

Exercise

Find all the local maxima, local minima, and saddle points of the function

$$f(x, y) = x^2 + xy + y^2 + 3x - 3y + 4$$

Solution

$$f_x = 2x + y + 3 = 0 \quad f_y = x + 2y - 3 = 0$$

$$\begin{cases} 2x + y = -3 \\ x + 2y = 3 \end{cases} \rightarrow x = -3 \quad y = 3$$

Therefore, the critical point is $(-3, 3)$

$$f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = 1$$

$$f_{xx}f_{yy} - f_{xy}^2 = (2)(2) - 1^2 = 3 > 0 \quad \text{and} \quad f_{xx} = 2 > 0$$

The function f has a **local minimum** at $(-3, 3)$ and the value is

$$\begin{aligned} f(-3, 3) &= (-3)^2 + (-3)(3) + 3^2 + 3(-3) - 3(3) + 4 \\ &= -5 \end{aligned}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$$f(x, y) = x^2 - xy + y^2 + 2x + 2y - 4$$

Solution

$$f_x = 2x - y + 2 = 0 \quad f_y = -x + 2y + 2 = 0$$

$$\begin{cases} 2x - y = -2 \\ x - 2y = 2 \end{cases} \quad \Delta = \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} = -3 \quad \Delta_x = \begin{vmatrix} -2 & -1 \\ 2 & -2 \end{vmatrix} = 6$$

$$x = \frac{6}{-3} = -2 \quad y = -4 + 2 = -2$$

Therefore, the critical point is $(-2, -2)$

$$f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = -1$$

At $(-2, -2)$: $f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = -1$

$$f_{xx}f_{yy} - f_{xy}^2 = (2)(2) - 1 = 3 > 0 \quad \text{and} \quad f_{xx} = 2 > 0$$

The function f has a **local minimum** at $(-2, -2)$ and the value is

$$f(-2, -2) = 4 - 4 + 4 - 4 - 4 - 4 \\ = -8$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$$f(x, y) = x^3 + y^3 - 3xy + 15$$

Solution

$$f_x = 3x^2 - 3y = 0 \quad f_y = 3y^2 - 3x = 0$$

$$\begin{cases} x^2 = y \\ y^2 = x \end{cases} \quad (x^2)^2 = x \rightarrow x^4 = x$$

$$x(x^3 - 1) = 0 \rightarrow \underline{x = 0, 1}$$

$$\begin{cases} x = 0 & y = 0 \\ x = 1 & y = 1 \end{cases}$$

Therefore, the critical point is $(0, 0)$ & $(1, 1)$

$$f_{xx} = 6x \quad f_{yy} = 6y \quad f_{xy} = -3$$

At $(0, 0)$

$$f_{xx} = 0 \quad f_{yy} = 0 \quad f_{xy} = -3$$

$$f_{xx}f_{yy} - f_{xy}^2 = 0 - 9 = -9 < 0$$

The function f has a **saddle point** at $(0, 0)$ and the value is $\underline{f(0, 0) = 15}$

At $(1, 1)$

$$f_{xx} = 6 \quad f_{yy} = 6 \quad f_{xy} = -3$$

$$f_{xx}f_{yy} - f_{xy}^2 = 36 - 9 = 27 > 0 \quad \text{and} \quad f_{xx} = 6 > 0$$

The function f has a **local minimum** at $(1, 1)$ and the value is $\underline{f(1, 1) = 14}$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$$f(x, y) = x^4 - 8x^2 + 3y^2 - 6y$$

Solution

$$f_x = 4x^3 - 16x = 0 \quad f_y = 6y - 6 = 0$$

$$\begin{cases} 4x(x^2 - 4) = 0 \rightarrow \underline{x = 0, \pm 2} \\ y = 1 \end{cases}$$

Therefore, the critical point is $(0, 1)$ & $(\pm 2, 1)$

$$f_{xx} = 12x^2 - 16 \quad f_{yy} = 6 \quad f_{xy} = 0$$

At $(0, 1)$

$$f_{xx} = -16 \quad f_{yy} = 6 \quad f_{xy} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = -96 < 0$$

The function f has a **saddle point** at $(0, 1)$ and the value is $\underline{f(0, 1) = -3}$

At $(2, 1)$

$$f_{xx} = 32 \quad f_{yy} = 6 \quad f_{xy} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = 192 > 0 \quad \text{and} \quad f_{xx} = 32 > 0$$

The function f has a **local minimum** at $(2, 1)$ and the value is $\underline{f(2, 1) = -19}$

At $(-2, 1)$

$$f_{xx} = 32 \quad f_{yy} = 6 \quad f_{xy} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = 192 > 0 \quad \text{and} \quad f_{xx} = 32 > 0$$

The function f has a **local minimum** at $(-2, 1)$ and the value is $\underline{f(-2, 1) = -19}$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$$f(x, y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$$

Solution

$$f_x = 2y - 10x + 4 = 0 \quad f_y = 2x - 4y + 4 = 0$$

$$\begin{cases} -5x + y = -2 \\ x - 2y = -2 \end{cases} \rightarrow x = \frac{2}{3} \quad y = \frac{4}{3}$$

Therefore, the critical point is $\left(\frac{2}{3}, \frac{4}{3}\right)$

$$f_{xx} \left| \left(\frac{2}{3}, \frac{4}{3}\right) \right. = -10 \quad f_{yy} \left| \left(\frac{2}{3}, \frac{4}{3}\right) \right. = -4 \quad f_{xy} \left| \left(\frac{2}{3}, \frac{4}{3}\right) \right. = 2$$

$$f_{xx}f_{yy} - f_{xy}^2 = (-10)(-4) - 2^2 = 36 > 0 \quad \text{and} \quad f_{xx} = -10 < 0$$

The function f has a local maximum at $\left(\frac{2}{3}, \frac{4}{3}\right)$ and the value is

$$f\left(\frac{2}{3}, \frac{4}{3}\right) = 2\left(\frac{2}{3}\right)\left(\frac{4}{3}\right) - 5\left(\frac{2}{3}\right)^2 - 2\left(\frac{4}{3}\right)^2 + 4\left(\frac{2}{3}\right) + 4\left(\frac{4}{3}\right) - 4 \\ = 0$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$$f(x, y) = x^2 - 4xy + y^2 + 6y + 2$$

Solution

$$f_x = 2x - 4y = 0 \quad f_y = -4x + 2y + 6 = 0$$

$$\begin{cases} x - 2y = 0 \\ -2x + y = -3 \end{cases} \rightarrow x = 2 \quad y = 1$$

Therefore, the critical point is $(2, 1)$

$$f_{xx} \left| (2, 1) \right. = 2, \quad f_{yy} \left| (2, 1) \right. = 2, \quad f_{xy} \left| (2, 1) \right. = -4$$

$$f_{xx}f_{yy} - f_{xy}^2 = (2)(2) - (-4)^2 = -12 < 0 \Rightarrow \text{Saddle point}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$$f(x, y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$$

Solution

$$f_x = 4x + 3y - 5 = 0 \quad f_y = 3x + 8y + 2 = 0$$

$$\begin{cases} 4x + 3y = 5 \\ 3x + 8y = -2 \end{cases} \rightarrow x = 2 \quad y = -1$$

Therefore, the critical point is $(2, -1)$

$$f_{xx}|_{(2,-1)} = 4, \quad f_{yy}|_{(2,-1)} = 8, \quad f_{xy}|_{(2,-1)} = 3$$

$$f_{xx}f_{yy} - f_{xy}^2 = (4)(8) - 3^2 = 23 > 0 \quad \text{and} \quad f_{xx} = 4 > 0$$

The function f has a local minimum at $(2, -1)$ and the value is

$$\begin{aligned} f(2, -1) &= 2(2)^2 + 3(2)(-1) + 4(-1)^2 - 5(2) + 2(-1) \\ &= -6 \end{aligned}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$$f(x, y) = x^2 - y^2 - 2x + 4y + 6$$

Solution

$$f_x = 2x - 2 = 0 \quad f_y = -2y + 4 = 0$$

$$\begin{cases} 2x = 2 \\ 2y = 4 \end{cases} \rightarrow x = 1 \quad y = 2$$

Therefore, the critical point is $(1, 2)$

$$f_{xx}|_{(1,2)} = 2, \quad f_{yy}|_{(1,2)} = -2, \quad f_{xy}|_{(1,2)} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = (2)(-2) - 0^2 = -4 < 0 \rightarrow \text{Saddle Point}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$$f(x, y) = \sqrt{56x^2 - 8y^2 - 16x - 31} + 1 - 8x$$

Solution

$$f_x = \frac{1}{2} \frac{112x - 16}{\sqrt{56x^2 - 8y^2 - 16x - 31}} - 8 = 0 \quad f_y = \frac{1}{2} \frac{-16y}{\sqrt{56x^2 - 8y^2 - 16x - 31}} = 0$$

$$\begin{cases} 56x - 8 = 8\sqrt{56x^2 - 8y^2 - 16x - 31} \\ -8y = 0 \end{cases} \rightarrow \begin{matrix} x = \frac{16}{7} & \cancel{x = -2} \\ y = 0 \end{matrix}$$

Therefore, the critical point is $\left(\frac{16}{7}, 0\right)$

$$\begin{aligned} f_{xx} \bigg|_{\left(\frac{16}{7}, 0\right)} &= \frac{56\sqrt{56x^2 - 8y^2 - 16x - 31} - (56x - 8)(56x - 8)(56x^2 - 8y^2 - 16x - 31)^{-1/2}}{56x^2 - 8y^2 - 16x - 31} \\ &= -\frac{8}{15} \end{aligned}$$

$$\begin{aligned} f_{yy} \bigg|_{\left(\frac{16}{7}, 0\right)} &= \frac{-8\sqrt{56x^2 - 8y^2 - 16x - 31} - (-8y)(56x^2 - 8y^2 - 16x - 31)^{-1/2}(-8y)}{56x^2 - 8y^2 - 16x - 31} \\ &= -\frac{8}{15} \end{aligned}$$

$$f_{xy} \bigg|_{\left(\frac{16}{7}, 0\right)} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = \left(-\frac{8}{15}\right)\left(-\frac{8}{15}\right) - 0 = \frac{34}{225} > 0 \quad \text{and} \quad f_{xx} = -\frac{8}{15} < 0$$

The function f has a local maximum at $\left(\frac{16}{7}, 0\right)$ and the value is

$$\begin{aligned} f\left(\frac{16}{7}, 0\right) &= \sqrt{56\left(\frac{16}{7}\right)^2 - 8(0)^2 - 16\left(\frac{16}{7}\right) - 31} + 1 - 8\left(\frac{16}{7}\right) \\ &= -\frac{16}{7} \end{aligned}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x, y) = 1 - \sqrt[3]{x^2 + y^2}$

Solution

$$\begin{aligned} f_x &= -\frac{1}{3}2x(x^2 + y^2)^{-2/3} \\ &= \frac{-2x}{3(x^2 + y^2)^{2/3}} = 0 \end{aligned}$$

$$f_y = -\frac{1}{3}2y(x^2 + y^2)^{-2/3}$$

$$= \frac{-2y}{3(x^2 + y^2)^{2/3}} = 0$$

There are no solutions to the system $f_x(x, y) = 0$ and $f_y(x, y) = 0$, however, this occurs when $x = 0$ $y = 0$. The critical point is $(0, 0)$

We cannot use the second derivative test, but this is the only possible local maximum, local minimum, or saddle point. $f(x, y)$ has a local maximum of $f(0, 0) = 1$ since

$$f(x, y) = 1 - \sqrt[3]{x^2 + y^2} \leq 1 \quad \forall (x, y) - \{(0, 0)\}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$$f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$$

Solution

$$f_x = 3x^2 + 6x = 0 \quad f_y = 3y^2 - 6y = 0$$

$$\begin{cases} 3x(x+2) = 0 \\ 3y(y-2) = 0 \end{cases} \rightarrow \begin{matrix} x = 0, -2 \\ y = 0, 2 \end{matrix}$$

