Solution Section 1.6 – Surface Area

Exercise

Find the lateral (side) surface area of the cone generated by revolving the line segment $y = \frac{x}{2}, \quad 0 \le x \le 4$, about the x-axis. Check your answer with the geometry formula

$$Lateral\ surface\ area = \frac{1}{2} \times base\ circumference \times slant\ height$$

Solution

$$\frac{dy}{dx} = \frac{1}{2}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$$

$$S = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

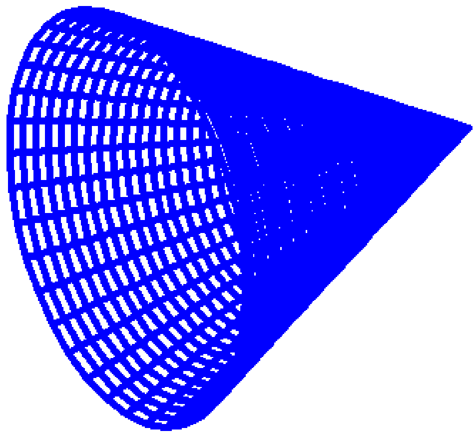
$$= 2\pi \int_0^4 \left(\frac{x}{2}\right)\frac{\sqrt{5}}{2}\, dx$$

$$= \frac{\pi\sqrt{5}}{2}\int_0^4 x dx$$

$$= \frac{\pi\sqrt{5}}{2}\frac{1}{2}x^2 \Big|_0^4$$

$$= \frac{\pi\sqrt{5}}{4}\left(4^2 - 0\right)$$

$$\boxed{= 4\pi\sqrt{5}\ \ unit^2}$$



$$base\ circumference = 2\pi r = 2\pi(2) = 4\pi$$

$$slant\ height = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$$

$$Lateral\ surface\ area = \frac{1}{2} \times base\ circumference \times slant\ height$$

$$= \frac{1}{2} \times (4\pi) \times \left(2\sqrt{5}\right)$$

$$\boxed{= 4\pi\sqrt{5}}$$

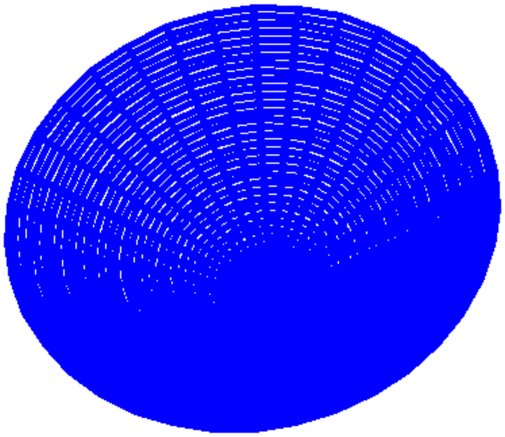
Exercise

Find the lateral surface area of the cone generated by revolving the line segment $y=\frac{x}{2}, \quad 0 \le x \le 4$, about the *y*-axis. Check your answer with the geometry formula

$$Lateral\ surface\ area = \frac{1}{2} \times base\ circumference \times slant\ height$$

Solution

$$y=\frac{x}{2} \quad \Rightarrow x=2y \to \begin{cases} x=0 & \to y=0 \\ x=4 & \to y=2 \end{cases}$$



$$\frac{dx}{dy}=2$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+4}=\sqrt{5}$$

$$S=2\pi\int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy$$

$$=2\pi\int_0^2 2y\sqrt{5}dy$$

$$=4\pi\sqrt{5}\int_0^2 ydy$$

$$=4\pi\sqrt{5}\frac{1}{2}y^2\Big|_0^2$$

$$=2\pi\sqrt{5}(4-0)$$

$$=8\pi\sqrt{5}\ unit^2$$

$$base\ circumference=2\pi(4)=8\pi$$

$$slant\ height=\sqrt{4^2+2^2}=\sqrt{20}=2\sqrt{5}$$

$$Lateral\ surface\ area=\frac{1}{2}\times base\ circumference\times slant\ height$$

$$=\frac{1}{2}\times(8\pi)\times(2\sqrt{5})$$

$$=8\pi\sqrt{5}$$

Exercise

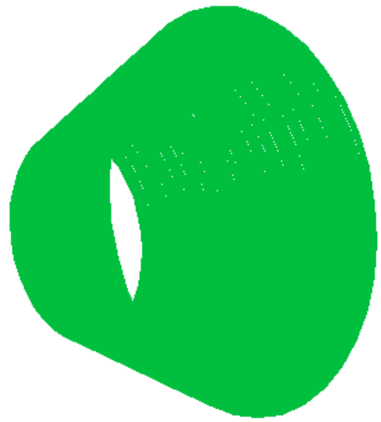
Find the lateral surface area of the cone frustum generated by revolving the line segment $y=\frac{x}{2}+\frac{1}{2},\quad 1\le x\le 3$, about the x-axis. Check your answer with the geometry formula

$$\textit{Frustum surface area}=\pi\left(r_1+r_2\right)\times \textit{slant height}$$

Solution

$$\frac{dy}{dx}=\frac{1}{2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\frac{1}{4}}=\frac{\sqrt{5}}{2}$$



$$S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=2\pi\int_1^3\left(\frac{x}{2}+\frac{1}{2}\right)\left(\frac{\sqrt{5}}{2}\right)dx$$

$$=\pi\frac{\sqrt{5}}{2}\int_1^3(x+1)\,dx$$

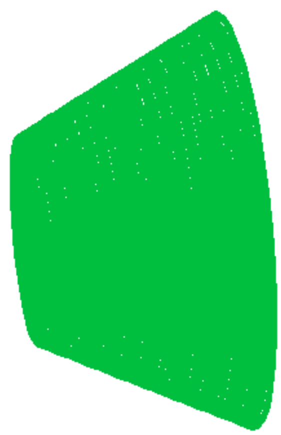
$$=\pi\frac{\sqrt{5}}{2}\left[\frac{1}{2}x^2+x\right]_1^3$$

$$=\pi\frac{\sqrt{5}}{2}\left[\frac{1}{2}(3)^2+(3)-\left(\frac{1}{2}(1)^2+(1)\right)\right]$$

$$=\pi\frac{\sqrt{5}}{2}\left[\frac{9}{2}+3-\frac{3}{2}\right]$$

$$=\pi\frac{\sqrt{5}}{2}(6)$$

$$=3\pi\sqrt{5}\ \textit{unit}^2$$



$$r_1=\frac{1}{2}+\frac{1}{2}=1\qquad r_2=\frac{3}{2}+\frac{1}{2}=2$$

$$\textit{slant height}=\sqrt{(2-1)^2+(3-1)^2}=\sqrt{5}$$

$$\textit{Frustum surface area}=\pi\left(r_1+r_2\right)\times \textit{slant height}$$

$$=\pi(1+2)\sqrt{5}$$

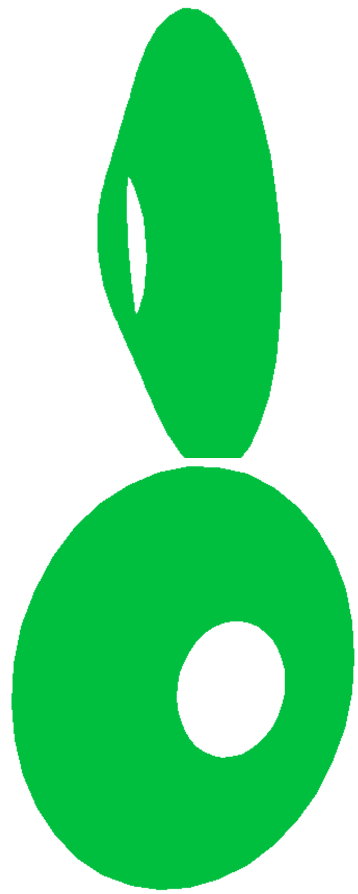
$$=3\pi\sqrt{5}$$

Exercise

Find the lateral surface area of the cone frustum generated by revolving the line segment $y=\frac{x}{2}+\frac{1}{2},\quad 1\le x\le 3$, about the *y*-axis. Check your answer with the geometry formula

$$\textit{Frustum surface area} = \pi\left(r_1+r_2\right)\times \textit{slant height}$$

Solution



$y=\frac{x}{2}+\frac{1}{2} \quad \to 2y=x+1 \Rightarrow \boxed{x=2y-1}$

$\frac{dx}{dy}=2$

$\sqrt{1+\left(\frac{dx}{dy}\right)^2}=\sqrt{1+4}=\sqrt{5}$

$S=2\pi\int_1^2 (2y-1)\left(\sqrt{5}\right)dy$

$=2\pi\sqrt{5}\int_1^2 (2y-1)dy$

$=2\pi\sqrt{5}\left[y^2-y\right]_1^2$

$=2\pi\sqrt{5}\left[(2)^2-2-\left(1^2-1\right)\right]$

$=4\pi\sqrt{5}\ \textit{unit}^2$

$r_1=1 \quad r_2=3$

$\textit{slant height}=\sqrt{(2-1)^2+(3-1)^2}=\sqrt{5}$

$\textit{Frustum surface area}=\pi\left(r_1+r_2\right)\times\textit{slant height}$

$=\pi(1+3)\sqrt{5}$

$=4\pi\sqrt{5}$

Exercise

Find the area of the surface generated by $y = \frac{x^3}{9}, \quad 0 \le x \le 2, \quad x - axis$

Solution

$$\frac{dy}{dx} = \frac{1}{3}x^2$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{9}x^4} = \frac{1}{3}\sqrt{9+x^4}$$

$$S = 2\pi \int_0^2 \frac{x^3}{9}\frac{1}{3}\sqrt{9+x^4}\,dx$$

$$= \frac{2\pi}{27}\int_0^2 x^3\sqrt{9+x^4}\,dx \qquad u = 9+x^4 \ \to du = 4x^3dx \Rightarrow \frac{1}{4}du = x^3dx$$

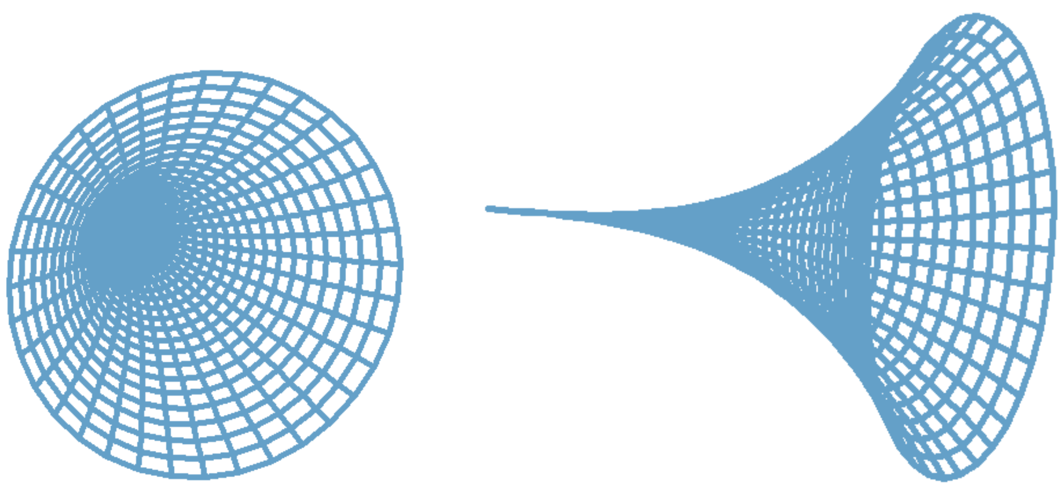
$$= \frac{2\pi}{27}\int_9^{25} u^{1/2}\left(\frac{1}{4}du\right) \qquad \to \begin{cases} x=2 & \to u = 25 \\ x=0 & \to u=9 \end{cases}$$

$$= \frac{\pi}{54}\int_9^{25} u^{1/2}du$$

$$= \frac{\pi}{54}\frac{2}{3}u^{3/2}\Big|_9^{25}$$

$$= \frac{\pi}{81}\left(25^{3/2} - 9^{3/2}\right)$$

$$= \boxed{\frac{98\pi}{81}\ unit^2}$$



Exercise

Find the area of the surface generated by $y=\sqrt{x+1}, \quad 1\le x\le 5, \quad x-axis$

Solution

$$y=\sqrt{x+1}=(x+1)^{1/2}$$

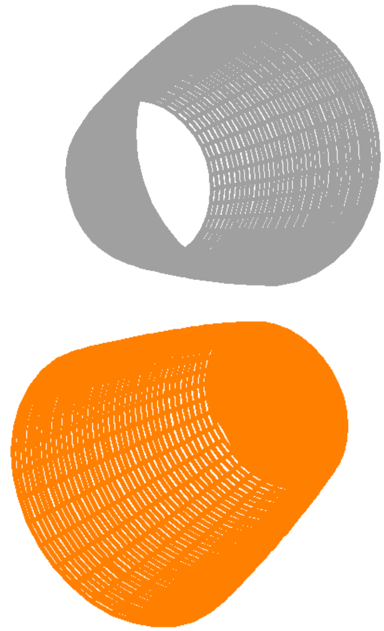
$$\frac{dy}{dx}=\frac{1}{2}(x+1)^{-1/2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\frac{1}{4}(x+1)^{-1}}$$

$$=\sqrt{1+\frac{1}{4(x+1)}}$$

$$=\sqrt{\frac{4x+4+1}{4(x+1)}}$$

$$=\frac{1}{2}\sqrt{\frac{4x+5}{x+1}}$$



$$S=2\pi\int_1^5 \sqrt{x+1}\,\frac{1}{2}\frac{\sqrt{4x+5}}{\sqrt{x+1}}dx$$

$$=\pi\int_1^5 \sqrt{4x+5}\ dx$$

$$=\frac{\pi}{4}\int_1^5 (4x+5)^{1/2}\ d(4x+5)$$

$$=\frac{\pi}{6}(4x+5)^{3/2}\Big|_1^5$$

$$=\frac{\pi}{6}\left(25^{3/2}-9^{3/2}\right)$$

$$=\frac{\pi}{6}(98)$$

$$=\frac{49\pi}{3}\ unit^2$$

Exercise

Find the area of the surface generated by $y=\sqrt{2x-x^2}, \quad 0.5 \le x \le 1.5, \quad x-axis$

Solution

$$\frac{dy}{dx}=\frac{1}{2}\left(2x-x^2\right)^{-1/2}(2-2x)=(1-x)\left(2x-x^2\right)^{-1/2}$$

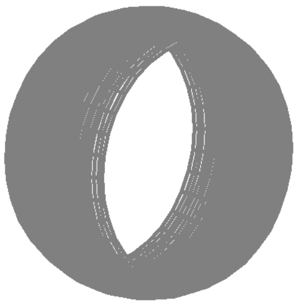
$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+(1-x)^2\left(2x-x^2\right)^{-1}}$$

$$=\sqrt{1+\frac{1-2x+x^2}{2x-x^2}}$$

$$=\sqrt{\frac{2x-x^2+1-2x+x^2}{2x-x^2}}$$

$$=\sqrt{\frac{1}{2x-x^2}}$$

$$=\frac{1}{\sqrt{2x-x^2}}$$



$$S=2\pi\int_{.5}^{1.5}\sqrt{2x-x^2}\,\frac{1}{\sqrt{2x-x^2}}dx$$

$$=2\pi\int_{.5}^{1.5}dx$$

$$=2\pi x\Big|_{.5}^{1.5} \qquad =2\pi(1.5-.5)$$

$$\underline{=2\pi\ unit^2}$$

Exercise

Find the area of the surface generated by $y=3x+4, \quad 0\le x\le 6, \quad revolved\ about\ x-axis$

Solution

$y'=3$

$$S=2\pi\int_0^6(3x+4)\sqrt{1+9}\ dx \qquad\qquad S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$=2\pi\sqrt{10}\left(\frac{3}{2}x^2+4x\right)_0^6$$

$$=2\pi\sqrt{10}(54+24)$$

$$\underline{=156\pi\sqrt{10}\ \ unit^2}$$

Exercise

Find the area of the surface generated by $y = 12 - 3x, \quad 1 \le x \le 3, \quad revolved\ about\ x-axis$

Solution

$$y' = -3$$

$$S = 2\pi \int_1^3 (12 - 3x)\sqrt{1+9}\ dx \qquad\qquad S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= 2\pi\sqrt{10}\left(12x - \frac{3}{2}x^2\right)\Big|_1^3$$

$$= 2\pi\sqrt{10}\left(36 - \frac{27}{2} - 12 + \frac{3}{2}\right)$$

$$= 24\pi\sqrt{10}\ \ unit^2$$

Exercise

Find the area of the surface generated by $y = 8\sqrt{x}, \quad 9 \le x \le 20, \quad revolved\ about\ x-axis$

Solution

$$y' = \frac{4}{\sqrt{x}}$$

$$S = 2\pi \int_9^{20} 8\sqrt{x}\sqrt{1+\frac{16}{x}}\ dx \qquad\qquad S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= 16\pi \int_9^{20} \sqrt{x}\,\frac{\sqrt{x+16}}{\sqrt{x}}\ dx$$

$$= 16\pi \int_9^{20} (x+16)^{1/2}\ d(x+16)$$

$$= \frac{32\pi}{3}(x+16)^{3/2}\Big|_9^{20}$$

$$= \frac{32\pi}{3}\left((36)^{3/2} - (25)^{3/2}\right)$$

$$= \frac{32\pi}{3}(216 - 125)$$

$$= \frac{2912\pi}{3}\ \ unit^2$$

Exercise

Find the area of the surface generated by $y = x^3, \quad 0 \le x \le 1, \quad revolved\ about\ x-axis$

Solution

$$y' = 3x^2$$

$$S=2\pi\int_0^1 x^3\sqrt{1+9x^4}\ dx \qquad\qquad S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$=\frac{\pi}{18}\int_0^1\left(1+9x^4\right)^{1/2}\ d\left(1+9x^4\right)$$

$$=\frac{\pi}{27}\left(1+9x^4\right)^{3/2}\Big|_0^1$$

$$=\frac{\pi}{27}\left((10)^{3/2}-1\right)$$

$$=\frac{\pi}{27}\left(10\sqrt{10}-1\right)\ \ unit^2$$

Exercise

Find the area of the surface generated by $y=x^{3/2}-\frac{1}{3}x^{1/2},\quad 1\le x\le 2,\quad revolved\ \ about\ \ x-axis$

Solution

$$y'=\frac{3}{2}x^{1/2}-\frac{1}{6}x^{-1/2}$$

$$=\frac{1}{2}\left(3\sqrt{x}-\frac{1}{3\sqrt{x}}\right)$$

$$=\frac{9x-1}{6\sqrt{x}}$$

$$S=2\pi\int_1^2\left(x^{3/2}-\frac{1}{3}x^{1/2}\right)\sqrt{1+\frac{(9x-1)^2}{36x}}\ dx \qquad\qquad S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$=\frac{2}{3}\pi\int_1^2\left(3x^{3/2}-x^{1/2}\right)\frac{\sqrt{36x+81x^2-18x+1}}{6\sqrt{x}}\ dx$$

$$=\frac{\pi}{9}\int_1^2(3x-1)\sqrt{81x^2+18x+1}\ dx$$

$$=\frac{\pi}{9}\int_1^2(3x-1)\sqrt{(9x+1)^2}\ dx$$

$$=\frac{\pi}{9}\int_1^2(3x-1)(9x+1)\ dx$$

$$=\frac{\pi}{9}\int_1^2\left(27x^2-6x-1\right)\ dx$$

$$=\frac{\pi}{9}\left(9x^3-3x^2-x\right)\Big|_1^2 \qquad\qquad =\frac{\pi}{9}(72-12-2-9+3+1)$$

$$=\frac{53\pi}{9}\ \ unit^2$$

Exercise

Find the area of the surface generated by $y=\sqrt{4x+6}, \quad 0\le x\le 5, \quad revolved\ about\ x-axis$

Solution

$$y'=\frac{2}{\sqrt{4x+6}}$$

$$S=2\pi\int_0^5 \sqrt{4x+6}\sqrt{1+\frac{4}{4x+6}}\ dx \qquad\qquad S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$=2\pi\int_0^5 \sqrt{4x+6}\sqrt{\frac{4x+6+4}{4x+6}}\ dx$$

$$=2\pi\int_0^5 (4x+10)^{1/2}\ dx$$

$$=\frac{\pi}{2}\int_0^5 (4x+10)^{1/2}\ d(4x+10)$$

$$=\frac{\pi}{3}(4x+10)^{3/2}\Big|_0^5$$

$$=\frac{\pi}{3}\left(30^{3/2}-10^{3/2}\right)$$

$$=\frac{\pi}{3}\left(30\sqrt{30}-10\sqrt{10}\right)$$

$$=\frac{10\pi\sqrt{10}}{3}\left(3\sqrt{3}-1\right)\ unit^2$$

Exercise

Find the area of the surface generated by $y=\frac{1}{4}\left(e^{2x}+e^{-2x}\right), \quad -2\le x\le 2, \quad revolved\ about\ x-axis$

Solution

$$y'=\frac{1}{2}\left(e^{2x}-e^{-2x}\right)$$

$$S=2\pi\int_{-2}^2 \frac{1}{4}\left(e^{2x}+e^{-2x}\right)\sqrt{1+\frac{1}{4}\left(e^{2x}-e^{-2x}\right)^2}\ dx \qquad\qquad S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$=\frac{\pi}{2}\int_{-2}^2 \left(e^{2x}+e^{-2x}\right)\frac{1}{2}\sqrt{4+e^{4x}-2+e^{-4x}}\ dx$$

$$=\frac{\pi}{4}\int_{-2}^2 \left(e^{2x}+e^{-2x}\right)\sqrt{e^{4x}+2+e^{-4x}}\ dx$$

$$=\frac{\pi}{4}\int_{-2}^2 \left(e^{2x}+e^{-2x}\right)\sqrt{\left(e^{2x}+e^{-2x}\right)^2}\ dx$$

$$= \frac{\pi}{4}\int_{-2}^{2}\left(e^{2x}+e^{-2x}\right)^2\, dx$$

$$= \frac{\pi}{4}\int_{-2}^{2}\left(e^{4x}+2+e^{-4x}\right)\, dx$$

$$= \frac{\pi}{4}\left(\frac{1}{4}e^{4x}+2x-\frac{1}{4}e^{-4x}\right)\Big|_{-2}^{2}$$

$$= \frac{\pi}{4}\left(\frac{1}{4}e^{8}+4-\frac{1}{4}e^{-8}-\frac{1}{4}e^{-8}+4+\frac{1}{4}e^{8}\right)$$

$$= \frac{\pi}{4}\left(\frac{1}{2}e^{8}+8-\frac{1}{2}e^{-8}\right)$$

$$\boxed{= \frac{\pi}{8}\left(e^{8}+16-e^{-8}\right)\ unit^2}$$

Exercise

Find the area of the surface generated by $y=\frac{1}{8}x^4+\frac{1}{4x^2}, \quad 1\le x\le 2, \quad revolved\ about\ x-axis$

