Solution   *Section* 4.4 – Area and Lengths in Polar Coordinates

Exercise

Find the slopes of the curves at the given points. Sketch the curves along with their tangents at these points.   *Cardioid* $r=-1+\cos\theta;\quad \theta=\pm\frac{\pi}{2}$

Solution

$$\theta=\frac{\pi}{2}\quad\Rightarrow\quad r=-1+\cos\frac{\pi}{2}=-1\quad\rightarrow\quad\left(-1,\ \frac{\pi}{2}\right)$$

$$\theta=-\frac{\pi}{2}\quad\Rightarrow\quad r=-1+\cos\left(-\frac{\pi}{2}\right)=-1\quad\rightarrow\quad\left(-1,\ -\frac{\pi}{2}\right)$$

$$r'=\frac{dr}{d\theta}=-\sin\theta$$

$$Slope=\frac{r'\sin\theta+r\cos\theta}{r'\cos\theta-r\sin\theta}$$

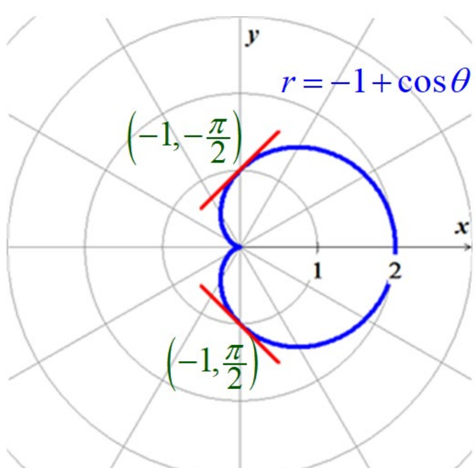
$$=\frac{-\sin^2\theta+r\cos\theta}{-\sin\theta\cos\theta-r\sin\theta}$$

$$Slope\Big|_{\left(-1,\frac{\pi}{2}\right)}=\frac{-\sin^2\frac{\pi}{2}+(-1)\cos\frac{\pi}{2}}{-\sin\frac{\pi}{2}\cos\frac{\pi}{2}+\sin\frac{\pi}{2}}$$

$$\underline{=-1}$$

$$Slope\Big|_{\left(-1,-\frac{\pi}{2}\right)}=\frac{-\sin^2\left(-\frac{\pi}{2}\right)+(-1)\cos\left(-\frac{\pi}{2}\right)}{-\sin\left(-\frac{\pi}{2}\right)co\left(-\frac{\pi}{2}\right)+\sin\left(-\frac{\pi}{2}\right)}$$

$$\underline{=1}$$



Exercise

Find the slopes of the curves at the given points. Sketch the curves along with their tangents at these points. *Cardioid* $r=-1+\sin\theta; \quad \theta=0,\ \pi$

Solution

$$\theta=0 \quad\Rightarrow\quad r=-1+\sin 0=-1 \quad\to\quad (-1,\ 0)$$

$$\theta=\pi \quad\Rightarrow\quad r=-1+\sin \pi=-1 \quad\to\quad (-1,\ \pi)$$

$$r'=\frac{dr}{d\theta}=\cos\theta$$

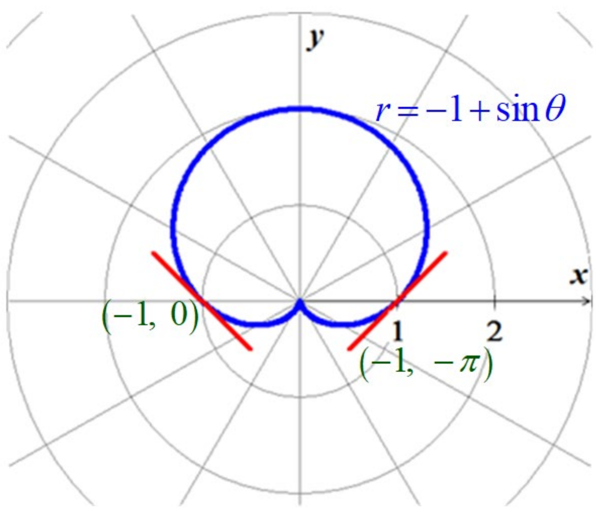
$$Slope=\frac{\cos\theta\sin\theta+r\cos\theta}{\cos^2\theta-r\sin\theta} \qquad\qquad Slope=\frac{r'\sin\theta+r\cos\theta}{r'\cos\theta-r\sin\theta}$$

$$Slope\Big|_{(-1,0)}=\frac{\cos(0)\sin(0)+(-1)\cos(0)}{\cos^2(0)-(-1)\sin(0)}$$

$$=-1$$

$$Slope\Big|_{(-1,\pi)}=\frac{\cos(\pi)\sin(\pi)+(-1)\cos(\pi)}{\cos^2(\pi)-(-1)\sin(\pi)}$$

$$=1$$



Exercise

Find the slopes of the curves at the given points. Sketch the curves along with their tangents at these points. $\quad$ *Four – leaved rose* $\quad r = \sin 2\theta; \quad \theta = \pm\frac{\pi}{4}, \ \pm\frac{3\pi}{4}$

Solution

$$\theta = -\frac{\pi}{4} \quad \Rightarrow \quad r = \sin\left(-\frac{\pi}{2}\right) = -1 \quad \rightarrow \quad \left(-1, \ -\frac{\pi}{4}\right)$$

$$\theta = \frac{\pi}{4} \quad \Rightarrow \quad r = \sin\left(\frac{\pi}{2}\right) = 1 \quad \rightarrow \quad \left(1, \ \frac{\pi}{4}\right)$$

$$\theta = -\frac{3\pi}{4} \quad \Rightarrow \quad r = \sin\left(-\frac{3\pi}{2}\right) = 1 \quad \rightarrow \quad \left(1, \ -\frac{3\pi}{4}\right)$$

$$\theta = \frac{3\pi}{4} \quad \Rightarrow \quad r = \sin\left(\frac{3\pi}{2}\right) = -1 \quad \rightarrow \quad \left(-1, \ \frac{3\pi}{4}\right)$$

$$r' = \frac{dr}{d\theta} = 2\cos 2\theta$$

$$Slope = \frac{2\cos 2\theta \sin\theta + r\cos\theta}{2\cos 2\theta\cos\theta - r\sin\theta} \qquad\qquad Slope = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$$

$$Slope\Big|_{\left(-1,-\frac{\pi}{4}\right)} = \frac{2\cos\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{4}\right) + (-1)\cos\left(-\frac{\pi}{4}\right)}{2\cos\left(-\frac{\pi}{2}\right)\cos\left(-\frac{\pi}{4}\right) - (-1)\sin\left(-\frac{\pi}{4}\right)}$$

$$= \frac{-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}}$$

$$= 1$$

$$Slope\Big|_{\left(1,\frac{\pi}{4}\right)} = \frac{2\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right) + (1)\cos\left(\frac{\pi}{4}\right)}{2\cos\left(\frac{\pi}{2}\right)\cos\left(-\frac{\pi}{4}\right) - (1)\sin\left(\frac{\pi}{4}\right)}$$

$$= \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}}$$

$$= -1$$

$$Slope\Big|_{\left(-1,\frac{3\pi}{4}\right)} = \frac{2\cos\left(\frac{3\pi}{2}\right)\sin\left(\frac{3\pi}{4}\right) + (-1)\cos\left(-\frac{3\pi}{4}\right)}{2\cos\left(\frac{3\pi}{2}\right)\cos\left(\frac{3\pi}{4}\right) - (-1)\sin\left(\frac{3\pi}{4}\right)}$$

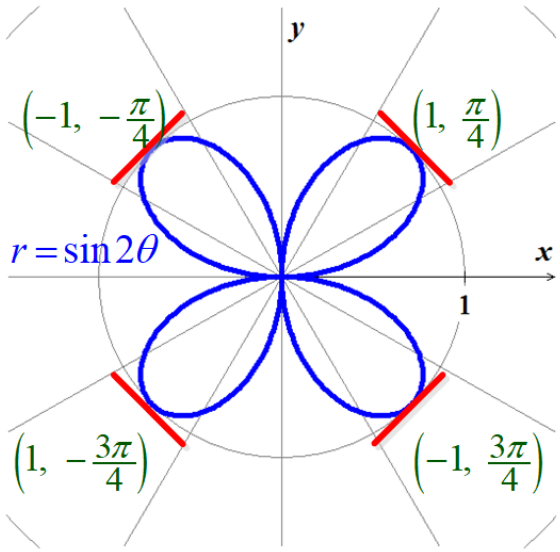
$$= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$$

$$= 1$$

$$Slope\Big|_{\left(1,-\frac{3\pi}{4}\right)} = \frac{2\cos\left(-\frac{3\pi}{2}\right)\sin\left(-\frac{3\pi}{4}\right)+(1)\cos\left(-\frac{3\pi}{4}\right)}{2\cos\left(-\frac{3\pi}{2}\right)\cos\left(-\frac{3\pi}{4}\right)-(1)\sin\left(-\frac{3\pi}{4}\right)}$$

$$= \frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$$

$$\underline{= -1}$$



Exercise

Find the slopes of the curves at the given points. Sketch the curves along with their tangents at these points. *Four – leaved rose* $r = \cos 2\theta;\quad \theta = 0,\ \pm\frac{\pi}{2},\ \pi$

Solution

$$\theta = 0 \;\Rightarrow\; r = \cos(0) = 1 \;\rightarrow\; (1,\ 0)$$

$$\theta = \frac{\pi}{2} \;\Rightarrow\; r = \cos(\pi) = -1 \;\rightarrow\; \left(-1,\ \frac{\pi}{2}\right)$$

$$\theta = -\frac{\pi}{2} \;\Rightarrow\; r = \cos(-\pi) = -1 \;\rightarrow\; \left(-1,\ -\frac{\pi}{2}\right)$$

$$\theta = \pi \;\Rightarrow\; r = \cos(2\pi) = 1 \;\rightarrow\; (1,\ \pi)$$

$$r' = \frac{dr}{d\theta} = -2\sin 2\theta$$

$$Slope = \frac{-2\sin 2\theta\sin\theta + r\cos\theta}{-2\sin 2\theta\cos\theta - r\sin\theta} \qquad Slope = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$$

$$Slope\Big|_{(1,0)} = \frac{-2\sin(0)\sin(0)+(1)\cos(0)}{-2\sin(0)\cos(0)-(1)\sin(0)}$$

$$= \frac{1}{0}$$

$$= undefined$$

$$Slope\Big|_{\left(-1,\frac{\pi}{2}\right)} = \frac{-2\sin(\pi)\sin\left(\frac{\pi}{2}\right)+(1)\cos\left(\frac{\pi}{2}\right)}{-2\sin(\pi)\cos\left(\frac{\pi}{2}\right)-(1)\sin\left(\frac{\pi}{2}\right)}$$

$$= \frac{0}{-1}$$

$$\underline{= 0}$$

$$Slope\Big|_{\left(-1,-\frac{\pi}{2}\right)} = \frac{-2\sin(-\pi)\sin\left(-\frac{\pi}{2}\right)+(1)\cos\left(-\frac{\pi}{2}\right)}{-2\sin(-\pi)\cos\left(-\frac{\pi}{2}\right)-(1)\sin\left(-\frac{\pi}{2}\right)}$$

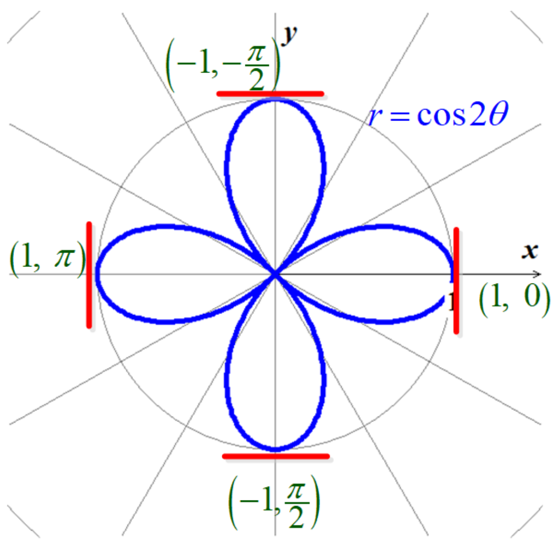
$$= \frac{0}{1}$$

$$\underline{= 0}$$

$$Slope\Big|_{(1,\pi)} = \frac{-2\sin(2\pi)\sin(\pi)+(1)\cos(\pi)}{-2\sin(2\pi)\cos(\pi)-(1)\sin(\pi)}$$

$$= \frac{-1}{0}$$

$$= undefined$$



Exercise

Find the area of the region bounded by the spiral $r=\theta \quad for \quad 0\le\theta\le\pi$

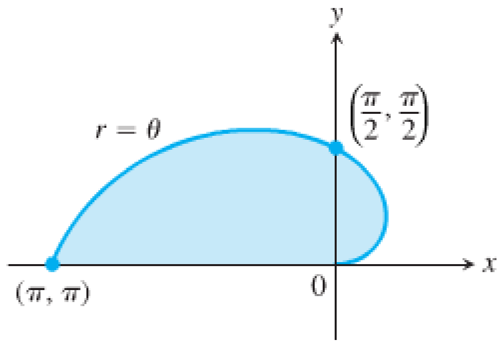
Solution

$$A=\frac{1}{2}\int_0^{\pi} r^2\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi} \theta^2\,d\theta$$

$$=\frac{1}{6}\theta^3\Big|_0^{\pi}$$

$$=\frac{\pi^3}{6}\ unit^2$$



Exercise

Find the area of the region bounded by the circle $r=2\sin\theta \quad for \quad \frac{\pi}{4}\le\theta\le\frac{\pi}{2}$

Solution

$$A=\frac{1}{2}\int_{\pi/4}^{\pi/2}(2\sin\theta)^2\,d\theta$$

$$=2\int_{\pi/4}^{\pi/2}\sin^2\theta\,d\theta$$

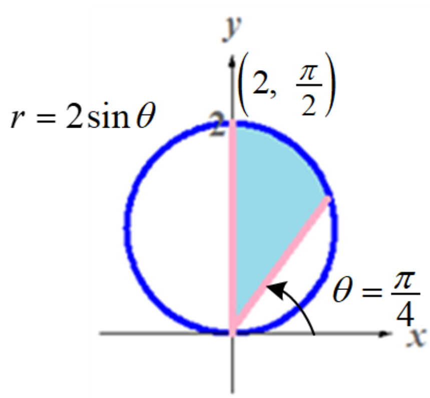
$$=2\int_{\pi/4}^{\pi/2}\frac{1-\cos 2\theta}{2}\,d\theta$$

$$=\theta-\frac{1}{2}\sin 2\theta\Big|_{\pi/4}^{\pi/2}$$

$$=\left(\frac{\pi}{2}-\frac{1}{2}\sin\pi\right)-\left(\frac{\pi}{4}-\frac{1}{2}\sin\frac{\pi}{2}\right)$$

$$=\frac{\pi}{2}-\frac{\pi}{4}+\frac{1}{2}\left(-\frac{1}{2}\sin\pi\right)-\left(-\sin\frac{\pi}{2}\right)$$

$$=\frac{\pi}{4}+\frac{1}{2}\ unit^2$$



Exercise

Find the area of the region inside the oval limaçon $r = 4 + 2\sin\theta$

Solution

$$A = \frac{1}{2}\int_0^{2\pi} (4 + 2\sin\theta)^2\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\left(16 + 16\sin\theta + 4\sin^2\theta\right) d\theta$$

$$= \int_0^{2\pi} \left(8 + 8\sin\theta + 2\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \int_0^{2\pi} (8 + 8\sin\theta + 1 - \cos 2\theta)\, d\theta$$

$$= \int_0^{2\pi} (9 + 8\sin\theta - \cos 2\theta)\, d\theta$$

$$= 9\theta - 8\cos\theta - \frac{1}{2}\sin 2\theta \Big|_0^{2\pi}$$

$$= 18\pi - 8\cos 2\pi - \frac{1}{2}\sin 4\pi - \left(0 - 8\cos 0 - \frac{1}{2}\sin 0\right)$$

$$= 18\pi - 8 + 8$$

$$= 18\pi \ \ unit^2$$

Exercise

Find the area of the region inside the cardioid $r = a(1 + \cos\theta), \quad a > 0$

Solution

$$A = \frac{1}{2}\int_0^{2\pi} a^2 (1 + \cos\theta)^2\, d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{a^2}{2}\left(\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta \right|_0^{2\pi}$$

$$= \frac{a^2}{2}(3\pi)$$

$$\boxed{= \frac{3}{2}\pi a^2 \quad unit^2}$$

Exercise

Find the area of the region inside one leaf of the three-leaved rose $r = \cos 3\theta$

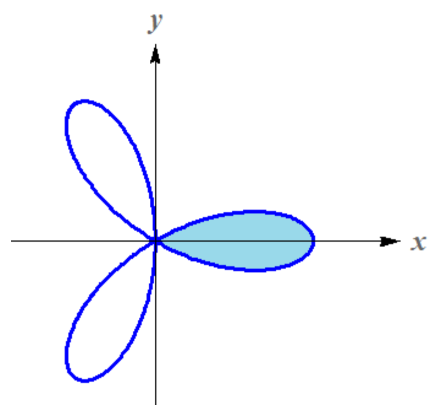
Solution

$$A = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (\cos 3\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2 3\theta \, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1+\cos 6\theta}{2} \, d\theta$$

$$= \frac{1}{4}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (1+\cos 6\theta) \, d\theta$$



$$= \frac{1}{4}\left(\theta + \frac{1}{6}\sin 6\theta \right|_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$= \frac{1}{4}\left[\left(\frac{\pi}{6} + \frac{1}{6}\sin\pi\right) - \left(-\frac{\pi}{6} + \frac{1}{6}\sin(-\pi)\right)\right]$$

$$= \frac{1}{4}\left(\frac{\pi}{6} + \frac{\pi}{6}\right)$$

$$\boxed{= \frac{\pi}{12} \quad unit^2}$$

Exercise

Find the area of the region inside the six-leaved rose $r^2 = 2\sin 3\theta$

Solution

$$A = \frac{1}{2}\int_0^{2\pi} r^2 d\theta$$

$$= (6)(2)\frac{1}{2}\int_0^{\pi/2} 2\sin 3\theta \, d\theta$$

$$= 12\int_0^{\pi/2} \sin 3\theta \, d\theta$$

$$= 12\left(-\frac{1}{3}\cos 3\theta \right|_0^{\pi/2}$$

$$= -4\cos\frac{3\pi}{2}$$

$$= 4 \quad unit^2$$

Exercise

Find the area of the region inside the curve $r = \sqrt{\cos\theta}$

Solution

$$r = \sqrt{\cos\theta} \geq 0$$

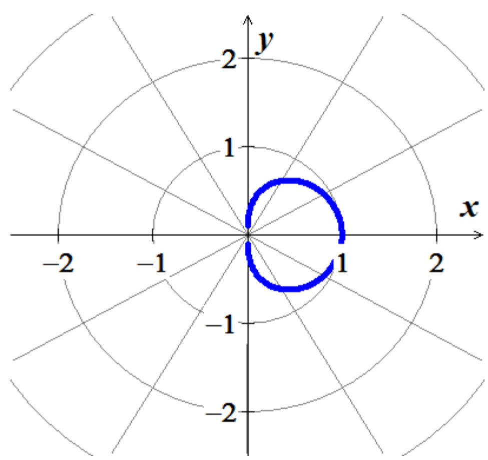
$$\cos\theta \geq 0 \quad \Rightarrow -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$A = \frac{1}{2}\int_0^{\pi/2} 2\left(\sqrt{\cos\theta}\right)^2 d\theta \qquad A = \int_\alpha^\beta \frac{1}{2} r^2 d\theta$$

$$= \int_0^{\pi/2} \cos\theta \, d\theta$$

$$= \sin\theta \Big|_0^{\pi/2}$$

$$= 1 \ unit^2$$



Exercise

Find the area of the region inside the right lobe of $r = \sqrt{\cos 2\theta}$

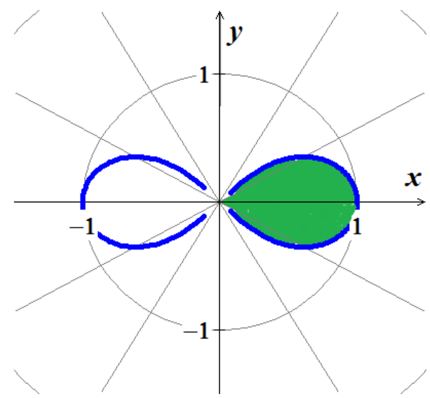
Solution

$$r = \sqrt{\cos 2\theta} \geq 0$$

$$\cos 2\theta \geq 0$$

$$\rightarrow -\frac{\pi}{2} \leq 2\theta \leq \frac{\pi}{2}$$

$$\boxed{-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}}$$



$$A = \frac{1}{2}\int_0^{\pi/4} 2\left(\sqrt{\cos 2\theta}\right)^2 d\theta \qquad A = \int_\alpha^\beta \frac{1}{2} r^2 d\theta$$

$$= \int_0^{\pi/2} \cos 2\theta \, d\theta$$

$$= \frac{1}{2}\sin 2\theta \Big|_0^{\pi/4}$$

$$\boxed{= \frac{1}{2} \quad unit^2}$$

Exercise

Find the area of the region inside the cardioid $r = 4 + 4\sin\theta$

Solution

$$A = \frac{1}{2}\int_0^{2\pi} (4 + 4\sin\theta)^2 \, d\theta \qquad A = \frac{1}{2}\int_\alpha^\beta r^2 d\theta$$

