Section 4.4 – Equivalence Relations

Definition

A relation on a set A is called an ***equivalence relation*** if it is reflexive, symmetric, and transitive.

Definition

Two elements a and b that related by an equivalence relation are called ***equivalent***. The notation $a \sim b$ is often used to denotes that a and b are equivalent elements with respect to a particular equivalence relation.

Example

Let R be the relation on the set of integers such that aRb if and only if $a = b$ or $a = -b$. It follows that R is an equivalence relation.

Example

Let R be the relation on the set of real numbers such that aRb if and only if $a - b$ is an integer . Is R an equivalence relation?

Solution

Because $a - a = 0$ is an integer for all real numbers a, aRa for all real numbers a. Hence, R is reflexive
Suppose that aRb, then $a - b$ is an integer, so $b - a$ and $b - c$ are integers.
Therefore, $a - c = (a - b) + (b - c)$ is also an integer. Hence, aRc.
Thus, R is transitive. Consequently, R is an equivalence relation.

Example

Let m be an integer with $m > 1$. Show that the relation $R = \{(a,\ b) \mid a \equiv b \ (\mathbf{mod}\ m)\}$ is an equivalence relation on the set of integers.

Solution

$a \equiv b \ (\mathbf{mod}\ m)$ *iff* m divides $a - b$.
Since $0 = 0 \cdot m$ then $a - a = 0$ is divisible by m. Hence, $a \equiv a \ (\mathbf{mod}\ m)$, so congruence modulo m is reflexive.
Suppose that $a \equiv b \ (\mathbf{mod}\ m)$, then $a - b$ is divisible by m, so $a - b = km$, where k is an integer. It follows that $b - a = (-k)m$, so $b \equiv a \ (\mathbf{mod}\ m)$. Hence, congruence modulo m is symmetric.
Suppose that $a \equiv b \ (\mathbf{mod}\ m)$ and $b \equiv c \ (\mathbf{mod}\ m)$, then m divides both $a - b$ and $b - c$. so $a - b = km$ and $b - c = lm$, where k and l are integers. It follows that $a - c = (a - b) + (b - c) = km + lm = (k + l)m$, so $a \equiv c \ (\mathbf{mod}\ m)$. Hence, congruence modulo m is transitive.
The congruence modulo m is an equivalence relation.

Example

Let n be a positive integer and S a set of strings. Suppose that R_n is the relation on S such that $s\ R_n t$ if and only if $s = t$, or both s and t have at least n characters and the first n characters of s and t are the same. That is, a string of fewer than n characters is related only to itself; a string s with at least n characters is related to a string t if and only if t has at least n characters and t begins with the n characters at the start of s. For example, let $n = 3$ and let S be the set of all bit strings. Then $s\ R_3 t$ either when $s = t$ or both s and t are bit strings of length 3 or more that begin with the same three bits. For instant, $01\ R_3\ 01$ and

$00111\ R_3\ 00101$ but $01 \not R_3\ 010$ and $01011 \not R_3\ 01110$

Show that every set S of strings and every positive integer n, R_n is an equivalence relation on S.

Solution

The relation R_n is reflexive because $s = s$, so that $s\ R_n s$ whenever s is a string in S.

If $s\ R_n t$, then either $s = t$ or s and t are both at least n characters long that begin with same n characters. This means that $t\ R_n s$. Therefore, R_n is symmetric.

Suppose that $s\ R_n t$ and $t\ R_n u$. Then either $s = t$ or s and t are both at least n characters long s and t begin with same n characters, and either $t = u$ or t and u are both at least n characters long t and u begin with same n characters. From this, we can deduce that either $s = u$ or s and u are both at least n characters long s and u begin with same n characters.

Because s, t and u are all at least n characters long s and u begin with same n characters as t does. Therefore, R_n is transitive.

It follows that R_n is an equivalence relation.

Example

Let R be the relation on the set of real numbers such that $x\ R\ y$ if and only if x and y are real numbers that differ by less than 1, that is $|x - y| < 1$. Show that R is not an equivalence relation.