Solution

$$y'=\frac{1}{2}x^3-\frac{1}{2x^3}=\frac{1}{2}\frac{x^6-1}{x^3}$$

$$S=2\pi\int_{1}^{2}\left(\frac{1}{8}x^4+\frac{1}{4x^2}\right)\sqrt{1+\left(\frac{x^6-1}{2x^3}\right)^2}\, dx \qquad S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \frac{\pi}{4}\int_{1}^{2}\left(\frac{x^6+2}{x^2}\right)\sqrt{1+\frac{x^{12}-2x^6+1}{4x^6}}\, dx$$

$$= \frac{\pi}{4}\int_{1}^{2}\left(\frac{x^6+2}{x^2}\right)\sqrt{\frac{x^{12}+2x^6+1}{4x^6}}\, dx$$

$$= \frac{\pi}{4}\int_{1}^{2}\left(\frac{x^6+2}{x^2}\right)\frac{\sqrt{\left(x^6+1\right)^2}}{2x^3}\, dx$$

$$= \frac{\pi}{4}\int_{1}^{2}\frac{\left(x^6+2\right)\left(x^6+1\right)}{2x^5}\, dx$$

$$= \frac{\pi}{4}\int_{1}^{2}\frac{x^{12}+3x^6+2}{2x^5}\, dx$$

$$= \frac{\pi}{8}\int_{1}^{2}\left(x^7+3x+2x^{-5}\right)\, dx$$

$$= \frac{\pi}{8}\left(\frac{1}{8}x^8 + \frac{3}{2}x^2 - \frac{1}{2}x^{-4}\right)\Big|_1^2$$

$$= \frac{\pi}{8}\left(32 + 6 - \frac{1}{32} - \frac{1}{8} - \frac{3}{2} + \frac{1}{2}\right)$$

$$= \frac{\pi}{8}\left(37 - \frac{5}{32}\right)$$

$$= \frac{1179\pi}{256} \quad unit^2$$

Exercise

Find the area of the surface generated by $y = \frac{1}{3}x^3 + \frac{1}{4x}, \quad \frac{1}{2} \le x \le 2, \quad revolved\ about\ x-axis$

Solution

$$y' = x^2 - \frac{1}{4x^2} = \frac{4x^4 - 1}{4x^2}$$

$$S = 2\pi\int_{1/2}^{2}\left(\frac{1}{3}x^3 + \frac{1}{4x}\right)\sqrt{1 + \left(\frac{4x^4 - 1}{4x^2}\right)^2}\, dx \qquad S = 2\pi\int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi\int_{1/2}^{2}\left(\frac{4x^4 + 3}{12x}\right)\sqrt{1 + \frac{16x^8 - 8x^4 + 1}{16x^4}}\, dx$$

$$= \frac{\pi}{6}\int_{1/2}^{2}\left(\frac{4x^4 + 3}{x}\right)\sqrt{\frac{16x^8 + 8x^4 + 1}{16x^4}}\, dx$$

$$= \frac{\pi}{6}\int_{1/2}^{2}\left(\frac{4x^4 + 3}{x}\right)\frac{\sqrt{\left(4x^4 + 1\right)^2}}{4x^2}\, dx$$

$$= \frac{\pi}{24}\int_{1/2}^{2}\left(\frac{4x^4 + 3}{x^3}\right)\left(4x^4 + 1\right)\, dx$$

$$= \frac{\pi}{24}\int_{1/2}^{2}\left(4x + 3x^{-3}\right)\left(4x^4 + 1\right)\, dx$$

$$= \frac{\pi}{24}\int_{1/2}^{2}\left(16x^5 + 16x + 3x^{-3}\right)\, dx$$

$$= \frac{\pi}{24}\left(\frac{8}{3}x^6 + 8x^2 - \frac{3}{2}x^{-2}\right)\Big|_{1/2}^{2}$$

$$= \frac{\pi}{24}\left(\frac{512}{3} + 32 - \frac{3}{8} - \frac{1}{24} - 2 + 6\right)$$

$$= \frac{\pi}{24}\left(\frac{4086}{24} + 36\right)$$

$$=\frac{\pi}{24}\left(\frac{681}{4}+36\right)$$

$$=\frac{\pi}{24}\left(\frac{825}{4}\right)$$

$$\boxed{=\frac{275\pi}{32}\quad unit^2}$$

Exercise

Find the area of the surface generated by $y=\sqrt{5x-x^2}, \quad 1\le x\le 4, \quad revolved\ about\ x-axis$

Solution

$$y'=\frac{5-2x}{2\sqrt{5x-x^2}}$$

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{(5-2x)^2}{4\left(5x-x^2\right)}$$

$$=\frac{20x-4x^2+25-20.x+4x^2}{4\left(5x-x^2\right)}$$

$$=\frac{25}{4\left(5x-x^2\right)}$$

$$S=2\pi\int_1^4\sqrt{5x-x^2}\sqrt{\frac{25}{4\left(5x-x^2\right)}}\,dx \qquad\qquad S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=5\pi\int_1^4 dx$$

$$=5\pi x\Big|_1^4$$

$$\boxed{=15\pi\ \ unit^2}$$

Exercise

Find the area of the surface generated by $y=(3x)^{1/3}; \quad 0\le x\le \frac{8}{3} \quad about\ y\text{-}axis$

Solution

$$3x=y^3\to x=\frac{1}{3}y^3 \quad\Rightarrow\ x'=y^2$$

$$\begin{cases} x=0 & \to y=0 \\ x=\frac{8}{3} & \to y=\left(3\frac{8}{3}\right)^{1/3}=2 \end{cases}$$

$$S = 2\pi\int_0^2 \frac{1}{3}y^3\sqrt{1+y^4}\ dy \qquad\qquad S = 2\pi\int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

$$= \frac{\pi}{6}\int_0^2 \left(1+y^4\right)^{1/2}\ d\left(1+y^4\right)$$

$$= \frac{\pi}{9}\left(1+y^4\right)^{3/2}\Big|_0^2$$

$$= \frac{\pi}{9}\left((17)^{3/2}-1\right)$$

$$\boxed{= \frac{\pi}{9}\left(17\sqrt{17}-1\right)}$$

Exercise

Find the area of the surface generated of the curve $y = 4x-1$ between the points $(1,\ 3)$ and $(4,\ 15)$ *about y-axis*

Solution

$$y = 4x-1 \to x = \frac{1}{4}(y+1) \quad \Rightarrow\ x' = \frac{1}{4}$$

$$S = 2\pi\int_3^{15} \frac{1}{4}(y+1)\sqrt{1+\frac{1}{16}}\ dy \qquad\qquad S = 2\pi\int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

$$= \frac{\pi}{2}\int_3^{15} (y+1)\sqrt{\frac{17}{16}}\ dy$$

$$= \frac{\pi\sqrt{17}}{8}\left(\frac{1}{2}y^2+y\right)\Big|_3^{15}$$

$$= \frac{\pi\sqrt{17}}{8}\left(\frac{225}{2}+15-\frac{9}{2}-3\right)$$

$$= \frac{\pi\sqrt{17}}{8}(120)$$

$$\boxed{= 15\pi\sqrt{17}}$$

Exercise

Find the area of the surface generated of the curve $y = \frac{1}{2}\ln\left(2x+\sqrt{4x^2-1}\right)$ between the points $\left(\frac{1}{2},\ 0\right)$ and $\left(\frac{17}{16},\ \ln 2\right)$ *about y-axis*

Solution

$$2y = \ln\left(2x+\sqrt{4x^2-1}\right) \ \to\ \left(2x+\sqrt{4x^2-1}\right)^2 = \left(e^{2y}\right)^2$$

$$4x^2+4x\sqrt{4x^2-1}+4x^2-1=e^{4y}$$

$$4x\left(2x+\sqrt{4x^2-1}\right)=e^{4y}+1 \qquad\qquad 2x+\sqrt{4x^2-1}=e^{2y}$$

$$4x\left(e^{2y}\right)=e^{4y}+1$$

$$x=\frac{e^{4y}+1}{4e^{2y}}=\frac{1}{4}\left(e^{2y}+e^{-2y}\right)$$

$$x'=\frac{1}{2}\left(e^{2y}-e^{-2y}\right)$$

$$S=2\pi\int_0^{\ln 2}\frac{1}{4}\left(e^{2y}+e^{-2y}\right)\sqrt{1+\frac{1}{4}\left(e^{2y}-e^{-2y}\right)^2}\,dy \qquad\qquad S=2\pi\int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$=\frac{\pi}{4}\int_0^{\ln 2}\left(e^{2y}+e^{-2y}\right)\sqrt{4+e^{4y}-2+e^{-4y}}\,dy$$

$$=\frac{\pi}{4}\int_0^{\ln 2}\left(e^{2y}+e^{-2y}\right)\sqrt{\left(e^{2y}+e^{-2y}\right)^2}\,dy$$

$$=\frac{\pi}{4}\int_0^{\ln 2}\left(e^{2y}+e^{-2y}\right)^2\,dy$$

$$=\frac{\pi}{4}\int_0^{\ln 2}\left(e^{4y}+2+e^{-4y}\right)\,dy$$

$$=\frac{\pi}{4}\left(\frac{1}{4}e^{4y}+2y-\frac{1}{4}e^{-4y}\right)\Big|_0^{\ln 2}$$

$$=\frac{\pi}{4}\left(\frac{1}{4}e^{4\ln 2}+2\ln 2-\frac{1}{4}e^{-4\ln 2}-\frac{1}{4}+\frac{1}{4}\right)$$

$$=\frac{\pi}{4}\left(\frac{1}{4}e^{\ln 2^4}+2\ln 2-\frac{1}{4}e^{\ln 2^{-4}}\right)$$

$$=\frac{\pi}{4}\left(\frac{1}{4}2^4+2\ln 2-\frac{1}{4}2^{-4}\right)$$

$$=\frac{\pi}{4}\left(4+2\ln 2-\frac{1}{64}\right)$$

$$\boxed{=\frac{\pi}{4}\left(\frac{255}{64}+2\ln 2\right)\ unit^2}$$

Exercise

Find the area of the surface generated by $x=\sqrt{12y-y^2};\quad 2\le y\le 10$ *about y-axis*

Solution

$$x'=\frac{6-y}{\sqrt{12y-y^2}}$$

$$S=2\pi\int_2^{10}\sqrt{12y-y^2}\sqrt{1+\frac{(6-y)^2}{12y-y^2}}\,dy \qquad\qquad S=2\pi\int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$=2\pi\int_2^{10}\sqrt{12y-y^2+36-12y+y^2}\,dy$$

$$=12\pi\int_2^{10}dy$$

$$=12\pi y\Big|_2^{10}$$

$$=96\pi\ \ unit^2$$

Exercise

Find the area of the surface generated by $x=4y^{3/2}-\frac{1}{12}y^{1/2};\quad 1\le y\le 4\quad$ *about y-axis*

Solution

$$x'=6y^{1/2}-\frac{1}{24\sqrt{y}}=\frac{144y-1}{24\sqrt{y}}$$

$$S=2\pi\int_1^4\left(4y^{3/2}-\frac{1}{12}y^{1/2}\right)\sqrt{1+\frac{(144y-1)^2}{576y}}\,dy \qquad\qquad S=2\pi\int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$=2\pi\int_1^4\left(4y^{3/2}-\frac{1}{12}y^{1/2}\right)\sqrt{\frac{576y+(144y)^2-288y+1}{576y}}\,dy$$

$$=\frac{\pi}{12}\int_1^4\left(4y^{3/2}-\frac{1}{12}y^{1/2}\right)\frac{1}{\sqrt{y}}\sqrt{(144y+1)^2}\,dy$$

$$=\frac{\pi}{144}\int_1^4(48y-1)(144y+1)\,dy$$

$$=\frac{\pi}{144}\int_1^4\left(6,912y^2-96y-1\right)dy$$

$$=\frac{\pi}{144}\left(2304y^3-48y^2-y\right)\Big|_1^4$$

$$=\frac{\pi}{144}(147,456-768-4-2304+48+1)$$

$$=\frac{144,429\pi}{144}$$

$$=\frac{48,143\pi}{48}\ \ unit^2$$

Exercise

Find the area of the surface generated by the curve $y = 1 + \sqrt{1-x^2}$ between the points $(1,\ 1)$ and $\left(\frac{\sqrt{3}}{1},\ \frac{3}{2}\right)$ *about y-axis*

Solution

$$\left(\sqrt{1-x^2}\right)^2 = (y-1)^2 \quad \Rightarrow \quad 1-x^2 = y^2-2y+1$$

$$x = \sqrt{2y-y^2} \ \rightarrow \ x' = \frac{1-y}{\sqrt{2y-y^2}}$$

$$S = 2\pi \int_1^{3/2} \sqrt{2y-y^2}\sqrt{1+\frac{(1-y)^2}{2y-y^2}}\ dy \qquad\qquad S = 2\pi \int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

$$= 2\pi \int_1^{3/2} \sqrt{2y-y^2+1-2y+y^2}\ dy$$

$$= 2\pi \int_1^{3/2} dy$$

$$= 2\pi y \Big|_1^{3/2}$$

$$= \pi \ \ unit^2$$

Exercise

Find the area of the surface generated by $x = 2\sqrt{4-y}$ $\quad 0 \le y \le \frac{15}{4}, \quad y - axis$

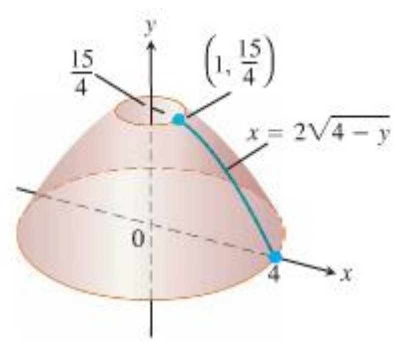
Solution

$$\frac{dy}{dx} = 2\frac{1}{2}(4-y)^{-1/2}(-1) = \frac{-1}{\sqrt{4-y}}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{4-y}}$$

$$= \sqrt{\frac{4-y+1}{4-y}}$$

$$= \sqrt{\frac{5-y}{4-y}}$$



$$S = 2\pi \int_0^{15/4} 2\sqrt{4-y}\,\frac{\sqrt{5-y}}{\sqrt{4-y}}\,dy$$

$$= 4\pi \int_0^{15/4} \sqrt{5-y}\ dy \qquad\qquad d(5-y) = -dy$$

$$= 4\pi \int_0^{15/4} (5-y)^{1/2}\ \left(-d(5-y)\right)$$

$$= -4\pi \tfrac{2}{3}(5-y)^{3/2} \Big|_0^{15/4}$$

$$= -\frac{8\pi}{3}\left[\left(5-\tfrac{15}{4}\right)^{3/2} - (5-0)^{3/2}\right]$$

$$= -\frac{8\pi}{3}\left[\left(\tfrac{5}{4}\right)^{3/2} - 5^{3/2}\right]$$

$$= -\frac{8\pi}{3}\left[\frac{5\sqrt{5}}{8} - 5\sqrt{5}\right]$$

$$= -\frac{8\pi}{3} 5\sqrt{5}\left(\tfrac{1}{8} - 1\right)$$

$$= -\frac{8\pi}{3} 5\sqrt{5}\left(-\tfrac{7}{8}\right)$$

$$\boxed{= \frac{35\pi\sqrt{5}}{3}\ unit^2}$$

Exercise

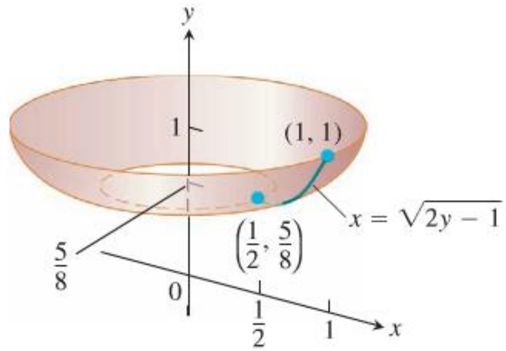
Find the area of the surface generated by $x=\sqrt{2y-1}\quad \frac{5}{8}\le y\le 1,\quad y-axis$

Solution

$$\frac{dy}{dx}=\frac{1}{2}(2y-1)^{-1/2}(2)=\frac{1}{\sqrt{2y-1}}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\frac{1}{2y-1}}$$

$$=\sqrt{\frac{2y}{2y-1}}$$



$$S=2\pi\int_{5/8}^{1}\sqrt{2y-1}\frac{\sqrt{2y}}{\sqrt{2y-1}}dy$$

$$=2\pi\int_{5/8}^{1}\sqrt{2y}\ dy \qquad\qquad u=2y\to du=2dy$$

$$=2\pi\int_{5/8}^{1}u^{1/2}\ \left(\frac{1}{2}du\right)$$

$$=\pi\int_{5/8}^{1}u^{1/2}\ du$$

$$=\frac{2\pi}{3}(2y)^{3/2}\Big|_{5/8}^{1}$$

$$=\frac{2\pi}{3}\left((2)^{3/2}-\left(\frac{5}{4}\right)^{3/2}\right)$$

$$=\frac{2\pi}{3}\left(2\sqrt{2}-\frac{5\sqrt{5}}{8}\right)$$

$$=\frac{2\pi}{3}\left(\frac{16\sqrt{2}-5\sqrt{5}}{8}\right)$$

$$=\frac{\pi}{12}\left(16\sqrt{2}-5\sqrt{5}\right)\ unit^2$$

Exercise

$y=\frac{1}{3}\left(x^2+2\right)^{3/2}, \quad 0\le x\le \sqrt{2}; \quad y-axis$ (Hint: Express $ds=\sqrt{dx^2+dy^2}$ in terms of dx, and evaluate the integral $S=\int 2\pi x\ ds$ with appropriate limits.)

Solution

$$dy=\frac{1}{3}\frac{3}{2}\left(x^2+2\right)^{1/2}(2x)dx=x\sqrt{x^2+2}\ dx$$

$$\begin{aligned}ds&=\sqrt{dx^2+\left(x\sqrt{x^2+2}\ dx\right)^2}\\&=\sqrt{dx^2+x^2\left(x^2+2\right)dx^2}\\&=\sqrt{1+x^4+2x^2}dx\\&=\sqrt{\left(1+x^2\right)^2}dx\\&=\left(1+x^2\right)dx\end{aligned}$$

$$\begin{aligned}S&=\int 2\pi x\ ds\\&=2\pi\int_0^{\sqrt{2}} x\left(1+x^2\right)\ dx \qquad\qquad d\left(1+x^2\right)=2xdx\\&=\pi\int_0^{\sqrt{2}}\left(1+x^2\right)\ d\left(1+x^2\right)\\&=\pi\frac{1}{2}u^2\Big|_1^3\\&=\frac{\pi}{2}\left(3^2-1^2\right)\\&=\frac{\pi}{2}(8)\\&=4\pi\ unit^2\end{aligned}$$

Exercise

Find the area of the surface generated by revolving the curve $x=\frac{1}{2}\left(e^{y}+e^{-y}\right), \quad 0\le y\le \ln 2,$ about y-axis

Solution

$$S=2\pi\int_0^{\ln 2}\frac{1}{2}\left(e^{y}+e^{-y}\right)\sqrt{1+\left(\frac{e^{y}-e^{-y}}{2}\right)^2}\,dy$$

$$=\pi\int_0^{\ln 2}\left(e^{y}+e^{-y}\right)\sqrt{1+\frac{e^{2y}+e^{-2y}-2}{4}}\,dy$$

$$=\pi\int_0^{\ln 2}\left(e^{y}+e^{-y}\right)\sqrt{\frac{4+e^{2y}+e^{-2y}-2}{4}}\,dy$$

$$=\frac{\pi}{2}\int_0^{\ln 2}\left(e^{y}+e^{-y}\right)\sqrt{e^{2y}+e^{-2y}+2}\,dy$$

$$=\frac{\pi}{2}\int_0^{\ln 2}\left(e^{y}+e^{-y}\right)\sqrt{\left(e^{y}+e^{-y}\right)^2}\,dy$$

$$=\frac{\pi}{2}\int_0^{\ln 2}\left(e^{y}+e^{-y}\right)^2\,dy$$

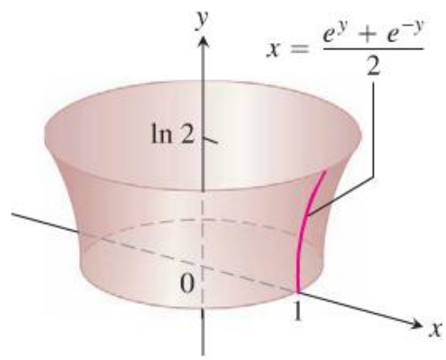
$$=\frac{\pi}{2}\int_0^{\ln 2}\left(e^{2y}+e^{-2y}+2\right)\,dy$$

$$=\frac{\pi}{2}\left[\frac{1}{2}e^{2y}-\frac{1}{2}e^{-2y}+2y\right]_0^{\ln 2}$$

$$=\frac{\pi}{2}\left[\left(\frac{1}{2}e^{2\ln 2}-\frac{1}{2}e^{-2\ln 2}+2\ln 2\right)-\left(\frac{1}{2}e^{0}-\frac{1}{2}e^{0}+0\right)\right]$$

$$=\frac{\pi}{2}\left(\frac{1}{2}\cdot 4-\frac{1}{2}\cdot\frac{1}{4}+2\ln 2\right)$$

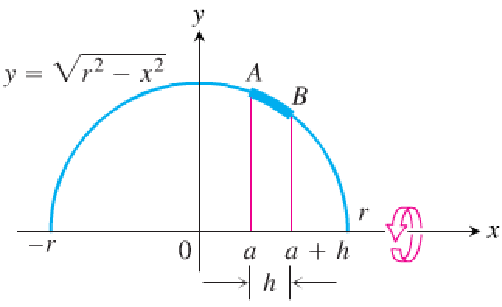
$$=\frac{\pi}{2}\left(\frac{15}{8}+2\ln 2\right)\ unit^2$$



Exercise

Did you know that if you can cut a spherical loaf of bread into slices of equal width, each slice will have the same amount of crust? To see why, suppose the semicircle $y=\sqrt{r^2-x^2}$ shown here is revolved about the x-axis to generate a sphere. Let ***AB*** be an arc of the semicircle that lies above an interval of length ***h*** on the x-axis. Show that the area swept out by ***AB*** does not depend on the location of the interval. (It does depend on the length of the interval.)

