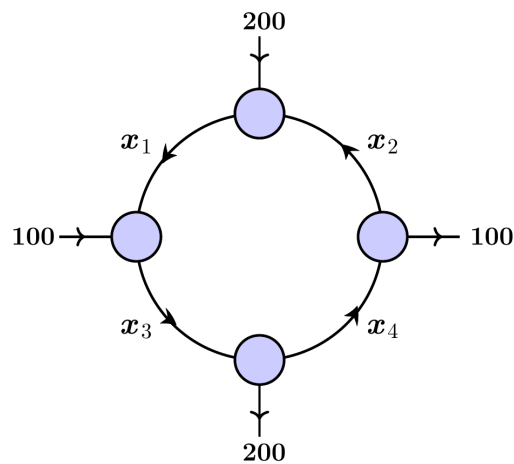


## Solution

### Section 1.8 – Applications

#### Exercise

The flow of traffic, in vehicles per hour, through a network of streets as is shown below



- a) Solve this system for  $x_i$ ,  $i = 1, 2, 3, 4$ .
- b) Find the traffic flow when  $x_4 = 0$ .
- c) Find the traffic flow when  $x_4 = 100$ .
- d) Find the traffic flow when  $x_1 = 2x_2$ .

#### Solution

$$a) \begin{cases} x_1 + 100 = x_3 \\ x_2 + 200 = x_1 \\ x_2 + 100 = x_4 \\ x_4 + 200 = x_3 \end{cases}$$

$$\begin{cases} -x_1 + x_3 = 100 \\ x_1 - x_2 = 200 \\ -x_2 + x_4 = 100 \\ x_3 - x_4 = 200 \end{cases}$$

$$\left( \begin{array}{cccc|c} -1 & 0 & 1 & 0 & 100 \\ 1 & -1 & 0 & 0 & 200 \\ 0 & -1 & 0 & 1 & 100 \\ 0 & 0 & 1 & -1 & 200 \end{array} \right)$$

$R_2 + R_1$

$$\left( \begin{array}{cccc|c} -1 & 0 & 1 & 0 & 100 \\ 1 & -1 & 0 & 0 & 200 \\ 0 & -1 & 0 & 1 & 100 \\ 0 & 0 & 1 & -1 & 200 \end{array} \right) = -1 \left( \begin{array}{ccc|c} -1 & 0 & 0 & 100 \\ -1 & 0 & 1 & 100 \\ 0 & 1 & -1 & 200 \end{array} \right) -1 \left( \begin{array}{ccc|c} 0 & 1 & 0 & 100 \\ -1 & 0 & 1 & 100 \\ 0 & 1 & -1 & 200 \end{array} \right)$$

$$\begin{pmatrix} -1 & 0 & 1 & 0 & | & 100 \\ 0 & -1 & 1 & 0 & | & 300 \\ 0 & -1 & 0 & 1 & | & 100 \\ 0 & 0 & 1 & -1 & | & 200 \end{pmatrix} \quad R_3 - R_2$$

$$\begin{pmatrix} -1 & 0 & 1 & 0 & | & 100 \\ 0 & -1 & 1 & 0 & | & 300 \\ 0 & 0 & -1 & 1 & | & -200 \\ 0 & 0 & 1 & -1 & | & 200 \end{pmatrix} \quad R_4 + R_3$$

$$\begin{pmatrix} -1 & 0 & 1 & 0 & | & 100 \\ 0 & -1 & 1 & 0 & | & 300 \\ 0 & 0 & -1 & 1 & | & -200 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \quad \begin{array}{l} \rightarrow -x_1 + x_3 = 100 \\ \rightarrow -x_2 + x_3 = 100 \\ \rightarrow -x_3 + x_4 = 100 \end{array}$$

Let  $x_4$  be the free variable

$$\begin{cases} \underline{x_3 = x_4 + 200} \\ \underline{x_2 = x_4 - 100} \\ \underline{x_1 = 200 + x_2 = x_4 + 100} \end{cases}$$

**Solution:**  $(x_4 + 100, x_4 - 100, x_4 + 200, x_4)$

**OR**

$$\begin{vmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} = -1 \begin{vmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= -1(1) - 1(-1)$$

$$= -1 + 1$$

$$= 0$$

$$\begin{cases} -x_1 + x_3 = 100 & \rightarrow x_1 = x_3 - 100 = \underline{x_4 + 100} \\ x_1 - x_2 = 200 \\ -x_2 + x_4 = 100 & \rightarrow \underline{x_2 = x_4 - 100} \\ x_3 - x_4 = 200 & \rightarrow \underline{x_3 = x_4 + 200} \end{cases}$$

b) The traffic flow when  $x_4 = 0$  is:

