Solution Section 5.6 – Mathematical Induction

Exercise

Find all positive integers n for which the given statement is not true

$a)\quad 3^n > 6n \qquad b)\quad 3^n > 2n+1 \qquad c)\quad 2^n > n^2 \qquad d)\quad n! > 2n$

Solution

a) $n=1 \quad 3<6$

$n=2 \quad 3^2 < 18$

$n=3, \quad 27 > 18$

The statement is true for all $n \ge 3 \quad 3^n > 6n$

The statement is not true for $\boxed{n=1,\ 2}$

b) $n=1; \quad 3=3$

$n=2; \quad 9>5$

The statement is true for all $n \ge 2 \quad 3^n > 2n+1$

The statement is not true for $\boxed{n=1}$

c) $n=1; \quad 2<4$

$n=2; \quad 4=4$

$n=3; \quad 8<9$

$n=4; \quad 16=16$

$n=5; \quad 32>25$

The statement is true for all $n \ge 5; \quad 2^n > n^2$

The statement is not true for $\boxed{n=1,\ 2,\ 3,\ 4}$

d) $n=1; \quad 1<2$

$n=2; \quad 2<4$

$n=3; \quad 6=6$

$n=4; \quad 12>8$

The statement is true for all $n \ge 4; \quad n! > 2n$

The statement is not true for $\boxed{n=1,\ 2,\ 3}$

Exercise

Prove that the statement is true for every positive integer n. $2+4+6+\ldots+2n=n(n+1)$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 2=1(1+1)=2$; hence P_1 is true.

(2) Assume $2+4+6+\ldots+2k=k(k+1)$ is true

$$\Rightarrow 2+4+6+\ldots+2k+2(k+1)=(k+1)(k+1+1)?$$

$$\begin{aligned} 2+4+6+\ldots+2k+2(k+1) &= 2+4+6+\ldots+2k+2(k+1) \\ &= k(k+1)+2(k+1) \qquad \textit{Factor } (\mathbf{k+1}) \\ &= (k+1)(k+2) \\ &= (k+1)(k+1+1) \ \checkmark \end{aligned}$$

Hence P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $1+3+5+\ldots+(2n-1)=n^2$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 1\overset{?}{=}1^2=1$; hence P_1 is true.

(2) Assume $1+3+5+\ldots+(2k-1)=k^2$ is true

$$\Rightarrow 1+3+5+\ldots+(2(k+1)-1)=(k+1)^2 ?$$

$$\begin{aligned} 1+3+5+\ldots+(2k-1)+(2(k+1)-1) &= 1+3+5+\ldots+(2k-1)+(2k+2-1) \\ &= k^2+(2k+1) \\ &= k^2+2k+1 \\ &= (k+1)^2 \ \checkmark \end{aligned}$$

Hence P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $2+7+12+\ldots+(5n-3)=\frac{1}{2}n(5n-1)$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 2\overset{?}{=}\frac{1}{2}(1)(5(1)-1)=\frac{1}{2}(4)=2$; hence P_1 is true.

(2) Assume $2+7+12+\ldots+(5k-3)=\frac{1}{2}k(5k-1)$ is true

$$2+7+12+\ldots+(5(k+1)-3)=\frac{1}{2}(k+1)(5(k+1)-1)?$$

$$\begin{aligned}2+7+12+\ldots+(5k-3)+(5(k+1)-3) &= 2+7+12+\ldots+(5k-3)+(5k+5-3)\\ &=\frac{1}{2}k(5k-1)+(5k+2)\frac{2}{2}\\ &=\frac{1}{2}\left[5k^2-k+10k+4\right]\\ &=\frac{1}{2}\left[5k^2-k+5k+5k+5-1\right]\\ &=\frac{1}{2}\left[k(5k-1+5)+5k+5-1\right]\\ &=\frac{1}{2}\left[(k+1)(5k+5-1)\right]\\ &=\frac{1}{2}\left[(k+1)(5(k+1)-1)\right] \quad \checkmark \qquad P_{k+1} \text{ is also true.}\end{aligned}$$

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $1+2.2+3.2^2+\ldots+n.2^{n-1}=1+(n-1).2^n$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 1\overset{?}{=}1+(1-1)2^1=1-0=1$; hence P_1 is true.