Solution



$$y=\sqrt{r^2-x^2}$$

$$\frac{dy}{dx}=\frac{-2x}{2\sqrt{r^2-x^2}}=\frac{-x}{\sqrt{r^2-x^2}}$$

$$\begin{aligned}\sqrt{1+\left(\frac{dy}{dx}\right)^2}&=\sqrt{1+\frac{x^2}{r^2-x^2}}\\&=\sqrt{\frac{r^2}{r^2-x^2}}\\&=\frac{r}{\sqrt{r^2-x^2}}\end{aligned}$$

$$\begin{aligned}S&=2\pi\int_a^{a+h}\sqrt{r^2-x^2}\,\frac{r}{\sqrt{r^2-x^2}}\,dx\\&=2\pi r\int_a^{a+h}dx\\&=2\pi r x\Big|_a^{a+h}\\&=2\pi r(a+h-a)\\&=2\pi rh\ unit^2\end{aligned}$$

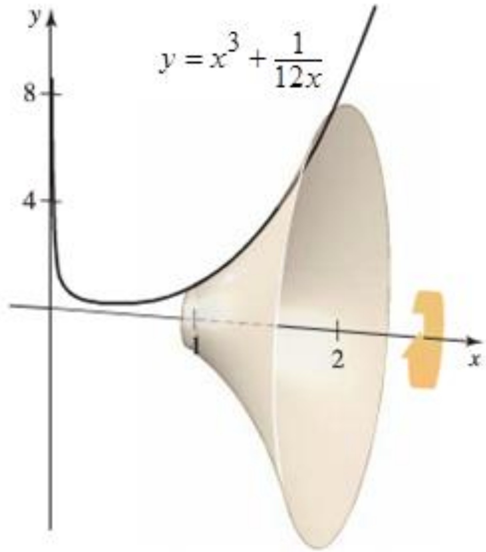
Example

The curved surface of a funnel is generated by revolving the graph of $y=f(x)=x^3+\frac{1}{12x}$ on the interval $[1,\ 2]$ about the x-*axis*. Approximately what volume of paint is needed to cover the outside of the funnel with a layer of paint 0.05 *cm* thick? Assume that x and y measured in centimeters.

Solution

$$f'(x)=3x^2-\frac{1}{12x^2}$$

$$1+f'(x)^2 = 1+\left(3x^2 - \frac{1}{12x^2}\right)^2$$
$$= 1+9x^4 - \frac{1}{2} + \frac{1}{144x^4}$$
$$= 9x^4 + \frac{1}{2} + \frac{1}{144x^4}$$
$$= \left(3x^2 + \frac{1}{12x^2}\right)^2$$



$$S = 2\pi\int_1^2 \left(x^3 + \frac{1}{12x}\right)\sqrt{\left(3x^2 + \frac{1}{12x^2}\right)^2}\,dx$$
$$= 2\pi\int_1^2 \left(x^3 + \frac{1}{12x}\right)\left(3x^2 + \frac{1}{12x^2}\right)dx$$
$$= 2\pi\int_1^2 \left(3x^5 + \frac{x}{3} + \frac{1}{144}x^{-3}\right)dx$$
$$= 2\pi\left(\frac{1}{2}x^6 + \frac{1}{6}x^2 - \frac{1}{288}x^{-2}\right)\Big|_1^2$$
$$= 2\pi\left(32 + \frac{2}{3} - \frac{1}{1152} - \frac{1}{2} - \frac{1}{6} + \frac{1}{288}\right)$$
$$= 2\pi\left(\frac{36864 + 768 - 1 - 576 - 192 + 4}{1152}\right)$$
$$\boxed{= \frac{12,289}{192}\pi \ \ cm^2}$$

Because the paint layer is 0.05 cm thick, the approximate volume of paint needed is

$$= \left(\frac{12,289}{192}\pi\ cm^2\right)(0.05\ \ cm)\ \boxed{\approx 10.1\ \ cm^3}$$

Exercise

When the circle $x^2 + (y-a)^2 = r^2$ on the interval $[-r,\ r]$ is revolved about the x-*axis*, the result is the surface of a torus, where $0 < r < a$. Show that the surface area of the torus is $S = 4\pi^2 ar$.

Solution

$$x^2 + (y-a)^2 = r^2 \quad \Rightarrow \quad (y-a)^2 = r^2 - x^2$$
$$y = a \pm \sqrt{r^2 - x^2}$$

$$f(x) = a + \sqrt{r^2 - x^2}$$

$$1+f'(x)^2 = 1+\left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2$$

$$=1+\frac{x^2}{r^2-x^2}$$

$$=\frac{r^2}{r^2-x^2}$$

$$S_1=2\pi\int_{-r}^{r}\left(a+\sqrt{r^2-x^2}\right)\frac{r}{\sqrt{r^2-x^2}}\,dx$$

$$=4\pi\int_0^r\left(\frac{ar}{\sqrt{r^2-x^2}}+r\right)dx$$

$$=4\pi\left[ar\sin^{-1}\left(\frac{x}{r}\right)+rx\right]_0^r$$

$$=4\pi\left(ar\frac{\pi}{2}+r^2\right)$$

$$\underline{=2\pi^2ar+4\pi r^2}$$

$$S_2=2\pi\int_{-r}^{r}\left(a-\sqrt{r^2-x^2}\right)\frac{r}{\sqrt{r^2-x^2}}\,dx$$

$$=4\pi\int_0^r\left(\frac{ar}{\sqrt{r^2-x^2}}-r\right)dx$$

$$=4\pi\left[ar\sin^{-1}\left(\frac{x}{r}\right)-rx\right]_0^r$$

$$=4\pi\left(ar\frac{\pi}{2}-r^2\right)$$

$$\underline{=2\pi^2ar-4\pi r^2}$$

$$S=2\pi^2ar+4\pi r^2+2\pi^2ar-4\pi r^2\ \underline{=4\pi^2ar\ \ unit^2}$$

Exercise

A 1.5–*mm* layer of paint is applied to one side. Find the approximate volume of paint needed of the spherical zone generated when the curve $y=\sqrt{8x-x^2}$ on the interval $[1,\ 7]$ is revolved about the *x-axis* Assume *x* and *y* are in *meters*.

Solution

$$y'=\frac{4-x}{\sqrt{8x-x^2}}$$

$$S=2\pi\int_1^7\sqrt{8x-x^2}\sqrt{1+\frac{(4-x)^2}{8x-x^2}}\,dx \qquad S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi \int_1^7 \sqrt{8x - x^2} \frac{\sqrt{8x - x^2 + 16 - 8x + x^2}}{\sqrt{8x - x^2}} \, dx$$

$$= 2\pi \int_1^7 \sqrt{16} \, dx$$

$$= 8\pi x \Big|_1^7$$

$$\boxed{= 48\pi \ \ m^2}$$

The volume of paint required to cover the surface to a thickness 0.0015 m is

$$V = 48\pi(0.0015) \ \boxed{\approx 0.226195 \ \ m^3} \qquad 1\, m^3 = 264.172052\, gal$$

$$= 0.226195 \times 264.172052 \ \boxed{\approx 59.75 \ \ gal}$$

Exercise

A 1.5–*mm* layer of paint is applied to one side. Find the approximate volume of paint needed of the spherical zone generated when the upper portion of the circle $x^2 + y^2 = 100$ on the interval $[-8,\ 8]$ is revolved about the x-*axis*. Assume x and y are in *meters*.

Solution

$$y = \sqrt{100 - x^2} \quad \Rightarrow \quad y' = \frac{-x}{\sqrt{100 - x^2}}$$

$$S = 2\pi \int_{-8}^{8} \sqrt{100 - x^2} \sqrt{1 + \frac{x^2}{100 - x^2}} \, dx \qquad S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$= 2\pi \int_{-8}^{8} \sqrt{100 - x^2} \frac{\sqrt{100 - x^2 + x^2}}{\sqrt{100 - x^2}} \, dx$$

$$= 20\pi \int_{-8}^{8} dx$$

$$= 20\pi x \Big|_{-8}^{8}$$

$$\boxed{= 320\pi \ \ m^2}$$

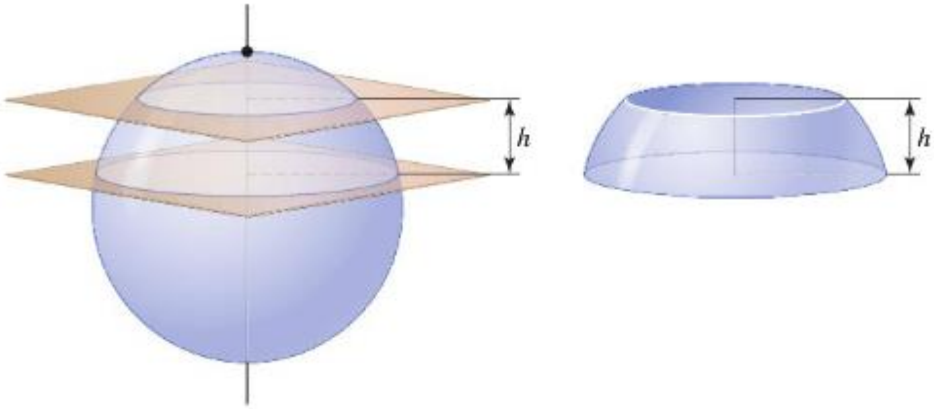
The volume of paint required to cover the surface to a thickness 0.0015 m is

$$V = 320\pi(0.0015) \ \boxed{\approx 1.507965 \ \ m^3} \qquad 1\, m^3 = 264.172052\, gal$$

$$= 1.507965 \times 264.172052 \ \boxed{\approx 398.36 \ \ gal}$$

Exercise

Suppose a sphere of radius r is sliced by two horizontal planes h units apart. Show that the surface area of the resulting zone on the sphere is $2\pi h$, independent of the location of the cutting planes.



Solution

$$f(x)=\sqrt{r^2-x^2}$$

$$1+f'(x)^2=1+\left(\frac{x}{\sqrt{r^2-x^2}}\right)^2$$

$$=1+\frac{x^2}{r^2-x^2}$$

$$=\frac{r^2}{r^2-x^2}$$

$$S=2\pi\int_a^{a+h}\sqrt{r^2-x^2}\,\frac{r}{\sqrt{r^2-x^2}}\,dx \qquad\qquad S=2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}dx$$

$$=2\pi rx\Big|_a^{a+h}$$

$$=2\pi r(a+h-a)$$

$$=2\pi rh\ \ unit^2$$

Solution Section 1.7 – Physical Applications

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = 1 + \sin x; \quad 0 \le x \le \pi$

Solution

$$m = \int_0^\pi (1 + \sin x)\,dx \qquad\qquad m = \int_a^b \rho(x)\,dx$$

$$= x - \cos x \Big|_0^\pi$$

$$= \pi + 1 + 1$$

$$= \pi + 2$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = 1 + x^3; \quad 0 \le x \le 1$

Solution

$$m = \int_0^1 \left(1 + x^3\right)dx \qquad\qquad m = \int_a^b \rho(x)\,dx$$

$$= x + \frac{1}{4}x^4 \Big|_0^1$$

$$= 1 + \frac{1}{4}$$

$$= \frac{5}{4}$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = 2 - \frac{x}{2}; \quad 0 \le x \le 2$

Solution

$$m = \int_0^2 \left(2 - \frac{x}{2}\right)dx \qquad\qquad m = \int_a^b \rho(x)\,dx$$

$$= 2x - \frac{1}{4}x^2 \Big|_0^2$$

$$= 4 - 1$$

$$= 3$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = 5e^{-2x}; \quad 0 \le x \le 4$

Solution

$$m = \int_0^4 5e^{-2x} dx \qquad\qquad m = \int_a^b \rho(x)\,dx$$

$$= -\frac{5}{2}e^{-2x}\Big|_0^4$$

$$= -\frac{5}{2}\left(e^{-8} - 1\right)$$

$$= \frac{5}{2}\left(1 - e^{-8}\right)$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = x\sqrt{2 - x^2}; \quad 0 \le x \le 1$

Solution

$$m = \int_0^1 x\sqrt{2 - x^2}\, dx \qquad\qquad m = \int_a^b \rho(x)\,dx$$

$$= -\frac{1}{2}\int_0^1 \left(2 - x^2\right)^{1/2}\, d\left(2 - x^2\right)$$

$$= -\frac{1}{3}\left(2 - x^2\right)^{3/2}\Big|_0^1$$

$$= -\frac{1}{3}\left(1 - 2\sqrt{2}\right)$$

$$= \frac{1}{3}\left(2\sqrt{2} - 1\right)$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = \begin{cases} 1 & if \ \ 0 \le x \le 2 \\ 2 & if \ \ 2 < x \le 3 \end{cases}$

Solution

$$m = \int_0^2 1\, dx + \int_2^3 2\, dx \qquad\qquad m = \int_a^b \rho(x)\,dx$$

$$= x\Big|_0^2 + (2x)\Big|_2^3$$

$$= 2 + (6 - 4)$$

$$= 4$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x)=\begin{cases} 1 & if\ \ 0\le x\le 2 \\ 1+x & if\ \ 2<x\le 4 \end{cases}$

Solution

$$m=\int_0^2 1\,dx+\int_2^4 (1+x)\,dx \qquad\qquad m=\int_a^b \rho(x)\,dx$$

$$=x\Big|_0^2+\left(x+\frac{1}{2}x^2\right)\Big|_2^4$$

$$=2+(4+8-2-2)$$

$$\underline{=10}$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x)=\begin{cases} x^2 & if\ \ 0\le x\le 1 \\ x(2-x) & if\ \ 1<x\le 2 \end{cases}$

Solution

$$m=\int_0^1 x^2\,dx+\int_1^2 \left(2x-x^2\right)\,dx \qquad\qquad m=\int_a^b \rho(x)\,dx$$

$$=\frac{1}{3}x^3\Big|_0^1+\left(x^2-\frac{1}{3}x^3\right)\Big|_1^2$$

$$=\frac{1}{3}+4-\frac{8}{3}-1+\frac{1}{3}$$

$$\underline{=1}$$

Exercise

A heavy-duty shock absorber is compressed 2 *cm* from its equilibrium position by a mass of 500 *kg*. How much work is required tocompress the shock absorber 4 *cm* from its equilibrium position? (A mass of 500 *kg* exerts a force (in newtons) of 500 *g*)

Solution

Given: $F(0.02)=500 \quad g=9.8\ m/s^2$

$$F(0.02)=0.02k=500\times 9.8 \qquad\qquad F(x)=kx=mg$$

$$k=\frac{4900}{0.02}\ \underline{=245{,}000}$$

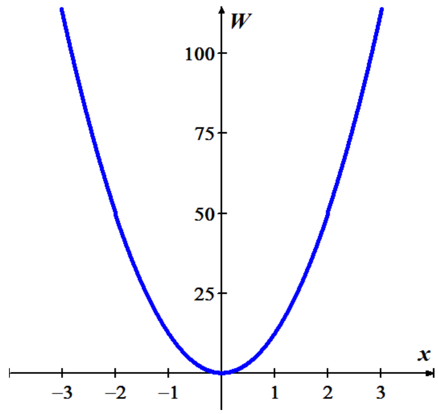
$$W=\int_0^{.04} 245{,}000x\,dx \qquad\qquad W=\int_a^b F(x)\,dx$$

$$=122{,}500x^2\Big|_0^{0.04}$$

$$\underline{=195\ \ J}$$

Exercise

A spring has a restoring force given by $F(x) = 25x$. Let $W(x)$ be the work required to stretch the spring from its equilibrium position $(x = 0)$ to a variable distance *x*. Graph the work function. Compare the work required to stretch the spring *x* units from equilibrium to the work required to compress the spring *x* units from equilibrium.



Solution

$$W = \int_0^x 25t\, dt \qquad W = \int_a^b F(x)\,dx$$

$$= \frac{25}{2}t^2 \Big|_0^x$$

$$= \boxed{\frac{25}{2}x^2}$$

Since $W(x)$ is an even function.

So that $W(-x) = W(x)$, and thus the work is the same to compress or stretch the spring a given distance from its equilibrium position,

Exercise

A swimming pool has the shape of a box with a base that measures 25 *m* by 15 *m* and a depth of 2.5 *m*. How much work is required to pump the water out of the pool when it is full?

Solution

$$W = \int_0^{2.5} \rho g A(y)(2.5 - y)\,dy$$

$$= \int_0^{2.5} (1000)(9.8)(25 \times 15)(2.5 - y)\,dy$$

$$= 3{,}675{,}000\left(2.5y - \frac{1}{2}y^2\right)\Big|_0^{2.5}$$

$$= 3{,}675{,}000\left(6.25 - \frac{6.25}{2}\right)$$

$$= \boxed{11{,}484{,}375 \ \ J}$$

Exercise

It took 1800 J of work to stretch a spring from its natural length of 2 m to a length of 5 m. Find the spring's force constant

Solution

$$W = \int_0^3 F(x)\,dx$$

$$1800 = \int_0^3 kx\,dx$$

$$1800 = \frac{1}{2}kx^2 \Big|_0^3$$

$$1800 = \frac{1}{2}k(9-0)$$

$$1800 = \frac{9}{2}k$$

$$1800\left(\frac{2}{9}\right) = k$$

$$\boxed{k = 400 \ \ N/m}$$

Exercise

How much work is required to move am object from $x=1$ to $x=5$ (measired in meters) in the presence of a constant force at 5 N acting along the x-$axis$.

Solution

$$W = \int_1^5 5\,dx$$

$$= 5(5-1)$$

$$= 20 \ \ J$$

Exercise

How much work is required to move am object from $x=0$ to $x=3$ (measired in meters) with a force (in N) is given by $F(x) = \frac{2}{x^2}$ acting along the x-$axis$.

Solution

$$W = \int_1^3 \frac{2}{x^2}\,dx$$

$$= -2\frac{1}{x}\Big|_1^3$$

$$= -2\left(\frac{1}{3} - 1\right)$$

$$\boxed{= \frac{4}{3}\ J}$$

Exercise

A spring on a horizontal surface can be stretched and held 0.5 *m* from its equilibrium position with a force of 50 *N*.

a) How much work is done in stretching the spring 1.5 *m* from its equilibrium position?

b) How much work is done in compressing the spring 0.5 *m* from its equilibrium position?

Solution

a) $f(x) = kx \quad \to \quad f(0.5) = 50 = 0.5k \ \Rightarrow \ \boxed{k = 100}$

$$W = \int_0^{1.5} 100x\, dx$$

$$= 50x^2 \Big|_0^{1.5}$$

$$\boxed{= 112.5\ J}$$

b) $$W = \int_0^{-.5} 100x\, dx$$

$$= 50x^2 \Big|_0^{-0.5}$$

$$\boxed{= 12.5\ J}$$

Exercise

Suppose a force of 10 *N* is required to stretch a spring 0.1 *m* from its equilibrium position and hold it in that position.

a) Assuming that the spring obeys Hooke's law, find the spring constant *k*.

b) How much work is needed to ***compress*** the spring 0.5 *m* from its equilibrium position?

c) How much work is needed to ***stretch*** the spring 0.25 *m* from its equilibrium position?

d) How much additional work is required to stretch the spring 0.25 *m* if it has already been stretched 0.1 *m* from its equilibrium position?

Solution

a) $F(0.1) = k(0.1) = 10$

$k = \frac{10}{0.1}\ \boxed{= 100\ N/m}$ Therfore, Hooke's law for this spring: $F(x) = 100x$

b) Work is needed to ***compress*** the spring

$$W = \int_0^{-0.5} 100x\, dx$$

$$= 50x^2 \Big|_0^{-0.5}$$

$$= 50(-0.5)^2$$

$$\underline{= 12.5\ \ J}$$

c) Work is needed to ***stretch*** the spring

$$W = \int_0^{0.25} 100x\, dx$$

$$= 50x^2 \Big|_0^{0.25}$$

$$= 50(.25)^2$$

$$\underline{= 3.125\ \ J}$$

d) Work is required to ***stretch*** the spring

$$W = \int_{0.1}^{0.35} 100x\, dx$$

$$= 50x^2 \Big|_{0.1}^{0.35}$$

$$= 50\left(0.35^2 - 0.1^2\right)$$

$$\underline{= 5.625\ \ J}$$

Exercise

A force of 200 N will stretch a garage door spring 0.8-*m* beyond its unstressed length.

a) How far will a 300-N-force stretch the spring?

b) How much work does it take to stretch the spring this far?

Solution

$$k = \frac{F}{x} = \frac{200}{0.8} \underline{= 250\ \ N/m}$$

a) $300 = 250x \quad \rightarrow \quad \underline{x = 1.2\ m}$

b) $$W = \int_0^{1.2} 250x\, dx$$

$$= 125x^2 \Big|_0^{1.2}$$

$$\underline{= 180\ \ J\ \ (N-m)}$$

Exercise

A spring has a natural length of 10 *in*. An 800-*lb* force stretches the spring to 14 *in*.

a) Find the force constant.

b) How much work is done in stretching the spring from 10 *in* to 12 *in*?

c) How far beyond its natural length will a 1600-*lb* force stretch the spring?

Solution

a) $k = \frac{F}{x} = \frac{800}{14-10} = \frac{800}{4}$

$k \ \underline{= 200 \ \text{lb} / in}$

b) $\Delta x = 12 - 10 = 2 \ in$

$$W = k\int_0^2 x dx$$

$$= 200\frac{1}{2}x^2 \Big|_0^2$$

$$= 100(4-0)$$

$$= 400 \ \ in.lb$$

$$= 400\frac{1 \ ft}{12 \ in} \ \ in.lb$$

$$\underline{= 33.3 \ \ ft \cdot lb}$$

c) $F = 200x$

$1600 = 200x$

$\frac{1600}{200} = x$

$x \ \underline{= 8 \ \ in}$

Exercise

It takes a force of 21,714 *lb* to compress a coil spring assembly on a Transit Authority subway car from its free height of 8 *in*. to its fully compressed height of 5 *in*.

a) What is the assembly's force constant?

b) How much work does it take to compress the assembly the first half inch? The second half inch? Answer to the nearest *in*-*lb*.

Solution

a) $F = kx$

$21714 = k(8-5)$

$21714 = 3k$

$k \ \underline{= 7238 \ \text{lb} / in}$

b) $W = k\int_0^{0.5} x dx$

$$= 7238\left[\frac{1}{2}x^2\right]_0^{0.5}$$

$$= 7238\left[\frac{1}{2}(0.5)^2 - 0\right]$$

$$\underline{= 905 \ \ in \cdot lb}$$

$$W = 7238\int_{0.5}^{1} x dx$$

$$= 7238\left[\frac{1}{2}x^2\right]_{0.5}^{1}$$

$$= 3619\left[1^2 - 0.5^2\right]_{0.5}^{1}$$

$$\underline{= 2714 \ \ in \cdot lb}$$

Exercise

A mountain climber is about to haul up a 50 *m* length of hanging rope. How much work will it take if the rope weighs 0.624 *N/m*?

Solution

$$W = 0.624\int_0^{50} x dx$$

$$= 0.624\left[\frac{1}{2}x^2\right]_0^{50}$$

$$= \frac{0.624}{2}\left[(50)^2 - 0\right]$$

$$\underline{= 780 \ \ J}$$

Exercise

A bag of sand originally weighing 144 *lb* was lifted a constant rate. As it rose, sand also leaked out at a constant rate. The sand was half gone by the time the bag had been lifted to 18 *ft*. How much work was done lifting the sand this far? (Neglect the weight of the bag and lifting equipment.)

Solution

The weight of sands decreases by $\frac{1}{2}144 = 72\ lb$ over the 18 ft. at rate $\frac{72}{18} = 4\ lb\ /\ ft$

$$F(x) = 144 - 4x$$

$$W = \int_0^{18} (144 - 4x)\, dx$$

$$= \left[144x - 2x^2\right]_0^{18}$$

$$= 144(18) - 2(18)^2 - (0)$$

$$\underline{= 1944 \ \ ft \cdot lb|}$$

Exercise

An electric elevator with a motor at the top has a multistrand cable weighing 4.5 *lb/ft*. When the car is at the first floor, 180 *ft* of cable are paid out, and effectively 0 *ft* are out when the car is at the top floor. How much work does the motor do just lifting the cable when it takes the car from the first floor to the top,?

