

$$\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

if $p^2 - 4q > 0 \Rightarrow 2 \text{ } \mathbb{R} \text{ roots}$

$p^2 - 4q < 0 \Rightarrow 2 \text{ } \mathbb{C}$

$p^2 - 4q = 0 \rightarrow 1 \text{ repeated root.}$

Case 1 Distinct Real Roots

$$y_1 = C_1 e^{\lambda_1 t}, y_2 = C_2 e^{\lambda_2 t}$$

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

Ex $y'' - 3y' + 2y = 0$

$$\begin{cases} y(0) = 2 \\ y'(0) = 1 \end{cases}$$

$$\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda_{1,2} = 1, 2$$

$$y(t) = C_1 e^t + C_2 e^{2t}$$

$$y(0) = 2 = C_1 + C_2$$

$$y'(t) = C_1 e^t + 2C_2 e^{2t}$$

$$y'(0) = C_1 + 2C_2 = 1$$

$$\begin{cases} C_1 + C_2 = 2 \\ C_1 + 2C_2 = 1 \end{cases}$$

$$\Rightarrow \underline{C_2 = -1, C_1 = 3}$$

$$y(t) = 3e^t - e^{2t}$$

2. Complex Roots $\lambda = a \pm ib$

$$y(t) = e^{at} (C_1 \cos bt + C_2 \sin bt)$$

Ex $y'' + 2y' + 2y = 0$ $y(0) = 2$ $y'(0) = 3$

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\lambda_{1,2} = -\overset{a}{1} \pm \overset{b}{i}$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y(t) = e^{-t} (C_1 \cos t + C_2 \sin t)$$

$$y(0) = C_1 = 2$$

$$y'(t) = e^{-t} [-C_1 \cos t - C_2 \sin t - C_1 \sin t + C_2 \cos t]$$

$$y'(0) = -C_1 + C_2 = 3 \Rightarrow C_2 = 5$$

$$\underline{y(t) = e^{-t} (2 \cos t + 5 \sin t)}$$

Ex $y'' - 4y' + 13y = 0$

$$\lambda_{1,2} = -2 \pm 3i$$

$$\underline{y(t) = e^{-2t} (C_1 \cos 3t + C_2 \sin 3t)}$$

3. Repeated Roots.

$$\lambda = -\frac{b}{2a}$$

$$y(t) = (C_1 + C_2 t) e^{\lambda t}$$

Ex

$$y'' - 2y' + y = 0$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$\lambda_{1,2} = 1$$

$$y(t) = (C_1 + C_2 t) e^t$$

$$y(0) = 2 \quad y'(0) = -1$$

$$y(0) = C_1 = 2$$

$$y'(t) = (C_2 + C_1 + C_2 t) e^t$$

$$y'(0) = C_2 + C_1 = -1 \Rightarrow C_2 = -3$$

$$y(t) = (2 - 3t) e^t$$

Ex

$$y'' - 10y' + 25y = 0$$

$$\lambda^2 - 10\lambda + 25 = 0$$

$$\lambda_{1,2} = 5$$

$$y(t) = (C_1 + C_2 t) e^{5t}$$

Ex

$$y''' + 3y'' - 4y = 0$$

$$\lambda^3 + 3\lambda^2 - 4 = 0$$

$$\lambda_1 = 1$$

$$\lambda_1 = 1, \lambda_{2,3} = -2$$

$$\begin{array}{r|rrrr} 1 & 1 & 3 & 0 & -4 \\ & & 1 & 4 & 4 \\ \hline & & 1 & 4 & 0 \end{array}$$

$$\lambda^2 + 4\lambda + 4 = 0$$

$$y(x) = C_1 e^t + (C_2 + C_3 t) e^{-2t}$$

Ex

$$\lambda^4 (\lambda + 1) (\lambda + 2)^2 (\lambda^2 + 4) = 0$$

Roots: $\lambda = \underbrace{0, 0, 0, 0}_{\downarrow}, -1, -2, -2, \pm 2i$

$$y(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 e^{-x}$$

$$+ (C_6 + C_7 x) e^{-2x} + C_8 \cos 2x + C_9 \sin 2x$$

Ex

$$m = 4, k = 169 \quad y(0) = 0.1 = \frac{1}{10}$$

$$y'(0) = 130 \text{ cm/s} = \frac{13}{10}$$

Soln

$$m y'' + \mu y' + k y = f(t)$$

$$4 y'' + 169 y = 0$$

$$y'' + \frac{169}{4} y = 0$$

$$\text{natural freq. : } \omega_0 = \sqrt{\frac{169}{4}} = \frac{13}{2}$$

$$\text{Period: } T = \frac{2\pi}{\omega_0} = \frac{4\pi}{13}$$

$$\lambda^2 + \frac{169}{4} = 0 \Rightarrow \lambda_{1,2} = \pm \frac{13}{2} i$$

$$y(t) = C_1 \cos \frac{13}{2} t + C_2 \sin \frac{13}{2} t$$

$$\frac{1}{10} = C_1$$

$$y'(t) = -\frac{13}{2} C_1 \sin \frac{13}{2} t + \frac{13}{2} C_2 \cos \frac{13}{2} t$$

$$y'(0) = \frac{13}{2} C_2 = \frac{13}{10} \Rightarrow C_2 = \frac{1}{5}$$

$$y(t) = \frac{1}{10} \cos \frac{13}{2} t + \frac{1}{5} \sin \frac{13}{2} t$$

$$\begin{aligned} \text{Amplitude: } A &= \sqrt{\left(\frac{1}{10}\right)^2 + \left(\frac{1}{5}\right)^2} \\ &= \sqrt{\frac{1}{100} + \frac{1}{25}} \\ &= \frac{\sqrt{5}}{10} \text{ m} \end{aligned}$$

$$\begin{aligned} \phi &= \tan^{-1} \frac{1/5}{1/10} \\ &= \tan^{-1} 2 \end{aligned}$$

$$y(t) = \frac{\sqrt{5}}{10} \cos \left(\frac{13}{2} t - \tan^{-1} 2 \right)$$