

## Section 4.2 – Calculus with Parametric Curves

### Tangents and Areas

A parametrized curve  $x = f(t)$  and  $y = g(t)$  is differentiable at  $t$  if  $f$  and  $g$  are differentiable at  $t$ .

### Parametric Formula for $dy/dx$

If all three derivatives exist and  $\frac{dx}{dt} \neq 0$ ,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

The derivatives  $\frac{dy}{dt}$ ,  $\frac{dx}{dt}$ , and  $\frac{dy}{dx}$  are related by the Chain Rule  $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

### Parametric Formula for $d^2y/dx^2$

If the equations  $x = f(t)$ ,  $y = g(t)$  define  $y$  as a twice-differentiable function of  $x$ , then at any point where  $\frac{dx}{dt} \neq 0$  and  $y' = \frac{dy}{dx}$ ,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

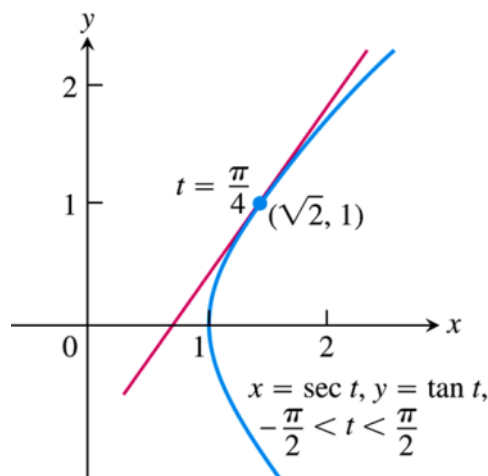
### Example

Find the tangent to the curve  $x = \sec t$ ,  $y = \tan t$ ,  $-\frac{\pi}{2} < t < \frac{\pi}{2}$ , at the point  $(\sqrt{2}, 1)$ , where  $t = \frac{\pi}{4}$

### Solution

The slope of the curve at  $t$  is:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{\sec^2 t}{\sec t \tan t} \\ &= \frac{\sec t}{\tan t} \\ m &= \left. \frac{dy}{dx} \right|_{t=\frac{\pi}{4}} \end{aligned}$$



$$= \frac{\sec \frac{\pi}{4}}{\tan \frac{\pi}{4}}$$

$$= \sqrt{2}$$

The tangent line is

$$y = \sqrt{2}(x - \sqrt{2}) + 1$$

$$y = m(x - x_1) + y_1$$

$$= \sqrt{2}x - 2 + 1$$

$$= \sqrt{2}x - 1$$

### Example

Find  $\frac{d^2y}{dx^2}$  as a function of  $t$  if  $x = t - t^2$ ,  $y = t - t^3$

### Solution

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{1 - 3t^2}{1 - 2t}$$

$$\frac{dy'}{dt} = \frac{d}{dt} \left( \frac{-3t^2 + 1}{-2t + 1} \right)$$

$$= \frac{-6t(1 - 2t) - (-2)(1 - 3t^2)}{(1 - 2t)^2}$$

$$= \frac{-6t + 12t^2 + 2 - 6t^2}{(1 - 2t)^2}$$

$$= \frac{6t^2 - 6t + 2}{(1 - 2t)^2}$$

$$u = 1 - 3t^2 \quad v = 1 - 2t$$

$$u' = -6t \quad v' = -2$$

$$\frac{d}{dx} \left( \frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{(ae - bd)x^2 + 2(af - cd)x + bf - ce}{(dx^2 + ex + f)^2}$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$= \frac{6t^2 - 6t + 2}{(1 - 2t)^2} \div (1 - 2t)$$

$$= \frac{6t^2 - 6t + 2}{(1 - 2t)^3}$$

### Example

Find the area enclosed by the asteroid:  $x = \cos^3 t$ ,  $y = \sin^3 t$ ,  $0 \leq t \leq 2\pi$

### Solution

By symmetry, the enclosed area is 4 times the area beneath the curve in the first quadrant where  $0 \leq t \leq \frac{\pi}{2}$ .

