Solution *Section* 4.1 – Inverse Functions

Exercise

Determine whether the function is one-to-one: $f(x) = 3x - 7$

Solution

$$f(a) = f(b)$$
$$3a - 7 = 3b - 7$$
$$3a = 3b - 7 + 7$$
$$3a = 3b$$

Divide both sides by 3

$$a = b$$

$\therefore$ The function is one-to-one

Exercise

Determine whether the function is one-to-one: $f(x) = x^2 - 9$

Solution

$1 \neq -1$ $1^2 - 9 \neq (-1)^2 - 9$ $-8 = -8 \rightarrow$ Contradict the definition	$f(a) = f(b)$ $a^2 - 9 = b^2 - 9$ $a^2 = b^2$ $a = \pm b$

The function is not one-to-one

Exercise

Determine whether the function is one-to-one: $f(x) = \sqrt{x}$

Solution

$$f(a) = f(b)$$
$$\sqrt{a} = \sqrt{b}$$
$$\left(\sqrt{a}\right)^2 = \left(\sqrt{b}\right)^2$$

Square both sides

$$a = b$$

$\therefore$ The function is one-to-one

Exercise

Determine whether the function is one-to-one: $f(x)=\sqrt[3]{x}$

Solution

$f(a)=f(b)$

$\sqrt[3]{a}=\sqrt[3]{b}$

$\left(\sqrt[3]{a}\right)^3=\left(\sqrt[3]{b}\right)^3$ *cube both sides*

$a=b$

$\therefore$ The function is one-to-one

Exercise

Determine whether the function is one-to-one: $f(x)=|x|$

Solution

$1\neq -1$

$|1|\neq |-1|$

$1\neq 1$ (false)

$\therefore$ The function is not one-to-one

Exercise

Given the function *f* described by $f(x)=\dfrac{2}{x+3}$, prove that f is one-to-one.

Solution

$f(a)=f(b)$

$\dfrac{2}{a+3}=\dfrac{2}{b+3}$

$(a+3)(b+3)\dfrac{2}{a+3}=\dfrac{2}{b+3}(a+3)(b+3)$

$2(b+3)=2(a+3)$

$b+3=a+3$

$a=b$ *f* is one-to-one

Exercise

Given the function *f* described by $f(x)=(x-2)^3$, prove that f is one-to-one.

Solution

$$f(a)=f(b)$$

$$(a-2)^3=(b-2)^3$$

$$\left[(a-2)^3\right]^{1/3}=\left[(b-2)^3\right]^{1/3}$$

$a-2=b-2$ *Add 2 on both sides*

$$a=b$$

Exercise

Given the function *f* described by $y=x^2+2$, prove that f is one-to-one.

Solution

$$f(a)=f(b)$$

$a^2+2=b^2+2$ *Subtract 2*

$$a^2=b^2$$

$a=\pm\sqrt{b^2}$ Function is not a one-to-one

The inverse function doesn't exist.

Exercise

Given the function *f* described by $f(x)=\frac{x+1}{x-3}$, prove that f is one-to-one.

Solution

$$f(a)=f(b)$$

$\frac{a+1}{a-3}=\frac{b+1}{b-3}$ ***Cross multiplication***

$$(a+1)(b-3)=(b+1)(a-3)$$

$$ab-3a+b-3=ab-3b+a-3$$

$$-3a-a=ab-3b-3-b+3-ab$$

$-4a=-4b$ ***Divide by -4***

$a=b$ Function is one-to-one

Exercise

Find the inverse of $f(x)=(x-2)^3$

Solution

$$y=(x-2)^3$$

$$x=(y-2)^3$$

$$x^{1/3}=\left[(y-2)^3\right]^{1/3}$$

$$x^{1/3}=y-2$$

$$\sqrt[3]{x}+2=y$$

$$\Rightarrow \boxed{f^{-1}(x)=\sqrt[3]{x}+2}$$

Exercise

Find the inverse of $f(x)=\dfrac{x+1}{x-3}$

Solution

$$y=\frac{x+1}{x-3}$$

$$x=\frac{y+1}{y-3}$$

$$x(y-3)=y+1$$

$$xy-3x=y+1$$

$$xy-y=3x+1$$

$$y(x-1)=3x+1$$

$$y=\boxed{\frac{3x+1}{x-1}=f^{-1}(x)}$$

Exercise

Let $f(x)=x^3-1$ and $g(x)=\sqrt[3]{x+1}$, is g the inverse function of f?

