

## ***Solution***      **Section 3.6 – Optimization**

### ***Exercise***

The product of two numbers is 72. Minimize the sum of the second number and twice the first number

### **Solution**

$$xy = 72 \quad (1)$$

$$S = 2x + y$$

$$\text{From (1)} \Rightarrow y = \frac{72}{x}$$

$$S = 2x + \frac{72}{x}$$

$$\frac{dS}{dx} = S' = 2 - \frac{72}{x^2}$$

$$2 - \frac{72}{x^2} = 0$$

$$-\frac{72}{x^2} = -2$$

$$72 = 2x^2$$

$$\Rightarrow x^2 = 36$$

$$x = \pm 6 \rightarrow x = 6$$

$$y = \frac{72}{x} = \frac{72}{6} = 12$$

### ***Exercise***

Verify the function  $V = 27x - \frac{1}{4}x^3$  has an absolute maximum when  $x = 6$ . What is the maximum volume?

### **Solution**

$$V' = 27 - \frac{3}{4}x^2 = 0$$

$$-\frac{3}{4}x^2 = -27$$

$$\Rightarrow x^2 = 27 \cdot \frac{4}{3} = 36$$

$$\rightarrow x = \pm 6$$

$$\Rightarrow x = 6 \text{ (only)}$$

$$27x - \frac{1}{4}x^3 = 0$$

$$108x - x^3 = 0$$

$$x(108 - x^2) = 0$$

$$\Rightarrow x = 0, \sqrt{108}$$

$$V(6) = 27(6) - \frac{1}{4}(6)^3 = \underline{108} \text{ is the maximum volume}$$

### ***Exercise***

A net enclosure for golf practice is open at one end. The volume of the enclosure is  $83\frac{1}{3}$  cubic meters.

Find the dimensions that require the least amount of netting.

### **Solution**

$$V = x^2y = 83\frac{1}{3} = \frac{250}{3}$$

$$y = \frac{250}{3x^2}$$

$$S = x^2 + 3xy$$

$$S = x^2 + 3x \frac{250}{3x^2} = x^2 + \frac{250}{x} = x^2 + 250x^{-1}$$

$$S' = 2x - 250x^{-2} = 2x - \frac{250}{x^2}$$

$$S' = 2x - \frac{250}{x^2} = 0$$

$$x^2 \left( 2x - \frac{250}{x^2} \right) = 0 \cdot x^2$$

$$\Rightarrow 2x^3 - 250 = 0$$

$$2x^3 = 250$$

$$\rightarrow x = 5m$$

$$y = \frac{250}{3x^2}$$

$$= \frac{250}{3(5)^2}$$

$$\approx 3.33m$$

### ***Exercise***

Find two numbers  $x$  and  $y$  such that their sum is 480 and  $x^2y$  is maximized.

### **Solution**

$$\text{Given: } x + y = 480; \Rightarrow y = 480 - x$$

$$f(x) = x^2y = x^2(480 - x) = 480x^2 - x^3$$

$$f'(x) = 960x - 3x^2 = 0$$

$$\Rightarrow 3x(320 - x) = 0$$

$$\Rightarrow x = 0, x = 320$$

$$\Rightarrow y = 480, y = 160$$

$x = 0, y = 320$  actually gives a minimum for the function, while the solution that maximizes the function is  **$x = 320, y = 160$** .

### ***Exercise***

If the price charged for a candy bar is  $p(x)$  cents, then  $x$  thousand candy bars will be sold in a certain city, where  $p(x) = 82 - \frac{x}{20}$ . How many candy bars must be sold to maximize revenue?

### **Solution**

$$\text{Recall that: } R = xp = 82x - \frac{x^2}{20}$$

$$R'(x) = 82 - \frac{x}{10} = 0$$

$$\Rightarrow x = 820 \text{ (thousand)}$$

**$x = \$820,000$**  candy bars gives a maximum

### Exercise

$S(x) = -x^3 + 6x^2 + 288x + 4000$ ;  $4 \leq x \leq 20$  is an approximation to the number of salmon swimming upstream to spawn, where  $x$  represents the water temperature in degrees Celsius. Find the temperature that produces the maximum number of salmon.

### Solution

$$S'(x) = -3x^2 + 12x + 288 = 0$$

$$-3(x^2 - 4x - 96) = 0$$

$$-3(x - 12)(x + 8) = 0$$

$$x = 12, x = -8$$

Temperature that maximizes the number of salmon is **12** degrees C.

### Exercise

A company wishes to manufacture a box with a volume of 52 cubic feet that is open on top and is twice as long as it is wide. Find the width of the box that can be produced using the minimum amount of material.

### Solution

Given:  $l = 2w$ ,  $V = lwh = 52$

$$\text{Substituting: } (2w)wh = 52, \Rightarrow 2w^2h = 52 \Rightarrow h = \frac{26}{w^2}$$

Surface Area ( 2 long sides, 2 short sides, bottom):

$$A = 2lh + 2wh + lw$$

$$A = 2(2w)\left(\frac{26}{w^2}\right) + 2w\left(\frac{26}{w^2}\right) + (2w)w$$

$$A = \frac{104}{w} + \frac{52}{w} + 2w^2 = 156w^{-1} + 2w^2$$

$$A'(w) = -156w^{-2} + 4w = 0$$

$$\left(-156w^{-2} + 4w\right)w^2 = 0(w^2)$$

$$\Rightarrow -156 + 4w^3 = 0$$

$$-4(39 - w^3) = 0$$

$$39 - w^3 = 0$$

$$w^3 = 39$$

$$w = \sqrt[3]{39}$$

### ***Exercise***

A rectangular field is to be enclosed on four sides with a fence. Fencing costs \$3 per foot for two opposite sides, and \$4 per foot for the other two sides. Find the dimensions of the field of area 730 square feet that would be the cheapest to enclose.

### **Solution**

$$A = lw = 730 \Rightarrow l = \frac{730}{w}$$

It makes the most sense to let the short sides ( $w$ ) cost \$4 per foot.

$$C = 2l(3) + 2w(4)$$

$$\begin{aligned} C &= 6\frac{730}{w} + 8w \\ &= 4380w^{-1} + 8w \end{aligned}$$

$$C'(w) = -4380w^{-2} + 8 = 0$$

$$\Rightarrow w^2(-4380w^{-2} + 8) = w^2(0)$$

$$\Rightarrow -4380 + 8w^2 = 0$$

$$\Rightarrow 8w^2 = 4380 \Rightarrow w^2 = \frac{4380}{8} = 547.5$$

$$w = \sqrt{547.5}.$$

$$\approx 23.4 \text{ ft @ } \$4 \text{ per ft.}$$

$$l = \frac{730}{23.4}$$

$$\approx 31.2 \text{ ft @ } \$3 \text{ per ft.}$$

### ***Exercise***

A page is to contain 30 square inches of print. The margins at the top and bottom of the page are 2 inches wide. The margins on the sides are 1 inch wide. What dimensions will minimize the amount of paper used?

