Lecture Two – Second & Higher Order Equations

Section 2.1- Definitions of Second and Higher Order Equations

A second-order differential equation is an equation involving the independent variable t and unknown function y.

$$y'' = f(t, y, y')$$

Linear equation: $$y'' + p(t)y' + q(t)y = g(t)$$

The coefficient $p, q, and\ g$ can be arbitrary functions.

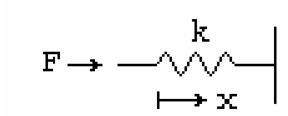
The equation is said to be ***homogeneous*** when:

$$y'' + p(t)y' + q(t)y = 0$$

2.1-1 *Newton's* - Hooke's Law for Springs: *F = kx*

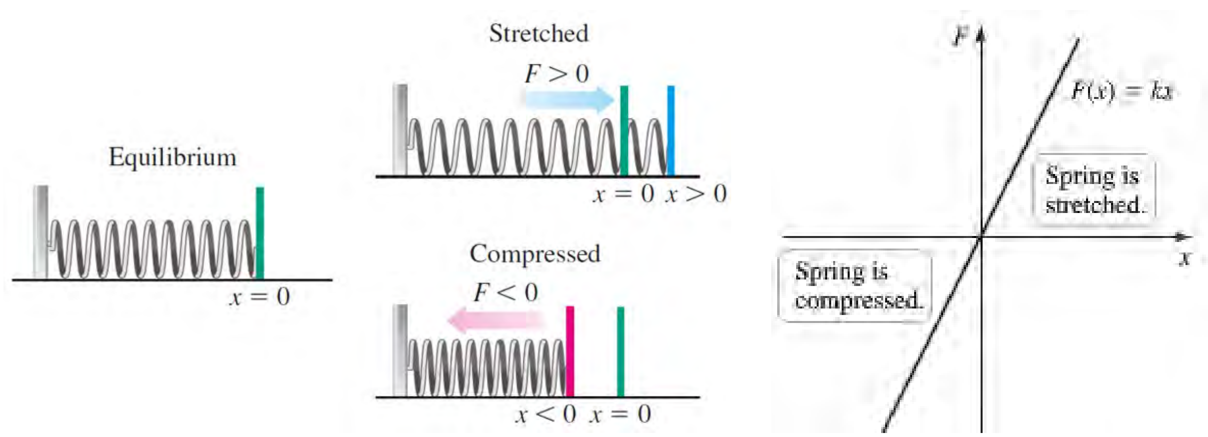
Hooke's Law is a principle of physics that states that the force (F) needed to stretch or compress a spring x units from its natural (unstressed) length is proportional to x. In symbols, that is

$$F = kx$$



The constant $\boldsymbol{k}$, measured in force units per unit length, is a characteristic of the spring, called ***the force constant*** (or ***spring constant***) of the spring.

The law is named after 17th-century British physicist ***Robert Hooke***.



- To stretch the spring to a position $x > 0$, a force $F > 0$ (in the ***positive*** direction) is required.
- To compress the spring to a position $x < 0$, a force $F < 0$ (in the ***negative*** direction) is required.

  $1\ kg.m/s^2 = 1\ N$ (*Newton*)

Example 1

A 4-*kg* weight is suspended from a spring. The displacement of the spring-mass equilibrium from the spring equilibrium is measured to be 49 *cm*. What is the spring constant?

Solution

$$mg = kx_0$$

$$k = \frac{mg}{x_0}$$

$$= \frac{4(9.8)}{0.49}$$

$$\underline{= 80 \;\; N/m}$$

2.1-2 *Proposition*

$$y'' + p(t)y' + q(t)y = 0$$

Solutions: $y = C_1 y_1 + C_2 y_2$

$C_1,\ C_2$ are any constant.

$y_1(t)\ \&\ y_2(t)$ are linearly independent solutions forming a ***fundamental set of solutions***.

2.1-3 *Definition*

A linear combination of the two functions $u\ \&\ v$ is any function of the form

$$w = Au + Bv$$

2.1-4 *Definition*

Two functions $u\ \&\ v$ are said to be linearly independent on the interval $(\alpha,\ \beta)$, if neither is a constant multiple of the order on that interval. If one is a constant multiple of the other on $(\alpha,\ \beta)$, they said to be linearly dependent there.

2.1-5 *Wronskian*

The Wronskian is a determinant introduced and named after the Polish mathematician Józef Hoene-Wroński (1776) and it is used to determine whether a set of differentiable functions (solutions) is ***linearly independent*** on a given interval.

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_n(x) \\ f_1{}'(x) & f_2{}'(x) & f_n{}'(x) \\ & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & f_n^{(n-1)}(x) \end{vmatrix}$$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - vu'$$

If $W = 0 \;\Rightarrow\; u \;\&\; v$ are *linearly dependent*.

If $W \neq 0 \;\Rightarrow\; u \;\&\; v$ are *linearly independent* (***LI***).

2.1-6 *Theorem*

Suppose that y_1 and y_2 are two solutions of the homogeneous second-order linear equation

$$y'' + p(x)y' + q(x)y = 0$$

On an open interval *I* on which *p* and *q* are continuous

1. If y_1 and y_2 are linearly dependent, then $W(y_1, y_2) \equiv 0$ on *I*.
2. If y_1 and y_2 are linearly independent, then $W(y_1, y_2) \neq 0$ at each point of *I*.

Example 2

Use the Wronskian to show that $\mathbf{f}_1 = x, \quad \mathbf{f}_2 = \sin x$ are linearly independence

Solution

The Wronskian is

$$W(x) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix}$$

$$= x\cos x - \sin x \neq 0$$

This function is not identically zero. Thus, the functions are linearly independent.

2.1-7 System Equations

A Planar System of first-order equations is a set of two first-order differential equations involving two unknowns

$$x' = f(t,x,y)$$
$$y' = g(t,x,y)$$

where f *and* g are functions of the independent variable t and the unknown x and y.

2.1-8 *Second-Order* Equations and Planar Systems

$$y'' = f(t,\ y,\ y')$$

Let's re-write in first-order system:

$$y' = v$$
$$v' = F(t,y,v)$$

$$y'' + p(t)y' + q(t)y = F(t)$$

$$y'' = F(t) - p(t)y' - q(t)y$$

$$v' = F(t) - p(t)v - q(t)y$$

$$\begin{cases} y' = v \\ v' = F(t) - p(t)v - q(t)y \end{cases}$$

Example 3

Consider a damped unforced spring: $y'' + 0.4y' + 3y = 0$; which satisfies the initial conditions $y(0) = 2$ and $v(0) = y'(0) = -1$

Solution

$$\begin{cases} y' = v \\ v' = -0.4v - 3y \end{cases}$$

$$v' + 0.4v = -3y$$

$$e^{\int 0.4dy} = e^{0.4y}$$

$$\int -3ye^{0.4y}dy = -\frac{3}{.16}e^{0.4y}(0.4y-1) \qquad \int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1)$$

$$v(y) = \boxed{-18.75e^{0.4y}(0.4y-1)+C}$$

$$v = -7.5y + 18.75 + Ce^{-0.4y}$$

$$v(0) = -7.5(0) + 18.75 + Ce^{-0.4(0)}$$

$$-1 = 18.75 + C$$

$$C = -19.75$$

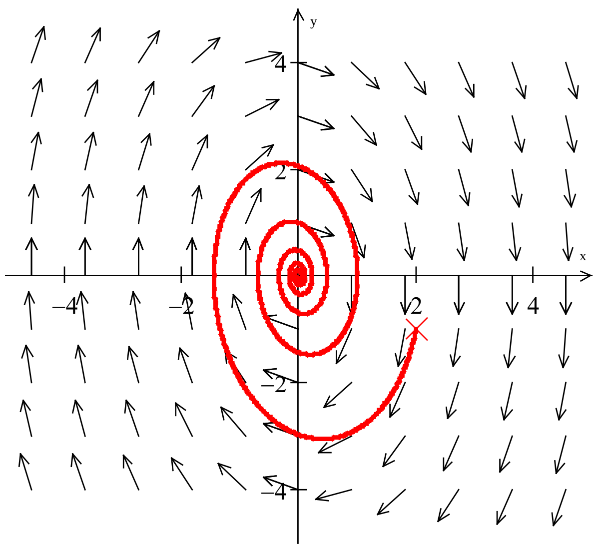
$$v(y) = -7.5y + 18.75 - 19.75e^{-0.4y}$$

$$y' = v = -7.5y + 18.75 - 19.75e^{-0.4y}$$

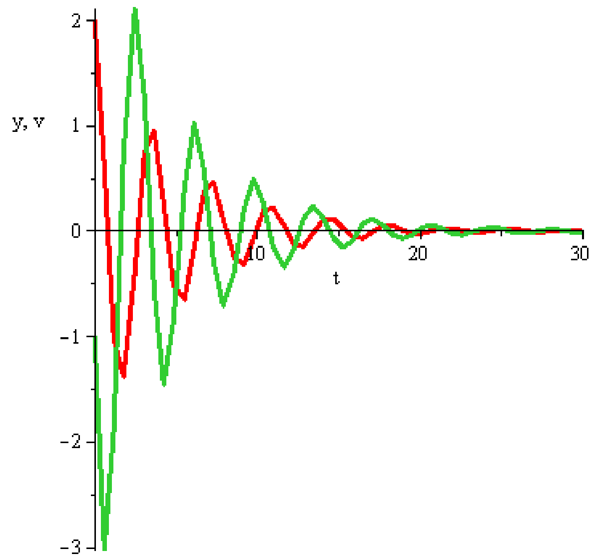
$$\frac{dy}{dt} = -7.5y + 18.75 - 19.75e^{-0.4y}$$

$$y(t) = -\frac{3\sqrt{74}}{74}e^{-t/5}\sin\left(\frac{\sqrt{74}}{5}t\right) + 2e^{-t/5}\cos\left(\frac{\sqrt{74}}{5}t\right)$$

The *yv*-plane is called the ***phase plane***.



Phase Plane Plot



Displacement y and the velocity v

Exercises *Section* 2.1 – Definitions of 2nd and Higher Order Equations

(1 – 4) Decide whether the equation is linear or nonlinear. For the linear equation, state whether the equation is homogeneous or inhomogeneous.

1. $t^2y'' = 4y' - \sin t$

2. $ty'' + (\sin t)y' = 4y - \cos 5t$

3. $t^2y'' + 4yy' = 0$

4. $y'' + 4y' + 7y = 3e^{-t}\sin t$

(5 – 6) Show by direct substitution that the given functions $y_1(t)$ and $y_2(t)$ are solutions of the given differential equation. Then verify by direct substitution, that any linear combination $C_1y_1(t) + C_2y_2(t)$ of the 2 given solutions is also a solution.

5. $y'' + 4y = 0;\quad y_1(t) = \cos 2t \quad y_2(t) = \sin 2t$

6. $y'' - 2y' + 2y = 0;\quad y_1(t) = e^t\cos t \quad y_2(t) = e^t\sin t$

7. Explain why $y_1(t)$ and $y_2(t)$ are linearly independent solutions. Calculate Wronskian and use it to explain the independence of the given solutions.

$$y'' + 9y = 0;\quad y_1(t) = \cos 3t \quad y_2(t) = \sin 3t$$

8. Show that $y_1(t) = e^t$ and $y_2(t) = e^{-3t}$ form a fundamental set of solutions for $y'' + 2y' - 3y = 0,$ then find a solution satisfying $y(0) = 1$ and $y'(0) = -2$.

(9 – 14) Use the Wronskian to show that are linearly independence

9. $y_1(x) = e^{-3x},\quad y_2(x) = e^{3x}$

10. $\mathbf{f}_1 = 1,\quad \mathbf{f}_2 = e^x,\quad \mathbf{f}_3 = e^{2x}$

11. $\left\{e^x,\ xe^x,\ (x+1)e^x\right\}$

12. $y_1(x) = e^{-3x},\quad y_2(x) = \cos 2x,\quad y_3(x) = \sin 2x$

13. $y_1(x) = e^x,\quad y_2(x) = e^{2x},\quad y_3(x) = e^{3x}$

14. $y_1(x) = \cos^2 x,\quad y_2(x) = \sin^2 x,\quad y_3(x) = \sec^2 x,\quad y_4(x) = \tan^2 x$

(15 – 20) Determine whether the functions $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $(0,\ 1)$

15. $y_1(t) = \cos t\sin t,\quad y_2(t) = \sin 2t$

16. $y_1(t) = e^{3t},\quad y_2(t) = e^{-4t}$

17. $y_1(t) = te^{2t},\quad y_2(t) = e^{2t}$

18. $y_1(t) = t^2\cos(\ln t),\quad y_2(t) = t^2\sin(\ln t)$

19. $y_1(t)=\tan^2 t-\sec^2 t, \quad y_2(t)=3$

20. $y_1(t)\equiv 0, \quad y_2(t)=e^t$

(21 – 25) Use the substitution $v=y'$ to write each second-order equation as a system of two first-order differential equation.

21. $y''+2y'-3y=0$

22. $y''+3y'+4y=2\cos 2t$

23. $y''+2y'+2y=2\sin 2\pi t$

24. $y''+\mu\left(t^2-1\right)y'+y=0$

25. $4y''+4y'+y=0$

(26 – 44) Find a particular solution satisfying the given initial conditions

26. $y''-4y=0\ ; \quad y_1(t)=e^{2t},\ y_2(t)=2e^{-2t}; \quad y(0)=1,\ y'(0)=-2$

27. $y''-y=0\ ; \quad y_1(t)=2e^{t},\ y_2(t)=e^{-t+3}; \quad y(-1)=1,\ y'(-1)=0$

28. $y''+y=0\ ; \quad y_1(t)=0,\ y_2(t)=\sin t; \quad y\left(\frac{\pi}{2}\right)=1,\ y'\left(\frac{\pi}{2}\right)=1$

29. $y''+y=0\ ; \quad y_1(t)=\cos t,\ y_2(t)=\sin t; \quad y\left(\frac{\pi}{2}\right)=1,\ y'\left(\frac{\pi}{2}\right)=1$

30. $y''-4y'+4y=0\ ; \quad y_1(t)=e^{2t},\ y_2(t)=te^{2t}; \quad y(0)=2,\ y'(0)=0$

31. $2y''-y'=0\ ; \quad y_1(t)=1,\ y_2(t)=e^{t/2}; \quad y(2)=0,\ y'(2)=2$

32. $y''-3y'+2y=0\ ; \quad y_1(t)=2e^{t},\ y_2(t)=e^{2t}; \quad y(-1)=1,\ y'(-1)=0$

33. $ty''+y'=0\ ; \quad y_1(t)=\ln t,\ y_2(t)=\ln 3t; \quad y(3)=0,\ y'(3)=3$

34. $t^2y''-ty'-3y=0\ ; \quad y_1(t)=t^3,\ y_2(t)=-\frac{1}{t}; \quad y(-1)=0,\ y'(-1)=-2 \quad (t<0)$

35. $y''+\pi^2y=0\ ; \quad y_1(t)=\sin\pi t+\cos\pi t,\ y_2(t)=\sin\pi t-\cos\pi t; \quad y\left(\frac{1}{2}\right)=1,\ y'\left(\frac{1}{2}\right)=0$

36. $x^3y^{(3)}-x^2y''+2xy'-2y=0$

$y(1)=3, \quad y'(1)=2, \quad y''(1)=1 \qquad y_1(x)=x, \quad y_2(x)=x\ln x, \quad y_3(x)=x^2$

37. $y^{(3)}+2y''-y'-2y=0$

$y(0)=1, \quad y'(0)=2, \quad y''(0)=0 \qquad y_1(x)=e^{x}, \quad y_2(x)=e^{-x}, \quad y_3(x)=e^{-2x}$

38. $y^{(3)}-6y''+11y'-6y=0$

$y(0)=0, \quad y'(0)=0, \quad y''(0)=3 \qquad y_1(x)=e^{x}, \quad y_2(x)=e^{2x}, \quad y_3(x)=e^{3x}$

39. $y^{(3)}-3y''+3y'-y=0$

$y(0)=2, \quad y'(0)=0, \quad y''(0)=0 \qquad y_1(x)=e^{x}, \quad y_2(x)=xe^{x}, \quad y_3(x)=x^2e^{x}$

40. $y^{(3)} - 5y'' + 8y' - 4y = 0$

$y(0) = 1, \quad y'(0) = 4, \quad y''(0) = 0 \qquad y_1(x) = e^x, \quad y_2(x) = e^{2x}, \quad y_3(x) = xe^{2x}$

41. $y^{(3)} + 9y'' = 0$

$y(0) = 3, \quad y'(0) = -1, \quad y''(0) = 2 \qquad y_1(x) = 1, \quad y_2(x) = \cos 3x, \quad y_3(x) = \sin 3x$

42. $y^{(3)} - 3y'' + 4y' - 2y = 0$

$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0 \qquad y_1(x) = e^x, \quad y_2(x) = e^x \cos x, \quad y_3(x) = e^x \sin x$

43. $x^3 y^{(3)} - 3x^2 y'' + 6xy' - 6y = 0$

$y(1) = 6, \quad y'(1) = 14, \quad y''(1) = 1 \qquad y_1(x) = x, \quad y_2(x) = x^2, \quad y_3(x) = x^3$

44. $x^3 y^{(3)} + 6x^2 y'' + 4xy' - 4y = 0$

$y(1) = 1, \quad y'(1) = 5, \quad y''(1) = -11 \qquad y_1(x) = x, \quad y_2(x) = x^{-2}, \quad y_3(x) = x^{-2} \ln x$

(45 – 46) Given the mass, damping, and spring constants of an undriven spring-mass system

$$my'' + \mu y' + ky = 0$$

a) Provide separate plots of the position versus time (y vs. t) and the velocity versus time (v vs. t)
b) Provide a combined plot of both position and velocity versus time
c) Provide a plot of the velocity versus position (v vs. y) in the yv phase plane.

45. $m = 1\ kg, \quad \mu = 0\ kg/s, \quad k = 4 kg/s^2, \quad y(0) = -2\ m, \quad y'(0) = -2\ m/s$

46. $m = 1\ kg, \quad \mu = 2\ kg/s, \quad k = 1 kg/s^2, \quad y(0) = -3\ m, \quad y'(0) = -2\ m/s$

47. When the values of a solution to a differential equation are specified at two different points, these conditions. (In contrast, initial conditions specify the values of a function and its derivative at the same point). The purpose of this is to show that for boundary value problems there is no existence-uniqueness theorem. Given that every solution to

$$y'' + y = 0 \qquad \text{is of the form} \qquad y(t) = c_1 \cos t + c_2 \sin t$$

Where c_1 and c_2 are arbitrary constants, show that

a) There is a unique solution to the given differential equation that satisfies the boundary conditions $y(0) = 2$ and $y\left(\frac{\pi}{2}\right) = 0$

b) There is no solution to given equation that satisfies $y(2) = 0$ and $y(\pi) = 0$

c) There are infinitely many solutions to the given DE equation that satisfy $y(0) = 2$ and $y(\pi) = -2$

Section 2.2 - Linear, Homogeneous Equations with Constant Coefficients

2.2-1 Introduction

The equations of the form: $y'' + py' + qy = 0$

This is a class of equations that we can solve easily.

The analogous first-order, linear, homogeneous equation:

$$y' + py = 0$$

It is separable and easily solved, its general solution is

$$y(t) = Ce^{-pt}$$

Let look for a solution of the type

$$y(t) = e^{\lambda t}$$

$$y' = \lambda e^{\lambda t}$$

$$y'' = \lambda^2 e^{\lambda t}$$

$$\begin{aligned} y'' + py' + qy &= \lambda^2 e^{\lambda t} + p\lambda e^{\lambda t} + qe^{\lambda t} \\ &= \left(\lambda^2 + p\lambda + q\right)e^{\lambda t} \\ &= 0 \end{aligned}$$

$\lambda^2 + p\lambda + q = 0$ *This is called the* ***characteristic equation***

We can rewrite the differential equation and its characteristic equations

$$y'' + py' + qy = 0$$

$$\lambda^2 + p\lambda + q = 0$$

The roots are: $\lambda_{1,2} = \dfrac{-p \pm \sqrt{p^2 - 4q}}{2}$

$$\textit{If} \quad p^2 - 4q > 0 \implies \textit{Two distinct real roots}$$

$$\textit{If} \quad p^2 - 4q < 0 \implies \textit{Two distinct complex roots}$$

$$\textit{If} \quad p^2 - 4q = 0 \implies \text{One } \textit{repeated real root}$$

2.2-2 *Case* 1: Distinct Real Root

$y_1 = C_1 e^{\lambda_1 t}$ *and* $y_2 = C_2 e^{\lambda_2 t}$ are both solutions.

2.2-3 *Proposition*

If the characteristic equations $\lambda^2 + p\lambda + q = 0$ has two distinct real roots λ_1 and λ_2, then the ***general solution*** to $y'' + py' + qy = 0$ is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

Where C_1 and C_2 are arbitrary constants.

Example 1

Find the general solution to the equation $y'' - 3y' + 2y = 0$

Find the unique solution corresponding to the initial conditions $y(0) = 2$ and $y'(0) = 1$

Solution

The characteristic equation:

$$y'' - 3y' + 2y = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

The solution: $\boxed{\lambda_{1,2} = 1,\ 2}$

The general solution

$$y(t) = C_1 e^t + C_2 e^{2t}$$

$$y' = C_1 e^t + 2C_2 e^{2t}$$

$y(0) = 2 \quad y(0) = C_1 e^0 + C_2 e^{2(0)}$

$$2 = C_1 + C_2$$

$y'(0) = 1 \quad y'(0) = C_1 e^0 + 2C_2 e^{2(0)}$

$$1 = C_1 + 2C_2$$

$$\begin{aligned} C_1 + C_2 &= 2 \\ C_1 + 2C_2 &= 1 \end{aligned} \Rightarrow C_2 = -1 \quad C_1 = 3$$

The unique solution is: $\boxed{y(t) = 3e^t - e^{2t}}$

2.2-4 *Case* 2: Complex Roots

2.2-5 *Proposition*

If the characteristic equations $\lambda^2 + p\lambda + q = 0$ has two complex conjugate roots $\lambda = a + ib$ and $\overline{\lambda} = a - ib$.

1. The functions

$$z = e^{(a+ib)t} \text{ and } \overline{z} = e^{(a-ib)t}$$

So, the general solution is

$$w(t) = C_1 e^{(a+ib)t} + C_2 e^{(a-ib)t}$$

Where C_1 and C_2 are arbitrary complex constants.

2. The functions

$$y_1(t) = e^{at}\cos(bt) \text{ and } y_2(t) = e^{at}\sin(bt)$$

So, the general solution is

$$y(t) = e^{at}\left(A_1 \cos bt + A_2 \sin bt\right)$$

Where A_1 and A_2 are constants.

Example 2

Find the general solution to the equation $y'' + 2y' + 2y = 0$

Find the unique solution corresponding to the initial conditions $y(0) = 2$ and $y'(0) = 3$

Solution

$$y'' + 2y' + 2y = 0$$

The characteristic equation: $\lambda^2 + 2\lambda + 2 = 0$

The solution: $\lambda_{1,2} = -1 \pm i = a \pm ib$

$\therefore\ a = -1;\ \ b = 1$

The general solution: $y(t) = e^{-t}\left(C_1 \cos t + C_2 \sin t\right)$

$$y(0) = e^{-(0)}\left(C_1 \cos(0) + C_2 \sin(0)\right)$$

$$2 = 1\left(C_1 + C_2(0)\right)$$

$$\Rightarrow\ C_1 = 2$$

$$y' = -e^{-t}\left(C_1\cos t + C_2\sin t\right) + e^{-t}\left(-C_1\sin t + C_2\cos t\right)$$

$$y'(0) = -e^{-(0)}\left(C_1\cos(0) + C_2\sin(0)\right) + e^{-(0)}\left(-C_1\sin(0) + C_2\cos(0)\right)$$

$$3 = -\left(C_1\right) + \left(C_2\right)$$

$$C_2 - C_1 = 3$$

$$C_2 = 3 + 2 = 5$$

$$y(t) = e^{-t}\left(2\cos t + 5\sin t\right)$$

***Example* 3**

Find the general solution to the equation $y'' - 4y' + 13y = 0$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 13 = 0$

The solutions: $\lambda_{1,2} = \dfrac{4 \pm \sqrt{16-52}}{2} = \dfrac{4 \pm \sqrt{-36}}{2}$

$$= \frac{4 \pm 6i}{2}$$

$$= 2 \pm 3i$$

$a = 2;\ \ b = 3$

The general solution:

$$y(x) = e^{2x}\left(C_1\cos 3x + C_2\sin 3x\right)$$

2.2-6 *Case* 3: Repeated Roots

If the roots of the characteristic equations are repeated

$$\lambda^2 + p\lambda + q = 0$$

$$(\lambda - \lambda_1)^2 = 0$$

$$\lambda_{1,2} = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

$$p^2 - 4q = 0 \;\Rightarrow\; q = \frac{p^2}{4}$$

$$\lambda_{1,2} = -\frac{p}{2}$$

$$\begin{aligned} y_1 &= C_1 e^{\lambda_1 t} \\ &= C_1 e^{-pt/2} \end{aligned}$$

$$\begin{aligned} y_2 &= v(t) y_1(t) \\ &= v(t) e^{-pt/2} \end{aligned}$$

$$y'' + py' + qy = 0$$

$$y'' + py' + \frac{p^2}{4} y = 0$$

$$y_2' = v' e^{-pt/2} - \frac{P}{2} v e^{-pt/2}$$

$$y_2'' = v'' e^{-pt/2} - \frac{P}{2} v' e^{-pt/2} - \frac{P}{2} v' e^{-pt/2} + \frac{P^2}{4} v e^{-pt/2}$$

$$v'' e^{-pt/2} - \frac{P}{2} v' e^{-pt/2} - \frac{P}{2} v' e^{-pt/2} + \frac{P^2}{4} v e^{-pt/2} + p\left(v' e^{-pt/2} - \frac{P}{2} v e^{-pt/2}\right) + \frac{P^2}{4} v e^{-pt/2} = 0$$

$$v'' e^{-pt/2} = 0$$

$$v'' = 0$$

$$\Rightarrow v' = a$$

$$\Rightarrow v = at + b$$

$$v = t$$

$$\boxed{y_2 = t e^{-pt/2}}$$

2.2-7 *Proposition*

If the characteristic equations $\lambda^2 + p\lambda + q = 0$ has one double root λ_1, then the ***general solution*** to $y'' + py' + qy = 0$ is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t}$$
$$= \left(C_1 + C_2 t\right) e^{\lambda_1 t}$$

Where C_1 and C_2 are arbitrary constants.

