Solution   *Section* **6.7 – Trigonometric Form**

Exercise

Write $-\sqrt{3}+i$ in trigonometric form. (Use radian measure)

Solution

$$-\sqrt{3}+i \implies \begin{cases} x=-\sqrt{3} \\ y=1 \end{cases}$$

$$r = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2}$$

$$\underline{=2}$$

$$\hat{\theta} = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) \qquad\qquad \theta = \tan^{-1}\frac{y}{x}$$

$$\underline{= \frac{\pi}{6}}$$

The angle is in quadrant *II*, therefore;

$$\theta = \pi - \frac{\pi}{6}$$

$$\underline{= \frac{5\pi}{6}}$$

$$\underline{-\sqrt{3}+i = 2 \; cis \frac{5\pi}{6}}$$

Exercise

Write $3-4i$ in trigonometric form.

Solution

$$3-4i \implies \begin{cases} x=3 \\ y=-4 \end{cases}$$

$$r = \sqrt{3^2 + (-4)^2}$$

$$\underline{=2}$$

$$\hat{\theta} = \tan^{-1}\left(\frac{4}{3}\right) \qquad\qquad \theta = \tan^{-1}\frac{y}{x}$$

$$\underline{\approx 53°}$$

The angle is in quadrant *II*, therefore;

$$\theta = 180° - 53°$$

$$\underline{= 127°}$$

$$\underline{3-4i = 5 \; cis 127°}$$

Exercise

Write $-21 - 20i$ in trigonometric form.

Solution

$$-21 - 20i \implies \begin{cases} x = -21 \\ y = -20 \end{cases}$$

$$r = \sqrt{(-21)^2 + (-20)^2}$$
$$\underline{= 29}$$

$$\hat{\theta} = \tan^{-1}\left(\frac{20}{21}\right) \qquad\qquad \theta = \tan^{-1}\frac{y}{x}$$
$$\underline{\approx 43.6°}$$

The angle is in quadrant *III*, therefore;

$$\theta = 180° + 43.6°$$
$$\underline{= 223.6°}$$

$$\underline{-21 - 20i = 29\ cis\,223.6°}$$

Exercise

Write $11 + 2i$ in trigonometric form.

Solution

$$11 + 2i \implies \begin{cases} x = 11 \\ y = 2 \end{cases}$$

$$r = \sqrt{11^2 + 2^2}$$
$$= \sqrt{125}$$
$$\underline{= 5\sqrt{5}}$$

$$\theta = \tan^{-1}\left(\frac{2}{11}\right) \qquad\qquad \theta = \tan^{-1}\frac{y}{x}$$
$$\underline{\approx 10.3°}$$

$$\underline{11 + 2i = 5\sqrt{5}\ cis\,10.3°}$$

Exercise

Write $\sqrt{3}-i$ in trigonometric form.

Solution

$$\sqrt{3}-i \Rightarrow \begin{cases} x=\sqrt{3} \\ y=-1 \end{cases}$$

$$r=\sqrt{3+1}$$
$$=2$$

$$\hat{\theta}=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \qquad\qquad \theta=\tan^{-1}\frac{y}{x}$$

$$\approx 30°$$

The angle is in quadrant *IV*, therefore;

$$\theta=360°-30°$$
$$=330°$$

$$\sqrt{3}-i=2\ cis330°$$

Exercise

Write $1-\sqrt{3}i$ in trigonometric form.

Solution

$$1-\sqrt{3}i \Rightarrow \begin{cases} x=1 \\ y=-\sqrt{3} \end{cases}$$

$$r=\sqrt{1+3}$$
$$=2$$

$$\hat{\theta}=\tan^{-1}\left(\frac{\sqrt{3}}{1}\right) \qquad\qquad \theta=\tan^{-1}\frac{y}{x}$$

$$=60°$$

The angle is in quadrant *IV*, therefore;

$$\theta=360°-60°$$
$$=300°$$

$$1-\sqrt{3}i=2\ cis300°$$

Exercise

Write $9\sqrt{3}+9i$ in trigonometric form.

