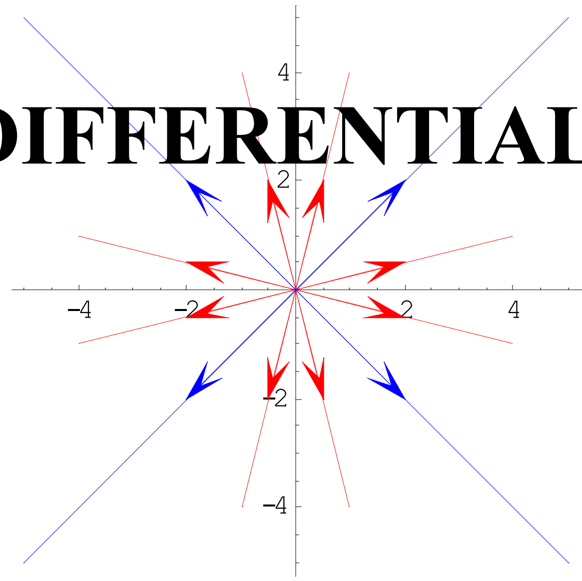

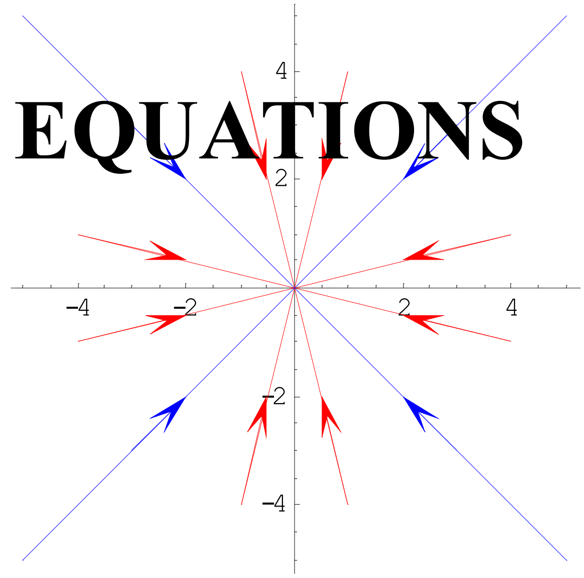

DIFFERENTIAL EQUATIONS

Solution Manual

Volume 4

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Lecture 4 – Series

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Solution Section 4.1 – Introduction and Review of Power Series

Exercise

Determine the centre, radius, and interval of convergencae of the power series

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{\sqrt{n+1}}$$

Solution

$$R = \lim_{n\to\infty} \left| \frac{1}{\sqrt{n+2}} \cdot \sqrt{n+1} \right| \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}}$$

$$= 1$$

The radius of convergence is 1.
The centre of convergence is 0.
The interval of convergence is (–1, 1).
The series does not converge at $x = -1$ or $x = 1$

Exercise

Determine the centre, radius, and interval of convergence of the power series

$$\sum_{n=0}^{\infty} 3n(x+1)^n$$

Solution

$$R = \lim_{n\to\infty} \left| \frac{3n}{3(n+1)} \right| \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{3n}{3n}$$

$$= 1$$

The radius of convergence is 1, and the centre of convergence is –1. $(x+1=0)$

$a - R < x < a + R \quad \Rightarrow \quad -1-1 < x < -1+1$

Therefore, the given series convergences absolutely on $(-2,\ 0)$

At $x = -2$, the series is $\sum_{n=0}^{\infty} 3n(-1)^n$ which diverges.

At $x=0$, the series is $\sum\limits_{n=0}^{\infty} 3n(1)^n = \sum\limits_{n=0}^{\infty} 3n$ which diverges.

Hence, the interval of convergence is $(-2,\ 0)$.

Exercise

Determine the centre, radius, and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} x^n$$

Solution

$$R = \lim_{n\to\infty} \left| \frac{(n+1)^4 2^{2n+2}}{n^4 2^{2n}} \right| \qquad\qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= 4 \lim_{n\to\infty} \left| \left(\frac{n+1}{n} \right)^4 \right|$$

$$\underline{= 4}$$

The radius of convergence is 4, and the centre of convergence is 0.

$a - R < x < a + R \quad \Rightarrow \quad -4 < x < 4$, the given series convergences absolutely on $(-4,\ 4)$

At $x = -4$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} (-4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} (-1)^n 2^{2n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^4} \qquad \text{which converges (}p\text{-series).}$$

At $x = 4$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} (4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} 2^{2n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$$

which also converges.

Hence, the interval of convergence is $[-4,\ 4]$.

Exercise

Determine the centre, radius, and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{e^n}{n^3}(4-x)^n$$

Solution

$$R = \lim_{n\to\infty}\left|\frac{e^n}{n^3}\cdot\frac{(n+1)^3}{e^{n+1}}\right| \qquad\qquad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= \frac{1}{e}\lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^3\right|$$

$$= \frac{1}{e}$$

The radius of convergence is $\frac{1}{e}$.

The centre of convergence is 4. $(4-x=0 \;\Rightarrow\; x=4)$

$a-R<x<a+R \quad\Rightarrow\quad 4-\frac{1}{e}<x<4+\frac{1}{e}$, which the given series convergences absolutely

At $x=4-\frac{1}{e}$

The series is $\sum_{n=1}^{\infty} \frac{e^n}{n^3}\left(\frac{1}{e}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ which converges (p-series).

At $x=4+\frac{1}{e}$,

the series is $\sum_{n=1}^{\infty} \frac{e^n}{n^3}\left(-\frac{1}{e}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ which also converges (p-series).

Hence, the interval of convergence is $\left[4-\frac{1}{e},\ 4+\frac{1}{e}\right]$.

Exercise

Determine the centre, radius, and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n}$$

Solution

$$\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n} = \sum_{n=1}^{\infty} \frac{4^n\left(x-\frac{1}{4}\right)^n}{n^n}$$

$$R = \lim_{n\to\infty}\left|\frac{4^n}{n^n}\cdot\frac{(n+1)^{n+1}}{4^{n+1}}\right| \qquad\qquad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= \frac{1}{4}\lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^n (n+1)\right|$$

$$\boxed{=\infty}$$

The radius of convergence is ∞ .

The centre of convergence is $\boxed{x = \frac{1}{4}}$.

The interval of convergence is the real line $(-\infty,\ \infty)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{1+5^n}{n!} x^n$

Solution

$$R = \lim_{n\to\infty}\left|\frac{1+5^n}{n!}\cdot\frac{(n+1)!}{1+5^{n+1}}\right| \qquad\qquad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1+5^n}{1+5^{n+1}}(n+1)\right|$$

$$\boxed{=\infty}$$

The radius of convergence is ∞ .
The centre of convergence is 0.
The interval of convergence is the real line $(-\infty,\ \infty)$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a: $f(x) = e^{2x}, \quad a = 0$

Solution

$$f(x) = e^{2x} \quad\to\quad f(0) = e^{2(0)} = 1$$

$$f'(x) = 2e^{2x} \quad\to\quad f'(0) = 2$$

$f''(x) = 4e^{2x} \quad \to \quad f''(0) = 4$

$f'''(x) = 8e^{2x} \quad \to \quad f'''(0) = 8$

$P_0(x) = f(0)$

$\quad = 1$

$P_1(x) = f(0) + f'(0)(x-0)$

$\quad = 1 + 2x$

$P_2(x) = f(0) + f'(0)(x-0) + \dfrac{f''(0)}{2!}(x-0)^2$

$\quad = 1 + 2x + 2x^2$

$P_3(x) = f(0) + f'(0)(x-0) + \dfrac{f''(0)}{2!}(x-0)^2 + \dfrac{f'''(0)}{3!}(x-0)^3$

$\quad = 1 + 2x + 2x^2 + \frac{4}{3}x^3$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a: $f(x) = \sin x, \quad a = 0$

Solution

$f(x) = \sin x \quad \to \quad f(0) = 0$

$f'(x) = \cos x \quad \to \quad f'(0) = 1$

$f''(x) = -\sin x \quad \to \quad f''(0) = 0$

$f'''(x) = -\cos x \quad \to \quad f'''(0) = -1$

$P_0(x) = f(0)$

$\quad = 0$

$P_1(x) = f(0) + f'(0)(x-0)$

$\quad = x$

$P_2(x) = f(0) + f'(0)(x-0) + \dfrac{f''(0)}{2!}(x-0)^2$

$\quad = x$

$P_3(x) = f(0) + f'(0)(x-0) + \dfrac{f''(0)}{2!}(x-0)^2 + \dfrac{f'''(0)}{3!}(x-0)^3$

$\quad = x - \frac{1}{6}x^3$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\ln(1+x)$, $a=0$

Solution

$$f(x)=\ln(1+x) \quad\rightarrow\quad f(0)=0$$

$$f'(x)=\frac{1}{1+x} \quad\rightarrow\quad f'(0)=1$$

$$f''(x)=-\frac{1}{(1+x)^2} \quad\rightarrow\quad f''(0)=-1$$

$$f'''(x)=\frac{2}{(1+x)^3} \quad\rightarrow\quad f'''(0)=2$$

$$P_0(x)=f(0)$$
$$\quad =0$$

$$P_1(x)=f(0)+f'(0)(x-0)$$
$$\quad =x$$

$$P_2(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2$$
$$\quad =x-\frac{1}{2}x^2$$

$$P_3(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3$$
$$\quad =x-\frac{1}{2}x^2+\frac{1}{3}x^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\frac{1}{x+2}$, $a=0$

Solution

$$f(x)=(x+2)^{-1} \quad\rightarrow\quad f(0)=\frac{1}{2}$$

$$f'(x)=-(x+2)^{-2} \quad\rightarrow\quad f'(0)=-\frac{1}{4}$$

$$f''(x)=2(x+2)^{-3} \quad\rightarrow\quad f''(0)=\frac{1}{4}$$

$$f'''(x)=-6(x+2)^{-4} \quad\rightarrow\quad f'''(0)=-\frac{3}{8}$$

$$P_0(x)=f(0)$$

$$= \frac{1}{2}$$

$$P_1(x) = f(0) + f'(0)(x-0)$$

$$= \frac{1}{2} - \frac{1}{4}x$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$= \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$$= \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = \sqrt{1-x}, \quad a = 0$

Solution

$$f(x) = (1-x)^{1/2} \quad \rightarrow \quad f(0) = 1$$

$$f'(x) = -\tfrac{1}{2}(1-x)^{-1/2} \quad \rightarrow \quad f(0) = -\tfrac{1}{2}$$

$$f''(x) = -\tfrac{1}{4}(1-x)^{-3/2} \quad \rightarrow \quad f(0) = -\tfrac{1}{4}$$

$$f'''(x) = -\tfrac{3}{8}(1-x)^{-5/2} \quad \rightarrow \quad f(0) = -\tfrac{3}{8}$$

$$P_0(x) = f(0)$$

$$= 1$$

$$P_1(x) = f(0) + f'(0)(x-0)$$

$$= 1 - \frac{1}{2}x$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a: $f(x)=x^3, \quad a=1$

Solution

$$f(x)=x^3 \quad \to \quad f(1)=1$$

$$f'(x)=3x^2 \quad \to \quad f'(1)=3$$

$$f''(x)=6x \quad \to \quad f''(1)=6$$

$$f'''(x)=6 \quad \to \quad f'''(1)=6$$

$$\boxed{P_0(x)=1} \qquad P_0(x)=f(a)$$

$$\boxed{P_1(x)=1+3(x-1)} \qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$\boxed{P_2(x)=1+3(x-1)+3(x-1)^2} \qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{P_3(x)=1+3(x-1)+3(x-1)^2+(x-1)^3}$$

$$P_3(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a: $f(x)=8\sqrt{x}, \quad a=1$

Solution

$$f(x)=8x^{1/2} \quad \to \quad f(1)=8$$

$$f'(x)=4x^{-1/2} \quad \to \quad f'(1)=4$$

$$f''(x)=-2x^{-3/2} \quad \to \quad f''(1)=-2$$

$$f'''(x)=3x^{-5/2} \quad \to \quad f'''(1)=3$$

$$\boxed{P_0(x)=8} \qquad P_0(x)=f(a)$$

$$\boxed{P_1(x)=8+4(x-1)} \qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$\boxed{P_2(x)=8+4(x-1)-(x-1)^2} \qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{P_3(x)=8+4(x-1)-(x-1)^2+3(x-1)^3} \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a: $f(x)=\sin x, \quad a=\frac{\pi}{4}$

Solution

$$f(x)=\sin x \quad \to \quad f\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

$$f'(x)=\cos x \quad \to \quad f'\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

$$f''(x)=-\sin x \quad \to \quad f''\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$$

$$f'''(x)=-\cos x \quad \to \quad f'''\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$$

$$\boxed{P_0(x)=\frac{\sqrt{2}}{2}} \qquad P_0(x)=f(a)$$

$$\boxed{P_1(x)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)} \qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$\boxed{P_2(x)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2}$$

$$P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{P_3(x)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2-\frac{\sqrt{2}}{12}\left(x-\frac{\pi}{4}\right)^3} \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a: $f(x)=\cos x, \quad a=\frac{\pi}{6}$

Solution

$$f(x)=\cos x \quad \to \quad f\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$$

$$f'(x)=-\sin x \quad \to \quad f'\left(\frac{\pi}{6}\right)=-\frac{1}{2}$$

$$f''(x)=-\cos x \quad \to \quad f''\left(\frac{\pi}{6}\right)=-\frac{\sqrt{3}}{2}$$

$$f'''(x)=\sin x \quad \to \quad f'''\left(\frac{\pi}{6}\right)=\frac{1}{2}$$

$$\boxed{P_0(x)=\frac{\sqrt{3}}{2}} \qquad P_0(x)=f(a)$$

$$\boxed{P_1(x)=\frac{\sqrt{2}}{2}-\frac{1}{2}\left(x-\frac{\pi}{6}\right)} \qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$\boxed{P_2(x)=\frac{\sqrt{2}}{2}-\frac{1}{2}\left(x-\frac{\pi}{6}\right)-\frac{\sqrt{3}}{4}\left(x-\frac{\pi}{6}\right)^2}$$

$$P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{P_3(x)=\frac{\sqrt{3}}{2}-\frac{1}{2}\left(x-\frac{\pi}{6}\right)-\frac{\sqrt{3}}{4}\left(x-\frac{\pi}{6}\right)^2+\frac{1}{12}\left(x-\frac{\pi}{6}\right)^3} \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by *f at a*: $f(x)=\sqrt{x}, \quad a=9$

Solution

$$f(x)=x^{1/2} \quad \to \quad f(9)=3$$

$$f'(x)=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}} \quad \to \quad f'(9)=\frac{1}{6}$$

$$f''(x)=-\frac{1}{4}x^{-3/2}=-\frac{1}{4x\sqrt{x}} \quad \to \quad f''(9)=-\frac{1}{4\times 3^3}$$

$$f'''(x)=\frac{3}{8}x^{-5/2}=\frac{3}{8x^2\sqrt{x}} \quad \to \quad f'''(9)=\frac{1}{2^3\times 3^4}$$

$$\boxed{P_0(x)=3} \qquad P_0(x)=f(a)$$

$$\boxed{P_1(x)=3+\frac{1}{6}(x-9)} \qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$\boxed{P_2(x)=3+\frac{1}{2\cdot 3}(x-9)-\frac{1}{2^3\cdot 3^3}(x-9)^2} \qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{P_3(x)=3+\frac{1}{2\cdot 3}(x-9)-\frac{1}{2^2\cdot 3^3}(x-9)^2+\frac{1}{2^4\cdot 3^5}(x-9)^3} \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by *f at a*: $f(x)=\sqrt[3]{x}, \quad a=8$

Solution

$$f(x)=x^{1/3} \quad \to \quad f(8)=2$$

$$f'(x)=\frac{1}{3}x^{-2/3}=\frac{1}{3x^{2/3}} \quad \to \quad f'(8)=\frac{1}{2^2\times 3}$$

$$f''(x) = -\frac{2}{9}x^{-5/3} = -\frac{2}{3^2 x^{5/3}} \quad \rightarrow \quad f''(8) = -\frac{1}{3^2 \times 2^4}$$

$$f'''(x) = \frac{10}{3^3}x^{-8/3} = \frac{2 \cdot 5}{3^3 x^{8/3}} \quad \rightarrow \quad f'''(8) = \frac{5}{2^7 \times 3^3}$$

$$\boxed{P_0(x) = 2} \qquad P_0(x) = f(a)$$

$$\boxed{P_1(x) = 2 + \frac{1}{2^2 \cdot 3}(x-8)} \qquad P_1(x) = f(a) + f'(a)(x-a)$$

$$\boxed{P_2(x) = 2 + \frac{1}{2^2 \cdot 3}(x-8) - \frac{1}{2^5 \cdot 3^2}(x-8)^2} \qquad P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{P_3(x) = 2 + \frac{1}{2^2 \cdot 3}(x-8) - \frac{1}{2^5 \cdot 3^2}(x-8)^2 + \frac{1}{2^8 \cdot 3^4}(x-8)^3} \qquad P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = \ln x, \quad a = e$

Solution

$$f(x) = \ln x \quad \rightarrow \quad f(e) = 1$$

$$f'(x) = \frac{1}{x} \quad \rightarrow \quad f'(e) = \frac{1}{e}$$

$$f''(x) = -\frac{1}{x^2} \quad \rightarrow \quad f''(e) = -\frac{1}{e^2}$$

$$f'''(x) = \frac{2}{x^3} \quad \rightarrow \quad f'''(e) = \frac{2}{e^3}$$

$$\boxed{P_0(x) = 1} \qquad P_0(x) = f(a)$$

$$\boxed{P_1(x) = 1 + \frac{1}{e}(x-e)} \qquad P_1(x) = f(a) + f'(a)(x-a)$$

$$\boxed{P_2(x) = 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2} \qquad P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{P_3(x) = 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3} \qquad P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = \sqrt[4]{x}, \quad a = 8$

Solution

$$f(x)=x^{1/4} \quad\rightarrow\quad f(8)=\sqrt[4]{8}$$

$$f'(x)=\frac{1}{4}x^{-3/4}=\frac{1}{4x^{3/4}} \quad\rightarrow\quad f'\left(8=2^3\right)=\frac{1}{2^2\times 2^{9/4}}=\frac{1}{2^4\sqrt[4]{2}}$$

$$f''(x)=-\frac{3}{16}x^{-7/4}=-\frac{3}{2^4x^{7/4}} \quad\rightarrow\quad f''(8)=-\frac{3}{2^4\,2^{21/4}}=-\frac{3}{2^9\sqrt[4]{2}}$$

$$f'''(x)=\frac{21}{2^6}x^{-11/4}=\frac{21}{2^6x^{11/4}} \quad\rightarrow\quad f'''(8)=\frac{21}{2^6\times 2^{33/4}}=\frac{21}{2^{14}\sqrt[4]{2}}$$

$$\boxed{P_0(x)=\sqrt[4]{8}} \qquad\qquad P_0(x)=f(a)$$

$$\boxed{P_1(x)=\sqrt[4]{8}+\frac{1}{2^4\cdot\sqrt[4]{2}}(x-8)} \qquad\qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{P_2(x)=\sqrt[4]{8}+\frac{1}{2^4\cdot\sqrt[4]{2}}(x-8)-\frac{3}{2^{10}\cdot\sqrt[4]{2}}(x-8)^2}$$

$$P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

$$\boxed{P_3(x)=\sqrt[4]{8}+\frac{1}{2^4\cdot\sqrt[4]{2}}(x-8)-\frac{3}{2^{10}\cdot\sqrt[4]{2}}(x-8)^2+\frac{7}{2^{15}\cdot\sqrt[4]{2}}(x-8)^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a:

$$f(x)=\tan^{-1}x+x^2+1, \quad a=1$$

Solution

$$f(x)=\tan^{-1}x+x^2+1 \quad\rightarrow\quad f(1)=\frac{\pi}{4}+2$$

$$f'(x)=\frac{1}{x^2+1}+2x \quad\rightarrow\quad f'(1)=\frac{5}{2}$$

$$f''(x)=-\frac{2x}{\left(x^2+1\right)^2}+2 \quad\rightarrow\quad f''(1)=-\frac{1}{2}+2=\frac{3}{2}$$

$$f'''(x)=-\frac{2x^2+2-8x^2}{\left(x^2+1\right)^3} \qquad\qquad \left(U^nV^m\right)'=U^{n-1}V^{m-1}\left(nU'V+mUV'\right)$$

$$=-\frac{2-2x^2}{\left(x^2+1\right)^3} \quad\rightarrow\quad f'''(1)=0$$

$$\boxed{P_0(x) = \frac{\pi}{4} + 2} \qquad P_0(x) = f(a)$$

$$\boxed{P_1(x) = \frac{\pi}{4} + 2 + \frac{5}{2}(x-1)} \qquad P_1(x) = f(a) + f'(a)(x-a)$$

$$\boxed{P_2(x) = \frac{\pi}{4} + 2 + \frac{5}{2}(x-1) - \frac{3}{4}(x-1)^2} \qquad P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{P_3(x) = \frac{\pi}{4} + 2 + \frac{5}{2}(x-1) - \frac{3}{4}(x-1)^2} \qquad P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a: $f(x) = e^x, \quad a = \ln 2$

Solution

$$f(x) = e^x \quad \to \quad f(\ln 2) = 2$$

$$f'(x) = e^x \quad \to \quad f'(\ln 2) = 2$$

$$f''(x) = e^x \quad \to \quad f''(\ln 2) = 2$$

$$f'''(x) = e^x \quad \to \quad f'''(\ln 2) = 2$$

$$\boxed{P_0(x) = 2} \qquad P_0(x) = f(a)$$

$$\boxed{P_1(x) = 2 + 2(x - \ln 2)} \qquad P_1(x) = f(a) + f'(a)(x-a)$$

$$\boxed{P_2(x) = 2 + 2(x - \ln 2) + (x - \ln 2)^2} \qquad P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{P_3(x) = 2 + 2(x - \ln 2) + (x - \ln 2)^2 + \frac{1}{3}(x - \ln 2)^3} \qquad P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x) = e^{4x}, \quad n = 4$

Solution

$$f(x) = e^{4x} \quad \to \quad f(0) = 1$$

$$f'(x) = 4e^{4x} \quad \to \quad f'(0) = 4$$

$$f''(x) = 16e^{4x} \quad \to \quad f''(0) = 16$$

$$f'''(x) = 64e^{4x} \quad \to \quad f'''(0) = 64$$

$$f^{(4)}(x) = 256e^{4x} \quad \to \quad f^{(4)}(0) = 256$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\underline{P_4(x) = 1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4}$$

Exercise

Find the nth Maclaurin polynomial for the function $f(x) = e^{-x}, \quad n = 5$

Solution

$$f(x) = e^{-x} \quad \to \quad f(0) = 1$$

$$f'(x) = -e^{-x} \quad \to \quad f'(0) = -1$$

$$f''(x) = e^{-x} \quad \to \quad f''(0) = 1$$

$$f'''(x) = -e^{-x} \quad \to \quad f'''(0) = -1$$

$$f^{(4)}(x) = e^{-x} \quad \to \quad f^{(4)}(0) = 1$$

$$f^{(5)}(x) = -e^{-x} \quad \to \quad f^{(5)}(0) = -1$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$\underline{P_5(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5}$$

Exercise

Find the nth Maclaurin polynomial for the function $f(x) = e^{-x/2}, \quad n = 4$

Solution

$$f(x) = e^{-x/2} \quad \to \quad f(0) = 1$$

$$f'(x) = -\frac{1}{2}e^{-x/2} \quad \to \quad f'(0) = -\frac{1}{2}$$

$$f''(x) = \frac{1}{4}e^{-x/2} \quad \to \quad f''(0) = \frac{1}{4}$$

$$f'''(x) = -\frac{1}{8}e^{-x/2} \quad \to \quad f'''(0) = -\frac{1}{8}$$

$$f^{(4)}(x) = \frac{1}{16}e^{-x/2} \quad \to \quad f^{(4)}(0) = \frac{1}{16}$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\boxed{P_4(x) = 1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4}$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x) = e^{x/3}, \quad n = 4$

Solution

$$f(x) = e^{x/3} \quad \to \quad f(0) = 1$$

$$f'(x) = \frac{1}{3}e^{x/3} \quad \to \quad f'(0) = \frac{1}{3}$$

$$f''(x) = \frac{1}{9}e^{x/3} \quad \to \quad f''(0) = \frac{1}{9}$$

$$f'''(x) = \frac{1}{27}e^{x/3} \quad \to \quad f'''(0) = \frac{1}{27}$$

$$f^{(4)}(x) = \frac{1}{81}e^{x/3} \quad \to \quad f^{(4)}(0) = \frac{1}{81}$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\boxed{P_4(x) = 1 + \frac{1}{3}x + \frac{1}{18}x^2 + \frac{1}{162}x^3 + \frac{1}{1944}x^4}$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x) = \sin x, \quad n = 5$

Solution

$$f(x) = \sin x \quad \to \quad f(0) = 0$$

$$f'(x) = \cos x \quad \to \quad f'(0) = 1$$

$$f''(x) = -\sin x \quad \to \quad f''(0) = 0$$

$$f'''(x) = -\cos x \quad \to \quad f'''(0) = -1$$

$$f^{(4)}(x) = \sin x \quad \to \quad f^{(4)}(0) = 0$$

$$f^{(5)}(x) = \cos x \quad \to \quad f^{(5)}(0) = 1$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$\boxed{P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5}$$

Exercise

Find the *n*th Maclaurin polynomial for the function $f(x) = \cos \pi x, \quad n = 4$

Solution

$f(x) = \cos \pi x \quad \rightarrow \quad f(0) = 1$

$f'(x) = -\pi \sin \pi x \quad \rightarrow \quad f'(0) = 0$

$f''(x) = -\pi^2 \cos \pi x \quad \rightarrow \quad f''(0) = -\pi^2$

$f'''(x) = \pi^3 \sin \pi x \quad \rightarrow \quad f'''(0) = 0$

$f^{(4)}(x) = \pi^4 \cos \pi x \quad \rightarrow \quad f^{(4)}(0) = \pi^4$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\boxed{P_4(x) = 1 - \frac{\pi^2}{2}x^2 + \frac{\pi^4}{24}x^4}$$

Exercise

Find the *n*th Maclaurin polynomial for the function $f(x) = xe^x, \quad n = 4$

Solution

$f(x) = xe^x \quad \rightarrow \quad f(0) = 0$

$f'(x) = e^x + xe^x \quad \rightarrow \quad f'(0) = 1$

$f''(x) = 2e^x + xe^x \quad \rightarrow \quad f''(0) = 2$

$f'''(x) = 3e^x + xe^x \quad \rightarrow \quad f'''(0) = 3$

$f^{(4)}(x) = 4e^x + xe^x \quad \rightarrow \quad f^{(4)}(0) = 4$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\boxed{P_4(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4}$$

Exercise

Find the *n*th Maclaurin polynomial for the function $f(x) = x^2 e^{-x}$, $n = 4$

Solution

$$f(x) = x^2 e^{-x} \quad \to \quad f(0) = 0$$

$$f'(x) = 2xe^{-x} - x^2 e^{-x} \quad \to \quad f'(0) = 0$$

$$f''(x) = 2e^{-x} - 4xe^{x} + x^2 e^{-x} \quad \to \quad f''(0) = 2$$

$$f'''(x) = -6e^{-x} + 6xe^{-x} - x^2 e^{-x} \quad \to \quad f'''(0) = -6$$

$$f^{(4)}(x) = 12e^{-x} - 8xe^{-x} + x^2 e^{-x} \quad \to \quad f^{(4)}(0) = 12$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\boxed{P_4(x) = x^2 - x^3 + \frac{1}{2}x^4}$$

Exercise

Find the *n*th Maclaurin polynomial for the function $f(x) = \frac{1}{x+1}$, $n = 5$

Solution

$$f(x) = \frac{1}{x+1} \quad \to \quad f(0) = 1$$

$$f'(x) = -(x+1)^{-2} \quad \to \quad f'(0) = -1$$

$$f''(x) = 2(x+1)^{-3} \quad \to \quad f''(0) = 2$$

$$f'''(x) = -6(x+1)^{-4} \quad \to \quad f'''(0) = -6$$

$$f^{(4)}(x) = 24(x+1)^{-5} \quad \to \quad f^{(4)}(0) = 24$$

$$f^{(5)}(x) = -120(x+1)^{-6} \quad \to \quad f^{(5)}(0) = -120$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$\boxed{P_5(x) = 1 - x + x^2 - x^3 + x^4 - x^5}$$

Exercise

Find the *n*th Maclaurin polynomial for the function $f(x) = \frac{x}{x+1}$, $n = 4$

Solution

$$f(x)=\frac{x}{x+1}=1-\frac{1}{x+1} \quad\to\quad f(0)=0$$

$$f'(x)=(x+1)^{-2} \quad\to\quad f'(0)=1$$

$$f''(x)=-2(x+1)^{-3} \quad\to\quad f''(0)=-2$$

$$f'''(x)=6(x+1)^{-4} \quad\to\quad f'''(0)=6$$

$$f^{(4)}(x)=-24(x+1)^{-5} \quad\to\quad f^{(4)}(0)=-24$$

$$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$

$$\boxed{P_4(x)=x-x^2+x^3-x^4}$$

Exercise

Find the nth Maclaurin polynomial for the function $f(x)=\sec x,\quad n=2$

Solution

$$f(x)=\sec x \quad\to\quad f(0)=1$$

$$f'(x)=\sec x\tan x \quad\to\quad f'(0)=0$$

$$f''(x)=\sec x\tan^2 x+\sec^3 x \quad\to\quad f''(0)=1$$

$$P_2(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2$$

$$\boxed{P_2(x)=1+\frac{1}{2}x^2}$$

Exercise

Find the nth Maclaurin polynomial for the function $f(x)=\tan x,\quad n=3$

Solution

$$f(x)=\tan x \quad\to\quad f(0)=0$$

$$f'(x)=\sec^2 x \quad\to\quad f'(0)=1$$

$$f''(x)=2\sec^2 x\tan x \quad\to\quad f''(0)=0$$

$$f'''(x)=4\sec^2 x\tan^2 x+2\sec^4 x \quad\to\quad f'''(0)=2$$

$$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3$$

$$\boxed{P_4(x) = x + \frac{1}{3}x^3}$$

Exercise

Find the Maclaurin series for: xe^x

Solution

$$f(x) = xe^x \quad \to \quad f(0) = 0$$

$$f'(x) = e^x + xe^x \quad \to \quad f'(0) = 1$$

$$f''(x) = 2e^x + xe^x \quad \to \quad f''(0) = 2$$

$$f'''(x) = 3e^x + xe^x \quad \to \quad f'''(0) = 3$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$f^{(n)}(x) = ne^x + xe^x \quad \to \quad f^{(n)}(0) = n$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$xe^x = x + x^2 + \frac{1}{2}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{(n-1)!}x^n$$

Exercise

Find the Maclaurin series for: $5\cos \pi x$

Solution

$$f(x) = 5\cos \pi x \quad \to \quad f(0) = 5$$

$$f'(x) = -5\pi \sin \pi x \quad \to \quad f'(0) = 0$$

$$f''(x) = -5\pi^2 \cos \pi x \quad \to \quad f''(0) = -5\pi^2$$

$$f'''(x) = 5\pi^3 \sin \pi x \quad \to \quad f'''(0) = 0$$

$$5\cos \pi x = 5 - \frac{5\pi^2 x^2}{2!} + \frac{5\pi^4 x^4}{4!} - \frac{5\pi^6 x^6}{6!} + \cdots$$

$$= 5\sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!}$$

Exercise

Find the Maclaurin series for: $\frac{x^2}{x+1}$

Solution

$$f(x)=\frac{x^2}{x+1} \quad \rightarrow \quad f(0)=0$$

$$f'(x)=\frac{2x(x+1)-x^2}{(x+1)^2}=\frac{x^2+2x}{(x+1)^2} \quad \rightarrow \quad f'(0)=0$$

$$f''(x)=\frac{(2x+2)(x+1)^2-2(x+1)\left(x^2+2x\right)}{(x+1)^4}$$

$$=\frac{2x^2+4x+2-2x^2-4x}{(x+1)^3}$$

$$=\frac{2}{(x+1)^3} \quad \rightarrow \quad f''(0)=2$$

$$f'''(x)=\frac{-6}{(x+1)^4} \quad \rightarrow \quad f'''(0)=-6$$

$$\frac{x^2}{x+1}=\frac{2}{2!}x^2-\frac{6x^3}{3!}+\frac{24x^4}{4!}-\cdots=x^2-x^3+x^4-\cdots$$

$$=\sum_{n=2}^{\infty}(-1)^n x^n$$

Exercise

Find the Maclaurin series for: e^{3x+1}

Solution

$$e^{3x+1}=e\cdot e^{3x}$$

$$=e\left(\sum_{n=0}^{\infty}\frac{(3x)^n}{n!}\right)$$

$$=\sum_{n=0}^{\infty}\frac{e3^n x^n}{n!} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\cos\left(2x^3\right)$

Solution

$$\cos\left(2x^3\right) = 1 - \frac{\left(2x^3\right)^2}{2!} + \frac{\left(2x^3\right)^4}{4!} - \frac{\left(2x^3\right)^6}{6!} + \cdots$$

$$= 1 - \frac{2^2 x^3}{2!} + \frac{2^4 x^{12}}{4!} - \frac{2^6 x^{18}}{6!} + \cdots$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n)!} x^{6n}} \quad (\textit{for all } x)$$

Exercise

Find the Maclaurin series for: $\cos(2x - \pi)$

Solution

$$\cos(2x - \pi) = \cos(2x)\cos\pi + \sin(2x)\sin\pi$$

$$= -\cos(2x)$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x)^{2n}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^{n+1} 4^n}{(2n)!} x^{2n}} \quad (\textit{for all } x)$$

Exercise

Find the Maclaurin series for: $x^2 \sin\left(\frac{x}{3}\right)$

Solution

$$x^2 \sin\left(\frac{x}{3}\right) = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{x}{3}\right)^{2n+1}}{(2n+1)!}$$

$$= x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{3^{2n+1}(2n+1)!}$$

$$=\boxed{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+3}}{3^{2n+1}(2n+1)!}}\quad (\textit{for all } x)$$

Exercise

Find the Maclaurin series for: $\cos^2\left(\frac{x}{2}\right)$

Solution

$$\cos^2\left(\frac{x}{2}\right)=\frac{1}{2}(1+\cos x)$$

$$=\frac{1}{2}\left(1+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}\right)$$

$$=\frac{1}{2}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$

$$=\boxed{1+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}}\quad (\textit{for all } x)$$

Exercise

Find the Maclaurin series for: $\sin x\cos x$

Solution

$$\sin x\cos x=\frac{1}{2}\sin(2x)$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2x)^{2n+1}$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}2^{2n+1}x^{2n+1}$$

$$=\boxed{\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}}{(2n+1)!}x^{2n+1}}\quad (\textit{for all } x)$$

Exercise

Find the Maclaurin series for: $\tan^{-1}\left(5x^2\right)$

Solution

$$\tan^{-1}\left(5x^2\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(5x^2\right)^{2n+1} \qquad 5x^2 \le 1 \ \rightarrow\ x^2 \le \frac{1}{5} \ \Rightarrow\ -\frac{1}{\sqrt{5}} \le x \le \frac{1}{\sqrt{5}}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n\, 5^{2n+1}}{2n+1} x^{4n+2}} \quad \left(for \ \ -\frac{1}{\sqrt{5}} \le x \le \frac{1}{\sqrt{5}} \right)$$

Exercise

Find the Maclaurin series for: $\ln\left(2+x^2\right)$

Solution

$$\ln\left(2+x^2\right) = \ln 2\left(1+\frac{x^2}{2}\right)$$

$$= \ln 2 + \ln\left(1+\frac{x^2}{2}\right)$$

$$= \ln 2 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n}\left(\frac{x^2}{2}\right)^n \qquad \frac{x^2}{2} \le 1 \ \rightarrow\ x^2 \le 2 \ \Rightarrow\ -\sqrt{2} \le x \le \sqrt{2}$$

$$\boxed{= \ln 2 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} \frac{x^{2n}}{2^n}} \quad \left(for \ \ -\sqrt{2} \le x \le \sqrt{2} \right)$$

Exercise

Find the Maclaurin series for: $\frac{1+x^3}{1+x^2}$

Solution

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

$$\frac{1+x^3}{1+x^2} = \left(1+x^3\right)\left(1-x^2+x^4-x^6+\cdots\right)$$

$$= 1 - x^2 + x^4 - x^6 + \cdots + x^3 - x^5 + x^7 - x^9 + \cdots$$

$$= 1 - x^2 + x^3 + x^4 - x^5 - x^6 + x^7 + x^8 - x^9 - \cdots$$

$$\boxed{= 1 - x^2 + \sum_{n=2}^{\infty} (-1)^n \left(x^{2n-1} + x^{2n}\right)} \quad \left(for \quad |x| < 1 \right)$$

Exercise

Find the Maclaurin series for: $\ln \frac{1+x}{1-x}$

Solution

$$\ln \frac{1+x}{1-x} = \ln(1+x) - \ln(1-x)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \right)$$

$$= 2x + 2\frac{x^3}{3} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$= \sum_{n=0}^{\infty} \left((-1)^n + 1 \right) \frac{x^{n+1}}{n+1}$$

$$\boxed{= 2\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}} \quad (-1 < x < 1)$$

Exercise

Find the Maclaurin series for: $\dfrac{e^{2x^2} - 1}{x^2}$

Solution

$$\frac{e^{2x^2} - 1}{x^2} = \frac{1}{x^2}\left(e^{2x^2} - 1 \right)$$

$$= \frac{1}{x^2}\left(1 + 2x^2 + \frac{\left(2x^2\right)^2}{2!} + \frac{\left(2x^2\right)^3}{3!} + \cdots - 1 \right)$$

$$= \frac{1}{x^2}\left(2x^2 + \frac{2^2 x^4}{2!} + \frac{2^3 x^6}{3!} + \cdots\right)$$

$$= 2 + \frac{2^2 x^2}{2!} + \frac{2^3 x^4}{3!} + \frac{2^4 x^6}{4!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{2^{n+1}}{(n+1)!} x^{2n} \qquad (\text{for all } x \neq 0)$$

Exercise

Find the Maclaurin series for: $\cosh x - \cos x$

Solution

$$\cosh x - \cos x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) = 2\frac{x^2}{2!} + 2\frac{x^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} \left(1 - (-1)^n\right) \frac{x^{2n}}{(2n)!}$$

$$= 2\sum_{n=0}^{\infty} \frac{x^{4n+2}}{(4n+2)!} \qquad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\sinh x - \sin x$

Solution

$$\sinh x - \sin x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \left(1 - (-1)^n\right) \frac{x^{2n+1}}{(2n+1)!}$$

$$= 2\sum_{n=0}^{\infty} \frac{x^{4n+3}}{(4n+3)!} \qquad (\text{for all } x)$$

Exercise

Finding Taylor and Maclaurin Series generated by f at $x = a$:

$$f(x) = x^3 - 2x + 4, \quad a = 2$$

Solution

$$f(x) = x^3 - 2x + 4 \quad \to \quad f(2) = 8$$

$$f'(x) = 3x^2 - 2 \quad \to \quad f'(2) = 10$$

$$f''(x) = 6x \quad \to \quad f''(2) = 12$$

$$f'''(x) = 6 \quad \to \quad f'''(2) = 6$$

$$f^{(n)}(x) = 0 \quad \to \quad f^{(n)}(2) = 0 \quad (n > 3)$$

$$P_n(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \cdots$$

$$x^3 - 2x + 4 = 8 + 10(x-2) + 6(x-2)^2 + (x-2)^3$$

Exercise

Finding Taylor and Maclaurin Series generated by f at $x = a$:

$$f(x) = 2x^3 + x^2 + 3x - 8, \quad a = 1$$

Solution

$$f(x) = 2x^3 + x^2 + 3x - 8 \quad \to \quad f(1) = -2$$

$$f'(x) = 6x^2 + 2x + 3 \quad \to \quad f'(1) = 11$$

$$f''(x) = 12x + 2 \quad \to \quad f''(1) = 14$$

$$f'''(x) = 12 \quad \to \quad f'''(1) = 12$$

$$f^{(n)}(x) = 0 \quad \to \quad f^{(n)}(1) = 0 \quad (n \ge 4)$$

$$P_n(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots$$

$$\underline{2x^3+x^2+3x-8=-2+11(x-1)+7(x-1)^2+2(x-1)^3}$$

Exercise

Finding Taylor and Maclaurin Series generated by f at $x = a$:

$$f(x)=3x^5-x^4+2x^3+x^2-2, \quad a=-1$$

Solution

$$f(x)=3x^5-x^4+2x^3+x^2-2 \quad \rightarrow \quad f(-1)=-7$$

$$f'(x)=15x^4-4x^3+6x^2+2x \quad \rightarrow \quad f'(-1)=23$$

$$f''(x)=60x^3-12x^2+12x+2 \quad \rightarrow \quad f''(-1)=-82$$

$$f'''(x)=180x^2-24x+12 \quad \rightarrow \quad f'''(-1)=216$$

$$f^{(4)}(x)=360x-24 \quad \rightarrow \quad f^{(4)}(-1)=-384$$

$$f^{(5)}(x)=360 \quad \rightarrow \quad f^{(5)}(-1)=360$$

$$f^{(n)}(x)=0 \quad \rightarrow \quad f^{(n)}(-1)=0 \quad (n\geq 6)$$

$$P_n(x)=f(-1)+f'(-1)(x+1)+\frac{f''(-1)}{2!}(x+1)^2+\frac{f'''(-1)}{3!}(x+1)^3+\frac{f^{(4)}(-1)}{4!}(x+1)^2+\frac{f^{(4)}(-1)}{5!}(x+1)^3$$

$$3x^5-x^4+2x^3+x^2-2=-7+23(x+1)-\frac{82}{2!}(x+1)^2+\frac{216}{3!}(x+1)^3-\frac{384}{4!}(x+1)^4+\frac{360}{5!}(x+1)^3$$

$$\underline{=-7+23(x+1)-41(x+1)^2+36(x+1)^3-16(x+1)^4+3(x+1)^3}$$

Exercise

Finding Taylor and Maclaurin Series generated by f at $x = a$: $\quad f(x)=\cos\left(2x+\frac{\pi}{2}\right), \quad a=\frac{\pi}{4}$

Solution

$$f(x)=\cos\left(2x+\frac{\pi}{2}\right) \quad \rightarrow \quad f\left(\frac{\pi}{4}\right)=-1$$

$$f'(x)=-2\sin\left(2x+\frac{\pi}{2}\right) \quad \rightarrow \quad f'\left(\frac{\pi}{4}\right)=0$$

$$f''(x)=-4\cos\left(2x+\frac{\pi}{2}\right) \quad \rightarrow \quad f''\left(\frac{\pi}{4}\right)=4$$

$$f'''(x)=8\sin\left(2x+\frac{\pi}{2}\right) \quad \rightarrow \quad f'''\left(\frac{\pi}{4}\right)=0$$

$$f^{(4)}(x)=16\cos\left(2x+\frac{\pi}{2}\right) \quad \rightarrow \quad f^{(4)}\left(\frac{\pi}{4}\right)=-16$$

$$f^{(5)}(x)=-32\sin\left(2x+\frac{\pi}{2}\right) \quad \rightarrow \quad f^{(5)}\left(\frac{\pi}{4}\right)=0$$

$$\rightarrow \quad f^{(2n)}\left(\frac{\pi}{4}\right) = (-1)^n\, 2^{2n}$$

$$\begin{aligned}
\cos\left(2x+\frac{\pi}{2}\right) &= -1+\frac{4}{2!}\left(x-\frac{\pi}{4}\right)^2 - \frac{16}{4!}\left(x-\frac{\pi}{4}\right)^4 + \cdots \\
&= -1 + 2\left(x-\frac{\pi}{4}\right)^2 - \frac{2}{3}\left(x-\frac{\pi}{4}\right)^4 + \cdots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n}}{(2n)!}\left(x-\frac{\pi}{4}\right)^{2n}
\end{aligned}$$

Solution Section 4.2 – Series Solutions near Ordinary Points

Exercise

Find a power series solution. $y' = 3y$

Solution

The equation $y' = 3y$ is separable with solution

$$\frac{dy}{dx} = 3y$$

$$\int \frac{dy}{y} = \int 3\,dx$$

$$y = Ce^{3x}$$

$$\ln(y) = 3x + C$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y' - 3y = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 3\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1) a_{n+1} - 3a_n\right] x^n = 0$$

$$(n+1) a_{n+1} - 3a_n = 0$$

$$a_{n+1} = \frac{3a_n}{n+1}; \quad n \ge 0$$

With $y(0) = 3a_0$

$$a_1 = 3a_0$$

$$a_2 = \frac{3}{2} a_1 = \frac{3 \cdot 3}{2} a_0$$

$$a_3 = \frac{3}{3}a_2 = \frac{3\cdot 3\cdot 3}{2\cdot 3}a_0$$

$$a_4 = \frac{3}{4}a_3 = \frac{3\cdot 3\cdot 3\cdot 3}{2\cdot 3\cdot 4}a_0$$

$$\vdots \qquad \vdots$$

$$\boxed{a_n = \frac{3^n}{n!}a_0}$$

$$y(x) = \sum_{n=0}^{\infty} \frac{3^n}{n!}a_0\,x^n \qquad\qquad y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$$

$$\boxed{= a_0 e^{3x}} \quad \checkmark$$

Exercise

Find a power series solution. $y' = 4y$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y' = 4y$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = 4\sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n = \sum_{n=0}^{\infty} 4a_n x^n$$

$$(n+1)a_{n+1}x^n = 4a_n x^n$$

$$(n+1)a_{n+1} = 4a_n$$

$$\boxed{a_{n+1} = \frac{4}{n+1}a_n}$$

$$n=0 \quad \to \quad a_1 = 4a_0$$

$$n=1 \quad \to \quad a_2 = \frac{4}{2}a_1 = \frac{4^2}{2!}a_0$$

$$n=2 \quad \to \quad a_3 = \frac{4}{3}a_2 = \frac{4^3}{3!}a_0$$

$$n=3 \quad \to \quad a_4 = \frac{4}{4}a_3 = \frac{4^4}{4!}a_0$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\boxed{a_n = \frac{4^n}{n!}a_0}$$

$$y(x) = \sum_{n=0}^{\infty} \frac{4^n}{n!} a_0 x$$

$$= a_0\left(1 + 4x + \frac{4^2}{2!}x^2 + \frac{4^3}{3!}x^3 + \ \ldots\right)$$

$$\boxed{= a_0 e^{4x}}$$

Exercise

Find a power series solution. $y' = x^2 y$

Solution

$$\frac{dy}{dx} = x^2 y$$

$$\int \frac{dy}{y} = \int x^2\, dx$$

$$\ln y = \frac{1}{3}x^3 + C_1$$

$$y = e^{\frac{1}{3}x^3 + C_1}$$

$$y = Ce^{\frac{1}{3}x^3}$$

$$y(0) = C(1) = a_0$$

$$\boxed{C = a_0}$$

$$\boxed{y = a_0 e^{x^3/3}}$$

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}$$

$$y'-x^2y=0$$

$$\sum_{n=1}^{\infty}na_n x^{n-1}-x^2\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=1}^{\infty}na_n x^{n-1}-\sum_{n=0}^{\infty}a_n x^{n+2}=0$$

$$\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{k=-2}^{\infty}(k+3)a_{k+3}\,x^{k+2}$$

$$=\sum_{n=-2}^{\infty}(n+3)a_{n+3}\,x^{n+2}$$

$$\sum_{n=-2}^{\infty}(n+3)a_{n+3}x^{n+2}-\sum_{n=0}^{\infty}a_n x^{n+2}=0$$

$$a_1+2a_2x+\sum_{n=-2}^{\infty}(n+3)a_{n+3}x^{n+2}-\sum_{n=0}^{\infty}a_n x^{n+2}=0$$

$$a_1+2a_2x+\sum_{n=-2}^{\infty}\left[(n+3)a_{n+3}-a_n\right]x^{n+2}=0$$

If we set $a_1=a_2=0$, then

$$(n+3)a_{n+3}-a_n=0$$

$$\Rightarrow\quad a_{n+3}=\frac{a_n}{n+3},\quad n\ge 0$$

$$a_3=\frac{1}{3}a_0$$

$$a_4=\frac{1}{4}a_1=0$$

$$a_5 = \frac{1}{5}a_2 = 0$$

$$a_6 = \frac{1}{6}a_3 = \frac{1}{3\cdot 6}a_0$$

$$a_7 = \frac{1}{7}a_4 = 0$$

$$a_9 = \frac{1}{9}a_6 = \frac{1}{3\cdot 6\cdot 9}a_0 = \frac{1}{3^3(1\cdot 2\cdot 3)}a_0$$

$$a_{12} = \frac{1}{12}a_9 = \frac{1}{3\cdot 6\cdot 9\cdot 12}a_0 = \frac{1}{3^4(1\cdot 2\cdot 3\cdot 4)}a_0$$

$$\boxed{a_{3n} = \frac{1}{3^n\cdot n!}a_0}$$

$$y(x) = \sum_{n=0}^{\infty}\frac{1}{3^n\cdot n!}a_0x^{3n}$$

$$= a_0\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{x^3}{3}\right)^n$$

$$\boxed{= a_0e^{x^3/3}} \quad \checkmark \qquad\qquad P = \lim_{n\to\infty}\left|\frac{3^k k!}{1}\right| \boxed{= \infty}$$

Exercise

Find a power series solution. $y' + 2xy = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty}a_nx^n$$

$$y'(x) = \sum_{n=1}^{\infty}na_nx^{n-1}$$

$$y' + 2xy = 0$$

$$\sum_{n=1}^{\infty}na_nx^{n-1} + 2x\sum_{n=0}^{\infty}a_nx^n = 0$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}2a_nx^{n+1} = 0$$

$$a_1 + \sum_{n=0}^{\infty}(n+2)a_{n+2}x^{n+1} + \sum_{n=0}^{\infty}2a_n x^{n+1} = 0$$

$$a_1 + \sum_{n=0}^{\infty}\left[(n+2)a_{n+2} + 2a_n\right]x^{n+1} = 0$$

$$\begin{cases} \boxed{a_1 = 0} \\ (n+2)a_{n+2} + 2a_n = 0 \quad \to \quad \boxed{a_{n+2} = -\frac{2}{n+2}a_0} \end{cases}$$

$$n=0 \quad \to \quad a_2 = -a_1 \qquad\qquad n=1 \quad \to \quad a_3 = -\frac{2}{3}a_1 = 0$$

$$n=2 \quad \to \quad a_4 = -\frac{1}{2}a_2 = \frac{1}{2}a_0 \qquad\qquad n=3 \quad \to \quad a_5 = -\frac{2}{7}a_3 = 0$$

$$n=4 \quad \to \quad a_6 = -\frac{1}{3}a_4 = -\frac{1}{2\cdot 3}a_0 \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n=6 \quad \to \quad a_8 = -\frac{1}{4}a_6 = \frac{1}{4!}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_{2k} = \frac{(-1)^k}{k!}a_0}$$

$$y(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}a_0 x^{2k}$$

$$= a_0\left(1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \cdots\right)$$

$$\boxed{= a_0 e^{-x^2}} \qquad\qquad P = \lim_{n\to\infty}\left|\frac{n+2}{-2}\right| \boxed{= \infty}$$

Exercise

Find a power series solution. $(x-2)y' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty}a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty}na_n x^{n-1}$$

$$(x-2)\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$x\sum_{n=1}^{\infty} na_n x^{n-1} - 2\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} na_n x^n - 2\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} na_n x^n - 2\sum_{n=0}^{\infty} (n+1)a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[na_n - 2(n+1)a_{n+1} + a_n \right] x^n = 0$$

$$na_n - 2(n+1)a_{n+1} + a_n = 0$$

$$2(n+1)a_{n+1} = (n+1)a_n$$

$$\boxed{a_{n+1} = \frac{1}{2}a_n}$$

$$n = 0 \quad \rightarrow \quad a_1 = \frac{1}{2}a_0$$

$$n = 1 \quad \rightarrow \quad a_2 = \frac{1}{2}a_1 = \frac{1}{2^2}a_0$$

$$n = 2 \quad \rightarrow \quad a_3 = \frac{1}{2}a_2 = \frac{1}{2^3}a_0$$

$$n = 3 \quad \rightarrow \quad a_4 = \frac{1}{2}a_3 = \frac{1}{2^4}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_n = \frac{1}{2^n}a_0}$$

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} \frac{1}{2^n}a_0 x^n \\ &= a_0\left(1 + \frac{1}{2}x + \frac{1}{2^2}x^2 + \frac{1}{2^3}x^3 + \cdots\right) \\ &= a_0\left(1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + \cdots\right) \end{aligned}$$

$$= a_0 \frac{1}{1-\frac{x}{2}}$$

$$\boxed{= \frac{2a_0}{2-x}}$$

Exercise

Find a power series solution. $(2x-1)y' + 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$(2x-1)\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$0 + \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[2n a_n - (n+1)a_{n+1} + 2a_n\right] x^n = 0$$

$$2n a_n - (n+1)a_{n+1} + 2a_n = 0$$

$$(n+1)a_{n+1} = 2(n+1)a_n$$

$$\boxed{a_{n+1} = 2a_n}$$

$$n = 0 \quad \to \quad a_1 = 2a_0$$

$$n = 1 \quad \to \quad a_2 = 2a_1 = 2^2 a_0$$

$$n = 2 \quad \to \quad a_3 = 2a_2 = 2^3 a_0$$

$$n = 3 \quad \to \quad a_4 = 2a_3 = 2^4 a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{a_n = 2^n a_0 \,\big|}$$

$$y(x) = \sum_{n=0}^{\infty} 2^n a_0 x^n$$

$$= a_0\left(1 + 2x + 2^2x^2 + 2^3x^3 + \cdots\right)$$

$$= a_0\left(1 + 2x + (2x)^2 + (2x)^3 + \cdots\right)$$

$$\underline{= \frac{a_0}{1-2x} \,\big|}$$

Exercise

Find a power series solution. $2(x-1)y' = 3y$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$(2x-2)\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} 3a_n x^n$$

$$2x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} 2n a_n x^{n-1} = \sum_{n=0}^{\infty} 3a_n x^n$$

$$0 + \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=1}^{\infty} 2n a_n x^{n-1} = \sum_{n=0}^{\infty} 3a_n x^n$$

$$\sum_{n=0}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} 2(n+1)a_{n+1} x^n = \sum_{n=0}^{\infty} 3a_n x^n$$

$$\sum_{n=0}^{\infty}\left[2na_n - 2(n+1)a_{n+1}\right]x^n = \sum_{n=0}^{\infty} 3a_n x^n$$

$$2na_n - 2(n+1)a_{n+1} = 3a_n$$

$$-2(n+1)a_{n+1} = (3-2n)a_n$$

$$\boxed{a_{n+1} = \frac{2n-3}{2n+2}a_n} \qquad \rho = \lim_{n\to\infty}\frac{2n-3}{2n+2} = \boxed{1}$$

$$n=0 \quad \rightarrow \quad a_1 = -\frac{3}{2}a_0$$

$$n=1 \quad \rightarrow \quad a_2 = -\frac{1}{4}a_1 = \frac{3}{8}a_0$$

$$n=2 \quad \rightarrow \quad a_3 = \frac{1}{6}a_2 = \frac{1}{16}a_0$$

$$n=3 \quad \rightarrow \quad a_4 = \frac{3}{8}a_3 = \frac{3}{128}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y(x) = a_0\left(1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \frac{3}{128}x^4 + \cdots\right)}$$

$$\frac{y'}{y} = \frac{3}{2}\frac{1}{x-1}$$

$$\int \frac{dy}{y} = \frac{3}{2}\int \frac{1}{x-1}dx$$

$$\ln y = \frac{3}{2}\ln(x-1) + \ln C$$

$$\ln y = \ln C(x-1)^{3/2}$$

$$\boxed{y(x) = C(x-1)^{3/2}}$$

Exercise

Find a power series solution. $(1+x)y' - y = 0$

Solution

$$(1+x)\frac{dy}{dx} = y$$

$$\frac{dy}{y} = \frac{dx}{1+x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{1+x}$$

$$\ln(y) = \ln(x+1) + C \quad \Rightarrow \quad \underline{y = C(x+1)}$$

With $y(0) = C(0+1) = a_0 \quad \rightarrow \quad C = a_0$

$$\Rightarrow \quad \underline{y = a_0(x+1)}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$(1+x)y' - y = 0$$

$$(1+x)\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left((n+1) a_{n+1} x^n + n a_n x^n - a_n x^n \right) = 0$$

$$(n+1) a_{n+1} + (n-1) a_n = 0$$

$$\rightarrow \ a_{n+1} = \frac{1-n}{n+1} a_n; \quad n \ge 0$$

$$a_1 = a_0 \qquad a_2 = 0 a_1 = 0 \qquad a_3 = \frac{-1}{3} a_2 = 0$$

$$a_n = 0 \quad \textit{for} \quad n \ge 2$$

$$y(x) = a_0 + a_1 x$$

$$= a_0 + a_0 x$$

$$\boxed{= a_0 (1+x)} \quad \checkmark$$

Exercise

Find a power series solution. $(2-x)y' + 2y = 0$

Solution

$$(2-x)\frac{dy}{dx} + 2y = 0$$

$$(2-x)\frac{dy}{dx} = -2y$$

$$\int \frac{dy}{y} = \int \frac{2d(2-x)}{2-x}$$

$$\ln y = 2\ln(2-x) + C_1$$

$$\ln y = \ln(2-x)^2 + C_1$$

$$\ln y = \ln C(2-x)^2$$

$$y = C(2-x)^2$$

$$y(0) = C(2-0)^2 = a_0$$

$$\boxed{C = \frac{1}{4}a_0}$$

$$\boxed{y = \frac{1}{4}a_0 (2-x)^2}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$(2-x)y' + 2y = 0$$

$$(2-x)\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$2\sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=1}^{\infty} na_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} na_n x^n = 0 + \sum_{n=1}^{\infty} na_n x^n = \sum_{n=0}^{\infty} na_n x^n$$

$$2\sum_{n=0}^{\infty} (n+1)a_{n+1} x^n - \sum_{n=0}^{\infty} na_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[2(n+1)a_{n+1} - na_n + 2a_n\right]x^n = 0$$

$$2(n+1)a_{n+1} - (n-2)a_n = 0$$

$$2(n+1)a_{n+1} = (n-2)a_n$$

$$a_{n+1} = \frac{n-2}{2(n+1)}a_n, \qquad n \ge 0$$

$$a_1 = \frac{-2}{2}a_0 = -a_0$$

$$a_2 = \frac{-1}{4}a_1 = \frac{1}{4}a_0$$

$$a_3 = \frac{0}{6}a_0 = 0$$

$$\vdots$$

$$a_n = 0$$

$$y(x) = a_0 - a_0 x + \frac{1}{4}a_0 x^2$$

$$= a_0\left(1 - x + \frac{1}{4}x^2\right)$$

$$= \frac{1}{4}a_0\left(4 - 4x + x^2\right)$$

$$\boxed{= \frac{1}{4}a_0(2-x)^2}\ \checkmark$$

Exercise

Find a power series solution. $(x-4)y' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$(x-4)y' + y = 0$$

$$(x-4)\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty} 4(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} \left[a_n - 4(n+1)a_{n+1}\right]x^n = 0$$

$$\sum_{n=0}^{\infty} na_n x^n + a_0 - 4a_1 + \sum_{n=1}^{\infty} \left[a_n - 4(n+1)a_{n+1}\right]x^n = 0$$

$$a_0 - 4a_1 + \sum_{n=1}^{\infty} \left[(n+1)a_n - 4(n+1)a_{n+1}\right]x^n = 0$$

$$a_0 - 4a_1 = 0 \quad \rightarrow \quad \boxed{a_1 = \frac{1}{4}a_0}$$

$$(n+1)a_n - 4(n+1)a_{n+1} = 0 \quad \rightarrow \quad \boxed{a_{n+1} = \frac{1}{4}a_n}$$

$$a_2 = \frac{1}{4}a_1 = \frac{1}{4^2}a_0$$

$$a_3 = \frac{1}{4}a_2 = \frac{1}{4^3}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_n = \frac{1}{4^n}a_0}$$

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{4^n}a_0 x^n$$

$$= a_0 \sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$$

$$= a_0\left(\frac{1}{1-\frac{x}{4}}\right)$$

$$= a_0\left(\frac{4}{4-x}\right)$$

$$\boxed{= \frac{-4a_0}{x-4}} \quad \checkmark$$

Exercise

Find a power series solution. $x^2y' = y - x - 1$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$x^2y' = y - x - 1$$

$$x^2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = -x - 1 + \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+2} = -x - 1 + \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=2}^{\infty}(n-1)a_{n-1}x^n = -x - 1 + \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=2}^{\infty}(n-1)a_{n-1}x^n = -x - 1 + a_0 + a_1x + \sum_{n=2}^{\infty} a_n x^n$$

$$-x - 1 + a_0 + a_1x = 0$$

$$a_0 + a_1x = 1 + x \quad \Rightarrow \quad a_0 = 1; \;\; a_1 = 1$$

$$(n-1)a_{n-1} = a_n$$

$$a_2 = a_1 = 1$$

$$a_3 = 2a_2 = 2$$
$$a_4 = 3a_3 = 1\cdot 2\cdot 3$$
$$\vdots \quad \vdots$$
$$a_n = (n-1)!$$

$$\boxed{y(x) = 1 + x + x^2 + 2!x^3 + 3!x^4 + \cdots}$$

Exercise

Find a power series solution. $(x-3)y' + 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$(x-3)y' + 2y = 0$$

$$(x-3)\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} 3(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 3(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 3(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[n a_n - 3(n+1) a_{n+1} + 2a_n \right] x^n = 0$$

$$-3(n+1)a_{n+1} + (n+2)a_n = 0 \quad \rightarrow \quad \boxed{a_{n+1} = \frac{n+2}{3(n+1)} a_n} \quad n = 0,\ 1,\ 2,\ \cdots$$

$$a_1 = \frac{2}{3}a_0 \qquad a_2 = \frac{3}{3\cdot 2}a_1 = \frac{3}{3^2}a_0$$

$$a_3 = \frac{4}{3\cdot 3}a_2 = \frac{4}{3^3}a_0 \qquad a_4 = \frac{5}{3\cdot 4}a_3 = \frac{5}{3^4}a_0$$

$$\vdots \quad \vdots$$

$$a_n = \frac{n+1}{3^n}a_0 \qquad n \ge 1$$

$$y(x) = \left(1 + \frac{2}{3}x + \frac{3}{3^2}x^2 + \frac{4}{3^3}x^3 + \dots\right)$$

$$= a_0 \sum_{n=0}^{\infty} \frac{n+1}{3^n}x^n$$

$$\rho = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{3n+3}{n+2}$$

$$\underline{= 3 \,|}$$

$$\underline{y(x) = \frac{1}{(3-x)^2} \,|}$$

Exercise

Find a power series solution. $xy' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$xy' + y = 0$$

$$x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_n x^n = 0$$

$$(n+1)a_n = 0 \quad \rightarrow \quad \underline{a_n = 0|}$$

$$\underline{y(x) \equiv 0|}$$

∴ The equation has no non-trivial power series.

