Solution Section 3.2 – Graphing Functions

Exercise

Find the open intervals on which the function $f(x) = x^3 + 3x^2 - 9x + 4$ is increasing or decreasing

Solution

$$f'(x) = 3x^2 + 6x - 9$$

$$3x^2 + 6x - 9 = 0$$

$CN: \; x = -3, \; 1$

$-\infty$ $\quad$ -3	1	∞
$f'(-4) > 0$	$f'(0) < 0$	$f'(2) > 0$
Increasing	*Decreasing*	*Increasing*

Increasing: $(-\infty, \; -3) \cup (1, \; \infty)$

Decreasing: $(-3, \; 1)$

Exercise

Find the critical numbers and decide on which the function $f(x) = (x-1)^{2/3}$ is increasing or decreasing

Solution

$$f'(x) = \frac{2}{3}(x-1)^{-1/3}$$

$$= \frac{2}{3(x-1)^{1/3}} = 0$$

$f'(x) \neq 0$

$x - 1 = 0$

$CN: \; x = 1$

$-\infty$ $\quad$ 1	∞
$f'(0) < 0$	$f'(2) > 0$
Decreasing	*Increasing*

Decreasing: $(-\infty, \; 1)$

Increasing: $(1, \; \infty)$

Exercise

Find the critical numbers and the open intervals on which the function is increasing or decreasing.

$f(x) = x\sqrt{x+1}$

Solution

$$f'(x) = \sqrt{x+1} + \frac{1}{2}x(x+1)^{-1/2} \qquad (uv)' = u'v + v'u$$

$$= \sqrt{x+1} + \frac{x}{2\sqrt{x+1}}$$

$$= \frac{2x+2+x}{2\sqrt{x+1}}$$

$$= \frac{3x+2}{2\sqrt{x+1}} = 0$$

$CN: \ x = -1, \ -\frac{2}{3}$ but the domain is $[-1, \infty)$

-1	$-\frac{2}{3}$	∞
Decreasing	$f'(0) > 0$ *Increasing*	

Decreasing $\left(-1, \ -\frac{2}{3}\right)$

Increasing $\left(-\frac{2}{3}, \ \infty\right)$

Exercise

Find the critical numbers and the open intervals on which the function is increasing or decreasing.

$$f(x) = \frac{x}{x^2+4}$$

Solution

$$f'(x) = \frac{-x^2+4}{\left(x^2+4\right)^2} = 0$$

$$\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 4 \end{matrix}$$

$$\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)' = \frac{(ae-bd)x^2+2(af-cd)x+bf-ce}{\left(dx^2+ex+f\right)^2}$$

$CN: \ x = \pm 2$

Decreasing: $(-\infty, -2) \cup (2, \infty)$

Increasing: $(-2, \ 2)$

$-\infty$	-2	2	∞
$-$	$+$	$-$	
Decreasing	***Increasing***	***Decreasing***	

Exercise

Find the critical numbers and the open intervals on which the function is increasing or decreasing.

$$f(x) = \frac{x}{x^2+1}$$

Solution

$$f'(x) = \frac{-x^2+1}{(x^2+1)^2} = 0$$

$$\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}$$

$$\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)' = \frac{(ae-bd)x^2+2(af-cd)x+bf-ce}{\left(dx^2+ex+f\right)^2}$$

$CN: \ x = \pm 1$

Decreasing: $\boxed{(-\infty,-1)\cup(1,\infty)}$

Increasing: $\boxed{(-1,\ 1)}$

$-\infty$	-1	1	∞
$-$	$+$	$-$	
Decreasing	*Increasing*	*Decreasing*	

Exercise

Find the critical numbers and the open intervals on which the function is increasing or decreasing.

$f(x)=x^3-12x$

Solution

$$f'(x)=3x^2-12=0$$

$$\boxed{x^2=4}$$

$CN:\ \boxed{x=\pm 2}$

$-\infty$	-2	2	∞
$+$	$-$	$+$	
Increasing	*Decreasing*	*Increasing*	

Decreasing: $\boxed{(-2,\ 2)}$

Increasing: $\boxed{(-\infty,-2)\cup(2,\infty)}$

Exercise

Find the open intervals on which the function $f(x)=x^{2/3}$ is increasing or decreasing

Solution

$$f'(x)=\frac{2}{3}x^{-1/3}$$

$$=\frac{2}{3x^{1/3}}=0$$

$\Rightarrow$ *Undefined*

$CN:\ \boxed{x=0}$

$-\infty$	0	∞
$f'(-1)<0$	$f'(1)>0$	
Decreasing	*Increasing*	

Decreasing: $\boxed{(-\infty,\ 0)}$

Increasing: $\boxed{(0,\ \infty)}$

Exercise

Find the open intervals on which the function is increasing and decreasing. Then, identify the function's local and absolute extreme values, if any, saying where they occur.

$g(t) = -t^2 - 3t + 3$

Solution

$$g'(t) = -2t - 3 = 0$$

$$CN: \quad \boxed{t = -\frac{3}{2}}$$

Decreasing: $\boxed{\left(-\frac{3}{2},\ \infty\right)}$

Increasing: $\boxed{\left(-\infty,\ -\frac{3}{2}\right)}$

$-\infty$	$-\frac{3}{2}$	∞
$f'(-2) > 0$		$f'(2) < 0$
Increasing		***Decreasing***

$$g\left(-\frac{3}{2}\right) = -\left(-\frac{3}{2}\right)^2 - 3\left(-\frac{3}{2}\right) + 3$$

$$= \boxed{\frac{21}{4}}$$

LMAX: $\boxed{\left(-\frac{3}{2},\ \frac{21}{4}\right)}$

Exercise

Find the open intervals on which the function is increasing and decreasing. Then, identify the function's local and absolute extreme values, if any, saying where they occur.

$h(x) = 2x^3 - 18x$

Solution

$$h'(x) = 6x^2 - 18 = 0$$

$$CN: \quad \boxed{x = \pm\sqrt{3}}$$

$$\begin{cases} x = -\sqrt{3} & \rightarrow \quad h = -6\sqrt{3} + 18\sqrt{3} \ \boxed{= 12\sqrt{3}} \\ x = \sqrt{3} & \rightarrow \quad h = 6\sqrt{3} - 18\sqrt{3} \ \boxed{= -12\sqrt{3}} \end{cases}$$

$-\infty$	$-\sqrt{3}$		$\sqrt{3}$	∞
+		−		+
Increasing		*Decreasing*		*Increasing*

Decreasing: $\boxed{\left(-\sqrt{3},\ \sqrt{3}\right)}$

Increasing: $\boxed{\left(-\infty,\ -\sqrt{3}\right) \text{ and } \left(\sqrt{3},\ \infty\right)}$

LMAX: $\boxed{\left(-\sqrt{3},\ 12\sqrt{3}\right)}$ ***LMIN***: $\boxed{\left(\sqrt{3},\ -12\sqrt{3}\right)}$

Exercise

Find the open intervals on which the function is increasing and decreasing. $f(\theta) = 3\theta^2 - 4\theta^3$

Then, identify the function's local and absolute extreme values, if any, saying where they occur.

Solution

$$f'(\theta) = 6\theta - 12\theta^2$$

$$= 6\theta(1 - 2\theta) = 0$$

$CN: \quad \boxed{\theta = 0, \ \frac{1}{2}}$

$$\begin{cases} \theta = 0 & f(0) = 0 \\ \theta = \frac{1}{2} & f\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right)^3 = \frac{1}{4} \end{cases}$$

$-\infty$	0	$\frac{1}{2}$	∞
$-$	$+$	$-$	
Decreasing	*Increasing*	*Decreasing*	

Decreasing: $\boxed{(-\infty, \ 0) \cup \left(\frac{1}{2}, \ \infty\right)}$

Increasing: $\boxed{(-\infty, \ 0) \cup \left(\frac{1}{2}, \ \infty\right)}$

LMAX: $\boxed{\left(\frac{1}{2}, \ \frac{1}{4}\right)}$

LMIN: $\boxed{(0, \ 0)}$

Exercise

Find the open intervals on which the function is increasing and decreasing $g(x) = x^4 - 4x^3 + 4x^2$.

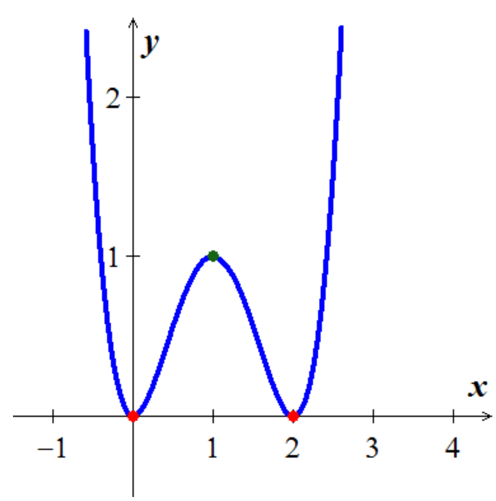
Then, identify the function's local and absolute extreme values, if any, saying where they occur.

Solution

$$g'(x) = 4x^3 - 12x^2 + 8x$$

$$= 4x\left(x^2 - 3x + 2\right) = 0$$

$CN: \quad \boxed{x = 0, \ 1, \ 2}$



$-\infty$	0	1	2	∞
$-$	$+$	$-$	$+$	
Decreasing	*Increasing*	*Decreasing*	*Increasing*	

Decreasing: $\boxed{(-\infty, \ 0) \ \cup \ (1, \ 2)}$

Increasing: $(0,\ 1)\cup(2,\ \infty)$

LMAX: $(1,\ 1)$

LMIN: $(0,\ 0),\ (2,\ 0)$

Abs. minimum: $(0,\ 0),\ (2,\ 0)$

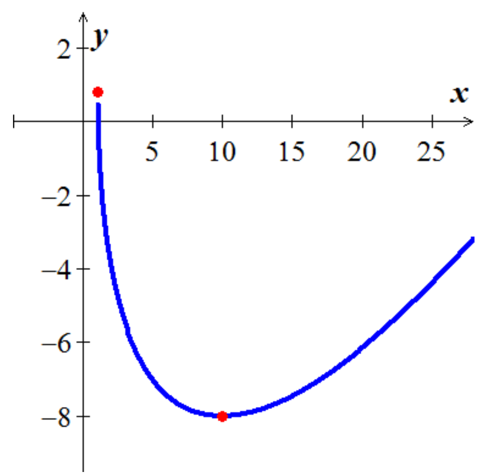
Exercise

Find the open intervals on which the function is increasing and decreasing. $f(x)=x-6\sqrt{x-1}$

Then, identify the function's local and absolute extreme values, if any, saying where they occur.

Solution

Domain: $x > 1$



$$f'(x)=1-6\frac{\frac{1}{2}}{\sqrt{x-1}}$$

$$=\frac{\sqrt{x-1}-3}{\sqrt{x-1}}=0$$

$\sqrt{x-1}=3$

$x-1=3^2$

$x=9+1$

$=10$

$CN:\ x=1,\ 10$

1	10	∞
−	+	
Decreasing	*Increasing*	

Decreasing: $(1,\ 10)$ ***Increasing***: $(10,\ \infty)$

Local minimum: $(10,\ -8)$ ***Local maximum***: $(1,\ 1)$

Absolute minimum: $(10,\ -8)$ ***Absolute maximum:*** $(1,\ 1)$

Exercise

Find the open intervals on which the function is increasing and decreasing. $f(x)=\dfrac{x^3}{3x^2+1}$

Then, identify the function's local and absolute extreme values, if any, saying where they occur.

Solution

$$f'(x)=\frac{3x^2\left(3x^2+1\right)-6x\left(x^3\right)}{\left(3x^2+1\right)^2} \qquad \left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$=\frac{9x^4+3x^2-6x^4}{\left(3x^2+1\right)^2}$$

$$=\frac{3x^4+3x^2}{\left(3x^2+1\right)^2}$$

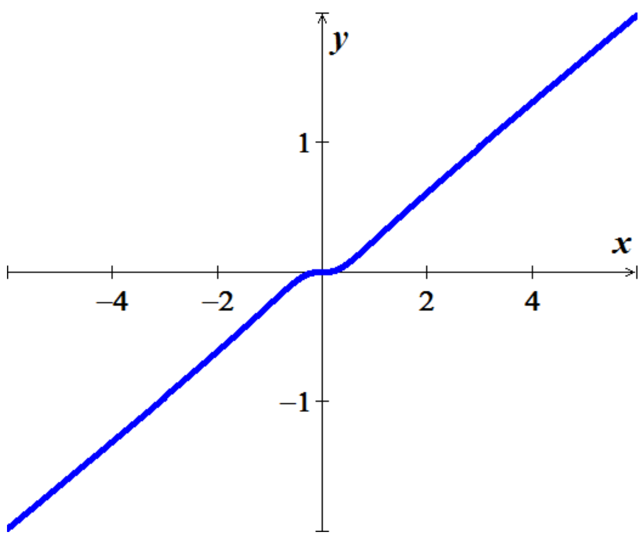
$$=\frac{3x^2\left(x^2+1\right)}{\left(3x^2+1\right)^2}=0$$

$-\infty$	0	∞
$f'(-1)>0$		$f'(1)>0$
Increasing		***Increasing***

$CN:\quad \underline{x=0}$

Increasing: $\quad (-\infty,\ 0)\cup(0,\ \infty)$

No local extrema, no absolute extrema



Exercise

Find the open intervals on which the function is increasing and decreasing. Then, identify the function's local and absolute extreme values, if any, saying where they occur.