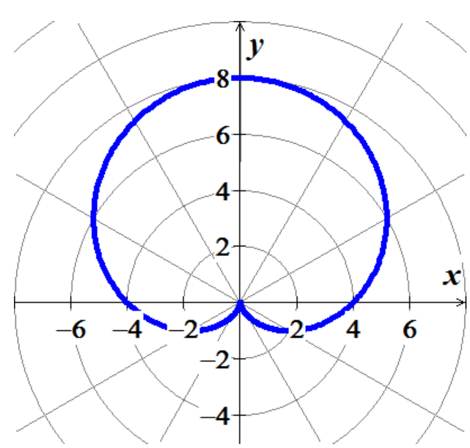
$$= 8\int_0^{2\pi} \left(1 + 2\sin\theta + \sin^2\theta\right) d\theta$$

$$= 8\int_0^{2\pi} \left(1 + 2\sin\theta + \frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= 8\left(\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right) \Big|_0^{2\pi}$$

$$= 8(3\pi - 2 + 2)$$

$$\boxed{= 24\pi \quad unit^2}$$



Exercise

Find the area of the region inside the limaçon $r = 2 + \cos\theta$

Solution

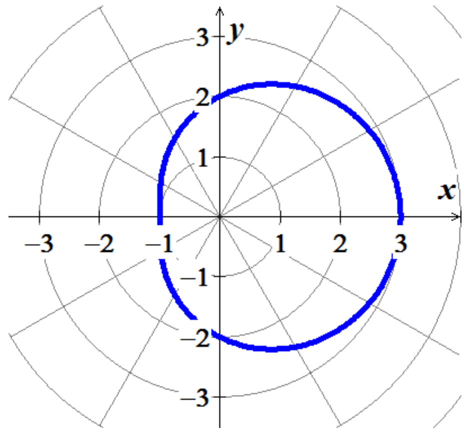
$$A = \frac{1}{2}\int_0^{\pi} 2(2+\cos\theta)^2\, d\theta \qquad\qquad A = \frac{1}{2}\int_\alpha^\beta r^2 d\theta$$

$$= \int_0^{\pi} \left(4 + 4\cos\theta + \cos^2\theta\right) d\theta$$

$$= \int_0^{\pi} \left(4 + 4\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \left(\frac{9}{2}\theta + 4\sin\theta + \frac{1}{4}\sin 2\theta \right|_0^{\pi}$$

$$= \frac{9\pi}{2} \quad unit^2$$



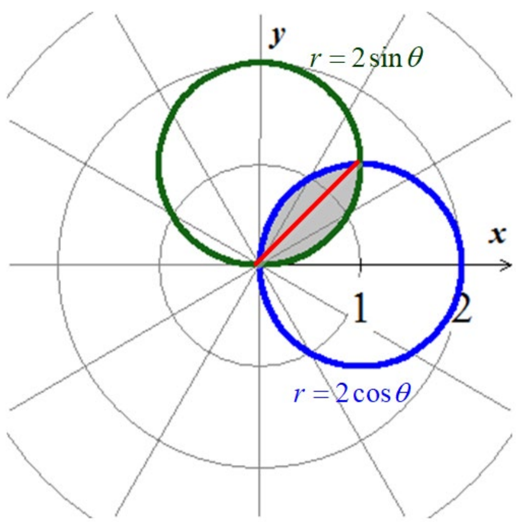
Exercise

Find the area of the region shared by the circles $r = 2\cos\theta$ *and* $r = 2\sin\theta$

Solution

$$r = 2\cos\theta = 2\sin\theta$$

$$\cos\theta = \sin\theta \quad \to \quad \theta = \frac{\pi}{4}$$



$$A = 2\int_0^{\pi/4} \frac{1}{2}(2\sin\theta)^2\, d\theta$$

$$= \int_0^{\pi/4} 4\sin^2\theta\, d\theta$$

$$= \int_0^{\pi/4} 2(1-\cos 2\theta)\, d\theta$$

$$= \left(2\theta - \sin 2\theta \right|_0^{\pi/4}$$

$$= 2\frac{\pi}{4} - \sin\frac{\pi}{2} - 0$$

$$= \frac{\pi}{2} - 1 \quad unit^2$$

Exercise

Find the area of the region shared by the circle $r = 2$ and the cardioid $r = 2(1-\cos\theta)$

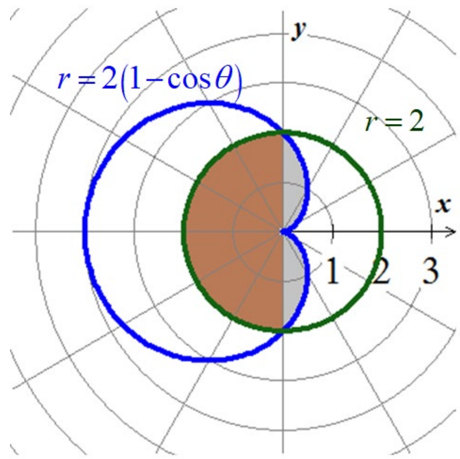
Solution

$$r = 2 - 2\cos\theta = 2$$

$$\cos\theta = 0 \;\Rightarrow\; \boxed{\theta = \pm\frac{\pi}{2}}$$

$$A = \frac{1}{2}\,\textit{Area of the circle} \;+\; 2\int_0^{\pi/2} \frac{1}{2}\big[2(1-\cos\theta)\big]^2\,d\theta$$

$$= \frac{1}{2}\pi\left(2^2\right) + 4\int_0^{\pi/2}\left(1 - 2\cos\theta + \cos^2\theta\right)d\theta$$



$$= 2\pi + \int_0^{\pi/2}(4 - 8\cos\theta + 2 + 2\cos 2\theta)\,d\theta$$

$$= 2\pi + \int_0^{\pi/2}(6 - 8\cos\theta + 2\cos 2\theta)\,d\theta$$

$$= 2\pi + \left(6\theta - 8\sin\theta + \sin 2\theta\right)\Big|_0^{\pi/2}$$

$$= 2\pi + 3\pi - 8$$

$$\boxed{= 5\pi - 8 \quad unit^2}$$

Exercise

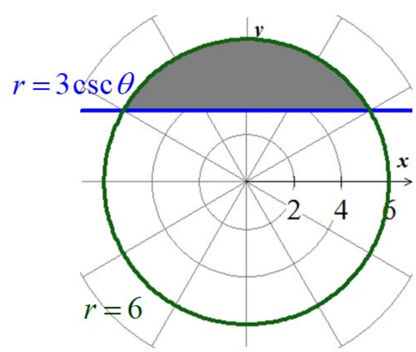
Find the area of the region inside the circle $r = 6$ above the line $r = 3\csc\theta$

Solution

$$r = 3\csc\theta = \frac{3}{\sin\theta} = 6$$

$$\sin\theta = \frac{3}{6} = \frac{1}{2}$$

$$\boxed{\theta = \frac{\pi}{6},\ \frac{5\pi}{6}}$$



$$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}\left[6^2 - (3\csc\theta)^2\right]d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6}\left(36 - 9\csc^2\theta\right)d\theta$$

$$= \frac{9}{2}\left(4\theta + \cot\theta\right|_{\pi/6}^{5\pi/6}$$

$$= \frac{9}{2}\left[\left(\frac{10\pi}{3} - \sqrt{3}\right) - \left(\frac{2\pi}{3} + \sqrt{3}\right)\right]$$

$$= \frac{9}{2}\left(\frac{8\pi}{3} - 2\sqrt{3}\right)$$

$$= 12\pi - 9\sqrt{3} \quad unit^2$$

Exercise

Find the area of the region in the plane enclosed by the four-leaf rose $r = f(\theta) = 2\cos 2\theta$

Solution

The curve is symmetric about the x-axis:

$$r = 2\cos(-2\theta) = 2\cos 2\theta$$

$$(r,\theta) = (r,-\theta)$$

The curve is symmetric about the y-axis:

$$-r = 2\cos 2(\pi - \theta) = -2\cos 2\theta$$

$$r = 2\cos 2\theta$$

$$(r,\theta) = (-r, \pi - \theta)$$

The graph of the rose appears to be symmetric about x- and y-axes.

$$A = 8\int_0^{\pi/4} \frac{1}{2} r^2 d\theta$$

$$= 4\int_0^{2\pi} (2\cos 2\theta)^2\, d\theta$$

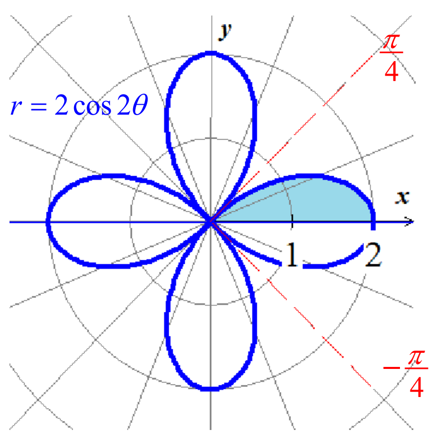
$$= 16\int_0^{\pi/4} \cos^2 2\theta\, d\theta$$

$$= 8\int_0^{\pi/4} (1 + \cos 4\theta)\, d\theta$$

$$= 8\left(\theta + \frac{1}{4}\sin 4\theta\right|_0^{\pi/4}$$

$$= 8\left[\frac{\pi}{4} + 0 - (0+0)\right]$$

$$= 2\pi \quad unit^2$$



Exercise

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 + \cos\theta$

Solution

$$A = \int_{\pi/2}^{3\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta$$

$$= 2\int_{\pi/2}^{\pi} \frac{1}{2}\left(1^2 - (1+\cos\theta)^2\right) d\theta$$

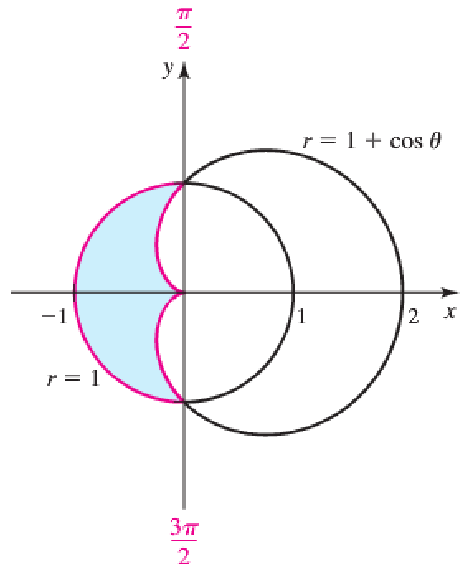
$$= \int_{\pi/2}^{\pi} \left(1 - \left(1 + 2\cos\theta + \cos^2\theta\right)\right) d\theta$$

$$= \int_{\pi/2}^{\pi} \left(-2\cos\theta - \frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= -2\sin\theta - \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta \Big|_{\pi/2}^{\pi}$$

$$= -\frac{\pi}{2} + 2 + \frac{\pi}{4}$$

$$= 2 - \frac{\pi}{4} \quad unit^2$$



Exercise

Find the area of the region inside the inner loop $r = \cos\theta - \frac{1}{2}$

Solution

$$A = \frac{1}{2}\int_0^{\pi/3} 2\left(\cos\theta - \frac{1}{2}\right)^2 d\theta \qquad A = \frac{1}{2}\int_\alpha^\beta r^2 d\theta$$

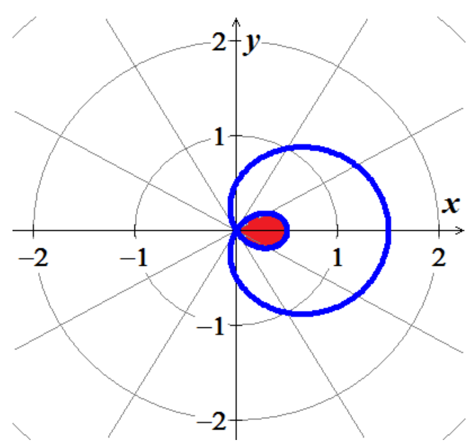
$$= \int_0^{\pi/3} \left(\cos^2\theta - \cos\theta + \frac{1}{4}\right) d\theta$$

$$= \int_0^{\pi/3} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta - \cos\theta + \frac{1}{4}\right) d\theta$$

$$= \left(\frac{3}{4}\theta + \frac{1}{4}\sin 2\theta - \sin\theta\right)\Big|_0^{\pi/3}$$

$$= \frac{\pi}{4} + \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{4} - \frac{3\sqrt{3}}{8} \quad unit^2$$



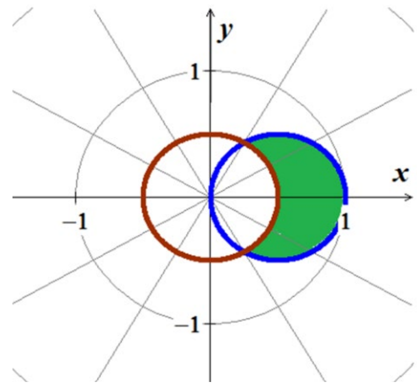
Exercise

Find the area of the region outside the circle $r=\frac{1}{2}$ and inside the circle $r=\cos\theta$

Solution

$$r=\cos\theta=\frac{1}{2} \quad \to \quad \theta=\pm\frac{\pi}{3}$$

$$A=\frac{1}{2}\int_0^{\pi/3} 2\left(\cos^2\theta-\frac{1}{4}\right)d\theta \qquad\qquad A=\frac{1}{2}\int_\alpha^\beta\left(r_2^2-r_1^2\right)d\theta$$



$$=\int_0^{\pi/3}\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta-\frac{1}{4}\right)d\theta$$

$$=\frac{1}{4}\theta+\frac{1}{4}\sin 2\theta \Big|_0^{\pi/3}$$

$$=\frac{\pi}{12}+\frac{\sqrt{3}}{8} \quad unit^2$$

Exercise

Find the area of the region outside the circle $r=\frac{1}{\sqrt{2}}$ and inside the curve $r=\sqrt{\cos\theta}$

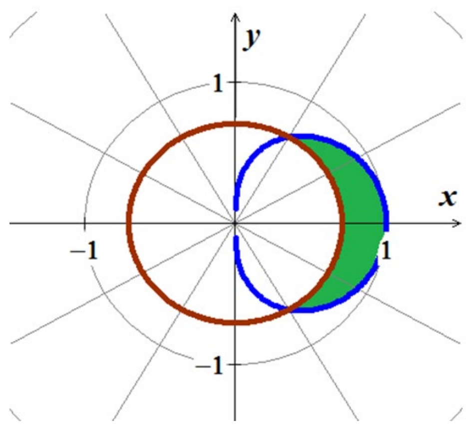
Solution

$$r=\sqrt{\cos\theta}=\frac{1}{\sqrt{2}}$$

$$\cos\theta=\frac{1}{2}$$

$$\theta=\pm\frac{\pi}{3}$$

$$A=\frac{1}{2}\int_0^{\pi/3} 2\left(\cos\theta-\frac{1}{2}\right)d\theta \qquad\qquad A=\frac{1}{2}\int_\alpha^\beta\left(r_2^2-r_1^2\right)d\theta$$



$$=\sin\theta-\frac{1}{2}\theta \Big|_0^{\pi/3}$$

$$=\frac{\sqrt{3}}{2}-\frac{\pi}{6} \quad unit^2$$

Exercise

Find the area of the region inside the circle $r = \frac{1}{\sqrt{2}}$ in QI and inside the right lobe of $r = \sqrt{\cos 2\theta}$

Solution

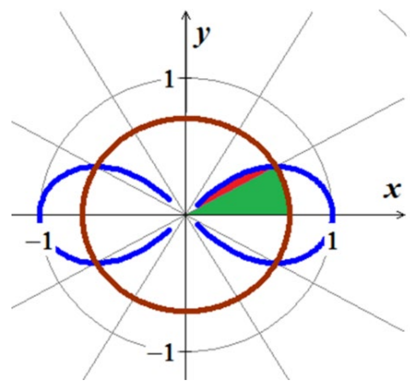
$$r = \sqrt{\cos 2\theta} = \frac{1}{\sqrt{2}}$$

$$\cos\theta = \frac{1}{2} \quad \Rightarrow \theta = \pm\frac{\pi}{3}$$

$$\sqrt{\cos 2\theta} = 0$$

$$\cos 2\theta = \frac{\pi}{2} \quad \Rightarrow \theta = \frac{\pi}{4}$$

$$A = A_1 + A_2$$



$$A = \frac{1}{2}\int_0^{\pi/6} \left(\frac{1}{\sqrt{2}}\right)^2 d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/4} \left(\sqrt{\cos 2\theta}\right)^2 d\theta$$

$$= \frac{1}{2}\int_0^{\pi/6} \frac{1}{2}d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/4} \cos 2\theta d\theta$$

$$= \frac{1}{4}\theta \Big|_0^{\pi/6} + \frac{1}{4}\sin 2\theta \Big|_{\pi/6}^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}}{2} - \frac{\pi}{12}\right) + \frac{1}{4}\left(1 - \frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{24} + \frac{1}{4} - \frac{\sqrt{3}}{8} \quad unit^2$$

Exercise

Find the area of the region inside the rose $r = 4\sin 2\theta$ and inside the circle $r = 2$

Solution

$$r = 4\sin 2\theta = 2$$

$$\sin 2\theta = \frac{1}{2}$$

$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

The area (**1**) inside one leaf but outside the circle is:

$$A = \frac{1}{2}\int_{\pi/12}^{5\pi/12} \left(16\sin^2 2\theta - 4\right) d\theta$$

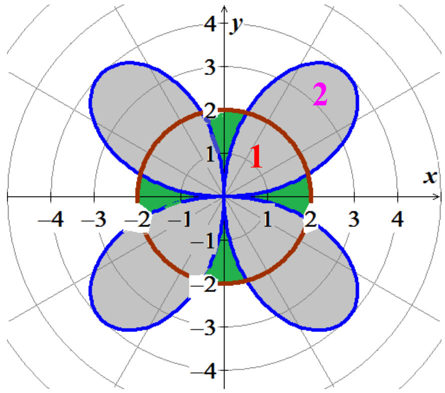
$$= \frac{1}{2}\int_{\pi/12}^{5\pi/12} \left(8 - 8\cos 4\theta - 4\right) d\theta$$

$$= \int_{\pi/12}^{5\pi/12} \left(2 - 4\cos 4\theta\right) d\theta$$

$$= 2\theta - \sin 4\theta \Big|_{\pi/12}^{5\pi/12}$$

$$= \frac{5\pi}{6} + \frac{\sqrt{3}}{2} - \frac{\pi}{6} + \frac{\sqrt{3}}{2}$$

$$= \frac{2\pi}{3} + \sqrt{3}$$



Area inside one leaf (2) is:

$$A = \frac{1}{2}\int_{0}^{\pi/2} 16\sin^2(2\theta)\, d\theta$$

$$= \int_{0}^{\pi/2} \left(4 - 4\cos 4\theta\right) d\theta$$

$$= 4\theta - \sin 4\theta \Big|_{0}^{\pi/2}$$

$$= 2\pi$$

$$\text{Total Area } = 4\left(2\pi - \frac{2\pi}{3} - \sqrt{3}\right)$$

$$= \frac{16\pi}{3} - 4\sqrt{3} \quad unit^2$$

Exercise

Find the area of the region inside the lemniscate $r^2 = 2\sin 2\theta$ and outside the circle $r = 1$

Solution

$$r^2 = 2\sin 2\theta = 1$$

$$\sin 2\theta = \frac{1}{2}$$

$$2\theta = \frac{\pi}{6},\ \frac{5\pi}{6}$$

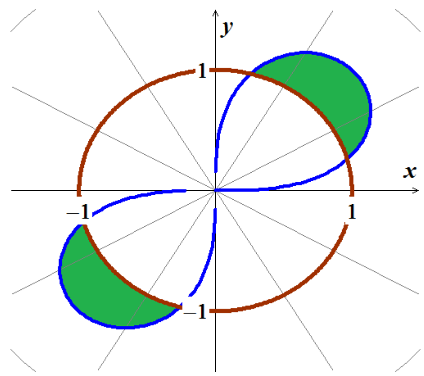
$$\Rightarrow \theta = \frac{\pi}{12},\ \frac{5\pi}{12}$$

$$A = \frac{1}{2}2\int_{\pi/12}^{5\pi/12}(2\sin 2\theta - 1)\,d\theta$$

$$= -\cos 2\theta - \theta \Big|_{\pi/12}^{5\pi/12}$$

$$= \frac{\sqrt{3}}{2} - \frac{5\pi}{12} + \frac{\sqrt{3}}{2} + \frac{\pi}{12}$$

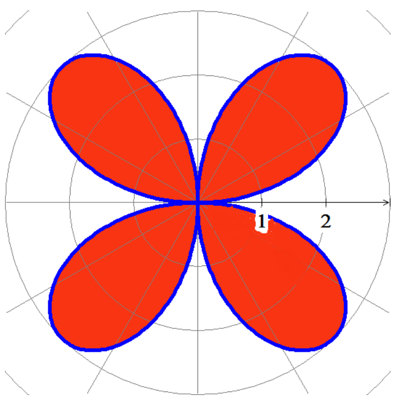
$$= \sqrt{3} - \frac{\pi}{3} \quad unit^2$$



Exercise

Find the area of the region inside all the leaves of the rose $r = 3\sin 2\theta$

Solution



Using symmetry $\frac{1}{2} - leaf$

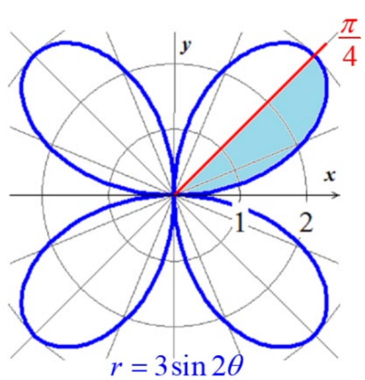
$$A = \frac{1}{2}(8)\int_0^{\pi/4} 9\sin^2 2\theta\, d\theta$$

$$= 18\int_0^{\pi/4}(1 - \cos 4\theta)\, d\theta$$

$$= 18\left(\theta - \frac{1}{4}\sin 4\theta\right)\Big|_0^{\pi/4}$$

$$= 18\left(\frac{\pi}{4}\right)$$

$$= \frac{9\pi}{2} \quad unit^2$$



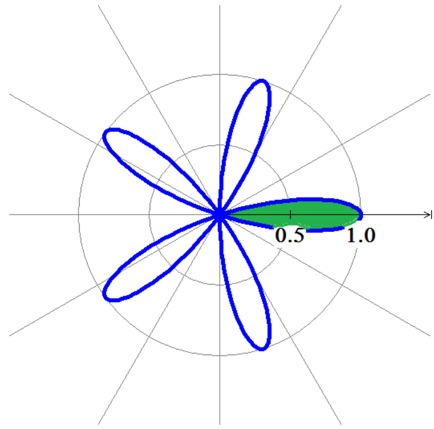
Exercise

Find the area of the region inside one leaf of the rose $r = \cos 5\theta$

Solution

$$0 \le 5\theta \le \frac{\pi}{2}$$

$$0 \le \theta \le \frac{\pi}{10}$$



Using symmetry $\frac{1}{2} - leaf$

$$A = \frac{1}{2}(2)\int_0^{\pi/10} \cos^2 5\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/10} (1 + \cos 10\theta) \, d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{10}\sin 10\theta\right)\Big|_0^{\pi/10}$$