Exercise

Find a power series solution. $x^3 y' - 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$$

$$x^3 y' - 2y = 0$$

$$x^3 \sum_{n=1}^{\infty} na_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} na_n x^{n+2} - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=3}^{\infty} (n-2) a_{n-2} x^n - 2\left(a_0 + a_1 x + a_2 x^2\right) - \sum_{n=3}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=3}^{\infty} \left[(n-2) a_{n-2} - 2a_n\right] x^n - 2\left(a_0 + a_1 x + a_2 x^2\right) = 0$$

$$\begin{cases} a_0 = a_1 = a_2 = 0 \\ (n-2)a_{n-2} = 2a_n \end{cases} \rightarrow ka_k = 2a_{k+2} \quad (k = n-2)$$

$$\underline{a_{k+2} = \frac{k}{2}a_k}$$

$$a_3 = \frac{1}{2}a_1 = 0$$

$$a_4 = a_2 = 0$$

$$a_n \equiv 0$$

$$\underline{y(x) \equiv 0}$$

∴ The equation has no non-trivial power series.

Exercise

Find a power series solution. $y'' = 4y$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$y'' = 4y$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = 4\sum_{n=0}^{\infty} a_n x^n$$

$$(n+2)(n+1)a_{n+2} = 4a_n$$

$$a_{n+2} = \frac{4}{(n+2)(n+1)}a_n$$

$$n = 0 \rightarrow a_2 = 2a_0 = \frac{4}{2}a_0 \qquad n = 1 \rightarrow a_3 = \frac{4}{2\cdot 3}a_1$$

$$n = 2 \rightarrow a_4 = \frac{4}{3\cdot 4}a_2 = \frac{4^2}{4!}a_0 \qquad n = 3 \rightarrow a_5 = \frac{4}{4\cdot 5}a_3 = \frac{4^2}{5!}a_1$$

$n=4 \quad \rightarrow \quad a_6 = \frac{4}{5\cdot 6}a_4 = \frac{4^3}{6!}a_0 \qquad\qquad n=5 \quad \rightarrow \quad a_7 = \frac{4}{6\cdot 7}a_5 = \frac{4^3}{7!}a_1$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{a_{2k} = \frac{2^{2k}}{(2k)!}a_0} \qquad\qquad \boxed{a_{2k+1} = \frac{2^{2k}}{(2k+1)!}a_1}$$

$$y(x) = a_0\left(1 + \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 + \frac{2^6}{6!}x^6 + \cdots\right) + a_1\left(x + \frac{2^3}{3!}x^3 + \frac{2^5}{5!}x^5 + \cdots\right)$$

$$= a_0\left(1 + \frac{1}{2!}(2x)^2 + \frac{1}{4!}(2x)^4 + \frac{1}{6!}(2x)^6 + \cdots\right) + a_1\left(x + \frac{1}{3!}(2x)^3 + \frac{1}{5!}(2x)^5 + \cdots\right)$$

$$\boxed{= a_0 \cosh 2x + a_1 \sinh 2x}$$

Exercise

Find a power series solution. $y'' = 9y$

Solution

The equation $y'' = 9y$ has a characteristic equation:

$\lambda^2 - 9 = 0$

$\boxed{\lambda = \pm 3}$

$\therefore$ The general solution: $y(x) = C_1 e^{3x} + C_2 e^{-3x}$

With $y(0) = a_0 \quad and \quad y'(0) = a_1$

$$y(0) = C_1 + C_2 = a_0$$

$$y'(x) = 3C_1 e^{3x} - 3C_2 e^{-3x}$$

$$y(0) = 3C_1 e^{3(0)} - 3C_2 e^{-3(0)} \quad \rightarrow \quad 3C_1 - 3C_2 = a_1$$

$$\begin{cases} C_1 + C_2 = a_0 \\ 3C_1 - 3C_2 = a_1 \end{cases} \rightarrow \begin{cases} 3C_1 + 3C_2 = 3a_0 \\ 3C_1 - 3C_2 = a_1 \end{cases}$$

$$6C_1 = 3a_0 + a_1$$

$$\boxed{C_1 = \frac{3a_0 + a_1}{6}}$$

$$C_2 = a_0 - C_1$$

$$C_2 = a_0 - \frac{3a_0 + a_1}{6}$$

$$=\boxed{\frac{3a_0-a_1}{6}}$$

$$\boxed{y(x)=\frac{3a_0+a_1}{6}e^{3x}+\frac{3a_0-a_1}{6}e^{-3x}}$$

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y''-9y=0$

$$\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-9\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-9\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-9a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}-9a_n=0$$

$$a_{n+2}=\frac{9}{(n+2)(n+1)}a_n,\quad n\ge 0$$

$$a_2=\frac{9}{(2)(1)}a_0=\frac{9}{2}a_0 \qquad a_3=\frac{9}{(3)(2)}a_1=\frac{9}{2\cdot 3}a_1$$

$$a_4=\frac{3^2}{(4)(3)}a_2=\frac{9\cdot 9}{2\cdot 3\cdot 4}a_0=\frac{3^4}{2\cdot 3\cdot 4}a_0 \qquad a_5=\frac{9}{(5)(4)}a_3=\frac{3^4}{2\cdot 3\cdot 4\cdot 5}a_1$$

$$a_6=\frac{3^2}{(6)(5)}a_4=\frac{3^6}{6!}a_0 \qquad a_7=\frac{9}{(7)(6)}a_5=\frac{3^6}{7!}a_1$$

$$a_{2n}=\frac{3^{2n}}{(2n)!}a_0 \qquad a_{2n+1}=\frac{3^{2n}}{(2n+1)!}a_1$$

$$y(x)=a_0\left(1+\frac{3^2}{2!}x^2+\frac{3^4}{4!}x^4+\frac{3^6}{6!}x^6+\cdots\right)+a_1\left(x+\frac{3^2}{3!}x^3+\frac{3^4}{5!}x^5+\frac{3^6}{7!}x^7+\cdots\right)$$

$$y(x)=\frac{3a_0+a_1}{6}e^{3x}+\frac{3a_0-a_1}{6}e^{-3x}$$

$$=\frac{3a_0+a_1}{6}\left[1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\cdots\right]+\frac{3a_0-a_1}{6}\left[1-3x+\frac{(-3x)^2}{2!}+\frac{(-3x)^3}{3!}+\cdots\right]$$

$$=\frac{3a_0}{6}\left[1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\frac{(3x)^4}{4!}+\cdots\right]+\frac{a_1}{6}\left[1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\frac{(3x)^4}{4!}+\cdots\right]$$

$$+\frac{3a_0}{6}\left[1-3x+\frac{(3x)^2}{2!}-\frac{(3x)^3}{3!}+\cdots\right]-\frac{a_1}{6}\left[1-3x+\frac{(3x)^2}{2!}-\frac{(3x)^3}{3!}+\cdots\right]$$

$$=\frac{1}{2}a_0\left[1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\frac{(3x)^4}{4!}+\cdots+1-3x+\frac{(3x)^2}{2!}-\frac{(3x)^3}{3!}+\frac{(3x)^4}{4!}+\cdots\right]$$

$$+\frac{a_1}{6}\left[1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\frac{(3x)^4}{4!}+\cdots-1+3x-\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}-\frac{(3x)^4}{4!}+\cdots\right]$$

$$=\frac{1}{2}a_0\left[2+2\frac{(3x)^2}{2!}+2\frac{(3x)^4}{4!}+\cdots\right]+\frac{a_1}{6}\left[6x+2\frac{(3x)^3}{3!}+2\frac{(3x)^5}{5!}+\cdots\right]$$

$$=a_0\left(1+\frac{(3x)^2}{2!}+\frac{(3x)^4}{4!}+\cdots\right)+a_1\left(x+\frac{3^2x^3}{3!}+\frac{3^4x^5}{5!}+\cdots\right)$$

Which are identical.

Exercise

Find a power series solution. $y''+y=0$

Solution

The equation $y''+y=0$ has a characteristic equation:

$\lambda^2+1=0 \quad\Rightarrow\quad \lambda=\pm i$

$\therefore$ The general solution: $y(x)=C_1\sin x+C_2\cos x$

With $y(0)=a_0 \quad and \quad y'(0)=a_1$

$$y(0) = C_2 = a_0$$

$$y'(x) = C_1 \cos x - C_2 \sin x$$

$$y(0) = C_1 = a_1$$

$$y(x) = a_1 \sin x + a_0 \cos x$$

$$\boxed{= a_0 \cos x + a_1 \sin x}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} + a_n = 0 \quad \to \quad a_{n+2} = \frac{-1}{(n+2)(n+1)} a_n, \quad n \ge 0$$

$$a_2 = \frac{1}{(2)(1)} a_0 = -\frac{1}{2} a_0 \qquad a_3 = \frac{1}{(3)(2)} a_1 = -\frac{1}{2 \cdot 3} a_1$$

$$a_4 = -\frac{1}{(4)(3)} a_2 = \frac{1}{2 \cdot 3 \cdot 4} a_0 \qquad a_5 = -\frac{1}{(5)(4)} a_3 = \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} a_1$$

$$a_6 = -\frac{1}{(6)(5)} a_4 = -\frac{1}{6!} a_0 \qquad a_7 = -\frac{1}{(7)(6)} a_5 = -\frac{1}{7!} a_1$$

$$a_{2n} = \frac{(-1)^n}{(2n)!} a_0 \qquad a_{2n+1} = \frac{(-1)^n}{(2n+1)!} a_1$$

$$y(x) = a_0 \left(1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \cdots + \frac{(-1)^n}{(2n)!} x^{2n} + \cdots \right) + a_1 \left(x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + \cdots \right)$$

$$\boxed{= a_0 \cos x + a_1 \sin x} \quad \checkmark$$

Exercise

Find a power series solution. $y'' - y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_n \right] x^n = 0$$

$$\boxed{a_{n+2} = \frac{1}{(n+2)(n+1)} a_n}$$

$$n = 0 \rightarrow a_2 = \frac{1}{1 \cdot 2} a_0 \qquad n = 1 \rightarrow a_3 = \frac{1}{2 \cdot 3} a_1$$

$$n = 2 \rightarrow a_4 = \frac{1}{3 \cdot 4} a_2 = \frac{1}{4!} a_0 \qquad n = 3 \rightarrow a_5 = \frac{1}{4 \cdot 5} a_3 = \frac{1}{5!} a_1$$

$$n = 4 \rightarrow a_6 = \frac{1}{5 \cdot 6} a_4 = \frac{1}{6!} a_0 \qquad n = 5 \rightarrow a_7 = \frac{1}{6 \cdot 7} a_5 = \frac{1}{7!} a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_{2k} = \frac{1}{(2k)!} a_0} \qquad \boxed{a_{2k+1} = \frac{1}{(2k+1)!} a_1}$$

$$y(x) = a_0 \left(1 + \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \frac{1}{6!} x^6 + \cdots \right) + a_1 \left(x + \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots \right)$$

$$\boxed{= a_0 \cosh x + a_1 \sinh x}$$

Exercise

Find a power series solution. $y'' + y = x$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$y'' + y = x$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = x$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n + a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n - x = 0$$

$$\sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n \right] x^n + a_0 + 2a_2 + \left(6a_3 + a_1 - 1 \right) x = 0$$

$$\begin{cases} a_0 + 2a_2 = 0 \\ 6a_3 + a_1 - 1 = 0 \\ (n+2)(n+1) a_{n+2} + a_n = 0 \end{cases}$$

$$\begin{cases} a_2 = -\frac{1}{2} a_0 \\ a_3 = -\frac{1}{6}\left(a_1 - 1 \right) \\ a_{n+2} = -\dfrac{1}{(n+2)(n+1)} a_n \end{cases}$$

$n = 2 \rightarrow a_4 = -\frac{1}{3 \cdot 4} a_2 = \frac{1}{4!} a_0$ $\quad$ $n = 3 \rightarrow a_5 = -\frac{1}{4 \cdot 5} a_3 = \frac{1}{5!}\left(a_1 - 1 \right)$

$n = 4 \rightarrow a_6 = -\frac{1}{5 \cdot 6} a_4 = -\frac{1}{6!} a_0$ $\quad$ $n = 5 \rightarrow a_7 = -\frac{1}{6 \cdot 7} a_5 = -\frac{1}{7!}\left(a_1 - 1 \right)$

$\vdots \quad \vdots \quad \vdots \quad \vdots$ $\qquad$ $\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{a_{2k} = \frac{(-1)^k}{(2k)!}a_0} \qquad\qquad \boxed{a_{2k+1} = \frac{(-1)^k}{(2k+1)!}\left(a_1 - 1\right)}$$

$$y(x) = a_0 + a_1 x + a_0 \sum_{k=2}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + \left(a_1 - 1\right)\sum_{k=3}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$= a_0 + a_0\left(-\frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right) + \left(a_1 - 1\right)\left(-\frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right) + a_1 x - x + x$$

$$= a_0\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right) + \left(a_1 - 1\right)\left(-\frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right) + \left(a_1 - 1\right)x + x$$

$$= x + a_0\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right) + \left(a_1 - 1\right)\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right)$$

$$\boxed{= x + a_0 \cos x + \left(a_1 - 1\right)\sin x}$$

Exercise

Find a power series solution. $y'' - xy = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$y'' - xy = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}a_{n-1}x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - a_{n-1}\right]x^n = 0$$

$$2a_2 = 0 \rightarrow \underline{a_2 = 0}$$

$$(n+2)(n+1)a_{n+2} - a_{n-1} = 0 \rightarrow a_{n+2} = \frac{a_{n-1}}{(n+1)(n+2)}$$

$$a_3 = \frac{a_0}{2\cdot 3} = \frac{1}{6}a_0 \qquad a_4 = \frac{a_1}{3\cdot 4} = \frac{1}{12}a_1 \qquad a_5 = \frac{a_2}{4\cdot 5} = 0$$

$$a_6 = \frac{a_3}{5\cdot 6} = \frac{1}{180}a_0 \qquad a_7 = \frac{a_3}{6\cdot 7} = \frac{1}{504}a_1 \qquad a_8 = \frac{a_5}{7\cdot 8} = 0$$

$$\vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots$$

$$\begin{cases} y_1(x) = \left(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots\right)a_0 \\ y_2(x) = \left(x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \cdots\right)a_1 \end{cases}$$

Exercise

Find a power series solution. $y'' + xy = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty}a_nx^n$$

$$y'(x) = \sum_{n=1}^{\infty}na_nx^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty}n(n-1)a_nx^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y'' + xy = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=0}^{\infty}a_nx^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}a_n x^{n+1}=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}a_{n-1}x^n=0$$

$$2a_2+\sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}a_{n-1}x^n=0$$

$$2a_2+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_{n-1}\right]x^n=0$$

$$\begin{cases} 2a_2=0 \ \rightarrow \ \boxed{a_2=0} \\ (n+2)(n+1)a_{n+2}+a_{n-1}=0 \end{cases}$$

$$\boxed{a_{n+2}=-\frac{a_{n-1}}{(n+1)(n+2)}}$$

a_0 $\qquad$ a_1 $\qquad$ $a_2=0$

$n=1 \ \rightarrow \ a_3=-\frac{a_0}{2\cdot 3}=-\frac{1}{6}a_0$ $\qquad$ $n=2 \ \rightarrow \ a_4=-\frac{1}{3\cdot 4}a_1$

$n=3 \ \rightarrow \ a_5=-\frac{a_2}{20}=0$

$n=4 \ \rightarrow \ a_6=-\frac{a_3}{5\cdot 6}=\frac{1}{180}a_0$ $\qquad$ $n=5 \ \rightarrow \ a_7=-\frac{a_3}{6\cdot 7}=\frac{1}{504}a_1$

$n=6 \ \rightarrow \ a_8=-\frac{a_5}{56}=0$

$n=7 \ \rightarrow \ a_9=-\frac{a_6}{8\cdot 9}=-\frac{1}{12{,}960}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$ $\qquad$ $\vdots \quad \vdots \quad \vdots \quad \vdots$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{\begin{cases} y_1(x)=\left(1-\frac{1}{6}x^3+\frac{1}{180}x^6-\frac{1}{12{,}960}x^9+\cdots\right)a_0 \\ y_2(x)=\left(x-\frac{1}{12}x^4+\frac{1}{504}x^7-\cdots\right)a_1 \end{cases}}$$

$$\boxed{y(x)=\left(1-\frac{1}{6}x^3+\frac{1}{180}x^6-\frac{1}{12{,}960}x^9+\cdots\right)a_0+\left(x-\frac{1}{12}x^4+\frac{1}{504}x^7-\cdots\right)a_1}$$

Exercise

Find a power series solution. $y'' + xy' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$y'' + xy' + y = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n+1) a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} + (n+1) a_n = 0$$

$$\boxed{a_{n+2} = -\frac{a_n}{n+2}}$$

a_0	a_1
$n = 0 \rightarrow a_2 = -\frac{1}{2} a_0$	$n = 1 \rightarrow a_3 = -\frac{1}{3} a_1$
$n = 2 \rightarrow a_4 = -\frac{1}{4} a_2 = \frac{1}{4 \cdot 2} a_0$	$n = 3 \rightarrow a_5 = -\frac{a_3}{5} = \frac{1}{3 \cdot 5} a_1$
$n = 4 \rightarrow a_6 = -\frac{a_4}{6} = -\frac{1}{6 \cdot 4 \cdot 2} a_0$	$n = 5 \rightarrow a_7 = -\frac{a_5}{7} = -\frac{1}{7 \cdot 5 \cdot 3} a_1$

$$n=6 \;\to\; a_8 = -\frac{a_6}{8} = \frac{1}{8\cdot 6\cdot 4\cdot 2}a_0 \qquad\qquad n=7 \;\to\; a_9 = -\frac{a_7}{9} = \frac{1}{9\cdot 7\cdot 5\cdot 3}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n} = \frac{(-1)^n a_0}{(2n)(2n-2)\cdots 4\cdot 2} = \frac{(-1)^n}{n!\ 2^n}a_0$$

$$a_{2n} = \frac{(-1)^n a_1}{(2n+1)(2n-1)\cdots 5\cdot 3} = \frac{(-1)^n}{(2n+1)!!}a_1$$

$$\boxed{y(x) = a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\ 2^n}x^{2n} + a_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!!}x^{2n+1}}$$

$$\boxed{y(x) = a_0\left(1-\tfrac{1}{2}x^2+\tfrac{1}{8}x^4-\tfrac{1}{48}x^6+\cdots\right)+a_1\left(x-\tfrac{1}{3}x^3+\tfrac{1}{15}x^5-\tfrac{1}{105}x^7+\cdots\right)}$$

Exercise

Find a power series solution. $y''-xy'-y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''-xy'-y=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x\sum_{n=1}^{\infty}na_nx^{n-1} - \sum_{n=0}^{\infty}a_nx^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}na_nx^n - \sum_{n=0}^{\infty}a_nx^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}na_nx^n - \sum_{n=0}^{\infty}a_nx^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-(n+1)a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}-(n+1)a_n=0$$

$$\boxed{a_{n+2}=\frac{a_n}{n+2}}$$

$$\begin{array}{ll}
\qquad a_0 & \qquad a_1 \\
n=0 \;\rightarrow\; a_2=\frac{1}{2}a_0 & n=1 \;\rightarrow\; a_3=\frac{1}{3}a_1 \\
n=2 \;\rightarrow\; a_4=\frac{1}{4}a_2=\frac{1}{4\cdot 2}a_0 & n=3 \;\rightarrow\; a_5=\frac{a_3}{5}=\frac{1}{3\cdot 5}a_1 \\
n=4 \;\rightarrow\; a_6=\frac{a_4}{6}=\frac{1}{6\cdot 4\cdot 2}a_0 & n=5 \;\rightarrow\; a_7=\frac{a_5}{7}=\frac{1}{7\cdot 5\cdot 3}a_1 \\
n=6 \;\rightarrow\; a_8=\frac{a_6}{8}=\frac{1}{8\cdot 6\cdot 4\cdot 2}a_0 & n=7 \;\rightarrow\; a_9=\frac{a_7}{9}=\frac{1}{9\cdot 7\cdot 5\cdot 3}a_1 \\
\vdots \quad \vdots \quad \vdots \quad \vdots & \vdots \quad \vdots \quad \vdots \quad \vdots
\end{array}$$

$$a_{2n}=\frac{a_0}{(2n)(2n-2)\cdots 4\cdot 2}=\frac{1}{n!\,2^n}a_0$$

$$a_{2n}=\frac{a_1}{(2n+1)(2n-1)\cdots 5\cdot 3}=\frac{1}{(2n+1)!!}a_1$$

$$\boxed{y(x)=a_0\sum_{n=0}^{\infty}\frac{1}{n!\,2^n}x^{2n}+a_1\sum_{n=0}^{\infty}\frac{1}{(2n+1)!!}x^{2n+1}}$$

$$\boxed{y(x)=a_0\left(1+\frac{x^2}{2}+\frac{x^4}{2^2\cdot 2}+\cdots+\frac{x^{2n}}{2^n\cdot n!}\right)+a_1\left(x+\frac{x^3}{3}+\frac{x^5}{5\cdot 3}+\cdots+\frac{2^n\cdot n!}{(2n+1)!!}x^{2n+1}\right)}$$

Exercise

Find a power series solution. $y''+x^2y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y''+x^2y=0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_{n-2} \right] x^n = 0$$

$$2a_2 = 0 \rightarrow a_2 = 0$$

$$6a_3 = 0 \rightarrow a_3 = 0$$

$$(n+2)(n+1) a_{n+2} + a_{n-2} = 0 \rightarrow a_{n+2} = \frac{-a_{n-2}}{(n+1)(n+2)}$$

$$a_4 = \frac{-a_0}{3\cdot 4} = -\frac{1}{12} a_0 \qquad a_5 = \frac{-a_1}{4\cdot 5} = -\frac{1}{20} a_1 \qquad a_6 = \frac{-a_2}{5\cdot 6} = 0 \qquad a_7 = \frac{-a_3}{6\cdot 7} = 0$$

$$a_8 = \frac{-a_4}{7\cdot 8} = \frac{1}{672} a_0 \qquad a_9 = \frac{-a_5}{8\cdot 9} = \frac{1}{1,440} a_1 \qquad a_{10} = \frac{-a_6}{9\cdot 10} = 0 \qquad a_{11} = \frac{a_7}{10\cdot 11} = 0$$

$$\vdots \quad \vdots \qquad \vdots \quad \vdots \qquad \vdots \quad \vdots \qquad \vdots \quad \vdots$$

$$\begin{cases} y_1(x) = \left(1 - \frac{1}{12}x^4 + \frac{1}{672}x^8 - \cdots\right) a_0 \\ y_2(x) = \left(x - \frac{1}{20}x^5 + \frac{1}{1,440}x^9 + \cdots\right) a_1 \end{cases}$$

Exercise

Find a power series solution. $y'' + k^2x^2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$y'' + k^2x^2y = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + k^2x^2\sum_{n=0}^{\infty}a_n x^n = 0$$

$$2a_2 + 6a_3x + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty}k^2a_{n-2}x^n = 0$$

$$2a_2 + 6a_3x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + k^2a_{n-2}\right]x^n = 0$$

$$\begin{cases} a_2 = 0 \\ a_3 = 0 \\ (n+2)(n+1)a_{n+2} + k^2a_{n-2} = 0 \end{cases}$$

$$\underline{a_{n+2} = -\frac{k^2}{(n+1)(n+2)}a_{n-2}} \quad (n \ge 2)$$

$n=2 \ \rightarrow \ a_4 = -\frac{k^2}{3\cdot 4}a_0$

$n=3 \rightarrow \ a_5 = -\frac{k^2}{4\cdot 5}a_1$

$n=6 \ \rightarrow \ a_8 = -\frac{k^2}{7\cdot 8}a_4 = \frac{k^4}{3\cdot 4\cdot 7\cdot 8}a_0$

$n=7 \rightarrow \ a_9 = -\frac{k^2}{8\cdot 9}a_5 = \frac{k^4}{4\cdot 5\cdot 8\cdot 9}a_1$

$n=10 \ \rightarrow \ a_{12} = -\frac{k^2}{11\cdot 12}a_8 = \frac{k^6}{3\cdot 4\cdot 7\cdot 8\cdot 11\cdot 12}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{4m} = -\frac{k^2}{(4m)(4m-1)} a_{4m-4} \qquad a_{4m+1} = -\frac{k^2}{(4m)(4m+1)} a_{4m-3}$$

$$n = 4 \to a_6 = -\frac{k^2}{5 \cdot 6} a_2 = 0 \qquad n = 5 \to a_7 = -\frac{k^2}{6 \cdot 7} a_3 = 0$$

$$n = 8 \to a_{10} = -\frac{k^2}{7 \cdot 8} a_6 = 0 \qquad n = 9 \to a_{11} = -\frac{k^2}{10 \cdot 11} a_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{\begin{aligned} y(x) = a_0 &\left(1 - \frac{k^2}{3 \cdot 4} x^4 + -\frac{k^4}{3 \cdot 4 \cdot 7 \cdot 8} x^8 - \frac{k^6}{3 \cdot 4 \cdot 7 \cdot 8 \cdot 11 \cdot 12} x^{12} + \cdots\right) \\ &+ a_1\left(x - \frac{k^2}{4 \cdot 5} x^5 + -\frac{k^4}{4 \cdot 5 \cdot 8 \cdot 9} x^9 - \frac{k^6}{4 \cdot 5 \cdot 8 \cdot 9 \cdot 12 \cdot 13} x^{13} + \cdots\right) \end{aligned}}$$

Exercise

Find a power series solution. $y'' + 3xy' + 3y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + 3xy' + 3y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 3x \sum_{n=1}^{\infty} n a_n x^{n-1} + 3 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 3n a_n x^n + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + (3n+3) a_n\right] x^n = 0$$

$$(n+2)(n+1)a_{n+2}+3(n+1)a_n=0$$

$$a_{n+2}=-\frac{3}{n+2}a_n$$

a_0

$n=2 \rightarrow a_2=-\frac{3}{2}a_0$

$n=4 \rightarrow a_4=-\frac{3}{4}a_2=\frac{3^2}{2^2\cdot 2}a_0$

$n=6 \rightarrow a_6=-\frac{3}{6}a_3=\frac{3^3}{2^3\cdot 2\cdot 3}a_0$

$n=8 \rightarrow a_8=-\frac{3}{8}a_6=\frac{3^4}{2^4\cdot 2\cdot 3\cdot 4}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$a_{2k}=\frac{(-3)^k}{2^k k!}a_0$$

a_1

$n=3 \rightarrow a_3=-\frac{3}{3}a_1$

$n=5 \rightarrow a_5=-\frac{3}{5}a_3=\frac{3^2}{3\cdot 5}a_1$

$n=7 \rightarrow a_7=-\frac{3}{7}a_5=\frac{3^3}{3\cdot 5\cdot 7}a_1$

$n=9 \rightarrow a_9=-\frac{3}{9}a_7=\frac{3^3}{3\cdot 5\cdot 7\cdot 9}a_1$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$a_{2k+1}=\frac{(-3)^k}{3\cdot 5\cdot 7\cdots(2k+1)}a_1$$

$$y(x)=a_0\left(1+\sum_{k=1}^{\infty}\frac{(-3)^k}{2^k k!}x^{2k}\right)+a_1\left(x+\sum_{k=1}^{\infty}\frac{(-3)^k}{3\cdot 5\cdot 7\cdots(2k+1)}x^{2k+1}\right)$$

$$=a_0\left(1-\frac{3}{2}x^2+\frac{9}{8}x^4-\frac{27}{56}x^6+\cdots\right)+a_1\left(x-x^3+\frac{3^2}{3\cdot 5}x^5-\frac{27}{3\cdot 5\cdot 7}x^7+\cdots\right)$$

Exercise

Find a power series solution. $y''-2xy'+y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y''-2xy'+y=0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - 2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}2na_n x^n + \sum_{n=0}^{\infty}a_n x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}2na_n x^n + a_0 + \sum_{n=1}^{\infty}a_n x^n = 0$$

$$2a_2 + a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - 2na_n + a_n\right]x^n = 0$$

$$2a_2 + a_0 = 0 \to \underline{a_2 = -\frac{1}{2}a_0}$$

$$(n+2)(n+1)a_{n+2} - (2n-1)a_n = 0 \to a_{n+2} = \frac{(2n-1)a_n}{(n+1)(n+2)}$$

$$a_3 = \frac{a_1}{2\cdot 3} = \frac{1}{6}a_1 \qquad a_4 = \frac{3a_2}{3\cdot 4} = -\frac{1}{4}\frac{1}{4}a_0 = -\frac{1}{8}a_0$$

$$a_5 = \frac{5a_3}{4\cdot 5} = \frac{1}{2\cdot 3\cdot 4}a_1 = \frac{1}{4!}a_1 \qquad a_6 = \frac{7a_4}{5\cdot 6} = -\frac{7}{240}a_0$$

$$\vdots \quad \vdots$$

$$a_7 = \frac{9a_5}{6\cdot 7} = \frac{1}{112}a_1$$

$$\vdots \quad \vdots$$

$$\begin{cases} y_1(x) = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{240}x^6 - \cdots\right)a_0 \\ y_2(x) = \left(x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{1}{112}x^7 + \cdots\right)a_1 \end{cases}$$

Exercise

Find a power series solution. $y'' - xy' + 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - xy' + 2y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n + 2a_0 + \sum_{n=1}^{\infty} 2 a_n x^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - n a_n + 2a_n \right] x^n = 0$$

$$2a_2 + 2a_0 = 0 \rightarrow \underline{a_2 = -a_0}\,|$$

$$(n+2)(n+1) a_{n+2} - (n-2) a_n = 0$$

$$\rightarrow a_{n+2} = \frac{(n-2) a_n}{(n+1)(n+2)}$$

$$a_3 = \frac{-a_1}{2 \cdot 3} = -\frac{1}{6} a_1 \qquad\qquad a_4 = 0$$

$$a_5 = \frac{a_3}{4 \cdot 5} = -\frac{1}{5!} a_1 \qquad\qquad a_6 = \frac{2a_4}{5 \cdot 6} = 0$$

$$a_7 = \frac{3a_5}{6 \cdot 7} = \frac{3}{7!} a_1 \qquad\qquad a_8 = \frac{4a_6}{5 \cdot 6} = 0$$

$$\vdots \quad \vdots$$

$$a_9 = \frac{5a_7}{8\cdot 9} = \frac{3\cdot 5}{9!}a_1$$

$$\vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x) = 1 - x^2 \\ y_2(x) = \left(x - \frac{1}{6}x^3 + \frac{1}{5!}x^5 + \frac{3}{7!}x^7 + \cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $y'' - xy' - x^2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$y'' - xy' - x^2y = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x\sum_{n=1}^{\infty}na_n x^{n-1} - x^2\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}na_n x^n - \sum_{n=0}^{\infty}a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}na_n x^n - \sum_{n=2}^{\infty}a_{n-2} x^n = 0$$

$$2a_2 + 6a_3x + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n - a_1x - \sum_{n=2}^{\infty}na_n x^n - \sum_{n=2}^{\infty}a_{n-2}x^n = 0$$

$$2a_2 + 6a_3x - a_1x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} - na_n - a_{n-2}\right]x^n = 0$$

$$\begin{cases} 2a_2 = 0 \ \rightarrow \ \boxed{a_2 = 0} \\ (6a_3 - a_1)x = 0 \ \rightarrow \ \boxed{a_3 = \frac{1}{6}a_1} \\ (n+1)(n+2)a_{n+2} - na_n - a_{n-2} = 0 \end{cases}$$

$$\boxed{a_{n+2} = \frac{na_n + a_{n-2}}{(n+1)(n+2)}}$$

$$a_0 \qquad\qquad a_1$$

$$a_2 = 0 \qquad\qquad a_3 = \frac{1}{6}a_1$$

$$n = 2 \ \rightarrow \ a_4 = \frac{2a_2 + a_0}{3 \cdot 4} = \frac{1}{12}a_0$$

$$n = 4 \ \rightarrow \ a_6 = \frac{4a_4 + a_2}{5 \cdot 6} = \frac{1}{90}a_0$$

$$n = 6 \ \rightarrow \ a_8 = \frac{6a_6 + a_4}{7 \cdot 8} = \frac{1}{56}\left(\frac{1}{15} + \frac{1}{12}\right)a_0 = \frac{3}{1120}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n = 3 \ \rightarrow \ a_5 = \frac{3a_3 + a_1}{20} = \frac{1}{20}\left(\frac{3}{6} + 1\right)a_1 = \frac{1}{12}a_1$$

$$n = 5 \ \rightarrow \ a_7 = \frac{5a_5 + a_3}{6 \cdot 7} = \frac{1}{42}\left(\frac{5}{12} + \frac{1}{6}\right)a_1 = \frac{1}{72}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y(x) = a_0\left(1 + \frac{1}{12}x^2 + \frac{1}{90}x^4 + \frac{3}{1120}x^6 + \cdots\right) + a_1\left(x + \frac{1}{12}x^3 + \frac{1}{72}x^5 + \cdots\right)}$$

Exercise

Find a power series solution. $y'' + x^2y' + xy = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$y'' + x^2y' + xy = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x^2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + x\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+2}+\sum_{n=0}^{\infty}a_nx^{n+1}=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}(n-1)a_{n-1}x^n+\sum_{n=1}^{\infty}a_{n-1}x^n=0$$

$$2a_2+6a_3x+\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}(n-1)a_{n-1}x^n+a_0x+\sum_{n=2}^{\infty}a_{n-1}x^n=0$$

$$2a_2+\left(6a_3+a_0\right)x+\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2}+(n-1)a_{n-1}+a_{n-1}\right]x^n=0$$

$$2a_2+\left(6a_3+a_0\right)x=0\ \Rightarrow\begin{cases}a_2=0\\ a_3=-\frac{1}{6}a_0\end{cases}$$

$$(n+2)(n+1)a_{n+2}+na_{n-1}=0\ \Rightarrow a_{n+2}=-\frac{n}{(n+1)(n+2)}a_{n-1}$$

$$a_4=-\frac{2}{3\cdot 4}a_1=-\frac{1}{6}a_1\qquad a_5=-\frac{3}{4\cdot 5}a_2=0\qquad a_6=-\frac{4}{5\cdot 6}a_3=\frac{1}{45}a_0$$

$$a_7=-\frac{5}{6\cdot 7}a_4=\frac{5}{252}a_1\qquad a_8=-\frac{6}{7\cdot 8}a_5=0\qquad a_9=-\frac{7}{8\cdot 9}a_3=-\frac{7}{3,240}a_0$$

$$\vdots\ \vdots\qquad\qquad \vdots\ \vdots\qquad\qquad \vdots\ \vdots$$

$$\begin{cases}y_1(x)=\left(1-\frac{1}{6}x^3+\frac{1}{45}x^6-\frac{7}{3,240}x^9+\cdots\right)a_0\\ y_2(x)=\left(x-\frac{1}{6}x^4+\frac{5}{252}x^7-\cdots\right)a_1\end{cases}$$

Exercise

Find a power series solution. $y''+x^2y'+2xy=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$y''+x^2y'+2xy=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x^2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 2x\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+2} + \sum_{n=0}^{\infty}2a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty}(n-1)a_{n-1}x^n + \sum_{n=1}^{\infty}2a_{n-1}x^n = 0$$

$$2a_2+6a_3x+\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty}(n-1)a_{n-1}x^n + 2a_0x+\sum_{n=2}^{\infty}2a_{n-1}x^n = 0$$

$$2a_2+(6a_3+2a_0)x+\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2}+(n-1)a_{n-1}+2a_{n-1}\right]x^n = 0$$

$$2a_2+(6a_3+2a_0)x=0 \Rightarrow \begin{cases} a_2=0 \\ a_3=-\frac{1}{3}a_0 \end{cases}$$

$$(n+2)(n+1)a_{n+2}+(n+1)a_{n-1}=0$$

$$\Rightarrow a_{n+2}=-\frac{a_{n-1}}{n+2} \qquad \boxed{a_{n+3}=-\frac{a_n}{n+3}}$$

a_0	a_1	$a_2=0$
$a_3=-\frac{1}{3}a_0$	$n=1 \to a_4=-\frac{1}{4}a_1$	$n=2 \to a_5=-\frac{a_2}{5}=0$
$n=3 \to a_6=-\frac{a_3}{6}=\frac{1}{2\cdot 3^2}a_0$	$n=4 \to a_7=-\frac{a_4}{7}=\frac{1}{7\cdot 4}a_1$	$n=5 \to a_8=-\frac{a_5}{8}=0$
$n=6 \to a_9=-\frac{a_6}{9}=-\frac{1}{3!\,3^3}a_0$	$n=7 \to a_{10}=-\frac{a_7}{10}=-\frac{1}{10\cdot 7\cdot 4\cdot 1}a_1$	$n=8 \to a_{11}=0$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$
$a_{3n}=\frac{(-1)^n}{n!\,3^n}$	$a_{3n+1}=\frac{(-1)^n}{1\cdot 4\cdot 7\cdots(3n+1)}$	$a_{3n+2}=0$

$$\boxed{y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,3^n} x^{3n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{1\cdot 4\cdot 7\cdots(3n+1)} x^{3n+1}}$$

$$\boxed{y(x) = a_0\left(1 - \frac{1}{3}x^3 + \frac{1}{18}x^6 - \frac{1}{162}x^9 + \cdots\right) + a_1\left(x - \frac{1}{4}x^4 + \frac{1}{28}x^7 - \frac{1}{280}x^{10} + \cdots\right)}$$

Exercise

Find a power series solution. $y'' - x^2 y' - 3xy = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - x^2 y' - 3xy = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x^2 \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 3x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+2} - \sum_{n=0}^{\infty} 3 a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n - \sum_{n=1}^{\infty} 3 a_{n-1} x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n - 3a_0 x - \sum_{n=2}^{\infty} 3 a_{n-1} x^n = 0$$

$$2a_2 + 3(2a_3 - a_0)x + \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} - (n+2) a_{n-1}\right] x^n = 0$$

$$\begin{cases} 2a_2 = 0 & \rightarrow \boxed{a_2 = 0} \\ 2a_3 - a_0 = 0 & \rightarrow \boxed{a_3 = \frac{1}{2}a_0} \end{cases}$$

$$(n+2)(n+1)a_{n+2} = (n+2)a_{n-1}$$

$$a_{n+2} = \frac{a_{n-1}}{n+1} \quad \rightarrow \quad \boxed{a_{n+3} = \frac{a_n}{n+2}}$$

a_0	a_1	$a_2 = 0$
$a_3 = \frac{1}{2}a_0$	$n=1 \rightarrow a_4 = \frac{1}{3}a_1$	$n=2 \rightarrow a_5 = \frac{a_2}{5} = 0$
$n=3 \rightarrow a_6 = \frac{a_3}{5} = \frac{1}{2\cdot 5}a_0$	$n=4 \rightarrow a_7 = \frac{a_4}{6} = \frac{1}{2\cdot 3^2}a_1$	$n=5 \rightarrow a_8 = 0$
$n=6 \rightarrow a_9 = \frac{a_6}{8} = -\frac{1}{2\cdot 5\cdot 8}a_0$	$n=7 \rightarrow a_{10} = \frac{a_7}{9} = \frac{1}{2\cdot 3\cdot 3^3}a_1$	$n=8 \rightarrow a_{11} = 0$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$
$a_{3n} = \frac{1}{2\cdot 5\cdot 8\cdots(3n-1)}$	$a_{3n+1} = \frac{1}{n!\ 3^n}$	$a_{3n+2} = 0$

$$\boxed{y(x) = a_0\sum_{n=0}^{\infty}\frac{x^{3n}}{2\cdot 5\cdot 8\cdots\cdot(3n-1)} + a_1\sum_{n=0}^{\infty}\frac{x^{3n+1}}{n!\ 3^n}}$$

$$\boxed{y(x) = a_0\left(1+\frac{1}{2}x^3+\frac{1}{10}x^6+\frac{1}{80}x^9+\cdots\right)+a_1\left(x+\frac{1}{3}x^4+\frac{1}{18}x^7+\frac{1}{162}x^{10}+\cdots\right)}$$

Exercise

Find a power series solution. $y'' + 2xy' + 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y'' + 2xy' + 2y = 0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + 2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 2\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}2(n+1)a_{n+1}x^{n+1} + \sum_{n=0}^{\infty}2a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty}2na_n x^n + \sum_{n=0}^{\infty}2a_n x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty}2na_n x^n + 2a_0 + \sum_{n=1}^{\infty}2a_n x^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + 2na_n + 2a_n\right]x^n = 0$$

$$2a_2 + 2a_0 = 0 \quad \rightarrow \quad \underline{a_2 = -a_0}|$$

$$(n+2)(n+1)a_{n+2} + 2(n+1)a_n = 0$$

$$\rightarrow \quad \underline{a_{n+2} = -\frac{2}{n+2}a_n}| \qquad n = 1,\ 2,\ \cdots$$

$$\begin{array}{ll} a_3 = -\frac{2}{3}a_1 & a_4 = -\frac{2}{4}a_2 = \frac{1}{2}a_0 \\ a_5 = -\frac{2}{5}a_3 = \frac{4}{15}a_1 & a_6 = -\frac{2}{6}a_4 = -\frac{1}{6}a_0 \\ a_7 = -\frac{2}{7}a_3 = -\frac{8}{105}a_1 & a_8 = -\frac{2}{8}a_6 = \frac{1}{24}a_0 \\ \vdots \quad \vdots & \vdots \quad \vdots \end{array}$$

$$\begin{cases} y_1(x) = \left(1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \frac{1}{4!}x^8 - \cdots\right)a_0 \\ y_2(x) = \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7 + \cdots\right)a_1 \end{cases}$$

Exercise

Find a power series solution. $2y'' + xy' + y = 0$

Solution

$$y(x)=\sum_{n=0}^{\infty} a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$2y''+xy'+y=0$$

$$2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}na_n x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$4a_2+\sum_{n=1}^{\infty}2(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}na_n x^n+a_0+\sum_{n=1}^{\infty}a_n x^n=0$$

$$4a_2+a_0+\sum_{n=1}^{\infty}\left[2(n+2)(n+1)a_{n+2}+(n+1)a_n\right]x^n=0$$

$$4a_2+a_0=0 \quad\rightarrow\quad \boxed{a_2=-\frac{1}{4}a_0}$$

$$2(n+2)(n+1)a_{n+2}+(n+1)a_n=0$$

$$\boxed{a_{n+2}=-\frac{1}{2(n+2)}a_n}$$

a_0	a_1
$n=0 \rightarrow a_2=-\frac{1}{4}a_0$	$n=1 \rightarrow a_3=-\frac{1}{6}a_1$
$n=2 \rightarrow a_4=-\frac{1}{8}a_2=\frac{1}{2^4\cdot 2}a_0$	$n=3 \rightarrow a_5=-\frac{1}{2\cdot 5}a_3=\frac{1}{2^2\cdot 3\cdot 5}a_1$
$n=4 \rightarrow a_6=-\frac{1}{2\cdot 6}a_4=-\frac{1}{2^6\cdot 2\cdot 3}a_0$	$n=5 \rightarrow a_7=-\frac{1}{2\cdot 7}a_5=-\frac{1}{2^3\cdot 3\cdot 5\cdot 7}a_1$

$$n=6 \;\to\; a_8 = -\frac{1}{2\cdot 8}a_6 = \frac{1}{2^8\cdot 4!}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n\ge 1 \quad a_{2n} = \frac{(-1)^n}{2^{2n}\ n!}a_0$$

$$n\ge 1 \quad a_{2n+1} = \frac{(-1)^n}{2^n\ 1\cdot 3\cdot 5\cdots(2n+1)}a_1 = \frac{(-1)^n}{2^n\ (2n+1)!!}a_1$$

$$y(x) = a_0\left(1-\frac{1}{4}x^2+\frac{1}{32}x^4-\dots\right)+a_1\left(x-\frac{1}{6}x^3+\frac{1}{60}x^5-\cdots\right)$$

$$y(x) = a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}\ n!}x^{2n} + a_1\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n\ (2n+1)!!}x^{2n+1}$$

Exercise

Find a power series solution. $3y''+xy'-4y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$3y''+xy'-4y=0$

$$3\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-4\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}3(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}-\sum_{n=0}^{\infty}4a_n x^n=0$$

$$\sum_{n=0}^{\infty} 3(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$6a_2 + \sum_{n=1}^{\infty} 3(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} na_n x^n - 4a_0 - \sum_{n=1}^{\infty} 4a_n x^n = 0$$

$$6a_2 - 4a_0 + \sum_{n=1}^{\infty} \left[3(n+2)(n+1)a_{n+2} + (n-4)a_n\right]x^n = 0$$

$$6a_2 - 4a_0 \quad \rightarrow \quad \boxed{a_2 = \frac{2}{3}a_0}$$

$$3(n+2)(n+1)a_{n+2} + (n-4)a_n = 0$$

$$\boxed{a_{n+2} = -\frac{(n-4)}{3(n+1)(n+2)}a_n}$$

a_0	a_1
$a_2 = \frac{2}{3}a_0$	$n=1 \rightarrow a_3 = \frac{3}{2\cdot 3\cdot 3}a_1 = \frac{1}{2\cdot 3}a_1$
$n=2 \rightarrow a_4 = \frac{2}{36}a_2 = \frac{1}{27}a_0$	$n=3 \rightarrow a_5 = \frac{1}{4\cdot 5\cdot 3}a_3 = \frac{1}{5!\cdot 3}a_1$
$n=4 \rightarrow a_6 = 0$	$n=5 \rightarrow a_7 = -\frac{1}{3\cdot 6\cdot 7}a_5 = -\frac{1}{7!\cdot 3^2}a_1$
	$n=7 \rightarrow a_9 = -\frac{3}{3\cdot 9\cdot 8}a_7 = \frac{3}{9!\cdot 3^3}a_1$
	$n=9 \rightarrow a_{11} = -\frac{5}{3\cdot 11\cdot 10}a_9 = -\frac{3\cdot 5}{11!\cdot 3^4}a_1$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$n \ge 3 \quad a_{2n+1} = \frac{1\cdot 3\cdot 5\cdots(2n-5)(-1)^n}{(2n+1)!\ 3^n}a_1 = \frac{(2n-5)!!(-1)^n}{(2n+1)!\ 3^n}a_1$$

$$\boxed{y(x) = a_0\left(1 + \frac{2}{3}x^2 + \frac{1}{27}x^4\right) + a_1\left(x + \frac{1}{6}x^3 + \frac{1}{360}x^5 + \sum_{n=3}^{\infty}\frac{(2n-5)!!(-1)^n}{(2n+1)!\ 3^n}\right)}$$

$$\boxed{= a_0\left(1 + \frac{2}{3}x^2 + \frac{1}{27}x^4\right) + a_1\left(x + \frac{1}{6}x^3 + \frac{1}{360}x^5 - \frac{1}{45,360}x^7 + \cdots\right)}$$

Exercise

Find a power series solution. $5y'' - 2xy' + 10y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$5y'' - 2xy' + 10y = 0$$

$$5\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2x\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 10\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 5(n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} 10 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 5(n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 10 a_n x^n = 0$$

$$10a_2 + \sum_{n=1}^{\infty} 5(n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + 10a_0 + \sum_{n=1}^{\infty} 10 a_n x^n = 0$$

$$10a_2 + 10a_0 + \sum_{n=1}^{\infty} \left[5(n+2)(n+1) a_{n+2} - 2(n-5) a_n\right] x^n = 0$$

$$10a_2 + 10a_0 \quad \to \quad \boxed{a_2 = -a_0}$$

$$5(n+2)(n+1) a_{n+2} - 2(n-5) a_n = 0$$

$$\boxed{a_{n+2} = \frac{2(n-5)}{5(n+1)(n+2)} a_n}$$

a_0 | a_1

$a_2 = -a_0$

$n = 1 \ \to \ a_3 = -\frac{8}{30} a_1 = -\frac{4}{15} a_1$

$n=2 \rightarrow a_4 = -\frac{6}{60}a_2 = \frac{1}{10}a_0$

$n=4 \rightarrow a_6 = -\frac{2}{5\cdot5\cdot6}a_4 = -\frac{1}{750}a_0$

$n=6 \rightarrow a_8 = \frac{2}{5\cdot7\cdot8}a_6 = -\frac{2}{8!\cdot5^2}a_0$

$n=8 \rightarrow a_{10} = \frac{2\cdot3}{5\cdot9\cdot10}a_8 = -\frac{2^2\cdot3}{10!\cdot5^3}a_0$

$n=10 \rightarrow a_{12} = \frac{2\cdot5}{5\cdot11\cdot12}a_8 = -\frac{2^3\cdot3\cdot5}{12!\cdot5^4}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$n=3 \rightarrow a_5 = -\frac{4}{100}a_3 = \frac{4}{375}a_1$

$n=5 \rightarrow a_7 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$n\geq 4 \quad a_{2n} = -15\cdot\frac{2^n(2n-7)!!}{5^n\ (2n)!}a_0$$

$$y(x) = a_0\left(1-x^2+\frac{1}{10}x^4-\frac{1}{750}x^6-\frac{1}{105,000}x^8-\cdots\right)+a_1\left(x-\frac{4}{15}x^3+\frac{4}{375}x^5\right)$$

Exercise

Find a power series solution. $(x-1)y''+y'=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$(x-1)y''+y'=0$

$$(x-1)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n=0$$

$$x\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+1}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n}+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n}=0$$

$$\sum_{n=1}^{\infty}n(n+1)a_{n+1}x^{n}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n}+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n}=0$$

$$\sum_{n=1}^{\infty}n(n+1)a_{n+1}x^{n}-2a_2-\sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^{n}+a_1+\sum_{n=1}^{\infty}(n+1)a_{n+1}x^{n}=0$$

$$a_1-2a_2+\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1}-(n+2)(n+1)a_{n+2}+(n+1)a_{n+1}\right]x^{n}=0$$

$$a_1-2a_2+\sum_{n=1}^{\infty}\left[-(n+2)(n+1)a_{n+2}+(n+1)^2a_{n+1}\right]x^{n}=0$$

$$a_1-2a_2=0 \quad \rightarrow \quad \boxed{a_2=\frac{1}{2}a_1}$$

$$-(n+2)(n+1)a_{n+2}+(n+1)^2a_{n+1}$$

$$\rightarrow \quad \boxed{a_{n+2}=\frac{n+1}{n+2}a_{n+1}} \quad n=1,\ 2,\ \ldots$$

$$a_3=\frac{2}{3}a_2=\frac{1}{3}a_1$$

$$a_4=\frac{3}{4}a_3=\frac{1}{4}a_1$$

$$a_5=\frac{4}{5}a_4=\frac{1}{5}a_1$$

$$\vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x)=a_0 \\ y_2(x)=\left(x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $(x+2)y'' + xy' - y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x+2)y'' + xy' - y = 0$$

$$(x+2)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+1} + \sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=1}^{\infty}n(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty}na_n x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=1}^{\infty}n(n+1)a_{n+1}x^n + 4a_2 + \sum_{n=1}^{\infty}2(n+2)(n+1)a_{n+2}x^n + 4a_2 + \sum_{n=1}^{\infty}na_n x^n - a_0 - \sum_{n=1}^{\infty}a_n x^n = 0$$

$$4a_2 - a_0 + \sum_{n=1}^{\infty}\left(n(n+1)a_{n+1} + 2(n+2)(n+1)a_{n+2} + (n-1)a_n\right)x^n = 0$$

$$4a_2 - a_0 = 0 \quad \to \quad \boxed{a_2 = \frac{1}{4}a_0} \quad \boxed{a_1}$$

$$n(n+1)a_{n+1} + 2(n+2)(n+1)a_{n+2} + (n-1)a_n = 0$$

$$\boxed{a_{n+2} = -\frac{n-1}{2(n+2)(n+1)}a_n - \frac{n}{2(n+2)}a_{n+1}} \quad n = 1,\ 2,\ \ldots$$

$$a_3 = -\frac{1}{6}a_2 = -\frac{1}{24}a_0$$

$$a_4 = -\frac{1}{24}a_2 - \frac{1}{4}a_3 = -\frac{1}{96}a_0 + \frac{1}{96}a_0 = 0$$

$$a_5 = -\frac{1}{20}a_3 - \frac{3}{10}a_4 = \frac{1}{480}a_0$$

$$a_6 = -\frac{3}{60}a_4 - \frac{1}{3}a_5 = -\frac{1}{1,440}a_0$$

$$\vdots \quad \vdots \qquad\qquad \vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x) = a_1 \\ y_2(x) = \left(1 + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{480}x^5 - \cdots\right)a_0 \end{cases}}$$

Exercise

Find a power series solution. $y'' - (x+1)y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - (x+1)y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_n\right] x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 - a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_n\right] x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 - a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_n - a_{n-1}\right] x^n = 0$$

$$2a_2 - a_0 = 0 \quad \rightarrow \quad \boxed{a_2 = \frac{1}{2}a_0}$$

$$(n+2)(n+1)a_{n+2} - a_n - a_{n-1} = 0$$

$$\boxed{a_{n+2} = \frac{a_n + a_{n-1}}{(n+1)(n+2)}}$$

$$a_0 \qquad a_1$$

$$a_2 = \frac{1}{2}a_0$$

$$n=1 \rightarrow a_3 = \frac{1}{6}(a_1 + a_0)$$

$$n=2 \rightarrow a_4 = \frac{1}{12}(a_2 + a_1) = \frac{1}{12}\left(\frac{1}{2}a_0 + a_1\right)$$

$$n=3 \rightarrow a_5 = \frac{1}{20}(a_3 + a_2) = \frac{1}{20}\left(\frac{2}{3}a_0 + \frac{1}{6}a_1\right)$$

$$n=4 \rightarrow a_6 = \frac{1}{30}(a_4 + a_3) = \frac{1}{30}\left(\frac{1}{2}a_0 + a_1 + \frac{1}{6}a_0 + \frac{1}{6}a_1\right) = \frac{1}{30}\left(\frac{2}{3}a_0 + \frac{7}{6}a_1\right)$$

$a_0 \neq 0 \quad a_1 = 0$	$a_0 = 0 \quad a_1 \neq 0$
$a_2 = \frac{1}{2}a_0$	$a_2 = 0$
$a_3 = \frac{1}{6}a_0$	$a_3 = \frac{1}{6}a_1$
$a_4 = \frac{1}{24}a_0$	$a_4 = \frac{1}{12}a_1$
$a_5 = \frac{1}{30}a_0$	$a_5 = \frac{1}{120}a_1$
$a_6 = \frac{1}{45}a_0$	$a_6 = \frac{7}{180}a_1$

$$\boxed{\begin{cases} y_1(x) = \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^5 + \frac{1}{45}x^6 + \cdots\right)a_0 \\ y_2(x) = \left(x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 + \frac{7}{180}x^6 + \cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $y'' - (x+1)y' - y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty} na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''-(x+1)y'-y=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-(x+1)\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}-\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=1}^{\infty}na_n x^n-\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-\sum_{n=0}^{\infty}a_n x^n=0$$

$$2a_2+\sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=1}^{\infty}na_n x^n-a_1-\sum_{n=1}^{\infty}(n+1)a_{n+1}x^n-a_0-\sum_{n=1}^{\infty}a_n x^n=0$$

$$2a_2-a_1-a_0+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}-na_n-(n+1)a_{n+1}-a_n\right]x^n=0$$

$$2a_2-a_1-a_0=0 \quad\rightarrow\quad a_2=\frac{1}{2}a_0+\frac{1}{2}a_1$$

$$(n+2)(n+1)a_{n+2}-(n+1)a_{n+1}-(n+1)a_n=0$$

$$\boxed{a_{n+2}=\frac{1}{n+2}a_{n+1}+\frac{1}{n+2}a_n}$$

$$a_3=\frac{1}{3}a_2+\frac{1}{3}a_1=\frac{1}{6}a_0+\frac{1}{2}a_1$$

$$a_4=\frac{1}{4}a_3+\frac{1}{4}a_2=\frac{1}{24}a_0+\frac{1}{8}a_1+\frac{1}{8}a_0+\frac{1}{8}a_1=\frac{1}{6}a_0+\frac{1}{4}a_1$$

$$a_5=\frac{1}{5}a_4+\frac{1}{5}a_3=\frac{1}{30}a_0+\frac{1}{20}a_1+\frac{1}{30}a_0+\frac{1}{10}a_1=\frac{1}{15}a_0+\frac{3}{20}a_1$$

$$\vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x)=\left(1+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{6}x^4+\cdots\right)a_0 \\ y_2(x)=\left(x+\frac{1}{2}x^2+\frac{1}{2}x^3+\frac{1}{4}x^4+\cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $\left(x^2+1\right)y''-6y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+1\right)y''-6y=0$$

$$\left(x^2+1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-6\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2}+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-6\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_nx^n+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-6\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_nx^n+2a_2+6a_3x+\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n-6a_0-6a_1x-6\sum_{n=2}^{\infty}a_nx^n=0$$

$$2a_2-6a_0+\left(6a_3-6a_1\right)x+\sum_{n=2}^{\infty}\left[\left(n^2-n-6\right)a_n+(n+2)(n+1)a_{n+2}\right]x^n=0$$

$$2a_2-6a_0+\left(6a_3-6a_1\right)x=0 \quad\rightarrow\quad \begin{cases}a_2=3a_0\\ a_3=a_1\end{cases}$$

$$(n+2)(n-3)a_n+(n+2)(n+1)a_{n+2}=0$$

$$\Rightarrow\quad a_{n+2}=-\frac{n-3}{n+1}a_n \qquad n=2,3,\ldots$$

$$a_4=\frac{1}{3}a_2=a_0 \qquad\qquad a_5=0$$

$$a_6 = -\frac{1}{5}a_4 = -\frac{1}{5}a_0 \qquad\qquad a_7 = -\frac{1}{3}a_5 = 0$$

$$a_8 = -\frac{3}{7}a_6 = \frac{3}{35}a_0$$

$$\vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x) = \left(1 + 3x^2 + x^4 - \frac{1}{5}x^6 + \cdots\right)a_0 \\ y_2(x) = \left(x + x^3\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $\left(x^2+2\right)y'' + 3xy' - y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(x^2+2\right)y'' + 3xy' - y = 0$$

$$\left(x^2+2\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + 3x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2} + \sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}3(n+1)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n + \sum_{n=0}^{\infty}\left[2(n+2)(n+1)a_{n+2} - a_n\right]x^n + \sum_{n=1}^{\infty}3na_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + 4a_2 - a_0 + \left(12a_3 - a_1\right)x + \sum_{n=2}^{\infty}\left[2(n+2)(n+1)a_{n+2} - a_n\right]x^n$$

$$+3a_1 x + \sum_{n=2}^{\infty} 3na_n x^n = 0$$

$$4a_2 - a_0 + \left(12a_3 + 2a_1\right)x + \sum_{n=2}^{\infty}\left[n(n-1)a_n + 2(n+2)(n+1)a_{n+2} + (3n-1)a_n\right]x^n = 0$$

$$4a_2 - a_0 + \left(12a_3 + 2a_1\right)x$$

$$\boxed{a_2 = \frac{1}{4}a_0} \qquad \boxed{a_3 = -\frac{1}{6}a_1}$$

$$2(n+2)(n+1)a_{n+2} + \left(n^2 + 2n - 1\right)a_n = 0$$

$$\rightarrow \quad a_{n+2} = -\frac{n^2 + 2n - 1}{2(n+2)(n+1)}a_n \qquad n = 2, 3, \ldots$$

$$a_4 = -\frac{7}{24}a_2 = -\frac{7}{96}a_0 \qquad a_5 = -\frac{7}{20}a_3 = \frac{7}{120}a_1$$

$$a_6 = -\frac{23}{60}a_4 = \frac{161}{5760}a_0 \qquad a_7 = -\frac{17}{42}a_5 = -\frac{17}{720}a_1$$

$$\vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x) = \left(1 + \frac{1}{4}x^2 - \frac{7}{96}x^4 + \frac{161}{5760}x^6 - \cdots\right)a_0 \\ y_2(x) = \left(1 - \frac{1}{6}x^3 + \frac{7}{120}x^5 - \frac{17}{720}x^7 + \cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $\left(x^2 - 1\right)y'' + xy' - y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2-1\right)y''+xy'-y=0$$

$$\left(x^2-1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2} - \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n - \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_n\right]x^n + \sum_{n=1}^{\infty}na_n x^n = 0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n - \left(2a_2+a_0\right) - \left(6a_3+a_1\right)x - \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_n\right]x^n$$

$$+a_1 x + \sum_{n=2}^{\infty}na_n x^n = 0$$

$$-2a_2 - a_0 - 6a_3 x + \sum_{n=2}^{\infty}\left[n(n-1)a_n - (n+2)(n+1)a_{n+2} + (n-1)a_n\right]x^n = 0$$

$$-2a_2 - a_0 - 6a_3 x = 0$$

$$\boxed{a_2 = -\tfrac{1}{2}a_0} \qquad \boxed{a_3 = 0}$$

$$-(n+2)(n+1)a_{n+2} + (n+1)(n-1)a_n = 0$$

$$\to \quad a_{n+2} = \frac{n-1}{n+2}a_n \qquad n=2,3,\ldots$$

$$a_4 = \tfrac{1}{4}a_2 = -\tfrac{1}{8}a_0 \qquad\qquad a_5 = -\tfrac{2}{5}a_3 = 0$$

$$a_6 = \tfrac{1}{2}a_4 = -\tfrac{1}{16}a_0 \qquad\qquad a_7 = \tfrac{4}{7}a_5 = 0$$

$$\vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x) = \left(1-\tfrac{1}{2}x^2-\tfrac{1}{8}x^4-\tfrac{1}{16}x^6-\cdots\right)a_0 \\ y_2(x) = a_1 x \end{cases}}$$

$$\begin{cases} y_1(x) = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \cdots \\ y_2(x) = x \end{cases}$$

