SOLUTIONS

Chapter 1

1.1 **a)** $(1 + x)dy - ydx = 0 \Rightarrow (1 + x)dy = ydx \Rightarrow \frac{dy}{y} = \frac{dx}{1+x} \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{1+x}$

$\Rightarrow \ln|y| = \ln|1+x| + c_1 \Rightarrow y = e^{\ln|1+x|+c_1} = |1 + x|\ e^{c_1} \Rightarrow y = \pm\, c(1 + x)$

b) $\frac{dy}{dx} = e^{3x + 5y} = e^{3x}\, e^{5y} \Rightarrow \frac{dy}{e^{5y}} = e^{3x}dx \Rightarrow e^{-5y}dy = e^{3x}dx \Rightarrow \int e^{-5y}dy = \int e^{3x}dx$

$\Rightarrow -\frac{1}{5}e^{-5y} = \frac{1}{3}e^{3x} + c_1 \Rightarrow e^{-5y} = -\frac{5}{3}e^{3x} + c_2$

c) $\frac{dy}{dx}\, e^{x}y = e^{-y} + e^{-2x-y} = e^{-y}\,(1 + e^{-2x}) \Rightarrow \frac{ydy}{e^{-y}} = e^{-x}(1 + e^{-2x})dx$

$\Rightarrow \int ye^{y}dy = \int (e^{-x} + e^{-3x})dx \Rightarrow -(y+1)\, e^{-y} = -e^{-x} - \frac{1}{3}e^{-3x} + c$

$\Rightarrow (y+1)\, e^{-y} = e^{-x} + \frac{1}{3}e^{-3x} + c$

d) $\frac{dy}{dx} - y^2 = -4 \Rightarrow \frac{dy}{dx} = y^2 - 4 \Rightarrow \frac{dy}{y^2 - 4} = dx$

(Encounter the divisors at $y = \pm 2$ could be zero)

$\Rightarrow \int \left(\frac{1/4}{y-2} - \frac{1/4}{y+2} \right) dy = \int dx \Rightarrow$ From the Appendix A (Integrals)

$\Rightarrow \frac{1}{4}\ln\left|\frac{y-2}{y+2}\right| = x + c_1 \Rightarrow \ln\left|\frac{y-2}{y+2}\right| = 4x + c_2$

$\Rightarrow \left|\frac{y-2}{y+2}\right| = e^{4x+c_2} \Rightarrow \frac{y-2}{y+2} = \pm\, ce^{4x} \Rightarrow y = 2\frac{1+ce^{4x}}{1-ce^{4x}}$

If $y = -2 \Rightarrow -2 = 2\frac{1+ce^{4x}}{1-ce^{4x}} \Rightarrow -(1 - ce^{4x}) = 1 + ce^{4x} \Rightarrow -1 = 1$ (impossible)

$\Rightarrow$ There is no solution at $y = -2$.

If $y = 2 \Rightarrow 2 = 2\frac{1+ce^{4x}}{1-ce^{4x}} \Rightarrow 1 - ce^{4x} = 1 + ce^{4x} \Rightarrow -ce^{4x} = ce^{4x}$ only when $c = 0$.

$\Rightarrow$ Therefore there is a solution when $y = 2$.

e) $\frac{dy}{dx} - 3y = 0 \Rightarrow \frac{dy}{dx} = 3y \Rightarrow \int \frac{dy}{3y} = \int dx \Rightarrow 1/3\ \ln(y) = x + c_1$

$\Rightarrow \ln(y) = 3x + c_2 \Rightarrow y = c\, e^{3x}$

f) $\frac{dy}{dx} - 0.2xy = 0 \Rightarrow \int \frac{dy}{y} = \int .2xdx \Rightarrow \ln(y) = 0.1\, x^2 + c_1 \Rightarrow y = c\, e^{.1x^2}$

g) $\frac{dP}{dt} = P(a - bP) \Rightarrow \int \frac{dP}{P(a-bP)} = \int dt \Rightarrow \frac{1}{a}\ln\left(\frac{P}{a-bP}\right) = t + c_1 \Rightarrow \frac{P}{a-bP} = c\, e^{at}$

h) $\frac{dy}{dx} = y^2 - 3y \Rightarrow \int \frac{dy}{y(y-3)} = \int dx \Rightarrow -$ $(\ln y - \ln(y-3)) = x + c_1 \Rightarrow \ln\left|\frac{y}{y-3}\right| = -3x + c_2$

$\Rightarrow \frac{y}{y-3} = ce^{-3x} \Rightarrow y = \frac{-3ce^{-3x}}{1-ce^{-3x}}$ (with $y \neq 3$)

1.2 **a**) $(x + 2y - 1)\,dx + 3(x + 2y)\,dy = 0 \Rightarrow$ Assume: $x + 2y = v \Rightarrow dx + 2dy = dv$

$\Rightarrow (v - 1)(dv - 2dy) + 3v\,dy = 0 \Rightarrow (v - 1)\,dv + (v - 2)\,dy = 0$

$\Rightarrow \left(\frac{\upsilon-1}{\upsilon+2}\right)dv + dy = 0 \Rightarrow \left(v - \frac{3}{v+2}\right)dv + dy = 0 \Rightarrow v - 3\ln|v + 2| + y + c = 0$

$\Rightarrow x + 2y - 3\ln|x + 2y + 2| + y + c = 0 \Rightarrow x + 3y - 3\ln|x + 2y + 2| + c = 0$

b) $(1 + 3x \sin y)\,dx - x^2 \cos y\,dy = 0 \Rightarrow$ Assume: $\sin y = v \Rightarrow dv = \cos y\,dy$

$\Rightarrow (1 + 3xv)\,dx - x^2\,dv = 0 \Rightarrow dx = -3xv\,dx + x^2\,dv \Rightarrow x^{-5}\,dx = -3x^{-4}\,v\,dx + x^{-3}\,dv \Rightarrow$

$\Rightarrow \int x^{-5}dx = \int (x^{-3}dv - 3x^{-4}vdx = \int (vx^{-3})' = vx^{-3} \Rightarrow -\frac{1}{4}x^{-4} + c_1 = vx^{-3}$

$\Rightarrow -1 + cx^4 = 4x \sin y$

1.3 **a**) $y(6y^2 - x - 1)\,dx + 2x\,dy = 0 \Rightarrow$ using Bernoulli method (Appendix B)

$6y^3dx - y(x+1)dx + 2xdy \Rightarrow$ Divide by $(2xdx) \Rightarrow \dot{y} - \frac{x+1}{2x}y = -\frac{3}{x}y^3$

$\Rightarrow n = 3,\ P(x) = -\frac{x+1}{2x};\ e^{-2\int Pdx} = e^{\int \frac{x+1}{x}dx} = e^{x+\ln x} = e^x x$, and $Q(x) = -\frac{3}{x}$

$\Rightarrow y^{-2}\,xe^x = -2\int \frac{-3}{x}xe^x dx = 6\int e^x dx + c = 6e^x + c \Rightarrow y^{-2}\,xe^x = 6e^x + c \Rightarrow y^2 = \frac{xe^x}{6e^x + c}$

b) $6y^2\,dx - x(2x^3 + y)\,dy = 0 \Rightarrow$ multiply by $x^2 \Rightarrow 6y^2x^2\,dx - x^3(2x^3 + y)\,dy = 0$

$\Rightarrow$ Assume $v = x^3 \Rightarrow dv = 3x^2\,dx \Rightarrow 2y^2dv - v(2v + y)\,dy = 0$

Let $v = wy \Rightarrow dv = ydw + wdy \Rightarrow 2y^2(ydw+wdy) - wy(2wy+y)\,dy = 0$

$\Rightarrow 2y^2(ydw+wdy) - wy^2(2w+1)\,dy = 0$

$\Rightarrow 2ydw + 2wdy - 2w^2dy - wdy = 2ydw - w(2w - 1)dy = 0$

$\Rightarrow \frac{2dw}{w(2w-1)} - \frac{dy}{y} = \left(\frac{4}{2w-1} - \frac{2}{w}\right)dw - \frac{dy}{y} = 0 \Rightarrow \int\left(\frac{4}{2w-1} - \frac{2}{w}\right)dw = \int \frac{dy}{y}$

$\Rightarrow 2\ln|2w - 1| - 2\ln|w| = \ln|y| + c \Rightarrow \ln\left(\frac{(2w-1)^2}{w^2}\right) = \ln|y| + \ln|c| = \ln|cy|$

$\Rightarrow \frac{(2w-1)^2}{w^2} = cy \Rightarrow cyw^2 = (2w-1)^2 \Rightarrow w = \frac{v}{y} = \frac{x^3}{y} \Rightarrow c\frac{x^6}{y} = (2\frac{x^3}{y} - 1)^2$

$\Rightarrow cyx^6 = (2x^3 - y)^2$

1.4 **a**) $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0 \Rightarrow a = 1, b = -2, c = -4 \Rightarrow m^2 - 3m - 4 = (m+1)(m-4) = 0$

$\Rightarrow m = -1, 4$ (Real & distinct) $\Rightarrow y = c_1x^{-1} + c_2x^4$

b) $4x^2\frac{d^2y}{dx^2} + 8x\frac{dy}{dx} + y = 0 \Rightarrow a = 4, b = 8, c = 1 \Rightarrow 4m^2 + 4m + 1 = (2m+1)^2 = 0$

$\Rightarrow m = -1/2$ (Real & equals) $\Rightarrow y = c_1x^{-1/2} + c_2x^{-1/2}\ln x$

c) $4x^2\frac{d^2y}{dx^2} + 17y = 0 \Rightarrow a = 4, b = 17, c = 0 \Rightarrow 4m^2 + 13m = 0 \Rightarrow m = -1/2 \pm 2i$

$\Rightarrow y = x^{1/2}\,[c_1\cos(2\ln x) + c_2\sin(2\ln x)]$

d) $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y = 2x^4e^x \Rightarrow a = 1, b = -3, c = 3 \Rightarrow m^2 - 4m + 3 = 0$

$\Rightarrow m = 1, 3$ (Real & equals) $\Rightarrow y_c = c_1x + c_2x^3$ and $y_p = u_1y_1 + u_2y_2$

$\Rightarrow \ddot{y} - \frac{3}{x}\dot{y} + \frac{3}{x^2}y = 2x^2e^x$

$$\Rightarrow \dot{u}_1 = \frac{W_1}{W} = \frac{\begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix}}{\begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}} = \frac{-2x^5e^x}{2x^3} = -x^2e^x$$

$\Rightarrow u_1 = -x^2e^x + 2xe^x - 2e^x$

$$\dot{u}_2 = \frac{W_2}{W} = \frac{\begin{vmatrix} x & 0 \\ 1 & 2x^2e^x \end{vmatrix}}{2x^3} = e^x \quad \Rightarrow u_2 = e^x$$

$\Rightarrow y_p = (-x^2e^x + 2xe^x - 2e^x)x + e^xx^3 = 2x^2e^x - 2xe^x$

$\Rightarrow y = c_1x + c_2x^3 + 2x^2e^x - 2xe^x$

e) $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \ln x$ Substitute $x = e^t \Rightarrow t = \ln x$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt} \Rightarrow \frac{d^2y}{dt^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\left(-\frac{1}{x^2}\right) = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t$ (Not Cauchy-Euler Form)

$\Rightarrow m^2 - 2m + 1 = (m-1)^2 = 0 \Rightarrow y_c = c_1e^t + c_2te^t$

$y_p = A + Bt \Rightarrow \dot{y}_p = B \Rightarrow \ddot{y}_p = 0 \Rightarrow -2B + A + BT = t \Rightarrow A = 2, B = 1$

$y = c_1e^t + c_2te^t + 2 + t = c_1x + c_2x\ln x + 2 + \ln x$

1.5 **a)** $\frac{dx}{dt} = 3t^2(1 + x^2) \Rightarrow \frac{dx}{1+x^2} = 3t^2\,dt \Rightarrow \tan^{-1} = t^3 + c \Rightarrow x(t) = \tan(t^3 + c)$

$x(0) = \tan(0 + c) = \tan(c) = 0 \Rightarrow c = 0$

$\Rightarrow x(t) = \tan(t^3)$; with $-\frac{\pi}{2} < t^3 < \frac{\pi}{2} \Rightarrow -\left(\frac{\pi}{2}\right)^{1/3} < t < \left(\frac{\pi}{2}\right)^{1/3}$

b) $\dot{y} + y\tan(t) = \cos(t) \Rightarrow p(t) = -\tan(t) \Rightarrow e^{-\int p(t)dt} = e^{\int \tan(t)dt} = e^{-\ln(\cos(t))} = \frac{1}{\cos t}$

$\frac{\dot{y}}{\cos t} + y\frac{\sin t}{\cos^2 t} = 1 \Rightarrow \left(\frac{y}{\cos t}\right)' = 1 \Rightarrow \frac{y}{\cos t} = t + c \Rightarrow y(t) = (t + c)\cos t$

$y(0) = c = 1 \Rightarrow y(t) = (t + 1)\cos t$

$\therefore$ $t \in$Ñ, however since tan(t) is given, therefore the solution $-\frac{\pi}{2} < t < \frac{\pi}{2}$

c) $\ddot{x} - 6\dot{x} + 25x = 0$; let $x = e^{rt} \Rightarrow (t^2 - 6t + 25)e^{rt} = 0 \Rightarrow t = 3 \pm 4i$

$\Rightarrow x = e^{3t}(c_1\cos 4t + c_2\sin 4t)$

$x(0) = c_1 = 0$

$x(\frac{\pi}{8}) = e^{3\pi/8}c_2 = 1 \Rightarrow c_2 = e^{-3\pi/8}$

1.6 **a)** $\dot{x} - \frac{2}{t}x = 2t^2 \Rightarrow e^{-\int \frac{2}{t}dt} = e^{-2\ln t} = t^{-2} \Rightarrow (t^{-2})' = t^{-2}(2t^2) = 2 \Rightarrow t^{-2}x = 2t + c$

$\Rightarrow x = 2t^3 + ct^2 \Rightarrow x(1) = 2 + c = 0 \Rightarrow c = -2 \Rightarrow x = 2t^3 - 2t^2$

b) If $x(0) = 3 \Rightarrow t\dot{x} - 2x = 2t^3 \Rightarrow 0 - 2(3) = 0$

$\Rightarrow -6 = 0$ (impossible); therefore there is no solution with this initial condition.

Or for the general solution: $x = 2t^3 + ct^2 \Rightarrow 3 = 0 + 0(c) = 0$ (impossible); therefore there is no solution for c of the general equation.

1.7 **a)** $\ddot{y} - 4\dot{y} + 13y = 0 \Rightarrow (D^2 - 4D + 13)y = 0 \Rightarrow D^2 - 4D + 13 = 0 \Rightarrow \lambda_{1,2} = 2 \pm 3i$

$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x) = e^{2x}(c_1\cos 3x + c_2\sin 3x)$

b) $y(0) = 1 \Rightarrow y = e^0(c_1\cos 0 + c_2\sin 0) = c_1 = 1$

and $\dot{y}(0) = 2 \Rightarrow \dot{y} = 2e^{2x}(c_1\cos 3x + c_2\sin 3x) + e^{2x}(-3c_1\sin 3x + 3c_2\cos 3x)$

$\Rightarrow \dot{y}(0) = 2e^0(c_1\cos 0x + c_2\sin 0) + e^0(-3c_1\sin 0 + 3c_2\cos 0) = 2 \Rightarrow 2c_1 + 3c_2 = 2$

$\Rightarrow 3c_2 = 2 - 2c_1 = 2 - 2 = 0 \Rightarrow c_2 = 0$

1.8 **a)** $y'' + y = \sin x \Rightarrow (D^2 + 1)y = 0 \Rightarrow D = \pm i \Rightarrow y_c = c_1\cos x + c_2\sin x$

$\Rightarrow D^2(\sin x) = -\sin x \Rightarrow D = \pm i \Rightarrow y_p = Ax\cos x + Bx\sin x.$

$\Rightarrow y_p' = A\cos x + B\sin x - Ax\sin x + Bx\cos x$

$\Rightarrow y_p'' = -A\sin x + B\cos x - A\sin x + B\cos x - Ax\cos x - Bx\sin x$

$= -A(2\sin x + x\cos x) + B(2\cos x - x\sin x)$

$\Rightarrow y'' + y = \sin x \Rightarrow y'' + y = -2A\sin x + 2B\cos x = \sin x$

$$\begin{cases} -2A = 1 \\ 2B = 0 \end{cases} \Rightarrow A = -\tfrac{1}{2};\ B = 0 \Rightarrow y_p = -1/2\, x\cos x$$

$\therefore$ The solution is: $y = c_1\cos x + c_2\sin x - \tfrac{1}{2}x\cos x$

b) $y''' - y' = 4e^{-x} + 3e^{2x} \Rightarrow y''' - y' = D^3 - D = D(D^2 - 1)y = 0 \Rightarrow D = 0, \pm 1$

$\Rightarrow y_c = c_1 + c_2e^{x} + c_3e^{-x}$

$y_p = Axe^{-x} + Be^{2x} \quad \Rightarrow y_p' = A(-xe^{-x} + e^{-x}) + 2Be^{2x}$

$\Rightarrow y_p'' = A(xe^{-x} - 2e^{-x}) + 4Be^{2x}$

$\Rightarrow y_p''' = A(-x + 3)e^{-x} + 8Be^{2x}$

$\Rightarrow y''' - y' = A(-x + 3)e^{-x} + 8Be^{2x} - A(-xe^{-x} + e^{-x}) - 2Be^{2x}$

$= 2Ae^{-x} + 6Be^{2x} = 4e^{-x} + 3e^{2x}$

$\Rightarrow 2A = 4;\ A = 2$ and $6B = 3$; $B = ½$

$\Rightarrow y = c_1 \cos x + c_2 \sin x + 2xe^{-x} + 1/2\, e^{2x}$

c) $y'' + y = 12\cos^2 x = 6(1 + \cos 2x) \Rightarrow y'' + y = D^2 + 1 = 0$

$\Rightarrow D = \pm i \Rightarrow y_c = c_1 \cos x + c_2 \sin x$

$y_p = D + A\cos 2x + B\sin 2x$

$\Rightarrow y_p' = -2A\sin 2x + 2B\cos 2x$

$\Rightarrow y_p''' = -4A\cos 2x - 4B\sin 2x$

$\Rightarrow y'' + y = -4A\cos 2x - 4B\sin 2x + A\cos 2x + B\sin 2x + D$

$= D - 3A\cos 2x - 3B\sin 2x$

$= 6 + \cos 2x$

$\Rightarrow -3B = 0;\ B = -2$ and $-3A = 6;\ A = -2$ and $D = 6$

$\Rightarrow y = c_1 \cos x + c_2 \sin x + 6 - 2\cos 2x$

Chapter 2

2.1 **a**) Equilibrium point: $\begin{cases} -5x-4y=0 \\ 2x+y=0 \end{cases} \Rightarrow x = 0$ and $y = 0$.

Therefore the origin point (0, 0) is the only equilibrium point.

$$|\lambda I - A| = \begin{vmatrix} \lambda+5 & 4 \\ -2 & \lambda-1 \end{vmatrix} = (\lambda+5)(\lambda-1) + 8 = \lambda^2 + 4\lambda + 3 = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = -3.$$

For $\lambda_1 = -1 \Rightarrow (A - \lambda_1 I)V_1 = 0 \Rightarrow \left[\begin{pmatrix} -5 & -4 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$

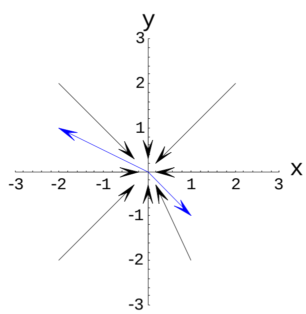
$$\begin{pmatrix} -4 & -4 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -4x_1-4y_1=0 \\ 2x_1+2y_1=0 \end{cases} \Rightarrow y_1 = -x_1$$

Assume $x_1 = 1 \Rightarrow V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

For $\lambda_2 = -3 \Rightarrow (A - \lambda_2 I)V_2 = 0 \Rightarrow \left[\begin{pmatrix} -5 & -4 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right]\begin{pmatrix} x2 \\ y2 \end{pmatrix} = 0$

$$\begin{pmatrix} -2 & -4 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -2x_2-4y_2=0 \\ 2x_2+4y_2=0 \end{cases} \Rightarrow x_2 = -2y_2$$

Let $y_2 = 1 \Rightarrow V_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$



The dynamic behavior of this system at the equilibrium point (0,0) is an asymptotically stable. And the general equation can be written as:

$$X = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-3t}$$

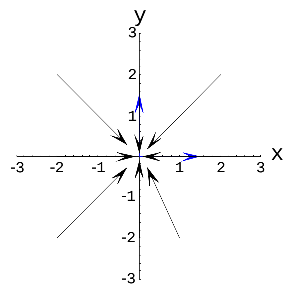
b) $A = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} \Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda+1 & 0 \\ 0 & \lambda+3 \end{vmatrix} = (\lambda+3)(\lambda+1) = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = -3.$

For $\lambda_1 = -1 \Rightarrow (A - \lambda_1 I)V_1 = 0 \Rightarrow \left[\begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$

$$\begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 0x_1 = 0 \\ 3y_1 = 0 \end{cases} \Rightarrow \forall x_1 \text{ (assume } x_1 = 1\text{) and } y_1 = 0 \Rightarrow V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For $\lambda_2 = -3 \Rightarrow (A - \lambda_2 I)V_2 = 0 \Rightarrow \left[\begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right]\begin{pmatrix} x2 \\ y2 \end{pmatrix} = 0$

$$\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_2 = 0 \text{ and } \forall\, y_2 \text{ (assume } y_2 = 1\text{)} \Rightarrow V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



The dynamic behavior of this system at the equilibrium point (0,0) is an asymptotically stable. And the general equation can be written as:

$$X = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-3t}$$

c) $\begin{cases} \dot{x} = 3x \\ \dot{y} = 2x + y \end{cases} \Rightarrow \begin{cases} x = 0 \\ 2x + y = 0 \end{cases} \Rightarrow$ The origin point O (0, 0) is the only equilibrium point.

$A = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} \Rightarrow |\lambda I - A| = (\lambda - 3)(\lambda - 1) = 0 \Rightarrow \lambda = 1, \lambda = 3.$

When $\lambda_1 = 1 \Rightarrow (A - \lambda_1 I)V_1 = 0 \Rightarrow \left[\begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

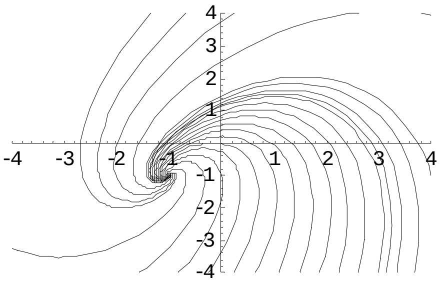
$\Rightarrow x_1 = 0$ and " $y_1 \Rightarrow V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda_2 = 3 \Rightarrow (A - \lambda_2 I)V_2 = 0 \Rightarrow \left[\begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right]\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 0 & 0 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$X = c_1\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-3t}$$

d) $\begin{cases} \dot{x} = x + 3y + 4 \\ \dot{y} = -6x + 5y - 1 \end{cases} \Rightarrow \begin{cases} x + 3y = -4 \\ -6x + 5y = 1 \end{cases} \Rightarrow$ (-1, -1) is the only equilibrium point.

$A = \begin{pmatrix} 1 & 3 \\ -6 & 5 \end{pmatrix} \Rightarrow |\lambda I - A| = (\lambda - 1)(\lambda - 5) + 18 = 0 \Rightarrow \lambda^2 - 6\lambda + 23 = 0 \Rightarrow \lambda = 3 \pm i\sqrt{14}$



e) $\begin{cases} \dot{x} = 3x + 4y \\ \dot{y} = 2x + y \end{cases} \Rightarrow \begin{cases} 3x = \ 4y \\ y = -2x \end{cases} \Rightarrow$ O (0, 0) is the only equilibrium point.

$A = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} \Rightarrow |\lambda I - A| = (\lambda - 3)\ (\lambda - 1)\ - 8 = 0 \Rightarrow \lambda^2 - 4\lambda - 5 = 0 \Rightarrow \lambda_1 = -1,\ \lambda_2 = 5.$

When $\lambda_1 = -1 \ \Rightarrow (A - \lambda_1 I)V_1 = 0 \ \Rightarrow \left[\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 4 & 4 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow x_1 = -y_1 \Rightarrow V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda_2 = 5 \Rightarrow (A - \lambda_2 I)V_2 = 0 \ \Rightarrow \left[\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\right]\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \Rightarrow V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$X = c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{-3t}$$

f) $\begin{cases} \dot{x} = x + 2y \\ \dot{y} = -2x + 5y \end{cases} \Rightarrow \begin{cases} x = -2y \\ 2x = 5y \end{cases} \Rightarrow$ O (0, 0) is the only equilibrium point.

$A = \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix} \Rightarrow |\lambda I - A| = (\lambda - 1)\ (\lambda - 5)\ + 4 = 0 \Rightarrow \lambda^2 - 6\lambda + 9 = 0 \Rightarrow \lambda_{1,2} = 3.$

When $\lambda = 3 \ \Rightarrow (A - \lambda_1 I)V_1 = 0 \ \Rightarrow \left[\begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow x_1 = y_1 \Rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ \Rightarrow \ X = c\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-3t}$

g) The origin point (0, 0, 0) is the only equilibrium point.

$A = \begin{pmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{pmatrix} \Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda+2 & 0 & 0 \\ -1 & \lambda+2 & 0 \\ 0 & -1 & \lambda+2 \end{vmatrix} = (\lambda + 2)^3 = 0 \Rightarrow \lambda = -2, -2, -2.$

For $\lambda = -2 \Rightarrow (A - \lambda_1 I)V = 0 \ \Rightarrow \begin{vmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix} = 0 \ \Rightarrow x = 0,\ y = 0,\ \forall z \Rightarrow V = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

The equilibrium point (0, 0, 0) is a sink which is an asymptotically stable.

2.2 Given that the two-dimensional system has a non-trivial periodic solution; which means A has pure imaginary eigenvalues. A: $\lambda = \pm\beta i$

The matrix A can be rewritten as: $\begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix}$ i.e. $\begin{cases} \dot{x} = \beta y \\ \dot{y} = -\beta x \end{cases}$

$\therefore$ Any circle with a center at the origin point (0, 0) is a solution of the linear system and they are all periodic solutions.

$(x = r\cos\theta,\ y = r\sin\theta)$

$\Rightarrow \dot{r} = 0 \Rightarrow r(t) = r_0 \in \tilde{N}$

$\Rightarrow \dot{\theta} = -\beta \Rightarrow \theta(t) = -\beta t + \theta_0$

$u(t) = (r_0\cos(-\beta t + \theta_0), r_0\sin(-\beta t + \theta_0)) = r_0\,\mathrm{cis}(-\beta t + \theta_0 + 2\pi)$

$= r_0\,\mathrm{cis}(-\beta(t - \frac{2\pi}{\beta}) + \theta_0) = u(t - \frac{2\pi}{\beta})$

$\Rightarrow \rho = -\frac{2\pi}{\beta}$ is the period.

2.3 **a)** $\dot{y} - y = -t^2 \Rightarrow (e^{-t}y)' = -t^2e^{-t} \Rightarrow \int(e^{-t}y)' = \int -t^2e^{-t}dt$

$\Rightarrow e^{-t}y = e^{-t}(t^2 + 2t + 2) + c \Rightarrow y = t^2 + 2t + 2 + ce^t$ (where c is an arbitrary constant)

b) $\dot{y} + 2y = 3e^t \Rightarrow$ Multiply both side by $e^{2t} \Rightarrow e^{2t}\dot{y} + 2e^{2t}y = 3e^{3t}$

$(e^{2t}y)' = 3e^{3t} \Rightarrow \int(e^{2t}y)'dt = \int 3e^{3t}dt \Rightarrow e^{2t}y = e^{3t} + c \Rightarrow y = ce^{-2t} + e^t\ (\forall\, t)$.

c) $\dot{y} = 5 + \cos t \Rightarrow \int dy = \int(5 + \cos t)dt \Rightarrow y = 5t + \sin t + c$

d) $\dot{y} + (\sin t)y = \sin t$

$\Rightarrow$ By using the linear first order equation (Appendix C), then:

$y\,e^{\int \sin t\,dt} = \int \sin t\ e^{\int \sin t\,dt}dt + c \Rightarrow e^{\int \sin t\,dt} = e^{-\cos t}$

$\Rightarrow y\,e^{-\cos t} = \int \sin t\ e^{-\cos t}dt + c = e^{-\cos t} + c$

$\Rightarrow y = 1 + \frac{c}{e^{-\cos t}} = 1 + c\,e^{\cos t}$

e) $\dot{y} - 2ty = t \Rightarrow e^{\int -2t\,dt} = e^{-t^2} \Rightarrow y\,e^{-t^2} = \int t\,e^{-t^2}dt = -\tfrac{1}{2}e^{-t^2} + c \Rightarrow y = c\,e^{t^2} - \tfrac{1}{2}$

f) $t\dot{y} + 2y = \sin t \Rightarrow \dot{y} + \frac{2}{t}y = \frac{\sin t}{t} \Rightarrow e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = e^{\ln t^2} = t^2$

$\Rightarrow y\,t^2 = \int \frac{\sin t}{t}t^2dt = \int t\sin t\,dt = \sin t - t\cos t + c \Rightarrow y = \frac{\sin t}{t^2} - \frac{\cos t}{t} + \frac{c}{t^2}$

2.4 $\begin{cases}\dot{x} = y \\ \dot{y} = x - 2x^3\end{cases} \Rightarrow \begin{cases} y = 0 \\ x(1-2x^2) = 0\end{cases} \Rightarrow x = 0,\ \pm\frac{\sqrt{2}}{2}$ and $y = 0$.