$\therefore (100, -100, 200, 0)$

c) The traffic flow when  $x_4 = 100$  is:

$$\therefore (200, 0, 300, 100)$$

d) The traffic flow when  $x_1 = 2x_2$  :

$$x_4 + 100 = 2(x_4 - 100)$$

$$x_4 + 100 = 2x_4 - 200$$

$$x_4 = 300$$

$$\therefore (400, 200, 500, 300)$$

### Exercise

Through a network, Express  $x_n$  's in terms of the parameters  $s$  and  $t$ .

### Solution

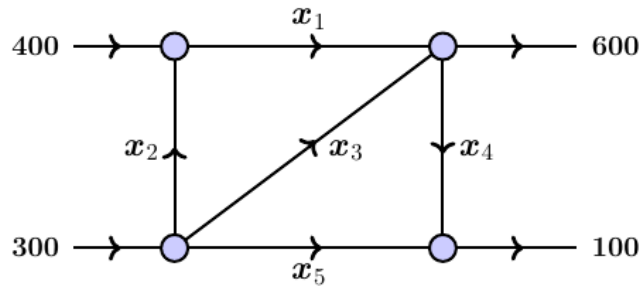
$$\begin{cases} x_1 = x_2 + 400 \\ x_1 + x_3 = x_4 + 600 \\ x_4 + x_5 = 100 \\ x_2 + x_3 + x_5 = 300 \end{cases}$$

$$\begin{cases} x_1 - x_2 = 400 \\ x_2 + x_3 - x_4 = 600 \\ x_4 + x_5 = 100 \\ x_2 + x_3 + x_5 = 300 \end{cases}$$

$$\left( \begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 1 & 0 & 1 & -1 & 0 & 600 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 1 & 1 & 0 & 1 & 300 \end{array} \right) \quad \begin{array}{l} R_2 - R_1 \\ R_3 \leftrightarrow R_4 \end{array}$$

$$\left( \begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 0 & 1 & 1 & -1 & 0 & 200 \\ 0 & 1 & 1 & 0 & 1 & 300 \\ 0 & 0 & 0 & 1 & 1 & 100 \end{array} \right) \quad R_3 - R_2$$

$$\left( \begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 0 & 1 & 1 & -1 & 0 & 200 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 100 \end{array} \right) \quad R_4 - R_3$$



$$\left( \begin{array}{cccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 0 & 1 & 1 & -1 & 0 & 200 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 - x_2 = 400 \quad \rightarrow x_1 = 400 + x_2 \\ x_2 + x_3 - x_4 = 200 \quad \rightarrow x_2 = 200 - x_3 + x_4 \\ x_4 + x_5 = 100 \quad \rightarrow \underline{x_4 = 100 - t} \end{array}$$

Let  $x_5 = t$  &  $x_3 = s$

$$x_2 = 200 - s + 100 - t = \underline{300 - s - t}$$

$$x_1 = 400 + 300 - s - t = \underline{700 - s - t}$$

### Exercise

Water is flowing through a network of pipes. Express  $x_n$ 's in terms of the parameters  $s$  and  $t$ .

### Solution

$$x_1 + x_3 = 900$$

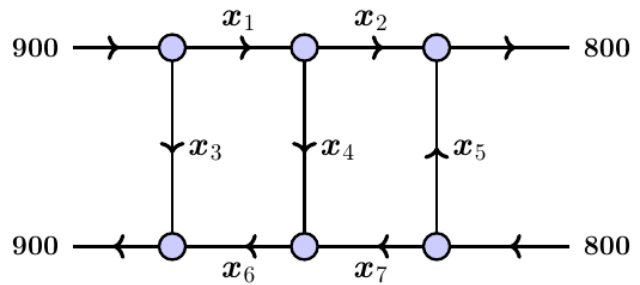
$$x_1 = x_2 + x_4 \quad \rightarrow \quad x_1 - x_2 - x_4 = 0$$

$$x_2 + x_5 = 800$$

$$x_5 + x_7 = 800$$

$$x_6 = x_4 + x_7 \quad \rightarrow \quad x_4 - x_6 + x_7 = 0$$

$$x_3 + x_6 = 900$$



$$\left[ \begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 900 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 800 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 900 \end{array} \right] \quad \begin{array}{l} \\ R_2 - R_1 \\ \\ \\ \end{array}$$