Therefore, the critical point is $(0, 0)$, $(0, 2)$, $(-2, 0)$, and $(-2, 2)$

$$f_{xx} = 6x + 6, \quad f_{yy} = 6y - 6, \quad f_{xy} = 0$$

$$\text{For } (0, 0) \quad f_{xx}|_{(0,0)} = 6, \quad f_{yy}|_{(0,0)} = -6, \quad f_{xy}|_{(0,0)} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = (6)(-6) - 0^2 = -36 < 0 \Rightarrow \text{Saddle Point}$$

$$\text{For } (0, 2) \quad f_{xx}|_{(0,2)} = 6, \quad f_{yy}|_{(0,2)} = 6, \quad f_{xy}|_{(0,2)} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = (6)(6) - 0^2 = 36 > 0 \quad \text{and} \quad f_{xx} > 0$$

The function f has a local minimum at $(0, 2)$ and the value is $f(0, 2) = -12$

$$\text{For } (-2, 0) \quad f_{xx}|_{(-2,0)} = -6, \quad f_{yy}|_{(-2,0)} = -6, \quad f_{xy}|_{(-2,0)} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = (-6)(-6) - 0^2 = 36 > 0 \quad \text{and} \quad f_{xx} < 0$$

The function f has a local maximum at $(-2, 0)$ and the value is $f(-2, 0) = -4$

For $(-2, 2)$ $f_{xx}|_{(-2,2)} = 6$, $f_{yy}|_{(-2,2)} = 6$, $f_{xy}|_{(-2,2)} = 0$

$$f_{xx}f_{yy} - f_{xy}^2 = (-6)(6) - 0^2 = -36 < 0 \Rightarrow \text{Saddle Point}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x, y) = 4xy - x^4 - y^4$

Solution

$$f_x = 4y - 4x^3 = 0 \quad f_y = 4x - 4y^3 = 0$$

$$\begin{cases} y - x^3 = 0 \\ x - y^3 = 0 \end{cases} \Rightarrow x = y \rightarrow x - x^3 = 0 \rightarrow x(1 - x^2) = 0 \rightarrow x = 0, \pm 1$$

Therefore, the critical point is $(0, 0)$, $(1, 1)$, and $(-1, -1)$

$$f_{xx} = -12x^2, \quad f_{yy} = -12y^2, \quad f_{xy} = 4$$

For $(0, 0)$ $f_{xx}|_{(0,0)} = 0$, $f_{yy}|_{(0,0)} = 0$, $f_{xy}|_{(0,0)} = 4$

$$f_{xx}f_{yy} - f_{xy}^2 = 0 - 4^2 = -16 < 0 \Rightarrow \text{Saddle Point}$$

For $(1, 1)$ $f_{xx}|_{(1,1)} = -12$, $f_{yy}|_{(1,1)} = -12$, $f_{xy}|_{(1,1)} = 4$

$$f_{xx}f_{yy} - f_{xy}^2 = (-12)(-12) - 4^2 = 128 > 0 \text{ and } f_{xx} < 0$$

The function has a local maximum at $(1, 1)$ and the value is $f(1, 1) = 2$

For $(-1, -1)$ $f_{xx}|_{(-1,-1)} = -12$, $f_{yy}|_{(-1,-1)} = -12$, $f_{xy}|_{(-1,-1)} = 4$

$$f_{xx}f_{yy} - f_{xy}^2 = (-12)(-12) - 4^2 = 128 > 0 \text{ and } f_{xx} < 0$$

The function f has a local maximum at $(-1, -1)$ and the value is $f(-1, -1) = 2$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x, y) = \frac{1}{x^2 + y^2 - 1}$

Solution

$$f_x = \frac{-2x}{(x^2 + y^2 - 1)^2} = 0 \quad f_y = \frac{-2y}{(x^2 + y^2 - 1)^2} = 0$$

$\Rightarrow x = y = 0$ Therefore, the critical point is $(0,0)$

$$\begin{aligned} f_{xx} &= \frac{-2(x^2 + y^2 - 1)^2 - (-2x)(4x)(x^2 + y^2 - 1)}{(x^2 + y^2 - 1)^4} \\ &= \frac{-2x^2 - 2y^2 + 2 + 8x^2}{(x^2 + y^2 - 1)^3} \\ &= \frac{6x^2 - 2y^2 + 2}{(x^2 + y^2 - 1)^3} \end{aligned}$$

$$\begin{aligned} f_{yy} &= \frac{-2(x^2 + y^2 - 1)^2 - (-2y)(4y)(x^2 + y^2 - 1)}{(x^2 + y^2 - 1)^4} \\ &= \frac{-2x^2 - 2y^2 + 2 + 8y^2}{(x^2 + y^2 - 1)^3} \\ &= \frac{-2x^2 + 6y^2 + 2}{(x^2 + y^2 - 1)^3} \end{aligned}$$

$$\begin{aligned} f_{xy} &= \frac{-2x(4y)(x^2 + y^2 - 1)}{(x^2 + y^2 - 1)^4} \\ &= \frac{-8xy}{(x^2 + y^2 - 1)^3} \end{aligned}$$

$$f_{xx} \Big|_{(0,0)} = -2, \quad f_{yy} \Big|_{(0,0)} = -2, \quad f_{xy} \Big|_{(0,0)} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - 0^2 = 4 > 0 \quad \text{and} \quad f_{xx} < 0$$

The function f has a local maximum at $(0,0)$ and the value is $\underline{f(0,0) = -1}$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x, y) = \frac{1}{x} + xy + \frac{1}{y}$

Solution

$$f_x = -\frac{1}{x^2} + y = 0 \quad f_y = x - \frac{1}{y^2} = 0$$

$$\rightarrow \begin{cases} y = \frac{1}{x^2} & (x \neq 0) \\ x = \frac{1}{y^2} & (y \neq 0) \end{cases} \quad x = x^4 \Rightarrow x = 1 = y$$

Therefore, the critical point is $(1,1)$

$$f_{xx} \Big|_{(1,1)} = \left(\frac{2}{x^3} \right) \Big|_{(1,1)} = 2, \quad f_{yy} \Big|_{(1,1)} = \left(\frac{2}{y^3} \right) \Big|_{(1,1)} = -2, \quad f_{xy} \Big|_{(1,1)} = (1) \Big|_{(1,1)} = 1$$

$$f_{xx}f_{yy} - f_{xy}^2 = (2)(-2) - 1^2 = -3 < 0 \quad \text{and} \quad f_{xx} > 0$$

The function f has a local minimum at $(1,1)$ and the value is $f(1, 1) = 3$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x, y) = y \sin x$

Solution

$$f_x = y \cos x = 0 \quad f_y = \sin x = 0$$

$$\rightarrow \begin{cases} y \cos x = 0 \\ \sin x = 0 \end{cases} \quad x = n\pi \quad y = 0 \quad \text{Therefore, the critical point is } (n\pi, 0)$$

$$f_{xx} \Big|_{(n\pi, 0)} = -y \sin x \Big|_{(n\pi, 0)} = 0$$

$$f_{yy} \Big|_{(n\pi, 0)} = 0$$

$$f_{xy} \Big|_{(n\pi, 0)} = \cos x \Big|_{(n\pi, 0)} = \pm 1$$

$$\text{If } n \text{ is even: } f_{xx}f_{yy} - f_{xy}^2 = 0 - 1^2 = -1 < 0 \Rightarrow \text{Saddle Point}$$

$$\text{If } n \text{ is odd: } f_{xx}f_{yy} - f_{xy}^2 = 0 - (-1)^2 = -1 < 0 \Rightarrow \text{Saddle Point}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x, y) = e^{2x} \cos y$

Solution

$$f_x = 2e^{2x} \cos y = 0 \quad f_y = -e^{2x} \sin y = 0$$

Since $e^{2x} \neq 0 \quad \forall x$, the functions $\cos y$ and $\sin y$ cannot equal to zero for the same y .

\therefore No critical points \Rightarrow no extrema and no saddle points.

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x, y) = e^y - ye^x$

Solution

$$f_x = -ye^x = 0 \quad f_y = e^y - e^x = 0$$

$$\rightarrow \begin{cases} -ye^x = 0 \\ e^y - e^x = 0 \end{cases} \quad y = 0 \quad e^x = e^y = 1 = e^0 \Rightarrow x = 0$$

\therefore The critical point is $(0, 0)$

$$f_{xx} \Big|_{(0,0)} = -ye^x \Big|_{(0,0)} = 0$$

$$f_{yy} \Big|_{(0,0)} = e^y = 1$$

$$f_{xy} \Big|_{(0,0)} = -e^x \Big|_{(0,0)} = -1$$

$$f_{xx}f_{yy} - f_{xy}^2 = 0(1) - (-1)^2 = -1 < 0 \Rightarrow \text{Saddle Point}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x, y) = e^{-y}(x^2 + y^2)$

Solution

$$f_x = 2xe^{-y} = 0$$

$$f_y = -e^{-y}(x^2 + y^2) + 2ye^{-y} = e^{-y}(2y - x^2 - y^2) = 0$$

$$\rightarrow \begin{cases} 2xe^{-y} = 0 \\ e^{-y}(2y - x^2 - y^2) = 0 \end{cases} \quad \rightarrow \boxed{x=0} \quad 2y - x^2 - y^2 = 0 \rightarrow y(2 - y) = 0 \quad \boxed{y=0, 2}$$

\therefore The critical point is $(0, 0)$ and $(0, 2)$

$$f_{xx} = 2e^{-y}$$

$$f_{yy} = -e^{-y}(2y - x^2 - y^2) + e^{-y}(2 - 2y) = e^{-y}(2 - 4y + x^2 + y^2)$$

$$f_{xy} = -2xye^{-y}$$

$$\text{For } (0,0) \quad f_{xx} \Big|_{(0,0)} = 2, \quad f_{yy} \Big|_{(0,0)} = 2, \quad f_{xy} \Big|_{(0,0)} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = (2)(2) - 0^2 = 4 > 0 \quad \text{and} \quad f_{xx} > 0$$

The function f has a local minimum at $(0,0)$ and the value is $\underline{f(0,0) = 0}$

$$\text{For } (0,2) \quad f_{xx} \Big|_{(0,2)} = \frac{2}{e^2}, \quad f_{yy} \Big|_{(0,2)} = -\frac{2}{e^2}, \quad f_{xy} \Big|_{(0,2)} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = \frac{2}{e^2} \left(-\frac{2}{e^2} \right) - 0^2 = -\frac{4}{e^4} < 0 \Rightarrow \text{Saddle Point}$$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x, y) = 2 \ln x + \ln y - 4x - y$

Solution

$$f_x = \frac{2}{x} - 4 = 0 \quad f_y = \frac{1}{y} - 1 = 0$$

$$\rightarrow \begin{cases} 2 = 4x \\ 1 = y \end{cases} \quad x = \frac{1}{2}$$

\therefore The critical point is $\left(\frac{1}{2}, 1\right)$

$$f_{xx} \Big|_{\left(\frac{1}{2}, 1\right)} = \left(-\frac{2}{x^2} \right) \Big|_{\left(\frac{1}{2}, 1\right)} = -8$$

$$f_{yy} \Big|_{\left(\frac{1}{2}, 1\right)} = \left(-\frac{1}{y^2} \right) \Big|_{\left(\frac{1}{2}, 1\right)} = -1$$

$$f_{xy} \Big|_{\left(\frac{1}{2}, 1\right)} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = (-8)(-1) - 0^2 = 8 > 0 \quad \text{and} \quad f_{xx} < 0$$

The function f has a local maximum at $\left(\frac{1}{2}, 1\right)$ and the value is $\underline{f\left(\frac{1}{2}, 1\right) = -3 - 2 \ln 2}$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x, y) = \ln(x + y) + x^2 - y$

Solution

$$f_x = \frac{1}{x+y} + 2x = 0$$

$$f_y = \frac{1}{x+y} - 1 = 0$$

$$\rightarrow \begin{cases} \frac{1}{x+y} = -2x \rightarrow -2x(x+y) = 1 \\ \frac{1}{x+y} = 1 \rightarrow 1 = x+y \end{cases} \Rightarrow -2x(1) = 1 \rightarrow x = -\frac{1}{2} \quad y = \frac{3}{2}$$

\therefore The critical point is $\left(-\frac{1}{2}, \frac{3}{2}\right)$

$$f_{xx} = -\frac{1}{(x+y)^2} + 2, \quad f_{yy} = -\frac{1}{(x+y)^2}, \quad f_{xy} = -\frac{1}{(x+y)^2}$$

$$f_{xx} \Big|_{\left(-\frac{1}{2}, \frac{3}{2}\right)} = 1$$

$$f_{yy} \Big|_{\left(-\frac{1}{2}, \frac{3}{2}\right)} = -1$$

$$f_{xy} \Big|_{\left(-\frac{1}{2}, \frac{3}{2}\right)} = -1$$

$$f_{xx}f_{yy} - f_{xy}^2 = (1)(-1) - (-1)^2 = -2 < 0 \quad \text{and} \quad \text{Saddle Point}$$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x, y) = 1 + x^2 + y^2$