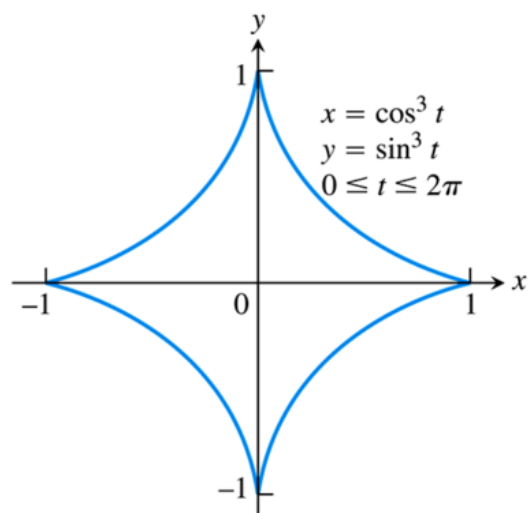
$$\begin{aligned} A &= 4 \int_0^1 |y| dx \\ &= 4 \int_0^{\pi/2} \sin^3 t \left| d(\cos^3 t) \right| & d(\cos^3 t) &= -3 \cos^2 t \sin t dt \\ &= 4 \int_0^{\pi/2} \sin^3 t \cdot 3 \cos^2 t \sin t dt \\ &= 12 \int_0^{\pi/2} \sin^4 t \cdot \cos^2 t dt \\ &= 12 \int_0^{\pi/2} \left( \frac{1 - \cos 2t}{2} \right)^2 \left( \frac{1 + \cos 2t}{2} \right) dt \\ &= \frac{3}{2} \int_0^{\pi/2} (1 - 2 \cos 2t + \cos^2 2t)(1 + \cos 2t) dt \\ &= \frac{3}{2} \int_0^{\pi/2} (1 - 2 \cos 2t + \cos^2 2t + \cos 2t - 2 \cos^2 2t + \cos^3 2t) dt \\ &= \frac{3}{2} \int_0^{\pi/2} (1 - \cos 2t - \cos^2 2t + \cos^3 2t) dt \\ &= \frac{3}{2} \left( t - \frac{1}{2} \sin 2t \right) \Big|_0^{\pi/2} - \frac{3}{2} \int_0^{\pi/2} \cos^2 2t dt + \frac{3}{2} \int_0^{\pi/2} \cos^3 2t dt & \cos^2 \alpha &= \frac{1 + \cos 2\alpha}{2} \\ &= \frac{3}{2} \left( \frac{\pi}{2} - 0 \right) - \frac{3}{2} \int_0^{\pi/2} \frac{1 + \cos 2t}{2} dt + \frac{3}{2} \int_0^{\pi/2} (1 - \sin^2 2t) \cos 2t dt \end{aligned}$$

$$= \frac{3\pi}{4} - \frac{3}{4} \left( t + \frac{1}{2} \sin 2t \right) \Big|_0^{\pi/2} + \frac{3}{4} \int_0^{\pi/2} (1 - \sin^2 2t) d(\sin 2t)$$

$$= \frac{3\pi}{4} - \frac{3}{4} \left( \frac{\pi}{2} - 0 \right) + \frac{3}{4} \left( \sin 2t - \frac{1}{3} \sin^3 2t \right) \Big|_0^{\pi/2}$$