(2) $1+2.2+3.2^2+\ldots+k.2^{k-1}=1+(k-1).2^k$ is true

$$1+2.2+3.2^2+\ldots+k.2^{k-1}+(k+1).2^{(k+1)-1}=1+((k+1)-1).2^{k+1}?$$

$$\begin{aligned}1+2.2+3.2^2+\ldots+k.2^{k-1}+(k+1).2^{(k+1)-1} &= 1+(k-1).2^k+(k+1).2^{k+1-1}\\ &=1+k.2^k-1.2^k+(k+1).2^k\\ &=1+k.2^k-1.2^k+k.2^k+1.2^k\\ &=1+2^1k.2^k\\ &=1+(k+0).2^{k+1}\\ &=1+((k+1)-1).2^{k+1} \quad \checkmark \qquad P_{k+1} \text{ is also true.}\end{aligned}$$

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 1^2 \stackrel{?}{=} \frac{1(1+1)(2(1)+1)}{6}=\frac{1(2)(3)}{6}=\frac{6}{6}=1\ \checkmark$; hence P_1 is true.

(2) $1^2+2^2+3^2+\ldots+k^2=\frac{k(k+1)(2k+1)}{6}$ is true

$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}?$$

$$\begin{aligned}1^2+2^2+3^2+\ldots+k^2+(k+1)^2&=\frac{k(k+1)(2k+1)}{6}+(k+1)^2\\&=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}\\&=\frac{(k+1)\left[k(2k+1)+6(k+1)\right]}{6}\\&=\frac{(k+1)\left[2k^2+k+6k+6\right]}{6}\\&=\frac{(k+1)\left[2k^2+7k+6\right]}{6}\\&=\frac{(k+1)((k+2)(2k+3))}{6}\\&=\frac{(k+1)((k+1+1)(2k+2+1))}{6}\\&=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}\ \checkmark\end{aligned}$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $\frac{1}{1\bullet 2}+\frac{1}{2\bullet 3}+\frac{1}{3\bullet 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow \frac{1}{1.2} \stackrel{?}{=} \frac{1}{1+1}=\frac{1}{2}=\frac{1}{1.2}\ \checkmark$; hence P_1 is true.

(2) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$ is true

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{(k+1)+1}?$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$=\frac{(k+1)(k+1)}{(k+1)(k+2)}$$

$$=\frac{k+1}{(k+1+1)}$$

$$=\frac{k+1}{(k+1)+1} \quad \surd$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$

Solution

(1) For $\boldsymbol{n=1}$ $\Rightarrow$ $\frac{1}{2}\overset{?}{=}1-\frac{1}{2}=\frac{1}{2}$ $\surd$; P_1 is true.

(2) $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}=1-\frac{1}{2^k}$ is true

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^{k+1}}?$$

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^k}+\frac{1}{2^{k+1}}$$

$$=1-\frac{1}{2^k}+\frac{1}{2^k\cdot 2}$$

$$=\frac{2^{k+1}-2+1}{2^{k+1}}$$

$$=\frac{2^{k+1}-1}{2^{k+1}}$$

$$=\frac{2^{k+1}}{2^{k+1}}-\frac{1}{2^{k+1}}$$

$$=1-\frac{1}{2^{k+1}} \quad \checkmark$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots+\frac{1}{(3n-2)\cdot(3n+1)}=\frac{n}{3n+1}$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow \frac{1}{1\cdot 4}\overset{?}{=}\frac{1}{3(1)+1}=\frac{1}{4}$ $\checkmark$; P_1 is true.

