

Solution ***Section 8.7 – Trigonometric Form***

Exercise

Write $-\sqrt{3} + i$ in trigonometric form. (Use radian measure)

Solution

$$-\sqrt{3} + i \Rightarrow \begin{cases} x = -\sqrt{3} \\ y = 1 \end{cases}$$

$$\begin{aligned} r &= \sqrt{(-\sqrt{3})^2 + 1^2} \\ &= 2 \end{aligned}$$

$$\begin{aligned} \hat{\theta} &= \tan^{-1} \left(\frac{1}{-\sqrt{3}} \right) & \theta &= \tan^{-1} \frac{y}{x} \\ &= \frac{\pi}{6} \end{aligned}$$

The angle is in quadrant *II*, therefore;

$$\begin{aligned} \theta &= \pi - \frac{\pi}{6} \\ &= \frac{5\pi}{6} \\ -\sqrt{3} + i &= 2 \operatorname{cis} \frac{5\pi}{6} \end{aligned}$$

Exercise

Write $3 - 4i$ in trigonometric form.

Solution

$$3 - 4i \Rightarrow \begin{cases} x = 3 \\ y = -4 \end{cases}$$

$$\begin{aligned} r &= \sqrt{3^2 + (-4)^2} \\ &= 5 \end{aligned}$$

$$\begin{aligned} \hat{\theta} &= \tan^{-1} \left(\frac{-4}{3} \right) & \theta &= \tan^{-1} \frac{y}{x} \\ &\approx 53^\circ \end{aligned}$$

The angle is in quadrant *IV*, therefore;

$$\begin{aligned} \theta &= 360^\circ - 53^\circ \\ &= 307^\circ \\ 3 - 4i &= 5 \operatorname{cis} 307^\circ \end{aligned}$$

Exercise

Write $-21 - 20i$ in trigonometric form.

Solution

$$-21 - 20i \Rightarrow \begin{cases} x = -21 \\ y = -20 \end{cases}$$

$$\begin{aligned} r &= \sqrt{(-21)^2 + (-20)^2} \\ &= 29 \end{aligned}$$

$$\begin{aligned} \hat{\theta} &= \tan^{-1}\left(\frac{20}{21}\right) & \theta &= \tan^{-1} \frac{y}{x} \\ &\approx 43.6^\circ \end{aligned}$$

The angle is in quadrant *III*, therefore;

$$\begin{aligned} \theta &= 180^\circ + 43.6^\circ \\ &= 223.6^\circ \end{aligned}$$

$$\underline{-21 - 20i = 29 \operatorname{cis} 223.6^\circ}$$

Exercise

Write $11 + 2i$ in trigonometric form.

Solution

$$11 + 2i \Rightarrow \begin{cases} x = 11 \\ y = 2 \end{cases}$$

$$\begin{aligned} r &= \sqrt{11^2 + 2^2} \\ &= \sqrt{125} \\ &= 5\sqrt{5} \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{2}{11}\right) & \theta &= \tan^{-1} \frac{y}{x} \\ &\approx 10.3^\circ \end{aligned}$$

$$\underline{11 + 2i = 5\sqrt{5} \operatorname{cis} 10.3^\circ}$$

Exercise

Write $\sqrt{3} - i$ in trigonometric form.

Solution

$$\sqrt{3} - i \Rightarrow \begin{cases} x = \sqrt{3} \\ y = -1 \end{cases}$$

$$\begin{aligned} r &= \sqrt{3+1} \\ &= 2 \end{aligned}$$

$$\begin{aligned} \hat{\theta} &= \tan^{-1} \left(\frac{1}{\sqrt{3}} \right) & \theta &= \tan^{-1} \frac{y}{x} \\ &\approx 30^\circ \end{aligned}$$

The angle is in quadrant *IV*, therefore;

$$\begin{aligned} \theta &= 360^\circ - 30^\circ \\ &= 330^\circ \end{aligned}$$

$$\sqrt{3} - i = 2 \operatorname{cis} 330^\circ$$

Exercise

Write $1 - \sqrt{3}i$ in trigonometric form.