Solution

$$F(x) = k\Delta x = 4.5(180 - x)$$

$$W = \int_0^{180} 4.5(180 - x)\, dx$$

$$= 4.5\left[180x - \frac{1}{2}x^2\right]_0^{180}$$

$$= 4.5\left[180(180) - \frac{1}{2}(180)^2 - 0\right]$$

$$\underline{= 72,900 \ \ ft \cdot lb|}$$

Exercise

The rectangular cistern (storage tank for rainwater) shown has its top 10 *ft* below ground level. The cistern, currently full, is to be emptied for inspection by pumping its contents to ground level. Assume that the water weighs $62.4\ lb / ft^3$

a) How much work will it take to empty the cistern?

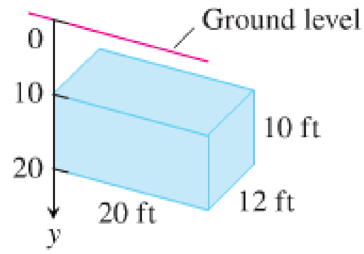
b) How long will it take a 1/-*hp* pump, rated at 275 *ft-lb/sec*, to pump the tank dry?

c) How long will it take the pump in part (*b*) to empty the tank hallway? (It will be less than half the time required to empty the tank completely)

d) What are the answers to parts (*a*) through (*c*) in a location where water weighs 62.6 lb / ft^3? 62.59 lb / ft^3?

Solution

a) $\Delta V = (20)(12)\Delta y = 240\Delta y$



$$F = 62.4(\Delta V)$$
$$= (62.4)240\Delta y$$
$$= 14976\Delta y$$

$$\Delta W = \textit{force} \times \textit{distance}$$
$$= 14976\ \Delta y \times y$$

$$W = 14976\int_{10}^{20} y\,dy$$
$$= 14976\left[\frac{1}{2}y^2\right]_{10}^{20}$$
$$= \frac{14976}{2}\left(20^2 - 10^2\right)$$
$$\underline{= 2,246,400\ \ ft\cdot lb}$$

b) $t = \dfrac{W}{275\frac{ft.lb}{\sec}}$

$$= \frac{2,246,400\ ft.lb}{275}\frac{\sec}{ft.lb}$$
$$\approx 8,168.73\ sec\frac{1\ \ hr}{3600\ sec}$$
$$\underline{\approx 2.27\ \ hrs} \quad 2\ hrs\ \ \&\ 16.1\text{min}$$

c) $W = 14976\displaystyle\int_{10}^{15} y\,dy$

$$= 14976\left[\frac{1}{2}y^2\right]_{10}^{15}$$
$$= \frac{14976}{2}\left(15^2 - 10^2\right)$$
$$\underline{= 936,000\ \ ft\cdot lb}$$

$$t = \frac{W}{275\frac{ft.lb}{\sec}}$$
$$= \frac{936,000\ ft.lb}{275}\frac{\sec}{ft.lb}$$
$$\approx 3403.64\ sec\frac{1\ \ \text{min}}{60\ sec}$$
$$\underline{\approx 56.7\ \ \text{min}}$$

d) ***Water weighs*** $62.26\ lb/\ ft^3$

$$W = (62.26)(240)(150)$$

$$\boxed{= 2,214,360 \ \ ft \cdot lb}$$

$$t = \frac{W}{275 \frac{ft.lb}{\sec}}$$

$$= \frac{2,241,360 \ ft.lb}{275} \frac{\sec}{ft.lb}$$

$$\approx 8,150.4 \ sec \frac{1 \ \ hr}{3600 \ sec}$$

$$\boxed{\approx 2.264 \ \ hrs} \quad 2 \ hrs \ \ \& \ 15.8 \min$$

$$W = (62.26)(240)\left(\frac{150}{2}\right)$$

$$\boxed{= 933,900 \ \ ft \cdot lb}$$

$$t = \frac{W}{275 \frac{ft.lb}{\sec}}$$

$$= \frac{933,900 \ ft.lb}{275} \frac{\sec}{ft.lb}$$

$$\approx 3396 \ sec \frac{1 \ \ \min}{60 \ sec}$$

$$\boxed{\approx 56.6 \ \ \min}$$

Water weighs $62.59 \ lb / \ ft^3$

$$W = (62.59)(240)(150)$$

$$\boxed{= 2,253,240 \ \ ft \cdot lb}$$

$$t = \frac{W}{275 \frac{ft.lb}{\sec}}$$

$$= \frac{2,253,240 \ ft.lb}{275} \frac{\sec}{ft.lb}$$

$$\approx 8,193.60 \ sec \frac{1 \ \ hr}{3600 \ sec}$$

$$\boxed{\approx 2.276 \ \ hrs} \quad 2 \ hrs \ \ \& \ 16.56 \min$$

$$W = (62.59)(240)\left(\frac{150}{2}\right) \quad \boxed{= 938,850 \ \ ft \cdot lb}$$

$$t = \frac{W}{275 \frac{ft.lb}{\sec}}$$

$$= \frac{938,850 \ ft.lb}{275} \frac{\sec}{ft.lb}$$

$$\approx 3414 \ sec \frac{1 \ \ \min}{60 \ sec}$$

$$\boxed{\approx 56.9 \ \ \min}$$

Exercise

When a particle of mass *m* is at $(x, 0)$, it is attracted toward the origin with a force whose magnitude is $\frac{k}{x^2}$. If the particle starts from rest at $x = b$ and is acted on by no other forces, find the work done on it by the time reaches $x = a$, $0 < a < b$.

Solution

$$F(x) = -\frac{k}{x^2}$$

$$W = \int_a^b -\frac{k}{x^2}\,dx$$

$$= k\int_a^b -\frac{1}{x^2}\,dx \qquad \int \frac{1}{x^2}\,dx = -\frac{1}{x}$$

$$= k\left[\frac{1}{x}\right]_a^b$$

$$= k\left(\frac{1}{b} - \frac{1}{a}\right)$$

$$\boxed{= \frac{k(a-b)}{ab}}$$

Exercise

The strength of Earth's gravitation field varies with the distance *r* from Earth's center, and the magnitude of the gravitational force experienced by a satellite of mass *m* during and after launch is

$$F(r) = \frac{mMG}{r^2}$$

Here, $M = 5.975 \times 10^{24}\ kg$ is Earth's mass, $G = 6.6720 \times 10^{-11}\ N \cdot m^2 kg^{-2}$ is the universal gravitational constant, and *r* is measured in meters. The work it takes to lift a 1000-*kg* satellite from Earth's surface to a circular orbit 35,780 *km* above Earth's center is therefore given by the integral

$$W = \int_{6{,}370{,}000}^{35{,}780{,}000} \frac{1000MG}{r^2}\,dr \ \ joules$$

Evaluate the integral. The lower limit of integration is Earth's radius in meters at the launch site. (This calculation does not take into account energy spend lifting the launch vehicle or energy spent bringing the satellite to orbit velocity.)

Solution

$$W = 1000MG\int_{6{,}370{,}000}^{35{,}780{,}000} \frac{dr}{r^2}$$

$$=1000MG\left[-\frac{1}{r}\right]_{6,370,000}^{35,780,000}$$

$$=1000\left(5.975\times10^{24}\right)\left(6.6720\times10^{-11}\right)\left(\frac{1}{6,370,000}-\frac{1}{35,780,000}\right)$$

$$\approx 5.144\times10^{10}\ J$$

Exercise

You drove an 800-*gal* truck from the base of Mt. Washington to the summit and discovered on arrival that the tank was only half full. You started with a full tank, climbed at a steady rate, and accomplished the 4750-*ft* elevation change in 50 minutes.
Assuming that the water leaked out at a steady rate, how much work was spent in carrying the water to the top? Do not count the work done in getting yourself and the truck there. Water weighs 8- *lb./gal*.

Solution

The force required to lift the water is equal to the water's weight which varies $8(800)$ *lbs.* to $8(400)$ *lbs.* over the 4750 *ft* change in elevation. Since it looses half of the water when the truck reaches its destination, it would loose all of the water if it went twice the distance. When the truck is *x* feet of the base of Mt. Washington, the water's weight is the following proportion.

$$F(x)=8(800)\left(\frac{2(4750)-x}{2(4750)}\right)=6400\left(1-\frac{x}{9500}\right)$$

$$W=6400\int_0^{4750}\left(1-\frac{x}{9500}\right)dx$$

$$=6400\left(x-\frac{x^2}{19000}\right)\Big|_0^{4750}$$

$$=22,800,000\ \ ft-lbs$$

Exercise

A cylindrical water tank has height 8 *m* and radius 2 *m*

a) If the tank is full of water, how much work is required to pump the water to the level of the top of the tank and out of the tank?

b) Is it true it takes half as much work to pump the water out of the tank when it is half full as when it is full? Explain.

Solution

a) $W=\rho g\int_0^a (a-y)w(y)dy$

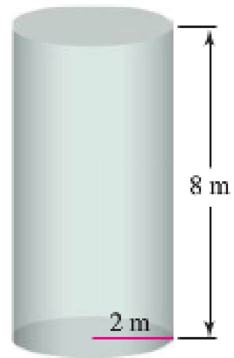
$$=1,000(9.8)\int_0^8 (8-y)\left(2^2\pi\right)dy$$

$$=39,200\pi\left(8y-\frac{1}{2}y^2\right)\Big|_0^8$$

$$=39,200\pi(64-32)$$

$$=125,400\pi$$

$$\boxed{\approx 3.941\times 10^6\ J}$$



b) The work done pumping the water from a half-full tank

$$W=9,800(4\pi)\int_4^8 (8-y)dy$$

$$=39,200\pi\left(8y-\frac{1}{2}y^2\right)\Big|_4^8$$

$$=39,200\pi(32-32+8)$$

$$\boxed{\approx 985203\ J}$$

To empty a half-full tank, the work is

$$W=39,200\pi\left(8y-\frac{1}{2}y^2\right)\Big|_0^4$$

$$=39,200\pi(32-16)$$

$$\boxed{\approx 2.9556\times 10^6\ J}$$

NO, it is not true.

Exercise

A water tank is shaped like an inverted cone with height 6 *m* and base radius 1.5 *m*.

a) If the tank is full of water, how much work is required to pump the water to the level of the top of the tank and out of the tank?

b) Is it true it takes half as much work to pump the water out of the tank when it is half full as when it is full? Explain.

Solution

$$\frac{x}{1.5}=\frac{y}{6} \quad\rightarrow\quad x=\frac{y}{4}$$

$$Area=\pi x^2=\frac{\pi}{16}y^2$$

a) $W=\rho g\int_0^a A(y)(a-y)dy$

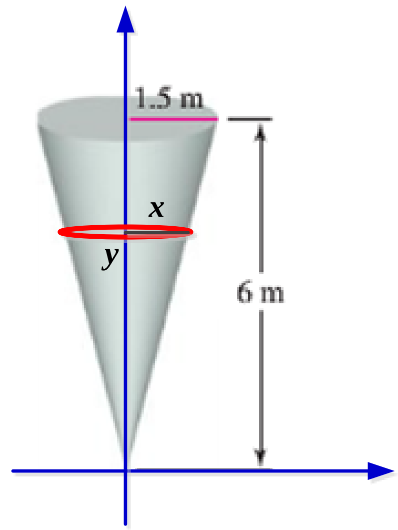
$$= 9{,}800 \int_0^6 \frac{\pi}{16} y^2 (6-y)\,dy$$

$$= 612.5\pi \int_0^6 \left(6y^2 - y^3\right) dy$$

$$= 612.5\pi \left(2y^3 - \frac{1}{4}y^4\right)\Big|_0^6$$

$$= 612.5\pi (432 - 324)$$

$$\underline{= 66{,}150\pi\ J}$$



b) $W = 9{,}800 \int_0^3 \frac{\pi}{16} y^2 (6-y)\,dy$

$$= 612.5\pi \left(2y^3 - \frac{1}{4}y^4\right)\Big|_0^3$$

$$= 612.5\pi (54 - 20.25)$$

$$\underline{\approx 20{,}672\pi\ J}$$

The work done is less than half the half amount from part (*a*).

It is not true, while the water must be raised further than water in the top half, due to the shape of the tank, there is far less water in the bottom half than in the top.

Exercise

A spherical water tank with an inner radius of 8 *m* has its lowest point 2 *m* above the ground. It is filled by a pipe that feed the tank at its lowest point.

a) Neglecting the volume of the inflow pipe, how much work is required to fill the tank if it is initially empty?

b) Now assume that the inflow pipe feeds the tank at the top of the tank. Neglecting the volume of the inflow pipe, how much work is required to fill the tank if it is initially empty?

Solution

a) Equation of the tank: $x^2 + (y-8)^2 = 64 \quad \to \quad x^2 = 16y - y^2$

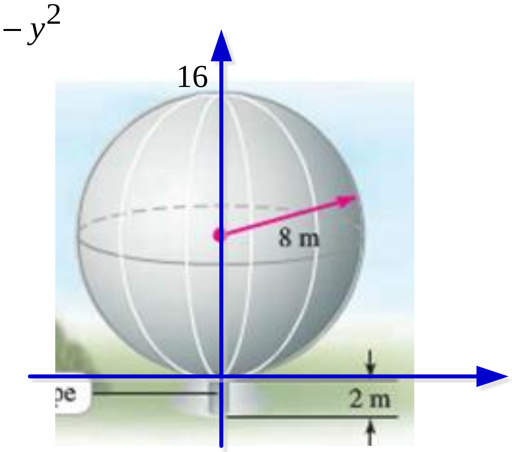
$$A(y) = \pi x^2 = \pi\left(16y - y^2\right)$$

$$W = \rho g \pi \int_0^{16} \left(16y - y^2\right) dy$$

$$= 9800\pi \left(8y^2 - \frac{1}{3}y^3\right)\Big|_0^{16}$$

$$= 9800\pi \left(16^2\right)\left(8 - \frac{16}{3}\right)$$

$$\underline{\approx 2.102 \times 10^8\ J}$$



b) The toal weight of the water lifted up for 18 *m* is

$$W = \frac{4\pi}{3} R^3 \rho g h$$

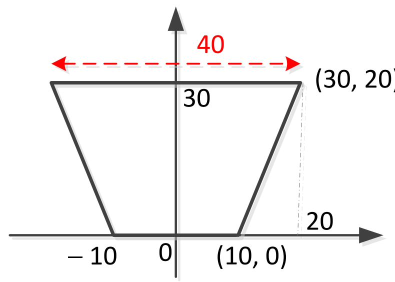
$$= \frac{4\pi}{3} 8^3 (9800)(18)$$

$$\approx 3.783 \times 10^8 \ J$$

Exercise

A large vertical dam in the shape of a symmetric trapezoid has a height of 30 *m*. a width of 20 *m*. at its base, and a width of 40 *m*. at the top. What is the total force on the face of the dam when the reservoir is full? $\left(\rho = 1000 \frac{kg}{m^3},\ g = 9.8 \frac{m}{s^2} \right)$

Solution

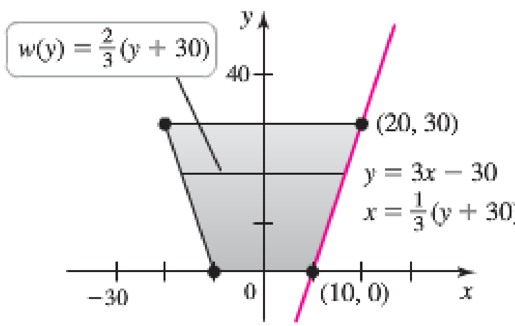


$$y - 0 = \frac{30}{10}(x - 10)$$

$$y = 3x - 30 \quad \rightarrow \quad x = \frac{1}{3}(y + 30)$$

Depth: $0 \le y \le 30$

Width: $w(y) = 2x = \frac{2}{3}(y + 30)$



$$F = \int_0^a \rho g (a - y) w(y)\, dy$$

$$= \int_0^{30} \left(10^3\right)(9.8)(30 - y)\frac{2}{3}(y + 30)\, dy$$

$$= \frac{19600}{3} \int_0^{30} \left(900 - y^2\right) dy$$

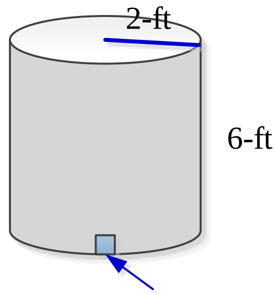
$$= \frac{19600}{3}\left(900y - \frac{1}{3}y^3\right)\Big|_0^{30}$$

$$= \frac{19600}{3}(27000 - 9000)$$

$$= 1.176 \times 10^8 \ kg$$

Exercise

Pumping water from a lake 15-*ft* below the bottom of the tank can fill the cylindrical tank shown here.



There are two ways to go about it. One is to pump the water through a hose attached to a valve in the bottom of the tank. The other is to attach the hose to the rim of the tank and let the water pour in. Which way will be faster? Give reason for answer

Solution

The water is being pumped from a lake that is 15-*ft* below the tank. That is the distance it takes to get the water from the lake to the valve, but this is not the total distance that the water is moved. We are forcing the water up into the tank, so the water travels a distance of y in the tank.

The total distance the water travels is $15 + y$.

The cross-section is a circle: $A(y) = \pi r^2 = 4\pi$

$$W = 62.4 \int_0^6 (4\pi)(15 + y)\,dy$$

$$= 249.6\pi \left(15y + \frac{1}{2}y^2\right)\Big|_0^6$$

$$= 249.6\pi(90 + 18)$$

$$\approx 84,687.3 \ \ ft - lbs$$

Now we are pumping the water to the top of the tank and letting it pour in.
Therefore, the distance that the water is pumped is $15 + 6 = 21\ ft$

$$W = 62.4 \int_0^6 21(4\pi)\,dy$$

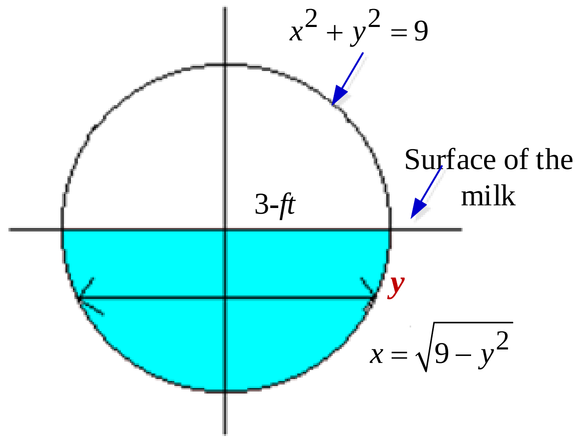
$$= 16,466.97(y)\Big|_0^6$$

$$\approx 98,801.83 \ \ ft - lbs$$

Exercise

A tank truck hauls milk in a 6-*ft* diameter horizontal right circular cylindrical tank. How must force does the milk exert on each end of the tank when the tank is half full?

Solution



Diameter = 6 $\rightarrow r = 3$

Circular cylinder: $x^2 + y^2 = 9$

$L(y) = 2x = 2\sqrt{9-y^2}$

Weight density of milk is $64.5\ lbs/ft^3$

$$F = 64.5\int_{-3}^{0} 2\sqrt{9-y^2}\,(0-y)\,dy$$

$$= 64.5\int_{-3}^{0} \left(9-y^2\right)^{1/2}\, d\left(9-y^2\right)$$

$$= 64.5\left(\frac{2}{3}\right)\left(9-y^2\right)^{3/2}\Big|_{-3}^{0}$$

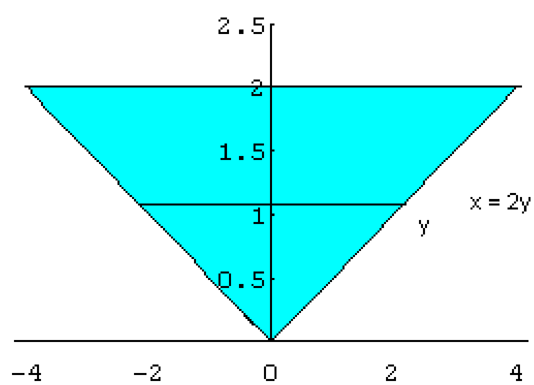
$$= 43(27)$$

$$\underline{= 1{,}161\ \ lbs}$$

Exercise

The vertical triangular plate shown here is the end plate of a trough full of water. What is the fluid force against the plate?

Solution



$L(y) = 2x = 4y$

$$F = 62.4\int_{0}^{2} (2-y)\cdot(4y)\,dy$$

$$= 249.6\int_{0}^{2} \left(2y-y^2\right)dy$$

$$= 249.6\left(y^2 - \frac{1}{3}y^3\right)\Big|_{0}^{2}$$

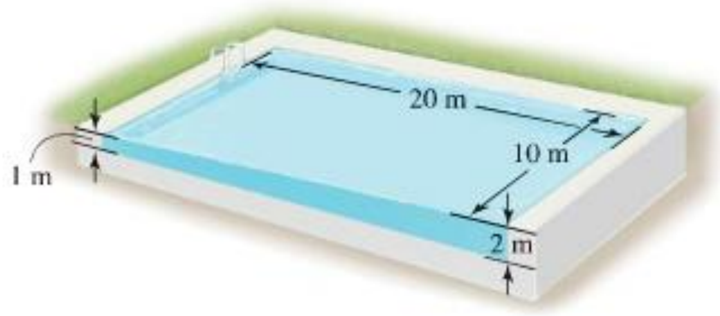
$$= 249.6\left(4-\frac{8}{3}\right)$$

$$\underline{= 332.8\ \ lbs}$$

Exercise

A swimming pool is 20 *m* leg long and 10 *m* wide, with a bottom that slopes uniformly from a depth of 1 *m* at one end to a depth of 2 m at the other end.
Assuming the pool is full, how much work is required to pump the water to a level 0.2 *m* above the top of the pool?