Solution

$$\begin{aligned}(f\circ g)(x)&=f(g(x))\\&=f\left(\sqrt[3]{x+1}\right)\\&=\left(\sqrt[3]{x+1}\right)^3-1\\&=x+1-1\\&=x\end{aligned}$$

$$\begin{aligned}(g\circ f)(x)&=g(f(x))\\&=g\left(x^3-1\right)\\&=\sqrt[3]{x^3-1+1}\\&=\sqrt[3]{x^3}\\&=x\end{aligned}$$

g is the inverse function of f

Exercise

Given that $f(x)=5x+8$, use composition of functions to show that $f^{-1}(x)=\frac{x-8}{5}$

Solution

$$\begin{aligned}\left(f^{-1}\circ f\right)(x)&=f^{-1}(f(x))\\&=f^{-1}(5x+8)\\&=\frac{(5x+8)-8}{5}\\&=\frac{5x+8-8}{5}\\&=\frac{5x}{5}=x\end{aligned}$$

$$\begin{aligned}\left(f\circ f^{-1}\right)(x)&=f(f^{-1}(x))=f^{-1}(\tfrac{x-8}{5})\\&=5(\tfrac{x-8}{5})+8=x-8+8=x\end{aligned}$$

Exercise

Given the function $f(x)=(x+8)^3$

a) Find $f^{-1}(x)$

b) Graph f and f^{-1} in the same rectangular coordinate system

c) Find the domain and the range of f and f^{-1}

Solution

a) $y=(x+8)^3$ **Replace f(*x*) with *y***

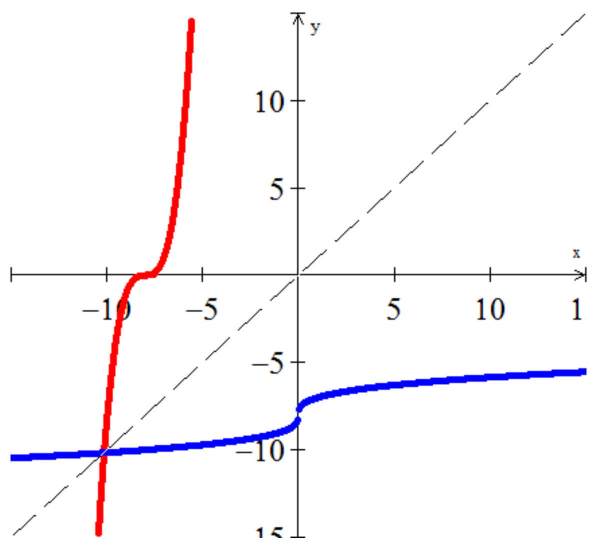
$x=(y+8)^3$ **Interchange *x* and *y***

$(x)^{1/3}=\left((y+8)^3\right)^{1/3}$

$x^{1/3}=y+8$ ***Subtract 8 from both sides***.

$\boxed{x^{1/3}-8}=y=f^{-1}(x)$

b)



c) Domain of f = Range of f^{-1}: $(-\infty,\infty)$

Range of f = Domain of f^{-1}: $(-\infty,\infty)$

Exercise

Find the inverse of $f(x)=\dfrac{2x+1}{x-3}$

Solution

$$y=\frac{2x+1}{x-3}$$

$$x=\frac{2y+1}{y-3}$$

$$x(y-3)=2y+1$$

$$xy-3x=2y+1$$

$$xy-2y=3x+1$$

$$y(x-2)=3x+1$$

$$y=\frac{3x+1}{x-2}=f^{-1}(x)$$

Exercise

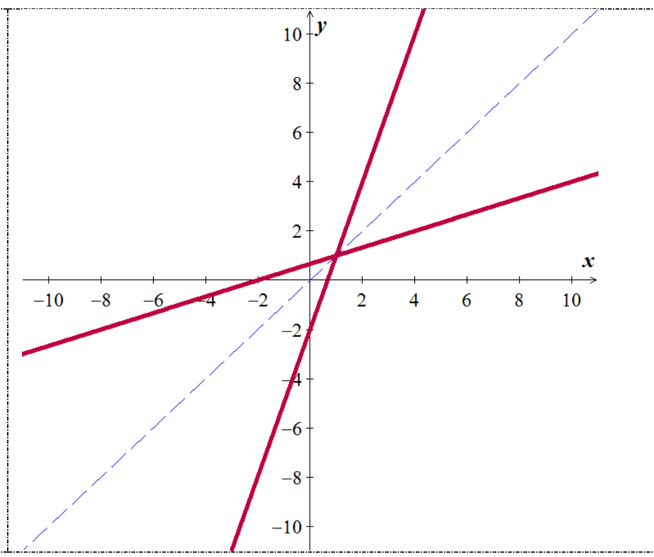
Prove the f *and* g are inverse functions of each other, and sketch the graphs of f *and* g:

