9. The electrostatic potential due to charges within a metal box

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a) Introduction

We found that the electrostatic potential due to a distribution of charges could be found by superposing the potential due to a point charge (i.e., the Green's function).

We found that we could determine these Green's functions for point sources in free space without having to directly solve Poisson's equation. We could do it by appealing to symmetry and Guass's law.

For sources not in free space, however, more general methods will be required to find the potential.

b) Boundary and continuity conditions

(1) Condition at the surface of a metal

Within a metal, charges are freely mobile. A static potential difference cannot exist between any two points on or in a metal.

If it did, then charge would flow redistributing itself in such a way to extinguish the potential difference.

Therefore, the POTENTIAL MUST BE THE SAME CONSTANT AT EVERY POINT IN OR ON A METAL.

It is often convenient to take that constant potential to be ZERO. We can always do this since a potential is only unique up to an additive constant.

(2) Continuity conditions

The potential function is constructed from the line integral of the electric field

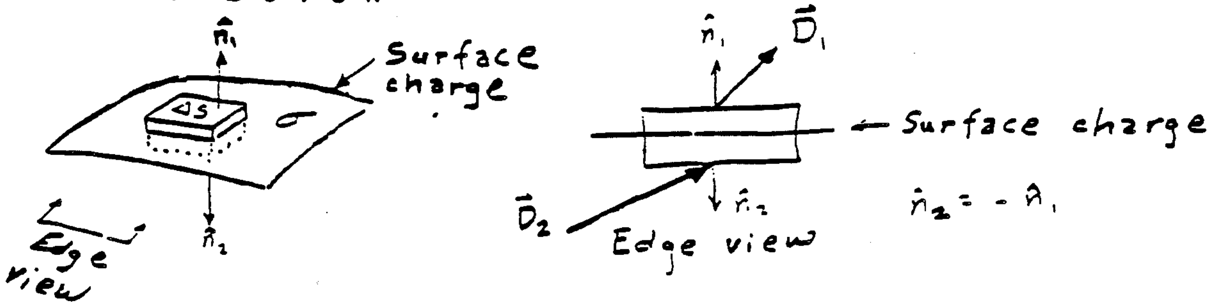
$$\varphi(x,y,z) = -\int_{P_0}^{P} \vec{E}\cdot d\vec{r} \qquad \begin{array}{l} P = (x,y,z) \\ P_0 = (x_0,y_0,z_0) \end{array}$$

Even if the electric field had a "jump" discontinuity, its potential would nevertheless be continuous.

For any physically meaningful finite electric field, the associated ELECTROSTATIC POTENTIAL MUST BE CONTINUOUS.

The electric field, on the other hand, need not be continuous.

Consider the electric field associated with the SURFACE charge shown below.



According to Guass's law, the total electric flux leaving the small volume is equal to the charge enclosed.

If σ is the surface charge density, then the charge enclosed is just $dS \cdot \sigma$.

If we let the little box get flatter and flatter, the total flux leaving approaches

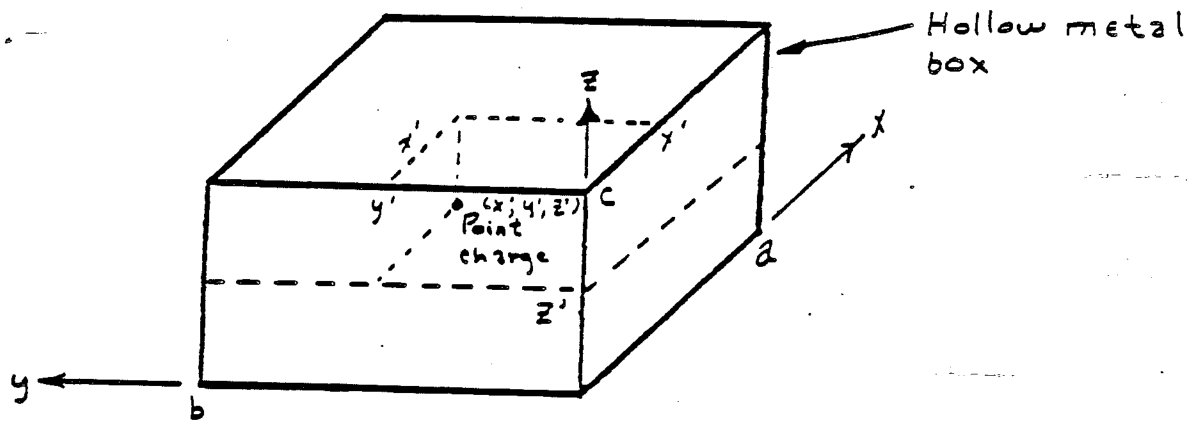
$$(\hat{n}_1 \cdot \vec{D}_1 + \hat{n}_2 \cdot \vec{D}_2)\Delta S = \Delta S \hat{n}_1 \cdot (\vec{D}_1 - \vec{D}_2) = \sigma \Delta S \rightarrow$$

