Solution   Section 4.7 – Stokes' Theorem

Exercise

Verify that the line integral and the surface integral of Stokes' Theorem are equal for the following vector fields, surfaces *S*, and closed curves *C*. Assume that *C* has counterclockwise orientation and *S* has a consistent orientation.

$\vec{F} = \langle y,\ -x,\ 10\rangle$; *S* is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ and *C* is the circle $x^2 + y^2 = 1$ in the *xy*-plane

Solution

$$\begin{aligned}\vec{F} &= \langle y,\ -x,\ 10\rangle \\ &= \langle \sin t,\ -\cos t,\ 10\rangle\end{aligned}$$

$$x^2 + y^2 = 1 = r^2$$

$$\vec{r}(t) = \langle \cos t,\ \sin t,\ 0\rangle$$

$$\vec{r}'(t) = \langle -\sin t,\ \cos t,\ 0\rangle$$

$$\begin{aligned}\oint_C \vec{F} \bullet d\vec{r} &= \iint_R \langle \sin t,\ -\cos t,\ 10\rangle \cdot \langle -\sin t,\ \cos t,\ 0\rangle\, dA \\ &= \int_0^{2\pi} \left(-\sin^2 t - \cos^2 t\right) dt \qquad \sin^2 t + \cos^2 t = 1 \\ &= -\int_0^{2\pi} dt \\ &= -2\pi\end{aligned}$$

$$\begin{aligned}\nabla \times \vec{F} &= \nabla \times \langle y,\ -x,\ 10\rangle \\ &= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 10 \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(10) + \frac{\partial}{\partial z}(x)\right)\hat{\boldsymbol{i}} + \left(\frac{\partial}{\partial z}(y) - \frac{\partial}{\partial x}(10)\right)\hat{\boldsymbol{j}} + \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y)\right)\hat{\boldsymbol{k}} \\ &= \langle 0,\ 0,\ -2\rangle\end{aligned}$$

$$\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\, dS = \iint_R \langle 0,\ 0,\ -2\rangle \cdot \left\langle \frac{x}{z},\ \frac{y}{z},\ 1\right\rangle dA$$

$$= \int_0^{2\pi} d\theta \int_0^1 -2r\,dr$$

$$= -(2\pi)\left[r^2 \right]_0^1$$

$$\underline{= -2\pi}$$

Or

Using the standard parametrization of the sphere

$\rightarrow\ \boldsymbol{n} = \left\langle \sin^2\phi\cos\theta,\ \sin^2\phi\cos\theta,\ \cos\phi\sin\phi \right\rangle$

$$\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS = \iint_R \langle 0,\ 0,\ -2\rangle \cdot \left\langle \sin^2\phi\cos\theta,\ \sin^2\phi\cos\theta,\ \cos\phi\sin\phi \right\rangle dA$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \left(-2\cos\phi\sin\phi\right) d\phi\, d\theta$$

$$= -\int_0^{2\pi} d\theta \int_0^{\pi/2} \sin 2\phi\, d\phi$$

$$= -(2\pi)\left[-\frac{1}{2}\cos 2\phi \right]_0^{\pi/2}$$

$$\underline{= -2\pi}$$

Exercise

Verify that the line integral and the surface integral of Stokes' Theorem are equal for the following vector fields, surfaces *S*, and closed curves *C*. Assume that *C* has counterclockwise orientation and *S* has a consistent orientation.

$\vec{F} = \langle 0,\ -x,\ y\rangle$; *S* is the upper half of the sphere $x^2 + y^2 + z^2 = 4$ and *C* is the circle $x^2 + y^2 = 4$ in the *xy*-plane

Solution

$x^2 + y^2 = 4 = r^2$

$\vec{r}(t) = \langle 2\cos t,\ 2\sin t,\ 0\rangle$

$\vec{r}\,'(t) = \langle -2\sin t,\ 2\cos t,\ 0\rangle$

$\vec{F} = \langle 0,\ -x,\ y\rangle$

$\quad = \langle 0,\ -2\cos t,\ 2\sin t\rangle$

$$\oint_C \vec{F} \bullet d\vec{r} = \iint_R \langle 0,\ -2\cos t,\ 2\sin t\rangle \cdot \langle -2\sin t,\ 2\cos t,\ 0\rangle\, dA$$

$$= \int_0^{2\pi} \left(-4\cos^2 t\right) dt$$

$$= -2\int_0^{2\pi} (1+\cos 2t)\, dt$$

$$= -2\left[t + \frac{1}{2}\sin 2t \right]_0^{2\pi}$$

$$\underline{= -4\pi}\,\rfloor$$

$$\nabla \times \vec{F} = \nabla \times \langle 0,\ -x,\ y \rangle$$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -x & y \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) \right)\hat{\boldsymbol{i}} + \left(\frac{\partial}{\partial z}(0) - \frac{\partial}{\partial x}(y) \right)\hat{\boldsymbol{j}} + \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(0) \right)\hat{\boldsymbol{k}}$$

$$\underline{= \langle 1,\ 0,\ -1 \rangle}\,\rfloor$$

$$\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS = \iint_R \langle 1,\ 0,\ -1 \rangle \cdot \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle dA$$

$$= \iint_R \left(\frac{x}{z} - 1 \right) dA$$

$$= \int_0^{2\pi} \int_0^2 \left(\frac{r\cos\theta}{\sqrt{4-r^2}} - 1 \right) r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^2 \left(\frac{r^2\cos\theta}{\sqrt{4-r^2}} - r \right) dr\,d\theta \qquad \int \frac{x^2}{\sqrt{a^2-x^2}}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{x}{2}\sqrt{a^2-x^2}$$

$$= \int_0^{2\pi} \left(\left(2\sin^{-1}\left(\frac{r}{2}\right) - \frac{r}{2}\sqrt{4-r^2} \right)\cos\theta - \frac{1}{2}r^2 \right)_0^2 d\theta$$

$$= \int_0^{2\pi} (\pi\cos\theta - 2)\, d\theta$$

$$= \left[\pi\sin\theta - 2\theta \right]_0^{2\pi}$$

$$\underline{= -4\pi}\,\rfloor$$

Exercise

Verify that the line integral and the surface integral of Stokes' Theorem are equal for the following vector fields, surfaces S, and closed curves C. Assume that C has counterclockwise orientation and S has a consistent orientation.

$\vec{F} = \langle x,\ y,\ z\rangle$; S is the paraboloid $z = 8 - x^2 - y^2$ for $0 \le z \le 8$ and C is the circle $x^2 + y^2 = 8$ in the xy-plane

Solution

$$x^2 + y^2 = 8 = r^2$$

$$\vec{r}(t) = \left\langle 2\sqrt{2}\cos t,\ 2\sqrt{2}\sin t,\ 0\right\rangle$$

$$\vec{r}'(t) = \left\langle -2\sqrt{2}\sin t,\ 2\sqrt{2}\cos t,\ 0\right\rangle$$

$$\vec{F} = \langle x,\ y,\ z\rangle$$

$$= \left\langle 2\sqrt{2}\cos t,\ 2\sqrt{2}\sin t,\ 0\right\rangle$$

$$\oint_C \vec{F}\bullet d\vec{r} = \iint_R \left\langle 2\sqrt{2}\cos t,\ 2\sqrt{2}\sin t,\ 0\right\rangle \cdot \left\langle -2\sqrt{2}\sin t,\ 2\sqrt{2}\cos t,\ 0\right\rangle dA$$

$$= \int_0^{2\pi} \left(-8\cos t\sin t + 8\cos t\sin t\right) dt$$

$$= 0$$

Surface integral: $\iint_S \left(\nabla\times\vec{F}\right)\bullet\vec{n}\ dS = 0$

Exercise

Verify that the line integral and the surface integral of Stokes' Theorem are equal for the following vector fields, surfaces S, and closed curves C. Assume that C has counterclockwise orientation and S has a consistent orientation.

$\vec{F} = \langle 2z,\ -4x,\ 3y\rangle$; S is the cap of the sphere $x^2 + y^2 + z^2 = 169$ above the plane $z = 12$ and C is the boundary of S.

Solution

$$x^2 + y^2 + 12^2 = 169$$

$\to\quad x^2 + y^2 = 25$ is the intersection of the sphere with the plane $z = 12$.

$$\vec{r}(t) = \langle 5\cos t,\ 5\sin t,\ 0\rangle$$

$$\vec{r}'(t) = \langle -5\sin t,\ 5\cos t,\ 0\rangle$$

$$\vec{F} = \langle 2z,\ -4x,\ 3y\rangle$$

$$= \langle 2(12),\ -4\times 5\cos t,\ 3\times 5\sin t\rangle$$
$$= \langle 24,\ -20\cos t,\ 15\sin t\rangle$$

$$\oint_C \vec{F}\bullet d\vec{r} = \iint_R \langle 24,\ -20\cos t,\ 15\sin t\rangle\cdot\langle -5\sin t,\ 5\cos t,\ 0\rangle\, dA$$
$$= \int_0^{2\pi}\left(-120\sin t - 100\cos^2 t\right)dt$$
$$= 10\int_0^{2\pi}\left(-12\sin t - 5 - 5\cos 2t\right)dt$$
$$= 10\left[12\cos t - 5t - \frac{5}{2}\sin 2t\right]_0^{2\pi}$$
$$= 10(12 - 10\pi - 12)$$
$$\boxed{= -100\pi}$$

$$\nabla\times\vec{F} = \nabla\times\langle 2z,\ -4x,\ 3y\rangle$$
$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & -4x & 3y \end{vmatrix}$$
$$= (3+0)\hat{\boldsymbol{i}} + (2-0)\hat{\boldsymbol{j}} + (-4-0)\hat{\boldsymbol{k}}$$
$$\boxed{= \langle 3,\ 2,\ -4\rangle}$$

$$\iint_S \left(\nabla\times\vec{F}\right)\bullet\vec{n}\, dS = \iint_R \langle 3,\ 2,\ -4\rangle\cdot\left\langle \frac{x}{z},\ \frac{y}{z},\ 1\right\rangle dA$$
$$= \iint_R \left(\frac{3x}{z} + \frac{2y}{z} - 4\right)dA$$
$$= \int_0^{2\pi}\int_0^5 \left(\frac{3r\cos\theta}{\sqrt{169-r^2}} + \frac{2r\sin\theta}{\sqrt{169-r^2}} - 4\right) r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^5 \left(\frac{3r^2\cos\theta}{\sqrt{169-r^2}} + \frac{2r^2\sin\theta}{\sqrt{169-r^2}} - 4r\right) dr\,d\theta$$

$$\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{x}{2}\sqrt{a^2-x^2}$$

$$= \int_0^{2\pi} \left(3\left(\frac{169}{2}\sin^{-1}\left(\frac{r}{13}\right) - \frac{r}{2}\sqrt{169-r^2} \right)\cos\theta + 2\left(\frac{169}{2}\sin^{-1}\left(\frac{r}{13}\right) - \frac{r}{2}\sqrt{169-r^2} \right)\sin\theta - 2r^2 \right)\Bigg|_0^5 d\theta$$

$$= \int_0^{2\pi} \left(\left(\frac{507}{2}\sin^{-1}\left(\frac{5}{13}\right) - 90 \right)\cos\theta + \left(169\sin^{-1}\left(\frac{5}{13}\right) - 60 \right)\sin\theta - 50 \right) d\theta$$

$$= \left[\left(\frac{507}{2}\sin^{-1}\left(\frac{5}{13}\right) - 90 \right)\sin\theta - \left(169\sin^{-1}\left(\frac{5}{13}\right) - 60 \right)\cos\theta - 50\theta \right]_0^{2\pi}$$

$$= -\left(169\sin^{-1}\left(\frac{5}{13}\right) - 60 \right) - 100\pi + \left(169\sin^{-1}\left(\frac{5}{13}\right) - 60 \right)$$

$$= -100\pi$$

Exercise

Verify that the line integral and the surface integral of Stokes' Theorem are equal for the following vector fields, surfaces *S*, and closed curves *C*. Assume that *C* has counterclockwise orientation and *S* has a consistent orientation.