### **Solution**

Let the dimensions of the original sheet have width  $x$  and height  $y$ .

The dimensions of the print area would then be,  $(x - 2)$  and  $(y - 4)$  respectively.

Area of the print space:  $A_1 = (x - 2)(y - 4) = 30$

$$y - 4 = \frac{30}{x - 2} \Rightarrow y = \frac{30}{x - 2} + 4$$

Area of the entire page:  $A = xy = x \left( \frac{30}{x - 2} + 4 \right)$

$$A = \frac{30x}{x - 2} + 4x$$

$$A' = \frac{(x - 2)(30) - (30x)}{(x - 2)^2} + 4$$

$$A' = \frac{-60}{(x - 2)^2} + 4 = 0$$

$$\frac{4}{1} = \frac{60}{(x - 2)^2}$$

$$\Rightarrow 4(x - 2)^2 = 60$$

$$(x - 2)^2 = 15$$

$$x = 2 - \sqrt{15} < 0$$

$$x = 2 + \sqrt{15} \approx 5.9 \text{ in.}$$

$$y = \frac{30}{x - 2} + 4 \approx \frac{30}{5.9 - 2} + 4 \approx \mathbf{4.8 \text{ in.}}$$

**Answer:** Approximately 5.9 in.  $\times$  4.8 in.

### Exercise

Find the points of  $y = 4 - x^2$  that are closet to  $(0, 3)$

### Solution

$$\begin{aligned}d &= \sqrt{(x-0)^2 + (y-3)^2} \\&= \sqrt{x^2 + (4-x^2-3)^2} \\&= \sqrt{x^2 + (1-x^2)^2} \\&= \sqrt{x^2 + 1 - 2x^2 + x^4} \\&= \sqrt{x^4 - x^2 + 1}\end{aligned}$$

$$f(x) = x^4 - x^2 + 1$$

$$f'(x) = 4x^3 - 2x = 2x(2x^2 - 1)$$

$$2x(2x^2 - 1) = 0 \rightarrow x = 0, \pm\sqrt{\frac{1}{2}}$$

$$\left(\sqrt{\frac{1}{2}}, \frac{7}{2}\right) \quad \left(-\sqrt{\frac{1}{2}}, \frac{7}{2}\right)$$

### Exercise

A manufacturer wants to design an open box that has a square base and a surface area of  $108 \text{ in}^2$ . What dimensions will produce a box with a maximum volume?

### Solution

$$V = x^2h$$

Surface Area (S) = area of base + 4 (area of each side)

$$108 = x^2 + 4xh$$

$$108 - x^2 = 4xh$$

$$h = \frac{108 - x^2}{4x}$$

$$V = x^2h$$

$$= x^2 \frac{108 - x^2}{4x}$$

$$= x \frac{108 - x^2}{4}$$

$$= \frac{108x}{4} - \frac{x^3}{4}$$

$$\begin{aligned}
&= 27x - \frac{x^3}{4} \\
27x - \frac{1}{4}x^3 &= 0 \\
\Rightarrow 108x - x^3 &= 0 \\
x(108 - x^2) &= 0 \\
x = 0, \sqrt{108} \\
V' = 27 - \frac{3}{4}x^2 &= 0 \\
-\frac{3}{4}x^2 &= -27 \\
\Rightarrow x^2 = 27\left(\frac{4}{3}\right) &= 36 \\
\rightarrow x = \pm 6 \Rightarrow x = 6 \text{ (only)} \\
h = \frac{108 - x^2}{4x} \\
&= \frac{108 - 6^2}{46} \\
&= 3
\end{aligned}$$

### ***Exercise***

Suppose the spawner-recruit function for Idaho rabbits is  $f(S) = 2.17\sqrt{S} \ln(S+1)$ , where  $S$  is measured in thousands of rabbits. Find  $S_0$  and the maximum sustainable harvest,  $H(S_0)$ .

### **Solution**

$$\begin{aligned}
H(S) &= f(S) - S \\
&= 2.17\sqrt{S} \ln(S+1) - S \\
&= 2.17S^{1/2} \ln(S+1) - S \\
H'(s) &= 2.17\left(\frac{1}{2}S^{-1/2} \ln(S+1) + \frac{\sqrt{S}}{S+1}\right) - 1 \\
0 &= 2.17\left(\frac{1}{2}S^{-1/2} \ln(S+1) + \frac{\sqrt{S}}{S+1}\right) - 1
\end{aligned}$$

Using a program:  $S = 36.5578$



### Exercise

A company wants to manufacture cylinder aluminum can with a volume  $1000\text{cm}^3$ . What should the radius and height of the can be to minimize the amount of aluminum used?

### Solution

$$V = \pi r^2 h = 1000$$

$$h = \frac{1000}{\pi r^2}$$

$$\text{Surface Area } S = 2\pi r^2 + 2\pi r h$$

$$S = 2\pi r^2 + 2\pi r \frac{1000}{\pi r^2}$$

$$= 2\pi r^2 + \frac{2000}{r}$$

$$S' = 4\pi r - \frac{2000}{r^2} = 0$$

$$4\pi r = \frac{2000}{r^2}$$

$$r^3 = \frac{2000}{4\pi}$$

$$r = \left(\frac{2000}{4\pi}\right)^{1/3} \approx 5.419$$

$$h = \frac{1000}{\pi r^2}$$

$$= \frac{1000}{\pi \cdot 5.419^2} \text{ s}$$

$$\approx 10.84$$

### Exercise

Find the maximum sustainable harvest of the given function  $f(S) = 28 \cdot S^{0.25}$ , where S is measured in thousands.

### Solution

$$H = f(S) - S$$

$$= 28 \cdot S^{0.25} - S$$

$$H' = 28(0.25)S^{-0.75} - 1$$

$$= 7S^{-0.75} - 1 = 0$$



### Exercise

When a wholesaler sold a product at \$40 per unit, sales were 300 units per week. After a price increase of \$5, however, the average number of units sold dropped to 275 per week. Assuming that the demand function is linear, what price per unit will yield maximum total revenue?

### Solution

$$(40, 300) \quad (45, 275)$$

$$\text{Rate} = \frac{275 - 300}{45 - 40} = -5$$

$$x - 300 = -5(p - 40)$$

$$x = -5p + 500$$

$$R = xp$$

$$= (-5p + 500)p$$

$$= -5p^2 + 500p$$

$$R' = -10p + 500 = 0$$

$$\Rightarrow p = 50$$

$$R'' = -10$$

Since  $R'' < 0$ ,  $R$  is maximum when  $p = \$50$

### Exercise

Find the number of units that must be produced to maximize the revenue function

$R = -x^3 + 150x^2 + 9375x$ . What is the maximum revenue?

