Solution Section 2.4 – Chain Rule

Exercise

Express $\frac{dw}{dt}$ as a function of t, then evaluate $\frac{dw}{dt}$ at the given value of t.

$w = x^2 + y^2, \quad x = \cos t, \quad y = \sin t, \quad t = \pi$

Solution

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\
&= \frac{\partial}{\partial x}\left(x^2 + y^2\right)\frac{d}{dt}(\cos t) + \frac{\partial f}{\partial y}\left(x^2 + y^2\right)\frac{d}{dt}(\sin t) \\
&= 2x(-\sin t) + 2y\cos t \\
&= -2(\cos t)\sin t + 2(\sin t)\cos t \\
&\underline{= 0}
\end{aligned}$$

$$\begin{aligned}
w &= x^2 + y^2 \\
&= \cos^2 t + \sin^2 t \\
&= 1
\end{aligned}$$

$$\frac{dw}{dt} = 0$$

$$\frac{dw}{dt}(t = \pi) \ \underline{= 0}$$

Exercise

Express $\frac{dw}{dt}$ as a function of t, then evaluate $\frac{dw}{dt}$ at the given value of t.

$w = x^2 + y^2, \quad x = \cos t + \sin t, \quad y = \cos t - \sin t, \quad t = 0$

Solution

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\
&= \frac{\partial}{\partial x}\left(x^2 + y^2\right)\frac{d}{dt}(\cos t + \sin t) + \frac{\partial f}{\partial y}\left(x^2 + y^2\right)\frac{d}{dt}(\cos t - \sin t) \\
&= (2x)(-\sin t + \cos t) + (2y)(-\sin t - \cos t) \\
&= 2(\cos t + \sin t)(\cos t - \sin t) - 2(\cos t - \sin t)(\sin t + \cos t) \\
&\underline{= 0}
\end{aligned}$$

$$\frac{dw}{dt}(t = 0) \ \underline{= 0}$$

Exercise

Express $\frac{dw}{dt}$ as a function of t, then evaluate $\frac{dw}{dt}$ at the given value of t.

$w = \ln\left(x^2 + y^2 + z^2\right), \quad x = \cos t, \quad y = \sin t, \quad z = 4\sqrt{t}, \quad t = 3$

Solution

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \frac{2x}{x^2+y^2+z^2}(-\sin t) + \frac{2y}{x^2+y^2+z^2}(\cos t) + \frac{2z}{x^2+y^2+z^2}\left(2\frac{1}{\sqrt{t}}\right)$$

$$= \frac{-2\cos t \sin t + 2\sin t \cos t + 4\left(4\sqrt{t}\right)\left(t^{-1/2}\right)}{\cos^2 t + \sin^2 t + 16t}$$

$$= \frac{16}{1+16t}$$

$$w = \ln\left(x^2 + y^2 + z^2\right)$$

$$= \ln\left(\cos^2 t + \sin^2 t + 16t\right)$$

$$= \ln(1+16t)$$

$$\frac{dw}{dt} = \frac{16}{1+16t}$$

$$\frac{dw}{dt}(3) = \frac{16}{1+16(3)}$$

$$= \frac{16}{49}$$

Exercise

Express $\frac{dw}{dt}$ as a function of t, then evaluate $\frac{dw}{dt}$ at the given value of t.

$w = z - \sin xy, \quad x = t, \quad y = \ln t, \quad z = e^{t-1}, \quad t = 1$

Solution

$$\frac{\partial w}{\partial t} = (-y\cos xy)(1) + (-x\cos xy)\left(\frac{1}{t}\right) + (1)\left(e^{t-1}\right)$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= -(\ln t)\cos(t\ln t) - \cos(t\ln t) + e^{t-1}$$

$$= -(\ln t + 1)\cos(t\ln t) + e^{t-1}$$

$$w = z - \sin xy$$

$$= e^{t-1} - \sin(t\ln t)$$

$$\frac{\partial w}{\partial t} = e^{t-1} - \cos(t\ln t)\left[\ln t + t\left(\frac{1}{t}\right)\right]$$

$$= e^{t-1} - (\ln t + 1)\cos(t\ln t)$$

$$\frac{\partial w}{\partial t}(1) = -(\ln 1 + 1)\cos(1\ln 1) + e^{1-1}$$

$$= -1\cos 0 + 1$$

$$\underline{= 0}$$

Exercise

Express $\frac{dw}{dt}$ as a function of t, then evaluate $\frac{dw}{dt}$ at the given value of t.

$$w = \sin(xy + \pi), \quad x = e^t, \quad y = \ln(t+1) \quad t = 0$$

Solution

$$\frac{\partial w}{\partial t} = y\cos(xy+\pi)e^t + x\cos(xy+\pi)\frac{1}{t+1} \qquad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= e^t \ln(t+1)\cos\left(e^t\ln(t+1)+\pi\right) + e^t\cos\left(e^t\ln(t+1)+\pi\right)\frac{1}{t+1}$$

$$\underline{= e^t\left(\ln(t+1) + \frac{1}{t+1}\right)\cos\left(e^t\ln(t+1)+\pi\right)}$$

$$\left.\frac{\partial w}{\partial t}\right|_{t=0} = \cos\pi$$

$$\underline{= -1}$$

Exercise

Express $\frac{dw}{dt}$ as a function of t, then evaluate $\frac{dw}{dt}$ at the given value of t.

$$w = xe^y + y\sin z - \cos z, \quad x = 2\sqrt{t}, \quad y = t - 1 + \ln t, \quad z = \pi t, \quad t = 1$$

Solution

$$\frac{\partial w}{\partial t} = e^y\frac{1}{\sqrt{t}} + \left(xe^y + \sin z\right)\left(1+\frac{1}{t}\right) + (y\cos z + \sin z)(\pi) \qquad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\underline{= \frac{1}{\sqrt{t}}e^{t-1+\ln t} + \left(2\sqrt{t}e^{t-1+\ln t} + \sin(\pi t)\right)\left(1+\frac{1}{t}\right) + \pi\left(e(t-1+\ln t)\cos(\pi t) + \sin(\pi t)\right)}$$

$$\left.\frac{\partial w}{\partial t}\right|_{t=1} = 1 + (2+\sin\pi)(2) + \pi\sin(\pi) \qquad = 1 + 4 + 0$$

$$\underline{= 5}$$

Exercise

Express $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ as functions of u and v if $z = 4e^x \ln y, \quad x = \ln(u\cos v), \quad y = u\sin v$, then evaluate $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ at the point $(u,v) = \left(2, \frac{\pi}{4}\right)$.