$\vec{F} = \langle y-z,\ z-x,\ x-y \rangle$; *S* is the cap of the sphere $x^2+y^2+z^2=16$ above the plane $z=\sqrt{7}$ and *C* is the boundary of *S*.

Solution

$$x^2+y^2+7=16$$

$\rightarrow\ x^2+y^2=9$ is the intersection of the sphere with the plane $z=\sqrt{7}$.

$$\vec{r}(t) = \langle 3\cos t,\ 3\sin t,\ 0 \rangle$$

$$\vec{r}\,'(t) = \langle -3\sin t,\ 3\cos t,\ 0 \rangle$$

$$\vec{F} = \langle y-z,\ z-x,\ x-y \rangle$$

$$= \left\langle 3\sin t - \sqrt{7},\ \sqrt{7} - 3\cos t,\ 3\cos t - 3\sin t \right\rangle$$

$$\oint_C \vec{F}\bullet d\vec{r} = \iint_R \left\langle 3\sin t - \sqrt{7},\ \sqrt{7} - 3\cos t,\ 3\cos t - 3\sin t \right\rangle \cdot \langle -3\sin t,\ 3\cos t,\ 0 \rangle\, dA$$

$$= \int_0^{2\pi} \left(-9\sin^2 t + 3\sqrt{7}\sin t + 3\sqrt{7}\cos t - 9\cos^2 t \right) dt \qquad \sin^2 t + \cos^2 t = 1$$

$$= \int_0^{2\pi} \left(3\sqrt{7}\sin t + 3\sqrt{7}\cos t - 9 \right) dt$$

$$= \left[-3\sqrt{7}\cos t + 3\sqrt{7}\sin t - 9t \right]_0^{2\pi}$$

$$= -3\sqrt{7} - 18\pi + 3\sqrt{7}$$

$$\boxed{= -18\pi}$$

$$\nabla \times \vec{F} = \nabla \times \langle y - z,\ z - x,\ x - y \rangle$$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z & z - x & x - y \end{vmatrix}$$

$$\boxed{= \langle -2,\ -2,\ -2 \rangle}$$

$$\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS = \iint_R \langle -2,\ -2,\ -2 \rangle \cdot \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle dA$$

$$= \iint_R \left(-2\frac{x}{z} - 2\frac{y}{z} - 2 \right) dA$$

$$= -2\int_0^{2\pi} \int_0^3 \left(\frac{r\cos\theta}{\sqrt{16 - r^2}} + \frac{r\sin\theta}{\sqrt{16 - r^2}} + 1 \right) r\,dr\,d\theta$$

$$= -2\int_0^{2\pi} \int_0^3 \left(\frac{r^2\cos\theta}{\sqrt{16 - r^2}} + \frac{r^2\sin\theta}{\sqrt{16 - r^2}} + r \right) dr\,d\theta$$

$$\int \frac{x^2}{\sqrt{a^2 - x^2}}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{x}{2}\sqrt{a^2 - x^2}$$

$$= -2\int_0^{2\pi} \left(\left(\frac{16}{2}\sin^{-1}\left(\frac{r}{4}\right) - \frac{r}{2}\sqrt{16 - r^2} \right)\cos\theta + \left(\frac{16}{2}\sin^{-1}\left(\frac{r}{4}\right) - \frac{r}{2}\sqrt{16 - r^2} \right)\sin\theta + \frac{1}{2}r^2 \right)\Bigg|_0^3 d\theta$$

$$= -2\int_0^{2\pi} \left(\left(8\sin^{-1}\left(\frac{3}{4}\right) - \frac{3\sqrt{7}}{2} \right)\cos\theta + \left(8\sin^{-1}\left(\frac{3}{4}\right) - \frac{3\sqrt{7}}{2} \right)\sin\theta + \frac{9}{2} \right) d\theta$$

$$= -2\left[-\left(8\sin^{-1}\left(\frac{3}{4}\right) - \frac{3\sqrt{7}}{2} \right)\sin\theta + \left(8\sin^{-1}\left(\frac{3}{4}\right) - \frac{3\sqrt{7}}{2} \right)\cos\theta + \frac{9}{2}\theta \right]_0^{2\pi}$$

$$= -2\left[8\sin^{-1}\left(\frac{3}{4}\right) - \frac{3\sqrt{7}}{2} - 9\pi - 8\sin^{-1}\left(\frac{3}{4}\right) + \frac{3\sqrt{7}}{2} \right]$$

$$\boxed{= -18\pi}$$

Exercise

Verify that the line integral and the surface integral of Stokes' Theorem are equal for the following vector fields, surfaces *S*, and closed curves *C*. Assume that *C* has counterclockwise orientation and *S* has a consistent orientation.

$\vec{F}=\langle -y,\ -x-z,\ y-x\rangle$; *S* is the part of the plane $z=6-y$ that lies in the cylinder $x^2+y^2=16$ and *C* is the boundary of *S*.

Solution

$$\vec{r}(t)=\langle 4\cos t,\ 4\sin t,\ 6-4\sin t\rangle \qquad \vec{r}(t)=\langle x,\ y,\ z\rangle$$

$$d\vec{r}=\langle -4\sin t,\ 4\cos t,\ -4\cos t\rangle$$

$$\begin{aligned}\vec{F}&=\langle -y,\ -x-z,\ y-x\rangle\\ &=\langle -4\sin t,\ -4\cos t-6+4\sin t,\ 4\sin t-4\cos t\rangle\end{aligned}$$

$$\begin{aligned}\oint_C \vec{F}\bullet d\vec{r}&=\int_0^{2\pi}\langle -4\sin t,\ -4\cos t-6+4\sin t,\ 4\sin t-4\cos t\rangle\bullet\langle -4\sin t,\ 4\cos t,\ -4\cos t\rangle\, dt\\ &=\int_0^{2\pi}\left(16\sin^2 t-16\cos^2 t-24\cos t+16\sin t\cos t-16\sin t\cos t+16\cos^2 t\right) dt\\ &=\int_0^{2\pi}\left(16\sin^2 t-24\cos t\right) dt\\ &=\int_0^{2\pi}\left(8-8\cos 2t-24\cos t\right) dt\\ &=\left(8t-4\sin 2t-24\sin t\right)\Big|_0^{2\pi}\\ &=16\pi\end{aligned}$$

Exercise

Evaluate the line integral $\oint_C \vec{F}\bullet d\vec{r}$ by evaluating the surface integral in Stokes' Theorem with an appropriate choice of *S*. Assume that *C* has a counterclockwise orientation,

$\vec{F}=\langle 2y,\ -z,\ x\rangle$; *C* is the circle $x^2+y^2=12$ in the plane $z=0$.

Solution

$$\nabla\times\vec{F}=\nabla\times\langle 2y,\ -z,\ x\rangle$$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & -z & x \end{vmatrix}$$

$$\boxed{= \langle 1,\ -1,\ -2 \rangle}$$

$z = 0 \quad (0x + 0y) \quad \to \quad \vec{n} = \langle 0,\ 0,\ 1 \rangle$

$$\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS = \iint_R \langle 1,\ -1,\ -2 \rangle \cdot \langle 0,\ 0,\ 1 \rangle\, dA$$

$$= \iint_R (-2)\, dA$$

$$= -2\int_0^{2\pi} d\theta \int_0^{2\sqrt{3}} r\,dr$$

$$= -2(2\pi)\left[\frac{1}{2}r^2\right]_0^{2\sqrt{3}}$$

$$\boxed{= -24\pi}$$

Exercise

Evaluate the line integral $\oint_C \vec{F} \bullet d\vec{r}$ by evaluating the surface integral in Stokes' Theorem with an appropriate choice of S. Assume that C has a counterclockwise orientation,

$\vec{F} = \langle y,\ xz,\ -y \rangle$; C is the ellipse $x^2 + \frac{y^2}{4} = 1$ in the plane $z = 1$.

Solution

$$\nabla \times \vec{F} = \nabla \times \langle y,\ xz,\ -y \rangle$$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & -y \end{vmatrix}$$

$$\boxed{= \langle -1 - x,\ 0,\ z - 1 \rangle}$$

$z = 1 \quad (+0x + 0y) \quad \to \quad \vec{n} = \langle 0,\ 0,\ 1 \rangle$

$$\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS = \iint_R \langle -1 - x,\ 0,\ z - 1 \rangle \cdot \langle 0,\ 0,\ 1 \rangle\, dA$$

$$= \iint_R (z-1)\,dA \qquad \text{Because } z=1$$

$$= \iint_R (0)\,dA$$

$$\underline{=0}$$

Exercise

Evaluate the line integral $\oint_C \vec{F} \cdot d\vec{r}$ by evaluating the surface integral in Stokes' Theorem with an appropriate choice of S. Assume that C has a counterclockwise orientation,

$\vec{F} = \left\langle x^2 - z^2,\ y,\ 2xz \right\rangle$; C is the boundary of the plane $z = 4 - x - y$ in the plane first octant.

