Section 1.4 – Limits at Infinity

Notation	*Terminology*
$f(x) \to \infty$	$f(x)$ increases without bound (can be made as large positive as desired)
$f(x) \to -\infty$	$f(x)$ decreases without bound (can be made as large negative as desired)

Horizontal Asymptote (*HA*)

The line $y = b$ is a ***horizontal asymptote*** for the graph of a function f if

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b$$

Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0} = \dfrac{a_n x^n}{b_m x^m}$ be a rational function. (***Proof*** !)

1. If the degree of numerator is less than of denominator $(n < m) \Rightarrow y = 0$

$$y = \frac{2x+1}{4x^2+5} \quad \Rightarrow \boxed{y = 0}$$

2. If the degree of numerator is equal of denominator $(n = m) \Rightarrow y = \dfrac{a_n}{b_m}$

$$y = \frac{2x^2+1}{4x^2+5} \quad \Rightarrow y = \frac{2}{4} = \frac{1}{2}$$

3. If the degree of numerator is greater than of denominator $(n > m) \Rightarrow$ No horizontal asymptote

$$y = \frac{2x^3+1}{4x^2+5} \quad \Rightarrow \textit{No HA}$$

Example

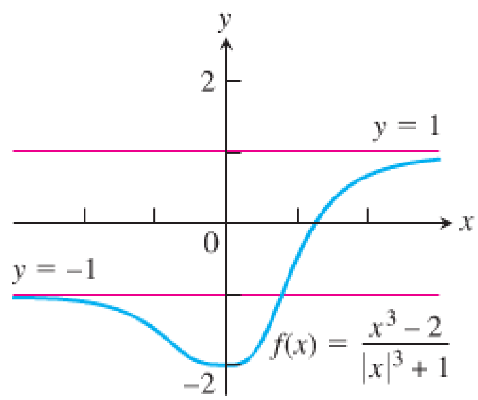
Find the horizontal asymptotes of the graph of $f(x) = \dfrac{x^3 - 2}{|x|^3 + 1}$

Solution

For $x \ge 0$ $\quad \displaystyle\lim_{x\to\infty} \frac{x^3-2}{|x|^3+1} = \lim_{x\to\infty} \frac{x^3}{x^3} = 1$

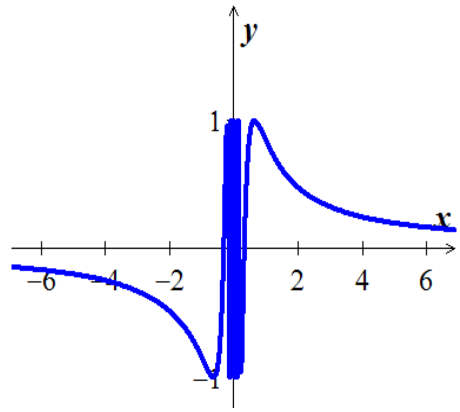
For $x \le 0$ $\quad \displaystyle\lim_{x\to\infty} \frac{x^3-2}{|x|^3+1} = \lim_{x\to-\infty} \frac{x^3}{(-x)^3} = -1$

The ***HA*** are $y = -1$ and $y = 1$.



Example

Find $\lim\limits_{x\to\infty} \sin\left(\frac{1}{x}\right)$



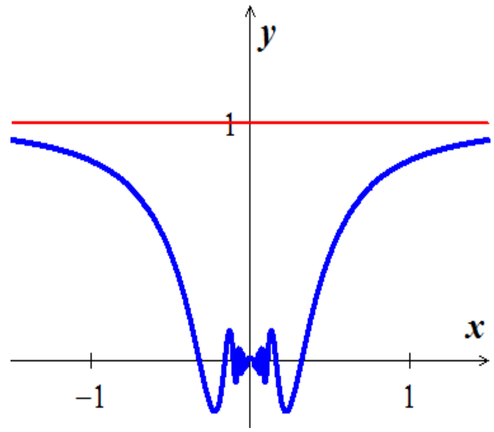
Solution

Let $t = \frac{1}{x} \Rightarrow \ t \to 0 \ \ as \ \ x \to \infty$

$\lim\limits_{x\to\infty} \sin\left(\frac{1}{x}\right) = \lim\limits_{t\to 0} \sin t \ \underline{= 0}|$

