Solution *Section* 4.5 – The Area between Two Curves

Exercise

Find the area of the region bounded by the graphs of $y = x^2 - x - 2$ and x-axis

Solution

Determine the intersection points:

$$x^2 - x - 2 = 0 \Rightarrow \boxed{x = -1, 2}$$

$$A = \int_{-1}^{2} [0 - (x^2 - x - 2)]dx$$

$$= -\frac{x^3}{3} + \frac{x^2}{2} + 2x \Big|_{-1}^{2}$$

$$= -\frac{2^3}{3} + \frac{2^2}{2} + 2(2) - \left[-\frac{(-1)^3}{3} + \frac{(-1)^2}{2} + 2(-1) \right]$$

$$= -\frac{8}{3} + 2 + 4 - \left[\frac{1}{3} + \frac{1}{2} - 2 \right]$$

$$= 4.5 \ unit^2$$

Exercise

Find the area of the region bounded by the graphs of $f(x) = x^3 + 2x^2 - 3x$ and $g(x) = x^2 + 3x$

Solution

$$x^3 + 2x^2 - 3x = x^2 + 3x$$

$$x^3 + x^2 - 6x = 0$$

$$x(x^2 + x - 6) = 0 \Rightarrow \begin{cases} x = 0 \\ x^2 + x - 6 = 0 \end{cases}$$

$$x^2 + x - 6 = 0 \rightarrow x = -3, 2$$

$$\rightarrow x = -3, 0, 2$$

$$A = \int_{-3}^{0} (f - g)dx + \int_{0}^{2} (g - f)dx$$

$$= \int_{-3}^{0} (x^3 + 2x^2 - 3x - (x^2 + 3x))dx + \int_{0}^{2} (x^2 + 3x - (x^3 + 2x^2 - 3x))dx$$

$$= \int_{-3}^{0} (x^3 + x^2 - 6x)dx + \int_{0}^{2} (-x^3 - x^2 + 6x)dx$$

$$= \frac{x^4}{4} + \frac{x^3}{3} - 3x^2 \Big|_{-3}^{0} + \left[-\frac{x^4}{4} - \frac{x^3}{3} + 3x^2 \right]_0^2$$

$$= 0 - \left(\frac{(-3)^4}{4} + \frac{(-3)^3}{3} - 3(-3)^2 \right) + \left[\left(-\frac{2^4}{4} - \frac{2^3}{3} + 32^2 \right) - 0 \right]$$

$$= \frac{253}{12}$$

$$\approx 21.083$$

Exercise

Find the area bounded by $f(x) = -x^2 + 1$, $g(x) = 2x + 4$, $x = -1$, and $x = 2$

