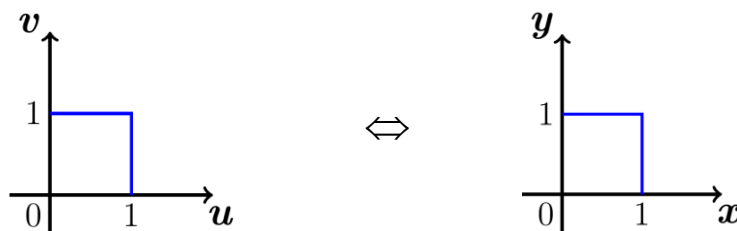


Solution **Section 3.7 – Change of Variables in Multiple Integrals**

Exercise

Let $S = \{0 \leq u \leq 1, 0 \leq v \leq 1\}$ be a unit square in the uv -plane. Find the image of S in the xy -plane under the following transformations. $T: x = v, y = u$

Solution

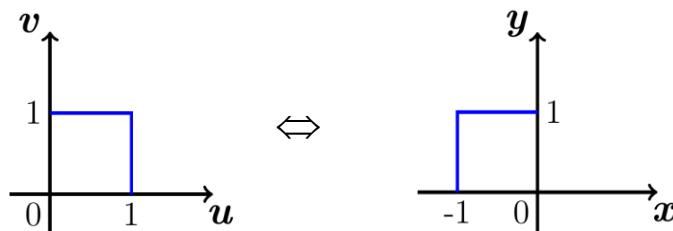


The transformation just switches the coordinates. Image xy is unit square.

Exercise

Let $S = \{0 \leq u \leq 1, 0 \leq v \leq 1\}$ be a unit square in the uv -plane. Find the image of S in the xy -plane under the following transformations. $T: x = -v, y = u$

Solution



$$T: x = -v, y = u$$

$$\begin{aligned} T &= \{(x, y): 0 \leq -x \leq 1, 0 \leq y \leq 1\} \\ &= \{(x, y): -1 \leq x \leq 0, 0 \leq y \leq 1\} \end{aligned}$$

T is a unit square in QII with one vertex at origin.

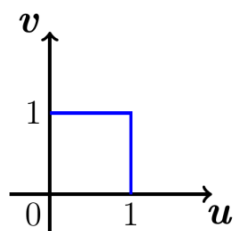
Exercise

Let $S = \{0 \leq u \leq 1, 0 \leq v \leq 1\}$ be a unit square in the uv -plane. Find the image of S in the xy -plane under the following transformations. $T: x = \frac{u+v}{2}, y = \frac{u-v}{2}$

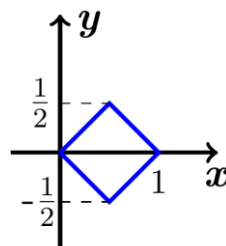
Solution

$$T: x = \frac{u+v}{2}, y = \frac{u-v}{2}$$

(u, v)	(x, y)
$(0, 0) \rightarrow (1, 0)$	$(0, 0) \rightarrow \left(\frac{1}{2}, \frac{1}{2}\right)$
$(1, 0) \rightarrow (1, 1)$	$\left(\frac{1}{2}, \frac{1}{2}\right) \rightarrow (1, 0)$
$(1, 1) \rightarrow (0, 1)$	$(1, 0) \rightarrow \left(\frac{1}{2}, -\frac{1}{2}\right)$



\Leftrightarrow



Diamond shape

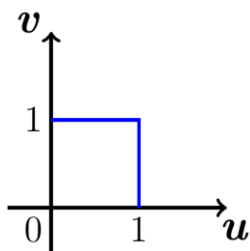
Exercise

Let $S = \{0 \leq u \leq 1, 0 \leq v \leq 1\}$ be a unit square in the uv -plane. Find the image of S in the xy -plane under the following transformations. $T: x = u, y = 2v + 2$

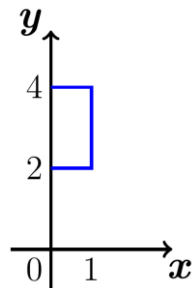
Solution

$$T: x = u, y = 2v + 2$$

(u, v)	(x, y)
$(0, 0) \rightarrow (1, 0)$	$(0, 2) \rightarrow (1, 2)$
$(1, 0) \rightarrow (1, 1)$	$(1, 2) \rightarrow (1, 4)$
$(1, 1) \rightarrow (0, 1)$	$(1, 4) \rightarrow (0, 4)$



\Leftrightarrow



Exercise

a) Solve the system $u = x - y$, $v = 2x + y$ for x and y in terms of u and v . Then find the value of

the Jacobian $\frac{\partial(x, y)}{\partial(u, v)}$

b) Find the image under the transformation $u = x - y$, $v = 2x + y$ of the triangular region with vertices $(0, 0)$, $(1, 1)$, and $(1, -2)$ in the xy -plane. Sketch the transformed region in the uv -plane.

Solution

$$a) \begin{cases} u = x - y \\ v = 2x + y \end{cases} \rightarrow \begin{cases} x = \frac{1}{3}u + \frac{1}{3}v \\ y = -\frac{2}{3}u + \frac{1}{3}v \end{cases}$$

$$\begin{aligned} \frac{\partial(x, y)}{\partial(u, v)} &= \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix} \\ &= \frac{1}{9} + \frac{2}{9} \\ &= \frac{1}{3} \end{aligned}$$

b) From $(0, 0)$ to $(1, 1) \Rightarrow y = x \rightarrow u = x - y = 0$

From $(0, 0)$ to $(1, -2) \Rightarrow y = -2x \rightarrow u = 2x + y = 0$

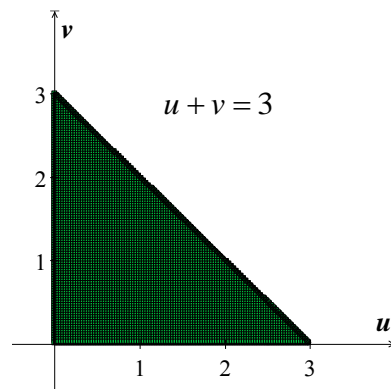
From $(1, 1)$ to $(1, -2) \Rightarrow x = 1$

$$\begin{aligned} \rightarrow x &= \frac{1}{3}u + \frac{1}{3}v = 1 \\ u + v &= 3 \end{aligned}$$

$$\text{OR: } (0, 0) \rightarrow \begin{cases} u = 0 \\ v = 0 \end{cases}$$

$$(1, 1) \rightarrow \begin{cases} u = 0 \\ v = 3 \end{cases}$$

$$(1, -2) \rightarrow \begin{cases} u = 3 \\ v = 0 \end{cases}$$



Exercise

Let R be the region in the first quadrant of the xy -plane bounded by the hyperbolas $xy = 1$, $xy = 9$ and the lines $y = x$, $y = 4x$. Use the transformation $x = \frac{u}{v}$, $y = uv$ with $u > 0$, and $v > 0$ to rewrite

$$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy} \right) dx dy$$

As an integral over an appropriate region G in the uv -plane. Then evaluate the uv -integral over G .

Solution

$$\begin{aligned} x = \frac{u}{v} &\rightarrow u = xv \\ y = uv &\rightarrow y = xv^2 \end{aligned} \quad \begin{cases} \frac{y}{x} = v^2 \\ xy = u^2 \end{cases}$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{u}{v} + \frac{u}{v} = \underline{2\frac{u}{v}}$$

$$\begin{aligned} xy = 1 = u^2 &\rightarrow \begin{cases} u = 1 \\ u = 3 \end{cases} \\ xy = 9 = u^2 &\end{aligned}$$

$$\begin{aligned} y = x &\Rightarrow \frac{y}{x} = 1 = v^2 \\ y = 4x &\Rightarrow \frac{y}{x} = 4 = v^2 \end{aligned} \quad \rightarrow \quad \begin{cases} v = 1 \\ v = 2 \end{cases}$$

$$\begin{aligned} \iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy} \right) dx dy &= \int_1^3 \int_1^2 (v + u) \frac{2u}{v} dv du \\ &= 2 \int_1^3 \int_1^2 \left(u + \frac{u^2}{v} \right) dv du \\ &= 2 \int_1^3 \left[uv + u^2 \ln v \right]_1^2 du \\ &= 2 \int_1^3 (2u + u^2 \ln 2 - u) du \\ &= 2 \int_1^3 (u + u^2 \ln 2) du \\ &= 2 \left[\frac{1}{2} u^2 + \frac{1}{3} u^3 \ln 2 \right]_1^3 \\ &= 2 \left(\frac{9}{2} + 9 \ln 2 - \frac{1}{2} - \frac{1}{3} \ln 2 \right) \\ &= \underline{8 + \frac{52}{3} \ln 2} \end{aligned}$$

Exercise

The area πab of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be found by integrating the function $f(x, y) = 1$ over the region bounded by the ellipse in the xy -plane. Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate the integral is to use the transformation $x = au$, $y = bv$ and evaluate the transformed integral over the disk G : $u^2 + v^2 \leq 1$ in the uv -plane. Find the area this way.

Solution

$$x = au, y = bv$$

$$\frac{\partial(x, y)}{\partial(u, v)} = J(u, v) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = \underline{ab}$$

$$u^2 + v^2 \leq 1 \rightarrow -1 \leq u \leq 1$$

$$u^2 + v^2 \leq 1 \rightarrow v^2 \leq 1 - u^2 \Rightarrow -\sqrt{1 - u^2} \leq v \leq \sqrt{1 - u^2}$$

$$\iint_R dx dy = \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} ab \, dv du$$

$$= ab \int_{-1}^1 \left(\sqrt{1-u^2} + \sqrt{1-u^2} \right) du$$

$$= 2ab \int_{-1}^1 (1-u^2)^{1/2} du$$

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$$

$$= 2ab \left[\frac{u}{2} \sqrt{1-u^2} + \frac{1}{2} \sin^{-1} u \right]_{-1}^1$$

$$= 2ab \left[\frac{1}{2} \sin^{-1} 1 - \left(\frac{1}{2} \sin^{-1}(-1) \right) \right]$$

$$= 2ab \left[\frac{1}{2} \frac{\pi}{2} - \left(-\frac{1}{2} \frac{\pi}{2} \right) \right]$$

$$= 2ab \left(\frac{\pi}{2} \right)$$

$$= \underline{ab\pi}$$

Exercise

Use the transformation $x = u + \frac{1}{2}v$, $y = v$ to evaluate the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3 (2x-y) e^{(2x-y)^2} dx dy$$

By first writing it as an integral over a region G in the uv -plane.