Example 4

Find the general solution to the equation $y'' - 2y' + y = 0$

Then find the unique solution corresponding to the initial conditions $y(0) = 2$ and $y'(0) = -1$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 1 = 0$

The solution: $\lambda_{1,2} = 1$

$$y(t) = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t}$$
$$= C_1 e^{t} + C_2 t e^{t}$$

$$y(0) = C_1 e^{(0)} + C_2 (0) e^{(0)} \quad \Rightarrow \quad 2 = C_1$$

$$y' = C_1 e^{t} + C_2 e^{t} + C_2 t e^{t}$$

$$y'(0) = 2e^{(0)} + C_2 e^{(0)} + C_2 (0) e^{(0)}$$

$$-1 = 2 + C_2 \quad \Rightarrow \quad C_2 = -3$$

$$y(t) = 2e^{t} - 3te^{t}$$

Example 5

Find the general solution to the equation $y'' - 10y' + 25y = 0$

Solution

The characteristic equation: $\lambda^2 - 10\lambda + 25 = \left(\lambda - 5\right)^2 = 0$

The solutions are: $\lambda_{1,2} = 5$

The general solution: $y(t) = C_1 e^{5t} + C_2 t e^{5t}$

2.2-8 *Higher*-Order Equations

In general, to solve an nth-order differential equation, we must solve an nth degree characteristic polynomial equation

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

If all roots are real and distinct, then the general solution is

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \cdots + C_n e^{\lambda_n x}$$

If all roots are equal to λ, then the general solution is

$$y(x) = C_1 e^{\lambda x} + C_2 x e^{\lambda x} + C_3 x^2 e^{\lambda x} + \cdots + C_n x^{n-1} e^{\lambda x}$$

Example 6

Find the general solution of $\quad y''' + 3y'' - 4y = 0$

Solution

$\lambda^3 + 3\lambda^2 - 4 = 0$ ***Solve for λ***

$\lambda_1 = 1, \quad \lambda_{2,\,3} = -2$

$$\boxed{y(x) = C_1 e^{x} + \left(C_2 + C_3 x\right) e^{-2x}}$$

Rational zero theorem: $\pm\left\{\frac{4}{1}\right\} = \pm\{1,\ 2,\ 4\}$

$\lambda_1 = 1, \quad \lambda_2 = -2$

$(\lambda - 1)(\lambda + 2)(\lambda - a) = 0$

$(-1)(2)(-a) = -4 \quad \Rightarrow \quad a = -2$

Example 7

Find the general solution of $\quad \lambda^4(\lambda + 1)(\lambda + 2)^2\left(\lambda^2 + 4\right) = 0$

Solution

$\lambda^2 + 4 = 0 \quad \Rightarrow \quad \lambda^2 = -4 \ \rightarrow \ \lambda = \pm 2i$

The solution: $\lambda = 0,\ 0,\ 0,\ 0,\ -1,\ -2,\ -2,\ \pm 2i$

$$\boxed{y(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 e^{-x} + \left(C_6 + C_7 x\right) e^{-2x} + C_8 \cos 2x + C_9 \sin 2x}$$

2.2-9 *Summary*

The equation: $y'' + py' + qy = 0$

The characteristic equations $\lambda^2 + p\lambda + q = 0$

If $p^2 - 4q > 0$	$y_1(t) = C_1 e^{\lambda_1 t}$ *and* $y_1(t) = C_2 e^{\lambda_2 t}$	$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$
If $p^2 - 4q < 0$	$y_1(t) = e^{at}\cos bt$ *and* $y_1(t) = e^{at}\sin bt$	$y(t) = e^{at}\left(C_1 \cos bt + C_2 \sin bt\right)$
If $p^2 - 4q = 0$	$y_1 = e^{\lambda t}$ *and* $y_1 = te^{\lambda t}$	$y(t) = \left(C_1 + C_2 t\right)e^{\lambda_1 t}$

Exercises *Section* 2.2 – Linear, Homogeneous Equations with Constant Coefficients

(1 – 86) Find the general solution of the second order differential equation

1. $y'' + y' = 0$
2. $y'' - 4y = 0$
3. $y'' + 8y = 0$
4. $y'' - 36y = 0$
5. $y'' + 9y = 0$
6. $y'' - 9y = 0$
7. $y'' + 16y = 0$
8. $y'' + 25y = 0$
9. $y'' - 64y = 0$
10. $y'' + y' + y = 0$
11. $y'' + y' - y = 0$
12. $y'' - y' - 2y = 0$
13. $y'' - y' - 6y = 0$
14. $y'' + y' - 6y = 0$
15. $y'' - y' - 11y = 0$
16. $y'' - y' - 12y = 0$
17. $y'' + 2y' + y = 0$
18. $y'' + 2y' + 3y = 0$
19. $y'' + 2y' - 3y = 0$
20. $y'' - 2y' - 3y = 0$
21. $y'' - 2y' + 3y = 0$
22. $y'' + 2y' + 4y = 0$
23. $y'' - 2y' + 5y = 0$
24. $y'' + 2y' - 15y = 0$
25. $y'' + 2y' + 17y = 0$
26. $y'' - 3y' + 2y = 0$
27. $y'' + 3y' - 4y = 0$
28. $y'' + 4y' - y = 0$
29. $y'' - 4y' + 4y = 0$
30. $y'' + 4y' + 4y = 0$
31. $y'' - 4y' + 5y = 0$
32. $y'' + 4y' + 5y = 0$
33. $y'' + 4y' - 5y = 0$
34. $y'' + 4y' + 7y = 0$
35. $y'' + 4y' + 9y = 0$
36. $y'' + 5y' = 0$
37. $y'' + 5y' + 6y = 0$
38. $y'' + 6y' + 9y = 0$
39. $y'' - 6y' + 9y = 0$
40. $y'' - 6y' + 25y = 0$
41. $y'' + 8y' + 16y = 0$
42. $y'' + 8y' - 16y = 0$
43. $y'' - 9y' + 20y = 0$
44. $y'' - 10y' + 25y = 0$
45. $y'' + 14y' + 49y = 0$
46. $2y'' - y' - 3y = 0$
47. $2y'' + y' - y = 0$
48. $2y'' + 2y' + y = 0$
49. $2y'' + 2y' + 3y = 0$
50. $2y'' - 3y' - 2y = 0$
51. $2y'' - 3y' + 4y = 0$
52. $2y'' - 4y' + 8y = 0$
53. $2y'' + 5y' = 0$
54. $2y'' - 5y' - 3y = 0$
55. $2y'' + 7y' - 4y = 0$
56. $3y'' + y = 0$
57. $3y'' - y' = 0$
58. $3y'' + 2y' + y = 0$
59. $3y'' + 11y' - 7y = 0$
60. $3y'' - 20y' + 12y = 0$
61. $4y'' + y' = 0$
62. $4y'' + 4y' + y = 0$
63. $4y'' - 4y' + y = 0$
64. $4y'' + 4y' + 2y = 0$
65. $4y'' - 4y' + 13y = 0$
66. $4y'' - 8y' + 7y = 0$
67. $4y'' - 12y' + 9y = 0$
68. $4y'' + 20y' + 25y = 0$
69. $6y'' + 5y' - 6y = 0$
70. $6y'' + y' - 2y = 0$
71. $6y'' - 7y' - 20y = 0$
72. $6y'' + 13y' - 5y = 0$
73. $6y'' + 13y' + 7y = 0$
74. $6y'' - 13y' + 7y = 0$
75. $8y'' - 10y' - 3y = 0$
76. $9y'' - y = 0$
77. $9y'' + 6y' + y = 0$
78. $9y'' - 12y' + 4y = 0$
79. $9y'' + 24y' + 16y = 0$
80. $12y'' - 5y' - 2y = 0$
81. $16y'' - 8y' + 7y = 0$
82. $16y'' - 12y' - 4y = 0$
83. $16y'' - 24y' + 9y = 0$
84. $25y'' + 10y' + y = 0$
85. $25y'' - 10y' + y = 0$
86. $35y'' - y' - 12y = 0$

(87 – 128) Find the general solution of the given higher-order differential equation

87. $y''' + 3y'' + 3y' + y = 0$

88. $y''' + 3y'' - y' - 3y = 0$

89. $y^{(3)} + 3y'' - 4y = 0$

90. $3y''' - 19y'' + 36y' - 10y = 0$

91. $y''' - 6y'' + 12y' - 8y = 0$

92. $y''' + 5y'' + 7y' + 3y = 0$

93. $y^{(3)} + y' - 10y = 0$

94. $y''' + y'' - 6y' + 4y = 0$

95. $y''' - 6y'' - y' + 6y = 0$

96. $y''' + 2y'' - 4y' - 8y = 0$

97. $y''' - 7y'' + 7y' + 15y = 0$

98. $y''' + 3y'' - 4y' - 12y = 0$

99. $y''' - 4y'' - 5y' = 0$

100. $y''' - y = 0$

101. $y''' - 5y'' + 3y' + 9y = 0$

102. $y''' + 3y'' - 4y' - 12y = 0$

103. $y''' + y'' - 2y = 0$

104. $y''' - y'' - 4y = 0$

105. $y''' + 3y'' + 3y' + y = 0$

106. $y''' - 6y'' + 12y' - 8y = 0$

107. $y^{(4)} + y''' + y'' = 0$

108. $y^{(4)} - 2y'' + y = 0$

109. $16y^{(4)} + 24y'' + 9y = 0$

110. $y^{(4)} - 7y'' - 18y = 0$

111. $y^{(4)} + 2y'' + y = 0$

112. $y^{(4)} + y''' + y'' = 0$

113. $y^{(4)} + 4y = 0$

114. $y^{(4)} + 2y''' + 9y'' - 2y' - 10y = 0$

115. $x^{(4)} - 4x^{(3)} + 7x'' - 4x' + 6x = 0$

116. $x^{(4)} + 8x^{(3)} + 24x'' + 32x' + 16x = 0$

117. $x^{(4)} - 4x'' + 16x' + 32x = 0$

118. $x^{(4)} + 4x^{(3)} + 6x'' + 4x' + x = 0$

119. $y^{(4)} - y^{(3)} + y'' - 3y' - 6y = 0$

120. $y^{(4)} + y^{(3)} - 3y'' - 5y' - 2y = 0$

121. $x^{(5)} - x^{(4)} - 2x^{(3)} + 2x'' + x' - x = 0$

122. $x^{(5)} + 5x^{(4)} + 10x^{(3)} + 10x'' + 5x' + x = 0$

123. $y^{(5)} + 5y^{(4)} - 2y''' - 10y'' + y' + 5y = 0$

124. $2y^{(5)} - 7y^{(4)} + 12y''' + 8y'' = 0$

125. $y^{(5)} - 2y^{(4)} + 17y''' = 0$

126. $x^{(6)} - 5x^{(4)} + 16x^{(3)} + 36x'' - 16x' - 32x = 0$

127. $\left(D^2 + 6D + 13\right)^2 y = 0$

128. $\lambda^3(\lambda - 1)(\lambda - 2)^3\left(\lambda^2 + 9\right) = 0$

(129 – 193) Find the solution of the given initial value problem.

129. $y'' + y = 0$; $y\left(\frac{\pi}{3}\right) = 0, \quad y'\left(\frac{\pi}{3}\right) = 2$

130. $y'' + y = 0$; $y(0) = 0, \quad y'\left(\frac{\pi}{2}\right) = 0$

131. $y'' + y' = 0$; $y(0) = 2, \quad y'(0) = 1$

132. $y'' - y' - 2y = 0$; $y(0) = -1, \quad y'(0) = 2$

133. $y'' + y' + 2y = 0$; $y(0) = 0, \quad y'(0) = 0$

134. $y'' + 2y' + y = 0$; $y(0) = 1, \quad y'(0) = -3$

135. $y'' - 2y' + y = 0$, $y(0) = 5, \quad y'(0) = 10$

136. $y'' - 2y' - 2y = 0$; $y(0) = 0, \quad y'(0) = 3$

137. $y'' - 2y' + 2y = 0$; $y(0) = 1, \quad y(\pi) = 1$

138. $y'' - 2y' - 3y = 0$; $y(0) = 2, \quad y'(0) = -3$

139. $y'' + 2y' - 8y = 0$; $y(0) = 3, \quad y'(0) = -12$

140. $y'' - 2y' + 17y = 0$; $y(0) = -2, \quad y'(0) = 3$

141. $y'' + 2\sqrt{2}y' + 2y = 0$; $y(0) = 1, \quad y'(0) = 0$

142. $y'' + 3y' - 10y = 0$; $y(0) = 4, \quad y'(0) = -2$

143. $y'' + 4y = 0$; $y(0) = 0, \quad y(\pi) = 0$

144. $y''+4y=0$; $y\left(\frac{\pi}{4}\right)=-2,\quad y\left(\frac{\pi}{4}\right)=1$

145. $y''+4y'+2y=0$; $y(0)=-1,\quad y'(0)=2$

146. $y''-4y'+3y=0$; $y(0)=1,\quad y'(0)=\frac{1}{3}$

147. $y''-4y'+4y=0,\quad y(1)=1,\quad y'(1)=1$

148. $y''+4y'+4y=0,\quad y(0)=1,\quad y'(0)=3$

149. $y''-4y'+5y=0;\quad y(0)=1,\quad y'(0)=5$

150. $y''+4y'+5y=0;\quad y(0)=1,\quad y'(0)=0$

151. $y''+4y'+5y=0;\quad y\left(\frac{\pi}{2}\right)=\frac{1}{2},\quad y'\left(\frac{\pi}{2}\right)=-2$

152. $y''-4y'-5y=0,\quad y(1)=0,\quad y'(1)=2$

153. $y''-4y'-5y=0,\quad y(-1)=3,\quad y'(-1)=9$

154. $y''-4y'+9y=0,\quad y(0)=0,\quad y'(0)=-8$

155. $y''-4y'+13y=0;\quad y(0)=-1,\quad y'(0)=2$

156. $y''-5y'+6y=0;\quad y(1)=e^2,\quad y'(1)=3e^2$

157. $y''+6y'+9y=0;\quad y(0)=2,\quad y'(0)=-2$

158. $y''+6y'+5y=0,\quad y(1)=0,\quad y'(0)=3$

159. $y''-6y'+5y=0;\quad y(0)=3,\quad y'(0)=11$

160. $y''-6y'+9y=0,\quad y(0)=2,\quad y'(0)=\frac{25}{3}$

161. $y''-6y'+9y=0;\quad y(0)=0,\quad y'(0)=5$

162. $y''+8y'-9y=0;\quad y(1)=2,\quad y'(1)=0$

163. $y''-8y'+17y=0;\quad y(0)=4,\quad y'(0)=-1$

164. $y''-9y=0,\quad y(0)=2,\quad y'(0)=-1$

165. $y''-10y'+25y=0,\quad y(0)=1,\quad y'(1)=0$

166. $y''+10y'+25y=0;\quad y(0)=2,\quad y'(0)=-1$

167. $y''+11y'+24y=0;\quad y(0)=0,\quad y'(0)=-7$

168. $y''+12y=0,\quad y(0)=0,\quad y'(0)=1$

169. $y''+16y=0,\quad y(0)=2,\quad y'(0)=-2$

170. $y''+16y=0,\quad y(\pi)=2,\quad y'(0)=-2$

171. $y''+16y=0,\quad y\left(\frac{\pi}{2}\right)=-10,\quad y'\left(\frac{\pi}{2}\right)=3$

172. $y''+25y=0;\quad y(0)=1,\quad y'(0)=-1$

173. $2y''-2y'+y=0;\quad y(-\pi)=1,\quad y'(-\pi)=-1$

174. $3y''+y'-14y=0,\quad y(0)=2,\quad y'(0)=-1$

175. $3y''+2y'-8y=0,\quad y(0)=-6,\quad y'(0)=-18$

176. $4y''-4y'+y=0,\quad y(0)=4,\quad y'(0)=4$

177. $4y''-4y'+y=0,\quad y(1)=-4,\quad y'(1)=0$

178. $4y''-4y'-3y=0,\quad y(0)=1,\quad y'(0)=5$

179. $4y''+4y'+5y=0,\quad y(\pi)=1,\quad y'(\pi)=0$

180. $4y''+4y'+17y=0,\quad y(0)=-1,\quad y'(0)=2$

181. $4y''-5y'=0,\quad y(-2)=0,\quad y'(-2)=7$

182. $4y''+12y'+9y=0,\quad y(0)=2,\quad y'(0)=1$

183. $4y''+24y'+37y=0,\quad y(\pi)=1,\quad y'(\pi)=0$

184. $9y''+y=0;\quad y\left(\frac{\pi}{2}\right)=4,\quad y'\left(\frac{\pi}{2}\right)=0$

185. $9y''+\pi^2y=0;\quad y(3)=2,\quad y'(3)=-\pi$

186. $9y''-6y'+y=0;\quad y(3)=-2,\quad y'(3)=-\frac{5}{3}$

187. $9y''+6y'+2y=0$; $y(3\pi)=0,\quad y'(3\pi)=\frac{1}{3}$

188. $9y''-12y'+4y=0,\quad y(0)=-1,\quad y'(0)=1$

189. $12y''+5y'-2y=0,\quad y(0)=1,\quad y'(0)=-1$

190. $16y''-8y'+y=0$; $y(0)=-4,\quad y'(0)=3$

191. $25y''+20y'+4y=0$; $y(5)=4e^{-2},\quad y'(5)=-\frac{3}{5}e^{-2}$

192. $y'''+12y''+36y'=0,\quad y(0)=0,\quad y'(0)=1,\quad y''(0)=-7$

193. $y'''+2y''-5y'-6y=0,\quad y(0)=0,\quad y'(0)=0,\quad y''(0)=1$

194. The roots of the characteristic equation of a certain differential equation are:

$$3, -5, 0, 0, 0, 0, -5,\ 2\pm 3i \text{ and } 2\pm 3i$$

Write a general solution of this homogeneous differential equation.

195. $y(x)=C_1e^{2x}+C_2e^{-2x}+C_3\cos 2x+C_4\sin 2x$ is the general solution of a homogeneous equation. What is the equation? $y(x)=C_1e^{2x}+C_2e^{-2x}+C_3\cos 2x+C_4\sin 2x$

196. Show that the second differential equation $y''+4y=0$

a) Has no solution to the boundary value $y(0)=0, \quad y(\pi)=1$

b) There are infinitely many solutions to the boundary value $y(0)=0, \quad y(\pi)=0$

197. Show that the general solution of the equation

$$y''+Py'+Qy=0$$

(where P and Q are constant) approaches 0 as $x\to\infty$ if and only if P and Q are both positive.

Section 2.3 – Harmonic Motion

Simple ***harmonic motion*** refers to the periodic sinusoidal oscillation of an object and denotes the ***second derivative*** of with respect to the angular frequency of oscillation.

2.3-1 Hooke's Law

The restoring force of a spring is proportional to the displacement

$$F = -ky, \quad k > 0 \quad (k : \textit{Spring constant})$$

2.3-2 Newton's Second Law

Force equals mass times acceleration

$$F = ma = m\frac{d^2y}{dt^2}$$

Mathematical model:

$$m\frac{d^2y}{dt^2} = -ky$$

$$m\frac{d^2y}{dt^2} + ky = 0$$

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

$$\frac{d^2y}{dt^2} + \omega_0^2 y = 0; \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

ω_0 is called natural frequency of the system

2.3-3 Damped, Free Vibrations

Simple Harmonic Motion with a ***Damping Force*** can be used to describe the motion of a mass at the end of a spring under the influence of friction.

When a damped oscillator is subject to a damping force which is linearly dependent upon the velocity, such as viscous damping, the oscillation will have exponential decay terms which depend upon a damping coefficient.

A resistance force R (e.g. friction) proportional to the velocity $v = y'$ and acting in a direction opposite to the motion

$$R = -cy', \quad c > 0$$

Force equation: $\quad F = -ky(t) - cy'(t)$

Mathematical model: $my'' = -ky - cy'$

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0 \qquad (c,\ m,\ k \text{ are constants})$$

The equation for the motion of a vibrating spring is given by

$$my'' + \mu y' + ky = F(t)$$

Where the constant coefficients are:

m	mass
μ	damping constant
k	spring constant
$F(t)$	external force

The differential equation that modeled simple RLC circuits is given by

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}$$

The charge on the capacitor is denoted by $q(t)$ and is related to the current $i(t)$ by $i = \frac{dq}{dt}$, so

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

If the electrical vibrations of a circuit are said to be free, when $E(t) = 0$

$$L\lambda^2 + R\lambda + \frac{1}{C} = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L}$$

If $R^2 - \frac{4L}{C} > 0$ ***overdamped***

If $R^2 - \frac{4L}{C} = 0$ ***critically damped***

If $R^2 - \frac{4L}{C} < 0$ ***underdamped***

Comparing the *Motion* to *electrical* systems are almost identical.

Combine the two systems:

$$y'' + \frac{\mu}{m}y' + \frac{k}{m}y = \frac{1}{m}F(t)$$

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = \frac{1}{L}\frac{dE}{dt}$$

If we let:
$$\begin{cases} & \text{Spring} & \text{Electrical} \\ c = & \frac{\mu}{2m} & \frac{R}{2L} \\ \omega_0 = & \sqrt{k/m} & \sqrt{1/LC} \\ f(t) = & \frac{1}{m}F(t) & \frac{1}{L}\frac{dE}{dt} \\ x = & y & I \end{cases}$$

$$x'' + 2cx' + \omega_0^2 x = f(t)$$

Where $c \ge 0$ *and* $\omega_0 > 0$ are constants.

This equation called ***harmonic motion***.

c ***damping*** constant
f ***forcing term***
ω_0 ***natural frequency***.

