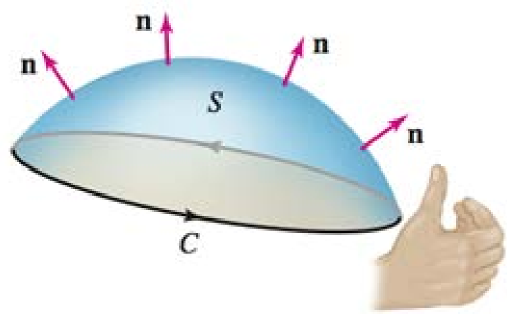
Section 4.7 – Stokes' Theorem

Stokes' Theorem

Stokes' Theorem is the three-dimensional version of the circulation from of Green's Theorem.

If C is a closed simple piecewise-smooth oriented curve in the xy-plane enclosing a region R and $\vec{F} = \langle f,\ g \rangle$ is a differentiable vector field on R. Green's Theorem says that

$$\underbrace{\oint \boldsymbol{F} \cdot d\boldsymbol{r}}_{circulation} = \underbrace{\iint_R \left(g_x - f_y\right) dA}_{curl\ or\ rotation}$$



If the fingers of your right hand curl in the positive direction around C, then your right thumb points in the direction of the vectors normal to S.

Theorem

Let S be an oriented surface in $\mathbb{R}^3$ with a piecewise-smooth closed boundary C whose orientation is consistent with that of S. Assume that $\vec{F} = \langle f,\ g,\ h \rangle$ is a vector field whose components have continuous first partial derivatives on S. Then

$$\oint_C \vec{F} \bullet d\vec{r} = \iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ \ dS$$

Where $\vec{n}$ is the unit vector normal to S determined by the orientation of S.

Example

Confirm that Stokes' Theorem holds for the vector field $\vec{F} = \langle z - y,\ x,\ -x\rangle$ where S is the hemisphere $x^2 + y^2 + z^2 = 4$, for $z \ge 0$, and C is the circle $x^2 + y^2 = 4$ oriented counterclockwise.

Solution

The orientation of C says that the vectors normal to S point in the outward direction. The vector field is a rotation field $\vec{a} \times \vec{r}$, where $\vec{a} = \langle 0,\ 1,\ 1\rangle$ and $\vec{r} = \langle x,\ y,\ z\rangle$, so the axis of rotation points in the direction of the vector $\langle 0,\ 1,\ 1\rangle$.

Compute first the circulation integral in Stokes' Theorem. The curve C with the given orientation is parametrized as $\vec{r}(t) = \langle 2\cos t,\ 2\sin t,\ 0\rangle$, for $0 \le t \le 2\pi$

$$\frac{d}{dt}\vec{r}(t) = \langle -2\sin t,\ 2\cos t,\ 0\rangle$$

$$\vec{F} = \langle -2\sin t,\ 2\cos t,\ -2\cos t\rangle$$

$$\vec{F} \bullet \vec{r}\,'(t) = 4\sin^2 t + 4\cos^2 t$$

$$\oint_C \vec{F} \bullet d\vec{r} = \int_0^{2\pi} \vec{F} \bullet \vec{r}\,'(t)\ dt$$

$$= \int_0^{2\pi} 4\left(\sin^2 t + \cos^2 t\right)\ dt \qquad \sin^2 t + \cos^2 t = 1$$

$$= 4\int_0^{2\pi} dt$$

$$= 8\pi$$

$$\nabla \times \vec{F} = \nabla \times \langle z - y,\ x,\ -x\rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z - y & x & -x \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(-x) - \frac{\partial}{\partial z}(x)\right)\hat{i} + \left(\frac{\partial}{\partial z}(z-y) - \frac{\partial}{\partial x}(-x)\right)\hat{j} + \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(z-y)\right)\hat{k}$$

$$= 0\hat{i} + 2\hat{j} + 2\hat{k}$$

$$= \langle 0,\ 2,\ 2\rangle$$

The region of integration is the base of the hemisphere in the *xy*-plane, which is

$$R = \left\{(x,y): x^2 + y^2 \le 4\right\} = \left\{(r,\theta):\ 0 \le r \le 2,\ 0 \le \theta \le 2\pi\right\}$$

$$z = \pm\sqrt{4 - \left(x^2 + y^2\right)}$$

The normal vector from the table: $\vec{n} = \left\langle -z_x,\ -z_y,\ 1\right\rangle$

$$x^2 + y^2 + z^2 = 4$$

$$2x + 2zz_x = 0 \quad \to \quad z_x = -\frac{x}{z}$$

$$2y + 2zz_x = 0 \quad \to \quad z_y = -\frac{y}{z}$$

$$\vec{n} = \left\langle \frac{x}{z},\ \frac{y}{z},\ 1\right\rangle$$

$$\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS = \iint_R \langle 0,\ 2,\ 2\rangle \bullet \left\langle \frac{x}{z},\ \frac{y}{z},\ 1\right\rangle dA$$