Solution

$$\nabla \times \boldsymbol{F} = \nabla \times \left\langle x^2 - z^2,\ y,\ 2xz \right\rangle$$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - z^2 & y & 2xz \end{vmatrix}$$

$$\underline{= \langle 0,\ -4z,\ 0 \rangle}$$

$$x + y + z = 4 \quad \to \quad \vec{n} = \langle 1,\ 1,\ 1 \rangle$$

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ dS = \iint_R \langle 0,\ -4z,\ 0 \rangle \cdot \langle 1,\ 1,\ 1 \rangle\, dA$$

$$= \iint_R (-4z)\,dA$$

$$= -4 \int_0^4 \int_0^{4-x} (4 - x - y)\,dx\,dy$$

$$= -4 \int_0^4 \left(4y - xy - \frac{1}{2}y^2\right)\Big|_0^{4-x} dx$$

$$= -4 \int_0^4 \left(16 - 4x - 4x + x^2 - \frac{1}{2}\left(16 - 8x + x^2\right)\right) dx$$

$$= -4\int_0^4 \left(\frac{1}{2}x^2 - 4x + 8\right) dx$$

$$= -4\left[\frac{1}{6}x^3 - 2x^2 + 8x\right]_0^4$$

$$= -4\left(\frac{32}{3} - 32 + 32\right)$$

$$\boxed{= -\frac{128}{3}}$$

Exercise

Evaluate the line integral $\oint_C \vec{F} \bullet d\vec{r}$ by evaluating the surface integral in Stokes' Theorem with an appropriate choice of S. Assume that C has a counterclockwise orientation,

$\vec{F} = \left\langle y^2,\ -z^2,\ x\right\rangle$; C is the circle $\boldsymbol{r}(t) = \langle 3\cos t,\ 4\cos t,\ 5\sin t\rangle$ for $0 \le t \le 2\pi$.

Solution

$$\nabla \times \vec{F} = \nabla \times \left\langle y^2,\ -z^2,\ x\right\rangle$$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & -z^2 & x \end{vmatrix}$$

$$\boxed{= \langle -2z,\ -1,\ -2y\rangle}$$

S is the disk $\boldsymbol{t} = \langle 3r\cos t,\ 4r\cos t,\ 5r\sin t\rangle$

$\boldsymbol{t}_r = \langle 3\cos t,\ 4\cos t,\ 5\sin t\rangle$ & $\boldsymbol{t}_t = \langle -3r\sin t,\ -4r\sin t,\ 5r\cos t\rangle$

$$\vec{n} = \boldsymbol{t}_r \times \boldsymbol{t}_t = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 3\cos t & 4\cos t & 5\sin t \\ -3r\sin t & -4r\sin t & 5r\cos t \end{vmatrix}$$

$$\boxed{= \langle 20r,\ -15r,\ 0\rangle}$$

$$\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS = \iint_R \langle -2z,\ -1,\ -2y\rangle \cdot \langle 20r,\ -15r,\ 0\rangle\, dA$$

$$= \int_0^{2\pi}\int_0^1 (-40rz + 15r)\ dr\ dt$$

$$= \int_0^{2\pi} \int_0^1 (-200r\sin t + 15r)\, dr\, dt$$

$$= \int_0^{2\pi} \left(-100r^2 \sin t + \frac{15}{2}r^2\right)\Big|_0^1\, dt$$

$$= \int_0^{2\pi} \left(-100\sin t + \frac{15}{2}\right)\, dt$$

$$= \left[100\cos t + \frac{15}{2}t\right]_0^{2\pi}$$

$$= 100 + 15\pi - 100$$

$$= 15\pi$$

Exercise

Evaluate the line integral $\oint_C \vec{F} \bullet d\vec{r}$ by evaluating the surface integral in Stokes' Theorem with an appropriate choice of S. Assume that C has a counterclockwise orientation,

$\vec{F} = \left\langle 2xy\sin z,\ x^2 \sin z,\ x^2 y\cos z \right\rangle$; C is the boundary of the plane $z = 8 - 2x - 4y$ in the first octant.

Solution

$$\nabla \times \vec{F} = \nabla \times \left\langle 2xy\sin z,\ x^2 \sin z,\ x^2 y\cos z \right\rangle$$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy\sin z & x^2 \sin z & x^2 y\cos z \end{vmatrix}$$

$$= \left\langle x^2 \cos z - x^2 \cos z,\ 2xy\cos z - 2xy\cos z,\ 2x\sin z - 2x\sin z \right\rangle$$

$$= \langle 0,\ 0,\ 0 \rangle$$

$$\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\, dS = 0$$

Exercise

Evaluate the line integral $\oint_C \vec{F} \cdot d\vec{r}$ using Stokes' Theorem, where $\vec{F} = \langle xz,\ yz,\ xy \rangle$; *C*: is the circle $x^2 + y^2 = 4$ in the *xy*-plane. Assume *C* has counterclockwise orientation.

Solution

$$\vec{r}(t) = \langle 2\cos t,\ 2\sin t,\ 0 \rangle$$

$$\vec{r}'(t) = \langle -2\sin t, 2\cos t,\ 0 \rangle$$

$$\begin{aligned}\vec{F} &= \langle xz,\ yz,\ xy \rangle \\ &= \langle 0,\ 0,\ 4\cos t \sin t \rangle\end{aligned}$$

$$\begin{aligned}\oint_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \langle 0,\ 0,\ 4\cos t \sin t \rangle \cdot \langle -2\sin t,\ 2\cos t,\ 0 \rangle\ dt \\ &= 0\end{aligned}$$

Exercise

Evaluate the line integral $\oint_C \vec{F} \cdot d\vec{r}$ using the Stoke's Theorem $\vec{F} = \left\langle x^2 - y^2,\ x,\ 2yz \right\rangle$; *C* is the boundary of the plane $z = 6 - 2x - y$ in the first octant and has counterclockwise orientation.

Solution

$$2x + y + z = 6 \quad \to \quad \vec{n} = \langle 2,\ 1,\ 1 \rangle$$

$$\begin{aligned}\nabla \times \vec{F} &= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & x & 2yz \end{vmatrix} \\ &= \langle 2z,\ 0,\ 1 + 2y \rangle\end{aligned}$$

$$z = 6 - 2x - y = 0 \quad \to \quad 0 \le y \le 2x - 6$$

$$y = 2x - 6 = 0 \quad \to \quad 0 \le x \le 3$$

$$\begin{aligned}\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ dS &= \iint_R \langle 2z,\ 0,\ 1 + 2y \rangle \cdot \langle 2,\ 1,\ 1 \rangle\, dA \\ &= \iint_R \langle 12 - 4x - 2y,\ 0,\ 1 + 2y \rangle \cdot \langle 2,\ 1,\ 1 \rangle\, dA\end{aligned}$$

$$= \iint_R (24 - 8x - 4y + 1 - 2y)\, dA$$

$$= \int_0^3 \int_0^{6-2x} (25 - 8x - 2y)\, dydx$$

$$= \int_0^3 \left(25y - 8xy - y^2\right)\Big|_0^{6-2x}\, dx$$

$$= \int_0^3 \left(150 - 50x - 48x + 16x^2 - \left(36 - 24x + 4x^2\right)\right) dx$$

$$= \int_0^3 \left(114 - 74x + 12x^2\right) dx$$

$$= 114x - 37x^2 + 4x^3 \Big|_0^3$$

$$\underline{= 117\,}|$$

Exercise

Evaluate the line integral $\oint_C \vec{F} \bullet d\vec{r}$ by evaluating the surface integral in Stokes' Theorem with an appropriate choice of S. Assume that C has a counterclockwise orientation

$\vec{F} = \left\langle x^2 - y^2,\ z^2 - x^2,\ y^2 - z^2 \right\rangle$; C is the boundary of the square $|x| \le 1,\quad |y| \le 1$ in the plane $z = 0$

Solution

Square bounded by $|x| \le 1,\quad |y| \le 1$, then $\vec{n} = \langle 0,\ 0,\ 1\rangle$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & z^2 - x^2 & y^2 - z^2 \end{vmatrix}$$

$$= \langle 2y - 2z,\ 0,\ -2x + 2y\rangle$$

$$\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS = \iint_R \langle 2y - 2z,\ 0,\ -2x + 2y\rangle \bullet \langle 0,\ 0,\ 1\rangle\ dA$$

$$= \iint_R (2y - 2x)\ dA$$

$$= \int_{-1}^{1}\int_{-1}^{1}(2y-2x)\,dydx$$

$$= \int_{-1}^{1}\left(y^2-2xy\right)\Big|_{-1}^{1}\,dx$$

$$= \int_{-1}^{1}(1-2x-1+2x)\,dx$$

$$\underline{=0}$$

Exercise

Evaluate the line integral in Stokes' Theorem to evaluate the surface integral $\iint_S \left(\nabla\times\vec{F}\right)\cdot\vec{n}\ dS$.

Assume that $\vec{n}$ points in an upward direction,

$\vec{F}=\langle x,\ y,\ z\rangle$; S is the upper half of the ellipsoid $\frac{x^2}{4}+\frac{y^2}{9}+z^2=1$

Solution

$$\nabla\times\vec{F}=\nabla\times\langle x,\ y,\ z\rangle$$

$$=\begin{vmatrix}\hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z\end{vmatrix}$$

$$\underline{=\langle 0,\ 0,\ 0\rangle}$$

$$\iint_S \left(\nabla\times\vec{F}\right)\cdot\vec{n}\ dS\ \underline{=0}$$

Let $z=0 \quad\to\quad \frac{x^2}{4}+\frac{y^2}{9}=1$

$\vec{r}(t)=\langle 2\cos t,\ 3\sin t,\ 0\rangle$

$\vec{r}\,'(t)=\langle -2\sin t,\ 3\cos t,\ 0\rangle$

$\vec{F}=\langle x,\ y,\ z\rangle=\langle 2\cos t,\ 3\sin t,\ 0\rangle$

$$\oint_C \vec{F}\cdot d\vec{r}=\iint_R \langle 2\cos t,\ 3\sin t,\ 0\rangle\cdot\langle -2\sin t,\ 3\cos t,\ 0\rangle\,dA$$

$$=\int_0^{2\pi}(-4\cos t\sin t+9\sin t\cos t)\,dt$$

$$= \int_0^{2\pi} (5\sin t \cos t)\, dt$$

$$= \frac{5}{2}\int_0^{2\pi} \sin 2t \ dt$$

$$= \frac{5}{4}\left[-\cos 2t\right]_0^{2\pi}$$

$$= \frac{5}{2}(-1+1)$$

$$= 0$$

Exercise

Evaluate the line integral in Stokes' Theorem to evaluate the surface integral $\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS$.