Example 1

For a circuit without resistance $(R = 0)$ and no source voltage, then the equation simplifies to

$$L\frac{d^2I}{dt^2} + \frac{1}{C}I = 0 \qquad \textit{Divide by L}$$

$$\frac{d^2I}{dt^2} + \frac{1}{LC}I = 0$$

$$\lambda^2 + \frac{1}{LC} = 0$$

$$\lambda^2 = -\frac{1}{LC}$$

$$\lambda = \pm i\frac{1}{\sqrt{LC}} \qquad \lambda = a \pm ib$$

$$\Rightarrow a = 0; \quad b = \frac{1}{\sqrt{LC}}$$

The general solution: $y(t) = e^{at}\left(A_1 \cos bt + A_2 \sin bt\right)$

$$\underline{I(t) = C_1 \cos\left(\frac{t}{\sqrt{LC}}\right) + C_2 \sin\left(\frac{t}{\sqrt{LC}}\right)}$$

2.3-4 Linear Constant-Coefficient Models

Mechanical System	***Electrical System***
$my'' + \mu y' + ky = F(t)$	$Lq'' + Rq' + \frac{1}{C}q = E(t)$
y: *displacement*	q: *charge*
y': *velocity*	q': *current*
y'': *acceleration*	q'': *change in current*
m: *mass*	L: *inductance*
μ: *damping constant*	R: *resistance*
k: *spring constant*	$\frac{1}{C}$: *where C is the capacitance*
$F(t)$: *forcing function*	$E(t)$: *voltage source*

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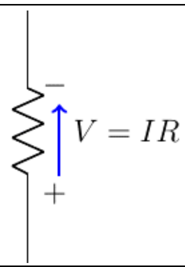
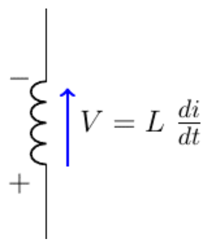
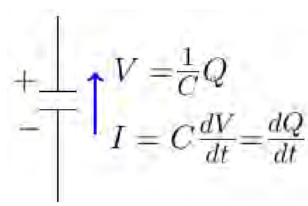
<tr><td>
</td><td>
</td><td>
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2.3-5 Simple Harmonic Motion

In the special case when there is no damping $(c = 0)$ the motion is called ***simple harmonic motion***.

$$x'' + \omega_0^2 x = 0$$

The characteristic equation is: $\lambda^2 + \omega_0^2 = 0$

The roots are $\lambda^2 = -\omega_0^2 \to \lambda = \pm i\omega_0$

$$x(t) = a\cos\omega_0 t + b\sin\omega_0 t$$

If we define period: $T = \frac{2\pi}{\omega_0} \Rightarrow T\omega_0 = 2\pi$

Then the periodic of the trigonometry functions implies that $x(t+T) = x(t)$ for all t.

Thus, the solution x is periodic with period T.

ω_0 is called the ***natural frequency***.

2.3-6 Amplitude and Phase Angle

$x(t) = a\cos\omega_0 t + b\sin\omega_0 t$

Consider the point $(a,\ b)$, we can rewrite this in polar coordinates with a length of A.

$$a = A\cos\phi \quad b = A\sin\phi$$

$$\begin{aligned} x(t) &= a\cos\omega_0 t + b\sin\omega_0 t \\ &= A\cos\phi\cos\omega_0 t + A\sin\phi\sin\omega_0 t \\ &= A\left(\cos\phi\cos\omega_0 t + \sin\phi\sin\omega_0 t\right) \\ &= A\cos\left(\omega_0 t - \phi\right) \end{aligned}$$

Where A ***amplitude*** of the oscillation $\quad A = \sqrt{a^2 + b^2}$

ϕ ***Phase*** of the oscillation $\quad \tan\phi = \dfrac{b}{a} \quad \left(-\dfrac{\pi}{2},\ \dfrac{\pi}{2}\right)$

Period: $T = \dfrac{2\pi}{\omega_0}$

Frequency: $\nu = \dfrac{1}{T}$

Time lag of the motion is: $\delta = \dfrac{\varphi}{\omega_0}$

Example 2

A mass of 4 *kg* is attached to a spring with a spring constant of $k = 169\ kg/s^2$. It is then stretched 10 *cm* from the spring mass equilibrium and set to oscillating with an initial velocity is 130 *cm/s*. Assuming it oscillates without damping, find the natural frequency, period, amplitude, and phase of the vibration.

Solution

$my'' + \mu y' + ky = F(t)$

$4y'' + 169y = 0$ *Divide by 4*

$y'' + \dfrac{169}{4}y = 0$

The ***natural frequency***:

$$\omega_0 = \sqrt{\frac{169}{4}}$$

$$\boxed{= \frac{13}{2}}$$

Period:

$$T = \frac{2\pi}{\frac{13}{2}} \qquad\qquad T = \frac{2\pi}{\omega_0}$$

$$= \frac{4\pi}{13}$$

$$y(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$$

$$= C_1 \cos \frac{13}{2} t + C_2 \sin \frac{13}{2} t$$

Stretched 10 $cm \rightarrow y(0) = 10\ cm = .1\ m$

$$y(0) = C_1 \cos\left(\frac{13}{2}(0)\right) + C_2 \sin\left(\frac{13}{2}(0)\right)$$

$$\frac{1}{10} = C_1$$

Initial velocity is 130 $cm/s \rightarrow y'(0) = 1.3\ m/s$

$$y'(t) = -\frac{13}{2} C_1 \sin \frac{13}{2} t + \frac{13}{2} C_2 \cos \frac{13}{2} t$$

$$y'(0) = -\frac{13}{2} C_1 \sin\left(\frac{13}{2}(0)\right) + \frac{13}{2} C_2 \cos\left(\frac{13}{2}(0)\right)$$

$$\frac{13}{10} = \frac{13}{2} C_2$$

$$C_2 = \frac{1}{5}$$

$$y(t) = \frac{1}{10} \cos \frac{13}{2} t + \frac{1}{5} \sin \frac{13}{2} t$$

$$A = \sqrt{\frac{1}{100} + \frac{1}{25}}$$

$$= \sqrt{\frac{5}{100}}$$

$$= \frac{\sqrt{5}}{10}\ m \qquad \approx 0.2236\ m$$

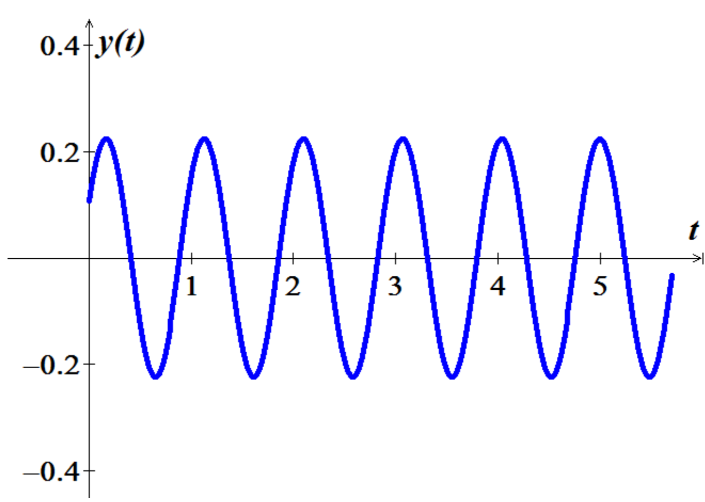
$$\phi = \tan^{-1} \frac{b}{a}$$

$$= \tan^{-1} \frac{.2}{.1}$$

$$= \tan^{-1} 2$$

$$\approx 1.1071$$

$$y(t) = \frac{\sqrt{5}}{10} \cos\left(\frac{13}{2} t - 1.1071\right)$$



2.3-7 Damped Harmonic Motion

In this case, $c > 0$.

$$x'' + 2cx' + \omega_0^2 x = 0$$

The characteristic equation is: $\lambda^2 + 2c\lambda + \omega_0^2 = 0$

The roots are
$$\lambda_{1,2} = \frac{-2c \pm \sqrt{4c^2 - 4\omega_0^2}}{2}$$
$$= -c \pm \sqrt{c^2 - \omega_0^2}$$

There are 3 cases to consider damping and depend on the sign of the discriminant $c^2 - \omega_0^2$

1. $c^2 - \omega_0^2 < 0 \Rightarrow c < \omega_0$. This is the ***underdamped*** case. The roots are distinct complex numbers.

The general solution is:

$$x(t) = e^{-ct}\left(C_1 \cos \omega t + C_2 \sin \omega t\right) \qquad \text{Where } \omega = \sqrt{\omega_0^2 - c^2}$$

2. $c^2 - \omega_0^2 > 0 \Rightarrow c > \omega_0$. This is the ***overdamped*** case. The roots are distinct and real numbers.

The general solution is:

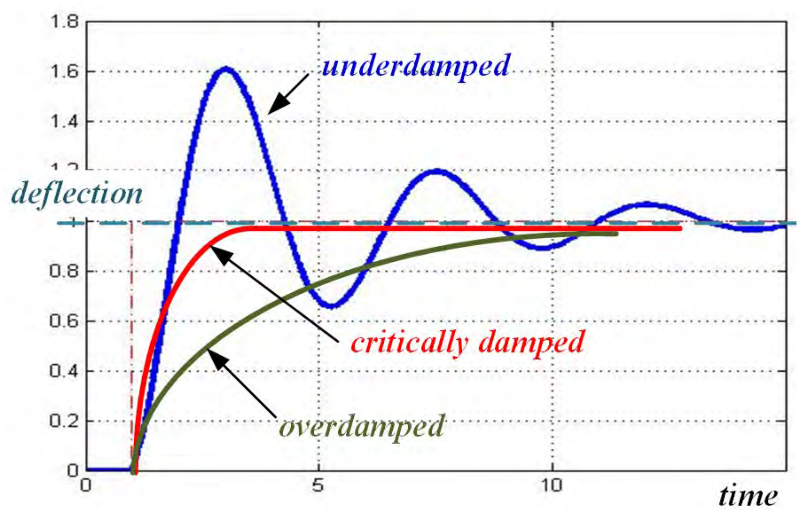
$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

Where $\sqrt{\omega_0^2 - c^2} < \sqrt{c^2} < c \qquad \left(\lambda_1 < \lambda_2 < 0\right)$

3. $c^2 - \omega_0^2 = 0 \Rightarrow c = \omega_0$. This is the ***damped*** case. The root is a double root.

The general solution is:

$$x(t) = C_1 e^{-ct} + C_2 t e^{-ct} \qquad \text{Where } \lambda = -c$$



Example 3

A mass of 4 *kg* is attached to a spring with a spring constant of $k = 169\ kg/s^2$ and damping constant $\mu = 12.8\ kg/s$. With initial values of $y(0) = 0.1\ m$ and $y'(0) = 1.3\ m/s$. Find the frequency, amplitude, and phase of the vibration.

Solution

$$my'' + \mu y' + ky = F(t)$$

$$4y'' + 12.8y' + 169y = 0$$

$$y'' + 3.2y' + 42.25y = 0$$

$$\lambda^2 + 3.2\lambda + 42.25 = 0$$

$$\lambda_{1,2} = \frac{3.2 \pm \sqrt{10.24 - 169}}{2}$$

$$= -1.6 \pm 6.3i$$

The general solution:

$$y(t) = e^{-1.6t}\left(C_1 \cos 6.3t + C_2 \sin 6.3t\right)$$

$$y(0) = e^{-1.6(0)}\left(C_1 \cos 6.3(0) + C_2 \sin 6.3(0)\right)$$

$$0.1 = C_1$$

$$y'(t) = -1.6e^{-1.6t}\left(C_1 \cos 6.3t + C_2 \sin 6.3t\right) + e^{-1.6t}\left(-6.3C_1 \sin 6.3t + 6.3C_2 \cos 6.3t\right)$$

$$y'(0) = -1.6e^{-1.6(0)}\left(C_1 \cos 6.3(0) + C_2 \sin 6.3(0)\right) + e^{-1.6(0)}\left(-6.3C_1 \sin 6.3(0) + 6.3C_2 \cos 6.3(0)\right)$$

$$1.3 = -1.6(0.1 + 0) + (1)\left(-0 + 6.3C_2\right)$$

$$1.3 = -0.16 + 6.3C_2$$

$$6.3C_2 = 1.46$$

$$C_2 \approx 0.2317$$

$$y(t) = e^{-1.6t}\left(0.1\cos 6.3t + 0.2317\sin 6.3t\right)$$

The *natural frequency*:

$$\omega_0 = \sqrt{42.25}$$

$$= 6.5$$

Period:

$$T = \frac{2\pi}{6.5} \qquad\qquad T = \frac{2\pi}{\omega_0}$$

$$\approx 0.9666$$

Frequency:

$$\nu = \frac{1}{.9666} \qquad\qquad \nu = \frac{1}{T}$$

$$\approx 1.03455$$

Amplitude:

$$A = \sqrt{.1^2 + .2317^2}$$

$$\approx 0.2524 \ \ m$$

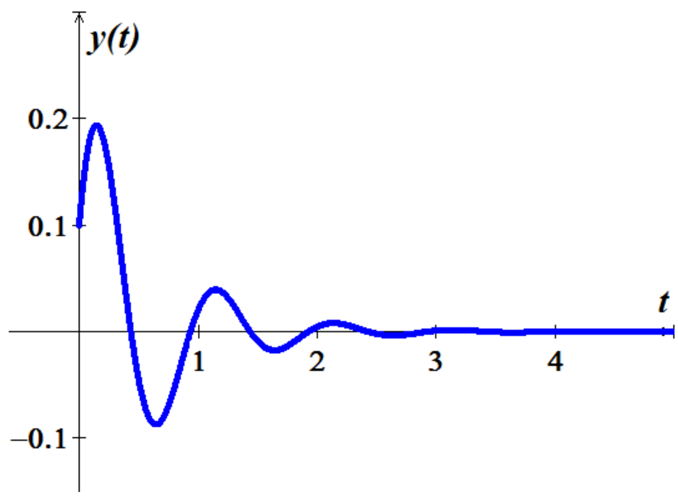
Phase of the vibration:

$$\phi = \tan^{-1}\frac{b}{a}$$

$$= \tan^{-1}\frac{.2317}{.1}$$

$$\approx 1.1634$$

$$y(t) = 0.2524e^{-1.6t}\cos(6.3t - 1.1634)$$



Example 4

A mass of 4 kg is attached to a spring with a spring constant of $k = 169\ kg/s^2$ and damping constant $\mu = 77.6\ kg/s$. With initial values of $y(0) = 0.1\ m$ and $y'(0) = 1.3\ m/s$. Find the general solution.

Solution

$$my'' + \mu y' + ky = F(t)$$

$$4y'' + 77.6y' + 169y = 0$$

$$y'' + 19.4y' + 42.25y = 0$$

$$\lambda^2 + 19.4\lambda + 42.25 = 0$$

$$\underline{\lambda_1 = -16.9,\ \lambda_2 = -2.5}\,\Big|$$

The general solution:

$$y(t) = C_1 e^{-16.9t} + C_2 e^{-2.5t}$$

$$0.1 = C_1 e^{-16.9(0)} + C_2 e^{-2.5(0)}$$

$$0.1 = C_1 + C_2$$

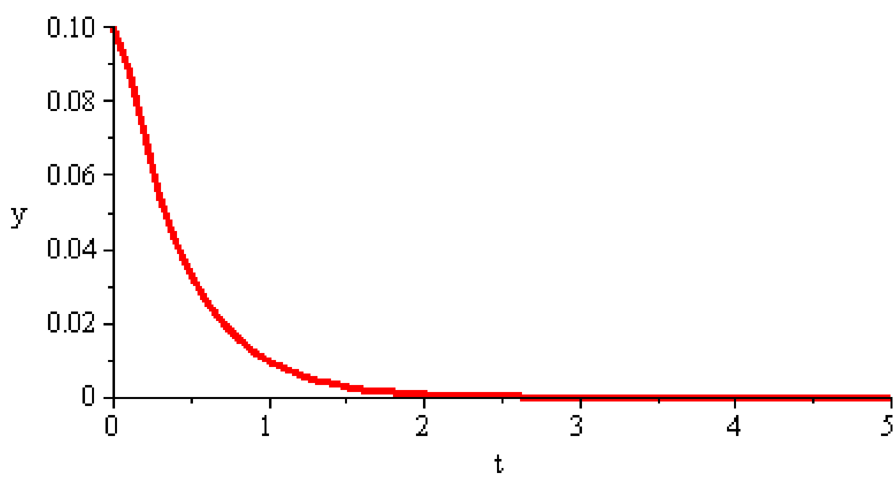
$$y' = -16.9C_1 e^{-16.9t} - 2.5C_2 e^{-2.5t}$$

$$1.3 = -16.9C_1 e^{-16.9(0)} - 2.5C_2 e^{-2.5(0)}$$

$$1.3 = -16.9C_1 - 2.5C_2$$

$$\left.\begin{aligned} 0.1 &= C_1 + C_2 \\ 1.3 &= -16.9C_1 - 2.5C_2 \end{aligned}\right\} \rightarrow C_1 = -\frac{31}{288},\ C_2 = \frac{299}{1440}$$

$$\underline{y(t) = -\frac{31}{288} e^{-16.9t} + \frac{299}{1440} e^{-2.5t}}\,\Big|$$



Example 5

A mass of 4 *kg* is attached to a spring with a spring constant of $k = 169\ kg/s^2$; with initial values of $y(0) = 0.1\ m$ and $y'(0) = 1.3\ m/s$. Find the damping constant μ for which there is critical damping

Solution

Critical damping occurs when $c = \omega_0$

Since $c = \frac{\mu}{2m} = \omega_0$

$$\mu = 2m\omega_0$$

$$= 2m\sqrt{\frac{k}{m}}$$

$$= 2(4)\sqrt{\frac{169}{4}}$$

$$\boxed{= 52\ \ kg/s}$$

$$4y'' + 52y' + 169y = 0$$

$$\lambda^2 + 13\lambda + 42.25 = 0$$

$$y(t) = C_1 e^{-6.5t} + C_2 te^{-6.5t}$$

$$0.1 = C_1$$

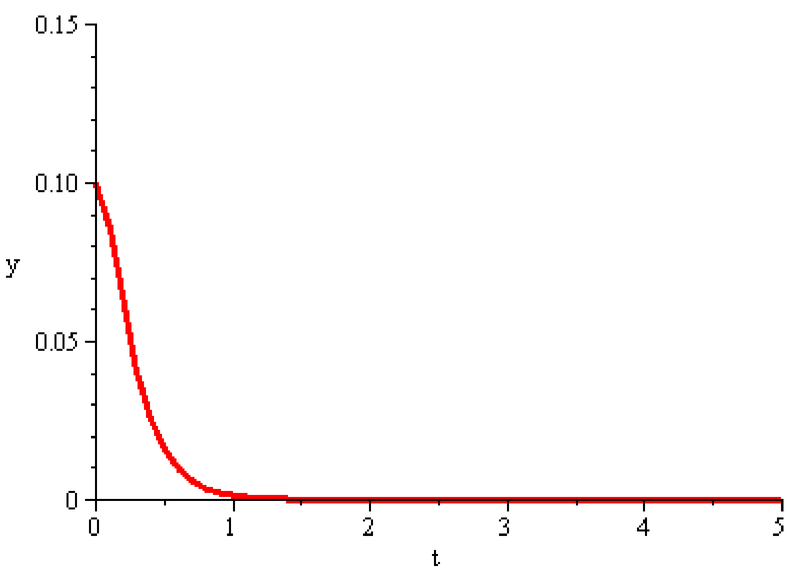
$$y' = -6.5C_1 e^{-6.5t} + C_2 e^{-6.5t} - 6.5C_2 te^{-6.5t}$$

$$1.3 = -6.5C_1 + C_2$$

$$1.3 = -6.5(0.1) + C_2$$

$$C_2 = 1.95$$

$$\boxed{y(t) = 0.1e^{-6.5t} + 1.95te^{-6.5t}}$$



Important facts that the differential equations for electrical and mechanical (Translation and Rotational) are identical in some forms.

TABLE *A*: Relationships between the variables of the analog system components.

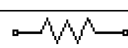
Electrical	***Mechanical Translation***	***Mechanical Rotational***
$i = C\frac{dv}{dt}$ $= Gv$ $= N\phi$ $= r^2\left(1 - r^2\right)$ $= \frac{1}{L}\int vdt$	$f = M\frac{dv}{dt}$ $= Dv$ $= kx$ $= k\int vdt$	$\text{T} = \text{J}$ $= D\omega$ $= k\theta$ $= k\int \omega dt$

Engineers sometimes utilize the similarity by determining the properties of a proposed mechanical system with a simple electrical analog.

TABLE *B*: Analogous between electrical and mechanical systems.

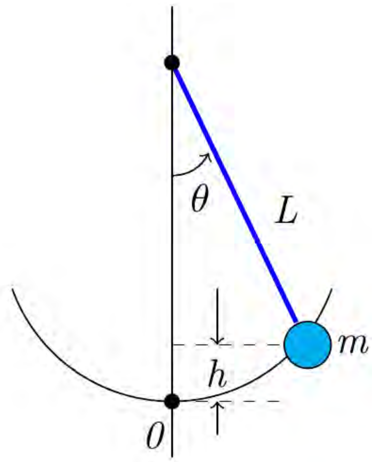
Electrical	***Mechanical Translation***	***Mechanical Rotational***
Current, ***i***	Force, ***f*** *N, lb*	Torque, ***T*** *N-m, lb-ft*
Voltage, ***V***	Velocity, ***v***	Angular velocity, $\boldsymbol{\omega}$
Flux linkages	Displacement	Angular displacement, $N\phi$, *xh* or θ *rad*
Capacitance. ***C***	Mass, **M** *kg, slug*	Moment of inertia, **J** *kg-m*^{2}*, lb-ft/sec*^{2}**.**
Conductance **G = 1/R**	Damping coefficient (of dash pot) **D** or **B** **N/m/sec**, *lb/ft/sec*	Rotational damping Coefficient friction: **D** or **B**
Inductance, ***L***	Compliance $\boldsymbol{\tau} = \frac{1}{k}$ of spring	Torsional compliance $\boldsymbol{\tau} = \frac{1}{k}$ of spring $k \to N \cdot m / rad$

Summary

	Abv.		***Unit***
Capacitor	*C*		Farad (**F**)
Current	*I*		Ampere (**A**)
Electric Charge	*q*		Coulomb (**C**)
Electromotive Force	*emf*		**Emf**
Inductor	*L*		Henry (**H**)
Resistor	*R*		Ohm (**Ω**)
Time	*t*		Second (**s**)
Voltage	*V*		Volt (**V**)

2.3-8 Pendulum

A simple Pendulum consists of a mass m swinging back and forth on the end if a string of length L.



We specify the position of the mass at time t by giving the counterclockwise angle $\theta = \theta(t)$ that the string or rod makes with the vertical at time t. To analyze the motion of the mass m, we apply the law of the conservation of mechanical energy, according to which the sum of the kinetic energy and the potential energy of m remains constant.

The distance along the circular arc from 0 to m is $s = L\theta$, so the velocity of the mass is

$$v = \frac{ds}{dt} = L\frac{d\theta}{dt}$$

Therefore, its kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m\left(\frac{ds}{dt}\right)^2 \\ &= \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 \end{aligned}$$

Then its potential energy V is the product of its weight mg and its vertical height $h = L(1-\cos\theta)$ above O, so

$$\begin{aligned} V &= mgh \\ &= mgL(1-\cos\theta) \end{aligned}$$

The sum of T and V is constant C, therefore

$$\frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 + mgL(1-\cos\theta) = C$$

$$\frac{d}{dt}\left(\frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 + mgL(1-\cos\theta) = C\right)$$

$$mL^2\left(\frac{d\theta}{dt}\right)\frac{d^2\theta}{dt^2} + mgL\sin\theta\frac{d\theta}{dt} = 0$$

$$mL^2 \frac{d^2\theta}{dt^2} + mgL\sin\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

Exercises *Section* 2.3 – Harmonic Motion

(1 – 2)

i. Plot the function

ii. Place the solution in the form $y = A\cos\left(\omega_0 t - \phi\right)$ and compare the graph with the plot in (*i*)

1. $y = \cos 2t + \sin 2t$

2. $y = \cos 4t + \sqrt{3}\sin 4t$

3. A 1-*kg* mass, when attached to a large spring, stretches the spring a distance of 4.9 *m*.

a) Calculate the spring constant.

b) The system is placed in a viscous medium that supplies a damping constant $\mu = 3\ kg/s$. The system is allowed to come to rest. Then the mass is displaced 1 *m* in the downward direction and given a sharp tap, imparting an instantaneous velocity of 1 *m/s* in the downward direction. Find the position of the mass as a function of time and plot the solution.

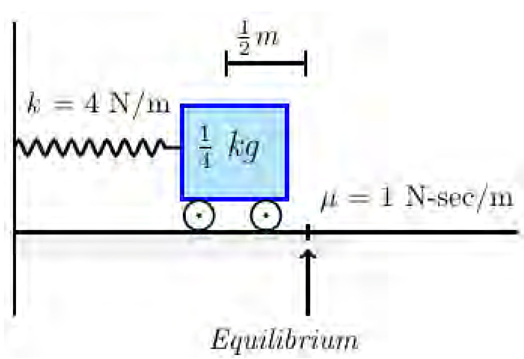
4. The undamped system

$$\frac{2}{5}x'' + kx = 0, \quad x(0) = 2 \quad x'(0) = v_0$$

is observed to have period $\frac{\pi}{2}$ and amplitude 2. Find k and v_0

5. A body with mass *m* = 0.5 *kg* is attached to the end of a spring that is stretched 2 *m* by a force of 100 *N*. It is set in motion with initial position $x_0 = 1m$ and initial velocity $v_0 = -5m/s$. (Note that these initial conditions indicate that he body is displaced to the right and is moving to the left at time *t* = 0.) Find the position function of the body as well as the amplitude, frequency, period of oscillation, and time lag of its motion.

