SOLUTION *Section* 4.1 – Relations and Their Properties

Exercise

List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$ where $(a,\ b) \in R$ if and only if

a) $a = b$ **b)** $a + b = 4$ **c)** $a > b$

d) $a \mid b$ **e)** $gcd(a,b) = 1$ **f)** $lcm(a,b) = 2$

Solution

a) {(0, 0), (1, 1), (2, 2), (3, 3)}

b) {(4, 0), (1, 3), (3, 1), (2, 2)}

c) {(1, 0), (2, 0), (2,1), (3, 0), (3, 1), (3, 2), (4, 0), (4, 1), (4, 2), (4, 3)}

d) {(1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 0) , (3, 3) , 4, 0)} **(**means b is multiple of $a \neq 0$**)**

e) {(1, 0), (0, 1), (1,1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (4, 1), (4, 3)} **(**means *relatively prime***)**

f) {(1, 2), (2, 1), (2, 2)} **(**Mean *least common multiple* is 2**)**.

Exercise

a) List all the ordered pairs in the relation $R = \{(a,\ b) \mid a \text{ divides } b\}$ on the set {1, 2, 3, 4, 5, 6}

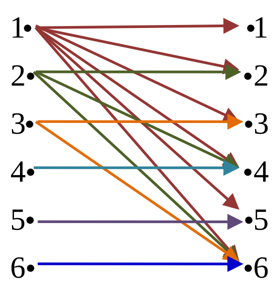
b) Display this relation graphically.

c) Display this relation in tabular form.

Solution

a) {(1, 1), (1, 2), (1,3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)}

b)



c)

R	1	2	3	4	5	6
1	×	×	×	×	×	×
2		×		×		×
3			×			×
4				×		
5					×	
6						×

Exercise

For each of these relations on the set {1, 2, 3, 4}, decide whether it is reflexive, symmetric, antisymmetric and transitive

- ***a)*** {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}
- ***b)*** {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}
- ***c)*** {(2, 4), (4, 2)}
- ***d)*** {(1, 2), (2, 3), (3, 4)}
- ***e)*** {(1, 1), (2, 2), (3, 3), (4, 4)}
- ***f)*** {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}

Solution

a) This relation is not reflexive, since (1, 1) is not included
It is not symmetric, since (2, 4) is included but not (4, 2)
It is not antisymmetric, since it includes (2, 3) and (3, 2) but $2 \neq 3$

$$\begin{matrix} (2,\ 3)\ \&\ (3,\ 4) \to (2,\ 4) \\ (2,\ 3)\ \&\ (3,\ 2) \to (2,\ 2) \end{matrix}\ \&\ \begin{matrix} (3,\ 2)\ \&\ (2,\ 4) \to (3,\ 4) \\ (3,\ 2)\ \&\ (2,\ 3) \to (3,\ 3) \end{matrix}$$ It is transitive.

b) This relation is reflexive, since (1, 1), (2, 2), (3, 3), and (4, 4)} are included
It is symmetric, since (2, 1) and (1, 2) are included
It is not antisymmetric, since it includes (2, 1) and (1, 2) but $2 \neq 1$

$$\begin{matrix} (2,\ 1)\ \&\ (1,\ 2) \to (2,\ 2) \\ (1,\ 2)\ \&\ (2,\ 1) \to (1,\ 1) \end{matrix}$$ It is transitive.

c) This relation is not reflexive, since (1, 1) is not included
It is symmetric, since (2, 4) and (4, 2) are included
It is not antisymmetric, since it includes (2, 4) and (4, 2) but $2 \neq 4$
It is not transitive, since it includes (2, 4) and (4, 2) but not (2, 2)

d) This relation is not reflexive, since (1, 1) is not included
It is not symmetric, since (1, 2) is included but not (2, 1)
It is antisymmetric, since no cases of (a, b) and (b, a) both being in the relation
It is not transitive, since it includes (1, 2) and (2, 3) but not (1, 3)

e) This relation is reflexive, since (1, 1), (2, 2), (3, 3), and (4, 4)} are included and it is *symmetric*
It is antisymmetric, since no cases of (a, b) and (b, a) both being in the relation
It is transitive, since the only time the hypothesis $(a,\ b) \in R \wedge (b,\ c) \in R$ is met is when $a \equiv b \equiv c$

f) This relation is not reflexive, since (1, 1) is not included
It is not symmetric, since (1, 4) is included but not (4, 1)
It is not antisymmetric, since it includes (1, 3) and (3, 1)
It is not transitive, since it includes (2, 3) and (3, 1) but not (2, 1)

Exercise

Determine whether the relation *R* on the set of all people is reflexive, symmetric, antisymmetric, and/or transitive, where $(a,\ b) \in R$ if and only if

a) *a* is taller than *b*.
b) *a* and *b* were born on the same day
c) *a* has the same first name as *b*.
d) *a* and *b* have a common grandparent.

Solution

a) I am *not* taller than myself, therefore *being taller* is not reflexive
It is *not* symmetric, since I am taller than my kid but my kid is not
It is antisymmetric since we never have *a* taller than *b* and *b* taller than *a* even if *a* = *b*
It is transitive since if *a* taller than *b* and *b* taller than *c* that implies that *A* taller then *c*

b) The relation is reflexive since *a* is born on the same day
It is symmetric, since *a* and *b* were born on the same day
It is *not* antisymmetric since *a* and *b* were born on the same day but $a \neq b$
It is transitive since if *a* and *b* were born on the same day and *b* and *c* were born on the same day that implies that *a* and *c* were born on the same day

c) The relation is reflexive since *a* has the same first name as *a*
It is symmetric, since *a* has the same first name as *b* than *b* has the same first name as *a*
It is *not* antisymmetric since *a* has the same first name as *b* but $a \neq b$
It is transitive since if *a* has the same first name as *b* and *c* has the same first name as *c* that implies that *a* has the same first name as *c*

d) The relation is reflexive since *a* and *a* have a common grandparent
It is symmetric, since *a* and *b* have a common grandparent than *b* and *a* have a common grandparent
It is *not* antisymmetric since *a* and *b* have a common grandparent but $a \neq b$
It is transitive since if *a* and *b* have a common grandparent and *b* and *c* have a common grandparent that implies that *a* and *c* have a common grandparent

Exercise

Determine whether the relation ***R*** on the set of all ***real numbers*** is reflexive, symmetric, antisymmetric, and/or transitive, where $(x,\ y) \in \boldsymbol{R}$ if and only if

a) $x+y=0$ ***b)*** $x=\pm y$ ***c)*** $x-y$ *is a rational number* ***d)*** $x=2y$
e) $xy \ge 0$ ***f)*** $xy=0$ ***g)*** $x=1$ ***h)*** $x=1$ *or* $y=1$

Solution

a) The relation is *not* reflexive since $1+1 \neq 0$
It is symmetric, since $x+y=0$ then $y+x=0$ because $x+y=y+x$
It is *not* antisymmetric since $(1, -1) \in \boldsymbol{R}$ and $(-1, 1) \in \boldsymbol{R}$ but $1 \neq -1$

It is *not* transitive since $(1, -1)$ and $(-1, 1) \in \boldsymbol{R}$ but $(1,\ 1) \notin \boldsymbol{R}$

b) The relation is reflexive since $x = \pm x$

It is symmetric, since $x = \pm y$ *iff* $y = \pm x$

It is *not* antisymmetric since $(1, -1) \in \boldsymbol{R}$ and $(-1, 1) \in \boldsymbol{R}$ but $1 \neq -1$

It is transitive since the product 1's and -1's is ± 1

c) The relation is reflexive since $x - x = 0$ is a rational number

It is symmetric, since $x - y$ is rational, then $-(x-y) = y - x$

It is *not* antisymmetric since $(1, -1) \in \boldsymbol{R}$ but $(-1, 1) \in \boldsymbol{R}$ but $1 \neq -1$

It is transitive since $(x, y) \in \boldsymbol{R}$ then $x - y$ is a rational number $(y, z) \in \boldsymbol{R}$ then $x - y$ is a rational number, therefore $x - z$ is rational that means that $(x, z) \in \boldsymbol{R}$

d) The relation is *not* reflexive since $1 \neq 2 \cdot 1$

It is *not* symmetric, since $(2,\ 1) \in \boldsymbol{R}$ then $2 = 2 \cdot 1$ but $1 \neq 2 \cdot 2$ therefore $(1,\ 2) \notin \boldsymbol{R}$

It is antisymmetric since $x = 2y$ and $y = 2x$ that implies to $y = 2(2y) = 4y$ which $y = 0$

It is *not* transitive since $2 = 2 \cdot 1$ and $4 = 2 \cdot 2 \ \Rightarrow\ 4 \neq 2 \cdot 1$ so $(4,\ 1) \notin \boldsymbol{R}$

e) The relation is reflexive since $x^2 \geq 0$ always positive

It is symmetric, since $xy \geq 0 \quad \Rightarrow \quad yx \geq 0$

It is *not* antisymmetric since $(2, 3) \in \boldsymbol{R}$ and $(3, 2) \in \boldsymbol{R}$ but $2 \neq 3$

It is *not* transitive since $(1,\ 0) \in \boldsymbol{R} \Rightarrow 1 \cdot 0 \geq 0 \quad (0,\ -1) \in \boldsymbol{R} \Rightarrow 0 \cdot (-1) \geq 0$ but $1 \cdot (-1) \not\geq 0 \Rightarrow (1,\ -1) \notin \boldsymbol{R}$

f) $\boxed{xy = 0}$ The relation is *not* reflexive since $(1,\ 1) \notin \boldsymbol{R}$

It is symmetric, since $xy = 0 \quad \rightarrow \quad yx = 0$

It is antisymmetric since $(2, 0) \in \boldsymbol{R}$ and $(0, 2) \in \boldsymbol{R}$ but $2 \neq 0$

It is *not* transitive since $2 \cdot 0 = 0$ $(2, 0) \in \boldsymbol{R}$ and $0 \cdot (-2) = 0$ $(0, -2) \in \boldsymbol{R} \Rightarrow$ $2 \cdot (-2) \neq 0$ so $(2,\ -2) \notin \boldsymbol{R}$

g) The relation is *not* reflexive since $(2,\ 2) \notin \boldsymbol{R}$

It is *not* symmetric, since $(1,\ 2) \in \boldsymbol{R}$ but $(2,\ 1) \notin \boldsymbol{R}$

It is antisymmetric since $(x, y) \in \boldsymbol{R}$ and $(y, x) \in \boldsymbol{R}$ then $x = 1$ and $y = 1$, so $x = y$

It is transitive since $(x, y) \in \boldsymbol{R}$ and $(y, z) \in \boldsymbol{R}$ then $x = 1$ and $y = 1$, so $(x, z) \in \boldsymbol{R}$

h) The relation is *not* reflexive since $(2,\ 2) \notin \boldsymbol{R}$

It is symmetric, since $(1,\ 2) \in \boldsymbol{R}$ and $(2,\ 1) \in \boldsymbol{R}$

It is *not* antisymmetric since $(1,\ 2) \in \boldsymbol{R}$ and $(2,\ 1) \in \boldsymbol{R}$ but $1 \neq 2$

It is *not* transitive since $(2,\ 1) \in \boldsymbol{R}$ and $(1,\ 3) \in \boldsymbol{R}$ but $(2,\ 3) \notin \boldsymbol{R}$

Exercise

Determine whether the relation *R* on the set of all ***integers numbers*** is reflexive, symmetric, antisymmetric, and/or transitive, where $(x,\ y)\in R$ if and only if

a) $x \ne y$ ***b***) $xy \ge 1$ ***c***) $x = y+1\ \ or\ \ x = y-1$ ***d***) $x \equiv y\left(\boldsymbol{mod}\ \ 7\right)$

e) $x\ is\ a\ multiple\ of\ \ y$ ***f***) $x = y^2$ ***g***) $x \ge y^2$

Solution

a) This relation is not reflexive, since 1 ≠ 1 for instance

It is symmetric, if $x \ne y \Rightarrow y \ne x$

It is *not* antisymmetric since $1 \ne 2 \Rightarrow 2 \ne 1$

It is *not* transitive since $1 \ne 2\ \ and\ \ 2 \ne 1 \Rightarrow 1 \ne 1$

b) This relation is not reflexive, since (0, 0) is not included $\left(0 \not\geq 1\right)$

It is symmetric, because $xy = yx$ (commutative property of multiplication)

It is *not* antisymmetric since (2, 3) and (3, 2) are both included

It is transitive holds between *x* and *y* if and only if either *x* and *y* are both positive or *x* and *y* are both negative

c) This relation is *not* reflexive, since (1, 1) is not included $\left(1 \ne 1+1\right)$

It is symmetric, because $x = y-1$ is equivalent to $y = x+1$

It is *not* antisymmetric since (1, 2) and (2, 1) are in the relation

It is *not* transitive since (1, 2) and (2, 1) are in the relation but (1, 1) is not

d) $x \equiv y\left(\boldsymbol{mod}\ \ 7\right)$ means that $x-y = 7t$ for some *t*.

This relation is reflexive since $x - x = 7\cdot 0$

It is symmetric since is $x \equiv y\left(\boldsymbol{mod}\ \ 7\right)$ then $x-y = 7t$, therefore $y-x = 7\left(-t\right)$ so $y \equiv x\left(\boldsymbol{mod}\ \ 7\right)$

It is *not* antisymmetric since $1 \equiv 8\left(\boldsymbol{mod}\ \ 7\right)$ and $8 \equiv 1\left(\boldsymbol{mod}\ \ 7\right)$

It is transitive since $x \equiv y\left(\boldsymbol{mod}\ \ 7\right)$ means $x-y = 7t$ and $y \equiv z\left(\boldsymbol{mod}\ \ 7\right)$ means $y-z = 7s$

$x-y = x-y+y-z = 7t+7s = 7\left(t+s\right)$; therefore $x \equiv z\left(\boldsymbol{mod}\ \ 7\right)$

e) $x\ is\ a\ multiple\ of\ \ y$ means that $x = ty$ for some *t*.

