Lecture Three – Exponential and Logarithmic Functions

Section 3.1 – Inverse Functions

Inverse Relations

Interchanging the first and second coordinates of each ordered pair in a relation produces the inverse relation.

If a relation is defined by an equation, interchanging the variables produces an equation of the inverse relation

Given the relation: $\{(Zambia,\ 4.2),\ (Columbia,\ 4.5),\ (Poland,\ 3.3),\ (Italy,\ 3.3),\ (US,\ 2.5)\}$

Inverse Relation: $\{(4.2,\ Zambia),\ (4.5,\ Columbia),\ (3.3,\ Poland),\ (3.3,\ Italy),\ (2.5,\ US)\}$

Example

Consider the relation g given by: $G = \{(2,\ 4),\ (-1,\ 3),\ (-2,\ 0)\}$

Solution

The inverse relation: $G = \{(4,\ 2),\ (3,\ -1),\ (0,\ -2)\}$

Example

Consider the relation given by: $F = \{(-2,\ 2),\ (-1,\ 1),\ (0,\ 0),\ (1,\ 3),\ (2,\ 5)\}$

Solution

The inverse relation: $G = \{(2,\ -2),\ (1,\ -1),\ (0,\ 0),\ (3,\ 1),\ (5,\ 2)\}$

One-to-One **Functions**

A function *f* is one-to-one (1 – 1) if different inputs have different outputs that is,

$$\text{if } a \neq b, \quad \text{then} \quad f(a) \neq f(b)$$

A function *f* is one-to-one (1 – 1) if different outputs the same, the inputs are the same – that is,

$$\text{if } f(a) = f(b), \quad \text{then } a = b$$

Example

Given the function *f* described by $f(x) = 2x - 3$, prove that *f* is one-to-one.

Solution

$f(a) = f(b)$

$2a - 3 = 2b - 3$ *Add 3 on both sides*

$2a = 2b$ *Divide by 2*

$a = b$

$\therefore f$ is one-to-one

Example

Given the function *f* described by $f(x) = -4x + 12$, prove that *f* is one-to-one.

Solution

$f(a) = f(b)$

$-4a + 12 = -4b + 12$ *Subtract 12 from both sides*

$-4a = -4b$ *Divide by -4*

$a = b$

$\therefore f$ is one-to-one

Example

Given the function *f* described by $f(x) = x^2$, prove that *f* is one-to-one.

Solution

$-1 \neq 1$

$$\begin{cases} f(-1) = 1 \\ f(1) = 1 \end{cases} \Rightarrow f(-1) = f(1)$$

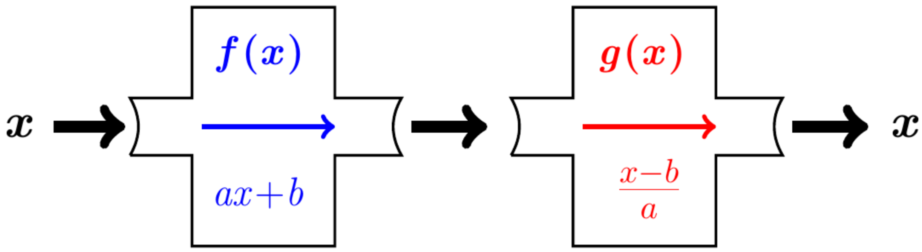
$\therefore f$ is ***not*** one-to-one

Definition of the Inverse of a Function

Let f and g be two functions such that

$$f(g(x)) = x \qquad \textit{and} \qquad g(f(x)) = x$$

$$x \underset{g=f^{-1}}{\overset{f}{\rightleftarrows}} f(x) \qquad\qquad g(f(x)) = f^{-1}(f(x)) = x$$



If the inverse of a function f is also a function, it is named f^{-1} read "f – inverse"

The –1 in f^{-1} is not an exponent! And is not equal to $\cancel{\frac{1}{f(x)}}$

***Domain* and *Range* of** f *and* f^{-1}

$$\textit{domain of } f^{-1} = \textit{range of } f$$

$$\textit{range of } f^{-1} = \textit{domain of } f$$

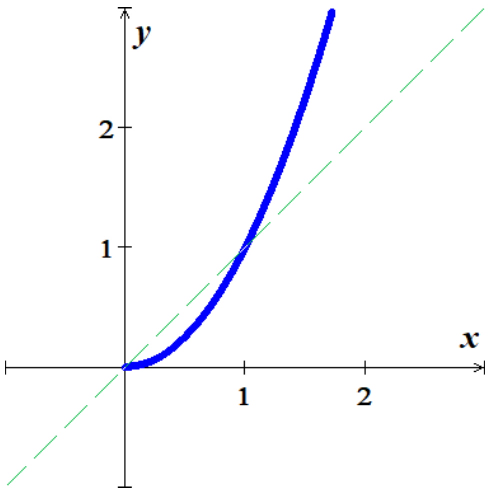
If a function f is one-to-one, then f^{-1} is the unique function such that each of the following holds.

