

## ***Solution***      **Section 3.4 - Concavity and the Second Derivative Test**

### ***Exercise***

Determine the intervals on which the graph of the function is concave upward or concave downward.

$$f(x) = \frac{x^2 - 1}{2x + 1}$$

### **Solution**

$$\begin{aligned} f'(x) &= \frac{(2x+1)(2x) - (x^2-1)(2)}{(2x+1)^2} \\ &= \frac{4x^2 + 2x - 2x^2 + 2}{(2x+1)^2} \\ &= \frac{2x^2 + 2x + 2}{(2x+1)^2} \\ &= \frac{2(x^2 + x + 1)}{(2x+1)^2} \end{aligned}$$

$$\begin{aligned} f''(x) &= 2 \frac{(2x+1)^2(2x+1) - (x^2+x+1)(2)(2x+1)(2)}{(2x+1)^4} \\ &= 2 \frac{(2x+1)^3 - 4(x^2+x+1)(2x+1)}{(2x+1)^4} \\ &= 2 \frac{(2x+1)[(2x+1)^2 - 4(x^2+x+1)]}{(2x+1)^4} \\ &= 2 \frac{4x^2 + 4x + 1 - 4x^2 - 4x - 4}{(2x+1)^3} \\ &= 2 \frac{-3}{(2x+1)^3} \\ &= -\frac{6}{(2x+1)^3} \end{aligned}$$

$$2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$$

$f$  is concave upward on  $\left(-\infty, -\frac{1}{2}\right)$

$f$  is concave downward on  $\left(-\frac{1}{2}, \infty\right)$

$-\infty$	$-\frac{1}{2}$	$\infty$
$f''(-1) > 0$		$f''(0) < 0$
<b><i>Upward</i></b>		<b><i>Downward</i></b>

### Exercise

Find the points of inflection.  $f(x) = x^3 - 9x^2 + 24x - 18$

### Solution

$$f'(x) = 3x^2 - 18x + 24$$

$$f''(x) = 6x - 18 = 0 \Rightarrow x = 3$$

$$x = 3 \Rightarrow f(3) = 0$$

→ Point of inflection (3, 0)

### Exercise

Find the second derivative of  $f(x) = -2\sqrt{x}$  and discuss the concavity of the graph

### Solution

$$f'(x) = -x^{-1/2}$$

$$\Rightarrow f''(x) = \frac{1}{2}x^{-3/2}$$

$$= \frac{1}{2x^{3/2}} > 0 \text{ for all } x > 0$$

$f$  is concave upward for all  $x > 0$ .

### Exercise

Determine the intervals on which the graph of the function  $f(x) = -4x^3 - 8x^2 + 32$  is concave upward or concave downward

### Solution

$$f'(x) = -12x^2 - 16x$$

$$f''(x) = -24x - 16$$

$$f''(x) = -24x - 16 = 0$$

$$\Rightarrow -24x = 16$$

$$\Rightarrow x = \frac{16}{-24} = -\frac{2}{3}$$

$-\infty$	$-\frac{2}{3}$	$\infty$
$f''(-1) > 0$		$f''(0) < 0$
<b>Upward</b>		<b>Downward</b>

Concave upward on  $(-\infty, -2/3)$  and concave downward on  $(-2/3, \infty)$

### Exercise

Determine the intervals on which the graph of the function  $f(x) = \frac{12}{x^2 + 4}$  is concave upward or concave downward.

### Solution

$$f(x) = 12(x^2 + 4)^{-1}$$

$$f'(x) = -12(x^2 + 4)^{-2}(2x) = -\frac{12x}{(x^2 + 4)^2}$$

$$\begin{aligned} f''(x) &= -\frac{12(x^2 + 4)^2 - 12x(2)(x^2 + 4)(2x)}{(x^2 + 4)^4} \\ &= -\frac{12(x^2 + 4)^2 - 48x^2(x^2 + 4)}{(x^2 + 4)^4} \\ &= -\frac{12(x^2 + 4)\left[(x^2 + 4) - 4x^2\right]}{(x^2 + 4)^4} \\ &= -\frac{12(x^2 + 4)\left[x^2 + 4 - 4x^2\right]}{(x^2 + 4)^4} \\ &= -\frac{12(x^2 + 4)(-3x^2 + 4)}{(x^2 + 4)^4} \\ &= -\frac{12(-3x^2 + 4)}{(x^2 + 4)^3} \end{aligned}$$

