Section 1.6 – Proof Methods and Strategy

Introduction

The strategy behind constructing proofs includes selecting a proof method and then successfully constructing an argument step by step, based on this method.

Exhaustive Proof and Proof by Cases

To prove a conditional statement of the form $\left(p_1 \vee p_2 \vee \cdots \vee p_n\right) \rightarrow q$

The tautology $\left[\left(p_1 \vee p_2 \vee \cdots \vee p_n\right) \rightarrow q\right] \leftrightarrow \left[\left(p_1 \rightarrow q\right) \wedge \left(p_2 \rightarrow q\right) \wedge \cdots \wedge \left(p_n \rightarrow q\right)\right]$

can be used as a rule of inference

Such an argument is called a ***proof by cases***. Sometimes to prove that a conditional statement $p \rightarrow q$ is true, it is convenient to use a disjunction $p_1 \vee p_2 \vee \cdots \vee p_n$ instead of *p* as the hypothesis of the conditional statement, where *p* and $p_1 \vee p_2 \vee \cdots \vee p_n$ are equivalent.

Exhaustive Proof

Also known as ***proof by cases***, ***perfect induction***, or the ***brute force method***, is a method of mathematical proof in which the statement to be proved is split into a finite number of cases and each case is checked to see if the proposition in question holds.

Theorem

A proposition that has been proved to be true

- Two special kinds of theorems: Lemma and Corollary.
- Lemma: A theorem that is usually not too interesting in its own right but is useful in proving another theorem.
- Corollary: A theorem that follows quickly from another theorem.

Example

Prove that $(n+1)^3 \geq 3^n$ if *n* is a positive integer with $n \leq 4$

Solution

Using a proof by exhaustion:

For $n = 1$: $(n+1)^3 = 2^3 = 8 \geq 3^1 = 3$

For $n = 2$: $(n+1)^3 = 3^3 = 27 \ge 3^2 = 9$

For $n = 3$: $(n+1)^3 = 4^3 = 64 \ge 3^3 = 27$

For $n = 4$: $(n+1)^3 = 5^3 = 125 \ge 3^4 = 81$

We have used the method of exhaustion to prove that $(n+1)^3 \ge 3^n$ if n is a positive integer with $n \le 4$

Example

Prove that if n is an integer, then $n^2 \ge n$

<u>Solution</u>

Case 1: When $n = 0$, that implies to $0^2 \ge 0$. It follows that $n^2 \ge n$ is true.

Case 2: When $n \ge 1$, $\Rightarrow n \cdot n \ge 1 \cdot n$, we obtain $n^2 \ge n$ It follows that $n^2 \ge n$ is true.

Case 3: When $n \le -1$, but $n^2 \ge 0$. It follows that $n^2 \ge n$ is true.

Because the inequality $n^2 \ge n$ holds in all three cases, we can conclude that if n is an integer, then $n^2 \ge n$.

Example

Show that if x and y are integers and both xy and $x + y$ are even, then both x and y are even.

<u>Solution</u>

Using the proof by contraposition:
Suppose that x and y are not both even. That is, x is odd or y is odd (or both).
Assume that x is odd, so that $x = 2k+1$ for some integer k.
Case 1: y even $\Rightarrow y = 2n$

$$x + y = 2k + 1 + 2n = 2(k+n) + 1 \text{ is } odd$$

Case 2: y odd $\Rightarrow y = 2n+1$

$$xy = (2k+1)(2n+1) = 4kn + 2k + 2n + 1 = 2(2kn + k + n) + 1 \text{ is } odd$$

This completes the proof by contraposition.

Existence Proofs

A statement $\exists x\, P(x)$ is called an ***existence proof***. There are several ways to proof a theorem of this type.

- ***Constructive***: Find a specific value of c for which $P(c)$ exists
- ***Nonconstructive***: Show that such a c exists, but don't actually find it. Assume it does not exist, and show a contradiction

Example

Show that a square exists that is the sum of two other squares

Solution

Proof: $3^2 + 4^2 = 5^2$

Because we have displayed a positive integer that can be written as the sum of two squares, we are done.

Example

Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.

Solution

Proof: $1729 = 10^3 + 9^3 = 12^3 + 1^3$

We proved that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.

Example

Show that a cube exists that is the sum of three other cubes

Solution

Proof: $3^3 + 4^3 + 5^3 = 6^3$

We proved that a cube exists that is the sum of three other cubes.

