Solution *Section* 3.2 – Angle and Orthogonality in Inner Product Spaces

Exercise

Which of the following form orthonormal sets?

a) $(1, 0), (0, 2)$ in $\boldsymbol{R}^2$

b) $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in $\boldsymbol{R}^2$

c) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in $\boldsymbol{R}^2$

d) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ in $\boldsymbol{R}^3$

e) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ in $\boldsymbol{R}^3$

f) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ in $\boldsymbol{R}^3$

Solution

a) $(1, 0) \cdot (0, 2) = 1(0) + 0(2) = 0$, they are ***orthonormal*** sets

b) $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \bullet \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} = \frac{1}{2} - \frac{1}{2} = 0$, they are orthonormal sets

c) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \bullet \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} = -\frac{1}{2} - \frac{1}{2} \underline{= -1}$

They are ***not orthonormal***

d) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \bullet \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \bullet \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{2}}\right) + 0 + \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{3}}\right)\frac{1}{\sqrt{2}}$$

$$= -\frac{1}{2}\frac{1}{\sqrt{3}} - \frac{1}{2}\frac{1}{\sqrt{3}}$$

$$= -\frac{1}{\sqrt{3}} \neq 0$$

They are ***not orthonormal*** sets

e) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \bullet \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) \bullet \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

$$= \frac{2}{3}\frac{2}{3}\frac{1}{3} + \left(-\frac{2}{3}\right)\frac{1}{3}\frac{2}{3} + \frac{1}{3}\left(-\frac{2}{3}\right)\frac{2}{3}$$

$$= \frac{4}{27} - \frac{4}{27} - \frac{4}{27}$$

$$= -\frac{4}{27} \neq 0$$

They are ***not orthonormal*** sets

f) $\left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right) \bullet \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right) = \frac{1}{\sqrt{6}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}}\left(-\frac{1}{\sqrt{2}}\right) + 0 \ \underline{= 0}$

They are ***orthonormal*** sets

Exercise

Find the cosine of the angle between ***u*** and ***v***.

a) $\boldsymbol{u} = (1, -3), \quad \boldsymbol{v} = (2, 4)$

b) $\boldsymbol{u} = (-1, 0), \quad \boldsymbol{v} = (3, 8)$

c) $\boldsymbol{u} = (-1, 5, 2), \quad \boldsymbol{v} = (2, 4, -9)$

d) $\boldsymbol{u} = (4, 1, 8), \quad \boldsymbol{v} = (1, 0, -3)$

e) $\boldsymbol{u} = (1, 0, 1, 0), \quad \boldsymbol{v} = (-3, -3, -3, -3)$

f) $\boldsymbol{u} = (2, 1, 7, -1), \quad \boldsymbol{v} = (4, 0, 0, 0)$

Solution

a) $\boldsymbol{u} = (1, -3), \quad \boldsymbol{v} = (2, 4)$

$$\|\boldsymbol{u}\| = \sqrt{1^2 + (-3)^2} = \underline{\sqrt{10}}$$

$$\|\boldsymbol{v}\| = \sqrt{2^2 + 4^2} = \underline{\sqrt{20}}$$

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 1(2) + (-3)(4) = \underline{-10}$$

$$\cos\theta = \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\|\boldsymbol{u}\| . \|\boldsymbol{v}\|}$$

$$= \frac{-10}{\sqrt{10}\sqrt{20}}$$

$$= -\frac{10}{\sqrt{200}}$$

$$\underline{= -\frac{1}{\sqrt{2}}}$$

b) $\boldsymbol{u}=(-1,0), \quad \boldsymbol{v}=(3,8)$

$$\|\boldsymbol{u}\|=\sqrt{(-1)^2+0^2}=1$$

$$\|\boldsymbol{v}\|=\sqrt{3^2+8^2}=\sqrt{73}$$

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle=(-1)(3)+(0)(8)=-3$$

$$\cos\theta=\frac{\langle \boldsymbol{u},\boldsymbol{v}\rangle}{\|\boldsymbol{u}\|.\|\boldsymbol{v}\|}=\frac{-3}{1\sqrt{73}}=-\frac{3}{\sqrt{73}}$$