Solve for  $x$ :

$$f''(x) = -\frac{12(-3x^2 + 4)}{(x^2 + 4)^3} = 0$$

$$\Rightarrow -3x^2 + 4 = 0$$

$$\rightarrow -3x^2 = -4$$

$$\rightarrow x^2 = \frac{4}{3}$$

$$\Rightarrow x = \pm\sqrt{\frac{4}{3}}$$

$$= \pm\frac{\sqrt{4}}{\sqrt{3}}$$

$$= \pm\frac{2}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}}$$

$$= \pm\frac{2\sqrt{3}}{3}$$

$-\infty$	$-\frac{2\sqrt{3}}{3}$	$\frac{2\sqrt{3}}{3}$	$\infty$
$f''(-2) > 0$		$f''(0) < 0$	$f''(2) > 0$
<i>upward</i>		<i>downward</i>	<i>upward</i>

$f$  is concave upward on  $\left(-\infty, -\frac{2\sqrt{3}}{3}\right)$  and  $\left(\frac{2\sqrt{3}}{3}, \infty\right)$

$f$  is concave downward on  $\left(-\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right)$

### Exercise

Find the extrema using the second derivative test  $f(x) = \frac{4}{x^2 + 1}$

### Solution

$$f'(x) = \frac{-8x}{(x^2 + 1)^2} \quad \text{CN is } x = 0$$

$$f''(x) = \frac{8(3x^2 - 1)}{(x^2 + 1)^3}$$

$$f''(0) = -8 < 0 \Rightarrow f(0) = 4 \text{ is a relative maximum (RMAX)}$$

### Exercise

Find all relative extrema of  $f(x) = x^4 - 4x^3 + 1$

### Solution

$$f'(x) = 4x^3 - 12x^2$$

$$f'(x) = 4x^2(x - 3) = 0 \rightarrow \boxed{x = 0, 3}$$

$$f''(x) = 12x^2 - 12x$$

Points:  $(0, 1)$        $f''(0) = 0$  Test fails  
 $(3, -26)$        $f''(3) > 0 \Rightarrow$  relative Minimum (RMIN)

### Exercise

Discuss the concavity of the graph of  $f$  and find its points of inflection.  $f(x) = x^4 - 2x^3 + 1$

### Solution

$$f'(x) = 4x^3 - 6x^2$$

$$f''(x) = 12x^2 - 12x = 0$$

$$12x(x-1) = 0 \Rightarrow x = 0, 1$$

$$\text{For } x = 0 \Rightarrow f(0) = 0^4 - 2(0)^3 + 1 = 1 \rightarrow (0, 1)$$

$$\text{For } x = 1 \Rightarrow f(1) = 1^4 - 2(1)^3 + 1 = 0 \rightarrow (1, 0)$$

$-\infty$	<b>0</b>	<b>1</b>	$\infty$
$f''(-1) > 0$	$f''(1/2) < 0$	$f''(2) > 0$	
<i>upward</i>	<i>downward</i>	<i>upward</i>	

$f$  is concave upward on  $(-\infty, 0)$  and  $(1, \infty)$

$f$  is concave downward on  $(0, 1)$

Points of inflection:  $(0, 1), (1, 0)$

### Exercise

The revenue  $R$  generated from sales of a certain product is related to the amount  $x$  spent on advertising by

$$R(x) = \frac{1}{15,000} (600x^2 - x^3), \quad 0 \leq x \leq 600$$

Where  $x$  and  $R$  are in thousands of dollars. Is there a point of diminishing returns for this function?

### Solution

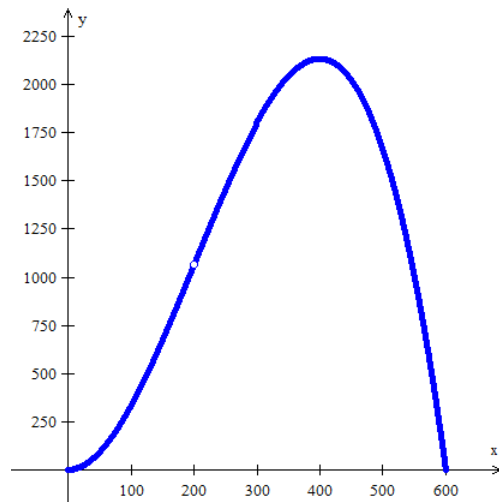
$$R' = \frac{1}{15,000} (1200x - 3x^2)$$

$$R' = \frac{1}{15,000} (1200 - 6x) = 0$$

$$\Rightarrow x = \frac{1200}{6} = 200$$

$x = 200$  (or \$200,000) is a **diminishing point**

An increased investment beyond this point is usually considered a poor use of capital