Assume that $\vec{n}$ points in an upward direction,

$\vec{F} = \langle 2y,\ -z,\ x-y-z \rangle$; S is the cap of the sphere $x^2+y^2+z^2=25$ *for* $3 \le x \le 5$

Solution

The boundary of the surface is the intersection of the plane $x=3$ and $x^2+y^2+z^2=25$

At $x=3 \quad \to \quad y^2+z^2=16$

$\vec{r}(t) = \langle 3,\ 4\cos t,\ 4\sin t \rangle$

$\vec{r}(t) = \langle 0,\ -4\sin t,\ 4\cos t \rangle$

$\vec{F} = \langle 2y,\ -z,\ x-y-z \rangle$

$\quad = \langle 8\cos t,\ -4\sin t,\ 3-4\cos t-4\sin t \rangle$

$$\oint_C \vec{F} \bullet d\vec{r} = \iint_R \langle 8\cos t,\ -4\sin t,\ 3-4\cos t-4\sin t \rangle \cdot \langle 0,\ -4\sin t,\ 4\cos t \rangle\, dA$$

$$= \int_0^{2\pi} \left(16\sin^2 t + 12\cos t - 16\cos^2 t - 16\sin t\cos t\right) dt \qquad \cos 2t = \cos^2 t - \sin^2 t$$

$$= \int_0^{2\pi} (12\cos t - 16\cos 2t - 8\sin 2t)\, dt$$

$$= \left[12\sin t - 8\sin 2t + 4\cos 2t\right]_0^{2\pi}$$

$$= (0-8+4-0+8-4)$$

$$= 0$$

$\nabla \times \vec{F} = \nabla \times \langle x,\ y,\ z \rangle$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & -z & x-y-z \end{vmatrix}$$

$$\underline{= \langle 0,\ -1,\ -2 \rangle}$$

$$x = 3 \quad \rightarrow \quad \vec{n} = \langle 3,\ 0,\ 0 \rangle$$

$$\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS = \iint_R \langle 0,\ -1,\ -2 \rangle \cdot \langle 3,\ 0,\ 0 \rangle\, dA$$

$$\underline{= 0}$$

Exercise

Evaluate the line integral in Stokes' Theorem to evaluate the surface integral $\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS$.

Assume that $\vec{n}$ points in an upward direction,

$\vec{F} = \langle x+y,\ y+z,\ x+z \rangle$; S is the tilted disk enclosed $\boldsymbol{r}(t) = \left\langle \cos t,\ 2\sin t,\ \sqrt{3}\cos t \right\rangle$

Solution

$$\vec{r}(t) = \left\langle \cos t,\ 2\sin t,\ \sqrt{3}\cos t \right\rangle$$

$$\boldsymbol{r}'(t) = \left\langle -\sin t,\ 2\cos t,\ -\sqrt{3}\sin t \right\rangle$$

$$\vec{F} = \langle x+y,\ y+z,\ x+z \rangle$$

$$= \left\langle \cos t + 2\sin t,\ 2\sin t + \sqrt{3}\cos t,\ \cos t + \sqrt{3}\cos t \right\rangle$$

$$\oint_C \vec{F} \bullet d\vec{r} = \iint_R \left\langle \cos t + 2\sin t,\ 2\sin t + \sqrt{3}\cos t,\ \cos t + \sqrt{3}\cos t \right\rangle \bullet \left\langle -\sin t,\ 2\cos t,\ -\sqrt{3}\sin t \right\rangle dA$$

$$= \int_0^{2\pi} \left(-\cos t \sin t - 2\sin^2 t + 4\cos t \sin t + 2\sqrt{3}\cos^2 t - \sqrt{3}\sin t \cos t - 3\cos t \sin t \right) dt$$

$$= \int_0^{2\pi} \left(-2\sin^2 t + 2\sqrt{3}\cos^2 t - \sqrt{3}\sin t \cos t \right) dt$$

$$= \int_0^{2\pi} \left(-2\left(\frac{1-\cos 2t}{2}\right) + 2\sqrt{3}\left(\frac{1+\cos 2t}{2}\right) - \frac{\sqrt{3}}{2}\sin 2t \right) dt$$

$$= \int_0^{2\pi} \left(-1 + \cos 2t + \sqrt{3} + \sqrt{3}\cos 2t - \frac{\sqrt{3}}{2}\sin 2t \right) dt$$

$$= \left(\left(\sqrt{3} - 1\right)t + \frac{1}{2}\sin 2t + \frac{\sqrt{3}}{2}\sin 2t + \frac{\sqrt{3}}{4}\cos 2t \right)\Big|_0^{2\pi}$$

$$= \left(\sqrt{3} - 1\right)(2\pi) + \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}$$

$$\boxed{= 2\pi\left(\sqrt{3} - 1\right)}$$

$$\nabla \times \vec{F} = \nabla \times \langle x,\ y,\ z \rangle$$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y & y + z & x + z \end{vmatrix}$$

$$\boxed{= \langle -1,\ -1,\ -1 \rangle}$$

S is the disk $\boldsymbol{t} = \left\langle r\cos t,\ 2r\sin t,\ \sqrt{3}r\cos t \right\rangle$

$$\boldsymbol{t}_r = \left\langle \cos t,\ 2\sin t,\ \sqrt{3}\cos t \right\rangle$$

$$\boldsymbol{t}_t = \left\langle -r\sin t,\ 2r\cos t,\ -r\sqrt{3}\sin t \right\rangle$$

$$\vec{n} = \boldsymbol{t}_r \times \boldsymbol{t}_t = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \cos t & 2\sin t & \sqrt{3}\cos t \\ -r\sin t & 2r\cos t & -r\sqrt{3}\sin t \end{vmatrix}$$

$$\boxed{= \left\langle -2r\sqrt{3},\ 0,\ 2r \right\rangle}$$

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ dS = \iint_R \langle -1,\ 0,\ -1 \rangle \cdot \left\langle -2r\sqrt{3},\ -r\sqrt{3},\ 2r \right\rangle dA$$

$$= \int_0^{2\pi} \int_0^1 \left(2r\sqrt{3} - 2r\right)\ dr\ dt$$

$$= \int_0^{2\pi} dt \int_0^1 \left(2r\sqrt{3} - 2r\right)\ dr$$

$$= (2\pi)\left[\sqrt{3}\,r^2 - r^2\right]_0^1$$

$$\boxed{= 2\pi\left(\sqrt{3} - 1\right)}$$

Exercise

Evaluate the line integral in Stokes' Theorem to evaluate the surface integral $\iint_S \left(\nabla\times\vec{F}\right)\cdot\vec{n}\ dS$.

Assume that $\vec{n}$ points in an upward direction

$\vec{F}=\frac{\vec{r}}{|\vec{r}|}$; S is the paraboloid $x=9-y^2-z^2$ *for* $0\le x\le 9$ (excluding its base), and $\vec{r}(t)=\langle x,\ y,\ z\rangle$

Solution

$$x=9-y^2-z^2=0 \quad\to\quad y^2+z^2=9$$

$$\vec{r}(t)=\langle 0,\ 3\cos t,\ 3\sin t\rangle$$

$$d\vec{r}=\langle 0,\ -3\sin t,\ 3\cos t\rangle$$

$$\begin{aligned}\vec{F}&=\frac{\vec{r}}{|\vec{r}|}\\ &=\frac{\langle x,\ y,\ z\rangle}{\sqrt{x^2+y^2+z^2}}\\ &=\frac{1}{3}\langle 0,\ 3\cos t,\ 3\sin t\rangle\end{aligned}$$

$$\begin{aligned}\oint_C \vec{F}\cdot d\vec{r}&=\frac{1}{3}\iint_R \langle 0,\ 3\cos t,\ 3\sin t\rangle\cdot\langle 0,\ -3\sin t,\ 3\cos t\rangle dA\\ &=\frac{1}{3}\int_0^{2\pi}\left(-9\sin t\cos t+9\sin t\cos t\right)dt\\ &=0\end{aligned}$$

Exercise

Use Stoke's Theorem to evaluate the surface integral $\iint_S \left(\nabla\times\vec{F}\right)\cdot\vec{n}\ dS$; $\vec{F}=\langle -z,\ x,\ y\rangle$, where S is the hyperboloid $z=10-\sqrt{1+x^2+y^2}$ *for* $z\ge 0$. Assume that $\vec{n}$ is the *outward normal*.

Solution

$$z=10-\sqrt{1+x^2+y^2}\ge 0$$

$$\sqrt{1+x^2+y^2}=10$$

$$1+x^2+y^2=100$$

$$x^2+y^2=99=r^2 \quad\to\quad r=\sqrt{99}$$

$$\vec{r}(t)=\left\langle \sqrt{99}\cos t,\ \sqrt{99}\sin t,\ 0\right\rangle$$

$$d\vec{r}=\left\langle -\sqrt{99}\sin t,\ \sqrt{99}\cos t,\ 0\right\rangle$$

$$\vec{F} = \langle -z,\ x,\ y\rangle$$
$$= \left\langle 0,\ \sqrt{99}\cos t,\ \sqrt{99}\sin t\right\rangle$$

$$\iint_S \left(\nabla\times\vec{F}\right)\cdot\vec{n}\ dS = \oint_C \vec{F}\cdot d\vec{r}$$
$$= \oint_C \left\langle 0,\ \sqrt{99}\cos t,\ \sqrt{99}\sin t\right\rangle \cdot \left\langle -\sqrt{99}\sin t,\ \sqrt{99}\cos t,\ 0\right\rangle\ dt$$
$$= \int_0^{2\pi} 99\cos^2 t\ dt$$
$$= \frac{99}{2}\int_0^{2\pi} (1+\cos 2t)\ dt$$
$$= \frac{99}{2}\left(t+\frac{1}{2}\sin 2t\right)\Big|_0^{2\pi}$$
$$\boxed{= 99\pi}$$

Exercise

Use Stokes' Theorem to evaluate the surface integral $\iint_S \left(\nabla\times\vec{F}\right)\cdot\vec{n}\ dS$, given $\vec{F} = \left\langle x^2 - z^2,\ y^2,\ xz\right\rangle$, where S is the hemisphere $x^2 + y^2 + z^2 = 4$, for $y \ge 0$. Assume that $\vec{n}$ is the outward normal.