c) $\boldsymbol{u}=(-1,5,2), \quad \boldsymbol{v}=(2,4,-9)$

$$\|\boldsymbol{u}\|=\sqrt{(-1)^2+5^2+2^2}=\sqrt{30}$$

$$\|\boldsymbol{v}\|=\sqrt{2^2+4^2+(-9)^2}=\sqrt{101}$$

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle=(-1)(2)+(5)(4)+(2)(-9)=0$$

$$\cos\theta=\frac{\langle \boldsymbol{u},\boldsymbol{v}\rangle}{\|\boldsymbol{u}\|.\|\boldsymbol{v}\|}=0$$

d) $\boldsymbol{u}=(4,1,8), \quad \boldsymbol{v}=(1,0,-3)$

$$\|\boldsymbol{u}\|=\sqrt{4^2+1^2+8^2}=9$$

$$\|\boldsymbol{v}\|=\sqrt{1^2+0^2+(-3)^2}=\sqrt{10}$$

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle=(4)(1)+(1)(0)+(8)(-3)=-20$$

$$\cos\theta=\frac{\langle \boldsymbol{u},\boldsymbol{v}\rangle}{\|\boldsymbol{u}\|.\|\boldsymbol{v}\|}=-\frac{20}{9\sqrt{10}}$$

e) $\boldsymbol{u}=(1,0,1,0), \quad \boldsymbol{v}=(-3,-3,-3,-3)$

f) $\boldsymbol{u}=(2,1,7,-1), \quad \boldsymbol{v}=(4,0,0,0)$

$$\|\boldsymbol{u}\|=\sqrt{2^2+1^2+7^2+(-1)^2}=\sqrt{55}$$

$$\|\boldsymbol{v}\|=\sqrt{4^2+0}=4$$

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle=(2)(4)+(1)(0)+(7)(0)+(-1)(0)=8$$

$$\cos\theta=\frac{\langle \boldsymbol{u},\boldsymbol{v}\rangle}{\|\boldsymbol{u}\|.\|\boldsymbol{v}\|}=\frac{8}{4\sqrt{55}}=\frac{2}{\sqrt{55}}$$

Exercise

Find the cosine of the angle between ***A*** and ***B***.

a) $A=\begin{pmatrix}2 & 6\\ 1 & -3\end{pmatrix}$ $B=\begin{pmatrix}3 & 2\\ 1 & 0\end{pmatrix}$

b) $A=\begin{pmatrix}2 & 4\\ -1 & 3\end{pmatrix}$ $B=\begin{pmatrix}-3 & 1\\ 4 & 2\end{pmatrix}$

Solution

a)

$$\|A\|=\sqrt{\langle A,A\rangle}$$
$$=\sqrt{2^2+6^2+1^2+(-3)^2}$$
$$=\sqrt{50}$$
$$=5\sqrt{2}$$

$$\|B\|=\sqrt{\langle B,B\rangle}$$
$$=\sqrt{3^2+2^2+1^2+0^2}$$
$$=\sqrt{14}$$

$$\langle A,B\rangle=2(3)+6(2)+1(1)+(-3)(0)=19$$

$$\cos\theta=\frac{\langle \boldsymbol{A},\boldsymbol{B}\rangle}{\|\boldsymbol{A}\|.\|B\|}=\frac{19}{5\sqrt{2}\sqrt{14}}=\underline{\frac{19}{10\sqrt{7}}}$$

b)

$$\|A\|=\sqrt{\langle A,A\rangle}$$
$$=\sqrt{2^2+4^2+(-1)^2+3^2}$$
$$=\sqrt{30}$$

$$\|B\|=\sqrt{\langle B,B\rangle}$$
$$=\sqrt{(-3)^2+1^2+4^2+2^2}$$
$$=\sqrt{30}$$

$$\langle A,B\rangle=2(-3)+4(1)+(-1)(4)+3(2)=0$$

$$\cos\theta=\frac{\langle \boldsymbol{A},\boldsymbol{B}\rangle}{\|\boldsymbol{A}\|.\|B\|}=\frac{0}{30}=\underline{0}$$

