Section 2.2 – Techniques for Finding Derivatives

Notations for the Derivative

The derivative of $y = f(x)$ may be written in any of the following ways:

1st *derivative*	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}[f(x)]$	$D_x[y]$

Constant Rule

$$\frac{d}{dx}[c] = f'(c) = 0 \qquad c \text{ is constant}$$

Proof:

Let $f(x) = c$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{c-c}{\Delta x}$$

$$= 0$$

So, $\frac{d}{dx}[c] = 0$

Example

Find the derivative

a) $f(x) = 9$

$f' = 0$

b) $h(t) = \pi$

$D_t[h(t)] = 0$

c) $y = 2^3$

$\frac{dy}{dx} = 0$

Power Rule

$$f(x) = x^n \quad \Rightarrow \quad f'(x) = nx^{n-1}$$ $\quad$ ***n*** is any real number

Proof

Let $f(x) = x^n$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2}x^{n-2}(\Delta x)^2 + \ldots + (\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} nx^{n-1}\Delta x + \frac{n(n-1)}{2}x^{n-2}\Delta x + \ldots + (\Delta x)^{n-1}$$

$$= nx^{n-1}$$

Example

Find the derivative

a) $y = x^6$

$y' = 6x^{6-1}$

$= 6x^5$

b) $y = t$

$y' = t^{1-1}$

$= t^0$

$= 1$

c) $y = \dfrac{1}{x^3}$

$y = x^{-3}$

$y' = -3x^{-3-1}$

$= -3x^{-4}$ $\qquad$ *or* $\; -\dfrac{3}{x^4}$

e) $D_x\left(x^{4/3}\right)$

$$D_x\left(x^{4/3}\right)=\frac{4}{3}x^{1/3}$$

f) $y=\sqrt{z}$

$$\frac{dy}{dz}=\frac{d}{dz}\left[z^{1/2}\right]$$

$$=\frac{1}{2}z^{1/2-1}$$

$$=\frac{1}{2}z^{-1/2} \qquad \frac{1}{2z^{1/2}} \qquad \frac{1}{2\sqrt{z}}$$

Constant Times a Function

If f is a differentiable function of x, and c is a real number, then

$$\text{If } f(x) = k.g(x) \qquad \Rightarrow f' = k.g'$$

Example

a) If $y = 8x^4$, find $\frac{dy}{dx}$

$$\frac{dy}{dx} = 8\left(4x^3\right)$$

$$= 32x^3$$

b) If $y = -\frac{3}{4}x^{12}$, find $\frac{dy}{dx}$

$$\frac{dy}{dx} = -\frac{3}{4}\left(12x^{11}\right)$$

$$= -9x^{11}$$

c) If $D_t(-8t)$

$$D_t(-8t) = -8$$

d) If $y = \frac{6}{x}$, find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d}{dx}\left[6x^{-1}\right]$$

$$= -6x^{-2}$$

$$= -\frac{6}{x^2}$$

e) $y = \dfrac{9}{4x^2}$

$$= \frac{9}{4}x^{-2}$$

$$\rightarrow y' = \frac{9}{4}(-2)x^{-3}$$

$$= -\frac{9}{2x^3}$$

Sum or Difference Rule

The derivative of the sum or difference of two differentiable functions is the sum or difference of their derivatives.

$$f(x) = u(x) \pm v(x) \qquad\qquad f'(x) = u'(x) \pm v'(x)$$

Example

Find the derivative of each function

a) $y = 6x^3 + 15x^2$

$$y' = 18x^2 + 30x$$

b) $p(t) = 12t^4 - 6\sqrt{t} + \frac{5}{t}$

$$p(t) = 12t^4 - 6t^{1/2} + 5t^{-1}$$

$$p' = 48t^3 - 3t^{-1/2} - 5t^{-2}$$

$$= 48t^3 - \frac{3}{t^{1/2}} - \frac{5}{t^2}$$

c) $f(x) = \dfrac{x^3 + 3\sqrt{x}}{x}$

$$f(x) = \frac{x^3}{x} + 3\frac{x^{1/2}}{x}$$

$$= x^2 + 3x^{-1/2}$$

$$f'(x) = 2x - \frac{3}{2}x^{-3/2}$$

$$= 2x - \frac{3}{2x^{3/2}}$$

$$= 2x - \frac{3}{2\sqrt{x^3}}$$

d) $f(x) = \left(4x^2 - 3x\right)^2$ $\qquad\qquad (a+b)^2 = a^2 + 2ab + b^2$

$$= 16x^4 - 24x^3 + 9x^2$$

$$f' = 64x^3 - 72x^2 + 18x$$

Example

Find the slope of the graph of $f(x) = x^2 - 5x + 1$ at the point (2, -5)