Example

Find $\lim\limits_{x\to\pm\infty} x\sin\left(\frac{1}{x}\right)$



Solution

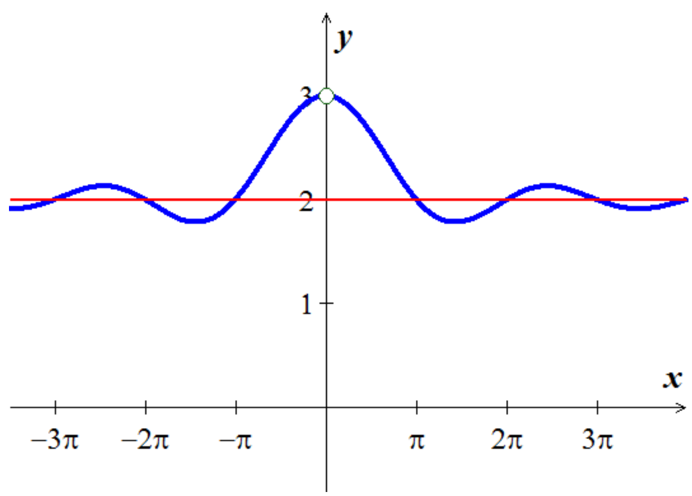
Let $t = \frac{1}{x} \Rightarrow x = \frac{1}{t}$

$\lim\limits_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim\limits_{t\to 0^+} \frac{\sin t}{t} \ \underline{=1}|$

$\lim\limits_{x\to-\infty} x\sin\left(\frac{1}{x}\right) = \lim\limits_{t\to 0^-} \frac{\sin t}{t} \ \underline{=1}|$

Example

Find the horizontal asymptote of $y = 2 + \frac{\sin x}{x}$



Solution

Since $0 \le \left|\frac{\sin x}{x}\right| \le \left|\frac{1}{x}\right|$

$\lim\limits_{x\to\pm\infty} \left|\frac{1}{x}\right| = 0 \quad \Rightarrow \quad \lim\limits_{x\to\pm\infty} \frac{\sin x}{x} = 0$

$\lim\limits_{x\to\pm\infty} \left(2 + \frac{\sin x}{x}\right) = 2 + 0 \ \underline{= 2}|$

The ***HA*** are $y = 2$

Example

Find $\lim\limits_{x\to\infty}\left(x-\sqrt{x^2+16}\right)$

Solution

$$\lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right)=\lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right)\frac{x+\sqrt{x^2+16}}{x+\sqrt{x^2+16}} \qquad (a-b)(a+b)=a^2-b^2$$

$$=\lim_{x\to\infty}\frac{x^2-\left(x^2+16\right)}{x+\sqrt{x^2+16}}$$

$$=\lim_{x\to\infty}\frac{x^2-x^2-16}{x+\sqrt{x^2+16}}$$

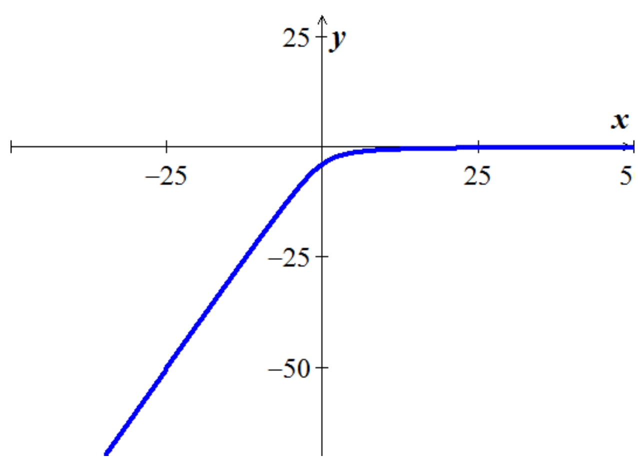
$$=\lim_{x\to\infty}\frac{-16}{x+\sqrt{x^2+16}}$$

$$=\lim_{x\to\infty}\frac{-\frac{16}{x}}{\frac{x}{x}+\sqrt{\frac{x^2}{x^2}+\frac{16}{x^2}}}$$