$$= \frac{3\pi}{4} - \frac{3\pi}{8} + \frac{3}{4} (0 - 0)$$

$$= \frac{3\pi}{8} \text{ unit}^2$$

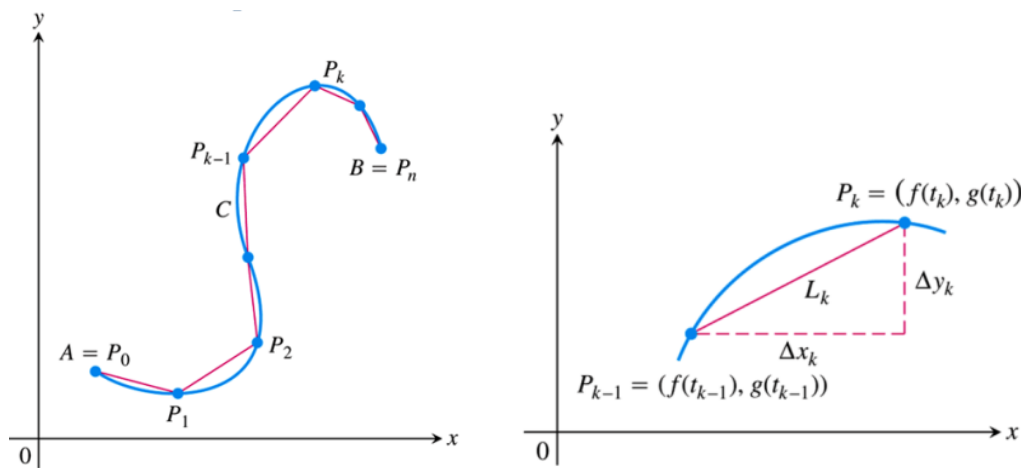


## Length of a Parametrically Defined Curve

### Definition

If a curve  $C$  is defined parametrically by  $x = f(t)$  and  $y = g(t)$ ,  $a \leq t \leq b$ , where  $f'$  and  $g'$  are continuous and not simultaneously zero on  $[a, b]$ , and  $C$  is traversed exactly once as  $t$  increases from  $t = a$  to  $t = b$ , then the length of  $C$  is the definite integral

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2} dt$$



### Example

Find the length of the circle of radius  $r$  defined parametrically by  $x = r \cos t$ ,  $y = r \sin t$ ,  $0 \leq t \leq 2\pi$

### Solution

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^{2\pi} \sqrt{(-r \sin t)^2 + (r \cos t)^2} dt \\ &= \int_0^{2\pi} \sqrt{r^2 \sin^2 t + r^2 \cos^2 t} dt \\ &= \int_0^{2\pi} \sqrt{r^2 (\sin^2 t + \cos^2 t)} dt \\ &= \int_0^{2\pi} r dt \end{aligned}$$

$$= rt \Big|_0^{2\pi}$$

$$= \underline{2\pi r \text{ unit}}$$

### Example

Find the length of the asteroid:  $x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \leq t \leq 2\pi$

### Solution

Because of the curve's symmetry with respect to the coordinate axes, its length is 4 times the length of the first quadrant.

$$\left(\frac{dx}{dt}\right)^2 = \left[3\cos^2 t(-\sin t)\right]^2$$

$$= 9\cos^4 t \sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = \left[3\sin^2 t(\cos t)\right]^2$$

$$= 9\sin^4 t \cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}$$

$$= 3|\cos t \sin t| \sqrt{\cos^2 t + \sin^2 t}$$

$$= \underline{3\cos t \sin t}$$

$$\cos^2 t + \sin^2 t = 1$$

$$\cos t \sin t \geq 0, \quad 0 \leq t \leq \frac{\pi}{2}$$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= 4 \int_0^{\pi/2} 3\cos t \sin t dt$$

$$\sin 2t = 2\cos t \sin t$$

$$= \frac{12}{2} \int_0^{\pi/2} \sin 2t dt$$

$$= -\frac{6}{2} \cos 2t \Big|_0^{\pi/2}$$

$$= -3(-1-1)$$

$$= -3(-2)$$

$$= \underline{6 \text{ unit}}$$

## Area of Surface of Revolution for Parametrized Curves

If a smooth curve  $x = f(t)$  and  $y = g(t)$ ,  $a \leq t \leq b$ , is traversed exactly once as  $t$  increases from  $a$  to  $b$ , then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

### 1. Revolution about the $x$ -axis ( $y \geq 0$ ):

$$S = 2\pi \int_a^b y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

### 2. Revolution about the $y$ -axis ( $x \geq 0$ ):

$$S = 2\pi \int_a^b x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

### Example

The standard parametrization of the circle of radius 1 centered at the point  $(0, 1)$  in the  $xy$ -plane is

$$x = \cos t, \quad y = 1 + \sin t, \quad 0 \leq t \leq 2\pi$$

Use the parametrization to find the area of the surface swept out by revolving the circle about the  $x$ -axis.