$f(x) = x^{1/3}(x+8)$

Solution

$$
\begin{aligned}
f'(x) &= \tfrac{1}{3}x^{-2/3}(x+8) + x^{1/3} \\
&= \frac{1}{3}x^{1/3} + \frac{8}{3}x^{-2/3} + x^{1/3} \\
&= \frac{4}{3}x^{1/3} + \frac{8}{3x^{2/3}} \\
&= \frac{4x+8}{3x^{2/3}} = 0
\end{aligned}
$$

$$
\rightarrow \begin{cases} 4x+8=0 & \Rightarrow x=-2 \\ x^{2/3}=0 & \Rightarrow x=0 \end{cases}
$$

$CN: \ \underline{x=-2,\ 0}$

$-\infty$ -2	0	∞
$f'(-3)<0$	$f'(-1)>0$	$f'(1)>0$
Decreasing	*Increasing*	*Increasing*

Decreasing: $(-\infty,\ 0)$

Increasing: $(-2,\ 0)\cup(0,\ \infty)$

Local minimum: $\left(-2,\ -6\sqrt[3]{2}\right)$

Local maximum: ***None***

Absolute minimum: $\left(-2,\ -6\sqrt[3]{2}\right)$

Absolute maximum: ***None***

Exercise

Find all relative Extrema as well as where the function is increasing and decreasing $f(x) = 2x^3 - 6x + 1$

Solution

$f'(x) = 6x^2 - 6 = 0$

$6x^2 = 6$

$x^2 = 1$

$CN: \ \underline{x=\pm 1}$

$$
\begin{cases} x=1 \rightarrow y=f(1)=-3 \\ x=-1 \rightarrow y=f(-1)=5 \end{cases} \qquad (-1,\ 5),\ \ (1,\ -3)
$$

RMAX: $\boxed{(-1,\ 5)}$

RMIN: $\boxed{(1,\ -3)}$

Increasing: $\boxed{(-\infty,\ -1) \text{ and } (1,\ \infty)}$

Decreasing: $\boxed{(-1,\ 1)}$

$-\infty$	-1	1	∞
$f'(-2) > 0$	$f'(0) < 0$	$f'(2) > 0$	
Increasing	*Decreasing*	*Increasing*	

Exercise

Find all relative Extrema of $f(x) = 6x^{2/3} - 4x$ and Find the open intervals on which is increasing or decreasing

Solution

$$f'(x) = 4x^{-1/3} - 4$$

$$= 4\left(\frac{1}{x^{1/3}} - 1\right) = 0 \qquad \boxed{x \neq 0}$$

$$\frac{1}{x^{1/3}} - 1 = 0$$

$$\frac{1}{x^{1/3}} = 1 \qquad \textit{Multiply both sides by } x^{1/3}$$

$$1 = x^{1/3}$$

$$\boxed{x = 1^3 = 1}$$

CN : $\boxed{x = 0,\ 1}$

$$\begin{cases} x = 0 \to y = 0 \\ x = 1 \to y = 2 \end{cases} \qquad (0, 0) \text{ and } (1, 2)$$

$-\infty$	0	1	∞
+	–	+	
Increasing	*Decreasing*	*Increasing*	

RMIN: $\boxed{(0,\ 0)}$

RMAX: $\boxed{(1,\ 1)}$

Increasing: $\boxed{(0,\ 1)}$

Decreasing: $\boxed{(-\infty,\ 0) \text{ and } (1,\ \infty)}$

Exercise

Find all relative Extrema as well as where the function is increasing and decreasing

$f(x) = x^4 - 4x^3$

Solution

$$f'(x) = 4x^3 - 12x^2$$

$$= 4x^2(x-3) = 0$$

$-\infty$	**0**	**3**	∞
$f'(-1) < 0$ Decreasing	$f'(1) < 0$ Decreasing	$f'(4) > 0$ Increasing	

$CN:$ $\boxed{x = 0,\ 3}$

$x = 3 \rightarrow y = f(3) = -27$

RMIN: $\boxed{(3,\ -27)}$

Decreasing: $\boxed{(-\infty,\ 3)}$ ***Increasing***: $\boxed{(3,\ \infty)}$

Exercise

Find all relative Extrema as well as where the function is increasing and decreasing $f(x) = 3x^{2/3} - 2x$

Solution

$$f'(x) = 2x^{-1/3} - 2$$

$$= 2\left(\frac{1}{x^{1/3}} - 1\right) = 0$$

$$\Rightarrow \begin{cases} x^{1/3} = 0 \rightarrow x = 0 \\ 1 - x^{1/3} = 0 \rightarrow x^{1/3} = 1 \Rightarrow x = 1 \end{cases}$$

$CN:$ $\boxed{x = 1}$

$\begin{cases} x = 0 \rightarrow y = 0 \\ x = 1 \rightarrow y = 1 \end{cases}$ (0, 0) and (1, 1)

RMAX: $\boxed{(0,\ 0)}$

RMIN: $\boxed{(1,\ 1)}$

Decreasing: $\boxed{(0,\ 1)}$

Increasing: $\boxed{(-\infty,\ 0)\ \textit{and}\ (1,\ \infty)}$

$-\infty$	0	1	∞
$+$	$-$	$+$	
Increasing	*Decreasing*	*Increasing*	

Exercise

Find all relative Extrema as well as where the function is increasing and decreasing $y=\sqrt{4-x^2}$

Solution

$$f'(x)=\frac{-x}{\sqrt{4-x^2}}$$

The critical values are $x=0,\ \pm 2$, but the domain of the function is [–2, 2].

We can't go outside of that interval to test.

–2	0	2
	+	–
	Increasing	*Decreasing*

The function has a RMAX of $f(0)=2$ @ $x=0$. Some texts also consider $f(-2)=0$ and $f(2)=0$ as RMIN

Exercise

Find all relative Extrema as well as where the function is increasing and decreasing $f(x)=x\sqrt{x+1}$

Solution

$$f'(x)=\sqrt{x+1}+\frac{1}{2}x(x+1)^{-1/2} \qquad (uv)'=u'v+v'u$$

$$=\sqrt{x+1}+\frac{x}{2\sqrt{x+1}}$$

$$=\frac{2x+2+x}{2\sqrt{x+1}}$$

$$=\frac{3x+2}{2\sqrt{x+1}}=0$$

Critical points are $x=-\frac{2}{3}$ and $x=-1$, but the domain is $[-1,\ \infty)$.

Decreasing $\left(-1,\ -\frac{2}{3}\right)$

Increasing $\left(-\frac{2}{3},\ \infty\right)$

–1	$-\frac{2}{3}$	∞
	–	+
	Decreasing	*Increasing*

Exercise

Find all relative Extrema as well as where the function is increasing and decreasing $f(x)=\dfrac{x}{x^2+1}$

Solution

$$f'(x)=\frac{x^2+1-2x^2}{\left(x^2+1\right)^2}$$

$$\left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$=\frac{-x^2+1}{\left(x^2+1\right)^2}=0$$

$$-x^2+1=0\Rightarrow x^2=1\rightarrow x=\pm 1$$

$-\infty$ -1	1	∞
$-$	$+$	$-$
Decreasing	*Increasing*	*Decreasing*

Critical numbers are $x=\pm 1$

DECR: $(-\infty,-1)\cup(1,\infty)$ ***INCR:*** $(-1,\ 1)$

RMAX: $\left(1,\ \frac{1}{2}\right)$ ***RMIN***: $\left(-1,\ -\frac{1}{2}\right)$

Exercise

Find all relative Extrema as well as where the function is increasing and decreasing $f(x)=x^4-8x^2+9$

Solution

$$f'(x)=4x^3-16x$$

$$=4x\left(x^2-4\right)=0$$

$x=0$ $x^2-4=0$

$x^2=4\ \Rightarrow\ x=\pm 2$

$CN:$ $x=-2,\ 0,\ 2$

$x=-2\ \rightarrow f(-2)=-7$

$x=0\ \rightarrow f(0)=9$

$x=2\ \rightarrow f(2)=-7$

$-\infty$ -2	0	2	∞
$f'(-3)<0$	$f'(-1)>0$	$f'(1)<0$	$f'(3)>0$
decreasing	*increasing*	*decreasing*	*increasing*

DECR: $(-\infty,\ -2)\cup(0,\ 2)$ ***INCR:*** $(-2,\ 0)\cup(2,\ \infty)$

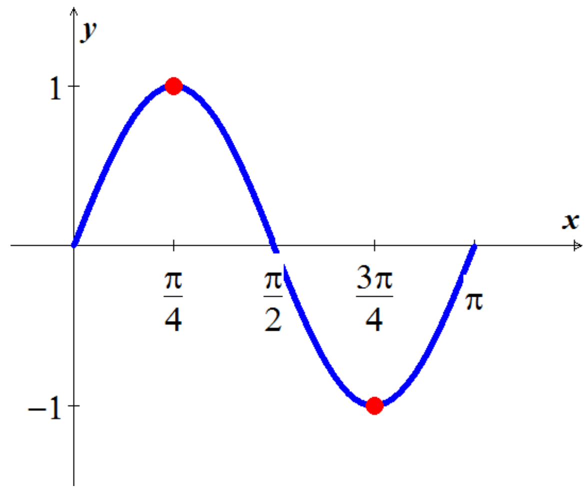
RMAX: $(0,\ 9)$ ***RMIN***: $(-2,\ -7)$ *and* $(2,\ -7)$

Exercise

Find the local extrema of the function on the given interval, and say where they occur

$f(x) = \sin 2x \quad 0 \le x \le \pi$

Solution



$$f'(x) = 2\cos 2x = 0$$

$$\rightarrow \begin{cases} 2x = \frac{\pi}{2} & \Rightarrow x = \frac{\pi}{4} \\ 2x = \frac{3\pi}{2} & \Rightarrow x = \frac{3\pi}{4} \end{cases}$$

$CN: \quad x = \frac{\pi}{4}, \ \frac{3\pi}{4}$

$$\begin{cases} x = 0 \Rightarrow f(x) = 0 & x = \frac{3\pi}{4} \Rightarrow f(x) = -1 \\ x = \frac{\pi}{4} \Rightarrow f(x) = 1 & x = \pi \Rightarrow f(x) = 0 \end{cases}$$

0 — $\frac{\pi}{4}$	$\frac{\pi}{4}$ — $\frac{3\pi}{4}$	$\frac{3\pi}{4}$ — π
$f'\left(\frac{\pi}{6}\right) > 0$	$f'\left(\frac{\pi}{3}\right) < 0$	$f'\left(\frac{5\pi}{6}\right) > 0$
Increasing	*Decreasing*	*Increasing*

DECR: $\left(\frac{\pi}{4}, \ \frac{3\pi}{4}\right)$ ***INCR:*** $\left(0, \ \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \ \pi\right)$

LMAX: $\left(\frac{\pi}{4}, \ 1\right) \ \ (\pi, \ 0)$ ***LMIN:*** $\left(\frac{3\pi}{4}, \ -1\right) \ \ (0, \ 0)$

Exercise

Find the local extrema of the function on the given interval, and say where they occur

$f(x) = \sqrt{3}\cos x + \sin x \quad 0 \le x \le 2\pi$

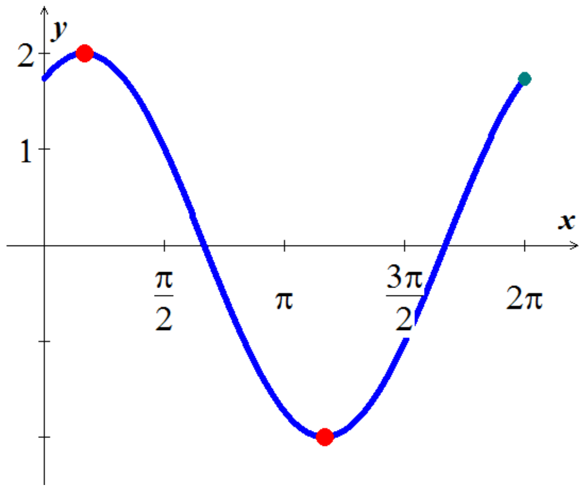
Solution

$$f'(x) = -\sqrt{3}\sin x + \cos x = 0$$

$$\sqrt{3}\sin x = \cos x$$

$CN: \quad x = \frac{\pi}{6}, \ \frac{7\pi}{6}$

$$\begin{cases} x = 0 \Rightarrow f(x) = \sqrt{3} & x = \frac{7\pi}{6} \Rightarrow f(x) = -2 \\ x = \frac{\pi}{6} \Rightarrow f(x) = 2 & x = 2\pi \Rightarrow f(x) = \sqrt{3} \end{cases}$$



Increasing: $\left(0,\ \frac{\pi}{6}\right)\cup\left(\frac{7\pi}{6},\ 2\pi\right)$

Decreasing: $\left(\frac{\pi}{6},\ \frac{7\pi}{6}\right)$

LMIN: $\left(\frac{7\pi}{6},\ -2\right)$

LMAX: $\left(\frac{\pi}{6},\ 2\right)$

0 $\quad\frac{\pi}{6}$	$\frac{7\pi}{6}$	2π
$f'\left(\frac{\pi}{12}\right)>0$	$f'\left(\frac{\pi}{2}\right)<0$	$f'\left(\frac{3\pi}{2}\right)>0$
Increasing	*Decreasing*	*Increasing*

Exercise

Find the local extrema of the function on the given interval, and say where they occur

$f(x)=\frac{x}{2}-2\sin\frac{x}{2} \quad 0\le x\le 2\pi$

Solution

$$f'(x)=\frac{1}{2}-2\left(\frac{1}{2}\right)\cos\frac{x}{2}=0$$

$$\cos\frac{x}{2}=\frac{1}{2} \quad \to \begin{cases}\frac{x}{2}=\frac{\pi}{3}\\ \frac{x}{2}=\frac{5\pi}{3}\end{cases} \quad \boxed{x=\frac{2\pi}{3},\ \cancel{\frac{10\pi}{3}}\ (>2\pi)}$$

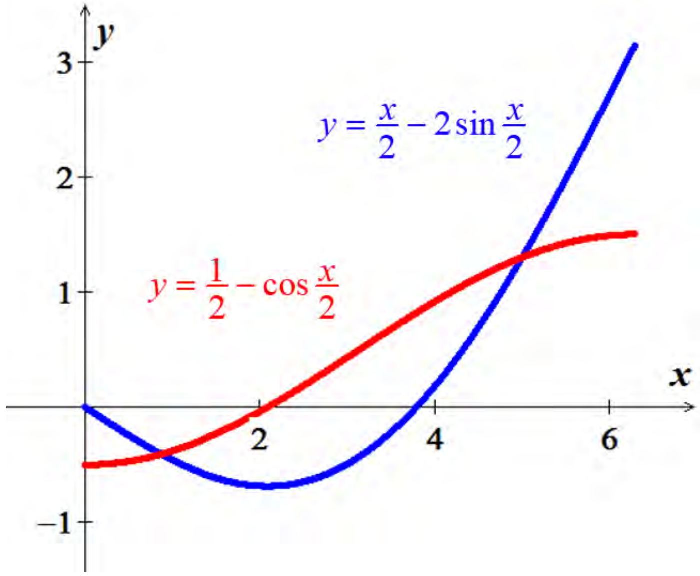
$CN:\ \ x=\frac{2\pi}{3}$

$$\begin{cases}x=0 & \Rightarrow f(x)=0\\ x=\frac{2\pi}{3} & \Rightarrow f(x)=\frac{\pi}{3}-\sqrt{3}\\ x=2\pi & \Rightarrow f(x)=\pi\end{cases}$$

0 $\quad\frac{2\pi}{3}$	2π
$f'\left(\frac{\pi}{2}\right)<0$	$f'(\pi)>0$

INCR: $\left(\frac{2\pi}{3},\ 2\pi\right)$ ***DECR:*** $\left(0,\ \frac{2\pi}{3}\right)$

LMIN: $\boxed{\left(\frac{2\pi}{3},\ \frac{\pi}{3}-\sqrt{3}\right)}$ ***LMAX:*** $\boxed{(2\pi,\ \pi)}$



Exercise

Find the local extrema of the function on the given interval, and say where they occur

$$f(x)=\sec^2 x-2\tan x \quad -\frac{\pi}{2}\le x\le\frac{\pi}{2}$$

Solution

$$f'(x)=2\sec x\cdot\sec x\cdot\tan x-2\sec^2 x$$

$$=2\sec^2 x(\tan x-1)=0$$

$$\begin{cases}\sec 2x\neq 0\\ \tan x-1=0 \quad\Rightarrow \tan x=1\rightarrow \boxed{x=\frac{\pi}{4}} \quad (CN)\end{cases}$$

$-\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$f'\left(\frac{\pi}{6}\right)<0$		$f'\left(\frac{\pi}{3}\right)>0$

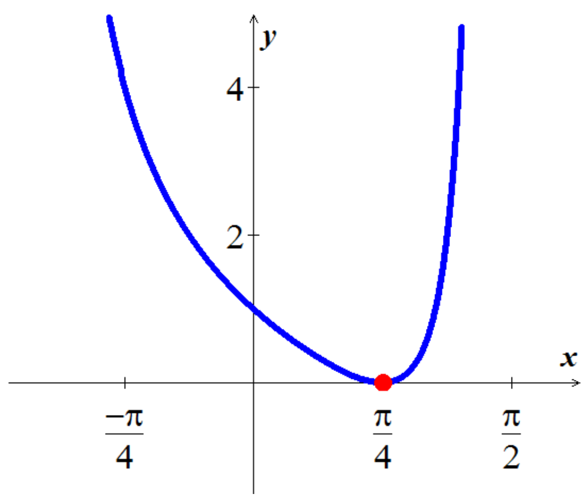
$$\begin{cases}x=\pm\frac{\pi}{2}\\ x=\frac{\pi}{4}\end{cases} \Rightarrow f(x)=0$$