Solution

$$9\sqrt{3}+9i \implies \begin{cases} x=9\sqrt{3} \\ y=9 \end{cases}$$

$$r=9\sqrt{3+1}$$
$$\underline{=18}$$

$$\theta=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$\underline{=30°}$$

$$9\sqrt{3}+9i =18\ cis30°$$

Exercise

Write $-2+3i$ in trigonometric form.

Solution

$$-2+3i \implies \begin{cases} x=-2 \\ y=3 \end{cases}$$

$$r=\sqrt{4+9}$$
$$\underline{=\sqrt{13}}$$

$$\hat{\theta}=\tan^{-1}\left(\frac{3}{2}\right) \qquad\qquad \theta=\tan^{-1}\frac{y}{x}$$
$$\underline{\approx 56.31°}$$

The angle is in quadrant *II*, therefore;

$$\theta=180°-56.31°$$
$$\underline{\approx 123.69°}$$

$$-2+3i =\sqrt{13}\ cis123.69°$$

Exercise

Write $4\left(\cos 30°+i\sin 30°\right)$ in standard form.

Solution

$$4\left(\cos 30°+i\sin 30°\right)=4\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)$$
$$\underline{=2\sqrt{3}+2i}$$

Exercise

Write $\sqrt{2}\ cis\frac{7\pi}{4}$ in standard form.

Solution

$$\sqrt{2}\ cis\frac{7\pi}{4} = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

$$= 1 - i$$

Exercise

Write $3cis\,210°$ in standard form.

Solution

$$3cis\,210° = 3\left(\cos 210° + i\sin 210°\right)$$

$$= -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$$

Exercise

Write $4\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$ in standard form.

Solution

$$4\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = 4\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$= 2\sqrt{2} - 2i\sqrt{2}$$

Exercise

Write $4cis\frac{\pi}{2}$ in standard form.

Solution

$$4cis\frac{\pi}{2} = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= 4i$$

Exercise

Find the quotient $\dfrac{20cis\left(75°\right)}{4cis\left(40°\right)}$. Write the result in rectangular form.

Solution

$$\frac{20cis(75°)}{4cis(40°)} = \frac{20}{4}cis(75° - 40°)$$

$$= 5cis(35°)$$

$$= 5(\cos 35° + i\sin 35°)$$

$$\underline{= 4.1 + 2.87i}$$

Exercise

Divide $\quad z_1 = 1 + i\sqrt{3}\quad by \quad z_2 = \sqrt{3} + i$. Write the result in rectangular form.

Solution

$$\frac{z_1}{z_2} = \frac{1 + i\sqrt{3}}{\sqrt{3} + i}$$

$$= \frac{1 + i\sqrt{3}}{\sqrt{3} + i} \frac{\sqrt{3} - i}{\sqrt{3} - i}$$

$$= \frac{\sqrt{3} - i + 3i - \sqrt{3}\ i^2}{3 + 1}$$

$$= \frac{2\sqrt{3} + 2i}{4}$$

$$= \frac{2\sqrt{3}}{4} + \frac{2i}{4}$$

$$\underline{= \frac{\sqrt{3}}{2} + \frac{i}{2}}$$

$\boxed{or}$

$$1 + i\sqrt{3} \quad \rightarrow \quad \begin{cases} r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} \\ \theta = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3} \end{cases}$$

$$\sqrt{3} + i \quad \rightarrow \quad \begin{cases} r = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} \\ \theta = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6} \end{cases}$$

$$\frac{z_1}{z_2} = \frac{2cis\frac{\pi}{3}}{2cis\frac{\pi}{6}}$$

$$= \frac{2}{2}cis\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$= \frac{2}{2}cis\left(\frac{\pi}{6}\right)$$

144

$$= cis\left(\frac{\pi}{6}\right)$$

$$= \frac{\sqrt{3}}{2} + \frac{i}{2}$$

Exercise

Find $(1+i)^8$ and express the result in rectangular form.