$$= \iint_R \left(\frac{2y}{\sqrt{4 - x^2 - y^2}} + 2\right) dA \qquad x^2 + y^2 + z^2 = 4 \ \to\ z = \sqrt{4 - x^2 - y^2}$$

$$= \int_0^{2\pi}\int_0^2 \left(\frac{2r\sin\theta}{\sqrt{4 - r^2}} + 2\right) r\,dr\,d\theta$$

$$= \int_0^2\int_0^{2\pi} \left(\frac{2r^2}{\sqrt{4 - r^2}}\sin\theta + 2r\right) d\theta\ dr$$

$$= \int_0^2 \left(-\frac{2r^2}{\sqrt{4 - r^2}}\cos\theta + 2r\theta\right)\Bigg|_0^{2\pi} dr$$

$$= \int_0^2 \left(-\frac{2r^2}{\sqrt{4 - r^2}} + 4\pi r + \frac{2r^2}{\sqrt{4 - r^2}}\right) dr$$

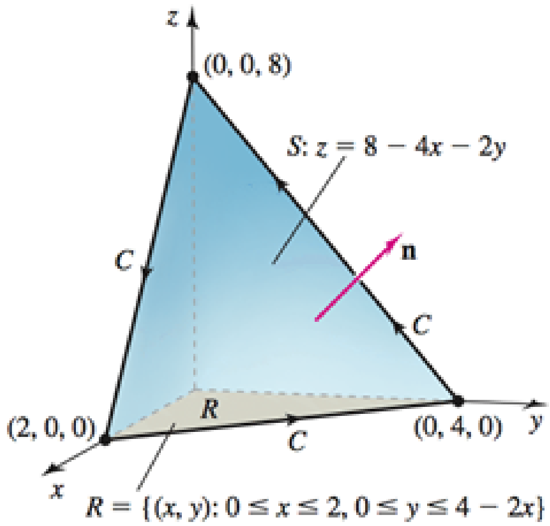
$$= 4\pi\int_0^2 r\ dr$$

$$= 2\pi r^2 \Big|_0^2$$

$$\boxed{= 8\pi}$$

Example

Evaluate the line integral $\oint_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \left\langle z,\ -z,\ x^2 - y^2 \right\rangle$ and C consists of the three line segments that bound the plane $z = 8 - 4x - 2y$ in the first octant, oriented as shown



Solution

$$\nabla \times \vec{F} = \nabla \times \left\langle z,\ -z,\ x^2 - y^2 \right\rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -z & x^2 - y^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}\left(x^2 - y^2\right) - \frac{\partial}{\partial z}(-z) \right)\hat{i} + \left(\frac{\partial}{\partial z}(z) - \frac{\partial}{\partial x}\left(x^2 - y^2\right) \right)\hat{j} + \left(\frac{\partial}{\partial x}(-z) - \frac{\partial}{\partial y}(z) \right)\hat{k}$$

$$= \langle 1 - 2y,\ 1 - 2x,\ 0 \rangle$$

$$z = 8 - 4x - 2y \ \Rightarrow\ 4x + 2y + z = 8$$

$$\vec{n} = \langle 4,\ 2,\ 1 \rangle$$

$$\iint_S \left(\nabla \times \vec{F} \right) \cdot \vec{n}\ dS = \iint_R \left(\langle 1 - 2y,\ 1 - 2x,\ 0 \rangle \cdot \langle 4,\ 2,\ 1 \rangle \right) dA$$

$$= \int_0^2 \int_0^{4-2x} (6 - 4x - 8y)\ dydx$$

$$= \int_0^2 \left(6y - 4xy - 4y^2 \,\Big|_0^{4-2x} \right. dx$$

$$= \int_0^2 \left(24 - 12x - 16x + 8x^2 - 4\left(16 - 16x + 4x^2\right)\right)\, dx$$

$$= \int_0^2 \left(-8x^2 + 36x - 40\right)\, dx$$

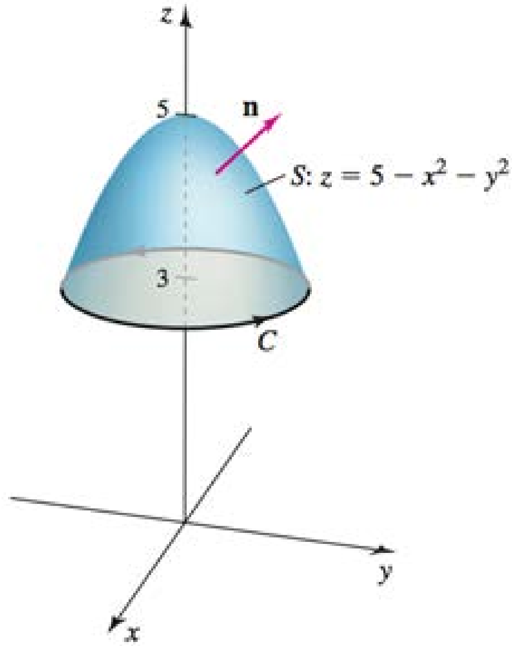
$$= -\frac{8}{3}x^3 + 18x^2 - 40x \,\Big|_0^2$$

$$= -\frac{64}{3} + 72 - 80$$

$$= -\frac{88}{3}$$

Example

Evaluate the line integral $\iint\limits_S \left(\nabla\times\vec{F}\right)\bullet\vec{n}\ dS$ where $\vec{F}=\left\langle -xz,\ yz,\ xye^{z}\right\rangle$ and S is the cap of the paraboloid $z=5-x^2-y^2$ above the plane $z=3$.