$$=\lim_{x\to\infty}\frac{-\frac{16}{x}}{1+\sqrt{1+\frac{16}{x^2}}}$$

$$=\frac{0}{1+\sqrt{1+0}}$$

$$\underline{=0}$$



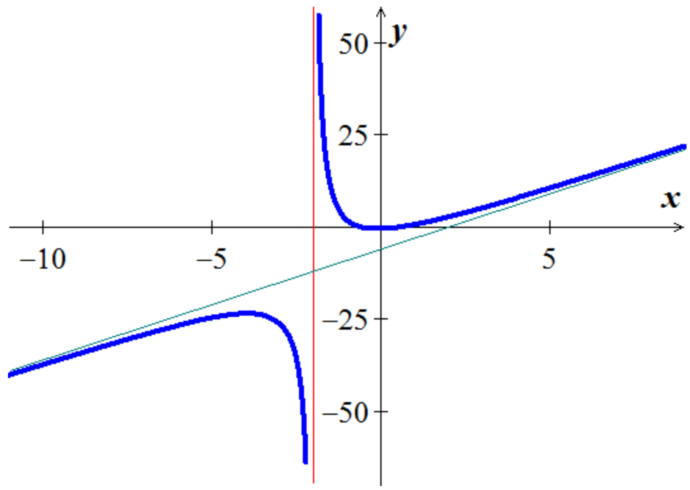
Slant or *Oblique* Asymptotes

When the degree of the numerator is one greater than the degree of the numerator, the graph has a ***slant*** or ***oblique*** asymptote and it is a line $y = ax + b, \quad a \neq 0$. To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

$$y = \frac{3x^2 - 1}{x + 2}$$

$$\begin{array}{r|l} & \quad\ \ 3x - 6 \\ \hline x + 2 & 3x^2 + 0x - 1 \\ & \underline{3x^2 + 6x} \\ & \quad\ -6x - 1 \\ & \quad\ \underline{-6x - 12} \\ & \qquad\ R = 11 \end{array}$$

$$y = \frac{3x^2 - 1}{x + 2} \;\boxed{= (3x - 6) + \frac{11}{x + 2}}$$



The ***oblique asymptote*** is the line ***y* = 3*x* - 6**

Example

Find the horizontal and vertical asymptotes of the curve $y = \frac{x + 3}{x + 2}$

Solution

HA: $y \to \frac{x}{x} = 1 \quad \Rightarrow \quad \boxed{y = 1}$ $\qquad\qquad$ VA: $x + 2 = 0 \quad \Rightarrow \quad \boxed{x = -2}$

Example

Find the horizontal and vertical asymptotes of the curve $f(x) = -\frac{8}{x^2 - 4}$

Solution

HA: $y \to \lim\limits_{x \to \infty} -\frac{8}{x^2} = 0 \quad \Rightarrow \quad \boxed{y = 0}$

VA: $x^2 - 4 = 0 \quad \Rightarrow \quad \boxed{x = \pm 2}$

$$\lim_{x \to 2^+} f(x) = -\infty \quad \textit{and} \quad \lim_{x \to 2^-} f(x) = \infty$$

Infinite Limits

The limit has a value of infinity or minus infinity, such a function $f(x)=\frac{1}{x}$. It is convenient to describe the behavior of f by saying that $f(x)$ approaches ∞ as $x\to 0^+$.

Definition

We say $\lim\limits_{x\to 0^+} f(x)=\infty$

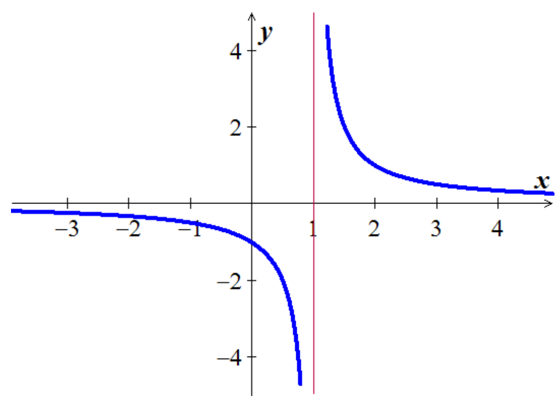
That $\lim\limits_{x\to 0^+}\frac{1}{x}$ doesn't exist because $\frac{1}{x}$ becomes arbitrary large and positive as $x\to 0^+$.

We say $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}\frac{1}{x}=-\infty$

That $\lim\limits_{x\to 0^-}\frac{1}{x}$ doesn't exist because $\frac{1}{x}$ becomes arbitrary large and negative as $x\to 0^-$.