<u>*Solution*</u>

$$f'(x) = 2x - 5$$

$$\begin{aligned} \text{Slope} &= f'(2) \\ &= 2(2) - 5 \\ &= -1 \end{aligned}$$

Example

Researchers have determined that the daily energy requirements of female beagles who are at least 1 year old change with respect to age according to the function

$$E(t) = 753t^{-0.1321}$$

where $E(t)$ is the daily energy requirements $\left(in\ kJ/W^{0.67}\right)$ for a dog that is t years old.

a) Find $E'(t)$

$$\begin{aligned} E' &= 753(-0.1321)t^{-0.1321-1} \\ &= -99.4713t^{-1.1321} \end{aligned}$$

b) Determine the rate of change of the daily energy requirements of a 2-year old female beagle

$$\begin{aligned} E'(2) &= -99.4713(2)^{-1.1321} \\ &= -45.4\ \ kJ/W^{0.67} \end{aligned}$$

The daily energy requirements of a 2-year old female beagle are decreasing at the rate

Example

From 1998 through 2005, the revenue per share *R* (in dollars) for McDonald's Corporation can be modeled by

$$R = 0.0598t^2 - 0.379t + 8.44 \qquad\qquad 8 \le t \le 15$$

Where t represents the year, with $t = 8$ corresponding to 1998. At what rate was McDonald's revenue per share changing in 2003?

<u>*Solution*</u>

$2003 \Rightarrow t = 13$

$$R' = 0.1196t - 0.379$$

$$\begin{aligned} \to R' &= 0.1196(13) - 0.379 \\ &= 1.1758 \end{aligned}$$

Marginal Analysis

Profit = P Revenue = R Cost = C $P = R - C$

The derivatives of these quantities are called ***Marginal***

$$\frac{dP}{dx} = \text{Marginal Profit}$$

$$\frac{dR}{dx} = \text{Marginal Revenue}$$

$$\frac{dC}{dx} = \text{Marginal Cost}$$

Marginal Cost

Example

Suppose that the total cost in hundreds of dollars to produce x thousand barrels of a beverage is given by

$$C(x) = 4x^2 + 100x + 500$$

Find the marginal cost for the following values of x.

a) $x = 5$

Solution

$$C' = 8x + 100$$
$$C'(5) = 8(5) + 100$$
$$= 140$$

After 5 thousand barrels, the cost will be 140 (hundred dollars) or \$14,000.00

b) $x = 30$

Solution

$$C'(30) = 8(30) + 100$$
$$= 340$$

After 30 thousand barrels, the cost will be \$34,000.00

Demand Functions

The numbers of unit q that are willing to purchase at a given price per unit p

$$p = f(q)$$

Total Revenue ***R***

Related to the price per unit and the quantity demanded (or sold): $R(q) = q.p$

Example

The demand function for a certain product is given by $p = \dfrac{50,000 - q}{25,000}$

Find the marginal revenue when $q = 10,000$ units and p is in dollars.

<u>*Solution*</u>

$$\begin{aligned}
R &= q.p \\
&= q\left(\frac{50,000 - q}{25,000}\right) \\
&= \frac{50,000q - q^2}{25,000} \\
&= \frac{50,000q}{25,000} - \frac{q^2}{25,000} \\
&= 2q - \frac{1}{25,000}q^2
\end{aligned}$$

$$R' = 2 - \frac{2}{25,000}q$$

$$\begin{aligned}
R'(10000) &= 2 - \frac{2}{25,000}(10000) \\
&= 1.2
\end{aligned}$$

When $q = 10,000$ units; the marginal revenue is \$1.20 per unit.