Assume $\vec{n}$ points in the upward direction on S.

Solution

$$z=5-x^2-y^2=3 \quad\Rightarrow\quad x^2+y^2=2$$

$$\vec{r}(t)=\left\langle \sqrt{2}\cos t,\ \sqrt{2}\sin t,\ 3\right\rangle \qquad\qquad \vec{r}(t)=\left\langle r\cos t,\ r\sin t,\ z\right\rangle$$

$$\vec{r}'(t)=\left\langle -\sqrt{2}\sin t,\ \sqrt{2}\cos t,\ 0\right\rangle$$

$$\begin{aligned}\left\langle -xz,\ yz,\ xye^{z}\right\rangle\cdot\left\langle -\sqrt{2}\sin t,\sqrt{2}\cos t,0\right\rangle &= xz\sqrt{2}\sin t+yz\sqrt{2}\cos t\\ &=3\sqrt{2}\cos t\sqrt{2}\sin t+3\sqrt{2}\sin t\sqrt{2}\cos t\\ &=12\sin t\cos t\\ &=6\sin 2t\end{aligned}$$

$$\begin{aligned}\iint\limits_S \left(\nabla\times\vec{F}\right)\bullet\vec{n}\ dS &= \int_0^{2\pi}\left\langle -xz,yz,\ xye^{z}\right\rangle\bullet\left\langle -\sqrt{2}\sin t,\sqrt{2}\cos t,0\right\rangle\ dt\\ &=\int_0^{2\pi} 6\sin 2t\,dt\\ &=-3\cos 2t\Big|_0^{2\pi}\\ &=0\end{aligned}$$

Interpreting the Curl

Stokes' Theorem leads to another interpretation of the curl at a point in a vector field. We need the idea of the average circulation. If *C* is the boundary of an oriented surface *C*, we define the average circulation of $\vec{F}$ over *S* as

$$\frac{1}{area(S)}\oint_C \vec{F}\bullet d\vec{r} = \frac{1}{area(S)}\iint_S \left(\nabla\times\vec{F}\right)\bullet\vec{n}\, dS$$

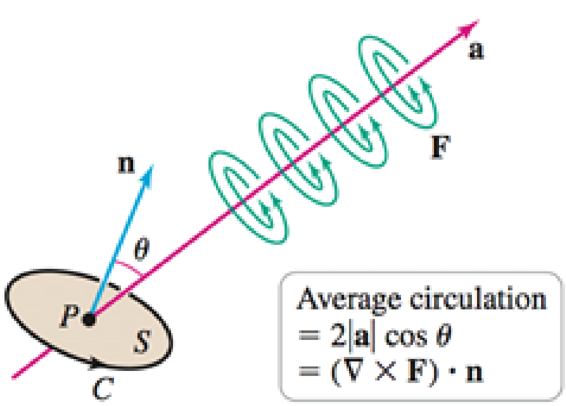
Where Stokes' Theorem is used to convert the circulation integral to a surface integral.

Example

Consider the vector field $\vec{F}=\vec{a}\times\vec{r}$, where $\vec{a}=\left(a_1,\ a_2,\ a_3\right)$ is a nonzero vector and $\vec{r}=(x,\ y,\ z)$

$$\begin{aligned}\vec{F}&=\vec{a}\times\vec{r}\\ &=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ x & y & z\end{vmatrix}\\ &=\left(a_2z-a_3y\right)\hat{i}+\left(a_3x-a_1z\right)\hat{j}+\left(a_1y-a_2x\right)\hat{k}\end{aligned}$$

$$\begin{aligned}curl\ \vec{F}&=\nabla\times\vec{F}\\ &=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ a_2z-a_3y & a_3x-a_1z & a_1y-a_2x\end{vmatrix}\\ &=\left(2a_1\right)\hat{i}+\left(2a_2\right)\hat{j}+\left(2a_3\right)\hat{k}\\ &=2\vec{a}\end{aligned}$$



Let S to be a small circular disk centered at a point P, whose normal vector $\vec{n}$ makes an angle θ with the axis $\vec{a}$.

Let C be the boundary of S with a counterclockwise orientation.