Exercise

Find a power series solution. $\left(x^2+1\right)y'' + xy' - y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(x^2+1\right)y'' + xy' - y = 0$$

$$\left(x^2+1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$x^2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2} + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty}na_n x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n + \sum_{n=1}^{\infty}na_n x^n + \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - a_n\right]x^n = 0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n + a_1 x + \sum_{n=2}^{\infty}na_n x^n + \left(2a_2 - a_0\right) + \left(6a_3 - a_1\right)x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} - a_n\right]x^n = 0$$

$$2a_2 - a_0 + 6a_3 x + \sum_{n=2}^{\infty}\left[\left(n^2 - n + n - 1\right)a_n + (n+2)(n+1)a_{n+2}\right]x^n = 0$$

$$\begin{cases} 2a_2 - a_0 = 0 & \rightarrow \boxed{a_2 = \frac{1}{2}a_0} \\ 6a_3 x = 0 & \rightarrow \boxed{a_3 = 0} \end{cases}$$

$$(n-1)(n+1)a_n + (n+2)(n+1)a_{n+2} = 0$$

$$\boxed{a_{n+2} = -\frac{n-1}{n+2}a_n}$$

a_0	a_1
$n=0 \rightarrow a_2 = \frac{1}{2}a_0$	$n=1 \rightarrow a_3 = 0$
$n=2 \rightarrow a_4 = -\frac{1}{4}a_2 = -\frac{1}{2^3}a_0$	$n=3 \rightarrow a_5 = -\frac{2}{5}a_3 = 0$
$n=4 \rightarrow a_6 = -\frac{3}{6}a_4 = \frac{1\cdot 3}{2^3\cdot 3!}a_0$	$n=5 \rightarrow a_7 = 0$
$n=6 \rightarrow a_8 = -\frac{5}{8}a_6 = -\frac{1\cdot 3\cdot 5}{2^4\cdot 4!}a_0$	$\vdots \quad \vdots \quad \vdots \quad \vdots$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	

$$a_{2n} = (-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n\cdot n!}a_0 \quad (n \ge 2)$$

$$\boxed{y(x) = a_0\left(1 + \frac{1}{2}x^2 - \frac{1}{2^2\cdot 2!}x^4 + \frac{1\cdot 3}{2^3\cdot 3!}x^6 - \frac{1\cdot 3\cdot 5}{2^4\cdot 4!}x^8 + \cdots\right) + a_1 x}$$

$$\boxed{= a_0\left(1 + \frac{1}{2}x^2 + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{(2n-3)!!}{2^n\cdot n!}x^{2n}\right) + a_1 x}$$

Exercise

Find a power series solution. $\left(x^2+1\right)y'' - xy' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty} na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} =\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+1\right)y''-xy'+y=0$$

$$\left(x^2+1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$x^2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2}+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=1}^{\infty}na_n x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n-\sum_{n=1}^{\infty}na_n x^n+\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_n\right]x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n-a_1x-\sum_{n=2}^{\infty}na_n x^n+\left(2a_2+a_0\right)+\left(6a_3+a_1\right)x$$

$$+\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_n\right]x^n=0$$

$$2a_2+a_0+6a_3x+\sum_{n=2}^{\infty}\left[\left(n^2-2n+1\right)a_n+(n+2)(n+1)a_{n+2}\right]x^n=0$$

$$\begin{cases}2a_2+a_0=0 & \rightarrow \quad \boxed{a_2=-\frac{1}{2}a_0}\\ 6a_3x=0 & \rightarrow \quad \boxed{a_3=0}\end{cases}$$

$$(n-1)^2a_n+(n+1)(n+2)a_{n+2}=0$$

$$\boxed{a_{n+2} = -\frac{(n-1)^2}{(n+1)(n+2)} a_n}$$

a_0

$n = 0 \rightarrow a_2 = -\frac{1}{2} a_0$

$n = 2 \rightarrow a_4 = -\frac{1}{12} a_2 = \frac{1}{4!} a_0$

$n = 4 \rightarrow a_6 = -\frac{3^2}{5 \cdot 6} a_4 = -\frac{3^2}{6!} a_0$

$n = 6 \rightarrow a_8 = -\frac{5^2}{7 \cdot 8} a_6 = \frac{1 \cdot 3^2 \cdot 5^2}{8!} a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

a_1

$n = 1 \rightarrow a_3 = 0$

$n = 3 \rightarrow a_5 = -\frac{2}{5} a_3 = 0$

$n = 5 \rightarrow a_7 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$a_{2n} = (-1)^{n-1} \frac{1 \cdot 3^2 \cdot 5^2 \cdots (2n-3)^2}{(2n)!} a_0 \quad (n \ge 3)$$

$$\boxed{y(x) = a_0 \left(1 - \frac{1}{2} x^2 + \frac{1}{24} x^4 - \frac{9}{6!} x^6 - \frac{1 \cdot 3^2 \cdot 5^2}{8!} x^8 - \cdots \right) + a_1 x}$$

$$\boxed{= a_0 \left(1 - \frac{1}{2} x^2 + \frac{1}{24} x^4 + \sum_{n=3}^{\infty} (-1)^n \frac{(2n-3)^2 !!}{(2n)!} x^{2n} \right) + a_1 x}$$

Exercise

Find a power series solution. $\left(1 - x^2\right) y'' - 6xy' - 4y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(1 - x^2\right) y'' - 6xy' - 4y = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x^2\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2} - 6x\sum_{n=1}^{\infty}na_n x^{n-1} - 4\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty}n(n-1)a_n x^n - \sum_{n=1}^{\infty}6na_n x^n - \sum_{n=0}^{\infty}4a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}n(n-1)a_n x^n - \sum_{n=0}^{\infty}6na_n x^n - \sum_{n=0}^{\infty}4a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - \big(n(n-1)+6n+4\big)a_n\right]x^n = 0$$

$$(n+2)(n+1)a_{n+2} - \left(n^2+5n+4\right)a_n = 0$$

$$(n+2)(n+1)a_{n+2} = (n+4)(n+1)a_n$$

$$\boxed{a_{n+2} = \frac{n+4}{n+2}a_n}$$

a_0	a_1
$n=2 \rightarrow a_2 = 2a_0$	$n=3 \rightarrow a_3 = \frac{5}{3}a_1$
$n=4 \rightarrow a_4 = \frac{6}{4}a_2 = 3a_0$	$n=5 \rightarrow a_5 = \frac{7}{5}a_3 = \frac{7}{3}a_1$
$n=6 \rightarrow a_6 = \frac{8}{6}a_3 = 4a_0$	$n=7 \rightarrow a_7 = \frac{11}{7}a_5 = \frac{11}{3}a_1$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$
$a_{2k} = (k+1)a_0$	$a_{2k+1} = \frac{2k+3}{3}a_1$

$$y(x) = a_0\left(1+2x^2+3x^4+4x^6+\cdots\right) + a_1\left(x+\frac{5}{3}x^3+\frac{7}{3}x^5+\frac{11}{3}x^7+\cdots\right)$$

$$\boxed{= \frac{a_0}{\left(1-x^2\right)^2} + \frac{3x-x^3}{3\left(1-x^2\right)^2}a_1}$$

Exercise

Find a power series solution. $y'' + (x-1)^2 y' - 4(x-1)y = 0$

Solution

Let $z = x-1 \rightarrow dz = dx$

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''+z^2y'-4zy=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n+z^2\sum_{n=1}^{\infty}na_n z^{n-1}-4z\sum_{n=0}^{\infty}a_n z^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n+\sum_{n=1}^{\infty}na_n z^{n+1}-\sum_{n=0}^{\infty}4a_n z^{n+1}=0$$

$$2a_2+\sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}z^n+\sum_{n=1}^{\infty}(n-4)a_n z^{n+1}=0$$

$$2a_2+\sum_{n=0}^{\infty}(n+2)(n+3)a_{n+3}z^{n+1}+\sum_{n=1}^{\infty}(n-4)a_n z^{n+1}=0$$

$$2a_2+\sum_{n=0}^{\infty}\left[(n+2)(n+3)a_{n+3}+(n-4)a_n\right]z^{n+1}=0$$

$$\underline{a_2=0}\,\Big|$$

$$(n+2)(n+3)a_{n+3}+(n-4)a_n=0$$

$$\underline{a_{n+3}=-\frac{n-4}{(n+2)(n+3)}a_n}\,\Big|$$

a_0	a_1	$a_2=0$
$n=0\to\ a_3=\frac{4}{2\cdot 3}a_0$	$n=1\to\ a_4=\frac{3}{3\cdot 4}a_1$	$n=2\ \to\ a_5=\frac{2}{20}a_2=0$
$n=3\to\ a_6=\frac{1}{5\cdot 6}a_3=\frac{4}{2\cdot 3\cdot 5\cdot 6}a_0$	$n=4\to\ a_7=0$	$n=5\ \to\ a_8=0$
$n=6\to\ a_9=-\frac{2}{8\cdot 9}a_6=-\frac{8}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}a_0$	$n=7\to\ a_{10}=0$	$n=8\ \to\ a_5=0$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad\qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y(x) = a_0\left(1 + \frac{4}{2\cdot 3}z^3 + \frac{4}{2\cdot 3\cdot 5\cdot 6}z^6 - \frac{8}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 8}z^9 + \cdots\right) + a_1\left(z + \frac{1}{4}z^4\right)$$

$$= a_0\left(1 + \frac{2}{3}(x-1)^3 + \frac{1}{45}(x-1)^6 - \frac{1}{1,620}(x-1)^9 + \cdots\right) + a_1\left(x - 1 + \frac{1}{4}(x-1)^4\right)$$

Exercise

Find a power series solution. $\left(2-x^2\right)y'' - xy' + 16y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(2-x^2\right)y'' - xy' + 16y = 0$$

$$2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x^2\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty}na_n x^{n-1} + 16\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty}n(n-1)a_n x^n - \sum_{n=1}^{\infty}na_n x^n + \sum_{n=0}^{\infty}16a_n x^n = 0$$

$$\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}n(n-1)a_n x^n - \sum_{n=0}^{\infty}na_n x^n + \sum_{n=0}^{\infty}16a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[2(n+2)(n+1)a_{n+2} - \left(n^2 - n + n - 16\right)a_n\right]x^n = 0$$

$$2(n+1)(n+2)a_{n+2} - \left(n^2 - 16\right)a_n = 0$$

$$2(n+1)(n+2)a_{n+2} = (n+4)(n-4)a_n$$

$$\boxed{a_{n+2} = \frac{(n+4)(n-4)}{2(n+1)(n+2)} a_n}$$

a_0

$n = 0 \rightarrow a_2 = -4a_0$

$n = 2 \rightarrow a_4 = -\frac{1}{2}a_2 = 2a_0$

$n = 4 \rightarrow a_6 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

a_1

$n = 1 \rightarrow a_3 = -\frac{5}{4}a_1$

$n = 3 \rightarrow a_5 = -\frac{7}{40}a_3 = \frac{7}{32}a_1$

$n = 5 \rightarrow a_7 = \frac{9}{70}a_5 = \frac{9}{320}a_1$

$n = 7 \rightarrow a_9 = \frac{33}{144}a_7 = \frac{33}{5,120}a_1$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$a_{2n+1} = \frac{(2n-5)!!\ (2n+3)!!}{2^n\ (2n+1)!} a_1$$

$$\boxed{y(x) = a_0\left(1 - 4x^2 + 2x^4\right) + a_1\left(x - \frac{5}{4}x^3 + \frac{7}{32}x^5 + \frac{9}{320}x^7 + \cdots\right)}$$

$$\boxed{= a_0\left(1 - 4x^2 + 2x^4\right) + a_1 \sum_{n=0}^{\infty} \frac{(2n-5)!!\ (2n+3)!!}{2^n\ (2n+1)!} x^{2n+1}}$$

Exercise

Find a power series solution. $\left(x^2+1\right)y'' - y' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(x^2+1\right)y'' - y' + y = 0$$

$$x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2} - (n+1)a_{n+1} + \left(n^2 - n + 1\right)a_n\right]x^n = 0$$

$$(n+1)(n+2)a_{n+2} - (n+1)a_{n+1} + \left(n^2 - n + 1\right)a_n = 0$$

$$a_{n+2} = \frac{(n+1)a_{n+1} - \left(n^2 - n + 1\right)a_n}{(n+1)(n+2)}$$

$$n = 0 \rightarrow\ a_2 = \tfrac{1}{2}\left(a_1 - a_0\right)$$

$$n = 1 \rightarrow\ a_3 = \tfrac{1}{6}\left(2a_2 - a_1\right) = \frac{1}{6}\left(a_1 - a_0 - a_1\right) = -\tfrac{1}{6}a_0$$

$$n = 2 \rightarrow\ a_4 = \tfrac{1}{12}\left(3a_3 - 3a_2\right) = \frac{1}{4}\left(-\tfrac{1}{6}a_0 - \tfrac{1}{2}a_1 + \tfrac{1}{2}a_0\right) = \tfrac{1}{12}a_0 - \tfrac{1}{8}a_1$$

$$n = 3 \rightarrow\ a_5 = \tfrac{1}{20}\left(4a_4 - 7a_3\right) = \frac{1}{20}\left(\tfrac{1}{3}a_0 - \tfrac{1}{2}a_1 + \tfrac{7}{6}a_0\right) = \tfrac{3}{40}a_0 - \tfrac{1}{40}a_1$$

$$n = 4 \rightarrow\ a_6 = \tfrac{1}{30}\left(5a_5 - 13a_4\right) = \frac{1}{30}\left(\tfrac{3}{8}a_0 - \tfrac{1}{8}a_1 - \tfrac{13}{12}a_0 + \tfrac{13}{8}a_1\right) = -\tfrac{17}{720}a_0 + \tfrac{1}{20}a_1$$

$$\begin{aligned} y(x) = a_0 + a_1 x + \left(\tfrac{1}{2}a_0 - \tfrac{1}{2}a_1\right)x^2 - \tfrac{1}{6}a_0 x^3 + \left(\tfrac{1}{12}a_0 - \tfrac{1}{8}a_1\right)x^4 + \left(\tfrac{3}{40}a_0 - \tfrac{1}{40}a_1\right)x^5 \\ + \left(-\tfrac{17}{720}a_0 + \tfrac{1}{20}a_1\right)x^6 + \cdots \end{aligned}$$

$$\begin{aligned} y(x) = a_0\left(1 + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \tfrac{1}{12}x^4 + \tfrac{3}{40}x^5 - \tfrac{17}{720}x^6 + \cdots\right) \\ + a_1\left(x - \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 - \tfrac{1}{40}x^5 + \tfrac{1}{20}x^6 + \cdots\right) \end{aligned}$$

Exercise

Find a power series solution. $\left(x^2+1\right)y''+6xy'+4y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+1\right)y''+6xy'+4y=0$$

$$x^2\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+6x\sum_{n=1}^{\infty}na_n x^{n-1}+4\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}6na_n x^n+\sum_{n=0}^{\infty}4a_n x^n=0$$

$$\sum_{n=0}^{\infty}n(n-1)a_n x^n+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}6na_n x^n+\sum_{n=0}^{\infty}4a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+\left(n^2+5n+4\right)a_n\right]x^n=0$$

$$(n+1)(n+2)a_{n+2}+(n+1)(n+4)a_n=0$$

$$\boxed{a_{n+2}=-\frac{n+4}{n+2}a_n}$$

a_0	a_1
$n=0\to\ a_2=-2a_0$	$n=1\ \to\ a_3=-\frac{5}{3}a_1$
$n=2\ \to\ a_4=-\frac{3}{2}a_2=3a_0$	$n=3\ \to\ a_5=-\frac{7}{5}a_3=\frac{7}{3}a_1$
$n=4\ \to\ a_6=-\frac{8}{6}a_4=-4a_0$	$n=5\ \to\ a_7=-\frac{9}{7}a_5=-\frac{9}{3}a_1$
$\vdots\quad\vdots\quad\vdots\quad\vdots$	$\vdots\quad\vdots\quad\vdots\quad\vdots$

$$a_{2n} = (-1)^n (n+1) a_0 \qquad\qquad a_{2n+1} = (-1)^n (2n+3) a_1$$

$$\boxed{y(x) = a_0 \sum_{n=0}^{\infty} (-1)^n (n+1) x^{2n} + \tfrac{1}{3} a_1 \sum_{n=0}^{\infty} (-1)^n (2n+3) x^{2n+1}}$$

$$\boxed{y(x) = a_0 \left(1 - 2x^2 + 3x^4 - 4x^6 + \cdots\right) + \tfrac{1}{3} a_1 \left(x - \tfrac{5}{3}x^3 + \tfrac{7}{3}x^5 - \tfrac{9}{3}x^7 + \cdots\right)}$$

Exercise

Find a power series solution. $\left(x^2 - 1\right) y'' - 6xy' + 12y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(x^2 - 1\right) y'' - 6xy' + 12y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 6x \sum_{n=1}^{\infty} n a_n x^{n-1} + 12 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} 6n a_n x^n + \sum_{n=0}^{\infty} 12 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} 6n a_n x^n + \sum_{n=0}^{\infty} 12 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[-(n+2)(n+1) a_{n+2} + \left(n^2 - 7n + 12\right) a_n \right] x^n = 0$$

$$-(n+1)(n+2) a_{n+2} + (n-3)(n-4) a_n = 0$$

$$\boxed{a_{n+2} = \frac{(n-3)(n-4)}{(n+1)(n+2)} a_n}$$

a_0	a_1
$n=0 \to a_2 = 6a_0$	$n=1 \to a_3 = a_1$
$n=2 \to a_4 = \frac{1}{6}a_2 = a_0$	$n=3 \to a_5 = 0$
$n=4 \to a_6 = 0$	$\vdots \quad \vdots \quad \vdots \quad \vdots$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	

$$\boxed{y(x) = a_0\left(1+6x^2+x^4\right) + a_1\left(x+x^3\right)}$$

Exercise

Find a power series solution. $\left(x^2-1\right)y'' + 8xy' + 12y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2-1\right)y'' + 8xy' + 12y = 0$$

$$x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + 8x\sum_{n=1}^{\infty} na_n x^{n-1} + 12\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} 8na_n x^n + \sum_{n=0}^{\infty} 12a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} 8na_n x^n + \sum_{n=0}^{\infty} 12a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[-(n+2)(n+1)a_{n+2}+\left(n^2+7n+12\right)a_n\right]x^n=0$$

$$-(n+1)(n+2)a_{n+2}+(n+3)(n+4)a_n=0$$

$$\boxed{a_{n+2}=\frac{(n+3)(n+4)}{(n+1)(n+2)}a_n}$$

a_0	a_1
$n=0 \to a_2=6a_0$	$n=1 \to a_3=\frac{10}{3}a_1$
$n=2 \to a_4=\frac{5}{2}a_2=15a_0$	$n=3 \to a_5=\frac{42}{20}a_3=\frac{21}{3}a_1$
$n=4 \to a_6=\frac{5}{2}a_4=28a_0$	$n=5 \to a_7=\frac{12}{7}a_5=\frac{36}{3}a_1$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$
$a_{2n}=(n+1)(2n+1)a_0$	$a_{2n+1}=\frac{1}{3}(n+1)(2n+3)a_0$

$$\boxed{y(x)=a_0\left(1+6x^2+15x^4+28x^6+\cdots\right)+a_1\left(x+\frac{10}{3}x^3+7x^5+12x^7+\cdots\right)}$$

$$\boxed{=a_0\sum_{n=0}^{\infty}(n+1)(2n+1)x^{2n}+\frac{1}{3}a_1\sum_{n=0}^{\infty}(n+1)(2n+3)x^{2n+1}}$$

Exercise

Find a power series solution. $\left(x^2-1\right)y''+4xy'+2y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$\left(x^2-1\right)y''+4xy'+2y=0$

$$x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + 4x\sum_{n=1}^{\infty} na_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} 4na_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} 4na_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[-(n+2)(n+1)a_{n+2} + \left(n^2+3n+2\right)a_n\right]x^n = 0$$

$$-(n+1)(n+2)a_{n+2} + (n+1)(n+2)a_n = 0$$

$$\boxed{a_{n+2} = a_n}$$

a_0	a_1
$n=0 \to a_2 = a_0$	$n=1 \to a_3 = a_1$
$n=2 \to a_4 = a_2 = a_0$	$n=3 \to a_5 = a_1$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x) = a_0\left(1+x^2+x^4+x^6+\cdots\right) + a_1\left(x+x^3+x^5+x^7+\cdots\right)}$$

$$= a_0\sum_{n=0}^{\infty} x^{2n} + a_1\sum_{n=0}^{\infty} x^{2n+1}$$

$$\boxed{= \frac{a_0 + a_1 x}{1-x^2}}$$

Exercise

Find a power series solution. $\left(x^2+1\right)y'' - 4xy' + 6y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty} na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+1\right)y''-4xy'+6y=0$$

$$x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-4x\sum_{n=1}^{\infty} na_n x^{n-1}+6\sum_{n=0}^{\infty} a_n x^n=0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n+\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n-\sum_{n=1}^{\infty} 4na_n x^n+\sum_{n=0}^{\infty} 6a_n x^n=0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n+\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n-\sum_{n=0}^{\infty} 4na_n x^n+\sum_{n=0}^{\infty} 6a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}+\left(n^2-5n+6\right)a_n\right]x^n=0$$

$$(n+1)(n+2)a_{n+2}+(n-2)(n-3)a_n=0$$

$$\boxed{a_{n+2}=-\frac{(n-2)(n-3)}{(n+1)(n+2)}a_n}$$

a_0	a_1
$n=0\to\ a_2=-3a_0$	$n=1\ \to\ a_3=-\frac{1}{3}a_1$
$n=2\ \to\ a_4=0$	$n=3\ \to\ a_5=0$
$\vdots\quad\vdots\quad\vdots\quad\vdots$	$\vdots\quad\vdots\quad\vdots\quad\vdots$

$$\boxed{y(x)=a_0\left(1-3x^2\right)+a_1\left(x-\tfrac{1}{3}x^3\right)}$$

Exercise

Find a power series solution. $\left(x^2+2\right)y''+4xy'+2y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+2\right)y''+4xy'+2y=0$$

$$x^2\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}+2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+4x\sum_{n=1}^{\infty}na_n x^{n-1}+2\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}4na_n x^n+\sum_{n=0}^{\infty}2a_n x^n=0$$

$$\sum_{n=0}^{\infty}n(n-1)a_n x^n+\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}4na_n x^n+\sum_{n=0}^{\infty}2a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[2(n+2)(n+1)a_{n+2}+\left(n^2+3n+2\right)a_n\right]x^n=0$$

$$2(n+1)(n+2)a_{n+2}+(n+1)(n+2)a_n=0$$

$$\boxed{a_{n+2}=-\frac{1}{2}a_n}$$

a_0	a_1
$n=0\rightarrow\ a_2=-\frac{1}{2}a_0$	$n=1\ \rightarrow\ a_3=-\frac{1}{2}a_1$
$n=2\ \rightarrow\ a_4=-\frac{1}{2}a_2=\frac{1}{2^2}a_0$	$n=3\ \rightarrow\ a_5=-\frac{1}{2}a_3=\frac{1}{2^2}a_1$
$n=4\ \rightarrow\ a_6=-\frac{1}{2}a_4=-\frac{1}{2^3}a_0$	$n=5\ \rightarrow\ a_7=-\frac{1}{2}a_5=-\frac{1}{2^3}a_1$
$\vdots\quad\vdots\quad\vdots\quad\vdots$	$\vdots\quad\vdots\quad\vdots\quad\vdots$
$a_{2n}=(-1)^n\frac{1}{2^n}a_0$	$a_{2n+1}=(-1)^n\frac{1}{2^n}a_1$

$$\boxed{y(x)=a_0\left(1-\frac{1}{2}x^2+\frac{1}{4}x^4-\frac{1}{8}x^6+\cdots\right)+a_1\left(x-\frac{1}{2}x^3+\frac{1}{4}x^5-\frac{1}{8}x^7+\cdots\right)}$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^{2n+1}$$

Exercise

Find a power series solution. $\left(x^2 - 3\right) y'' + 2xy' = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(x^2 - 3\right) y'' + 2xy' = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 3\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 2x \sum_{n=1}^{\infty} n a_n x^{n-1} = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} 3(n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} 2n a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} 3(n+1)(n+2) a_{n+2} x^n + \sum_{n=0}^{\infty} 2n a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[-3(n+1)(n+2) a_{n+2} + \left(n^2 + n\right) a_n \right] x^n = 0$$

$$-3(n+1)(n+2) a_{n+2} + n(n+1) a_n = 0$$

$$a_{n+2} = \frac{1}{3} \frac{n}{n+2} a_n$$

a_0

$n = 0 \rightarrow a_2 = 0$

a_1

$n = 1 \rightarrow a_3 = \frac{1}{3^2} a_1$

$n=2 \;\rightarrow\; a_4 = \frac{2}{12}a_2 = 0$

$n=4 \;\rightarrow\; a_6 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$n=3 \;\rightarrow\; a_5 = \frac{1}{3}\frac{3}{5}a_3 = \frac{1}{3^2 \cdot 5}a_1$

$n=5 \;\rightarrow\; a_7 = \frac{1}{3}\frac{5}{7}a_5 = \frac{1}{3^3 \cdot 7}a_1$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$a_{2n+1} = (-1)^n \frac{1}{(2n+1)3^n}a_1$$

$$y(x) = a_0 + a_1\left(x + \frac{1}{9}x^3 + \frac{1}{45}x^5 + \frac{1}{189}x^7 + \cdots\right)$$

$$= a_0 + a_1\sum_{n=0}^{\infty}\frac{1}{(2n+1)3^n}x^{2n+1}$$

Exercise

Find a power series solution. $\left(x^2+3\right)y'' - 7xy' + 16y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+3\right)y'' - 7xy' + 16y = 0$$

$$x^2\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2} + 3\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - 7x\sum_{n=1}^{\infty}na_n x^{n-1} + 16\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n + \sum_{n=0}^{\infty}3(n+1)(n+2)a_{n+2}x^n - \sum_{n=1}^{\infty}7na_n x^n + \sum_{n=0}^{\infty}16a_n x^n = 0$$

$$\sum_{n=0}^{\infty}n(n-1)a_n x^n + \sum_{n=0}^{\infty}3(n+1)(n+2)a_{n+2}x^n - \sum_{n=0}^{\infty}7na_n x^n + \sum_{n=0}^{\infty}16a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[3(n+1)(n+2)a_{n+2}+\left(n^2-8n+16\right)a_n\right]x^n=0$$

$$3(n+1)(n+2)a_{n+2}+(n-4)^2a_n=0$$

$$\boxed{a_{n+2}=-\frac{(n-4)^2}{3(n+1)(n+2)}a_n}$$

a_0	a_1
$n=0\to\ a_2=-\frac{16}{6}a_0=-\frac{8}{3}a_0$	$n=1\ \to\ a_3=-\frac{9}{18}a_1=-\frac{1}{2}a_1$
$n=2\ \to\ a_4=-\frac{1}{9}a_2=\frac{8}{27}a_0$	$n=3\ \to\ a_5=-\frac{1}{60}a_3=\frac{1}{120}a_1$
$n=4\ \to\ a_6=0$	$n=5\ \to\ a_7=-\frac{1}{126}a_5=-\frac{1}{560\cdot 3^3}a_1$
$\vdots\quad\vdots\quad\vdots\quad\vdots$	$\vdots\quad\vdots\quad\vdots\quad\vdots$

$$\boxed{y(x)=a_0\left(1-\frac{8}{3}x^2+\frac{8}{27}x^4\right)+a_1\left(x-\frac{1}{2}x^3+\frac{1}{120}x^5+\frac{1}{15,120}x^7+\cdots\right)}$$

Exercise

Find the series solution to the initial value problem $y''+4y=0$; $y(0)=0,\quad y'(0)=3$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y''+4y=0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+4\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+4a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}+4a_n=0$$

$$\boxed{a_{n+2}=-\frac{4}{(n+1)(n+2)}a_n}$$

Given: $y(0)=0=a_0,\quad y'(0)=3=a_1$

$a_0=0$

$n=0 \rightarrow a_2=-2a_0=0$

$n=2 \rightarrow a_4=-\frac{4}{12}a_2=0$

$n=4 \rightarrow a_6=0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$a_1=3$

$n=1 \rightarrow a_3=-\frac{4}{6}a_1=-\frac{2^2}{3!}a_1=-2$

$n=3 \rightarrow a_5=-\frac{4}{20}a_3=-\frac{2^4}{5!}a_1=\frac{2}{5}$

$n=5 \rightarrow a_7=-\frac{4}{42}a_5=-\frac{2^6}{7!}a_1=-\frac{4}{105}$

$$a_{2k+1}=\frac{(-1)^k 2^{2k}}{(2k+1)!}a_1$$

$$\begin{aligned}
y(x)&=3x-2x^3+\frac{2}{5}x^5-\frac{4}{105}x^7+\cdots\\
&=3\left(x-\frac{2^2}{3!}x^3+\frac{2^4}{5!}x^5-\frac{2^6}{7!}x^7+\cdots\right)\\
&=\frac{3}{2}\left((2x)-\frac{1}{3!}(2x)^3+\frac{1}{5!}(2x)^5-\frac{1}{7!}(2x)^7+\cdots\right)\\
&\boxed{=\frac{3}{2}\sin 2x}
\end{aligned}$$

Exercise

Find the series solution to the initial value problem $y''+x^2y=0$; $y(0)=1,\quad y'(0)=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y'' + x^2 y = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x^2\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty}a_{n-2} x^{n} = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty}a_{n-2} x^{n} = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty}\left[(n+1)(n+2)a_{n+2} + a_{n-2}\right]x^n = 0$$

$$(n+1)(n+2)a_{n+2} + a_{n-2} = 0$$

$$\boxed{a_{n+2} = -\frac{1}{(n+1)(n+2)}a_{n-2}}$$

Given: $y(0) = 1 = a_0, \quad y'(0) = 0 = a_1$

$2a_2 + 6a_3 x = 0 \quad \rightarrow \quad \boxed{a_2 = a_3 = 0}$

$a_0 = 1$

$n = 2 \quad \rightarrow \quad a_4 = -\frac{1}{12}a_0 = -\frac{1}{12}$

$a_1 = a_2 = a_3 = 0$

$n = 3 \rightarrow a_5 = -\frac{1}{20}a_1 = 0$

$n = 4 \rightarrow a_6 = {}^*a_2 = 0$

$n = 5 \rightarrow a_7 = {}^*a_3 = 0$

$n = 6 \rightarrow a_8 = -\frac{1}{56}a_4 = \frac{1}{672}$

$n = 10 \rightarrow a_{12} = -\frac{1}{132}a_8 = \frac{1}{88,704}$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x) = 1 - \frac{1}{12}x^4 + \frac{1}{672}x^8 - \frac{1}{88,704}x^{12} + \cdots}$$

Exercise

Find the series solution to the initial value problem $y'' - 2xy' + 8y = 0; \quad y(0) = 3, \quad y'(0) = 0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''-2xy'+8y=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+8\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}2(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}8a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+8a_n\right]x^n-\sum_{n=1}^{\infty}2na_n x^n=0$$

$$2a_2+8a_0+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}+8a_n\right]x^n-\sum_{n=1}^{\infty}2na_n x^n=0$$

$$2a_2+8a_0+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}+(8-2n)a_n\right]x^n=0$$

Given: $y(0)=3=a_0,\quad y'(0)=0=a_1$

$2a_2+8a_0=0 \quad\rightarrow\quad \boxed{a_2=-4a_0=-12}$

$(n+2)(n+1)a_{n+2}+(8-2n)a_n=0$

$\rightarrow a_{n+2}=\dfrac{2n-8}{(n+1)(n+2)}a_n \qquad n=1,2,\ldots$

$a_3=-a_1=0$	$a_4=-\frac{1}{3}a_2=4$
$a_5=-\frac{1}{10}a_3=0$	$a_6=0a_4=0$
$a_7=\frac{1}{21}a_5=0$	$a_6=0a_4=0$

$$\boxed{y(x)=3-12x^2+4x^4}$$

Exercise

Find the series solution to the initial value problem $y'' + y' - 2y = 0$; $y(0) = 1,\quad y'(0) = -2$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + y' - 2y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n+1) a_{n+1} - 2a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} + (n+1) a_{n+1} - 2a_n = 0$$

$$\boxed{a_{n+2} = \frac{2a_n - (n+1) a_{n+1}}{(n+1)(n+2)}}$$

Given: $y(0) = 1 = a_0,\quad y'(0) = -2 = a_1$

$a_0 = 1 \qquad\qquad a_1 = -2$

$n = 0 \ \rightarrow\ a_2 = \dfrac{2a_0 - a_1}{2} = 2 \qquad\qquad n = 1 \ \rightarrow\ a_3 = \dfrac{2a_1 - 2a_2}{6} = -\dfrac{8}{6}$

$n = 2 \ \rightarrow\ a_4 = \dfrac{2a_2 - 3a_3}{12} = \dfrac{2}{3} \qquad\qquad n = 3 \ \rightarrow\ a_5 = \dfrac{2a_3 - 4a_4}{20} = -\dfrac{4}{15}$

$n = 4 \ \rightarrow\ a_6 = \dfrac{2a_4 - 5a_5}{30} = \dfrac{4}{45} \qquad\qquad n = 5 \ \rightarrow\ a_7 = \dfrac{2a_5 - 6a_6}{42} = \dfrac{8}{315}$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x) = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 - \frac{4}{15}x^5 + \frac{4}{45}x^6 - \frac{8}{315}x^7 + \cdots}$$

$$= 1 + (-2x) + \frac{1}{2!}(-2x)^2 + \frac{1}{3!}(-2x)^3 + \frac{1}{4!}(-2x)^4 + \frac{1}{5!}(-2x)^5 + \frac{1}{6!}(-2x)^6 + \frac{1}{7!}(-2x)^7 + \cdots$$

$$\boxed{= e^{-2x}}$$

Exercise

Find the series solution to the initial value problem $y'' - 2y' + y = 0$; $y(0) = 0,\quad y'(0) = 1$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - 2y' + y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - 2(n+1) a_{n+1} + a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} - 2(n+1) a_{n+1} + a_n = 0$$

$$\boxed{a_{n+2} = \frac{2(n+1) a_{n+1} - a_n}{(n+1)(n+2)}}$$

Given: $y(0) = 0 = a_0, \quad y'(0) = 1 = a_1$

$a_0 = 0 \qquad\qquad a_1 = 1$

$n = 0 \ \rightarrow \ a_2 = \frac{2a_1 - a_0}{2} = 1 \qquad\qquad n = 1 \ \rightarrow \ a_3 = \frac{4a_2 - a_1}{6} = \frac{1}{2}$

$n = 2 \ \rightarrow \ a_4 = \frac{6a_3 - a_2}{12} = \frac{1}{6} \qquad\qquad n = 3 \ \rightarrow \ a_5 = \frac{8a_4 - a_3}{20} = \frac{1}{20}\left(\frac{4}{3} - \frac{1}{2}\right) = \frac{1}{24}$

$n = 4 \ \rightarrow \ a_6 = \frac{10a_5 - a_4}{30} = \frac{1}{30}\left(\frac{5}{12} - \frac{1}{6}\right) = \frac{1}{120} \qquad\qquad n = 5 \ \rightarrow \ a_7 = \frac{12a_6 - a_5}{42} = \frac{1}{42}\left(\frac{1}{10} - \frac{1}{24}\right) = \frac{1}{720}$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$y(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \frac{1}{120}x^6 + \frac{1}{720}x^7 + \cdots$$

$$= x\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots\right)$$

$$\boxed{= xe^x}$$

Exercise

Find the series solution to the initial value problem $y'' + xy' + y = 0 \quad y(0) = 1 \quad y'(0) = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + xy' + y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + n a_n + a_n\right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} + (n+1) a_n = 0$$

$$(n+2)(n+1) a_{n+2} = -(n+1) a_n$$

$$a_{n+2} = -\frac{1}{n+2} a_n$$

$$a_0 = y(0) = 1 \qquad\qquad a_1 = y'(0) = 0$$

$$a_2 = -\frac{1}{2} a_0 = -\frac{1}{2} \qquad\qquad a_3 = -\frac{1}{3} a_1 = 0$$

$$a_4 = -\frac{1}{4}a_2 = \frac{1}{2\cdot 4} = \frac{1}{2^2\cdot 1\cdot 2} \qquad a_5 = -\frac{1}{5}a_3 = 0$$

$$a_6 = -\frac{1}{6}a_4 = -\frac{1}{2^3\cdot 1\cdot 2\cdot 3} \qquad a_7 = -\frac{1}{7}a_7 = 0$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + +a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{2^2 2!}x^4 - \frac{1}{2^3 3!}x^6 + \cdots$$

Exercise

Find the series solution to the initial value problem $y'' - xy' - y = 0 \quad y(0) = 2 \quad y'(0) = 1$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - xy' - y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - n a_n - a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} - (n+1) a_n = 0$$

$$(n+2)(n+1) a_{n+2} = -(n+1) a_n$$

$$a_{n+2} = \frac{1}{n+2} a_n$$

$$a_0 = y(0) = 2 \qquad\qquad a_1 = y'(0) = 1$$
$$a_2 = \frac{1}{2}a_0 = 1 \qquad\qquad a_3 = \frac{1}{3}a_1 = \frac{1}{3}$$
$$a_4 = \frac{1}{4}a_2 = \frac{1}{4} \qquad\qquad a_5 = \frac{1}{5}a_3 = \frac{1}{3\cdot 5}$$
$$a_6 = \frac{1}{6}a_4 = \frac{1}{4}\frac{1}{6} = \frac{1}{24} \qquad\qquad a_7 = \frac{1}{7}a_7 = \frac{1}{3\cdot 5\cdot 7}$$
$$\vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y(x) = 2 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{15}x^5 + \frac{1}{24}x^6 + \cdots}$$

Exercise

Find the series solution to the initial value problem $y'' - xy' - y = 0$; $y(0) = 1$ $y'(0) = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - xy' - y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - x\sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - na_n - a_n\right]x^n = 0$$

$$(n+2)(n+1)a_{n+2} - (n+1)a_n = 0$$

$$(n+2)(n+1)a_{n+2} = -(n+1)a_n$$

$$\boxed{a_{n+2}=\frac{1}{n+2}a_n}$$

Given: $a_0=y(0)=1$ $\qquad a_1=y'(0)=0$

$a_2=\frac{1}{2}a_0=\frac{1}{2}$ $\qquad a_3=\frac{1}{3}a_1=0$

$a_4=\frac{1}{4}a_2=\frac{1}{2\cdot 2^2}$ $\qquad a_5=\frac{1}{5}a_3=0$

$a_6=\frac{1}{6}a_4=\frac{1}{2^3\cdot 3!}$ $\qquad \vdots \quad \vdots \quad \vdots$

$\vdots \quad \vdots \quad \vdots$

$$\boxed{y(x)=1+\frac{1}{2}x^2+\frac{1}{8}x^4+\frac{1}{48}x^6+\cdots}$$

Exercise

Find a power series solution. $\qquad y''+xy'-2y=0$; $\quad y(0)=1 \quad y'(0)=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y''+xy'-2y=0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+x\sum_{n=1}^{\infty}na_nx^{n-1}-\sum_{n=0}^{\infty}2a_nx^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}na_nx^n-\sum_{n=0}^{\infty}2a_nx^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}+(n-2)a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2} + (n-2)a_n = 0$$

$$a_{n+2} = -\frac{n-2}{(n+1)(n+2)}a_n$$

$$a_0 = y(0) = 1 \qquad\qquad a_1 = y'(0) = 0$$

$$n=0 \rightarrow a_2 = \frac{2}{2}a_0 = 1 \qquad\qquad n=1 \rightarrow a_3 = \frac{1}{6}a_1 = 0$$

$$n=2 \rightarrow a_4 = 0 \qquad\qquad n=3 \rightarrow a_5 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y(x) = 1 + x^2$$

Exercise

Find the series solution to the initial value problem $y'' + (x-1)y' + y = 0 \quad y(1) = 2 \quad y'(1) = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$y'' + (x-1)y' + y = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n + (x-1)\sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n + \sum_{n=1}^{\infty} n a_n (x-1)^{n} + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + na_n + a_n\right]x^{n-1} = 0$$

$$(n+2)(n+1)a_{n+2} + (n+1)a_n = 0$$

$$a_{n+2} = -\frac{1}{n+2}a_n$$

$$a_0 = y(1) = 2 \qquad a_1 = y'(1) = 0$$

$$a_2 = -\frac{1}{2}a_0 = -1 \qquad a_3 = -\frac{1}{3}a_1 = 0$$

$$a_4 = -\frac{1}{4}a_2 = \frac{1}{2\bullet 4}a_0 = \frac{1}{4} \qquad a_5 = -\frac{1}{5}a_3 = 0$$

$$a_6 = -\frac{1}{6}a_4 = -\frac{1}{24} \qquad a_7 = -\frac{1}{7}a_5 = 0$$

$$\vdots \quad \vdots \quad \vdots$$

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$= a_0 + a_1(x-1) + a_2(x-1) + a_3(x-1)^3 + a_4(x-1)^4 + \cdots$$

$$= 2 - (x-1)^2 + \frac{1}{4}(x-1)^4 - \frac{1}{24}(x-1)^6 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{2n}}{n!\ 2^n}$$

Exercise

Find the series solution to the initial value problem $(x-1)y'' - xy' + y = 0; \quad y(0) = -2, \quad y'(0) = 6$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x-1)y'' - xy' + y = 0$$

$$(x-1)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+1}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n}-\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=1}^{\infty}n(n+1)a_{n+1}x^{n}-\sum_{n=1}^{\infty}na_nx^{n}+\sum_{n=0}^{\infty}\left[a_n-(n+2)(n+1)a_{n+2}\right]x^n=0$$

$$\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1}-na_n\right]x^{n}+a_0-2a_2+\sum_{n=1}^{\infty}\left[a_n-(n+2)(n+1)a_{n+2}\right]x^n=0$$

$$a_0-2a_2+\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1}-(n-1)a_n-(n+2)(n+1)a_{n+2}\right]x^n=0$$

Given: $y(0)=-2=a_0,\quad y'(0)=6=a_1$

$$a_0-2a_2=0\quad\rightarrow\quad a_2=\frac{1}{2}a_0=-1$$

$$n(n+1)a_{n+1}-(n-1)a_n-(n+2)(n+1)a_{n+2}=0$$

$$\rightarrow\ a_{n+2}=\frac{n}{n+2}a_{n+1}-\frac{n-1}{(n+2)(n+1)}a_n$$

$$a_3=\frac{1}{3}a_2-0a_1=-\frac{1}{3}\qquad\qquad a_4=\frac{1}{2}a_3-\frac{1}{12}a_2=-\frac{1}{6}+\frac{1}{12}=-\frac{1}{12}$$

$$a_5=\frac{3}{5}a_4-\frac{1}{10}a_3=-\frac{3}{60}+\frac{1}{30}=-\frac{1}{60}\qquad\qquad a_6=\frac{2}{3}a_5-\frac{1}{10}a_4=-\frac{1}{90}+\frac{1}{120}=-\frac{1}{360}$$

$$\begin{aligned}y(x)&=\left(-2-x^2-\frac{1}{3}x^3-\frac{1}{12}x^4-\dots\right)+a_1x\\&=-2\left(1+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\dots\right)+6x\\&=-2\left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\dots\right)+6x+2x\\&=8x-2e^x\end{aligned}$$

Exercise

Find the series solution to the initial value problem

$$(x+1)y''-(2-x)y'+y=0;\quad y(0)=2,\quad y'(0)=-1$$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x+1)y'' - (2-x)y' + y = 0$$

$$(x+1)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - (2-x)\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+1} + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}2(n+1)a_{n+1}x^n$$

$$+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} + \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + (n+1)a_{n+1}\right]x^{n+1} + \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - 2(n+1)a_{n+1} + a_n\right]x^n = 0$$

$$\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1} + na_n\right]x^n + \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - 2(n+1)a_{n+1} + a_n\right]x^n = 0$$

$$\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1} + na_n\right]x^n + 2a_2 - 2a_1 + a_0$$

$$+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - 2(n+1)a_{n+1} + a_n\right]x^n = 0$$

$$2a_2 - 2a_1 + a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + (n-2)(n+1)a_{n+1} + (n+1)a_n\right]x^n = 0$$

Given: $y(0) = 2 = a_0$, $\quad y'(0) = -1 = a_1$

$$2a_2 - 2a_1 + a_0 = 0$$

$$a_2 = \frac{1}{2}\left(2a_1 - a_0\right) \underline{= -2 \,|}$$

$$(n+2)(n+1)a_{n+2} + (n-2)(n+1)a_{n+1} + (n+1)a_n = 0$$

$$\rightarrow\ a_{n+2} = -\frac{n-2}{n+2}a_{n+1} - \frac{1}{n+2}a_n$$

$$a_3 = \frac{1}{3}a_2 - \frac{1}{3}a_1 = \frac{2}{3} + \frac{1}{3} = 1 \qquad a_4 = 0a_3 - \frac{1}{4}a_2 = \frac{1}{2}$$

$$a_5 = -\frac{1}{5}a_4 - \frac{1}{5}a_3 = -\frac{1}{10} - \frac{1}{5} = -\frac{3}{10} \qquad a_6 = -\frac{1}{3}a_5 - \frac{1}{6}a_4 = \frac{1}{10} - \frac{1}{12} = \frac{1}{60}$$

$$y(x) = 2 - x - 2x^2 + x^3 + \frac{1}{2}x^4 - \frac{3}{10}x^5 + \dots$$

Exercise

Find the series solution to the initial value problem

$$(1-x)y'' + xy' - 2y = 0\ ; \quad y(0) = 0, \quad y'(0) = 1$$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$(1-x)y'' + xy' - 2y = 0$$

$$(1-x)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - 2\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+1} + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty}2a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - 2a_n\right]x^n - \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - (n+1)a_{n+1}\right]x^{n+1} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - 2a_n\right]x^n - \sum_{n=1}^{\infty}\left[n(n+1)a_{n+1} - na_n\right]x^n = 0$$

$$2a_2 - 2a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - 2a_n\right]x^n - \sum_{n=1}^{\infty}\left[n(n+1)a_{n+1} - na_n\right]x^n = 0$$

$$2a_2 - 2a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - n(n+1)a_{n+1} + (n-2)a_n\right]x^n = 0$$

$$(n+2)(n+1)a_{n+2} - n(n+1)a_{n+1} + (n-2)a_n = 0$$

$$\boxed{a_{n+2} = \frac{n(n+1)a_{n+1} - (n-2)a_n}{(n+1)(n+2)}}$$

Given: $y(0) = 0 = a_0$, $y'(0) = 1 = a_1$

$$2a_2 - 2a_0 = 0 \quad \rightarrow \quad \boxed{a_2 = a_0 = 0}$$

$$n = 1 \rightarrow a_3 = \frac{2a_2 + a_1}{6} = \frac{1}{6}$$

$$n = 2 \rightarrow a_4 = \frac{6a_3}{12} = \frac{1}{12}$$

$$n = 3 \rightarrow a_5 = \frac{1}{20}\left(12a_4 - a_3\right) = \frac{1}{20}\left(1 - \frac{1}{6}\right) = \frac{1}{24}$$

$$n = 4 \rightarrow a_6 = \frac{1}{30}\left(20a_5 - 2a_4\right) = \frac{1}{30}\left(\frac{5}{6} - \frac{1}{6}\right) = \frac{1}{45}$$

$$n = 5 \rightarrow a_7 = \frac{1}{42}\left(30a_6 - 3a_5\right) = \frac{1}{30}\left(\frac{2}{3} - \frac{1}{8}\right) = \frac{13}{1008}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y(x) = x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{24}x^5 + \frac{1}{45}x^6 + \frac{13}{1008}x^7 + \cdots}$$

Exercise

Find the series solution to the initial value problem

$$\left(x^2 + 1\right)y'' + 2xy' = 0; \quad y(0) = 0, \quad y'(0) = 1$$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+1\right)y''+2xy'=0$$

$$\left(x^2+1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2}+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}2(n+1)a_{n+1}x^{n+1}=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}2na_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+2a_2+6a_3x+\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n+2a_1x+\sum_{n=2}^{\infty}2na_n x^n=0$$

$$2a_2+\left(6a_3+2a_1\right)x+\sum_{n=2}^{\infty}\left[\left(n(n-1)+2n\right)a_n+(n+2)(n+1)a_{n+2}\right]x^n=0$$

Given: $y(0)=0=a_0,\quad y'(0)=1=a_1$

$$2a_2+\left(6a_3+2a_1\right)x=0 \quad\to\quad \begin{cases} a_2=0 \\ a_3=-\frac{1}{3}a_1=-\frac{1}{3}\end{cases}$$

$$n(n+1)a_n+(n+2)(n+1)a_{n+2}=0$$

$$\to\quad a_{n+2}=-\frac{n}{n+2}a_n \qquad n=2,3,\ldots$$

$$a_4=-\frac{1}{2}a_2=0 \qquad\qquad a_5=-\frac{3}{5}a_3=\frac{1}{5}$$

$$a_6=-\frac{2}{3}a_4=0 \qquad\qquad a_7=-\frac{5}{7}a_5=-\frac{1}{7}$$

$$\boxed{y(x)=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\cdots}$$

Exercise

Find the series solution to the initial value problem

$$\left(x^2-1\right)y''+3xy'+xy=0\ ; \qquad y(0)=4, \qquad y'(0)=6$$

Solution

$$y(x)=\sum_{n=0}^{\infty} a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty} na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2-1\right)y''+3xy'+xy=0$$

$$\left(x^2-1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+3x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+x\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$+\sum_{n=0}^{\infty}3(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}a_n x^{n+1}=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^{n+2}+\sum_{n=0}^{\infty}\left[(3n+3)a_{n+1}+a_n\right]x^{n+1}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+\sum_{n=1}^{\infty}\left(3na_n+a_{n-1}\right)x^n-\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+\left(3a_1+a_0\right)x+\sum_{n=2}^{\infty}\left(3na_n+a_{n-1}\right)x^n$$

$$-2a_2-6a_3x-\sum_{n=2}^{\infty}(n+1)(n+2)a_{n+2}x^n=0$$

$$-2a_2+\left(3a_1+a_0-6a_3\right)x+\sum_{n=2}^{\infty}\left[\left(n^2+2n\right)a_n+a_{n-1}-(n+2)(n+1)a_{n+2}\right]x^n=0$$

Given: $y(0)=4=a_0,\quad y'(0)=6=a_1$

$$\begin{cases}-2a_2=0 & \rightarrow\ \boxed{a_2=0}\\ 3a_1+a_0-6a_3=0 & \rightarrow\ \boxed{a_3=\frac{22}{6}=\frac{11}{3}}\end{cases}$$

$$\left(n^2+2n\right)a_n+a_{n-1}-(n+2)(n+1)a_{n+2}=0$$

$$\boxed{a_{n+2}=\frac{\left(n^2+2n\right)a_n+a_{n-1}}{(n+1)(n+2)}}$$

$$n=2\ \rightarrow\ a_4=\frac{8a_2+a_1}{12}=\frac{6}{12}=\frac{1}{2}$$

$$n=3\rightarrow\ a_5=\frac{1}{20}\left(15a_3+a_2\right)=\frac{1}{20}(55)=\frac{11}{4}$$

$$n=4\rightarrow\ a_6=\frac{1}{30}\left(24a_4+a_3\right)=\frac{1}{30}\left(12+\frac{11}{3}\right)=\frac{47}{90}$$

$$\vdots\quad\vdots\quad\vdots\quad\vdots$$

$$\boxed{y(x)=4+6x+\frac{11}{3}x^3+\frac{1}{2}x^4+\frac{11}{4}x^5+\frac{47}{90}x^6+\cdots}$$

Exercise

Find the series solution to the initial value problem

$$\left(2+x^2\right)y''-xy'+4y=0\qquad y(0)=-1\qquad y'(0)=3$$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(2+x^2\right)y''-xy'+4y=0$$

$$\left(2+x^2\right)\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}-x\sum_{n=1}^{\infty}na_n x^{n-1}+4\sum_{n=0}^{\infty}a_n x^n=0$$

$$2\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}+x^2\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}-\sum_{n=0}^{\infty}na_n x^{n-1}+4\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}n(n-1)a_n x^n-\sum_{n=0}^{\infty}na_n x^{n-1}+\sum_{n=0}^{\infty}4a_n x^n=0$$

$$2(n+2)(n+1)a_{n+2}+n(n-1)a_n-na_n+4a_n=0$$

$$2(n+2)(n+1)a_{n+2}+\left(n^2-2n+4\right)a_n=0$$

$$\boxed{a_{n+2}=-\frac{n^2-2n+4}{2(n+2)(n+1)}a_n}$$

$a_0=y(0)=-1$ $\qquad$ $a_1=y'(0)=3$

$n=0\to\quad a_2=-\frac{4}{4}a_0=1$ $\qquad$ $n=1\to\quad a_3=-\frac{3}{12}a_1=-\frac{1}{4}(3)=-\frac{3}{4}$

$n=2\to\quad a_4=-\frac{4}{24}a_2=-\frac{1}{6}$ $\qquad$ $n=3\to\quad a_5=-\frac{7}{40}a_3=-\frac{7}{40}\left(-\frac{3}{4}\right)=\frac{21}{160}$

$$\boxed{y(x)=-1+3x+x^2-\frac{3}{4}x^3-\frac{1}{6}x^4+\frac{21}{160}x^5+\cdots}$$

Exercise

Find the series solution to the initial value problem

$$\left(2-x^2\right)y''-xy'+4y=0 \qquad y(0)=1 \qquad y'(0)=0$$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \ =\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$\left(2-x^2\right)y''-xy'+4y=0$$

$$\left(2-x^2\right)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}-x\sum_{n=1}^{\infty} na_n x^{n-1}+4\sum_{n=0}^{\infty} a_n x^n=0$$

$$2\sum_{n=2}^{\infty} n(n+1)a_n x^{n-2}-x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}-\sum_{n=0}^{\infty} na_n x^{n-1}+4\sum_{n=0}^{\infty} a_n x^n=0$$

$$\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty} n(n-1)a_n x^n-\sum_{n=0}^{\infty} na_n x^{n-1}+\sum_{n=0}^{\infty} 4a_n x^n=0$$

$$\sum_{n=0}^{\infty} \left[2(n+2)(n+1)a_{n+2}-\left(n^2-n+n-4\right)a_n\right]x^n=0$$

$$\sum_{n=0}^{\infty} \left[2(n+2)(n+1)a_{n+2}-\left(n^2-4\right)a_n\right]x^n=0$$

$$2(n+2)(n+1)a_{n+2}-(n-2)(n+2)a_n=0$$

$$\boxed{a_{n+2}=\frac{n-2}{2(n+1)}a_n}$$

$a_0=y(0)=1$ $\qquad\qquad$ $a_1=y'(0)=0$

$n=0\rightarrow \quad a_2=\frac{-2}{2}a_0=-1$ $\qquad\qquad$ $n=1\rightarrow \quad a_3=-\frac{1}{4}a_1=0$

$n=2\rightarrow \quad a_4=0$ $\qquad\qquad$ $n=3\rightarrow \quad a_5=*a_3=0$

$\vdots \quad \vdots \quad \vdots$ $\qquad\qquad$ $\vdots \quad \vdots \quad \vdots$

$$\boxed{y(x)=1-x^2}$$

Exercise

Find the series solution to the initial value problem

$$\left(4-x^2\right)y''+2y=0 \qquad y(0)=0 \qquad y'(0)=1$$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(4-x^2\right)y''+2y=0$$

$$\left(4-x^2\right)\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}+2\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}4n(n-1)a_n x^{n-2}-x^2\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}+2\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}4n(n-1)a_n x^{n-2}-\sum_{n=2}^{\infty}n(n-1)a_n x^n+2\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}4(n+1)(n+2)a_{n+2}x^n-\sum_{n=0}^{\infty}n(n-1)a_n x^n+2\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[4(n+1)(n+2)a_{n+2}-\left(n^2-n-2\right)a_n\right]x^n=0$$

$$4(n+1)(n+2)a_{n+2}-(n+1)(n-2)a_n=0$$

$$\boxed{a_{n+2}=\frac{n-2}{4(n+2)}a_n}$$

$a_0=y(0)=0$ $\qquad$ $a_1=y'(0)=1$

$n=0\rightarrow \quad a_2=\frac{-2}{8}a_0=0$ $\qquad$ $n=1\rightarrow \quad a_3=-\frac{1}{12}a_1=-\frac{1}{12}$

$n=2\rightarrow \quad a_4=0$ $\qquad$ $n=3\rightarrow \quad a_5=\frac{1}{20}a_3=-\frac{1}{240}$

$\vdots \quad \vdots \quad \vdots$ $\qquad$ $n=5\rightarrow \quad a_7=\frac{3}{28}a_5=-\frac{1}{2{,}240}$

$\vdots \quad \vdots \quad \vdots$

$$y(x)=x-\frac{1}{12}x^3-\frac{1}{240}x^5-\frac{1}{2240}x^7-\frac{1}{16,128}x^9-\cdots$$

Exercise

Find a power series solution. $\left(x^2-4\right)y''+3xy'+y=0$; $y(0)=4,$ $y'(0)=1$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2-4\right)y''+3xy'+y=0$$

$$x^2\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}-4\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}+3x\sum_{n=1}^{\infty}na_n x^{n-1}+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n-\sum_{n=2}^{\infty}4n(n-1)a_n x^{n-2}+\sum_{n=1}^{\infty}3na_n x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}n(n-1)a_n x^n-\sum_{n=0}^{\infty}4(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}3na_n x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[\left(n^2-n+3n+1\right)a_n-4(n+2)(n+1)a_{n+2}\right]x^n=0$$

$$\left(n^2+2n+1\right)a_n-4(n+2)(n+1)a_{n+2}=0$$

$$4(n+2)(n+1)a_{n+2}=(n+1)^2a_n$$

$$a_{n+2}=\frac{n+1}{4(n+2)}a_n$$

$$a_0=y(0)=4 \qquad\qquad a_1=y'(0)=1$$

$$n=0 \quad \rightarrow \quad a_2 = \frac{1}{8}a_0 = \frac{1}{2} \qquad\qquad n=1 \quad \rightarrow \quad a_3 = \frac{2}{4\cdot 3}a_1 = \frac{1}{6}$$

$$n=2 \quad \rightarrow \quad a_4 = \frac{3}{16}a_2 = \frac{3}{32} \qquad\qquad n=3 \quad \rightarrow \quad a_5 = \frac{1}{5}a_3 = \frac{1}{30}$$

$$n=4 \quad \rightarrow \quad a_6 = \frac{5}{24}a_4 = \frac{5}{256} \qquad\qquad n=5 \quad \rightarrow \quad a_7 = \frac{6}{4\cdot 7}a_5 = \frac{1}{140}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y(x) = 4 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{3}{32}x^4 + \frac{1}{30}x^5 + \frac{5}{256}x^6 + \frac{1}{140}x^7 + \cdots \big|}$$

Exercise

Find a power series solution. $\left(x^2+1\right)y'' + 2xy' - 2y = 0$; $y(0)=0, \quad y'(0)=1$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+1\right)y'' + 2xy' - 2y = 0$$

$$x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2x\sum_{n=1}^{\infty} n a_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2)a_{n+2} + \left(n^2+n-2\right)a_n\right]x^n = 0$$

$$(n+1)(n+2)a_{n+2}+(n-1)(n+2)a_n=0$$

$$\boxed{a_{n+2}=-\frac{n-1}{n+1}a_n}$$

$$a_0=y(0)=0 \qquad\qquad a_1=y'(0)=1$$

$$n=0 \quad\rightarrow\quad a_2=a_0 \qquad\qquad n=1 \quad\rightarrow\quad a_3=0$$

$$n=2 \quad\rightarrow\quad a_4=-\tfrac{1}{3}a_2=-\tfrac{1}{3}a_0 \qquad\qquad n=3 \quad\rightarrow\quad a_5=0$$

$$n=4 \quad\rightarrow\quad a_6=-\tfrac{3}{5}a_4=\tfrac{1}{5}a_0 \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n=6 \quad\rightarrow\quad a_8=-\tfrac{5}{7}a_6=-\tfrac{1}{7}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n}=\frac{(-1)^n}{2n-1}a_0$$

$$\boxed{y(x)=a_1x+a_0\left(1+x^2-\tfrac{1}{3}x^4+\tfrac{1}{5}x^6-\tfrac{1}{7}x^8+\cdots\right)}$$

$$y(x)=a_1x+a_0\left(1+x\left(x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5-\tfrac{1}{7}x^7+\cdots\right)\right)$$

$$\boxed{=a_1x+a_0\left(1+x\tan^{-1}x\right)}$$

Exercise

Find a power series solution. $\left(2x-x^2\right)y''-6(x-1)y'-4y=0\ ;\quad y(1)=0,\quad y'(1)=1$

