Lecture One – First Order Equations

Section 1.1 – Introduction to Differential Equations

Basic Terminology

A differential equation is an equation that contains an unknown function together with one or more of its derivatives.

Examples

1. $y' = 3x + \sin x$

2. $xy'' + 2y' + 3y = 5x^4$

3. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} = 0$

4. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \quad (Laplace's\ equation)$

Type

If the unknown function depends on a single independent variable, then the equation is an ***ordinary differential equation*** (ODE); if the unknown function depends on more than one independent variable, then the equation is a ***partial differential equation*** (PDE).

Order

The ***order*** of a differential equation is the order of the highest derivative of the unknown function appearing in the equation.

Examples

1. $y' = 3y + \sin x \qquad order\ 1$

2. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} = 0 \qquad order\ \mathbf{3}$

3. $xy'' + 2y' + 3y = 5x^4 \qquad order\ \mathbf{2}$

4. $\dfrac{d^2y}{dx^2} - 3x\cos\left(\dfrac{dy}{dx}\right) - 5x = \dfrac{d^3}{dx^3}\left(e^{4x}\right) \qquad order\ \mathbf{2}$

Since the unknown function is $y(x)$ not in *x*.

Ordinary Differential Equations

Involve an unknown function of a single variable with one or more of its derivatives.

$$\frac{dy}{dt} = y - t$$

y: $y(t)$ is unknown function

t: independent variable

Some other example:

$$y' = y^2 - t$$
$$ty' = y$$
$$y' + 4y = e^{-3t}$$
$$yy'' + t^2 y = \cos t$$
$$y' = \cos(ty)$$

$\therefore$ The order of a differential equation is the order of the highest derivative that occurs in the equation.

y'' : *second order*

$$\frac{\partial^2 \omega}{\partial t^2} = c^2 \frac{\partial^2 \omega}{\partial x^2}$$ is not an ODE (ω is dependent on x and t)

This equation is called a ***partial differential equation***.

Definition

A first-order differential equation of the form $\frac{dy}{dt} = y' = f(t,\ y)$ is said to be in normal form.

$y^{(n)} = f\left(t,\ y, y', \dots, y^{(n-1)}\right)$ is said to be in normal form.

f: is a given function of 2 variables t & y (***rate function***)

Solutions

A solution of a differential equation is a function defined on some domain D such that the equation reduces to an identity when the function is substituted into the equation.

A solution of the first-order, ordinary differential equation $f(t, y, y') = 0$ is a differentiable function $y(t)$ such that $f(t, y(t), y'(t)) = 0$ for all t in the interval where $y(t)$ is defined.

1. Can be found in explicit and implicit form by applying manipulation (integration)
2. No real solution.

Examples

1. $y' = 3x^2 + \sin x \quad \Rightarrow \quad y = x^3 - \cos x + C$

2. $y' = f(x) \quad \Rightarrow \quad y = \int f(x) + C = F(x) + C$

Example

Show that $y(t) = Ce^{-t^2}$ is a solution of the 1st order equation $y' = -2ty$

Solution

$$y(t) = Ce^{-t^2} \Rightarrow y' = -2tCe^{-t^2}$$

$$y' = -2tCe^{-t^2}$$

$$y' = -2t\, y(t) \qquad \text{True; it is a solution}$$

$y(t)$ is called the ***general solution***.

The solutions from the graph are called ***solution curves***.

Example

Is the function $y(t) = \cos t$ a solution to the differential equation $y' = 1 + y^2$

Solution

$$y' = -\sin t$$

$$y' = 1 + y^2 = -\sin t$$

$$1 + \cos^2 t \overset{?}{=} -\sin t \qquad \text{False; it is not a solution.}$$

Example

Find values of r such that $y(t) = e^{rt}$ is a solution of $y'' - 2y' - 15y = 0$

Solution

$$y'(t) = re^{rt}$$

$$y'' = r^2 e^{rt}$$

$$y'' - 2y' - 15y = r^2 e^{rt} - 2re^{rt} - 15e^{rt} = 0$$

$$r^2 - 2r - 15 = 0 \quad \Rightarrow \quad \underline{r = -3,\ 5}$$

n-Parameter Family of Solutions

To find a set of solutions of an n-th order differential equation we ***integrate*** n times, with each integration step producing an arbitrary constant of integration. Thus, "*in theory*", an n-th order differential equation has an ***n-parameter family of solutions***.

Example

Solve the differential equation: $y''' - 12x + 6e^{2x} = 0$

Solution

$$y''' = 12x - 6e^{2x}$$

$$\int y''' dx = \int \left(12x - 6e^{2x}\right) dx$$

$$y'' = 6x^2 - 3e^{2x} + C_1$$

$$\int y'' dx = \int \left(6x^2 - 3e^{2x} + C_1\right) dx$$

$$y' = 2x^3 - \frac{3}{2}e^{2x} + C_1 x + C_2$$

$$\int y' dx = \int \left(2x^3 - \frac{3}{2}e^{2x} + C_1 x + C_2\right) dx$$

$$\underline{y(x) = \frac{1}{2}x^4 - \frac{3}{4}e^{2x} + \frac{1}{2}C_1 x^2 + C_2 x + C_3}$$

General Solution/Singular Solutions

An "*n-parameter family of solutions*" is also called the ***general solution***.

Solutions of an *n*-th order differential equation which are not included in n-parameter family of solutions are called ***singular solutions***.

Example

Given the differential equation $y' = (4x+2)(y-2)^{1/3}$ has a general solution $(y-2)^{2/3} = \frac{4}{3}x^2 + \frac{4}{3}x + C$.

Is the singular solution $y \equiv 2$?

Solution

$$\frac{2}{3}(y-2)^{-1/3}\frac{dy}{dx} = \frac{8}{3}x + \frac{4}{3}$$

$$\frac{3}{2}(y-2)^{1/3} \quad \left[\frac{2}{3}\frac{1}{(y-2)^{1/3}}\frac{dy}{dx} = \frac{8}{3}x + \frac{4}{3}\right]$$

$$\frac{dy}{dx} = (4x+2)(y-2)^{1/3} \qquad y \neq 2$$

$y = 2$ is not a part of the general solution.

Particular Solutions

In many applications of integrations, we are given enough information to determine a particular solution. To do this, we need to know the value of *F(x)* for one value of *x*. This information is called an initial condition.

Example

Solve the differential equation: $y' = te^t$ that satisfies $y(0) = 2$

Solution

$$y = \int te^t dt$$

$$y = te^t - e^t + C$$

$$y(0) = (0)e^0 - e^0 + C = 2$$

$$-1 + C = 2$$

$$C = 3$$

$$\boxed{y(t) = e^t(t-1) + 3}$$

	$\int e^t dt$
t	e^t
1	e^t

Example

Solve the differential equation: $y' = \frac{1}{x}$ that satisfies $y(1) = 3$

Solution

$$y = \int \frac{1}{x} dx$$

$$= \ln|x| + C$$

$$y(1) = \ln|1| + C = 3$$

$$\boxed{C = 3}$$

$$y(x) = \ln x + 3 \qquad \textit{with } x > 0$$

n–th Order initial-Value Problems

Example

Find a solution of $y' = 3x^2 + 2x + 1$ which passes through the point $(-2,\ 4)$

Solution

$$y = \int \left(3x^2 + 2x + 1\right) dx$$

$$= x^3 + x^2 + x + C$$

$$4 = (-2)^3 + (-2)^2 - 2 + C$$

$$4 = -8 + 4 - 2 + C \quad \Rightarrow \quad C = 10$$

$$y = x^3 + x^2 + x + 10$$

Example

$y = C_1 \cos 3x + C_2 \sin 3x$ is the general solution of $y'' + 9y = 0$.

a) Find a solution which satisfies $y(0) = 3$

b) Find a solution which satisfies $y(0) = 4, \quad y(\pi) = 4$

c) Find a solution which satisfies $y\left(\frac{\pi}{4}\right) = 1, \quad y'\left(\frac{\pi}{4}\right) = 2$

Solution

a) $3 = C_1 \cos(0) + C_2 \sin(0) \quad \Rightarrow \quad \underline{C_1 = 3|}$

$\underline{y = 3\cos 3x + C_2 \sin 3x|}$ *for any* C_2

b) $4 = C_1 \cos(0) + C_2 \sin(0) \quad \Rightarrow \quad \underline{C_1 = 4|}$

$4 = C_1 \cos(3\pi) + C_2 \sin(3\pi) \quad \Rightarrow \quad \underline{C_1 = -4|}$

$\therefore$ No Solution

c) $1 = C_1 \cos\left(\frac{3\pi}{4}\right) + C_2 \sin\left(\frac{3\pi}{4}\right) \quad \Rightarrow \quad -\frac{1}{\sqrt{2}} C_1 + \frac{1}{\sqrt{2}} C_2 = 1 \rightarrow \boxed{-C_1 + C_2 = \sqrt{2}} \ (1)$

$y' = -3C_1 \sin 3x + 3C_2 \cos 3x$

$2 = -3C_1 \sin\left(\frac{3\pi}{4}\right) + 3C_2 \cos\left(\frac{3\pi}{4}\right) \quad \Rightarrow \quad -\frac{3}{\sqrt{2}} C_1 - \frac{3}{\sqrt{2}} C_2 = 2 \rightarrow \boxed{-3C_1 - 3C_2 = 2\sqrt{2}} \ (2)$

$$\begin{cases} -3C_1 + 3C_2 = 3\sqrt{2} \\ -3C_1 - 3C_2 = 2\sqrt{2} \end{cases} \rightarrow \quad C_1 = -\frac{5\sqrt{2}}{6} \quad C_2 = \sqrt{2} - \frac{5\sqrt{2}}{6} = \frac{\sqrt{2}}{6}$$

$\underline{y = -\frac{5\sqrt{2}}{6} \cos 3x + \frac{\sqrt{2}}{6} \sin 3x|}$

Example

Suppose a ball thrown into the air with initial velocity $v_0 = 20\ ft/\sec$. Assuming the ball thrown from a height of $x_0 = 6\ ft$, how long does it take for the ball to hit the ground?

Solution

$$\frac{dv}{dt} = -g \Rightarrow dv = -gdt$$

$$v(t) = -gt + C_1$$

$$v(t=0) = -g(0) + C_1 = 20$$

$$C_1 = 20$$

$$v(t) = -32t + 20$$

$$\frac{dx}{dt} = v \Rightarrow dx = vdt$$

$$\int dx = \int vdt$$

$$x(t) = \int (-32t + 20)dt$$

$$= -16t^2 + 20t + C_2$$

$$x(t=0) = -16(0)^2 + 20(0) + C_2 = 6$$

$$C_2 = 6$$

$$\underline{x(t) = -16t^2 + 20t + 6}$$

Exercises *Section* 1.1 – Introduction to Differential Equations

1. Show that $y(t) = Ce^{-(1/2)t^2}$ is a solution of the 1st order equation $y' = -ty$ *for* $-3 \le C \le 3$

2. Show that $y(t) = \dfrac{4}{1+Ce^{-4t}}$ is a solution of the 1st order equation $y' = y(4-y)$

3. Show that $y(x) = x^{-3/2}$ is a solution of $4x^2y'' + 12xy' + 3y = 0$ for $x > 0$

4. A general solution may fail to produce all solutions of a differential equation $y(t) = \dfrac{4}{1+Ce^{-4t}}$. Show that $y = 0$ is a solution of the differential equation, but no value of C in the given general solution will produce this solution.

5. Use the given general solution to find a solution of the differential equation having the given initial condition. $ty' + y = t^2, \quad y(t) = \frac{1}{3}t^2 + \frac{C}{t}, \quad y(1) = 2$

6. Show that $y(t) = 2t - 2 + Ce^{-t}$ is a solution of the 1st order equation $y' + y = 2t$ *for* $-3 \le C \le 3$

7. Use the given general solution to find a solution of the differential equation having the given initial condition. $y' + 4y = \cos t, \quad y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t + Ce^{-4t}, \quad y(0) = -1$

8. Use the given general solution to find a solution of the differential equation having the given initial condition. $ty' + (t+1)y = 2te^{-t}, \quad y(t) = e^{-t}\left(t + \frac{C}{t}\right), \quad y(1) = \frac{1}{e}$

9. Use the given general solution to find a solution of the differential equation having the given initial condition. $y' = y(2+y), \quad y(t) = \dfrac{2}{-1+Ce^{-2t}}, \quad y(0) = -3$

10. Find the values of $\boldsymbol{m}$ so that the function $y = e^{mx}$ is a solution of the given differential equation

a) $y' + 2y = 0$

b) $5y' - 2y = 0$

c) $y'' - 5y' + 6y = 0$

d) $2y'' + 7y' - 4y = 0$

11. Let $x = c_1 \cos t + c_2 \sin t$ is 2-parameter family solutions of the second order differential equation of $x'' + x = 0$. Find a solution of the second-order consisting of this differential equation and the given initial conditions.

a) $x(0) = -1, \quad x'(0) = 8$

b) $x\left(\frac{\pi}{2}\right) = 0, \quad x'\left(\frac{\pi}{2}\right) = 1$

c) $x\left(\frac{\pi}{6}\right) = \frac{1}{2}, \quad x'\left(\frac{\pi}{6}\right) = 0$

d) $x\left(\frac{\pi}{4}\right) = \sqrt{2}, \quad x'\left(\frac{\pi}{4}\right) = 2\sqrt{2}$

12. Find values of r such that $y(x) = x^r$ is a solution of $x^2y'' - 4xy' + 6y = 0$

Solve the differential equation:

13. $y' = 3x^2 - 2x + 4$

14. $y'' = 2x + \sin 2x$

15. Given the differential equation $x^2 y'' - 2xy' + 2y = 4x^3$, is the given equation a solution?

a) $y = 2x^3 + x^2$

b) $y = 2x + x^2$

Section 1.2 – Separable Equations

Separable Equation

Separable equation is an equation that can be written with its variables separated and then easily solved.

If f is independent of $y \Rightarrow y' = \frac{dy}{dx} = f(x, y)$ is separable equation if f has the form

$$f(x, y) = g(x)h(y)$$

Definition

A 1st order differential equation of the form $\frac{dy}{dx} = g(x)h(y)$ is said to be separable or to have separable variables.

$$\frac{dy}{h(y)} = g(x)dx$$

$$\frac{dy}{dx} = y^2 xe^{3x+4y} = \left(xe^{3x}\right)\left(y^2 e^{4y}\right)$$

$$\frac{dy}{dx} = y + \sin x \qquad \textit{not separable}$$

Example

At time t the sample contains $N(t)$ radioactive nuclei and is given by the differential equation:

$$N' = -\lambda N$$

This is called the ***exponential equation***.

$$N' = -\lambda N$$

$$\frac{dN}{dt} = -\lambda N \qquad \textit{Separable equation}$$

$$\frac{dN}{N} = -\lambda dt$$

$$\int \frac{dN}{N} = -\int \lambda dt$$

$$\ln|N| = -\lambda t + C$$

$$|N(t)| = e^{-\lambda t + C}$$

$$= e^C e^{-\lambda t}$$

$$N(t)=\begin{cases} e^{C}e^{-\lambda t} & \text{if } N>0 \\ -e^{C}e^{-\lambda t} & \text{if } N<0 \end{cases}$$

$$N(t)=Ae^{-\lambda t} \qquad A=\begin{cases} e^{C} & \text{if } N>0 \\ -e^{C} & \text{if } N<0 \end{cases}$$

Example

Solve the differential equation $y'=ty^2$

Solution

$$\frac{dy}{dt}=ty^2$$

$$\frac{dy}{y^2}=tdt$$

$$\int y^{-2}dy=\int tdt$$

$$-y^{-1}=\frac{1}{2}t^2+C$$

$$-\frac{1}{y}=\frac{t^2+2C}{2} \qquad \textit{Cross multiplication}$$

$$-\frac{2}{t^2+2C}=y$$

$$y(t)=-\frac{2}{t^2+2C}$$

General Method

Step 1: Establish that the equation is separate $\frac{dy}{dx}=g(x)h(y)$

Step 2: Divide both sides by $h(y)$ to separate the variables $\frac{dy}{h(y)}=g(x)dx$

Step 3: Integrate both sides $\int\frac{dy}{h(y)}=\int g(x)dx$

Step 4: Solve for the solution $y(t)$, if possible

Losing a solution

When we use separate variables, the variable divisors could be zero at a point.