Solution

$$1 - \sqrt{3}i \Rightarrow \begin{cases} x = 1 \\ y = -\sqrt{3} \end{cases}$$

$$\begin{aligned} r &= \sqrt{1+3} \\ &= 2 \end{aligned}$$

$$\begin{aligned} \hat{\theta} &= \tan^{-1} \left(\frac{\sqrt{3}}{1} \right) & \theta &= \tan^{-1} \frac{y}{x} \\ &= 60^\circ \end{aligned}$$

The angle is in quadrant *IV*, therefore;

$$\begin{aligned} \theta &= 360^\circ - 60^\circ \\ &= 300^\circ \end{aligned}$$

$$1 - \sqrt{3}i = 2 \operatorname{cis} 300^\circ$$

Exercise

Write $9\sqrt{3} + 9i$ in trigonometric form.

Solution

$$9\sqrt{3} + 9i \Rightarrow \begin{cases} x = 9\sqrt{3} \\ y = 9 \end{cases}$$

$$r = 9\sqrt{3+1} \\ = 18$$

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \\ = 30^\circ$$

$$9\sqrt{3} + 9i = 18 \operatorname{cis} 30^\circ$$

Exercise

Write $-2 + 3i$ in trigonometric form.

Solution

$$-2 + 3i \Rightarrow \begin{cases} x = -2 \\ y = 3 \end{cases}$$

$$r = \sqrt{4+9} \\ = \sqrt{13}$$

$$\hat{\theta} = \tan^{-1}\left(\frac{3}{2}\right) \qquad \theta = \tan^{-1} \frac{y}{x} \\ \approx 56.31^\circ$$

The angle is in quadrant *II*, therefore;

$$\theta = 180^\circ - 56.31^\circ \\ \approx 123.69^\circ$$

$$-2 + 3i = \sqrt{13} \operatorname{cis} 123.69^\circ$$

Exercise

Write $4(\cos 30^\circ + i \sin 30^\circ)$ in standard form.

Solution

$$4(\cos 30^\circ + i \sin 30^\circ) = 4\left(\frac{\sqrt{3}}{2} + i \frac{1}{2}\right) \\ = 2\sqrt{3} + 2i$$

Exercise

Write $\sqrt{2} \operatorname{cis} \frac{7\pi}{4}$ in standard form.

Solution

$$\begin{aligned}\sqrt{2} \operatorname{cis} \frac{7\pi}{4} &= \sqrt{2} \left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} \right) \\ &= \sqrt{2} \left(\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \right) \\ &= \underline{1 - i}\end{aligned}$$

Exercise

Write $3\operatorname{cis}210^\circ$ in standard form.

Solution

$$\begin{aligned}3\operatorname{cis}210^\circ &= 3(\cos 210^\circ + i \sin 210^\circ) \\ &= \underline{-\frac{3\sqrt{3}}{2} - \frac{3}{2}i}\end{aligned}$$

Exercise

Write $4\left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}\right)$ in standard form.

Solution

$$\begin{aligned}4\left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}\right) &= 4\left(\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2}\right) \\ &= \underline{2\sqrt{2} - 2i\sqrt{2}}\end{aligned}$$

Exercise

Write $4\operatorname{cis} \frac{\pi}{2}$ in standard form.

Solution

$$\begin{aligned}4\operatorname{cis} \frac{\pi}{2} &= 4\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right) \\ &= \underline{4i}\end{aligned}$$

Exercise

Find the quotient $\frac{20\operatorname{cis}(75^\circ)}{4\operatorname{cis}(40^\circ)}$. Write the result in rectangular form.

Solution

$$\begin{aligned}
 \frac{20\text{cis}(75^\circ)}{4\text{cis}(40^\circ)} &= \frac{20}{4}\text{cis}(75^\circ - 40^\circ) \\
 &= 5\text{cis}(35^\circ) \\
 &= 5(\cos 35^\circ + i \sin 35^\circ) \\
 &= \underline{4.1 + 2.87i}
 \end{aligned}$$

Exercise

Divide $z_1 = 1 + i\sqrt{3}$ by $z_2 = \sqrt{3} + i$. Write the result in rectangular form.