### Solution

$$R' = -3x^2 + 300x + 9375 = 3(-x^2 + 100x + 3125) = 0$$

$$-x^2 + 100x + 3125 = 0 \quad \text{Solve for } x$$

$$x = \frac{-100 \pm \sqrt{100^2 - 4(-1)(3125)}}{2(-1)}$$

$$x = \begin{cases} \frac{-100 - 150}{-2} = \frac{250}{2} = 125 \\ \frac{-100 + 150}{-2} = \frac{50}{-2} < 0 \end{cases}$$

$$f(x) = ax^2 + bx + c$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

125 units to maximize the revenue

$$\begin{aligned} R(125) &= -(125)^3 + 150(125)^2 + 9375(125) \\ &= \$1,562,500.00 \end{aligned}$$

### Exercise

A manufacturer has a steady demand for 14,112 cases of sugar. It costs \$4 to store 1 case for 1 year, \$36 in set up cost to produce each batch, and \$25 to produce each case. Find the number of cases per batch that should be produced to minimize cost.

### Solution

**Given:**  $k = 4$ ,  $f = 36$ ,  $g = 25$ ,  $M = 14,112$

$$\text{Number of cases: } q = \sqrt{\frac{2fM}{k}} = \sqrt{\frac{2(36)(14,122)}{4}} = \underline{504}$$

The manufacturer should produce **504** cases per batch.

### Exercise

A restaurant has an annual for 920 bottles of California wine. It costs \$2 to store 1 bottle for 1 year and it costs \$4 to place a reorder. Find the optimum number of bottles per order.

### Solution

**Given:**  $k = 2$ ,  $f = 4$ ,  $M = 920$

$$q = \sqrt{\frac{2fM}{k}} = \sqrt{\frac{2(4)(920)}{2}} \approx \underline{60.66}$$

The optimum number of bottles per order is **61**.

### Exercise

Every year, Dan sells 213,696 cases of Cookie Mix. It costs \$1 per year in electricity to store a case, plus he must pay annual warehouse fees of \$4 per case for the maximum number of cases he will store. If it costs him \$742 to set up a production run, plus \$7 per case to manufacture a single case, how many production runs should he have each year to minimize his total costs/

### Solution

**Given:**  $k_1 = 1$ ,  $k_2 = 4$ ,  $f = 742$ ,  $g = 7$ ,  $M = 213,696$

$$x = \sqrt{\frac{(k_1 + 2k_2)M}{2f}} = \sqrt{\frac{(1+2(4))(213696)}{2(742)}} = \underline{36}$$

The optimum number of runs is **36**.

### Exercise

A certain company produces potting soil and sells it in 50 lb. bags. Suppose that 100,000 bags are to be produced each year. It costs \$6 per year to store a bag of potting soil, and it costs \$3,000 to set up the facility to produce a batch of bags. Find the number of bags per batch that should be produced.

### Solution

**Given:**  $k = 6$ ,  $f = 3,000$ ,  $M = 100,000$

$$q = \sqrt{\frac{2fM}{k}} = \sqrt{\frac{2(3,000)(100,000)}{6}} = \underline{10,000}$$

The manufacturer should produce **10,000 bags** per batch.

### Exercise

Find the approximate number of batches (to the nearest whole number) of an item that should be produced annually if 100,000 units are to be made. It costs \$3 to store a unit for one year, and it costs \$460 to setup the factory to produce each batch.

### Solution

**Given:**  $k = 3$ ,  $f = 460$ ,  $M = 100,000$

$$x = \sqrt{\frac{kM}{2f}} = \sqrt{\frac{3(100,000)}{2(460)}} = \underline{18}$$

The manufacturer should produce **18 batches**.

### Exercise

A bookstore has an annual demand for 67,000 copies of a best-selling book. It costs \$0.40 to store one copy for one year and it costs \$25 to place an order. Find the optimum number of copies per order

### Solution

**Given:**  $k = .4$ ,  $f = 25$ ,  $M = 67,000$

$$q = \sqrt{\frac{2fM}{k}} = \sqrt{\frac{2(25)(67,000)}{.4}} = \underline{2,894}$$

The optimum number of copies per order is **2,894**.

### Exercise

Find the approximate number of batches (to the nearest whole number of an item that should be produced annually if 170,000 units are to be made. It costs \$3 to store a unit for one year, and it costs \$520 to set up the factory to produce each batch.

### Solution

The cost per batch:  $520 + 3x$

Total annual manufacturing cost:  $T(x) = (520 + 3x) \frac{170,000}{x} = \frac{88,400,00}{x} + 2,100,00$

**Given:**  $k = 3$ ,  $f = 520$ ,  $M = 170,000$

$$x = \sqrt{\frac{kM}{2f}} = \sqrt{\frac{3(170,000)}{2(520)}} = \underline{22}$$

The manufacturer should produce 22 batches.

### Exercise

The homeowner judges that an area of  $800 \text{ ft}^2$  for the garden is too small and decides to increase the area to  $1,250 \text{ ft}^2$ . What is the minimum cost of building a fence that will enclose a garden with an area of  $1,250 \text{ ft}^2$ ? What are the dimensions of this garden? Assume that the cost of fencing remains unchanged.

$$\text{Minimize } C = 8x + 4y \text{ subject to } xy = 1,250$$

Since  $x$  and  $y$  represent distances, we know that  $x > 0$  and  $y > 0$ .

### Solution

$$xy = 1,250 \Rightarrow y = \frac{1,250}{x}$$

$$\begin{aligned} C(x) &= 8x + 4 \frac{1,250}{x} \\ &= 8x + \frac{5,000}{x} \end{aligned} \quad x > 0$$

$$C'(x) = 8 - \frac{5,000}{x^2} = 0 \quad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$\text{Solve for } x: \quad 8 = \frac{5,000}{x^2}$$

$$x^2 = \frac{5,000}{8} = 625$$

$$x = \pm 25 \rightarrow \boxed{x = 25} \text{ (CN)}$$

0	25
$C'(1) < 0$	$C'(50) > 0$
Decreasing	Increasing

$C(x)$  has a local minimum at  $x = 25$ .

$$C(25) = 8(25) + \frac{5,000}{25} = \$400.00$$

$$y = \frac{1,250}{25} = 50$$

The minimal cost for enclosing a  $1,250 \text{ ft}^2$  is \$400.00, and the dimensions are 25 feet by 50 feet, with one 25-foot side of wood fencing.

