$$\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n}$$

$$\cos n\pi = (-1)^n$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{5^n} = \sum_{n=0}^{\infty} \left(-\frac{1}{5}\right)^n$$

3.5

Ratio Tests

$\sum a_n$ $\qquad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$

1 - if $\rho < 1 \Rightarrow$ series converges

$\rho > 1$ " diverges

$\rho = 1$ inconclusive.

Ex $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}+5}{3^{n+1}} \cdot \frac{3^n}{2^n+5}$$

$$= \frac{1}{3} \frac{2^{n+1}+5}{2^n+5}$$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

$$= \frac{1}{3} \lim_{n\to\infty} \frac{2^{n+1}+5}{2^n+5}$$

$$= \frac{1}{3}(2)$$

$$= \frac{2}{3} < 1$$

By the Ratio Test, the given series converges

$\frac{2^n+5}{3^n} = \left(\frac{2}{3}\right)^n + 5\left(\frac{1}{3}\right)^n$ $\quad$ $\frac{2}{3}<1$, $\frac{1}{3}<1$

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$

$n! = 1\,2\cdots(n-1)n$

$(n+1)! = 1\cdot 2\cdots n\cdot(n+1)$

$$\frac{a_{n+1}}{a_n} = \frac{(2(n+1))!}{(n+1)!\,(n+1)!}\cdot\frac{n!\,n!}{(2n)!}$$

$$= \frac{(2n+2)!}{(n+1)\,(n+1)}\;\frac{1}{(2n)!}$$

$$= \frac{(2n+1)\,(2n+2)}{(n+1)\,(n+1)}$$

$$\rho = \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)(n+1)}$$

$$= \lim_{n\to\infty} \frac{4n^2+\cdots}{n^2+\cdots}$$

$$= 4 > 1$$

$\therefore$ By the Ratio test, the given series diverges

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{4^n n!\, n!}{(2n)!}$

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}(n+1)!\,(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{4^n n!\, n!}$$

$$= 4\,\frac{(n+1)(n+1)}{(2n+1)(2n+2)}$$

$$\rho = 4 \lim_{n\to\infty} \cdot \frac{n^2+2n+1}{4n^2+6n+2}$$

$$= 4\left(\frac{1}{4}\right)$$
$$= 1$$

$a_{k+1} \; ? \; a_k$

$1 \to \frac{4}{2} = 2$

$2 \to \frac{16(4)}{24} = \frac{8}{3}$

$$\frac{4^{k+1}(k+1)!\,(k+1)!}{(2k+2)!} \; ? \; \frac{4^k k!\, k!}{(2k)!}$$

$$\frac{4^{k+1}(k+1)!(k+1)!}{4^k k!\, k!} \; ? \; \frac{(2k+2)!}{(2k)!}$$

$$4(k+1)(k+1) \; ? \; (2k+1)(2k+2)$$
$$\underline{\underline{4}}(k+1)\underline{(k+1)} \; ? \; \underline{\underline{2}}(2k+1)\underline{(k+1)}$$
$$2k+2 > 2k+1$$

$a_{k+1} > a_k$

By the Ratio Test is inconclusive,
since $a_{k+1} > a_k$, the given series diverges.

Root Test

$\sum a_n \qquad \lim_{n\to\infty} \sqrt[n]{a_n} = \rho$

$\rho < 1$, series converges

$\rho > 1$ " diverges

$\rho = 1$ inconclusive

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

$$\sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{2}$$

$$= \frac{n^{2/n}}{2}$$

$$\rho = \lim_{n\to\infty} \frac{n^{2/n}}{2}$$

$$= \frac{1}{2}\,\infty^0$$

$$= \frac{1}{2} < 1$$

$\therefore$ By the Root Test, the given series converges

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}$$

$$= \frac{1}{2}\left(\frac{n+1}{n}\right)^2$$

$$\sum_{n=1}^{\infty} \frac{2^n}{n^3}$$

$$\sqrt[n]{\frac{2^n}{n^3}} = \frac{2}{n^{3/n}} \qquad \left(n^{\frac{1}{n}}\right)^3 \to 1$$

$$\rho = \lim_{n\to\infty} \frac{2}{n^{3/n}}$$

$$= 2 > 1$$

By the Root Test, the given series diverges

$$\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$$

$$\sqrt[n]{\left(\frac{1}{1+n}\right)^n} = \frac{1}{n+1}$$

$$\rho = \lim_{n\to\infty} \frac{1}{n+1}$$

$$= 0 < 1$$

$\therefore$ By the Root Test, the given series converges

($\therefore$ Therefore

3.5.

#1 Ratio Test.

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}$$

$$= \frac{2}{n+1}$$

$$\rho = \lim_{n\to\infty} \frac{2}{n+1}$$

$$= 0 < 1$$

$\therefore$ By the Ratio Tests the given series converges.