Example

Find a general solution to $\frac{dy}{dx}=y^2-4$

Solution

$$\frac{dy}{y^2-4}=dx$$

$$\left(\frac{1/4}{y-2}-\frac{1/4}{y+2}\right)dy=dx \qquad y=\pm 2 \text{ Critical points}$$

$$\frac{1}{4}\left(\int\frac{dy}{y-2}-\int\frac{dy}{y+2}\right)=\int dx$$

$$\frac{1}{4}\left[\ln|y-2|-\ln|y+2|\right]=x+c_1$$

$$\ln\left|\frac{y-2}{y+2}\right|=4x+c_2$$

$$\left|\frac{y-2}{y+2}\right|=e^{4x+c_2}$$

$$\frac{y-2}{y+2}=\pm e^{c_2}e^{4x}$$

$$y-2=Ce^{4x}(y+2)$$

$$y-Ce^{4x}y=2Ce^{4x}+2$$

$$\left(1-Ce^{4x}\right)y=2\left(Ce^{4x}+1\right)$$

$$\boxed{y=2\frac{1+Ce^{4x}}{1-Ce^{4x}}}$$

If $y=-2\Rightarrow -2=2\frac{1+Ce^{4x}}{1-Ce^{4x}}$

$$-1=\frac{1+Ce^{4x}}{1-Ce^{4x}}$$

$$-1+Ce^{4x}=1+Ce^{4x} \quad\Rightarrow\quad -1=1 \quad \textit{impossible}$$

If $y=2 \quad \Rightarrow 2=2\frac{1+Ce^{4x}}{1-Ce^{4x}}$

$$1-Ce^{4x}=1+Ce^{4x}$$

$$-Ce^{4x}=Ce^{4x} \quad\Rightarrow\quad -C=C$$

$$\boxed{y=2\Rightarrow C=0}$$

Implicitly Defined Solutions

Example

Find the solutions of the equation $y' = \dfrac{e^x}{1+y}$, having initial conditions $y(0)=1$ and $y(0)=-4$

Solution

$$\frac{dy}{dx} = \frac{e^x}{1+y}$$

$$(1+y)\,dy = e^x dx$$

$$\int (1+y)\,dy = \int e^x dx$$

$$y + \tfrac{1}{2}y^2 = e^x + c$$

$$y^2 + 2y - 2\left(e^x + c\right) = 0$$

$$y(x) = \tfrac{1}{2}\left(-2 \pm \sqrt{4 + 8\left(e^x + c\right)}\right) \qquad \textit{Quadratic Formula}$$

$$= -1 \pm \sqrt{1 + 2\left(e^x + c\right)} \qquad \textit{Implicit}$$

$$y(0) = -1 + \sqrt{1 + 2\left(e^0 + c\right)} = 1$$

$$\sqrt{1 + 2(1+c)} = 2$$

$$1 + 2 + 2c = 4$$

$$2c = 1$$

$$\boxed{c = \frac{1}{2}}$$

$$y(0) = -1 - \sqrt{1 + 2\left(e^0 + c\right)} = -4$$

$$-\sqrt{1 + 2 + 2c} = -3$$

$$1 + 2 + 2c = 9$$

$$2c = 6$$

$$\boxed{c = 3}$$

$$\begin{cases} y(t) = -1 + \sqrt{2 + 2e^x} \\ y(t) = -1 - \sqrt{7 + 2e^x} \end{cases}$$

$\therefore y \ne -1$ *from y'*, but it never it will be.

Explicit Solutions: $y = -1 + \sqrt{}$

Notes

1. $Q(y) = P(x) + C$ is the general solution. Typically, this is an ***implicit*** relation; we may or may not be able to solve it for y.

2. $h(y) = 0$ is a source of singular solutions:

If k is a number such that $h(k) = 0$, *then* $y = k$

Might be a ***singular*** ***solution***.

Example

Find the general solution and any singular solutions: $y' - xy^2 = x$

Solution

$$y' = x + xy^2$$

$$\frac{dy}{dx} = x\left(1 + y^2\right)$$

$$\frac{dy}{1 + y^2} = xdx$$

$$\int \frac{dy}{1 + y^2} = \int xdx \qquad 1 + y^2 \neq 0$$

$$\tan^{-1} y = \frac{1}{2}x^2 + C$$

$$\boxed{y = \tan\left(\frac{1}{2}x^2 + C\right)} \qquad \textit{No singular solutions}$$

Example

Find the general solution and any singular solutions: $\frac{1}{x}y' = e^x\sqrt{y + 1}$

Solution

	$\int e^x$
x	e^x
1	e^x

$$\int \frac{dy}{\sqrt{y + 1}} = \int xe^x dx$$

$$\int \frac{d(y + 1)}{\sqrt{y + 1}} = xe^x + e^x + C$$

$$\boxed{2\sqrt{y + 1} = xe^x + e^x + C}$$

$$h(y) = y + 1 = 0 \quad \Rightarrow \quad y = -1 \textit{ is a singular solution}$$

Example

Find the general solution and any singular solutions: $y' = \dfrac{xy^2 - x}{y}$

Solution

$$\frac{dy}{dx} = \frac{x\left(y^2 - 1\right)}{y}$$

$$\frac{y}{y^2 - 1}dy = xdx$$

$$\frac{1}{2}\int \frac{1}{y^2 - 1}d\left(y^2 - 1\right) = \int xdx$$

$$\frac{1}{2}\ln\left|y^2 - 1\right| = \frac{1}{2}x^2 + \frac{1}{2}\ln C$$

$$\ln\left|y^2 - 1\right| - \ln C = x^2$$

$$\ln\frac{\left|y^2 - 1\right|}{C} = x^2$$

$$\frac{\left|y^2 - 1\right|}{C} = e^{x^2}$$

$$y^2 - 1 = Ce^{x^2}$$

$$\boxed{y^2 = Ce^{x^2} + 1}$$

Singular Solutions:

$$\frac{y^2 - 1}{y} = 0 \quad \Rightarrow \quad y = \pm 1$$

For $y = 1$: $\quad if\ \ C = 0 \quad y = 1$

For $y = -1$: $\quad No\ \ C$

No singular solution.

Example

Find the solutions to the differential equation $x' = \dfrac{2tx}{1+x}$, having $x(0) = 1,\ -2,\ 0$

Solution

$$\frac{dx}{dt} = \frac{2tx}{1+x}$$

$$\frac{1+x}{x}dx = 2tdt$$

$$\left(\frac{1}{x}+1\right)dx = 2tdt$$

$$\int\left(\frac{1}{x}+1\right)dx = \int 2tdt$$

$$\ln|x| + x = t^2 + c$$

For $x(0) = 1$

$$1 = 0^2 + c$$

$$c = 1$$

$$\ln|x| + x = t^2 + c \qquad x > 0$$

We can't solve for $x(t)$

$\Rightarrow$ This solution is defined as implicit.

For $x(0) = -2$

$$\ln|-2| + (-2) = 0^2 + c$$

$$c = -2 + \ln 2$$

$$\ln|x| + x = t^2 - 2 + \ln 2$$

Since the initial condition < 0, then:

$$x + \ln(-x) = t^2 - 2 + \ln 2$$

For $x(0) = 0$

$$0 = 0^2 + c \qquad \textit{True statement}$$

$y' = 0 \ \Rightarrow\ x(t) = 0$ is a solution

Exercises *Section* 1.2 – Separable Equations

Find the general solution of the differential equation.

1. $y' = xy$

2. $xy' = 2y$

3. $y' = e^{x-y}$

4. $y' = \left(1+y^2\right)e^x$

5. $y' = xy + y$

6. $y' = ye^x - 2e^x + y - 2$

7. $y' = \dfrac{x}{y+2}$

8. $y' = \dfrac{xy}{x-1}$

9. $y' = \dfrac{y^2 + ty + t^2}{t^2}$

10. $\dfrac{dy}{dx} = \dfrac{4x - x^3}{4 + y^3}$

11. $y' = \dfrac{2xy + 2x}{x^2 - 1}$

12. $\dfrac{dy}{dx} = \sin 5x$

13. $\dfrac{dy}{dx} = (x+1)^2$

14. $dx + e^{3x}dy = 0$

15. $dy - (y-1)^2\, dx = 0$

16. $x\dfrac{dy}{dx} = 4y$

17. $\dfrac{dx}{dy} = y^2 - 1$

18. $\dfrac{dy}{dx} = e^{2y}$

19. $\dfrac{dy}{dx} + 2xy^2 = 0$

20. $\dfrac{dy}{dx} = e^{3x+2y}$

21. $e^x y\dfrac{dy}{dx} = e^{-y} + e^{-2x-y}$

22. $y\ln x\dfrac{dx}{dy} = \left(\dfrac{y+1}{x}\right)^2$

23. $\dfrac{dy}{dx} = \left(\dfrac{2y+3}{4x+5}\right)^2$

24. $\csc y dx + \sec^2 x dy = 0$

25. $\sin 3x dx + 2y\cos^3 3x dy = 0$

26. $\left(e^y + 1\right)^2 e^{-y}dx + \left(e^x + 1\right)^3 e^{-x}dy = 0$

27. $x\left(1+y^2\right)^{1/2} dx = y\left(1+x^2\right)^{1/2} dy$

28. $\dfrac{dy}{dx} = y\sin x$

29. $(1+x)\dfrac{dy}{dx} = 4y$

30. $2\sqrt{x}\dfrac{dy}{dx} = \sqrt{1-y^2}$

31. $\dfrac{dy}{dx} = 3\sqrt{xy}$

32. $\dfrac{dy}{dx} = (64xy)^{1/3}$

33. $\dfrac{dy}{dx} = 2x\sec y$

34. $\left(1-x^2\right)\dfrac{dy}{dx} = 2y$

35. $(1+x)^2\dfrac{dy}{dx} = (1+y)^2$

36. $\dfrac{dy}{dx} = xy^3$

37. $y\dfrac{dy}{dx} = x\left(y^2 + 1\right)$

38. $y^3\dfrac{dy}{dx} = \left(y^4 + 1\right)\cos x$

39. $\dfrac{dy}{dx} = \dfrac{1+\sqrt{x}}{1+\sqrt{y}}$

40. $\dfrac{dy}{dx} = \dfrac{(x-1)y^5}{x^2\left(2y^3 - y\right)}$

41. $\left(x^2 + 1\right)(\tan y)y' = x$

42. $x^2y' = 1 - x^2 + y^2 - x^2y^2$

43. $xy' + 4y = 0$

44. $\left(x^2+1\right)y' + 2xy = 0$

45. $\dfrac{y'}{\left(x^2+1\right)y} = 3$

46. $y + e^x y' = 0$

47. $\dfrac{dx}{dt} = 3xt^2$

48. $x\dfrac{dy}{dx} = \dfrac{1}{y^3}$

49. $\dfrac{dy}{dx} = \dfrac{x}{y^2\sqrt{x+1}}$

50. $\dfrac{dx}{dt} - x^3 = x$

51. $\dfrac{dy}{dx} = \dfrac{x}{ye^{x+2y}}$

52. $\dfrac{dy}{dx} = \dfrac{\sec^2 y}{1+x^2}$

53. $x\dfrac{dv}{dx} = \dfrac{1-4v^2}{3v}$

54. $\dfrac{dy}{dx} = 3x^2\left(1+y^2\right)^{3/2}$

55. $\dfrac{1}{y}dy + ye^{\cos x}\sin x dx = 0$

56. $\left(x+xy^2\right)dx + e^{x^2}ydy = 0$

Find the exact solution of the initial value problem. Indicate the interval of existence.

57. $y' = \dfrac{y}{x}, \quad y(1) = -2$

58. $y' = -\dfrac{2t\left(1+y^2\right)}{y}, \quad y(0) = 1$

59. $y' = \dfrac{\sin x}{y}, \quad y\left(\frac{\pi}{2}\right) = 1$

60. $4tdy = \left(y^2 + ty^2\right)dt, \quad y(1) = 1$

61. $y' = \dfrac{1-2t}{y}, \quad y(1) = -2$

62. $y' = y^2 - 4, \quad y(0) = 0$

63. $\dfrac{dy}{dx} = \dfrac{3x^2+4x+2}{2(y-1)}, \quad y(0) = -1$

64. $y' = \dfrac{x}{1+2y}, \quad y(-1) = 0$

65. $\left(e^{2y} - y\right)\cos x\dfrac{dy}{dx} = e^y \sin 2x, \quad y(0) = 0$

66. $\dfrac{dy}{dx} = e^{-x^2}, \quad y(3) = 5$

67. $\dfrac{dy}{dx} + 2y = 1, \quad y(0) = \dfrac{5}{2}$

68. $\sqrt{1-y^2}dx - \sqrt{1-x^2}dy = 0, \quad y(0) = \dfrac{\sqrt{3}}{2}$

69. $\left(1+x^4\right)dy + x\left(1+4y^2\right)dx = 0, \quad y(1) = 0$

70. $\dfrac{1}{t^2}\dfrac{dy}{dt} = y, \quad y(0) = 1$

71. $\dfrac{dy}{dt} = -y^2e^{2t}; \quad y(0) = 1$

72. $\dfrac{dy}{dt} - (2t+1)y = 0; \quad y(0) = 2$

73. $\dfrac{dy}{dt} + 4ty^2 = 0; \quad y(0) = 1$

74. $\dfrac{dy}{dx} = ye^x; \quad y(0) = 2e$

75. $\dfrac{dy}{dx} = 3x^2\left(y^2+1\right); \quad y(0) = 1$

76. $2y\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2-16}}; \quad y(5) = 2$

77. $\dfrac{dy}{dx} = 4x^3y - y; \quad y(1) = -3$

78. $\dfrac{dy}{dx} + 1 = 2y; \quad y(1) = 1$

79. $(\tan x)\dfrac{dy}{dx} = y; \quad y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

80. $e^{-2t}\dfrac{dy}{dt} = \dfrac{1+e^{-2t}}{y}, \quad y(0) = 0$

81. $\dfrac{dy}{dt} = y\cos t + y, \quad y(0) = 2$

82. $\dfrac{dy}{dt} = \dfrac{t+2}{y}, \quad y(0) = 2$

83. $x\dfrac{dy}{dx} - y = 2x^2 y;\quad y(1) = 1$

84. $\dfrac{dy}{dx} = 2xy^2 + 3x^2 y^2;\quad y(1) = -1$

85. $\dfrac{dy}{dx} = 6e^{2x-y};\quad y(0) = 0$

86. $2\sqrt{x}\dfrac{dy}{dx} = \cos^2 y;\quad y(4) = \frac{\pi}{4}$

87. $y' + 3y = 0;\quad y(0) = -3$

88. $2y' - y = 0;\quad y(-1) = 2$

89. $2xy - y' = 0;\quad y(1) = 3$

90. $y\dfrac{dy}{dx} - \sin x = 0;\quad y\left(\frac{\pi}{2}\right) = -2$

91. $\dfrac{dy}{dt} = \dfrac{1}{y^2};\quad y(1) = 2$

92. $y' + \dfrac{1}{y+1} = 0;\quad y(1) = 0$

93. $y' + e^y t = e^y \sin t;\quad y(0) = 0$

94. $y' - 2ty^2 = 0;\quad y(0) = -1$

95. $\dfrac{dy}{dx} = 1 + y^2;\quad y\left(\frac{\pi}{4}\right) = -1$

96. $\dfrac{dy}{dt} = t - ty^2;\quad y(0) = \frac{1}{2}$

97. $3y^2\dfrac{dy}{dt} + 2t = 1;\quad y(-1) = -1$

98. $e^x y' + (\cos y)^2 = 0;\quad y(0) = \frac{\pi}{4}$

99. $(2y - \sin y)y' + x = \sin x;\quad y(0) = 0$

100. $e^y y' + \dfrac{x}{y+1} = \dfrac{2}{y+1};\quad y(1) = 2$

101. $(\ln y)y' + x = 1;\quad y(3) = e$

102. $y' = x^3(1-y);\quad y(0) = 3$

103. $y' = \left(1 + y^2\right)\tan x;\quad y(0) = \sqrt{3}$

104. $\dfrac{1}{2}\dfrac{dy}{dx} = \sqrt{1+y}\,\cos x;\quad y(\pi) = 0$

105. $x^2\dfrac{dy}{dx} = \dfrac{4x^2 - x - 2}{(x+1)(y+1)};\quad y(1) = 1$

106. $\dfrac{1}{\theta}\dfrac{dy}{d\theta} = \dfrac{y\sin\theta}{y^2+1}\quad y(\pi) = 1$

107. $x^2 dx + 2y dy = 0;\quad y(0) = 2$

108. $\dfrac{1}{t}\dfrac{dy}{dt} = 2\cos^2 y;\quad y(0) = \frac{\pi}{4}$

109. $\dfrac{dy}{dx} = 8x^3 e^{-2y};\quad y(1) = 0$

110. $\dfrac{dy}{dx} = x^2(1+y);\quad y(0) = 3$

111. $\sqrt{y}\,dx + (1+x)\,dy = 0;\quad y(0) = 1$

112. $\dfrac{dy}{dx} = 6y^2 x,\quad y(1) = \frac{1}{25}$

113. $\dfrac{dy}{dx} = \dfrac{3x^2 + 4x - 4}{2y - 4},\quad y(1) = 3$

114. $y' = e^{-y}(2x - 4)\quad y(5) = 0$

115. $\dfrac{dr}{d\theta} = \dfrac{r^2}{\theta},\quad r(1) = 2$

116. $\dfrac{dy}{dt} = e^{y-t}\left(1 + t^2\right)\sec y,\quad y(0) = 0$

Section 1.3 – Linear Differential Equations

Basic Assumption

The equation can be solved for y'; that is, the equation can be written in the form $y' = f(x, y)$

A linear differential equation of order ***n*** has the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x)$$

A ***first order*** linear equation is given by the form:

$$y' + p(x)y = f(x)$$

If $f(x) = 0 \rightarrow y' = p(x)y$. This linear equation is said to be ***homogeneous***. (Otherwise it is ***nonhomogeneous or inhomogeneous***).

