# Solution   *Section* 4.6 – Orthogonal Diagonalization

### *Exercise*

Determine whether the matrix is orthogonal $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

**Solution**

Let $P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\begin{aligned} PP^T &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I_2 \quad \checkmark \end{aligned}$$

Therefore, the given matrix is orthogonal.

### *Exercise*

Determine whether the matrix is orthogonal $\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

**Solution**

Let $P = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

$$\begin{aligned} PP^T &= \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I_2 \quad \checkmark \end{aligned}$$

Therefore, the given matrix is orthogonal.

### ***Exercise***

Determine whether the matrix is orthogonal $\begin{pmatrix} \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$

***Solution***

Let $P = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$

$$PP^T = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{8}{9} & \\ & \end{pmatrix}$$

$$\neq I_2$$

Therefore, the given matrix is ***not*** orthogonal.

### ***Exercise***

Determine whether the matrix is orthogonal

$$\begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

***Solution***

Let $P = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$

$$PP^T = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= I_3 \quad \checkmark$$

Therefore, the given matrix is orthogonal.

## *Exercise*

Determine whether the matrix is orthogonal

$$\begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}$$

***Solution***

Let $P = \begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}$

$$PP^T = \begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}\begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= I_3 \quad \checkmark$$

Therefore, the given matrix is orthogonal.

## *Exercise*

Determine whether the matrix is orthogonal

$$\begin{pmatrix} -4 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 4 \end{pmatrix}$$

***Solution***

Let $P = \begin{pmatrix} -4 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 4 \end{pmatrix}$

$$PP^T = \begin{pmatrix} -4 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 4 \end{pmatrix}\begin{pmatrix} -4 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 25 & & \\ & & \\ & & \end{pmatrix}$$

$$\neq I_3$$

Therefore, the given matrix is ***not*** orthogonal.

## *Exercise*

Determine whether the matrix is orthogonal

$$\begin{pmatrix} 4 & -1 & -4 \\ -1 & 0 & -17 \\ 1 & 4 & -1 \end{pmatrix}$$

***Solution***

Let $P = \begin{pmatrix} 4 & -1 & -4 \\ -1 & 0 & -17 \\ 1 & 4 & -1 \end{pmatrix}$

$$PP^T = \begin{pmatrix} 4 & -1 & -4 \\ -1 & 0 & -17 \\ 1 & 4 & -1 \end{pmatrix}\begin{pmatrix} 4 & -1 & -4 \\ -1 & 0 & -17 \\ 1 & 4 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & & \\ & & \\ & & \end{pmatrix}$$

$\neq I_3$

Therefore, the given matrix is ***not*** orthogonal.

## *Exercise*

Determine whether the matrix is orthogonal

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \end{pmatrix}$$

***Solution***

$$\text{Let } P = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \end{pmatrix}$$

$$PP^T = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= I_3 \quad \checkmark$$

Therefore, the given matrix is orthogonal.

### *Exercise*

Determine whether the matrix is orthogonal

$$\begin{pmatrix} \frac{\sqrt{2}}{3} & 0 & \frac{\sqrt{5}}{2} \\ 0 & \frac{2\sqrt{5}}{5} & 0 \\ -\frac{\sqrt{2}}{6} & -\frac{\sqrt{5}}{5} & \frac{1}{2} \end{pmatrix}$$

***Solution***

$$\text{Let } P = \begin{pmatrix} \frac{\sqrt{2}}{3} & 0 & \frac{\sqrt{5}}{2} \\ 0 & \frac{2\sqrt{5}}{5} & 0 \\ -\frac{\sqrt{2}}{6} & -\frac{\sqrt{5}}{5} & \frac{1}{2} \end{pmatrix}$$

$$PP^T = \begin{pmatrix} \frac{\sqrt{2}}{3} & 0 & \frac{\sqrt{5}}{2} \\ 0 & \frac{2\sqrt{5}}{5} & 0 \\ -\frac{\sqrt{2}}{6} & -\frac{\sqrt{5}}{5} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{3} & 0 & -\frac{\sqrt{2}}{6} \\ 0 & \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \\ \frac{\sqrt{5}}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{53}{36} & & \\ & & \\ & & \end{pmatrix}$$

$$\neq I_3$$

Therefore, the given matrix is ***not*** orthogonal.

### *Exercise*

Determine whether the matrix is orthogonal

$$\begin{pmatrix} \frac{1}{8} & 0 & 0 & \frac{3\sqrt{7}}{8} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{3\sqrt{7}}{8} & 0 & 0 & \frac{1}{8} \end{pmatrix}$$

***Solution***

Let $P = \begin{pmatrix} \frac{1}{8} & 0 & 0 & \frac{3\sqrt{7}}{8} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{3\sqrt{7}}{8} & 0 & 0 & \frac{1}{8} \end{pmatrix}$

$$PP^T = \begin{pmatrix} \frac{1}{8} & 0 & 0 & \frac{3\sqrt{7}}{8} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{3\sqrt{7}}{8} & 0 & 0 & \frac{1}{8} \end{pmatrix}\begin{pmatrix} \frac{1}{8} & 0 & 0 & \frac{3\sqrt{7}}{8} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{3\sqrt{7}}{8} & 0 & 0 & \frac{1}{8} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= I_3 \quad \checkmark$$

Therefore, the given matrix is orthogonal.

## *Exercise*

Find a matrix $P$ that orthogonally diagonalizes $A$, and determine $P^{-1}AP$ $\qquad A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

***Solution***

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)^2 - 1$$

$$= \lambda^2 - 6\lambda + 8 = 0$$

The eigenvalues are: $\lambda_1 = 2 \quad and \quad \lambda_2 = 4$

For $\lambda_1 = 2$, we have: $(A - \lambda_1 I)\vec{v}_1 = 0$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad x + y = 0$$

$$\Rightarrow \quad x = -y$$

Therefore, the eigenvector $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

For $\lambda_2 = 4$, we have: $\left(A - \lambda_2 I\right)\vec{v}_2 = 0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad -x + y = 0$$

$$\Rightarrow \quad \underline{x = y}$$

Therefore, the eigenvector $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \frac{(-1,\ 1)}{\sqrt{(-1)^2 + 1^2}}$$

$$= \frac{(-1,\ 1)}{\sqrt{2}}$$

$$= \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

$$\vec{u}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \frac{(1,\ 1)}{\sqrt{1+1}}$$

$$= \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$P^T AP = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} & \frac{4}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

### *Exercise*

Find a matrix $P$ that orthogonally diagonalizes $A$, and determine $P^{-1}AP$ $\qquad A=\begin{pmatrix} 6 & 2\sqrt{3} \\ 2\sqrt{3} & 7 \end{pmatrix}$

***Solution***

$$\det(A-\lambda I)=\begin{vmatrix} 6-\lambda & 2\sqrt{3} \\ 2\sqrt{3} & 7-\lambda \end{vmatrix}$$

$$=(6-\lambda)(7-\lambda)-12$$

$$=\lambda^2-13\lambda+30=0$$

The eigenvalues are: $\boxed{\lambda_1=3 \quad and \quad \lambda_2=10}$

For $\lambda_1=3$, we have: $(A-\lambda_1 I)\vec{v}_1=0$

$$\begin{pmatrix} 3 & 2\sqrt{3} \\ 2\sqrt{3} & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \quad 3x+2\sqrt{3}\,y=0$$

$\Rightarrow \ \boxed{3x=-2\sqrt{3}\,y}$

Therefore, the eigenvector $\boxed{\vec{v}_1=\begin{pmatrix} -2\sqrt{3} \\ 3 \end{pmatrix}}$

For $\lambda_2=10$, we have: $(A-\lambda_2 I)\vec{v}_2=0$

$$\begin{pmatrix} -4 & 2\sqrt{3} \\ 2\sqrt{3} & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \quad -4x+2\sqrt{3}\,y=0$$

$\Rightarrow \ \boxed{2x=\sqrt{3}\,y}$

Therefore, the eigenvector $\boxed{\vec{v}_2=\begin{pmatrix} \sqrt{3} \\ 2 \end{pmatrix}}$

$$\vec{u}_1=\frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$=\frac{\left(-\frac{2}{\sqrt{3}},\ 1\right)}{\sqrt{\left(-\frac{2}{\sqrt{3}}\right)^2+1^2}}$$

$$=\frac{\sqrt{3}}{\sqrt{7}}\left(-\frac{2}{\sqrt{3}},\ 1\right)$$

$$\boxed{=\left(-\frac{2}{\sqrt{7}},\ \frac{\sqrt{3}}{\sqrt{7}}\right)}$$

$$\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{(\sqrt{3},\ 2)}{\sqrt{3+4}}$$

$$= \left(\frac{\sqrt{3}}{\sqrt{7}},\ \frac{2}{\sqrt{7}}\right)$$

$$P = \begin{pmatrix} -\frac{2}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} \\ \frac{\sqrt{3}}{\sqrt{7}} & \frac{2}{\sqrt{7}} \end{pmatrix}$$

$$P^{-1} = P^T$$

$$= \begin{pmatrix} -\frac{2}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} \\ \frac{\sqrt{3}}{\sqrt{7}} & \frac{2}{\sqrt{7}} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -\frac{2}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} \\ \frac{\sqrt{3}}{\sqrt{7}} & \frac{2}{\sqrt{7}} \end{pmatrix} \begin{pmatrix} 6 & 2\sqrt{3} \\ 2\sqrt{3} & 7 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} \\ \frac{\sqrt{3}}{\sqrt{7}} & \frac{2}{\sqrt{7}} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{6}{\sqrt{7}} & \frac{3\sqrt{3}}{\sqrt{7}} \\ \frac{10\sqrt{3}}{\sqrt{7}} & \frac{20}{\sqrt{7}} \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} \\ \frac{\sqrt{3}}{\sqrt{7}} & \frac{2}{\sqrt{7}} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & 10 \end{pmatrix} \qquad \checkmark$$