Solution



Depth: $(2+.2)-y=2.2-y$

From 0–1 *m*:

$$y=\frac{1}{20}(10-x) \quad \rightarrow \quad 10-x=2y$$

$$A(y)=10(20y)=200y$$

From 1–2 *m*: $A(y)=10(20)=200$

$$W=\rho g\int_0^1 200y(2.2-y)dy+\rho g\int_1^2 200(2.2-y)dy$$

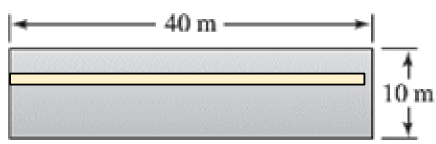
$$=(200\rho g)\left\{\left(1.1y^2-\frac{1}{3}y^3\right)\Big|_0^1+\left(2.2y-\frac{1}{2}y^2\right)\Big|_1^2\right\}$$

$$=\left(1.96\times10^6\right)\left(1.1-\frac{1}{3}+4.4-2-2.2+\frac{1}{2}\right)$$

$$\approx 2.875\times10^6 \ J$$

Exercise

Find the total force on the face of the given dam



Solution

Freshwater Weight density: $\rho=62.4\ lb/ft^3=10^3\ kg/m^3$

$$F=\int_0^{10}\left(10^3\right)(9.8)(10-y)(40)dy \qquad\qquad F=\int_0^a \rho g(a-y)w(y)dy$$

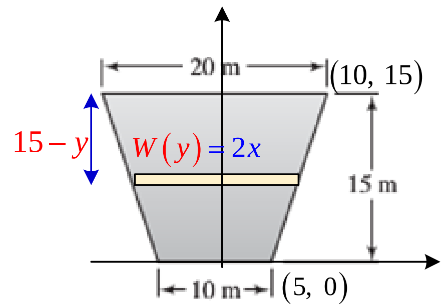
$$=392\times10^3\left(10y-\frac{1}{2}y^2\right)\Big|_0^{10}$$

$$=392\times10^3(100-50)$$

$$=196\times10^5\ \ N$$

Exercise

Find the total force on the face of the given dam



Solution

Freshwater Weight density: $\rho = 10^3\ kg/m^3$

$y = \frac{15-0}{10-5}(x-5) = 3x - 15$

$x = \frac{1}{3}(y+15) \Rightarrow \boxed{2x = \frac{2}{3}(y+15)}$

$F = \int_0^{15}\left(10^3\right)(9.8)(15-y)\frac{2}{3}(y+15)\,dy \qquad F = \int_0^a \rho g(a-y)w(y)\,dy$

$= \frac{19.6}{3}\times 10^3\int_0^{15}\left(225-y^2\right)dy$

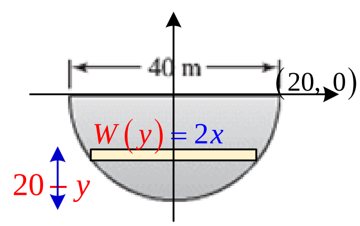
$= \frac{19.6}{3}\times 10^3\left(225y - \frac{1}{3}y^3\right)\Big|_0^{15}$

$= \frac{19.6}{3}\times 10^3\left(15^5 - \frac{1}{3}15^3\right)$

$\boxed{= 1.47\times 10^7\ \ N}$

Exercise

Find the total force on the face of the given dam



Solution

Freshwater Weight density: $\rho = 10^3\ kg/m^3$

$x^2 + (y-20)^2 = 20^2$

$x = \sqrt{400 - \left(y^2 - 40y + 400\right)} \quad \to \ 2x = 2\sqrt{40y - y^2}$

$F = \int_0^{20}\left(10^3\right)(9.8)(20-y)(2)\sqrt{40y-y^2}\,dy \qquad F = \int_0^a \rho g(a-y)w(y)\,dy$

$= 9.8\times 10^3\int_0^{20}\left(40y-y^2\right)^{1/2} d\left(40y-y^2\right).$

$= \frac{19.6}{3}\times 10^3\left(40y-y^2\right)^{3/2}\Big|_0^{20}$

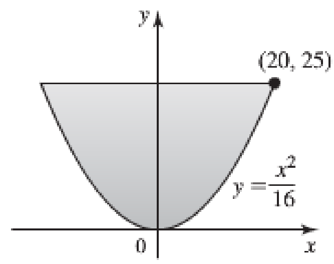
$= \frac{19.6}{3}\times 10^3(800-400)^{3/2}$

$= \frac{19.6}{3}\times 10^3(20)^3$

$\boxed{= 5.227\times 10^7\ \ N}$

Exercise

Find the total force on the face of the given dam



Solution

Freshwater Weight density: $\rho = 10^3\ kg/m^3$

$x^2 = 16y \to x = 4\sqrt{y} \quad \Rightarrow \boxed{2x = 8\sqrt{y}}$

$$F = \int_0^{25} \left(10^3\right)(9.8)(25 - y)(8)\sqrt{y}\ dy \qquad F = \int_0^a \rho g(a - y)w(y)dy$$

$$= 78.4 \times 10^3 \int_0^{25} \left(25y^{1/2} - y^{3/2}\right)dy$$

$$= 78.4 \times 10^3 \left(\frac{50}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right)\Big|_0^{25}$$

$$= 78.4 \times 10^3 \left(\frac{2}{3}5^5 - \frac{2}{5}5^5\right)$$

$$= 78.4 \times 5^5 \times 10^3 \left(\frac{4}{15}\right)$$

$$\boxed{= 6.533 \times 10^7\ \ N}$$

Exercise

A large building shaped like a box is 50 m high with a face that is 80 m wide. A strong wind blows directly at the face of the building, exerting a pressure of 150 N/m^2 at the ground and increasing with height according to $P(y) = 150 + 2y$, where y is the height above the ground. Calculate the total force on the building, which is a meaure of the resistance that must be included in the design of the building.

Solution

$$F = \int_0^{50} (150 + 2y)(80)\ dy \qquad F = \int_0^a P(y)w(y)dy$$

$$= 80\left(150y + y^2\right)\Big|_0^{50}$$

$$= 80(7500 + 2500)$$

$$\boxed{= 8 \times 10^5\ N}$$

Exercise

Adiving pool that is 4 *m* deep full of water has a viewing window on one of its vertical walls. Find the force of the a square window, 0.5 *m* on side, with the lower edge of the window on the bottom of the pool.

Solution

Freshwater Weight density: $\rho = 10^3\ kg/m^3$

$$F = \int_0^{0.5} 10^3 (9.8)(4-y)(0.5)\ dy \qquad\qquad F = \int_0^a \rho g(a-y)w(y)dy$$

$$= 4.9\times 10^3 \left(4y - \frac{1}{2}y^2\right)\Big|_0^{0.5}$$

$$= 4.9\times 10^3 \left(2 - \frac{1}{8}\right)$$

$$= 4.9\times 10^3 \left(\frac{15}{8}\right)$$

$$= 9187.5\ N$$

Exercise

Adiving pool that is 4 *m* deep full of water has a viewing window on one of its vertical walls. Find the force of the a square window, 0.5 *m* on side, with the lower edge of the window 1 *m* from the bottom of the pool.

Solution

Freshwater Weight density: $\rho = 10^3\ kg/m^3$

$$F = \int_1^{1.5} 10^3 (9.8)(4-y)(0.5)\ dy \qquad\qquad F = \int_0^a \rho g(a-y)w(y)dy$$

$$= 4.9\times 10^3 \left(4y - \frac{1}{2}y^2\right)\Big|_1^{1.5}$$

$$= 4.9\times 10^3 \left(6 - \frac{9}{8} - 4 + \frac{1}{2}\right)$$

$$= 4.9\times 10^3 \left(2 - \frac{5}{8}\right)$$

$$= 4.9\times 10^3 \left(\frac{11}{8}\right)$$

$$= 6737.5\ N$$

Exercise

Adiving pool that is 4 *m* deep full of water has a viewing window on one of its vertical walls. Find the force of the a circle window, with a radius of 0.5 *m*, tangent to the bottom of the pool.

Solution

Equation of the circle: $x^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4}$

$$x^2=\frac{1}{4}-y^2+y-\frac{1}{4}=y-y^2$$

$$x=\sqrt{y-y^2}\quad\Rightarrow\quad 2x=2\sqrt{y-y^2}$$

$$F=\int_0^1 10^3(9.8)(4-y)\left(2\sqrt{y-y^2}\right)dy \qquad F=\int_0^a \rho g\,(a-y)\,w(y)\,dy$$

$$=19.6\times10^3\int_0^1\left(\frac{7}{2}+\frac{1}{2}-y\right)\sqrt{y-y^2}\,dy \qquad d\left(y-y^2\right)=1-2y=2\left(\frac{1}{2}-y\right)$$

$$=19.6\times10^3\int_0^1\frac{7}{2}\sqrt{y-y^2}\,dy+19.6\times10^3\int_0^1\left(\frac{1}{2}-y\right)\sqrt{y-y^2}\,dy$$

$$=7\times9.8\times10^3\int_0^1\sqrt{y-y^2}\,dy+39.2\times10^3\int_0^1\sqrt{y-y^2}\,d\left(y-y^2\right) \qquad \textit{Area of semicircle}:\ \frac{1}{2}\pi r^2$$

$$=7\times9.8\times10^3\left(\frac{1}{2}\frac{\pi}{4}\right)+39.2\times10^3\left(\frac{1}{2}y^2-\frac{1}{3}y^3\right)\Big|_0^1$$

$$=\frac{7\pi}{8}\times9.8\times10^3+39.2\times10^3\left(\frac{1}{2}-\frac{1}{3}\right)$$

$$=\frac{7\pi}{8}\times9.8\times10^3+39.2\times10^3\left(\frac{1}{6}\right)$$

$$\approx 2.694\times10^4\ N$$

Solution Section 1.8 – Exponential Models

Exercise

Find the derivative of $y = \ln\left(\dfrac{\sqrt{\sin\theta\cos\theta}}{1+2\ln\theta}\right)$

Solution

$$y = \ln(\sin\theta\cos\theta)^{1/2} - \ln(1+2\ln\theta)$$

$$= \tfrac{1}{2}\big(\ln(\sin\theta) + \ln(\cos\theta)\big) - \ln(1+2\ln\theta)$$

$$y' = \frac{1}{2}\left(\frac{(\sin\theta)'}{\sin\theta} + \frac{(\cos\theta)'}{\cos\theta}\right) - \frac{(1+2\ln\theta)'}{1+2\ln\theta} = \frac{1}{2}\left(\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}\right) - \frac{\frac{2}{\theta}}{1+2\ln\theta}$$

$$\boxed{= \frac{1}{2}(\cot\theta - \tan\theta) - \frac{2}{\theta(1+2\ln\theta)}}$$

Exercise

Find the derivative of $f(x) = e^{\left(4\sqrt{x}+x^2\right)}$

Solution

$$\frac{d}{dx}e^{\left(4\sqrt{x}+x^2\right)} = e^{\left(4\sqrt{x}+x^2\right)}\frac{d}{dx}\left(4\sqrt{x}+x^2\right) \boxed{= \left(\frac{2}{\sqrt{x}} + 2x\right)e^{\left(4\sqrt{x}+x^2\right)}}$$

Exercise

Find the derivative of $f(t) = \ln\left(3te^{-t}\right)$

Solution

$$\frac{d}{dt}\ln\left(3te^{-t}\right) = \frac{\left(3te^{-t}\right)'}{3te^{-t}}$$

$$= 3\frac{e^{-t} - te^{-t}}{3te^{-t}}$$

$$\frac{e^{-t}(1-t)}{te^{-t}}$$

$$\boxed{= \frac{1-t}{t}}$$

$$\ln\left(3te^{-t}\right) = \ln 3 + \ln t + \ln e^{-t}$$

$$= \ln 3 + \ln t - t$$

$$\left(\ln\left(3te^{-t}\right)\right)' = \frac{1}{t} - 1$$

$$\boxed{= \frac{1-t}{t}}$$

Exercise

Find the Derivative of $f(x)=\dfrac{e^{\sqrt{x}}}{\ln\left(\sqrt{x}+1\right)}$

Solution

$$f=e^{\sqrt{x}} \qquad U=\sqrt{x}=x^{1/2}\Rightarrow U'=\tfrac{1}{2}x^{-1/2} \qquad f'=\tfrac{1}{2}x^{-1/2}e^{\sqrt{x}}=\frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

$$g=\ln\left(\sqrt{x}+1\right) \qquad U=x^{1/2}+1\Rightarrow U'=\tfrac{1}{2}x^{-1/2} \qquad g'=\frac{\frac{1}{2}x^{-1/2}}{\sqrt{x}+1}=\frac{1}{2x^{1/2}\left(\sqrt{x}+1\right)}$$

$$f'(x)=\frac{\frac{e^{\sqrt{x}}}{2\sqrt{x}}\ln\left(\sqrt{x}+1\right)-\frac{1}{2\sqrt{x}\left(\sqrt{x}+1\right)}e^{\sqrt{x}}}{\left(\ln\left(\sqrt{x}+1\right)\right)^2}$$

$$=\frac{\frac{\left(\sqrt{x}+1\right)e^{\sqrt{x}}\ln\left(\sqrt{x}+1\right)-e^{\sqrt{x}}}{2\sqrt{x}\left(\sqrt{x}+1\right)}}{\left(\ln\left(\sqrt{x}+1\right)\right)^2}$$

$$\boxed{=\frac{e^{\sqrt{x}}\left[\left(\sqrt{x}+1\right)\ln\left(\sqrt{x}+1\right)-1\right]}{2\sqrt{x}\left(\sqrt{x}+1\right)\left(\ln\left(\sqrt{x}+1\right)\right)^2}}$$

Exercise

Find the Derivative of $y=\sqrt{\dfrac{(x+1)^{10}}{(2x+1)^5}}$

Solution

$$y=\left(\frac{(x+1)^{10}}{(2x+1)^5}\right)^{1/2}$$

$$\ln y=\ln\left(\frac{(x+1)^{10}}{(2x+1)^5}\right)^{1/2}$$

$$\ln y=\frac{1}{2}\ln\left(\frac{(x+1)^{10}}{(2x+1)^5}\right)$$

$$=\frac{1}{2}\left(\ln(x+1)^{10}-\ln(2x+1)^5\right)$$

$$=\frac{1}{2}\left(10\ln(x+1)-5\ln(2x+1)\right)$$

$$= 5\ln(x+1) - \frac{5}{2}\ln(2x+1)$$

$$\frac{y'}{y} = 5\frac{1}{x+1} - \frac{5}{2}\frac{2}{2x+1}$$

$$\frac{y'}{y} = \frac{5}{x+1} - \frac{5}{2x+1}$$

$$y' = y\left(\frac{5}{x+1} - \frac{5}{2x+1}\right)$$

$$\boxed{y' = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}}\left(\frac{5}{x+1} - \frac{5}{2x+1}\right)}$$

Exercise

Find the derivative of $f(x) = (2x)^{4x}$

Solution

$$\ln f(x) = 4x\ln(2x)$$

$$\frac{f'}{f} = 4\left(\ln 2x + x\frac{2}{2x}\right)$$

$$\boxed{f'(x) = 4(\ln 2x + 1)(2x)^{4x}}$$

Exercise

Find the derivative of $f(x) = 2^{x^2}$

Solution

$$\boxed{f'(x) = 2x \cdot 2^{x^2}\ln 2}$$

Exercise

Find the derivative of $h(y) = y^{\sin y}$

Solution

$$\ln h = \ln y^{\sin y} = \sin y \ln y$$

$$\frac{h'}{h} = \cos y \ln y + \frac{\sin y}{y}$$

$$h'(y) = y^{\sin y}\left(\cos y \ln y + \frac{\sin y}{y}\right)$$

Exercise

Find the derivative of $f(x) = x^{\pi}$

Solution

$$\ln f = \pi \ln x$$

$$\frac{f'}{f} = \frac{\pi}{x}$$

$$\boxed{f'(x) = \pi x^{\pi - 1}}$$

Exercise

Find the derivative of $h(t) = (\sin t)^{\sqrt{t}}$

Solution

$$\ln h = \ln(\sin t)^{\sqrt{t}} = \sqrt{t}\ln(\sin t)$$

$$\frac{h'}{h} = \frac{1}{2\sqrt{t}}\ln \sin t + \sqrt{t}\,\frac{\cos t}{\sin t}$$

$$\boxed{h'(t) = \frac{1}{2\sqrt{t}}(\ln \sin t + 2t\cot t)(\sin t)^{\sqrt{t}}}$$

Exercise

Find the derivative of $p(x) = x^{-\ln x}$

Solution

$$\ln p(x) = \ln x^{-\ln x} = -(\ln x)^2$$

$$\frac{p'}{p} = -\frac{2\ln x}{x}$$

$$p'(x) = -\frac{2\ln x}{x}x^{-\ln x} \boxed{= -\frac{2\ln x}{x^{1+\ln x}}}$$

Exercise

Find the derivative of $f(x) = x^{2x}$

Solution

$$\ln f = \ln x^{2x} = 2x\ln x$$

$$\frac{f'}{f} = 2\ln x + 2\frac{x}{x}$$

$$\boxed{f'(x) = 2(1 + \ln x)x^{2x}}$$

Exercise

Find the derivative of $f(x) = x^{\tan x}$

Solution

$$\ln f(x) = \ln x^{\tan x} = \tan x \ln x$$

$$\frac{f'}{f} = \sec^2 x \ln x + \frac{\tan x}{x}$$

$$\boxed{f'(x) = \left(\sec^2 x \ln x + \frac{\tan x}{x}\right) x^{\tan x}}$$

Exercise

Find the derivative of $f(x) = x^e + e^x$

Solution

$$\boxed{f'(x) = ex^{e-1} + e^x}$$

Exercise

Find the derivative of $f(x) = x^{x^{10}}$

Solution

$$\ln f = x^{10} \ln x$$

$$\frac{f'}{f} = 10x^9 \ln x + \frac{x^{10}}{x}$$

$$f'(x) = x^{x^{10}}\left(10x^9 \ln x + x^9\right)$$

$$\boxed{= x^{9+x^{10}}(10\ln x + 1)}$$

Exercise

Find the derivative of $f(x) = \left(1 + \frac{4}{x}\right)^x$

Solution

$$\ln f = x \ln\left(1 + \frac{4}{x}\right)$$

$$\frac{f'}{f} = \ln\left(1 + \frac{4}{x}\right) + x\frac{-\frac{4}{x^2}}{1 + \frac{4}{x}}$$

$$\boxed{f'(x) = \left(1 + \frac{4}{x}\right)^x \left(\ln\left(1 + \frac{4}{x}\right) - \frac{4}{x+4}\right)}$$

Exercise

Find the derivative of $f(x)=\cos\left(x^{2\sin x}\right)$

Solution

$$f'=-\left(x^{2\sin x}\right)'\sin\left(x^{2\sin x}\right)$$

$$Let\ y=x^{2\sin x} \quad \to \quad \ln y=(2\sin x)\ln x$$

$$\frac{y'}{y}=2\cos x\ln x+\frac{2\sin x}{x}$$

$$\boxed{f'=-x^{2\sin x}\left(2\cos x\ln x+\frac{2\sin x}{x}\right)\sin\left(x^{2\sin x}\right)}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{2ydy}{y^2-25}$

Solution

$$\int \frac{2ydy}{y^2-25}=\int \frac{d\left(y^2-25\right)}{y^2-25} \qquad d\left(y^2-25\right)=2ydy$$

$$\boxed{=\ln\left|y^2-25\right|+C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sec y\tan y}{2+\sec y}dy$

Solution

$$\int \frac{\sec y\tan y}{2+\sec y}dy=\int \frac{d(2+\sec y)}{2+\sec y} \qquad d(2+\sec y)=\sec y\tan ydy$$

$$\boxed{=\ln|2+\sec y|+C}$$

Exercise

Find the integral $\displaystyle\int \frac{5}{e^{-5x}+7}dx$

Solution

$$\int \frac{5}{e^{-5x}+7}\frac{e^{5x}}{e^{5x}}dx=\int \frac{5e^{5x}}{1+7e^{5x}}dx \qquad d\left(1+7e^{5x}\right)=35e^{5x}dx$$

$$= \frac{1}{7}\int \frac{d\left(1+7e^{5x}\right)}{1+7e^{5x}}$$

$$\boxed{= \frac{1}{7}\ln\left|1+7e^{5x}\right| + C}$$

Exercise

Find the integral $\int \frac{e^{2x}}{4+e^{2x}}dx$

Solution

$$\int \frac{e^{2x}}{4+e^{2x}}dx = \frac{1}{2}\int \frac{d\left(4+e^{2x}\right)}{4+e^{2x}} \qquad d\left(4+e^{2x}\right) = 2e^{2x}dx$$

$$\boxed{= \frac{1}{2}\ln\left(4+e^{2x}\right) + C}$$

Exercise

Find the integral $\int \frac{dx}{x\ln x\ln(\ln x)}$

Solution

$$\int \frac{dx}{x\ln x\ln(\ln x)} = \int \frac{d\left(\ln(\ln x)\right)}{\ln(\ln x)} \qquad d\left(\ln(\ln x)\right) = \frac{1/x}{\ln x} = \frac{1}{x\ln x}$$

$$\boxed{= \ln\ln(\ln x) + C}$$

Exercise

Find the integral $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

Solution

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = 2\int e^{\sqrt{x}}d\left(\sqrt{x}\right) \qquad d\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}dx$$

$$\boxed{= 2e^{\sqrt{x}} + C}$$

Exercise

Find the integral $\int \frac{e^{\sin x}}{\sec x}dx$

Solution

$$\int \frac{e^{\sin x}}{\sec x}dx = \int e^{\sin x}d(\sin x) \qquad d(\sin x) = \cos x dx = \frac{dx}{\sec x}$$

$$\boxed{= e^{\sin x} + C}$$

Exercise

Find the integral $\int \frac{e^x + e^{-x}}{e^x - e^{-x}}dx$

Solution

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}}dx = \int \frac{1}{e^x - e^{-x}}d\left(e^x - e^{-x}\right) \qquad d\left(e^x - e^{-x}\right) = \left(e^x + e^{-x}\right)dx$$

$$\boxed{= \ln\left(e^x - e^{-x}\right) + C}$$

Exercise

Find the integral $\int \frac{4^{\cot x}}{\sin^2 x}dx$

Solution

$$\int \frac{4^{\cot x}}{\sin^2 x}dx = -\int 4^{\cot x}d(\cot x) \qquad d(\cot x) = -\csc^2 x dx = -\frac{dx}{\sin^2 x}$$

$$\boxed{= \frac{4^{\cot x}}{\ln 4} + C} \qquad \int a^x dx = \frac{a^x}{\ln a} + C$$

Exercise

Find the integral $\int \frac{4x^2 + 2x + 4}{x+1}dx$

Solution

$$\int \frac{4x^2 + 2x + 4}{x+1}dx = \int \left(4x + 2 + \frac{6}{x+1}\right)dx$$

$$= \int (4x-2)dx + \int \frac{6}{x+1}dx$$

$$= \int (4x-2)dx + 6\int \frac{d(x+1)}{x+1} \qquad \int \frac{d(U)}{U} = \ln|U|$$

$$\boxed{= 2x^2 - 2x + 6\ln|x+1| + C}$$

Exercise

Evaluate the integral $\int_{\ln 4}^{\ln 9} e^{x/2}dx$

Solution

$$\int_{\ln 4}^{\ln 9} e^{x/2}dx = 2e^{x/2}\Big|_{\ln 2^2}^{\ln 3^2}$$

$$= 2\left(e^{(2\ln 3)/2} - e^{(2\ln 2)/2}\right)$$

$$= 2\left(e^{\ln 3} - e^{\ln 2}\right)$$

$$= 2(3-2)$$

$$\underline{= 2}$$

Exercise

Evaluate the integral $\int_0^3 \frac{2x-1}{x+1}dx$

Solution

$$\int_0^3 \frac{2x-1}{x+1}dx = \int_0^3 \left(2 - \frac{3}{x+1}\right)dx$$

$$= \left(2x - 3\ln|x+1|\right)\Big|_0^3$$

$$\underline{= 6 - 3\ln 4}$$

$$\begin{array}{r|l} & 2 \\ \hline x+1 & 2x-1 \\ & -2x-2 \\ \hline & -3 \end{array}$$

