

Section 1.3 – The Algebra of Matrices

Matrices

$$\begin{array}{ccc} & \text{Column} & \\ & C_1 & C_2 & C_3 \\ & \downarrow & \downarrow & \downarrow \\ \text{Row 1} \rightarrow R_1 & \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \end{array} \right] \\ \text{Row 2} \rightarrow R_2 & \left[\begin{array}{ccc} a_{21} & a_{22} & a_{23} \end{array} \right] \\ \text{Row 3} \rightarrow R_3 & \left[\begin{array}{ccc} a_{31} & a_{32} & a_{33} \end{array} \right] \end{array}$$

This is called Matrix (*Matrices*)

Each number in the array is an *element* or *entry*

The matrix is said to be of order $m \times n$

m : numbers of rows,

n : number of columns

When $m = n$, then matrix is said to be *square*.

Given the system equations

$$3x + y + 2z = 31$$

$$x + y + 2z = 19$$

$$x + 3y + 2z = 25$$

Write into an *augmented matrix* form

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 31 \\ 1 & 1 & 2 & 19 \\ 1 & 3 & 2 & 25 \end{array} \right]$$

The Matrix: $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix}$ is called the *coefficient matrix* of the system.

The matrix A above has 3 rows and 3 columns, therefore the order of the matrix A is (3×3)

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Equality of Matrices

Definition of Equality of Matrices

Two matrices **A** and **B** are equal if and only if they have the same order (size) $m \times n$ and if each pair corresponding elements is equal

$$a_{ij} = b_{ij} \text{ for } i = 1, 2, \dots, m \text{ and } j = 1, 2, \dots, n$$

Example

Find the values of the variables for which each statement is true, if possible.

$$a) \begin{bmatrix} 2 & 1 \\ p & q \end{bmatrix} = \begin{bmatrix} x & y \\ -1 & 0 \end{bmatrix}$$

$$x = 2, y = 1, p = -1, q = 0$$

$$b) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$$

can't be true

$$c) \begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} w = 9 & x = 17 \\ 8 = y & -12 = z \end{bmatrix}$$

Addition and Subtraction of Matrices

Definition

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices, their sum $A + B$, is the $m \times n$ matrix obtained by adding the corresponding entries; that is

$$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

Matrices can be added if their shapes are the same, meaning have the same **order**.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 4 & 4 \\ 9 & 9 \end{bmatrix} = \begin{bmatrix} 1+2 & 2+2 \\ 3+4 & 4+4 \\ 0+9 & 0+9 \end{bmatrix} \\ = \begin{bmatrix} 3 & 4 \\ 7 & 8 \\ 9 & 9 \end{bmatrix}$$

Scalar Multiplication Matrices

Definition

If k is a scalar and $A = [a_{ij}]$ is an $m \times n$ matrices, then scalar product kA is the $m \times n$ matrix obtained by multiplying each entry of A by k ; that is

$$k[a_{ij}] = [ka_{ij}]$$

$$kA = k \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}$$

Example

$$2 \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (2)1 & (2)2 \\ (2)3 & (2)4 \\ (2)0 & (2)0 \end{bmatrix} \\ = \begin{bmatrix} 2 & 4 \\ 6 & 8 \\ 0 & 0 \end{bmatrix}$$

Definition

If A_1, A_2, \dots, A_n are matrices of the same size, and if c_1, c_2, \dots, c_n are scalars, then expression of the form

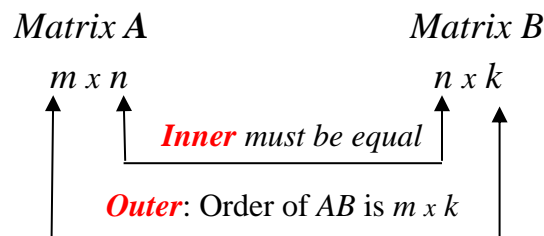
$$c_1 A_1 + c_2 A_2 + \dots + c_n A_n$$

Is called a **linear combination** of A_1, A_2, \dots, A_n with *coefficients* c_1, c_2, \dots, c_n .

Matrix Multiplication

Product of Two Matrices

Let A be an $m \times n$ matrix and let B be an $n \times k$ matrix. To find the element in the i^{th} row and j^{th} column of the product matrix AB , multiply each element in the i^{th} row of A by the corresponding element in the j^{th} column of B , and then add these products. The product matrix AB is an $m \times k$ matrix.