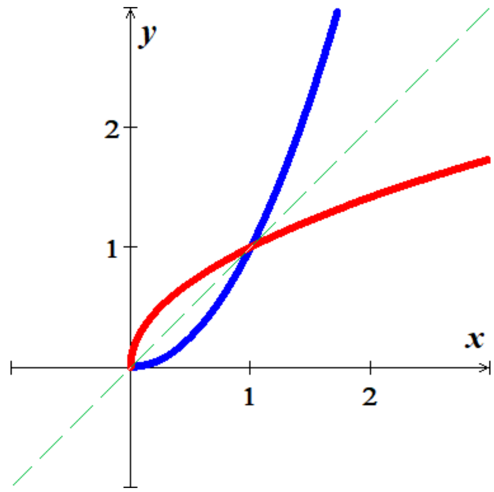
$\boxed{\left(f^{-1} \circ f\right)(x) = f^{-1}(f(x)) = x}$ for each x in the *domain* of f, and

$\boxed{\left(f \circ f^{-1}\right)(x) = f(f^{-1}(x)) = x}$ for each x in the *domain* of f^{-1}

The condition that f is one-to-one in the definition of inverse function is important; otherwise, g will not define a function

Graphing





Example

Let $f(x)=x^3-1$ and $g(x)=\sqrt[3]{x+1}$, is g the inverse function of f?

Solution

$$\begin{aligned}(f\circ g)(x)&=f(g(x))\\&=f\left(\sqrt[3]{x+1}\right)\\&=\left(\sqrt[3]{x+1}\right)^3-1\\&=x+1-1\\&=x\end{aligned}$$

$$\begin{aligned}(g\circ f)(x)&=g\left(f(x)\right)\\&=g\left(x^3-1\right)\\&=\sqrt[3]{x^3-1+1}\\&=\sqrt[3]{x^3}\\&=x\end{aligned}$$

g is the inverse function of f

Example

Show that each function is the inverse of the other: $f(x)=4x-7$ *and* $g(x)=\frac{x+7}{4}$

Solution

$$\begin{aligned}f(g(x))&=f\left(\frac{x+7}{4}\right)\\&=4\left(\frac{x+7}{4}\right)-7\\&=x+7-7\\&=x\end{aligned}$$

$$\begin{aligned}g(f(x))&=g(4x-7)\\&=\frac{4x-7+7}{4}\\&=\frac{4x}{4}\\&=x\end{aligned}$$

Finding the *Inverse Function*

Example

Finding an Inverse Function $f(x) = 2x + 7$

1. **Replace** $f(x)$ with y $\quad y = 2x + 7$
2. **Interchange** x and y $\quad x = 2y + 7$
3. **Solve for** y $\quad x - 7 = 2y$

$$\frac{x-7}{2} = y$$

4. **Replace** y with $f^{-1}(x)$ $\quad f^{-1}(x) = \frac{x-7}{2}$

Example

Find the inverse of $f(x) = 4x^3 - 1$

Solution

$$y = 4x^3 - 1$$

$$x = 4y^3 - 1$$

$$x + 1 = 4y^3$$

$$\frac{x+1}{4} = y^3$$

$$y = \left(\frac{x+1}{4}\right)^{1/3}$$

$$f^{-1}(x) = \sqrt[3]{\frac{x+1}{4}}$$

Example

Find a formula for the inverse $f(x) = \frac{5x-3}{2x+1}$

Solution

$$y = \frac{5x-3}{2x+1}$$

$$x = \frac{5y-3}{2y+1}$$

$$x(2y+1) = 5y - 3$$

$$2xy + x = 5y - 3$$

$$2xy-5y=-x-3$$

$$y(2x-5)=-x-3$$

$$y=\frac{-x-3}{2x-5}$$

$$f^{-1}(x)=-\frac{x+3}{2x-5}$$

Exercise   *Section* 3.1 – Inverse Functions

(**1 – 9**) Find the inverse relation of the given sets:

1. $A=\{(-2,\ 2),\ (1,\ -1),\ (0,\ 4),\ (1,\ 3)\}$

2. $B=\{(1,\ -1),\ (2,\ -2),\ (3,\ -3),\ (4,\ -4)\}$

3. $C=\{(a,\ -a),\ (b,\ -b),\ (c,\ -c)\}$

4. $D=\{(0,\ 0),\ (1,\ 1),\ (2,\ 2),\ (3,\ 3),\ (4,\ 4)\}$

5. $E=\{(-a,\ a),\ (-b,\ b),\ (-c,\ c),\ (-d,\ d)\}$

(**6 – 14**) Determine whether the function is one-to-one

6. $f(x)=3x-7$

7. $f(x)=x^2-9$

8. $f(x)=\sqrt{x}$

9. $f(x)=\sqrt[3]{x}$

10. $f(x)=|x|$

11. $f(x)=\frac{2}{x+3}$

12. $f(x)=(x-2)^3$

13. $y=x^2+2$

14. $f(x)=\frac{x+1}{x-3}$

15. Given that $f(x)=5x+8$, use composition of functions to show that $f^{-1}(x)=\frac{x-8}{5}$

16. Given the function $f(x)=(x+8)^3$

a) Find $f^{-1}(x)$

b) Graph f and f^{-1} in the same rectangular coordinate system

c) Find the domain and the range of f and f^{-1}

(**17 – 32**) Prove that $f(x)$ *and* $g(x)$ are inverse functions of each other.