INCR: $\boxed{\left(\frac{\pi}{4},\ \frac{\pi}{2}\right)}$

DECR: $\boxed{\left(-\frac{\pi}{2},\ \frac{\pi}{4}\right)}$

LMIN: $\boxed{\left(\frac{\pi}{4},\ 0\right)}$

LMAX: None



Exercise

Determine the intervals on which the graph of the function is concave upward or concave downward.

$$f(x)=\frac{x^2-1}{2x+1}$$

Solution

$$f'(x)=\frac{(2x+1)(2x)-2x^2+2}{(2x+1)^2} \qquad \left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$=\frac{2x^2+2x+2}{(2x+1)^2}$$

$$=\frac{2(x^2+x+1)}{(2x+1)^2}$$

$$f''(x)=2(2x+1)^{-3}\left((2x+1)^2-2(2)\left(x^2+x+1\right)\right) \qquad \left(U^mV^n\right)'=U^{m-1}V^{n-1}\left(mU'V+nUV'\right)$$

$$=2\frac{4x^2+4x+1-4x^2-4x-4}{(2x+1)^3}$$

$$=-\frac{6}{(2x+1)^3}=0$$

$2x=1=0 \ \Rightarrow\ x=-\frac{1}{2}$

$-\infty$	$-\frac{1}{2}$	∞
$f''(-1)>0$ **Upward**		$f''(0)<0$ **Downward**

f is ***concave upward*** on $\boxed{\left(-\infty,-\frac{1}{2}\right)}$

f is ***concave downward*** on $\boxed{\left(-\frac{1}{2},\infty\right)}$

Exercise

Determine the intervals on which the graph of the function is concave upward or concave downward.

$$f(x)=-4x^3-8x^2+32$$

Solution

$f'(x)=-12x^2-16x$

$f''(x)=-24x-16=0$

$\Rightarrow -24x=16$

$\boxed{x=\frac{16}{-24}=-\frac{2}{3}}$

$-\infty$	$-\frac{2}{3}$	∞
$f''(-1)>0$ **Upward**		$f''(0)<0$ **Downward**

Concave up : $\boxed{\left(-\infty,\ -\frac{2}{3}\right)}$ C***oncave down***: $\boxed{\left(-\frac{2}{3},\ \infty\right)}$

Exercise

Find the points of inflection. $f(x) = x^3 - 9x^2 + 24x - 18$

Solution

$$f'(x) = 3x^2 - 18x + 24$$

$$f''(x) = 6x - 18 = 0 \Rightarrow x = 3$$

$$x = 3 \Rightarrow f(3) = 0$$

Point of inflection: $\underline{(3,\ 0)}|$

Exercise

Does $f(x) = 2x^5 - 10x^4 + 20x^3 + x + 1$ have any inflection points? If so, identify them.

Solution

$$f'(x) = 10x^4 - 40x^3 + 60x^2 + 1$$

$$f''(x) = 40x^3 - 120x^2 + 120x = 0$$

$$40x\left(x^2 - 3x + 3\right) = 0$$

$$x^2 - 3x + 3 = 0 \quad \rightarrow \quad x = \frac{3 \pm 2i}{2} \in \mathbb{C}$$

Point of inflection: $\underline{(0,\ 1)}|$

Exercise

Find the second derivative of $f(x) = -2\sqrt{x}$ and discuss the concavity of the graph

Solution

$$f'(x) = -x^{-1/2}$$

$$\Rightarrow f''(x) = \tfrac{1}{2}x^{-3/2}$$

$$= \frac{1}{2x^{3/2}} > 0 \text{ for all } x > 0$$

f is concave up for all $x > 0$.

Exercise

Determine the intervals on which the graph of the function is concave upward or concave downward.

$f(x)=\dfrac{12}{x^2+4}$

Solution

$$f'(x)=-\frac{24x}{\left(x^2+4\right)^2}$$

$\begin{array}{ccc} 0 & 0 & 12 \\ 1 & 0 & 4 \end{array}$ $\quad \left(\dfrac{ax^2+bx+c}{dx^2+ex+f}\right)'=\dfrac{(ae-bd)x^2+2(af-cd)x+bf-ce}{\left(dx^2+ex+f\right)^2}$

$$f''(x)=-24\left(x^2+4\right)^{-3}\left(x^2+4-2(2x)x\right) \qquad \left(U^mV^n\right)'=U^{m-1}V^{n-1}\left(mU'V+nUV'\right)$$

$$=-\frac{24\left(-3x^2+4\right)}{\left(x^2+4\right)^3}=0$$

$$-3x^2+4=0$$

$$x=\pm\sqrt{\frac{4}{3}}$$

$$\boxed{=\pm\frac{2\sqrt{3}}{3}}$$

$-\infty$	$-\frac{2\sqrt{3}}{3}$	$\frac{2\sqrt{3}}{3}$	∞
$f''(-2)>0$	$f''(0)<0$	$f''(2)>0$	
upward	***downward***	***upward***	

Concave up on $\boxed{\left(-\infty,\ -\frac{2\sqrt{3}}{3}\right) \text{ and } \left(\frac{2\sqrt{3}}{3},\ \infty\right)}$

Concave down on $\boxed{\left(-\frac{2\sqrt{3}}{3},\ \frac{2\sqrt{3}}{3}\right)}$

Exercise

Find the extrema using the second derivative test $f(x)=\dfrac{4}{x^2+1}$

Solution

$$f'(x)=\frac{-8x}{\left(x^2+1\right)^2} \qquad \textbf{\textit{CN}} \text{ is } x=0 \qquad \left(\frac{1}{U}\right)'=-\frac{U'}{U^2}$$

$$f''(x)=-8\left(x^2+1\right)^{-3}\left(x^2+1-2(2x)x\right) \qquad \left(U^mV^n\right)'=U^{m-1}V^{n-1}\left(mU'V+nUV'\right)$$

$$=\frac{8(3x^2-1)}{\left(x^2+1\right)^3}$$

$f''(0)=-8<0 \Rightarrow f(0)=4$ is a ***local maximum*** *(LMAX)*

Exercise

Discuss the concavity of the graph of f and find its points of inflection. $f(x)=x^4-2x^3+1$

Solution

$$f'(x)=4x^3-12x^2$$

$$f'(x)=4x^2(x-3)=0 \quad \to \quad \underline{x=0,\ 3}$$

$$f''(x)=12x^2-12x$$

Points: (0, 1) $f''(0)=0$ Test fails

(3, −26) $f''(3)>0 \Rightarrow$ ***local Minimum*** *(LMIN)*

Exercise

Find all relative extrema of $f(x)=x^4-4x^3+1$

Solution

$$f'(x)=4x^3-6x^2$$

$$f''(x)=12x^2-12x=0$$

$$12x(x-1)=0 \Rightarrow x=0,1$$

For $x=0 \Rightarrow f(0)=0^4-2(0)^3+1=1 \to (0,1)$

For $x=0 \Rightarrow f(1)=1^4-2(1)^3+1=0 \to (1,0)$

$-\infty$ **0**	**1**	∞
$f''(-1)>0$	$f''(1/2)<0$	$f''(2)>0$
upward	***downward***	***upward***

Concave up on $(-\infty,\ 0)$ *and* $(1,\ \infty)$ ***concave down*** on $(0,\ 1)$

Points of inflection: (0, 1), (1, 0)

Exercise

Sketch the graph $f(x)=x^4-4x^3+5$

Solution

$$f'(x)=4x^3-12x^2=0$$

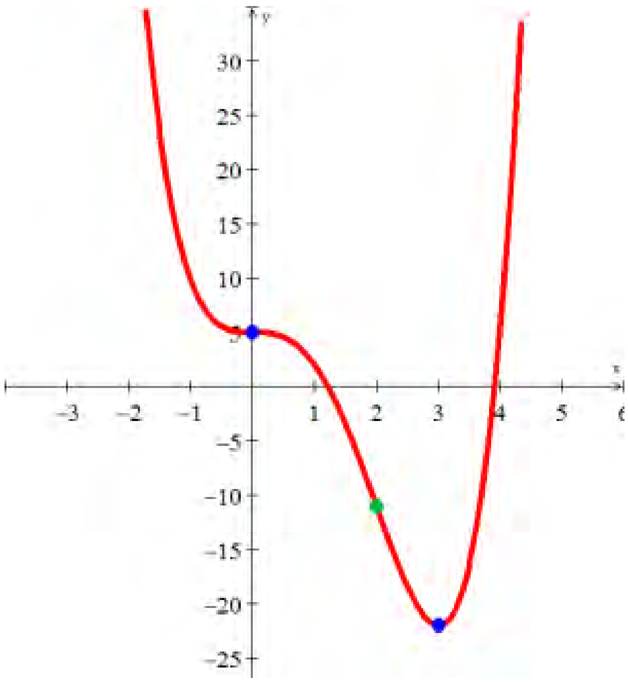
$$4x^2(x-3)=0$$

$$\Rightarrow x=0,\ 0,\ 3$$

$$f''(x)=12x^2-24x=0$$

$$12x(x-2)=0$$

$\Rightarrow x = 0,\ 2$



	f	f'	f''	
$(-\infty, 0)$		–	+	Decreasing, Concave up
$x = 0$	5	0	0	RMAX
$(0, 2)$		–	–	Decreasing, Concave down
$x = 2$	–11	–	0	Point of Inflection
$(2, 3)$		–	+	Decreasing, Concave up
$x = 3$	–22	0	+	RMIN
$(3, \infty)$		+	+	Increasing, Concave up

Exercise

Given $\quad f(x) = \dfrac{x^2+1}{x^2-1}$

Solution

VA: $x = \pm 1$ ***HA***: $y = 1$

$$f'(x) = \frac{(2x)(x^2-1)-(x^2+1)(2x)}{(x^2-1)^2} \qquad \left(\frac{u}{v}\right)' = \frac{u'v-v'u}{v^2}$$

$$= \frac{2x^3-2x-2x^3-2x}{\left(x^2-1\right)^2}$$

$$= -\frac{4x}{\left(x^2-1\right)^2} = 0$$

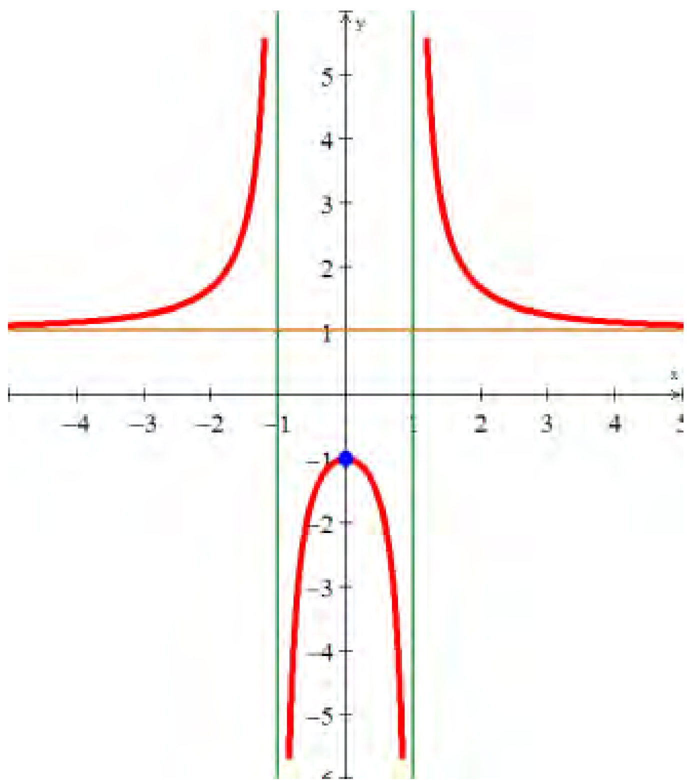
$\Rightarrow x = 0$

$$f'' = -4\left(x^2 - 1\right)^{-3}\left(x^2 - 1 - 2(2x)x\right) \qquad \left(U^m V^n\right)' = U^{m-1}V^{n-1}\left(mU'V + nUV'\right)$$

$$= \frac{4\left(3x^2+1\right)}{\left(x^2-1\right)^3} = 0$$

$\Rightarrow 3x^2 + 1 = 0 \ \Rightarrow 3x^2 = -1 \ \ (no\ zeros)$

	f	f'	f''	
$(-\infty, -1)$		+	–	Increasing, Concave up
$x = -1$	*Undef.*	*Undef.*	*Undef.*	Vertical Asymptote
$(-1, 0)$		+	–	Increasing, Concave down
$x = 0$	-1	0	–	RMAX
$(0, 1)$		–	–	Decreasing, Concave down
$x = 1$	*Undef.*	*Undef.*	*Undef.*	Vertical Asymptote
$(1, \infty)$		–	+	Decreasing, Concave up



Exercise

Given $f(x)=2x^{3/2}-6x^{1/2}$

Solution

$f'(x)=3x^{1/2}-3x^{-1/2}=0$

$x^{1/2}\left(3x^{1/2}-3x^{-1/2}\right)=0$

$3x-3=0$

$\Rightarrow x=1$

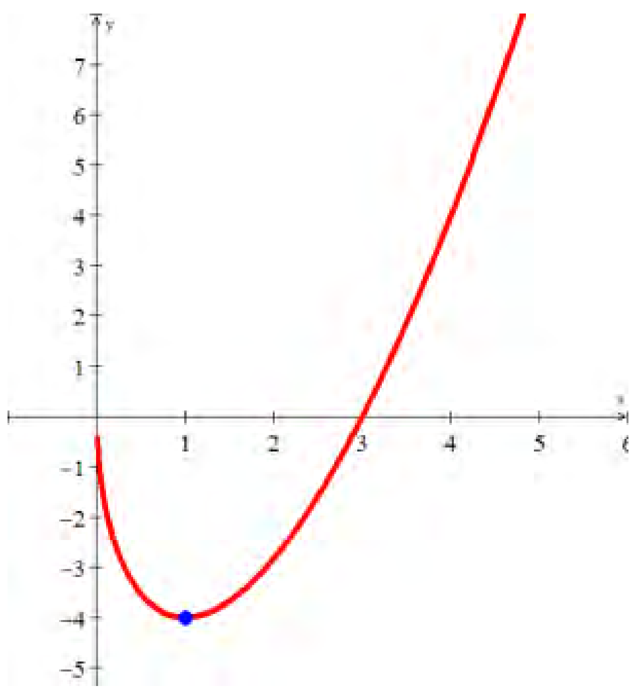
$f''(x)=\frac{3}{2}x^{-1/2}+\frac{3}{2}x^{-3/2}=0$

$\frac{2}{3}x^{3/2}\left(\frac{3}{2}x^{-1/2}+\frac{3}{2}x^{-3/2}\right)=0$

$x+1=0$

$\rightarrow x=-1<0$

x	f	f'	f''	
(0, 1)		–	+	Decreasing, Concave up
$x = 1$	– 4	0	+	RMIN
(1, ∞)		+	+	Increasing, Concave up