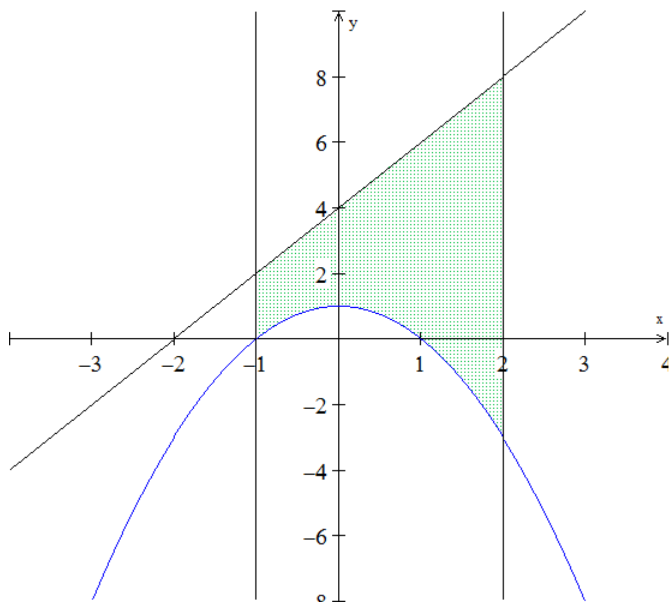
Solution

$$f \cap g \Rightarrow \quad -x^2 + 1 = 2x + 4$$

$$-x^2 - 2x - 3 = 0$$

$$x^2 + 2x + 3 = 0$$

$$\Rightarrow x = -1 \pm i\sqrt{2}$$



$$A = \int_{-1}^{2} (g - f)dx$$

$$= \int_{-1}^{2} \left(2x + 4 - \left(-x^2 + 1 \right) \right) dx$$

$$= \int_{-1}^{2} \left(x^2 + 2x + 3 \right) dx$$

$$= \frac{1}{3}x^3 + x^2 + 3x \Big|_{-1}^{2}$$

$$= \left(\frac{1}{3}(2)^3 + (2)^2 + 3(2) \right) - \left(\frac{1}{3}(-1)^3 + (-1)^2 + 3(-1) \right)$$

$$= \left(\frac{8}{3} + 4 + 6 \right) - \left(-\frac{1}{3} + 1 - 3 \right)$$

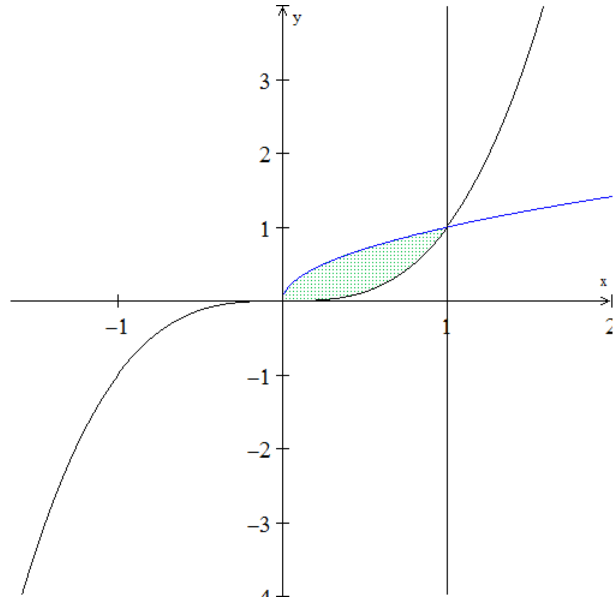
$$= \frac{8}{3} + 10 + \frac{1}{3} + 2$$

$$= 15$$

Exercise

Find the area between the curves $y = x^{1/2}$ *and* $y = x^3$

Solution



$$x^3 = x^{1/2}$$

Square both sides

$$x^6 = x$$

$$x^6 - x = 0$$

$$x\left(x^5 - 1\right) = 0$$

$$x = 0 \quad x^5 - 1 = 0 \Rightarrow x = 1$$

$$A = \int_0^1 \left(x^{1/2} - x^3\right) dx$$

$$= \frac{2}{3}x^{3/2} - \frac{1}{4}x^4 \Big|_0^1$$

$$= \frac{2}{3}1^{3/2} - \frac{1}{4}1^4 - 0$$

$$= \frac{2}{3} - \frac{1}{4}$$

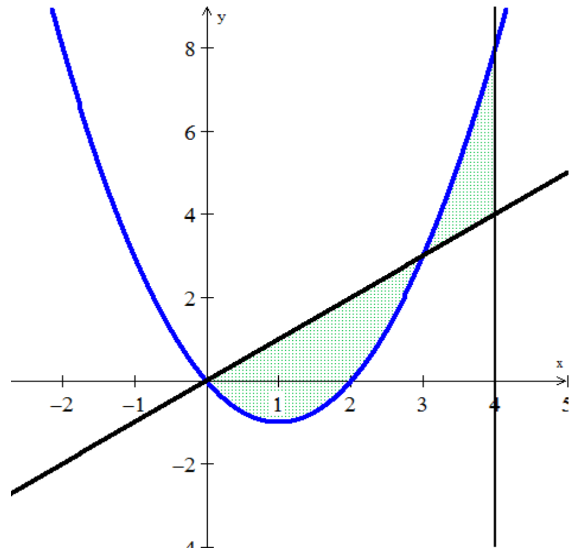
$$= \frac{8-3}{12}$$

$$= \frac{5}{12}$$

Exercise

Find the area of the region bounded by the graphs of $y = x^2 - 2x$ and $y = x$ on [0, 4].

