# *Section* 4.6 – Substitution Rule

## *Substitution*: Running the Chain Rule Backwards

The Chain rule formula is:

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right)=u^n\frac{du}{dx}$$

We can see that $\frac{u^{n+1}}{n+1}$ is an antiderivative of the function $u^n\frac{du}{dx}$. Therefore, if we integrate both sides

$$\int u^n\frac{du}{dx}dx=\int\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right)dx$$

$$\boxed{\int u^n du=\frac{u^{n+1}}{n+1}+C}$$

### *Example*

Find the integral $\int\left(x^3+x\right)^5\left(3x^2+1\right)dx$

***Solution***

Let: $u=x^3+x$

$$du=\frac{du}{dx}dx$$

$$=\left(3x^2+1\right)dx$$

$$\int\left(x^3+x\right)^5\left(3x^2+1\right)dx=\int u^5du$$

$$=\frac{u^6}{6}+C$$

$$=\frac{1}{6}\left(x^3+x\right)^6+C$$

$$d\left(x^3+x\right)=\left(3x^2+1\right)dx$$

$$\int\left(x^3+x\right)^5\left(3x^2+1\right)dx=\int\left(x^3+x\right)^5d\left(x^3+x\right)$$

$$=\frac{1}{6}\left(x^3+x\right)^6+C$$

### *Example*

Find the integral $\int \sqrt{2x+1}dx$

***Solution***

Let: $u = 2x+1 \quad \Rightarrow \quad du = \frac{du}{dx}dx = 2dx$

$$dx = \frac{1}{2}du$$

$$\int \sqrt{2x+1}dx = \frac{1}{2}\int u^{1/2}du$$

$$= \frac{1}{2}\frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{3}(2x+1)^{3/2} + C$$

$d(2x+1) = 2dx$

$$\int \sqrt{2x+1}\ dx = \frac{1}{2}\int (2x+1)^{1/2} d(2x+1)$$

$$= \frac{1}{3}(2x+1)^{3/2} + C$$

## *Theorem* – The Substitution Rule

If $u = g(x)$ is differentiable function whose range is an interval *I*, and *f* is continuous on *I*, then

$$\int f\left(\left(g(x)\right)\right)g'(x)dx = \int f(u)du$$

***Proof***

By the Chain Rule, $F\left(g(x)\right)$ is an antiderivative of $f\left(g(x)\right)\cdot g'(x)$ whenever *F* is an antiderivative of *f*:

$$\frac{d}{dx}F\left(g(x)\right) = F'\left(g(x)\right)\cdot g'(x)$$
$$= f\left(g(x)\right)\cdot g'(x)$$

If we make the substitution $u = g(x)$, then

$$\int f\left(g(x)\right)g'(x)dx = \int \frac{d}{dx}F\left(g(x)\right)dx$$
$$= F\left(g(x)\right) + C$$
$$= F(u) + C$$
$$= \int F'(u)du$$
$$= \int f(u)du$$

## Integral of $\int \frac{1}{u}du$

If $u$ is a differentiable function that is never zero $\quad \int \frac{1}{u}du = \ln|u| + C$

### *Example*

Evaluate the integral $\int \tan x dx$

***Solution***

$$\int \tan x dx = \int \frac{\sin x}{\cos x}dx \qquad u = \cos x > 0 \quad \to du = -\sin x dx \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\begin{aligned}\int \tan x dx &= -\int \frac{d(\cos x)}{\cos x} \\ &= -\ln|\cos x| + C \\ &= \ln\frac{1}{|\cos x|} + C \\ &= \ln|\sec x| + C\end{aligned}$$

### *Example*

Evaluate the integral $\int_0^2 \frac{2x}{x^2-5}dx$

***Solution***

$$u = x^2 - 5 \quad \to du = 2xdx \quad \begin{cases} x = 0 & \to u = -5 \\ x = 2 & \to u = -1 \end{cases}$$

$$\begin{aligned}\int_0^2 \frac{2x}{x^2-5}dx &= \int_{-5}^{-1} \frac{du}{u} \\ &= \ln|u| \Big|_{-5}^{-1} \\ &= \ln|-1| - \ln|-5| \\ &= \ln 1 - \ln 5 \\ &= -\ln 5\end{aligned}$$

$$d\left(x^2-5\right) = 2xdx$$

$$\begin{aligned}\int_0^2 \frac{2x}{x^2-5}dx &= \int_0^2 \frac{d\left(x^2-5\right)}{x^2-5} \\ &= \ln\left|x^2-5\right| \Big|_0^2 \\ &= \ln|-1| - \ln|-5| \\ &= -\ln 5\end{aligned}$$

***Example***

Find the integral $\int \sec^2(5t+1)\cdot 5dt$

***Solution***

$$\int \sec^2(5t+1)\cdot 5dt = \int \sec^2(5t+1)\,d(5t+1) \qquad d(5t+1) = 5dt$$

$$\boxed{= \tan(5t+1) + C} \qquad \frac{d}{du}\tan u = \sec^2 u$$

## The General Antiderivative of the Exponential Function

$$\int e^u du = e^u + C$$

***Example***

Evaluate the integral $\int_0^{\ln 2} e^{3x}dx$

***Solution***

$$u = 3x \quad du = 3dx \to dx = \frac{1}{3}dx \quad \begin{cases} x = 0 & \to u = 0 \\ x = \ln 2 & \to u = 3\ln 2 = \ln 2^3 = \ln 8 \end{cases}$$

