2.5 Calculus Topics

2.5.1 Limits

The study of limits of functions is a fundamental activity in calculus. Finding the limit of a function, f, as x approaches c, means that we analyze the values of $f(x)$ for values of x near c. If the values of $f(x)$ appear to be close to a value L, for values of x near c, then it may be that the limit of the function is L. This analysis of a function can be carried out by examining the graph and tables of values of a function as illustrated in the examples below. It is important to realize that the exact computation of a limit requires knowledge and techniques of calculus.

For example, the function $f(x) = \frac{\sin(5x)}{3x}$, has a numerator and denominator of zero when $x = 0$. Does f have a limit as x goes to zero? How do the values of this function behave when x is near zero?

Reset the `MODE` option to `FUNC` if it is not already there. In the `Y= Editor`, enter the function and construct a table of values for the function, selecting `TblStart=-0.03, ΔTbl=0.01`, and both `Indpnt` and `Depend` to `AUTO`. The calculator produces the table given in Figure 37. See Section 2.4.7 for more on tables.

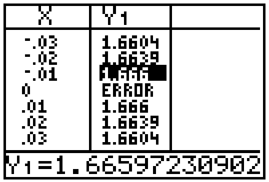


Figure 37: Values of the function near zero

As suspected, the calculator cannot compute the value of the function at $x = 0$, but it can compute the values of the function for x near zero. As the values of x approach zero, either from the negative numbers or the positive numbers, the values of the function are getting closer to $1.66666 \approx \frac{5}{3}$.

Now let's look at the graph of the function in Figures 38 and 39, where the trace feature is used to study the graph. Note that the viewing window is set at `Xmin=-2.5, Xmax=2.5, Xscl=1, Ymin=-2.5, Ymax=2.5, Yscl=1`.

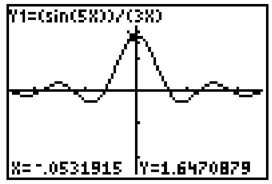


Figure 38: Graph of the function near zero

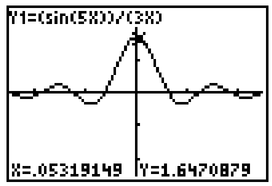
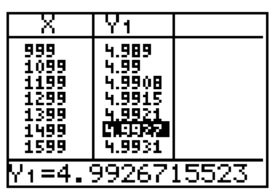


Figure 39: Graph of the function near zero

The graph also indicates that the limit of $f(x)$ as x approaches zero is $\frac{5}{3}$. This is definitely the correct answer and can be verified analytically. The table of values and the graph of a function help you to understand limits. As this example shows, the calculator may only compute an approximate value of a limit (1.66666). The limit of this function as x goes to zero is indeed $\frac{5}{3}$ and the value can be obtained using analytical methods.

You can also study the limit of a function, f, as x goes to infinity, that is, as x moves away from the origin in the positive direction with its values becoming very large. Use the same techniques as described above to examine a function for large positive values of x. Let $f(x) = \frac{5x-1}{x+2}$. When entering this function, remember to use parentheses around both the numerator and the denominator. Construct a table for large values of x by setting `TblStart=999`, `ΔTbl=100`, and both `Indpnt` and `Depend` to `AUTO`. Your calculator should produce the table given in Figure 40. As you continue to scroll down, observe that the values of the function approach 5.



X	Y1	
999	4.989	
1099	4.99	
1199	4.9908	
1299	4.9915	
1399	4.9921	
1499	4.9927	
1599	4.9931	

Y1=4.9926715523

Figure 40: Values of the function for large x

To look at the graph for large positive values of x, select the viewing window `Xmin=999, Xmax=1999, Xscl=100, Ymin=0, Ymax=10, Yscl=2`. Graph, and then press `TRACE` to verify that the values of the function get closer to 5 as the cursor moves away from the origin in the positive direction (Figure 41). The limit of this function as x goes to infinity is indeed 5, and the value can be obtained using analytical methods.

We can use similar techniques to examine functions as x moves away from the origin in the negative direction.

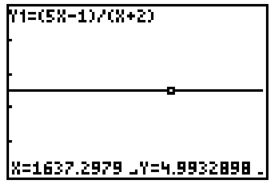


Figure 41: Graph of the function for large x

2.5.2 Maximum and Minimum

To find the maximum of a function on your calculator you can approach the subject from a graphical or numeric point of view. In the examples below use the function $f(x) = 2x^3 - 5x^2 + x - 3$ and the viewing window `Xmin=-4, Xmax=4, Xscl=1, Ymin=-10, Ymax=10, Yscl=2`. Geometrically, you can graph the function, then use the `TRACE` feature to move the cursor to the peaks and valleys of the graph to determine the x- and y-values at those points (Figures 42 and 43).