This relation is reflexive since $x = x\cdot 1$

It is *not* symmetric since is $6 = 3\cdot 2$ but $2 \ne 3\cdot 6$

It is *not* antisymmetric since 2 is multiple of –2 but $2 \ne -2$

It is transitive since $x = ty$ and $y = sz\ \ \Rightarrow\ x = ty = tsz = \left(ts\right)z$ therefore *x* is a multiply of *z*.

f) This relation is *not* reflexive, since $3 \ne 3^2$

It is *not* symmetric since is $9 = 3^2$ but $3 \ne 9^2$

It is antisymmetric since $x = y^2$ and $y = x^2$

$$\Rightarrow\ x = y^2 = x^4$$

$$x - x^4 = 0$$

$$x\left(1-x^3\right) = 0$$

$$x(1-x)\left(1+x+x^2\right)=0 \qquad \rightarrow x=0,\ 1$$

$x=y^2$ and $y=x^2$ when $x=y$

It is *not* transitive since $81=9^2$ and $9=3^2$ but $81\neq 3^2$

g) This relation is *not* reflexive, since $3\not\geq 3^2$

It is *not* symmetric since is $9\geq 3^2$ but $3\not\geq 9^2$

It is antisymmetric since $x\geq y^2$ and $y\geq x^2$, only when $x=0,\ 1$.

It is transitive since $x\geq y^2$ and $y\geq z^2 \Rightarrow \underline{x}\geq y^2\geq\left(z^2\right)^2=z^4\ \underline{\geq z^2}$

Exercise

Show that the relation $R=\varnothing$ on nonempty set *S* is symmetric and transitive, but not reflexive.

Solution

If $R=\varnothing$, then the hypothesis of the conditional statements in the definitions of symmetric and transitive are never true, so those statements are always true by definition.

$S\neq\varnothing$ the statement $(a,a)\in R$ is false for an element of *S*, so $\forall a\ (a,a)\in R$ is not true, thus *R* is not reflexive.

Exercise

Show that the relation $R=\varnothing$ on nonempty set $S=\varnothing$ is reflexive, symmetric and transitive.

Solution

Since the domain is empty, then the relation is vacuously reflexive, symmetric and transitive s

Exercise

Give an example of a relation on a set that is

a) both symmetric and antisymmetric

b) neither symmetric nor antisymmetric

Solution

a) The empty set on $\{a\}$ – vacuously symmetric and antisymmetric

b) $\{(a, b), (b, a), (a, c)\}$ on $\{a, b, c\}$

Exercise

A relation R is called ***asymmetric*** if $(a,\ b) \in R$ implies that $(b,\ a) \notin R$. Explore the notion of an asymmetric relation to the following

a) {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}
b) {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}
c) {(2, 4), (4, 2)}
d) {(1, 2), (2, 3), (3, 4)}
e) {(1, 1), (2, 2), (3, 3), (4, 4)}
f) {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}
g) *a* is taller than *b*.
h) *a* and *b* were born on the same day
i) *a* has the same first name as *b*.
j) *a* and *b* have a common grandparent.

Solution

The relations (***a***), (***b***), and (***c***) are not *asymmetric* since they contain pairs of the form (*x*, *x*)
The relation (***f***) is not *asymmetric* since both (1, 3) and (3, 1) are in the relation
The relation (***d***) is not *asymmetric*
The relation (***g***) is *asymmetric* since if *a* taller than *b*, then *b* can't be taller than *a*.
The relation (***h***) is not *asymmetric* since *a* and *b* were born on the same day but $a \neq b$
The relation (***i***) is not *asymmetric* since *a* has the same first name as *b* but $a \neq b$
The relation (***j***) is not *asymmetric* since *a* and *b* have a common grandparent but $a \neq b$

Exercise

Let R be the relation $R = \{(a,\ b) \mid a < b\}$ on the set of integers. Find

a) R^{-1} ***b)*** $\bar{R}$

Solution

a) $R^{-1} = \{(b,a) \mid (a,b) \in R\} = \{(b,a) \mid a < b\} = \{(a,b) \mid a > b\}$

b) $\bar{R} = \{(b,a) \mid (a,b) \notin R\} = \{(b,a) \mid a \not< b\} = \{(a,b) \mid a \geq b\}$

Exercise

Let R be the relation $R = \{(a,\ b) \mid a \ divides\ b\}$ on the set of positive integers. Find

a) R^{-1} ***b)*** $\bar{R}$

Solution

a) $R^{-1} = \{(a,\ b) \mid b \text{ divides } a\}$

b) $\bar{R} = \{(a,\ b) \mid a \ does\ not\ divide\ b\}$

Exercise

Let R be the relation on the set of all states in the U.S. consisting of pairs (a, b) where state a borders state b. Find

a) R^{-1} ***b)*** $\bar{R}$

Solution

a) Since this relation is symmetric, $R^{-1} = R$

b) This relation consists of all pairs (a, b) in which state a does not border state b.

Exercise

Let $R_1 = \{(1, 2), (2, 3), (3, 4)\}$ and

$R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4)\}$ be relation from {1, 2, 3} to {1, 2, 3, 4}. Find

a) $R_1 \cup R_2$ ***b***) $R_1 \cap R_2$ ***c***) $R_1 - R_2$ ***d***) $R_2 - R_1$

Solution

a) $R_1 \cup R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4)\} = R_2$

b) $R_1 \cap R_2 = \{(1, 2), (2, 3), (3, 4)\} = R_1$

c) $R_1 - R_2 = \varnothing$

d) $R_2 - R_1 = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)\}$

Exercise

Let the relation R = {(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)} and the relation S = {(2, 1), (3, 1), (3, 2), (4, 2)}
Find $S \circ R$

Solution

$(1, 2) \in R \text{ and } (2, 1) \in S \Rightarrow (1, 1) \in S \circ R$

$(1, 3) \in R \text{ and } (3, 2) \in S \Rightarrow (1, 2) \in S \circ R$

$(2, 3) \in R \text{ and } (3, 1) \in S \Rightarrow (2, 1) \in S \circ R$

$(2, 4) \in R \text{ and } (4, 2) \in S \Rightarrow (2, 2) \in S \circ R$

$S \circ R = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

Exercise

$$R_1 = \{(a,b) \in \mathbf{R}^2 | a > b\} \qquad R_3 = \{(a,b) \in \mathbf{R}^2 | a < b\} \qquad R_5 = \{(a,b) \in \mathbf{R}^2 | a = b\}$$

$$R_2 = \{(a,b) \in \mathbf{R}^2 | a \ge b\} \qquad R_4 = \{(a,b) \in \mathbf{R}^2 | a \le b\} \qquad R_6 = \{(a,b) \in \mathbf{R}^2 | a \ne b\}$$

Find the following:

a) $R_1 \cup R_3$ ***b***) $R_1 \cup R_5$ ***c***) $R_2 \cap R_4$ ***d***) $R_3 \cap R_5$ ***e***) $R_1 - R_2$

f) $R_2 - R_1$ ***g***) $R_1 \oplus R_3$ ***h***) $R_2 \oplus R_4$ ***i***) $R_1 \circ R_1$ ***j***) $R_1 \circ R_2$

k) $R_1 \circ R_3$ ***l***) $R_1 \circ R_4$ ***m***) $R_1 \circ R_5$ ***n***) $R_1 \circ R_6$ ***o***) $R_2 \circ R_3$

Solution

a)
$$\begin{aligned} R_1 \cup R_3 &= \{(a,b) \in \mathbf{R}^2 \mid a > b \text{ or } a < b\} \\ &= \{(a,b) \in \mathbf{R}^2 \mid a \ne b\} \\ &= R_6 \end{aligned}$$

b)
$$\begin{aligned} R_1 \cup R_5 &= \{(a,b) \in \mathbf{R}^2 \mid a > b \text{ or } a = b\} \\ &= \{(a,b) \in \mathbf{R}^2 \mid a \le b\} \\ &= R_2 \end{aligned}$$

c)
$$\begin{aligned} R_2 \cap R_4 &= \{(a,b) \in \mathbf{R}^2 \mid a \ge b \text{ and } a \le b\} \\ &= \{(a,b) \in \mathbf{R}^2 \mid a = b\} \\ &= R_5 \end{aligned}$$

d)
$$\begin{aligned} R_3 \cap R_5 &= \{(a,b) \in \mathbf{R}^2 \mid a < b \text{ and } a = b\} \\ &= \varnothing \end{aligned}$$

e)
$$\begin{aligned} R_1 - R_2 &= R_1 \cap \bar{R}_2 \\ &= \{(a,b) \in \mathbf{R}^2 \mid a > b \text{ and } a < b\} \\ &= \varnothing \end{aligned}$$

f)
$$\begin{aligned} R_2 - R_1 &= R_2 \cap \bar{R}_1 \\ &= \{(a,b) \in \mathbf{R}^2 \mid a \ge b \text{ and } a \le b\} \\ &= \{(a,b) \in \mathbf{R}^2 \mid a = b\} \\ &= R_5 \end{aligned}$$

g)
$$R_1 \oplus R_3 = \left(R_1 \cap \bar{R}_3\right) \cup \left(R_3 \cap \bar{R}_1\right)$$

$$= \left\{(a,b) \in \boldsymbol{R}^2 \mid a > b \ and \ a \geq b\right\} \cup \left\{(a,b) \in \boldsymbol{R}^2 \mid a < b \ and \ a \leq b\right\}$$

$= R_1 \cup R_3$ **(From part *a*)**

$= R_6$

h) $R_2 \oplus R_4 = \left(R_2 \cap \bar{R}_4\right) \cup \left(R_4 \cap \bar{R}_2\right)$

$$= \left\{(a,b) \in \boldsymbol{R}^2 \mid a \geq b \ and \ a > b\right\} \cup \left\{(a,b) \in \boldsymbol{R}^2 \mid a \leq b \ and \ a < b\right\}$$

$= R_1 \cup R_3$ **(From part *a*)**

$= R_6$

i) $R_1 \circ R_1 = \left\{(a,b) \in R_1 \ and \ (b,c) \in R_1\right\}$

$a > b \ and \ b > c \Rightarrow a > c$ (*clearly*) that means $(a,c) \in R_1$ (*Transitive*).

Therefore, $R_1 \circ R_1 = R_1$

j) $R_1 \circ R_2 = \left\{(a,b) \in R_2 \ and \ (b,c) \in R_1\right\}$

$a \geq b \ and \ b > c \Rightarrow a > c$ (*clearly*) that means $(a,c) \in R_1$. Therefore, $R_1 \circ R_2 = R_1$

k) $R_1 \circ R_3 = \left\{(a,b) \in R_3 \ and \ (b,c) \in R_1\right\}$

$a < b \ and \ b > c$. Therefore, $R_1 \circ R_3 = \boldsymbol{R}^2$

l) $R_1 \circ R_4 = \left\{(a,b) \in R_4 \ and \ (b,c) \in R_1\right\}$

$a \leq b \ and \ b > c$. Clearly this can always be done simply by choosing b to be large enough.

Therefore, $R_1 \circ R_4 = \boldsymbol{R}^2$

m) $R_1 \circ R_5 = \left\{(a,b) \in R_5 \ and \ (b,c) \in R_1\right\}$

$a = b \ and \ b > c$ *iff* $a > c$. Therefore, $R_1 \circ R_5 = R_1$

n) $R_1 \circ R_6 = \left\{(a,b) \in R_6 \ and \ (b,c) \in R_1\right\}$

$a \neq b \ and \ b > c$. Clearly this can always be done simply by choosing b to be large enough.

Therefore, $R_1 \circ R_6 = \boldsymbol{R}^2$

o) $R_2 \circ R_3 = \left\{(a,b) \in R_3 \ and \ (b,c) \in R_2\right\}$

$a < b \ and \ b \geq c$. Clearly this can always be done simply by choosing b to be large enough.

Therefore, $R_2 \circ R_3 = \boldsymbol{R}^2$

Exercise

Let R_1 *and* R_2 be the "*divides*" and "*is a multiple of*" relations on the set of all positive integers, respectively. That is $R_1 = \{(a,b) \mid a \text{ divides } b\}$ and $R_2 = \{(a,b) \mid a \text{ is a multiple of } b\}$

Find the following:

a) $R_1 \cup R_2$ ***b***) $R_1 \cap R_2$ ***c***) $R_1 - R_2$ ***d***) $R_2 - R_1$ ***e***) $R_1 \oplus R_2$

Solution

a) $(a,\ b) \in R_1 \cup R_2$ if and only if $a|b$ or $b|a$

b) $(a,\ b) \in R_1 \cup R_2$ if and only if $a|b$ and $b|a$ with $a = \pm b$ and $a \neq 0$

c) $R_1 - R_2 = R_1 \cap \bar{R}_2$ this relation holds between 2 integers if R_1 holds and R_2 does not hold.

$(a,\ b) \in R_1 \cap R_2$ if and only if $a|b$ and $b \nmid a$ $(a \neq \pm b)$

d) $R_2 - R_1 = R_2 \cap \bar{R}_1$ this relation holds between 2 integers if R_2 holds and R_1 does not hold.