(2) $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1}$ is true

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3(k+1)-2)(3(k+1)+1)}\overset{?}{=}\frac{k+1}{3(k+1)+1}$$

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3(k+1)-2)(3(k+1)+1)}=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$$

$$=\frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$=\frac{(3k+1)(k+1)}{(3k+1)(3k+3+1)}$$

$$=\frac{k+1}{3(k+1)+1} \quad \checkmark$$

P_{k+1} is also true

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\frac{4}{5}+\frac{4}{5^2}+\frac{4}{5^3}+\cdots+\frac{4}{5^n}=1-\frac{1}{5^n}$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow \frac{4}{5}\overset{?}{=}1-\frac{1}{5}=\frac{4}{5}$ $\checkmark$; P_1 is true.

(2) $\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}=1-\frac{1}{5^k}$ is true

$$\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}+\frac{4}{5^{k+1}}\overset{?}{=}1-\frac{1}{5^{k+1}}$$

$$\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}+\frac{4}{5^{k+1}}=1-\frac{1}{5^k}+\frac{4}{5^{k+1}}$$
$$=1-\left(\frac{1}{5^k}-\frac{4}{5^{k+1}}\right)$$
$$=1-\frac{5-4}{5^{k+1}}$$
$$=1-\frac{1}{5^{k+1}} \quad \checkmark$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $1^3+2^3+3^3+\cdots+n^3=\dfrac{n^2(n+1)^2}{4}$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 1^3 \overset{?}{=} \dfrac{1^2(1+1)^2}{4}=\dfrac{4}{4}=1$ ✓ ; P_1 is true.

(2) $\dfrac{4}{5}+\dfrac{4}{5^2}+\cdots+\dfrac{4}{5^k}=1-\dfrac{1}{5^k}$ is true

$$1^3+2^3+\cdots+k^3+(k+1)^3 \overset{?}{=} \frac{(k+1)^2\left((k+1)+1\right)^2}{4}$$

$$1^3+2^3+\cdots+k^3+(k+1)^3=\frac{k^2(k+1)^2}{4}+(k+1)^3$$
$$=\frac{k^2(k+1)^2+4(k+1)^3}{4}$$
$$=\frac{(k+1)^2\left[k^2+4(k+1)\right]}{4}$$
$$=\frac{(k+1)^2\left(k^2+4k+4\right)}{4}$$
$$=\frac{(k+1)^2(k+2)^2}{4}$$
$$=\frac{(k+1)^2\left((k+1)+1\right)^2}{4} \quad \checkmark$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $3+3^2+3^3+\ldots+3^n=\frac{3}{2}\left(3^n-1\right)$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 3\overset{?}{=}\frac{3}{2}\left(3^1-1\right)=\frac{3}{2}2=3$ √ ; P_1 is true.

(2) $3+3^2+\cdots+3^k=\frac{3}{2}\left(3^k-1\right)$ is true → Is $3+3^2+\cdots+3^k+3^{k+1}\overset{?}{=}\frac{3}{2}\left(3^{k+1}-1\right)$

$$\begin{aligned}3+3^2+\cdots+3^k+3^{k+1}&=\frac{3}{2}\left(3^k-1\right)+3^{k+1}\\&=\frac{1}{2}3^{k+1}-\frac{3}{2}+3^{k+1}\\&=\frac{3}{2}3^{k+1}-\frac{3}{2}\\&=\frac{3}{2}\left(3^{k+1}-1\right)\quad √\end{aligned}$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $x^{2n}+x^{2n-1}y+\cdots+xy^{2n-1}+y^{2n}=\dfrac{x^{2n+1}-y^{2n+1}}{x-y}$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow x^2+xy+y^2\overset{?}{=}\dfrac{x^3-y^3}{x-y}$

$$=\frac{(x-y)\left(x^2+xy+y^2\right)}{x-y}$$

$$=x^2+xy+y^2 \quad √$$; P_1 is true.