Exercise

Sketch the graph $\quad y = x^3 - 3x + 3$

Solution

$y' = 3x^2 - 3 = 0$

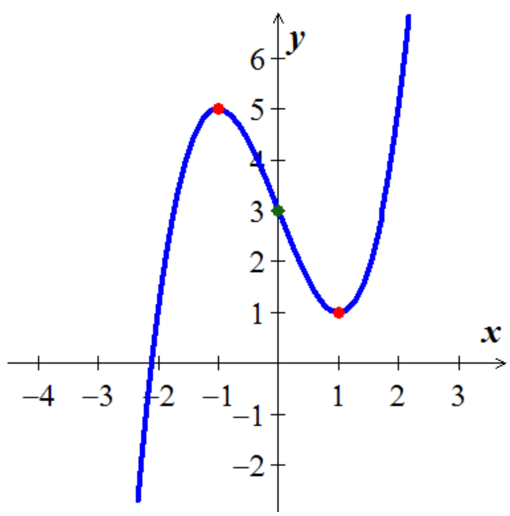
$x^2 = 1 \Rightarrow \boxed{x = \pm 1} \quad (CP)$

$$\begin{cases} x = -1 & \to y = 5 \\ x = 1 & \to y = 1 \end{cases}$$

$y'' = 6x = 0 \quad \Rightarrow \boxed{x = 0}$

$(x = 0 \quad \to y = 3$

x	f	f'	f''	
$(-\infty, -1)$		+	+	Increasing, Concave Up
$x = -1$	5	0	+	Concave Up
$(-1, 0)$		–	+	Decreasing, Concave Up
$x = 0$	3	–	0	Decreasing, Pt. of Inflection
$(0, 1)$		–	–	Decreasing, Concave Down
$x = 1$	1	0	–	Concave Down
$(1, \infty)$		+	–	Increasing, Concave Down



Decreasing: $(-1,\ 1)$

Increasing: $(-\infty, -1) \cup (1, \infty)$

Concave Down: $(0,\ \infty)$

Concave Up: $(-\infty,\ 0)$

Local Minimum: $(-1,\ 5)$

Local Maximum: $(1,\ 5)$

Points of inflection: $(0,\ 3)$

Exercise

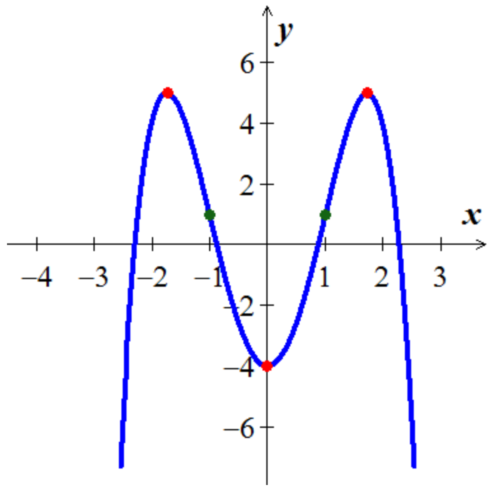
Sketch the graph $\quad y=-x^4+6x^2-4$

Solution

$$y'=-4x^3+12x$$
$$=-4x\left(x^2-3\right)=0$$

$$\begin{cases} x=0 \\ x^2=3 \quad \rightarrow x=\pm\sqrt{3} \end{cases} \quad \boxed{x=0,\ \pm\sqrt{3}} \quad (CP)$$

$$\begin{cases} x=-\sqrt{3} & \rightarrow y=5 \\ x=0 & \rightarrow y=-4 \\ x=\sqrt{3} & \rightarrow y=5 \end{cases}$$



$$y''=-12x^2+12=0$$

$$x^2=1 \quad \rightarrow \boxed{x=\pm 1} \quad (\textbf{\textit{Points of Inflection}})$$

$$\begin{cases} x=-1 & \rightarrow y=1 \\ x=1 & \rightarrow y=1 \end{cases}$$

x	***f***	***f′***	***f″***	
$\left(-\infty,\ -\sqrt{3}\right)$		+	–	Increasing, Concave Down
$x=-\sqrt{3}$	5	0	–	Concave Down
$\left(-\sqrt{3},\ -1\right)$		–	–	Decreasing, Concave Down
$x=-1$	1	–	0	Decreasing, Pt. of Inflection
(–1, 0)		–	+	Decreasing, Concave Up
$x=0$	–4	0	+	Concave Up
(0, 1)		+	+	Increasing, Concave Up
$x=1$	1	+	0	Increasing, Pt. of Inflection
$\left(1,\ \sqrt{3}\right)$		+	–	Increasing, Concave Down
$x=\sqrt{3}$	5	0	–	Concave Down
$\left(\sqrt{3},\ \infty\right)$		–	–	Decreasing, Concave Down

Decreasing: $\left(-\sqrt{3},\ 0\right)\cup\left(\sqrt{3},\ \infty\right)$ ***Increasing:*** $\left(-\infty,\ -\sqrt{3}\right)\cup\left(0,\ \sqrt{3}\right)$

Concave Down: $(-1,\ 1)$ ***Concave Up:*** $(-\infty,\ -1)\quad(1,\ \infty)$

Local Minimum: $(0,\ -4)$ ***Local Maximum***: $\left(-\sqrt{3},\ 5\right)\quad\left(\sqrt{3},\ 5\right)$

Points of inflection: $(-1,\ 1)\quad(1,\ 1)$

Exercise

Sketch the graph $\quad y = x\left(\frac{x}{2}-5\right)^4$

Solution

$$y' = \left(\frac{x}{2}-5\right)^4 + 4x\left(\frac{1}{2}\right)\left(\frac{x}{2}-5\right)^3 \qquad (uv)' = u'v + v'u$$

$$= \left(\frac{x}{2}-5\right)^3\left(\frac{x}{2}-5+2x\right)$$

$$= \left(\frac{x}{2}-5\right)^3\left(\frac{5x}{2}-5\right) = \underline{0}$$

$$\begin{cases} \frac{x}{2}-5=0 & \rightarrow \boxed{x=10} \quad (CP) \\ \frac{5x}{2}-5=0 & \rightarrow \boxed{x=2} \quad (CP) \end{cases} \Rightarrow \begin{cases} x=2 & \rightarrow y=512 \\ x=10 & \rightarrow y=0 \end{cases}$$

$$y'' = 3\left(\frac{1}{2}\right)\left(\frac{x}{2}-5\right)^2\left(\frac{5x}{2}-5\right) + \frac{5}{2}\left(\frac{x}{2}-5\right)^3$$

$$= \frac{1}{2}\left(\frac{x}{2}-5\right)^2\left(3\left(\frac{5x}{2}-5\right)+5\left(\frac{x}{2}-5\right)\right)$$

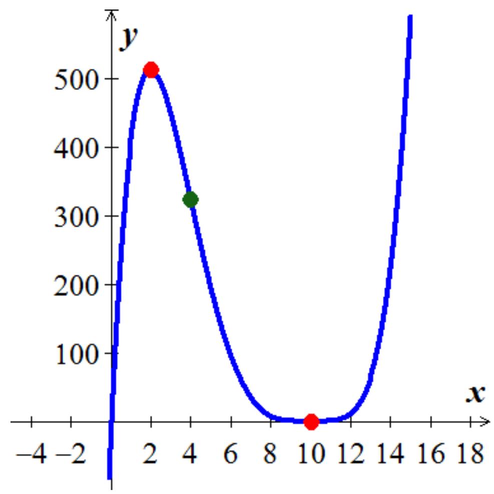
$$= \frac{1}{2}\left(\frac{x}{2}-5\right)^2\left(\frac{15x}{2}-15+\frac{5x}{2}-25\right)$$

$$= \frac{1}{2}\left(\frac{x}{2}-5\right)^2\left(\frac{20x}{2}-40\right)$$

$$= \frac{1}{2}\left(\frac{x}{2}-5\right)^2(10x-40)$$

$$= \frac{1}{2}\left(\frac{x}{2}-5\right)^2(10)(x-4)$$

$$= 5\left(\frac{x}{2}-5\right)^2(x-4) = \underline{0}$$



$$\begin{cases} \frac{x}{2}-5=0 & \rightarrow x=10 \\ x-4=0 & \rightarrow x=4 \end{cases} \Rightarrow \begin{cases} x=10 & \rightarrow y=0 \\ x=4 & \rightarrow y=324 \end{cases}$$

x	***f***	***f′***	***f″***	
$(-\infty,\ 2)$		+	–	Increasing, Concave Down
$x=2$	512	0	–	Concave Down
$(2,\ 4)$		–	–	Decreasing, Concave Down
$x=4$	324	–	0	Decreasing, Pt. of Inflection
(1, 10)		–	+	Decreasing, Concave Up
$x = 10$	0	0	0	Pt. of Inflection
$(10,\ \infty)$		+	+	Increasing, Concave Up

Exercise

Sketch the graph $y = x + \sin x \quad 0 \le x \le 2\pi$

Solution

$y' = 1 + \cos x = 0$

$\cos x = -1 \quad \to \quad \boxed{x = \pi} \quad (CP)$

$$\begin{cases} x = 0 & \to y = 0 \\ x = \pi & \to y = \pi \\ x = 2\pi & \to y = 2\pi \end{cases}$$

$y'' = -\sin x = 0 \quad \to \boxed{x = 0,\ \pi,\ 2\pi}$

x	***f***	***f′***	***f″***	
$x = 0$	0	+	0	
$(0,\ \pi)$		+	–	Increasing, Concave Down
$x = \pi$	π	0	0	Pt. of Inflection
$(\pi,\ 2\pi)$		+	+	Increasing, Concave Up
$x = 2\pi$	2π	+	0	

Decreasing:

Increasing: $(0,\ 2\pi)$

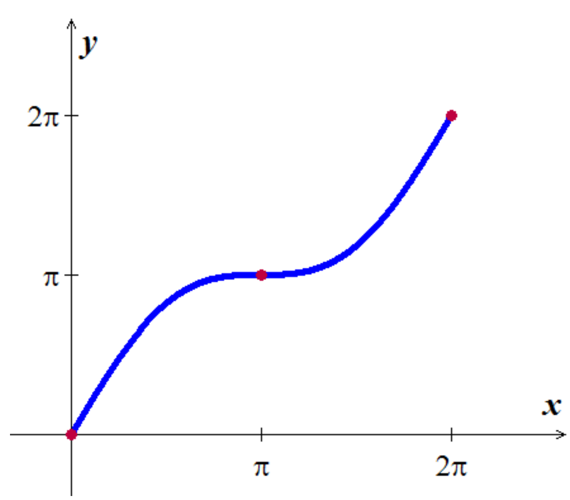
Concave Down: $(0,\ \pi)$

Concave Up: $(\pi,\ 2\pi)$

Local and Absolute Minimum: $(0,\ 0)$

Local and Absolute Maximum: $(2\pi,\ 2\pi)$

Points of inflection: $x = \pi$



Exercise

Sketch the graph $\quad y = \cos x + \sqrt{3}\sin x \quad 0 \le x \le 2\pi$

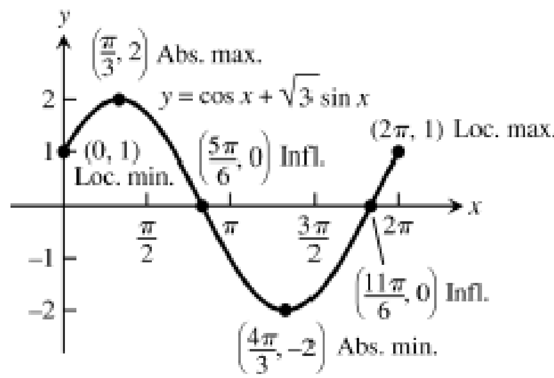
Solution

$$y' = -\sin x + \sqrt{3}\cos x = 0$$

$$\sin x = \sqrt{3}\cos x$$

$$\frac{\sin x}{\cos x} = \sqrt{3} = \tan x \quad \to \boxed{x = \frac{\pi}{3},\ \frac{4\pi}{3}} \quad (CN)$$

$$\to \begin{cases} x = 0 & \to y = 1 \\ x = \frac{\pi}{3} & \to y = 2 \\ x = \frac{4\pi}{3} & \to y = -2 \\ x = 2\pi & \to y = 1 \end{cases}$$



$$y'' = -\cos x - \sqrt{3}\sin x = 0$$

$$\sqrt{3}\sin x = -\cos x$$

$$\frac{\sin x}{\cos x} = -\frac{1}{\sqrt{3}} = \tan x$$

$$\to \boxed{x = \frac{5\pi}{6},\ \frac{11\pi}{6}} \quad (\textbf{\textit{Points of Inflection}})$$

x	***f***	***f'***	***f''***	
$x = 0$	1			***Absolute Min.***
$\left(0, \frac{\pi}{3}\right)$		+	–	Increasing, Concave Down
$x = \frac{\pi}{3}$	2	0	–	***LMAX***, Concave Down
$\left(\frac{\pi}{3}, \frac{5\pi}{6}\right)$		–	–	Decreasing, Concave Down
$x = \frac{5\pi}{6}$	0	–	0	Decreasing, ***Pt. of Inflection***
$\left(\frac{5\pi}{6}, \frac{4\pi}{3}\right)$		–	+	Decreasing, Concave Up
$x = \frac{4\pi}{3}$	–2	0	+	***LMIN***, Concave Up
$\left(\frac{4\pi}{3}, \frac{11\pi}{6}\right)$		+	+	Increasing, Concave Up
$x = \frac{11\pi}{6}$	0	+	0	***Pt. of Inflection***
$\left(\frac{11\pi}{6}, 2\pi\right)$		+	–	Increasing, Concave Down
$x = 2\pi$	1			***Absolute Max.***

Exercise

Sketch the graph $\quad y=\dfrac{x}{\sqrt{x^2+1}}$

Solution

$$y'=\left(x^2+1\right)^{3/2}\left(x^2+1-\frac{1}{2}(2x)x\right) \qquad \left(U^mV^n\right)'=U^{m-1}V^{n-1}\left(mU'V+nUV'\right)$$

$$=\frac{1}{\left(x^2+1\right)^{3/2}}\neq 0$$

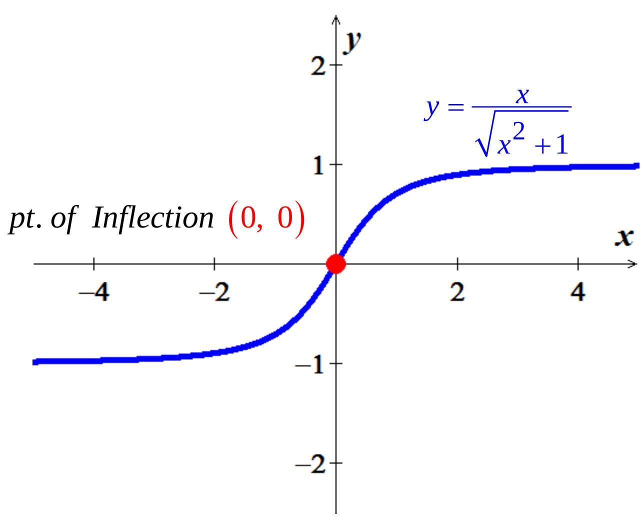
$$y''=-\frac{3}{2}\frac{2x}{\left(x^2+1\right)^{5/2}} \qquad \left(\frac{1}{U^n}\right)'=-\frac{n\cdot U'}{U^{n+1}}$$

$$=\frac{-3x}{\left(x^2+1\right)^{5/2}}=0 \quad \rightarrow \boxed{x=0}$$

$-\infty$	0	∞
$f''(-1)>0$		$f'(1)<0$
Concave Up		***Concave Down***

Concave Down: $(0,\ \infty)$ ***Concave Up:*** $(-\infty,\ 0)$

No Local or Absolute Extrema ***Points of inflection***: $x=0$

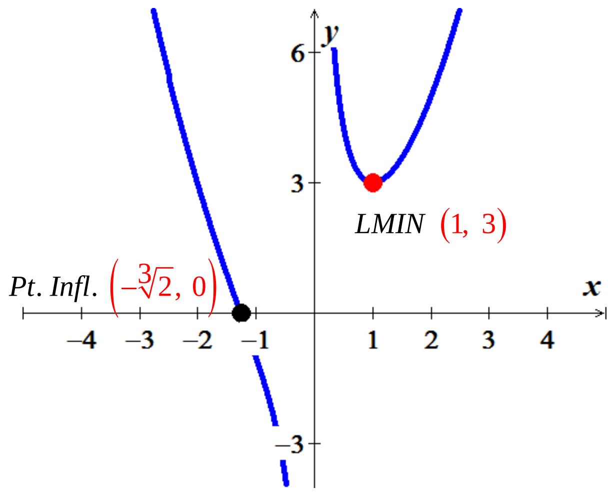


Exercise

Sketch the graph $\quad y = x^2 + \frac{2}{x}$

Solution

Vertical Asymptote: $x = 0$



$$y' = 2x - \frac{2}{x^2} = \frac{2x^3 - 2}{x^2} = 0$$

$y = x^2 + \frac{2}{x}$ $\quad 2x^3 - 2 = 0 \Rightarrow x^3 = 1 \quad \boxed{x = 1} \quad (CN)$

$$\{x = 1 \quad \rightarrow y = 3$$

$$y'' = 2 \cdot \frac{3x^2\left(x^2\right) - (2x)\left(x^3 - 1\right)}{x^4}$$

$$= 2 \cdot \frac{3x^4 - 2x^4 + 2x}{x^4}$$

$$= 2 \cdot \frac{x^4 + 2x}{x^4}$$

$$= 2 \cdot \frac{x^3 + 2}{x^3} = 0$$

$x^3 + 2 = 0 \quad \boxed{x = -\sqrt[3]{2}}$

x	f	f'	f''	
$\left(-\infty, \ -\sqrt[3]{2}\right)$		–	+	Decreasing, Concave Up
$x = -\sqrt[2]{3}$	0	–	0	Decreasing, ***Pt. of Inflection***
$\left(-\sqrt[2]{3}, \ 0\right)$		–	–	Decreasing, Concave Down
$x = 0$				***V.A.***
$(0, 1)$		–	+	Decreasing, Concave Up
$x = 1$	3	0	+	***LMIN***
$(1, \infty)$		+	+	Increasing, Concave Up