The equilibrium points are: (0, 0), $(\frac{\sqrt{2}}{2}, 0)$, and $(-\frac{\sqrt{2}}{2}, 0)$

The Jacobian is given by: $DF = \begin{pmatrix} 0 & 1 \\ 1-6x^2 & 0\end{pmatrix}$

At (0, 0) $\Rightarrow DF|_{(0,0)} = A = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \Rightarrow$ the system is a saddle (unstable).

At $(\pm\frac{\sqrt{2}}{2}, 0) \Rightarrow DF| = A = \begin{pmatrix} 0 & 1 \\ -2 & 0\end{pmatrix} \Rightarrow \lambda^2 + 2 = 0 \Rightarrow \lambda = \pm 2i \Rightarrow$ the system is a center stable.

2.5 **a)** $\begin{cases}\dot{x} = x \\ \dot{y} = y\end{cases}$ The point (0,0) is the only critical points

b) $\frac{dx}{dy} = \frac{x}{y} \Rightarrow \int\frac{dx}{x} = \int\frac{dy}{y} \Rightarrow \ln x = \ln y + c \Rightarrow x = e^{\ln y + c} = e^{\ln y}e^{c} = c_1 y$

c) Hamilton function: $\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 1 + 1 = 2 \neq 0$; which it diverges to 2. Since is not equal to zero, therefore it does not satisfy Hamilton criteria, therefore the point (0, 0) is a negative attraction.

2.6 **a)** $\begin{pmatrix}\dot{x} \\ \dot{y}\end{pmatrix} = \begin{pmatrix} 1 & a \\ -1 & -2\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix} \Rightarrow \begin{cases} x + ay = 0 \\ -x - 2y = 0\end{cases} \Rightarrow \begin{cases} x = -ay \\ x = -2y\end{cases}$

If a = 2, then $\dot{x} = -\dot{y} \Rightarrow x = -y$; therefore the system is a line.

If $a \neq 2$, then the point O(0, 0) is the only equilibrium point.

$\Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -a \\ 1 & \lambda + 2\end{vmatrix} = (\lambda - 1)(\lambda + 2) + a = 0 \Rightarrow \lambda^2 + \lambda + a - 2 = 0$

$\Rightarrow \lambda_{1,2} = \frac{-1 \pm \sqrt{9-4a}}{2}$

i. If $9 - 4a > 0 \Rightarrow a < \frac{9}{4} \Rightarrow \lambda_1 > 0$ & $\lambda_2 < 0 \Rightarrow$ The system is a unstable saddle.

ii. If $9 - 4a = 0 \Rightarrow a = \frac{9}{4} \Rightarrow \lambda_1 = \lambda_2 = -\frac{1}{2} < 0 \Rightarrow$ The system is an asymptotically stable.

iii. If $9 - 4a < 0 \Rightarrow a > \frac{9}{4} \Rightarrow \lambda_1$ & $\lambda_2 = \alpha + i\beta \in$ Â

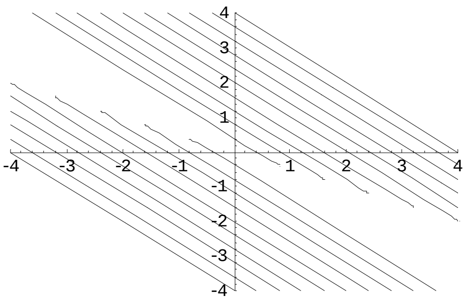
$\Rightarrow$ The system is a non-hyperbolic (spiral in).

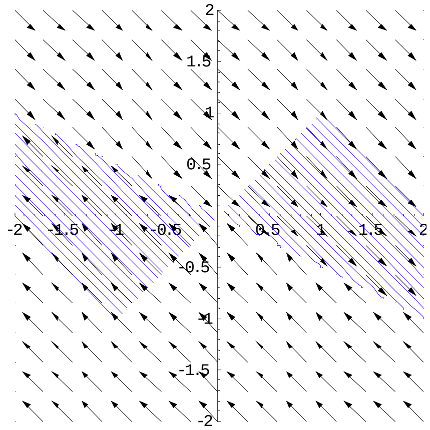
b) When a = 2, $\Rightarrow A = \begin{pmatrix} 1 & 2 \\ -1 & -2\end{pmatrix} \Rightarrow \lambda_1 = 0$ & $\lambda_2 = -1$.

For $\lambda_1 = 0 \Rightarrow (A - \lambda_1 I)V_1 = 0 \Rightarrow \begin{pmatrix} 1 & 2 \\ -1 & -2\end{pmatrix}\begin{pmatrix} x_1 \\ y_1\end{pmatrix} = \begin{pmatrix} 0 \\ 0\end{pmatrix} \Rightarrow V_1 = \begin{pmatrix} -2 \\ 1\end{pmatrix}$

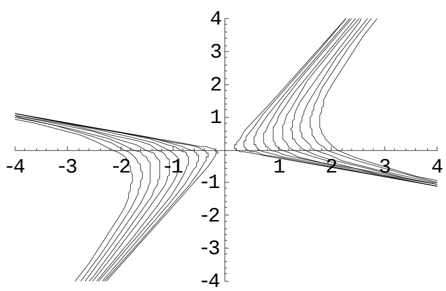
For λ_2 = -1 $\Rightarrow$ (A - λ_2I)V_2 = 0 $\Rightarrow \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

c)

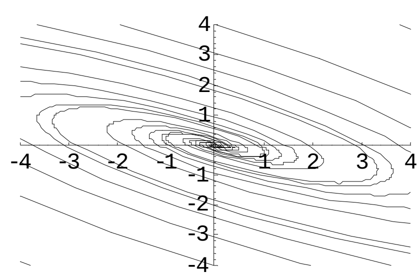




d) Phase portrait for different value of a. (a < 2 and a > 2).



a < 2



a > 2

2.7 **a)** $\begin{cases} \dot{x} = -2x - y + 3 \\ \dot{y} = -x + y^2 \end{cases} \Rightarrow \begin{cases} -2x - y + 3 = 0 \\ -x + y^2 = 0 \end{cases} \Rightarrow x = y^2 \Rightarrow 2y^2 + y - 3 = 0 \Rightarrow y = 1, -\frac{3}{2}$

Therefore, the equilibrium points are: (1, 1) and $(\frac{9}{4}, -\frac{3}{2})$

The Jacobian is given by: DF = $\begin{pmatrix} -2 & -1 \\ -1 & 2y \end{pmatrix}$

At (1, 1) $\Rightarrow$ $DF|_{(1,1)} = A = \begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix} \Rightarrow |\lambda I - A| = \lambda^2 - 5 = 0 \Rightarrow \lambda = \pm\sqrt{5}$

For $\lambda_1 = \sqrt{5}$ $\Rightarrow$ (A - λ_1I)V_1 = 0 $\Rightarrow \begin{pmatrix} -2-\sqrt{5} & -1 \\ -1 & 2-\sqrt{5} \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow V_1 = \begin{pmatrix} 2-\sqrt{5} \\ 1 \end{pmatrix}$

For $\lambda_2 = -\sqrt{5}$ $\Rightarrow$ (A - λ_2I)V_2 = 0 $\Rightarrow \begin{pmatrix} -2+\sqrt{5} & -1 \\ -1 & 2+\sqrt{5} \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow V_2 = \begin{pmatrix} 2+\sqrt{5} \\ 1 \end{pmatrix}$

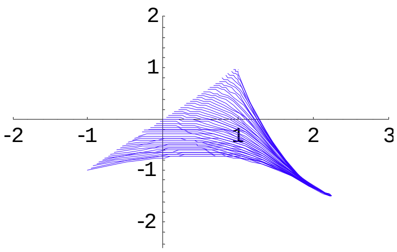
At $(\frac{9}{4}, -\frac{3}{2}) \Rightarrow DF| = A = \begin{pmatrix} -2 & -1 \\ -1 & -3 \end{pmatrix} \Rightarrow \lambda^2 + 5\lambda + 5 = 0 \Rightarrow \lambda = \frac{-5\pm\sqrt{5}}{2} < 0.$

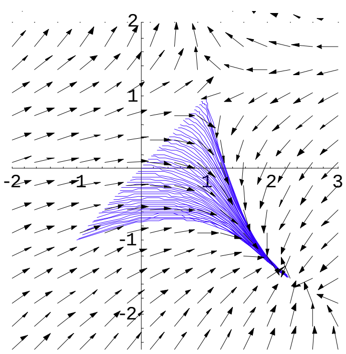
For $\lambda_3 = \frac{-5+\sqrt{5}}{2} \Rightarrow (A - \lambda_3 I)V_1 = 0 \Rightarrow \begin{pmatrix} \frac{1}{2}(1-\sqrt{5}) & -1 \\ -1 & -\frac{1}{2}(2+\sqrt{5}) \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow V_1 = \begin{pmatrix} -\frac{1}{2}(1+\sqrt{5}) \\ 1 \end{pmatrix}$

For $\lambda_4 = \frac{-5-\sqrt{5}}{2} \Rightarrow (A - \lambda_4 I)V_2 = 0 \Rightarrow \begin{pmatrix} \frac{1}{2}(1+\sqrt{5}) & -1 \\ -1 & -\frac{1}{2}(2-\sqrt{5}) \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow V_2 = \begin{pmatrix} \frac{1}{2}(1+\sqrt{5}) \\ 1 \end{pmatrix}$





b) At (1, 1) $\Rightarrow$ the equilibrium point is a saddle and the system is an unstable.

At $(\frac{9}{4}, -\frac{3}{2}) \Rightarrow$ the system is an asymptotically stable and it sinks into the equilibrium point.

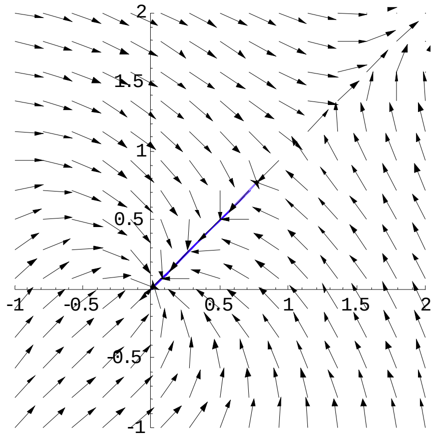
2.8 a) $\begin{cases} \dot{x} = y^2 - x \\ \dot{y} = x^2 - y \end{cases} \Rightarrow \begin{cases} y^2 - x = 0 \\ x^2 - y = 0 \end{cases} \Rightarrow \begin{cases} x = y^2 \\ y = x^2 \end{cases} \Rightarrow y = y^4 \Rightarrow y(1 - y^3) = 0 \Rightarrow y = 0, 1$

The equilibrium points are: (0,0) and (1,1)

The Jacobian is given by: $DF = \begin{pmatrix} -1 & 2y \\ 2x & -1 \end{pmatrix}$

<u>At the point (0, 0)</u> $\Rightarrow DF|_{(0,0)} = A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow |\lambda I - A| = (\lambda + 1)^2 = 0 \Rightarrow \lambda_{1,2} = -1$

The system sinks into the origin point (0, 0).



<u>At the point (1, 1)</u> $\Rightarrow DF|_{(1,1)} = A = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \Rightarrow \lambda^2 + 2\lambda - 3 = 0 \Rightarrow \lambda_{3,4} = 1, -3.$

For $\lambda_3 = 1 \Rightarrow (A - \lambda_3 I)V_3 = 0 \Rightarrow \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow V_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

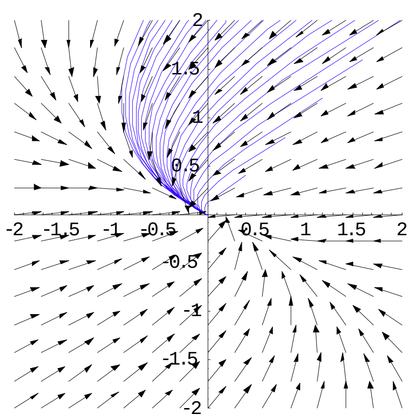
For $\lambda_4 = -3 \Rightarrow (A - \lambda_4 I)V_4 = 0 \Rightarrow \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow V_4 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

2.9 **a)** $\begin{cases} \dot{x} = -8x - 6y = 0 \\ \dot{y} = -x - 7y = 0 \end{cases} \Rightarrow \begin{cases} 4x = -3y \\ x = -7y \end{cases} \Rightarrow$ The origin point (0, 0) is the only equilibrium point.

$DF = \begin{pmatrix} -8 & -6 \\ -1 & -7 \end{pmatrix} \Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda + 8 & 6 \\ 1 & \lambda + 7 \end{vmatrix} = (\lambda + 8)(\lambda + 7) - 6 = \lambda^2 + 15\lambda + 50 = 0$

$\Rightarrow \lambda_1 = -5, \lambda_2 = -10$; since $\lambda_{1,2} < 0$, therefore the system is asymptotically stable (sink).

b)



c) For $\lambda_1 = -5 \Rightarrow (A - \lambda_1 I)V_1 = 0 \Rightarrow \begin{pmatrix} -3 & -6 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow V_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

For $\lambda_2 = -10 \Rightarrow (A - \lambda_2 I)V_2 = 0 \Rightarrow \begin{pmatrix} 2 & -6 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow V_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\Rightarrow y = V_1\ e^{\lambda 1} + V_2 e^{\lambda 2} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-5} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-10}$

Chapter 3

3.1 **a)** The equilibrium points are: $\begin{cases} x(10 - x - \frac{1}{2}y) = 0 \\ y(16 - y - x) = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \quad \text{or} \quad 10\text{-}x\text{-}\frac{1}{2}\ y = 0 \\ y = 0 \quad \text{or} \quad 16\text{-}y\text{-}x = 0 \end{cases}$

$\Rightarrow$ The origin point (0, 0);
If x = 0 $\Rightarrow$ 16 – y – x = 0 $\Rightarrow$ 16 – y = 0 $\Rightarrow$ y = 16 $\rightarrow$ Point (0, 16)
If y = 0 $\Rightarrow$ 10 – x – ½ y = 0 $\Rightarrow$ 10 – x = 0 $\Rightarrow$ x = 10 $\rightarrow$ Point (10, 0)

If $\begin{cases} 10 - x - \frac{1}{2}y = 0 \\ 16 - y - x = 0 \end{cases} \Rightarrow \begin{cases} x + \frac{1}{2}y = 10 \\ x + y = 16 \end{cases}$

$\Rightarrow$ By using elimination: ½ y = 6 $\Rightarrow$ y =12 & x =4 $\rightarrow$ Point (4, 12)
$\therefore$ The points (0, 0), (0, 16), (10, 0), and (4, 12) are the equilibrium points for this system.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \approx \begin{pmatrix} 10 - 2x - \frac{1}{2}y & -\frac{1}{2}x \\ -y & 16 - 2y - x \end{pmatrix}_{(x_0, y_0)} \begin{pmatrix} x \\ y \end{pmatrix} = A\Big|_{(x_0, y_0)} \begin{pmatrix} x \\ y \end{pmatrix}$$

<u>At the point (0, 0)</u>: $DF|_{(0,0)} = \begin{pmatrix} 10 & 0 \\ 0 & 16 \end{pmatrix}$

$\Rightarrow |\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda - 10 & 0 \\ 0 & \lambda - 16 \end{vmatrix} = (\lambda - 10)(\lambda - 16) = 0 \Rightarrow \lambda_1 = 10$ and $\lambda_2 = 16$

$\rightarrow$ Since the eigenvalues $\lambda_{1,2} > 0$; therefore the system is an unstable source (repeller) at the origin point (0, 0).

<u>At the point (0, 16)</u>: $A = DF|_{(0,16)} = \begin{pmatrix} 2 & 0 \\ -16 & -16 \end{pmatrix}$

$\Rightarrow |\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda - 2 & 0 \\ 16 & \lambda + 16 \end{vmatrix} = (\lambda - 2)(\lambda + 16) = 0 \Rightarrow \lambda_1 = 2$ and $\lambda_2 = -16$

$\rightarrow$ Since the eigenvalues $\lambda_1 > 0$ and $\lambda_2 < 0$; therefore the system is a saddle at the point (0, 16).

For $\lambda_1 = 2 \Rightarrow \begin{pmatrix} 0 & 0 \\ -16 & -18 \end{pmatrix} V_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -16x - 18y = 0$

$\Rightarrow$ If x = 1, then $y = -\frac{8}{9} \rightarrow V_1 = \begin{pmatrix} 1 \\ -\frac{8}{9} \end{pmatrix}$

For $\lambda_2 = -16 \Rightarrow \begin{pmatrix} -14 & 0 \\ -16 & 0 \end{pmatrix} V_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = 0$ and $\forall y \rightarrow V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

<u>At the point (10, 0)</u>: $A = DF|_{(10,0)} = \begin{pmatrix} -10 & -5 \\ 0 & 6 \end{pmatrix}$

$\Rightarrow |\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda + 10 & 5 \\ 0 & \lambda - 6 \end{vmatrix} = (\lambda + 10)(\lambda - 6) = 0 \Rightarrow \lambda_1 = -10$ and $\lambda_2 = 6$

$\rightarrow$ Since the eigenvalues $\lambda_1 < 0$ and $\lambda_2 > 0$; therefore the system is a saddle at the point (10, 0).

For $\lambda_1 = -10 \Rightarrow \begin{pmatrix} 0 & -5 \\ 0 & 16 \end{pmatrix} V_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow y = 0$ and $\forall x \to V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

For $\lambda_2 = 6 \Rightarrow \begin{pmatrix} -16 & -5 \\ 0 & 0 \end{pmatrix} V_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -16x - 5y = 0$

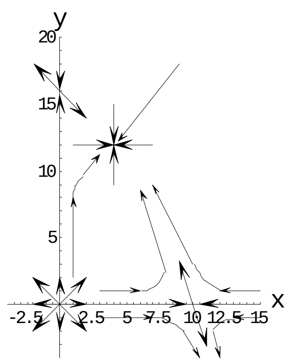
$\Rightarrow$ If $x = 1$, then $y = -\frac{16}{5} \to V_2 = \begin{pmatrix} 0 \\ -\frac{16}{5} \end{pmatrix}$

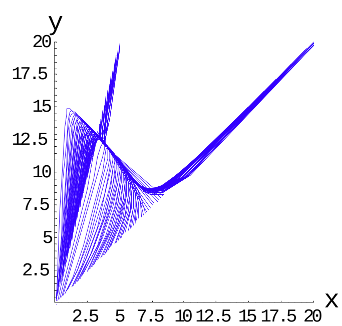
<u>At the point (4, 12)</u>: $A = DF|_{(4,12)} = \begin{pmatrix} -4 & -2 \\ -12 & -12 \end{pmatrix}$

$\Rightarrow |\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda+4 & 2 \\ 12 & \lambda+12 \end{vmatrix} = (\lambda + 4)(\lambda + 12) - 24 = 0 \Rightarrow \lambda^2 + 16\lambda + 24 = 0$

$\Rightarrow \lambda_{1,2} = -8 \pm 2\sqrt{10} < 0$

$\to$ Since the eigenvalues $\lambda_{1,2} < 0$; therefore the system is a sink source at the point (4, 12).





b) The equilibrium points are determined by: $\begin{cases} y - x^2 + 2 = 0 & (1) \\ x^2 - xy = x(x-y) = 0 & (2) \end{cases}$

(2) $\to x = 0$ or $x - y = 0$ $(x = y)$
If $x = 0 \Rightarrow (1) \Rightarrow y = -2 \Rightarrow$ Point (0, -2)
If $x = y \Rightarrow (1) \Rightarrow y - y^2 + 2 = 0 \Rightarrow y = -1, 2 \Rightarrow$ Points (-1, -1) and (2, 2).
$\therefore$ The points (0, -2), (-1, -1), and (2, 2) are the equilibrium points of the system.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \approx \begin{pmatrix} -2x & 1 \\ 2x - y & -x \end{pmatrix}_{(x_0,y_0)} \begin{pmatrix} x \\ y \end{pmatrix} = A|_{(x_0,y_0)} \begin{pmatrix} x \\ y \end{pmatrix}$$

<u>At the point (0, -2)</u>: $A = DF|_{(0,-2)} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$

$\Rightarrow |\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda & -1 \\ -2 & \lambda \end{vmatrix} = \lambda^2 - 2 = 0 \Rightarrow \lambda_{1,2} = \pm\sqrt{2}$

$\to$ The system is a saddle at the point (0, -2).
The eigenvectors are determined as:

For $\lambda_1 = \sqrt{2} \Rightarrow \begin{pmatrix} \sqrt{2} & -1 \\ -2 & \sqrt{2} \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0 \Rightarrow \begin{cases} \sqrt{2}x_1 - y_1 = 0 \\ -2x_1 + \sqrt{2}y_1 = 0 \end{cases} \Rightarrow y_1 = \sqrt{2}\, x_1$ (assume $x_1 = 1$)

$\Rightarrow V_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$

For $\lambda_2 = -\sqrt{2} \Rightarrow \begin{pmatrix} -\sqrt{2} & -1 \\ -2 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = 0 \Rightarrow \begin{cases} -\sqrt{2}x_2 - y_2 = 0 \\ -2x_2 - \sqrt{2}y_2 = 0 \end{cases}$

$\Rightarrow y_2 = -\sqrt{2}\, x_2$ (assume $x_2 = 1$) $\qquad \Rightarrow V_2 = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$

<u>At the point (-1, -1)</u>: $A = DF|_{(-1,-1)} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$

$\Rightarrow |\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda - 2 & -1 \\ 1 & \lambda - 1 \end{vmatrix} = (\lambda - 2)(\lambda - 1) + 1 = 0 \Rightarrow \lambda_{1,2} = \frac{3}{2} \pm i\frac{\sqrt{3}}{2} = \alpha + i\beta$

$\rightarrow$ The system is a spiral out at the point (-1, -1), since $\alpha > 0$.

<u>At the point (2, 2)</u>: $A = DF|_{(2,2)} = \begin{pmatrix} -4 & 1 \\ 2 & -2 \end{pmatrix}$

$\Rightarrow |\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda + 4 & -1 \\ -2 & \lambda + 2 \end{vmatrix} = (\lambda + 4)(\lambda + 2) - 2 = 0 \Rightarrow \lambda_{1,2} = -3 \pm \sqrt{3} < 0$

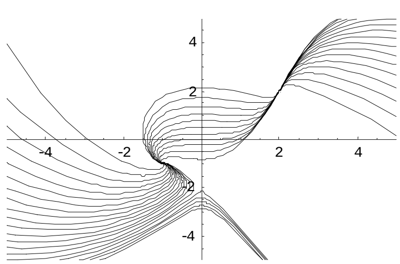
$\rightarrow$ The system is a sink at the point (2, 2).

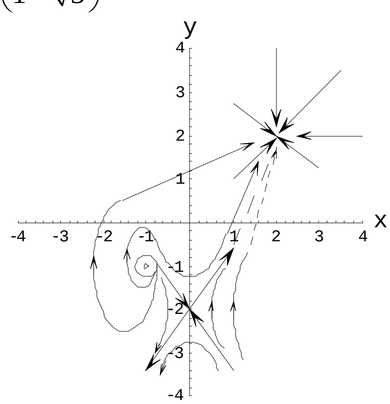
For $\lambda_1 = -3 + \sqrt{3} \Rightarrow \begin{pmatrix} -4+3-\sqrt{3} & 1 \\ 2 & -2+3-\sqrt{3} \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0 \Rightarrow \begin{cases} (-1-\sqrt{3})x_1 + y_1 = 0 \\ 2x_1 + (1-\sqrt{3})y_1 = 0 \end{cases}$

$\Rightarrow y_1 = (1 + \sqrt{3})\, x_1$ (assume $x_1 = 1$) $\Rightarrow V_1 = \begin{pmatrix} 1 \\ 1+\sqrt{3} \end{pmatrix}$

For $\lambda_2 = -3 - \sqrt{3} \Rightarrow \begin{pmatrix} -1+\sqrt{3} & 1 \\ 2 & 1+\sqrt{3} \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = 0 \Rightarrow \begin{cases} (-1+\sqrt{3})x_2 + y_2 = 0 \\ 2x_2 + (1+\sqrt{3})y_2 = 0 \end{cases}$

$\Rightarrow y_2 = (1 - \sqrt{3})x_2$ (assume $x_2 = 1$) $\quad \Rightarrow V_2 = \begin{pmatrix} 1 \\ 1-\sqrt{3} \end{pmatrix}$





c) F(x, y) = 0

$$\Rightarrow \quad \sqrt{x^2+y^2}\,.\,x.\sin\left(\frac{\pi}{\sqrt{x^2+y^2}}\right)\Bigg|_{(0,0)} = 0 \quad \text{and} \quad \sqrt{x^2+y^2}\,.\,y.\sin\left(\frac{\pi}{\sqrt{x^2+y^2}}\right)\Bigg|_{(0,0)} = 0$$

Obviously, the point (0, 0) is the only equilibrium point.

The linearized system is in the form of: $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$

$$\Rightarrow \; A = \begin{pmatrix} 0 & -y \\ x & 0 \end{pmatrix} \Rightarrow DF = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow DF\big|_{(0,0)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x(x^2+y^2)\sin\left(\frac{\pi}{\sqrt{x^2+y^2}}\right) \\ y(x^2+y^2)\sin\left(\frac{\pi}{\sqrt{x^2+y^2}}\right) \end{pmatrix}$$

$$\Rightarrow |\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda & 1 \\ -1 & \lambda \end{vmatrix} = 0 \Rightarrow \lambda_{1,2} = \pm\, i$$

$\therefore$ The equilibrium point (0, 0) with $\lambda_{1,2} = \pm i$ is non-hyperbolic. Therefore to prove the stability of the system, we have to consider the polar coordinates: $r^2 = x^2 + y^2 \Rightarrow r\dot{r} = x\dot{x} + y\dot{y}$

$$\Rightarrow r\dot{r} = x\,(-y + r^2.x.\sin(\tfrac{\pi}{r})) + y\,(x + r^2.y.\sin(\tfrac{\pi}{r})) = -xy + r^2.x^2.\sin(\tfrac{\pi}{r}) + xy + r^2.y^2.\sin(\tfrac{\pi}{r})$$

$$= r^2.(x^2+y^2)\sin(\tfrac{\pi}{r}) \;=\; r^4 \sin(\tfrac{\pi}{r})$$

$$\Rightarrow \; \dot{r} = r^3 \sin(\tfrac{\pi}{r})$$

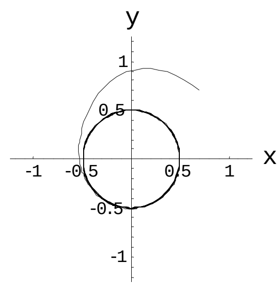
$$\theta = \tan^{-1}(\tfrac{y}{x})$$

$$\Rightarrow \dot{\theta} \;=\; \frac{\frac{\dot{y}x-\dot{x}y}{x^2}}{1+\frac{y^2}{x^2}} = \frac{\dot{y}x-\dot{x}y}{r^2} = \frac{x(x+r^2y\sin\frac{\pi}{r})-y(-y+r^2x\sin\frac{\pi}{r})}{r^2} = \frac{1}{r^2}(x^2+y^2) = \frac{r^2}{r^2} = 1$$

$$\begin{cases} \dot{r}=r^3\sin\frac{\pi}{r} \\ \dot{\theta}=1 \end{cases} \Rightarrow \text{if } \dot{r} = 0 \Rightarrow \sin\frac{\pi}{r} = 0 \Rightarrow \frac{\pi}{r} = 2n\pi \Rightarrow r = \frac{1}{2n} \quad (n \in Ù)$$

$\forall$ r, if $r = \frac{1}{2n}$, then $\dot{r} = 0 \Rightarrow r(t) = c \in Ñ^{+} + \{0\}$

In this case, the non-linear system has infinitely many periodic solutions.



d) $\begin{cases} \dot{x} = -3x + y^2 + 2 = 0 \\ \dot{y} = x^2 - y^2 = 0 \end{cases} \Rightarrow x^2 = y^2 \Rightarrow x = \pm y \Rightarrow \begin{cases} x = y \Rightarrow -3y + y^2 + 2 = 0 \Rightarrow y = 1,\ 2 \\ x = \text{-}y \Rightarrow 3y + y^2 + 2 = 0 \Rightarrow y = -1,\ \text{-}2 \end{cases}$

$\therefore$ (1, 1), (2, 2), (2, -2), and (1, -1) are the equilibrium points of the system.

$$\Rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3 & 2y \\ 2x & -2y \end{pmatrix}_{(x_0, y_0)} \begin{pmatrix} x \\ y \end{pmatrix}$$

<u>At the point (1, 1)</u>:

Then: $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{vmatrix} \lambda + 3 & -2 \\ -2 & \lambda + 2 \end{vmatrix} = 0$

$\Rightarrow (\lambda+3)(\lambda+2) - 4 = \lambda^2+5\lambda+2=0$

$\Rightarrow \lambda_{1,2} = \dfrac{-5 \pm \sqrt{17}}{2}$ $\therefore$ The system is sink since $\lambda_{1,2} < 0$.