Solution

Let $y = 0 \quad \to \quad x^2 + z^2 = 4$

$\vec{r}(t) = \langle 2\cos t,\ 0,\ 2\sin t\rangle$

$d\vec{r} = \langle -2\sin t,\ 0,\ 2\cos t\rangle$

$$\vec{F} = \left\langle x^2 - z^2,\ y^2,\ xz\right\rangle$$
$$= \left\langle 4\cos^2 t - 4\sin^2 t,\ 0,\ 4\cos t\sin t\right\rangle$$

$$\iint_S \left(\nabla\times\vec{F}\right)\cdot\vec{n}\ dS = \oint_C \vec{F}\cdot d\vec{r}$$
$$= \oint_C \left\langle 4\cos^2 t - 4\sin^2 t,\ 0,\ 4\cos t\sin t\right\rangle \cdot \langle -2\sin t,\ 0,\ 2\cos t\rangle\ dt$$
$$= \int_0^{2\pi} \left(-8\cos^2 t\sin t + 8\sin^3 t + 8\cos^2 t\sin t\right)\ dt$$

$$= 8\int_0^{2\pi} \sin^2 t \sin t \, dt$$

$$= -8\int_0^{2\pi} \left(1-\cos^2 t\right) \, d\left(\cos t\right)$$

$$= 8\left(\frac{1}{3}\cos^3 t - \cos t\right)\Big|_0^{2\pi}$$

$$= 8\left(\frac{1}{3} - 1 - \frac{1}{3} + 1\right)$$

$$= 0$$

Exercise

Use Stokes' Theorem to find the circulation of the following vector fields around any simple closed smooth curve *C*. $\vec{F} = \langle 2x, \ -2y, \ 2z \rangle$

Solution

This is a conservative vector field, and since it is around closed curve, then

$$\oint_C \vec{F} \bullet d\vec{r} = 0 \qquad \text{(for any closed curve)}$$

Exercise

Use Stokes' Theorem to find the circulation of the following vector fields around any simple closed smooth curve *C*. $\vec{F} = \nabla\left(x \sin y e^z\right)$

Solution

This is a conservative vector field, and since it is around closed curve, then

$$\oint_C \vec{F} \bullet d\vec{r} = 0$$

Exercise

Use Stokes' Theorem to find the circulation of the following vector fields around any simple closed smooth curve *C*. $\vec{F} = \left\langle 3x^2 y, \ x^3 + 2yz^2, \ 2y^2 z \right\rangle$

Solution

This is a conservative vector field with $\varphi = x^3 y + y^2 z^2$, and since it is around closed curve, then

$$\boxed{\oint_C \vec{F} \bullet d\vec{r} = 0}$$

Exercise

Use Stokes' Theorem to find the circulation of the following vector fields around any simple closed smooth curve *C*. $\vec{F} = \left\langle y^2z^3,\ 2xyz^3,\ 3xy^2z^2 \right\rangle$

Solution

This is a conservative vector field with $\varphi = xy^2z^3$, and since it is around closed curve, then

$$\boxed{\oint_C \vec{F} \bullet d\vec{r} = 0}$$

Exercise

Use Stokes' Theorem and a surface integral to find the circulation on *C* of the vector field $\vec{F} = \langle -y,\ x,\ 0 \rangle$ as a function of φ. For what value of φ is the circulation a maximum?

Solution

$$\nabla \times \vec{F} = \nabla \times \langle x,\ y,\ z \rangle$$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix}$$

$$\boxed{= \langle 0,\ 0,\ 2 \rangle}$$

$$\boldsymbol{t} = \langle r\cos\varphi\cos t,\ r\sin t,\ r\sin\varphi\cos t \rangle$$

$$\boldsymbol{t}_r = \langle \cos\varphi\cos t,\ \sin t,\ \sin\varphi\cos t \rangle$$

$$\boldsymbol{t}_t = \langle -r\cos\varphi\sin t,\ r\cos t,\ -r\sin\varphi\sin t \rangle$$

$$\vec{n} = \boldsymbol{t}_r \times \boldsymbol{t}_t = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \cos\varphi\cos t & \sin t & \sin\varphi\cos t \\ -r\cos\varphi\sin t & r\cos t & -r\sin\varphi\sin t \end{vmatrix}$$

$$= \left\langle -r\sin\varphi\sin^2 t - r\sin\varphi\cos^2 t,\ 0,\ r\cos\varphi\cos^2 t + r\cos\varphi\sin^2 t \right\rangle$$

$$\boxed{= \langle -r\sin\varphi,\ 0,\ r\cos\varphi \rangle}$$

$$\iint_S \left(\nabla\times\vec{F}\right)\cdot\vec{n}\ dS = \iint_R \langle 0,\ 0,\ 2\rangle\cdot\langle -r\sin\varphi,\ 0,\ r\cos\varphi\rangle\, dA$$

$$= \int_0^{2\pi}\int_0^1 \left(2r\cos\varphi\right)\, dr\ dt$$

$$= \left(2\pi\right)\left[r^2\cos\varphi\right]_0^1$$

$$\underline{= 2\pi\cos\varphi\ |}$$

The maximum value of the circulation when $\cos\varphi = 1 \quad\Rightarrow\quad \varphi = 0$ which is 2π

Exercise

A circle C in the plane $x+y+z=8$ has a radius of 4 and center (2, 3, 3). Evaluate $\oint_C \vec{F}\cdot d\vec{r}$ for $\vec{F} = \langle 0,\ -z,\ 2y\rangle$ where C has a counterclockwise orientation when viewed from above. Does the circulation depend on the radius of the circle? Does it depend on the location of the center of the circle?

Solution

$$\nabla\times\vec{F} = \nabla\times\langle 0,\ -z,\ 2y\rangle$$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -z & 2y \end{vmatrix}$$

$$\underline{= \langle 3,\ 0,\ 0\rangle\ |}$$

$$x+y+z=8 \quad\to\quad \vec{n} = \langle 1,\ 1,\ 1\rangle$$

$$\iint_S \left(\nabla\times\vec{F}\right)\cdot\vec{n}\ dS = \iint_R \langle 3,\ 0,\ 0\rangle\cdot\langle 1,\ 1,\ 1\rangle\, dA$$

$$= \int_0^{2\pi}\int_0^4 (3)\ r dr\ dt$$

$$= \left(2\pi\right)\left[\frac{3}{2}r^2\right]_0^4$$

$$\underline{= 48\pi\ |}$$

Exercise

Begin with the paraboloid $z = x^2 + y^2$, for $0 \le z \le 4$, and slice it with the plane $y = 0$. Let S be the surface that remains for $y \ge 0$ (including the planar surface in the xz-plane). Let C be the semicircle and line segment that bound the cap of S in the plane $z = 4$ with counterclockwise orientation. Let $\vec{F} = \langle 2z + y,\ 2x + z,\ 2y + x \rangle$

a) Describe the direction of the vectors normal to the surface that are consistent with the orientation of C.

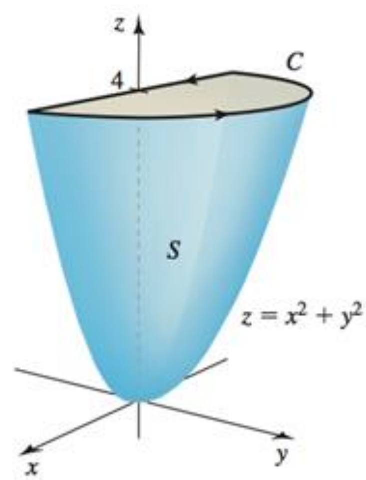
b) Evaluate $\iint_S (\nabla \times \vec{F}) \bullet \vec{n}\ dS$

c) Evaluate $\oint_C \vec{F} \bullet d\vec{r}$ and check for argument with part (b).

Solution

a) The normal vector point toward the z-axis on the curved surface of S and in the direction $\langle 0,\ 1,\ 0 \rangle$ on the flat surface of S.

b) $\nabla \times \vec{F} = \nabla \times \langle 2z + y,\ 2x + z,\ 2y + x \rangle$



$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z + y & 2x + z & 2y + x \end{vmatrix}$$

$$= \langle 1,\ 1,\ 1 \rangle$$

The planar surface in the xz-plane, then let S_1 be the surface parameterized by $\langle x,\ 0,\ z \rangle$.

Where, since $y = 0$,

$z = x^2 + 0^2 \quad \Rightarrow \quad x^2 \le z \le 4$

and $z = 4 = x^2 \quad \Rightarrow \quad -2 \le x \le 0$

$\boldsymbol{t} = \langle x,\ 0,\ z \rangle$

$\boldsymbol{t}_x = \langle 1,\ 0,\ 0 \rangle \quad \& \quad \boldsymbol{t}_z = \langle 0,\ 0,\ 1 \rangle$

$\boldsymbol{n} = \boldsymbol{t}_x \times \boldsymbol{t}_z$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \langle 0,\ -1,\ 0 \rangle$$

$$\iint_S \left(\nabla\times\vec{F}\right)\bullet\vec{n}\ dS = \iint_{S_1} \langle 1,\ 1,\ 1\rangle \bullet \langle 0,\ -1,\ 0\rangle\, dS$$

$$= \int_{-2}^{2}\int_{x^2}^{4} (-1)\ dz\ dx$$

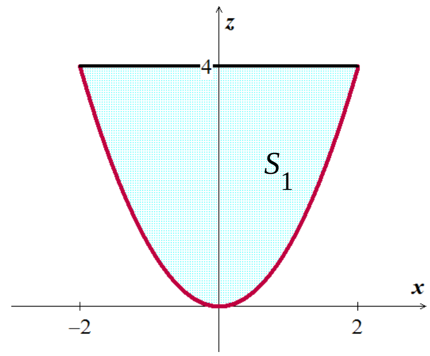
$$= -\int_{-2}^{2} z\Big|_{x^2}^{4}\ dx$$

$$= -\int_{-2}^{2} \left(4-x^2\right) dx$$

$$= -\left(4x-\frac{1}{3}x^3\right)\Big|_{-2}^{2}$$

$$= -\left(8-\frac{8}{3}+8-\frac{8}{3}\right)$$

$$\boxed{= -\frac{32}{3}}$$



Let S_2 be the surface of the half of the paraboloid for $y \ge 0$, parametrized as

$$\boldsymbol{t} = \left\langle r\cos\phi,\ r\sin\phi,\ r^2\right\rangle;\quad 0\le r\le 2;\quad -\pi\le\phi\le 0$$

$$\boldsymbol{t}_r = \langle \cos\phi,\ \sin\phi,\ 2r\rangle$$

$$\boldsymbol{t}_\phi = \langle -r\sin\phi,\ r\cos\phi,\ 0\rangle$$

$$\vec{n} = \boldsymbol{t}_r \times \boldsymbol{t}_\phi$$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \cos\phi & \sin\phi & 2r \\ -r\sin\phi & r\cos\phi & 0 \end{vmatrix}$$