$$\int_0^{\ln 2} e^{3x}dx = \int_0^{\ln 8} e^u \cdot \frac{1}{3}du$$

$$= \frac{1}{3}\int_0^{\ln 8} e^u du$$

$$= \frac{1}{3}e^u \Big|_0^{\ln 8}$$

$$= \frac{1}{3}\left(e^{\ln 8} - e^0\right)$$

$$= \frac{1}{3}(8-1)$$

$$\boxed{= \frac{7}{3}}$$

***Example***

Find the integral $\int \cos(7\theta+3)\,d\theta$

***Solution***

$$\int \cos(7\theta+3)\,d\theta = \frac{1}{7}\int \cos(7\theta+3)\ d(7\theta+3) \qquad\qquad d(7\theta+3)=7d\theta$$

$$= \frac{1}{7}\sin(7\theta+3)+C$$

***Example***

Find the integral $\int x^2 \sin\left(x^3\right)dx$

***Solution***

$$\int x^2 \sin\left(x^3\right)dx = \frac{1}{3}\int \sin\left(x^3\right)d\left(x^3\right)$$

$$= -\frac{1}{3}\cos\left(x^3\right)+C$$

***Example***

Find the integral $\int x\sqrt{2x+1}\ dx$

***Solution***

Let: $u = 2x+1$

$du = 2dx$

$dx = \frac{1}{2}du$

$u = 2x+1$

$2x = u-1$

$\Rightarrow\ \ x = \frac{u-1}{2}$

$$\int x\sqrt{2x+1}\ dx = \int \frac{1}{2}(u-1)\sqrt{u}\ \ \frac{1}{2}du$$

$$= \frac{1}{4}\int (u-1)u^{1/2}\ du$$

$$=\frac{1}{4}\int\left(u^{3/2}-u^{1/2}\right)\,du$$

$$=\frac{1}{4}\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)+C$$

$$\boxed{=\frac{1}{10}(2x+1)^{5/2}-\frac{1}{6}(2x+1)^{3/2}+C}$$

### *Example*

Find the integral $\displaystyle\int\frac{2z\ dz}{\sqrt[3]{z^2+1}}$

***Solution***

Let: $u=z^2+1 \quad\Rightarrow\quad du=2zdz$

$$\int\frac{2z\ dz}{\sqrt[3]{z^2+1}}=\int\frac{du}{u^{1/3}}$$

$$=\int u^{-1/3}du$$

$$=\frac{3}{2}u^{2/3}+C$$

$$\boxed{=\frac{3}{2}\left(z^2+1\right)^{2/3}+C}$$

***Or*** let: $u=\sqrt[3]{z^2+1} \quad\to\quad u^3=z^2+1$

$3u^2du=2zdz$

$$\int\frac{2z\ dz}{\sqrt[3]{z^2+1}}=\int\frac{3u^2du}{u}$$

$$=3\int u\ du$$

$$=3\cdot\frac{u^2}{2}+C$$

$$\boxed{=\frac{3}{2}\left(z^2+1\right)^{2/3}+C}$$

### *Definition*

If $a>0$ and $u$ is a differentiable of $x$, then $a^u$ is a differentiable function of $x$ and

$$\frac{d}{dx}a^u=a^u\ln a\frac{du}{dx}\quad\to\quad\int a^u du=\frac{a^u}{\ln a}+C$$

### *Example*

- $\displaystyle\int 2^x dx\ \boxed{=\frac{2^x}{\ln 2}+C}$

- $\displaystyle\int 2^{\sin x}\cos x dx=\int 2^{\sin x}d(\sin x)$

$$\boxed{=\frac{2^{\sin x}}{\ln 2}+C}$$

$$\int \frac{\log_2 x}{x}dx = \int \frac{1}{x}\frac{\ln x}{\ln 2}dx$$

$$= \frac{1}{\ln 2}\int \frac{\ln x}{x}dx$$

$$= \frac{1}{\ln 2}\int \ln x \, d(\ln x)$$

$$= \frac{1}{\ln 2}\cdot\frac{1}{2}(\ln x)^2 + C$$

$$= \frac{(\ln x)^2}{2\ln 2} + C$$

## Substitution Formula

### *Theorem*

If $g'$ is continuous on the interval $[a, b]$ and $f$ is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x))\cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

### *Proof*

Let $F$ denote any antiderivative of $f$. Then

$$\int_a^b f(g(x))\cdot g'(x)dx = F(g(x))\Big|_{x=a}^{x=b}$$

$$= F(g(b)) - F(g(a))$$

$$= F(u)\Big|_{u=g(a)}^{u=g(b)}$$

$$= \int_{g(a)}^{g(b)} f(u)du$$

### *Example*

Evaluate $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$

***Solution***

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \int_{-1}^{1}\left(x^3+1\right)^{1/2} d\left(x^3+1\right) \qquad d\left(x^3+1\right) = 3x^2dx$$

$$= \frac{2}{3}\left(x^3+1\right)^{3/2} \Big|_{-1}^{1}$$

$$= \frac{2}{3}\left(2^{3/2}-0^{3/2}\right)$$

$$= \frac{2}{3}2^{3/2} \qquad 2^{5/2} = \sqrt{2^5} = 2^2\sqrt{2}$$

$$\boxed{= \frac{4\sqrt{2}}{3}}$$

### *Example*

Evaluate $\displaystyle\int_{\pi/4}^{\pi/2} \cot\theta\csc^2\theta\, d\theta$

***Solution***

Let $u = \cot\theta \quad \Rightarrow \quad du = -\csc^2\theta d\theta \to -du = \csc^2\theta d\theta$

$$\to \begin{cases} \theta = \frac{\pi}{4} & u = \cot\frac{\pi}{4} = 1 \\ \theta = \frac{\pi}{2} & u = \cot\frac{\pi}{2} = 0 \end{cases}$$

$$\int_{\pi/4}^{\pi/2} \cot\theta\csc^2\theta\, d\theta = \int_{1}^{0} u\cdot(-du)$$