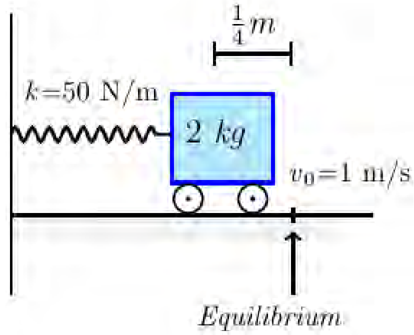
6. A $\frac{1}{4}$-*kg* mass is attached to a spring with a stiffness 4 *N/m*. The damping constant $1\ N\text{-}sec/m$. If the mass is displaced $x_0 = \frac{1}{2}\ m$ to the left and given an initial velocity of 1 *m/s* to the left.



a) Find the equation of motion.

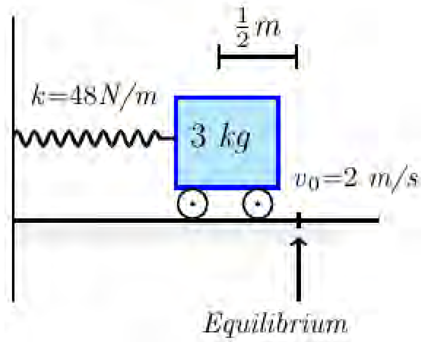
b) What is the maximum displacement that the mass will attain?

7. A 2-*kg* mass is attached to a spring with a stiffness $k = 50\ N/m$. The mass is displaced $\frac{1}{4}\ m$ to the left of the equilibrium point and given a velocity of $1\ m/s$ to the left. Neglecting the damping,



a) Find the equation of motion of the mass along with the amplitude, period, and frequency.
b) How long after release does the mass pass through the equilibrium position?

8. A 3-*kg* mass is attached to a spring with a stiffness $k = 48\ N/m$. The mass is displaced $\frac{1}{2}\ m$ to the left of the equilibrium point and given a velocity of $2\ m/s$ to the left. Neglecting the damping.



c) Find the equation of motion of the mass
d) Find the amplitude, period, and frequency.
e) How long after release does the mass pass through the equilibrium position?

9. A 20-*kg* mass is attached to a spring with a stiffness $k = 200\ N/m$. The damping constant $\mu = 140\ N\text{-}sec/m$. If the mass is pulled $25\ cm$ to the right of the equilibrium point and given an initial velocity of $1\ m/s$. Neglecting the damping,
a) Find the equation of motion.
b) When will it first return to its equilibrium position?

10. A $\frac{1}{4}$-*kg* mass is attached to a spring with a stiffness $k = 8\ N/m$. The damping constant $\mu = \frac{1}{4}\ N\text{-}sec/m$. If the mass is displaced $x_0 = 1\ m$ to the left of equilibrium and released, what is the maximum displacement to the right that the mass will attain?

11. A $\frac{1}{4}$-*kg* mass is attached to a spring with a stiffness $k = 8\ N/m$. The damping constant $\mu = 2\ N\text{-}sec/m$. If the mass is pushed $50\ cm$ to the left of equilibrium and given a leftward velocity of $2\ m/sec$, when will the mass attain its maximum displacement to the left?

12. A 8-*lb* mass weight stretches a spring 2 *feet*. Assuming that a damping force numerically equal to 2 times the instantaneous velocity acts on the system, determine the equation if motion if the mass released from the equilibrium position with an upward velocity of 3 *ft/s*

13. A 8-*lb* mass weight is attached to the end of a spring, causing the spring to stretch a spring 6 *in*. beyond its natural length. The block is then pulled down 3 *in*. and released. Determine the motion of the block, assuming there is no damping or external applied force.

14. A 8-*lb* mass weight is attached to the end of a spring, causing the spring to stretch a spring 6 *in*. beyond its natural length. The block is then pulled down 3 *in*. and released. Determine the motion of the block, assuming there damping is present and that the damping coefficient is $\mu = 1$ *lb-sec/ft* and external applied force.

15. A 16-*lb* mass weight is attached to a 5-*foot* spring. At equilibrium the spring measures 5.2 *feet*. If the mass is initially released from rest at a point $x_0 = 2\ ft$ above the equilibrium position, find the displacements $x(t)$ if it is further known that the surrounding medium offers a resistance numerically equal to the instantaneous velocity.

16. A 16-*lb* mass weight is attached to a spring, stretches $\frac{8}{9}\ ft$ by itself. There is no damping and no external forces acting on the system. The spring is initially displaced 6 *inches* upwards from its equilibrium position and given an initial velocity of 1 *ft/sec* downward. Find the displacement $y(t)$ at any time *t*.

17. A 16-*lb* mass weight is attached to a spring, stretches $\frac{8}{9}\ ft$ by itself. A damper to the mass that will exert of 12 *lbs*. when the velocity is 2 *ft/sec* . The spring is initially displaced 6 *inches* upwards from its equilibrium position and given an initial velocity of 1 *ft/sec* downward. Find the displacement $y(t)$ at any time *t*.

18. A 16-*lb* mass weight is attached to a spring, stretches $\frac{8}{9}\ ft$ by itself. A damper to the mass that will exert of 5 *lbs*. when the velocity is 2 *ft/sec* . The spring is initially displaced 6 *inches* upwards from its equilibrium position and given an initial velocity of 1 *ft/sec* downward. Find the displacement $y(t)$ at any time *t*.

19. A mass weighing 4-*lb* is attached to a spring whose spring constant is 16 *lb/ft*.

a) Find the equation of motion.
b) What is the period of simple harmonic motion?

20. A 20-*kg* mass is attached to a spring. If the frequency of simple harmonic motion is $\frac{2}{\pi}$ *cycles/s* .

a) What is the spring constant *k*?
b) Find the equation of motion.

c) What is the frequency of simple harmonic motion if the original mass is replaced with an 80-*kg* mass.?

21. A 24-*lb* mass weight is attached to the end of a spring, stretches it 4 *inches*. Initially, the mass is released from rest from a point 3 *inches* above the equilibrium position.

a) Find the equation of the motion.

b) If the mass is initially released from the equilibrium position with a downward velocity of 2 *ft/s*

22. The motion of a mass-spring system with damping is given by:

$$y'' + 4y' + ky = 0\ ; \quad y(0) = 1, \quad y'(0) = 0$$

Find the equation of motion and sketch its graph for $k = 2,\ 4,\ 6,\ and\ 8$.

23. A 10-*lb* mass weight is attached to the end of a spring, stretches it 3 *inches*. This mass is removed and replaced with a mass of 1.6 *slugs*, which initially released from a point 4 inches above the equilibrium position with a downward velocity of. $\frac{5}{4}$ *ft/s*

a) Find the equation of the motion.

b) Find the amplitude, phase angle, period and the frequency.

c) Express the motion equation in amplitude and phase angle form.

d) Determine the times the mass attains a displacement below the equilibrium position numerically equal to $\frac{1}{2}$ the amplitude of motion.

24. A 64-*lb* mass weight is attached to the end of a spring, stretches it 0.32 *foot*. This mass is initially released from a point 8 *inches* above the equilibrium position with a downward velocity of 5 *ft/s* .

a) Find the equation of the motion.

b) Find the amplitude, phase angle, period and the frequency.

c) Write the motion equation with phase angle form.

d) How many complete cycles will the mass have completed at the end of 3π *sec* .

e) At what time does the mass pass through the equilibrium position heading downward for the second time?

f) At what times does the mass attain its extreme displacements on either side of the equilibrium position?

g) What is the position of the mass at $t = 3\ sec$?

h) What is the instantaneous velocity at $t = 3\ sec$?

i) What is the acceleration at $t = 3\ sec$?

j) What is the instantaneous velocity at the times when the mass passes through the equilibrium position?

k) At what times is the mass 5 inches below the equilibrium position?

l) At what times is the mass 5 inches below the equilibrium position heading in the upward direction?

(**25 – 31**) If it is underdamped, write the position function in the form $x(t)=C_1e^{-pt}\cos\left(\omega_1 t-\alpha_1\right)$. Also, find the undamped position function $u(t)=C_0\cos\left(\omega_0 t-\alpha_0\right)$ that would result if the mass on the spring were set in motion with the same initial position and velocity, but with the dashpot disconnected (so $c=0$). Then, construct a figure that illustrates the effect of damping by comparing the graphs of $x(t)$ and $u(t)$

25. $m=\frac{1}{2},\quad c=3,\quad k=4;\quad x_0=2,\quad v_0=0$

26. $m=1,\quad c=8,\quad k=16;\quad x_0=5,\quad v_0=-10$

27. $m=1,\quad c=10,\quad k=125;\quad x_0=6,\quad v_0=50$

28. $m=2,\quad c=12,\quad k=50;\quad x_0=0,\quad v_0=-8$

29. $m=2,\quad c=16,\quad k=40;\quad x_0=5,\quad v_0=4$

30. $m=3,\quad c=30,\quad k=63;\quad x_0=2,\quad v_0=2$

31. $m=4,\quad c=20,\quad k=169;\quad x_0=4,\quad v_0=16$

32. Suppose that the mass in a mass–spring–dashpot system with $m=10,\ c=9,\ and\ k=2$ is set in motion with $x(0)=0$ and $x'(0)=5$

a) Find the position function $x(t)$ and graph the function

b) Find how far the mass moves to the right before starting back toward the origin.

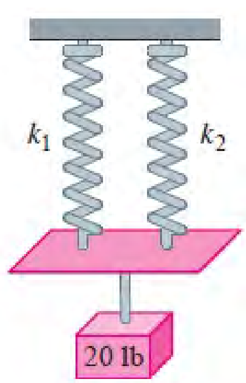
33. Suppose that the mass in a mass–spring–dashpot system with $m=25,\ c=10,\ and\ k=226$ is set in motion with $x(0)=20$ and $x'(0)=41$

a) Find the position function $x(t)$ and graph the function

b) Find the pseudoperiod of the oscillations and the equations of the "envelope curves" that are dashed.

34. A mass of 1 *slug* is suspended from a spring, the spring constant is 9 *lb/ft*. The mass is initially released from a point 1 *foot* above the equilibrium position with an upward velocity of $\sqrt{3}$ *ft/s*. Find the times at which the mass is heading downward at a velocity of 3 *ft/s*

35. Two parallel springs, with constants k_1 and k_2, support a single mass, the effective spring constant of the system is given by $k=\dfrac{4k_1k_2}{k_1+k_2}$.



A mass weight 20 *pounds* stretches one spring 6 *inches* and another spring 2 *inches*. The springs are attached to a common rigid support and then to a metal plate. The mass is attached to the center of the plate in the double-spring constant arrangement.

a) Determine the effective spring constant of this system.
b) Find the equation of motion if the mass is initially released from the equilibrium position with a downward velocity of $2\ ft/s$.

36. A 12–*lb* weight is attached both to a vertically suspended spring that it stretches 6 *in*. and to a dashpot that provides 3 *lb*. of resistance for every foot per second of velocity.
a) If the weight is pulled down 1 *foot*. below its static equilibrium position and then released from rest at time $t=0$, find its position function $x(t)$.
b) Find the frequency, time-varying amplitude, and phase angle of the motion.

37. A $\frac{1}{8}$-kg mass is attached to a spring with a spring constant $k = 16\ N/m$. The mass is displaced $\frac{1}{2}\ m$ to the right of the equilibrium point and given an outward velocity (to the right) of $\sqrt{2}\ m/sec$. Neglecting any damping or external forces that may be present,
a) Determine the equation of motion of the mass
b) Determine the equation of motion amplitude, period, and natural frequency.
c) How long after release does the mass pass through the equilibrium position?

38. A 3-kg mass is attached to a spring with a spring constant $75\ N/m$. The mass is displaced $\frac{1}{4}\ m$ to the left and given a velocity of $1\ m/sec$ to the right. The damping force is negligible.
a) Determine the equation of motion of the mass
b) Determine the equation of motion amplitude, period, and natural frequency.
c) How long after release does the mass pass through the equilibrium position?

39. A 3-kg mass is attached to a spring with a spring constant $300\ N/m$. The mass is pulled down 10 cm and released with downward velocity of $1\ m/sec$. The damping force is negligible.
a) Determine the equation of motion of the mass
b) Solve the equation to find the time when the maximum downward displacement of the mass from its equilibrium position is first achieved.
c) What is the maximum downward displacement?

40. A 10-kg mass is attached to the end of a spring hanging vertically, stretches the spring 0.03 m. The mass is pulled down another 7 cm and released (with no initial velocity).
a) Determine the spring constant *k*.
b) Determine the equation of motion of the mass

41. A 10-kg mass is attached to a spring with spring constant $k = 300\ N/m$. At time $t=0$, the mass is pulled down another 10 cm and released with a downward velocity of $100\ cm/sec$.
a) Determine the equation of motion.
b) What is the maximum downward displacement?

42. A 10-*kg* mass is attached to the end of a spring hanging vertically at rest. The mass is pulled down another 7 *cm* and released (with no initial velocity).

a) Determine the spring constant *k*.

b) Determine the equation of motion of the mass

43. A 10-*kg* mass is attached to the end of a spring hanging vertically, stretches the spring 0.7 *m*. The mass is started in motion from the equilibrium position with an initial velocity 1 *m/sec* in the upward direction. IF the force due to air resistance is $-90y'$ *N*

c) Determine the spring constant *k*.

d) Determine the equation of motion of the mass

44. A $\frac{1}{4}$-*slug* mass is attached to the end of a spring hanging vertically, stretches the spring 1.28 *ft*. The mass is started in motion from the equilibrium position with an initial velocity 4 *ft/sec* in the downward direction. If the force due to air resistance is $-2y'$ *lb*

a) Determine the spring constant *k*.

b) Determine the equation of motion of the mass

45. A 20-*kg* mass is attached to the end of a spring hanging vertically at rest. When given an initial downward velocity of 2 *m/s* from its equilibrium position the mass was observed to attain a maximum displacement of 0.2 *m* from its equilibrium position.

a) Determine the spring constant *k*.

b) Determine the equation of motion of the mass

46. A steel ball weighing 128-*lb* is attached to the end of a spring, stretches 2 *ft* from its natural length. The ball is started in motion with no initial velocity by displacing it 6 *in* above the equilibrium position. Assuming no air resistance.

a) Determine the spring constant *k*.

c) Find the equation of the ball position at time *t*.

d) Find the position of the ball at $t = \frac{\pi}{12}$ *sec*

47. A 9-*lb* mass is attached to the end of a spring hanging vertically with spring constant $k = 32$ *lb/ft*, is perturbed from its equilibrium position with a certain upward initial velocity. The amplitude of the resulting vibrations is observed to be 4 *in*.

a) Determine the equation of motion.

b) What is the initial velocity?

c) Determine the period and frequency of the vibrations?

48. A 2-*kg* mass is suspended from a spring with a spring constant of 10 *N/m*. The mass is started in motion from the equilibrium position with an initial velocity 1.5 *m/sec*. Assuming no air resistance

a) Determine the equation of motion of the mass.

b) Determine the circular frequency, natural frequency, and period.

49. A $\frac{1}{4}$-*slug* mass is attached to a spring having a spring constant of 1 *lb/ft*. The mass is started in motion initially displacing it 2 *ft* in the downward direction with an initial velocity 2 *ft/sec* in the upward direction. If the force due to air resistance is $-1x'$ *lb*. Find the subsequent motion of the mass

50. A spring with a mass of 2-*kg* has natural length 0.5 *m*. A force of 25.6 *N* is required to maintain it stretched to a length of 0.7 *m*. If the spring is stretched to a length of 0.7 *m* and then released with initial velocity zero. Find the position of the mass at any time *t*.

51. A spring with a mass of 2-*kg* has natural length 0.5 *m*. A force of 25.6 *N* is required to maintain it stretched to a length of 0.7 *m*. The spring is immersed in a fluid with damping constant $c = 40$. If the spring is started from the equilibrium position and is given a push to start it with initial velocity 0.6 *m/s*. Find the position of the mass at any time *t*.

52. A spring with a mass of 3-*kg* is held stretched 0.6 *m* beyond its natural length by a force of 20 *N*. If the spring begins at its equilibrium and with initial velocity 1.2 *m/s*. Find the position of the mass at any time *t*.

53. A spring with a mass of 2-*kg* is held stretched 0.5 *m*, has damping constant 14, and a force of 6 *N*. If the spring is stretched 1 *m* beyond at its equilibrium and with no initial velocity.

a) Find the position of the mass at any time *t*.

b) Find the mass that would produce critical damping.

54. A spring has a mass of 1-*kg* and its spring constant $k = 100$. The spring is released at a point 0.1 *m* above its equilibrium position. Graph the position function for the following values of damping constant *c*: 10, 15, 20, 25, 30. What type of damping occurs each case

55. A 4-*kg* mass is attached to a spring and set in motion. A record of the displacements was made and found to be described by $y(t) = 25\cos\left(2t - \frac{\pi}{6}\right)$, with displacement measured in centimeters and time in seconds.

a) Determine the displacement y_0.

b) Determine the initial velocity y'_0?

c) Determine the spring constant *k*.

d) Determine the period and frequency of the vibrations?

56. A 10-*kg* mass is attached to a spring with a spring constant $k = 100$ *N/m*; the dashpot has damping constant 7 *kg/sec*. At time $t = 0$, the system is set into motion by pulling the mass down 0.5 *m* from its equilibrium rest position while simultaneously giving it an initial downward velocity of 1 *m/s*

a) Solve the equation of motion.

b) What is the $\lim\limits_{t\to\infty} y(t)$

c) Plot the solution.

d) How long it takes for the magnitude of the vibrations to be reduced to 0.1 *m*. (Estimate the smallest time, τ, for which $|y(t)| \le 0.1\ m,\quad \tau \le t < \infty$)

57. A spring and dashpot system is to be designed for a 32-*lb* weight so that the overall system is critically damped

a) How must the damping constant c and the spring constant k be related?

b) Assume the system is to be designed so that the mass, when given initial velocity of 4 *ft/sec* from its rest position, will have a maximum displacement of 6 *in*. What values of damping constant c and spring constant k are required?

c) It is observed that the time interval between successive zero crossing is 20% larger for the damped vibration displacement than for the undamped vibration displacement. What is the damping constant c? (Spring constant k remains same from part (*b*)).

58. Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit when $L = 0.25\ H$, $R = 10\ \Omega$, $C = 0.001\ F$, $E(t) = 0$, $q(0) = q_0\ C$, and $i(0) = 0$.

59. Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit at $t = 0.01\ sec$ when $L = 0.05\ h$, $R = 2\ \Omega$, $C = 0.01\ f$, $E(t) = 0$, $q(0) = 5\ C$, and $i(0) = 0\ A$. Determine the first time at which the charge on the capacitor is equal to zero.

60. Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit when $L = 0.25\ h$, $R = 20\ \Omega$, $C = \frac{1}{300}\ f$, $E(t) = 0$, $q(0) = 4\ C$, and $i(0) = 0\ A$. Is the charge on the capacitor ever equal to zero.

61. Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit when $L = \frac{5}{3}\ h$, $R = 10\ \Omega$, $C = \frac{1}{30}\ f$, $E(t) = 0$, $q(0) = 0\ C$, and $i(0) = 0\ A$.

Section 2.4 - Inhomogeneous Equations; the Method of Undetermined Coefficients

The method of undetermined coefficients is a technique for determining the particular solution to linear constant-coefficient differential equations.

The second order ***nonhomogeneous*** equation is given by: $y'' + p(x)y' + q(x)y = f(x) \quad (N)$

The corresponding ***homogeneous*** equation: $y'' + p(x)y' + q(x)y = 0 \quad (H)$

2.4-1 *Theorem*

Suppose that y_p is a particular solution to the nonhomogeneous (or inhomogeneous) equation $y'' + py' + qy = f$ and that y_1 and y_2 form a fundamental set of solutions to the homogeneous equation $y'' + py' + qy = 0.$ Then the general solution to the inhomogeneous equation is given by

$$y = y_p + C_1 y_1 + C_2 y_2$$

C_1 *and* C_2 are arbitrary constants.

2.4-2 *Theorem*

Let $y = y_1(x)$ *and* $y = y_2(x)$ be ***linearly independent*** $(W(x) \neq 0)$ solutions of the reduced equation (*H*) and let $y_p(x)$ be a ***particular solution*** of (*N*). Then the general solution of (*N*) consists of the general solution of the reduced equation (*H*) ***plus*** a particular solution of (*N*):

$$y(x) = \underbrace{y_p(x)}_{\text{a Particular Solution}} + \underbrace{C_1 y_1(x) + C_2 y_2(x)}_{\text{General Solution}}$$

2.4-3 *Forcing Term*

If the forcing term f has a form that is replicated under differentiation, then look for a solution with the same general form as the forcing term.

Example 1

Find a particular solution to the equation $y'' - y' - 2y = 2e^{-2t}$

Solution

The forcing term $f(t) = 2e^{-2t} \Rightarrow$ the particular solution $y = ae^{-2t}$

$$y' = -2ae^{-2t}$$
$$y'' = 4ae^{-2t}$$

$$4ae^{-2t} + 2ae^{-2t} - 2ae^{-2t} = 2e^{-2t}$$
$$4ae^{-2t} = 2e^{-2t}$$
$$4a = 2$$
$$a = \frac{1}{2}$$
$$y_p(t) = \frac{1}{2}e^{-2t}$$

2.4-4 Trigonometric Forcing Term

$f(t) = A\cos\omega t + B\sin\omega t$

The general solution: $y(t) = a\cos\omega t + b\sin\omega t$

Example 2

Find a particular solution to the equation $y'' + 2y' - 3y = 5\sin 3t$

Solution

The particular solution: $y(t) = a\cos 3t + b\sin 3t$

$$y' = -3a\sin 3t + 3b\cos 3t$$
$$y'' = -9a\cos 3t - 9b\sin 3t$$

$$\begin{aligned} y'' + 2y' - 3y &= -9a\cos 3t - 9b\sin 3t + 2(-3a\sin 3t + 3b\cos 3t) - 3(a\cos 3t + b\sin 3t) \\ &= -9a\cos 3t - 9b\sin 3t - 6a\sin 3t + 6b\cos 3t - 3a\cos 3t - 3b\sin 3t \\ &= (-12a + 6b)\cos 3t - (6a + 12b)\sin 3t \\ &= 5\sin 3t \end{aligned}$$

$$\begin{cases} -12a + 6b = 0 \\ -(6a + 12b) = 5 \end{cases}$$

$$\begin{cases} -2a + b = 0 \\ -6a - 12b = 5 \end{cases}$$

$$a = \frac{\begin{vmatrix} 0 & 1 \\ 5 & 12 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ -6 & -12 \end{vmatrix}} = \frac{-5}{30} = -\frac{1}{6} \qquad b = \frac{\begin{vmatrix} -2 & 0 \\ -6 & 5 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ -6 & -12 \end{vmatrix}} = \frac{-10}{30} = -\frac{1}{3}$$

$$y_p(t) = -\frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t$$

2.4-5 Complex Method

Example 3

Find a particular solution to the equation $y''+2y'-3y=5\sin 3t$

Solution

$$5e^{3it}=5\cos 3t+5i\sin 3t=5cis3t$$

$$z''+2z'-3z=5e^{3it}$$

The particular solution: $z(t)=x(t)+i\,y(t)$

$$\begin{aligned} z''+2z'-3z &= (x+iy)''+2(x+iy)'-3(x+iy) \\ &= (x''+2x'-3x)+i(y''+2y'-3y) \\ &= 5\cos 3t+i\ 5\sin 3t \end{aligned}$$

$$x''+2x'-3x=5\cos 3t$$

$$z(t)=ae^{3it}$$

$$z'=3iae^{3it}$$

$$z''=9i^2ae^{3it}=-9ae^{3it}$$

$$\begin{aligned} z''+2z'-3z &= -9ae^{3it}+2(3i)ae^{3it}-3ae^{3it} \\ &= -12ae^{3it}+6iae^{3it} \\ &= -6(2-i)ae^{3it} \\ &= 5e^{3it} \end{aligned}$$

$$-6(2-i)a=5$$

$$\begin{aligned} a &= -\frac{5}{6(2-i)}\frac{2+i}{2+i} \\ &= -\frac{5(2+i)}{6(4+1)} \\ &= -\frac{2+i}{6} \end{aligned}$$

$$\begin{aligned} z(t) &= -\tfrac{1}{6}(2+i)e^{3it} \\ &= -\tfrac{1}{6}(2+i)(\cos 3t+i\sin 3t) \\ &= -\tfrac{1}{6}\left[(2\cos 3t-\sin 3t)+i(\cos 3t+2\sin 3t)\right] \end{aligned}$$

$$\boxed{y_p(t)=-\tfrac{1}{6}(\cos 3t+2\sin 3t)}$$

2.4-6 Polynomial Forcing Term

If the nonhomogeneous term is a polynomial of degree n, then an *initial* guess for the particular solution should be a polynomial of degree n:

$$f(t) = a_0 t^n + a_1 t^{n-1} + \ldots + a_{n-1} t + a_n$$

Example 4

Find a particular solution to the equation $y'' + 2y' - 3y = 3t + 4$

Solution

The right-hand side is a polynomial of degree 1.