$p(x)$ & $f(x)$ are called the coefficients and continuous function on some interval *I*.

Linear	***Non-linear***
$x' = \sin(t)x$	$x' = t\sin x$
$y' = e^{2t}y + \cos t$	$y' = 1 - y^2$
$x' = (3t+2)x + t^2 - 1$	

Solution of the homogenous equation

$$\frac{dx}{dt} = a(t)x \quad \Rightarrow \quad \frac{dx}{x} = a(t)dt$$

$$\int \frac{dx}{x} = \int a(t)dt$$

$$\ln|x| = \int a(t)dt + C$$ *Convert to exponential form*

$$|x| = e^{\int a(t)dt + C} = e^C e^{\int a(t)dt}$$ *Let* $A = e^C$

$$\boxed{x(t) = Ae^{\int a(t)dt}}$$

Example

Solve: $x' = \sin(t)\ x$

Solution

$$\frac{dx}{dt} = \sin(t)\ x$$

$$x(t) = A.e^{\int \sin(t)dt}$$

$$\boxed{= A.e^{-\cos t}}$$

$$\frac{dx}{x} = \sin(t)\ dt$$

$$\int \frac{dx}{x} = \int \sin(t)\ dt$$

$$\ln|x| = \int \sin(t)dt + C$$

$$\ln|x| = -\cos(t) + C$$

$$\boxed{x = e^{-\cos(t)+C}}$$

Solving a linear first-order Equation (*Properties*)

1. Put a linear equation into a standard form $y' + p(x)y = f(x)$
2. Identify $p(x)$ then find $y_h = e^{-\int p dx}$
3. Multiply the standard form by y_h
4. Integrate both sides

Solution of the Inhomogeneous Equation

$$x' = p(t)x + f(t)$$

$$x' - px = f$$

$$u(t) = e^{-\int p(t)dt}$$

$$(ux)' = u\left(x' - px\right) = uf$$

$$u(t)x(t) = \int u(t) f(t)dt + C$$

1st Method

Example

Find the general solution to: $x' = x + e^{-t}$

Solution

$$x' - x = e^{-t} \qquad\qquad x' - p(t)x = f(t)$$

$$e^{-\int 1dt} = e^{-t} \qquad\qquad e^{\int p(t)dt}$$

$$e^{-t}(x' - x) = e^{-t}e^{-t} \qquad\qquad e^{\int p(t)dt}x' - e^{\int p(t)dt}p(t)x = e^{\int p(t)dt}f(t)$$

$$\left(e^{-t}x\right)' = e^{-2t} \qquad\qquad \left(e^{\int p(t)dt}x\right)' = f(t)\, e^{\int p(t)dt}$$

$$e^{-t}x(t) = \int e^{-2t}dt \qquad\qquad e^{\int p(t)dt}x = \int f(t)\, e^{\int p(t)dt}$$

$$e^{-t}x(t) = -\tfrac{1}{2}e^{-2t} + C$$

$$\boxed{x(t) = -\tfrac{1}{2}e^{-t} + Ce^{t}}$$

Solution of the Nonhomogeneous Equation $y' + p(x)y = f(x)$

Let assume: $y = y_h + y_p$ *where* $\begin{cases} y_h & \text{Homogeneous Solution} \\ y_p & \text{Paticular Solution} \end{cases}$

The homogeneous equation is given by $y'_h + p(x)y_h = 0$

$$y'_h = -p(x)y_h$$

$$\boxed{y_h = e^{-\int pdx}}$$

$$y_p = u(x)y_h = u.e^{-\int pdx}$$

$$y'_p + p(x)y_p = f(x)$$

$$(uy_h)' + puy_h = f$$

$$u'y_h + uy'_h + puy_h = f$$

$$u'y_h + u(y'_h + py_h) = f \qquad \textit{Since } y'_h + py_h = 0 \qquad \textit{homogeneous}$$

$$u'y_h = f$$

$$\frac{du}{dx} = \frac{f}{y_h}$$

$$du = \frac{f}{e^{-\int pdx}}dx$$

$$= f.e^{\int pdx}dx$$

$$u = \int f.e^{\int pdx}dx$$

$$y_p = u.e^{-\int pdx}$$

$$u = \left(\int f.e^{\int pdx}dx\right)e^{-\int pdx}$$

$$\boxed{y_p = e^{-\int pdx}\int f.e^{\int pdx}dx}$$

$$\boxed{y = Ce^{-\int pdx} + e^{-\int pdx}\int f.e^{\int pdx}dx} \qquad y = y_h + y_p \qquad \boxed{y = e^{-\int pdx}\left(C + \int f.e^{\int pdx}dx\right)}$$

Example

Find the general solution of $x' = x\sin t + 2te^{-\cos t}$ and the particular solution that satisfies $x(0) = 1$.

Solution

$$x' - x\sin t = 2te^{-\cos t} \qquad P(t) = \sin t, \quad Q(t) = 2te^{-\cos t}$$

$$x_h = e^{-\int \sin t dt} = e^{\cos t}$$

$$\int Q(t)x_h dt = \int 2te^{-\cos t}e^{\cos t}dt = \int 2tdt = t^2$$

$$x(t) = e^{-\cos t}\left(t^2 + C\right) \qquad x = \frac{1}{e^{\int Pdt}}\left(\int Q.e^{\int Pdt}dt + C\right)$$

$$x(0) = \left((0)^2 + C\right)e^{-\cos 0} = 1$$

$$Ce^{-1} = 1$$

$$C = e$$

$$\boxed{x(t) = \left(t^2 + e\right)e^{-\cos t}}$$

Example

Find the general solution of $x' = x\tan t + \sin t$ and the particular solution that satisfies $x(0) = 2$..

Solution

$$x' - (\tan t)x = \sin t \qquad P(t) = -\tan t, \quad Q(t) = \sin t$$

$$e^{-\int \tan t dt} = e^{\ln(\cos t)} \boxed{= \cos t}$$

$$\int(\sin t)(\cos t)\ dt = -\int \cos t\ d(\cos t) = -\frac{1}{2}\cos^2 t$$

$$x(t) = \frac{1}{\cos t}\left(-\frac{1}{2}\cos^2 t + C\right) = -\frac{1}{2}\cos t + \frac{1}{\cos t}C$$

$$\boxed{= -\frac{1}{2}\cos t + \frac{1}{\cos t}C}$$

$$x(0) = -\frac{1}{2}\cos(0) + \frac{C}{\cos(0)} = 2$$

$$-\frac{1}{2} + C = 2 \quad \Rightarrow \quad C = \frac{5}{2}$$

$$\boxed{x(t) = -\frac{1}{2}\cos t + \frac{5}{2\cos t}}$$

Linear Differential Operators

$L[y] = y' + p(x)y$ is a linear operator.

- $L[f+g] = L[f] + L[g]$

  Proof

$$\begin{aligned} L[f]+L[g] &= f' + p(x)f + g' + p(x)g \\ &= (f'+g') + p(x)(f+g) \\ &= (f+g)' + p(x)(f+g) \\ &= L[f+g] \end{aligned}$$

- $L[cf] = cL[f]$

  Proof

$$\begin{aligned} L[cf] &= (cf)' + p(x)(cf) \\ &= cf' + cp(x)f \\ &= c\left(f' + p(x)f\right) \\ &= cL[f] \end{aligned}$$

Any operation L that has the two properties

$$\begin{cases} L\left[y_1 + y_2\right] = L\left[y_1\right] + L\left[y_2\right] \\ L[cy] = cL[y], & c \text{ is constant} \end{cases}$$

is a ***linear operation***.

Differential is a linear operation; ***integration*** is a linear operation.

Notes

1. Integrating an expression that is not the differential of any elementary function is called non-elementary.

$$\int e^{x^2}dx \qquad \int x\tan x\,dx \qquad \int \frac{e^{-x}}{x}dx$$

$$\int \sin x^2 dx \qquad \int \cos x^2 dx \qquad \int \frac{\sin x}{x}dx \qquad \int \frac{\cos x}{x}dx$$

2. In math some important functions are defined in terms of non-elementary integrals. Two such functions are the error function and the complementary error function.

$$erf\,(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \qquad erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$$

Exercises *Section* 1.3 – Linear Differential Equations

Find the general solution of the first-order, linear equation.

1. $y' - y = 3e^t$
2. $y' + y = \sin t$
3. $y' + y = \dfrac{1}{1+e^t}$
4. $y' - y = e^{2t} - 1$
5. $y' + y = te^{-t} + 1$
6. $y' + y = 1 + e^{-x}\cos 2x$
7. $y' + y\cot x = \cos x$
8. $y' + y\sin t = \sin t$
9. $y' = \cos x - y\sec x$
10. $y' + (\tan x)y = \cos^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$
11. $y' + (\cot t)y = 2t\csc t$
12. $y' + (1+\sin t)y = 0$
13. $y' + \left(\frac{1}{2}\cos x\right)y = -\frac{3}{2}\cos x$
14. $\dfrac{dy}{dx} + y = e^{3x}$
15. $y' - ty = t$
16. $y' = 2y + x^2 + 5$
17. $xy' + 2y = 3$
18. $\dfrac{dy}{dt} - 2y = 4 - t$
19. $y' + 2y = 1$
20. $y' + 2y = e^{-t}$
21. $y' + 2y = e^{-2t}$
22. $y' - 2y = e^{3t}$
23. $y' + 2y = e^{-x} + x + 1$
24. $y' + 2xy = x$
25. $y' - 2ty = t$
26. $y' + 2ty = 5t$
27. $y' - 2xy = e^{x^2}$
28. $y' + 2xy = x^3$
29. $y' - 2y = t^2e^{2t}$
30. $x' - 2\dfrac{x}{t+1} = (t+1)^2$
31. $y' + \dfrac{2}{t}y = \dfrac{\cos t}{t^2}$
32. $y' - 2(\cos 2t)y = 0$
33. $y' + 2y = \cos 3t$
34. $y' - 3y = 5$
35. $y' + 3y = 2xe^{-3x}$
36. $y' + 3t^2y = t^2$
37. $y' + 3x^2y = x^2$
38. $y' + \dfrac{3}{t}y = \dfrac{\sin t}{t^3}, \quad (t \neq 0)$
39. $y' + \dfrac{3}{x}y = 1 + \dfrac{1}{x}$
40. $y' + \dfrac{3}{2}y = \dfrac{1}{2}e^x$
41. $y' + 5y = t + 1$
42. $xy' - y = x^2\sin x$
43. $x\dfrac{dy}{dx} + y = e^x, \quad x > 0$
44. $x\dfrac{dy}{dx} + 2y = 1 - \dfrac{1}{x}, \quad x > 0$
45. $y\dfrac{dx}{dy} + 2x = 5y^3$
46. $ty' + y = \cos t$
47. $xy' + 2y = x^2$
48. $xy' = 2y + x^3\cos x$
49. $xy' + 2y = x^{-3}$
50. $ty' + 2y = t^2$
51. $xy' + 2\left(y + x^2\right) = \dfrac{\sin x}{x}$
52. $xy' + 4y = x^3 - x$
53. $xy' + (x+1)y = e^{-x}\sin 2x$

54. $xy' + (3x+1)y = e^{3x}$

55. $xy' + (2x-3)y = 4x^4$

56. $2xy'' - 3y = 9x^3$

57. $2y' + 3y = e^{-t}$

58. $2y' + 2ty = t$

59. $3xy' + y = 10\sqrt{x}$

60. $3xy' + y = 12x$

61. $x^2y' + xy = 1$

62. $x^2y' + x(x+2)y = e^x$

63. $y^2 + (y')^2 = 1$

64. $(1+x)y' + y = \sqrt{x}$

65. $(1+x)y' + y = \cos x$

66. $(x+1)y' + (x+2)y = 2xe^{-x}$

67. $(x+1)y' - xy = x + x^2$

68. $\left(1+x^3\right)y' = 3x^2y + x^2 + x^5$

69. $(t+1)\dfrac{ds}{dt} + 2s = 3(t+1) + \dfrac{1}{(t+1)^2}, \quad t > -1$

70. $(x+2)^2\, y' = 5 - 8y - 4xy$

71. $\left(x^2-1\right)y' + 2y = (x+1)^2$

72. $\left(x^2+4\right)y' + 2xy = x^2\left(x^2+4\right)$

73. $\left(1+e^t\right)y' + e^t y = 0$

74. $\left(t^2+9\right)y' + ty = 0$

75. $e^{2x}y' + 2e^{2x}y = 2x$

76. $\tan\theta \dfrac{dr}{d\theta} + r = \sin^2\theta, \quad 0 < \theta < \frac{\pi}{2}$

77. $(\cos t)y' + (\sin t)y = 1$

78. $\cos x \dfrac{dy}{dx} + (\sin x)y = 1$

79. $\cos^2 x \sin x \dfrac{dy}{dx} + \left(\cos^3 x\right)y = 1$

80. $\dfrac{dr}{d\theta} + r\sec\theta = \cos\theta$

81. $\dfrac{dr}{d\theta} + r\tan\theta = \sec\theta$

82. $\dfrac{dP}{dt} + 2tP = P + 4t - 2$

83. $ydx - 4\left(x + y^6\right)dy = 0$

84. $ydx = \left(ye^y - 2x\right)dy$

85. $(x+y+1)dx - dy = 0$

86. $\dfrac{dy}{dx} = x^2e^{-4x} - 4y$

87. $\left(x^2+1\right)y' + xy - x = 0$

88. $\dfrac{dx}{dt} = 9.8 - 0.196x$

89. $\dfrac{di}{dt} + 500i = 10\sin\omega t$

90. $2\dfrac{dQ}{dt} + 100Q = 10\sin 60t$

Find the solution of the initial value problem

91. $y' - 3y = 4\,; \quad y(0) = 2$

92. $y' = y + 2xe^{2x}; \quad y(0) = 3$

93. $\left(x^2+1\right)y' + 3xy = 6x; \quad y(0) = -1$

94. $t\dfrac{dy}{dt} + 2y = t^3, \quad t > 0, \quad y(2) = 1$

95. $\theta\dfrac{dy}{d\theta} + y = \sin\theta, \quad \theta > 0, \quad y\left(\frac{\pi}{2}\right) = 1$