## *Exercise*

Find a matrix $P$ that orthogonally diagonalizes $A$, and determine $P^{-1}AP$ $\qquad A = \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}$

***Solution***

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}$$

$$= -2 + \lambda + \lambda^2 - 4$$

$$= \lambda^2 + \lambda - 6 = 0$$

The eigenvalues are: $\lambda_1 = -3 \quad and \quad \lambda_2 = 2$

For $\lambda_1 = -3$, we have: $\left(A-\lambda_1 I\right)\vec{v}_1 = 0$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x+2y=0$$

$\Rightarrow \quad \underline{x=-2y}|$

Therefore, the eigenvector $\boxed{\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$

For $\lambda_2 = 2$, we have: $\left(A-\lambda_2 I\right)\vec{v}_2 = 0$

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad 2x-y=0$$

$\Rightarrow \quad \underline{2x=y}|$

Therefore, the eigenvector $\boxed{\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \frac{(-2,\ 1)}{\sqrt{4+1}}$$

$$\boxed{= \left(-\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right)}$$

$$\vec{u}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \frac{(1,\ 2)}{\sqrt{1+4}}$$

$$\boxed{= \left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right)}$$

$$P = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$P^{-1} = P^T$$

$$P^{-1}AP = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{6}{\sqrt{5}} & -\frac{3}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{4}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \qquad \checkmark$$

### *Exercise*

Find a matrix $P$ that orthogonally diagonalizes $A$, and determine $P^{-1}AP$ $\qquad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

***Solution***

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= 1 - 2\lambda + \lambda^2 - 1$$

$$= \lambda^2 - 2\lambda = 0$$

The eigenvalues are: $\lambda_1 = 0 \quad and \quad \lambda_2 = 2$

For $\lambda_1 = 0$, we have: $(A - \lambda_1 I)\vec{v}_1 = 0$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x + y = 0$$

$\Rightarrow \ x = -y$

Therefore, the eigenvector $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

For $\lambda_2 = 2$, we have: $(A - \lambda_2 I)\vec{v}_2 = 0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad -x + y = 0$$

$\Rightarrow \ x = y$

Therefore, the eigenvector $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(-1,\ 1)}{\sqrt{2}}$$

$$= \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

$$\vec{u}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \frac{(1,\ 1)}{\sqrt{2}}$$

$$\boxed{= \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)}$$

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$P^T AP = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \quad \checkmark$$

## *Exercise*

Find a matrix $P$ that orthogonally diagonalizes $A$, and determine $P^{-1}AP$ $\quad A = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$

***Solution***

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & \sqrt{2} \\ \sqrt{2} & 1-\lambda \end{vmatrix}$$

$$= 2 - 3\lambda + \lambda^2 - 2$$

$$\boxed{= \lambda^2 - 3\lambda = 0}$$

The eigenvalues are: $\boxed{\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = 3}$

For $\lambda_1 = 0$, we have: $(A - \lambda_1 I)\vec{v}_1 = 0$

$$\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad 2x + \sqrt{2}\,y = 0$$

$$\Rightarrow \quad \boxed{2x = -\sqrt{2}\,y}$$

Therefore, the eigenvector $\vec{v}_1 = \begin{pmatrix} -\sqrt{2} \\ 2 \end{pmatrix}$

For $\lambda_2 = 3$, we have: $\left(A - \lambda_2 I\right)\vec{v}_2 = 0$

$$\begin{pmatrix} -1 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad -x + \sqrt{2}y = 0$$

$$\Rightarrow \quad x = \sqrt{2}\,y$$

Therefore, the eigenvector $\vec{v}_2 = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$

$$\begin{aligned} \vec{u}_1 &= \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|} \\ &= \frac{\left(-\sqrt{2},\ 2\right)}{\sqrt{2+4}} \\ &= \left(-\frac{\sqrt{2}}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right) \end{aligned}$$

$$\begin{aligned} \vec{u}_2 &= \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|} \\ &= \frac{\left(\sqrt{2},\ 1\right)}{\sqrt{3}} \\ &= \left(\frac{\sqrt{2}}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) \end{aligned}$$

$$P = \begin{pmatrix} -\frac{\sqrt{2}}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$\begin{aligned} P^T AP &= \begin{pmatrix} -\frac{\sqrt{2}}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} -\frac{\sqrt{2}}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ \frac{3\sqrt{2}}{\sqrt{3}} & \frac{3}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} -\frac{\sqrt{2}}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \quad \checkmark \end{aligned}$$

### *Exercise*

Find a matrix $P$ that orthogonally diagonalizes $A$, and determine $P^{-1}AP$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

***Solution***

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix}$$

$$= -\lambda^3 + 2 + 3\lambda$$

$$= -\lambda^3 + 3\lambda + 2 = 0$$

$\lambda = 2$

$$\begin{array}{c|cccc} 2 & -1 & 0 & 3 & 2 \\ & & -2 & -4 & -2 \\ \hline & -1 & -2 & -1 & 0 \end{array} \quad \rightarrow \quad -\lambda^2 - 2\lambda - 1 = 0$$

The eigenvalues are: $\lambda_{1,2} = -1$ & $\lambda_3 = 2$

For $\lambda_{1,2} = -1$, we have: $(A - \lambda_1 I)\vec{v}_1 = 0$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x + y + z = 0 \quad (1)$$

If $z = 0 \quad \Rightarrow \quad x = -y$

Therefore, the eigenvector $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

If $y = 0 \quad \Rightarrow \quad x = -z$

Therefore, the eigenvector $\vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

For $\lambda_3 = 2$, we have: $(A - \lambda_3 I)\vec{v}_3 = 0$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{matrix} \\ 2R_2 + R_1 \\ 2R_3 + R_1 \end{matrix}$$

$$\begin{pmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix} \begin{matrix} 3R_1 + R_2 \\ \\ R_3 + R_2 \end{matrix}$$

$$\begin{pmatrix} -6 & 0 & 6 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} -\frac{1}{6}R_1 \\ -\frac{1}{3}R_2 \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \to & x - z = 0 \\ \to & y - z = 0 \\ \to & \end{matrix}$$

$$\Rightarrow \quad \boxed{x = y = z}$$

Therefore, the eigenvector $\boxed{\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$

$$\begin{aligned} \vec{u}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{(-1,\ 1,\ 0)}{\sqrt{2}} \\ &\boxed{= \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)} \end{aligned}$$

$$\begin{aligned} \vec{w}_2 &= \vec{v}_2 - \left(\vec{v}_2 \bullet \vec{u}_1\right)\vec{u}_1 \\ &= (-1,\ 0,\ 1) - \left((-1,\ 0,\ 1) \bullet \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)\right)\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right) \\ &= (-1,\ 0,\ 1) - \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right) \\ &= (-1,\ 0,\ 1) - \left(-\frac{1}{2},\ \frac{1}{2},\ 0\right) \end{aligned}$$

$$= \left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{\left(-\frac{1}{2}, -\frac{1}{2}, 1\right)}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}}$$

$$= \frac{2}{\sqrt{6}}\left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)$$

$$= \left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)$$

$$\vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{(1, 1, 1)}{\sqrt{3}}$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$P^T AP = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \checkmark$$

## ***Exercise***

Find a matrix $P$ that orthogonally diagonalizes $A$, and determine $P^{-1}AP$

$$A=\begin{pmatrix}0&10&10\\10&5&0\\10&0&-5\end{pmatrix}$$

***Solution***

$$\det(A-\lambda I)=\begin{vmatrix}-\lambda&10&10\\10&5-\lambda&0\\10&0&-5-\lambda\end{vmatrix}$$

$$=-\lambda\left(-25+\lambda^2\right)-100(5-\lambda)-100(-5-\lambda)$$

$$=25\lambda-\lambda^3-500+100\lambda+500+100\lambda$$

$$=-\lambda^3+225\lambda$$

$$=-\lambda\left(\lambda^2-225\right)=0$$

The eigenvalues are: $\lambda_1=-15 \quad \lambda_2=0 \quad \lambda_3=15$

For $\lambda_1=-15$, we have: $\left(A-\lambda_1 I\right)\vec{v}_1=0$

$$\begin{pmatrix}15&10&10\\10&20&0\\10&0&10\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}15&10&10\\10&20&0\\10&0&10\end{pmatrix}\quad\begin{matrix}\\3R_2-2R_1\\3R_3-2R_1\end{matrix}$$

$$\begin{pmatrix}15&10&10\\0&40&-20\\0&-20&10\end{pmatrix}\quad\begin{matrix}4R_1-R_2\\ \\2R_3+R_2\end{matrix}$$

$$\begin{pmatrix}60&0&60\\0&40&-20\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}60&0&60\\0&40&-20\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad\begin{matrix}\to\ 60x+60z=0 & (1)\\ \to\ 40y-20z=0 & (2)\\ & \end{matrix}$$

$(1)\ \Rightarrow\ x=-z$

$(2) \Rightarrow \underline{2y = z}$

Therefore, the eigenvector $\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$

For $\lambda_2 = 0$, we have: $(A - \lambda_2 I)\vec{v}_2 = 0$

$$\begin{pmatrix} 0 & 10 & 10 \\ 10 & 5 & 0 \\ 10 & 0 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 0 & -5 \\ 10 & 5 & 0 \\ 0 & 10 & 10 \end{pmatrix} \quad R_2 - R_1$$

$$\begin{pmatrix} 10 & 0 & -5 \\ 0 & 5 & 5 \\ 0 & 10 & 10 \end{pmatrix} \quad R_3 - 2R_2$$

$$\begin{pmatrix} 10 & 0 & -5 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{array}{l} \rightarrow 10x - 5z = 0 \quad (1) \\ \rightarrow 5y + 5z = 0 \quad (2) \end{array}$$