Solution

$$f_x = 2x = 0 \rightarrow \underline{x = 0}$$

$$f_y = 2y = 0 \rightarrow \underline{y = 0}$$

\therefore The critical point is $(0, 0)$

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 \Big|_{(0,0)} = 4 > 0 \quad \text{and} \quad f_{xx} > 0$$

The function f has a local minimum at $(0, 0)$ and the value is $\underline{f(0, 0) = 1}$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x, y) = x^2 - 6x + y^2 + 8y$

Solution

$$f_x = 2x - 6 = 0 \rightarrow \underline{x = 3}$$

$$f_y = 2y + 8 = 0 \rightarrow \underline{y = -4}$$

\therefore The critical point is $(3, -4)$

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 \Big|_{(3, -4)} = 4 > 0 \quad \text{and} \quad f_{xx} > 0$$

The function f has a local minimum at $(3, -4)$ and the value is

$$\underline{f(3, -4) = 9 - 18 + 16 - 32 = -25}$$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x, y) = (3x - 2)^2 + (y - 4)^2$

Solution

$$f_x = 6(3x - 2) = 0 \rightarrow \underline{x = \frac{2}{3}}$$

$$f_y = 2(y - 4) = 0 \rightarrow \underline{y = 4}$$

\therefore The critical point is $(\frac{2}{3}, 4)$

$$f_{xx} = 18, \quad f_{yy} = 2, \quad f_{xy} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 \Big|_{(\frac{2}{3}, 4)} = 36 > 0 \quad \text{and} \quad f_{xx} > 0$$

The function f has a local minimum at $(\frac{2}{3}, 4)$ and the value is $\underline{f(\frac{2}{3}, 4) = 0}$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x, y) = 3x^2 - 4y^2$

Solution

$$f_x = 6x = 0 \rightarrow \underline{x = 0}$$

$$f_y = -8y = 0 \rightarrow y = 0$$

∴ The critical point is (0, 0)

$$f_{xx} = 6, \quad f_{yy} = -8, \quad f_{xy} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 \Big|_{(0,0)} = -48 < 0 \text{ and Saddle point}$$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x, y) = x^4 + y^4 - 16xy$

Solution

$$f_x = 4x^3 - 16y = 0$$

$$f_y = 4y^3 - 16x = 0$$

$$\begin{cases} x^3 = 4y \\ y^3 = 4x \end{cases} \rightarrow x = \frac{y^3}{4} \rightarrow \left(\frac{y^3}{4}\right)^3 = 4y$$

$$y^9 = 4^4 y$$

$$y(y^8 - 2^8) = 0 \rightarrow y = 0, \pm 2$$

∴ The critical point is (0, 0), (-2, -2), (2, 2)

$$f_{xx} = 12x^2 > 0, \quad f_{yy} = 12y^2, \quad f_{xy} = -16$$

$$f_{xx}f_{yy} - f_{xy}^2 = 144x^2y^2 - 256$$

@ (0, 0)

$$f_{xx}f_{yy} - f_{xy}^2 = -256 < 0 \text{ and Saddle point}$$

@ (-2, -2)

$$f_{xx}f_{yy} - f_{xy}^2 = 2,048 > 0 \text{ and } f_{xx} > 0$$

The function f has a local minimum at (-2, -2) and the value is $f(-2, -2) = -32$

@ (2, 2)

$$f_{xx}f_{yy} - f_{xy}^2 = 2,048 > 0 \text{ and } f_{xx} > 0$$

The function f has a local minimum at (2, 2) and the value is $f(2, 2) = -32$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x, y) = \frac{1}{3}x^3 - \frac{1}{3}y^3 + 3xy$

Solution

$$f_x = x^2 + 3y = 0$$

$$f_y = -y^2 + 3x = 0$$

$$\begin{cases} x^2 = -3y \\ y^2 = 3x \rightarrow x = \frac{y^2}{3} \end{cases} \rightarrow \left(\frac{y^2}{3}\right)^2 = -3y$$

$$y^4 = -3^3 y$$

$$y(y^3 + 3^3) = 0 \rightarrow y = 0, -3$$

\therefore The critical point is $(0, 0), (3, -3)$

$$f_{xx} = 2x, \quad f_{yy} = -2y, \quad f_{xy} = 3$$

$$f_{xx}f_{yy} - f_{xy}^2 = -4xy - 9$$

@ $(0, 0)$

$$f_{xx}f_{yy} - f_{xy}^2 = -9 < 0 \text{ and Saddle point}$$

@ $(3, -3)$

$$f_{xx}f_{yy} - f_{xy}^2 = 36 - 9 = 27 > 0 \text{ and } f_{xx} = 6 > 0$$

The function f has a local minimum at $(3, -3)$ and the value is

$$\underline{f(3, -3) = 9 + 9 - 27 = -9}$$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x, y) = x^4 - 2x^2 + y^2 - 4y + 5$

Solution

$$f_x = 4x^3 - 4x = 4x(x^2 - 1) = 0 \rightarrow x = 0, \pm 1$$

$$f_y = 2y - 4 = 0 \rightarrow \underline{y = 2}$$

\therefore The critical point is $(0, 2), (-1, 2), (1, 2)$

$$f_{xx} = 12x^2 - 4, \quad f_{yy} = 2, \quad f_{xy} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = 24x^2 - 8$$

@ (0, 2)

$$f_{xx}f_{yy} - f_{xy}^2 = -8 < 0 \quad \text{and} \quad \text{Saddle point}$$

@ (-1, 2)

$$f_{xx}f_{yy} - f_{xy}^2 = 16 > 0 \quad \text{and} \quad f_{xx} = 8 > 0$$

The function f has a local minimum at (-1, 2) and the value is

$$\underline{f(-1, 2) = 1 - 2 + 4 - 8 + 5 = 0}$$

@ (1, 2)

$$f_{xx}f_{yy} - f_{xy}^2 = 16 > 0 \quad \text{and} \quad f_{xx} = 8 > 0$$

The function f has a local minimum at (1, 2) and the value is

$$\underline{f(1, 2) = 1 - 2 + 4 - 8 + 5 = 0}$$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x, y) = x^2 + xy - 2x - y + 1$

Solution

$$f_x = 2x + y - 2 = 0$$

$$f_y = x - 1 = 0 \rightarrow \underline{x = 1}$$

$$y = 2 - 2x = \underline{0}$$

∴ The critical point is (1, 0)

$$f_{xx} = 2, \quad f_{yy} = 0, \quad f_{xy} = 1$$

$$f_{xx}f_{yy} - f_{xy}^2 = -1 < 0 \quad \text{and} \quad \text{Saddle point}$$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x, y) = x^2 + 6x + y^2 + 8$

Solution

$$f_x = 2x + 6 = 0 \rightarrow x = -3$$

$$f_y = 2y = 0 \rightarrow y = 0$$

∴ The critical point is $(-3, 0)$

$$f_{xx} = 2 > 0, \quad f_{yy} = 2, \quad f_{xy} = 0$$

$$f_{xx} f_{yy} - f_{xy}^2 \Big|_{(-3, 0)} = 4 > 0 \quad \text{and} \quad f_{xx} > 0$$

The function f has a local minimum at $(-3, 0)$ and the value is

$$\underline{f(-3, 0) = 9 - 18 + 8 = -1}$$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x, y) = e^{x^2 y^2 - 2xy^2 + y^2}$

Solution

$$f_x = (2xy^2 - 2y^2) e^{x^2 y^2 - 2xy^2 + y^2} = 0 \rightarrow 2(x-1)y^2 = 0$$

$$f_y = (2x^2 y - 4xy + 2y) e^{x^2 y^2 - 2xy^2 + y^2} = 0 \rightarrow 2y(x^2 - 2x + 1) = 0$$

$$\begin{cases} 2(x-1)y^2 = 0 & \rightarrow y = 0, x = 1 \\ 2y(x^2 - 2x + 1) = 0 & \rightarrow y = 0, x = 1 \end{cases}$$

∴ The critical point is $(1, 0), (x, 0), (1, y)$

@ $(1, 0)$

$$\begin{aligned} f_{xx} &= (2y^2 + 2xy^2 - 2y^2) e^{x^2 y^2 - 2xy^2 + y^2} \\ &= 2xy^2 e^{x^2 y^2 - 2xy^2 + y^2} \Big|_{(1,0)} \\ &= 0 \end{aligned}$$

$$\begin{aligned} f_{yy} &= (2x^2 - 4x + 2 + 2x^2 y - 4xy + 2y) e^{x^2 y^2 - 2xy^2 + y^2} \Big|_{(1,0)} \\ &= 0 \end{aligned}$$

$$\begin{aligned} f_{xy} &= (4xy - 4y + 2x^2 y - 4xy + 2y) e^{x^2 y^2 - 2xy^2 + y^2} \\ &= (2x^2 y - 2y) e^{x^2 y^2 - 2xy^2 + y^2} \Big|_{(1,0)} \\ &= 0 \end{aligned}$$

$$f_{xx}f_{yy} - f_{xy}^2 \Big|_{(1, 0)} = 0$$

Inconclusive. No extreme values.

Exercise

Identify the critical points of the functions. Then determine whether each critical point corresponds to a local maximum, local minimum, or saddle point. State when your analysis is inconclusive.

$$f(x, y) = x^4 + y^4 - 16xy$$

Solution

$$f_x = 4x^3 - 16y = 0 \quad (1)$$

$$f_y = 4y^3 - 16x = 0 \quad (2)$$

$$\begin{cases} (1) \rightarrow x^3 = 4y \\ (2) \rightarrow y^3 = 4x \end{cases}$$

$$\left(\frac{y^3}{4}\right)^3 = 4y$$

$$y^9 = 4^4 y$$

$$y(y^8 - 2^8) = 0 \rightarrow \underline{y = 0, \pm 2}$$

C.P: $(0, 0), (2, 2), (-2, -2)$

$$f_{xx} = 12x^2 \quad f_{yy} = 12y^2 \quad f_{xy} = -16$$

$$f_{xx}f_{yy} - f_{xy}^2 = 144x^2y^2 - 256$$

@ $(0, 0)$

$$f_{xx}f_{yy} - f_{xy}^2 = -256 < 0$$

$(0, 0)$ is a *saddle point*.

@ $\pm(2, 2)$

$$f_{xx}f_{yy} - f_{xy}^2 = 144(4)(4) - 256 = 2,048 > 0$$

f has a *local Min* @ $(2, 2)$ & $(-2, -2)$

Exercise

Identify the critical points of the functions. Then determine whether each critical point corresponds to a local maximum, local minimum, or saddle point. State when your analysis is inconclusive.

$$f(x, y) = \frac{1}{3}x^3 - \frac{1}{3}y^3 + 2xy$$

Solution

$$f_x = x^2 + 2y = 0 \quad (1)$$

$$f_y = -y^2 + 2x = 0 \quad (2)$$

$$(1) \rightarrow y = -\frac{1}{2}x^2$$

$$(2) \rightarrow -\left(-\frac{1}{2}x^2\right)^2 + 2x = 0$$

$$-\frac{1}{4}x^4 + 2x = 0$$

$$-\frac{1}{4}x(x^3 - 8) = 0 \quad \rightarrow \begin{cases} x = 0 & \rightarrow y = 0 \\ x = 2 & \rightarrow y = -2 \end{cases}$$

C.P: $(0, 0)$ & $(2, -2)$

$$f_{xx} = 2x \qquad f_{yy} = -2y \qquad f_{xy} = 2 > 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = -4xy - 4$$

@ $(0, 0)$

$$f_{xx}f_{yy} - f_{xy}^2 = -4 < 0$$

$(0, 0)$ is a *saddle point*.