Exercise

Evaluate the integral $\int_e^{e^2} \frac{dx}{x\ln^3 x}$

Solution

$$\int_e^{e^2} \frac{dx}{x\ln^3 x} = \int_e^{e^2} \ln^{-3} x \, d(\ln x)$$

$$= -\frac{1}{2}\ln^{-2} x\Big|_e^{e^2}$$

$$= -\frac{1}{2}(2-1)$$

$$\underline{= -\frac{1}{2}}$$

Exercise

Evaluate the integral $\displaystyle\int_{e^2}^{e^3} \frac{dx}{x\ln x\ln^2(\ln x)}$

Solution

$$\int_{e^2}^{e^3} \frac{dx}{x\ln x\ln^2(\ln x)} = \int_{e^2}^{e^3} (\ln(\ln x))^{-2}\, d(\ln(\ln x)) \qquad d(\ln(\ln x)) = \frac{1/x}{\ln x} = \frac{1}{x\ln x}$$

$$= -\frac{1}{\ln(\ln x)}\Big|_{e^2}^{e^3}$$

$$= -\frac{1}{\ln\left(\ln e^3\right)} + \frac{1}{\ln\left(\ln e^2\right)}$$

$$= -\frac{1}{\ln 3} + \frac{1}{\ln 2}$$

Exercise

Evaluate the integral $\displaystyle\int_0^1 \frac{y\ln^4\left(y^2+1\right)}{y^2+1}dy$

Solution

$$\int_0^1 \frac{y\ln^4\left(y^2+1\right)}{y^2+1}dy = \frac{1}{2}\int_0^1 \ln^4\left(y^2+1\right)d\left(\ln\left(y^2+1\right)\right) \qquad d\left(\ln\left(y^2+1\right)\right) = \frac{2y}{y^2+1}dy$$

$$= \frac{1}{10}\ln^5\left(y^2+1\right)\Big|_0^1$$

$$= \frac{1}{10}(\ln 2)^5$$

Exercise

Evaluate the integral $\displaystyle\int_{\ln 2}^{\ln 3} \frac{e^x+e^{-x}}{e^{2x}-2+e^{-2x}}dx$

Solution

$$\int_{\ln 2}^{\ln 3} \frac{e^x+e^{-x}}{e^{2x}-2+e^{-2x}}dx = \int_{\ln 2}^{\ln 3} \frac{e^x+e^{-x}}{\left(e^x-e^{-x}\right)^2}dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{1}{\left(e^x - e^{-x}\right)^2} d\left(e^x - e^{-x}\right)$$

$$= -\frac{1}{e^x - e^{-x}} \Big|_{\ln 2}^{\ln 3}$$

$$= -\frac{1}{e^{\ln 3} - e^{-\ln 3}} + \frac{1}{e^{\ln 2} - e^{-\ln 2}}$$

$$= \frac{1}{2 - \frac{1}{2}} - \frac{1}{3 - \frac{1}{3}}$$

$$= \frac{2}{3} - \frac{3}{8}$$

$$= \frac{7}{24}$$

Exercise

Evaluate the integral $\int_{-2}^{2} \frac{e^{z/2}}{e^{z/2}+1} dz$

Solution

$$\int_{-2}^{2} \frac{e^{z/2}}{e^{z/2}+1} dz = 2\int_{-2}^{2} \frac{1}{e^{z/2}+1} d\left(e^{z/2}+1\right)$$

$$d\left(e^{z/2}+1\right) = \frac{1}{2}e^{z/2}dz$$

$$= 2\ln\left(e^{z/2}+1\right)\Big|_{-2}^{2}$$

$$= 2\left(\ln(e+1) - \ln\left(e^{-1}+1\right)\right)$$

Exercise

Evaluate the integral $\int_{0}^{\pi/2} 4^{\sin x} \cos x \, dx$

Solution

$$\int_{0}^{\pi/2} 4^{\sin x} \cos x \, dx = \int_{0}^{\pi/2} 4^{\sin x} \, d(\sin x)$$

$$\int a^x dx = \frac{a^x}{\ln a}$$

$$= \frac{1}{\ln 4} 4^{\sin x} \Big|_{0}^{\pi/2}$$

$$= \frac{1}{\ln 4}(4-1)$$

$$= \frac{3}{\ln 4}$$

Exercise

Evaluate the integral $\int_{1/3}^{1/2} \frac{10^{1/p}}{p^2} dp$

Solution

$$\int_{1/3}^{1/2} \frac{10^{1/p}}{p^2} dp = -\int_{1/3}^{1/2} 10^{1/p} d\left(\frac{1}{p}\right)$$

$$= -\frac{1}{\ln 10} 10^{1/p} \Big|_{1/3}^{1/2}$$

$$= -\frac{1}{\ln 10}\left(10^2 - 10^3\right)$$

$$= \frac{900}{\ln 10}$$

Exercise

Evaluate the integral $\int_1^2 (1+\ln x)x^x dx$

Solution

$$y = x^x \quad \rightarrow \quad \ln y = x \ln x$$

$$\frac{y'}{y} = 1 + \ln x \quad \Rightarrow \quad \left(x^x\right)' = x^x(1+\ln x)$$

$$\int_1^2 (1+\ln x)x^x dx = \int_1^2 d\left(x^x\right)$$

$$= x^x \Big|_1^2$$

$$= 2^2 - 1$$

$$= 3$$

Exercise

Find a curve through the origin in the *xy*-plane whose length from $x = 0$ to $x = 1$ is $L = \int_0^1 \sqrt{1 + \frac{1}{4}e^x}\, dx$

Solution

$$L = \int_0^1 \sqrt{1 + \frac{1}{4}e^x}\, dx \quad \Rightarrow \frac{dy}{dx} = \frac{e^{x/2}}{2}$$

$$dy = \frac{e^{x/2}}{2}dx$$

$$y = \int \frac{e^{x/2}}{2}dx = e^{x/2} + C$$

$$0 = e^{0/2} + C$$

$$0 = 1 + C$$

$$C = -1$$

$$\boxed{y = e^{x/2} - 1}$$

Exercise

Find the length of the curve $y = \ln\left(e^x - 1\right) - \ln\left(e^x + 1\right)$ *from* $x = \ln 2$ *to* $x = \ln 3$

Solution

$$y = \ln\left(e^x - 1\right) - \ln\left(e^x + 1\right) \Rightarrow \frac{dy}{dx} = \frac{e^x}{e^x - 1} - \frac{e^x}{e^x + 1}$$

$$= \frac{e^{2x} + e^x - e^{2x} + e^x}{e^{2x} - 1}$$

$$= \frac{2e^x}{e^{2x} - 1}$$

$$L = \int_{\ln 2}^{\ln 3} \sqrt{1 + \left(\frac{2e^x}{e^{2x} - 1}\right)^2}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{1 + \frac{4e^{2x}}{e^{4x} - 2e^{2x} + 1}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{e^{4x} - 2e^{2x} + 1 + 4e^{2x}}{\left(e^{2x} - 1\right)^2}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{e^{4x} + 2e^{2x} + 1}{\left(e^{2x} - 1\right)^2}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{\left(e^{2x} + 1\right)^2}{\left(e^{2x} - 1\right)^2}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{e^{2x}+1}{e^{2x}-1}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{\frac{e^{2x}+1}{e^x}}{\frac{e^{2x}-1}{e^x}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{\frac{e^{2x}}{e^x}+\frac{1}{e^x}}{\frac{e^{2x}}{e^x}-\frac{1}{e^x}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}\, dx$$

Let $u = e^x - e^{-x} \quad \Rightarrow du = \left(e^x + e^{-x}\right)dx \quad \rightarrow \begin{cases} x = \ln 2 & u = e^{\ln 2} - e^{-\ln 2} = 2 - \frac{1}{2} = \frac{3}{2} \\ x = \ln 3 & u = e^{\ln 3} - e^{-\ln 3} = 3 - \frac{1}{3} = \frac{8}{3} \end{cases}$

$$L = \int_{3/2}^{8/3} \frac{du}{u}$$

$$= \left[\ln|u|\ \right]_{3/2}^{8/3}$$

$$= \ln\frac{8}{3} - \ln\frac{3}{2}$$

$$= \ln\frac{8/3}{3/2}$$

$$= \ln\left(\frac{16}{9}\right)$$

Exercise

Find the length of the curve $y = \ln(\cos x)$ *from* $x = 0$ *to* $x = \frac{\pi}{4}$

Solution

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$$

$$L = \int_0^{\pi/4} \sqrt{1+\tan^2 x}\, dx$$

$$= \int_0^{\pi/4} \sqrt{\sec^2 x}\ dx$$

$$= \int_0^{\pi/4} \sec x\ dx$$

$$= \left[\ln\left|\sec x + \tan x\right| \right]_0^{\pi/4}$$

$$= \ln\left|\sec\tfrac{\pi}{4} + \tan\tfrac{\pi}{4}\right| - \ln\left|\sec 0 + \tan 0\right|$$

$$= \ln\left|\sqrt{2}+1\right| - \ln\left|1+0\right|$$

$$= \ln\left|\sqrt{2}+1\right| - 0$$

$$= \ln\left(\sqrt{2}+1\right)$$

Exercise

Find the area of the surface generated by revolving the curve $x = \frac{1}{2}\left(e^{y} + e^{-y}\right), \quad 0 \le y \le \ln 2,$ about y-axis

Solution

$$S = 2\pi \int_0^{\ln 2} \tfrac{1}{2}\left(e^{y}+e^{-y}\right)\sqrt{1+\left(\frac{e^{y}-e^{-y}}{2}\right)^2}\,dy$$

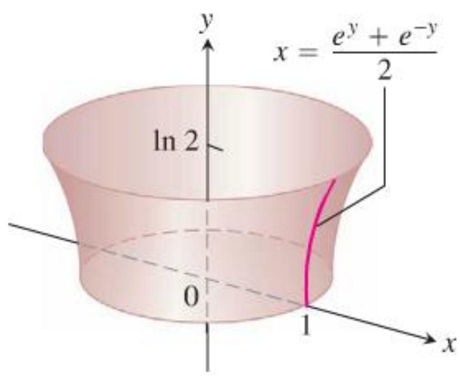
$$= \pi \int_0^{\ln 2} \left(e^{y}+e^{-y}\right)\sqrt{1+\frac{e^{2y}+e^{-2y}-2}{4}}\,dy$$

$$= \pi \int_0^{\ln 2} \left(e^{y}+e^{-y}\right)\sqrt{\frac{4+e^{2y}+e^{-2y}-2}{4}}\ dy$$

$$= \frac{\pi}{2} \int_0^{\ln 2} \left(e^{y}+e^{-y}\right)\sqrt{e^{2y}+e^{-2y}+2}\ dy$$

$$= \frac{\pi}{2} \int_0^{\ln 2} \left(e^{y}+e^{-y}\right)\sqrt{\left(e^{y}+e^{-y}\right)^2}\ dy$$

$$= \frac{\pi}{2} \int_0^{\ln 2} \left(e^{y}+e^{-y}\right)^2\ dy$$



$$= \frac{\pi}{2} \int_0^{\ln 2} \left(e^{2y} + e^{-2y} + 2\right) dy$$

$$= \frac{\pi}{2} \left[\frac{1}{2} e^{2y} - \frac{1}{2} e^{-2y} + 2y \right]_0^{\ln 2}$$

$$= \frac{\pi}{2} \left[\left(\frac{1}{2} e^{2\ln 2} - \frac{1}{2} e^{-2\ln 2} + 2\ln 2 \right) - \left(\frac{1}{2} e^0 - \frac{1}{2} e^0 + 0 \right) \right]$$

$$= \frac{\pi}{2} \left(\frac{1}{2} \cdot 4 - \frac{1}{2} \cdot \frac{1}{4} + 2\ln 2 \right)$$

$$= \frac{\pi}{2} \left(\frac{15}{8} + 2\ln 2 \right)$$

Exercise

The population of a town with a 2010 population of 90,000 grows at a rate of 2.4% /yr. In what year will the population coudle its initial value (to 180,000)?

Solution

$$k = \frac{\ln \frac{1.024(90,000)}{90,000}}{1} = \ln(1.024)$$

$$T_2 = \frac{\ln 2}{\ln 1.024} \approx 29.226 \ \ yrs$$

It reaches 180,000 around the year 2039.

Exercise

How long will it take an initial deposit of \$1500 to increase in value to \$2500 in a saving account with an APY of 3.1%? Assume the interest rate reamins constant and no additional deposits or withdrawals are made.

Solution

$$y(t) = 1500\, e^{kt}$$

$$k = \frac{\ln 1.031}{1} = \ln(1.031)$$

$$T = \frac{\ln\left(\frac{2500}{1500}\right)}{\ln 1.031} \approx 16.7 \ \ yrs \qquad\qquad kT = \ln\left(\frac{y}{y_0}\right)$$

Exercise

The number of cells in a tumor doubles every 6 weeks starting with 8 cells. After how many weeks doses the tumor have 1500 cells?

Solution

$$k = \frac{\ln 2}{T_2} = \frac{\ln 2}{6} \quad \Rightarrow \quad y(t) = 8\, e^{(t \ln 2)/6}$$

$$t = 6\frac{\ln\left(\frac{1500}{8}\right)}{\ln 2} \approx 45.3 \ \ weeks \qquad\qquad kT = \ln\left(\frac{y}{y_0}\right)$$

Exercise

According to the 2010 census, the U.S. population was 309 million with an estimated growth rate of 0.8% /*yr*.

a) Based on these figures, find the doubling time and project the population in 2050.

b) Suppose the actual growth rate is just 0.2 percentage point lower than 0.8% /*yr* (0.6%). What are the resulting doubling time and projected 2050 population? Repeat these calculstions assuming the growth rate is 0.2 percentage point higher than 0.8% /yr.

c) Comment on th sensitivity of these projections to the growth rate.

Solution

a) $T_2 = \frac{\ln 2}{\ln 1.008} \approx 87 \ \ yrs$

The population in 2050: $P(50) = 309e^{40 \ln 1.008} \approx 425 \ \ million$

b) If the growth rate is 0.6%: $T_2 = \frac{\ln 2}{\ln 1.006} \approx 116 \ \ yrs$

The population in 2050: $P(50) = 309e^{40 \ln 1.006} \approx 392.5 \ \ million$

If the growth rate is 1%: $T_2 = \frac{\ln 2}{\ln 1.01} \approx 69.7 \ \ yrs$

The population in 2050: $P(50) = 309e^{40 \ln 1.01} \approx 460.1 \ \ million$

c) A growth rate of just 0.2% produces large differences in population growth.

Exercise

The homicide rate decreases at a rate of 3%/yr in a city that had 800 homicides /yr in 2010. At this rate, when will the homicide rate reach 600 homicides/yr?

Solution

The homicide rate is modeled by: $H(t) = 800e^{-kt}$

$k = \ln(1 - .03) \approx -0.03$

$H(t) = 800e^{-0.03t}$

$$t=\frac{\ln(6/8)}{-0.03}\ \underline{\approx 9.6\ \ yrs|}\qquad\qquad kT=\ln\left(\frac{y}{y_0}\right)$$

So it should achieve this rate in 2019.

Exercise

A drug is eliminated from the body at a rate of 15% /hr. after how many hours does the amount of drug reach 10% of the initial dose?

Solution

$$k=\ln(1-.15)\ \underline{\approx -\ln(.85)|}$$

$$t=\frac{\ln(.1)}{\ln(.85)}\ \underline{\approx 14.17\ \ hrs|}\qquad\qquad kT=\ln\left(\frac{y}{y_0}\right)$$

Exercise

A large die-casting machine used to make automobile engine blocks is purchased for $2.5 million. For tax purposes, the value of the machine can be depreciated by 6.8% of its current value each year.

a) What is the value of the machine after 10 years?

b) After how many years is the value of the machine 10% of its original value?

Solution

a) $V(t)=2.5e^{-kt}$

$$k=\frac{\ln(1-.068)}{1}\ \underline{\approx \ln(.932)|}\qquad\qquad kT=\ln\left(\frac{y}{y_0}\right)$$

$$V(t)=2.5e^{-t\ln .932}\ \rightarrow\ V(10)=2.5e^{-10\cdot\ln .932}\ \underline{\approx \$1.2\ million|}$$

b) $t=\dfrac{\ln(.1)}{\ln(.932)}\ \underline{\approx 32.7\ \ yrs|}$

Exercise

Roughly 12,000 Americans are diagmosed with thyroid cancer every year, which accounts for 1% of all cancer cases. It occurs in women three times as frequency as in men. Fortunately, thyroid cancer can be treated successfully in many cases with radioactive iodine, or I-131. This unstable form of iodine has a half-life of 8 days and is given in small doses meansured in millicuries.

a) Suppose a patient is given an initial dose of 100 millicuries. Find the function that gives the amount of I-131 in the body after $t \ge 0$ days.

b) How long does it take the amount of I-131 to reach 10% of the initial dose?

c) Finding the initial dose to give a particular patient is a critical calculation. How does the time reach 10% of the initial dose change if the initial dose is increased by 5%?

Solution

a) $\boxed{k = \frac{\ln 2}{8}}$ $\qquad kT = \ln\left(y / y_0\right)$

After t days would be: $y = 100e^{-(t\ln 2)/8}$ millicuries.

b) $t = \dfrac{-8\ln\left(\frac{10}{100}\right)}{\ln(2)}$ $\boxed{\approx 26.58\ \ days}$

c) $t = \dfrac{-8\ln\left(\frac{10}{105}\right)}{\ln(2)}$ $\boxed{\approx 27.14\ \ days}$

Exercise

City ***A*** has a current population of 500,000 people and grows at a rate of 3% /*yr*. City ***B*** has a cuurent population of 300,000 and grows at a rate of 5%/*yr*.

a) When will the cities have the same population?

b) Suppose City ***C*** has a current population of $y_0 < 500{,}000$ and a growth rate of $p > 3\% / yr$. What is the relationship between y_0 and *p* such that the Cities ***A*** and ***C*** have the same population in 10 years?

Solution

a) $500{,}000e^{\ln(1.03)t} = 300{,}000e^{\ln(1.05)t}$

$5e^{\ln(1.03)t} = 3e^{\ln(1.05)t}$

$\frac{5}{3} = e^{(\ln(1.05) - \ln 1.03)t}$

$\ln\frac{5}{3} = \left(\ln\frac{1.05}{1.03}\right)t \quad \rightarrow \quad t = \dfrac{\ln(5/3)}{\ln(1.05/1.03)}$ $\boxed{\approx 26.56\ yrs}$

b) $500{,}000e^{\ln(1.03)(10)} = y_0 e^{\ln(1+p)(10)}$

$y_0 = 500{,}000e^{10(\ln(1.03) - \ln(1+p))}$

$= 500{,}000e^{\ln\left(\frac{1.03}{1+p}\right)^{10}}$

$\boxed{= 500{,}000\left(\frac{1.03}{1+p}\right)^{10}}$

Exercise

Suppose the acceleration of an object moving along a line is given by $a(t) = -kv(t)$, where k is a positive constant and v is the object's velocity. Assume that the initial velocity and position are given by $v(0) = 10$ and $s(0) = 0$, respectively.

a) Use $a(t) = v'(t)$ to find the velocity of the object as a function of time.

b) Use $v(t) = s'(t)$ to find the position of the object as a function of time.

c) Use the fact that $\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt}$ (by the *Chain Rule*) to find the velocity as a function of position.

Solution

a) If $a(t) = \frac{dv}{dt} = -kv \quad \rightarrow \quad \frac{dv}{v} = -kdt$

$$\int \frac{dv}{v} = -k\int dt$$

$\ln v = -kt + C$ Since $v(0) = 10$

$$\underline{\ln 10 = C}$$

$$\ln v = -kt + \ln 10$$

$$v = e^{-kt+\ln 10} = e^{-kt}e^{\ln 10}$$

$$\underline{v(t) = 10e^{-kt}}$$

b) $v(t) = \frac{ds}{dt} = 10e^{-kt}$

$$\int ds = 10\int e^{-kt}dt$$

$s(t) = -\frac{10}{k}e^{-kt} + C$ Since $s(0) = 0$

$$0 = -\frac{10}{k} + C \quad \rightarrow \quad \underline{C = \frac{10}{k}}$$

$$\underline{s(t) = -\frac{10}{k}e^{-kt} + \frac{10}{k}}$$

c) $\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt}$

$$-10ke^{-kt} = \frac{dv}{ds}\left(10e^{-kt}\right)$$

$$-k = \frac{dv}{ds}$$

$$\int dv = -k\int ds$$

$v = -ks + C$ Since $v(0) = 10$

$$\underline{v = 10 - ks}$$

Exercise

On the first day of the year $(t=0)$, a city uses electricity at a rate of 2000 MW. That rate is projected to increase at a rate of 1.3% per *year*.

a) Based on these figures, find an exponential growth function for the power (rate of electricity use) for the city.

b) Find the total energy (in MW-*yr*) used by the city over four full years beginning at $t=0$

c) Find a function that gives the total energy used (in MW-*yr*) between $t=0$ and any future time $t>0$

Solution

a) $P(t)=2000e^{kt}$

At a rate of 1.3% per year: $k=\ln(1.013)$

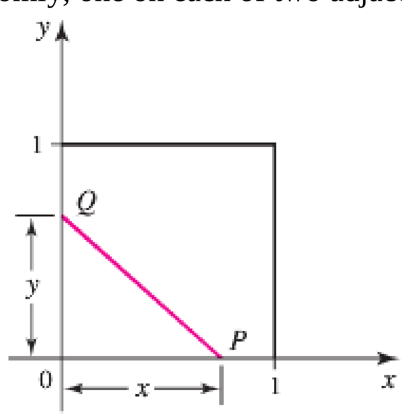
$$\boxed{P(t)=2000e^{t\ln 1.013}}$$

b)
$$\int_0^4 P(t)\,dt = 2000\int_0^4 e^{t\ln 1.013}dt$$
$$= \frac{2000}{\ln 1.013}e^{t\ln 1.013}\Big|_0^4$$
$$\boxed{\approx 8210.3}$$

c)
$$\int_0^t P(s)\,ds = 2000\int_0^t e^{s\ln 1.013}ds$$
$$= \frac{2000}{\ln 1.013}e^{s\ln 1.013}\Big|_0^t$$
$$\boxed{=-154{,}844\left(1+e^{t\ln(1.013)}\right)}$$

Exercise

Two points P and Q are chosen randomly, one on each of two adjacent sides of a unit square.



What is the probability that the area of the triangle formed by the sides of the square and the line segment

PQ is less than one-fourth the area of the square? Begin by showing that x and y must satisfy $xy < \frac{1}{2}$ in order for the area consition to be met. Then argue that the required probability is $\frac{1}{2} + \int_{1/2}^{1} \frac{dx}{2x}$ and evaluate the integral.

Solution

The area of the triangle is $\frac{1}{2}xy$

If $xy < \frac{1}{2}$, then if we let $0 < x < \frac{1}{2}$ we have $0 < y < 1$

Because there is a probability of $\frac{1}{2}$ of choosing $0 < x < \frac{1}{2}$, the probability we seek is at least $\frac{1}{2}$.

