

Lecture Three – Multiple Integrals

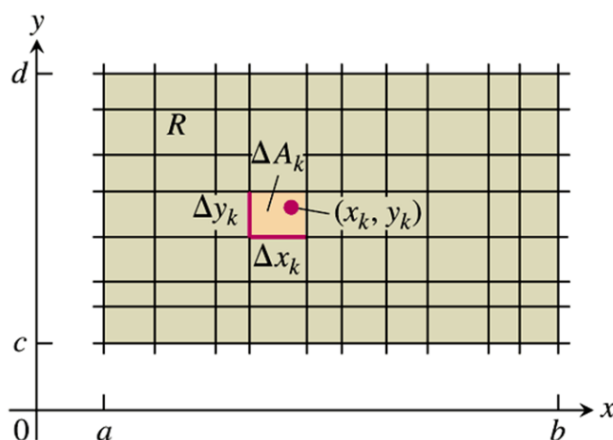
Section 3.1 – Double Integrals over Rectangular Regions

Double Integrals

Consider a function $f(x, y)$ defined on a rectangular region R ,

$$R: a \leq x \leq b, \quad c \leq y \leq d$$

A small rectangular piece of width Δx and height Δy has area $\Delta A = \Delta x \Delta y$.



To form a Riemann sum over R , select a point (x_k, y_k) in the k^{th} small rectangle, multiply the value of f at that point by the area ΔA_k and add together the products:

$$S_n = \sum_{k=1}^n f(x_k, y_k) \Delta A_k$$

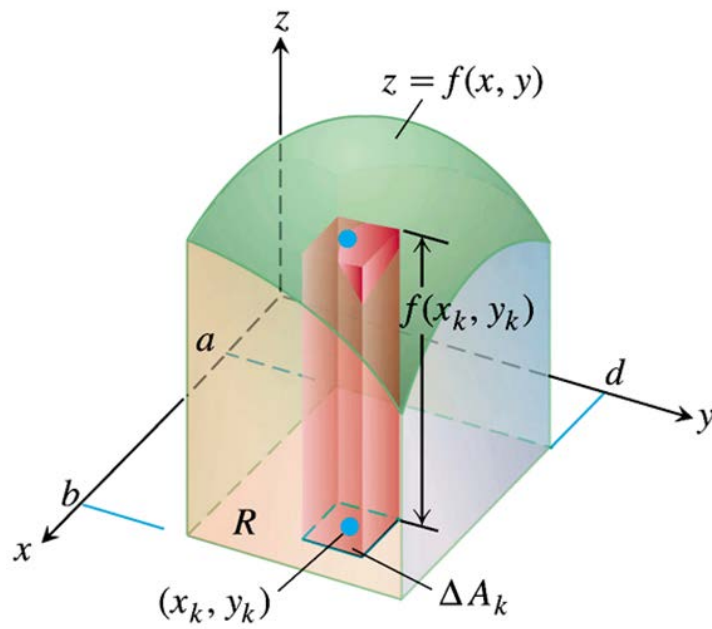
As the rectangles get narrow and short, their number n increases, therefore

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k, y_k) \Delta A_k$$

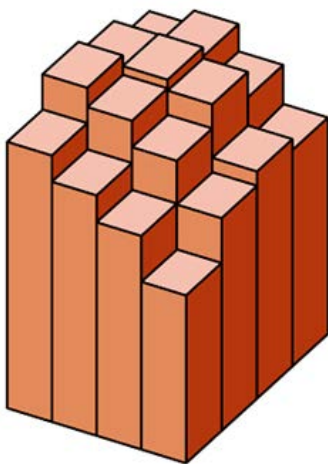
Then the function f is said to be integrable and the limit is called double integral of f over R ,

$$\iint_R f(x, y) dA \quad \text{or} \quad \iint_R f(x, y) dx dy$$

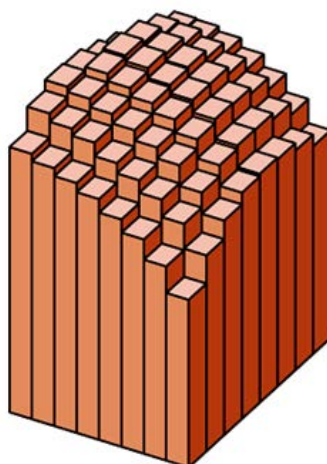
Double Integrals as Volumes



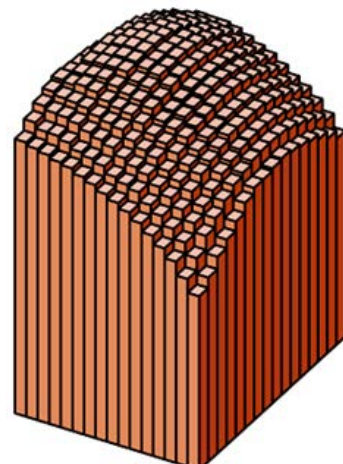
$$\text{Volume} = \lim_{n \rightarrow \infty} S_n = \iint_R f(x, y) dA, \text{ where } \Delta A_k \rightarrow 0 \text{ as } n \rightarrow \infty$$