(2) $x^{2k}+x^{2k-1}y+\cdots+xy^{2k-1}+y^{2k}=\dfrac{x^{2k+1}-y^{2k+1}}{x-y}$ is true

$$x^{2(k+1)}+x^{2(k+1)-1}y+\cdots+xy^{2(k+1)-1}+y^{2(k+1)}\overset{?}{=}\frac{x^{2(k+1)+1}-y^{2(k+1)+1}}{x-y}$$

$$\begin{aligned}x^{2k+2}+x^{2k+1}y+\cdots+xy^{2k+1}+y^{2k+2}&=x^2\left(x^{2k}+x^{2k-1}y+\cdots+y^{2k}\right)+xy^{2k+1}+y^{2k+2}\\&=x^2\left(\frac{x^{2k+1}-y^{2k+1}}{x-y}\right)+xy^{2k+1}+y^{2k+2}\\&=\frac{x^{2k+3}-x^2y^{2k+1}+x^2y^{2k+1}+xy^{2k+2}-xy^{2k+2}-y^{2(k+1)+1}}{x-y}\end{aligned}$$

$$= \frac{x^{2(k+1)+1} - y^{2(k+1)+1}}{x-y} \quad \checkmark$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\quad 5 \cdot 6 + 5 \cdot 6^2 + 5 \cdot 6^3 + \cdots + 5 \cdot 6^n = 6\left(6^n - 1\right)$

Solution

(1) For $\boldsymbol{n = 1} \Rightarrow 5 \cdot 6 \overset{?}{=} 6\left(6^1 - 1\right) = 6(5)$ $\checkmark$; P_1 is true.

(2) $5 \cdot 6 + 5 \cdot 6^2 + \cdots + 5 \cdot 6^k = 6\left(6^k - 1\right)$ is true

$$5 \cdot 6 + 5 \cdot 6^2 + \cdots + 5 \cdot 6^k + 5 \cdot 6^{k+1} \overset{?}{=} 6\left(6^{k+1} - 1\right)$$

$$\begin{aligned} 5 \cdot 6 + 5 \cdot 6^2 + \cdots + 5 \cdot 6^k + 5 \cdot 6^{k+1} &= 6\left(6^k - 1\right) + 5 \cdot 6^{k+1} \\ &= 6^{k+1} - 6 + 5 \cdot 6^{k+1} \\ &= 6^{k+1}(1+5) - 6 \\ &= 6 \cdot 6^{k+1} - 6 \\ &= 6\left(6^{k+1} - 1\right) \quad \checkmark \end{aligned}$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\quad 7 \cdot 8 + 7 \cdot 8^2 + 7 \cdot 8^3 + \cdots + 7 \cdot 8^n = 8\left(8^n - 1\right)$

Solution

(1) For $\boldsymbol{n = 1} \Rightarrow 7 \cdot 8 \overset{?}{=} 8\left(8^1 - 1\right) = 8(7)$ $\checkmark$; P_1 is true.

(2) $7 \cdot 8 + 7 \cdot 8^2 + \cdots + 7 \cdot 8^k = 8\left(8^k - 1\right)$ is true

$$7 \cdot 8 + 7 \cdot 8^2 + \cdots + 7 \cdot 8^k + 7 \cdot 8^{k+1} \overset{?}{=} 8\left(8^{k+1} - 1\right)$$

$$7 \cdot 8 + 7 \cdot 8^2 + \cdots + 7 \cdot 8^k + 7 \cdot 8^{k+1} = 8\left(8^k - 1\right) + 7 \cdot 8^{k+1}$$

$$
\begin{aligned}
&= 8^{k+1} - 8 + 7 \cdot 8^{k+1} \\
&= 8^{k+1}(1+7) - 8 \\
&= 8 \cdot 8^{k+1} - 8 \\
&= 8\left(8^{k+1} - 1\right) \quad \surd
\end{aligned}
$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $3 + 6 + 9 + \cdots + 3n = \dfrac{3n(n+1)}{2}$

Solution

(1) For $\boldsymbol{n = 1} \Rightarrow 3 \overset{?}{=} \dfrac{3(1)(1+1)}{2} = 3$ $\surd$; P_1 is true.