<u>At the point (1, -1)</u>:

Then: $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{vmatrix} \lambda + 3 & 2 \\ -2 & \lambda - 2 \end{vmatrix} = 0 \Rightarrow (\lambda+3)(\lambda\text{-}2) + 4 = \lambda^2+\lambda\text{-}2 = 0$

$\Rightarrow \lambda_{1,2} = \dfrac{-1 \pm 3}{2} = \begin{cases} 1 \\ -2 \end{cases}$

$\lambda_1 = 1 \Rightarrow \begin{pmatrix} 4 & 2 \\ -2 & -1 \end{pmatrix} V_1 = 0 \Rightarrow 4x + 2y = 0 \Rightarrow 2x = \text{-}y$ (let $x = 1$) $\Rightarrow V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\lambda_2 = \text{-}2 \Rightarrow \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} V_2 = 0 \Rightarrow x + 2y = 0 \Rightarrow$ let $y = 1 \Rightarrow V_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\therefore$ The system is unstable (saddle point), since $\lambda_1 > 0$ and $\lambda_2 < 0$.

<u>At the point (2, 2)</u>:

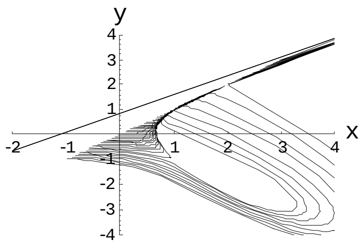
Then: $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{vmatrix} \lambda + 3 & -4 \\ -4 & \lambda + 4 \end{vmatrix} = 0 \Rightarrow (\lambda+3)(\lambda+4) - 16 = \lambda^2+7\lambda\text{-}4 = 0$

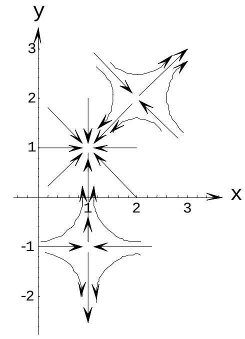
$\Rightarrow \lambda_{1,2} = \dfrac{-7 \pm \sqrt{65}}{2}$ $\therefore$ The system is unstable (saddle point), since $\lambda_1 < 0$ and $\lambda_2 < 0$.

<u>At the point (2, -2)</u>:

Then: $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3 & -4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{vmatrix} \lambda + 3 & 4 \\ -4 & \lambda + 4 \end{vmatrix} = 0 \Rightarrow (\lambda+3)(\lambda+4) + 16 = \lambda^2\text{-}\lambda+4 = 0$

$\Rightarrow \lambda_{1,2} = \frac{1 \pm i\sqrt{15}}{2}$ $\therefore$ The system is unstable spiral.





e) The equilibrium points are:

$$\begin{cases} \dot{x} = -y - \sqrt{x^2 + y^2}.x = 0 \\ \dot{y} = x - \sqrt{x^2 + y^2}.y = 0 \end{cases} \Rightarrow \text{The point (0, 0) is the only equilibrium point.}$$

The system can be rewritten as: $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x\sqrt{x^2 + y^2} \\ y\sqrt{x^2 + y^2} \end{pmatrix}$

The eigenvalues are determined by:

$|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda & 1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$

Since $\lambda_{1,2} = \pm i$ (imaginary number only $Re(\lambda) = 0$), which is a non-hyperbolic case. That means; we can not draw any conclusion of the behavior near the origin point (0, 0).

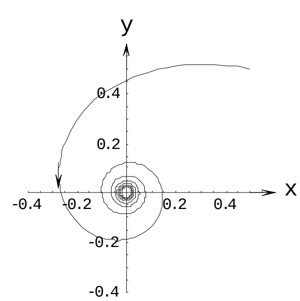
Let assume: $r^2 = x^2 + y^2 \Rightarrow 2r\dot{r} = 2x\dot{x} + 2y\dot{y} \Rightarrow r\dot{r} = x\dot{x} + y\dot{y}$

$\rightarrow r\dot{r} = x(-y - rx) + y(x - ry) = -xy - rx^2 + xy - ry^2 = -r(x^2 + y^2) = -r^3 \Rightarrow \dot{r} = -r^2 < 0.$

$\rightarrow \theta = \tan^{-1}(\frac{y}{x})$

$$\Rightarrow \dot{\theta} = \frac{\frac{\dot{y}x - \dot{x}y}{x^2}}{1 + \frac{y^2}{x^2}} = \frac{\dot{y}x - \dot{x}y}{r^2} = \frac{x(x - ry) - y(-y - rx)}{r^2} = \frac{1}{r^2}(x^2 + y^2) = \frac{r^2}{r^2} = 1$$

From that we can prove that it is a stable spiral point.



f) The equilibrium points are determined by:

$$\begin{cases} \dot{x} = y = 0 \\ \dot{y} = 2x(x^2 - 1) = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ x = 0 \text{ or } x = \pm 1 \end{cases}$$

$\therefore$ The equilibrium points are (0, 0), (-1, 0), and (1,0).

$$\Rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \approx \begin{pmatrix} 0 & 1 \\ 6x^2 - 2 & 0 \end{pmatrix}_{(x_0, y_0)} \begin{pmatrix} x \\ y \end{pmatrix}$$

<u>At the point (0, 0)</u>: $A = DF|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$

The eigenvalues are determined by:

$$|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda & -1 \\ 2 & \lambda \end{vmatrix} = \lambda^2 + 2 = 0 \Rightarrow \lambda_{1,2} = \pm i\sqrt{2}$$

Since $\lambda_{1,2} = \pm i\sqrt{2}$ (imaginary number only $Re(\lambda) = 0$), which is a non-hyperbolic case. That means, we can not draw any conclusion (i.e. the behavior near the origin point (0, 0) is unknown). $\rightarrow$ The system is a stable (periodic) at the origin point (0, 0).

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{2x^3 - 2x}{y} \Rightarrow y\,dy = (2x^3 - 2x)\,dx \Rightarrow \int y\,dy = \int (2x^3 - 2x)\,dx$$

$$\Rightarrow \tfrac{1}{2} y^2 = \tfrac{1}{2} x^4 - x^2 + c \Rightarrow y^2 = x^4 - 2x^2 + c \Rightarrow y = \pm\sqrt{x^4 - 2x^2 + c}$$

<u>At the point (1, 0)</u>: $A = DF|_{(1,0)} = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$

The eigenvalues are determined by: $|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda & -1 \\ -4 & \lambda \end{vmatrix} = \lambda^2 - 4 = 0 \Rightarrow \lambda_{1,2} = \pm 2$

The eigenvectors are determined as:

For $\lambda_1 = 2 \Rightarrow \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0 \Rightarrow \begin{cases} 2x_1 - y_1 = 0 \\ -4x_1 + 2y_1 = 0 \end{cases} \Rightarrow y_1 = 2x_1$

Assume $x_1 = 1 \Rightarrow V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

For $\lambda_2 = -2 \Rightarrow \begin{pmatrix} -2 & -1 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = 0 \Rightarrow \begin{cases} -2x_2 - y_2 = 0 \\ -4x_2 - 2y_2 = 0 \end{cases} \Rightarrow y_2 = -2x_2$

Assume $x_2 = 1 \Rightarrow V_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

The behavior of the system is a saddle (unstable) at the equilibrium point (1, 0).

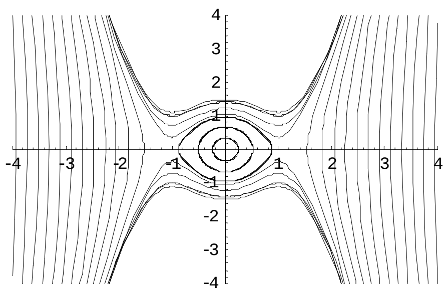
<u>At the point (-1, 0)</u>: $A = DF|_{(-1,0)} = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$

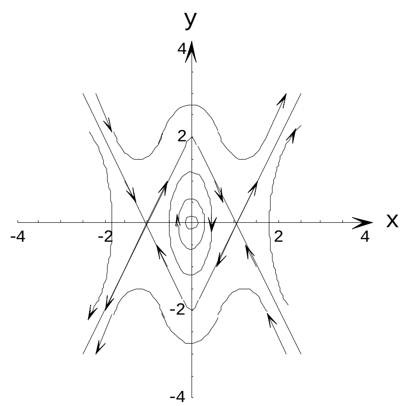
The eigenvalues are determined by:

$|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda & -1 \\ -4 & \lambda \end{vmatrix} = \lambda^2 - 4 = 0 \Rightarrow \lambda_{1,2} = \pm 2$ (are same as for the point (1,0))

Therefore, the eigenvectors are the same: $V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $V_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

The behavior of the system is a saddle (unstable) at the equilibrium point (-1, 0).





3.2 **a)** $\begin{cases} \dot{x} = 16x^2 + 9y^2 - 25 = 0 \\ \dot{y} = 16x^2 - 16y^2 \quad = 0 \end{cases} \Rightarrow x^2 = y^2 \Rightarrow x = y = \pm 1$

The critical points are: (1, 1), (1, -1), (-1, 1), and (-1, -1)

$DF = \begin{pmatrix} 32x & 18y \\ 32x & -32y \end{pmatrix}$

<u>At the point (1, 1)</u>: $A = DF|_{(1,1)} = \begin{pmatrix} 32 & 18 \\ 32 & -32 \end{pmatrix}$

The eigenvalues are: $|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda - 32 & -18 \\ -32 & \lambda + 32 \end{vmatrix} = \lambda^2 - 1600 = 0 \Rightarrow \lambda_{1,2} = \pm 40$

The behavior of the system is a saddle (unstable) at the equilibrium point (1, 1).

<u>At the point (1, -1)</u>: $A = DF|_{(1,1)} = \begin{pmatrix} 32 & -18 \\ 32 & 32 \end{pmatrix}$

$|\lambda I - A| = \begin{vmatrix} \lambda - 32 & 18 \\ -32 & \lambda - 32 \end{vmatrix} = \lambda^2 - 64\lambda + 1600 = 0 \Rightarrow \lambda_{1,2} = 32 \pm 24i$

The behavior of the system is an unstable spiral at the equilibrium point (1, -1).

<u>At the point (-1, 1)</u>: $A = DF|_{(-1,1)} = \begin{pmatrix} -32 & 18 \\ -32 & -32 \end{pmatrix}$

The eigenvalues are: $|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda + 32 & -18 \\ 32 & \lambda + 32 \end{vmatrix} = \lambda^2 + 64\lambda + 1600 = 0 \Rightarrow \lambda_{1,2} = -32 \pm 24i$

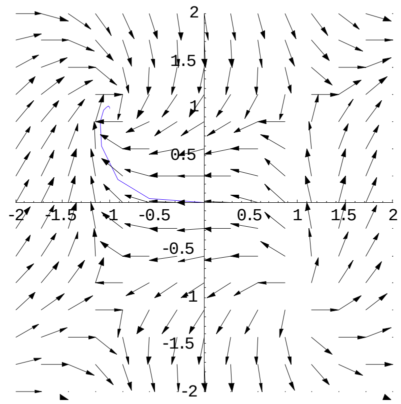
The behavior of the system is an unstable spiral at the equilibrium point (-1, 1).

<u>At the point (-1, -1)</u>: $A = DF|_{(-1,-1)} = \begin{pmatrix} -32 & -18 \\ -32 & 32 \end{pmatrix}$

$|\lambda I - A| = \begin{vmatrix} \lambda+32 & 18 \\ 32 & \lambda-32 \end{vmatrix} = \lambda^2 - 1600 = 0 \Rightarrow \lambda_{1,2} = \pm 40$

The behavior of the system is a saddle (unstable) at the equilibrium point (-1, -1).

b)



3.3 $\ddot{x} + c\dot{x} - x(1-x) = 0$, with $\lim_{t\to-\infty} x(t) = 0$, $\lim_{t\to+\infty} x(t) = 1$.

Assume: $\dot{x} = y \Rightarrow \ddot{x} = \dot{y} = x(1-x) - cy$

The equilibrium points are (0, 0) and (1, 0)

$DF = \begin{pmatrix} 0 & 1 \\ 1-2x & -c \end{pmatrix}$

<u>At the point (0, 0)</u>: $A = DF|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix}$

The eigenvalues are: $|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda & -1 \\ -1 & \lambda+c \end{vmatrix} = \lambda(\lambda + c) - 1 = 0 \Rightarrow \lambda_{1,2} = \frac{-c\pm\sqrt{c^2+4}}{2}$

$\sqrt{c^2+4} > c \Rightarrow$ One $\lambda > 0$ and the other $\lambda < 0$

$\Rightarrow$ The behavior of the system is a saddle (unstable) at the origin point (0, 0).

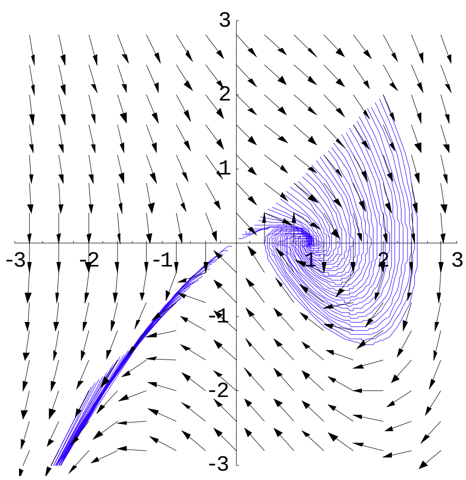
<u>At the point (1, 0)</u>: $A = DF|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix}$

The eigenvalues are: $|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda & -1 \\ 1 & \lambda+c \end{vmatrix} = \lambda(\lambda + c) + 1 = 0 \Rightarrow \lambda_{1,2} = \frac{-c\pm\sqrt{c^2-4}}{2}$

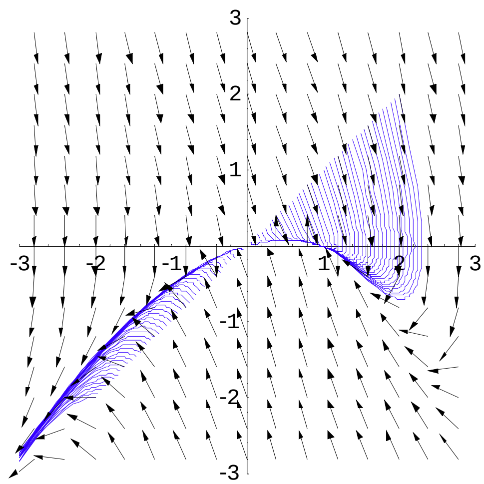
The system behavior is depending on $\sqrt{c^2-4}$ $(c > 0)$.

If $0 < c < 2$, $\sqrt{c^2-4}$ is complex and the system is a spiral near the equilibrium point (1, 0).

If $c \geq 2$, the system is an attractor and has a monotonic behavior of x(t).



c = 1



c = 3

3.4 $\begin{cases} \dot{x} = x(1-x^2-6y^2) = 0 \\ \dot{y} = y(1-3x^2-3y^2) = 0 \end{cases} \Rightarrow$ (0, 0) is an equilibrium point.

$\begin{cases} 1-x^2-6y^2 = 0 \\ 1-3x^2-3y^2 = 0 \end{cases} \Rightarrow \begin{cases} 1-6y^2 = x^2 \\ 1-3(1-6y^2)-3y^2 = 0 \end{cases} \Rightarrow -2 + 15y^2 = 0 \Rightarrow y = \pm\frac{\sqrt{30}}{15}$

If $y = \frac{\sqrt{30}}{15} \Rightarrow x = \pm\frac{\sqrt{5}}{5}$ and for $y = -\frac{\sqrt{30}}{15} \Rightarrow x = \pm\frac{\sqrt{5}}{5}$

If x = 0 (1) $\Rightarrow$ (2); $y(1-3y^2) = 0 \Rightarrow y = \pm\frac{1}{\sqrt{3}}$

If y = 0 (2) $\Rightarrow$ (1); $x(1-x^2) = 0 \Rightarrow x = \pm 1$.

The equilibrium points are: (0, 0), (±1, 0), $(0, \pm\frac{1}{\sqrt{3}})$, $\left(\frac{\sqrt{5}}{5}, \frac{\sqrt{30}}{15}\right)$, $\left(\frac{\sqrt{5}}{5}, -\frac{\sqrt{30}}{15}\right)$, $\left(-\frac{\sqrt{5}}{5}, \frac{\sqrt{30}}{15}\right)$,

and $\left(\frac{\sqrt{5}}{5}, -\frac{\sqrt{30}}{15}\right)$

$$DF = \begin{pmatrix} 1-3x^2-6y^2 & -6xy \\ -12xy & 1-3x^2-9y^2 \end{pmatrix}$$

<u>At the point (0, 0)</u>: $A = DF|_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

The eigenvalues are: $|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda-1 & 0 \\ 0 & \lambda-1 \end{vmatrix} = (\lambda - 1)^2 = 0 \Rightarrow \lambda_{1,2} = 1$

$\rightarrow$ Since the $\lambda_{1,2} > 0$; therefore the system is an unstable source (repeller) at the origin point (0, 0).

<u>At the point (±1, 0)</u>: $A = DF|_{(\pm 1,0)} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

$\Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda+2 & 0 \\ 0 & \lambda+2 \end{vmatrix} = (\lambda + 2)^2 = 0 \Rightarrow \lambda_{3,4} = -2\ (<0)$

$\rightarrow$ The system is an stable sink at the origin point (±1, 0).

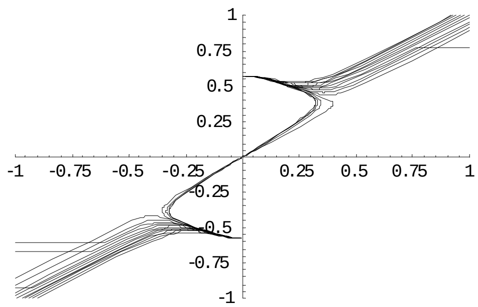
At the point $(0, \pm\frac{1}{\sqrt{3}})$: $A = DF| = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$|\lambda I - A| = (\lambda + 1)^2 = 0 \Rightarrow \lambda_{1,2} = -1 < 0$;

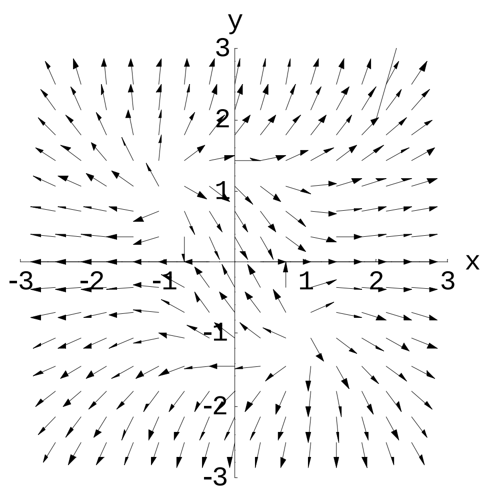
The system is an stable sink at the origin point $(0, \pm\frac{1}{\sqrt{3}})$.

At the point $\left(\frac{\sqrt{5}}{5}, \frac{\sqrt{30}}{15}\right)$: $DF| = \begin{pmatrix} -\frac{3}{5} & -\frac{4\sqrt{3}}{5} \\ -\frac{8\sqrt{3}}{5} & -\frac{8}{5} \end{pmatrix}$

The eigenvalues are: $|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda + \frac{3}{5} & \frac{4\sqrt{3}}{5} \\ \frac{8\sqrt{3}}{5} & \lambda + \frac{8}{5} \end{vmatrix} = 5\lambda^2 + 11\lambda - 62 = 0 \Rightarrow \lambda = \frac{-11 \pm \sqrt{1361}}{10}$



3.5 $\begin{cases} \dot{x} = x^3 + y = 0 \\ \dot{y} = y(x^2 + y^2 - 2) = 0 \end{cases} \Rightarrow$ (0, 0), (1, -1), and (-1, 1) are the equilibrium points.



3.6 The system can be rewritten: $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x\left(x^2 + y^2\right)\sin\sqrt{x^2 + y^2} \\ y\left(x^2 + y^2\right)\sin\sqrt{x^2 + y^2} \end{pmatrix}$

The eigenvalues are determined by: $|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda & 1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$

Since $\lambda_{1,2} = \pm i$ (imaginary number only $Re(\lambda) = 0$), which is a non-hyperbolic case. That means; we can not draw any conclusion of the behavior near the origin point (0, 0).

Let assume: $r^2 = x^2 + y^2 \Rightarrow 2r\dot{r} = 2x\dot{x} + 2y\dot{y} \Rightarrow r\dot{r} = x\dot{x} + y\dot{y}$

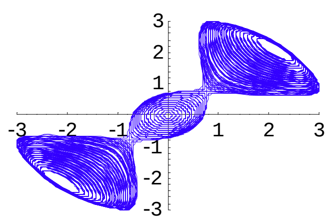
$\rightarrow r\dot{r} = x(-y - xr^2\sin(r)) + y(x - y\,r^2\sin(r)) = -xy - x^2 r^2\sin(r) + xy - y^2 r^2\sin(r) = -r^2(x^2 + y^2)$ sin(r)

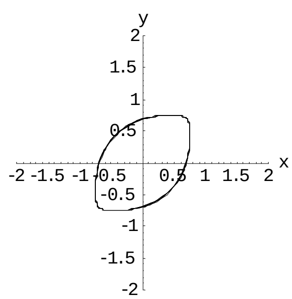
$= -r^4 \sin(r) \Rightarrow \dot{r} = -r^3 \sin(r)$

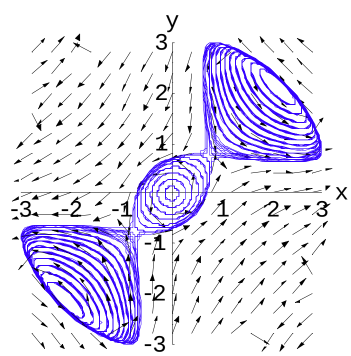
$\rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\Rightarrow \dot{\theta} = \frac{\frac{\dot{y}x - \dot{x}y}{x^2}}{1 + \frac{y^2}{x^2}} = \frac{\dot{y}x - \dot{x}y}{r^2} = \frac{x^2 + xyr^2\sin r + y^2 - xyr^2\sin r}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

Therefore; the system is a stable spiral at the origin point (0, 0).



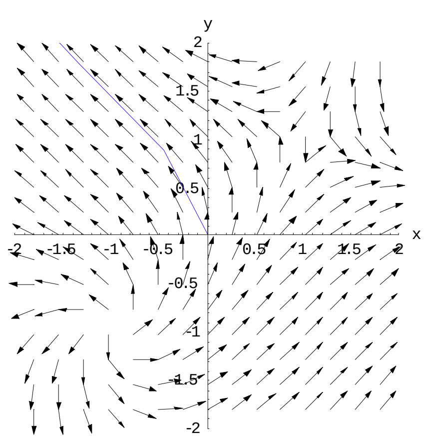


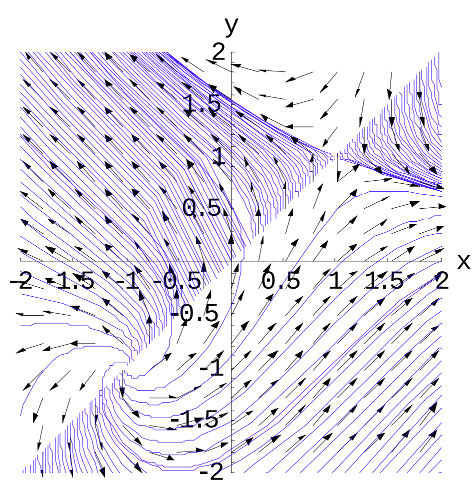


3.7 **a)** $\begin{cases} \dot{x} = x - y = 0 \\ \dot{y} = 1 - xy = 0 \end{cases} \Rightarrow \begin{cases} x = y \\ 1 - x^2 = 0 \end{cases} \Rightarrow x = \pm 1 = y$

$\Rightarrow$ The critical points are (1, 1) and (-1, -1).

b)

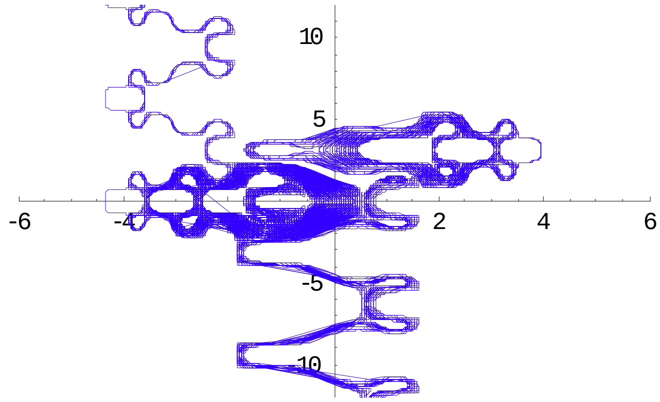




3.8 **a)** $(\sin y - 2\sin x^2 \sin 2y)\dfrac{dy}{dx} + \cos x + 2x\cos x^2 \cos 2y = 0$

Assume $\dot{x} = y \Rightarrow \dot{y} = \dfrac{-\cos x - 2x\cos x^2 \cos 2y}{\sin y - 2\sin x^2 \sin 2y}y$

b)



3.9 **a)** $\begin{cases} \ddot{x} = x^2 - y = 0 \\ \ddot{y} = y - x = 0 \end{cases} \Rightarrow \begin{cases} x^2 = y \\ x^2 - x = 0 \end{cases}$ $\Rightarrow$ the critical points are (0, 0) and (1, 1)

b) $D = \begin{bmatrix} 2x & -1 \\ -1 & 1 \end{bmatrix}$

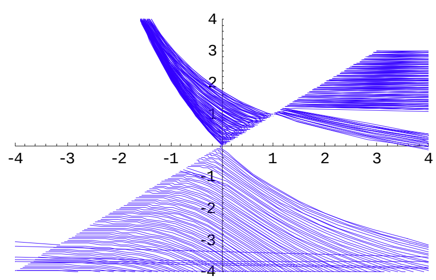
<u>At the point</u> (0, 0): $A = D|_{(0,0)} = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$

$|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda & 1 \\ 1 & \lambda - 1 \end{vmatrix} = \lambda^2 - \lambda + 1 = 0 \Rightarrow \lambda_{1,2} = \dfrac{1 \pm \sqrt{5}}{2} \Rightarrow$ The system is a saddle at the origin.

<u>At the point</u> (0, 0): $A = D|_{(1,1)} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow \begin{vmatrix} \lambda - 2 & 1 \\ 1 & \lambda - 1 \end{vmatrix} = 0 \Rightarrow = \lambda^2 - 3\lambda + 1 = 0 \Rightarrow \lambda_{1,2} =$

$\dfrac{3 \pm \sqrt{5}}{2}$

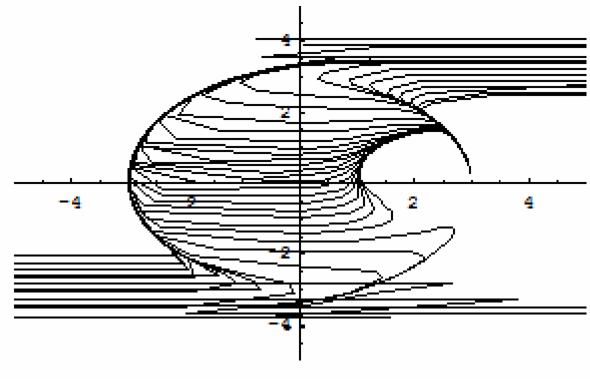
$\Rightarrow$ The system is an unstable at the point (1, 1).