Solution

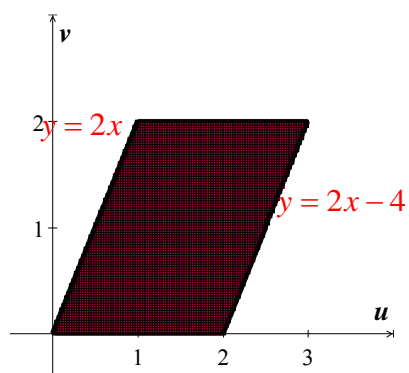
$$\frac{\partial(x, y)}{\partial(u, v)} = J(u, v) = \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = \underline{1}$$

$$\begin{aligned} x = u + \frac{1}{2}v & \rightarrow u = x - \frac{1}{2}v \\ y = v & \rightarrow v = y \end{aligned}$$

$$x = \frac{y}{2} \rightarrow y = 2x$$

$$x = \frac{y+4}{2} \rightarrow y = 2x - 4$$

$$0 \leq x \leq 2$$



$x = \frac{y}{2}$	$u = x - \frac{y}{2} = \frac{y}{2} - \frac{y}{2} = 0$	$u = 0$
$x = \frac{y}{2} + 2$	$u = x - \frac{y}{2} = \frac{y}{2} + 2 - \frac{y}{2} = 2$	$u = 2$
$y = 0$	$v = 0$	$v = 0$
$y = 2$	$v = 2$	$v = 2$

$$\begin{aligned} \int_0^2 \int_{y/2}^{(y+4)/2} y^3 (2x - y) e^{(2x-y)^2} dx dy &= \int_0^2 \int_0^2 v^3 (2u) e^{4u^2} du dv & d(4u^2) &= 8u du \\ &= \frac{1}{4} \int_0^2 \int_0^2 v^3 e^{4u^2} d(4u^2) dv \\ &= \frac{1}{4} \int_0^2 v^3 \left[e^{4u^2} \right]_0^2 dv \\ &= \frac{1}{4} (e^{16} - 1) \int_0^2 v^3 dv \\ &= \frac{1}{4} (e^{16} - 1) \left[\frac{1}{4} v^4 \right]_0^2 \\ &= \underline{e^{16} - 1} \end{aligned}$$

Exercise

Use the transformation $x = \frac{u}{v}$, $y = uv$ to evaluate the integral

$$\int_1^2 \int_{1/y}^y (x^2 + y^2) dx dy + \int_2^4 \int_{y/4}^{4/y} (x^2 + y^2) dx dy$$

Solution

$$\begin{aligned} x = \frac{u}{v} &\rightarrow u = xv \\ y = uv &\rightarrow y = xv^2 \end{aligned} \quad \begin{cases} \frac{y}{x} = v^2 \\ xy = u^2 \end{cases}$$

$$J(u, v) = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{u}{v} + \frac{u}{v} = \frac{2u}{v}$$

$x = y$	$\frac{y}{x} = 1 = v^2$	$v = 1$
$x = \frac{1}{y}$	$xy = 1 = u^2$	$u = 1$
$x = \frac{4}{y}$	$xy = 4 = u^2$	$u = 2$
$x = \frac{y}{4}$	$\frac{y}{x} = 4 = v^2$	$v = 2$

$$\begin{aligned} \int_1^2 \int_{1/y}^y (x^2 + y^2) dx dy + \int_2^4 \int_{y/4}^{4/y} (x^2 + y^2) dx dy &= \int_1^2 \int_1^2 \left(\frac{u^2}{v^2} + u^2 v^2 \right) \left(\frac{2u}{v} \right) du dv \\ &= 2 \int_1^2 \int_1^2 \left(\frac{u^3}{v^3} + u^3 v \right) du dv \\ &= 2 \int_1^2 \left(\frac{1}{v^3} + v \right) \left[\frac{1}{4} u^4 \right]_1^2 dv \\ &= \frac{1}{2} (16 - 1) \int_1^2 (v^{-3} + v) dv \\ &= \frac{15}{2} \left[-\frac{1}{2} v^{-2} + \frac{1}{2} v^2 \right]_1^2 \\ &= \frac{15}{4} \left[-\frac{1}{4} + 4 - (-1 + 1) \right] \\ &= \frac{15}{4} \left(\frac{15}{4} \right) \\ &= \frac{225}{16} \end{aligned}$$

Exercise

Find the Jacobian $\frac{\partial(x, y)}{\partial(u, v)}$ of the transformation

a) $x = u \cos v, \quad y = u \sin v$

b) $x = u \sin v, \quad y = u \cos v$

Solution

$$\begin{aligned}
 a) \quad \frac{\partial(x, y)}{\partial(u, v)} &= \begin{vmatrix} \cos v & -u \sin v \\ \sin v & u \cos v \end{vmatrix} \\
 &= u \cos^2 v + u \sin^2 v \\
 &= \underline{u}
 \end{aligned}$$

$$\begin{aligned}
 b) \quad \frac{\partial(x, y)}{\partial(u, v)} &= \begin{vmatrix} \sin v & u \cos v \\ \cos v & -u \sin v \end{vmatrix} \\
 &= -u \sin^2 v - u \cos^2 v \\
 &= \underline{-u}
 \end{aligned}$$

Exercise

Find the Jacobian $\frac{\partial(x, y)}{\partial(u, v)}$ of the transformation

$$a) \quad x = u \cos v, \quad y = u \sin v, \quad z = w$$

$$b) \quad x = 2u - 1, \quad y = 3v - 4, \quad z = \frac{1}{2}(w - 4)$$

Solution

$$\begin{aligned}
 a) \quad \frac{\partial(x, y, z)}{\partial(u, v, w)} &= \begin{vmatrix} \cos v & -u \sin v & 0 \\ \sin v & u \cos v & 0 \\ 0 & 0 & 1 \end{vmatrix} \\
 &= u \cos^2 v + u \sin^2 v \\
 &= \underline{u}
 \end{aligned}$$

$$b) \quad \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{vmatrix} = \underline{3}$$

Exercise

Evaluate the appropriate determinant to show that the Jacobian of the transformation from Cartesian $\rho\phi\theta$ -space to Cartesian xyz -space is $\rho^2 \sin \phi$

Solution

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi$$

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{vmatrix} \quad \begin{matrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta \\ \cos \phi & -\rho \sin \phi \end{matrix}$$

$$\begin{aligned}
&= \rho^2 \cos^2 \phi \sin \phi \cos^2 \theta + \rho^2 \sin^3 \phi \sin^2 \theta + \rho^2 \sin \phi \cos^2 \phi \sin^2 \theta + \rho^2 \sin^3 \phi \cos^2 \theta \\
&= \rho^2 \cos^2 \phi \sin \phi (\cos^2 \theta + \sin^2 \theta) + \rho^2 \sin^3 \phi (\sin^2 \theta + \cos^2 \theta) \\
&= \rho^2 \cos^2 \phi \sin \phi + \rho^2 \sin^3 \phi \\
&= \rho^2 \sin \phi (\cos^2 \phi + \sin^2 \phi) \\
&= \rho^2 \sin \phi
\end{aligned}$$

Exercise

How can substitutions in single definite integrals be viewed as transformations of regions? What is the Jacobian in such a case? Illustrate with an example.

Solution

$$\text{Let } u = g(x) \Rightarrow J(x) = \frac{du}{dx} = g'(x)$$

$$\int_a^b f(u) du = \int_{g(a)}^{g(b)} f(g(x)) g'(x) dx$$

$g'(x)$ represents the Jacobian of the transformation $u = g(x)$ or $x = g^{-1}(u)$

Exercise

Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

(Hint: Let $x = au$, $y = bv$, and $z = cw$. Then find the volume of an appropriate region in uvw -space)

Solution

$$J(u, v, w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \Rightarrow \frac{(au)^2}{a^2} + \frac{(bv)^2}{b^2} + \frac{(cw)^2}{c^2} \leq 1 \rightarrow u^2 + v^2 + w^2 \leq 1$$

$$u^2 + v^2 + w^2 \leq 1 \Rightarrow V = \frac{4\pi}{3} = \iiint_G du dv dw$$

$$V = \iiint_R dx dy dz$$

$$\begin{aligned}
&= \iiint_G abc \, dudvdw \\
&= abc \iiint_G dudvdw \\
&= \frac{4\pi abc}{3}
\end{aligned}$$

Exercise

Use the transformation $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2} \, dydx$$

(Hint: Show that the image of the triangular region G with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ in the uv -plane is the region of integration R in the xy -plane defined by the limits of integration.)

Solution

$$x = u^2 - v^2, \quad y = 2uv$$

$$\begin{aligned}
J(u, v) &= \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} & J(u, v) &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \\
&= 4u^2 + 4v^2 \\
&= 4(u^2 + v^2)
\end{aligned}$$

$y = 2\sqrt{1-x}$	$2uv = 2\sqrt{1-u^2+v^2} \rightarrow u^2v^2 = 1-u^2+v^2$ $u^2v^2 + u^2 = 1+v^2 \Rightarrow u^2(v^2+1) = 1+v^2$	$u = \pm 1$
$y = 0$	$2uv = 0$	$u = 0, v = 0$
$x = 0$	$u^2 - v^2 = 0$	$u = \pm v$

$$\begin{aligned}
\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2} \, dydx &= \int_0^1 \int_0^u \sqrt{(u^2 - v^2)^2 + (2uv)^2} \cdot 4(u^2 + v^2) \, dvdu \\
&= 4 \int_0^1 \int_0^u \sqrt{u^4 + v^4 - 2u^2v^2 + 4u^2v^2} \cdot (u^2 + v^2) \, dvdu
\end{aligned}$$

$$\begin{aligned}
&= 4 \int_0^1 \int_0^u \sqrt{u^4 + v^4 + 2u^2v^2} \cdot (u^2 + v^2) dv du \\
&= 4 \int_0^1 \int_0^u \sqrt{(u^2 + v^2)^2} \cdot (u^2 + v^2) dv du \\
&= 4 \int_0^1 \int_0^u (u^2 + v^2)^2 dv du \\
&= 4 \int_0^1 \int_0^u (u^4 + v^4 + 2u^2v^2) dv du \\
&= 4 \int_0^1 \left[u^4v + \frac{1}{5}v^5 + \frac{2}{3}u^2v^3 \right]_0^u du \\
&= 4 \int_0^1 \left(u^5 + \frac{1}{5}u^5 + \frac{2}{3}u^5 \right) du \\
&= \frac{112}{15} \int_0^1 u^5 du \\
&= \frac{112}{15} \left[\frac{1}{6}u^6 \right]_0^1 \\
&= \frac{56}{45}
\end{aligned}$$

Exercise

Evaluate $\iint_R y^4 dA$; R is the region bounded by the hyperbolas $xy = 1$ and $xy = 4$ and the lines $\frac{y}{x} = 1$