$(1) \Rightarrow \underline{2x = z}$

$(2) \Rightarrow \underline{y = -z}$

Therefore, the eigenvector $\vec{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

For $\lambda_3 = 15$, we have: $(A - \lambda_3 I)\vec{v}_3 = 0$

$$\begin{pmatrix} -15 & 10 & 10 \\ 10 & -10 & 0 \\ 10 & 0 & -20 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -15 & 10 & 10 \\ 10 & -10 & 0 \\ 10 & 0 & -20 \end{pmatrix} \quad \begin{array}{l} \\ 3R_2 + 2R_1 \\ 3R_3 + 2R_1 \end{array}$$

$$\begin{pmatrix} -15 & 10 & 10 \\ 0 & -10 & 20 \\ 0 & 20 & -40 \end{pmatrix} \quad \begin{array}{l} R_1 + R_2 \\ \\ R_3 + 2R_2 \end{array}$$

$$\begin{pmatrix} -15 & 0 & 30 \\ 0 & -10 & 20 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \rightarrow \quad -x + 2z = 0 \quad (1) \\ \rightarrow \quad -y + 2z = 0 \quad (2) \\ \end{array}$$

$(1) \quad \Rightarrow \quad \underline{x = 2z}$

$(2) \quad \Rightarrow \quad \underline{y = 2z}$

Therefore, the eigenvector $\vec{v}_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(-2,\ 1,\ 2)}{\sqrt{4+1+4}}$$

$$= \left(-\frac{2}{3},\ \frac{1}{3},\ \frac{2}{3}\right)$$

$$\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{(1,\ -2,\ 2)}{\sqrt{1+4+4}}$$

$$= \left(\frac{1}{3},\ -\frac{2}{3},\ \frac{2}{3}\right)$$

$$\vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{(2,\ 2,\ 1)}{\sqrt{9}}$$

$$= \left(\frac{2}{3},\ \frac{2}{3},\ \frac{1}{3}\right)$$

$$P = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$P^T AP = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & 10 & 10 \\ 10 & 5 & 0 \\ 10 & 0 & -5 \end{pmatrix} \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 10 & -5 & -10 \\ 0 & 0 & 0 \\ 10 & 10 & 5 \end{pmatrix} \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -15 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 15 \end{pmatrix} \qquad \checkmark$$

## *Exercise*

Find a matrix $P$ that orthogonally diagonalizes $A$, and determine $P^{-1}AP$

$$A = \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}$$

***Solution***

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 3 & 0 \\ 3 & -\lambda & 4 \\ 0 & 4 & -\lambda \end{vmatrix}$$

$$= -\lambda^3 + 16\lambda + 9\lambda$$

$$= -\lambda\left(\lambda^2 - 25\right) = 0$$

The eigenvalues are: $\lambda_1 = -5 \quad \lambda_2 = 0 \quad \lambda_3 = 5$

For $\lambda_1 = -5$, we have: $\left(A - \lambda_1 I\right)\vec{v}_1 = 0$

$$\begin{pmatrix} 5 & 3 & 0 \\ 3 & 5 & 4 \\ 0 & 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 3 & 0 \\ 3 & 5 & 4 \\ 0 & 4 & 5 \end{pmatrix} \quad 5R_2 - 3R_1$$

$$\begin{pmatrix} 5 & 3 & 0 \\ 0 & 16 & 20 \\ 0 & 4 & 5 \end{pmatrix} \begin{matrix} 16R_1 - 3R_2 \\ \\ 4R_3 - R_2 \end{matrix}$$

$$\begin{pmatrix} 80 & 0 & -60 \\ 0 & 16 & 20 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \frac{1}{20}R_1 \\ \frac{1}{4}R_2 \\ \\ \end{matrix}$$

$$\begin{pmatrix} 4 & 0 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \rightarrow & 4x - 3z = 0 & (1) \\ \rightarrow & 4y + 5z = 0 & (2) \\ & & \end{matrix}$$

$(1) \Rightarrow \underline{4x = 3z}$

$(2) \Rightarrow \underline{4y = -5z}$

Therefore, the eigenvector $\vec{v}_1 = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$

For $\lambda_2 = 0$, we have: $(A - \lambda_2 I)\vec{v}_2 = 0$

$$\begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 4 & 0 \end{pmatrix} \begin{matrix} \\ \\ 3R_3 - 4R_2 \end{matrix}$$

$$\begin{pmatrix} 3 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \rightarrow & 3x + 4z = 0 & (1) \\ \rightarrow & \underline{y = 0} & \\ & & \end{matrix}$$

$(1) \Rightarrow \underline{3x = -4z}$

Therefore, the eigenvector $\vec{v}_2 = \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix}$

For $\lambda_3 = 5$, we have: $\left(A - \lambda_3 I\right)\vec{v}_3 = 0$

$$\begin{pmatrix} -5 & 3 & 0 \\ 3 & -5 & 4 \\ 0 & 4 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -5 & 3 & 0 \\ 3 & -5 & 4 \\ 0 & 4 & -5 \end{pmatrix} \quad \begin{array}{l} \\ 5R_2 + 3R_1 \\ \\ \end{array}$$

$$\begin{pmatrix} -5 & 3 & 0 \\ 0 & -16 & 20 \\ 0 & 4 & -5 \end{pmatrix} \quad \begin{array}{l} 16R_1 + 3R_2 \\ \\ 4R_3 + R_2 \end{array}$$

$$\begin{pmatrix} -80 & 0 & 60 \\ 0 & -16 & 20 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} -\frac{1}{20}R_1 \\ -\frac{1}{4}R_2 \\ \\ \end{array}$$

$$\begin{pmatrix} 4 & 0 & -3 \\ 0 & 4 & -5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{array}{l} \to \quad 4x - 3z = 0 \quad (1) \\ \to \quad 4y - 5z = 0 \quad (2) \\ \\ \end{array}$$

$(1) \quad \Rightarrow \quad 4x = 3z$

$(2) \quad \Rightarrow \quad 4y = 5z$

Therefore, the eigenvector $\vec{v}_3 = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} \qquad \vec{v}_1 = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$$

$$= \frac{(3,\ -5,\ 4)}{\sqrt{9+25+16}}$$

$$= \frac{1}{5\sqrt{2}}(3,\ -5,\ 4)$$

$$= \left(\frac{3}{5\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ \frac{4}{5\sqrt{2}}\right)$$

$$\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} \qquad \vec{v}_2 = \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix}$$

$$= \frac{(-4,\ 0,\ 3)}{\sqrt{16+9}}$$

$$\boxed{= \left(-\frac{4}{5},\ 0,\ \frac{3}{5}\right)}$$

$$\vec{u}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}\ \boxed{\vec{v}_3 = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}}$$

$$= \frac{(3,\ 5,\ 4)}{\sqrt{9+25+16}}$$

$$\boxed{= \left(\frac{3}{5\sqrt{2}},\ \frac{1}{\sqrt{2}},\ \frac{4}{5\sqrt{2}}\right)}$$

$$P = \begin{pmatrix} \frac{3}{5\sqrt{2}} & -\frac{4}{5} & \frac{3}{5\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{4}{5\sqrt{2}} & \frac{3}{5} & \frac{4}{5\sqrt{2}} \end{pmatrix}$$

$$P^T AP = \begin{pmatrix} \frac{3}{5\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{4}{5\sqrt{2}} \\ -\frac{4}{5} & 0 & \frac{3}{5} \\ \frac{3}{5\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{4}{5\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{5\sqrt{2}} & -\frac{4}{5} & \frac{3}{5\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{4}{5\sqrt{2}} & \frac{3}{5} & \frac{4}{5\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3}{\sqrt{2}} & \frac{5}{\sqrt{2}} & -\frac{4}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{3}{\sqrt{2}} & \frac{5}{\sqrt{2}} & \frac{4}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{3}{5\sqrt{2}} & -\frac{4}{5} & \frac{3}{5\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{4}{5\sqrt{2}} & \frac{3}{5} & \frac{4}{5\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad \checkmark$$

### *Exercise*

Find a matrix $P$ that orthogonally diagonalizes $A$, and determine $P^{-1}AP$

$$A=\begin{pmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{pmatrix}$$

***Solution***

$$\det(A-\lambda I)=\begin{vmatrix} 2-\lambda & 2 & -2 \\ 2 & -1-\lambda & 4 \\ -2 & 4 & -1-\lambda \end{vmatrix}$$

$$=(2-\lambda)\left(1+2\lambda+\lambda^2\right)-32+4+4\lambda-32+16\lambda+4+4\lambda$$

$$=2+3\lambda-\lambda^3+24\lambda-56$$

$$=-\lambda^3+27\lambda-54=0$$

$\lambda=3$

| | | | | |
|---|---|---|---|---|
| 3 | $-1$ | 0 | 27 | $-54$ |
| | | $-3$ | $-9$ | 54 |
| | $-1$ | $-3$ | 18 | 0 |

$\rightarrow\ -\lambda^2-3\lambda+18=0$

The eigenvalues are: $\lambda_{1,2}=3 \quad \& \quad \lambda_3=-6$

For $\lambda_{1,2}=3$, we have: $\left(A-\lambda_1 I\right)\vec{v}_1=0$

$$\begin{pmatrix} -1 & 2 & -2 \\ 2 & -4 & 4 \\ -2 & 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad -x+2y-2z=0 \quad (1)$$

If $z=0 \quad \Rightarrow \quad x=2y$

Therefore, the eigenvector $\vec{v}_1=\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

If $y=0 \quad \Rightarrow \quad x=-2z$

Therefore, the eigenvector $\vec{v}_2=\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