3.10 **a)** $\begin{cases} \dot{x} = [(x-2)^2 + y^2 - 1][x^2 + y^2 - 9] = 0 \\ \dot{y} = (x-1)^2 + y^2 - 4 = 0 \end{cases}$

$\Rightarrow$ The point (3, 0) is the only critical point.

b)



3.11 **a)** $\begin{cases} \dot{x} = x + y \\ \dot{y} = y - x^2 \\ \dot{z} = 1 \end{cases} \Rightarrow x = -y \Rightarrow y - y^2 = 0 \Rightarrow y = 0 \text{ or } 1$

The equilibrium points are: (0, 0, 1) and (-1, 1, 1).

$$DF = \begin{bmatrix} 1 & -2x & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

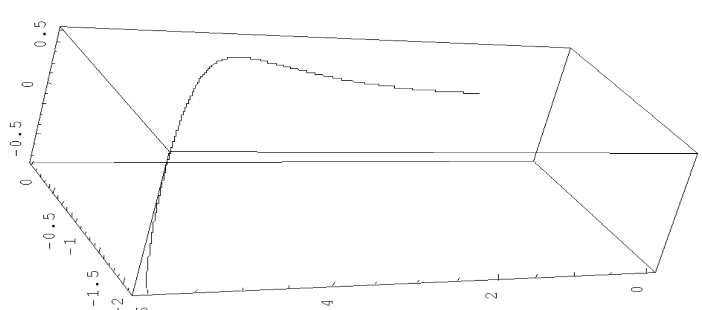
<u>At the point (0, 0, 1)</u>: $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda - 1 & 0 & 0 \\ -1 & \lambda - 1 & 0 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda(\lambda - 1)(\lambda - 1) = 0 \Rightarrow \lambda = 0, 1, 1.$$

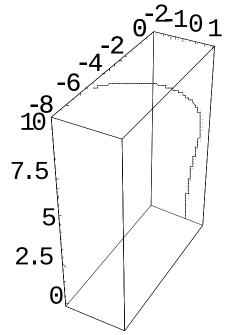
<u>At the point (-1, 1, 1)</u>: $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda - 1 & -2 & 0 \\ -1 & \lambda - 1 & 0 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda(\lambda - 1)(\lambda - 1) - 2\lambda = 0 \Rightarrow \lambda = 0, 1 \pm \sqrt{2}\ .$$

$\Rightarrow$ The system is a saddle node.



b) $\begin{cases} \dot{x} = x + y \\ \dot{y} = y - x^2 \\ \dot{z} = 2 + x \end{cases} \Rightarrow \begin{cases} x + y = 0 \\ y - x^2 = 0 \\ 2 + x = 0 \end{cases} \Rightarrow$ No equilibrium point.



c) $\begin{cases} \dot{x} = -y - 2 \\ \dot{y} = x + .2y \\ \dot{z} = .2 - 5.7z + xz \end{cases} \Rightarrow \begin{cases} y = -2 \\ x + .2y = 0 \\ .2 - 5.7z + xz = 0 \end{cases} \Rightarrow x = -.4 \Rightarrow z = .033$

The point (-.4, -2, 0.033) is the only equilibrium point.

$$DF = \begin{bmatrix} 0 & 1 & z \\ -1 & .2 & 0 \\ 0 & 0 & -5.7 + x \end{bmatrix}$$

<u>At the point (-.4, -2, .033)</u>: $A = \begin{bmatrix} 0 & 1 & .033 \\ -1 & .2 & 0 \\ 0 & 0 & -6.1 \end{bmatrix}$

$|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda & -1 & -.033 \\ 1 & \lambda - .2 & 0 \\ 0 & 0 & \lambda + 6.1 \end{vmatrix} = \lambda(\lambda - .2)(\lambda + 6.1) + \lambda + 6.1 = 0 \Rightarrow \lambda = -6.1,\ 0.15 \pm .9887i.$

$\Rightarrow$ The system is a negative attraction away from the equilibrium point (-.4, -2, 0.033).

3.12 **a)** $\begin{cases} \alpha x + y - x(x^2 + y^2) = 0 \\ -x + \alpha y - y(x^2 + y^2) = 0 \end{cases} \Rightarrow$ The origin point (0, 0) is the only equilibrium point.

The Jacobian matrix is: $A = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix}$

The eigenvalues for the system are: $|\lambda I - A| = 0$

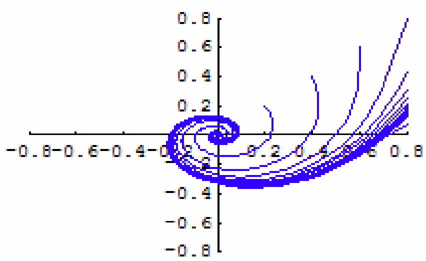
$$\Rightarrow \begin{vmatrix} \lambda - \alpha & -1 \\ 1 & \lambda - \alpha \end{vmatrix} = (\lambda - \alpha)^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \alpha \pm i.$$

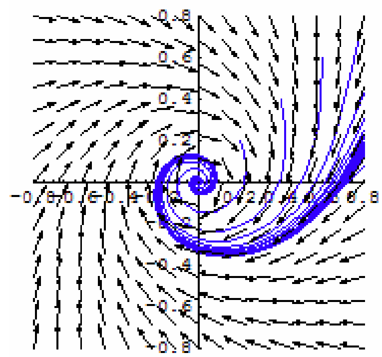
Which the system is a non-hyperbolic case. That means; we can not draw any conclusion of the behavior near the origin point (0, 0). Let assume: $r^2 = x^2 + y^2 \Rightarrow r\dot{r} = x\dot{x} + y\dot{y}$

$\Rightarrow r\dot{r} = x[\alpha x + y - x(x^2 + y^2)] + y[-x + \alpha y - y(x^2 + y^2)] = 2(x^2 + y^2)\,[\alpha - (x^2 + y^2)]$

$\Rightarrow \dot{r} = 2[\alpha - (x^2 + y^2)]$

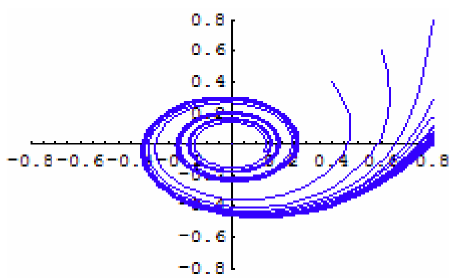
b) For $\alpha = -\frac{1}{4}$, the eigenvalues are $-\frac{1}{4} \pm i \Rightarrow \dot{r} = 2[\alpha - (x^2 + y^2)] = 2[-\frac{1}{4} - (x^2 + y^2)] \leq 0$.

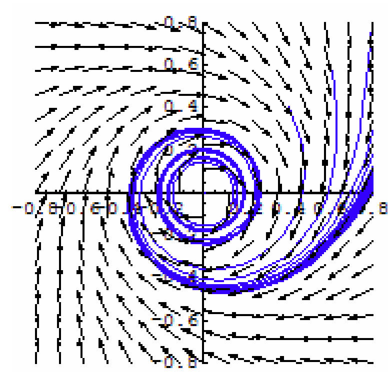




Therefore, the behavior of the system is a spiral-in towards the origin point (0, 0) as $t \rightarrow \infty$.

c) For $\alpha = 0$, the eigenvalues are $\pm i$ (Imaginary number only) $\Rightarrow \dot{r} = -2(x^2 + y^2) \leq 0$



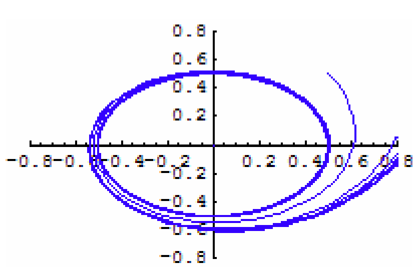


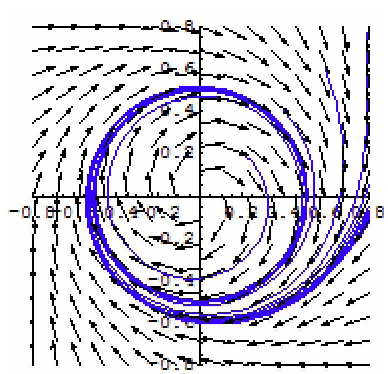
Therefore, the behavior of the system is a spiral-in towards the origin point (0, 0) as $t \rightarrow \infty$. However, the behavior circles near zero (well short) versus what we have seen in part-b.

d) For the case $\alpha = \frac{1}{4}$, the eigenvalues are $\frac{1}{4} \pm i \Rightarrow$ The origin is spiral source.

$\Rightarrow \dot{r} = 2[\alpha - (x^2 + y^2)] = 2[\frac{1}{4} - (x^2 + y^2)]$ **?** 0.

The inequality depends on the circle $x^2 + y^2$ and the radius $\frac{1}{2}$.





$$\begin{cases} \text{if } x^2+y^2>\frac{1}{4} \;\Rightarrow\; \dot{r}<0 \;\Rightarrow\; \text{Start from outside towards the circle} \\ \text{if } x^2+y^2<\frac{1}{4} \;\Rightarrow\; \dot{r}>0 \;\Rightarrow\; \text{Start intside the circle and is drawn out} \\ \text{if } x^2+y^2=\frac{1}{4} \;\Rightarrow\; \dot{r}=0 \;\Rightarrow\; \text{solutions curve around the circle with } r=\frac{1}{2} \end{cases}$$

e) The solution curve is a circle of radius $\frac{1}{2}$ and along this circle. This is a reference to a limit cycle.

Chapter 4

4.1 $\begin{cases} \dot{x} = -x + y + x(x^2 + y^2) \\ \dot{y} = -x - y + y(x^2 + y^2) \end{cases}$

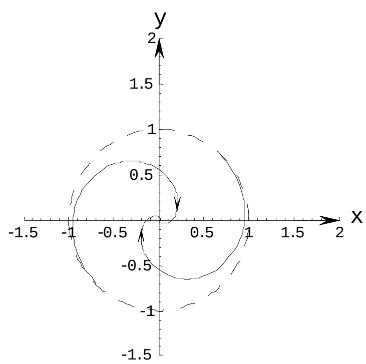
The equilibrium point(s):

$\begin{cases} \dot{x} = -x + y + x(x^2 + y^2) = 0 \\ \dot{y} = -x - y + y(x^2 + y^2) = 0 \end{cases}$ $\Rightarrow$ The point (0, 0) is the only equilibrium point.

$V(x, y) = x^2 + y^2$

$$\begin{aligned}
\dot{V} &= (2x, 2y)\,(-x + y + x(x^2 + y^2),\ - x - y + y(x^2 + y^2)) \\
&= -2x^2 + 2xy + 2x^4 + 2x^2y^2 - 2xy - 2y^2 + 2y^2x^3 - 2y^4 \\
&= -2(x^2 + y^2) + 2(x^2 + y^2)^2 \\
&= (x^2 + y^2)(-2 + 2(x^2 + y^2)) \\
&= 2(x^2 + y^2)\,(x^2 + y^2 - 1) < 0
\end{aligned}$$

If $(x^2 + y^2 - 1) < 0$ $\therefore$ the point (0, 0) is a local asymptotically stable equilibrium point $\forall\ x^2 + y^2 < 1$



$x^2 + y^2 < 1$

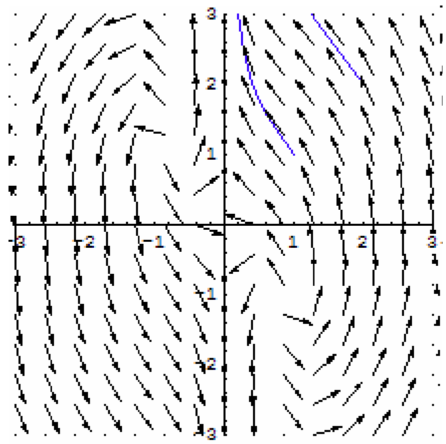
4.2 $\begin{cases} \dot{x} = -x - x^2 y = 0 \\ \dot{y} = y + 2x^3 = 0 \end{cases} \Rightarrow \begin{cases} -x(1+xy) = 0 \\ y = -2x^3 \end{cases} \Rightarrow x(1 - 2x^4) = 0 \Rightarrow x = 0 \text{ or } x = \pm\sqrt[4]{\frac{1}{2}}$

If $x = 0 \Rightarrow y = 0$

If $x = \sqrt[4]{\frac{1}{2}} \Rightarrow y = -\sqrt[4]{2}$

If $x = -\sqrt[4]{\frac{1}{2}} \Rightarrow y = \sqrt[4]{2}$

The equilibrium points (0, 0), $(\sqrt[4]{\frac{1}{2}}, -\sqrt[4]{2})$, and $(-\sqrt[4]{\frac{1}{2}}, -\sqrt[4]{2})$



4.3 **a)** the equilibrium points:

$$\begin{cases} \dot{x} = x(x^2 + y^2 - 2x - 3) - y = 0 \\ \dot{y} = y(x^2 + y^2 - 2x - 3) + x = 0 \end{cases} \Rightarrow \text{From (1); } x^2 + y^2 - 2x - 3 = \frac{y}{x}.$$

We can rewrite part (2): $y\,\frac{y}{x} + x = 0 \Rightarrow y^2 + x^2 = 0$.

Therefore; the origin point (0, 0) is the only critical point for this system.

b) This is a spiral point with a positive attraction. Then, we apply the criterion of Bendixon's theorem.

$$4x^2 + 4y^2 - 6x - 6 = 4[(x - \tfrac{3}{4})^2 + y^2 - \tfrac{33}{16}]$$

We can determine that the divergence of the vector function is on the right hand-side.

And inside the Bendixon circle with the center $(\frac{3}{4}, 0)$ and with a radius of $\frac{\sqrt{33}}{4}$, the expression is sign definite and no closed orbits can be contained in the interior of this circle. Closed orbits are possible which enclose or which intersect this Bendixon-circle.

We can transform this system to the polar coordinates by using ($x = r.\cos\theta$, and $y = r.\sin\theta$), then: $x^2 + y^2 = r^2 \Rightarrow 2x\dot{x} + 2y\dot{y} = 2r\dot{r} \Rightarrow r\dot{r} = x\dot{x} + y\dot{y}$

$\Rightarrow r\dot{r} = x[x(x^2 + y^2 - 2x - 3) - y] + y[y(x^2 + y^2 - 2x - 3) + x]$

$\Rightarrow = x[x^3 + xy^2 - 2x^2 - 3x - y] + y[yx^2 + y^3 - 2xy - 3y + x]$

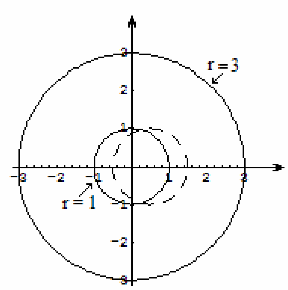
$\Rightarrow = x^4 + x^2y^2 - 2x^3 - 3x^2 - xy + y^2x^2 + y^4 - 2xy^2 - 3y^2 + xy$

$\Rightarrow = x^4 + y^4 + 2x^2y^2 - 2x(x^2 + y^2) - 3(x^2 + y^2)$

$\Rightarrow = (x^2 + y^2)^2 - (x^2 + y^2)(2x + 3)$

$\Rightarrow r\dot{r} = (r^2)^2 - r^2(2r.\cos\theta + 3) = r^2(r^2 - 2r.\cos\theta - 3)$

$\Rightarrow \dot{r} = r(r^2 - 2r.\cos\theta - 3)$



And $\dot{\theta} = 1$.

Therefore; the polar system is depending on r.

If $r < 1$, then $\dot{r} < 0$.
If $r > 3$, then $\dot{r} > 0$.

According to the Poincaré – Bendixon theorem, the annulus $1 < r < 3$ must contain one or more limit cycles.

4.4 **a)** The divergence of the vector field is: $\frac{\partial}{\partial x}(y - x^3 + \mu x) = -3x^2 + \mu$; so there are no periodic solution if $\mu < 0$ (by Bendixon).

Let $r^2 = x^2 + y^2 \Rightarrow r\dot{r} = x\dot{x} + y\dot{y} = xy - x^4 + \mu x^2 - xy = x^2(\mu - x^2) \Rightarrow \dot{r} = -\frac{1}{r}x^2(x^2 - \mu)$

If $\mu = 0 \Rightarrow \dot{r} \le 0$ $(r > 0)$; the phase flow is contracting except when passing the y-axis. Therefore, there is no periodic solution when $\mu = 0$.

If $\mu > 0$, we can differentiate the equation for x to get: $\ddot{x} = \dot{y} - 3x^2\dot{x} + \mu\dot{x} = -x - 3x^2\dot{x} + \mu\dot{x}$

$\Rightarrow \ddot{x} + (3x^2 - \mu)\dot{x} + x = 0 \leftarrow$ in the form of: $\ddot{x} + f(x)\dot{x} + x = 0$ (Lienard equation).

$$\begin{cases}\dot{x} = y - f(x)\\ \dot{y} = -x\end{cases}$$

b) As $\mu \to 0$, the limit cycle contracts around the origin point (0, 0) and vanishes into (0, 0) as $\mu = 0$.

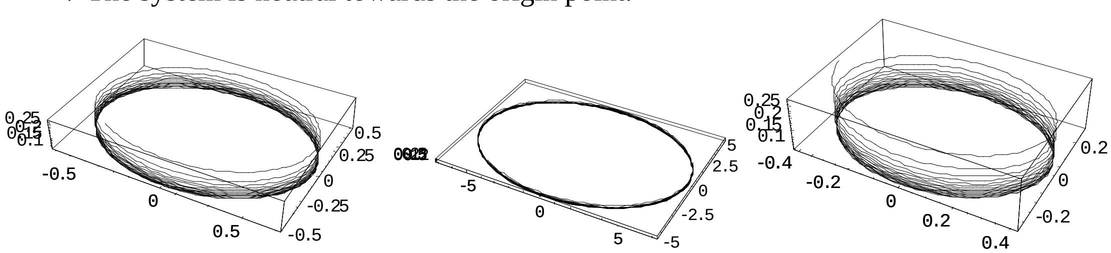
4.5 $\begin{cases}\dot{x} = 2y(z-1)\\ \dot{y} = -x(z-1)\\ \dot{z} = -z^3\end{cases} \Rightarrow -z^3 = 0 \Rightarrow z = 0,\ -x(z-1) = -x(-1) = 0 \Rightarrow x = 0 = y.$

The origin point (0, 0, 0) is the only equilibrium point.

$$DF = \begin{bmatrix} 0 & 2z-2 & 2y \\ -z+1 & 0 & -x \\ 0 & 0 & -3z^2 \end{bmatrix} \Rightarrow A = DF|_{(0,0,0)} = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda & 2 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda(\lambda^2 + 2) = 0 \Rightarrow \lambda_{1,2,3} = 0, \pm 2i$$

$\Rightarrow$ The system is neutral towards the origin point.



intial points at = 0.4 4.0 .4, .04, .4

Chapter 5

5.1 $m\ddot{x} + c\dot{x} + kx = 0. \Rightarrow \ddot{x} = -\frac{c}{m}\dot{x} - \frac{k}{m}x$

Assume that: $y = \dot{x} \Rightarrow \dot{y} = \ddot{x} = -\frac{c}{m}\dot{x} - \frac{k}{m}x$

Therefore; we can rewrite the system in the form of: $\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{k}{m}x - \frac{c}{m}y \end{cases}$

The equilibrium point is the origin point (0, 0).

$DF = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} = DF|_{(0,0)}$

$|\lambda I - DF|_{(0,0)}| = \begin{vmatrix} \lambda & -1 \\ \frac{k}{m} & \lambda + \frac{c}{m} \end{vmatrix} = \lambda(\lambda + \frac{c}{m}) + \frac{k}{m} = 0 \Rightarrow m\lambda^2 + c\lambda + k = 0$

The eigenvalues are: $\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$

a) If $c^2 - 4km \geq 0 \Rightarrow c^2 \geq 4km \Rightarrow c \leq -2\sqrt{km}$ and $c \geq 2\sqrt{km}$ (since $c \geq 0$ and $k, m > 0$)
$\Rightarrow \sqrt{km} > 0$, then $c \leq -2\sqrt{km}$ does not exist since $c > 0$.
$\rightarrow c > 2\sqrt{km}$:

i) $\lambda_1 = \frac{-c - \sqrt{c^2 - 4km}}{2m} < 0$

ii) $\lambda_2 = \frac{-c + \sqrt{c^2 - 4km}}{2m} < 0$

To prove that $\lambda_2 < 0$; if NOT $c = 2n\sqrt{km} \Rightarrow c^2 - 4km = 4km(n^2 - 1)$

$\lambda_2 = \frac{-2n\sqrt{km} + 2\sqrt{km}\sqrt{n^2 - 1}}{2m} > 0 \Rightarrow -n + \sqrt{n^2 - 1} > 0 \Rightarrow -n > -\sqrt{n^2 - 1}$

$n < \sqrt{n^2 - 1} \Rightarrow n^2 < n^2 - 1 \Rightarrow 0 < -1$ (contradiction) $\Rightarrow \lambda_2 < 0$

For $c > 2\sqrt{km}$, that imply $\lambda_{1,2} < 0$; therefore the system is stable at the point (0, 0).

If $c = 2\sqrt{km} \Rightarrow \lambda_{1,2} = -\frac{c}{2m} < 0 \Rightarrow$ the system is stable at the point (0, 0).

b) If $c^2 - 4km < 0 \Rightarrow c^2 < 4km \Rightarrow 0 < c < 2\sqrt{km}$

$\lambda_{1,2} = -\frac{c}{2m} \pm \frac{\sqrt{c^2 - 4km}}{2m}$ with $\mathrm{Re}(\lambda_{1,2}) = -\frac{c}{2m} < 0 \Rightarrow$ the system is stable at (0, 0).

From part (a) and (b), we can conclude that the system is always stable at the origin point (0, 0), for $c^2 - 4km \geq 0$ or < 0.

By using Lyapunov functions, let assume that $V(x, y) = ax^2 + by^2$.

i- $V(0, 0) = 0 < V(x, y), \forall (x, y) \in$ Ñ$^2 - \{(0, 0)\}$

ii- $\dot{V} = (2ax, 2by).(y, -\frac{k}{m}x - \frac{c}{m}y) = 2axy - 2b\frac{k}{m}xy - 2b\frac{c}{m}y^2 = 2xy(a - b\frac{k}{m}) - 2b\frac{c}{m}y^2$

$\Rightarrow -2b\frac{c}{m}y^2 < 0$ and let $a - b\frac{k}{m} = 0 \Rightarrow a = b\frac{k}{m} \Rightarrow V(x, y) = \frac{k}{m}x^2 + y^2$

$\therefore$ The origin point (0, 0) is the global asymptotically stable for the equilibrium point.

5.2 The characteristic behavior of the dynamic system is:

$$\begin{cases} F + k_1 x_2 = M_1 \ddot{x}_2 + f_1 \dot{x}_2 + k_1 x_2 \\ k_1 x_1 = M_2 \ddot{x}_2 + f_2 \dot{x}_2 + (k_1 + k_2) x_2 \end{cases}$$

5.3 $\begin{cases} m\ddot{x} = -kx - k_1(x - x_1) - c(\dot{x} - x_1) + f \\ m_1\ddot{x}_1 = k_1(x - x_1) + c(\dot{x} - \dot{x}_1) \end{cases} \Rightarrow \begin{cases} m\ddot{x} + c\dot{x} + (k + k_1)x = c\dot{x}_1 + k_1 x_1 + f \\ m_1\ddot{x}_1 + c\dot{x}_1 + k_1 x_1 = c\dot{x} + k_1 x \end{cases}$

Assume $y = x_1 \Rightarrow \dot{y} = \dot{x}_1$ and $\ddot{y} = \ddot{x}_1$

$$\Rightarrow \begin{cases} m\ddot{x} + c\dot{x} + (k + k_1)x = c\dot{y} + k_1 y + f \\ m_1\ddot{y} + c\dot{y} + k_1 y = c\dot{x} + k_1 x \end{cases}$$

5.4 $\ddot{x} + \mu\dot{x} + x + ax^2 + bx^3 = 0$

The damping $\mu\dot{x}$ causes the trivial solution to be asymptotically stable.

The energy of the non-linear oscillator without damping:

$$V(x, \dot{x}) = ½ \dot{x}^2 + ½ x^2 + 1/3\, ax^3 + ¼\, bx^4$$

Where the neighborhood region D of the point (0, 0) is given by:

$$L_T V = \dot{x}\ddot{x} + x\dot{x} + x + ax^2\dot{x} + bx^3\dot{x} = -\mu\dot{x}^3$$

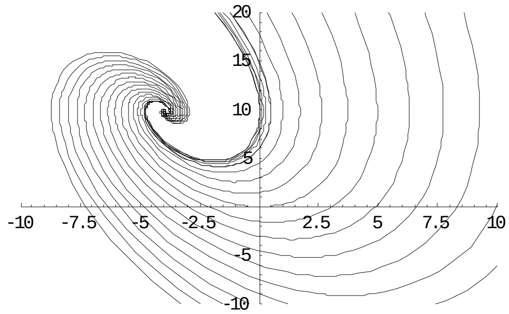
The point (0, 0) is a Lyapunov-stable, as $L_T V \le 0$ in D the solutions stay inside the region D and can not leave D. Also, we can determine that $L_T V = 0$ only if $\dot{x} = 0$.
However, if $x \neq 0$ and $\dot{x} = 0$ the system is transversal solution of the phase-flow so we can conclude that D is a domain of attraction of the trivial solution.

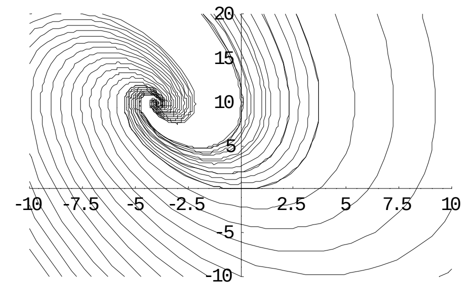
5.5 **a)** $m\ddot{x} + c\dot{x} + kx = 0$. Assume $y = \dot{x} \Rightarrow \dot{y} = \ddot{x} = -\frac{c}{m}y - \frac{k}{m}x$

The origin point (0, 0) is the only equilibrium point.

b) m =1, k = 5, and c = 2. $\Rightarrow \ddot{x} + 2\dot{x} + 5x = 0 \Rightarrow \lambda^2 + 2\lambda + 5 = 0 \Rightarrow \lambda = -1 \pm 2i$
$x(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t) \Rightarrow$ at initial point $x(0) = 0 = c_1$ and $\dot{x}(0) = 4 = 2c_2 \Rightarrow c_2 = 2$
$\Rightarrow x(t) = 2e^{-t}\sin 2t$

c) $\lambda^2 + c\lambda + 5 = 0 \Rightarrow c^2 - 20 = 0$ or $c = \sqrt{20}$ will be the critical damped.





5.6 $\ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y = -g + \frac{1}{m}f(t) \Rightarrow \ddot{y} + .2\dot{y} + 10y = -9.8$

Assume: $\dot{y} = x \Rightarrow \ddot{y} = \dot{x}$

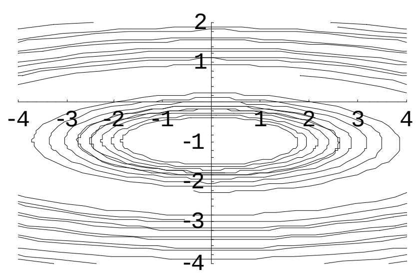
$\Rightarrow \begin{cases} \dot{x} = -.2x - 10y - 9.8 \\ \dot{y} = x \end{cases} \Rightarrow$ The point (0, -.98) is the equilibrium point.

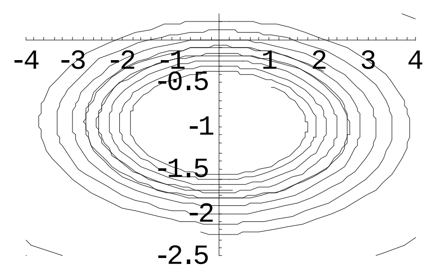
$DF = \begin{bmatrix} -.2 & -10 \\ 1 & 0 \end{bmatrix} \Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda + .2 & 10 \\ -1 & \lambda \end{vmatrix} = \lambda(\lambda + .2) + 10 = 0 \Rightarrow \lambda_{1,2} = -0.1 \pm 3.16i$

This is hyperbolic solution and we can determine the solution from. However, we can conclude that the system is periodic.