@ $(2, -2)$

$$f_{xx}f_{yy} - f_{xy}^2 = 16 - 4 = 12 > 0$$

f has a *local Min* @ $(2, -2)$

Exercise

Identify the critical points of the functions. Then determine whether each critical point corresponds to a local maximum, local minimum, or saddle point. State when your analysis is inconclusive.

$$f(x, y) = xy(2+x)(y-3)$$

Solution

$$f(x, y) = 2xy^2 - 6xy + x^2y^2 - 3x^2y$$

$$f_x = 2y^2 - 6y + 2xy^2 - 6xy = 0$$

$$= 2y(y - 3) + 2xy(y - 3)$$

$$= 2y(y - 3)(x + 1) = 0 \quad (1)$$

$$\rightarrow \underline{y = 0, \quad y = 3, \quad x = -1}$$

$$f_y = 4xy - 6x + 2x^2y - 3x^2$$

$$= 2x(2y - 3) + x^2(2y - 3)$$

$$= x(2y - 3)(x + 2) = 0 \quad (2)$$

$$\rightarrow \underline{x = 0, \quad x = -2, \quad y = \frac{2}{3}}$$

$$y = 0 \quad (2) \rightarrow x(-3)(x + 2) = 0 \Rightarrow \underline{x = 0, -2}$$

$$y = 3 \quad (2) \rightarrow x(3)(x + 2) = 0 \Rightarrow \underline{x = 0, -2}$$

$$x = -1 \quad (2) \rightarrow -(2y - 3) = 0 \Rightarrow \underline{y = \frac{3}{2}}$$

$$\text{C.P: } (0, 0), (-2, 0), (0, 3), (-2, 3), \text{ \& } \left(1, \frac{3}{2}\right)$$

$$f_{xx} = 2y^2 - 6y$$

$$f_{xy} = 4y - 6 + 4xy - 6x$$

$$f_{yy} = 4x + 2x^2$$

$$f_{xx}f_{yy} - f_{xy}^2 = (2y^2 - 6y)(4x + 2x^2) - (4y - 6 + 4xy - 6x)^2$$

$$@ \quad (0, 0)$$

$$f_{xx}f_{yy} - f_{xy}^2 = -36 < 0$$

$(0, 0)$ is a *saddle point*.

$$@ \quad (-2, 0)$$

$$f_{xx}f_{yy} - f_{xy}^2 = -36 < 0$$

$(-2, 0)$ is a *saddle point*.

$$@ \quad (0, 3)$$

$$f_{xx}f_{yy} - f_{xy}^2 = -36 < 0$$

$(0, 3)$ is a *saddle point*.

@ $(-2, 3)$

$$f_{xx}f_{yy} - f_{xy}^2 = -64 < 0$$

$(-2, 3)$ is a *saddle point*.

@ $(-1, \frac{3}{2})$

$$f_{xx}f_{yy} - f_{xy}^2 = 9 > 0 \quad f_{xx} = -\frac{9}{2} < 0$$

Function has a *local max* @ $(-1, \frac{3}{2})$

Exercise

Identify the critical points of the functions. Then determine whether each critical point corresponds to a local maximum, local minimum, or saddle point. State when your analysis is inconclusive.

$$f(x, y) = 10 - x^3 - y^3 - 3x^2 + 3y^2$$

Solution

$$\begin{aligned} f_x &= -3x^2 - 6x = 0 \\ &= -3x(x + 2) = 0 \quad (1) \end{aligned}$$

$$\rightarrow \underline{x = 0, \quad x = -2}$$

$$\begin{aligned} f_y &= -3y^2 + 6y \\ &= -3y(y - 2) = 0 \quad (2) \end{aligned}$$

$$\rightarrow \underline{y = 0, \quad y = 2}$$

C.P: $(0, 0)$, $(-2, 0)$, $(0, 2)$, & $(-2, 2)$

$$f_{xx} = -6x - 6$$

$$f_{xy} = 0$$

$$f_{yy} = -6y + 6$$

$$\begin{aligned} f_{xx}f_{yy} - f_{xy}^2 &= (-6x - 6)(-6y + 6) \\ &= -36(-xy + x - y + 1) \end{aligned}$$

@ $(0, 0)$

$$f_{xx}f_{yy} - f_{xy}^2 = -36 < 0$$

$(0, 0)$ is a *saddle point*.

@ $(0, 2)$

$$f_{xx}f_{yy} - f_{xy}^2 = 36 > 0 \quad f_{xx} = -6 < 0$$

Function has a *local min* @ $(0, 2)$

@ $(-2, 0)$

$$f_{xx}f_{yy} - f_{xy}^2 = 36 > 0 \quad f_{xx} = 6 > 0$$

Function has a *local max* @ $(-2, 0)$

@ $(-2, 2)$

$$f_{xx}f_{yy} - f_{xy}^2 = -36 < 0$$

$(-2, 2)$ is a *saddle point*.

Exercise

Find the absolute maximum and minimum values of the function on the specified region R .

$$f(x, y) = \frac{1}{3}x^3 - \frac{1}{3}y^3 + 2xy \text{ on the rectangle } R = \{(x, y) : 0 \leq x \leq 3, -1 \leq y \leq 1\}$$

Solution

$$f_x = x^2 + 2y = 0 \rightarrow y = -\frac{1}{2}x^2$$

$$f_y = -y^2 + 2x = 0$$

$$-\frac{1}{4}x^4 + 2x = 0$$

$$-\frac{1}{4}x(x^3 - 8) = 0 \rightarrow \underline{x = 0, x = 2}$$

$$\rightarrow \begin{cases} x = 0 & y = 0 \\ x = 2 & y = -2 \end{cases}$$

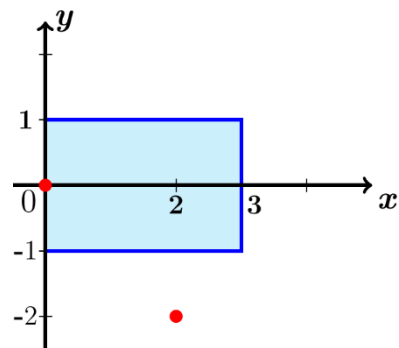
C.P: $(0, 0)$ & $(2, -2)$ neither in the interior of R .

$$0 \leq x \leq 3 \rightarrow y = -1$$

$$g(x) = f(x, -1) = \frac{1}{3}x^3 - 2x + \frac{1}{3}$$

$$f_x = x^2 - 2 = 0 \rightarrow \underline{x = \sqrt{2}}, \quad \cancel{x = -\sqrt{2}}$$

$$g(\sqrt{2}) = \frac{2}{3}\sqrt{2} - 2\sqrt{2} + \frac{1}{3}$$



$$\left. = \frac{1-4\sqrt{2}}{3} \right|$$

$$\rightarrow y = 1$$

$$g(x) = f(x, 1) = \frac{1}{3}x^3 + 2x - \frac{1}{3}$$

$$f_x = x^2 + 2 \neq 0$$

$$-1 \leq y \leq 1 \rightarrow x = 0$$

$$h(y) = f(0, y) = -\frac{1}{3}y^3$$

$$f_y = y^2 = 0 \rightarrow \underline{y = 0}$$

$$\rightarrow x = 3$$

$$h(y) = f(3, y) = -\frac{1}{3}y^3 + 6y + 9$$

$$f_y = -y^2 + 6 = 0 \rightarrow \underline{y = \sqrt{6}}$$

$$h(\sqrt{6}) = f(3, \sqrt{6}) = \underline{4\sqrt{6} + 9}$$

$$\text{Absolute minimum: } \left. f(\sqrt{2}, 1) = \frac{1-4\sqrt{2}}{3} \right|$$

$$\text{Absolute maximum: } \left. f(3, \sqrt{6}) = 4\sqrt{6} + 9 \right|$$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R .

$$f(x, y) = x^4 + y^4 - 4xy + 1 \text{ on the square } R = \{(x, y) : -2 \leq x \leq 2, -2 \leq y \leq 2\}$$

Solution

$$f_x = 4x^3 - 4y = 0 \rightarrow y = x^3$$

$$f_y = 4y^3 - 4x = 0$$

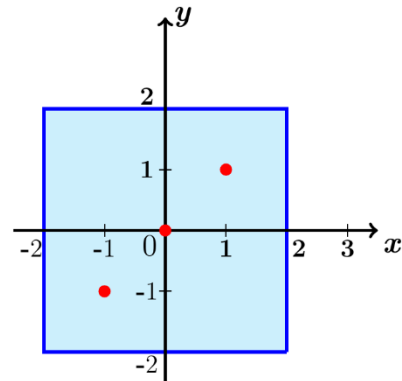
$$x^9 - x = 0 \rightarrow \underline{x = 0, \pm 1}$$

$$x = 0 \rightarrow y = 0$$

$$x = -1 \rightarrow y = -1$$

$$x = 1 \rightarrow y = 1$$

$$\text{C.P.: } (0, 0), (1, 1) \text{ \& } (-1, -1)$$



	$f(x, y)$
$(0, 0)$	1
$(-1, -1)$	$1 + 1 - 4 + 1 = -1$
$(1, 1)$	$1 + 1 - 4 + 1 = -1$
$(2, 2)$	$16 + 16 - 16 + 1 = 17$
$(-2, 2)$	$16 + 16 + 16 + 1 = 49$
$(2, -2)$	$16 + 16 + 16 + 1 = 49$
$(-2, -2)$	$16 + 16 - 16 + 1 = 17$

Absolute minimum: $f(1, 1) = f(-1, -1) = -1$

Absolute maximum: $f(-2, 2) = f(2, -2) = 49$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R .

$f(x, y) = x^2y - y^3$ on the triangle $R = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 2 - x\}$

Solution

$$f_x = 2xy = 0 \quad (0, 0)$$

$$f_y = x^2 - 3y^2 = 0 \quad (2)$$

C.P: None inside the triangle

$$y = 2 - x$$

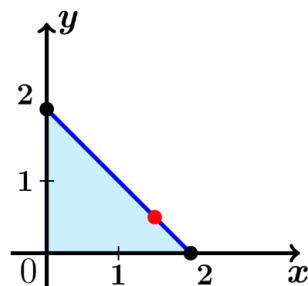
$$\begin{aligned} f(x, y) &= x^2(2 - x) - (2 - x)^3 \\ &= (2 - x)(x^2 - 4 + 4x - x^2) \\ &= (2 - x)(4x - 4) \\ &= 12x - 8 - 4x^2 \end{aligned}$$

$$g' = 12 - 8x = 0 \rightarrow x = \frac{3}{2}$$

$$x = \frac{3}{2} \rightarrow y = 2 - \frac{3}{2} = \frac{1}{2}$$

Absolute minimum: $f(0, 2) = -8$

Absolute maximum: $f\left(\frac{3}{2}, \frac{1}{2}\right) = 1$



	$f(x, y)$
$(0, 0)$	0
$(2, 0)$	0
$(0, 2)$	-8
$\left(\frac{3}{2}, \frac{1}{2}\right)$	$\frac{9}{8} - \frac{1}{8} = 1$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R .

$$f(x, y) = xy \text{ on the semicircular disk } R = \{(x, y): -1 \leq x \leq 1, 0 \leq y \leq \sqrt{1-x^2}\}$$

Solution

$$f_x = y = 0$$

$$f_y = x = 0$$

$$\text{C.P.: } (0, 0)$$

$$y = \sqrt{1-x^2} \rightarrow y^2 + x^2 = 1$$

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$$

$$\begin{aligned} f(x, y) &= g(t) = \cos t \sin t \\ &= \frac{1}{2} \sin 2t \end{aligned}$$

$$g' = \cos 2t = 0 \rightarrow 2t = \frac{\pi}{2}, \frac{3\pi}{2}$$

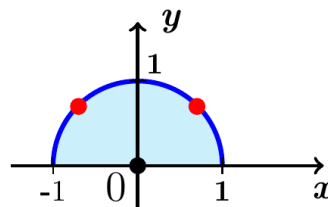
$$t = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$g\left(\frac{\pi}{4}\right) = \frac{1}{2}$$

$$g\left(\frac{3\pi}{4}\right) = -\frac{1}{2}$$

$$\text{Absolute minimum: } \underline{f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = -\frac{1}{2}}$$

$$\text{Absolute maximum: } \underline{f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{1}{2}}$$



Exercise

Find the absolute maximum and minimum values of the function on the specified region R .