Solution

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{dx}{du} + \frac{\partial z}{\partial y}\frac{dy}{du}$$

$$= \left(4e^x \ln y\right)\left(\frac{\cos v}{u\cos v}\right) + \left(4\frac{e^x}{y}\right)(\sin v)$$

$$= 4e^x\left(\frac{\ln y}{u} + \frac{\sin v}{y}\right)$$

$$= 4e^{\ln(u\cos v)}\left(\frac{\ln(u\sin v)}{u} + \frac{\sin v}{u\sin v}\right)$$

$$= 4(u\cos v)\left(\frac{\ln(u\sin v)}{u} + \frac{1}{u}\right)$$

$$= 4\cos v\ln(u\sin v) + 4\cos v$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{dx}{dv} + \frac{\partial z}{\partial y}\frac{dy}{dv}$$

$$= \left(4e^x \ln y\right)\left(\frac{-u\sin v}{u\cos v}\right) + \left(4\frac{e^x}{y}\right)(u\cos v)$$

$$= 4e^{\ln(u\cos v)}\left[\frac{-\ln(u\sin v)(u\sin v)}{u\cos v} + \frac{u\cos v}{u\sin v}\right]$$

$$= 4u\cos v\left(\frac{-u\sin^2 v\cdot\ln(u\sin v) + u\cos^2 v}{u\cos v\sin v}\right)$$

$$= 4\left(\frac{-u\sin^2 v\cdot\ln(u\sin v) + u\cos^2 v}{\sin v}\right)$$

$$= -4u\sin v\cdot\ln(u\sin v) + 4u\frac{\cos^2 v}{\sin u}$$

$$\frac{\partial z}{\partial u}\left(2, \frac{\pi}{4}\right) = 4\cos\frac{\pi}{4}\ln\left(2\sin\frac{\pi}{4}\right) + 4\cos\frac{\pi}{4}$$

$$= 2\sqrt{2}\ln\sqrt{2} + 2\sqrt{2}$$

$$= 2\sqrt{2}\left(\frac{1}{2}\ln 2 + 1\right)$$

$$= \sqrt{2}(\ln 2 + 2)$$

$$\frac{\partial z}{\partial v}\left(2,\ \frac{\pi}{4}\right) = -8\sin\left(\frac{\pi}{4}\right)\cdot\ln\left(2\sin\left(\frac{\pi}{4}\right)\right) + 8\frac{\cos^2\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)}$$

$$= -4\sqrt{2}\ln\left(\sqrt{2}\right) + 8\cdot\frac{1}{2}\cdot\sqrt{2}$$

$$\boxed{= -2\sqrt{2}\ln 2 + 4\sqrt{2}}$$

Exercise

Express $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ as functions of u and v if $w = xy + yz + xz, \quad x = u + v, \quad y = u - v, \quad z = uv$, then evaluate $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ at the point $(u, v) = \left(\frac{1}{2},\ 1\right)$.

Solution

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{dx}{du} + \frac{\partial w}{\partial y}\frac{dy}{du} + \frac{\partial w}{\partial z}\frac{dz}{du}$$

$$= (y+z)(1) + (x+z)(1) + (y+x)(v)$$

$$= y + z + x + z + (y+x)(v)$$

$$= y + x + 2z + yv + xv$$

$$= u - v + u + v + 2uv + uv - v^2 + uv + v^2$$

$$\boxed{= 2u + 4uv}$$

$$\frac{\partial w}{\partial u}\left(\frac{1}{2},\ 1\right) = 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right)(1)$$

$$\boxed{= 3}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{dx}{dv} + \frac{\partial w}{\partial y}\frac{dy}{dv} + \frac{\partial w}{\partial z}\frac{dz}{dv}$$

$$= (y+z)(1) + (x+z)(-1) + (y+x)(u)$$

$$= y + z - x - z + yu + xu$$

$$= y - x + yu + xu$$

$$= u - v - u - v + u^2 - uv + u^2 + uv$$

$$\boxed{= -2v + 2u^2}$$

$$\frac{\partial w}{\partial v}\left(\frac{1}{2},\ 1\right) = -2(1) + 2\left(\frac{1}{2}\right)^2$$

$$\boxed{= -\frac{3}{2}}$$

Exercise

Express $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ as functions of *x, y* and *z* if $u = e^{qr}\sin^{-1} p, \quad p = \sin x, \quad q = z^2 \ln y, \quad r = \frac{1}{z}$, then evaluate $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ at the point $(x, y, z) = \left(\frac{\pi}{4},\ \frac{1}{2},\ -\frac{1}{2}\right)$.