Solution

Let $\quad z=x-1 \quad\Rightarrow\quad \begin{cases} x=z+1 \\ dz=dx \end{cases}$

$$\left(2x-x^2\right)y''-6(x-1)y'-4y=0$$

$$\left(2z+2-z^2-2z-1\right)y''-6zy'-4y=0$$

$$\left(1-z^2\right)y''-6zy'-4y=0$$

$$y(z)=\sum_{n=0}^{\infty}a_nz^n$$

$$y'(z)=\sum_{n=1}^{\infty}na_nz^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n$$

$$y''(z)=\sum_{n=2}^{\infty}n(n-1)a_nz^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n$$

$$\left(1-z^2\right)y''-6zy'-4y=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_nz^{n-2}-z^2\sum_{n=2}^{\infty}n(n-1)a_nz^{n-2}-6z\sum_{n=1}^{\infty}na_nz^{n-1}-4\sum_{n=0}^{\infty}a_nz^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_nz^{n-2}-\sum_{n=2}^{\infty}n(n-1)a_nz^n-\sum_{n=1}^{\infty}6na_nz^n-4\sum_{n=0}^{\infty}a_nz^n=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}z^n-\sum_{n=0}^{\infty}n(n-1)a_nz^n-\sum_{n=0}^{\infty}6na_nz^n-4\sum_{n=0}^{\infty}a_nz^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}-\left(n^2+5n+4\right)a_n\right]z^n=0$$

$$(n+1)(n+2)a_{n+2}-(n+1)(n+4)a_n=0$$

$$\boxed{a_{n+2}=\frac{n+4}{n+2}a_n}$$

Given: $y(1)=0=a_0$, $y'(1)=1=a_1$

$a_0=0$

$n=0 \quad \rightarrow \quad a_2=2a_0=0$

$n=2 \quad \rightarrow \quad a_4=\frac{3}{2}a_2=0$

$n=4 \quad \rightarrow \quad a_6=0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$a_1=1$

$n=1 \quad \rightarrow \quad a_3=\frac{5}{3}a_1=\frac{5}{3}$

$n=3 \quad \rightarrow \quad a_5=\frac{7}{5}a_3=\frac{7}{3}$

$n=5 \quad \rightarrow \quad a_7=\frac{9}{7}a_5=\frac{9}{3}$

$n=7 \quad \rightarrow \quad a_9=\frac{11}{9}a_7=\frac{11}{3}$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$a_{2n+1}=\frac{2n+3}{3}$$

$$y(z) = z + \frac{5}{3}z^3 + \frac{7}{3}z^5 + 3z^7 + \frac{11}{3}z^9 + \cdots + \frac{2n+3}{3}z^{2n+1} + \cdots$$

$$\boxed{y(x) = (x-1) + \frac{5}{3}(x-1)^3 + \frac{7}{3}(x-1)^5 + 3(x-1)^7 + \frac{11}{3}(x-1)^9 + \cdots}$$

Exercise

Find a power series solution. $\left(x^2 - 6x + 10\right)y'' - 4(x-3)y' + 6y = 0$; $y(3) = 2, \quad y'(3) = 0$

Solution

Let $z = x - 3 \quad \Rightarrow \begin{cases} x = z + 3 \\ dz = dx \end{cases}$

$$\left(x^2 - 6x + 10\right)y'' - 4(x-3)y' + 6y = 0$$

$$\left(z^2 + 6z + 9 - 6z - 18 + 10\right)y'' - 4zy' + 6y = 0$$

$$\boxed{\left(z^2 + 1\right)y'' - 4zy' + 6y = 0}$$

$$y(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$y'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$$

$$y''(z) = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n$$

$$\left(z^2 + 1\right)y'' - 4zy' + 6y = 0$$

$$z^2 \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} - 4z \sum_{n=1}^{\infty} n a_n z^{n-1} + 6 \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n z^n + \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} - \sum_{n=1}^{\infty} 4n a_n z^n + \sum_{n=0}^{\infty} 6 a_n z^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n z^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n - \sum_{n=0}^{\infty} 4n a_n z^n + \sum_{n=0}^{\infty} 6 a_n z^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}+\left(n^2-5n+6\right)a_n\right]z^n=0$$

$$(n+1)(n+2)a_{n+2}+(n-2)(n-3)a_n=0$$

$$\boxed{a_{n+2}=-\frac{(n-2)(n-3)}{(n+1)(n+2)}a_n}$$

Given: $y(3)=2=a_0,\quad y'(3)=0=a_1$

$a_0=2$ | $a_1=0$
$n=0 \;\to\; a_2=-3a_0=-6$ | $n=1 \;\to\; a_3=-\frac{2}{4}a_1=0$
$n=2 \;\to\; a_4=0$ | $n=3 \;\to\; a_5=0$
$n=4 \;\to\; a_6=0$ | $n=5 \;\to\; a_7=0$
$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$ | $\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$y(z)=2-6z^2$$

$$\boxed{y(x)=2-6(x-3)^2}$$

Exercise

Find a power series solution. $\left(4x^2+16x+17\right)y''-8y=0\ ;\quad y(-2)=1,\quad y'(-2)=0$

Solution

Let $z=x+2 \;\Rightarrow\; \begin{cases} x=z-2 \\ dz=dx \end{cases}$

$$\left(4z^2-16z+16+16z-32+17\right)y''-8y=0$$

$$\boxed{\left(4z^2+1\right)y''-8y=0}$$

$$y(z)=\sum_{n=0}^{\infty}a_nz^n$$

$$y'(z)=\sum_{n=1}^{\infty}na_nz^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n$$

$$y''(z)=\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n$$

$$\left(4z^2+1\right)y''-8y=0$$

$$4z^2\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}+\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}-8\sum_{n=0}^{\infty} a_n z^n=0$$

$$\sum_{n=2}^{\infty} 4n(n-1)a_n z^n+\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}-8\sum_{n=0}^{\infty} a_n z^n=0$$

$$\sum_{n=0}^{\infty} 4n(n-1)a_n z^n+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n-8\sum_{n=0}^{\infty} a_n z^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}+\left(4n^2-4n-8\right)a_n\right]z^n=0$$

$$(n+1)(n+2)a_{n+2}+4(n+1)(n-2)a_n=0$$

$$\boxed{a_{n+2}=-\frac{4(n-2)}{n+2}a_n}$$

Given: $y(-2)=1=a_0,\quad y'(-2)=0=a_1$

$a_0=1$	$a_1=0$
$n=0 \quad\rightarrow\quad a_2=\frac{8}{2}a_0=4$	$n=1 \quad\rightarrow\quad a_3=\frac{4}{3}a_1=0$
$n=2 \quad\rightarrow\quad a_4=-0a_2=0$	$n=3 \quad\rightarrow\quad a_5=0$
$n=4 \quad\rightarrow\quad a_6=0$	$n=5 \quad\rightarrow\quad a_7=0$
$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$y(z)=1+4z^2$$

$$\boxed{y(x)=1+4(x+2)^2}$$

Exercise

Find a power series solution. $\left(x^2+6x\right)y''+(3x+9)y'-3y=0\ ;\quad y(-3)=0,\quad y'(-3)=2$

Solution

Let $z=x+3 \quad \Rightarrow \quad \begin{cases} x=z-3 \\ dz=dx \end{cases}$

$$\left(x^2+6x\right)y''+(3x+9)y'-3y=0$$

$$\left(z^2-6z+9+6z-18\right)y''+3zy'-3y=0$$

$$\boxed{\left(z^2-9\right)y''+3zy'-3y=0}$$

$$y(z)=\sum_{n=0}^{\infty}a_n z^n$$

$$y'(z)=\sum_{n=1}^{\infty}na_n z^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n$$

$$y''(z)=\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n$$

$$\left(z^2-9\right)y''+3zy'-3y=0$$

$$z^2\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}-9\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}+3z\sum_{n=1}^{\infty}na_n z^{n-1}-3\sum_{n=0}^{\infty}a_n z^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n z^n-\sum_{n=2}^{\infty}9n(n-1)a_n z^{n-2}+\sum_{n=1}^{\infty}3na_n z^n-\sum_{n=0}^{\infty}3a_n z^n=0$$

$$\sum_{n=0}^{\infty}n(n-1)a_n z^n-\sum_{n=0}^{\infty}9(n+2)(n+1)a_{n+2}z^n+\sum_{n=0}^{\infty}3na_n z^n-\sum_{n=0}^{\infty}3a_n z^n=0$$

$$\sum_{n=0}^{\infty}\left[-9(n+2)(n+1)a_{n+2}+\left(n^2-n+3n-3\right)a_n\right]z^n=0$$

$$-9(n+2)(n+1)a_{n+2}+(n+3)(n-1)a_n=0$$

$$\boxed{a_{n+2}=\frac{(n+3)(n-1)}{9(n+1)(n+2)}a_n}$$

Given: $y(-3)=0=a_0,\quad y'(-3)=2=a_1$

$a_0=0$	$a_1=2$
$n=0 \;\rightarrow\; a_2=\frac{-3}{18}a_0=0$	$n=1 \;\rightarrow\; a_3=0a_1=0$
$n=2 \;\rightarrow\; a_4=\frac{5}{108}a_2=0$	$n=3 \;\rightarrow\; a_5=0$
$n=4 \;\rightarrow\; a_6=0$	$n=5 \;\rightarrow\; a_7=0$
$\vdots \;\; \vdots \;\; \vdots \;\; \vdots \;\; \vdots$	$\vdots \;\; \vdots \;\; \vdots \;\; \vdots \;\; \vdots$

$y(z)=2z$

$$\boxed{y(x)=2x+6}$$

Exercise

Find a power series solution. $y''+x^3y'+2y=0;\quad y(0)=1,\quad y'(0)=-2$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y''+x^3y'+2y=0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+x^3\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}2a_nx^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+2a_n\right]x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+3}=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+2a_n\right]x^n+\sum_{n=3}^{\infty}(n-2)a_{n-2}x^n=0$$

$$2a_2+2a_0+\left(6a_3+2a_1\right)x+\left(12a_4+2a_2\right)x^2$$
$$+\sum_{n=3}^{\infty}\left[(n+2)(n+1)a_{n+2}+2a_n\right]x^n+\sum_{n=3}^{\infty}(n-2)a_{n-2}x^n=0$$

$$2a_2+2a_0+\left(6a_3+2a_1\right)x+\left(12a_4+2a_2\right)x^2$$
$$+\sum_{n=3}^{\infty}\left[(n+2)(n+1)a_{n+2}+2a_n+(n-2)a_{n-2}\right]x^n=0$$

$y(0)=1=a_0 \quad y'(0)=-2=a_1$

$$2a_2+2a_0=0 \quad \Rightarrow \quad \boxed{a_2=-1}$$

$$6a_3+2a_1=0 \quad \Rightarrow \quad \boxed{a_3=\frac{2}{3}}$$

$$12a_4+2a_2=0 \quad \Rightarrow \quad \boxed{a_4=\frac{1}{6}}$$

$$(n+2)(n+1)a_{n+2}+2a_n+(n-2)a_{n-2}=0$$

$$a_{n+2}=-\frac{2a_n+(n-2)a_{n-2}}{(n+1)(n+2)}$$

$n=3 \qquad a_5=-\frac{2a_3+a_1}{(4)(5)}$

$$=-\frac{1}{20}\left(\frac{4}{3}-2\right)$$

$$\boxed{=\frac{1}{30}}$$

$n=4 \qquad a_6=-\frac{2a_4+2a_2}{(5)(6)}$

$$=-\frac{1}{30}\left(\frac{1}{3}-2\right)$$

$$\boxed{=\frac{1}{18}}$$

$$n=5 \qquad a_7 = -\frac{2a_5 + 3a_3}{(6)(7)}$$

$$= -\frac{1}{42}\left(\frac{1}{15} + 2\right)$$

$$= -\frac{31}{630}$$

$$n=6 \qquad a_8 = -\frac{2a_6 + 4a_4}{(7)(8)}$$

$$= -\frac{1}{56}\left(\frac{1}{9} + \frac{2}{3}\right)$$

$$= -\frac{1}{72}$$

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

$$= 1 - 2x - x^2 + \frac{2}{3}x^3 + \frac{1}{6}x^4 + \frac{1}{30}x^5 + \frac{1}{18}x^6 - \frac{31}{630}x^7 - \frac{1}{72}x^8 + \cdots$$

Exercise

Find a power series solution. $\left(x^2+2\right)y'' + 2xy' + 3y = 0 \quad y(0)=1, \quad y'(0)=2$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(x^2+2\right)y'' + 2xy' + 3y = 0$$

$$x^2 y'' + 2y'' + 2xy' + 3y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + 3\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} 2n\,a_n x^n + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} 2n\,a_n x^n + \sum_{n=0}^{\infty} \left[2(n+2)(n+1)a_{n+2} + 3a_n\right]x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + 2a_1 x + \sum_{n=2}^{\infty} 2n\,a_n x^n + (4a_2 + 3a_0) + (12a_3 + 3a_1)x$$

$$+ \sum_{n=2}^{\infty} \left[2(n+2)(n+1)a_{n+2} + 3a_n\right]x^n = 0$$

$$(4a_2 + 3a_0) + (12a_3 + 5a_1)x + \sum_{n=2}^{\infty} \left[\left(n^2 - n\right)a_n + 2n\,a_n + 2(n+2)(n+1)a_{n+2} + 3a_n\right]x^n = 0$$

$$(4a_2 + 3a_0) + (12a_3 + 5a_1)x + \sum_{n=2}^{\infty} \left[\left(n^2 + n + 3\right)a_n + 2(n+2)(n+1)a_{n+2}\right]x^n = 0$$

$y(0) = 1 = a_0 \qquad y'(0) = 2 = a_1$

$4a_2 + 3a_0 = 0 \quad \Rightarrow \quad \boxed{a_2 = -\frac{3}{4}}$

$12a_3 + 5a_1 = 0 \quad \Rightarrow \quad \boxed{a_3 = -\frac{5}{6}}$

$\left(n^2 + n + 3\right)a_n + 2(n+2)(n+1)a_{n+2} = 0$

$$a_{n+2} = -\frac{n^2 + n + 3}{2(n+2)(n+1)}a_n$$

$a_2 = -\frac{3}{4} \qquad\qquad a_3 = -\frac{5}{6}$

$n = 2 \qquad a_4 = -\frac{9}{24}a_2 = -\frac{3}{8}\left(-\frac{3}{4}\right) = \boxed{\frac{9}{32}}$

$n = 3 \qquad a_5 = -\frac{15}{40}a_3 = -\frac{3}{8}\left(-\frac{5}{6}\right) = \boxed{\frac{5}{16}}$

$n = 4 \qquad a_6 = -\frac{23}{60}a_4 = -\frac{23}{60}\left(\frac{9}{32}\right) = \boxed{-\frac{69}{640}}$

$n = 5 \qquad a_7 = -\frac{33}{84}a_5 = -\frac{11}{28}\,\frac{5}{16} = \boxed{-\frac{55}{448}}$

$$y(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+\cdots$$

$$\boxed{=1+2x-\frac{3}{4}x^2-\frac{5}{6}x^3+\frac{9}{32}x^4+\frac{5}{16}x^5-\frac{69}{640}x^6-\frac{55}{118}x^7+\cdots}$$

Exercise

Find a power series solution. $\left(1+2x^2\right)y''+10xy'+8y=0 \quad y(0)=2, \quad y'(0)=-3$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(1+2x^2\right)y''+10xy'+8y=0$$

$$y''+2x^2y''+10xy'+8y=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+2x^2\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+10x\sum_{n=1}^{\infty}na_nx^{n-1}+8\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}2n(n-1)a_nx^n+\sum_{n=1}^{\infty}10na_nx^n+\sum_{n=0}^{\infty}8a_nx^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+8a_n\right]x^n+\sum_{n=2}^{\infty}2n(n-1)a_nx^n+\sum_{n=1}^{\infty}10na_nx^n=0$$

$$2a_2+8a_0+\left(6a_3+8a_1\right)x+\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2}+8a_n\right]x^n$$

$$+\sum_{n=2}^{\infty}2n(n-1)a_nx^n+10a_1x+\sum_{n=2}^{\infty}10na_nx^n=0$$

$$2a_2 + 8a_0 + \left(6a_3 + 18a_1\right)x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} + 8a_n + 2n(n-1)a_n + 10na_n\right]x^n = 0$$

$$2a_2 + 8a_0 + \left(6a_3 + 18a_1\right)x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} + \left(8 + 2n^2 - 2n + 10n\right)a_n\right]x^n = 0$$

$$2a_2 + 8a_0 + \left(6a_3 + 18a_1\right)x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} + \left(2n^2 + 8n + 8\right)a_n\right]x^n = 0$$

$$y(0) = 2 = a_0 \qquad y'(0) = -3 = a_1$$

$$2a_2 + 8a_0 = 0 \quad \Rightarrow \quad \underline{a_2 = -8}$$

$$6a_3 + 18a_1 = 0 \quad \Rightarrow \quad \underline{a_3 = 9}$$

$$(n+2)(n+1)a_{n+2} + \left(2n^2 + 8n + 8\right)a_n = 0$$

$$\begin{aligned} a_{n+2} &= -\frac{2\left(n^2 + 4n + 4\right)}{(n+1)(n+2)}a_n \\ &= -\frac{2(n+2)^2}{(n+1)(n+2)}a_n \\ &= \underline{-\frac{2(n+2)}{n+1}a_n} \end{aligned}$$

$$n = 2 \qquad \begin{aligned} a_4 &= -\frac{2(4)}{3}a_2 \\ &= -\frac{8}{3}(-8) \\ &= \underline{\frac{64}{3}} \end{aligned}$$

$$n = 3 \qquad \begin{aligned} a_5 &= -\frac{2(5)}{4}a_3 \\ &= -\frac{5}{2}(9) \\ &= \underline{-\frac{45}{2}} \end{aligned}$$

$$n = 4 \qquad \begin{aligned} a_6 &= -\frac{2(6)}{5}a_4 \\ &= -\frac{12}{5}\left(\frac{64}{3}\right) \end{aligned}$$

$$= -\frac{256}{5}$$

$n = 5$ $\quad a_7 = -\frac{2(7)}{6}a_5$

$$= -\frac{7}{3}\left(-\frac{45}{2}\right)$$

$$= \frac{105}{2}$$

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots$$

$$= 2 - 3x - 8x^2 + 9x^3 + \frac{64}{3}x^4 - \frac{45}{2}x^5 - \frac{256}{5}x^6 + \frac{105}{2}x^7 + \cdots$$

Exercise

Find a power series solution. $\left(x^2 - 2x - 3\right)y'' + 3(x-1)y' + y = 0 \quad y(0) = 4, \quad y'(0) = 1$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=1}^{\infty} n(n+1)a_{n+1}x^{n-1} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2 - 2x - 3\right)y'' + 3(x-1)y' + y = 0$$

$$x^2y'' - 2xy'' - 3y'' + 3xy' - 3y' + y = 0$$

$$x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n(n+1)a_{n+1}x^{n-1} - 3\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$+ 3x\sum_{n=1}^{\infty} na_n x^{n-1} - 3\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 2n(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty} 3(n+2)(n+1)a_{n+2}x^n$$

$$+\sum_{n=1}^{\infty} 3na_n x^n - \sum_{n=0}^{\infty} 3(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} \left(3na_n - 2n(n+1)a_{n+1}\right)x^n$$

$$+\sum_{n=0}^{\infty} \left(a_n - 3(n+1)a_{n+1} - 3(n+2)(n+1)a_{n+2}\right)x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \left(3a_1 - 4a_2\right)x + \sum_{n=2}^{\infty} \left(3na_n - 2n(n+1)a_{n+1}\right)x^n + \left(a_0 - 3a_1 - 6a_2\right)$$

$$+\left(a_1 - 6a_2 - 18a_3\right)x + \sum_{n=2}^{\infty} \left(a_n - 3(n+1)a_{n+1} - 3(n+2)(n+1)a_{n+2}\right)x^n = 0$$

$$\left(a_0 - 3a_1 - 6a_2\right) + \left(4a_1 - 10a_2 - 18a_3\right)x$$

$$+\sum_{n=2}^{\infty} \left(-\left(3n+3+2n^2+2n\right)a_{n+1} - 3(n+2)(n+1)a_{n+2} + \left(n^2 - n + 3n + 1\right)a_n\right)x^n = 0$$

$$\left(a_0 - 3a_1 - 6a_2\right) + \left(4a_1 - 10a_2 - 18a_3\right)x$$

$$+\sum_{n=2}^{\infty} \left(-\left(2n^2+5n+3\right)a_{n+1} - 3(n+2)(n+1)a_{n+2} + \left(n^2 + 2n + 1\right)a_n\right)x^n = 0$$

$$y(0) = 4 = a_0 \qquad y'(0) = 1 = a_1$$

$$a_0 - 3a_1 - 6a_2 = 0 \quad \Rightarrow \quad \boxed{a_2 = \frac{1}{6}}$$

$$2a_1 - 5a_2 - 9a_3 = 0 \quad \Rightarrow \quad a_3 = \frac{1}{9}\left(2 - \frac{5}{6}\right) \boxed{= \frac{7}{54}}$$

$$-(2n+3)(n+1)a_{n+1} - 3(n+2)(n+1)a_{n+2} + (n+1)^2 a_n = 0$$

$$3(n+2)a_{n+2} = (n+1)a_n - (2n+3)a_{n+1}$$

$$a_{n+2} = \frac{1}{3(n+2)}\left((n+1)a_n - (2n+3)a_{n+1}\right)$$

$n=2 \quad a_4 = \frac{1}{12}\left(3a_2 - 7a_3\right)$

$$= \frac{1}{12}\left(\frac{1}{2} - \frac{49}{54}\right)$$

$$= -\frac{11}{324}$$

$n=3 \quad a_5 = \frac{1}{15}\left(4a_3 - 9a_4\right)$

$$= \frac{1}{15}\left(\frac{14}{27} + \frac{11}{36}\right)$$

$$= \frac{1}{15}\left(\frac{89}{108}\right)$$

$$= \frac{89}{1,620}$$

$n=4 \quad a_6 = \frac{1}{18}\left(5a_4 - 11a_5\right)$

$$= \frac{1}{18}\left(-\frac{55}{324} - \frac{979}{1,620}\right)$$

$$= -\frac{1}{18}\left(\frac{1,254}{1,620}\right)$$

$$= -\frac{209}{4,860}$$

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

$$= 4 + x + \frac{1}{6}x^2 + \frac{7}{54}x^3 - \frac{11}{324}x^4 + \frac{89}{1,620}x^5 - \frac{209}{4,860}x^6 + \frac{617}{14,580}x^7 + \cdots$$

Exercise

Find the series solution near the given value $\quad y'' - (x-2)y' + 2y = 0$; near $x = 2$

Solution

$$y = \sum_{n=0}^{\infty} a_n (x-2)^n$$

$$y' = \sum_{n=1}^{\infty} n a_n (x-2)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-2)^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x-2)^n$$

$$y'' - (x-2)y' + 2y = 0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-2)^n - (x-2)\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-2)^n + 2\sum_{n=0}^{\infty}a_n(x-2)^n = 0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-2)^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-2)^{n+1} + \sum_{n=0}^{\infty}2a_n(x-2)^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2} + 2a_n\right](x-2)^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-2)^{n+1} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2} + 2a_n\right](x-2)^n - \sum_{n=1}^{\infty}na_n(x-2)^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty}\left[(n+1)(n+2)a_{n+2} + 2a_n\right](x-2)^n - \sum_{n=1}^{\infty}na_n(x-2)^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty}\left[(n+1)(n+2)a_{n+2} - (n-2)a_n\right](x-2)^n = 0$$

For $n=0 \quad \rightarrow \quad 2a_2 + 2a_0 = 0 \quad \Rightarrow \quad \boxed{a_2 = -a_0}$

$$(n+1)(n+2)a_{n+2} - (n-2)a_n = 0$$

$$\boxed{a_{n+2} = \frac{n-2}{(n+1)(n+2)}a_n}$$

a_0	a_1
$n=0 \rightarrow a_2 = -a_0$	$n=1 \rightarrow a_3 = -\frac{1}{6}a_1$
$n=2 \rightarrow a_4 = 0$	$n=3 \rightarrow a_5 = \frac{1}{20}a_3 = -\frac{1}{120}a_1$
$n=4 \rightarrow a_6 = 0$	$n=5 \rightarrow a_7 = \frac{3}{42}a_5 = -\frac{1}{1680}a_1$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x) = a_0\left(1-(x-2)^2\right) + a_1\left((x-2) - \tfrac{1}{6}(x-2)^3 - \tfrac{1}{120}(x-2)^5 - \tfrac{1}{1680}(x-2)^7 - \cdots\right)}$$

Exercise

Find the series solution near the given value $y'' + (x-1)^2 y' - 4(x-1)y = 0$; near $x = 1$

Solution

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x-1)^n$$

$$y'' + (x-1)^2 y' - 4(x-1)y = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x-1)^n + (x-1)^2 \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n$$
$$-4(x-1) \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x-1)^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^{n+2} - \sum_{n=0}^{\infty} 4 a_n (x-1)^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x-1)^n + \sum_{n=2}^{\infty} (n-1) a_{n-1} (x-1)^n - \sum_{n=1}^{\infty} 4 a_{n-1} (x-1)^n = 0$$

$$2a_2 + 6a_3 (x-1) + \sum_{n=2}^{\infty} (n+1)(n+2) a_{n+2} (x-1)^n + \sum_{n=2}^{\infty} (n-1) a_{n-1} (x-1)^n$$
$$-4a_0 (x-1) - \sum_{n=2}^{\infty} 4 a_{n-1} (x-1)^n = 0$$

$$2a_2 + (6a_3 - 4a_0)(x-1) + \sum_{n=2}^{\infty} \left[(n+1)(n+2) a_{n+2} + (n-5) a_{n-1} \right] (x-1)^n = 0$$

$$\begin{cases} 2a_2 = 0 & \rightarrow \boxed{a_2 = 0} \\ 6a_3 - 4a_0 = 0 & \rightarrow \boxed{a_3 = \frac{2}{3}a_0} \end{cases}$$

$$(n+1)(n+2)a_{n+2} + (n-5)a_{n-1} = 0$$

$$\boxed{a_{n+2} = -\frac{(n-5)}{(n+1)(n+2)}a_{n-1}}$$

a_0	a_1	$a_2 = 0$
$n=1 \rightarrow a_3 = \frac{2}{3}a_0$	$n=2 \rightarrow a_4 = \frac{1}{4}a_1$	$n=3 \rightarrow a_5 = \frac{2}{20}a_2 = 0$
$n=4 \rightarrow a_6 = \frac{1}{30}a_3 = \frac{1}{45}a_0$	$n=5 \rightarrow a_7 = 0$	$n=6 \rightarrow a_8 = 0$
$n=7 \rightarrow a_9 = -\frac{2}{8\cdot 9}a_6 = -\frac{1}{1,620}a_0$	$n=8 \rightarrow a_{10} = 0$	$n=9 \rightarrow a_{11} = 0$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x) = a_0\left(1 + \frac{2}{3}(x-1)^3 + \frac{1}{45}(x-1)^6 - \frac{1}{1,620}(x-1)^9 + \cdots\right) + a_1\left((x-1) + \frac{1}{4}(x-1)^4\right)}$$

$$\boxed{y(x) = a_0\sum_{n=0}^{\infty}\frac{(-1)^n 4(x-1)^{3n}}{3^n(3n-1)(3n-4)n!} + a_1\left((x-1) + \frac{1}{4}(x-1)^4\right)}$$

Exercise

Find the series solution near the given value $y'' + (x-1)y = e^x$; *near* $x = 1$

Solution

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$y' = \sum_{n=1}^{\infty} na_n (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}(x-1)^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n (x-1)^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}(x-1)^n$$

$$y'' + (x-1)y = e^{x-1+1}$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n+(x-1)\sum_{n=0}^{\infty}a_n(x-1)^n=e\cdot e^{x-1} \qquad e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n+\sum_{n=0}^{\infty}a_n(x-1)^{n+1}=e\cdot\sum_{n=0}^{\infty}\frac{(x-1)^n}{n!}$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n+\sum_{n=1}^{\infty}a_{n-1}(x-1)^n=e\cdot\sum_{n=0}^{\infty}\frac{(x-1)^n}{n!}$$

$$2a_2+\sum_{n=1}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n+\sum_{n=1}^{\infty}a_{n-1}(x-1)^n=e+e\cdot\sum_{n=1}^{\infty}\frac{(x-1)^n}{n!}$$

$$2a_2+\sum_{n=1}^{\infty}\left[(n+1)(n+2)a_{n+2}+a_{n-1}\right](x-1)^n=e+e\cdot\sum_{n=1}^{\infty}\frac{(x-1)^n}{n!}$$

$$2a_2=e \quad\rightarrow\quad \boxed{a_2=\frac{e}{2}}$$

$$(n+1)(n+2)a_{n+2}+a_{n-1}=\frac{e}{n!}$$

$$\boxed{a_{n+2}=\frac{e}{(n+1)(n+2)\,n!}-\frac{1}{(n+1)(n+2)}a_{n-1}}$$

a_0

$$n=1 \ \rightarrow\ a_3=\frac{e}{6}-\frac{1}{6}a_0$$

$$n=4 \ \rightarrow\ a_6=\frac{e}{720}-\frac{1}{30}a_3=-\frac{11e}{720}+\frac{1}{180}a_0$$

a_1

$$n=2 \rightarrow\ a_4=\frac{e}{24}-\frac{1}{12}a_1$$

$$n=5 \rightarrow\ a_7=\frac{e}{5040}-\frac{1}{42}a_4=\frac{e}{1260}+\frac{1}{504}a_0$$

$a_2=\frac{e}{2}$

$$n=3 \ \rightarrow\ a_5=\frac{e}{120}-\frac{1}{20}a_2=\frac{e}{120}-\frac{e}{40}=-\frac{e}{60}$$

$$n=6 \ \rightarrow\ a_8=\frac{e}{40,320}-\frac{1}{56}a_5=\frac{e}{40,320}+\frac{e}{3360}=\frac{13e}{40,320}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y(x)=a_0+(x-1)a_1+\frac{e}{2}(x-1)^2+\left(\frac{e}{6}-\frac{1}{6}a_0\right)(x-1)^3+\left(\frac{e}{24}-\frac{1}{12}a_1\right)(x-1)^4-\frac{e}{60}(x-1)^5+\cdots$$

$$= a_0 + (x-1)a_1 + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + -\frac{1}{6}a_0(x-1)^3 + \frac{e}{24}(x-1)^4$$
$$-\frac{1}{12}a_1(x-1)^4 - \frac{e}{60}(x-1)^5 + \cdots$$

$$y(x) = e\left(\frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4 - \frac{1}{60}(x-1)^5 + \cdots\right)$$
$$+ a_0\left(1 - \frac{1}{6}(x-1)^3 + \cdots\right) + a_1\left((x-1) - \frac{1}{12}(x-1)^4 + \cdots\right)$$

Exercise

Find the series solution near the given value:

$$\left(-2x^2 + 4x + 2\right)y'' - (12x - 12)y' - 12y = 0 \ ; \quad \textit{near } x = 1$$

Solution

$$2 - \left(x^2 - 2x - 1\right)y'' - 12(x-1)y' - 12y = 0$$

$$\left((x-1)^2 - 2\right)y'' + 6(x-1)y' + 6y = 0$$

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}(x-1)^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n (x-1)^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}(x-1)^n$$

$$\left((x-1)^2 - 2\right)y'' + 6(x-1)y' + 6y = 0$$

$$(x-1)^2 y'' - 2y'' + 6(x-1)y' + 6y = 0$$

$$(x-1)^2 \sum_{n=2}^{\infty} n(n-1)a_n (x-1)^{n-2} - 2\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}(x-1)^n$$
$$+ 6(x-1)\sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + 6\sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^n - 2\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n + \sum_{n=1}^{\infty} 6n\,a_n(x-1)^n$$
$$+\sum_{n=0}^{\infty} 6a_n(x-1)^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^n + \sum_{n=1}^{\infty} 6n\,a_n(x-1)^n + \sum_{n=0}^{\infty}\left[6a_n - 2(n+1)(n+2)a_{n+2}\right](x-1)^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^n + 6a_1(x-1) + \sum_{n=2}^{\infty} 6n\,a_n(x-1)^n$$
$$+6a_0 - 4a_2 + \left(6a_1 - 12a_3\right)(x-1) + \sum_{n=2}^{\infty}\left[6a_n - 2(n+1)(n+2)a_{n+2}\right](x-1)^n = 0$$

$$6a_0 - 4a_2 + \left(12a_1 - 12a_3\right)(x-1)$$
$$+\sum_{n=2}^{\infty}\left[n(n-1)a_n + 6n\,a_n + 6a_n - 2(n+1)(n+2)a_{n+2}\right](x-1)^n = 0$$

$$6a_0 - 4a_2 + 12\left(a_1 - a_3\right)(x-1)$$
$$+\sum_{n=2}^{\infty}\left[\left(n^2+5n+6\right)a_n - 2(n+1)(n+2)a_{n+2}\right](x-1)^n = 0$$

$$6a_0 - 4a_2 = 0 \quad\Rightarrow\quad \boxed{a_2 = \tfrac{3}{2}a_0}$$

$$a_1 - a_3 = 0 \quad\Rightarrow\quad \boxed{a_3 = a_1}$$

$$\left(n^2+5n+6\right)a_n - 2(n+1)(n+2)a_{n+2} = 0$$

$$2(n+1)(n+2)a_{n+2} = (n+2)(n+3)a_n$$

$$a_{n+2} = \frac{1}{2}\,\frac{n+3}{n+1}\,a_n$$

$$a_2 = \tfrac{3}{2}a_0 \qquad\qquad a_3 = a_1$$

$n=2$ $\quad a_4=\frac{1}{2}\frac{5}{3}a_2$ $=\frac{5}{6}\left(\frac{3}{2}\right)a_0$ $=\frac{5}{2^2}a_0$	$n=3$ $\quad a_5=\frac{1}{2}\frac{6}{4}a_3$ $=\frac{3}{4}a_1$
$n=4$ $\quad a_6=\frac{1}{2}\frac{7}{5}a_4$ $=\frac{7}{10}\left(\frac{5}{2^2}\right)a_0$ $=\frac{7}{2^3}a_0$	$n=5$ $\quad a_7=\frac{1}{2}\frac{8}{6}a_5$ $=\frac{2}{3}\frac{3}{4}a_1$ $=\frac{1}{2}a_1$
$n=6$ $\quad a_8=\frac{1}{2}\frac{9}{7}a_6$ $=\frac{9}{14}\left(\frac{7}{2^3}\right)a_0$ $=\frac{9}{2^4}a_0$	$n=7$ $\quad a_9=\frac{1}{2}\frac{10}{8}a_7$ $=\frac{5}{2^3}\frac{1}{2}a_1$ $=\frac{5}{2^4}a_1$
$\vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots$ $\frac{3}{2^2} \quad \frac{1}{2}\frac{4}{4} \quad \frac{5}{2^4}$
$a_{2k}=\frac{2k+1}{2^k}a_0$	$a_{2k+1}=\frac{k+1}{2^k}a_1$

$$y(x)=a_0\sum_{k=0}^{\infty}\frac{2k+1}{2^k}(x-1)^{2k}+a_1\sum_{k=0}^{\infty}\frac{k+1}{2^k}(x-1)^{2k+1}$$

Exercise

Find the series solution near the given value

$$y''+xy'+(2x-1)y=0\;; \qquad \textit{near } x=-1 \qquad y(-1)=2, \qquad y'(-1)=-2$$

Solution

$$t=x+1 \quad\to\quad x=t-1$$

$$y=\sum_{n=0}^{\infty}a_n t^n$$

$$y'=\sum_{n=1}^{\infty}na_n t^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}t^n$$

$$y'' + xy' + (2x-1)y = 0$$

$$y'' + (t-1)y' + (2t-3)y = 0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}t^n + (t-1)\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n + (2t-3)\sum_{n=0}^{\infty}a_n t^n = 0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}t^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}t^{n+1} - \sum_{n=0}^{\infty}(n+1)a_{n+1}t^n$$
$$+\sum_{n=0}^{\infty}2a_n t^{n+1} - \sum_{n=0}^{\infty}3a_n t^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2} - (n+1)a_{n+1} - 3a_n\right]t^n + \sum_{n=0}^{\infty}\left[(n+1)a_{n+1} + 2a_n\right]t^{n+1} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2} - (n+1)a_{n+1} - 3a_n\right]t^n + \sum_{n=1}^{\infty}\left[na_n + 2a_{n-1}\right]t^n = 0$$

$$2a_2 - a_1 - 3a_0 + \sum_{n=1}^{\infty}\left[(n+1)(n+2)a_{n+2} - (n+1)a_{n+1} - 3a_n\right]t^n$$
$$+\sum_{n=1}^{\infty}\left(na_n + 2a_{n-1}\right)t^n = 0$$

$$2a_2 - a_1 - 3a_0 + \sum_{n=1}^{\infty}\left[(n+1)(n+2)a_{n+2} - (n+1)a_{n+1} + (n-3)a_n + 2a_{n-1}\right]t^n = 0$$

$$2a_2 - a_1 - 3a_0 = 0 \quad\rightarrow\quad a_2 = \frac{1}{2}\left(a_1 + 3a_0\right)$$

$$(n+1)(n+2)a_{n+2} - (n+1)a_{n+1} + (n-3)a_n + 2a_{n-1} = 0$$

$$a_{n+2} = \frac{1}{n+2}a_{n+1} - \frac{n-3}{(n+1)(n+2)}a_n - \frac{2}{(n+1)(n+2)}a_{n-1}$$

Given: $t = x+1$

$$y(x=-1)=y(t=0)=2=a_0, \qquad y'(x=-1)=y(t=0)=-2=a_1$$

$$\underline{\left| a_2 \right.}=\frac{1}{2}\left(a_1+3a_0\right)=\frac{1}{2}(-2+6)\ \underline{=2|}$$

$$n=1 \ \rightarrow\ a_3=\frac{1}{3}a_2+\frac{1}{3}a_1-\frac{1}{3}a_0$$

$$=\frac{2}{3}-\frac{2}{3}-\frac{2}{3}$$

$$=-\frac{2}{3}$$

$$n=2 \ \rightarrow\ a_4=\frac{1}{4}a_3+\frac{1}{12}a_2-\frac{1}{6}a_1$$

$$=-\frac{1}{6}+\frac{1}{6}+\frac{1}{3}$$

$$=\frac{1}{3}$$

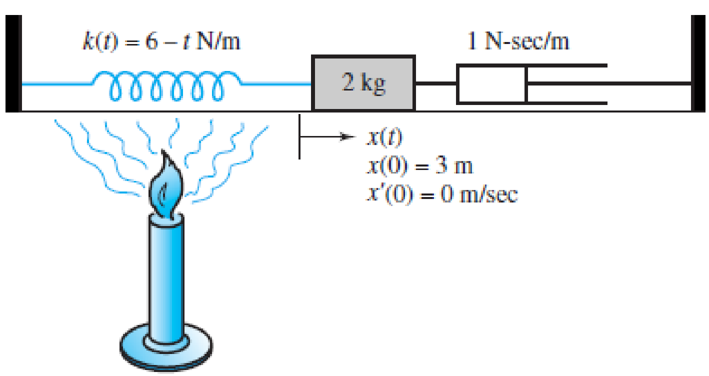
$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y(t)=2-2t+3t^2-\frac{1}{3}t^3+\frac{1}{3}t^4+\cdots$$

$$y(x)=2-2(x+1)+3(x+1)^2-\frac{1}{3}(x+1)^3+\frac{1}{3}(x+1)^4+\cdots$$

Exercise

As a spring is heated, its spring "constant" decreases. Suppose the spring is heated so that the spring "constant" at time *t* is $k(t)=6-t\ \ N/m$.



If the unforced mass-spring system has mass $m=2\ kg$ and a damping constant $b=1\ N\text{-}sec/m$ with initial conditions $x(0)=3\ m$ and $x'(0)=0\ m/sec$, then the displacement $x(t)$ is governed by the initial value problem

$$2x''(t)+x'(t)+(6-t)x(t)=0\ ; \quad x(0)=3, \quad x'(0)=0$$

Find at least the first four nonzero terms in a power series expansion about $t=0$ for the displacement.

Solution

$$x(t)=\sum_{n=0}^{\infty} a_n t^n$$

$$x'(t)=\sum_{n=1}^{\infty} n a_n t^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n$$

$$x''(t)=\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n$$

$$2x''+x'+(6-t)x=0$$

$$2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n+(6-t)\sum_{n=0}^{\infty}a_n t^n=0$$

$$\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}t^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n+\sum_{n=0}^{\infty}6a_n t^n-\sum_{n=0}^{\infty}a_n t^{n+1}=0$$

$$\sum_{n=0}^{\infty}\left[2(n+2)(n+1)a_{n+2}+(n+1)a_{n+1}+6a_n\right]t^n-\sum_{n=1}^{\infty}a_{n-1}t^n=0$$

$$4a_2+a_1+6a_0+\sum_{n=1}^{\infty}\left[2(n+2)(n+1)a_{n+2}+(n+1)a_{n+1}+6a_n\right]t^n-\sum_{n=1}^{\infty}a_{n-1}t^n=0$$

$$4a_2+a_1+6a_0+\sum_{n=1}^{\infty}\left[2(n+1)(n+2)a_{n+2}+(n+1)a_{n+1}+6a_n-a_{n-1}\right]t^n=0$$

Given: $x(0)=3=a_0,\quad x'(0)=0=a_1$

$$4a_2+a_1+6a_0=0 \quad\rightarrow\quad \boxed{a_2=-\frac{9}{2}}$$

$$2(n+1)(n+2)a_{n+2}+(n+1)a_{n+1}+6a_n-a_{n-1}=0$$

$$\boxed{a_{n+2}=\frac{a_{n-1}-6a_n-(n+1)a_{n+1}}{2(n+1)(n+2)}}$$

$$n = 1 \;\rightarrow\; a_3 = \frac{1}{12}\left(a_0 - 6a_1 - 2a_2\right) = \frac{1}{12}(3 + 9) = 1$$

$$n = 2 \;\rightarrow\; a_4 = \frac{1}{24}\left(a_1 - 6a_2 - 3a_3\right) = \frac{1}{24}(27 - 3) = 1$$

$$x(t) = 3 - \frac{9}{2}t^2 + t^3 + t^4 + \cdots$$

Solution Section 4.3 – Legendre's Equation

Exercise

Establish the recursion formula using the following two steps

a) Differentiate both sides of equation

$$g(t,\ x)=\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n \ \text{ with respect to } t \text{ to show that}$$

$$(x-t)\sum_{n=0}^{\infty}P_n(x)t^n=\left(1-2xt+t^2\right)\sum_{n=1}^{\infty}nP_n(x)t^{n-1}$$

b) Equate the coefficients of t^n in this equation to show that

$$P_1(x)=xP_0(x) \qquad \textit{and} \qquad (n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x) \quad \textit{for } \ n\ge 1$$

Solution

a) Let: $\left(1-2xt+t^2\right)^{-1/2}=\sum\limits_{n=0}^{\infty}P_n(x)t^n$

Differentiate both sides with respect to t:

$$\left(\left(1-2xt+t^2\right)^{-1/2}\right)'=\left(\sum_{n=0}^{\infty}P_n(x)t^n\right)'$$

$$-\frac{1}{2}(-2x+2t)\left(1-2xt+t^2\right)^{-3/2}=\sum_{n=1}^{\infty}nP_n(x)t^{n-1}$$

$$(x-t)\left(1-2xt+t^2\right)^{-1/2}=\left(1-2xt+t^2\right)\sum_{n=1}^{\infty}nP_n(x)t^{n-1}$$

Multiply both sides by: $1-2xt+t^2$

$$(x-t)\sum_{n=0}^{\infty}P_n(x)t^n=\left(1-2xt+t^2\right)\sum_{n=1}^{\infty}nP_n(x)t^{n-1}$$

b)

$$(x-t)\sum_{n=0}^{\infty}P_n(x)t^n=\sum_{n=0}^{\infty}xP_n(x)t^n-\underbrace{\sum_{n=0}^{\infty}P_n(x)t^{n+1}}_{n=n+1}$$

$$= \sum_{n=0}^{\infty} xP_n(x)t^n - \sum_{n=1}^{\infty} P_{n-1}(x)t^n$$

$$\left(1-2xt+t^2\right)\sum_{n=1}^{\infty} nP_n(x)t^{n-1} = \sum_{n=1}^{\infty} nP_n(x)t^{n-1} - \sum_{n=1}^{\infty} 2xnP_n(x)t^n + \sum_{n=1}^{\infty} nP_n(x)t^{n+1}$$

$$= \sum_{n=0}^{\infty}(n+1)P_{n+1}(x)t^n - \sum_{n=1}^{\infty} 2nxP_n(x)t^n + \sum_{n=2}^{\infty}(n-1)P_{n-1}(x)t^n$$

Thus,

$$\sum_{n=0}^{\infty} xP_n(x)t^n - \sum_{n=1}^{\infty} P_{n-1}(x)t^n = \sum_{n=0}^{\infty}(n+1)P_{n+1}(x)t^n - \sum_{n=1}^{\infty} 2nxP_n(x)t^n + \sum_{n=2}^{\infty}(n-1)P_{n-1}(x)t^n$$

Therefore;

$$0 = \left[xP_0(x) - P_1(x)\right]t^0 + \left[xP_1(x) - P_0(x) - 2P_2(x) + 2xP_1(x)\right]t^1 + \sum_{n=0}^{\infty}\left[xP_n(x) - P_{n-1}(x) - (n+1)P_{n+1}(x) + 2nxP_n(x) - (n-1)P_{n-1}(x)\right]t^n$$

$$0 = \left[xP_0(x) - P_1(x)\right]t^0 + \left[3xP_1(x) - P_0(x) - 2P_2(x)\right]t^1 + \sum_{n=0}^{\infty}\left[(2n+1)xP_n(x) - nP_{n-1}(x) - (n+1)P_{n+1}(x)\right]t^n$$

That implies:

$$xP_0(x) - P_1(x) = 0 \quad \Rightarrow \quad P_1(x) = xP_0(x)$$

$$3xP_1(x) - P_0(x) - 2P_2(x) = 0 \quad \Rightarrow \quad 2P_2(x) = P_0(x) - 3xP_1(x)$$

$$(2n+1)xP_n(x) - nP_{n-1}(x) - (n+1)P_{n+1}(x) = 0$$

$$\Rightarrow \quad (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

If $n = 1$ then: $2P_2(x) = 3xP_1(x) - P_0(x)$ √

Exercise

Show that $P_{2n+1}(0)=0 \quad \text{and} \quad P_{2n}(0)=(-1)^n \dfrac{2n!}{2^{2n}(n!)^2}$

Solution

Given the formula:

$$nP_n(x)=(2n-1)xP_{n-1}(x)-(n-1)P_{n-2}(x) \qquad \text{for } n\ge 2$$

By letting $x=0$, then the formula can be:

$$nP_n(0)=-(n-1)P_{n-2}(0)$$

Replacing n with $2n$, then

$$2nP_{2n}(0)=-(2n-1)P_{2n-2}(0)$$

$$P_{2n}(0)=\frac{1-2n}{2n}P_{2n-2}(0)$$

$$P_2(0)=\frac{1-2}{2}P_0(0)=-\frac{1}{2}P_0(0)$$

$$P_4(0)=\frac{1-4}{4}P_2(0)=\frac{1-2}{2}\cdot\frac{1-4}{4}P_0(0)=\frac{1\cdot 3}{2^2\cdot 1\cdot 2}P_0(0)$$

$$P_6(0)=\frac{1-6}{6}P_4(0)=\frac{1-2}{2}\cdot\frac{1-4}{4}\cdot\frac{1-6}{6}P_0(0)=-\frac{1\cdot 3\cdot 5}{2^3\cdot 1\cdot 2\cdot 3}P_0(0)$$

$$\vdots \qquad \vdots \quad \vdots$$

$$P_{2n}(0)=\frac{1-2}{2}\cdot\frac{1-4}{4}\cdot\frac{1-6}{6}\cdots\frac{1-2n}{2n}P_0(0)$$

$$=(-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\cdot 1\cdot 2\cdot 3\cdots n}P_0(0)$$

$$1\cdot 3\cdot 5\cdots(2n-1)=\frac{1\cdot 2\cdot 3\cdot 4\cdots(2n-1)(2n)}{2\cdot 4\cdot 6\cdots(2n)}$$

$$=\frac{(2n)!}{2^n n!}$$

$$=(-1)^n\frac{(2n)!}{2^n\cdot(n!)^2}P_0(0)$$

With $P_0(0)=1$

$$\boxed{P_{2n}(0)=(-1)^n\frac{(2n)!}{2^n\cdot(n!)^2}}$$

Exercise

Show that $P_n'(0) = (-1)^{n+1} P_n'(-1) = \frac{n(n+1)}{2}$

Hint: Use Legendre's equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$

Solution

Because $P_n(x)$ is a solution of Legendre's equation, then

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$$

Let $x=1$, then

$$-2P_n'(1) + n(n+1)P_n(1) = 0$$

$$P_n'(1) = \frac{n(n+1)}{2}P_n(1)$$

Let $x=-1$, then

$$2P_n'(-1) + n(n+1)P_n(-1) = 0$$

$$P_n'(-1) = -\frac{n(n+1)}{2}P_n(-1)$$

However, $P_n(1) = P_n(-1) = 1$

$$\boxed{(-1)^{n+1} P_n'(-1) = \frac{n(n+1)}{2}}$$

Exercise

The differential equation $y'' + xy = 0$ is called ***Airy's equation***, and its solutions are called ***Airy functions***. Find the series for the solutions y_1 and y_2 where $y_1(0)=1$ and $y_1'(0)=0$, while $y_2(0)=0$ and $y_2'(0)=1$. What is the radius of convergence for these two series?

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$y'' + xy = 0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty}a_{n-1} x^{n} = 0$$

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty}a_{n-1} x^{n} = 0$$

$$2a_2 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + a_{n-1}\right]x^n = 0$$

$$2a_2 = 0 \quad or \quad (n+2)(n+1)a_{n+2} + a_{n-1} = 0$$

$$a_2 = 0 \quad or \quad a_{n+2} = \frac{-a_{n-1}}{(n+2)(n+1)} \qquad n \ge 1$$

$$a_3 = -\frac{a_0}{3\cdot 2} \qquad\qquad a_4 = -\frac{a_1}{4\cdot 3}$$

$$a_6 = -\frac{a_3}{6\cdot 5} = \frac{a_0}{6\cdot 5\cdot 3\cdot 2} \qquad\qquad a_7 = -\frac{a_4}{7\cdot 6} = \frac{a_1}{7\cdot 6\cdot 4\cdot 3}$$

$$a_9 = -\frac{a_6}{9\cdot 8} = -\frac{a_0}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2} \qquad\qquad a_{10} = -\frac{a_7}{10\cdot 9} = -\frac{a_1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}$$

$$\vdots \quad \vdots \qquad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \qquad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots$$

$$a_5 = -\frac{a_2}{5\cdot 4} = 0$$

$$a_8 = -\frac{a_5}{8\cdot 7} = 0$$

$$a_{11} = 0$$

$$a_{3n} = \frac{(-1)^n a_0}{(2\cdot 3)\cdot(5\cdot 6)\cdot\cdots\cdot(3n-1)(3n)} \qquad a_{3n+1} = \frac{(-1)^n a_1}{(3\cdot 4)\cdot(6\cdot 7)\cdot\cdots\cdot(3n)(3n+1)} \qquad a_{3n+2} = 0$$

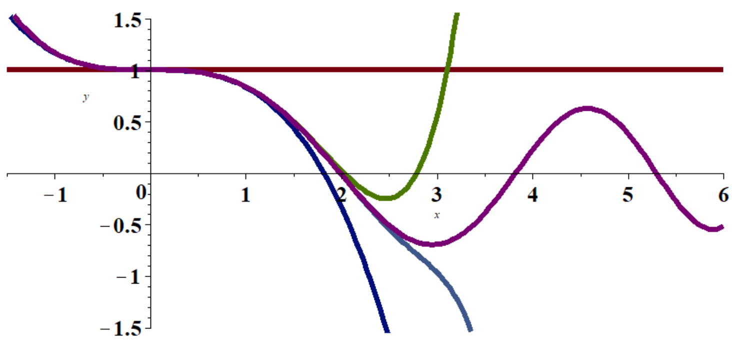
$$y(x) = a_0\left[1 - \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6}x^6 - \cdots\right] + a_1\left[x - \frac{1}{3\cdot 4}x^4 + \frac{1}{3\cdot 4\cdot 6\cdot 7}x^7 - \cdots\right]$$

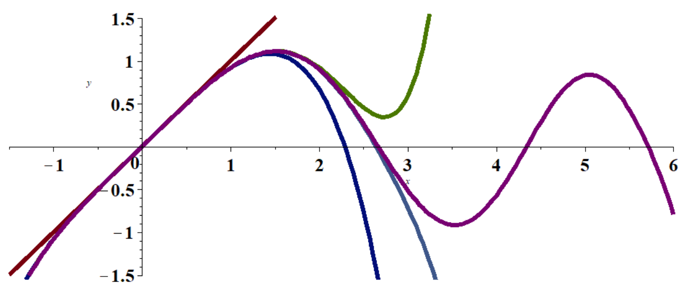
$$y_1(x) = 1 - \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6}x^6 - \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{3n}}{(2\cdot 3)\cdot(5\cdot 6)\cdot\cdots(3n-1)(3n)}$$

$$y_2(x) = x - \frac{1}{3\cdot 4}x^4 + \frac{1}{3\cdot 4\cdot 6\cdot 7}x^7 - \cdots$$

$$= x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{3n+1}}{(3\cdot 4)\cdot(6\cdot 7)\cdot\cdots(3n)(3n+1)}$$





Exercise

The Hermite equation of order α is $\quad y'' - 2xy' + 2\alpha y = 0$

a) Find the general solution is $y(x) = a_0 y_1(x) + a_1 y_2(x)$

Show that $y_1(x)$ is a polynomial if α is an even integer, whereas $y_2(x)$ is a polynomial if α is an odd integer.

b) When $\alpha = n$, use $y_1(x)$ to find polynomial solutions for $n = 0$, $n = 2$, and $n = 4$, then use $y_2(x)$ to find polynomial solutions for $n = 1$, $n = 3$, and $n = 5$.

c) The Hermite polynomial of degree *n* is denoted by $H_n(x)$. It is the *n*th-degree polynomial solution of Hermite's equation, multiplied by a suitable constant so that the coefficient of x^n is 2^n. Use part (b) to show the first six Hermite polynomials are

$$H_0(x) = 1$$
$$H_1(x) = 2x$$
$$H_2(x) = 4x^2 - 2$$
$$H_3(x) = 8x^3 - 12x$$
$$H_4(x) = 16x^4 - 48x^2 + 12$$
$$H_5(x) = 32x^5 - 160x^3 + 120x$$

d) A general formula for the Hermite polynomials is

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

Verify that this formula does in fact give an *n*th-degree polynomial.

Solution

a) $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$y'' - 2xy' + 2\alpha y = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 2\alpha \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} 2\alpha a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 2\alpha a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}-2(n-\alpha)a_n\right]x^n=0$$

$$(n+1)(n+2)a_{n+2}-2(n-\alpha)a_n=0$$

$$\boxed{a_{n+2}=\frac{2(n-\alpha)}{(n+1)(n+2)}a_n}$$

a_0

$$n=0\ \rightarrow\ a_2=-\frac{2\alpha}{2}a_0$$

$$n=2\ \rightarrow\ a_4=\frac{2(2-\alpha)}{3\cdot 4}a_2=-\frac{2^2\alpha(2-\alpha)}{4!}a_0$$

$$n=4\ \rightarrow\ a_6=\frac{2(4-\alpha)}{5\cdot 6}a_4=-\frac{2^3\alpha(2-\alpha)(4-\alpha)}{6!}a_0$$

$$\vdots\quad\vdots\quad\vdots\quad\vdots$$

$$y_1(x)=1-\frac{2\alpha}{2!}x^2-\frac{2^2(2-\alpha)}{4!}x^4-\frac{2^3\alpha(2-\alpha)(4-\alpha)}{6!}x^6-\cdots$$

$$=1-\frac{2\alpha}{2!}x^2+\frac{2^2(\alpha-2)}{4!}x^4-\frac{2^3\alpha(\alpha-2)(\alpha-4)}{6!}x^6+\cdots$$

a_1

$$n=1\ \rightarrow\ a_3=\frac{2(1-\alpha)}{6}a_1=\frac{2(1-\alpha)}{3!}a_1$$

$$n=3\ \rightarrow\ a_5=\frac{2(3-\alpha)}{4\cdot 5}a_3=\frac{2^2(1-\alpha)(3-\alpha)}{5!}a_1$$

$$n=5\ \rightarrow\ a_7=\frac{2(3-\alpha)}{6\cdot 7}a_5=\frac{2^3(1-\alpha)(3-\alpha)(5-\alpha)}{7!}a_1$$

$$\vdots\quad\vdots\quad\vdots\quad\vdots$$

$$y_2(x)=x+\frac{2(1-\alpha)}{3!}x^3+\frac{2^2(1-\alpha)(3-\alpha)}{5!}x^5+\frac{2^3(1-\alpha)(3-\alpha)(5-\alpha)}{7!}x^7+\cdots$$

$$=x-\frac{2(\alpha-1)}{3!}x^3+\frac{2^2(\alpha-1)(\alpha-3)}{5!}x^5-\frac{2^3(\alpha-1)(\alpha-3)(\alpha-5)}{7!}x^7+\cdots$$

$$y(x)=a_0y_1(x)+a_1y_2(x)$$

$$= a_0\left(1-\frac{2\alpha}{2!}x^2+\frac{2^2\alpha(\alpha-2)}{4!}x^4-\frac{2^3\alpha(\alpha-2)(\alpha-4)}{6!}x^6+\cdots\right)$$

$$+a_1\left(x-\frac{2(\alpha-1)}{3!}x^3+\frac{2^2(\alpha-1)(\alpha-3)}{5!}x^5-\frac{2^3(\alpha-1)(\alpha-3)(\alpha-5)}{7!}x^7+\cdots\right)$$

$$= a_0 + a_0\sum_{m=1}^{\infty}(-1)^m\frac{2^m\alpha(\alpha-2)\cdots(\alpha-2m+2)}{(2m)!}x^{2m}$$

$$+a_1x+a_1\sum_{m=1}^{\infty}(-1)^m\frac{2^m(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)!}x^{2m+1}$$

$$\boxed{y_1(x)=1+\sum_{m=1}^{\infty}(-1)^m\frac{2^m\alpha(\alpha-2)\cdots(\alpha-2m+2)}{(2m)!}x^{2m}}$$

$$\boxed{y_2(x)=x+\sum_{m=1}^{\infty}(-1)^m\frac{2^m(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)!}x^{2m+1}}$$

b) $n=\alpha=0 \quad\rightarrow\quad y_1(x)\ \boxed{=1}$

$n=\alpha=2 \quad\rightarrow\quad y_1(x)=1-\alpha x^2\ \boxed{=1-2x^2}$

$n=\alpha=4 \quad\rightarrow\quad y_1(x)=1-\alpha x^2+\frac{\alpha(\alpha-2)}{6}x^4$

$$\boxed{=1-4x^2+\frac{4}{3}x^4}$$

$n=\alpha=1 \quad\rightarrow\quad y_2(x)\ \boxed{=x}$

$n=\alpha=3 \quad\rightarrow\quad y_2(x)=x-\frac{2(\alpha-1)}{3!}x^3\ \boxed{=x-\frac{2}{3}x^3}$

$n=\alpha=5 \quad\rightarrow\quad y_2(x)=x-\frac{2(\alpha-1)}{3!}x^3+\frac{2^2(\alpha-1)(\alpha-3)}{5!}x^5$

$$\boxed{y_2(x)=x-\frac{4}{3}x^3+\frac{4}{15}x^5}$$

c) $H_0(x)=2^0\cdot 1=1$

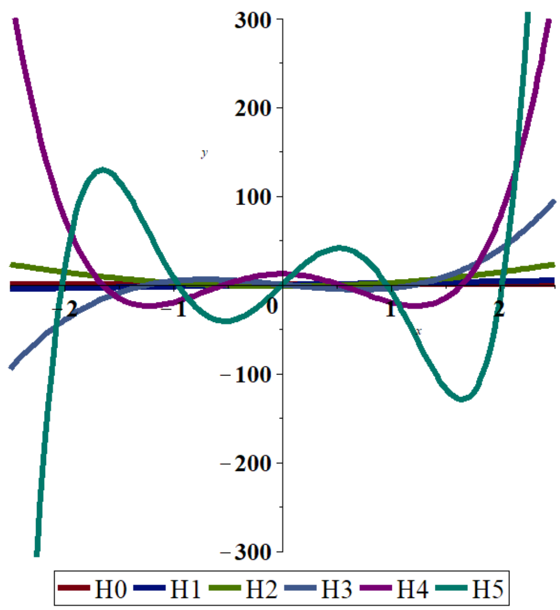
$H_1(x)=2^1\cdot x=2x$

$$H_2(x) = -2\left(1-2x^2\right) = 4x^2 - 2$$

$$H_3(x) = -2^2 \cdot 3\left(x - \frac{2}{3}x^3\right) = 8x^3 - 12x$$

$$H_4(x) = 2^2 \cdot 3\left(1 - 4x^2 + \frac{4}{3}x^4\right) = 16x^4 - 48x^2 + 12$$

$$H_5(x) = 2^3 \cdot 3 \cdot 5\left(\frac{4}{15}x^5 - \frac{4}{3}x^3 + x\right) = 32x^5 - 160x^3 + 120x$$



d) $\dfrac{d}{dx}\left(e^{-x^2}\right) = -2xe^{-x^2}$

$$\frac{d^2}{dx^2}\left(e^{-x^2}\right) = -2\frac{d}{dx}\left(xe^{-x^2}\right) = -2\left(1-2x^2\right)e^{-x^2}$$

$$\frac{d^3}{dx^3}\left(e^{-x^2}\right) = 2\frac{d}{dx}\left(\left(2x^2-1\right)e^{-x^2}\right) = 2\left(4x - 4x^3 + 2x\right)e^{-x^2} = \left(12x - 8x^3\right)e^{-x^2}$$

$$\frac{d^4}{dx^4}\left(e^{-x^2}\right) = 4\frac{d}{dx}\left(\left(3x-2x^3\right)e^{-x^2}\right) = 4\left(3 - 6x^2 - 6x^2 + 4x^4\right)e^{-x^2} = \left(16x^4 - 48x^2 + 12\right)e^{-x^2}$$

$$H_1(x) = -e^{x^2}\frac{d}{dx}\left(e^{-x^2}\right) = 2xe^{x^2}e^{-x^2} = 2x \quad \checkmark$$

$$H_2(x) = e^{x^2}\frac{d^2}{dx^2}\left(e^{-x^2}\right) = -2e^{x^2}\left(1-2x^2\right)e^{-x^2} = 4x^2 - 2 \quad \checkmark$$

$$H_3(x) = e^{x^2}\frac{d^3}{dx^3}\left(e^{-x^2}\right) = e^{x^2}\left(12x - 8x^3\right)e^{-x^2} = 12x - 8x^3 \quad \checkmark$$

$$H_4(x) = e^{x^2}\frac{d^4}{dx^4}\left(e^{-x^2}\right) = e^{x^2}\left(16x^4 - 48x^2 + 12x\right)e^{-x^2} = 16x^4 - 48x^2 + 12 \quad \checkmark$$

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

Exercise

Rodrigues's Formula is given by: $\quad P_n(x) = \dfrac{1}{n!\ 2^n}\dfrac{d^n}{dx^n}\left(x^2-1\right)^n$

For the nth-degree Legendre polynomial.

a) Show that $v = \left(x^2-1\right)^n$ satisfies the differential equation $\left(1-x^2\right)v' + 2nxv = 0$

Differentiate each side of this equation to obtain

$$\left(1-x^2\right)v'' + 2(n-1)xv' + 2nv = 0$$

b) Differentiate each side of the last equation n times in succession to obtain

$$\left(1-x^2\right)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} = 0$$

which satisfies Legendre's equation of order n.

c) Show that the coefficient of x^n in u is $\dfrac{(2n)!}{n!}$; then state why this proves Rodrigues.