- ✓ To multiply AB or dot product, if A has **n** columns, B must have **n** rows.
- ✓ Squares matrices can be multiplied if and only if (*iff*) they have the same size.
- ✓ The entry in row i and column j of AB is $(\text{row } i \text{ of } A) \cdot (\text{col } j \text{ of } B)$

The result: $\sum a_{ik} b_{kj}$

$$\begin{matrix}
 \begin{bmatrix} * & * & & & \\ a_{i1} & a_{i2} & \cdots & \cdots & a_{i5} \\ * & & & & \\ * & & & & \end{bmatrix} &
 \begin{bmatrix} * & * & b_{1j} & * & * & * \\ & b_{2j} & & & & \\ & \vdots & & & & \\ & \vdots & & & & \\ & b_{5j} & & & & \end{bmatrix} &
 = &
 \begin{bmatrix} & * & & & & \\ * & * & (AB)_{ij} & * & * & * \\ & * & & & & \\ & * & & & & \end{bmatrix} \\
 \text{4 by 5} & \text{5 by 6} & & \text{4 by 6}
 \end{matrix}$$

$$\mathbf{AB} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$\begin{matrix} 2 \times 2 & & 2 \times 2 & \rightarrow & 2 \times 2 \end{matrix}$

$$a_{11} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & - \\ - & - \end{bmatrix}$$

$$a_{12} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} - & af+bh \\ - & - \end{bmatrix}$$

$$a_{21} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} - & - \\ ce+dg & - \end{bmatrix}$$

$$a_{22} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} - & - \\ - & cf+dh \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Example

Find: $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1(5)+1(1) & 1(6)+1(0) \\ 2(5)-1(1) & 2(6)-1(0) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 6 \\ 9 & 12 \end{bmatrix}$$

Special Case

When A is a square matrix, then

$$A \quad \text{times} \quad A = A^2 \quad \text{times} \quad A = \underline{A^3}$$

$$A^p = AA \cdots A \quad (p \text{ factors})$$

$$(A^p)(A^q) = A^{p+q}$$

$$(A^p)^q = A^{pq}$$

Block Multiplication

If the cuts between columns of \mathbf{A} match the cuts between rows of \mathbf{B} , then the block multiplication of \mathbf{AB} allowed.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \left[\begin{array}{c|c} a_{11}b_{11} + a_{12}b_{21} & a_{12}b_{21} + a_{12}b_{22} \\ \hline a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{21} + a_{22}b_{22} \end{array} \right]$$

Important special case

$$\begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 5 & 0 \end{bmatrix}$$

Matrix Form of the Equations

The coefficient matrix is $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}$

The equivalent matrix equation is in the form $AX = b$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Multiplication by **rows** $AX = \begin{bmatrix} (\text{row 1}).X \\ (\text{row 2}).X \\ (\text{row 3}).X \end{bmatrix}$

Multiplication by **columns** $AX = x (\text{column 1}) + y (\text{column 2}) + z (\text{column 3})$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Identity Matrix

The identity matrix is given by the form: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \boxed{Ix = x}$

Properties of Matrix

Addition and Scalar Multiplication

$A + B = B + A$	<i>Commutative Property of Addition</i>
$A + (B + C) = (A + B) + C$	<i>Associative Property of Addition</i>
$(kl)A = k(lA)$	<i>Associative Property of Scalar Multiplication</i>
$k(A + B) = kA + kB$	<i>Distributive Property</i>
$k(A - B) = kA - kB$	<i>Distributive Property</i>
$(k + l)A = kA + lA$	<i>Distributive Property</i>
$(k - l)A = kA - lA$	<i>Distributive Property</i>
$A + 0 = 0 + A = A$	<i>Additive Identity Property</i>
$A + (-A) = (-A) + A = 0$	<i>Additive Inverse Property</i>
$k(AB) = kA(B) = A(kB)$	

Multiplication

$AB \neq BA$	<i>Commutative “law” is usually broken</i>
$A(BC) = (AB)C$	<i>Associative Property of Multiplication (Parentheses not needed)</i>
$A(B + C) = AB + AC$	<i>Distributive Property</i>
$(B + C)A = BA + CA$	<i>Distributive Property</i>
$A(B - C) = AB - AC$	<i>Distributive Property</i>
$(B - C)A = BA - CA$	<i>Distributive Property</i>

Consider the three vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} :

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The linear combinations in three-dimensional space are $c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$

$$\textbf{Combination} \quad c \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ d - c \\ e - d \end{bmatrix}$$

Combine the three vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} into on matrix A .

