4.3 Polar.

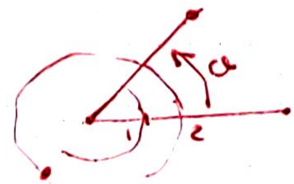
$f(\theta) =$

$r = f(\theta)$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$r = \sqrt{x^2+y^2}$$

$$\hat{\theta} = \tan^{-1}\frac{y}{x}$$



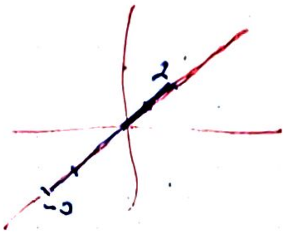
$(-2, 60^\circ)$

$(x, y) \to (r, \theta)$

$x, y \in \mathbb{R}$

(r, θ_{rad}) ? no(pi) π

$-3 \le r \le 2 \quad \theta = \frac{\pi}{4}$



$r\cos\theta = 2 \Rightarrow x = 2$

$r^2\cos\theta\sin\theta = 4$

$(r\cos\theta)(r\sin\theta) = 4$

$xy = 4$

$r = 1 - \cos\theta \qquad r \neq 0$

$r^2 = r - r\cos\theta$

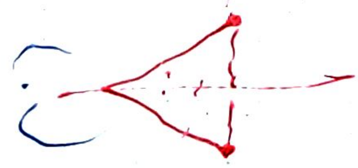
$x^2+y^2 = \sqrt{x^2+y^2} - x$

$x^4+y^4+2x^2y^2+2x^3+2xy^2-y^2=0$

$0, 1, \frac{1}{2}$ $\Big\}$ $\cos 60^\circ = \frac{1}{2}$, $\sin 30^\circ = \frac{1}{2}$

Symmetry x-axis $(-r, \theta)$

$(r, -\theta)$, $(-r, \pi - \theta)$



$r = f(\theta) = \cos\theta$

$f(-\theta) = \cos(-\theta)$

$= \cos(\theta)$

Symmetry y-axis

$(-r, -\theta)$ $(r, \pi - \theta)$

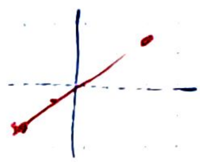
$r = \sin\theta$

$f(-\theta) = \sin(-\theta)$

$= -\sin\theta$

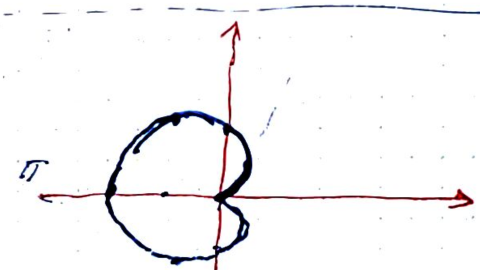
$= -r$

Origin : $(-r, \theta)$ $(r, \pi + \theta)$



$r = 1 - \cos\theta$

$f(\theta) = 1 - \cos\theta$



Cardioid.

11.4. Calculus

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

55 $f(\theta) = 1 - \cos\theta = r$

$-\pi \le \theta \le \pi$

$$\frac{dy}{dx} = \frac{\sin^2\theta + (1-\cos\theta)\cos\theta}{\sin\theta\cos\theta - (1-\cos\theta)\sin\theta}$$

$$= \frac{\sin^2\theta - \cos^2\theta + \cos\theta}{2\sin\theta\cos\theta - \sin\theta}$$

$$= \frac{1 - 2\cos^2\theta + \cos\theta}{\sin\theta(2\cos\theta - 1)} = 0$$

$\sin\theta \ne 0 \Rightarrow \theta \ne 0, \pm\pi$, $\cos\theta \ne \frac{1}{2}$