Uniqueness Proofs

A theorem may state that only one such value exists. Theorem statements that involve the word ``unique" are known as ***uniqueness theorems***. Typically the proof of such a statement follows the idea that we assume there are two elements that satisfy the conclusion of the statement and then show that these elements are identical.

Existence: We show that an element x with the desired property exists.
Uniqueness: We show that if $y \neq x$, then y does not have the desired property.

Equivalently, we can show that if x and y both have the desired property, then $x = y$.

Example

Show that if x and y are real numbers and $x \neq 0$, then there is a unique real number r such that $xr + y = 0$

Solution

The solution of $xr + y = 0$ is $r = -\frac{y}{x}$ because $x\left(-\frac{y}{x}\right) + y = -y + y = 0$. Consequently, a real number r exists for which $xr + y = 0$. This is the existence part of the proof.

Suppose that s is a real number such that $xs + y = 0$, then $xr + y = xs + y$, where $r = -\frac{y}{x}$.

$xr + y = xs + y \quad \Rightarrow \quad xr = xs \quad (x \neq 0) \quad \rightarrow r = s$

This means that if $s \neq r$, then $xs + y \neq 0$. This establishes the uniqueness part of the proof.

Proof Strategies

Usually, when you are working on a proof, you should use the logical forms of the givens and goals to guide you in choosing what proof strategies to use. Generally, if the statement is a conditional statement, we should try a direct proof; if this fails, we can try an indirect proof. If neither of these approaches works, you might try a proof by contradiction.

Example

Given two positive numbers x and y, their ***arithmetic mean*** is $\frac{x+y}{2}$ and their ***geometric mean*** is $\sqrt{xy}$.

When we compare the arithmetic and geometric means of pairs of distinct positive real numbers, we find that the arithmetic mean is always greater than the geometric mean. For example, when $x = 4$ and $y = 6$, we have $\frac{4+6}{2} = 5 > \sqrt{4 \cdot 6} = \sqrt{24}$. Can we prove that this inequality is always true?

Solution

To prove $\frac{x+y}{2} > \sqrt{xy}$

$$\left(\frac{x+y}{2}\right)^2 > \left(\sqrt{xy}\right)^2$$

$$\frac{x^2+2xy+y^2}{4} > xy$$

$$x^2+2xy+y^2 > 4xy$$

$$x^2-2xy+y^2 > 0$$

$$(x-y)^2 > 0$$

It is true inequality, since $(x-y)^2 > 0$ when $x \neq y$, it follows that $\frac{x+y}{2} > \sqrt{xy}$.

Suppose that x and y are distinct positive real numbers. Then $(x-y)^2 > 0$ because the square of a nonzero real number is positive.

$$x^2-2xy+y^2 > 0$$

$$x^2-2xy+y^2+4xy > 4xy$$

$$x^2+2xy+y^2 > 4xy$$

$$(x+y)^2 > 4xy$$

divide both sides by 4

$$\frac{(x+y)^2}{4} > xy$$

Square roots both sides

$$\frac{x+y}{2} > \sqrt{xy}$$

We conclude that if x and y are distinct positive real numbers, then their arithmetic mean $\frac{x+y}{2}$ is greater than the geometric mean $\sqrt{xy}$

Fermat's Last Theorem

The eqaution $\quad x^n + y^n = z^n$

Has no solutions in integers x, y, and z with $xyz \neq 0$ whenever n is an integer with $n > 2$.

Exercises *Section* 1.6 – Proof Methods and Strategy

1. Prove that $n^2+1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$

2. Prove that there are no positive perfect cubes less than 1000 that are the sum of the cubes of two positive integers.

3. Prove that if x and y are real numbers, then $\max(x, y)+\min(x, y)=x+y$. (*Hint*: Use a proof by cases, with the two cases corresponding to $x \geq y$ and $x < y$, respectively.)

4. Prove the triangle inequality, which states that if x and y are real numbers, then $|x|+|y| \geq |x+y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \geq 0$ and equals $-x$ if $x < 0$

5. Prove that either $2 \cdot 10^{500}+15$ or $2 \cdot 10^{500}+16$ is not a perfect square

6. Prove that there exists a pair of consecutive integers such that one of these integers is a perfect square and the other is a perfect cube.

7. Suppose that a and b are odd integers with $a \neq b$. Show there is a unique integer c such that $|a-c|=|b-c|$