Solution

$$\begin{aligned}
 \frac{z_1}{z_2} &= \frac{1 + i\sqrt{3}}{\sqrt{3} + i} \\
 &= \frac{1 + i\sqrt{3}}{\sqrt{3} + i} \frac{\sqrt{3} - i}{\sqrt{3} - i} \\
 &= \frac{\sqrt{3} - i + 3i - \sqrt{3}i^2}{3 + 1} \\
 &= \frac{2\sqrt{3} + 2i}{4} \\
 &= \frac{2\sqrt{3}}{4} + \frac{2i}{4} \\
 &= \underline{\frac{\sqrt{3}}{2} + \frac{i}{2}}
 \end{aligned}$$

or

$$1 + i\sqrt{3} \rightarrow \begin{cases} r = \sqrt{1^2 + (\sqrt{3})^2} \\ \theta = \tan^{-1} \frac{\sqrt{3}}{1} = \frac{\pi}{3} \end{cases}$$

$$\sqrt{3} + i \rightarrow \begin{cases} r = \sqrt{(\sqrt{3})^2 + 1^2} \\ \theta = \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6} \end{cases}$$

$$\begin{aligned}
 \frac{z_1}{z_2} &= \frac{2\text{cis}\frac{\pi}{3}}{2\text{cis}\frac{\pi}{6}} \\
 &= \frac{2}{2}\text{cis}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) \\
 &= \frac{2}{2}\text{cis}\left(\frac{\pi}{6}\right)
 \end{aligned}$$

$$= cis\left(\frac{\pi}{6}\right)$$

$$= \frac{\sqrt{3}}{2} + \frac{i}{2}$$

Exercise

Find $(1+i)^8$ and express the result in rectangular form.

Solution

$$1+i \Rightarrow \begin{cases} x=1 \\ y=1 \end{cases}$$

$$r = \sqrt{1^2 + 1^2}$$

$$= \sqrt{2}$$

$$\theta = \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

$$1+i = \sqrt{2} cis \frac{\pi}{4}$$

$$(1+i)^8 = \left(\sqrt{2} cis \frac{\pi}{4}\right)^8$$

$$= (\sqrt{2})^8 cis \left[8\left(\frac{\pi}{4}\right)\right]$$

$$= 16 cis 2\pi$$

$$= 16(\cos 2\pi + i \sin 2\pi)$$

$$= 16(1+i0)$$

$$= 16$$

Exercise

Find $(1+i)^{10}$ and express the result in rectangular form.

Solution

$$1+i \Rightarrow \begin{cases} x=1 \\ y=1 \end{cases}$$

$$r = \sqrt{1^2 + 1^2}$$

$$= \sqrt{2}$$

$$\theta = \tan^{-1}(1)$$

$$\begin{aligned}
&= \frac{\pi}{4} \Big| \\
1+i &= \sqrt{2} \operatorname{cis} \frac{\pi}{4} \Big| \\
(1+i)^{10} &= \left(\sqrt{2} \operatorname{cis} \frac{\pi}{4} \right)^{10} \\
&= (\sqrt{2})^{10} \operatorname{cis} \left[10 \left(\frac{\pi}{4} \right) \right] \\
&= 32 \operatorname{cis} \frac{5\pi}{2} \\
&= 32 \operatorname{cis} \frac{\pi}{2} \\
&= 32 \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) \\
&= 32(0+i) \\
&= 32i \Big|
\end{aligned}$$

Exercise

Find and express the result in rectangular form $(1-i)^5$

Solution

$$1-i \Rightarrow \begin{cases} x=1 \\ y=-1 \end{cases}$$

$$\begin{aligned}
r &= \sqrt{1^2 + 1^2} \\
&= \sqrt{2} \Big|
\end{aligned}$$

$$\begin{aligned}
\hat{\theta} &= \tan^{-1}(1) \\
&= \frac{\pi}{4} \Big|
\end{aligned}$$