### Solution

$$x = \cos t \Rightarrow \left(\frac{dx}{dt}\right)^2 = (-\sin t)^2$$

$$y = 1 + \sin t \Rightarrow \left(\frac{dy}{dt}\right)^2 = (\cos t)^2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\sin^2 t + \cos^2 t} = 1$$

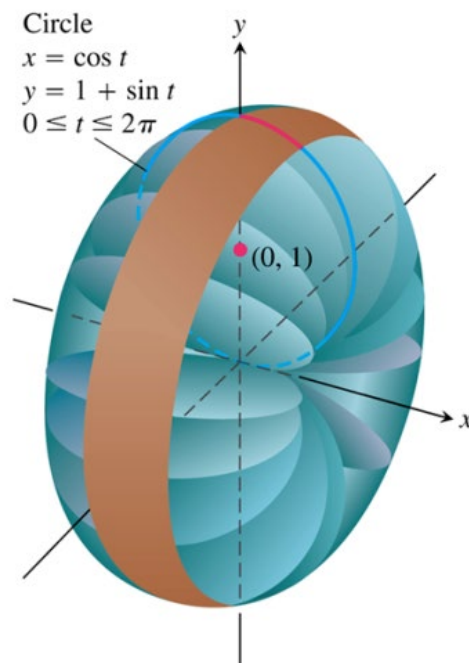
$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= \int_0^{2\pi} 2\pi(1 + \sin t) dt$$

$$= 2\pi[t - \cos t]_0^{2\pi}$$

$$= 2\pi(2\pi - 1 - (0 - 1))$$

$$= 4\pi^2 \text{ unit}^2$$



## Exercises      Section 4.2 – Calculus with Parametric Curves

(1 – 4) Find all the points at which the curve has the given slope.

1.  $x = 4 \cos t, \quad y = 4 \sin t; \quad \text{slope} = \frac{1}{2}$

3.  $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}; \quad \text{slope} = 1$

2.  $x = 2 \cos t, \quad y = 8 \sin t; \quad \text{slope} = -1$

4.  $x = 2 + \sqrt{t}, \quad y = 2 - 4t; \quad \text{slope} = -8$

(5 – 12) Find an equation of the line tangent to the curve at the point corresponding to the given value of  $t$ .

5.  $x = \sin t, \quad y = \cos t, \quad t = \frac{\pi}{4}$

9.  $x = 6t, \quad y = t^2 + 4, \quad t = 1$

6.  $x = t^2 - 1, \quad y = t^3 + t, \quad t = 2$

10.  $x = t - 2, \quad y = \frac{1}{t} + 3, \quad t = 1$

7.  $x = e^t, \quad y = \ln(t + 1), \quad t = 0$

11.  $x = t^2 - t + 2, \quad y = t^3 - 3t, \quad t = -1$

8.  $x = \cos t + t \sin t, \quad y = \sin t - t \cos t, \quad t = \frac{\pi}{4}$

12.  $x = -t^2 + 3t, \quad y = 2t^{3/2}, \quad t = \frac{1}{4}$

(13 – 33) Find the tangent to the curve at the point defined by the given value of  $t$ . Also find the value of

$$\frac{d^2y}{dx^2} \text{ at this point}$$

13.  $x = \sin 2\pi t, \quad y = \cos 2\pi t, \quad t = -\frac{1}{6}$

24.  $x = \cos \theta, \quad y = 3 \sin \theta, \quad \theta = 0$

14.  $x = \cos t, \quad y = \sqrt{3} \cos t, \quad t = \frac{2\pi}{3}$

25.  $x = 2 + \sec \theta, \quad y = 1 + 2 \tan \theta, \quad \theta = \frac{\pi}{6}$

