3.4 L'Hôpital Rule

$\frac{0}{0}$: $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$

Ex

derivative

$$\lim_{x \to 0} \frac{3x - \sin x}{x} = \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{3 - \cos x}{1}$$

$$= 3 - 1$$

$$= \underline{2}$$

$$\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} = \frac{1-1}{0} = \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1+x}}}{1}$$

$$= \underline{\frac{1}{2}}$$

$$\lim_{x \to 0} \frac{\sqrt{1+x} - 1 - \frac{x}{2}}{x^2} = \frac{1-1}{0} = \frac{0}{0} \rightarrow \frac{1}{2}(1+x)^{-1/2}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1+x}} - \frac{1}{2}}{2x} = \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{-\frac{1}{4}(1+x)^{-3/2}}{2}$$

$$= \underline{-\frac{1}{8}}$$

Ex

$$\lim_{x\to 0} \frac{x-\sin x}{x^3} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{1-\cos x}{3x^2} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin x}{6x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\cos x}{6}$$

$$= \frac{1}{6}$$

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = \frac{0}{0}$$

$$= \lim_{\theta\to 0} \frac{\cos\theta}{1}$$

$$= 1$$

$$\lim_{x\to 0} \frac{1-\cos x}{x+x^2} = \frac{1-1}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin x}{1+2x}$$

$$= \frac{0}{1}$$

$$= 0$$

$$\lim_{x \to 0} \frac{\sin x}{x^2} = \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\cos x}{2x}$$

$$= \frac{1}{0}$$

$$= \infty$$

$$\lim_{x \to 0^-} \frac{\sin x}{x^2} = \frac{0}{0}$$

$$= \lim_{x \to 0^-} \frac{\cos x}{2x}$$

$$= \frac{1}{0^-}$$

$$= -\infty$$

$$\frac{\infty}{\infty} \qquad \infty - 0 \qquad \infty - \infty$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\sec x}{1 + \tan x} = \frac{\infty}{\infty}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\sec x \tan x}{\sec^2 x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\sec x}$$

$$= \lim_{x \to \frac{\pi}{2}} \underbrace{\left\{\frac{\sin x}{\cos x}\right\}}_{\tan x} \cdot \overbrace{\cos x}^{\frac{1}{\sec x}}$$

$$= \lim_{x \to \frac{\pi}{2}} \sin x$$

$$= 1$$

$$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}} \quad \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x}}$$

$$= \frac{1}{\infty}$$

$$= 0$$

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{e^x}{2x} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{e^x}{2}$$

$$= \infty$$

$$\lim_{x\to\infty} \left(x \cdot \sin\frac{1}{x}\right) = \infty \cdot 0 \qquad \sin\frac{1}{\infty} = \sin 0$$

$$= \lim_{\frac{1}{x}\to 0} \frac{\sin\frac{1}{x}}{\frac{1}{x}} \qquad x\to\infty, \ \frac{1}{x}\to 0$$

$$= 1$$

$$\lim_{x\to 0^+} \sqrt{x}\ln x = 0(-\infty)$$

$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}\frac{1}{x^{3/2}}} \qquad \frac{x^{3/2}}{x}$$

$$= -2\lim_{x\to 0^+} x^{1/2}$$

$$= 0$$

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \infty - \infty$$

$$= \lim_{x\to 0} \frac{x - \sin x}{x\sin x}$$

$$= \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin x}{\cos x + \cos x - x\sin x}$$

$$= \lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x}$$

$$= \frac{0}{2}$$

$$= 0$$

Indeterminate Power $\quad 1^{\infty}$

If $\lim_{x \to a} \ln f(x) = L$

$$\lim_{x \to a} f(x) = \lim_{x \to a} e^{\ln f(x)}$$
$$= e^{L}$$

$\lim_{x \to 0^+} (1+x)^{1/x} = e$ $\quad$ Prove

$$\lim_{x \to 0^+} (1+x)^{1/x} = 1^{\infty}$$

$$f(x) = (1+x)^{1/x}$$

$$\lim_{x \to 0^+} \ln f(x) = \lim_{x \to 0^+} \ln (1+x)^{\frac{1}{x}}$$
$$= \lim_{x \to 0^+} \left. \tfrac{1}{x} \cdot \ln(1+x) \right\} \#$$
$$= \lim_{x \to 0^+} \frac{\ln(1+x)}{x} = \frac{0}{0}$$
$$= \lim_{x \to 0^+} \frac{\frac{1}{1+x}}{1}$$
$$= 1$$