17. $f(x)=4x;\quad g(x)=\frac{x}{4}$

18. $f(x)=2x;\quad g(x)=\frac{1}{2x}$

19. $f(x)=4x-1;\quad g(x)=\frac{x+1}{4}$

20. $f(x)=\frac{1}{2}x-\frac{3}{2};\quad g(x)=2x+3$

21. $f(x)=-\frac{1}{2}x-\frac{1}{2};\quad g(x)=-2x+1$

22. $f(x)=3x+2;\quad g(x)=\frac{1}{3}(x-2)$

23. $f(x)=\frac{5}{x+3};\quad g(x)=\frac{5}{x}-3$

24. $f(x)=\frac{2x}{x+1};\quad g(x)=\frac{-x}{x-2}$

25. $f(x)=\frac{3x}{x-1};\quad g(x)=\frac{x}{x-3}$

26. $f(x)=x^3+2;\quad g(x)=\sqrt[3]{x-2}$

27. $f(x)=x^3-1;\quad g(x)=\sqrt[3]{x+1}$

28. $f(x)=(x+4)^3;\quad g(x)=\sqrt[3]{x}-4$

29. $f(x)=x^3-1\quad g(x)=\sqrt[3]{x+1}$

30. $f(x)=3x-2\quad g(x)=\frac{x+2}{3}$

31. $f(x)=x^2+5, x\le 0\quad g(x)=-\sqrt{x-5},\ \ x\ge 5$

32. $f(x)=x^3-4;\quad g(x)=\sqrt[3]{x+4}$

(**33 – 35**) Find the inverse of

33. $f(x)=(x-2)^3$

34. $f(x)=\dfrac{x+1}{x-3}$

35. $f(x)=\dfrac{2x+1}{x-3}$

(**36 – 38**) Determine the domain and range of f^{-1} (Hint: first find the domain and range of f)

36. $f(x)=-\dfrac{2}{x-1}$

37. $f(x)=\dfrac{5}{x+3}$

38. $f(x)=\dfrac{4x+5}{3x-8}$

(**39 – 66**) For the given functions

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

39. $f(x)=\dfrac{2x}{x-1}$

40. $f(x)=\dfrac{x}{x-2}$

41. $f(x)=\dfrac{x+1}{x-1}$

42. $f(x)=\dfrac{2x+1}{x+3}$

43. $f(x)=\dfrac{3x-1}{x-2}$

44. $f(x)=\dfrac{2x}{x-1}$

45. $f(x)=\dfrac{x}{x-2}$

46. $f(x)=\dfrac{x+1}{x-1}$

47. $f(x)=\dfrac{2x+1}{x+3}$

48. $f(x)=\dfrac{3x-1}{x-2}$

49. $f(x)=\dfrac{3x-2}{x+4}$

50. $f(x)=\dfrac{-3x-2}{x+4}$

51. $f(x)=\sqrt{x-1}\quad x\ge 1$

52. $f(x)=\sqrt{2-x}\quad x\le 2$

53. $f(x)=x^2+4x\quad x\ge -2$

54. $f(x)=3x+5$

55. $f(x)=\dfrac{1}{3x-2}$

56. $f(x)=\dfrac{3x+2}{2x-5}$

57. $f(x)=\dfrac{4x}{x-2}$

58. $f(x)=2-3x^2;\quad x\le 0$

59. $f(x)=2x^3-5$

60. $f(x)=\sqrt{3-x}$

61. $f(x)=\sqrt[3]{x}+1$

62. $f(x)=\left(x^3+1\right)^5$

63. $f(x)=x^2-6x;\quad x\ge 3$

64. $f(x)=(x-2)^3$

65. $f(x)=\dfrac{x+1}{x-3}$

66. $f(x)=\dfrac{2x+1}{x-3}$

67. The function $w(x)=2x+24$ can be used to convert a U.S. women's shoe size into an Italian women's shoe size. Determine the function $w^{-1}(x)$ that can use to convert an Italian women's shoe size to its equivalent U.S. shoe size.

68. The function $m(x) = 1.3x - 4.7$ can be used to convert a U.S. men's shoe size into an U.K. women's shoe size. Determine the function $m^{-1}(x)$ that can used to convert an U.K. men's shoe size to its equivalent U.S. shoe size.

69. A catering service use the function $c(x) = \frac{300 + 12x}{x}$ to determine the amount, in *dollars*, it charges per person for a sit-down dinner, where *x* is the number of people in attendance.

a) Find $c(30)$ and explain what it represents

b) Find $c^{-1}(x)$

c) Use $c^{-1}(x)$ to determine how many people attended a dinner for which the cost per person was $15.00

70. A landscaping service use the function $c(x) = \frac{600 + 140x}{x}$ to determine the amount, in *dollars*, it charges per tree to deliver, where *x* is the number of trees.

a) Find $c(5)$ and explain what it represents

b) Find $c^{-1}(x)$

c) Use $c^{-1}(x)$ to determine how many trees were delivered for which the cost per tree was $160.00