The angle is in quadrant *IV*, therefore;

$$\begin{aligned}
\theta &= 2\pi - \frac{\pi}{4} \\
&= \frac{7\pi}{4} \Big|
\end{aligned}$$

$$1-i = \sqrt{2} \operatorname{cis} \frac{7\pi}{4} \Big|$$

$$\begin{aligned}
(1-i)^5 &= \left(\sqrt{2} \operatorname{cis} \frac{7\pi}{4} \right)^5 \\
&= 4\sqrt{2} \left(\operatorname{cis} \left(5 \times \frac{7\pi}{4} \right) \right)
\end{aligned}$$

$$\begin{aligned}
&= 4\sqrt{2} \left(\operatorname{cis} \frac{35\pi}{4} \right) & 35\pi - 8 \times 2\pi = 3\pi \\
&= 4\sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) \\
&= 4\sqrt{2} \left(-\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) \\
&= \underline{-4 + 4i}
\end{aligned}$$

Exercise

Find and express the result in rectangular form $(1 - \sqrt{5}i)^8$

Solution

$$1 - \sqrt{5}i \Rightarrow \begin{cases} x = 1 \\ y = -\sqrt{5} \end{cases}$$

$$\begin{aligned}
r &= \sqrt{1 + 5} \\
&= \underline{\sqrt{6}}
\end{aligned}$$

$$\begin{aligned}
\hat{\theta} &= \tan^{-1} \left(\frac{\sqrt{5}}{1} \right) \\
&\approx \underline{66^\circ}
\end{aligned}$$

The angle is in quadrant *IV*, therefore;

$$\begin{aligned}
\theta &= 360^\circ - 66^\circ \\
&= \underline{294^\circ}
\end{aligned}$$

$$1 - \sqrt{5}i = \sqrt{6} \operatorname{cis} 294^\circ$$

$$\begin{aligned}
(1 - \sqrt{5}i)^8 &= (\sqrt{6} \operatorname{cis} 294^\circ)^8 \\
&= (\sqrt{6})^8 (\operatorname{cis} 2352^\circ) & 2352^\circ - 6 \times 360^\circ = 192^\circ \\
&= 1296 (\cos 192^\circ + i \sin 192^\circ) \\
&= 1296 (-.978 - 0.208i) \\
&= \underline{-1267.488 - 269.568 i}
\end{aligned}$$

Exercise

Find and express the result in rectangular form $(3cis80^\circ)^3$

Solution

$$\begin{aligned}(3cis80^\circ)^3 &= 3^3 (cis240^\circ) \\ &= 27 (\cos 240^\circ + i \sin 240^\circ) \\ &= 27 \left(-\frac{1}{2} - i \frac{\sqrt{3}}{2} \right) \\ &= \underline{-\frac{27}{2} - i \frac{27\sqrt{3}}{2}}\end{aligned}$$

Exercise

Find and express the result in rectangular form $(\sqrt{3}cis10^\circ)^6$

Solution

$$\begin{aligned}(\sqrt{3}cis10^\circ)^6 &= 27 (cis60^\circ) \\ &= 27 (\cos 60^\circ + i \sin 60^\circ) \\ &= \underline{\frac{27}{2} + i \frac{27\sqrt{3}}{2}}\end{aligned}$$

Exercise

Find and express the result in rectangular form $(\sqrt{2} - i)^6$

Solution

$$\sqrt{2} - i \Rightarrow \begin{cases} x = \sqrt{2} \\ y = -1 \end{cases}$$

$$\begin{aligned}r &= \sqrt{2+1} \\ &= \underline{\sqrt{3}}\end{aligned}$$

$$\begin{aligned}\hat{\theta} &= \tan^{-1} \left(\frac{1}{\sqrt{2}} \right) \\ &= \underline{\approx 35.26^\circ}\end{aligned}$$

The angle is in quadrant *IV*, therefore;

$$\begin{aligned}\theta &= 360^\circ - 35.26^\circ \\ &= \underline{324.74^\circ}\end{aligned}$$

$$\underline{\sqrt{2} - i = \sqrt{3} cis 324.74^\circ}$$

$$\begin{aligned}
 (\sqrt{2}-i)^6 &= (\sqrt{3} \operatorname{cis} 324.74^\circ)^6 \\
 &= 27 (\operatorname{cis} 1948.44^\circ) & 1948.44^\circ - 5 \times 360^\circ = 148.44^\circ \\
 &= 27 (\cos 148.44^\circ + i \sin 148.44^\circ) \\
 &= \underline{-23 + 14.142i}
 \end{aligned}$$