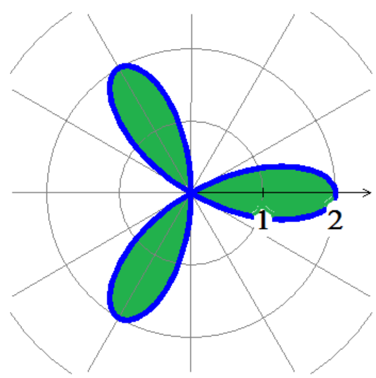
$$= \frac{\pi}{20} \quad unit^2$$

Exercise

Find the area of the region of a complete three-leaf rose $r = 2\cos 3\theta$

Solution

$$0 \le 3\theta \le \frac{\pi}{2} \quad \to \quad 0 \le \theta \le \frac{\pi}{6}$$



Using symmetry $\frac{1}{2} - leaf$

$$A = (6)\frac{1}{2}\int_0^{\pi/6} 4\cos^2 3\theta \, d\theta$$

$$= 6\int_0^{\pi/6} (1 + \cos 6\theta) \, d\theta$$

$$= 6\left(\theta + \frac{1}{6}\sin 6\theta\right)\Big|_0^{\pi/6}$$

$$= \pi \quad unit^2$$

Exercise

Find the area of the region inside the rose $r = 4\cos 2\theta$ and outside the circle $r = 2$

Solution

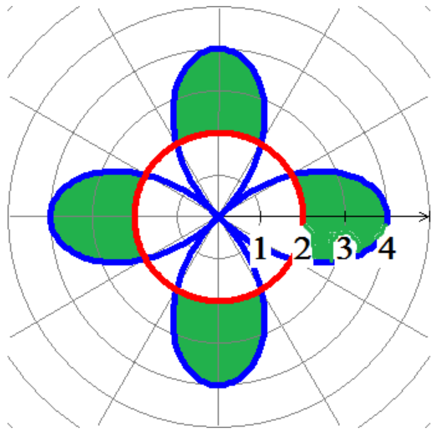
$$r = 4\cos 2\theta = 2$$

$$\cos 2\theta = \frac{1}{2}$$

$$2\theta = \frac{\pi}{3}$$

$$0 \le \theta \le \frac{\pi}{6}$$

Using symmetry $\frac{1}{2} - leaf$



$$A = (8)\frac{1}{2}\int_0^{\pi/6} \left(16\cos^2 2\theta - 4\right) d\theta$$

$$= 4\int_0^{\pi/6} (8 + 8\cos 4\theta - 4)\ d\theta$$

$$= 4\int_0^{\pi/6} (4 + 8\cos 4\theta)\ d\theta$$

$$= 4\ (4\theta + 2\sin 4\theta\ \Big|_0^{\pi/6}$$

$$= 4\left(\frac{2\pi}{3} + \sqrt{3}\right)$$

$$= \frac{8\pi}{3} + 4\sqrt{3} \quad unit^2$$

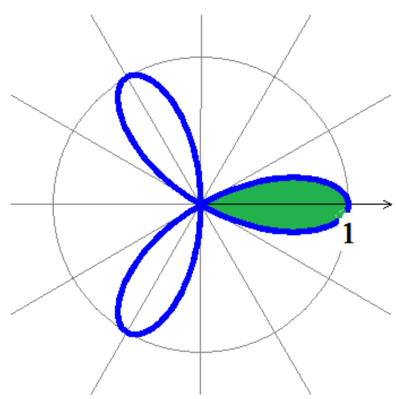
Exercise

Find the area of the region inside one leave of $r = \cos 3\theta$

Solution

$$0 \le 3\theta \le \frac{\pi}{2} \quad \rightarrow \ 0 \le \theta \le \frac{\pi}{6}$$

Using symmetry $\frac{1}{2} - leaf$



$$A = (2)\frac{1}{2}\int_0^{\pi/6} \cos^2 3\theta\ d\theta$$

$$= \frac{1}{2}\int_0^{\pi/6} (1 + \cos 6\theta)\ d\theta$$

$$=\frac{1}{2}\left(\theta+\frac{1}{6}\sin 6\theta\right|_{0}^{\pi/6}$$

$$=\frac{\pi}{12}\quad unit^2$$

Exercise

Find the area of the region bounded by the lemniscate $r^2=6\sin 2\theta$

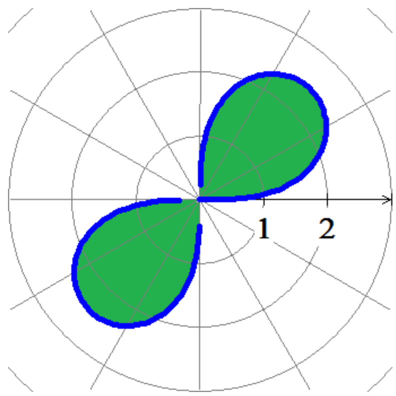
Solution

Using symmetry

$$A=(2)\frac{1}{2}\int_0^{\pi/2} 6\sin 2\theta\, d\theta$$

$$=6\left(-\frac{1}{2}\cos 2\theta\right|_{0}^{\pi/2}$$

$$=6\quad unit^2$$



Exercise

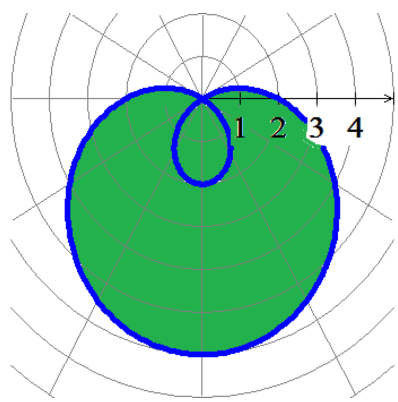
Find the area of the region bounded by the limaçon $r=2-4\sin\theta$

Solution

$$2-4\sin\theta=0$$

$$\sin\theta=\frac{1}{2}$$

$$\theta=\frac{\pi}{6}$$



$$A=(2)\frac{1}{2}\int_{-\pi/2}^{\pi/6}(2-4\sin\theta)^2\, d\theta$$

$$=\int_{-\pi/2}^{\pi/6}\left(4-16\sin\theta+16\sin^2\theta\right)d\theta$$

$$=\int_{-\pi/2}^{\pi/6}(4-16\sin\theta+8-8\cos 2\theta)\, d\theta$$

$$=12\theta+16\cos\theta-4\sin 2\theta\Big|_{-\pi/2}^{\pi/6}$$

$$=2\pi+8\sqrt{3}-2\sqrt{3}+6\pi$$

$$=8\pi+6\sqrt{3}\quad unit^2$$

Exercise

Find the area of the region bounded by the limaçon $r = 4 - 2\cos\theta$

Solution

$$A = \frac{1}{2}\int_0^{2\pi} (4 - 2\cos\theta)^2 \, d\theta$$

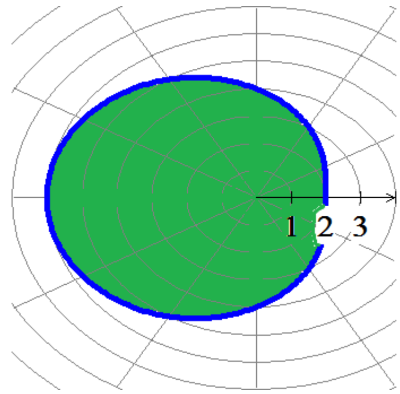
$$= \frac{1}{2}\int_0^{2\pi} \left(16 - 16\cos\theta + 4\cos^2\theta\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (16 - 16\cos\theta + 2 + 2\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left(18\theta - 16\sin\theta - \sin 2\theta\right.\Big|_0^{2\pi}$$

$$= 18\pi \quad unit^2$$

ϕ	r
0	-3
$\frac{\pi}{3}$	0
$\frac{\pi}{2}$	3
π	9
$\frac{5\pi}{3}$	0
2π	-3



Exercise

Find the area of the region inside one leaf: $r = 2\sin 6\phi$

Solution

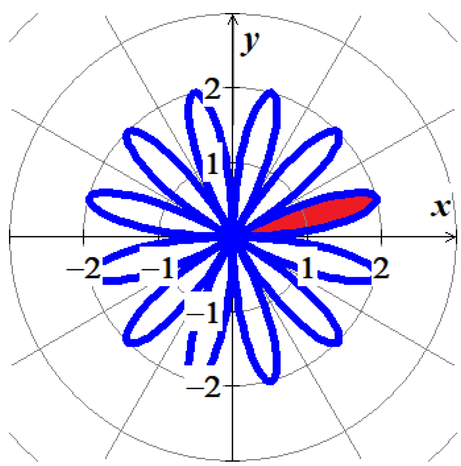
$$A = \frac{1}{2}\int_0^{\frac{\pi}{6}} (2\sin 6\phi)^2 \, d\phi$$

$$= 2\int_0^{\frac{\pi}{6}} \sin^2 6\phi \, d\phi$$

$$= \int_0^{\frac{\pi}{6}} (1 - \cos 12\phi)\, d\phi$$

$$= \phi - \frac{1}{12}\sin 12\phi \Big|_0^{\frac{\pi}{6}}$$

$$= \frac{\pi}{6} \quad unit^2$$



Exercise

Find the area of the region between inner and outer: $r = 3 - 6\cos\phi = 3(1 - 2\cos\phi)$

Solution

Area inside of the *inner* loop:

$$A_1 = 2 \cdot \frac{1}{2} \int_0^{\pi/3} 9(1 - 2\cos\phi)^2 \, d\phi$$

$$= 9 \int_0^{\pi/3} \left(1 - 4\cos\phi + 4\cos^2\phi\right) d\phi$$

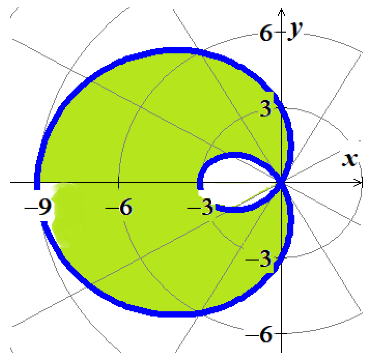
$$= 9 \int_0^{\pi/3} (1 - 4\cos\phi + 2 + 2\cos 2\phi) \, d\phi$$

$$= 9 \int_0^{\pi/3} (3 - 4\cos\phi + 2\cos 2\phi) \, d\phi$$

$$= 9 \left(3\phi - 4\sin\phi + \sin 2\phi\right) \Big|_0^{\pi/3}$$

$$= 9\left(\pi - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right)$$

$$= 9\pi - \frac{27\sqrt{3}}{2}$$



$$A_2 = 2 \cdot \frac{1}{2} \int_{\pi/3}^{\pi} 9(1 - 2\cos\phi)^2 \, d\phi$$

$$= 9 \left(3\phi - 4\sin\phi + \sin 2\phi\right) \Big|_{\pi/3}^{\pi}$$

$$= 9\left(3\pi - \pi + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right)$$

$$= 18\pi + \frac{27\sqrt{3}}{2}$$

The area between the loops $= 18\pi + \frac{27\sqrt{3}}{2} - \left(9\pi - \frac{27\sqrt{3}}{2}\right)$

$$= 9\pi + 27\sqrt{3} \quad unit^2$$

Exercise

Find the area of the given region inner loop of $r = 1 + 2\cos\theta$

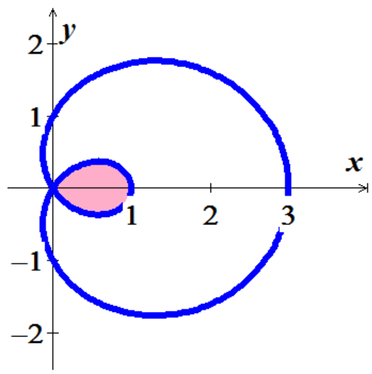
Solution

$$r = 1 + 2\cos\theta = 0$$

$$\cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}$$

$$r = 1 + 2\cos\theta = -1$$

$$\cos\theta = -1 \Rightarrow \theta = \pi$$



$$A = 2\cdot\frac{1}{2}\int_{2\pi/3}^{\pi} (1 + 2\cos\theta)^2\, d\theta$$

$$= \int_{2\pi/3}^{\pi} \left(1 + 4\cos\theta + 4\cos^2\theta\right) d\theta$$

$$= \int_{2\pi/3}^{\pi} (1 + 4\cos\theta + 2 + 2\cos 2\theta)\, d\theta$$

$$= 3\theta + 4\sin\theta + \sin 2\theta \Big|_{2\pi/3}^{\pi}$$

$$= 3\pi - 2\pi - 2\sqrt{3} + \frac{\sqrt{3}}{2}$$

$$= \pi - \frac{3\sqrt{3}}{2} \quad unit^2$$

Exercise

Find the area of the given region Inner loop of $r = 2 - 4\cos\theta$

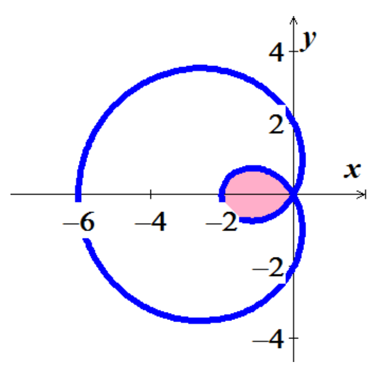
Solution

$$r = 2 - 4\cos\theta = 0$$

$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

$$r = 2 - 4\cos\theta = -2$$

$$\cos\theta = 1 \Rightarrow \theta = 0$$



$$A = 2\frac{1}{2}\int_{0}^{\pi/3} (2 - 4\cos\theta)^2\, d\theta$$

$$= \int_{0}^{\pi/3} \left(4 - 16\cos\theta + 16\cos^2\theta\right) d\theta$$

$$= \int_0^{\pi/3} \left(4 - 16\cos\theta + 8 + 8\cos 2\theta\right) d\theta$$

$$= 12\theta - 16\sin\theta + 4\sin 2\theta \Big|_0^{\pi/3}$$

$$= 4\pi - 8\sqrt{3} + 2\sqrt{3}$$

$$= 4\pi - 6\sqrt{3} \quad unit^2$$

Exercise

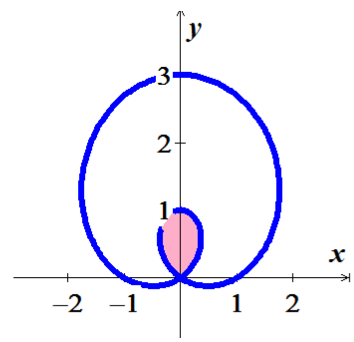
Find the area of the given region Inner loop of $r = 1 + 2\sin\theta$

Solution

$r = 1 + 2\sin\theta = 0$

$\sin\theta = -\frac{1}{2} \Rightarrow \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

$\frac{7\pi}{6} \le \theta \le \frac{11\pi}{6}$



$$A = \frac{1}{2}\int_{7\pi/6}^{11\pi/6} (1 + 2\sin\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_{7\pi/6}^{11\pi/6} \left(1 + 4\sin\theta + 4\sin^2\theta\right) d\theta$$

$$= \frac{1}{2}\int_{7\pi/6}^{11\pi/6} \left(1 + 4\sin\theta + 2 - 2\cos 2\theta\right) d\theta$$

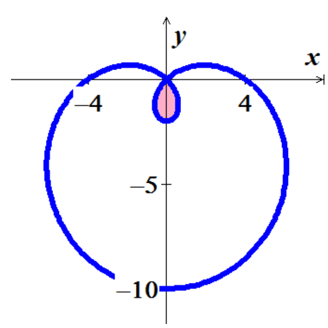
$$= \frac{1}{2}\left(3\theta - 4\cos\theta - \sin 2\theta \right|_{7\pi/6}^{11\pi/6}$$

$$= \frac{1}{2}\left(\frac{11\pi}{2} - 2\sqrt{3} + \frac{\sqrt{3}}{2} - \frac{7\pi}{2} - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2}\left(2\pi - 3\sqrt{3}\right) \quad unit^2$$

Exercise

Find the area of the given region Inner loop of $r = 4 - 6\sin\theta$



Solution

$r = 4 - 6\sin\theta = 0$

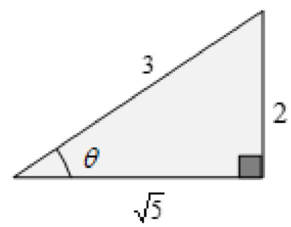
$\sin\theta = \frac{2}{3} \Rightarrow \theta = \arcsin\frac{2}{3}$

$r = 4 - 6\sin\theta = -2$

$$\sin\theta = 1 \Rightarrow \boxed{\theta = \frac{\pi}{2}}$$

$$A = 2\cdot\frac{1}{2}\int_{\arcsin 2/3}^{\pi/2}(4-6\sin\theta)^2\,d\theta$$

$$= \int_{\arcsin 2/3}^{\pi/2}\left(16-48\sin\theta+36\sin^2\theta\right)d\theta$$



$$= \int_{\arcsin 2/3}^{\pi/2}\left(16-48\sin\theta+18-18\cos 2\theta\right)d\theta$$

$$= 34\theta + 48\cos\theta - 9\sin 2\theta \Big|_{\arcsin 2/3}^{\pi/2}$$

$$= 17\pi - 34\arcsin\frac{2}{3} + 48\cos\left(\arcsin 2/3\right) - 18\sin\left(\arcsin 2/3\right)\cos\left(\arcsin 2/3\right)$$

$$= 17\pi - 34\arcsin\frac{2}{3} + 16\sqrt{5} - 4\sqrt{5}$$

$$\boxed{= 17\pi - 34\arcsin\frac{2}{3} + 12\sqrt{5} \quad unit^2}$$

Exercise

Find the area of the given region between the loops $r = 1 + 2\cos\theta$

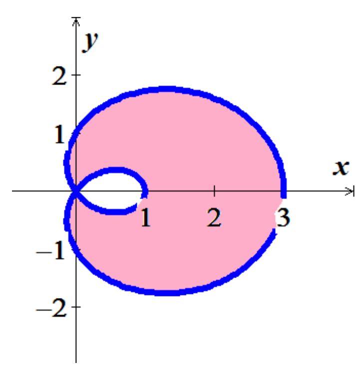
Solution

$$r = 1 + 2\cos\theta = 0$$

$$\cos\theta = -\frac{1}{2} \Rightarrow \boxed{\theta = \frac{2\pi}{3}}$$

$$r = 1 + 2\cos\theta = -1$$

$$\cos\theta = -1 \Rightarrow \boxed{\theta = \pi}$$



Area of the inner loop:

$$A_1 = \int_{2\pi/3}^{\pi}(1+2\cos\theta)^2\,d\theta$$

$$= \int_{2\pi/3}^{\pi}\left(1+4\cos\theta+2+2\cos 2\theta\right)d\theta$$

$$= 3\theta + 4\sin\theta + \sin 2\theta \Big|_{2\pi/3}^{\pi}$$

$$= 3\pi - 2\pi - 2\sqrt{3} + \frac{\sqrt{3}}{2}$$

$$= \pi - \frac{3\sqrt{3}}{2}$$

Area of the outer loop:

$$A_2 = 2\frac{1}{2}\int_0^{2\pi/3} (1+2\cos\theta)^2\, d\theta$$

$$= \int_0^{2\pi/3} (1+4\cos\theta+2+2\cos 2\theta)\, d\theta$$

$$= 3\theta + 4\sin\theta + \sin 2\theta \Big|_0^{2\pi/3}$$

$$= 2\pi + 2\sqrt{3} - \frac{\sqrt{3}}{2}$$

$$= 2\pi + \frac{3\sqrt{3}}{2}$$

Are between the loops: $= 2\pi + \frac{3\sqrt{3}}{2} - \left(\pi - \frac{3\sqrt{3}}{2}\right)$

$$= \pi + 3\sqrt{3} \quad unit^2$$

Exercise

Find the area of the given region between the loops $r = 2(1+2\sin\theta)$

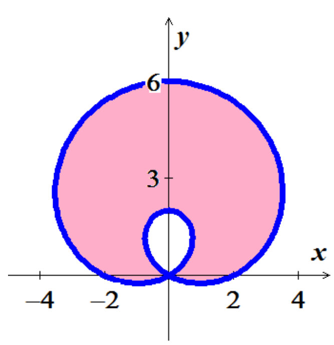
Solution

$r = 1+2\sin\theta = 0$

$$\sin\theta = -\frac{1}{2} \Rightarrow \theta = \frac{7\pi}{6},\ \frac{11\pi}{6}$$

$r = 2(1+2\sin\theta) = -2$

$$\sin\theta = -1 \Rightarrow \theta = \frac{3\pi}{2}$$



Area inside of the inner loop:

$$A_1 = 2\cdot\frac{1}{2}\int_{7\pi/6}^{3\pi/2} (2+4\sin\theta)^2\, d\theta$$

$$= 4\int_{7\pi/6}^{3\pi/2} \left(1+4\sin\theta+4\sin^2\theta\right) d\theta$$

$$= 4\int_{7\pi/6}^{3\pi/2} (1+4\sin\theta+2-2\cos 2\theta)\, d\theta$$

$$= 4\left(3\theta - 4\cos\theta - \sin 2\theta \right|_{7\pi/6}^{3\pi/2}$$

$$= 4\left(\frac{9\pi}{2} - \frac{7\pi}{2} - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right)$$

$$\underline{= 4\pi - 6\sqrt{3}}$$

Area inside of the outer loop:

$r = 2(1 + 2\sin\theta) = 3$

$$\sin\theta = 1 \Rightarrow \underline{\theta = \frac{\pi}{2}}$$

$$A_2 = 2\frac{1}{2}\int_{-\pi/6}^{\pi/2} 4(1 + 2\sin\theta)^2\, d\theta$$

$$= 4\int_{-\pi/6}^{\pi/2} \left(1 + 4\sin\theta + 4\sin^2\theta\right) d\theta$$

$$= 4\int_{-\pi/6}^{\pi/2} (1 + 4\sin\theta + 2 - 2\cos 2\theta)\, d\theta$$

$$= 4\left(3\theta - 4\cos\theta - \sin 2\theta \right|_{-\pi/6}^{\pi/2}$$

$$= 4\left(\frac{3\pi}{2} + \frac{\pi}{2} + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right)$$

$$\underline{= 8\pi + 6\sqrt{3}}$$

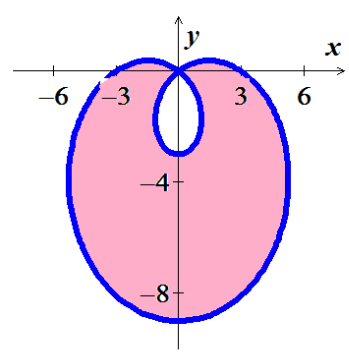
The area between the loops $= 8\pi + 6\sqrt{3} - \left(4\pi - 6\sqrt{3}\right)$

$$\underline{= 4\pi + 12\sqrt{3}\ \ unit^2}$$

Exercise

Find the area of the given region between the loops $r = 3 - 6\sin\theta$

Solution



$r = 3 - 6\sin\theta = 0$

$$\sin\theta = \frac{1}{2} \Rightarrow \underline{\theta = \frac{\pi}{6}, \frac{5\pi}{6}}$$

