Solution Section 3.5 – Applications of Recurrence Relations

Exercise

a) Find a recurrence relation for the number of permutation of a set with *n* elements

b) Use the recurrence relation to find the number of permutations of a set with *n* elements using iteration.

Solution

a) A permutation of a set with *n* elements of a choice of a first element, followed by a permutation of a set of $n-1$ elements. Therefore $P_n = nP_{n-1} \quad with \quad P_0 = 1$

b) $P_n = nP_{n-1}$

$$= n(n-1)P_{n-2}$$

$$= n(n-1)\cdots 2\cdot 1\cdot P_0$$

$$\underline{= n!}$$

Exercise

A vending machine dispensing books of stamps accepts only one-dollar coins, \$1 bills, and \$5 bills.

a) Find a recurrence relation for the number of ways to deposit n dollars in the vending machine, where the order in which the coins and bills are deposited matter.

b) What are the initial conditions?

c) How many ways are there to deposit \$10 for a book of stamps?

Solution

a) Let a_n be the number of ways to deposit *n* dollars in the vending machine. We must express a_n in terms of earlier terms in the sequence. If we want to deposit *n* dollars, we may start with a dollar coin and then deposit *n* – 1 dollars. This gives us a_{n-1} ways to deposit *n* dollars.

We can start with a dollar bill and then deposit *n* – 1 dollars. This gives us a_{n-1} more ways to deposit *n* dollars.

Finally, we can deposit a five-dollar bill and follow that with *n* – 5 dollars; there are a_{n-5} ways to do this, Therefore the recurrence relation is $a_n = 2a_{n-1} + a_{n-5} \quad for \quad n \ge 5$

b) We need initial conditions for all *n* from 0 to 4. Clearly, $a_0 = 1$ (deposit nothing) and $a_1 = 2$ (deposit either the dollar coin or the dollar bill)

$a_2 = 2^2 = 4; \quad a_3 = 2^3 = 8 \quad and \quad a_4 = 2^4 = 16$

c) $a_5 = 2a_4 + a_0 = 2(16) + 1 = 33$

$$a_6 = 2a_5 + a_1 = 2(33) + 2 = 68$$

$$a_7 = 2a_6 + a_2 = 2(68) + 4 = 140$$

$a_8 = 2a_7 + a_3 = 2\cdot 140 + 8 = 288$

$a_9 = 2a_8 + a_4 = 2\cdot 288 + 16 = 592$

$a_{10} = 2a_8 + a_5 = 2\cdot 592 + 33 = 1217$

Therefore, there are 1217 ways to deposit \$10.

Exercise

a) Find a recurrence relation for the number of bit strings of length *n* that contain three consecutive 0s.

b) What are the initial conditions?

c) How many bit strings of length seven contain three consecutive 0s?

Solution

a) Let a_n be the number of bit strings of length *n* containing three consecutive 0's. In order to construct a bit string of length *n* containing three consecutive 0's we could start with 1 and follow with a string of length *n* – 1 three consecutive 0's, or we could start with a 01 and follow with a string of length *n* – 2 three consecutive 0's, or we could start with a 001 and follow with a string of length *n* – 3 three consecutive 0's, or we could start with a 000 and follow with a string of length *n* – 3.

These 4 cases are mutually exclusive and exhaust the possibilities for how the string might start. We can write down the recurrence relation, valid for all $n \ge 3$: $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$

b) There are no bit strings of length 0, 1, or 2 containing 3 consecutive 0's, so the initial conditions are $a_0 = a_1 = a_2 = 0$

c) $a_3 = a_2 + a_1 + a_0 + 2^0$

$= 0 + 0 + 0 + 1$

$\underline{= 1}$

$a_4 = a_3 + a_2 + a_1 + 2^1$

$= 1 + 0 + 0 + 2$

$\underline{= 3}$

$a_5 = a_4 + a_3 + a_2 + 2^2$

$= 3 + 1 + 0 + 4$

$\underline{= 8}$

$a_6 = a_5 + a_4 + a_3 + 2^3$

$= 8 + 3 + 1 + 8$

$\underline{= 20}$

$a_7 = a_6 + a_5 + a_4 + 2^4$

$$= 20 + 8 + 3 + 16$$

$$\underline{= 47}$$

Therefore, there are 47 bits of length 7 containing three consecutive 0's.

Exercise

a) Find a recurrence relation for the number of bit strings of length *n* that do not contain three consecutive 0s.

b) What are the initial conditions?

c) How many bit strings of length seven do not contain three consecutive 0s?

Solution

a) Let a_n be the number of bit strings of length *n* do not contain three consecutive 0's. In order to construct a bit string of length *n* of this type we could start with 1 and follow with a string of length $n - 1$ not containing three consecutive 0's, or we could start with 01 and follow with a string of length $n - 2$ not containing three consecutive 0's, or we could start with a 001 and follow with a string of length $n - 3$ not containing three consecutive 0's.

These 3 cases are mutually exclusive and exhaust the possibilities for how the string might start since it cannot start 000.

We can write down the recurrence relation, valid for all $n \ge 3$: $a_n = a_{n-1} + a_{n-2} + a_{n-3}$

b) There are no bit strings of length 0, 1, or 2 containing 3 consecutive 0's, so the initial conditions are $a_0 = 1; \quad a_1 = 2 \quad and \quad a_2 = 4$

c) $a_3 = a_2 + a_1 + a_0$

$$= 4 + 2 + 1$$

$$\underline{= 7}$$

$$a_4 = a_3 + a_2 + a_1$$

$$= 7 + 4 + 2$$

$$\underline{= 13}$$

$$a_5 = a_4 + a_3 + a_2$$

$$= 13 + 7 + 4$$

$$\underline{= 24}$$

$$a_6 = a_5 + a_4 + a_3$$

$$= 24 + 13 + 7$$

$$\underline{= 44}$$

$$a_7 = a_6 + a_5 + a_4$$

$$= 44 + 24 + 13$$

$$\underline{= 81}$$

Therefore, there are 81 bits of length 7 that do not contain three consecutive 0's.

Exercise

a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time.

b) What are the initial conditions?

c) In how many can this person climb a flight of eight stairs

Solution

a) Let a_n be the number of ways to climb n stairs. In order to climb n stairs, a person must either start with a step of one stair and climb $n-1$ stairs $\left(a_{n-1}\right)$ or else start with a step of two stairs and then climb $n-2$ stairs $\left(a_{n-2}\right)$ or else start with a step of two stairs and then climb $n-3$ stairs $\left(a_{n-3}\right)$.

From this analysis we can immediately write down the recurrence relation, valid for all $n \ge 3$: $a_n = a_{n-1} + a_{n-2} + a_{n-3}$

b) The initial conditions are $a_0 = 1, \quad a_1 = 1 \quad and \quad a_2 = 2$, since there is one way to climb no stairs (do nothing), clearly only one way to climb one stair, and two ways to climb stairs (one step twice or two steps at once).

c) Each term in our sequence $\left\{a_n\right\}$ is the sum of the previous three terms, so the sequence begins

$a_0 = 1,\ a_1 = 1,\ a_2 = 2,\ a_3 = 4,\ a_4 = 7,\ a_5 = 13,\ a_6 = 24,\ a_7 = 44,\ a_8 = 81$

Thus, a person can climb a flight of 8 stairs in 81 ways under the restrictions in this problem.