$\ddot{y} + .2\dot{y} + 10y = -9.8 \Rightarrow y = e^{-.1t}(\cos 3.16t + \sin 3.16t) - 0.98$

$\Rightarrow z = h - y = -e^{-.1t}(\cos 3.16t + \sin 3.16t) + h + 0.98$





5.7 $\begin{cases} \dot{\theta} = y & = 0 \\ \dot{y} = -k\sin\theta - \varepsilon y = 0 \end{cases} \Rightarrow y = 0$, then $\sin\theta = 0 \Rightarrow \theta = n\pi \to (0, 0)$ is the equilibrium point.

$DF = \begin{pmatrix} 0 & 1 \\ -k\cos\theta & -\varepsilon \end{pmatrix}$

a) $k = 1 \Rightarrow A = DF|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & -\varepsilon \end{pmatrix}$

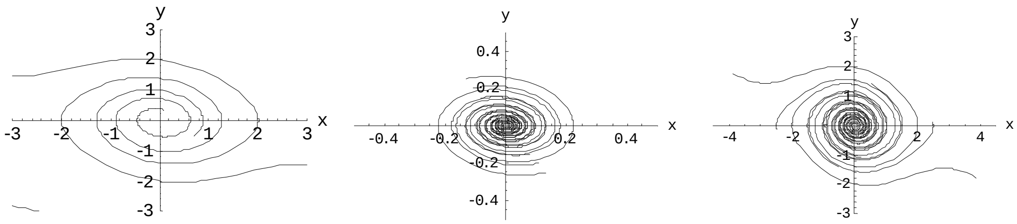
$\Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda + \varepsilon \end{vmatrix} = \lambda(\lambda + \varepsilon) + 1 = 0 \Rightarrow \lambda_{1,2} = \frac{-\varepsilon \pm \sqrt{\varepsilon^2 - 4}}{2}$

If $\varepsilon^2 - 4 > 0 \Rightarrow \varepsilon < -2$ or $\varepsilon > 2 \Rightarrow \lambda_1 < 0$, $\lambda_2 > 0 \Rightarrow$ The system is unstable (saddle point).

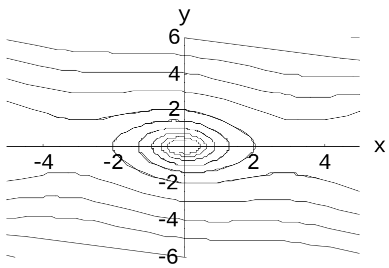
If $\varepsilon^2 - 4 \;\; 0 \Rightarrow \varepsilon = 2 \Rightarrow \lambda_{1,2} = -1 \Rightarrow$ the system is asymptotically stable.

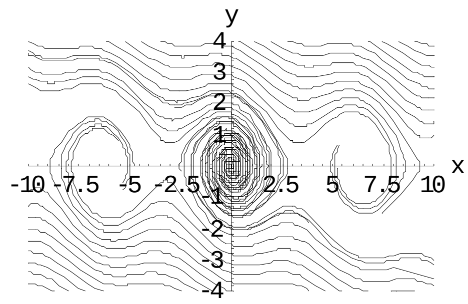
If $\varepsilon^2 - 4 < 0 \Rightarrow -2 < \varepsilon < 2 \Rightarrow \lambda_{1,2} \in$ Â

b) $\varepsilon = .2 \Rightarrow \lambda_{1,2} = \frac{-.2 \pm .63i}{2} = -.1 \pm .3146\, i$. The system is an asymptotically stable - spiral in.



Initial value = .01 .1 1





5.8 $F(x) = -\,k(x)\,x = -\,x\,e^{x^2}$

Sketch the phase portrait of this mechanical system.

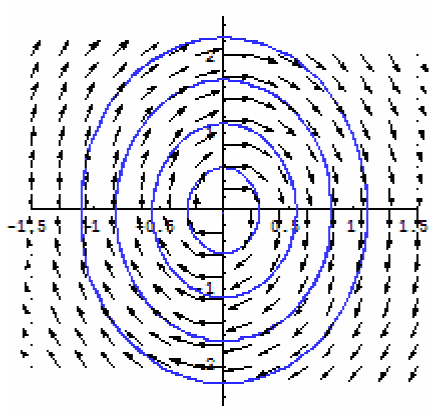
I the displacement held of l > ½ then released, what will its velocity be the first time?

5.9 $m = 1$, $k_d = 0$, and $k_s = 4 \Rightarrow \dot{v} = -4x$

a) Equilibrium points: $\frac{d^2x}{dt^2} = \ddot{x} = \dot{v} = -4x \Rightarrow \dot{x} = v$

$$\Rightarrow \begin{cases} \dot{x} = v \\ \dot{v} = -4x \end{cases} \Rightarrow \text{The origin point (0, 0) is the only equilibrium system.}$$

b) The phase portrait of the system:



c) Obviously, the solution is periodic from the phase portrait.

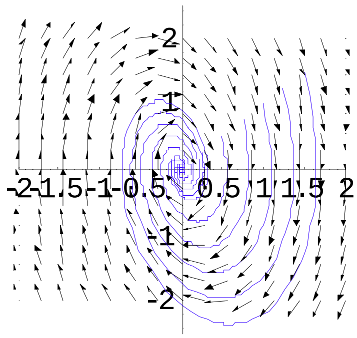
d) The frequency turns out to be $\frac{1}{\pi}$ but even with getting an exact solution.

e) $\frac{dv}{dt} = -4x = \frac{d^2x}{dt^2} \Rightarrow x(t) = c.\cos 2t. \Rightarrow (x, v) = (c.\cos 2t, -2c.\sin 2t)$

f) Since the origin point is the only equilibrium point. We can conclude that the building is vertical and stationary. From the phase portrait, we can conclude that the solutions trace out to closed curves in ellipse motion. Which we can determine that the building oscillates back and forth at some fixed frequency and constant amplitude (changes from different solutions).

g) $k_d = 1 \Rightarrow \ddot{x} = \dot{v} = -v - 4x \Rightarrow \begin{cases} \dot{x} = v \\ \dot{v} = -v - 4x \end{cases}$

$\Rightarrow$ The origin point (0, 0) is the only equilibrium.



From the phase portrait, we can determine that the solutions spiral around the origin. We can take the frequency of a typical solution to be the reciprocal of the amount of time elapsed between successive maxima or minima of x(t). The frequency $= \frac{1}{\pi}$.

5.10 **a)** $m = 1$, $k_d = 0$, $k_s = 4$, and $c = 3 \Rightarrow \dot{v} = -4x + 3x^3 = 0 \Rightarrow x = 0$ or $\pm\frac{2\sqrt{3}}{3}$

The equilibrium points are (0, 0), $(\frac{2\sqrt{3}}{3}, 0)$, and $(-\frac{2\sqrt{3}}{3}, 0)$

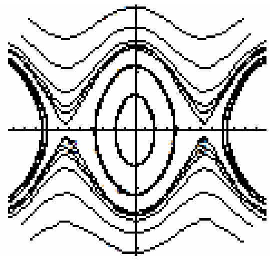
b) $\begin{cases} \dot{x} = v \\ \dot{v} = 3x^3 - 4x \end{cases} \Rightarrow D = \begin{bmatrix} 0 & 1 \\ 9x^2 - 4 & 0 \end{bmatrix}$

<u>At the point</u> (0, 0): $A = D_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 4 & \lambda \end{vmatrix} = \lambda^2 + 4 = 0 \Rightarrow \lambda_{1,2} = 2i.$

The system is non-hyperbolic at the point (0, 0). It is a periodic, which corresponds to the building is vertical and stationary.

<u>At the points</u> $(\pm\frac{2\sqrt{3}}{3}, 0)$: $A = D_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 8 & 0 \end{bmatrix} \Rightarrow \begin{vmatrix} \lambda & -1 \\ -8 & \lambda \end{vmatrix} = \lambda^2 - 8 = 0 \Rightarrow \lambda_{1,2} = \pm 2\sqrt{2}$

The system is unstable at these two points. The building is stationary but tipped a certain amount so that the gravitational force exactly balances the restoring force attempting to return the building to vertical (oscillated).



In the term of the phase plane this means that there are solutions which start arbitrarily close to the equilibrium points and head away (infinity) as t increases.

c) For $t \to \infty$, there are two solutions:

i. Any solution starts inside the interior of the region spirals towards (0,0).

ii. Any solution outside the region heads to ∞.

d) When k_d is increased. The equilibrium points occur when v = 0, therefore there is no effect to the equilibrium points. However, the only changes will occur when k_d is increased is the inside region increase too but not the behavior of the system in general.

5.11 $\ddot{x} - (1 - x^2 - \dot{x}^2)\dot{x} + x = 0$

$\Rightarrow$ Assume: $y = \dot{x} \Rightarrow \dot{y} = \ddot{x} = (1 - x^2 - \dot{x}^2)\dot{x} - x = (1 - x^2 - y^2)y - x$

We can rewrite the system in the form:

$$\begin{cases} \dot{x} = y \\ \dot{y} = (1 - x^2 - y^2)y - x \end{cases}$$

a) The origin point (0, 0) is the only critical point for the system, and it is an unstable.

b) If we assume that $(1 - x^2 - \dot{x}^2) = 0$, then $\ddot{x} + x = 0$

$\Rightarrow$ $x = \cos\theta$ and $\dot{x} = \sin\theta$ is a periodic solution.

Transferring the system to the polar coordinates by letting: $x = r\cos\theta \Rightarrow y = \dot{x} = r\sin\theta$)

$\Rightarrow r\dot{r} = x\dot{x} + y\dot{y} = r\cos\theta(r\sin\theta) + r\sin\theta[(1 - r^2\cos^2\theta - r^2\sin^2\theta)\, r\sin\theta - r\cos\theta]$

$\Rightarrow r\dot{r} = r^2\cos\theta\sin\theta + (1 - r^2)\, r^2\sin^2\theta - r^2\cos\theta\sin\theta = (1 - r^2)\, r^2\sin^2\theta$

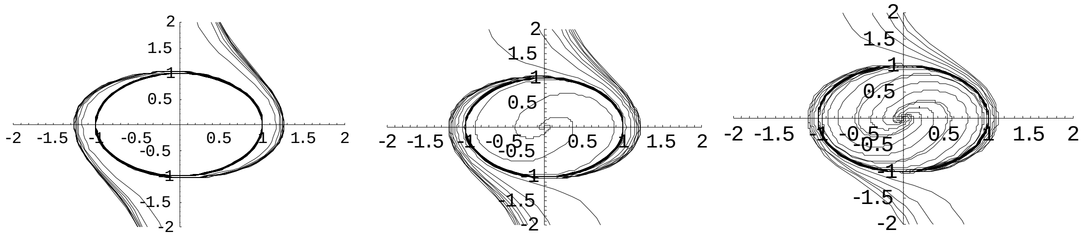
$\Rightarrow \dot{r} = (1 - r^2)\, r\sin^2\theta$

And $\dot{\theta} = -1 + (1 - r^2)\sin\theta\cos\theta$

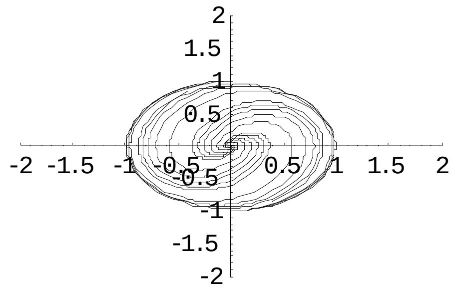
b) $\lim_{t\to\infty} r(t) = 1$; therefore r is near 1 as $\theta(t)$ is monotonic. From part (a), the system has a periodic solution explicitly. Assume that: $x = \cos t + y \Rightarrow \dot{x} = \sin t + \dot{y}$

$\Rightarrow \ddot{y} + 2\sin^2 t\, \dot{y} - 2\sin t\cos t\, y = \ldots$ (nonlinear terms)

We can find one characteristic exponent zero and the other is (-1). So we have stability.



initial points: 1 .5 .1



Initial point 0.001

Chapter 6

6.1 $I'' + 4I' + 5I = 5\cos 2t$

Let assume: $I = y$, then $x = y' \Rightarrow x' = y''$;

Therefore the system equation can be rewritten as follow:

$$\Rightarrow \begin{cases} \dot{x} = -4x - 5y + 5\cos 2t \\ \dot{y} = x \end{cases}$$

The equilibrium point(s) can be determined as: $\begin{cases} -5y + 5\cos 2t = 0 \\ x = 0 \end{cases}$

$\Rightarrow$ (0, cos2t) is the equilibrium point, which is periodic of period $n\pi$.

The Jacobian of the system is: $DF = \begin{pmatrix} -4 & -5 \\ 1 & 0 \end{pmatrix}$

At the equilibrium point (0, cos2t), the Jacobian can be rewritten as: $DF|_{(0,\cos 2t)} = \begin{pmatrix} -4 & -5 \\ 1 & 0 \end{pmatrix}$

The eigenvalues for the system are:

$|\lambda I - DF|_{(0,\cos 2t)} = \begin{vmatrix} \lambda + 4 & 5 \\ -1 & \lambda \end{vmatrix} = \lambda(\lambda + 4) + 5 = \lambda^2 + 4\lambda + 5 = 0$

$\Rightarrow \lambda_{1,2} = -2 \pm i \Rightarrow$ Therefore, we can not determine the behavior of the system.

$$\frac{dy}{dx} = \frac{x}{-4x - 5y + 5\cos 2t}$$

The solutions of the system: $I'' + 4I' + 5I = 0$ have the form:

$c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t$ (c_1, c_2 are real constants).

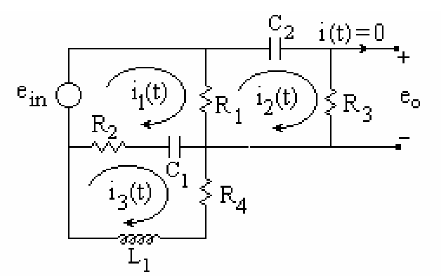
$5\cos 2t = Re(5e^{2it}) \Rightarrow y'' + 4y' + 5y = 5e^{2it} \Rightarrow y = \dfrac{5e^{2it}}{(2i)^2 + 4(2i) + 5} = \dfrac{5e^{2it}}{1+8i}$

So that $I_d(t) = Re(y) = \frac{1}{13}(\cos 2t + 8\sin 2t)$

I is a periodic forced oscillation. At any time t the current I(t) has the form:

$I(t) = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t + \frac{1}{13}(\cos 2t + 8\sin 2t)$

6.2



a) Translate this electrical network into differential equation.
b) Find the equilibrium points.
c) Sketch the phase portrait of the system and determine the dynamic behavior.

a) Loop 1: $E_{in} = R_1(i_1 - i_2) + R_2(i_1 - i_2) + \frac{1}{C_1} Q_1$ [1]

Loop 2: $R_1(i_2 - i_1) + R_3\, i_2 + \frac{1}{C_2} Q_2 = 0$ [2]

Loop 3: $R_2\,(i_3 - i_1) + \frac{1}{C_1} Q_1 + R_4\, i_3 + L_1 \frac{di_3}{dt} = 0$ [3]

Loop 4: $E_0 = R_3\, i_2$ [4]

$Q_1 = \int i_1 dt$ or $i_1 = \frac{dQ_1}{dt}$ and $Q_2 = \int i_2 dt$

Assume that $x = i_1$, $y = i_2$, and $z = i_3$.

$[1] \rightarrow (R_1 + R_2)\,\dot{x} - (R_1 + R_2)\,\dot{y} + \frac{1}{C_1} x = 0$

$[2] \rightarrow -R_1 \frac{di_1}{dt} + (R_1 + R_3) \frac{di_2}{dt} + \frac{1}{C_2} i_2 = 0 \Rightarrow -R_1\,\dot{x} + (R_1 + R_3)\,\dot{y} + \frac{1}{C_2} y = 0$

$[3] \rightarrow -R_2 \frac{di_1}{dt} + \frac{1}{C_1} i_1 + (R_2 + R_4) \frac{di_3}{dt} + L_1 \frac{d^2 i_3}{dt^2} = 0 \Rightarrow -R_2\,\dot{x} + \frac{1}{C_1} x + (R_2 + R_4)\,\dot{z} + L_1\,\ddot{z} = 0$

$[4] \rightarrow E_0 = R_3\, y$

From [3] $\rightarrow \dfrac{I_3(s)}{I_1(s)} = \dfrac{R_2 + \frac{1}{sC_1}}{R_2 + R_4 + \frac{1}{sC_1} + sL_1} = G_6$

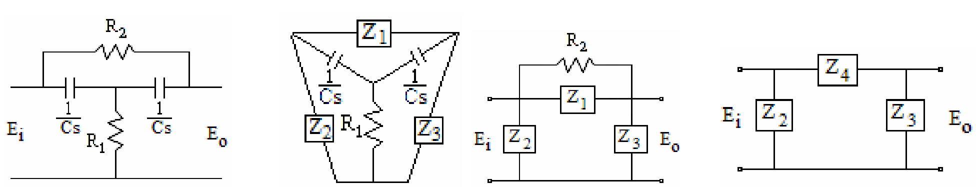
From [2] $\rightarrow \dfrac{I_2(s)}{I_1(s)} = \dfrac{R_1}{R_1 + R_3 + \frac{1}{sC_2}} = G_2$

From [1] $\rightarrow E_{in}(s) - (-R_1)I_2(s) + (R_2 + \frac{1}{sC_1})I_3(s) = (R_1 + R_2 + \frac{1}{sC_1})I_1(s)$

$\Rightarrow G_4 = R_1,\ G_5 = R_2 + \dfrac{1}{sC_1},\ G_1 = \dfrac{1}{R_1 + R_2 + \frac{1}{s\,C_1}}$ $\quad \dfrac{E_o(s)}{E_i(s)} = \dfrac{G_1 G_2 G_3}{1 - G_5 G_6 G_1 - G_1 G_2 G_4}$

6.3 Consider the bridged-T RC network is

a) Translate the bridged-T (Fig.6.15-i) to a differential equation.
b) Find the equilibrium points.
c) Sketch the phase portrait of the system
d) Determine the dynamic behavior.
e) Repeat (a thru d) for Fig.6.15-ii.



$Z_1 = \dfrac{\frac{1}{Cs}\frac{1}{Cs} + \frac{1}{Cs}R_1 + \frac{1}{Cs}R_1}{R_1} = \dfrac{1 + 2R_1 Cs}{R_1 C^2 s^2}$, and $\quad Z_2 = Z_3 = \dfrac{\frac{1}{Cs}\frac{1}{Cs} + \frac{1}{Cs}R_1 + \frac{1}{Cs}R_1}{\frac{1}{Cs}} = \dfrac{1 + 2R_1 Cs}{Cs}$

$Z_4 = Z_1 \,//\, R_2 \Rightarrow Z_4 = \dfrac{R_2 + 2R_1 R_2 Cs}{1 + 2R_1 Cs + R_1 R_2 C^2 s^2}$

$$\rightarrow \quad \frac{E_0(s)}{E_i(s)} = \frac{Z_3}{Z_3 + Z_4} = \frac{1+2R_1Cs}{Cs} \div \left(\frac{1+2R_1Cs}{Cs} + \frac{R_2+2R_1R_2Cs}{1+2R_1Cs+R_1R_2C^2s^2} \right)$$

$$= \frac{1+2R_1Cs}{CS} \cdot \frac{Cs(1+2R_1Cs+R_1R_2C^2s^2)}{(1+2R_1Cs)(1+2R_1Cs+R_1R_2C^2s^2+R_2Cs)}$$

$$= \frac{1 + 2R_1Cs + R_1R_2C^2s^2}{1 + (2R_1 + R_2)Cs + R_1R_2C^2s^2}$$

The equation for the zeros is:

$$\frac{E_0(s)}{E_i(s)} = \frac{1 - 2RC_2s + R^2C_1C_2s^2}{1 + R(C_1 + 2C_2)s + R^2C_1C_2s^2}$$

6.4 **a)** $\ddot{x} - (1 - x^2 - \dot{x}^2)\dot{x} + x = 0$

Let $\dot{x} = y \Rightarrow \ddot{x} = \dot{y}$

$$\Rightarrow \begin{cases} \dot{x} = y = 0 \\ \dot{y} = (1 - x^2 - y^2)y - x = 0 \end{cases} \Rightarrow \text{The origin point is the only critical point.}$$

$$DF = \begin{bmatrix} 0 & 1 \\ -2xy - 1 & 1 - x^2 - 3y^2 \end{bmatrix} \Rightarrow A = DF|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

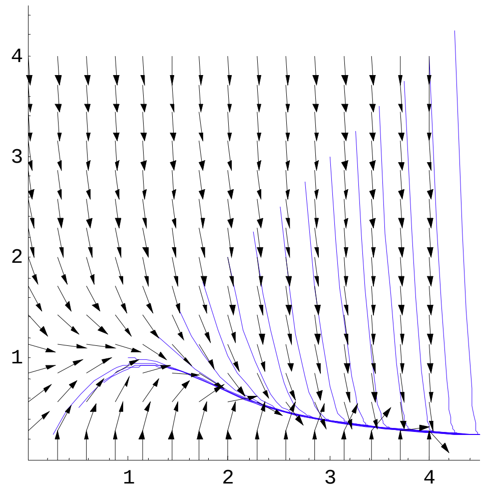
$$\Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda - 1 \end{vmatrix} = \lambda^2 - \lambda + 1 = 0 \Rightarrow \lambda = \frac{1 \pm i\sqrt{3}}{2} \Rightarrow \text{the system is unstable}$$

b) If $\dot{x} = 0$, then $\ddot{x} = -x \Rightarrow$ The system is an harmonic oscillator equation.
$\Rightarrow x(t) = \cos t \Rightarrow \dot{x}(t) = -\sin t \Rightarrow \ddot{x} = -x \Rightarrow$ the system is a periodic solution.

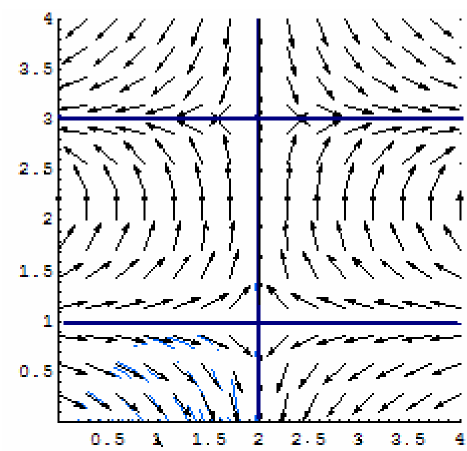
Transforming to polar coordinates system: $x = r\cos\theta$ and $\dot{x} = -r\sin\theta$
So that: $r\dot{r} = x\dot{x} + y\dot{y} = xy + y((1 - r^2)y - x) = xy + (1 - r^2)y^2 - xy = (1 - r^2)r^2\sin^2\theta$
$\Rightarrow \dot{r} = (1 - r^2)r\sin^2\theta$
and $\dot{\theta} = -1 + (1 - r^2)\sin\theta\cos\theta$

c) The stability and the limit cycle of the periodic solution can obtained in 2 ways:

1. r is near 1 and $\theta(t)$ is monotonic $\Rightarrow \lim\limits_{t\to\infty} r(t) = 1$.
2. Since the periodic solution is explicit, then assume that $x = \cos t + y \Rightarrow \dot{x} = -\sin t + \dot{y}$
   $\Rightarrow \ddot{y} + 2\sin^2 t\,\dot{y} - 2\sin t\cos t\, y$ (non-linear terms)
   One characteristic exponent is zero and the other -1. $\Rightarrow$ The system is stable.



6.5 **a)** $\begin{cases} \dot{x} = xy - 2x - 2y + 4 \\ \dot{y} = -y^2 + 4y - 3 \end{cases} \Rightarrow \begin{cases} \dot{x} = (x-2)(y-2) = 0 \\ \dot{y} = -(y-1)(y-3) = 0 \end{cases} \Rightarrow \begin{cases} y = 2, x = 2 \\ y = 1, y = 3 \end{cases} \Rightarrow$ (2, 1) and (2, 3)



b) The chemist will live, because as is shown in the portrait that the line x = 2 is vertical line and asymptote (null cline) any solution starting in the region with x < 2 must remain in that region and will not hit 3.

6.6 Assume: t: Time in minutes
V(t): Total volume in the pot (oz)
a(t) = Total amount of salt in the pot (oz)
c(t) = Concentration of salt in the pot

The concentration and the amount are given by the relation: $c(t) = \frac{a(t)}{V(t)}$

Since the rate at 2 oz/min and the out at 1 oz/min $\Rightarrow$ V(t) = 3 + (2-1)t = 3 + t
We obtain the differential equation for a(t) by considering the rate of flow in & out. There is no salt flowing in, so:

$\dot{a}$ (t) = - rate out = -1 (oz/min) * t = - $\frac{a(t)}{V(t)}$ $\Rightarrow$ $\dot{a}$ (t) = = - $\frac{a(t)}{3+t}$

p(t) = $e^{\int \frac{dt}{3+t}}$ = 3 + t

Hence, (a(t) (3+t))′ = 0 ⇒ a(t). (3 + t) = c 0 ⇒ a(t) = $\frac{c}{3+t}$

The initial condition is c(0) = 0.04, V(0) = 3 ⇒ a(0) = 0.12 = $\frac{c}{3}$ ⇒ c = 0.36

The solution is: a(t) = $\frac{.36}{3+t}$

The value of t for which c(t) = $\frac{a(t)}{V(t)}$ = $\frac{.36}{(3+t)^2}$ = 0.04 ⇒ $(3 + t)^2$ = 9 ⇒ t = 3

Chapter 7

7.1 **a)** The equilibrium points:

$$\begin{cases} \dot{x} = 60x - 4x^2 - 3xy = 0 \\ \dot{y} = 42y - 2y^2 - 3xy = 0 \end{cases} \Rightarrow \begin{cases} x(60 - 4x - 3y) = 0 \quad (1) \\ y(42 - 2y - 3x) = 0 \quad (2) \end{cases}$$

From (1): If x = 0 $\Rightarrow$ (2) $\Rightarrow$ y(42 – 2y) = 0 $\Rightarrow \begin{cases} y = 0 \\ y = 21 \end{cases} \Rightarrow$ (0, 0), (0, 21)

From (2): If y = 0 $\Rightarrow$ (1) $\Rightarrow$ x(60 – 4x) = 0 $\Rightarrow \begin{cases} x = 0 \\ x = 15 \end{cases} \Rightarrow$ (0,0), (15, 0)

From (1) and (2): $\begin{cases} 60 - 4x - 3y = 0 \\ 42 - 2y - 3x = 0 \end{cases} \Rightarrow \begin{cases} 4x + 3y = 60 \\ 3x + 2y = 42 \end{cases} \Rightarrow$ x = 6 and y = 12

Then the equilibrium points are: (0, 0), (15, 0), (0, 21) and (6, 12).