Example

Find the revenue function and marginal revenue for a demand function of $P = 2000 - 4x$

<u>*Solution*</u>

Revenue = quantity * price

Revenue: $R = x.P$

$$\begin{aligned}
&= x(2000 - 4x) \\
&= 2000x - 4x^2
\end{aligned}$$

Marginal: $R' = 2000 - 8x$

Marginal Profit

Example

Suppose that the function for the product $p = \dfrac{50,000 - x}{25,000}$ is given by

$$C(x) = 2100 + 0.25x \qquad \text{where } 0 \le x \le 30,000$$

Find the marginal profit from the production of the following numbers of units.

a) $x = 15,000$

b) $x = 25,000$

<u>*Solution*</u>

a) $R(x) = 2x - \dfrac{1}{25,000}x^2$

The profit is given by: $P = R - C$

$$P = 2x - \frac{1}{25,000}x^2 - (2100 + 0.25x)$$

$$= 2x - \frac{1}{25,000}x^2 - 2100 - 0.25x$$

$$= -\frac{1}{25,000}x^2 + 1.75x - 2100$$

$$P' = -\frac{2}{25,000}x + 1.75$$

$$P'(15000) = -\frac{2}{25,000}(15000) + 1.75$$

$$= 0.55$$

The marginal profit is \$0.55 per unit

b) $P'(25000) = -\dfrac{2}{25,000}(25000) + 1.75$

$$= -0.25$$

The marginal profit is $= -\$0.25$ per unit, which will reduce the profit.

Exercises *Section* 2.2 – Techniques for Finding Derivatives

Find the derivative of

1. $f(x) = -2$

2. $y = \pi$

3. $y = \sqrt{5}$

4. $f(x) = x^4$

5. $s(t) = \dfrac{1}{t}$

6. $y = 4x^2$

7. $y = \dfrac{9}{4x^2}$

8. $y = \dfrac{9}{(4x)^2}$

9. $y = \sqrt{5x}$

10. $f(x) = 16x^{1/2}$

11. $y = \sqrt[3]{x}$

12. $y = \dfrac{t}{4}$

13. $y = \dfrac{0.4}{\sqrt{x^3}}$

14. $y = -\dfrac{2}{\sqrt[3]{x}}$

15. $y = \dfrac{1}{\sqrt[3]{x}}$

16. $y = \dfrac{x^3 - 4x}{\sqrt{x}}$

17. $f(x) = 3x^2 + 2x$

18. $f(x) = 4 + 2x^3 - 3x^{-1}$

19. $f(x) = \dfrac{5}{3x^2} - \dfrac{2}{x^4} + \dfrac{x^3}{9}$

20. $f(x) = \dfrac{3}{x^{3/5}} - \dfrac{6}{x^{1/2}}$

21. $f(x) = \dfrac{5}{x^{1/5}} - \dfrac{8}{x^{3/2}}$

22. $y = \dfrac{1.2}{\sqrt{x}} - 3.2x^{-2} + x$

23. $f(x) = x^2 - 3x - 4\sqrt{x}$

24. $f(x) = 3\sqrt[3]{x^4} - 2x^3 + 4x$

25. $f(x) = 0.05x^4 + 0.1x^3 - 1.5x^2 - 1.6x + 3$

26. $y = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$

27. $f(t) = -3t^2 + 2t - 4$

28. $g(x) = 4\sqrt[3]{x} + 2$

29. $f(x) = x\left(x^2 + 1\right)$

30. $f(x) = \frac{2x^2 - 3x + 1}{x}$

31. $f(x) = \frac{4x^3 - 3x^2 + 2x + 5}{x^2}$

32. $f(x) = \frac{-6x^3 + 3x^2 - 2x + 1}{x}$

33. Find the slope of the graph of $f(x) = x^2 - 5x + 1$ at the point (2, –5)

34. Find an equation of the tangent line to the graph of $f(x) = -x^2 + 3x - 2$ at the point (2, 0)

35. Find the slope of the graph of $f(x) = x^3$ when $x = -1$, 0, and 1.

36. The height h (in feet) of a free-falling object at time (in seconds) is given by $h = -16t^2 + 180$. Find the average velocity of the object over each interval.
   a. [0, 1]
   b. [1, 2]

37. Give the position function of a diver who jumps from a board 12 feet high with initial velocity 16 feet per second. Then find the diver's velocity function.

38. An analyst has found that a company's costs and revenues in dollars for one product are given by

$$C(x) = 2x \qquad\qquad R(x) = 6x - \frac{x^2}{1000}$$

Respectively, where x is the number of items produced.

   a) Find the marginal cost function
   b) Find the marginal revenue function
   c) Using the fact that profit is the difference between revenue and costs, find the marginal profit function.
   d) What value of x makes the marginal profit is 0.
   e) Find the profit when the marginal profit is 0.

39. A business sells 2000 units per month at a price \$10 each. If monthly sales increases 200 units for each \$0.10 reduction in price.