$-2\cos^2\theta + \cos\theta + 1 = 0$

$\cos\theta = 1$ $\quad$ $\cos\theta = -\frac{1}{2}$

$\theta = 0$ # $\quad$ $\theta = \pm\frac{2\pi}{3}$

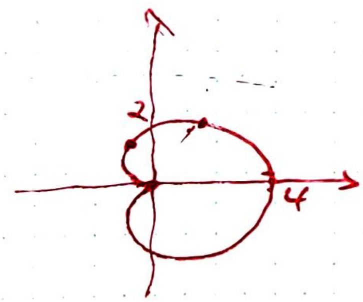
denominator

$\theta = \pm\pi$, $\theta = \pm\frac{\pi}{3}$

$A = \frac{1}{2} r^2 \theta$ (θ rad)

$$\text{Area} = \frac{1}{2} \int_{\alpha}^{\beta} (f(\theta))^2 \, d\theta$$

$$= \frac{1}{2} \int_{\alpha}^{\beta} r^2 \, d\theta$$



Ex $\quad r = 2(1 + \cos\theta)$

$$A = \frac{1}{2} \int_0^{2\pi} r^2 \, d\theta$$

$$= 2 \int_0^{2\pi} (1 + \cos\theta)^2 \, d\theta \qquad + \int_0^{\pi}$$

$$= 2 \int_0^{2\pi} (1 + \cos^2\theta + 2\cos\theta) \, d\theta \qquad \cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

$$= 2 \int_0^{2\pi} \left(\frac{3}{2} + \frac{1}{2}\cos 2\theta + 2\cos\theta\right) d\theta$$

$$= 2 \left(\frac{3}{2}\theta + \frac{1}{4}\sin 2\theta + 2\sin\theta \right|_0^{2\pi}$$

$$= 2(3\pi)$$

$$= 6\pi \text{ unit}^2$$

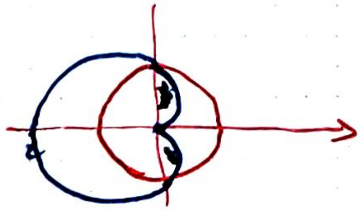
$$\text{Area} = \frac{1}{2}\int_{\alpha}^{\beta} (r_2^2 - r_1^2)\, d\theta$$

Ex: in $r = 1$ (circle) out $r = 1 - \cos\theta$

$1 - \cos\theta = 1$

$\cos\theta = 0$

$\theta = \pm\frac{\pi}{2}$



$$A = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(1^2 - (1-\cos\theta)^2\right) d\theta$$

$$= 2\left(\frac{1}{2}\right)\int_0^{\pi/2} (2\cos\theta - \cos^2\theta)\, d\theta$$

$$= \int_0^{\pi/2} \left(2\cos\theta - \frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= 2\sin\theta - \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta \Big|_0^{\pi/2}$$

$$= 2 - \frac{\pi}{4} \text{ unit}^2$$

Length

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Ex L? $r = 1 - \cos\theta$

$$\frac{dr}{d\theta} = \sin\theta$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \sqrt{2 - 2\cos\theta}$$

$$= \sqrt{4\sin^2\frac{\theta}{2}}$$

$$= 2\sin\frac{\theta}{2}$$

$\sin^2 x = \frac{1 - \cos 2x}{2}$

$2\sin^2\frac{\theta}{2} = 1 - \cos\theta$

$$L = 2\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta$$

$$= -4\cos\frac{\theta}{2}\Big|_0^{2\pi}$$

$$= -4(-1-1)$$

$$= 8 \text{ unit}$$

Surface

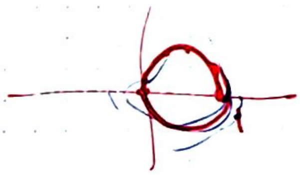
$$S = 2\pi \int_{\alpha}^{\beta} f(\theta)\sin\theta \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta$$

polar axis

$$S = 2\pi \int_{\alpha}^{\alpha} f(\theta)\cos\theta \sqrt{f^2 + (f')^2}\, d\theta$$

line $\theta = \frac{\pi}{2}$

Ex $f(\theta) = \cos\theta$ about line $\theta = \frac{\pi}{2}$



$$\sqrt{r^2 + (r')^2} = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$S = 2\pi \int_{0}^{\pi} \cos^2\theta \, d\theta$$

$$= \pi \int_{0}^{\pi} (1 + \cos 2\theta)\, d\theta$$

$$= \pi \left(\theta + \frac{1}{2}\sin 2\theta \right|_{0}^{\pi}$$

$$= \pi^2 \text{ unit}^2$$

#17 A? 1 leaf $\qquad r = \cos 3\theta$

$$A = \frac{1}{2}\left(\frac{1}{3}\right)\int_0^{2\pi} \cos^2 3\theta \, d\theta$$

