Linear 9/8

Matrix Multiplication

$$\underset{A}{m \times n} \quad \text{inner} = \quad \underset{B}{n \times k}$$

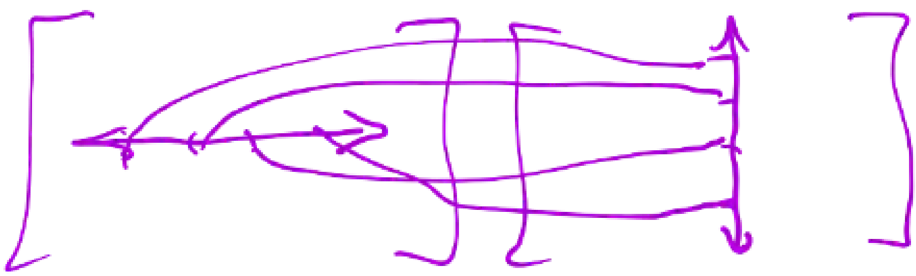
1st

outer: $m \times k$

$AB =$

- AB or dot product
  A: n col $\qquad B$: n rows.
- Square matrices can be multiplied iff have same size
- (row i of A) (col j of B)

  $$\sum a_{ik} b_{kj}$$



Ex $AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$= \begin{pmatrix} ae + bg & af + b \\ ce + dg & cf + dh \end{pmatrix}$$

Ex $\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 5+1 & 6+0 \\ 10-1 & 12-0 \end{pmatrix}$

$$= \begin{pmatrix} 6 & 6 \\ 9 & 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{2\times 1} \underbrace{\begin{bmatrix} 3 & 2 \end{bmatrix}}_{1\times 2} + \begin{bmatrix} 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 5 & 0 \end{bmatrix}$$

Identity matrix I $\underline{n \times n}$ (square matrix)

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow IA = A$$

∴ <u>note</u> matrix 1's: $n \times m$ all <u>one</u>

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$A + B = B + A$

$A + (B + C) = (A + B) + C$

$k(lA) = (kl)A$

$k(A \pm B) = kA \pm kB$

$(k + l)A = kA + lA$

Ak $\quad A^k$

$\boxed{kA}$ k times A

$A + 0 = 0 + A = A$

$A + (-A) = (-A) + A = 0 \to$ matrix

$AB \neq BA$

<u>Ex</u> $A_{2\times 3}$ $B_{3\times 3}$

$\underset{2\times 3\ \ 3\times 3}{AB} = 2\times 3$ $\underset{3\times 3\ \ 2\times 3}{BA}$ $\nexists$

$\vec{u} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = (1, -1, 0)$

$\boxed{\neq (1 \quad -1 \quad 0)}$

<u>1.4</u> A^{-1}

<u>Defn</u> matrix A is invertible, A^{-1} $\exists$

$\boxed{A \text{ invertible} \Rightarrow AA^{-1} = A^{-1}A = I}$

$a \cdot a^{-1} = a\frac{1}{a} = 1$

→ A has to be square matrix

Not All matrices have inverses.

- Inverse $\exists$ iff n pivots
- A cannot have 2 different inverses.
- A invertible $\Rightarrow$ $AX = B$

  $X = A^{-1}B$

Proof

$$\overset{\text{Given}}{AX = B}$$

A is invertible $\Rightarrow AA^{-1} = A^{-1}A = I$

$$A^{-1}(AX) = A^{-1}(B)$$

$$(A^{-1}A)X = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B \checkmark$$

A invertible $\Rightarrow AA^{-1} = A^{-1}A = I$

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if $\underbrace{Ax = 0}_{\text{homogeneous.}} \Rightarrow A^{-1} \nexists$

$2x + y = 5$

$\rightarrow 2x + y = 0$

Finding A^{-1}?? $AA^{-1} = I$

Only 2×2

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$\therefore$ if $\underbrace{ad - bc}_{\text{determinant.}} = 0 \Rightarrow A^{-1} \nexists$

Can't have 2 different inverses.

Inverse ($\exists$) is unique(!)

Suppose A has A^{-1} and $B \ni BA = I$

Proof

$$\begin{aligned} B &= BI \qquad A \text{ invers } I = AA^{-1} \\ &= B(AA^{-1}) \\ &= (BA)A^{-1} \\ &= IA^{-1} \\ &= A^{-1} \end{aligned}$$

$\therefore$ the inverse is unique.

never use numbers to prove equality

$(AB)^{-1} = B^{-1}A^{-1}$ A & B are invertible

Given: A inv $\Rightarrow AA^{-1} = A^{-1}A = I$

B . . . .

$$\boxed{\begin{matrix} (AB)(AB)^{-1} = \\ M \quad M^{-1} = I \end{matrix}}$$

$$\begin{aligned} (AB)(AB)^{-1} &= (AB)(B^{-1}A^{-1}) \\ &= A(BB^{-1})A^{-1} \qquad B \text{ invertible } BB^{-1} = I \\ &= (AI)A^{-1} \qquad AI = A \\ &= AA^{-1} \qquad A \text{ invertible} \\ &= I \end{aligned}$$

$(AB)^{-1} = B^{-1}A^{-1}$

$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

A is invertible:

$(A^{-1})^{-1} = A$

-1

$$(A^{-1})^n = A^{-n} = (A^n)$$

$$(kA)^{-1} = k^{-1}A^{-1} \boxed{= \frac{1}{k}A^{-1}}$$

$$(kA)(kA)^{-1} = (kA)(k^{-1}A^{-1}) =$$

$$\begin{aligned}(kA)(k^{-1}A^{-1}) &= k^{-1}(kA)A^{-1}\\ &= (k^{-1}k)AA^{-1}\\ &= 1\,(I)\\ &= I \checkmark\end{aligned}$$

$$AA^{-1} = I$$

$$A^{-1}A = I\ ?$$

A^{-1} ?

$$[A \mid I] \xrightarrow[\text{Jordan}]{\text{Gauss-}} [I \mid A^{-1}]$$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{pmatrix} \qquad |A| = 0 \ \nexists A^{-1}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right) -\frac{1}{2}R_2$$

$\cancel{0 - 0 \quad 0}$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right) -\frac{1}{3}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right) \begin{array}{l} R_1 - R_3 \\ R_2 - \frac{3}{2}R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right]$$

$$A^{-1} = \begin{pmatrix} 0 & 0 & 1/3 \\ -1/2 & -1/2 & 1/2 \\ 1 & 0 & -1/3 \end{pmatrix}$$

$A^{-1}\ \exists$ $\quad$ $A\ n \times n$

n-pivots (not)

row $\to$ zero row

matrix is invertible $\Rightarrow$ $\begin{cases} \text{zero row} \\ n\text{-pivots} \end{cases}$

Elementary Matrices

$E - eI$

$$\begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \quad R_2 \text{ of } I \text{ by } -3$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow -5 \times R_3 \text{ of } I$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \overset{\text{add}}{-3} R_1 + R_2 = R_2$$

A & B are equivalent. ($A \neq B$)

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \quad \begin{matrix} R_1 + R_2 \\ R_2 - 2R_1 \end{matrix}$$

$$= \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

$$= B$$

$$A_{2\times 2} \rightarrow A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$A \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$ ← only 2×2

if $ad - bc = 0 \Rightarrow A^{-1} \nexists$

$[A|I] \rightarrow [I|A^{-1}]$

$[A|I] \to [\ \ \ ]$

$\hookrightarrow I \ \nexists \Rightarrow A^{-1}, \nexists$

$\hookrightarrow$ zero row

A is invertible $\Rightarrow AA^{-1} = A^{-1}A = I$

$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$