40. From 1998 through 2005, the revenue per share R (in dollars) for McDonald's Corporation can be modeled by

$$R = 0.0598t^2 - 0.379t + 8.44 \qquad\qquad 8 \le t \le 15$$

Where t represents the year, with $t = 8$ corresponding to 1998. At what rate was McDonald's revenue per share changing in 2003?

41. The cost **C** (in dollars) of producing x units of a product is given by $C = 3.6\sqrt{x} + 500$

a) Find the additional cost when the production increases from 9 to 10 units.
b) Find the marginal cost when $x = 9$
c) Compare the results of parts (*a*) and (*b*)

42. The revenue **R** (in dollars) of renting x apartments can be modeled by $R = 2x\left(900 + 32x - x^2\right)$

a) Find the additional revenue when the number of rentals is increased from 14 to 15
b) Find the marginal revenue when $x = 14$
c) Compare the results of parts (*a*) and (*b*)

43. The profit **P** (in dollars) of selling x units of calculus textbooks is given by

$$P = -0.05x^2 + 20x - 1000$$

a) Find the additional profit when the sales increase from 150 to 151 units.
b) Find the marginal profit when $x = 150$
c) Compare the results of parts (*a*) and (*b*)

44. From 1998 through 2005, the revenue per share R (in dollars) for McDonald's Corporation can be modeled by

$$R = 0.0598t^2 - 0.379t + 8.44 \qquad\qquad 8 \le t \le 15$$

Where t represents the year, with $t = 8$ corresponding to 1998. At what rate was McDonald's revenue per share changing in 2003?

45. The profit derived from selling x units, is given by $P = 0.0002x^3 + 10x$, find the marginal profit for a production level of 100 units. Compare this with the actual gain in profit by increasing production from 100 to 101 units.

46. The Cost of producing x hamburgers is $C = 5000 + 0.56x, \quad 0 \le x \le 50{,}000$ and the revenue function is given by

$$R = \frac{1}{20000}\left(60000x - x^2\right)$$

Compare the marginal profit when 10,000 units are produced with the actual increase in profit from 10,000 units to 10,001 units

47. An object moves along the y-axis (marked in feet) so that its position at time x (in seconds) is

$$f(x) = x^3 - 6x^2 + 9x$$

a) Find the instantaneous velocity function v.

b) Find the velocity at $x = 2$ and $x = 5$ seconds

c) Find the time(s) when the velocity is 0.

48. A company's total sales (in millions of dollars) t months from now are given by

$$S(t) = 0.03t^3 + 0.5t^2 + 2t + 3$$

a) Find $S'(t)$.

b) Find $S(5)$ and $S'(5)$ (to two decimal places). Write a brief verbal interpretation of these results.

c) Find $S(10)$ and $S'(10)$ (to two decimal places). Write a brief verbal interpretation of these results.

49. A company's total sales (in millions of dollars) t months from now are given by

$$S(t) = 0.015t^4 + 0.4t^3 + 3.4t^2 + 10t - 3$$

a) Find $S'(t)$.

b) Find $S(4)$ and $S'(4)$ (to two decimal places). Write a brief verbal interpretation of these results.

c) Find $S(8)$ and $S'(8)$ (to two decimal places). Write a brief verbal interpretation of these results.

50. A marine manufacturer will sell $N(x)$ power boats after spending \$$x$ thousand on advertising, as given by

$$N(x) = 1{,}000 - \frac{3{,}780}{x} \quad 5 \le x \le 30$$

a) Find $N'(x)$.

b) Find $N(20)$ and $N'(20)$ (to two decimal places). Write a brief verbal interpretation of these results.

51. A company manufactures and sells x transistor radios per week. If the weekly cost and revenue equations are

$$C(x) = 5{,}000 + 2x \quad R(x) = 10x - \frac{x^2}{1{,}000} \quad 0 \le x \le 8{,}000$$

Then find the approximate changes in revenue and profit if production is increased from 2,000 to 2,010 per week.

52. A company manufactures fuel tanks for cars. The total weekly cost (in dollars) of producing x tanks given by

$$C(x) = 10,000 + 90x - 0.05x^2$$

a) Find the marginal cost function.
b) Find the marginal cost at a production level of 500 tanks per week.
c) Interpret the result of part b.
d) Find the exact cost of producing the 501^{st} item.

