Lecture Two – Techniques of Integration

Section 2.1 – Integration by Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)\,dx$$

Example: $\int x\cos x\,dx,\quad \int x^2e^x\,dx,\quad and \quad \int x\ln x\,dx$

Integration by Parts Formula

$$\int f(x)g'(x) = f(x)g(x) - \int f'(x)g(x)\,dx$$

Let u and v be differentiable functions of x. $\int u\,dv = uv - \int v\,du$

Guidelines for integration by Parts

1. Let dv be the most complicated portion of the integrand that fits a basic integration formula. Let u be the remaining factor.
2. Let u be the portion of the integrand whose derivative is a function simpler than u. Let dv be the remaining factor.

Example

Evaluate: $\int x\cos x\,dx$

Solution

Let:

$u = x \qquad dv = \cos x\,dx$

$du = dx \qquad v = \int dv = \int \cos x\,dx = \sin x$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx \qquad\qquad \int u\,dv = uv - \int v\,du$$

$$= x\sin x + \cos x + C$$

Example

Evaluate: $\int \ln x \, dx$

Solution

Let:

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \quad v = \int dx = x$

$$\int \ln x \, dx = x \ln x - \int x \frac{1}{x} dx \qquad \int u dv = uv - \int v du$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + C$$

Tabular Integration

Example

Evaluate $\int x^2 e^x dx$

Solution

$f(x) = x^2 \quad and \quad g(x) = e^x$

$f(x)$ & **derivatives**		$\int g(x) = \int e^x$
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x

It is called ***tabular integration***

$$\int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x + C$$

$u = x^2 \qquad dv = e^x dx$

$du = 2xdx \quad v = \int e^x dx = e^x$

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx$$

Let: $u = x \qquad dv = e^x dx$

$du = dx \quad v = \int e^x dx = e^x$

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx$$

$$= x^2 e^x - 2\left[xe^x - \int e^x dx \right]$$

$$= x^2 e^x - 2\left(xe^x - e^x \right) + C$$

$$= x^2 e^x - 2xe^x + 2e^x + C$$

Example

Evaluate $\int x^3 \sin x\, dx$

Solution

$$\int x^3 \sin x dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6\sin x + C$$

		$\int \sin x$
+	x^3	$-\cos x$
−	$3x^2$	$-\sin x$
+	$6x$	$\cos x$
−	6	$\sin x$

Example

Evaluate $\int e^x \cos x\, dx$

Solution

$$\int e^x \cos x\, dx = e^x \sin x + e^x \cos x - \int e^x \cos x\, dx$$

$$2\int e^x \cos x\, dx = e^x (\sin x + \cos x)$$

$$\int e^x \cos x\, dx = \frac{1}{2} e^x (\sin x + \cos x) + C$$

		$\int \cos x\, dx$
+	e^x	$\sin x$
-	e^x	$-\cos x$
+	e^x	$-\int \cos x\, dx$

Let: $u = e^x \qquad dv = \cos x dx$

$du = e^x dx \qquad v = \int \cos x dx = \sin x$

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$$

Let: $u = e^x \qquad dv = \sin x dx$

$du = e^x dx \qquad v = \int \sin x dx = -\cos x$

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$$

$$= e^x \sin x - \left[-e^x \cos x - \int (-\cos x) e^x dx \right]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x dx$$

$$\int e^x \cos x dx + \int e^x \cos x dx = e^x \sin x + e^x \cos x - \int e^x \cos x dx + \int e^x \cos x dx$$

$$2\int e^x \cos x dx = e^x \sin x + e^x \cos x + C_1$$

$$\int e^x \cos x\, dx = \frac{1}{2} e^x \sin x + \frac{1}{2} e^x \cos x + C$$

Example

Obtain a formula that expresses the integral $\int \cos^n x dx$

Solution

Let: $u = \cos^{n-1} x$ $\qquad$ $dv = \cos x dx$

$$du = (n-1)\cos^{n-2} x(-\sin x dx) = -(n-1)\cos^{n-2} x \sin x dx \qquad v = \int \cos x dx = \sin x$$