(2) $3 + 6 + 9 + \cdots + 3k = \dfrac{3k(k+1)}{2}$ is true

$$3 + 6 + 9 + \cdots + 3k + 3(k+1) \overset{?}{=} \frac{3(k+1)(k+2)}{2}$$

$$
\begin{aligned}
3 + 6 + 9 + \cdots + 3k + 3(k+1) &= \frac{3k(k+1)}{2} + 3(k+1) \\
&= \frac{3k(k+1) + 6(k+1)}{2} \\
&= \frac{(k+1)(3k+6)}{2} \\
&= \frac{3(k+1)(k+2)}{2} \quad \surd
\end{aligned}
$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $5 + 10 + 15 + \cdots + 5n = \dfrac{5n(n+1)}{2}$

Solution

(1) For $\boldsymbol{n = 1} \Rightarrow 5 \overset{?}{=} \dfrac{5(1)(1+1)}{2} = 5$ $\surd$; P_1 is true.

(2) $5 + 10 + 15 + \cdots + 5k = \dfrac{5k(k+1)}{2}$ is true

$$5+10+15+\cdots+5k+5(k+1)\overset{?}{=}\frac{5(k+1)(k+2)}{2}$$

$$\begin{aligned}5+10+15+\cdots+5k+5(k+1)&=\frac{5k(k+1)}{2}+5(k+1)\\&=\frac{5k(k+1)+10(k+1)}{2}\\&=\frac{(k+1)(5k+10)}{2}\\&=\frac{5(k+1)(k+2)}{2}\quad\checkmark\end{aligned}$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $1+3+5+\cdots+(2n-1)=n^2$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 1\overset{?}{=}1^2=1$ $\checkmark$; P_1 is true.

(2) $1+3+5+\cdots+(2k-1)=k^2$ is true

$$1+3+5+\cdots+(2k-1)+(2(k+1)-1)\overset{?}{=}(k+1)^2$$

$$\begin{aligned}1+3+5+\cdots+(2k-1)+(2(k+1)-1)&=k^2+2k+2-1\\&=k^2+2k+1\\&=(k+1)^2\quad\checkmark\end{aligned}$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $4+7+10+\cdots+(3n+1)=\dfrac{n(3n+5)}{2}$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 4\overset{?}{=}\dfrac{1(3+5)}{2}=4$ √ ; P_1 is true.

(2) $4+7+10+\cdots+(3k+1)=\dfrac{k(3k+5)}{2}$ is true

$$4+7+10+\cdots+(3k+1)+(3(k+1)+1)\overset{?}{=}\frac{(k+1)(3(k+1)+5)}{2}=\frac{(k+1)(3k+8)}{2}$$

$$\begin{aligned}4+7+10+\cdots+(3k+1)+(3k+4)&=\frac{k(3k+5)}{2}+3k+4\\&=\frac{3k^2+5k+6k+8}{2}\\&=\frac{3k^2+5k+3k+3k+8}{2}\\&=\frac{k(3k+8)+(3k+8)}{2}\\&=\frac{(3k+8)(k+1)}{2}\quad √\end{aligned}$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $\left(a^m\right)^n=a^{mn}$ (a and m are constant)

Solution

- For $\boldsymbol{n=1} \Rightarrow \left(a^m\right)^1\overset{?}{=}a^{m(1)} \quad \rightarrow \quad a^m=a^m$ √ ; P_1 is true.

