

# Lecture R – Introduction to Differential Equation

## **Solution**      **Section R.1 – Derivative**

### **Exercise**

Find the derivative of  $f(t) = -3t^2 + 2t - 4$

### **Solution**

$$f'(t) = \underline{-6t + 2}$$

### **Exercise**

Find the derivative of  $g(x) = 4\sqrt[3]{x} + 2$

### **Solution**

$$g(x) = 4x^{1/3} + 2$$

$$g'(x) = \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3x^{2/3}}$$

$$= \underline{\frac{4}{3\sqrt[3]{x^2}}}$$

### **Exercise**

Find the derivative of  $f(x) = x(x^2 + 1)$

### **Solution**

$$f(x) = x^3 + x$$

$$f'(x) = \underline{3x^2 + 1}$$

### **Exercise**

Find the derivative of  $f(x) = \frac{2x^2 - 3x + 1}{x}$

### **Solution**

$$f(x) = \frac{2x^2}{x} - \frac{3x}{x} + \frac{1}{x}$$

$$= 2x - 3 + \frac{1}{x}$$

$$f'(x) = \underline{2 - \frac{1}{x^2}}$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

### Exercise

Find the derivative of  $f(x) = \frac{4x^3 - 3x^2 + 2x + 5}{x^2}$

### Solution

$$f(x) = 4x - 3 + \frac{2}{x} + 5x^{-2} \qquad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$\begin{aligned} f'(x) &= 4 - \frac{2}{x^2} - 10x^{-3} \\ &= 4 - \frac{2}{x^2} - \frac{10}{x^3} \end{aligned}$$

### Exercise

Find the derivative of  $f(x) = \frac{-6x^3 + 3x^2 - 2x + 1}{x}$

### Solution

$$f(x) = -6x^2 + 3x - 2 + \frac{1}{x} \qquad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$f'(x) = -12x + 3 - \frac{1}{x^2}$$

### Exercise

Find the derivative of  $f(x) = x\left(1 - \frac{2}{x+1}\right)$

### Solution

$$f(x) = x - \frac{2x}{x+1}$$

$$\left(\frac{2x}{x+1}\right)' \Rightarrow \begin{array}{ll} f = 2x & f' = 2 \\ g = x+1 & g' = 1 \end{array}$$

$$\begin{aligned} f'(x) &= 1 - \frac{2(x+1) - 2x}{(x+1)^2} \\ &= 1 - \frac{2x + 2 - 2x}{(x+1)^2} \\ &= 1 - \frac{2}{(x+1)^2} \end{aligned}$$

### Exercise

Find the derivative of  $g(s) = \frac{s^2 - 2s + 5}{\sqrt{s}}$

### Solution

$$\begin{aligned}g(s) &= \frac{s^2}{s^{1/2}} - 2\frac{s}{s^{1/2}} + \frac{5}{s^{1/2}} \\&= s^{3/2} - 2s^{1/2} + 5s^{-1/2} \\g'(s) &= \frac{3}{2}s^{1/2} - 2\frac{1}{2}s^{-1/2} + 5\left(-\frac{1}{2}\right)s^{-3/2} \\&= \frac{3}{2}s^{1/2} - s^{-1/2} - \frac{5}{2}s^{-3/2} \\&= \frac{3}{2}\sqrt{s} - \frac{1}{\sqrt{s}} - \frac{5}{2s^{3/2}} \\&= \frac{\frac{3}{2}\sqrt{s} - \frac{1}{\sqrt{s}} - \frac{5}{2s\sqrt{s}}}{\quad}\end{aligned}$$

### Exercise

Find the derivative of  $f(x) = \frac{x+1}{\sqrt{x}}$

### Solution

$$\begin{aligned}f(x) &= \frac{x}{x^{1/2}} + \frac{1}{x^{1/2}} \\&= x^{1/2} + x^{-1/2} \\f'(x) &= \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} \\&= \frac{\frac{1}{2x^{1/2}} - \frac{1}{2x^{3/2}}}{\quad}\end{aligned}$$

### Exercise

Find the derivative to the following functions  $y = 3x(2x^2 + 5x)$

### Solution

$$\begin{aligned}y &= 6x^3 + 15x^2 \\&\Rightarrow \underline{y' = 18x^2 + 30x}\end{aligned}$$