$$= -\int_{1}^{0} u\, du$$

$$= -\frac{u^2}{2}\Big|_{1}^{0}$$

$$= -\left(0-\frac{1}{2}\right)$$

$$\boxed{= \frac{1}{2}}$$

***Example***

Evaluate the integral $\int_0^{\pi/6} \tan 2x dx$

***Solution***

$$\int_0^{\pi/6} \tan 2x dx = \int_0^{\pi/6} \tan u \cdot \left(\frac{du}{2}\right) \qquad u = 2x \quad \rightarrow du = 2dx \quad \Rightarrow dx = \frac{du}{2}$$

$$= \frac{1}{2}\int_0^{\pi/6} \tan u \cdot du$$

$$= \frac{1}{2}\ln\left|\sec 2x\right| \Big|_0^{\pi/6}$$

$$= \frac{1}{2}\left[\ln\left|\sec 2\frac{\pi}{6}\right| - \ln\left|\sec 0\right|\right]$$

$$= \frac{1}{2}(\ln 2 - \ln 1)$$

$$= \frac{1}{2}\ln 2$$

## *Integrals* of $\sin^2 x$ *and* $\cos^2 x$

### *Example*

Find the integral $\int \sin^2 x\, dx$

***Solution***

$$\int \sin^2 x\, dx = \frac{1}{2}\int (1-\cos 2x)\,dx \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + C$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

### *Example*

Find the integral $\int \cos^2 x\, dx$

***Solution***

$$\int \cos^2 x\, dx = \int \frac{1+\cos 2x}{2}\,dx \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right] + C$$

$$= \frac{x}{2} + \frac{\sin 2x}{4} + C$$

## *Integration Formulas*

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C \qquad \left(\text{Valid for } u^2 < a^2\right)$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \qquad \left(\text{Valid for all } u\right)$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C \qquad \left(\text{Valid for } |u| > a > 0\right)$$

# *Exercises* Section 4.6 – Substitution Rule

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

**1.** $\int 2(2x+4)^5\,dx, \quad u=2x+4$

**2.** $\int \frac{4x^3}{(x4+1)^2}dx, \quad u=x^4+1$

**3.** $\int x\sin\left(2x^2\right)dx, \quad u=2x^2$

**4.** $\int 12\left(y^4+4y^2+1\right)^2\left(y^3+2y\right)dx, \quad u=y^4+4y^2+1$

**5.** $\int \csc^2 2\theta\cot 2\theta\; d\theta \quad \rightarrow \begin{cases} a)\ U\sin g\quad u=\cot 2\theta \\ b)\ U\sin g\quad u=\csc 2\theta \end{cases}$