$$\left[ \begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 900 \\ 0 & -1 & -1 & -1 & 0 & 0 & -900 \\ 0 & 1 & 0 & 0 & 1 & 0 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 800 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 900 \end{array} \right] \quad \begin{array}{l} \\ R_3 + R_2 \\ R_6 \\ R_4 \end{array}$$

$$\left[ \begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 900 \\ 0 & -1 & -1 & -1 & 0 & 0 & -900 \\ 0 & 0 & -1 & -1 & 1 & 0 & -100 \\ 0 & 0 & 1 & 0 & 0 & 1 & 900 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 800 \end{array} \right] \quad \begin{array}{l} -R_2 \\ \\ R_4 + R_3 \end{array}$$

$$\left[ \begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 900 \\ 0 & 1 & 1 & 1 & 0 & 0 & 900 \\ 0 & 0 & -1 & -1 & 1 & 0 & -100 \\ 0 & 0 & 0 & -1 & 1 & 1 & 800 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 800 \end{array} \right] \quad R_5 + R_4$$

$$\left[ \begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 900 \\ 0 & 1 & 1 & 1 & 0 & 0 & 900 \\ 0 & 0 & -1 & -1 & 1 & 0 & -100 \\ 0 & 0 & 0 & -1 & 1 & 1 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 800 \end{array} \right] \quad R_6 - R_5$$

$$\left[ \begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 900 \\ 0 & 1 & 1 & 1 & 0 & 0 & 900 \\ 0 & 0 & -1 & -1 & 1 & 0 & -100 \\ 0 & 0 & 0 & -1 & 1 & 1 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 800 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} x_2 = 900 - x_3 \quad (5) \\ x_2 = 900 - x_3 - x_4 \quad (4) \\ x_3 = 100 - x_4 + x_5 \quad (3) \\ -x_4 = 800 - x_5 - x_6 \quad (2) \\ x_5 = 800 - x_7 \quad (1) \end{array}$$

Let  $x_6 = s$  &  $x_7 = t$

$$(1) \rightarrow x_5 = 800 - t$$

$$(2) \rightarrow x_4 = s - t$$

$$(3) \rightarrow x_3 = 900 - s$$

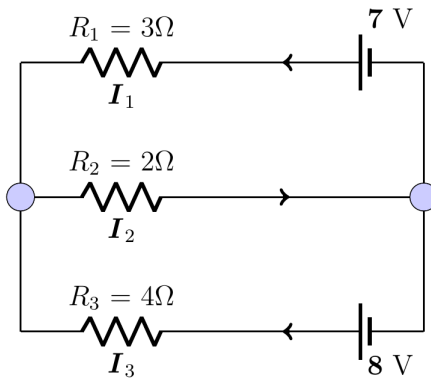
$$(2) \rightarrow x_2 = t$$

$$(1) \rightarrow x_2 = s$$

**Solution:**  $(s, t, 900-s, s-t, 800-t, s, t)$

### Exercise

Determine the currents  $I_1$ ,  $I_2$ , and  $I_3$  for the electrical network shown below



### Solution

$$I_2 = I_1 + I_3$$

$$3I_1 + 2I_2 = 7$$

$$2I_2 + 4I_3 = 8$$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 3I_1 + 2I_2 = 7 \\ I_2 + 2I_3 = 4 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 3 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 13$$

$$D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 7 & 2 & 0 \\ 4 & 1 & 2 \end{vmatrix} = 13$$

$$D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 3 & 7 & 0 \\ 0 & 4 & 2 \end{vmatrix} = 26$$

$$D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 3 & 2 & 7 \\ 0 & 1 & 4 \end{vmatrix} = 13$$

$$\underline{I_1 = 1 \text{ A}} \quad \underline{I_2 = 2 \text{ A}} \quad \underline{I_3 = 1 \text{ A}}$$

**OR**

$$\left( \begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 3 & 2 & 0 & 7 \\ 0 & 1 & 2 & 4 \end{array} \right) \quad R_2 - 3R_1$$