The average circulation of this vector field on S is

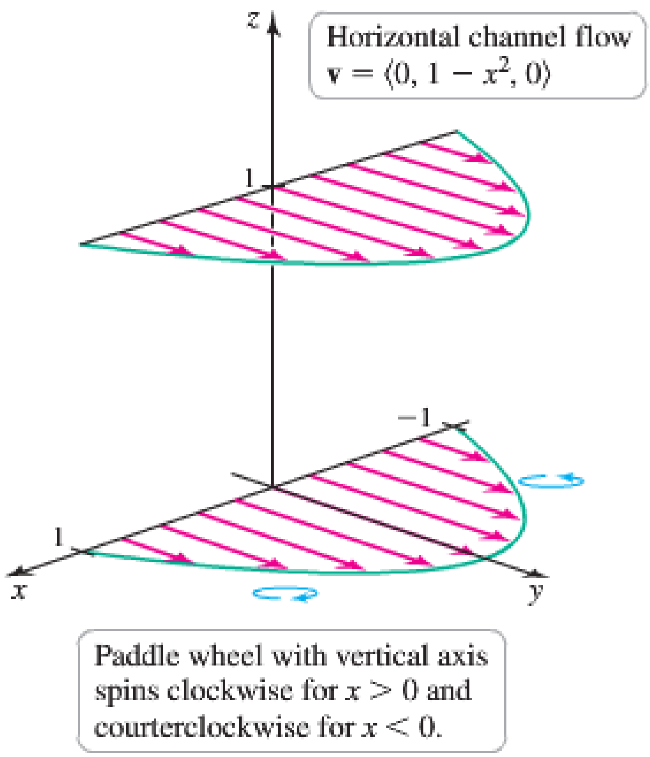
$$\frac{1}{area(S)}\iint_S (\nabla\times\vec{F})\bullet\vec{n}\ dS = \frac{1}{area(S)}(\nabla\times\vec{F})\bullet\vec{n}\ \cdot area(S)$$

$$=(2\vec{a})\cdot\vec{n}$$

$$=2|\vec{a}|\cos\theta$$

Example

Consider the velocity field $\vec{v}=\left\langle 0,\ 1-x^2,\ 0\right\rangle$, for $|x|\le 1$ and $|z|\le 1$, which represents a horizontal flow in the y-direction.



a) Suppose you place a paddle wheel at the point $P\left(\frac{1}{2},0,0\right)$. Using physical arguments, in which of the coordinate directions should the axis of the wheel point in order for the wheel to spin? In which direction does it spin? What happens if you place the wheel at $Q\left(-\frac{1}{2},0,0\right)$?

b) Compute and graph the curl of $\vec{v}$ and provide an interpretation.

Solution

a) If the axis of the wheel is aligned with the x-axis at P, the flow strikes the upper and lower halves of the wheel symmetrically and the wheel does not spin. If the axis of the wheel is aligned with the z-axis at P, the flow in the y-direction is greater for $x < \frac{1}{2}$ than it is for $x > \frac{1}{2}$.

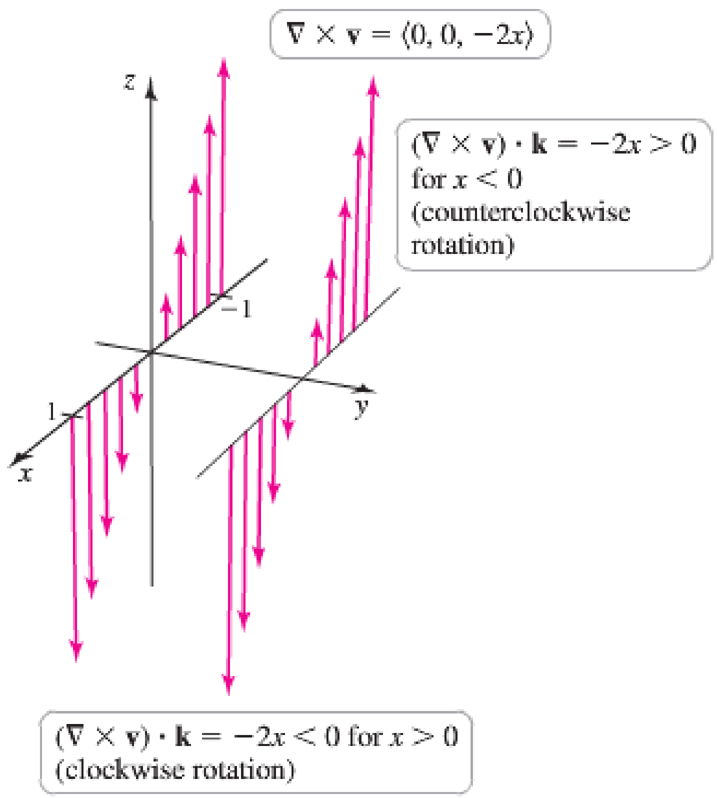
Therefore, a wheel located at $\left(\frac{1}{2}, 0, 0\right)$ spins in the clockwise direction, looking from above.