$(a,\ b) \in R_1 \cap R_2$ if and only if $b|a$ and $a \nmid b$ $(a \neq \pm b)$

e) $R_1 \oplus R_2 = (R_1 - R_2) \cup (R_2 - R_1)$ this relation holds between 2 integers if R_2 holds and R_1 does not hold and R_2 holds and R_1 does not hold. if and only if $a|b$ or $b|a$ $(a \neq \pm b)$

SOLUTION   Section 4.2 – Representing Relations

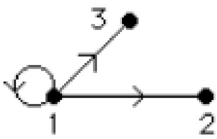
Exercise

Represent each of these relations on {1, 2, 3} with a matrix (with the elements of this set listed in increasing order). Then draw the directed graphs representing each relation

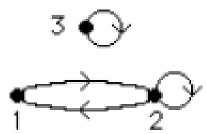
a) {(1, 1), (1, 2), (1, 3)}
b) {(1, 2), (2, 1), (2, 2), (3, 3)}
c) {(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)}
d) {(1, 3), (3, 1)}

Solution

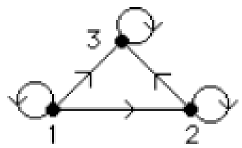
a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$



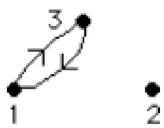
b) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$



c) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$



d) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$



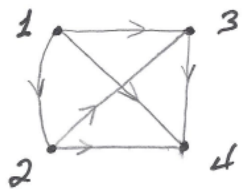
Exercise

Represent each of these relations on {1, 2, 3, 4} with a matrix (with the elements of this set listed in increasing order). Then draw the directed graphs representing each relation

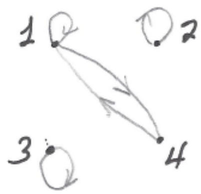
a) {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}
b) {(1, 1), (1, 4), (2, 2), (3, 3), (4, 1)}
c) { (1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)}
d) {(2, 4), (3, 1), (3, 2), (3, 4)}

Solution

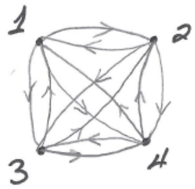
a) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$



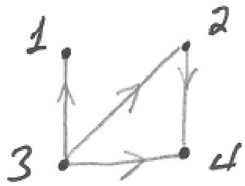
b) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$



c) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$



d) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$



Exercise

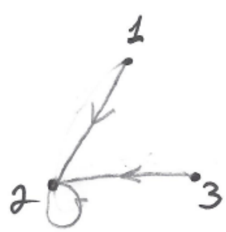
List the ordered pairs in the relations on {1, 2, 3} corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order). Then draw the directed graphs representing each relation

$a)\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad b)\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad c)\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Solution

a) {(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)}

b) {(1, 2), (2, 2), (3, 2)}



c) {(1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (3, 3)}

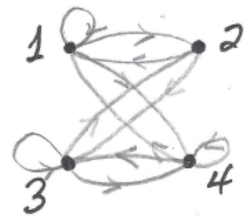
Exercise

List the ordered pairs in the relations on {1, 2, 3, 4} corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order). Then draw the directed graphs representing each relation

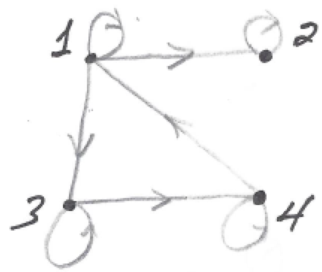
$$a)\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \qquad b)\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad c)\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Solution

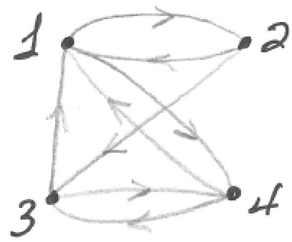
a) {(1, 1), (1, 2), (1, 4), (2, 1), (2, 3), (3, 2), (3, 3), (3, 4), (4, 1), (4, 3), (4, 4)}



b) {(1, 1), (1, 2), (1, 3), (2, 2), (3, 3), (3, 4), (4, 1), (4, 4)}



c) {(1, 2), (1, 4), (2, 1), (2, 3), (3, 1), (3, 4), (4, 1), (4, 3)}



Exercise

Let R be the relation represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Find: $a)\ R^2 \qquad b)\ R^3 \qquad c)\ R^4$

Solution

a) $M_{R^2} = M^2{}_R = M_R \odot M_R$

$$R^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

b) $M_{R^3} = M^3{}_R = M_R \odot M^2{}_R$

$$R^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

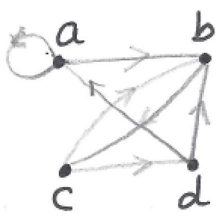
c) $M_{R^4} = M^{(4)}{}_R = M_R \odot M^{(3)}{}_R$

$$R^4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Exercise

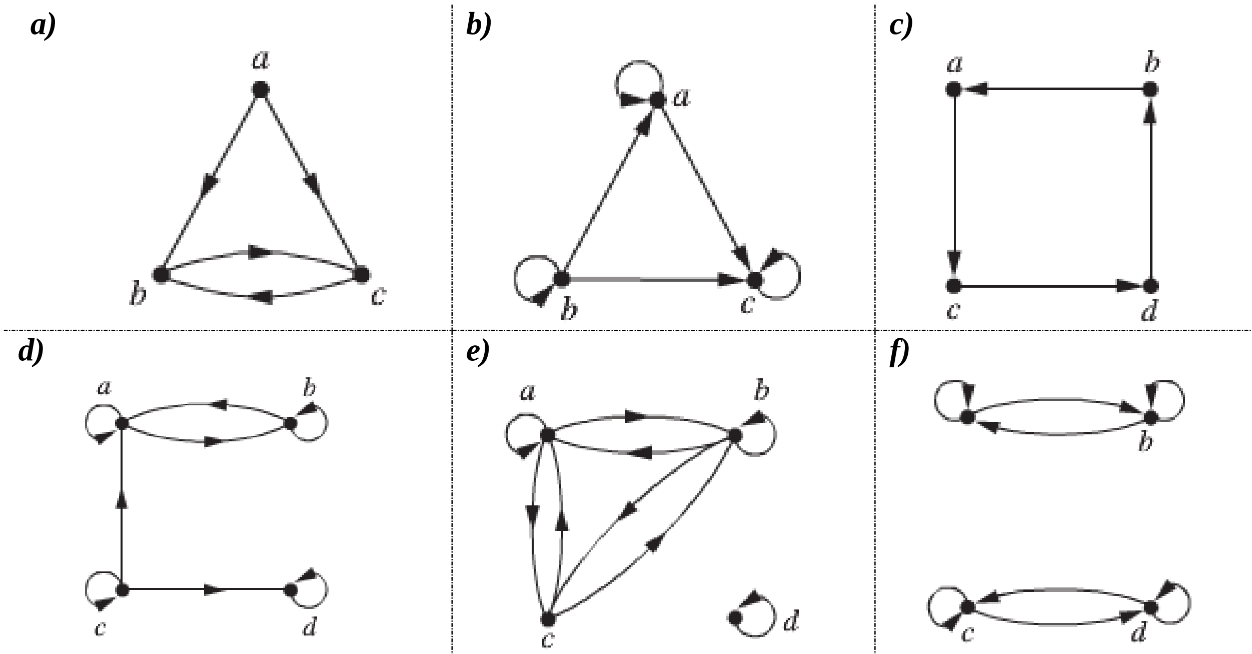
Draw the directed graph that represents the relation $\{(a, a), (a, b), (b, c), (c, b), (c, d), (d, a), (d, b)\}$

Solution



Exercise

Determine whether the relations represented by the directed graphs are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive



Solution

a) $\{(a, b), (a, c), (b, c), (c, b)\}$
It is not reflexive since (a, a) doesn't exist
It is not symmetric
It is transitive since $(a, b), (b, c) \Rightarrow (a, c)$

b) $\{(a, a), (a, c), (b, a), (b ,b), (b, c), (c, c)\}$
It is reflexive
It is not symmetric
It is transitive since $(b, a), (a, c) \Rightarrow (b, c)$

c) $\{ (a, c), (b, a), (c, d), (d, b)\}$
It is not reflexive; it is not symmetric, and not transitive since

d) $\{(a, a), (a, b), (b, a), (b, b), (c, a), (c, c), (c, d), (d, d)\}$
It is reflexive, not symmetric (no (a, c)), and not transitive (c, a), (a, b) but no (c, b)

e) $\{(a, a), (a, b) , (a, c), (b, a), (b, b), (b, c), (c, a) , (c, b), (d, d)\}$
It is not reflexive; it is symmetric and transitive

f) $\{(a, a), (a, b), (b, a), (b, b), (c, c), (c, d), (d, d), (d, c)\}$
It is reflexive; it is symmetric and transitive

SOLUTION *Section* 4.3 – Closures of Relations

Exercise

Let R be the relation on the set {0, 1, 2, 3} containing the ordered pairs (0, 1), (1, 1), (1, 2), (2, 0), (2, 2), and (3, 0). Find the

a) Reflexive closure of R.
b) Symmetric closure of R.

Solution

a) The reflexive closure of R is R with all (a, a). In this case the closure of R is
{(0, 0), (0, 1), (1, 1), (1, 2), (2,0), (2, 2), (3, 0), (3, 3)}

b) The symmetric closure of R is R with (b, a) for which (a, b) is in R. In this case the symmetric of R is
{(0, 1), (0, 2), (0, 3), (1, 0) (1, 1), (1, 2), (2,0), (2, 1) (2, 2), (3, 0) }

Exercise

Let R be the relation $\{(a,\ b) \mid a \neq b\}$ on the set of integers. What is the reflexive closure of R?

Solution

When we add all the pairs (x, x) to the given relation we have all of $\mathbb{Z}\times\mathbb{Z}$, which the relation will always holds.

Exercise

Let R be the relation $\{(a,\ b) \mid a \text{ divides } b\}$ on the set of integers. What is the symmetric closure of R?

Solution

To form the symmetric closure, we need to add all the pairs (b, a) such that (a, b) is in R.
We need to include pairs (b, a) such that a divides b, which is equivalent to saying that we need to include all the pairs (a, b) such that b divides a.
Thus the closure is $\{(a, b) \mid a$ divides b **or** b divides $a\}$

Exercise

How can the directed graph representing the reflexive closure of a relation on a finite set be constructed from the directed graph of the relation?

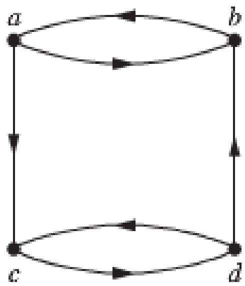
Solution

To form a reflexive closure, we simply need to add a loop at each vertex that does not already have one.

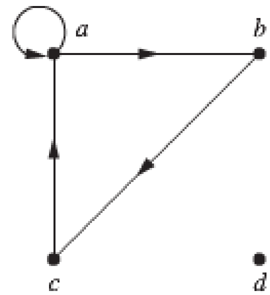
Exercise

Draw the directed graph of the *reflexive*, *symmetric*, and *both reflexive and symmetric* closure of the relations with the directed graph shown

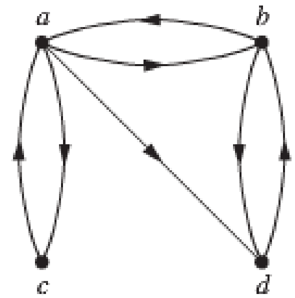
a)



b)

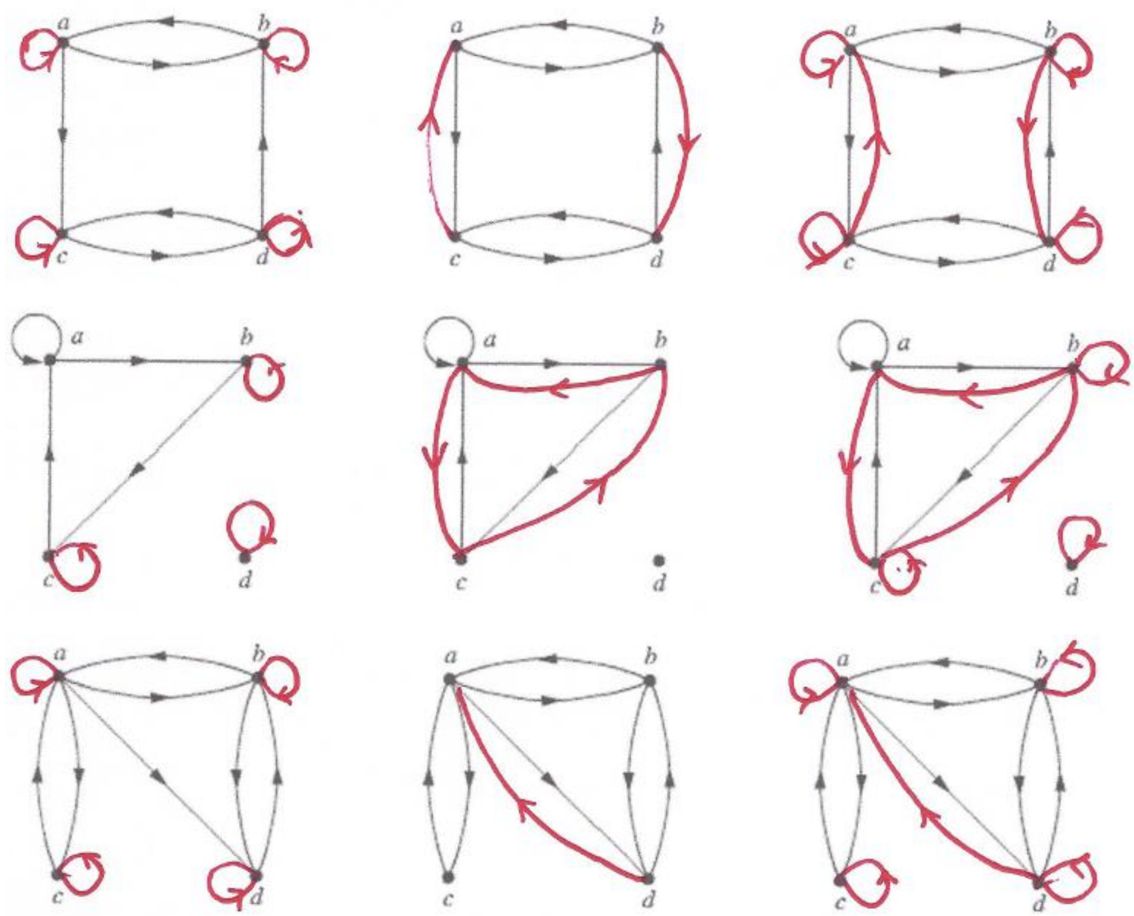


c)



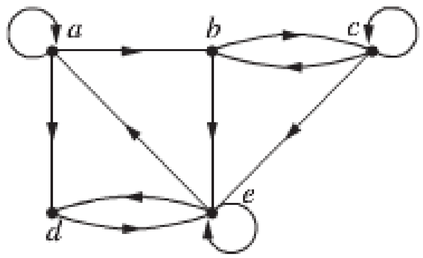
Solution

Reflexive	***Symmetric***	***Reflexive** and **Symmetric***



Exercise

1. Determine whether these sequences of vertices are paths in this directed graph



a) *a, b, c, e*
b) *b, e, c, b, e*
c) *a, a, b, e, d, e*
d) *b, c, e, d, a, a, b*
e) *b, c, c, b, e, d, e, d*
f) *a, a, b, b, c, ,c, b, e, d*

2. Find all circuits of length three in the directed graph

Solution

a) This is a path
b) This is not a path (no edge from *e* to *c*)
c) This is a path
d) This is not a path (no edge from *d* to *a*)
e) This is a path
f) This is not a path (no loop at *b*)

2. A circuit of length 3 can be written as a sequence of 4 vertices.
Start @ ***b***: *bccb* and *beab*
Start @ ***c***: *ccbc* and *cbcc*
Start @ ***d***: *deed*, *eede* and *edee*
eabe, dead, eade, abea, adea, aaaa, cccc, and eeee

Exercise

Let *R* be the relation on the set {1, 2, 3, 4, 5} containing the ordered pairs (1, 3), (2, 4), (3, 1), (3, 5), (4, 3), (5, 1), and (5, 2). Find

a) R^2 ***b***) R^3 ***c***) R^4 ***d***) R^5 ***e***) R^6 ***f***) R^*

Solution

$$M_R = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

a) $$M_{R^2} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

b) $M_{R^3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$

c) $M_{R^4} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$

d) $M_{R^5} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$

e) $M_{R^6} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$

f) $M_{R^*} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$

Exercise

Let R be the relation on the pair (a, b) if a and b are cities such that there is a direct non-stop airline flight from a to b. When is (a, b) in

a) R^2 ***b***) R^3 ***c***) R^*

Solution

a) The pair (a, b) is in R^2 precisely when there is a city c such that there is a direct flight from a to c and a direct flight from c to b – when it is possible to fly from a to b with a scheduled stop in some intermediate city.

b) The pair (a, b) is in R^3 precisely when there are cities c and d such that there is a direct flight from a to c, a direct flight from c to d, and a direct flight from d to b – when it is possible to fly from a to b with two scheduled stops in some intermediate cities.

c) The pair (a, b) is in $R*$ precisely when it is possible to fly from a to b.