$f(x)=3x-2 \quad g(x)=\dfrac{x+2}{3}$

Solution

$$\begin{aligned} f(g(x)) &= f\left(\frac{x+2}{3}\right) \\ &= 3\frac{x+2}{3}-2 \\ &= x+2-2 \\ &= x \end{aligned}$$

$$\begin{aligned} g(f(x)) &= g(3x-2) \\ &= \frac{3x-2+2}{3} \\ &= \frac{3x}{3} \\ &= x \end{aligned}$$



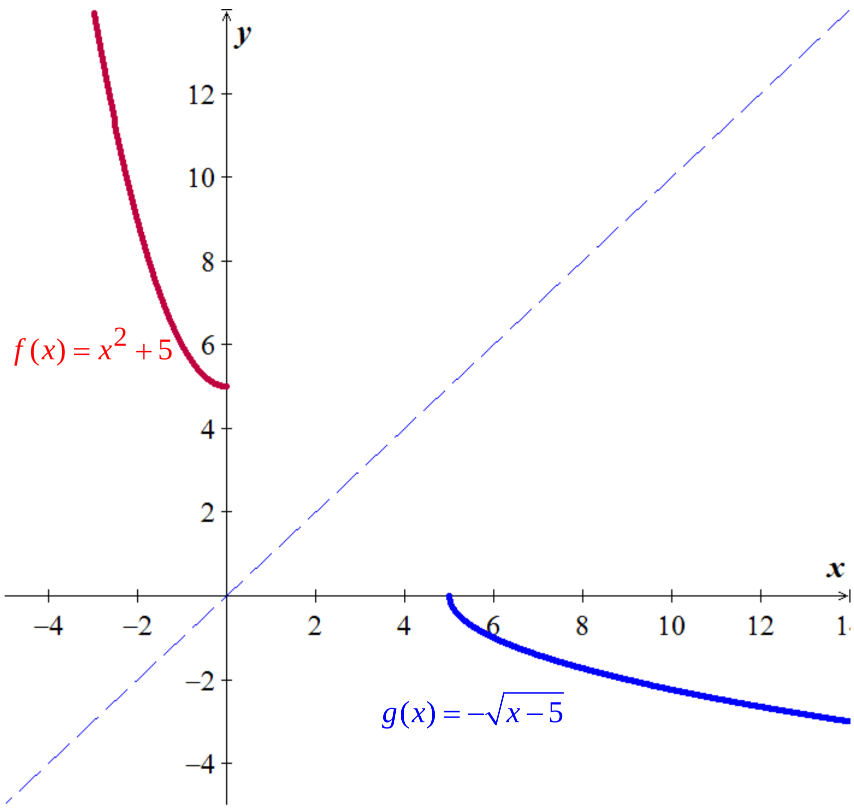
Exercise

Prove the f *and* g are inverse functions of each other, and sketch the graphs of f *and* g:

$f(x)=x^2+5,\ x\le 0 \quad g(x)=-\sqrt{x-5}, x\ge 5$

Solution

$f(g(x))=f\left(-\sqrt{x-5}\right)$ $=\left(-\sqrt{x-5}\right)^2+5$ $=x-5+5$ $\underline{=x}$	$g(f(x))=g\left(x^2+5\right)$ $=-\sqrt{x^2+5-5}$ $=-\sqrt{x^2}$ $=-\|x\|$ $=-(-x)$ since $x<0$ $\underline{=x}$