, and $\frac{y}{x} = 3$

Solution

$$\text{Let } \begin{cases} u = xy & \rightarrow x = \frac{u}{y} \\ v = \frac{y}{x} & \rightarrow x = \frac{y}{v} \end{cases}$$

$$x = \frac{y}{v} = \frac{u}{y} \rightarrow y^2 = uv \Rightarrow \underline{y = \sqrt{uv}}$$

$$x = \frac{\sqrt{uv}}{v} = \underline{\sqrt{\frac{u}{v}}}$$

$$x = u^{1/2}v^{-1/2} \quad y = u^{1/2}v^{1/2}$$

$$J(u, v) = \begin{vmatrix} \frac{1}{2}u^{-1/2}v^{-1/2} & -\frac{1}{2}u^{1/2}v^{-3/2} \\ \frac{1}{2}u^{-1/2}v^{1/2} & \frac{1}{2}u^{1/2}v^{-1/2} \end{vmatrix}$$

$$= \frac{1}{4}v^{-1} + \frac{1}{4}v^{-1}$$

$$= \frac{1}{2v}$$

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$xy = 1$	$u = xy = 1$	$u = 1$
$xy = 4$	$u = xy = 4$	$u = 4$
$\frac{y}{x} = 1$	$v = \frac{y}{x} = 1$	$v = 1$
$\frac{y}{x} = 3$	$v = \frac{y}{x} = 3$	$v = 3$

$$\iint_R y^4 dA = \int_1^4 \int_1^3 \frac{1}{2v} (\sqrt{uv})^4 dv du$$

$$\iint_R f(x, y) dx dy = \iint_G f(g(u, v), h(u, v)) |J(u, v)| du dv$$

$$= \frac{1}{2} \int_1^4 u^2 du \int_1^3 v dv$$

$$= \frac{1}{12} u^3 \Big|_1^4 v^2 \Big|_1^3$$

$$= \frac{1}{12} (64 - 1)(9 - 1)$$

$$= \frac{504}{12}$$

$$= 42$$

Exercise

Evaluate $\iint_R (y^2 + xy - 2x^2) dA$; R is the region bounded by the lines $y = x$, $y = x - 3$, $y = -2x + 3$,

and $y = -2x - 3$

Solution

$$\begin{cases} y - x = 0 & y - x = -3 \\ y + 2x = 3 & y + 2x = -3 \end{cases}$$

Let $\begin{cases} u = y - x \\ v = y + 2x \end{cases}$

$$\begin{aligned} y - x &= u \\ \frac{y + 2x = v}{x = \frac{1}{3}(v - u)} &\rightarrow y = \frac{1}{3}(v + 2u) \end{aligned}$$

$$J(u, v) = \begin{vmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{vmatrix} \\ = -\frac{1}{3}$$

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$y = x$	$\frac{1}{3}(v + 2u) = \frac{1}{3}(v - u) \rightarrow 2u = -u$	$u = 0$
$y = x - 3$	$\frac{1}{3}(v + 2u) = \frac{1}{3}(v - u) - 3 \rightarrow 2u = -u - 9$	$u = -3$
$y = -2x + 3$	$\frac{1}{3}(v + 2u) = -\frac{2}{3}(v - u) + 3 \rightarrow v = -2v + 9$	$v = 3$
$y = -2x - 3$	$\frac{1}{3}(v + 2u) = -\frac{2}{3}(v - u) - 3 \rightarrow v = -2v - 9$	$v = -3$

$$\begin{aligned} y^2 + xy - 2x^2 &= \frac{1}{9}(v + 2u)^2 + \frac{1}{9}(v + 2u)(v - u) - \frac{2}{9}(v - u)^2 \\ &= \frac{1}{9}(v^2 + 4uv + 4u^2 + v^2 + uv - 2u^2 - 2v^2 + 4uv - 2u^2) \\ &= uv \end{aligned}$$

$$\begin{aligned} \iint_R (y^2 + xy - 2x^2) dA &= \int_{-3}^0 \int_{-3}^3 \left(-\frac{1}{3}\right) uv \, dv du \\ &= -\frac{1}{3} \int_{-3}^0 u \, du \int_{-3}^3 v \, dv \\ &= -\frac{1}{12} u^2 \Big|_{-3}^0 v^2 \Big|_{-3}^3 \\ &= -\frac{1}{12} (-9)(9 - 9) \\ &= 0 \end{aligned}$$

Exercise

Evaluate $\iiint_D x \, dV$; R is bounded by the planes $y - 2x = 0$, $y - 2x = 1$, $z - 3y = 0$, $z - 3y = 1$,
 $z - 4x = 0$ and $z - 4x = 3$

Solution

$$\text{Let } \begin{cases} u = y - 2x & \rightarrow & 0 \leq u \leq 1 \\ v = z - 3y & \rightarrow & 0 \leq v \leq 1 \\ w = z - 4x & \rightarrow & 0 \leq w \leq 3 \end{cases}$$

$$\begin{cases} u = y - 2x & \rightarrow & y = u + 2x \\ w = z - 4x & \rightarrow & z = w + 4x \end{cases}$$

$$\begin{aligned} v &= z - 3y \\ &= w + 4x - 3u - 6x \end{aligned}$$

$$2x = w - 3u - v$$

$$\begin{cases} x = -\frac{3}{2}u - \frac{1}{2}v + \frac{1}{2}w \\ y = -2u - v + w \\ z = -6u - 2v + 3w \end{cases}$$

$$\begin{aligned} J(u, v, w) &= \begin{vmatrix} -2 & -1 & 1 \\ -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ -6 & -2 & 3 \end{vmatrix} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \iint_D x \, dV &= \int_0^3 \int_0^1 \int_0^1 \left(\frac{1}{2}\right) \frac{1}{2} (-3u - v + w) \, du \, dv \, dw \\ &= \frac{1}{4} \int_0^3 \int_0^1 \left(-\frac{3}{2}u^2 - vu + wu\right) \Big|_0^1 \, dv \, dw \\ &= \frac{1}{4} \int_0^3 \int_0^1 \left(-\frac{3}{2} - v + w\right) \, dv \, dw \\ &= \frac{1}{4} \int_0^3 \left(-\frac{3}{2}v - \frac{1}{2}v^2 + wv\right) \Big|_0^1 \, dw \\ &= \frac{1}{4} \int_0^3 \left(-\frac{3}{2} - \frac{1}{2} + w\right) \, dw \\ &= \frac{1}{4} \left(-2w + \frac{1}{2}w^2\right) \Big|_0^3 \\ &= \frac{1}{4} \left(-6 + \frac{9}{2}\right) \\ &= -\frac{3}{8} \end{aligned}$$

Exercise

Let R be the region bounded by the lines $x + y = 1$; $x + y = 4$; $x - 2y = 0$; $x - 2y = -4$

Evaluate the integral $\iint_R 3xy dA$

Solution

$$\text{Let } \begin{cases} u = x + y \\ v = x - 2y \end{cases}$$

$$\overline{u - v = 3y} \rightarrow \underline{y = \frac{1}{3}(u - v)}$$

$$x = u - y$$

$$= u - \frac{1}{3}u + \frac{1}{3}v$$

$$= \underline{\frac{2}{3}u + \frac{1}{3}v}$$

$$J(u, v) = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix}$$

$$= \underline{-\frac{1}{3}}$$

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$x + y = 1$	$\frac{2}{3}u + \frac{1}{3}v + \frac{1}{3}u - \frac{1}{3}v = 1$	$u = 1$
$x + y = 4$	$\frac{2}{3}u + \frac{1}{3}v + \frac{1}{3}u - \frac{1}{3}v = 4$	$u = 4$
$x - 2y = 0$	$\frac{2}{3}u + \frac{1}{3}v - \frac{2}{3}u + \frac{2}{3}v = 0$	$v = 0$
$x - 2y = -4$	$\frac{2}{3}u + \frac{1}{3}v - \frac{2}{3}u + \frac{2}{3}v = -4$	$v = -4$

$$\iint_R 3xy dA = \int_1^4 \int_{-4}^0 3 \frac{1}{3}(2u + v) \frac{1}{3}(u - v) \left| -\frac{1}{3} \right| dv du$$

$$= \frac{1}{9} \int_1^4 \int_{-4}^0 (2u^2 - uv - v^2) dv du$$

$$= \frac{1}{9} \int_1^4 \left(2u^2v - \frac{1}{2}uv^2 - \frac{1}{3}v^3 \right) \Big|_{-4}^0 du$$

$$= \frac{1}{9} \int_1^4 \left(8u^2 + 8u - \frac{64}{3} \right) du$$

$$= \frac{8}{9} \left(\frac{1}{3}u^3 + \frac{1}{2}u^2 - \frac{8}{3}u \right) \Big|_1^4$$

$$\begin{aligned}
&= \frac{8}{9} \left(\frac{64}{3} + 8 - \frac{32}{3} - \frac{1}{3} - \frac{1}{2} + \frac{8}{3} \right) \\
&= \frac{8}{9} \left(\frac{39}{3} + \frac{15}{2} \right) \\
&= \frac{8}{9} \left(\frac{123}{6} \right) \\
&= \frac{164}{9}
\end{aligned}$$

Exercise

Let R be the region bounded by the square with vertices $(0, 1)$, $(1, 2)$, $(2, 1)$, & $(1, 0)$.

Evaluate the integral $\iint_R (x+y)^2 \sin^2(x-y) dA$

Solution

$$(0, 1) \text{ \& \; } (1, 2) \rightarrow m = \frac{2-1}{1-0} = 1 \Rightarrow \underline{y = x+1}$$

$$(0, 1) \text{ \& \; } (1, 0) \rightarrow m = \frac{0-1}{1-0} = -1 \Rightarrow \underline{y = -x+1}$$

$$(2, 1) \text{ \& \; } (1, 0) \rightarrow m = 1 \Rightarrow \underline{y = x-1}$$

$$(2, 1) \text{ \& \; } (1, 2) \rightarrow m = -1 \Rightarrow \underline{y = -x+3}$$

$$\begin{cases} y-x=1 & y+x=1 \\ y-x=-1 & y+x=3 \end{cases}$$

Let $\begin{cases} u = y-x \\ v = y+x \end{cases}$

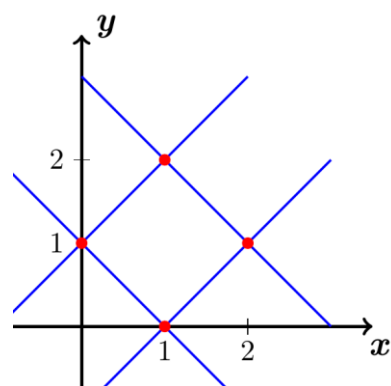
$$\underline{u+v=2y \rightarrow y = \frac{1}{2}u + \frac{1}{2}v}$$

$$x = y - u$$

$$\underline{= -\frac{1}{2}u + \frac{1}{2}v}$$

$$\begin{aligned} J(u, v) &= \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} \\ &= -\frac{1}{2} \end{aligned}$$

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$



$y - x = 1$	$\frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}u - \frac{1}{2}v = 1$	$u = 1$
$y - x = -1$	$\frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}u - \frac{1}{2}v = -1$	$u = -1$
$x + y = 1$	$\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}u + \frac{1}{2}v = 1$	$v = 1$