Using the similar argument, we conclude that a vertically oriented paddle wheel placed at $Q\left(-\frac{1}{2}, 0, 0\right)$ spins in the counterclockwise direction (when viewing from above).

b) $$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 1 - x^2 & 0 \end{vmatrix}$$

$$= -2x\hat{k}$$

The curl points in the z-direction, which is the direction of the paddle wheel axis that gives the maximum angular speed of the wheel. Consider the z-component of the curl, which is $(\nabla \times \vec{v}) \bullet \hat{k} = -2x$

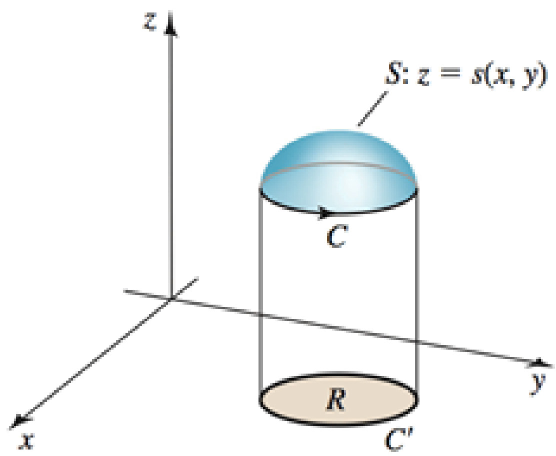


At $x = 0$, this component is zero, meaning the wheel does not spin at any point along the y-axis when its axis of the wheel is aligned with the z-axis. For $x > 0$, we see that $(\nabla \times \vec{v}) \bullet \hat{k} < 0$, which corresponds to clockwise rotation of the vector field.

For $x < 0$, we see that $(\nabla \times \vec{v}) \bullet \hat{k} > 0$, which corresponds to counterclockwise rotation.

Proof of Stokes' Theorem

Consider the case in which the surface S is the graph of the function $z = s(x, y)$, defined on a region in the xy-plane. Let C be the curve that bounds S with a counterclockwise orientation, let R be the projection of S in the xy-plane, and let C' the projection of C in the xy-plane.



C' is the projection of C in the xy-plane

Let $\vec{F} = \langle f, g, h \rangle$ the line integral in Stokes' Theorem is

$$\oint_C \vec{F} \bullet d\vec{r} = \oint_C f\ dx + g\ dy + h\ dz$$

$$= \oint_C f\ dx + g\ dy + h\left(z_x dx + z_y dy\right)$$

$$= \oint_C \underbrace{\left(f + hz_x\right)}_{M(x,y)} dx + \underbrace{\left(g + hz_y\right)}_{N(x,y)} dy$$

Where $M(x, y) = f + hz_x \quad N(x, y) = g + hz_y$

Applying Green's Theorem: $\displaystyle\oint_{C'} Mdx + Ndy = \iint_R \left(N_x - M_y\right) dA$

$M(x,y) = f + hz_x \quad \rightarrow M_y = f_y + f_z z_y + hz_{xy} + z_x\left(h_y + h_z z_y\right) \qquad \frac{df}{dy} = f_x x_y + f_y y_y + f_z z_y$

$N(x,y) = g + hz_y \quad \rightarrow N_x = g_x + g_z z_x + hz_{yx} + z_y\left(h_x + h_z z_x\right)$

$$N_x - M_y = g_x + g_z z_x + hz_{yx} + h_x z_y + h_z z_x z_y - f_y - f_z z_y - hz_{xy} - h_y z_x - h_z z_y z_x$$

$$= g_x - f_y + z_x\left(g_z - h_y\right) + z_y\left(h_x - f_z\right)$$

$$\oint_C \vec{F} \bullet d\vec{r} = \iint_R \left(g_x - f_y + z_x \left(g_z - h_y \right) + z_y \left(h_x - f_z \right) \right) dA$$

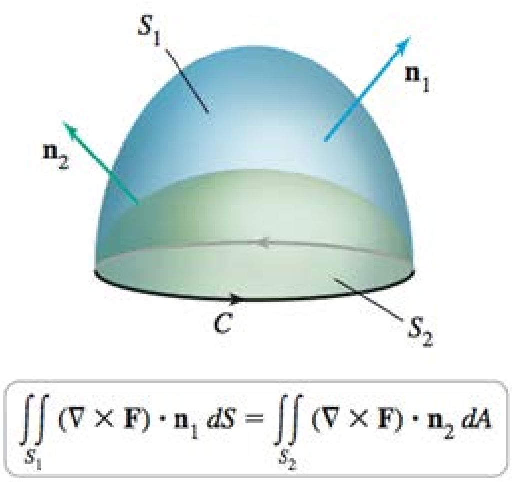
$$\iint_S \left(\nabla \times \vec{F} \right) \bullet \vec{n}\ dS = \iint_R \left(\left(h_y - g_z \right)\left(-z_x \right) + \left(f_z - h_x \right)\left(-z_y \right) + \left(g_x - f_y \right) \right) dA$$