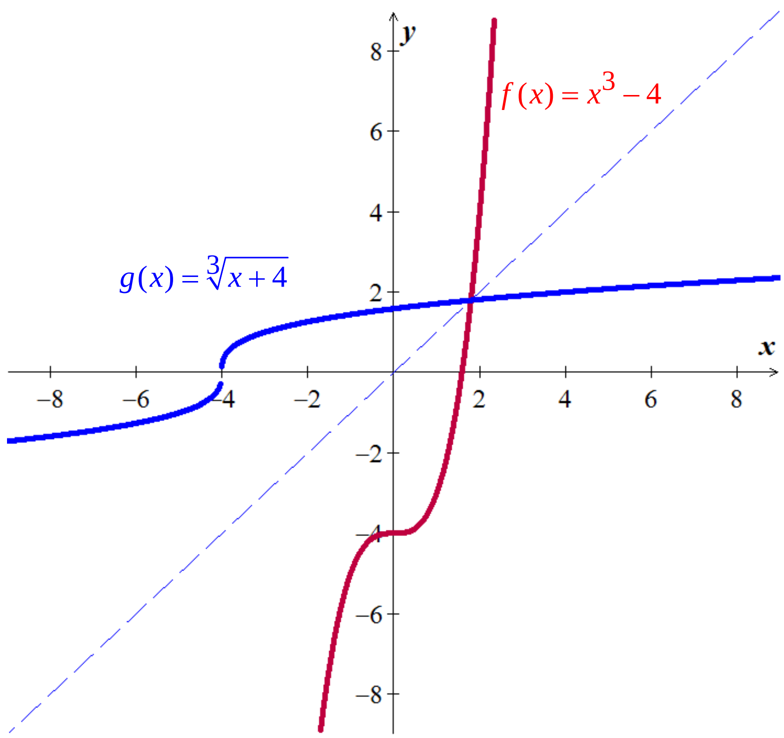
Exercise

Prove the f *and* g are inverse functions of each other, and sketch the graphs of f *and* g:

$f(x)=x^3-4; \quad g(x)=\sqrt[3]{x+4}$

Solution

$f\left(g(x)\right)=f\left(\sqrt[3]{x+4}\right)$ $=\left(\sqrt[3]{x+4}\right)^3-4$ $=x+4-4$ $\underline{=x}$	$g\left(f(x)\right)=g\left(x^3-4\right)$ $=\sqrt[3]{x^3-4+4}$ $=\sqrt[3]{x^3}$ $\underline{=x}$



Exercise

Determine the domain and range of f^{-1}: $f(x)=-\dfrac{2}{x-1}$ (Hint: first find the domain and range of f)

Solution

$x-1\neq 0 \Rightarrow x\neq 1$

Range of f^{-1} = Domain of f: $\mathbb{R}-\{1\}$ $\qquad (-\infty,\ 1)\cup(1,\ \infty)$

Domain of f^{-1} = Range of f: $\mathbb{R}-\{0\}$ $\qquad (-\infty,\ 0)\cup(0,\ \infty)$

Exercise

Determine the domain and range of f^{-1}: $f(x)=\dfrac{5}{x+3}$ (Hint: first find the domain and range of *f*)

Solution

Domain of f^{-1} = Range of f: $\mathbb{R}-\{0\}$ $\quad (-\infty,\ 0)\cup(0,\ \infty)$

Range of f^{-1} = Domain of f: $\mathbb{R}-\{-3\}$ $\quad (-\infty,\ -3)\cup(-3,\ \infty)$

Exercise

Determine the domain and range of f^{-1}: $f(x)=\dfrac{4x+5}{3x-8}$ (Hint: first find the domain and range of *f*)

Solution

Domain of f^{-1} = Range of f: $\mathbb{R}-\left\{\frac{8}{3}\right\}$ $\quad \left(-\infty,\ \frac{8}{3}\right)\cup\left(\frac{8}{3},\ \infty\right)$

Range of f^{-1} = Domain of f: $\mathbb{R}-\left\{\frac{4}{3}\right\}$ $\quad \left(-\infty,\ \frac{4}{3}\right)\cup\left(\frac{4}{3},\ \infty\right)$

