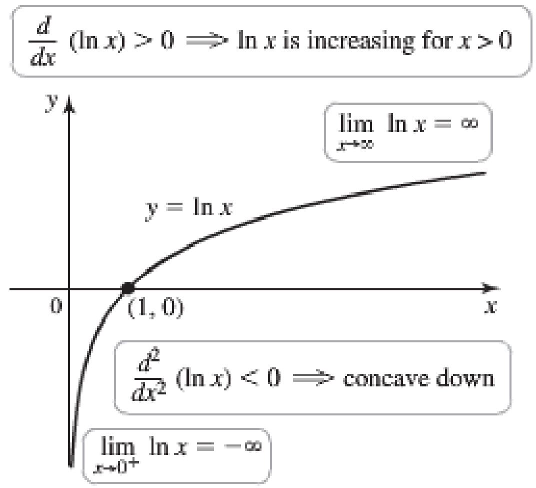
Section 1.8 – Exponential Models

Review

Definition

The ***number e*** is that number in the domain of the ***natural logarithm*** satisfying

$$\ln e = 1 \quad \text{and} \quad \int_1^e \frac{1}{t}\,dt = 1$$



The ***natural logarithm*** of a number $x > 0$, denoted by $\ln x$, is defined as

$$\ln x = \int_1^x \frac{1}{t}\,dt$$

Example

Evaluate $\displaystyle\int_0^4 \frac{x}{x^2+9}\,dx$

Solution

$$\begin{aligned}\int_0^4 \frac{x}{x^2+9}\,dx &= \frac{1}{2}\int_0^4 \frac{1}{x^2+9}\,d\left(x^2+9\right)\\ &= \frac{1}{2}\ln\left(x^2+9\right)\Big|_0^4\\ &= \frac{1}{2}(\ln 25 - \ln 9)\\ &= \frac{1}{2}(2\ln 5 - 2\ln 3)\\ &= \ln\frac{5}{3}\end{aligned}$$

The inverse of ln*x* and the Number *e*

The function $\ln x$, being *increasing* function of *x*. *Domain* $(0, \infty)$ and *range* $(-\infty, \infty)$

The inverse function $\ln^{-1} x$ with *Domain* $(-\infty, \infty)$ and *range* $(0, \infty)$

The function $\ln^{-1} x$ is usually denoted as $\exp x$ $\left(e^x\right)$

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \quad (\text{all } x > 0) \qquad\qquad \ln\left(e^x\right) = x \quad (\text{all } x)$$

The Derivative and Integral of e^x

The natural exponential function is differentiable because it is the inverse of a differentiable function whose derivative is never zero.

$\ln\left(e^x\right) = x$ ***Inverse relationship***

$\frac{d}{dx}\ln\left(e^x\right) = 1$ ***Differentiate both sides***.

$\frac{1}{e^x}\frac{d}{dx}\left(e^x\right) = 1$ $\frac{d}{dx}\ln u = \frac{1}{u}\cdot\frac{du}{dx}$

$\frac{d}{dx}e^x = e^x$

Theorem

For real numbers *x*,

$$\frac{d}{dx}\left(e^{u(x)}\right) = u'(x)e^{u(x)} \quad \text{and} \quad \int e^x dx = e^x + C$$

Example

Evaluate $\int \frac{e^x}{1+e^x}dx$

Solution

$$\int \frac{e^x}{1+e^x}dx = \int \frac{1}{1+e^x}d\left(1+e^x\right)$$

$$= \ln\left(1+e^x\right) + C$$

Definition

If $a > 0$ and u is a differentiable of x, then a^u is a differentiable function of x and

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx} \quad \& \quad \frac{d}{dx}\left(\log_a u\right) = \frac{1}{u}\cdot\frac{1}{\ln a}\frac{du}{dx}$$

$$\int a^u du = \frac{a^u}{\ln a} + C$$

Example

Evaluate $\int x3^{x^2}\,dx$

Solution

$$\int x3^{x^2}\,dx = \frac{1}{2}\int 3^{x^2}\,d\left(x^2\right)$$

$$= \frac{1}{2}\frac{1}{\ln 3}3^{x^2} + C$$

Example

Evaluate $\int_1^4 \frac{6^{-\sqrt{x}}}{\sqrt{x}}\,dx$

Solution

$$\int_1^4 \frac{6^{-\sqrt{x}}}{\sqrt{x}}\,dx = -2\int_1^4 6^{-\sqrt{x}}\,d\left(-\sqrt{x}\right) \qquad d\left(-\sqrt{x}\right) = -\frac{1}{2\sqrt{x}}\,dx$$