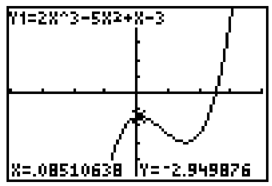


Figure 42: The maximum

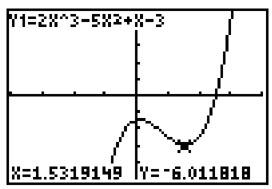


Figure 43: The minimum

Numerically, you can compute a table of values for the function, and analyze the outputs of the function (Figures 44 and 45).

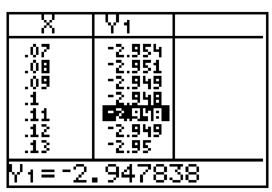


Figure 44: The maximum

Notice that the answers can be different. Geometrically (with the graph), you found the maximum of $f(x) = -2.949876$ to occur at $x = .08510638$. Numerically (with the table), you found that the maximum of $f(x) = -2.947838$ to occur at $x = 0.11$. Although both are good approximations, the exact value can only be obtained analytically. (The exact value of x is $\frac{5-\sqrt{19}}{6}$, and the exact value of y is $\frac{19\sqrt{19}}{54} - \frac{121}{27}$.) The minimum obtained geometrically is also different from the minimum obtained numerically.

The TI-83+/84+ also has built-in functions that allow you to find the maximum and minimum of a function. The `minimum` and `maximum` features are in the `CALCULATE` menu (from the graph screen, press

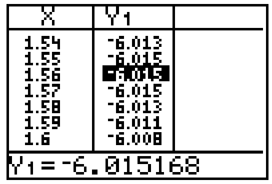


Figure 45: The minimum

2nd [CALC]), and the fMin(and fMax(features are in the MATH menu (press the MATH key from the HOME screen). These features are described below with the function $f(x) = 2x^3 - 5x^2 + x - 3$ in the viewing window Xmin=-4, Xmax=4, Xscl=1, Ymin=-10, Ymax=6, Yscl=2.

To use the minimum and maximum features, enter and graph a function. Press 2nd [CALC] to access the CALCULATE menu, select minimum by pressing 3. Use the arrows to move the cursor to select the left bound, the right bound and a guess, as prompted by the calculator, press ENTER to save each value. The cursor will move to the lowest point of the function within the bounds selected, and the calculator will display the values of x and y at that point (Figures 46–48). The commands for finding the maximum of the function are similar, and will not be discussed here.

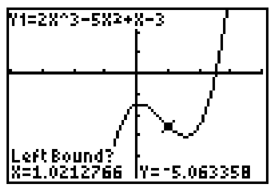


Figure 46: Left bound

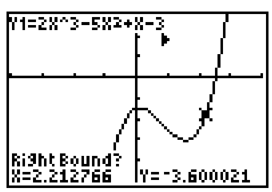


Figure 47: Right bound

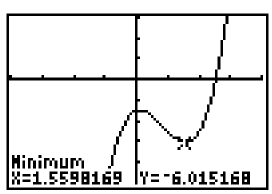


Figure 48: The minimum

For fMin(and fMax(, press MATH. Select fMax(by pressing 7. This will take you to the Home Screen where the command fMax(is copied onto the screen. Enter the equation, the variable (X), a lower bound, and an upper bound (separated by commas), to complete the command (if you entered the function into Y1, press VARS Y-VARS Function Y1 ENTER to copy Y1 onto the screen). Close the parenthesis and press ENTER. The calculator will return the value of x at which the maximum occurs within the search area. Once the calculator gives you the x-value you can compute the y-value by having the calculator compute the value of the function at the given x-value as in Figure 49. The computations for the minimum are similar and will not be displayed here.

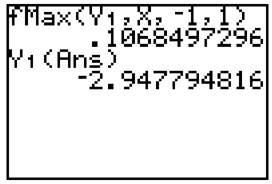


Figure 49: The maximum

Even though you you could use `fMax(` without graphing, it is important to graph the function first to get reasonable numbers for the lower and upper bound.

2.5.3 Derivative

The TI-83+/84+ has two built-in functions that allow you to find the numerical derivative of a function, `dy/dx` and `nDeriv(`. We illustrate how to use these features below using the function $f(x) = x^4 + 2x^3 - x^2 + 1$, and the viewing window `Xmin=-4, Xmax=4, Xscl=1, Ymin=-6, Ymax=6, Yscl=1`.