15.  $x = t, \quad y = \sqrt{t}, \quad t = \frac{1}{4}$

26.  $x = \sqrt{t}, \quad y = \sqrt{t-1}, \quad t = 2$

16.  $x = \sec^2 t - 1, \quad y = \tan t, \quad t = -\frac{\pi}{4}$

27.  $x = \cos^3 \theta, \quad y = \sin^3 \theta, \quad \theta = \frac{\pi}{4}$

17.  $x = \frac{1}{t+1}, \quad y = \frac{t}{t-1}, \quad t = 2$

28.  $x = \theta - \sin \theta, \quad y = 1 - \cos \theta, \quad \theta = \pi$

18.  $x = t + e^t, \quad y = 1 - e^t, \quad t = 0$

29.  $x = 5t + 2, \quad y = 1 - 4t, \quad t = 3$

19.  $x = 4t, \quad y = 3t - 2, \quad t = 3$

30.  $x = t - 6, \quad y = t^2, \quad t = 5$

20.  $x = \sqrt{t}, \quad y = 3t - 1, \quad t = 1$

31.  $x = \frac{1}{t}, \quad y = 2t + 3, \quad t = -1$

21.  $x = t + 1, \quad y = t^2 + 3t, \quad t = -1$

32.  $x = \frac{1}{t}, \quad y = t^2, \quad t = -2$

22.  $x = t^2 + 5t + 4, \quad y = 4t, \quad t = 0$

33.  $x = e^t, \quad y = e^{-t}, \quad t = 1$

23.  $x = 4 \cos \theta, \quad y = 4 \sin \theta, \quad \theta = \frac{\pi}{4}$



(33 – 37) Find the equations of the tangent lines at the point where the curve crosses itself

34.  $x = 2 \sin 2t, \quad y = 3 \sin t$

36.  $x = t^2 - t, \quad y = t^3 - 3t - 1$

35.  $x = 2 - \pi \cos t, \quad y = 2t - \pi \sin t$

37.  $x = t^3 - 6t, \quad y = t^2$

(38 – 40) Find the slope of the curve  $x = f(t), y = g(t)$  at the given value of  $t$ . Define  $x$  and  $y$  as differentiable functions.

38.  $x^3 + 2t^2 = 9, \quad 2y^3 - 3t^2 = 4, \quad t = 2$

39.  $x + 2x^{3/2} = t^2 + t, \quad y\sqrt{t+1} + 2t\sqrt{y} = 4, \quad t = 0$

40.  $t = \ln(x - t), \quad y = te^t, \quad t = 0$

(41 – 37) Find  $\frac{d^2y}{dx^2}$

41.  $x(t) = t - t^2 \quad y(t) = t - t^3$

43.  $x(t) = t^2 + 1 \quad y(t) = 2t - 1$

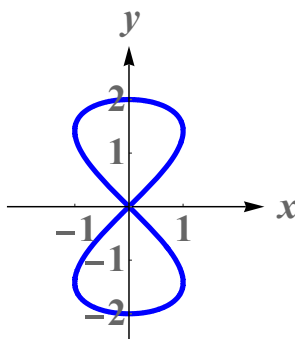
42.  $x(t) = 2 \sec t \quad y(t) = 4 \tan t + 2$

44.  $x(t) = 2t^2 - 1 \quad y(t) = 2t^3 + t$

45. Find an equation of the line tangent to cycloid  $x(t) = t - \sin t, \quad y(t) = 2 - \cos t$  at the points corresponding to  $t = \frac{\pi}{6}$  and  $t = \frac{2\pi}{3}$ .

46. Consider Lissajous curve, estimate the coordinates of the points on the curve at which there is

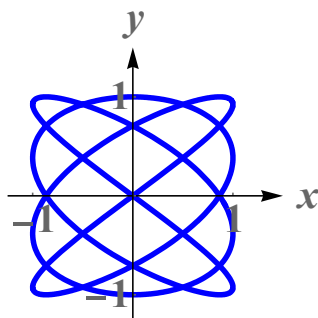
$$x = \sin 2t, \quad y = 2 \sin t; \quad 0 \leq t \leq 2\pi$$



- a) A horizontal tangent line
- b) A vertical tangent line.