Formula.

Note: That the coefficient of x^n in $P_n(x)$ is $\dfrac{(2n)!}{2^n(n!)^2}$

$$u = v^{(n)} = D^n\left(x^2-1\right)^n$$

Solution

a) $v = \left(x^2-1\right)^n \quad \rightarrow \quad v' = 2nx\left(x^2-1\right)^{n-1}$

$$v' = 2nx\left(x^2-1\right)^{n-1}$$

$$\begin{aligned}\left(1-x^2\right)v' + 2nxv &= -2nx\left(x^2-1\right)\left(x^2-1\right)^{n-1} + 2nx\left(x^2-1\right)^n \\ &= -2nx\left(x^2-1\right)^n + 2nx\left(x^2-1\right)^n \\ &= 0\ \rfloor\end{aligned}$$

$$\frac{d}{dx}\left(\left(1-x^2\right)v'+2nxv\right)=0$$

$$\left(1-x^2\right)v''-2xv'+2nxv'+2nv=0$$

$$\boxed{\left(1-x^2\right)v''+2(n-1)xv'+2nv=0}$$

b) $$\frac{d}{dx}\left(\left(1-x^2\right)v''-2xv'+2nxv'+2nv\right)=0$$

$$\left(1-x^2\right)v^{(3)}-2xv''-2v'-2xv''+2nxv''+2nv'+2nv'$$

$$\boxed{\left(1-x^2\right)v^{(3)}+2(n-2)xv''+2(2n-1)v'=0}$$

$n=1 \quad \rightarrow \quad \left(1-x^2\right)v^{(3)}-2xv''+2v'=0$

$$\boxed{\left(1-x^2\right)v^{(n+2)}-2xv^{(n+1)}+n(n+1)v^{(n)}=0} \quad \checkmark$$

$$\frac{d}{dx}\left(\left(1-x^2\right)v^{(3)}+2x(n-2)v''+2(2n-1)v'\right)=0$$

$$\left(1-x^2\right)v^{(4)}-2xv^{(3)}+2x(n-2)v^{(3)}+2(n-2)v''+2(2n-1)v''=0$$

$$\left(1-x^2\right)v^{(4)}+2x(n-3)v^{(3)}+6(n-1)v''=0$$

$$\boxed{\left(1-x^2\right)v^{(4)}+2(n-3)xv^{(3)}+3(2n-2)v''=0}$$

$n=2 \quad \rightarrow \quad \left(1-x^2\right)v^{(4)}-2xv^{(3)}+3\cdot 2v''=0$

$$\boxed{\left(1-x^2\right)v^{(n+2)}-2xv^{(n+1)}+n(n+1)v^{(n)}=0} \quad \checkmark$$

$$\frac{d}{dx}\left(\left(1-x^2\right)v^{(4)}+2(n-3)xv^{(3)}+3(2n-2)v''\right)=0$$

$$\left(1-x^2\right)v^{(5)}-2xv^{(4)}+2(n-3)xv^{(4)}+2(n-3)v^{(3)}+3(2n-2)v^{(3)}=0$$

$$\left(1-x^2\right)v^{(5)}+(2n-6-2)xv^{(4)}+(2n-6+6n-6)v^{(3)}=0$$

$$\left(1-x^2\right)v^{(5)}+(2n-8)xv^{(4)}+(8n-12)v^{(3)}=0$$

$$\boxed{\left(1-x^2\right)v^{(5)}+2(n-4)xv^{(4)}+4(2n-3)v^{(3)}=0}$$

$n=3 \quad \rightarrow \quad \left(1-x^2\right)v^{(5)}-2xv^{(4)}+4\cdot 3v^{(3)}=0$

$$\boxed{\left(1-x^2\right)v^{(n+2)}-2xv^{(n+1)}+n(n+1)v^{(n)}=0} \quad \checkmark$$

After m differentiations:

$$\left(1-x^2\right)v^{(m+2)}+2\left(n-(m+1)\right)xv^{(m+1)}+(m+1)(2n-m)v^{(m)}=0$$

If we let $m=n$, then

$$\left(1-x^2\right)v^{(n+2)}-2xv^{(n+1)}+n(n+1)v^{(n)}=0$$

Let assume that $\left(1-x^2\right)v^{(n+2)}-2xv^{(n+1)}+n(n+1)v^{(n)}=0$ is true.

We need to prove that next derivative is also true.

$$\frac{d}{dx}\left(\left(1-x^2\right)v^{(n+2)}-2xv^{(n+1)}+n(n+1)v^{(n)}\right)=0$$

$$\left(1-x^2\right)v^{(n+3)}-2xv^{(n+2)}-2v^{(n+1)}-2xv^{(n+2)}+(2n-n)(n+1)v^{(n+1)}=0$$

$$\left(1-x^2\right)v^{(n+3)}-4xv^{(n+2)}+(2n-n-2)(n+1)v^{(n+1)}=0$$

$$\left(1-x^2\right)v^{(n+3)}-2(2)xv^{(n+2)}+(n-1)(n+2)v^{(n+1)}=0$$

$$\left(1-x^2\right)v^{(n+3)}+2(n-n-2)xv^{(n+2)}+(n-1)(n+2)v^{(n+1)}=0$$

$$\left(1-x^2\right)v^{(n+3)}+2\left(n-(n+2)\right)xv^{(n+2)}+\left(2n-(n+1)\right)\left((n+1)+1\right)v^{(n+1)}=0$$

If we let $m=n+1$, then

$$\left(1-x^2\right)v^{(m+2)}+2\left(n-(m+1)\right)xv^{(m+1)}+(2n-m)(m+1)v^{(m)}=0$$

If we let $m=n$, then

$$\left(1-x^2\right)v^{(n+2)}+2(n-n-1)xv^{(m+1)}+(2n-n)(n+1)v^{(n)}=0$$

$$\left(1-x^2\right)v^{(n+2)}-2xv^{(n+1)}+n(n+1)v^{(n)}=0$$ ✓

c) $u=v^{(n)}=D^n\left(x^2-1\right)^n$

$$=\frac{d^n}{dx^n}\left(x^{2n}-nx^{2n-1}+\cdots-1\right)$$

$$=2n(2n-1)\cdots\left(2n-(n-1)\right)x^n-\frac{d^n}{dx^n}\left(nx^{2n-1}+\cdots-1\right)$$

$$=\frac{(2n)!}{n!}x^n-\frac{d^n}{dx^n}\left(nx^{2n-1}+\cdots-1\right)$$

Since $u = v^{(n)}$ satisfies Legendre's equation of order n, $\dfrac{u}{2^n n!}$

From the notes:

The solution of Legendre polynomial of degree ***n*** is

$$P_n(x) = \sum_{k=0}^{K} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}$$

For the highest order, the result when:

$$k = 0 \quad \rightarrow \quad P_n(x) = \frac{(2n)!}{2^n (n!)^2} x^n$$

$$\frac{u}{2^n n!} = \frac{(2n)!}{2^n (n!)^2} x^n + \cdots$$

$$P_n(x) = \frac{1}{n!\, 2^n} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n$$

Solution Section 4.4 – Solution about Singular Points

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x^2y'' + 3y' - xy = 0$$

Solution

$$y'' + \frac{3}{x^2}y' - \frac{x}{x^2}y = 0$$

$$P(x) = \frac{3}{x^2} \qquad Q(x) = -\frac{x}{x^2}$$

For $P(x) = \frac{3}{x^2} \quad \to \quad \underline{x = 0}$

$\therefore\ p(x)$ is analytic except at $\underline{x = 0}$

For $Q(x) = -\frac{x}{x^2} = -\frac{1}{x}$

$\therefore\ q(x)$ is not analytic at $\underline{x = 0}$

The singular point is: $\underline{x = 0}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^2 + x\right)y'' + 3y' - 6xy = 0$$

Solution

$$y'' + \frac{3}{x(x+1)}y' - \frac{6}{x+1}y = 0$$

$$P(x) = \frac{3}{x(x+1)} \qquad Q(x) = -\frac{6x}{x(x+1)}$$

For $P(x) = \frac{3}{x(x+1)} \quad \to \quad \underline{x = 0, -1}$

$\therefore\ p(x)$ is analytic except at $\underline{x = 0, -1}$

For $q(x) = -\frac{6x}{x(x+1)} \quad \to \quad x = 0,\ -1$

$q(x) = -\frac{6x}{x(x+1)} = -\frac{6}{x+1}$; is actually analytic at $x = 0$

$\therefore\ q(x)$ is analytic except at $\underline{x=-1}$

The singular points are: $\underline{x=0,\ -1}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^2-1\right)y''+(1-x)y'+\left(x^2-2x+1\right)y=0$$

Solution

$$(x-1)(x+1)y''+(1-x)y'+(x-1)^2 y=0$$

$$y''-\frac{1}{x+1}y'+\frac{x-1}{x+1}y=0$$

$$P(x)=\frac{1-x}{x^2-1} \qquad Q(x)=\frac{(x-1)^2}{x^2-1}$$

For $p(x)=\dfrac{1-x}{(x-1)(x+1)} \quad \rightarrow \quad \underline{x=-1,\ 1}$

$p(x)=\dfrac{1-x}{(x-1)(x+1)}=-\dfrac{1}{x+1}$; is actually analytic at $x=1$

$\therefore\ p(x)$ is analytic except at $\underline{x=-1}$

For $q(x)=\dfrac{(x-1)^2}{(x-1)(x+1)} \quad \rightarrow \quad \underline{x=-1,\ 1}$

$q(x)=\dfrac{(x-1)^2}{(x-1)(x+1)}=\dfrac{x-1}{x+1}$; is actually analytic at $x=1$

$\therefore\ q(x)$ is analytic except at $\underline{x=-1}$

The singular point is: $\underline{x=-1}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$e^x y''-\left(x^2-1\right)y'+2xy=0$$

Solution

$$y''-\frac{x^2-1}{e^x}y'+\frac{2x}{e^x}y=0$$

$$P(x) = -\frac{x^2 - 1}{e^x} \qquad Q(x) = \frac{2x}{e^x}$$

Since $e^x \neq 0$, there are ***no*** singular points.

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\ln(x-1)y'' + (\sin 2x)y' - e^x y = 0$$

Solution

$$y'' + \frac{\sin 2x}{\ln(x-1)}y' - \frac{e^x}{\ln(x-1)}y = 0$$

$$P(x) = \frac{\sin 2x}{\ln(x-1)} \qquad Q(x) = -\frac{e^x}{\ln(x-1)}$$

$$\ln(x-1) = 0$$

$$x - 1 = 1$$

$$\underline{x = 2|}$$

The singular point is: $\underline{x \le 1,\ x = 2|}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$xy'' + x(1-x)^{-1}y' + (\sin x)y = 0$$

Solution

$$y'' + \frac{x}{x(1-x)}y' + \frac{\sin x}{x}y = 0$$

$$P(x) = \frac{x}{x(1-x)} \qquad Q(x) = \frac{\sin x}{x}$$

For $p(x) = \frac{x}{x(1-x)} \quad \rightarrow \quad \underline{x = 0,\ 1|}$

$p(x) = \frac{x}{x(1-x)} = \frac{1}{1-x}$; is actually analytic at $x = 0$

$\therefore\ p(x)$ is analytic except at $\underline{x = 1|}$

For $q(x) = \frac{\sin x}{x}$

$$=\frac{x-\frac{1}{3!}x^3+\cdots}{x}$$

$$=1-\frac{1}{3!}x^2+\cdots \quad \text{is analytic everywhere } (x=0 \text{ is removable}).$$

The only singular point is $\underline{x=1}\rfloor$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x^3y''+4x^2y+3y=0$$

Solution

$$y''+\frac{4}{x}y+\frac{3}{x^3}y=0$$

$$P(x)=\frac{4}{x} \qquad Q(x)=\frac{3}{x^3}$$

For $p(x)=x\frac{4}{x} \quad \rightarrow \quad \underline{x=0}\rfloor$

$\therefore\ p(x)$ is analytic at $x=0$

For $q(x)=x^2\frac{3}{x^3} \quad \rightarrow \quad \underline{x=0}\rfloor$

$\therefore\ q(x)$ is analytic except at $x=0$

The singular point: $x=0$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x(x+3)^2y''-y=0$$

Solution

$$y''-\frac{1}{x(x+3)^2}y=0$$

$$P(x)=0 \qquad Q(x)=-\frac{1}{x(x+3)^2}$$

For $q(x)=-\frac{1}{x(x+3)^2} \quad \rightarrow \quad \underline{x=0,\ -3}\rfloor,$

$q(x)$ is analytic elsewhere

The *Regular* singular points are $\underline{x = 0, \ -3}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^2-9\right)^2 y'' + (x+3)y' + 2y = 0$$

Solution

$$y'' + \frac{x+3}{\left(x^2-9\right)^2} y' + \frac{2}{\left(x^2-9\right)^2} y = 0$$

$$P(x) = \frac{x+3}{\left(x^2-9\right)^2} \qquad Q(x) = \frac{2}{\left(x^2-9\right)^2}$$

For $P(x) = \frac{x+3}{\left(x^2-9\right)^2} \quad \rightarrow \quad \underline{x = \pm 3}$

$$p(x) = \frac{x+3}{\left((x+3)(x-3)\right)^2}$$

$$= \frac{(x+3)(x-3)}{(x+3)(x-3)^2}; \qquad \text{is analytic at } x = -3$$

For $Q(x) = \frac{2\left(x^2-9\right)^2}{\left(x^2-9\right)^2} \quad \rightarrow \quad \underline{x = \pm 3}$

$\therefore \ q(x)$ is analytic at $\underline{x = \pm 3}$

The *Regular* singular point: $\underline{x = -3}$, and *Irregular* singular point: $\underline{x = 3}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$y'' - \frac{1}{x} y' + \frac{1}{(x-1)^3} y = 0$$

Solution

$$y'' - \frac{1}{x}y' + \frac{1}{(x-1)^3}y = 0$$

$$P(x) = -\frac{1}{x} \qquad Q(x) = \frac{1}{(x-1)^3}$$

For $P(x) = -\frac{1}{x} \quad \rightarrow \quad \underline{x = 0}\,|$

$p(x) = \frac{x}{x} = 1$ is analytic at $\underline{x = 0}\,|$

For $Q(x) = \frac{1}{(x-1)^3} \quad \rightarrow \quad \underline{x = 1}\,|$

$q(x) = \frac{(x-1)^2}{(x-1)^3} = \frac{1}{x-1}$ is *not* an analytic at $x = 1$

The *Regular* singular point: $\underline{x = 0}\,|$, and *Irregular* singular point: $\underline{x = 1}\,|$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^3 + 4x\right)y'' - 2xy' + 6y = 0$$

Solution

$$y'' - \frac{2x}{x\left(x^2+4\right)}y' + \frac{6}{x\left(x^2+4\right)}y = 0$$

$$P(x) = -\frac{2x}{x\left(x^2+4\right)} \qquad Q(x) = \frac{6}{x\left(x^2+4\right)}$$

For $P(x) = -\frac{2x}{x\left(x^2+4\right)} \quad \rightarrow \quad x = 0, \ \pm 2i$

$p(x) = -\frac{2}{x^2+4}$ is analytic at $x = \pm 2i$

For $Q(x) = \frac{6}{x\left(x^2+4\right)} \quad \rightarrow \quad x = 0, \ \pm 2i$ is analytic

The *Regular* singular points: $\underline{x = 0, \ \pm 2i}\,|$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x^2(x-5)^2 y'' + 4xy' + \left(x^2-25\right)y = 0$$

Solution

$$y'' + \frac{4x}{x^2(x-5)^2}y' + \frac{x^2-25}{x^2(x-5)^2}y = 0$$

$$P(x) = \frac{4x}{x^2(x-5)^2} \qquad Q(x) = \frac{x^2-25}{x^2(x-5)^2}$$

For $P(x) = \dfrac{4x}{x^2(x-5)^2} \quad \to \quad x = 0,\ 5$

$p(x) = \dfrac{4x}{x^2(x-5)^2} = x(x-5)\dfrac{4}{x(x-5)^2}$ is not an analytic at $x=5$

For $Q(x) = \dfrac{(x-5)(x+5)}{x^2(x-5)^2} \quad \to \quad x = 0,\ 5$

$q(x) = x^2(x-5)^2\dfrac{x+5}{x^2(x-5)}$ is an analytic at $x = 0,\ 5$

The *Regular* singular point: $\underline{x=0}\,|$, and *Irregular* singular point: $\underline{x=5}\,|$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^2+x-6\right)y'' + (x+3)y' + (x-2)y = 0$$

Solution

$$y'' + \frac{x+3}{x^2+x-6}y' + \frac{x-2}{x^2+x-6}y = 0$$

$$P(x) = \frac{x+3}{(x+3)(x-2)} \qquad Q(x) = \frac{x-2}{(x+3)(x-2)}$$

For $P(x) = \dfrac{x+3}{(x+3)(x-2)} \quad \to \quad x = -3,\ 2$

$p(x) = \dfrac{1}{x-2}$ is an analytic at $x = 2$

For $Q(x) = \dfrac{x-2}{(x+3)(x-2)} \quad \to \quad x = -3,\ 2$

$$q(x)=\frac{1}{x+3} \text{ is an analytic at } x=-3$$

The *Regular* singular points: $\underline{x=-3,\ 2}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x\left(x^2+1\right)^2 y''+y=0$$

Solution

$$y''+\frac{1}{x\left(x^2+1\right)^2}y=0$$

$$P(x)=0 \qquad Q(x)=\frac{1}{x\left(x^2+1\right)^2}$$

For $Q(x)=\dfrac{1}{x\left(x^2+1\right)^2} \quad\to\quad x=0,\ \pm i$

$$q(x)=x^2\left(x^2+1\right)^2\frac{1}{x\left(x^2+1\right)^2} \text{ is an analytic at } x=0,\ \pm i$$

The *Regular* singular points: $\underline{x=0,\ \pm i}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x^3\left(x^2-25\right)(x-2)^2 y''+3x(x-2)y'+7(x+5)y=0$$

Solution

$$y''+\frac{3x(x-2)}{x^3\left(x^2-25\right)(x-2)^2}y'+\frac{7(x+5)}{x^3\left(x^2-25\right)(x-2)^2}y=0$$

$$P(x)=\frac{3x(x-2)}{x^3(x-5)(x+5)(x-2)^2} \qquad Q(x)=\frac{7(x+5)}{x^3(x-5)(x+5)(x-2)^2}$$

For $P(x)=\dfrac{3x(x-2)}{x^3(x-5)(x+5)(x-2)^2} \quad \rightarrow \quad x=0,\ \pm 5,\ 2$

$$p(x)=\frac{3x(x-5)(x+5)(x-2)}{x^2(x-5)(x+5)(x-2)} \text{ is not an analytic at } x=0$$

For $Q(x)=\dfrac{7(x+5)}{x^3(x-5)(x+5)(x-2)^2} \quad \rightarrow \quad x=0,\ \pm 5,\ 2$

$$q(x)=\frac{3x^2(x-5)^2(x+5)^2(x-2)^2}{x^3(x-5)(x-2)^2} \text{ is not an analytic at } x=0$$

The *Regular* singular point: $\underline{x=2,\ \pm 5}\,|$, and *Irregular* singular point: $\underline{x=0}\,|$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^3-2x^2-3x\right)^2 y''+x(x-3)^2 y'-(x+1)y=0$$

Solution

$$y''+\frac{x(x-3)^2}{\left(x^3-2x^2-3x\right)^2}y'-\frac{x+1}{\left(x^3-2x^2-3x\right)^2}y=0$$

$$P(x)=\frac{x(x-3)^2}{x^2(x-3)^2(x+1)^2} \qquad Q(x)=-\frac{x+1}{x^2(x-3)^2(x+1)^2}$$

For $P(x)=\dfrac{x(x-3)^2}{x^2(x-3)^2(x+1)^2} \quad \rightarrow \quad x=0,\ -1,\ 3$

$$p(x)=\frac{1}{x(x+1)^2} \text{ is not an analytic at } x=-1$$

For $Q(x)=\dfrac{x+1}{x^2(x-3)^2(x+1)^2} \quad \rightarrow \quad x=0,\ -1,\ 3$

$$q(x)=\frac{x^2(x-3)^2(x+1)^2}{x^2(x-3)^2(x+1)} \text{ is an analytic at } x=0,\ -1,\ 3$$

The *Regular* singular point: $\underline{x=0,\ 3}\,|$, and *Irregular* singular point: $\underline{x=-1}\,|$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(1-x^2\right)y''+(\tan x)y'+x^{5/3}y=0$$

Solution

$$y''+\frac{\tan x}{1-x^2}y'+\frac{x^{5/3}}{1-x^2}y=0$$

$$P(x)=\frac{\tan x}{1-x^2} \qquad Q(x)=\frac{x^{5/3}}{1-x^2}$$

For $P(x)=\dfrac{\tan x}{1-x^2} \quad \to \quad x=\pm 1$

$\tan x=\pm\infty \quad \to \quad x=\pm\frac{\pi}{2}$ (Vertical Asymptotes).

For $Q(x)=\dfrac{x^{5/3}}{1-x^2} \quad \to \quad x=\pm 1$ is not analytic

The second derivatices doesn't exist at $x=0$

The *Regular* singular point: $x=0,\ \pm 1,\ \pm\frac{\pi}{2}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x(x-1)^2(x+2)y''+x^2y'-\left(x^3+2x-1\right)y=0$$

Solution

$$y''+\frac{x^2}{x(x-1)^2(x+2)}y'-\frac{x^3+2x-1}{x(x-1)^2(x+2)}y=0$$

$$P(x)=\frac{x}{(x-1)^2(x+2)} \quad \& \quad Q(x)=-\frac{x^3+2x-1}{x(x-1)^2(x+2)}$$

For $P(x)=\dfrac{x}{(x-1)^2(x+2)} \quad \to \quad x=1,\ -2$

$$p_0=\lim_{x\to 1}(x-1)P(x)=\lim_{x\to 1}\frac{x}{(x-1)(x+2)}=\infty \text{ is not analytic}$$

$$p_0=\lim_{x\to -2}(x+2)P(x)$$

$$=\lim_{x\to -2}\frac{x}{(x-1)^2}$$

$$= -\frac{2}{9}$$

For $Q(x) = -\dfrac{x^3+2x-1}{x(x-1)^2(x+2)} \quad \to \quad x = 0,\ 1,\ -2$

$$q_0 = \lim_{x\to 0} x^2 Q(x)$$

$$= \lim_{x\to 0} \frac{x\left(x^3+2x-1\right)}{(x-1)^2(x+2)}$$

$$= 0$$

$$q_0 = \lim_{x\to 1} (x-1)^2 Q(x)$$

$$= \lim_{x\to 1} \frac{x^3+2x-1}{x(x+2)}$$

$$= \frac{2}{3}$$

$$q_0 = \lim_{x\to -2} (x+2)^2 Q(x)$$

$$= -\lim_{x\to -2} \frac{\left(x^3+2x-1\right)(x+2)}{x(x-1)^2}$$

$$= 0$$

The *Regular* singular point: $x = 0,\ -2$, and *Irregular* singular point: $x = 1$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x^4\left(x^2+1\right)(x-1)^2 y'' + 4x^3(x-1)y' + (x+1)y = 0$$

Solution

$$y'' + \frac{4x^3(x-1)}{x^4\left(x^2+1\right)(x-1)^2} y' + \frac{x+1}{x^4\left(x^2+1\right)(x-1)^2} y = 0$$

$$P(x) = \frac{4}{x\left(x^2+1\right)(x-1)} \quad \& \quad Q(x) = \frac{x+1}{x^4\left(x^2+1\right)(x-1)^2}$$

For $P(x)=\dfrac{4}{x\left(x^2+1\right)(x-1)} \quad \rightarrow \quad \underline{x=0,\ 1,\ \pm i}\,\rfloor$

$$p_0=\lim_{x\to 0} xP(x)$$

$$=\lim_{x\to 0}\frac{4}{\left(x^2+1\right)(x-1)}$$

$$\underline{=-4}\,\rfloor$$

$$p_0=\lim_{x\to 1}(x-1)P(x)$$

$$=\lim_{x\to 1}\frac{4}{x\left(x^2+1\right)}$$

$$\underline{=2}\,\rfloor$$

$$p_0=\lim_{x\to i}(x-i)P(x)$$

$$=\lim_{x\to i}\frac{4}{x(x-1)(x+i)}$$

$$=-\frac{2}{i-1}$$

$$=-\frac{2}{i-1}\frac{i+1}{i+1}$$

$$\underline{=i+1}\,\rfloor$$

$$p_0=\lim_{x\to -i}(x+i)P(x)$$

$$=\lim_{x\to -i}\frac{4}{x(x-1)(x-i)}$$

$$=\frac{2}{i-1}$$

$$=\frac{2}{i-1}\frac{i+1}{i+1}$$

$$\underline{=-i-1}\,\rfloor$$

For $Q(x)=\dfrac{x+1}{x^4\left(x^2+1\right)(x-1)^2} \quad \rightarrow \quad x=0,\ 1,\ \pm i$

$$q_0=\lim_{x\to 0} x^2Q(x)$$

$$=\lim_{x\to 0}\frac{x+1}{x^2\left(x^2+1\right)(x-1)^2}$$

$\underline{=\infty}\,\rfloor$ is not analytic

$$q_0 = \lim_{x \to 1} (x-1)^2 Q(x)$$

$$= \lim_{x \to 0} \frac{x+1}{x^4\left(x^2+1\right)}$$

$$\underline{=1}$$

$$q_0 = \lim_{x \to \pm i} \left(x^2+1\right)^2 Q(x)$$

$$= \lim_{x \to \pm i} \frac{(x+1)\left(x^2+1\right)}{x^2(x-1)^2}$$

$$\underline{=0}$$

The *Regular* singular point: $\underline{x=0, \ \ \pm i}$, and *Irregular* singular point: $\underline{x=0}$

Exercise

Determine whether $x=0$ is an ordinary point, singular point, or irregular singular point of the given differential equation $\quad xy'' + (1-\cos x)y' + x^2 y = 0$

Solution

$$y'' + \frac{1-\cos x}{x}y' + xy = 0$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\frac{1-\cos x}{x} = \frac{1}{x}\left(1 - 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right)$$

$$\underline{= \frac{x}{2} - \frac{x^3}{24} + \frac{x^5}{720} - \cdots}, \quad \text{is analytic at } x=0$$

$\therefore \ x=0$ is an ordinary point of the differential equation.

Exercise

Determine whether $x=0$ is an ordinary point, singular point, or irregular singular point of the given differential equation $\quad \left(e^x - 1 - x\right)y'' + xy = 0$

Solution

$$x^2 y'' + x^2 \frac{x}{e^x - 1 - x} y = 0$$

$$x^2 y'' + \frac{x^3}{e^x - 1 - x} y = 0$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^x - 1 - x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots - 1 - x$$

$$= \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$\frac{x^3}{e^x - 1 - x} = \frac{1}{\frac{1}{x^3}\left(\frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)}$$

$$= \frac{1}{\frac{1}{2x} + \frac{1}{6} + \frac{x}{24} + \cdots}$$

$\therefore$ $x = 0$ is a regular singular point of the differential equation.

Exercise

Find the Frobenius series solutions of $2x^2 y'' + 3xy' - \left(1 + x^2\right) y = 0$

Solution

$$y'' + \frac{3}{2}\frac{1}{x} y' - \frac{1}{2}\frac{1 + x^2}{x^2} y = 0 \qquad \textit{\textbf{Divide each term by }} 2x^2$$

Therefore, $x = 0$ is a regular singular point, and that $p_0 = \frac{3}{2}, \quad q_0 = -\frac{1}{2}$

$p(x) \equiv \frac{3}{2}, \quad q(x) = -\frac{1}{2} - \frac{1}{2}x^2$ are polynomials.

The Frobenius series will converge for all $x > 0$. The indicial equation is

$$r(r-1) + \frac{3}{2}r - \frac{1}{2} = r^2 + \frac{1}{2}r - \frac{1}{2} = (r+1)\left(r - \frac{1}{2}\right) = 0$$

So the roots are $r_1 = \frac{1}{2}$ *and* $r_2 = -1$.

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/2} \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{-1} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x^2 y'' + 3xy' - \left(1+x^2\right) y = 0$$

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + 3x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$- \sum_{n=0}^{\infty} a_n x^{n+r} - x^2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(2n+2r-2) a_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r-2) + 3(n+r) - 1 \right] a_n x^{n+r} - \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r+1) - 1 \right] a_n x^{n+r} - \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$\left(2r^2 + r - 1\right) a_0 + \left(2r^2 + 5r + 2\right) a_1 x +$$

$$\sum_{n=2}^{\infty} \left[(n+r)(2n+2r+1) - 1 \right] a_n x^{n+r} - \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$\left(2r^2 + r - 1\right) a_0 + \left(2r^2 + 5r + 2\right) a_1 x + \sum_{n=2}^{\infty} \left(\left[(n+r)(2n+2r+1) - 1 \right] a_n - a_{n-2} \right) x^{n+r} = 0$$

For $n = 0 \quad \rightarrow \quad \left(2r^2 + r - 1\right) a_0 = 0$

$$r = -1 \quad or \quad r = \frac{1}{2} \quad \checkmark$$

For $n = 1 \quad \rightarrow \quad \left(2r^2 + 5r + 2\right) a_1 = 0$

$\cancel{r=-2 \quad or \quad r=-\frac{1}{2}}$ Therefore $a_1 = 0$

$$\left[(n+r)(2n+2r+1)-1\right]a_n - a_{n-2} = 0$$

$$a_n = \frac{1}{(n+r)(2(n+r)+1)-1}a_{n-2}$$

$$= \frac{1}{2(n+r)^2+(n+r)-1}a_{n-2} \qquad \text{for } n \ge 2$$

$r = \frac{1}{2}$

$$a_n = \frac{1}{2\left(n+\frac{1}{2}\right)^2+\left(n+\frac{1}{2}\right)-1}a_{n-2} = \frac{1}{2n^2+3n}a_{n-2}$$

$$a_2 = \frac{1}{14}a_0$$

$$a_3 = \frac{1}{24}a_1 = 0$$

$$a_4 = \frac{1}{44}a_2 = \frac{1}{616}a_0$$

$$a_5 = 0$$

$$a_6 = \frac{1}{90}a_4 = \frac{1}{55440}a_0$$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_1(x) = a_0 x^{1/2}\left(1+\frac{x^2}{14}+\frac{x^4}{616}+\frac{x^6}{55,440}+\cdots\right)$$

$r = -1$

$$b_n = \frac{1}{2(n-1)^2+(n-1)-1}b_{n-2}$$

$$b_2 = \frac{1}{2}b_0$$

$$b_3 = \frac{1}{9}b_1 = 0$$

$$b_4 = \frac{1}{20}b_2 = \frac{1}{40}b_0$$

$$b_5 = 0$$

$$b_6 = \frac{1}{54}b_4 = \frac{1}{2160}b_0$$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_2(x) = b_0 x^{-1}\left(1+\frac{x^2}{2}+\frac{x^4}{40}+\frac{x^6}{2160}+\cdots\right)$$

$$y(x) = C_1\left(1+\sum_{n=0}^{\infty}\frac{4}{n!\ 1\cdot 3\cdot 5\cdots(2n-1)}x^n\right)+C_2 x^{1/2}\left(1+\sum_{n=0}^{\infty}\frac{4^n}{n!\ 3\cdot 5\cdots(2n+1)}x^n\right)$$

Exercise

Find the Frobenius series solutions of $2x^2y'' - xy' + \left(1+x^2\right)y = 0$

Solution

$$y'' - \frac{1}{2}\frac{1}{x}y' + \frac{1}{2}\frac{1+x^2}{x^2}y = 0 \qquad \textit{Divide each term by } 2x^2$$

Therefore, $x=0$ is a *regular singular* point, and that $p_0 = -\frac{1}{2}, \quad q_0 = \frac{1}{2}$

$p(x) = -\frac{1}{2}, \quad q(x) = \frac{1}{2} + \frac{1}{2}x^2$ are polynomials.

The Frobenius series will converge for all $x > 0$. The indicial equation is

$$r(r-1) - \frac{1}{2}r + \frac{1}{2} = r^2 - \frac{3}{2}r + \frac{1}{2} = \frac{1}{2}\left(2r^2 - 3r + 1\right) = 0$$

So the roots are $r_1 = \frac{1}{2} \quad and \quad r_2 = 1$.

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/2}\sum_{n=0}^{\infty} a_n x^n \quad and \quad y_2(x) = x^1 \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x^2 y'' - xy' + \left(1 + x^2\right) y = 0$$

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} - x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} + x^2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r-2) - (n+r) + 1\right] a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r-3) + 1\right] a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$\left(2r^2 - 3r + 1\right) a_0 + \left((1+r)(2r-1) + 1\right) a_1 x +$$

$$\sum_{n=2}^{\infty} \left[(n+r)(2n+2r-3) + 1\right] a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$\left(2r^2-3r+1\right)a_0+\left(2r^2+r\right)a_1x+\sum_{n=2}^{\infty}\left[\left((n+r)(2n+2r-3)+1\right)a_n+a_{n-2}\right]x^{n+r}=0$$

For $n=0 \quad \rightarrow \quad \left(2r^2-3r+1\right)a_0=0 \quad \Rightarrow \quad \boxed{r=1 \ \ or \ r=\frac{1}{2}}$ √

For $n=1 \quad \rightarrow \quad \left(2r^2+r\right)a_1=0 \quad \Rightarrow r \ \ \boxed{\cancel{=0,-\frac{1}{2}}}$

Therefore $\boxed{a_1=0}$

$$\left((n+r)(2n+2r-3)+1\right)a_n+a_{n-2}=0$$

$$\boxed{a_n=-\frac{1}{(n+r)(2n+2r-3)+1}a_{n-2}} \qquad for \ \ n\geq 2$$

$r=\frac{1}{2}$

$$a_n=-\frac{1}{\left(n+\frac{1}{2}\right)(2n-2)+1}a_{n-2}$$

$$=\boxed{-\frac{1}{2n^2-n}a_{n-2}}$$

$n=2 \ \rightarrow \ a_2=-\frac{1}{6}a_0 \qquad\qquad n=3 \ \rightarrow \ a_3=-\frac{1}{15}a_1=0$

$n=4 \ \rightarrow \ a_4=-\frac{1}{28}a_2=\frac{1}{168}a_0 \qquad\qquad n=5 \ \rightarrow \ a_5=0$

$n=6 \ \rightarrow \ a_6=-\frac{1}{66}a_4=-\frac{1}{11,088}a_0$

$$y_1(x)=a_0x^{1/2}\left(1-\frac{x^2}{6}+\frac{x^4}{168}-\frac{x^6}{11,088}+\cdots\right)$$

$$\boxed{=a_0\left(x^{1/2}-\frac{1}{6}x^{5/2}+\frac{1}{168}x^{9/2}-\frac{1}{11,088}x^{13/2}+\cdots\right)}$$

$r=1$

$$b_n=-\frac{1}{(n+1)(2n-1)+1}b_{n-2}$$

$$=\boxed{-\frac{1}{2n^2+n}b_{n-2}}$$

$n=2 \ \rightarrow \ b_2=-\frac{1}{10}b_0 \qquad\qquad n=3 \ \rightarrow \ b_3=-\frac{1}{21}b_1=0$

$$n=4 \ \to\ b_4 = -\tfrac{1}{36}b_2 = \tfrac{1}{360}b_0 \qquad\qquad n=5 \ \to\ b_5 = 0$$

$$n=6 \ \to\ b_6 = -\tfrac{1}{78}b_4 = -\tfrac{1}{28,080}b_0 \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x\left(1 - \frac{x^2}{10} + \frac{x^4}{360} - \frac{x^6}{28,080} + \cdots\right)$$

$$= b_0\left(x - \frac{1}{10}x^3 + \frac{1}{360}x^5 - \frac{1}{28,080}x^7 + \cdots\right)$$

$$y(x) = a_0\left(x^{1/2} - \frac{1}{6}x^{5/2} + \frac{1}{168}x^{9/2} - \frac{1}{11,088}x^{13/2} + \cdots\right) + b_0\left(x - \frac{1}{10}x^3 + \frac{1}{360}x^5 - \frac{1}{28,080}x^7 + \cdots\right)$$

Exercise

Find the general solution to the equation $2xy'' + (1+x)y' + y = 0$

Solution

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$2xy'' + (1+x)y' + y = 0$$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + (1+x)\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}$$

$$+\sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)c_n x^{n-1}x^r + \sum_{n=0}^{\infty}(n+r)c_n x^{n-1}x^r$$

$$+\sum_{n=0}^{\infty}(n+r)c_n x^n x^r + \sum_{n=0}^{\infty} c_n x^n x^r = 0$$

$$x^r\left(\sum_{n=0}^{\infty}(n+r)(2n+2r-2+1)c_n x^{n-1} + \sum_{n=0}^{\infty}(n+r+1)c_n x^n\right) = 0$$

$$x^r\left(\sum_{n=0}^{\infty}(n+r)(2n+2r-1)c_n x^{n-1} + \sum_{n=0}^{\infty}(n+r+1)c_n x^n\right) = 0$$

$$x^r\left(c_0 r(2r-1)x^{-1} + \underbrace{\sum_{n=1}^{\infty} c_n (n+r)(2n+2r-1)x^{n-1}}_{k=n-1} + \underbrace{\sum_{n=0}^{\infty}(n+r+1)c_n x^n}_{k=n}\right) = 0$$

$$x^r\left(c_0 r(2r-1)x^{-1} + \sum_{k=0}^{\infty} c_{k+1}(r+k+1)(2k+2r+1)x^k + \sum_{k=0}^{\infty} c_k(r+k+1)x^k\right) = 0$$

$$x^r\left(c_0 r(2r-1)x^{-1} + \sum_{k=0}^{\infty}\left[c_{k+1}(r+k+1)(2k+2r+1) + c_k(r+k+1)\right]x^k\right) = 0$$

$$\begin{cases} c_0 r(2r-1) = 0 & \Rightarrow \quad \boxed{r=0} \quad \boxed{r=\frac{1}{2}} \\ c_{k+1}(r+k+1)(2k+2r+1) + c_k(r+k) = 0 & \Rightarrow \quad \boxed{c_{k+1} = -\frac{r+k+1}{(r+k+1)(2k+2r+1)}c_k} \end{cases}$$

$r=0$

$$c_{k+1} = -\frac{1}{2k+1}c_k$$

$$c_1 = -\frac{1}{1}c_0$$

$$c_2 = -\frac{1}{3}c_1 = \frac{1}{3}c_0$$

$r=\frac{1}{2}$

$$c_{k+1} = -\frac{k+\frac{3}{2}}{\left(k+\frac{3}{2}\right)(2k+2)}c_k = -\frac{1}{2(k+1)}c_k$$

$$c_1 = -\frac{1}{2}c_0$$

$$c_2 = -\frac{1}{2\cdot 2}c_1 = \frac{1}{2\cdot 2\cdot 2}c_0$$

$$c_3 = -\frac{1}{5}c_2 = -\frac{1}{1\cdot 3\cdot 5}c_0 \qquad\qquad c_3 = -\frac{1}{2\cdot 3}c_2 = -\frac{1}{2^3(2\cdot 3)}c_0 = -\frac{1}{2^3\cdot 3!}c_0$$

$$c_4 = -\frac{1}{7}c_3 = \frac{1}{1\cdot 3\cdot 5\cdot 7}c_0 \qquad\qquad c_4 = -\frac{1}{2\cdot 4}c_3 = \frac{1}{2^4\cdot 4!}c_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$c_n = \frac{(-1)^n}{1\cdot 3\cdot 5\cdots(2n-1)}c_0 \qquad\qquad c_n = \frac{(-1)^n}{2^n n!}c_0$$

$$y_1(x) = c_0\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{1\cdot 3\cdot 5\cdots(2n-1)}x^n\right) \qquad y_2(x) = c_0x^{1/2}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n n!}x^n\right)$$

$$y(x) = C_1y_1(x) + C_2y_2(x)$$

$$= C_1\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{1\cdot 3\cdot 5\cdots(2n-1)}x^n\right) + C_2\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}x^{n+1/2}\right)$$

Exercise

Find the Frobenius series solutions of $xy'' + 2y' + xy = 0$

Solution

$$\frac{1}{x}\left(xy'' + 2y' + xy = 0\right)$$

$$y'' + 2\frac{1}{x}y' + \frac{x^2}{x^2}y = 0$$

$\therefore\ x = 0$ is a regular singular point with $p_0 = 2$ and $q_0 = 0$

The indicial equation is: $r(r-1) + 2r = r(r+1) = 0 \rightarrow \underline{r = 0,\ -1}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty}a_nx^n = \sum_{n=0}^{\infty}a_nx^n \quad \textit{and} \quad y_2(x) = x^{-1}\sum_{n=0}^{\infty}b_nx^n$$

$$y = \sum_{n=0}^{\infty}a_nx^{n+r}$$

$$y' = \sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$xy'' + 2y' + xy = 0$$

$$x\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} + 2\sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} + x\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) + 2(n+r)\right]a_n x^{n+r-1} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} = 0$$

$$r(r+1)a_0 x^{r-1} + (r+1)(r+2)a_1 x^r + \sum_{n=2}^{\infty} (n+r)(n+r+1)a_n x^{n+r-1} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} = 0$$

For $n=0$

$$r(r+1)a_0 = 0$$

$$\underline{r = 0 \ \ or \ r = -1} \ \checkmark$$

For $n=1$

$$(r+1)(r+2)a_1 = 0$$

$$\cancel{r = -1, \ -2}$$

$$\therefore \ \underline{a_1 = 0}$$

$$(n+r)(n+r+1)a_n + a_{n-2} = 0$$

$$\boxed{a_n = -\frac{1}{(n+r)(n+r+1)} a_{n-2}}$$

$$r = 0 \ \rightarrow \ \boxed{a_n = -\frac{1}{n(n+1)} a_{n-2}}$$

$n=2 \ \rightarrow \ a_2 = -\frac{1}{2\cdot 3} a_0$

$n=3 \ \rightarrow \ a_3 = -\frac{1}{12} a_1 = 0$

$n=4 \ \rightarrow \ a_4 = -\frac{1}{4\cdot 5} a_2 = \frac{1}{5!} a_0$

$n=5 \ \rightarrow \ a_5 = 0$

$n=6 \ \rightarrow \ a_6 = -\frac{1}{6\cdot 7} a_4 = -\frac{1}{7!} a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_1(x) = a_0\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right)$$

$$\boxed{= \frac{a_0}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)}$$

$$r = -1 \quad \rightarrow \quad \boxed{b_n = -\frac{1}{n(n-1)}b_{n-2}}$$

$n = 2 \ \rightarrow \ b_2 = -\frac{1}{2\cdot 1}b_0$

$n = 3 \ \rightarrow \ b_3 = -\frac{1}{6}b_1 = 0$

$n = 4 \ \rightarrow \ b_4 = -\frac{1}{4\cdot 3}b_2 = \frac{1}{4!}b_0$

$n = 5 \ \rightarrow \ b_5 = 0$

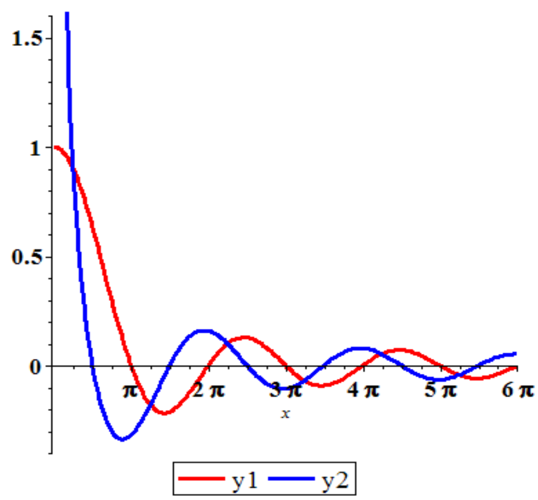
$n = 6 \ \rightarrow \ b_6 = -\frac{1}{6\cdot 5}b_4 = -\frac{1}{6!}b_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_2(x) = b_0 x^{-1}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)}$$

$$y(x) = \frac{a_0}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) + \frac{b_0}{x}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$\boxed{= a_0\frac{\sin x}{x} + b_0\frac{\cos x}{x}}$$



Exercise

Find the Frobenius series solutions of $2xy'' - y' + 2y = 0$

Solution

$$\left(\frac{x}{2}\right)2xy'' - \left(\frac{x}{2}\right)y' + \left(\frac{x}{2}\right)2y = 0$$

$x^2y'' - \frac{1}{2}xy' + xy = 0$

That implies to $p(x) = -\frac{1}{2}$ and $q(x) = x$, both are analytic.

Hence, $x_0 = 0$ is a regular point

The indicial equation is: $r(r-1) - \frac{1}{2}r = r\left(r - \frac{3}{2}\right) = 0 \quad \rightarrow \quad \boxed{r = 0,\ \frac{3}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{3/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$2xy'' - y' + 2y = 0$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + 2\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[2(n+r)(n+r-1) - (n+r)\right]a_n x^{n+r-1} + \sum_{n=1}^{\infty}2a_{n-1}x^{n+r-1} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-3)a_n + 2a_{n-1}\right] x^{n+r-1} = 0$$

$$(n+r)(2n+2r-3)a_n + 2a_{n-1} = 0$$

$$\boxed{a_n = -\frac{2}{(n+r)(2n+2r-3)}a_{n-1}}$$

$$r = 0 \quad \rightarrow \quad \boxed{a_n = -\frac{2}{n(2n-3)}a_{n-1}}$$

$$n = 1 \;\to\; a_1 = 2a_0$$

$$n = 2 \;\to\; a_2 = -a_1 = -2a_0$$

$$n = 3 \;\to\; a_3 = -\frac{2}{9}a_2 = \frac{4}{9}a_0$$

$$n = 4 \;\to\; a_4 = -\frac{1}{10}a_3 = -\frac{2}{45}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_1(x) = a_0\left(1 + 2x - 2x^2 + \frac{4}{9}x^3 - \frac{2}{45}x^4 + \cdots\right)$$

$$r = \frac{3}{2} \;\to\; b_n = -\frac{1}{n\left(n + \frac{3}{2}\right)}b_{n-1}$$

$$n = 1 \;\to\; b_1 = -\frac{2}{5}b_0$$

$$n = 2 \;\to\; b_2 = -\frac{1}{7}b_1 = \frac{2}{35}b_0$$

$$n = 3 \;\to\; b_3 = -\frac{2}{27}b_2 = -\frac{4}{945}b_0$$

$$n = 4 \;\to\; b_4 = -\frac{1}{22}b_3 = \frac{2}{20,790}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{3/2}\left(1 - \frac{2}{5}x + \frac{2}{35}x^2 - \frac{4}{945}x^3 + \frac{2}{20,790}x^4 - \cdots\right)$$

$$= b_0\sqrt{x}\left(x - \frac{2}{5}x^2 + \frac{2}{35}x^3 - \frac{4}{945}x^4 + \frac{2}{20,790}x^5 - \cdots\right)$$

$$y(x) = a_0\left(1 + 2x - 2x^2 + \frac{4}{9}x^3 - \frac{2}{45}x^4 + \cdots\right) + b_0\sqrt{x}\left(x - \frac{2}{5}x^2 + \frac{2}{35}x^3 - \frac{4}{945}x^4 + \frac{2}{20,790}x^5 - \cdots\right)$$

Exercise

Find the Frobenius series solutions of $2xy'' + 5y' + xy = 0$

Solution

$$\left(\frac{x}{2}\right)2xy'' + 5\left(\frac{x}{2}\right)y' + \left(\frac{x}{2}\right)xy = 0$$

$$x^2y'' + \frac{5}{2}xy' + \frac{1}{2}x^2y = 0$$

That implies to $p(x) = \frac{5}{2}$ and $q(x) = \frac{1}{2}x^2$, both are analytic.

Hence, $x_0 = 0$ is a regular point

The indicial equation is: $r(r-1)+\frac{5}{2}r = r^2+\frac{3}{2}r = 0 \quad \to \quad \underline{r = 0,\ -\frac{3}{2}}|$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{-3/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$2xy'' + 5y' + xy = 0$

$$2x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \sum_{n=0}^{\infty} 5(n+r) a_n x^{n+r-1} + x \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 5(n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty} \left[2(n+r)(n+r-1) + 5(n+r)\right] a_n x^{n+r-1} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} = 0$$

$$r(2r+3)a_0 + (r+1)(2r+5)a_1 + \sum_{n=2}^{\infty} (n+r)(2n+2r+3) a_n x^{n+r-1} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} = 0$$

$$r(2r+3)a_0 + (r+1)(2r+5)a_1 + \sum_{n=2}^{\infty} \left[(n+r)(2n+2r+3) a_n + a_{n-2}\right] x^{n+r-1} = 0$$

For $n = 0$

$$r(2r+3)a_0 = 0$$

$$\underline{r = 0 \ \ or\ r = -\frac{3}{2}}| \quad \surd$$

For $n = 1$

$$(r+1)(2r+5)a_1 = 0$$

$$\cancel{r = -1,\ -\frac{5}{2}}$$

$$\rightarrow \boxed{a_1 = 0}$$

$$(n+r)(2n+2r+3)a_n + a_{n-2} = 0$$

$$\boxed{a_n = -\frac{1}{(n+r)(2n+2r+3)}a_{n-2}}$$

$$r = 0 \quad \rightarrow \quad \boxed{a_n = -\frac{1}{n(2n+3)}a_{n-2}}$$

$n = 2 \ \rightarrow\ a_2 = -\frac{1}{14}a_0$ $n = 3 \ \rightarrow\ a_3 = -\frac{1}{27}a_1 = 0$

$n = 4 \ \rightarrow\ a_4 = -\frac{1}{88}a_2 = \frac{1}{616}a_0$ $n = 5 \ \rightarrow\ a_5 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$ $\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0\left(1 - \frac{1}{14}x^2 + \frac{1}{616}x^4 - \cdots\right)}$$

$$r = -\frac{3}{2}$$

$$b_n = -\frac{1}{2n\left(n-\frac{3}{2}\right)}b_{n-2}$$

$$\boxed{= -\frac{1}{n(2n-3)}b_{n-2}}$$

$n = 2 \ \rightarrow\ b_2 = -\frac{1}{2}b_0$ $n = 3 \ \rightarrow\ b_3 = -\frac{1}{9}b_1 = 0$

$n = 4 \ \rightarrow\ b_4 = -\frac{1}{20}b_2 = \frac{1}{40}b_0$ $n = 5 \ \rightarrow\ b_5 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$ $\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_2(x) = b_0 x^{-3/2}\left(1 - \frac{1}{2}x^2 + \frac{1}{40}x^3 - \cdots\right)$$

$$\boxed{= b_0\left(x^{-3/2} - \frac{1}{2}x^{1/2} + \frac{1}{40}x^{3/2} - \cdots\right)}$$

$$\boxed{y(x) = a_0\left(1 - \frac{1}{14}x^2 + \frac{1}{616}x^4 - \cdots\right) + b_0\left(x^{-3/2} - \frac{1}{2}x^{1/2} + \frac{1}{40}x^{3/2} - \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $4xy'' + \frac{1}{2}y' + y = 0$

Solution

$$\left(\frac{x}{4}\right)4xy'' + \frac{1}{2}\left(\frac{x}{4}\right)y' + \left(\frac{x}{4}\right)y = 0$$

$$x^2y'' + \frac{1}{8}xy' + \frac{1}{4}x^2y = 0$$

$$y'' + \frac{1}{8x}y' + \frac{1}{4}y = 0$$

That implies to $p(x) = \frac{1}{8x}$ and $q(x) = \frac{1}{4}$

$$\begin{aligned} p_0 &= \lim_{x\to 0} xp(x) \\ &= \lim_{x\to 0} x\frac{1}{8x} \\ &= \frac{1}{8} \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x\to 0} x^2q(x) \\ &= \lim_{x\to 0} x^2\frac{1}{8} \\ &= 0 \end{aligned}$$

The indicial equation is:

$$r(r-1) + \frac{1}{8}r = r^2 - \frac{7}{8}r = 0$$

$$r = 0,\ \frac{7}{8}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{7/8}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$8xy'' + y' + 2y = 0$$

$$8x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+2\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}8(n+r)(n+r-1)a_n x^{n+r-1}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\sum_{n=0}^{\infty}2a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left[8(n+r)(n+r-1)+(n+r)\right]a_n x^{n+r-1}+\sum_{n=1}^{\infty}2a_{n-1}x^{n+r-1}=0$$

$$r(8r-7)a_0+\sum_{n=0}^{\infty}(n+r)(8n+8r-7)a_n x^{n+r-1}+\sum_{n=1}^{\infty}2a_{n-1}x^{n+r-1}=0$$

$$r(8r-7)a_0+\sum_{n=0}^{\infty}\left[(n+r)(8n+8r-7)a_n+2a_{n-1}\right]x^{n+r-1}=0$$

For $n=0$

$$r(8r-7)a_0=0$$

$$\boxed{r=0,\ \frac{7}{8}}\ \checkmark$$

$$(n+r)(8n+8r-7)a_n+2a_{n-1}=0$$

$$\boxed{a_n=-\frac{2}{(n+r)(8n+8r-7)}a_{n-1}}$$

$$r=0\ \rightarrow\ \boxed{a_n=-\frac{2}{n(8n-7)}a_{n-1}}$$

$n=1\ \rightarrow\ a_1=-2a_0$

$n=2\ \rightarrow\ a_2=-\frac{1}{9}a_1=\frac{2}{9}a_0$

$n=3\ \rightarrow\ a_3=-\frac{2}{51}a_2=-\frac{4}{459}a_0$

$\vdots\quad\vdots\quad\vdots\quad\vdots$

$$\boxed{y_1(x)=a_0\left(1-2x+\frac{2}{9}x^2-\frac{4}{459}x^3+\cdots\right)}$$

$$r=\frac{7}{8}$$

$$b_n=-\frac{2}{\left(n+\frac{7}{8}\right)(8n)}b_{n-1}$$

$$=-\frac{2}{n(8n+7)}b_{n-1}$$

$$n=1 \ \rightarrow \ b_1 = -\frac{2}{15}b_0$$

$$n=2 \ \rightarrow \ b_2 = -\frac{1}{23}b_1 = \frac{2}{345}b_0$$

$$n=3 \ \rightarrow \ b_3 = -\frac{2}{93}b_2 = -\frac{4}{32,085}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{7/8}\left(1-\frac{2}{15}x+\frac{2}{345}x^2-\frac{4}{32,085}x^3+\cdots\right)$$

$$y(x) = a_0\left(1-2x+\frac{2}{9}x^2-\frac{4}{459}x^3+\cdots\right)+b_0 x^{7/8}\left(1-\frac{2}{15}x+\frac{2}{345}x^2-\frac{4}{32,085}x^3+\cdots\right)$$

Exercise

Find the Frobenius series solutions of $2x^2y''-xy'+\left(x^2+1\right)y=0$

Solution

$$\frac{1}{2}2x^2y''-\frac{1}{2}xy'+\frac{1}{2}\left(x^2+1\right)y=0$$

$$x^2y''-\frac{1}{2}xy'+\frac{1}{2}\left(x^2+1\right)y=0$$

$$y''-\frac{1}{2x}y'+\left(\frac{1}{2}+\frac{1}{2x^2}\right)y=0$$

That implies to $p(x)=-\frac{1}{2x}$ and $q(x)=\frac{1}{2}+\frac{1}{2x^2}$.

$$p_0 = \lim_{x\to 0} xp(x)$$

$$= -\lim_{x\to 0} x\frac{1}{2x}$$

$$= -\frac{1}{2}$$

$$q_0 = \lim_{x\to 0} x^2 q(x)$$

$$= \lim_{x\to 0} x^2\left(\frac{1}{2}+\frac{1}{2x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{1}{2}x^2+\frac{1}{2}\right)$$

$$=\frac{1}{2}\Big|$$

The indicial equation is:

$$r(r-1)-\frac{1}{2}r+\frac{1}{2}=r^2-\frac{3}{2}r+\frac{1}{2}=0$$

$$\boxed{r=1,\ \frac{1}{2}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x)=x^1\sum_{n=0}^{\infty}a_nx^n \quad \text{and} \quad y_2(x)=x^{1/2}\sum_{n=0}^{\infty}b_nx^n$$

$$y=\sum_{n=0}^{\infty}a_nx^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}$$

$$2x^2y''-xy'+\left(x^2+1\right)y=0$$

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}-x\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}+\left(x^2+1\right)\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_nx^{n+r}-\sum_{n=0}^{\infty}(n+r)a_nx^{n+r}+\sum_{n=0}^{\infty}a_nx^{n+r+2}+\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

$$\sum_{n=0}^{\infty}\big(2(n+r)(n+r-1)-(n+r)+1\big)a_nx^{n+r}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r}$$

$$\big(r(2r-3)+1\big)a_0+\big((r+1)(2r-1)+1\big)a_1+\sum_{n=2}^{\infty}\big((n+r)(2n+2r-3)+1\big)a_nx^{n+r}$$

$$+\sum_{n=2}^{\infty}a_{n-2}x^{n+r}=0$$

$$\left(2r^2-3r+1\right)a_0+\left(2r^2+r\right)a_1+\sum_{n=2}^{\infty}\left[\left((n+r)(2n+2r-3)+1\right)a_n+a_{n-2}\right]x^{n+r}=0$$

For $n=0$

$$\left(2r^2-3r+1\right)a_0=0$$

$$\boxed{r=1,\ \frac{1}{2}}\ \checkmark$$

For $n=1$

$$\left(2r^2+r\right)a_1=0$$

$$\cancel{r=0,\ -\frac{1}{2}}$$

$$\to\ \boxed{a_1=0}$$

$$\left((n+r)(2n+2r-3)+1\right)a_n+a_{n-2}=0$$

$$\boxed{a_n=-\frac{1}{(n+r)(2n+2r-3)+1}a_{n-2}}$$

$r=1$

$$a_n=-\frac{1}{(n+1)(2n-1)+1}a_{n-2}$$

$$\boxed{=-\frac{1}{2n^2+n}a_{n-2}}$$

$n=2\ \to\ a_2=-\frac{1}{10}a_0$ $\qquad$ $n=3\ \to\ a_3=-\frac{1}{21}a_1=0$

$n=4\ \to\ a_4=-\frac{1}{36}a_2=\frac{1}{360}a_0$ $\qquad$ $n=5\ \to\ a_5=0$

$\vdots\quad\vdots\quad\vdots\quad\vdots$ $\qquad$ $\vdots\quad\vdots\quad\vdots\quad\vdots$

$$y_1(x)=a_0x\left(1-\frac{1}{10}x^2+\frac{1}{360}x^4-\cdots\right)$$

$$\boxed{=a_0\left(x-\frac{1}{10}x^3+\frac{1}{360}x^5-\cdots\right)}$$

$r=\frac{1}{2}$

$$b_n=-\frac{1}{\left(n+\frac{1}{2}\right)(2n-2)+1}b_{n-2}$$

$$\boxed{=-\frac{1}{2n^2-n}b_{n-2}}$$

$n=2 \;\rightarrow\; b_2 = -\frac{1}{6}b_0$ $\qquad$ $n=3 \;\rightarrow\; a_3 = -\frac{1}{15}a_1 = 0$

