircles.



a) Express the region formed by the channel as a set in polar coordinates.
b) Express the inflow and outflow regions of the channel as sets in polar coordinates.
c) Suppose the tangential velocity of the water in *m/s* is given by $v(r) = 10r$, for $1 \le r \le 2$. Is the velocity greater at $\left(1.5,\ \frac{\pi}{4}\right)$ or $\left(1.2,\ \frac{3\pi}{4}\right)$? Explain.
d) Suppose the tangential velocity of the water is given by $v(r) = \frac{20}{r}$, for . Is the velocity greater $\left(1.8,\ \frac{\pi}{6}\right)$ or $\left(1.3,\ \frac{2\pi}{3}\right)$? Explain.
e) The total amount of water that flows through the channel (across a cross section of the channel $\theta = \theta_0$) is proportional to $\int_1^2 v(r)\,dr$. Is the total flow through the channel greater for the flow in part (*c*) or (*d*)?

49. A simplified model assumes that the orbits of Earth and Mars are circular with radii of 2 and 3, respectively, and that Earth completes one orbit in one year while Mars takes two years. When $t = 0$

. Earth is at $(2,\ 0)$ and Mars is at $(3,\ 0)$; both orbit the Sum (at $(0,\ 0)$) in the counterclockwise direction.

The position of Mars relative to Earth is given by the parametric equations

$$x=(3-4\cos\pi t)\cos\pi t+2, \quad y=(3-4\cos\pi t)\sin\pi t$$

a) Graph the parametric equations, for $0\le t\le 2$

b) Letting $r=3-4\cos\pi t$, explain why the path of Mars relative to Earth is a limaçon.

Section 4.4 – Calculus in Polar Coordinates

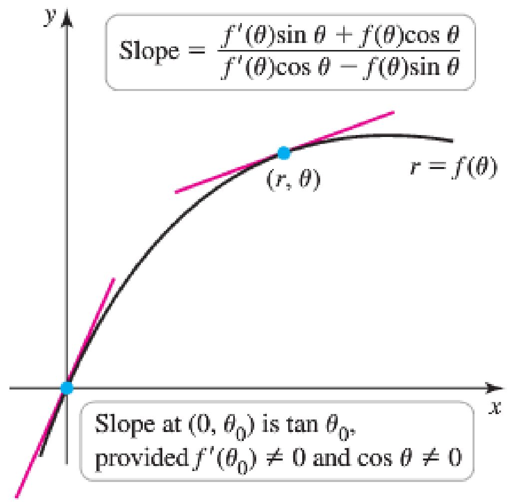
Slope

The slope of a polar curve $r = f(\theta)$ in the xy-plane is still given by $\frac{dy}{dx}$, which is not $r' = \frac{df}{d\theta}$

$$x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} \\ &= \frac{\frac{d}{d\theta}\left(f(\theta)\cdot\sin\theta\right)}{\frac{d}{d\theta}\left(f(\theta)\cdot\cos\theta\right)} \\ &= \frac{\frac{df}{d\theta}\cdot\sin\theta + f(\theta)\cdot\cos\theta}{\frac{df}{d\theta}\cdot\cos\theta + f(\theta)\cdot\sin\theta}\end{aligned}$$

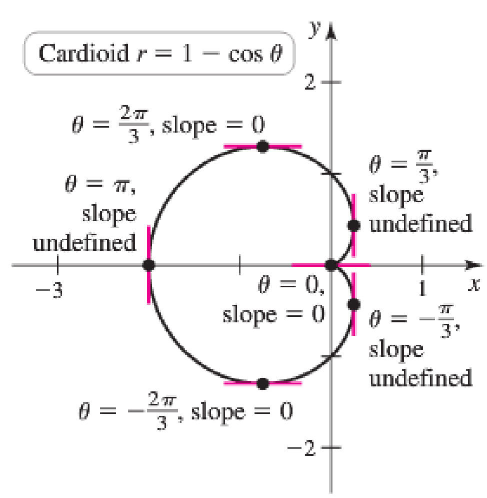
$$\boxed{\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}}$$



Example

Find the points on the interval $-\pi \le \theta \le \pi$ at which the cardioid $r = f(\theta) = 1 - \cos\theta$ has a vertical or horizontal tangent line.

