Solution *Section* 3.3 – Double Integrals in Polar Coordinates

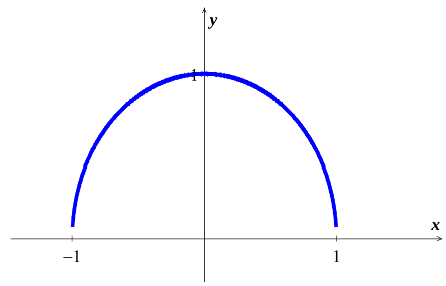
Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dydx$$

Solution

$$y=\sqrt{1-x^2} \quad \Rightarrow \quad y^2=1-x^2 \to x^2+y^2=1=r^2$$



$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dydx = \int_{0}^{\pi}\int_{0}^{1} rdrd\theta$$

$$= \int_{0}^{\pi} \frac{1}{2}\left[r^2\right]_0^1 d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi} d\theta \qquad = \frac{1}{2}[\theta]_0^{\pi}$$

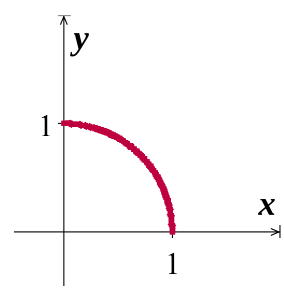
$$= \frac{\pi}{2}$$

Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} \left(x^2+y^2\right)dxdy$$

Solution



$$\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} \left(x^2+y^2\right)dxdy = \int_{0}^{\pi/2}\int_{0}^{1} r^2\, rdrd\theta$$

$$= \frac{1}{4}\int_{0}^{\pi/2} \left[r^4\right]_0^1 d\theta$$

$$= \frac{1}{4}\int_{0}^{\pi/2} d\theta \qquad = \frac{1}{4}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{8}$$

Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dydx$$

Solution

$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dydx = \int_0^{2\pi}\int_0^{a} rdrd\theta$$

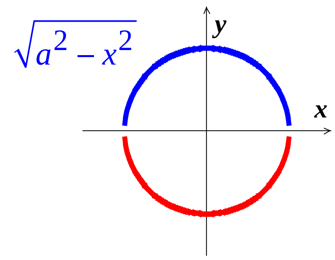
$-\sqrt{a^2-x^2}$

$$= \frac{1}{2}\int_0^{2\pi}\left[r^2\right]_0^a d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} d\theta$$

$$= \frac{a^2}{2}\left[\theta\right]_0^{2\pi}$$

$$= \pi a^2$$



Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_0^6\int_0^y xdxdy$$

Solution

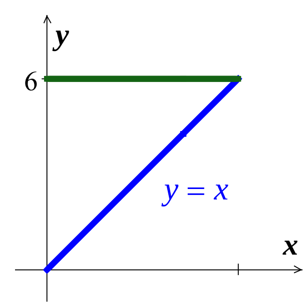
$\theta \quad x = r\cos\theta, \quad \sin\theta = \frac{6}{r} \to r = \frac{6}{\sin\theta} = 6\csc\theta \quad \frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$$\int_0^6\int_0^y xdxdy = \int_{\pi/4}^{\pi/2}\int_0^{6\csc\theta} r^2\cos\theta \, drd\theta$$

$$= \frac{1}{3}\int_{\pi/4}^{\pi/2}\cos\theta\left[r^3\right]_0^{6\csc\theta} d\theta$$

$$= \frac{216}{3}\int_{\pi/4}^{\pi/2}\cos\theta\csc^3\theta \, d\theta$$

$$= 72\int_{\pi/4}^{\pi/2}\cot\theta\csc^2\theta \, d\theta \qquad d(\cot\theta) = -\csc^2\theta \, d\theta$$



$$= -72\int_{\pi/4}^{\pi/2} \cot\theta \; d\left(\cot\theta\right)$$

$$= -36\left[\cot^2\theta\right]_{\pi/4}^{\pi/2}$$

$$= -36(0-1)$$

$$\underline{= 36}$$

Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\,dydx$$

Solution

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\,dydx = \int_{\pi}^{3\pi/2}\int_{0}^{1} \frac{2}{1+r}r\; drd\theta$$

$$= 2\int_{\pi}^{3\pi/2}\int_{0}^{1} \left(1-\frac{1}{1+r}\right)\, drd\theta$$

$$= 2\int_{\pi}^{3\pi/2} \left[1-\ln(1+r)\right]_0^1\, d\theta$$

$$= 2\int_{\pi}^{3\pi/2} (1-\ln 2)\,d\theta$$

$$= 2(1-\ln 2)\left[\theta\right]_{\pi}^{3\pi/2}$$

$$= 2(1-\ln 2)\left(\frac{3\pi}{2}-\pi\right)$$

$$\underline{= (1-\ln 2)\pi}$$

Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_0^{\ln 2}\int_0^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}\, dxdy$$