Exercise

Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

a) $\boldsymbol{u}=(-1,3,2), \quad \boldsymbol{v}=(4,2,-1)$

b) $\boldsymbol{u}=(a,b), \quad \boldsymbol{v}=(-b,a)$

c) $\boldsymbol{u}=(-2,-2,-2), \quad \boldsymbol{v}=(1,1,1)$

d) $\boldsymbol{u}=(-4,\ 6,\ -10,\ 1), \quad \boldsymbol{v}=(2,\ 1,\ -2,\ 9)$

e) $\boldsymbol{u}=(-4,\ 6,\ -10,\ 1), \quad \boldsymbol{v}=(2,\ 1,\ -2,\ 9)$

Solution

a) $\langle \mathbf{u},\mathbf{v}\rangle=(-1)(4)+3(2)+2(-1)=\underline{0}$ Therefore the given vectors are orthogonal.

b) $\langle \mathbf{u},\mathbf{v}\rangle=a(-b)+b(a)=\underline{0}$ Therefore the given vectors are orthogonal.

c) $\langle \mathbf{u},\mathbf{v}\rangle=(-2)(1)+(-2)(1)+(-2)(1)=\underline{-6}$ Therefore the given vectors are ***not*** orthogonal.

d) $\langle \mathbf{u},\mathbf{v}\rangle=(-4)(2)+(6)(1)+(-10)(-2)+(1)(9)=\underline{27}$ Therefore the given vectors are ***not*** orthogonal.

e) $\|\mathbf{u}\|=\sqrt{(-4)^2+6^2+(-10)^2+1^2}=\sqrt{153}=3\sqrt{17}$

$$\|\mathbf{v}\|=\sqrt{2^2+1^2+(-2)^2+9^2}=\sqrt{90}=3\sqrt{10}$$

$$\langle \boldsymbol{u},\ \boldsymbol{v}\rangle=(-4)(2)+6(1)-10(-2)+1(9)=\underline{27}$$

$$\cos\theta=\frac{\langle \boldsymbol{u},\ \boldsymbol{v}\rangle}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}$$

$$=\frac{27}{3\sqrt{17}\left(3\sqrt{10}\right)}$$

$$=\frac{3}{\sqrt{170}}$$

The vectors ***u*** and ***v*** are NOT orthogonal with respect to the Euclidean

Exercise

Do there exist scalars k and l such that the vectors $\boldsymbol{u}=(2,k,6), \quad \boldsymbol{v}=(l,5,3), \quad and \quad \boldsymbol{w}=(1,2,3)$ are mutually orthogonal with respect to the Euclidean inner product?

Solution

$$\langle \boldsymbol{u},\boldsymbol{w}\rangle=(2)(1)+(k)(2)+(6)(3)=20+2k=0 \quad \Rightarrow \quad \boxed{k=-10}$$

$$\langle \boldsymbol{v},\boldsymbol{w}\rangle=(l)(1)+(5)(2)+(3)(3)=l+19=0 \quad \Rightarrow \quad \boxed{l=-19}$$

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle=(2)(l)+(k)(5)+(6)(3)=2l+5k+18=0$$

$$2(-19)+5(-10)+18=-70\neq 0$$

Thus, there are no scalars such that the vectors are mutually orthogonal

Exercise

Let $\boldsymbol{R}^3$ have the Euclidean inner product. For which values of k are $\boldsymbol{u}$ and $\boldsymbol{v}$ orthogonal?

a) $\boldsymbol{u}=(2,1,3), \quad \boldsymbol{v}=(1,7,k)$

b) $\boldsymbol{u}=(k,k,1), \quad \boldsymbol{v}=(k,5,6)$

Solution

a) $\langle \boldsymbol{u},\boldsymbol{v}\rangle=(2)(1)+(1)(7)+(3)(k)$
$$=9+3k=0$$

$\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal for $\boxed{k=-3}$

b) $\langle \boldsymbol{u},\boldsymbol{v}\rangle=(k)(k)+(k)(5)+(1)(6)$
$$=k^2+5k+6=0$$

$\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal for $\boxed{k=-2,\ -3}$

Exercise

Let V be an inner product space. Show that if $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal unit vectors in V, then $\|\boldsymbol{u}-\boldsymbol{v}\|=\sqrt{2}$