Exercise

Find and express the result in rectangular form $(4\operatorname{cis} 40^\circ)^6$

Solution

$$\begin{aligned}
 (4\operatorname{cis} 40^\circ)^6 &= 4^6 (\operatorname{cis} (6 \times 40^\circ)) \\
 &= 4^6 (\cos 240^\circ + i \sin 240^\circ) \\
 &= 4096 \left(-\frac{1}{2} + i \frac{\sqrt{3}}{2} \right) \\
 &= \underline{-2048 + 2048i\sqrt{3}}
 \end{aligned}$$

Exercise

Find and express the result in rectangular form $(2\operatorname{cis} 30^\circ)^5$

Solution

$$\begin{aligned}
 (2\operatorname{cis} 30^\circ)^5 &= 2^5 \operatorname{cis} (5(30^\circ)) \\
 &= 32 (\cos 150^\circ + i \sin 150^\circ) \\
 &= 32 \left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i \right) \\
 &= \underline{-16\sqrt{3} + 16i}
 \end{aligned}$$

Exercise

Find and express the result in rectangular form $\left(\frac{1}{2}\operatorname{cis} 72^\circ\right)^5$

Solution

$$\begin{aligned}
 \left(\frac{1}{2}\operatorname{cis} 72^\circ\right)^5 &= \frac{1}{2^5} \operatorname{cis} (5 \times 72^\circ) \\
 &= \frac{1}{32} \operatorname{cis} (\cos 360^\circ + i \sin 360^\circ) \\
 &= \underline{\frac{1}{32}}
 \end{aligned}$$

Exercise

Find **fifth** roots of $z = 1 + i\sqrt{3}$ and express the result in rectangular form.

Solution

$$1 + i\sqrt{3} \Rightarrow \begin{cases} x = 1 \\ y = \sqrt{3} \end{cases}$$

$$r = \sqrt{1+3} \\ = 2$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) \\ = 60^\circ$$

$$1 + i\sqrt{3} = 2 \operatorname{cis} 60^\circ$$

$$\begin{aligned} (1 + i\sqrt{3})^{1/5} &= (2 \operatorname{cis} 60^\circ)^{1/5} \\ &= \sqrt[5]{2} \left(\operatorname{cis} \frac{60^\circ}{5} + \frac{360^\circ k}{5} \right) \\ &= \sqrt[5]{2} \operatorname{cis} (12^\circ + 72^\circ k) \quad k = 0, 1, 2, 3, 4 \end{aligned}$$

For $k = 0$

$$\sqrt[5]{2} \operatorname{cis} (12^\circ + 72^\circ \cdot 0) = \sqrt[5]{2} \operatorname{cis} 12^\circ$$

For $k = 1$

$$\sqrt[5]{2} \operatorname{cis} (12^\circ + 72^\circ \cdot (1)) = \sqrt[5]{2} \operatorname{cis} 84^\circ$$

For $k = 2$

$$\sqrt[5]{2} \operatorname{cis} (12^\circ + 72^\circ \cdot (2)) = \sqrt[5]{2} \operatorname{cis} 156^\circ$$

For $k = 3$

$$\sqrt[5]{2} \operatorname{cis} (12^\circ + 72^\circ \cdot (3)) = \sqrt[5]{2} \operatorname{cis} 228^\circ$$

For $k = 4$

$$\sqrt[5]{2} \operatorname{cis} (12^\circ + 72^\circ \cdot (4)) = \sqrt[5]{2} \operatorname{cis} 300^\circ$$

Exercise

Find the *fourth* roots of $z = 16\text{cis}60^\circ$

Solution

$$\begin{aligned}\sqrt[4]{z} &= \sqrt[4]{16} \text{cis} \left(\frac{60^\circ}{4} + \frac{360^\circ}{4} k \right) \\ &= \underline{2\text{cis}(15^\circ + 90^\circ k)} \quad | \quad k = 0, 1, 2, 3\end{aligned}$$