Solution

$$1+i \implies \begin{cases} x=1 \\ y=1 \end{cases}$$

$$r = \sqrt{1^2 + 1^2}$$
$$= \sqrt{2}$$

$$\theta = \tan^{-1}(1)$$
$$= \frac{\pi}{4}$$

$$1+i = \sqrt{2}\,cis\,\frac{\pi}{4}$$

$$(1+i)^8 = \left(\sqrt{2}\,cis\,\frac{\pi}{4}\right)^8$$

$$= \left(\sqrt{2}\right)^8 cis\left[8\left(\frac{\pi}{4}\right)\right]$$

$$= 16cis\,2\pi$$

$$= 16\left(\cos 2\pi + i\sin 2\pi\right)$$

$$= 16\left(1 + i0\right)$$

$$= 16$$

Exercise

Find $(1+i)^{10}$ and express the result in rectangular form.

Solution

$$1+i \implies \begin{cases} x=1 \\ y=1 \end{cases}$$

$$r = \sqrt{1^2 + 1^2}$$
$$= \sqrt{2}$$

$$\theta = \tan^{-1}(1)$$

145

$$= \frac{\pi}{4}$$

$$1+i = \sqrt{2}cis\frac{\pi}{4}$$

$$(1+i)^{10} = \left(\sqrt{2}cis\frac{\pi}{4}\right)^{10}$$

$$= \left(\sqrt{2}\right)^{10}cis\left[10\left(\frac{\pi}{4}\right)\right]$$

$$= 32cis\frac{5\pi}{2}$$

$$= 32cis\frac{\pi}{2}$$

$$= 32\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$

$$= 32(0+i)$$

$$= 32i$$

Exercise

Find and express the result in rectangular form $(1-i)^5$

Solution

$$1-i \implies \begin{cases} x = 1 \\ y = -1 \end{cases}$$

$$r = \sqrt{1^2 + 1^2}$$

$$= \sqrt{2}$$

$$\hat{\theta} = \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

The angle is in quadrant IV, therefore;

$$\theta = 2\pi - \frac{\pi}{4}$$

$$= \frac{7\pi}{4}$$

$$1-i = \sqrt{2}\;cis\frac{7\pi}{4}$$

$$(1-i)^5 = \left(\sqrt{2}\;cis\frac{7\pi}{4}\right)^5$$

$$= 4\sqrt{2}\left(cis\left(5\times\frac{7\pi}{4}\right)\right)$$

$$= 4\sqrt{2}\left(cis\,\frac{35\pi}{4}\right)$$

$35\pi - 8 \times 2\pi = 3\pi$

$$= 4\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$= 4\sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}\right)$$

$$\underline{= -4 + 4i}\;|$$

Exercise

Find and express the result in rectangular form $\left(1-\sqrt{5}i\right)^8$

Solution

$$1-\sqrt{5}i \;\Rightarrow\; \begin{cases} x = 1 \\ y = -\sqrt{5} \end{cases}$$

$$r = \sqrt{1+5}$$

$$\underline{= \sqrt{6}}\;|$$

$$\hat{\theta} = \tan^{-1}\left(\frac{\sqrt{5}}{1}\right)$$

$$\underline{\approx 66°}\;|$$

The angle is in quadrant *IV*, therefore;

$$\theta = 360° - 66°$$

$$\underline{= 294°}\;|$$

$$\underline{1-\sqrt{5}i = \sqrt{6}\;cis\,294°}\;|$$

$$\left(1-\sqrt{5}i\right)^8 = \left(\sqrt{6}cis\,294°\right)^8$$

$$= \left(\sqrt{6}\right)^8\left(cis\,2352°\right)$$

$2352° - 6 \times 360° = 192°$

$$= 1296\left(\cos 192° + i\sin 192°\right)$$

$$= 1296\left(-.978 - 0.208i\right)$$

$$\underline{= -1267.488 - 269.568\;i}\;|$$

Exercise

Find and express the result in rectangular form $(3cis80°)^3$

Solution

$$(3cis80°)^3 = 3^3 (cis240°)$$

$$= 27(\cos 240° + i\sin 240°)$$

$$= 27\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$= -\frac{27}{2} - i\frac{27\sqrt{3}}{2}$$

