Section 1.9 - Existence and Uniqueness of Solutions

The questions of existence and uniqueness

- When can we be sure that a solution exists**?**
- How many different solutions are there

- ✓ ***Existence***: Under what conditions does the Initial Value Problem (**IVP**) have at least one solution?
- ✓ ***Uniqueness***: Under what conditions does the **IVP** have at most one solution?
- ✓ ***Extension and Long-Term Bevaior***: How far ahead into the future and back into the past does a solution extend? How does a solution behave as t gets large?
- ✓ ***Continuity***: Suppose the data f and y_0 change. Can the corresponding change in solution be limited by limiting the change in the data f and y_0.
- ✓ ***Desctiption***: How can a solution and its behavior be described?

Existence of Solutions

Example

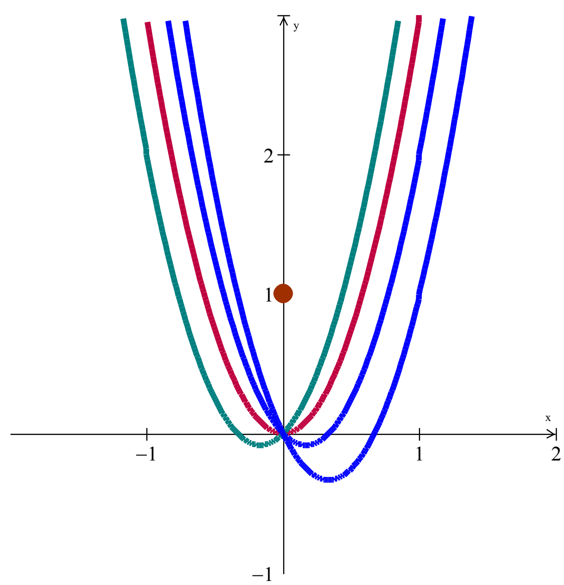
Consider the initial value problem: $tx' = x + 3t^2$ with $x(0) = 1$

Solution

$$x' = \frac{1}{t}x + 3t$$

$$x' = \frac{1}{t}x + 3t \qquad t \neq 0$$

There is ***no solution*** to the given initial value



$$u(t) = e^{-\int \frac{1}{t}dt}$$
$$= e^{-\ln t}$$
$$= \frac{1}{t}$$

$$\left[\frac{x}{t}\right]' = 3$$

$$\frac{x}{t} = \int 3dt$$
$$= 3t + C$$

$$\Rightarrow \boxed{x(t) = 3t^2 + Ct}$$

Theorem: Existence of Solutions

Suppose the function $f(t,x)$ is defined and continuous on the rectangle $\boldsymbol{R}$ in the *tx*-plane. Then given any point $\left(t_0,\ x_0\right) \in \boldsymbol{R}$, the initial value problem

$$x' = f(t,x) \quad \textit{and} \quad x(t_0) = x_0$$

has a solution $x(t)$ defined in an interval containing x_0. Furthermore, the solution will be defined at least until the solution curve $t \to \left(t,\ x(t)\right)$ leaves the rectangle $\boldsymbol{R}$.

Interval of Existence of a Solution

Example

Consider the initial value problem $x' = 1 + x^2$ with $x(0) = 0$. Find the solution and its interval of existence.

Solution

The right-hand side is $f(t,x) = 1 + x^2$ which is continuous on the entire *tx*-plane.

The solution to the initial value problem is:

$$\frac{dx}{dt} = 1 + x^2$$

$$\frac{dx}{1+x^2} = dt$$

$$\int \frac{dx}{1+x^2} = \int dt$$

$$\tan^{-1} x = t$$

$$x(t) = \tan t$$

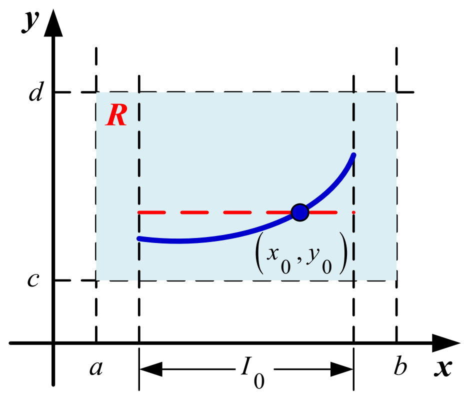
$x(t)$ is discontinuous at $t = \pm\frac{\pi}{2}$.