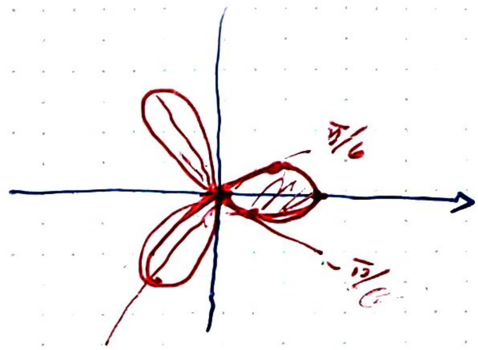
$$= \frac{1}{12}\int_0^{2\pi} (1+\cos 6\theta)\, d\theta$$

$$= \frac{1}{12}\left(\theta + \frac{1}{6}\sin 6\theta \right|_0^{2\pi}$$

$$= \frac{1}{12}(2\pi)$$

$$= \frac{\pi}{6}$$

$\cos 3\theta = \frac{\pi}{2}$



$$A = \frac{1}{2}\int_{-\pi/6}^{\pi/6} \cos^2 3\theta \, d\theta$$

$$= \frac{1}{4}(2)\int_0^{\pi/6} (1+\cos 6\theta)\, d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{6}\sin 6\theta\right)\Big|_0^{\pi/6}$$

$$= \frac{1}{2}\left(\frac{\pi}{6}\right)$$

$$= \underline{\frac{\pi}{12} \text{ unit}^2}$$

#8. A? $r = 4 + 2\sin\theta$

$$A = \frac{1}{2}\int_0^{2\pi} 4(2+\sin\theta)^2\,d\theta$$

$$= 2\int_0^{2\pi} (4 + 4\sin\theta + \sin^2\theta)\,d\theta$$

$$= 2\int_0^{2\pi} \left(\frac{9}{2} + 4\sin\theta - \frac{1}{2}\cos 2\theta\right)d\theta$$

$$= 2\left(\frac{9}{2}\theta - 4\cos\theta - \frac{1}{4}\sin 2\theta\right|_0^{2\pi}$$