Solution

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial p}\frac{dp}{dx} + \frac{\partial u}{\partial q}\frac{dq}{dx} + \frac{\partial u}{\partial r}\frac{dr}{dx} \\
&= \left(\frac{e^{qr}}{\sqrt{1-p^2}}\right)(\cos x) + \left(re^{qr}\sin^{-1} p\right)(0) + \left(qe^{qr}\sin^{-1} p\right)(0) \\
&= \frac{e^{z\ln y}\cos x}{\sqrt{1-\sin^2 x}} \\
&= e^{\ln y^z}\frac{\cos x}{|\cos x|} \\
&= \boxed{y^z} \quad \text{if} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial u}{\partial x}\left(\frac{\pi}{4},\ \frac{1}{2},\ -\frac{1}{2}\right) &= \left(\frac{1}{2}\right)^{-1/2} \\
&= 2^{1/2} \\
&= \boxed{\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial p}\frac{dp}{dy} + \frac{\partial u}{\partial q}\frac{dq}{dy} + \frac{\partial u}{\partial r}\frac{dr}{dy} \\
&= \left(\frac{e^{qr}}{\sqrt{1-p^2}}\right)(0) + \left(re^{qr}\sin^{-1} p\right)\left(\frac{z^2}{y}\right) + \left(qe^{qr}\sin^{-1} p\right)(0) \\
&= \frac{z^2}{y}\frac{1}{z}e^{z\ln y}\sin^{-1}(\sin x) \\
&= \frac{z}{y}e^{\ln y^z}(x) \\
&= \frac{xz}{y}y^z \\
&= \boxed{xzy^{z-1}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial u}{\partial y}\left(\frac{\pi}{4},\ \frac{1}{2},\ -\frac{1}{2}\right) &= \left(\frac{\pi}{4}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)^{-1/2-1} \\
&= -\left(\frac{\pi}{8}\right)2^{3/2}
\end{aligned}$$

$$= -\frac{\pi\sqrt{2}}{4}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{dp}{dz} + \frac{\partial u}{\partial q}\frac{dq}{dz} + \frac{\partial u}{\partial r}\frac{dr}{dz}$$

$$= \left(\frac{e^{qr}}{\sqrt{1-p^2}}\right)(0) + \left(re^{qr}\sin^{-1}p\right)(2z\ln y) + \left(qe^{qr}\sin^{-1}p\right)\left(-\frac{1}{z^2}\right)$$

$$= 2z\ln y\left(\frac{1}{z}y^z\sin^{-1}(\sin x)\right) - \frac{1}{z^2}\left(z^2(\ln y)y^z\sin^{-1}(\sin x)\right)$$

$$= 2xy^z\ln y - xy^z\ln y$$

$$= xy^z\ln y$$

$$\frac{\partial u}{\partial z}\left(\frac{\pi}{4},\ \frac{1}{2},\ -\frac{1}{2}\right) = \left(\frac{\pi}{4}\right)\left(\frac{1}{2}\right)^{-1/2}\ln\left(\frac{1}{2}\right)$$

$$= \left(\frac{\pi}{4}\right)\left(\sqrt{2}\right)(-\ln 2)$$

$$= -\frac{\pi\sqrt{2}}{4}\ln 2$$

Exercise

Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z^3 - xy + yz + y^3 - 2 = 0$ at the point $(1,\ 1,\ 1)$

Solution

$$F(x,y,z) = z^3 - xy + yz + y^3 - 2$$

$$F_x = -y, \quad F_y = -x + z + 3y^2, \quad F_z = 3z^2 + y$$

$$\frac{\partial z}{\partial x} = -\frac{-y}{3z^2+y} \qquad\qquad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

$$= \frac{y}{3z^2+y}$$

$$\frac{\partial z}{\partial x}(1,\ 1,\ 1) = \frac{1}{3(1)^2+1}$$

$$= \frac{1}{4}$$

$$\frac{\partial z}{\partial y} = -\frac{-x+z+3y^2}{3z^2+y} \qquad\qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

$$= \boxed{\frac{x-z-3y^2}{3z^2+y}}$$

$$\frac{\partial z}{\partial y}(1,\ 1,\ 1) = \frac{1-1-3(1)^2}{3(1)^2+1}$$

$$= \boxed{-\frac{3}{4}}$$

Exercise

Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $\sin(x+y)+\sin(y+z)+\sin(x+z)=0$ at the point $(\pi,\ \pi,\ \pi)$

Solution

$$F(x,y,z) = \sin(x+y)+\sin(y+z)+\sin(x+z)$$

$$F_x = \cos(x+y)+\cos(x+z)$$

$$F_y = \cos(x+y)+\cos(y+z)$$

$$F_z = \cos(y+z)+\cos(x+z)$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\cos(x+y)+\cos(x+z)}{\cos(y+z)+\cos(x+z)}$$

$$\frac{\partial z}{\partial x}(\pi,\ \pi,\ \pi) = -\frac{\cos(2\pi)+\cos(2\pi)}{\cos(2\pi)+\cos(2\pi)}$$

$$= \boxed{-1}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\cos(x+y)+\cos(y+z)}{\cos(y+z)+\cos(x+z)}$$

$$\frac{\partial z}{\partial x}(\pi,\ \pi,\ \pi) = -\frac{\cos(2\pi)+\cos(2\pi)}{\cos(2\pi)+\cos(2\pi)}$$

$$= \boxed{-1}$$

Exercise

Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0$ at the point $(1,\ \ln 2,\ \ln 3)$

Solution

$$F(x,y,z) = xe^y + ye^z + 2\ln x - 2 - 3\ln 2$$

$$F_x = e^y + \frac{2}{x} \qquad F_y = xe^y + e^z \qquad F_z = ye^z$$

$$\frac{\partial z}{\partial x} = -\frac{e^y + \frac{2}{x}}{ye^z} \qquad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

$$= -\frac{xe^y + 2}{xye^z}$$

$$\frac{\partial z}{\partial x}(1, \ln 2, \ln 3) = -\frac{(1)e^{\ln 2} + 2}{\ln 2e^{\ln 3}}$$

$$= -\frac{2+2}{3\ln 2}$$

$$= -\frac{4}{3\ln 2}$$

$$\frac{\partial z}{\partial y} = -\frac{xe^y + e^z}{ye^z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

$$\frac{\partial z}{\partial x}(1, \ln 2, \ln 3) = -\frac{e^{\ln 2} + e^{\ln 3}}{\ln 2e^{\ln 3}}$$

$$= -\frac{2+3}{3\ln 2}$$

$$= -\frac{5}{3\ln 2}$$

Exercise

Find $\frac{\partial w}{\partial r}$ when $r=1,\ s=-1$ if $w=(x+y+z)^2, \quad x=r-s, \quad y=\cos(r+s), \quad z=\sin(r+s)$