For $\lambda_3=-6$, we have: $\left(A-\lambda_3 I\right)\vec{v}_3=0$

$$\begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix} \quad \begin{matrix} \\ 4R_2 - R_1 \\ 4R_3 + R_1 \end{matrix}$$

$$\begin{pmatrix} 8 & 2 & -2 \\ 0 & 18 & 18 \\ 0 & 18 & 18 \end{pmatrix} \quad \begin{matrix} 9R_1 - R_2 \\ \\ R_3 - R_2 \end{matrix}$$

$$\begin{pmatrix} 72 & 0 & -36 \\ 0 & 18 & 18 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} \frac{1}{36}R_1 \\ \frac{1}{18}R_2 \\ \\ \end{matrix}$$

$$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \rightarrow & 2x - z = 0 & (1) \\ \rightarrow & y + z = 0 & (2) \\ \rightarrow & & \end{matrix}$$

$(1) \Rightarrow \underline{2x = z}$

$(2) \Rightarrow \underline{y = -z}$

Therefore, the eigenvector $\vec{v}_3 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(2,\ 1,\ 0)}{\sqrt{4+1}}$$

$$= \left(\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

$$= (-2,\ 0,\ 1) - \left((-2,\ 0,\ 1) \cdot \left(\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)\right)\left(\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)$$

$$= (-2,\ 0,\ 1) + \frac{4}{\sqrt{5}}\left(\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)$$

$$= (-2,\ 0,\ 1) + \left(\frac{8}{5},\ \frac{4}{5},\ 0\right)$$

$$=\left(-\frac{2}{5},\ \frac{4}{5},\ 1\right)$$

$$\vec{u}_2=\frac{\vec{w}_2}{\left\|\vec{w}_2\right\|}$$

$$=\frac{\left(-\frac{2}{5},\ \frac{4}{5},\ 1\right)}{\sqrt{\frac{4}{25}+\frac{16}{25}+1}}$$

$$=\frac{5}{\sqrt{45}}\left(-\frac{2}{5},\ \frac{4}{5},\ 1\right)$$

$$=\frac{5}{3\sqrt{5}}\left(-\frac{2}{5},\ \frac{4}{5},\ 1\right)$$

$$=\left(-\frac{2}{3\sqrt{5}},\ \frac{4}{3\sqrt{5}},\ \frac{5}{3\sqrt{5}}\right)$$

$$\vec{u}_3=\frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$=\frac{(1,\ -2,\ 2)}{\sqrt{1+4+4}}$$

$$=\left(\frac{1}{3},\ -\frac{2}{3},\ \frac{2}{3}\right)$$

$$P=\begin{pmatrix}\frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} & \frac{1}{3}\\ \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} & -\frac{2}{3}\\ 0 & \frac{5}{3\sqrt{5}} & \frac{2}{3}\end{pmatrix}$$

$$P^TAP=\begin{pmatrix}\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0\\ -\frac{2}{3\sqrt{5}} & \frac{4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}}\\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}2 & 2 & -2\\ 2 & -1 & 4\\ -2 & 4 & -1\end{pmatrix}\begin{pmatrix}\frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} & \frac{1}{3}\\ \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} & -\frac{2}{3}\\ 0 & \frac{5}{3\sqrt{5}} & \frac{2}{3}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{6}{\sqrt{5}} & \frac{3}{\sqrt{5}} & 0\\ -\frac{2}{\sqrt{5}} & \frac{4}{\sqrt{5}} & \frac{5}{\sqrt{5}}\\ -2 & 4 & -4\end{pmatrix}\begin{pmatrix}\frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} & \frac{1}{3}\\ \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} & -\frac{2}{3}\\ 0 & \frac{5}{3\sqrt{5}} & \frac{2}{3}\end{pmatrix}$$

$$=\begin{pmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & -6\end{pmatrix} \quad \checkmark$$

### ***Exercise***

Find a matrix $P$ that orthogonally diagonalizes $A$, and determine $P^{-1}AP$

$$A=\begin{pmatrix}-2&0&-36\\0&-3&0\\-36&0&-23\end{pmatrix}$$

***Solution***

$$\det(A-\lambda I)=\begin{vmatrix}-2-\lambda&0&-36\\0&-3-\lambda&0\\-36&0&-23-\lambda\end{vmatrix}$$

$$=(-2-\lambda)(-3-\lambda)(-23-\lambda)-(36)(-3-\lambda)(36)$$

$$=-\left(6+5\lambda+\lambda^2\right)(23+\lambda)+3888+1296\lambda$$

$$=-138-115\lambda-23\lambda-6\lambda-5\lambda^2-\lambda^3+3888+1296\lambda$$

$$=-\lambda^3-28\lambda^2+1175\lambda+3750=0$$

$\lambda=-3$

| | | | | |
|---|---|---|---|---|
| $-3$ | $-1$ | $-28$ | $1175$ | $3750$ |
| | | $3$ | $75$ | $-3750$ |
| | $-1$ | $-25$ | $1250$ | $0$ |

$\rightarrow\ -\lambda^2-25\lambda+1250=0$

$$\lambda=\frac{25\pm\sqrt{625+5,000}}{-2}$$

$$=\frac{25\pm 75}{-2}$$

The eigenvalues are: $\lambda_1=25,\quad \lambda_2=-3,\quad \textit{and}\quad \lambda_3=-50$

For $\lambda_1=25$, we have: $\left(A-\lambda_1 I\right)\vec{v}_1=0$

$$\begin{pmatrix}-27&0&-36\\0&-28&0\\-36&0&-48\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\rightarrow\begin{cases}-27x-36z=0\\ y=0\\ -36x-48z=0\end{cases}$$

$$\Rightarrow\begin{cases}3x+4z=0\\3x+4z=0\end{cases}$$

$$\Rightarrow 3x=-4z$$

Therefore, the eigenvector $\vec{v}_1=\begin{pmatrix}-4\\0\\3\end{pmatrix}$

For $\lambda_2=-3$, we have: $\left(A-\lambda_2 I\right)\vec{v}_2=0$

$$\begin{pmatrix} 1 & 0 & -36 \\ 0 & 0 & 0 \\ -36 & 0 & -20 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \to & x-36z=0 \\ & \\ \to & -36x-20z=0 \end{matrix}$$

$$\Rightarrow \begin{cases} x-36z=0 \\ 9x+5z=0 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & -36 \\ 9 & 5 \end{vmatrix} = 329 \neq 0$$

$$\Rightarrow \underline{x=z=0}$$

Therefore, the eigenvector $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda_3 = -25$, we have: $\left(A-\lambda_3 I\right)\vec{v}_3 = 0$

$$\begin{pmatrix} 48 & 0 & -36 \\ 0 & 47 & 0 \\ -36 & 0 & 27 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \to & 48x-36z=0 \\ \to & \underline{y=0} \\ \to & -36x+27z=0 \end{matrix}$$

$$\Rightarrow \begin{cases} 4x-3z=0 \\ -4x+3z=0 \end{cases}$$

$$\Rightarrow \underline{4x=3z}$$

Therefore, the eigenvector $\vec{v}_3 = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \frac{(-4,\ 0,\ 3)}{\sqrt{16+9}}$$

$$= \frac{(-4,\ 0,\ 3)}{5}$$

$$= \left(-\frac{4}{5},\ 0,\ \frac{3}{5}\right)$$

$$\vec{u}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \frac{(0,1,\ 0)}{\sqrt{1^2}}$$

$$= (0,\ 1,\ 0)$$

$$\vec{u}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$= \frac{(3,\ 0,\ 4)}{\sqrt{3^2+4^2}}$$

$$= \frac{(3,\ 0,\ 4)}{5}$$

$$= \left(\frac{3}{5},\ 0,\ \frac{4}{5}\right)$$

$$P = \begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}$$

$$P^T AP = \begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix} \begin{pmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{pmatrix} \begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -20 & 0 & 15 \\ 0 & -3 & 0 \\ -30 & 0 & -40 \end{pmatrix} \begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 25 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -50 \end{pmatrix}$$

## *Exercise*

Find a matrix $P$ that orthogonally diagonalizes $A$, and determine $P^{-1}AP$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

***Solution***

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix}$$

$$= -\lambda\left[(1-\lambda)^2 - 1\right]$$

$$= -\lambda\left(\lambda^2 - 2\lambda\right) = 0$$

The eigenvalues are: $\lambda_{1,2} = 0 \quad and \quad \lambda_3 = 2$

For $\lambda_{1,2} = 0$, we have: $\left(A - \lambda_1 I\right)\vec{v}_1 = 0$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x + y = 0$$

$$\Rightarrow \quad x = -y$$

Therefore, the eigenvector $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

For $\lambda_3 = 2$, we have: $\left(A - \lambda_3 I\right)\vec{v}_3 = 0$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{array}{l} \rightarrow \quad -x + y = 0 \ (1) \\ \\ \rightarrow \quad z = 0 \end{array}$$

$$(1) \Rightarrow \quad x = y$$

Therefore; the eigenvector $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \frac{(-1,\ 1,\ 0)}{\sqrt{1^2 + 1^2}}$$

$$= \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$\vec{u}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \frac{(0,\ 0,\ 1)}{\sqrt{1^2}}$$

$$= (0,\ 0,\ 1)$$

$$\vec{u}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$= \frac{(1,\ 1,\ 0)}{\sqrt{1^2+1^2}}$$

$$\boxed{= \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)}$$

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

$$P^{-1} = P^T = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \checkmark$$