The Jacobian: DF = $\begin{pmatrix} 60 - 8x - 3y & -3x \\ -3y & 42 - 4y - 3x \end{pmatrix}$

The behavior of the system is as follow:

i- At the point (0, 0) $\Rightarrow$ $DF|_{(0,0)} = \begin{pmatrix} 60 & 0 \\ 0 & 42 \end{pmatrix}$

$\Rightarrow |\lambda I - DF|_{(0,0)} = \begin{vmatrix} \lambda - 60 & 0 \\ 0 & \lambda - 42 \end{vmatrix} = (\lambda - 60)(\lambda - 42) = 0 \Rightarrow \lambda_1 = 60 > 0$ and $\lambda_2 = 42 > 0$

The behavior of this system at the point (0, 0) is a source (unstable).

ii- At the point (0, 21) $\Rightarrow$ $DF|_{(0,21)} = \begin{pmatrix} -3 & 0 \\ -63 & -42 \end{pmatrix}$

$\Rightarrow |\lambda I - DF|_{(0,21)} = \begin{vmatrix} \lambda + 3 & 0 \\ 63 & \lambda = 42 \end{vmatrix} = (\lambda + 3)(\lambda + 42) = 0 \Rightarrow \lambda_1 = -3 < 0$ & $\lambda_2 = -42 < 0$

The behavior of this system at the point (0, 21) is a sink (stable).

iii- At the point (15, 0) $\Rightarrow$ $DF|_{(15,0)} = \begin{pmatrix} -60 & -45 \\ 0 & -3 \end{pmatrix}$

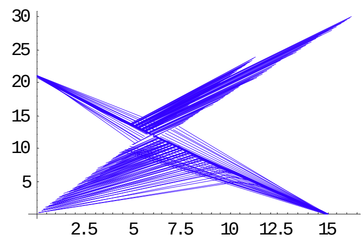
$\Rightarrow |\lambda I - DF|_{(15,0)} = \begin{vmatrix} \lambda + 60 & 45 \\ 0 & \lambda + 3 \end{vmatrix} = (\lambda + 60)(\lambda + 3) = 0 \Rightarrow \lambda_1 = -60 < 0$ & $\lambda_2 = -3 < 0$

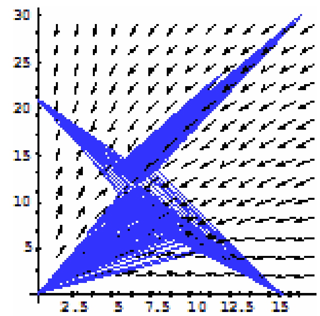
The behavior of this system at the point (15, 0) is a sink (stable).

iv- At the point (6, 12) $\Rightarrow$ $DF|_{(6,12)} = \begin{pmatrix} -24 & -18 \\ -36 & -24 \end{pmatrix}$

$\Rightarrow |\lambda I - DF|_{(6,12)} = \begin{vmatrix} \lambda + 24 & 18 \\ 36 & \lambda + 24 \end{vmatrix} = (\lambda + 24)^2 - 18(36) = 0$

$\Rightarrow \lambda_{1,2} = -24 \pm 8\sqrt{2} \Rightarrow$ The behavior of this system at the point (6, 12) is a saddle.





b) The equilibrium points:

$$\begin{cases} \dot{x}=60x-3x^2-4xy=0 \\ \dot{y}=42y-3y^2-2xy=0 \end{cases} \Rightarrow \begin{cases} x(60-3x-4y)=0 & (1) \\ y(42-3y-2x)=0 & (2) \end{cases}$$

From (1): If $x = 0 \Rightarrow (2) \Rightarrow y(42 - 3y) = 0 \Rightarrow \begin{cases} y=0 \\ y=14 \end{cases} \Rightarrow (0, 0), (0, 14)$

From (2): If $y = 0 \Rightarrow (1) \Rightarrow x(60 - 3x) = 0 \Rightarrow \begin{cases} x=0 \\ x=20 \end{cases} \Rightarrow (0,0), (20, 0)$

From (1) and (2): $\begin{cases} 60-3x-4y=0 \\ 42-3y-2x=0 \end{cases} \Rightarrow \begin{cases} 3x+4y=60 \\ 2x+3y=42 \end{cases} \Rightarrow x = 12$ and $y = 6$

Then the equilibrium points are: (0, 0), (20, 0), (0, 14) and (12, 6).

The Jacobian: $DF = \begin{pmatrix} 60-6x-4y & -4x \\ -2y & 42-6y-2x \end{pmatrix}$

The behavior of the system is as follow:

i- At the point (0, 0) $\Rightarrow DF|_{(0,0)} = \begin{pmatrix} 60 & 0 \\ 0 & 42 \end{pmatrix}$

$\Rightarrow |\lambda I - DF|_{(0,0)} = \begin{vmatrix} \lambda-60 & 0 \\ 0 & \lambda-42 \end{vmatrix} = (\lambda - 60)(\lambda - 42) = 0 \Rightarrow \lambda_1 = 60 > 0$ and $\lambda_2 = 42 > 0$

The behavior of this system at the point (0, 0) is a source (unstable).

ii- At the point (0, 14) $\Rightarrow DF|_{(0,14)} = \begin{pmatrix} 4 & 0 \\ -28 & -42 \end{pmatrix}$

$\Rightarrow |\lambda I - DF|_{(0,14)} = \begin{vmatrix} \lambda-4 & 0 \\ 28 & \lambda+42 \end{vmatrix} = (\lambda - 4)(\lambda + 42) = 0 \Rightarrow \lambda_1 = 4 > 0$ & $\lambda_2 = - 42 < 0$

The behavior of this system at the point (0, 14) is a saddle.

iii- At the point (20, 0) $\Rightarrow DF|_{(20,0)} = \begin{pmatrix} -60 & -80 \\ 0 & 2 \end{pmatrix}$

$\Rightarrow |\lambda I - DF|_{(20,0)} = \begin{vmatrix} \lambda+60 & 80 \\ 0 & \lambda-2 \end{vmatrix} = (\lambda + 60)(\lambda - 2) = 0 \Rightarrow \lambda_1 = -60 < 0$ & $\lambda_2 = 2 > 0$

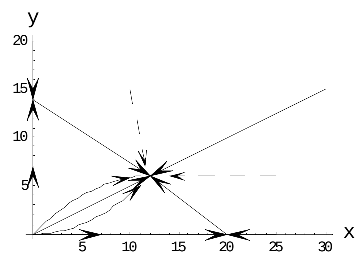
The behavior of this system at the point (20, 0) is a saddle.

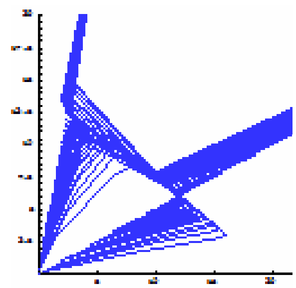
iv- At the point (12, 6) $\Rightarrow$ $DF|_{(6,12)} = \begin{pmatrix} -36 & -48 \\ -12 & -18 \end{pmatrix}$

$\Rightarrow |\lambda I - DF|_{(12,6)} = \begin{vmatrix} \lambda+36 & 48 \\ 12 & \lambda+18 \end{vmatrix} = (\lambda + 36)(\lambda + 18) - 576 = 0$

$\Rightarrow \lambda_{1,2} = -27 \pm 3\sqrt{27} < 0$

$\Rightarrow$ The behavior of this system at the point (6, 12) is a sink stable.





7.2

a) $\begin{cases} \dot{x} = (2 - 2x - y)x = 0 \\ \dot{y} = (2 - x - 2y)y = 0 \end{cases} \Rightarrow \begin{cases} x=0,\ y = 0 \\ 2\text{-}2x\text{-}y = 0 \\ 2\text{-}x\text{-}2y = 0 \end{cases}$

$\Rightarrow \begin{cases} x = 0 \to 2 - x - 2y = 2 - 2y = 0 \Rightarrow y = 1 \\ y = 0 \to 2 - 2x - y = 2 - 2x = 0 \Rightarrow x = 1 \end{cases}$

$\Rightarrow \begin{cases} 2 - 2x - y = 0 \\ 2 - x - 2y = 0 \end{cases} \Rightarrow \left(\frac{2}{3},\ \frac{2}{3}\right)$

The equilibrium points are: (0, 0), (0, 1), (1, 0), and $\left(\frac{2}{3},\ \frac{2}{3}\right)$

$DF = \begin{bmatrix} 2 - 4x - y & -x \\ -y & 2 - x - 4y \end{bmatrix}$

<u>At the point (0, 0)</u>: $A = DF|_{(0,0)} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda-2 & 0 \\ 0 & \lambda-2 \end{vmatrix} = (\lambda - 2)^2 = 0 \Rightarrow \lambda_1\ \&\ \lambda_2 = 2 > 0$

$\Rightarrow$ The behavior at the point (0, 0) is a source (unstable).

<u>At the point (1, 0)</u>: $A = DF|_{(1,0)} = \begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix} \Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda+2 & 1 \\ 0 & \lambda-1 \end{vmatrix} = (\lambda + 2)(\lambda - 1) = 0$

$\Rightarrow \lambda_1 = -2\ \&\ \lambda_2 = 1$ $\Rightarrow$ The behavior at the point (1, 0) is a saddle.

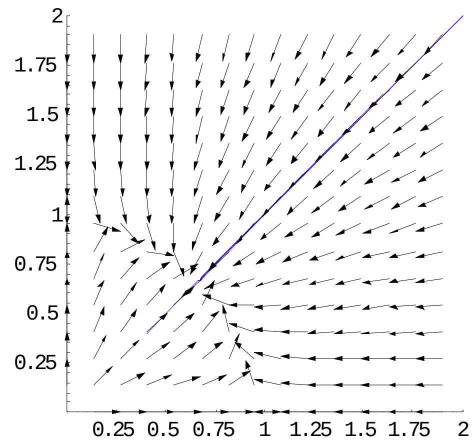
<u>At the point (0, 1)</u>: $A = DF|_{(0,1)} = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix} \Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda-1 & 0 \\ 1 & \lambda+2 \end{vmatrix} = (\lambda + 2)(\lambda - 1) = 0$

$\Rightarrow \lambda_1 = -2\ \&\ \lambda_2 = 1$ $\Rightarrow$ The behavior at the point (0, 1) is a saddle.

<u>At the point $\left(\frac{2}{3},\ \frac{2}{3}\right)$</u>: $A = DF| = \begin{bmatrix} -\frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{4}{3} \end{bmatrix} \Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda+\frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \lambda+\frac{4}{3} \end{vmatrix} = \left(\lambda + \frac{4}{3}\right)^2 - \frac{4}{9} = 0$

$\Rightarrow \lambda_1 = -2$ & $\lambda_2 = -\frac{2}{3}$ $\Rightarrow$ The behavior at the point $\left(\frac{2}{3}, \frac{2}{3}\right)$ is a sink stable.

b)



The species of this system is tend towards the stable, attracting equilibrium point $\left(\frac{2}{3}, \frac{2}{3}\right)$.

7.3 $\begin{cases} \dot{x} = (2 - x - 2y)x \\ \dot{y} = (2 - 2x - y)y \end{cases}$ $\Rightarrow$ The equilibrium points are: (0, 0), (0, 2), (2, 0), and $\left(\frac{2}{3}, \frac{2}{3}\right)$

$$DF = \begin{bmatrix} 2-2x-2y & -2x \\ -2y & 2-2x-2y \end{bmatrix}$$

<u>At the point (0, 0)</u>: $A = DF|_{(0,0)} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda-2 & 0 \\ 0 & \lambda-2 \end{vmatrix} = (\lambda - 2)^2 = 0 \Rightarrow \lambda_1$ & $\lambda_2 = 2 > 0$

$\Rightarrow$ The behavior at the point (0, 0) is a source (unstable).

<u>At the point (2, 0)</u>: $A = DF|_{(2,0)} = \begin{bmatrix} -2 & -4 \\ 0 & -2 \end{bmatrix} \Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda+2 & 4 \\ 0 & \lambda+2 \end{vmatrix} = (\lambda + 2)^2 = 0$

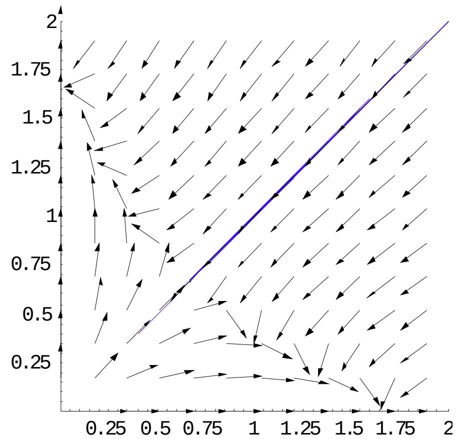
$\Rightarrow \lambda_{1,2} = -2$ $\Rightarrow$ The behavior at the point (2, 0) is a sink stable.

<u>At the point (0, 2)</u>: $A = DF|_{(0,2)} = \begin{bmatrix} -2 & 0 \\ -4 & -2 \end{bmatrix} \Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda+2 & 0 \\ 4 & \lambda+2 \end{vmatrix} = (\lambda + 2)^2 = 0$

$\Rightarrow \lambda_{1,2} = -2$ $\Rightarrow$ The behavior at the point (0, 2) is a sink stable.

<u>At the point $\left(\frac{2}{3}, \frac{2}{3}\right)$</u>: $A = DF| = \begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} \\ -\frac{4}{3} & -\frac{2}{3} \end{bmatrix} \Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda+\frac{2}{3} & \frac{4}{3} \\ \frac{4}{3} & \lambda+\frac{2}{3} \end{vmatrix} = \left(\lambda + \frac{2}{3}\right)^2 - \frac{16}{9} = 0$

$\Rightarrow \lambda_1 = -2$ & $\lambda_2 = \frac{2}{3}$ $\Rightarrow$ The behavior at the point $\left(\frac{2}{3}, \frac{2}{3}\right)$ is a saddle.



The line y = x divides the system into 2-orbits (regions) that are symmetries and each one of the species tends towards its equilibrium point of the y = x, but the other dies out if it passes the other region.

7.4 **a)** $\begin{cases}(a+by)x = 0\\(-c+dx-ey)y = 0\end{cases} \Rightarrow$ The equilibrium points are: (0, 0), $\left(0,-\frac{c}{e}\right)$, and $\left(\frac{bc-ae}{bd},-\frac{a}{b}\right)$

$$DF = \begin{bmatrix} a+by & bx \\ dy & -c+dx-2ey \end{bmatrix}$$

<u>At the point (0, 0)</u>: $A = DF|_{(0,0)} = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix} \Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda-a & 0 \\ 0 & \lambda+c \end{vmatrix} = (\lambda - a)(\lambda + c) = 0$

$\Rightarrow \lambda_1 = a > 0$ & $\lambda_2 = -c < 0 \Rightarrow$ The system behavior at the point (0, 0) is a saddle.

<u>At the point $\left(0,-\frac{c}{e}\right)$</u>: $A = \begin{bmatrix} a-\frac{bc}{e} & 0 \\ -\frac{cd}{e} & c \end{bmatrix} \Rightarrow |\lambda I - A| = \left(\lambda - a + \frac{bc}{e}\right)(\lambda - c) = 0$

$\Rightarrow \lambda_1 = a - \frac{bc}{e}$ & $\lambda_2 = c > 0$

<u>At the point $\left(\frac{bc-ae}{bd},-\frac{a}{b}\right)$</u>: $A = \begin{bmatrix} 0 & \frac{bc-ae}{d} \\ -\frac{ad}{b} & \frac{ae}{b} \end{bmatrix} \Rightarrow |\lambda I - A| = \lambda\left(\lambda - \frac{ae}{b}\right) + \frac{abc-a^2e}{b} = 0$

b) $\begin{cases}(-1+y)x = 0\\(1-x)y = 0\end{cases} \Rightarrow$ The equilibrium points are: (0, 0), (1, 1)

$$DF = \begin{bmatrix} -1+y & x \\ -y & 1-x \end{bmatrix}$$

<u>At the point (0, 0)</u>: $A = DF|_{(0,0)} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow |\lambda I - A| = (\lambda - 1)(\lambda + 1) = 0$

$\Rightarrow \lambda_1 = 1 > 0$ & $\lambda_2 = -1 < 0 \Rightarrow$ The system behavior at the point (0, 0) is a saddle.

<u>At the point (1, 1)</u>: $A = DF|_{(1,1)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow |\lambda I - A| = \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm\, i$.

7.5 **a)** $P' = \frac{1}{2} P\left(1 - \frac{P}{10}\right) - \frac{AP}{1+P} = 0 \Rightarrow \frac{p(-p^2+9p+10-20A)}{20(1+p)} = 0$

$p(-p^2 + 9p + 10 - 20A) = 0 \Rightarrow$ the equilibrium points are: $p = 0$, $\frac{9 \pm \sqrt{121-80A}}{2}$

If $121 - 80A < 0 \Rightarrow A > \frac{121}{80} \approx 1.5125$, the system is non-hyperbolic

b) <u>A = 0.3</u> $\Rightarrow$ the equilibrium points are: $p = 0$, $p = \frac{9 \pm \sqrt{97}}{2} = \begin{cases} -0.39 \\ 9.39 \end{cases}$

at the equilibrium point p = 0, the system is unstable and it is stable when p = 9.39

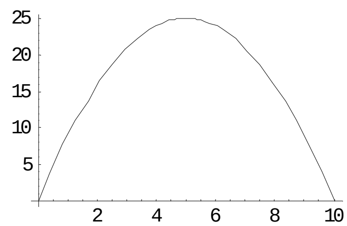
c) $A = 0.9 \Rightarrow$ the equilibrium points are: $p = 0$, $p = \frac{9 \pm \sqrt{49}}{2} = \begin{cases} 1 \\ 8 \end{cases}$

d) $A = 1.6 \Rightarrow$ the equilibrium points are: $p = 0$, $p = \frac{9 \pm \sqrt{-7}}{2} = \frac{9 \pm i\sqrt{7}}{2}$

7.6 **a)** $\frac{dP}{dt} = P(10 - P) - C = 0 \Rightarrow P^2 - 10P + C = 0$

$\Rightarrow P = 5 \pm \sqrt{25 - C}$ are the equilibrium points.

b) i. For case no fishing, $C = 0 \Rightarrow$ The equilibrium points are P = 0 & 10



ii. For case $C = 16 \Rightarrow$ The equilibrium points are P = 2 & 8

c) Given $P(t_0) = 1$, then the equilibrium points lies between P_1 & P_2.
From part (b), we have $P_1 = 0, 2$ and $P_2 = 8, 10$ (> 0), then we give us that $P_2 > 5$ always.
However, $P_1 = 5 - \sqrt{25-C} < 1 \Rightarrow \sqrt{25-C} > 4 \Rightarrow 25 - C > 16 \Rightarrow C < 9$.

7.7 **a)** $P' = A(P).P = 0 \Rightarrow P = 0$ or $A(P) = 0$.
Since A(P) is strictly decreasing function of P, A(0) = 0, and $A(b) = 0 \Rightarrow P = b$.

Therefore, the equilibrium points are at P = 0 or b.

b) A(0) = 0 and positive, A will be positive for small positive values of P, and will be negative for small negative values of P. Then the behavior of the function A(P) near:

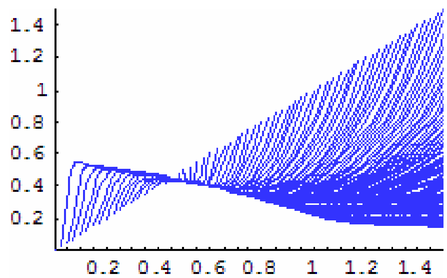
I. P = 0, the behavior will be unstable system.

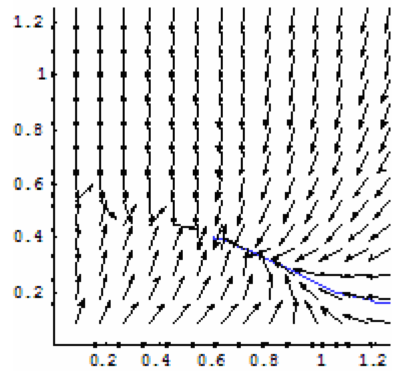
II. P =b, A(P) is positive for P < b, and A(P) is negative for P > b. Since P is positive number on either side of the equilibrium point. Then the system is stable.

c) The behavior of this system is the same as the a logistic equation, which is $(A(P) = r(1 - \frac{P}{r})$. This shows that the special form of A(P) in the logistic case was not crucial. We can conclude that the point 'b' is carrying the capacity for this population.

7.8 $\begin{cases} \dot{x} = (1-x-y)x \\ \dot{y} = (a-bx-cy)y \end{cases}$

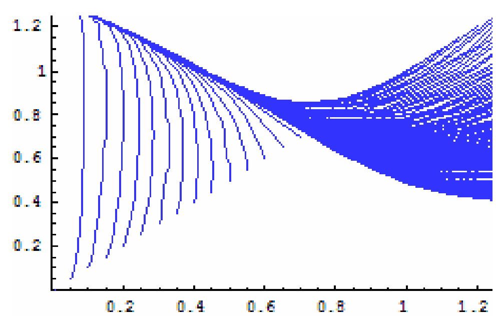
a) a = 4, b = 2, and c = 7 $\Rightarrow \begin{cases} (1-x-y)x \quad = 0 \\ (4-2x-7y)y = 0 \end{cases} \Rightarrow (0, 0), (\frac{3}{5}, \frac{2}{5}), (0, \frac{4}{7})$, and $(1, 0)$

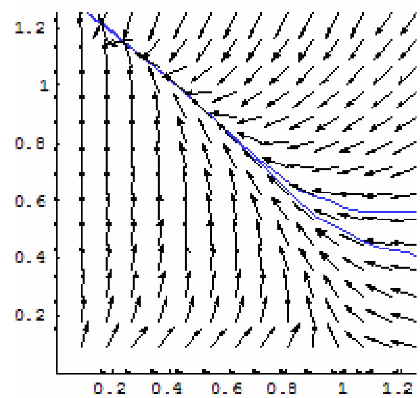




The behavior of the system approach the point $(\frac{3}{5}, \frac{2}{5})$, as $t \to \infty$

b) a = 4, b = 2, and c = 3 $\Rightarrow \begin{cases} \dot{x} = (1-x-y)x \\ \dot{y} = (4-2x-3y)y \end{cases} \Rightarrow (0, 0), (-1, 2), (0, \frac{4}{3})$, and $(1, 0)$





The behavior of the system approach the point $(0, \frac{4}{3})$, as $t \to \infty$ the population dies out over time.

7.9 **a)** $\begin{cases} \dot{R} = 2R - 1.5RF = 0 \\ \dot{F} = -F + .5RF = 0 \end{cases} \Rightarrow \begin{cases} R(2-1.5F) = 0 \\ F(-1+.5R) = 0 \end{cases} \Rightarrow (0, 0)$ and $(2, \frac{4}{3})$

The Jacobian: $DF = \begin{pmatrix} 2-1.5F & -1.5R \\ .5F & -1+.5R \end{pmatrix}$, where you can assume that R corresponds to x-axis and F to y.

The behavior of the system is as follow:

i- At the point $(0, 0) \Rightarrow DF|_{(0,0)} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$

$\Rightarrow |\lambda I - DF|_{(0,0)} = \begin{vmatrix} \lambda-2 & 0 \\ 0 & \lambda+1 \end{vmatrix} = (\lambda - 2)(\lambda + 1) = 0 \Rightarrow \lambda_1 = 2 > 0$ and $\lambda_2 = -1 < 0$

The behavior of this system at the point (0, 0) is a saddle.

ii- At the point $(2, \frac{4}{3}) \Rightarrow DF|_{(2, \frac{4}{3})} = \begin{pmatrix} 0 & -3 \\ \frac{2}{3} & 0 \end{pmatrix}$

$\Rightarrow |\lambda I - DF|_{(0,0)} = \lambda^2 + 2 = 0 \Rightarrow \lambda_1 = \pm 2i$ (imaginary number only $Re(\lambda) = 0$), which is a non-hyperbolic case. That means, we can not draw any conclusion (i.e. the behavior near the equilibrium point $(2, \frac{4}{3})$ is unknown). → The system is a stable (periodic) at the point $(2, \frac{4}{3})$.

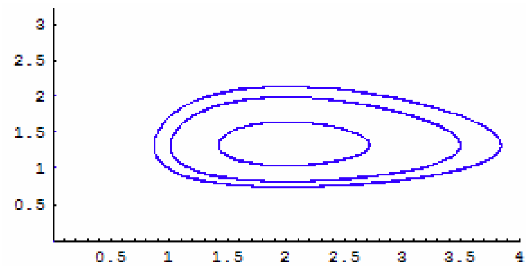
$\frac{dR}{dF} = \frac{R(2-1.5F)}{F(-1+.5R)} \Rightarrow \frac{(-1+.5R)}{R} dR = \frac{2-1.5F}{F} dF \Rightarrow (-\frac{1}{R} + \frac{1}{2}) dR = (\frac{2}{F} - \frac{3}{2}) dF$

$\Rightarrow -\ln R + \frac{1}{2} = 2\ln F - \frac{3}{2} + c \Rightarrow -2\ln F + \ln R = 2 + c$

$\Rightarrow \ln \frac{R}{F^2} = 2 + c \Rightarrow e^{\ln(R/F^2)} = e^{2+c}$

$\Rightarrow \frac{R}{F^2} = c_1 e^2.$

b)



c) Given: $R(t_0) = 2$ and $F(t_0) = 2$. From the phase portrait in part (b), the point (2, 2) at t_0 is inside the curve solution.

I. Since the point (2, 2) can not cross the curve by the uniqueness theorem. Therefore; the rabbit populations exceed 5 because R always is less than 5.

II. The rabbit population will never die, from the plot we can determine that the population is keeping going along the curve.

7.10 **a)** $\begin{cases} \dot{r} = 4r - 2f \\ \dot{w} = r + f \end{cases} \Rightarrow$ The origin point (0, 0) is the only equilibrium point.

b) $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \Rightarrow |\lambda - AI| = \begin{vmatrix} \lambda-4 & 2 \\ -1 & \lambda-1 \end{vmatrix} = \lambda^2 - 5\lambda + 6 = 0 \Rightarrow \lambda_{1,2} = 2, 3.$

For $\lambda_1 = 2 \Rightarrow (A - \lambda I)V_1 = \begin{pmatrix} 4-2 & -2 \\ 1 & 1-2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \Rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For $\lambda_2 = 3 \Rightarrow (A - \lambda I)V_2 = \begin{pmatrix} 4-3 & -2 \\ 1 & 1-3 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \Rightarrow V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Given that: $\begin{pmatrix} r(0) \\ f(0) \end{pmatrix} = \begin{pmatrix} 30 \\ 20 \end{pmatrix} = c_1 V_1 e^0 + c_2 V_2 e^0 = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} c_1 + 2c_2 = 30 \\ c_1 + c_2 = 20 \end{cases} \Rightarrow c_1 = c_2 = 10$

$x(t) = 10\begin{pmatrix}1\\1\end{pmatrix}e^{2t} + 10\begin{pmatrix}2\\1\end{pmatrix}e^{3t}.$

c) $\begin{cases} r(t) = 10(e^{2t} + 2e^{3t}) \\ f(t) = 10(e^{2t} + e^{3t}) \end{cases}$

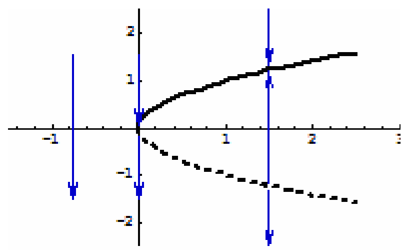
d) $\lim \frac{r(t)}{f(t)} = \lim \frac{e^{2t}+2e^{3t}}{e^{2t}+e^{3t}} = \lim \frac{2e^{3t}}{e^{3t}} = 2$

Chapter 8

8.1 **a)** $\dot{x} = \alpha - x^2 = 0 \Rightarrow x = \pm\sqrt{\alpha}$ are the equilibrium points (with $\alpha \geq 0$)

$DF = -2x \Rightarrow DF|_{\sqrt{\alpha}} = -2\sqrt{\alpha} \leq 0 \Rightarrow$ the system is stable.

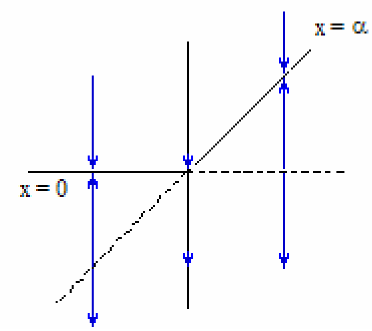
$\Rightarrow DF|_{-\sqrt{\alpha}} = 2\sqrt{\alpha} \geq 0 \Rightarrow$ the system is unstable.



b) $\dot{x} = \alpha x - x^2 \Rightarrow x = 0, \alpha$ are the equilibrium points.

$DF = -2x + \alpha \Rightarrow DF|_0 = \alpha \Rightarrow$ the system is dependent on the value of α.

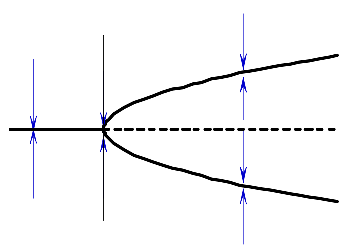
$\Rightarrow DF|_\alpha = -\alpha$



c) $\dot{x} = \alpha x - x^3 \Rightarrow x = 0, \pm\sqrt{\alpha}$ are the equilibrium points (with $\alpha \geq 0$)

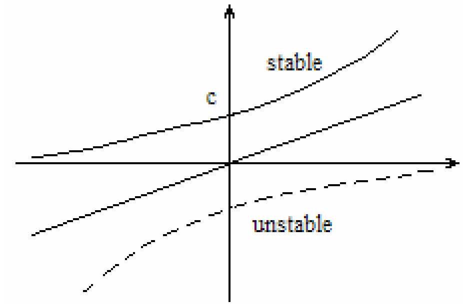
$DF = \alpha - 3x^2 \Rightarrow DF|_{\pm\sqrt{\alpha}} = -2\alpha \leq 0 \Rightarrow$ the system is stable.

$\Rightarrow DF|_0 = \alpha \geq 0 \Rightarrow$ the system is unstable.