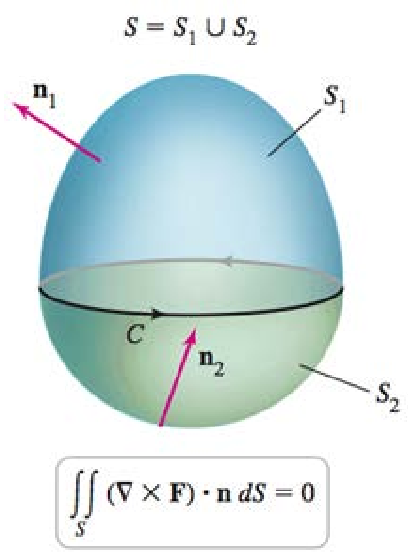
Where the upward vector normal $\vec{n} = \left\langle -z_x,\ -z_y,\ 1 \right\rangle$

Notes on Stokes' *Theorem*

1. Stokes' Theorem allows a surface integral $\iint_S \left(\nabla \times \vec{F} \right) \bullet \vec{n}\ dS$ to be evaluated using only the values of the vector field in the boundary *C*.

$$\iint_{S_1} \left(\nabla \times \boldsymbol{F} \right) \cdot \boldsymbol{n}_1\ dS = \iint_{S_2} \left(\nabla \times \boldsymbol{F} \right) \cdot \boldsymbol{n}_2\ dS$$





Since $\vec{n}_1$ and $\vec{n}_2$ are equal in magnitude and of opposite sign; therefore

$$\iint_S \left(\nabla \times \vec{F} \right) \bullet \vec{n}\ dS = \iint_{S_1} \left(\nabla \times \vec{F} \right) \bullet \vec{n}_1\ dS + \iint_{S_2} \left(\nabla \times \vec{F} \right) \bullet \vec{n}_2\ dS$$

$$\underline{= 0}\,|$$

2. If $\vec{F}$ is conservative vector field, then $\nabla \times \vec{F} = 0$.

Theorem Curl $\vec{F} = 0$ Implies $\vec{F}$ is Conservative

Suppose that $\nabla \times \vec{F} = 0$ throughout an open simply connected region *D* of $\mathbb{R}^3$. Then $\oint_C \vec{F} \bullet d\vec{r} = 0$ on all closed simple smooth curves *C* in *D* and $\vec{F}$ is a conservative vector field on *D*.

$$\oint_C \vec{F} \bullet d\vec{r} = \iint_S \underbrace{\left(\nabla \times \vec{F}\right)}_{0} \bullet \vec{n}\ dS$$

$$\underline{= 0\ }\rfloor$$

Exercises Section 4.7 – Stokes' Theorem

(**1–6**) Verify that the line integral and the surface integral of Stokes' Theorem are equal for the following vector fields, surfaces S, and closed curves C. Assume that C has counterclockwise orientation and S has a consistent orientation.

1. $\vec{F} = \langle y,\ -x,\ 10 \rangle$; S is the upper half of the sphere $x^2+y^2+z^2=1$ and C is the circle $x^2+y^2=1$ in the xy-plane

2. $\vec{F} = \langle 0,\ -x,\ y \rangle$; S is the upper half of the sphere $x^2+y^2+z^2=4$ and C is the circle $x^2+y^2=4$ in the xy-plane

3. $\vec{F} = \langle x,\ y,\ z \rangle$; S is the paraboloid $z=8-x^2-y^2$ for $0\le z\le 8$ and C is the circle $x^2+y^2=8$ in the xy-plane

4. $\vec{F} = \langle 2z,\ -4x,\ 3y \rangle$; S is the cap of the sphere $x^2+y^2+z^2=169$ above the plane $z=12$ and C is the boundary of S.

5. $\vec{F} = \langle y-z,\ z-x,\ x-y \rangle$; S is the cap of the sphere $x^2+y^2+z^2=16$ above the plane $z=\sqrt{7}$ and C is the boundary of S.

6. $\vec{F} = \langle -y,\ -x-z,\ y-x \rangle$; S is the part of the plane $z=6-y$ that lies in the cylinder $x^2+y^2=16$ and C is the boundary of S.

(**7–14**) Evaluate the line integral $\oint_C \vec{F}\bullet d\vec{r}$ by evaluating the surface integral in Stokes' Theorem with an appropriate choice of S. Assume that C has a counterclockwise orientation

7. $\vec{F} = \langle 2y,\ -z,\ x \rangle$; C is the circle $x^2+y^2=12$ in the plane $z=0$.

8. $\vec{F} = \langle y,\ xz,\ -y \rangle$; C is the ellipse $x^2+\frac{y^2}{4}=1$ in the plane $z=1$.

9. $\vec{F} = \left\langle x^2-z^2,\ y,\ 2xz \right\rangle$; C is the boundary of the plane $z=4-x-y$ in the plane first octant.

10. $\vec{F} = \left\langle y^2,\ -z^2,\ x \right\rangle$; C is the circle $\vec{r}(t)=\langle 3\cos t,\ 4\cos t,\ 5\sin t \rangle$ for $0\le t\le 2\pi$.