Example

Find $\lim\limits_{x\to 1^+}\frac{1}{x-1}$ *and* $\lim\limits_{x\to 1^-}\frac{1}{x-1}$



Solution

As $x\to 1^+ \quad\Rightarrow\quad x-1\to 0^+$

$$\lim_{x\to 1^+}\frac{1}{x-1}=\infty$$

$$\lim_{x\to 1^-}\frac{1}{x-1}=-\infty$$

- $\lim\limits_{x\to 2}\frac{(x-2)^2}{x^2-4}=\lim\limits_{x\to 2}\frac{(x-2)^2}{(x-2)(x+2)}=\lim\limits_{x\to 2}\frac{(x-2)}{(x+2)}=\frac{0}{4}\ \underline{=0}$
- $\lim\limits_{x\to 2}\frac{x-2}{x^2-4}=\lim\limits_{x\to 2}\frac{x-2}{(x-2)(x+2)}=\lim\limits_{x\to 2}\frac{1}{x+2}\ \underline{=\frac{1}{4}}$
- $\lim\limits_{x\to 2^+}\frac{x-3}{x^2-4}=\lim\limits_{x\to 2^+}\frac{x-3}{(x-2)(x+2)}\ \underline{=-\infty}$
- $\lim\limits_{x\to 2^-}\frac{x-3}{x^2-4}=\lim\limits_{x\to 2^-}\frac{x-3}{(x-2)(x+2)}\ \underline{=\infty}$
- $\lim\limits_{x\to 2}\frac{x-3}{x^2-4}=\lim\limits_{x\to 2}\frac{x-3}{(x-2)(x+2)}\ \underline{=\textit{doesn't exist}}$

