Solution Section 4.1 – System of linear Equations

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 3x+2y=-4 \\ 2x-y=-5 \end{cases}$$

Solution

$$\begin{cases} \quad 3x+2y=-4 \\ 2\times \quad 2x-y=-5 \end{cases}$$

$$\begin{array}{r} 3x+2y=-4 \\ \underline{4x-2y=-10} \\ 7x=-14 \end{array}$$

$$\underline{x=-2}|$$

$$\begin{aligned} y &= 2x+5 \\ &= -4+5 \\ &= 1 \end{aligned}$$

Solution: $\underline{(-2,\ 1)}|$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 2x+5y=7 \\ 5x-2y=-3 \end{cases}$$

Solution

$$\begin{cases} -5\times \quad 2x+5y=7 \\ 2\times \quad 5x-2y=-3 \end{cases}$$

$$\begin{array}{r} -10x-25y=-35 \\ \underline{10x-4y=-6} \\ -29y=-41 \end{array}$$

$$\underline{y=\frac{41}{29}}|$$

$$x=\frac{1}{2}\left(7-5\left(\frac{41}{29}\right)\right)$$

$$x=\frac{1}{2}\left(-\frac{2}{29}\right)$$

$$= -\frac{1}{29}$$

$\therefore$ ***Solution***: $\quad \left(-\frac{1}{29},\ \frac{41}{29}\right)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$$

Solution

$$\begin{cases} \qquad 4x - 7y = -16 \\ -2\times \quad 2x + 5y = 9 \end{cases}$$

$$\begin{array}{r} 4x - 7y = -16 \\ -4x - 10y = -18 \\ \hline -17y = -34 \end{array}$$

$$y = 2$$

$$x = \frac{9 - 5y}{2}$$

$$= \frac{9 - 10}{2}$$

$$= -\frac{1}{2}$$

$\therefore$ ***Solution***: $\quad \left(-\frac{1}{2},\ 2\right)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$$

Solution

$$\begin{cases} 3x + 2y = 4 & (1) \\ 2x + y = 1 & (2) \end{cases}$$

$$(2) \rightarrow \quad y = 1 - 2x \quad (3)$$

$$(1) \rightarrow \quad 3x + 2 - 4x = 4$$

$\underline{x = -2}|$

$(3) \rightarrow \quad y = 1 + 4$

$\quad\quad = 5|$

$\therefore$ ***Solution***: $\quad \underline{(-2,\ 5)}|$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 3x + 4y = 2 \\ 2x + 5y = -1 \end{cases}$$

Solution

$$\begin{cases} -2\times \quad 3x + 4y = 2 \\ 3\times \quad 2x + 5y = -1 \end{cases}$$

$$\begin{array}{r} -6x - 8y = -4 \\ \underline{6x + 15y = -3} \\ 7y = -7 \end{array}$$

$\underline{y = -1}|$

$2x = -1 + 5$

$x = \dfrac{4}{2}$

$\quad = 2|$

$\therefore$ ***Solution***: $\quad \underline{(2,\ -1)}|$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method) $\begin{cases} 5x - 2y = 4 \\ -10x + 4y = 7 \end{cases}$

Solution

$$\begin{cases} 2\times \quad 5x - 2y = 4 \\ \quad -10x + 4y = 7 \end{cases}$$

$$\begin{array}{r} 10x - 4y = 8 \\ \underline{-10x + 4y = 7} \\ 0 = 15 \end{array} \quad (impossible)$$

$\therefore$ ***No Solution***

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} x-4y=-8 \\ 5x-20y=-40 \end{cases}$$

Solution

$$\begin{cases} x-4y=-8 & (1) \\ 5x-20y=-40 & (2) \end{cases}$$

$(1) \rightarrow \quad x=4y-8$

$(2) \rightarrow \quad 5(4y-8)-20y=-40$

$20y-40-20y=-40$

$-40=-40 \quad (True)$

$\therefore$ ***Solution***: $\quad \underline{x-4y=-8}$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$$

Solution

$$\begin{cases} 2x+y=3 & (1) \\ x-y=3 & (2) \end{cases}$$

$(2) \rightarrow \quad x=3+y \quad (3)$

$(1) \rightarrow \quad 6+2y+y=3$

$3y=-3$

$\underline{y=-1}$

$(3) \rightarrow \quad \underline{x=2}$

$\therefore$ ***Solution***: $\quad \underline{(2,\ -1)}$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 2x + 10y = -14 \\ 7x - 2y = -16 \end{cases}$$

Solution

$$\begin{cases} \quad 2x + 10y = -14 \\ 5\times \quad 7x - 2y = -16 \end{cases}$$

$$\begin{array}{r} 2x + 10y = -14 \\ \underline{35x - 10y = -80} \\ 37x = -94 \end{array}$$

$$\underline{x = -\frac{94}{37}|}$$

$$2y = 7\left(-\frac{94}{37}\right) + 16$$

$$y = -\frac{329}{37} + 8$$

$$\underline{= -\frac{33}{37}|}$$

$\therefore$ ***Solution***: $\underline{\left(-\frac{94}{37},\ -\frac{33}{37}\right)|}$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$$

Solution

$$\begin{cases} 3\times \quad 4x - 3y = 24 \\ \quad -3x + 9y = -1 \end{cases}$$

$$\begin{array}{r} 12x - 9y = 72 \\ \underline{-3x + 9y = -1} \\ -9x = -71 \end{array}$$

$$\underline{x = \frac{71}{9}|}$$

$$3y = 4\left(\frac{71}{9}\right) - 24$$

$$y = \frac{284}{27} - 8$$

$= \dfrac{68}{27}$

$\therefore$ ***Solution***: $\left(\dfrac{71}{9},\ \dfrac{68}{27}\right)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$$

Solution

$$\begin{array}{r} 4x + 2y = 12 \\ \underline{3x - 2y = 16} \\ 7x = 28 \end{array}$$

$x = 4$

$2y = 12 - 4(4)$

$y = -\dfrac{4}{2}$

$\quad = -2$

$\therefore$ ***Solution***: $(4,\ -2)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$$

Solution

$$\begin{array}{r} x + 2y = -1 \\ \underline{4x - 2y = 6} \\ 5x = 5 \end{array}$$

$x = 1$

$2y = -x - 1$

$y = -\dfrac{2}{2}$

$\quad = -1$

$\therefore$ ***Solution***: $(1,\ -1)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} x-2y=5 \\ -10x+2y=4 \end{cases}$$

Solution

$$\begin{array}{r} x-2y=5 \\ \underline{-10x+2y=4} \\ -9x=9 \end{array}$$

$\underline{x=-1}|$

$2y=x-5$

$y=-\frac{6}{2}$

$\quad \underline{=-3}|$

$\therefore$ ***Solution***: $\quad \underline{(-1,\ -3)}|$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 12x+15y=-27 \\ 30x-15y=-15 \end{cases}$$

Solution

$$\begin{array}{r} 12x+15y=-27 \\ \underline{30x-15y=-15} \\ 42x=-42 \end{array}$$

$\underline{x=-1}|$

$15y=-27-12(-1)$

$y=-\frac{15}{15}$

$\quad \underline{=-1}|$

$\therefore$ ***Solution***: $\quad \underline{(-1,\ -1)}|$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$$

Solution

$$\begin{array}{r} 4x - 4y = -12 \\ \underline{4x + 4y = -20} \\ 8x = -32 \end{array}$$

$$\underline{x = -4|}$$

$$4y = 4(-4) + 12$$

$$y = -\frac{4}{4}$$

$$\underline{= -1|}$$

$\therefore$ ***Solution***: $\quad \underline{(-4,\ -1)\ |}$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 1 & 4 & 7 \\ 3 & 5 & 0 \end{array}\right] \quad R_2 - 3R_1$$

Solution

$$\begin{array}{ccc} 3 & 5 & 0 \\ -3 & -12 & -21 \\ \hline 0 & -7 & -21 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 4 & 7 \\ 0 & -7 & -21 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -5 \end{array}\right] \quad R_2 - 2R_1$$

Solution

$$\begin{array}{ccc} 2 & 1 & -5 \\ -2 & 6 & -2 \\ \hline 0 & 7 & -7 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 7 & -7 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 5 & 2 & 19 \end{array}\right] \quad R_2 - 5R_1$$

Solution

$$\begin{array}{ccc} 5 & 2 & 19 \\ -5 & 15 & -15 \\ \hline 0 & 17 & -4 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 17 & -4 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 2 & -3 & 8 \\ -6 & 9 & 4 \end{array}\right] \quad R_2 + 3R_1$$

Solution

$$\begin{array}{ccc} -6 & 9 & 4 \\ 6 & -9 & 24 \\ \hline 0 & 0 & 28 \end{array}$$

$$\left[\begin{array}{cc|c} 2 & -3 & 8 \\ 0 & 0 & 28 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 2 & 3 & 11 \\ 1 & 2 & 8 \end{array}\right] \quad 2R_2 - R_1$$

Solution

$$\begin{array}{ccc} 2 & 4 & 16 \\ -2 & -3 & -11 \\ \hline 0 & 1 & 5 \end{array}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 11 \\ 0 & 1 & 5 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 3 & 5 & -13 \\ 2 & 3 & -9 \end{array}\right] \quad 3R_2 - 2R_1$$

Solution

$$\begin{array}{ccc} 6 & 9 & -27 \\ -6 & -10 & 26 \\ \hline 0 & -1 & -1 \end{array}$$

$$\left[\begin{array}{cc|c} 3 & 5 & -13 \\ 0 & -1 & -1 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 5 & 4 & 1 \end{array}\right] \quad R_3 - 5R_2$$

Solution

$$\begin{array}{cccc} 0 & 5 & 4 & 1 \\ 0 & -5 & 5 & -10 \\ \hline 0 & 0 & 9 & -9 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 9 & -9 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array}\right] \quad \begin{array}{c} R_2 - 3R_1 \\ R_3 - R_1 \end{array}$$

Solution

$$\begin{array}{cccc} 3 & 3 & -1 & 10 \\ -3 & 3 & -15 & 18 \\ \hline 0 & 6 & -16 & 28 \end{array} \qquad\qquad \begin{array}{cccc} 1 & 3 & 2 & 5 \\ -1 & 1 & -5 & 6 \\ \hline 0 & 4 & -3 & 11 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 0 & 4 & -3 & 11 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 2 & 4 & 4 & 22 \\ -1 & -2 & 3 & 15 \end{array}\right] \quad \begin{array}{c} 3R_2 - 2R_1 \\ 3R_3 + R_1 \end{array}$$

Solution

$$\begin{array}{cccc} 6 & 12 & 12 & 66 \\ -6 & -4 & -2 & -2 \\ \hline 0 & 8 & 10 & 64 \end{array} \qquad\qquad \begin{array}{cccc} -3 & -6 & 9 & 45 \\ 3 & 2 & 1 & 1 \\ \hline 0 & -4 & 10 & 46 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 0 & 8 & 10 & 64 \\ 0 & -4 & 10 & 46 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & -4 & 2 & -7 \end{array}\right] \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

Solution

$$\begin{array}{cccc} 2 & 1 & 1 & 3 \\ -2 & -2 & -2 & -4 \\ \hline 0 & -1 & -1 & -1 \end{array} \qquad \begin{array}{cccc} 3 & -4 & 2 & -7 \\ -3 & -3 & -3 & -6 \\ \hline 0 & -7 & -1 & -13 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & -1 \\ 0 & -7 & -1 & -13 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & -2 \\ 2 & -3 & 5 & -1 & 0 \\ 1 & 0 & 3 & 1 & -4 \\ -4 & 3 & 2 & -1 & 3 \end{array}\right] \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 + 4R_1 \end{array}$$

Solution

$$\begin{array}{ccccc} 2 & -3 & 5 & -1 & 0 \\ -2 & 4 & -2 & -6 & 4 \\ \hline 0 & 1 & 3 & -7 & 4 \end{array} \qquad \begin{array}{ccccc} 1 & 0 & 3 & 1 & -4 \\ -1 & 2 & -1 & -3 & 2 \\ \hline 0 & 2 & 2 & -2 & -2 \end{array} \qquad \begin{array}{ccccc} -4 & 3 & 2 & -1 & 3 \\ 4 & -8 & 4 & 12 & -8 \\ \hline 0 & -5 & 6 & 11 & -5 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & -2 \\ 0 & 1 & 3 & -7 & 4 \\ 0 & 2 & 2 & -2 & -2 \\ 0 & -5 & 6 & 11 & -5 \end{array}\right]$$

Exercise

Use the Gauss-Jordan method to solve the system
$$\begin{aligned} x-y+5z&=-6\\ 3x+3y-z&=10\\ x+3y+2z&=5\end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6\\ 3 & 3 & -1 & 10\\ 1 & 3 & 2 & 5\end{array}\right] \begin{array}{l} \\ R_2-3R_1\\ R_3-R_1\end{array}$$

$$\begin{array}{cccc} 3 & 3 & -1 & 10\\ -3 & 3 & -15 & 18\\ \hline 0 & 6 & -16 & 28\end{array} \qquad \begin{array}{cccc} 1 & 3 & 2 & 5\\ -1 & 1 & -5 & 6\\ \hline 0 & 4 & -3 & 11\end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6\\ 0 & 6 & -16 & 28\\ 0 & 4 & -3 & 11\end{array}\right] \ \tfrac{1}{6}R_2 \qquad 0 \quad 1 \quad -\tfrac{8}{3} \quad \tfrac{14}{3}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6\\ 0 & 1 & -\frac{8}{3} & \frac{14}{3}\\ 0 & 4 & -3 & 11\end{array}\right] \begin{array}{l} R_1+R_2\\ \\ R_3-4R_2\end{array}$$

$$\begin{array}{cccc} 0 & 4 & -3 & 11\\ 0 & -4 & \frac{32}{3} & -\frac{56}{3}\\ \hline 0 & 0 & \frac{23}{3} & -\frac{23}{3}\end{array} \qquad \begin{array}{cccc} 1 & -1 & 5 & -6\\ 0 & 1 & -\frac{8}{3} & \frac{14}{3}\\ \hline 1 & 0 & \frac{7}{3} & -\frac{4}{3}\end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{3} & -\frac{4}{3}\\ 0 & 1 & -\frac{8}{3} & \frac{14}{3}\\ 0 & 0 & \frac{23}{3} & -\frac{23}{3}\end{array}\right] \begin{array}{l} \\ \\ \frac{3}{23}R_3\end{array} \qquad 0 \quad 0 \quad 1 \quad -1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{3} & -\frac{4}{3}\\ 0 & 1 & -\frac{8}{3} & \frac{14}{3}\\ 0 & 0 & 1 & -1\end{array}\right] \begin{array}{l} R_1-\frac{7}{3}R_3\\ R_2+\frac{8}{3}R_3\\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & \frac{7}{3} & -\frac{4}{3}\\ 0 & 0 & -\frac{7}{3} & \frac{7}{3}\\ \hline 1 & 0 & 0 & 1\end{array} \qquad \begin{array}{cccc} 0 & 1 & -\frac{8}{3} & \frac{14}{3}\\ 0 & 0 & \frac{8}{3} & -\frac{8}{3}\\ \hline 0 & 1 & 0 & 2\end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & -1\end{array}\right]$$

S***olution***: $(1,\ 2,\ -1)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} 2x-y+4z=-3 \\ x-2y-10z=-6 \\ 3x \qquad +4z=7 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & -1 & 4 & -3 \\ 1 & -2 & -10 & -6 \\ 3 & 0 & 4 & 7 \end{array}\right] \begin{array}{l} \frac{1}{2}R_1 \\ \\ \\ \end{array} \qquad \begin{array}{cccc} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 1 & -2 & -10 & -6 \\ 3 & 0 & 4 & 7 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 3R_1 \end{array} \qquad \begin{array}{cccc} 1 & -2 & -10 & -6 \\ -1 & \frac{1}{2} & -2 & \frac{3}{2} \\ \hline 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \end{array} \qquad \begin{array}{cccc} 3 & 0 & 4 & 7 \\ -3 & \frac{3}{2} & -6 & \frac{9}{2} \\ \hline 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \\ 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}\right] \begin{array}{l} \\ -\frac{2}{3}R_2 \\ \\ \end{array} \qquad \begin{array}{cccc} 0 & 1 & 8 & 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & 1 & 8 & 3 \\ 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}\right] \begin{array}{l} R_1 + \frac{1}{2}R_2 \\ \\ R_3 - \frac{3}{2}R_2 \end{array} \qquad \begin{array}{cccc} 0 & \frac{3}{2} & -2 & \frac{23}{2} \\ 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \\ \hline 0 & 0 & -14 & 7 \end{array} \qquad \begin{array}{cccc} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & \frac{1}{2} & 4 & \frac{3}{2} \\ \hline 1 & 0 & 6 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & -14 & 7 \end{array}\right] \begin{array}{l} \\ \\ -\frac{1}{14}R_3 \end{array} \qquad \begin{array}{cccc} 0 & 0 & 1 & -\frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right] \begin{array}{l} R_1 - 6R_3 \\ R_2 - 8R_3 \\ \\ \end{array} \qquad \begin{array}{cccc} 1 & 0 & 6 & 0 \\ 0 & 0 & -6 & 3 \\ \hline 1 & 0 & 0 & 3 \end{array} \qquad \begin{array}{cccc} 0 & 1 & 8 & 3 \\ 0 & 0 & -8 & 4 \\ \hline 0 & 1 & 0 & 7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right]$$

Solution: $\left(3,\ 7,\ -\frac{1}{2}\right)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} 4x+3y-5z=-29 \\ 3x-7y-z=-19 \\ 2x+5y+2z=-10 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 4 & 3 & -5 & -29 \\ 3 & -7 & -1 & -19 \\ 2 & 5 & 2 & -10 \end{array}\right] \quad \frac{1}{4}R_1$$

$$1 \quad \frac{3}{4} \quad -\frac{5}{4} \quad -\frac{29}{4}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 3 & -7 & -1 & -19 \\ 2 & 5 & 2 & -10 \end{array}\right] \quad \begin{array}{l} R_2-3R_1 \\ R_3-2R_1 \end{array}$$

$$\begin{array}{cccc} 3 & -7 & -1 & -19 \\ -3 & -\frac{9}{4} & \frac{15}{4} & \frac{87}{4} \\ \hline 0 & -\frac{37}{4} & \frac{11}{4} & \frac{11}{4} \end{array} \qquad \begin{array}{cccc} 2 & 5 & 2 & -10 \\ -2 & -\frac{3}{2} & \frac{5}{2} & \frac{29}{2} \\ \hline 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & -\frac{37}{4} & \frac{11}{4} & \frac{11}{4} \\ 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}\right] \quad -\frac{4}{37}R_2$$

$$0 \quad 1 \quad -\frac{11}{37} \quad -\frac{11}{37}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}\right] \quad \begin{array}{l} R_1-\frac{3}{4}R_2 \\ \\ R_3-\frac{7}{2}R_2 \end{array}$$

$$\begin{array}{cccc} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & -\frac{3}{4} & \frac{33}{148} & \frac{33}{148} \\ \hline 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \end{array} \qquad \begin{array}{cccc} 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \\ 0 & -\frac{7}{2} & \frac{77}{72} & \frac{77}{72} \\ \hline 0 & 0 & \frac{401}{72} & \frac{401}{72} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & \frac{401}{72} & \frac{401}{72} \end{array}\right] \quad \frac{72}{401}R_3$$

$$0 \quad 0 \quad 1 \quad 1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_1+\frac{38}{37}R_3 \\ R_2+\frac{11}{37}R_3 \end{array}$$

$$\begin{array}{cccc} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 0 & \frac{38}{37} & \frac{38}{37} \\ \hline 1 & 0 & 0 & -6 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & \frac{11}{37} & \frac{11}{37} \\ \hline 0 & 1 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Solution: $(-6,\ 0,\ 1)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} x+2y-3z=-15 \\ 2x-3y+4z=18 \\ -3x+y+z=1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 2 & -3 & 4 & 18 \\ -3 & 1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 2 & -3 & 4 & 18 \\ -3 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3+3R_1 \end{array}$$

$$\begin{array}{cccc} -2 & -4 & 6 & 30 \\ 2 & -3 & 4 & 18 \\ \hline 0 & -7 & 10 & 48 \end{array} \qquad \begin{array}{cccc} 3 & 6 & -9 & -45 \\ -3 & 1 & 1 & 1 \\ \hline 0 & 7 & -8 & -44 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 0 & -7 & 10 & 48 \\ 0 & 7 & -8 & -44 \end{array}\right] \quad -\frac{1}{7}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 7 & -8 & -44 \end{array}\right] \begin{array}{l} R_1-2R_2 \\ \\ R_3-7R_2 \end{array}$$

$$\begin{array}{cccc} 1 & 2 & -3 & -15 \\ 0 & -2 & \frac{20}{7} & \frac{96}{7} \\ \hline 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \end{array} \qquad \begin{array}{cccc} 0 & -7 & 10 & 48 \\ 0 & 7 & -8 & -44 \\ \hline 0 & 0 & 2 & 4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & 2 & 4 \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{2}R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1+\frac{1}{7}R_3 \\ R_2+\frac{10}{7}R_3 \\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 0 & \frac{1}{7} & \frac{2}{7} \\ \hline 1 & 0 & 0 & -1 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & \frac{10}{7} & \frac{20}{7} \\ \hline 0 & 1 & 0 & -4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solution: $(-1,\ -4,\ 2)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} x+2y+3z=10 \\ 4x+5y+6z=11 \\ 7x+8y+9z=12 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 4 & 5 & 6 & 11 \\ 7 & 8 & 9 & 12 \end{array}\right] \begin{array}{l} \\ R_2-4R_1 \\ R_3-7R_1 \end{array}$$

$$\begin{array}{cccc} -4 & -8 & -12 & -40 \\ 4 & 5 & 6 & 11 \\ \hline 0 & -3 & -6 & -29 \end{array} \qquad \begin{array}{cccc} -7 & -14 & -21 & -70 \\ 7 & 8 & 9 & 12 \\ \hline 0 & -6 & -12 & -58 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & -3 & -6 & -29 \\ 0 & -6 & -12 & -58 \end{array}\right] \begin{array}{l} \\ \frac{1}{3}R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & \frac{29}{3} \\ 0 & -6 & -12 & -58 \end{array}\right] \begin{array}{l} \\ \\ R_3+6R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -6 & -12 & -58 \\ 0 & 6 & 12 & 58 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & \frac{29}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

let z be the variable

From Row 1 $\Rightarrow y+2z=\frac{29}{3}$

$$\boxed{y=\frac{29}{3}-2z}$$

From Row 1 $\Rightarrow x+2y+3z=10$

$$x=10-2y-3z$$

$$x=10-2\left(\frac{29}{3}-2z\right)-3z$$

$$x=10-\frac{58}{3}+4z-3z$$

$$\boxed{x=z-\frac{28}{3}}$$

Solution: $\boxed{\left(z-\frac{28}{3},\ \frac{29}{3}-2z,\ z\right)}$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} 2x + y + 2z = 4 \\ 2x + 2y \qquad = 5 \\ 2x - y + 6z = 2 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 4 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array}\right] \begin{array}{l} \frac{1}{2}R_1 \\ \\ \\ \end{array} \qquad \begin{array}{cccc} 1 & \frac{1}{2} & 1 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{array} \qquad \begin{array}{cccc} 2 & 2 & 0 & 5 \\ -2 & -1 & -2 & -4 \\ \hline 0 & 1 & -2 & 1 \end{array} \qquad \begin{array}{cccc} 2 & -1 & 6 & 2 \\ -2 & -1 & -2 & -4 \\ \hline 0 & -2 & 4 & -2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 4 & -2 \end{array}\right] \begin{array}{l} R_1 - \frac{1}{2}R_2 \\ \\ R_3 + 2R_2 \end{array} \qquad \begin{array}{cccc} 1 & \frac{1}{2} & 1 & 2 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ \hline 1 & 0 & 2 & \frac{3}{2} \end{array} \qquad \begin{array}{cccc} 0 & -2 & 4 & -2 \\ 0 & 2 & -4 & 2 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & \frac{3}{2} \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From Row 3: $0 = 0$ is a true statement. Let z be the variable.