Solution

$$\int_0^{\ln 2}\int_0^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}\, dxdy = \int_0^{\pi/2}\int_0^{\ln 2} e^r r\, drd\theta$$

		$\int e^r$
+	r	e^r
−	1	e^r

$$= \int_0^{\pi/2} \left[re^r - e^r\right]_0^{\ln 2} d\theta$$

$$= \int_0^{\pi/2} \left(\ln 2 e^{\ln 2} - e^{\ln 2} + 1\right) d\theta$$

$$= \int_0^{\pi/2} (2\ln 2 - 2 + 1)\, d\theta$$

$$= \int_0^{\pi/2} (2\ln 2 - 1)\, d\theta$$

$$= (2\ln 2 - 1)\left(\frac{\pi}{2} - 0\right)$$

$$\boxed{= \frac{\pi}{2}(2\ln 2 - 1)}$$

Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right)dxdy$$

Solution

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right)dxdy = \int_0^{2\pi}\int_0^1 \ln\left(r^2+1\right)rdrd\theta$$

$$= 4\int_0^{\pi/2}\int_0^1 \ln\left(r^2+1\right)\tfrac{1}{2}d\left(r^2+1\right)d\theta$$

$$= 2\int_0^{\pi/2}\left[\left(\ln\left(r^2+1\right)\right)^2\right]_0^1 d\theta \qquad \int \ln axdx = x\ln ax - x$$

$$= 2\int_0^{\pi/2}(\ln 4-1)d\theta$$

$$= 2(\ln 4-1)[\theta]_0^{\pi/2}$$

$$= 2(\ln 4-1)\left(\tfrac{\pi}{2}-0\right)$$

$$\underline{= \pi(\ln 4-1)}$$

Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_1^2\int_0^{\sqrt{2x-x^2}} \frac{1}{\left(x^2+y^2\right)^2}dydx$$

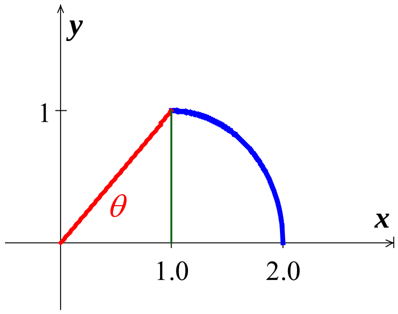
Solution

$$y^2 = 2x - x^2 \Rightarrow x^2 - 2x + 1 - 1 + y^2 = 0 \quad (x-1)^2 + y^2 = 1$$

$$r = \frac{x}{\cos\theta} = \frac{1}{\cos\theta} = \sec\theta$$

$$y = \sqrt{2x - x^2} \quad \rightarrow \quad y^2 = 2x - x^2 \Rightarrow x^2 + y^2 = 2x$$

$$r^2 = 2r\cos\theta \rightarrow r = 2\cos\theta$$



$$\int_1^2\int_0^{\sqrt{2x-x^2}}\frac{1}{\left(x^2+y^2\right)^2}dydx=\int_0^{\pi/4}\int_{\sec\theta}^{2\cos\theta}\frac{1}{r^4}rdrd\theta$$

$$=\int_0^{\pi/4}\int_{\sec\theta}^{2\cos\theta}r^{-3}drd\theta$$

$$=\int_0^{\pi/4}\left[-\frac{1}{2r^2}\right]_{\sec\theta}^{2\cos\theta}d\theta$$

$$=\int_0^{\pi/4}\left(-\frac{1}{8\cos^2\theta}+\frac{1}{2\sec^2\theta}\right)d\theta$$