The particular solution: $y(t) = at + b$

$$y' = a$$
$$y'' = 0$$

$$\begin{aligned} y'' + 2y' - 3y &= 0 + 2a - 3(at + b) \\ &= 2a - 3b - 3at \\ &= 3t + 4 \end{aligned}$$

$$\rightarrow \begin{cases} -3a = 3 \\ 2a - 3b = 4 \end{cases} \quad \Rightarrow a = -1; \quad b = -2$$

$$\boxed{y_p(t) = -t - 2}$$

2.4-7 Exceptional Cases

Example 5

Find a particular solution to the equation $y'' - y' - 2y = 3e^{-t}$

Solution

The particular solution $y = ae^{-t}$

$$y'' - y' - 2y = ae^{-t} + ae^{-t} - 2ae^{-t} = 0$$

The particular solution $y = ate^{-t}$ or $y = at^2 e^{-t}$

$$y' = ae^{-t} - ate^{-t} = ae^{-t}(1 - t)$$

$$y'' = -ae^{-t} - ae^{-t} + ate^{-t} = ate^{-t} - 2ae^{-t}$$

$$\begin{aligned} y'' - y' - 2y &= ate^{-t} - 2ae^{-t} - ae^{-t} + ate^{-t} - 2ate^{-t} \\ &= -3ae^{-t} \end{aligned}$$

$-3ae^{-t} = 3e^{-t}$

$\boxed{a = -1}$

The particular solution $\boxed{y_p = -te^{-t}}$

2.4-8 *Summary*

$f(t)$	y_p
Any Constant	A
$at+b$	$At+B$
at^2+c	At^2+Bt+C
$at^3+\cdots+b$	At^3+Bt^2+Ct+E
$\sin at$ or $\cos at$	$A\cos at+B\sin at$
e^{at}	Ae^{at}
$(at+b)e^{at}$	$(At+B)e^{at}$
t^2e^{at}	$\left(At^2+Bt+C\right)e^{at}$
$e^{at}\sin bt$	$(A\cos bt+B\sin bt)e^{at}$
$t^2\sin bt$	$\left(At^2+Bt+C\right)\cos bt+\left(Et^2+Ft+G\right)\sin bt$
$te^{at}\cos bt$	$((At+B)\cos bt+(Ct+E)\sin bt)\ e^{at}$

- Check the homogeneous solution first, then start with next power for the particular solution.

  If we use the same power, we can add the constants which result with another constant. Therefore, we do not need to use the same power or the same terms as the homogeneous solution.

Examples 6

$y''-y'-12y=e^{4x}$	$y_h=C_1e^{-3x}+C_2e^{4x}$	$y_p=Axe^{4x}$
$y''-2y'=12x-10$	$y_h=C_1+C_2e^{2x}$	$y_p=Ax^2+Bx$
$y''+2y'+y=x^2e^{-x}$	$y_h=\left(C_1+C_2x\right)e^{-x}$	$y_p=\left(Ax^4+Bx^3+Cx^2\right)e^{-x}$
$y''+6y'+13y=e^{-3x}\cos 2x$	$y_h=e^{-3x}\left(C_1\cos 2x+C_2\sin 2x\right)$	$y_p=e^{-3x}\left(Ax\cos 2x+Bx\sin 2x\right)$

Example 7

Find the general solution to the equation $2y''-5y'+3y=4e^{3t}$, given initial conditions $y(0)=1,\quad y'(0)=0$

Solution

The Characteristic Equation: $2\lambda^2-5\lambda+3=0$

$$\boxed{\lambda_{1,2}=1,\ \frac{3}{2}}$$

The homogeneous solution: $\boxed{y_h=C_1e^t+C_2e^{\frac{3}{2}t}}$

The particular solution $y_p=ae^{3t}$

$$y'=3ae^{3t}$$
$$y''=9ae^{3t}$$

$$18ae^{3t}-15ae^{3t}+3ae^{3t}=4e^{3t}$$
$$6a=4$$
$$a=\frac{2}{3}$$
$$\boxed{y_p(t)=\frac{2}{3}e^{3t}}$$

$$y(t)=C_1e^t+C_2e^{\frac{3}{2}t}+\frac{2}{3}e^{3t}$$

$$y(0)=1$$
$$1=C_1+C_2+\frac{2}{3}\quad\rightarrow\quad C_1+C_2=\frac{1}{3}$$

$$y'(t)=C_1e^t+\frac{3}{2}C_2e^{\frac{3}{2}t}+2e^{3t}$$

$$y'(0)=0$$
$$0=C_1+\frac{3}{2}C_2+2\quad\rightarrow\quad 2C_1+3C_2=-4$$

$$\begin{cases}3C_1+3C_2=1\\-2C_1-3C_2=4\end{cases}$$
$$\boxed{C_1=5}$$

$$\Rightarrow\quad\boxed{C_2=-\frac{14}{3}}$$

$$\boxed{y(t)=5e^t-\frac{14}{3}e^{\frac{3}{2}t}+\frac{2}{3}e^{3t}}$$

Exercises Section 2.4 - Inhomogeneous Equations; the Method of Undetermined Coefficients

1. Show that the 3 solutions $y_1 = x, \quad y_2 = x\ln x, \quad y_3 = x^2$ of the 3rd order equation $x^3y''' - x^2y'' + 2xy' - 2y = 0$ are linearly independent on an open interval $x > 0$. Then find a particular solution that satisfies the initial conditions $y(1) = 3, \quad y'(1) = 2, \quad y''(1) = 1$

(**2 – 14**) Find the particular solution for the given differential equation

2. $y'' + 3y' + 2y = 4e^{-3t}$

3. $y'' + 6y' + 8y = -3e^{-t}$

4. $y'' + 2y' + 5y = 12e^{-t}$

5. $y'' + 3y' - 18y = 18e^{2t}$

6. $y'' + 4y = \cos 3t$

7. $y'' + 7y' + 6y = 3\sin 2t$

8. $y'' + 5y' + 4y = 2 + 3t$

9. $y'' + 6y' + 8y = 2t - 3$

10. $y'' + 3y' + 4y = t^3$

11. $y'' + 2y' + 2y = 2 + \cos 2t$

12. $y'' - y = t - e^{-t}$

13. $y'' - 2y' + y = 10e^{-2t}\cos t$

14. $y''' - 4y'' + 4y' = 5t^2 - 6t + 4t^2e^t + 3e^{5t}$

(**15 – 16**) Use the ***complex method*** to find the particular solution for

15. $y'' + 4y' + 3y = \cos 2t + 3\sin 2t$

16. $y'' + 4y = \cos 3t$

(**17 – 124**) Find the general solution for the given differential equation

17. $y'' + y = 2\cos x$

18. $y'' + y = \cos 3x$

19. $y'' + y = 2x\sin x$

20. $y'' - y = x^2e^x + 5$

21. $y'' - y' = -3$

22. $y'' - y' = 2\sin x$

23. $y'' - y' = \sin x$

24. $y'' - y' = -8x + 3$

25. $y'' + y = 2x + 3e^x$

26. $y'' - y = x^2 + e^x$

27. $y'' + y' = 10x^4 + 2$

28. $y'' - y' = 5e^x - \sin 2x$

29. $y'' + y = x\cos x - \cos x$

30. $y'' + y = e^x \sin x$

31. $y'' - y' - 2y = 20\cos x$

32. $y'' - y' + \frac{1}{4}y = 3 + e^{x/2}$

33. $y'' + y' + \frac{1}{4}y = e^x(\sin 3x - \cos 3x)$

34. $y'' - y' - 2y = e^{3x}$

35. $y'' - y' - 6y = 20e^{-2x}$

36. $y'' + y' - 6y = 2x$

37. $y'' - y' - 6y = e^{-x} - 7\cos x$

38. $y'' + y' + 8y = x\cos 3x + (10x^2 + 21x + 9)\sin 3x$

39. $y'' - y' - 12y = e^{4x}$

40. $y'' + 2y' = 2x + 5 - e^{-2x}$

41. $y'' - 2y' = 12x - 10$

42. $y'' + 2y' + y = \sin x + 3\cos 2x$

43. $y'' - 2y' + y = 6e^{x}$

44. $y'' + 2y' + y = x^{2}$

45. $y'' + 2y' + y = x^{2}e^{-x}$

46. $y'' - 2y' + y = x^{3} + 4x$

47. $y'' + 2y' + y = 6\sin 2x$

48. $y'' - 2y' + y = \left(x^{2} - 1\right)e^{2x} + (3x + 4)e^{x}$

49. $y'' + 2y' + 2y = 5e^{6x}$

50. $y'' + 2y' + 2y = x^{3}$

51. $y'' + 2y' + 2y = \cos x + e^{-x}$

52. $y'' - 2y' + 2y = e^{x}\sin x$

53. $y'' - 2y' + 2y = e^{2x}(\cos x - 3\sin x)$

54. $y'' - 2y' - 3y = 1 - x^{2}$

55. $y'' - 2y' - 3y = 4e^{x} - 9$

56. $y'' - 2y' - 3y = 2e^{-x}\cos x + x^{2} + xe^{3x}$

57. $y'' - 2y' + 5y = 25x^{2} + 12$

58. $y'' - 2y' + 5y = e^{x}\cos 2x$

59. $y'' - 2y' + 5y = e^{x}\sin x$

60. $y'' + 2y' - 24y = 16 - (x + 2)e^{4x}$

61. $y'' + 3y = -48x^{2}e^{3x}$

62. $y'' - 3y' = e^{3x} - 12x$

63. $y'' + 3y' = 4x - 5$

64. $y'' - 3y' = 8e^{3x} + 4\sin x$

65. $y'' + 3y' + 2y = 6$

66. $y'' + 3y' + 2y = 4x^{2}$

67. $y'' - 3y' + 2y = 5e^{x}$

68. $y'' - 3y' + 2y = 2x^{2} + e^{x} + 2xe^{x} + 4e^{3x}$

69. $y'' - 3y' + 2y = 14\sin 2x - 18\cos 2x$

70. $y'' + 3y' + 2y = e^{-x} + e^{-2x} - x$

71. $y''' - 3y'' + 3y' - y = 3e^{x}$

72. $y'' - 3y' - 10y = -3$

73. $y'' - 3y' - 10y = 2x - 3$

74. $y'' + 3y' - 10y = 6e^{4x}$

75. $y'' + 3y' - 10y = x\left(e^{x} + 1\right)$

76. $y'' - 4y = 4x^{2}$

77. $y'' + 4y = 3x^{3}$

78. $y'' + 4y = 3\sin x$

79. $y'' + 4y = 3\sin 2x$

80. $y'' + 4y = 4\cos x + 3\sin x - 8$

81. $y'' - 4y = \left(x^{2} - 3\right)\sin 2x$

82. $y'' + 4y' + 4y = 2x + 6$

83. $y'' + 4y' + 5y = 5x + e^{-x}$

84. $y'' + 4y' + 5y = 2e^{-2x} + \cos x$

85. $y'' + 5y' = 15x^{2}$

86. $y'' - 5y' = 2x^{3} - 4x^{2} - x + 6$

87. $y'' + 6y' + 8y = 3e^{-2x} + 2x$

88. $y'' - 6y' + 9y = e^{3x}$

89. $y'' + 6y' + 9y = -xe^{4x}$

90. $y'' + 6y' + 13y = e^{-3x}\cos 2x$

91. $y'' - 7y' = -3$

92. $y'' + 7y' = 42x^{2} + 5x + 1$

93. $y'' + 8y = 5x + 2e^{-x}$

94. $y'' - 8y' + 20y = 100x^{2} - 26xe^{x}$

95. $y'' - 9y = 54$

96. $y'' + 9y = x^{2}\cos 3x + 4\sin x$

97. $y'' + 10y' + 25y = 14e^{-5x}$

98. $y'' - 10y' + 25y = 30x + 3$

99. $y'' - 16y = 2e^{4x}$

100. $y'' + 25y = 6\sin x$

101. $y'' + 25y = 20\sin 5x$

102. $\frac{1}{4}y'' + y' + y = x^{2} - 2x$

103. $2y'' - 5y' + 2y = -6e^{x/2}$

104. $2y'' - 7y' + 5y = -29$

105. $4y'' + 9y = 15$

106. $4y'' - 4y' - 3y = \cos 2x$

107. $9y'' - 6y' + y = 9xe^{x/3}$

108. $y^{(3)} + y'' = 8x^2$

109. $y^{(3)} - y'' - 4y' + 4y = 5 - e^x + e^{2x}$

110. $y^{(3)} + y'' = 3e^x + 4x^2$

111. $y^{(3)} + 2y'' + y' = 10$

112. $y^{(3)} - 2y'' - 4y' + 8y = 6xe^{2x}$

113. $y^{(3)} - 3y'' + 3y' - y = x - 4e^x$

114. $y^{(3)} - 4y'' + y' + 6y = 4\sin 2x$

115. $y^{(3)} - 3y'' + 3y' - y = e^x - x + 16$

116. $y^{(3)} - 6y'' = 3 - \cos x$

117. $y^{(3)} - 6y'' + 11y' - 6y = 2xe^{-x}$

118. $y^{(3)} + 8y'' = -6x^2 + 9x + 2$

119. $y^{(4)} + y'' = 3x^2 + 4\sin x - 2\cos x$

120. $y^{(4)} + 2y'' + y = (x-2)^2$

121. $y^{(4)} - y'' = 4x + 2xe^{-x}$

122. $\left(D^2 + D - 2\right)y = 2x - 40\cos 2x$

123. $\left(D^2 - 3D + 2\right)y = 2\sin x$

124. $(D-2)^3\left(D^2 + 9\right)y = x^2 e^{2x} + x\sin 3x$

(125 – 174) Find the general solution that satisfy the given initial conditions

125. $y'' + y = \cos x;\quad y(0) = 1,\quad y'(0) = -1$

126. $y'' + y' = x\ ;\quad y(1) = 0,\quad y'(1) = 1$

127. $y'' + y' = -x\ ;\quad y(0) = 1,\quad y'(0) = 0$

128. $y'' + y = 8\cos 2t - 4\sin t\quad y\left(\frac{\pi}{2}\right) = -1,\quad y'\left(\frac{\pi}{2}\right) = 0$

129. $y'' - y' - 2y = 4x^2;\quad y(0) = 1,\quad y'(0) = 4$

130. $y'' - y' - 2y = e^{3x};\quad y(1) = 2,\quad y'(1) = 1$

131. $y'' - y' - 2y = e^{3x};\quad y(0) = 1,\quad y'(0) = 2$

132. $y'' - y' - 2y = e^{3x};\quad y(0) = 2,\quad y'(0) = 1$

133. $y'' + 2y' + y = 2\cos t\ ;\quad y(0) = 3,\quad y'(0) = 0$

134. $y'' - 2y' + y = t^3;\quad y(0) = 1,\quad y'(0) = 0$

135. $y'' - 2y' + y = -3 - x + x^2\ ;\quad y(0) = -2,\quad y'(0) = 1$

136. $y'' - 2y' + 2y = x + 1;\quad y(0) = 3,\quad y'(0) = 0$

137. $y'' + 2y' + 2y = \sin 3x;\quad y(0) = 2,\quad y'(0) = 0$

138. $y'' + 2y' + 2y = 2\cos 2t;\quad y(0) = -2,\quad y'(0) = 0$

139. $y'' - 2y' - 3y = 2e^x - 10\sin x\ ;\quad y(0) = 2,\quad y'(0) = 4$

140. $y'' + 2y' + 10y = 4 + 26x + 6x^2 + 10x^3\ ;\quad y(0) = 2,\quad y'(0) = 9$

141. $y'' - 2y' + 10y = 6\cos 3t - \sin 3t$; $y(0) = 2, \quad y'(0) = -8$

142. $y'' + 3y' + 2y = e^{x}$; $y(0) = 0, \quad y'(0) = 3$

143. $y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x \quad y(0) = 1 \quad y'(0) = 2$

144. $y'' + 4y = -2$; $y\left(\frac{\pi}{8}\right) = \frac{1}{2}, \quad y'\left(\frac{\pi}{8}\right) = 2$

145. $y'' + 4y = 2x$; $y(0) = 1, \quad y'(0) = 2$

146. $y'' + 4y = \sin^2 2t$; $x\left(\frac{\pi}{8}\right) = 0 \quad x'\left(\frac{\pi}{8}\right) = 0$

147. $y'' - 4y' + 8y = x^3$; $y(0) = 2, \quad y'(0) = 4$

148. $y'' + 4y' + 4y = (3 + x)e^{-2x}$; $y(0) = 2, \quad y'(0) = 5$

149. $y'' + 4y' + 4y = 4 - t$; $y(0) = -1, \quad y'(0) = 0$

150. $y'' - 4y' + 4y = e^{x}$; $y(0) = 2, \quad y'(0) = 0$

151. $y'' - 4y' - 5y = 4e^{-2t}$; $y(0) = 0, \quad y'(0) = -1$

152. $y'' + 4y' + 5y = 35e^{-4x}$; $y(0) = -3, \quad y'(0) = 1$

153. $y'' + 4y' + 8y = \sin t$; $y(0) = 1, \quad y'(0) = 0$

154. $y'' - 4y' - 12y = 3e^{5t}$; $y(0) = \frac{18}{7}, \quad y'(0) = -\frac{1}{7}$

155. $y'' - 4y' - 12y = \sin 2t$; $y(0) = 0, \quad y'(0) = 0$

156. $y'' - 5y' = t - 2$; $y(0) = 0, \quad y'(0) = 2$

157. $y'' + 5y' - 6y = 10e^{2x}$; $y(0) = 1, \quad y'(0) = 1$

158. $y'' + 6y' + 10y = 22 + 20x$; $y(0) = 2, \quad y'(0) = -2$

159. $y'' + 7y' + 12y = -2\cos 2x + 36\sin 2x$; $y(0) = -3, \quad y'(0) = 3$

160. $y'' + 8y' + 7y = 10e^{-2x}$; $y(0) = -2, \quad y'(0) = 10$

161. $y'' + 9y = \sin 2x$; $y(0) = 1, \quad y'(0) = 0$

162. $y'' - 64y = 16$; $y(0) = 1, \quad y'(0) = 0$

163. $2y'' + 3y' - 2y = 14x^2 - 4x + 11$; $y(0) = 0, \quad y'(0) = 0$

164. $5y'' + y' = -6x$; $y(0) = 0, \quad y'(0) = -10$

165. $x'' + 9x = 10\cos 2t$; $x(0) = x'(0) = 0$

166. $x'' + 4x = 5\sin 3t$; $x(0) = x'(0) = 0$

167. $x'' + 100x = 225\cos 5t + 300\sin 5t$; $x(0) = 375, \quad x'(0) = 0$

168. $x'' + 25x = 90\cos 4t$; $x(0) = 0, \quad x'(0) = 90$

169. $y^{(3)} - y' = 4e^{-x} + 3e^{2x}$; $y(0) = 0, \quad y'(0) = -1, \quad y''(0) = 2$

170. $y^{(3)} + y'' = x + e^{-x}$; $y(0) = 1$, $y'(0) = 0$, $y''(0) = 1$

171. $y^{(3)} - 2y'' + y' = 1 + xe^{x}$; $y(0) = y'(0) = 0$, $y''(0) = 1$

172. $y^{(4)} - 4y'' = x^2$; $y(0) = y'(0) = 1$, $y''(0) = y^{(3)}(0) = -1$

173. $y^{(4)} - y = 5$; $y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0$

174. $y^{(4)} - y''' = x + e^{x}$; $y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0$

175. If k and b are positive constants, then find the general solution of $y'' + k^2 y = \sin bx$

Section 2.5 – Variation of Parameters

In this section, we will introduce a technique called ***variation of parameters***, which is a general method that we can use in many more cases.

The inhomogeneous equation is given by: $y'' + p(t)y' + q(t)y = g(t)$

A fundamental set of solutions y_1 and y_2 to associated homogeneous equation $y'' + py' + qy = 0.$

Then the general solution to the inhomogeneous equation is given by

$$y_k = C_1 y_1 + C_2 y_2$$

C_1 *and* C_2 are arbitrary constants.

2.5-1 General Case

A differential system can be written in a form:

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

y_1 and y_2 fundamental set of solution to the homogeneous equation, they are linearly independent. Then the determinant will be recognized as the Wronskian:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

Which we can obtain:

$$v_1' = -\frac{y_2 g(t)}{y_1 y_2' - y_1' y_2} = -\frac{y_2}{W} g(t) \qquad \Rightarrow v_1(t) = -\int \frac{y_2 g(t)}{W} dt$$

$$v_2' = \frac{y_1 g(t)}{y_1 y_2' - y_1' y_2} = \frac{y_1}{W} g(t) \qquad \Rightarrow v_2(t) = \int \frac{y_1 g(t)}{W} dt$$

$$\boxed{y_p = v_1 y_1 + v_2 y_2}$$

Example 1

$\left\{y_1(x)=x^4,\ y_2(x)=x^2\right\}$ is a fundamental set of solutions of $y''-\frac{5}{x}y'+\frac{8}{x^2}y=4x^3$.

Find a general solution of the equation?

Solution

$$W=\begin{vmatrix} x^4 & x^2 \\ 4x^3 & 2x \end{vmatrix}$$

$$=-2x^5 \neq 0$$

$$v_1(x)=-\int \frac{x^2\left(4x^3\right)}{-2x^5}dx \qquad v_1(x)=-\int \frac{y_2 g(x)}{W}dx$$

$$=\int 2dx$$

$$=2x$$

$$v_2(x)=\int \frac{x^4\left(4x^3\right)}{-2x^5}dx \qquad v_2(x)=\int \frac{y_1 g(x)}{W}dx$$

$$=-2\int x^2dx$$

$$=-\frac{2}{3}x^3$$

The particular solution:

$$y_p=v_1y_1+v_2y_2$$

$$=(2x)\left(x^4\right)-\frac{2}{3}x^3\left(x^2\right)$$

$$=\frac{4}{3}x^5$$

The general solution:

$$y(x)=C_1x^4+C_2x^2+\frac{4}{3}x^5$$

Example 2

$\left\{y_1(x)=e^{2x},\ y_2(x)=xe^{2x}\right\}$ is a fundamental set of solutions of $y''-4y'+4y=\dfrac{e^{2x}}{x}$.

Find a general solution of the equation?

Solution

$$W=\begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}+2xe^{2x} \end{vmatrix}$$

$$=e^{4x}+2xe^{4x}-2xe^{4x}$$

$$=e^{4x}\neq 0$$

$$v_1(x)=-\int \frac{xe^{2x}}{e^{4x}}\frac{e^{2x}}{x}dx \qquad v_1(x)=-\int \frac{y_2 g(x)}{W}dx$$

$$=-\int dx$$

$$=-x$$

$$v_2(x)=\int \frac{e^{2x}}{e^{4x}}\frac{e^{2x}}{x}dx \qquad v_2(x)=\int \frac{y_1 g(x)}{W}dx$$

$$=\int \frac{1}{x}dx$$

$$=\ln|x|$$

The particular solution:

$$y_p=-xe^{2x}+\ln|x|\left(xe^{2x}\right) \qquad y_p=v_1y_1+v_2y_2$$

The general solution:

$$y(x)=C_1e^{2x}+C_2xe^{2x}-xe^{2x}+xe^{2x}\ln|x|$$

$$=C_1e^{2x}+\left(C_2-1\right)xe^{2x}+xe^{2x}\ln|x|$$

$$=C_1e^{2x}+C_3xe^{2x}+xe^{2x}\ln|x|$$

Example 3

Find the particular solution for $y'' + y = \tan t$

Solution

The homogeneous equation for the differential equation

$$\lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm i$$

Therefore; $y_1 = \cos t$ *and* $y_2 = \sin t$

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1 \neq 0$$

The system has a solution

$$\underline{|v_1} = \int (\sin t \;\tan t)\, dt \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= \int \frac{\sin^2 t}{\cos t} dt$$

$$= \int \frac{1 - \cos^2 t}{\cos t}\, dt$$

$$= \int \left(\frac{1}{\cos t} - \cos t\right) dt$$

$$= \int (\sec t - \cos t)\, dt$$

$$\underline{= -\ln|\sec t + \tan t| + \sin t\,|}$$

$$\underline{|v_2} = \int (\cos t \;\tan t)\, dt \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \int \sin t\, dt$$

$$\underline{= -\cos t\,|}$$

$$y_p = v_1 \cos t + v_2 \sin t \qquad\qquad y_p = v_1 y_1 + v_2 y_2$$

$$= (-\ln|\sec t + \tan t| + \sin t)\cos t + (-\cos t)\sin t$$

$$= -\cos t \ln|\sec t + \tan t| + \sin t \cos t - \cos t \sin t$$

$$\underline{= -(\cos t)\; \ln|\sec t + \tan t|\;|}$$

2.5-2 *Higher*-Order Equations

The nonhomogeneous higher-order equation is given by:

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \ldots + p_1(t)y' + p_0(t)y = g(t)$$

A fundamental set of solutions $y_1,\ y_2,\ \ldots,\ y_n$ to associated homogeneous equation

$$y_h = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$$

Then the particular solution is $y_p = u_1 y_1 + u_2 y_2 + \ldots + u_n y_n$

Where u_k', $k = 1, 2, \ldots, n$ are determined by the n equations:

$$\begin{aligned} u_1' y_1 + \ u_2' y_2 \ &+ \ldots + \ u_n' y_n = 0 \\ u_1' y_1' + \ u_2' y_2' \ &+ \ldots + \ u_n' y_n' = 0 \\ \vdots \qquad \vdots \quad &\quad \vdots \qquad\quad \vdots \\ u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} &+ \ldots + u_n' y_n^{(n-1)} = g(t) \end{aligned}$$

Using Cramer's Rule give: $u_k' = \dfrac{W_k}{W}$

For the 3rd-order differential equation:

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ g(t) & y_2'' & y_3'' \end{vmatrix} \qquad W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & g(t) & y_3'' \end{vmatrix} \qquad W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & g(t) \end{vmatrix}$$

$$u_1 = \int \frac{W_1}{W} \qquad u_2 = \int \frac{W_2}{W} \qquad u_3 = \int \frac{W_3}{W}$$

Exercises *Section* 2.5 – Variation of Parameters

1. $\left\{y_1(x)=e^{2x},\ y_2(x)=e^{-3x}\right\}$ is a fundamental set of solutions of $y''+y'-6y=3e^{2x}$.
Find a particular solution of the equation?