### Exercise

Find the point of diminishing returns  $(x, y)$  for the function

$$R(x) = -x^3 + 45x^2 + 400x + 8000, \quad 0 \leq x \leq 20$$

where  $R(x)$  represents revenue in thousands of dollars and  $x$  represents the amount spent on advertising in tens of thousands of dollars.

### Solution

$$R'(x) = -3x^2 + 90x + 400$$

$$R''(x) = -6x + 90 = 0$$

$$-6x = -90$$

$$|x = \frac{-90}{-6} = 15|$$

$$\begin{aligned} R(x = 15) &= -(15)^3 + 45(15)^2 + 400(15) + 8000 \\ &= 20,750 \end{aligned}$$

The point of diminishing returns is  $(15, 20,750)$

### Exercise

The population of a certain species of fish introduced into a lake is described by the logistic equation

$$G(t) = \frac{12,000}{1 + 19e^{-1.2t}}$$

where  $G(t)$  is the population after  $t$  years. Find the point at which the growth rate of this population begins to decline.

### Solution

$$G'(t) = -\frac{12,000((-1.2)19e^{-1.2t})}{(1 + 19e^{-1.2t})^2}$$

$$= 273,600 \frac{e^{-1.2t}}{(1 + 19e^{-1.2t})^2}$$

$$f = e^{-1.2t}$$

$$f' = -1.2e^{-1.2t}$$

$$\begin{aligned} g &= (1 + 19e^{-1.2t})^2 & g' &= 2(-1.2)(19e^{-1.2t})(1 + 19e^{-1.2t}) \\ & & &= -45.6e^{-1.2t}(1 + 19e^{-1.2t}) \end{aligned}$$

$$G = \frac{1}{U} \quad G' = -\frac{U'}{U^2}$$

$$\begin{aligned}
G''(t) &= 273,600 \frac{-1.2e^{-1.2t} \left(1 + 19e^{-1.2t}\right)^2 - e^{-1.2t} \left(-45.6e^{-1.2t} \left(1 + 19e^{-1.2t}\right)\right)}{\left(1 + 19e^{-1.2t}\right)^4} \\
&= 273,600 \frac{-1.2e^{-1.2t} \left(1 + 19e^{-1.2t}\right)^2 + 45.6e^{-2.4t} \left(1 + 19e^{-1.2t}\right)}{\left(1 + 19e^{-1.2t}\right)^4} \\
&= 273,600 \frac{\left(1 + 19e^{-1.2t}\right) \left[-1.2e^{-1.2t} \left(1 + 19e^{-1.2t}\right) + 45.6e^{-2.4t}\right]}{\left(1 + 19e^{-1.2t}\right)^4} \\
&= 273,600 \frac{\left[-1.2e^{-1.2t} - 22.8e^{-2.4t} + 45.6e^{-2.4t}\right]}{\left(1 + 19e^{-1.2t}\right)^3} \\
&= 273,600 \frac{\left(-1.2e^{-1.2t} + 22.8e^{-2.4t}\right)}{\left(1 + 19e^{-1.2t}\right)^3} \\
&= 273,600 \frac{1.2e^{-1.2t} \left(-1 + 19e^{-1.2t}\right)}{\left(1 + 19e^{-1.2t}\right)^3} = 0
\end{aligned}$$

$$\rightarrow \begin{cases} e^{-1.2t} = 0 & \text{never equals to zero} \\ -1 + 19e^{-1.2t} = 0 & e^{-1.2t} = \frac{1}{19} \end{cases}$$

$$\ln e^{-1.2t} = \ln \frac{1}{19}$$

$$-1.2t = \ln \frac{1}{19}$$

$$\lfloor t = \frac{\ln\left(\frac{1}{19}\right)}{-1.2} = 2.45 \rfloor$$

$$G(t = 2.45) = \frac{12,000}{1 + 19e^{-1.2(2.45)}} = 5,986.68$$

The point at which the growth rate of this population begins to decline: (2.45, 5,987)