96. $\dfrac{dy}{dx} + xy = x, \quad y(0) = -6$

97. $ty' + 2y = 4t^2, \quad y(1) = 2$

98. $\left(1+t^2\right)y' + 4ty = \left(1+t^2\right)^{-2}, \quad y(1) = 0$

99. $y' + y = e^t, \quad y(0) = 1$

100. $y' + \frac{1}{2}y = t, \quad y(0) = 1$

101. $y' = x + 5y, \quad y(0) = 3$

102. $y' = 2x - 3y, \quad y(0) = \frac{1}{3}$

103. $xy' + y = e^x, \quad y(1) = 2$

104. $y\frac{dx}{dy} - x = 2y^2, \quad y(1) = 5$

105. $xy' + y = 4x + 1, \quad y(1) = 8$

106. $y' + 4xy = x^3 e^{x^2}, \quad y(0) = -1$

107. $(x+1)y' + y = \ln x, \quad y(1) = 10$

108. $x(x+1)y' + xy = 1, \quad y(e) = 1$

109. $y' - (\sin x)y = 2\sin x, \quad y\left(\frac{\pi}{2}\right) = 1$

110. $y' + (\tan x)y = \cos^2 x, \quad y(0) = -1$

111. $L\frac{di}{dt} + RI = E, \quad i(0) = i_0$

112. $\frac{dT}{dt} = k\left(T - T_m\right) \quad T(0) = T_0$

113. $y' + y = 2, \quad y(0) = 0$

114. $xy' + 2y = 3x, \quad y(1) = 5$

115. $y' - 2y = 3e^{2x}, \quad y(0) = 0$

116. $xy' + 5y = 7x^2, \quad y(2) = 5$

117. $xy' - y = x, \quad y(1) = 7$

118. $xy' + y = 3xy, \quad y(1) = 0$

119. $xy' + 3y = 2x^5, \quad y(2) = 1$

120. $y' + y = e^x, \quad y(0) = 1$

121. $xy' - 3y = x^3, \quad y(1) = 10$

122. $y' + 2xy = x, \quad y(0) = -2$

123. $y' = (1-y)\cos x, \quad y(\pi) = 2$

124. $(1+x)y' + y = \cos x, \quad y(0) = 1$

125. $y' = 1 + x + y + xy, \quad y(0) = 0$

126. $xy' = 3y + x^4 \cos x, \quad y(2\pi) = 0$

127. $y' = 2xy + 3x^2 e^{x^2}, \quad y(0) = 5$

128. $\left(x^2 + 4\right)y' + 3xy = x, \quad y(0) = 1$

129. $y' - 2y = e^{3x}; \quad y(0) = 3$

130. $y' - 3y = 6; \quad y(0) = 1$

131. $2y' + 3y = e^x; \quad y(0) = 0$

132. $\left(x^2 + 1\right)y' + 3x^3 y = 6xe^{-3x^2/2}, \quad y(0) = 1$

133. $y' + y = 1 + e^{-x}\cos 2x; \quad y\left(\frac{\pi}{2}\right) = 0$

134. $2y' + (\cos x)y = -3\cos x; \quad y(0) = -4$

135. $y' + 2y = e^{-x} + x + 1; \quad y(-1) = e$

136. $y' + \frac{y}{x} = xe^{-x}; \quad y(1) = e - 1$

137. $y' + 4y = e^{-x}; \quad y(1) = \frac{4}{3}$

138. $x^2 y' + 3xy = x^4 \ln x + 1; \quad y(1) = 0$

139. $y' + \frac{3}{x}y = 3x - 2 \quad y(1) = 1$

140. $(\cos x)y' + y\sin x = 2x\cos^2 x \quad y\left(\frac{\pi}{4}\right) = \frac{-15\sqrt{2}\pi^2}{32}$

141. $(\cos x)y' + (\sin x)y = 2\cos^3 x \sin x - 1 \quad y\left(\frac{\pi}{4}\right) = 3\sqrt{2}$

142. $t\,y' + 2y = t^2 - t + 1 \quad y(1) = \frac{1}{2}$

143. $t\,y' - 2y = t^5 \sin 2t - t^3 + 4t^4 \quad y(\pi) = \frac{3}{2}\pi^4$

144. $2y' - y = 4\sin 3t \quad y(0) = y_0$

145. $y' + 2y = 2 - e^{-4t} \quad y(0) = 1$

146. $y' - y = -\frac{1}{2}e^{t/2}\sin 5t + 5e^{t/2}\cos 5t \quad y(0) = 0$

147. $y' + 2y = 3; \quad y(0) = -1$

148. $y' + (\cos t)y = \cos t; \quad y(\pi) = 2$

149. $y' + 2ty = 2t; \quad y(0) = 1$

150. $y' + y = \frac{e^{-t}}{t^2}; \quad y(1) = 0$

151. $ty' + 2y = \sin t; \quad y(\pi) = \frac{1}{\pi}$

152. $y' + \frac{2}{t}y = \frac{\cos t}{t^2}; \quad y(\pi) = 0$

153. $(\sin t)y' + (\cos t)y = 0; \quad y\left(\frac{3\pi}{4}\right) = 2$

154. $y' + 3t^2 y = t^2$; $y(0) = 2$

155. $ty' + y = t\sin t$; $y(\pi) = -1$

156. $y' + y = \sin t$; $y(\pi) = 1$

157. $y' + y = \cos 2t$; $y(0) = 5$

158. $y' + 3y = \cos 2t$; $y(0) = -1$

159. $y' - 2y = 7e^{2t}$; $y(0) = 3$

160. $y' - 2y = 3e^{-2t}$; $y(0) = 10$

161. $y' + 2y = t^2 + 2t + 1 + e^{4t}$; $y(0) = 0$

162. $y' - 3y = 2t - e^{4t}$; $y(0) = 0$

163. $y' + y = t^3 + \sin 3t$; $y(0) = 0$

164. $y' + 2y = \cos 2t + 3\sin 2t + e^{-t}$; $y(0) = 0$

165. $y' + y = e^{3t}$; $y(0) = y_0$

166. $t^2 y' - ty = 1$; $y(1) = y_0$

167. $y' + ay = e^{at}$; $y(0) = y_0$, $a \neq 0$

168. $3y' + 12y = 4$; $y(0) = y_0$

Find a solution to the initial value problem that is continuous on the given interval $[a,\ b]$

169. $y' + \frac{1}{x} y = f(x)$, $y(1) = 1$ $\quad f(x) = \begin{cases} 3x, & 1 \le x \le 2 \\ 0, & 2 < x \le 3 \end{cases}$ $\quad [a,\ b] = [1,\ 3]$

170. $y' + (\sin x) y = f(x)$, $y(0) = 3$ $\quad f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ -\sin x, & \pi < x \le 2\pi \end{cases}$ $\quad [a,\ b] = [0,\ 2\pi]$

171. $y' + p(t) y = 2$, $y(0) = 1$ $\quad p(t) = \begin{cases} 0, & 0 \le t \le 1 \\ \frac{1}{t}, & 1 < t \le 2 \end{cases}$ $\quad [a,\ b] = [0,\ 2]$

172. $y' + p(t) y = 0$, $y(0) = 3$ $\quad p(t) = \begin{cases} 2t - 1, & 0 \le t \le 1 \\ 0, & 1 < t \le 3 \\ -\frac{1}{t}, & 3 < t \le 4 \end{cases}$ $\quad [a,\ b] = [0,\ 4]$

Find the solution of the initial value problem. Discuss the interval of existence and provide a sketch of your solution

173. $xy' + 2y = \sin x$; $y\left(\frac{\pi}{2}\right) = 0$

174. $(2x+3)y' = y + (2x+3)^{1/2}$; $y(-1) = 0$

175. The following system of differential equations is encountered in the study of the decay of a special type of radioactive series of elements

$$\frac{dx}{dt} = -\lambda_1 x \qquad \frac{dy}{dt} = \lambda_1 x - \lambda_2 y$$

Where λ_1 *and* λ_2 are constants.

Discuss how to solve this system subject to $x(0) = x_0$, $y(0) = y_0$

Section 1.4 – Exact Differential Equations

A class of equations known as exact equations for which there is also a well-defined method of solution

Theorem

Let the function M, N, M_y *and* N_x, where M_y *and* N_x are partial derivatives, be continuous in the rectangular region $R:\ \alpha < x < \beta,\ \ \gamma < y < \delta$ then

$$M(x, y) + N(x, y)y' = 0$$

Is an exact differential equation in *R*, if and only if

$$M_y(x, y) = N_x(x, y)$$

At each point in *R*. That is, there exists a function ψ satisfying

$$\psi_y(x, y) = M(x, y) \quad \text{and} \quad \psi_x(x, y) = N(x, y)$$

Iff $M_y(x, y) = N_x(x, y)$

Example

Solve the differential equation: $\qquad 2x + y^2 + 2xyy' = 0$

Solution

$$\frac{\partial\psi}{\partial x} = M = 2x + y^2 \quad \Rightarrow \quad M_y = 2y$$

$$\frac{\partial\psi}{\partial y} = N = 2xy \quad \Rightarrow \quad N_x = 2y$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial\psi}{\partial x} = 2x + y^2 \quad \Rightarrow \quad \psi = \int\left(2x + y^2\right)dx = x^2 + xy^2 + h(y)$$

$$\psi_y = 2xy + h'(y) = 2xy \quad \Rightarrow \quad h'(y) = 0$$

$$\Rightarrow h(y) = C$$

$$\psi(x, y) = x^2 + xy^2 + C$$

$$\boxed{x^2 + xy^2 = C}$$

Example

Solve the differential equation: $y\cos x + 2xe^{y} + \left(\sin x + x^{2}e^{y} - 1\right)y' = 0$

Solution

$$M = y\cos x + 2xe^{y} = \frac{\partial\psi}{\partial x} \quad\Rightarrow\quad M_{y} = \cos x + 2xe^{y}$$

$$\frac{\partial\psi}{\partial y} = N = \sin x + x^{2}e^{y} - 1 \quad\Rightarrow\quad N_{x} = \cos x + 2xe^{y}$$

$$\Rightarrow M_{y} = N_{x}$$

$$\psi = \int\left(y\cos x + 2xe^{y}\right)dx = y\sin x + x^{2}e^{y} + h(y)$$

$$\psi_{y} = \sin x + x^{2}e^{y} + h'(y) = \sin x + x^{2}e^{y} - 1 \quad\Rightarrow\quad h'(y) = -1$$

$$\Rightarrow h(y) = -y$$

$$\psi(x, y) = y\sin x + x^{2}e^{y} - y = C$$

$$\boxed{y\sin x + x^{2}e^{y} - y = C}$$

Example

Solve the differential equation: $3xy + y^{2} + \left(x^{2} + xy\right)y' = 0$

Solution

$$M = 3xy + y^{2} = \frac{\partial\psi}{\partial x} \quad\Rightarrow\quad M_{y} = 3x + 2y$$

$$N = x^{2} + xy = \frac{\partial\psi}{\partial y} \quad\Rightarrow\quad N_{x} = 2x + y$$

$$\Rightarrow M_{y} \neq N_{x}$$

Can be solved by this procedure.

Integrating Factors

It is sometimes possible to convert a differential equation that is not exact equation by multiplying the equation by a suitable integrating factor.

Definition

An integrating factor for the differential equation $\omega = Mdx + Ndy = 0$ is a function $\mu(x,\ y)$ such that the form $\mu\omega = \mu(x,\ y)M(x,\ y)dx + \mu(x,\ y)N(x,\ y)dy$ is exact.

$$(\mu M)_y = (\mu N)_x$$

$$M\mu_y - N\mu_x + \left(M_y - N_x\right)\mu = 0$$

Assuming that μ is a function of x only, we have

$$(\mu M)_y = \mu M_y \quad \& \quad (\mu N)_x = \mu N_x + N\frac{d\mu}{dx}$$

$$\Rightarrow \quad \mu M_y = \mu N_x + N\frac{d\mu}{dx}$$

$$\boxed{\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu}$$

Example

Find an integrating factor for the equation $\left(3xy + y^2\right) + \left(x^2 + xy\right)y' = 0$

And then solve the equation.

Solution

$$M_y = \frac{\partial}{\partial y}\left(3xy + y^2\right) = 3x + 2y \qquad N_x = \frac{\partial}{\partial x}\left(x^2 + xy\right) = 2x + y$$

$$\Rightarrow M_y \neq N_x$$

$$\frac{M_y - N_x}{N} = \frac{3x + 2y - 2x - y}{x^2 + xy} = \frac{x + y}{x(x + y)} = \frac{1}{x}$$

$$\frac{d\mu}{dx} = \frac{\mu}{x} \quad \Rightarrow \quad \int\frac{d\mu}{\mu} = \int\frac{dx}{x}$$

$$\ln\mu = \ln x$$

$$\mu = x$$

$$x\left(3xy+y^2\right)+x\left(x^2+xy\right)y'=0$$

$$M_y=\frac{\partial}{\partial y}\left(3x^2y+xy^2\right)=3x^2+2xy \qquad N_x=\frac{\partial}{\partial x}\left(x^3+x^2y\right)=3x^2+2xy$$

$$\Rightarrow M_y=N_x$$

$$\psi=\int\left(3x^2y+xy^2\right)dx=x^3y+\frac{1}{2}x^2y^2+h(y)$$

$$\psi_y=x^3+x^2y+h'(y)=x^3+x^2y \quad\Rightarrow\quad h'(y)=0$$

$$\Rightarrow h(y)=C$$

$$\psi(x,\,y)=x^3y+\frac{1}{2}x^2y^2=C$$

$$\boxed{x^3y+\frac{1}{2}x^2y^2=C}$$

Bernoulli Equations

An equation of the form $y' + P(x)y = Q(x)y^n, \quad n \neq 0,\ 1$ is called a ***Bernoulli equation***.

If $n=0 \quad \Rightarrow \quad y' + Py = Q$ First–order linear differential equation

If $n=1 \quad \Rightarrow \quad y' + Py = Qy \ \rightarrow \ y' + (P-Q)y = 0$ Separable equation.

For $n \neq 0,\ 1$, the Bernoulli equation can be written as $y^{-n}\dfrac{dy}{dx} + Py^{1-n} = Q \qquad (1)$

Let $u = y^{1-n} \quad \Rightarrow \quad \dfrac{du}{dx} = (1-n)y^{-n}\dfrac{dy}{dx}$

$$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{du}{dx}$$

$$(1) \quad \Rightarrow \quad \frac{1}{1-n}\frac{du}{dx} + Pu = Q$$

$\underline{u' + (1-n)Pu = (1-n)Q}$ Which is 1st–order linear differential equation.

Example

Find the general solution $y' - 4y = 2e^x\sqrt{y}$

Solution

$$\sqrt{y} = y^{1/2} \quad \Rightarrow \quad n = \frac{1}{2}$$

Let $u = y^{1-\frac{1}{2}} = y^{1/2} \quad \Rightarrow \quad y = u^2$

$$\frac{du}{dx} = \frac{1}{2}y^{-1/2}\frac{dy}{dx} \quad \Rightarrow \quad 2y^{1/2}\frac{du}{dx} = \frac{dy}{dx}$$

$$\frac{dy}{dx} - 4y = 2e^x u$$

$2u\dfrac{du}{dx} - 4u^2 = 2ue^x$ ***Divide by*** $2u$

$$u' - 2u = e^x$$

$$e^{\int -2dx} = e^{-2x}$$

$$\int e^x e^{-2x}dx = \int e^{-x}dx = -e^{-x}$$

$$u = \frac{1}{e^{-2x}}\left(-e^{-x} + C\right)$$

$$y^{1/2} = -e^x + Ce^{2x}$$

$$\underline{y = \left(Ce^{2x} - e^x\right)^2}$$

Example

Find the general solution $xy' + y = 3x^3y^2$

Solution

$$y' + \frac{1}{x}y = 3x^2y^2$$

Let $u = y^{1-2} = y^{-1} \quad \Rightarrow \quad y = \frac{1}{u}$

$$\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx} \quad \Rightarrow \quad y' = -y^2u' = -\frac{1}{u^2}u'$$

$$-\frac{1}{u^2}u' + \frac{1}{x}\frac{1}{u} = 3x^2\frac{1}{u^2}$$ ***Multiply both sides by*** $-u^2$

$$u' - \frac{1}{x}u = -3x^2$$

$$e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

$$\int -3x^2x^{-1}dx = -3\int xdx = -\frac{3}{2}x^2$$

$$u = x\left(-\frac{3}{2}x^2 + C_1\right)$$

$$\frac{1}{y} = \frac{-3x^3 + 2C_1x}{2}$$

$$\boxed{y = \frac{2}{Cx - 3x^3}}$$

Homogeneous Equations $\frac{dy}{dx} = f(x, y)$

The form of a homogeneous equation suggests that it may be simplified by using a variable denoted by 'v', to represent the ratio of y to x. This

$$y = xv \quad \Rightarrow \quad \frac{dy}{dx} = F(v)$$

Let assume that v is a function of x, then

$$\frac{dy}{dx} = x\frac{dv}{dx} + v \quad \text{or} \quad y' = xv' + v$$

The most significant fact about this equation is that the variables x & v can always be ***separated***, regardless of the form of the function F.

$$\frac{dx}{x} = \frac{dv}{F(v) - v}$$

Solving this equation and then replacing v by $\frac{y}{x}$ gives the solution of the original equation.