**Exercise**

Find the derivative to the following functions  $y = 3(2x^2 + 5x)$

**Solution**

$$y = 6x^2 + 15x$$

$$\Rightarrow \boxed{y' = 12x + 15}$$

**Exercise**

Find the derivative to the following functions  $y = \frac{x^2 + 4x}{5}$

**Solution**

$$\boxed{y' = \frac{1}{5}(2x + 4)}$$

**Exercise**

Find the derivative to the following functions  $y = \frac{3x^4}{5}$

**Solution**

$$\boxed{y' = \frac{12}{5}x^3}$$

**Exercise**

Find the derivative to the following functions  $y = \frac{x^2 - 4}{2x + 5}$

**Solution**

$$\begin{aligned} y' &= \frac{(2x+5)(2x) - (x^2-4)(2)}{(2x+5)^2} \\ &= \frac{4x^2 + 10x - 2x^2 + 8}{(2x+5)^2} \\ &= \boxed{\frac{2x^2 + 10x + 8}{(2x+5)^2}} \end{aligned}$$

**Exercise**

Find the derivative to the following functions  $y = \frac{(1+x)(2x-1)}{x-1}$

**Solution**

$$\begin{aligned}
y' &= \frac{(x-1) \frac{d}{dx}[(1+x)(2x-1)] - (1+x)(2x-1) \frac{d}{dx}[x-1]}{(x-1)^2} \\
&= \frac{(x-1)[(1)(2x-1) + 2(1+x)] - (1+x)(2x-1)(1)}{(x-1)^2} \\
&= \frac{(x-1)(2x-1+2+2x) - (2x-1+2x^2-x)}{(x-1)^2} \\
&= \frac{(x-1)(4x+1) - 2x+1-2x^2+x}{(x-1)^2} \\
&= \frac{4x^2+x-4x-1-2x+1-2x^2+x}{(x-1)^2} \\
&= \frac{2x^2-4x}{(x-1)^2}
\end{aligned}$$

*Or*

$$\begin{aligned}
y &= \frac{(1+x)(2x-1)}{x-1} \\
&= \frac{2x-1+2x^2-x}{x-1} \\
&= \frac{2x^2+x-1}{x-1} \\
y' &= \frac{(x-1)(4x+1) - (2x^2+x-1)(1)}{(x-1)^2} \\
&= \frac{4x^2+x-4x-1-2x^2-x+1}{(x-1)^2} \\
&= \frac{2x^2-4x}{(x-1)^2}
\end{aligned}$$

### **Exercise**

Find the derivative to the following functions  $y = \frac{4}{2x+1}$

**Solution**

$$y = 4(2x+1)^{-1}$$

$$y' = -4(2x+1)^{-2}(2)$$

$$= -8(2x+1)^{-2}$$

$$= -\frac{8}{(2x+1)^2}$$

### ***Exercise***

Find the derivative to the following functions  $y = \frac{2}{(x-1)^3} = 2(x-1)^{-3}$

#### **Solution**

$$y = 2(x-1)^{-3}$$

$$y' = 2(-3)(x-1)^{-4}(1)$$

$$= -\frac{6}{(x-1)^4}$$

### ***Exercise***

Find the derivative to the following functions  $y = \sqrt[3]{(x+4)^2}$

#### **Solution**

$$y = (x+4)^{2/3}$$

$$y' = \frac{2}{3}(x+4)^{-1/3}$$

$$= \frac{2}{3} \frac{1}{(x+4)^{1/3}}$$

$$= \frac{2}{3 \sqrt[3]{x+4}}$$

### ***Exercise***

Find the derivative of  $f(x) = \sqrt{2t^2 + 5t + 2}$

#### **Solution**

$$f(t) = (2t^2 + 5t + 2)^{1/2}$$

$$f'(t) = \frac{1}{2}(4t+5)(2t^2 + 5t + 2)^{-1/2}$$

$$= \frac{1}{2} \frac{4t+5}{\sqrt{2t^2 + 5t + 2}}$$

$$U = 2t^2 + 5t + 2 \rightarrow U' = 4t + 5$$

$$(U^n)' = nU'U^{n-1}$$

**Exercise**

Find the derivative of  $f(x) = \frac{1}{(x^2 - 3x)^2}$

**Solution**

$$\begin{aligned}
 f(x) &= (x^2 - 3x)^{-2} \\
 f'(x) &= -2(2x - 3)(x^2 - 3x)^{-3} \\
 &= -\frac{2(2x - 3)}{(x^2 - 3x)^3}
 \end{aligned}$$