Solution



$$\begin{aligned}\frac{dy}{dx} &= \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} \\ &= \frac{\sin\theta\sin\theta + (1-\cos\theta)\cos\theta}{\sin\theta\cos\theta - (1-\cos\theta)\sin\theta} \\ &= \frac{\sin^2\theta + \cos\theta - \cos^2\theta}{\sin\theta\cos\theta - \sin\theta + \cos\theta\sin\theta} \\ &= \frac{1-\cos^2\theta + \cos\theta - \cos^2\theta}{2\sin\theta\cos\theta - \sin\theta} \\ &= \frac{-2\cos^2\theta + \cos\theta + 1}{\sin\theta(2\cos\theta - 1)} \\ &= \frac{-(2\cos\theta + 1)(\cos\theta - 1)}{\sin\theta(2\cos\theta - 1)}\end{aligned}$$

The points with a horizontal tangent line:

$$\frac{dy}{dx}=0 \quad \Rightarrow \quad (2\cos\theta+1)(\cos\theta-1)=0 \rightarrow \begin{cases} \cos\theta=-\frac{1}{2} & \rightarrow \ \boxed{\theta=\pm\frac{2\pi}{3}} \\ \cos\theta=1 & \rightarrow \ \boxed{\cancel{\theta=0}} \end{cases} \quad \textit{numerator is } 0$$

The points with a Vertical tangent line:

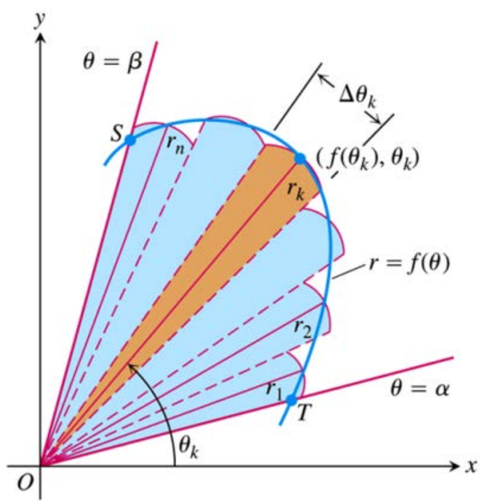
$$\sin\theta(2\cos\theta-1)=0 \rightarrow \begin{cases} \sin\theta=0 & \rightarrow \ \boxed{\theta=\cancel{0},\pm\pi} \\ \cos\theta=\frac{1}{2} & \rightarrow \ \boxed{\theta=\pm\frac{\pi}{3}} \end{cases} \quad \textit{denominator is } 0$$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\theta\to0^+}\left(\frac{-2\cos^2\theta+\cos\theta+1}{2\cos\theta\sin\theta-\sin\theta}\right)=\frac{0}{0} \\ &= \lim_{\theta\to0^+}\left(\frac{4\cos\theta\sin\theta-\sin\theta}{-2\sin^2\theta+2\cos^2\theta-\cos\theta}\right) \\ &= \frac{0}{1} \\ &= \boxed{0} \end{aligned}$$

Therefore, the curve has a slope of 0 at origin.

Area in the plane

The region OTS is bounded by the rays $\theta=\alpha$ *and* $\theta=\beta$ and the curve $r=f(\theta)$.