From Row 2: $y - 2z = 1$

$\underline{y = 1 + 2z}$

From Row 1: $x + 2z = \frac{3}{2}$

$\underline{x = -2z + \frac{3}{2}}$

$\therefore$ ***Solution***: $\underline{\left(-2z + \frac{3}{2},\ 2z + 1,\ z\right)}$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array} \qquad \begin{array}{cccc} -1 & -2 & 3 & 1 \\ 1 & 1 & 2 & 8 \\ \hline 0 & -1 & 5 & 9 \end{array} \qquad \begin{array}{cccc} 3 & -7 & 4 & 10 \\ -3 & -3 & -6 & -24 \\ \hline 0 & -10 & -2 & -14 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right] -R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 10R_2 \end{array} \qquad \begin{array}{cccc} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ \hline 1 & 0 & 7 & 17 \end{array} \qquad \begin{array}{cccc} 0 & -10 & -2 & -14 \\ 0 & 10 & -50 & -90 \\ \hline 0 & 0 & -52 & -104 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right] -\frac{1}{52}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 - 7R_3 \\ R_2 + 5R_3 \\ \\ \end{array} \qquad \begin{array}{cccc} 1 & 0 & 7 & 17 \\ 0 & 0 & -7 & -14 \\ \hline 1 & 0 & 0 & 3 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -5 & -9 \\ 0 & 0 & 5 & 10 \\ \hline 0 & 1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

∴ ***Solution***: $\underline{(3,\ 1,\ 2)}$

Exercise

Use augmented elimination to solve linear system
$$\begin{aligned} 2x-5y+3z&=1 \\ x-2y-2z&=8 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 2 & -5 & 3 & 1 \end{array}\right] \quad R_2-2R_1$$

$$\begin{array}{cccc} 2 & -5 & 3 & 1 \\ -2 & 4 & 4 & -16 \\ \hline 0 & -1 & 7 & -15 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 0 & -1 & 7 & -15 \end{array}\right] \quad -R_2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 0 & 1 & -7 & 15 \end{array}\right] \quad R_1+2R_2$$

$$\begin{array}{cccc} 1 & -2 & -2 & 8 \\ 0 & 2 & -14 & 30 \\ \hline 1 & 0 & -16 & 38 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -16 & 38 \\ 0 & 1 & -7 & 15 \end{array}\right] \begin{array}{l} \rightarrow \quad x-16z=38 \\ \rightarrow \quad y-7z=15 \end{array}$$

$$\begin{cases} x=16z+38 \\ y=7z+15 \end{cases}$$

$\therefore$ ***Solution***: $\underline{(16z+38,\ 7z+15,\ z)}$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} x+y+z=2 \\ 2x+y-z=5 \\ x-y+z=-2 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 5 \\ 1 & -1 & 1 & -2 \end{array}\right] \quad \begin{array}{l} R_2-2R_1 \\ R_3-R_1 \end{array}$$

$$\begin{array}{cccc} 2 & 1 & -1 & 5 \\ -2 & -2 & -2 & -4 \\ \hline 0 & -1 & -3 & 1 \end{array} \qquad \begin{array}{cccc} 1 & -1 & 1 & -2 \\ -1 & -1 & -1 & -2 \\ \hline 0 & -2 & 0 & -4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & 1 \\ 0 & -2 & 0 & -4 \end{array}\right] \quad \begin{array}{l} (2) \\ (1) \\ -2y=-4 \end{array}$$

$\underline{y=2}$

$(1) \rightarrow \quad -y-3z=1$

$3z=-1-2$

$\underline{z=-1}$

$(2) \to \quad x + y + z = 2$

$x = 2 - 2 + 1$

$= 1$

$\therefore$ ***Solution***: $(1,\ 2,\ -1)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ -1 & -1 & 1 & 1 \\ 3 & -1 & 1 & 9 \end{array}\right] \begin{array}{l} \\ 2R_2 + R_1 \\ 2R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} -2 & -2 & 2 & 2 \\ 2 & 1 & 1 & 9 \\ \hline 0 & -1 & 3 & 11 \end{array} \qquad \begin{array}{cccc} 6 & -2 & 2 & 18 \\ -6 & -3 & -3 & -27 \\ \hline 0 & -5 & -1 & -9 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ 0 & -1 & 3 & 11 \\ 0 & -5 & -1 & -9 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 5R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -5 & -1 & -9 \\ 0 & 5 & -15 & -55 \\ \hline 0 & 0 & -16 & -64 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ 0 & -1 & 3 & 11 \\ 0 & 0 & -16 & -64 \end{array}\right] \begin{array}{c} (2) \\ (1) \\ -16z = -64 \end{array}$$

$z = 4$

$(1) \to \quad -y + 3z = 11$

$y = 12 - 11$

$= 1$

$(2) \to \quad 2x + y + z = 9$

$2x = 9 - 1 - 4$

$x = 2$

$\therefore$ ***Solution***: $(2,\ 1,\ 4)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ -3 & 6 & 2 & 11 \end{array}\right] \quad R_3 + 3R_1$$

$$\begin{array}{cccc} -3 & 6 & 2 & 11 \\ 3 & 15 & -3 & -12 \\ \hline 0 & 21 & -1 & -1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ 0 & 21 & -1 & -1 \end{array}\right] \quad R_3 - 7R_2$$

$$\begin{array}{cccc} 0 & 21 & -1 & -1 \\ 0 & -21 & 7 & 7 \\ \hline 0 & 0 & 6 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ 0 & 0 & 6 & 6 \end{array}\right] \quad \begin{array}{ll} \to x + 5y - z = -4 & (2) \\ \to 3y - z = -1 & (1) \\ \to 6z = 6 & \end{array}$$

$\underline{z = 1}$

$(1) \to \quad 3y = -1 + 1$

$\qquad \underline{y = 0}$

$(2) \to \quad x = -4 + 1$

$\qquad \underline{x = -3}$

∴ ***Solution***: $\underline{(-3,\ 0,\ 1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 2 & -3 & 2 & 10 \\ 3 & -1 & 1 & 9 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -3 & 2 & 10 \\ -2 & -6 & -8 & -28 \\ \hline 0 & -9 & -6 & -18 \end{array} \qquad \begin{array}{cccc} 3 & -1 & 1 & 9 \\ -3 & -9 & -12 & -42 \\ \hline 0 & -10 & -11 & -33 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & -9 & -6 & -18 \\ 0 & -10 & -11 & -33 \end{array}\right] \quad 9R_3 - 10R_2$$

$$\begin{array}{cccc} 0 & -90 & -99 & -297 \\ 0 & 90 & 60 & 180 \\ \hline 0 & 0 & -39 & -117 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & -9 & -6 & -18 \\ 0 & 0 & -39 & -117 \end{array}\right] \quad \begin{array}{ll} x + 3y + 4z = 14 & (3) \\ -9y - 6z = -18 & (2) \\ -39z = -117 & (1) \end{array}$$

$(1) \to \quad z = \dfrac{117}{39}$

$\qquad \underline{= 3}$

$(2) \to \quad 9y = 18 - 6(3)$

$9y = 0$

$\underline{y = 0}|$

$(3) \to \quad x = 14 - 12$

$\underline{x = 2}|$

$\therefore$ ***Solution***: $\underline{(2,\ 0,\ 3)}\ |$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 3 & 2 & 1 & 8 \\ 2 & -3 & 2 & -16 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 2 & 1 & 8 \\ -3 & -12 & 3 & -60 \\ \hline 0 & -10 & 4 & -52 \end{array} \qquad \begin{array}{cccc} 2 & -3 & 2 & -16 \\ -2 & -8 & 2 & -40 \\ \hline 0 & -11 & 4 & -56 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 0 & -10 & 4 & -52 \\ 0 & -11 & 4 & -56 \end{array}\right] \begin{array}{l} \\ \\ 10R_3 - 11R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -110 & 40 & -560 \\ 0 & 110 & -44 & 572 \\ \hline 0 & 0 & -4 & 12 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 0 & -10 & 4 & -52 \\ 0 & 0 & -4 & 12 \end{array}\right] \quad \begin{array}{rl} x + 4y - z = 20 & (3) \\ -10y + 4z = -52 & (2) \\ -4z = 12 & (1) \end{array}$$

$(1) \to \quad \underline{z = -3}|$

$(2) \to \quad -10y = -52 + 12$

$-10y = -40$

$\underline{y = 4}|$

$(3) \to \quad x = 20 - 16 - 3$

$\underline{x = 1}|$

$\therefore$ ***Solution***: $\underline{(1,\ 4,\ -3)}\ |$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 17 \\ 0 & 2 & -1 & 7 \\ 2 & -3 & 2 & -1 \end{array}\right] \begin{matrix} \\ \\ R_3 - 2R_1 \end{matrix} \qquad \begin{array}{cccc} 2 & -3 & 2 & -1 \\ -2 & -4 & -2 & -34 \\ \hline 0 & -7 & 0 & -35 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 17 \\ 0 & 2 & -1 & 7 \\ 0 & -7 & 0 & -35 \end{array}\right] \quad \begin{array}{rl} x + 2y + z = 17 & (3) \\ 2y - z = 7 & (2) \\ -7y = -35 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{y = 5}$

$(2) \rightarrow \quad z = 10 - 7$

$\qquad \underline{= 3}$

$(3) \rightarrow \quad x = 17 - 10 - 3$

$\qquad \underline{= 4}$

$\therefore$ ***Solution***: $\underline{(4,\ 5,\ 3)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ -4 & 5 & 3 & 7 \\ -6 & 3 & 5 & -4 \end{array}\right] \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \qquad \begin{array}{cccc} -4 & 5 & 3 & 7 \\ 4 & -12 & -14 & -6 \\ \hline 0 & -7 & -11 & 1 \end{array} \qquad \begin{array}{cccc} -6 & 3 & 5 & -4 \\ 6 & -18 & -21 & -9 \\ \hline 0 & -15 & -16 & -13 \end{array}$$

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ 0 & -7 & -11 & 1 \\ 0 & -15 & -16 & -13 \end{array}\right] \begin{matrix} \\ \\ 7R_3 - 15R_1 \end{matrix} \qquad \begin{array}{cccc} 0 & -105 & -112 & -91 \\ 0 & 105 & 165 & -15 \\ \hline 0 & 0 & 53 & -106 \end{array}$$

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ 0 & -7 & -11 & 1 \\ 0 & 0 & 53 & -106 \end{array}\right] \quad \begin{array}{rl} -2x + 6y + 7z = 3 & (3) \\ -7y - 11z = 1 & (2) \\ 53z = -106 & (1) \end{array}$$

$(1) \to \quad \underline{z = -2}|$

$(2) \to \quad -7y = 1 - 22$

$-7y = -21$

$\underline{y = 3}|$

$(3) \to \quad -2x = 3 - 18 + 14$

$-2x = -1$

$\underline{x = \frac{1}{2}}|$

$\therefore$ ***Solution***: $\underline{\left(\frac{1}{2},\ 3,\ -2\right)}\,|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 3 & -3 & 4 & 5 \\ 4 & -2 & 3 & 4 \end{array}\right] \quad \begin{array}{l} 2R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

6	−6	8	10
−6	3	−3	−3
0	−3	5	7

4	−2	3	4
−4	2	−2	−2
0	0	1	2

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 0 & -3 & 5 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \begin{array}{ll} 2x - y + z = 1 & (2) \\ -3y + 5z = 7 & (1) \\ \underline{z = 2}| & \end{array}$$

$(1) \to \quad -3y = 7 - 10$

$-3y = -3$

$\underline{y = 1}|$

$(3) \to \quad 2x = 1 + 1 - 2$

$\underline{x = 0}|$

$\therefore$ ***Solution***: $\underline{(0,\ 1,\ 2)}\,|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x-4y+4z=7 \\ x-y-2z=2 \\ 2x-3y+6z=5 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 3 & -4 & 4 & 7 \\ 2 & -3 & 6 & 5 \end{array}\right] \begin{array}{l} \\ R_2-3R_1 \\ R_3-2R_1 \end{array}$$

$$\begin{array}{cccc} 3 & -4 & 4 & 7 \\ -3 & 3 & 6 & -6 \\ \hline 0 & -1 & 10 & 1 \end{array} \qquad \begin{array}{cccc} 2 & -3 & 6 & 5 \\ -2 & 2 & 4 & -4 \\ \hline 0 & -1 & 10 & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & -1 & 10 & 1 \\ 0 & -1 & 10 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3=R_2 \end{array} \qquad \begin{array}{ll} \to x-y-2z=2 & (2) \\ \to -y+10z=1 & (1) \end{array}$$

$(1) \to \quad \underline{y=10z-1}$

$(2) \to \quad x=2+10z-1+2z$

$\qquad\qquad \underline{=12z+1}$

$\therefore$ ***Solution***: $\underline{(12z+1,\ 10z-1,\ z)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x-2y-z=2 \\ 2x-y+z=4 \\ -x+y+z=4 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 2 & -1 & 1 & 4 \\ -1 & 1 & 1 & 4 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3+R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -1 & 1 & 4 \\ -2 & 4 & 2 & -4 \\ \hline 0 & 3 & 3 & 0 \end{array} \qquad \begin{array}{cccc} -1 & 1 & 1 & 4 \\ 1 & -2 & -1 & 2 \\ \hline 0 & -1 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 3 & 3 & 0 \\ 0 & -1 & 0 & 6 \end{array}\right] \quad \begin{array}{rl} x-2y-z=2 & (3) \\ 3y+3z=0 & (2) \\ -y=6 & (1) \end{array}$$

$(1) \to \quad \underline{y=-6}$

$(2) \to \quad z=-y$

$\qquad\qquad \underline{=6}$

$(3) \to \quad x=2-12+6$

$\qquad\qquad \underline{=-4}$

$\therefore$ ***Solution***: $\underline{(-4,\ -6,\ 6)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x+y+z=3 \\ -y+2z=1 \\ -x+z=0 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ -1 & 0 & 1 & 0 \end{array}\right] \quad R_3+R_1$$

$$\begin{array}{cccc} -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 3 \\ \hline 0 & 1 & 2 & 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 1 & 2 & 3 \end{array}\right] \quad R_3+R_2$$

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \\ 0 & -1 & 2 & 1 \\ \hline 0 & 0 & 4 & 4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 4 & 4 \end{array}\right] \quad \begin{array}{rl} x+y+z=3 & (3) \\ -y+2z=1 & (2) \\ 4z=4 & (1) \end{array}$$

$(1) \to \quad \underline{z=1}$

$(2) \to \quad -y=1-2$

$\qquad \underline{y=1}$

$(3) \to \quad x=3-1-1$

$\qquad \underline{=1}$

$\therefore$ ***Solution***: $\underline{(1,\ 1,\ 1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x+y+3z=14 \\ 7x+5y+8z=37 \\ x+3y+2z=9 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 3 & 1 & 3 & 14 \\ 7 & 5 & 8 & 37 \end{array}\right] \quad \begin{array}{l} R_2-3R_1 \\ R_3-7R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 1 & 3 & 14 \\ -3 & -9 & -6 & -27 \\ \hline 0 & -8 & -3 & -13 \end{array} \qquad \begin{array}{cccc} 7 & 5 & 8 & 37 \\ -7 & -21 & -14 & -63 \\ \hline 0 & -16 & -6 & -26 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 0 & -8 & -3 & -13 \\ 0 & -16 & -6 & -26 \end{array}\right] \quad R_3-2R_2$$

$$\begin{array}{cccc} 0 & -16 & -6 & -26 \\ 0 & 16 & 6 & 26 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 0 & -8 & -3 & -13 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{ll} x+3y+2z=9 & (2) \\ -8y-3z=-13 & (1) \end{array}$$

$(1)\to \quad -8y = 3z - 13$

$$\underline{y = -\frac{3}{8}z + \frac{13}{8}}$$

$(3)\to \quad x = 9 - 3\left(\frac{13}{8} - \frac{3}{8}z\right) - 2z$

$$= 9 - \frac{39}{8} + \frac{9}{8}z - 2z$$

$$\underline{= \frac{33}{8} - \frac{7}{8}z}$$

$\therefore$ ***Solution***: $\underline{\left(\frac{33}{8} - \frac{7}{8}z,\ \ \frac{13}{8} - \frac{3}{8}z,\ \ z\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 4x-2y+z=7 \\ x+y+z=-2 \\ 4x+2y+z=3 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 4 & -2 & 1 & 7 \\ 4 & 2 & 1 & 3 \end{array}\right] \quad \begin{array}{l} \\ R_2 - 4R_1 \\ R_3 - 4R_1 \end{array}$$

$$\begin{array}{cccc} 4 & -2 & 1 & 7 \\ -4 & -4 & -4 & 8 \\ \hline 0 & -6 & -3 & 15 \end{array} \qquad \begin{array}{cccc} 4 & 2 & 1 & 3 \\ -4 & -4 & -4 & 8 \\ \hline 0 & -2 & -3 & 11 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -6 & -3 & 15 \\ 0 & -2 & -3 & 11 \end{array}\right] \quad -3R_3 + R_2$$

$$\begin{array}{cccc} 0 & 6 & 9 & -33 \\ 0 & -6 & -3 & 15 \\ \hline 0 & 0 & 6 & -18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -6 & -3 & 15 \\ 0 & 0 & 6 & -18 \end{array}\right] \quad \begin{array}{ll} x+y+z=-2 & (3) \\ -6y-3z=15 & (2) \\ 6z=-18 & (1) \end{array}$$

$(1)\to \quad \underline{z = -3}$

$(2)\to \quad -6y = 15 - 9$

$$\underline{y = -1}$$

$(3)\to \quad x = -2 + 1 + 3$

$$\underline{= 2}$$

$\therefore$ ***Solution***: $\underline{(2,\ -1,\ -3)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x-2y+z=-4 \\ 6x+4y-3z=-24 \\ x-2y+2z=1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 2 & -2 & 1 & -4 \\ 6 & 4 & -3 & -24 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-6R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -2 & 1 & -4 \\ -2 & 4 & -4 & -2 \\ \hline 0 & 2 & -3 & -6 \end{array} \qquad \begin{array}{cccc} 6 & 4 & -3 & -24 \\ -6 & 12 & -12 & -6 \\ \hline 0 & 16 & -15 & -30 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 2 & -3 & -6 \\ 0 & 16 & -15 & -30 \end{array}\right] \begin{array}{l} \\ \\ R_3-8R_2 \end{array}$$

$$\begin{array}{cccc} 0 & 16 & -15 & -30 \\ 0 & -16 & 24 & 48 \\ \hline 0 & 0 & 9 & 18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 2 & -3 & -6 \\ 0 & 0 & 9 & 18 \end{array}\right] \quad \begin{array}{rl} x-2y+2z=1 & (3) \\ 2y-3z=-6 & (2) \\ 9z=18 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z=2}$

$(2) \rightarrow \quad 2y=-6+6$

$\qquad \underline{y=0}$

$(3) \rightarrow \quad x=1-4$

$\qquad \underline{=-3}$

$\therefore$ ***Solution***: $\underline{(-3,\ 0,\ 2)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 9x+3y+z=4 \\ 16x+4y+z=2 \\ 25x+5y+z=2 \end{cases}$