### Exercise

An office supply company sells  $x$  permanent markers per year at  $\$p$  per marker. The price-demand equation for these markers is  $p = 10 - 0.001x$ . The total annual cost of manufacturing  $x$  permanent markers for the office supply company is.

- What price should the company charge for the markers to maximize revenue?
- What is the maximum revenue?
- What is the company's maximum profit?
- What should the company charge for each marker, and how many markers should be produced?
- The government decides to tax the company \$2 for each marker produced. Taking into account this additional cost, how many markers should the company manufacture each week to maximize its weekly profit?
- What is the maximum weekly profit?
- How much should the company charge for the markers to realize the maximum weekly profit?

### Solution

- a)  $\text{Revenue} = \text{price} \times \text{demand}$

$$R(x) = (10 - 0.001x)x$$

$$= 10x - 0.001x^2$$

$$R'(x) = 10 - 0.002x = 0$$

$$10 = 0.002x$$

$$x = \frac{10}{.002} = 5,000 \quad (CN)$$

$R(x)$  has a local maximum at  $x = 5,000$ .

$$p = 10 - 0.001(5000) = \$5.00$$

- b)  $R(5,000) = (10 - 0.001(5,000))(5,000) = \$25,000$

The price of a marker should be \$5.00 for the company to realize maximum revenue of \$25,000.

- c)  $\text{Profit} = \text{Revenue} - \text{Cost}$

$$P(x) = R(x) - C(x)$$

$$= 10x - 0.001x^2 - 5,000 - 2x$$

$$= -0.001x^2 + 8x - 5,000$$

$$P'(x) = -0.002x + 8 = 0$$

0	5,000
$R'(1) > 0$	$R'(10,000) < 0$
Increasing	Decreasing

0	4,000
$P'(1) > 0$	$P'(5,000) < 0$
Increasing	Decreasing

$$8 = 0.002x \rightarrow \underline{x = \frac{8}{0.002} = 4,000} \quad (CN)$$

$$P(4,000) = -0.001(4,000)^2 + 8(4,000) - 5,000 = \underline{\$11,000}$$

A maximum profit of \$11,000 is realized when 4,000 markers are manufactured.

d)  $\underline{p = 10 - 0.001(4000) = \$6.00}$

The company should charge \$6.00 per marker to maximize the profit.

- e) The tax of \$2 per unit changes the company's cost equation:

$$C(x) = \text{Original cost} + \text{tax}$$

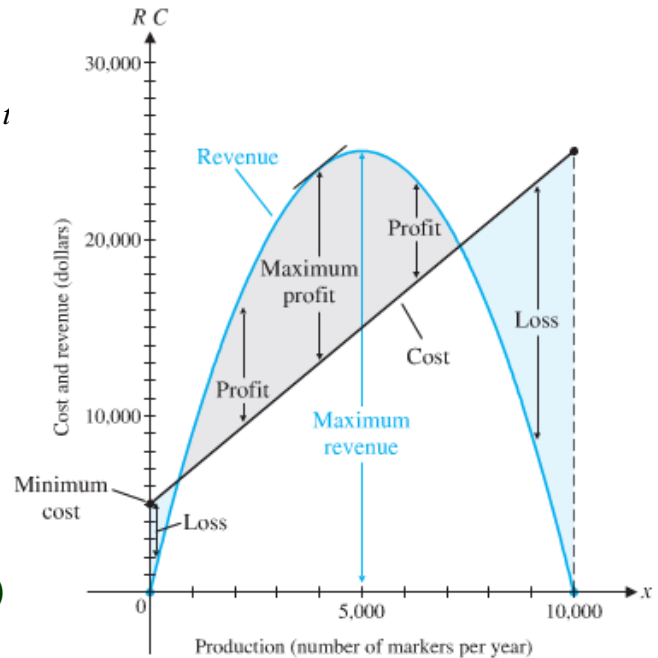
$$\begin{aligned} C(x) &= 5,000 + 2x + 2x \\ &= 5,000 + 4x \end{aligned}$$

The new profit function is:

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 10x - 0.001x^2 - 5,000 - 4x \\ &= -0.001x^2 + 6x - 5,000 \end{aligned}$$

$$P'(x) = -0.002x + 6 = 0$$

$$6 = 0.002x \rightarrow \underline{x = \frac{6}{0.002} = 3,000} \quad (CN)$$



The company should manufacture 4,000 markers each week to maximize its weekly profit.

f)  $P(3,000) = -0.001(3,000)^2 + 6(3,000) - 5,000 = \underline{\$4,000}$

A maximum weekly profit of \$4,000 is realized when 3,000 markers are manufactured.

g)  $\underline{p = 10 - 0.001(3000) = \$7.00}$

The company should charge \$7.00 per marker to maximize the new profit profit.

The company increases the charge \$1.00 per unit with a resulting decrease of \$7,000 in maximum profit.



## Exercise

When a management training company prices its seminar on management techniques at \$400 per person, 1,000 people will attend the seminar. The company estimates that for each \$5 reduction in price, an additional 20 people will attend the seminar.

- How much should the company charge for the seminar in order to maximize its revenue?
- What is the maximum revenue?
- After additional analysis, the management decides that its estimate of attendance was too high. Its new estimate is that only 10 additional people will attend the seminar for each \$5 decrease in price. How much should the company charge for the seminar now in order to maximize revenue?
- What is the new maximum revenue?

## Solution

- a) Let  $x$  represent the number of \$5 price reductions.

$$\text{Price per customer} = 400 - 5x$$

$$\text{Number of customers} = 1,000 + 20x$$

$$\text{Revenue} = (\text{price per customer})(\text{number of customers})$$

Since price cannot be negative, then

$$400 - 5x \geq 0$$

$$400 \geq 5x$$

$$80 \geq x$$

$$R(x) = (400 - 5x)(1,000 + 20x) \quad 0 \leq x \leq 80$$

$$= 400,000 + 3,000x - 100x^2$$

$$R'(x) = 3,000 - 200x = 0$$

$$200x = 3,000$$

$$x = 15 \quad (CN)$$

$$\text{Price per customer} = 400 - 5(15) = \$325.00$$

The company should charge \$325.00 for the seminar.

$$\begin{aligned} b) \quad R(325) &= 400,000 + 3,000(325) - 100(325)^2 \\ &= \$422,500.00 \end{aligned}$$

The maximum revenue is \$422,500.