Exercise

Find and express the result in rectangular form $\left(\sqrt{3}cis10°\right)^6$

Solution

$$\left(\sqrt{3}cis10°\right)^6 = 27(cis60°)$$

$$= 27(\cos 60° + i\sin 60°)$$

$$= \frac{27}{2} + i\frac{27\sqrt{3}}{2}$$

Exercise

Find and express the result in rectangular form $\left(\sqrt{2} - i\right)^6$

Solution

$$\sqrt{2} - i \implies \begin{cases} x = \sqrt{2} \\ y = -1 \end{cases}$$

$$r = \sqrt{2+1}$$

$$= \sqrt{3}$$

$$\hat{\theta} = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\approx 35.26°$$

The angle is in quadrant IV, therefore;

$$\theta = 360° - 35.26°$$

$$= 324.74°$$

$$\sqrt{2} - i = \sqrt{3}\ cis324.74°$$

$$\left(\sqrt{2}-i\right)^6 = \left(\sqrt{3}\ cis\,324.74°\right)^6$$

$$= 27\left(cis1948.44°\right) \qquad\qquad 1948.44° - 5\times360° = 148.44°$$

$$= 27\left(\cos148.44° + i\sin148.44°\right)$$

$$\underline{= -23 + 14.142i}\ |$$

Exercise

Find and express the result in rectangular form $\left(4cis40°\right)^6$

Solution

$$\left(4cis40°\right)^6 = 4^6\left(cis\left(6\times40°\right)\right)$$

$$= 4^6\left(\cos240° + i\sin240°\right)$$

$$= 4096\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$\underline{= -2048 + 2048\ i\sqrt{3}}\ |$$

Exercise

Find and express the result in rectangular form $\left(2cis30°\right)^5$

Solution

$$\left(2cis30°\right)^5 = 2^5\,cis\left(5(30°)\right)$$

$$= 32\left(\cos150° + i\sin150°\right)$$

$$= 32\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$\underline{= -16\sqrt{3} + 16i}\ |$$

Exercise

Find and express the result in rectangular form $\left(\frac{1}{2}cis72°\right)^5$

Solution

$$\left(\frac{1}{2}cis72°\right)^5 = \frac{1}{2^5}\,cis\left(5\times72°\right)$$

$$= \frac{1}{32}\,cis\left(\cos360° + i\sin360°\right)$$

$$\underline{= \frac{1}{32}}\ |$$

Exercise

Find ***fifth*** roots of $z = 1 + i\sqrt{3}$ and express the result in rectangular form.

Solution

$$1 + i\sqrt{3} \implies \begin{cases} x = 1 \\ y = \sqrt{3} \end{cases}$$

$$r = \sqrt{1 + 3}$$
$$\underline{= 2}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$$
$$\underline{= 60°}$$

$$\underline{1 + i\sqrt{3} = 2 \ cis60°}$$

$$\left(1 + i\sqrt{3}\right)^{1/5} = \left(2 \ cis60°\right)^{1/5}$$

$$= \sqrt[5]{2}\left(cis \ \frac{60°}{5} + \frac{360°k}{5}\right)$$

$$\underline{= \sqrt[5]{2} \ cis\left(12° + 72°k\right)} \qquad k = 0,\ 1,\ 2,\ 3,\ 4$$

For $k = 0$

$$\sqrt[5]{2} \ cis\left(12° + 72°0\right) = \underline{\sqrt[5]{2} \ cis12°}$$

For $k = 1$

$$\sqrt[5]{2} \ cis\left(12° + 72°.(1)\right) = \underline{\sqrt[5]{2} \ cis84°}$$