To use the `dy/dx` feature, you must enter and graph the function. Press $\boxed{\text{2nd}}$ `[CALC]` to access the `CALCULATE` menu, and select `dy/dx` by pressing $\boxed{6}$. Use the arrows to move the cursor to select the point at which the derivative will be computed or enter a value for x. Press $\boxed{\text{ENTER}}$. The cursor will move to the point and the calculator will display the value of `dy/dx` at the selected point (Figure 50).

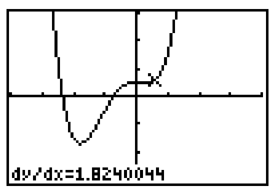


Figure 50: `dy/dx` computed at $x = 0.6$

To use `nDeriv(`, first return to the Home screen if necessary by pressing $\boxed{\text{2nd}}$ `QUIT`, and then press $\boxed{\text{MATH}}$. Select `nDeriv(` by pressing $\boxed{8}$. Complete the command by entering the function, the variable (`X`), and a value for `X`, separated by commas (if you entered the function into `Y1`, press $\boxed{\text{VARS}}$ `Y-VARS Function Y1` $\boxed{\text{ENTER}}$ to copy `Y1` onto the screen). Close the parenthesis and press $\boxed{\text{ENTER}}$. The calculator will return a value for the derivative of the function at the specified value of x (Figure 51).

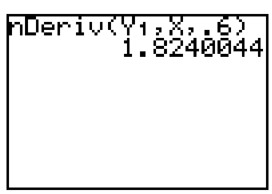


Figure 51: `nDeriv(` computed at $x = 0.6$

The derivative of a function is a function itself, and therefore can be entered into the calculator where it can be graphed and analyzed. However, you do not need to actually compute the derivative. Enter the function $y_1 = x^3 - 5x^2 + 6x - 4$ into the calculator. Define the derivative $\mathtt{Y_2} = \mathtt{nDeriv(Y_1(X),X,X)}$ as shown in Figure 52. The graphs of the original function and its derivative are shown in Figure 53 using the viewing window `Xmin=-5, Xmax=5, Xscl=1, Ymin=-5, Ymax=5, Yscl=1`. Don't worry if it takes a while for your calculator to generate this graph.

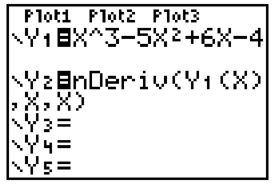


Figure 52: The derivative entered

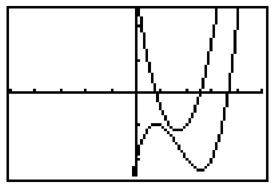


Figure 53: The graphs of the function and its derivative

2.5.4 Integrals

The TI-83+/84+ has two built-in functions that allow you to compute a value for the definite integral of a function, ∫`f(x)dx` and `fnInt(`. We illustrate how to use these features below with the function $f(x) = 5e^{-(x-3)^2}$, and the viewing window `Xmin=-2, Xmax=6, Xscl=1, Ymin=-2, Ymax=6, Yscl=1`.

To use the ∫`f(x)dx` feature in the `CALCULATE` menu, you must enter and graph the function. Press `2nd` `[CALC]`, and select ∫`f(x)dx` by pressing `7`. Use the arrows to move the cursor to select the lower and upper limits of integration as prompted by the calculator, or enter values for these limits, then press `ENTER`. The calculator will shade the area represented by the definite integral and display the value of ∫`f(x)dx` (Figures 54–56).

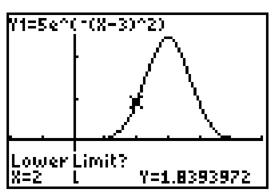


Figure 54: Lower limit

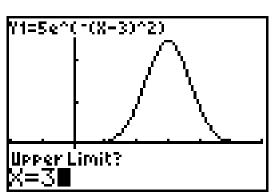


Figure 55: Upper limit

To use the `fnInt(` feature, first exit to the Home Screen if necessary, and press `MATH`. Select `fnInt(` by pressing `9`. Complete the command by entering the function, the variable (`X`), and the limits of

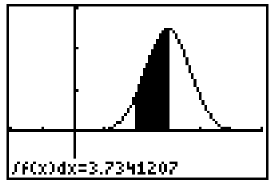


Figure 56: The definite integral

integration, separated by commas (if you entered the function into `Y1`, press [VARS] `Y-VARS Function Y1` [ENTER] to copy `Y1` onto the screen). Close the parenthesis and press [ENTER]. The calculator will return a value for the definite integral of the function within the interval specified (Figure 57).