In addtion, for $\frac{1}{2} < x < 1$, if $y < \frac{1}{2x}$,

$$\int_{1/2}^{1} \frac{dx}{2x} = \frac{1}{2} \ln x \Big|_{1/2}^{1} = \frac{\ln 2}{2}$$

$$\frac{1}{2} + \int_{1/2}^{1} \frac{dx}{2x} = \frac{1}{2}(1 + \ln 2)$$

Solution   *Section* 1.9 – Hyperbolic Functions

Exercise

Rewrite the expression $\cosh 3x - \sinh 3x$ in terms of exponentials and simplify the results as much as you can.

Solution

$$\begin{aligned}
\cosh 3x - \sinh 3x &= \frac{e^{3x}+e^{-3x}}{2} - \frac{e^{3x}-e^{-3x}}{2} \\
&= \frac{e^{3x}+e^{-3x}-e^{3x}+e^{-3x}}{2} \\
&= \frac{2e^{-3x}}{2} \\
&= e^{-3x}
\end{aligned}$$

Exercise

Rewrite the expression $\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x)$ in terms of exponentials and simplify the results as much as you can.

Solution

$$\begin{aligned}
\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x) &= \ln\left[(\cosh x + \sinh x)(\cosh x - \sinh x)\right] \\
&= \ln\left(\cosh^2 x - \sinh^2 x\right) \\
&= \ln(1) \\
&= 0
\end{aligned}$$

Exercise

Prove the identities

a) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

b) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

Solution

a)
$$\begin{aligned}
\sinh x \cosh y + \cosh x \sinh y &= \frac{e^{x}-e^{-x}}{2}\frac{e^{y}+e^{-y}}{2} + \frac{e^{x}+e^{-x}}{2}\frac{e^{y}-e^{-y}}{2} \\
&= \frac{e^{x}e^{y}+e^{x}e^{-y}-e^{-x}e^{y}-e^{-x}e^{-y}+e^{x}e^{y}-e^{x}e^{-y}+e^{-x}e^{y}-e^{-x}e^{-y}}{4} \\
&= \frac{2e^{x}e^{y}-2e^{-x}e^{-y}}{4} \\
&= \frac{e^{x+y}-e^{-x-y}}{2}
\end{aligned}$$

$$= \frac{e^{x+y} - e^{-(x+y)}}{2}$$

$$\boxed{= \sinh(x+y)}$$

b) $\cosh x \cosh y + \sinh x \sinh y = \frac{e^y + e^{-y}}{2}\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\frac{e^y - e^{-y}}{2}$

$$= \frac{e^x e^y + e^x e^{-y} + e^{-x}e^y + e^{-x}e^{-y} + e^x e^y - e^x e^{-y} - e^{-x}e^y + e^{-x}e^{-y}}{4}$$

$$= \frac{2e^x e^y + 2e^{-x}e^{-y}}{4}$$

$$= \frac{e^{x+y} + e^{-x-y}}{2}$$

$$= \frac{e^{x+y} + e^{-(x+y)}}{2}$$

$$\boxed{= \cosh(x+y)}$$

Exercise

Find the derivative of $y = \frac{1}{2}\sinh(2x+1)$

Solution

$$y' = \frac{1}{2}\left[\cosh(2x+1)\right](2) \quad \boxed{= \cosh(2x+1)}$$

Exercise

Find the derivative of $y = 2\sqrt{t}\tanh\sqrt{t}$

Solution

$$y' = 2\left(\frac{1}{2}t^{-1/2}\tanh\sqrt{t} + t^{1/2}\left(\frac{1}{2}\right)\sec^2\sqrt{t}\right)$$

$$\boxed{= \frac{\tanh\sqrt{t}}{\sqrt{t}} + \sqrt{t}\sec^2\sqrt{t}}$$

Exercise

Find the derivative of $y = \ln(\cosh z)$

Solution

$$y' = \frac{\sinh z}{\cosh z} = \boxed{\tanh z}$$

Exercise

Find the derivative of $y = \operatorname{csch}\theta(1 - \ln \operatorname{csch}\theta)$

Solution

$$y' = (-\operatorname{csch}\theta \coth\theta)(1 - \ln \operatorname{csch}\theta) + \operatorname{csch}\theta\left(-\frac{-\operatorname{csch}\theta \coth\theta}{\operatorname{csch}\theta}\right)$$

$$= -\operatorname{csch}\theta \coth\theta + \operatorname{csch}\theta \coth\theta(\ln \operatorname{csch}\theta) + \operatorname{csch}\theta \coth\theta$$

$$\boxed{= \operatorname{csch}\theta \coth\theta(\ln \operatorname{csch}\theta)}$$

Exercise

Find the derivative of $y = \ln \sinh v - \frac{1}{2}\coth^2 v$

Solution

$$y' = \frac{\cosh v}{\sinh v} - \frac{1}{2}2\coth v\left(-\operatorname{csch}^2 v\right)$$

$$\boxed{= \coth v + (\coth v)\left(\operatorname{csch}^2 v\right)}$$

Exercise

Find the derivative of $y = \left(x^2 + 1\right)\operatorname{sech}(\ln x)$

Solution

$$y = \left(x^2 + 1\right)\left(\frac{2}{e^{\ln x} + e^{-\ln x}}\right)$$

$$= \left(x^2 + 1\right)\left(\frac{2}{x + x^{-1}}\right)$$

$$= \left(x^2 + 1\right)\left(\frac{2x}{x^2 + 1}\right)$$

$$= 2x$$

$$\boxed{y' = 2}$$

Exercise

Find the derivative of $y = \left(4x^2 - 1\right)\operatorname{csch}\left(\ln 2x\right)$

Solution

$$y = \left(4x^2 - 1\right)\left(\frac{2}{e^{\ln 2x} - e^{-\ln 2x}}\right)$$

$$= \left(4x^2 - 1\right)\left(\frac{2}{2x - (2x)^{-1}}\right)$$

$$= \left(4x^2 - 1\right)\left(\frac{4x}{4x^2 - 1}\right)$$

$$= 4x$$

$$\boxed{y' = 4}$$

Exercise

Find the derivative of $y = \cosh^{-1} 2\sqrt{x+1}$

Solution

$$y = \cosh^{-1} 2\sqrt{x+1} = \cosh^{-1} 2(x+1)^{1/2}$$

$$y' = \frac{2\left(\frac{1}{2}\right)(x+1)^{-1/2}}{\sqrt{\left(2(x+1)^{1/2}\right)^2 - 1}}$$

$$= \frac{1}{(x+1)^{1/2}\sqrt{4(x+1) - 1}}$$

$$= \frac{1}{\sqrt{x+1}\sqrt{4x+3}}$$

$$\boxed{= \frac{1}{\sqrt{4x^2 + 7x + 3}}}$$

Exercise

Find the derivative of $y = \left(\theta^2 + 2\theta\right)\tanh^{-1}(\theta + 1)$

Solution

$$y' = (2\theta + 2)\tanh^{-1}(\theta + 1) + \left(\theta^2 + 2\theta\right)\left(\frac{1}{1 - (\theta + 1)^2}\right)$$

$$= (2\theta + 2)\tanh^{-1}(\theta + 1) + \frac{\theta^2 + 2\theta}{1 - \left(\theta^2 + 2\theta + 1\right)}$$

$$= (2\theta + 2)\tanh^{-1}(\theta + 1) + \frac{\theta^2 + 2\theta}{-\theta^2 - 2\theta}$$

$$\boxed{= (2\theta + 2)\tanh^{-1}(\theta + 1) - 1}$$

Exercise

Find the derivative of $y = (1-t)\coth^{-1}\sqrt{t}$

Solution

$$y' = -\coth^{-1}\sqrt{t} + (1-t)\frac{\frac{1}{2}t^{-1/2}}{1-\left(t^{1/2}\right)^2}$$

$$= -\coth^{-1}\sqrt{t} + (1-t)\frac{1}{2\sqrt{t}(1-t)}$$

$$\boxed{= -\coth^{-1}\sqrt{t} + \frac{1}{2\sqrt{t}}}$$

Exercise

Find the derivative of $y = \ln x + \sqrt{1-x^2}\ \operatorname{sech}^{-1}x$

Solution

$$y' = \frac{1}{x} + \frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x)\ \operatorname{sech}^{-1}x + \sqrt{1-x^2}\left(\frac{-1}{x\sqrt{1-x^2}}\right)$$

$$= \frac{1}{x} - \frac{x}{\left(1-x^2\right)^{1/2}}\ \operatorname{sech}^{-1}x - \frac{1}{x}$$

$$\boxed{= -\frac{x}{\sqrt{1-x^2}}\ \operatorname{sech}^{-1}x}$$

Exercise

Find the derivative of $y = \operatorname{csch}^{-1}\left(\frac{1}{2}\right)^{\theta}$

Solution

$$y' = -\frac{\left[\ln\left(\frac{1}{2}\right)\right]\left(\frac{1}{2}\right)^{\theta}}{\left(\frac{1}{2}\right)^{\theta}\sqrt{1+\left[\left(\frac{1}{2}\right)^{\theta}\right]^2}}$$

$$= -\frac{-\ln 2}{\sqrt{1+\left(\frac{1}{2}\right)^{2\theta}}}$$

$$= \boxed{\dfrac{\ln 2}{\sqrt{1+\left(\frac{1}{2}\right)^{2\theta}}}}$$

Exercise

Find the derivative of $y = \cosh^{-1}(\sec x)$

Solution

$$y' = \frac{(\sec x)(\tan x)}{\sqrt{\sec^2 x - 1}}$$

$$= \frac{(\sec x)(\tan x)}{\sqrt{\tan^2 x}}$$

$$= \frac{(\sec x)(\tan x)}{|\tan x|}$$

$$\boxed{= \sec x} \quad 0 < x < \frac{\pi}{2}$$

Exercise

Find the derivative of $y = -\sinh^3 4x$

Solution

$$\boxed{y' = -12\left(\sinh^2 4x\right)(\cosh 4x)}$$

Exercise

Find the derivative of $y = \sqrt{\coth 3x}$

Solution

$$y' = \boxed{\frac{-3\,\text{csch}^2 3x}{2\sqrt{\coth 3x}}}$$

$$\left(\sqrt{u}\right)' = \frac{u'}{2\sqrt{u}}$$

Exercise

Find the derivative of $y = \dfrac{x}{\text{csch} x}$

Solution

$$y' = \frac{\text{csch} x + x\,\text{csch} x \coth x}{\text{csch}^2 x}$$

$$\frac{d}{dx}(\text{csch}\, u) = -u'\,\text{csch}\, u\ \coth u$$

$$= \frac{1 + x\coth x}{\operatorname{csch} x}$$

$$= \frac{1}{\operatorname{csch} x} + x\frac{\coth x}{\operatorname{csch} x}$$

$$\boxed{= \sinh x + x\cosh x}$$

Exercise

Find the derivative of $\quad y = \tanh^2 x$

Solution

$$\boxed{y' = 2\tanh x\ \operatorname{sech}^2 x} \qquad \frac{d}{dx}(\tanh u) = u'\operatorname{sech}^2 u$$

Exercise

Find the derivative of $\quad y = \ln\ \operatorname{sech} 2x$

Solution

$$y' = \frac{-2\operatorname{sech} 2x\ \tanh 2x}{\operatorname{sech} 2x}\ \boxed{= -2\tanh 2x} \qquad \frac{d}{dx}(\operatorname{sech} u) = -u'\operatorname{sech} u\ \tanh u$$

Exercise

Find the derivative of $\quad y = x^2\cosh^2 3x$

Solution

$$y' = 2x\cosh^2 3x + 6x^2\cosh 3x\ \sinh 3x \qquad \frac{d}{dx}(\sinh u) = u'\cosh u$$

$$\boxed{= 2x\cosh 3x(\cosh 3x + 3x\ \sinh 3x)}$$

Exercise

Find the derivative of $\quad f(t) = 2\tanh^{-1}\sqrt{t}$

Solution

$$f'(t) = 2\frac{\frac{1}{2\sqrt{t}}}{1 - \left(\sqrt{t}\right)^2}\ \boxed{= \frac{1}{\sqrt{t}(1-t)}} \qquad \frac{d}{dx}\left(\tanh^{-1} u\right) = \frac{u'}{1-u^2}$$

Exercise

Find the derivative of $f(x) = \sinh^{-1} x^2$

Solution

$$f'(x) = \boxed{\frac{2x}{\sqrt{x^4+1}}}$$

$$\frac{d}{dx}\left(\sinh^{-1} u\right) = \frac{u'}{\sqrt{1+u^2}}$$

Exercise

Find the derivative of $f(x) = \operatorname{csch}^{-1}\left(\frac{2}{x}\right)$

Solution

$$f'(x) = \frac{-1}{\left|\frac{2}{x}\right|\sqrt{1+\frac{4}{x^2}}}\left(\frac{-2}{x^2}\right) = \boxed{\frac{1}{\sqrt{x^2+4}}}$$

$$\frac{d}{dx}\left(\operatorname{csch}^{-1} u\right) = -\frac{u'}{|u|\sqrt{1+u^2}}$$

Exercise

Find the derivative of $f(x) = x\sinh^{-1} x - \sqrt{x^2+1}$

Solution

$$f'(x) = \sinh^{-1} x + x\frac{1}{\sqrt{x^2+1}} - \frac{2x}{2\sqrt{x^2+1}}$$

$$\boxed{= \sinh^{-1} x}$$

$$\frac{d}{dx}\left(\sinh^{-1} u\right) = \frac{u'}{\sqrt{1+u^2}}$$

Exercise

Find the derivative of $f(x) = \sinh^{-1}(\tan x)$

Solution

$$f'(x) = \frac{\sec^2 x}{\sqrt{1+\tan^2 x}}$$

$$= \frac{\sec^2 x}{\sqrt{\sec^2 x}}$$

$$\boxed{= |\sec x|}$$

$$\frac{d}{dx}\left(\sinh^{-1} u\right) = \frac{u'}{\sqrt{1+u^2}}$$

Exercise

Verify the integration $\int x\operatorname{sech}^{-1} x\,dx = \frac{x^2}{2}\operatorname{sech}^{-1} x - \frac{1}{2}\sqrt{1-x^2} + C$

Solution

If $y=\frac{x^2}{2}\operatorname{sech}^{-1}x-\frac{1}{2}\sqrt{1-x^2}+C$

$$dy=\left[x\operatorname{sech}^{-1}x+\frac{x^2}{2}\left(\frac{-1}{x\sqrt{1-x^2}}\right)-\frac{1}{2}\left(\frac{1}{2}\right)\frac{-2x}{\sqrt{1-x^2}}\right]dx$$

$$dy=\left[x\operatorname{sech}^{-1}x-\frac{x}{2\sqrt{1-x^2}}+\frac{x}{2\sqrt{1-x^2}}\right]dx$$

$\boxed{dy=\left(x\operatorname{sech}^{-1}x\right)dx}$ ✓ Which verifies the formula

Exercise

Verify the integration $\int \tanh^{-1}x\,dx=x\tanh^{-1}x+\frac{1}{2}\ln\left(1-x^2\right)+C$

Solution

If $y=x\tanh^{-1}x+\frac{1}{2}\ln\left(1-x^2\right)+C$

$$\frac{dy}{dx}=\tanh^{-1}x+x\left(\frac{1}{1-x^2}\right)+\frac{1}{2}\frac{-2x}{1-x^2}$$

$$=\tanh^{-1}x+\frac{x}{1-x^2}-\frac{x}{1-x^2}$$

$\boxed{=\tanh^{-1}x}$ ✓ which verifies the formula

Exercise

Evaluate the integral: $\int \sinh 2x\,dx$

Solution

$$\int \sinh 2x\;dx=\frac{1}{2}\int \sinh 2x\;d(2x)$$

$$\boxed{=\frac{1}{2}\cosh 2x+C}$$

Exercise

Evaluate the integral: $\int 4\cosh(3x-\ln 2)\,dx$

Solution

$$\int 4\cosh(3x-\ln 2)\,dx = \frac{4}{3}\int \cosh(3x-\ln 2)\ d(3x-\ln 2) \qquad d(3x-\ln 2) = 3dx$$

$$= \frac{4}{3}\sinh(3x-\ln 2)+C$$

Exercise

Evaluate the integral: $\int \tanh\frac{x}{7}\,dx$

Solution

$$\int \tanh\frac{x}{7}\ dx = \int \frac{\sinh\frac{x}{7}}{\cosh\frac{x}{7}}\ d(x) \qquad d\left(\cosh\frac{x}{7}\right) = \frac{1}{7}\left(\sinh\frac{x}{7}\right)dx$$

$$= 7\int \frac{1}{\cosh\frac{x}{7}}\ d\left(\cosh\frac{x}{7}\right)$$

$$= 7\ln\left|\cosh\frac{x}{7}\right| + C$$

$$= 7\ln\left[\frac{e^{x/7}+e^{-x/7}}{2}\right] + C$$

$$= 7\left[\ln\left(e^{x/7}+e^{-x/7}\right) - \ln 2\right] + C$$

$$= 7\ln\left(e^{x/7}+e^{-x/7}\right) - 7\ln 2 + C \qquad C_1 = -7\ln 2 + C$$

$$= 7\ln\left(e^{x/7}+e^{-x/7}\right) + C_1$$

Exercise

Evaluate the integral: $\int \coth\frac{\theta}{\sqrt{3}}\,d\theta$

Solution

$$\int \coth\frac{\theta}{\sqrt{3}}\ d\theta = \int \frac{\cosh\frac{\theta}{\sqrt{3}}}{\sinh\frac{\theta}{\sqrt{3}}}\ d\theta \qquad d\left(\sinh\frac{\theta}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}\left(\cosh\frac{\theta}{\sqrt{3}}\right)d\theta$$

$$= \sqrt{3}\int \frac{1}{\sinh\frac{\theta}{\sqrt{3}}}\ d\left(\sinh\frac{\theta}{\sqrt{3}}\right)$$

$$= \sqrt{3}\ln\left|\sinh\frac{\theta}{\sqrt{3}}\right| + C_1$$

$$=\sqrt{3}\ln\left|\frac{e^{\theta/\sqrt{3}}-e^{-\theta/\sqrt{3}}}{2}\right|+C_1$$

$$=\sqrt{3}\ln\left|e^{\theta/\sqrt{3}}-e^{-\theta/\sqrt{3}}\right|-\sqrt{3}\ln 2+C_1 \qquad C=-\sqrt{3}\ln 2+C_1$$

$$\underline{=\sqrt{3}\ln\left|e^{\theta/\sqrt{3}}-e^{-\theta/\sqrt{3}}\right|+C}$$

Exercise

Evaluate the integral: $\int \operatorname{csch}^2(5-x)\,dx$

Solution

$$\int \operatorname{csch}^2(5-x)\ dx=-\int \operatorname{csch}^2(5-x)\ d(5-x) \qquad \int \operatorname{csch}^2 u\ du=-\coth u$$

$$\underline{=\coth(5-x)+C}$$

Exercise

Evaluate the integral: $\int \frac{\operatorname{sech}\sqrt{t}\ \tanh\sqrt{t}}{\sqrt{t}}\,dt$

Solution

$$\int \frac{\operatorname{sech}\sqrt{t}\ \tanh\sqrt{t}}{\sqrt{t}}\,dt=2\int \operatorname{sech}\sqrt{t}\ \tanh\sqrt{t}\ d\left(\sqrt{t}\right) \qquad d\left(\sqrt{t}\right)=\frac{1}{2\sqrt{t}}\,dt$$

$$\underline{=-2\operatorname{sec}h\sqrt{t}+C} \qquad \int \operatorname{sech} u\tanh u\ du=-\operatorname{sech} u$$

Exercise

Evaluate the integral: $\int \frac{\operatorname{csch}(\ln t)\ \coth(\ln t)}{t}\,dt$

Solution

$$\int \frac{\operatorname{csch}(\ln t)\ \coth(\ln t)}{t}\,dt=\int \operatorname{csch}(\ln t)\ \coth(\ln t)\ d(\ln t) \qquad d(\ln t)=\frac{dt}{t}$$

$$\underline{=-\operatorname{csc}h(\ln t)+C} \qquad \int \operatorname{csch} u\coth u\ du=-\operatorname{csch} u$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sinh x}{1+\cosh x}\,dx$

Solution

$$\int \frac{\sinh x}{1+\cosh x}\,dx = \int \frac{d(1+\cosh x)}{1+\cosh x} \qquad d(1+\cosh x) = \sinh x\;dx$$

$$\boxed{= \ln|1+\cosh x| + C}$$

Exercise

Evaluate the integral $\displaystyle\int \operatorname{sech}^2 x\;\tanh x\;dx$

Solution

$$\int \operatorname{sech}^2 x\;\tanh x\;dx = \int \tanh x\;d(\tanh x) \qquad d(\tanh x) = \operatorname{sech}^2 x\;dx$$

$$\boxed{= \frac{1}{2}\tanh^2 x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \coth^2 x\;\operatorname{csch}^2 x\;dx$

Solution

$$\int \coth^2 x\;\operatorname{csch}^2 x\;dx = -\int \coth^2 x\;d(\coth x) \qquad d(\coth x) = -\operatorname{csch}^2 x\;dx$$

$$\boxed{= -\frac{1}{3}\coth^3 x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \tanh^2 x\;dx$

Solution

$$\int \tanh^2 x\;dx = \int \left(1-\operatorname{sech}^2 x\right)dx \qquad \tanh^2 x = 1-\operatorname{sech}^2 x$$

$$\boxed{= x - \tanh x + C} \qquad \int \operatorname{sech}^2 u\,du = \tanh u$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sinh(\ln x)}{x}\,dx$

Solution

$$\int \frac{\sinh(\ln x)}{x}\,dx = \int \sinh(\ln x)\; d(\ln x)$$
$$= \cosh(\ln x) + C$$
$$= \frac{e^{\ln x} + e^{-\ln x}}{2} + C$$
$$= \frac{1}{2}\left(x + \frac{1}{x}\right) + C$$
$$\boxed{= \frac{x^2+1}{2x} + C}$$

$$\sinh(\ln x) = \frac{e^{\ln x} - e^{-\ln x}}{2} \quad = \frac{1}{2}\left(x - \frac{1}{x}\right)$$
$$\int \frac{\sinh(\ln x)}{x}\,dx = \frac{1}{2}\int \frac{1}{x}\left(x - \frac{1}{x}\right)\, d(x)$$
$$= \frac{1}{2}\int \left(1 - \frac{1}{x^2}\right)\, d(x)$$
$$= \frac{1}{2}\left(x + \frac{1}{x}\right) + C$$
$$\boxed{= \frac{x^2+1}{2x} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{8-x^2} \qquad x > 2\sqrt{2}$

Solution

$$\int \frac{dx}{8-x^2} \boxed{= \frac{1}{2\sqrt{2}}\tanh^{-1}\left(\frac{x}{2\sqrt{2}}\right) + C}$$

$$\int \frac{du}{a^2-u^2} = \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right)$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\sqrt{x^2-16}}$