### *Exercise*

Find a matrix $P$ that orthogonally diagonalizes $A$, and determine $P^{-1}AP$

$$A=\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

***Solution***

$$\det(A-\lambda I)=\begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix}$$

$$=(2-\lambda)^3-1-1-(2-\lambda)-(2-\lambda)-(2-\lambda)$$

$$=8-12\lambda+6\lambda^2-\lambda^3-2-6+3\lambda$$

$$=-\lambda^3+6\lambda^2-9\lambda=0$$

The eigenvalues are: $\lambda_1=0 \quad and \quad \lambda_{2,3}=3$

For $\lambda_1=0$, we have: $(A-\lambda_1 I)\vec{v}_1=0$

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{array}{l} \\ 2R_2+R_1 \\ 2R_3+R_1 \end{array}$$

$$\begin{pmatrix} 2 & -1 & -1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{pmatrix} \begin{array}{l} 3R_1+R_2 \\ \\ R_3+R_2 \end{array}$$

$$\begin{pmatrix} 6 & 0 & -6 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \frac{1}{6}R_1 \\ \frac{1}{3}R_2 \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{array}{l} \rightarrow \quad x_1-z_1=0 \\ \rightarrow \quad y_1-z_1=0 \\ \end{array}$$

$\Rightarrow \boxed{x_1 = z_1 = y_1}$

Therefore; the eigenvector $\boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$

For $\lambda_{2,3} = 3$, we have: $(A - \lambda_1 I)\vec{v}_1 = 0$

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \; -x_1 - y_1 - z_1 = 0$$

$\Rightarrow \boxed{x_1 = -y_1 - z_1}$

Therefore; the eigenvector $\boxed{\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$

$$\begin{aligned} \vec{u}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{(1,\ 1,\ 1)}{\sqrt{3}} \\ &= \boxed{\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)} \end{aligned}$$

$$\begin{aligned} \vec{w}_2 &= \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 \\ &= (-1,\ 1,\ 0) - \left[(-1,\ 1,\ 0) \cdot \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) \\ &= (-1,\ 1,\ 0) - 0\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) \\ &= \boxed{(-1,\ 1,\ 0)} \end{aligned}$$

$$\begin{aligned} \vec{u}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\ &= \frac{(-1,\ 1,\ 0)}{\sqrt{2}} \\ &= \boxed{\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)} \end{aligned}$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$$

$$=(-1,\ 0,\ 1)-\left[(-1,\ 0,\ 1)\bullet\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)\right]\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$=(-1,\ 0,\ 1)-\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$=(-1,\ 0,\ 1)-\left(-\frac{1}{2},\ \frac{1}{2},\ 0\right)$$

$$=\left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)$$

$$\vec{u}_3=\frac{\vec{w}_3}{\left\|\vec{w}_3\right\|}$$

$$=\frac{\left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}}$$

$$=\frac{1}{\frac{\sqrt{6}}{2}}\left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)$$

$$=\left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)$$

$$P=\begin{pmatrix}\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}\end{pmatrix}$$

$$P^TAP=\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}\end{pmatrix}$$

$$=\begin{pmatrix}0 & 0 & 0\\ -\frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 0\\ -\frac{3}{\sqrt{6}} & -\frac{3}{\sqrt{6}} & \frac{4}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}\end{pmatrix}$$

$$=\begin{pmatrix}0 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 2\end{pmatrix} \quad \checkmark$$

## *Exercise*

Find a matrix $P$ that orthogonally diagonalizes $A$, and determine $P^{-1}AP$

$$A = \begin{pmatrix} -7 & 24 & 0 & 0 \\ 24 & 7 & 0 & 0 \\ 0 & 0 & -7 & 27 \\ 0 & 0 & 24 & 7 \end{pmatrix}$$

**Solution**

$$\det(A - \lambda I) = \begin{vmatrix} -7-\lambda & 24 & 0 & 0 \\ 24 & 7-\lambda & 0 & 0 \\ 0 & 0 & -7-\lambda & 24 \\ 0 & 0 & 24 & 7-\lambda \end{vmatrix}$$

$$= (-7-\lambda)(7-\lambda)\left[(-7-\lambda)(7-\lambda)-24^2\right] - 24^2\left[(-7-\lambda)(7-\lambda)-24^2\right]$$

$$= \left(\lambda^2-49\right)\left(\lambda^2-625\right) - 576\left(\lambda^2-625\right)$$

$$= \left(\lambda^2-625\right)\left(\lambda^2-49-576\right)$$

$$= \left(\lambda^2-625\right)^2 = 0$$

The eigenvalues are: $\lambda_{1,2} = 25 \quad and \quad \lambda_{3,4} = -25$

For $\lambda_{1,2} = 25$, we have:

$$\begin{pmatrix} -32 & 24 & 0 & 0 \\ 24 & -18 & 0 & 0 \\ 0 & 0 & -32 & 24 \\ 0 & 0 & 24 & -18 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -32 & 24 & 0 & 0 \\ 24 & -18 & 0 & 0 \\ 0 & 0 & -32 & 24 \\ 0 & 0 & 24 & -18 \end{pmatrix} \begin{array}{l} \\ 4R_2 + 3R_1 \\ \\ 4R_4 + 3R_3 \end{array}$$

$$\begin{pmatrix} -32 & 24 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -32 & 24 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} -\frac{1}{8}R_1 \\ \\ -\frac{1}{8}R_3 \\ \\ \end{array}$$

$$\begin{pmatrix} 4 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \rightarrow & 4x_1 - 3x_2 = 0 \\ \\ \rightarrow & 4x_3 - 3x_4 = 0 \\ \\ \end{matrix}$$

$$\Rightarrow \begin{cases} \boxed{4x_1 = 3x_2} \\ \boxed{4x_3 = 3x_4} \end{cases}$$

Therefore; the eigenvector $\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 4 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 3 \\ 4 \\ 0 \\ 0 \end{pmatrix}$

For $\lambda_{3,4} = -25$, we have:

$$\begin{pmatrix} 18 & 24 & 0 & 0 \\ 24 & 32 & 0 & 0 \\ 0 & 0 & 18 & 24 \\ 0 & 0 & 24 & 32 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 18 & 24 & 0 & 0 \\ 24 & 32 & 0 & 0 \\ 0 & 0 & 18 & 24 \\ 0 & 0 & 24 & 32 \end{pmatrix} \quad \begin{matrix} \\ 3R_2 - 4R_1 \\ \\ 3R_4 - 4R_3 \end{matrix}$$

$$\begin{pmatrix} 18 & 24 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 18 & 24 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} \frac{1}{6}R_1 \\ \\ \frac{1}{6}R_3 \\ \\ \end{matrix}$$

$$\begin{pmatrix} 3 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \rightarrow & 3x_1 + 4x_2 = 0 \\ \\ \rightarrow & 3x_3 + 4x_4 = 0 \\ \\ \end{matrix}$$

$$\Rightarrow \begin{cases} \boxed{3x_1 = 4x_2} \\ \boxed{3x_3 = 4x_4} \end{cases}$$

Therefore; the eigenvector $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ -4 \\ 3 \end{pmatrix} \quad \vec{v}_4 = \begin{pmatrix} -4 \\ 3 \\ 0 \\ 0 \end{pmatrix}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(0,\ 0,\ 3,\ 4)}{\sqrt{3^2+4^2}}$$

$$= \left(0,\ 0,\ \frac{3}{5},\ \frac{4}{5}\right)$$

$$\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{(3,\ 4,\ 0,\ 0)}{\sqrt{3^2+4^2}}$$

$$= \left(\frac{3}{5},\ \frac{4}{5},\ 0,\ 0\right)$$

$$\vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{(0,\ 0,\ -4,\ 3)}{\sqrt{(-4)^2+3^2}}$$

$$= \left(0,\ 0,\ -\frac{4}{5},\ \frac{3}{5}\right)$$

$$\vec{u}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

$$= \frac{(-4,\ 3,\ 0,\ 0)}{\sqrt{25}}$$

$$= \left(-\frac{4}{5},\ \frac{3}{5},\ 0,\ 0\right)$$

$$P = \begin{pmatrix} 0 & \frac{3}{5} & 0 & -\frac{4}{5} \\ 0 & \frac{4}{5} & 0 & \frac{3}{5} \\ \frac{3}{5} & 0 & -\frac{4}{5} & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} & 0 & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & -\frac{4}{5} & \frac{3}{5} \\ 0 & 0 & \frac{3}{5} & \frac{4}{5} \end{pmatrix}$$

$$P^T AP = \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} & 0 & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & -\frac{4}{5} & \frac{3}{5} \\ 0 & 0 & \frac{3}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} -7 & 24 & 0 & 0 \\ 24 & 7 & 0 & 0 \\ 0 & 0 & -7 & 24 \\ 0 & 0 & 24 & 7 \end{pmatrix} \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} & 0 & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & -\frac{4}{5} & \frac{3}{5} \\ 0 & 0 & \frac{3}{5} & \frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 20 & -15 & 0 & 0 \\ 15 & 20 & 0 & 0 \\ 0 & 0 & 20 & -15 \\ 0 & 0 & 15 & 20 \end{pmatrix} \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} & 0 & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & -\frac{4}{5} & \frac{3}{5} \\ 0 & 0 & \frac{3}{5} & \frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -25 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 \\ 0 & 0 & -25 & 0 \\ 0 & 0 & 0 & 25 \end{pmatrix} \quad \checkmark$$

### *Exercise*

Find a matrix $P$ that orthogonally diagonalizes $A$, and determine $P^{-1}AP$

$$A = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & 0 & 0 \\ 1 & 3-\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix}$$

$$= \lambda^2 (3-\lambda)^2 - \lambda^2$$

$$= \lambda^2 \left(9 - 6\lambda + \lambda^2 - 1\right)$$

$$= \lambda^2 \left(\lambda^2 - 6\lambda + 8\right)$$

$$\underline{= \lambda^2 (\lambda - 2)(\lambda - 4) = 0}$$

Therefore, the matrix has eigenvalues $\underline{\lambda_{1,2,3,4} = 0,\ 0,\ 2,\ 4}$