$$= -\frac{2}{\ln 6}6^{-\sqrt{x}}\Big|_1^4$$

$$= -\frac{2}{\ln 6}\left(\frac{1}{36} - \frac{1}{6}\right)$$

$$= \frac{5}{18\ln 6}$$

Power Rule – *Definition*

For any $x > 0$ and for any real number n, $\quad x^n = e^{n\ln x}$

Example

Evaluate the derivative $f(x) = x^{2x}$

Solution

$$\frac{d}{dx}\left(x^{2x}\right) = \frac{d}{dx}\left(e^{2x\ln x}\right)$$

$$= e^{2x\ln x}\left(2x\ln x\right)'$$

$$= 2e^{2x\ln x}\left(\ln x + 1\right)$$

$$= 2x^{2x}\left(\ln x + 1\right)$$

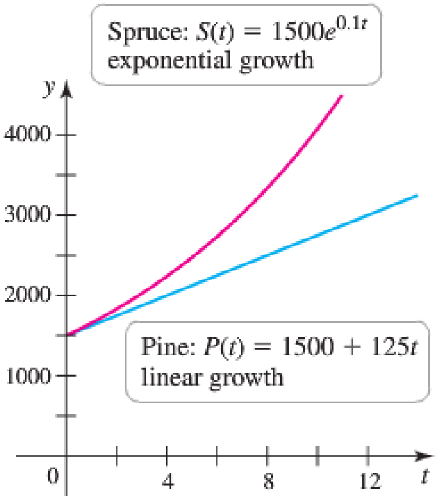
Exponential Models

Exponential Growth Functions

Exponential growth is described by functions of the form $y(t) = y_0(t)e^{kt}$. The ***initial value*** of y at $t = 0$ is $y(0) = y_0$ and the ***rate constant*** $k > 0$ determines the rate of the growth. Exponential growth is characterized by a constant relative growth rate.

Example

Suppose the population of the town of Pine is given by $P(t) = 1500 + 125t$, while the population of the town of Spruce is given by $S(t) = 1500e^{0.1t}$, where $t \ge 0$ is measured in years. Find the growth rate and the relative growth rate of each town.



Solution

$$\frac{dP}{dt} = 125$$

$$\frac{dS}{dt} = 150e^{0.1t}$$

The relative growth rate of Pine is

$\frac{1}{P}\frac{dP}{dt} = \frac{125}{1500 + 125t}$, which decreases in time.

The relative growth rate of Spruce is

$\frac{1}{S}\frac{dS}{dt} = \frac{150e^{0.1t}}{1500e^{0.1t}} = 0.1$ Contant for all times

The linear population function has a constant absolute growth rate and the exponential population function has a constant relative growth rate.

Definition

The quantity described by the function $y(t) = y_0 e^{kt}$ *for* $k > 0$, has a constant doubling time of $T_2 = \frac{\ln 2}{k}$, with the same units as *t*.

Formula To find either *k* or *T*: $A = A_0 e^{kt} \Rightarrow kT = \ln\frac{A}{A_0}$

Proof

$$A = A_0 e^{kt} \quad \Rightarrow \quad \frac{A}{A_0} = e^{kt}$$

$$\ln\frac{A}{A_0} = \ln e^{kt} \qquad \boxed{\ln\frac{A}{A_0} = kt} \quad \checkmark$$

Example

Human population growth rates vary geographically and fluctuate over time. The overall growth rate for world population peaked at an annual rate of 2.1% per year in the 1960s. Assume a world population of 6.0 billion in 1999 $(t=0)$ and 6.9 billion in 2009 $(t=10)$

a) Find an exponential growth function for the world population that fits the two data points.
b) Find the doubling time for the world population using the model in part (a).
c) Find the (absolute) growth rate $y'(t)$ and graph it, for $0 \le t \le 50$.
d) How fast was the population growing in 2014 $(t=15)$?