Solution

$$\begin{cases} z+9x+3y=4 \\ z+16x+4y=2 \\ z+25x+5y=2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 1 & 16 & 4 & 2 \\ 1 & 25 & 5 & 2 \end{array}\right] \begin{array}{l} \\ R_2-R_1 \\ R_3-R_1 \end{array}$$

$$\begin{array}{cccc} 1 & 16 & 4 & 2 \\ -1 & -9 & -3 & -4 \\ \hline 0 & 7 & 1 & -2 \end{array} \qquad \begin{array}{cccc} 1 & 25 & 5 & 2 \\ -1 & -9 & -3 & -4 \\ \hline 0 & 16 & 2 & -2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 0 & 7 & 1 & -2 \\ 0 & 16 & 2 & -2 \end{array}\right] \quad 7R_3 - 16R_3$$

$$\begin{array}{cccc} 0 & 112 & 14 & -14 \\ 0 & -112 & -16 & 32 \\ \hline 0 & 0 & -2 & 18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 0 & 7 & 1 & -2 \\ 0 & 0 & -2 & 18 \end{array}\right] \quad \begin{array}{rl} z + 9x + 3y = 4 & (3) \\ 7x + y = -2 & (2) \\ -2y = 18 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{y = -9}|$

$(2) \rightarrow \quad 7x = -2 + 9$

$\qquad \underline{= 1}|$

$(3) \rightarrow \quad z = 4 - 9 + 27$

$\qquad \underline{= 22}|$

$\therefore$ ***Solution***: $\underline{(1, \ -9, \ 22)}|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x - y + 2z = -8 \\ x + 2y - 3z = 9 \\ 3x - y - 4z = 3 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 2 & -8 \\ 3 & -1 & -4 & 3 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -1 & 2 & -8 \\ -2 & -4 & 6 & -18 \\ \hline 0 & -5 & 8 & -26 \end{array} \qquad \begin{array}{cccc} 3 & -1 & -4 & 3 \\ -3 & -6 & 9 & -27 \\ \hline 0 & -7 & 5 & -24 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 0 & -7 & 5 & -24 \end{array}\right] \quad 5R_3 - 7R_2$$

$$\begin{array}{cccc} 0 & -35 & 25 & -120 \\ 0 & 35 & -56 & 182 \\ \hline 0 & 0 & -31 & 62 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 0 & 0 & -31 & 62 \end{array}\right] \quad \begin{array}{rl} x + 2y - 3z = 9 & (3) \\ -5y + 8z = -26 & (2) \\ -31z = 62 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z = -2}|$

$(2) \rightarrow \quad -5y = -26 + 16$

$\qquad -5y = 10$

$\qquad \underline{y = 2}|$

$(3) \rightarrow \quad x = 9 - 4 - 6$

$$=-1$$

$\therefore$ ***Solution***: $(-1,\ 2,\ -2)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x \quad -3z=-5 \\ 2x-y+2z=16 \\ 7x-3y-5z=19 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 2 & -1 & 2 & 16 \\ 7 & -3 & -5 & 19 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-7R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -1 & 2 & 16 \\ -2 & 0 & 6 & 10 \\ \hline 0 & -1 & 8 & 26 \end{array} \qquad \begin{array}{cccc} 7 & -3 & -5 & 19 \\ -7 & 0 & 21 & 35 \\ \hline 0 & -3 & 16 & 54 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 0 & -1 & 8 & 26 \\ 0 & -3 & 16 & 54 \end{array}\right] \begin{array}{l} \\ \\ R_3-3R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -3 & 16 & 54 \\ 0 & 3 & -24 & -78 \\ \hline 0 & 0 & -8 & -24 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 0 & -1 & 8 & 26 \\ 0 & 0 & -8 & -24 \end{array}\right] \quad \begin{array}{rr} x-3z=-5 & (3) \\ -y+8z=26 & (2) \\ -8z=-24 & (1) \end{array}$$

$(1) \rightarrow \quad z=3$

$(2) \rightarrow \quad -y=26-24$

$y=-2$

$(3) \rightarrow \quad x=-5+9$

$=4$

$\therefore$ ***Solution***: $(4,\ -2,\ 3)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x+2y-z=5 \\ 2x-y+3z=0 \\ \quad 2y+z=1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 2 & -1 & 3 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ \\ \end{array}$$

$$\begin{array}{cccc} 2 & -1 & 3 & 0 \\ -2 & -4 & 2 & -10 \\ \hline 0 & -5 & 5 & -10 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 5 & -10 \\ 0 & 2 & 1 & 1 \end{array}\right] \quad 5R_3 + 2R_2$$

$$\begin{array}{cccc} 0 & 10 & 5 & 5 \\ 0 & -10 & 10 & -20 \\ \hline 0 & 0 & 15 & -15 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 5 & -10 \\ 0 & 0 & 15 & -15 \end{array}\right] \quad \begin{array}{rl} x + 2y - z = 5 & (3) \\ -5y + 5z = -10 & (2) \\ 15z = -15 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z = -1}$

$(2) \rightarrow \quad -5y = -10 + 5$

$\qquad \underline{y = 1}$

$(3) \rightarrow \quad x = 5 - 2 - 1$

$\qquad \underline{= 2}$

$\therefore$ ***Solution***: $\underline{(2,\ 1,\ -1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + y + z = 6 \\ 3x + 4y - 7z = 1 \\ 2x - y + 3z = 5 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & 4 & -7 & 1 \\ 2 & -1 & 3 & 5 \end{array}\right] \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 4 & -7 & 1 \\ -3 & -3 & -3 & -18 \\ \hline 0 & 1 & -10 & -17 \end{array} \qquad \begin{array}{cccc} 2 & -1 & 3 & 5 \\ -2 & -2 & -2 & -12 \\ \hline 0 & -3 & 1 & -7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -10 & -17 \\ 0 & -3 & 1 & -7 \end{array}\right] \quad R_3 + 3R_2$$

$$\begin{array}{cccc} 0 & -3 & 1 & -7 \\ 0 & 3 & -30 & -51 \\ \hline 0 & 0 & -29 & -58 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -10 & -17 \\ 0 & 0 & -29 & -58 \end{array}\right] \quad \begin{array}{rl} x + y + z = 6 & (3) \\ y - 10z = -17 & (2) \\ -29z = -58 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z = 2}$

$(2) \rightarrow \quad y = -17 + 20$

$\qquad \underline{= 3}$

$(3) \rightarrow \quad x = 6 - 3 - 2$

$\qquad \underline{= 1}$

$\therefore$ ***Solution***: $\underline{(1,\ 3,\ 2)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x+2y+3z=3 \\ 4x-5y+7z=1 \\ 2x+3y-2z=6 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 4 & -5 & 7 & 1 \\ 2 & 3 & -2 & 6 \end{array}\right] \begin{array}{l} \\ 3R_2-4R_1 \\ 3R_3-2R_1 \end{array} \qquad \begin{array}{cccc} 12 & -15 & 21 & 3 \\ -12 & -8 & -12 & -12 \\ \hline 0 & -23 & 9 & -9 \end{array} \qquad \begin{array}{cccc} 6 & 9 & -6 & 18 \\ -6 & -4 & -6 & -6 \\ \hline 0 & 5 & -12 & 12 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 0 & -23 & 9 & -9 \\ 0 & 5 & -12 & 12 \end{array}\right] \begin{array}{l} \\ \\ 23R_3+5R_2 \end{array} \qquad \begin{array}{cccc} 0 & 115 & -276 & 276 \\ 0 & -115 & 45 & -45 \\ \hline 0 & 0 & -231 & 231 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 0 & -23 & 9 & -9 \\ 0 & 0 & -231 & 231 \end{array}\right] \begin{array}{rl} 3x+2y+3z=3 & (3) \\ -23y+9z=-9 & (2) \\ -231z=231 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z=-1}$

$(2) \rightarrow \quad -23y=-9+9$

$\qquad \underline{y=0}$

$(3) \rightarrow \quad 3x=3+3$

$\qquad \underline{x=2}$

$\therefore$ ***Solution***: $\underline{(2,\ 0,\ -1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x-3y+z=2 \\ 4x-12y+4z=8 \\ -2x+6y-2z=-4 \end{cases}$

Solution

$$\begin{cases} \qquad x-3y+z=2 \\ \frac{1}{4}\times \quad 4x-12y+4z=8 \\ -\frac{1}{2}\times \quad -2x+6y-2z=-4 \end{cases}$$

$$\begin{cases} x-3y+z=2 \\ x-3y+z=2 \\ x-3y+z=2 \end{cases}$$

Since all three equations are the same.

$\therefore$ ***Solution***: is the plane $\underline{x-3y+z=2}\,\big|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x-2y+z=-1 \\ x+2y-z=2 \\ 6x+4y+3z=5 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & -2 & 1 & -1 \\ 6 & 4 & 3 & 5 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-6R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -2 & 1 & -1 \\ -2 & -4 & 2 & -4 \\ \hline 0 & -6 & 3 & -5 \end{array} \qquad \begin{array}{cccc} 6 & 4 & 3 & 5 \\ -6 & -12 & 6 & -12 \\ \hline 0 & -8 & 9 & -7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -6 & 3 & -5 \\ 0 & -8 & 9 & -7 \end{array}\right] \begin{array}{l} \\ \\ 3R_3-4R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -24 & 27 & -21 \\ 0 & 24 & -12 & 20 \\ \hline 0 & 0 & 15 & -1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -6 & 3 & -5 \\ 0 & 0 & 15 & -1 \end{array}\right] \begin{array}{rl} x+2y-z=2 & (3) \\ -6y+3z=-5 & (2) \\ 15z=-1 & (1) \end{array}$$

$(1) \to \quad \underline{z=-\frac{1}{15}}\,\big|$

$(2) \to \quad -6y=-5+\frac{1}{5}$

$-6y=-\frac{24}{5}$

$\underline{y=\frac{4}{5}}\,\big|$

$(3) \to \quad x=2-\frac{8}{5}-\frac{1}{15}$

$\underline{=\frac{1}{3}}\,\big|$

$\therefore$ ***Solution***: $\underline{\left(\frac{1}{3},\ \frac{4}{5},\ -\frac{1}{15}\right)}\,\big|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 \\ \quad x_2 - 3x_3 - x_4 = 0 \\ 3x_1 \quad + 2x_3 - x_4 = 6 \\ -4x_1 + x_2 + 4x_3 + 2x_4 = -3 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 3 & 0 & 2 & -1 & 6 \\ -4 & 1 & 4 & 2 & -3 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 3R_1 \\ R_4 + 4R_1 \end{array}$$

$$\begin{array}{ccccc} 3 & 0 & 2 & -1 & 6 \\ -3 & 15 & -6 & 6 & -12 \\ \hline 0 & 15 & -4 & 5 & -6 \end{array} \qquad \begin{array}{ccccc} -4 & 1 & 4 & 2 & -3 \\ 4 & -20 & 8 & -8 & 16 \\ \hline 0 & -19 & 12 & -6 & 13 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 15 & -4 & 5 & -6 \\ 0 & -19 & 12 & -6 & 13 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 15R_2 \\ R_4 + 19R_2 \end{array}$$

$$\begin{array}{ccccc} 0 & 15 & -4 & 5 & -6 \\ 0 & -15 & 45 & 15 & 0 \\ \hline 0 & 0 & 41 & 20 & -6 \end{array} \qquad \begin{array}{ccccc} 0 & -19 & 12 & -6 & 13 \\ 0 & 19 & -57 & -19 & 0 \\ \hline 0 & 0 & -45 & -25 & 13 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 41 & 20 & -6 \\ 0 & 0 & -45 & -25 & 13 \end{array}\right] \begin{array}{l} \\ \\ \\ 41R_4 + 45R_2 \end{array}$$

$$\begin{array}{ccccc} 0 & 0 & -1845 & -1025 & 533 \\ 0 & 0 & 1845 & 900 & -270 \\ \hline 0 & 0 & 0 & -125 & 263 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 41 & 20 & -6 \\ 0 & 0 & 0 & -125 & 263 \end{array}\right] \quad \begin{array}{rr} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 & (4) \\ x_2 - 3x_3 - x_4 = 0 & (3) \\ 41x_3 + 20x_4 = -6 & (2) \\ -125x_4 = 263 & (1) \end{array}$$

$(1) \rightarrow \quad \boxed{x_4 = -\frac{263}{125}}$

$(2) \rightarrow \quad 41x_3 = -6 + \frac{1,052}{25}$

$\qquad = \frac{902}{25}$

$\boxed{x_3 = \frac{22}{25}}$

$(3) \rightarrow \quad x_2 = \frac{66}{25} - \frac{263}{125}$

$\qquad \boxed{= \frac{67}{125}}$

$(4) \rightarrow \quad x_1 = 4 + \frac{67}{25} - \frac{44}{25} - \frac{526}{125}$

$$= 4 + \frac{23}{25} - \frac{526}{125}$$

$$= \frac{500 + 115 - 526}{125}$$

$$= \frac{89}{125}$$

$\therefore$ ***Solution***: $\left(\frac{89}{125}, \frac{67}{125}, \frac{22}{25}, -\frac{263}{125} \right)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x_1 + x_2 + x_3 + x_4 = 5 \\ x_1 + 2x_2 - x_3 - 2x_4 = -1 \\ x_1 - 3x_2 - 3x_3 - x_4 = -1 \\ 2x_1 - x_2 + 2x_3 - x_4 = -2 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 1 & 2 & -1 & -2 & -1 \\ 1 & -3 & -3 & -1 & -1 \\ 2 & -1 & 2 & -1 & -2 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - 2R_1 \end{array}$$

$$\begin{array}{ccccc} 1 & 2 & -1 & -2 & -1 \\ -1 & -1 & -1 & -1 & -5 \\ \hline 0 & 1 & -2 & -3 & -6 \end{array} \qquad \begin{array}{ccccc} 1 & -3 & -3 & -1 & -1 \\ -1 & -1 & -1 & -1 & -5 \\ \hline 0 & -4 & -4 & -2 & -6 \end{array} \qquad \begin{array}{ccccc} 2 & -1 & 2 & -1 & -2 \\ -2 & -2 & -2 & -2 & -10 \\ \hline 0 & -3 & 0 & -3 & -12 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & -4 & -4 & -2 & -6 \\ 0 & -3 & 0 & -3 & -12 \end{array}\right] \begin{array}{l} \\ \\ R_3 + 4R_2 \\ R_4 + 3R_2 \end{array}$$

$$\begin{array}{ccccc} 0 & -4 & -4 & -2 & -6 \\ 0 & 4 & -8 & -12 & -24 \\ \hline 0 & 0 & -12 & -14 & -30 \end{array}$$

$$\begin{array}{ccccc} 0 & -3 & 0 & -3 & -12 \\ 0 & 3 & -6 & -9 & -18 \\ \hline 0 & 0 & -6 & -12 & -30 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & 0 & -12 & -14 & -30 \\ 0 & 0 & -6 & -12 & -30 \end{array}\right] \begin{array}{l} \\ \\ \\ -2R_4 + R_3 \end{array}$$

$$\begin{array}{ccccc} 0 & 0 & 12 & 24 & 60 \\ 0 & 0 & -12 & -14 & -30 \\ \hline 0 & 0 & 0 & 10 & 30 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & 0 & -12 & -14 & -30 \\ 0 & 0 & 0 & 10 & 30 \end{array}\right] \quad \begin{array}{rl} x_1 + x_2 + x_3 + x_4 = 5 & (4) \\ x_2 - 2x_3 - 3x_4 = -6 & (3) \\ -12x_3 - 14x_4 = -30 & (2) \\ 10x_4 = 30 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{x_4 = 3}|$

$(2) \rightarrow \quad -12x_3 = -30 + 42$

$= 12$

$\underline{x_3 = -1}|$

$(3) \rightarrow \quad x_2 = -6 - 2 + 9$

$\underline{= 1}|$

$(4) \rightarrow \quad x_1 = 5 - 1 + 1 - 3$

$\underline{= 2}|$

∴ ***Solution***: $\underline{(2,\ 1,\ -1,\ 3)}|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x + 8y - z + w = 0 \\ 4x + 16y - 3z - w = -10 \\ -2x + 4y - z + 3w = -6 \\ -6x + 2y + 5z + w = 3 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 4 & 16 & -3 & -1 & -10 \\ -2 & 4 & -1 & 3 & -6 \\ -6 & 2 & 5 & 1 & 3 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 + 3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - \frac{13}{6}R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \qquad \textit{Interchange } R_2 \textit{ and } R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & \frac{19}{3} & -\frac{14}{3} & 16 \end{array}\right] \quad R_4 + \frac{19}{3} R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & 0 & -\frac{71}{3} & -\frac{142}{3} \end{array}\right] \quad \begin{array}{ll} 2x + 8y - z + w = 0 & (3) \\ 12y - 2z + 4w = -6 & (2) \\ -z - 3w = -10 & (1) \\ -\frac{71}{3} w = -\frac{142}{3} \quad \to \quad \underline{w = 2} & \end{array}$$

$(1) \to \quad z = 10 - 3w \underline{= 4}$

$(2) \to \quad 12y = 2z - 4w - 6$

$\underline{y = -\frac{1}{2}}$

$(3) \to \quad 2x = -8y + z - w$

$2x = 4 + 4 - 2$

$2x = 6$

$\underline{x = 3}$

$\therefore$ ***Solution***: $\underline{\left(3,\ -\frac{1}{2},\ 4,\ 2\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 + 2x_2 \qquad = 0 \\ \qquad x_2 + x_3 = 0 \end{cases}$

Solution

$$\begin{cases} x_1 = -2x_2 \\ x_3 = -x_2 \end{cases}$$

$2x_1 + x_2 + 3x_3 = 0$

$-4x_2 + x_2 - 3x_2 = 0 \quad \to \quad \underline{x_2 = 0}$

$\therefore$ ***Solution***: $\underline{(0,\ 0,\ 0)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x+2y+4z \quad\;\; =0 \\ \quad -y-3z+w=0 \\ 3x+y+z+2w=0 \\ x+3y-2z-2w=0 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & -1 & -3 & 1 & 0 \\ 3 & 1 & 1 & 2 & 0 \\ 1 & 3 & -2 & -2 & 0 \end{array}\right] \begin{array}{l} \\ -R_2 \\ 2R_3-3R_1 \\ 2R_4-R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & -4 & -10 & 4 & 0 \\ 0 & 4 & -8 & -4 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3+4R_2 \\ R_4-4R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & -20 & 0 & 0 \end{array}\right] \begin{array}{ll} 2x+2y-4z=0 & (1) \\ y+3z-w=0 & (2) \\ \to \underline{z=0} & \\ & \end{array}$$

$(2) \quad \to \quad \underline{y=w}$

$(1) \quad \to 2x=-2y \quad \underline{x=-w}$

$\therefore$ ***Solution***: $\underline{(-w,\ w,\ 0,\ w)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x \quad\; +z+w=5 \\ \quad y \quad\;\; -w=-1 \\ 3x \quad\; -z-w=0 \\ 4x+y+2z+w=9 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 3 & 0 & -1 & -1 & 0 \\ 4 & 1 & 2 & 1 & 9 \end{array}\right] \begin{array}{l} \\ \\ 2R_3-3R_1 \\ 2R_4-4R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -5 & -5 & -15 \\ 0 & 2 & 0 & -2 & -2 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - 2R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -5 & -5 & -15 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{rl} 2x + z + w = 5 & (1) \\ y - w = -1 & (2) \\ -5z - 5w = -15 & (3) \\ & \end{array}$$

$(2) \rightarrow \quad \underline{y = 1 + w}$

$(3) \rightarrow \quad \underline{z = 3 - w}$

$(1) \rightarrow \quad 2x = 5 - (3 - w) - w \quad \Rightarrow \quad \underline{x = 1}$

$\therefore$ ***Solution***: $\underline{(1,\ 1+w,\ 3-w,\ w)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} \quad 4y + z = 20 \\ 2x - 2y + z = 0 \\ x \quad\quad + z = 5 \\ x + y - z = 10 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 10 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 1 & 5 \\ 0 & 4 & 1 & 20 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - R_1 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 10 \\ 0 & -4 & 3 & -20 \\ 0 & -1 & 2 & -5 \\ 0 & 4 & 1 & 20 \end{array}\right] \begin{array}{l} \\ \\ 4R_3 - R_2 \\ R_4 + R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 10 \\ 0 & -4 & 3 & -20 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \begin{array}{l} x + y = 10 \\ \rightarrow -4y = -20 \\ \rightarrow z = 0 \\ \\ \end{array}$$

$\therefore$ ***Solution***: $\underline{(5,\ 5,\ 0)}$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x \qquad\qquad - 3w = -3 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ -1 & 3 & -2 & 1 \\ 3 & 4 & -7 & 10 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 5 & -1 & 9 \\ 0 & -2 & -10 & -14 \end{array}\right] \begin{array}{l} \\ \\ 5R_3 + 2R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 5 & -1 & 9 \\ 0 & 0 & -52 & -52 \end{array}\right] \quad \begin{array}{rl} x + 2y + z = 8 & (3) \\ 5y - z = 9 & (2) \\ -52z = -52 & (1) \end{array}$$

$(1) \quad \Rightarrow \quad z = 1$

$(2) \quad \Rightarrow \quad 5y = 9 + 1 = 10 \rightarrow y = 2$

$(3) \quad \Rightarrow \quad x = 8 - 4 - 1 = 3$

$\therefore$ ***Solution***: $(3,\ 2,\ 1)$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} 2u - 3v + w - x + y = 0 \\ 4u - 6v + 2w - 3x - y = -5 \\ -2u + 3v - 2w + 2x - y = 3 \end{cases}$$