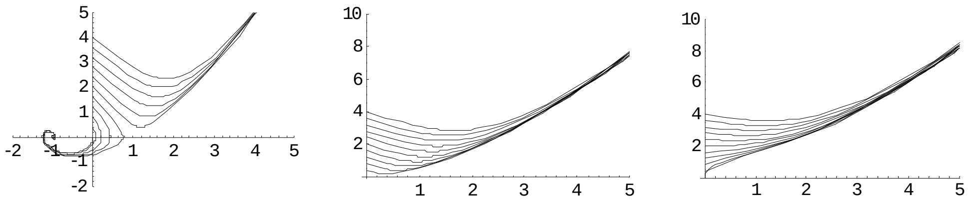


d) $\dot{x} = \alpha x - x^2 + c^2 \Rightarrow x = \frac{1}{2}\left(\alpha \pm \sqrt{\alpha^2 + 4c^2}\right) \Rightarrow \sqrt{\alpha^2 + 4c^2} \geq \alpha$

$DF = -2x + \alpha \Rightarrow DF = \pm\sqrt{\alpha^2 + 4c^2}$



8.2 **a)** $\begin{cases} \dot{x} = y & = 0 \\ \dot{y} = x^2 - y + \alpha = 0 \end{cases} \Rightarrow y = 0 \text{ and } x = \pm\sqrt{\alpha} \;\Rightarrow\; \alpha \geq 0.$

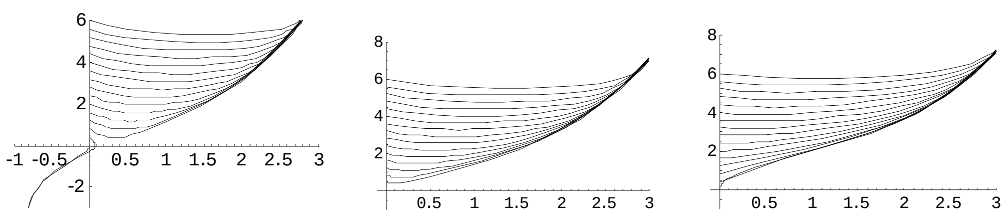


α = -1.0 0.0 2.0

b) $\begin{cases} \dot{x} = y - x^3 + xy = 0 \\ \dot{y} = 4x - y - \alpha \ = 0 \end{cases} \Rightarrow x^3 - 4x^2 + (\alpha - 4)x - \alpha = 0$

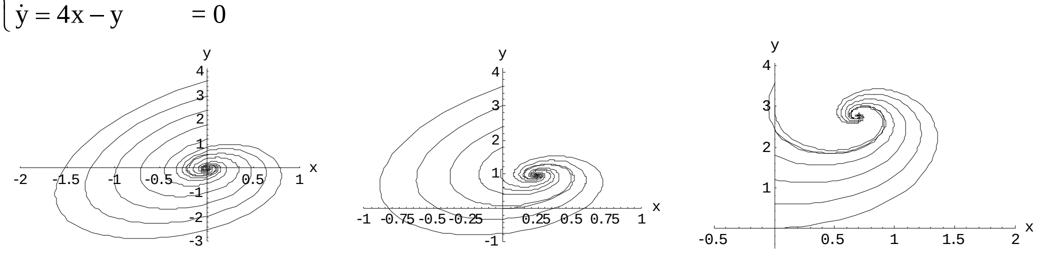
If $\alpha = 0$, then the equilibrium points are (0, 0), $(-2\pm 2\sqrt{2}, -8\pm 8\sqrt{2})$

$$DF = \begin{pmatrix} -3x^2 + y & x+1 \\ 4 & -1 \end{pmatrix}$$



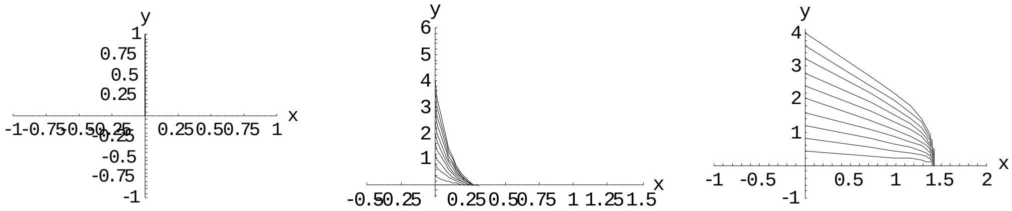
α = -1.0 0.0 2.0

c) $\begin{cases} \dot{x} = \cos x - y + \alpha = 0 \\ \dot{y} = 4x - y \quad = 0 \end{cases}$



α = -1.0 0.0 2.0

d) $\begin{cases} \dot{x} = \alpha - x^2 = 0 \\ \dot{y} = -y \quad\quad = 0 \end{cases} \Rightarrow$ y = 0 and x = $\pm\sqrt{\alpha} \;\; \Rightarrow \alpha \geq 0$.



$\alpha = 0.0$ 0.1 2.0

8.3 a) $\begin{cases} \dot{x} = y + xy^2 = (1+xy)y = 0 \\ \dot{y} = -x + xy - y^2 = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \quad \text{or} \quad y = -\frac{1}{x} \rightarrow \end{cases} \begin{matrix} \text{if } y = 0 \Rightarrow x = 0 \\ \text{if } y = -\frac{1}{x} \Rightarrow x^3 + x^2 + 1 = 0 \end{matrix}$

x = -1.46, .23 ± .8i

The equilibrium points are: (0, 0), (-1.46, .67), (.23 ± .8i, -.28 ± .96i)

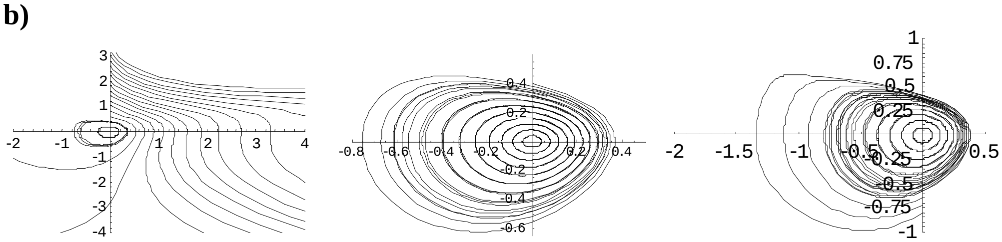
$DF = \begin{pmatrix} y^2 & 1+2xy \\ y-1 & x-2y \end{pmatrix}$

<u>At the point (0, 0):</u> $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

<u>At the point (-1.5, .67):</u> $A = \begin{pmatrix} .45 & -1.01 \\ -.33 & -2.84 \end{pmatrix} \Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda - .45 & 1.01 \\ .33 & \lambda + 2.84 \end{vmatrix} = \lambda^2 + 2.39\lambda - 1.61$

$= 0 \Rightarrow \lambda = -2.94, 0.55$

b)



8.4 a) $\begin{cases} \dot{x} = (1+\alpha)x - 2y + x^2 = 0 \\ \dot{y} = x - y \quad\quad\quad\quad = 0 \end{cases} \Rightarrow \begin{cases} (\alpha - 1)x + x^2 = 0 \\ x = y \end{cases} \Rightarrow x = 0 = y$ *or* $x = 1 - \alpha = y$

The equilibrium points are: (0, 0) and (1-α, 1-α).

$DF = \begin{pmatrix} 1+\alpha+2x & -2 \\ 1 & -1 \end{pmatrix}$

b) If α = 0; at the point (0, 0): $A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} \Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda - 1 & 2 \\ -1 & \lambda + 1 \end{vmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

The system is hyperbolic and can not be determined from this conclusion

c) By using Liapunov; assume E(x, y) = ax + by
E(0, 0) = 0 < E(x, y)
$E(x, y) = (a, b)(\dot{x}, \dot{y}) = a(x-2y+x^2) + b(x-y) = ax - 2ay + ax^2 + bx - by$ (let a = 1 = -b)

$E(x, y) = x-2y+x^2 - x+y = x^2 - y > 0$ if $y < x^2$.

d) If E > 0, then the equation has an unstable limit cycle for which the origin is sink.

If $\alpha = -1$; $A|_{(0,0)} = \begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix} \Rightarrow |\lambda I - A| = \lambda^2 + \lambda + 2 = 0$

$\Rightarrow \lambda = \frac{1}{2}(-1 \pm 2i\sqrt{2}) \Rightarrow (0, 0)$ is a sink.

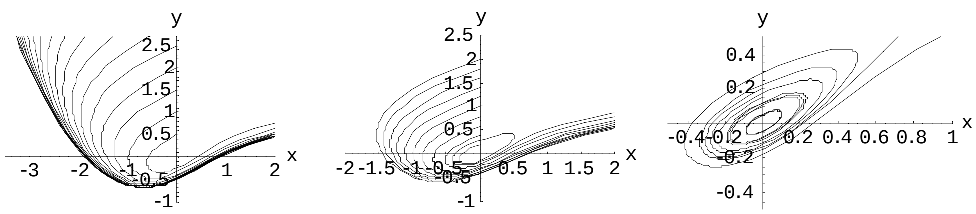
If $\alpha = -1/2$; $A|_{(0,0)} = \begin{pmatrix} 1/2 & -2 \\ 1 & -1 \end{pmatrix} \Rightarrow |\lambda I - A| = 2\lambda^2 + \lambda + 3 = 0$

$\Rightarrow \lambda = \frac{1}{4}(-1 \pm i\sqrt{23}) \Rightarrow (0, 0)$ is a sink.

If $\alpha = 1$; $A|_{(0,0)} = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} \Rightarrow |\lambda I - A| = \lambda^2 - \lambda = 0$

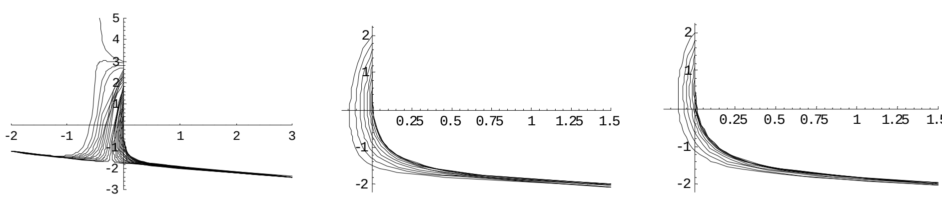
$\Rightarrow \lambda = 0,1 \Rightarrow (0, 0)$ is a source.

$\therefore$ The limit cycle is unstable for small negative α value.



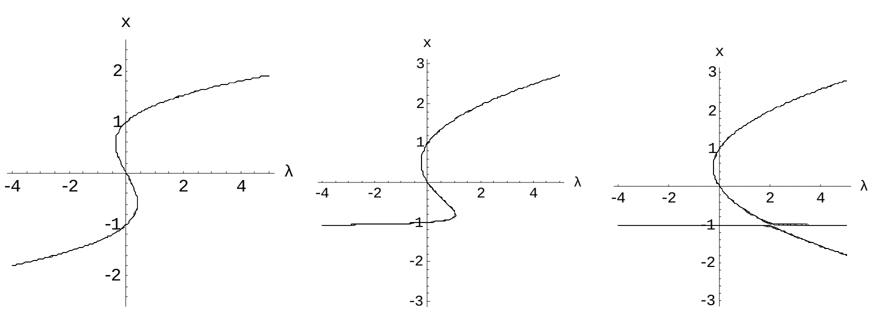
8.5 **a)** $\begin{cases} \dot{x} = x^2 - xy + x - 0.4y + \alpha \\ \dot{y} = y^2 + xy - y - 5 \end{cases} \Rightarrow \begin{cases} x^2 - xy + x - 0.4y + \alpha = 0 \\ y^2 + xy - y - 5 = 0 \end{cases}$

When α = -0.1, 0.1, and 0.2.

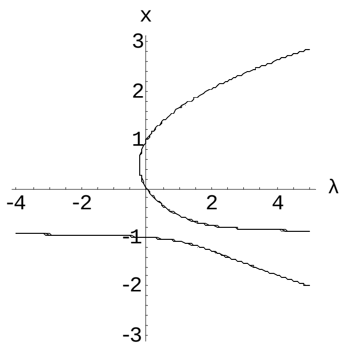


α = -0.1 0.1 0.2

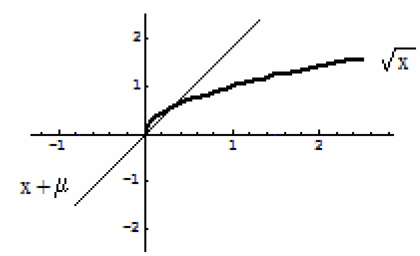
8.6



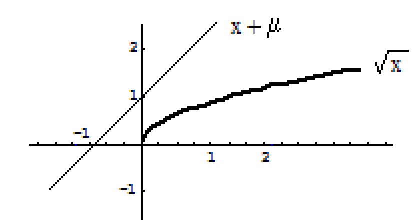
m = -1, -0.1 , 0.0



8.7 **a)** $\dot{x} = x + \mu - \sqrt{x}$, $\mu \in [0, 1]$



$\mu = 0$



$\mu = 1$

b) Assume $y = \sqrt{x} \Rightarrow y^2 - y + \mu = 0 \Rightarrow y = \frac{1}{2}(1 \pm \sqrt{1-4\mu}) = \sqrt{x}$

$\Rightarrow x = \frac{1}{2}(1 - 2\mu \pm \sqrt{1-4\mu})$

The stability of the system depends on $\sqrt{1-4\mu}$.

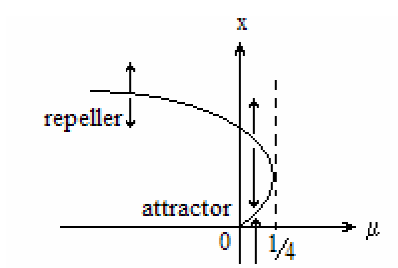
i. If $1 - 4\mu > 0 \Rightarrow \mu < \frac{1}{4} \Rightarrow$ there are 2 equilibrium points.

ii. If $1 - 4\mu = 0 \Rightarrow \mu = \frac{1}{4} \Rightarrow$ there is 1 equilibrium point $x = \frac{1}{4}$.

iii. If $1 - 4\mu < 0 \Rightarrow \mu > \frac{1}{4} \Rightarrow$ there is no equilibrium point.

$DF = 1 - \frac{1}{2\sqrt{x}} < 0 \Rightarrow x > \frac{1}{4}$.

c)



8.8 **a)** $P(x) = x(1-x) + \mu = 0 \Rightarrow x = \frac{1 \pm \sqrt{1+4\mu}}{2}$

i. If $1 + 4\mu = 0 \Rightarrow \mu = -\frac{1}{4} \Rightarrow$ the system has 1 equilibrium point at $x = \frac{1}{2}$

ii. If $1 + 4\mu > 0 \Rightarrow$ the system has 2 equilibrium points.

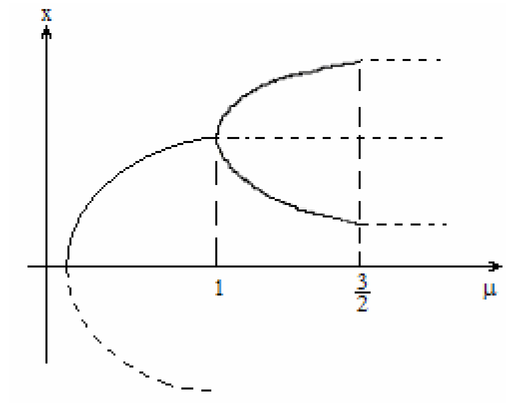
iii. If $1 + 4\mu < 0 \Rightarrow$ the system has no equilibrium point.

b) $|\dot{P}(x)| < 1 \Rightarrow |-2x + 1| < 1$

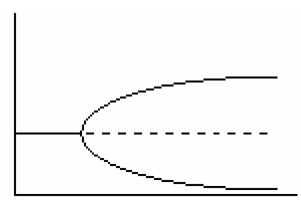
At $x = \frac{1\pm\sqrt{1+4\mu}}{2} \Rightarrow |\dot{P}(x)| = |\pm\sqrt{1+4\mu}\,| < 1 \Rightarrow -1 < 1 + 4\mu < 1 \Rightarrow -\frac{1}{2} < \mu < 0$

c) $\frac{P(P(x))-x}{P(x)-x} = \frac{P(-x^2+x+\mu)-x}{-x^2+x+\mu-x} = \frac{-x^4+2x^3+2(\mu-1)x^2-2\mu x-\mu(\mu-2)}{-x^2+\mu} = x^2 - 2x - \mu + 2$

d)



8.9 **a)** $F(x) = xe^{r(1-x)} = 0 \Rightarrow x = 0$ is the only equilibrium point.
b) $F' = e^{r(1-x)} - rx\,e^{r(1-x)} = (1 - rx)e^{r(1-x)} \Rightarrow F'|_{x=0} = e^r > 1.$
c)



8.10 **a)** $\begin{pmatrix}\varepsilon_{n+1}\\ \delta_{n+1}\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\varepsilon_n\\ \delta_n\end{pmatrix} = \begin{pmatrix}a\varepsilon_n + b\delta_n\\ c\varepsilon_n + d\delta_n\end{pmatrix}$

b) $A = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \Rightarrow |\lambda I - A| = \begin{vmatrix}\lambda - a & -b\\ -c & \lambda - d\end{vmatrix} = 0$

$\Rightarrow \lambda^2 - (a + d)\lambda + (ad - bc) = 0$

c) $\begin{cases} y = x \\ x + y - y^2 = x \end{cases} \Rightarrow y - y^2 = 0 \Rightarrow$ (0, 0) and (1, 1) are the fixed points.

d) $DF = \begin{pmatrix}0 & 1\\ 1 & 1-2y\end{pmatrix}$

At he fixed point (0, 0) $\Rightarrow A = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$ $|\lambda I - A| = \begin{vmatrix}\lambda & -1\\ -1 & \lambda - 1\end{vmatrix} = \lambda(\lambda - 1) - 1 = 0 \Rightarrow \lambda = \frac{1}{2}(1\pm\sqrt{5})$

At he fixed point (1, 1) $\Rightarrow$ A = $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ $|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda+1 \end{vmatrix} = \lambda(\lambda + 1) - 1 = 0$

$\Rightarrow \lambda = \frac{1}{2}(-1 \pm \sqrt{5})$

8.11 Consider the dynamical system given by the vector field: $\dot{x} = \lambda e^x - x$

a) Find the equilibrium points.
b) Show for $\lambda = \lambda_0$ that the system has of a saddle-node bifurcation.
c) Find the value λ_0 and x_0 of part-a.
d) Sketch the bifurcation diagram.

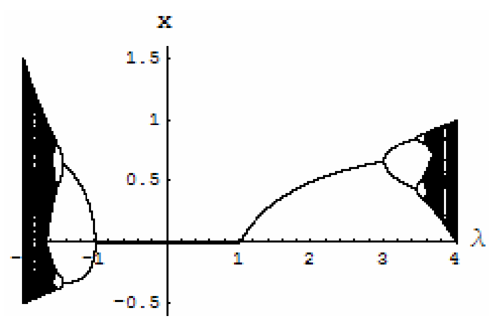
8.12 Consider the system given by the vector field: $\dot{x} = \lambda x - \sin x$

a) Find the equilibrium points.
b) Show that the system has a pitchfork bifurcation.
c) Graph the bifurcation diagram.
d) Determine that the system has more than one saddle-node bifurcation

Chapter 9

9.1 Consider the logistic function $f(x) = \lambda x(1 - x)$

a) Show that when $\lambda \geq 4$ has infinitely many fixed points?
b) If $\lambda > 4$, determine the periodic points number in term of 2^n.
c) If $\lambda > 2 + \sqrt{5}$, determine the behavior of the system.
d) Then determine the number of periodic points.
e) Show that the system of f(x) has an attracting orbit with a prime two when $3 < \lambda < 1 + \sqrt{6}$.
    Hint for part (e): Assume 2 prime numbers (p_1 and p_2), then:
    $$(f^2)'(p_1) = (f^2)'(p_2) = f'(p_1) \cdot f'(p_2)$$
    First solve the equation $f^2(x) = x$ to get expressions for p_1 and p_2 in terms of λ. Find the roots of the equation of 4^{th} degree**.**
f) When $\lambda = 1 + \sqrt{6}$, show that the system has a period-doubling bifurcation.
g) Compute the fixed point when $|f'(x)| < 1$, then show that the logistic function has an attracting point when $0 < \lambda < 3$.
h) To determine the period-two orbit, find the p(x) first from:
    $f(f(x) - x = p(x)\,(f(x) - x)$
    Then find the roots p_1 and p_2 of p(x) where they are the period-two orbit.
i) Compute $|f'(p_1)f'(p_2)| < 1$ to show that the period-two is attracting for $3 < \lambda < 7/2$
j) Sketch the bifurcation diagram of this system.



9.2 **a)** $x^2 + \mu = x \Rightarrow x^2 - x + \mu = 0 \Rightarrow$ the fixed points are at $x = \frac{1 \pm \sqrt{1-4\mu}}{2}$

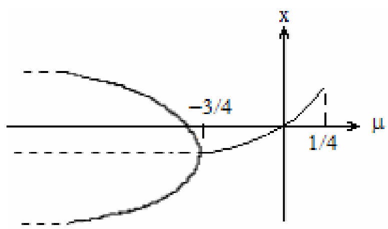
b) $|g'(x)| = |2x| < 1$

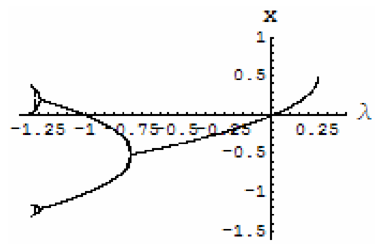
At $x = \frac{1+\sqrt{1-4\mu}}{2} \Rightarrow |1 + \sqrt{1-4\mu}| > 1$ (always)

At $x = \frac{1-\sqrt{1-4\mu}}{2} \Rightarrow |1 - \sqrt{1-4\mu}| < 1 \Rightarrow -1 < 1 - \sqrt{1-4\mu} < 1 \Rightarrow -2 < \sqrt{1-4\mu} < 0$

$\Rightarrow -4 < 1 - 4\mu < 0 \Rightarrow -3 < 4\mu < 1 \Rightarrow -\frac{3}{4} < \mu < \frac{1}{4}$

Show that the only periodic points of g(x) are a single attracting fixed point and a repelling fixed point when $-\frac{3}{4} < \mu < \frac{1}{4}$.

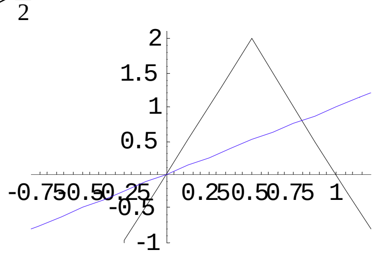




Show the system is chaos when $\mu < -\frac{3}{4}$.

Determine the μ value at which the period 2^n attracting periodic point of g(x) splits into a period 2^{n+1} attracting periodic point.

9.3 **a)** $f(x)\begin{cases} 4x & \text{if } x \le \frac{1}{2} \\ 4-4x & \text{if } x > \frac{1}{2} \end{cases}$



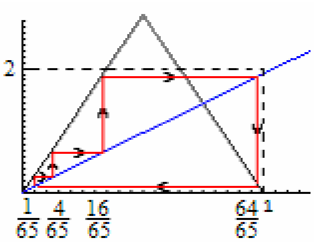
Plot then determine the iteration of this system by using any initial point for x < 0 and x > 1.

b) $x = \frac{1}{4^k+1} \Rightarrow$ For $f(x) = 4x \rightarrow \frac{4}{4^k+1}, \frac{4^2}{4^k+1}, \ldots \frac{4^{k-1}}{4^k+1}$

$\Rightarrow$ For $f(x) = 4 - 4x \rightarrow 4 - \frac{4}{4^k+1}, \quad 4 - \frac{4^{k+1}}{4^k+1} = \frac{4}{4^k+1}$

Therefore; the point will be trapped in an orbit of period k.

c) The orbit for k = 3: $\frac{1}{4^k+1} = \frac{1}{4^3+1} = \frac{1}{65} \rightarrow \frac{4}{65} \rightarrow \frac{16}{65} \rightarrow \frac{64}{65}$



Determine the behavior of the system for $0 \le x \le 1$. (Give an argument)

9.4 $\dot{x} = -x^2 + x + \lambda$

a) $-x^2 + x + \lambda = 0 \Rightarrow x = \frac{1 \pm \sqrt{1+4\lambda}}{2}$ are the equilibrium points.

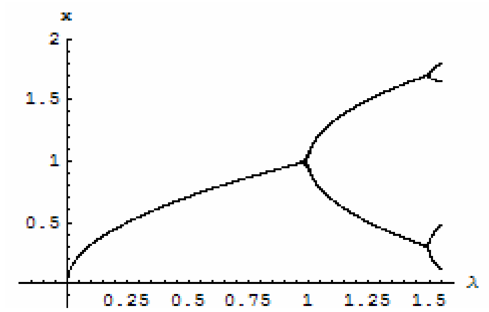
b) The behavior of the system depends on: $\sqrt{1+4\lambda}$

If $\lambda > -\frac{1}{4}$, then there are two real equilibrium points.

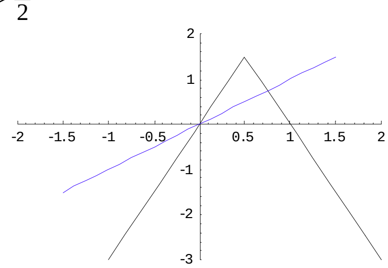
If $\lambda = -\frac{1}{4}$, then $x = \frac{1}{2}$.

If $\lambda < -\frac{1}{4}$, then the equilibrium points are complex numbers.

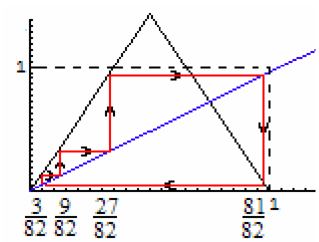
c)



9.5 **a)** $f(x) = \begin{cases} 3x & \text{if } x \le \frac{1}{2} \\ 3-3x & \text{if } x > \frac{1}{2} \end{cases}$



b) For $k = 4 \Rightarrow \dfrac{3}{3^k+1} = \dfrac{3}{82} \to \dfrac{9}{82} \to \dfrac{27}{82} \to \dfrac{81}{82} \Rightarrow$ Period 4.



Determine the behavior of the system for $0 \le x \le 1$. (Give an argument)

9.6 **a)** $x_1 = f(x_2) = \lambda x_2(1-x_2)$

$x_2 = f(x_1) = \lambda x_1(1-x_1)$

$\Rightarrow x_1 = \lambda x_2(1-x_2) = \lambda^2 x_1(1-x_1)[1 - \lambda x_1(1-x_1)] = \lambda^3 x_1^4 - 2\lambda^3 x_1^3 + \lambda^2(1+\lambda) x_1^2 + \lambda^2 x_1$

$\Rightarrow x_1 - \lambda^3 x_1^4 - 2\lambda^3 x_1^3 + \lambda^2(1+\lambda) x_1^2 + \lambda^2 x_1 = 0$

$\Rightarrow x_1(\lambda x_1 + (1-\lambda))[\lambda^2 x_1^2 - \lambda(1+\lambda)x_1 + (1+\lambda)] = 0$

The solutions are: $x_1 = 0$, $\frac{\lambda-1}{\lambda}$, $\frac{1+\lambda \pm \sqrt{(\lambda+1)(\lambda-3)}}{2}$

b) For 2-cycles to exist, the value of $\sqrt{(\lambda+1)(\lambda-3)} \geq 0 \rightarrow \lambda \leq -1$ & $\lambda \geq 3$

9.7 Consider the system given below:

$$f(x) = \begin{cases} 3x & \text{if } 0 \leq x < \frac{1}{3} \\ 3x-1 & \text{if } \frac{1}{3} \leq x < \frac{2}{3} \\ 3x-2 & \text{if } \frac{2}{3} \leq x < 1 \end{cases}$$

Assume that an infinite sequence s(x) is given by $s(x) = (a_0\ a_1\ a_2\ \ldots)$

where
$$a_i = \begin{cases} 0 & \text{if } 0 \leq f'(x) < \frac{1}{3} \\ 1 & \text{if } \frac{1}{3} \leq f'(x) < \frac{2}{3} \\ 2 & \text{if } \frac{2}{3} \leq f'(x) < 1 \end{cases}$$

a) Determine the $\frac{1}{2s}$ orbit under iteration of f? (Use fractions for this computation)

b) Sketch the iteration $\frac{1}{2s}$ orbit.

c) What is the itinerary of $\frac{1}{2s}$?

d) What is the itinerary of a point of the form $\frac{2}{3^k}$?