$$f(x, y) = x^2 + y^2 - 2y + 1; \quad R = \{(x, y): x^2 + y^2 \leq 4\}$$

Solution

$$f_x = 2x = 0 \rightarrow \underline{x = 0}$$

$$f_y = 2y - 2 = 0 \rightarrow \underline{y = 1}$$

$$\text{C.P.: } (0, 1)$$

$$y^2 + x^2 = 4$$

$$\begin{cases} x = 2 \cos t \\ y = 2 \sin t \end{cases}$$

$$\begin{aligned} f(x, y) = g(t) &= 4 \cos^2 t + 4 \sin^2 t - 4 \sin t + 1 \\ &= 5 - 4 \sin t \end{aligned}$$

$$g' = -4 \cos t = 0 \rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$g\left(\frac{\pi}{2}\right) = 1 = f(0, 2)$$

$$g\left(\frac{3\pi}{2}\right) = 9 = f(0, -2)$$

$$f(0, 1) = 0$$

$$\text{Absolute minimum: } \underline{f(0, 1) = 0}$$

$$\text{Absolute maximum: } \underline{f(0, -2) = 9}$$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R .

$$f(x, y) = 2x^2 + y^2; \quad R = \{(x, y) : x^2 + y^2 \leq 16\}$$

Solution

$$f_x = 4x = 0 \rightarrow x = 0$$

$$f_y = 2y = 0 \rightarrow y = 0$$

$$\text{C.P.: } (0, 0)$$

$$y^2 + x^2 = 16 \quad \begin{cases} x = 4 \cos t \\ y = 4 \sin t \end{cases}$$

$$\begin{aligned} f(x, y) = g(t) &= 32 \cos^2 t + 16 \sin^2 t \\ &= 16 \cos^2 t + 16 \end{aligned}$$

$$\begin{aligned} g' &= -32 \sin t \cos t \\ &= -16 \sin 2t = 0 \end{aligned}$$

$$2t = n\pi \rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

$$\text{Absolute minimum: } \underline{f(0, 0) = 0}$$

$$\text{Absolute maximum: } \underline{f(\pm 4, 0) = 32}$$

t	(x, y)	$f(x, y)$
	$(0, 0)$	0
0	$(4, 0)$	32
$\frac{\pi}{2}$	$(0, 4)$	16
π	$(-4, 0)$	32
$\frac{3\pi}{2}$	$(0, -4)$	16

Exercise

Find the absolute maximum and minimum values of the function on the specified region R .

$$f(x, y) = 4 + 2x^2 + y^2; \quad R = \{(x, y): -1 \leq x \leq 1, -1 \leq y \leq 1\}$$

Solution

$$f_x = 4x = 0 \rightarrow \underline{x = 0}$$

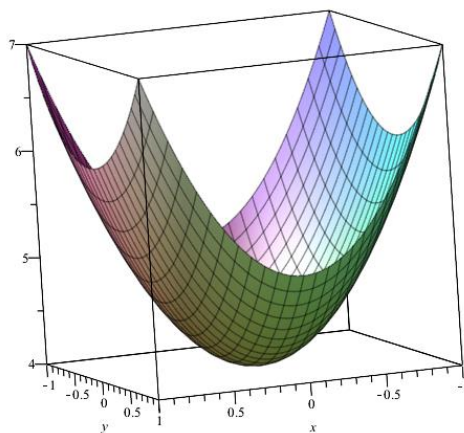
$$f_y = 2y = 0 \rightarrow \underline{y = 0}$$

C.P.: $(0, 0)$

	$f(x, y)$
$(0, 0)$	4
$(\pm 1, \pm 1)$	$4 + 2 + 1 = 7$

Absolute minimum: $\underline{f(0, 0) = 4}$

Absolute maximum: $\underline{f(\pm 1, \pm 1) = 7}$



Exercise

Find the absolute maximum and minimum values of the function on the specified region R .

$$f(x, y) = 6 - x^2 - 4y^2; \quad R = \{(x, y): -2 \leq x \leq 2, -1 \leq y \leq 1\}$$

Solution

$$f_x = -2x = 0 \rightarrow \underline{x = 0}$$

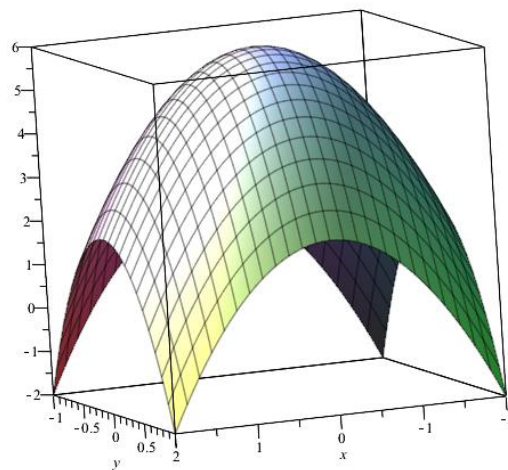
$$f_y = -8y = 0 \rightarrow \underline{y = 0}$$

C.P.: $(0, 0)$

	$f(x, y)$
$(0, 0)$	6
$(\pm 2, \pm 1)$	$6 - 4 - 4 = -2$

Absolute minimum: $\underline{f(\pm 2, \pm 1) = -2}$

Absolute maximum: $\underline{f(0, 0) = 6}$



Exercise

Find the absolute maximum and minimum values of the function on the specified region R .

$$f(x, y) = 2x^2 - 4x + 3y^2 + 2; \quad R = \{(x, y): (x-1)^2 + y^2 \leq 1\}$$

Solution

$$f_x = 4x - 4 = 0 \rightarrow x = 1$$

$$f_y = 6y = 0 \rightarrow y = 0$$

C.P.: (1, 0)

$$(x-1)^2 + y^2 = 1 \quad \begin{cases} x-1 = \cos t \rightarrow x = 1 + \cos t \\ y = \sin t \end{cases}$$

$$\begin{aligned} f(x, y) = g(t) &= 2(1 + \cos t)^2 - 4 - 4\cos t + 3\sin^2 t + 2 \\ &= 2\cos^2 t + 3\sin^2 t \\ &= 2 + \sin^2 t \end{aligned}$$

$$\begin{aligned} g' &= 2\sin t \cos t \\ &= \sin 2t = 0 \end{aligned}$$

$$2t = n\pi \rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

t	(x, y)	$f(x, y)$
	(0, 0)	2
0	(2, 0)	2
$\frac{\pi}{2}$	(1, 1)	3
π	(0, 0)	2
$\frac{3\pi}{2}$	(1, -1)	3

Absolute minimum: $f(0, 0) = f(2, 0) = 2$

Absolute maximum: $f(1, \pm 1) = 3$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R .

$$f(x, y) = -2x^2 + 4x - 3y^2 - 6y - 1; \quad R = \{(x, y): (x-1)^2 + (y+1)^2 \leq 1\}$$

Solution

$$f_x = -4x + 4 = 0 \rightarrow x = 1$$

$$f_y = -6y - 6 = 0 \rightarrow \underline{y = -1}$$

$$\text{C.P.: } (1, -1)$$

$$(x-1)^2 + (y+1)^2 = 1 \quad \begin{cases} x-1 = \cos t \rightarrow x = 1 + \cos t \\ y+1 = \sin t \rightarrow y = \sin t - 1 \end{cases}$$

$$\begin{aligned} f(x, y) = g(t) &= -2(1 + \cos t)^2 + 4 + 4 \cos t - 3(\sin t - 1)^2 - 6 \sin t + 6 - 1 \\ &= 4 - 2 \cos^2 t - 3 \sin^2 t \\ &= 1 + \cos^2 t \end{aligned}$$

$$\begin{aligned} g' &= -2 \sin t \cos t \\ &= -\sin 2t = 0 \end{aligned}$$

$$2t = n\pi \rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

t	(x, y)	$f(x, y)$
	$(1, -1)$	$-2 + 4 - 3 + 6 - 1 = 4$
0	$(2, -1)$	$-8 + 8 - 3 + 6 - 1 = 2$
$\frac{\pi}{2}$	$(1, 0)$	1
π	$(0, -1)$	2
$\frac{3\pi}{2}$	$(1, -2)$	1

$$\text{Absolute minimum: } \underline{f(1, 0) = f(1, -2) = 1}$$

$$\text{Absolute maximum: } \underline{f(1, -1) = 4}$$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R .

$$f(x, y) = \sqrt{x^2 + y^2 - 2x + 2}; \quad R = \{(x, y): x^2 + y^2 \leq 4, y \geq 0\}$$

Solution

$$g(x, y) = x^2 + y^2 - 2x + 2$$

$$g_x = 2x - 2 = 0 \rightarrow \underline{x = 1}$$

$$g_y = 2y = 0 \rightarrow \underline{y = 0}$$

$$\text{C.P.: } (1, 0)$$

$$y^2 + x^2 = 4 \quad \begin{cases} x = 2 \cos t \\ y = 2 \sin t \end{cases}$$

$$g(x, y) = h(t) = 4 - 4\cos t + 2$$

$$= 6 - \cos t$$

$$g' = \sin t = 0$$

$$t = 0, \pi$$

t	(x, y)	$f(x, y)$
	$(1, 0)$	1
0	$(2, 0)$	$\sqrt{2}$
π	$(-2, 0)$	$\sqrt{10}$

Absolute **minimum**: $f(1, 0) = 1$

Absolute **maximum**: $f(-2, 0) = \sqrt{10}$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R .

$$f(x, y) = \frac{-x^2 + 2y^2}{2 + 2x^2y^2}; R \text{ is the closed region bounded by the lines } y = x, y = 2x, \text{ and } y = 2$$

Solution

$$f_x = \frac{2(-2 - 4y^4)x}{(2 + 2x^2y^2)^2}$$

$$= -\frac{(2y^4 + 1)x}{(1 + x^2y^2)^2} = 0 \rightarrow \underline{x = 0}$$

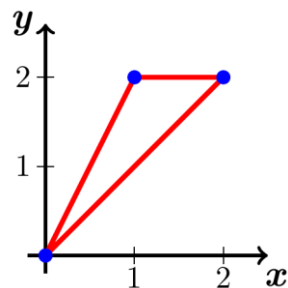
$$f_y = \frac{2(4 + 2x^4)y}{(2 + 2x^2y^2)^2}$$

$$= \frac{(2 + x^4)y}{(1 + x^2y^2)^2} = 0 \rightarrow \underline{y = 0}$$

C.P.: $(0, 0)$

@ $y = x$ $f(x, y) = \frac{x^2}{2 + 2x^4} = 0$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$



$$f' = \frac{4x - 4x^5}{(2 + 2x^4)^2}$$

$$= \frac{4x(1 - x^4)}{(2 + 2x^4)^2} = 0 \rightarrow x = 0, \pm 1$$

$$y = x = 0 \rightarrow f(0, 0) = 0$$

$$y = x = \pm 1 \rightarrow f(1, 1) = \frac{1}{4}$$

$$@ \text{ } y = 2x \quad f(x, y) = \frac{7x^2}{2 + 8x^4}$$

$$f' = \frac{28x - 112x^5}{(2 + 8x^4)^2}$$

$$= \frac{28x(1 - 8x^4)}{(2 + 8x^4)^2} = 0 \rightarrow x = 0, \pm \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$y = 2x = \sqrt{2} \rightarrow f\left(\frac{\sqrt{2}}{2}, \sqrt{2}\right) = \frac{7}{8}$$

$$@ \text{ } y = 2 \quad f(x, y) = \frac{-x^2 + 8}{8x^2 + 2}$$

$$f' = \frac{-132x}{(8x^2 + 2)^2} = 0 \rightarrow x = 0$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

$$y = x = 2 \rightarrow f(2, 2) = \frac{4}{34} = \frac{2}{17}$$

$$y = 2x = 2 \Rightarrow x = 1 \rightarrow f(1, 2) = \frac{7}{10}$$

$$\text{Absolute minimum: } \underline{f(0, 0) = 0}$$

$$\text{Absolute maximum: } \underline{f\left(\frac{\sqrt{2}}{2}, \sqrt{2}\right) = \frac{7}{8}}$$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R .

$$f(x, y) = \sqrt{x^2 + y^2}; R \text{ is the closed region bounded by the ellipse } \frac{x^2}{4} + y^2 = 1$$