Solution

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 4 & -6 & 2 & -3 & -1 & -5 \\ -2 & 3 & -2 & 2 & -1 & 3 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -3 & -5 \\ 0 & 0 & -1 & 1 & 0 & 3 \end{array}\right] \qquad \begin{array}{rl} 2u-3v+w-x+y=0 & (3) \\ -x-3y=-5 & (2) \\ -w+x=3 & (1) \end{array}$$

$$(2) \Rightarrow x = 5-3y$$

$$(1) \Rightarrow w = x-3 = 2-3y$$

$$(3) \Rightarrow 2u = 3v-2+3y+5-3y-y = 3v-y+3$$

$$u = \tfrac{3}{2}v - \tfrac{1}{2}y + \tfrac{3}{2}$$

$\therefore$ ***Solution***: $\underline{\left(\tfrac{3}{2}v-\tfrac{1}{2}y+\tfrac{3}{2}, v, 2-3y, 5-3y, y\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 6x_3+2x_4-4x_5-8x_6=8 \\ 3x_3+x_4-2x_5-4x_6=4 \\ 2x_1-3x_2+x_3+4x_4-7x_5+x_6=2 \\ 6x_1-9x_2+11x_4-19x_5+3x_6=1 \end{cases}$

Solution

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 6 & -9 & 0 & 11 & -19 & 3 & 1 \end{array}\right] \quad R_4 - 3R_1$$

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & -3 & -1 & 2 & 0 & -5 \end{array}\right] \quad \begin{array}{l} R_3 - 2R_2 \\ R_4 + R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & -1 \end{array}\right] \qquad \begin{array}{r} 2x_1-3x_2+x_3+4x_4-7x_5=2-x_6 \\ 3x_3+x_4-2x_5=4+4x_6 \\ \\ \to \underline{x_6=\tfrac{1}{4}} \end{array}$$

$$\begin{cases} 3x_3 = 5-x_4+2x_5 \\ 2x_1 = \tfrac{7}{4}+3x_2-\tfrac{1}{3}\left(5-x_4+2x_5\right)-4x_4+7x_5 \end{cases}$$

$$\begin{cases} \boxed{x_3 = \frac{5}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5} \\ 2x_1 = \frac{1}{12} + 3x_2 - \frac{11}{3}x_4 + \frac{19}{3}x_5 \end{cases}$$

∴ ***Solution***: $\boxed{\left(\frac{1}{24} + \frac{3}{2}x_2 - \frac{11}{6}x_4 + \frac{19}{6}x_5,\ x_2,\ \frac{5}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5,\ x_4,\ x_5,\ \frac{1}{4}\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x_1 + 2x_2 - x_3 = -15 \\ 5x_1 + 3x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 3x_3 = 11 \\ -6x_1 - 4x_2 + 2x_3 = 30 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ -6 & -4 & 2 & 30 \end{array}\right] \begin{array}{l} \\ 3R_2 - 5R_1 \\ R_3 - R_1 \\ R_4 + 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & -1 & 4 & 26 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & 0 & -7 & -49 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{rl} 3x_1 + 2x_2 - x_3 = -15 & (3) \\ -x_2 + 11x_3 = 75 & (2) \\ -7x_3 = -49 & (1) \end{array}$$

$(1) \rightarrow x_3 = 7$

$(2) \rightarrow x_2 = 77 - 75 = 2$

$(1) \rightarrow 3x_1 = -15 - 4 + 7 = 12$

$\boxed{x_1 = -4}$

∴ ***Solution***: $\boxed{(-4, 2, 7)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1 \\ 5x_3 + 10x_4 + 15x_6 = 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6 \end{cases}$

Solution

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ \\ R_4 - 2R_1 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \begin{array}{l} \\ -R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 5R_2 \\ R_4 - 4R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right] \begin{array}{l} \\ \\ \\ \frac{1}{6}R_4 \quad \textit{then interchanging row3 and row4} \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_3 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{cases} x_1 + 3x_2 + 4x_4 + 2x_5 = 0 \\ x_3 + 2x_4 = 0 \\ +x_6 = \frac{1}{3} \end{cases}$$

The general solution of the system: $\underline{x_6 = \frac{1}{3}, \quad x_3 = -2x_4, \quad x_1 = -3x_2 - 4x_4 - 2x_5}$

$\therefore$ ***Solution***: $\underline{\left(-3x_2 - 4x_4 - 2x_5, \ x_2, \ -2x_4, \ x_4, \ x_5, \ \frac{1}{3}\right)}$

Exercise

At SnackMix, caramel corn worth \$2.50 per *pound* is mixed with honey roasted missed nuts worth \$7.50 per *pound* in order to get 20 *lbs*. of a mixture worth \$4.50 per *pound*. How much of each snack is used?

Solution

$$x + y = 20 \qquad (1)$$
$$2.50x + 7.50y = 90 \qquad (2)$$

(1) $y = 20 - x$

(2) $2.5x + 7.5(20 - x) = 90$

$2.5x + 150 - 7.5x = 90$

$-5x = 90 - 150$

$-5x = -60$

$x = \frac{-60}{-5} = 12$

$y = 20 - x$
$= 20 - 12$
$= 8$

The mixture consists of 12 *lbs*. of caramel and 8 *lbs*. of nuts

Exercise

Find values for the variables so that the matrices are equal. $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

Solution

$$\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$$

$$\Rightarrow \begin{cases} w = 9 & x = 17 \\ y = 8 & z = -12 \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal. $\begin{bmatrix} x & y+3 \\ 2z & 8 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 6 & 8 \end{bmatrix}$

Solution

$$\begin{cases} \underline{x = 12} \\ y + 3 = 5 \rightarrow \underline{y = 2} \\ 2z = 6 \rightarrow \underline{z = 3} \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal. $\begin{bmatrix} 5 & x-4 & 9 \\ 2 & -3 & 8 \\ 6 & 0 & 5 \end{bmatrix} = \begin{bmatrix} y+3 & 2 & 9 \\ z+4 & -3 & 8 \\ 6 & 0 & w \end{bmatrix}$

Solution

$$\begin{bmatrix} 5 = y+3 & x-4 = 2 & 9 = 9 \\ 2 = z+4 & -3 = -3 & 8 = 8 \\ 6 = 6 & 0 = 0 & 5 = w \end{bmatrix}$$

$$\rightarrow \begin{cases} y = 2 & z = -2 \\ x = 6 & w = 5 \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} a+2 & 3b & 4c \\ d & 7f & 8 \end{bmatrix} + \begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$$

Solution

$$\begin{bmatrix} a-5 & 5b & 4c+6 \\ -2d & 7f-6 & 6 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$$

$$\begin{cases} a-5=15 & \rightarrow \quad a=20 \\ 5b=25 & \rightarrow \quad b=5 \\ 4c+6=6 & \rightarrow \quad 4c=0 \rightarrow c=0 \\ -2d=-8 & \rightarrow \quad d=4 \\ 7f-6=1 & \rightarrow \quad 7f=7 \rightarrow f=1 \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} a+11 & 12z+1 & 5m \\ 11k & 3 & 1 \end{bmatrix} + \begin{bmatrix} 9a & 9z & 4m \\ 12k & 5 & 3 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

Solution

$$\begin{bmatrix} a+11+9a & 12z+1+9z & 5m+4m \\ 11k+12k & 3+5 & 1+3 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 10a+11 & 21z+1 & 9m \\ 23k & 8 & 4 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

$$10a+11=41 \quad \rightarrow \quad 10a=30$$

$$\underline{a=3}$$

$$21z+1=-62 \quad \rightarrow \quad 21z=-63$$

$$\underline{z=-3}$$

$$9m=72 \quad \rightarrow \quad \underline{m=8}$$

$$23k=92 \quad \rightarrow \quad \underline{k=\frac{92}{23}=4}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} x+2 & 3y+1 & 5z \\ 8w & 2 & 3 \end{bmatrix} + \begin{bmatrix} 3x & 2y & 5z \\ 2w & 5 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 7 & -2 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 4x+2 & 5y+1 & 10z \\ 10w & 7 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 7 & -2 \end{bmatrix}$$

$$\begin{cases} 4x+2=10 & \rightarrow \quad \underline{x=2} \\ 5y+1=-14 & \rightarrow \quad \underline{y=-3} \\ 10z=80 & \rightarrow \quad \underline{z=8} \\ 10w=10 & \rightarrow \quad \underline{w=1} \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} 2x-3 & y-2 & 2z+1 \\ 5 & 2w & 7 \end{bmatrix} + \begin{bmatrix} 3x-3 & y+2 & z-1 \\ -5 & 5w+1 & 3 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 9 \\ 0 & 8 & 10 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 5x-6 & 2y & 3z \\ 0 & 7w+1 & 10 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 9 \\ 0 & 8 & 10 \end{bmatrix}$$

$$\begin{cases} 5x-6=20 & \rightarrow \quad \underline{x=\frac{26}{5}} \\ 2y=8 & \rightarrow \quad \underline{y=4} \\ 3z=9 & \rightarrow \quad \underline{z=3} \\ 7w+1=8 & \rightarrow \quad \underline{w=1} \end{cases}$$

Exercise

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -3 & 6 \\ -3 & 1 & -4 \end{bmatrix}$$

Solution

$$\begin{aligned} A-B &= \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \end{bmatrix} - \begin{bmatrix} 2 & -3 & 6 \\ -3 & 1 & -4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 & -5 \\ 2 & 1 & 9 \end{bmatrix} \end{aligned}$$

$$3A + 2B = 3\begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \end{bmatrix} + 2\begin{bmatrix} 2 & -3 & 6 \\ -3 & 1 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 3 & 3 \\ -3 & 6 & 15 \end{bmatrix} + \begin{bmatrix} 4 & -6 & 12 \\ -6 & 2 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & -3 & 15 \\ -9 & 8 & 7 \end{bmatrix}$$

Exercise

Given $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ $\quad F = \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix}$ Find $\quad 3F + 2A$

Solution

$$3F + 2A = 3\begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix} + 2\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3(3) & 3(3) \\ 3(-1) & 3(-1) \end{bmatrix} + \begin{bmatrix} 2(1) & 2(2) \\ 2(4) & 2(3) \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 9 \\ -3 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 8 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 9+2 & 9+4 \\ -3+8 & -3+6 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 13 \\ 5 & 3 \end{bmatrix}$$

Exercise

Evaluate $\quad \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + \begin{bmatrix} -6 \\ 3 \\ 12 \end{bmatrix}$

Solution

$$\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + \begin{bmatrix} -6 \\ 3 \\ 12 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \\ 20 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 5 & 8 \\ 6 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 9 & 1 \\ 4 & 2 & 5 \end{bmatrix}$

Solution

$$\begin{bmatrix} 5 & 8 \\ 6 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 9 & 1 \\ 4 & 2 & 5 \end{bmatrix}$$

It is ***impossible****;* 2×2 and 2×3 are not the same size.

Exercise

Evaluate $\begin{bmatrix} -5 & 0 \\ 4 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & 3 \end{bmatrix}$

Solution

$$\begin{bmatrix} -5 & 0 \\ 4 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -5+6 & 0+(-3) \\ 4+2 & \frac{1}{2}+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 \\ 6 & \frac{7}{2} \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix}$

Solution

$$\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix} = \begin{bmatrix} 5-4 & -6+6 \\ 8+8 & 9-3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 16 & 6 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} -5 & 6 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -3 & 2 \\ 5 & -8 \end{bmatrix}$

Solution

$$\begin{bmatrix} -5 & 6 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -3 & 2 \\ 5 & -8 \end{bmatrix} = \begin{bmatrix} -5-(-3) & 6-2 \\ 2-5 & 4-(-8) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 4 \\ -3 & 12 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 8 & 6 & -4 \end{bmatrix} - \begin{bmatrix} 3 & 5 & -8 \end{bmatrix}$

Solution

$$\begin{bmatrix} 8 & 6 & -4 \end{bmatrix} - \begin{bmatrix} 3 & 5 & -8 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 4 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1(4)+3(1) & 1(6)+3(0) \\ 2(4)+5(1) & 2(6)+5(0) \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 6 \\ 13 & 12 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{bmatrix}$

Solution

$$\begin{bmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -3(-6)+4(2)+2(3) & -3(4)+4(3)+2(-2) \\ 5(-6)+0(2)+4(3) & 5(4)+0(3)+4(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 32 & -4 \\ -18 & 12 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix} . \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix} . \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1(1)-1(1)+4(1) & 1(1)-1(2)+4(-1) & 1(0)-1(4)+4(3) \\ 4(1)-1(1)+3(1) & 4(1)-1(2)+3(-1) & 4(0)-1(4)+3(3) \\ 2(1)+0(1)-2(1) & 2(1)+0(2)-2(-1) & 2(0)+0(4)-2(3) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -5 & 8 \\ 6 & -1 & 5 \\ 0 & 4 & -6 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix} . \begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix} . \begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1(1)+1(4)+0(2) & 1(-1)+1(-1)+0(0) & 1(4)+1(3)+0(-2) \\ 1(1)+2(4)+4(2) & 1(-1)+2(-1)+4(0) & 1(4)+2(3)+4(-2) \\ 1(1)-1(4)+3(2) & 1(-1)-1(-1)+3(0) & 1(4)-1(3)+3(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -2 & 7 \\ 17 & -3 & 2 \\ 3 & 0 & -5 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} -2 & -3 & -4 \\ 2 & -1 & 0 \\ 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix}$

Solution

$$\begin{bmatrix} -2 & -3 & -4 \\ 2 & -1 & 0 \\ 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix} = \begin{bmatrix} -3-12 & -2-6-8 & -8+3+8 \\ -1 & 2-2 & 8+1 \\ -2+9 & 4-4+6 & 16+2-6 \end{bmatrix}$$

$$= \begin{bmatrix} -15 & -16 & 3 \\ -1 & 0 & 9 \\ 7 & 6 & 12 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} \sqrt{2} & \sqrt{2} & -\sqrt{18} \\ \sqrt{3} & \sqrt{27} & 0 \end{bmatrix} \begin{bmatrix} 8 & -10 \\ 9 & 12 \\ 0 & 2 \end{bmatrix}$

Solution

$$\begin{bmatrix} \sqrt{2} & \sqrt{2} & -\sqrt{18} \\ \sqrt{3} & \sqrt{27} & 0 \end{bmatrix} \begin{bmatrix} 8 & -10 \\ 9 & 12 \\ 0 & 2 \end{bmatrix} = \begin{pmatrix} 17\sqrt{2} & -4\sqrt{2} \\ 35\sqrt{3} & 26\sqrt{3} \end{pmatrix}$$

Exercise

Evaluate $\begin{bmatrix} x & 2x+1 & 4 \\ 5 & x-1 & 8 \\ -2 & 3x & 2x+1 \end{bmatrix} + \begin{bmatrix} 2x-1 & -2x-1 & 4x \\ -5 & 6 & x+1 \\ -5 & 2 & -2x \end{bmatrix}$

Solution

$$\begin{bmatrix} x & 2x+1 & 4 \\ 5 & x-1 & 8 \\ -2 & 3x & 2x+1 \end{bmatrix} + \begin{bmatrix} 2x-1 & -2x-1 & 4x \\ -5 & 6 & x+1 \\ -5 & 2 & -2x \end{bmatrix} = \begin{bmatrix} 3x-1 & 0 & 4x+4 \\ 0 & x+5 & x+9 \\ -7 & 3x+2 & 1 \end{bmatrix}$$

Exercise

Given $A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ *and* $B = \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}$. Find AB and BA.

Solution

$$AB = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 13 \\ 4 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} -16 & 29 \\ -4 & 10 \end{bmatrix}$$

Note: $AB \neq BA$

Exercise

Given $A=\begin{pmatrix}2 & -3\\ 1 & 4\end{pmatrix}$ $B=\begin{pmatrix}-2 & 4\\ 2 & -3\end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix}2 & -3\\ 1 & 4\end{pmatrix}\begin{pmatrix}-2 & 4\\ 2 & -3\end{pmatrix}$$

$$=\begin{pmatrix}-6 & 17\\ 6 & -8\end{pmatrix}$$

$$BA=\begin{pmatrix}-2 & 4\\ 2 & -3\end{pmatrix}\begin{pmatrix}2 & -3\\ 1 & 4\end{pmatrix}$$

$$=\begin{pmatrix}0 & 14\\ 1 & -20\end{pmatrix}$$

Exercise

Given $A=\begin{pmatrix}3 & -2\\ 4 & 1\end{pmatrix}$ $B=\begin{pmatrix}-1 & -1\\ 0 & 4\end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix}3 & -2\\ 4 & 1\end{pmatrix}\begin{pmatrix}-1 & -1\\ 0 & 4\end{pmatrix}$$

$$=\begin{pmatrix}-3 & -11\\ 4 & 0\end{pmatrix}$$

$$BA=\begin{pmatrix}-1 & -1\\ 0 & 4\end{pmatrix}\begin{pmatrix}3 & -2\\ 4 & 1\end{pmatrix}$$

$$=\begin{pmatrix}-7 & 1\\ 16 & 4\end{pmatrix}$$

Exercise

Given $A=\begin{pmatrix}3 & -1\\ 2 & 3\end{pmatrix}$ $B=\begin{pmatrix}4 & 1\\ 2 & -3\end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix}3 & -1\\ 2 & 3\end{pmatrix}\begin{pmatrix}4 & 1\\ 2 & -3\end{pmatrix}$$

$$=\begin{pmatrix}10 & 6\\ 14 & -7\end{pmatrix}$$

$$BA = \begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 14 & -1 \\ 0 & -11 \end{pmatrix}$$

Exercise

Given $A = \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}$$
$$= \begin{pmatrix} -4 & 2 \\ 4 & -4 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix}$$
$$= \begin{pmatrix} 4 & -4 \\ 14 & -12 \end{pmatrix}$$

Exercise

Given $A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -4 & 5 \\ 6 & 0 & 3 \\ -3 & -2 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}$$
$$= \begin{pmatrix} 5 & -13 \\ 3 & -4 \end{pmatrix}$$

Exercise

Given $A=\begin{pmatrix}-1 & 3\\ 2 & 1\\ -3 & 2\end{pmatrix}$ $B=\begin{pmatrix}1 & -2 & 3\\ 0 & 1 & 2\end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix}-1 & 3\\ 2 & 1\\ -3 & 2\end{pmatrix}\begin{pmatrix}1 & -2 & 3\\ 0 & 1 & 2\end{pmatrix}$$

$$=\begin{pmatrix}-1 & 5 & 4\\ 2 & -3 & 8\\ -3 & 8 & -5\end{pmatrix}$$

$$BA=\begin{pmatrix}1 & -2 & 3\\ 0 & 1 & 2\end{pmatrix}\begin{pmatrix}-1 & 3\\ 2 & 1\\ -3 & 2\end{pmatrix}$$

$$=\begin{pmatrix}-14 & 7\\ -4 & 5\end{pmatrix}$$

Exercise

Given $A=\begin{pmatrix}2 & 4\\ 0 & -1\\ -3 & 2\end{pmatrix}$ $B=\begin{pmatrix}3 & 0 & -2\\ -2 & 6 & 2\end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix}2 & 4\\ 0 & -1\\ -3 & 2\end{pmatrix}\begin{pmatrix}3 & 0 & -2\\ -2 & 6 & 2\end{pmatrix}$$

$$=\begin{pmatrix}-2 & 24 & 4\\ 2 & -6 & -2\\ -13 & 12 & 10\end{pmatrix}$$

$$BA=\begin{pmatrix}3 & 0 & -2\\ -2 & 6 & 2\end{pmatrix}\begin{pmatrix}2 & 4\\ 0 & -1\\ -3 & 2\end{pmatrix}$$

$$=\begin{pmatrix}12 & 8\\ -10 & 10\end{pmatrix}$$

Exercise

Given $\quad A=\begin{pmatrix}2 & -1 & 3\\ 0 & 2 & -1\\ 0 & 1 & 2\end{pmatrix} \quad A=\begin{pmatrix}3 & 0 & 0\\ 1 & -1 & 0\\ 2 & -1 & -2\end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix}2 & -1 & 3\\ 0 & 2 & -1\\ 0 & 1 & 2\end{pmatrix}\begin{pmatrix}3 & 0 & 0\\ 1 & -1 & 0\\ 2 & -1 & -2\end{pmatrix}$$

$$=\begin{pmatrix}11 & -2 & -6\\ 0 & -1 & 2\\ 5 & -3 & -4\end{pmatrix}$$

$$BA=\begin{pmatrix}3 & 0 & 0\\ 1 & -1 & 0\\ 2 & -1 & -2\end{pmatrix}\begin{pmatrix}2 & -1 & 3\\ 0 & 2 & -1\\ 0 & 1 & 2\end{pmatrix}$$

$$=\begin{pmatrix}6 & -3 & 9\\ 2 & -3 & 4\\ 4 & -6 & 3\end{pmatrix}$$

Exercise

Given $\quad A=\begin{pmatrix}-1 & 2 & 0\\ 2 & -1 & 1\\ -2 & 2 & -1\end{pmatrix} \quad B=\begin{pmatrix}2 & -1 & 0\\ 1 & 5 & -1\\ 0 & -1 & 3\end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix}-1 & 2 & 0\\ 2 & -1 & 1\\ -2 & 2 & -1\end{pmatrix}\begin{pmatrix}2 & -1 & 0\\ 1 & 5 & -1\\ 0 & -1 & 3\end{pmatrix}$$

$$=\begin{pmatrix}0 & 8 & -2\\ 3 & -8 & 4\\ -2 & 13 & -5\end{pmatrix}$$

$$BA=\begin{pmatrix}2 & -1 & 0\\ 1 & 5 & -1\\ 0 & -1 & 3\end{pmatrix}\begin{pmatrix}-1 & 2 & 0\\ 2 & -1 & 1\\ -2 & 2 & -1\end{pmatrix}$$