We approximate the region with n non-overlapping fan-shaped circular sectors based on a partition P of angle TOS. The typical sector has radius $r_k=f(\theta_k)$ and central angle of radian measure $\Delta\theta_k$. Its area is $\frac{\Delta\theta_k}{2\pi}$ times the area of a circle of radius r_k, or

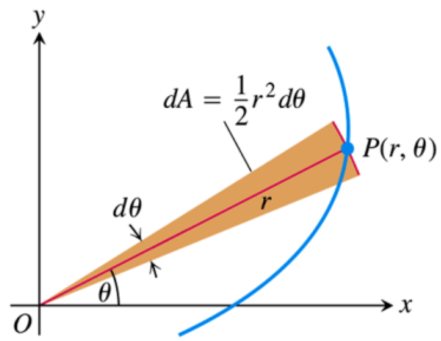
$$A_k=\frac{1}{2}r_k^2\Delta\theta_k=\frac{1}{2}\left(f(\theta_k)\right)^2\Delta\theta_k$$

Area of the Fan-Shaped Region between the Origin and the curve $r=f(\theta), \ \alpha\le\theta\le\beta$

$$A = \int_{\alpha}^{\beta} \tfrac{1}{2}\left(f(\theta)\right)^2 d\theta$$

This is the integral of the ***area differential***

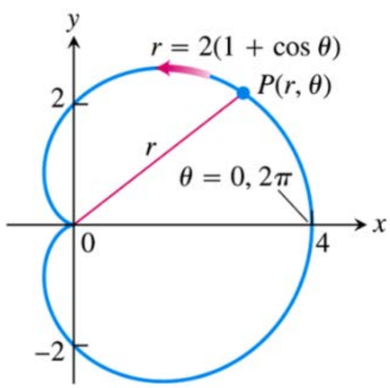
$$dA = \tfrac{1}{2} r^2 d\theta = \tfrac{1}{2}\left(f(\theta)\right)^2 d\theta$$



Example

Find the area of the region in the plane enclosed by the cardioid $r = 2(1+\cos\theta)$

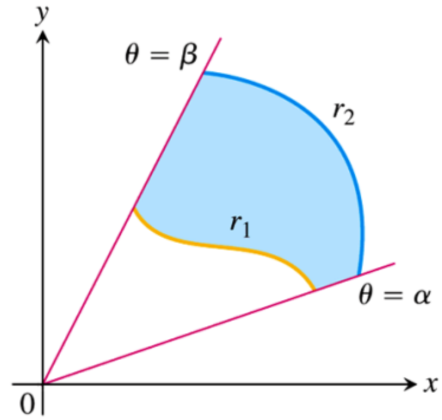
Solution



$$\begin{aligned}
A &= \int_0^{2\pi} \tfrac{1}{2} r^2 d\theta = \int_0^{2\pi} \tfrac{1}{2}\cdot 4(1+\cos\theta)^2 d\theta \\
&= \int_0^{2\pi} 2\left(1+2\cos\theta+\cos^2\theta\right) d\theta \\
&= \int_0^{2\pi} \left(2+4\cos\theta+2\frac{1+\cos 2\theta}{2}\right) d\theta \\
&= \int_0^{2\pi} (3+4\cos\theta+\cos 2\theta)\, d\theta \\
&= \left[3\theta+4\sin\theta+\tfrac{1}{2}\sin 2\theta\right]_0^{2\pi} \\
&= \left[3(2\pi)+4\sin(2\pi)+\tfrac{1}{2}\sin(2\pi)-\left(3(0)+4\sin(0)+\tfrac{1}{2}\sin(0)\right)\right] \\
&= 6\pi
\end{aligned}$$

Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta), \quad \alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta$$



Example

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$

Solution

$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta$$

$$= 2\int_{0}^{\pi/2} \frac{1}{2}\left(1^2 - (1-\cos\theta)^2\right) d\theta$$

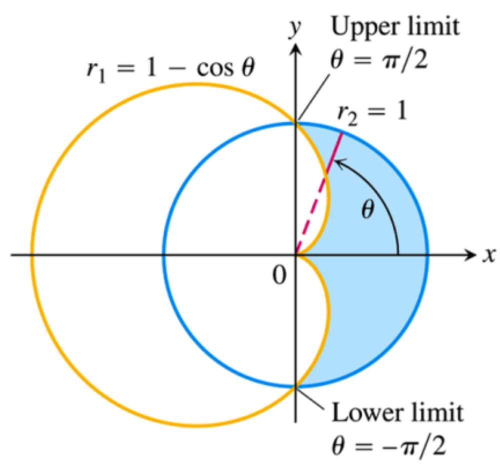
$$= \int_{0}^{\pi/2} \left(1 - \left(1 - 2\cos\theta + \cos^2\theta\right)\right) d\theta$$