$x + y = 3$	$\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}u + \frac{1}{2}v = 3$	$v = 3$
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$$\begin{aligned}
\iint_R (x+y)^2 \sin^2(x-y) dA &= \int_{-1}^1 \int_1^3 \left(-\frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}u + \frac{1}{2}v\right)^2 \sin^2\left(-\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}u - \frac{1}{2}v\right) dv du \\
&= \int_{-1}^1 \int_1^3 v^2 \sin^2(-u) dv du \\
&= \int_{-1}^1 \frac{1}{2}(1 - \cos 2u) du \left(\frac{1}{3}v^3\right)\bigg|_1^3 \\
&= \frac{1}{6}(27 - 1) \left(u - \frac{1}{2}\sin 2u\right)\bigg|_{-1}^1 \\
&= \frac{13}{3}\left(1 - \frac{1}{2}\sin 2 + 1 - \frac{1}{2}\sin 2\right) \\
&= \frac{13}{3}(2 - \sin 2)
\end{aligned}$$

Exercise

Evaluate $\iiint_D yz dV$ D is bounded by the planes: $x + 2y = 1$, $x + 2y = 2$, $x - z = 0$, $x - z = 2$, $2y - z = 0$, and $2y - z = 3$

Solution

$$\text{Let } \begin{cases} u = x + 2y \\ v = x - z \\ w = 2y - z \end{cases} \rightarrow v - w = x - 2y$$

$$\begin{cases} u = x + 2y \\ v - w = x - 2y \end{cases} \rightarrow u + v - w = 2x$$

$$2y = \frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}w - v + u$$

$$z = \frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}w - v$$

$$\begin{cases} x = \frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}w \\ y = \frac{1}{4}u - \frac{1}{4}v + \frac{1}{4}w \\ z = \frac{1}{2}u - \frac{1}{2}v + \frac{1}{2}w \end{cases}$$

$$\begin{aligned}
J(u, v, w) &= \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{vmatrix} \\
&= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} - \frac{1}{16} + \frac{1}{16} + \frac{1}{16} \\
&= \frac{1}{4}
\end{aligned}$$

$x + 2y = 1$	$\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}w + \frac{1}{2}u - \frac{1}{2}v + \frac{1}{2}w = 1$	$u = 1$
$x + 2y = 2$	$\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}w + \frac{1}{2}u - \frac{1}{2}v + \frac{1}{2}w = 2$	$u = 2$
$x - z = 0$	$\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}w - \frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}w = 0$	$v = 0$
$x - z = 2$	$\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}w - \frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}w = 2$	$v = 2$
$2y - z = 0$	$\frac{1}{2}u - \frac{1}{2}v + \frac{1}{2}w - \frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}w = 0$	$w = 0$
$2y - z = 3$	$\frac{1}{2}u - \frac{1}{2}v + \frac{1}{2}w - \frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}w = 3$	$w = 3$

$$\begin{aligned}
\iiint_D yz dV &= \int_1^2 \int_0^2 \int_0^3 \frac{1}{8}(u-v+w)(u-v-w) \frac{1}{4} dw dv du \\
&= \frac{1}{32} \int_1^2 \int_0^2 \int_0^3 \left((u-v)^2 + w^2 \right) dw dv du \\
&= \frac{1}{32} \int_1^2 \int_0^2 \left((u^2 - 2uv + v^2)w + \frac{1}{3}w^3 \right) \Big|_0^3 dv du \\
&= \frac{1}{32} \int_1^2 \int_0^2 \left(3u^2 - 6uv + 3v^2 + 9 \right) dv du \\
&= \frac{1}{32} \int_1^2 \left(3u^2v - 3uv^2 + v^3 + 9v \right) \Big|_0^2 du \\
&= \frac{1}{32} \int_1^2 \left(6u^2 - 12u + 8 + 18 \right) du \\
&= \frac{1}{32} \int_1^2 \left(6u^2 - 12u + 26 \right) du
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{32} \left(2u^3 - 6u^2 + 26u \right) \Big|_1^2 \\
&= \frac{1}{16} (8 - 12 + 26 - 1 + 3 - 13) \\
&= \frac{11}{16}
\end{aligned}$$

Exercise

Evaluate $\iint_R xy \, dA$; R is the square with vertices $(0, 0)$, $(1, 1)$, $(2, 0)$, and $(1, -1)$

Solution

$$(0, 0) \text{ \& } (1, 1) \rightarrow y = x \quad \Rightarrow y - x = 0$$

$$(0, 0) \text{ \& } (2, 0) \rightarrow y = 0$$

$$(0, 0) \text{ \& } (1, -1) \rightarrow y = -x \quad \Rightarrow y + x = 0$$

$$(1, 1) \text{ \& } (2, 0) \rightarrow y = -x + 2 \quad \Rightarrow y + x = 2$$

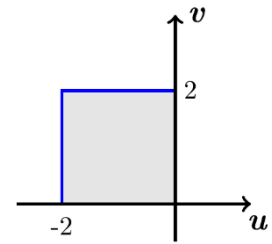
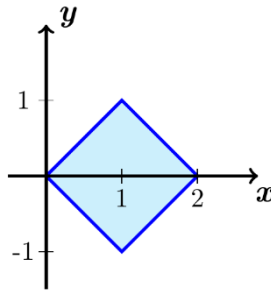
$$(1, 1) \text{ \& } (1, -1) \rightarrow x = 1$$

$$(2, 0) \text{ \& } (1, -1) \rightarrow y = x - 2 \quad \Rightarrow y - x = -2$$

$$\begin{aligned}
\text{Let } \begin{cases} u = y - x \\ v = y + x \end{cases} \\
\frac{u + v}{2} = y \rightarrow \underline{y = \frac{1}{2}(u + v)}
\end{aligned}$$

$$\begin{aligned}
x &= v - y \\
&= v - \frac{1}{2}u - \frac{1}{2}v \\
&= \underline{\frac{1}{2}v - \frac{1}{2}u}
\end{aligned}$$

$$\begin{cases} x = -\frac{1}{2}u + \frac{1}{2}v \\ y = \frac{1}{2}u + \frac{1}{2}v \end{cases}$$



$y - x = 0$	$\frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}u - \frac{1}{2}v = 0$	$u = 0$
$y - x = -2$	$\frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}u - \frac{1}{2}v = -2$	$u = -2$
$y + x = 0$	$\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}u + \frac{1}{2}v = 0$	$v = 0$
$y + x = 2$	$\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}u + \frac{1}{2}v = 2$	$v = 2$

$$J(u, v) = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\begin{aligned} \iint_R xy \, dA &= \int_0^2 \int_{-2}^2 \left(-\frac{1}{2}u + \frac{1}{2}v\right) \left(\frac{1}{2}u + \frac{1}{2}v\right) \left|-\frac{1}{2}\right| du dv \\ &= \frac{1}{8} \int_0^2 \int_{-2}^2 (v-u)(v+u) du dv \\ &= \frac{1}{8} \int_0^2 \int_{-2}^2 (v^2 - u^2) du dv \\ &= \frac{1}{8} \int_0^2 \left(v^2u - \frac{1}{3}u^3\right) \Big|_{-2}^2 dv \\ &= \frac{1}{8} \int_0^2 \left(2v^2 - \frac{8}{3} + 2v^2 - \frac{8}{3}\right) dv \\ &= \frac{1}{8} \int_0^2 \left(4v^2 - \frac{16}{3}\right) dv \\ &= \frac{1}{2} \left(\frac{1}{3}v^3 - \frac{4}{3}v\right) \Big|_0^2 \\ &= \frac{1}{6}(8-8) \\ &= 0 \end{aligned}$$

Exercise

Evaluate $\iint_R x^2 y \, dA$; $R = \{(x, y): 0 \leq x \leq 2, x \leq y \leq x+4\}$

Solution

$$y = x \rightarrow y - x = 0$$

$$y = x + 4 \rightarrow y - x = 4$$

$$\text{Let } \begin{cases} u = x \\ v = y - x \end{cases} \quad y = u + v$$

$$\begin{cases} x = u \\ y = u + v \end{cases}$$

$$J(u, v) = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= 1$$

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$x = 0$		$u = 0$
$x = 2$		$u = 2$
$y - x = 0$	$u + v - u = 0$	$v = 0$
$y - x = 4$	$u + v - u = 4$	$v = 4$

$$\iint_R x^2 y \, dA = \int_0^4 \int_0^2 u^2 (u + v) |1| \, du dv$$

$$= \int_0^4 \int_0^2 (u^3 + vu^2) \, du dv$$

$$= \int_0^4 \left(\frac{1}{4} u^4 + \frac{1}{3} vu^3 \right) \Big|_0^2 \, dv$$

$$= \int_0^4 \left(4 + \frac{8}{3} v \right) \, dv$$

$$= \left(4v + \frac{4}{3} v^2 \right) \Big|_0^4$$

$$= 16 + \frac{64}{3}$$

$$= \frac{112}{3}$$

Exercise

Evaluate $\iint_R x^2 \sqrt{x+2y} \, dA$; $R = \left\{ (x, y) : 0 \leq x \leq 2, -\frac{x}{2} \leq y \leq 1-x \right\}$

Solution

$$y = -\frac{1}{2}x \rightarrow 2y + x = 0$$

$$y = 1-x \rightarrow y + x = 1$$

$$\text{Let } \begin{cases} u = \frac{1}{2}x \\ v = y + x \end{cases}$$

$$\begin{cases} x = 2u \\ y = -2u + v \end{cases}$$

$$J(u, v) = \begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} = 2$$

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$x = 0$	$2u = 0$	$u = 0$
$x = 2$	$2u = 2$	$u = 1$
$y + x = 1$	$v = 1$	$v = 1$
$y = -\frac{1}{2}x$	$-2u + v = -u$	$v = u$

$$\begin{aligned} \iint_R x^2 \sqrt{x+2y} \, dA &= \int_0^1 \int_u^1 4u^2 \sqrt{2u-4u+2v} |2| \, dv du \\ &= 4 \int_0^1 \int_u^1 u^2 (-2u+2v)^{1/2} \, d(-2u+2v) du \\ &= \frac{8}{3} \int_0^1 u^2 (-2u+2v)^{3/2} \Big|_u^1 \, du \\ &= \frac{8}{3} \int_0^1 u^2 \left((-2u+2)^{3/2} - 0 \right) du \\ &= \frac{16\sqrt{2}}{3} \int_0^1 u^2 (1-u)^{3/2} \, du \\ &\quad \text{Let } w = 1-u \rightarrow dw = -du \\ &\quad \quad u = 1-w \\ &= -\frac{16\sqrt{2}}{3} \int_0^1 (1-w)^2 w^{3/2} dw \\ &= -\frac{16\sqrt{2}}{3} \int_0^1 (1-2w+w^2) w^{3/2} dw \\ &= -\frac{16\sqrt{2}}{3} \int_0^1 (w^{3/2} - 2w^{5/2} + w^{7/2}) dw \end{aligned}$$

$$\begin{aligned}
&= -\frac{16\sqrt{2}}{3} \left(\frac{2}{5}(1-u)^{5/2} - \frac{4}{7}(1-u)^{7/2} + \frac{2}{9}(1-u)^{9/2} \right) \Big|_0^1 \\
&= -\frac{16\sqrt{2}}{3} \left(-\frac{2}{5} + \frac{4}{7} - \frac{2}{9} \right) \\
&= -\frac{16\sqrt{2}}{3} \left(\frac{-126+184-70}{315} \right) \\
&= -\frac{16\sqrt{2}}{3} \left(-\frac{16}{315} \right) \\
&= \frac{256\sqrt{2}}{945} \Big|
\end{aligned}$$

Exercise

Evaluate $\iint_R xy \, dA$; where R is bounded by the ellipse $9x^2 + 4y^2 = 36$.