47. Consider Lissajous curve, estimate the coordinates of the points on the curve at which there is

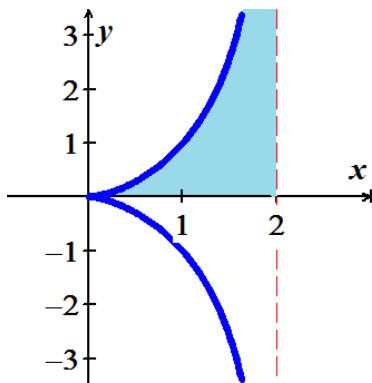
$$x = \sin 4t, \quad y = \sin 3t; \quad 0 \leq t \leq 2\pi$$



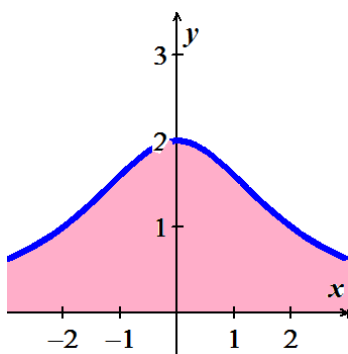
- a) A horizontal tangent line  
b) A vertical tangent line.

- (48 – 58) Find the area of the region

48.  $x = 2 \sin^2 \theta, \quad y = 2 \sin^2 \theta \tan \theta, \quad 0 \leq \theta < \frac{\pi}{2}$



49.  $x = 2 \cot \theta, \quad y = 2 \sin^2 \theta, \quad 0 \leq \theta < \pi$

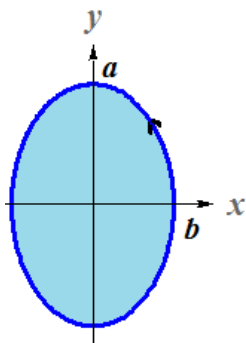


50. Find the area under one arch of the cycloid  $x = a(t - \sin t), \quad y = a(1 - \cos t)$

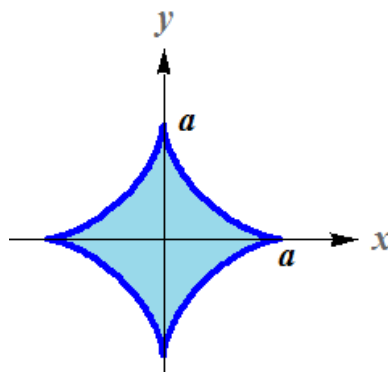
51. Find the area enclosed by the y-axis and the curve  $x = t - t^2, \quad y = 1 + e^{-t}$

52. Find the area enclosed by the ellipse  $x = a \cos t, \quad y = b \sin t, \quad 0 \leq t \leq 2\pi$

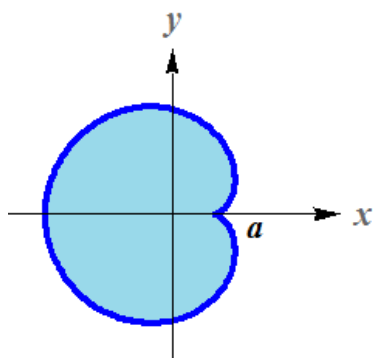
53. *Ellipse*  $\begin{cases} x = b \cos t \\ y = a \sin t \end{cases} \quad 0 \leq t \leq 2\pi$



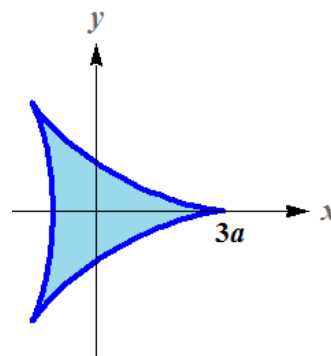
54. *Astroid*  $\begin{cases} x = a \cos^3 t \\ y = a \sin^3 t \end{cases} \quad 0 \leq t \leq 2\pi$