$$\vec{D}_1 \cdot \hat{n}_1 = \vec{D}_2 \cdot n_1 \therefore \sigma$$

Therefore, we find that the components of $\vec{D}$ normal to a surface charge are DISCONTINUOUS by an amount just equal to the surface charge density.

c) Green's function for a point charge inside a metal box --- separation of variables

We will now find the potential due to a unit point charge inside the metal box drawn below:



The Green's function, $G(x,y,z|x',y',z')$ satisfies Poisson's equation:

$$\nabla^2 G = -\delta(x-x')\,\delta(y-y')\,\delta(z-z')/\epsilon$$

We will divide the volume in the box into two regions:

Let Region I be all points with $z < z'$.

Let Region II be all points with $z > z'$.

Since the charge density is zero for $z>z'$ and $z<z'$, the Green's function satisfies Laplace's equation in Regions I and II.

$$\nabla^2 G_{II}(x,y,z\,|\,x',y',z') = 0 \quad \text{for } z>z', \text{ Region II}$$
$$\nabla^2 G_{I}(x,y,z\,|\,x',y',z') = 0 \quad \text{for } z<z', \text{ Region I}$$
$$G(x,y,z\,|\,x',y',z') = \begin{cases} G_I & \text{in I} \\ G_{II} & \text{in II}. \end{cases}$$

We will attempt to find solutions to this equation, each valid in its own region, by separation of variables.

Then we will connect the solutions in the two regions by the continuity conditions.

Assume a solution to Laplace's equation in Region I of the form $X(x)Y(y)Z(z)$. The boundary conditions that the soltution in Region I must satisfy are:

$$X(0)Y(y)Z(z) = 0 \quad \text{--->} \quad X(0) = 0$$
$$X(a)Y(y)Z(z) = 0 \quad \text{--->} \quad X(a) = 0$$
$$X(x)Y(0)Z(z) = 0 \quad \text{--->} \quad Y(0) = 0$$
$$X(x)Y(b)Z(z) = 0 \quad \text{--->} \quad Y(b) = 0$$
$$X(x)Y(y)Z(0) = 0 \quad \text{--->} \quad Z(0) = 0$$

Substituting this assumed solution into the differential equation, we get

$$\nabla^2 X(x) Y(y) Z(z) = 0$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) X(x) Y(y) Z(z) = X'' Y Z + X Y'' Z + X Y Z'' = 0$$

Multiplying by $\frac{1}{XYZ}$, $\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0 \longrightarrow$

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)}$$

We have a function of x only equal to a function of y and z only and therefore,

$$-\frac{X''(x)}{X(x)} = \mu = \text{constant} = \underbrace{K_x^2 \quad (\text{for convenience})}$$

Solving for X(x), we obtain

$$X'' + K_x^2 X = 0 \longrightarrow X(x) = A\cos(K_x x) + B\sin(K_x x)$$

Applying the first boundary condition, we find that A=0.