$$\lim_{x \to 0^+} \ln (1+x)^{1/x} = 1$$
$$\lim_{x \to 0^+} (1+x)^{1/x} = e^{1}$$
$$= e$$

$$\lim_{x \to \infty} x^{1/x} = \infty^0$$

$$\lim_{x \to \infty} \ln x^{1/x} = \lim_{x \to \infty} \frac{\ln x}{x} = \frac{\infty}{\infty}$$

$$= \lim_{x \to \infty} \frac{\frac{1}{x}}{1}$$

$$= 0$$

$$\lim_{x \to \infty} x^{1/x} = e^0$$

$$= \boxed{1}$$

#1 $$\lim_{x \to -2} \frac{x+2}{x^2-4} = \frac{0}{0}$$

$$= \lim_{x \to -2} \frac{1}{2x}$$

$$= \boxed{-\frac{1}{4}}$$

#5 $$\lim_{\theta \to -\frac{\pi}{3}} \frac{3\theta+\pi}{\sin(\theta+\frac{\pi}{3})} = \frac{0}{0}$$

$$= \lim_{\theta \to -\frac{\pi}{3}} \frac{3}{\cos(\theta+\frac{\pi}{3})}$$

$$= \boxed{3}$$

$$\lim_{\theta \to -\frac{\pi}{3}} \frac{3(\theta+\frac{\pi}{3})}{\sin(\theta+\frac{\pi}{3})} = 3 \lim_{\theta+\frac{\pi}{3} \to 0} \frac{1}{\frac{\sin(\theta+\pi/3)}{\theta+\frac{\pi}{3}}}$$

$$= \boxed{3}$$

#6 $\lim\limits_{x\to 0} \dfrac{x^2}{\ln(\sec x)} = \dfrac{0}{0}$

$$= \lim_{x\to 0} \frac{2x}{\frac{\sec x \tan x}{\sec x}}$$

$$= \lim_{x\to 0} \frac{2x}{\tan x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{2}{\sec^2 x}$$

$$= \boxed{2}$$

#7 $\lim\limits_{\theta\to 0} \dfrac{3^{\sin\theta}-1}{\theta} = \dfrac{1-1}{0} = \dfrac{0}{0}$ $\quad (a^u)' = u' \ln a (a^u)$

$$= \lim_{\theta\to 0} \frac{(\cos\theta)(\ln 3)\, 3^{\sin\theta}}{1}$$

$$= \boxed{\ln 3}$$

#10 $\lim\limits_{x\to 0} \dfrac{(e^x-1)^2}{x \sin x} = \dfrac{0}{0}$

$$= \lim_{x\to 0} \frac{2e^x(e^x-1)}{\sin x + x\cos x} = \frac{0}{0}$$

$$= 2 \lim_{x\to 0} \frac{e^{2x}-e^x}{\sin x + x\cos x}$$

$$= 2 \lim_{x\to 0} \frac{2e^{2x}-e^x}{2\cos x - x\sin x}$$

$$= 2\ \frac{2-1}{2}$$

$$= \boxed{1}$$

#18 $\lim\limits_{\substack{x \to 1 \\ n>0}} \dfrac{x^n - 1}{x - 1} = \dfrac{0}{0}$

$= \lim\limits_{x \to 1} \dfrac{n x^{n-1}}{1}$

$= \boxed{n}$

#28 $\lim\limits_{x \to \infty} \dfrac{4x^3 - 2x^2 + 6}{\pi x^3 + \cancel{4}} = \dfrac{\infty}{\infty}$

$= \lim\limits_{x \to \infty} \dfrac{4x^3}{\pi x^3}$

$= \boxed{\dfrac{4}{\pi}}$

#39 $\lim\limits_{x \to e^+} (\ln x)^{\frac{1}{x-e}} = (\ln e)^{\frac{1}{0}} = 1^\infty$