$$\int u dv = uv - \int v du$$

$$\int \cos^n x dx = \cos^{n-1} x \sin x - \int \sin x\left(-(n-1)\cos^{n-2} x \sin x dx\right)$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\left(1 - \cos^2 x\right) dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \left(\cos^{n-2} x - \cos^n x\right) dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx - (n-1)\int \cos^n x dx$$

$$\int \cos^n x dx + (n-1)\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx$$

$$(1+n-1)\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx$$

$$n\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx$$

$$\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx$$

Example: $\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\int \cos x dx$

$$= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$$

Evaluating Definite Integrals by Parts

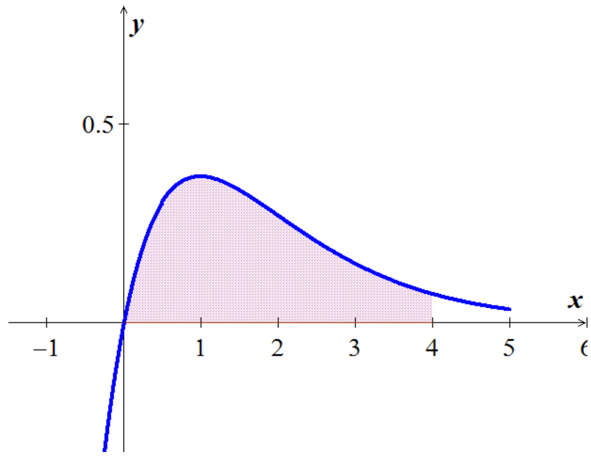
Example

Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

Solution

$$A = \int_0^4 xe^{-x}dx$$

		$\int e^{-x}$
$+$	x	$-e^{-x}$
$-$	1	e^{-x}



$$A = (-x-1)e^{-x}\Big|_0^4$$

$$= -5e^{-4} + 1$$

$$\underline{\approx 0.91 \ unit^2}$$

2nd *Method*

Let: $u = x \qquad dv = e^{-x}dx$

$du = dx \qquad v = \int e^{-x}dx = -e^{-x}$

$$\int udv = uv - \int vdu$$

$$\int_0^4 xe^{-x}dx = -xe^{-x}\Big]_0^4 - \int_0^4 \left(-e^{-x}\right)dx$$

$$= -\left[4e^{-4} - 0\right] + \int_0^4 e^{-x}dx$$

$$= -4e^{-4} + \left[-e^{-x}\right]_0^4$$

$$= -4e^{-4} - \left[e^{-4} - 1\right]$$

$$= -4e^{-4} - e^{-4} + 1$$

$$= 1 - 5e^{-4}$$

$$\underline{\approx 0.91 \ unit^2}$$

Formula

Evaluate $\int x^n e^{ax}\, dx$

		$\int e^{ax}$
+	x^n	$\frac{1}{a}e^{ax}$
−	nx^{n-1}	$\frac{1}{a^2}e^{ax}$
+	$n(n-1)x^{n-2}$	$\frac{1}{a^3}e^{ax}$
−	$n(n-1)(n-2)x^{n-3}$	$\frac{1}{a^4}e^{ax}$
	$\vdots\ \vdots$	$\vdots\ \vdots$

$$\int x^n e^{ax}\, dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a^2}x^{n-1}e^{ax} + \frac{n(n-1)}{a^3}x^{n-2}e^{ax} - \frac{n(n-1)(n-2)}{a^4}x^{n-3}e^{ax} + \ldots$$

$$= e^{ax}\sum_{k=0}^{n}(-1)^k \cdot \frac{n!}{(n-k)!}\cdot\frac{1}{a^{k+1}}\cdot x^{n-k}$$