Exercises *Section* 1.4 – Limits at Infinity

Find the limit as $x \to \infty$ and as $x \to -\infty$ of

1. $h(x) = \dfrac{-5 + \frac{7}{x}}{3 - \frac{1}{x^2}}$

2. $f(x) = \dfrac{2x+3}{5x+7}$

3. $f(x) = \dfrac{2x^3+7}{x^3 - x^2 + x + 7}$

4. $f(x) = \dfrac{x+1}{x^2+3}$

5. $f(x) = \dfrac{7x^3}{x^3 - 3x^2 + 6x}$

6. $f(x) = \dfrac{9x^4 + x}{2x^4 + 5x^2 - x + 6}$

7. $f(x) = \dfrac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x}$

Evaluate

8. $\lim\limits_{x\to\infty} x^{12}$

9. $\lim\limits_{x\to-\infty} 3x^9$

10. $\lim\limits_{x\to-\infty} x^{-8}$

11. $\lim\limits_{x\to-\infty} x^{-9}$

12. $\lim\limits_{x\to-\infty} 2x^{-6}$

13. $\lim\limits_{x\to\infty} \left(3x^{12} - 9x^7\right)$

14. $\lim\limits_{x\to-\infty} \left(3x^7 + x^2\right)$

15. $\lim\limits_{x\to-\infty} \left(-2x^{16} + 2\right)$

16. $\lim\limits_{x\to-\infty} \left(2x^{-6} + 4x^5\right)$

17. $\lim\limits_{x\to-\infty} \dfrac{\cos x}{3x}$

18. $\lim\limits_{x\to\infty} \dfrac{x + \sin x}{2x + 7 - 5\sin x}$

19. $\lim\limits_{x\to\infty} \sqrt{\dfrac{8x^2-3}{2x^2+x}}$

20. $\lim\limits_{x\to-\infty} \left(\dfrac{x^2+x-1}{8x^2-3}\right)^{1/3}$

21. $\lim\limits_{x\to\infty} \dfrac{2\sqrt{x} + x^{-1}}{3x-7}$

22. $\lim\limits_{x\to\infty} \dfrac{x^{-1} + x^{-4}}{x^{-2} + x^{-3}}$

23. $\lim\limits_{x\to-\infty} \dfrac{4 - 3x^3}{\sqrt{x^6+9}}$

24. $\lim\limits_{x\to\infty} \left(\sqrt{x^2+3x} - \sqrt{x^2-2x}\right)$

25. $\lim\limits_{x\to-\infty} \left(\sqrt{x^2+3} + x\right)$

26. $\lim\limits_{x\to\infty} \dfrac{2x-3}{4x+10}$

27. $\lim\limits_{x\to\infty} \dfrac{x^4-1}{x^5+2}$

28. $\lim\limits_{x\to-\infty} \left(-3x^3 + 5\right)$

29. $\lim\limits_{x\to\infty} \left(e^{-2x} + \dfrac{2}{x}\right)$

30. $\lim\limits_{x\to\infty} \dfrac{1}{\ln x + 1}$

31. $\lim_{x\to\infty}\left(3+\frac{10}{x^2}\right)$

32. $\lim_{x\to\infty}\left(5+\frac{1}{x}+\frac{10}{x^2}\right)$

33. $\lim_{x\to\infty}\frac{4x^2+2x+3}{x^2}$

34. $\lim_{x\to\infty}\left(5+\frac{100}{x}+\frac{\sin^4 x^3}{x^2}\right)$

35. $\lim_{\theta\to\infty}\frac{\cos\theta}{\theta^2}$

36. $\lim_{\theta\to\infty}\frac{\cos\theta^5}{\sqrt{\theta}}$

37. $\lim_{x\to\infty}\frac{4x}{20x+1}$

38. $\lim_{x\to-\infty}\frac{4x}{20x+1}$

39. $\lim_{x\to\infty}\frac{3x^2-7}{x^2+5x}$

40. $\lim_{x\to-\infty}\frac{3x^2-7}{x^2+5x}$

41. $\lim_{x\to\infty}\frac{6x^2-9x+8}{3x^2+2}$

42. $\lim_{x\to-\infty}\frac{6x^2-9x+8}{3x^2+2}$

43. $\lim_{x\to\infty}\frac{4x^2-7}{8x^2+5x+2}$

44. $\lim_{x\to-\infty}\frac{4x^2-7}{8x^2+5x+2}$

45. $\lim_{x\to\infty}\frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4}$

46. $\lim_{x\to-\infty}\frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4}$

47. $\lim_{x\to\infty}\frac{3x^4+3x^3-36x^2}{x^4-25x^2+144}$

48. $\lim_{x\to-\infty}\frac{3x^4+3x^3-36x^2}{x^4-25x^2+144}$

49. $\lim_{x\to\infty}16x^2\left(4x^2-\sqrt{16x^4+1}\right)$

50. $\lim_{x\to-\infty}16x^2\left(4x^2-\sqrt{16x^4+1}\right)$

51. $\lim_{x\to\infty}\frac{x-1}{x^{2/3}-1}$

52. $\lim_{x\to-\infty}\frac{x-1}{x^{2/3}-1}$

53. $\lim_{x\to\infty}\frac{\sqrt{x^2+2x+6}-3}{x-1}$

54. $\lim_{x\to\infty}\frac{\left|1-x^2\right|}{x(x+1)}$

55. $\lim_{x\to\infty}\left(\sqrt{|x|}-\sqrt{|x-1|}\right)$

56. $\lim_{x\to\infty}\frac{\tan^{-1}x}{x}$

57. $\lim_{x\to\infty}\frac{\cos x}{e^{3x}}$

58. $\lim_{x\to 0}\frac{2e^x+10e^{-x}}{e^x+e^{-x}}$

59. $\lim_{x\to\infty}\frac{2e^x+10e^{-x}}{e^x+e^{-x}}$

60. $\lim_{x\to-\infty}\frac{2e^x+10e^{-x}}{e^x+e^{-x}}$

Graph the rational function and include the equations of the asymptotes

61. $y=\frac{1}{2x+4}$ **62.** $y=\frac{2x}{x+1}$ **63.** $y=\frac{x^2}{x-1}$ **64.** $y=\frac{x^3+1}{x^2}$

65. Let $f(x)=\dfrac{x^2-5x+6}{x^2-2x}$

a) Analyze $\lim\limits_{x\to 0^-} f(x)$, $\lim\limits_{x\to 0^+} f(x)$, $\lim\limits_{x\to 2^-} f(x)$, and $\lim\limits_{x\to 2^+} f(x)$

b) Does the graph of f have any vertical asymptotes? Explain?

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of

66. $y=\dfrac{3x}{1-x}$

67. $y=\dfrac{x^2}{x^2+9}$

68. $y=\dfrac{x-2}{x^2-4x+3}$

69. $y=\dfrac{5x-1}{1-3x}$

70. $y=\dfrac{3}{x-5}$

71. $y=\dfrac{x^3-1}{x^2+1}$

72. $y=\dfrac{3x^2-27}{(x+3)(2x+1)}$

73. $y=\dfrac{x^3+3x^2-2}{x^2-4}$

74. $y=\dfrac{x-3}{x^2-9}$

75. $y=\dfrac{6}{\sqrt{x^2-4x}}$

76. $f(x)=\dfrac{4x^3+1}{1-x^3}$

77. $f(x)=\dfrac{x+1}{\sqrt{9x^2+x}}$

78. $f(x)=1-e^{-2x}$

79. $f(x)=\dfrac{1}{\ln x^2}$

80. $f(x)=\dfrac{1}{\tan^{-1}x}$

81. $f(x)=\dfrac{2x^2+6}{2x^2+3x-2}$

82. $f(x)=\dfrac{3x^2+2x-1}{4x+1}$

83. $f(x)=\dfrac{9x^2+4}{(2x-1)^2}$

84. $f(x)=\dfrac{1+x-2x^2-x^3}{x^2+1}$

85. $f(x)=\dfrac{x(x+2)^3}{3x^2-4x}$