$$= \int_{0}^{\pi/2} \left(2\cos\theta - \frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \left[2\sin\theta - \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_{0}^{\pi/2}$$

$$= \left[2\sin\frac{\pi}{2} - \frac{1}{2}\frac{\pi}{2} - \frac{1}{4}\sin 2\frac{\pi}{2} - 0\right]$$

$$= 2 - \frac{\pi}{4} \ \ unit^2$$



Length of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

Example

Find the length of the cardioid $r = 1 - \cos\theta$

Solution

$r = 1 - \cos\theta \quad \Rightarrow \quad \frac{dr}{d\theta} = \sin\theta$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1-\cos\theta)^2 + \sin^2\theta$$

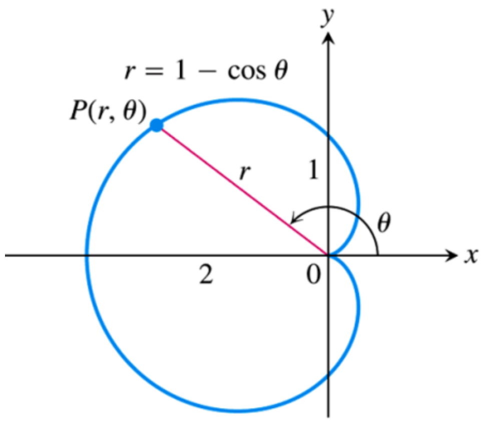
$$= 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta \qquad \cos^2\theta + \sin^2\theta = 1$$

$$= 2 - 2\cos\theta$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\ d\theta \qquad 1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

$$= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\ d\theta$$

$$= \int_0^{2\pi} 2\sin\frac{\theta}{2}\ d\theta$$



$$= \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi}$$

$$= -4\left(\cos\frac{2\pi}{2} - \cos 0\right)$$

$$= -4(-1-1)$$

$$\underline{= 8\ \ unit}$$

Area of a surface of Revolution

Theorem

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$.

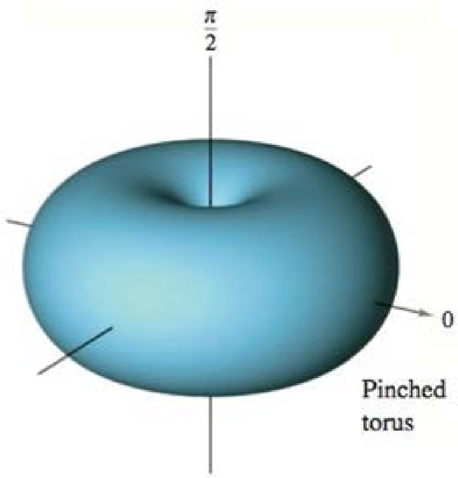
The area of the surface formed by revolving the graph of $r = f(\theta)$ about the indicated line

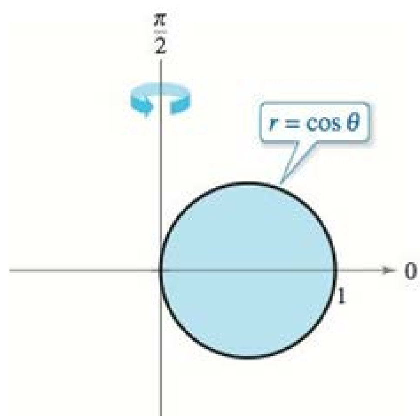
$$S = 2\pi \int_{\alpha}^{\beta} f(\theta)\sin\theta \sqrt{\left(f(\theta)\right)^2 + \left(f'(\theta)\right)^2}\, d\theta \qquad \textit{About the polar axis}$$

$$S = 2\pi \int_{\alpha}^{\beta} f(\theta)\cos\theta \sqrt{\left(f(\theta)\right)^2 + \left(f'(\theta)\right)^2}\, d\theta \qquad \textit{About the line } \theta = \tfrac{\pi}{2}$$