$r = 3 - 6\sin\theta = -3$

$$\sin\theta = 1 \Rightarrow \underline{\theta = \frac{\pi}{2}}$$

$r = 3 - 6\sin\theta = 6$

$$\sin\theta = -1 \;\Rightarrow\; \underline{\theta = \frac{3\pi}{2}}$$

Area inside of the *inner* loop:

$$A_1 = 2\cdot\frac{1}{2}\int_{\pi/6}^{\pi/2} 9(1-2\sin\theta)^2\, d\theta$$

$$= 9\int_{\pi/6}^{\pi/2} \left(1-4\sin\theta+4\sin^2\theta\right) d\theta$$

$$= 9\int_{\pi/6}^{\pi/2} (1-4\sin\theta+2-2\cos 2\theta)\, d\theta$$

$$= 9\,(3\theta+4\cos\theta-\sin 2\theta\;\Big|_{\pi/6}^{\pi/2}$$

$$= 9\left(\frac{3\pi}{2}-\frac{\pi}{2}-2\sqrt{3}+\frac{\sqrt{3}}{2}\right)$$

$$\underline{= 9\pi-\frac{27\sqrt{3}}{2}}$$

Area inside of the *outer* loop:

$$A_2 = 2\cdot\frac{1}{2}\int_{5\pi/6}^{3\pi/2} 9(1-2\sin\theta)^2\, d\theta$$

$$= 9\int_{5\pi/6}^{3\pi/2} \left(1-4\sin\theta+4\sin^2\theta\right) d\theta$$

$$= 9\int_{5\pi/6}^{3\pi/2} (1-4\sin\theta+2-2\cos 2\theta)\, d\theta$$

$$= 9\,(3\theta+4\cos\theta-\sin 2\theta\;\Big|_{5\pi/6}^{3\pi/2}$$

$$= 9\left(\frac{9\pi}{2}-\frac{5\pi}{2}+2\sqrt{3}-\frac{\sqrt{3}}{2}\right)$$

$$\underline{= 18\pi+\frac{27\sqrt{3}}{2}}$$

The area between the loops $= 18\pi+\dfrac{27\sqrt{3}}{2}-\left(9\pi-\dfrac{27\sqrt{3}}{2}\right)$

$$\underline{= 9\pi+27\sqrt{3}\quad unit^2}$$

Exercise

Find the area of the given region between the loops $r=\frac{1}{2}+\cos\theta$

Solution

$$r=\frac{1}{2}+\cos\theta=0$$

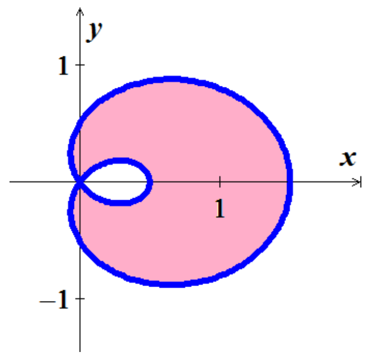
$$\cos\theta=-\frac{1}{2} \Rightarrow \underline{\theta=\frac{2\pi}{3},\ \frac{4\pi}{3}}$$

$$r=\frac{1}{2}+\cos\theta=-\frac{1}{2}$$

$$\cos\theta=-1 \Rightarrow \underline{\theta=\pi}$$

$$r=\frac{1}{2}+\cos\theta=\frac{3}{2}$$

$$\cos\theta=\frac{1}{2} \Rightarrow \underline{\theta=\frac{\pi}{3},\ \frac{5\pi}{3}}$$



Area inside of the *inner* loop:

$$A_1=2\cdot\frac{1}{2}\int_{2\pi/3}^{\pi}\left(\frac{1}{2}+\cos\theta\right)^2 d\theta$$

$$=\int_{2\pi/3}^{\pi}\left(\frac{1}{4}+\cos\theta+\cos^2\theta\right)d\theta$$

$$=\int_{2\pi/3}^{\pi}\left(\frac{1}{4}+\cos\theta+\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$=\frac{3}{4}\theta+\sin\theta+\frac{1}{4}\sin 2\theta\ \Big|_{2\pi/3}^{\pi}$$

$$=\frac{3\pi}{4}-\frac{\pi}{2}-\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{8}$$

$$=\underline{\frac{\pi}{4}-\frac{3\sqrt{3}}{8}}$$

Area inside of the *outer* loop:

$$A_2=2\cdot\frac{1}{2}\int_{0}^{2\pi/3}\left(\frac{1}{2}+\cos\theta\right)^2 d\theta$$

$$=\int_{0}^{2\pi/3}\left(\frac{1}{4}+\cos\theta+\cos^2\theta\right)d\theta$$

$$= \int_0^{2\pi/3} \left(\frac{1}{4} + \cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{3}{4}\theta + \sin\theta + \frac{1}{4}\sin 2\theta \,\Big|_0^{2\pi/3}$$

$$= \frac{\pi}{2} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8}$$

$$= \frac{\pi}{2} + \frac{3\sqrt{3}}{8}$$

The area between the loops $= \frac{\pi}{2} + \frac{3\sqrt{3}}{8} - \left(\frac{\pi}{4} - \frac{3\sqrt{3}}{8}\right)$

$$= \frac{\pi}{4} + \frac{3\sqrt{3}}{4} \quad unit^2$$

Exercise

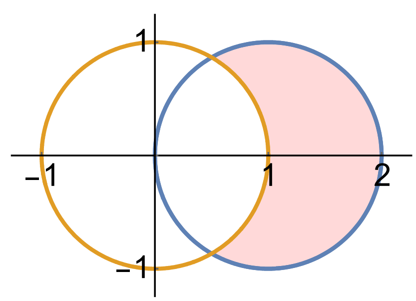
Find the area of the given region inside $r = 2\cos\theta$ and outside $r = 1$

Solution

$$r = 2\cos\theta = 1$$

$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

$$A = 2 \cdot \frac{1}{2} \int_0^{\pi/3} \left[(2\cos\theta)^2 - 1\right] d\theta$$



$$= \int_0^{\pi/3} \left(4\cos^2\theta - 1\right) d\theta$$

$$= \int_0^{\pi/3} (2 + 2\cos 2\theta - 1)\, d\theta$$

$$= \theta + \sin 2\theta \,\Big|_0^{\pi/3}$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2} \quad unit^2$$

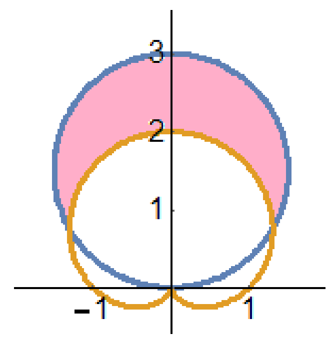
Exercise

Find the area of the given region inside $r = 3\sin\theta$ and outside $r = 1+\sin\theta$

Solution

$$r = 3\sin\theta = 1+\sin\theta$$

$$\sin\theta = \frac{1}{2} \;\Rightarrow \boxed{\theta = \frac{\pi}{6}}$$



$$A = 2\cdot\frac{1}{2}\int_{\pi/6}^{\pi/2}\left[(3\sin\theta)^2 - (1+\sin\theta)^2\right]d\theta$$

$$= \int_{\pi/6}^{\pi/2}\left(9\sin^2\theta - 1 - 2\sin\theta - \sin^2\theta\right)d\theta$$

$$= \int_{\pi/6}^{\pi/2}\left(4 - 4\cos 2\theta - 1 - 2\sin\theta\right)d\theta$$

$$= 3\theta - 2\sin 2\theta + 2\cos\theta \Big|_{\pi/6}^{\pi/2}$$

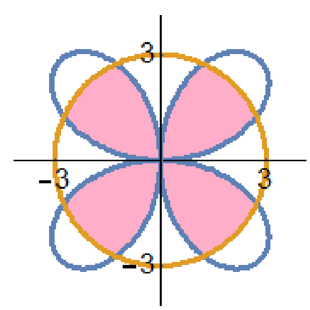
$$= \frac{3\pi}{2} - \frac{\pi}{2} + \sqrt{3} - \sqrt{3}$$

$$\boxed{= \pi \quad unit^2}$$

Exercise

Find the area of the given region common interior of $r = 4\sin 2\theta$ *and* $r = 2$

Solution



$$r = 4\sin 2\theta = 2$$

$$\sin 2\theta = \frac{1}{2}$$

$$2\theta = \frac{\pi}{6},\ \frac{5\pi}{6}$$

$$\boxed{\theta = \frac{\pi}{12},\ \frac{5\pi}{12}}$$

$$A = 4\cdot\frac{1}{2}\int_{0}^{\pi/12}(4\sin 2\theta)^2\,d\theta + 4\cdot\frac{1}{2}\int_{\pi/12}^{5\pi/12}(2)^2\,d\theta + 4\cdot\frac{1}{2}\int_{5\pi/12}^{\pi/2}(4\sin 2\theta)^2\,d\theta$$

$$= 32\int_{0}^{\pi/12}\sin^2 2\theta\,d\theta + (8\theta \Big|_{\pi/12}^{5\pi/12} + 32\int_{5\pi/12}^{\pi/2}\sin^2 2\theta\,d\theta$$

$$= 16\int_0^{\pi/12} (1-\cos 4\theta)\, d\theta \;+\; 8\left(\frac{5\pi}{12}-\frac{\pi}{12}\right) \;+\; 16\int_{5\pi/12}^{\pi/2} (1-\cos 4\theta)\, d\theta$$

$$= \frac{8\pi}{3} + 16\left(\theta - \frac{1}{4}\sin 4\theta\right)\Big|_0^{\pi/12} + 16\left(\theta - \frac{1}{4}\sin 4\theta\right)\Big|_{5\pi/12}^{\pi/2}$$

$$= \frac{8\pi}{3} + 16\left(\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right) + 16\left(\frac{\pi}{2} - \frac{5\pi}{12} - \frac{\sqrt{3}}{8}\right)$$

$$= \frac{8\pi}{3} + \frac{4\pi}{3} - 2\sqrt{3} + \frac{4\pi}{3} - 2\sqrt{3}$$

$$= \frac{16\pi}{3} - 4\sqrt{3} \quad unit^2$$

Exercise

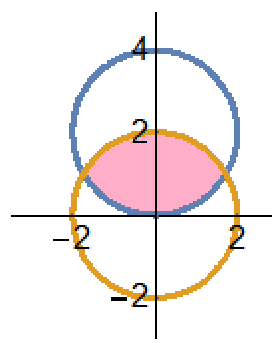
Find the area of the given region common interior of $r = 4\sin\theta$ *and* $r = 2$

Solution

$$r = 4\sin\theta = 2$$

$$\sin\theta = \frac{1}{2} \;\Rightarrow\; \theta = \frac{\pi}{6}$$

$$A = 2\cdot\frac{1}{2}\int_0^{\pi/6} 16\sin^2\theta\, d\theta \;+\; 2\cdot\frac{1}{2}\int_{\pi/6}^{\pi/2} 4\, d\theta$$



$$= 8\int_0^{\pi/6} (1-\cos 2\theta)\, d\theta \;+\; 4\left(\theta\right)\Big|_{\pi/6}^{\pi/2}$$

$$= 8\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/6} + 4\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$$

$$= 8\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) + \frac{4\pi}{3}$$

$$= \frac{8\pi}{3} - 2\sqrt{3} \quad unit^2$$

Exercise

Find the area of the given region common interior of $r = 2\cos\theta$ *and* $r = 2\sin\theta$

Solution

$$r = 2\cos\theta = 2\sin\theta \;\rightarrow\; \theta = \frac{\pi}{4}$$

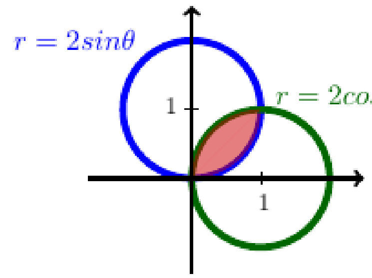
$$A = 2 \cdot \frac{1}{2} \int_{\pi/4}^{\pi/2} 4\cos^2\theta \, d\theta$$

$$= 2\int_{\pi/4}^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= 2\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_{\pi/4}^{\pi/2}$$

$$= 2\left(\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right)$$

$$= \frac{\pi}{2} - 1 \quad unit^2$$



Exercise

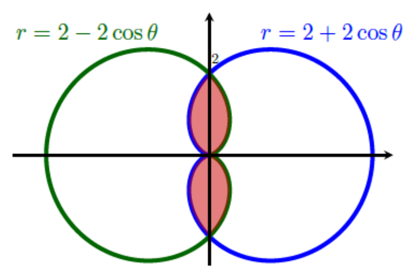
Find the area of the given region common interior of $r = 2(1+\cos\theta)$ *and* $r = 2(1-\cos\theta)$

Solution

$$r = 2(1+\cos\theta) = 2(1-\cos\theta)$$

$$\cos\theta = -\cos\theta$$

$$\theta = \frac{\pi}{2}$$



$$A = 4 \cdot \frac{1}{2}\int_0^{\pi/2} 4(1-\cos\theta)^2 \, d\theta$$

$$= 8\int_0^{\pi/2} \left(1 - 2\cos\theta + \cos^2\theta\right) d\theta$$

$$= 8\int_0^{\pi/2} \left(1 - 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= 8\left(\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/2}$$

$$= 8\left(\frac{3\pi}{4} - 2\right)$$

$$= 6\pi - 16 \quad unit^2$$

Exercise

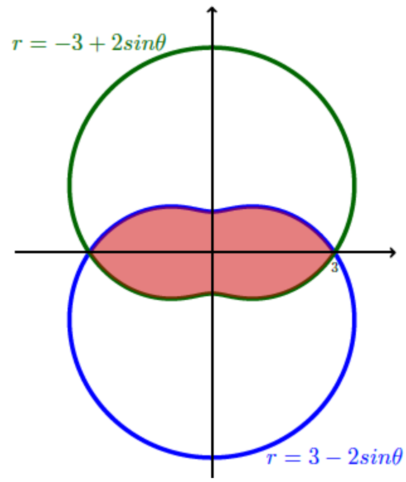
Find the area of the given region common interior of $r = 3 - 2\sin\theta$ *and* $r = -3 + 2\sin\theta$

Solution

$$r = 3 - 2\sin\theta = -3 + 2\sin\theta$$

$$4\sin\theta = 6$$

$$\sin\theta = \frac{3}{2} > 1 \quad \times$$



$$A = 4 \cdot \frac{1}{2}\int_0^{\pi/2} (3 - 2\sin\theta)^2 \, d\theta$$

$$= 2\int_0^{\pi/2} \left(9 - 12\sin\theta + 4\sin^2\theta\right) d\theta$$

$$= 2\int_0^{\pi/2} (9 - 12\sin\theta + 2 - 2\cos 2\theta) \, d\theta$$

$$= 2\left(11\theta + 12\cos\theta - \sin 2\theta\right) \Big|_0^{\pi/2}$$

$$= 2\left(\frac{11\pi}{2} - 12\right)$$

$$= 11\pi - 24 \quad unit^2$$

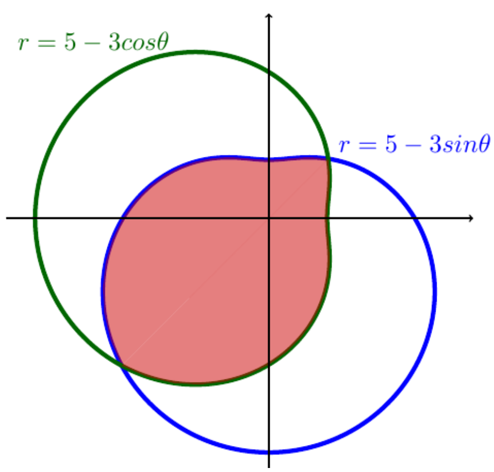
Exercise

Find the area of the given region common interior of $r = 5 - 3\sin\theta$ *and* $r = 5 - 3\cos\theta$

Solution

$$r = 5 - 3\sin\theta = 5 - 3\cos\theta$$

$$\sin\theta = \cos\theta \quad \Rightarrow \quad \theta = \frac{\pi}{4}, \ \frac{5\pi}{4}$$



$$A = 2 \cdot \frac{1}{2}\int_{\pi/4}^{5\pi/4} (5 - 3\sin\theta)^2 \, d\theta$$

$$= \int_{\pi/4}^{5\pi/4} \left(25 - 30\sin\theta + 9\sin^2\theta\right) d\theta$$

$$= \int_{\pi/4}^{5\pi/4} \left(25 - 30\sin\theta + \frac{9}{2} - \frac{9}{2}\cos 2\theta\right) d\theta$$

$$= \frac{59}{2}\theta + 30\cos\theta - \frac{9}{4}\sin 2\theta \Big|_{\pi/4}^{5\pi/4}$$

$$= 5\cdot\frac{59\pi}{8} - 15\sqrt{2} - \frac{9}{4} - \frac{59\pi}{8} - 15\sqrt{2} + \frac{9}{4}$$

$$= \frac{59\pi}{2} - 30\sqrt{2} \quad unit^2$$

Exercise

Find the area of the region inside $r = a(1+\cos\theta)$ and outside $r = a\cos\theta$

Solution

$$A = 2\cdot\frac{1}{2}\int_0^{\pi} a^2(1+\cos\theta)^2\,d\theta \;-\; (\text{area of a circle})$$

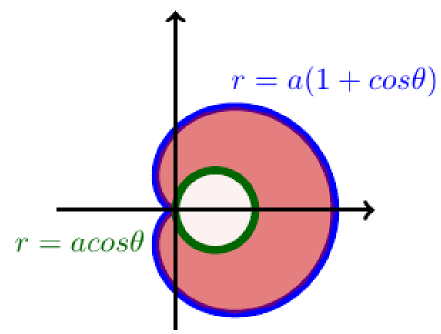
$$= a^2\int_0^{\pi}\left(1+2\cos\theta+\cos^2\theta\right)d\theta \;-\; \pi\left(\frac{a}{2}\right)^2$$

$$= a^2\int_0^{\pi}\left(1+2\cos\theta+\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta \;-\; \frac{\pi a^2}{4}$$

$$= a^2\left(\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right.\Big|_0^{\pi} \;-\; \frac{\pi a^2}{4}$$

$$= a^2\left(\frac{3\pi}{2}\right) - \frac{\pi a^2}{4}$$

$$= \frac{5\pi a^2}{4} \quad unit^2$$



Exercise

Find the area of the region inside $r = 2a\cos\theta$ and outside $r = a$

Solution

$$A = Area(2a\cos\theta) - Area\ of\ sector\ - 2\times Area(between\ r = 2a\cos\theta\ \&\ lines)$$

$$= \pi a^2 \;-\; \frac{\pi}{3}a^2 \;-\; 2\cdot\frac{1}{2}\int_{\pi/3}^{\pi/2}(2a\cos\theta)^2\,d\theta$$

$$= \frac{2\pi a^2}{3} \;-\; 2a^2\int_{\pi/3}^{\pi/2}(1+\cos 2\theta)\,d\theta$$

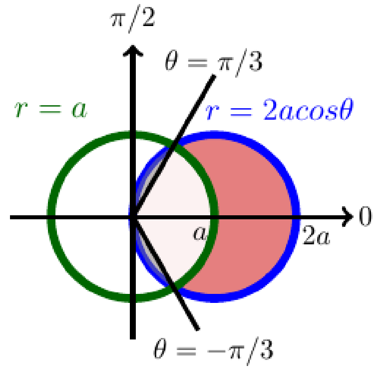
$$= \frac{2\pi a^2}{3} - 2a^2\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_{\pi/3}^{\pi/2}$$

$$= \frac{2\pi a^2}{3} - 2a^2\left(\frac{\pi}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$$

$$= \frac{2\pi a^2}{3} - \frac{\pi a^2}{3} + \frac{\sqrt{3}a^2}{2}$$

$$= \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)a^2$$

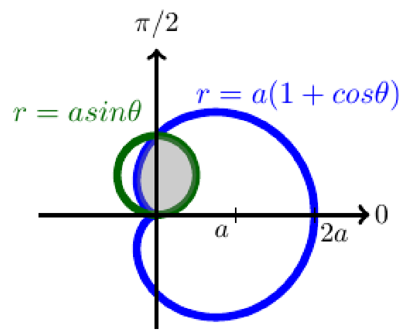
$$= \left(\frac{2\pi + 3\sqrt{3}}{6}\right)a^2 \quad unit^2$$



Exercise

Find the area of the region common interior of $r = a(1+\cos\theta)$ and $r = a\sin\theta$

Solution



$$A = \frac{1}{2}\int_0^{\pi/2} a^2\sin^2\theta\, d\theta + \frac{1}{2}\int_{\pi/2}^{\pi} a^2(1+\cos\theta)^2\, d\theta$$

$$= \frac{a^2}{4}\int_0^{\pi/2}(1-\cos 2\theta)\, d\theta + \frac{a^2}{2}\int_{\pi/2}^{\pi}\left(1 + 2\cos\theta + \cos^2\theta\right)d\theta$$

$$= \frac{a^2}{4}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/2} + \frac{a^2}{2}\int_{\pi/2}^{\pi}\left(1 + 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \frac{\pi a^2}{8} + \frac{a^2}{2}\left(\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right.\Big|_{\pi/2}^{\pi}$$

$$= \frac{\pi a^2}{8} + \frac{a^2}{2}\left(\frac{3\pi}{2} - \frac{3\pi}{4} - 2\right)$$

$$= \frac{\pi a^2}{8} + \frac{3\pi a^2}{8} - a^2$$

$$= (\pi - 2)\frac{a^2}{2} \quad unit^2$$

Exercise

Find the area of the region common interior of $r = a\cos\theta$ and $r = a\sin\theta$, where $a > 0$