Applying the second boundary condition, we obtain

$$B\sin(K_x a) = 0 \qquad \text{so either } B = 0 \text{ or}$$

$$\sin(K_x a) = 0$$

The only non-trivial solution that this admits is that the separation constant must take on the DISCRETE values,

$$\sin(k_x a) = 0 \longrightarrow k_x a = m\pi, \quad m = 0, 1, 2, \ldots$$

$$\longrightarrow k_x = k_{xm} = \frac{m\pi}{a} \longrightarrow X(x) = \sin\left(\frac{m\pi x}{a}\right) \text{ (with B taken to be 1)}$$

Returning to the result,

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = +k_x^2, \longrightarrow \frac{Y''}{Y} = +k_x^2 - \frac{Z''}{Z} = -k_y^2$$

we see that $Y''(y)/Y(y)$ must also be a constant.

Therefore,

$$\frac{Y''}{Y} = -k_y^2 \longrightarrow Y'' + k_y^2 Y = 0 \longrightarrow$$

$$Y(y) = A\cos(k_y y) + B\sin(k_y y).$$

Applying the boundary conditions, we find that

$$Y(y) = \sin\left(\frac{n\pi y}{b}\right)$$

with the separation constant for y given by

$$k_y = k_{yn} = \frac{n\pi}{b}, \quad n = 0, 1, 2, \ldots$$

Finally, we can conclude that

$$\frac{Z''(z)}{Z} = k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \equiv \gamma_{mn}^2 \longrightarrow$$

$$Z(z) = A\cosh(\gamma_{mn} z) + B\sinh(\gamma_{mn} z)$$

For Z(z), we have only one boundary condition in Region I to apply. It requires that

$$Z(0) = 0 = A$$

$$\therefore Z(z) = \sinh(\gamma_{mn} z) \qquad (\text{taking } B = 1).$$

Therefore, the potential in Region I is given by

$$XYZ = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sinh(\gamma_{mn} z).$$

We can do the same thing in Region II.

In this region, all the boundary conditions are the same except for the condition on Z(z).

In Region II, Z(c) = 0.

In this region, the X(x) and Y(y) solutions are therefore the same as for Region I.

However, Z(z) must satisfy a different condition and hence will be different from what we obtained in Region I:

A very convenient alternate form of the solution for Z(z) in Region II to

$$Z(z) = A\cosh(\gamma_{mn} z) + B\sinh(\gamma_{mn} z)$$

is

$$Z(z) = A\cosh[\gamma_{mn}(c-z)] + B\sinh[\gamma_{mn}(c-z)]$$

since upon application of the boundary condition,

$A = 0$ and $Z(z) = \sinh[\gamma_{mn}(c-z)]$ with $B = 1$.

Therefore, the potential in Region II is

$$XYZ = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sinh[\gamma_{mn}(c-z)]$$

The solutions found in Regions I and II are valid for any m and n.

Therefore, any linear combination of these potentials also satisfy Laplace's equation and the boundary conditions in the two regions:

$$G_{I} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn}^{I} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sinh\left(\gamma_{mn} z\right)$$

$$G_{II} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn}^{II} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sinh\left[\gamma_{mn}(c-z)\right]$$

The only physical conditions remaining to be applied are the continuity conditions which will relate the solutions in Regions I and II.

Applying the continuity of the potential function at z=z',

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn}^{I} \sinh\left(\gamma_{mn} z'\right) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$
$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn}^{II} \sinh\left[\gamma_{mn}(c-z')\right] \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right).$$

What information about the coefficients in these expansions can we obtain from this equation?

To answer this, we must digress briefly to discuss the concept of orthogonality.

Consider the integral

$$\int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{k\pi x}{a}\right)dx.$$

To evaluate this integral, recall the trigonometric identity,

$$\sin(A)\sin(B) = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B).$$

Therefore,

$$\int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{k\pi x}{a}\right)dx$$

$$= \frac{1}{2}\int_0^a \cos\left[(m-k)\frac{\pi x}{a}\right]dx - \frac{1}{2}\int_0^a \cos\left[(m+k)\frac{\pi x}{a}\right]dx.$$