Solution

Given: $y(0)=6, \quad y(10)=6.9$

a) $k=\frac{1}{T}\ln\left(\frac{y}{y_0}\right)=\frac{1}{10}\ln\frac{6.9}{6} \approx 0.014$

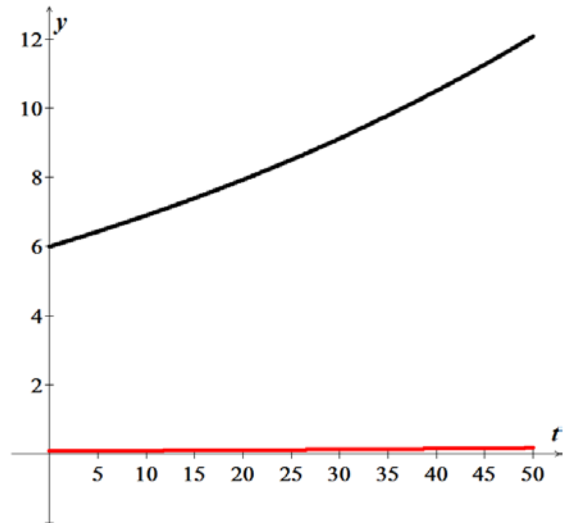
The growth function is: $y(t)=6e^{0.014t}$

b) $T_2=\frac{\ln 2}{k}=\frac{\ln 2}{0.014} \approx 50\, years$

c) $y'(t)=0.084e^{0.014t} \quad (billion\ of\ people\ /year)$

The growth rate itself increases exponentially

d) $y'(t=15)=0.084e^{0.014(15)} \approx 0.104\ \ bil/yr$



Financial Model

The balance in the account increases exponentially at a rate that can be determined from the advertised ***annual percentage yield*** (or **APY**) of the account.

Effective Rate

The ***effective rate*** corresponding to a started rate of interest r compounded m times per year is

$$r_e=\left(1+\frac{r}{m}\right)^m-1$$

APY is also referred to as ***effective rate*** or true interest rate.

Example

The APY of a savings account is the percentage increase in the balance over the course of a year. Suppose you deposit \$500 in a savings account that has an APY of 6.18% per year. Assume that the interest rate remains constant and that no additional deposits or withdrawals are made. How long will it take the balance to reach \$2500?

Solution

In one year the balance: $y(1) = (1+.0618)y_0 = 1.0618y_0$

$$k = \frac{1}{T}\ln\left(\frac{y(1)}{y_0}\right) = \ln 1.0618 \approx 0.05997$$

$$y(t) = 500e^{0.05997t}$$

$$T = \frac{1}{k}\ln\left(\frac{y}{y_0}\right) = \frac{1}{0.05997}\ln\left(\frac{2500}{500}\right) \approx 26.8\ yrs$$

Resource Consumption

The rate at which energy is conssumed is called ***power***.
The basic unit power is the ***watt*** (**W**).
The basic unit energy is the ***joule*** (**J**).

$$1\ W = 1\ J/s$$

$$\text{Total energy used } = \int_a^b E'(t) = \int_a^b P(t)\,dt$$

$E(t)$: the total energy used

$P(t)$: Power is the rate at which energy used

Example

At the beginning of 2010, the rate energy consumption for the city of Denver was 7,000 megawatts (*MW*), where $1\ MW = 10^6\ W$. That rate is expected to increase at an annual growth rate of 2% per year.

a) Find the function that gives the power or rate of energy consumption for all times after the beginning of 2010.
b) Find the total amount of energy used during 2014.
c) Find the function that gives the total (cumulative) amount of energy used by the city between 2010 and any time $t \ge 0$.

Solution

a) Let $t \ge 0$, be the number of years after the brginning of 2010.

$$k = \frac{1}{T}\ln\left(\frac{P(1)}{P_0}\right) = \ln 1.02 \approx 0.0198$$

$$P(t) = 7{,}000e^{0.0198t}, \quad t \ge 0$$

b) Entire year 2014 $\to$ $4 \le t \le 5$

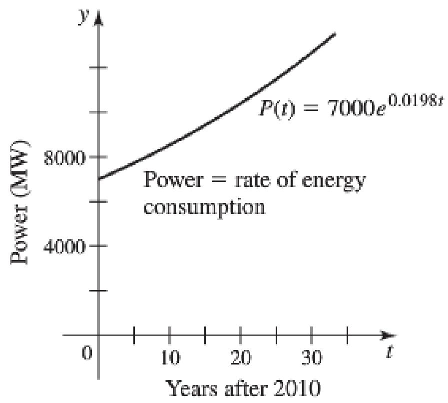
$$Total\ energy = \int_4^5 P(t)\,dt = \int_4^5 7{,}000e^{0.0198t}\,dt$$