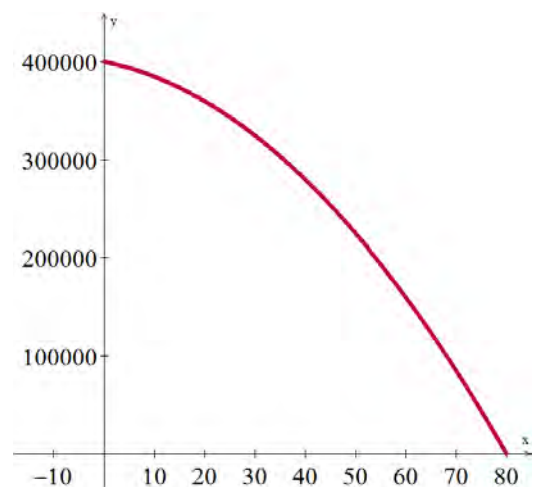
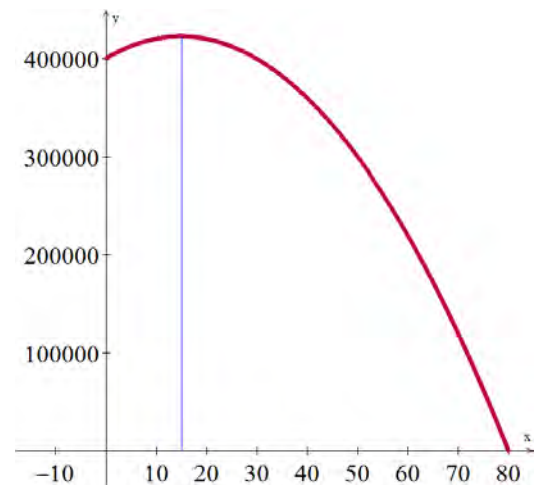
$$c) \quad R(x) = (400 - 5x)(1,000 + 10x) \quad 0 \leq x \leq 80$$

$$= 400,000 - 1,000x - 50x^2$$

$$R'(x) = -1,000 - 100x = 0$$

$$-100x = -1,000$$

$$x = -10 \quad (CN)$$



Since  $x$  is not in the interval  $[0, 80]$

Price per customer should be \$400.00

$$\begin{aligned} d) \quad R(0) &= 400,000 - 1,000(0) - 50(0)^2 \\ &= \underline{\$400,000.00} \end{aligned}$$

From the graph, it would be better to charge \$400 per customer to maximize the revenue of \$400,000.00.

## Exercise

A multimedia company anticipates that there will be a demand for 20,000 copies of a certain DVD during the next year. It costs the company \$0.50 to store a DVD for one year. Each time it must make additional DVDs, it costs \$200 to set up the equipment. How many DVDs should the company make during each production run to minimize its total storage and setup costs?

## Solution

Let assume that there are 250 working days and then the daily demand is:  $\frac{20,000}{250} = 80$  DVDs

Let:  $x$ : number of DVDs manufactured during each production run.

$y$ : number of production runs.

The number of DVDs in storage between production runs will decrease from  $x$  to 0, and the average number in storage each day is  $\frac{x}{2}$ .

Since it costs \$0.50 to store a DVD for one year, the total storage cost is  $(0.5)\frac{x}{2} = 0.25x$ .

The total cost is:

Total cost = setup cost + storage cost

$$C = 200y + 0.25x$$

The total number of DVDs produced is  $xy$ .  $xy = 20,000$

$$y = \frac{20,000}{x}$$

$$C(x) = 200\left(\frac{20,000}{x}\right) + 0.25x \quad 1 \leq x \leq 20,000$$

$$= \frac{4,000,000}{x} + 0.25x$$

$$C'(x) = -\frac{4,000,000}{x^2} + 0.25 = 0$$

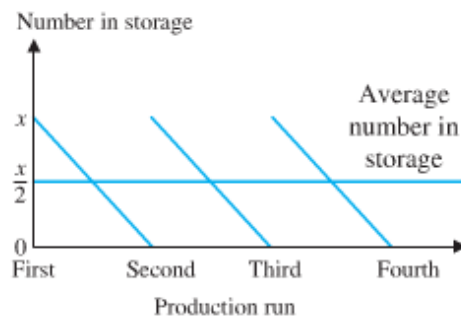
$$0.25 = \frac{4,000,000}{x^2} \rightarrow x^2 = \frac{4,000,000}{0.25}$$

$$\Rightarrow x = \sqrt{\frac{4,000,000}{0.25}} = \underline{4,000}$$

$$C(4,000) = \frac{4,000,000}{4,000} + 0.25(4,000) = \underline{\$2,000}$$

$$y = \frac{20,000}{4,000} = \underline{5}$$

The company will minimize its total cost by making 4,000 DVDs five times during the year.



0	4,000
$C'(1) < 0$	$C'(10,000) > 0$
<i>Decreasing</i>	<i>Increasing</i>

### Exercise

A company manufactures and sells  $x$  digital cameras per week. The weekly price-demand and cost equations are, respectively:

$$p = 400 - 0.4x \quad \text{and} \quad C(x) = 2,000 + 160x$$

- What price should the company charge for the cameras, and how many cameras should be produced to maximize the weekly revenue? What is the maximum revenue?
- What is the maximum weekly profit? How much should the company charge for the cameras, and how many cameras should be produced to realize the maximum weekly profit?

### Solution

a) Revenue:  $R(x) = x \cdot p(x)$

$$= x \cdot (400 - .4x)$$

$$= 400x - .4x^2$$

$$R'(x) = 400 - 0.8x = 0$$

$$0.8x = 400 \rightarrow \underline{x = \frac{400}{0.8} = 500} \quad (CN)$$

0	500
$R'(1) > 0$	$R'(600) < 0$
<i>Increasing</i>	<i>Decreasing</i>

$$R(500) = 400(500) - .4(500)^2 = \underline{\$100,000.00}$$

$$p = 400 - .4(500) = \underline{\$200}$$

The maximum revenue is \$100,000 when 500 cameras are produced and sold for \$200 each.

b)  $P(x) = R(x) - C(x)$

$$= 400x - .4x^2 - 2,000 - 160x$$

$$= -.4x^2 + 240x - 2,000$$

$$P'(x) = -.8x + 240 = 0$$

$$.8x = 240 \rightarrow \underline{x = \frac{240}{0.8} = 300} \quad (CN)$$

$$P(300) = -.4(300)^2 + 240(300) - 2,000 = \underline{\$34,000}$$

$$p = 400 - .4(300) = \underline{\$280}$$

The price that the company should charge is \$280 to maximize the profit of \$34,000.

### Exercise

A company manufactures and sells  $x$  television sets per month. The monthly cost and price-demand equations are:

$$C(x) = 60,000 + 60x, \quad p = 200 - \frac{x}{50} \quad 0 \leq x \leq 10,000$$

- Find the maximum revenue.
- Find the maximum profit, the production level that will realize the maximum profit, and the price the company should charge for each television set.
- If the government decides to tax the company \$5 for each set it produces, how many sets should the company manufacture each month to maximize the profit? What should the company charge for each set?