$n = 16$



$n = 64$



$n = 256$

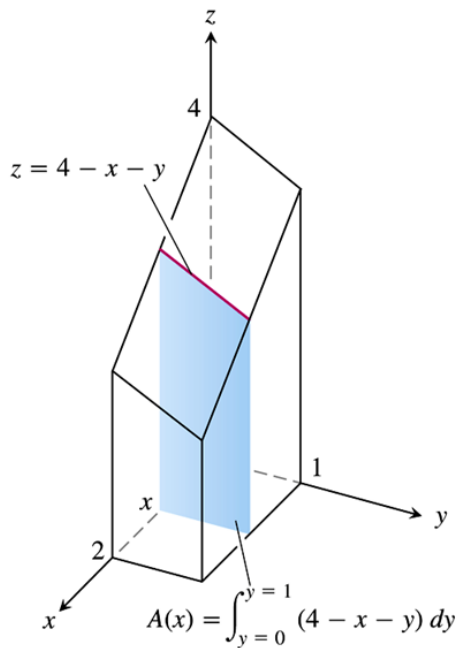
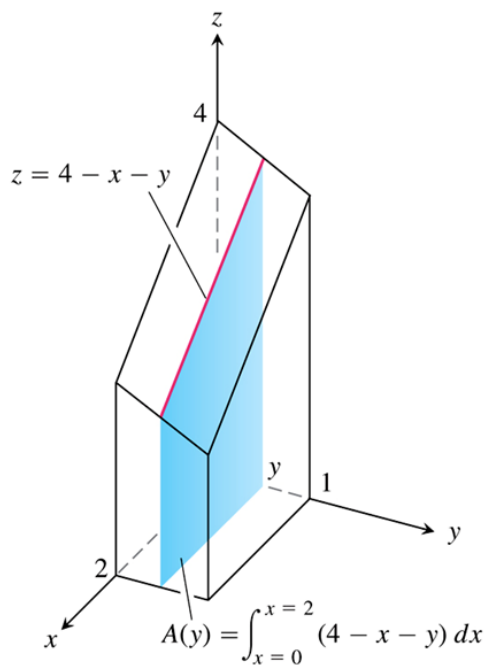
As n increases, the **Riemann sum** approximations approach the total volume of the solid

Example

Calculate the volume under the plane $z = 4 - x - y$ over the rectangular region R : $0 \leq x \leq 2$, $0 \leq y \leq 1$ in the xy -plane.

Solution

$$\begin{aligned} \text{Volume} &= \int_{x=0}^{x=2} A(x) dx \\ &= \int_{x=0}^{x=2} \int_{y=0}^{y=1} (4 - x - y) dy dx \\ &= \int_{x=0}^{x=2} \left[4y - xy - \frac{1}{2}y^2 \right]_{y=0}^{y=1} dx \\ &= \int_{x=0}^{x=2} \left(4 - x - \frac{1}{2} \right) dx \\ &= \int_{x=0}^{x=2} \left(\frac{7}{2} - x \right) dx \\ &= \left[\frac{7}{2}x - \frac{1}{2}x^2 \right]_0^2 \\ &= 7 - 2 \\ &= \underline{5} \text{ unit}^3 \end{aligned}$$



$$\text{Volume} = \int_0^1 \int_0^2 (4 - x - y) dx dy$$

Theorem – Fubini's Theorem

If $f(x, y)$ is continuous throughout the rectangular region R : $a \leq x \leq b$, $c \leq y \leq d$, then

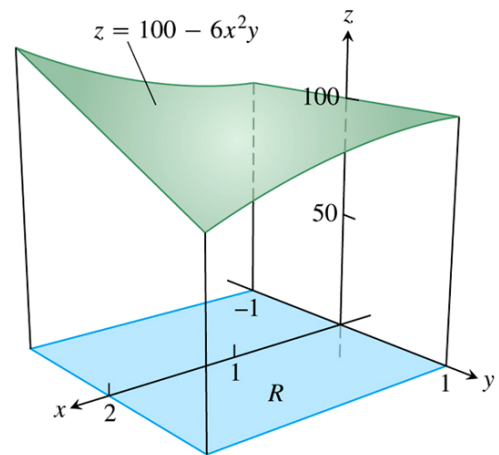
$$\iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx$$

Example

Calculate $\iint_R f(x, y) dA$ for $f(x, y) = 100 - 6x^2y$ and $R: 0 \leq x \leq 2, -1 \leq y \leq 1$