$$= \frac{7000}{0.0198} e^{0.0198t} \Big|_4^5$$

$$\approx 7652\ MW - yr$$

$$\approx 7652\ (MW \cdot yr) \times 8760 \frac{hr}{yr}$$

$$\approx 6.7 \times 10^7\ MWh$$



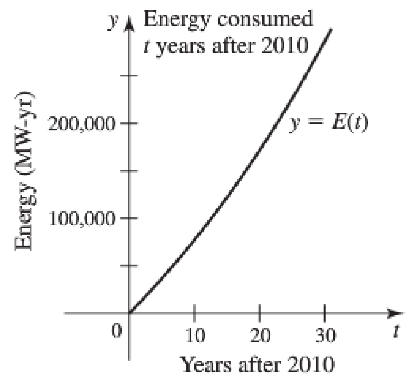
c) The total (cumulative) amount of energy used $t \ge 0$ is given by

$$E(t) = E(0) + \int_0^t E'(s)\,ds$$

$$= E(0) + \int_0^t P(s)\,ds$$

$$= 0 + \int_0^t 7000e^{0.0198s}\,ds$$

$$\approx 353{,}535\left(e^{0.0198t} - 1\right)$$



The total amount of energy consumed increases expotentially.

Exponential Decay Function

Exponential decay is described by functions of the form $y(t) = y_0 e^{-kt}$.

Rate constant: $k > 0$.

Initial value: y_0

Half-life is $T_{1/2} = \frac{\ln 2}{k}$

Example

Researchers determine that a fossilized bone has 30% of the C-14 of a live bone. Estimate the age of the bone. Assume a half-life for C-14 of ~5730 yrs.

Solution

$$k=\frac{\ln 2}{T_{1/2}}=\frac{\ln 2}{5730}\ \underline{\approx 0.000121}$$

$$T=\frac{\ln\frac{y}{y_0}}{k}=\frac{\ln 0.3}{-0.000121}\ \underline{\approx 9950\ yrs}$$

Example

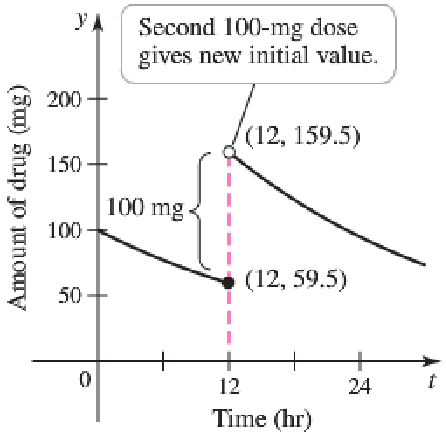
An exponential decay function $y(t)=y_0e^{-kt}$ models he amount of drug in the blood *t hr* after an initial dose of $y_0=100\ mg$ is administered. Assume the half-life of the drug is 16 hours.

a) Find the exponential decay function that governs the amount of drug in the blood.

b) How much time is required for the drug to reach 1% of the initial dose (1 *mg*)?

c) If a second 100-*mg* dose is given 12 *hr* after the first dose, how much time is required for the drug level to reach 1 *mg*?

Solution



a) $T_{1/2}=\frac{\ln 2}{k}=\frac{\ln 2}{16}\ \underline{\approx 0.0433}$

$\therefore\ \underline{y(t)=100e^{-0.0433t}}$

b) $T=\dfrac{\ln\frac{1}{100}}{-0.0433}\ \underline{\approx 106\ \ hrs}$

It takes more than 4 days for the drug to be reduced to 1% of the initial dose.

c) $y(t=12)=100e^{-0.0433(12)}\ \underline{\approx 59.5\ \ mg}$

The second 100-*mg* dose given after 12 *hr* increases the amount of drug to 159.5 *mg* (new initial value)

$\rightarrow\ y(t)=159.5\,e^{-0.0433t}$

The amount of drug reaches 1 mg in

$$t=\frac{\ln\frac{1}{159.5}}{-0.0433}\ \underline{\approx 117.1\ \ hrs}$$

Approximately 117 *hr* after the second dose (or 129 *hr* after the first dose), the amount of drug reaches 1 *mg*.