For $k = 2$

$$\sqrt[5]{2} \ cis\left(12° + 72°.(2)\right) = \underline{\sqrt[5]{2} \ cis156°}$$

For $k = 3$

$$\sqrt[5]{2} \ cis\left(12° + 72°.(3)\right) = \underline{\sqrt[5]{2} \ cis228°}$$

For $k = 4$

$$\sqrt[5]{2} \ cis\left(12° + 72°.(4)\right) = \underline{\sqrt[5]{2} \ cis300°}$$

Exercise

Find the **fourth** roots of $z = 16cis60°$

Solution

$$\sqrt[4]{z} = \sqrt[4]{16} \, cis\left(\frac{60°}{4} + \frac{360°}{4}k\right)$$

$$= 2cis\left(15° + 90°k\right) \mid \qquad k = 0,\ 1,\ 2,\ 3$$

For $k = 0$

$$2\,cis\left(15° + 90°(0)\right) = 2cis15° \mid$$

For $k = 1$

$$2\,cis\left(15° + 90°(1)\right) = 2cis105° \mid$$

For $k = 2$

$$2\,cis\left(15° + 90°(2)\right) = 2cis195° \mid$$

For $k = 3$

$$2\,cis\left(15° + 90°(3)\right) = 2cis285° \mid$$

Exercise

Find the **fourth** roots of $\sqrt{3} - i$

Solution

$$\sqrt{3} - i \quad \Rightarrow \quad \begin{cases} x = \sqrt{3} \\ y = -1 \end{cases}$$

$$r = \sqrt{3 + 1}$$

$$\underline{= 2 \mid}$$

$$\hat{\theta} = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\underline{= \frac{\pi}{6}} \mid$$

The angle is in quadrant *IV*, therefore;

$$\theta = 2\pi - \frac{\pi}{6}$$

$$\underline{= \frac{11\pi}{6}} \mid$$

$$\underline{\sqrt{3} - i = 2\,cis\frac{11\pi}{6}} \mid$$

$$\sqrt[4]{\sqrt{3} - i} = \sqrt[4]{2}\,cis\frac{11\pi}{6}$$

151

$$= \sqrt[4]{2} \ cis \left(\frac{1}{4} \frac{11\pi}{6} + \frac{2\pi k}{4} \right)$$

$$= \sqrt[4]{2} \ cis \left(\frac{11\pi}{24} + \frac{\pi k}{2} \right) \Bigg| \qquad k = 0, \ 1, \ 2, \ 3$$

For $k = 0$

$$\sqrt[4]{2} \ cis \left(\frac{11\pi}{24} + 0 \right) = \sqrt[4]{2} \ cis \frac{11\pi}{24} \Bigg|$$

For $k = 1$

$$\sqrt[4]{2} \ cis \left(\frac{11\pi}{24} + \frac{\pi}{2} \right) = \sqrt[4]{2} \ cis \frac{23\pi}{24} \Bigg|$$

For $k = 2$

$$\sqrt[4]{2} \ cis \left(\frac{11\pi}{24} + \pi \right) = \sqrt[4]{2} \ cis \frac{35\pi}{24} \Bigg|$$

For $k = 3$

$$\sqrt[4]{2} \ cis \left(\frac{11\pi}{24} + \frac{3\pi}{2} \right) = \sqrt[4]{2} \ cis \frac{47\pi}{24} \Bigg|$$