Exercise

Find the inverse function of: $f(x)=3x+5$

Solution

$y=3x+5$

$x=3y+5$ ***Interchange x and y***

$x-5=3y$ ***Solve for y***

$\frac{x-5}{3}=y \quad \to f^{-1}(x)=\dfrac{x-5}{3}$

Exercise

Find the inverse function of: $f(x) = \dfrac{1}{3x-2}$

Solution

$$y = \frac{1}{3x-2}$$

$$x = \frac{1}{3y-2}$$ ***Interchange x and y***

$$x(3y-2) = 1$$ ***Solve for y***

$$3xy - 2x = 1$$

$$3xy = 1 + 2x$$

$$y = \frac{1+2x}{3x} = f^{-1}(x)$$

Exercise

Find the inverse function of: $f(x) = \dfrac{3x+2}{2x-5}$

Solution

$$y = \frac{3x+2}{2x-5}$$

$$x = \frac{3y+2}{2y-5}$$ ***Interchange x and y***

$$x(2y-5) = 3y+2$$ ***Solve for y***

$$2xy - 5x = 3y + 2$$

$$2xy - 3y = 5x + 2$$

$$(2x-3)y = 5x + 2$$

$$y = \frac{5x+2}{2x-3} = f^{-1}(x)$$

Exercise

Find the inverse function of: $f(x)=\dfrac{4x}{x-2}$

Solution

$$y=\frac{4x}{x-2}$$

$$x=\frac{4y}{y-2}$$

$$x(y-2)=4y$$

$$xy-2x=4y$$

$$xy-4y=2x$$

$$(x-4)y=2x$$

$$y=\boxed{\frac{2x}{x-4}=f^{-1}(x)}$$

Exercise

Find the inverse function of: $f(x)=2-3x^2; \quad x\le 0$

Solution

$$y=2-3x^2$$

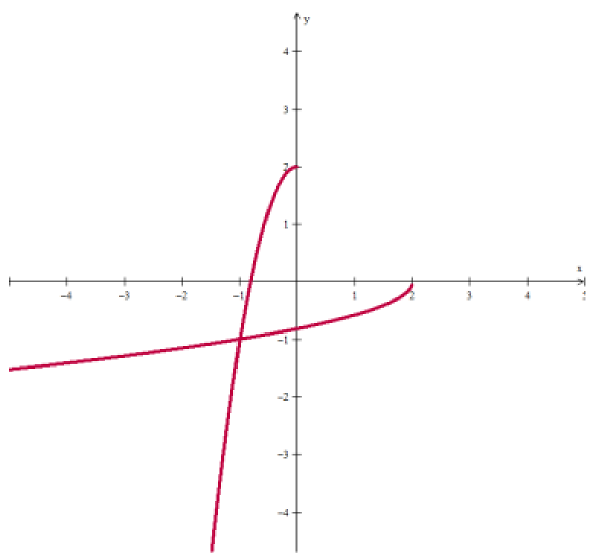
$$x=2-3y^2$$

$$3y^2=2-x$$

$$y^2=\frac{2-x}{3}$$

$$y=\boxed{-\sqrt{\frac{2-x}{3}}=f^{-1}(x)}$$

Since x < 0



Exercise

Find the inverse function of: $f(x) = 2x^3 - 5$

Solution

$$y = 2x^3 - 5$$

$$y + 5 = 2x^3$$

$$\frac{y+5}{2} = x^3$$

$$x = \sqrt[3]{\frac{y+5}{2}}$$

$$f^{-1}(x) = \sqrt[3]{\frac{x+5}{2}}$$

Exercise

Find the inverse function of: $f(x) = \sqrt{3-x}$

Solution

$$y = \sqrt{3-x} \qquad \boldsymbol{y \ge 0}$$

$$y^2 = 3 - x$$

$$x = 3 - y^2 \qquad \boldsymbol{x \ge 0}$$

$$\boxed{f^{-1}(x) = 3 - x^2}$$

Exercise

Find the inverse function of: $f(x) = \sqrt[3]{x} + 1$

Solution

$$y = \sqrt[3]{x} + 1$$

$$y - 1 = \sqrt[3]{x}$$

$$(y-1)^3 = x$$

$$\boxed{f^{-1}(x) = (x-1)^3}$$

Exercise

Find the inverse function of: $f(x) = \left(x^3 + 1\right)^5$

Solution

$$y = \left(x^3 + 1\right)^5$$

$$\sqrt[5]{y} = x^3 + 1$$

$$\sqrt[5]{y} - 1 = x^3$$

$$x = \sqrt[3]{\sqrt[5]{y} - 1}$$

$$\boxed{f^{-1}(x) = \sqrt[3]{\sqrt[5]{y} - 1}}$$

Exercise

Find the inverse function of: $f(x) = x^2 - 6x; \quad x \ge 3$

Solution

$$y = x^2 - 6x$$

$$x^2 - 6x - y = 0$$

$$\begin{aligned} x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-y)}}{2(1)} \\ &= \frac{6 \pm \sqrt{36 + 4y}}{2} \\ &= \frac{6 \pm 4\sqrt{9 + y}}{2} \\ &= 3 \pm \sqrt{9 + y} \end{aligned}$$

Since $x \ge 3 \Rightarrow$ we can select $x = 3 + \sqrt{y + 9}$

$$\therefore \boxed{f^{-1}(x) = 3 + \sqrt{x + 9}}$$