Evaluate the integrals

**6.** $\int \frac{1}{\sqrt{5s+4}}ds$

**7.** $\int \theta\sqrt[4]{1-\theta^2}\; d\theta$

**8.** $\int \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2}dx$

**9.** $\int \tan^2 x\sec^2 x\; dx$

**10.** $\int \sin^5\frac{x}{3}\cos\frac{x}{3}\; dx$

**11.** $\int \tan^7\frac{x}{2}\sec^2\frac{x}{2}\; dx$

**12.** $\int r^4\left(7-\frac{r^5}{10}\right)^3 dr$

**13.** $\int x^{1/2}\sin\left(x^{3/2}+1\right)dx$

**14.** $\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv$

**15.** $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}dt$

**16.** $\int \frac{\sec z\ \tan z}{\sqrt{\sec z}}dz$

**17.** $\int \frac{1}{\sqrt{t}}\cos\left(\sqrt{t}+3\right)dt$

**18.** $\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\; d\theta$

**19.** $\int t^3\left(1+t^4\right)^3 dt$

**20.** $\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}dx$

**21.** $\int x^3\sqrt{x^2+1}\; dx$

**22.** $\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}d\theta$

**23.** $\int 2x\sqrt{x^2-2}\; dx$

**24.** $\int \frac{x}{\left(x^2-4\right)^3}dx$

**25.** $\int x^3\left(3x^4+1\right)^2 dx$

**26.** $\int 2\left(3x^4+1\right)^2 dx$

**27.** $\int 5x\sqrt{x^2-1}\; dx$

**28.** $\int \left(x^2-1\right)^3(2x)dx$

**29.** $\int \frac{6x}{\left(1+x^2\right)^3}dx$

**30.** $\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}dr$

**31.** $\int u^3\sqrt{u^4+2}\ du$

**32.** $\int \frac{t+2t^2}{\sqrt{t}}\ dt$

**33.** $\int \left(1+\frac{1}{t}\right)^3 \frac{1}{t^2}\ dt$

**34.** $\int \left(7-3x-3x^2\right)(2x+1)\ dx$

**35.** $\int \sqrt{x}\left(4-x^{3/2}\right)^2\ dx$

**36.** $\int \frac{1}{\sqrt{x}+\sqrt{x+1}}dx$

**37.** $\int \sqrt{1-x}\ dx$

**38.** $\int x\sqrt{x^2+4}\ dx$

**39.** $\int \sin^2\left(\theta+\frac{\pi}{6}\right)d\theta$

**40.** $\int \cos^2(8\theta)\,d\theta$

**41.** $\int \sin^2(2\theta)\,d\theta$

**42.** $\int 8\cos^4 2\pi x\ dx$

**43.** $\int \sec x dx$

**44.** $\int \frac{dx}{\sqrt{1-4x^2}}$

**45.** $\int \frac{dx}{\sqrt{3-4x^2}}$

**46.** $\int \frac{dx}{\sqrt{e^{2x}-6}}$

**47.** $\int \frac{dx}{\sqrt{4x-x^2}}$

**48.** $\int \frac{dx}{4x^2+4x+2}$

**49.** $\int \frac{1}{6x-5}dx$

**50.** $\int \frac{x^2+2x+3}{x^3+3x^2+9x+1}dx$

**51.** $\int \frac{1}{x(\ln x)^2}dx$

**52.** $\int \frac{x-3}{x+3}dx$

**53.** $\int \frac{3x}{x^2+4}dx$

**54.** $\int \frac{dx}{2\sqrt{x}+2x}$

**55.** $\int \frac{\sec x dx}{\sqrt{\ln(\sec x+\tan x)}}$

**56.** $\int 8e^{(x+1)}dx$

**57.** $\int 4xe^{x^2}dx$

**58.** $\int (2x+1)e^{x^2+x}dx$

**59.** $\int \frac{e^{-\sqrt{r}}}{\sqrt{r}}dr$

**60.** $\int t^3e^{t^4}dt$

**61.** $\int e^{\sec \pi t}\sec\pi\ \ \tan\pi t\ dt$

**62.** $\int (2x+1)e^{x^2+x}dx$

**63.** $\int \frac{dx}{1+e^x}$

**64.** $\int \frac{e^x}{1+e^x}dx$

**65.** $\int \frac{2}{e^{-x}+1}dx$

**66.** $\int \frac{1}{x^3}e^{1/4x^2}dx$

**67.** $\int \frac{e^{1/\sqrt{x}}}{x^{3/2}}dx$

**68.** $\int \frac{-e^{3x}}{2-e^{3x}}dx$

**69.** $\int \frac{7e^{7x}}{3+e^{7x}}dx$

**70.** $\int \frac{2(e^x-e^{-x})}{(e^x+e^{-x})^2}dx$

**71.** $\int \frac{3^x}{3-3^x}dx$

**72.** $\int \frac{x\ 2^{x^2}}{1+2^{x^2}}dx$

**73.** $\int (6x+e^x)\sqrt{3x^2+e^x}\,dx$

**74.** $\int \frac{dx}{x\left(\log_8 x\right)^2}$

**75.** $\int \frac{dx}{x\sqrt{25x^2-2}}$

**76.** $\int \frac{6dr}{\sqrt{4-(r+1)^2}}$

**77.** $\int \frac{dx}{2+(x-1)^2}$

**78.** $\int \frac{\sec^2 y\,dy}{\sqrt{1-\tan^2 y}}$

**79.** $\int \frac{dx}{\sqrt{-x^2+4x-3}}$

**80.** $\int \frac{dx}{\sqrt{2x-x^2}}$

**81.** $\int \frac{x-2}{x^2-6x+10}dx$

**82.** $\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$

**83.** $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$

**84.** $\int \frac{e^{\cos^{-1}x}dx}{\sqrt{1-x^2}}$

**85.** $\int \frac{\left(\sin^{-1}x\right)^2\,dx}{\sqrt{1-x^2}}$

**86.** $\int \frac{dy}{\left(\sin^{-1}y\right)\sqrt{1+y^2}}$

**87.** $\int \frac{1}{\sqrt{x}\,(x+1)\left(\left(\tan^{-1}\sqrt{x}\right)^2+9\right)}dx$

**88.** $\int 2x\left(x^2+1\right)^4\,dx$

**89.** $\int 8x\cos\left(4x^2+3\right)dx$

**90.** $\int \sin^3 x\cos x\,dx$

**91.** $\int (6x+1)\sqrt{3x^2+x}\,dx$

**92.** $\int 2x\left(x^2-1\right)^{99}\,dx$

**93.** $\int xe^{x^2}\,dx$

**94.** $\int \frac{2x^2}{\sqrt{1-4x^3}}dx$

**95.** $\int \frac{\left(\sqrt{x}+1\right)^4}{2\sqrt{x}}dx$

**96.** $\int \left(x^2+x\right)^{10}(2x+1)\,dx$

**97.** $\int \frac{dx}{10x-3}$

**98.** $\int x^3\left(x^4+16\right)^6\,dx$

**99.** $\int \sin^{10}\theta\,\cos\theta\,d\theta$

**100.** $\int \frac{dx}{\sqrt{1-9x^2}}$

**101.** $\int x^9\sin x^{10}dx$

102. $\int \left(x^6 - 3x^2\right)^4 \left(x^5 - x\right) dx$