$$=\begin{pmatrix}-4 & 5 & -1\\ 11 & -5 & 6\\ -8 & 7 & -4\end{pmatrix}$$

Exercise

Given $A=\begin{pmatrix}1&-2&0\\2&0&1\\2&-2&-1\end{pmatrix}$ $B=\begin{pmatrix}-3&1&0\\1&4&-1\\0&0&2\end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix}1&-2&0\\2&0&1\\2&-2&-1\end{pmatrix}\begin{pmatrix}-3&1&0\\1&4&-1\\0&0&2\end{pmatrix}$$

$$=\begin{pmatrix}-5&-7&2\\-6&2&2\\-8&-6&0\end{pmatrix}$$

$$BA=\begin{pmatrix}-3&1&0\\1&4&-1\\0&0&2\end{pmatrix}\begin{pmatrix}1&-2&0\\2&0&1\\2&-2&-1\end{pmatrix}$$

$$=\begin{pmatrix}-1&6&1\\7&0&5\\4&-4&-2\end{pmatrix}$$

Exercise

Given $A=\begin{bmatrix}-3&4\\2&-3\\-1&0\end{bmatrix}$ $B=\begin{bmatrix}4&1\\1&-2\\3&-4\end{bmatrix}$, Find

a) $A+B$
b) $A-B$
c) $3A$
d) $-2B$
e) $2A+3B$
f) A^2
g) AB
h) BA

Solution

a) $A+B=\begin{bmatrix}-3&4\\2&-3\\-1&0\end{bmatrix}+\begin{bmatrix}4&1\\1&-2\\3&-4\end{bmatrix}$

$$=\begin{bmatrix}1&5\\3&-5\\2&-4\end{bmatrix}$$

b) $A-B=\begin{bmatrix}-3&4\\2&-3\\-1&0\end{bmatrix}-\begin{bmatrix}4&1\\1&-2\\3&-4\end{bmatrix}$

$$=\begin{bmatrix} -7 & 3 \\ 1 & -1 \\ -4 & 4 \end{bmatrix}$$

c) $3A=3\begin{bmatrix} -3 & 4 \\ 2 & -3 \\ -1 & 0 \end{bmatrix}$

$$=\begin{bmatrix} -9 & 12 \\ 6 & -9 \\ -3 & 0 \end{bmatrix}$$

d) $-2B=-2\begin{bmatrix} 4 & 1 \\ 1 & -2 \\ 3 & -4 \end{bmatrix}$

$$=\begin{bmatrix} -8 & -2 \\ -2 & 4 \\ -6 & 8 \end{bmatrix}$$

e) $2A+3B=2\begin{bmatrix} -3 & 4 \\ 2 & -3 \\ -1 & 0 \end{bmatrix}+3\begin{bmatrix} 4 & 1 \\ 1 & -2 \\ 3 & -4 \end{bmatrix}$

$$=\begin{bmatrix} -6 & 8 \\ 4 & -6 \\ -2 & 0 \end{bmatrix}+\begin{bmatrix} 12 & 3 \\ 3 & -6 \\ 9 & -12 \end{bmatrix}$$

$$=\begin{bmatrix} 6 & 11 \\ 7 & -12 \\ 7 & -12 \end{bmatrix}$$

f) $A^2 = \textit{doesn't exist}$ (not a square matrix)

g) $AB = \nexists$ $(2\times 3 \quad 2\times 3)$ the inner not equal

h) $BA = \nexists$ $(2\times 3 \quad 2\times 3)$ the inner not equal

Exercise

Given $A=\begin{bmatrix}2 & -2\\ 3 & 4\\ 1 & 0\end{bmatrix}$ $B=\begin{bmatrix}-1 & 8\\ 2 & -2\\ -4 & 3\end{bmatrix}$, Find

a) $A+B$
b) $A-B$
c) $3A$
d) $-2B$
e) $2A+3B$
f) A^2
g) AB
h) BA

Solution

a) $A+B=\begin{bmatrix}2 & -2\\ 3 & 4\\ 1 & 0\end{bmatrix}+\begin{bmatrix}-1 & 8\\ 2 & -2\\ -4 & 3\end{bmatrix}$

$=\begin{bmatrix}1 & 6\\ 5 & 2\\ -3 & 3\end{bmatrix}$

b) $A-B=\begin{bmatrix}2 & -2\\ 3 & 4\\ 1 & 0\end{bmatrix}-\begin{bmatrix}-1 & 8\\ 2 & -2\\ -4 & 3\end{bmatrix}$

$=\begin{bmatrix}3 & -10\\ 1 & 6\\ 5 & -3\end{bmatrix}$

c) $3A=3\begin{bmatrix}2 & -2\\ 3 & 4\\ 1 & 0\end{bmatrix}$

$=\begin{bmatrix}6 & -6\\ 9 & 12\\ 3 & 0\end{bmatrix}$

d) $-2B=-2\begin{bmatrix}-1 & 8\\ 2 & -2\\ -4 & 3\end{bmatrix}$

$=\begin{bmatrix}2 & -16\\ -4 & 4\\ 8 & -6\end{bmatrix}$

e) $2A+3B=2\begin{bmatrix}2 & -2\\ 3 & 4\\ 1 & 0\end{bmatrix}+3\begin{bmatrix}-1 & 8\\ 2 & -2\\ -4 & 3\end{bmatrix}$

$$= \begin{bmatrix} 4 & -4 \\ 6 & 8 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} -3 & 24 \\ 6 & -6 \\ -12 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 20 \\ 12 & 2 \\ -10 & 9 \end{bmatrix}$$

f) $A^2 = doesn't\ exist$ (not a square matrix)

g) $AB = \nexists$ $\quad (2\times 3 \quad 2\times 3)$ the inner not equal

h) $BA = \nexists$ $\quad (2\times 3 \quad 2\times 3)$ the inner not equal

Exercise

Given $A = \begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$, Find

a) $A+B$
b) $A-B$
c) $3A$
d) $-2B$
e) $2A+3B$
f) A^2
g) AB
h) BA

Solution

a) $A+B = \begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 1 & -1 \\ 2 & 2 & 1 \\ -1 & 2 & 5 \end{bmatrix}$$

b) $A-B = \begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} -3 & 5 & -1 \\ -2 & -4 & 3 \\ -7 & 4 & 1 \end{bmatrix}$$

c) $3A = 3\begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -6 & 9 & -3 \\ 0 & -3 & 6 \\ -12 & 9 & 9 \end{bmatrix}$$

d) $-2B = -2\begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} -2 & 4 & 0 \\ -4 & -6 & 2 \\ -6 & 2 & -4 \end{bmatrix}$$

e) $2A + 3B = 2\begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix} + 3\begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} -4 & 6 & -2 \\ 0 & -2 & 4 \\ -8 & 6 & 6 \end{bmatrix} + \begin{bmatrix} 3 & -6 & 0 \\ 6 & 9 & -3 \\ 9 & -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & -2 \\ 6 & 7 & 1 \\ 1 & 3 & 12 \end{bmatrix}$$

f) $A^2 = \begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}\begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 4+4 & -6-3-3 & 2+6-3 \\ -8 & 1+6 & -2+6 \\ 8-12 & -12-3+9 & 4+6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -12 & 5 \\ -8 & 7 & 4 \\ -4 & -6 & 19 \end{bmatrix}$$

g) $AB = \begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} -2+6-3 & 4+9+1 & -3-2 \\ -2+6 & -3-2 & 1+4 \\ -4+6+9 & 8+9-3 & -3+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 14 & -5 \\ 4 & -5 & 5 \\ 11 & 14 & 3 \end{bmatrix}$$

h) $BA = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}\begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -2 & 3+2 & -1-4 \\ -4+4 & 6-3-3 & -2+6-3 \\ -6-8 & 9+1+6 & -3-2+6 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 5 & -5 \\ 0 & 0 & 1 \\ -14 & 16 & 1 \end{bmatrix}$$

Exercise

Given $A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix}$, Find

a) $A+B$
b) $A-B$
c) $3A$
d) $-2B$
e) $2A+3B$
f) A^2
g) AB
h) BA

Solution

a) $A+B = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 4 & 4 \\ 4 & 0 & 1 \\ 1 & 8 & 1 \end{bmatrix}$$

b) $A-B = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & -4 \\ -2 & -6 & 5 \\ 9 & 0 & -5 \end{bmatrix}$$

c) $3A = 3\begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}$

$$=\begin{bmatrix} 0 & 6 & 0 \\ 3 & -9 & 9 \\ 15 & 12 & -6 \end{bmatrix}$$

d) $-2B=-2\begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix}$

$$=\begin{bmatrix} 2 & -4 & -8 \\ -6 & -6 & 4 \\ 8 & -8 & -6 \end{bmatrix}$$

e) $2A+3B=2\begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}+3\begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix}$

$$=\begin{bmatrix} 0 & 4 & 0 \\ 2 & -6 & 6 \\ 10 & 8 & -4 \end{bmatrix}+\begin{bmatrix} -3 & 6 & 12 \\ 9 & 9 & -6 \\ -12 & 12 & 9 \end{bmatrix}$$

$$=\begin{bmatrix} -3 & 10 & 12 \\ 11 & 3 & 0 \\ -2 & 20 & 5 \end{bmatrix}$$

f) $A^2=\begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}$

$$=\begin{bmatrix} 2 & -6 & 6 \\ -3+15 & 2+9+12 & -9-6 \\ 4-10 & 10-12-8 & 12+4 \end{bmatrix}$$

$$=\begin{bmatrix} 2 & -6 & 6 \\ 12 & 23 & -15 \\ -6 & -10 & 16 \end{bmatrix}$$

g) $AB=\begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}\begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix}$

$$=\begin{bmatrix} 6 & 6 & -4 \\ -1-9-12 & 2-9+12 & 4+6+9 \\ -5+12+8 & 10+12-8 & 20-8-6 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 6 & -4 \\ -22 & 5 & 19 \\ 15 & 14 & 6 \end{bmatrix}$$

h) $BA = \begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 2+10 & -2-6+16 & 6-8 \\ 3-10 & 6-9-8 & 9+4 \\ 4+15 & -8-12+12 & 12-6 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 8 & -2 \\ -7 & -11 & 13 \\ 19 & -8 & 6 \end{bmatrix}$$

Exercise

Given $A = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$ $C = \begin{pmatrix} 4 & 3 & 2 \\ -1 & 2 & 1 \end{pmatrix}$ $D = \begin{pmatrix} -2 & 3 \\ 2 & -1 \\ 3 & 2 \end{pmatrix}$, Find

a) $4A-2B$
b) $3A+C$
c) $3A+B$
d) $2A-3B$
e) AB
f) BA
g) A^2
h) B^3
i) AC
j) CA
k) CD
l) DC

Solution

a) $4A-2B = 4\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$

$$= \begin{pmatrix} -4 & 8 \\ -8 & 4 \end{pmatrix} - \begin{pmatrix} 2 & -4 \\ 4 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & 12 \\ -12 & 6 \end{pmatrix}$$

b) $3A + C = \nexists$

They are not the same order.

c) $3A+B = 3\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$

$$= \begin{pmatrix} -3 & 6 \\ -6 & 3 \end{pmatrix} + \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 4 \\ -4 & 2 \end{pmatrix}$$

d) $2A-3B=2\begin{pmatrix}-1 & 2\\ -2 & 1\end{pmatrix}-3\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}$

$=\begin{pmatrix}-2 & 4\\ -4 & 2\end{pmatrix}-\begin{pmatrix}3 & -6\\ 6 & -3\end{pmatrix}$

$=\begin{pmatrix}-5 & 10\\ -10 & 5\end{pmatrix}$

e) $AB=\begin{pmatrix}-1 & 2\\ -2 & 1\end{pmatrix}\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}$

$=\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}$

f) $BA=\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}\begin{pmatrix}-1 & 2\\ -2 & 1\end{pmatrix}$

$=\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}$

g) $A^2=\begin{pmatrix}-1 & 2\\ -2 & 1\end{pmatrix}\begin{pmatrix}-1 & 2\\ -2 & 1\end{pmatrix}$

$=\begin{pmatrix}-3 & 0\\ 0 & -3\end{pmatrix}$

h) $B^3=\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}$

$=\begin{pmatrix}-3 & 0\\ 0 & -3\end{pmatrix}\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}$

$=\begin{pmatrix}-3 & 6\\ -6 & 3\end{pmatrix}$

i) $AC=\begin{pmatrix}-1 & 2\\ -2 & 1\end{pmatrix}\begin{pmatrix}4 & 3 & 2\\ -1 & 2 & 1\end{pmatrix}$ $\qquad 2\times 2 \quad 2\times 3 \quad \rightarrow 2\times 3$

$=\begin{pmatrix}-6 & 1 & 0\\ -9 & -4 & -3\end{pmatrix}$

j) $CB=\nexists$ $\qquad 2\times 3 \quad 2\times 2$

C and B are not the same order.

k) $CD=\begin{pmatrix}4 & 3 & 2\\ -1 & 2 & 1\end{pmatrix}\begin{pmatrix}-2 & 3\\ 2 & -1\\ 3 & 2\end{pmatrix}$ $\qquad 2\times 3 \quad 3\times 2 \quad \rightarrow 2\times 2$

$$=\begin{pmatrix} -8+6+6 & 12-3+4 \\ 2+4+3 & -3+2+2 \end{pmatrix}$$

$$=\begin{pmatrix} 4 & 13 \\ 9 & 1 \end{pmatrix}$$

l) $DC = \begin{pmatrix} -2 & 3 \\ 2 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 4 & 3 & 2 \\ -1 & 2 & 1 \end{pmatrix}$ $\quad 3\times 2 \quad 2\times 3 \quad \rightarrow 3\times 3$

$$=\begin{pmatrix} -8-3 & -6+6 & -4+3 \\ 8+1 & 6-2 & 4-1 \\ 12-2 & 9+4 & 6+2 \end{pmatrix}$$

$$=\begin{pmatrix} -11 & 0 & -1 \\ 9 & 4 & 3 \\ 10 & 13 & 8 \end{pmatrix}$$

Exercise

Given $A=\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$ $B=\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$ $C=\begin{pmatrix} 1 & 4 & 5 \\ -2 & 3 & 4 \\ -1 & 0 & -2 \end{pmatrix}$ $D=\begin{pmatrix} 2 & 4 & -2 \\ 0 & 3 & 5 \\ -3 & 1 & 1 \end{pmatrix}$, Find

a) $4A-2B$
b) $3A+C$
c) $3A+B$
d) $2A-3B$
e) AB
f) BA
g) A^2
h) B^3
i) AC
j) CB
k) CD
l) DC

Solution

a) $4A-2B = 4\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix} - 2\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

$$=\begin{pmatrix} 8 & 16 \\ 12 & -4 \end{pmatrix} - \begin{pmatrix} -2 & 6 \\ 4 & -2 \end{pmatrix}$$

$$=\begin{pmatrix} 10 & 10 \\ 8 & -2 \end{pmatrix}$$

b) $3A+C = \nexists$

They are not the same order.

c) $3A+B = 3\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix} + \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

$$=\begin{pmatrix} 6 & 12 \\ 9 & -3 \end{pmatrix} + \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 15 \\ 11 & -4 \end{pmatrix}$$

d) $2A - 3B = 2\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix} - 3\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 4 & 8 \\ 6 & -2 \end{pmatrix} - \begin{pmatrix} -3 & 9 \\ 6 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -1 \\ 0 & 1 \end{pmatrix}$$

e) $AB = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 6 & 2 \\ -5 & 10 \end{pmatrix}$$

f) $BA = \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 7 & -7 \\ 1 & 9 \end{pmatrix}$$

g) $A^2 = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 14 & 4 \\ 3 & 13 \end{pmatrix}$$

h) $B^3 = \left(\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}\right)\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 7 & -6 \\ -4 & 7 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -19 & 27 \\ 18 & -19 \end{pmatrix}$$

i) $AC = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 & 3 & 2 \\ -1 & 2 & 1 \end{pmatrix}$ $\qquad 2\times 2 \quad 2\times 3 \quad \to 2\times 3$

$$= \begin{pmatrix} -6 & 1 & 0 \\ -9 & -4 & -3 \end{pmatrix}$$

j) $CB = \nexists$ $\qquad 2\times 3 \quad 2\times 2$

C and B are not the same order.

k) $CD = \begin{pmatrix} 1 & 4 & 5 \\ -2 & 3 & 4 \\ -1 & 0 & -2 \end{pmatrix}\begin{pmatrix} 2 & 4 & -2 \\ 0 & 3 & 5 \\ -3 & 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} -12 & 21 & 13 \\ -16 & 5 & 23 \\ 4 & -6 & 0 \end{pmatrix}$

l) $DC = \begin{pmatrix} 2 & 4 & -2 \\ 0 & 3 & 5 \\ -3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 & 5 \\ -2 & 3 & 4 \\ -1 & 0 & -2 \end{pmatrix}$

$= \begin{pmatrix} 2-8+2 & 8+12 & 10+16+4 \\ -6-5 & 9 & 12-10 \\ -3-2-1 & -12+3 & -15+4-2 \end{pmatrix}$

$= \begin{pmatrix} -4 & 20 & 30 \\ -11 & 9 & 2 \\ -6 & -9 & -12 \end{pmatrix}$

Exercise

A contractor builds three kinds of houses, models *A, B*, and *C*, with a choice of two styles, Spanish and contemporary. Matrix *P* shows the number of each kind of house planned for a new 100-home subdivision. The amounts for each of the exterior materials depend primarily on the style of the house. These amounts are shown in matrix *Q*. (concrete is in cubic yards, lumber in units of 1000 board feet, brick in 1000s, and shingles in units of 100 ft^2.) Matrix *R* gives the cost in dollars for each kind of material.

a) What is the total cost of these materials for each model?
b) How much of each of four kinds of material must be ordered
c) What is the total cost for exterior materials?

Solution

$$\begin{array}{l} \\ \text{Model } A \\ \text{Model } B \\ \text{Model } C \end{array} \begin{array}{c} \begin{array}{cc} \textit{Spanish} & \textit{Contemporary} \end{array} \\ \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix} \end{array} = P$$

$$\begin{array}{l} \\ \textit{Spanish} \\ \textit{Contemporary} \end{array} \begin{array}{c} \begin{array}{cccc} \textit{Concrete} & \textit{Lumber} & \textit{Brick} & \textit{Shingles} \end{array} \\ \begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix} \end{array} = Q$$

$$\begin{array}{l} \\ Concrete \\ Lumber \\ Brick \\ Shingles \end{array} \begin{array}{c} \textit{Cost per unit} \\ \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix} \end{array} = R$$

a) What is the total cost of these materials for each model?

$$PQ = \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix} \begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix}$$

$$\begin{array}{cccc} Concrete & Lumber & Brick & Shingles \end{array}$$

$$= \begin{bmatrix} 1500 & 30 & 600 & 60 \\ 100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix} \begin{array}{l} Model\ A \\ Model\ \text{B} \\ Model\ \text{C} \end{array}$$

$$(PQ)R = \begin{bmatrix} 1500 & 30 & 600 & 60 \\ 100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix} \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix}$$

$$= \begin{bmatrix} 72,900 \\ 54,700 \\ 60,800 \end{bmatrix} \begin{array}{l} Model\ A \\ Model\ B \\ Model\ C \end{array}$$

The total cost of materials is \$72,900 for model A, \$54,700 for model B, \$60,800 for model C.

b) How much of each of four kinds of material must be ordered

$$\begin{array}{c} \begin{bmatrix} 1500 & 30 & 600 & 60 \\ 100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix} \\ \hline 3800 \quad 130 \quad 1400 \quad 200 \end{array}$$

$$T = \begin{bmatrix} 3800 & 130 & 1400 & 200 \end{bmatrix}$$

3800 yd^3 of concrete, 130,000 board feet of lumber, 1,400,000 bricks, and 20,000 ft^2 of shingles are needed.

c) What is the total cost for exterior materials?

$$TR = \begin{bmatrix} 3800 & 130 & 1400 & 200 \end{bmatrix} \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix}$$

$$= \begin{bmatrix} 188,400 \end{bmatrix}$$

The total cost for exterior materials is $188,400.

Exercise

Mitchell Fabricators manufactures three styles of bicycle frames in its two plants. The following table shows the number of each style produced at each plant

	Mountain Bike	***Racing Bike***	***Touring Bike***
North Plant	150	120	100
South Plant	180	90	130

a) Write a 2 *x* 3 matrix ***A*** that represents the information in the table

b) The manufacturer increased production of each style by 20%. Find a Matrix ***M*** that represents the increased production figures.

c) Find the matrix ***A*** + ***M*** and tell what it represents

Solution

a) $A = \begin{bmatrix} 150 & 120 & 100 \\ 180 & 90 & 130 \end{bmatrix}$

b) The 20% production will represent

$\boldsymbol{A} + 20\%(\boldsymbol{A})$

$\rightarrow \boldsymbol{A} + .2\boldsymbol{A} = 1.2\boldsymbol{A}$

$$M = (1.2)\begin{bmatrix} 150 & 120 & 100 \\ 180 & 90 & 130 \end{bmatrix}$$

$$= \begin{bmatrix} 180 & 144 & 120 \\ 216 & 108 & 156 \end{bmatrix}$$

c) $A + M = \begin{bmatrix} 150 & 120 & 100 \\ 180 & 90 & 130 \end{bmatrix} + \begin{bmatrix} 180 & 144 & 120 \\ 216 & 108 & 156 \end{bmatrix}$

$$= \begin{bmatrix} 330 & 264 & 220 \\ 396 & 198 & 286 \end{bmatrix}$$

The matrix $A + M$ represents the total production of each style at each plant for the time period (2 months)

Exercise

Sal's Shoes and Fred's Footwear both have outlets in California and Arizona. Sal's sells shoes for \$80, sandals for \$40, and boots for \$120. Fred's prices are \$60, \$30, and \$150 for shoes, sandals and boots, respectively. Half of all sales in California stores are shoes, 1/4 are *sandals*, and 1/4 are *boots*. In Arizona the fractions are 1/5 *shoes*, 1/5 are *sandals*, and 3/5 are *boots*.

a) Write a 2 x 3 matrix called P representing prices for the two stores and three types of footwear.

b) Write a 2 x 3 matrix called F representing fraction of each type of footwear sold in each state.

c) Only one of the two products *PF* and *FP* is meaningful. Determine which one it is, calculate the product, and describe what the entries represent.