Exercise

Find the ***fourth*** roots of $4 - 4\sqrt{3}i$

Solution

$$4 - 4\sqrt{3}i \ \Rightarrow \ \begin{cases} x = 4 \\ y = -4\sqrt{3} \end{cases}$$

$$r = 4\sqrt{3+1}$$

$$\underline{= 8} \ \Big|$$

$$\hat{\theta} = \tan^{-1} \left(\frac{4\sqrt{3}}{4} \right)$$

$$= \tan^{-1} \left(\sqrt{3} \right)$$

$$\underline{= \frac{\pi}{3}} \ \Big|$$

The angle is in quadrant *IV*, therefore;

$$\theta = 2\pi - \frac{\pi}{3}$$

$$\underline{= \frac{5\pi}{3}} \ \Big|$$

$$\underline{4 - 4\sqrt{3}i = 8 \ cis \frac{5\pi}{3}} \ \Big|$$

$$\sqrt[4]{4 - 4\sqrt{3}i} = \sqrt[4]{8 \ cis \frac{5\pi}{3}}$$

152

$$= \sqrt[4]{8}\ cis\left(\frac{5\pi}{12} + \frac{\pi k}{2}\right) \qquad k = 0,\ 1,\ 2,\ 3$$

For k = 0

$$\sqrt[4]{8}\ cis\left(\frac{5\pi}{12} + 0\right) = \sqrt[4]{8}\ cis\frac{5\pi}{12}$$

For k = 1

$$\sqrt[4]{8}\ cis\left(\frac{5\pi}{12} + \frac{\pi}{2}\right) = \sqrt[4]{8}\ cis\frac{11\pi}{12}$$

For k = 2

$$\sqrt[4]{8}\ cis\left(\frac{5\pi}{12} + \pi\right) = \sqrt[4]{8}\ cis\frac{17\pi}{12}$$

For k = 3

$$\sqrt[4]{8}\ cis\left(\frac{5\pi}{12} + \frac{3\pi}{2}\right) = \sqrt[4]{8}\ cis\frac{23\pi}{12}$$

Exercise

Find the **fourth** roots of $-16i$

Solution

$$-16i \implies \begin{cases} x = 0 \\ y = -16 \end{cases}$$

$$r = 16$$

$$\theta = \frac{3\pi}{2}$$

$$-16i = 16\ cis\frac{3\pi}{2}$$

$$\sqrt[4]{-16i} = \sqrt[4]{16\ cis\frac{3\pi}{2}}$$

$$= 2\ cis\left(\frac{3\pi}{8} + \frac{\pi k}{2}\right) \qquad k = 0,\ 1,\ 2,\ 3$$

For k = 0

$$2\ cis\left(\frac{3\pi}{8} + 0\right) = 2\ cis\frac{3\pi}{8}$$

For k = 1

$$2\ cis\left(\frac{3\pi}{8} + \frac{\pi}{2}\right) = 2\ cis\frac{7\pi}{8}$$

For k = 2

$$2\ cis\left(\frac{3\pi}{8} + \pi\right) = 2\ cis\frac{11\pi}{8}$$

For $k = 3$

$$2 \ cis\left(\frac{3\pi}{8} + \frac{3\pi}{2}\right) = 2 \ cis\frac{15\pi}{8}$$

Exercise

Find the ***cube*** roots of 27.

<u>*Solution*</u>

$$\sqrt[3]{27} = \left(27 cis 0°\right)^{1/3}$$

$$= \sqrt[3]{27} \ cis\left(\frac{0°}{3} + \frac{360°}{3}k\right)$$

$$= 3 \ cis\left(0° + 120°k\right) \qquad k = 0, \ 1, \ 2$$

For $k = 0$

$$z = 3 \ cis\left(0° + 120°(0)\right) = 2 \ cis 0°$$

For $k = 1$

$$z = 3 \ cis\left(0° + 120°(1)\right) = 2 \ cis 120°$$

For $k = 2$

$$z = 3 \ cis\left(0° + 120°(2)\right) = 2 \ cis 240°$$

Exercise

Find the ***cube*** roots of $8 - 8i$

<u>*Solution*</u>

$$8 - 8i \quad \Rightarrow \quad \begin{cases} x = 8 \\ y = -8 \end{cases}$$

$$r = 8\sqrt{1 + 1}$$

$$= 8\sqrt{2}$$

$$\hat{\theta} = \tan^{-1}\left(\frac{8}{8}\right)$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