103. $\int \frac{x}{x-2} dx$

104. $\int \frac{dx}{1+4x^2}$

105. $\int \frac{3}{1+25y^2} dy$

106. $\int \frac{2}{x\sqrt{4x^2-1}} dx \quad \left(x > \frac{1}{2}\right)$

107. $\int \frac{8x+6}{2x^2+3x} dx$

108. $\int \frac{x}{\sqrt{x-4}} dx$

109. $\int \frac{x^2}{(x+1)^4} dx$

110. $\int \frac{x}{\sqrt[3]{x+4}} dx$

111. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

112. $\int x \sqrt[3]{2x+1}\, dx$

113. $\int (x+1)\sqrt{3x+2}\, dx$

114. $\int \sin^2 x\, dx$

115. $\int \sin^2\left(\theta + \frac{\pi}{6}\right) d\theta$

116. $\int x \cos^2\left(x^2\right) dx$

117. $\int \sec 4x \,\tan 4x\, dx$

118. $\int \sec^2 10x\, dx$

119. $\int \left(\sin^5 x + 3\sin^3 x - \sin x\right) \cos x\, dx$

120. $\int \frac{\csc^2 x}{\cot^3 x} dx$

121. $\int \left(x^{3/2} + 8\right)^5 \sqrt{x}\, dx$

122. $\int \sin x \,\sec^8 x\, dx$

123. $\int \frac{e^{2x}}{e^{2x}+1} dx$

124. $\int \sec^3\theta \,\tan\theta\, d\theta$

125. $\int x \,\sin^4 x^2 \,\cos x^2\, dx$

126. $\int \frac{dx}{\sqrt{1+\sqrt{1+x}}}$

127. $\int \tan^{10} 4x \,\sec^2 4x\, dx$

128. $\int \frac{x^2}{x^3+27} dx$

129. $\int y^2 \left(3y^3 + 1\right)^4 dy$

130. $\int x \sin x^2 \cos^8 x^2\, dx$

131. $\int \frac{\sin 2x}{1+\cos^2 x} dx$

**132.** $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$

**133.** $\int \frac{dx}{\left(\tan^{-1} x\right)\left(1+x^2\right)}$

**134.** $\int \frac{\left(\tan^{-1} x\right)^5}{1+x^2}\,dx$

**135.** $\int \frac{1}{x^2}\sin\frac{1}{x}\,dx$

**136.** Evaluate the integral $\int \frac{18\tan^2 x\sec^2 x}{\left(2+\tan^3 x\right)^2}\,dx$

*a)* $u=\tan x$, followed by $v=u^3$ *then by* $w=2+v$

*b)* $u=\tan^3 x$, followed by $v=2+u$

*c)* $u=2+\tan^3 x$

Evaluate the integrals

**137.** $\int_0^1 (2t+3)^3\,dt$

**138.** $\int_0^2 \sqrt{4-x^2}\,dx$

**139.** $\int_0^3 \sqrt{y+1}\,dy$

**140.** $\int_{-1}^1 r\sqrt{1-r^2}\,dr$

**141.** $\int_0^{\pi/4} \tan x\sec^2 x\,dx$

**142.** $\int_{2\pi}^{3\pi} 3\cos^2 x\sin x\,dx$

**143.** $\int_0^1 t^3\left(1+t^4\right)^3\,dt$

**144.** $\int_0^1 \frac{r}{\left(4+r^2\right)^2}\,dr$

**145.** $\int_0^1 \frac{10\sqrt{v}}{\left(1+v^{3/2}\right)^2}\,dv$

**146.** $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$

**147.** $\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$

**148.** $\int_0^{\pi/6} (1-\cos 3t)\sin 3t\,dt$

**149.** $\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$

**150.** $\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$

**151.** $\int_{-\pi/2}^{0} \frac{\sin w}{(3+2\cos w)^2}\,dw$

**152.** $\int_0^1 \sqrt{t^5+2t}\,\left(5t^4+2\right)dt$

**153.** $\int_1^4 \frac{dy}{2\sqrt{y}\left(1+\sqrt{y}\right)^2}$

**154.** $\int_0^1 \left(4y-y^2+4y^3+1\right)^{-2/3}\left(12y^2-2y+4\right)dy$

**155.** $\int_0^5 |x-2|\,dx$

**156.** $\int_0^{\pi/2} e^{\sin x}\cos x\,dx$

**157.** $\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$

**158.** $\int_0^{\pi/3} \frac{4\sin\theta}{1-4\cos\theta}\,d\theta$

**159.** $\int_1^2 \frac{2\ln x}{x}\,dx$

**160.** $\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$

**161.** $\int_0^{\pi/2} \tan\frac{x}{2}\,dx$

**162.** $\int_{\pi/4}^{\pi/2} \cot x\,dx$

**163.** $\int_{-\ln 2}^{0} e^{-x}dx$

**164.** $\int_{\pi/4}^{\pi/2} \left(1+e^{\cot\theta}\right)\csc^2\theta\,d\theta$

**165.** $\int_0^{\sqrt{\ln\pi}} 2x\,e^{x^2}\cos\left(e^{x^2}\right)dx$

**166.** $\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx$

**167.** $\int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan t}\sec^2 t\,dt$

**168.** $\int_1^e x^{(\ln 2)-1}dx$

**169.** $\int_0^9 \frac{2\log_{10}(x+1)}{x+1}\,dx$

**170.** $\int_1^e \frac{2\ln 10\log_{10} x}{x}\,dx$

**171.** $\int_1^{e^x} \frac{1}{t}\,dt$

**172.** $\frac{1}{\ln a}\int_1^x \frac{1}{t}\,dt \quad x>0$

**173.** $\int_0^{3\sqrt{2}/4} \frac{dx}{\sqrt{9-4x^2}}$

**174.** $\int_{\pi/6}^{\pi 4} \frac{\csc^2 x dx}{1+(\cot x)^2}$

**175.** $\int_1^{e^{\pi/4}} \frac{4dt}{t\left(1+\ln^2 t\right)}$

**176.