$n=4 \;\rightarrow\; b_4 = -\frac{1}{28}b_2 = \frac{1}{168}b_0$ $\qquad$ $n=5 \;\rightarrow\; a_5 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_2(x) = b_0 x^{1/2}\left(1 - \frac{1}{6}x^2 + \frac{1}{168}x^4 - \cdots\right)}$$

$$\boxed{y(x) = a_0\left(x - \frac{1}{10}x^3 + \frac{1}{360}x^5 - \cdots\right) + b_0 x^{1/2}\left(1 - \frac{1}{6}x^2 + \frac{1}{168}x^4 - \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $3xy'' + (2-x)y' - y = 0$

Solution

$$\frac{x}{3}3xy'' + \frac{x}{3}(2-x)y' - \frac{x}{3}y = 0$$

$$x^2y'' + \left(\frac{2}{3}x - \frac{1}{3}x^2\right)y' - \frac{x}{3}y = 0$$

$$y'' + \left(\frac{2}{3x} - \frac{1}{3}\right)y' - \frac{1}{3x}y = 0$$

That implies to $p(x) = \frac{2}{3x} - \frac{1}{3}$ and $q(x) = -\frac{1}{3x}$.

$$\begin{aligned} p_0 &= \lim_{x\to 0} xp(x) \\ &= \lim_{x\to 0} x\left(\frac{2}{3x} - \frac{1}{3}\right) \\ &= \lim_{x\to 0}\left(\frac{2}{3} - \frac{1}{3}x\right) \\ &= \boxed{\frac{2}{3}} \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x\to 0} x^2 q(x) \\ &= -\lim_{x\to 0} x^2\frac{1}{3x} \\ &= \lim_{x\to 0}\frac{x}{3} \\ &= \boxed{0} \end{aligned}$$

The indicial equation is: $r(r-1) + \frac{2}{3}r = r^2 - \frac{1}{3}r = 0$

$$\boxed{r = 0,\ \frac{1}{3}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/3} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$3xy'' + (2-x)y' - y = 0$$

$$3x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + (2-x) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 3(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}$$
$$- \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(3n+3r-3) + 2(n+r)\right] a_n x^{n+r-1} - \sum_{n=0}^{\infty} (n+r+1) a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(3n+3r-1) a_n x^{n+r-1} - \sum_{n=1}^{\infty} (n+r) a_{n-1} x^{n+r-1} = 0$$

$$r(3r-1)a_0 + \sum_{n=1}^{\infty} (n+r)(3n+3r-1) a_n x^{n+r-1} - \sum_{n=1}^{\infty} (n+r) a_{n-1} x^{n+r-1} = 0$$

$$r(3r-1)a_0 + \sum_{n=1}^{\infty}\left[(n+r)(3n+3r-1)a_n - (n+r)a_{n-1}\right]x^{n+r-1} = 0$$

For $n=0$

$$r(3r-1)a_0 = 0$$

$$\boxed{r = 0,\ \frac{1}{3}} \quad \checkmark$$

$$(n+r)(3n+3r-1)a_n - (n+r)a_{n-1} = 0$$

$$\boxed{a_n = \frac{1}{3n+3r-1}a_{n-1}}$$

$r = 0$

$$\boxed{a_n = \frac{1}{3n-1}a_{n-1}}$$

$n=1 \rightarrow a_1 = \frac{1}{2}a_0$

$n=2 \rightarrow a_2 = \frac{1}{5}a_1 = \frac{1}{10}a_0$

$n=3 \rightarrow a_3 = \frac{1}{8}a_2 = \frac{1}{80}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0\left(1+\frac{1}{2}x+\frac{1}{10}x^2+\frac{1}{80}x^3+\cdots\right)}$$

$r = \frac{1}{3}$

$$\boxed{b_n = \frac{1}{3n}b_{n-1}}$$

$n=1 \rightarrow b_1 = \frac{1}{3}b_0$

$n=2 \rightarrow b_2 = \frac{1}{6}b_1 = \frac{1}{18}b_0$

$n=3 \rightarrow b_3 = \frac{1}{9}b_2 = \frac{1}{162}b_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_2(x) = b_0x^{1/3}\left(1+\frac{1}{3}x+\frac{1}{18}x^2+\frac{1}{162}x^3+\cdots\right)}$$

$$\boxed{y(x) = a_0\left(1+\frac{1}{2}x+\frac{1}{10}x^2+\frac{1}{80}x^3+\cdots\right)+b_0x^{1/3}\left(1+\frac{1}{3}x+\frac{1}{18}x^2+\frac{1}{162}x^3+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2xy'' - (3+2x)y' + y = 0$

Solution

$$\frac{x}{2}2xy'' - \frac{x}{2}(3+2x)y' + \frac{x}{2}y = 0$$

$$x^2y'' - \left(\frac{3}{2}x + x^2\right)y' + \frac{1}{2}xy = 0$$

$$y'' - \left(\frac{3}{2x} + 1\right)y' + \frac{1}{2x}y = 0$$

That implies to $p(x) = -\frac{3}{2x} - 1$ and $q(x) = \frac{1}{2x}$.

$$\begin{aligned} p_0 &= \lim_{x\to 0} xp(x) \\ &= \lim_{x\to 0} x\left(-\frac{3}{2x} - 1\right) \\ &= \lim_{x\to 0} \left(-\frac{3}{2} - x\right) \\ &= -\frac{3}{2} \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x\to 0} x^2q(x) \\ &= \lim_{x\to 0} x^2\frac{1}{2x} \\ &= \lim_{x\to 0} \frac{x}{2} \\ &= 0 \end{aligned}$$

The indicial equation is: $r(r-1) - \frac{3}{2}r = r^2 - \frac{5}{2}r = 0$

$$\Rightarrow \quad r = 0,\ \frac{5}{2}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{5/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$2xy'' - (3+2x)y' + y = 0$$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - (3+2x)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r-1} - \sum_{n=0}^{\infty}3(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty}2(n+r)a_n x^{n+r}$$
$$+\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[2(n+r)(n+r-1) - 3(n+r)\right]a_n x^{n+r-1} - \sum_{n=0}^{\infty}(2n+2r-1)a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(2n+2r-5)a_n x^{n+r-1} - \sum_{n=1}^{\infty}(2n+2r-3)a_{n-1}x^{n+r-1} = 0$$

$$r(2r-5)a_0 + \sum_{n=1}^{\infty}(n+r)(2n+2r-5)a_n x^{n+r-1} - \sum_{n=1}^{\infty}(2n+2r-3)a_{n-1}x^{n+r-1} = 0$$

$$r(2r-5)a_0 + \sum_{n=1}^{\infty}\left[(n+r)(2n+2r-5)a_n - (2n+2r-3)a_{n-1}\right]x^{n+r-1} = 0$$

For $n = 0$

$$r(2r-5)a_0 = 0$$

$$\boxed{r = 0,\ \frac{5}{2}} \quad \checkmark$$

$$(n+r)(2n+2r-5)a_n - (2n+2r-3)a_{n-1} = 0$$

$$\boxed{a_n = \frac{2n+2r-3}{(n+r)(2n+2r-5)}a_{n-1}}$$

$$\boxed{r = 0 \quad \rightarrow \quad a_n = \frac{2n-3}{n(2n-5)}a_{n-1}}$$

$$n = 1 \rightarrow a_1 = \frac{1}{3}a_0$$

$$n=2 \ \rightarrow \ a_2 = -\tfrac{1}{2}a_1 = -\tfrac{1}{6}a_0$$

$$n=3 \ \rightarrow \ a_3 = -a_2 = -\tfrac{1}{6}a_0$$

$$n=4 \ \rightarrow \ a_4 = \tfrac{5}{12}a_3 = -\tfrac{5}{72}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0\left(1+\tfrac{1}{3}x-\tfrac{1}{6}x^2-\tfrac{1}{6}x^3-\tfrac{5}{72}x^4-\cdots\right)}$$

$$r=\tfrac{5}{2}$$

$$b_n = \frac{2n+2}{2n\left(n+\frac{5}{2}\right)}b_{n-1}$$

$$\underline{= \frac{2n+2}{n(2n+5)}b_{n-1}}$$

$$n=1 \ \rightarrow \ b_1 = \tfrac{4}{7}b_0$$

$$n=2 \ \rightarrow \ b_2 = \tfrac{1}{3}b_1 = \tfrac{4}{21}b_0$$

$$n=3 \ \rightarrow \ b_3 = \tfrac{8}{33}b_2 = \tfrac{32}{693}b_0$$

$$n=4 \ \rightarrow \ b_4 = \tfrac{5}{26}b_3 = \tfrac{80}{9,009}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = b_0x^{5/2}\left(1+\tfrac{4}{7}x+\tfrac{4}{21}x^2+\tfrac{32}{693}x^3+\tfrac{80}{9,009}x^4+\cdots\right)}$$

$$\underline{y(x) = a_0\left(1+\tfrac{1}{3}x-\tfrac{1}{6}x^2-\tfrac{1}{6}x^3-\tfrac{5}{72}x^4-\cdots\right)+b_0x^{5/2}\left(1+\tfrac{4}{7}x+\tfrac{4}{21}x^2+\tfrac{32}{693}x^3+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $xy''+(x-6)y'-3y=0$

Solution

$$xxy''+x(x-6)y'-3xy=0$$

$$x^2y''+\left(x^2-6x\right)y'-3xy=0$$

$$y''+\left(1-\tfrac{6}{x}\right)y'-\tfrac{3}{x}y=0$$

That implies to $p(x)=1-\frac{6}{x}$ and $q(x)=-\frac{3}{x}$.

$$\begin{aligned} p_0 &= \lim_{x\to 0} xp(x) \\ &= \lim_{x\to 0} x\left(1-\frac{6}{x}\right) \\ &= \lim_{x\to 0} (x-6) \\ &= -6 \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x\to 0} x^2 q(x) \\ &= -\lim_{x\to 0} x^2 \frac{3}{x} \\ &= -\lim_{x\to 0} 3x \\ &= 0 \end{aligned}$$

The indicial equation is: $r(r-1)-6r = r^2-7r = 0$

$\rightarrow \quad r = 0,\ 7$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^7 \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$xy'' + (x-6)y' - 3y = 0$$

$$x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + (x-6)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - 3\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}-\sum_{n=0}^{\infty}6(n+r)a_n x^{n+r-1}$$
$$-\sum_{n=0}^{\infty}3a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)-6(n+r)\right]a_n x^{n+r-1}+\sum_{n=0}^{\infty}(n+r-3)a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-7)a_n x^{n+r-1}+\sum_{n=1}^{\infty}(n+r-4)a_{n-1}x^{n+r-1}=0$$

$$r(r-7)a_0+\sum_{n=1}^{\infty}(n+r)(n+r-7)a_n x^{n+r-1}+\sum_{n=1}^{\infty}(n+r-4)a_{n-1}x^{n+r-1}=0$$

$$r(r-7)a_0+\sum_{n=1}^{\infty}\left[(n+r)(n+r-7)a_n+(n+r-4)a_{n-1}\right]x^{n+r-1}=0$$

For $n=0$

$$r(r-7)a_0=0$$

$$\Rightarrow \boxed{r=0,\ \ 7}\ \surd$$

$$(n+r)(n+r-7)a_n+(n+r-4)a_{n-1}=0$$

$$\boxed{a_n=-\frac{n+r-4}{(n+r)(n+r-7)}a_{n-1}}$$

$$\boxed{r=0\ \ \rightarrow\ \ a_n=-\frac{n-4}{n(n-7)}a_{n-1}}$$

$$n=1\ \rightarrow\ a_1=-\frac{1}{2}a_0$$

$$n=2\ \rightarrow\ a_2=-\frac{1}{5}a_1=\frac{1}{10}a_0$$

$$n=3\ \rightarrow\ a_3=-\frac{1}{12}a_2=-\frac{1}{120}a_0$$

$$n=4\ \rightarrow\ a_4=0a_3=0$$

$$\vdots\quad\vdots\quad\vdots\quad\vdots$$

$$\boxed{y_1(x)=a_0\left(1-\frac{1}{2}x+\frac{1}{10}x^2-\frac{1}{120}x^3\right)}$$

$$r=7 \quad \to \quad b_n = -\frac{n+3}{n(n+7)} b_{n-1}$$

$$n=1 \to b_1 = -\frac{1}{2} b_0$$

$$n=2 \to b_2 = -\frac{5}{18} b_1 = \frac{5}{36} b_0$$

$$n=3 \to b_3 = -\frac{1}{5} b_2 = -\frac{1}{36} b_0$$

$$n=4 \to b_4 = -\frac{7}{44} b_3 = \frac{7}{1,584} b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^7 \left(1 - \frac{1}{2}x + \frac{5}{36}x^2 - \frac{1}{36}x^3 + \frac{7}{1,584}x^4 - \cdots\right)$$

$$y(x) = a_0\left(1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3\right) + b_0 x^7 \left(1 - \frac{1}{2}x + \frac{5}{36}x^2 - \frac{1}{36}x^3 + \frac{7}{1,584}x^4 - \cdots\right)$$

Exercise

Find the Frobenius series solutions of $x(x-1)y'' + 3y' - 2y = 0$

Solution

$$\frac{1}{x}x(x-1)y'' + 3\frac{1}{x}y' - 2\frac{1}{x}y = 0$$

$$(x-1)y'' + \frac{3}{x}y' - \frac{2}{x}y = 0$$

That implies to $p(x) = \frac{3}{x}$ and $q(x) = -\frac{2}{x}$.

$$p_0 = \lim_{x\to 0} xp(x)$$

$$= \lim_{x\to 0} x\frac{3}{x}$$

$$= 3$$

$$q_0 = \lim_{x\to 0} x^2 q(x)$$

$$= -\lim_{x\to 0} x^2 \frac{2}{x}$$

$$= -\lim_{x\to 0} 2x$$

$$= 0$$

The indicial equation is: $-r(r-1) + 3r = -r^2 + 4r = 0$

$$\to r = 0,\ 4$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^4 \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x(x-1)y'' + 3y' - 2y = 0$$

$$\left(x^2 - x\right) \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + 3\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1}$$

$$+\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) - 2\right] a_n x^{n+r} - \sum_{n=0}^{\infty} \left[(n+r)(n+r-1) - 3(n+r)\right] a_n x^{n+r-1} = 0$$

$$\sum_{n=1}^{\infty} \left[(n-1+r)(n+r-2) - 2\right] a_{n-1} x^{n+r-1} - \sum_{n=0}^{\infty} (n+r)(n+r-4) a_n x^{n+r-1} = 0$$

$$\sum_{n=1}^{\infty} \left[(n-1+r)(n+r-2) - 2\right] a_{n-1} x^{n+r-1} - r(r-4) a_0$$

$$-\sum_{n=1}^{\infty} (n+r)(n+r-4) a_n x^{n+r-1} = 0$$

$$-r(r-4)a_0 + \sum_{n=1}^{\infty}\left[\left((n+r-1)(n+r-2)-2\right)a_{n-1} - (n+r)(n+r-4)a_n\right]x^{n+r-1} = 0$$

For $n=0$

$$-r(r-4)a_0 = 0$$

$$\Rightarrow \underline{r = 0,\ \ 4} \quad \checkmark$$

$$\left((n+r-1)(n+r-2)-2\right)a_{n-1} - (n+r)(n+r-4)a_n = 0$$

$$a_n = \frac{(n+r-1)(n+r-2)-2}{(n+r)(n+r-4)}a_{n-1}$$

$r=0$

$$a_n = \frac{(n-1)(n-2)-2}{n(n-4)}a_{n-1}$$

$n=1 \ \rightarrow\ a_1 = \frac{2}{3}a_0$

$n=2 \ \rightarrow\ a_2 = \frac{1}{2}a_1 = \frac{1}{3}a_0$

$n=3 \ \rightarrow\ a_3 = \frac{0}{3}a_2 = 0$

$n=4 \ \rightarrow\ a_4 = 0a_3 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_1(x) = a_0\left(1+\frac{2}{3}x+\frac{1}{3}x^2\right)$$

$r=4$

$$b_n = \frac{(n+3)(n+2)-2}{n(n+4)}b_{n-1}$$

$n=1 \ \rightarrow\ b_1 = 2b_0$

$n=2 \ \rightarrow\ b_2 = \frac{3}{2}b_1 = 3b_0$

$n=3 \ \rightarrow\ b_3 = \frac{28}{21}b_2 = 4b_0$

$n=4 \ \rightarrow\ b_4 = \frac{5}{4}b_3 = 5b_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_2(x) = b_0x^4\left(1+2x+3x^2+4x^3+5x^4+\cdots\right)$$

$$y(x) = a_0\left(1+\frac{2}{3}x+\frac{1}{3}x^2\right) + b_0\left(x^4+2x^5+3x^6+4x^7+5x^8+\cdots\right)$$

Exercise

Find the Frobenius series solutions of $x^2y'' - \left(x - \frac{2}{9}\right)y = 0$

Solution

$$x^2y'' - \left(x - \frac{2}{9}\right)y = 0$$

$$y'' - \left(\frac{1}{x} - \frac{2}{9x^2}\right)y = 0$$

That implies to $p(x) = 0$ and $q(x) = \frac{2}{9x^2} - \frac{1}{x}$.

$$\begin{aligned} q_0 &= \lim_{x\to 0} x^2 q(x) \\ &= \lim_{x\to 0} x^2\left(\frac{2}{9x^2} - \frac{1}{x}\right) \\ &= \lim_{x\to 0} \left(\frac{2}{9} - x\right) \\ &= \frac{2}{9} \end{aligned}$$

The indicial equation is: $r(r-1) + \frac{2}{9} = r^2 - r + \frac{2}{9} = 0$

$$9r^2 - 9r + 2 = 0$$

$$r = \frac{9 \pm 3}{18} = \frac{1}{3}, \frac{2}{3}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/3}\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{2/3}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$x^2y'' - \left(x - \frac{2}{9}\right)y = 0$$

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}-\left(x-\tfrac{2}{9}\right)\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}-\sum_{n=0}^{\infty}a_n x^{n+r+1}+\sum_{n=0}^{\infty}\tfrac{2}{9}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)+\tfrac{2}{9}\right]a_n x^{n+r}-\sum_{n=1}^{\infty}a_{n-1}x^{n+r}=0$$

$$\left(r^2-r+\tfrac{2}{9}\right)a_0+\sum_{n=1}^{\infty}\left[(n+r)(n+r-1)+\tfrac{2}{9}\right]a_n x^{n+r}-\sum_{n=1}^{\infty}a_{n-1}x^{n+r}=0$$

$$\left(r^2-r+\tfrac{2}{9}\right)a_0+\sum_{n=1}^{\infty}\left[\left((n+r)(n+r-1)+\tfrac{2}{9}\right)a_n-a_{n-1}\right]x^{n+r}=0$$

For $n=0$

$$\left(r^2-r+\tfrac{2}{9}\right)a_0=0$$

$$\Rightarrow \boxed{r=\tfrac{1}{3},\ \tfrac{2}{3}}\ \checkmark$$

$$\left((n+r)(n+r-1)+\tfrac{2}{9}\right)a_n-a_{n-1}=0$$

$$\boxed{a_n=\frac{1}{(n+r)(n+r-1)+\frac{2}{9}}a_{n-1}}$$

$r=\frac{1}{3}$

$$a_n=\frac{1}{\left(n+\frac{1}{3}\right)\left(n-\frac{2}{3}\right)+\frac{2}{9}}a_{n-1}$$

$$=\frac{1}{n^2-\frac{1}{3}n}a_{n-1}$$

$$=\boxed{\frac{3}{3n^2-n}a_{n-1}}$$

$n=1 \rightarrow a_1=\frac{3}{2}a_0$

$n=2 \rightarrow a_2=\frac{3}{10}a_1=\frac{9}{20}a_0$

$n=3 \rightarrow a_3=\frac{1}{8}a_2=\frac{9}{160}a_0$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_1(x) = a_0 x^{1/3}\left(1+\frac{3}{2}x+\frac{9}{20}x^2+\frac{9}{160}x^3+\cdots\right)$$

$$r = \frac{2}{3}$$

$$b_n = \frac{1}{\left(n+\frac{2}{3}\right)\left(n-\frac{1}{3}\right)+\frac{2}{9}}b_{n-1}$$

$$= \frac{3}{3n^2+n}b_{n-1}$$

$$n=1 \rightarrow b_1 = \frac{3}{4}b_0$$

$$n=2 \rightarrow b_2 = \frac{3}{14}b_1 = \frac{9}{56}b_0$$

$$n=3 \rightarrow b_3 = \frac{1}{10}b_2 = \frac{9}{560}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{2/3}\left(1+\frac{3}{4}x+\frac{9}{56}x^2+\frac{9}{560}x^3+\cdots\right)$$

$$y(x) = a_0 x^{1/3}\left(1+\frac{3}{2}x+\frac{9}{20}x^2+\frac{9}{160}x^3+\cdots\right)+b_0 x^{2/3}\left(1+\frac{3}{4}x+\frac{9}{56}x^2+\frac{9}{560}x^3+\cdots\right)$$

Exercise

Find the Frobenius series solutions of $x^2y''+x(3+x)y'-3y=0$

Solution

$$\frac{1}{x^2}x^2y''+\frac{1}{x^2}x(3+x)y'-3\frac{1}{x^2}y=0$$

$$y''+\left(\frac{3}{x}+1\right)y'-\frac{3}{x^2}y=0$$

That implies to $p(x)=\frac{3}{x}+1$ and $q(x)=-\frac{3}{x^2}$.

$$p_0 = \lim_{x\to 0} xp(x)$$

$$= \lim_{x\to 0} x\left(\frac{3}{x}+1\right)$$

$$= \lim_{x\to 0} (3+x)$$

$$=3$$

$$\begin{aligned} q_0 &= \lim_{x\to 0} x^2 q(x) \\ &= -\lim_{x\to 0} x^2 \frac{3}{x^2} \\ &= -3 \end{aligned}$$

The indicial equation is: $r(r-1)+3r-3=r^2+2r-3=0$

$$\rightarrow \ r=1, \ -3$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x)=x\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x)=x^{-3}\sum_{n=0}^{\infty} b_n x^n$$

$$y=\sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x^2y''+x(3+x)y'-3y=0$$

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\left(3x+x^2\right)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}-3\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}+\sum_{n=0}^{\infty}3(n+r)a_n x^{n+r}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1}-\sum_{n=0}^{\infty}3a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)+3(n+r)-3\right]a_n x^{n+r}+\sum_{n=1}^{\infty}(n+r-1)a_{n-1}x^{n+r}=0$$

$$\left(r^2+2r-3\right)a_0+\sum_{n=1}^{\infty}\left[(n+r)(n+r+2)-3\right]a_n x^{n+r}+\sum_{n=1}^{\infty}(n+r-1)a_{n-1}x^{n+r}=0$$

$$\left(r^2+2r-3\right)a_0+\sum_{n=1}^{\infty}\left[\left((n+r)(n+r+2)-3\right)a_n+(n+r-1)a_{n-1}\right]x^{n+r}=0$$

For $n=0$

$$\left(r^2+2r-3\right)a_0=0$$

$$\Rightarrow \underline{r=1,\ -3}\rfloor \quad \checkmark$$

$$\left((n+r)(n+r+2)-3\right)a_n+(n+r-1)a_{n-1}=0$$

$$\underline{a_n=-\frac{n+r-1}{(n+r)(n+r+2)-3}a_{n-1}}\rfloor$$

$r=1$

$$a_n=-\frac{n}{(n+1)(n+3)-3}a_{n-1}$$

$$\underline{=-\frac{n}{n^2+4n}a_{n-1}}\rfloor$$

$$n=1 \rightarrow a_1=-\frac{1}{5}a_0$$

$$n=2 \rightarrow a_2=-\frac{2}{12}a_1=\frac{1}{30}a_0$$

$$n=3 \rightarrow a_3=-\frac{3}{21}a_2=-\frac{1}{210}a_0$$

$$n=4 \rightarrow a_4=-\frac{4}{32}a_3=\frac{1}{1,680}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x)=a_0x\left(1-\frac{1}{5}x+\frac{1}{30}x^2-\frac{1}{210}x^3+\frac{1}{1,680}x^4-\cdots\right)}\rfloor$$

$r=-3$

$$b_n=-\frac{n-4}{(n-3)(n-1)-3}b_{n-1}$$

$$\underline{=-\frac{n-4}{n^2-4n}b_{n-1}}\rfloor$$

$$n=1 \rightarrow b_1=-b_0$$

$$n=2 \rightarrow b_2=-\frac{2}{4}b_1=\frac{1}{2}b_0$$

$$n=3 \rightarrow b_3=-\frac{1}{3}b_2=-\frac{1}{6}b_0$$

$$n=4 \rightarrow b_4=0b_3=0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_2(x)=b_0x^{-3}\left(1-x+\frac{1}{2}x^2-\frac{1}{6}x^3\right)}\rfloor$$

$$y(x)=a_0x\left(1-\frac{1}{5}x+\frac{1}{30}x^2-\frac{1}{210}x^3+\frac{1}{1,680}x^4-\cdots\right)+b_0x^{-3}\left(1-x+\frac{1}{2}x^2-\frac{1}{6}x^3\right)$$

Exercise

Find the Frobenius series solutions of $x^2y''+\left(x^2-2x\right)y'+2y=0$

Solution

$$\frac{1}{x^2}x^2y''+\frac{1}{x^2}\left(x^2-2x\right)y'+2\frac{1}{x^2}y=0$$

$$y''+\left(1-\frac{2}{x}\right)y'+\frac{2}{x^2}y=0$$

That implies to $p(x)=1-\frac{2}{x}$ and $q(x)=\frac{2}{x^2}$.

$$\begin{aligned}p_0&=\lim_{x\to 0}xp(x)\\&=\lim_{x\to 0}x\left(1-\frac{2}{x}\right)\\&=\lim_{x\to 0}(x-2)\\&=-2\end{aligned}$$

$$\begin{aligned}q_0&=\lim_{x\to 0}x^2q(x)\\&=\lim_{x\to 0}x^2\frac{2}{x^2}\\&=2\end{aligned}$$

The indicial equation is: $r(r-1)-2r+2=r^2-3r+2=0$

$$\rightarrow r=1,\ 2$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x)=x\sum_{n=0}^{\infty}a_nx^n \quad \text{and} \quad y_2(x)=x^2\sum_{n=0}^{\infty}b_nx^n$$

$$y=\sum_{n=0}^{\infty}a_nx^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$x^2y'' + \left(x^2 - 2x\right)y' + 2y = 0$$

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \left(x^2-2x\right)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + 2\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1} - \sum_{n=0}^{\infty}2(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty}2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)-2(n+r)+2\right]a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-3)+2\right]a_n x^{n+r} + \sum_{n=1}^{\infty}(n+r-1)a_{n-1} x^{n+r} = 0$$

$$\left(r^2-3r+2\right)a_0 + \sum_{n=1}^{\infty}\left[(n+r)(n+r-3)+2\right]a_n x^{n+r} + \sum_{n=1}^{\infty}(n+r-1)a_{n-1} x^{n+r} = 0$$

$$\left(r^2-3r+2\right)a_0 + \sum_{n=1}^{\infty}\left[\left((n+r)(n+r-3)+2\right)a_n + (n+r-1)a_{n-1}\right]x^{n+r} = 0$$

For $n=0$

$$\left(r^2-3r+2\right)a_0 = 0$$

$$\Rightarrow \underline{r = 1,\ 2} \quad \checkmark$$

$$\left((n+r)(n+r-3)+2\right)a_n + (n+r-1)a_{n-1} = 0$$

$$a_n = -\frac{n+r-1}{(n+r-1)(n+r-2)}a_{n-1}$$

$$= -\frac{1}{n+r-2}a_{n-1}$$

$r = 2$

$$a_n = -\frac{1}{n}a_{n-1}$$

$$n = 1 \ \rightarrow\ a_1 = -a_0$$

$$n=2 \;\to\; a_2 = -\frac{1}{2}a_1 = \frac{1}{2}a_0$$

$$n=3 \;\to\; a_3 = -\frac{1}{3}a_2 = -\frac{1}{3!}a_0$$

$$n=4 \;\to\; a_4 = -\frac{1}{4}a_3 = \frac{1}{4!}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 x\left(1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \cdots\right)}\Big|$$

$r=1$

$$b_n = -\frac{1}{n-1}b_{n-1}$$

Since $n \neq 1$

$$y(x) = a_0 x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \cdots\right) + b_0 x\ln x\left(-x + x^2 - \frac{x^3}{2} + \frac{x^4}{6} - \cdots\right)$$
$$+ x\ln x\left(1 - x + \frac{x^3}{4} - \frac{5}{36}x^4 + \cdots\right)$$

Exercise

Find the Frobenius series solutions of $x^2y'' + \left(x^2 + 2x\right)y' - 2y = 0$

Solution

$$\frac{1}{x^2}x^2y'' + \frac{1}{x^2}\left(x^2 + 2x\right)y' - 2\frac{1}{x^2}y = 0$$

$$y'' + \left(1 + \frac{2}{x}\right)y' - \frac{2}{x^2}y = 0$$

That implies to $p(x) = 1 + \frac{2}{x}$ and $q(x) = -\frac{2}{x^2}$.

$$p_0 = \lim_{x\to 0} xp(x)$$
$$= \lim_{x\to 0} x\left(1 + \frac{2}{x}\right)$$
$$= \lim_{x\to 0} (x+2)$$
$$\underline{= 2}\,\rfloor$$

$$q_0 = \lim_{x\to 0} x^2 q(x)$$
$$= -\lim_{x\to 0} x^2 \frac{2}{x^2}$$

$$\underline{= -2}\,\big|$$

The indicial equation is: $r(r-1)+2r-2=r^2+r-2=0$

$$\to \underline{r=1,\ -2}\,\big|$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x)=x\sum_{n=0}^{\infty}a_n x^n \quad \textit{and} \quad y_2(x)=x^{-2}\sum_{n=0}^{\infty}b_n x^n$$

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$x^2y''+\left(x^2+2x\right)y'-2y=0$$

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\left(x^2+2x\right)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}-2\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1}+\sum_{n=0}^{\infty}2(n+r)a_n x^{n+r}-2\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)+2(n+r)-2\right]a_n x^{n+r}+\sum_{n=1}^{\infty}(n+r-1)a_{n-1}x^{n+r}=0$$

$$\left(r^2+r-2\right)a_0+\sum_{n=1}^{\infty}\left[(n+r)(n+r+1)-2\right]a_n x^{n+r}+\sum_{n=1}^{\infty}(n+r-1)a_{n-1}x^{n+r}=0$$

$$\left(r^2+r-2\right)a_0+\sum_{n=0}^{\infty}\left[\left((n+r)(n+r+1)-2\right)a_n+(n+r-1)a_{n-1}\right]x^{n+r}=0$$

For $n=0$

$$\left(r^2+r-2\right)a_0=0$$

$$\Rightarrow \underline{r=1,\ 2}\,\big| \quad \checkmark$$

$$\left((n+r)^2+(n+r)-2\right)a_n+(n+r-1)a_{n-1}=0$$

$$a_n=-\frac{n+r-1}{(n+r-1)(n+r+2)}a_{n-1}$$

$$\boxed{=-\frac{1}{n+r+2}a_{n-1}}$$

$r=1$

$$a_n=-\frac{1}{n+3}a_{n-1}$$

$$n=1\ \rightarrow\ a_1=-\frac{1}{4}a_0$$

$$n=2\ \rightarrow\ a_2=-\frac{1}{5}a_1=\frac{1}{20}a_0$$

$$n=3\ \rightarrow\ a_3=-\frac{1}{6}a_2=-\frac{1}{120}a_0$$

$$n=4\ \rightarrow\ a_4=-\frac{1}{7}a_3=\frac{1}{840}a_0$$

$$\vdots\quad\vdots\quad\vdots\quad\vdots$$

$$\boxed{y_1(x)=a_0x\left(1-\frac{1}{4}x+\frac{1}{20}x^2-\frac{1}{120}x^3+\frac{1}{840}x^4-\cdots\right)}$$

$r=2$

$$b_n=-\frac{1}{n+4}b_{n-1}$$

$$n=1\ \rightarrow\ b_1=-\frac{1}{5}b_0$$

$$n=2\ \rightarrow\ b_2=-\frac{1}{6}b_1=\frac{1}{30}b_0$$

$$n=3\ \rightarrow\ b_3=-\frac{1}{7}b_2=-\frac{1}{210}b_0$$

$$n=4\ \rightarrow\ b_4=-\frac{1}{8}b_3=\frac{1}{1,680}b_0$$

$$\vdots\quad\vdots\quad\vdots\quad\vdots$$

$$\boxed{y_2(x)=b_0x^2\left(1-\frac{x}{5}+\frac{x^2}{30}-\frac{x^3}{210}+\frac{x^4}{1,680}-\cdots\right)}$$

$$\boxed{y(x)=a_0x\left(1-\frac{x}{4}+\frac{x^2}{20}-\frac{x^3}{120}+\frac{x^4}{840}-\cdots\right)+b_0x^2\left(1-\frac{x}{5}+\frac{x^2}{30}-\frac{x^3}{210}+\frac{x^4}{1,680}-\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2xy'' + 3y' - y = 0$

Solution

$$\frac{x}{2}2xy'' + 3\frac{x}{2}y' - \frac{x}{2}y = 0$$

$$x^2y'' + \frac{3}{2}xy' - \frac{1}{2}xy = 0$$

$$y'' + \frac{3}{2x}y' - \frac{1}{2x}y = 0$$

That implies to $p(x) = \frac{3}{2x}$ and $q(x) = -\frac{1}{2x}$

$$p_0 = \lim_{x\to 0} xp(x)$$

$$= \lim_{x\to 0} x\frac{3}{2x}$$

$$= \frac{3}{2}$$

$$q_0 = \lim_{x\to 0} x^2q(x)$$

$$= \lim_{x\to 0} x^2\frac{1}{2x}$$

$$= 0$$

The indicial equation is: $r(r-1) + \frac{3}{2}r = r^2 + \frac{1}{2}r = 0$

$$\to r = 0,\ -\frac{1}{2}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{-1/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$2xy'' + 3y' - y = 0$$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\sum_{n=0}^{\infty}3(n+r)a_n x^{n+r-1}-\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r-1}+\sum_{n=0}^{\infty}3(n+r)a_n x^{n+r-1}-\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left[2(n+r)(n+r-1)a_n+3(n+r)\right]x^{n+r-1}-\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}=0$$

$$r(2r+1)a_0+\sum_{n=1}^{\infty}(n+r)(2n+2r+1)a_n x^{n+r-1}-\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}=0$$

$$r(2r+1)a_0+\sum_{n=1}^{\infty}\left[(n+r)(2n+2r+1)a_n-a_{n-1}\right]x^{n+r-1}=0$$

For $n=0$

$$r(2r+1)a_0=0$$

$$\Rightarrow \boxed{r=0,\quad -\tfrac{1}{2}} \quad \checkmark$$

$$(n+r)(2n+2r+1)a_n-a_{n-1}=0$$

$$\boxed{a_n=\frac{1}{(n+r)(2n+2r+1)}a_{n-1}}$$

$r=0$

$$a_n=\frac{1}{n(2n+1)}a_{n-1}$$

$$n=1 \;\rightarrow\; a_1=\tfrac{1}{3}a_0$$

$$n=2 \;\rightarrow\; a_2=\tfrac{1}{15}a_1=\tfrac{1}{30}a_0$$

$$n=3 \;\rightarrow\; a_3=\tfrac{1}{21}a_2=\tfrac{1}{630}a_0$$

$$n=4 \;\rightarrow\; a_4=\tfrac{1}{36}a_3=\tfrac{1}{22,680}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x)=a_0\left(1+\tfrac{1}{3}x+\tfrac{1}{30}x^2+\tfrac{1}{630}x^3+\tfrac{1}{22,680}x^4+\cdots\right)}$$

$r=-\frac{1}{2}$

$b_n=\frac{1}{n(2n-1)}b_{n-1}$

$n=1 \ \rightarrow \ b_1=b_0$

$n=2 \ \rightarrow \ b_2=\frac{1}{6}b_1=\frac{1}{6}b_0$

$n=3 \ \rightarrow \ b_3=\frac{1}{15}b_2=\frac{1}{90}b_0$

$n=4 \ \rightarrow \ b_4=\frac{1}{28}b_3=\frac{1}{2,520}b_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_2(x)=b_0x^{-1/2}\left(1+x+\frac{1}{6}x^2+\frac{1}{90}x^3+\frac{1}{2,520}x^4+\cdots\right)$$

$$y(x)=a_0\left(1+\frac{1}{3}x+\frac{1}{30}x^2+\frac{1}{630}x^3+\frac{1}{22,680}x^4+\cdots\right)+b_0x^{-1/2}\left(1+x+\frac{1}{6}x^2+\frac{1}{90}x^3+\frac{1}{2,520}x^4+\cdots\right)$$

$$y(x)=a_0\sum_{n=0}^{\infty}\frac{x^n}{n!(2n+1)!!}+\frac{b_0}{\sqrt{x}}\left(1+\sum_{n=0}^{\infty}\frac{x^n}{n!(2n-1)!!}\right)$$

Exercise

Find the Frobenius series solutions of $2xy''-y'-y=0$

Solution

$\frac{1}{2x}2xy''-\frac{1}{2x}y'-\frac{1}{2x}y=0$

$y''-\frac{1}{2x}y'-\frac{1}{2x}y=0$

That implies to $p(x)=-\frac{1}{2x}$ and $q(x)=\frac{1}{2x}$

$$\begin{aligned} p_0 &= \lim_{x\to 0} xp(x) \\ &= \lim_{x\to 0} x\frac{-1}{2x} \\ &= -\frac{1}{2} \end{aligned}$$

$$q_0 = \lim_{x\to 0} x^2q(x)$$

$$= \lim_{x\to 0} x^2 \frac{-1}{2x}$$

$$= -\lim_{x\to 0} \frac{x}{2}$$

$$= 0$$

The indicial equation is: $r(r-1)-\frac{1}{2}r = r^2-\frac{3}{2}r = 0$

$$\rightarrow \boxed{r = 0, \ \frac{3}{2}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{3/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2xy'' - y' - y = 0$$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[2(n+r)(n+r-1)a_n - (n+r)\right]x^{n+r-1} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

$$r(2r-3)a_0 + \sum_{n=1}^{\infty}(n+r)(2n+2r-3)a_n x^{n+r-1} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

$$r(2r-3)a_0 + \sum_{n=1}^{\infty}\left[(n+r)(2n+2r-3)a_n - a_{n-1}\right]x^{n+r-1} = 0$$

For $n=0$

$$r(2r-3)a_0 = 0$$

$$\Rightarrow \boxed{r = 0,\ \frac{3}{2}}\ \checkmark$$

$$(n+r)(2n+2r-3)a_n - a_{n-1} = 0$$

$$\boxed{a_n = \frac{1}{(n+r)(2n+2r-3)}a_{n-1}}$$

$r=0$

$$a_n = \frac{1}{n(2n-3)}a_{n-1}$$

$n=1 \rightarrow a_1 = -a_0$

$n=2 \rightarrow a_2 = \frac{1}{2}a_1 = -\frac{1}{2}a_0$

$n=3 \rightarrow a_3 = \frac{1}{9}a_2 = -\frac{1}{18}a_0$

$n=4 \rightarrow a_4 = \frac{1}{20}a_3 = -\frac{1}{360}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0\left(1 - x - \frac{1}{2}x^2 - \frac{1}{18}x^3 - \frac{1}{360}x^4 - \cdots\right)}$$

$r = \frac{3}{2}$

$$b_n = \frac{1}{n(2n+3)}b_{n-1}$$

$n=1 \rightarrow b_1 = \frac{1}{5}b_0$

$n=2 \rightarrow b_2 = \frac{1}{14}b_1 = \frac{1}{70}b_0$

$n=3 \rightarrow b_3 = \frac{1}{27}b_2 = \frac{1}{1890}b_0$

$n=4 \rightarrow b_4 = \frac{1}{44}b_3 = \frac{1}{83{,}160}b_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_2(x) = b_0 x^{3/2}\left(1 + \frac{1}{5}x + \frac{1}{70}x^2 + \frac{1}{1890}x^3 + \frac{1}{83{,}160}x^4 + \cdots\right)}$$

$$\boxed{y(x) = a_0\left(1 - x - \frac{1}{2}x^2 - \frac{1}{18}x^3 - \frac{1}{360}x^4 + \cdots\right) + b_0 x^{3/2}\left(1 + \frac{1}{5}x + \frac{1}{70}x^2 + \frac{1}{1{,}890}x^3 + \frac{1}{83{,}160}x^4 - \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2xy'' + (1+x)y' + y = 0$

Solution

$$\frac{1}{2x}2xy'' + \frac{1}{2x}(1+x)y' + \frac{1}{2x}y = 0$$

$$y'' + \left(\frac{1}{2x} + \frac{1}{2}\right)y' + \frac{1}{2x}y = 0$$

That implies to $p(x) = \frac{1}{2x} + \frac{1}{2}$ and $q(x) = \frac{1}{2x}$

$$\begin{aligned} p_0 &= \lim_{x\to 0} xp(x) \\ &= \lim_{x\to 0} x\left(\frac{1}{2x} + \frac{1}{2}\right) \\ &= \lim_{x\to 0} \left(\frac{1}{2} + \frac{1}{2}x\right) \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x\to 0} x^2 q(x) \\ &= \lim_{x\to 0} x^2 \frac{1}{2x} \\ &= \lim_{x\to 0} \frac{x}{2} \\ &= 0 \end{aligned}$$

The indicial equation is: $r(r-1) + \frac{1}{2}r = r^2 - \frac{1}{2}r = 0$

$$\rightarrow \underline{r = 0,\ \frac{1}{2}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{1/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$2xy'' + (1+x)y' + y = 0$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + (1+x)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r}$$

$$+\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[2(n+r)(n+r-1)+(n+r)\right]a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r+1)a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-1)\right]a_n x^{n+r-1} + \sum_{n=1}^{\infty}(n+r)a_{n-1}x^{n+r-1} = 0$$

$$r(2r-1)a_0 + \sum_{n=1}^{\infty}(n+r)(2n+2r-1)a_n x^{n+r-1} + \sum_{n=1}^{\infty}(n+r)a_{n-1}x^{n+r-1} = 0$$

$$r(2r-1)a_0 + \sum_{n=1}^{\infty}\left[(n+r)(2n+2r-1)a_n + (n+r)a_{n-1}\right]x^{n+r-1} = 0$$

For $n = 0$

$$r(2r-1)a_0 = 0$$

$$\Rightarrow \boxed{r = 0,\ \tfrac{1}{2}}\ \checkmark$$

$$(n+r)(2n+2r-1)a_n + (n+r)a_{n-1} = 0$$

$$\boxed{a_n = -\frac{1}{2n+2r-1}a_{n-1}}$$

$r = 0$

$$a_n = -\frac{1}{2n-1}a_{n-1}$$

$n=1 \;\to\; a_1 = -a_0$

$n=2 \;\to\; a_2 = \frac{1}{3}a_1 = -\frac{1}{3}a_0$

$n=3 \;\to\; a_3 = -\frac{1}{5}a_2 = \frac{1}{15}a_0$

$n=4 \;\to\; a_4 = -\frac{1}{7}a_3 = -\frac{1}{105}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0\left(1 - x + \frac{1}{3}x^2 - \frac{1}{15}x^3 + \frac{1}{105}x^4 - \cdots\right)}$$

$r = \frac{1}{2}$

$b_n = -\frac{1}{2n}b_{n-1}$

$n=1 \;\to\; b_1 = -\frac{1}{2}b_0$

$n=2 \;\to\; b_2 = -\frac{1}{4}b_1 = \frac{1}{8}b_0$

$n=3 \;\to\; b_3 = -\frac{1}{6}b_2 = -\frac{1}{48}b_0$

$n=4 \;\to\; b_4 = -\frac{1}{8}b_3 = -\frac{1}{384}b_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_2(x) = b_0 x^{1/2}\left(1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4 + \cdots\right)}$$

$$\boxed{y(x) = a_0\left(1 - x + \frac{1}{3}x^2 - \frac{1}{15}x^3 + \frac{1}{105}x^4 - \cdots\right) + b_0 x^{1/2}\left(1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4 + \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2xy'' + \left(1 - 2x^2\right)y' - 4xy = 0$

Solution

$\frac{x}{2}2xy'' + \frac{x}{2}\left(1 - 2x^2\right)y' - \frac{x}{2}4xy = 0$

$x^2y'' + \left(\frac{1}{2}x - x^3\right)y' + 2x^2y = 0$

$y'' + \left(\frac{1}{2x} - x\right)y' + 2y = 0$

That implies to $p(x) = \frac{1}{2x} - x$ and $q(x) = 2$

$$\begin{aligned} p_0 &= \lim_{x\to 0} xp(x) \\ &= \lim_{x\to 0} x\left(\frac{1}{2x}-x\right) \\ &= \lim_{x\to 0} \left(\frac{1}{2}-x^2\right) \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x\to 0} x^2 q(x) \\ &= \lim_{x\to 0} 2x^2 \\ &= 0 \end{aligned}$$

The indicial equation is: $r(r-1)+\frac{1}{2}r = r^2-\frac{1}{2}r = 0$

$$\rightarrow \underline{r = 0,\ \frac{1}{2}}\Big|$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$2xy'' + \left(1-2x^2\right)y' - 4xy = 0$$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \left(1-2x^2\right)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - 4x\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty} 2(n+r)a_n x^{n+r+1}$$
$$-\sum_{n=0}^{\infty} 4a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-2)+(n+r)\right]a_n x^{n+r-1} - \sum_{n=0}^{\infty}(2n+2r+4)a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-2)+(n+r)\right]a_n x^{n+r-1} - \sum_{n=2}^{\infty}(2n+2r)a_{n-2} x^{n+r-1} = 0$$

$$r(2r-1)a_0 + (r+2)(2r+1)a_1 + \sum_{n=2}^{\infty}(n+r)(2n+2r-1)a_n x^{n+r-1}$$
$$-\sum_{n=2}^{\infty}(2n+2r)a_{n-2} x^{n+r-1} = 0$$

$$r(2r-1)a_0 + (r+2)(2r+1)a_1 + \sum_{n=2}^{\infty}\left[(n+r)(2n+2r-1)a_n - 2(n+r)a_{n-2}\right]x^{n+r-1} = 0$$

For $n=0$

$$r(2r+1)a_0 = 0$$

$$\Rightarrow \boxed{r = 0, \quad -\frac{1}{2}} \quad \checkmark$$

For $n=1$

$$(r+2)(2r+1)a_1 = 0$$

$$\Rightarrow \boxed{\cancel{r = -2, \quad -\frac{1}{2}}}$$

$$\boxed{a_1 = 0}$$

$$(n+r)(2n+2r-1)a_n - 2(n+r)a_{n-2} = 0$$

$$\boxed{a_n = \frac{2}{2n+2r-1}a_{n-2}}$$

$r = 0$

$$a_n = \frac{2}{2n-1}a_{n-2}$$

$n=2 \rightarrow a_2 = \frac{2}{3}a_0$

$n=3 \rightarrow a_3 = \frac{2}{5}a_1 = 0$

$n=4 \rightarrow a_4 = \frac{2}{7}a_2 = \frac{4}{21}a_0$

$n=5 \rightarrow a_5 = \frac{2}{9}a_3 = 0$

$n=6 \rightarrow a_6 = \frac{2}{11}a_4 = \frac{8}{231}a_0$

$n=7 \rightarrow a_7 = 0$

$n=8 \rightarrow a_8 = \frac{2}{15}a_6 = \frac{16}{3,465}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_1(x) = a_0\left(1+\frac{2}{3}x^2+\frac{4}{21}x^4+\frac{8}{231}x^6+\frac{16}{3465}x^8+\cdots\right)$$

$r=\frac{1}{2}$

$b_n = \frac{1}{n}b_{n-2}$

$n=2 \rightarrow b_2 = \frac{1}{2}b_0$

$n=3 \rightarrow b_3 = \frac{1}{3}b_1 = 0$

$n=4 \rightarrow b_4 = \frac{1}{4}b_2 = \frac{1}{8}b_0$

$n=5 \rightarrow b_5 = \frac{1}{5}b_3 = 0$

$n=6 \rightarrow b_6 = \frac{1}{6}b_4 = \frac{1}{48}b_0$

$n=7 \rightarrow b_7 = 0$

$n=8 \rightarrow b_8 = \frac{1}{8}b_6 = \frac{1}{384}b_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_2(x) = b_0 x^{1/2}\left(1+\frac{1}{2}x^2+\frac{1}{8}x^4+\frac{1}{48}x^6+\frac{1}{384}x^8+\cdots\right)$$

$$\begin{aligned} y(x) = a_0&\left(1+\frac{2}{3}x^2+\frac{4}{21}x^4+\frac{8}{231}x^6+\frac{16}{3465}x^8+\cdots\right) \\ &+b_0x^{1/2}\left(1+\frac{1}{2}x^2+\frac{1}{8}x^4+\frac{1}{48}x^6+\frac{1}{384}x^8+\cdots\right)\end{aligned}$$

Exercise

Find the Frobenius series solutions of $2x^2y''+xy'-\left(1+2x^2\right)y=0$

Solution

$$\frac{1}{2}2x^2y''+\frac{1}{2}xy'-\frac{1}{2}\left(1+2x^2\right)y=0$$

$$x^2y''+\frac{1}{2}xy'-\frac{1}{2}\left(1+2x^2\right)y=0$$

$$y''+\frac{1}{2x}y'-\left(\frac{1}{2x^2}+1\right)y=0$$

That implies to $p(x)=\frac{1}{2x}$ and $q(x)=-\frac{1}{2x^2}-1$

$$\begin{aligned}
p_0 &= \lim_{x\to 0} xp(x) \\
&= \lim_{x\to 0} x\frac{1}{2x} \\
&= \lim_{x\to 0} \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
q_0 &= \lim_{x\to 0} x^2 q(x) \\
&= \lim_{x\to 0} x^2\left(-\frac{1}{2x^2}-1\right) \\
&= \lim_{x\to 0} x^2\left(-\frac{1}{2}-x^2\right) \\
&= -\frac{1}{2}
\end{aligned}$$

The indicial equation is: $r(r-1)+\frac{1}{2}r-\frac{1}{2}=r^2-\frac{1}{2}r-\frac{1}{2}=0$

$$\rightarrow \underline{r=1,\ -\frac{1}{2}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x)=x^1\sum_{n=0}^{\infty}a_n x^n \quad \textit{and} \quad y_2(x)=x^{-1/2}\sum_{n=0}^{\infty}b_n x^n$$

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$2x^2y''+xy'-\left(1+2x^2\right)y=0$$

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}-\left(1+2x^2\right)\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}-\sum_{n=0}^{\infty}a_n x^{n+r}-\sum_{n=0}^{\infty}2a_n x^{n+r+2}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-2)+(n+r)-1\right]a_n x^{n+r}-\sum_{n=2}^{\infty}2a_{n-2}x^{n+r}=0$$

$$\left(2r^2-r-1\right)a_0+\left((r+1)(2r+1)-1\right)a_1+\sum_{n=2}^{\infty}\left[(n+r)(2n+2r-1)-1\right]a_n x^{n+r}$$

$$-\sum_{n=2}^{\infty}2a_{n-2}x^{n+r}=0$$

$$\left(2r^2-r-1\right)a_0+\left(2r^2+3r\right)a_1+\sum_{n=2}^{\infty}\left[\left((n+r)(2n+2r-1)-1\right)a_n-2a_{n-2}\right]x^{n+r}=0$$

For $n=0$

$$\left(2r^2-r-1\right)a_0=0$$

$$\Rightarrow \boxed{r=1,\quad -\frac{1}{2}}\ \checkmark$$

For $n=1$

$$r(2r+3)a_1=0$$

$$\Rightarrow \boxed{\cancel{r=0,\ -\frac{3}{2}}}$$

$$\boxed{a_1=0}$$

$$\left((n+r)(2n+2r-1)-1\right)a_n-2a_{n-2}=0$$

$$\boxed{a_n=\frac{2}{(n+r)(2n+2r-1)-1}a_{n-2}}$$

$r=1$

$$a_n=\frac{2}{(n+1)(2n+1)-1}a_{n-2}$$

$$=\frac{2}{2n^2+3n}a_{n-2}$$

$n=2 \;\rightarrow\; a_2 = \frac{2}{14}a_0 = \frac{1}{7}a_0$ $\qquad$ $n=3 \;\rightarrow\; a_3 = \frac{2}{27}a_1 = 0$

$n=4 \;\rightarrow\; a_4 = \frac{2}{44}a_2 = \frac{1}{154}a_0$ $\qquad$ $n=5 \;\rightarrow\; a_5 = \frac{2}{65}a_3 = 0$

$n=6 \;\rightarrow\; a_6 = \frac{2}{90}a_4 = \frac{1}{6,390}a_0$ $\qquad$ $n=7 \;\rightarrow\; a_7 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\underline{y_1(x) = a_0 x\left(1+\frac{1}{7}x^2+\frac{1}{154}x^4+\frac{1}{6,930}x^6+\cdots\right)}$$

$r=-\frac{1}{2}$

$$b_n = \frac{2b_{n-2}}{\left(n-\frac{1}{2}\right)(2n-2)-1}$$

$$= \frac{2}{2n^2-3n}b_{n-2}$$

$n=2 \;\rightarrow\; b_2 = \frac{2}{2}b_0 = b_0$ $\qquad$ $n=3 \;\rightarrow\; b_3 = \frac{2}{9}b_1 = 0$

$n=4 \;\rightarrow\; b_4 = \frac{2}{20}b_2 = \frac{1}{10}b_0$ $\qquad$ $n=5 \;\rightarrow\; b_5 = \frac{2}{35}b_3 = 0$

$n=6 \;\rightarrow\; b_6 = \frac{2}{54}b_4 = \frac{1}{270}b_0$ $\qquad$ $n=7 \;\rightarrow\; b_7 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\underline{y_2(x) = b_0 x^{-1/2}\left(1+x^2+\frac{1}{10}x^4+\frac{1}{270}x^6+\cdots\right)}$$

$$\underline{y(x) = a_0 x\left(1+\frac{1}{7}x^2+\frac{1}{154}x^4+\frac{1}{6,930}x^6+\cdots\right)+b_0 x^{-1/2}\left(1+x^2+\frac{1}{10}x^4+\frac{1}{270}x^6+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2x^2y''+xy'-\left(3-2x^2\right)y=0$

Solution

$$\frac{1}{2x^2}2x^2y''+\frac{1}{2x^2}xy'-\frac{1}{2x^2}\left(3-2x^2\right)y=0$$

$$y''+\frac{1}{2x}y'-\frac{1}{2x^2}\left(3-2x^2\right)y=0$$

That implies to $p(x)=\frac{1}{2x}$ and $q(x)=-\frac{1}{2x^2}\left(3-2x^2\right)$

$$p_0 = \lim_{x\to 0} xp(x)$$
$$= \lim_{x\to 0} x\frac{1}{2x}$$
$$= \frac{1}{2}$$

$$q_0 = \lim_{x\to 0} x^2 q(x)$$
$$= -\lim_{x\to 0} x^2\left(\frac{1}{2x^2}\left(3-2x^2\right)\right)$$
$$= -\lim_{x\to 0} \left(\frac{1}{2}\left(3-2x^2\right)\right)$$
$$= \frac{3}{2}$$

The indicial equation is: $r(r-1)+\frac{1}{2}r-\frac{3}{2} = r^2-\frac{1}{2}r-\frac{3}{2} = 0$

$$\to \boxed{r=-1,\ \frac{3}{2}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{-1}\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{3/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x^2y'' + xy' - \left(3-2x^2\right)y = 0$$

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \left(3-2x^2\right)\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty}3a_n x^{n+r} + \sum_{n=0}^{\infty}2a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-2)+(n+r)-3\right]a_n x^{n+r}+\sum_{n=2}^{\infty}2a_{n-2}x^{n+r}=0$$

$$\left(2r^2-r-3\right)a_0+\left((r+1)(2r+1)-3\right)a_1+\sum_{n=2}^{\infty}\left[(n+r)(2n+2r-1)-3\right]a_n x^{n+r}$$

$$+\sum_{n=2}^{\infty}2a_{n-2}x^{n+r}=0$$

$$\left(2r^2-r-3\right)a_0+\left(2r^2+3r-2\right)a_1+\sum_{n=2}^{\infty}\left[\left((n+r)(2n+2r-1)-3\right)a_n+2a_{n-2}\right]x^{n+r}=0$$

For $n=0$

$$\left(2r^2-r-3\right)a_0=0$$

$$\Rightarrow \boxed{r=-1,\ \frac{3}{2}}\ \checkmark$$

For $n=1$

$$\left(2r^2+3r-2\right)a_1=0$$

$$\Rightarrow \boxed{\cancel{r=-2,\ \frac{1}{2}}}$$

$$\boxed{a_1=0}$$

$$\left((n+r)(2n+2r-1)-3\right)a_n+2a_{n-2}=0$$

$$\boxed{a_n=-\frac{2}{(n+r)(2n+2r-1)-3}a_{n-2}}$$

$r=-1$

$$a_n=-\frac{2}{(n-1)(2n-3)-3}a_{n-2}$$

$$=-\frac{2}{2n^2-5n}a_{n-2}$$

$n=2 \rightarrow a_2=a_0$

$n=3 \rightarrow a_3=-\frac{2}{3}a_1=0$

$n=4 \rightarrow a_4=-\frac{2}{12}a_2=-\frac{1}{6}a_0$

$n=5 \rightarrow a_5=-\frac{2}{25}a_3=0$

$n=6 \rightarrow a_6=-\frac{2}{252}a_4=\frac{1}{126}a_0$

$n=7 \rightarrow a_7=0$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0 x^{-1}\left(1 + x^2 - \frac{1}{6}x^4 + \frac{1}{126}x^6 - \cdots\right)}$$

$$r = \frac{3}{2}$$

$$b_n = -\frac{2b_{n-2}}{\left(n+\frac{3}{2}\right)(2n+2)-3}$$

$$= -\frac{2}{2n^2+5n}b_{n-2}$$

$$n = 2 \ \rightarrow \ b_2 = -\frac{2}{18}b_0 = -\frac{1}{9}b_0 \qquad\qquad n = 3 \ \rightarrow \ b_3 = -\frac{2}{33}b_1 = 0$$

$$n = 4 \ \rightarrow \ b_4 = -\frac{2}{568}b_2 = \frac{1}{234}b_0 \qquad\qquad n = 5 \ \rightarrow \ b_5 = 0$$

$$n = 6 \ \rightarrow \ b_6 = \frac{2}{102}b_4 = \frac{1}{11{,}934}b_0 \qquad\qquad n = 7 \ \rightarrow \ b_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{3/2}\left(1 - \frac{1}{9}x^2 + \frac{1}{234}x^4 - \frac{1}{11{,}934}x^6 + \cdots\right)}$$

$$\boxed{y(x) = a_0 x^{-1}\left(1 + x^2 - \frac{1}{6}x^4 + \frac{1}{126}x^6 - \cdots\right) + b_0 x^{3/2}\left(1 - \frac{1}{9}x^2 + \frac{1}{234}x^4 - \frac{1}{11{,}934}x^6 + \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $3xy'' + 2y' + 2y = 0$

Solution

$$\frac{x}{3}3xy'' + 2\frac{x}{3}y' + 2\frac{x}{3}y = 0$$

$$x^2y'' + \frac{2}{3}xy' + \frac{2}{3}xy = 0$$

$$y'' + \frac{2}{3x}y' + \frac{2}{3x}y = 0$$

That implies to $p(x) = \frac{2}{3x}$ and $q(x) = \frac{2}{3x}$

$$p_0 = \lim_{x\to 0} xp(x)$$

$$= \lim_{x\to 0} x\frac{2}{3x}$$

$$= \frac{2}{3}$$

$$q_0 = \lim_{x\to 0} x^2 q(x)$$
$$= \lim_{x\to 0} x^2 \frac{2}{3x}$$
$$= \lim_{x\to 0} \frac{2}{3}x$$
$$= 0$$

The indicial equation is: $r(r-1)+\frac{2}{3}r = r^2 - \frac{1}{3}r = 0$

$$\to \ r = 0, \ \frac{1}{3}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/3} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$3xy'' + 2y' + 2y = 0$$

$$3x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + 2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + 2\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}3(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}2(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(3n+3r-3)+2(n+r)\right]a_n x^{n+r-1} + \sum_{n=1}^{\infty}2a_{n-1}x^{n+r-1} = 0$$

$$r(3r-1)a_n + \sum_{n=1}^{\infty}(n+r)(3n+3r-1)a_n x^{n+r-1} + \sum_{n=1}^{\infty}2a_{n-1}x^{n+r-1} = 0$$

$$r(3r-1)a_n + \sum_{n=1}^{\infty}\left[(n+r)(3n+3r-1)a_n + 2a_{n-1}\right]x^{n+r-1} = 0$$