Exercise

Let R be the relation on the set of all students containing the ordered pair (a, b) if a and b are in at least one common class and $a \neq b$. When is (a, b) in

a) R^2 ***b***) R^3 ***c***) R^*

Solution

a) The pair $(a,\ b) \in R^2$ if there is a person c other than a or b who is in a class with a and a class with b. $(a,\ a) \in R^2$ as long a is taking a class that has at least one other person in it, that person serves as the "c".

b) The pair $(a,\ b) \in R^3$ if there are persons c different from a and d different from b ***and*** c such that c is in a class with a, c is in class with d, and d is in class with b.

c) The pair $(a,\ b) \in R^*$ if there is a sequence of persons $c_0,\ c_1,\ c_2,\ \ldots,\ c_n$, with $n \geq 1$ such that $c_0 = a,\ c_n = b$, and for each i from 1 to n, $c_{i-1} \neq c_i$ and c_{i-1} is at least one class with c_i

Exercise

Suppose that the relation R is reflexive. Show that R^* is reflexive.

Solution

Since $R \subseteq R^*$, clearly if $\Delta \subseteq R$, then $\Delta \subseteq R^*$

Exercise

Suppose that the relation R is symmetric. Show that R^* is symmetric.

Solution

Suppose $(a,\ b) \in R^*$, then there is a path from a to b in R. Given such a path, if R is symmetric, then the reverse of every edge in the path is also in R; therefore there is a path from b to a in R.
This means that $(b,\ a) \in R^*$ whenever (a, b) is.

Exercise

Suppose that the relation R is irreflexive. Is the relation R^2 necessarily irreflexive.

Solution

It is certainly possibly for R^2 to contain some pairs (a, a).
For example: $R = \{(1, 2), (2, 1)\}$

SOLUTION ***Section* 4.4 – Equivalence Relations**

Exercise

Which of these relations on {0, 1, 2, 3} are equivalence relations?
Determine the properties of an equivalence relation that the others lack.
What are the equivalence classes of the equivalence relations?

a) {(0, 0), (1, 1), (2, 2), (3, 3)}
b) {(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}
c) {(0, 0), (1, 1), (1, 2), (2, 1), (3, 2), (3, 3)}
d) {(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}
e) {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}

Solution

a) This is an equivalence relation. It is reflexive, symmetric and transitive.
The equivalence classes all have just one element.
Each element is in an equivalence class by itself.

b) This is not reflexive since pair (1, 1) is missing. It is symmetric and it is not transitive since the pairs (0, 2) and (2, 3) are there, but not (0, 3).
This is not an equivalence relation.

c) This is an equivalence relation. It is reflexive, symmetric and transitive.
The elements 1 and 2 are in one equivalence class, and 0 and 3 are each in their own equivalence class.

d) This is reflexive and symmetric and it is not transitive since the pairs (1, 3) and (3, 2) are there, but not (1, 2).
This is not an equivalence relation.

e) This is reflexive, it is not symmetric since (2, 1) is missing and it is not transitive since the pairs (2, 0) and (0, 1) are there, but not (2, 1).
This is not an equivalence relation.

Exercise

Which of these relations on the set of all people are equivalence relations?
Determine the properties of an equivalence relation that the others lack.
What are the equivalence classes of the equivalence relations?

a) $\{(a,\ b) \mid a \text{ and } b \text{ are the same age}\}$
b) $\{(a,\ b) \mid a \text{ and } b \text{ have the same parents}\}$
c) $\{(a,\ b) \mid a \text{ and } b \text{ share a common parent}\}$
d) $\{(a,\ b) \mid a \text{ and } b \text{ have met}\}$
e) $\{(a,\ b) \mid a \text{ and } b \text{ speak a common language}\}$

Solution

a) This relation is reflexive, since *a* is the same person (same age).
If *a* is the same age as *b*, then *b* has to be the same age as *a*. this relation is symmetric.
If *a* is the same age as *b* and *b* is the same age as *c*, then *a* has to be the same age as *c*. this relation is transitive.
An equivalence class is the set of all people who are the same age. To really identify the equivalence class and the equivalence relation itself, one would need to specify exactly what ine meant by the "same age". For example, we could define two people to be the same age if their official dates of birth were identical.

b) For each pair (*m*, *w*) of a man and a woman, the set of offspring of their union, if nonempty, is an equivalence class. In many cases, then, an equivalence class consists of all the children in a nuclear family with children.

c) Let assume the relation is biological parentage. It is possible that *a* to be the the child of *W* and *X*, *b* is the child of *X* and *Y*, and *c* is the child of *Y* and *Z*. Then *a* is related to *b*, and *b* is related to *c*, but *a* is not related to *c*. This is not an equivalence relation, since it is not transitive. Therefore, this is not an equivalence relation.

d) If *a* met *b* and *b* met *c*, then it is not neccesary that *a* met *c*. This is not an equivalence relation, since it is not transitive. Therefore, this is not an equivalence relation.

e) If *a* speaks the same language (english) as *b* and *b* speaks the same language (spanish) as *c*, then it is not neccesary that *a* can speak spanish as *c*. This is not an equivalence relation, since it is not transitive.

Exercise

Which of these relations on the set of all functions from $\mathbb{Z}$ to $\mathbb{Z}$ are equivalence relations?
Determine the properties of an equivalence relation that the others lack.
What are the equivalence classes of the equivalence relations?

a) $\{(f, g) \mid f(1) = g(1)\}$

b) $\{(f, g) \mid f(0) = g(0) \text{ or } f(1) = g(1)\}$

c) $\{(f, g) \mid f(x) - g(x) = 1 \text{ for all } x \in \mathbb{Z}\}$

d) $\{(f, g) \mid \text{for some } C \in \mathbb{Z}, \text{ for all } x \in \mathbb{Z}, f(x) - g(x) = C\}$

e) $\{(f, g) \mid f(0) = g(1) \text{ or } f(1) = g(0)\}$

Solution

a) This is an equivalence relation, one of the general form that 2 things are considered equivalent if they have the same "something" (is 1).
$\{(f, f) \mid f(1) = f(1)\}$ This relation is reflexive.
If $\{(f, g) \mid f(1) = g(1)\}$ then $\{(g, f) \mid g(1) = f(1)\}$ $\therefore$ this relation is symmetric
If $\{(f, g) \mid f(1) = g(1)\}$ and $\{(g, h) \mid g(1) = h(1)\}$, then $\{(f, h) \mid f(1) = h(1)\}$ $\therefore$ this relation is transitive.

There is one equivalence class for each $n \in \mathbb{Z}$ and it contains all those functions whose value at 1 is *n*.

b) Let $f(x)=0, \quad g(x)=x, \quad and \quad h(x)=1$ for all $x \in \mathbb{Z}$. Then *f* is related to *g* since $f(0)=g(0)$ and *g* is related to *h* since $g(1)=h(1)$, but $f(0) \neq h(1)$, therefore *f* is related to *h* since that have no values in common. Hence, this is not an equivalence relation because it is not transitive.

c) It is not reflexive relation since $f(x)-f(x)=0 \neq 1$.

It is not symmetric since if $f(x)-g(x)=1$, then $g(x)-f(x)=-1 \neq 1$

It is not transitive since $f(x)-g(x)=1$ and $g(x)-h(x)=1 \quad \Rightarrow \quad f(x)-h(x)=2 \neq 1$,

This is not an equivalence relation.

d) This relation is reflexive, $f(x)-f(x)=0 \in \mathbb{Z}$

$f(x)-g(x)=C \quad \Rightarrow \quad g(x)-f(x)=-C \in \mathbb{Z}$, this relation is symmetric

$f(x)-g(x)=C_1 \quad g(x)-h(x)=C_2 \quad f(x)-h(x)=C_1+C_2 \in \mathbb{Z}$, this relation is transitive.

This is an equivalence relation.

The set of equivalence classes is uncountable. For each function $f:\mathbb{Z} \to \mathbb{Z}$, there is the equivalence class consisting of all functions *g* for which there is a constant *C* such that $g(n)=f(n)+C$ for all $n \in \mathbb{Z}$.

e) It is not reflexive relation since $f(0) \neq f(1)$ since it not given. This is not an equivalence relation.

Exercise

Define three equivalence relations on the set of students in your discrete mathematics class different from the relations discussed in the text. Determine the equivalence classes for each of these equivalence relations.

Solution

One relation is that *a* and *b* are related if they were born in the same state. Here the equivalence classes are the nonempty sets of students from each state.

Another example is for *a* to be related to *b* if *a* and *b* have lived the same number of complete decades. The equivalence classes are the set of all 10 to 19 years olds. The set of all 20 to 29 year olds, and so on.

A Third example is *a* to be related to *b* if 10 is a divisor of the difference between *a*'s age and *b*'s age, where "age" means the whole number of years since birth, as of the first day of class.

For each *i* = 0, 1, 2, …, 9, there is the equivalence class (if it is nonempty) of those students whose age ends with the digit *i*.

Exercise

Define three equivalence relations on the set of buildings on a college campus. Determine the equivalence classes for each of these equivalence relations.

Solution

Two buildings are equivalent, if they were opened during the same year; an equivalence class consists of the set of buildings opened in a given year.

For another example, we can define 2 buildings to be equivalent if they have the same number of stories; the equivalence classes are the set of 1-story buildings, the set of 2-story buildings, and so on.

The third example, partition the set of all buildings into 2 classes – those in which you do have a class this semester and those in which you don't. Every building in which you have a class is equivalent to every building in which you have a class (including itself), and every building in which you don't have a class is equivalent to every building in which you don't have a class.

Exercise

Let R be the relation on the set of all sets of real numbers such that $S\ R\ T$ if and only if S and T have the same cardinality. Show that R is an equivalence relation. What are the equivalence classes of the sets {0, 1, 2} and $\boldsymbol{Z}$?

Solution

Two sets have the same cardinality if there is a bijection (1−1 and onto function) from one set to the other.

We need to prove that R is reflexive, symmetric, and transitive.

Every set has the same cardinality as itself because of the identity function.

If f is a bijection from S to T, then f^{-1} is a bijection from T tot S, so R is symmetric.

If f is a bijection from S to T, and g is a bijection from T to U, then $g \circ f$ is a bijection from S to U, so R is transitive.

The equivalence class {1, 2, 3} is the set of all 3-element sets of real numbers, including such sets as {4, 25, 1948} and $\left(e,\ \pi,\ \sqrt{2}\right)$.

Similarly, $[\boldsymbol{Z}]$ is the set of all infinite countable sets of real numbers, such as the set of natural numbers, the set of rational numbers, and the set of the prime numbers, but not including the set {1, 2, 3} (it's too small) or the set of all real numbers (too big).

Exercise

Suppose that A is a nonempty set, and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x, y) such that $f(x) = f(y)$

a) Show that R is an equivalence relation on A.

b) What are the equivalence classes of R?

Solution

a) It is reflexive since $f(x) = f(x)$ for all $x \in A$

It is symmetric since $f(x) = f(y)$, then $f(y) = f(x)$

It is transitive since $f(x) = f(y)$ and $f(y) = f(z)$ then $f(x) = f(z)$

b) The equivalence class of x is the set of all $y \in A$ such that $f(y) = f(x)$ (definition of inverse).

Thus the equivalence classes are precisely the sets $f^{-1}(b)$ for every b in the range of f.

Exercise

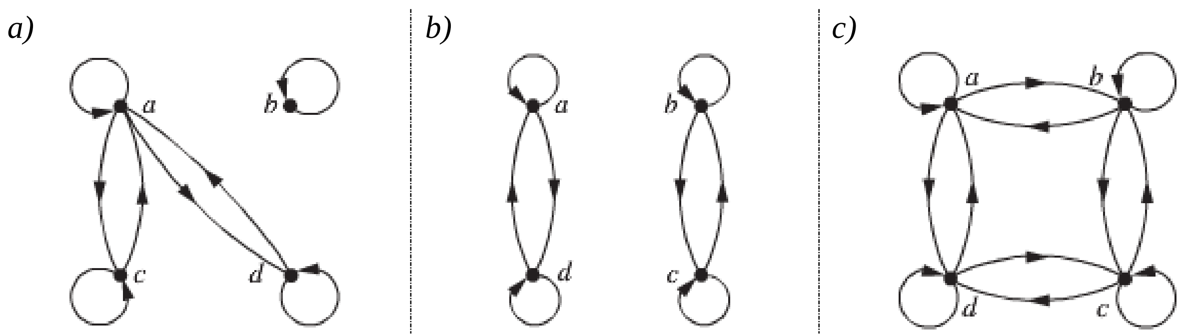
Suppose that A is a nonempty set, and R is an equivalence relation on A. Show that there is a function f with A as its domain such that $(x, y) \in R$ if and only if $f(x) = f(y)$

Solution

The function that sends each $x \in A$ to its equivalence class $[x]$ is obviously such a function.