53. A company's market research department recommends the manufacture and marketing of a new headphone set for MP3 players. After suitable test marketing, the research department presents the following *price-demand* equation:

$$x = 10,000 - 1,000p \quad \rightarrow \quad p = 10 - 0.001x$$

Where x is the number of headphones that retailers are likely to buy at \$$p$ per set.
The financial department provides the cost function

$$C(x) = 7,000 + 2x$$

Where \$7,000 is the estimate of fixed costs (tooling and overhead) and \$2 is the estimate of variable costs per headphone set (materials, labor, marketing, transportation, storage, etc.).
a) Find the domain of the function defined by the price demand function.
b) Find and interpret the marginal cost function $C'(x)$.
c) Find the revenue function as a function of x and find its domain.
d) Find the marginal revenue at x = 2,000, 5,000, and 7,000. Interpret these results.
e) Graph the cost function and the revenue function in the same coordinate system, Find the intersection points of these two graphs and interpret the results.
f) Find the profit function and its domain and sketch the graph of the function.
g) Find the marginal profit at x = 1,000, 4,000, and 6,000. Interpret these results.

54. A small machine shop manufactures drill bits used in the petroleum industry. The manager estimates that the total daily cost (in dollars) of producing x bits is

$$C(x) = 1,000 + 25x - 0.1x^2$$

a) Find $\overline{C}(x)$ and $\overline{C}'(x)$
b) Find $\overline{C}(10)$ and $\overline{C}'(10)$. Interpret these quantities.
c) Use the results in part (***b***) to estimate the average cost per bit at a production level of 11 bits per day.

55. The total profit (in dollars) from the sale of x calendars is

$$P(x) = 22x - 0.2x^2 - 400 \qquad 0 \le x \le 100$$

a) Find the exact profit from the sale of the 41^{st} calendar.
b) Use the marginal profit to approximate the profit from the sale of the 41^{st} calendar.

56. The total profit (in dollars) from the sale of x cameras is

$$P(x) = 12x - 0.02x^2 - 1{,}000 \qquad 0 \le x \le 600$$

Evaluate the marginal profit at the given values of x, and interpret the results.

a) $x = 200$.

b) $x = 350$.

57. The total profit (in dollars) from the sale of x gas grills is

$$P(x) = 20x - 0.02x^2 - 320 \qquad 0 \le x \le 1{,}000$$

a) Find the average profit per grill if 40 grills are produced.

b) Find the marginal average profit at a production level of 40 grills and interpret the results.

c) Use the results from parts (*a*) and (*b*) to estimate the average profit per grill if 41 grills are produced.

58. The price p (in dollars) and the demand x for a particular steam iron are related by the equation

$$x = 1{,}000 - 20p$$

a) Express the price p in terms of the demand x, and find the domain of this function.

b) Find the revenue $R(x)$ from the sale of x steam irons. What is the domain of R?

c) Find the marginal revenue at a production level of 400 steam irons and interpret the results.

d) Find the marginal revenue at a production level of 650 steam irons and interpret the results.

59. The price-demand equation and the cost function for the production of TVs are given respectively, by

$$x = 9{,}000 - 30p \quad \textit{and} \quad C(x) = 150{,}000 + 30x$$

Where x is the number of TVs that can be sold at a price of \$$p$ per TV and $C(x)$ is the total cost (in dollars) of producing x TVs.

a) Express the price p as a function of the demand x, and find the domain of this function.

b) Find the marginal cost.

c) Find the revenue function and state its domain.

d) Find the marginal revenue.

e) Find $R'(3{,}000)$ and $R'(6{,}000)$ and interpret these quantities.

f) Graph the cost function and the revenue function on the same coordinate system for $0 \le x \le 9{,}000$. Find the break–even points and indicate regions of loss and profit.

g) Find the profit function in terms of x.

h) Find the marginal profit.

i) Find $P'(1{,}500)$ and $P'(4{,}500)$ and interpret these quantities

60. The total cost and the total revenue (in dollars) for the production and sale of x hair dryers are given, respectively, by

$$C(x) = 5x + 2{,}340 \quad \textit{and} \quad R(x) = 40x - 0.1x^2 \quad 0 \le x \le 400$$

a) Find the value of x where the graph of $R(x)$ has a horizontal tangent line.

b) Find the profit function $P(x)$.

c) Find the value of x where the graph of $P(x)$ has a horizontal tangent line.

d) Graph $C(x)$, $R(x)$, and $P(x)$ on the same coordinate system for $0 \le x \le 400$. Find the break–even points. Find the x intercept of the graph of $P(x)$.