Hence the solution to the initial value problem is defined only for $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

The interval: $\boxed{\left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)}$

Theorem: Existence of a Unique Solution

Let R be a rectangular region in the *xy*-plane defined by $a \le x \le b, \quad c \le y \le d$ that contains the point $\left(x_0, y_0\right)$ in its interior. If $f\left(x, y\right)$ and $\dfrac{\partial f}{\partial y}$ are continuous on R, then there exists some interval $I_0 : \left(x_0 - h, x_0 + h\right), \quad h > 0$, contained in [*a, b*], and a unique function $y\left(x\right)$, defined on I_0 that is a solution of the initial-value problem (IVP)



Mathematics & Theorems

Any theorem is a logical statement which has hypotheses (when it's true) and conclusions (true)

The Hypotheses of the Uniqueness of Solutions Theorem

1. The equation is in normal form $y' = f(t, y)$
2. The right-hand side $f(t, y)$ and its derivative $\dfrac{\partial f}{\partial y}$ are both continuous in the rectangle ***R***.
3. The initial point $\left(t_0, y_0\right)$ is in the rectangle ***R***.

For the Uniqueness Theorem, the conclusions are as follows:

1- There is one and only one solution to the initial value problem.
2- The solution exists until the solution curve $t \to \left(t, y(t)\right)$ leaves the rectangle ***R***.

Example

Consider the initial value problem $tx' = x + 3t^2$. Is there a solution to this equation with initial condition $x(1) = 2$? If so, is the solution unique?

Solution

$$x' = \frac{x}{t} + 3t$$

The right-hand side: $f(t, x) = \frac{x}{t} + 3t$ is continuous except where $t = 0$.

We can take ***R*** to be any rectangle which contains the point $(1,\ 2)$ to avoid $t = 0$, we can choose

$\frac{1}{2} < t < 2$ and $0 < x < 4$

Then f is continuous everywhere in ***R*** $\Rightarrow$ hypotheses of the existence theorem are satisfied.

Since $\frac{\partial f}{\partial x} = \frac{1}{t}$ is also continuous in ***R***.

There is only one solution.

Exercises *Section* 1.9 - Existence and Uniqueness of Solutions

Which of the initial value problems are guaranteed a unique solution

1. $y' = 4 + y^2, \quad y(0) = 1$

2. $y' = \sqrt{y}, \quad y(4) = 0$

3. $y' = t\tan^{-1} y, \quad y(0) = 2$

4. $\omega' = \omega \sin\omega + s, \quad \omega(0) = -1$

5. $x' = \dfrac{t}{x+1}, \quad x(0) = 0$

6. $y' = \dfrac{1}{x}y + 2, \quad y(0) = 1$

7. $y' = e^t y - y^3, \quad y(0) = 0$

8. $y' = ty^2 - \dfrac{1}{3y+t}, \quad y(0) = 1$

9. $y' = xy, \quad y(0) = 1$

10. $y' = -\dfrac{t^2}{1-y^2}, \quad y(-1) = \dfrac{1}{2}$

11. $y' = \dfrac{y}{\sin t}, \quad y\left(\dfrac{\pi}{2}\right) = 1$

12. $y' = \sqrt{1-y^2}, \quad y(0) = 1$

13. Show that $y(t) = 0$ and $y(t) = t^3$ are both solutions of the initial value problem $y' = 3y^{2/3}$, where $y(0) = 0$. Explain why this fact doesn't contradict Theorem

14. Use a numerical solver to sketch the solution of the given initial value problem

$$\frac{dy}{dt} = \frac{t}{y+1}, \qquad y(2) = 0$$

a) Where does your solver experience difficulty? why? Use the image of your solution to estimate the interval of existence.

b) Find an explicit solution; then use your formula to determine the interval of existence. How does it compare with the approximation found in part (*a*).