**Exercise**

Find the derivative of  $y = t^2 \sqrt{t - 2}$

**Solution**

$$\begin{aligned}
 y' &= 2t\sqrt{t-2} + t^2 \frac{1}{2}(t-2)^{-1/2} & \begin{aligned} f &= t^2 & f' &= 2t \\ g &= (t-2)^{1/2} & g' &= \frac{1}{2}(t-2)^{-1/2} \end{aligned} \\
 &= \left[ 2t(t-2)^{1/2} + t^2 \frac{1}{2}(t-2)^{-1/2} \right] \frac{2(t-2)^{1/2}}{2(t-2)^{1/2}} \\
 &= \frac{4t(t-2) + t^2}{2(t-2)^{1/2}} \\
 &= \frac{4t^2 - 8t + t^2}{2\sqrt{t-2}} \\
 &= \frac{5t^2 - 8t}{2\sqrt{t-2}}
 \end{aligned}$$

**Exercise**

Find the derivative of  $y = \left( \frac{6-5x}{x^2-1} \right)^2$

**Solution**

$$\begin{aligned}
 f &= 6 - 5x & f' &= -5 \\
 g &= x^2 - 1 & g' &= 2x
 \end{aligned}$$

$$\begin{aligned}
 y' &= 2 \frac{-5(x^2 - 1) - 2x(6 - 5x) \left( \frac{6 - 5x}{x^2 - 1} \right)}{(x^2 - 1)^2} \\
 &= 2 \frac{-5x^2 + 5 - 12x + 10x^2}{(x^2 - 1)^3} (6 - 5x) \\
 &= \frac{2(5x^2 - 12x + 5)(6 - 5x)}{(x^2 - 1)^3}
 \end{aligned}
 \qquad
 \left( U^n \right)' = nU'U^{n-1}$$

### Exercise

Find the derivative to the following functions  $y = x^2 \sqrt{x^2 + 1}$

### Solution

$$\begin{aligned}
 y &= x^2 (x^2 + 1)^{1/2} \\
 y' &= x^2 \frac{d}{dx} \left[ (x^2 + 1)^{1/2} \right] + (x^2 + 1)^{1/2} \frac{d}{dx} \left[ x^2 \right] \\
 &= x^2 \left[ \frac{1}{2} (x^2 + 1)^{-1/2} (2x) \right] + (x^2 + 1)^{1/2} [2x] \\
 &= x^3 (x^2 + 1)^{-1/2} + 2x (x^2 + 1)^{1/2} \\
 &= \frac{(x^2 + 1)^{1/2}}{(x^2 + 1)^{1/2}} \left[ x^3 (x^2 + 1)^{-1/2} + 2x (x^2 + 1)^{1/2} \right] \\
 &= \frac{x^3 (x^2 + 1)^{-1/2} (x^2 + 1)^{1/2} + 2x (x^2 + 1)^{1/2} (x^2 + 1)^{1/2}}{(x^2 + 1)^{1/2}} \\
 &= \frac{x^3 + 2x(x^2 + 1)}{(x^2 + 1)^{1/2}} \\
 &= \frac{x^3 + 2x^3 + 2x}{\sqrt{x^2 + 1}} \\
 &= \frac{3x^3 + 2x}{\sqrt{x^2 + 1}} \\
 &= \frac{x(3x^2 + 2)}{\sqrt{x^2 + 1}}
 \end{aligned}$$



### Exercise

Find the derivative to the following functions  $y = \left(\frac{x+1}{x-5}\right)^2$

#### Solution

$$\begin{aligned}y' &= 2\left(\frac{x+1}{x-5}\right) \frac{d}{dx} \left[ \frac{x+1}{x-5} \right] \\&= 2\left(\frac{x+1}{x-5}\right) \left[ \frac{(1)(x-5) - (1)(x+1)}{(x-5)^2} \right] \\&= 2\left(\frac{x+1}{x-5}\right) \left( \frac{x-5-x-1}{(x-5)^2} \right) \\&= 2\left(\frac{x+1}{x-5}\right) \left( \frac{-6}{(x-5)^2} \right) \\&= -\frac{12(x+1)}{(x-5)^3}\end{aligned}$$

### Exercise

Find the derivative to the following functions  $y = x^2 \sin x$

#### Solution

$$y' = \underline{2x \sin x + x^2 \cos x} \quad \begin{array}{ll} u = x^2 & v = \sin x \\ u' = 2x & v' = \cos x \end{array}$$