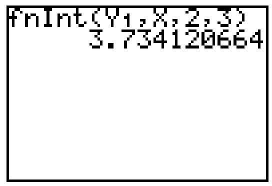


Figure 57: `fnInt(` computed in [2, 3]

The integral $\int_0^x f(t)dt$ of a function f is a function itself (an antiderivative of f), and therefore can be entered into the calculator, where it can be graphed and analyzed, but you do not need to actually compute the integral. Enter the function $y_1 = \frac{1}{1+x^2}$ into the calculator, then define y_2, the integral, as shown in Figure 58. The graphs of the original function and its integral are shown in Figure 59 using the viewing window `Xmin=-3, Xmax=3, Xscl=1, Ymin=-1, Ymax=3, Yscl=1`. This will take some time to graph.

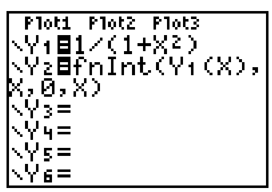


Figure 58: The integral entered

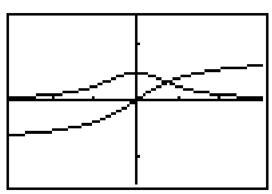


Figure 59: The graphs of the function and its integral

2.5.5 `DRAW` menu

The TI-83+/84+ has three features in the `DRAW` menu that are useful in calculus applications, we describe these here. For additional topics on the `DRAW` menu, consult the guidebook that came with your calculator. First, press 2nd `[DRAW]` 1 to select `ClrDraw`. Press ENTER. This will erase any drawings already in the graph window and graph only selected functions and plots.

Tangent Line. To draw the tangent line to the graph of $f(x) = 2x^3 - 5x^2 + x - 3$ at the point $(2,\ -5)$, enter the function into the calculator. Make sure this function is the only one selected, and set the viewing window to `Xmin=-4, Xmax=4, Xscl=1, Ymin=-8, Ymax=6, Yscl=1`. While you are viewing the graph press 2nd `[DRAW]` 5 to select `Tangent(`. The trace cursor will appear on the graph. You can move the cursor to the point or enter the x-value. In either case, press ENTER. The cursor will rest at the point $(2,\ -5)$, the tangent line at the given point will be drawn on the screen, and the equation of the line will be displayed (Figure 60).

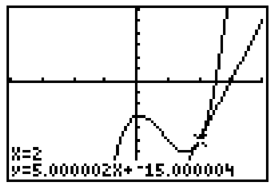


Figure 60: The tangent line

Inverse. To draw the graph of the inverse of $f(x) = \ln(2x - 1)$, enter the function (make sure this function is the only one selected) and set the viewing window to `Xmin=-10.6, Xmax=10.6, Xscl=1, Ymin=-8, Ymax=6, Yscl=1`. While viewing the graph press 2nd `[DRAW]` 8 to select `DrawInv`. Complete the command with the name of the function as in Figure 61. Assuming that you saved the function in `Y1` press VARS `Y-VARS Function Y1` ENTER.

Figure 61: `DrawInv` entered

Press ENTER to view the graph of the inverse function drawn on the screen (Figure 62).

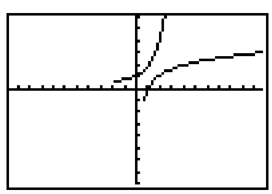


Figure 62: The function and its inverse

Shade. Use the `Shade(` command to view the area between the graphs of two functions in a specified interval. Let $y_1 = -2\sin(x)$ and $y_2 = -2\cos(x)$. Enter these functions and graph them in the viewing window `Xmin=-4, Xmax=4, Xscl=1, Ymin=-4, Ymax=4, Yscl=1`. Note that in the interval $[-\frac{3\pi}{4},\ \frac{\pi}{4}]$, $y_2(x) \leq y_1(x)$. While you are viewing the graphs, press 2nd `[DRAW]` 7 to select `Shade(`. Complete the command as in Figure 63, giving the `lower function, upper function, left endpoint, right endpoint`, and then pressing ENTER.

The graphs will be shown again. The shaded area is the region above `Y2`, below `Y1`, and in the interval $[-\frac{3\pi}{4},\ \frac{\pi}{4}]$ (Figure 64).

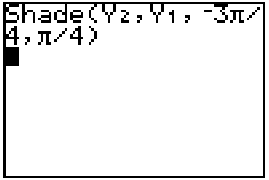


Figure 63: The `Shade(` command

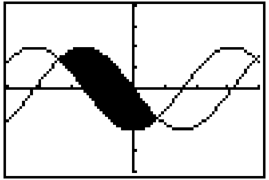


Figure 64: Area between two graphs