For $m \neq k$, $m, k = 1, 2, 3, \ldots$,

$$\int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{k\pi x}{a}\right)dx$$

$$= \frac{a}{2\pi(m-k)}\sin\left[(m-k)\frac{\pi}{a}x\right]\Bigg|_0^a$$

$$- \frac{a}{2\pi(m+k)}\sin\left[(m+k)\frac{\pi}{a}x\right]\Bigg|_0^a = 0$$

For $m = k > 0$,

$$\int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx$$

$$= \frac{1}{2}\int_0^a dx - \frac{1}{2}\int_0^a \cos\left(2m\frac{\pi}{a}x\right)dx$$

$$= \frac{a}{2} - \frac{a}{4m\pi}\sin\left(\frac{2m\pi}{a}x\right)\Bigg|_0^a = \frac{a}{2}.$$

Thus

$$\int_0^a \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{k\pi}{a}x\right)dx = \frac{a}{2}\delta_{mk},$$

where the "Kronecker δ" is defined as

$$\delta_{mk} = \begin{cases} 1 & m = k \\ 0 & m \neq k \end{cases}$$

Similarly,

$$\int_0^b \sin\left(\frac{n\pi}{b}y\right)\sin\left(\frac{\ell\pi}{b}y\right)dy = \frac{b}{2}\delta_{n\ell}.$$

Therefore, if

$$\varphi_{mn}(x,y) \equiv \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

and $A = \{(x,y) \mid 0 \leq x \leq a,\ 0 \leq y \leq b\}$,

$$\iint_A \varphi_{mn}(x,y)\,\varphi_{k\ell}(x,y)\,dx\,dy = \frac{ab}{4}\delta_{mk}\delta_{n\ell}$$

The set of functions, $\{\varphi_{mn}(x,y) \mid m,n = 1,2,3,\ldots\}$ is said to be ORTHOGONAL.

The continuity of potential at $z=z'$ requires that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^{I} \sinh(\gamma_{mn} z')\, \varphi_{mn}(x,y)$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^{II} \sinh[\gamma_{mn}(c-z')]\, \varphi_{mn}(x,y)$$

Multiplying both sides by $\varphi_{k\ell}(x,y)$ and integrating over A, the orthogonality requires

$$A_{k\ell}^{I} \sinh(\gamma_{k\ell} z') = A_{k\ell}^{II} \sinh[\gamma_{k\ell}(c-z')],$$

for $k, l = 1, 2, 3, \ldots$.

For each (k, ℓ), this represents one equation in two unknowns,

$$A_{k\ell}^{I} \text{ and } A_{k\ell}^{II}.$$

We have, however, one remaining continuity equation left. Namely,

$$D_z(z'+) - D_z(z'-) = \sigma$$

where D_z is the z component of electric flux density, and σ is the surface charge density function.

The notation $D_z(z'\pm)$ means

$$D_z(z'\pm) = \lim_{\substack{a \to 0 \\ a > 0}} D_z(z' \pm a).$$

But $\vec{D} = \epsilon \vec{E} = -\epsilon \nabla G$ so that

$$D_z = -\epsilon \frac{\partial G}{\partial z}.$$

Therefore,

$$\left.\frac{\partial G}{\partial z}\right|_{z=z'_+} - \left.\frac{\partial G}{\partial z}\right|_{z=z'_-} = -\frac{\sigma}{\epsilon}.$$

The surface charge density, σ, is

$$\sigma(x,y) = \int_{z'_-}^{z'_+} \rho(x,y,z)\,dz =$$

$$= \delta(x-x')\,\delta(y-y') \int_{z'_-}^{z'_+} \delta(z-z')\,dz$$

$$= \delta(x-x')\,\delta(y-y').$$

Therefore, we must enforce

$$\left.\frac{\partial G_{II}}{\partial z}\right|_{z=z'} - \left.\frac{\partial G_{I}}{\partial z}\right|_{z=z'} = -\frac{1}{\epsilon}\,\delta(x-x')\,\delta(y-y')$$

[Recall $G = \begin{cases} G_I & z < z' \\ G_{II} & z > z' \end{cases}$ so that

$$\left.\frac{\partial G}{\partial z}\right|_{z=z'_+} = \left.\frac{\partial G_{II}}{\partial z}\right|_{z=z'}, \text{ etc.}$$]

Another way we could have arrived at this continuity condition is by integrating

$$\nabla^2 G = -\rho/\epsilon \quad \longleftrightarrow$$

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2} = -\frac{1}{\epsilon}\,\delta(x-x')\,\delta(y-y')\,\delta(z-z')$$