Solution

$$f_x = \frac{x}{\sqrt{x^2 + y^2}} = 0 \rightarrow x = 0$$

$$f_y = \frac{y}{\sqrt{x^2 + y^2}} = 0 \rightarrow y = 0$$

C.P.: (0, 0)

$$\frac{x^2}{4} + y^2 = 1 \quad \begin{cases} x = 2 \cos t \\ y = \sin t \end{cases}$$

$$f(x, y) = g(t) = \sqrt{4 \cos^2 t + \sin^2 t} \\ = \sqrt{3 \cos^2 t + 1}$$

$$g' = \frac{-3 \cos t \sin t}{\sqrt{3 \cos^2 t + 1}} \\ = -\frac{3}{2} \frac{\sin 2t}{\sqrt{3 \cos^2 t + 1}} = 0$$

$$2t = n\pi \rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

t	(x, y)	$f(x, y)$
	(0, 0)	0
0	(2, 0)	2
$\frac{\pi}{2}$	(0, 1)	1
π	(-2, 0)	2
$\frac{3\pi}{2}$	(0, -1)	1

Absolute **minimum**: $f(0, 0) = 0$

Absolute **maximum**: $f(-2, 0) = f(2, 0) = 2$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R .

$$f(x, y) = x^2 + y^2 - 4; \quad R = \{(x, y) : x^2 + y^2 < 4\}$$

Solution

$$f_x = 2x = 0 \rightarrow x = 0$$

$$f_y = 2y = 0 \rightarrow y = 0$$

$$C.P.: (0, 0)$$

$$f(0, 0) = -4$$

$$y^2 + x^2 = 4 \quad \begin{cases} x = 2 \cos t \\ y = 2 \sin t \end{cases}$$

$$f(x, y) = g(t) = 4 \cos^2 t + 4 \sin^2 t - 4 = 0 \quad \text{No extreme points.}$$

$$f(x, y) \geq -4$$

$$\text{Absolute minimum: } \underline{f(0, 0) = -4}$$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R .

$$f(x, y) = x + 3y; \quad R = \{(x, y): |x| < 1, |y| < 2\}$$

Solution

$$f_x = 1 \neq 0$$

$$f_y = 3 \neq 0$$

$$C.P.: \text{None}$$

$$-1 < x < 1 \quad -2 < y < 2$$

(x, y)	$f(x, y)$
$(-1, -2)$	-7
$(1, -2)$	-5
$(1, 2)$	7
$(-1, 2)$	5

The range of the function $f(x, y)$ on R is the interval $(-7, 7)$.

$\therefore f(x, y)$ has **neither** an absolute minimum or maximum on R .

Exercise

Find the absolute maximum and minimum values of the function on the specified region R .

$$f(x, y) = 2e^{-x-y}; \quad R = \{(x, y): x \geq 0, y \geq 0\}$$

Solution

$$f_x = -2e^{-x-y} \neq 0$$

$$f_y = -2e^{-x-y} \neq 0$$

C.P.: None

$$R = \{(x, y) : x \geq 0, y \geq 0\}$$

$$f(0, 0) = 2$$

$$(x, y) \rightarrow \infty \Rightarrow f(x, y) \rightarrow 0$$

Absolute **minimum**: None

Absolute **maximum**: $f(0, 0) = 2$

Exercise

Find the absolute maxima and minima of the function $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0$, $y = 2$, $y = 2x$ in the first quadrant.

Solution

$$f_x = 4x - 4 = 0 \quad f_y = 2y - 4 = 0$$

$$x = 1 \quad y = 2$$

The critical point is $(1, 2)$ and the value is $f(1, 2) = -5$

i. On the segment OA . The function $f(0, y) = y^2 - 4y + 1$

This function is defined on the closed interval $0 \leq y \leq 2$.

$$f'(0, y) = 2y - 4 = 0 \rightarrow y = 2$$

$$\begin{cases} y = 0 & \rightarrow f(0, 0) = 1 \\ y = 2 & \rightarrow f(0, 2) = -3 \end{cases}$$

ii. On the segment OB

$$f(x, 2x) = 2x^2 - 4x + (2x)^2 - 4(2x) + 1 = 6x^2 - 12x + 1 \quad 0 \leq x \leq 1$$

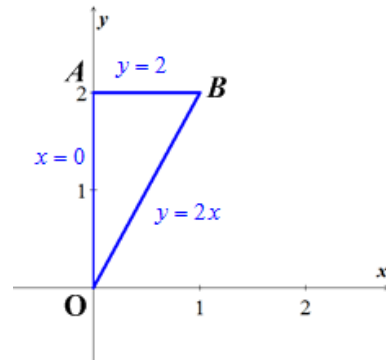
$$f'(x, 2x) = 12x - 12 = 0 \rightarrow x = 1$$

$$\begin{cases} x = 0 & \rightarrow f(0, 0) = 1 \\ x = 1 & \rightarrow f(1, 2) = -5 \end{cases} \quad \therefore (1, 2) \text{ is not interior point of } OB$$

iii. On the segment AB

$$f(x, 2) = 2x^2 - 4x + (2)^2 - 4(2) + 1 = 2x^2 - 4x - 3 \quad 0 \leq x \leq 1$$

$$f'(x, 2) = 4x - 4 = 0 \rightarrow x = 1$$



$$\begin{cases} x=0 & \rightarrow f(0, 2) = \underline{-3} \\ x=1 & \rightarrow f(1, 2) = \underline{-5} \end{cases}$$

$\Rightarrow (1, 2)$ is not interior point of triangular region.

Therefore; the absolute **maximum** is 1 at $(0, 0)$ and the absolute **minimum** is -5 at $(1, 2)$

Exercise

Find the absolute maxima and minima of the function $D(x, y) = x^2 - xy + y^2 + 1$ on the closed triangular plate bounded by the lines $x = 0$, $y = 4$, $y = x$ in the first quadrant.

Solution

$$D_x = 2x - y = 0, \quad D_y = -x + 2y = 0, \quad \Rightarrow x = y = 0$$

The critical point is $(0, 0)$ and the value is $D(0, 0) = \underline{1}$

i. On the segment OA .

$$D(0, y) = y^2 + 1, \quad 0 \leq y \leq 4$$

$$D'(0, y) = 2y = 0 \rightarrow y = 0$$

$$\begin{cases} y=0 & \rightarrow D(0, 0) = \underline{1} \\ y=4 & \rightarrow D(0, 4) = \underline{17} \end{cases}$$

ii. On the segment OB

$$D(x, x) = x^2 + 1 \quad 0 \leq x \leq 4$$

$$D'(x, x) = 2x = 0 \rightarrow x = 0$$

$$x = 0 \rightarrow D(0, 0) = \underline{1}$$

iii. On the segment AB

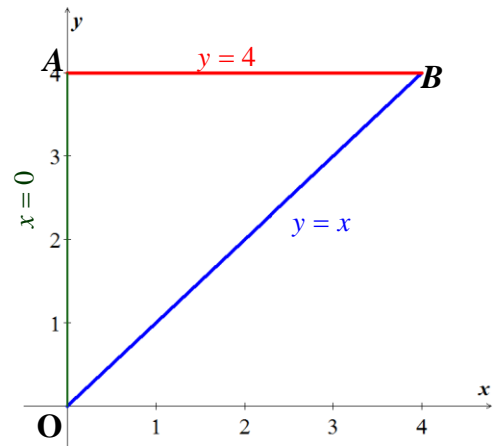
$$D(x, 4) = x^2 - 4x + 17 \quad 0 \leq x \leq 4$$

$$D'(x, 4) = 2x - 4 = 0 \rightarrow x = 2$$

$$\begin{cases} x=2 & \rightarrow D(2, 4) = 13 \\ x=4 & \rightarrow D(4, 4) = \underline{17} \end{cases}$$

$\Rightarrow (0, 0)$ is not interior point of triangular region.

Therefore; the absolute **maximum** is 17 at $(0, 4)$ and $(4, 4)$ and the absolute **minimum** is 1 at $(0, 0)$



Exercise

Find the absolute maxima and minima of the function $T(x, y) = x^2 + xy + y^2 - 6x + 2$ on the triangular plate $0 \leq x \leq 5$, $-3 \leq y \leq 0$.

Solution

$$T_x = 2x + y - 6 = 0, \quad T_y = x + 2y = 0$$

$$\begin{cases} 2x + y = 6 \\ x + 2y = 0 \end{cases} \rightarrow \boxed{x = 4, y = -2}$$

The critical point is $(4, -2)$ and the value is $T(4, -2) = -10$

i. On the segment OA .

$$T(0, y) = y^2 + 2, \quad -3 \leq y \leq 0$$

$$T'(0, y) = 2y = 0 \rightarrow y = 0$$

$$\begin{cases} y = 0 \rightarrow T(0, 0) = 2 \\ y = -3 \rightarrow T(0, -3) = 11 \end{cases}$$

ii. On the segment AB

$$T(x, -3) = x^2 - 9x + 11 \quad 0 \leq x \leq 5$$

$$T'(x, -3) = 2x - 9 = 0 \rightarrow x = \frac{9}{2}$$

$$\begin{cases} x = \frac{9}{2} \rightarrow T\left(\frac{9}{2}, -3\right) = -\frac{37}{4} \\ x = 0 \rightarrow T(0, -3) = 11 \end{cases}$$

iii. On the segment BC

$$T(5, y) = y^2 + 5y - 3 \quad -3 \leq y \leq 0$$

$$T'(5, y) = 2y + 5 = 0 \rightarrow y = -\frac{5}{2}$$

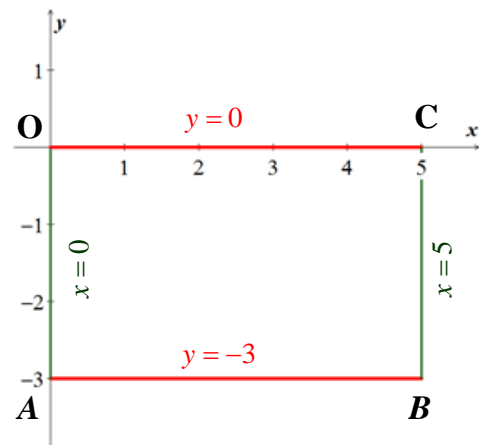
$$\begin{cases} y = 0 \rightarrow T(5, 0) = -3 \\ y = -\frac{5}{2} \rightarrow T\left(5, -\frac{5}{2}\right) = -\frac{37}{4} \\ y = -3 \rightarrow T(5, -3) = -9 \end{cases}$$

iv. On the segment CO

$$T(x, 0) = x^2 - 6x + 2 \quad 0 \leq x \leq 5$$

$$T'(x, 0) = 2x - 6 = 0 \rightarrow x = 3$$

$$(3, 0) \rightarrow T(3, 0) = -7$$



Therefore; the absolute **maximum** is 11 at $(0, -3)$ and the absolute **minimum** is -10 at $(4, -2)$

Exercise

Find the absolute maxima and minima of the function $f(x, y) = (4x - x^2) \cos y$ on the triangular plate $1 \leq x \leq 3, -\frac{\pi}{4} \leq y \leq \frac{\pi}{4}$.