Exercise

Determine whether the relation with the directed graph shown is an equivalence relation



Solution

a) The relation is reflexive since there is a loop at each vertex.
It is symmetric since every edge has 2 vertices and pointing in the both direction.
It is not transitive since we have {d, a} and {a, c} but not {d, a}

b) The relation is reflexive since there is a loop at each vertex.
It is symmetric since every edge has 2 vertices and pointing in the both direction.

It is transitive since paths of length 2 are accompanied by the path of length 1, edge between the same 2 vertices in the same direction.
This relation is an equivalence relation. The equivalence classes are $\{a, d\}$ and $\{b, c\}$

c) The relation is reflexive since there is a loop at each vertex.
It is symmetric since every edge has 2 vertices and pointing in the both direction.
It is not transitive (a, b) and (b, c) but not (a, c).

Exercise

Which of these collections of subsets are partitions of $\{1, 2, 3, 4, 5, 6\}$

a) $\{1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}$
b) $\{1\}, \{2, 3, 6\}, \{4\}, \{5\}$
c) $\{2, 4, 6\}, \{1, 3, 5\}$
d) $\{1, 4, 5\}, \{2, 6\}$

Solution

a) This is not a partition, since the sets are not pairwise disjoint. 2 and 4 appear in 2 of the sets.
b) This is a partition
c) This is a partition
d) This is not a partition, since element 3 is missing from the sets

Exercise

Which of these collections of subsets are partitions of $\{-3, -2, -1, 0, 1, 2, 3\}$

a) $\{-3, -1, 1, 3\}, \{-2, 0, 2\}$
b) $\{-3, -2, -1, 0\}, \{0, 1, 2, 3\}$
c) $\{-3, 3\}, \{-2, 2\}, \{-1, 1\}, \{0\}$
d) $\{-3, -2, 2, 3\}, \{-1, 1\}$

Solution

a) This a partition, since it satisfies the definition
b) This is not a partition, since the subsets are not disjoint
c) This a partition, since it satisfies the definition
d) This is not a partition, since the union of the subsets leaves out 0

SOLUTION Section 4.5 – Partial Orderings

Exercise

Which of these relations on {0, 1, 2, 3} are partial orderings? Determine the properties of a partial ordering that the others lack.

a) {(0, 0), (1, 1), (2, 2), (3, 3)}
b) {(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}
c) {(0, 0), (1, 1), (1, 2), (2, 2), (3, 3)}
d) {(0, 0), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)}
e) {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}
f) {(0, 0), (2, 2), (3, 3)}
g) {(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 3)}
h) {(0, 0), (1, 1), (1, 2), (2, 2), (3, 1), (3, 3)}
i) {(0, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (2, 3), (3, 0), (3, 3)}
j) {(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 3)}

Solution

a) This relation is reflexive because each of 0, 1, 2, 3 is related to itself.
This relation is antisymmetric because *a* to be related to *b* is for *a* to be equal to *b*
Since *a* is related to *b* and *b* related to *c* and $a = b = c$, then *a* is related to *c*. So the relation is transitive.
The equality relation on any set satisfies all three conditions, therefore is a partial ordering.

b) It is reflexive but it is not antisymmetric since we have $2R3$ and $3R2$ but $2 \neq 3$. Therefore this is not a partial ordering.

c) This relation is reflexive because each of 0, 1, 2, 3 is related to itself.
This relation is antisymmetric because *a* to be related to *b* is for *a* to be equal to *b*
It is transitive for the same reason and $1R1 \text{ and } 1R2 \quad \Rightarrow 1R2$
Therefore is a partial ordering.

d) This relation is reflexive because each of 0, 1, 2, 3 is related to itself.
This relation is antisymmetric because *a* to be related to *b* is for *a* to be equal to *b*
It is transitive for the same reason and $1R1 \text{ and } 1R2 \quad \Rightarrow 1R2$, $1R3 \text{ and } 3R3 \quad \Rightarrow 1R3$, and
$2R3 \text{ and } 3R3 \quad \Rightarrow 2R3$
Therefore is a partial ordering.

e) It is reflexive but it is not antisymmetric since we have $0R1$ and $1R0$ but $0 \neq 1$. Therefore this is not a partial ordering.

f) Since 1 is not related to itself, so this relation is not reflexive. Therefore *R* is not a partial ordering.

g) This relation is reflexive because each of 0, 1, 2, 3 is related to itself.
This relation is antisymmetric because *a* to be related to *b* is for *a* to be equal to *b*
It is transitive for the same reason and $2R0 \text{ and } 0R0 \quad \Rightarrow 2R0$, and $2R3 \text{ and } 3R3 \quad \Rightarrow 2R3$
Therefore is a partial ordering.

h) Since $3R1 \text{ and } 1R2 \quad \Rightarrow \cancel{3R2}$, so this relation is not transitive. Therefore *R* is not a partial ordering.

i) Since $1R2 \text{ and } 2R0 \quad \Rightarrow \cancel{1R0}$, so this relation is not transitive. Therefore *R* is not a partial ordering.

j) Since $0R1 \text{ and } 1R0$ but $0 \neq 1$, so this relation is not antisymmetric and it is not transitive because $2R0 \text{ and } 0R1 \quad \Rightarrow \cancel{2R1}$. Therefore *R* is not a partial ordering.

Exercise

Is (*S*, *R*) a poset If *S* is the set of all people in the world and $(a, b) \in R$, where *a* and *b* are people, if

a) a is a taller than *b*?

b) a is not taller than *b*?

c) $a = b$ or a is an ancestor of *b*?

d) a and b have a common friend?

e) a is a shorter than *b*?

f) a weighs more than *b*?

g) $a = b$ or a is a descendant of *b*?

h) a and b do not have a common friend?

Solution

a) Since nobody is taller than himself, this relation is not reflexive, so (*S*, *R*) is not a poset.

b) To be not a taller means exactly the same height or shorter. 2 different people *x* and *y* could have the same height, in which case $xRy \text{ and } yRx \text{ but } x \neq y$, so *R* is not antisymmetric.
Therefore, this relation is not a poset.

c) The equality clause in the given of *R* guarantees that *R* is reflexive.
If *a* is ancestor to *b*, then *b* can't be ancestor to *a*, so the relation is vacuously antisymmetric.
If *a* is ancestor to *b* and *b* is ancestor to *c*, then *a* is ancestor to *c*, thus *R* is transitive.
Therefore, this relation is a poset.

d) Let *x* and *y* be any 2 distinct friends, $xRy \text{ and } yRx \text{ but } x \neq y$, so *R* is not antisymmetric.
Therefore, this relation is not a poset.

e) Let 2 people can be the same height since are not the same person, so *R* is not antisymmetric.
Therefore, this relation is not a poset.

f) Since nobody is weight more than himself, this relation is not reflexive, so this relation is not a poset.

g) Since $a = a$, then the *R* is reflexive.
Given that $a = b$ but if *a* is a descendant of *b*, then *b* cannot be a descendant of *a*. So the relation is vacuously antisymmetric.
if *a* is a descendant of *b* and *b* is a descendant of *c*, then *a* is a descendant of *c*. So the *R* is transitive.
Therefore, this relation is a poset.

h) Since anyone and himself have a common friend, then this relation is not reflexive, so this relation is not a poset.

Exercise

Which of these are posets?

a) $(Z, =)$	*b)* $(Z,\ \neq)$	*c)* $(Z,\ \geq)$	*d)* $(Z,\ \nmid)$
e) $(R,\ =)$	*f)* $(R,\ <)$	*g)* $(R,\ \leq)$	*h)* $(R,\ \neq)$

Solution

a) The equality relation of any set satisfies all three conditions. Therefore a partial order.

b) This is not a poset since the relation is not reflexive $(a \neq a)$

c) The relation is reflexive since the relation involved the equality sign.

d) This is not a poset since the relation is not reflexive $(2 \nmid 2)$

e) The equality relation of any set satisfies all three conditions. Therefore a partial order.

f) This is not a poset since the relation is not reflexive $(2 \not< 2)$

g) The relation is reflexive since the relation involved the equality sign.

h) This is not a poset since the relation is not reflexive $(2 = 2)$

It is not antisymmetric since $1R2$ *and* $2R1$ *but* $1 \neq 2$

It is not transitive $1R2$ *and* $2R1$ *but* $1 = 1 \Rightarrow 1 \not R 1$

Exercise

Determine whether the relations represented by these zero-one matrices are partial orders

a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *b)* $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *c)* $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *d)* $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ *f)* $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

Solution

a) The relation is $\{(1,\ 1),\ (1,\ 2),\ (1,\ 3),\ (2,\ 1),\ (2,\ 2),\ (3,\ 3)\}$

This is not antisymmetric because $1R2$ *and* $2R1$ *but* $1 \neq 2$.

Therefore this matrix is not a partial order.

b) The relation is $\{(1,\ 1),\ (1,\ 2),\ (1,\ 3),\ (2,\ 2),\ (3,\ 3)\}$

It is clearly reflexive. The pairs (1, 2) and (1, 3) are in the relation that neither can be part of a counterexample to antisymmetry or transitivity.

c) The relation is $\{(1,\ 1),\ (1,\ 3),\ (2,\ 1),\ (2,\ 2),\ (3,\ 3)\}$

It is clearly reflexive. The pairs (1, 3) and (2, 1) are in the relation that neither can be part of a counterexample to antisymmetry.
It is not transitive since, (2, 1) and (1, 3) that will lead to (2, 3) which is not in the relation.
Therefore this matrix is not a partial order.

d) The relation is $\{(1,\ 1),\ (2,\ 2),\ (3,\ 1),\ (3,\ 3)\}$

It is clearly reflexive. The pair (3, 1) is in the relation that can't be part of a counterexample to antisymmetry or transitivity.

e) The relation is $\{(1,\ 1),\ (1,\ 2),\ (1,\ 3),\ (2,\ 2),\ (2,\ 3),\ (3,\ 3),\ (3,\ 4),\ (4,\ 1),\ (4,\ 2),\ (4,\ 4)\}$

It is not transitive since, (4, 1) and (1, 3) are in the relation but not (4, 3).
Therefore this matrix is not a partial order.

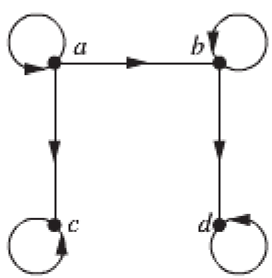
f) The relation is $\{(1,\ 1),\ (1,\ 3),\ (2,\ 2),\ (2,\ 3),\ (3,\ 3),\ (3,\ 4),\ (4,\ 1),\ (4,\ 2),\ (4,\ 4)\}$

It is not transitive since, (4, 1) and (1, 3) are in the relation but not (4, 3).
Therefore this matrix is not a partial order.

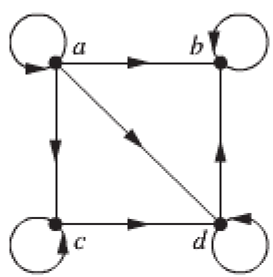
Exercise

Determine whether the relation with the directed graph shown is a partial order.

a)



b)



c)

Solution

a) This is relation is not transitive since there no relation (arrow) between *a* and *d*.

$aRb \text{ and } bRd \quad \Rightarrow \cancel{aRd}$

b) This is relation is not transitive since there no relation (arrow) from *c* and *b*.

c) This relation is reflexive since all points have an arrow to itself.
This relation is antisymmetric since no pair of arrows going in opposite directions between 2 different points.
Therefore this relation is a partial order.

Exercise

Let (S, R) be a poset. Show that $\left(S, R^{-1}\right)$ is also a poset, where R^{-1} is the inverse of *R*. The poset $\left(S, R^{-1}\right)$ is called the dual of (S, R).

Solution

Since *R* is reflexive, then R^{-1} is clearly reflexive.

Suppose that $(a,b) \in R^{-1}$ and $a \neq b$. Then $(b,a) \in R$, so $(a,b) \notin R$, so $(b,a) \notin R^{-1}$

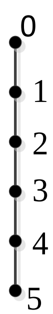
If $(a,b) \in R^{-1}$ and $(b,c) \in R^{-1}$, then $(b,a) \in R$ and $(c,b) \in R$, since *R* is transitive, so $(c,a) \in R$, therefore $(a,c) \in R^{-1}$, thus R^{-1} is transitive.

Therefore $\left(S, R^{-1}\right)$ is a poset

Exercise

Draw the Hasse diagram for the "greater than or equal to" relation on {0, 1, 2, 3, 4, 5}

Solution

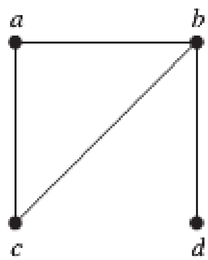


SOLUTION *Section* 4.6 – Graphs: Definitions and Basic Properties

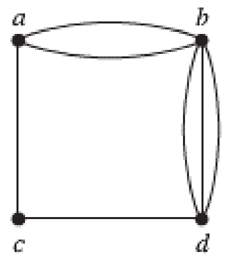
Exercise

Determine whether the graph shown has directed or undirected edges, whether it has multiple edges, and whether it has one or more loops.