Solution

$$r = a\cos\theta = a\sin\theta \quad \rightarrow \quad \theta = \frac{\pi}{4}$$

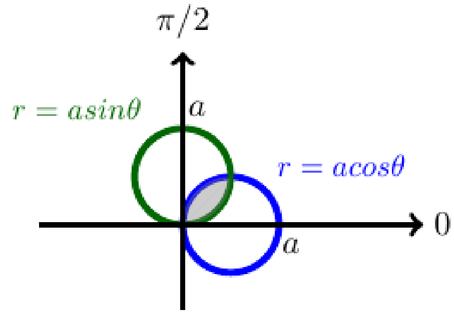
$$A = 2 \cdot \frac{1}{2} \int_0^{\pi/4} a^2 \sin^2 \theta \, d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi/4} (1 - \cos 2\theta) \, d\theta$$

$$= \frac{a^2}{2} \left(\theta - \frac{1}{2} \sin 2\theta \right|_0^{\pi/4}$$

$$= \frac{a^2}{2} \left(\frac{\pi}{4} - \frac{1}{2} \right)$$

$$\boxed{= (\pi - 2) \frac{a^2}{8} \quad unit^2}$$



Exercise

Find the area of the region enclosed by all the leaves of the rose $r = 3\sin 4\theta$

Solution

$$0 \le 4\theta \le \pi$$

$$0 \le \theta \le \frac{\theta}{4}$$

$$A = 8 \frac{1}{2} \int_0^{\frac{\pi}{4}} (3\sin 4\theta)^2 \, d\theta$$

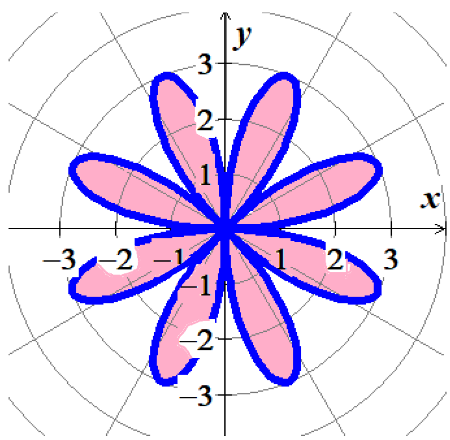
$$= 36 \int_0^{\frac{\pi}{4}} \sin^2 4\theta \, d\theta$$

$$= 18 \int_0^{\frac{\pi}{4}} (1 - \cos 8\theta) \, d\theta$$

$$= 18 \left(\theta - \frac{1}{8} \sin 8\theta \right|_0^{\frac{\pi}{4}}$$

$$= 18 \left(\frac{\pi}{4} - \frac{1}{8} \sin 2\pi \right)$$

$$\boxed{= \frac{9\pi}{2} \quad unit^2}$$



Exercise

Find the area of the region enclosed by the limaçon $r = 2 - \cos\theta$

Solution

$$A = \frac{1}{2}\int_0^{2\pi} (2 - \cos\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(4 - 4\cos\theta + \cos^2\theta\right) d\theta$$

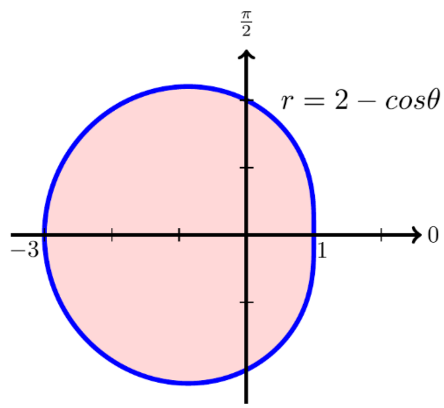
$$= \frac{1}{2}\int_0^{2\pi} \left(4 - 4\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(\frac{9}{2} - 4\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{1}{2}\left(\frac{9}{2}\theta - 4\sin\theta + \frac{1}{4}\sin 2\theta \right. \Big|_0^{2\pi}$$

$$= \frac{1}{2}\left(\frac{9}{2}2\pi\right)$$

$$= \frac{9\pi}{2} \ unit^2$$



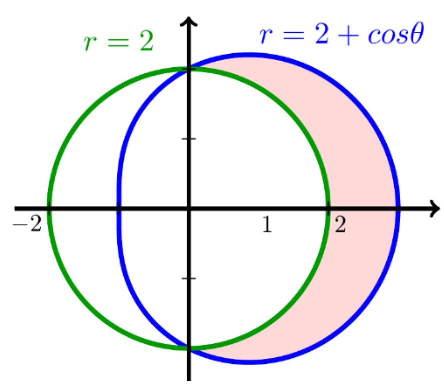
Exercise

Find the area of the region inside limaçon $r = 2 + \cos\theta$ and outside the circle $r = 2$

Solution

$r = 2 + \cos\theta = 2$

$\cos\theta = 0 \quad \Rightarrow \quad \theta = \pm\frac{\pi}{2}$



$$A = 2 \cdot \frac{1}{2}\int_0^{\pi/2} \left((2 + \cos\theta)^2 - 4\right) d\theta$$

$$= \int_0^{\pi/2} \left(4 + 4\cos\theta + \cos^2\theta - 4\right) d\theta$$

$$= \int_0^{\pi/2} \left(4\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \left(4\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \right. \Big|_0^{\pi/2}$$

$= 4 + \frac{\pi}{4} \quad unit^2$

Exercise

Find the area of the region inside lemniscate $r^2 = 4\cos 2\theta$ and outside the circle $r = \frac{1}{2}$

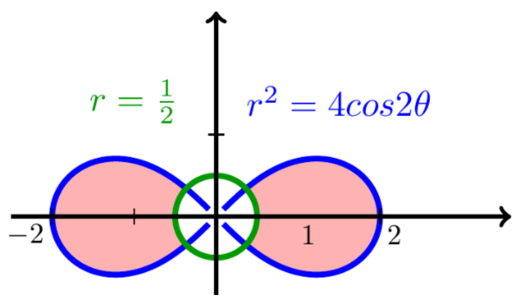
Solution

$r^2 = 4\cos 2\theta \ge 0$

$\cos 2\theta \ge 0$

$0 \le 2\theta \le \frac{\pi}{2}$

$\boxed{0 \le \theta \le \frac{\pi}{4}}$



$r^2 = 4\cos 2\theta = \frac{1}{4}$

$\cos 2\theta = \frac{1}{16} > 0$

$2\theta = \arccos\frac{1}{16} < \frac{\pi}{2}$

$$A = 4\cdot\frac{1}{2}\int_0^{\frac{\pi}{4}}\left(4\cos 2\theta - \frac{1}{4}\right)d\theta$$

$$= 2\left(2\sin 2\theta - \frac{1}{4}\theta\right)\Big|_0^{\frac{\pi}{4}}$$

$$= 2\left(2 - \frac{\pi}{16}\right)$$

$$= 4 - \frac{\pi}{8} \quad unit^2$$

Exercise

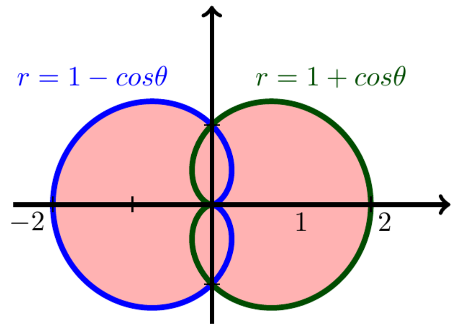
Find the area of the region inside both cardioids $r = 1 - \cos\theta$ and $r = 1 + \cos\theta$

Solution

$$A = 4\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}}(1+\cos\theta)^2\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}}\left(1 + 2\cos\theta + \cos^2\theta\right)\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left(1 + 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right)\, d\theta$$



$$= 2\int_0^{\frac{\pi}{2}} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right)\, d\theta$$

$$= 3\theta + 4\sin\theta + \frac{1}{2}\sin 2\theta \,\Big|_0^{\frac{\pi}{2}}$$

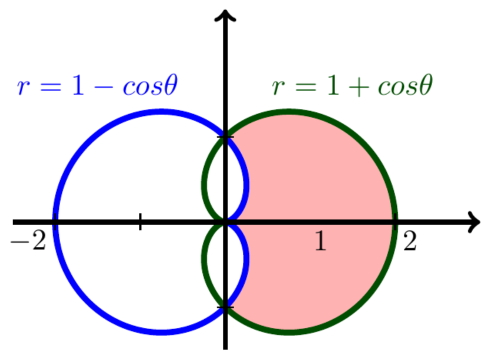
$$= \frac{3\pi}{2} + 4 \quad unit^2$$

Exercise

Find the area of the region inside the cardioid $r = 1 + \cos\theta$ and outside the cardioid $r = 1 - \cos\theta$

Solution

$$A = 2 \cdot \frac{1}{2}\int_0^{\frac{\pi}{2}} \left((1 + \cos\theta)^2 - (1 - \cos\theta)^2\right) d\theta$$



$$= \int_0^{\frac{\pi}{2}} \left(1 + 2\cos\theta + \cos^2\theta - 1 + 2\cos\theta - \cos^2\theta\right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} 4\cos\theta\, d\theta$$

$$= 4\sin\theta \,\Big|_0^{\frac{\pi}{2}}$$

$$= 4 \quad unit^2$$

Exercise

Find the area of the region inside both cardioids $r = 2 - 2\sin\theta$ and $r = 2 + 2\sin\theta$

Solution

$$A = 4 \cdot \frac{1}{2}\int_0^{\frac{\pi}{2}} (2 + 2\sin\theta)^2\, d\theta$$

$$= 8\int_0^{\frac{\pi}{2}} \left(1 + 2\sin\theta + \sin^2\theta\right)\, d\theta$$

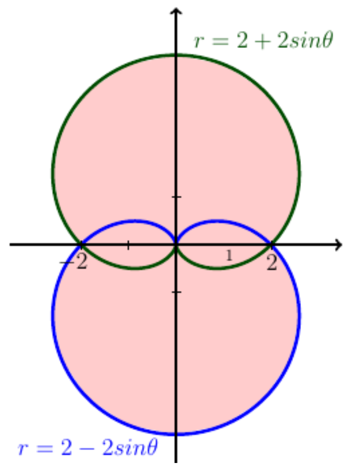
$$= 8\int_0^{\frac{\pi}{2}} \left(1 + 2\sin\theta + \frac{1}{2} - \frac{1}{2}\cos 2\theta\right)\, d\theta$$

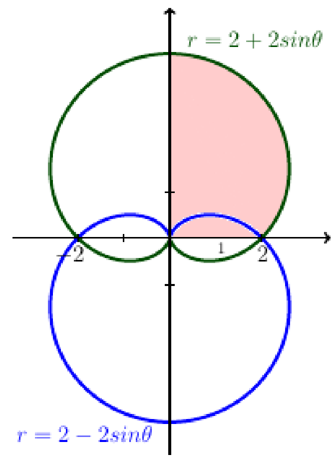
$$= 8\int_0^{\frac{\pi}{2}} \left(\frac{3}{2} + 2\sin\theta - \frac{1}{2}\cos 2\theta\right)\, d\theta$$

$$= 8\left(\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right.\Big|_0^{\frac{\pi}{2}}$$

$$= 8\left(\frac{3\pi}{4} + 2\right)$$

$$\underline{= 6\pi + 16 \quad unit^2}$$





Exercise

Find the area of the region common interior of $r = 2 - 2\sin\theta$ and $r = 2 + 2\sin\theta$

Solution

$$A = 4 \cdot \frac{1}{2}\int_0^{\frac{\pi}{2}} (2 - 2\sin\theta)^2\, d\theta$$

$$= 8\int_0^{\frac{\pi}{2}} \left(1 - 2\sin\theta + \sin^2\theta\right)\, d\theta$$

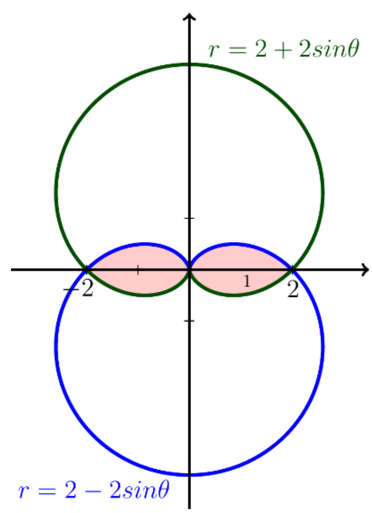
$$= 8\int_0^{\frac{\pi}{2}} \left(1 - 2\sin\theta \frac{1}{2} - \frac{1}{2}\cos 2\theta\right)\, d\theta$$

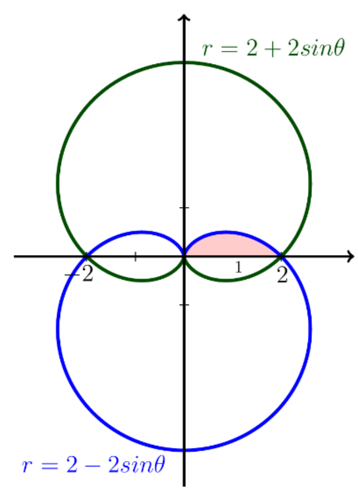
$$= 8\int_0^{\frac{\pi}{2}} \left(\frac{3}{2} - 2\sin\theta - \frac{1}{2}\cos 2\theta\right)\, d\theta$$

$$= 8\left(\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\right.\Big|_0^{\frac{\pi}{2}}$$

$$= 8\left(\frac{3\pi}{4} - 2\right)$$

$$\underline{= 6\pi - 16 \quad unit^2}$$

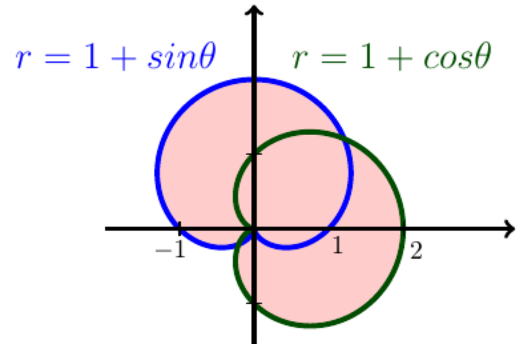




Exercise

Find the area of the region inside both cardioids $r = 1 + \sin\theta$ and $r = 1 + \cos\theta$

Solution



$$r = 1 + \sin\theta = 1 + \cos\theta$$

$$\sin\theta = \cos\theta \quad \Rightarrow \quad \theta = \frac{\pi}{4}, \ \frac{5\pi}{4}$$

$$A = 2 \cdot \frac{1}{2} \int_{\pi/4}^{5\pi/4} (1 + \cos\theta)^2 \, d\theta$$

$$= \int_{\pi/4}^{5\pi/4} \left(1 + 2\cos\theta + \cos^2\theta\right) \, d\theta$$

$$= \int_{\pi/4}^{5\pi/4} \left(1 + 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right) \, d\theta$$

$$= \int_{\pi/4}^{5\pi/4} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) \, d\theta$$

$$= \frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta \Big|_{\pi/4}^{5\pi/4}$$

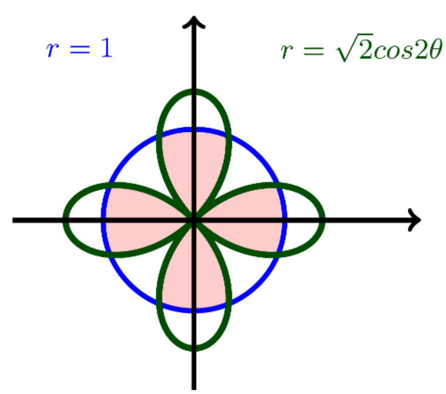
$$= \frac{15\pi}{8} - \sqrt{2} + \frac{1}{4} - \left(\frac{3\pi}{8} + \sqrt{2} + \frac{1}{4}\right)$$

$$= \frac{3\pi}{2} - 2\sqrt{2} \ \ unit^2$$

Exercise

Find the area of the region common interior $r = 1$ and $r = \sqrt{2}\ \cos 2\theta$

Solution



$$r = \sqrt{2}\ \cos 2\theta = 1$$

$$\cos 2\theta = \frac{1}{\sqrt{2}}$$

$$2\theta = \frac{\pi}{4}, \ \frac{7\pi}{4} \quad \Rightarrow \quad \theta = \frac{\pi}{8}, \ \frac{7\pi}{8}, \ \frac{9\pi}{8}, \ \frac{13\pi}{8}$$

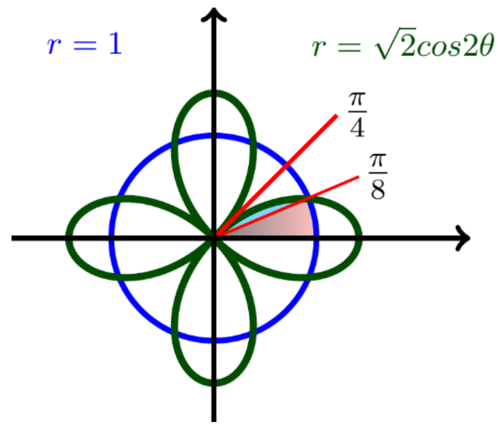
$$\sqrt{2}\ \cos 2\theta = 0$$

$$2\theta = \frac{\pi}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{4}$$

By symmetry:

$r=\sqrt{2}\ \cos 2\theta \quad \to \quad 0\le\theta\le\frac{\pi}{8}$

$r=1 \quad \to \quad \frac{\pi}{8}\le\theta\le\frac{\pi}{4}$



$$A=8\cdot\frac{1}{2}\int_0^{\frac{\pi}{8}}1\,d\theta \;+\; 8\cdot\frac{1}{2}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}}\left(\sqrt{2}\ \cos 2\theta\right)^2\ d\theta$$

$$=4\theta\Big|_0^{\frac{\pi}{8}} \;+\; 8\int_{\frac{\pi}{8}}^{\frac{\pi}{4}}\cos^2 2\theta\,d\theta$$

$$=4\left(\frac{\pi}{8}-0\right)+4\int_{\frac{\pi}{8}}^{\frac{\pi}{4}}(1+\cos 4\theta)\ d\theta$$

$$=\frac{\pi}{2}+4\left(\theta+\frac{1}{4}\sin 4\theta\right.\Big|_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

$$=\frac{\pi}{2}+4\left(\frac{\pi}{4}-\frac{\pi}{8}-\frac{1}{4}\right)$$

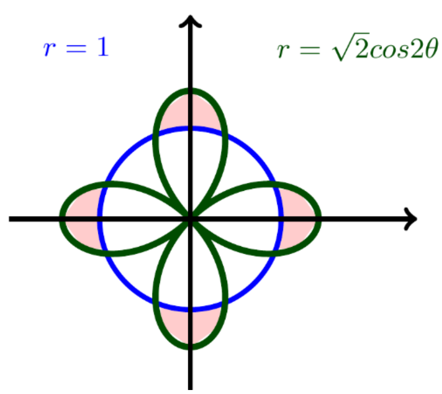
$$=\frac{\pi}{2}+\frac{\pi}{2}-1$$

$$=\pi-1\quad unit^2$$

Exercise

Find the area of the region outside $r=1$ and inside $r=\sqrt{2}\ \cos 2\theta$

Solution



$r=\sqrt{2}\ \cos 2\theta=1$

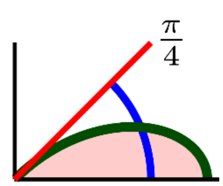
$\cos 2\theta=\frac{1}{\sqrt{2}}$

$2\theta=\frac{\pi}{4} \quad\Rightarrow\quad \theta=\frac{\pi}{8}$

$\sqrt{2}\ \cos 2\theta=0$

$2\theta=\frac{\pi}{2} \quad\Rightarrow\quad \theta=\frac{\pi}{4}$

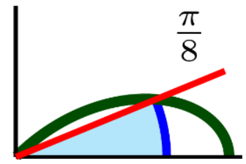
By symmetry:

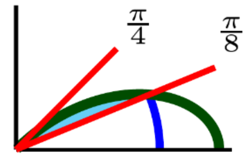


$$A=8\cdot\frac{1}{2}\int_0^{\frac{\pi}{8}}\left(\sqrt{2}\ \cos 2\theta\right)^2\ d\theta-8\cdot\frac{1}{2}\int_0^{\frac{\pi}{8}}1\,d\theta-8\cdot\frac{1}{2}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}}\left(\sqrt{2}\ \cos 2\theta\right)^2\ d\theta$$

$$= 8\int_0^{\frac{\pi}{8}} \cos^2 2\theta \, d\theta \; - \; 4\left(\theta \right|_0^{\frac{\pi}{8}} \; - \; 8\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta$$

$$= 4\int_0^{\frac{\pi}{8}} (1+\cos 4\theta) \, d\theta \; - \; \frac{\pi}{2} \; - \; 4\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} (1+\cos 4\theta) \, d\theta$$





$$= 4\left(\theta + \frac{1}{4}\sin 4\theta \right|_0^{\frac{\pi}{8}} - 4\left(\theta + \frac{1}{4}\sin 4\theta \right|_{\frac{\pi}{8}}^{\frac{\pi}{4}} - \frac{\pi}{2}$$

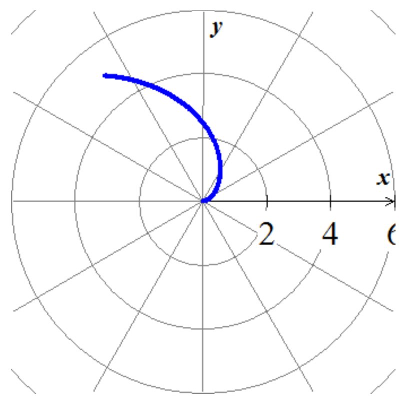
$$= 4\left(\frac{\pi}{8} + \frac{1}{4}\right) - 4\left(\frac{\pi}{4} - \frac{\pi}{8} - \frac{1}{4}\right) - \frac{\pi}{2}$$

$$= \frac{\pi}{2} + 1 - \frac{\pi}{2} + 1 - \frac{\pi}{2}$$

$$= 2 - \frac{\pi}{2} \quad unit^2$$

Exercise

Find the length of the spiral $r = \theta^2, \quad 0 \le \theta \le \sqrt{5}$



Solution

$$r = \theta^2$$

$$\frac{dr}{d\theta} = 2\theta$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{\theta^4 + 4\theta^2}$$

$$= |\theta| \sqrt{\theta^2 + 4}$$

$$L = \int_0^{\sqrt{5}} \theta \sqrt{\theta^2 + 4} \, d\theta \qquad\qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

$$= \frac{1}{2}\int_0^{\sqrt{5}} \left(\theta^2 + 4\right)^{1/2} d\left(\theta^2 + 4\right)$$

$$= \frac{1}{3}\left(\theta^2 + 4\right)^{3/2} \Big|_0^{\sqrt{5}}$$

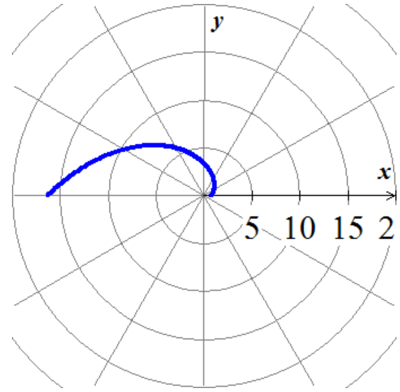
$$= \frac{1}{3}\left(9^{3/2} - 4^{3/2}\right)$$

$$= \frac{1}{3}(27 - 8)$$

$$= \frac{19}{3} \quad unit$$

Exercise

Find the length of the spiral $r = \frac{e^{\theta}}{\sqrt{2}}, \quad 0 \le \theta \le \pi$