For $\lambda_{1,2} = 0$ , then $(A - 0)\vec{v}_1 = 0$

$$\begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \to \ 3x_1 + x_2 = 0 \\ \to \ x_1 + 3x_2 = 0 \\ \\ \\ \end{array}$$

$$\Rightarrow \underline{x_1 = x_2 = 0}$$

The eigenvectors are: $\underline{\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$

For $\lambda_3 = 2$ , then $(A - 2I)\vec{v}_3 = 0$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{cases} x_3 + y_3 = 0 \\ x_3 + y_3 = 0 \\ 2z_3 = 0 \\ 2w_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_3 = -y_3 \\ z_3 = w_3 = 0 \end{cases}$$

The eigenvectors are: $V_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ or $V_3 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}$

For $\lambda_4 = 4$ , then $(A - 4I)V_4 = 0$

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} -x_4 + y_4 = 0 \\ x_4 - y_4 = 0 \\ -4z_4 = 0 \\ -4w_4 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_4 = y_4 \\ z_4 = w_4 = 0 \end{cases}$$

The eigenvectors are: $V_4 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}$

$$P = \begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$P^{-1} = P^T = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} & 0 & 0 \\ \frac{4}{\sqrt{2}} & \frac{4}{\sqrt{2}} & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad \checkmark$$

### *Exercise*

Find the eigenvalues of $A$ and $B$ and check the Orthogonality of their first two eigenvectors. Graph these eigenvectors to see the discrete sines and cosines:

$$A=\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad B=\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

The $-1, 2, -1$ pattern in both matrices is a "second derivative. Then $A\vec{x}=\lambda\vec{x}$ and $B\vec{x}=\lambda\vec{x}$ are like $\dfrac{d^2\vec{x}}{dt^2}=\lambda\vec{x}$. This has eigenvectors $x=\sin kt$ and $x=\cos kt$ that are the bases for Fourier series. The matrices lead to "discrete sines" and "discrete cosines" that are the bases for the discrete Fourier Transform. This DFT is absolutely central to all areas of digital signal processing.

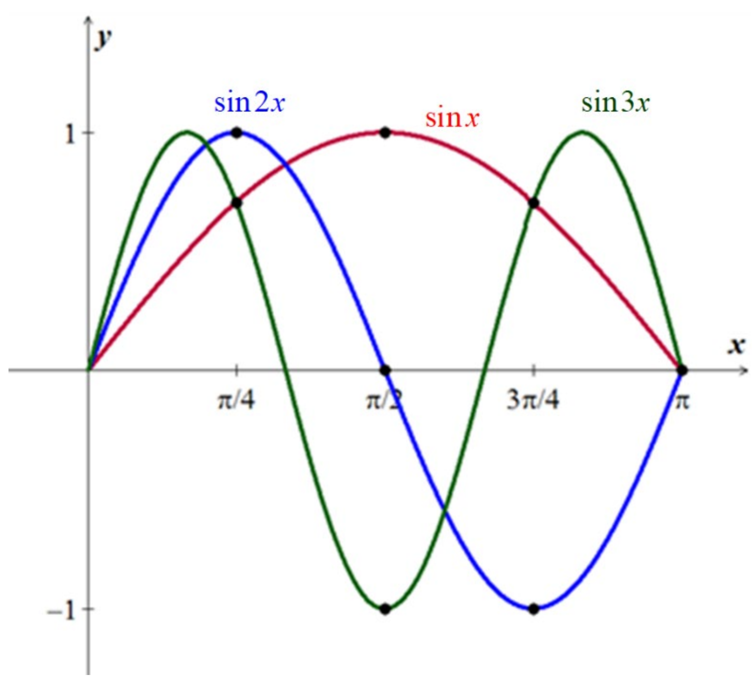
***Solution***

The eigenvalues of $A$ are $\lambda = 2\pm\sqrt{2}$ *and* $2$.

Their sum is 6 (the trace of $A$) and their product is 4 (the determinant).

The eigenvector matrix $S$ gives the "Discrete Sine Transform".

$$S=\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$



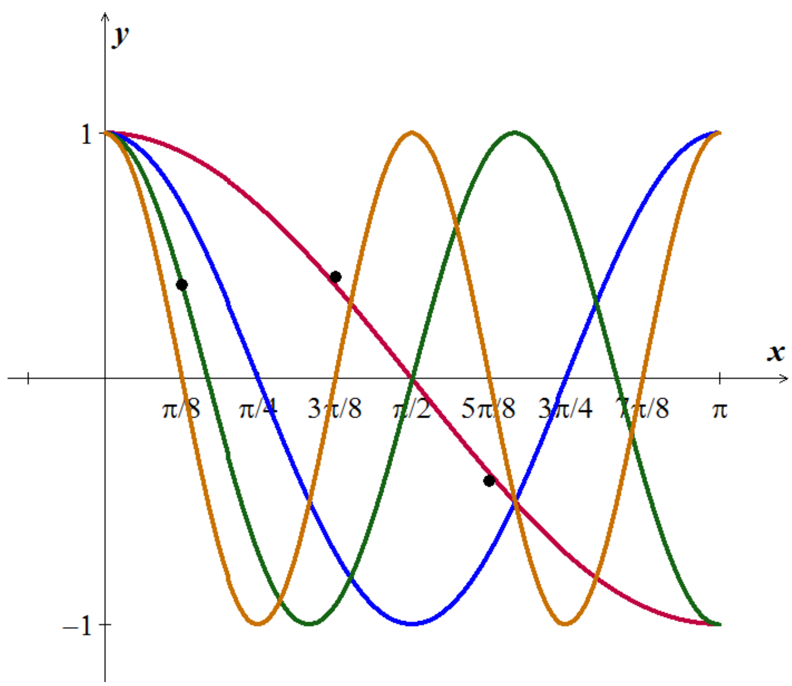
$$V_1 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} \frac{\sqrt{2}}{2} \\ 1 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$V_2 = \begin{pmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} \frac{\sqrt{2}}{2} \\ -1 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

The eigenvalues of $B$ are $\lambda = 2 \pm \sqrt{2},\ \ 2,\ \ 0$.

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \sqrt{2}-1 & -1 & 1-\sqrt{2} \\ 1 & 1-\sqrt{2} & -1 & \sqrt{2}-1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$



### *Exercise*

Suppose $A\vec{x} = \lambda\vec{x}$ and $A\vec{y} = 0\vec{y}$ and $\lambda \neq 0$. Then y is in the nullspace and $\vec{x}$ is in the column space. They are perpendicular because _______. (why are these subspaces orthogonal?) If the second eigenvalue is a nonzero number β, apply this argument to $A - \beta I$. The eigenvalue moves to zero and the eigenvectors stay the same – so they are perpendicular.

***Solution***

Suppose that $A = A^T$ and $Ax = \lambda x$, $Ay = 0y$, and $\lambda \neq 0$. Then $x$ is in the column space of $A$, and $y$ is in the left nullspace of $A$ since $N(A) = N\left(A^T\right)$. But $C(A)$ and $N\left(A^T\right)$ are orthogonal complements, so $x$ and $y$ are perpendicular.

If $Ay = \beta y$ with $\beta \neq \lambda$ then $(A - \beta I)x = (\lambda - \beta)x$ and $(A - \beta I)y = 0$. Since $\lambda - \beta \neq 0$ it follows that $x$ is in the column space $A$- $\beta I$ and $y$ is in the nullspace of $A$- $\beta I$, and $(A - \beta I)^T = A^T - \beta I^T = A - \beta I$, Therefore we can replace A with $A - \beta I$ in the argument of previous paragraph and it follows that $x$ and $y$ are perpendicular.

### *Exercise*

Which of these classes of matrices do $A$ and $B$ belong to: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad B = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Which of these factorizations are possible for $A$ and $B$: $LU$, $QR$, $ADP^{-1}$, $QDQ^T$?

***Solution***

Matrix $A$ is invertible, orthogonal, a permutation matrix, diagonalizable, and Markov! (Everything but a projection).

Matrix $A$ satisfies $A^2 = I$ , $A = A^T$, and also $AA^T = I$, This means it is invertible, symmetric, and orthogonal. Since it is symmetric, it is diagonalizable (with real eigenvalues!). It is a permutation matrix by just looking at it. It is Markov since the columns add to 1. It is not a projection since $A^2 = I \neq A$.

All of the factorization are possible for it: $LU$ and $QR$ are always possible. $PDP^{-1}$ is possible since it is diagonalizable, and $QDQ^T$ is possible since it is symmetric.

Matrix $B$ is a projection, diagonalizable, and Markov. It is not invertible, not orthogonal, and not a permutation.

$B$ is a projection since $B^2 = B$, it is symmetric and thus diagonalizable, and it is Markov since the columns add to 1. It is not invertible since the columns are visibly linearly dependent, it is not orthogonal since the columns are far from orthonormal, and it's clearly not a permutation.

All the factorizations are possible for it: $LU$ and $QR$ are always possible. $PDP^{-1}$ is possible since it is diagonalizable, and $QDQ^T$ is possible since it is symmetric.

### *Exercise*

True or false. Give a reason or a counterexample.