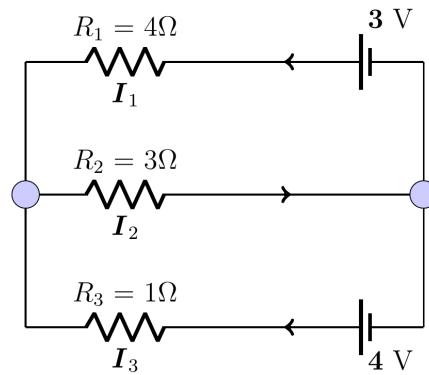
$$\left( \begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 5 & -3 & 7 \\ 0 & 1 & 2 & 4 \end{array} \right) \quad -5R_3 + R_2$$

$$\left( \begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 5 & -3 & 7 \\ 0 & 0 & -13 & -13 \end{array} \right) \quad \begin{array}{l} I_1 = I_2 - I_3 \\ 5I_2 = 3I_3 + 7 \\ \underline{I_3 = 1} \end{array}$$

$$\underline{I_2 = 2} \quad \underline{I_1 = 1}$$

### Exercise

Determine the currents  $I_1$ ,  $I_2$ , and  $I_3$  for the electrical network shown below



### Solution

$$I_2 = I_1 + I_3$$

$$4I_1 + 3I_2 = 3$$

$$3I_2 + I_3 = 4$$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 4I_1 + 3I_2 = 3 \\ 3I_2 + I_3 = 4 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 4 & 3 & 0 \\ 0 & 3 & 1 \end{vmatrix} = 19$$

$$D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 3 & 3 & 0 \\ 4 & 3 & 1 \end{vmatrix} = 0$$

$$D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 4 & 3 & 0 \\ 0 & 4 & 1 \end{vmatrix} = 19$$

$$D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 4 & 3 & 3 \\ 0 & 3 & 4 \end{vmatrix} = 19$$

$$\underline{I_1 = 0 \text{ A}} \quad \underline{I_2 = 1 \text{ A}} \quad \underline{I_3 = 1 \text{ A}}$$

**OR**

$$\left( \begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 4 & 3 & 0 & 3 \\ 0 & 3 & 1 & 4 \end{array} \right) \quad R_2 - 4R_1$$

$$\left( \begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 7 & -4 & 3 \\ 0 & 3 & 1 & 4 \end{array} \right) \quad 7R_3 - 3R_2$$

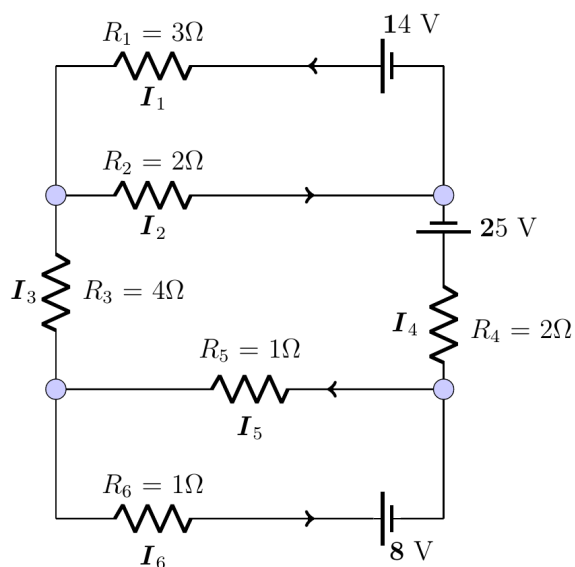
$$\left( \begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 7 & -4 & 3 \\ 0 & 0 & 19 & 19 \end{array} \right) \quad \begin{array}{l} \rightarrow I_1 = I_2 - I_3 \quad (2) \\ \rightarrow 7I_2 = 4I_3 + 3 \quad (1) \end{array}$$

$$\underline{I_3 = 1}$$

$$\underline{I_2 = 1} \quad \underline{I_1 = 0}$$

## Exercise

Determine the currents  $I_1$ ,  $I_2$ ,  $I_3$ ,  $I_4$ ,  $I_5$ , and  $I_6$  for the electrical network shown below