Solution



$x^2 - 2x = x$

$x^2 - 3x = 0$

$x(x-3) = 0 \Rightarrow \boxed{x = 0, 3}$

$$A = \int_0^3 \left(x - \left(x^2 - 2x\right)\right)dx + \int_3^4 \left(x^2 - 2x - x\right)dx$$

$$= \int_0^3 \left(-x^2 + 3x\right)dx + \int_3^4 \left(x^2 - 3x\right)dx$$

$$= \left(-\frac{1}{3}x^3 + \frac{3}{2}x^2\right)\Big|_0^3 + \left(\frac{1}{3}x^3 - \frac{3}{2}x^2\right)\Big|_3^4$$

$$= \left(-\frac{1}{3}3^3 + \frac{3}{2}3^2\right) + \left[\left(\frac{1}{3}4^3 - \frac{3}{2}4^2\right) - \left(\frac{1}{3}3^3 - \frac{3}{2}3^2\right)\right]$$

$$= \left(\frac{9}{2}\right) + \left[\left(-\frac{8}{3}\right) - \left(-\frac{9}{2}\right)\right]$$

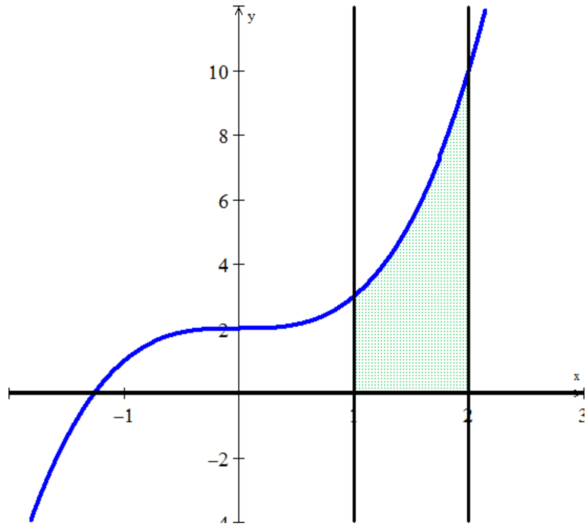
$$= \frac{9}{2} - \frac{8}{3} + \frac{9}{2}$$

$$= \frac{19}{3}$$

Exercise

Find the area between the curves $x = 1, \quad x = 2, \quad y = x^3 + 2, \quad y = 0$