Solution

Let $x = 2.5$, $y = 1.8$, and $z = 1.1$, so that

$|x - y| = |2.5 - 1.8| = .7 < 1$ and $|y - z| = |1.8 - 1.1| = .7 < 1$

But $|x - z| = |2.5 - 1.1| = 1.4 > 1$. That is $2.5 R\ 1.8$, $1.8 R\ 1.1$, but $2.5 \not R\ 1.1$

Equivalence Classes

Definition

Let R be an equivalent relation on a set A. The set of all elements that are related to an element a of A is called the ***equivalence class*** of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.

$$[a]_R = \{s \mid (a, s) \in R\}$$

$b \in [a]_R$, then b called a ***representative*** of this equivalence class.

Example

Let R be the relation on the set of integers such that aRb if and only if $a = b$ or $a = -b$. What is the equivalence class for this relation?

Solution

Because an integer is equivalent to itself and its negative in this equivalence relation, it follows that $[a] = \{-a, a\}$. This set contains two distinct integers unless $a = 0$.

For instance, $[7] = \{-7, 7\}$, $[5] = \{-5, 5\}$, and $[0] = \{0\}$

Example

What is the equivalence class of 0 and 1 for congruence modulo 4?

Solution

The equivalence class of 0 contains all integers a such that $a \equiv 0(\mathbf{mod}\ 4)$. The integers in this class are those divisible by 4, Hence, the equivalence class of 0 for this relation is

$[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$

The equivalence class of 1 contains all integers a such that $a \equiv 1(\mathbf{mod}\ 4)$. The integers in this class are those that have a remainder of 1when divided by 4, Hence, the equivalence class of 1 for this relation is

$[1] = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$

Example

What is the equivalence class of the string 0111 with respect to the equivalence relation R_3 on the set of all bit strings?

Recall that $s\ R_3 t$ if and only if s and t are bit strings with $s = t$ or s and t are strings of at least three bits that start with the same three bits.

Solution

The bit strings equivalent to 0111 are the bit strings with at least three bits that begin with 011. These are the bit strings 011, 0110, 0111, 01100, 01101, 01110, 01111, and so on …

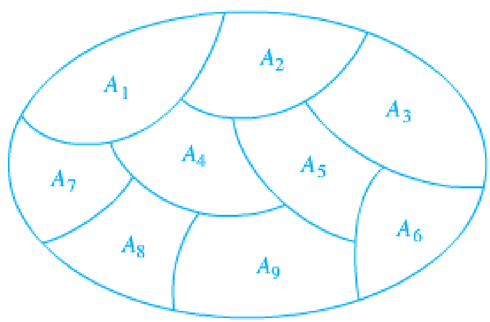
$$[011]_{R_3} = \{011,\ 0110,\ 0111,\ 01100,\ 01101,\ 01110,\ 01111,\ \ldots\}$$

Equivalence Classes and Partitions

Theorem

Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

$(\boldsymbol{i})\ aRb \qquad (\boldsymbol{ii})\ [a]=[b] \qquad (\boldsymbol{iii})\ [a]\cap[b]\neq\varnothing$



$$\bigcup_{i\in I} A_i = S$$

Example

Suppose that $S = \{1, 2, 3, 4, 5, 6\}$.

The collection of sets $A_1 = \{1,\ 2,\ 3\}$, $A_2 = \{4,\ 5\}$, and $A_3 = \{6\}$ forms a partition of S.

Because these sets are disjoint and their union is S.

Theorem

Let R be an equivalent relation on a set S. Then the equivalence classes of R form a partition of S, Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Example

List the ordered pairs in the equivalence relation R produced by the partition $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ of $S = \{1, 2, 3, 4, 5, 6\}$.

Solution

The subsets in the partition are the equivalence classes of R. The pair $(a, b) \in R$ if and only if a and b are in the same subset of the partition.

The pairs (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), and (3, 3) belong to R because $A_1 = \{1, 2, 3\}$ is an equivalence class.

The pairs (4, 4), (4, 5), (5, 4), and (5, 5) belong to R because $A_2 = \{4, 5\}$ is an equivalence class.

The pair (6, 6) belong to R because $A_3 = \{6\}$ is an equivalence class

Example

What are the sets in the partition of the integers arising from congruence modulo 4?