For $k = 0$

$$2 \text{cis}(15^\circ + 90^\circ(0)) = \underline{2\text{cis}15^\circ}$$

For $k = 1$

$$2 \text{cis}(15^\circ + 90^\circ(1)) = \underline{2\text{cis}105^\circ}$$

For $k = 2$

$$2 \text{cis}(15^\circ + 90^\circ(2)) = \underline{2\text{cis}195^\circ}$$

For $k = 3$

$$2 \text{cis}(15^\circ + 90^\circ(3)) = \underline{2\text{cis}285^\circ}$$

Exercise

Find the *fourth* roots of $\sqrt{3} - i$

Solution

$$\sqrt{3} - i \Rightarrow \begin{cases} x = \sqrt{3} \\ y = -1 \end{cases}$$

$$\begin{aligned}r &= \sqrt{3+1} \\ &= \underline{2}\end{aligned}$$

$$\begin{aligned}\hat{\theta} &= \tan^{-1} \left(\frac{1}{\sqrt{3}} \right) \\ &= \underline{\frac{\pi}{6}}\end{aligned}$$

The angle is in quadrant *IV*, therefore;

$$\begin{aligned}\theta &= 2\pi - \frac{\pi}{6} \\ &= \underline{\frac{11\pi}{6}}\end{aligned}$$

$$\sqrt{3} - i = 2 \text{cis} \frac{11\pi}{6} \quad |$$

$$\sqrt[4]{\sqrt{3} - i} = \sqrt[4]{2 \text{cis} \frac{11\pi}{6}}$$

$$\begin{aligned}
 &= \sqrt[4]{2} \operatorname{cis} \left(\frac{1}{4} \frac{11\pi}{6} + \frac{2\pi k}{4} \right) \\
 &= \sqrt[4]{2} \operatorname{cis} \left(\frac{11\pi}{24} + \frac{\pi k}{2} \right) \quad \left| \quad k = 0, 1, 2, 3 \right.
 \end{aligned}$$

For $k = 0$

$$\sqrt[4]{2} \operatorname{cis} \left(\frac{11\pi}{24} + 0 \right) = \sqrt[4]{2} \operatorname{cis} \frac{11\pi}{24}$$

For $k = 1$

$$\sqrt[4]{2} \operatorname{cis} \left(\frac{11\pi}{24} + \frac{\pi}{2} \right) = \sqrt[4]{2} \operatorname{cis} \frac{23\pi}{24}$$

For $k = 2$

$$\sqrt[4]{2} \operatorname{cis} \left(\frac{11\pi}{24} + \pi \right) = \sqrt[4]{2} \operatorname{cis} \frac{35\pi}{24}$$

For $k = 3$

$$\sqrt[4]{2} \operatorname{cis} \left(\frac{11\pi}{24} + \frac{3\pi}{2} \right) = \sqrt[4]{2} \operatorname{cis} \frac{47\pi}{24}$$

Exercise

Find the *fourth* roots of $4 - 4\sqrt{3}i$

Solution

$$4 - 4\sqrt{3}i \Rightarrow \begin{cases} x = 4 \\ y = -4\sqrt{3} \end{cases}$$

$$\begin{aligned}
 r &= 4\sqrt{3+1} \\
 &= 8
 \end{aligned}$$

$$\begin{aligned}
 \hat{\theta} &= \tan^{-1} \left(\frac{4\sqrt{3}}{4} \right) \\
 &= \tan^{-1} (\sqrt{3}) \\
 &= \frac{\pi}{3}
 \end{aligned}$$

The angle is in quadrant *IV*, therefore;

$$\begin{aligned}
 \theta &= 2\pi - \frac{\pi}{3} \\
 &= \frac{5\pi}{3}
 \end{aligned}$$

$$4 - 4\sqrt{3}i = 8 \operatorname{cis} \frac{5\pi}{3}$$

$$\sqrt[4]{4 - 4\sqrt{3}i} = \sqrt[4]{8} \operatorname{cis} \frac{5\pi}{3}$$

$$= \sqrt[4]{8} \operatorname{cis}\left(\frac{5\pi}{12} + \frac{\pi k}{2}\right) \quad k = 0, 1, 2, 3$$