For $n = 0$

$$r(3r-1)a_0 = 0$$

$$\Rightarrow \boxed{r = 0,\ \frac{1}{3}}\ \checkmark$$

$$(n+r)(3n+3r-1)a_n + 2a_{n-1} = 0$$

$$\boxed{a_n = -\frac{2}{(n+r)(3n+3r-1)}a_{n-1}}$$

$r = 0$

$$a_n = -\frac{2}{3n^2 - n}a_{n-1}$$

$n = 1 \rightarrow a_1 = -a_0$

$n = 2 \rightarrow a_2 = -\frac{1}{5}a_1 = \frac{1}{5}a_0$

$n = 3 \rightarrow a_3 = -\frac{2}{24}a_2 = -\frac{1}{60}a_0$

$n = 4 \rightarrow a_4 = -\frac{2}{44}a_3 = \frac{1}{1320}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0 x^0\left(1 - x + \frac{1}{5}x^2 - \frac{1}{60}x^3 + \frac{1}{1320}x^4 - \cdots\right)}$$

$r = \frac{1}{3}$

$$b_n = -\frac{2}{3n^2 + n}b_{n-1}$$

$n = 1 \rightarrow b_1 = -\frac{1}{2}b_0$

$n = 2 \rightarrow b_2 = -\frac{2}{14}b_1 = \frac{1}{14}b_0$

$n = 3 \rightarrow b_3 = -\frac{2}{30}b_2 = -\frac{1}{210}b_0$

$n = 4 \rightarrow b_4 = -\frac{2}{52}b_3 = \frac{1}{5460}b_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_2(x) = b_0 x^{1/3}\left(1 - \frac{1}{2}x + \frac{1}{14}x^2 - \frac{1}{210}x^3 + \frac{1}{5460}x^4 - \cdots\right)}$$

$$\boxed{y(x) = a_0\left(1 - x + \frac{1}{5}x^2 - \frac{1}{60}x^3 + \frac{1}{1320}x^4 - \cdots\right) + b_0 x^{1/3}\left(1 - \frac{1}{2}x + \frac{1}{14}x^2 - \frac{1}{210}x^3 + \frac{1}{5460}x^4 - \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $3x^2y'' + 2xy' + x^2y = 0$

Solution

$$\frac{1}{3}3x^2y'' + 2\frac{1}{3}xy' + \frac{1}{3}x^2y = 0$$

$$x^2y'' + \frac{2}{3}xy' + \frac{1}{3}x^2y = 0$$

$$y'' + \frac{2}{3x}y' + \frac{1}{3}y = 0$$

That implies to $p(x) = \frac{2}{3x}$ and $q(x) = \frac{1}{3}$.

$$\begin{aligned} p_0 &= \lim_{x\to 0} xp(x) \\ &= \lim_{x\to 0} x\frac{2}{3x} \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x\to 0} x^2q(x) \\ &= \lim_{x\to 0} x^2\frac{1}{3} \\ &= 0 \end{aligned}$$

The indicial equation is: $r(r-1) + \frac{2}{3}r = r^2 - \frac{1}{3}r = 0$

$$\rightarrow \boxed{r = 0,\ \frac{1}{3}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/3}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$3x^2y'' + 2xy' + x^2y = 0$$

$$3x^2\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} + 2x\sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} + x^2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 3(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} 2(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(3n+3r-3) + 2(n+r)\right]a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$r(3r-1)a_0 + (r+1)(3r+2)a_1 + \sum_{n=2}^{\infty} (n+r)(3n+3r-1)a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$r(3r-1)a_0 + (r+1)(3r+2)a_1 + \sum_{n=2}^{\infty} \left[(n+r)(3n+3r-1)a_n + a_{n-2}\right]x^{n+r} = 0$$

For $n = 0$

$$r(3r-1)a_0 = 0$$

$$\Rightarrow \boxed{r = 0,\ \frac{1}{3}} \quad \checkmark$$

For $n = 1$

$$(r+1)(3r+2)a_1 = 0$$

$$\Rightarrow \boxed{\cancel{r = -1,\ -\frac{2}{3}}}$$

$$\boxed{a_1 = 0}$$

$$(n+r)(3n+3r-1)a_n + a_{n-2} = 0$$

$$\boxed{a_n = -\frac{1}{(n+r)(3n+3r-1)}a_{n-2}}$$

$r = 0$

$$a_n = -\frac{1}{n(3n-1)}a_{n-2}$$

$$n=2 \to a_2 = -\frac{1}{10}a_0 \qquad n=3 \to a_3 = -\frac{1}{24}a_1 = 0$$

$$n=4 \to a_4 = -\frac{1}{44}a_2 = \frac{1}{440}a_0 \qquad n=5 \to a_5 = -\frac{1}{70}a_3 = 0$$

$$n=6 \to a_6 = -\frac{1}{102}a_4 = -\frac{1}{44,880}a_0 \qquad n=7 \to a_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 x^0\left(1 - \frac{1}{10}x^2 + \frac{1}{440}x^4 - \frac{1}{44,880}x^6 + \cdots\right)}$$

$$r = \frac{1}{3}$$

$$b_n = -\frac{1}{\left(n+\frac{1}{3}\right)(3n)}b_{n-2}$$

$$= -\frac{1}{n(3n+1)}b_{n-2}$$

$$n=2 \to b_2 = -\frac{1}{14}b_0 \qquad n=3 \to b_3 = -\frac{1}{30}b_1 = 0$$

$$n=4 \to b_4 = -\frac{1}{72}b_2 = \frac{1}{728}b_0 \qquad n=5 \to b_5 = 0$$

$$n=6 \to b_6 = -\frac{1}{114}b_4 = -\frac{1}{82,992}b_0 \qquad n=7 \to b_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_2(x) = b_0 x^{1/3}\left(1 - \frac{1}{14}x^2 + \frac{1}{728}x^4 - \frac{1}{82,992}x^6 + \cdots\right)}$$

$$\underline{y(x) = a_0\left(1 - \frac{1}{10}x^2 + \frac{1}{440}x^4 - \frac{1}{44,880}x^6 + \cdots\right) + b_0 x^{1/3}\left(1 - \frac{1}{14}x^2 + \frac{1}{728}x^4 - \frac{1}{82,992}x^6 + \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $3x^2y'' - xy' + y = 0$

Solution

$$\frac{1}{3}3x^2y'' - \frac{1}{3}xy' + \frac{1}{3}y = 0$$

$$x^2y'' - \frac{1}{3}xy' + \frac{1}{3}y = 0$$

$$y'' - \frac{1}{3x}y' + \frac{1}{3x^2}y = 0$$

That implies to $p(x) = -\frac{1}{3x}$ and $q(x) = \frac{1}{3x^2}$.

$$p_0 = \lim_{x\to 0} xp(x)$$
$$= -\lim_{x\to 0} x\frac{1}{3x}$$
$$= -\frac{1}{3}$$

$$q_0 = \lim_{x\to 0} x^2 q(x)$$
$$= \lim_{x\to 0} x^2 \frac{1}{3x^2}$$
$$= \frac{1}{3}$$

The indicial equation is: $r(r-1)-\frac{1}{3}r+\frac{1}{3}=r^2-\frac{4}{3}r+\frac{1}{3}=0$

$$\rightarrow r = 1,\ \frac{1}{3}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/3}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$3x^2y'' - xy' + y = 0$$

$$3x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 3(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(3n+3r-3)-(n+r)+1\right]a_n x^{n+r} = 0$$

Since neither of λ, then let assume $a_n = 0, \quad n \ge 1$

$$y_1(x) = x\sum_{n=0}^{\infty} a_n x^n \boxed{= a_0 x}$$

$$y_2(x) = x^{1/3}\sum_{n=0}^{\infty} b_n x^n \boxed{= b_0 x^{1/3}}$$

$$\boxed{y(x) = a_0 x + b_0 x^{1/3}}$$

Exercise

Find the Frobenius series solutions of $4xy'' + 2y' + y = 0$

Solution

$$\frac{x}{4}4xy'' + 2\frac{x}{4}y' + \frac{x}{4}y = 0$$

$$x^2 y'' + \frac{1}{2}xy' + \frac{x}{4}y = 0$$

$$y'' + \frac{1}{2x}y' + \frac{1}{4x}y = 0$$

That implies to $p(x) = \frac{1}{2x}$ and $q(x) = \frac{1}{4x}$.

$$\begin{aligned} p_0 &= \lim_{x\to 0} xp(x) \\ &= \lim_{x\to 0} x\frac{1}{2x} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x\to 0} x^2 q(x) \\ &= \lim_{x\to 0} x^2\frac{1}{4x} \\ &= \lim_{x\to 0} \frac{x}{4} \\ &= 0 \end{aligned}$$

The indicial equation is: $r(r-1) + \frac{1}{2}r = r^2 - \frac{1}{2}r = 0$

$$\rightarrow \boxed{r = 0, \ \frac{1}{2}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad and \quad y_2(x) = x^{1/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$4xy'' + 2y' + y = 0$

$$4x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + 2 \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 4(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(4n+4r-4) + 2(n+r)\right] a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

$$r(4r-2) a_n + \sum_{n=1}^{\infty} (n+r)(4n+4r-2) a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

$$2r(2r-1) a_n + \sum_{n=1}^{\infty} \left[2(n+r)(2n+2r-1) a_n + a_{n-1}\right] x^{n+r-1} = 0$$

For $n = 0$

$$2r(2r-1) a_0 = 0$$

$$\Rightarrow r = 0, \ \frac{1}{2} \quad \checkmark$$

$$2(n+r)(2n+2r-1) a_n + a_{n-1} = 0$$

$$a_n = -\frac{1}{2(n+r)(2n+2r-1)} a_{n-1}$$

$r = 0$

$$a_n = -\frac{1}{2n(2n-1)}a_{n-1}$$

$$n=1 \rightarrow a_1 = -\frac{1}{2}a_0$$

$$n=2 \rightarrow a_2 = -\frac{1}{12}a_1 = \frac{1}{24}a_0$$

$$n=3 \rightarrow a_3 = -\frac{1}{30}a_2 = -\frac{1}{720}a_0$$

$$n=4 \rightarrow a_4 = -\frac{1}{42}a_3 = \frac{1}{30,240}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_1(x) = a_0 x^0\left(1-\frac{1}{2}x+\frac{1}{24}x^2-\frac{1}{720}x^3+\frac{1}{30,240}x^4-\cdots\right)}$$

$$r=\frac{1}{2}$$

$$b_n = -\frac{1}{2\left(n+\frac{1}{2}\right)(2n)}b_{n-1}$$

$$= -\frac{1}{4n^2+2n}b_{n-1}$$

$$n=1 \rightarrow b_1 = -\frac{1}{6}b_0$$

$$n=2 \rightarrow b_2 = -\frac{1}{20}b_1 = \frac{1}{120}b_0$$

$$n=3 \rightarrow b_3 = -\frac{1}{42}b_2 = -\frac{1}{5040}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y_2(x) = b_0 x^{1/2}\left(1-\frac{1}{6}x+\frac{1}{120}x^2-\frac{1}{5,040}x^3+\cdots\right)}$$

$$\underline{y(x) = a_0\left(1-\frac{1}{2}x+\frac{1}{24}x^2-\frac{1}{720}x^3+\frac{1}{30,240}x^4-\cdots\right)+b_0 x^{1/2}\left(1-\frac{1}{6}x+\frac{1}{120}x^2-\frac{1}{5,040}x^3+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $6x^2y''+7xy'-\left(x^2+2\right)y=0$

Solution

$$\frac{1}{6}6x^2y''+\frac{1}{6}7xy'-\frac{1}{6}\left(x^2+2\right)y=0$$

$$x^2y''+\frac{7}{6}xy'-\frac{1}{6}\left(x^2+2\right)y=0$$

$$y''+\frac{7}{6x}y'-\frac{1}{6x^2}\left(x^2+2\right)y=0$$

That implies to $p(x)=\frac{7}{6x}$ and $q(x)=-\frac{1}{6x^2}\left(x^2+2\right)$.

$$\begin{aligned} p_0 &= \lim_{x\to 0} xp(x) \\ &= \lim_{x\to 0} x\frac{7}{6x} \\ &= \frac{7}{6} \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x\to 0} x^2q(x) \\ &= -\lim_{x\to 0} x^2\frac{1}{6x^2}\left(x^2+2\right) \\ &= -\lim_{x\to 0} \left(\frac{1}{6}x^2+\frac{1}{3}\right) \\ &= -\frac{1}{3} \end{aligned}$$

The indicial equation is: $r(r-1)+\frac{7}{6}r-\frac{1}{3}=r^2+\frac{1}{6}r-\frac{1}{3}=0$

$$6r^2+r-2=0$$

$$\boxed{r=\frac{1}{2},\ -\frac{2}{3}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x)=x^{1/2}\sum_{n=0}^{\infty}a_nx^n \quad \text{and} \quad y_2(x)=x^{-2/3}\sum_{n=0}^{\infty}b_nx^n$$

$$y=\sum_{n=0}^{\infty}a_nx^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}$$

$$6x^2y''+7xy'-\left(x^2+2\right)y=0$$

$$6x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}+7x\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}-x^2\sum_{n=0}^{\infty}a_nx^{n+r}$$
$$-2\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

$$\sum_{n=0}^{\infty}6(n+r)(n+r-1)a_nx^{n+r}+\sum_{n=0}^{\infty}7(n+r)a_nx^{n+r}-\sum_{n=0}^{\infty}a_nx^{n+r+2}-\sum_{n=0}^{\infty}2a_nx^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left[6(n+r)(n+r-1)+7(n+r)-2\right]a_nx^{n+r}-\sum_{n=2}^{\infty}a_{n-2}x^{n+r}=0$$

$$\left(6r^2+r-2\right)a_0+\left((r+1)(6r+7)-2\right)a_1+\sum_{n=2}^{\infty}\left[(n+r)(6n+6r+1)-2\right]a_nx^{n+r}$$
$$-\sum_{n=2}^{\infty}a_{n-2}x^{n+r}=0$$

$$\left(6r^2+r-2\right)a_0+\left(6r^2+13r+5\right)a_1+\sum_{n=2}^{\infty}\left[\left((n+r)(6n+6r+1)-2\right)a_n-a_{n-2}\right]x^{n+r}=0$$

For $n=0$

$$\left(6r^2+r-2\right)a_0=0$$

$$\Rightarrow \ \boxed{r=\frac{1}{2},\ \ -\frac{2}{3}}\ \ \checkmark$$

For $n=1$

$$\left(6r^2+13r+5\right)a_1=0$$

$$\Rightarrow \ r=\frac{-13\pm 7}{12}=\cancel{-\frac{1}{2},\ -\frac{5}{3}}$$

$$\boxed{a_1=0}$$

$$\left((n+r)(6n+6r+1)-2\right)a_n-a_{n-2}=0$$

$$\boxed{a_n=\frac{1}{(n+r)(6n+6r+1)-2}a_{n-2}}$$

$r = \frac{1}{2}$

$$a_n = \frac{1}{n(6n+7)} a_{n-2}$$

$n = 2 \rightarrow a_2 = \frac{1}{38} a_0$

$n = 3 \rightarrow a_3 = \frac{1}{75} a_1 = 0$

$n = 4 \rightarrow a_4 = \frac{1}{124} a_2 = \frac{1}{4,712} a_0$

$n = 5 \rightarrow a_5 = \frac{1}{185} a_3 = 0$

$n = 6 \rightarrow a_6 = \frac{1}{258} a_4 = \frac{1}{1,215,696} a_0$

$n = 7 \rightarrow a_7 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_1(x) = a_0 x^{1/2} \left(1 + \frac{1}{38} x^2 + \frac{1}{4,712} x^4 + \frac{1}{1,215,696} x^6 + \cdots \right)$$

$r = -\frac{2}{3}$

$$b_n = \frac{1}{n(6n-7)} b_{n-2}$$

$n = 2 \rightarrow b_2 = \frac{1}{10} b_0$

$n = 3 \rightarrow b_3 = \frac{1}{33} b_1 = 0$

$n = 4 \rightarrow b_4 = \frac{1}{68} b_2 = \frac{1}{680} b_0$

$n = 5 \rightarrow b_5 = 0$

$n = 6 \rightarrow b_6 = \frac{1}{174} b_4 = \frac{1}{118,320} b_0$

$n = 7 \rightarrow b_7 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_2(x) = b_0 x^{-2/3} \left(1 + \frac{1}{10} x^2 + \frac{1}{680} x^4 + \frac{1}{118,320} x^6 + \cdots \right)$$

$$y(x) = a_0 x^{1/2} \left(1 + \frac{x^2}{38} + \frac{x^4}{4,712} + \frac{x^6}{1,215,696} + \cdots \right) + b_0 x^{-2/3} \left(1 + \frac{x^2}{10} + \frac{x^4}{680} + \frac{x^6}{118,320} + \cdots \right)$$

Exercise

Find the Frobenius series solutions of $xy'' + y' + 2y = 0$

Solution

$x \times \quad xy'' + y' + 2y = 0$

$x^2 y'' + xy' + 2xy = 0$

$y'' + \frac{1}{x} y' + \frac{2}{x} y = 0$

That implies to $p(x) = \frac{1}{x}$ and $q(x) = \frac{2}{x}$.

$$\begin{aligned} p_0 &= \lim_{x\to 0} xp(x) \\ &= \lim_{x\to 0} x\frac{1}{x} \\ &= 1 \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x\to 0} x^2 q(x) \\ &= \lim_{x\to 0} x^2\frac{2}{x} \\ &= \lim_{x\to 0} 2x \\ &= 0 \end{aligned}$$

The indicial equation is: $r^2 + (1-1)r = 0$

$$\rightarrow \underline{r_{1,2} = 0}|$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = y_1(x)\ln|x| + \sum_{n=0}^{\infty} c_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=0}^{\infty} a_n x^n \qquad (r = r_1 = 0)$$

$$y' = \sum_{n=0}^{\infty} na_n x^{n+r-1} = \sum_{n=0}^{\infty} na_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$$

$$xy'' + y' + 2y = 0$$

$$x\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n^2 a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n^2 a_n x^{n-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n-1} = 0$$

$$\sum_{n=1}^{\infty} n^2 a_n x^{n-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n-1} = 0$$

$$\sum_{n=1}^{\infty} \left(n^2 a_n + 2a_{n-1}\right) x^{n-1} = 0$$

$$n^2 a_n + 2a_{n-1} = 0$$

$$\boxed{a_n = -\frac{2}{n^2} a_{n-1}}$$

$$n = 1 \rightarrow a_1 = -2a_0$$

$$n = 2 \rightarrow a_2 = -\frac{2}{2^2} a_1 = a_0 = \frac{2^2}{2^2} a_0$$

$$n = 3 \rightarrow a_3 = -\frac{2}{9} a_2 = -\frac{2^3}{(2\cdot 3)^2} a_0 = -\frac{2}{9} a_0$$

$$n = 4 \rightarrow a_4 = -\frac{2}{4^2} a_3 = \frac{2^4}{(4!)^2} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_n = \frac{(-1)^n 2^n}{(n!)^2} a_0}$$

$$\boxed{y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{(n!)^2} x^n}$$

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n \quad (a_0 = 1)$$

$$y_2(x) = y_1(x) \ln x + \sum_{n=0}^{\infty} c_n x^n$$

$$y_2(x) = y_1(x) \ln x + x^{r_1} \sum_{n=0}^{\infty} c_n x^n$$

$$xy_2'' + y_2' + 2y_2 = 0$$

$$x\left(y_1(x)\ln x + \sum_{n=0}^{\infty} c_n x^n \right)'' + \left(y_1(x)\ln x + \sum_{n=0}^{\infty} c_n x^n \right)' + 2\left(y_1(x)\ln x + \sum_{n=0}^{\infty} c_n x^n \right) = 0$$

$$x\left(y_1'\ln x+\frac{1}{x}y_1\right)'+x\sum_{n=0}^{\infty}n(n-1)c_nx^{n-2}+y_1'\ln x+\frac{1}{x}y_1+\sum_{n=0}^{\infty}nc_nx^{n-1}+2y_1\ln x$$
$$+\sum_{n=0}^{\infty}2c_nx^n=0$$

$$x\left(y_1''\ln x+\frac{1}{x}y_1'-\frac{1}{x^2}y_1+\frac{1}{x}y_1'\right)+y_1'\ln x+\frac{1}{x}y_1+2y_1\ln x$$
$$+\sum_{n=0}^{\infty}n(n-1)c_nx^{n-1}+\sum_{n=0}^{\infty}nc_nx^{n-1}+\sum_{n=0}^{\infty}2c_nx^n=0$$

$$xy_1''\ln x+2y_1'-\frac{1}{x}y_1+\frac{1}{x}y_1+\left(y_1'+2y_1\right)\ln x+\sum_{n=0}^{\infty}n^2c_nx^{n-1}+\sum_{n=0}^{\infty}2c_nx^n=0$$

$$\left(xy_1''+y_1'+2y_1\right)\ln x+2y_1'+\sum_{n=0}^{\infty}n^2c_nx^{n-1}+\sum_{n=1}^{\infty}2c_{n-1}x^{n-1}=0$$

Since: $xy_1''+y_1'+2y_1=0$

$$2y_1'+\sum_{n=1}^{\infty}n^2c_nx^{n-1}+\sum_{n=1}^{\infty}2c_{n-1}x^{n-1}=0$$

$$2y_1'+\sum_{n=1}^{\infty}\left(n^2c_n+2c_{n-1}\right)x^{n-1}=0$$

$$2\sum_{n=1}^{\infty}(-1)^n\frac{2^n}{(n!)^2}nx^{n-1}+\sum_{n=1}^{\infty}\left(n^2c_n+2c_{n-1}\right)x^{n-1}=0$$

$$\sum_{n=1}^{\infty}\left[(-1)^n\frac{2^{n+1}}{(n-1)!n!}+n^2c_n+2c_{n-1}\right]x^{n-1}=0$$

$$n^2c_n+2c_{n-1}+(-1)^n\frac{2^{n+1}}{(n-1)!n!}=0$$

$$\boxed{c_n=-\frac{2}{n^2}c_{n-1}+(-1)^{n+1}\frac{2^{n+1}}{(n-1)!n!\,n^2}}$$

$n=1\ \rightarrow\ c_1=-2c_0+4 \qquad\qquad c_0=0\ \rightarrow\ c_1=4$

$n=2 \to c_2 = -\frac{1}{2}c_1 - 1$ $\qquad c_2 = -3$

$n=3 \to c_3 = -\frac{2}{9}c_2 + \frac{16}{12(9)}$ $\qquad c_3 = \frac{22}{27}$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_2(x) = y_1(x)\ln|x| + 4x - 3x^2 + \frac{22}{27}x^3 - \cdots \qquad y_2(x) = y_1(x)\ln x + x^{r_1}\sum_{n=0}^{\infty} c_n x^n$$

Exercise

Find the Frobenius series solutions of $xy'' - y = 0$

Solution

$x\times \quad xy'' - y = 0$

$x^2y'' - xy = 0$

$y'' - \frac{1}{x}y = 0$

That implies to $p(x) = 0$ and $q(x) = -\frac{1}{x}$.

$$p_0 = \lim_{x\to 0} xp(x)$$
$$= 0$$

$$q_0 = \lim_{x\to 0} x^2 q(x)$$
$$= \lim_{x\to 0} x^2\left(-\frac{1}{x}\right)$$
$$= -\lim_{x\to 0} x$$
$$= 0$$

The indicial equation is: $r^2 - r = 0$

$$\to r_1 = 1,\ r_2 = 0$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1} \quad \text{and} \quad y_2(x) = \alpha y_1(x)\ln|x| + \sum_{n=0}^{\infty} c_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=0}^{\infty} a_n x^{n+1} \qquad (r = r_1 = 1)$$

$$y' = \sum_{n=0}^{\infty} na_n x^{n+r-1} = \sum_{n=0}^{\infty} na_n x^n$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} = \sum_{n=0}^{\infty} n(n+1)a_n x^{n-1}$$

$xy'' - y = 0$

$$x\sum_{n=0}^{\infty} n(n+1)a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} n(n+1)a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} n(n+1)a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\sum_{n=1}^{\infty} n(n+1)a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\sum_{n=1}^{\infty} \left[n(n+1)a_n - a_{n-1} \right] x^n = 0$$

$$n(n+1)a_n - a_{n-1} = 0$$

$$\boxed{a_n = \frac{1}{n(n+1)}a_{n-1}}$$

$n = 1 \ \rightarrow \ a_1 = \frac{1}{2}a_0$

$n = 2 \ \rightarrow \ a_2 = \frac{1}{6}a_1 = a_0 = \frac{1}{(2)3!}a_0$

$n = 3 \ \rightarrow \ a_3 = \frac{1}{3\cdot 4}a_2 = \frac{1}{(2\cdot 3)4!}a_0$

$n = 4 \ \rightarrow \ a_4 = \frac{1}{4\cdot 5}a_3 = \frac{1}{4!5!}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{a_n = \frac{1}{n!(n+1)!}a_0}$$

$$y_1(x) = x\sum_{n=1}^{\infty}\frac{1}{n!(n+1)!}x^n \qquad\qquad y_1(x) = x^{r_1}\sum_{n=0}^{\infty}a_n x^n \quad (a_0 = 1)$$

$$= \sum_{n=1}^{\infty}\frac{1}{n!(n+1)!}x^{n+1}$$

$$y_2(x) = \alpha y_1 \ln x + \sum_{n=0}^{\infty}d_n x^n$$

$$y_2(x) = \alpha y_1(x)\ln x + \sum_{n=0}^{\infty}d_n x^n \quad (d_0 = 1)$$

$$y_2' = \alpha y_1' \ln x + \frac{\alpha}{x}y_1 + \sum_{n=0}^{\infty}nd_n x^{n-1}$$

$$xy_2'' - y_2 = 0$$

$$x\left(\alpha y_1' \ln x + \frac{\alpha}{x}y_1 + \sum_{n=0}^{\infty}nd_n x^{n-1}\right)' - \left(\alpha y_1 \ln x + \sum_{n=0}^{\infty}d_n x^n\right) = 0$$

$$x\left(\alpha y_1' \ln x + \frac{\alpha}{x}y_1\right)' + x\sum_{n=0}^{\infty}n(n-1)d_n x^{n-2} - \alpha y_1 \ln x - \sum_{n=0}^{\infty}d_n x^n = 0$$

$$x\left(\alpha y_1'' \ln x + \frac{\alpha}{x}y_1' - \frac{\alpha}{x^2}y_1 + \frac{\alpha}{x}y_1'\right) - \alpha y_1(x)\ln x + \sum_{n=0}^{\infty}n(n-1)d_n x^{n-1} - \sum_{n=0}^{\infty}d_n x^n = 0$$

$$\alpha\left(2y_1' - \frac{1}{x}y_1\right) + \alpha\left(xy_1'' - y_1\right)\ln x + \sum_{n=0}^{\infty}n(n+1)d_{n+1}x^n - \sum_{n=0}^{\infty}d_n x^n = 0$$

Since: $xy_1'' - y_1 = 0$

$$y_1(x) = \sum_{n=1}^{\infty}\frac{1}{n!(n+1)!}x^{n+1}$$

$$y_1' = \sum_{n=1}^{\infty}\frac{n+1}{n!(n+1)!}x^n$$

$$\alpha\left(2y_1' - \frac{1}{x}y_1\right) + \sum_{n=0}^{\infty}\left[n(n+1)d_{n+1} - d_n\right]x^n = 0$$

$$\alpha\left(2\sum_{n=1}^{\infty}\frac{n+1}{n!(n+1)!}x^n - \frac{1}{x}\sum_{n=1}^{\infty}\frac{1}{n!(n+1)!}x^{n+1}\right) + \sum_{n=0}^{\infty}\left[n(n+1)d_{n+1} - d_n\right]x^n = 0$$

$$\alpha\left(\sum_{n=1}^{\infty}\frac{2n+2}{n!(n+1)!}x^n - \sum_{n=1}^{\infty}\frac{1}{n!(n+1)!}x^n\right) + \sum_{n=0}^{\infty}\left[n(n+1)d_{n+1} - d_n\right]x^n = 0$$

$$\alpha\sum_{n=1}^{\infty}\frac{2n+1}{n!(n+1)!}x^n + \sum_{n=0}^{\infty}\left[n(n+1)d_{n+1} - d_n\right]x^n = 0$$

$$\sum_{n=0}^{\infty}\left[n(n+1)d_{n+1} - d_n\right]x^n = -\alpha\sum_{n=1}^{\infty}\frac{2n+1}{n!(n+1)!}x^n$$

$$n(n+1)d_{n+1} - d_n = -\alpha\frac{2n+1}{n!(n+1)!} \qquad (d_0 = 1)$$

$$n = 0 \ \rightarrow\ -d_0 = -\alpha \qquad \Rightarrow \alpha = d_0 = 1$$

$$d_{n+1} = \frac{1}{n(n+1)}\left(d_n - \frac{2n+1}{n!(n+1)!}\right)$$

$$n = 1 \ \rightarrow\ d_2 = \frac{1}{2}\left(d_1 - \frac{3}{2}\right)$$

$$= \frac{1}{2}d_1 - \frac{3}{4}$$

$$n = 2 \ \rightarrow\ d_3 = \frac{1}{6}\left(d_2 - \frac{5}{12}\right)$$

$$= \frac{1}{6}\left(\frac{1}{2}d_1 - \frac{3}{4} - \frac{5}{12}\right)$$

$$= \frac{1}{12}d_1 - \frac{7}{36}$$

$$n = 3 \ \rightarrow\ d_4 = \frac{1}{12}\left(d_3 - \frac{7}{144}\right)$$

$$= \frac{1}{12}\left(\frac{1}{12}d_1 - \frac{7}{36} - \frac{7}{144}\right)$$

$$= \frac{1}{144}d_1 - \frac{35}{1{,}728}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

If we let $d_1 = 0$

$$\boxed{y_2(x) = y_1(x)\ln x + 1 - \frac{3}{4}x^2 - \frac{7}{36}x^3 - \cdots}$$

$$y_2(x) = y_1(x)\ln x + \sum_{n=0}^{\infty} d_n x^n$$

Exercise

Find the Frobenius series solutions of $2x(1-x)y'' + (1+x)y' - y = 0$

Solution

$$xy'' + \frac{x+1}{2(1-x)}y' - \frac{1}{2(1-x)}y = 0$$

$$x^2y'' + \frac{1}{2}\frac{x(x+1)}{1-x}y' - \frac{x}{2(1-x)}y = 0$$

$$y'' + \frac{1}{2}\frac{x+1}{x(1-x)}y' - \frac{1}{2x(1-x)}y = 0$$

That implies to $p(x) = \frac{1}{2}\frac{x+1}{x(1-x)}$ and $q(x) = -\frac{1}{2x(1-x)}$.

$$\begin{aligned} p_0 &= \lim_{x\to 0} xp(x) \\ &= \frac{1}{2}\lim_{x\to 0} x\frac{x+1}{x(1-x)} \\ &= \frac{1}{2}\lim_{x\to 0} \frac{x+1}{1-x} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x\to 0} x^2 q(x) \\ &= -\frac{1}{2}\lim_{x\to 0} \frac{x}{1-x} \\ &= 0 \end{aligned}$$

The indicial equation is: $r^2 + \left(\frac{1}{2} - 1\right)r = r^2 - \frac{1}{2}r = 0$

$$\rightarrow \boxed{r = 0,\ \frac{1}{2}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x(1-x)y'' + (1+x)y' - y = 0$$

$$\left(2x - 2x^2\right) \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + (1+x) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} - \sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$+ \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(2n+2r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} \left[-2(n+r)(n+r-1) + n + r - 1\right] a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(2n+2r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r-1)(-2(n+r)+1) a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(2n+2r-1) a_n x^{n+r-1} + \sum_{n=1}^{\infty} (n+r-2)(-2n-2r+3) a_{n-1} x^{n+r-1} = 0$$

$$r(2r-1) a_0 x^{r-1} + \sum_{n=1}^{\infty} (n+r)(2n+2r-1) a_n x^{n+r-1}$$

$$- \sum_{n=1}^{\infty} (n+r-2)(2n+2r-3) a_{n-1} x^{n+r-1} = 0$$

$$r(2r-1) a_0 x^{r-1} + \sum_{n=1}^{\infty} \left[(n+r)(2n+2r-1) a_n - (n+r-2)(2n+2r-3) a_{n-1}\right] x^{n+r-1} = 0$$

For $n = 0$

$$r(2r-1) a_0 = 0$$

$$\Rightarrow \underline{r = 0,\ \frac{1}{2}} \quad \checkmark$$

$$(n+r)(2n+2r-1)a_n - (n+r-2)(2n+2r-3)a_{n-1} = 0$$

$$\boxed{a_n = \frac{(n+r-2)(2n+2r-3)}{(n+r)(2n+2r-1)}a_{n-1}}$$

$r = 0$

$$a_n = \frac{(n-2)(2n-3)}{n(2n-1)}a_{n-1}$$

$n = 1 \rightarrow a_1 = a_0$

$n = 2 \rightarrow a_2 = 0a_1 = 0$

$n = 3 \rightarrow a_3 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0(1+x)}$$

$r = \frac{1}{2}$

$$b_n = \frac{\left(n-\frac{3}{2}\right)(2n-2)}{2n\left(n+\frac{1}{2}\right)}b_{n-1}$$

$$= \frac{(2n-3)(n-1)}{n(2n+1)}b_{n-1}$$

$n = 1 \rightarrow b_1 = 0b_0 = 0$

$n = 2 \rightarrow b_2 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_2(x) = b_0 x^{1/2}}$$

$$\boxed{y(x) = a_0(1+x) + b_0\sqrt{x}}$$

Exercise

Find the Frobenius series solutions of $x^2y'' + \left(x^2 + \frac{1}{2}x\right)y' + xy = 0$

Solution

$$y'' + \left(1 + \frac{1}{2x}\right)y' + \frac{1}{x}y = 0$$

That implies to $p(x) = 1 + \frac{1}{2x}$ and $q(x) = \frac{1}{x}$.

$$\begin{aligned} p_0 &= \lim_{x\to 0} xp(x) \\ &= \lim_{x\to 0} x\left(1 + \frac{1}{2x}\right) \\ &= \lim_{x\to 0} \left(x + \frac{1}{2}\right) \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x\to 0} x^2 q(x) \\ &= \lim_{x\to 0} x \\ &= 0 \end{aligned}$$

The indicial equation is: $r^2 + \left(\frac{1}{2} - 1\right)r = r^2 - \frac{1}{2}r = 0$

$$\rightarrow \; r = 0, \; \frac{1}{2}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x^2 y'' + \left(x^2 + \frac{1}{2}x\right)y' + xy = 0$$

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \left(x^2 + \frac{1}{2}x\right) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + x \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1}\sum_{n=0}^{\infty}\frac{1}{2}(n+r)a_n x^{n+r}$$

$$+\sum_{n=0}^{\infty}a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty}(n+r+1)a_n x^{n+r+1} + \sum_{n=0}^{\infty}\left[(n+r)(n+r-1)+\frac{1}{2}(n+r)\right]a_n x^{n+r} = 0$$

$$\sum_{n=1}^{\infty}(n+r)a_{n-1}x^{n+r} + \sum_{n=0}^{\infty}(n+r)\left(n+r-\frac{1}{2}\right)a_n x^{n+r} = 0$$

$$\sum_{n=1}^{\infty}(n+r)a_{n-1}x^{n+r} + r\left(r-\frac{1}{2}\right)a_0 + \sum_{n=1}^{\infty}(n+r)\left(n+r-\frac{1}{2}\right)a_n x^{n+r} = 0$$

$$r\left(r-\frac{1}{2}\right)a_0 + \sum_{n=1}^{\infty}\left[(n+r)\left(n+r-\frac{1}{2}\right)a_n + (n+r)a_{n-1}\right]x^{n+r} = 0$$

For $n=0$

$$r(2r-1)a_0 = 0$$

$$\Rightarrow \boxed{r = 0,\ \frac{1}{2}}\ \checkmark$$

$$(n+r)\left(n+r-\frac{1}{2}\right)a_n + (n+r)a_{n-1} = 0$$

$$\boxed{a_n = -2\frac{(n+r)}{(n+r)(2n+2r-1)}a_{n-1}}$$

$r=0$

$$a_n = -\frac{2n}{n(2n-1)}a_{n-1}$$

$$= -\frac{2}{2n-1}a_{n-1}$$

$$n=1 \ \rightarrow\ a_1 = -2a_0$$

$$n=2 \ \rightarrow\ a_2 = -\frac{2}{3}a_1 = \frac{4}{3}a_0$$

$$n=3 \ \rightarrow\ a_3 = -\frac{2}{5}a_2 = -\frac{8}{15}a_0$$

$$n=4 \ \rightarrow \ a_4 = -\frac{2}{7}a_3 = \frac{16}{105}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_1(x) = a_0\left(1-2x+\frac{4}{3}x^2-\frac{8}{15}x^3+\frac{16}{105}x^4-\cdots\right)$$

$$r=\frac{1}{2}$$

$$b_n = -\frac{2\left(n+\frac{1}{2}\right)}{\left(n+\frac{1}{2}\right)(2n)}b_{n-1}$$

$$= -\frac{1}{n}b_{n-1}$$

$$n=1 \ \rightarrow \ b_1 = -b_0$$

$$n=2 \ \rightarrow \ b_2 = -\frac{1}{2}b_1 = \frac{1}{2}b_0$$

$$n=3 \ \rightarrow \ b_3 = -\frac{1}{3}b_2 = -\frac{1}{6}b_0$$

$$n=4 \ \rightarrow \ b_4 = -\frac{1}{4}b_3 = \frac{1}{24}b_0$$

$$n=5 \ \rightarrow \ b_5 = -\frac{1}{5}b_4 = -\frac{1}{120}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0x^{1/2}\left(1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{120}x^5+\cdots\right)$$

$$y(x) = a_0\left(1-2x+\frac{4}{3}x^2-\frac{8}{15}x^3+\frac{16}{105}x^4-\cdots\right) + b_0\sqrt{x}\left(1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{120}x^5+\cdots\right)$$

Exercise

Find the Frobenius series solutions of $18x^2y''+3x(x+5)y'-(10x+1)y=0$

Solution

$$y''+\frac{x+5}{6x}y'-\frac{10x+1}{18x^2}y=0$$

That implies to $p(x)=\frac{x+5}{6x}$ and $q(x)=-\frac{10x+1}{18x^2}$.

$$p_0 = \lim_{x\to 0} xp(x)$$

$$= \lim_{x\to 0} x\left(\frac{x+5}{6x}\right)$$

$$= \lim_{x\to 0} \frac{x+5}{6}$$

$$= \frac{5}{6}$$

$$q_0 = \lim_{x\to 0} x^2 q(x)$$

$$= -\lim_{x\to 0} \frac{10x+1}{18}$$

$$= -\frac{1}{18}$$

The indicial equation is: $r^2 + \left(\frac{5}{6} - 1\right)r - \frac{1}{18} = r^2 - \frac{1}{6}r - \frac{1}{18} = 0$

$$18r^2 - 3r - 1 = 0 \quad \rightarrow \quad r = -\frac{1}{6}, \ \frac{1}{3}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{-1/6} \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/3} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$18x^2 y'' + 3x(x+5)y' - (10x+1)y = 0$$

$$18x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \left(3x^2 + 15x\right) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$-(10x+1) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}18(n+r)(n+r-1)a_n x^{n+r}+\sum_{n=0}^{\infty}3(n+r)a_n x^{n+r+1}+\sum_{n=0}^{\infty}15(n+r)a_n x^{n+r}$$
$$-\sum_{n=0}^{\infty}10a_n x^{n+r+1}-\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left[18(n+r)(n+r-1)+15(n+r)-1\right]a_n x^{n+r}+\sum_{n=0}^{\infty}(3n+3r-10)a_n x^{n+r+1}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(18n+18r-3)-1\right]a_n x^{n+r}+\sum_{n=1}^{\infty}(3n+3r-13)a_{n-1}x^{n+r}=0$$

$$\left(r(18r-3)-1\right)a_0x^r+\sum_{n=1}^{\infty}\left[(n+r)(18n+18r-3)-1\right]a_n x^{n+r}+\sum_{n=1}^{\infty}(3n+3r-13)a_{n-1}x^{n+r}=0$$

$$\left(r(18r-3)-1\right)a_0x^r+\sum_{n=1}^{\infty}\left[\left((n+r)(18n+18r-3)-1\right)a_n+(3n+3r-13)a_{n-1}\right]x^{n+r}=0$$

For $n=0$

$$\left(18r^2-3r-1\right)a_0=0$$

$$\Rightarrow \boxed{r=-\frac{1}{6},\ \frac{1}{3}}\ \checkmark$$

$$\left((n+r)(18n+18r-3)-1\right)a_n+(3n+3r-13)a_{n-1}=0$$

$$\boxed{a_n=-\frac{3n+3r-13}{(n+r)(18n+18r-3)-1}a_{n-1}}$$

$$r=-\frac{1}{6}$$

$$a_n=-\frac{3n-\frac{1}{2}-13}{\left(n-\frac{1}{6}\right)(18n-6)-1}a_{n-1}$$
$$=-\frac{1}{2}\frac{6n-27}{(6n-1)(3n-1)-1}a_{n-1}$$

$$n=1\ \to\ a_1=-\frac{1}{2}\frac{-21}{9}a_0=\frac{7}{6}a_0$$

$$n=2\ \to\ a_2=-\frac{1}{2}\frac{-15}{54}a_1=\frac{5}{36}\frac{7}{6}a_0=\frac{35}{216}a_0$$

$$n=3 \rightarrow a_3 = -\frac{1}{2}\frac{-9}{135}a_2 = \frac{1}{30}\frac{35}{216}a_0 = \frac{7}{1,296}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0 x^{-1/6}\left(1+\frac{7}{6}x+\frac{35}{216}x^2+\frac{7}{1,296}x^3+\cdots\right)}$$

$$r = \frac{1}{3}$$

$$\begin{aligned} b_n &= -\frac{3n-12}{\left(n+\frac{1}{3}\right)(18n+3)-1}b_{n-1} \\ &= -\frac{3(n-4)}{(3n+1)(6n+1)-1}b_{n-1} \end{aligned}$$

$$n=1 \rightarrow b_1 = -\frac{-9}{27}b_0 = \frac{1}{3}b_0$$

$$n=2 \rightarrow b_2 = \frac{6}{90}b_1 = \frac{1}{15}\frac{1}{3}b_0 = \frac{1}{45}b_0$$

$$n=3 \rightarrow b_3 = \frac{3}{189}b_2 = \frac{1}{63}\frac{1}{45}b_0 = \frac{1}{2,835}b_0$$

$$n=4 \rightarrow b_4 = 0b_3 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{1/3}\left(1+\frac{1}{3}x+\frac{1}{45}x^2+\frac{1}{2835}x^3\right)}$$

$$\boxed{y(x) = a_0\frac{1}{x^{1/6}}\left(1+\frac{7}{6}x+\frac{35}{216}x^2+\frac{7}{1296}x^3+\cdots\right)+b_0 x^{1/3}\left(1+\frac{1}{3}x+\frac{1}{45}x^2+\frac{1}{2835}x^3\right)}$$

Exercise

Find the Frobenius series solutions of $2x^2y''+7x(x+1)y'-3y=0$

Solution

$$y''+\frac{7}{2}\frac{x+1}{x}y'-\frac{3}{2x^2}y=0$$

That implies to $p(x)=\frac{7}{2}\frac{x+1}{x}$ and $q(x)=-\frac{3}{2x^2}$.

$$\begin{aligned} p_0 &= \lim_{x\to 0} xp(x) \\ &= \frac{7}{2}\lim_{x\to 0} x\left(\frac{x+1}{x}\right) \\ &= \frac{7}{2}\lim_{x\to 0}(x+1) \end{aligned}$$

$$=\frac{7}{2}$$

$$q_0 = \lim_{x\to 0} x^2 q(x)$$

$$=-\lim_{x\to 0} \frac{3}{2}$$

$$=-\frac{3}{2}$$

The indicial equation is: $r^2+\left(\frac{7}{2}-1\right)r-\frac{3}{2}=r^2+\frac{5}{2}r-\frac{3}{2}=0$

$$2r^2+5r-3=0 \quad \rightarrow \quad \underline{r=-3,\ \frac{1}{2}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x)=x^{-3}\sum_{n=0}^{\infty}a_n x^n \quad \textit{and} \quad y_2(x)=x^{1/2}\sum_{n=0}^{\infty}b_n x^n$$

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$2x^2y''+7x(x+1)y'-3y=0$$

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\left(7x^2+7x\right)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}-3\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r}+\sum_{n=0}^{\infty}7(n+r)a_n x^{n+r+1}+\sum_{n=0}^{\infty}7(n+r)a_n x^{n+r}$$
$$-\sum_{n=0}^{\infty}3a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-2)+7(n+r)-3\right]a_n x^{n+r}+\sum_{n=0}^{\infty}7(n+r)a_n x^{n+r+1}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r+5)-3\right]a_n x^{n+r}+\sum_{n=1}^{\infty}7(n+r-1)a_{n-1}x^{n+r}=0$$

$$\left(2r^2+5r-3\right)a_0x^r+\sum_{n=1}^{\infty}\left[(n+r)(2n+2r+5)-3\right]a_n x^{n+r}+\sum_{n=1}^{\infty}7(n+r-1)a_{n-1}x^{n+r}=0$$

$$\left(2r^2+5r-3\right)a_0x^r+\sum_{n=1}^{\infty}\left[\left((n+r)(2n+2r+5)-3\right)a_n+7(n+r-1)a_{n-1}\right]x^{n+r}=0$$

For $n=0$

$$\left(2r^2+5r-3\right)a_0=0$$

$$\Rightarrow \boxed{r=-3,\ \tfrac{1}{2}}\ \checkmark$$

$$\left((n+r)(2n+2r+5)-3\right)a_n+7(n+r-1)a_{n-1}=0$$

$$\boxed{a_n=-\frac{7(n+r-1)}{(n+r)(2n+2r+5)-3}a_{n-1}}$$

$r=-3$

$$a_n=-\frac{7(n-4)}{(n-3)(2n-1)-3}a_{n-1}$$

$$n=1\ \to\ a_1=-\frac{21}{5}a_0$$

$$n=2\ \to\ a_2=-\frac{14}{6}a_1=-\frac{7}{3}\left(-\frac{21}{5}\right)a_0=\frac{49}{5}a_0$$

$$n=3\ \to\ a_3=-\frac{-7}{-3}a_2=-\frac{7}{3}\frac{49}{5}a_0=-\frac{343}{15}a_0$$

$$n=4\ \to\ a_4=0$$

$$\vdots\quad\vdots\quad\vdots\quad\vdots$$

$$\boxed{y_1(x)=a_0x^{-3}\left(1-\frac{21}{5}x+\frac{49}{5}x^2-\frac{343}{15}x^3\right)}$$

$r=\frac{1}{2}$

$$b_n=-\frac{7\left(n-\frac{1}{2}\right)}{\left(n+\frac{1}{2}\right)(2n+6)-3}b_{n-1}$$

$$=-\frac{7}{2}\frac{2n-1}{(2n+1)(n+3)-3}b_{n-1}$$

$$n=1 \;\rightarrow\; b_1 = -\frac{7}{2}\frac{1}{9}b_0 = -\frac{7}{18}b_0$$

$$n=2 \;\rightarrow\; b_2 = -\frac{7}{2}\frac{3}{22}b_1 = -\frac{21}{44}\frac{-7}{18}b_0 = \frac{49}{264}b_0$$

$$n=3 \;\rightarrow\; b_3 = -\frac{7}{2}\frac{5}{39}b_2 = -\frac{35}{78}\frac{49}{264}b_0 = -\frac{1,715}{20,592}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{1/2}\left(1 - \frac{7}{18}x + \frac{49}{264}x^2 - \frac{1,715}{20,592}x^3 + \cdots\right)}$$

$$\boxed{y(x) = a_0\frac{1}{x^3}\left(1 - \frac{21}{5}x + \frac{49}{5}x^2 - \frac{343}{15}x^3\right) + b_0\sqrt{x}\left(1 - \frac{7}{18}x + \frac{49}{264}x^2 - \frac{1,715}{20,592}x^3 + \cdots\right)}$$

Exercise

Find the Frobenius series solutions:

$$x(1-x)y'' + \left[c - (a+b+1)x\right]y' - aby = 0 \qquad (\textit{Gauss' Hypergeometric})$$

Solution

$$y'' + \frac{c-(a+b+1)x}{x(1-x)}y' - \frac{ab}{x(1-x)}y = 0$$

That implies to $p(x) = \dfrac{c-(a+b+1)x}{x(1-x)}$ and $q(x) = -\dfrac{ab}{x(1-x)}$.

$$\begin{aligned} p_0 &= \lim_{x\to 0} xp(x) \\ &= \lim_{x\to 0} x\left(\frac{c-(a+b+1)x}{x(1-x)}\right) \\ &= \lim_{x\to 0}\left(\frac{c-(a+b+1)x}{1-x}\right) \\ &\boxed{= c} \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x\to 0} x^2 q(x) \\ &= -\lim_{x\to 0} x^2\frac{ab}{x(1-x)} \\ &= -\lim_{x\to 0}\frac{abx}{1-x} \\ &\boxed{= 0} \end{aligned}$$

$$p_1 = \lim_{x\to 1}(x-1)p(x)$$

$$= \lim_{x\to 1}(x-1)\left(\frac{c-(a+b+1)x}{x(1-x)}\right)$$

$$= \lim_{x\to 1}\left(-\frac{c-(a+b+1)x}{x}\right)$$

$$\underline{= a+b+1-c\,|}$$

$$q_1 = \lim_{x\to 1}(x-1)^2 q(x)$$

$$= -\lim_{x\to 1}(x-1)^2\frac{ab}{x(1-x)}$$

$$= \lim_{x\to 1}\frac{ab}{x}(x-1)$$

$$\underline{= 0\,|}$$

The *Regular* singular points: $\underline{x = 0,\ 1\,|}$

The indicial equation is: $r(r-1)-cr = r^2+(c-1)r = 0 \quad \to \quad \underline{r = 0,\ 1-c\,|}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1-c}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$x(1-x)y'' + \left[c-(a+b+1)x\right]y' - aby = 0$$

$$\left(x-x^2\right)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \left[c-(a+b+1)x\right]\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$-ab\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1}-\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}+\sum_{n=0}^{\infty}c(n+r)a_n x^{n+r-1}$$

$$-\sum_{n=0}^{\infty}(a+b+1)(n+r)a_n x^{n+r}-ab\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)+c(n+r)\right]a_n x^{n+r-1}$$

$$-\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)+(a+b+1)(n+r)+ab\right]a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1+c)a_n x^{n+r-1}-\sum_{n=1}^{\infty}\left[(n+r-1)(n+r-2+a+b+1)+ab\right]a_{n-1}x^{n+r-1}=0$$

$$r(r+c-1)a_0x^{r-1}+\sum_{n=1}^{\infty}(n+r)(n+r-1+c)a_n x^{n+r-1}$$

$$-\sum_{n=1}^{\infty}\left[(n+r-1)(n+r-1+a+b)+ab\right]a_{n-1}x^{n+r-1}=0$$

$$r(r+c-1)a_0x^{r-1}$$

$$+\sum_{n=1}^{\infty}\left[(n+r)(n+r-1+c)a_n-\left((n+r-1)(n+r-1+a+b)+ab\right)a_{n-1}\right]x^{n+r-1}=0$$

For $n=0$

$$r(r+c-1)a_0=0$$

$$\Rightarrow \underline{r=0,\ 1-c} \quad \surd$$

$$(n+r)(n+r-1+c)a_n-\left((n+r-1)(n+r-1+a+b)+ab\right)a_{n-1}=0$$

$$(n+r)(n+r-1+c)a_n=\left((n+r-1)(n+r-1+a+b)+ab\right)a_{n-1}$$

$$\boxed{a_n=\frac{(n+r-1)(n+r-1+a+b)+ab}{(n+r)(n+r-1+c)}a_{n-1}}$$

$r=0$

$$a_n = \frac{(n-1)(n-1+a+b)+ab}{n(n-1+c)} a_{n-1}$$

$$n=1 \rightarrow a_1 = \frac{ab}{c} a_0$$

$$n=2 \rightarrow a_2 = \frac{1+a+b+ab}{2\cdot(c+1)} a_1 = \frac{(a+1)(b+1)}{2\cdot(c+1)} a_1 = \frac{ab(a+1)(b+1)}{2\cdot c\cdot(c+1)} a_0$$

$$n=3 \rightarrow a_3 = \frac{4+2a+2b+ab}{3\cdot(c+2)} a_2 = \frac{(a+2)(b+2)}{3\cdot(c+2)} a_2 = \frac{a(a+1)(a+2)\cdot b(b+1)(b+2)}{2\cdot 3\cdot c(c+1)(c+2)} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\rightarrow a_n = \frac{a(a+1)(a+2)\cdots(a+n-1)\bullet b(b+1)(b+2)\cdots(b+n-1)}{n!\cdot c(c+1)(c+2)\cdots(c+n-1)} a_0$$

$$y_1(x) = a_0\left(1+\frac{ab}{c}x+\frac{ab(a+1)(b+1)}{2\cdot c\cdot(c+1)}x^2+\frac{a(a+1)(a+2)\cdot b(b+1)(b+2)}{2\cdot 3\cdot c(c+1)(c+2)}x^3+\cdots\right)$$

$$\underline{= a_0\left(1+\sum_{n=0}^{\infty}\frac{a(a+1)\cdots(a+n-1)\bullet b(b+1)\cdots(b+n-1)}{n!\cdot c(c+1)\cdots(c+n-1)}x^n\right)}\Bigg|$$

$$r = 1-c$$

$$b_n = \frac{(n-c)(n-c+a+b)+ab}{n(n+1-c)} b_{n-1}$$

$$n=1 \rightarrow b_1 = \frac{(1-c)(1-c+a+b)+ab}{2-c} b_0$$

$$n=2 \rightarrow b_2 = \frac{(2-c)(2-c+a+b)+ab}{2(3-c)} b_1 = \frac{(1-c)(1-c+a+b)+ab}{2-c} b_0$$

$$= \frac{\big((2-c)(2-c+a+b)+ab\big)\big((1-c)(1-c+a+b)+ab\big)}{2(2-c)(3-c)} b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\rightarrow b_n = \frac{\big((n-c)(n-c+a+b)+ab\big)\cdots\big((1-c)(1-c+a+b)+ab\big)}{2(2-c)(3-c)\ \cdots(n+1-c)} b_0$$

$$\underline{y_2(x) = b_0 x^{1-c}\left(\begin{array}{l} 1+\dfrac{(1-c)(1-c+a+b)+ab}{2-c}x \\ +\dfrac{\big((2-c)(2-c+a+b)+ab\big)\big((1-c)(1-c+a+b)+ab\big)}{2(2-c)(3-c)}x^2+\cdots \end{array}\right)}\Bigg|$$

$$y(x) = a_0\left(1 + \sum_{n=0}^{\infty} \frac{a(a+1)\cdots(a+n-1)\bullet b(b+1)\cdots(b+n-1)}{n!\cdot c(c+1)\cdots(c+n-1)}x^n\right)$$
$$+ b_0 x^{1-c}\left(1 + \sum_{n=0}^{\infty} \frac{((n-c)(n-c+a+b)+ab)\cdots((1-c)(1-c+a+b)+ab)}{2(2-c)(3-c)\ \cdots(n+1-c)}x^n\right)$$

Solution Section 4.5 – Bessel's Equation and Bessel Functions

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2y'' + xy' + \left(x^2 - \upsilon^2\right)y = 0$: $\qquad x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0$

Solution

$$\upsilon^2 = \frac{1}{9} \rightarrow \upsilon = \frac{1}{3}$$

The general solution is: $\boxed{y(x) = c_1 J_{1/3}(x) + c_2 J_{-1/3}(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2y'' + xy' + \left(x^2 - \upsilon^2\right)y = 0$: $\qquad x^2y'' + xy' + \left(x^2 - 1\right)y = 0$

Solution

$$\upsilon^2 = 1 \rightarrow \upsilon = 1$$

The general solution is: $\boxed{y(x) = c_1 J_1(x) + c_2 Y_1(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2y'' + xy' + \left(x^2 - \upsilon^2\right)y = 0$: $\qquad 4x^2y'' + 4xy' + \left(4x^2 - 25\right)y = 0$

Solution

$$\upsilon^2 = \frac{25}{4} \rightarrow \upsilon = \pm\frac{5}{2}$$

The general solution is: $\boxed{y(x) = c_1 J_{5/2}(x) + c_2 J_{-5/2}(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2y'' + xy' + \left(x^2 - \upsilon^2\right)y = 0$: $\qquad 16x^2y'' + 16xy' + \left(16x^2 - 1\right)y = 0$

Solution

$\upsilon^2 = \frac{1}{15} \rightarrow \upsilon = \pm\frac{1}{4}$

The general solution is: $\boxed{y(x) = c_1 J_{1/4}(x) + c_2 J_{-1/4}(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2 y'' + xy' + \left(x^2 - \upsilon^2\right) y = 0$: $\quad xy'' + y' + xy = 0$

Solution

$\upsilon^2 = 0 \rightarrow \upsilon = 0$

The general solution is: $\boxed{y(x) = c_1 J_0(x) + cY_0(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2 y'' + xy' + \left(x^2 - \upsilon^2\right) y = 0$: $\quad xy'' + y' + \left(x - \frac{4}{x}\right) y = 0$

Solution

$\upsilon^2 = 4 \rightarrow \upsilon = 2$

The general solution is: $\boxed{y(x) = c_1 J_2(x) + c_2 Y_2(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2 y'' + xy' + \left(\alpha^2 x^2 - \upsilon^2\right) y = 0$: $\quad x^2 y'' + xy' + \left(9x^2 - 4\right) y = 0$

Solution

$$\begin{cases} \alpha^2 = 9 \rightarrow \alpha = 3 \\ \upsilon^2 = 4 \rightarrow \upsilon = 2 \end{cases}$$

The general solution is: $\boxed{y(x) = c_1 J_2(3x) + c_2 Y_2(3x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2y'' + xy' + \left(\alpha^2x^2 - \upsilon^2\right)y = 0$: $\quad x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right)y = 0$

Solution

$$\begin{cases} \alpha^2 = 36 \ \rightarrow \ \alpha = 6 \\ \upsilon^2 = \frac{1}{4} \ \rightarrow \ \upsilon = \frac{1}{2} \end{cases}$$

The general solution is: $\quad y(x) = c_1 J_{1/2}(6x) + c_2 J_{-1/2}(6x)$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2y'' + xy' + \left(\alpha^2x^2 - \upsilon^2\right)y = 0$: $\quad x^2y'' + xy' + \left(25x^2 - \frac{4}{9}\right)y = 0$

Solution

$$\begin{cases} \alpha^2 = 25 \ \rightarrow \ \alpha = 5 \\ \upsilon^2 = \frac{4}{9} \ \rightarrow \ \upsilon = \frac{2}{3} \end{cases}$$

The general solution is: $\quad y(x) = c_1 J_{2/3}(5x) + c_2 J_{-2/3}(5x)$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2y'' + xy' + \left(\alpha^2x^2 - \upsilon^2\right)y = 0$: $\quad x^2y'' + xy' + \left(2x^2 - 64\right)y = 0$

Solution

$$\begin{cases} \alpha^2 = 2 \ \rightarrow \ \alpha = \sqrt{2} \\ \upsilon^2 = 64 \ \rightarrow \ \upsilon = 8 \end{cases}$$

The general solution is: $\quad y(x) = c_1 J_8\left(\sqrt{2}x\right) + c_2 Y_8\left(\sqrt{2}x\right)$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$4x^2y'' + 8xy' + \left(x^4 - 3\right)y = 0$$

Solution

$$\frac{1}{4}\times \quad 4x^2y'' + 8xy' + \left(x^4 - 3\right)y = 0$$

$$x^2y'' + 2xy' + \left(-\frac{3}{4} + \frac{1}{4}x^4\right)y = 0 \qquad\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = 2, \quad B = -\frac{3}{4}, \quad C = \frac{1}{4}, \quad p = 4$$

$$\begin{cases} \alpha = \frac{1-2}{2} = -\frac{1}{2} \\ \beta = \frac{4}{2} = 2 \\ k = \frac{2\sqrt{\frac{1}{4}}}{4} = \frac{1}{4} \\ \upsilon = \frac{\sqrt{1+3}}{4} = \frac{1}{2} \end{cases}$$

$$y(x) = x^{-1/2}\left[c_1J_{1/2}\left(\frac{1}{4}x^2\right) + c_2J_{-1/2}\left(\frac{1}{4}x^2\right)\right]$$

$$y(x) = x^{\alpha}\left[c_1J_{\upsilon}\left(kx^{\beta}\right) + c_2J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

$$= x^{-1/2}\left(c_1\sqrt{\frac{2}{\pi z}}\sin z + c_2\sqrt{\frac{2}{\pi z}}\cos z\right)$$

$$= c_1\left(\frac{2}{\pi x}\right)^{1/2}\sin x + c_2\left(\frac{2}{\pi x}\right)^{1/2}\cos x$$

$$= x^{-1/2}\left(c_1\frac{2}{x}\sqrt{\frac{2}{\pi}}\sin\frac{x^2}{4} + c_2\frac{2}{x}\sqrt{\frac{2}{\pi}}\cos\frac{x^2}{4}\right) \qquad\qquad z = kx^{\beta} = \frac{x^2}{4}$$

$$\boxed{= x^{-3/2}\left(C_1\sqrt{\frac{2}{\pi}}\sin\frac{x^2}{4} + C_2\sqrt{\frac{2}{\pi}}\cos\frac{x^2}{4}\right)}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$y'' + 9xy = 0$$