Solution

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{dx}{dr} + \frac{\partial w}{\partial y}\frac{dy}{dr} + \frac{\partial w}{\partial z}\frac{dz}{dr}$$

$$= 2(x+y+z)(1) + 2(x+y+z)(-\sin(r+s)) + 2(x+y+z)(\cos(r+s))$$

$$= 2(x+y+z)\left[1 - \sin(r+s) + \cos(r+s)\right]$$

$$= 2(r - s + \cos(r+s) + \sin(r+s))(1 - \sin(r+s) + \cos(r+s))$$

$$\frac{\partial w}{\partial r}(1,-1) = 2\left(1-(-1)+\cos(1-1)+\sin(1-1)\right)\left(1-\sin(1-1)+\cos(1-1)\right)$$
$$= 2(1+1+1+0)(1-0+1)$$
$$= 2(3)(2)$$
$$\underline{= 12}$$

Exercise

Find $\frac{\partial z}{\partial u}$ when $u=0$, $v=1$ if $z=\sin xy + x\sin y$, $\quad x=u^2+v^2$, $\quad y=uv$

Solution

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{dx}{du} + \frac{\partial z}{\partial y}\frac{dy}{du}$$
$$= (y\cos x + \sin y)(2u) + (x\cos xy + x\cos y)(v)$$
$$= 2u\left(uv\cos\left(u^2+v^2\right)+\sin uv\right) + v\left(\left(u^2+v^2\right)\cos\left(u^3v+uv^3\right)+\left(u^2+v^2\right)\cos uv\right)$$
$$= 2u\left(uv\cos\left(u^2+v^2\right)+\sin uv\right) + v\left(u^2+v^2\right)\left(\cos\left(u^3v+uv^3\right)+\cos uv\right)$$

$$\left.\frac{\partial z}{\partial u}\right|_{u=0,\ v=1} = 2(0)\left(0\cos(1)+\sin 0\right)+1(1)\left(\cos(0)+\cos 0\right)$$
$$= 0+1(1+1)$$
$$\underline{= 2}$$

Exercise

Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ when $u=\ln 2$, $v=1$ if $z=5\tan^{-1}x$, $\quad x=e^u+\ln v$

Solution

$$\frac{\partial z}{\partial u} = \frac{dz}{dx}\frac{\partial x}{\partial u} = \left(\frac{5}{1+x^2}\right)e^u = \left(\frac{5}{1+\left(e^u+\ln v\right)^2}\right)e^u$$

$$\left.\frac{\partial z}{\partial u}\right|_{u=\ln 2,\ v=1} = \left(\frac{5}{1+\left(e^{\ln 2}+\ln 1\right)^2}\right)e^{\ln 2}$$
$$= \left(\frac{5}{1+(2+0)^2}\right)(2)$$

$$= 2\left(\frac{5}{5}\right)$$

$$\boxed{= 2}$$

Exercise

Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ when $u = 1,\ v = -2$ if $z = \ln q, \quad q = \sqrt{v+3}\ \tan^{-1} u$

Solution

$$\frac{\partial z}{\partial u} = \frac{dz}{dq}\frac{\partial q}{\partial u}$$

$$= \left(\frac{1}{q}\right)\left(\sqrt{v+3}\,\frac{1}{1+u^2}\right)$$

$$= \frac{1}{\sqrt{v+3}\ \tan^{-1} u} \cdot \frac{\sqrt{v+3}}{1+u^2}$$

$$\boxed{= \frac{1}{\left(1+u^2\right)\tan^{-1} u}}$$

$$\left.\frac{\partial z}{\partial u}\right|_{u=1,\ v=-2} = \frac{1}{\left(1+1^2\right)\tan^{-1} 1}$$

$$= \frac{1}{2 \cdot \frac{\pi}{4}}$$

$$\boxed{= \frac{2}{\pi}}$$

$$\frac{\partial z}{\partial v} = \frac{dz}{dq}\frac{\partial q}{\partial v}$$

$$= \left(\frac{1}{q}\right)\left(\frac{1}{2\sqrt{v+3}}\tan^{-1} u\right)$$

$$= \left(\frac{1}{\sqrt{v+3}\ \tan^{-1} u}\right)\left(\frac{\tan^{-1} u}{2\sqrt{v+3}}\right)$$

$$\boxed{= \frac{1}{2(v+3)}}$$

$$\left.\frac{\partial z}{\partial v}\right|_{u=1,\ v=-2} = \frac{1}{2(-2+3)}$$

$$\boxed{= \frac{1}{2}}$$

Exercise

Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ when $r=\pi$ and $s=0$ if $w=\sin(2x-y)$, $\quad x=r+\sin s$, $\quad y=rs$

Solution

$$\frac{\partial w}{\partial x}=2\cos(2x-y)$$

$$=2\cos(2r+2\sin s-rs)\Big|_{r=\pi \quad s=0}$$

$$=2\cos(2\pi)$$

$$\underline{=2}$$

$$\frac{\partial w}{\partial y}=-\cos(2x-y)$$

$$=-\cos(2r+2\sin s-rs)\Big|_{r=\pi \quad s=0}$$

$$=-\cos(2\pi)$$

$$\underline{=-1}$$

$$\boxed{\frac{dx}{dr}=1}$$

$$\frac{dy}{dr}=s\Big|_{r=\pi \quad s=0} \quad \underline{=0}$$

$$\frac{\partial w}{\partial r}=2(1)+(-1)(0) \qquad \frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

$$\underline{=2}$$

$$\frac{dx}{ds}=\cos s\Big|_{s=0} \quad \underline{=1}$$

$$\frac{dy}{ds}=r\Big|_{r=\pi} \quad \underline{=\pi}$$

$$\frac{\partial w}{\partial s}=2(1)+(-1)(\pi) \qquad \frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

$$\underline{=2-\pi}$$

Exercise

Assume that $w=f\left(s^3+t^2\right)$ and $f'(x)=e^x$. Find $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$