** $\int_{1/2}^1 \frac{6dx}{\sqrt{-4x^2+4x+3}}$

**177.** $\displaystyle\int_{2/\sqrt{3}}^{2} \frac{\cos\left(\sec^{-1} x\right)dx}{x\sqrt{x^2-1}}$

**178.** $\displaystyle\int_0^3 \frac{x}{\sqrt{25-x^2}}\,dx$

**179.** $\displaystyle\int_0^{\pi} \sin^2 5\theta \, d\theta$

**180.** $\displaystyle\int_0^{\pi} \left(1-\cos^2 3\theta\right) d\theta$

**181.** $\displaystyle\int_2^3 \frac{x^2+2x-2}{x^3+3x^2-6x}\,dx$

**182.** $\displaystyle\int_0^{\ln 2} \frac{e^x}{1+e^{2x}}\,dx$

**183.** $\displaystyle\int_1^3 x\sqrt[3]{x^2-1}\,dx$

**184.** $\displaystyle\int_0^2 (x+3)^3\,dx$

**185.** $\displaystyle\int_{-2}^{2} e^{4x+8}dx$

**186.** $\displaystyle\int_0^1 \sqrt{x}\left(\sqrt{x}+1\right)dx$

**187.** $\displaystyle\int_0^1 \frac{dx}{\sqrt{4-x^2}}$

**188.** $\displaystyle\int_0^2 \frac{2x}{\left(x^2+1\right)^2}\,dx$

**189.** $\displaystyle\int_0^{\pi/2} \sin^2\theta\, \cos\theta\, d\theta$

**190.** $\displaystyle\int_0^{\pi/4} \frac{\sin x}{\cos^2 x}\,dx$

**191.** $\displaystyle\int_{1/3}^{1/\sqrt{3}} \frac{4}{9x^2+1}\,dx$

**192.** $\displaystyle\int_0^{\ln 4} \frac{e^x}{3+2e^x}\,dx$

**193.** $\displaystyle\int_{-\pi}^{\pi} \cos^2 x\, dx$

**194.** $\displaystyle\int_0^{\pi/4} \cos^2 8\theta\, d\theta$

**195.** $\displaystyle\int_{-\pi/4}^{\pi/4} \sin^2 2\theta\, d\theta$

**196.** $\displaystyle\int_0^{\pi/6} \frac{\sin 2x}{\sin^2 x+2}\,dx$

**197.** $\displaystyle\int_0^{\pi/2} \sin^4\theta\, d\theta$

**198.** $\displaystyle\int_0^1 x\sqrt{1-x^2}\, dx$

**199.** $\displaystyle\int_0^{1/4} \frac{x}{\sqrt{1-16x^2}}\,dx$

**200.** $\displaystyle\int_2^3 \frac{x}{\sqrt[3]{x^2-1}}\,dx$

**201.** $\displaystyle\int_0^{6/5} \frac{dx}{25x^2+36}$

**202.** $\displaystyle\int_0^2 x^3\sqrt{16-x^4}\, dx$

**203.** $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^2 x}\,dx$

**204.** $\displaystyle\int_{-1}^{1} (x-1)\left(x^2-2x\right)^7\,dx$

**205.** $\displaystyle\int_{-\pi}^{0} \frac{\sin x}{2+\cos x}\,dx$

**206.** $\displaystyle\int_{0}^{1} \frac{(x+1)(x+2)}{2x^3+9x^2+12x+36}\,dx$

**207.** $\displaystyle\int_{1}^{2} \frac{4}{9x^2+6x+1}\,dx$

**208.** $\displaystyle\int_{0}^{\pi/4} e^{\sin^2 x}\sin 2x\,dx$

**209.** $\displaystyle\int_{0}^{1} x\sqrt{x+a}\,dx \quad (a>0)$

**210.** $\displaystyle\int_{0}^{1} x\sqrt[p]{x+a}\,dx \quad (a>0)$

**211.** $\displaystyle\int_{0}^{1} x\sqrt{1-\sqrt{x}}\,dx$

**212.** $\displaystyle\int_{0}^{1} \sqrt{x-x\sqrt{x}}\,dx$

**213.** $\displaystyle\int_{0}^{\pi/2} \frac{\cos\theta\sin\theta}{\sqrt{\cos^2\theta+16}}\,d\theta$

**214.** $\displaystyle\int_{\frac{2}{5\sqrt{3}}}^{\frac{2}{5}} \frac{dx}{x\sqrt{25x^2-1}}$

**215.** $\displaystyle\int_{0}^{4} \frac{x}{\sqrt{9+x^2}}\,dx$

**216.** $\displaystyle\int_{0}^{\pi/4} \frac{\sin\theta}{\cos^3\theta}\,d\theta$

**217.** $\displaystyle\int_{0}^{1} 2x\left(4-x^2\right)\,dx$

**218.** $\displaystyle\int_{0}^{3} \frac{x^2+1}{\sqrt{x^3+3x+4}}\,dx$

**219.** $\displaystyle\int_{0}^{4} \frac{x}{x^2+1}\,dx$

**220.** $\displaystyle\int_{1}^{e^2} \frac{\ln x}{x}\,dx$

**221.** $\displaystyle\int_{0}^{3} \frac{x^2+1}{\sqrt{x^3+3x+4}}\,dx$

**222.** $\displaystyle\int_{0}^{1} \left(y^3+6y^2-12y+9\right)^{-1/2}\left(y^2+4y-4\right)dy$

**223.** $\displaystyle\int_{-1}^{2} x^2e^{x^3+1}\,dx$

**224.** $\displaystyle\int_{0}^{2} x^2e^{x^3}\,dx$

**225.** $\displaystyle\int_{0}^{4} \frac{x}{x^2+1}\,dx$

Solve the initial value problem

**226.** $\frac{dy}{dt} = e^t \sin\left(e^t - 2\right), \quad y(\ln 2) = 0$

**227.** $\frac{dy}{dt} = e^{-t} \sec^2\left(\pi e^{-t}\right), \quad y(\ln 4) = \frac{2}{\pi}$

**228.** Verify the integration formula: $\int \frac{\tan^{-1} x}{x^2} dx = \ln x - \frac{1}{2}\ln\left(1 + x^2\right) - \frac{\tan^{-1} x}{x} + C$

**229.** Verify the integration formula: $\int \ln\left(a^2 + x^2\right) dx = x\ln\left(a^2 + x^2\right) - 2x + 2a\tan^{-1}\frac{x}{a} + C$

**230.** Find the area of the region bounded by the graphs of $x = 3\sin y\sqrt{\cos y}$, *and* $x = 0, \quad 0 \le y \le \frac{\pi}{2}$

**231.** Find the area of the region bounded by the graph of $f(x) = \frac{x}{\sqrt{x^2 - 9}}$ *on* $3 \le x \le 4$

**232.** Find the area of the region bounded by the graph of $f(x) = \frac{x}{\sqrt{x^2 - 9}}$ and the *x-axis* between $x = 4$ and $x = 5$.