Chapter 10

10.1 $\begin{aligned}\dot{x} &= -10x + 10y = 0\\ \dot{y} &= rx - y - xz = 0\\ \dot{z} &= xy - \tfrac{8}{3}z \quad = 0\end{aligned} \Rightarrow \begin{cases} x = y \\ rx - x - \frac{3}{8}x^3 = -x(\frac{3}{8}x^2 + 1 - r) = 0 \\ z = \frac{3}{8}x^2 \end{cases}$

$\Rightarrow x = 0,\ x = \pm\sqrt{\frac{8}{3}(r-1)}$

$$DF = \begin{bmatrix} -10 & 10 & 0 \\ r - z & -1 & -x \\ y & x & -\frac{8}{3} \end{bmatrix}$$

<u>When r = 28</u> $\Rightarrow x = 0,\ x = \pm 6\sqrt{2}$

$\rightarrow$ The equilibrium points are: (0, 0, 0), ($6\sqrt{2}$, $6\sqrt{2}$, 27), and (- $6\sqrt{2}$, - $6\sqrt{2}$, 27)

At (0, 0, 0) $\Rightarrow A = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix}$

$\Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda + 10 & -10 & 0 \\ -28 & \lambda + 1 & 0 \\ 0 & 0 & \lambda + \frac{8}{3} \end{vmatrix} = (\lambda + \frac{8}{3})(\lambda^2 + 11\lambda - 270) = 0 \Rightarrow \lambda = -\frac{8}{3},\ \frac{-11 \pm \sqrt{1201}}{2}$

$\Rightarrow\ x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + c_3 e^{\lambda_3 t}$ $\Rightarrow$ the system is an unstable.

At ($\pm 6\sqrt{2}$, $\pm 6\sqrt{2}$, 27) $\Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda + 10 & -10 & 0 \\ 1 & \lambda + 1 & \pm 6\sqrt{2} \\ \mp 6\sqrt{2} & \mp 6\sqrt{2} & \lambda + \frac{8}{3} \end{vmatrix} = \frac{1}{3}(3\lambda^3 + 41\lambda^2 + 464\lambda) = 0$

$\Rightarrow \lambda = 0,\ \frac{-41 \pm 13i\sqrt{23}}{6}$ $\rightarrow$ the system is positive attractor towards the origin point O.

<u>When r = 200</u> $\Rightarrow x = 0,\ x = \pm\sqrt{\frac{1592}{3}}$

$\rightarrow$ The equilibrium points are: (0, 0, 0), and ($\pm\sqrt{\frac{1592}{3}}$, $\pm\sqrt{\frac{1592}{3}}$, 27)

At (0, 0, 0) $\Rightarrow A = \begin{bmatrix} -10 & 10 & 0 \\ 200 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix}$

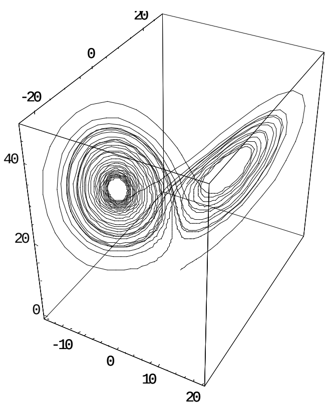
$$\Rightarrow\ |\lambda I - A| = \begin{vmatrix} \lambda+10 & -10 & 0 \\ -200 & \lambda+1 & 0 \\ 0 & 0 & \lambda+\frac{8}{3} \end{vmatrix} = (\lambda + \tfrac{8}{3})\,(\lambda^2 + 11\lambda - 1990) = 0$$

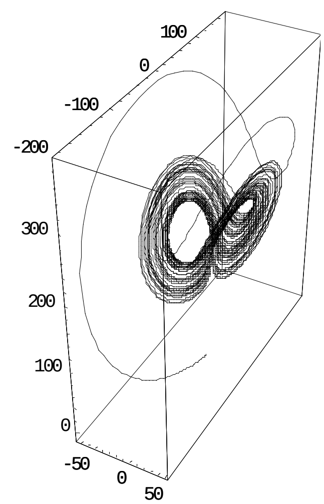
$$\Rightarrow \lambda = -\tfrac{8}{3},\ -50.45,\ 39.45$$

$$\Rightarrow\ x = c_1 e^{-2.6t} + c_2 e^{-50.45t} + c_3 e^{39.45t} \Rightarrow \text{the system is an unstable.}$$

$$\text{At } \left(\pm\sqrt{\tfrac{1592}{3}},\ \pm\sqrt{\tfrac{1592}{3}},\ 199\right) \Rightarrow\ |\lambda I - A| = \begin{vmatrix} \lambda+10 & -10 & 0 \\ 1 & \lambda+1 & \pm\sqrt{\frac{1592}{3}} \\ \mp\sqrt{\frac{1592}{3}} & \mp\sqrt{\frac{1592}{3}} & \lambda+\frac{8}{3} \end{vmatrix} = \lambda^3 + \tfrac{41}{3}\lambda^2 + 580\lambda = 0$$

$\Rightarrow \lambda = 0,\ -6.83 \pm 23.1i \rightarrow$ The system is positive attractor towards the origin point O.





10.2 **a)** $\begin{cases} \dot{x} = -y - z = 0 \\ \dot{y} = x + ay = 0 \\ \dot{z} = b - cz + xz = 0 \end{cases} \Rightarrow \begin{cases} z = -y \\ x = -ay \\ b + cy + ay^2 \end{cases} \Rightarrow y = \dfrac{-c \pm \sqrt{c^2 - 4ab}}{2a}$

Therefore the equilibrium points are: $P_1 = \left(\dfrac{c-\sqrt{c^2-4ab}}{2}, \dfrac{-c+\sqrt{c^2-4ab}}{2a}, \dfrac{c-\sqrt{c^2-4ab}}{2a}\right)$

And $P_2 = \left(\dfrac{c+\sqrt{c^2-4ab}}{2}, \dfrac{-c-\sqrt{c^2-4ab}}{2a}, \dfrac{c+\sqrt{c^2-4ab}}{2a}\right)$

The stabilities: $DF = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x-c \end{bmatrix}$

$$\Rightarrow |\lambda I - A| = \begin{vmatrix} \lambda & 1 & 1 \\ -1 & \lambda - a & 0 \\ -z & 0 & \lambda - x + c \end{vmatrix} = \lambda^3 + (c\text{-}a\text{-}x)\lambda^2 + (ax\text{-}ac\text{+}z\text{+}1)\lambda + c - x - az = 0$$

With b = 2, and c = 4.

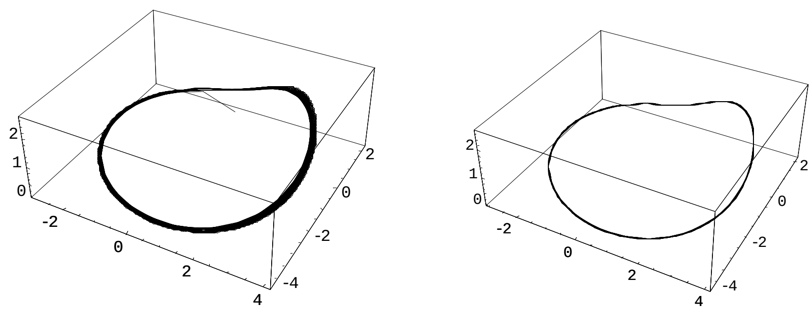
At P_1: $|\lambda I - A| = \lambda^3 + (2\text{-}a\text{-}\sqrt{4-2a})\lambda^2 + (\frac{2}{a}\text{-}2a\text{+} (a\text{+}\frac{1}{a})\sqrt{4-2a} + 1)\lambda - \sqrt{4-2a} = 0$

At P_2: $|\lambda I - A| = \lambda^3 + (2\text{-}a\text{-}\sqrt{4-2a})\lambda^2 + (\frac{2}{a}\text{-}2a\text{+} (a\text{+}\frac{1}{a})\sqrt{4-2a} + 1)\lambda - \sqrt{4-2a} = 0$

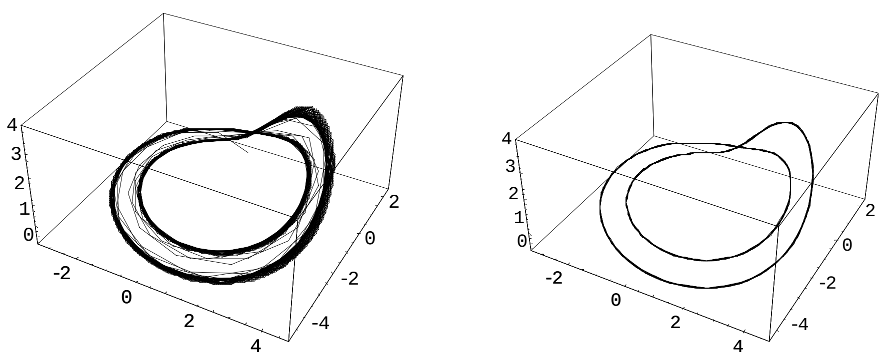
$(4 - 2a \geq 0 \Rightarrow a \leq 2)$

For x(0) = z(0) = 0, y(0) = 2.

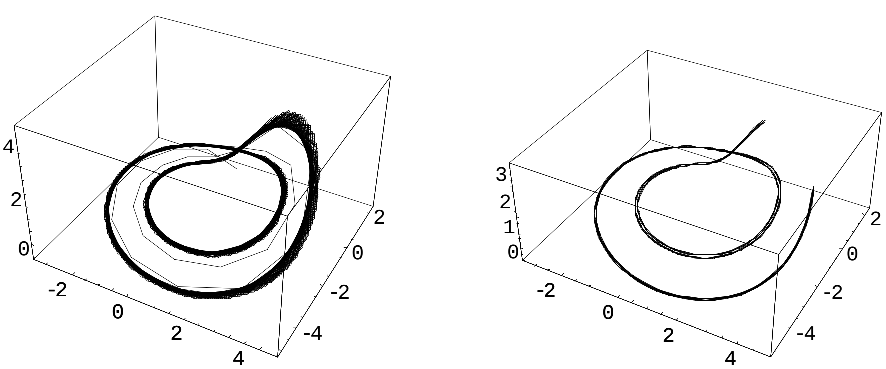
a = 0.3



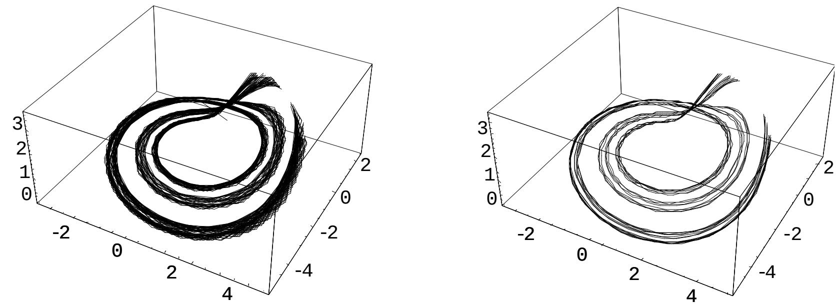
a = 0.35



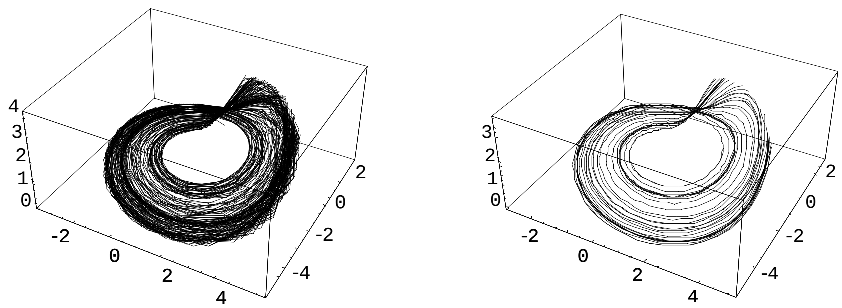
a = 0.375



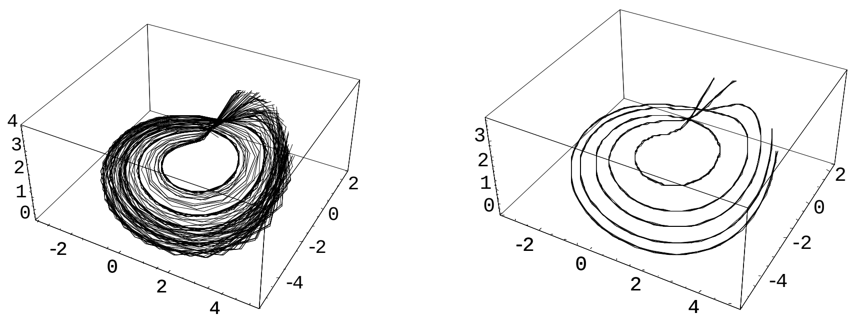
a = 0.386



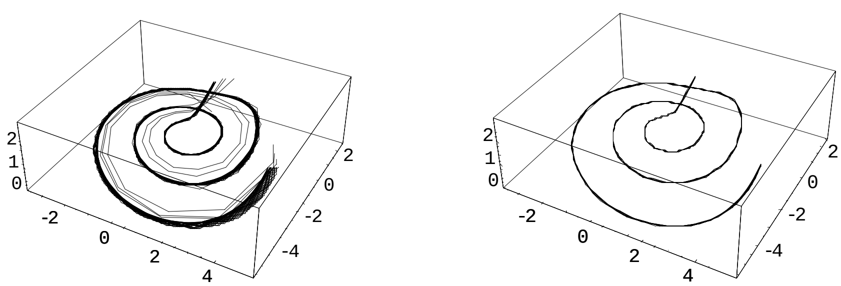
a = 0.395



a = 0.4



a = 0.41



$0 \le t \le 1000$ $\qquad$ $800 \le t \le 1000.$

10.3 The Duffing's equation is given by: $\ddot{x} + c\dot{x} - ax + bx^3 = A \cos \omega t$
Determine the behavior of this system.

10.4 Consider a sinusoidal driven pendulum with a constant torque and has the form:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -ay - b\sin x + c\,\cos\omega t + d \end{cases}$$

Plot this system and explain the chaotic behavior of the pendulum.

Chapter 11

11.1 **a)** $x''(t)+6x'(t)+9x(t) = \delta(t) \rightarrow sX^2(s)+6sX(s)+9X(s) = 1 \rightarrow X(s)[s^2+6s+9] = 1$

$X(s)=\frac{1}{s^2+6s+9}=\frac{1}{(s+3)^2}$; Using Laplace transform $x(t) = e^{-3t}u(t)$

b) $x''(t)+3x'(t)+2x(t) = u(t) \rightarrow s^2X(s)+3sX(s)+2X(s) = 1/s$

$$X(s) = \frac{1}{s(s^2+3s+2)} = \frac{1}{2s} + \frac{-1}{s+1} + \frac{1}{2}\frac{1}{(s+2)}$$

$\rightarrow$ £ $[X(s)] = x(t) = (1/2 - e^{-t} + 1/2\, e^{-2t})u(t)$

c) $x''(t)+3x'(t)+2x(t) = \cos 4t \rightarrow s^2X(s)+3sX(s)+2X(s) = \frac{s}{s^2+16}$

$\rightarrow X(s)(s^2+3s+2) = \frac{s}{s^2+16}$

$$X(s) = \frac{s}{(s^2+16)(s^2+3s+2)} = \frac{s}{(s+j4)(s-j4)(s+1)(s+2)}$$

$$\rightarrow \phi(ju) = \left.\frac{s}{(s+1)(s+2)}\right|_{s=j4} = .141 - j(.167)$$

$$k_1 = \frac{1}{10},\ k_2 = -\frac{1}{17} \Rightarrow X(s) = \frac{-.141 - j.165}{s^2+16} + \frac{-1/17}{s+1} + \frac{1/10}{s+2}$$

£ $[X(s)] = x(t) = \frac{1}{4}e^{-at}(-.165\cos 4t + .141\sin 4t) + \frac{1}{10}e^{-2t} - \frac{1}{17}e^{-t}$

$= [-.04125\cos 4t + .03525\sin 4t - \frac{1}{17}e^{-2t} + .1\, e^{-2t}]u(t)$

d) $D^2x(t)+5Dx(t)+4x(t) = 8 \rightarrow s^2X(s)+5sX(s)+4X(s) = \frac{8}{s}$

$$\rightarrow X(s)[s^2+5s+4] = \frac{8}{s} \rightarrow X(s)=\frac{8}{s(s+1)(s+4)} = \frac{2}{s} - \frac{8/3}{s+1} + \frac{2/3}{s+4}$$

£ $[X(s)] = x(t) = (2 - (8/3)e^{-t} + (2/3)e^{-4t})\, u(t)$

11.2 **a)** $(s^2+2s+5)X(s) - 2s - 4 = \frac{10}{s}$

b) $(s^3 + 4s^2 + 8s + 4)X(s) + 4s^2 + 15s + 28 = \frac{5}{s^2+25}$

11.3 $$X(s) = \frac{10}{(s+2)\left[(s+2.035)^2+.886\right]\left[(s-1.035)^2+1.88\right]}$$

$$= \frac{7.7}{s+2} + \frac{.742(s+1.99)}{(s+2.035)^2+.886} + \frac{.0267(s+.953)}{(s-1.035)^2+1.88}$$

$\pounds\,[X(s)] = x(t) = 7.7e^{-2t} + 0.742e^{-2.035t}\sin(.785t\text{-}87.2°) + .267e^{1.035t}\sin(3.54t\text{-}136.6°)$

11.4 $F(s)=\int f(t)e^{-st}dt = \int_1^3 (t+1)e^{-st}dt + \int_3^4 4e^{-st}dt$

$$= \int_1^3 te^{-st}dt + \int_1^3 e^{-st}dt + 4\int_3^4 e^{-st}dt = -e^{-st}\left(\frac{t}{s}+\frac{1}{s^2}\right)\Big|_1^3 - \frac{e^{-st}}{s}\Big|_1^3 - 4\frac{e^{-st}}{s}\Big|_3^4$$

$$= -4\frac{e^{-4s}}{s} + \frac{e^{-3s}}{s^2} + 2\frac{e^{-s}}{s} + \frac{e^{-s}}{s^2}$$

11.5 **a)** $G(s) = \frac{1}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$ $\left\{\begin{array}{l} A+B=0 \to B=-A=-1 \\ 3A+2B=1 \to A=1 \end{array}\right\} \Rightarrow G(s) = \frac{1}{s+2} - \frac{1}{s+3}$

$\pounds^{-1}[G(s)] = g(t) = e^{-2t} - e^{-3t}$

b) $G(s) = \frac{1}{(s+1)^2(s+4)} = \frac{A}{s+4} + \frac{Cs+d}{(s+1)^2}$

$$\to \left\{\begin{array}{l} A + C = 0 \\ 2A+4C+D=0 \\ A + 4D = 1 \end{array}\right\} \begin{array}{l} A=1/9 \\ C=-1/9 \\ D=2/9 \end{array} \Rightarrow G(s) = \frac{1/9}{s+4} + \frac{1}{9}\frac{2-s}{(s+1)^2} = \frac{1}{9}\frac{1}{s+4} + \frac{2}{9}\frac{1}{(s+1)^2} - \frac{1}{9}\frac{s}{(s+1)^2}$$

$\to \pounds^{-1}[G(s)] = g(t) = \frac{1}{9}e^{-4t} + \frac{2}{9}te^{-t} - \frac{1}{9}$

c) $G(s) = \frac{A}{s+4} + \frac{B}{s+2} + \frac{Cs+D}{(s+2)^2} + \frac{Es^2+Fs+G}{(s+2)^3} \Rightarrow \left\{\begin{array}{l} A = -\frac{5}{4} \\ B = \frac{5}{4} \\ D = -\frac{5}{2} \\ G = \frac{5}{2} \end{array}\right.$

$g(t) = \frac{5}{2}t^2e^{-2t} - \frac{5}{2}te^{-2t} + \frac{5}{4}e^{-2t} - \frac{5}{4}e^{-4t}$

d) $G(s) = \frac{2(s+1)}{s(s^2+s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+2}$

or this function will be in form of: $\frac{\omega_n^2(1+as)}{s(s^2+2\xi\omega_n+\omega_n^2)}$

Where: $a = 1$, $\omega_n^2 = 2$, and $\xi = \frac{1}{2\sqrt{2}} = .353$

$$\text{£}^{-1}[G(s)] = g(t) = 1 + \frac{1}{\sqrt{1-\xi^2}}\sqrt{1-2a\xi\omega_n + a^2\omega_n^2}\; e^{-\xi\omega_n t}\; \sin(\omega_n\sqrt{1-\xi^2}\; t + \phi)$$

$$\phi = \tan^{-1}\frac{a\omega_n\sqrt{1-\xi^2}}{1-a\xi\omega_n} - \tan^{-1}\frac{\sqrt{1-\xi^2}}{-\xi} = \tan^{-1}\frac{1.322}{.5} - \tan^{-1}\frac{.9354}{-.353} = 69.3° + 69.3° \cong 138.6°$$

$$g(t) = 1 + 1.51\, e^{-.5t}\sin(1.323t + 138.6°)$$

e) $G(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} \rightarrow$ A = 1.5; B = -3.0; C = 2.5 $\Rightarrow$ $g(t) = 1.5e^{-t} - 3e^{-2t} + 2.5e^{-3t}$

11.6 Time delay: $\text{£}^{-1}\left[\frac{2}{s^2-6s+3} = \frac{2}{(s-3)^2+2^2}\right] = e^{3t}\sin 2t$

$$\text{£}^{-1}\left[\frac{s-1}{(s-1)^2+1}\right] = e^t\cos t$$

$$f(t) = \begin{cases} -e^t\cos t & 0 < t \le .5 \\ e^{3(t-.5)}\sin 2(t-.5) - e^t\cos t & t > .5 \end{cases}$$

11.7 $\text{£}\{y'''(t) + 2y''(t) + 11y'(t) + 4y(t)\} = 0$

$s^3Y(s) - s^2y(0) - sy(0) - y'(0) + 2s^2Y(s) - 2sy(0) - 2y'(0) + 11sY(s)$

$\Rightarrow$ $11y(0) + 4Y(s) = 0$

$\rightarrow Y(s)(s^3 + 2s^2 + 11s + 4) = \alpha_1 s^2 + (\alpha_2 + 2\alpha_1)s + \alpha_3 + 2\alpha_2 + 11\alpha_1$

$$Y(s) = \frac{\alpha_1 s^2 + (\alpha_2 + 2\alpha_1)s + \alpha_3 + 2\alpha_2 + 11\alpha_1}{s^3+2s^2+11s+4} = \frac{5s^2+6s+2}{s^3+2s^2+11s+4}$$

$$\begin{cases} \alpha_1 = 5 & \\ 2\alpha_1 + \alpha_2 = 6 & \rightarrow \alpha_2 = 4 \\ 11\alpha_1 + 2\alpha_2 + \alpha_3 = 2 & \rightarrow \alpha_3 = -45 \end{cases}$$

11.8 $\text{£}^{-1}\left[\frac{1}{s}\right] = u(t);\quad \text{£}^{-1}\left(\frac{1}{s+2}\right)$

The convolution integral: $\text{£}^{-1}\left\{\frac{1}{s}\frac{1}{s+2}\right\} = \int_0^t u(t-\tau)e^{-2\tau}d\tau = -\frac{1}{2}e^{-2\tau}\Big|_0^t = \frac{1}{2}(1-e^{2t})$

11.9 $f(\infty) = \lim_{s \to 0} s\,F(s) = \frac{0}{0} \Rightarrow f(\infty) = \lim_{s \to 0} \left[\frac{2}{4s^3+39s^2+115s+75}\right] = \frac{2}{75}$

11.10 $f(0) = \lim_{s \to \infty} s\,F(s) = \frac{1}{\infty} = 0$

11.11 For the response to initial conditions, the input r can be taken to be zero, and the transform is:

$[s^2Y - sy(0) - y'(0)] + 7\,[sY - y(0)] + 6Y = 0 \Rightarrow s^2Y - s - 2 + 7sY - 7 + 6Y = 0$

$$Y(s) = \frac{s+9}{s^2+7s+6} = \frac{s+9}{(s+1)(s+6)} = \frac{A}{s+1} + \frac{B}{s+6} \Rightarrow \begin{cases} A = 1.6 \\ B = -.6 \end{cases}$$

$$Y(s) = \frac{1.6}{s+1} - \frac{.6}{s+6} \Leftrightarrow y(t) = 1.6\,e^{-t} - 0.6\,e^{-6t}$$

11.12 The forced response is given by:

$$y_b(t) = \int_0^t \omega(t-\tau)\left[3\frac{dx}{dt} + 2x\right]d\tau = 3\int_0^t \omega(t-\tau)\frac{dx}{dt}d\tau + 2\int_0^t \omega(t-\tau)x\,d\tau$$

$$= \int_0^t \omega(t-\tau)\frac{dx}{d\tau}d\tau = \omega(0)x(t) - \omega(t)x(0) - \int_0^t \frac{\partial\omega(t-\tau)}{\partial\tau}x\,d\tau \;;\; \omega(0) = 0.$$

$$y_b(t) = \int_0^t \left[-3\frac{\partial\omega(t-\tau)}{\partial\tau} + 2\omega(t-\tau)\right]x(\tau)d\tau - 3\omega(t)\,x(0)$$

The forced response is:

$$y_b(t) = 3e^{-2t}\int_0^t e^{2\tau}e^{-3\tau}d\tau - 4te^{-2t}\int_0^t e^{2\tau}e^{-3\tau}d\tau + 4e^{-2t}\int_0^t \tau e^{2\tau}e^{-3\tau}d\tau - 3te^{-2t} = 7\,(e^{-2t} - e^{-3t} - te^{-2t})$$

11.13 $F(s) = -\frac{2}{9s} + \frac{1}{3s^2} + \frac{1}{5(s+1)} + \frac{1}{45(s+6)}$

11.14 $F(s) = E\left[\frac{1}{Ts^2} - \frac{e^{-Ts}}{s(1-e^{-Ts})}\right]$

11.15 **a)** $F(z) = \frac{z(z-\cosh 2T)}{z^2-2z\cosh 2T+1}$

b) $F(z) = \frac{Tz}{(z-1)^2} - \frac{1}{3}\frac{z\sin 2T}{z^2-2z\cos^2 T+1}$

c) $F(s) = \frac{A}{s} + \frac{Bs}{s^2+2} \Rightarrow A = 1, B = -1 \Rightarrow F(z) = \frac{z}{z-1} - \frac{z(z-\cos\sqrt{2}T)}{z^2-2z\cos\sqrt{2}\,T+1}$

d) $F(z) = \frac{e^{-1}z+1-2e^{-1}}{z^2-(1+e^{-1})z+e^{-1}}$; for T = 1.

e) $\boldsymbol{z}[ak] = \frac{1}{1-az^{-1}}$ and $\boldsymbol{z}[x(k-1)] = \boldsymbol{z}^{-1} X(z) \Rightarrow F(z) = \boldsymbol{z}^{-1}\frac{1}{1-az^{-1}} = \frac{k}{1-az^{-1}}$; k=1,2,3...

f) Using the complex integration: $g(k) = \frac{x(k)}{k} = k$

$$\boldsymbol{z}\left[\frac{x(k)}{k}\right] = G(z) = \sum_{k=0}^{\infty}\frac{x(k)}{k}z^{-1} = \sum_{k=0}^{\infty}kz^{-k}$$

$$\frac{d}{dz}G(z) = -\sum_{k=0}^{\infty}k^2z^{-k-1} = -z^{-1}\sum_{k=0}^{\infty}k^2z^{-k} = -\frac{X(z)}{z}$$

$$G(z) = \boldsymbol{z}\left[\frac{x(k)}{k}\right] = \int_z^{\infty}\frac{X(z_1)}{z_1}dz_1 + G(\infty) = \int_z^{\infty}\sum_{k=0}^{\infty}k^2z_1^{-k-1}dz_1 + G(\infty)$$

$$= \sum_{k=0}^{\infty}k^2\left.\frac{z_1^{-k}}{-k}\right|_z^{\infty} + G(\infty) = \sum_{k=0}^{\infty}kz^{-k} + \lim_{k\to 0}\frac{x(k)}{k} = \frac{z^{-1}}{\left(1-z^{-1}\right)^2} + \lim_{k\to 0}k$$

$$= \frac{z^{-1}}{\left(1-z^{-1}\right)^2}$$

11.16 **a)** $\frac{E(z)}{z} = \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1}+\frac{1}{z-2} \Rightarrow E(z) = \frac{-z}{z-1}+\frac{z}{z-2} \Rightarrow e(t) = (-1 + 2^t)\, u(t)$

b) $\frac{E(z)}{z} = \frac{1}{z-1} + \frac{1}{z-e^{-aT}} \Rightarrow E(z) = \frac{z}{z-1} + \frac{z}{z-e^{-aT}} \Rightarrow e(kT) = 1-e^{-akT}$