Solution

$$f_x = (4 - 2x) \cos y = 0, \quad f_y = (x^2 - 4x) \sin y = 0$$

$$\begin{cases} (4 - 2x) \cos y = 0 & \rightarrow x = 2, y = \frac{(n+1)\pi}{2} \\ x(x - 4) \sin y = 0 & \rightarrow x = 0, 4, y = n\pi \end{cases}$$

$$\boxed{x = 2, y = 0} \quad \text{because } 1 \leq x \leq 3, \quad -\frac{\pi}{4} \leq y \leq \frac{\pi}{4}$$

The critical point is $(2, 0)$ and the value is $f(2, 0) = 4$

Values of all 4 corner points:

$$A\left(1, -\frac{\pi}{4}\right) \rightarrow f\left(1, -\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$$

$$B\left(1, \frac{\pi}{4}\right) \rightarrow f\left(1, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$$

$$C\left(3, \frac{\pi}{4}\right) \rightarrow f\left(3, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$$

$$A\left(3, -\frac{\pi}{4}\right) \rightarrow f\left(3, -\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$$

i. On the segment AB

$$f(1, y) = 3 \cos y \quad -\frac{\pi}{4} \leq y \leq \frac{\pi}{4}$$

$$f'(1, y) = -3 \sin y = 0 \rightarrow y = 0$$

$$x = 1 \rightarrow f(1, 0) = 3$$

ii. On the segment BC

$$f\left(x, \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} (4x - x^2) \quad 1 \leq x \leq 3$$

$$f'\left(x, \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} (4 - 2x) = 0 \Rightarrow x = 2$$

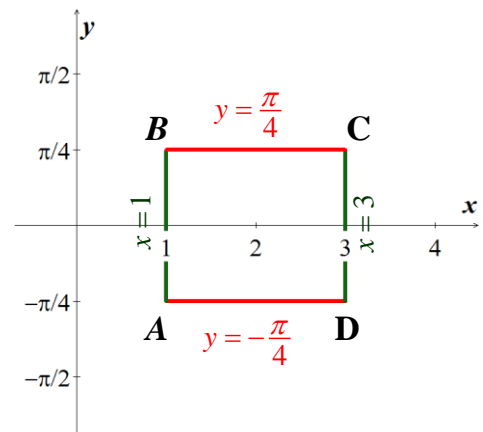
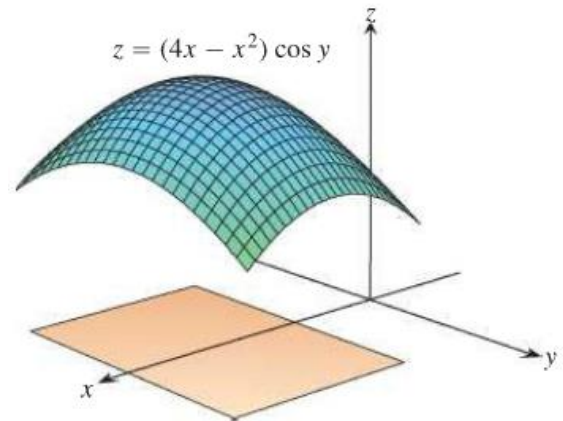
$$x = 2 \rightarrow f\left(2, \frac{\pi}{4}\right) = 2\sqrt{2}$$

iii. On the segment CD

$$f(3, y) = 3 \cos y \quad -\frac{\pi}{4} \leq y \leq \frac{\pi}{4}$$

$$f'(3, y) = -3 \sin y = 0 \rightarrow y = 0$$

iv. On the segment DA



$$f\left(x, -\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(4x - x^2) \quad 1 \leq x \leq 3$$

$$f'\left(x, -\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(4 - 2x) = 0 \Rightarrow x = 2$$

Therefore; the absolute **maximum** is 4 at $(2, 0)$ and the absolute **minimum** is $\frac{3\sqrt{2}}{2}$ at

$$\left(1, -\frac{\pi}{4}\right), \left(1, \frac{\pi}{4}\right), \left(3, -\frac{\pi}{4}\right), \text{ and } \left(3, \frac{\pi}{4}\right)$$

Exercise

Find the point on the graph of $z = x^2 + y^2 + 10$ nearest the plane $x + 2y - z = 0$

Solution

The point on $z = x^2 + y^2 + 10$ where the tangent plane is parallel to the plane $x + 2y - z = 0$.

Let $w = z - x^2 - y^2 - 10 \rightarrow \nabla w = -2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$ is normal to $z = x^2 + y^2 + 10$ at (x, y) .

The vector ∇w is parallel to $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ which is normal to the plane if $x = \frac{1}{2}$ and $y = 1$

$$(-2x = -1 \text{ and } -2y = -2), z = \left(\frac{1}{2}\right)^2 + 1^2 + 10 = \frac{45}{4}$$

Thus, the point $\left(\frac{1}{2}, 1, \frac{45}{4}\right)$ is the point on the surface $z = x^2 + y^2 + 10$ nearest the plane $x + 2y - z = 0$

Exercise

Find the minimum distance from the point $(2, -1, 1)$ to the plane $x + y - z = 2$

Solution

$$d(x, y, z) = \sqrt{(x-2)^2 + (y+1)^2 + (z-1)^2}$$

$$x + y - z = 2 \Rightarrow z = x + y - 2$$

$$\text{Let: } D(x, y, z) = (x-2)^2 + (y+1)^2 + (z-1)^2$$

$$\begin{aligned} D(x, y) &= (x-2)^2 + (y+1)^2 + (x+y-2-1)^2 \\ &= (x-2)^2 + (y+1)^2 + (x+y-3)^2 \end{aligned}$$

$$\begin{aligned} D_x &= 2(x-2) + 2(x+y-3) \\ &= 4x + 2y - 10 = 0 \end{aligned}$$

$$\begin{aligned} D_y &= 2(y+1) + 2(x+y-3) \\ &= 2x + 4y - 4 = 0 \end{aligned}$$

$$\begin{cases} 4x + 2y = 10 \\ 2x + 4y = 4 \end{cases} \Rightarrow \boxed{x = \frac{8}{3}, y = -\frac{1}{3}}$$

∴ The critical point is $\left(\frac{8}{3}, -\frac{1}{3}\right)$.

$$\underline{z} = \frac{8}{3} - \frac{1}{3} - 2 = \underline{\frac{1}{3}}$$

$$D_{xx} \left| \left(\frac{8}{3}, -\frac{1}{3}\right) \right. = 4, \quad D_{yy} \left| \left(\frac{8}{3}, -\frac{1}{3}\right) \right. = 4, \quad D_{xy} \left| \left(\frac{8}{3}, -\frac{1}{3}\right) \right. = 2$$

$$D_{xx}D_{yy} - D_{xy}^2 = (4)(4) - 2^2 = 12 > 0 \quad \text{and} \quad D_{xx} > 0$$

Therefore, the local **minimum** of the distance is

$$\begin{aligned} d\left(\frac{8}{3}, -\frac{1}{3}, \frac{1}{3}\right) &= \sqrt{\left(\frac{8}{3} - 2\right)^2 + \left(-\frac{1}{3} + 1\right)^2 + \left(\frac{1}{3} - 1\right)^2} \\ &= \underline{\frac{2}{\sqrt{3}}} \end{aligned}$$

Exercise

Find the maximum value of $s = xy + yz + xz$ where $x + y + z = 6$

Solution

$$x + y + z = 6 \Rightarrow z = 6 - x - y$$

$$s(x, y, z) = xy + yz + xz$$

$$\begin{aligned} s(x, y) &= xy + y(6 - x - y) + x(6 - x - y) \\ &= xy + 6y - xy - y^2 + 6x - x^2 - xy \\ &= -x^2 - y^2 + 6y + 6x - xy \end{aligned}$$

$$s_x = -2x + 6 - y = 0 \quad s_y = -2y + 6 - x = 0$$

$$\begin{cases} 2x + y = 6 \\ x + 2y = 6 \end{cases} \Rightarrow \boxed{x = 2, y = 2}$$

∴ The critical point is $(2, 2)$.

$$\underline{z} = 6 - 2 - 2 = \underline{2}$$

$$s_{xx} \left| (2, 2) \right. = -2, \quad s_{yy} \left| (2, 2) \right. = -2, \quad s_{xy} \left| (2, 2) \right. = -1$$

$$s_{xx}s_{yy} - s_{xy}^2 = (-2)(-2) - (-1)^2 = 3 > 0 \quad \text{and} \quad s_{xx} < 0$$

Therefore, the local *maximum* of the distance is

$$s(2, 2, 2) = (2)(2) + (2)(2) + (2)(2) \\ = 12$$

Exercise

Among all triangles with a perimeter of 9 *units*, find the dimensions of the triangle with the maximum area. It may be easiest to use Heron's formula, which states that the area of a triangle with side length a , b , and c is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $2s$ is the perimeter of the triangle.

Solution

The semi-perimeter is: $s = \frac{a+b+c}{2}$

$$c = 2s - a - b$$

$$A^2 = s(s-a)(s-b)(s-2s+a+b) \\ = s(s-a)(s-b)(a+b-s)$$

$$f(a, b) = s(s-a)(s-b)(a+b-s)$$

$$f'_a = s(s-b)(-a-b+s) + s(s-a)(s-b) \\ = s(s-b)(-a-b+s+s-a) \\ = s(s-b)(2s-2a-b) = 0$$

$$f'_b = s(s-a)(-a-b+s) + s(s-a)(s-b) \\ = s(s-a)(-a-b+s+s-b) \\ = s(s-a)(2s-a-2b) = 0$$

$$\begin{cases} (s-b)(2s-2a-b) = 0 & \rightarrow b = s & 2a+b = 2s \\ (s-a)(2s-a-2b) = 0 & \rightarrow a = s & a+2b = 2s \end{cases}$$

$$\begin{cases} 2a+b = 2s \\ a+2b = 2s \end{cases} \quad \Delta = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 \quad \Delta_a = \begin{vmatrix} 2s & 1 \\ 2s & 2 \end{vmatrix} = 2s \quad \Delta_b = \begin{vmatrix} 2 & 2s \\ 1 & 2s \end{vmatrix} = 2s$$

$$\rightarrow a = b = \frac{2}{3}s$$

$$c = 2s - 2\frac{2}{3}s \\ = \frac{2}{3}s$$

$$a = b = c = \frac{2}{3}s \quad \therefore \text{Equilateral triangle}$$

The maximum area is obtained when all three sides are equal with each side length is 3 *units* (since the perimeter is 9 *units*).

Exercise

Let P be a plane tangent to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ at a point in the first octant. Let T be the tetrahedron in the first octant bounded by P and the coordinate planes $x = 0$, $y = 0$, and $z = 0$. Find the minimum volume T . (the volume of a tetrahedron is one-third the area of the base times the height.)

Solution

Let $Q(x_0, y_0, z_0)$ be a point on the ellipsoid.

The tangent plane P at the point Q has an equation:

$$\frac{x_0}{a^2}x + \frac{y_0}{b^2}y + \frac{z_0}{c^2}z = 1$$

The intersection points of the plane with the axes are:

$$\left(\frac{a^2}{x_0}, 0, 0\right), \left(0, \frac{b^2}{y_0}, 0\right), \text{ and } \left(0, 0, \frac{c^2}{z_0}\right)$$

\therefore The tetrahedron T has base area

$$A = \frac{a^2 b^2}{2x_0 y_0}$$

$$\text{Height: } h = \frac{c^2}{z_0}$$

$$\begin{aligned}\therefore V &= \frac{1}{3} \frac{a^2 b^2 c^2}{2x_0 y_0 z_0} \\ &= \frac{1}{6} \frac{a^2 b^2 c^2}{x_0 y_0 z_0}\end{aligned}$$

$$\frac{x_0}{a^2}x + \frac{y_0}{b^2}y + \frac{z_0}{c^2}z = 1 \rightarrow \frac{1}{a^2}x_0^2 + \frac{1}{b^2}y_0^2 + \frac{1}{c^2}z_0^2 = 1$$

$$z_0^2 = c^2 \left(1 - \frac{1}{a^2}x_0^2 - \frac{1}{b^2}y_0^2\right)$$

$$\begin{aligned}V &= \frac{1}{6} \frac{a^2 b^2 c^2}{(x_0 y_0 c) \sqrt{\frac{a^2 b^2 - b^2 x_0^2 - a^2 y_0^2}{a^2 b^2}}} \\ &= \frac{a^3 b^3 c}{6 x_0 y_0} \frac{1}{\sqrt{a^2 b^2 - b^2 x_0^2 - a^2 y_0^2}}\end{aligned}$$

$$= \frac{a^3 b^3 c}{6} \left(a^2 b^2 x_0^2 y_0^2 - b^2 x_0^4 y_0^2 - a^2 x_0^2 y_0^4\right)^{-1/2}$$

$$V_{x_0} = -\frac{a^3 b^3 c}{12} \frac{2a^2 b^2 x_0 y_0^2 - 4b^2 x_0^3 y_0^2 - 2a^2 x_0 y_0^4}{\left(a^2 b^2 x_0^2 y_0^2 - b^2 x_0^4 y_0^2 - a^2 x_0^2 y_0^4\right)^{3/2}} = 0$$

$$V_{y_0} = -\frac{a^3 b^3 c}{12} \frac{2a^2 b^2 x_0^2 y_0 - 2b^2 x_0^4 y_0 - 4a^2 x_0^2 y_0^3}{\left(a^2 b^2 x_0^2 y_0^2 - b^2 x_0^4 y_0^2 - a^2 x_0^2 y_0^4\right)^{3/2}} = 0$$

$$\begin{cases} 2x_0 y_0^2 (a^2 b^2 - 2b^2 x_0^2 - a^2 y_0^2) = 0 \\ 2x_0^2 y_0 (a^2 b^2 - b^2 x_0^2 - 2a^2 y_0^2) = 0 \end{cases}$$

$$\begin{cases} 2b^2 x_0^2 + a^2 y_0^2 = a^2 b^2 \\ b^2 x_0^2 + 2a^2 y_0^2 = a^2 b^2 \end{cases}$$

$$\Delta = \begin{vmatrix} 2b^2 & a^2 \\ b^2 & 2a^2 \end{vmatrix} = 3a^2 b^2 \quad \Delta_x = \begin{vmatrix} a^2 b^2 & a^2 \\ a^2 b^2 & 2a^2 \end{vmatrix} = a^4 b^2 \quad \Delta_y = \begin{vmatrix} 2b^2 & a^2 b^2 \\ b^2 & a^2 b^2 \end{vmatrix} = a^2 b^4$$

$$x_0^2 = \frac{a^4 b^2}{3a^2 b^2} = \frac{a^2}{3}, \quad y_0^2 = \frac{a^2 b^4}{3a^2 b^2} = \frac{b^2}{3}$$

$$\begin{aligned} z_0^2 &= c^2 \left(1 - \frac{1}{a^2} \frac{a^2}{3} - \frac{1}{b^2} \frac{b^2}{3} \right) \\ &= c^2 \left(1 - \frac{1}{3} - \frac{1}{3} \right) \\ &= \frac{1}{3} c^2 \end{aligned}$$

$$\underline{x_0 = \frac{a}{\sqrt{3}}, \quad y_0 = \frac{b}{\sqrt{3}}, \quad z_0 = \frac{c}{\sqrt{3}}}$$

$$\begin{aligned} V &= \frac{1}{6} \frac{a^2 b^2 c^2}{\frac{a}{\sqrt{3}} \frac{b}{\sqrt{3}} \frac{c}{\sqrt{3}}} \\ &= \frac{\sqrt{3}}{2} abc \end{aligned}$$