$$\boxed{= \left\langle -2r^2\cos\phi,\ -2r^2\sin\phi,\ r\right\rangle}$$

$$\iint_{S_2} \left(\nabla\times\vec{F}\right)\bullet\vec{n}\ dS = \iint_{S_2} \langle 1,\ 1,\ 1\rangle \bullet \left\langle -2r^2\cos\phi,\ -2r^2\sin\phi,\ r\right\rangle dS$$

$$= \int_{-\pi}^{0}\int_{0}^{2} \left(-2r^2\cos\phi - 2r^2\sin\phi + r\right) dr\,d\phi$$

$$= \int_{-\pi}^{0} \left(-\frac{2}{3}r^3\cos\phi - \frac{2}{3}r^3\sin\phi + \frac{1}{2}r^2\right)\Big|_{0}^{2}\ d\phi$$

$$= \int_{-\pi}^{0} \left(-\frac{16}{3}\cos\phi - \frac{16}{3}\sin\phi + 2\right)\ d\phi$$

$$= \left(-\frac{16}{3}\sin\phi + \frac{16}{3}\cos\phi + 2\phi\right)\Big|_{-\pi}^{0}$$

$$= \frac{16}{3} + \frac{16}{3} + 2\pi$$

$$\boxed{= \frac{32}{3} + 2\pi}$$

$$\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS = \iint_{S_1} \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS + \iint_{S_2} \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS$$

$$= -\frac{32}{3} + \frac{32}{3} + 2\pi$$

$$\boxed{= 2\pi}$$

c) $\displaystyle \oint_C \vec{F} \bullet d\vec{r} = \oint_{C_1} \vec{F} \bullet d\vec{r}_1 + \oint_{C_2} \vec{F} \bullet d\vec{r}_2$

$\vec{F} = \langle 2z + y,\ 2x + z,\ 2y + x \rangle$

$C_1:\ \vec{r}_1 = \langle t,\ 0,\ 4 \rangle = \langle x,\ y,\ z \rangle \quad$ *for* $\quad -2 \le t \le 2$

$\vec{r}_1{}' = \langle 1,\ 0,\ 0 \rangle$

$C_2:\ \vec{r}_2 = \langle 2\cos t,\ 2\sin t,\ 4 \rangle = \langle x,\ y,\ z \rangle \quad$ *for* $\quad -\pi \le t \le 0$

$\vec{r}_2{}' = \langle -2\sin t,\ 2\cos t,\ 0 \rangle$

$$\oint_{C_1} \vec{F} \bullet d\vec{r}_1 = -\int_{-2}^{2} \langle 2z + y,\ 2x + z,\ 2y + x \rangle \bullet \langle 1,\ 0,\ 0 \rangle\, dt$$

$$= -\int_{-2}^{2} (2z + y)\, dt$$

$$= -\int_{-2}^{2} (2(4) + 0)\, dt$$

$$= -\int_{-2}^{2} (8)\, dt$$

$$= -8t\Big|_{-2}^{2}$$

$$\boxed{= -32}$$

$$\oint_{C_2} \vec{F} \bullet d\vec{r}_2 = \int_{-\pi}^{0} \langle 2z + y,\ 2x + z,\ 2y + x \rangle \bullet \langle -2\sin t,\ 2\cos t,\ 0 \rangle\, dt$$

$$= \int_{-\pi}^{0} \langle 8+2\sin t,\ 4\cos t+4,\ 4\sin t+2\cos t\rangle \cdot \langle -2\sin t,\ 2\cos t,\ 0\rangle\, dt$$

$$= \int_{-\pi}^{0} \left(-16\sin t - 4\sin^2 t + 8\cos^2 t + 8\cos t\right) dt \qquad \sin^2 t = 1-\cos^2 t$$

$$= \int_{-\pi}^{0} \left(-16\sin t - 4\left(1-\cos^2 t\right) + 8\cos^2 t + 8\cos t\right) dt$$

$$= \int_{-\pi}^{0} \left(-16\sin t - 4 + 12\cos^2 t + 8\cos t\right) dt \qquad \cos^2 t = \frac{1+\cos 2t}{2}$$

$$= \int_{-\pi}^{0} \left(-16\sin t + 2 + 6\cos 2t + 8\cos t\right) dt$$

$$= \left[16\cos t + 2t + 3\sin 2t + 8\sin t\right]_{-\pi}^{0}$$

$$= 32 + 2\pi$$

$$\oint_C \vec{F} \bullet d\vec{r} = \oint_{C_1} \vec{F} \bullet d\vec{r}_1 + \oint_{C_2} \vec{F} \bullet d\vec{r}_2$$

$$= -32 + 32 + 2\pi$$

$$= 2\pi$$

Exercise

The French Physicist André-Marie Ampère discovered that an electrical current I in a wire produces a magnetic field B. A special case of Ampère's Law relates the current to the magnetic field through the equation $\oint_C \boldsymbol{B} \cdot d\boldsymbol{r} = \mu I$, where C is any closed curve through which the wire passes and μ is a physical constant. Assume that the current I is given in terms of the current density $\boldsymbol{J}$ as $I = \iint_S \boldsymbol{J} \bullet \vec{n}\ dS$, where S is an oriented surface with C as a boundary. Use Stokes' Theorem to show that an equivalent form of Ampère's Law is $\nabla \times \boldsymbol{B} = \mu \boldsymbol{J}$.

Solution

$$\iint_S (\nabla \times B) \bullet \vec{n}\ dS = \oint_C \boldsymbol{B}\vec{r}_\varphi \times \vec{r}_\theta\, d\boldsymbol{r}$$

$$= \mu I$$

$$= \mu \iint_S \boldsymbol{J} \bullet \vec{n}\ dS$$

$$\iint_S (\nabla \times B) \bullet \vec{n}\ dS - \mu \iint_S \boldsymbol{J} \bullet \vec{n}\ dS = 0$$

Thus $$\iint_S \left[(\nabla \times B) - \mu \boldsymbol{J} \right] \bullet \vec{n}\ dS = 0$$

For all surfaces *S* bounded by any given closed curve *C*.

It is clear that given the freedom to choose *C* and *S*, that it follows that the integrand is identically zero, i.e. that for any surface *S*, $\left((\nabla \times B) - \mu \boldsymbol{J} \right) \bullet \vec{n} \ = 0$.

From this, it is easy to see that we must have $(\nabla \times B) = \mu \boldsymbol{J}$, since we are free to make normal vector point in any direction at any given point by choosing *S* appropriately.

Exercise

Let S be the paraboloid $z = a\left(1 - x^2 - y^2\right)$, for $z \ge 0$, where *a* > 0 is a real number. Let $\boldsymbol{F} = \langle x - y,\ y + z,\ z - x \rangle$. For what value(s) of *a* (if any) does $\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS$ have its maximum value?

Solution

$$For \quad z = a\left(1 - x^2 - y^2\right) = 0 \quad \Rightarrow \quad x^2 + y^2 = 1$$

$$\vec{r}(t) = \langle \cos t,\ \sin t,\ 0 \rangle$$

$$\vec{r}\,' = \langle -\sin t,\ \cos t,\ 0 \rangle$$

$$\begin{aligned}\vec{F} &= \langle x - y,\ y + z,\ z - x \rangle \\ &= \langle \cos t - \sin t,\ \sin t,\ -\cos t \rangle\end{aligned}$$

$$\begin{aligned}\oint_C \vec{F} \bullet d\vec{r} &= \int_0^{2\pi} \langle \cos t - \sin t,\ \sin t,\ -\cos t \rangle \bullet \langle -\sin t,\ \cos t,\ 0 \rangle\ dt \\ &= \int_0^{2\pi} \left(-\cos t \sin t + \sin^2 t + \cos t \sin t \right)\ dt \\ &= \int_0^{2\pi} \sin^2 t\ dt \\ &= \frac{1}{2} \int_0^{2\pi} (1 - \cos 2t)\ dt\end{aligned}$$

$$= \frac{1}{2}\left[t - \frac{1}{2}\sin 2t\right]_0^{2\pi}$$

$$\underline{= \pi}$$

$\therefore$ The integral is independent of a.

Exercise

The goal is to evaluate $A = \iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS$, where $\vec{F} = \langle yz,\ -xz,\ xy\rangle$ and S ids the surface of the upper half of the ellipsoid $x^2 + y^2 + 8z^2 = 1\ \ (z \ge 0)$

a) Evaluate a surface integral over a more convenient surface to find the value of A.

b) Evaluate A using a line integral.

Solution

a) The boundary of this surface is the circle $x^2 + y^2 = 0$ at $z = 0$

$$\nabla \times \vec{F} = \nabla \times \langle yz,\ -xz,\ xy\rangle$$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -xz & xy \end{vmatrix}$$

$$\underline{= \langle 2x,\ 0,\ -2z\rangle}$$

$$\nabla \times \vec{F}\Big|_{z=0} \ \ \underline{= \langle 2x,\ 0,\ 0\rangle}$$

At $z = 0 \ \ \rightarrow \ \ \vec{n} = \langle 0,\ 0,\ 1\rangle$

$$\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS = \iint_S \langle 2x,\ 0,\ 0\rangle \bullet \langle 0,\ 0,\ 1\rangle\ dS$$

$$= \iint_S (0)\ dS$$

$$\underline{= 0}$$

b) With the parameterization of the boundary circle and $z = 0$, we have

$$\oint_C \vec{F} \bullet d\vec{r} = \int_0^{2\pi} \langle 0,\ 0,\ 1\rangle \bullet \langle -\sin t,\ \cos t,\ 0\rangle\ dt$$

$$= \int_0^{2\pi} 0\ dt$$

$$\underline{= 0}$$

Exercise

Let $\vec{F} = \langle 2z,\ z,\ \ x+2y \rangle$ and let S be the hemisphere of radius a with its base in the xy-plane and center at the origin.

a) Evaluate $\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS$ by computing $\nabla \times \vec{F}$ and appealing to symmetry.

b) Evaluate the line integral using Stokes' Theorem to check part (*a*).