Example

Find the area of the surface formed by revolving the circle $f(\theta) = \cos\theta$ about the line $\theta = \frac{\pi}{2}$





Solution

$$\sqrt{r^2 + (r')^2} = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$S = 2\pi \int_0^{\pi} \cos^2\theta \, d\theta$$

$$= \pi \int_0^{\pi} (1 + \cos 2\theta) \, d\theta$$

$$= \pi \left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi}$$

$$= \pi^2 \ \ unit^2$$

Exercises Section 4.4 – Calculus in Polar Coordinates

Find the slopes of the curves at the given points. Sketch the curves along with their tangents at these points.

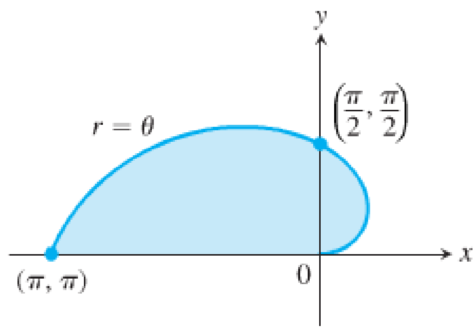
1. *Cardioid* $r = -1 + \cos\theta; \quad \theta = \pm\frac{\pi}{2}$

2. *Cardioid* $r = -1 + \sin\theta; \quad \theta = 0, \ \pi$

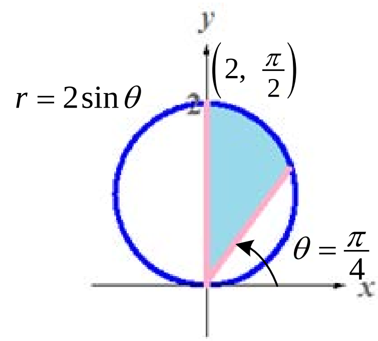
3. *Four – leaved rose* $r = \sin 2\theta; \quad \theta = \pm\frac{\pi}{4}, \ \pm\frac{3\pi}{4}$

4. *Four – leaved rose* $r = \cos 2\theta; \quad \theta = 0, \ \pm\frac{\pi}{2}, \ \pi$

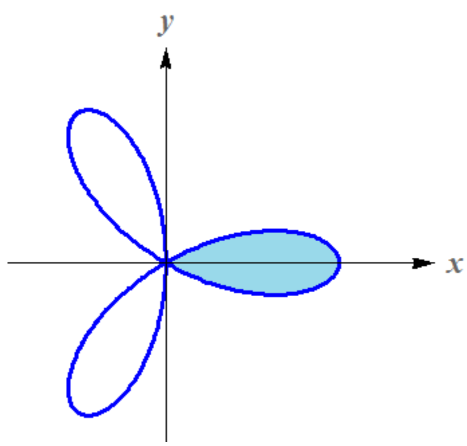
5. Find the area of the region bounded by the spiral $r = \theta$ *for* $0 \le \theta \le \pi$



6. Find the area of the region bounded by the circle $r = 2\sin\theta$ *for* $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$