The angle is in quadrant *IV*, therefore;

$$\theta = 2\pi - \frac{\pi}{4}$$

$$= \frac{7\pi}{4}$$

154

$$\sqrt[3]{8-8i} = \sqrt[3]{8\sqrt{2} \ cis \ \frac{7\pi}{4}}$$

$$= 2\sqrt[3]{2} \ cis\left(\frac{7\pi}{12} + \frac{2\pi k}{3}\right) \qquad k = 0, \ 1, \ 2$$

For $k = 0$

$$z = 2\sqrt[3]{2} \ cis\left(\frac{7\pi}{12} + 0\right) = 2\sqrt[3]{2} \ cis \ \frac{7\pi}{12}$$

For $k = 1$

$$z = 2\sqrt[3]{2} \ cis\left(\frac{7\pi}{12} + \frac{2\pi}{3}\right) = 2\sqrt[3]{2} \ cis \ \frac{15\pi}{12}$$

For $k = 2$

$$z = 2\sqrt[3]{2} \ cis\left(\frac{7\pi}{12} + \frac{4\pi}{3}\right) = 2\sqrt[3]{2} \ cis \ \frac{23\pi}{12}$$

Exercise

Find the **cube** roots of -8

Solution

$$r = 8$$

$$\theta = \frac{3\pi}{2}$$

$$\sqrt[3]{-8} = \sqrt[3]{8 \ cis \ \frac{3\pi}{2}}$$

$$= 2 \ cis\left(\frac{\pi}{2} + \frac{2\pi k}{3}\right) \qquad k = 0, \ 1, \ 2$$

For $k = 0$

$$z = 2 \ cis\left(\frac{\pi}{2} + 0\right) = 2 \ cis \ \frac{\pi}{2}$$

For $k = 1$

$$z = 2 \ cis\left(\frac{\pi}{2} + \frac{2\pi}{3}\right) = 2 \ cis \ \frac{7\pi}{6}$$

For $k = 2$

$$z = 2 \ cis\left(\frac{\pi}{2} + \frac{4\pi}{3}\right) = 2 \ cis \ \frac{11\pi}{6}$$

Exercise

Find all complex number solutions of $x^3 + 1 = 0$.

Solution

$x^3 + 1 = 0$

$x^3 = -1$

$\underline{r = 1 \,|}$

$\theta = \tan^{-1}\left(\dfrac{0}{-1}\right)$

$\underline{= \pi \,|}$

$x^3 = 1 \, cis\left(\pi\right)$

$x = \left(1 \, cis\pi\right)^{1/3}$

$\quad = \left(1\right)^{1/3} \, cis\left(\dfrac{\pi}{3} + \dfrac{2\pi}{3}k\right)$

$\underline{\quad = cis\left(\dfrac{\pi}{3} + \dfrac{2\pi}{3}k\right)\,|} \qquad k = 0,\ 1,\ 2$

For $k = 0$

$\quad x = cis\left(\dfrac{\pi}{3} + \dfrac{2\pi}{3}(0)\right)$

$\qquad = cis\,\dfrac{\pi}{3}$

$\qquad = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$

$\underline{\qquad = \dfrac{1}{2} + i\,\dfrac{\sqrt{3}}{2}\,|}$

For $k = 1$

$\quad x = cis\left(\dfrac{\pi}{3} + \dfrac{2\pi}{3}(1)\right)$

$\qquad = cis\left(\dfrac{3\pi}{3}\right)$

$\qquad = cis\pi$

$\qquad = \cos\pi + i\sin\pi$

$\underline{\qquad = -1 \,|}$

For $k = 2$

$\quad x = cis\left(\dfrac{\pi}{3} + \dfrac{2\pi}{3}(2)\right)$

$\qquad = cis\,\dfrac{5\pi}{3}$

$$= \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}$$

$$= \frac{1}{2} - i \frac{\sqrt{3}}{2}$$