Solution

$$u^2 + v^2 = 1 \rightarrow v = \pm\sqrt{1-u^2}$$

$$\underline{-1 \leq u \leq 1}$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \rightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

$$\begin{cases} \frac{x}{2} = u \rightarrow x = 2u \\ \frac{y}{3} = v \rightarrow y = 3v \end{cases}$$

$$J(u, v) = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix}$$

$$\underline{= 6}$$

$$\iint_R xy \, dA = \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} (2u)(3v)|6| \, dvdu$$

$$= 18 \int_{-1}^1 uv^2 \Big|_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} du$$

$$= 36 \int_{-1}^1 u(1-u^2) du$$

$$\begin{aligned}
&= 36 \int_{-1}^1 (u - u^3) du \\
&= 36 \left(\frac{1}{2} u^2 - \frac{1}{4} u^4 \right) \Big|_{-1}^1 \\
&= 36 \left(\frac{1}{2} - \frac{1}{4} - \frac{1}{2} + \frac{1}{4} \right) \\
&= \underline{0}
\end{aligned}$$

Exercise

Evaluate $= \int_0^1 \int_{2u-2}^{2u} \sqrt{u+u-v} |1| dv du$

Solution

$$x = y, \quad x = y + 2$$

$$0 \leq x \leq 1$$

$$\begin{cases} u = x \\ v = x + y \end{cases} \rightarrow \begin{cases} x = u \\ y = -u + v \end{cases}$$

$$J(u, v) = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = \underline{1}$$

$x = 0$		$u = 0$
$x = 1$		$u = 1$
$x = y$	$u = -u + v$	$v = 2u$
$x = y + 2$	$u = -u + v + 2$	$v = 2u - 2$

$$\begin{aligned}
\int_0^1 \int_y^{y+2} \sqrt{x-y} dx dy &= \int_0^1 \int_{2u-2}^{2u} \sqrt{u+u-v} |1| dv du \\
&= - \int_0^1 \int_{2u-2}^{2u} (2u-v)^{1/2} d(2u-v) du \\
&= -\frac{2}{3} \int_0^1 (2u-v)^{3/2} \Big|_{2u-2}^{2u} du \\
&= \frac{2}{3} \int_0^1 (2)^{3/2} du \\
&= \underline{\frac{4\sqrt{2}}{3}}
\end{aligned}$$

Exercise

Evaluate $\iint_R \sqrt{y^2 - x^2} \, dA$; where R is the diamond bounded by $y - x = 0$, $y - x = 2$, $y + x = 0$, and $y + x = 2$

Solution

$$\begin{cases} u = y - x \\ v = y + x \end{cases} \rightarrow \begin{cases} x = -\frac{1}{2}u + \frac{1}{2}v \\ y = \frac{1}{2}u + \frac{1}{2}v \end{cases}$$

$$J(u, v) = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} \\ = -\frac{1}{2}$$

$y - x = 0$	$\frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}u - \frac{1}{2}v = 0$	$u = 0$
$y - x = 2$	$\frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}u - \frac{1}{2}v = 2$	$u = 2$
$y + x = 0$	$\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}u + \frac{1}{2}v = 0$	$v = 0$
$y + x = 2$	$\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}u + \frac{1}{2}v = 2$	$v = 2$

$$\begin{aligned} \iint_R \sqrt{y^2 - x^2} \, dA &= \int_0^2 \int_0^2 \left| -\frac{1}{2} \right| \sqrt{\left(\frac{1}{2}u + \frac{1}{2}v \right)^2 - \left(-\frac{1}{2}u + \frac{1}{2}v \right)^2} \, dvdu \\ &= \frac{1}{8} \int_0^2 \int_0^2 \sqrt{u^2 + 2uv + v^2 - u^2 + 2uv - v^2} \, dvdu \\ &= \frac{1}{4} \int_0^2 \int_0^2 \sqrt{uv} \, dvdu \\ &= \frac{1}{4} \int_0^2 \int_0^2 \frac{1}{u} (uv)^{1/2} \, d(uv) \, du \\ &= \frac{1}{6} \int_0^2 \frac{1}{u} (uv)^{3/2} \Big|_0^2 \, du \\ &= \frac{\sqrt{2}}{3} \int_0^2 \frac{1}{u} (u)^{3/2} \, du \end{aligned}$$

$$\begin{aligned}
&= \frac{\sqrt{2}}{3} \int_0^2 (u)^{1/2} du \\
&= \frac{2\sqrt{2}}{9} (u)^{3/2} \Big|_0^2 \\
&= \frac{16}{9}
\end{aligned}$$

Exercise

Evaluate $\iint_R \left(\frac{y-x}{y+2x+1} \right)^4 dA$; where R is the parallelogram bounded by $y-x=1$, $y-x=2$, $y+2x=0$, and $y+2x=4$

Solution

$$\text{Let } \begin{cases} u = y-x \\ v = y+2x \end{cases} \rightarrow \begin{cases} x = -\frac{1}{3}u + \frac{1}{3}v \\ y = \frac{2}{3}u + \frac{1}{3}v \end{cases}$$

$$\begin{aligned}
J(u, v) &= \begin{vmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} \\
&= -1
\end{aligned}$$

$y-x=1$	$\frac{2}{3}u + \frac{1}{3}v + \frac{1}{3}u - \frac{1}{3}v = 1$	$u=1$
$y-x=2$	$\frac{2}{3}u + \frac{1}{3}v + \frac{1}{3}u - \frac{1}{3}v = 2$	$u=2$
$y+2x=0$	$\frac{2}{3}u + \frac{1}{3}v - \frac{2}{3}u + \frac{2}{3}v = 0$	$v=0$
$y+2x=4$	$\frac{2}{3}u + \frac{1}{3}v - \frac{2}{3}u + \frac{2}{3}v = 4$	$v=4$

$$\begin{aligned}
\iint_R \left(\frac{y-x}{y+2x+1} \right)^4 dA &= \int_0^4 \int_1^2 |-1| \left(\frac{\frac{1}{3}(2u+v+u-v)}{\frac{1}{3}(2u+v-2u+2v)+1} \right)^4 dudv \\
&= \int_0^4 \int_1^2 \left(\frac{u}{v+1} \right)^4 dudv \\
&= \int_0^4 (v+1)^{-4} d(3v+1) \int_1^2 u^4 du
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{3}(v+1)^{-3} \Big|_0^4 \left(\frac{1}{5}u^5 \right) \Big|_1^2 \\
&= -\frac{1}{15} \left(\frac{1}{125} - 1 \right) (32) \\
&= -\frac{32}{15} \left(-\frac{124}{125} \right) \\
&= \frac{3,968}{1,875}
\end{aligned}$$

Exercise

Evaluate $\iint_R e^{xy} dA$; where R is the region bounded by $xy = 1$, $xy = 4$, $\frac{y}{x} = 1$, and $\frac{y}{x} = 3$

Solution

$$\text{Let } \begin{cases} u = xy \\ v = \frac{y}{x} \end{cases} \rightarrow \begin{cases} y = \frac{u}{x} \\ v = \frac{u}{x^2} \end{cases}$$

$$\begin{cases} x = \sqrt{\frac{u}{v}} \\ y = \sqrt{uv} \end{cases}$$

$$\begin{aligned}
J(u, v) &= \begin{vmatrix} \frac{1}{2\sqrt{uv}} & -\frac{\sqrt{u}}{v^{3/2}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{vmatrix} \\
&= \frac{1}{4v} + \frac{1}{4v} \\
&= \frac{1}{2v}
\end{aligned}$$

$xy = 1$	$\sqrt{\frac{u}{v}}\sqrt{uv} = 1$	$u = 1$
$xy = 4$	$\sqrt{\frac{u}{v}}\sqrt{uv} = 4$	$u = 4$
$\frac{y}{x} = 1$	$\sqrt{\frac{v}{u}}\sqrt{uv} = 1$	$v = 1$
$\frac{y}{x} = 3$	$\sqrt{\frac{v}{u}}\sqrt{uv} = 3$	$v = 3$

$$\iint_R e^{xy} dA = \int_1^3 \int_1^4 \left| \frac{1}{2v} \right| e^{\sqrt{\frac{u}{v}}\sqrt{uv}} du dv$$

$$\begin{aligned}
&= \frac{1}{2} \int_1^3 \frac{1}{v} dv \int_1^4 e^u du \\
&= \frac{1}{2} \ln v \Big|_1^3 e^u \Big|_1^4 \\
&= \frac{\ln 3}{2} (e^4 - e)
\end{aligned}$$

Exercise

Evaluate $\iint_R xy \, dA$; where R is the region bounded by the hyperbolas $xy = 1$, $xy = 4$, $y = 1$, and $y = 3$

Solution

$$\text{Let } \begin{cases} u = xy \\ v = y \end{cases} \rightarrow \begin{cases} x = \frac{u}{v} \\ y = v \end{cases}$$

$$\begin{aligned}
J(u, v) &= \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} \\
&= \frac{1}{v}
\end{aligned}$$

$xy = 1$	$u = 1$
$xy = 4$	$u = 4$
$y = 1$	$v = 1$
$y = 3$	$v = 3$

$$\begin{aligned}
\iint_R xy \, dA &= \int_1^3 \int_1^4 \left| \frac{1}{v} \right| u \, du dv \\
&= \int_1^3 \frac{1}{v} dv \int_1^4 u \, du \\
&= \ln v \Big|_1^3 \frac{1}{2} u^2 \Big|_1^4 \\
&= \frac{15}{2} \ln 3
\end{aligned}$$

Exercise

Evaluate $\iint_R (x-y)\sqrt{x-2y} \, dA$; where R is the triangular region bounded by $y=0$, $x-2y=0$, and $x-y=1$