11. $\vec{F} = \left\langle 2xy\sin z,\ x^2 \sin z,\ x^2 y\cos z \right\rangle$; C is the boundary of the plane $z=8-2x-4y$ in the first octant.

12. $\vec{F} = \langle xz,\ yz,\ xy \rangle$; C: is the circle $x^2+y^2=4$ in the xy-plane.

13. $\vec{F} = \left\langle x^2-y^2,\ x,\ 2yz \right\rangle$; C is the boundary of the plane $z=6-2x-y$ in the first octant.

14. $\vec{F} = \left\langle x^2-y^2,\ z^2-x^2,\ y^2-z^2 \right\rangle$; C is the boundary of the square $|x|\le 1,\ \ |y|\le 1$ in the plane $z=0$

(15–20) Evaluate the line integral in Stokes' Theorem to evaluate the surface integral $\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS$. Assume that $\vec{n}$ points in an upward direction.

15. $\vec{F} = \langle x,\ y,\ z\rangle$; S is the upper half of the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = 1$

16. $\vec{F} = \langle 2y,\ -z,\ x - y - z\rangle$; S is the cap of the sphere $x^2 + y^2 + z^2 = 25$ *for* $3 \le x \le 5$

17. $\vec{F} = \langle x + y,\ y + z,\ x + z\rangle$; S is the tilted disk enclosed $\vec{r}(t) = \left\langle \cos t,\ 2\sin t,\ \sqrt{3}\cos t\right\rangle$

18. $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|}$; S is the paraboloid $x = 9 - y^2 - z^2$ *for* $0 \le x \le 9$ (excluding its base), and $\vec{r}(t) = \langle x,\ y,\ z\rangle$

19. $\vec{F} = \langle -z,\ x,\ y\rangle$, where S is the hyperboloid $z = 10 - \sqrt{1 + x^2 + y^2}$ *for* $z \ge 0$. Assume that $\vec{n}$ is the *outward normal*.

20. $\vec{F} = \left\langle x^2 - z^2,\ y^2,\ xz\right\rangle$, where S is the hemisphere $x^2 + y^2 + z^2 = 4$, for $y \ge 0$. Assume that $\vec{n}$ is the outward normal.

(21–24) Use Stokes' Theorem to find the circulation of the following vector fields around any simple closed smooth curve C.

21. $\vec{F} = \langle 2x,\ -2y,\ 2z\rangle$

22. $\vec{F} = \nabla\left(x \sin ye^z\right)$

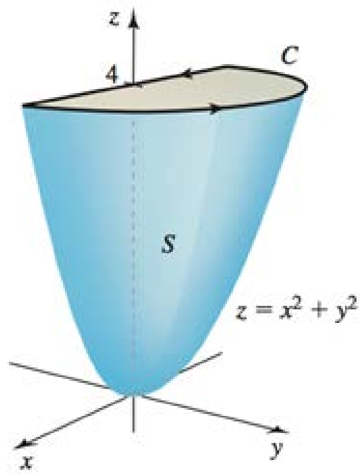
23. $\vec{F} = \left\langle 3x^2 y,\ x^3 + 2yz^2,\ 2y^2 z\right\rangle$

24. $\vec{F} = \left\langle y^2 z^3,\ 2xyz^3,\ 3xy^2 z^2\right\rangle$

25. Use Stokes' Theorem and a surface integral to find the circulation on C of the vector field $\vec{F} = \langle -y,\ x,\ 0\rangle$ as a function of ϕ. For what value of ϕ is the circulation a maximum?

26. A circle C in the plane $x + y + z = 8$ has a radius of 4 and center (2, 3, 3). Evaluate $\oint_C \vec{F} \bullet d\vec{r}$ for $\vec{F} = \langle 0,\ -z,\ 2y\rangle$ where C has a counterclockwise orientation when viewed from above. Does the circulation depend on the radius of the circle? Does it depend on the location of the center of the circle?

27. Begin with the paraboloid $z = x^2 + y^2$, for $0 \le z \le 4$, and slice it with the plane $y = 0$. Let *S* be the surface that remains for $y \ge 0$ (including the planar surface in the *xz*-plane). Let *C* be the semicircle and line segment that bound the cap of *S* in the plane $z = 4$ with counterclockwise orientation. Let $\vec{F} = \langle 2z + y,\ 2x + z,\ 2y + x \rangle$



a) Describe the direction of the vectors normal to the surface that are consistent with the orientation of *C*.

b) Evaluate $\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\, dS$

c) Evaluate $\oint_C \vec{F} \bullet d\vec{r}$ and check for argument with part (*b*).