Solution

$$w=f\left(s^3+t^2\right)=f(x) \quad \rightarrow \quad x=s^3+t^2$$

$$\frac{\partial w}{\partial t} = f'(x)\cdot 2t \qquad \frac{\partial w}{\partial t} = \frac{dw}{dx}\frac{\partial x}{\partial t}$$

$$= 2te^{x}$$

$$\boxed{= 2te^{s^3+t^2}}$$

$$\frac{\partial w}{\partial s} = \left(e^{x}\right)\left(3s^2\right) \qquad \frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$

$$\boxed{= 3s^2 e^{s^3+t^2}}$$

Exercise

Evaluate the derivatives $w'(t)$, where $w = xy\sin z,\quad x = t^2,\quad y = 4t^3,\quad and\quad z = t+1$

Solution

$$\frac{dw}{\partial t} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= y\sin z(2t) + x\sin z\left(12t^2\right) + xy\cos z$$

$$= 8t^4\sin(t+1) + 12t^4\sin(t+1) + 4t^5\cos(t+1)$$

$$\boxed{= 20t^4\sin(t+1) + 4t^5\cos(t+1)}$$

Or

$$w(t) = 4t^5\sin(t+1)$$

$$\boxed{w' = 20t^4\sin(t+1) + 4t^5\cos(t+1)}$$

Exercise

Evaluate the derivatives $w'(t)$, where $w = \sqrt{x^2+y^2+z^2},\quad x = \sin t,\quad y = \cos t,\quad and\quad z = \cos t$

Solution

$$\frac{dw}{\partial t} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}}\cos t - \frac{y}{\sqrt{x^2+y^2+z^2}}\sin t - \frac{z}{\sqrt{x^2+y^2+z^2}}\sin t$$

$$= \frac{\sin t\cos t - \cos t\sin t - \cos t\sin t}{\sqrt{\sin^2 t + \cos^2 t + \sin^2 t}}$$

$$\boxed{= -\frac{\cos t\sin t}{\sqrt{1+\sin^2 t}}}$$

Exercise

Evaluate the derivatives w_s and w_t, where $w = xyz, \quad x = 2st, \quad y = st^2, \quad and \quad z = s^2 t$

Solution

$$w_s = w_x x_s + w_y y_s + w_z z_s$$
$$= yz(2t) + xzt^2 + xy(2st)$$
$$= 2s^3t^4 + 2s^3t^4 + 4s^3t^4$$
$$\boxed{= 8s^3t^4}$$

$$w_t = w_x x_t + w_y y_t + w_z z_t$$
$$= yz(2s) + xz(2st) + xy\left(s^2\right)$$
$$= 2s^4t^3 + 4s^4t^3 + 2s^4t^3$$
$$\boxed{= 8s^4t^3}$$

Or

$$w = (2st)\left(st^2\right)\left(s^2t\right)$$
$$= 2s^4t^4$$
$$\boxed{w_s = 8s^3t^4}$$
$$\boxed{w_t = 8s^4t^3}$$

Exercise

Evaluate the derivatives w_r, w_s, and w_t, where $w = \ln\left(xy^2\right), \quad x = rst, \quad and \quad y = r + s$

Solution

$$w_r = w_x x_r + w_y y_r$$
$$= \frac{1}{y^2}(st) + \frac{2}{xy}$$
$$= \frac{st}{(r+s)^2} + \frac{2}{rst(r+s)}$$
$$\boxed{= \frac{rs^2t^2 + 2r + 2s}{rst(r+s)^2}}$$

$$w_s = w_x x_s + w_y y_s$$

$$= \frac{rt}{y^2} + \frac{2}{xy}$$

$$= \frac{rt}{(r+s)^2} + \frac{2}{rst(r+s)}$$

$$\boxed{= \frac{r^2 st^2 + 2r + 2s}{rst(r+s)^2}}$$

$$w_t = w_x x_t + w_y y_t$$

$$= \frac{rs}{y^2} + \frac{2}{xy}(0)$$

$$\boxed{= \frac{rs}{(r+s)^2}}$$

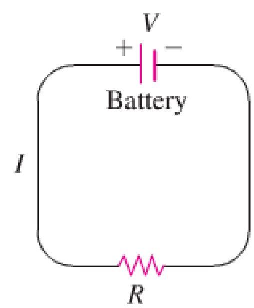
Or $\qquad w = \ln\left(rst(r+s)^2\right)$

Exercise

The voltage V in a circuit that satisfies the law $V = IR$ is slowly dropping as the battery wears out. At the same time, the resistance R is increasing as the resistor heats up. Use the equation

$$\frac{dV}{dt} = \frac{\partial V}{\partial I}\frac{dI}{dt} + \frac{\partial V}{\partial R}\frac{dR}{dt}$$

To find how the current is changing at the instant when $R = 600\ \Omega$, $I = 0.04A$, $\frac{dR}{dt} = 0.5\ ohm/\sec$, and $\frac{dV}{dt} = -0.01\ volt/\sec$



Solution

$$V = IR \quad \to \quad \frac{\partial V}{\partial I} = R, \quad \frac{\partial V}{\partial R} = I$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial I}\frac{dI}{dt} + \frac{\partial V}{\partial R}\frac{dR}{dt}$$

$$-0.01 = (600)\frac{dI}{dt} + (0.04)(0.5)$$

$$-0.02 - 0.01 = 600\frac{dI}{dt}$$

$$\frac{dI}{dt} = \boxed{-0.00005\ amps/\sec}$$

Exercise

The lengths *a*, *b*, and *c* of the edges of a rectangular box are changing with time. At the instant in question, $a = 1\ m, \quad b = 2\ m, \quad c = 3\ m$, $\frac{da}{dt} = \frac{db}{dt} = 1\ m/\sec$, and $\frac{dc}{dt} = -3\ m/\sec$. At what rates the box's volume *V* and surface area *S* changing at that instant? Are the box's interior diagonals increasing in length or decreasing?