Solution

$$\text{Let } \begin{cases} u = x-2y \\ v = y \end{cases} \rightarrow \begin{cases} x = u+2v \\ y = v \end{cases}$$

$$J(u, v) = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \\ = 1$$

$y = 0$		$v = 0$
$x - 2y = 0$		$u = 0$
$x - y = 1$	$u + 2v - v = 1$	$v = 1 - u$

$$\begin{aligned} \iint_R (x-y)\sqrt{x-2y} \, dA &= \int_0^1 \int_0^{1-u} (u+v)\sqrt{u} \, dv du \\ &= \int_0^1 \int_0^{1-u} (u^{3/2} + vu^{1/2}) \, dv du \\ &= \int_0^1 \left(u^{3/2}v + \frac{1}{2}u^{1/2}v^2 \right) \Big|_0^{1-u} du \\ &= \int_0^1 \left(u^{3/2}(1-u) + \frac{1}{2}u^{1/2}(1-2u+u^2) \right) du \\ &= \int_0^1 \left(u^{3/2} - u^{5/2} + \frac{1}{2}u^{1/2} - u^{3/2} + \frac{1}{2}u^{5/2} \right) du \\ &= \int_0^1 \left(\frac{1}{2}u^{1/2} - \frac{1}{2}u^{5/2} \right) du \\ &= \frac{1}{3}u^{3/2} - \frac{1}{7}u^{7/2} \Big|_0^1 \\ &= \frac{1}{3} - \frac{1}{7} \\ &= \frac{4}{21} \end{aligned}$$

Exercise

Evaluate $\iiint_D xy \, dV$: D is bounded by the planes: $y - x = 0$, $y - x = 2$, $z - y = 0$, $z - y = 2$,
 $z = 0$, and $z = 3$

Solution

$$\text{Let: } \begin{cases} u = y - x \\ v = z - y \\ w = z \end{cases} \rightarrow \begin{cases} x = -u - v + w \\ y = -v + w \end{cases}$$

$$J(u, v, w) = \begin{vmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \\ = 1$$

$y - x = 0$	$u = 0$
$y - x = 2$	$u = 2$
$z - y = 0$	$v = 0$
$z - y = 2$	$v = 2$
$z = 0$	$w = 0$
$z = 3$	$w = 3$

$$\begin{aligned} \iiint_D xy \, dV &= \int_0^3 \int_0^2 \int_0^2 (w - u - v)(w - v) \, du \, dv \, dw \\ &= \int_0^3 \int_0^2 \int_0^2 (w^2 - 2vw - uw + uv + v^2) \, du \, dv \, dw \\ &= \int_0^3 \int_0^2 \left(w^2 u - 2vwu - \frac{1}{2} w u^2 + \frac{1}{2} v u^2 + v^2 u \right) \Big|_0^2 \, dv \, dw \\ &= \int_0^3 \int_0^2 (2w^2 - 4vw - 2w + 2v + 2v^2) \, dv \, dw \\ &= \int_0^3 \left(2w^2 v - 2wv^2 - 2wv + v^2 + \frac{2}{3} v^3 \right) \Big|_0^2 \, dw \\ &= \int_0^3 \left(4w^2 - 12w + \frac{28}{3} \right) \, dw \\ &= \left(\frac{4}{3} w^3 - 6w^2 + \frac{28}{3} w \right) \Big|_0^3 \\ &= 36 - 54 + 28 \\ &= 10 \end{aligned}$$

Exercise

Evaluate $\iiint_D dV$: D is bounded by the planes: $y - 2x = 0$, $y - 2x = 1$, $z - 3y = 0$, $z - 3y = 1$,
 $z - 4x = 0$, and $z - 4x = 3$

Solution

$$\text{Let: } \begin{cases} u = y - 2x \\ v = z - 3y \\ w = z - 4x \end{cases} \rightarrow \begin{cases} x = \frac{1}{2}y - \frac{1}{2}u \\ z = v + 3y \end{cases}$$

$$w = v + 3y - 2y + 2u$$

$$\begin{cases} y = -2u - v + w \\ x = -\frac{3}{2}u - \frac{1}{2}v + \frac{1}{2}w \\ z = -6u - 2v + 3w \end{cases}$$

$$J(u, v, w) = \begin{vmatrix} -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ -2 & -1 & 1 \\ -6 & -2 & 3 \end{vmatrix}$$

$$= \frac{9}{2} + 3 + 2 - 3 - 3 - 3$$

$$= \frac{1}{2}$$

$y - 2x = 0$	$u = 0$
$y - 2x = 1$	$u = 1$
$z - 3y = 0$	$v = 0$
$z - 3y = 1$	$v = 1$
$z - 4x = 0$	$w = 0$
$z - 4x = 3$	$w = 3$

$$\iiint_D dV = \int_0^3 dw \int_0^1 dv \int_0^1 \frac{1}{2} du$$

$$= \frac{1}{2}(3)(1)(1)$$

$$= \frac{3}{2}$$

Exercise

Evaluate $\iiint_D z \, dV$: D is bounded by the paraboloid $z = 16 - x^2 - 4y^2$ and the xy -plane.

Solution

$$z = 16 - x^2 - 4y^2 = 0$$

$$x^2 + 4y^2 = 16 \rightarrow \left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

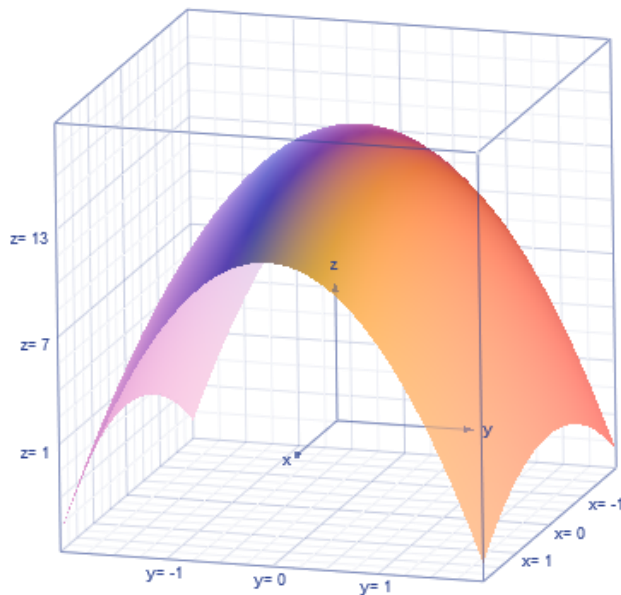
Let:
$$\begin{cases} u = \frac{x}{4} \\ v = \frac{y}{2} \\ w = z \end{cases} \quad \begin{cases} x = 4u \\ y = 2v \\ z = w \end{cases}$$

$$u^2 + v^2 = 1 \rightarrow v = \pm\sqrt{1-u^2}$$

$$\underline{-1 \leq u \leq 1}$$

$$w = z = 16 - 16u^2 - 16v^2$$

$$J(u, v, w) = \begin{vmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ = 8$$



$$\iiint_D z \, dV = \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \int_0^{16-16u^2-16v^2} 8w \, dw \, dv \, du$$

$$= 4 \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} w^2 \Big|_0^{16-16u^2-16v^2} dv \, du$$

$$= 4 \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} 16^2 \left(1 - (u^2 + v^2)\right)^2 dv \, du$$

$$= 1,024 \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \left(1 - 2(u^2 + v^2) + (u^2 + v^2)^2\right) dv \, du$$

$$= 1,024 \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \left(1 - 2u^2 - 2v^2 + u^4 + 2u^2v^2 + v^4\right) dv \, du$$

$$\begin{aligned}
&= 1,024 \int_{-1}^1 \left(v - 2u^2 v - \frac{2}{3} v^3 + u^4 v + \frac{2}{3} u^2 v^3 + \frac{1}{5} v^5 \right) \bigg|_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} du \\
&= 1,024 \int_{-1}^1 \left((1-2u^2+u^4)v - \frac{2}{3}(1-u^2)v^3 + \frac{1}{5}v^5 \right) \bigg|_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} du \\
&= 2,048 \int_{-1}^1 \left((1-u^2)^2 (1-u^2)^{1/2} - \frac{2}{3}(1-u^2)^{5/2} + \frac{1}{5}(1-u^2)^{5/2} \right) du \\
&= 2,048 \left(\frac{8}{15} \right) \int_{-1}^1 (1-u^2)^{5/2} du
\end{aligned}$$

$$u = \sin t \rightarrow du = \cos t dt$$

$$\begin{cases} u = 1 = \sin t & \rightarrow t = \frac{\pi}{2} \\ u = -1 = \sin t & \rightarrow t = -\frac{\pi}{2} \end{cases}$$

$$\begin{aligned}
&= 2,048 \left(\frac{8}{15} \right) \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^5 t \cos t dt \\
&= 2,048 \left(\frac{8}{15} \right) \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^6 t dt \\
&= \frac{2,048}{15} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (1 + \cos 2t)^3 dt \\
&= \frac{2,048}{15} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (1 + 3\cos 2t + 3\cos^2 2t + \cos^3 2t) dt \\
&= \frac{2,048}{15} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{5}{2} + 3\cos 2t + \frac{3}{2}\cos 4t + \cos^2 2t (\sin 2t) \right) dt \\
&\quad \int \cos^2 2t (\sin 2t) dt = -\frac{1}{2} \int \cos^2 2t d(\cos 2t) \\
&\quad = -\frac{1}{6} \cos^3 2t \\
&= \frac{2,048}{15} \left(\frac{5}{2}t + \frac{3}{2}\sin 2t + \frac{3}{8}\sin 4t - \frac{1}{6}\cos^3 2t \right) \bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{2,048}{15} \left(\frac{15\pi}{4} + \frac{1}{6} - \frac{5\pi}{4} - \frac{1}{6} \right) \\
&= \frac{2,048}{15} \left(\frac{5\pi}{2} \right) \\
&= \frac{1,024\pi}{3} \quad |
\end{aligned}$$

Exercise

Evaluate $\iiint_D dV$: D is bounded by the upper half of the ellipsoid $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$ and the xy -plane.

