Solution   *Section* 3.1 – Introduction to Linear Systems

Exercise

Find a solution for x, y, z to the system of equations

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3e+2\sqrt{2}+\pi \\ 6e+5\sqrt{2}+4\pi \\ 9e+8\sqrt{2}+7\pi \end{pmatrix}$$

Solution

$$\begin{pmatrix} x+2y+3z \\ 4x+5y+6z \\ 7x+8y+9z \end{pmatrix} = \begin{pmatrix} 3e+2\sqrt{2}+\pi \\ 6e+5\sqrt{2}+4\pi \\ 9e+8\sqrt{2}+7\pi \end{pmatrix}$$

$$\Rightarrow \begin{cases} x+2y+3z = \pi+2\sqrt{2}+3e \\ 4x+5y+6z = 4\pi+5\sqrt{2}+6e \\ 7x+8y+9z = 7\pi+8\sqrt{2}+9e \end{cases}$$

Solution: $\boxed{x=\pi \quad y=\sqrt{2} \quad z=e}$

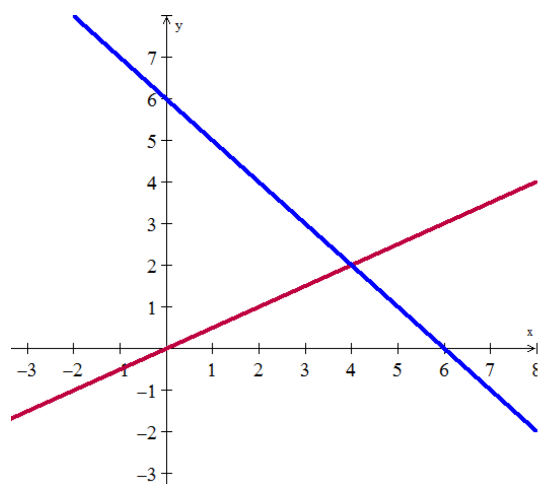
Exercise

Draw the two pictures in two planes for the equations: $x-2y=0, \quad x+y=6$

Solution

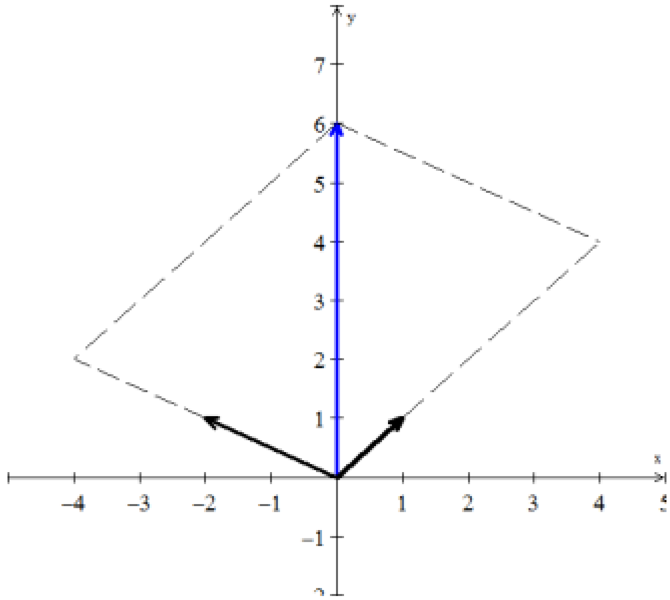
The matrix form of the 2 equations:

$$\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$



Row picture is the 2 lines from the given equations and their intersection is the point (4, 2) which is the solution for the system.

Column Picture is the column vectors $\begin{pmatrix}1 & 1\end{pmatrix}$ and $\begin{pmatrix}-2 & 1\end{pmatrix}$



$$x\begin{pmatrix}1\\1\end{pmatrix}+y\begin{pmatrix}-2\\1\end{pmatrix}=4\begin{pmatrix}1\\1\end{pmatrix}+2\begin{pmatrix}-2\\1\end{pmatrix}=\begin{pmatrix}0\\6\end{pmatrix}$$

The parallelogram show how the solution vector $(0 \quad 6)$ can be written as the linear combination of the column vectors.

Exercise

Normally 4 planes in 4-dimensional space meet at a ________. Normally 4 column vectors in 4-deimensional space can combine to produce b. what combinations of

$(1,\ 0,\ 0,\ 0),\quad (1,\ 1,\ 0,\ 0),\quad (1,\ 1,\ 1,\ 0),\quad (1,\ 1,\ 1,\ 1)$ produces $b=(3,\ 3,\ 3,\ 2)$?

What 4 equations for $x,\ y,\ z,\ w$ are you solving?

Solution

Normally 4 planes in 4-dimensional space meet at a ***point***.

The combination of the vectors producing b is:

$$0\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+0\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+1\begin{pmatrix}1\\1\\1\\0\end{pmatrix}+2\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}3\\3\\3\\2\end{pmatrix}$$

$$1\begin{pmatrix}1\\1\\1\\0\end{pmatrix}+2\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}3\\3\\3\\2\end{pmatrix}$$

The system of equations that satisfies the given vectors is:

$$\begin{cases} x+y+z+w=3 \\ \quad\ y+z+w=3 \\ \qquad\quad z+w=3 \\ \qquad\qquad\ w=2 \end{cases}$$

Exercise

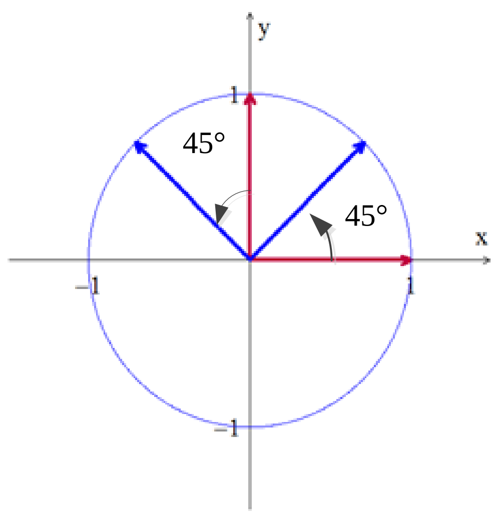
What 2 by 2 matrix A rotates every vector through $45°$?