Solution

$$V = abc \quad \Rightarrow \quad \frac{\partial V}{\partial t} = \frac{\partial V}{\partial a}\frac{da}{dt} + \frac{\partial V}{\partial b}\frac{db}{dt} + \frac{\partial V}{\partial c}\frac{dc}{dt}$$

$$\frac{\partial V}{\partial t} = (bc)\frac{da}{dt} + (ac)\frac{db}{dt} + (ab)\frac{dc}{dt}$$

$$= (2m)(3m)(1\ m/\sec) + (1m)(3m)(1\ m/\sec) + (1m)(2m)(-3\ m/\sec)$$

$$\underline{= 3\ m^3/\sec}$$

Exercise

Let $T = f(x, y)$ be the temperature at the point (x, y) on the circle $x = \cos t, \quad y = \sin t, \ 0 \le t \le 2\pi$ and suppose that

$$\frac{\partial T}{\partial x} = 8x - 4y, \quad \frac{\partial T}{\partial y} = 8y - 4x$$

a) Find where the maximum and minimum temperatures on the circle occur by examining the derivatives $\frac{dT}{dt}$ and $\frac{d^2T}{dt^2}$.

b) Suppose that $T = 4x^2 - 4xy + 4y^2$. Find the maximum and minimum values of *T* on the circle.

Solution

a) $$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$$

$$= (8x - 4y)(-\sin t) + (8y - 4x)(\cos t)$$

$$= (8\cos t - 4\sin t)(-\sin t) + (8\sin t - 4\cos t)(\cos t)$$

$$= -8\cos t \sin t + 4\sin^2 t + 8\cos t \sin t - 4\cos^2 t$$

$$\underline{= 4\sin^2 t - 4\cos^2 t}$$

$$\frac{dT}{dt} = 0 \quad \Rightarrow \quad 4\sin^2 t - 4\cos^2 t = 0$$

$$\sin^2 t = \cos^2 t$$
$$\sin t = \pm \cos t$$

$$\boxed{t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}} \quad \text{on the interval } 0 \le t \le 2\pi$$

$$\frac{d^2T}{dt^2} = 8\sin t\cos t + 8\cos t\sin t$$

$$\boxed{= 16\sin t\cos t}$$

$$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{\pi}{4}} = 16\sin\frac{\pi}{4}\cos\frac{\pi}{4} > 0 \quad \Rightarrow \text{T has a minimum at } (x,\ y) = \left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$$

$$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{3\pi}{4}} = 16\sin\frac{3\pi}{4}\cos\frac{3\pi}{4} < 0 \Rightarrow \text{T has a maximum at } (x,\ y) = \left(-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$$

$$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{5\pi}{4}} = 16\sin\frac{5\pi}{4}\cos\frac{5\pi}{4} > 0 \Rightarrow \text{T has a minimum at } (x,\ y) = \left(-\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\right)$$

$$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{7\pi}{4}} = 16\sin\frac{7\pi}{4}\cos\frac{7\pi}{4} < 0 \quad \Rightarrow \text{T has a maximum at } (x,\ y) = \left(\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\right)$$

b) $T = 4x^2 - 4xy + 4y^2$

$$T\left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right) = 4\left(\frac{\sqrt{2}}{2}\right)^2 - 4\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + 4\left(\frac{\sqrt{2}}{2}\right)^2 = 2 - 2 + 2 = \underline{2}$$

$$T\left(-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right) = 4\left(-\frac{\sqrt{2}}{2}\right)^2 - 4\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + 4\left(\frac{\sqrt{2}}{2}\right)^2 = 2 + 2 + 2 = \underline{6}$$

$$T\left(-\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\right) = 4\left(-\frac{\sqrt{2}}{2}\right)^2 - 4\left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) + 4\left(-\frac{\sqrt{2}}{2}\right)^2 = 2 - 2 + 2 = \underline{2}$$

$$T\left(\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\right) = 4\left(\frac{\sqrt{2}}{2}\right)^2 - 4\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) + 4\left(-\frac{\sqrt{2}}{2}\right)^2 = 2 + 2 + 2 = \underline{6}$$

The maximum value is 6 at $\left(-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$ and $\left(\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\right)$

The minimum value is 2 at $\left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$ and $\left(-\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\right)$

Exercise

Evaluate $\frac{dy}{dx}$: $\quad x^2 - 2y^2 - 1 = 0$

Solution

$$F(x,\ y) = x^2 - 2y^2 - 1$$

$$\frac{dy}{dx} = -\frac{2x}{-4y} \qquad\qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= \frac{x}{2y}$$

Exercise

Evaluate $\frac{dy}{dx}$: $\quad x^3 + 3xy^2 - y^5 = 0$

Solution

$$F(x,\ y) = x^3 + 3xy^2 - y^5$$

$$\frac{dy}{dx} = -\frac{3x^2 + 3y^2}{6xy - 5y^4} \qquad\qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

Exercise

Evaluate $\frac{dy}{dx}$: $\quad 2\sin xy = 1$

Solution

$$F(x,\ y) = 2\sin xy - 1$$

$$\frac{dy}{dx} = -\frac{2y\cos xy}{2x\cos xy} \qquad\qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= -\frac{y}{x}$$