*a)* A matrix with real eigenvalues and eigenvectors is symmetric.
*b)* A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.
*c)* The inverse of a symmetric matrix is symmetric
*d)* The eigenvector matrix $S$ of a symmetric matrix is symmetric.
*e)* A complex symmetric matrix has real eigenvalues.
*f)* If $A$ is symmetric, then $e^{iA}$ is symmetric.
*g)* If $A$ is Hermitian, then $e^{iA}$ is Hermitian.
*h)* An $n \times n$ matrix that is orthogonally diagonalizable must be symmetric.
*i)* If $A^T = A$ and if vectors $\vec{u}$ and $\vec{v}$ satisfy $A\vec{u} = 3\vec{u}$ and $A\vec{v} = 4\vec{v}$, then $\vec{u} \bullet \vec{v} = 0$
*j)* An $n \times n$ symmetric matrix has $n$ distinct real eigenvalues.
*k)* For nonzero $\vec{v}$ in $\mathbb{R}^n$, the matrix $\vec{v}\vec{v}^T$ is called a projection matrix.
*l)* Every symmetric matrix is orthogonally diagonalizable
*m)* If $B = PDP^T$, where $P^T = P^{-1}$ and $D$ is a diagonal matrix, then $B$ is a symmetric matrix.
*n)* An orthogonal matrix is orthogonally diagonalizable.
*o)* The dimension of an eigenspace of a symmetric matrix equals the multiplicity of the corresponding eigenvalue.

**Solution**

***a)*** ***False***. Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

Then $\det\left(A - \lambda I\right) = \begin{vmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = 0$

So, $A$ has eigenvalues $\lambda_1 = -1 \quad \lambda_2 = 2$

The eigenvectors are: $V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ so both the eigenvalues and eigenvectors are real but $A$ is not symmetric.

***b) True***. If the matrix $A$ has orthogonal eigenvectors $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n$ with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, we can define $\vec{s}_i = \frac{\vec{x}_i}{\|\vec{x}_i\|}$ for all $i$; then $A\vec{s}_i = \lambda_i \vec{s}_i$ for all $i$ and the $s_i$ are orthonormal. Then we can diagonalize $A$ as: $A = S\Lambda S^{-1}$ where the $i^{th}$ column of $S$ is $\vec{s}_i$, and $\Lambda$ is the diagonal matrix, so $S^T = S^{-1}$ and $A = S\Lambda S^T$.

$$A^T = \left(S^T\right)^T \Lambda^T S^T$$
$$= S\Lambda S^T$$
$$= A$$

So, $A$ is symmetric.

***c) True***. If $A$ is symmetric then it can be diagonalized by an orthogonal matrix Q, $A = QDQ^{-1}$, and then $A^{-1} = QD^{-1}Q^{-1} = QD^{-1}Q^T$. Since $D^{-1}$ is still a diagonal matrix, it follows:

$$\left(A^{-1}\right)^T = QD^{-1}Q^T$$
$$= A^{-1}$$

***d) False***. The eigenvalues of $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ are: $\lambda_1 = 0 \quad \lambda_2 = 5$ and the eigenvectors are:

$V_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

We can diagonalize $A$ with eigenvector matrix $S = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ which is not symmetric.

***e) False***. For example, $A = (i)$, the 1 by 1 matrix. The eigenvalue is ***i***, it is not a real number.

***f) True***. $\left(e^{iA}\right)^T = e^{(iA)^T} = e^{iA}$

***g) False***. $\left(e^{iA}\right)^H = e^{(iA)^H} = e^{-iA^H} = e^{-iA}$. It is typically not the same as $e^{iA}$.

Taking $A = (1)$, the 1 by 1 matrix, would be an enough example because $e^{iA} = e^i$ which is not a real number.

*p)* An $n \times n$ matrix that is orthogonally diagonalizable must be symmetric.

**a.** True. See Theorem 2 and the paragraph preceding the theorem.

*q)* If $A^T = A$ and if vectors $\vec{u}$ and $\vec{v}$ satisfy $A\vec{u} = 3\vec{u}$ and $A\vec{v} = 4\vec{v}$, then $\vec{u} \bullet \vec{v} = 0$

*r)* An $n \times n$ symmetric matrix has *n* distinct real eigenvalues.

*s)* For nonzero $\vec{v}$ in $\mathbb{R}^n$, the matrix $\vec{v}\vec{v}^T$ is called a projection matrix.

*t)* Every symmetric matrix is orthogonally diagonalizable

*u)* If $B = PDP^T$, where $P^T = P^{-1}$ and *D* is a diagonal matrix, then *B* is a symmetric matrix.

*v)* An orthogonal matrix is orthogonally diagonalizable.

*w)* The dimension of an eigenspace of a symmetric matrix equals the multiplicity of the corresponding eigenvalue.

**b.** True. This is a particular case of the statement in Theorem 1, where **u** and **v** are nonzero.

**c.** False. There are *n* real eigenvalues (Theorem 3), but they need not be distinct (Example 3).

**d.** False. See the paragraph following formula (2), in which each **u** is a unit vector.

**a.** True. See Theorem 2.

**b.** True. See the displayed equation in the paragraph before Theorem 2.

**c.** False. An orthogonal matrix can be symmetric (and hence orthogonally diagonalizable), but not every orthogonal matrix is symmetric. See the matrix *P* in Example 2.

**d.** True. See Theorem 3(b).

### *Exercise*

Find a symmetric matrix $\begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$ that has a negative eigenvalue.

*a)* How do you know it must have a negative pivot?

*b)* How do you know it can't have two negative eigenvalues?

***Solution***

***a)*** The eigenvalues of that matrix are $1 \pm b\,/$ so take any $b < -1 \quad b > 1$. In this case, the determinant is $1 - b^2 < 0$.

***b)*** The signs of the pivots coincide with the signs of the eigenvalues. Alternatively, the product of the pivots is the determinant, which is negative in this case. So, one of the two pivots must be negative.

***c)*** The product of the eigenvalues equals the determinant, which is negative in this case. So, precisely one numbers cannot have a negative product.

### *Exercise*

Prove that *A* is any $m \times n$ matrix, then $A^T A$ has an orthonormal set of *n* eigenvectors

***Solution***

$\left(A^T A\right)^T = A^T\left(A^T\right)^T = A^T A$, then $A^T A$ is symmetric, therefore there is an eigenvector $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ for $A^T A$.

Let $A\vec{v}_1 = \lambda_1 \vec{v}_1 \quad and \quad A\vec{v}_2 = \lambda_2 \vec{v}_2$

$A\vec{v}_1 \bullet \vec{v}_2 = \vec{v}_1 \bullet A^T \vec{v}_2$ $\qquad A\vec{x} \bullet \vec{y} = \vec{x} \bullet A^T \vec{y}$

$A\vec{v}_1 \bullet \vec{v}_2 = \vec{v}_1 \bullet A\vec{v}_2$ $\qquad A^T = A \ \ (Since\ A\ is\ a\ symmetric)$

$\lambda_1 \vec{v}_1 \bullet \vec{v}_2 = \vec{v}_1 \bullet \lambda_2 \vec{v}_2$

$\lambda_1\left(\vec{v}_1 \bullet \vec{v}_2\right) = \lambda_2\left(\vec{v}_1 \bullet \vec{v}_2\right)$

$\left(\lambda_1 - \lambda_2\right)\left(\vec{v}_1 \bullet \vec{v}_2\right) = 0$ $\qquad$ Since $\lambda_1 \neq \lambda_2$

Therefore; $\vec{v}_1 \bullet \vec{v}_2 = 0$

Then the vectors $A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_n$ are orthogonal

$$
\begin{aligned}
A\vec{v}_1 \bullet A\vec{v}_2 &= \left(A\vec{v}_i\right)^T A\vec{v}_j \\
&= \vec{v}_i^{\,T} A^T A\vec{v}_j \\
&= \vec{v}_i \bullet \left(A^T A\vec{v}_j\right) \\
&= \vec{v}_i \bullet \left(\lambda_j \vec{v}_j\right) \\
&= \lambda_j\left(\vec{v}_i \bullet \vec{v}_j\right) = 0
\end{aligned}
$$

### *Example*

Construct a 3 by 3 matrix $A$ with no zero entries whose columns are mutually perpendicular. Compute $A^T A$. Why is it a diagonal matrix?

***Solution***

Consider the matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ to be columns mutual perpendicular

Let assume $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{pmatrix}$ $\quad then \quad A^T = \begin{pmatrix} 1 & 2 & -2 \\ 2 & -2 & -1 \\ 2 & 1 & 2 \end{pmatrix}$ $\qquad or \ \ A = \begin{pmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ 2 & -2 & -1 \\ 2 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$\left(A^T A\right)_{ij} = (column\ i\ of\ A)(column\ j\ of\ A)$$

## *Exercise*

Assuming that $b \neq 0$, find a matrix that orthogonally diagonalizes $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$

***Solution***

$$\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix}$$

$$= (a-\lambda)^2 - b^2$$

$$= (a-\lambda-b)(a-\lambda+b)$$

$$= (a-b-\lambda)(a+b-\lambda) = 0$$

Therefore; the eigenvalues are: $\lambda_1 = a-b \quad and \quad \lambda_2 = a+b$

Assume that $b \neq 0$.

For $\lambda_1 = a-b$, then $\left(A-(a-b)I\right)V_1 = 0$

$$\begin{pmatrix} b & b \\ b & b \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad bx_1 = -by_1$$

The eigenvectors are: $V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

For $\lambda_2 = a+b$, then $\left(A-(a+b)I\right)V_2 = 0$

$$\begin{pmatrix} -b & b \\ b & -b \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad bx_2 = by_2$$

The eigenvectors are: $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Applying the Gram Schmidt process.

$$\vec{u}_1 = \frac{V_1}{\left\|V_1\right\|}$$

$$= \frac{(-1,\ 1)}{\sqrt{2}}$$

$$= \left.\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}\right|$$

$$\vec{w}_2 = V_2 - \frac{\left\langle \vec{u}_1, V_2 \right\rangle}{\left\|V_2\right\|^2}$$

$$= (1,\ 1) - \frac{(-1,\ 1) \bullet (1,\ 1)}{2}(1,\ 1)$$

$$= (1,\ 1)\ \Big|$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\left\|\vec{w}_2\right\|}$$

$$= \frac{(1,\ 1)}{\sqrt{2}}$$

$$= \left.\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}\right|$$

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

## *Exercise*

Suppose $A$ is a symmetric $n \times n$ matrix and $B$ is any $n \times m$ matrix. Show that $B^T AB$, $B^T B$, and $BB^T$ are symmetric matrices.