## Solution

$$I_1 + I_3 = I_2 \rightarrow I_1 - I_2 + I_3 = 0$$

$$I_1 + I_4 = I_2 \rightarrow I_1 - I_2 + I_4 = 0$$

$$I_3 + I_6 = I_5 \rightarrow I_3 - I_5 + I_6 = 0$$

$$\begin{cases} 3I_1 + 2I_2 = 14 \\ 2I_2 + 4I_3 + I_5 + 2I_4 = 25 \\ I_5 + I_6 = 8 \end{cases}$$

$$\left[ \begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 & 14 \\ 0 & 2 & 4 & 2 & 1 & 0 & 25 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array} \right] \quad \begin{array}{l} R_2 - R_1 \\ R_4 - 3R_1 \end{array}$$

$$\left[ \begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 2 & 4 & 2 & 1 & 0 & 25 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array} \right] \quad \begin{array}{l} R_4 \\ R_2 \\ R_3 \end{array}$$



$$\left[ \begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 2 & 4 & 2 & 1 & 0 & 25 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array} \right] \quad \begin{array}{l} 5R_3 - 2R_2 \\ \\ R_5 + R_4 \end{array}$$

$$\left[ \begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array} \right] \quad 26R_4 + R_3$$

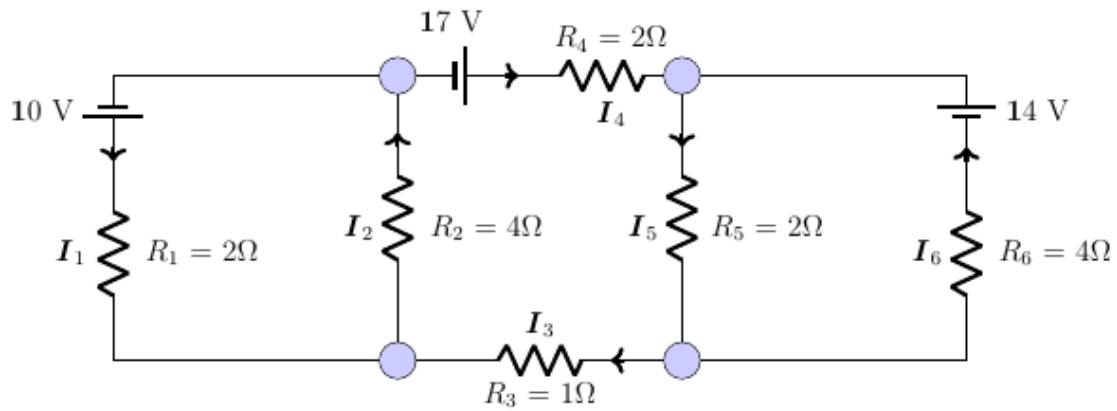
$$\left[ \begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & 0 & 36 & 5 & 0 & 97 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array} \right] \quad 36R_5 - R_4$$

$$\left[ \begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & 0 & 36 & 5 & 0 & 97 \\ 0 & 0 & 0 & 0 & -41 & 36 & -97 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array} \right] \quad 41R_6 + R_5$$

$$\left[ \begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & 0 & 36 & 5 & 0 & 97 \\ 0 & 0 & 0 & 0 & -41 & 36 & -97 \\ 0 & 0 & 0 & 0 & 0 & 77 & 231 \end{array} \right] \quad \begin{array}{ll} I_1 = 4 - 2 & \rightarrow \underline{I_1 = 2} \\ 5I_2 = 14 + 3(2) & \rightarrow \underline{I_2 = 4} \\ 26I_3 = 97 - 10(2) - 5(5) & \rightarrow \underline{I_3 = 2} \\ 36I_4 = 97 - 5(5) & \rightarrow \underline{I_4 = 2} \\ -41I_5 = -97 - 36(3) & \rightarrow \underline{I_5 = 5} \\ 77I_6 = 231 & \rightarrow \underline{I_6 = 3} \end{array}$$

## Exercise

Determine the currents  $I_1$ ,  $I_2$ ,  $I_3$ ,  $I_4$ ,  $I_5$ , and  $I_6$  for the electrical network shown below



## Solution

$$1 \rightarrow I_1 + I_3 = I_2$$

$$2 \rightarrow I_1 + I_4 = I_2$$

$$3 \rightarrow I_3 + I_6 = I_5$$

$$4 \rightarrow I_4 + I_6 = I_5$$

$$\left\{ \begin{array}{l} I_1 - I_2 + I_3 = 0 \\ I_1 - I_2 + I_4 = 0 \\ I_3 - I_5 + I_6 = 0 \\ I_4 - I_5 + I_6 = 0 \\ 2I_1 + 4I_2 = 10 \\ 4I_2 + I_3 + 2I_4 + 2I_5 = 17 \\ 2I_5 + 4I_6 = 14 \end{array} \right.$$