Exercise

Evaluate $\frac{dy}{dx}$: $\quad ye^{xy} - 2 = 0$

Solution

$$F(x,\ y) = ye^{xy} - 2$$

$$\frac{dy}{dx} = -\frac{y^2 e^{xy}}{e^{xy} + xye^{xy}} \qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= -\frac{y^2}{1+xy}$$

Exercise

Evaluate $\frac{dy}{dx}$: $\quad \sqrt{x^2 + 2xy + y^4} = 3$

Solution

$$F(x, y) = \sqrt{x^2 + 2xy + y^4} - 3$$

$$\frac{dy}{dx} = -\frac{\frac{1}{2}(2x+2y)\left(x^2 + 2xy + y^4\right)^{-1/2}}{\frac{1}{2}\left(2x + 4y^3\right)\left(x^2 + 2xy + y^4\right)^{-1/2}} \qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= -\frac{x+y}{x+2y^3}$$

Exercise

Evaluate $\frac{dy}{dx}$: $\quad y\ln\left(x^2 + y^2 + 4\right) = 3$

Solution

$$F(x, y) = y\ln\left(x^2 + y^2 + 4\right) - 3$$

$$\frac{dy}{dx} = -\frac{\frac{2xy}{x^2+y^2+4}}{\ln\left(x^2 + y^2 + 4\right) + \frac{2y^2}{x^2+y^2+4}} \qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= -\frac{2xy}{2y^2 + \left(x^2 + y^2 + 4\right)\ln\left(x^2 + y^2 + 4\right)}$$

Exercise

Evaluate $\frac{dy}{dx}$: $\quad y\ln\left(x^2 + y^2\right) = 4$

Solution

$$F(x,\ y) = y\ln\left(x^2+y^2\right) - 4$$

$$\frac{dy}{dx} = -\frac{\frac{2xy}{x^2+y^2}}{\ln\left(x^2+y^2\right)+\frac{2y^2}{x^2+y^2}} \qquad\qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= -\frac{2xy}{\left(x^2+y^2\right)\ln\left(x^2+y^2\right)+2y^2}$$

Exercise

Evaluate $\frac{dy}{dx}$: $\quad 2x^2+3xy-3y^4=2$

Solution

$$F(x,\ y) = 2x^2+3xy-3y^4-2$$

$$\frac{dy}{dx} = -\frac{4x+3y}{3x-12y^3} \qquad\qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

Exercise

Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ at the given point. $\quad z^3-xy+yz+y^3-2=0;\quad (1,\ 1,\ 1)$

Solution

$$F(x,\ y,\ z) = z^3-xy+yz+y^3-2$$

$$F_x = -y, \quad F_y = -x+z+3y^2, \quad \textit{and} \quad F_z = 3z^2+y\Big|_{(1,1,1)} = 4 \neq 0$$

$$\frac{dz}{dx} = -\frac{F_x}{F_z} = -\frac{-y}{3z^2+y} \qquad\qquad \frac{dz}{dy} = -\frac{F_y}{F_z} = -\frac{e^{xz}-z\sin y}{2z+xye^{xz}+\cos y}$$

$$\left.\frac{dz}{dx}\right|_{(1,1,1)} = -\frac{-1}{4} = \frac{1}{4} \qquad\qquad \left.\frac{dz}{dz}\right|_{(1,1,1)} = -\frac{3}{4}$$

Exercise

Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ at the given point. $\quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1=0;\quad (2,\ 3,\ 6)$

Solution

$$F(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1$$

$$F_x = -\frac{1}{x^2}\bigg|_{(2,3,6)} = -\frac{1}{4}$$

$$F_y = -\frac{1}{y^2}\bigg|_{(2,3,6)} = -\frac{1}{9}$$

$$F_z = -\frac{1}{z^2}\bigg|_{(2,3,6)} = -\frac{1}{36} \neq 0$$

$$\frac{dz}{dx}\bigg|_{(2,3,6)} = -\frac{F_x}{F_z}\bigg|_{(2,3,6)}$$

$$= -\frac{-\frac{1}{4}}{-\frac{1}{36}}$$

$$\underline{= -9}$$

$$\frac{dz}{dy}\bigg|_{(2,3,6)} = -\frac{F_y}{F_z}\bigg|_{(2,3,6)}$$

$$= -\frac{-\frac{1}{9}}{-\frac{1}{36}}$$

$$\underline{= -4}$$

Exercise

Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ at the given point. $\sin(x+y) + \sin(y+z) + \sin(x+z) = 0; \quad (\pi, \pi, \pi)$

Solution

$$F(x, y, z) = \sin(x+y) + \sin(y+z) + \sin(x+z)$$

$$F_x = \cos(x+y) + \cos(x+z)\Big|_{(\pi,\pi,\pi)}$$

$$= \cos 2\pi + \cos 2\pi$$

$$\underline{= 2}$$

$$F_y = \cos(x+y) + \cos(y+z)\Big|_{(\pi,\pi,\pi)}$$

$$= \cos 2\pi + \cos 2\pi$$

$$\underline{= 2}$$

$$F_z = \cos(y+z) + \cos(x+z)\Big|_{(\pi,\pi,\pi)}$$

$= \cos 2\pi + \cos 2\pi$

$\underline{= 2} \quad \neq 0$

$$\left.\frac{dz}{dx}\right|_{(\pi,\pi,\pi)} = -\left.\frac{F_x}{F_z}\right|_{(\pi,\pi,\pi)}$$

$$= -\frac{2}{2}$$

$$\underline{= -1}$$

$$\left.\frac{dz}{dy}\right|_{(\pi,\pi,\pi)} = -\left.\frac{F_y}{F_z}\right|_{(\pi,\pi,\pi)}$$

$$= -\frac{2}{2}$$

$$\underline{= -1}$$

Exercise

Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ at the given point. $\quad xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0; \quad (1,\ \ln 2,\ \ln 3)$