Solution

$$\begin{aligned} \int_{-1}^1 \int_0^2 (100 - 6x^2y) dx dy &= \int_{-1}^1 \left(100x - 2x^3y \right) \Big|_0^2 dy \\ &= \int_{-1}^1 (200 - 16y) dy \\ &= 200y - 8y^2 \Big|_{-1}^1 \\ &= 200 - 8 - (-200 - 8) \\ &= 400 \end{aligned}$$



Example

Evaluate $\iint e^{4x} y^3 dy dx$

Solution

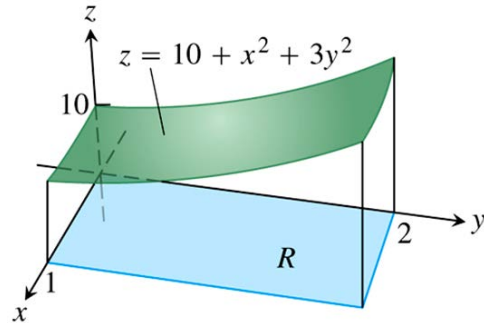
$$\begin{aligned} \iint e^{4x} y^3 dy dx &= \int e^{4x} dx \int y^3 dy \\ &= \frac{1}{4} e^{4x} \left(\frac{1}{4} y^4 \right) + C \\ &= \frac{1}{16} y^4 e^{4x} + C \end{aligned}$$

Example

Find the volume of the region bounded above the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below the rectangle $R: 0 \leq x \leq 1, 0 \leq y \leq 2$

Solution

$$\begin{aligned} \text{Volume} &= \int_0^1 \int_0^2 (10 + x^2 + 3y^2) dy dx \\ &= \int_0^1 \left(10y + yx^2 + y^3 \right) \Big|_0^2 dx \\ &= \int_0^1 (2x^2 + 28) dx \\ &= \left. \frac{2}{3}x^3 + 28x \right|_0^1 \\ &= \frac{2}{3} + 28 \\ &= \frac{86}{3} \text{ unit}^3 \end{aligned}$$



Example

Evaluate $\int_0^1 \int_y^1 ye^{-x^3} dx dy$

Solution

$$y \leq x \leq 1 \rightarrow \begin{cases} y = x \\ x = 1 \end{cases}$$

$$0 \leq y \leq 1$$

$$0 \leq y \leq x \quad \& \quad 0 \leq x \leq 1$$

$$\begin{aligned} \int_0^1 \int_y^1 ye^{-x^3} dx dy &= \int_0^1 \int_0^x ye^{-x^3} dy dx \\ &= \frac{1}{2} \int_0^1 e^{-x^3} y^2 \Big|_0^x dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \int_0^1 x^2 e^{-x^3} dx \\
&= -\frac{1}{6} \int_0^1 e^{-x^3} d(-x^3) \\
&= -\frac{1}{6} e^{-x^3} \Big|_0^1 \\
&= -\frac{1}{6} (e^{-1} - 1) \\
&= \frac{1}{6} \left(1 - \frac{1}{e}\right) \\
&= \frac{e-1}{6e}
\end{aligned}$$

Exercises Section 3.1 – Double Integrals over Rectangular Regions

(1 – 18) Evaluate the iterated integral

1. $\int_1^2 \int_0^4 2xy \, dydx$

2. $\int_0^2 \int_{-1}^1 (x - y) \, dydx$

3. $\int_0^1 \int_0^1 \left(1 - \frac{x^2 + y^2}{2}\right) dx dy$

4. $\int_0^3 \int_{-2}^0 (x^2 y - 2xy) dy dx$

5. $\int_0^1 \int_0^1 \frac{y}{1 + xy} dx dy$

6. $\int_0^{\ln 2} \int_1^{\ln 5} e^{2x+y} dy dx$

7. $\int_0^1 \int_1^2 xye^x dy dx$

8. $\int_{\pi}^{2\pi} \int_0^{\pi} (\sin x + \cos y) dx dy$

9. $\int_1^2 \int_1^4 \frac{xy}{(x^2 + y^2)^2} dx dy$

10. $\int_1^3 \int_1^{e^x} \frac{x}{y} dy dx$

11. $\int_1^2 \int_0^{\ln x} x^3 e^y dy dx$

12. $\int_1^{10} \int_0^{1/y} ye^{xy} dx dy$

13. $\int_0^1 \int_{\sqrt{y}}^{2-\sqrt{y}} xy \, dx dy$

14. $\int_0^1 \int_{x^2}^x \sqrt{x} \, dy dx$

15. $\int_0^{3/2} \int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y dx dy$

16. $\int_0^2 \int_0^{4-x^2} 2x \, dy dx$

17. $\int_0^1 \int_{2y}^2 4 \cos(x^2) \, dx dy$

18. $\int_0^1 \int_{\sqrt[3]{y}}^1 \frac{2\pi \sin \pi x^2}{x^2} dx dy$

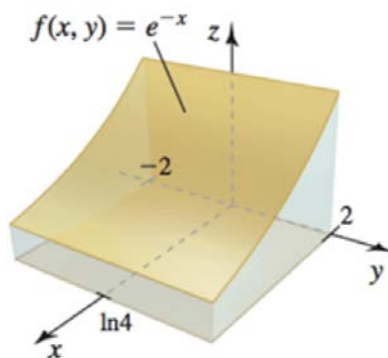
(19 – 26) Evaluate the double integral over the given region R .