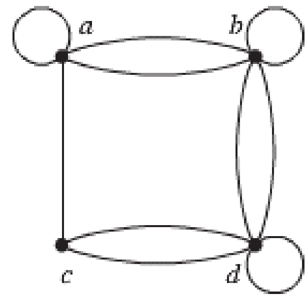
a)



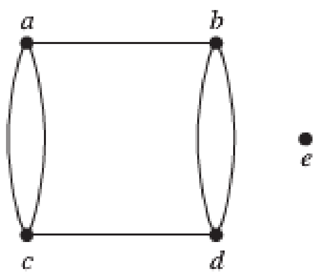
b)



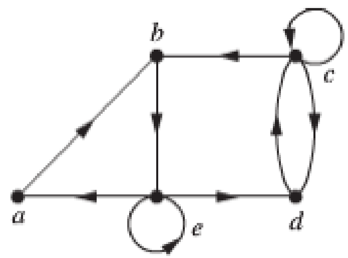
c)



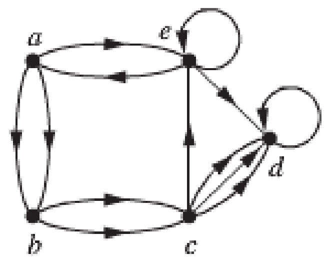
d)



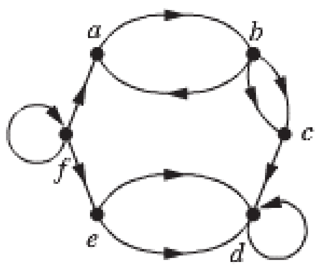
e)



f)



g)



Solution

a) This is a simple graph; the edges are undirected, and there are no parallel edges or loops.

b) This is a multigraph; the edges are undirected, and there are no loops, but there are parallel edges.

c) This is a pseudograph; the edges are undirected, and there are no parallel edges or loops.

d) This is a multigraph; the edges are undirected, and there are no loops, but there are parallel edges.

e) This is a directed graph; the edges are directed, and there are no parallel edges.

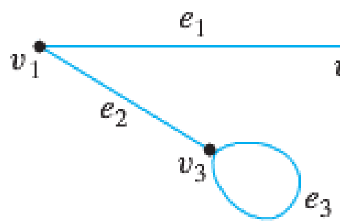
f) This is a directed multigraph; the edges are directed, and there are parallel edges.

g) This is a directed multigraph; the edges are directed, and there is a set of parallel edges.

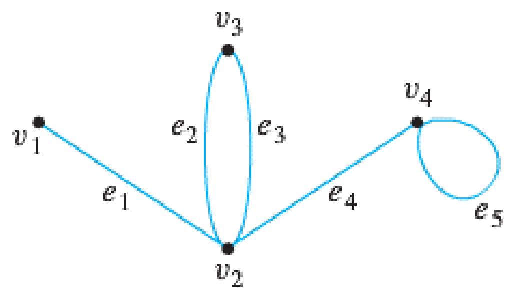
Exercise

Define each graph formally by specifying its vertex set, its edge set, and a table giving the edge-endpoint function

a)



b)



Solution

a) Vertex set $\{v_1, v_2, v_3, v_4\}$

Edge set $\{e_1, e_2, e_3\}$

Edge-endpoint function:

Edge	***Endpoints***
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_3\}$
e_3	$\{v_3\}$

b) Vertex set $\{v_1, v_2, v_3, v_4\}$

Edge set $\{e_1, e_2, e_3, e_4, e_5\}$

Edge-endpoint function:

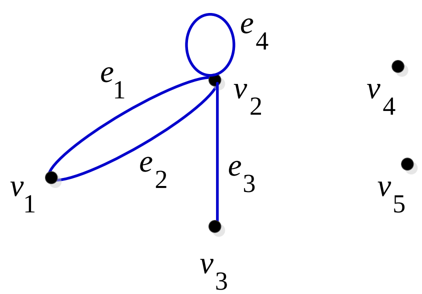
Edge	***Endpoints***
e_1	$\{v_1, v_2\}$
e_2	$\{v_2, v_3\}$
e_3	$\{v_2, v_3\}$
e_4	$\{v_2, v_4\}$
e_5	$\{v_4\}$

Exercise

Graph G has vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{e_1, e_2, e_3, e_4\}$, with edge-endpoint function as follow

Edge	*Endpoints*
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_2\}$
e_3	$\{v_2, v_3\}$
e_4	$\{v_2\}$

Solution

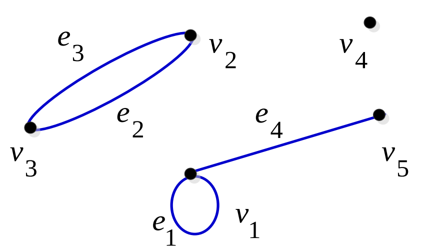


Exercise

Graph H has vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{e_1, e_2, e_3, e_4\}$, with edge-endpoint function as follow

Edge	*Endpoints*
e_1	$\{v_1\}$
e_2	$\{v_2, v_3\}$
e_3	$\{v_2, v_3\}$
e_4	$\{v_1, v_5\}$

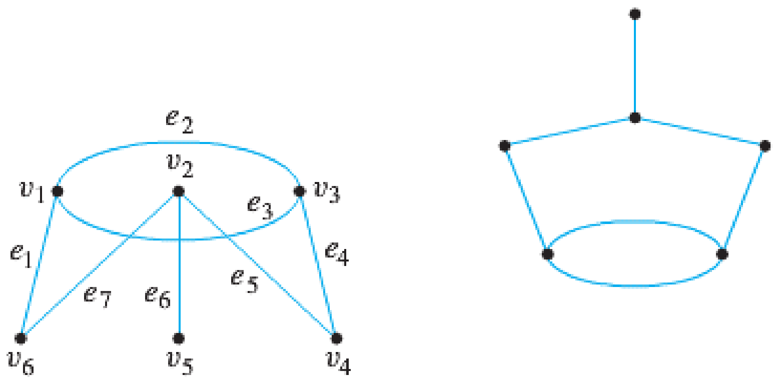
Solution



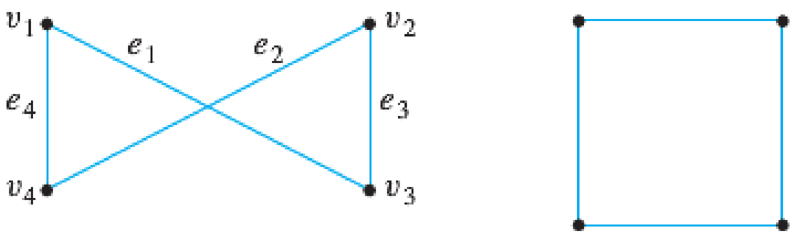
Exercise

Show that the 2 drawings represent the same graph by labeling the vertices and edges of the right-hand drawing to correspond to those of the left-hand drawing.

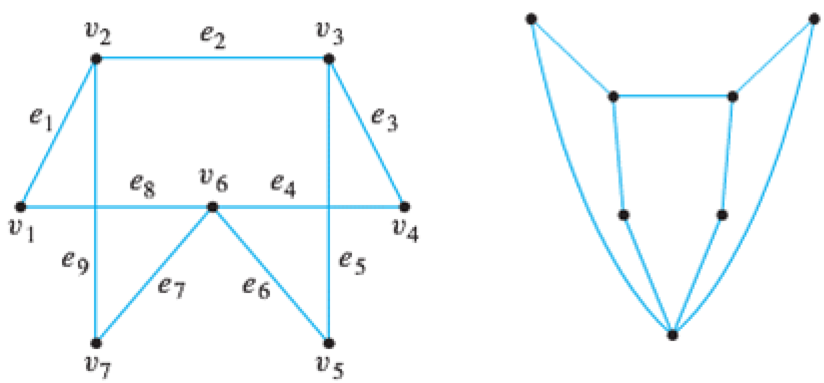
a)



b)

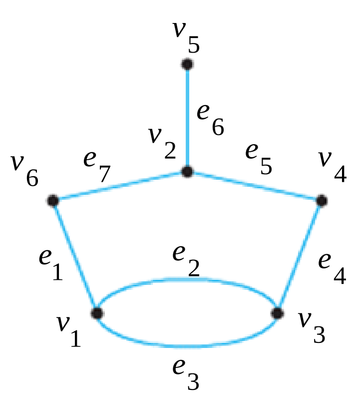


c)

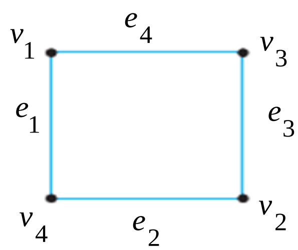


Solution

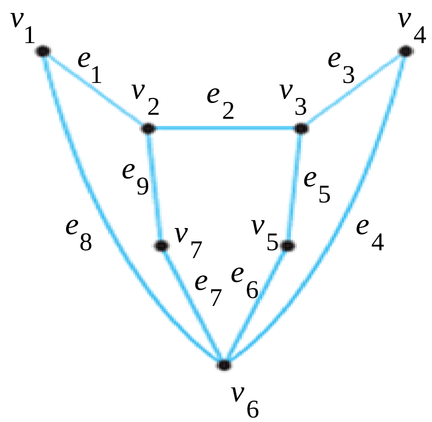
a) If you just hold the vertex v_5 turn it around to up position and stretch vertically little



b) Hold the edge e_4 and twisted as the vertices switch position.



c)



Exercise

For each of the graphs

i. Find all edges that are incident on v_1

ii. Find all vertices that are adjacent to v_3

iii. Find all edges that are adjacent to e_1

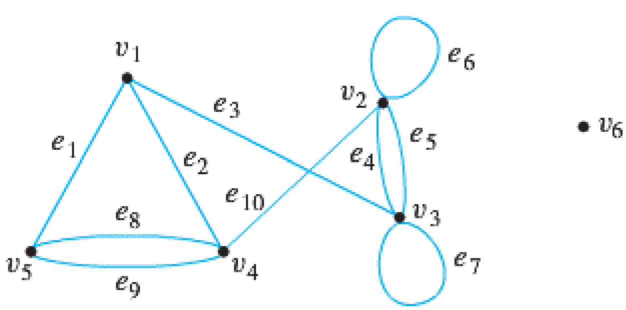
iv. Find all loops

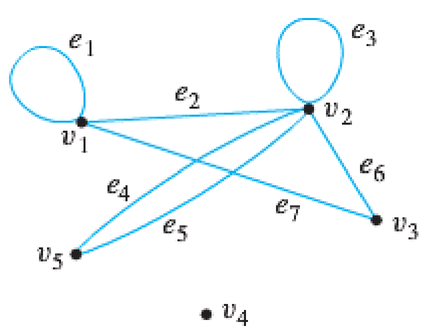
v. Find all parallel edges

vi. Find all isolated vertices

vii. Find the degree of v_3

viii. Find the total degree of the graph





<u>Solution</u>

a) e_1, e_2, and e_3 are incident on v_1

v_1, v_2 and v_3 are adjacent to v_3

e_2, e_3, e_8, and e_9 are adjacent to e_1

e_6 and e_7 are loops.

e_4 and e_5 are parallel; e_8 and e_9 are parallel

v_6 is an isolated vertex.

Degree of $v_3 = 5$

Total degree = 20

b) e_1, e_2, and e_7 are incident on v_1

v_1, v_2 and v_3 are adjacent to v_3

e_2 and e_7 are adjacent to e_1

e_1 and e_3 are loops.

e_4 and e_5 are parallel

Isolated vertex: none.

Degree of $v_3 = 2$

Total degree = 14

Exercise

Let G be a simple graph. Show that the relation R on the set of vertices of G such that uRv if and only if there is an edge associated to $\{u, v\}$ is a symmetric, irreflexive relation on G.

Solution

In a simple graph, edges are undirected.

If uRv, then there is edge associated with $\{u, v\}$. But $\{u, v\} = \{v, u\}$, so this edge is associated with $\{v, u\}$ and therefore. So, R is symmetric.

A simple graph does not allow loops; that is if there is an edge associated with $\{u, v\}$, then $u \neq v$.

Thus uRu never holds, and so by definition R is irreflexive.

Exercise

Let G be an undirected graph with a loop at every vertex. Show that the relation R on the set of vertices of G such that uRv if and only if there is an edge associated to $\{u, v\}$ is a symmetric, reflexive relation on G.

Solution

If uRv, then there is edge associated with $\{u, v\}$, and since the graph is undirected, this is also edge joining vertices $\{v, u\}$ and therefore. So, R is symmetric.

The relation is reflexive because the loops guarantees that uRu for each vertex u.

Exercise

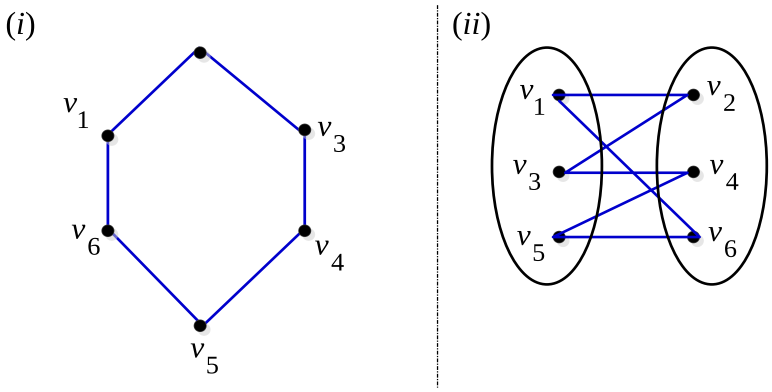
Explain how graphs can be used to model electronic mail messages in a network. Should the edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed? Describe a graph that models the electronic mail sent in a network in a particular week.