Integrate both sides from "z=z'−" to "z=z'+":

$$\int_{z'-}^{z'+} \frac{\partial^2 G}{\partial x^2}\,dz + \int_{z'-}^{z'+} \frac{\partial^2 G}{\partial y^2}\,dz + \int_{z'-}^{z'+} \frac{\partial^2 G}{\partial z^2}\,dz$$

$$= \frac{\partial^2}{\partial x^2} \cancelto{0}{\int_{z'-}^{z'+} G\,dz} + \frac{\partial^2}{\partial y^2} \cancelto{0}{\int_{z'-}^{z'+} G\,dz} + \int_{z'-}^{z'+} \frac{\partial}{\partial z}\left[\frac{\partial G}{\partial z}\right] dz$$

$$= \left.\frac{\partial G}{\partial z}\right|_{z'-}^{z'+} = \left.\frac{\partial G_{II}}{\partial z}\right|_{z=z'} - \left.\frac{\partial G_{I}}{\partial z}\right|_{z=z'}$$

$$= \frac{1}{\epsilon}\int_{z'-}^{z'+} \delta(x-x')\,\delta(y-y')\,\delta(z-z')\,dz$$

$$= -\frac{1}{\epsilon}\,\delta(x-x')\,\delta(y-y')$$

Now let's apply this continuity condition.

In Region I,

$$\left.\frac{\partial G_I}{\partial z}\right|_{z=z'} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\partial}{\partial z} A_{mn}^{I} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sinh(\gamma_{mn} z)$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \gamma_{mn} A_{mn}^{I} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cosh(\gamma_{mn} z').$$

In Region II,

$$\left.\frac{\partial G_{II}}{\partial z}\right|_{z=z'} = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \gamma_{mn} A_{mn}^{II} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cosh\left[\gamma_{mn}(c-z')\right]$$

Therefore

$$\left.\frac{\partial G_{II}}{\partial z}\right|_{z=z'} - \left.\frac{\partial G_I}{\partial z}\right|_{z=z'}$$

$$= -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \gamma_{mn}\left\{A_{mn}^{II}\cosh\left[\gamma_{mn}(c-z')\right] + A_{mn}^{I}\cosh(\gamma_{mn} z')\right\}\varphi_{mn}$$

$$= -\delta(x-x')\,\delta(y-y')/\epsilon$$

Let

$$B_{mn} = -\gamma_{mn}\left\{A_{mn}^{II}\cosh\left[\gamma_{mn}(c-z')\right] + A_{mn}^{I}\cosh(\gamma_{mn} z')\right\}$$

Then

$$\left.\frac{\partial G_{II}}{\partial z}\right|_{z=z'} - \left.\frac{\partial G_I}{\partial z}\right|_{z=z'}$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\,\varphi_{mn}(x,y) = -\frac{1}{\epsilon}\delta(x-x')\,\delta(y-y').$$

How do we determine B_{mn}?

We find B_{mn} using the orthogonality of
$$\left\{ \varphi_{mn}(x,y) = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \right\}.$$

Mulitplying by $\varphi_{k\ell}(x,y)$,
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \iint_A \varphi_{mn}\,\varphi_{k\ell}\,dx\,dy$$
$$= \frac{ab}{4} B_{k\ell} = -\frac{1}{\epsilon} \iint_A \sin\left(k\frac{\pi x}{a}\right)\sin\left(\frac{\ell\pi y}{b}\right)\delta(x-x')\,\delta(y-y')\,dx\,dy$$
$$= -\frac{1}{\epsilon}\sin\left(\frac{k\pi x'}{a}\right)\sin\left(\frac{\ell\pi y'}{b}\right).$$

Therefore,
$$B_{k\ell} = -\frac{4}{ab\epsilon}\sin\left(\frac{k\pi x'}{a}\right)\sin\left(\frac{\ell\pi y'}{b}\right).$$

We now have two equations in two unknowns for each $m, n = 1, 2, 3, \ldots$:

① $$-\gamma_{mn}\cosh[\gamma_{mn}(c-z')]A_{mn}^{II} - \gamma_{mn}\cosh(\gamma_{mn}z')A_{mn}^{I} = -\frac{4}{ab\epsilon}\sin\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)$$