The vector (1, 0) goes to $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. The vector (0, 1) goes to $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Those determine the matrix. Draw these particular vectors is the xy-plane and find A.

Solution

$$A=\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$



Exercise

What two vectors are obtained by rotating the plane vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ by 30° (cw) ?

Write a matrix A such that for every vector v in the plane, Av is the vector obtained by rotating v clockwise by 30°.

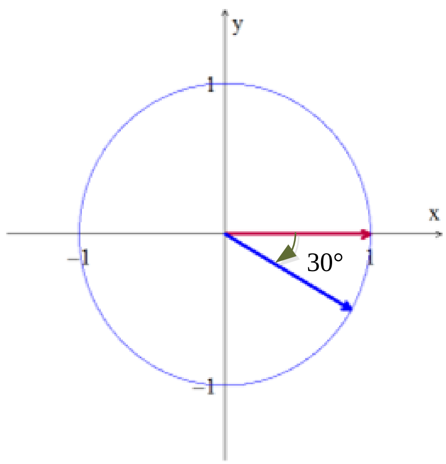
Find a matrix B such that for every 3-dimensional vector v, the vector Bv is the reflection of v through the plane $x+y+z=0$. $Hint: v=(1,\ 0,\ 0)$

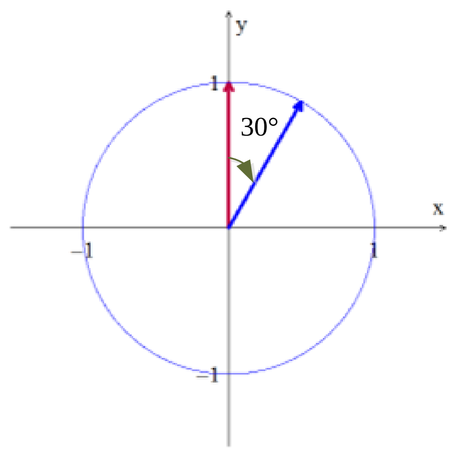
Solution

Rotating the vectors by 30° (cw) yields:

For the vector $\begin{bmatrix}1\\0\end{bmatrix}$ yields to $\begin{pmatrix}\cos(-30°)\\ \sin(-30°)\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{2}\\ -\frac{1}{2}\end{pmatrix}$

And for the vector $\begin{bmatrix}0\\1\end{bmatrix}$ yields to $\begin{pmatrix}\sin(30°)\\ \cos(30°)\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}$





The desired matrix is: $A=\begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}$

To get 1 from $\frac{\sqrt{3}}{2}$ is to multiply by $\frac{2}{\sqrt{3}}=2\frac{1}{\sqrt{3}}$

The unit vector to the plane $x+y+z=0$ is $\hat{u}=\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$

$$Bv=B\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}-\frac{2}{\sqrt{3}}\hat{u}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{2}{\sqrt{3}} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$B\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{\sqrt{3}}\hat{u} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}$$

$$B\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{2}{\sqrt{3}}\hat{u} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

The solution: $\begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$

Exercise

Find a system of linear equation corresponding to the given augmented matrix

$$\begin{bmatrix} 0 & 3 & -1 & -1 & -1 \\ 5 & 2 & 0 & -3 & -6 \end{bmatrix}$$

Solution

$$\begin{cases} \quad\quad 3x_2 - x_3 - x_4 = -1 \\ 5x_1 + 2x_2 \quad\quad - 3x_4 = -6 \end{cases}$$

Exercise

Find a system of linear equation corresponding to the given augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \\ 5 & -6 & 1 & 1 \\ -8 & 0 & 0 & 3 \end{bmatrix}$$

Solution

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ -4x_1 - 3x_2 - 2x_3 = -1 \\ 5x_1 - 6x_2 + x_3 = 1 \\ -8x_1 = 3 \end{cases}$$

Exercise

Find the augmented matrix for the given system of linear equations.

$$\begin{cases} -2x_1 = 6 \\ 3x_1 = 8 \\ 9x_1 = -3 \end{cases}$$

Solution

$$\left[\begin{array}{c|c} -2 & 6 \\ 3 & 8 \\ 9 & -3 \end{array}\right]$$

Exercise

Find the augmented matrix for the given system of linear equations.

$$\begin{cases} 3x_1 - 2x_2 = -1 \\ 4x_1 + 5x_2 = 3 \\ 7x_1 + 3x_2 = 2 \end{cases}$$

Solution

$$\left[\begin{array}{cc|c} 3 & -2 & -1 \\ 4 & 5 & 3 \\ 7 & 3 & 2 \end{array}\right]$$

Exercise

Find the augmented matrix for the given system of linear equations.

$$\begin{cases} 2x_1 \qquad\quad + 2x_3 = 1 \\ 3x_1 - x_2 + 4x_3 = 7 \\ 6x_1 + x_2 - x_3 = 0 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 1 \\ 3 & -1 & 4 & 7 \\ 6 & 1 & -1 & 0 \end{array}\right]$$