Solution

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 + z^2 = 1$$

$$\text{Let: } \begin{cases} u = \frac{x}{3} \\ v = \frac{y}{2} \\ w = z \end{cases} \quad \begin{cases} x = 3u \\ y = 2v \\ z = w \end{cases}$$

$$\begin{aligned}
J(u, v, w) &= \begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\
&= 6 \quad |
\end{aligned}$$

$$u^2 + v^2 + w^2 = 1$$

$$-\sqrt{1-v^2-w^2} \leq u \leq \sqrt{1-v^2-w^2}$$

$$-\sqrt{1-w^2} \leq v \leq \sqrt{1-w^2} \quad \& \quad 0 \leq w \leq 1$$

$$\begin{aligned}
\iiint_D dV &= \int_0^1 \int_{-\sqrt{1-w^2}}^{\sqrt{1-w^2}} \int_{-\sqrt{1-v^2-w^2}}^{\sqrt{1-v^2-w^2}} (6) \, du \, dv \, dw \\
&= 6 \int_0^1 \int_{-\sqrt{1-w^2}}^{\sqrt{1-w^2}} u \bigg|_{-\sqrt{1-v^2-w^2}}^{\sqrt{1-v^2-w^2}} dv \, dw \\
&= 12 \int_0^1 \int_{-\sqrt{1-w^2}}^{\sqrt{1-w^2}} \sqrt{1-(v^2+w^2)} \, dv \, dw
\end{aligned}$$

$$v^2 + w^2 = r^2 \rightarrow 0 \leq r \leq 1 \quad 0 \leq \theta \leq \pi$$

$$\begin{aligned}
 &= 12 \int_0^\pi \int_0^1 \sqrt{1-r^2} \, r \, dr d\theta \\
 &= -6 \int_0^\pi d\theta \int_0^1 (1-r^2)^{1/2} d(1-r^2) \\
 &= -6\pi \left. \frac{2}{3} (1-r^2)^{3/2} \right|_0^1 \\
 &= \underline{4\pi}
 \end{aligned}$$

Exercise

Evaluate $\iiint_D xz \, dV$: D is bounded by the planes: $y = x$, $y = x + 2$, $x - z = 0$, $z = x + 3$, $z = 0$,

and $z = 4$

Solution

$$y - x = 0 \quad y - x = 2$$

$$z - x = 0 \quad z - x = 3$$

$$\text{Let: } \begin{cases} u = y - x \\ v = z - x \\ w = z \end{cases} \rightarrow \begin{cases} y = u - v + w \\ x = -v + w \end{cases}$$

$$\begin{aligned}
 J(u, v, w) &= \begin{vmatrix} 0 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \\
 &= \underline{1}
 \end{aligned}$$

$y = x$	$u - v + w = w - v$	$u = 0$
$y = x + 2$	$u - v + w = w - v + 2$	$u = 2$
$x - z = 0$	$w - v - w = 0$	$v = 0$
$z = x + 3$	$w + v - w = 3$	$v = 3$
$z = 0$		$w = 0$
$z = 4$		$w = 4$

$$\begin{aligned}
 \iiint_D xz \, dV &= \int_0^2 \int_0^3 \int_0^4 (w-v)(w) \, dw dv du \\
 &= \int_0^2 du \int_0^3 \int_0^4 (w^2 - vw) \, dw dv
 \end{aligned}$$

$$\begin{aligned}
&= 2 \int_0^3 \left(\frac{1}{3} w^3 - \frac{1}{2} v w^2 \right) \Big|_0^4 dv \\
&= 2 \int_0^3 \left(\frac{64}{3} - 8v \right) dv \\
&= 2 \left(\frac{64}{3} v - 4v^2 \right) \Big|_0^3 \\
&= 18(64 - 36) \\
&= \underline{56}
\end{aligned}$$

Exercise

Let R be the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$ are real numbers.

Find the area of R .

Solution

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\text{Let: } \begin{cases} u = \frac{x}{a} \rightarrow x = au \\ v = \frac{y}{b} \rightarrow y = bv \end{cases}$$

$$J(u, v) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = \underline{ab}$$

$$u^2 + v^2 = 1 \rightarrow u = \sqrt{1-v^2}$$

$$-\sqrt{1-v^2} \leq u \leq \sqrt{1-v^2} \quad \& \quad -1 \leq v \leq 1$$

Since, $u^2 + v^2 = 1$ is a unit circle, then the area $\pi r^2 = \pi$. Therefore, the area of the ellipse is $ab\pi$

$$\begin{aligned}
\iint_R dA &= \int_{-1}^1 \int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} ab \, du \, dv \\
&= ab \int_{-1}^1 u \Big|_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} dv \\
&= 2ab \int_{-1}^1 \sqrt{1-v^2} \, dv
\end{aligned}$$

$$v = \sin t \rightarrow dv = \cos t \, dt$$

$$\begin{cases} v = 1 = \sin t & \rightarrow t = \frac{\pi}{2} \\ v = -1 = \sin t & \rightarrow t = -\frac{\pi}{2} \end{cases}$$

$$= 2ab \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t \, dt$$

$$= ab \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos 2t) \, dt$$

$$= ab \left(t + \frac{1}{2} \sin 2t \right) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= ab \left(\frac{\pi}{2} + \frac{\pi}{2} \right)$$

$$\underline{ab\pi}$$

Exercise

Let R be the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$ are real numbers.

Evaluates $\iint_R |xy| \, dA$

Solution

$$\left(\frac{x}{a} \right)^2 + \left(\frac{y}{b} \right)^2 = 1$$

$$\text{Let: } \begin{cases} u = \frac{x}{a} & \rightarrow x = au \\ v = \frac{y}{b} & \rightarrow y = bv \end{cases}$$

$$J(u, v) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} \\ = ab$$

$$u^2 + v^2 = 1 \rightarrow u = \sqrt{1-v^2}$$

$$-\sqrt{1-v^2} \leq u \leq \sqrt{1-v^2} \quad \& \quad -1 \leq v \leq 1$$

$$\iint_R |xy| \, dA = \int_{-1}^1 \int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} |abuv| ab \, du \, dv$$

$$\begin{aligned}
&= 4a^2b^2 \int_0^1 \int_0^{\sqrt{1-v^2}} uv \, dudv \\
&= 2a^2b^2 \int_0^1 vu^2 \bigg|_0^{\sqrt{1-v^2}} dv \\
&= 2a^2b^2 \int_0^1 v(1-v^2) dv \\
&= 2a^2b^2 \int_0^1 (v-v^3) dv \\
&= 2a^2b^2 \left(\frac{1}{2}v^2 - \frac{1}{4}v^4 \right) \bigg|_0^1 \\
&= 2a^2b^2 \left(\frac{1}{2} - \frac{1}{4} \right) \\
&= \frac{1}{2}a^2b^2
\end{aligned}$$

Exercise

Let R be the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$ are real numbers.

Find the center of mass of the upper half of R ($y \geq 0$) assuming it has a constant density.

Solution

$$\text{Let: } \begin{cases} u = \frac{x}{a} & \rightarrow x = au \\ v = \frac{y}{b} & \rightarrow y = bv \end{cases}$$

$$\begin{aligned}
J(u, v) &= \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} \\
&= ab
\end{aligned}$$

$$\begin{aligned}
u^2 + v^2 &= 1 \rightarrow v = \sqrt{1-u^2} \\
0 \leq v &\leq \sqrt{1-u^2} \quad \& \quad -1 \leq u \leq 1
\end{aligned}$$

Since, $u^2 + v^2 = 1$ is a unit circle, then the area $\pi r^2 = \pi$. Therefore, the area of the ellipse is $ab\pi$

Mass of the upper half is given by:

$$m = \frac{1}{2} \pi ab$$

By symmetry, $\bar{x} = 0$

$$\begin{aligned}
\bar{y} &= \frac{2}{\pi ab} \int_{-1}^1 \int_0^{\sqrt{1-u^2}} aby \, dv du \\
&= \frac{2}{\pi} \int_{-1}^1 \int_0^{\sqrt{1-u^2}} bv \, dv du \\
&= \frac{b}{\pi} \int_{-1}^1 v^2 \Big|_0^{\sqrt{1-u^2}} du \\
&= \frac{b}{\pi} \int_{-1}^1 (1-u^2) du \\
&= \frac{b}{\pi} \left(u - \frac{1}{3}u^3 \right) \Big|_{-1}^1 \\
&= \frac{2b}{\pi} \left(1 - \frac{1}{3} \right) \\
&= \frac{4b}{3\pi}
\end{aligned}$$

\therefore the center of mass of the upper half of R is $\left(0, \frac{4b}{3\pi} \right)$

Exercise

Let R be the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$ are real numbers.

Find the average square of the distance between points of R and the origin.

Solution

$$\text{Let: } \begin{cases} u = \frac{x}{a} & \rightarrow x = au \\ v = \frac{y}{b} & \rightarrow y = bv \end{cases}$$

The distance between points of R and the origin is $d = \sqrt{x^2 + y^2}$

$$d = \sqrt{a^2 u^2 + b^2 v^2}$$

$$\begin{aligned}
J(u, v) &= \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} \\
&= ab
\end{aligned}$$

$$u^2 + v^2 = 1 \rightarrow v = \sqrt{1-u^2}$$

$$0 \leq v \leq \sqrt{1-u^2} \quad \& \quad -1 \leq u \leq 1$$

Average square of the distance is:

$$\begin{aligned}
 avg &= \frac{1}{\pi ab} \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} ab \left(a^2 u^2 + b^2 v^2 \right) dv du \\
 &= \frac{1}{\pi} \int_{-1}^1 \left(a^2 u^2 v + \frac{1}{3} b^2 v^3 \right) \bigg|_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} du \\
 &= \frac{2}{\pi} \int_{-1}^1 \left(a^2 u^2 (1-u^2)^{1/2} + \frac{1}{3} b^2 (1-u^2)^{3/2} \right) du \\
 &\quad u = \sin t \rightarrow du = \cos t dt \\
 &\quad \begin{cases} u = 1 = \sin t & \rightarrow t = \frac{\pi}{2} \\ u = -1 = \sin t & \rightarrow t = -\frac{\pi}{2} \end{cases} \\
 &\quad \int u^2 (1-u^2)^{1/2} du = \int \sin^2 t \cos t \cos t dt \\
 &\quad = \frac{1}{4} \int (1 - \cos 2t)(1 + \cos 2t) dt \\
 &\quad = \frac{1}{4} \int (1 - \cos^2 2t) dt \\
 &\quad = \frac{1}{4} \int \left(\frac{1}{2} - \frac{1}{2} \cos 4t \right) dt \\
 &\quad = \frac{1}{8} \left(t - \frac{1}{4} \sin 4t \right) \\
 &\quad \int (1-u^2)^{3/2} du = \int \cos^4 t dt \\
 &\quad = \frac{1}{4} \int (1 + \cos 2t)^2 dt \\
 &\quad = \frac{1}{4} \int (1 + 2 \cos 2t + \cos^2 2t) dt \\
 &\quad = \frac{1}{4} \int \left(\frac{3}{2} + 2 \cos 2t + \frac{1}{2} \cos 4t \right) dt \\
 &\quad = \frac{1}{4} \left(\frac{3}{2} t + \sin 2t + \frac{1}{8} \sin 4t \right) \\
 &= \frac{2}{\pi} \left(\frac{a^2}{8} \left(t - \frac{1}{4} \sin 4t \right) + \frac{b^2}{12} \left(\frac{3}{2} t + \sin 2t + \frac{1}{8} \sin 4t \right) \right) \bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}
 \end{aligned}$$

$$= \frac{4}{\pi} \left(\frac{a^2}{8} \frac{\pi}{2} + \frac{b^2}{8} \frac{\pi}{2} \right)$$

$$= \frac{a^2 + b^2}{4}$$

Exercise

Let R be the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$ are real numbers.

Find the average distance between points in the upper half of R and the x -axis.