### Exercise

Find the derivative to the following functions  $y = \frac{\sin x}{x}$

#### Solution

$$y' = \underline{\frac{x \cos x - \sin x}{x^2}} \quad \begin{array}{ll} u = \sin x & v = x \\ u' = \cos x & v' = 1 \end{array}$$

### Exercise

Find the derivative to the following functions  $y = \frac{\cot x}{1 + \cot x}$

#### Solution

$$y' = \frac{-\csc^2 x (1 + \cot x) + \csc^2 x \cot x}{(1 + \cot x)^2} \quad \begin{array}{ll} u = \cot x & v = 1 + \cot x \\ u' = -\csc^2 x & v' = -\csc^2 x \end{array}$$

$$= \frac{-\csc^2 x - \csc^2 x \cot x + \csc^2 x \cot x}{(1 + \cot x)^2}$$

$$= \frac{-\csc^2 x}{(1 + \cot x)^2}$$

### Exercise

Find the derivative to the following functions  $y = x^2 \sin x + 2x \cos x - 2 \sin x$

#### Solution

$$y' = 2x \sin x + x^2 \cos x + 2 \cos x - 2x \sin x - 2 \cos x$$

$$= x^2 \cos x$$

### Exercise

Find the derivative to the following functions  $y = x^3 \sin x \cos x$

#### Solution

$$y' = (x^3)' \sin x \cos x + x^3 (\sin x)' \cos x + x^3 \sin x (\cos x)'$$

$$= 3x^2 \sin x \cos x + x^3 \cos^2 x - x^3 \sin^2 x$$

### Exercise

Find the derivative to the following functions  $y = \frac{4}{\cos x} + \frac{1}{\tan x}$

#### Solution

$$y' = \frac{-4 \sin x}{\cos^2 x} - \frac{\sec^2 x}{\tan^2 x} \quad \left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$$

$$= -4 \frac{\sin x}{\cos x} \frac{1}{\cos x} - \frac{1}{\cos^2 x} \frac{\cos^2 x}{\sin^2 x}$$

$$= -4 \tan x \sec x - \csc^2 x$$

### Exercise

Find the derivative to the following functions  $f(x) = x^2 e^x$

#### Solution

$$f'(x) = e^x \frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(e^x)$$

$$\begin{aligned}
 &= e^x(2x) + x^2 e^x \\
 &= \underline{xe^x(2+x)}
 \end{aligned}$$

### Exercise

Find the derivative to the following functions  $f(x) = \frac{e^x + e^{-x}}{2}$

#### Solution

$$\begin{aligned}
 f(x) &= \frac{e^x + e^{-x}}{2} \\
 &= \frac{1}{2}(e^x + e^{-x}) \\
 f'(x) &= \frac{1}{2} \left( \frac{d}{dx}[e^x] + \frac{d}{dx}[e^{-x}] \right) \\
 &= \underline{\frac{1}{2}(e^x - e^{-x})}
 \end{aligned}$$

### Exercise

Find the derivative to the following functions  $f(x) = \frac{e^x}{x^2}$

#### Solution

$$\begin{aligned}
 f'(x) &= \frac{x^2 e^x - e^x(2x)}{x^4} \\
 &= \frac{x^2 e^x - 2x e^x}{x^4} \\
 &= \frac{x e^x (x-2)}{x^4} \\
 &= \underline{\frac{e^x (x-2)}{x^3}}
 \end{aligned}$$

### Exercise

Find the derivative to the following functions  $f(x) = x^2 e^x - e^x$

#### Solution

$$\begin{aligned}
 f'(x) &= e^x \frac{d}{dx}[x^2] + x^2 \frac{d}{dx}[e^x] - \frac{d}{dx}[e^x] \\
 &= e^x(2x) + x^2 e^x - e^x \\
 &= \underline{e^x(x^2 + 2x - 1)}
 \end{aligned}$$

### Exercise

Find the derivative to the following functions  $f(x) = (1 + 2x)e^{4x}$

#### Solution

$$\begin{aligned}f'(x) &= (2)e^{4x} + (1 + 2x)(4e^{4x}) \\&= 2e^{4x} + (1 + 2x)(4e^{4x}) \\&= 2e^{4x}(1 + 2(1 + 2x)) \\&= 2e^{4x}(1 + 2 + 4x) \\&= \underline{2e^{4x}(3 + 4x)}\end{aligned}$$