For $k = 0$

$$\sqrt[4]{8} \operatorname{cis}\left(\frac{5\pi}{12} + 0\right) = \sqrt[4]{8} \operatorname{cis} \frac{5\pi}{12}$$

For $k = 1$

$$\sqrt[4]{8} \operatorname{cis}\left(\frac{5\pi}{12} + \frac{\pi}{2}\right) = \sqrt[4]{8} \operatorname{cis} \frac{11\pi}{12}$$

For $k = 2$

$$\sqrt[4]{8} \operatorname{cis}\left(\frac{5\pi}{12} + \pi\right) = \sqrt[4]{8} \operatorname{cis} \frac{17\pi}{12}$$

For $k = 3$

$$\sqrt[4]{8} \operatorname{cis}\left(\frac{5\pi}{12} + \frac{3\pi}{2}\right) = \sqrt[4]{8} \operatorname{cis} \frac{23\pi}{12}$$

Exercise

Find the *fourth* roots of $-16i$

Solution

$$-16i \Rightarrow \begin{cases} x = 0 \\ y = -16 \end{cases}$$

$$r = 16$$

$$\theta = \frac{3\pi}{2}$$

$$-16i = 16 \operatorname{cis} \frac{3\pi}{2}$$

$$\sqrt[4]{-16i} = \sqrt[4]{16 \operatorname{cis} \frac{3\pi}{2}}$$

$$= 2 \operatorname{cis}\left(\frac{3\pi}{8} + \frac{\pi k}{2}\right) \quad k = 0, 1, 2, 3$$

For $k = 0$

$$2 \operatorname{cis}\left(\frac{3\pi}{8} + 0\right) = 2 \operatorname{cis} \frac{3\pi}{8}$$

For $k = 1$

$$2 \operatorname{cis}\left(\frac{3\pi}{8} + \frac{\pi}{2}\right) = 2 \operatorname{cis} \frac{7\pi}{8}$$

For $k = 2$

$$2 \operatorname{cis}\left(\frac{3\pi}{8} + \pi\right) = 2 \operatorname{cis} \frac{11\pi}{8}$$

For $k=3$

$$2 \operatorname{cis}\left(\frac{3\pi}{8} + \frac{3\pi}{2}\right) = \underline{2 \operatorname{cis} \frac{15\pi}{8}}$$

Exercise

Find the **cube** roots of 27.

Solution

$$\begin{aligned} \sqrt[3]{27} &= (27 \operatorname{cis} 0^\circ)^{1/3} \\ &= \sqrt[3]{27} \operatorname{cis}\left(\frac{0^\circ}{3} + \frac{360^\circ}{3}k\right) \\ &= \underline{3 \operatorname{cis}(0^\circ + 120^\circ k)} \quad k = 0, 1, 2 \end{aligned}$$

For $k=0$

$$z = 3 \operatorname{cis}(0^\circ + 120^\circ(0)) = \underline{2 \operatorname{cis} 0^\circ}$$

For $k=1$

$$z = 3 \operatorname{cis}(0^\circ + 120^\circ(1)) = \underline{2 \operatorname{cis} 120^\circ}$$

For $k=2$

$$z = 3 \operatorname{cis}(0^\circ + 120^\circ(2)) = \underline{2 \operatorname{cis} 240^\circ}$$

Exercise

Find the **cube** roots of $8-8i$

Solution

$$8-8i \Rightarrow \begin{cases} x=8 \\ y=-8 \end{cases}$$

$$\begin{aligned} r &= 8\sqrt{1+1} \\ &= \underline{8\sqrt{2}} \end{aligned}$$

$$\begin{aligned} \hat{\theta} &= \tan^{-1}\left(\frac{8}{8}\right) \\ &= \tan^{-1}(1) \\ &= \underline{\frac{\pi}{4}} \end{aligned}$$

The angle is in quadrant *IV*, therefore;

$$\begin{aligned} \theta &= 2\pi - \frac{\pi}{4} \\ &= \underline{\frac{7\pi}{4}} \end{aligned}$$

$$\begin{aligned}\sqrt[3]{8-8i} &= \sqrt[3]{8\sqrt{2} \operatorname{cis} \frac{7\pi}{4}} \\ &= \sqrt[3]{2\sqrt{2} \operatorname{cis} \left(\frac{7\pi}{12} + \frac{2\pi k}{3} \right)} \quad \left| \quad k = 0, 1, 2 \right.\end{aligned}$$