Example

Solve the differential equation $\qquad \frac{dy}{dx} = \frac{y^2 + 2xy}{x^2}$

Solution

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + 2\frac{y}{x} = v^2 + 2v$$

$$x\frac{dv}{dx} + v = v^2 + 2v \qquad \Rightarrow x\frac{dv}{dx} = v^2 + v$$

$$xdv = v(v+1)dx$$

$$\int \frac{dx}{x} = \int \frac{dv}{v(v+1)}$$

$$\int \frac{dx}{x} = \int \left(\frac{1}{v} - \frac{1}{v+1}\right)dv$$

$$\ln x + \ln C = \ln v - \ln(v+1)$$

$$\ln(Cx) = \ln\frac{v}{v+1}$$

$$Cx = \frac{v}{v+1} = \frac{\frac{y}{x}}{\frac{y}{x} + 1} = \frac{y}{y+x} \qquad \Rightarrow Cxy + Cx^2 = y$$

$$Cx^2 = y - Cxy$$

$$\boxed{y = \frac{Cx^2}{1 - Cx}}$$

Example

Find the general solution $\qquad y' = \dfrac{x^2 e^{y/x} + y^2}{xy}$

Solution

Let $y = xv \quad \Rightarrow \quad y' = v + xv'$

$$v + xv' = \frac{x^2 e^{xv/x} + (xv)^2}{x(xv)}$$

$$xv' = \frac{x^2 e^v + x^2 v^2}{x^2 v} - v$$

$$x\frac{dv}{dx} = \frac{e^v + v^2}{v} - v$$

$$x\frac{dv}{dx} = \frac{e^v}{v}$$

$$\int \frac{v}{e^v} dv = \int \frac{dx}{x}$$

$$-ve^{-v} - e^{-v} = \ln x + C$$

$$-e^{-v}(v+1) = \ln x + C$$

$$-e^{-y/x}\left(\frac{y}{x} + 1\right) = \ln x + C$$

$$\underline{y + x = -xe^{y/x}(\ln x + C)}$$

Exercises *Section* 1.4 – Exact Differential Equations

Solve the differential equation

1. $(2x+y)dx+(x-6y)dy=0$

2. $(2x+3)dx+(2y-2)dy=0$

3. $(1-y\sin x)+(\cos x)y'=0$

4. $\dfrac{dy}{dx}=-\dfrac{ax+by}{bx+cy}$

5. $\dfrac{dy}{dx}=\dfrac{3x^2+y}{3y^2-x}$

6. $2xydx+\left(x^2-1\right)dy=0$

7. $y'=\dfrac{x^2+y^2}{2xy}$

8. $2xyy'=x^2+2y^2$

9. $xy'=y+2\sqrt{xy}$

10. $xy^2y'=x^3+y^3$

11. $x^2y'=xy+x^2e^{y/x}$

12. $x^2y'=xy+y^2$

13. $xyy'=x^2+3y^2$

14. $\left(x^2-y^2\right)y'=2xy$

15. $xyy'=y^2+x\sqrt{4x^2+y^2}$

16. $xy'=y+\sqrt{x^2+y^2}$

17. $y^2y'+2xy^3=6x$

18. $x^2y'+2xy=5y^4$

19. $2xy'+y^3e^{-2x}=2xy$

20. $y^2\left(xy'+y\right)\left(1+x^4\right)^{1/2}=x$

21. $3y^2y'+y^3=e^{-x}$

22. $3xy^2y'=3x^4+y^3$

23. $xe^yy'=2\left(e^y+x^3e^{2x}\right)$

24. $(2x\sin y\cos y)y'=4x^2+\sin^2 y$

25. $\left(x+e^y\right)y'=xe^{-y}-1$

26. $\left(x^2+y^2\right)dx+\left(x^2-xy\right)dy=0$

27. $x\dfrac{dy}{dx}+y=x^2y^2$

28. $\left(3x^2-2xy+2\right)+\left(6y^2-x^2+3\right)y'=0$

29. $\left(e^x\sin y-2y\sin x\right)dx+\left(e^x\cos y+2\cos x\right)dy=0$

30. $\left(\dfrac{y}{x}+6x\right)dx+(\ln x-2)dy=0, \qquad x>0$

31. $\left(e^{2y}-y\cos x\right)dx+\left(2xe^{2y}x\cos xy+2y\right)dy=0$

32. $\dfrac{xdx}{\left(x^2+y^2\right)^{3/2}}+\dfrac{ydy}{\left(x^2+y^2\right)^{3/2}}=0$

33. $(2x-1)dx+(3y+7)dy=0$

34. $(5x+4y)dx+\left(4x-8y^3\right)dy=0$

35. $(\sin y-y\sin x)dx+(\cos x+x\cos y-y)dy=0$

36. $\left(2xy^2-3\right)dx+\left(2x^2y+4\right)dy=0$

37. $\left(1+\ln x+\dfrac{y}{x}\right)dx-(1-\ln x)dy=0$

38. $\left(x-y^3+y^2\sin x\right)dx-\left(3xy^2+2y\cos x\right)dy=0$

39. $\left(x^3+y^3\right)dx+3xy^2dy=0$

40. $\left(3x^2y+e^y\right)dx+\left(x^3+xe^y-2y\right)dy=0$

41. $xdy+\left(y-2xe^x-6x^2\right)dx=0$

42. $\left(1-\dfrac{3}{y}+x\right)dy+\left(y-\dfrac{3}{x}+1\right)dx=0$

43. $\left(x^2y^3-\dfrac{1}{1+9x^2}\right)\dfrac{dx}{dy}+x^3y^2=0$

44. $(5y-2x)y'-2y=0$

45. $(x-y)dx-xdy=0$

46. $(x+y)\,dx + x\,dy = 0$

47. $\dfrac{dy}{dx} = -\dfrac{2xy^2+1}{2x^2y}$

48. $\left(1+e^x y + xe^x y\right)dx + \left(xe^x + 2\right)dy = 0$

49. $\left(2xy^3+1\right)dx + \left(3x^2y^2 - \frac{1}{y}\right)dy = 0$

50. $(2x+y)\,dx + (x-2y)\,dy = 0$

51. $e^x(y-x)\,dx + \left(1+e^x\right)dy = 0$

52. $\left(ye^{xy} - \frac{1}{y}\right)dx + \left(xe^{xy} + \frac{x}{y^2}\right)dy = 0$

53. $(\tan x - \sin x \sin y)\,dx + (\cos x \cos y)\,dy = 0$

54. $\left(2x^3 - xy^2 - 2y + 3\right)dx - \left(x^2y + 2x\right)dy = 0$

55. $(x + \sin y)\,dx + (x\cos y - 2y)\,dy = 0$

56. $\left(x + \dfrac{1}{\sqrt{y^2-x^2}}\right)dx + \left(1 - \dfrac{1}{y\sqrt{y^2-x^2}}\right)dy = 0$

57. $\left(2x + y^2 - \cos(x+y)\right)dx + \left(2xy - \cos(x+y) - e^y\right)dy = 0$

58. $\left(\dfrac{2}{\sqrt{1-x^2}} + y\cos(xy)\right)dx + \left(x\cos(xy) - y^{-1/3}\right)dy = 0$

59. $\left(2x + y\cos(xy)\right)dx + \left(x\cos(xy) - 2y\right)dy = 0$

60. $\left(e^x \sin y - 3x^2\right)dx + \left(e^x \cos y + \frac{1}{3}y^{-2/3}\right)dy = 0$

61. $\left(2y\sin x\cos x - y + 2y^2e^{xy^2}\right)dx = \left(x - \sin^2 x - 4xye^{xy^2}\right)dy$

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

62. $x^2y^3 + x\left(1+y^2\right)y' = 0, \qquad \mu(x, y) = \dfrac{1}{xy^3}$

63. $y^2 - xy + \left(x^2\right)y' = 0, \qquad \mu(x, y) = \dfrac{1}{xy^2}$

64. $x^2y^3 - y + x\left(1 + x^2y^2\right)y' = 0, \qquad \mu(x, y) = \dfrac{1}{xy}$

65. $\left(\dfrac{\sin y}{y} - 2e^{-x}\sin x\right)dx + \left(\dfrac{\cos y + 2e^{-x}\cos x}{y}\right)dy = 0, \qquad \mu(x, y) = ye^x$

66. $(x+2)\sin y\,dx + x\cos y\,dy = 0, \qquad \mu(x, y) = xe^x$

67. $\left(x^2 + y^2 - x\right)dx - y\,dy = 0, \qquad \mu(x, y) = \dfrac{1}{x^2+y^2}$

68. $(2y - 6x)\,dx + \left(3x - 4x^2y^{-1}\right)dy = 0, \qquad \mu(x, y) = xy^2$

Find the general solution of each homogenous equation

69. $\left(x^2+y^2\right)dx-2xydy=0$

70. $(x+y)dx+(y-x)dy=0$

71. $\dfrac{dy}{dx}=\dfrac{y\left(x^2+y^2\right)}{xy^2-2x^3}$

Find an integrating factor and solve the given equation

72. $\left(3x^2y+2xy+y^3\right)dx+\left(x^2+y^2\right)dy=0$

73. $dx+\left(\dfrac{x}{y}-\sin y\right)dy=0$

74. $e^x dx+\left(e^x\cot y+2y\csc y\right)dy=0$

75. $\left(3x+\dfrac{6}{y}\right)dx+\left(\dfrac{x^2}{y}+3\dfrac{y}{x}\right)dy=0$

76. $\left(x+3x^3\sin y\right)dx+\left(x^4\cos y\right)dy=0$

77. $\left(2x^2+y\right)dx+\left(x^2y-x\right)dy=0$

78. $\left(3x^2+y\right)dx+\left(x^2y-x\right)dy=0$

79. $\left(2y^2+2y+4x^2\right)dx+(2xy+x)dy=0$

80. $\left(x^4-x+y\right)dx-xdy=0$

81. $(2xy)dx+\left(y^2-3x^2\right)dy=0$

Solve the given initial-value problem

82. $\dfrac{dy}{dx}=\dfrac{xy^2-\cos x\sin x}{y\left(1-x^2\right)},\quad y(0)=2$

83. $(x+y)^2\,dx+\left(2xy+x^2-1\right)dy,\quad y(1)=1$

84. $\left(e^x+y\right)dx+\left(2+x+ye^y\right)dy,\quad y(0)=1$

85. $(2x-y)dx+(2y-x)dy,\quad y(1)=3$

86. $\left(9x^2+y-1\right)dx-(4y-x)dy,\quad y(1)=0$

87. $\left(x+y^3\right)y'+y+x^3=0,\quad y(0)=-2$

88. $y'=\left(3x^2+1\right)\left(y^2+1\right),\quad y(0)=1$

89. $\left(y^3+\cos t\right)y'=2+y\sin t,\quad y(0)=-1$

90. $\left(y^3-t^3\right)y'=3t^2y+1,\quad y(-2)=-1$

91. $\dfrac{dy}{dx}=(-2x+y)^2-7,\quad y(0)=0$

92. $(2y-x)y'-y+2x=0,\quad y(1)=0$

93. $\left(e^{2y}+t^2y\right)y'+ty^2+\cos t=0,\quad y\left(\frac{\pi}{2}\right)=0$

94. $y'=-\dfrac{y\cos(ty)+1}{t\cos(ty)+2ye^{y^2}},\quad y(\pi)=0$

95. $\left(2ty+\dfrac{1}{y}\right)y'+y^2=1,\quad y(1)=1$

96. $\left(ye^x+1\right)dx+\left(e^x-1\right)dy=0\quad y(1)=1$

97. $2xy^2+4=2\left(3-x^2y\right)y'\quad y(-1)=8$

98. $y'+\dfrac{4}{x}y=x^3y^2\quad y(2)=-1$

99. $y'=5y+e^{-2x}y^{-2}\quad y(0)=2$

100. $6y'-2y=xy^4\quad y(0)=-2$

101. $y'+\dfrac{y}{x}-\sqrt{y}=0\quad y(1)=0$

102. $xyy'+4x^2+y^2=0\quad y(2)=-7$

103. $xy'=y(\ln x-\ln y)\quad y(1)=4$

104. $y'-(4x-y+1)^2=0\quad y(0)=2$

105. $\left(e^{t+y}+2y\right)y'+\left(e^{t+y}+3t^2\right)=0,\quad y(0)=0$

106. $(4y+2x-5)\,dx+(6y+4x-1)\,dy,\quad y(-1)=2$

107. $\left(ye^{xy}-\frac{1}{y}\right)dx+\left(xe^{xy}+\frac{x}{y^2}\right)dy=0\quad y(1)=1$

108. $(2y\ln t-t\sin y)\,y'+\frac{1}{t}y^2+\cos y=0,\quad y(2)=0$

109. $(\tan y-2)\,dx+\left(x\sec^2 y+\frac{1}{y}\right)dy=0\quad y(0)=1$

110. $2xy-9x^2+\left(2y+x^2+1\right)\frac{dy}{dx}=0\quad y(0)=-3$

111. $\frac{2t}{t^2+1}y-2t+\left(2-\ln\left(t^2+1\right)\right)\frac{dy}{dt}=0\quad y(5)=0$

112. $3y^3e^{3xy}-1+\left(2ye^{3xy}+3xy^2e^{3xy}\right)y'=0\quad y(0)=1$

113. $2xydx+\left(1+x^2\right)dy=0\ ;\quad y(2)=-5$

114. $\frac{dy}{dx}=-\frac{2x\cos y+3x^2y}{x^3-x^2\sin y-y}\ ;\quad y(0)=2$

Find an integrating factor of the form $x^n y^m$ and solve the equation

115. $\left(2y^2-6xy\right)dx+\left(3xy-4x^2\right)dy=0$

116. $(12+5xy)\,dx+\left(6xy^{-1}+3x^2\right)dy=0$

117. $\left(3y+4xy^2\right)dx+\left(2x+3x^2y\right)dy=0$

Find the general solution by using Bernoulli

118. $\frac{dy}{dx}-5y=-\frac{5}{2}xy^3$

119. $\frac{dy}{dx}+\frac{y}{x}=x^2y^2$

120. $\frac{dy}{dx}-y=e^{2x}y^3$

121. $\frac{dy}{dx}+\frac{y}{x-2}=5(x-2)\,y^{1/2}$

122. $\frac{dy}{dx}+y=e^xy^{-2}$

123. $\frac{dy}{dx}+y^3x+y=0$

Find the general solution by using homogeneous equations.

124. $\left(xy+y^2\right)dx-x^2dy=0$

125. $\left(x^2+y^2\right)dx+2xydy=0$

126. $\left(y^2-xy\right)dx+x^2dy=0$

127. $\frac{dy}{d\theta}=\frac{\theta\sec\left(\frac{y}{\theta}\right)+y}{\theta}$

128. $\frac{dy}{dx}=\frac{y(\ln y-\ln x+1)}{x}$

Section 1.5 – Population: Exponential Growth/Decay

Modeling Population Growth

The mathematical model of the growth of a population is given by:

$$\frac{dP}{dt} = rP(t)$$

$P(t)$: *Population at time t.*

r: *Growth rate or reproductive rate* $\quad (r > 0)$

The natural of the predictions of the model depend on the nature of the reproductive rate r.