$$\left( \begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 5 \\ 0 & 4 & 1 & 2 & 2 & 0 & 17 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \end{array} \right) \begin{array}{l} \\ R_2 - R_1 \\ \\ R_5 - R_1 \\ \end{array}$$

$$\begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & | & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & | & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & | & 5 \\ 0 & 4 & 1 & 2 & 2 & 0 & | & 17 \\ 0 & 0 & 0 & 0 & 1 & 2 & | & 7 \end{pmatrix} \quad \begin{array}{l} R_3 + R_2 \\ \\ \\ 3R_6 - 4R_5 \end{array}$$

$$\begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & | & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & | & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & | & 5 \\ 0 & 0 & 7 & 6 & 6 & 0 & | & 31 \\ 0 & 0 & 0 & 0 & 1 & 2 & | & 7 \end{pmatrix} \quad R_4 - R_3$$

$$\begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & | & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & | & 5 \\ 0 & 0 & 7 & 6 & 6 & 0 & | & 31 \\ 0 & 0 & 0 & 0 & 1 & 2 & | & 7 \end{pmatrix} \quad R_6 + 7R_2$$

$$\begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 & | & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & | & 5 \\ 0 & 0 & -1 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & | & 0 \\ 0 & 0 & 0 & 13 & 6 & 0 & | & 31 \\ 0 & 0 & 0 & 0 & 1 & 2 & | & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \quad R_5 - 13R_4$$

$$\begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 & | & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & | & 5 \\ 0 & 0 & -1 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & 19 & -13 & | & 31 \\ 0 & 0 & 0 & 0 & 1 & 2 & | & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \quad 19R_6 - R_5$$

$$\left( \begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 19 & -13 & 31 \\ 0 & 0 & 0 & 0 & 0 & 51 & 102 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \begin{array}{l} (4) \\ (3) \\ (2) \\ (1) \\ I_5 = \frac{1}{19}(31 + 13I_6) \\ \underline{I_6 = 2} \end{array}$$

$$\underline{I_5 = 3}$$

$$(1) \rightarrow \underline{I_4 = I_5 - I_6 = 1}$$

$$(2) \rightarrow \underline{I_3 = I_4 = 1}$$

$$(3) \rightarrow \underline{I_2 = \frac{1}{3}(I_3 + 5) = 2}$$

$$(4) \rightarrow \underline{I_1 = I_2 - I_3 = 1}$$

### Exercise

Consider the invertible matrix:  $A = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{pmatrix}$

The message: **ICEBERG DEAD AHEAD**

- Write the uncoded for the message.
- Use the matrix  $A$  to encode the message.
- Decode a message from part  $b$ ) given the matrix  $A$ .

### Solution

$a)$

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

$$\begin{array}{cccccc} I & C & E & B & E & R & G & _ & D & E & A & D & _ & A & H & E & A & D \\ [9 & 3 & 5] & [2 & 5 & 18] & [7 & 0 & 4] & [5 & 1 & 4] & [0 & 1 & 8] & [5 & 1 & 4] \end{array}$$

- Let encode the message **ICEBERG DEAD AHEAD**

$$\begin{bmatrix} 9 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 29 & 80 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & 18 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} -37 & 3 & 175 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} -5 & 6 & 42 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} -4 & 9 & 47 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} -21 & -5 & 65 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} -4 & 9 & 47 \end{bmatrix}$$

The sequence of coded row matrices is

$$\begin{bmatrix} 3 & 29 & 80 \end{bmatrix} \begin{bmatrix} -37 & 3 & 175 \end{bmatrix} \begin{bmatrix} -5 & 6 & 42 \end{bmatrix} \begin{bmatrix} -4 & 9 & 47 \end{bmatrix} \begin{bmatrix} -21 & -5 & 65 \end{bmatrix} \begin{bmatrix} -4 & 9 & 47 \end{bmatrix}$$

The cryptogram:

$$3 \ 29 \ 80 \ -37 \ 3 \ 175 \ -5 \ 6 \ 42 \ -4 \ 9 \ 47 \ -21 \ -5 \ 65 \ -4 \ 9 \ 47$$

c) To decode a message given the matrix  $A$ .

$$|A| = \begin{vmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{vmatrix} = 1$$

$$A^{-1} = \begin{bmatrix} 67 & -18 & 4 \\ -48 & 13 & -3 \\ 15 & -4 & 1 \end{bmatrix}$$