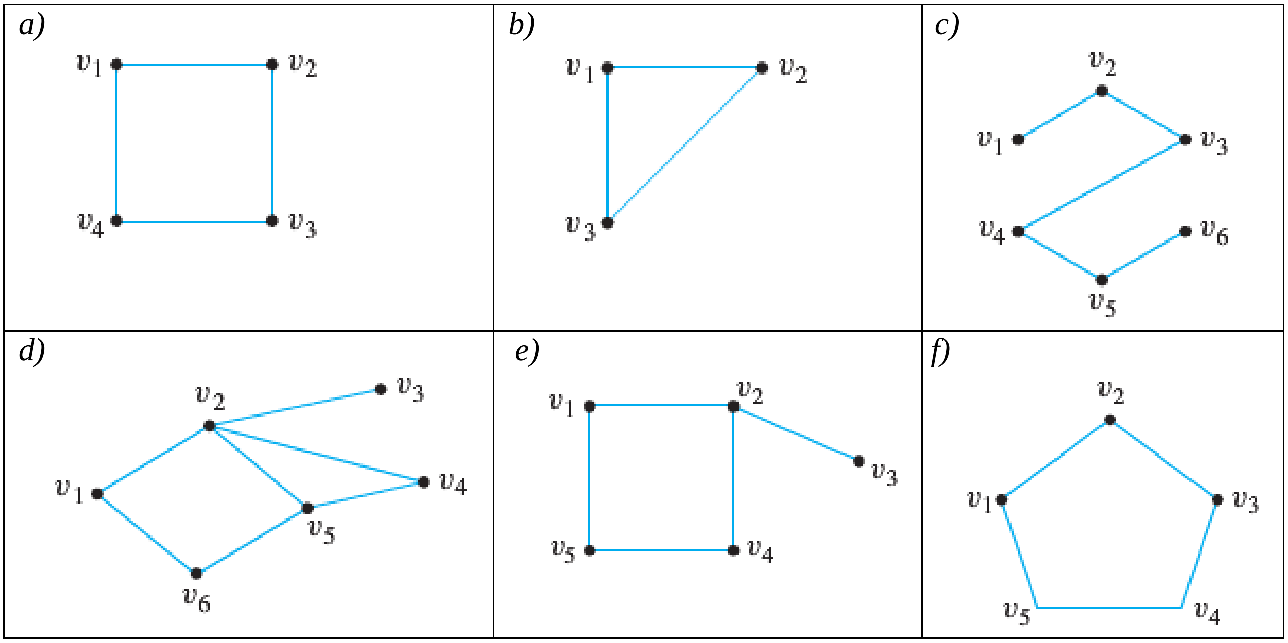
Solution

We can have a vertex for each mailbox or e-mail address in the network, with a directed edge between two vertices if a message is sent from the tail of the edge to the head.
We use directed edge for each message sent during the week.

Exercise

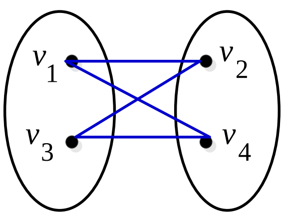
A bipartite graph G is a simple graph whose vertex set can be portioned into two disjoint nonempty subsets V_1 and V_2 such that vertices in V_1 may be connected to vertices in V_2, but no vertices in V_1 are connected to other vertices in V_1 and no vertices in V_2 are connected to other vertices in V_2. For example, the graph G illustrated in (i) can be redrawn as shown in (ii). From the drawing in (ii), you can see that G is bipartite with mutually disjoint vertex set $V_1 = \{v_1, v_3, v_5\}$ and $V_1 = \{v_2, v_4, v_6\}$





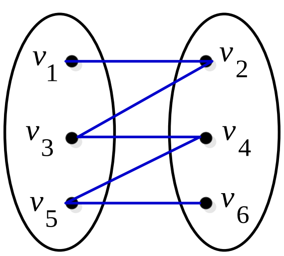
Solution

a)



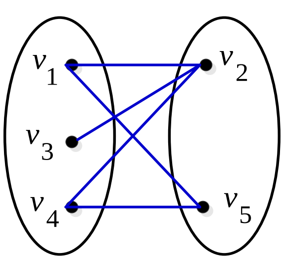
b) $\{v_1, v_2, v_3\}$ form a triangle, we can’t create a bipartite graph *G*.

c)



d) $\{v_2, v_4, v_5\}$ form a triangle, therefore we can’t create a bipartite graph *G*.

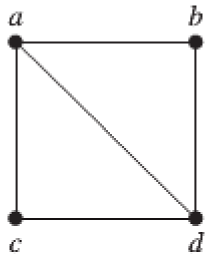
e)



SOLUTION *Section* 4.7 – Representing Graphs and Graph Isomorphism

Exercise

Use the adjacency list to represent the given graph, then represent with an adjacency matrix



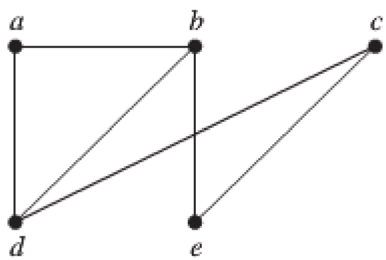
Solution

Vertex	*Adjacent Vertices*
a	*b, c*
b	*a, d*
c	*a, d*
d	*a, b, c*

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Exercise

Use the adjacency list to represent the given graph, then represent with an adjacency matrix



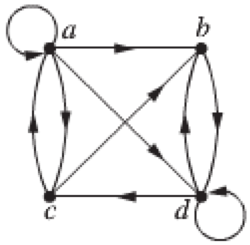
Solution

Vertex	*Adjacent Vertices*
a	*b, d*
b	*a, d, e*
c	*d, e*
d	*a, b, c*
e	*b, c*

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Exercise

Use the adjacency list to represent the given graph, then represent with an adjacency matrix



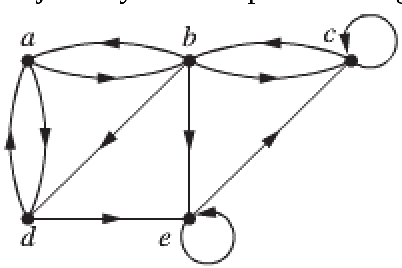
Solution

Initial Vertex	*Terminal Vertices*
a	*a, b , c, d*
b	*d*
c	*a, b*
d	*b, c, d*

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Exercise

Use the adjacency list to represent the given graph, then represent with an adjacency matrix



Solution

Initial Vertex	*Terminal Vertices*
a	*b, d*
b	*a, c, d, e*
c	*b, c*
d	*a, e*
e	*c , e*

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

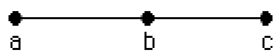
Exercise

Draw a graph with the given adjacency

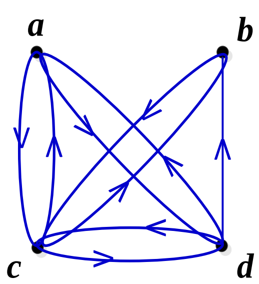
a) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ *b*) $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ *c*) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

Solution

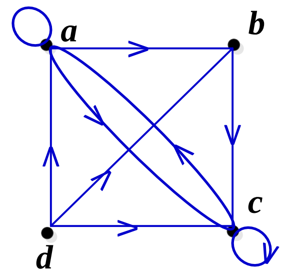
a)



b)



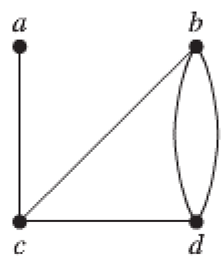
c)



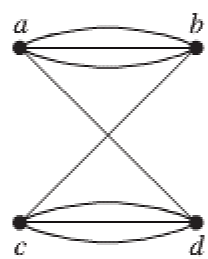
Exercise

Represent the given graph using adjacency matrix

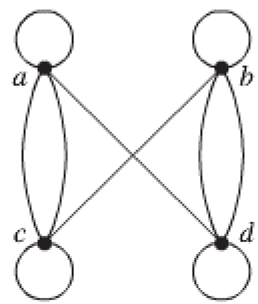
a)



b)



c)



Solution

a) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$ ***b***) $\begin{bmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 0 \end{bmatrix}$ ***c***) $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

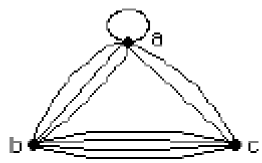
Exercise

Draw an undirected graph represented by the given adjacency

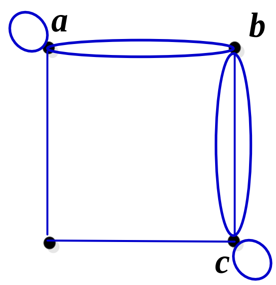
$$a)\ \begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix} \qquad b)\ \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \qquad c)\ \begin{bmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$$

Solution

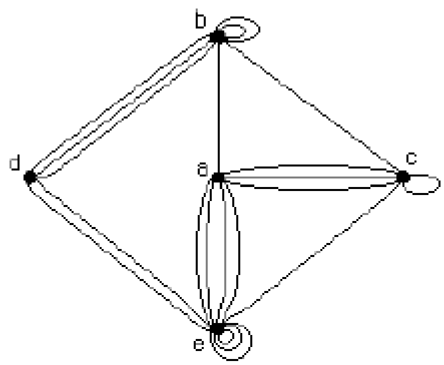
a)



b)



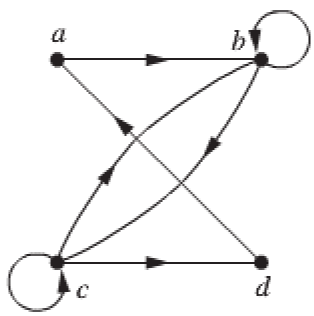
c)



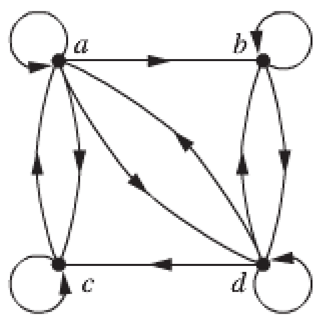
Exercise

Find the adjacency matrix of the given directed multigraph with respect to the vertices listed in alphabetic order.

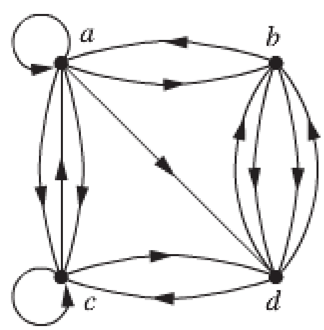
a)



b)



c)

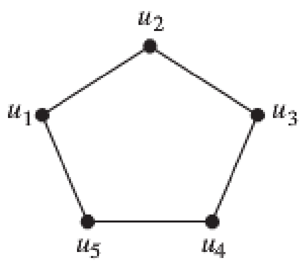


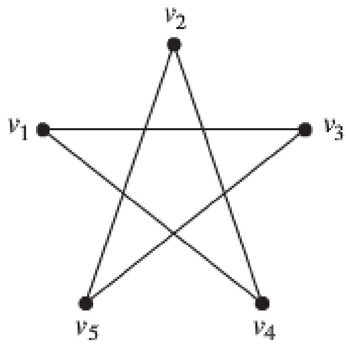
Solution

$$\boldsymbol{a})\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{b})\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad \boldsymbol{c})\ \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

Exercise

Determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.



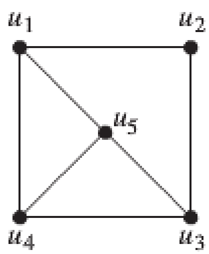


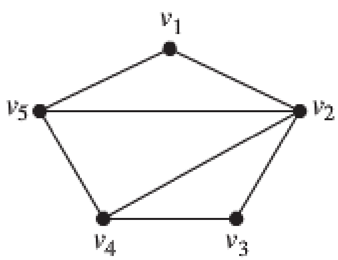
Solution

Both graphs have 5 vertices and 5 edges. However, each vertex in the second graph has of degree 2, whereas the first does not.

Exercise

Determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.



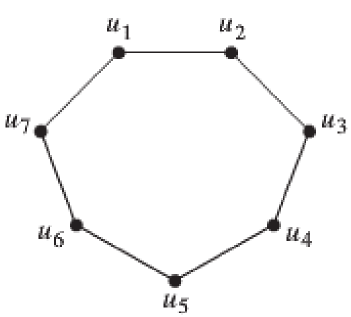


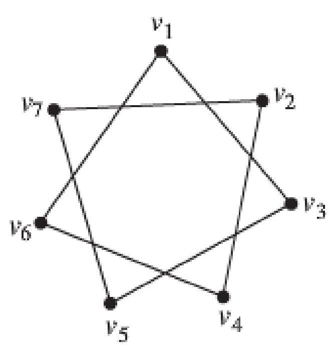
Solution

Both graphs have 5 vertices and 7 edges. However, the second graph has a vertex of degree 4, whereas the first does not.

Exercise

Determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.





Solution

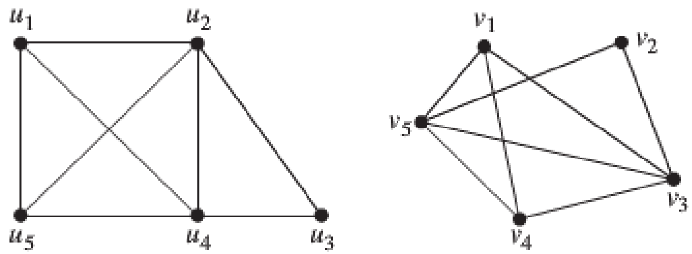
Both graphs have 7 vertices and 7 edges.

$f(u_1)=v_1$, $f(u_2)=v_3$, $f(u_3)=v_5$, $f(u_4)=v_7$, $f(u_5)=v_2$, $f(u_6)=v_4$, and $f(u_7)=v_6$

$\therefore$ The graphs are isomorphic.

Exercise

Determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.



Solution

Both graphs have 5 vertices and 8 edges.

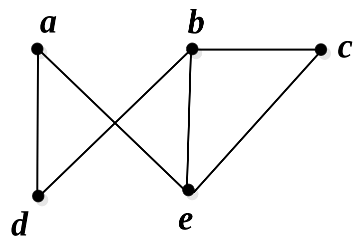
$f(u_1)=v_1$, $f(u_2)=v_3$, $f(u_3)=v_2$, $f(u_4)=v_5$, and $f(u_5)=v_4$

$\therefore$ The graphs are isomorphic.

SOLUTION Section 4.8 – Connectivity

Exercise

Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? Which are the lengths of those that are paths?



a) *a, e, b, c, b* ***b)*** *a, e, a, d, b, c, a* ***c)*** *e, b, a, d, b, e* ***d)*** *c, b , d, a, e, c*

Solution

a) This is a path of length 4, but it is not a circuit, since it ends at a vertex other than the one at which it began. It is not a simple, since it uses an edge more than once.