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Multiplies the matrix A by a vector \mathbf{x} , where c , d , e are the component of a vector \mathbf{x} .

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} c \\ d - c \\ e - d \end{bmatrix}$$

We can rewrite the form, matrix A times the vector \mathbf{x} , as the combination $c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$

$$A\mathbf{x} = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$$

Write the matrix in the form $A\mathbf{x} = \mathbf{b}$

$$\textcolor{blue}{A}\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} c \\ d - c \\ e - d \end{bmatrix} = \begin{bmatrix} \textcolor{red}{b}_1 \\ \textcolor{red}{b}_2 \\ \textcolor{red}{b}_3 \end{bmatrix} = \textcolor{blue}{b}$$

Where the \mathbf{x} is the input and \mathbf{b} is the output.

Cyclic Difference

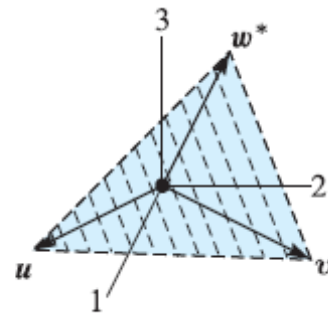
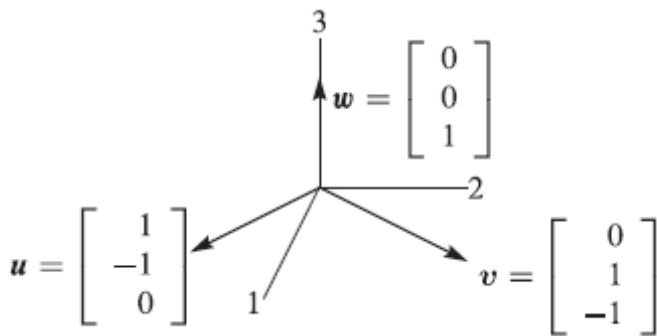
The linear combinations of three vectors u , v , and w^* lead to a cyclic difference matrix C and is given by:

$$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad w^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = b$$

The matrix C is not triangular. It is not easy to find the solution to $Cx = b$, because either we are going to have *infinitely many solution* or *no solution*.

Let looks at these problems geometrically.



Exercises Section 1.3 – The Algebra of Matrices

1. For the matrices: $A = \begin{bmatrix} p & 0 \\ q & r \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, when does $AB = BA$

2. Find a combination $x_1 w_1 + x_2 w_2 + x_3 w_3$ that gives the zero vector:

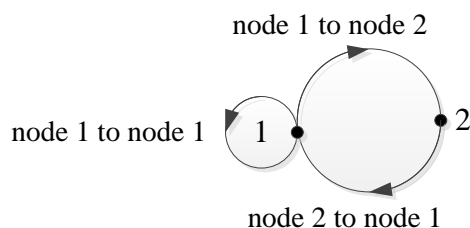
$$w_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}; \quad w_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}; \quad w_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Those vectors are independent or dependent?

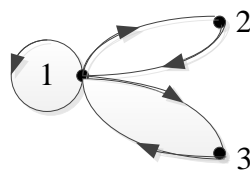
The vectors lie in a _____.

The matrix W with those columns is not invertible.

3. The very last words say that the 5 by 5 centered difference matrix is not invertible, Write down the 5 equations $Cx = b$. Find a combination of left sides that gives zero. What combination of b_1, b_2, b_3, b_4, b_5 must be zero?
4. A direct graph starts with n nodes. There are n^2 possible edges, each edge leaves one of the n nodes and enters one of the n nodes (possibly itself). The n by n adjacency matrix has $a_{ij} = 1$ when edge leaves node i and enter node j ; if no edge then $a_{ij} = 0$. Here are directed graphs and their adjacency matrices:



$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The i, j entry of A^2 is $a_{i1}a_{1j} + \dots + a_{in}a_{nj}$.

Why does that sum count the two-step paths from i to any node to j ?

The i, j entry of A^k counts k -steps paths:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{array}{l} \text{counts the paths} \\ \text{with two edges} \end{array} \quad \begin{bmatrix} 1 \text{ to } 2 \text{ to } 1, 1 \text{ to } 1 \text{ to } 1 & 1 \text{ to } 1 \text{ to } 2 \\ 2 \text{ to } 1 \text{ to } 1 & 2 \text{ to } 1 \text{ to } 2 \end{bmatrix}$$

List all 3-step paths between each pair of nodes and compare with A^3 . When A^k has **no zeros**, that number k is the diameter of the graph – the number of edges needed to connect the most pair of nodes. What is the diameter of the second graph?