② $$\sinh[\gamma_{mn}(c-z')]A_{mn}^{II} - \sinh(\gamma_{mn}z')A_{mn}^{I} = 0.$$

This pair of equations can be solved for A^{II}_{mn} and A^{I}_{mn} :

$$A^{I}_{mn} = \frac{1}{\epsilon} \cdot \frac{\begin{vmatrix} -\gamma_{mn}\cosh[\gamma_{mn}(c-z')] & -\frac{4}{ab}\sin\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right) \\ \sinh[\gamma_{mn}(c-z')] & 0 \end{vmatrix}}{\begin{vmatrix} -\gamma_{mn}\cosh[\gamma_{mn}(c-z')] & -\gamma_{mn}\cosh(\gamma_{mn}z') \\ \sinh[\gamma_{mn}(c-z')] & -\sinh(\gamma_{mn}z') \end{vmatrix}}$$

The denominator is

$$\Delta \equiv \gamma_{mn}\cosh[\gamma_{mn}(c-z')]\sinh(\gamma_{mn}z') + \gamma_{mn}\sinh[\gamma_{mn}(c-z')]\cosh(\gamma_{mn}z')$$

Recall the trigonometric identity,

$$\sinh(A+B) = \sinh(A)\cosh(B) + \cosh(A)\sinh(B)$$

Then

$$\Delta = \gamma_{mn}\sinh[\gamma_{mn}z' + \gamma_{mn}(c-z')] = \gamma_{mn}\sinh(\gamma_{mn}c)$$

Therefore,

$$A^{I}_{mn} = +\frac{4}{ab\epsilon} \cdot \frac{\sin\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)}{\gamma_{mn}\sinh(\gamma_{mn}c)}\sinh[\gamma_{mn}(c-z')]$$

Similarly,

$$A^{II}_{mn} = \frac{1}{\epsilon} \cdot \begin{vmatrix} -\frac{4}{ab}\sin\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{r\pi y'}{b}\right) & -\gamma_{mn}\cosh(\gamma_{mn} z') \\ 0 & -\sinh(\gamma_{mn} z') \end{vmatrix} \cdot \frac{1}{\Delta}$$

$$= +\frac{4}{ab\epsilon} \cdot \frac{\sin\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)}{\gamma_{mn}\sinh(\gamma_{mn} c)} \sinh(\gamma_{mn} z').$$

Therefore

$$G = \begin{cases} G_I = \dfrac{4}{ab\epsilon}\displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{n\pi y'}{b}\right)\sinh[\gamma_{mn}(c-z')]\sinh(\gamma_{mn} z}{\gamma_{mn}\sinh(\gamma_{mn} c)} \\ \quad \text{for } z < z' \\ G_{II} = \dfrac{4}{ab}\displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{n\pi y'}{b}\right)\sinh[\gamma_{mn}(c-z)]\sinh(\gamma_{mn} z')}{\gamma_{mn}\sinh(\gamma_{mn} c)} \\ \quad \text{for } z > z' \end{cases}$$

Or, more succinctly

$$G(x,y,z|x',y',z') = \frac{4}{ab\epsilon}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\varphi_{mn}(x,y)\,\varphi_{mn}(x',y')}{\gamma_{mn}\sinh(\gamma_{mn} c)} \cdot \begin{cases} \sinh(\gamma_{mn} z)\sinh[\gamma_{mn}(c-z' \\ \sinh(\gamma_{mn} z')\sinh[\gamma_{mn}(c-z \end{cases}$$

$$\text{for } z \lessgtr z'.$$

Notice the symmetry in $G(x,y,z|x',y',z')$:

$$G(x,y,z|x',y',z') = G(x',y',z'|x,y,z)$$

That is, the potential measured at (x, y, z) due to a unit charge at (x', y', z') is the same as the potential measured at (x', y', z') due to a unit charge at (x, y, z).

Compare this Green's function to the Green's function of "free space:"

$$G(x, y, z \mid x', y', z') = \frac{1}{4\pi\epsilon} \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$