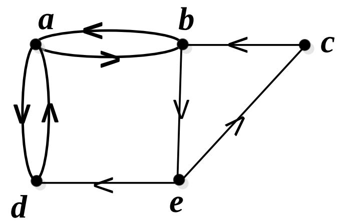
b) This is not a path, since there is no edge from *c* to *a*.

c) This is not a path, since there is no edge from *b* to *a*.

d) This is a path of length 5, which is a circuit. It is simple, since no edges are repeated.

Exercise

Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? Which are the lengths of those that are paths?



a) *a, b, e, c, b* ***b)*** *a, d, a, d, a* ***c)*** *a, d, b, e, a* ***d)*** *a, b , e, c, b, d, a*

Solution

e) This is a path of length 4, but it is not a circuit, since it ends at a vertex other than the one at which it began. It is simple, since no edges are repeated.

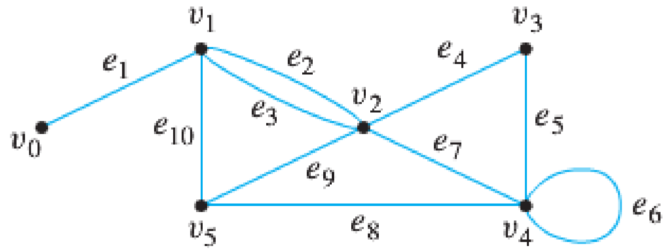
f) This is a path of length 4, which is a circuit. It is not simple, since it uses an edge more than once.

g) This is not a path, since there is no edge from *d* to *b*.

h) This is not a path, since there is no edge from *b* to *d*.

Exercise

Determine whether of the following walks are trails, paths, circuits, or simple circuits or just walk to the graph below.



a) $v_0e_1v_1e_{10}v_5e_9v_2e_2v_1$ *b)* $v_4e_7v_2e_9v_5e_{10}v_1e_3v_2e_9v_5$ *c)* v_2

d) $v_5v_2v_3v_4v_4v_5$ *e)* $v_2v_3v_4v_5v_2v_4v_3v_2$ *f)* $e_5e_8e_{10}e_3$

Solution

a) It is trail since no repeated edge. It is not a path, repeated vertex v_1, it is not a circuit, since it ends at a vertex other than the one at which it began v_0

b) It is a walk, it is not a trail since it has a repeated edge e_9. It is not a circuit, since it ends at a vertex other than the one at which it began v_4.

c) It is a closed walk, starts and ends at the same vertex v_2. It is a trail since no repeated edge. It is not a path or a circuit, since no edge.

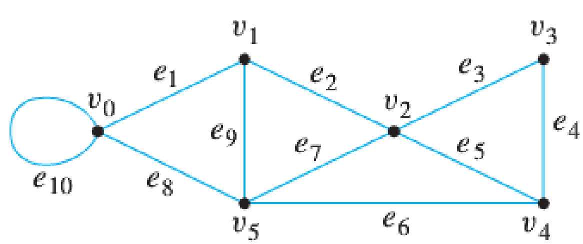
d) It is a path and it is circuit but not a simple circuit since it has a repeated vertex v_4

e) It is a closed walk, starts and ends at the same vertex v_2. It is not a trail since it has repeated edges $\{v_2, v_3\}$ & $\{v_3, v_4\}$.

f) It is a path, it is not a circuit, since it ends at a vertex other than the one at which it began.

Exercise

Determine whether of the following walks are trails, paths, circuits, or simple circuits or just walk to the graph below.



a) $v_1e_2v_2e_3v_3e_4v_4e_5v_2e_2v_1e_1v_0$ *b)* $v_2v_3v_4v_5v_2$ *c)* $v_4v_2v_3v_4v_5v_2v_4$

d) $v_2v_1v_5v_2v_3v_4v_2$ *e)* $v_0v_5v_2v_3v_4v_2v_1$ *f)* $v_5v_4v_2v_1$

Solution

a) It is not a trail since it has a repeated edge e_2. It is not a path, repeated vertex v_1, it is not a circuit, since it ends at a vertex other than the one at which it began v_1

b) It is a closed walk, starts and ends at the same vertex v_2. It is a trail since no repeated edge. It is a circuit.

c) It is not a trail since it has repeated edges $\{v_2, v_4\}$. It is a circuit, but not a simple circuit.

d) It is a path and it is circuit but not a simple circuit since it has a repeated vertex v_2

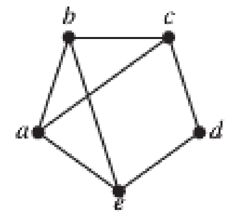
e) It is a trail since no repeated edge. It is not a circuit, since it ends at a vertex other than the one at which it began.

f) It is a path, it is not a circuit, since it ends at a vertex other than the one at which it began.

SOLUTION Section 4.9 – Euler and Hamilton Paths

Exercise

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

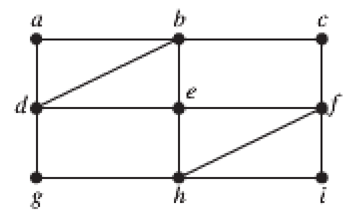


Solution

The vertices *a, b, c, e* have degree 3, therefore the graph has no Euler circuit.
It is not Euler path since there is more than 2 vertices with an odd degree.

Exercise

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

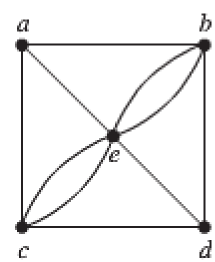


Solution

All the vertex degree are even, so there is an Euler circuit.
Circuit form: *a, b, c, f, i, h, g, d, e, h, f, e, b, d, a*

Exercise

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

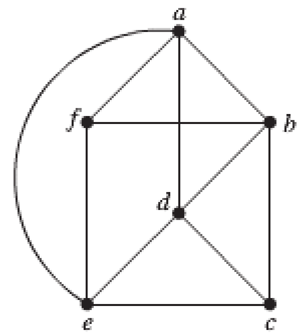


Solution

The vertices *a, d* have degree 3, therefore the graph has no Euler circuit.
It has an Euler path *a, e, c, e, b, e, d, b, a, c, d*. (it has exactly 2 vertices of odd degree)

Exercise

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

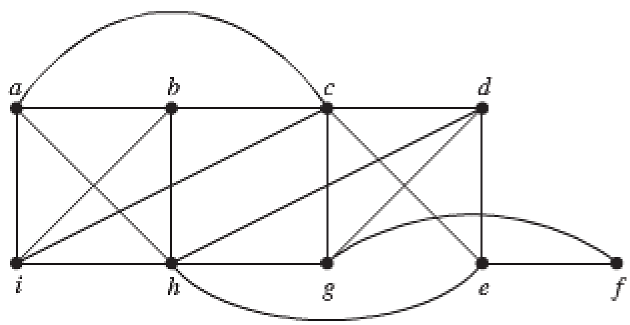


Solution

The vertices *c*, *f* have degree 3, therefore the graph has no Euler circuit.
There is an Euler path between the two vertices of odd degree.
One such path is: *f, a, b, c, d, e, f, b, d, a, e, c.*

Exercise

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

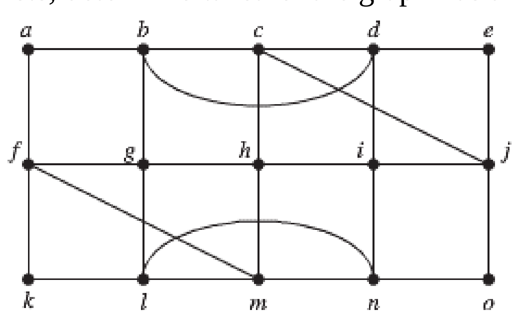


Solution

All the vertex degree are even, so there is an Euler circuit.
Form: *a, i, h, g, d, e, f, g, c, e, h, d, c, a, b, I, c, b, h, a*

Exercise

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

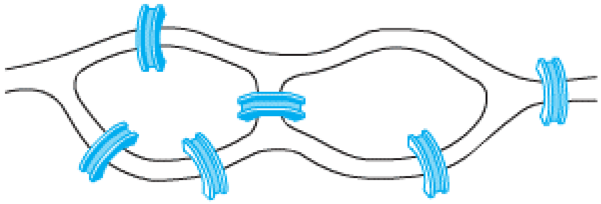


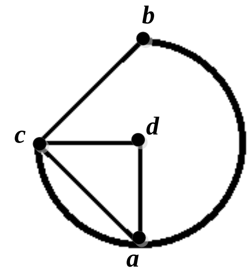
Solution

All the vertex degree are even, so there is an Euler circuit.
Circuit: *a, b, c, d, e, j, c, h, i, d, b, g, h, m, n, o, j, i, n, l, m, f, g, l, k, f, a*

Exercise

Can someone cross all the bridges shown in this map exactly once and return to the starting point?





Solution

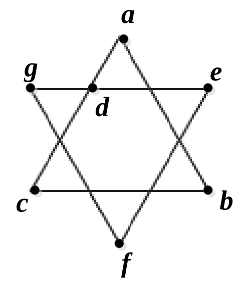
Vertices *a* and *b* are the banks of the river, and vertices *c* and *d* are the islands.

Each vertex has even degree, so the graph has an Euler circuit, such as: *a, c, b, a, d, c, a*.

Therefore a walk of the type described is possible.

Exercise

Determine whether the picture shown can be drawn with a pencil in a continuous motion without lifting the pencil or retracing part of the picture

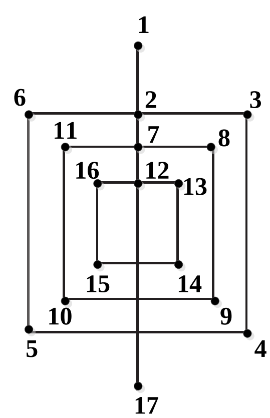


Solution

Yes, the path: *a, b, c, d, e, f, g, d, a*.

Exercise

Determine whether the picture shown can be drawn with a pencil in a continuous motion without lifting the pencil or retracing part of the picture

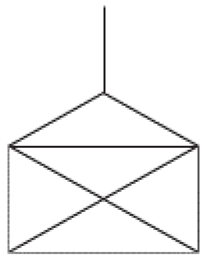


Solution

1, 2, 3, 4, 5, 6, 2, 7, 8, 9, 10, 11, 7, 12, 13, 14, 15, 16, 12, 17

Exercise

Determine whether the picture shown can be drawn with a pencil in a continuous motion without lifting the pencil or retracing part of the picture

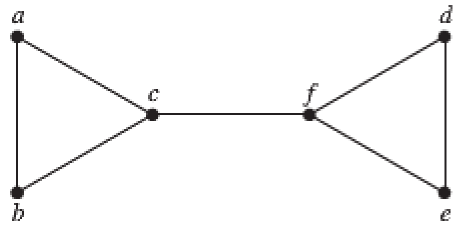


Solution

No

Exercise

Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.

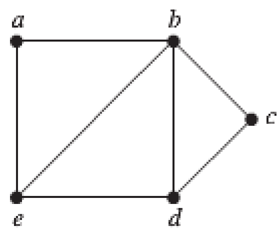


Solution

The graph is not a Hamilton circuit because of the cut edge $\{c, f\}$.
Every simple circuit must be confined to one of the 2 components obtained by deleting this edge.

Exercise

Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.

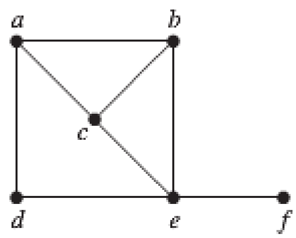


Solution

Hamilton circuit: *a, b, c, d, e, a*.

Exercise

Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.

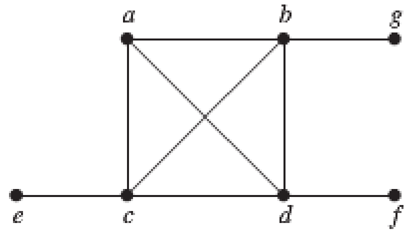


Solution

The graph is not a Hamilton circuit because of the cut edge $\{e, f\}$.

Exercise

Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.

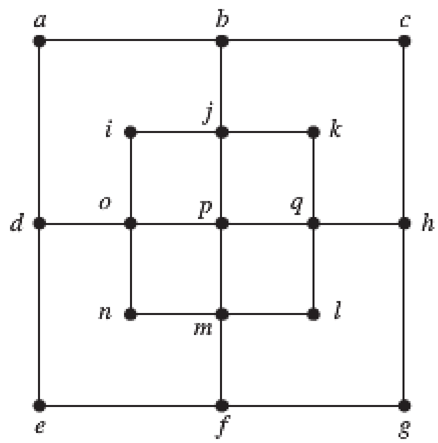


Solution

No Hamilton circuit exists, because once a purported circuit has reached *e* it would be nowhere to go.

Exercise

Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.

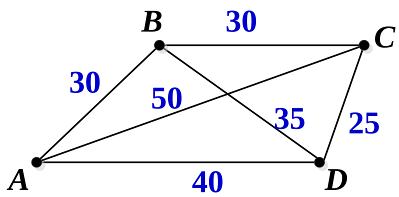


Solution

This graph has no Hamilton circuit.
If it did, then certainly the circuit would have to contain edges $\{d, a\}$ and $\{a, b\}$, since these are the only edges incident to vertex *a*. By the same reasoning, the circuit would have to contain the other six edges around the outside of the figure. These 8 edges already complete a circuit, and this circuit omits the 9 vertices on the inside.
Therefore, there is no Hamilton circuit.

Exercise

Imagine that the drawing below is a map showing 4 cities and the distances in kilometers between them. Suppose that a salesman must travel to each city exactly once, starting and ending in city *A*. Which route from city to city will minimize the total distance that must be traveled?



Solution

Route	*Total Distance (Km)*
ABCDA	30 + 30 + 25 + 40 = 125
ABDCA	30 + 35 + 25 + 50 = 140
ACBDA	50 + 30 + 35 + 40 = 155
ACDBA	50 + 25 + 35 + 30 = 140
ADBCA	40 + 35 + 30 + 50 = 155
ADCBA	40 + 25 + 30 + 30 = 125

Thus either route *ABCDA* or *ADCBA* fives the minimum total distance of 125 km.