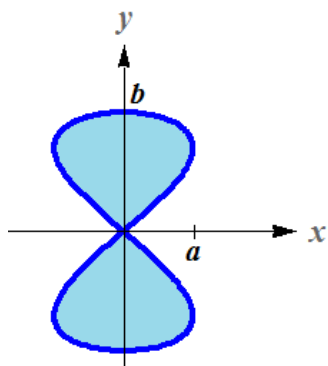
55. *Cardioid*  $\begin{cases} x = 2a \cos t - a \cos 2t \\ y = 2a \sin t - a \sin 2t \end{cases} \quad 0 \leq t \leq 2\pi$



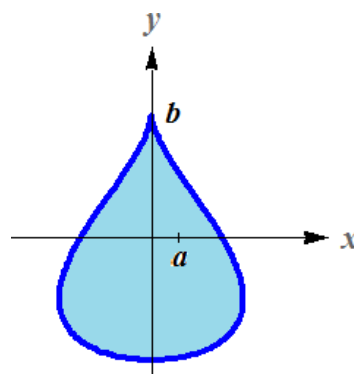
56. *Deltoid*  $\begin{cases} x = 2a \cos t + a \cos 2t \\ y = 2a \sin t - a \sin 2t \end{cases} \quad 0 \leq t \leq 2\pi$



57. *Hourglass*  $\begin{cases} x = a \sin 2t \\ y = b \sin t \end{cases} \quad 0 \leq t \leq 2\pi$



58. *Teardrop*  $\begin{cases} x = 2a \cos t - a \sin 2t \\ y = b \sin t \end{cases} \quad 0 \leq t \leq 2\pi$



(59 – 68) Find the lengths of the curves

59.  $x = \cos t, \quad y = t + \sin t, \quad 0 \leq t \leq \pi$

60.  $x = t^3, \quad y = \frac{3}{2}t^2, \quad 0 \leq t \leq \sqrt{3}$

61.  $x = 8 \cos t + 8t \sin t, \quad y = 8 \sin t - 8t \cos t, \quad 0 \leq t \leq \frac{\pi}{2}$

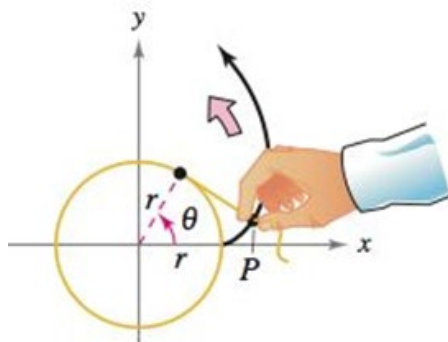
62.  $x = \ln(\sec t + \tan t) - \sin t, \quad y = \cos t, \quad 0 \leq t \leq \frac{\pi}{3}$
63. Hypocycloid perimeter curve:  $x = a \cos \theta, \quad y = a \sin \theta$
64. Circle circumference:  $x = a \cos^3 \theta, \quad y = a \sin^3 \theta$
65. Cycloid arch:  $x = a(\theta - \sin \theta), \quad y = a(1 - \cos \theta)$
66. Involute of a circle:  $x = \cos \theta + \theta \sin \theta, \quad y = \sin \theta - \theta \cos \theta$
67.  $x = t^2, \quad y = t^3, \quad 0 \leq t \leq 2$
68.  $x = 5 \sin t, \quad y = 5 \cos t, \quad -\frac{\pi}{3} \leq t \leq \frac{\pi}{2}$