Exercise

Sketch the graph $\qquad y = \dfrac{x^2 - 3}{x - 2}$

Solution

Vertical Asymptote: $x = 2$

$$y' = \frac{2x(x-2) - \left(x^2 - 3\right)}{(x-2)^2}$$

$$= \frac{2x^2 - 4x - x^2 + 3}{(x-2)^2}$$

$$= \frac{x^2 - 4x + 3}{(x-2)^2} = \underline{0|} \quad \Rightarrow \boxed{x = 1,\ 3} \quad (CN)$$

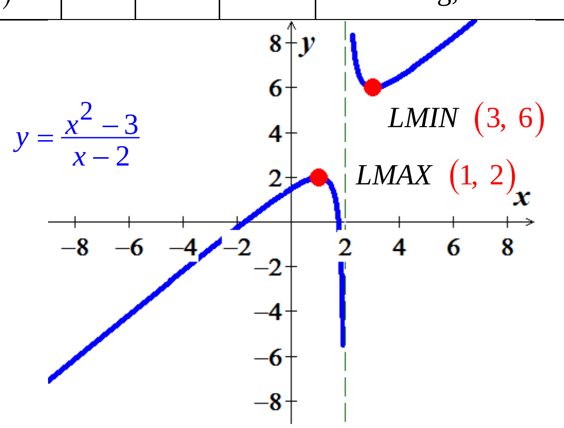
$$\rightarrow \begin{cases} x = 1 & \rightarrow y = 2 \\ x = 3 & \rightarrow y = 6 \end{cases}$$

$$y'' = (x-2)^{-3}\left((2x-4)(x-2) - 2\left(x^2 - 4x + 3\right)\right) \qquad \left(U^m V^n\right)' = U^{m-1} V^{n-1}\left(mU'V + nUV'\right)$$

$$= \frac{2x^2 - 8x + 8 - 2x^2 + 8x - 6}{(x-2)^3}$$

$$= \frac{2}{(x-2)^3} \quad \underline{\neq 0|}$$

x	f	f'	f''	
$(-\infty,\ 1)$		+	-	Increasing, Concave Up
$x = 1$	2	0		***LMAX***
$(1,\ 2)$		–	–	Decreasing, Concave Down
$x = 2$				***V.A.***
(2, 3)		–	+	Decreasing, Concave Up
$x = 3$	6	0	+	***LMIN***
$(3,\ \infty)$		+	+	Increasing, Concave Up



Exercise

Sketch the graph $y = \dfrac{5}{x^4+5}$

Solution

Horizontal Asymptote: $y = 0$

$$y' = \frac{-20x^3}{\left(x^4+5\right)^2} = 0 \quad \to x^3 = 0 \Rightarrow \boxed{x=0} \quad (CN) \qquad \to \{x = 0 \quad \to y = 1$$

$$y'' = -20\left(x^4+5\right)^{-3}\left(3x^2\left(x^4+5\right) - 2\left(4x^3\right)x^3\right) \qquad \left(U^m V^n\right)' = U^{m-1}V^{n-1}\left(mU'V + nUV'\right)$$

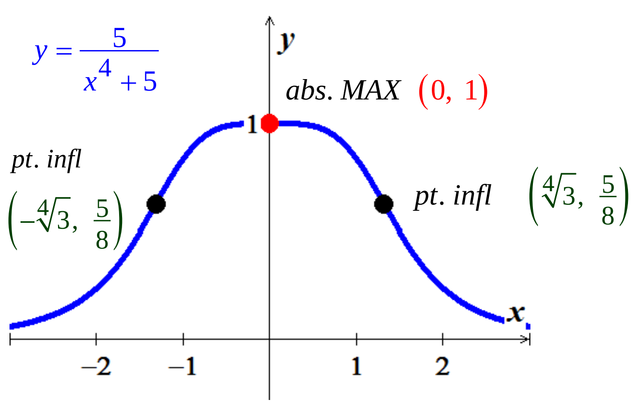
$$= -20\frac{3x^6 + 15x^2 - 8x^6}{\left(x^4+5\right)^3}$$

$$= \frac{100x^2\left(x^4-3\right)}{\left(x^4+5\right)^3} = 0$$

$$x^2\left(x^4-3\right) = 0 \to \begin{cases} x^2 = 0 & x = 0 \\ x^4 - 3 = 0 & \boxed{x = \pm\sqrt[4]{3}} \end{cases}$$

$$\to \begin{cases} x = -\sqrt[4]{3} & y = \frac{5}{8} \\ x = \sqrt[4]{3} & y = \frac{5}{8} \end{cases}$$

x	***f***	***f'***	***f''***	
$\left(-\infty, -\sqrt[4]{3}\right)$		+	+	Increasing, Concave Up
$x = -\sqrt[4]{3}$	2	+	0	Increasing, ***Pt. of Inflection***
$\left(-\sqrt[4]{3}, 0\right)$		+	–	Increasing, Concave Down
$x = 0$		0	0	***Abs. maximum, HA***
$\left(0, \sqrt[4]{3}\right)$		–	–	Decreasing, Concave Down
$x = \sqrt[4]{3}$	6	–	0	Decreasing, ***Pt. of Inflection***
$\left(\sqrt[4]{3}, \infty\right)$		–	+	Decreasing, Concave Up



Exercise

Sketch the graph $y=\dfrac{x^2-49}{x^2+5x-14}$

Solution

$$y=\frac{x^2-49}{x^2+5x-14}$$

$$=\frac{(x-7)(x+7)}{(x-2)(x+7)}$$

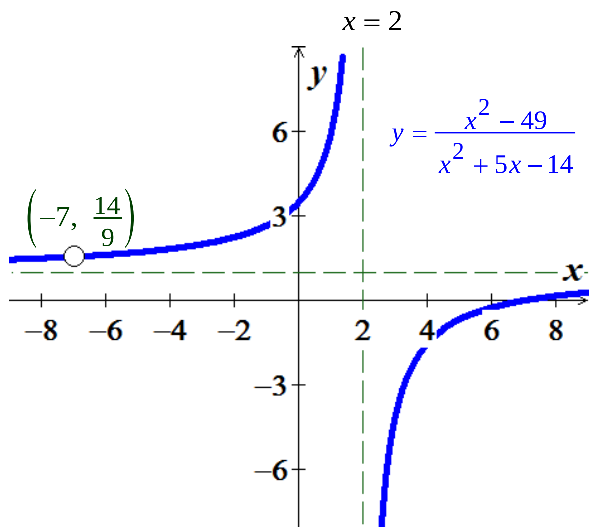
$y=1$ $$=\frac{x-7}{x-2}$$

$$=1-\frac{5}{x-2}$$

Hole: $x=-7$

Oblique Asymptote: $y=1$

Vertical Asymptote: $x=2$



$$y'=-5\frac{-1}{(x-2)^2}=\frac{5}{(x-2)^2}\neq 0$$

$$y''=\frac{5(-2)(x-2)}{(x-2)^4}=\frac{-10}{(x-2)^3}\neq 0$$

Exercise

Sketch the graph $\quad y=\dfrac{x^4+1}{x^2}$

Solution

$$y=\frac{x^4+1}{x^2}=\frac{x^4}{x^2}+\frac{1}{x^2}=x^2+\frac{1}{x^2}$$

Vertical Asymptote: $x=0$

Oblique Asymptote: $y=x^2$

$$y'=\frac{4x^3x^2-2x\left(x^4+1\right)}{x^4}$$

$$=\frac{2x\left(2x^4-x^4-1\right)}{x^4}$$

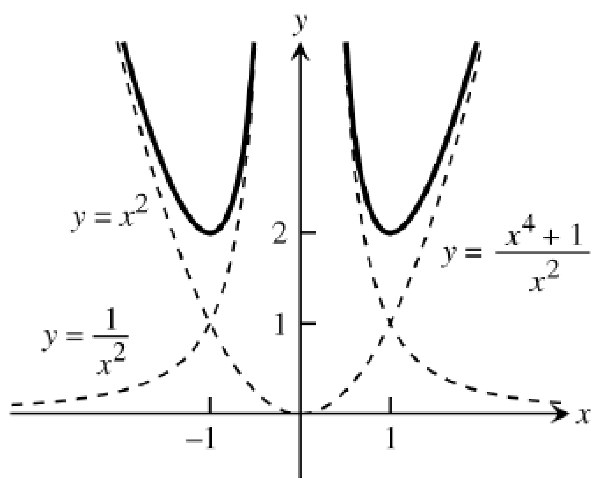
$$=\frac{2\left(x^4-1\right)}{x^3}=\underline{0|} \quad \to x^4-1=0 \quad \boxed{x=\pm 1} \quad (CN)$$

$$y'=\left(x^2+\frac{1}{x^2}\right)'$$

$$=2x-\frac{2x}{x^4}=2x-\frac{2}{x^3}$$

$$\to \begin{cases} x=-1 & \to y=2 \\ x=1 & \to y=2 \end{cases}$$

$-\infty$ to -1	-1 to 0	0 to 1	1 to ∞
$f'(-2)<0$	$f'(-0.5)>0$	$f'(0.5)<0$	$f'(2)>0$
Decreasing	***Increasing***	***Decreasing***	***Increasing***



Exercise

Sketch the graph $\qquad y=\dfrac{x^2-4}{x^2-2}$

Solution

$x^2-2=0 \;\Rightarrow\; x=\pm\sqrt{2}$

Vertical Asymptote: $x=\pm\sqrt{2}$

Horizontal Asymptote: $y=1$

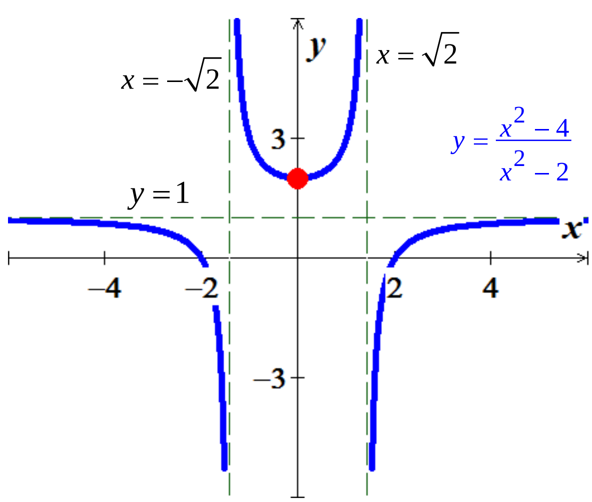
$$y'=\frac{2x\left(x^2-2\right)-2x\left(x^2-4\right)}{\left(x^2-2\right)^2}$$

$$=\frac{2x^3-4x-2x^3+8x}{\left(x^2-2\right)^2}$$

$$=\frac{4x}{\left(x^2-2\right)^2}=0 \quad\rightarrow\quad \boxed{x=0,\ \pm\sqrt{2}}\quad (CN)$$

$$\rightarrow\begin{cases} x=-\sqrt{2} \;\Rightarrow\; y=0 & \rightarrow\left(-\sqrt{2},0\right)\\ x=0 \;\Rightarrow\; y=2 & \rightarrow(0,2)\\ x=\sqrt{2} \;\Rightarrow\; y=0 & \rightarrow\left(\sqrt{2},0\right)\end{cases}$$

$-\infty$ $-\sqrt{2}$	0	$\sqrt{2}$	∞
$f'(-2)<0$	$f'(-1)<0$	$f'(1)>0$	$f'(2)>0$
Decreasing	*Decreasing*	*Increasing*	*Increasing*



Exercise

Sketch the graph $\quad y = -\dfrac{x^2 - x + 1}{x - 1}$

Solution

$$y = -\frac{x^2 - x + 1}{x - 1} = -\left(x + \frac{1}{x - 1}\right)$$

$$\begin{array}{r} x \\ x-1 \overline{\big) x^2 - x + 1} \\ \underline{x^2 - x} \\ 1 \end{array}$$

Vertical Asymptote: $x = 1$

Oblique Asymptote: $y = -x$

$$y' = -\left(1 - \frac{1}{(x-1)^2}\right)$$

$$= \frac{1}{(x-1)^2} - 1$$

$$= \frac{-x^2 + 2x}{(x-1)^2} = 0$$

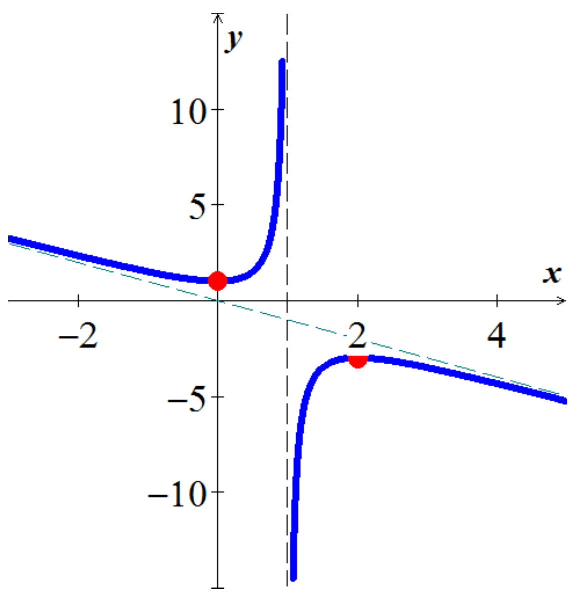
x	$f(x)$
0	1
2	−3

(CN) $\quad x = 0,\ 1,\ 2$

$-\infty$ 0	0 1	1 2	2 ∞
$f'(-2) < 0$	$f'(0.5) > 0$	$f'(1.5) > 0$	$f'(3) < 0$
Decreasing	*Increasing*	*Increasing*	*Decreasing*

Incr.: $(0,\ 1) \cup (1,\ 2)$ $\qquad$ ***Decr.***: $(-\infty,\ 0) \cup (2,\ \infty)$