Solution



$$A = \int_1^2 \left(x^3 + 2 - 0\right) dx$$

$$= \frac{1}{4}x^4 + 2x \Big|_1^2$$

$$= \left(\frac{1}{4}2^4 + 2(2)\right) - \left(\frac{1}{4}1^4 + 2(1)\right)$$

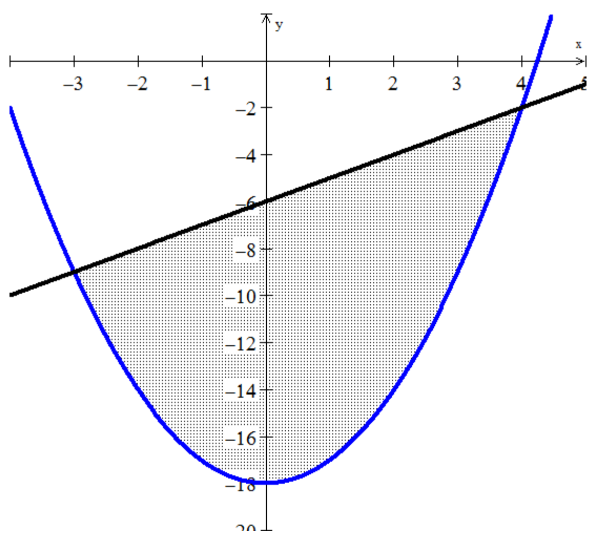
$$= (8) - \left(\frac{9}{4}\right)$$

$$= \frac{23}{4}$$

Exercise

Find the area between the curves $y = x^2 - 18, \quad y = x - 6$

Solution



$$x^2 - 18 = x - 6$$

$$x^2 - x - 12 = 0 \rightarrow \boxed{x = -3, 4}$$

$$A = \int_{-3}^4 \left(x^2 - 18 - (x - 6)\right) dx$$

$$= \int_{-3}^4 \left(x^2 - x - 12\right) dx$$

$$= \frac{1}{3}x^3 - \frac{1}{2}x^2 - 12x \Big|_{-3}^4$$

$$= \left(\frac{1}{3}4^3 - \frac{1}{2}4^2 - 12(4)\right) - \left(\frac{1}{3}(-3)^3 - \frac{1}{2}(-3)^2 - 12(-3)\right)$$

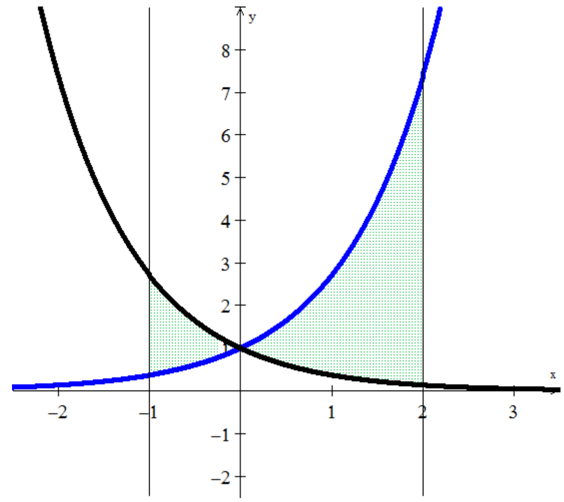
$$= \left(-\frac{104}{3}\right) - \left(\frac{45}{2}\right)$$

$$= \boxed{\frac{343}{6}}$$

Exercise

Find the area between the curves $x=-1, \quad x=2, \quad y=e^{-x}, \quad y=e^{x}$

Solution



$$e^{x}=e^{-x}$$

$$x=-x$$

$$\Rightarrow \boxed{x=0}$$

$$A=\int_{-1}^{0}\left(e^{-x}-e^{x}\right)dx+\int_{0}^{2}\left(e^{x}-e^{-x}\right)dx$$

$$=\left(-e^{-x}-e^{x}\right)\Big|_{-1}^{0}+\left(e^{x}+e^{-x}\right)\Big|_{0}^{2}$$

$$=\left(-e^{-0}-e^{0}\right)-\left(-e^{-(-1)}-e^{-1}\right)+\left[\left(e^{2}+e^{-2}\right)-\left(e^{0}+e^{-0}\right)\right]$$

$$=(-1-1)-\left(-e-e^{-1}\right)+\left[\left(e^{2}+e^{-2}\right)-(1+1)\right]$$

$$=-2+e+e^{-1}+e^{2}+e^{-2}-2$$

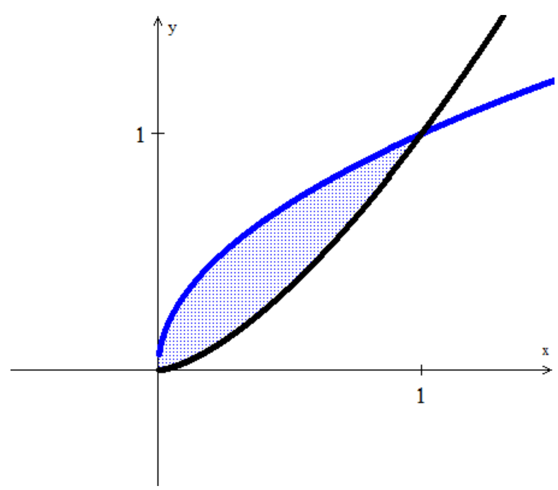
$$=e+e^{-1}+e^{2}+e^{-2}-4$$

$$\boxed{=6.61}$$

Exercise

Find the area between the curves $y=\sqrt{x}, \quad y=x\sqrt{x}$

Solution



$$x\sqrt{x}=\sqrt{x}$$

$$\left(x\sqrt{x}\right)^{2}=\left(\sqrt{x}\right)^{2}$$

$$x^{2}x=x$$

$$x\left(x^{2}-1\right)=0$$

$$\boxed{x=0} \quad x^{2}-1=0 \Rightarrow x=\pm 1 (\textit{no negative}) \quad \boxed{x=1}$$