With the cryptogram:

$$\begin{bmatrix} 3 & 29 & 80 \end{bmatrix} \begin{bmatrix} -37 & 3 & 175 \end{bmatrix} \begin{bmatrix} -5 & 6 & 42 \end{bmatrix} \begin{bmatrix} -4 & 9 & 47 \end{bmatrix} \begin{bmatrix} -21 & -5 & 65 \end{bmatrix} \begin{bmatrix} -4 & 9 & 47 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 29 & 80 \end{bmatrix} \begin{bmatrix} 67 & -18 & 4 \\ -48 & 13 & -3 \\ 15 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} -37 & 3 & 175 \end{bmatrix} \begin{bmatrix} 67 & -18 & 4 \\ -48 & 13 & -3 \\ 15 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 18 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 6 & 42 \end{bmatrix} \begin{bmatrix} 67 & -18 & 4 \\ -48 & 13 & -3 \\ 15 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 9 & 47 \end{bmatrix} \begin{bmatrix} 67 & -18 & 4 \\ -48 & 13 & -3 \\ 15 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -21 & -9 & 65 \end{bmatrix} \begin{bmatrix} 67 & -18 & 4 \\ -48 & 13 & -3 \\ 15 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 8 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 9 & 47 \end{bmatrix} \begin{bmatrix} 67 & -18 & 4 \\ -48 & 13 & -3 \\ 15 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 4 \end{bmatrix}$$

The message is:

9 3 5 2 5 18 7 0 1 5 1 4 0 1 8 5 1 4  
*I C E B E R G \_ D E A D \_ A H E A D*

## Exercise

You want to send the message: **LINEAR ALGEBRA** with a key word **MATH**

- Write the matrix  $A$ .
- Write the uncoded for the message.
- Use the matrix  $A$  to encode the message.
- Decode a message from part  $b$ ) given the matrix  $A$ .

## Solution

$a)$

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

$M \quad A \quad T \quad H$

13   1   20   8

$$A = \begin{pmatrix} 13 & 1 \\ 20 & 8 \end{pmatrix}$$

**b)**  $L \quad I \quad N \quad E \quad A \quad R \quad \_ \quad A \quad L \quad G \quad E \quad B \quad R \quad A$   
 12   9   14   5   1   18   0   1   12   7   5   2   18   1

$$\begin{bmatrix} 12 & 9 \end{bmatrix} \quad \begin{bmatrix} 14 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 18 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 12 & 7 \end{bmatrix} \quad \begin{bmatrix} 5 & 2 \end{bmatrix} \quad \begin{bmatrix} 18 & 1 \end{bmatrix}$$

**c)** Encoding the message

$$\begin{bmatrix} 12 & 9 \end{bmatrix} \begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = \begin{bmatrix} 336 & 84 \end{bmatrix}$$

$$\begin{bmatrix} 14 & 5 \end{bmatrix} \begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = \begin{bmatrix} 282 & 54 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 18 \end{bmatrix} \begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = \begin{bmatrix} 373 & 145 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = \begin{bmatrix} 20 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 12 & 7 \end{bmatrix} \begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = \begin{bmatrix} 296 & 68 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 2 \end{bmatrix} \begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = \begin{bmatrix} 105 & 21 \end{bmatrix}$$

$$\begin{bmatrix} 18 & 1 \end{bmatrix} \begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = \begin{bmatrix} 254 & 26 \end{bmatrix}$$

The cryptogram:

336   84   282   54   373   145   20   8   296   68   105   21   254   26

**d)** To decode a message given the matrix  $A$ .

$$A^{-1} = \frac{1}{84} \begin{bmatrix} 8 & -1 \\ -20 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix}$$

With the cryptogram:

$$\begin{bmatrix} 336 & 84 \end{bmatrix} \quad \begin{bmatrix} 282 & 54 \end{bmatrix} \quad \begin{bmatrix} 373 & 145 \end{bmatrix} \quad \begin{bmatrix} 20 & 8 \end{bmatrix} \quad \begin{bmatrix} 296 & 68 \end{bmatrix} \quad \begin{bmatrix} 105 & 21 \end{bmatrix} \quad \begin{bmatrix} 254 & 26 \end{bmatrix}$$

$$\begin{bmatrix} 336 & 84 \end{bmatrix} \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = \begin{bmatrix} 12 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 282 & 54 \end{bmatrix} \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = \begin{bmatrix} 14 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 373 & 145 \end{bmatrix} \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = \begin{bmatrix} 1 & 18 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 8 \end{bmatrix} \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 296 & 68 \end{bmatrix} \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = \begin{bmatrix} 12 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 105 & 21 \end{bmatrix} \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = \begin{bmatrix} 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 254 & 26 \end{bmatrix} \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = \begin{bmatrix} 18 & 1 \end{bmatrix}$$