19. $\iint_R (6y^2 - 2x) dA \quad R: 0 \leq x \leq 1, 0 \leq y \leq 2$

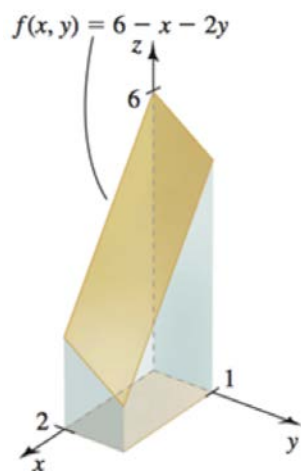
20. $\iint_R \left(\frac{\sqrt{x}}{y^2} \right) dA \quad R: 0 \leq x \leq 4, 1 \leq y \leq 2$

21. $\iint_R y \sin(x+y) dA$ $R: -\pi \leq x \leq 0, 0 \leq y \leq \pi$
22. $\iint_R e^{x-y} dA$ $R: 0 \leq x \leq \ln 2, 0 \leq y \leq \ln 2$
23. $\iint_R \frac{y}{x^2 y^2 + 1} dA$ $R: 0 \leq x \leq 1, 0 \leq y \leq 1$
24. $\iint_R x^{-1/2} e^y dA$; R is the region bounded by $x=1$, $x=4$, $y=\sqrt{x}$, and $y=0$
25. $\iint_R (x^2 + y^2) dA$; R is the region $\{(x, y): 0 \leq x \leq 2, 0 \leq y \leq x\}$
26. $\iint_R \frac{2y}{\sqrt{x^4 + 1}} dA$; R is the region bounded by $x=1$, $x=2$, $y=x^{3/2}$, $y=0$
27. Integrate $f(x, y) = \frac{1}{xy}$ over the **square** $1 \leq x \leq 2, 1 \leq y \leq 2$
28. Integrate $f(x, y) = y \cos xy$ over the **rectangle** $0 \leq x \leq \pi, 0 \leq y \leq 1$
29. Find the volume of the region bounded above the paraboloid $z = x^2 + y^2$ and below by the square $R: -1 \leq x \leq 1, -1 \leq y \leq 1$
30. Find the volume of the region bounded above the plane $z = \frac{y}{2}$ and below by the rectangle $R: 0 \leq x \leq 4, 0 \leq y \leq 2$
31. Find the volume of the region bounded above the surface $z = 4 - y^2$ and below by the rectangle $R: 0 \leq x \leq 1, 0 \leq y \leq 2$
32. Find the volume of the region bounded above the elliptical paraboloid $z = 16 - x^2 - y^2$ and below by the square $R: 0 \leq x \leq 2, 0 \leq y \leq 2$
33. Evaluate $\int_0^{1/2} (\sin^{-1}[2x] - \sin^{-1} x) dx$ by converting it to a double integral.

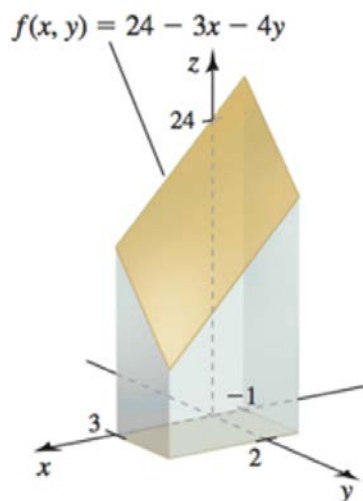
34. Find the volume of the solid beneath the cylinder $f(x, y) = e^{-x}$ and above the region $R = \{(x, y) : 0 \leq x \leq \ln 4, -2 \leq y \leq 2\}$



35. Find the volume of the solid beneath the plane $f(x, y) = 6 - x - 2y$ and above the region $R = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 1\}$



36. Find the volume of the solid beneath the plane $f(x, y) = 24 - 3x - 4y$ and above the region $R = \{(x, y) : -1 \leq x \leq 3, 0 \leq y \leq 2\}$



37. Find the volume of the solid beneath the paraboloid $f(x, y) = 12 - x^2 - 2y^2$ and above the region $R = \{(x, y) : 1 \leq x \leq 2, 0 \leq y \leq 1\}$