Solution

$$\int \frac{dx}{\sqrt{x^2-16}} \boxed{= \coth^{-1}\left(\frac{x}{4}\right) + C}$$

$$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right)$$

Exercise

Evaluate the integral $\displaystyle\int_0^1 \cosh^3 3x\; \sinh 3x\; dx$

Solution

$$\int_0^1 \cosh^3 3x\; \sinh 3x\; dx = \frac{1}{3}\int_0^1 \cosh^3 3x\; d(\cosh 3x)$$

$$d(\cosh 3x) = 3\sinh x\; dx$$

$$= \frac{1}{12}\cosh^4 3x \Big|_0^1$$

$$= \frac{1}{12}\left(\cosh^4 3 - \cosh^4 0\right)$$

$$= \frac{1}{12}\left(\cosh^4 3 - 1\right) \qquad \approx 856.034$$

Exercise

Evaluate the integral $\int_0^4 \frac{\operatorname{sech}^2\sqrt{x}}{\sqrt{x}}\,dx$

Solution

$$\int_0^4 \frac{\operatorname{sech}^2\sqrt{x}}{\sqrt{x}}\,dx = 2\int_0^4 \operatorname{sech}^2\sqrt{x}\; d\left(\sqrt{x}\right) \qquad d\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}dx$$

$$= 2\tanh\sqrt{x}\Big|_0^4$$

$$= 2\tanh 2 \qquad \approx 1.93$$

Exercise

Evaluate the integral $\int_{\ln 2}^{\ln 3} \operatorname{csch} x\; dx$

Solution

$$\int_{\ln 2}^{\ln 3} \operatorname{csch} x\; dx = \ln\left|\tanh\frac{x}{2}\right|\Big|_{\ln 2}^{\ln 3}$$

$$= \ln\left|\tanh\frac{\ln 3}{2}\right| - \ln\left|\tanh\frac{\ln 2}{2}\right| \qquad \approx 0.405$$

Exercise

Evaluate the integral: $\int_{\ln 2}^{\ln 4} \coth x\; dx$

Solution

$$\int_{\ln 2}^{\ln 4} \coth x\; dx = \int_{\ln 2}^{\ln 4} \frac{\cosh x}{\sinh x}\; dx$$

$$= \int_{\ln 2}^{\ln 4} \frac{1}{\sinh x}\; d\left(\sinh x\right) \qquad d\left(\sinh x\right) = \cosh x\; dx$$

$$= \ln|\sinh x| \Big|_{\ln 2}^{\ln 4}$$

$$= \ln|\sinh \ln 4| - \ln|\sinh \ln 2|$$

$$= \ln\left(\frac{e^{\ln 4} - e^{-\ln 4}}{2}\right) - \ln\left(\frac{e^{\ln 2} - e^{-\ln 2}}{2}\right)$$

$$= \ln\left(\frac{4 - \frac{1}{4}}{2}\right) - \ln\left(\frac{2 - \frac{1}{2}}{2}\right)$$

$$= \ln\left(\frac{15}{8}\right) - \ln\left(\frac{3}{4}\right)$$

$$= \ln\left(\frac{15}{8} \div \frac{3}{4}\right)$$

$$= \ln\left(\frac{5}{2}\right)$$

Exercise

Evaluate the integral: $\displaystyle\int_0^{\pi/2} 2\sinh(\sin\theta)\cos\theta \, d\theta$

Solution

$$\int_0^{\pi/2} 2\sinh(\sin\theta)\cos\theta \, d\theta = 2\int_0^{\pi/2} \sinh(\sin\theta) \, d(\sin\theta) \qquad d(\sin\theta) = \cos\theta \, d\theta$$

$$= 2\cosh(\sin\theta) \Big|_0^{\pi/2}$$

$$= 2(\cosh 1 - \cosh 0)$$

$$= 2\left(\frac{e + e^{-1}}{2} - 1\right)$$

$$= e + e^{-1} - 2$$

Exercise

Evaluate the integral: $\displaystyle\int_1^2 \frac{8\cosh\sqrt{x}}{\sqrt{x}} dx$

Solution

$$\int_1^2 \frac{8\cosh\sqrt{x}}{\sqrt{x}} dx = 16\int_1^2 \cosh\sqrt{x} \, d\left(\sqrt{x}\right)$$

$$= 16\sinh\sqrt{x} \Big|_1^2$$

$$=16\left(\sinh\sqrt{2}-\sinh 1\right)$$

$$=16\left(\frac{e^{\sqrt{2}}-e^{-\sqrt{2}}}{2}-\frac{e-e^{-1}}{2}\right)$$

$$\boxed{=8\left(e^{\sqrt{2}}-e^{-\sqrt{2}}-e+e^{-1}\right)}$$

Exercise

Evaluate the integral: $\displaystyle\int_{-\ln 2}^{0}\cosh^2\left(\frac{x}{2}\right)dx$

Solution

$$\int_{-\ln 2}^{0}\cosh^2\left(\frac{x}{2}\right)dx=\frac{1}{2}\int_{-\ln 2}^{0}(\cosh x+1)dx$$

$$=\frac{1}{2}(\sinh x+x)\Big|_{-\ln 2}^{0}$$

$$=\frac{1}{2}\left(-\sinh(-\ln 2)+\ln 2\right)$$

$$=\frac{1}{2}\left(-\frac{e^{-\ln 2}-e^{\ln 2}}{2}+\ln 2\right)$$

$$=\frac{1}{2}\left(-\frac{\frac{1}{2}-2}{2}+\ln 2\right)$$

$$=\frac{3}{8}+\frac{1}{2}\ln 2$$

$$\boxed{=\frac{3}{8}+\ln\sqrt{2}}$$

Exercise

Evaluate the integral: $\displaystyle\int_{0}^{\ln 2}4e^{-\theta}\sinh\theta\,d\theta$

Solution

$$\int_{0}^{\ln 2}4e^{-\theta}\sinh\theta d\theta=\int_{0}^{\ln 2}4e^{-\theta}\frac{e^{\theta}-e^{-\theta}}{2}d\theta$$

$$=2\int_{0}^{\ln 2}\left(1-e^{-2\theta}\right)d\theta$$

$$= 2\left[\theta + \frac{1}{2}e^{-2\theta}\right]_0^{\ln 2}$$

$$= 2\left[\ln 2 + \frac{1}{2}e^{-2\ln 2} - \left(0 + \frac{1}{2}\right)\right]$$

$$= 2\left[\ln 2 + \frac{1}{2}e^{\ln 2^{-2}} - \frac{1}{2}\right]$$

$$= 2\left(\ln 2 + \frac{1}{2}2^{-2} - \frac{1}{2}\right)$$

$$= 2\left(\ln 2 + \frac{1}{8} - \frac{1}{2}\right)$$

$$= 2\left(\ln 2 - \frac{3}{8}\right)$$

$$= 2\ln 2 - \frac{3}{4}$$

$$\boxed{= \ln 4 - \frac{3}{4}}$$

Exercise

Evaluate the integral: $\displaystyle\int_1^{e^2} \frac{dx}{x\sqrt{\ln^2 x + 1}}$

Solution

$$\int_1^{e^2} \frac{dx}{x\sqrt{\ln^2 x + 1}} = \int_1^{e^2} \frac{d(\ln x)}{\sqrt{\ln^2 x + 1}}$$

$$\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right)$$

$$= \sinh^{-1}(\ln x)\Big|_1^{e^2}$$

$$= \sinh^{-1} 2 - 0$$

$$\boxed{= \sinh^{-1} 2}$$

Exercise

Evaluate the integral: $\displaystyle\int_{1/8}^{1} \frac{dx}{x\sqrt{1 + x^{2/3}}}$

Solution

$$u = x^{1/3} \quad \rightarrow \quad du = \frac{1}{3}x^{-2/3}dx$$

$$u^3 = x \quad \& \quad dx = 3x^{2/3}du = 3u^2du$$

$$\int_{1/8}^{1} \frac{dx}{x\sqrt{1+x^{2/3}}} = 3\int_{1/8}^{1} \frac{u^2 du}{u^3\sqrt{1+u^2}}$$

$$= 3\int_{1/8}^{1} \frac{du}{u\sqrt{1+u^2}} \qquad \int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right|$$

$$= -3\operatorname{csch}^{-1}\left|x^{1/3}\right|\Big|_{1/8}^{1}$$

$$= -3\left(\operatorname{csch}^{-1}1 - \operatorname{csch}^{-1}\frac{1}{2}\right)$$

$$= 3\left(\sinh^{-1}2 - \sinh^{-1}1\right) \qquad x = \ln\left(y+\sqrt{y^2+1}\right)$$

$$= 3\left(\ln\left(2+\sqrt{5}\right) - \ln\left(1+\sqrt{2}\right)\right)$$

Exercise

Derive the formula $\sinh^{-1} x = \ln\left(x+\sqrt{x^2+1}\right)$ for all real x. Explain in your derivation why the plus sign is used with the square root instead of the minus sign

Solution

$$y = \sinh^{-1} x \quad \Rightarrow \quad x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - e^{-y}$$

$$2xe^y = e^y e^y - e^{-y} e^y$$

$$e^{2y} - 2xe^y - 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$y = \ln\left(x \pm \sqrt{x^2+1}\right)$$

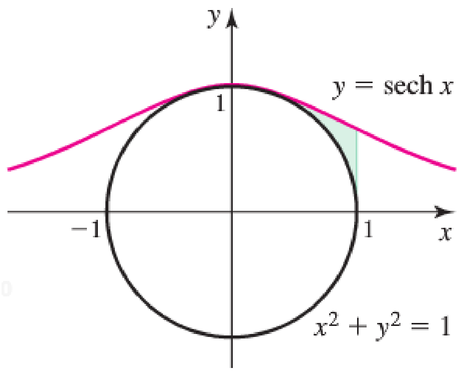
Since $x - \sqrt{x^2+1} < 0$ (***impossible***) $\Rightarrow \quad y = \ln\left(x - \sqrt{x^2+1}\right)$

$$\therefore\ y = \ln\left(x+\sqrt{x^2+1}\right)$$

Exercise

Find the area of the region bounded by $y = \text{sech}\, x, \;\; x = 1$, and the unit circle.

Solution



The area of a quarter circle $= \frac{1}{4}\left(\pi r^2\right) = \boxed{\frac{\pi}{4}}$

$$Area = \int_0^1 \text{sech}x\, dx - \frac{\pi}{4}$$

$$= \tan^{-1}\left|\sinh x\right| \Big|_0^1 - \frac{\pi}{4}$$

$$= \tan^{-1}(\sinh 1) - \frac{\pi}{4}$$

$$\approx 0.08$$

$$Area = \int_0^1 \left(\text{sech}x - \sqrt{1-x^2}\right) dx$$

$$= \left(\tan^{-1}\left|\sinh x\right| - \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\sin^{-1} x\right)\Big|_0^1$$

$$= \tan^{-1}(\sinh 1) - \frac{1}{2}\sin^{-1} 1$$

$$= \tan^{-1}(\sinh 1) - \frac{\pi}{4}$$

$$\approx 0.08$$

Exercise

A region in the first quadrant is bounded above the curve $y = \cosh x$, below by the curve $y = \sinh x$, and on the left and right by the y-axis and the line $x = 2$, respectively. Find the volume of the solid generated by revolving the region about the x-axis.

Solution

$$V = \pi \int_0^2 \left(\cosh^2 x - \sinh^2 x\right) dx$$

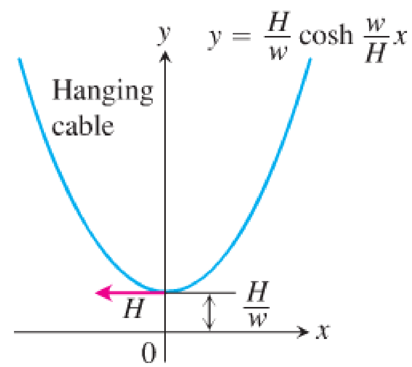
$$= \pi \int_0^2 dx$$

$$= \pi x \Big|_0^2$$

$$= 2\pi$$

Exercise

Imagine a cable, like a telephone line or TV cable, strung from one support to another and hanging freely. The cable's weight per unit length is a constant *w* and the horizontal tension at its lowest point is a vector of length *H*. If we choose a coordinate system for the plane of the cable in which the *x*-axis is horizontal, the force of gravity is straight down, the positive *y*-axis points straight up, and the lowest point of the cable lies at the point $y=\frac{H}{w}$ on the *y*-axis, then it can be shown that the cable lies along the graph of the hyperbolic cosine



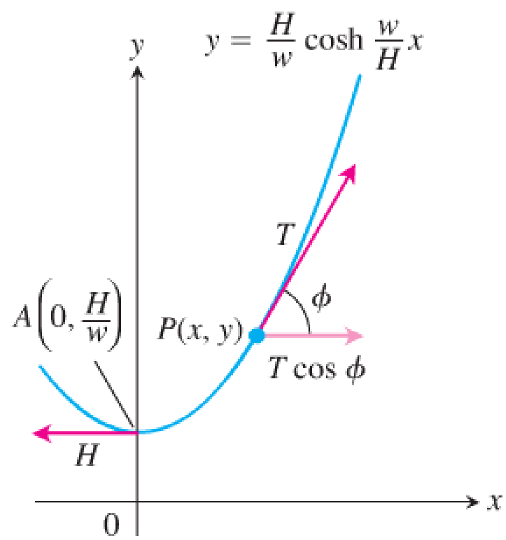
$$y=\frac{H}{w}\cosh\left(\frac{w}{H}x\right)$$

Such a curve is sometimes called a ***chain curve*** or a ***catenary***, the latter deriving from the Latin *catena*, meaning "*chain*".

a) Let $P(x,y)$ denote an arbitrary point on the cable. The next accompanying displays the tension *H* at the lowest point *A*. Show that the cable's slope at *P* is

$$\tan\phi=\frac{dy}{dx}=\sinh\left(\frac{w}{H}x\right)$$

b) Using the result in part (*a*) and the fact that the horizontal tension at *P* must equal *H* (the cable is not moving), show that $T=wy$. Hence, the magnitude of the tension at $P(x,y)$ is exactly equal to the weight of *y* units of cable.



c) The length of arc *AP* is $s=\frac{1}{a}\sinh ax$, where $a=\frac{w}{H}$. Show that the coordinates of *P* may be expressed in terms of *s* as $x=\frac{1}{a}\sinh^{-1}as,\quad y=\sqrt{s^2+\frac{1}{a^2}}$

Solution

a) $y=\frac{H}{w}\cosh\left(\frac{w}{H}x\right)$

$$\tan\phi=\frac{dy}{dx}=\frac{H}{w}\left[\frac{w}{H}\sinh\left(\frac{w}{H}x\right)\right]$$

$$=\sinh\left(\frac{w}{H}x\right)$$

b) The tension at *P* is given by $T\cos\phi=H$.

$$T=\frac{H}{\cos\phi}$$

$$=H\sec\phi$$

$$=H\sqrt{1+\tan^2\phi}$$

$$= H\sqrt{1+\sinh^2\left(\frac{w}{H}x\right)}$$ $\cosh^2 x - \sinh^2 x = 1 \;\to\; \cosh x = \sqrt{1+\sinh^2 x}$

$$= H\cosh\left(\frac{w}{H}x\right)$$ $yw = H\cosh\left(\frac{w}{H}x\right)$

$$\boxed{= wy}$$

c) $s = \frac{1}{a}\sinh ax \quad \to \sinh ax = as$

$$ax = \sinh^{-1} as \;\to\; x = \frac{1}{a}\sinh^{-1} as$$

$$y = \frac{H}{w}\cosh\left(\frac{w}{H}x\right)$$

$$= \frac{1}{a}\cosh(ax)$$ $a = \frac{w}{H}$

$$= \frac{1}{a}\sqrt{\cosh^2(ax)}$$

$$= \frac{1}{a}\sqrt{1+\sinh^2(ax)}$$

$$= \frac{1}{a}\sqrt{1+(as)^2}$$

$$\boxed{= \sqrt{\frac{1}{a^2}+s^2}}$$

Exercise

The portion of the curve $y = \frac{17}{15} - \cosh x$ that lies above the x-axis forms a catenary arch. Find the average height of the arch above the x-axis.

Solution

By symmetry;

$$I = 2\int_0^{\cosh^{-1}(17/15)}\left(\frac{17}{15} - \cosh x\right)dx$$

$$= 2\left(\frac{17}{15}x - \sinh x\right)\Big|_0^{\cosh^{-1}(17/15)}$$

$$= 2\left[\frac{17}{15}\cosh^{-1}\left(\frac{17}{15}\right) - \sinh\left(\cosh^{-1}\left(\frac{17}{15}\right)\right)\right]$$

$$= \frac{34}{15}\cosh^{-1}\left(\frac{17}{15}\right) - \frac{16}{15}$$

$$\textit{Average height} = \frac{I}{2\cosh^{-1}\left(\frac{17}{15}\right)}$$

$$\boxed{= \frac{17}{15} - \frac{8}{15\cosh^{-1}\left(\frac{17}{15}\right)}} \quad \boxed{\approx 0.09}$$

Exercise

A power line is attached at the same height to two utility poles that are separated by a distance of 100 *ft*; the power line follows the curve $f(x) = a\cosh\left(\frac{x}{a}\right)$. Use the following steps to find the value of a that produces a sag of 10 *ft*. midway between the poles. Use the coordinate system that places the poles at $x = \pm 50$

a) Show that a satisfies the equation $\cosh\left(\frac{50}{a}\right) - 1 = \frac{10}{a}$

b) Let $t = \frac{10}{a}$, confirm that the equation in part (a) reduces to $\cosh 5t - 1 = t$, and solve for t using a graphing utility. (2 decimal places)

c) Use the answer in part (b) to find a and then compute the length of the power line.

Solution

a) Let $a = 10\ ft\ \ (sag)$.

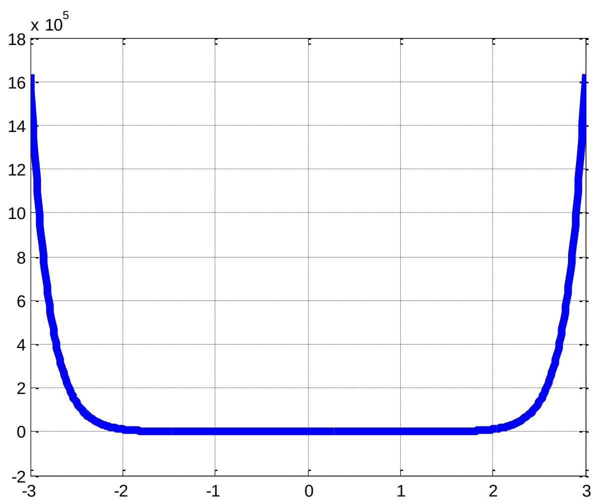
$$f(0) + sag = f(50)$$

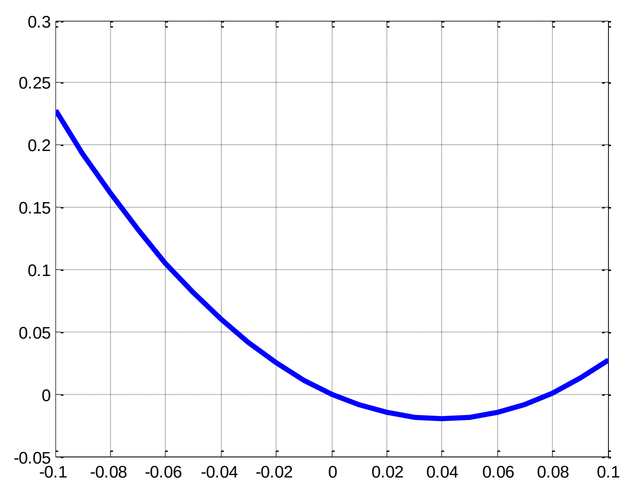
$$a + sag = a\cosh\left(\frac{50}{a}\right)$$

$$1 + \frac{sag}{a} = \cosh\left(\frac{50}{a}\right)$$

$$\cosh\left(\frac{50}{a}\right) - 1 = \frac{10}{a}$$

b) If $t = \frac{10}{a} \ \rightarrow\ \cosh(5t) - 1 = t$





$t \approx 0.08$

c) If $\frac{10}{a} = 0.08 \ \Rightarrow\ a = \frac{10}{0.08} = 125$

The length of the power line is:

$$L = 2\int_0^{50} \sqrt{1 - \sinh^2\left(\frac{x}{125}\right)}\ dx$$

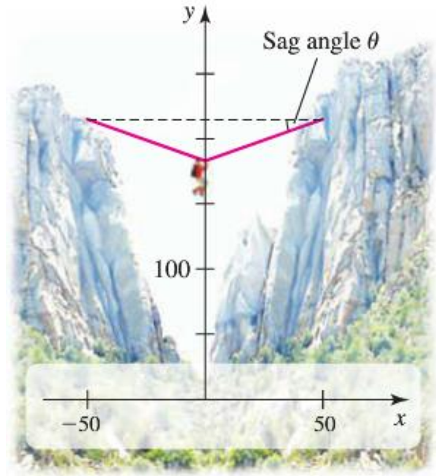
$$= 2\int_0^{50} \cosh\left(\frac{x}{125}\right)\, dx$$

$$= 250\sinh\left(\frac{x}{125}\right)\Big|_0^{50}$$

$$= 250\sinh\left(\frac{2}{5}\right) \approx 102.7\ ft$$

Exercise

Imagine a climber clipping onto the rope and pulling hinself to the rope's midpoint. Because the rope is supporting the weight of the climber, it no longer takes the shape of the catenary $y = 200\cosh\left(\frac{x}{200}\right)$. Instead, the rope (nearly) forms two sides of an isosceles triangle. Compute the sag angle illustrated in the figure, assuming that the rope does not stretch when weighted. Assume the length of the rope is 101 *feet*.



Solution

$$\theta = \cos^{-1}\left(\frac{50}{50.5}\right) \approx 0.14\ rad$$

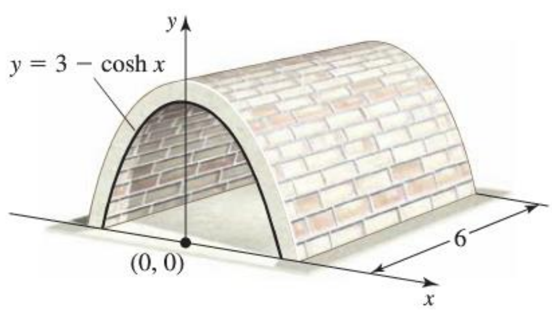
Exercise

Find the volume interior to the inverted catenary kiln (an oven used to fire pottery).

Solution

$$y = 3 - \cosh x = 0 \quad \Rightarrow \quad \cosh x = 3 \rightarrow x = \cosh^{-1}(\pm 3)$$

Therfore; the area is between $\cosh^{-1}(-3)$ and $\cosh^{-1}(3)$.



$$A = 2\int_0^{\cosh^{-1}(3)} (3 - \cosh x)\, dx$$

$$= 2(3x - \sinh x)\Big|_0^{\cosh^{-1}(3)}$$

$$= 2\left(3\cosh^{-1}(3) - \sinh x\left(\cosh^{-1}(3)\right)\right)$$

$$\approx 4.92$$

$$Volume \approx 6(4.92) \approx 29.5$$