$\ln e = 1$

$\lim\limits_{x \to e^+} \ln (\ln x)^{\frac{1}{x-e}} = \lim\limits_{x \to e^+} \dfrac{\ln(\ln x)}{x - e} = \dfrac{0}{0}$

$= \lim\limits_{x \to e^+} \dfrac{\frac{1/x}{\ln x}}{1}$

$= \lim\limits_{x \to e^+} \dfrac{1}{x \ln x}$

$= \boxed{\dfrac{1}{e}}$

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$$\lim_{x \to \frac{\pi}{2}^+} (\sin x)^{\tan x} = 1^{\infty}$$

$$\lim_{x \to \frac{\pi}{2}^+} \ln(\sin x)^{\tan x} = \lim_{x \to \frac{\pi}{2}^+} \tan x \ln(\sin x)$$

$$= \lim_{x \to \frac{\pi}{2}^+} \frac{\ln(\sin x)}{\cot x} = \frac{0}{0}$$

$$= \lim_{x \to \frac{\pi}{2}^+} \frac{\frac{\cos x}{\sin x}}{-\csc^2 x} \rightarrow \frac{1}{\sin^2 x}$$

$$= - \lim_{x \to \frac{\pi}{2}^+} \frac{\cos x}{\sin x} \cdot \sin^2 x$$

$$= - \lim_{x \to \frac{\pi}{2}^+} \cos x \sin x$$

$$= 0$$

$$\lim_{x \to \frac{\pi}{2}^+} (\sin x)^{\tan x} = e^0 = \underline{1}$$

Review

3.1 # 16 $f(x) = x^3 - 3x^2 \quad [0, 4]$

$$f'(x) = 3x^2 - 6x = 0$$

$$3x(x-2) = 0$$

C.N: $x = 0, 2$

x	$f(x)$	
0	0	
2	-4	$\rightarrow$ Abs. Min $(2, -4)$
4	16	$\rightarrow$ abs. Max $(4, 16)$

 # 18 $f(x) = \dfrac{1}{x+2} \quad [-4, 1]$ $\rightarrow x \neq -2 \rightarrow$ Asymptote

$$f'(x) = \frac{-1}{(x+2)^2} \neq 0$$

No abs. extrema

#20 $f(x) = \sin 2x + 3 \quad [-\pi, \pi]$

$$f'(x) = 2\cos 2x = 0$$

$$2x = \pm\frac{\pi}{2} \qquad \frac{(2n+1)\pi}{2}$$

$$x = \pm\frac{\pi}{4} \qquad x = \pm\frac{(2n+1)\pi}{4}$$

$$x = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, -\frac{3\pi}{4} \qquad x = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$$

x	$f(x)$
$-\pi$	3
$-\frac{\pi}{4}$	2
$-\frac{3\pi}{4}$	4
$\frac{\pi}{4}$	4
$\frac{3\pi}{4}$	2
π	3

abs Max: $(\frac{\pi}{4}, 4)$ $(-\frac{3\pi}{4}, 4)$

abs. Min: $(-\frac{\pi}{4}, 2)$ $(\frac{3\pi}{4}, 2)$

#11 $f(x) = x^3 e^{-x} \quad [-1, 5]$

$$f'(x) = 3x^2e^{-x} - x^3e^{-x}$$

$$= \underbrace{(3x^2 - x^3)}_{=0}\underbrace{e^{-x}}_{\neq 0} = 0$$

$$x^2(3-x) = 0 \Rightarrow CN: x = 0, 3$$

x	$f(x)$
-1	$-e$
0	0
3	$27/e^3$
5	$\frac{5^3}{e^3}$

abs. Min $(-1, -e)$

abs Max $(5, \frac{5^3}{e^3})$

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$f(x) = 2x \ln x + 10 \qquad (0, 4)$

$f'(x) = 2\ln x + 2x\frac{1}{x}$

$= 2\ln x + 2 = 0 \Rightarrow \ln x = -1$

C.N : $x = e^{-1}$

$f(\frac{1}{e}) = 2\frac{1}{e}\underbrace{\ln e^{-1}} + 10$

$= 10 - \frac{2}{e}$

$x = 0 \to 10$
$x = 1 \to 10$

abs Min: $(\frac{1}{e}, 10 - \frac{2}{e})$