$$A = \int_0^1 \left(\sqrt{x} - x\sqrt{x}\right)dx$$

$$= \int_0^1 \left(x^{1/2} - x^{3/2}\right)dx$$

$$= \frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2}\Big|_0^1$$

$$= \left(\frac{2}{3}1^{3/2} - \frac{2}{5}1^{5/2}\right) - \left(\frac{2}{3}0^{3/2} - \frac{2}{5}0^{5/2}\right)$$

$$= \left(\frac{2}{3} - \frac{2}{5}\right) - 0$$

$$\boxed{= \frac{4}{15}}$$

Exercise

A company is considering a new manufacturing process in one of its plants. The new process provides substantial initial savings, with the savings declining with time *t* (in years) according to the rate-of-savings function

$$S'(t) = 100 - t^2$$

where $S'(t)$ is in thousands of dollars per year. At the same time, the cost of operating the new process increases with time *t* (in years), according to the rate-of-cost function (in thousands of dollars per year)

$$C'(t) = t^2 + \frac{14}{3}t$$

a) For how many years will the company realize savings?

b) What will be the net total savings during this period?

Solution

a) $C'(t) = S'(t)$

$$C'(t) = S'(t)$$

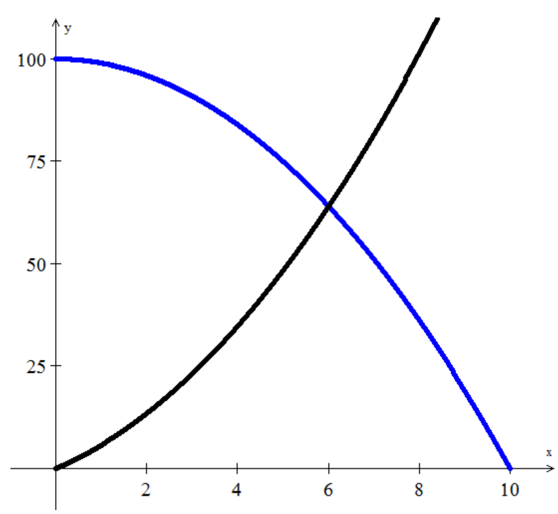
$$t^2 + \frac{14}{3}t = 100 - t^2$$

$$2t^2 + \frac{14}{3}t - 100 = 0$$

$$\to t = -\frac{25}{3} \; or \;\; 6$$

$$\boxed{t = 6}$$

The company should use this type for 6 years.



b) What will be the net total savings during this period?

$$\text{Total savings } = \int_0^6 \left[\left(100-t^2\right)-\left(t^2+\tfrac{14}{3}t\right)\right]dt$$

$$= \int_0^6 \left[100-2t^2-\tfrac{14}{3}t\right]dt$$

$$= 100t-\tfrac{2}{3}t^3-\tfrac{7}{3}t^2\Big|_0^6$$

$$= 100(6)-\tfrac{2}{3}(6)^3-\tfrac{7}{3}(6)^2-\left(100(0)-\tfrac{2}{3}(0)^3-\tfrac{7}{3}(0)^2\right)$$

$$\underline{= 372}$$

The company will save a total of \$372,000. Over the 6-year period

Exercise

Find the producers' surplus if the supply function for pork bellies is given by

$$S(x)=x^{5/2}+2x^{3/2}+50$$

Assume supply and demand are in equilibrium at $x=16$.