Solution

a) $\nabla \times \vec{F} = \nabla \times \langle 2z,\ z,\ \ x+2y \rangle$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & z & x+2y \end{vmatrix}$$

$$= \langle 1,\ 1,\ \ 0 \rangle$$

$S:\ x^2 + y^2 + z^2 = a^2 \quad \textit{with} \quad z \ge 0$

$2xdx + 2zdz = 0 \quad \rightarrow \quad z_x = -\frac{x}{z}$

$2ydy + 2zdz = 0 \quad \rightarrow \quad z_y = -\frac{y}{z}$

$\vec{n} = \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle$

$$\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS = \iint_S \langle 1,\ 1,\ \ 0 \rangle \bullet \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle\ dS$$

$$= \iint\limits_R \left(\frac{x}{z} + \frac{y}{z} \right)\ dA$$

$$= \iint\limits_R \left(\frac{x+y}{z} \right)\ dA$$

By symmetry, the integral vanishes on each level curve, so it vanishes altogether.

b) Let $z = 0 \quad \rightarrow \quad x^2 + y^2 = a^2$

$\vec{r}(t) = \langle a\cos t,\ a\sin t,\ \ 0 \rangle$

$d\vec{r} = \langle -a\sin t,\ a\cos t,\ \ 0 \rangle$

$\vec{F} = \langle 2z,\ z,\ \ x+2y \rangle$

$$= \langle 0,\ 0,\ a\cos t + 2a\sin t \rangle$$

$$\iint_S \left(\nabla\times\vec{F}\right)\bullet\vec{n}\ dS = \oint_C \vec{F}\bullet d\vec{r}$$

$$= \oint_C \left\langle 0,\ 0,\ a\cos t + 2a\sin t\right\rangle \bullet \left\langle -a\sin t,\ a\cos t,\ 0\right\rangle\ dt$$

$$\underline{=0}$$

Exercise

Let S be the disk enclosed by the curve $C:\ \vec{r}(t)=\langle\cos\varphi\cos t,\ \sin t,\ \sin\varphi\cos t\rangle$ fot $0\le t\le 2\pi$, where $0\le\varphi\le\frac{\pi}{2}$ is a fixed angle.

a) Find the a vector normal to *S*.

b) What is the areas of *S*?

c) Whant the length of *C*?

d) Use the Stokes' Theorem and a surface integral to find the ciurculation on *C* of the vector field $\vec{F}=\langle -y,\ x,\ 0\rangle$ as a function of φ. For what value of φ is the circulation a maximum?

e) What is the circulation on *C* of the vector field $\vec{F}=\langle -y,\ -z,\ x\rangle$ as a function of φ? For what value of φ is the circulation a maximum?

f) Consider the vector field $\vec{F}=\vec{a}\times\vec{r}$, where $\vec{a}=\langle a_1,\ a_2,\ a_3\rangle$ is a constant nonzero vector and $\vec{r}=\langle x,\ y,\ z\rangle$. Show that the circulation is a maximum when $\vec{a}$ points in the direection of the normal to *S*.

Solution

a) $\vec{r}(t)=\langle r\cos\varphi\cos t,\ r\sin t,\ r\sin\varphi\cos t\rangle$

$\boldsymbol{t}_r = \langle\cos\varphi\cos t,\ \sin t,\ \sin\varphi\cos t\rangle$

$\boldsymbol{t}_t = \langle -r\cos\varphi\sin t,\ r\cos t,\ -r\sin\varphi\sin t\rangle$

$$\boldsymbol{t}_\varphi\times\boldsymbol{t}_t = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \cos\varphi\cos t & \sin t & \sin\varphi\cos t \\ -r\cos\varphi\sin t & r\cos t & -r\sin\varphi\sin t \end{vmatrix}$$

$$\begin{aligned} &= \Big\langle -r\sin\varphi\sin^2 t - r\sin\varphi\cos^2 t, \\ &\qquad -r\sin\varphi\cos\varphi\cos t\sin t + r\sin\varphi\cos\varphi\cos t\sin t, \\ &\qquad r\cos\varphi\cos^2 t + r\cos\varphi\sin^2 t\Big\rangle \\ &= \left\langle -r\sin\varphi\left(\sin^2 t+\cos^2 t\right),\ 0,\ r\cos\varphi\left(\cos^2 t\sin^2 t\right)\right\rangle \\ &= \langle -r\sin\varphi,\ 0,\ r\cos\varphi\rangle \end{aligned}$$

$$\vec{n} = \boldsymbol{t}_\varphi \times \boldsymbol{t}_t \ \underline{= \langle -r\sin\varphi,\ 0,\ r\cos\varphi \rangle}$$

b) $$\left|\boldsymbol{t}_r \times \boldsymbol{t}_t\right| = \sqrt{r^2 \sin^2\varphi + r^2\cos^2\varphi}$$

$$\underline{= r}$$

$$Area = \int_0^{2\pi}\int_0^1 \left|\boldsymbol{t}_r \times \boldsymbol{t}_t\right| drdt \qquad\qquad Surface\ Area = \iint_S 1\, dS$$

$$= \int_0^{2\pi} dt \int_0^1 r\, dr$$

$$= (2\pi)\left(\frac{1}{2}r^2\right)\Big|_0^1$$

$$\underline{= \pi}$$

(this surface is simply the unit circle inclined at the angle φ to the xy-plane)

c) $$\vec{r}(t) = \langle \cos\varphi\cos t,\ \sin t,\ \sin\varphi\cos t \rangle$$

$$\vec{r}'(t) = \langle -\cos\varphi\sin t,\ \cos t,\ -\sin\varphi\sin t \rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{\cos^2\varphi\sin^2 t + \cos^2 t + \sin^2\varphi\sin^2 t}$$

$$= \sqrt{\left(\cos^2\varphi + \sin^2\varphi\right)\sin^2 t + \cos^2 t}$$

$$= \sqrt{\sin^2 t + \cos^2 t}$$

$$\underline{= 1}$$

$$L = \int_0^{2\pi} 1 dt$$

$$\underline{= 2\pi}$$

(Because it just the circumference of the unit circle)

d) $$\vec{F} = \langle -y,\ x,\ 0 \rangle$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix}$$

$$= \langle 0,\ 0,\ 2 \rangle$$

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\, dS = \iint_R \langle 0,\ 0,\ 2 \rangle \cdot \langle -r\sin\varphi,\ 0,\ r\cos\varphi \rangle\, dA$$

$$= \int_0^{2\pi} \int_0^1 2r\cos\varphi \; drdt$$

$$= \cos\varphi \int_0^{2\pi} dt \int_0^1 2r \; dr$$

$$= 2\pi\cos\varphi \; \left(r^2\right)\Big|_0^1$$

$$\underline{= 2\pi\cos\varphi}$$

The maximum when $\cos\varphi = 1 \quad \rightarrow \quad \underline{\varphi = 0}$

The circulation has a maximum of 2π at $\varphi = 0$.

e) $\vec{r}(t) = \langle \cos\varphi\cos t, \; \sin t, \; \sin\varphi\cos t \rangle$

$\vec{r}'(t) = \langle -\cos\varphi\sin t, \; \cos t, \; -\sin\varphi\sin t \rangle$

$\vec{F} = \langle -y, \; -z, \; x \rangle$

$= \langle -\sin t, \; -\sin\varphi\cos t, \; \cos\varphi\cos t \rangle$

$\vec{F} \cdot d\vec{r} = \langle -\sin t, \; -\sin\varphi\cos t, \; \cos\varphi\cos t \rangle \cdot \langle -\cos\varphi\sin t, \; \cos t, \; -\sin\varphi\sin t \rangle$

$= \cos\varphi\sin^2 t - \sin\varphi\cos^2 t - \cos\varphi\cos t\sin\varphi\sin t$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(\cos\varphi\sin^2 t - \sin\varphi\cos^2 t - \cos\varphi\cos t\sin\varphi\sin t \right) dt$$

$$= \frac{1}{2}\cos\varphi \int_0^{2\pi} (1 - \cos 2t)\, dt - \frac{1}{2}\sin\varphi \int_0^{2\pi} (1 + \cos 2t)\, dt$$

$$+ \cos\varphi\sin\varphi \int_0^{2\pi} \cos t \; d(\cos t)$$

$$= \frac{1}{2}\cos\varphi \left(t - \frac{1}{2}\sin 2t \right)\Big|_0^{2\pi} - \frac{1}{2}\sin\varphi \left(t + \frac{1}{2}\sin 2t \right)\Big|_0^{2\pi} + \frac{1}{2}\cos\varphi\sin\varphi\cos^2 t\Big|_0^{2\pi}$$

$$= \pi\cos\varphi - \pi\sin\varphi + \frac{1}{2}\cos\varphi\sin\varphi(1-1)$$

$$\underline{= \pi(\cos\varphi - \sin\varphi)}$$

The maximum when $\cos\varphi - \sin\varphi = 1 \; \rightarrow \; \varphi = 0, \; \frac{3\pi}{2}$

The maximum circulation is π at $\varphi = 0$.

f) $\vec{F} = \vec{a} \times \vec{r} \qquad \vec{a} = \langle a_1, \; a_2, \; a_3 \rangle$

$= \langle a_1, \; a_2, \; a_3 \rangle \times \langle x, \; y, \; z \rangle$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= \langle a_2 z - a_3 y,\ a_3 x - a_1 z,\ a_1 y - a_2 x \rangle$$

$$\nabla \times (\vec{a} \times \vec{r}) = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$= \langle 2a_1,\ 2a_2,\ 2a_3 \rangle$$

$$\vec{r}(t) = \langle r\cos\varphi\cos t,\ r\sin t,\ r\sin\varphi\cos t \rangle$$

$$\vec{n} = \langle -r\sin\varphi,\ 0,\ r\cos\varphi \rangle$$

$$\oint_C \vec{F} \bullet d\vec{r} = \iint_S \langle 2a_1,\ 2a_2,\ 2a_3 \rangle \bullet \langle -r\sin\varphi,\ 0,\ r\cos\varphi \rangle \ dS$$

$$= \int_0^{2\pi} \int_0^1 \left(-2a_1 r\sin\varphi + 2a_3 r\cos\varphi \right) drdt$$

$$= 2\int_0^{2\pi} dt \int_0^1 \left(a_3 \cos\varphi - a_1 \sin\varphi \right) r \ dr$$

$$= (2\pi)\left(a_3 \cos\varphi - a_1 \sin\varphi \right) r^2 \Big|_0^1$$

$$\boxed{= 2\pi\left(a_3 \cos\varphi - a_1 \sin\varphi \right)}$$

When $\vec{a}$ points in the direection of the normal to S their cross-product is zero.

$$\langle a_1,\ a_2,\ a_3 \rangle \times \langle -r\sin\varphi,\ 0,\ r\cos\varphi \rangle = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ -r\sin\varphi & 0 & r\cos\varphi \end{vmatrix}$$

$$= \left\langle ra_2 \cos\varphi,\ -r\left(a_3 \sin\varphi + a_1 \cos\varphi \right),\ ra_2 \sin\varphi \right\rangle = 0$$

$$\left\langle a_2 \cos\varphi,\ \left(a_3 \sin\varphi + a_1 \cos\varphi \right),\ a_2 \sin\varphi \right\rangle = 0$$

$\boxed{a_2 = 0}$ & $\boxed{a_3 \cos\varphi - a_1 \sin\varphi = 0}$

Exercise

Let R be a region in a plane that has a unit normal vector $\vec{n} = \langle a,\ b,\ c\rangle$ and boundary C. Let $\vec{F} = \langle bz,\ cx,\ ay\rangle$

a) Show that $\nabla\times\vec{F} = \vec{n}$

b) Use Stokes' Theorem to show that

$$Area\ of\ R\ = \oint_C \vec{F}\bullet d\vec{r}$$

c) Consider the curve C given by $\vec{r}(t) = \langle 5\sin t,\ 13\cos t,\ 12\sin t\rangle$, for $0 \le t \le 2\pi$. Prove that C lies in a plane by showing that $\vec{r}\times\vec{r}\,'$ is constant for all t.

d) Use part (b) to find the area of the region enclosed by C in part (c). (*Hint*: Find the unit normal vector that is consistent with the orientation of C.)