Solution

$$F(x,\ y,\ z) = xe^y + ye^z + 2\ln x - 2 - 3\ln 2$$

$$F_x = e^y + \left.\frac{2}{x}\right|_{(1,\ln 2,\ln 3)} = 2 + 2 \ \underline{= 4}$$

$$F_y = \left. xe^y + e^z \right|_{(1,\ln 2,\ln 3)}$$

$$= e^{\ln 2} + e^{\ln 3}$$

$$= 2 + 3$$

$$\underline{= 5}$$

$$F_z = \left. ye^z \right|_{(1,\ln 2,\ln 3)} = \ln 2 e^{\ln 3} \ \underline{= 3\ln 2} \neq 0$$

$$\left.\frac{dz}{dx}\right|_{(1,\ln 2,\ln 3)} = -\left.\frac{F_x}{F_z}\right|_{(1,\ln 2,\ln 3)}$$

$$\underline{= -\frac{4}{3\ln 2}}$$

$$\left.\frac{dz}{dy}\right|_{(1,\ln 2,\ln 3)} = -\left.\frac{F_y}{F_z}\right|_{(1,\ln 2,\ln 3)}$$

$$\underline{= -\frac{5}{3\ln 2}}$$

Exercise

Consider the surface and parameterized curves C in the xy-plane

$$z = 4x^2 + y^2 - 2; \quad C: \ x = \cos t, \quad y = \sin t, \quad \text{for} \ \ 0 \le t \le 2\pi$$

a) Find $z'(t)$ on C.

b) Imagine that you are walking on the surface directly above C consistent with the positive orientation of C. Find the values of ***t*** for which you are walking uphill.

Solution

a) $z'(t) = z_x x_t + z_y y_t$

$$= 8x(-\sin t) + 2y\cos t$$

$$= -8\cos t \sin t + 2\sin t \cos t$$

$$= -6\cos t \sin t$$

$$\boxed{= -3\sin 2t}$$

b) Walking uphill $\rightarrow \ z'(t) > 0$

$-3\sin 2t > 0 \quad \rightarrow \quad \sin 2t < 0$

$\sin 2t = 0 \quad \rightarrow \quad 2t = 0, \ \pi, \ 2\pi, 3\pi, \ 4\pi$

$t = 0, \ \frac{\pi}{2}, \ \pi, \ \frac{3\pi}{2}, \ 2\pi$

$\boxed{\frac{\pi}{2} \le t \le \pi \quad \& \quad \frac{3\pi}{2} \le t \le 2\pi}$

Exercise

Consider the surface and parameterized curves C in the xy-plane

$$z = 4x^2 - 2y^2 + 4; \quad C: \ x = 2\cos t, \quad y = 2\sin t, \quad \text{for} \ \ 0 \le t \le 2\pi$$

a) Find $z'(t)$ on C.

b) Imagine that you are walking on the surface directly above C consistent with the positive orientation of C. Find the values of ***t*** for which you are walking uphill.

Solution

a) $z'(t) = z_x x_t + z_y y_t$

$$= 8x(-2\sin t) - 4y(2\cos t)$$

$$= -16\cos t \sin t - 16\sin t \cos t$$

$$= -32\cos t \sin t$$

$$\boxed{= -16\sin 2t}$$

b) Walking uphill $\rightarrow \ z'(t) > 0$

$-16\sin 2t > 0 \quad \rightarrow \quad \sin 2t < 0$

$\sin 2t = 0 \quad \rightarrow \quad 2t = 0, \ \pi, \ 2\pi, 3\pi, \ 4\pi$

$t = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi$

$$\boxed{\frac{\pi}{2} \le t \le \pi \quad \& \quad \frac{3\pi}{2} \le t \le 2\pi}$$

Exercise

Find the value of the derivative of $f(x, y, z) = xy + yz + xz$ with respect to t on the curve $x = \cos t, \quad y = \sin t, \quad z = \cos 2t$ at $t = 1$

Solution

$$f_x = y + z = \sin t + \cos 2t$$

$$f_y = x + z = \cos t + \cos 2t$$

$$f_x = y + z = \cos t + \sin t$$

$$\frac{dx}{dt} = -\sin t \quad \frac{dy}{dt} = \cos t \quad \frac{dz}{dt} = -2\sin 2t$$

$$\begin{aligned}
\frac{df}{dt} &= f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} \\
&= (\sin t + \cos 2t)(-\sin t) + (\cos t + \cos 2t)(\cos t) + (\cos t + \sin t)(-2\sin 2t) \\
&= -\sin^2 t - \sin t \cos 2t + \cos^2 t + \cos t \cos 2t - 2\sin 2t \cos t - 2\sin 2t \sin t \\
&= \cos^2 t - \sin^2 t + (\cos t - \sin t)\cos 2t - 2\sin 2t(\cos t + \sin t)
\end{aligned}$$

$$\boxed{\left.\frac{df}{dt}\right|_{t=1} = \cos 2 + (\cos 1 - \sin 1)\cos 2 - 2(\cos 1 + \sin 1)\sin 2}$$

Exercise

Define y as a differentiable function of x for $2xy + e^{x+y} - 2 = 0$, find the values of $\frac{dy}{dx}$ at point $P(0,\ \ln 2)$

Solution

$$F(x, y) = 2xy + e^{x+y} - 2$$

$$F_x = 2y + e^{x+y} \qquad F_y = 2x + e^{x+y}$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2y + e^{x+y}}{2x + e^{x+y}}$$

$$\left.\frac{dy}{dx}\right|_{(0,\ \ln 2)} = -\frac{2\ln 2 + e^{\ln 2}}{0 + e^{\ln 2}}$$

$$= -\frac{2\ln 2 + 2}{2}$$

$$= -\ln 2 - 1$$