Solution

$$r = \frac{e^{\theta}}{\sqrt{2}}$$

$$\frac{dr}{d\theta} = \frac{1}{\sqrt{2}}e^{\theta}$$

$$\begin{aligned}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} &= \sqrt{\frac{1}{2}e^{2\theta} + \frac{1}{2}e^{2\theta}} \\ &= \sqrt{e^{2\theta}} \\ &= e^{\theta}\end{aligned}$$

$$L = \int_0^{\pi} e^{\theta}\, d\theta \qquad\qquad L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= e^{\theta} \Big|_0^{\pi}$$

$$\underline{= e^{\pi} - 1 \quad unit}$$

Exercise

Find the length of the curve $r = a\sin^2\left(\frac{\theta}{2}\right), \quad 0 \le \theta \le \pi, \quad a > 0$

Solution

$$r = a\sin^2\left(\frac{\theta}{2}\right)$$

$$\frac{dr}{d\theta} = a\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

$$\begin{aligned}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} &= \sqrt{a^2\sin^4\left(\frac{\theta}{2}\right) + a^2\sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)} \\ &= a\sin\left(\frac{\theta}{2}\right)\sqrt{\sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right)} \qquad \sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right) = 1 \\ &= a\left|\sin\left(\frac{\theta}{2}\right)\right|\end{aligned}$$

$$L = \int_0^{\pi} a\sin\left(\frac{\theta}{2}\right)\, d\theta \qquad\qquad L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= -2a\cos\left(\frac{\theta}{2}\right)\Bigg|_0^{\pi}$$

$$= -2a\left(\cos\left(\frac{\pi}{2}\right) - \cos 0\right)$$

$$\underline{= 2a \quad unit}$$

Exercise

Find the length of the parabolic segment $r = \frac{6}{1+\cos\theta}, \quad 0 \le \theta \le \frac{\pi}{2}$

Solution

$$r = \frac{6}{1+\cos\theta}$$

$$\frac{dr}{d\theta} = \frac{6\sin\theta}{(1+\cos\theta)^2}$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{\frac{36}{(1+\cos\theta)^2} + \frac{36\sin^2\theta}{(1+\cos\theta)^4}}$$

$$= \frac{6}{|1+\cos\theta|}\sqrt{1 + \frac{\sin^2\theta}{(1+\cos\theta)^2}}$$

$$= \frac{6}{|1+\cos\theta|}\sqrt{\frac{1+2\cos\theta+\cos^2\theta+\sin^2\theta}{(1+\cos\theta)^2}}$$

$$= \frac{6}{(1+\cos\theta)^2}\sqrt{2+2\cos\theta}$$

$$= \frac{6\sqrt{2}}{(1+\cos\theta)^2}(1+\cos\theta)^{1/2}$$

$$= \frac{6\sqrt{2}}{(1+\cos\theta)^{3/2}}$$

$$L = \int_0^{\pi/2} \frac{6\sqrt{2}}{(1+\cos\theta)^{3/2}}\, d\theta \qquad\qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 6\sqrt{2}\int_0^{\pi/2} \frac{d\theta}{\left(2\cos^2\frac{\theta}{2}\right)^{3/2}}$$

$$= 6\sqrt{2}\int_0^{\pi/2} \frac{d\theta}{2^{3/2}\cos^3\frac{\theta}{2}}$$

$$= \frac{6}{2}\int_0^{\pi/2} \sec^3\frac{\theta}{2}\, d\theta$$

$$= 3\int_0^{\pi/2} \sec^3\frac{\theta}{2}\, d\theta$$

$$= 6\int_0^{\pi/2} \sec^3\frac{\theta}{2}\, d\left(\frac{\theta}{2}\right)$$

Let: $u = \sec x \qquad dv = \sec^2 xdx$

$du = \sec x\tan xdx \qquad v = \tan x$

$$\int \sec^3 xdx = \sec x\tan x - \int \tan x\left(\sec x\tan xdx\right)$$

$$= \sec x\tan x - \int \tan^2 x\sec xdx$$

$$= \sec x\tan x - \int \left(\sec^2 x - 1\right)\sec x\, dx$$

$$= \sec x\tan x - \int \left(\sec^3 x - \sec x\right)\, dx$$

$$= \sec x\tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$2\int \sec^3 x\, dx = \sec x\tan x + \int \sec x\, dx$$

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right|$$

$$= 6\left(\frac{1}{2}\sec\frac{\theta}{2}\tan\frac{\theta}{2}\,\Bigg|_0^{\pi/2} + \frac{1}{2}\ln\left|\sec\frac{\theta}{2} + \tan\frac{\theta}{2}\right|\,\Bigg|_0^{\pi/2}\right)$$

$$= 6\left(\frac{1}{2}\sec\frac{\pi}{4}\tan\frac{\pi}{4} + \frac{1}{2}\ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \frac{1}{2}\ln 1\right)$$

$$= 6\left(\frac{\sqrt{2}}{2} + \frac{1}{2}\ln\left|\sqrt{2} + 1\right|\right)$$

$$= 3\sqrt{2} + 3\ln\left(\sqrt{2}+1\right)\ \textit{unit}$$

Exercise

Find the length of the curve $r = \cos^3\left(\frac{\theta}{3}\right), \quad 0 \le \theta \le \frac{\pi}{4}$

Solution

$$r = \cos^3\left(\frac{\theta}{3}\right)$$

$$\frac{dr}{d\theta} = -\cos^2\left(\frac{\theta}{3}\right)\sin\left(\frac{\theta}{3}\right)$$

$$\begin{aligned}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} &= \sqrt{\cos^6\left(\frac{\theta}{3}\right) + \cos^4\left(\frac{\theta}{3}\right)\sin^2\left(\frac{\theta}{3}\right)} \\ &= \left|\cos^2\left(\frac{\theta}{3}\right)\right|\sqrt{\cos^2\left(\frac{\theta}{3}\right) + \sin^2\left(\frac{\theta}{3}\right)} \\ &= \cos^2\left(\frac{\theta}{3}\right)\end{aligned}$$

$$L = \int_0^{\pi/4} \cos^2\left(\frac{\theta}{3}\right)\, d\theta \qquad\qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \left(1 + \cos\frac{2\theta}{3}\right) d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{3}{2}\sin\frac{2\theta}{3}\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{3}{2}\sin\frac{\pi}{6} - 0\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{3}{4}\right)$$

$$= \frac{\pi}{8} + \frac{3}{8} \quad \textit{unit}$$

Exercise

Find the length of the curve $r = \sqrt{1 + \sin 2\theta}, \quad 0 \le \theta \le \pi\sqrt{2}$

Solution

$$r = \sqrt{1 + \sin 2\theta}$$

$$\begin{aligned}\frac{dr}{d\theta} &= \frac{1}{2}(1 + \sin 2\theta)^{-1/2}(2\cos 2\theta) \\ &= \cos 2\theta (1 + \sin 2\theta)^{-1/2}\end{aligned}$$

$$\begin{aligned}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} &= \sqrt{1 + \sin 2\theta + \cos^2 2\theta (1 + \sin 2\theta)^{-1}} \\ &= \sqrt{1 + \sin 2\theta + \frac{\cos^2 2\theta}{1 + \sin 2\theta}}\end{aligned}$$

$$= \sqrt{\frac{1 + 2\sin 2\theta + \sin^2 2\theta + \cos^2 2\theta}{1 + \sin 2\theta}}$$

$$= \sqrt{\frac{2(1 + \sin 2\theta)}{1 + \sin 2\theta}}$$

$$= \sqrt{2}$$

$$L = \int_0^{\pi\sqrt{2}} \sqrt{2}\, d\theta \qquad\qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \sqrt{2}\ \theta \Big|_0^{\pi\sqrt{2}}$$

$$= \sqrt{2}\left(\pi\sqrt{2} - 0\right)$$

$$= 2\pi \quad unit$$

Exercise

Find the length of $\quad r = 8 \quad 0 \le \theta \le 2\pi$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{64 + 0}$$

$$= 8$$

$$L = \int_0^{2\pi} 8\, d\theta \qquad\qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 16\pi \ \ unit$$

Exercise

Find the length of $\quad r = a \quad 0 \le \theta \le 2\pi$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2 + 0}$$

$$= a$$

$$L = \int_0^{2\pi} a\, d\theta \qquad\qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 2\pi a \ \ unit$$

Exercise

Find the length of $r = 4\sin\theta \quad 0 \le \theta \le \pi$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{16\sin^2\theta + 16\cos^2\theta}$$

$$= 4\sqrt{\sin^2\theta + \cos^2\theta} \qquad \sin^2\theta + \cos^2\theta = 1$$

$$\underline{= 4}$$

$$L = \int_0^\pi 4\, d\theta \qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$\underline{= 4\pi \ \ unit}$$

Exercise

Find the length of $r = 2a\cos\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{4a^2\cos^2\theta + 4a^2\sin^2\theta} \qquad \sin^2\theta + \cos^2\theta = 1$$

$$\underline{= 2a}$$

$$L = 2a\int_{-\pi/2}^{\pi/2} d\theta \qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 2a\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$\underline{= 2\pi a \quad unit}$$

Exercise

Find the length of $r = 1 + \sin\theta \quad 0 \le \theta \le 2\pi$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{(1+\sin\theta)^2 + \cos^2\theta}$$

$$= \sqrt{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta} \qquad \sin^2\theta + \cos^2\theta = 1$$

$$= \sqrt{2 + 2\sin\theta}$$

$$L=\sqrt{2}\int_0^{2\pi}\sqrt{1+\sin\theta}\ d\theta \qquad\qquad L=\int_\alpha^\beta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$=\sqrt{2}\int_0^{2\pi}\sqrt{1+\sin\theta}\,\frac{\sqrt{1-\sin\theta}}{\sqrt{1-\sin\theta}}\ d\theta \qquad\qquad \cos\theta=\pm\sqrt{1-\sin^2\theta}$$

$$=2\sqrt{2}\int_{\pi/2}^{3\pi/2}\frac{-\cos\theta}{\sqrt{1-\sin\theta}}\ d\theta \qquad\qquad \frac{\pi}{2}<\theta<\frac{3\pi}{2}\ \ \rightarrow\cos\theta<0$$

$$=2\sqrt{2}\int_{\pi/2}^{3\pi/2}(1-\sin\theta)^{-1/2}\ d(1-\sin\theta)$$

$$=4\sqrt{2}\left(\sqrt{1-\sin\theta}\ \right|_{\pi/2}^{3\pi/2}$$

$$=4\sqrt{2}\left(\sqrt{2}-0\right)$$

$$=8\quad unit$$

Exercise

Find the length of $\quad r=8(1+\cos\theta)\quad 0\le\theta\le 2\pi$

Solution

$$\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}=\sqrt{64(1+\cos\theta)^2+64\sin^2\theta}$$

$$=8\sqrt{1+2\cos\theta+\cos^2\theta+\sin^2\theta} \qquad\qquad \sin^2\theta+\cos^2\theta=1$$

$$=8\sqrt{2+2\cos\theta}$$

$$L=8\sqrt{2}\int_0^{2\pi}\sqrt{1+\cos\theta}\ d\theta \qquad\qquad L=\int_\alpha^\beta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$=8\sqrt{2}\int_0^{2\pi}\sqrt{1+\cos\theta}\,\frac{\sqrt{1-\cos\theta}}{\sqrt{1-\cos\theta}}\ d\theta \qquad\qquad \sin\theta=\pm\sqrt{1-\cos^2\theta}$$

$$=16\sqrt{2}\int_0^{\pi}\frac{\sin\theta}{\sqrt{1-\cos\theta}}\ d\theta \qquad\qquad 0<\theta<\pi\ \ \rightarrow\sin\theta>0$$

$$=16\sqrt{2}\int_0^{\pi}(1-\cos\theta)^{-1/2}\ d(1-\cos\theta)$$

$$=32\sqrt{2}\left(\sqrt{1-\cos\theta}\ \right|_0^{\pi}$$

$= 32\sqrt{2}\left(\sqrt{2} - 0\right)$

$= 64 \quad unit$

Exercise

Find the length of $\quad r = 2\theta \quad 0 \le \theta \le \frac{\pi}{2}$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{4\theta^2 + 4}$$

$$= 2\sqrt{1+\theta^2}$$

$$L = 2\int_0^{\pi/2} \sqrt{1+\theta^2}\ d\theta \qquad\qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$\theta = \tan\alpha \qquad \sqrt{\theta^2 + 1} = \sec\alpha$$

$$d\theta = \sec^2\alpha\ d\alpha$$

$$= 2\int_0^{\pi/2} \sec^3\alpha\ d\alpha$$

Let: $\quad u = \sec x \qquad dv = \sec^2 xdx$

$\quad du = \sec x \tan xdx \qquad v = \tan x$

$$\int \sec^3 xdx = \sec x \tan x - \int \tan x\left(\sec x \tan xdx\right)$$

$$= \sec x \tan x - \int \tan^2 x \sec xdx$$

$$= \sec x \tan x - \int \left(\sec^2 x - 1\right)\sec x\ dx$$

$$= \sec x \tan x - \int \left(\sec^3 x - \sec x\right)\ dx$$

$$= \sec x \tan x - \int \sec^3 x\ dx + \int \sec x\ dx$$

$$2\int \sec^3 x\ dx = \sec x \tan x + \int \sec x\ dx$$

$$\int \sec^3 x\ dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right|$$

$$= 2\left(\frac{1}{2}\sec\alpha\tan\alpha + \frac{1}{2}\ln\left|\sec\alpha + \tan\alpha\right|\ \right|_0^{\pi/2}$$

$$= 2\left(\frac{1}{2}\theta\sqrt{1+\theta^2} + \frac{1}{2}\ln\left|\sqrt{1+\theta^2} + \theta\right|\ \right|_0^{\pi/2}$$

$$\boxed{= \frac{\pi}{2}\sqrt{1+\frac{\pi^2}{4}} + \ln\left(\sqrt{1+\frac{\pi^2}{4}} + \frac{\pi}{2}\right) \quad unit}$$

Exercise

Find the length of $\quad r = \sec\theta \quad 0 \le \theta \le \frac{\pi}{3}$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{\sec^2\theta + \sec^2\theta\tan^2\theta}$$

$$= \sec\theta\sqrt{1+\tan^2\theta}$$

$$\boxed{= \sec^2\theta}$$

$$L = \int_0^{\pi/3} \sec^2\theta\, d\theta \qquad\qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \tan\theta \Big|_0^{\pi/3}$$

$$\boxed{= \sqrt{3} \quad unit}$$

Exercise

Find the length of $\quad r = \frac{1}{\theta} \quad \pi \le \theta \le 2\pi$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{\frac{1}{\theta^2} + \frac{1}{\theta^4}}$$

$$\boxed{= \frac{1}{\theta^2}\sqrt{\theta^2+1}}$$

$$L = \int_\pi^{2\pi} \frac{1}{\theta^2}\sqrt{\theta^2+1}\, d\theta \qquad\qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \sinh^{-1}\theta - \frac{\sqrt{1+\theta^2}}{\theta}\ \Big|_\pi^{2\pi}$$

$$= \sinh^{-1} 2\pi - \frac{\sqrt{1+4\pi^2}}{2\pi} - \sinh^{-1} \pi + \frac{\sqrt{1+\pi^2}}{\pi}$$

$$= 2.5376 - 1.01259 - 1.8623 + 1.04944$$

$$\approx 0.71215$$

Exercise

Find the length of $r = e^{\theta} \quad 0 \le \theta \le \pi$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{e^{2\theta} + e^{2\theta}}$$

$$= \sqrt{2}e^{\theta}$$

$$L = \sqrt{2}\int_0^{\pi} e^{\theta}\, d\theta \qquad L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \sqrt{2}\left(e^{\pi} - 1\right) \quad \textit{unit}$$

Exercise

Find the length of $r = 5\cos\theta \quad \frac{\pi}{2} \le \theta \le \pi$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{25\cos^2\theta + 25\sin^2\theta}$$

$$= 5$$

$$L = \int_{\pi/2}^{\pi} 5\, d\theta \qquad L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 5\theta \Big|_{\pi/2}^{\pi}$$

$$= \frac{5\pi}{2} \quad \textit{unit}$$

Exercise

Find the length of $r = 3(1-\cos\theta) \quad 0 \le \theta \le \pi$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{9(1-\cos\theta)^2 + 9\sin^2\theta}$$

$$= 3\sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= 3\sqrt{2 - 2\cos\theta}$$

$$L = 3\int_0^{\pi} \sqrt{4\sin^2\theta}\, d\theta \qquad\qquad L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$L = 6\int_0^{\pi} \sin\theta\, d\theta$$

$$= -6\cos\theta \Big|_0^{\pi}$$

$$= -6(1-1)$$

$$= 12 \quad unit$$

Exercise

Find the length of one petal $r = 2\sin 6\phi$

Solution

$$\frac{dr}{d\phi} = 12\cos 6\phi$$

$$\sqrt{r^2 + \left(\frac{dr}{d\phi}\right)^2} = \sqrt{4\sin^2 6\phi + 144\cos^2 6\phi}$$

$$= 2\sqrt{\sin^2 6\phi + 36\cos^2 6\phi}$$

$$= 2\sqrt{1 + 35\cos^2 6\phi}$$

$$L = 2\int_0^{\pi/6} \sqrt{1 + 35\cos^2 6\phi}\, d\phi \qquad\qquad L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

≈ 6.28 You have to use ***Calculator*** or ***software*** to do this kind of integration.

Exercise

Find the length of inner loop $r = 3 - 6\cos\phi$

Solution

$$\frac{dr}{d\phi} = 6\sin\phi$$

$$\begin{aligned}\sqrt{r^2 + \left(\frac{dr}{d\phi}\right)^2} &= \sqrt{(3-6\cos\phi)^2 + 36\sin^2\phi} \\ &= \sqrt{9 - 36\cos\phi + 36\cos^2\phi + 36\sin^2\phi} \\ &= \sqrt{9 - 36\cos\phi + 36} \\ &= \sqrt{45 - 36\cos\phi}\end{aligned}$$

$$L = 2\int_0^{\pi/3} \sqrt{45 - 36\cos\phi}\ d\phi$$

$$\approx 6.28388 \quad unit$$

Exercise

Find the length of $r = e^{2\theta}$ $0 \le \theta \le 2$

Solution

$$\frac{dr}{d\theta} = 2e^{2\theta}$$

$$\begin{aligned}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} &= \sqrt{e^{4\theta} + 4e^{4\theta}} \\ &= \sqrt{5}\ e^{2\theta}\end{aligned}$$

$$L = \sqrt{5}\int_0^2 e^{2\theta}\ d\theta$$

$$= \frac{\sqrt{5}}{2}\ e^{2\theta}\Big|_0^2$$

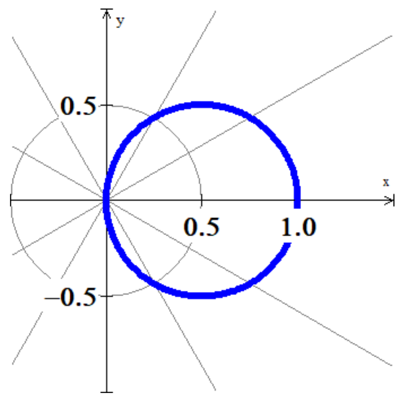
$$= \frac{\sqrt{5}}{2}\left(e^4 - 1\right) \quad unit$$

Exercise

Find the length of $r = \cos\theta$

Solution

θ	r
0	1
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	$-\frac{1}{2}$
π	-1



$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\frac{dr}{d\theta} = -\sin\theta$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{\cos^2\theta + \sin^2\theta}$$

$$= 1$$

$$L = \int_0^{2\pi} 1\, d\theta$$

$$= \theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi \quad unit$$

Exercise

Find the length of $r = a(1 - \cos\theta)$

Solution

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}$$

$$= a\sqrt{1-2\cos\theta+\cos^2\theta+\sin^2\theta}$$

$$= a\sqrt{2-2\cos\theta}$$

$$L = a\sqrt{2}\int_0^{2\pi}\sqrt{1-\cos\theta}\ d\theta$$

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$$

$$\sqrt{2}\ \sin\frac{\theta}{2} = \sqrt{1-\cos\theta}$$

$$= 2a\int_0^{2\pi}\sin\frac{\theta}{2}\ d\theta$$

$$= -4a\cos\frac{\theta}{2}\Big|_0^{2\pi}$$

$$= -4a(-1-1)$$

$$= 8a\ \ unit$$

Exercise

Find the surface area bounded by $r = 6\cos\theta \quad 0 \le \theta \le \frac{\pi}{2}$ *revolving about Polar axis*

Solution

$$\sqrt{r^2+(r')^2} = \sqrt{36\cos^2\theta+36\sin^2\theta}$$

$$= 6$$

$$S = 2\pi\int_0^{\pi/2} 6\cos\theta\sin\theta(6)\ d\theta \qquad S = 2\pi\int_\alpha^\beta f(\theta)\sin\theta\sqrt{(f(\theta))^2+(f'(\theta))^2}\ d\theta$$

$$= 36\pi\int_0^{\pi/2}\sin 2\theta\ d\theta \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= -18\pi\cos 2\theta\Big|_0^{\pi/2}$$

$$= -18\pi(-1-1)$$

$$= 36\pi\ \ \ unit$$

Exercise

Find the surface area bounded by $r = a\cos\theta \quad 0 \le \theta \le \frac{\pi}{2}$ *revolving about* $\theta = \frac{\pi}{2}$

Solution

$$\sqrt{r^2 + (r')^2} = \sqrt{a^2\cos^2\theta + a^2\sin^2\theta}$$

$$= a$$

$$S = 2\pi\int_0^{\pi/2} a^2\cos^2\theta\, d\theta \qquad S = 2\pi\int_\alpha^\beta f(\theta)\cos\theta\sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta$$

$$= a^2\pi\int_0^{\pi/2} (1+\cos 2\theta)\, d\theta \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \pi a^2\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/2}$$

$$= \pi a^2\left(\frac{\pi}{2}\right)$$

$$= \frac{1}{2}\pi^2 a^2 \quad unit$$

Exercise

Find the surface area bounded by $r = e^{a\theta} \quad 0 \le \theta \le \frac{\pi}{2}$ *revolving about* $\theta = \frac{\pi}{2}$