$$=\int_0^{\pi/4}\left(-\frac{1}{8}\sec^2\theta+\frac{1}{2}\cos^2\theta\right)d\theta \qquad \int\cos^2 axdx=\frac{x}{2}+\frac{\sin 2ax}{4a}$$

$$=\left[-\frac{1}{8}\tan\theta+\frac{1}{2}\left(\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right)\right]_0^{\pi/4}$$

$$=\left[\frac{1}{4}\theta+\frac{1}{8}\sin 2\theta-\frac{1}{8}\tan\theta\right]_0^{\pi/4}$$

$$=\frac{1}{4}\frac{\pi}{4}+\frac{1}{8}-\frac{1}{8}-(0)$$

$$=\frac{\pi}{16}$$

Exercise

Find the area of the region cut from the first quadrant by the curve $r=2(2-\sin 2\theta)^{1/2}$

Solution

$$\int_0^{\pi/2}\int_0^{2\sqrt{2-\sin 2\theta}}rdrd\theta=\frac{1}{2}\int_0^{\pi/2}\left[r^2\right]_0^{2\sqrt{2-\sin 2\theta}}d\theta$$

$$=\frac{1}{2}\int_0^{\pi/2}4(2-\sin 2\theta)d\theta$$

$$=2\left[2\theta+\frac{1}{2}\cos 2\theta\right]_0^{\pi/2}$$

$$=2\left[\pi-\frac{1}{2}-\left(\frac{1}{2}\right)\right]$$

$$=2(\pi-1)$$

Exercise

Find the area of the region lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$

Solution

$$A = 2\int_0^{\pi/2}\int_1^{1+\cos\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[r^2\right]_1^{1+\cos\theta} d\theta$$

$$= \int_0^{\pi/2}\left[(1+\cos\theta)^2 - 1\right] d\theta$$

$$= \int_0^{\pi/2}\left(1 + 2\cos\theta + \cos^2\theta - 1\right) d\theta$$

$$= \int_0^{\pi/2}\left(2\cos\theta + \cos^2\theta\right) d\theta \qquad \int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$= \left[2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$= 2 + \frac{\pi}{4}$$

Exercise

Find the area enclosed by one leaf of the rose $r = 12\cos 3\theta$

Solution

$$A = 2\int_0^{\pi/6}\int_0^{12\cos 3\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/6}\left[r^2\right]_0^{12\cos 3\theta} d\theta$$

$$= 144\int_0^{\pi/6} \cos^2 3\theta\,d\theta \qquad \int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$= 144\left[\frac{\theta}{2} + \frac{\sin 6\theta}{12}\right]_0^{\pi/6}$$

$$= 144\left(\frac{\pi}{12}\right)$$

$$= 12\pi$$

Exercise

Find the area of the region common to the interiors of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$

Solution

$$A = 4\int_0^{\pi/2}\int_0^{1-\cos\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\left[r^2\right]_0^{1-\cos\theta} d\theta$$

$$= 2\int_0^{\pi/2}(1-\cos\theta)^2\,d\theta$$

$$= 2\int_0^{\pi/2}\left(1 - 2\cos\theta + \cos^2\theta\right)d\theta \qquad \int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$= 2\left[\theta - 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$= 2\left(\frac{\pi}{2} - 2 + \frac{\pi}{4}\right)$$

$$= \frac{3\pi}{2} - 4$$

Exercise

Integrate $f(x,y) = \dfrac{\ln\left(x^2+y^2\right)}{\sqrt{x^2+y^2}}$ over the region $1 \le x^2 + y^2 \le e$

Solution

$$\int_0^{2\pi}\int_1^{\sqrt{e}}\left(\frac{\ln r^2}{r}\right) r\,dr\,d\theta = \int_0^{2\pi}\int_1^{\sqrt{e}} 2\ln r\; dr\,d\theta$$

$$= 2\int_0^{2\pi}\left[r\ln r - r\right]_1^{\sqrt{e}} d\theta$$

$$= 2\int_0^{2\pi}\left[\sqrt{e}\ln e^{1/2} - \sqrt{e} - (0-1)\right]d\theta$$

$$= 2\int_0^{2\pi}\left[\frac{1}{2}\sqrt{e} - \sqrt{e} + 1\right]d\theta$$

$$= 2\left(-\frac{1}{2}\sqrt{e} + 1\right)\left[\theta\right]_0^{2\pi}$$

$$= 2\pi\left(2 - \sqrt{e}\right)$$

Exercise

Evaluate the integral $\int_0^{\infty}\int_0^{\infty}\frac{1}{\left(1+x^2+y^2\right)^2}dxdy$