Solution

a) Write a 2 x 3 matrix called *P* representing prices for the two stores and three types of footwear.

$$P = \begin{array}{c} \begin{bmatrix} 80 & 40 & 120 \\ 60 & 30 & 150 \end{bmatrix} \end{array} \begin{array}{l} Sal's \\ Fred's \end{array}$$

b) Write a 2 *x* 3 matrix called F representing fraction of each type of footwear sold in each state.

$$F = \begin{array}{c} \begin{array}{cc} CA & AR \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{3}{5} \end{bmatrix} \end{array}$$

c) $PF = \begin{bmatrix} 80 & 40 & 120 \\ 60 & 30 & 150 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{3}{5} \end{bmatrix}$

$$= \begin{bmatrix} 80\frac{1}{2} + 40\frac{1}{4} + 120\frac{1}{4} & 80\frac{1}{5} + 40\frac{1}{5} + 120\frac{3}{5} \\ 60\frac{1}{2} + 30\frac{1}{4} + 150\frac{1}{4} & 60\frac{1}{5} + 30\frac{1}{5} + 150\frac{3}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 80 & 96 \\ 75 & 108 \end{bmatrix}$$

Exercise

Show that B is Multiplicative inverse of A

$$A = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

Solution

$$AB = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{pmatrix} -6 & \\ & \end{pmatrix}$$

$$\neq I$$

B is not multiplicative inverse of A

Exercise

Show that B is Multiplicative inverse of A

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \quad \& \quad B = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

Solution

$$AB = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I$$

$$BA = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I$$

$\therefore B$ is Multiplicative inverse of A

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{-4+6}\begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ -\frac{1}{3} & 1 \end{bmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{10-10}\begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$$

$$= \frac{1}{0}\begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$

Solution

$$\left[\begin{array}{cc|cc} -2 & 3 & 1 & 0 \\ -3 & 4 & 0 & 1 \end{array}\right] \quad -\frac{1}{2}R_1$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ -3 & 4 & 0 & 1 \end{array}\right] \quad R_2 + 3R_1$$

$$\begin{array}{cccc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ \\ -3 & 4 & 0 & 1 \\ 3 & -\frac{9}{2} & -\frac{3}{2} & 0 \\ \hline 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right] \quad -2R_2$$

$$\begin{array}{cccc} 0 & 1 & 3 & -2 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 3 & -2 \end{array}\right] \quad R_1 + \frac{3}{2}R_2$$

$$\begin{array}{cccc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & \frac{9}{2} & -3 \\ \hline 1 & 0 & 4 & -3 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 4 & -3 \\ 0 & 1 & 3 & -2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{bmatrix} a & b \\ 3 & 3 \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{3a-3b}\begin{bmatrix} 3 & -b \\ -3 & a \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{3(a-b)} & \frac{-b}{3(a-b)} \\ \frac{-3}{3(a-b)} & \frac{a}{3(a-b)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{a-b} & \frac{-b}{3(a-b)} \\ \frac{-1}{a-b} & \frac{a}{3(a-b)} \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{bmatrix} -2 & a \\ 4 & a \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{-2a-4a}\begin{bmatrix} a & -a \\ -4 & -2 \end{bmatrix}$$

$$=\begin{bmatrix} \frac{a}{-6a} & \frac{-a}{-6a} \\ \frac{-4}{-6a} & \frac{-2}{-6a} \end{bmatrix}$$

$$=\begin{bmatrix} -\frac{1}{6} & \frac{1}{6} \\ \frac{2}{3a} & \frac{1}{3a} \end{bmatrix}$$

Exercise

Find the inverse of $A=\begin{bmatrix} 4 & 4 \\ b & a \end{bmatrix}$

Solution

$$A^{-1}=\frac{1}{4a-4b}\begin{bmatrix} a & -4 \\ -b & 4 \end{bmatrix}$$

$$=\begin{bmatrix} \frac{a}{4(a-b)} & \frac{-4}{4(a-b)} \\ \frac{-b}{4(a-b)} & \frac{4}{4(a-b)} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{a}{4(a-b)} & \frac{-1}{a-b} \\ \frac{-b}{4(a-b)} & \frac{1}{a-b} \end{bmatrix}$$

Exercise

Find the inverse of $A=\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$

Solution

$$A^{-1}=\frac{1}{-1+4}\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{3}\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$

Solution

$$A^{-1} = -\frac{1}{14}\begin{pmatrix} -1 & -4 \\ -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{14} & \frac{2}{7} \\ \frac{3}{14} & -\frac{1}{7} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

Solution

$$A^{-1} = -\frac{1}{5}\begin{pmatrix} -1 & -3 \\ -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{11}\begin{pmatrix} 5 & -3 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{11} & -\frac{3}{11} \\ \frac{2}{11} & \frac{1}{11} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{0}\begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{pmatrix} -6 & 9 \\ 2 & -3 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{18-18}\begin{pmatrix} & \\ & \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{pmatrix} -2 & 7 \\ 0 & 2 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -7 \\ 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{7}{4} \\ 0 & -\frac{1}{2} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$

Solution

$$A = \frac{1}{-16+16}\begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Solution

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 2 & 1 \\ a & a \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{a}\begin{pmatrix} a & -1 \\ -a & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\frac{1}{a} \\ -1 & \frac{2}{a} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} b & 3 \\ b & 2 \end{pmatrix}$

Solution

$$A^{-1} = -\frac{1}{b}\begin{pmatrix} 2 & -3 \\ -b & b \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{b} & \frac{3}{b} \\ 1 & -1 \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & a \\ 3 & a \end{pmatrix}$

Solution

$$A^{-1} = -\frac{1}{2a}\begin{pmatrix} a & -a \\ -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2a} & -\frac{1}{2a} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{a^2 - 4}\begin{pmatrix} a & -2 \\ -2 & a \end{pmatrix}$$

$$= \begin{pmatrix} \frac{a}{a^2-4} & \frac{-2}{a^2-4} \\ \frac{-2}{a^2-4} & \frac{a}{a^2-4} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{0}\begin{pmatrix} & \\ & \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{pmatrix} -3 & \frac{1}{2} \\ 6 & -1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{0}\begin{pmatrix} & \end{pmatrix}$$

∴ Inverse ***doesn't exist***

Exercise

Find A^{-1} *if* $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccccc} 2 & -2 & -1 & 0 & 1 & 0 \\ -2 & 0 & -2 & -2 & 0 & 0 \\ \hline 0 & -2 & -3 & -2 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 3 & 0 & 0 & 0 & 0 & 1 \\ -3 & 0 & -3 & -3 & 0 & 0 \\ \hline 0 & 0 & -3 & -3 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] -\tfrac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] -\tfrac{1}{3}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 - \frac{3}{2}R_3 \\ \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & \frac{1}{3} \\ \hline 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{array} \qquad \begin{array}{cccccc} 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & 0 & \frac{1}{2} \\ \hline 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

Exercise

Find A^{-1}, where $A=\begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2-3R_1 \\ R_3+2R_1 \end{array}$$

$$\begin{array}{cccccc} 3 & 5 & 3 & 0 & 1 & 0 \\ -3 & -6 & 3 & -3 & 0 & 0 \\ \hline 0 & -1 & 6 & -3 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 2 & 4 & 3 & 0 & 0 & 1 \\ -2 & -4 & 2 & -2 & 0 & 0 \\ \hline 0 & 0 & 5 & -2 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} R_1-2R_2 \\ \\ \\ \end{array}$$

$$\begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 12 & -6 & 2 & 0 \\ \hline 1 & 0 & 11 & -5 & 2 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{5}R_3 \end{array}$$

$$\begin{array}{cccccc} 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}\right] \begin{array}{l} R_1-11R_3 \\ R_2+6R_3 \\ \\ \end{array}$$

$$\begin{array}{cccccc} 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 6 & -\frac{12}{5} & 0 & \frac{6}{5} \\ \hline 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \end{array} \qquad \begin{array}{cccccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 0 & -11 & \frac{22}{5} & 0 & -\frac{11}{5} \\ \hline 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}\right]$$

$$A^{-1}=\begin{bmatrix} -\frac{3}{5} & 2 & -\frac{11}{5} \\ \frac{3}{5} & -1 & \frac{6}{5} \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{bmatrix}$$

Exercise

Find A^{-1}, where $A=\begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2+2R_1 \\ R_3-R_1 \end{array}$$

$$\begin{array}{cccccc} -2 & 0 & 1 & 0 & 1 & 0 \\ 2 & 4 & -2 & 2 & 0 & 0 \\ \hline 0 & 4 & -1 & 2 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 1 & -1 & 0 & 0 & 0 & 1 \\ -1 & -2 & 1 & -1 & 0 & 0 \\ \hline 0 & -3 & 1 & -1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 4 & -1 & 2 & 1 & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array}\right] \tfrac{1}{4}R_2 \qquad 0 \quad 1 \quad -\tfrac{1}{4} \quad \tfrac{1}{2} \quad \tfrac{1}{4} \quad 0$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} R_1-2R_2 \\ \\ R_3+3R_2 \end{array}$$

$$\begin{array}{cccccc} 0 & -3 & 1 & -1 & 0 & 1 \\ 0 & 3 & -\frac{3}{4} & \frac{3}{2} & \frac{3}{4} & 0 \\ \hline 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{array} \qquad \begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & \frac{1}{2} & -1 & -\frac{1}{2} & 0 \\ \hline 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{array}\right] 4R_3 \qquad 0 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array}\right] \begin{array}{l} R_1+\frac{1}{2}R_3 \\ R_2+\frac{1}{4}R_3 \\ \\ \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 1 & \frac{3}{2} & 2 \\ \hline 1 & 0 & 0 & 1 & 1 & 2 \end{array} \qquad \begin{array}{cccccc} 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \\ \hline 0 & 1 & 0 & 1 & 1 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array}\right]$$

$$A^{-1}=\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

Exercise

Find A^{-1}, where $A=\begin{bmatrix} -2 & 5 & 3 \\ 4 & -1 & 3 \\ 7 & -2 & 5 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} -2 & 5 & 3 & 1 & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \qquad \frac{1}{-2}R_1 \qquad 1 \quad -\frac{5}{2} \quad -\frac{3}{2} \quad -\frac{1}{2} \quad 0 \quad 0$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \qquad R_2-4R_1 \qquad \begin{array}{cccccc} 4 & -1 & 3 & 0 & 1 & 0 \\ -4 & 10 & 6 & 2 & 0 & 0 \\ \hline 0 & 9 & 9 & 2 & 1 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \qquad R_3-7R_1 \qquad \begin{array}{cccccc} 7 & -2 & 5 & 0 & 0 & 1 \\ -7 & \frac{35}{2} & \frac{21}{2} & \frac{7}{2} & 0 & 0 \\ \hline 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}\right] \qquad \frac{1}{9}R_2 \qquad 0 \quad 1 \quad 1 \quad \frac{2}{9} \quad \frac{1}{9} \quad 0$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}\right] \qquad R_3-\frac{31}{2}R_2 \qquad \begin{array}{cccccc} 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \\ 0 & -\frac{31}{2} & -\frac{31}{2} & -\frac{31}{9} & -\frac{31}{18} & 0 \\ \hline 0 & 0 & 0 & \frac{1}{18} & -\frac{31}{18} & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{18} & -\frac{31}{18} & 1 \end{array}\right]$$

$\therefore$ The inverse matrix ***doesn't exist***

OR

$$\left[\begin{array}{ccc|ccc} -2 & 5 & 3 & 1 & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \qquad \begin{array}{c} R_2+2R_1 \\ -2R_3+7R_1 \end{array} \qquad \begin{array}{cccccc} 4 & -1 & 3 & 0 & 1 & 0 \\ -4 & 10 & 6 & 2 & 0 & 0 \\ \hline 0 & 9 & 9 & 2 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} -14 & 4 & -10 & 0 & 0 & -2 \\ 14 & 35 & 21 & 7 & 0 & 0 \\ \hline 0 & 39 & 11 & 7 & 0 & -2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} -2 & 5 & 3 & 1 & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 0 & 39 & 11 & 7 & 0 & -2 \end{array}\right] \begin{array}{l} 9R_1 - 5R_2 \\ \\ 9R_3 - 39R_2 \end{array}$$

$$\begin{array}{cccccc} -18 & 45 & 27 & 9 & 0 & 0 \\ 0 & -45 & -45 & -10 & -5 & 0 \\ \hline -18 & 0 & -18 & -1 & -5 & 0 \end{array}$$

$$\begin{array}{cccccc} 0 & 351 & -99 & 63 & 0 & -18 \\ 0 & -351 & 99 & -78 & -39 & 0 \\ \hline 0 & 0 & 0 & -15 & -39 & -18 \end{array}$$

$$\left[\begin{array}{ccc|ccc} -18 & 0 & -18 & -1 & -5 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 0 & 0 & 0 & -15 & -39 & -18 \end{array}\right]$$

$\therefore$ The inverse matrix ***doesn't exist***

Exercise

Find the inverse, if exists, of $A=\begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right) \quad R_2+R_1 \qquad \begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ \hline 0 & 4 & 4 & 1 & 1 & 0 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 4 & 4 & 1 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right) \quad \tfrac{1}{4}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{c} R_1-R_2 \\ \\ R_3-4R_2 \end{array} \qquad \begin{array}{cccccc} 0 & 4 & 3 & 0 & 0 & 1 \\ 0 & -4 & -4 & -1 & -1 & 0 \\ \hline 0 & 0 & -1 & -1 & -1 & 1 \end{array} \qquad \begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ \hline 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right) \quad -R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right) \quad \begin{array}{c} R_1+R_3 \\ R_2-R_3 \end{array} \qquad \begin{array}{cccccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ \hline 1 & 0 & 0 & \frac{7}{4} & \frac{3}{4} & -1 \end{array} \qquad \begin{array}{cccccc} 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \\ \hline 0 & 1 & 0 & -\frac{3}{4} & -\frac{3}{4} & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{4} & \frac{3}{4} & -1 \\ 0 & 1 & 0 & -\frac{3}{4} & -\frac{3}{4} & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right)$$

$$A^{-1}=\begin{pmatrix} \frac{7}{4} & \frac{3}{4} & -1 \\ -\frac{3}{4} & -\frac{3}{4} & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ -2 & -3 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ R_3 + 2R_1 \end{array} \qquad \begin{array}{cccccc} -2 & -3 & 0 & 0 & 0 & 1 \\ 2 & -2 & 2 & 2 & 0 & 0 \\ \hline 0 & -5 & 2 & 2 & 0 & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & -5 & 2 & 2 & 0 & 1 \end{array}\right) \begin{array}{l} \\ -\frac{1}{2}R_2 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & -5 & 2 & 2 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 + R_2 \\ \\ R_3 + 5R_2 \end{array} \qquad \begin{array}{cccccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ \hline 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \end{array} \qquad \begin{array}{cccccc} 0 & -5 & 2 & 2 & 0 & 1 \\ 0 & 5 & -\frac{5}{2} & 0 & -\frac{5}{2} & 0 \\ \hline 0 & 0 & -\frac{1}{2} & 2 & -\frac{5}{2} & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 2 & -\frac{5}{2} & 1 \end{array}\right) \begin{array}{l} \\ \\ -2R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -4 & 5 & -2 \end{array}\right) \begin{array}{l} R_1 - \frac{1}{2}R_3 \\ R_2 + \frac{1}{2}R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & -4 & 5 & -2 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 + R_1 \\ R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right) \tfrac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right) R_3 + R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}\right) 2R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right) \begin{array}{l} R_1 - 2R_3 \\ R_2 - \frac{5}{2}R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 1 & 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & -1 \\ 3 & 1 & 2 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & -1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -4 & -3 & 1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{array}\right) \begin{array}{l} \\ -R_2 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 3 & -1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 2R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -2 & 1 & 0 \\ 0 & 1 & 4 & 3 & -1 & 0 \\ 0 & 0 & 7 & 3 & -2 & 1 \end{array}\right) \begin{array}{l} \\ \\ \frac{1}{7}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -2 & 1 & 0 \\ 0 & 1 & 4 & 3 & -1 & 0 \\ 0 & 0 & 1 & \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{array}\right) \begin{array}{l} R_1 + 3R_3 \\ R_2 - 4R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{7} & \frac{1}{7} & \frac{3}{7} \\ 0 & 1 & 0 & \frac{9}{7} & \frac{1}{7} & -\frac{4}{7} \\ 0 & 0 & 1 & \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{5}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{9}{7} & \frac{1}{7} & -\frac{4}{7} \\ \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 3 & 3 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \frac{1}{3}R_1 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & -3 & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 3R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & \frac{7}{3} & -\frac{5}{3} & 3 & 1 \end{array}\right) \begin{array}{l} \\ \\ \frac{3}{7}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 & -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{array}\right) \begin{array}{l} R_1 + \frac{1}{3}R_3 \\ R_2 - \frac{2}{3}R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ 0 & 1 & 0 & \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ 0 & 0 & 1 & -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} -3 & 1 & -1 \\ 1 & -4 & -7 \\ 1 & 2 & 5 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} -3 & 1 & -1 & 1 & 0 & 0 \\ 1 & -4 & -7 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array}\right) \quad -\frac{1}{3}R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 1 & -4 & -7 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{11}{3} & -\frac{22}{3} & \frac{1}{3} & 1 & 0 \\ 0 & \frac{7}{3} & \frac{14}{3} & \frac{1}{3} & 0 & 1 \end{array}\right) \quad -\frac{3}{11}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 2 & -\frac{1}{11} & -\frac{3}{11} & 0 \\ 0 & \frac{7}{3} & \frac{14}{3} & \frac{1}{3} & 0 & 1 \end{array}\right) \quad R_3 - \frac{7}{3}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 2 & -\frac{1}{11} & -\frac{3}{11} & 0 \\ 0 & 0 & 0 & & & \end{array}\right)$$

$\therefore$ Inverse ***does not exist***

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 1 & -3 \\ 2 & -4 & 1 \\ -5 & 7 & 1 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 2 & -4 & 1 & 0 & 1 & 0 \\ -5 & 7 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + 5R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & -6 & 7 & -2 & 1 & 0 \\ 0 & 12 & -14 & 5 & 0 & 1 \end{array}\right) \quad -\frac{1}{6}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 12 & -14 & 5 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 12R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & & & \end{array}\right)$$

$\therefore$ Inverse ***does not exist***

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_3 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right) \quad -R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 - 2R_2 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ \frac{1}{5}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}\right) \begin{array}{l} R_1 - 11R_3 \\ R_2 + 6R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{3}{5} & 2 & -\frac{11}{5} \\ \frac{3}{5} & -1 & \frac{6}{5} \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} -2 & -3 & 4 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & -6 & 1 \\ -2 & -2 & 5 & 1 \end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc} -2 & -3 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & 5 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} -\frac{1}{2}R_1 \\ \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & 5 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 + 2R_1 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_4 - R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$\therefore$ Inverse ***does not exist***