- $\left(a^m\right)^k=a^{mk}$ is true

$$\left(a^m\right)^{(k+1)}\overset{?}{=}a^{m(k+1)}$$

$$\begin{aligned}\left(a^m\right)^{(k+1)}&=\left(a^m\right)^k a^m\\&=a^{km}a^m\\&=a^{km+m}\end{aligned}$$

$= a^{m(k+1)}$ √

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $n < 2^n$

Solution

***Step* 1**. For $\boldsymbol{n = 1} \Rightarrow 1 \overset{?}{<} 2^1$ √ $\Rightarrow P_1$ is true.

***Step* 2**. Assume that P_k is true $k < 2^k$

We need to prove that P_{k+1} is true, that is $k+1 < 2^{k+1}$

$$\begin{aligned} k+1 &< k+k = 2k \\ &< 2 \cdot 2^k \\ &= 2^{k+1} \quad √ \end{aligned}$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. 3 is a factor of $n^3 - n + 3$

Solution

- For $\boldsymbol{n = 1} \Rightarrow 1^3 - 1 + 3 = 3 = 3(1)$ √ $\Rightarrow P_1$ is true.
- Assume that P_k is true 3 is a factor of $k^3 - k + 3$

  We need to prove that P_{k+1} is true, that is $(k+1)^3 - (k+1) + 3$

$$\begin{aligned} (k+1)^3 - (k+1) + 3 &= k^3 + 3k^2 + 3k + 1 - k - 1 + 3 \\ &= \left(k^3 - k + 3\right) + 3k^2 + 3k \\ &= 3K + 3k^2 + 3k \\ &= 3\left(K + k^2 + k\right) \quad √ \end{aligned}$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. 4 is a factor of $5^n - 1$

Solution

- For $\boldsymbol{n = 1} \Rightarrow 5^1 - 1 = 4 = 4(1)$ √ $\Rightarrow P_1$ is true.
- Assume that P_k is true 4 is a factor of $5^k - 1$

  We need to prove that P_{k+1} is true, that is $5^{k+1} - 1$

  $$5^{k+1} - 1 = 5^k 5^1 - 5 + 4$$
  $$= 5\left(5^k - 1\right) + 4$$
  $$= 5\left(5^k - 1\right) + 4$$

By the induction hypothesis, 4 is a factor of $5^k - 1$ and 4 is a factor of 4, so 4 is a factor of the $(k+1)$ term. √

Thus, P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $2^n > 2n \quad if \quad n \ge 3$

Solution

- For $\boldsymbol{n = 3} \Rightarrow 2^3 \ge 2(3) \quad \Rightarrow \quad 8 \ge 6$ √ $\Rightarrow P_1$ is true.
- Assume that P_k is true: $2^k > 2k$;

  we need to prove that P_{k+1} : $2^{k+1} > 2(k+1)$ is true

  $$2^k > 2k$$
  $$2^k \cdot 2 > 2k \cdot 2$$
  $$2^{k+1} > 4k = 2k + 2k \qquad k \ge 3$$
  $$> 2k + 2$$
  $$= 2(k+1)$$ √

  P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: If $0 < a < 1$, then $a^n < a^{n-1}$

Solution

- For $\boldsymbol{n = 1} \Rightarrow a^1 < a^{1-1} \Rightarrow a < 1$ √ since $0 < a < 1 \Rightarrow P_1$ is true.

- Assume that P_k is true: $a^k < a^{k-1}$;

  We need to prove that P_{k+1} : $a^{k+1} < a^k$ is true

  $$a^k < a^{k-1} \rightarrow a^k \cdot a < a^{k-1} \cdot a$$
  $$a^{k+1} < a^k \quad \checkmark$$

  P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: If $n \geq 4$, then $n! > 2^n$

Solution

- For $\boldsymbol{n = 4} \Rightarrow 4! > 2^4 \Rightarrow 24 > 16$ √ $\Rightarrow P_1$ is true.
- Assume that P_k is true: $k! > 2^k$;

  We need to prove that P_{k+1} : $(k+1)! > 2^{k+1}$ is true

  $$\begin{aligned} (k+1)! &= k! \cdot (k+1) \\ &> 2^k (k+1) \qquad\qquad k \geq 4 \Rightarrow k+1 > 2 \\ &> 2^k \cdot 2 \\ &= 2^{k+1} \quad \checkmark \end{aligned}$$

  Thus, P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $3^n > 2n+1 \quad if \quad n \geq 2$

Solution

- For $\boldsymbol{n = 2} \Rightarrow 3^2 > 2(2)+1 \Rightarrow 9 > 5$ √ $\Rightarrow P_1$ is true.

