

## Section 3.4 – Properties of Logarithms

### Product Rule

$$\log_b MN = \log_b M + \log_b N \quad \underline{\text{For } M > 0 \text{ and } N > 0}$$

$$\begin{cases} \log_b M = x \Rightarrow M = b^x \\ \log_b N = y \Rightarrow N = b^y \end{cases} \Rightarrow MN = b^x b^y = b^{x+y}$$

Convert back to logarithmic form:  $\log_b MN = x + y$

$$\log_b MN = \log_b M + \log_b N$$

### Example

Use the product rule to expand the logarithmic expression

$$\log(100x) = \log 100 + \log x$$

### Power Rule

$$\log_b M^p = p \log_b M$$

### Example

Use the power rule to expand each logarithmic expression

$$\ln \sqrt[3]{x} = \ln(x)^{1/3} = \frac{1}{3} \ln x$$

### Quotient Rule

$$\log_b \frac{M}{N} = \log_b M - \log_b N$$

### Example

Use the quotient rule to expand the logarithmic expression

$$\ln \left( \frac{e^5}{11} \right) = \ln e^5 - \ln 11 \\ = 5 - \ln 11$$

### Example

Express each of the following in terms of sums and differences of logarithm:  $\log_6 (7 \times 9)$

#### Solution

$$\log_6 (7 \times 9) = \log_6 7 + \log_6 9 \quad \text{Product Rule}$$

### Example

Express each of the following in terms of sums and differences of logarithm:  $\log_9 \left( \frac{15}{7} \right)$

#### Solution

$$\log_9 \left( \frac{15}{7} \right) = \log_9 15 - \log_9 7 \quad \text{Quotient Rule}$$

### Example

Express each of the following in terms of sums and differences of logarithm:  $\log_5 \sqrt{8}$

#### Solution

$$\begin{aligned} \log_5 \sqrt{8} &= \log_5 \left( 2^3 \right)^{1/2} \\ &= \log_5 2^{3/2} \quad \text{Power Rule} \\ &= \frac{3}{2} \log_5 2 \end{aligned}$$

### Example

Express each of the following in terms of sums and differences of logarithm:  $\log_b \left( x^4 \sqrt[3]{y} \right)$

#### Solution

$$\begin{aligned} \log_b \left( x^4 \sqrt[3]{y} \right) &= \log_b \left( x^4 \right) + \log_b \left( \sqrt[3]{y} \right) \quad \text{Product Rule} \\ &= \log_b \left( x^4 \right) + \log_b \left( y^{1/3} \right) \quad \text{Power Rule} \\ &= 4 \log_b x + \frac{1}{3} \log_b y \end{aligned}$$

### Example

Express each of the following in terms of sums and differences of logarithm:

$$\log_a \left( \frac{mnq}{p^2 r^4} \right)$$

### Solution

$$\log_a \left( \frac{mnq}{p^2 r^4} \right) = \log_a (mnq) - \log_a (p^2 r^4)$$

*Quotient Rule*

$$= \log_a m + \log_a n + \log_a q - (\log_a p^2 + \log_a r^4)$$

*Product Rule*

$$= \log_a m + \log_a n + \log_a q - \log_a p^2 - \log_a r^4$$

$$= \log_a m + \log_a n + \log_a q - 2 \log_a p - 4 \log_a r$$

*Power Rule*

### Example

Express each of the following in terms of sums and differences of logarithm:

$$\log_5 \left( \frac{\sqrt{x}}{25y^3} \right)$$

### Solution

$$\log_5 \left( \frac{\sqrt{x}}{25y^3} \right) = \log_5 (x^{1/2}) - \log_5 (25y^3)$$

*Quotient Rule*

$$= \log_5 (x^{1/2}) - [\log_5 (5^2) + \log_5 (y^3)]$$

*Product Rule*

$$= \log_5 (x^{1/2}) - \log_5 (5^2) - \log_5 (y^3)$$

$$\log_5 (5^2) = 2$$

$$= \frac{1}{2} \log_5 x - 2 - 3 \log_5 y$$

### Example

Write as a single logarithmic  $\log(7x + 6) - \log x$

### Solution

$$\log(7x + 6) - \log x = \log \frac{7x + 6}{x} \quad \text{Quotient Rule}$$

### Example

Write as a single logarithmic  $\log_3(x + 2) + \log_3 x - \log_3 2$

### Solution

$$\begin{aligned} \log_3(x + 2) + \log_3 x - \log_3 2 &= \log_3 x(x + 2) - \log_3 2 && \text{Product Rule} \\ &= \log_3 \frac{x(x + 2)}{2} && \text{Quotient Rule} \end{aligned}$$

### Example

Write as a single logarithmic  $2 \ln x + \frac{1}{3} \ln(x + 5)$

### Solution

$$\begin{aligned} 2 \ln x + \frac{1}{3} \ln(x + 5) &= \ln x^2 + \ln(x + 5)^{1/3} && \text{Power Rule} \\ &= \ln x^2 (x + 5)^{1/3} && \text{Product Rule} \\ &= \ln \left( x^2 \sqrt[3]{x + 5} \right) && \end{aligned}$$

### Example

Write as a single logarithmic  $2 \log(x - 3) - \log x$

### Solution

$$\begin{aligned} 2 \log(x - 3) - \log x &= \log(x - 3)^2 - \log x && \text{Power Rule} \\ &= \log \frac{(x - 3)^2}{x} && \text{Quotient Rule} \end{aligned}$$