7. Find the area of the region inside one leaf of the three-leaved rose $r = \cos 3\theta$



Find the area of the region inside the

8. Oval limaçon $r = 4 + 2\sin\theta$

9. Cardioid $r = a(1 + \cos\theta), \quad a > 0$

10. Six-leaved rose $r^2 = 2\sin 3\theta$

11. Curve $r = \sqrt{\cos\theta}$

12. Right lobe of $r = \sqrt{\cos 2\theta}$

13. Cardioid $r = 4 + 4\sin\theta$

14. Limaçon $r = 2 + \cos\theta$

15. Circle $r = 6$ above the line $r = 3\csc\theta$

16. Inner loop $r = \cos\theta - \frac{1}{2}$

17. One leave of $r = \cos 3\theta$

18. Find the area of the region shared by the circles $r = 2\cos\theta$ *and* $r = 2\sin\theta$

19. Find the area of the region shared by the circle $r = 2$ and the cardioid $r = 2(1 - \cos\theta)$

20. Find the area of the region in the plane enclosed by the four-leaf rose $r = f(\theta) = 2\cos 2\theta$

21. Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 + \cos\theta$

22. Find the area of the region outside the circle $r = \frac{1}{2}$ and inside the circle $r = \cos\theta$

23. Find the area of the region outside the circle $r = \frac{1}{\sqrt{2}}$ and inside the curve $r = \sqrt{\cos\theta}$

24. Find the area of the region inside the circle $r = \frac{1}{\sqrt{2}}$ in QI and inside the right lobe of $r = \sqrt{\cos 2\theta}$

25. Find the area of the region inside the rose $r = 4\sin 2\theta$ and inside the circle $r = 2$

26. Find the area of the region inside the lemniscate $r^2 = 2\sin 2\theta$ and outside the circle $r = 1$

27. Find the area of the region inside all the leaves of the rose $r = 3\sin 2\theta$

28. Find the area of the region inside one leaf of the rose $r = \cos 5\theta$

29. Find the area of the region of a complete three-leaf rose $r = 2\cos 3\theta$

30. Find the area of the region inside the rose $r = 4\cos 2\theta$ and outside the circle $r = 2$

31. Find the area of the region bounded by the lemniscate $r^2 = 6\sin 2\theta$

32. Find the area of the region bounded by the limaçon $r = 2 - 4\sin\theta$

33. Find the area of the region bounded by the limaçon $r = 4 - 2\cos\theta$

Find the area of the given region

34. Inner loop of $r = 1 + 2\cos\theta$

35. Inner loop of $r = 2 - 4\cos\theta$

36. Inner loop of $r = 1 + 2\sin\theta$

37. Inner loop of $r = 4 - 6\sin\theta$

38. Between the loops $r = 1 + 2\cos\theta$

39. Between the loops $r = 2(1 + 2\sin\theta)$

40. Between the loops $r = 3 - 6\sin\theta$

41. Between the loops $r = \frac{1}{2} + \cos\theta$

42. Inside $r = 2\cos\theta$ and outside $r = 1$

43. Inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$

44. Common interior of $r = 4\sin 2\theta$ and $r = 2$

45. Common interior of $r = 4\sin\theta$ and $r = 2$

Find the area of the given region

46. Common interior of $r = 2\cos\theta$ and $r = 2\sin\theta$

47. Common interior of $r = 2(1+\cos\theta)$ *and* $r = 2(1-\cos\theta)$

48. Common interior of $r = 3-2\sin\theta$ *and* $r = -3+2\sin\theta$

49. Common interior of $r = 5-3\sin\theta$ *and* $r = 5-3\cos\theta$

50. Inside $r = a(1+\cos\theta)$ and outside $r = a\cos\theta$

51. Inside $r = 2a\cos\theta$ and outside $r = a$

52. Common interior of $r = a(1+\cos\theta)$ and $r = a\sin\theta$

53. Common interior of $r = a\cos\theta$ and $r = a\sin\theta$, where $a > 0$

Find the length of

54. $r = \theta^2, \quad 0 \le \theta \le \sqrt{5}$

55. $r = \dfrac{e^\theta}{\sqrt{2}}, \quad 0 \le \theta \le \pi$

56. $r = a\sin^2\left(\dfrac{\theta}{2}\right), \quad 0 \le \theta \le \pi, \quad a > 0$

57. $r = \dfrac{6}{1+\cos\theta}, \quad 0 \le \theta \le \dfrac{\pi}{2}$

58. $r = \cos^3\left(\dfrac{\theta}{3}\right), \quad 0 \le \theta \le \dfrac{\pi}{4}$

