$$y' + 3y = e^{2t} \qquad y(0) = -1$$

$$\mathcal{L}\{y' + 3y\} = \mathcal{L}\{e^{2t}\}$$

$$sY(s) - y(0) + 3Y(s) = \frac{1}{s-2}$$

$$(s+3)Y(s) = \frac{1}{s-2} - 1 = \frac{-s+3}{(s+3)(s-2)}$$

$$Y(s) = \frac{1}{(s-2)(s+3)} - \frac{1}{(s+3)}$$

$$\frac{A}{s-2} + \frac{B}{s+3} \Rightarrow As + 3A + Bs - 2B = 1$$

$$s \quad A + B = 0$$

$$s^0 \quad 3A - 2B = 1$$

$$A = +\frac{1}{5} \qquad B = -\frac{1}{5}$$

$$Y(s) = \frac{1}{5}\frac{1}{s-2} + \left(-\frac{1}{5} - 1\right)\frac{1}{s+3}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{6}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$\boxed{y(t) = \frac{1}{5}e^{2t} - \frac{6}{5}e^{-3t}}$$

Ex.

$$x'' + 4x' + 4x = t^2 \qquad x(0) = x'(0) = 0$$

$$\mathcal{L}\{x'' + 4x' + 4x\} = \mathcal{L}\{t^2\}$$

$$s^2 X(s) - sX(0) - X'(0) + 4sX(s) - 4X(0) + 4X(s) = \frac{2}{s^3}$$

$$(s^2 + 4s + 4)X(s) = \frac{2}{s^3}$$

$$X(s) = \frac{2}{s^3(s+2)^2}$$

$$\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+2} + \frac{E}{(s+2)^2} = \frac{2}{}$$

$$As^2(s^2 + 4s + 4) + Bs(s^2 + 4s + 4)$$
$$+ C(s^2 + 4s + 4) + Ds^3(s+2) + Es^3 = 2$$

s^4: $A + D = 0$

s^3: $4A + B + 2D + E = 0$ (2)

s^2: $4A + 4B + C = 0$ (1)

s^1: $4B + 4C = 0 \rightarrow \boxed{B = -\frac{1}{2}}$

s^0: $4C = 2 \rightarrow \boxed{C = \frac{1}{2}}$

$$\boxed{A = -B - \frac{1}{4}C = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}}$$

$$\boxed{D = -\frac{3}{8}}$$

$$(2) \rightarrow E = -\frac{3}{2} + \frac{1}{2} + \frac{3}{4} = \boxed{-\frac{1}{4}}$$

$$\mathcal{L}^{-1}\{X(s)\} = \frac{3}{8}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$
$$- \frac{3}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

$$x(t) = \frac{3}{8} - \frac{1}{2}t + \frac{1}{4}t^2 - \frac{3}{8}e^{-2t} - \frac{1}{4}te^{-2t}$$

$y'' + 4y' + 8y = \sin t \qquad y(0) = 1$
$y'(0) = 0$

$\mathcal{L}\{y'' + 4y' + 8y\} = \mathcal{L}\{\sin t\}$

$$s^2 Y(s) - s\,y(0) - \cancel{y'(0)} + 4s\,Y(s) - 4y(0) + 8Y(s) = \frac{1}{s^2+1}$$

$$(s^2 + 4s + 8)\,Y(s) = \frac{1}{s^2+1} + s + 4$$

$$Y(s) = \frac{s^3 + 4s^2 + s + 5}{(s^2+1)(s^2+4s+8)}$$

$$= \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4s+8} \to (s+2)^2+4$$

$Y_2(s)$

$s^3 \quad A + C = 1 \to C = 1 - A = \frac{69}{65}$

$s^2 \quad 4A + B + D = 4$

$s^1 \quad 8A + 4B + C = 1$

$s^0 \quad + 8B + D = 5 \to D = 5 - 8B$

$$\begin{cases} 4A + B + 5 - 8B = 4 \\ 8A + 4B + 1 - A = 1 \end{cases} \qquad \begin{cases} 4A - 7B = -1 \\ 7A + 4B = 0 \end{cases}$$

$A = \frac{-4}{65} \qquad B = \frac{7}{65}$

$D = 5 - \frac{56}{65} = \frac{269}{65}$

$\frac{325}{56}$

$$Y(s) = \frac{-4}{65}\frac{s}{s^2+1} + \frac{7}{65}\frac{1}{s^2+1} + \frac{1}{65}\frac{69s+269}{(s+2)^2+4}$$

$$\mathcal{L}^{-1}\{Y(s)\} = -\frac{4}{65}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{7}{65}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$+ \frac{1}{65}\mathcal{L}^{-1}\left(\frac{69(s+2)}{(s+2)^2+4} + \frac{131}{(s+2)^2+4}\right)\frac{2}{2}$$

$$\boxed{\begin{array}{r} 269 - 2(69) \\ -138 \\ \hline 131 \end{array}}$$ calculation

$$y(t) = -\frac{4}{65}\cos t + \frac{7}{65}\sin t + \frac{69}{65}e^{-2t}\cos 2t$$

$$+ \frac{131}{130}e^{-2t}\sin 2t$$

$$\begin{cases} x_1'(t) = 2x_1 + 2x_2 \\ x_2'(t) = x_1 + 3x_2 \end{cases}$$

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= \lambda^2 - 5\lambda + 4 = 0$$