LMIN: $(2,\ -3)$ $\qquad$ ***LMAX***: $(0,\ 1)$



Exercise

Sketch the graph $\quad y=\dfrac{x^3-3x^2+3x-1}{x^2+x-2}$

Solution

$$
\begin{aligned}
y&=\frac{x^3-3x^2+3x-1}{x^2+x-2}\\
&=\frac{(x-1)(x-1)(x-1)}{(x-1)(x+2)}\\
&=\frac{x^2-2x+1}{x+2}\\
&=x-4+\frac{9}{x+2}
\end{aligned}
$$

$$
\begin{array}{r}
x-4 \\
x+2\,\big)\,\overline{x^2-2x+1} \\
\underline{x^2+2x} \\
-4x+1 \\
\underline{-4x-8} \\
9
\end{array}
$$

Vertical Asymptote: $x=-2$

Hole: $x=1 \Rightarrow y=0$

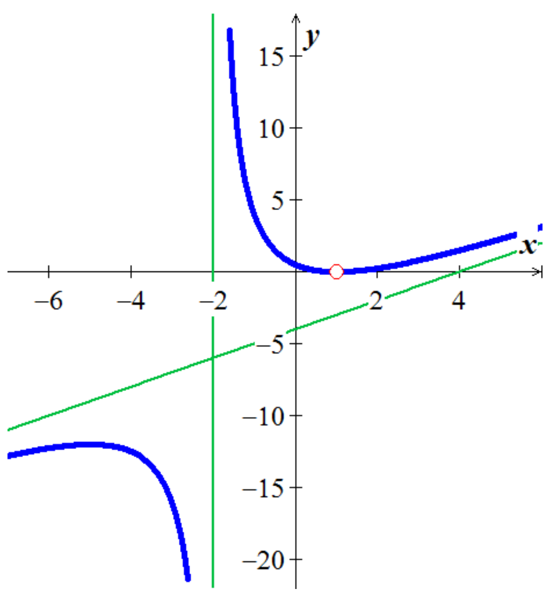
Oblique Asymptote: $y=x-4$

$$y'=1-\frac{9}{(x+2)^2} \quad =\frac{(x+2)^2-9}{(x+2)^2}=\underline{0}$$

$$(x+2)^2=9 \to x+2=\pm 3 \Rightarrow x=-2\pm 3 \to (x=-5,\ 1)$$

$$\to \begin{cases} x=-5 \Rightarrow y=1 & \to (-5,1) \\ x=1 \Rightarrow y=0 & \to (1,0) \end{cases}$$

$-\infty$	-5	-2	1	∞
$f'(-6)>0$	$f'(-3)<0$	$f'(0)<0$	$f'(2)>0$	
Increasing	***Decreasing***	***Decreasing***	***Increasing***	



Exercise

Sketch the graph $y = \dfrac{4x}{x^2+4}$

Solution

VA: *N/A* ***HA:*** $y=0$

$$y' = \frac{4\left(x^2+4\right)-(4x)(2x)}{\left(x^2+4\right)^2}$$

$$= \frac{16-4x^2}{\left(x^2+4\right)^2} = 0$$

$$16-4x^2=0 \rightarrow x^2=4$$

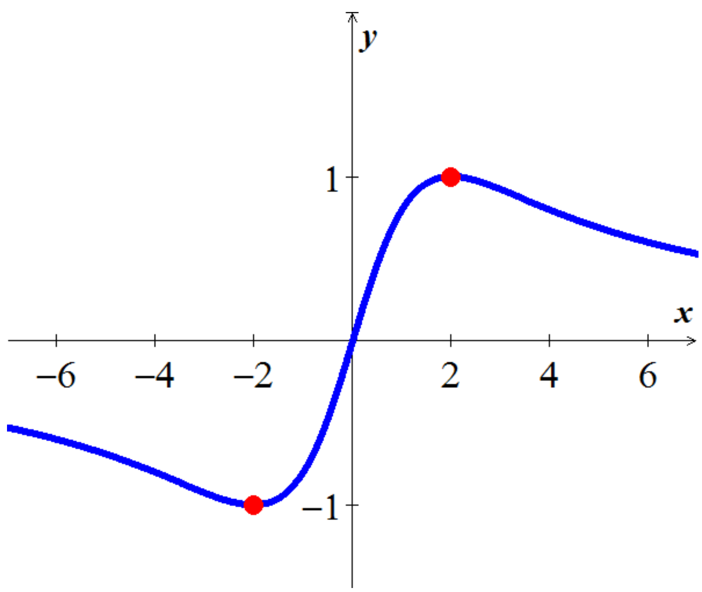
$CN:$ $\underline{x=\pm 2}$

x	$f(x)$
-2	-1
2	1

$-\infty$	-2	2	∞
$-$	$+$	$-$	
Decreasing	*Increasing*	*Decreasing*	

Incr.: $(-2,\ 2)$ ***Decr.:*** $(-\infty,\ -2)\cup(2,\ \infty)$

LMIN: $(-2,\ -1)$ ***LMAX:*** $(2,\ 1)$



Exercise

Sketch the graph of $f(x)=\dfrac{x^2+4}{2x}$

Solution

$$f(x)=\frac{x^2+4}{2x}=\frac{x}{2}+\frac{2}{x}$$

Oblique Asymptote: $\boxed{y=\dfrac{x}{2}}$

$$f'(x)=\frac{1}{2}-\frac{2}{x^2}$$

$$=\frac{x^2-4}{2x^2}=0$$

CN : $\boxed{x=\pm 2}$

$$f'(x)=\frac{1}{2}-\frac{2}{x^2} \quad \to \quad f''(x)=\frac{4}{x^3}$$

No point of inflection and when $\begin{cases} x>0 & \to f''>0 \\ x<0 & \to f''<0 \end{cases}$

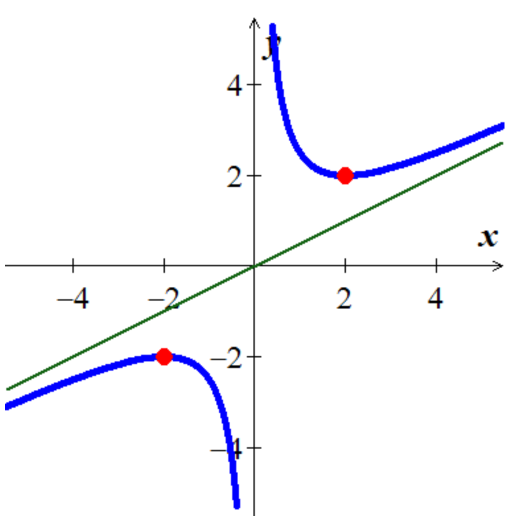
$-\infty$ -2	0	2 ∞
$f'(-3)>0$ ***Increasing***	$f'(0)<0$ ***Decreasing***	$f'(3)>0$ ***Increasing***
$f''(-1)<0$ ***Concave down***		$f''(1)>0$ ***Concave up***

RMIN: $(2,\ 2)$ ***Decreasing***: $(-2,\ 2)$

RMAX: $(-2,\ -2)$ ***Increasing***: $(-\infty,\ -2)\cup(2,\ \infty)$

Concave down: $(-\infty,\ 0)$

Concave up: $(0,\ \infty)$



Exercise

Sketch the graph of $\quad f(x)=\frac{1}{2}x^4-3x^2+4x+1$

Solution

$$f'(x)=2x^3-6x+4=0$$

$$x^3-3x+2=0 \quad \to \quad x=1$$

$$\begin{array}{c|cccc} 1 & 1 & 0 & -3 & 2 \\ & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & 0 \end{array} \quad \to x^2+x-2=0$$

CN : $x=1,\ 1,\ -2$

x	$f(x)$
-2	-11
1	$\frac{5}{2}$

$-\infty$ to -2	-2 to 1	1 to ∞
$-$	$+$	$+$
Decr.	*Incr.*	*Incr.*

$$f''(x)=6x^2-6=0 \quad \to \quad x=\pm 1$$

x	$f(x)$
-1	$-\frac{11}{2}$

$-\infty$ to -1	-1 to 1	1 to ∞
$+$	$-$	$+$
Up	*Down*	*Up*

Points of inflection: $\left(-1,\ -\frac{11}{2}\right)$ & $\left(1,\ \frac{5}{2}\right)$

Incr.: $(-2,\ \infty)$

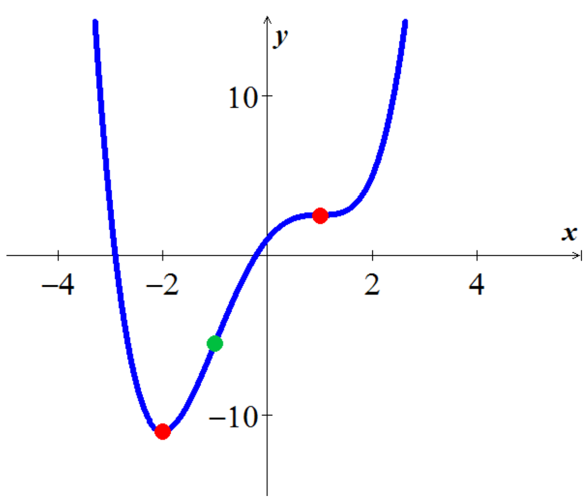
Decr.: $(-\infty,\ -2)$

LMIN: $(-2,\ -11)$

LMAX: $\left(1,\ \frac{5}{2}\right)$

Concave down: $(-1,\ 1)$

Concave up: $(-\infty,\ -1)\cup(1,\ \infty)$



Exercise

Sketch the graph of $f(x)=\dfrac{3x}{x^2+3}$

Solution

$$f'(x)=\frac{-3x^2+9}{\left(x^2+3\right)^2}=0$$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)=\frac{(ae-bd)x^2+2(af-dd)x+(bf-ce)}{\left(dx^2+ex+f\right)^2}$$

$$x^2=3 \quad\to\quad CN:\ \ \boxed{x=\pm\sqrt{3}}$$

$-\infty$ to $-\sqrt{3}$	$-\sqrt{3}$ to $\sqrt{3}$	$\sqrt{3}$ to ∞
$-$	$+$	$-$
Decr.	*Incr.*	*Decr.*

x	$f(x)$
$-\sqrt{3}$	$\frac{-3\sqrt{3}}{6}=-\frac{\sqrt{3}}{2}$
$\sqrt{3}$	$\frac{\sqrt{3}}{2}$

$$f''(x)=3\frac{-2x\left(x^2+3\right)-4x\left(-x^2+3\right)}{\left(x^2+3\right)^3}$$

$$=3\frac{2x^3-18x}{\left(x^2+3\right)^3}=0$$

$$2x^3-18x=2x\left(x^2-9\right)=0$$

$$\boxed{x=0,\ \pm 3}$$

x	$f(x)$
-3	$-\frac{3}{4}$
0	0
3	$\frac{3}{4}$

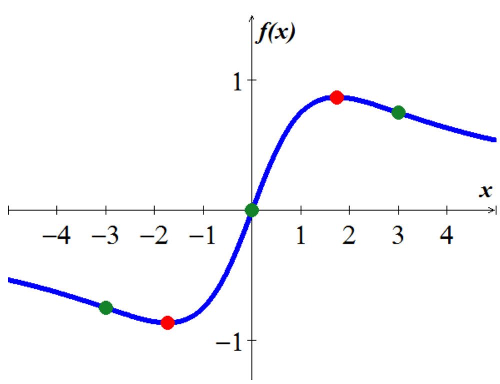
$-\infty$ to -3	-3 to 0	0 to 3	3 to ∞
$-$	$+$	$-$	$+$
Down	*Up*	*Down*	*Up*

Points of inflection: $\boxed{\left(-3,\ -\frac{3}{4}\right)\ (0,\ 0)\ \left(3,\ \frac{3}{4}\right)}$

Incr.: $\boxed{\left(-\sqrt{3},\ \sqrt{3}\right)}$ ***Decr.***: $\boxed{\left(-\infty,\ -\sqrt{3}\right)\cup\left(\sqrt{3},\ \infty\right)}$

LMIN: $\boxed{\left(-\sqrt{3},\ -\frac{\sqrt{3}}{2}\right)}$ ***LMAX***: $\boxed{\left(\sqrt{3},\ \frac{\sqrt{3}}{2}\right)}$

Concave down: $\boxed{(-\infty,\ -3)\cup(0,\ 3)}$ ***Concave up***: $\boxed{(-3,\ 0)\cup(3,\ \infty)}$



Exercise

Sketch the graph of $f(x) = 4\cos(\pi(x-1))$ *on* $[0,\ 2]$

Solution

$$f'(x) = -4\pi\sin(\pi(x-1)) = 0$$

$$\pi(x-1) = n\pi \to \begin{cases} \pi(x-1) = -\pi & \Rightarrow \quad x = 0 \\ \pi(x-1) = 0 & \Rightarrow \quad x = 1 \\ \pi(x-1) = \pi & \Rightarrow \quad x = 2 \end{cases}$$

CN: $\quad x = 0,\ 1,\ 2$

x	$f(x)$
0	−4
1	4
2	−4

$$f''(x) = -4\pi^2\cos(\pi(x-1)) = 0$$

$$\pi(x-1) = n\frac{\pi}{2} \to \begin{cases} \pi(x-1) = -\frac{\pi}{2} & \Rightarrow \quad x = \frac{1}{2} \\ \pi(x-1) = \frac{\pi}{2} & \Rightarrow \quad x = \frac{3}{2} \end{cases}$$

x	$f(x)$
$\frac{1}{2}$	0
$\frac{3}{2}$	0

0	$\frac{1}{2}$	$\frac{3}{2}$ 2
+	−	+
Up.	***Down.***	***Up .***

Points of inflection: $\left(\frac{1}{2},\ 0\right)\ \left(\frac{3}{2},\ 0\right)$

Incr.: $(0,\ 1)$

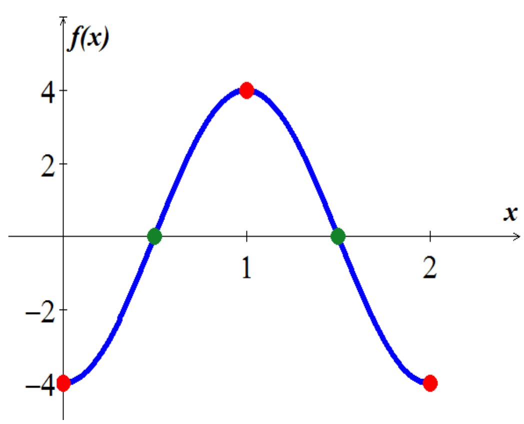
Decr.: $(1,\ 2)$

Abs. MIN: $(0,\ -4)\ (2,\ -4)$

Abs. MAX: $(1,\ 4)$

Concave down: $\left(\frac{1}{2},\ \frac{3}{2}\right)$

Concave up: $\left(0,\ \frac{1}{2}\right)\cup\left(\frac{3}{2},\ 2\right)$



Exercise

Sketch the graph of $\quad f(x)=\dfrac{x^2+x}{4-x^2}$

Solution

$\boldsymbol{VA}:\ x=\pm 2 \quad \boldsymbol{HA}:\ y=-1$

$$f'(x)=\frac{x^2+8x+4}{\left(4-x^2\right)^2}=0$$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)=\frac{(ae-bd)x^2+2(af-dd)x+(bf-ce)}{\left(dx^2+ex+f\right)^2}$$