(69 – 80) Find the areas of the surfaces generated by revolving the curves

69.  $x = \frac{1}{3}t^3, \quad y = t + 1, \quad 1 \leq t \leq 2, \quad y\text{-axis}$
70.  $x = \frac{2}{3}t^{3/2}, \quad y = 2\sqrt{t}, \quad 0 \leq t \leq \sqrt{3}; \quad x\text{-axis}$
71.  $x = t + \sqrt{2}, \quad y = \frac{t^2}{2} + \sqrt{2}t, \quad -\sqrt{2} \leq t \leq \sqrt{2}; \quad y\text{-axis}$
72.  $x = 2t, \quad y = 3t; \quad 0 \leq t \leq 3 \quad x\text{-axis}$
73.  $x = 2t, \quad y = 3t; \quad 0 \leq t \leq 3 \quad y\text{-axis}$
74.  $x = t, \quad y = 4 - 2t; \quad 0 \leq t \leq 2 \quad x\text{-axis}$
75.  $x = t, \quad y = 4 - 2t; \quad 0 \leq t \leq 2 \quad y\text{-axis}$
76.  $x = 5 \cos \theta, \quad y = 5 \sin \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad y\text{-axis}$
77.  $x = a \cos^3 \theta, \quad y = a \sin^3 \theta, \quad 0 \leq \theta \leq \pi, \quad x\text{-axis}$
78.  $x = a \cos \theta, \quad y = b \sin \theta, \quad 0 \leq \theta \leq 2\pi$   
     a)  $x\text{-axis}$                       b)  $y\text{-axis}$
79.  $x = 2t, \quad y = 3t, \quad 0 \leq t \leq 3$   
     a)  $x\text{-axis}$                       b)  $y\text{-axis}$
80.  $x = t, \quad y = 4 - 2t, \quad 0 \leq t \leq 2$   
     a)  $x\text{-axis}$                       b)  $y\text{-axis}$

81. Use the parametric equations  $x = t^2\sqrt{3}$  and  $y = 3t - \frac{1}{3}t^3$  to
- Graph the curve on the interval  $-3 \leq t \leq 3$ .
  - Find  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$
  - Find the equation of the tangent line at the point  $(\sqrt{3}, \frac{8}{3})$
  - Find the length of the curve
  - Find the surface area generated by revolving the curve about the  $x$ -axis
82. Use the parametric equations  $x = a(\theta - \sin \theta)$  and  $y = a(1 - \cos \theta)$   $a > 0$
- Find  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$
  - Find the equation of the tangent line at the point where  $\theta = \frac{\pi}{6}$
  - Find all points (if any) of horizontal tangency.
  - Determine where the curve is concave upward or concave downward.
  - Find the length of one arc of the curve

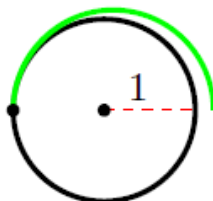
83. The involute of a circle is described by the endpoint  $P$  of a string that is held taut as it is unwound from a spool that does not turn.



Show that a parametric representation of the involute is

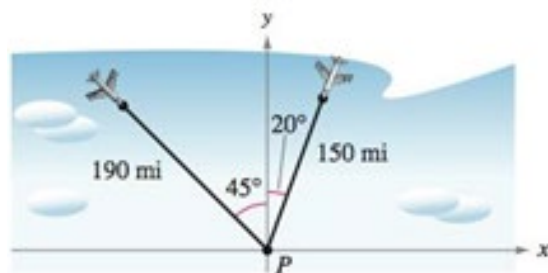
$$x = r(\cos \theta + \theta \sin \theta) \quad \text{and} \quad y = r(\sin \theta - \theta \cos \theta)$$

84. The figure shows a piece of string tied to a circle with a radius of *one* unit. The string is just long enough to reach the opposite side of the circle.



Find the area that is covered when the string is unwound counterclockwise.

85. An Air traffic controller spots two planes at the same altitude flying toward each other.



Their flight paths are  $20^\circ$  and  $315^\circ$ . One plane is 150 miles from point  $P$  with a speed of 375 miles per hour. The other is 190 miles from point  $P$  with a speed of 450 miles per hour.

- Find parametric equations for the path of each plane where  $t$  is the time in hours, with  $t = 0$  corresponding to the time at which the air traffic controller spots the planes.
- Use part (a) to write the distance between the planes as a function of  $t$ .
- Graph the function in part (b).
- When the distance between the planes be minimum?
- If the planes must keep a separation of at least 3 miles, is the requirement met?