28. The French Physicist André-Marie Ampère discovered that an electrical current *I* in a wire produces a magnetic field *B*. A special case of Ampère's Law relates the current to the magnetic field through the equation $\oint_C \boldsymbol{B} \bullet d\vec{r} = \mu I$, where *C* is any closed curve through which the wire passes and μ is a physical constant. Assume that the current *I* is given in terms of the current density ***J*** as $I = \iint_S \boldsymbol{J} \bullet \vec{n}\, dS$, where *S* is an oriented surface with *C* as a boundary. Use Stokes' Theorem to show that an equivalent form of Ampère's Law is $\nabla \times \boldsymbol{B} = \mu \boldsymbol{J}$.

29. Let S be the paraboloid $z = a\left(1 - x^2 - y^2\right)$, for $z \ge 0$, where $a > 0$ is a real number. Let $\vec{F} = \langle x - y,\ y + z,\ z - x \rangle$. For what value(s) of *a* (if any) does $\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\, dS$ have its maximum value?

30. The goal is to evaluate $A = \iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\, dS$, where $\vec{F} = \langle yz,\ -xz,\ xy \rangle$ and S ids the surface of the upper half of the ellipsoid $x^2 + y^2 + 8z^2 = 1\ \ (z \geq 0)$

a) Evaluate a surface integral over a more convenient surface to find the value of *A*.
b) Evaluate *A* using a line integral.

31. Let $\vec{F} = \langle 2z,\ z,\ x + 2y \rangle$ and let *S* be the hemisphere of radius a with its base in the *xy*-plane and center at the origin.

a) Evaluate $\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\, dS$ by computing $\nabla \times \vec{F}$ and appealing to symmetry.
b) Evaluate the line integral using Stokes' Theorem to check part (*a*).

32. Let *S* be the disk enclosed by the curve $C:\ \ \vec{r}(t) = \langle \cos\varphi \cos t,\ \sin t,\ \sin\varphi \cos t \rangle$ fot $0 \leq t \leq 2\pi$, where $0 \geq \varphi \leq \frac{\pi}{2}$ is a fixed angle.

a) Find the a vector normal to *S*.
b) What is the areas of *S*?
c) Whant the length of *C*?
d) Use the Stokes' Theorem and a surface integral to find the ciurculation on *C* of the vector field $\vec{F} = \langle -y,\ x,\ 0 \rangle$ as a function of φ. For what value of φ is the circulation a maximum?
e) What is the circulation on C of the vector field $\vec{F} = \langle -y,\ -z,\ x \rangle$ as a function of φ? For what value of φ is the circulation a maximum?
f) Consider the vector field $\vec{F} = \vec{a} \times \vec{r}$, where $\vec{a} = \langle a_1,\ a_2,\ a_3 \rangle$ is a constant nonzero vector and $\vec{r} = \langle x,\ y,\ z \rangle$. Show that the circulation is a maximum when $\vec{a}$ points in the direection of the normal to *S*.

33. Let *R* be a region in a plane that has a unit normal vector $\vec{n} = \langle a,\ b,\ c \rangle$ and boundary *C*. Let $\vec{F} = \langle bz,\ cx,\ ay \rangle$

a) Show that $\nabla \times \vec{F} = \vec{n}$
b) Use Stokes' Theorem to show that

$$\textit{Area of } R = \oint_C \vec{F} \bullet d\vec{r}$$

c) Consider the curve *C* given by $\vec{r}(t) = \langle 5\sin t,\ 13\cos t,\ 12\sin t \rangle$, for $0 \leq t \leq 2\pi$. Prove that *C* lies in a plane by showing that $\vec{r} \times \vec{r}'$ is constant for all *t*.
d) Use part (*b*) to find the area of the region enclosed by *C* in part (*c*). (*Hint*: Find the unit normal vector that is consistent with the orientation of *C*.)

34. Consider the radial vector fields $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^p}$, where p is a real number and $\vec{r} = \langle x,\ y,\ z \rangle$. Let C be any circle in the xy-plane centered at the origin.

a) Evaluate a line integral to show that the field has zero circulation on C.

b) For what values of p does Stokes' Theorem apply? For those values of p, use the surface integral in Stokes' Theorem to show that the field has zero circulation on C.

35. Consider the vector fierld $\vec{F} = \dfrac{-y}{x^2 + y^2}\hat{i} + \dfrac{x}{x^2 + y^2}\hat{j} + z\hat{k}$

a) Show that $\nabla \times \vec{F} = \vec{0}$

b) Show that $\oint_C \vec{F} \bullet d\vec{r}$ is not zero on circle C in the xy-plane enclosing the origin.

c) Explain why Stokes' Theorem does not apply in this case.

36. Let S be a small circular disk of radius R centered at the point P with a unit normal vector $\vec{n}$. Let C be the boundary of S.

a) Express the average circulation of the vector field $\vec{F}$ on S as a surface integral of $\nabla \times \vec{F}$

b) Argue for that small R, the average circulation approaches $\left(\nabla \times \vec{F}\right)\Big|_P \bullet \vec{n}$ (the component of $\nabla \times \vec{F}$ in the direction of $\vec{n}$ evaluated at P) with the approximation improving as $R \to 0$.