$CN:\quad \boxed{x=-4\pm 2\sqrt{3}}$

x	$f(x)$
$-4-2\sqrt{3}$	$\approx .933$
$-4+2\sqrt{3}$	$\approx -.067$

$-\infty$ $-4-2\sqrt{3}$	$-4-2\sqrt{3}$ -2	-2 $-4+2\sqrt{3}$	$-4+2\sqrt{3}$ 2	2 ∞
+	–	–	+	+
Incr	*Decr*	*Decr*	*Incr*	*Incr*

$$f''(x)=\frac{(2x+8)\left(4-x^2\right)+4x\left(x^2+8x+4\right)}{\left(4-x^2\right)^3}$$

$$=\frac{2x^3+24x^2+24x+32}{\left(4-x^2\right)^3}=0$$

$$x^3+12x^2+12x+16=0 \xrightarrow[\text{using software}]{} x=-2\sqrt[3]{9}-2\sqrt[3]{3}-4 \approx \boxed{-11.045} \quad 2\ \mathbb{C}$$

$f(-11.045) \approx \boxed{-.94}$

$-\infty$ -11.045	-11.045 -2	-2 2	2 ∞
+	–	+	–
Up	*Down*	Up	*Down*

Points of inflection: $\boxed{(-11.045,\ -.94)}$

Incr.: $\boxed{\left(-\infty,\ -4-2\sqrt{3}\right)\ \left(-4+2\sqrt{3},\ 2\right)\ (2,\ \infty)}$

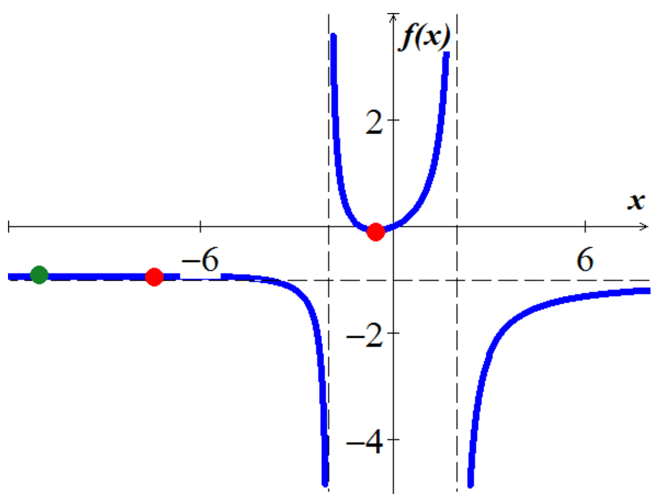
Decr.: $\boxed{\left(-4-2\sqrt{3},\ -2\right)\ \left(-2,\ -4+2\sqrt{3}\right)}$

LMIN: $\boxed{\left(-4-2\sqrt{3},\ -.933\right)}$

LMAX: $\boxed{\left(-4+2\sqrt{3},\ -.067\right)}$

Concave down: $\boxed{(-11.045,\ -2)\ (2,\ \infty)}$

Concave up: $\boxed{(-\infty,\ -11.045)\ (-2,\ 2)}$



Exercise

Sketch the graph of $f(x) = \sqrt[3]{x} - \sqrt{x} + 2$

Solution

Domain: $x \ge 0$

$$f'(x) = \frac{1}{3}x^{-2/3} - \frac{1}{2}x^{-1/2} = 0$$

$$\frac{1}{3x^{2/3}} = \frac{1}{2x^{1/2}}$$

$$\left(2x^{1/2}\right)^6 = \left(3x^{2/3}\right)^6$$

$$\left(\frac{2}{3}\right)^6 x^3 = x^4$$

$$x^4 - \left(\frac{2}{3}\right)^6 x^3 = 0$$

$$x^3\left(x - \left(\frac{2}{3}\right)^6\right) = 0$$

$CN:\quad \underline{x = 0,\ \left(\frac{2}{3}\right)^6}$

0	$\left(\frac{2}{3}\right)^6$
+	–
Incr.	***Decr.***

$f(0) = \underline{2}$

$$f\left(\left(\frac{2}{3}\right)^6\right) = \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3 + 2 = \frac{4}{9} - \frac{8}{27} + 2$$

$$= \frac{58}{27}$$

$$f''(x) = -\frac{2}{9}x^{-5/3} + \frac{1}{4}x^{-3/2} = 0$$

$$\frac{2}{9}x^{-5/3} = \frac{1}{4}x^{-3/2}$$

$$\left(\frac{8}{9}x^{-5/3}\right)^{-6} = \left(x^{-3/2}\right)^{-6}$$

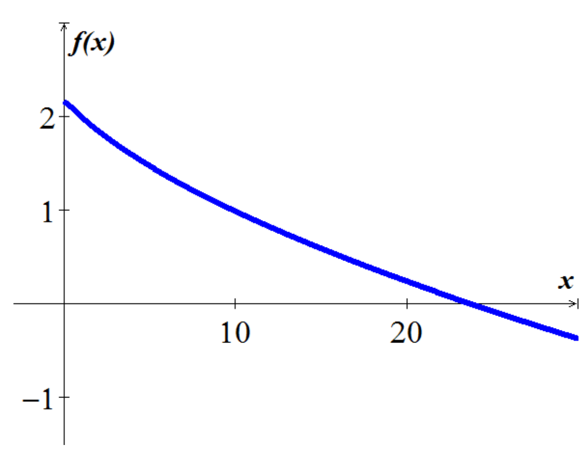
$$\left(\frac{8}{9}\right)^{-6} x^{10} = x^9$$

$$x^{10} - \left(\frac{8}{9}\right)^6 x^9 = 0$$

$$x^9\left(x - \left(\frac{8}{9}\right)^6\right) = 0$$

Point of inflection. $\quad \underline{x = 0,\ \left(\frac{8}{9}\right)^6}$

0	$\left(\frac{8}{9}\right)^6$
–	+
Down	***Up***



Incr.: $\boxed{\left(0,\ \left(\frac{2}{3}\right)^6\right)}$

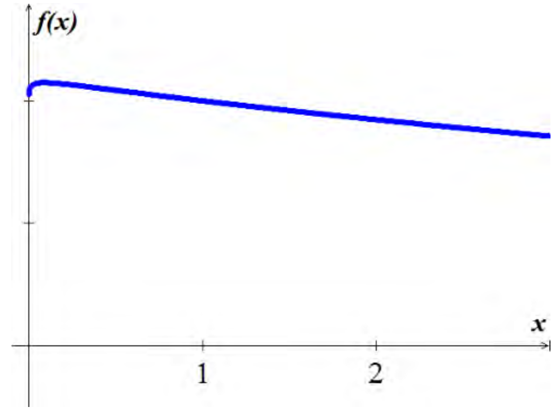
Decr.: $\boxed{\left(\left(\frac{2}{3}\right)^6,\ \infty\right)}$

Abs. MIN: *none*

Abs. MAX: $\boxed{\left(\left(\frac{2}{3}\right)^6,\ \frac{58}{27}\right)}$

Concave down: $\boxed{\left(0,\ \left(\frac{8}{9}\right)^6\right)}$

Concave up: $\boxed{\left(\left(\frac{8}{9}\right)^6,\ \infty\right)}$



Exercise

Sketch the graph of $f(x)=\dfrac{\cos \pi x}{1+x^2}$ *on* $[-2,\ 2]$

Solution

$$f'(x)=\frac{-\pi\left(1+x^2\right)\sin \pi x-2x\cos \pi x}{\left(1+x^2\right)^2}=0$$

$$\pi\left(1+x^2\right)\sin \pi x+2x\cos \pi x=0$$

Using software: *CN*: $\boxed{x=0,\ \pm.902,\ \pm 1.919}$

x	$f(x)$
-2	$.2$
-1.919	$\approx .21$
$-.902$	$\approx -.53$
0	1
$.902$	$\approx -.53$
1.919	$\approx .21$
2	$.2$

-2 $\quad -1.919$ $\quad -.902$ $\quad 0$ $\quad .902$ $\quad 1.919$ $\quad 2$

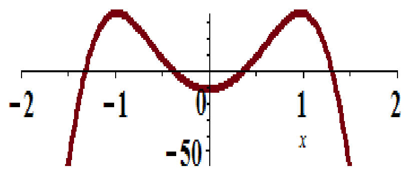
+	–	+	–	+	–
Incr	*Decr*	*Incr*	*Decr*	*Incr*	*Decr*

$$f''(x)=\frac{1}{\left(1+x^2\right)^3}\left(\begin{array}{l}\left(-2\pi x\sin \pi x-\pi^2\left(1+x^2\right)\cos \pi x-2\cos \pi x+2\pi x\sin \pi x\right)\left(1+x^2\right)\\ +4\pi x\left(1+x^2\right)\sin \pi x+8x^2\cos \pi x\end{array}\right)$$

$$= \frac{1}{\left(1+x^2\right)^3} \left(\begin{array}{l} -2\pi x\left(1+x^2\right)\sin \pi x - \pi^2\left(1+x^2\right)^2 \cos \pi x - 2\left(1+x^2\right)\cos \pi x + 2\pi x\left(1+x^2\right)\sin \pi x \\ +4\pi x\left(1+x^2\right)\sin \pi x + 8x^2 \cos \pi x \end{array} \right)$$

$$= \frac{1}{\left(1+x^2\right)^3}\left(\left(-\pi^2\left(1+x^2\right)^2 - 2\left(1+x^2\right) + 8x^2\right)\cos \pi x + 4\pi x\left(1+x^2\right)\sin \pi x\right) = 0$$

$$\left(-\pi^2\left(1+x^2\right)^2 - 2\left(1+x^2\right) + 8x^2\right)\cos \pi x + 4\pi x\left(1+x^2\right)\sin \pi x = 0$$



Using graph and software to find the roots:

Point of inflection: $x = \pm 0.3816, \quad \pm 1.307$

$-2 \qquad -1.307 \qquad -.3816 \qquad .3816 \qquad 1.307 \qquad 2$

–	+	–	+	–
Down	*Up*	*Down*	*Up*	*down*

Incr.: $(-2,\ -1.919) \cup (-.902,\ 0) \cup (.902,\ 1.919)$

Decr.: $(-1.919,\ -.902) \cup (0,\ .902) \cup (1.919,\ 2)$

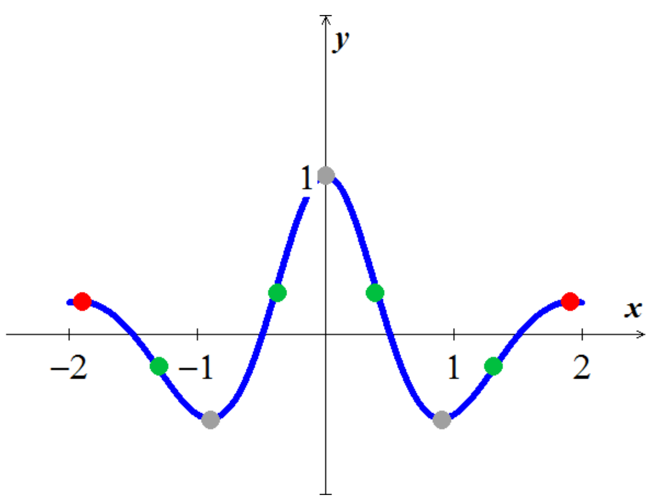
Abs. MIN: $(\pm .902,\ -.53)$

Abs. MAX: $(0,\ 1)$

L*MAX*: $(\pm 1.919,\ 0.21)$

Concave down: $(-2,\ -1.307)\ (-.3816,\ .386)\ (1.307,\ 2)$

Concave up: $(-1.307,\ -.3816)\ (.3816,\ 1.307)$



Exercise

Sketch the graph of $f(x) = x^{2/3} + (x+2)^{1/3}$

Solution

$$f'(x) = \frac{2}{3}x^{-1/3} + \frac{1}{3}(x+2)^{-2/3} = 0$$

$$\left(2x^{-1/3}\right)^3 = \left(-(x+2)^{-2/3}\right)^3$$

$$8x^{-1} = -(x+2)^{-2} \quad (x \neq 0,\ -2)$$

$$8(x+2)^2 = -x$$

$$8x^2 + 32x + 32 = -x$$

$$8x^2 + 33x + 32 = 0 \quad \rightarrow \quad \boxed{x = \frac{-33 \pm \sqrt{65}}{16}}$$

x	$f(x)$
$\frac{-33-\sqrt{65}}{16}$	≈ 1.05
-2	$\sqrt[3]{4}$
$\frac{-33+\sqrt{65}}{16}$	≈ 2.11
0	$\sqrt[3]{2}$

CN: $\boxed{x = 0,\ -2,\ \frac{-33 \pm \sqrt{65}}{16}}$

$\frac{-33-\sqrt{65}}{16}$	-2	$\frac{-33+\sqrt{65}}{16}$	0	
–	+	+	–	+
Decr.	*Incr.*	*Incr.*	*Decr.*	*Incr.*

$$f''(x) = -\frac{2}{9}x^{-4/3} - \frac{2}{9}(x+2)^{-5/3} = 0$$

$$x^{-4/3} = -(x+2)^{-5/3} \qquad (x \neq 0,\ -2)$$

$$\left(x^{4/3}\right)^3 = \left(-(x+2)^{5/3}\right)^3$$

$$x^4 = -(x+2)^5$$

$$(x+2)^5 + x^4 = 0 \xrightarrow[\text{software}]{} \boxed{x = -6.43375}$$

Point of inflection: $\boxed{x = 0,\ -2,\ -6.43375}$

$-.643375$	-2	0	
–	+	–	–
Down	*Up.*	*Down*	*Down*

$$f(-6.43375) \boxed{\approx 1.8164}$$

Incr.: $\boxed{\left(\frac{-3-\sqrt{65}}{16},\ \frac{-3+\sqrt{65}}{16}\right) \cup (0,\ \infty)}$

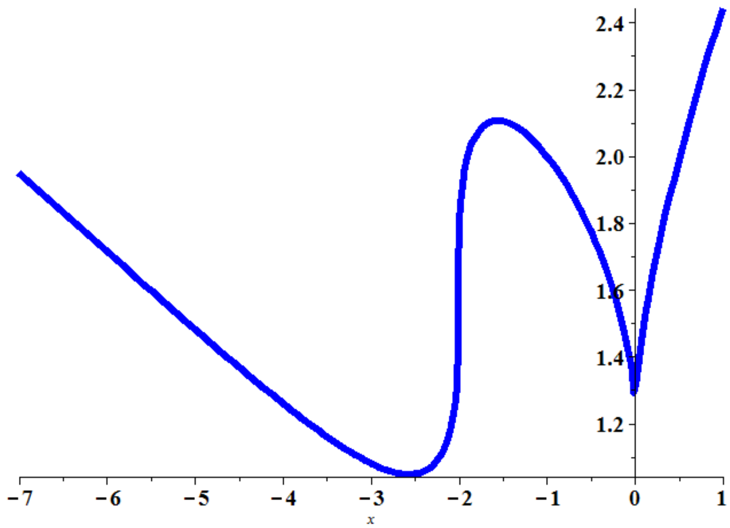
Decr.: $\left(-\infty, \frac{-3-\sqrt{65}}{16}\right) \cup \left(\frac{-3+\sqrt{65}}{16}, 0\right)$