## Exercises Section 3.4 – Properties of Logarithms

(1 – 31) Express the following in terms of sums and differences of logarithms

1.  $\log_3(ab)$
2.  $\log_7(7x)$
3.  $\log \frac{x}{1000}$
4.  $\log_5 \left( \frac{125}{y} \right)$
5.  $\log_b x^7$
6.  $\ln \sqrt[7]{x}$
7.  $\log_a \frac{x^2 y}{z^4}$
8.  $\log_b \frac{x^2 y}{b^3}$
9.  $\log_b \left( \frac{x^3 y}{z^2} \right)$
10.  $\log_b \left( \frac{\sqrt[3]{xy^4}}{z^5} \right)$
11.  $\log \left( \frac{100x^3 \sqrt[3]{5-x}}{3(x+7)^2} \right)$
12.  $\log_a \sqrt[4]{\frac{m^8 n^{12}}{a^3 b^5}}$
13.  $\log_p \sqrt[3]{\frac{m^5 n^4}{t^2}}$
14.  $\log_b \sqrt[n]{\frac{x^3 y^5}{z^m}}$
15.  $\log_a \sqrt[3]{\frac{a^2 b}{c^5}}$
16.  $\log_b \left( x^4 \sqrt[3]{y} \right)$
17.  $\log_5 \left( \frac{\sqrt{x}}{25y^3} \right)$
18.  $\log_a \frac{x^3 w}{y^2 z^4}$
19.  $\log_a \frac{\sqrt{y}}{x^4 \sqrt[3]{z}}$
20.  $\ln 4 \sqrt[4]{\frac{x^7}{y^5 z}}$
21.  $\ln x^3 \sqrt[3]{\frac{y^4}{z^5}}$
22.  $\log_b \sqrt[5]{\frac{m^4 n^5}{x^2 ab^{10}}}$
23.  $\log_b \frac{a^5 b^{10}}{c^2 \sqrt[4]{d^3}}$
24.  $\ln \left( x^2 \sqrt{x^2 + 1} \right)$
25.  $\ln \frac{x^2}{x^2 + 1}$
26.  $\ln \left( \frac{x^2 (x+1)^3}{(x+3)^{1/2}} \right)$
27.  $\ln \sqrt{\frac{(x+1)^5}{(x+2)^{20}}}$
28.  $\ln \frac{(x^2 + 1)^5}{\sqrt{1-x}}$
29.  $\ln \left( \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2 + 1)(2x+3)}} \right)$
30.  $\ln \left( \sqrt{\frac{1}{x(x+1)}} \right)$
31.  $\ln \left( \sqrt{(x^2 + 1)(x-1)^2} \right)$

(32 – 55) Write the expression as a single logarithm and simplify if necessary

32.  $\log(x+5) + 2 \log x$
33.  $3 \log_b x - \frac{1}{3} \log_b y + 4 \log_b z$
34.  $\frac{1}{2} \log_b (x+5) - 5 \log_b y$
35.  $\ln(x^2 - y^2) - \ln(x - y)$
36.  $\ln(xz) - \ln(x\sqrt{y}) + 2 \ln \frac{y}{z}$
37.  $\log(x^2 y) - \log z$
38.  $\log(z^2 \sqrt{y}) - \log z^{1/2}$
39.  $2 \log_a x + \frac{1}{3} \log_a (x-2) - 5 \log_a (2x+3)$

- 40.**  $5\log_a x - \frac{1}{2}\log_a (3x-4) - 3\log_a (5x+1)$       **48.**  $\frac{1}{2}\log_y p^3 q^4 - \frac{2}{3}\log_y p^4 q^3$   
**41.**  $\log(x^3 y^2) - 2\log(x\sqrt[3]{y}) - 3\log\left(\frac{x}{y}\right)$       **49.**  $\frac{1}{2}\log_a x + 4\log_a y - 3\log_a x$   
**42.**  $\ln y^3 + \frac{1}{3}\ln(x^3 y^6) - 5\ln y$       **50.**  $\frac{2}{3}\left[\ln(x^2 - 9) - \ln(x+3)\right] + \ln(x+y)$   
**43.**  $2\ln x - 4\ln\left(\frac{1}{y}\right) - 3\ln(xy)$       **51.**  $\frac{1}{4}\log_b x - 2\log_b 5 - 10\log_b y$   
**44.**  $4\ln x + 7\ln y - 3\ln z$       **52.**  $2\ln(x+4) - \ln x - \ln(x^2 - 3)$   
**45.**  $\frac{1}{3}\left[5\ln(x+6) - \ln x - \ln(x^2 - 25)\right]$       **53.**  $\ln x + \ln(y+3) + \ln(y+2) - \ln(y^2 + 5y + 6)$   
**46.**  $\frac{2}{3}\left[\ln(x^2 - 4) - \ln(x+2)\right] + \ln(x+y)$       **54.**  $\ln x + \ln(x+4) + \ln(x+1) - \ln(x^2 + 5x + 4)$   
**47.**  $\frac{1}{2}\log_b m + \frac{3}{2}\log_b 2n - \log_b m^2 n$       **55.**  $\ln(x^2 - 25) - 2\ln(x+5) + \ln(x-5)$
- 56.** Assume that  $\log_{10} 2 = .3010$ . Find each logarithm  $\log_{10} 4$ ,  $\log_{10} 5$
- 57.** Given that:  $\log_a 2 \approx 0.301$ ,  $\log_a 7 \approx 0.845$ , and  $\log_a 11 \approx 1.041$  find each of the following:
- |                          |                         |                          |
|--------------------------|-------------------------|--------------------------|
| a) $\log_a \frac{2}{11}$ | c) $\log_a 98$          | e) $\log_a 9$            |
| b) $\log_a 14$           | d) $\log_a \frac{1}{7}$ | f) $\log_a \frac{77}{8}$ |