### Solution

a) Revenue:  $R(x) = x \cdot p(x)$

$$\begin{aligned} &= x \left( 200 - \frac{x}{50} \right) \\ &= 200x - \frac{1}{50}x^2 \quad 0 \leq x \leq 10,000 \end{aligned}$$

$$R'(x) = 200 - \frac{1}{25}x = 0$$

$$\frac{1}{25}x = 200 \rightarrow \boxed{x = 5,000} \quad (CN)$$

$$R''(x) = -\frac{1}{25} < 0 \text{ that implies } R \text{ has an absolute maximum.}$$

$$\text{Maximum revenue: } R(5,000) = 200(5,000) - \frac{1}{50}(5,000)^2 = \underline{\$500,000.00}$$

b)  $P(x) = R(x) - C(x)$

$$\begin{aligned} &= 200x - \frac{1}{50}x^2 - (60,000 + 60x) \\ &= 140x - \frac{1}{50}x^2 - 60,000 \end{aligned}$$

$$P'(x) = 140 - \frac{1}{25}x = 0$$

$$\frac{1}{25}x = 140 \rightarrow \boxed{x = 3,500} \quad (CN)$$

$$P''(x) = -\frac{1}{25} < 0$$

Thus, the maximum profit occurs when 3,500 television sets are produced.

The maximum profit is:

$$P(3,500) = 140(3,500) - \frac{1}{50}(3,500)^2 - 60,000 = \underline{\$185,000.00}$$

The price that the company should charge is:

$$\boxed{p = 200 - \frac{3,500}{50} = \$130.00} \text{ per set.}$$

- c) The taxes \$5 for each set, then the profit function is given by

$$P(x) = 200x - \frac{1}{50}x^2 - (60,000 + 60x) - 5x$$

$$= 135x - \frac{1}{50}x^2 - 60,000$$

$$P'(x) = 135 - \frac{1}{25}x = 0$$

$$\frac{1}{25}x = 135 \rightarrow \boxed{x = 3,375} \quad (CN)$$

$$P''(x) = -\frac{1}{25}x < 0$$

Thus, the maximum profit occurs when 3,375 television sets are produced.

The maximum profit is:

$$P(3,375) = 135(3,375) - \frac{1}{50}(3,375)^2 - 60,000 = \underline{\$167,812.50}$$

The price that the company should charge is:

$$\underline{p} = 200 - \frac{3,375}{50} = \underline{\$132.50} \text{ per set.}$$

## Exercise

A university student center sells 1,600 cups of coffee per day at a price of \$2.40.

- A market survey shows that for every \$0.05 reduction in price, 50 more cups of coffee will be sold. How much should the student center charge for a cup of coffee in order to maximize revenue?
- A different market survey shows that for every \$0.10 reduction in the original \$2.40 price, 60 more cups of coffee will be sold. Now how much should the student center charge for a cup of coffee in order to maximize revenue?

## Solution

- Let  $x$ : number of price reductions

The price of a cup of coffee will be:  $p = 2.40 - 0.05x$

The number of cups sold will be:  $1,600 + 50x$

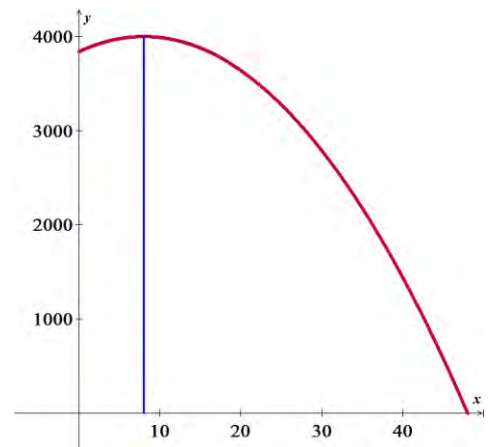
$$\begin{aligned} \text{Revenue: } R(x) &= x \cdot p(x) \\ &= (1,600 + 50x)(2.40 + 0.05x) \\ &= 3,840 + 40x - 2.5x^2 \end{aligned}$$

$$R'(x) = 40 - 5x = 0$$

$$5x = 40 \rightarrow \boxed{x = 8} \quad (CN)$$

$R''(x) = -5 < 0$  that implies  $R$  has an absolute maximum.

Maximum revenue:  $R(8) = 3,840 + 40(8) - 2.5(8)^2 = \$4,000.00$  when 1,600 cups of coffee sold at the price  $p = 2.40 - 0.05(8) = \$2.00$  per cup.



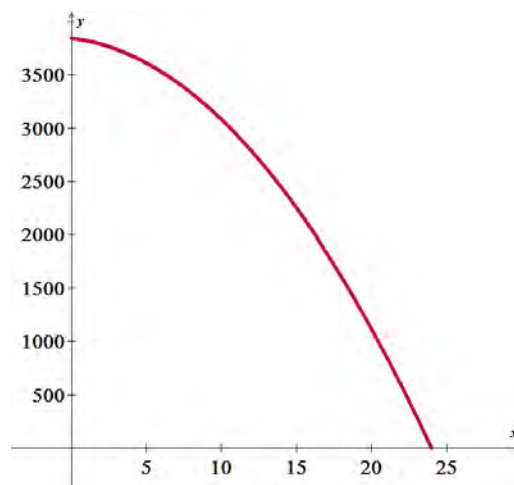
- Revenue:  $R(x) = (1,600 + 60x)(2.40 - .10x)$

$$= 3,840 - 16x - 6x^2$$

$$R'(x) = -16 - 12x = 0$$

$$12x = -16 \rightarrow \boxed{x = -\frac{4}{3} < 0} \quad (CN)$$

Thus,  $R(x)$  is decreasing and its maximum occurs at  $x = 0$  they should charge \$2.40 per cup.



### Exercise

A 300-room hotel in Las Vegas is filled to capacity every night at \$80 a room. For each \$1 increase in rent, 3 fewer rooms are rented. If each rented room costs \$10 to service per day, how much should be the management charge for each room to maximize gross profit? What is the maximum gross profit?

### Solution

Let  $x$ : number of dollar increases in the rate per night.