Exercises Section 1.8 – Exponential Models

Find the derivative of

1. $y=\ln\left(\dfrac{\sqrt{\sin\theta\cos\theta}}{1+2\ln\theta}\right)$
2. $f(x)=e^{\left(4\sqrt{x}+x^2\right)}$
3. $f(t)=\ln\left(3te^{-t}\right)$
4. $f(x)=\dfrac{e^{\sqrt{x}}}{\ln\left(\sqrt{x}+1\right)}$
5. $y=\sqrt{\dfrac{(x+1)^{10}}{(2x+1)^5}}$
6. $f(x)=(2x)^{4x}$
7. $f(x)=2^{x^2}$
8. $h(y)=y^{\sin y}$
9. $f(x)=x^{\pi}$
10. $h(t)=(\sin t)^{\sqrt{t}}$
11. $p(x)=x^{-\ln x}$
12. $f(x)=x^{2x}$
13. $f(x)=x^{\tan x}$
14. $f(x)=x^{e}+e^{x}$
15. $f(x)=x^{x^{10}}$
16. $f(x)=\left(1+\frac{4}{x}\right)^{x}$
17. $f(x)=\cos\left(x^{2\sin x}\right)$

Evaluate the integral

18. $\displaystyle\int \frac{2y}{y^2-25}dy$
19. $\displaystyle\int \frac{\sec y\tan y}{2+\sec y}dy$
20. $\displaystyle\int \frac{5}{e^{-5x}+7}dx$
21. $\displaystyle\int \frac{e^{2x}}{4+e^{2x}}dx$
22. $\displaystyle\int \frac{dx}{x\ln x\ln(\ln x)}$
23. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$
24. $\displaystyle\int \frac{e^{\sin x}}{\sec x}dx$
25. $\displaystyle\int \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}dx$
26. $\displaystyle\int \frac{4^{\cot x}}{\sin^2 x}dx$
27. $\displaystyle\int \frac{4x^2+2x+4}{x+1}dx$
28. $\displaystyle\int_{\ln 4}^{\ln 9} e^{x/2}dx$
29. $\displaystyle\int_{0}^{3} \frac{2x-1}{x+1}dx$
30. $\displaystyle\int_{e}^{e^2} \frac{dx}{x\ln^3 x}$
31. $\displaystyle\int_{e^2}^{e^3} \frac{dx}{x\ln x\ln^2(\ln x)}$
32. $\displaystyle\int_{0}^{1} \frac{y\ln^4\left(y^2+1\right)}{y^2+1}dy$
33. $\displaystyle\int_{\ln 2}^{\ln 3} \frac{e^{x}+e^{-x}}{e^{2x}-2+e^{-2x}}dx$
34. $\displaystyle\int_{-2}^{2} \frac{e^{z/2}}{e^{z/2}+1}dz$
35. $\displaystyle\int_{0}^{\pi/2} 4^{\sin x}\cos x\, dx$
36. $\displaystyle\int_{1/3}^{1/2} \frac{10^{1/p}}{p^2}dp$
37. $\displaystyle\int_{1}^{2} (1+\ln x)x^{x}dx$

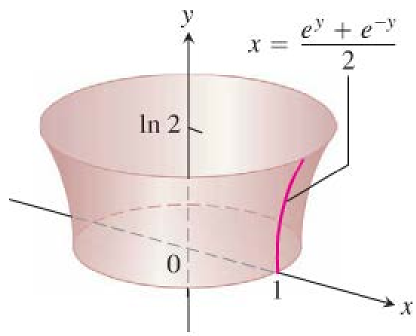
38. Find a curve through the origin in the *xy*-plane whose length from $x = 0$ to $x = 1$ is

$$L = \int_0^1 \sqrt{1 + \frac{1}{4}e^x}\, dx$$

39. Find the length of the curve $y = \ln\left(e^x - 1\right) - \ln\left(e^x + 1\right)$ *from* $x = \ln 2$ *to* $x = \ln 3$

40. Find the length of the curve $y = \ln(\cos x)$ *from* $x = 0$ *to* $x = \frac{\pi}{4}$

41. Find the area of the surface generated by revolving the curve $x = \frac{1}{2}\left(e^y + e^{-y}\right)$, $0 \le y \le \ln 2$, about *y*-axis



42. The population of a town with a 2010 population of 90,000 grows at a rate of 2.4% /yr. In what year will the population coudle its initial value (to 180,000)?