**233.** Find the area of the region bounded by the graph of $f(x) = x \sin x^2$ and the *x-axis* between $x = 0$ and $x = \sqrt{\pi}$.

**234.** Find the area of the region bounded by the graph of $f(\theta) = \cos\theta \sin\theta$ and the *θ-axis* between $\theta = 0$ and $\theta = \frac{\pi}{2}$.

**235.** Find the area of the region bounded by the graph of $f(x) = (x - 4)^4$ and the *x-axis* between $x = 2$ and $x = 6$.

**236.** Perhaps the simplest change of variables is the shift or translation given by $u = x + c$, where $c$ is a real number.

***a)*** Prove that shifting a function does not change the net area under the curve, in the sense that

$$\int_a^b f(x + c)dx = \int_{a+c}^{b+c} f(u)du$$

***b)*** Draw a picture to illustrate this change of variables in the case that $f(x) = \sin x$, $a = 0$, $b = \pi$, and $c = \frac{\pi}{2}$

**237.** Another change of variables that can be interpreted geometrically is the scaling $u = cx$, where $c$ is a real number. Prove and interpret the fact that

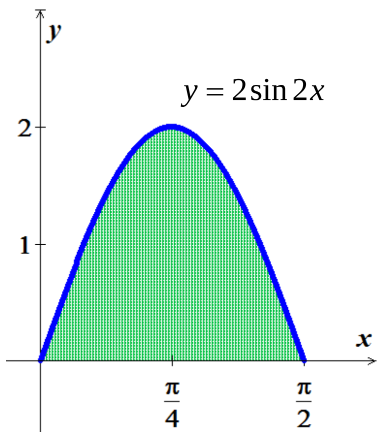
$$\int_a^b f(cx)\,dx = \frac{1}{c}\int_{ac}^{bc} f(u)\,du$$

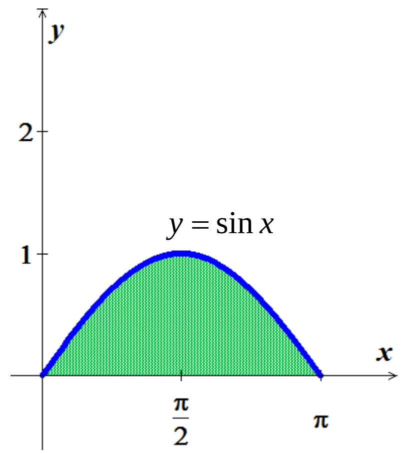
Draw a picture to illustrate this change of variables in the case that $f(x) = \sin x$, $a = 0$, $b = \pi$, and $c = \frac{1}{2}$

**238.** The function $f$ satisfies the equation $3x^4 - 48 = \int_2^x f(t)\,dt$. Find $f$ and check your answer by substitution.

**239.** Assume $f'$ is continuous on $[2,\ 4]$, $\int_1^2 f'(2x)\,dx = 10$, and $f(2) = 4$. Evaluate $f(4)$.

**240.** The area of the shaded region under the curve $y = 2\sin 2x$ in

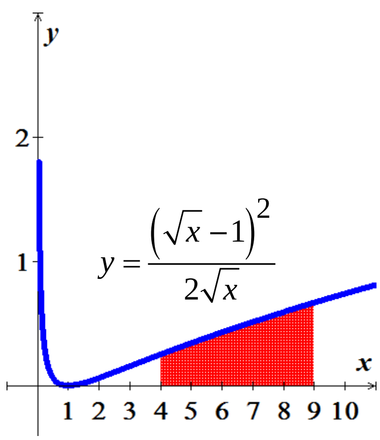


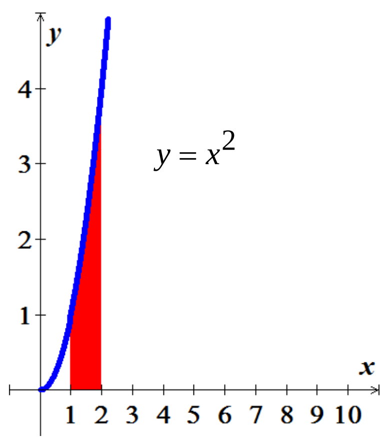




*a)* Equals the area on the shaded region under the curve $y = \sin x$

*b)* Explain why this is true without computation areas.

**241.** The area of the shaded region under the curve $y = \dfrac{\left(\sqrt{x}-1\right)^2}{2\sqrt{x}}$ on the interval $[4,\ 9]$



*a)* Equals the area on the shaded region under the curve $y = x^2$ on the interval $[1,\ 2]$

*b)* Explain why this is true without computation areas.

**242.** The family of parabolas $y = \frac{1}{a} - \frac{x^2}{a^3}$, where $a > 0$, has the property that for $x \ge 0$, the *x*-intercept is $(a,\ 0)$ and the *y*-intercept is $\left(0,\ \frac{1}{a}\right)$. Let $A(a)$ be the area of the region in the first quadrant bounded by the parabola and the *x-axis*. Find $A(a)$ and determine whether it is increasing, decreasing, or constant function of $a$.

**243.** Consider the right triangle with vertices $(0,\ 0)$, $(0,\ b)$, and $(a,\ 0)$, where $a > 0$ and $b > 0$. Show that the average vertical distance from points on the *x-axis* to the hypotenuse is $\frac{b}{2}$, for all $a > 0$.