59. $r = \sqrt{1+\sin 2\theta}, \quad 0 \le \theta \le \pi\sqrt{2}$

60. $r = 8 \quad 0 \le \theta \le 2\pi$

61. $r = a \quad 0 \le \theta \le 2\pi$

62. $r = 4\sin\theta \quad 0 \le \theta \le \pi$

63. $r = 2a\cos\theta \quad -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

64. $r = 1+\sin\theta \quad 0 \le \theta \le 2\pi$

65. $r = 8(1+\cos\theta) \quad 0 \le \theta \le 2\pi$

66. $r = 2\theta \quad 0 \le \theta \le \dfrac{\pi}{2}$

67. $r = \sec\theta \quad 0 \le \theta \le \dfrac{\pi}{3}$

68. $r = \dfrac{1}{\theta} \quad \pi \le \theta \le 2\pi$

69. $r = e^\theta \quad 0 \le \theta \le \pi$

70. $r = 5\cos\theta \quad \dfrac{\pi}{2} \le \theta \le \pi$

71. $r = 3(1-\cos\theta) \quad 0 \le \theta \le \pi$

Find the surface area bounded by

72. $r = 6\cos\theta \quad 0 \le \theta \le \dfrac{\pi}{2}$ *revolving about Polar axis*

73. $r = a\cos\theta \quad 0 \le \theta \le \dfrac{\pi}{2}$ *revolving about* $\theta = \dfrac{\pi}{2}$

74. $r = e^{a\theta} \quad 0 \le \theta \le \dfrac{\pi}{2}$ *revolving about* $\theta = \dfrac{\pi}{2}$

75. $r = a(1+\cos\theta) \quad 0 \le \theta \le \pi$ *revolving about polar axis*

76. $r = 1+4\cos\theta \quad 0 \le \theta \le \dfrac{\pi}{2}$ *revolving about Polar axis*

77. $r = 2\sin\theta \quad 0 \le \theta \le \dfrac{\pi}{2}$ *revolving about* $\theta = \dfrac{\pi}{2}$

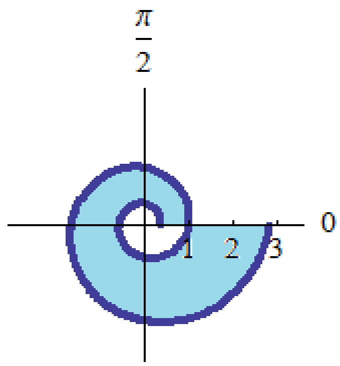
78. Find the surface area of the torus generated by revolving the circle given by $r = 2$ about the line $r = 5\sec\theta$

79. Find the surface area of the torus generated by revolving the circle given by $r = a$ about the line $r = b\sec\theta$, where $0 < a < b$

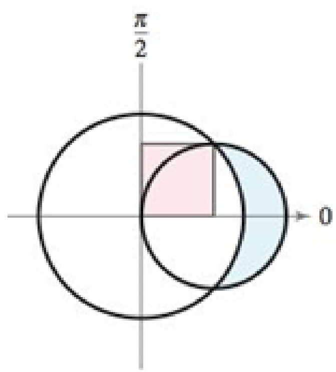
80. Let a and b be positive constants. Find the area of the region in the first quadrant bounded by the graph of the polar equation

$$r = \frac{ab}{a\sin\theta + b\cos\theta}, \quad 0 \le \theta \le \frac{\pi}{2}$$

81. The curve represented by the equation $r = ae^{b\theta}$, where a and b are constants, is called a logarithmic spiral. The figure shows the graph of $r = e^{\theta/6}$. $-2\pi \le \theta \le 2\pi$. Find the area of the shaded region.