Solution

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{v}\|^2 &= \langle \boldsymbol{u}-\boldsymbol{v},\boldsymbol{u}-\boldsymbol{v}\rangle\\ &= \langle \boldsymbol{u},\boldsymbol{u}-\boldsymbol{v}\rangle-\langle \boldsymbol{v},\boldsymbol{u}-\boldsymbol{v}\rangle\\ &= \langle \boldsymbol{u},\boldsymbol{u}\rangle-\langle \boldsymbol{u},\boldsymbol{v}\rangle-\langle \boldsymbol{v},\boldsymbol{u}\rangle+\langle \boldsymbol{v},\boldsymbol{v}\rangle\\ &= \|\boldsymbol{u}\|^2-0-0+\|\boldsymbol{v}\|^2\\ &= 2\end{aligned}$$

Thus $\|\boldsymbol{u}-\boldsymbol{v}\|=\sqrt{2}$

Exercise

Let $\mathbf{S}$ be a subspace of $\mathbb{R}^n$. Explain what $\left(\mathbf{S}^{\perp}\right)^{\perp}=\mathbf{S}$ means and why it is true.

Solution

$\left(\mathbf{S}^{\perp}\right)^{\perp}$ is the orthogonal complement of , $\mathbf{S}^{\perp}$, which is itself the orthogonal complement of **S**, so $\left(\mathbf{S}^{\perp}\right)^{\perp} = \mathbf{S}$ means that **S** is the orthogonal of its orthogonal complement.

We need to show that **S** is contained in $\left(\mathbf{S}^{\perp}\right)^{\perp}$ and, conversely, that $\left(\mathbf{S}^{\perp}\right)^{\perp}$ is contained in **S** to be true.

i. Suppose $\vec{v} \in \mathbf{S}$ and $\vec{w} \in \mathbf{S}^{\perp}$. Then $\langle \vec{v},\ \vec{w} \rangle = 0$ by definition of $\mathbf{S}^{\perp}$. Thus **S** is certainly contained is $\left(\mathbf{S}^{\perp}\right)^{\perp}$ (which consists of all vectors in $\mathbb{R}^n$ which are orthogonal to $\mathbf{S}^{\perp}$).

ii. Suppose $\vec{v} \in \left(\mathbf{S}^{\perp}\right)^{\perp}$ (means $\vec{v}$ is orthogonal to all vectors in $\mathbf{S}^{\perp}$); then we need to show that $\vec{v} \in \mathbf{S}$.

Let assume $\left\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\right\}$ be a basis for **S** and let $\left\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_q\right\}$ be a basis for $\mathbf{S}^{\perp}$. If $\vec{v} \notin \mathbf{S}$, then $\left\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p, \vec{v}\right\}$ is linearly independent set. Since each vector ifs that set is orthogonal to all of $\mathbf{S}^{\perp}$, the set $\left\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p,\ \vec{v},\ \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_q\right\}$ is linearly independent. Since there are $p+q+1$ vectors in this set, this means that $p+q+1 \le n \quad \Leftrightarrow \quad p+q \le n-1$. On the other hand, If A is the matrix whose i^{th} row is $\vec{u}_i^T$, then the row space of A is **S** and the nullspace of A is $\mathbf{S}^{\perp}$. Since **S** is p-dimensional, the rank of A is p, meaning that the dimension of $\text{nul}(A) = \mathbf{S}^{\perp}$ is $q = n - p$. Therefore,

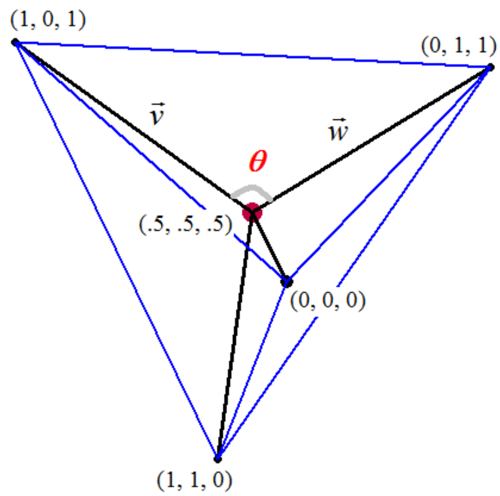
$$p + q = p + (n - p) = n$$

Which contradict the fact that $p+q \le n-1$. From this, we see that, if $\vec{v} \in \left(\mathbf{S}^{\perp}\right)^{\perp}$, it must be the case that $\vec{v} \in \mathbf{S}$.