Malthusian Method

Since r is a constant because the birth or death rates do not depend on time or on the size.
Therefore the solution to $P' = rP$ is given by:

$$\int \frac{dP}{P} = r\int dt$$

$$\ln|P| = rt + C$$

$$P = e^{rt+C}$$

$$\boxed{P(t) = P_0 e^{rt}}$$

The population at time $t = 0$ is P_0.

Doubling Time: $T = \dfrac{\ln 2}{r}$

Example

A biologist starts with 10 cells in a culture. Exactly 24 *hrs* later he counts 25. Assuming a Malthusian model, what the reproductive rate? What will be the number of cells of the end of 10 days?

Solution

$$P = P_0 e^{rt}$$

$$P = 10e^{rt}$$

$$25 = 10e^{r(1)}$$ **24 *hrs* = 1 *day P* = 25**

$$\frac{25}{10} = e^r$$

$$\ln \frac{25}{10} = \ln e^{r}$$

$$r = \ln 2.5$$
$$\approx 0.9163$$

$$P(t) = 10e^{0.9163t}$$

$$P(10) = 10e^{0.9163(10)}$$
$$\approx 95367 \ \ cells$$

Decay

The rate of decay of a material at time t is proportional to the amount of material present at time t.

$$\frac{dA}{dt} = -rA(t)$$

$A(t)$: *Amount of material present at time t.*

r: *Decay rate* $(r > 0)$

$$\int \frac{dA}{A} = -r\int dt$$

$$\ln|A| = -rt + C$$

$$A = e^{-rt+C}$$

$$\boxed{A(t) = A_0 e^{-rt}}$$

$$rT = \ln\frac{1}{2}$$

Half-life: $T = \dfrac{\ln 2}{r}$

Example

A certain radioactive material is decaying at a rate proportional to the amount present. If a sample of 50 grams of the material was present initially and after 2 hours the sample lost 10% of its mass, find:

a) An expression for the mass of the material remaining at any time.
b) The mass of the material after 4 hours.
c) How long will it take for 75% of the material to decay?
d) The half-life of the material.

Solution

Given: $A_0 = 50g$

$A(2) = 50 - .1(50) = 45g$

a) $A(t) = A(0)e^{-rt}$

$A(2) = 50e^{-2r}$

$45 = 50e^{-2r}$

$e^{-2r} = \frac{45}{50} = \frac{9}{10}$ *Convert to logarithm*

$-2r = \ln\frac{9}{10}$

$r = -\frac{1}{2}\ln\frac{9}{10}$

$A(t) = 50e^{\frac{1}{2}\ln\left(\frac{9}{10}\right)t}$

$= 50e^{\ln\left(\frac{9}{10}\right)^{t/2}}$

$= 50\left(\frac{9}{10}\right)^{t/2}$

b) $A(4) = 50\left(\frac{9}{10}\right)^2 = 40.5\ g$

c) $At\ 75\% \Rightarrow A(t) = 50 \times .25 = 12.5\ g$

$12.5 = 50\left(\frac{9}{10}\right)^{t/2}$

$\left(\frac{9}{10}\right)^{t/2} = \frac{12.5}{50}$

$\frac{t}{2}\ln\left(\frac{9}{10}\right) = \ln\left(\frac{12.5}{50}\right)$

$t = \dfrac{2\ln\left(\frac{12.5}{50}\right)}{\ln\left(\frac{9}{10}\right)} \approx 26.32\ hours$

d) $T = \dfrac{\ln 2}{-\frac{1}{2}\ln 0.9} \approx 13.16\ \text{hrs}$

Note

We can this formula to solve most of the questions

$$\boxed{rT = \ln\frac{A}{A_0}}$$

Logistic Model of Growth

Suppose an environment is capable of sustaining no more than a fixed number K of individuals in its populations. The quantity K is called the ***carrying capacity*** of the environment. In reality this model in unrealistic because environments impose limitations to population growth.

The logistic equation is given by:

$$P' = rP\left(1 - \frac{P}{K}\right) = KrP(K - P) \qquad P' = kP(M - P) \quad \textit{where} \quad k = \frac{r}{K} \ \ \& \ \ M = K$$

The logistic equation can be solved by separation of variables

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

$$\int \frac{dP}{P\left(1 - \frac{P}{K}\right)} = \int r\,dt \qquad\qquad \frac{1}{P\left(1 - \frac{P}{K}\right)} = \frac{K}{P(K-P)} = \frac{1}{P} + \frac{1}{K-P}$$

$$\int \frac{dP}{P} + \int \frac{dP}{K-P} = \int r\,dt$$

$$\ln|P| - \ln|K - P| = rt + C$$

$$\ln\left|\frac{P}{K-P}\right| = rt + C$$

$$\frac{P}{K-P} = e^{rt+C}$$

$$\frac{P}{K-P} = Ae^{rt}$$

$$t = 0 \quad \Rightarrow \quad A = \frac{P_0}{K - P_0}$$

$$P = KAe^{rt} - PAe^{rt}$$

$$P\left(1 + Ae^{rt}\right) = KAe^{rt}$$

$$P = \frac{KAe^{rt}}{1 + Ae^{rt}}$$

$$= \frac{KA}{e^{-rt} + A}$$

$$= \frac{K\dfrac{P_0}{K-P_0}}{e^{-rt} + \dfrac{P_0}{K-P_0}}$$

$$= \frac{KP_0}{\left(K - P_0\right)e^{-rt} + P_0}$$

$$P(t) = \frac{KP_0}{P_0 + \left(K - P_0\right)e^{-r(t-t_0)}} \quad or \quad P(t) = \frac{MP_0}{P_0 + \left(M - P_0\right)e^{-kMt}}$$

Example

Suppose we start at time $t_0 = 0$ with a sample of 1000 cells. One day later we see that the population has doubled, and sometime later we notice that the population has stabilized at 100,000.

Solution

Given: $K = 100,000 = 10^5, \quad P_0 = 1000$

$$P(t) = \frac{10^5 10^3}{10^3 + \left(10^5 - 10^3\right)e^{-r(t-0)}}$$

$$= \frac{10^8}{10^3 + 10^3(100-1)e^{-rt}}$$

$$= \frac{10^5}{1+99e^{-rt}}$$

$$2P_0 = \frac{10^5}{1+99e^{-r(1)}}$$

$$2\times 10^3 = \frac{10^5}{1+99e^{-r}}$$

$$1+99e^{-r} = \frac{10^5}{2\times 10^3}$$

$$1+99e^{-r} = 50$$

$$99e^{-r} = 49$$

$$e^{-r} = \frac{49}{99}$$

$$-r = \ln\left(\frac{49}{99}\right)$$

$$r \approx 0.7033$$

$$\boxed{P(t) = \frac{10^5}{1+99e^{-0.7033t}}}$$

Exercises *Section* 1.5 – Population: Exponential Growth/Decay

1. The rate of growth of bacteria in a petri dish is proportional to the number of bacteria in the dish.

2. The rate of growth of a population of field mice is inversely proportional to the square root of the population.

3. A biologist starts with 100 *cells* in a culture. After 24 *hrs*, he counts 300. Assuming a Malthusian model, what the reproductive rate? What will be the number of cells of the end of 5 *days*?

4. A biologist prepares a culture. After 1 *day* of growth, the biologist counts 1000 *cells*. After 2*days*, he counts 3000. Assuming a Malthusian model, what the reproductive rate and how many cells were present initially?

5. A population of bacteria is growing according to the Malthusian model. If the population is triples in 10 *hrs*, what is the reproduction rate? How often does the population double itself?

6. Consider a lake that is stocked with walleye pike and that the population of pike is governed by the logistic equation

$$P' = 0.1P\left(1 - \frac{P}{10}\right)$$

where time is measured in days and P in thousands of fish. Suppose that fishing is started in this lake and that 100 fish are removed each day.

a) Modify the logistic model to account for the fishing.
b) Find and classify the equilibrium points for your model.
c) Use qualitative analysis to completely discuss the fate of the fish population with this model. In particular, if the initial fish population is 1000, what happens to the fish as time passes? What will happen to an initial population having 2000 fish?

7. Suppose that in 1885 the population of a certain country was 50 million and was growing at the rate of 750,000 people per year at that time. Suppose also that in 1940 its population was 100 million and was then growing at the rate of 1 million per year. Assume that this population satisfies the logistic equation. Determine both the limiting population M and the predicted population for the year 2000.

8. The time rate of change of a rabbit population P is proportional to the square root of P. At time $t = 0$ (months) the population numbers 100 rabbits and is increasing at the rate of 20 rabbits per month. How many rabbits will there be one year later?

9. Suppose that the fish population $P(t)$ in a lake is attacked by a disease at time $t = 0$, with the result that the fish cease to reproduce (so that the birth rate is $\beta = 0$) and the death rate δ (deaths per week per fish) is thereafter proportional to $\frac{1}{\sqrt{P}}$. If there were initially 900 fish in the lake and 441 were left after 6 weeks, how long did it take all the fish in the lake to die?

10. Suppose that when a certain lake is stocked with fish, the birth and death rates β and δ are both inversely proportional to $\sqrt{P}$

a) Show that $P(t)=\left(\frac{1}{2}kt+\sqrt{P_0}\right)^2$, where k is a constant.

b) If $P_0=100$ and after 6 months there are 169 fish in the lake, how many will there be after 1 year?

11. The time rate of change of an alligator population P in a swamp is proportional to the square of P. The swamp contained a dozen alligators in 1988, two dozen in 1998.

a) When will there be four dozen alligators in the swamp?

b) What happens thereafter?

12. Consider a prolific breed of rabbits whose birth and death rates, β and δ, are each proportional to the rabbit population $P=P(t)$, with $\beta>\delta$

a) Show that $P(t)=\dfrac{P_0}{1-kP_0t}$, $\quad k$ *constant*

Note that $P(t)\to+\infty \quad as \quad t\to\dfrac{1}{kP_0}$. This is doomsday

b) Suppose that $P_0=6$ and that there are nine rabbits after ten months. When does doomsday occur?

c) With $\beta<\delta$, repeat part (*a*)

d) What now happens to the rabbit population in the long run?

13. Consider a population $P(t)$ satisfying the logistic equation $\dfrac{dP}{dt}=aP-bP^2$, where $B=aP$ is the time rate at which births occur and $D=bP^2$ is the rate at which deaths occur.

a) If the initial population is $P(0)=P_0$, and B_0 births per month and D_0 deaths per month are occurring at time $t=0$, show that the limiting population is $M=\dfrac{B_0P_0}{D_0}$.

b) If the initial population is 120 rabbits and there are 8 births per month and 6 deaths per month occurring at time $t=0$, how many months does it take for $P(t)$ to reach 95% of the limiting population *M*?

c) If the initial population is 240 rabbits and there are 9 births per month and 12 deaths per month occurring at time $t=0$, how many months does it take for $P(t)$ to reach 105% of the limiting population *M*?

14. The amount of drug in the blood of a patient (in *mg*) due to an intravenous line is governed by the initial value problem

$$y'(t)=-0.02y+3, \quad y(0)=0 \ \ for\ t\ge 0$$

Where t is measured in hours

a) Find and graph the solution of the initial value problem.

b) What is the steady-state level of the drug?

c) When does the drug level reach 90% of the steady-state value?

15. A fish hatchery has 500 *fish* at time $t = 0$, when harvesting begins at a rate of b *fish/yr*. where $b > 0$. The fish population is modeled by the initial value problem.

$$y'(t) = 0.1y - b, \quad y(0) = 500 \qquad for \ \ t \geq 0$$

Where t is measured in years.

a) Find the fish population for $t \geq 0$ in terms of the harvesting rate b.

b) Graph the solution in the case that $b = 40\ fish\ /\ yr$. Describe the solution.

c) Graph the solution in the case that $b = 60\ fish\ /\ yr$. Describe the solution.

16. A community of hares on an island has a population of 50 when observations begin at $t = 0$. The population for $t \geq 0$ is modeled by the initial value problem.

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{200}\right), \quad P(0) = 50$$

a) Find the solution of the initial value problem.

b) What is the steady-state population?

17. When an infected person is introduced into a closed and otherwise healthy community, the number of people who become infected with the disease (in the absence of any intervention) may be modeled by the logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{A}\right), \quad P(0) = P_0$$

Where k is a positive infection rate, A is the number of people in the community, and P_0 is the number of infected people at $t = 0$. The model assumes no recovery or intervention.

a) Find the solution of the initial value problem in terms of k, A, and P_0.

b) Graph the solution in the case that $k = 0.025, \quad A = 300, \quad and \ \ P_0 = 1$.

c) For fixed values of k and A, describe the long-term behavior of the solutions for any P_0 with $0 < P_0 < A$

18. The reaction of chemical compounds can often be modeled by differential equations. Let $y(t)$ be the concentration of a substance in reaction for $t \geq 0$ (typical units of y are *moles/L*). The change in the concentration of a substance, under appropriate conditions, is $\frac{dy}{dt} = -ky^n$, where $k > 0$ is a rate constant and the positive integer n is the order of the reaction.

a) Show that for a first-order reaction $(n = 1)$, the concentration obeys an exponential decay law.

b) Solve the initial value problem for a second-order reaction $(n = 2)$ assuming $y(0) = y_0$

c) Graph and compare the concentration for a first-order and second-order reaction with $k = 0.1$ and $y_0 = 1$

19. The growth of cancer turmors may be modeled by the Gomperts growth equation. Let $M(t)$ be the mass of the tumor for $t \geq 0$. The relevant intial value problem is

$$\frac{dM}{dt} = -aM \ln \frac{M}{K}, \quad M(0) = M_0$$

Where a and K are positive constants and $0 < M_0 < K$

a) Graph the growth rate function $R(M) = -aM \ln \frac{M}{K}$ assuming $a = 1$ and $K = 4$. For what values of M is the growth rate positive? For what values of M is maximum?

b) Solve the initial value problem and graph the solution for $a = 1$, $K = 4$, and $M_0 = 1$. Describe the growth pattern of the tumor. Is the growth unbounded? If not, what is the limiting size of the tumor?

c) In the general equation, what is the meaning of K?

20. The halibut fishery has been modeled by the differential equation

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right)$$

Where $y(t)$ is the biomass (the total mass of the members of the population) in kilograms at time t (measured in years), the carrying capacity is estimated to be $M = 8 \times 10^7\ kg$ and $k = 0.71\ per\ year$.

a) If $y(0) = 2 \times 10^7\ kg$, find the biomass a year later.

b) How long will it take for the biomass to reach $4 \times 10^7\ kg$.

21. Suppose a population $P(t)$ satisfies $\frac{dP}{dt} = 0.4P - 0.001P^2, \quad P(0) = 50$, where t is measured in years.

a) What is the carrying capacity?

b) What is $P'(0)$?

c) When will the population reach 50% of the carrying capacity?

22. The board of directors of a corporation is calculating the price to pay for a business that is forecast to yield a continuous flow of profit of $500,000 per *year*. The money will earn a nominal rate of 5% per year compounded continuously. What is the present value of the business?

a) For 20 *years*?

b) Forever (in perpetuity)?

23. The population of a community is known to increase at a rate proportional to the number of people present at a time *t*. If the population has doubled in 6 *years*, how long it will take to triple?

24. Let population of country be decreasing at the rate proportional to its population. If the population has decreased to 25% in 10 *years*, how long will it take to be half?

25. Suppose that we have an artifact, say a piece of fossilized wood, and measurements show that the ratio of $C\!-\!14$ to carbon in the sample is 37% of the current ratio. Let us assume that the wood died at time 0, then compute the time T it would take for one gram of the radioactive carbon to decay this amount.

26. A certain radioactive material is known to decay at a rate proportional to the amount present. If initially there is $50\ mg$ of the material present and after of its original mass, find

a) An expression for the mass of the material remaining at any time t.
b) The mass of the material after 4 *hours*
c) The time at which the material has decayed to one half of its initial mass.

27. The rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in given sample. Half of the original number of radioactive nuclei have undergone disintegration in a period of 1,500 *years*.

a) What percentage of the original radioactive nuclei will remain after 4,500 *years*?
b) In how many years will only one-tenth of the original number remain?