Solution

$$x^2 \times \quad y'' + 9xy = 0$$

$$x^2 y'' + 9x^3 y = 0$$

$$x^2 \frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$A = 0, \quad B = 0, \quad C = 9, \quad p = 3$

$\alpha = \frac{1}{2}, \quad \beta = \frac{3}{2}, \quad k = 2, \quad \upsilon = \frac{1}{3}$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$y(x) = x^{1/2}\left[c_1 J_{1/3}\left(2x^{3/2}\right) + c_2 J_{-1/3}\left(2x^{3/2}\right)\right]$$

$$y(x) = x^{\alpha}\left[c_1 J_{\upsilon}\left(kx^{\beta}\right) + c_2 J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' + (x-3)y = 0$$

Solution

$$x \times \quad xy'' - 3y + xy = 0$$

$$x^2 y'' - 3xy + x^2 y = 0$$

$$x^2 \frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$A = -3, \quad B = 0, \quad C = 1, \quad p = 2$

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{p}{2}, \quad k = \frac{2\sqrt{C}}{p}, \quad \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$\alpha = 2, \quad \beta = 1, \quad k = 1, \quad \upsilon = \frac{\sqrt{16}}{2} = 2$

$$y(x) = x^2\left[c_1 Y_2(x) + c_2 J_2(x)\right]$$

$$y(x) = x^{\alpha}\left[c_1 J_{\upsilon}\left(kx^{\beta}\right) + c_2 J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' + \left(4x^3 - 1\right)y = 0$$

Solution

$x \times \quad xy'' - y + 4x^3 y = 0$

$$x^2 y'' - xy + 4x^4 y = 0 \qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$A = -1, \quad B = 0, \quad C = 4, \quad p = 4$

$$\alpha = 1, \quad \beta = 2, \quad k = 1, \quad \upsilon = \frac{1}{2} \qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$y(x) = x\left[c_1 J_{1/2}\left(x^2\right) + c_2 J_{-1/2}\left(x^2\right)\right] \qquad y(x) = x^\alpha\left[c_1 J_\upsilon\left(kx^\beta\right) + c_2 J_{-\upsilon}\left(kx^\beta\right)\right]$$

$$= x\left(c_1 \frac{1}{x}\sqrt{\frac{2}{\pi}}\sin x^2 + c_2 \frac{1}{x}\sqrt{\frac{2}{\pi}}\cos x^2\right) \qquad y(z) = x^\alpha\left(c_1\left(\frac{2}{\pi z}\right)^{1/2}\sin z + c_2\left(\frac{2}{\pi z}\right)^{1/2}\cos z\right)$$

$$= c_1\sqrt{\frac{2}{\pi}}\sin x^2 + c_2\sqrt{\frac{2}{\pi}}\cos x^2 \qquad z = kx^\beta = x^2$$

$$\boxed{= C_1 \sin x^2 + C_2 \cos x^2}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^2 y'' + xy' - \left(\frac{1}{4} + x^2\right)y = 0$$

Solution

$$x^2 y'' + xy' + \left(-\frac{1}{4} - x^2\right)y = 0 \qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$A = 1, \quad B = -\frac{1}{4}, \quad C = -1, \quad p = 2$

$$\alpha = 0, \quad \beta = 1, \quad k = i, \quad \upsilon = \frac{1}{2} \qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$y(x) = c_1 I_{1/2}(x) + c_2 I_{-1/2}(x) \qquad y(x) = x^\alpha\left[c_1 I_\upsilon\left(kx^\beta\right) + c_2 I_{-\upsilon}\left(kx^\beta\right)\right]$$

$$\boxed{y(x) = c_1\sqrt{\frac{2}{\pi x}}\sinh x + c_2\sqrt{\frac{2}{\pi x}}\cosh x} \qquad I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sinh x$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' + (2x+1)y' + (2x+1)y = 0$$

Solution

$$x\times \quad xy'' + (2x+1)y' + (2x+1)y = 0$$

$$x^2y'' + x(2x+1)y' + \left(2x^2 + x\right)y = 0$$

Let $Y = ye^{x} \quad \to y = Ye^{-x}$

$$x^2\left(Y'' - 2Y' + Y\right)e^{-x} + x(2x+1)(Y' - Y)e^{-x} + \left(2x^2 + x\right)Ye^{-x} = 0$$

$$x^2Y'' - 2x^2Y' + x^2Y + \left(2x^2 + x\right)Y' - \left(2x^2 + x\right)Y + \left(2x^2 + x\right)Y = 0$$

$$x^2Y'' + xY' + x^2Y = 0 \qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$A = 1, \quad B = 0, \quad C = 1, \quad p = 2$

$$\alpha = 0, \quad \beta = 1, \quad k = 1, \quad \upsilon = 0 \qquad \alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$Y(x) = c_1J_0(x) + c_2Y_0(x) \qquad y(x) = x^{\alpha}\left[c_1J_0\left(kx^{\beta}\right) + c_2Y_0\left(kx^{\beta}\right)\right]$$

$$\boxed{y(x) = \left(c_1J_0(x) + c_2Y_0(x)\right)e^{-x}}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' - y' - xy = 0$$

Solution

$$x\times \quad xy'' - y' - xy = 0$$

$$x^2y'' - xy' - x^2y = 0 \qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$A = -1, \quad B = 0, \quad C = -1 = i, \quad p = 2$

Let $Y = \frac{y}{x}$ & $X = ix$

$y = xY$ & $x = -iX$

$$x^2\left(2Y' + xY''\right) - x\left(Y + xY'\right) - x^3Y = 0$$

$$x^3Y'' + x^2Y' - x\left(x^2+1\right)Y = 0$$

$$x^2Y'' + xY' - \left(x^2+1\right)Y = 0$$

$$-X^2Y'' - iXY' - \left(-X^2+1\right)Y = 0$$

$$X^2Y'' + XY' + \left(X^2-1\right)Y = 0 \qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$A = 1, \quad B = -1, \quad C = 1, \quad p = 2$

$$\alpha = 0, \quad \beta = 1, \quad k = 1, \quad \upsilon = 1 \qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2-4B}}{p}$$

$$Y = Z_1(X)$$

$$y(x) = xZ_1(ix) \qquad y(x) = x^\alpha\left[c_1J_0\left(kx^\beta\right) + c_2Y_0\left(kx^\beta\right)\right]$$

$$= x\left(c_1I_1(x) + c_2K_1(x)\right)$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^4y'' + a^2y = 0$$

Solution

$$\frac{1}{x^2}\times \quad x^4y'' + a^2y = 0$$

$$x^2y'' + \frac{a^2}{x^2}y = 0$$

Let $Y = \frac{y}{\sqrt{x}} \quad \to y = \sqrt{x}\ Y$

$$X = \frac{a}{x} \quad \to \quad x = \frac{a}{X}$$

$$X^2Y'' + XY' + \left(X^2 - K^2\right)Y = 0$$

$$Y = x^{-1/2}y$$

$$Y' = -\frac{1}{2}x^{-3/2}y + x^{-1/2}y'$$

$$Y'' = \frac{3}{4}x^{-5/2}y - x^{-3/2}y' + x^{-1/2}y''$$

$$x^2\left(x^{-1/2}y'' - x^{-3/2}y' + \frac{3}{4}x^{-5/2}y\right) + x\left(-\frac{1}{2}x^{-3/2}y + x^{-1/2}y'\right) + \left(x^2 - K^2\right)x^{-1/2}y = 0$$

$$x^{3/2}y'' - x^{1/2}y' + \frac{3}{4}x^{-1/2}y - \frac{1}{2}x^{-1/2}y + x^{1/2}y' + \left(x^2 - K^2\right)x^{-1/2}y = 0$$

$$x^{3/2}y'' + \left(x^2 - K^2 + \frac{1}{4}\right)x^{-1/2}y = 0 \quad \times x^{1/2}$$

$$x^2y'' + \left(x^2 - K^2 + \frac{1}{4}\right)y = 0$$

$$x^2 - K^2 + \frac{1}{4} = x^2$$

$$-K^2 + \frac{1}{4} = 0 \quad \rightarrow \quad K^2 = \frac{1}{4}$$

$$X^2Y'' + XY' + \left(X^2 - \frac{1}{4}\right)Y = 0 \qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = 1, \quad B = -\frac{1}{4}, \quad C = 1, \quad p = 2$$

$$\alpha = 0, \quad \beta = 1, \quad k = 1, \quad \upsilon = \frac{1}{2} \qquad \alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$Y = Z_{1/2}(X)$$

$$y(x) = \sqrt{x}Z_{1/2}\left(\frac{a}{x}\right) \qquad y(x) = x^{\alpha}\left[c_1J_{\upsilon}\left(kx^{\beta}\right) + c_2J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

$$= \sqrt{x}\left(c_1J_{1/2}\left(\frac{a}{x}\right) + c_2J_{-1/2}\left(\frac{a}{x}\right)\right)$$

$$y(z) = x^{\alpha}\left(c_1\left(\frac{2}{\pi z}\right)^{1/2}\sin z + c_2\left(\frac{2}{\pi z}\right)^{1/2}\cos z\right)$$

$$= \sqrt{x}\left(c_1\sqrt{\frac{2x}{\pi a}}\sin\frac{a}{x} + c_2\sqrt{\frac{2x}{\pi a}}\cos\frac{a}{x}\right) \qquad z = kX^{\beta} = \frac{a}{x}$$

$$= x\left(c_1\sqrt{\frac{2}{\pi a}}\sin\frac{a}{x} + c_2\sqrt{\frac{2}{\pi a}}\cos\frac{a}{x}\right)$$

$$\boxed{= x\left(C_1\sin\frac{a}{x} + C_2\cos\frac{a}{x}\right)}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$y'' - x^2y = 0$$

Solution

$$x^2 \times \quad y'' - x^2y = 0$$

$$x^2y'' - x^4y = 0$$

$$A = 0, \quad B = 0, \quad C = -1, \quad p = 4$$

$$\alpha = \frac{1}{2}, \quad \beta = 1, \quad k = \frac{i}{2}, \quad \upsilon = 0 \qquad\qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2-4B}}{p}$$

Let $Y = \frac{y}{\sqrt{x}} \quad \to y = \sqrt{x}\ Y$

$$X = \frac{1}{2}ix^2 \quad \to \quad x^2 = -2iX$$

$$X^2Y'' + XY' + \left(X^2 - K^2\right)Y = 0$$

$$Y = x^{-1/2}y$$

$$Y' = -\frac{1}{2}x^{-3/2}y + x^{-1/2}y'$$

$$Y'' = \frac{3}{4}x^{-5/2}y - x^{-3/2}y' + x^{-1/2}y''$$

$$x^2\left(x^{-1/2}y'' - x^{-3/2}y' + \frac{3}{4}x^{-5/2}y\right) + x\left(-\frac{1}{2}x^{-3/2}y + x^{-1/2}y'\right) + \left(x^2 - K^2\right)x^{-1/2}y = 0$$

$$x^{3/2}y'' - x^{1/2}y' + \frac{3}{4}x^{-1/2}y - \frac{1}{2}x^{-1/2}y + x^{1/2}y' + \left(x^2 - K^2\right)x^{-1/2}y = 0$$

$$x^{3/2}y'' + \left(x^2 - K^2 + \frac{1}{4}\right)x^{-1/2}y = 0 \quad \times x^{1/2}$$

$$x^2y'' + \left(x^2 - K^2 + \frac{1}{4}\right)y = 0$$

$$x^2 - K^2 + \frac{1}{4} = -x^4$$

$$K = \frac{1}{4} \quad \to \quad K^2 = \frac{1}{16}$$

$$X^2Y'' + XY' + \left(X^2 - \frac{1}{16}\right)Y = 0$$

$$A = 1, \quad B = -\frac{1}{16}, \quad C = 1, \quad p = 2$$

$$\alpha = 0, \quad \beta = 1, \quad k = 1, \quad \upsilon = \frac{1}{4} \qquad\qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2-4B}}{p}$$

$$Y = Z_{1/4}(X)$$

$$y(x) = \sqrt{x}Z_{\frac{1}{4}}\left(\frac{i}{2}x^2\right)$$

$$= \sqrt{x}\left(c_1 I_{\frac{1}{4}}\left(\frac{x^2}{2}\right) + c_2 I_{-\frac{1}{4}}\left(\frac{x^2}{2}\right)\right)$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^2y'' - xy' + \left(1+x^2\right)y = 0$$

Solution

$$x^2y'' - xy' + \left(1+x^2\right)y = 0 \qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = -1, \quad B = 1, \quad C = 1, \quad p = 2$$

$$\alpha = 1, \quad \beta = 1, \quad k = 1, \quad \upsilon = 0 \qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\boxed{y(x) = x\left[c_1J_0(x) + c_2Y_0(x)\right]}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' + 3y' + xy = 0$$

Solution

$$x \times \quad xy'' + 3y' + xy = 0$$

$$x^2y'' + 3xy' + x^2y = 0 \qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = 3, \quad B = 0, \quad C = 1, \quad p = 2$$

$$\alpha = -1, \quad \beta = 1, \quad k = 1, \quad \upsilon = 1 \qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\boxed{y(x) = x^{-1}\left[c_1J_1(x) + c_2Y_1(x)\right]}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' - y' + 36x^3y = 0$$

Solution

$$x \times \quad xy'' - y' + 36x^3y = 0$$

$$x^2y'' - xy' + 36x^4y = 0 \qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = -1, \quad B = 0, \quad C = 36, \quad p = 4$$

$$\alpha = 1, \quad \beta = 2, \quad k = 3, \quad \upsilon = \frac{1}{2} \qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\underline{y(x) = x\left[c_1J_{1/2}\left(3x^2\right) + c_2J_{-1/2}\left(3x^2\right)\right]} \qquad y(x) = c_1J_{1/2}(x) + c_2J_{-1/2}(x)$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^2y'' - 5xy' + (8 + x)y = 0$$

Solution

$$x^2y'' - 5xy' + (8 + x)y = 0 \qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = -5, \quad B = 8, \quad C = 1, \quad p = 1$$

$$\alpha = 3, \quad \beta = \frac{1}{2}, \quad k = 2, \quad \upsilon = 2 \qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\underline{y(x) = x^3\left[c_1J_2\left(2x^{1/2}\right) + c_2Y_2\left(2x^{1/2}\right)\right]} \qquad y(x) = x^\alpha\left[c_1J_\upsilon\left(kx^\beta\right) + c_2Y_\upsilon\left(kx^\beta\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$36x^2y'' + 60xy' + \left(9x^3 - 5\right)y = 0$$

Solution

$$x^2y'' + \frac{5}{3}xy' + \left(\frac{1}{4}x^3 - \frac{5}{36}\right)y = 0 \qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = \frac{5}{3}, \quad B = -\frac{5}{36}, \quad C = \frac{1}{4}, \quad p = 3$$

$$\alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\alpha = -\frac{1}{3}, \quad \beta = \frac{3}{2}, \quad k = \frac{1}{3}, \quad \upsilon = \frac{\sqrt{\left(-\frac{2}{3}\right)^2 + \frac{5}{9}}}{3} = \frac{1}{3}$$

$$y(x) = x^{\alpha}\left[c_1 J_{\upsilon}\left(kx^{\beta}\right) + c_2 J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

$$\boxed{y(x) = x^{-1/3}\left[c_1 J_{1/3}\left(\frac{1}{3}x^{3/2}\right) + c_2 J_{-1/3}\left(\frac{1}{3}x^{3/2}\right)\right]}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$16x^2 y'' + 24xy' + \left(1 + 144x^3\right)y = 0$$

Solution

$$x^2 y'' + \frac{3}{2}xy' + \left(\frac{1}{16} + 9x^3\right)y = 0 \qquad\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = \frac{3}{2}, \quad B = \frac{1}{16}, \quad C = 9, \quad p = 3$$

$$\alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\alpha = -\frac{1}{4}, \quad \beta = \frac{3}{2}, \quad k = 2, \quad \upsilon = \frac{\sqrt{\left(-\frac{1}{2}\right)^2 - \frac{1}{4}}}{3} = 0$$

$$\boxed{y(x) = x^{-1/4}\left[c_1 J_0\left(2x^{3/2}\right) + c_2 Y_0\left(2x^{3/2}\right)\right]} \qquad\qquad y(x) = x^{\alpha}\left[c_1 J_{\upsilon}\left(kx^{\beta}\right) + c_2 Y_{\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^2 y'' + 3xy' + \left(1 + x^2\right)y = 0$$

Solution

$$x^2 y'' + 3xy' + \left(1 + x^2\right)y = 0 \qquad\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = 3, \quad B = 1, \quad C = 1, \quad p = 2$$

$$\alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\alpha = -1, \quad \beta = 1, \quad k = 1, \quad \upsilon = \frac{\sqrt{(-2)^2 - 4}}{3} = 0$$

$$\boxed{y(x) = x^{-1}\left[c_1 J_0(x) + c_2 Y_0(x)\right]} \qquad y(x) = x^{\alpha}\left[c_1 J_{\upsilon}\left(kx^{\beta}\right) + c_2 Y_{\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$4x^2 y'' - 12xy' + (15 + 16x) y = 0$$

Solution

$$x^2 y'' - 3xy' + \left(\frac{15}{4} + 4x\right) y = 0 \qquad x^2 \frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right) y = 0$$

$$A = -3, \quad B = \frac{15}{4}, \quad C = 4, \quad p = 1$$

$$\alpha = 2, \quad \beta = \frac{1}{2}, \quad k = 4, \quad \upsilon = \frac{\sqrt{(4)^2 - 15}}{1} = 1 \qquad \alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\boxed{y(x) = x^2\left[c_1 J_1\left(4x^{1/2}\right) + c_2 Y_1\left(4x^{1/2}\right)\right]} \qquad y(x) = x^{\alpha}\left[c_1 J_{\upsilon}\left(kx^{\beta}\right) + c_2 Y_{\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$16x^2 y'' - \left(5 - 144x^3\right) y = 0$$

Solution

$$x^2 y'' + \left(9x^3 - \frac{5}{16}\right) y = 0 \qquad x^2 \frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right) y = 0$$

$$A = 0, \quad B = -\frac{5}{16}, \quad C = 9, \quad p = 3$$

$$\alpha = \frac{1}{2}, \quad \beta = \frac{3}{2}, \quad k = 2, \quad \upsilon = \frac{\sqrt{1 + \frac{5}{4}}}{3} = \frac{1}{2} \qquad \alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\boxed{y(x) = x^{1/2}\left[c_1 J_{1/2}\left(2x^{3/2}\right) + c_2 J_{-1/2}\left(2x^{3/2}\right)\right]} \qquad y(x) = x^{\alpha}\left[c_1 J_{\upsilon}\left(kx^{\beta}\right) + c_2 J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$2x^2y'' + 3xy' - \left(28 - 2x^5\right)y = 0$$

Solution

$$x^2y'' + \frac{3}{2}xy' + \left(x^5 - 14\right)y = 0 \qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$A = \frac{3}{2}, \quad B = -14, \quad C = 1, \quad p = 5$

$$\alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\alpha = -\frac{1}{4}, \quad \beta = \frac{5}{2}, \quad k = \frac{2}{5}, \quad \upsilon = \frac{\sqrt{\left(-\frac{1}{2}\right)^2 + 56}}{5} = \frac{\frac{15}{2}}{5} = \frac{3}{2}$$

$$\boxed{y(x) = x^{-1/4}\left[c_1 J_{3/2}\left(\frac{2}{5}x^{5/2}\right) + c_2 J_{-3/2}\left(\frac{2}{5}x^{5/2}\right)\right]} \qquad y(x) = x^{\alpha}\left[c_1 J_{\upsilon}\left(kx^{\beta}\right) + c_2 J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$y'' + x^4 y = 0$$

Solution

$$x^2 \times \quad y'' + x^4 y = 0$$

$$x^2y'' + x^6y = 0 \qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$A = 0, \quad B = 0, \quad C = 1, \quad p = 6$

$\alpha = \frac{1}{2}, \quad \beta = 3, \quad k = \frac{1}{3}, \quad \upsilon = \frac{1}{6}$

$$\alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\boxed{y(x) = x^{1/2}\left[c_1 J_{1/6}\left(\frac{1}{3}x^3\right) + c_2 J_{-1/6}\left(\frac{1}{3}x^3\right)\right]} \qquad y(x) = x^{\alpha}\left[c_1 J_{\upsilon}\left(kx^{\beta}\right) + c_2 J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$y'' + 4x^3 y = 0$$

Solution

$$x^2 \times \quad y'' + 4x^3 y = 0$$

$$x^2 y'' + 4x^5 y = 0 \qquad\qquad x^2 \frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = 0, \quad B = 0, \quad C = 4, \quad p = 5$$

$$\alpha = \frac{1}{2}, \quad \beta = \frac{5}{2}, \quad k = \frac{4}{5}, \quad \upsilon = \frac{1}{5} \qquad\qquad \alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\underline{y(x) = x^{1/2}\left[c_1 J_{1/5}\left(\tfrac{4}{5}x^{5/2}\right) + c_2 J_{-1/5}\left(\tfrac{4}{5}x^{5/2}\right)\right]} \qquad\qquad y(x) = x^{\alpha}\left[c_1 J_{\upsilon}\left(kx^{\beta}\right) + c_2 J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Find a Frobenius solution of Bessel's equation of order zero $x^2 y'' + xy' + x^2 y = 0$

Solution

$$y'' + \frac{1}{x}y' + y = 0$$

Therefore, $x = 0$ is a regular singular point, and that $p_0 = 1, \quad q_0 = 0$ and $p(x) \equiv 1, \quad q(x) = x^2$.

The indicial equation is: $r(r-1) + r = r^2 = 0 \quad \rightarrow \quad \underline{r = 0}$

There is only one Frobenius series solution: $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$x^2 y'' + xy' + x^2 y = 0$$

$$x^2\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=0}^{\infty} na_n x^{n-1} + x^2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} \left[n(n-1)+n\right]a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\sum_{n=0}^{\infty} n^2 a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0 \quad a_4 = -\frac{a_2}{4^2} = \frac{a_0}{2^2\cdot 4^2}$$

$$0 + a_1 x + \sum_{n=2}^{\infty} n^2 a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$a_1 x + \sum_{n=2}^{\infty} \left(n^2 a_n + a_{n-2}\right)x^n = 0$$

$$a_1 = 0 \quad \to \quad a_{n(odd)} = 0$$

$$n^2 a_n + a_{n-2} = 0$$

$$a_n = -\frac{a_{n-2}}{n^2} \qquad (n \ge 2)$$

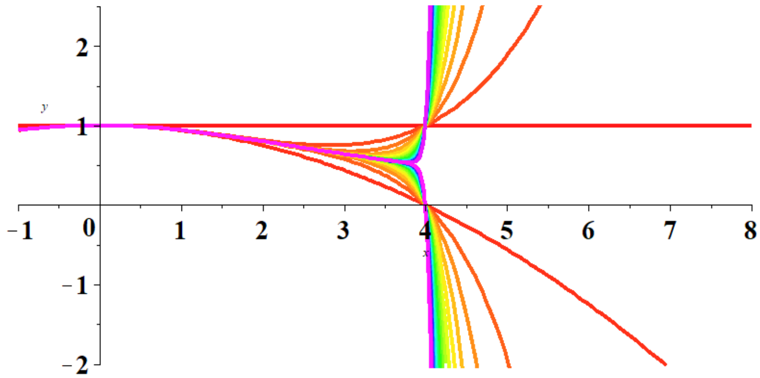
$$a_2 = -\frac{a_0}{4}$$

$$a_6 = -\frac{a_4}{6^2} = -\frac{a_0}{2^2\cdot 4^2\cdot 6^2}$$

$$a_{2n} = \frac{(-1)^n}{2^2\cdot 4^2 \cdots (2n)^2}a_0 = \frac{(-1)^n}{2^{2n}\cdot (n!)^2}a_0$$

The choice $a_0 = 1$ gives us the Bessel function of order zero of the first kind.

$$J_0(x) = \frac{(-1)^n x^{2n}}{2^{2n}\cdot (n!)^2} \quad = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots$$



Exercise

Derive the formula $\quad xJ'_\upsilon(x) = \upsilon J_\upsilon(x) - xJ_{\upsilon+1}(x)$

Solution

$$xJ_\upsilon(x) = x\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$xJ'_\upsilon(x) = x\sum_{n=0}^{\infty}\frac{(-1)^n(2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= 2\sum_{n=0}^{\infty}\frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \upsilon\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \upsilon J_v(x)$$

$$= \upsilon J_v(x) + x\sum_{n=1}^{\infty}\frac{(-1)^n}{(n-1)!\,\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= \upsilon J_v(x) + x\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n!\,\Gamma(2+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon+1}$$

$$\underline{= \upsilon J_\upsilon(x) - xJ_{\upsilon+1}(x)} \quad \checkmark$$

Exercise

Derive the formula $\quad x J_{\upsilon}'(x) = -\upsilon J_{\upsilon}(x) + x J_{\upsilon-1}(x)$

Solution

$$x J_{\upsilon}'(x) = x \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\upsilon)}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= 2\sum_{n=0}^{\infty} \frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon} + \upsilon \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$-\upsilon J_{\upsilon}(x) + x J_{\upsilon-1}(x) = -\upsilon \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon} + x \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n \upsilon}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n \upsilon}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon} + \sum_{n=0}^{\infty} \frac{(-1)^n (\upsilon+n)}{n!\Gamma(1+\upsilon+n)} 2\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (-\upsilon+2n+2\upsilon)}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\upsilon)}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= x \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\upsilon)}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= x J_{\upsilon}'(x) \quad \checkmark$$

Exercise

Derive the formula $\quad 2\upsilon J_{\upsilon}'(x) = x J_{\upsilon+1}(x) + x J_{\upsilon-1}(x)$

Solution

From previous proofs:

$$\begin{array}{rl} & xJ'_{\upsilon}(x) = \upsilon J_{\upsilon}(x) - xJ_{\upsilon+1}(x) \\ - & xJ'_{\upsilon}(x) = -\upsilon J_{\upsilon}(x) + xJ_{\upsilon-1}(x) \\ \hline & 0 = 2\upsilon J_{\upsilon}(x) - xJ_{\upsilon+1}(x) - xJ_{\upsilon-1}(x) \end{array}$$

$$\boxed{2\upsilon J'_{\upsilon}(x) = xJ_{\upsilon+1}(x) + xJ_{\upsilon-1}(x)} \quad \checkmark$$

Exercise

Prove that $\quad \frac{d}{dx}\left[x^{\upsilon+1}J_{\upsilon+1}(x)\right] = x^{\upsilon+1}J_{\upsilon}(x)$

Solution

$$\frac{d}{dx}\left[x^{\upsilon+1}J_{\upsilon+1}(x)\right] = \frac{d}{dx}\left[x^{\upsilon+1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+1+n)}\left(\frac{x}{2}\right)^{2n+\upsilon+1}\right]$$

$$= \frac{d}{dx}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\upsilon+n+2)}\left(\frac{x}{2}\right)^{2n+2\upsilon+2}\right]$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(2n+2\upsilon+2)}{n!\Gamma(\upsilon+n+2)}\left(\frac{x}{2}\right)^{2n+2\upsilon+1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n 2(n+\upsilon+1)}{n!2\Gamma(\upsilon+n+2)}\left(\frac{x}{2}\right)^{2n+2\upsilon+1}$$

$$2\Gamma(\upsilon+n+2) = 2(\upsilon+n+1)\Gamma(\upsilon+n+1)$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n 2(n+\upsilon+1)}{n!\,2(\upsilon+n+1)\Gamma(\upsilon+n+1)}\left(\frac{x}{2}\right)^{2n+2\upsilon+1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(\upsilon+n+1)}\left(\frac{x}{2}\right)^{2n+2\upsilon+1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(\upsilon+n+1)}\left(\frac{x}{2}\right)^{2n+\upsilon}\left(\frac{x}{2}\right)^{\upsilon+1}$$

$$\boxed{= x^{\upsilon+1}J_{\upsilon}(x)} \quad \checkmark$$

Exercise

Show that $y=\sqrt{x}\,J_{3/2}(x)$ is a solution of $x^2y''+\left(x^2-2\right)y=0$

Solution

$$x^2y''+\left(x^2-2\right)y=0$$

$J_{3/2}(x)$ is the solution of Bessel's equation of order $\frac{3}{2}$:

$$x^2J''_{3/2}(x)+xJ'_{3/2}(x)+\left(x^2-\frac{9}{4}\right)J_{3/2}(x)=0$$

$$x^2\left(\sqrt{x}\,J_{3/2}(x)\right)''+\left(x^2-2\right)\sqrt{x}\,J_{3/2}(x)=$$

$$=x^2\left[-\frac{1}{4}x^{-3/2}J_{3/2}(x)+x^{-1/2}J'_{3/2}(x)+x^{1/2}J''_{3/2}(x)\right]+\left(x^2-2\right)\sqrt{x}\,J_{3/2}(x)$$

$$=-\frac{1}{4}x^{1/2}J_{3/2}(x)+x^{3/2}J'_{3/2}(x)+x^{5/2}J''_{3/2}(x)+x^{5/2}J_{3/2}(x)-2\sqrt{x}\,J_{3/2}(x)$$

$$=\sqrt{x}\left[x^2J''_{3/2}(x)+xJ'_{3/2}(x)+\left(x^2-\frac{9}{4}\right)J_{3/2}(x)\right]$$

$$=0$$

Exercise

Show that $\quad 4J''_{\upsilon}(x)=J_{\upsilon-2}(x)-2J_{\upsilon}(x)+J_{\upsilon+2}(x)$

Solution

$$J_{\upsilon}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$xJ'_{\upsilon}(x)=x\sum_{n=0}^{\infty}\frac{(-1)^n(2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$=2\sum_{n=0}^{\infty}\frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}+\upsilon\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$-\upsilon J_{\upsilon}(x)+xJ_{\upsilon-1}(x)=-\upsilon\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}+x\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$=-\sum_{n=0}^{\infty}\frac{(-1)^n\upsilon}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n \upsilon}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \sum_{n=0}^{\infty} \frac{(-1)^n (\upsilon+n)}{n!\Gamma(1+\upsilon+n)} 2\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (-\upsilon+2n+2\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= x\sum_{n=0}^{\infty} \frac{(-1)^n (2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$\underline{= x\, J'_{\upsilon}(x)}$$

$$x\, J'_{\upsilon}(x) = x\sum_{n=0}^{\infty} \frac{(-1)^n (2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= 2\sum_{n=0}^{\infty} \frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \upsilon\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \upsilon J_v(x)$$

$$= \upsilon J_v(x) + x\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!\,\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= \upsilon J_v(x) + x\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n!\,\Gamma(2+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon+1}$$

$$\underline{= \upsilon J_{\upsilon}(x) - x\, J_{\upsilon+1}(x)}$$

$$x\, J'_{\upsilon}(x) = -\upsilon J_{\upsilon}(x) + x\, J_{\upsilon-1}(x)$$

$$\begin{array}{rl} & xJ'_\upsilon(x) = \upsilon J_\upsilon(x) - xJ_{\upsilon+1}(x) \\ + & xJ'_\upsilon(x) = -\upsilon J_\upsilon(x) + xJ_{\upsilon-1}(x) \\ \hline & 2xJ'_\upsilon(x) = -xJ_{\upsilon+1}(x) + xJ_{\upsilon-1}(x) \end{array}$$

$$J'_\upsilon(x) = \frac{1}{2}\left(J_{\upsilon-1}(x) - J_{\upsilon+1}(x)\right)$$

$$J''_\upsilon(x) = \frac{1}{2}\left(J'_{\upsilon-1}(x) - J'_{\upsilon+1}(x)\right)$$

$$J'_\upsilon(x) = \frac{1}{2}\left(J_{\upsilon-1}(x) - J_{\upsilon+1}(x)\right) \rightarrow (\upsilon = \upsilon - 1) \quad J'_{\upsilon-1}(x) = \frac{1}{2}\left(J_{\upsilon-2}(x) - J_\upsilon(x)\right)$$

$$J'_\upsilon(x) = \frac{1}{2}\left(J_{\upsilon-1}(x) - J_{\upsilon+1}(x)\right) \rightarrow (\upsilon = \upsilon + 1) \quad J'_{\upsilon+1}(x) = \frac{1}{2}\left(J_\upsilon(x) - J_{\upsilon+2}(x)\right)$$

$$\begin{aligned} J''_\upsilon(x) &= \frac{1}{2}\left(J'_{\upsilon-1}(x) - J'_{\upsilon+1}(x)\right) \\ &= \frac{1}{2}\left(\frac{1}{2}J_{\upsilon-2}(x) - \frac{1}{2}J_\upsilon(x) - \frac{1}{2}J_\upsilon(x) + \frac{1}{2}J_{\upsilon+2}(x)\right) \\ &= \frac{1}{4}\left(J_{\upsilon-2}(x) - 2J_\upsilon(x) + J_{\upsilon+2}(x)\right) \end{aligned}$$

$$\underline{4J''_\upsilon(x) = J_{\upsilon-2}(x) - 2J_\upsilon(x) + J_{\upsilon+2}(x)\,\big|} \quad \checkmark$$

Exercise

Show that $y = x^{1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right)$ is a solution of Airy's differential equation $y'' + \alpha^2 xy = 0$, $x > 0$, whenever w is a solution of Bessel's equation of order $\frac{2}{3}$, that is, $t^2w'' + tw' + \left(t^2 - \frac{1}{9}\right)w = 0$, $t > 0$.

[*Hint*: After differentiating, substituting, and simplifying, then let $t = \frac{2}{3}\alpha x^{3/2}$].

Solution

$$y = x^{1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right)$$

$$\begin{aligned} y' &= \frac{1}{2}x^{-1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right) + x^{1/2}\left(\alpha x^{1/2}\right)w'\left(\frac{2}{3}\alpha x^{3/2}\right) \\ &= \alpha x w'\left(\frac{2}{3}\alpha x^{3/2}\right) + \frac{1}{2}x^{-1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right) \end{aligned}$$

$$\begin{aligned} y'' &= \alpha x\left(\alpha x^{1/2}\right)w''\left(\frac{2}{3}\alpha x^{3/2}\right) + \alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right) + \frac{1}{2}x^{-1/2}\left(\alpha x^{1/2}\right)w'\left(\frac{2}{3}\alpha x^{3/2}\right) - \frac{1}{4}x^{-3/2}w\left(\frac{2}{3}\alpha x^{3/2}\right) \\ &= \alpha^2 x^{3/2}w''\left(\frac{2}{3}\alpha x^{3/2}\right) + \frac{3}{2}\alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right) - \frac{1}{4}x^{-3/2}w\left(\frac{2}{3}\alpha x^{3/2}\right) \end{aligned}$$

$$y'' + \alpha^2 xy = 0$$

$$\alpha^2 x^{3/2} w''\left(\frac{2}{3}\alpha x^{3/2}\right)+\frac{3}{2}\alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right)-\frac{1}{4}x^{-3/2}w\left(\frac{2}{3}\alpha x^{3/2}\right)+\alpha^2 x^{3/2} w\left(\frac{2}{3}\alpha x^{3/2}\right)=0$$

$$\alpha^2 x^{3/2} w''\left(\frac{2}{3}\alpha x^{3/2}\right)+\frac{3}{2}\alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right)+\left(\alpha^2 x^{3/2}-\frac{1}{4x^{3/2}}\right)w\left(\frac{2}{3}\alpha x^{3/2}\right)=0$$

$$t=\frac{2}{3}\alpha x^{3/2} \quad \to \quad \alpha x^{3/2}=\frac{3}{2}t$$

$$\frac{3}{2}\frac{\alpha}{t}\left[t^2 w''(t)+tw'(t)+\left(t^2-\frac{1}{9}\right)w(t)\right]=0$$

$$\boxed{t^2 w''+tw'+\left(t^2-\frac{1}{9}\right)w=0} \checkmark$$

Exercise

Use the relation $\Gamma(x+1)=x\Gamma(x)$ and if p is nonnegative integer, then show that

$$J_\upsilon(x)=\frac{1}{\Gamma(\upsilon+1)}\left(\frac{x}{2}\right)^\upsilon\left[1+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(\upsilon+1)(\upsilon+2)\cdots(\upsilon+n)}\left(\frac{x}{2}\right)^{2n}\right]$$

Solution

$$J_\upsilon(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(\upsilon+n+1)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

Given: $\Gamma(x+1)=x\Gamma(x)$

$$\Gamma(\upsilon+n+1)=(\upsilon+1)(\upsilon+2)\cdots(\upsilon+n)\Gamma(\upsilon+n)$$

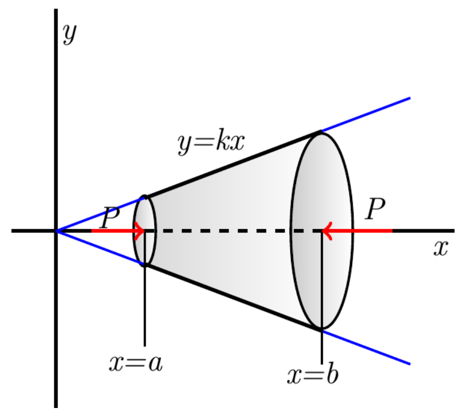
$$J_\upsilon(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(\upsilon+1)(\upsilon+2)\cdots(\upsilon+n)\Gamma(\upsilon+n)}\left(\frac{x}{2}\right)^{2n}\left(\frac{x}{2}\right)^{\upsilon}$$

$$=\frac{1}{\Gamma(\upsilon+1)}\left(\frac{x}{2}\right)^\upsilon\left[1+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(\upsilon+1)(\upsilon+2)\cdots(\upsilon+n)}\left(\frac{x}{2}\right)^{2n}\right] \checkmark$$

Exercise

A linearly tapered rod with circular cross section, subject to an axial force P of compression. Its deflection curve $y = y(x)$ satisfies the endpoint value problem

$$EIy'' + Py = 0\ ; \quad y(a) = y(b) = 0 \quad (1)$$



Here, however, the moment of inertia $I = I(x)$ of the cross section at x is given by

$$I(x) = \frac{1}{4}\pi(kx)^4 = I_0\left(\frac{x}{b}\right)^4 \quad (2)$$

Where $I_0 = I(b)$, the value of I at $x = b$. Substitution of $I(x)$ in the differential equation (1) yields to the eigenvalue problem

$$x^4y'' + \lambda y = 0\ ; \quad y(a) = y(b) = 0 \quad (3)$$

Where $\lambda = \mu^2 = \dfrac{Pb^4}{EI_0}$

a) Show that the general solution of $x^4y'' + \mu^2 y = 0$ is $\quad y(x) = x\left(A\cos\dfrac{\mu}{x} + B\sin\dfrac{\mu}{x}\right)$

b) Conclude that the nth eigenvalue is given by $\mu_n = n\pi\dfrac{ab}{L}$, where $L = b - a$ is the length of the rod, and hence that the nth buckling force is

$$P_n = \frac{n^2\pi^2}{L^2}\left(\frac{a}{b}\right)^2 EI_0$$

Solution

a) $x^{-2}\times \quad x^4y'' + \mu^2 y = 0$

$$x^2y'' + \mu^2x^{-2}y = 0$$

$$A = 0, \quad B = 0, \quad C = \mu^2, \quad p = -2$$

$$\alpha = \frac{1}{2}, \quad \beta = -1, \quad k = \mu, \quad \upsilon = \frac{1}{2} \qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$y(x) = x^{1/2}\left[c_1J_{1/2}\left(\mu x^{-1}\right) + c_2J_{-1/2}\left(\mu x^{-1}\right)\right] \qquad y(x) = x^{\alpha}\left(c_1J_{1/2}\left(kx^{\beta}\right) + c_2J_{-1/2}\left(kx^{\beta}\right)\right)$$

$$= \sqrt{x}\left(c_1\sqrt{\frac{2x}{\pi\mu}}\cos\left(\frac{\mu}{x}\right) + c_2\sqrt{\frac{2x}{\pi\mu}}\sin\left(\frac{\mu}{x}\right)\right)$$

$$= x^{\alpha}\left(c_1\left(\frac{2}{\pi k x^{\beta}}\right)^{1/2}\sin\left(kx^{\beta}\right) + c_2\left(\frac{2}{\pi k x^{\beta}}\right)^{1/2}\cos\left(kx^{\beta}\right)\right)$$

$$= x\left(c_1\sqrt{\frac{2}{\pi\mu}}\cos\left(\frac{\mu}{x}\right) + c_2\sqrt{\frac{2}{\pi\mu}}\sin\left(\frac{\mu}{x}\right)\right) \qquad A = c_1\sqrt{\frac{2}{\pi\mu}}, \quad B = c_2\sqrt{\frac{2}{\pi\mu}}$$

$$\boxed{= x\left(A\cos\left(\frac{\mu}{x}\right) + B\sin\left(\frac{\mu}{x}\right)\right)}$$

b) Given: $\mu_n = n\pi\frac{ab}{L}; \quad y(a) = y(b) = 0, \quad L = b - a$

$$\begin{cases} y(a) = a\left(A\cos\left(\frac{\mu}{a}\right) + B\sin\left(\frac{\mu}{a}\right)\right) = 0 \\ y(b) = b\left(A\cos\left(\frac{\mu}{b}\right) + B\sin\left(\frac{\mu}{b}\right)\right) = 0 \end{cases}$$

$$\begin{cases} A\cos\left(\frac{\mu}{a}\right) + B\sin\left(\frac{\mu}{a}\right) = 0 \\ A\cos\left(\frac{\mu}{b}\right) + B\sin\left(\frac{\mu}{b}\right) = 0 \end{cases} \qquad (a,\ b \neq 0)$$

$$\Delta = \begin{vmatrix} \cos\frac{\mu}{a} & \sin\frac{\mu}{a} \\ \cos\frac{\mu}{b} & \sin\frac{\mu}{b} \end{vmatrix}$$

$$= \cos\frac{\mu}{a}\sin\frac{\mu}{b} - \sin\frac{\mu}{a}\cos\frac{\mu}{b}$$

$$= \sin\left(\frac{\mu}{b} - \frac{\mu}{a}\right)$$

$$= \sin\left(\frac{b-a}{ab}\mu\right)$$

$$\boxed{= \sin\left(\frac{L}{ab}\mu\right)}$$

$$\lambda = \mu^2 = \frac{Pb^4}{EI_0}$$

$$P = \frac{EI_0}{b^4}\mu^2$$

$$= \frac{EI_0}{b^4}\left(n\pi\frac{ab}{L}\right)^2$$

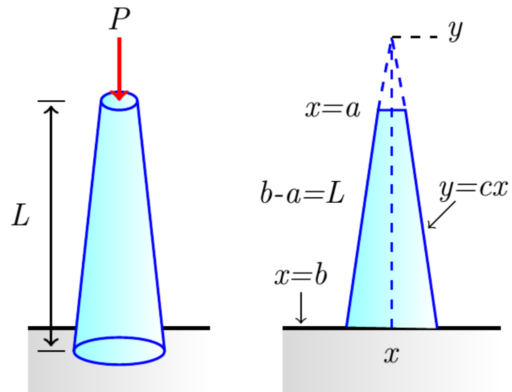
$$\boxed{= \frac{n^2\pi^2}{L^2}\left(EI_0\right)\left(\frac{a}{b}\right)^2}$$

Exercise

When a constant vertical compressive force or load P was applied to a thin column of uniform cross section, the deflection $y(x)$ was a solution of the boundary-value problem

$$EI\frac{d^2y}{dy^2}+Py=0\ ; \quad y(0)=0, \quad y(L)=0$$

The assumption here is that the column is hinged at both ends. The column will buckle or deflect only when the compression force is a critical load P_n



a) Let assume that the column is of length L, is hinged at both ends, has circular cross sections, and is tapered. If the column, a truncated cone, has a linear taper $y=cx$ in cross section, the moment of inertia of a cross section with respect to an axis perpendicular to the $xy-plane$ is $I=\frac{1}{4}\pi r^4$, where $r=y$ and $y=cx$. Hence we can write $I(x)=I_0(x\ b)^4$, where $I_0=I(b)=\frac{1}{4}\pi(cb)^4$. Substituting $I(x)$ into the differential equation, we see that the deflection in this case is determine from the BVP?

$$x^4\frac{d^2y}{dx^2}+\lambda y=0\ ; \quad y(a)=0, \quad y(b)=0$$

Where $\lambda=Pb^4EI_0$

Find the critical loads P_n for the tapered column. Use an appropriate identity to express the buckling modes $y_n(x)$ as a single function.

b) Plot the graph of the first buckling mode $y_1(x)$ corresponding to the Euler load P_1 when $b=11$ and $a=1$

Solution

a) $x^{-2}\times \quad x^4y''+\lambda y=0$

$$x^2y''+\lambda x^{-2}y=0$$

$$A=0, \quad B=0, \quad C=\lambda, \quad p=-2$$

$$\alpha = \frac{1}{2}, \quad \beta = -1, \quad k = \sqrt{\lambda}, \quad \upsilon = \frac{1}{2}$$

$$\alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$y(x) = x^{1/2}\left[c_1 J_{1/2}\left(\sqrt{\lambda}x^{-1}\right) + c_2 J_{-1/2}\left(\sqrt{\lambda}x^{-1}\right)\right]$$

$$\begin{aligned}
y(x) &= x^{\alpha}\left(c_1 J_{1/2}\left(kx^{\beta}\right) + c_2 J_{-1/2}\left(kx^{\beta}\right)\right) \\
&= \sqrt{x}\left(c_1\sqrt{\frac{2x}{\pi\sqrt{\lambda}}}\cos\left(\frac{\sqrt{\lambda}}{x}\right) + c_2\sqrt{\frac{2x}{\pi\sqrt{\lambda}}}\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right) \\
&= x^{\alpha}\left(c_1\sqrt{\frac{2}{\pi kx^{\beta}}}\sin\left(kx^{\beta}\right) + c_2\sqrt{\frac{2}{\pi kx^{\beta}}}\cos\left(kx^{\beta}\right)\right) \\
&= x\left(c_1\sqrt{\frac{2x}{\pi\sqrt{\lambda}}}\cos\left(\frac{\sqrt{\lambda}}{x}\right) + c_2\sqrt{\frac{2x}{\pi\sqrt{\lambda}}}\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right) \qquad A = c_1\sqrt{\frac{2}{\pi\sqrt{\lambda}}}, \quad B = c_2\sqrt{\frac{2}{\pi\sqrt{\lambda}}} \\
&= \boxed{x\left(A\cos\left(\frac{\sqrt{\lambda}}{x}\right) + B\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right)}
\end{aligned}$$

Given: $\lambda = Pb^4 EI_0$; $y(a) = y(b) = 0, \quad L = b - a$

$$\begin{cases} y(a) = a\left(A\cos\left(\frac{\sqrt{\lambda}}{a}\right) + B\sin\left(\frac{\sqrt{\lambda}}{a}\right)\right) = 0 \\ y(b) = b\left(A\cos\left(\frac{\sqrt{\lambda}}{b}\right) + B\sin\left(\frac{\sqrt{\lambda}}{b}\right)\right) = 0 \end{cases}$$

$$\begin{cases} A\cos\left(\frac{\sqrt{\lambda}}{a}\right) + B\sin\left(\frac{\sqrt{\lambda}}{a}\right) = 0 \\ A\cos\left(\frac{\sqrt{\lambda}}{b}\right) + B\sin\left(\frac{\sqrt{\lambda}}{b}\right) = 0 \end{cases} \qquad (a,\ b \neq 0)$$

$$\begin{aligned}
\Delta &= \begin{vmatrix} \cos\frac{\sqrt{\lambda}}{a} & \sin\frac{\sqrt{\lambda}}{a} \\ \cos\frac{\sqrt{\lambda}}{b} & \sin\frac{\sqrt{\lambda}}{b} \end{vmatrix} \\
&= \cos\frac{\sqrt{\lambda}}{a}\sin\frac{\sqrt{\lambda}}{b} - \sin\frac{\sqrt{\lambda}}{a}\cos\frac{\sqrt{\lambda}}{b} \\
&= \sin\left(\frac{\sqrt{\lambda}}{b} - \frac{\sqrt{\lambda}}{a}\right)
\end{aligned}$$

$$= \sin\left(\frac{b-a}{ab}\sqrt{\lambda}\right)$$

$$\boxed{= \sin\left(\frac{L}{ab}\sqrt{\lambda}\right) = 0}$$

$$\frac{L}{ab}\sqrt{\lambda} = n\pi \quad \rightarrow \quad \sqrt{\lambda} = \frac{n\pi ab}{L} \qquad (n \in \mathbb{N})$$

$$\lambda = \frac{n^2\pi^2a^2b^2}{L^2} = Pb^4EI_0$$

$$\boxed{P_n = \frac{n^2\pi^2}{L^2}\left(EI_0\right)\left(\frac{a}{b}\right)^2}$$

If we let $B = -A\dfrac{\sin\frac{\sqrt{\lambda}}{a}}{\cos\frac{\sqrt{\lambda}}{a}}$

$$y(x) = x\left(A\cos\left(\frac{\sqrt{\lambda}}{x}\right) + B\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right)$$

$$= x\left(A\cos\left(\frac{\sqrt{\lambda}}{x}\right) - A\frac{\sin\frac{\sqrt{\lambda}}{a}}{\cos\frac{\sqrt{\lambda}}{a}}\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right)$$

$$= \frac{A}{\cos\frac{\sqrt{\lambda}}{a}}x\left(\cos\frac{\sqrt{\lambda}}{a}\cos\left(\frac{\sqrt{\lambda}}{x}\right) - \sin\frac{\sqrt{\lambda}}{a}\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right) \qquad \frac{A}{\cos\frac{\sqrt{\lambda}}{a}} = C$$

$$= Cx\sin\left(\frac{\sqrt{\lambda}}{x} - \frac{\sqrt{\lambda}}{a}\right)$$

$$\boxed{= Cx\sin\sqrt{\lambda}\left(\frac{1}{x} - \frac{1}{a}\right)}$$

$$y_n(x) = Cx\sin\sqrt{\lambda}\left(\frac{1}{x} - \frac{1}{a}\right) \qquad \left(\sqrt{\lambda} = \frac{n\pi ab}{L}\right)$$

$$= Cx\sin\frac{n\pi ab}{L}\left(\frac{1}{x} - \frac{1}{a}\right)$$

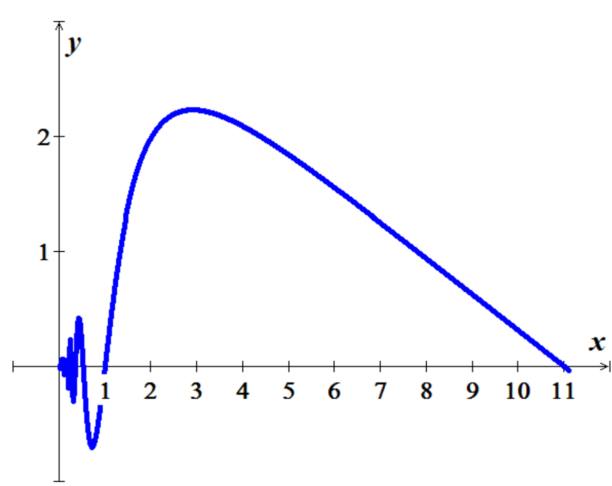
$$= Cx\sin\frac{n\pi b}{L}\left(\frac{a}{x} - 1\right)$$

$$\boxed{= C_1x\sin\frac{n\pi b}{L}\left(1 - \frac{a}{x}\right)}$$

b) Given: $n = 1, \quad a = 1, \quad b = 11$

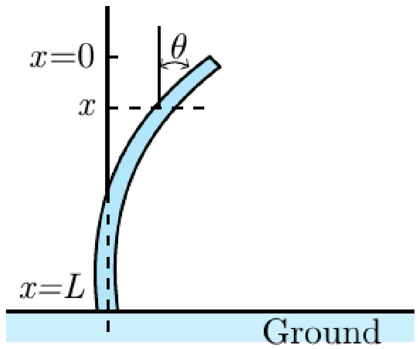
Let $C_1 = 1$

$$y_1(x) = x\sin\frac{11\pi}{10}\left(1 - \frac{1}{x}\right)$$



Exercise

For a practical application, we now consider the problem of determining when a uniform vertical column will buckle under its own weight (after, perhaps, being nudged laterally just a bit by a passing breeze). We take $x=0$ at the free top end of the column and $x=L>0$ at its bottom; we assume that the bottom is rigidly imbedded in the ground, perhaps in concrete.



Denote the angular deflection of the column at the point x by $\theta(x)$. From the theory of elasticity it follows that

$$EI\frac{d^2\theta}{dx^2}+g\rho x\theta=0$$

Where E is the Young's modulus of the material of the column,

I is its cross-sectional moment of inertia

ρ is the linear density of the column

g is gravitational acceleration.

For physical reasons – no bending at the free top of the column and no deflection at its imbedded bottom – the boundary conditions are $\theta'(0)=0, \quad \theta(L)=0$

Determine the general equation of the length L.

Solution

$EI\theta''+g\rho x\theta=0$

$\theta''+\frac{g\rho}{EI}x\theta=0$

Let $\lambda=\frac{g\rho}{EI}=\gamma^2$

$x^2\times \quad \theta''+\gamma^2x\theta=0$

$x^2\theta''+\gamma^2x^3\theta=0\ ; \quad \theta'(0)=0, \quad \theta(L)=0$

$A=0, \quad B=0, \quad C=\gamma^2, \quad p=3$

$\alpha=\frac{1}{2}, \quad \beta=\frac{3}{2}, \quad k=\frac{2\gamma}{3}, \quad \upsilon=\frac{1}{3}$ $\qquad \alpha=\frac{1-A}{2},\ \beta=\frac{p}{2},\ k=\frac{2\sqrt{C}}{p},\ \upsilon=\frac{\sqrt{(1-A)^2-4B}}{p}$

$$\theta(x) = x^{1/2}\left[c_1 J_{1/3}\left(\frac{2}{3}\gamma x^{3/2}\right) + c_2 J_{-1/3}\left(\frac{2}{3}\gamma x^{3/2}\right)\right]$$

$$y(x) = x^{\alpha}\left(c_1 J_{\nu}\left(kx^{\beta}\right) + c_2 J_{-\nu}\left(kx^{\beta}\right)\right)$$

$$J_{1/3}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma\left(1+\frac{1}{3}+n\right)}\left(\frac{x}{2}\right)^{2n+\frac{1}{3}}$$

$$J_{\upsilon}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma\left(n+\frac{4}{3}\right)}\left(\frac{x}{2}\right)^{2n+\frac{1}{3}}$$

$$J_{\upsilon}(x) = \frac{x^{\upsilon}}{2^{\upsilon}\Gamma(\upsilon+1)}\left\{1 - \frac{x^2}{2(2\upsilon+2)} + \frac{x^4}{2\cdot 4\cdot(2\upsilon+2)(2\upsilon+4)} - \cdots\right\}$$

$$= \frac{x^{1/3}}{2^{1/3}\Gamma\left(\frac{4}{3}\right)}\left\{1 - \frac{x^2}{2\left(\frac{2}{3}+2\right)} + \frac{x^4}{2\cdot 4\cdot\left(\frac{2}{3}+2\right)\left(\frac{2}{3}+4\right)} - \cdots\right\}$$

$$= \frac{x^{1/3}}{2^{1/3}\Gamma\left(\frac{4}{3}\right)}\left\{1 - \frac{3x^2}{2^3} + \frac{3^2 x^4}{112\times 2^3} - \cdots\right\}$$

$$J_{1/3}\left(\frac{2}{3}\gamma x^{3/2}\right) = \frac{1}{2^{1/3}\Gamma\left(\frac{4}{3}\right)}\left(\frac{2}{3}\gamma x^{3/2}\right)^{1/3}\left\{1 - \frac{3}{2^3}\left(\frac{2}{3}\gamma x^{3/2}\right)^2 + \frac{3^2}{896}\left(\frac{2}{3}\gamma x^{3/2}\right)^4 - \cdots\right\}$$

$$= \frac{\gamma^{1/3}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)} x^{1/2}\left\{1 - \frac{1}{12}\gamma^2 x^3 + \frac{1}{504}\gamma^4 x^6 - \cdots\right\}$$

$$J_{-1/3}\left(\frac{2}{3}\gamma x^{3/2}\right) = \frac{1}{2^{-1/3}\Gamma\left(\frac{4}{3}\right)}\left(\frac{2}{3}\gamma x^{3/2}\right)^{-1/3}\left\{1 - \frac{1}{2\left(2-\frac{1}{3}\right)}\left(\frac{2}{3}\gamma x^{3/2}\right)^2 + \frac{1}{8\left(2-\frac{1}{3}\right)\left(4-\frac{1}{3}\right)}\left(\frac{2}{3}\gamma x^{3/2}\right)^4 - \cdots\right\}$$

$$= \frac{3^{1/3}}{\gamma^{1/3}\Gamma\left(\frac{2}{3}\right)} x^{-1/2}\left\{1 - \frac{1}{6}\gamma^2 x^3 + \frac{1}{180}\gamma^4 x^6 - \cdots\right\}$$

$$\theta(x) = x^{1/2}\left[c_1 J_{1/3}\left(\frac{2}{3}\gamma x^{3/2}\right) + c_2 J_{-1/3}\left(\frac{2}{3}\gamma x^{3/2}\right)\right]$$

$$= x^{1/2}\left[c_1 \frac{\gamma^{1/3}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)} x^{1/2}\left\{1-\frac{1}{12}\gamma^2 x^3+\frac{1}{504}\gamma^4 x^6-\cdots\right\}+c_2 \frac{3^{1/3}}{\gamma^{1/3}\Gamma\left(\frac{2}{3}\right)} x^{-1/2}\left\{1-\frac{1}{6}\gamma^2 x^3+\frac{1}{180}\gamma^4 x^6-\cdots\right\}\right]$$

$$= c_1 \frac{\gamma^{1/3}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)}\left\{x-\frac{1}{12}\gamma^2 x^4+\frac{1}{504}\gamma^4 x^7-\cdots\right\}+c_2 \frac{3^{1/3}}{\gamma^{1/3}\Gamma\left(\frac{2}{3}\right)}\left\{1-\frac{1}{6}\gamma^2 x^3+\frac{1}{180}\gamma^4 x^6-\cdots\right\}$$

Given: $\theta(L)=0, \quad \theta'(0)=0$

$$\theta'(x) = c_1 \frac{\gamma^{1/3}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)}\left\{1-\frac{1}{3}\gamma^2 x^3+\frac{1}{72}\gamma^4 x^6-\cdots\right\}+\frac{3^{1/3}}{\gamma^{1/3}\Gamma\left(\frac{2}{3}\right)}\left\{\frac{1}{2}\gamma^2 x^2+\frac{1}{30}\gamma^4 x^5-\cdots\right\}$$

$$\theta'(0) = c_1 \frac{\gamma^{1/3}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)} = 0 \quad \rightarrow \quad c_1 = 0$$

$$\frac{3^{1/3} c_2}{\gamma^{1/3}\Gamma\left(\frac{2}{3}\right)}\left\{1-\frac{1}{6}\gamma^2 L^3+\frac{1}{180}\gamma^4 L^6-\cdots\right\}=0$$

$$c_2 J_{-1/3}\left(\frac{2}{3}\gamma L^{3/2}\right)=0 \quad \rightarrow \quad J_{-1/3}\left(\frac{2}{3}\gamma L^{3/2}\right)=0$$

$$J_{-1/3}\left(z=\frac{2}{3}\gamma L^{3/2}\right)=0$$

Using MatLab:

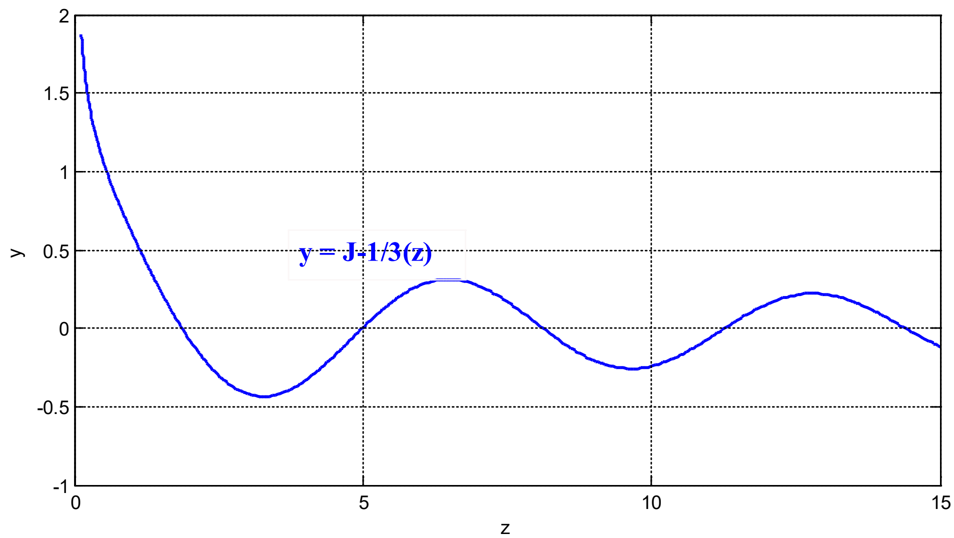
```
fzero('besselj(-1/3,z)',2)
syms z y
fplot(besselj(-1/3, z))
axis([0 10 -0.5 1.1])
grid on
ylabel('J_-1/3(z)')
legend('J_-1/3')
```

$z = 1.8664$

$$z=\frac{2}{3}\gamma L^{3/2} \quad \rightarrow \quad L=\left(\frac{3z}{2\gamma}\right)^{2/3}$$

$$L=\left(\frac{3(1.86635)}{2\sqrt{\frac{g\rho}{EI}}}\right)^{2/3}$$

$$\approx 1.986352\left(\frac{EI}{g\rho}\right)^{1/3}$$


y = J-1/3(z)
y
z
2
1.5
1
0.5
0
-0.5
-1
0
5
10
15