Solution

a) $\nabla\times\vec{F} = \nabla\times\langle bz,\ cx,\ ay\rangle$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bz & cx & ay \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}(ay) - \frac{\partial}{\partial z}(cx),\ \frac{\partial}{\partial z}(bz) - \frac{\partial}{\partial x}(ay),\ \frac{\partial}{\partial x}(cx) - \frac{\partial}{\partial y}(bz)\right\rangle$$

$$= \langle a,\ b,\ c\rangle$$

$$\underline{= \vec{n}\ |} \qquad \checkmark$$

b) $$Area\ of\ R\ = \iint_S \left(\nabla\times\vec{F}\right)\bullet\vec{n}\ dS$$

$$= \iint_S \vec{n}\bullet\vec{n}\ dS$$

$$= \iint_R |\vec{n}|^2\ dA \qquad \text{Since } |\vec{n}| = 1$$

$$= \iint_R dA$$

$$= Area\ of\ R$$

$$= \oint_C \vec{F}\bullet d\vec{r} \qquad \checkmark$$

c) $\vec{r}(t) = \langle 5\sin t,\ 13\cos t,\ 12\sin t\rangle$

$$\vec{r}'(t) = \langle 5\cos t,\ -13\sin t,\ 12\cos t\rangle$$

$$\vec{r}\times\vec{r}' = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 5\sin t & 13\cos t & 12\sin t \\ 5\cos t & -13\sin t & 12\cos t \end{vmatrix}$$

$$= \left\langle 156\cos^2 t + 156\sin^2 t,\ 70\cos t\sin t - 70\cos t\sin t,\ -65\sin^2 t - 65\cos^2 t\right\rangle$$

$$= \left\langle 156\left(\cos^2 t + \sin^2 t\right),\ 0,\ -65\left(\sin^2 t + \cos^2 t\right)\right\rangle$$

$$\boxed{= \langle 156,\ 0,\ -65\rangle}$$

$\therefore\ \vec{r}\times\vec{r}'$ is constant for all t, so that $\vec{r}$ must lie in a plane.

d) $\vec{r}\times\vec{r}' = \langle 156,\ 0,\ -65\rangle$

$$\left|\vec{r}\times\vec{r}'\right| = \sqrt{156^2 + 65^2}$$

$$= \sqrt{28,561}$$

$$\boxed{= 169}$$

$$\vec{n} = \frac{\vec{r}\times\vec{r}'}{\left|\vec{r}\times\vec{r}'\right|}$$

$$= \frac{1}{169}\langle 156,\ 0,\ -65\rangle$$

$$= \left\langle \frac{12}{13}\ 0,\ -\frac{5}{13}\right\rangle$$

$$\boxed{a = \frac{12}{13},\quad b = 0,\quad c = -\frac{5}{13}}$$

$$\vec{F} = \langle bz,\ cx,\ ay\rangle$$

$$= \left\langle 12(0)\sin t,\ 5\left(-\frac{5}{13}\right)\sin t,\ 13\left(\frac{12}{13}\right)\cos t\right\rangle$$

$$= \left\langle 0,\ \frac{25}{13}\sin t,\ 12\cos t\right\rangle$$

$$\vec{r}'(t) = \langle 5\cos t,\ -13\sin t,\ 12\cos t\rangle$$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left\langle 0,\ \frac{25}{13}\sin t,\ 12\cos t\right\rangle \cdot \langle 5\cos t,\ -13\sin t,\ 12\cos t\rangle\ dt$$

$$= \int_0^{2\pi} \left(25\sin^2 t + 144\cos^2 t\right)\ dt$$

$$= \int_0^{2\pi} \left(\frac{25}{2} - \frac{1}{2}\cos 2t + 72 + \frac{1}{2}\cos 2t\right)\ dt$$

$$= \int_0^{2\pi} \frac{169}{2}\, dt$$

$$= \frac{169}{2} t \Big|_0^{2\pi}$$

$$\underline{= 169\pi}$$

Exercise

Consider the radial vector fields $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^p}$, where p is a real number and $\vec{r} = \langle x,\ y,\ z \rangle$. Let C be any circle in the xy-plane centered at the origin.

a) Evaluate a line integral to show that the field has zero circulation on C.

b) For what values of p does Stokes' Theorem apply? For those values of p, use the surface integral in Stokes' Theorem to show that the field has zero circulation on C.

Solution

a) Let $C: \quad x^2 + y^2 = a^2$

$$\vec{r}(t) = \langle a\cos t,\ a\sin t,\ 0 \rangle$$

$$d\vec{r} = \langle -a\sin t,\ a\cos t,\ 0 \rangle$$

$$\vec{F} = \frac{\vec{r}}{|\vec{r}|^p}$$

$$= \frac{\langle a\cos t,\ a\sin t,\ 0 \rangle}{\left|a^2\cos^2 t + a^2\sin^2 t\right|^{p/2}}$$

$$= \frac{a\langle \cos t,\ \sin t,\ 0 \rangle}{\left|a^2\right|^{p/2}}$$

$$= \frac{a\langle \cos t,\ \sin t,\ 0 \rangle}{a^p}$$

$$= a^{1-p}\langle \cos t,\ \sin t,\ 0 \rangle$$

$$\oint_C \vec{F} \bullet d\vec{r} = a^{1-p}\int_0^{2\pi} \langle \cos t,\ \sin t,\ 0 \rangle \bullet \langle -a\sin t,\ a\cos t,\ 0 \rangle\, dt$$

$$= a^{2-p}\int_0^{2\pi} \left(-\cos t\sin t + \sin t\cos t\right)\, dt$$

$$\underline{= 0}$$

b) Stokes' Theorem will apply when the vector field is defined throughout the disk of radius a, which happens only $p \le 0$.

In this case, $\nabla \times \vec{F} = a^{-p}\langle 0,\ 0,\ 0\rangle$, so that the surface integral is zero.

Exercise

Consider the vector fierld $\vec{F} = \dfrac{-y}{x^2+y^2}\hat{\boldsymbol{i}} + \dfrac{x}{x^2+y^2}\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$

a) Show that $\nabla \times \vec{F} = \vec{0}$

b) Show that $\oint_C \vec{F} \bullet d\vec{r}$ is not zero on circle C in the xy-plane enclosing the origin.

c) Explain why Stokes' Theorem does not apply in this case.

Solution

a) $$\nabla \times \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & z \end{vmatrix}$$

$$= \left\langle 0,\ 0,\ \frac{y^2-x^2}{\left(x^2+y^2\right)^2} + \frac{x^2-y^2}{\left(x^2+y^2\right)^2} \right\rangle$$

$$= \langle 0,\ 0,\ 0\rangle \quad \checkmark$$

b) Let $C:\ \ x^2+y^2=1$

$\vec{r}(t) = \langle \cos t,\ \sin t,\ 0\rangle$

$d\vec{r} = \langle -\sin t,\ \cos t,\ 0\rangle$

$$\vec{F} = \left\langle \frac{-y}{x^2+y^2},\ \frac{x}{x^2+y^2},\ z \right\rangle$$

$$= \langle -\sin t,\ \cos t,\ 0\rangle$$

$$\oint_C \vec{F} \bullet d\vec{r} = \int_0^{2\pi} \langle -\sin t,\ \cos t,\ 0\rangle \bullet \langle -\sin t,\ \cos t,\ 0\rangle\ dt$$

$$= \int_0^{2\pi} \left(\sin^2 t + \cos^2 t\right)\ dt$$

$$= \int_0^{2\pi} dt$$

$$\underline{= 2\pi}$$

c) The Theroem does not apply because the vector field is not defined at the origin,, which is inside the curve *C*.
The limit of the *y*-coordinate is different depending on the direction.

Exercise

Let *S* be a small circular disk of radius *R* centered at the point *P* with a unit normal vector $\vec{n}$. Let *C* be the boundary of *S*.

a) Express the average circulation of the vector field $\vec{F}$ on *S* as a surface integral of $\nabla \times \vec{F}$

b) Argue for that small *R*, the average circulation approaches $\left(\nabla \times \vec{F}\right)\Big|_P \cdot \vec{n}$ (the component of $\nabla \times \vec{F}$ in the direction of $\vec{n}$ evaluated at *P*) with the approximation improving as $R \to 0$.

Solution

a) The circumference of the disk is $2\pi R$, so the average circulation is

$$\frac{1}{2\pi R} \iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ dS$$

b) As *R* becomes small, because the vector field $\vec{F}$ and thus $\nabla \times \vec{F}$ are continuous.
$\nabla \times \vec{F}$ can be made arbitrarily close to $\left(\nabla \times \vec{F}\right)\Big|_P$ everywhere on *S* by taking *R* small enough.
Approximately, then

$$\left(\nabla \times \vec{F}\right) \cdot \vec{n} \approx \left(\nabla \times \vec{F}\right)\Big|_P \cdot \vec{n}$$

So that

$$\begin{aligned} \frac{1}{2\pi R} \iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ dS &\approx \frac{1}{2\pi R} \iint_S \left(\nabla \times \vec{F}\right)_P \cdot \vec{n}\ dS \\ &= \frac{1}{2\pi R}\left(\nabla \times \vec{F}\right)_P \cdot \vec{n} \iint_S 1\ dS \\ &= \left(\nabla \times \vec{F}\right)_P \cdot \vec{n} \end{aligned}$$

As $R \to 0$, the approximation $\nabla \times \vec{F}$ becomes better, so the value of the integral does as well.