Solution

The equilibrium price: $\underline{p_0} = S(x=16)=16^{5/2}+2(16)^{3/2}+50\underline{=1202}$

$$\text{Producer's surplus } = \int_0^{x_0}\left[p_0-S(x)\right]dx$$

$$= \int_0^{16}\left[1202-\left(x^{5/2}+2x^{3/2}+50\right)\right]dx$$

$$= \int_0^{16}\left[1152-x^{5/2}-2x^{3/2}\right]dx$$

$$= 1152x-\tfrac{2}{7}x^{7/2}-\tfrac{4}{5}x^{5/2}\Big|_0^{16}$$

$$= \left(1152(16)-\tfrac{2}{7}(16)^{7/2}-\tfrac{4}{5}(16)^{5/2}\right)-\left(1152(0)-\tfrac{2}{7}(0)^{7/2}-\tfrac{4}{5}(0)^{5/2}\right)$$

$$\underline{= 12,931.66}$$

The producers' surplus is 12,931.66

Exercise

Suppose the supply function for concrete is given by

$$S(q) = 100 + 3q^{3/2} + q^{5/2}$$

And that supply and demand are in equilibrium at $q = 9$. Find the producers' surplus.

Solution

$$p_0 = S(9) = 100 + 3(9)^{3/2} + (9)^{5/2} = 424$$

$$\text{Producers' surplus } = \int_0^{q_0} \left| p_0 - S(q) \right| dq$$

$$= \int_0^9 \left[424 - \left(100 + 3q^{3/2} - q^{5/2} \right) \right] dq$$

$$= \int_0^9 \left[324 - 3q^{3/2} + q^{5/2} \right] dq$$

$$= 324q - \tfrac{6}{5} q^{5/2} + \tfrac{2}{7} q^{7/2} \Big|_0^9$$

$$= 324(9) - \tfrac{6}{5}(9)^{5/2} + \tfrac{2}{7}(9)^{7/2} - (0)$$

$$= 1999.54$$

The producers' surplus is 1999.54

Exercise

Find the consumers' surplus if the demand function for grass seed is given by

$$D(x) = \frac{200}{(3x+1)^2}$$

Assuming supply and demand are in equilibrium at $x = 3$.

Solution

$$p_0 = D(x) = \frac{200}{(3(3)+1)^2} = 2$$

$$\text{Consumers' surplus } = \int_0^{x_0} \left| D(x) - p_0 \right| dx$$

$$= \int_0^3 \left| \frac{200}{(3x+1)^2} - 2 \right| dx$$

$$= \int_0^3 \frac{200}{(3x+1)^2}dx - \int_0^3 2dx \qquad u = 3x+1 \Rightarrow du = 3dx \to \frac{1}{3}du = dx$$

$$= \frac{1}{3}\int_1^{10} \frac{200}{u^2}du - \int_0^3 2dx$$

$$= \frac{200}{3}\int_1^{10} u^{-2}du - \int_0^3 2dx$$

$$= \frac{200}{3}\left[\frac{u^{-1}}{-1}\right]_1^{10} - 2x\Big|_0^3$$

$$= \frac{200}{3}\left[-\frac{1}{u}\right]_1^{10} - 2(3-0)$$

$$= \frac{200}{3}\left(-\frac{1}{10}+\frac{1}{1}\right) - 6$$

$$\underline{= 54}$$

Exercise

Find the consumers' surplus if the demand function for olive oil is given by

$$D(x) = \frac{32{,}000}{(2x+8)^3}$$

And if supply and demand are in equilibrium at $x = 6$.

Solution

$$p_0 = D(6) = \frac{32000}{(2(6)+8)^3} = 4$$

$$\text{Consumers' surplus } = \int_0^{x_0} \left|D(x) - p_0\right|dx$$

$$= \int_0^6 \left(\frac{32{,}000}{(2x+8)^3} - 4\right)dx$$

$$= \int_0^6 32{,}000(2x+8)^{-3}dx - \int_0^6 4dx \qquad u = 2x+8 \Rightarrow du = 2dx \to \frac{1}{2}du = dx$$

$$= 32{,}000\int_8^{20} u^{-3}\frac{1}{2}du - \int_0^6 4dx$$

$$= 16000.\frac{u^{-2}}{-2}\Big|_{8}^{20} - 4x\Big|_{0}^{6}$$

$$= -8000\left(20^{-2} - 8^{-2}\right) - 4(6)$$

$$\underline{= 81}$$