***Solution***

$A$ is a symmetric, that implies to $A = A^T$

$$\left(B^T AB\right)^T = B^T A^T \left(B^T\right)^T$$

$$= B^T AB\ \Big| \qquad \checkmark$$

Since, $\left(B^T AB\right)^T = B^T AB$, then $B^T AB$ is symmetric.

$$\left(B^T B\right)^T = B^T \left(B^T\right)^T$$

$$\underline{= B^T B} \quad \checkmark$$

Therefore, $B^T B$ is symmetric.

$$\left(BB^T\right)^T = \left(B^T\right)^T B^T$$

$$\underline{= BB^T} \quad \checkmark$$

Therefore, $BB^T$ is symmetric.

## *Exercise*

Show that if $A$ is an $n \times n$ symmetric matrix, then $(A\vec{x}) \bullet \vec{y} = \vec{x} \bullet (A\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$

***Solution***

$A$ is a symmetric, that implies to $A = A^T$

$$(A\vec{x}) \bullet \vec{y} = (A\vec{x})^T \bullet \vec{y}$$

$$= \vec{x}^T A^T \bullet \vec{y}$$

$$= \vec{x}^T A \bullet \vec{y}$$

$$\underline{= \vec{x} \bullet (A\vec{y})} \quad \checkmark$$

## *Exercise*

Suppose $A$ is invertible and orthogonally diagonalizable. Explain why $A^{-1}$ is also orthogonally diagonalizable.

***Solution***

Since $A$ is invertible, then $AA^{-1} = A^{-1}A = I$

And $A$ is orthogonally diagonalizable, then $A = PDP^{-1}$

$$(A)^{-1} = \left(PDP^{-1}\right)^{-1}$$

$$A^{-1} = \left(P^{-1}\right)^{-1} D^{-1} P^{-1} \qquad D^{-1} = D \quad \text{(is a diagonal matrix)}$$

$$= PDP^{-1} \quad \checkmark$$

Therefore, $A^{-1}$ is also orthogonally diagonalizable.

### *Exercise*

Suppose $A$ and $B$ are both orthogonally diagonalizable and $AB = BA$. Explain why $AB$ is also orthogonally diagonalizable

***Solution***

Since $A$ and $B$ are both orthogonally diagonalizable, and $A$ and $B$ are symmetric, then

$$A = A^T \quad \& \quad B = B^T$$

If $AB = BA$, then

$$(AB)^T = (BA)^T$$

$$= A^T B^T \qquad A \text{ and } B \text{ are symmetric}$$

$$\underline{= AB} \quad \checkmark$$

Therefore, $AB$ is also orthogonally diagonalizable

### *Exercise*

Let $A = PDP^{-1}$, where $P$ is orthogonal and $D$ is diagonal, let $\lambda$ be an eigenvalue of $A$ of multiplicity $k$. Then $\lambda$ appears $k$ times on the diagonal of $D$. Explain why the dimension of the eigenspace for $\lambda$ is $k$.

***Solution***

The columns of $P$ are linearly independent eigenvectors by the ***diagonalization*** theorem corresponding to the eigenvalues $\lambda$ of $A$.

Since $D$ is a diagonal with the eigenvalues $\lambda$. when the eigenvalues $\lambda$ is of multiplicity $k$, then $\lambda$ appears $k$ times on the diagonal of $D$.

So, $P$ has exactly $k$ columns of eigenvectors corresponding to the eigenvalues $\lambda$.

Therefore, the $k$ columns form a basis for the eigenspace.

### *Exercise*

Suppose $A = PUP^{-1}$, where $P$ is orthogonal and $U$ is an upper triangular. Show that if $A$ is symmetric, then $U$ is symmetric and hence is actually a diagonal matrix.

***Solution***

Given: $A = PUP^{-1}$

If $A$ is symmetric, then $A = A^T$

$$A = PUP^{-1}$$

$$P^{-1}AP = P^{-1}PUP^{-1}P$$

$$U = P^{-1}AP$$

Since $P$ is orthogonal, then

$U = P^{-1}AP = P^T AP$

$U^T = \left(P^T AP\right)^T$

$= P^T A^T \left(P^T\right)^T$ *A is symmetric*

$\boxed{= P^T AP}$

That implies that $U$ is symmetric

Since $U$ is an upper triangular and symmetric, then the entries above and below the main diagonal must be equal to zeros.

Therefore, $U$ is a diagonal matrix.

## Exercise

Let $\vec{u}$ be a unit vector in $\mathbb{R}^n$, and let $B = \vec{u}\,\vec{u}^T$.

*a)* Given $\vec{x} \in \mathbb{R}^n$, compute $B\vec{x}$ and show that $B\vec{x}$ is the orthogonal projection of $\vec{x}$ onto $\vec{u}$.

*b)* Show that $B$ is a symmetric matrix and $B^2 = B$.

*c)* Show that $\vec{u}$ is an eigenvector of $B$. What is the corresponding eigenvalue?

**Solution**

***a)*** Given $\vec{x} \in \mathbb{R}^n$

$B\vec{x} = \left(\vec{u}\,\vec{u}^T\right)\vec{x}$

$= \vec{u}\left(\vec{u}^T\vec{x}\right)$

$= \left(\vec{u}^T\vec{x}\right)\vec{u}$ $\vec{u}^T\vec{x}$ : *scalar*

$= \left(\vec{x} \bullet \vec{u}\right)\vec{u}$

Given that $\vec{u}$ is a unit vector in $\mathbb{R}^n$, then $B\vec{x}$ is the orthogonal projection of $\vec{x}$ onto $\vec{u}$.

***b)*** $B^T = \left(\vec{u}\,\vec{u}^T\right)^T$

$= \left(\vec{u}^T\right)^T \vec{u}^T$

$= \vec{u}\,\vec{u}^T$

$\boxed{= B}$ ✓

Therefore, $B$ is symmetric.

$$B^2 = \left(\vec{u}\vec{u}^T\right)^2$$
$$= \left(\vec{u}\vec{u}^T\right)\left(\vec{u}\vec{u}^T\right)$$
$$= \vec{u}\left(\vec{u}^T\vec{u}\right)\vec{u}^T \qquad\qquad \vec{u}^T\vec{u} = 1$$
$$= \vec{u}\,\vec{u}^T$$
$$\underline{= B} \qquad \checkmark$$

Therefore, $B^2$ is symmetric.

***c)*** Since $\vec{u}^T\vec{u} = 1$, then

$$B\vec{u} = \left(\vec{u}\,\vec{u}^T\right)\vec{u}$$
$$B\vec{u} = \vec{u}\left(\vec{u}^T\,\vec{u}\right)$$
$$= \vec{u}\,(1)$$
$$\underline{= \vec{u}}$$

Therefore, $\vec{u}$ is an eigenvector of $B$ with the corresponding eigenvalue 1.

## *Exercise*

Let $B$ be an $n \times n$ symmetric matrix such that $B^2 = B$. Any such matrix is called a ***projection matrix*** (or an *orthogonal projection matrix*). Given any $\vec{y} \in \mathbb{R}^n$, let $\hat{y} = B\vec{y}$ and $\vec{z} = \vec{y} - \hat{y}$.

*a)* Show that $\vec{z}$ is orthogonal to $\hat{y}$.

*b)* Let $W$ be the column space of $B$. Show that $\vec{y}$ is the sum of a vector in $W$ and a vector in $W^\perp$. Why does this prove that $B\vec{y}$ is the orthogonal projection of $\vec{y}$ onto the column space of $B$?

***Solution***

Since $B$ is symmetric, then $B = B^T$

Given too, that $B^2 = B$ which is symmetric.

$\hat{y} = B\vec{y}$

***a)*** $\vec{z} \bullet \hat{y} = \left(\vec{y} - \hat{y}\right) \bullet B\vec{y}$

$$= \vec{y} \bullet (B\vec{y}) - \hat{y} \bullet (B\vec{y})$$
$$= \vec{y}^T \bullet (B\vec{y}) - (B\vec{y})^T \bullet (B\vec{y})$$
$$= \vec{y}^T \bullet B\vec{y} - \vec{y}^T B^T \bullet B\vec{y}$$
$$= \vec{y}^T \bullet B\vec{y} - \vec{y}^T B \bullet B\vec{y} \qquad\qquad B^2 = B$$

$= \vec{y}^T B\vec{y} - \vec{y}^T B\vec{y}$

$\underline{= 0}$

Therefore, $\vec{z}$ is orthogonal to $\hat{y}$

***b)*** Since $W$ be the column space of $B$, then $W = Col(B)$ has the form $B\vec{u}$ (for some $\vec{u}$ )

$$(\vec{y} - \hat{y}) \cdot (B\vec{u}) = B(\vec{y} - \hat{y}) \cdot \vec{u}$$

$$= (B\vec{y} - B\hat{y}) \cdot \vec{u}$$

$$= (B\vec{y} - BB\vec{y}) \cdot \vec{u} \qquad B^2 = B$$

$$= (B\vec{y} - B\vec{y}) \cdot \vec{u}$$

$$\underline{= 0}$$

Therefore, $(\vec{y} - \hat{y})$ is in $W^{\perp}$, and the decomposition $\vec{y} = \hat{y} + (\vec{y} - \hat{y})$ expresses $\vec{y}$ as the sum of a vector in $W$ and a vector in $W^{\perp}$.

By the orthogonal Decomposition, this decomposition is unique, and so $\hat{y}$ must be orthogonal projection of $\vec{y}$ onto the column space of $B$ ($W$)