12 9 14 5 1 18 0 1 12 7 5 2 18 1  
*L I N E A R \_ A L G E B R A*

The message is: *Linear Algebra*

### Exercise

Consider the invertible matrix:  $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

The message: *Math hard*

- Write the uncoded for the message.
- Use the matrix  $A$  to encode the message.
- Decode a message from part  $b$ ) given the matrix  $A$ .

### Solution



a)

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

<i>M</i>	<i>A</i>	<i>T</i>	<i>H</i>	<i>_</i>	<i>H</i>	<i>A</i>	<i>R</i>	<i>D</i>
13	1	20	8	0	8	1	18	4

b)  $[13 \ 1] \ [20 \ 8] \ [0 \ 8] \ [1 \ 18] \ [4 \ 0]$

$$[13 \ 1] \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = [27 \ 14]$$

$$[20 \ 8] \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = [48 \ 28]$$

$$[0 \ 8] \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = [8 \ 8]$$

$$[1 \ 18] \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = [20 \ 19]$$

$$[4 \ 0] \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = [8 \ 4]$$

The sequence of coded row matrices is

$$[27 \ 14] \ [48 \ 28] \ [8 \ 8] \ [20 \ 19] \ [8 \ 4]$$

The cryptogram: 27 14 48 28 8 8 20 19 8 4

c)  $A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

$$[27 \ 14] \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = [13 \ 1]$$

$$[48 \ 28] \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = [20 \ 8]$$

$$[8 \ 8] \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = [0 \ 8]$$

$$[20 \ 19] \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = [1 \ 18]$$

$$\begin{bmatrix} 8 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \end{bmatrix}$$

13 1 20 8 0 8 1 18 4 0  
*M A T H \_ H A R D \_*

The message is: *Math hard*

### ***Exercise***

Consider the invertible matrix:  $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Decode the cryptogram 27 14 48 28 5 5 21 20 50 25

### **Solution**

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\begin{bmatrix} 27 & 14 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 48 & 28 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 20 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 21 & 20 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 19 \end{bmatrix}$$

$$\begin{bmatrix} 50 & 25 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 0 \end{bmatrix}$$

13 1 20 8 0 5 1 19 25 0  
*M A T H \_ E A S Y \_*

The message is: *Math Easy*

### Exercise

Consider the invertible matrix:  $A = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix}$

Decode the cryptogram

$$\begin{array}{cccccccccccc} 1 & -5 & 11 & 19 & -25 & -45 & 11 & -16 & -28 & 20 & -29 & -27 \\ 12 & -12 & -53 & 40 & -61 & -35 & 8 & -17 & 7 & & & \end{array}$$

### Solution

$$|A| = 1$$

$$A^{-1} = \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix}$$

With the cryptogram:

$$\begin{array}{cccc} [1 & -5 & 11] & [19 & -25 & -45] & [11 & -16 & -28] & [20 & -29 & -27] \\ [12 & -12 & -53] & [40 & -61 & -35] & [8 & -17 & 7] & & & \end{array}$$

$$[1 \quad -5 \quad 11] \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix} = [4 \quad 9 \quad 6]$$

$$[19 \quad -25 \quad -45] \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix} = [6 \quad 5 \quad 18]$$

$$[11 \quad -16 \quad -28] \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix} = [5 \quad 14 \quad 20]$$

$$[20 \quad -29 \quad -27] \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix} = [9 \quad 1 \quad 12]$$

$$[12 \quad -12 \quad -53] \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix} = [0 \quad 5 \quad 17]$$

$$[40 \quad -61 \quad -35] \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix} = [21 \quad 1 \quad 20]$$

$$\begin{bmatrix} 8 & -17 & 7 \end{bmatrix} \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix} = \begin{bmatrix} 9 & 15 & 14 \end{bmatrix}$$

4 9 6 6 5 18 5 14 20 9 1 12 0 5 17 21 1 20 9 15 14  
*D I F F E R E N T I A L \_ E Q U A T I O N*

The message is: *Differential Equation*

### *Exercise*

### *Solution*