#5 $$\sum_{n=1}^{\infty} \frac{n\,5^n}{(2n+3)\ln(n+1)}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1).5^{n+1}}{(2n+5)\ln(n+2)} \cdot \frac{(2n+3)\ln(n+1)}{n.5^n}$$

$$= 5\ \frac{2n^2+5n+3}{2n^2+5n}\ \frac{\ln(n+1)}{\ln(n+2)}$$

$$\rho = 5 \lim_{n\to\infty} \frac{2n^2+5n+3}{2n^2+5n} \cdot \lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n+2)} \quad \frac{\infty}{\infty}$$

$$= 5 \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n+2}}$$

$$= 5 \lim_{n\to\infty} \frac{n+2}{n+1}$$

$$= 5 > 1$$

$\therefore$ By the Ratio Tests the given series diverges

Root Test

#24 $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$

$$\sqrt[n]{\left(\frac{4}{3n}\right)^n} = \frac{4}{3n}$$

$$\rho = \lim_{n\to\infty} \frac{4}{3n}$$

$$= 0 < 1$$

$\therefore$ By the Root Test, the given series converges

#26 $\displaystyle\sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$

$$\sqrt[n]{\sin^n \frac{1}{\sqrt{n}}} = \sin\frac{1}{\sqrt{n}}$$

$$\rho = \lim_{n\to\infty} \sin\frac{1}{\sqrt{n}}$$

$$= \sin 0$$

$$= 0 < 1$$

$\therefore$ By the Root Test, the given series converges.

Sec 3.6

Alternating Series. $+ - + - \cdots$

$- + - + \cdots \quad (-1)^?$

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - \cdots$$

Converges : (3 steps

(1) u_n's all positive 😊 #

* $u_n > u_{n+1}$
* $u_n \longrightarrow 0$ $\quad \lim_{n\to\infty} u_n = 0$

Ex $\quad \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ $\quad -l_n$

1) $u_n = \frac{1}{n}$

$n < n+1$

$\frac{1}{n} > \frac{1}{n+1}$

$u_n > u_{n+1}$ ✓

2) $\frac{1}{n} \longrightarrow 0$

$\therefore$ By the alternating series, the given series converges.

Absolute convergence

Defn $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

Ex $\sum (-1)^{n+1} \frac{1}{n^2}$

$$\sum \left|(-1)^{n+1}\frac{1}{n^2}\right| = \sum \frac{1}{n^2} \qquad p=2>1 \to \text{converges}$$

The given series converges because it converges absolutely

Defn A series converges but doesn't converge absolutely converges conditionally.

#1 $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$

1) $u_n = \frac{1}{\sqrt{n}}$

$$n < n+1$$
$$\sqrt{n} < \sqrt{n+1}$$
$$\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$$
$$u_n > u_{n+1} \checkmark$$

2) $\frac{1}{\sqrt{n}} \longrightarrow 0 \checkmark$

$\therefore$ By the alternating series, the given series converges.

#:2 $\sum_{n=2}^{\infty} (-1)^n \frac{4}{(\ln n)^2}$

$$u_n = \frac{4}{(\ln n)^2}$$

$$\ln n < \ln(n+1)$$

$$(\ln n)^2 < (\ln(n+1))^2$$

$$\frac{1}{(\ln n)^2} > \frac{1}{(\ln(n+1))^2}$$

$$\frac{4}{(\ln n)^2} > \frac{4}{(\ln(n+1))^2}$$

$$u_n > u_{n+1} \checkmark$$

$$\frac{4}{(\ln n)^2} \longrightarrow 0 \checkmark$$

$\therefore$ By the Alternating Series, the given series converges.

#71 $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

$$u_n = \frac{1}{n \ln n}$$

$$\ln n < \ln(n+1)$$

$$n \ln n < (n+1)\ln(n+1)$$

$$\frac{1}{n \ln n} > \frac{1}{(n+1)\ln(n+1)}$$

$$u_n > u_{n+1} \checkmark$$

$$\frac{1}{n \ln n} \longrightarrow 0 \checkmark$$

By the alternating series, the given series converges.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Geom #
p-series # n^{p}
Integral ✓
Comparison ✓
Ratio ✓
Root ?
alternating #

$$\int_2^{\infty} \frac{dx}{x \ln x} = \int_2^{\infty} \frac{d(\ln x)}{\ln x}$$

$$= \ln(\ln x) \Big|_2^{\infty}$$

$$= \infty$$

By the integral Test, the given series diverges

Comparison $\frac{1}{n \ln n}$

$n \ln n > n$

$\frac{1}{n \ln n} < \frac{1}{n}$

$\frac{1}{n} \to p = 1$ diverges by p-series

Ratio

$$\frac{a_{n+1}}{a_n} = \frac{1}{(n+1)\ln(n+1)} \cdot \frac{n \ln(n)}{1}$$

$$\rho = \lim_{n\to\infty} \frac{n}{n+1} \cdot \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}}$$

$$= \lim_{n\to\infty} \frac{n+1}{n}$$

$= 1$ inconclusive.

Root $\sqrt[n]{\frac{1}{n \ln n}} = \frac{1}{\sqrt[n]{n}} \cdot \frac{1}{\sqrt[n]{\ln n}}$ $(\ln n)^{1/n}$

$$\rho = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} \cdot \lim_{n\to\infty} \frac{1}{\sqrt[n]{\ln n}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt[n]{\ln n}}$$

$$\lim \ln\left((\ln n)^{1/n}\right) = \lim_{n\to\infty} \frac{\ln(\ln n)}{n}$$

$$\frac{\frac{1}{\ln n}}{1}$$

$$= 0$$

$$\lim (\ln n)^{1/n} = e^0$$

$$= 1$$

$\rho = 1$ inconclusive