Solution

Let:
$$\begin{cases} u = \frac{x}{a} & \rightarrow x = au \\ v = \frac{y}{b} & \rightarrow y = bv \end{cases}$$

The distance between points in the upper half of R and the x -axis is $d = y = bv$

$$J(u, v) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}$$

$$= ab$$

$$u^2 + v^2 = 1 \rightarrow v = \sqrt{1 - u^2}$$

$$0 \leq v \leq \sqrt{1 - u^2} \quad \& \quad -1 \leq u \leq 1$$

$$\begin{aligned} \text{avg} &= \frac{2}{\pi ab} \int_{-1}^1 \int_0^{\sqrt{1-u^2}} ab^2 v \, dv du \\ &= \frac{b}{\pi} \int_{-1}^1 v^2 \bigg|_0^{\sqrt{1-u^2}} du \\ &= \frac{b}{\pi} \int_{-1}^1 (1 - u^2) du \\ &= \frac{2b}{\pi} \left(u - \frac{1}{3} u^3 \right) \bigg|_0^1 \\ &= \frac{2b}{\pi} \left(1 - \frac{1}{3} \right) \\ &= \frac{4b}{3\pi} \end{aligned}$$

Exercise

Let D be the region bounded by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, where $a > 0$, $b > 0$ and $c > 0$ are real numbers. Find the Volume of D .

Solution

$$\text{Let: } \begin{cases} u = \frac{x}{a} & \rightarrow x = au \\ v = \frac{y}{b} & \rightarrow y = bv \\ w = \frac{z}{c} & \rightarrow z = cw \end{cases}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \rightarrow u^2 + v^2 + w^2 = 1$$

$$J(u, v) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} \\ = abc$$

Since, $u^2 + v^2 + w^2 = 1$ is a unit sphere, then the volume $\frac{4\pi r^2}{3} = \frac{4\pi}{3}$. Therefore, the volume of the ellipsoid is $\frac{4}{3}abc\pi$

Exercise

Let D be the region bounded by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, where $a > 0$, $b > 0$ and $c > 0$ are

real numbers. Evaluates $\iiint_V |xyz| dV$

Solution

$$\text{Let: } \begin{cases} u = \frac{x}{a} & \rightarrow x = au \\ v = \frac{y}{b} & \rightarrow y = bv \\ w = \frac{z}{c} & \rightarrow z = cw \end{cases}$$

$$J(u, v) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} \\ = abc$$

$$u^2 + v^2 + w^2 = 1 \rightarrow w = \sqrt{1 - u^2 - v^2}$$

$$-\sqrt{1 - u^2 - v^2} \leq w \leq \sqrt{1 - u^2 - v^2}$$

$$-\sqrt{1 - u^2} \leq v \leq \sqrt{1 - u^2} \quad \& \quad -1 \leq u \leq 1$$

$$\iint_R |xyz| dA = \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \int_{-\sqrt{1-u^2-v^2}}^{\sqrt{1-u^2-v^2}} (aubvcw) abc \, dw dv du$$

$$= 4a^2 b^2 c^2 \int_0^1 \int_0^{\sqrt{1-u^2}} uvw^2 \Big|_0^{\sqrt{1-u^2-v^2}} dv du$$

$$= 4a^2 b^2 c^2 \int_0^1 \int_0^{\sqrt{1-u^2}} (uv - u^3 v - uv^3) dv du$$

$$= 4a^2 b^2 c^2 \int_0^1 \left(\frac{1}{2} (u - u^3) v^2 - \frac{1}{4} uv^4 \right) \Big|_0^{\sqrt{1-u^2}} du$$

$$= a^2 b^2 c^2 \int_0^1 \left(2u(1-u^2)(1-u^2) - u(1-u^2)^2 \right) du$$

$$= a^2 b^2 c^2 \int_0^1 (1-u^2)^2 (2u-u) du$$

$$= a^2 b^2 c^2 \int_0^1 u(1-u^2)^2 du$$

$$= -\frac{1}{2} a^2 b^2 c^2 \int_0^1 (1-u^2)^2 d(1-u^2)$$

$$= -\frac{1}{6} a^2 b^2 c^2 (1-u^2)^3 \Big|_0^1$$

$$= \frac{1}{6} a^2 b^2 c^2$$

Exercise

Let D be the region bounded by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, where $a > 0$, $b > 0$ and $c > 0$ are real numbers. Find the center of mass of the upper half of D ($z \geq 0$) assuming it has a constant density.

Solution

$$\text{Let: } \begin{cases} u = \frac{x}{a} & \rightarrow x = au \\ v = \frac{y}{b} & \rightarrow y = bv \\ w = \frac{z}{c} & \rightarrow z = cw \end{cases}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \rightarrow u^2 + v^2 + w^2 = 1$$

$$J(u, v) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} \\ = abc$$

Since, $u^2 + v^2 + w^2 = 1$ is a unit sphere, then the volume $\frac{4\pi r^2}{3} = \frac{4\pi}{3}$. Therefore, the volume of the ellipsoid is $\frac{4}{3}abc\pi$

$$m = \frac{1}{2} \frac{4}{3} abc\pi \quad (\text{upper half of } D)$$

$$m = \frac{2}{3} abc\pi$$

By symmetry $\bar{x} = \bar{y} = 0$

$$\begin{aligned} \bar{z} &= \frac{3}{2\pi abc} \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \int_0^{\sqrt{1-u^2-v^2}} (abc)(cw) \, dw \, dv \, du \\ &= \frac{3c}{4\pi} \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} w^2 \Big|_0^{\sqrt{1-u^2-v^2}} \, dv \, du \\ &= \frac{3c}{4\pi} \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} (1-u^2-v^2) \, dv \, du \\ &= \frac{3c}{2\pi} \int_{-1}^1 \left((1-u^2)v - \frac{1}{3}v^3 \right) \Big|_0^{\sqrt{1-u^2}} \, du \end{aligned}$$

$$\begin{aligned}
&= \frac{3c}{2\pi} \int_{-1}^1 \left((1-u^2)^{3/2} - \frac{1}{3}(1-u^2)^{3/2} \right) du \\
&= \frac{c}{\pi} \int_{-1}^1 (1-u^2)^{3/2} du
\end{aligned}$$

$$u = \sin t \rightarrow du = \cos t dt$$

$$\begin{cases} u = 1 = \sin t & \rightarrow t = \frac{\pi}{2} \\ u = -1 = \sin t & \rightarrow t = -\frac{\pi}{2} \end{cases}$$

$$\int (1-u^2)^{3/2} du = \int \cos^4 t dt$$

$$= \frac{1}{4} \int (1 + \cos 2t)^2 dt$$

$$= \frac{1}{4} \int (1 + 2\cos 2t + \cos^2 2t) dt$$

$$= \frac{1}{4} \int \left(\frac{3}{2} + 2\cos 2t + \frac{1}{2}\cos 4t \right) dt$$

$$= \frac{1}{4} \left(\frac{3}{2}t + \sin 2t + \frac{1}{8}\sin 4t \right)$$

$$= \frac{c}{4\pi} \left(\frac{3}{2}t + \sin 2t + \frac{1}{8}\sin 4t \right) \Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{c}{4\pi} \left(2 \frac{3}{2} \frac{\pi}{2} \right)$$

$$= \frac{3c}{8}$$

Exercise

Let D be the region bounded by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, where $a > 0$, $b > 0$ and $c > 0$ are real numbers. Find the average square of the distance between points of D and the origin.

Solution

$$\text{Let: } \begin{cases} u = \frac{x}{a} & \rightarrow x = au \\ v = \frac{y}{b} & \rightarrow y = bv \\ w = \frac{z}{c} & \rightarrow z = cw \end{cases}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \rightarrow u^2 + v^2 + w^2 = 1$$

$$J(u, v) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

$$= abc$$

The distance between a point on D and the origin is

$$d = \sqrt{x^2 + y^2 + z^2}$$

$$= \sqrt{a^2 u^2 + b^2 v^2 + c^2 w^2}$$

Since, $u^2 + v^2 + w^2 = 1$ is a unit sphere, then the volume $\frac{4\pi r^2}{3} = \frac{4\pi}{3}$. Therefore, the volume of the ellipsoid is $\frac{4}{3}abc\pi$

$$m = \frac{1}{2} \frac{4}{3} abc\pi \quad (\text{upper half of } D)$$

$$m = \frac{2}{3} abc\pi$$

$$avg = \frac{1}{2} \frac{3}{2\pi abc} \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \int_{-\sqrt{1-u^2-v^2}}^{\sqrt{1-u^2-v^2}} (a^2 u^2 + b^2 v^2 + c^2 w^2) (abc) dw dv du$$

$$= \frac{3}{4\pi} \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \int_{-\sqrt{1-u^2-v^2}}^{\sqrt{1-u^2-v^2}} (a^2 u^2 + b^2 v^2 + c^2 w^2) dw dv du$$

Let $u = r \cos \theta$ $v = r \sin \theta$

$$avg = \frac{3}{4\pi} \int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} (a^2 r^2 \cos^2 \theta + b^2 r^2 \sin^2 \theta + c^2 z^2) r dz dr d\theta$$

$$= \frac{3}{2\pi} \int_0^{2\pi} \int_0^1 \left((a^2 \cos^2 \theta + b^2 \sin^2 \theta) r^3 z + \frac{1}{3} r c^2 z^3 \right) \Big|_0^{\sqrt{1-r^2}} dr d\theta$$

$$= \frac{3}{2\pi} \int_0^{2\pi} \int_0^1 \left((a^2 \cos^2 \theta + b^2 \sin^2 \theta) r^3 (1-r^2)^{1/2} + \frac{1}{3} r c^2 (1-r^2)^{3/2} \right) dr d\theta$$

$$r = \sin t \rightarrow dr = \cos t dt$$

$$\begin{cases} r = 1 = \sin t & \rightarrow t = \frac{\pi}{2} \\ r = 0 = \sin t & \rightarrow t = 0 \end{cases}$$

$$\begin{aligned}
\int_0^1 r^3 (1-r^2)^{1/2} dr &= \int_0^{\frac{\pi}{2}} \sin^3 t \cos^2 t dt \\
&= - \int_0^{\frac{\pi}{2}} (1 - \cos^2 t) \cos^2 t d(\cos t) \\
&= - \int_0^{\frac{\pi}{2}} (\cos^2 t - \cos^4 t) d(\cos t) \\
&= \frac{1}{5} \cos^5 t - \frac{1}{3} \cos^3 t \Big|_0^{\frac{\pi}{2}} \\
&= -\frac{1}{5} + \frac{1}{3} \\
&= \underline{\frac{2}{15}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{5\pi} \int_0^{2\pi} (a^2 \cos^2 \theta + b^2 \sin^2 \theta) d\theta - \frac{c^2}{4\pi} \int_0^{2\pi} d\theta \int_0^1 (1-r^2)^{3/2} d(1-r^2) \\
&= \frac{1}{10\pi} \int_0^{2\pi} (a^2 (1 + \cos 2\theta) + b^2 (1 - \cos 2\theta)) d\theta - \frac{c^2}{5} (1-r^2)^{5/2} \Big|_0^1 \\
&= \frac{1}{10\pi} \left(a^2 \left(\theta + \frac{1}{2} \sin 2\theta \right) + b^2 \left(\theta - \frac{1}{2} \sin 2\theta \right) \right) \Big|_0^{2\pi} + \frac{c^2}{5} \\
&= \frac{1}{10\pi} (2\pi a^2 + 2\pi b^2) + \frac{c^2}{5} \\
&= \underline{\frac{a^2 + b^2 + c^2}{5}}
\end{aligned}$$