Solution

$[0]_4 = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$

$[1]_4 = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$

$[2]_4 = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$

$[3]_4 = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

Exercises *Section* 4.4 – Equivalence Relations

1. Which of these relations on {0, 1, 2, 3} are equivalence relations?
Determine the properties of an equivalence relation that the others lack.
What are the equivalence classes of the equivalence relations?

a) {(0, 0), (1, 1), (2, 2), (3, 3)}
b) {(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}
c) {(0, 0), (1, 1), (1, 2), (2, 1), (3, 2), (3, 3)}
d) {(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}
e) {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}

2. Which of these relations on the set of all people are equivalence relations?
Determine the properties of an equivalence relation that the others lack.
What are the equivalence classes of the equivalence relations?

a) $\{(a, b) \mid a \text{ and } b \text{ are the same age}\}$
b) $\{(a, b) \mid a \text{ and } b \text{ have the same parents}\}$
c) $\{(a, b) \mid a \text{ and } b \text{ share a common parent}\}$
d) $\{(a, b) \mid a \text{ and } b \text{ have met}\}$
e) $\{(a, b) \mid a \text{ and } b \text{ speak a common language}\}$

3. Which of these relations on the set of all functions from $\mathbb{Z}$ to $\mathbb{Z}$ are equivalence relations?
Determine the properties of an equivalence relation that the others lack.
What are the equivalence classes of the equivalence relations?

a) $\{(f, g) \mid f(1) = g(1)\}$
b) $\{(f, g) \mid f(0) = g(0) \text{ or } f(1) = g(1)\}$
c) $\{(f, g) \mid f(x) - g(x) = 1 \text{ for all } x \in \mathbb{Z}\}$
d) $\{(f, g) \mid \text{for some } C \in \mathbb{Z}, \text{ for all } x \in \mathbb{Z}, f(x) - g(x) = C\}$
e) $\{(f, g) \mid f(0) = g(1) \text{ or } f(1) = g(0)\}$

4. Define three equivalence relations on the set of students in your discrete mathematics class different from the relations discussed in the text. Determine the equivalence classes for each of these equivalence relations.

5. Define three equivalence relations on the set of buildings on a college campus. Determine the equivalence classes for each of these equivalence relations.

6. Let R be the relation on the set of all sets of real numbers such that $S\ R\ T$ if and only if S and T have the same cardinality. Show that R is an equivalence relation. What are the equivalence classes of the sets {0, 1, 2} and $\boldsymbol{Z}$?

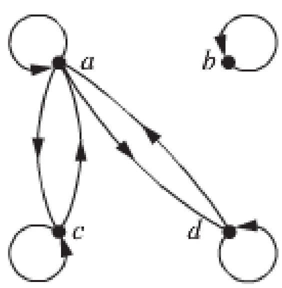
7. Suppose that A is a nonempty set, and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x, y) such that $f(x) = f(y)$

a) Show that R is an equivalence relation on A.
b) What are the equivalence classes of R?

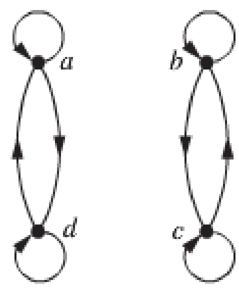
8. Suppose that A is a nonempty set, and R is an equivalence relation on A. Show that there is a function f with A as its domain such that $(x, y) \in R$ if and only if $f(x) = f(y)$

9. Determine whether the relation with the directed graph shown is an equivalence relation

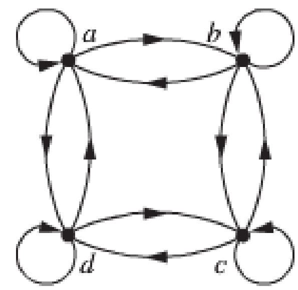
a)



b)



c)



10. Which of these collections of subsets are partitions of $\{1, 2, 3, 4, 5, 6\}$

a) $\{1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}$
b) $\{1\}, \{2, 3, 6\}, \{4\}, \{5\}$
c) $\{2, 4, 6\}, \{1, 3, 5\}$
d) $\{1, 4, 5\}, \{2, 6\}$

11. Which of these collections of subsets are partitions of $\{-3, -2, -1, 0, 1, 2, 3\}$

a) $\{-3, -1, 1, 3\}, \{-2, 0, 2\}$
b) $\{-3, -2, -1, 0\}, \{0, 1, 2, 3\}$
c) $\{-3, 3\}, \{-2, 2\}, \{-1, 1\}, \{0\}$
d) $\{-3, -2, 2, 3\}, \{-1, 1\}$