Total number of rooms rented:  $300 - 3x$

Rate per night:  $80 + x$

Total income = (total number of rooms rented) (rate - 10)

$$y(x) = (300 - 3x)(80 + x - 10) \quad 0 \leq x \leq 100$$

$$= (300 - 3x)(70 + x)$$

$$= 21,000 + 90x - 3x^2$$

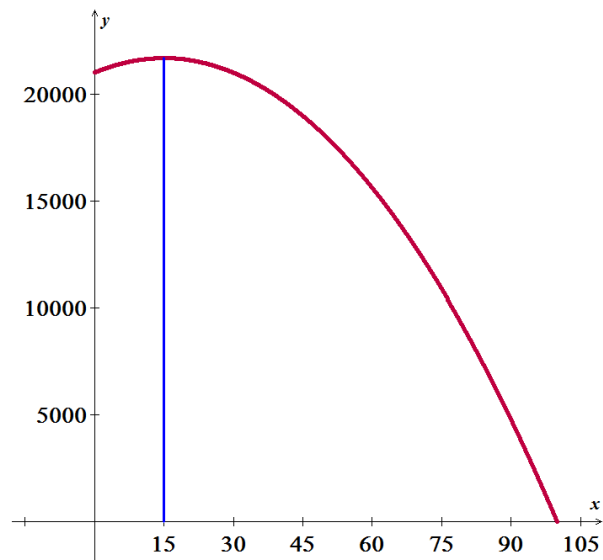
$$y'(x) = 90 - 6x = 0$$

$$6x = 90 \rightarrow \boxed{x=15} \quad (CN)$$

$$y''(x) = -6 < 0.$$

$$\text{Maximum income: } y(15) = 21,000 + 90(15) - 3(15)^2 = \underline{\$21,675.00}$$

$$\text{The rate per night: } 80 + 15 = \underline{\$95}$$





### Exercise

A commercial pear grower must decide on the optimum time to have fruit picked and sold. If the pears are picked now, they will bring 30¢ per pound, with each tree yielding an average of 60 pounds of salable pears. If the average yield per tree increases 6 pounds per tree per week for the next 4 weeks, but the price drops 2 ¢ per pound per week, when should the pears be picked to realize the maximum return per tree? What is the maximum return?

### Solution

Let  $x$ : number of additional weeks the grower waits to pick the pears.

Yield of a pear tree in pounds:  $60 + 6x$

Price per pound:  $0.3 - 0.02x$

The return will be:

$$y(x) = (60 + 6x)(0.3 - 0.02x) \quad 0 \leq x \leq 4$$
$$= 18 + .6x - .12x^2$$

$$y(x) = .6 - .24x = 0 \quad \Rightarrow .24x = .6 \rightarrow \boxed{x = 2.5} \quad (CN)$$

Maximum income:  $y(2.5) = 18 + .6(2.5) - .12(2.5)^2 = \underline{\$18.75}$  at  $x = 2.5$  weeks.

### Exercise

A pharmacy has a uniform annual demand for 200 bottles of a certain antibiotic. It costs \$10 to store one bottle for one year and \$40 to place an order. How many times during the year should the pharmacy order the antibiotic in order to minimize the total storage and reorder costs?

### Solution

Let  $x$  be the number of times the pharmacy places order.

Let  $y$  be the number of demands.

$$\text{The cost: } C = 40x + 10\left(\frac{y}{2}\right) = 40x + 5y$$

$$\text{We also have } xy = 200 \Rightarrow y = \frac{200}{x}$$

$$C(x) = 40x + 5\left(\frac{200}{x}\right) = 40x + \frac{1,000}{x}$$

$$C'(x) = 40 - \frac{1,000}{x^2} = 0$$

$$\frac{1,000}{x^2} = 40 \rightarrow x^2 = \frac{1,000}{40} = 25 \Rightarrow x = \pm 5 \rightarrow \boxed{x = 5} \text{ since } x > 0$$

$$C''(x) = \frac{2,000}{x^3} > 0 \text{ the function has an absolute minimum at } x = 5$$

$$C(5) = 40(5) + \frac{1,000}{5} = 400 \text{ is the minimum cost.}$$

### Exercise

A parcel delivery service will deliver a package only if the length plus girth (distance around) does not exceed 108 inches.

- Find the dimensions of a rectangular box with square ends that satisfies the delivery service's restriction and has maximum volume. What is the maximum volume?
- Find the dimensions (radius and height) of a cylinder container that meets the delivery service's requirement and has maximum volume. What is the maximum volume?



### Solution

- a) Let  $x$ : length of the side of the square.

$$L + 4x = 108 \rightarrow L = 108 - 4x$$

The volume of the box:

$$V = L \cdot x^2 = (108 - 4x) \cdot x^2$$

$$= 108x^2 - 4x^3$$

$$V' = 216x - 12x^2 = 0$$

$$x = 0 \quad |x = \frac{216}{12} = 18|$$

$$V(18) = 108(18)^2 - 4(18)^3 = 11,664$$

$$L = 108 - 4(18) = 36$$

Therefore, the volume is maximum at  $11,664 \text{ in}^3$  for an 18" by 18" by 36" container.

- b) Let  $x$ : radius and  $L$ : height of the cylindrical container.

$$L + 2\pi x = 108 \rightarrow L = 108 - 2\pi x$$

$$V = \pi x^2 L$$

$$= \pi x^2 (108 - 2\pi x)$$

$$= 108\pi x^2 - 2\pi^2 x^3$$

$$V' = 216\pi x - 6\pi^2 x^2$$

$$= 6\pi x(36 - \pi x)$$

$$x = 0 \quad |x = \frac{36}{\pi}|$$

$$V\left(\frac{36}{\pi}\right) = 108\pi\left(\frac{36}{\pi}\right)^2 - 2\pi^2\left(\frac{36}{\pi}\right)^3 = 14,851 \text{ in}^3$$

$$L = 108 - 2\pi\left(\frac{36}{\pi}\right) = 36$$

Therefore, the volume is maximum at  $14,851 \text{ in}^3$  for a container of radius  $\frac{36}{\pi}$  and height of 36".

### Exercise

The owner of a retail lumber store wants to construct a fence an outdoor storage are adjacent to the store, using all of the store as part of one side of the area. Find the dimensions that will enclose the largest area if

- a) 240 feet fencing material are used.
- b) 400 feet fencing material are used.