43. How long will it take an initial deposit of \$1500 to increase in value to \$2500 in a saving account with an APY of 3.1%? Assume the interest rate reamins constant and no additional deposits or withdrawals are made.

44. The number of cells in a tumor doubles every 6 weeks starting with 8 cells. After how many weeks doses the tumor have 1500 cells?

45. According to the 2010 census, the U.S. population was 309 million with an estimated growth rate of 0.8% /*yr*.

a) Based on these figures, find the doubling time and project the population in 2050.

b) Suppose the actual growth rate is just 0.2 percentage point lower than 0.8% /*yr* (0.6%). What are the resulting doubling time and projected 2050 population? Repeat these calculstions assuming the growth rate is 0.2 percentage point higher than 0.8% /*yr*.

c) Comment on th sensitivity of these projections to the growth rate.

46. The homicide rate decreases at a rate of 3%/yr in a city that had 800 homicides /yr in 2010. At this rate, when will the homicide rate reach 600 homicides/yr?

47. A drug is eliminated from the body at a rate of 15% /hr. after how many hours does the amount of drug reach 10% of the initial dose?

48. A large die-casting machine used to make automobile engine blocks is purchased for \$2.5 million. For tax purposes, the value of the machine can be depreciated by 6.8% of its current value each year.

a) What is the value of the machine after 10 years?

b) After how many years is the value of the machine 10% of its original value?

49. Roughly 12,000 Americans are diagmosed with thyroid cancer every year, which accounts for 1% of all cancer cases. It occurs in women three times as frequency as in men. Fortunately, thyroid cancer can be treated successfully in many cases with radioactive iodine, or I-131. This unstable form of iodine has a half-life of 8 days and is given in small doses meansured in millicuries.

a) Suppose a patient is given an initial dose of 100 millicuries. Find the function that gives the amount of I-131 in the body after $t \ge 0$ days.

b) How long does it take the amount of I-131 to reach 10% of the initial dose?

c) Finding the initial dose to give a particular patient is a critical calculation. How does the time reach 10% of the initial dose change if the initial dose is increased by 5%?

50. City ***A*** has a current population of 500,000 people and grows at a rate of 3% /*yr*. City ***B*** has a cuurent population of 300,000 and grows at a rate of 5%/*yr*.

a) When will the cities have the same population?

b) Suppose City ***C*** has a current population of $y_0 < 500,000$ and a growth rate of $p > 3\% / yr$. What is the relationship between y_0 and *p* such that the Cities ***A*** and ***C*** have the same population in 10 years?

51. Suppose the acceleration of an object moving along a line is given by $a(t) = -kv(t)$, where *k* is a positive constant and *v* is the object's velocity. Assume that the initial velocity and position are given by $v(0) = 10$ and $s(0) = 0$, respectively.

a) Use $a(t) = v'(t)$ to find the velocity of the object as a function of time.

b) Use $v(t) = s'(t)$ to find the position of the object as a function of time.

c) Use the fact that $\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt}$ (by the *Chain Rule*) to find the velocity as a function of position.

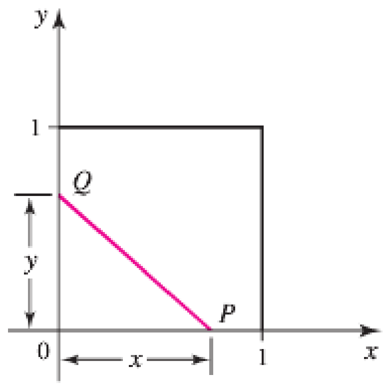
52. On the first day of the year $(t = 0)$, a city uses electricity at a rate of 2000 MW. That rate is projected to increase at a rate of 1.3% per year.

a) Based on these figures, find an exponential growth function for the power (rate of electricity use) for the city.

b) Find the total energy (in MW-*yr*) used by the city over four full years beginning at $t = 0$

c) Find a function that gives the total energy used (in MW-*yr*) between $t = 0$ and any future time $t > 0$

53. Two points P and Q are chosen randomly, one on each of two adjacent sides of a unit square.



What is the probability that the area of the triangle formed by the sides of the square and the line segment PQ is less than one-fourth the area of the square? Begin by showing that x and y must satisfy $xy < \frac{1}{2}$ in order for the area consition to be met. Then argue that the required probability is $\frac{1}{2} + \int_{1/2}^{1} \frac{dx}{2x}$ and evaluate the integral.