- Assume that P_k is true: $3^k > 2k+1$

  We need to prove that P_{k+1}: $3^{k+1} > 2(k+1)+1$ is true

  $$3^k > 2k+1 \quad \Rightarrow \quad 3^k \cdot 3 > (2k+1)\cdot 3$$
  $$3^{k+1} > 6k+3$$
  $$> 2k+2+1$$
  $$= 2(k+1)+1 \quad \checkmark$$

  Thus, P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $2^n > n^2 \quad for \quad n > 4$

Solution

- For $\boldsymbol{n = 5}$ $\Rightarrow 2^5 > 5^2 \quad \Rightarrow \quad 32 > 25 \checkmark \quad \Rightarrow P_1$ is true.

- Assume that P_k is true: $2^k > k^2$

  We need to prove that P_{k+1}: $2^{k+1} > (k+1)^2$ is true

  $$2^k > k^2 \quad \Rightarrow \quad 2^k \cdot 2 > k^2 \cdot 2$$
  $$2^{k+1} > 2k^2 \qquad\qquad k < k+1 \quad \Rightarrow \quad k^2 > 2k+1$$
  $$> (k+1)^2 \quad \checkmark$$

  Thus, P_{k+1} is also true

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $4^n > n^4 \quad for \quad n \ge 5$

Solution

- For $\boldsymbol{n = 5}$ $\Rightarrow 4^5 > 5^4 \quad \Rightarrow \quad 1024 > 625 \checkmark \quad \Rightarrow P_1$ is true.

- Assume that P_k is true: $4^k > k^4$

  We need to prove that P_{k+1}: $4^{k+1} > (k+1)^4$ is true

$$4^k > k^4 \quad \Rightarrow \quad 4^k \cdot 4 > k^4 \cdot 4$$

$$4^{k+1} > 4k^4 \qquad\qquad k < k+1 \quad \Rightarrow \quad k^2 > 2k+1$$

$$> (k+1)^4 \quad \surd$$

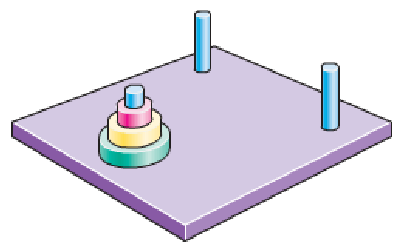
Thus, P_{k+1} is also true

∴ By the mathematical induction, the proof is completed.

Exercise

A pile of *n* rings, each smaller than the one below it, is on a peg on board. Two other pegs are attached to the board. In the game called the Tower of Hanoi puzzle, all the rings must moved, one at a time, to a different peg with no ring ever placed on top of a smaller ring. Find the least number of moves that would be required. Prove your result by mathematical induction.

Solution



With 1 ring, 1 move is required.

With 2 rings, 3 moves are required $\Rightarrow \quad 3 = 2+1$

With 3 rings, 7 moves are required $\Rightarrow \quad 7 = 2^2 + 2 + 1$

With *n* rings, $2^{n-1} + \cdots + 2^2 + 2^1 + 2^0 = 2^n - 1$ moves are required

- For $\boldsymbol{n = 1} \Rightarrow 2^0 = 2^1 - 1 = 1 \; \surd \quad \Rightarrow P_1$ is true.
- Assume that P_k is true: $2^{k-1} + \cdots + 2^2 + 2^1 + 2^0 = 2^k - 1$

$$2^k + 2^{k-1} + \cdots + 2^2 + 2^1 + 1 \overset{?}{=} 2^{k+1} - 1$$

$$\begin{aligned} 2^k + 2^{k-1} + \cdots + 2^2 + 2^1 + 1 &= 2^k + 2^k - 1 \\ &= 2 \cdot 2^k - 1 \\ &= 2^{k+1} - 1 \quad \surd \end{aligned}$$

∴ By the mathematical induction, the proof is completed.