### Exercise

Find the derivative to the following functions  $y = x^2e^{5x}$

#### Solution

$$\begin{aligned}y' &= x^2(5e^{5x}) + 2x(e^{5x}) \\&= \underline{xe^{5x}(5x + 2)}\end{aligned}$$

### Exercise

Find the derivative to the following functions  $y = e^{x^2+1}\sqrt{5x+2}$

#### Solution

$$\begin{aligned}y &= (2x)e^{x^2+1}\sqrt{5x+2} + e^{x^2+1}\frac{5}{2\sqrt{5x+2}} \\&= 2xe^{x^2+1}\sqrt{5x+2}\frac{2\sqrt{5x+2}}{2\sqrt{5x+2}} + \frac{5e^{x^2+1}}{2\sqrt{5x+2}} \\&= \frac{4xe^{x^2+1}(5x+2)}{2\sqrt{5x+2}} + \frac{5e^{x^2+1}}{2\sqrt{5x+2}} \\&= \frac{20x^2e^{x^2+1} + 8xe^{x^2+1} + 5e^{x^2+1}}{2\sqrt{5x+2}} \\&= \underline{\frac{e^{x^2+1}(20x^2 + 8x + 5)}{2\sqrt{5x+2}}}\end{aligned}$$

### Exercise

Find the derivative to the following functions  $f(x) = \ln \sqrt[3]{x+1}$

### Solution

$$f(x) = \ln(x+1)^{1/3}$$

$$= \frac{1}{3} \ln(x+1)$$

$$u = x+1 \Rightarrow \frac{du}{dx} = 1$$

$$f'(x) = \frac{1}{3} \frac{1}{x+1}$$
$$= \frac{1}{3(x+1)}$$

### Exercise

Find the derivative to the following functions  $f(x) = \ln \left[ x^2 \sqrt{x^2+1} \right]$

### Solution

$$f(x) = \ln(x^2) + \ln \sqrt{x^2+1} \quad \text{Product Property}$$

$$f(x) = \ln(x^2) + \ln(x^2+1)^{1/2}$$

$$f(x) = 2 \ln x + \frac{1}{2} \ln(x^2+1) \quad \text{Power Property}$$

$$f'(x) = 2 \frac{1}{x} + \frac{1}{2} \frac{2x}{x^2+1} \quad \text{Differentiate}$$
$$= \frac{2}{x} + \frac{x}{x^2+1}$$

### Exercise

Find the derivative to the following functions  $y = \ln \frac{x^2}{x^2+1}$

### Solution

$$y = \ln x^2 - \ln x^2 + 1$$

$$y' = \frac{2x}{x^2} - \frac{2x}{x^2+1}$$
$$= \frac{2}{x} - \frac{2x}{x^2+1}$$

### Exercise

Find the derivative to the following functions  $y = \ln \frac{1+e^x}{1-e^x}$

### Solution

$$y = \ln(1+e^x) - \ln(1-e^x)$$

$$\begin{aligned} y' &= \frac{e^x}{1+e^x} - \frac{-e^x}{1-e^x} \\ &= \frac{e^x}{1+e^x} + \frac{e^x}{1-e^x} \\ &= \frac{e^x - e^{2x} + e^x + e^{2x}}{(1+e^x)(1-e^x)} \\ &= \frac{2e^x}{(1+e^x)(1-e^x)} \end{aligned}$$

### Exercise

Find the derivative to the following functions  $y = x 3^{x+1}$

### Solution

$$\begin{aligned} y' &= 3^{x+1} + x 3^{x+1} \ln 3 \\ &= 3^{x+1}(1+x \ln 3) \end{aligned}$$

### Exercise

Find the derivative to the following functions  $f(t) = \frac{\log_8(t^{3/2}+1)}{t}$

### Solution

$$\begin{aligned} f' &= \frac{\frac{1}{\ln 8} \frac{3}{2} t^{1/2}}{t^{3/2}+1} \cdot t - \log_8(t^{3/2}+1) \\ &= \frac{\frac{1}{\ln 8} \frac{3}{2} t^{3/2} - \log_8(t^{3/2}+1)}{t^2} \cdot \frac{2 \ln 8(t^{3/2}+1)}{2 \ln 8(t^{3/2}+1)} \\ &= \frac{3t^{3/2} - 2(t^{3/2}+1)(\ln 8) \log_8(t^{3/2}+1)}{t^2(t^{3/2}+1) \ln 8} \end{aligned}$$