Section 1.6 – Applications

Newton's Law of Cooling

Newton's Law of Cooling states that the rate of change of an object's temperature (***T***) is proportional to the difference between its temperature and the ambient temperature (***A***) (i.e. the temperature of its surroundings).

$$\boxed{\frac{dT}{dt} = -k(T - A)}$$

$$\frac{dT}{T-A} = -kdt$$

$$\int_{T_0}^{T} \frac{dT}{T-A} = -\int_0^t kdt$$

$$\ln|T-A| \Big|_{T_0}^{T} = -kt \Big|_0^t$$

$$\ln|T-A| - \ln|T_0 - A| = -kt$$

$$\ln\frac{|T-A|}{|T_0-A|} = -kt \qquad \textit{Quotient Rule}$$

$$\frac{T-A}{T_0-A} = e^{-kt} \qquad \Rightarrow \quad T - A = (T_0 - A)e^{-kt}$$

$$\boxed{T(t) = A + (T_0 - A)e^{-kt}}$$

$$e^{-kt} = \frac{T-A}{T_0-A}$$

$$\boxed{-kt = \ln\frac{T-A}{T_0-A}}$$

Example

A can of beer at 40° F is placed into a room when the temperature is 70° F. After 10 minutes the temperature of the beer is 50° F. What is the temperature of the beer as a function of time? What is the temperature of the beer 30 minutes after the beer was placed into the room?

Solution

Given: $T_0 = 40°F \quad \& \quad A = 70°F$

$$T(t) = 70 + (40 - 70)e^{-kt} = \underline{70 - 30e^{-kt}}$$

$$T(t = 10) = 70 - 30e^{-10k} = 50$$

$$-30e^{-10k} = 50 - 70$$

$$-30e^{-10k} = -20$$

$$e^{-10k} = \frac{2}{3}$$

$$-10k = \ln\frac{2}{3}$$

$$k = \frac{\ln\frac{2}{3}}{-10} = 0.0405$$

$$T(t) = 70 - 30e^{-0.0405t}$$

$$T(t = 30) = 70 - 30e^{-0.0405(30)}$$

$$\underline{= 61.1 \ \ °F|}$$

Mathematical Modeling

Example

A 1000-*gallon* cylindrical tank, initially full of water, develops a leak at the bottom. Suppose that the eater drains off a rate proportional to the product of the time elapsed and the amount of water present. Let $A(t)$ be the amount of water in the tank at time *t*.

a) Give the mathematical model (initial-value problem) which describes the process.

b) Find the solution

c) Given that 200 *gallons* of water leak out in the first 10 *minutes*, find the amount of water, $A(t)$, left in the tank *t* minutes after the leak develops.

Solution

a) $\frac{dA}{dt} = ktA$

$$\int \frac{dA}{A} = \int ktdt$$

$$\ln A = \frac{1}{2}kt^2 + \ln C$$

$$\ln A - \ln C = \frac{1}{2}kt^2$$

$$\ln\frac{A}{C} = \frac{1}{2}kt^2$$

$$\frac{A}{C} = e^{\frac{1}{2}kt^2}$$

$$A(t) = Ce^{\frac{1}{2}kt^2}$$

b) $1000 = Ce^{\frac{1}{2}k(0)^2} \quad \rightarrow \quad C = 1000$

$$A(t) = 1{,}000\, e^{\frac{1}{2}kt^2}$$

c) $A(t = 10) = 1{,}000 - 200 = 800$

$$800 = 1{,}000\, e^{\frac{1}{2}k(10)^2}$$

$$0.8 = e^{50k}$$

$$50k = \ln 0.8$$

$$k = \frac{\ln 0.8}{50}$$

$$\boxed{A(t) = 1{,}000\, e^{(\ln 0.8)t^2/100}}$$

Orthogonal Trajectories

Given a one-parameter family of curves

$$F(x,\ y,\ C) = 0$$

A curve that intersects each member of the family at ***right*** angles (*orthogonally*) is called an ***orthogonal trajectory*** of the family.

If $F(x,\ y,\ C) = 0 \quad and \quad G(x,\ y,\ K) = 0$ are one-parameter families of curves such that each member of one family is an orthogonal trajectory of the other family, then the two families are said to be ***orthogonal trajectories***.

A ***procedure*** for finding a family of orthogonal trajectories

$$G(x,\ y,\ K) = 0$$

For a given family of curves $\qquad F(x,\ y,\ C) = 0$

1. Determine the differential equation for the given family $F(x,\ y,\ C) = 0$

2. Replace y' in that equation by $-\frac{1}{y'}$; the resulting equation is the differential equation for the family of orthogonal trajectories.

3. Find the general solution of the new differential equation. This is the family of orthogonal trajectories.

Example

Find the family of orthogonal trajectories of: $y^3 = Cx^2 + 2$

Solution

$$3y^2y' = 2Cx \quad \Rightarrow \quad C = \frac{3}{2}\frac{y^2y'}{x}$$

$$y^3 = \frac{3}{2}\frac{y^2y'}{x}x^2 + 2$$

$$y^3 = \frac{3}{2}xy^2y' + 2$$

Differential equation for the orthogonal family

$$y^3 = \frac{3}{2}xy^2\left(-\frac{1}{y'}\right) + 2$$

$$\left(y^3 - 2\right)\frac{dy}{dx} = -\frac{3}{2}xy^2$$

$$\int\left(y - \frac{2}{y^2}\right)dy = -\frac{3}{2}\int xdx$$

$$\frac{1}{2}y^2 + \frac{2}{y} = -\frac{3}{4}x^2 + C_1$$

$$\boxed{y^2 + \frac{8}{y} = -3x^2 + C}$$

Newton's 2^nd^ Law

The force acting on a mass is equal to the rate of change of momentum with respect to time. Momentum is defined as the product of mass and velocity ($m.v$).The force is equal to the derivative of the momentum

$$F = m\frac{dv}{dt} = ma$$

Position: $x(t) = -\frac{1}{2}gt^2 + v_0 t + x_0$

Air Resistance

$$R(x,v) = -r(x,v).v$$

R: resistance force (*has sign opposite of the velocity*)
r: is a function that is always nonnegative

- *when a ball is falling from a high altitude, the density of the air has to be taken into account.*

$$F = -mg + R(v)$$

$$m\frac{dv}{dt} = -mg - rv$$

$$\frac{dv}{dt} = -g - \frac{r}{m}v$$

$$dv = \left(-g - \frac{r}{m}v\right)dt$$

$$\frac{dv}{g + \frac{r}{m}v} = -dt$$

$$\int \frac{dv}{g + \frac{r}{m}v} = -\int dt$$

$$v(t) = Ce^{-rt/m} - \frac{mg}{r}$$

When $t \to \infty \Rightarrow v = -\frac{mg}{r}$ (*Terminal Velocity*)

$\boxed{x(t) = -\frac{mC}{r}e^{-rt/m} - \frac{mg}{r}t + A}$ (A: is a constant)

Example

Suppose you drop a brick from the top of a building that is 250 *m* high. The brick has a mass of 2 *kg*, and the resistance force is given by $R = -4v$. How long will it take the brick to reach the ground? what will be its velocity at that time?

Solution

$$v(t) = Ce^{-rt/m} - \frac{mg}{r}$$

$$v(0) = 0 = C - \frac{mg}{r}$$

$$\Rightarrow C = \frac{mg}{r}$$

$$= \frac{2(9.8)}{4}$$

$$= 4.9$$

$$\frac{dx}{dt} = v(t)$$

$$= 4.9\left(e^{-2t} - 1\right)$$

$$\int dx = \int 4.9\left(e^{-2t} - 1\right)dt$$

$$x(t) = 4.9\left(-\tfrac{1}{2}e^{-2t} - t\right) + A$$

$$x(0) = 250 = 4.9\left(-\tfrac{1}{2}e^{-2(0)} - (0)\right) + A$$

$$250 = 4.9\left(-\tfrac{1}{2}\right) + A$$

$$250 = -2.45 + A$$

$$A = 252.45$$

$$x(t) = 4.9\left(-\tfrac{1}{2}e^{-2t} - t\right) + 252.45$$

$x(t) = 0 \quad \Rightarrow \quad \underline{t = 51.52\ sec}$ (Using software to solve it)

$$v(t) = 4.9\left(e^{-2t} - 1\right)$$

$$\underline{v(t = 51.52) \approx -4.9\ m/s}$$

Finding the displacement

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}.v$$

$$F = -mg + R(v) \qquad R = -k|v|.v$$

$$m\frac{dv}{dt} = -mg - k|v|v$$

$$\frac{dv}{dt} = -g - \frac{k}{m}v^2 = \boxed{\frac{dv}{dx}.v}$$

$$v\frac{dv}{dx} = -g - \frac{k}{m}v^2 \; \boxed{= -\frac{mg + kv^2}{m}}$$

Example

A ball of mass $m = 0.2\ kg$ is protected from the surface of the earth, with velocity $v_0 = 50\ m/s$. Assume that the force of air resistance is given by $R = -k|v|.v$, where $k = 0.02$. What is the maximum height reached by the ball?

Solution

$$v\frac{dv}{dx} = -\frac{mg + kv^2}{m}$$

$$\frac{vdv}{mg + kv^2} = -\frac{dx}{m}$$

$$\int_{v_0}^{0} \frac{vdv}{mg + kv^2} = -\int_0^{x_{\max}} \frac{dx}{m}$$

$$d\left(mg + kv^2\right) = 2kvdv \quad \Rightarrow \quad \frac{d\left(mg+kv^2\right)}{2k} = vdv$$

$$\frac{1}{2k}\int_{v_0}^{0} \frac{d\left(mg + kv^2\right)}{mg + kv^2} = -\int_0^{x_{\max}} \frac{dx}{m}$$

$$\frac{1}{2k}\ln\left|mg + kv^2\right|_{50}^{0} = -\frac{x}{m}\Big|_0^{x_{max}}$$

$$\frac{1}{2k}\left[\ln\left(mg\right) - \ln\left(mg + kv^2\right)\right] = -\frac{x_{max}}{m}$$

$$x_{max} = \frac{m}{2k}\left[\ln\left(mg + kv^2\right) - \ln\left(mg\right)\right]$$

$$= \frac{0.2}{2(.02)}\left[\ln\left(\frac{0.2(9.8)+(.02)(50)^2}{0.2(9.8)}\right)\right]$$

$$\boxed{= 16.4\ m}$$

Exercises *Section* 1.6 – Applications

1. Suppose that a corpse was discovered in a motel room at midnight and its temperature was $80^\circ\ F$. The temperature of the room is kept constant at $60^\circ\ F$. Two hours later the temperature of the corpse dropped to $75^\circ\ F$. Find the time of death. (Temperature of a corpse at time of death is $98.6^\circ\ F$)

2. Suppose that a corpse was discovered at 10 PM and its temperature was $85^\circ\ F$. Two hours later, its temperature is $74^\circ\ F$. If the ambient temperature is $68^\circ\ F$. Estimate the time of death.

3. A thermometer reading 100°*F* is placed in a medium having a constant temperature of 70°*F*. After 6 *min*, the thermometer reads 80°*F*. What is the reading after 20 *min*?

4. Blood plasma is stored at 40°*F*. Before the plasma can be used, it must be at 90°*F*. When the plasma is placed in an oven at 120°*F*, it takes 45 *min* for the plasma to warm to 90°*F*. How long will it take for the plasma to warm to 90°*F* if the oven temperature is set at:

a) 100°*F*.
b) 140°*F*.
c) 80°*F*.

5. A pot of boiling water at 100°*C* is removed from a stove at time $t = 0$ and left to cool in the kitchen. After 5 *min*, the water temperature has decreased to 80°*C*, and another 5 *min* later it has dropped to 65°*C*. Assuming Newton's law for cooling, determine the (constant) temperature of the kitchen.

6. A murder victim is discovered at midnight and the temperature of the body is recorded at 31°*C*. One hour later, the temperature of the body is 29°*C*. Assume that the surrounding air temperature remains constant at 21°*C*. Use Newton's law of cooling to calculate the victim's time of death. *Note*: The normal temperature of a living human being is approximately 37°*C*.

7. Suppose a cold beer at 40°*F* is placed into a warn room at 70°*F*. suppose 10 *minutes* later, the temperature of the beer is 48°*F*. Use Newton's law of cooling to find the temperature 25 *minutes* after the beer was placed into the room.

8. A thermometer is removed from a room where the temperature is $70^\circ\ F$ and is taken outside, where the air temperature is $10^\circ\ F$. After one-half minute the thermometer reads $50^\circ\ F$.

a) What is the reading of the thermometer at $t = 1\ min$?
b) How long will it take for the thermometer to reach $15^\circ\ F$?

9. A thermometer is taken from an inside room to the outside, where the air temperature is $5^\circ\ F$. After 1 *minute* the thermometer reads $55^\circ\ F$, and after 5 *minutes* the thermometer reads $30^\circ\ F$. What is the initial temperature of the inside room?

10. The temperature inside a house is $70° F$. A thermometer is taken outside after being inside the house for enough time for it to read $70° F$. The outside air temperature is $10° F$. After three *minutes* the thermometer reading is found to be $25° F$. Find the reading on the thermometer as a function of time.

11. A metal bar at a temperature of $100° F$ is placed in a room at a constant temperature of $0° F$. If after 20 minutes the temperature of the bar is $50° F$.

a) Find the time it will take the bar to reach a temperature of $25° F$

b) Find the temperature of the bar after 10 *minutes*.

12. A small metal bar, whose initial temperature was $20° C$, is dropped into a large container of boiling water.

a) How long will it take the bar to reach $90° C$ if it is known that its temperature increases $2°$ in 1 *second*?

b) How long will it take the bar to reach $98° C$

13. Two large containers ***A*** and ***B*** of the same size are filled with different fluids. The fluids in containers ***A*** and ***B*** are maintained at $0° C$ and $100° C$, respectively. A small metal bar, whose initial temperature is $100° C$, is lowered into container ***A***. After 1 *minute* the temperature of the bar is $90° C$. After 2 *minutes* the bar is removed and instantly transferred to the other container. After 1 *minute* in container ***B*** the temperature of the bar rises $10° C$. How long, measured from the start of the entire process, will it take the bar to reach $99.9° C$?

14. A thermometer reading $70° F$ is placed in an oven preheated to a constant temperature. Through a glass window in the oven door, an observer records that the thermometer reads $110° F$ after $\frac{1}{2}$ *minute* and $145° F$ after 1 *minute*. How hot is the oven?

15. At $t = 0$ a sealed test tube containing a chemical is immersed in a liquid bath. The initial temperature of the chemical in the test tube is $80° F$. the liquid bath has a controlled temperature given by $T_m(t) = 100 - 40e^{-0.1t}$, $t \ge 0$, where t is measured in *minutes*.

a) Assume that $k = -0.1$, describe in words what you expect the temperature $T(t)$ of the chemical to be like in the short term. In the long term.

b) Solve the initial-value problem.

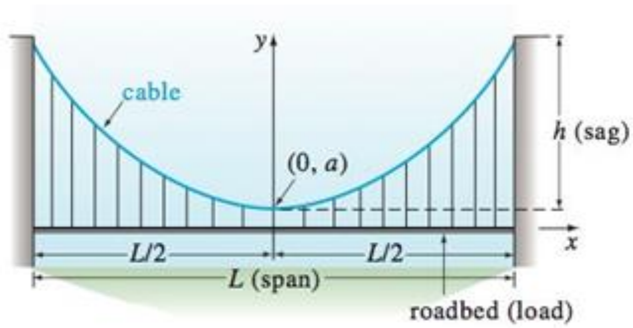
c) Graph $T(t)$.

16. The mathematical model for the shape of a flexible cable strung between two vertical supports is given by

$$\frac{dy}{dx} = \frac{W}{T_1}$$

Where W denotes the portion of the total vertical load between the points P_1 and P_2

The model is separable under the following conditions that describe a suspension bridge.



Let assume that the *x*-axis runs along the horizontal roadbed, and the *y*-axis passes through $(0,\ a)$, which is the lowest point on one cable over the span of the bridge, coinciding with the interval $\left[-\frac{L}{2},\ \frac{L}{2}\right]$.