(**2 – 9**) Find a particular solution to the given second-order differential equation (Use ***variation of parameters***):

2. $y''-y=t+3$

3. $y''-2y'+y=e^{t}$

4. $x''-4x'+4x=e^{2t}$

5. $x''+x=\tan^2 t$

6. $y''+25y=-2\tan(5x)$

7. $y''-6y'+9y=5e^{3x}$

8. $y''+4y=2\cos 2x$

9. $y''-5y'+6y=4e^{2x}+3$

10. Verify that $y_1(t)=t$ and $y_2(t)=t^{-3}$ are solution to the homogeneous equation

$$t^2y''(t)+3ty'(t)-3y(t)=0$$

Use variation of parameters to find the general solution to

$$t^2y''(t)+3ty'(t)-3y(t)=\frac{1}{t}$$

(**11 – 58**) Find the general solution to the given differential equation (Use ***variation of parameters***).

11. $y''-y=\frac{1}{x}$

12. $y''-y=\sinh 2x$

13. $y''-y=x$

14. $y''-y=\cosh x$

15. $y''-y=\sin x$

16. $y''-y=e^{x}$

17. $y''+y=\sec x$

18. $y''+y=\tan x$

19. $y''+y=\sin x$

20. $y''+y=\csc x$

21. $y''+y=\cos^2 x$

22. $y''+y=\csc^2 x$

23. $y''+y=\sec^2 x$

24. $y''+y=\sec x\tan x$

25. $y''+y'=x$

26. $y''-y'=e^{x}\cos x$

28. $y''+y'-2y=e^{3x}$

29. $y''+2y'+y=e^{-x}\ln x$

30. $y''-2y'+y=\dfrac{e^{x}}{1+x^2}$

31. $y''+2y'+y=e^{-x}$

32. $y''-2y'-8y=3e^{-2x}$

33. $y''+3y'+2y=\sin e^{x}$

34. $y''+3y'+2y=4e^{x}$

35. $y''+3y'+2y=\dfrac{1}{1+e^{x}}$

36. $y''-4y=\sinh 2x$

37. $y''+4y=\sec 2x$

38. $y''+4y=\cos 3x$

39. $y''+4y=\sin^2 x$

40. $y''+4y=\sin^2 2t$

27. $y'' + y' - 2y = xe^{x}$

42. $y'' - 4y = xe^{x}$

43. $y'' - 4y' + 4y = 2e^{2x}$

44. $y'' - 4y' + 4y = (x+1)e^{2x}$

45. $y'' + 4y' + 5y = 10$

46. $y'' - 9y = \dfrac{9x}{e^{3x}}$

47. $y'' + 9y = \csc 3x$

48. $y'' + 9y = 3\tan 3t$

49. $y'' + 9y = \sin 3x$

50. $y'' + 9y = \sec 3x$

51. $y'' + 9y = 2\sec 3x$

41. $y'' - 4y = \dfrac{e^{x}}{x}$

52. $4y'' + 36y = \csc 3x$

53. $\left(D^2 + 5D + 6\right)y = x^2 + 2x$

54. $\left(D^2 - 3D + 2\right)y = \dfrac{1}{1+e^{-x}}$

55. $y''' + y' = \sec x$

56. $y''' - 3y'' + 2y' = \dfrac{e^{x}}{1+e^{-x}}$

57. $y''' - 6y'' + 11y' - 6y = e^{x}$

58. $x^3 y^{(3)} - 4x^2 y'' + 8xy' - 8y = 4\ln x$

(**59 – 70**) Find the general solution by to ***variation of parameters*** with the given initial conditions.

59. $y'' + y = \sec t$; $y(0) = 1, \quad y'(0) = 2$

60. $y'' + y = \sec^3 t$; $y(0) = 1, \quad y'(0) = \frac{1}{2}$

61. $y'' - y = t + \sin t$; $y(0) = 2, \quad y'(0) = 3$

62. $y'' - 2y' + y = \dfrac{e^{x}}{x}$; $y(0) = 1, \quad y'(0) = 0$

63. $y'' + 2y' - 8y = 2e^{-2x} - e^{-x}$; $y(0) = 1, \quad y'(0) = 0$

64. $y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x$; $y(0) = 1, \quad y'(0) = 2$

65. $y'' + 4y = \sin^2 2t$; $y\left(\frac{\pi}{8}\right) = 0, \quad y'\left(\frac{\pi}{8}\right) = 0$

66. $y'' + 4y = \sin^2 2t$; $y(0) = 0, \quad y'(0) = 0$

67. $y'' - 4y' + 4y = \left(12x^2 - 6x\right)e^{2x}$; $y(0) = 1, \quad y'(0) = 0$

68. $2y'' + y' - y = x + 1$; $y(0) = 1, \quad y'(0) = 0$

69. $4y'' - y = xe^{x/2}$; $y(0) = 1, \quad y'(0) = 0$

70. $t^2 y'' - ty' + y = t$; $y(1) = 1, \quad y'(1) = 4$

Section 2.6 – Forced Harmonic Motion

In this section, we will add to the homogeneous equation of motion that we covered, an external force. This force, will be in general time-dependent, and it would lead to non-homogeneous equations of motion.

A sinusoidal forcing is giving by the model:

$$x'' + 2cx' + \omega_0^2 x = F_0 \cos \omega t$$

F_0 : Amplitude if the driving force (constant)

ω : driving frequency.

c: damping constant.

ω_0 : natural frequency.

2.6-1 Forced undamped harmonic motion

The undamped equation has $c = 0$ or

$$x'' + \omega_0^2 x = F_0 \cos \omega t$$

The homogeneous equation is: $x'' + \omega_0^2 x = 0$

The characteristic equation: $\lambda^2 + \omega_0^2 = 0 \;\Rightarrow\; \boxed{\lambda_{1,2} = \pm\omega_0 i}$

With general solution: $\boxed{x_h = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t}$

2.6-2 *Case* 1 $\omega \neq \omega_0$

The particular solution is given by the form: $x_p = a \cos \omega t + b \sin \omega t$

$$x'_p = -a\omega \sin \omega t + b\omega \cos \omega t$$

$$x''_p = -a\omega^2 \cos \omega t - b\omega^2 \sin \omega t$$

$$\begin{aligned}
x''_p + \omega_0^2 x_p &= -a\omega^2 \cos \omega t - b\omega^2 \sin \omega t + \omega_0^2 \left(a \cos \omega t + b \sin \omega t\right) \\
&= -a\omega^2 \cos \omega t - b\omega^2 \sin \omega t + \omega_0^2 a \cos \omega t + \omega_0^2 b \sin \omega t \\
&= a\left(\omega_0^2 - \omega^2\right)\cos \omega t + b\left(\omega_0^2 - \omega^2\right)\sin \omega t \\
&= \boxed{F_0 \cos \omega t}
\end{aligned}$$

$$F_0 = a\left(\omega_0^2 - \omega^2\right) \qquad b\left(\omega_0^2 - \omega^2\right) = 0$$

$$a = \frac{F_0}{\omega_0^2 - \omega^2} \qquad b = 0 \qquad \textit{since } \ \omega_0^2 - \omega^2 \neq 0$$

$$\boxed{x_p = \frac{F_0}{\omega_0^2 - \omega^2}\cos\omega t}$$

$$x(t) = x_h(t) + x_p(t)$$

$$\boxed{= C_1\cos\omega_0 t + C_2\sin\omega_0 t + \frac{F_0}{\omega_0^2 - \omega^2}\cos\omega t}$$

When the *motion starts at equilibrium*; this means $x(0) = x'(0) = 0$

$$C_1 + \frac{F_0}{\omega_0^2 - \omega^2} = 0$$

$$\boxed{C_1 = -\frac{F_0}{\omega_0^2 - \omega^2}}$$

$$x' = -C_1\omega_0\sin\omega_0 t + C_2\omega_0\cos\omega_0 t - \frac{F_0}{\omega_0^2 - \omega^2}\omega_0\sin\omega t$$

$$x'(0) = C_2\omega_0 = 0 \quad \Rightarrow \ \boxed{C_2 = 0}$$

$$x(t) = -\frac{F_0}{\omega_0^2 - \omega^2}\cos\omega_0 t + \frac{F_0}{\omega_0^2 - \omega^2}\cos\omega t$$

$$\boxed{x(t) = \frac{F_0}{\omega_0^2 - \omega^2}\left(\cos\omega t - \cos\omega_0 t\right)}$$

Example 1

Suppose $F_0 = 23,\ \omega_0 = 11,\ \omega = 12$ with these values of the parameters the solution becomes

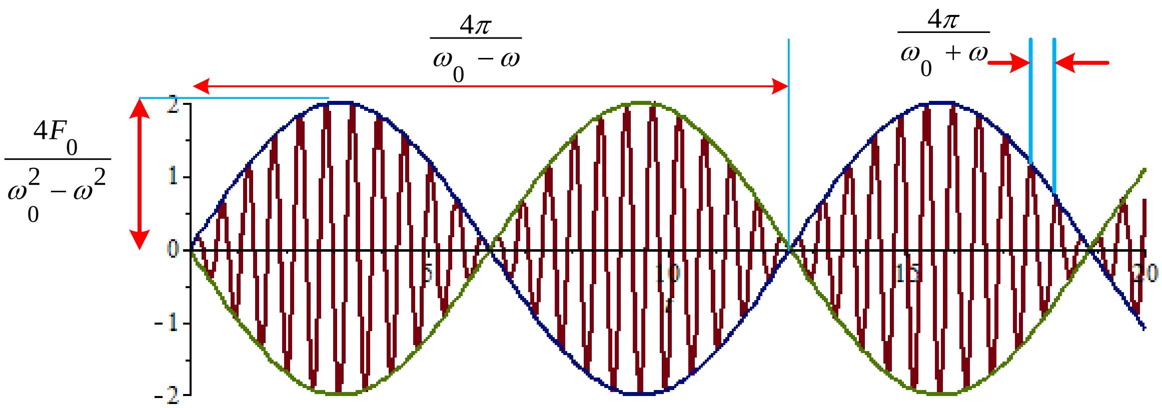
Solution

$$x(t) = \frac{F_0}{\omega_0^2 - \omega^2}\left(\cos\omega t - \cos\omega_0 t\right)$$

$$= \frac{23}{\left(11^2 - 12^2\right)}\left(\cos 12t - \cos 11t\right)$$

$$= -\left(\cos 12t - \cos 11t\right)$$

$$\boxed{= \cos 11t - \cos 12t}$$



Beat

Mean frequency: $\bar{\omega} = \frac{\omega_0 + \omega}{2}$ ***Half difference***: $\delta = \frac{\omega_0 - \omega}{2}$

2.6-3 *Case* 2 $\omega = \omega_0$

The particular solution is given by the form: $x_p = t\left(a\cos\omega_0 t + b\sin\omega_0 t\right)$

$$x_p' = a\cos\omega_0 t + b\sin\omega_0 t - at\omega_0 \sin\omega_0 t + bt\omega_0 \cos\omega_0 t$$

$$x_p'' = -a\omega_0 \sin\omega_0 t + b\omega_0 \cos\omega_0 t - a\omega_0 \sin\omega_0 t + b\omega_0 \cos\omega_0 t - at\omega_0^2 \cos\omega_0 t - bt\omega_0^2 \sin\omega_0 t$$

$$= -2\omega_0\left(a\sin\omega_0 t - b\cos\omega_0 t\right) - t\omega_0^2\left(a\cos\omega_0 t + b\sin\omega_0 t\right)$$

$$x_p'' + \omega_0^2 x_p = -2\omega_0\left(a\sin\omega_0 t - b\cos\omega_0 t\right) - t\omega_0^2\left(a\cos\omega_0 t + b\sin\omega_0 t\right) + t\omega_0^2\left(a\cos\omega_0 t + b\sin\omega_0 t\right)$$

$$= -2\omega_0\left(a\sin\omega_0 t - b\cos\omega_0 t\right)$$

$$= F_0 \cos\omega t$$

$$F_0 = 2b\omega_0 \qquad a = 0$$

$$b = \frac{F_0}{2\omega_0}$$

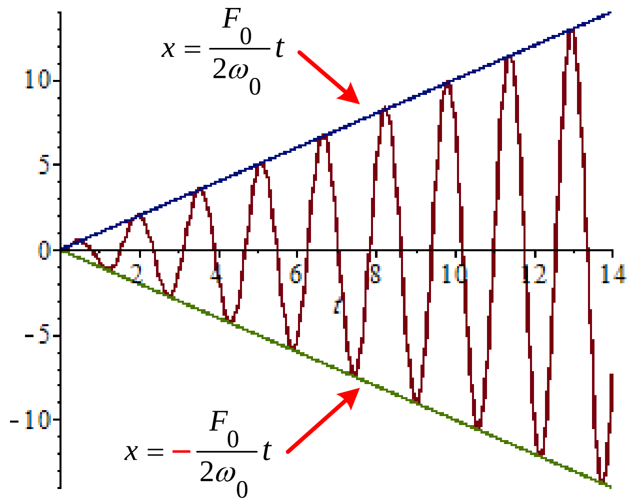
$$x_p = \frac{F_0}{2\omega_0} t \sin \omega_0 t$$

$$x(t) = x_h(t) + x_p(t)$$

$$\boxed{= C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_0}{2\omega_0} t \sin \omega_0 t}$$

If we let $x(0) = C_1 = 0, \quad x'(0) = C_2 = 0, \quad F_0 = 8, \quad \omega_0 = 4$

$$\boxed{x(t) = t \sin 4t}$$



Pure Resonance

2.6-4 *Forced Damped Harmonic Motion*

In real physical systems, there is always some damping, from frictional effects if nothing else. Let's add damping to the system

$$mx'' + cx' + kx = F_0 \cos \omega t \qquad x'' + 2cx' + \omega_0^2 x = F_0 \cos \omega t$$

The homogeneous equation (***transient solution***) is:

$$mx'' + cx' + kx = 0$$

$$m\lambda^2 + c\lambda + k = 0$$

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$x'' + 2cx' + \omega_0^2 x = 0$$

$$\lambda^2 + 2c\lambda + \omega_0^2 = 0$$

$$\lambda_{1,2} = -c \pm \sqrt{c^2 - \omega_0^2}$$

The particular solution is:
$$x_p = A\cos\omega t + B\sin\omega t$$
$$x'(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$
$$x''(t) = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$$

$$-Am\omega^2\cos\omega t - Bm\omega^2\sin\omega t - Ac\omega\sin\omega t + Bc\omega\cos\omega t + Ak\cos\omega t + Bk\sin\omega t = F_0\cos\omega t$$

$$\begin{cases} -Am\omega^2 + Bc\omega + Ak = F_0 \\ -Bm\omega^2 - Ac\omega + Bk = 0 \end{cases}$$

$$\begin{cases} \left(k - m\omega^2\right)A + c\omega B = F_0 \\ -c\omega A + \left(k - m\omega^2\right)B = 0 \end{cases}$$

$$D = \begin{vmatrix} k - m\omega^2 & c\omega \\ -c\omega & k - m\omega^2 \end{vmatrix} = \left(k - m\omega^2\right)^2 + (c\omega)^2$$

$$D_A = \begin{vmatrix} F_0 & -c\omega \\ 0 & k - m\omega^2 \end{vmatrix} = F_0\left(k - m\omega^2\right)$$

$$D_B = \begin{vmatrix} k - m\omega^2 & F_0 \\ -c\omega & 0 \end{vmatrix} = c\omega F_0$$

$$A = \frac{\left(k - m\omega^2\right)F_0}{\left(k - m\omega^2\right)^2 + (c\omega)^2}$$

$$B = \frac{c\omega F_0}{\left(k - m\omega^2\right)^2 + (c\omega)^2}$$

The *amplitude*:

$$C = \sqrt{A^2 + B^2}$$
$$= \sqrt{\frac{\left(k - m\omega^2\right)^2 F_0^2}{\left(\left(k - m\omega^2\right)^2 + (c\omega)^2\right)^2} + \frac{(c\omega)^2 F_0^2}{\left(\left(k - m\omega^2\right)^2 + (c\omega)^2\right)^2}}$$
$$= F_0\sqrt{\frac{\left(k - m\omega^2\right)^2 + (c\omega)^2}{\left(\left(k - m\omega^2\right)^2 + (c\omega)^2\right)^2}}$$

$$= \boxed{\frac{F_0}{\sqrt{\left(k - m\omega^2\right)^2 + (c\omega)^2}}}$$

The *phase shift*:

$$\varphi = \tan^{-1}\frac{B}{A}$$

$$= \tan^{-1}\frac{c\omega}{k - m\omega^2}$$

$$\begin{cases} \varphi = \tan^{-1}\dfrac{c\omega}{k - m\omega^2} & \text{if } \ k > m\omega^2 \\ \varphi = \pi + \tan^{-1}\dfrac{c\omega}{k - m\omega^2} & \text{if } \ k < m\omega^2 \end{cases}$$

2.6-5 *Underdamped Case*: $\quad c < \omega_0$

$$x_h = e^{-ct}\left(C_1 \cos \eta t + C_2 \sin \eta t\right)$$

Where $\eta = \sqrt{\omega_0^2 - c^2}$

To determine the inhomogeneous equation, it is better to use complex method.

$$z'' + 2cz' + \omega_0^2 z = Ae^{i\omega t}$$

However, $x(t) = \text{Re}(z)$

The particular solution: $z(t) = ae^{i\omega t}$

$$z'' + 2cz' + \omega_0^2 z = (i\omega)^2 ae^{i\omega t} + 2c(i\omega)ae^{i\omega t} + \omega_0^2 ae^{i\omega t}$$

$$= \left((i\omega)^2 + 2c(i\omega) + \omega_0^2\right)ae^{i\omega t}$$

$$P(i\omega) = (i\omega)^2 + 2c(i\omega) + \omega_0^2$$

$$P(i\omega)ae^{i\omega t} = Ae^{i\omega t}$$

$$\Rightarrow a = \frac{A}{P(i\omega)}$$

$$z(t) = \frac{A}{P(i\omega)}e^{i\omega t}$$

$$= H(i\omega)e^{i\omega t}$$

$H(i\omega)$ is called the ***transfer function.***

$$P(i\omega) = -\omega^2 + 2ic\omega + \omega_0^2$$

$$= \omega_0^2 - \omega^2 + 2ic\omega$$

$$P(i\omega) = Re^{i\phi}$$
$$= R(\cos\phi + i\sin\phi)$$

Polar Coordinates: $R\cos\phi = \omega_0^2 - \omega^2$ $\quad R\sin\phi = 2c\omega$

$$R = \sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4c^2\omega^2}$$

$$\cos\phi = \frac{\omega_0^2 - \omega^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4c^2\omega^2}}$$

$$\sin\phi = \frac{2c\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4c^2\omega^2}}$$

$$\cot\phi = \frac{\omega_0^2 - \omega^2}{2c\omega}$$

$$\phi(\omega) = arc\cot\left(\frac{\omega_0^2 - \omega^2}{2c\omega}\right) \qquad 0 < \phi < \pi$$

$$H(i\omega) = \frac{1}{P(i\omega)}$$
$$= \frac{1}{R}e^{-i\phi}$$

We will define the ***gain*** G by:

$$G(\omega) = \frac{1}{R} = \frac{1}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4c^2\omega^2}} \qquad H(i\omega) = G(\omega)e^{-i\phi}$$

The solution:

$$z(t) = H(i\omega)Ae^{i\omega t}$$
$$= G(\omega)Ae^{i(\omega t - \phi)}$$

$$x_p(t) = \operatorname{Re} z(t)$$
$$= G(\omega)A\cos(\omega t - \phi)$$

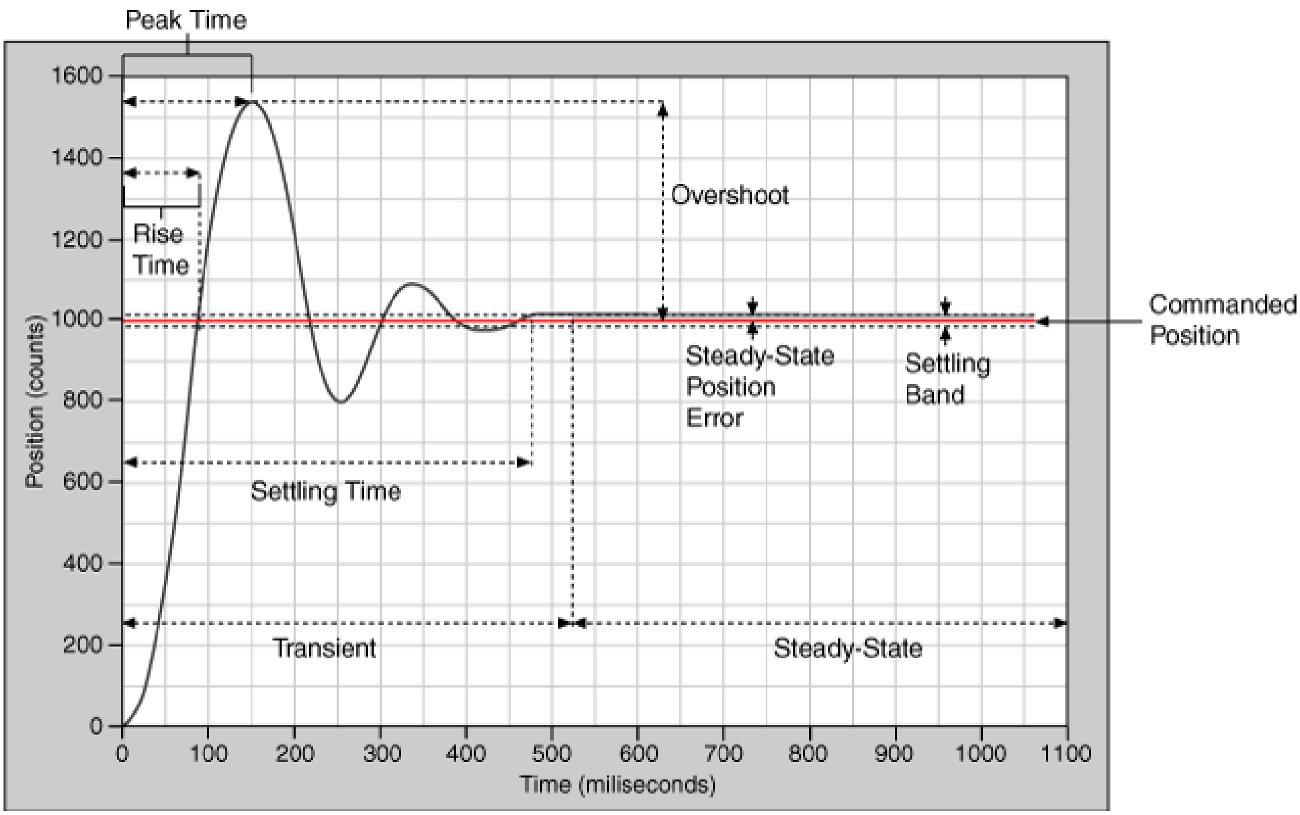
$$\boxed{x(t) = e^{-ct}\left(C_1\cos\eta t + C_2\sin\eta t\right) + G(\omega)A\cos(\omega t - \phi)}$$

e^{-ct} : ***transient*** term.