Exercise

The methane molecule CH_4 is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at $(0,\ 0,\ 0)$, $(1,\ 1,\ 0)$, $(1,\ 0,\ 1)$ and $(0,\ 1,\ 1)$ – (***note*** that all six edges have length $\sqrt{2}$, so the tetrahedron is regular). What is the cosine of the angle between the rays going from the center $\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$ to the vertices?

Solution



Let $\vec{v}$ be the vector of the segment (1, 0, 1) and $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

be the vector of the segment (0, 1, 1) and $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$$\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

We have:

$$\begin{aligned} \cos\theta &= \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \cdot \|\vec{w}\|} \\ &= \frac{\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) \cdot \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}{\sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}}\ \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}}} \\ &= \frac{-\frac{1}{4}}{\frac{3}{4}} \\ &= -\frac{1}{3} \end{aligned}$$

$$\theta \approx 109.47^\circ$$

Exercise

Determine if the given vectors are orthogonal.

$$\boldsymbol{x}_1=(1,\ 0,\ 1,\ 0),\quad \boldsymbol{x}_2=(0,\ 1,\ 0,\ 1),\quad \boldsymbol{x}_3=(1,\ 0,\ -1,\ 0),\quad \boldsymbol{x}_4=(1,\ 1,\ -1,\ -1)$$

Solution

$$\boldsymbol{x}_1\bullet\boldsymbol{x}_2=(1,\ 0,\ 1,\ 0)\bullet(0,\ 1,\ 0,\ 1)=0$$

$$\boldsymbol{x}_1\bullet\boldsymbol{x}_3=(1,\ 0,\ 1,\ 0)\bullet(1,\ 0,\ -1,\ 0)=1-1=0$$

$$\boldsymbol{x}_1\bullet\boldsymbol{x}_4=(1,\ 0,\ 1,\ 0)\bullet(1,\ 1,\ -1,\ -1)=1-1=0$$

$$\boldsymbol{x}_2\bullet\boldsymbol{x}_3=(0,\ 1,\ 0,\ 1)\bullet(1,\ 0,\ -1,\ 0)=0$$

$$\boldsymbol{x}_2\bullet\boldsymbol{x}_4=(0,\ 1,\ 0,\ 1)\bullet(1,\ 1,\ -1,\ -1)=1-1=0$$

$$\boldsymbol{x}_3\bullet\boldsymbol{x}_4=(1,\ 0,\ -1,\ 0)\bullet(1,\ 1,\ -1,\ -1)=1-1=0$$

The given vectors are orthogonal

Exercise

Which of the following sets of vectors are orthogonal with respect to the Euclidean inner

a) $\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\ \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)\ \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$

b) $\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right)\ \left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right)\ \left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right)$

Solution

a) $\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\bullet\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)=\frac{1}{\sqrt{6}}+0-\frac{1}{\sqrt{6}}\ \underline{=0}$

$\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\bullet\left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)=-\frac{1}{2}+0+\frac{1}{2}\ \underline{=0}$

$\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)\bullet\left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)=-\frac{1}{\sqrt{6}}+0-\frac{1}{\sqrt{6}}=-\frac{2}{\sqrt{6}}\ \underline{\neq 0}$

Therefore the given vectors are ***not*** orthogonal.

b) $\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right)\bullet\left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right)=\frac{4}{9}-\frac{2}{9}-\frac{2}{9}\ \underline{=0}$

$\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right)\bullet\left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right)=\frac{2}{9}-\frac{4}{9}+\frac{2}{9}\ \underline{=0}$

$\left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right)\bullet\left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right)=\frac{2}{9}+\frac{2}{9}-\frac{4}{9}\ \underline{=0}$

Therefore the given vectors are orthogonal.