Exercises   *Section* 2.1 – Integration by Parts

(1 – 92) Evaluate the integrals

1. $\int x \ln x \, dx$

2. $\int \ln x^2 \, dx$

3. $\int \ln(3x) \, dx$

4. $\int \frac{1}{x \ln x} dx$

5. $\int x (\ln x)^2 \, dx$

6. $\int x^2 (\ln x)^2 \, dx$

7. $\int \frac{(\ln x)^3}{x} dx$

8. $\int x^2 \ln x^3 \, dx$

9. $\int \ln\left(x + x^2\right) dx$

10. $\int x \ln(x+1) \, dx$

11. $\int \frac{(\ln x)^2}{x} dx$

12. $\int x^5 \ln 3x \, dx$

13. $\int x^5 \ln x \, dx$

14. $\int \ln(x+1) \, dx$

15. $\int \frac{\ln x}{x^{10}} dx$

16. $\int x e^{2x} dx$

17. $\int x^3 e^x dx$

18. $\int \frac{2x}{e^x} dx$

19. $\int \frac{x^3 e^{x^2}}{\left(x^2+1\right)^2} dx$

20. $\int x^2 e^{-3x} dx$

21. $\int \left(x^2 - 2x + 1\right) e^{2x} dx$

22. $\int x^5 e^{x^3} dx$

23. $\int x e^{-4x} dx$

24. $\int \frac{x e^{2x}}{(2x+1)^2} dx$

25. $\int \frac{5x}{e^{2x}} dx$

26. $\int \frac{e^{1/x}}{x^2} dx$

27. $\int x^2 e^{4x} \, dx$

28. $\int x^3 e^{-3x} \, dx$

29. $\int x^4 e^x dx$

30. $\int x^3 e^{4x} dx$

31. $\int (x+1)^2 e^x dx$

32. $\int 2x e^{3x} \, dx$

33. $\int x^2 \sin x \, dx$

34. $\int \theta \cos \pi\theta \, d\theta$

35. $\int 4x \sec^2 2x \, dx$

36. $\int x^3 \sin x \, dx$

37. $\int \left(x^3 - 2x\right) \sin 2x \, dx$

38. $\int x^2 \sin 2x \, dx$

39. $\int x^2 \sin(1-x) \, dx$

40. $\int x \sin x \cos x \, dx$

41. $\int x \cos x \, dx$

42. $\int x \csc x \cot x \, dx$

43. $\int x^2 \cos x \, dx$

44. $\int x^3 \cos 2x \, dx$

45. $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$

46. $\int x \sinh x \, dx$

47. $\int x^2 \cosh x \, dx$

48. $\int e^{2x} \cos 3x \, dx$

49. $\int e^{-3x} \sin 5x \, dx$

50. $\int e^{-x} \sin 4x \, dx$

51. $\int e^{-2\theta} \sin 6\theta \, d\theta$

52. $\int e^{-3x} \sin 4x \, dx$

53. $\int e^{4x} \cos 2x \, dx$

54. $\int e^{3x} \cos 3x \, dx$

55. $\int e^{3x} \cos 2x \, dx$

56. $\int e^{x} \sin x \, dx$

57. $\int e^{-2x} \sin 3x dx$

58. $\int \frac{x}{\sqrt{x-1}} dx$

59. $\int x\sqrt{x-5} \, dx$

60. $\int \frac{x}{\sqrt{6x+1}} dx$

61. $\int \frac{x}{2\sqrt{x+2}} dx$

62. $\int \frac{2x^2-3x}{(x-1)^3} dx$

63. $\int \frac{x^2+3x+4}{\sqrt[3]{2x+1}} dx$

64. $\int \frac{x}{\sqrt{x+1}} dx$

65. $\int \frac{x^5}{\sqrt{1-2x^3}} dx$

66. $\int x\sqrt{1-3x} \, dx$

67. $\int \sin(\ln x) \, dx$

68. $\int \tan^{-1} y \, dy$

69. $\int \sin^{-1} y \, dy$

70. $\int x \tan^{-1} x \, dx$

71. $\int \sinh^{-1} x \, dx$

72. $\int \tan^{-1} 3x \, dx$

73. $\int \cos^{-1}\left(\frac{x}{2}\right) dx$