**244.** Consider the integral $I = \int \sin^2 x\ \cos^2 x\ dx$

*a)* Find *I* using the identity $\sin 2x = 2\sin x \cos x$

*b)* Find *I* using the identity $\cos^2 x = 1 - \sin^2 x$

*c)* Confirm that the results in part (*a*) and (*b*) are consistent and compare the work involved in each method.

**245.** Let $H(x) = \int_0^x \sqrt{4 - t^2}\, dt$, for $-2 \le x \le 2$.

*a)* Evaluate $H(0)$

*b)* Evaluate $H'(1)$

*c)* Evaluate $H'(2)$

*d)* Use geometry to evaluate $H(2)$

*e)* Find the value of *s* such that $H(x) = sH(-x)$

Evaluate the limits

**246.** $\lim\limits_{x \to 2} \dfrac{\int_2^x e^{t^2}\, dt}{x - 2}$

**247.** $\lim\limits_{x \to 1} \dfrac{\int_1^{x^2} e^{t^3}\, dt}{x - 1}$

**248.** Prove that for nonzero constants $a$ and $b$, $\int \frac{dx}{a^2x^2 + b^2} = \frac{1}{ab} \tan^{-1}\left(\frac{ax}{b}\right) + C$

**249.** Let $a > 0$ be a real number and consider the family of functions $f(x) = \sin ax$ on the interval $\left[0, \frac{\pi}{a}\right]$.

*a)* Graph $f$, for $a$ = 1, 2, 3.

*b)* Let $g(a)$ be the area of the region bounded by the graph of $f$ and the *x-axis* on the interval $\left[0, \frac{\pi}{a}\right]$. Graph $g$ for $0 < a < \infty$. Is $g$ an increasing function, a decreasing function, or neither?

**250.** Explain why if a function $u$ satisfies the equation $u(x) + 2\int_0^x u(t)\,dt = 10$, then it also satisfies the equation $u'(x) + 2u(x) = 0$. Is it true that is $u$ satisfies the second equation, then it satisfies the first equation?

**251.** Let $f(x) = \int_0^x (t-1)^{15}(t-2)^9\,dt$

*a)* Find the interval on which $f$ is increasing and the intervals on which $f$ is decreasing.

*b)* Find the intervals on which $f$ is concave up and the intervals on which $f$ is concave down.

*c)* For what values of $x$ does $f$ have local minima? Local maxima?

*d)* Where are the inflection points of $f$?

**252.** A company is considering a new manufacturing process in one of its plants. The new process provides substantial initial savings, with the savings declining with time $t$ (in years) according to the rate-of-savings function

$$S'(t) = 100 - t^2$$

where $S'(t)$ is in thousands of dollars per year. At the same time, the cost of operating the new process increases with time $t$ (in years), according to the rate-of-cost function (in thousands of dollars per year)

$$C'(t) = t^2 + \tfrac{14}{3}t$$

*a)* For how many years will the company realize savings?

*b)* What will be the net total savings during this period?

**253.** Find the producers' surplus if the supply function for pork bellies is given by

$$S(x) = x^{5/2} + 2x^{3/2} + 50$$

Assume supply and demand are in equilibrium at $x = 16$.

**254.** An object moves along a line with a velocity in $m/s$ given by $v(t) = 8\cos\left(\frac{\pi t}{6}\right)$. Its initial position is $s(0) = 0$.

*a)* Graph the velocity function.

*b)* The position of the object is given by $s(t)=\int_0^t v(y)\,dy$, for $t \ge 0$. Find the position function, for $t \ge 0$.

*c)* What is the period of the motion – that is, starting at any point, how long does it take the object to return to that position?

**255.** The population of a culture of bacteria has a growth rate given by $p'(t)=\frac{200}{(t+1)^r}$ bacteria per hour, for $t \ge 0$, where $r > 1$ is a real number. It is shown that the increase in the population over time interval $[0,\ t]$ is given by $\int_0^t p'(s)\,ds$. (note that the growth rate decreases in time, reflecting competition for space and food.)

*a)* Using the population model with $r=2$, what is the increase in the population over the time interval $0 \le t \le 4$?

*b)* Using the population model with $r=3$, what is the increase in the population over the time interval $0 \le t \le 6$?

*c)* Let $\Delta P$ be the increase in the population over a fixed time interval $[0,\ T]$. For fixed *T*, does $\Delta P$ increase or decrease with the parameter *r*? Explain.

*d)* A lab technician measures an increase in the population of 350 bacteria over the 10-hr period $[0,\ 10]$. Estimate the value of *r* that best fits this data point.

*e)* Use the population model in part (*b*) to find the increase in population over time interval $[0,\ T]$, for any $T > 0$. If the culture is allowed to grow indefinitely $(T \to \infty)$, does the bacteria population increase without bound? Or does it approach a finite limit?

**256.** Consider the function $f(x)=x^2-5x+4$ and the area function $A(x)=\int_0^x f(t)\,dt$.

*a)* Graph $f$ on the interval [0, 6].

*b)* Compute and graph *A* on the interval [0, 6].

*c)* Show that the local extrema of *A* occur at the zeros of $f$.

*d)* Give a geometric and analytical explanation for the observation in part (c).

*e)* Find the approximate zeros of *A*, other than 0, and call them $x_1$ and $x_2$.

*f)* Find *b* such that the area bounded by the graph of $f$ and the *x*-axis on the interval $[0,\ x_1]$ equals the area bounded by the graph of $f$ and the *x*-axis on the interval $[x_1,\ b]$.

*g)* If $f$ is an integrable function and $A(x)=\int_0^x f(t)\,dt$, is it always true that the local extrema of *A* occur at the zeros of $f$? Explain