$T_c = \frac{1}{c}$: ***time constant***.

2.6-6 *Transient and Steady-State*

A motion or current is ***transient*** *solution* if the solution approaches zero as $t \to \infty$.
A ***steady-state*** motion or current is one that is not transient and does not become unbounded.
Free damped systems always yield transient motions, while forced damped systems (assuming the external force to be sinusoidal) yield both transient and steady-state motions.



Example 2

Determine the transient and steady-state to: $x'' + 2x' + 2x = 4\cos t + 2\sin t$; $x(0) = 0$, $x'(0) = v_0$

Solution

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\lambda_{1,2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$x_h = e^{-t}\left(C_1 \cos t + C_2 \sin t\right)$$

$$x_p = A\cos t + B\sin t$$

$$x'_p = -A\sin t + B\cos t$$

$$x''_p = -A\cos t - B\sin t$$

$$x'' + 2x' + 2x = 4\cos t + 2\sin t$$

$$-A\cos t - B\sin t - 2A\sin t + 2B\cos t + 2A\cos t + 2B\sin t = 4\cos t + 2\sin t$$

$$\begin{cases} \cos t & A + 2B = 4 \\ \sin t & -2A + B = 2 \end{cases} \rightarrow \boxed{B = 2, \ \ A = 0}$$

$$\boxed{x_p = 2\sin t}$$

$$x(t) = e^{-t}\left(C_1 \cos t + C_2 \sin t\right) + 2\sin t$$

$$x(0) = 0 \quad \rightarrow \quad \boxed{C_1 = 0}$$

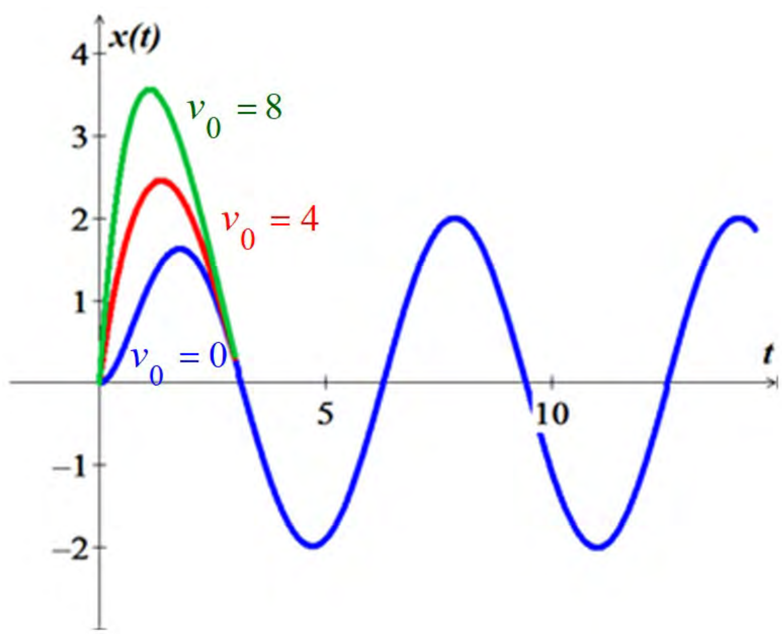
$$x'(t) = e^{-t}\left(-C_1 \cos t - C_2 \sin t - C_1 \sin t + C_2 \cos t\right) + 2\cos t$$

$$x(0) = v_0$$

$$C_1 + C_2 + 2 = v_0$$

$$\boxed{C_2 = v_0 - 2}$$

$$x(t) = \left(v_0 - 2\right)\underbrace{e^{-t}\sin t}_{transient} + \underbrace{2\sin t}_{steady-state}$$



Exercises Section 2.6 – Forced Harmonic Motion

1. A 1-*kg* mass is attached to a spring $k = 4kg / s^2$ and the system is allowed to come to rest. The spring-mass system is attached to a machine that supplies external driving force $f(t) = 4\cos \omega t$ *Newtons*. The system is started from equilibrium; the mass is having no initial displacement or velocity. Ignore any damping forces.

a) Find the position of the mass as a function of time

b) Place your answer in the form $s(t) = A\sin \delta t \sin \bar{\omega} t$. Select an ω near the natural frequency of the system to demonstrate the "beating" of the system. Sketch a plot shows the "beats:" and include the envelope of the beating motion in your plot.

(2 – 3) Find a particular solution to the differential equation using undetermined coefficients. Find and plot the solution of the initial value problem. Superimpose the plots of the transient response and the steady state solution.

2. $x'' + 7x' + 10x = 3\cos 3t \quad x(0) = -1, \ x'(0) = 0$

3. $x'' + 4x' + 5x = 3\sin t \quad x(0) = 0, \ x'(0) = -3$

4. Find a particular solution of $y'' - 2y' + 5y = 2\cos 3x - 4\sin 3x + e^{2x}$ given the set $y_p = A\cos 3x + B\sin 3x + Ce^{2x}$ where *A*, *B*, *C* are to be determined

(5 – 7) Find the general solution

5. $mx'' + kx = F_0 \cos \omega t \ ; \quad x(0) = x_0, \quad x'(0) = 0 \quad (\omega \neq \omega_0)$

6. $mx'' + kx = F_0 \cos \omega t \ ; \quad x(0) = 0, \quad x'(0) = v_0 \quad (\omega = \omega_0)$

7. $x'' + \omega_0^2 x = F_0 \sin \omega t \ ; \quad x(0) = 0, \quad x'(0) = 0 \quad (\omega \neq \omega_0)$

8. A forced mass–spring–dashpot system with equation $mx'' + cx' + kx = F_0 \cos \omega t$. Investigate the possibility of practical resonance of this system. In particular, find the amplitude $C(\omega)$ of steady state periodic forced oscillations with frequency ω. Sketch the graph $C(\omega)$ of and find the practical resonance frequency ω (if any).

a) $m = 1, \quad c = 2, \quad k = 2, \quad F_0 = 2$

b) $m = 1, \quad c = 4, \quad k = 5, \quad F_0 = 10$

c) $m = 1, \quad c = 6, \quad k = 45, \quad F_0 = 50$

d) $m = 1, \quad c = 10, \quad k = 650, \quad F_0 = 100$

9. A mass weighing 100 *lb*. (mass *m* = 3.125 *slugs* in *fps* units) is attached to the end of a spring that is stretched 1 *in*. by a force of 100 *lb*. A force $F_0 \cos \omega t$ acts on the mass. At what frequency (in hertz) will resonance oscillation occur? Neglect damping.

10. A mass weighing 16 *pounds* stretches a spring $\frac{8}{3}$ *ft*. The mass is initially released form rest from a point 2 *ft* below the equilibrium position, and the subsequent motion takes place in a medium that offers a damping force that is numerically equal to $\frac{1}{2}$ the instantaneous velocity. Find the equation of motion if the mass is driven by an external force equal to $f(t) = 10\cos 3t$

11. A mass of 32 *pounds* is attached to a spring with a constant spring 5 *lb/ft*. Initially, the mass is released 1 foot below the equilibrium position with a downward velocity of 5 *ft/s*, and the subsequent motion takes is numerically equal to 2 times the instantaneous velocity.

a) Find the equation of motion if the mass is driven by an external force equal to $f(t) = 12\cos 2t + 3\sin 2t$.

b) Graph the transient, steady-state, and the equation of motion solutions on the same coordinate axes.

12. A mass of 32 *pounds* is attached to a spring and stretched it 2 *feet* and then comes to rest in the equilibrium position. The surrounding medium offers a damping force that is numerically equal to 8 times the instantaneous velocity

a) Find the equation of motion if the mass is driven by an external force equal to $f(t) = 8\sin 4t$.

b) Graph the transient, steady-state, and the equation of motion solutions on the same coordinate axes.

13. A mass of 32 *pounds* is attached to a spring and stretched it 2 *feet* and then comes to rest in the equilibrium position. The surrounding medium offers a damping force that is numerically equal to 8 times the instantaneous velocity

a) Find the equation of motion with a starting external force equal to $f(t) = e^{-t}\sin 4t$ at $t = 0$

b) Graph the transient, steady-state, and the equation of motion solutions on the same coordinate axes.

14. A mass of 64 *pounds* is attached to a spring with a spring constant 32 *lb/ft* and then comes to rest in the equilibrium position.

a) Find the equation of motion with a starting external force equal to $f(t) = 68e^{-2t}\cos 4t$

b) Graph the transient, steady-state, and the equation of motion solutions on the same coordinate axes.

15. A 3-*kg* object is attached to a spring and stretches the spring 392 *mm* by itself. There is no damping in the system and a forcing function of the form $F(t) = 10\cos\omega t$ is attached to the object and the system will experience resonance. If the object is initially displaced 20 *cm* downward from its equilibrium position and given a velocity of 10 *cm/sec* upward find the displacement $y(t)$ at any time *t*.

16. A 8-*kg* mass is attached to a spring hanging vertically, thereby causing the spring to stretch 1.96 *m* upon coming to rest at equilibrium. The damping constant is given by 3 *N-sec/m*.

a) Find the equation of motion if the mass is driven by an external force equal to $f(t)=\cos 2t\ N$.

b) Determine the transient, steady-state solution of the motion

17. A 2-*kg* mass is attached to a spring hanging vertically, thereby causing the spring to stretch 0.2 *m* upon coming to rest at equilibrium. At $t=0$, the mass is displaced 5 *cm* below the equilibrium position and released. The damping constant is given by 5 *N-sec/m*.

a) Find the equation of motion if the mass is driven by an external force equal to $f(t)=0.3\cos t\ N$.

b) Determine the transient, steady-state solution of the motion.

18. A 8-*kg* mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by 3 *N-sec/m* and the spring constant is 40 *N/m*. Find steady-state solution if the mass is driven by an external force equal to $f(t)=2\sin\left(2t+\frac{\pi}{4}\right)\ N$.

19. A 32-*lb* mass weight is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by 2 *lb-sec/ft* and the spring constant is 5 *lb/ft*. If the mass is driven by an external force equal to $f(t)=3\cos 4t\ lb$ at time $t=0$.

a) Find steady-state solution.

b) Determine the amplitude and frequency

20. A 8-*kg* mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by 3 *N-sec/m* and the spring constant is 40 *N/m*. If the mass is driven by an external force equal to $f(t)=2\sin 2t\cos 2t\ N$.

a) Find steady-state solution.

b) Determine the amplitude, phase angle, period and frequency

21. A 10-*kg* mass is attached to a spring hanging vertically stretches the spring 0.098 *m* from its equilibrium rest position, measured positive in the downward direction. At time $t=0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t)=20\cos 10t\ N$. (t in *seconds*)

a) Determine the spring constant *k*.

b) Find the equation of motion.

c) Plot the equation of motion.

d) Determine the maximum excursion from equilibrium made of the object on the *t*-interval $0\le t<\infty$

22. A 10-*kg* mass is attached to a spring hanging vertically stretches the spring 0.098 *m* from its equilibrium rest position, measured positive in the downward direction. At time $t=0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t)=20\cos 8t\ N$. (t in *seconds*)

a) Determine the spring constant k.
b) Find the equation of motion.
c) Plot the equation of motion.
d) Determine the maximum excursion from equilibrium made of the object on the t-interval $0 \le t < \infty$

23. A 10-kg mass is attached to a spring hanging vertically stretches the spring $0.098\ m$ from its equilibrium rest position, measured positive in the downward direction. At time $t=0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t)=20e^{-t}\ N$. (t in *seconds*)

a) Determine the spring constant k.
b) Find the equation of motion.
c) Plot the equation of motion.
d) Determine the maximum excursion from equilibrium made of the object on the t-interval $0 \le t < \infty$

24. A 2-kg mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by $c = 8\ kg/sec$ and the spring constant is $k = 80\ N/m$. At time $t=0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t)=20\cos 8t\ N$. (t in *seconds*)

a) Find the equation of motion.
b) Plot the equation of motion.
c) Determine the long-time behavior of the system, as $t \to \infty$

25. A 2-kg mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by $c = 8\ kg/sec$ and the spring constant is $k = 80\ N/m$. At time $t=0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t)=20\sin 6t\ N$. (t in *seconds*)

a) Find the equation of motion.
b) Plot the equation of motion.
c) Determine the long-time behavior of the system, as $t \to \infty$

26. A 2-kg mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by $c = 8\ kg/sec$ and the spring constant is $k = 80\ N/m$. At time $t=0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t)=20e^{-t}\ N$. (t in *seconds*)

a) Find the equation of motion.
b) Plot the equation of motion.
c) Determine the long-time behavior of the system, as $t \to \infty$

27. A 10-kg mass is attached to a spring having a spring constant of $140\ N/m$. The mass is started in motion initially from the equilibrium position with an initial velocity 1 m/sec in the upward direction and with an applied external force $F(t)=5\sin t$. If the force due to air resistance is $-90y'\ N$.

a) Find the subsequent motion of the mass.
b) Plot the motion.

28. A 10-*kg* mass is attached to a spring having a spring constant of 140 *N/m*. The mass is started in motion initially from the equilibrium position with an initial velocity 1 *m/sec* in the upward direction and with an applied external force $F(t)=5\sin t$. If the force due to air resistance is $-90y'$ *N*.

a) Find the equation motion of the mass.
b) Plot the motion.
c) Determine the motion of the solution.

29. A 128-*lb* weight is attached to a spring having a spring constant of 64 *lb/ft*. The weight is started in motion initially by displacing it 6 *in* above the equilibrium position with no initial velocity and with an applied external force $F(t)=8\sin 4t$. Assume no air resistance.

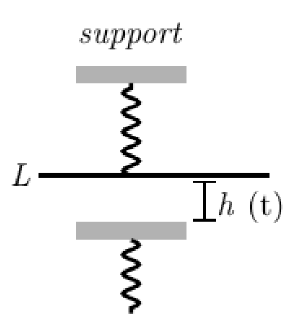
a) Find the equation motion of the mass.
b) Plot the motion.
c) Determine the motion of the solution.

30. A 3-*kg* object is attached to spring and stretches the spring 39.2 *cm* by itself. There is no damping in the system and a forcing function is given by $F(t)=10\cos \omega t$ is attached to the object and the system will experience resonance. If the object is initially displaced 20 *cm* downward from its equilibrium position and given a velocity of 10 *cm/sec* upward.

a) Find the spring constant *k*.
b) Find the natural frequency ω.
c) Find the displacement at any time *t*.
d) Sketch the displacement function.

31. Find the transient motion and steady periodic oscillations of a damped mass-and-spring system with $m=1$, $c=2$, and $k=26$ under the influence of an external force $F(t)=82\cos 4t$ with $x(0)=6$ and $x'(0)=0$. Also investigate the possibility of practical resonance for this system.

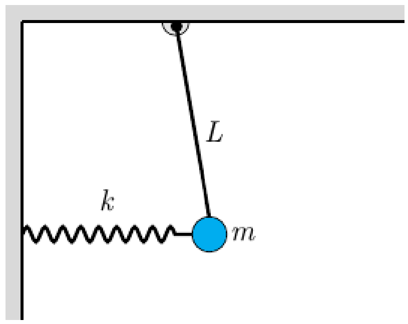
32. A mass *m* is attached to the end of a spring with a spring constant *k*. After the mass reaches equilibrium, its support begins to oscillate vertically about a horizontal line *L* according to a formula $h(t)$. The value of *h* represents the distance in feet measured from *L*.



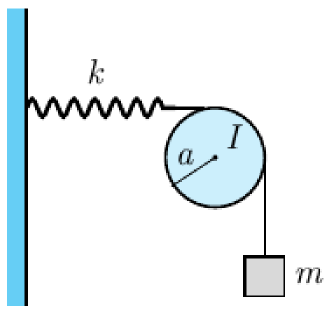
a) Determine the differential equation of motion if the entire system moves through a medium offering a damping force that is numerically equal to $\mu \frac{dx}{dt}$

b) Solve the differential equation in part (a) if the spring is stretched 4 *feet* by a mass weighing 16 *pounds* and $\mu = 2, \quad h(t) = 5\cos t, \quad x(0) = x'(0) = 0$

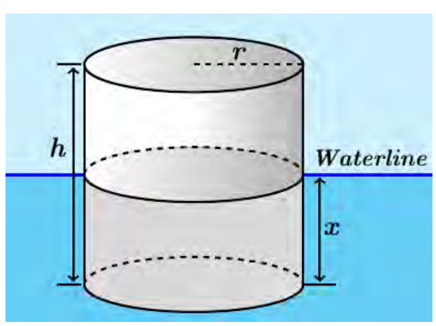
33. A mass *m* on the end of a pendulum (of length *L*) also attached to a horizontal spring (with constant *k*). Assume small oscillations of *m* so that the spring remains essentially horizontal and neglect damping. Find the natural circular frequency ω_0 of motion of the mass in terms of *L*, *k*, *m*, and the gravitational constant *g*.



34. A mass *m* hangs on the end of a cord around a pullet of radius *a* and moment of inertia *I*. The rim of the pulley is attached to a spring (with constant *k*). Assume small oscillations so that the spring remains essentially and neglect friction. Find the natural circular frequency in terms of *m*, *a*, *k*, *I*, and *g*.



35. Consider a floating cylindrical buoy with radius *r*, height *h*, and uniform density $\rho \le 0.5$ (recall that the density of water is $1\ g/cm^3$). The buoy is initially suspended at rest with its bottom at the top surface of the water and is released at time $t = 0$.

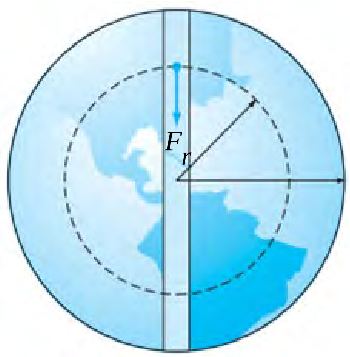


Therafter it is acted on by two forces: a downward gravitational force equal to its weight $mg = \pi r^2 hg$ and (by Archmedes' principle of buoyancy) an upward force equal to the weight $\pi r^2 xg$ of water displaced, where $x = x(t)$ is the depth of the bottom of the buoy beneath the surface at time *t*. Conclude that the buoy undergoes simple harmonic motion around its equilibrium position $x_e = \rho h$ with period $p = 2\pi\sqrt{\frac{\rho h}{g}}$.

a) Compute p and the amplitude of the motion if $\rho = 0.5\ g/cm^3$, $h = 200\ cm$, and $g = 980\ cm/s^2$

b) If the cylindrical buoy weighting 100 *lb* floats in water with its axis vertical. When depressed slightly and released, it oscillates up and down four times every 10 *sec*. assume that friction is negligible. Find the radius of the buoy.

36. Assume that the earth is a solid sphere of uniform density, with mass *M* and radius *R* = 3960 (*mi*). For a particle of mass *m* within the earth at distance *r* from the center of the earth, the gravitational force attracting *m* toward the center is $F_r = -\frac{GM_r m}{r^2}$, where M_r is the mass of the part of the earth within a sphere of radius *r*.



a) Show that $F_r = -\frac{GMmr}{R^3}$

b) Now suppose that a small hole is drilled straight through the center of the earth, thus connecting two antipodal points on its surface. Let a particle of mass *m* be dropped at time $t = 0$ into this hole with initial speed zero, and let $r(t)$ be its distance from the center of the earth at time *t*. conclude from Newton's second law and part (*a*) that $r''(t) = -k^2 r(t)$, where $k^2 = \frac{GM}{R^3} = \frac{g}{R}$.

c) Take $g = 32.2\ ft/s^2$, and conclude from part (*b*) that the particle undergoes simple harmonic motion back and forth between the ends of the hole, with a period of about 84 *min*.

d) Look up (or derive) the period of a satellite that just skims the surface of the earth; compare with the result in part (*c*). How do you explain the coincidence? Or is it a coincidence?

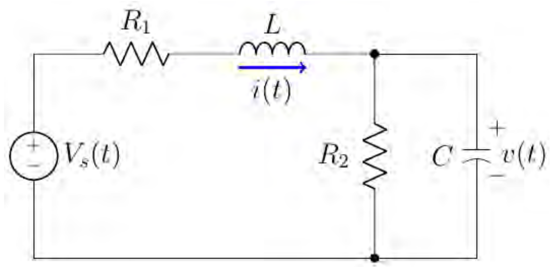
e) With what speed (in miles per hours) does the particle pass through the center of the earth?

f) Look up (or derive) the orbital velocity of a satellite that just skims the surface of the earth; compare with the result in part (*e*). How do you explain the coincidence? Or is it a coincidence?

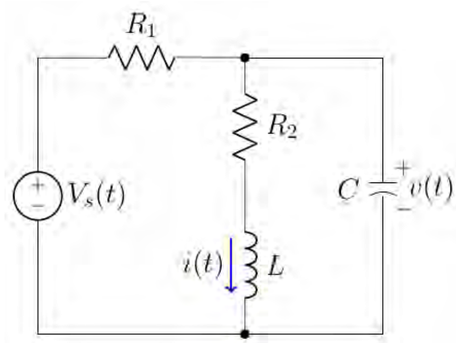
37. Find the steady-state solution $q_p(t)$ and the steady-state current in and *LRC*–series circuit when the source voltage is $E(t) = E_0 \sin \omega t$

(**38 – 39**) Express the given circuit in the second-order differential equation

38.



39.



40. Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit when $L = \frac{5}{3}\ h$, $R = 10\ \Omega$, $C = \frac{1}{30}\ f$, $E(t) = 300\ V$, $q(0) = 0\ C$, and $i(0) = 0\ A$. Find the maximum charge on the capacitor

41. Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit when $L = 1\ h$, $R = 100\ \Omega$, $C = 0.0004\ f$, $E(t) = 30\ V$, $q(0) = 0\ C$, and $i(0) = 2\ A$. Find the maximum charge on the capacitor.

42. Find the charge $q(t)$ and $i(t)$ son the capacitor in an *LRC*–series circuit when $L = \frac{1}{2}\ h$, $R = 10\ \Omega$, $C = 0.01\ f$, and $E(t) = 150\ V$, $q(0) = 1\ C$, and $i(0) = 0\ A$. What is the charge on the capacitor after a long time?

43. Find the charge $q(t)$ and $i(t)$ on the capacitor in an *LRC*–series circuit when $L = 1\ h$, $R = 50\ \Omega$, $C = 0.0002\ f$, $E(t) = 50\ V$, $q(0) = 0\ C$, and $i(0) = 0\ A$.

(**44 – 46**) Find the steady-state charge and the steady-state current in an *LRC*–series circuit when

44. $L = 1\ h, \quad R = 2\ \Omega, \quad C = 0.25\ f, \quad E(t) = 50\cos t\ V$

45. $L = \frac{1}{2}\ h, \quad R = 20\ \Omega, \quad C = 0.001\ f, \quad E(t) = 100\sin 60t\ V$

46. $L = \frac{1}{2}\ h, \quad R = 20\ \Omega, \quad C = 0.001\ f, \quad E(t) = 100\sin 60t + 200\cos 40t\ V$

(**47 – 51**) Find the charge $q(t)$ and $i(t)$ on the capacitor in an *LC*–series circuit when

47. $E(t) = E_0 \sin\omega t\ V, \quad q(0) = q_0\ C, \quad and \quad i(0) = i_0\ A$

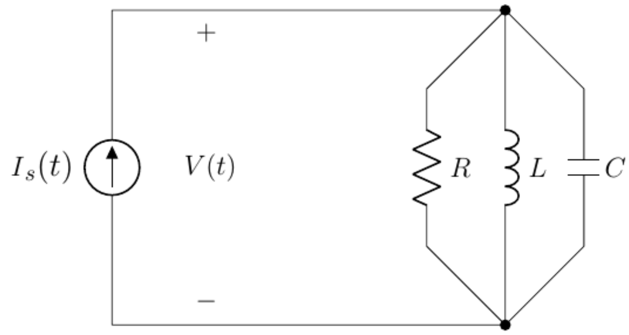
48. $E(t) = E_0 \cos\omega t\ V, \quad q(0) = q_0\ C, \quad and \quad i(0) = i_0\ A$

49. $L = 0.1\ h, \quad C = 0.1\ f, \quad E(t) = 100\sin\omega t\ V, \quad q(0) = 0\ C, \quad and \quad i(0) = 0\ A$

50. $L = 1\ H, \quad C = 4\ \mu F, \quad E(t) = 3\sin 3t\ V, \quad q(0) = 0\ C, \quad and \quad i(0) = 0\ A$

51. $L = 1\ H,\quad C = 4\ \mu F,\quad E(t) = 10te^{-t}\ V,\quad q(0) = 0\ C,\quad$ *and* $\quad i(0) = 0\ A$

(52 – 55) Consider the parallel *RLC* network. Assume that at time $t = 0$, the voltage $V(t)$ and its time rate of change are both zero. Determine the voltage $V(t)$ for



52. $R = 1\ k\Omega,\quad L = 1\ H,\quad C = \frac{1}{2}\ \mu F,\quad I_s(t) = 1 - e^{-t}\ mA$

53. $R = 1\ k\Omega,\quad L = 1\ H,\quad C = \frac{1}{2}\ \mu F,\quad I_s(t) = 5\sin t\ mA$

54. $R = 1\ k\Omega,\quad L = 1\ H,\quad C = \frac{1}{2}\ \mu F,\quad I_s(t) = 5\cos t\ mA$

55. $R = 2\ k\Omega,\quad L = 1\ H,\quad C = \frac{1}{4}\ \mu F,\quad I_s(t) = e^{-t}\ mA$

56. An *RCL* circuit connected in series has $R = 180\ \Omega,\quad C = \frac{1}{280}\ F,\quad L = 20\ H$, and applied voltage $E(t) = 10\sin t\ V$. Assuming no initial charge on the capacitor, but an initial current of $1\,A$ at $t = 0$ when the voltage is first applied.

a) Find the subsequent charge on the capacitor.
b) Plot the *transient*, *steady-state*, and the charge on the capacitor.
c) Find the current on the capacitor.