5. A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. All entries are 1. Which of these matrix operations are allowed, and what are the results?

- a) AB b) BA c) ABD d) DBA
 e) ABC f) $ABCD$ g) $A(B + C)$

6. What rows or columns or matrices do you multiply to find.

- a) The third column of AB ?
 b) The second column of AB ?
 c) The first row of AB ?
 d) The second row of AB ?
 e) The entry in row 3, column 4 of AB ?
 f) The entry in row 2, column 3 of AB ?

7. Add AB to AC and compare with $A(B + C)$:

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

8. True or False

- a) If A^2 is defined then A is necessarily square.
 b) If AB and BA are defined then A and B are squares.
 c) If AB and BA are defined then AB and BA are squares.
 d) If $AB = B$, then $A = I$

9. a) Find a nonzero matrix A such that $A^2 = 0$

- b) Find a matrix that has $A^2 \neq 0$ but $A^3 = 0$

10. Suppose you solve $Ax = b$ for three special right sides b :

$$Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

If the three solutions x_1, x_2, x_3 are the columns of a matrix X , what is A times X ?

11. Show that $(A+B)^2$ is different from $A^2 + 2AB + B^2$, when

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

Write down the correct rule for $(A+B)(A+B) = A^2 + \underline{\hspace{2cm}} + B^2$

12. Find the product of the 2 matrices by rows or by columns:

a) $\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

c) $\begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

13. Given $A = \begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix}$ $B = \begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix}$ Find $A+B$, $2A$, and $-B$

14. Given $A = \begin{bmatrix} 3 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ Find AB and BA if possible

15. Given $A = \begin{bmatrix} 5 & 3 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 4 & -2 \\ -2 & 0 \\ 9 & 1 \end{bmatrix}$ Find AB and BA if possible

16. Given $A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$ Find AB and BA if possible

17. Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

Compute the following (where possible):

a) $D+E$ b) $D-E$ c) $5A$ d) $-7C$ e) $2B-C$ f) $-3(D+2E)$

18. Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ -4 & 1 & 3 \end{bmatrix} \quad C = \begin{bmatrix} -3 & -1 \\ 2 & 1 \\ 4 & 3 \end{bmatrix} \quad D = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$$

Compute the following (where possible):

a) $A + B$ b) $A + C$ c) AB d) BA e) CD f) DC
g) BD h) DB i) A^2 j) B^2 k) D^2

19. Let $B = \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$, show that $B^4 = \begin{pmatrix} a^4 & 0 \\ a^3 + a^2b + ab^2 + b^3 & b^4 \end{pmatrix}$

20. Let $B = \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$, show that $B^n = \begin{pmatrix} a^n & 0 \\ \sum_{k=0}^{n-1} a^{n-1-k}b^k & b^n \end{pmatrix}$

21. Let $A = \begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix}$. Prove that $A^n = \begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix}$ if $n \geq 1$

22. Let $A = \begin{bmatrix} 2a & -a^2 \\ 1 & 0 \end{bmatrix}$. Prove that $A^n = \begin{bmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{bmatrix}$ if $n \geq 1$

23. The following system of recurrence relations holds for all $n \geq 0$

$$\begin{cases} x_{n+1} = 7x_n + 4y_n \\ y_{n+1} = -9x_n - 5y_n \end{cases}$$

Solve the system for x_n and y_n in terms of x_0 and y_0

24. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, prove that $A^2 - (a+d)A + (ad-bc)I_{2 \times 2} = 0$

25. If $A = \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}$, use the fact $A^2 = 4A - 3I$ and mathematical induction, to prove that

$$A^n = \frac{3^n - 1}{2}A + \frac{3 - 3^n}{2}I \quad \text{if } n \geq 1$$

- 26.** A sequence of numbers $x_1, x_2, \dots, x_n, \dots$ satisfies the recurrence relation $x_{n+1} = ax_n + bx_{n-1}$ for $n \geq 1$, where a and b are constants. Prove that

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = A \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

Where $A = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$ and hence express $\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$ in terms of $\begin{bmatrix} x_1 \\ x_0 \end{bmatrix}$.

If $a = 4$ and $b = -3$, use the previous question to find a formula for x_n in terms x_1 and x_0