Exercise

Given three distinct noncollinear points A , B , and C in the plane, find the point P in the plane such the sum of the distances $|AP| + |BP| + |CP|$ is a minimum. Here is how to proceed with three points, assuming that the triangle formed by the three points has no angle greater than $\left(120^\circ = \frac{2\pi}{3}\right)$

- Assume the coordinates of the three given points are $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$.
Let $d_1(x, y)$ be the distance between $A(x_1, y_1)$ and a variable point $P(x, y)$. Compute the gradient of d_1 and show that it is a unit vector pointing along the line between the two points.
- Define d_2 and d_3 in a similar way and show that ∇d_2 and ∇d_3 are also unit vectors in the direction of line between the two points.
- The goal is to minimize $f(x, y) = d_1 + d_2 + d_3$. Show that the condition $f_x = f_y = 0$ implies that $\nabla d_1 + \nabla d_2 + \nabla d_3 = 0$.
- Explain why part (c) implies that the optimal point P has the property the three line segments AP , BP , and CP all intersect symmetrically in angles of $\frac{2\pi}{3}$.
- What is the optimal solution if one of the angles in the triangle is greater than $\frac{2\pi}{3}$ (draw a picture)?
- Estimate the Steiner point for the three points $(0, 0)$, $(0, 1)$, $(2, 0)$

Solution

$$\begin{aligned}
 a) \quad d_1(x, y) &= \sqrt{(x - x_1)^2 + (y - y_1)^2} \\
 \nabla d_1(x, y) &= \frac{x - x_1}{\sqrt{(x - x_1)^2 + (y - y_1)^2}} \hat{i} + \frac{y - y_1}{\sqrt{(x - x_1)^2 + (y - y_1)^2}} \hat{j} \\
 &= \frac{x - x_1}{d_1(x, y)} \hat{i} + \frac{y - y_1}{d_1(x, y)} \hat{j} \\
 |\nabla d_1(x, y)| &= \frac{1}{d_1(x, y)} \sqrt{(x - x_1)^2 + (y - y_1)^2} \\
 &= \frac{d_1(x, y)}{d_1(x, y)} \\
 &= 1
 \end{aligned}$$

\therefore The gradient of d_1 is a unit vector

$$b) \quad d_2(x, y) = \sqrt{(x - x_2)^2 + (y - y_2)^2}$$

$$\nabla d_2(x, y) = \frac{x - x_2}{d_2(x, y)} \hat{i} + \frac{y - y_2}{d_2(x, y)} \hat{j}$$

$$\begin{aligned} |\nabla d_2(x, y)| &= \frac{1}{d_2(x, y)} \sqrt{(x - x_2)^2 + (y - y_2)^2} \\ &= \frac{d_2(x, y)}{d_2(x, y)} \\ &= 1 \end{aligned}$$

$\therefore \nabla d_2$ is a unit vector

$$d_3(x, y) = \sqrt{(x - x_3)^2 + (y - y_3)^2}$$

$$\nabla d_3(x, y) = \frac{x - x_3}{d_3(x, y)} \hat{i} + \frac{y - y_3}{d_3(x, y)} \hat{j}$$

$$\begin{aligned} |\nabla d_3(x, y)| &= \frac{1}{d_3(x, y)} \sqrt{(x - x_3)^2 + (y - y_3)^2} \\ &= \frac{d_3(x, y)}{d_3(x, y)} \\ &= 1 \end{aligned}$$

$\therefore \nabla d_3$ is a unit vector

c) $f(x, y) = d_1 + d_2 + d_3$

$$\nabla f = \nabla d_1 + \nabla d_2 + \nabla d_3$$

Given that $f_x = f_y = 0$

$$\nabla f = f_x \hat{i} + f_y \hat{j} = \mathbf{0}$$

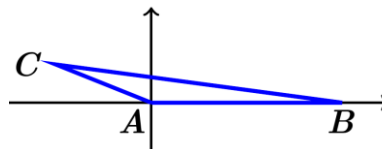
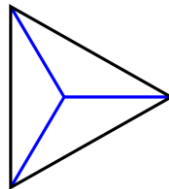
$$= \nabla d_1 + \nabla d_2 + \nabla d_3$$

$$\nabla d_1 + \nabla d_2 + \nabla d_3 = \mathbf{0} \quad \checkmark$$

d) Since $\nabla d_1 + \nabla d_2 + \nabla d_3 = \mathbf{0}$ that implies all the 3 unit vectors add to 0.

Therefore, all three divide the unit circle into 3 equal sectors, they must make angles of $\pm \frac{2\pi}{3}$.

e) The optimal point is the vertex at the large angle.



f) Three points $(0, 0)$, $(0, 1)$, $(2, 0)$

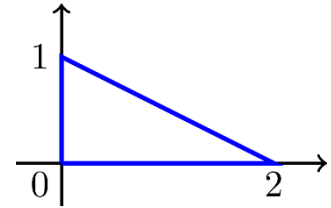
$$f_x = f_y = 0$$

$$f(x, y) = d_1 + d_2 + d_3$$

$$\begin{aligned} &= \sqrt{(x-x_1)^2 + (y-y_1)^2} + \sqrt{(x-x_2)^2 + (y-y_2)^2} + \sqrt{(x-x_3)^2 + (y-y_3)^2} \\ &= \sqrt{x^2 + y^2} + \sqrt{x^2 + (y-1)^2} + \sqrt{(x-2)^2 + y^2} \end{aligned}$$

$$f_x = \frac{x}{\sqrt{x^2 + y^2}} + \frac{x}{\sqrt{x^2 + (y-1)^2}} + \frac{x-2}{\sqrt{(x-2)^2 + y^2}} = 0$$

$$f_y = \frac{y}{\sqrt{x^2 + y^2}} + \frac{y-1}{\sqrt{x^2 + (y-1)^2}} + \frac{y}{\sqrt{(x-2)^2 + y^2}} = 0$$



Using maple:

$$\text{solvefor} \left\{ \left\{ \begin{aligned} \frac{x}{\sqrt{x^2 + y^2}} + \frac{x}{\sqrt{x^2 + (y-1)^2}} + \frac{x-2}{\sqrt{(x-2)^2 + y^2}} &= 0, \\ \frac{y}{\sqrt{x^2 + y^2}} + \frac{y-1}{\sqrt{x^2 + (y-1)^2}} + \frac{y}{\sqrt{(x-2)^2 + y^2}} &= 0 \end{aligned} \right. \right\}$$

$$x = \frac{1}{13} + \frac{4\sqrt{3}}{39} \approx 0.25456931$$

$$y = \frac{8}{13} - \frac{7\sqrt{3}}{39} \approx 0.30450371$$

Exercise

Show that the following two functions have two local maxima but no other extreme points (thus no saddle or basin between the mountains).

$$f(x, y) = -(x^2 - 1)^2 - (x^2 - e^y)^2$$

Solution

$$\begin{aligned} f_x &= -4x(x^2 - 1) - 4x(x^2 - e^y) \\ &= -4x^3 + 4x - 4x^3 + 4xe^y \\ &= -8x^3 + 4x(1 + e^y) = 0 \end{aligned}$$

$$f_y = 2e^y(x^2 - e^y) = 0$$

$$\begin{cases} -4x(2x^2 - 1 - e^y) = 0 \rightarrow x = 0, e^y = 2x^2 - 1 & (1) \\ 2e^y(x^2 - e^y) = 0 & e^y = x^2 & (2) \end{cases}$$

$$(1) \rightarrow e^y = 2x^2 - 1 \Big|_{x=0} = \cancel{-1}$$

$$\text{from } (2) \rightarrow (1): x^2 = 2x^2 - 1$$

$$x^2 = 1 \rightarrow \underline{x = \pm 1}$$

$$e^y = x^2 \Big|_{x=\pm 1} = 1 \rightarrow \underline{y = 0}$$

$$\therefore \text{C.P.: } (\pm 1, 0)$$

$$f_{xx} = -24x^2 + 4(1 + e^y)$$

$$f_{yy} = 2x^2 e^y - 4e^{2y}$$

$$f_{xy} = 4xe^y$$

$$f_{xx} f_{yy} - f_{xy}^2 = (-24x^2 + 4 + 4e^y)(2x^2 e^y - 4e^{2y}) - 16x^2 e^{2y}$$

$$@ \quad (-1, 0)$$

$$f_{xx} f_{yy} - f_{xy}^2 = (-24 + 4 + 4)(2 - 4) - 16 = 16 > 0$$

$$f_{xx} = -24 + 8 = -16 < 0$$

$$f \text{ has a } \text{local Max} @ (-1, 0)$$

$$@ \quad (1, 0)$$

$$f_{xx} f_{yy} - f_{xy}^2 = (-24 + 4 + 4)(2 - 4) - 16 = 16 > 0$$

$$f_{xx} = -24 + 8 = -16 < 0$$

$$f \text{ has a } \text{local Max} @ (1, 0)$$

Exercise

Show that the following two functions have two local maxima but no other extreme points (thus no saddle or basin between the mountains).

$$f(x, y) = 4x^2 e^y - 2x^4 - e^{4y}$$

Solution

$$f_x = 8xe^y - 8x^3 = 0$$

$$f_y = 4x^2 e^y - 4e^{4y} = 0$$

$$\begin{cases} 8x(e^y - x^2) = 0 & \rightarrow x = 0, e^y = x^2 \quad (1) \\ 4e^y(x^2 - e^{3y}) = 0 & e^{3y} = x^2 \quad (2) \end{cases}$$

$$(1) \rightarrow e^y = x^2 \Big|_{x=0} = \cancel{0}$$

$$\text{from } (2) \rightarrow (1): e^{3y} = e^y \Rightarrow 3y = y$$

$$\underline{y=0} \rightarrow e^y = x^2 = 1 \quad \underline{x=\pm 1}$$

$\therefore C.P.: (\pm 1, 0)$

$$f_{xx} = 8e^y - 24x^2$$

$$f_{yy} = 4x^2 e^y - 16e^{4y}$$

$$f_{xy} = 8xe^y$$

$$f_{xx}f_{yy} - f_{xy}^2 = (8e^y - 24x^2)(4x^2 e^y - 16e^{4y}) - 64x^2 e^{2y}$$

@ $(-1, 0)$

$$f_{xx}f_{yy} - f_{xy}^2 = (8 - 24)(4 - 16) - 64 = 128 > 0$$

$$f_{xx} = 8 - 24 = -16 < 0$$

f has a **local Max** @ $(-1, 0)$

@ $(1, 0)$

$$f_{xx}f_{yy} - f_{xy}^2 = (8 - 24)(4 - 16) - 64 = 128 > 0$$

$$f_{xx} = 8 - 24 = -16 < 0$$

f has a **local Max** @ $(1, 0)$