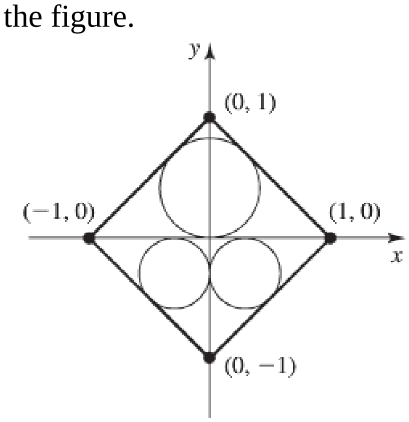


82. The larger circle in the figure is the graph of $r = 1$.



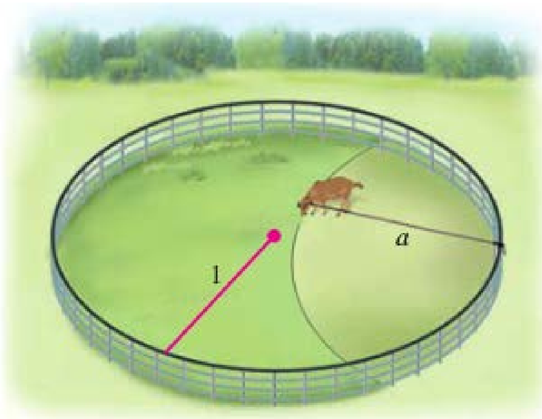
Find the polar equation of the smaller circle such that the shaded regrions are equal.

83. Find equations of the circles in the figure.



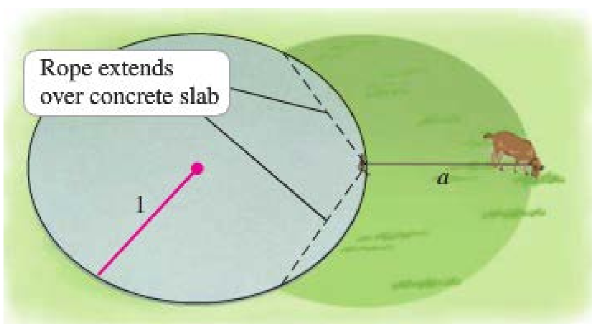
Determine whether the combined area of the circles is greater than or less than the area of the region inside the square but outside the circles.

84. A circular corral of unit radius is enclosed by a fence. A goat inside the corral is tied to the fence with a rope of length $0 \le a \le 2$.



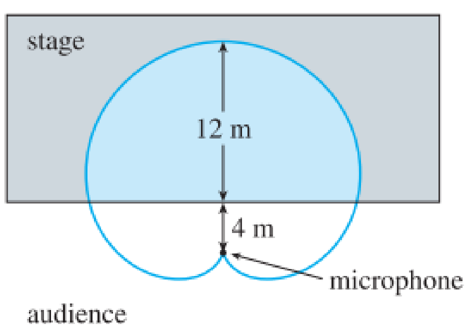
What is the area of the region (inside the corral) that the goat can graze? Check your answer with the special cases $a = 0$ and $a = 2$

85. A circular concrete slab of unit radius is surrounded by grass. A goat is tied to the edge of the slab with a rope of length $0 \le a \le 2$.



What is the area of the grassy region that the goat can graze? Note that the rope can extend over the concrete slab. Check your answer with the special cases $a = 0$ and $a = 2$

86. When recording live performance, sound engineers often use a microphone with a cardioid pickup pattern because it suppresses noise from the audience. Suppose the microphone is placed 4 *m* from the front of the stage and the boundary of the optimal pickup region is given by the cardioid $r = 8 + 8\sin\theta$, where r if measured in meters and the microphone is at the pole.



The musicians want to know the area they will have on stage within the optimal pickup range of the microphone, Answer their question.

87. The curve given by the parametric equations

$$x(t) = \frac{1-t^2}{1+t^2} \quad and \quad y(t) = \frac{t(1-t^2)}{1+t^2}$$

a) Find the rectangular equation of the strophoid.
b) Find a polar equation of the strophoid.
c) Sketch a graph of the strophoid.
d) Find the equations of the two tangent lines at the origin.
e) Find the points on the graph at which the tangent lines are horizontal.