Solution

$$\sqrt{r^2 + (r')^2} = \sqrt{e^{2a\theta} + a^2e^{2a\theta}}$$

$$= e^{a\theta}\sqrt{1+a^2}$$

$$S = 2\pi\sqrt{1+a^2}\int_0^{\pi/2} e^{a\theta}\cos\theta\left(e^{a\theta}\right)d\theta \qquad S = 2\pi\int_\alpha^\beta f(\theta)\cos\theta\sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta$$

$$= 2\pi\sqrt{1+a^2}\int_0^{\pi/2} e^{2a\theta}\cos\theta\, d\theta$$

		$\int\cos\theta d\theta$
+	$e^{2a\theta}$	$\sin\theta$
−	$2ae^{2a\theta}$	$-\cos\theta$
+	$4a^2e^{2a\theta}$	$-\int\cos\theta$

$$\int e^{2a\theta}\cos\theta\,d\theta = e^{2a\theta}\sin\theta + 2ae^{2a\theta}\cos\theta - 4a^2\int e^{2a\theta}\cos\theta\,d\theta$$

$$\left(1+4a^2\right)\int e^{2a\theta}\cos\theta\,d\theta = e^{2a\theta}\left(\sin\theta + 2a\cos\theta\right)$$

$$= \frac{2\pi\sqrt{1+a^2}}{1+4a^2}\left(e^{2a\theta}\left(\sin\theta+2a\cos\theta\right)\right)\Big|_0^{\pi/2}$$

$$\boxed{= \frac{2\pi\sqrt{1+a^2}}{1+4a^2}\left(e^{a\pi}-2a\right)\quad unit^2}$$

Exercise

Find the area surface bounded by $r = a\left(1+\cos\theta\right)$ $\quad 0 \le \theta \le \pi \quad$ *revolving about polar axis*

Solution

$$\sqrt{r^2+\left(r'\right)^2} = \sqrt{a^2\left(1+\cos\theta\right)^2 + a^2\sin^2\theta}$$

$$= a\sqrt{\left(1+2\cos\theta+\cos^2\theta\right)+\sin^2\theta}$$

$$\boxed{= a\sqrt{2+2\cos\theta}}$$

$$S = 2a^2\pi\sqrt{2}\int_0^{\pi/2}\left(1+\cos\theta\right)\sin\theta\left(\sqrt{1+\cos\theta}\right)d\theta \qquad S = 2\pi\int_\alpha^\beta f(\theta)\sin\theta\sqrt{\left(f(\theta)\right)^2+\left(f'(\theta)\right)^2}\,d\theta$$

$$= -2a^2\pi\sqrt{2}\int_0^{\pi/2}\left(1+\cos\theta\right)^{3/2}\,d\left(1+\cos\theta\right)$$

$$= -\frac{4\sqrt{2}}{5}a^2\pi\left(1+\cos\theta\right)^{5/2}\Big|_0^{\pi/2}$$

$$= -\frac{4\sqrt{2}}{5}a^2\pi\left(1-1-2^{5/2}\right)$$

$$= \frac{4\sqrt{2}}{5}a^2\pi\left(4\sqrt{2}\right)$$

$$\boxed{= \frac{32}{5}\pi a^2\quad unit^2}$$

Exercise

Find the surface area bounded by $r = 1 + 4\cos\theta \quad 0 \le \theta \le \frac{\pi}{2}$ *revolving about Polar axis*

Solution

$$\sqrt{r^2 + (r')^2} = \sqrt{(1+4\cos\theta)^2 + 16\sin^2\theta}$$

$$= \sqrt{1 + 8\cos\theta + 16\cos^2\theta + 16\sin^2\theta}$$

$$= \sqrt{17 + 8\cos\theta}$$

$$S = 2\pi \int_0^{\pi/2} (1 + 4\cos\theta)\sin\theta\left(\sqrt{17 + 8\cos\theta}\right) d\theta \qquad S = 2\pi \int_\alpha^\beta f(\theta)\sin\theta\sqrt{(f(\theta))^2 + (f'(\theta))^2}\ d\theta$$

$$= 2\pi \int_0^{\pi/2} \sin\theta(17 + 8\cos\theta)^{1/2}\ d\theta + 8\pi \int_0^{\pi/2} \cos\theta\sin\theta(17 + 8\cos\theta)^{1/2}\ d\theta$$

$$= -\frac{\pi}{4} \int_0^{\pi/2} (17 + 8\cos\theta)^{1/2}\ d(17 + 8\cos\theta) + 8\pi \int_0^{\pi/2} \cos\theta\sin\theta(17 + 8\cos\theta)^{1/2}\ d\theta$$

$$-\frac{\pi}{4} \int_0^{\pi/2} (17 + 8\cos\theta)^{1/2}\ d(17 + 8\cos\theta) = -\frac{\pi}{6}(17 + 8\cos\theta)^{3/2}\ \Big|_0^{\pi/2}$$

$$= -\frac{\pi}{6}\left(17\sqrt{17} - 125\right)$$

$$8\pi \int_0^{\pi/2} \cos\theta\sin\theta(17 + 8\cos\theta)^{1/2}\ d\theta$$

$$u = 17 + 8\cos\theta \qquad \cos\theta = \frac{1}{8}(u - 17)$$

$$du = -8\sin\theta\ d\theta$$

$$= -\frac{\pi}{8} \int_0^{\pi/2} (u - 17)u^{1/2}\ du$$

$$= -\frac{\pi}{8} \int_0^{\pi/2} \left(u^{3/2} - 17u^{1/2}\right) du$$

$$= -\frac{\pi}{8}\ \left(\frac{2}{5}(17 + 8\cos\theta)^{5/2} - \frac{34}{3}(17 + 8\cos\theta)^{3/2}\ \right|_0^{\pi/2}$$

$$= -\frac{\pi}{8}\left(\frac{578}{5}\sqrt{17} - \frac{578\sqrt{17}}{3} - 1{,}250 + \frac{4{,}250}{3}\right)$$

$$= -\frac{\pi}{8}\left(-\frac{1{,}156\sqrt{17}}{15} + \frac{500}{3}\right)$$

$$= -\frac{17\sqrt{17}\pi}{6} + \frac{125\pi}{6} + \frac{289\sqrt{17}\pi}{30} - \frac{125\pi}{6}$$

$$= \frac{34\pi\sqrt{17}}{5} \quad unit^2$$

Exercise

Find the surface area bounded by $r = 2\sin\theta \quad 0 \le \theta \le \frac{\pi}{2} \quad$ *revolving about* $\theta = \frac{\pi}{2}$

Solution

$$\sqrt{r^2 + (r')^2} = \sqrt{4\sin^2\theta + 4\cos^2\theta}$$

$$= 2$$

$$S = 2\pi \int_0^{\pi/2} 4\sin\theta\cos\theta \, d\theta \qquad\qquad S = 2\pi \int_\alpha^\beta f(\theta)\cos\theta\sqrt{(f(\theta))^2 + (f'(\theta))^2} \, d\theta$$

$$= 4\pi \int_0^{\pi/2} \sin 2\theta \, d\theta \qquad\qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= -2\pi \cos 2\theta \Big|_0^{\pi/2}$$

$$= -2\pi(-1-1)$$

$$= 4\pi \quad unit^2$$

Exercise

Find the surface area of the torus generated by revolving the circle given by $r = 2$ about the line $r = 5\sec\theta$

Solution

$$\sqrt{r^2 + (r')^2} = \sqrt{4+0}$$

$$= 2$$

$$S = 4\pi \int_0^{2\pi} \left(\frac{5}{\cos\theta} - 2\right)\cos\theta \, d\theta \qquad\qquad S = 2\pi \int_\alpha^\beta (r_2 - r)\cos\theta\sqrt{r^2 + (r')^2} \, d\theta$$

$$= 4\pi \int_0^{2\pi} (5 - 2\cos\theta) \, d\theta$$

$$= 4\pi \left(5\theta - 2\sin\theta\right)\Big|_0^{2\pi}$$

$$= 4\pi(10\pi)$$

$$\boxed{= 40\pi^2 \quad unit^2}$$

Exercise

Find the surface area of the torus generated by revolving the circle given by $r = a$ about the line $r = b\sec\theta$, where $0 < a < b$

Solution

$$\sqrt{r^2 + (r')^2} = \sqrt{a^2 + 0}$$

$$\boxed{= a}$$

$$S = 2\pi a \int_0^{2\pi} \left(\frac{b}{\cos\theta} - a\right)\cos\theta \, d\theta \qquad\qquad S = 2\pi \int_\alpha^\beta \left(r_2 - r\right)\cos\theta \sqrt{r^2 + (r')^2} \, d\theta$$

$$= 2\pi a \int_0^{2\pi} (b - a\cos\theta) \, d\theta$$

$$= 2\pi a \left(b\theta - a\sin\theta \right|_0^{2\pi}$$

$$= 2\pi a(2b\pi)$$

$$\boxed{= 4\pi^2 ab \quad unit^2}$$

Exercise

Let a and b be positive constants. Find the area of the region in the first quadrant bounded by the graph of the polar equation

$$r = \frac{ab}{a\sin\theta + b\cos\theta}, \quad 0 \le \theta \le \frac{\pi}{2}$$

Solution

$$r(a\sin\theta + b\cos\theta) = ab$$

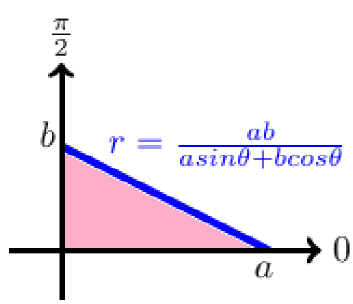
$$ar\sin\theta + br\cos\theta = ab$$

$$ax + by = ab \quad (line\ segment)$$

$$x = 0 \quad \to y = b$$

$$y = 0 \quad \to x = a$$

$$\boxed{Area = \frac{1}{2}ab \quad unit^2}$$



Exercise

Assume m is a positive integer

a) Even number of leaves: what is the relationship between the total area enclosed by the $4m$-leaf rose $r = \cos(2m\theta)$ and m?

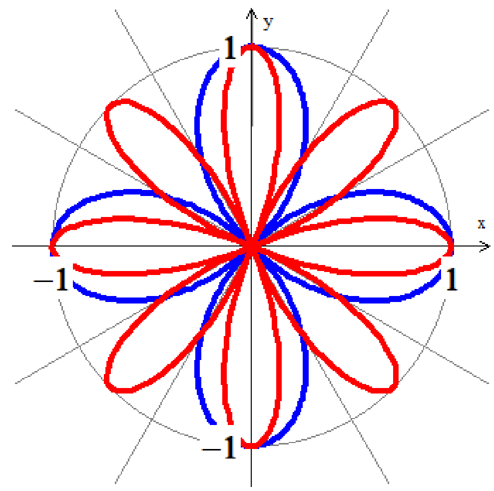
b) Odd number of leaves: what is the relationship between the total area enclosed by the $(2m+1)$-leaf rose $r = \cos((2m+1)\theta)$ and m?

Solution

a) For $m = 1 \quad \rightarrow \quad r = \cos 2\theta$

Which gives 4 equals leaves.

For any m, the numbers of leaves are $4m$.



Let: $r = \cos 2m\theta = 0$

$2m\theta = \frac{\pi}{2} \quad \rightarrow \quad \theta = \frac{\pi}{4m}$ (half a leaf)

Then the area of one half leaf is:

$$A = \frac{1}{2}\int_0^{\frac{\pi}{4m}} \cos^2(2m\theta)\, d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{4m}} (1+\cos 4m\theta)\, d\theta$$

$$= \frac{1}{4}\left(\theta + \frac{1}{4m}\sin 4m\theta\right) \Bigg|_0^{\frac{\pi}{4m}}$$

$$= \frac{1}{4}\left(\frac{\pi}{4m} + \frac{1}{4m}\sin\pi\right)$$

$$= \boxed{\frac{\pi}{16m}}$$

$$\text{Total area } = (8m)\frac{\pi}{16m}$$

$$= \boxed{\frac{\pi}{2} \quad unit^2}$$

Therefore; the area of any rose for any ***m*** is always $\frac{\pi}{2}$ and independent of m.

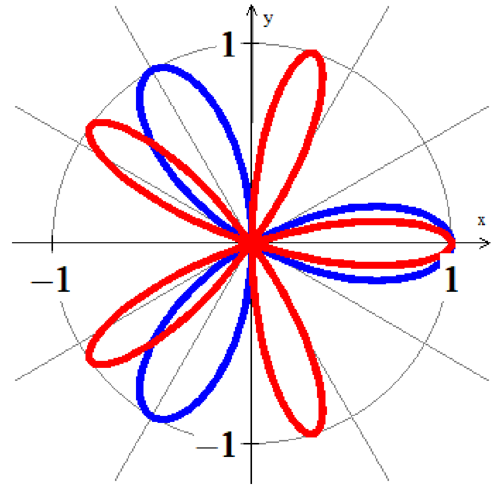
b) For $m = 1 \quad \rightarrow \quad r = \cos 3\theta$

For any m, the numbers of leaves are $(2m+1)$.

$r = \cos((2m+1)\theta) = 0$

$(2m+1)\theta = \frac{\pi}{2} \quad \rightarrow \quad \theta = \frac{\pi}{4m+2}$ (half a leaf)

Then the area of one half leaf is:



$$A=\frac{1}{2}\int_0^{\frac{\pi}{4m+2}} \cos^2(2m+1)\theta\, d\theta$$

$$=\frac{1}{4}\int_0^{\frac{\pi}{4m+2}} \left(1+\cos 2(2m+1)\theta\right)\, d\theta$$

$$=\frac{1}{4}\int_0^{\frac{\pi}{4m+2}} \left(1+\cos(4m+2)\theta\right)\, d\theta$$

$$=\frac{1}{4}\left(\theta+\frac{1}{4m+2}\sin(4m+2)\theta\right.\Bigg|_0^{\frac{\pi}{4m+2}}$$

$$=\frac{1}{4}\left(\frac{\pi}{4m+2}\right)$$

$$=\frac{\pi}{8(2m+1)}$$

Total area $=\left(2(2m+1)\right)\frac{\pi}{8(2m+1)}$

$$=\frac{\pi}{4}\ unit^2$$

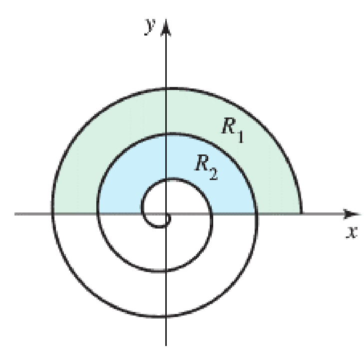
Exercise

Let R_n be the region bounded by the nth turn and the $(n+1)st$ turn of the spiral $r=e^{-\theta}$ in the first and second quadrants, for $\theta\ge 0$

a) Find the area A_n of R_n

b) Evaluate $\lim\limits_{n\to\infty} A_n$

c) Evaluate $\lim\limits_{n\to\infty} \frac{A_{n+1}}{A_n}$



Solution

a) $R_1 \to 0\le\theta\le 2\pi-\pi$

$R_n \to 2n\pi\le\theta\le 2n\pi+\pi$

$2n\pi\le\theta\le(2n+1)\pi$

$R_{n-1} \to 2(n-1)\pi\le\theta\le 2(n-1)\pi+\pi$

$(2n-2)\pi\le\theta\le(2n-1)\pi$

$$A_n = Area\left(R_{n-1}\right) - Area\left(R_n\right)$$

$$A_n = \frac{1}{2}\int_{(2n-2)\pi}^{(2n-1)\pi} e^{-2\theta}\, d\theta \; - \; \frac{1}{2}\int_{2n\pi}^{(2n+1)\pi} e^{-2\theta}\, d\theta$$

$$= -\frac{1}{4}\left(e^{-2\theta} \; \Big|_{(2n-2)\pi}^{(2n-1)\pi} \; + \; \frac{1}{4}\left(e^{-2\theta} \; \Big|_{2n\pi}^{(2n+1)\pi}\right.\right.$$

$$\boxed{= -\frac{1}{4}e^{-(4n-2)\pi} + \frac{1}{4}e^{-(4n-4)\pi} + \frac{1}{4}e^{-(4n+2)\pi} - \frac{1}{4}e^{-4n\pi}}$$

b) $$\lim_{n\to\infty} A_n = \lim_{n\to\infty}\left(-\frac{1}{4}e^{-(4n-2)\pi} + \frac{1}{4}e^{-(4n-4)\pi} + \frac{1}{4}e^{-(4n+2)\pi} - \frac{1}{4}e^{-4n\pi}\right)$$

$$\boxed{= 0}$$

c) $$\lim_{n\to\infty}\frac{A_{n+1}}{A_n} = \lim_{n\to\infty}\frac{-e^{-(4n+2)\pi} + e^{-(4n)\pi} + e^{-(4n+6)\pi} - e^{-(4n+4)\pi}}{-e^{-(4n-2)\pi} + e^{-(4n-4)\pi} + e^{-(4n+2)\pi} - e^{-4n\pi}}$$

$$= \lim_{n\to\infty}\frac{e^{-4n\pi}}{e^{-(4n-4)\pi}}$$

$$= \lim_{n\to\infty}\frac{e^{-4n\pi}}{e^{-4n\pi}e^{4\pi}}$$

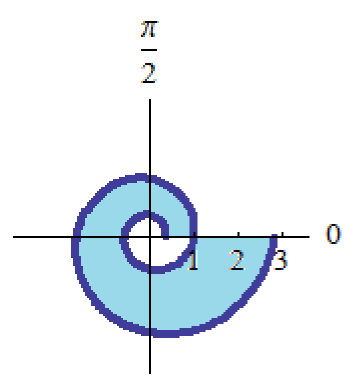
$$\boxed{= \frac{1}{e^{4\pi}}}$$

Exercise

The curve represented by the equation $r = ae^{b\theta}$, where a and b are constants, is called a logarithmic spiral. The figure shows the graph of $r = e^{\theta/6}$. $-2\pi \le \theta \le 2\pi$. Find the area of the shaded region.

Solution

$$r = e^{\theta/6}$$



$$A = \frac{1}{2}\int_0^{2\pi}\left(e^{\theta/6}\right)^2 d\theta \; - \; \frac{1}{2}\int_{-2\pi}^{0}\left(e^{\theta/6}\right)^2 d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} e^{\theta/3}\, d\theta - \frac{1}{2}\int_{-2\pi}^{0} e^{\theta/3}\, d\theta$$

$$= \frac{3}{2}\left(e^{\theta/3} \; \Big|_0^{2\pi} \; - \; \frac{3}{2}\left(e^{\theta/3} \; \Big|_{-2\pi}^{0}\right.\right.$$

$$= \frac{3}{2}\left(e^{2\pi/3} - 1\right) - \frac{3}{2}\left(1 - e^{-2\pi/3}\right)$$

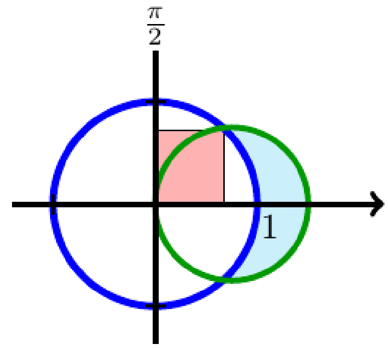
$$= \frac{3}{2}\left(e^{2\pi/3} - 2 + e^{-2\pi/3}\right) \quad unit^2 \qquad \approx 9.3655$$

Exercise

The larger circle in the figure is the graph of $r = 1$.

Find the polar equation of the smaller circle such that the shaded regrions are equal.

Solution



Small circle: $r = a\cos\theta$ with center at $\left(1\cos\frac{\pi}{4},\ 0\right) = \left(\frac{\sqrt{2}}{2},\ 0\right)$

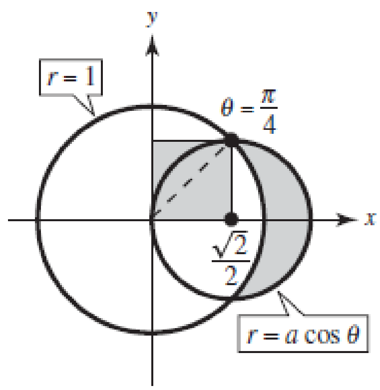
$$A = 2\frac{1}{2}\int_0^{\pi/4} \left[(a\cos\theta)^2 - 1\right] d\theta$$

$$= \int_0^{\pi/4} \left(a^2\cos^2\theta - 1\right) d\theta$$

$$= \int_0^{\pi/4} \left(\frac{a^2}{2}(1 + \cos 2\theta) - 1\right) d\theta$$

$$= \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) - \theta \Big|_0^{\pi/4}$$

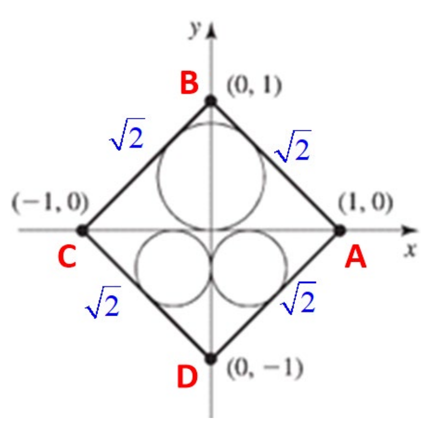
$$= \frac{a^2}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right) - \frac{\pi}{4} \quad unit^2$$



Exercise

Find equations of the circles in the figure. Determine whether the combined area of the circles is greater than or less than the area of the region inside the square but outside the circles.