Eigenvalues: $\lambda_{1,2} = 1, 4$

For $\lambda_1 = 1 \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow x = -2y$$

$$V_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 4 \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow x = y \qquad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x(t) = c_1\begin{pmatrix} -2 \\ 1 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{4t}$$

$$x(t) = \begin{cases} -2c_1 e^{t} + c_1 e^{4t} \\ c_1 e^{t} + c_2 e^{4t} \end{cases}$$

$$\begin{cases} x_1' = 3x_1 - x_2 \\ x_2' = x_1 + x_2 \end{cases} \qquad \begin{aligned} x_1(0) &= 2 \\ x_2(0) &= -1 \end{aligned}$$

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= \lambda^2 - 4\lambda + 4 = 0$$

Eigenvalues: $\lambda_{1,2} = 2$.

For $\lambda = 2 \Rightarrow (A - \lambda I)^2 V_2 = 0$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \cancel{x = y}$$

$$\cancel{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$(A - \lambda I) V_2 = V_1$$

$$V_1 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$X(t) = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + C_2\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) e^{2t}$$

$$= \begin{pmatrix} C_1 + C_2 t + 1 \\ C_2 \end{pmatrix} e^{2t}$$

$$\begin{aligned} x_1(0) &= 2 \\ x_2(0) &= -1 \end{aligned} \qquad X(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} c_1 + 1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \rightarrow \begin{matrix} c_1 = 1 \\ c_2 = -1 \end{matrix}$$

$$x(t) = \begin{pmatrix} -t + 2 \\ -1 \end{pmatrix} e^{2t}$$

$$(A - 2I) v_1 = 0$$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad x = y$$

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

$$(A - 2I) v_2 = v_1$$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{matrix} x - y = 1 \\ y = 0 \rightarrow x = 1 \end{matrix}$$

$$v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$x(t) = \left[c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \right] e^{2t}$$

$$= \begin{pmatrix} c_2 t + c_1 + c_2 \\ c_2 t + c_1 \end{pmatrix} e^{2t}$$

$$\begin{pmatrix} c_1 + c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad \begin{matrix} c_1 = -1 \\ c_2 = 3 \end{matrix}$$

$$x(t) = \begin{pmatrix} 3t + 2 \\ 3t - 1 \end{pmatrix} e^{2t} = \cancel{\begin{pmatrix} -t + 2 \\ -1 \end{pmatrix}} e^{2t}$$

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 5 \\ 2 \end{pmatrix} \qquad \cancel{\begin{pmatrix} 2 \\ -1 \end{pmatrix}} \begin{pmatrix} 1 \\ \end{pmatrix}$$

$$\begin{cases} x_1'(t) = 6x_1 - x_2 \\ x_2'(t) = 5x_1 + 2x_2 \end{cases}$$

$$A = \begin{pmatrix} 6 & -1 \\ 5 & 2 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 6-\lambda & -1 \\ 5 & 2-\lambda \end{vmatrix}$$

$$= \lambda^2 - 8\lambda + 17$$

$$\lambda_{1,2} = 4 \pm i$$

$x_1' = x_1 + 9x_2$

$x_2' = -2x_1 - 5x_2$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 9 \\ -2 & -5-\lambda \end{vmatrix}$$

$$= \lambda^2 + 4\lambda + 13 = 0$$

$$\lambda_{1,2} = -2 \pm 3i$$

For $\lambda = -2+3i \Rightarrow (A - \lambda I)V_1 = 0$

$$\begin{pmatrix} 3-3i & 9 \\ -2 & \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$3(1-i)x = -9y \qquad V_1 = \begin{pmatrix} -3 \\ 1-i \end{pmatrix}$$

$$(1-i)x = -3y$$

$$x(t) = \begin{pmatrix} -3 \\ 1-i \end{pmatrix} e^{(-2+3i)t} \quad / \quad e^{-2t}\, e^{3it} \quad \begin{pmatrix} 3 \\ i-1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 1-i \end{pmatrix} (\cos 3t + i \sin 3t) e^{-2t}$$

$$= \begin{pmatrix} -3\cos 3t - i\, 3\sin 3t \\ \cos 3t + \sin 3t + i(-\cos 3t + \sin 3t) \end{pmatrix} e^{-2t}$$

$$x(t) = \left[C_1 \begin{pmatrix} -3\cos 3t \\ \cos 3t + \sin 3t \end{pmatrix} + C_2 \begin{pmatrix} 3\sin 3t \\ \sin 3t - \cos 3t \end{pmatrix} \right] e^{-2t}$$