57. An *RCL* circuit connected in series has $R = 10\ \Omega,\quad C = 10^{-2}\ F,\quad L = \frac{1}{2}\ H$, and applied voltage $E(t) = 12\ V$. Assuming no initial charge and no initial current at $t = 0$ when the voltage is first applied.

a) Find the subsequent charge on the capacitor.
b) Plot the *transient*, *steady-state*, and the charge on the capacitor.
c) Find the current on the capacitor.

58. An *RCL* circuit connected in series has $R = 5\ \Omega,\quad C = 4\times 10^{-4}\ F,\quad L = 0.05\ H$, and applied voltage $E(t) = 200\cos 100t\ V$. Assuming no initial charge and no initial current at $t = 0$ when the voltage is first applied.

a) Find the subsequent charge on the capacitor.
b) Plot the *transient*, *steady-state*, and the charge on the capacitor.
c) Find the current flowing through this circuit.

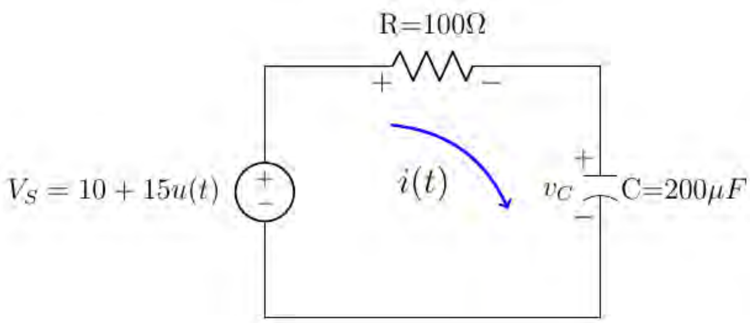
59. An *RCL* circuit connected in series has $R = 40\ \Omega$, $C = 16\times10^{-4}\ F$, $L = 1\ H$, and applied voltage $E(t) = 100\cos 10t\ V$. Assuming no initial charge and no initial current at $t = 0$ when the voltage is first applied.

a) Find the charge in the circuit at time *t*.
b) Find the current flowing through this circuit.
c) Find the linit of the charge as $t \to \infty$

60. A series circuit consists of a resistor with $R = 20\ \Omega$, an inductor with $L = 1\ H$, a capacitor with $C = 0.002\ F$, and a 12-*V* battery. If the initial charge and current are both 0, find the charge and current at time *t*.

61. A series circuit consists of a resistor with $R = 20\ \Omega$, an inductor with $L = 1\ H$, a capacitor with $C = 0.002\ F$, and $E(t) = 12\sin 10t$. If the initial charge and current are both 0, find the charge and current at time *t*.

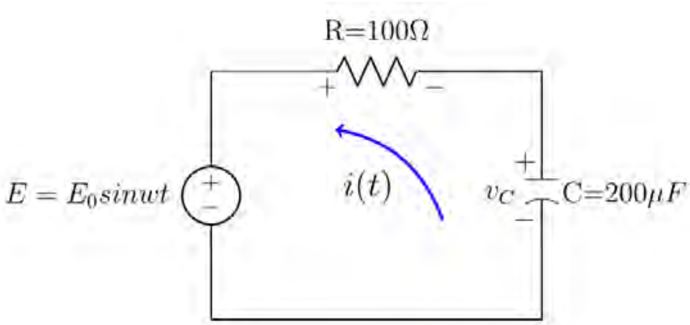
62. Consider the given circuit. Assuming that the voltage source changes from 10 to 25 *V* at time $t = 0$, $v_s = 10 + 15u(t)\ V$, where $u(t)$ is a unit step function.



Find the expressions that describe the voltage drop across the resistor across the capacitor and the current in the loop for $t > 0$

63. Find the steady-state solution $q_p(t)$ and the steady-state current in and *LRC*–series circuit when the source voltage is $E(t) = E_0 \sin \omega t$

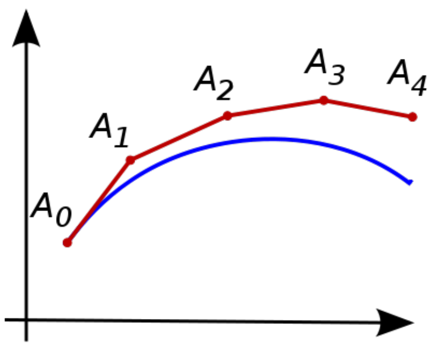
64. Consider the given *RC*-circuit with impressed *emf* is $E = E_0 sin\omega t\ V$. If no initial current is flowing at $t = 0$, find the current $i(t)$ for all $t > 0$.



Section 2.7 – Euler's & Runge-Kutta Methods

2.7-1 *Euler's method* named after ***Leonhard Euler*** is an example of a ***fixed-step*** solver.

Euler's method is a first-order numerical procedure for solving ordinary differential equations (***ODE***s) with a given initial value. It is the most basic kind of explicit method for numerical integration of ordinary differential equations. The Euler method often serves as the basis to construct more complex methods.
The Euler method is named after Leonhard Euler



Leonhard Euler (1768–70)

$y' = f(x,y) \qquad y(x_0) = y_0$

The setting size: $h = \dfrac{b-a}{k} > 0\ ;\quad k \in \mathbb{N}$

Then, $\quad x_0 = a$

$$x_1 = x_0 + h = a + h$$

$$x_k = x_{k-1} + h = a + kh$$

Last point $\quad x_k = a + kh = b$

By the definition of the derivative:

$$y'\left(x_k\right) \approx \frac{y\left(x_{k+1}\right) - y\left(x_k\right)}{h}$$

$$y'\left(x_k\right) \approx \frac{y_{k+1} - y_k}{h} = f\left(x_k, y_k\right) : \textit{slope}$$

The tangent line at the point $\left(x_0, y(x_0)\right)$ is:

$$y_{k+1} = y_k + h.f\left(x_k, y_k\right)$$

$$y_{k+1} = y_k + \Delta x_{step}.f\left(x_k, y_k\right)$$

This method is known as *Euler's Method* with step size h.

Example 1

Compute the first four step in the Euler's method approximation to the solution of $y' = y - x$ with $y(1) = 1$, using the step size $h = 0.1$. Compare the result with the actual solution to the initial value problem.

Solution

$y(1) = 1 \Rightarrow x_0 = 1$ *and* $y_0 = 1$

The *first* step:

$$\begin{aligned} y_1 &= y_0 + h\left(y_0 - x_0\right) \\ &= 1 + 0.1(1-1) \\ &= 1 \end{aligned}$$

$x_1 = x_0 + h = 1 + 0.1 = 1.1$

The *second* step:

$$\begin{aligned} y_2 &= y_1 + h\left(y_1 - x_1\right) \\ &= 1 + 0.1(1-1.1) \\ &= 0.99 \end{aligned}$$

$x_2 = x_1 + h = 1.1 + 0.1 = 1.2$

The *third* step:

$$\begin{aligned} y_3 &= y_2 + h\left(y_2 - x_2\right) \\ &= 0.99 + 0.1(0.99 - 1.2) \\ &= 0.969 \end{aligned}$$

$x_3 = x_2 + h = 1.2 + 0.1 = 1.3$

The *fourth* step:

$$\begin{aligned} y_4 &= y_3 + h\left(y_3 - x_3\right) \\ &= 0.969 + 0.1(0.969 - 1.3) \\ &= 0.9359 \end{aligned}$$

$x_3 = x_2 + h = 1.3 + 0.1 = 1.4$

The exact solution to $y' = y - x$ is $y(x) = 1 + x - e^{x-1}$

x_k	y_k : Euler's	y_k - exact	*Error*
1.0	1.0	1.0	0
1.1	1.0	0.9948	–0.0052
1.2	0.990	0.9786	–0.0114
1.3	0.969	0.9501	–0.0189
1.4	0.9359	0.9082	–0.0277

2.7-2 Runge-Kutta Methods

Like Euler's method, the Runge-Kutta methods are fixed-step solvers.

2.7-3 The Second-Order Runge-Kutta Method

The Second-Order Runge-Kutta method is also known as the improved Euler's method.
It is generally referred to as "***RK*2**"

Starting from the initial value point (x_0, y_0), we compute two slopes:

$$s_1 = f(t_0, y_0)$$

$$s_2 = f(t_0 + h, y_0 + hs_1)$$

$$y_1 = y_0 + h\frac{s_1 + s_2}{2}$$

But an analysis using Taylor's theorem reveals that there is an improvement in the estimate for the truncation error.
For the Second-Order Runge-Kutta method, we have

$$|y(t_1) - y_1| \le Mh^3$$

The constant M depends on the function $f(t, y)$.
The Second-Order Runge-Kutta method is controlled by the cube of the step size instead of the square.

Input t_0 *and* y_0
For k = 1 to N

$$s_1 = f(t_{k-1}, y_{k-1})$$

$$s_2 = f(t_{k-1} + h, y_{k-1} + hs_1)$$

$$y_k = y_{k-1} + h\frac{s_1 + s_2}{2}$$

$$t_k = t_{k-1} + h$$

Example 2

Compute the first four step in the Second-Order Runge-Kutta method approximation to the solution of $y' = y - t$ with $y(1) = 1$, using the step size $h = 0.1$. Compare the result with the actual solution to the initial value problem.

Solution

$\Rightarrow t_0 = 1 \ \ and \ \ y_0 = 1$

The *first* step:

$$
\begin{aligned}
s_1 &= f\left(t_0, y_0\right) \\
&= y_0 - t_0 \\
&= 1 - 1 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
s_2 &= f\left(t_0 + h, y_0 + hs_1\right) \\
&= \left(y_0 + hs_1\right) - \left(t_0 + h\right) \\
&= \left(1 + .1(0)\right) - \left(1 + .1\right) \\
&= 1 - 1.1 \\
&= -0.1
\end{aligned}
$$

$$
\begin{aligned}
y_1 &= y_0 + h\frac{s_1 + s_2}{2} \\
&= 1 + 0.1\left(\frac{0 - 0.1}{2}\right) \\
&= 0.995
\end{aligned}
$$

$$
\begin{aligned}
t_1 &= t_0 + h \\
&= 1 + 0.1 \\
&= 1.1
\end{aligned}
$$

The *second* step:

$$
\begin{aligned}
s_1 &= y_1 - t_1 \\
&= 0.995 - 1.1 \\
&= -0.105
\end{aligned}
$$

$$
\begin{aligned}
s_2 &= \left(y_1 + hs_1\right) - \left(t_1 + h\right) \\
&= \left(0.995 + .1(-0.105)\right) - \left(1.1 + .1\right) \\
&= -.2155
\end{aligned}
$$

$$
y_2 = y_1 + h\frac{s_1 + s_2}{2}
$$

$$= .995 + 0.1\left(\frac{-.105 - .2155}{2}\right)$$

$$\underline{= .978975}$$

$$t_2 = t_1 + h$$
$$= 1.1 + .1$$
$$\underline{= 1.2}$$

The *third* step:

$$s_1 = y_2 - t_2$$
$$= 0.978975 - 1.2$$
$$\underline{= -0.221025}$$

$$s_2 = \left(y_2 + hs_1\right) - \left(t_2 + h\right)$$
$$= \left(0.978975 + .1(-0.221025)\right) - \left(1.2 + .1\right)$$
$$\underline{= -0.3431275}$$

$$y_3 = y_2 + h\frac{s_1 + s_2}{2}$$
$$= .978975 + 0.1\left(\frac{-.221025 - .3431275}{2}\right)$$
$$\underline{= 0.9507673}$$

$$t_3 = t_2 + h$$
$$\underline{= 1.3}$$

The *fourth* step:

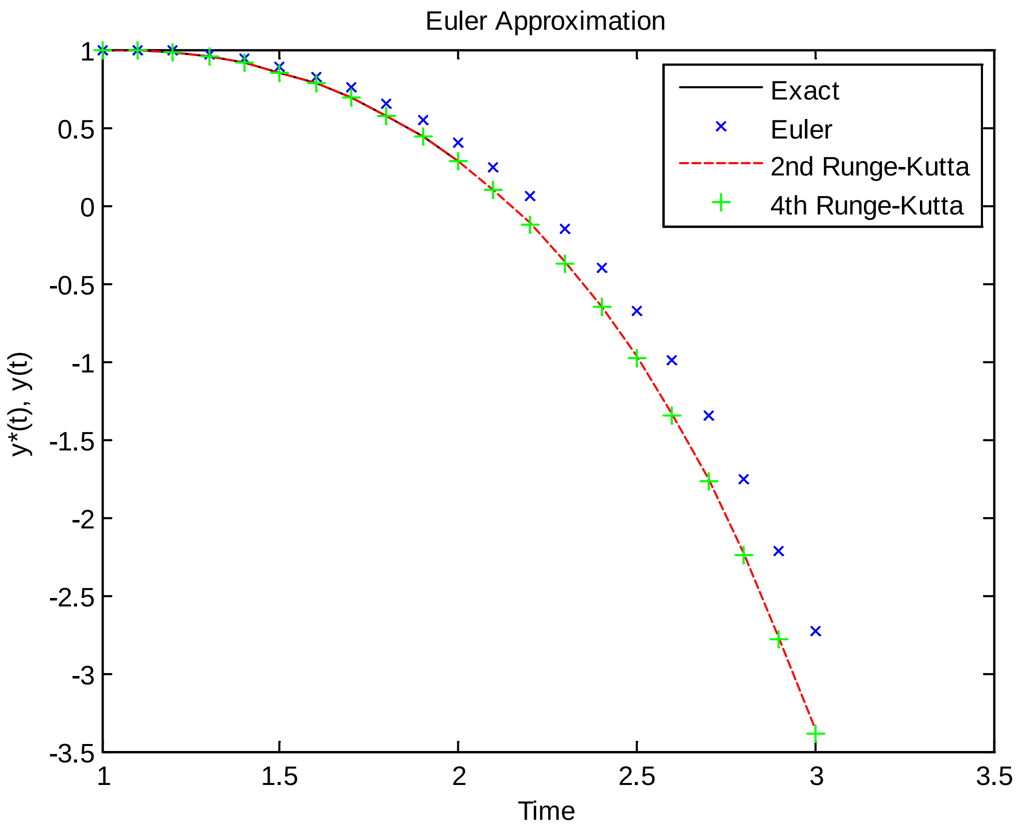
$$s_1 = y_3 - t_3$$
$$= 0.9507673 - 1.3$$
$$\underline{= -0.3492327}$$

$$s_2 = \left(y_3 + hs_1\right) - \left(t_3 + h\right)$$
$$= \left(0.9507673 + .1(-0.3492327)\right) - \left(1.3 + .1\right)$$
$$\underline{= -0.48415597}$$

$$y_4 = y_3 + h\frac{s_1 + s_2}{2}$$
$$= .9507673 + 0.1\left(\frac{-.3492327 - .48415597}{2}\right)$$
$$\underline{= 0.9090979}$$

$$t_4 = t_3 + h$$
$$\underline{= 1.4}$$

t_k	y_k : Runge-Kutta	y_k - Exact	***Runge-Kutta Error***	***Euler's Error***
1.0	1.0	1.0	**0**	**0**
1.1	0.9950000	0.994829081	**–0.000170918**	**–0.0052**
1.2	0.9789750	0.978597241	**–0.000377758**	**–0.0114**
1.3	0.9507673	0.950141192	**–0.000626182**	**–0.0189**
1.4	0.9090979	0.908175302	**–0.000922647**	**–0.0277**



2.7-4 *Fourth-Order* Runge-Kutta Method

This method is the most commonly used solution algorithm. For most equations and systems, it is suitably fast, simple and accurate. It is generally referred to as "***RK*4**".
It is the best for numerical solution of differential equations when combined with an intelligent adaptive step-size routine.

Starting from the initial value point $\left(t_0\ ,y_0\right)$, we compute two slopes:

$$s_1 = f\left(t_0, y_0\right)$$

$$s_2 = f\left(t_0 + \frac{h}{2},\ y_0 + \frac{h}{2}s_1\right)$$

$$s_3 = f\left(t_0 + \frac{h}{2},\ y_0 + \frac{h}{2}s_2\right)$$

$$s_4 = f\left(t_0 + h,\ y_0 + hs_3\right)$$

$$y_1 = y_0 + h\frac{s_1 + 2s_2 + 2s_3 + s_4}{6}$$

Example 3

Compute the first four step in the Second-Order Runge-Kutta method approximation to the solution of $y' = y - t$ with $y(1) = 1$, using the step size $h = 0.1$. Compare the result with the actual solution to the initial value problem.

Solution

$\Rightarrow t_0 = 1$ *and* $y_0 = 1$

The *first* step:

$$s_1 = f\left(t_0, y_0\right)$$

$$= 1 - 1$$

$$\underline{= 0\ |}$$

$$s_2 = f\left(t_0 + \frac{h}{2},\ y_0 + \frac{h}{2}s_1\right)$$

$$= f\left(1.05\ ,1\right)$$

$$= 1 - 1.05$$

$$\underline{= -0.05\ |}$$

$$s_3 = f\left(t_0 + \frac{h}{2},\ y_0 + \frac{h}{2}s_2\right)$$

$$= f\left(1.05\ ,.9975\right)$$

$$= .9975 - 1.05$$

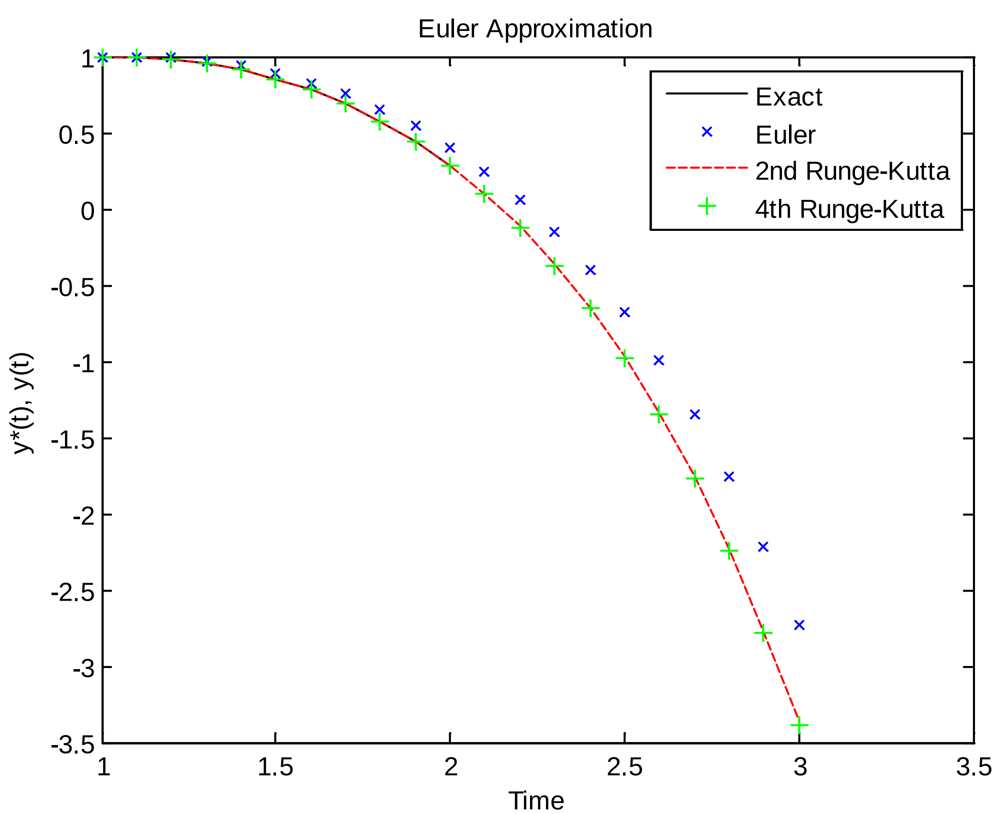
$$\underline{=-0.0525}$$

$$s_4 = f\left(t_0+h,\ y_0+hs_3\right)$$
$$= f(1.1\ ,.99475)$$
$$= .99475-1.1$$
$$\underline{=-0.10525}$$

$$y_1 = y_0 + h\frac{s_1+2s_2+2s_3+s_4}{6}$$
$$=1+0.1\left(\frac{0+2(-.05)+2(-.0525)+(-.10525)}{6}\right)$$
$$\underline{=0.99482916667}$$

$$t_1 = t_0 + h$$
$$\underline{=1.1}$$

t_k	y_k : Runge-Kutta	y_k - Exact	***Runge-Kutta Error***
1.0	1.0	1.0	**0**
1.1	0.994829167	0.994829081	**–0.000000086**
1.2	0.978597429	0.978597241	**0.000000295**
1.3	0.950141502	0.950141192	**–0.000000310**
1.4	0.908175759	0.908175302	**–0.000000457**



Exercises *Section* 2.7 – Euler's & Runge-Kutta Methods

(1 – 3) Calculate the first five iterations of Euler's method with step $h = 0.1$ of

1. $y' = ty \quad y(0) = 1$

2. $z' = x - 2z \quad z(0) = 1$

3. $z' = 5 - z \quad z(0) = 0$

4. Given: $y' + 2xy = x \quad y(0) = 8$

a) Use a computer and Euler's method to calculate three separate approximate solutions on the interval $[0, 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

5. Given: $y' + 2y = 2 - e^{-4t} \quad y(0) = 1$

a) Solve the differential equation

b) Use Euler's method and Runge-Kutta methods to calculate three separate approximate solutions on the interval $[0, 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$. Plot the exact solution and approximate solutions as discrete points.

6. Given: $z' - 2z = xe^{2x} \quad z(0) = 1$

a) Use a computer and Euler's method to calculate three separate approximate solutions on the interval $[0, 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a third with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

7. Consider the initial value problem $y' = 12y(4 - y) \quad y(0) = 1$

Use Euler's method with step size $h = 0.04$ to sketch solution on the interval $[0, 2]$

8. You've seen that the error in Euler's method varies directly as the first power of the step size $\left(i.e \ E_h \approx \lambda h\right)$. This makes Euler's method an order to halve the error? How does this affect the number of required iterations?

9. Use Euler's method to provide an approximate solution over the given time interval using the given steps sizes. Provide a plot of ***v*** versus ***y*** for each step size

$$y'' + 4y = 0, \quad y(0) = 4, \quad y'(0) = 0, \quad [0, 2\pi]; \quad h = 0.1,\ 0.01,\ 0.001$$

10. Given $z' + z = \cos x \quad z(0) = 1$

a) Use a computer and Runge-Kutta method to calculate three separate approximate solutions on the interval $[0,\ 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

11. Given $x' = \frac{t}{x} \quad x(0) = 1$

a) Use a computer and Runge-Kutta method to calculate three separate approximate solutions on the interval $[0,\ 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

12. Consider the initial value problem $y' = \frac{t}{y^2} \quad y(0) = 1$

Use Runge-Kutta method with step size $h = 0.04$ to sketch solution on the interval $[0,\ 2]$

13. Consider the initial value problem $y' - y = -\frac{1}{2}e^{t/2}\sin 5t + 5e^{t/2}\cos 5t$; $y(0) = 0$

Use Runge-Kutta method with step size $h = 0.05$ to sketch solution on the interval $[0,\ 5]$