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 1 & -14 & 7 & 38 \\ -1 & 2 & 1 & -2 \\ 1 & 2 & -1 & -6 \\ 1 & -2 & 3 & 6 \end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 2 & -1 & -6 & 0 & 0 & 1 & 0 \\ 1 & -2 & 3 & 6 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ 0 & -12 & 8 & 36 & 1 & 1 & 0 & 0 \\ 0 & 16 & -8 & -44 & -1 & 0 & 1 & 0 \\ 0 & 12 & -4 & -32 & -1 & 0 & 0 & 1 \end{array}\right] \quad -\frac{1}{12}R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 16 & -8 & -44 & -1 & 0 & 1 & 0 \\ 0 & 12 & -4 & -32 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + 14R_2 \\ \\ R_3 - 16R_2 \\ R_4 - 12R_2 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -\frac{7}{3} & -4 & -\frac{1}{6} & -\frac{7}{6} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & \frac{8}{3} & 4 & \frac{1}{3} & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 & 1 & 0 & 1 \end{array}\right] \quad \frac{3}{8}R_3$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -\frac{7}{3} & -4 & -\frac{1}{6} & -\frac{7}{6} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 4 & 4 & 0 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + \frac{7}{3}R_3 \\ R_2 + \frac{2}{3}R_3 \\ \\ R_4 - 4R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ 0 & 1 & 0 & -2 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & -2 & -\frac{1}{2} & -1 & -\frac{3}{2} & 1 \end{array}\right] -\frac{1}{2}R_4$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ 0 & 1 & 0 & -2 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{array}\right] \begin{array}{l} R_1 + \frac{1}{2}R_4 \\ R_2 + 2R_4 \\ R_3 - \frac{3}{2}R_4 \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{5}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{5}{4} & \frac{7}{4} & -1 \\ 0 & 0 & 1 & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{5}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{5}{4} & \frac{7}{4} & -1 \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{bmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 10 & 20 & -30 & 15 \\ 3 & -7 & 14 & -8 \\ -7 & -2 & -1 & 2 \\ 4 & 4 & -3 & 1 \end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc} 10 & 20 & -30 & 15 & 1 & 0 & 0 & 0 \\ 3 & -7 & 14 & -8 & 0 & 1 & 0 & 0 \\ -7 & -2 & -1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 4 & -3 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad \frac{1}{10}R_1$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0 \\ 3 & -7 & 14 & -8 & 0 & 1 & 0 & 0 \\ -7 & -2 & -1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 4 & -3 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 + 7R_1 \\ R_4 - 4R_1 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0 \\ 0 & -13 & 23 & -\frac{25}{2} & -\frac{3}{10} & 1 & 0 & 0 \\ 0 & 12 & -22 & \frac{25}{2} & \frac{7}{10} & 0 & 1 & 0 \\ 0 & -4 & 9 & -5 & -\frac{2}{5} & 0 & 0 & 1 \end{array}\right] \quad -\frac{1}{13}R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 12 & -22 & \frac{25}{2} & \frac{7}{10} & 0 & 1 & 0 \\ 0 & -4 & 9 & -5 & -\frac{2}{5} & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 - 12R_2 \\ R_4 + 4R_2 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \frac{7}{13} & -\frac{11}{26} & \frac{7}{130} & \frac{2}{13} & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 0 & -\frac{10}{13} & \frac{25}{26} & \frac{11}{26} & \frac{12}{13} & 1 & 0 \\ 0 & 0 & \frac{25}{13} & -\frac{15}{13} & -\frac{4}{13} & -\frac{4}{13} & 0 & 1 \end{array}\right] \quad -\frac{13}{10}R_3$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \frac{7}{13} & -\frac{11}{26} & \frac{7}{130} & \frac{2}{13} & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 0 & 1 & -\frac{5}{4} & -\frac{11}{20} & -\frac{6}{5} & -\frac{13}{10} & 0 \\ 0 & 0 & \frac{25}{13} & -\frac{15}{13} & -\frac{4}{13} & -\frac{4}{13} & 0 & 1 \end{array}\right] \begin{array}{l} R_1 - \frac{7}{13}R_3 \\ R_2 + \frac{23}{13}R_3 \\ \\ R_4 - \frac{25}{13}R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & -\frac{5}{4} & -\frac{19}{20} & -\frac{11}{5} & -\frac{23}{10} & 0 \\ 0 & 0 & 1 & -\frac{5}{4} & -\frac{11}{20} & -\frac{6}{5} & -\frac{13}{10} & 0 \\ 0 & 0 & 0 & \frac{5}{4} & \frac{3}{4} & 2 & \frac{5}{2} & 1 \end{array}\right] \begin{array}{l} R_2 + R_4 \\ R_3 + R_4 \end{array}.$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & \frac{5}{4} & \frac{3}{4} & 2 & \frac{5}{2} & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ \frac{4}{5}R_4 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{array}\right] \begin{array}{l} R_1 - \frac{1}{4}R_4 \\ \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{bmatrix}$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1} $\quad A=\begin{bmatrix}x\end{bmatrix}$

Solution

For A^{-1} exists, $x \neq 0$

$$AA^{-1}=I$$

$$\begin{bmatrix}x\end{bmatrix}\begin{bmatrix}a\end{bmatrix}=\begin{bmatrix}1\end{bmatrix}$$

$$xa=1$$

$$\boxed{a=\frac{1}{x}}$$

$$A^{-1}=\begin{bmatrix}\frac{1}{x}\end{bmatrix}$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1} $\quad A=\begin{bmatrix}x & 0\\ 0 & y\end{bmatrix}$

Solution

For A^{-1} exists, $\boxed{x, y \neq 0}$

$$\begin{bmatrix}x & 0\\ 0 & y\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad AA^{-1}=I$$

$$\begin{cases}ax=1 & bx=0\\ cy=0 & dy=1\end{cases}$$

$$\begin{cases}a=\frac{1}{x} & b=0\\ c=0 & d=\frac{1}{y}\end{cases}$$

$$A^{-1}=\begin{bmatrix}\frac{1}{x} & 0\\ 0 & \frac{1}{y}\end{bmatrix}$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1} $\quad A=\begin{bmatrix}0&0&x\\0&y&0\\z&0&0\end{bmatrix}$

Solution

For A^{-1} exists, $x, y, z \neq 0$

$$\begin{pmatrix}0&0&x\\0&y&0\\z&0&0\end{pmatrix}\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \qquad AA^{-1}=I$$

$$\begin{pmatrix}xg&xh&xi\\yd&ye&yf\\za&zb&zc\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\begin{cases}xg=1 & xh=0 & xi=0\\ yd=0 & ye=1 & yf=0\\ za=0 & zb=0 & zc=1\end{cases}$$

$$\begin{cases}g=\frac{1}{x} & h=0 & i=0\\ d=0 & e=\frac{1}{y} & f=0\\ a=0 & b=0 & c=\frac{1}{z}\end{cases}$$

$$A^{-1}=\begin{pmatrix}0&0&\frac{1}{z}\\0&\frac{1}{y}&0\\\frac{1}{z}&0&0\end{pmatrix}$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1} $\quad A=\begin{bmatrix}x&1&1&1\\0&y&0&0\\0&0&z&0\\0&0&0&w\end{bmatrix}$

Solution

For A^{-1} exists, $x, y, z, w \neq 0$

$$\begin{pmatrix} x & 1 & 1 & 1 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad AA^{-1} = I$$

$$\begin{cases} xa_{11} + xa_{21} + xa_{31} + xa_{41} = 1 \\ xa_{12} + xa_{22} + xa_{32} + xa_{42} = 0 \\ xa_{13} + xa_{23} + xa_{33} + xa_{43} = 0 \\ xa_{14} + xa_{24} + xa_{34} + xa_{44} = 0 \end{cases}$$

$$\begin{cases} ya_{21} = 0 & a_{21} = 0 \\ ya_{22} = 1 & a_{22} = \frac{1}{y} \\ ya_{23} = 0 & a_{23} = 0 \\ ya_{24} = 0 & a_{24} = 0 \end{cases}$$

$$\begin{cases} za_{31} = 0 & a_{31} = 0 \\ za_{32} = 0 & a_{32} = 0 \\ za_{33} = 1 & a_{33} = \frac{1}{z} \\ za_{34} = 0 & a_{34} = 0 \end{cases}$$

$$\begin{cases} wa_{41} = 0 & a_{41} = 0 \\ wa_{42} = 0 & a_{42} = 0 \\ wa_{43} = 0 & a_{43} = 0 \\ wa_{44} = 1 & a_{44} = \frac{1}{w} \end{cases}$$

$$\rightarrow \begin{cases} xa_{11} = 1 & a_{11} = \frac{1}{x} \\ xa_{12} + \frac{x}{y} = 0 & a_{12} = -\frac{1}{y} \\ xa_{13} + \frac{x}{z} = 0 & a_{13} = -\frac{1}{z} \\ xa_{14} + \frac{x}{w} = 0 & a_{14} = -\frac{1}{w} \end{cases}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{x} & -\frac{1}{y} & -\frac{1}{z} & -\frac{1}{w} \\ 0 & \frac{1}{y} & 0 & 0 \\ 0 & 0 & \frac{1}{z} & 0 \\ 0 & 0 & 0 & \frac{1}{w} \end{pmatrix}$$

Exercise

Solve the system using A^{-1} $\begin{cases} x \quad\quad +2z = 6 \\ -x+2y+3z=-5 \\ x-y \quad\quad = 6 \end{cases}$ Given $A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

Solution

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3(6)-2(-5)-4(6) \\ 3(6)-2(-5)-5(6) \\ -1(6)+1(-5)+2(6) \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

Solution: $(4,\ -2,\ 1)$

Exercise

Solve the system using A^{-1} $\begin{cases} x+2y+5z=2 \\ 2x+3y+8z=3 \\ -x+\ y+2z=3 \end{cases}$

a) Write the linear system as a matrix equation in the form $AX = B$

b) Solve the system using the inverse that is given for the coefficient matrix

the inverse of $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{bmatrix}$ *is* $\begin{bmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{bmatrix}$

Solution

a) $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$

b) $\begin{bmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}=\begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$

Exercise

Solve the system using A^{-1} $\begin{cases} x-y+z=8 \\ \quad 2y-z=-7 \\ 2x+3y \quad =1 \end{cases}$

a) Write the linear system as a matrix equation in the form $AX=B$

b) Solve the system using the inverse that is given for the coefficient matrix

the inverse is $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

Solution

a) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 8 \\ -7 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -7 \\ 1 \end{bmatrix}=\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 3x+2y=-4 \\ 2x-y=-5 \end{cases}$

Solution

$A=\begin{pmatrix} 3 & 2 \\ 2 & -1 \end{pmatrix} \quad B=\begin{pmatrix} -4 \\ 5 \end{pmatrix}$

$A^{-1}=-\frac{1}{7}\begin{pmatrix} -1 & -2 \\ -2 & 3 \end{pmatrix}$

$$= \begin{pmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{pmatrix} \begin{pmatrix} -4 \\ 5 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{6}{7} \\ -\frac{23}{7} \end{pmatrix}$$

$\therefore$ ***Solution***: $\left(\frac{6}{7},\ -\frac{23}{7}\right)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$

Solution

$$A = \begin{pmatrix} 2 & 5 \\ 5 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{29} \begin{pmatrix} -2 & -5 \\ -5 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{29} & \frac{5}{29} \\ \frac{5}{29} & -\frac{2}{29} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{2}{29} & \frac{5}{29} \\ \frac{5}{29} & -\frac{2}{29} \end{pmatrix} \begin{pmatrix} 7 \\ -3 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{29} \\ -\frac{41}{7} \end{pmatrix}$$

$\therefore$ ***Solution***: $\left(-\frac{1}{29},\ -\frac{41}{29}\right)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$

Solution

$$A = \begin{pmatrix} 4 & -7 \\ 2 & 5 \end{pmatrix} \quad B = \begin{pmatrix} -16 \\ 9 \end{pmatrix}$$

$$A^{-1} = \frac{1}{34}\begin{pmatrix} 5 & 7 \\ -2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{34} & \frac{7}{34} \\ -\frac{1}{17} & \frac{2}{17} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{5}{34} & \frac{7}{34} \\ -\frac{1}{17} & \frac{2}{17} \end{pmatrix}\begin{pmatrix} -16 \\ 9 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 2 \end{pmatrix}$$

$\therefore$ ***Solution***: $\left(-\frac{1}{2},\ 2\right)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$

Solution

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$A^{-1} = -1\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$$

$$X = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

$\therefore$ ***Solution***: $(-2,\ 5)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 3x+4y=2 \\ 2x+5y=-1 \end{cases}$

Solution

$$A=\begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} \quad B=\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$A^{-1}=\frac{1}{7}\begin{pmatrix} 5 & -4 \\ -2 & 3 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{5}{7} & -\frac{4}{7} \\ -\frac{2}{7} & \frac{3}{7} \end{pmatrix}$$

$$X=\begin{pmatrix} \frac{5}{7} & -\frac{4}{7} \\ -\frac{2}{7} & \frac{3}{7} \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

∴ ***Solution***: $(2,\ -1)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 5x-2y=4 \\ -10x+4y=7 \end{cases}$

Solution

$$A=\begin{pmatrix} 5 & -2 \\ -10 & 4 \end{pmatrix} \quad B=\begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

$$A^{-1}=\frac{1}{0}\begin{pmatrix} 4 & 2 \\ 10 & 5 \end{pmatrix}$$

Inverse matrix ***doesn't exist***.

$$\begin{cases} 5x-2y=4 \\ -\frac{1}{2}\quad 5x-2y=-\frac{7}{2} \end{cases}$$

$$4\neq -\frac{7}{2}$$

∴ ***No Solution***

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x - 4y = -8 \\ 5x - 20y = -40 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & -4 \\ 5 & -20 \end{pmatrix} \quad B = \begin{pmatrix} -8 \\ -40 \end{pmatrix}$$

$$A^{-1} = \frac{1}{0}\begin{pmatrix} & \end{pmatrix}$$

Inverse matrix ***doesn't exist***.

$$\frac{1}{5}\begin{cases} x - 4y = -8 \\ x - 4y = -8 \end{cases}$$

$\therefore$ ***Solution***: $\quad \underline{(4y-8,\ y)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$

Solution

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{3}\begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}\begin{pmatrix} 3 \\ 3 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$\therefore$ ***Solution***: $\quad \underline{(2,\ -1)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x+10y=-14 \\ 7x-2y=-16 \end{cases}$

Solution

$$A=\begin{pmatrix} 2 & 10 \\ 7 & -2 \end{pmatrix} \quad B=\begin{pmatrix} -14 \\ -16 \end{pmatrix}$$

$$\begin{aligned} A^{-1} &= -\frac{1}{74}\begin{pmatrix} -2 & -10 \\ -7 & 2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{37} & \frac{5}{37} \\ \frac{7}{74} & -\frac{1}{37} \end{pmatrix} \end{aligned}$$

$$X=\begin{pmatrix} \frac{1}{37} & \frac{5}{37} \\ \frac{7}{74} & -\frac{1}{37} \end{pmatrix}\begin{pmatrix} -14 \\ -16 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -\frac{94}{37} \\ -\frac{33}{37} \end{pmatrix}$$

$\therefore$ ***Solution***: $\left(-\frac{94}{37},\ -\frac{33}{37}\right)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 4x-3y=24 \\ -3x+9y=-1 \end{cases}$

Solution

$$A=\begin{pmatrix} 4 & -3 \\ -3 & 9 \end{pmatrix} \quad B=\begin{pmatrix} 24 \\ -1 \end{pmatrix}$$

$$\begin{aligned} A^{-1} &= \frac{1}{27}\begin{pmatrix} 9 & 3 \\ 3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} & \frac{1}{9} \\ \frac{1}{9} & \frac{4}{27} \end{pmatrix} \end{aligned}$$

$$X=\begin{pmatrix} \frac{1}{3} & \frac{1}{9} \\ \frac{1}{9} & \frac{4}{27} \end{pmatrix}\begin{pmatrix} 24 \\ -1 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{71}{9} \\ \frac{68}{27} \end{pmatrix}$$

$\therefore$ ***Solution***: $\left(\frac{71}{9},\ \frac{68}{27}\right)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$

Solution

$$A = \begin{pmatrix} 4 & 2 \\ 3 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 12 \\ 16 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{14}\begin{pmatrix} -2 & -2 \\ -3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{3}{14} & -\frac{2}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{3}{14} & -\frac{2}{7} \end{pmatrix}\begin{pmatrix} 12 \\ 16 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$\therefore$ ***Solution***: $(4,\ -2)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & 2 \\ 4 & -2 \end{pmatrix} \quad B = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{10}\begin{pmatrix} -2 & -2 \\ -4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{10} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{10} \end{pmatrix} \begin{pmatrix} -1 \\ 6 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\therefore$ ***Solution***: $\quad \underline{(1,\ -1)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x - 2y = 5 \\ -10x + 2y = 4 \end{cases}$

Solution

$$\frac{1}{2}\begin{cases} x - 2y = 5 \\ -5x + y = 2 \end{cases}$$

$$A = \begin{pmatrix} 1 & -2 \\ -5 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{9}\begin{pmatrix} 1 & 2 \\ 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{9} & -\frac{2}{9} \\ -\frac{5}{9} & -\frac{1}{9} \end{pmatrix}$$

$$X = \begin{pmatrix} -\frac{1}{9} & -\frac{2}{9} \\ -\frac{5}{9} & -\frac{1}{9} \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

$\therefore$ ***Solution***: $\quad \underline{(-1,\ -3)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 12x+15y=-27 \\ 30x-15y=-15 \end{cases}$

Solution

$$\begin{array}{c} \frac{1}{3} \\ \frac{1}{15} \end{array} \to \begin{cases} 4x+5y=-9 \\ 2x-y=-1 \end{cases}$$

$$A=\begin{pmatrix} 4 & 5 \\ 2 & -1 \end{pmatrix} \quad B=\begin{pmatrix} -9 \\ -1 \end{pmatrix}$$

$$A^{-1}=-\frac{1}{14}\begin{pmatrix} -1 & -5 \\ -2 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{14} & \frac{5}{14} \\ \frac{1}{7} & -\frac{2}{7} \end{pmatrix}$$

$$X=\begin{pmatrix} \frac{1}{14} & \frac{5}{14} \\ \frac{1}{7} & -\frac{2}{7} \end{pmatrix}\begin{pmatrix} -9 \\ -1 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$\therefore$ ***Solution***: $(-1,\ -1)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 4x-4y=-12 \\ 4x+4y=-20 \end{cases}$

Solution

$$\frac{1}{4} \to \begin{cases} x-y=-3 \\ x+y=-5 \end{cases}$$

$$A=\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad B=\begin{pmatrix} -3 \\ -5 \end{pmatrix}$$

$$A^{-1}=\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -3 \\ -5 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$$

$\therefore$ ***Solution***: $\underline{(-4,\ -1)}|$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} -2x+3y=4 \\ -3x+4y=5 \end{cases}$

Solution

$$A = \begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}$$

$$X = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\therefore$ ***Solution***: $\underline{(1,\ 2)}|$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x-2y=6 \\ 4x+3y=2 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & -2 \\ 4 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{11} \begin{pmatrix} 3 & 2 \\ -4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{11} & \frac{2}{11} \\ -\frac{4}{11} & \frac{1}{11} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{3}{11} & \frac{2}{11} \\ -\frac{4}{11} & \frac{1}{11} \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$\therefore$ ***Solution***: $\quad (2,\ -2)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x - 3y = 7 \\ 4x + y = -7 \end{cases}$

Solution

$$A = \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 7 \\ -7 \end{pmatrix}$$

$$A^{-1} = \frac{1}{14}\begin{pmatrix} 1 & 3 \\ -4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{14} & \frac{3}{14} \\ -\frac{2}{7} & \frac{1}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{14} & \frac{3}{14} \\ -\frac{2}{7} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} 7 \\ -7 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

$\therefore$ ***Solution***: $\quad (-1,\ -3)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} \\ -R_2 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & -1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 2R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & 3 & 2 & -1 & 0 \\ 0 & 0 & 6 & 3 & -2 & 1 \end{array}\right) \begin{array}{l} \\ \\ \frac{1}{6}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & 3 & 2 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{3} & \frac{1}{6} \end{array}\right) \begin{array}{l} R_1 + 2R_3 \\ R_2 - 3R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{3} & \frac{1}{6} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$\therefore$ ***Solution***: $\underline{(1,\ 2,\ -1)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x+y+z=9 \\ -x-y+z=1 \\ 3x-y+z=9 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & -1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -1 \\ 9 \\ 9 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2-2R_1 \\ R_2-3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 0 & -4 & 4 & -3 & 0 & 1 \end{array}\right) \begin{array}{l} \\ -R_2 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & -4 & 4 & -3 & 0 & 1 \end{array}\right) \begin{array}{l} R_1-R_2 \\ \\ R_3+4R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & -8 & 5 & -4 & 1 \end{array}\right) \begin{array}{l} \\ \\ -\frac{1}{8}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{array}\right) \begin{array}{l} R_1-2R_3 \\ R_2+3R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{8} & \frac{1}{2} & -\frac{3}{8} \\ 0 & 0 & 1 & -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & -\frac{3}{8} \\ -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & -\frac{3}{8} \\ -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{pmatrix} \begin{pmatrix} -1 \\ 9 \\ 9 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$$

∴ ***Solution***: $(2,\ 1,\ 4)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & 5 & -1 \\ 0 & 3 & -1 \\ -3 & 6 & 2 \end{pmatrix} \quad B = \begin{pmatrix} -4 \\ -1 \\ 11 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 5 & -1 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ -3 & 6 & 2 & 0 & 0 & 1 \end{array}\right) \quad R_3 + 3R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 5 & -1 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 0 & 21 & -1 & 3 & 0 & 1 \end{array}\right) \quad \frac{1}{3}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 5 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 21 & -1 & 3 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 - 5R_2 \\ \\ R_3 - 21R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & 1 & -\frac{5}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 6 & 3 & -7 & 1 \end{array}\right) \quad \frac{1}{6}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & 1 & -\frac{5}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{array}\right) \quad \begin{array}{l} R_1 - \frac{2}{3}R_3 \\ R_2 + \frac{1}{3}R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & \frac{8}{9} & -\frac{1}{9} \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{1}{18} & \frac{1}{18} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{8}{9} & -\frac{1}{9} \\ \frac{1}{6} & -\frac{1}{18} & \frac{1}{18} \\ \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{2}{3} & \frac{8}{9} & -\frac{1}{9} \\ \frac{1}{6} & -\frac{1}{18} & \frac{1}{18} \\ \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} -4 \\ -1 \\ 11 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore$ ***Solution***: $\underline{(-3,\ 0,\ 1)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x+3y+4z=14 \\ 2x-3y+2z=10 \\ 3x-y+z=9 \end{cases}$