Solution



$$Area(\Delta ABC) = \frac{1}{2}bh$$

$$= \frac{1}{2}(2)(1)$$

$$= 1$$

The radius of a circle inscribed in the triangle ABC is

For the bigger circle, the radius is:

$$R = \frac{Area}{\frac{1}{2}\,perimeter}$$

$$= \frac{2}{2+\sqrt{2}+\sqrt{2}}$$

$$= \frac{1}{1+\sqrt{2}}$$

$$Area(\Delta AOD) = Area(\Delta COD)$$

$$= \frac{1}{2}(1)(1)$$

$$= \frac{1}{2}$$

The radius of the small circle inscribed in the triangle *COD & AOD* is

$$R_S = \frac{Area}{\frac{1}{2}\left(1+1+\sqrt{2}\right)}$$

$$= \frac{1}{2+\sqrt{2}}$$

The area inside the 3 circles is:

$$Area = \pi\left(\frac{1}{1+\sqrt{2}}\right)^2 + 2\pi\left(\frac{1}{2+\sqrt{2}}\right)^2$$

$$= \frac{\pi}{\left(1+\sqrt{2}\right)^2} + \frac{2\pi}{\left(2+\sqrt{2}\right)^2} \quad unit^2$$

$$\approx 1.078 \quad unit^2$$

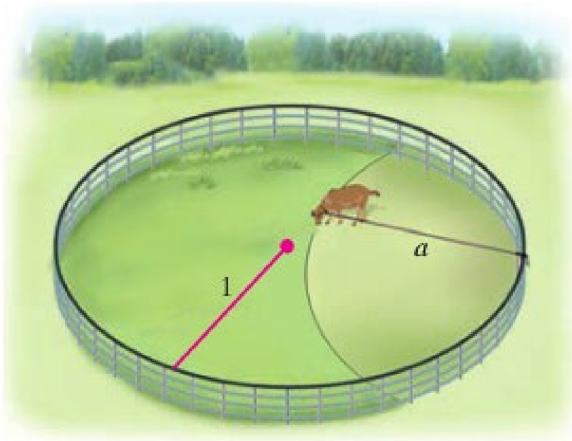
The area of the square is $= \left(\sqrt{2}\right)^2 \underline{= 2}$

The area outside the circle but inside the square is $\approx 2 - 1.078 \approx 0.922 \quad unit^2$

Therefore, the area inside the circles is more than outside the circles but inside the square.

Exercise

A circular corral of unit radius is enclosed by a fence. A goat inside the corral is tied to the fence with a rope of length $0 \le a \le 2$.



What is the area of the region (inside the corral) that the goat can graze? Check your answer with the special cases $a=0$ and $a=2$

Solution

Suppose that the goal is tethered at the origin, and that the center of the coral is $(1,\ \pi)$.

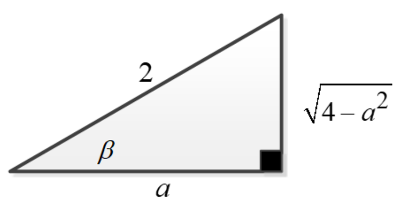
The circle that the goat can graze is $r=a$, and the corral is given by $r=-2\cos\theta$.

The intersection occurs for $\theta=\cos^{-1}\left(-\frac{a}{2}\right)$

The area grazed by the goat is twice the area of the sector of the circle $r=a$ between $\cos^{-1}\left(-\frac{a}{2}\right)$ and π, plus twice the area of the circle $r=-2\cos\theta$ between $\frac{\pi}{2}$ and $\cos^{-1}\left(-\frac{a}{2}\right)$.

$$A=\int_{\cos^{-1}\left(-\frac{a}{2}\right)}^{\pi} a^2\,d\theta\ +\int_{\pi/2}^{\cos^{-1}\left(-\frac{a}{2}\right)} 4\cos^2\theta\,d\theta$$

$$=a^2\,\theta\,\Big|_{\cos^{-1}\left(-\frac{a}{2}\right)}^{\pi}\ +\ 2\int_{\pi/2}^{\cos^{-1}\left(-\frac{a}{2}\right)}(1+\cos 2\theta)\,d\theta$$



$$=a^2\left(\pi-\cos^{-1}\left(-\frac{a}{2}\right)\right)+(2\theta+\sin 2\theta)\Big|_{\pi/2}^{\cos^{-1}\left(-\frac{a}{2}\right)}$$

$\sin 2\beta=2\sin\beta\cos\beta$

$$=a^2\left(\pi-\cos^{-1}\left(-\frac{a}{2}\right)\right)+2\cos^{-1}\left(-\frac{a}{2}\right)+\sin\left(2\cos^{-1}\left(-\frac{a}{2}\right)\right)-\pi$$

$\sin 2\beta=2\frac{\sqrt{4-a^2}}{2}\frac{a}{2}$

$$=a^2\left(\pi-\cos^{-1}\left(-\frac{a}{2}\right)\right)+2\cos^{-1}\left(-\frac{a}{2}\right)-\frac{1}{2}a\sqrt{4-a^2}-\pi$$

Case $a=0$:

$A = \pi - \pi$

$\boxed{= 0 \quad unit^2}$

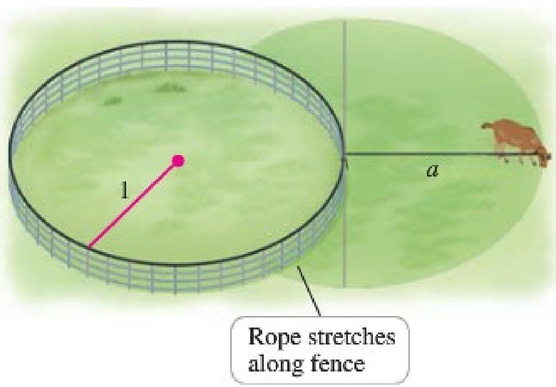
Case $a = 2$:

$A = 4(\pi - \pi) + 2\pi - \pi$

$\boxed{= \pi \quad unit^2}$

Exercise

A circular corral of unit radius is enclosed by a fence. A goat outside the corral is tied to the fence with a rope of length $0 \le a \le \pi$.



What is the area of the grassy region (outside the corral) that the goat can reach?

Solution

$$A = \frac{1}{2}\int_0^a (a - \phi)^2 \, d\phi$$

$$= \frac{1}{2}\int_0^a \left(a^2 - 2a\phi + \phi^2\right) d\phi$$

$$= \frac{1}{2}\left(a^2\phi - a\phi^2 + \frac{1}{3}\phi^3\right)\Big|_0^a$$

$$= \frac{1}{2}\left(a^3 - a^3 + \frac{1}{3}a^3\right)$$

$$\boxed{= \frac{1}{6}a^3 \quad unit^2}$$

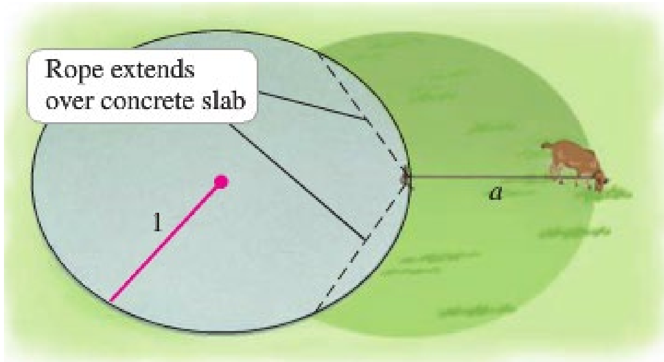
The goat can graze about half-circle of area $= \frac{1}{2}\pi a^2$

Total area $= \frac{1}{2}\pi a^2 + 2\left(\frac{1}{6}a^3\right)$

$$\boxed{= \frac{1}{2}\pi a^2 + \frac{1}{3}a^3 \quad unit^2}$$

Exercise

A circular concrete slab of unit radius is surrounded by grass. A goat is tied to the edge of the slab with a rope of length $0 \le a \le 2$.



What is the area of the grassy region that the goat can graze? Note that the rope can extend over the concrete slab. Check your answer with the special cases $a=0$ and $a=2$

Solution

$$A = \int_{\cos^{-1}\left(\frac{a}{2}\right)}^{\pi} a^2\, d\theta + \int_{\pi/2}^{\cos^{-1}\left(\frac{a}{2}\right)} 4\cos^2\theta\, d\theta$$

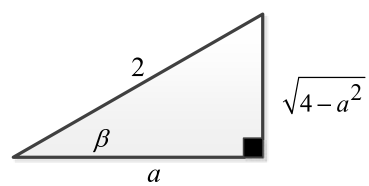
$$= a^2\left(\pi - \cos^{-1}\left(\frac{a}{2}\right)\right) + \left(2\theta + \sin 2\theta\right)\Big|_{\pi/2}^{\cos^{-1}\left(\frac{a}{2}\right)}$$

$\sin 2\beta = 2\sin\beta\cos\beta$

$$= a^2\pi - a^2\cos^{-1}\left(\frac{a}{2}\right) + 2\cos^{-1}\left(\frac{a}{2}\right) + \sin\left(2\cos^{-1}\left(\frac{a}{2}\right)\right) - \pi$$

$\sin 2\beta = 2\frac{\sqrt{4-a^2}}{2}\frac{a}{2}$

$$= \pi\left(a^2 - 1\right) + \left(2 - a^2\right)\cos^{-1}\left(\frac{a}{2}\right) + \frac{1}{2}a\sqrt{4-a^2}$$



Case $a=0$:

$$A = -\pi + 2\frac{\pi}{2}$$

$$= 0 \quad unit^2$$

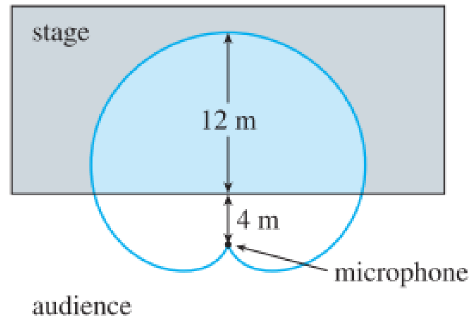
Case $a=2$:

$$A = 3\pi \ \ unit^2$$

Exercise

When recording live performance, sound engineers often use a microphone with a cardioid pickup pattern because it suppresses noise from the audience. Suppose the microphone is placed 4 m from the front of the stage and the boundary of the optimal pickup region is given by the cardioid $r = 8 + 8\sin\theta$, where r if measured in meters and the microphone is at the pole.

The musicians want to know the area they will have on stage within the optimal pickup range of the microphone, Answer their question.



Solution

At $y=4=r\sin\theta$, the line represents the front stage with angle $\theta=\alpha$. $\Leftrightarrow\ r=\dfrac{4}{\sin\theta}$

The line intersects the curve:

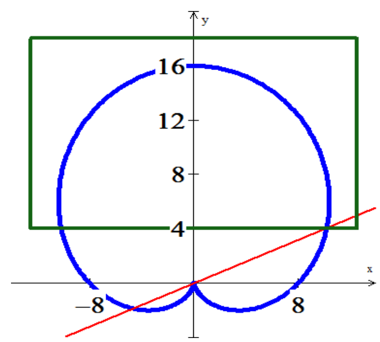
$$r=8+8\sin\theta=\frac{4}{\sin\theta}$$

$$2\sin^2\theta+2\sin\theta-1=0$$

$$\sin\theta=\frac{-2\pm\sqrt{12}}{4}$$

$$=\frac{-1+\sqrt{3}}{2}$$

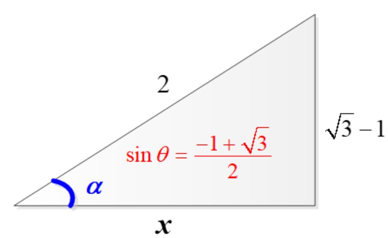
$$\theta=\alpha=\sin^{-1}\frac{\sqrt{3}-1}{2}$$



$$A=2\int_{\alpha}^{\pi/2}\frac{1}{2}\left[(8+8\sin\theta)^2-\left(\frac{4}{\sin\theta}\right)^2\right]d\theta$$

$$=\int_{\alpha}^{\pi/2}\left(64+128\sin\theta+64\sin^2\theta-16\csc^2\theta\right)d\theta$$

$$=16\int_{\alpha}^{\pi/2}\left(4+8\sin\theta+2-2\cos2\theta-\csc^2\theta\right)d\theta$$



$$=16\left(6\theta-8\cos\theta-\sin2\theta-\cot\theta\right.\Big|_{\alpha}^{\pi/2}$$

$$=16\left(3\pi-6\alpha+8\cos\alpha+\sin2\alpha+\cot\alpha\right)$$

$$x^2+\left(\sqrt{3}-1\right)^2=4$$

$$x=\sqrt{4-3+2\sqrt{3}-1}=\sqrt{2\sqrt{3}}=\sqrt{\sqrt{4}\sqrt{3}}=\left(\sqrt{12}\right)^{1/2}=\sqrt[4]{12}$$

$$=16\left(3\pi-6\sin^{-1}\frac{\sqrt{3}-1}{2}+4\sqrt[4]{12}+\frac{\sqrt[4]{12}\sqrt{3}-1}{2}+\frac{\sqrt{3}-1}{\sqrt[4]{12}}\right)$$

$$\approx 204.16\quad m^2$$

Exercise

The curve given by the parametric equations

$$x(t)=\frac{1-t^2}{1+t^2} \quad \textit{and} \quad y(t)=\frac{t\left(1-t^2\right)}{1+t^2}$$

a) Find the rectangular equation of the strophoid.
b) Find a polar equation of the strophoid.
c) Sketch a graph of the strophoid.
d) Find the equations of the two tangent lines at the origin.
e) Find the points on the graph at which the tangent lines are horizontal.

Solution

a) $x^2(t)=\dfrac{\left(1-t^2\right)^2}{\left(1+t^2\right)^2} \qquad y^2(t)=\dfrac{t^2\left(1-t^2\right)^2}{\left(1+t^2\right)^2}$

$$\frac{1-x}{1+x}=\frac{1-\frac{1-t^2}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}$$

$$=\frac{1+t^2-1+t^2}{1+t^2+1-t^2}$$

$$=\frac{2t^2}{2}$$

$$=t^2$$

$$y^2(t)=t^2\left(\frac{1-t^2}{1+t^2}\right)^2$$

$$\boxed{=\left(\frac{1-x}{1+x}\right)x^2}$$

b) $y^2=\left(\dfrac{1-x}{1+x}\right)x^2$

$$r^2\sin^2\theta=r^2\cos^2\theta\left(\frac{1-r\cos\theta}{1+r\cos\theta}\right) \qquad (r\neq 0)$$

$$\sin^2\theta+r\cos\theta\sin^2\theta=\cos^2\theta-r\cos^3\theta$$

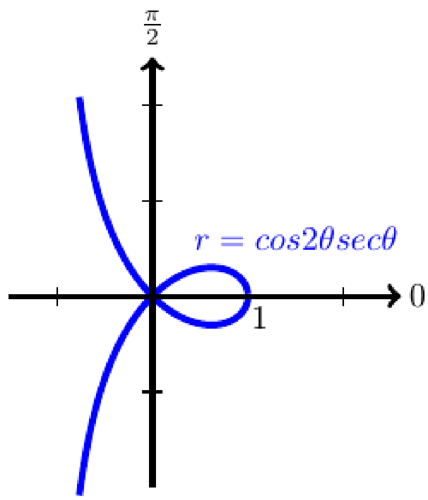
$$r\cos\theta\sin^2\theta+r\cos^3\theta=\cos^2\theta-\sin^2\theta$$

$$r\cos\theta\left(\sin^2\theta+\cos^2\theta\right)=\cos 2\theta$$

$$r\cos\theta=\cos 2\theta$$

$$\boxed{r=\cos 2\theta\sec\theta}$$

c)



d) $r = \cos 2\theta \sec\theta = 0 \quad \rightarrow \quad \cos 2\theta = 0$

$2\theta = \frac{\pi}{2}, \ \frac{3\pi}{2} \quad \rightarrow \quad \theta = \frac{\pi}{4}, \ \frac{3\pi}{4}$

The tangent lines to curve at origin at the origin:

$\underline{y = x} \quad$ *and* $\quad \underline{y = -x}$

e) $y' = \dfrac{\left(1-3t^2\right)\left(1+t^2\right) - 2t\left(t-t^3\right)}{\left(1+t^2\right)^2}$

$= \dfrac{1-2t^2-3t^4-2t^2+2t^4}{\left(1+t^2\right)^2}$

$= \dfrac{1-4t^2-t^4}{\left(1+t^2\right)^2} = 0$

$t^4 + 4t^2 - 1 = 0$

$t^2 = \dfrac{-4 \pm 2\sqrt{5}}{2}$

$= -2 \pm \sqrt{5}$

$x = \dfrac{1-t^2}{1+t^2} = \dfrac{1-\left(-2+\sqrt{5}\right)}{1+\left(-2+\sqrt{5}\right)}$

$= \dfrac{3-\sqrt{5}}{-1+\sqrt{5}} \dfrac{-1-\sqrt{5}}{-1-\sqrt{5}}$

$= \dfrac{-3-2\sqrt{5}+5}{-4}$

$$= \boxed{\frac{\sqrt{5}-1}{2}}$$

$$y = \pm x\sqrt{\frac{1-x}{1+x}}$$

$$= \pm\frac{\sqrt{5}-1}{2}\sqrt{\frac{1-\frac{\sqrt{5}-1}{2}}{1+\frac{\sqrt{5}-1}{2}}}$$

$$= \pm\frac{\sqrt{5}-1}{2}\sqrt{\frac{3-\sqrt{5}}{\sqrt{5}+1}}$$

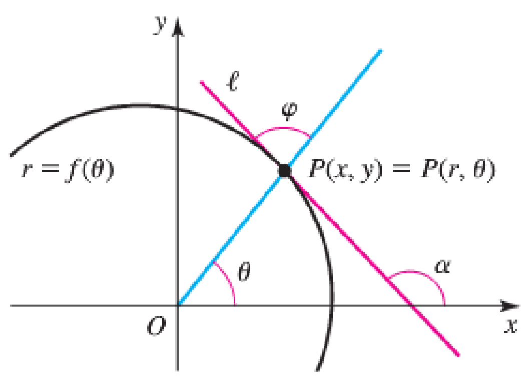
$$= \pm\frac{\sqrt{5}-1}{2}\sqrt{\frac{4\sqrt{5}-8}{4}}$$

$$= \boxed{\pm\frac{\sqrt{5}-1}{2}\sqrt{\sqrt{5}-2}}$$

$$\boxed{\left(\frac{\sqrt{5}-1}{2},\quad \pm\frac{\sqrt{5}-1}{2}\sqrt{\sqrt{5}-2}\right)}$$

Exercise

Let a polar curve be described by $r = f(\theta)$ and let ℓ be the line tangent to the curve at the point $P(x, y) = P(r, \theta)$



a) Explain why $\tan\alpha = \dfrac{dy}{dx}$

b) Explain why $\tan\theta = \dfrac{y}{x}$

c) Let φ be the angle between ℓ and the line *O* and *P*. Prove that $\tan\varphi = \dfrac{f(\theta)}{f'(\theta)}$

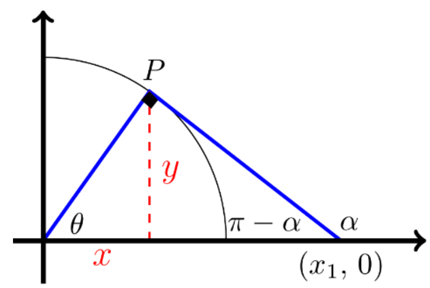
d) Prove that the value of θ for which ℓ is parallel to the *x*-axis satisfy $\tan\theta = -\dfrac{f(\theta)}{f'(\theta)}$

e) Prove that the value of θ for which ℓ is parallel to the y-axis satisfy $\tan\theta = \dfrac{f'(\theta)}{f(\theta)}$

Solution

a) The slope of the line tangent to the circle $r = f(\theta)$ at the point P is $\dfrac{dy}{dx}\Big|_P$

$\tan(\pi - \alpha) = -\tan\alpha$



Slope at P:

$$m = \frac{y-0}{x-x_1}$$

$$= \frac{y}{x-x_1}$$

$$\tan(\pi-\alpha) = \frac{y}{x-x_1}$$

$$-\tan\alpha = m$$

Therefore; the $\tan\alpha = \dfrac{dy}{dx}$

b) From the graph, after we project the point P coordinates the proper axis.
From the right triangle:

$$\boxed{\tan\theta = \frac{y}{x}}$$

c) Prove: $\tan\varphi = \dfrac{f(\theta)}{f'(\theta)}$

From part (a): $f'(\theta) = \tan\theta$

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

$$= \frac{\dfrac{f'(\theta)\sin\theta}{f'(\theta)\cos\theta} + \dfrac{f(\theta)\cos\theta}{f'(\theta)\cos\theta}}{\dfrac{f'(\theta)\cos\theta}{f'(\theta)\cos\theta} - \dfrac{f(\theta)\sin\theta}{f'(\theta)\cos\theta}}$$

$$= \frac{\tan\theta + \dfrac{f(\theta)}{f'(\theta)}}{1 - \dfrac{f(\theta)}{f'(\theta)}\tan\theta}$$

$$= \tan\alpha$$

From inside triangle:

$$\pi - \alpha + \varphi + \theta = \pi$$

$$\alpha = \varphi + \theta$$

$$\tan\alpha = \tan(\varphi+\theta)$$

$$= \frac{\tan\varphi + \tan\theta}{1-\tan\varphi\tan\theta}$$

$$\frac{\tan\theta + \frac{f(\theta)}{f'(\theta)}}{1-\frac{f(\theta)}{f'(\theta)}\tan\theta} = \frac{\tan\varphi + \tan\theta}{1-\tan\varphi\tan\theta}$$

$$\tan\varphi = \frac{f(\theta)}{f'(\theta)} \quad \checkmark$$

d) Prove: $\tan\theta = -\dfrac{f(\theta)}{f'(\theta)}$ when $\ell \parallel x-axis$

$$\ell \parallel x-axis: \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = 0$$

$$f'(\theta)\sin\theta + f(\theta)\cos\theta = 0$$

$$f'(\theta)\sin\theta = -f(\theta)\cos\theta$$

$$\frac{\sin\theta}{\cos\theta} = -\frac{f(\theta)}{f'(\theta)}$$

$$\tan\theta = -\frac{f(\theta)}{f'(\theta)} \quad \checkmark$$

e) Prove: $\tan\theta = \dfrac{f'(\theta)}{f(\theta)}$ when $\ell \parallel y-axis$

$$\ell \parallel y-axis: \frac{dy}{dx} = \infty$$

$$\frac{dx}{dy} = 0$$

$$\frac{f'(\theta)\cos\theta - f(\theta)\sin\theta}{f'(\theta)\sin\theta + f(\theta)\cos\theta} = 0$$

$$f'(\theta)\cos\theta - f(\theta)\sin\theta = 0$$

$$f(\theta)\sin\theta = f'(\theta)\cos\theta$$

$$\frac{\sin\theta}{\cos\theta} = \frac{f'(\theta)}{f(\theta)}$$

$$\tan\theta = \frac{f'(\theta)}{f(\theta)} \quad \checkmark$$