Abs. MIN: $\left(\frac{-3-\sqrt{65}}{16}, 1.05\right)$

LMAX: $\left(\frac{-3+\sqrt{65}}{16}, 2.11\right)$ $(0, 1.26)$

Concave down: $(-\infty, -6.43375)$ $(-2, 0)$ $(0, \infty)$

Concave up: $(-6.43375, -2)$



Exercise

Sketch the graph of $f(x) = x(x-1)e^{-x}$

Solution

$$f(x) = \left(x^2 - x\right)e^{-x}$$

$$f'(x) = \left(2x - 1 - x^2 + x\right)e^{-x}$$

$$= -\left(x^2 - 3x + 1\right)e^{-x} = 0$$

$x^2 - 3x + 1 = 0 \quad \to \quad CN: \quad x = \dfrac{3 \pm \sqrt{5}}{2}$

x	$f(x)$
$\frac{3-\sqrt{5}}{2}$	≈ -0.16
$\frac{3+\sqrt{5}}{2}$	≈ 2.31

	$\frac{3-\sqrt{5}}{2}$	$\frac{3+\sqrt{5}}{2}$
$-$	$+$	$-$
Decr.	*Incr.*	*Decr.*

$$f''(x) = -\left(2x - 3 - x^2 + 3x - 1\right)e^{-x}$$

$$= \left(x^2 - 5x + 4\right)e^{-x} = 0$$

$x^2 - 5x + 4 = 0 \quad \to \quad$ *Pt. infl.*: $x = 1,\ 4$

x	$f(x)$
1	0
4	≈ 0.22

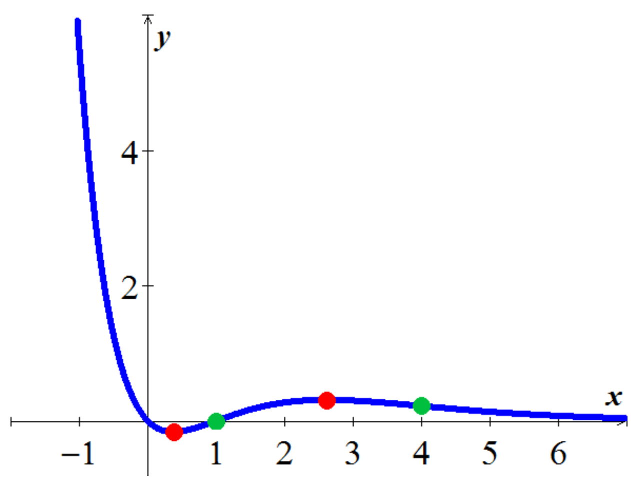
	1	4
$+$	$-$	$+$
Up	*Down*	*Up*

Incr.: $\left(\dfrac{3-\sqrt{5}}{2}, \dfrac{3+\sqrt{5}}{2}\right)$

Decr.: $\left(-\infty, \dfrac{3-\sqrt{5}}{2}\right) \cup \left(\dfrac{3+\sqrt{5}}{2}, \infty\right)$

Abs. MIN: $\left(\dfrac{3-\sqrt{5}}{2}, -0.16\right)$

LMAX: $\left(\dfrac{3+\sqrt{5}}{2}, 0.31\right)$



Concave down: $(1,\ 4)$

Concave up: $(-\infty,\ 1)\ (4,\ \infty)$

Exercise

The revenue R generated from sales of a certain product is related to the amount x spent on advertising by

$$R(x)=\frac{1}{15,000}\left(600x^2-x^3\right), \qquad 0\le x\le 600$$

Where x and R are in thousands of dollars. Is there a point of diminishing returns for this function?

Solution

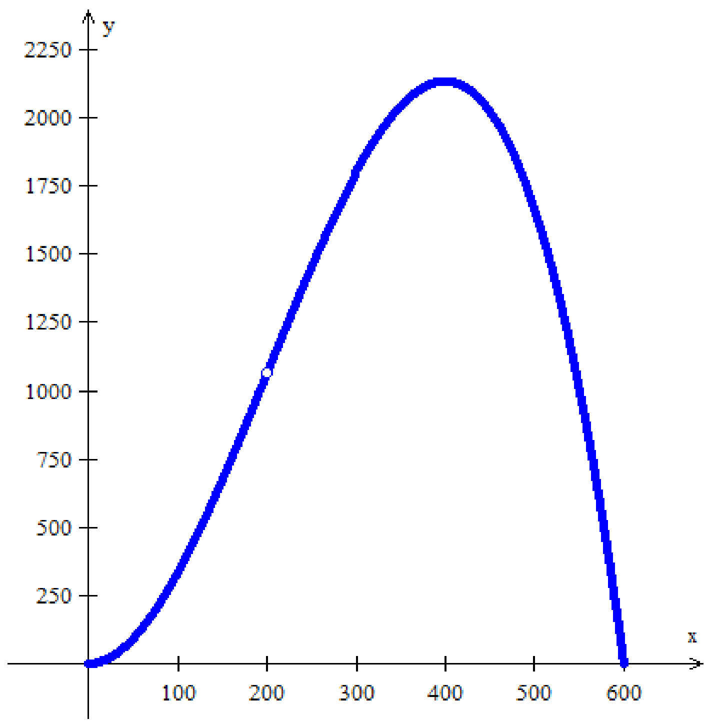
$$R'=\frac{1}{15,000}\left(1200x-3x^2\right)$$

$$R'=\frac{1}{15,000}(1200-6x)=0$$

$$\Rightarrow\ x=\frac{1200}{6}=200$$

x = 200 (or \$200,000) is a ***diminishing point***

An increased investment beyond this point is usually considered a poor use of capital



Exercise

Find the point of diminishing returns (x, y) for the function

$$R(x) = -x^3 + 45x^2 + 400x + 8000, \quad 0 \le x \le 20$$

where $R(x)$ represents revenue in thousands of dollars and x represents the amount spent on advertising in tens of thousands of dollars.

Solution

$$R'(x) = -3x^2 + 90x + 400$$

$$R''(x) = -6x + 90 = 0$$

$$-6x = -90$$

$$\underline{x = \frac{-90}{-6} = 15}$$

$$R(x = 15) = -(15)^3 + 45(15)^2 + 400(15) + 8000$$

$$\underline{= 20,750}$$

The point of diminishing returns is $\boxed{(15, \ 20,750)}$

Exercise

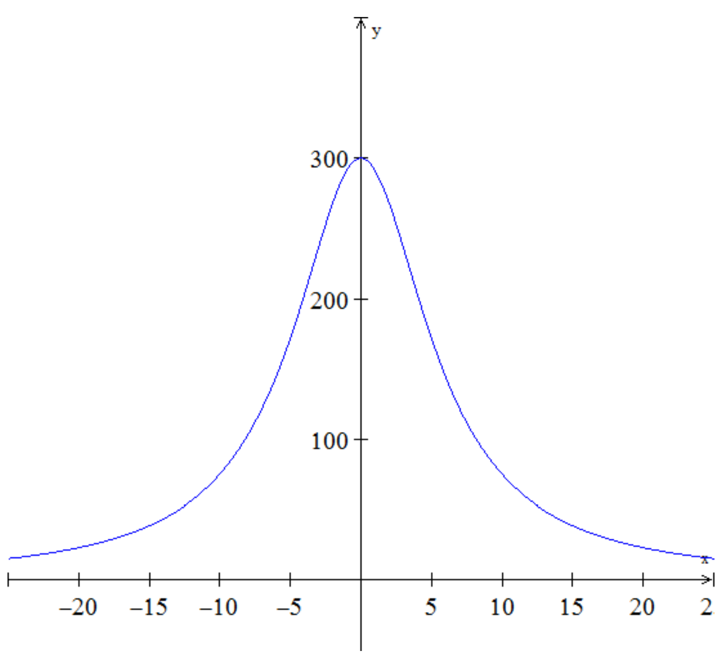
A county realty group estimates that the number of housing starts per year over the next three years will be

$$H(r) = \frac{300}{1 + 0.03r^2}$$

Where r is the mortgage rate (in percent).

a) Where is $H(r)$ increasing?

b) Where is $H(r)$ decreasing?



Solution

$$H'(r) = \frac{-300(0.06r)}{\left(1 + 0.03r^2\right)^2}$$

$$H'(r) = \frac{-18r}{\left(1 + 0.03r^2\right)^2}$$

$$-18r = 0 \Rightarrow \boxed{r = 0} \quad (CN)$$

a) $H(r)$ is ***increasing*** on the interval $(-\infty, \ 0)$

b) $H(r)$ is ***decreasing*** on the interval $(0, \ \infty)$

Exercise

Suppose the total cost $C(x)$ to manufacture a quantity x of insecticide (in hundreds of liters) is given by $C(x)=x^3-27x^2+240x+750$. Where is $C(x)$ decreasing?

Solution

$$C'(x)=3x^2-54x+240=0$$

$$\Rightarrow x=8,\ 10$$

$C(x)$ is ***decreasing*** $(8,\ 10)$

0 — 8	8 — 10	10 —
$C'(1)=189>0$	$C'<0$	$C'>0$
Increasing	***Decreasing***	***Increasing***

Exercise

The cost of a computer system increases with increased processor speeds. The cost C of a system as a function of processor speed is estimated as $C(x)=14x^2-4x+1200$, where x is the processor speed in MHz. Determine the intervals where the cost function $C(x)$ is decreasing.

Solution

$$C'(x)=28x-4=\mathbf{0}$$

$$\Rightarrow x=\frac{4}{28}=\frac{1}{7}$$

The cost function $C(x)$ is decreasing $\left(0,\ \frac{1}{7}\right)$

— $\frac{1}{7}$	$\frac{1}{7}$ —
$C'(0)=-4<0$	$C'>0$
Decreasing	***Increasing***

Exercise

The percent of concentration of a drug in the bloodstream t hours after the drug is administered is given by $K(t)=\dfrac{t}{t^2+36}$. On what time interval is the concentration of the drug increasing?

Solution

$$K'(t)=\frac{1\left(t^2+36\right)-2t(t)}{\left(t^2+36\right)^2}$$

$$K=\frac{f}{g}\Rightarrow K'=\frac{f'g+g'f}{g^2}\qquad f=t\quad f'=1\qquad g=t^2+36\quad g'=2t$$

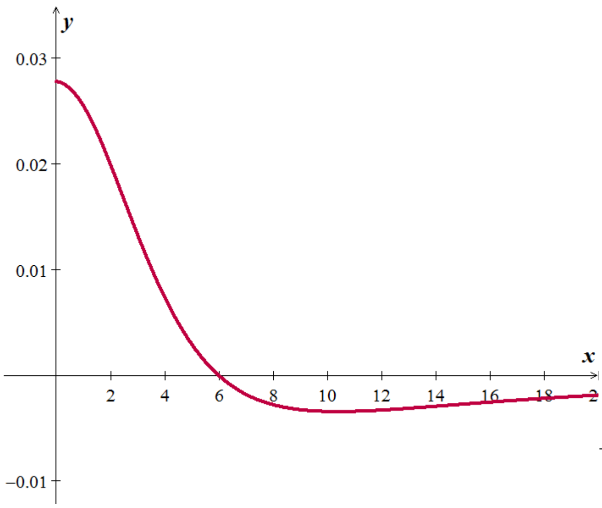
$$=\frac{t^2+36-2t^2}{\left(t^2+36\right)^2}$$

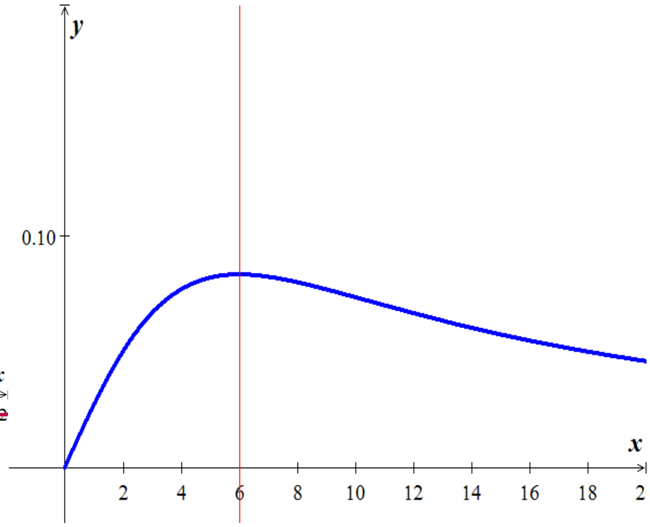
$$=\frac{36-t^2}{\left(t^2+36\right)^2}=0$$

$$t=\pm\sqrt{36}=\pm 6\quad\Rightarrow\boxed{t=6}$$

0 — 6	6 —
$K'(1)=\frac{35}{37^2}>0$	$K'(7)<0$
Increasing	***Decreasing***

The concentration of the drug is increasing over $\boxed{(0,\ 6)}$





Exercise

Coughing forces the trachea to contract, this in turn affects the velocity of the air through the trachea. The velocity of the air during coughing can be modeled by: $v = k(R-r)r^2, \quad 0 \le r < R$ where k is a constant, R is the normal radius of the trachea (also a constant) and r is the radius of the trachea during coughing. What radius r will produce the maximum air velocity?

Solution

$$v = k\left(Rr^2 - r^3\right)$$

$$v' = k\left(2Rr - 3r^2\right) = kr(2R - 3r) = 0$$

$$r = 0 \quad or \quad 2R - 3r = 0$$

$$r = 0 \quad or \quad r = (2/3)R$$

A trachea radius of zero minimizes air velocity (duh!). And a radius of 2/3 its normal size maximizes air flow.

Exercise

$P(x) = -x^3 + 15x^2 - 48x + 450, \quad x \ge 3$ is an approximation to the total profit (in thousands of dollars) from the sale of x hundred thousand tires. Find the number of hundred thousands of tires that must be sold to maximize profit.

Solution

$$P'(x) = -3x^2 + 30x - 48 = 0$$

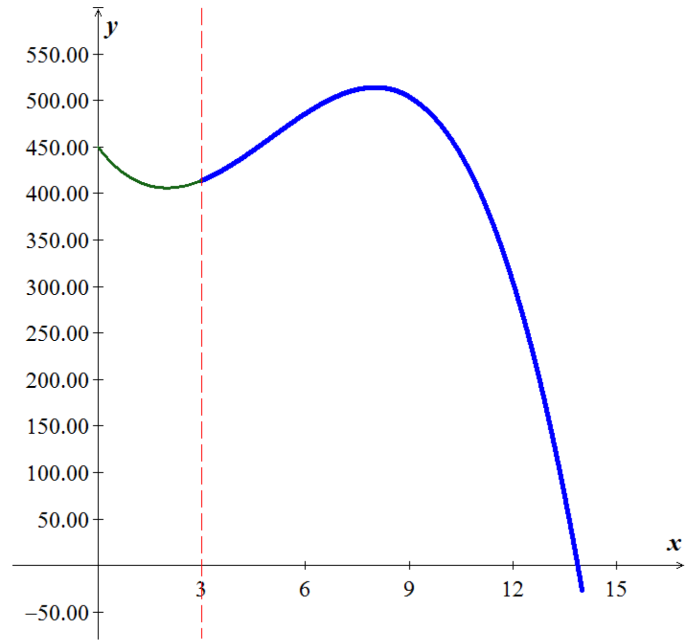
$$\Rightarrow x = 2,\ 8$$

$$Since\ x \ge 3 \quad \Rightarrow \boxed{x = 8}$$

$$P(x = 8) = -(8)^3 + 15(8)^2 - 48(8) + 450$$

$$= 541$$

The number of tires that must be sold to maximize profit is 800,000 tires



Exercise

$P(x) = -x^3 + 3x^2 + 360x + 5000; \quad 6 \le x \le 20$ is an approximation to the number of salmon swimming upstream to spawn, where x represents the water temperature in degrees Celsius. Find the temperature that produces the maximum number of salmon.

Solution

$$P'(x) = -3x^2 + 6x + 360 = 0$$

$$\Rightarrow x = 12, \quad -10\,(not\ \ in\ the\ interval)$$

$$P(x = 6) = -(6)^3 + 3(6)^2 + 360(6) + 5000$$

$$= 7052$$

$$P(x = 20) \ = 5400$$

$$P(x = 12) \ = 8024$$

12° is the temperature that produces the maximum number of salmon