$$= 2(9\pi - 4 + 4)$$

$$= 18\pi \text{ unit}^2$$

#10 A? $r^2 = 2\sin 3\theta$ $\frac{\pi}{2}$

$$A = \frac{1}{2}\int_0^{2\pi} r^2\,d\theta$$

$$= (6)\,\frac{1}{2}\,\frac{(2)}{(r^2)}\int_0^{\pi/2} 2\sin 3\theta\,d\theta$$

$$= -\frac{12}{3}\cos 3\theta \Big|_0^{\pi/2}$$

$$= -4(0 - 1)$$

$$= 4 \text{ unit}^2$$

Limaçon $a + b\begin{cases}\cos\theta\\ \sin\theta\end{cases}$

Cardioid. $a\left(1 + \begin{matrix}\cos\theta\\ \sin\theta\end{matrix}\right)$

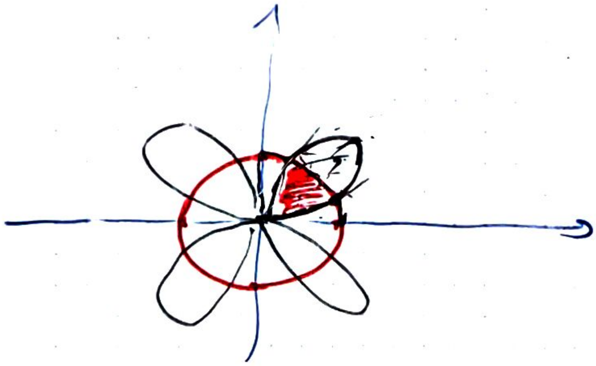
#25 $r = 4\sin 2\theta \qquad r = 2$

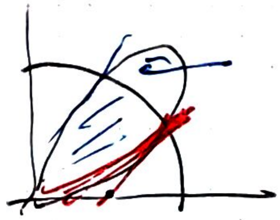
$r = 4\sin 2\theta = 2$

$\sin 2\theta = \frac{1}{2}$

$2\theta = \frac{\pi}{6}, \quad \frac{5\pi}{6}$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$





$A = \frac{1}{2}\int_{\pi/12}^{5\pi/12} (16\sin^2 2\theta - 4)\, d\theta$

$= \frac{1}{2}\int_{\pi/12}^{5\pi/12} (8 - 8\cos 4\theta - 4)\, d\theta$

$= 2\int_{\pi/12}^{5\pi/12} (1 - 2\cos 4\theta)\, d\theta$

$= 2\left(\theta - \frac{1}{2}\sin 4\theta\right)\Big|_{\pi/12}^{5\pi/12}$

$= 2\left(\frac{5\pi}{12} + \frac{\sqrt{3}}{4} - \frac{\pi}{12} + \frac{\sqrt{3}}{4}\right)$

$= \frac{2\pi}{3} + \sqrt{3}$

$A = \frac{1}{4}\pi r^2$

$= \pi$

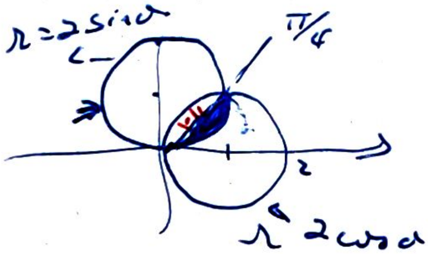
$A = 4\left(\pi + \frac{2\pi}{3} + \sqrt{3}\right)$

$= \frac{20\pi}{3} + 4\sqrt{3}$

#18 A? $r = 2\cos\theta$ $r = 2\sin\theta$

$r = 2\cos\theta = 2\sin\theta$

$\theta = \frac{\pi}{4}$



$$A = 2\,\frac{1}{2}\int_0^{\pi/4} 4\sin^2\theta \, d\theta$$

$$= 2\int_0^{\pi/4} (1-\cos 2\theta)\, d\theta$$

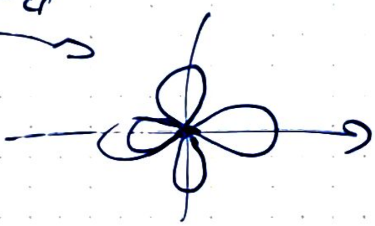
$$= 2\left(\theta - \frac{1}{2}\sin 2\theta \right|_0^{\pi/4}$$

$$= 2\left(\frac{\pi}{4} - \frac{1}{2}\right)$$

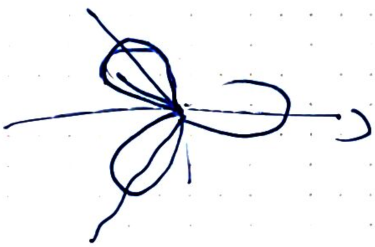
$$= \frac{\pi}{2} - 1 \quad \text{unit}^2$$

$\frac{360°}{4} = 90°$

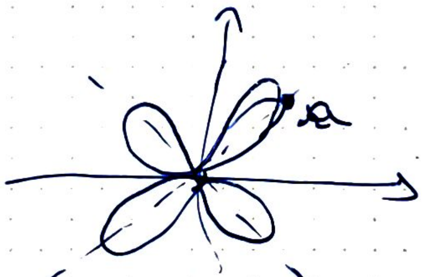
$2\cos 2\theta \longrightarrow$



$\cos 3\theta$



$\sin 2\theta$



$\sin 3\theta$