74. $\int x \sec^{-1} x \, dx$

75. $\int_{-1}^{0} 2x^2 \sqrt{x+1} \, dx$

76. $\int_{0}^{1/\sqrt{2}} x \tan^{-1} x^2 \, dx$

77. $\int_{1}^{e} x^2 \ln x \, dx$

78. $\int_{-1}^{\ln 2} \frac{3t}{e^t} dt$

79. $\int_{\pi}^{2\pi} \cot \frac{x}{3} \, dx$

80. $\int_{0}^{1/\sqrt{2}} 2x \sin^{-1}\left(x^2\right) dx$

81. $\int_{1}^{e} x^3 \ln x \, dx$

82. $\int_{0}^{1} x\sqrt{1-x} \, dx$

83. $\int_{0}^{\pi/3} x \tan^2 x \, dx$

84. $\int_{0}^{\pi} x \sin x \, dx$

85. $\int_{1}^{e} \ln 2x \, dx$

86. $\int_{0}^{\pi/2} x \cos 2x \, dx$

87. $\int_{0}^{\ln 2} xe^{x} \, dx$

88. $\int_{1}^{e^2} x^2 \ln x \, dx$

89. $\int_{0}^{3} xe^{x/2} dx$

90. $\int_{0}^{2} x^2 e^{-2x} dx$

91. $\int_{0}^{\pi/4} x \cos 2x \, dx$

92. $\int_{0}^{\pi} x \sin 2x \, dx$

93. $\int_{1}^{4} e^{\sqrt{x}} \, dx$

(94 – 98) Use integration by parts to establish the reduction formula

94. $\displaystyle\int x^n \sin dx = -x^n \cos x + n\int x^{n-1} \cos x dx$

95. $\displaystyle\int x^n e^{ax} dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a}\int x^{n-1} e^{ax} dx, \quad a \neq 0$

96. $\displaystyle\int (\ln x)^n \, dx = x(\ln x)^n - n\int (\ln x)^{n-1} \, dx$

97. $\displaystyle\int_a^b \left(\int_x^b f(t)\,dt \right) dx = \int_a^b (x-a) f(x)\,dx$

98. $\displaystyle\int \sqrt{1-x^2}\,dx = \frac{1}{2} x\sqrt{1-x^2} + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}} dx$

99. Find the indefinite integral: $\displaystyle\int 5x^n \ln ax \, dx \quad a \neq 0, \; n \neq -1$

100. Find the volume of the solid generated by the region bounded by $f(x) = x\ln x$, and the $x-axis$ on $\left[1, \; e^2\right]$ is revolved about the $y-axis$.

101. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate aces, the cure $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$

102. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the cure $y = e^{-x}$, and the line $x = 1$, about

a) the line $y-axis$

b) the line $x = 1$

103. Find the volume of the solid that is generated by the region bounded by $f(x) = e^{-x}$, $x = \ln 2$, and the coordinate axes is revolved about the y-$axis$.

104. Find the volume of the solid that is generated by the region bounded by $f(x) = e^{-x}$, and the $x-axis$ on $[1, \; \ln 2]$ is revolved about the line $x = \ln 2$.

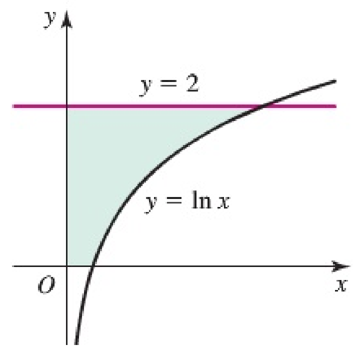
105. Find the volume of the solid that is generated by the region bounded by $f(x) = \sin x$, and the x-$axis$ on $[0, \; \pi]$ is revolved about the y-$axis$.