For $k = 0$

$$z = \sqrt[3]{2\sqrt{2} \operatorname{cis} \left(\frac{7\pi}{12} + 0 \right)} = \sqrt[3]{2\sqrt{2} \operatorname{cis} \frac{7\pi}{12}}$$

For $k = 1$

$$z = \sqrt[3]{2\sqrt{2} \operatorname{cis} \left(\frac{7\pi}{12} + \frac{2\pi}{3} \right)} = \sqrt[3]{2\sqrt{2} \operatorname{cis} \frac{15\pi}{12}}$$

For $k = 2$

$$z = \sqrt[3]{2\sqrt{2} \operatorname{cis} \left(\frac{7\pi}{12} + \frac{4\pi}{3} \right)} = \sqrt[3]{2\sqrt{2} \operatorname{cis} \frac{23\pi}{12}}$$

Exercise

Find the **cube** roots of -8

Solution

$$\begin{aligned}r &= 8 \\ \theta &= \frac{3\pi}{2}\end{aligned}$$

$$\begin{aligned}\sqrt[3]{-8} &= \sqrt[3]{8 \operatorname{cis} \frac{3\pi}{2}} \\ &= \sqrt[3]{2 \operatorname{cis} \left(\frac{\pi}{2} + \frac{2\pi k}{3} \right)} \quad \left| \quad k = 0, 1, 2 \right.\end{aligned}$$

For $k = 0$

$$z = \sqrt[3]{2 \operatorname{cis} \left(\frac{\pi}{2} + 0 \right)} = \sqrt[3]{2 \operatorname{cis} \frac{\pi}{2}}$$

For $k = 1$

$$z = \sqrt[3]{2 \operatorname{cis} \left(\frac{\pi}{2} + \frac{2\pi}{3} \right)} = \sqrt[3]{2 \operatorname{cis} \frac{7\pi}{6}}$$

For $k = 2$

$$z = \sqrt[3]{2 \operatorname{cis} \left(\frac{\pi}{2} + \frac{4\pi}{3} \right)} = \sqrt[3]{2 \operatorname{cis} \frac{11\pi}{6}}$$

Exercise

Find all complex number solutions of $x^3 + 1 = 0$.

Solution

$$x^3 + 1 = 0$$

$$x^3 = -1$$

$$\underline{r = 1}$$

$$\theta = \tan^{-1}\left(\frac{0}{-1}\right)$$

$$\underline{= \pi}$$

$$x^3 = 1 \operatorname{cis}(\pi)$$

$$x = (1 \operatorname{cis}\pi)^{1/3}$$

$$= (1)^{1/3} \operatorname{cis}\left(\frac{\pi}{3} + \frac{2\pi}{3}k\right)$$

$$\underline{= \operatorname{cis}\left(\frac{\pi}{3} + \frac{2\pi}{3}k\right)} \quad k = 0, 1, 2$$

For $k = 0$

$$x = \operatorname{cis}\left(\frac{\pi}{3} + \frac{2\pi}{3}(\mathbf{0})\right)$$

$$= \operatorname{cis} \frac{\pi}{3}$$

$$= \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$$

$$\underline{= \frac{1}{2} + i \frac{\sqrt{3}}{2}}$$

For $k = 1$

$$x = \operatorname{cis}\left(\frac{\pi}{3} + \frac{2\pi}{3}(\mathbf{1})\right)$$

$$= \operatorname{cis}\left(\frac{3\pi}{3}\right)$$

$$= \operatorname{cis}\pi$$

$$= \cos \pi + i \sin \pi$$

$$\underline{= -1}$$

For $k = 2$

$$x = \operatorname{cis}\left(\frac{\pi}{3} + \frac{2\pi}{3}(\mathbf{2})\right)$$

$$= \operatorname{cis} \frac{5\pi}{3}$$

$$= \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}$$

$$= \frac{1}{2} - i \frac{\sqrt{3}}{2}$$