### Solution

- a) Let  $x$  and  $y$  be the width and the length of the rectangle respectively.

$$2x + 2y - 100 = 240$$

$$2x + 2y = 340$$

$$x + y = 170 \rightarrow y = 170 - x$$

The area:  $A = xy$

$$= x(170 - x)$$

$$= 170x - x^2 \quad 100 \leq x \leq 170$$

$$A' = 170 - 2x = 0$$

$|x = \frac{170}{2} = 85|$  which is not in the domain.

$$A(100) = 170(100) - (100)^2 = 7,000$$

$$A(170) = 170(170) - (170)^2 = 0$$

Thus, the maximum occurs when  $x = 100$  and  $y = 170 - 100 = 70|$

- b)  $2x + 2y - 100 = 400$

$$2x + 2y = 500$$

$$x + y = 250 \rightarrow y = 250 - x$$

$$A = xy = x(250 - x)$$

$$= 250x - x^2 \quad 100 \leq x \leq 250$$

$$A' = 250 - 2x = 0 \rightarrow |x = 125|$$

$$A(125) = 250(125) - (125)^2 = 15,625$$

Thus, the maximum occurs when  $x = 125 \text{ ft}$  and  $y = 250 - 125 = 125 \text{ ft}|$

### Exercise

The cost per hour for fuel to run a train is  $\frac{v^2}{4}$  dollars, where  $v$  is the speed of the train in miles per hour.

(Note that the cost goes up as the square of the speed.) Other costs, including labor are \$300 per hour.

How fast should the train travel on a 360-mile trip to minimize the total cost for the trip?

### Solution

$x$  = number of hours it take the train to travel 360 miles.

$$\text{Then } 360 = xv \rightarrow x = \frac{360}{v}$$

$$\text{Cost: } C = \left( 300 + \frac{v^2}{4} \right) \left( \frac{360}{v} \right)$$

$$C(v) = \frac{108,000}{v} + 90v$$

$$C'(v) = -\frac{108,000}{v^2} + 90 = 0$$

$$\frac{108,000}{v^2} = 90 \rightarrow v^2 = \frac{108,000}{90} = 1,200$$

$$|v = \sqrt{1200} \approx 34.64|$$

$$C''(v) = \frac{216,000}{v^3} > 0$$

The cost has an absolute minimum at  $x = 34.64$

$$C(34.64) = \frac{108,000}{34.64} + 90(34.64) = 6,235.38$$

### Exercise

A large grocery chain found that, on average, a checker can recall  $P\%$  of a given price list  $x$  hours after starting work, as given approximately by

$$P(x) = 96x - 24x^2 \quad 0 \leq x \leq 3$$

At what time  $x$  does the checker recall a maximum percentage? What is the maximum?

### Solution

$$P'(x) = 96 - 48x = 0 \rightarrow |x = 2|$$

$$P''(x) = -48 < 0 \rightarrow P(x) \text{ has a local maximum at } x = 2.$$

$$P(0) = 0$$

$$P(2) = 96(2) - 24(2)^2 = 96$$

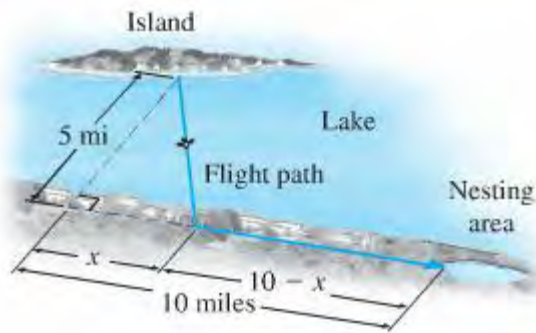
$$P(3) = 96(3) - 24(3)^2 = 72$$

$P(x)$  has its absolute maximum at  $x = 2$  with 96%

## Exercise

Some birds tend to avoid flights over large bodies of water during daylight hours. Suppose that an adult bird with this tendency is taken from its nesting area on the edge of a large lake to an island 5 miles offshore and is then released.

- If it takes only 1.4 times as much energy to fly over water as land, how far up the shore ( $x$ , in miles) should the bird head to minimize the total energy expended in returning to the nesting area?
- If it takes only 1.1 times as much energy to fly over water as land, how far up the shore should the bird head to minimize the total energy expended in returning to the nesting area?



## Solution

- Let the energy to fly over land be 1 unit; then the energy to fly over the water is 1.4 units.

$$(\text{flight path})^2 = x^2 + 5^2$$

Total Energy:

$$E(x) = (1.4)\sqrt{x^2 + 25} + (1)(10 - x)$$

$$= 1.4\sqrt{x^2 + 25} + 10 - x$$

$$E'(x) = \frac{1.4(2x)}{2\sqrt{x^2 + 25}} - 1$$

$$= \frac{1.4x}{\sqrt{x^2 + 25}} - 1 = 0$$

$$\frac{1.4x}{\sqrt{x^2 + 25}} = 1$$

$$1.4x = \sqrt{x^2 + 25}$$

$$(1.4x)^2 = (\sqrt{x^2 + 25})^2$$

$$1.96x^2 = x^2 + 25$$

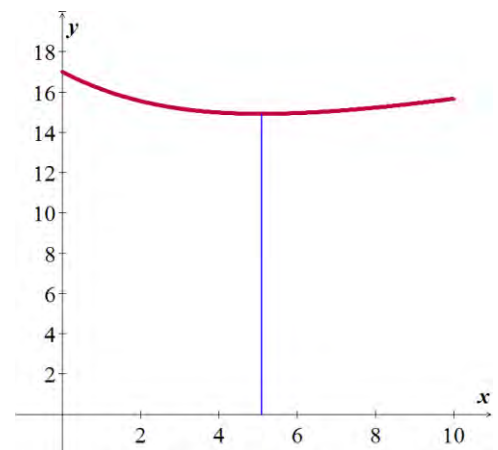
$$0.96x^2 = 25$$

$$x^2 = 26.04 \rightarrow x = \pm 5.1$$

Thus the critical value is  $x = 5.1$

Thus, the energy will be minimum when  $x = 5.1$ .

$$E(5.1) = 1.4\sqrt{(5.1)^2 + 25} + 10 - 5.1 = 14.9$$



0	5.1
$E'(1) < 0$	$E'(6) > 0$
Decreasing	Increasing

$$E(10) = 1.4\sqrt{(10)^2 + 25} + 10 - 10 = 15.65$$

Thus, the absolute minimum occurs when  $x = 5.1$  miles.

$$b) \quad E(x) = 1.1\sqrt{x^2 + 25} + 10 - x \quad 0 \leq x \leq 10$$

$$E'(x) = \frac{1.1x}{\sqrt{x^2 + 25}} - 1 = 0$$

$$1.1x = \sqrt{x^2 + 25} \quad (1.1x)^2 = \left(\sqrt{x^2 + 25}\right)^2$$

$$1.21x^2 = x^2 + 25$$

$$0.21x^2 = 25$$

$$x^2 = 119.05 \rightarrow x = \pm 10.91$$

The critical value  $x = 10.91 > 10$

$$E(0) = 1.1\sqrt{(0)^2 + 25} + 10 - 0 = 15.5$$

$$E(10) = 1.1\sqrt{(10)^2 + 25} + 10 - 10 = 12.30$$

Therefore, the absolute minimum occurs when  $x = 10$