106. Find the area of the region generated when the region bounded by $y = \sin x$ and $y = \sin^{-1} x$ on the interval $\left[0,\ \frac{1}{2}\right]$.

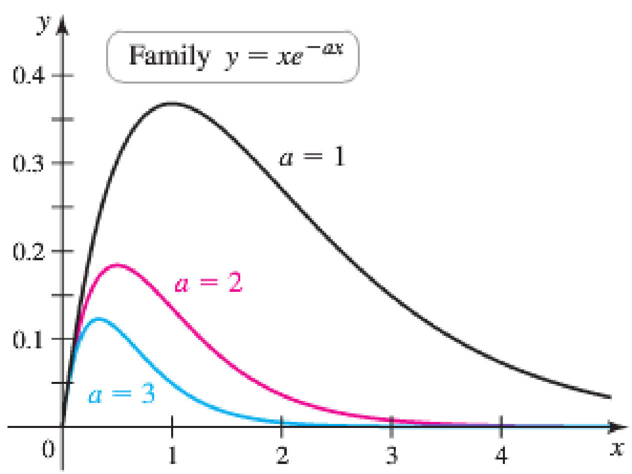
107. Find the area between the curves $y = \ln x^2$, $y = \ln x$, *and* $x = e^2$

108. Determine the area of the shaded region bounded by

$y = \ln x, \quad y = 2, \quad y = 0, \quad and\ x = 0$



109. The curves $y = xe^{-ax}$ are shown in the figure for $a = 1,\ 2,\ and\ 3$.



a) Find the area of the region bounded by $y = xe^{-x}$ and the x-axis on the interval [0, 4].

b) Find the area of the region bounded by $y = xe^{-ax}$ and the x-axis on the interval [0, 4] where $a > 0$

c) Find the area of the region bounded by $y = xe^{-ax}$ and the x-axis on the interval [0, b]. Because this area depends on a and b, we call it $A(a,\ b)$ where $a > 0$ and $b > 0$.

d) Use part (*c*) to show that $A(1,\ \ln b) = 4A\left(2,\ \frac{1}{2}\ln b\right)$

e) Does this pattern continue? Is it true that $A(1,\ \ln b) = a^2 A\left(a,\ \frac{1}{a}\ln b\right)$

110. Suppose a mass on a spring that is slowed by friction has the position function $s(t) = e^{-t}\sin t$

a) Graph the position function. At what times does the oscillator pass through the position $s = 0$?

b) Find the average value of the position on the interval $[0,\ \pi]$.

c) Generalize part (*b*) and find the average value of the position on the interval $[n\pi,\ (n+1)\pi]$, for $n = 0,\ 1,\ 2,\ \ldots$

111. Given the region bounded by the graphs of $y = x\sin x, \quad y = 0, \quad x = 0, \quad x = \pi$, find

a) The area of the region.

b) The volume of the solid generated by revolving the region about the *x-axis*

c) The volume of the solid generated by revolving the region about the *y-axis*

d) The centroid of the region

112. The region *R* is bounded by the curve $y = \ln x$ and the *x-axis* on the interval $[1,\ e]$. Find the volume of the solid that is generated when *R* is revolved in the following ways

a) About the *x-axis*

b) About the *y-axis*

c) About the line $x = 1$

d) About the line $y = 1$

113. A string stretched between the two points $(0,\ 0)$ and $(2,\ 0)$ is plucked by displacing the string h units at its midpoint. The motion of the string is modeled by a ***Fourier Sine series*** whose coefficients are given by

$$b_n = h\int_0^1 x\sin\frac{n\pi x}{2}\,dx + h\int_1^2 (-x+2)\sin\frac{n\pi x}{2}\,dx$$

Find b_n