Solution

$$A=\begin{pmatrix} 1 & 3 & 4 \\ 2 & -3 & 2 \\ 3 & -1 & 1 \end{pmatrix} \quad B=\begin{pmatrix} 14 \\ 10 \\ 9 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 2 & -3 & 2 & 0 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2-2R_1 \\ R_3-3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & -9 & -6 & -2 & 1 & 0 \\ 0 & -10 & -11 & -3 & 0 & 1 \end{array}\right) -\tfrac{1}{9}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{9} & -\frac{1}{9} & 0 \\ 0 & -10 & -11 & -3 & 0 & 1 \end{array}\right) \begin{array}{l} R_1-3R_2 \\ \\ R_3+10R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{9} & -\frac{1}{9} & 0 \\ 0 & 0 & -\frac{13}{3} & -\frac{7}{9} & -\frac{10}{9} & 1 \end{array}\right) \begin{array}{l} \\ \\ -\frac{3}{13}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{9} & -\frac{1}{9} & 0 \\ 0 & 0 & 1 & \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{array}\right) \begin{array}{l} R_2-2R_3 \\ R_2-\frac{2}{3}R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{39} & -\frac{7}{39} & \frac{6}{13} \\ 0 & 1 & 0 & \frac{4}{39} & -\frac{11}{39} & \frac{2}{13} \\ 0 & 0 & 1 & \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{1}{39} & -\frac{7}{39} & \frac{6}{13} \\ \frac{4}{39} & -\frac{11}{39} & \frac{2}{13} \\ \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{pmatrix}$$

$$X = \begin{pmatrix} -\frac{1}{39} & -\frac{7}{39} & \frac{6}{13} \\ \frac{4}{39} & -\frac{11}{39} & \frac{2}{13} \\ \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{pmatrix} \begin{pmatrix} 14 \\ 10 \\ 9 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

∴ ***Solution***: $(2,\ 0,\ 3)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & -3 & 2 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -10 & 4 & -3 & 1 & 0 \\ 0 & -11 & 4 & -2 & 0 & 1 \end{array}\right) \quad \begin{array}{l} 5R_1 + 2R_2 \\ \\ 10R_3 - 11R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 5 & 0 & 3 & -1 & 2 & 0 \\ 0 & -10 & 4 & -3 & 1 & 0 \\ 0 & 0 & -4 & 13 & -11 & 10 \end{array}\right) \quad \begin{array}{l} 4R_1 + 3R_3 \\ R_2 + R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 20 & 0 & 0 & 35 & -25 & 30 \\ 0 & -10 & 0 & 10 & -10 & 10 \\ 0 & 0 & -4 & 13 & -11 & 10 \end{array}\right) \quad \begin{array}{l} \frac{1}{20}R_1 \\ -\frac{1}{10}R_2 \\ -\frac{1}{4}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{4} & -\frac{5}{4} & \frac{3}{2} \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -\frac{13}{4} & \frac{11}{4} & -\frac{5}{2} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{7}{4} & -\frac{5}{4} & \frac{3}{2} \\ -1 & 1 & -1 \\ -\frac{13}{4} & \frac{11}{4} & -\frac{5}{2} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{7}{4} & -\frac{5}{4} & \frac{3}{2} \\ -1 & 1 & -1 \\ -\frac{13}{4} & \frac{11}{4} & -\frac{5}{2} \end{pmatrix}\begin{pmatrix} 20 \\ 8 \\ -16 \end{pmatrix} \qquad X = A^{-1}B$$

$$= \begin{pmatrix} 35-10-24 \\ -20+8+16 \\ -65+22+40 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

$\therefore$ ***Solution***: $(1,\ 4,\ -3)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 2 & -3 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 17 \\ 7 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 2 & -3 & 2 & 0 & 0 & 1 \end{array}\right) \quad R_3 - 2R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & -7 & 0 & -2 & 0 & 1 \end{array}\right) \quad \frac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -7 & 0 & -2 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 7R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{7}{2} & -2 & \frac{7}{2} & 1 \end{array}\right) \begin{array}{l} \\ \\ -\frac{2}{7}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{4}{7} & -1 & -\frac{2}{7} \end{array}\right) \begin{array}{l} R_1 - 2R_3 \\ R_2 + \frac{1}{2}R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{7} & 1 & \frac{4}{7} \\ 0 & 1 & 0 & \frac{2}{7} & 0 & -\frac{1}{7} \\ 0 & 0 & 1 & \frac{4}{7} & -1 & -\frac{2}{7} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{1}{7} & 1 & \frac{4}{7} \\ \frac{2}{7} & 0 & -\frac{1}{7} \\ \frac{4}{7} & -1 & -\frac{2}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} -\frac{1}{7} & 1 & \frac{4}{7} \\ \frac{2}{7} & 0 & -\frac{1}{7} \\ \frac{4}{7} & -1 & -\frac{2}{7} \end{pmatrix} \begin{pmatrix} 17 \\ 7 \\ -1 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$$

∴ ***Solution***: $\underline{(4,\ 5,\ 3)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} -2x+6y+7z=3 \\ -4x+5y+3z=7 \\ -6x+3y+5z=-4 \end{cases}$

Solution

$$A=\begin{pmatrix} -2 & 6 & 7 \\ -4 & 5 & 3 \\ -6 & 3 & 5 \end{pmatrix} \quad B=\begin{pmatrix} 3 \\ 7 \\ -4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} -2 & 6 & 7 & 1 & 0 & 0 \\ -4 & 5 & 3 & 0 & 1 & 0 \\ -6 & 3 & 5 & 0 & 0 & 1 \end{array}\right) \quad -\frac{1}{2}R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & -3 & -\frac{7}{2} & -\frac{1}{2} & 0 & 0 \\ -4 & 5 & 3 & 0 & 1 & 0 \\ -6 & 3 & 5 & 0 & 0 & 1 \end{array}\right) \quad \begin{matrix} \\ R_2+4R_1 \\ R_3+6R_1 \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -3 & -\frac{7}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & -7 & -11 & -2 & 1 & 0 \\ 0 & -15 & -16 & -3 & 0 & 1 \end{array}\right) \quad -\frac{1}{7}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & -3 & -\frac{7}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{11}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & -15 & -16 & -3 & 0 & 1 \end{array}\right) \quad \begin{matrix} R_1+3R_2 \\ \\ R_3+15R_2 \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{17}{14} & \frac{5}{14} & -\frac{3}{7} & 0 \\ 0 & 1 & \frac{11}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & \frac{53}{7} & \frac{9}{7} & -\frac{15}{7} & 1 \end{array}\right) \quad \begin{matrix} \\ \\ \frac{7}{53}R_3 \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{17}{14} & \frac{5}{14} & -\frac{3}{7} & 0 \\ 0 & 1 & \frac{11}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & 1 & \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{array}\right) \quad \begin{matrix} R_1-\frac{17}{14}R_3 \\ R_2-\frac{11}{7}R_3 \\ \\ \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{8}{53} & -\frac{9}{106} & -\frac{17}{106} \\ 0 & 1 & 0 & \frac{1}{53} & \frac{16}{53} & -\frac{11}{53} \\ 0 & 0 & 1 & \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{8}{53} & -\frac{9}{106} & -\frac{7}{106} \\ \frac{1}{53} & \frac{16}{53} & -\frac{11}{53} \\ \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{8}{53} & -\frac{9}{106} & -\frac{7}{106} \\ \frac{1}{53} & \frac{16}{53} & -\frac{11}{53} \\ \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{pmatrix}\begin{pmatrix} 3 \\ 7 \\ -4 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 3 \\ -2 \end{pmatrix}$$

$\therefore$ ***Solution***: $\left(\frac{1}{2},\ 3,\ -2\right)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$

Solution

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -3 & 4 \\ 4 & -2 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 1 & 1 & 0 & 0 \\ 3 & -3 & 4 & 0 & 1 & 0 \\ 4 & -2 & 3 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} \\ 2R_2 - 3R_1 \\ 2R_3 - 4R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 1 & 1 & 0 & 0 \\ 0 & -3 & 5 & -3 & 2 & 0 \\ 0 & 0 & 2 & -4 & 0 & 2 \end{array}\right) \quad \begin{array}{l} 3R_1 - R_2 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 6 & 0 & -2 & 6 & -2 & 0 \\ 0 & -3 & 5 & -3 & 2 & 0 \\ 0 & 0 & 2 & -4 & 0 & 2 \end{array}\right) \quad \begin{array}{l} R_1 + R_3 \\ 2R_2 - 5R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 6 & 0 & 0 & 2 & -2 & 2 \\ 0 & -6 & 0 & 14 & 4 & -10 \\ 0 & 0 & 2 & -4 & 0 & 2 \end{array}\right) \quad \begin{array}{l} \frac{1}{6}R_1 \\ -\frac{1}{6}R_2 \\ \frac{1}{2}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{7}{6} & -\frac{2}{3} & \frac{5}{3} \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{7}{6} & -\frac{2}{3} & \frac{5}{3} \\ -2 & 0 & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{7}{6} & -\frac{2}{3} & \frac{5}{3} \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

∴ ***Solution***: $\underline{(0,\ 1,\ 2)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x - 2y - z = 2 \\ 2x - y + z = 4 \\ -x + y + z = 4 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 3 & -2 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{array}\right) \quad \frac{1}{3}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 + 2R_2 \\ \\ R_3 + R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 1 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & 1 \end{array}\right) \quad \begin{array}{l} R_1 - R_3 \\ R_2 - R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{3} & \frac{1}{3} & -1 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & -1 \\ -1 & 0 & -1 \\ \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & -1 \\ -1 & 0 & -1 \\ \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \\ 6 \end{pmatrix}$$

∴ ***Solution***: $\underline{(-4,\ -6,\ 6)}$

Solution   *Section* 4.4 – Determinants

Exercise

Evaluate $\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix}$

Solution

$$\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix} = -9-(-6)$$

$$\underline{=-3}$$

Exercise

Evaluate $\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix} = -6-(0)$$

$$\underline{=-6}$$

Exercise

Evaluate $\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix} = x(8x)-4x(2x)$$

$$=8x^2-8x^2$$

$$\underline{=0}$$

Exercise

Evaluate $\begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix} = 3x - 2x(4)$$

$$= 3x - 8x$$

$$\underline{= -5x}$$

Exercise

Evaluate $\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix}$

Solution

$$\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix} = \underline{-3x^4 - 2x}$$

Exercise

Evaluate $\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix}$

Solution

$$\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix} = \underline{-8a + 5b}$$

Exercise

Evaluate $\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix}$

Solution

$$\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix} = 15 - 14$$

$$\underline{= 1}$$

Exercise

Evaluate $\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix} = 5 - 20$$

$$\underline{= -16}$$

Exercise

Evaluate $\begin{vmatrix} 5 & 3 \\ -2 & 3 \end{vmatrix}$

Solution

$$\begin{vmatrix} 5 & 3 \\ -2 & 3 \end{vmatrix} = 15 + 6$$

$$\underline{= 21}$$

Exercise

Evaluate $\begin{vmatrix} -4 & -1 \\ 5 & 6 \end{vmatrix}$

Solution

$$\begin{vmatrix} -4 & -1 \\ 5 & 6 \end{vmatrix} = -24 + 5$$

$$\underline{= -19}$$

Exercise

Evaluate $\begin{vmatrix} \sqrt{3} & -2 \\ -3 & \sqrt{3} \end{vmatrix}$

Solution

$$\begin{vmatrix} \sqrt{3} & -2 \\ -3 & \sqrt{3} \end{vmatrix} = 3 - 6$$

$$\underline{= -3}$$

Exercise

Evaluate $\begin{vmatrix} \sqrt{7} & 6 \\ -3 & \sqrt{7} \end{vmatrix}$

Solution

$$\begin{vmatrix} \sqrt{7} & 6 \\ -3 & \sqrt{7} \end{vmatrix} = 7 + 18$$

$$\underline{= 25}$$

Exercise

Evaluate $\begin{vmatrix} \sqrt{5} & 3 \\ -2 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} \sqrt{5} & 3 \\ -2 & 2 \end{vmatrix} \underline{= 2\sqrt{5} + 6}$$

Exercise

Evaluate $\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{vmatrix}$

Solution

$$\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{vmatrix} = -\frac{3}{8} - \frac{1}{16}$$

$$\underline{= -\frac{7}{16}}$$

Exercise

Evaluate $\begin{vmatrix} \frac{1}{5} & \frac{1}{6} \\ -6 & -5 \end{vmatrix}$

Solution

$$\begin{vmatrix} \frac{1}{5} & \frac{1}{6} \\ -6 & -5 \end{vmatrix} = -1 + 1$$

$$\underline{= 0}$$

Exercise

Evaluate $\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{2} & \frac{3}{4} \end{vmatrix}$

Solution

$$\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{2} & \frac{3}{4} \end{vmatrix} = \frac{1}{2} + \frac{1}{6}$$

$$= \frac{2}{3}$$

Exercise

Evaluate $\begin{vmatrix} x & x^2 \\ 4 & x \end{vmatrix}$

Solution

$$\begin{vmatrix} x & x^2 \\ 4 & x \end{vmatrix} = x^2 - 4x^2$$

$$= -3x^2$$

Exercise

Evaluate $\begin{vmatrix} x & x^2 \\ x & 9 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & x^2 \\ x & 9 \end{vmatrix} = 9x - x^3$$

Exercise

Evaluate $\begin{vmatrix} x^2 & x \\ -3 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} x^2 & x \\ -3 & 2 \end{vmatrix} = 2x^2 + 3x$$

Exercise

Evaluate $\begin{vmatrix} x+2 & 6 \\ x-2 & 4 \end{vmatrix}$

Solution

$$\begin{vmatrix} x+2 & 6 \\ x-2 & 4 \end{vmatrix} = 4(x+2) - 6(x-2)$$

$$= 4x + 8 - 6x + 12$$

$$\underline{= -2x + 20}$$

Exercise

Evaluate $\begin{vmatrix} x+1 & -6 \\ x+3 & -3 \end{vmatrix}$

Solution

$$\begin{vmatrix} x+1 & -6 \\ x+3 & -3 \end{vmatrix} = -3x - 3 + 6x + 18$$

$$\underline{= -2x + 20}$$

Exercise

Evaluate $\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & -5 \\ 2 & 5 & -1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & -5 \\ 2 & 5 & -1 \end{vmatrix} \begin{matrix} 3 & 0 \\ 2 & 1 \\ 2 & 5 \end{matrix}$$

$$= -3 + 0 + 0 - 0 + 75 - 0$$

$$\underline{= 72}$$

Exercise

Evaluate $\begin{vmatrix} 4 & 0 & 0 \\ 3 & -1 & 4 \\ 2 & -3 & 6 \end{vmatrix}$

Solution

$$\begin{vmatrix} 4 & 0 & 0 \\ 3 & -1 & 4 \\ 2 & -3 & 6 \end{vmatrix} \begin{matrix} 4 & 0 \\ 3 & -1 \\ 2 & -3 \end{matrix} \qquad or \quad = 4\begin{vmatrix} -1 & 4 \\ -3 & 6 \end{vmatrix}$$

$$= -24 + 48$$

$$\underline{= 24}$$

Exercise

Evaluate $\begin{vmatrix} 3 & 1 & 0 \\ -3 & -4 & 0 \\ -1 & 3 & 5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 3 & 1 & 0 \\ -3 & -4 & 0 \\ -1 & 3 & 5 \end{vmatrix} \begin{matrix} 3 & 1 \\ -3 & -4 \\ -1 & 3 \end{matrix}$$

$$= -60 + 15$$

$$\underline{= -45}$$

Exercise

Evaluate $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & -4 & 5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & -4 & 5 \end{vmatrix} \begin{matrix} 1 & 1 \\ 2 & 2 \\ 3 & -4 \end{matrix}$$

$$= 10 + 6 - 8 - 6 + 8 - 10$$

$$\underline{= 0}$$

Exercise

Evaluate $\begin{vmatrix} x & 0 & -1 \\ 2 & 1 & x^2 \\ -3 & x & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 0 & -1 \\ 2 & 1 & x^2 \\ -3 & x & 1 \end{vmatrix} \begin{matrix} x & 0 \\ 2 & 1 \\ -3 & x \end{matrix}$$

$$= x - 2x - 3 - x^4$$

$$\underline{= -x^4 - x - 3}$$

Exercise

Evaluate $\begin{vmatrix} x & 1 & -1 \\ x^2 & x & x \\ 0 & x & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 1 & -1 \\ x^2 & x & x \\ 0 & x & 1 \end{vmatrix} \begin{matrix} x & 1 \\ x^2 & x \\ 0 & x \end{matrix}$$

$$= x^2 - x^3 - x^3 - x^2$$

$$\underline{= -2x^3}$$

Exercise

Evaluate $\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix}$

Solution

$$\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix} = 0 + 126 + 48 - (-48 + 36 + 0)$$

$$\underline{= 90}$$

Exercise

Evaluate $\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix} = 0 - 4 - 16 - (-14 - 16 + 0)$$

$$= 10$$

Exercise

Evaluate $\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix}$

Solution

$$\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix} \begin{matrix} 3 & 1 \\ -2 & 3 \\ 3 & 4 \end{matrix}$$

$$= -54 + 3 - 16 - 18 - 12 - 12$$

$$= -109$$

Exercise

Evaluate $\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix} \begin{matrix} 2x & 1 \\ 0 & 4 \\ 3 & 0 \end{matrix}$$

$$= 16x + 3x + 12$$

$$= 19x + 12$$

Exercise

Evaluate $\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix} \begin{matrix} 0 & x \\ x & x^2 \\ x & 7 \end{matrix}$$

$$= 5x^2 + 7x^2 - x^4 + 5x^2$$

$$= 17x^2 - x^4$$

Exercise

Evaluate $\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix} \begin{matrix} 2 & x \\ -3 & 1 \\ 2 & 1 \end{matrix}$$

$$= 8 - 3 - 2 + 12x$$

$$= 12x + 3$$

Exercise

Evaluate $\begin{vmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix} \begin{matrix} 1 & x \\ 3 & 1 \\ 0 & -2 \end{matrix}$$

$$= 2 + 12 + 2 - 6x$$

$$= -6x + 16$$

Exercise

Solve for x. $\begin{vmatrix} x & 3 \\ 2 & 1 \end{vmatrix} = 12$

Solution

$$\begin{vmatrix} x & 3 \\ 2 & 1 \end{vmatrix} = x - 6 = 12$$

$\therefore$ ***Solution***: $\underline{x = 18}$

Exercise

Solve for x. $\begin{vmatrix} x & 1 \\ 2 & x \end{vmatrix} = -1$

Solution

$$\begin{vmatrix} x & 1 \\ 2 & x \end{vmatrix} = x^2 - 2 = -1$$

$$x^2 = 1$$

$\therefore$ ***Solution***: $\underline{x = \pm 1}$

Exercise

Solve for x. $\begin{vmatrix} 3 & x \\ x & 4 \end{vmatrix} = -13$

Solution

$$\begin{vmatrix} 3 & x \\ x & 4 \end{vmatrix} = 12 - x^2 = -13$$

$$x^2 = 25$$

$\therefore$ ***Solution***: $\underline{x = \pm 5}$

Exercise

Solve for x. $\begin{vmatrix} x & 2 \\ 3 & x \end{vmatrix} = x$

Solution

$$\begin{vmatrix} x & 2 \\ 3 & x \end{vmatrix} = x^2 - 6 = x$$

$$x^2 - x - 6 = 0$$

$\therefore$ ***Solution***: $\underline{x = -2,\ 3}$

Exercise

Solve for x. $\begin{vmatrix} 4 & 6 \\ -2 & x \end{vmatrix} = 32$

Solution

$$\begin{vmatrix} 4 & 6 \\ -2 & x \end{vmatrix} = 4x + 12 = 32$$

$$4x = 20$$

$\therefore$ ***Solution***: $\underline{x = 5}$

Exercise

Solve for x. $\begin{vmatrix} x+2 & -3 \\ x+5 & -4 \end{vmatrix} = 3x - 5$

Solution

$$\begin{vmatrix} x+2 & -3 \\ x+5 & -4 \end{vmatrix} = -4x - 8 + 3x + 15 = 3x - 5$$

$$-4x = -12$$

$\therefore$ ***Solution***: $\underline{x = 3}$

Exercise

Solve for x. $\begin{vmatrix} x+3 & -6 \\ x-2 & -4 \end{vmatrix} = 28$

Solution

$$\begin{vmatrix} x+3 & -6 \\ x-2 & -4 \end{vmatrix} = -4x - 12 + 6x - 12 = 28$$

$$2x = 52$$

$\therefore$ ***Solution***: $\underline{x = 26}$

Exercise

Solve for x. $\begin{vmatrix} x & -3 \\ -1 & x \end{vmatrix} \ge 0$

Solution

$$\begin{vmatrix} x & -3 \\ -1 & x \end{vmatrix} = x^2 - 3 \geq 0$$

$$x^2 \geq 3$$

$\therefore$ ***Solution***: $\underline{x \leq -\sqrt{3} \quad x \geq \sqrt{3}}$

Exercise

Solve for x. $\begin{vmatrix} 2 & x & 1 \\ 1 & 2 & -1 \\ 3 & 4 & -2 \end{vmatrix} = -6$

Solution

$$\begin{vmatrix} 2 & x & 1 \\ 1 & 2 & -1 \\ 3 & 4 & -2 \end{vmatrix} = -8 - 3x + 4 - 6 + 8 + 2x = -6$$

$$-x = -4$$

$\therefore$ ***Solution***: $\underline{x = 4}$

Exercise

Solve for x. $\begin{vmatrix} 1 & x & -3 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix} = 8$

Solution

$$\begin{vmatrix} 1 & x & -3 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix} = 2 + 18 + 2 - 6x = 8$$

$$-6x = -14$$

$\therefore$ ***Solution***: $\underline{x = \frac{7}{3}}$

Exercise

Solve for x. $\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix} = 39$

Solution

$$\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix} = 8-3-2+12x = 39$$

$12x = 36$

$\therefore$ ***Solution***: $\boxed{x=3}$

Exercise

Solve for x. $\begin{vmatrix} x & 0 & 0 \\ 7 & x & 1 \\ 7 & 2 & 1 \end{vmatrix} = -1$

Solution

$$\begin{vmatrix} x & 0 & 0 \\ 7 & x & 1 \\ 7 & 2 & 1 \end{vmatrix} = x^2 - 2x = -1$$

$x^2 - 2x + 1 = 0$

$\therefore$ ***Solution***: $\boxed{x=1}$