Solution

$$\int_0^{\infty}\int_0^{\infty}\frac{1}{\left(1+x^2+y^2\right)^2}dxdy=\int_0^{\pi/2}\int_0^{\infty}\frac{1}{\left(1+r^2\right)^2}rdrd\theta$$

$$=\int_0^{\infty}\int_0^{\pi/2}d\theta\frac{rdr}{\left(1+r^2\right)^2}$$

$$=\int_0^{\infty}\left[\theta\right]_0^{\pi/2}\frac{rdr}{\left(1+r^2\right)^2} \qquad d\left(1+r^2\right)=2rdr$$

$$=\frac{\pi}{2}\int_0^{\infty}\left(1+r^2\right)^{-2}\frac{1}{2}d\left(1+r^2\right)$$

$$=\frac{\pi}{4}\left[-\frac{1}{1+r^2}\right]_0^{\infty} \qquad \frac{1}{\infty}=0$$

$$=-\frac{\pi}{4}(0-1)$$

$$=\frac{\pi}{4}$$

Exercise

The region enclosed by the lemniscates $r^2=2\cos 2\theta$ is the base of a solid right cylinder whose top is bounded by the sphere $z=\sqrt{2-r^2}$. Find the cylinder's volume.

Solution

$$V=4\int_0^{\pi/4}\int_0^{\sqrt{2\cos 2\theta}}r\sqrt{2-r^2}drd\theta \qquad d\left(2-r^2\right)=-2rdr$$

$$=-2\int_0^{\pi/4}\int_0^{\sqrt{2\cos 2\theta}}\left(2-r^2\right)^{1/2}d\left(2-r^2\right)d\theta$$

$$=-2\int_0^{\pi/4}\left[\frac{2}{3}\left(2-r^2\right)^{3/2}\right]_0^{\sqrt{2\cos 2\theta}}d\theta$$

$$= -\frac{4}{3}\int_0^{\pi/4}\left[\left(2-2\cos 2\theta\right)^{3/2} - 2^{3/2}\right]d\theta$$

$$= -\frac{4}{3}\int_0^{\pi/4}\left[2^{3/2}\left(1-\cos 2\theta\right)^{3/2}\right]d\theta + \frac{4}{3}\int_0^{\pi/4} 2^{3/2}\,d\theta$$

$$= -\frac{4}{3}2\sqrt{2}\int_0^{\pi/4}\left(2\sin^2\theta\right)^{3/2} d\theta + \frac{4}{3}2\sqrt{2}\left[\theta\right]_0^{\pi/4}$$

$$= -\frac{8\sqrt{2}}{3}\int_0^{\pi/4} 2\sqrt{2}\sin^3\theta d\theta + \frac{8}{3}\sqrt{2}\left(\frac{\pi}{4}\right)$$

$$= -\frac{32}{3}\int_0^{\pi/4}\sin^2\theta\sin\theta d\theta + \frac{2\pi\sqrt{2}}{3}$$

$$= \frac{32}{3}\int_0^{\pi/4}\left(1-\cos^2\theta\right)d\left(\cos\theta\right) + \frac{2\pi\sqrt{2}}{3}$$

$$= \frac{32}{3}\left[\cos\theta - \frac{1}{3}\cos^3\theta\right]_0^{\pi/4} + \frac{2\pi\sqrt{2}}{3}$$

$$= \frac{32}{3}\left[\frac{\sqrt{2}}{2} - \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^3 - \left(1-\frac{1}{3}\right)\right] + \frac{2\pi\sqrt{2}}{3}$$

$$= \frac{32}{3}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{12} - \frac{2}{3}\right) + \frac{2\pi\sqrt{2}}{3}$$

$$= \frac{32}{3}\left(\frac{5\sqrt{2}-8}{12}\right) + \frac{2\pi\sqrt{2}}{3}$$

$$= 8\left(\frac{5\sqrt{2}-8}{9}\right) + \frac{2\pi\sqrt{2}}{3}$$

$$= \frac{40\sqrt{2}-64+6\pi\sqrt{2}}{9}$$