In the case of a suspension bridge, the usual assumption is that the vertical load in the given equation is only a uniform roadbed, distributed along the horizontal axis. In other words, it is assumed that the weight of all cables is negligible in comparison to the weight of the roadbed and that the weight per unit length of the roadbed (lb/ft) is a constant ρ. Use this information to set up and solve an appropriate initial-value problem from which the shape (a curve with equation $y = \varphi(x)$) of each of the two cables in a suspension bridge is determined.

Express the solution of the IVP in terms of the sag h and span L.

17. The Brentano-Stevens Law in psychology models the way that a subject reacts to a stimulus. It states that if *R* represents the reaction to an amount *S* of stimulus, then the relative rates of increase are proportional:

$$\frac{1}{R}\frac{dR}{dt} = \frac{k}{S}\frac{dS}{dt}$$

Where k is a positive constant. Find *R* as a function of *S*.

18. Barbara weighs 60 *kg* and is on a diet of 1600 *calories* per day, of which 850 are used automatically by basal metabolism. She spends about 15 *cal/kg/day* times her weight doing exercises. If 1 *kg* of fat contains 10,000 *cal*. and we assume that the storage of calories in the form of fat is 100% efficient, formulate a differential equation and solve it to find her weight as a function of time. Does her weight ultimately approach an equilibrium weight?

19. When a chicken is removed from an oven, its temperature is measured at 300° *F*. Three minutes later its temperature is 200° *F*. How long will it take for the chicken to cool off to a room temperature of 70° *F*.

1. A body of mass *m* falls from rest subject to gravity in a medium offering resistance proportional to the square of the velocity. Determine the velocity and position of the body at *t seconds.*

2. A body of mass *m*, with initial velocity v_0 , falls vertically. If the initial position is denoted s_0 . Determine the velocity and position of the body at *t* seconds.

Assume the body acted upon by gravity alone and the air resistance proportional to the square of the velocity.

3. A body falls from a height of 300 *ft*. What distance has it traveled after 4 *sec*. if subject to *g*, the earth's acceleration?

4. A body falls from an initial velocity of $1,000$ *ft/s*. What distance has it traveled after 3 *sec*. if subject to $g = 32$ ft/s^2, the earth's acceleration?

5. A projectile is fired straight upwards with an initial velocity of $1,600$ *ft/s*. What is its velocity at $40,000$ *ft*. $\left(g = 32 \ ft/s^2\right)$

6. A projectile is fired straight upwards with an initial velocity of $1,000$ *ft/s*. What is its velocity at $8,000$ *ft*. $\left(g = 32 \ ft/s^2\right)$

7. An 8 *lb*. weight falls from rest toward earth. Assuming that the weight is acted upon by air resistance, numerically equal to $2v$, but measured in pounds, find the velocity and distance fallen after *t* seconds.

(The variable *v* represents the velocity measured in *ft/sec*.)

8. A stone is released from rest and dropped into a deep well. Eight seconds later, the sound of the stone splashing into the water at the bottom of the well returns to the ear of the person who released the stone. How long does it take the stone to drop to the bottom of the well? How deep is the well? Ignore air resistance. The speed of sound is 340 *m/s*.

9. A rocket is fired vertically and ascends with constant acceleration $a = 100 \ m/s^2$ for 1.0 *min*. At that point, the rocket motor shuts off and the rocket continues upward under the influence of gravity. Find the maximum altitude acquired by the rocket and the total time elapsed from the take-off until the rocket returns to the earth. *Ignore air resistance*.

10. A ball having mass $m = 0.1 \ kg$ falls from rest under the influence of gravity in a medium that provides a resistance that is proportional to its velocity. For a velocity of $0.2 \ m/s$ the force due to the resistance of the medium is -1 *N*. Find the terminal velocity of the ball.

1 *N* is the force required to accelerate a 1 *kg* mass at a rate of $1 \ m/s^2$: $1N = 1 \ \mathrm{kg} \cdot m/s^2$

11. A ball is projected vertically upward with initial velocity v_0 from ground level. Ignore air resistance.

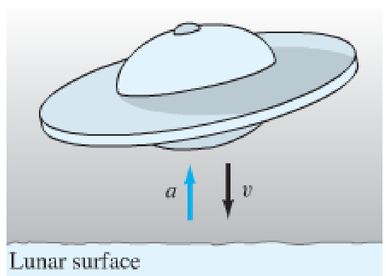
a) What is the maximum height acquired by the ball?
b) How long does it take the ball to reach its maximum height? How long does it take the ball to return to the ground? Are these times identical?
c) What is the speed of the ball when it impacts with the ground on its return?

12. An object having mass 70 *kg* falls from rest under the influence of gravity. The terminal velocity of the object is $-20\ m/s$. Assume that the air resistance is proportional to the velocity.

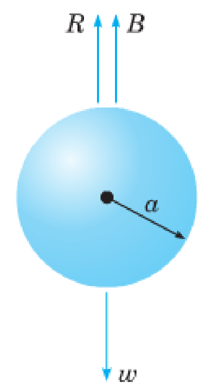
a) Find the velocity and distance traveled at the end of 2 seconds.

b) How long does it take the object to reach 80% of its terminal velocity?

13. A lunar lander is falling freely toward the surface of the moon at a speed of 450 *m/s*. Its retrorockets, when fired, provide a constant deceleration of 2.5 m/s^2 (the gravitational acceleration produced by the moon is assumed to be included in the given acceleration). At What height above the lunar surface should the retrorockets be activated to ensure a "soft touchdown? ($v = 0$ at impact)?



14. A body falling in a relatively dense fluid, oil for example, is acted on by three forces: a resistance force *R*, a buoyant force *B*, and its weight *w* due to gravity. The buoyant force is equal to the weight of the fluid displaced by the object. For a slowly moving spherical body of radius a, the resistive force is given by Stokes's law $R = 6\pi \dfrac{\mu a}{v}$, where v is the velocity of the body, and μ is the coefficient of viscosity of the surrounding fluid?

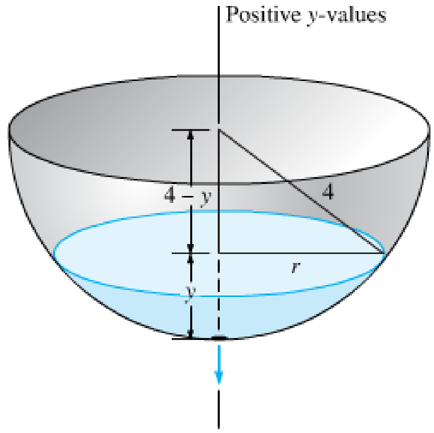


a) Find the limiting velocity of a solid sphere of radius a and density ρ falling freely in a medium of density ρ' and coefficient of viscosity μ.

b) In 1910 R. A. Millikan studied the motion of tiny droplets of oil falling in an electric field. A field of strength *E* exerts a force E_e on a droplet with charge e. Assume that *E* has been adjusted so the droplet is held stationary $(v = 0)$ and that *w* and *B* are as given. Find an expression for *e*.

15. Suppose that the tank has a radius of 3 *ft*. and that its bottom hole is circular with radius 1 *in*. How long will it take the water (initially 9 *ft*. deep) to drain completely?

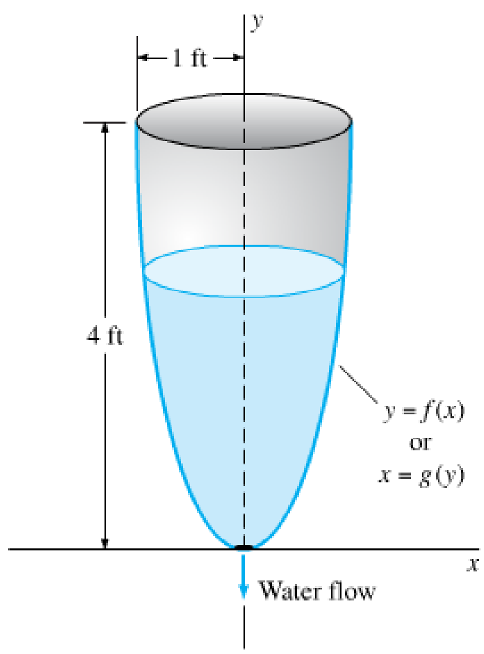
16. A hemispherical bowl has top radius of 4 *ft*. and at time $t = 0$ is full of water. At that moment a circular hole with diameter 1 *in*. is opened in the bottom of the tank. How long will it take for all the water to drain from the tank?



17. At time $t = 0$ the bottom plug (at the vertex) of a full conical water tank 16 *feet* high is removed. After 1 *hr* the water in the tank is 9 *feet* deep. When will the tank be empty?

18. Suppose that a cylindrical tank initially containing V_0 gallons of water drains (through a bottom hole) in T minutes. Use Torricelli's law to show that the volume of water in the tank after $t \le T$ minutes is $V = V_0\left(1 - \frac{t}{T}\right)^2$

19. The clepsydra, or water clock – A 12-*hr* water clock is to be designed with the dimensions, shaped like the surface obtained by revolving the curve $y = f(x)$ around the y-axis. What should be this curve, and what should be the radius of the circular bottom hole, in order that the water level will fall at the constant rate of 4 inches per hour?



20. Many chemical reactions are the result of the interaction of 2 molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If *a* is the amount of substance *A* and *b* is the substance *B* at time *t* = 0, and if *x* is the amount of product at time *t*, then the rate of formation of *x* may be given by the differential equation

$$\frac{dx}{dt} = k(a-x)(b-x) \quad or \quad \frac{1}{(a-x)(b-x)}\frac{dx}{dt} = k$$

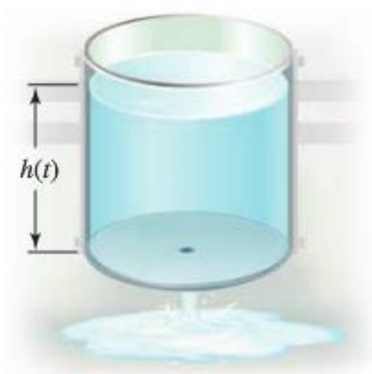
Where *k* is a constant for the reaction. Integrate both sides of this equation to obtain a relation between *x* and *t*.

a) If $a = b$

b) If $a \neq b$

Assume in each case that $x = 0$ when $t = 0$

21. An open cylindrical tank initially filled with water drains through a hole in the bottom of the tank according to Torricelli's Law. If $h(t)$ is the depth of water in the tank for $t \geq 0$, then Torricelli's Law implies $h'(t) = -2k\sqrt{h}$, where *k* is a constant that includes the acceleration due to gravity, the radius of the tank, and the radius of the drain. Assume that the initial depth of the water is $h(0) = H$



a) Find the solution of the initial value problem.

b) Find the solution in the case that $k = 0.1$ and $H = 0.5\ m$.

c) In general, how long does it take the tank to drain in terms of *k* and *H*?

22. An object in free fall may be modeled by assuming that the only forces at work are the gravitational force and resistance (friction due to the medium in which the objects falls). By Newton's second law (mass × acceleration = the sum of the external forces), the velocity of the object satisfies the differential equation

$$\underset{mass}{m} \cdot \underset{acceleration}{v'(t)} = \underbrace{mg + f(v)}_{external\ force}$$

Where f is a function that models the resistance and the positive direction is downward. One common assumption (often used for motion in air) is that $f(v) = -kv^2$, where $k > 0$ is a drag coefficient.

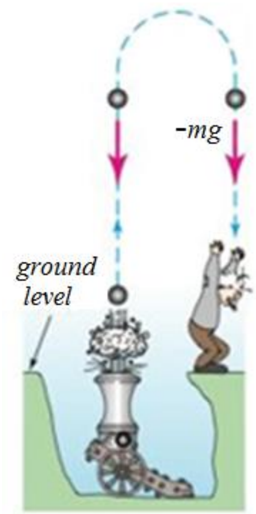
a) Show that the equation can be written in the form $v'(t) = g - av^2$ where $a = \frac{k}{m}$

b) For what (positive) value of v is $v'(t)=0$? (This equilibrium solution is called the ***terminal velocity***.)

c) Find the solution of this separable equation assuming $v(0)=0$ *and* $0<v(t)^2<\dfrac{g}{a}$ for $t\geq 0$

d) Graph the solution found in part (***c***) with $g=9.8\ m/s^2,\ m=1\ kg,\ and\ k=0.1\ kg/m$, and verify the terminal velocity agrees with the value found in part (***b***).

23. Suppose a small cannonball weighing 16 *pounds* is shot vertically upward, with an initial velocity $v_0 = 300$ *ft/s*



The answer to the question "How high does the cannonball go?" depends on whether we take air resistance into account.

a) Suppose air resistance is ignored. If the positive direction is upward, then a model for the state of the cannonball is given by $\dfrac{d^2s}{dt^2}=-g$. Since $\dfrac{ds}{dt}=v(t)$ the last differential equation is the same as $\dfrac{dv}{dt}=-g$, where we take $g=32$ *ft/s*^{2}. Find the velocity $v(t)$ of the cannonball at time *t*.

b) Use the result in part (*a*) to determine the height $s(t)$ of the cannonball measured from ground level. Find the maximum height attained by the cannonball.

24. Two chemicals *A* and *B* are combined to form a chemical *C*. The resulting reaction between the two chemicals is such that for each *gram* of *A*, 4 *grams* of *B* is used. It is observed that 30 *grams* of the compound *C* is formed in 10 *minutes*.

a) Determine the amount of *C* at time *t* if the rate of the reaction is proportional to the amounts of *A* and *B* remaining and if initially there are 50 *grams* of *A* and 32 *grams* of *B*.

b) How much of the compound *C* is present at 15 *minutes*.

c) Interpret the solution as $t\to\infty$

25. Two chemicals *A* and *B* are combined to form a chemical *C*. The rate, or velocity, of the reaction is proportional to the product of the instantaneous amounts of *A* and *B* not converted to chemical *C*. Initially, there are 40 *grams* of *A* and 50 *grams* of *B*, and each gram of *B*, 2 *grams* of *A* is used. It is observed that 10 *grams* of *C* is formed in 5 *minutes*.

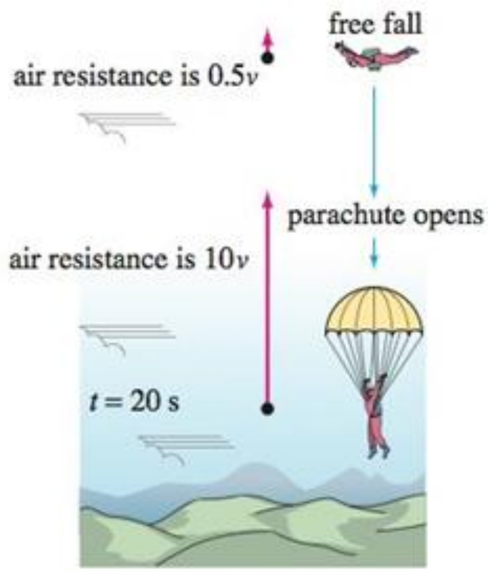
a) How much is formed in 20 *minutes*?

b) What is the limiting amount of *C* after a long time?

c) How much of chemicals *A* and *B* remains after a long time?

d) If 100 *grams* of chemical *A* is present initially, at what time is chemical *C* half-formed?

26. A skydiver weighs 125 *pounds*, and her parachute and equipment combined weigh another 35 *pounds*. After exiting from a plane at an altitude of 15,000 *feet*, she waits 15 *seconds* and opens her parachute. Assume that the constant of proportionality has the value $k=0.5$ during free fall and $k=10$ after the parachute is opened.



Assume that her initial velocity on leaving the plane is *zero*.

a) What is her velocity and how far has she traveled 20 *seconds* after leaving the plane?
b) How does her velocity at 20 *seconds* compare with her terminal velocity?
c) How long does it take her to reach the ground?

27. A tank in the form of a right-circular cylinder standing on end is leaking water through a circular hole in its bottom. When friction and contraction of water at the hole are ignored, the height h of water in the tank is described by

$$\frac{dh}{dt} = -\frac{A_h}{A_w}\sqrt{2gh}$$

Where A_w and A_h are the cross-sectional areas of the water and the hole, respectively.

a) Find $h(t)$ if the initial height of the water is *H*.

b) Sketch the graph $h(t)$ and give the interval *I* of definition in terms of the symbols A_w, A_h, and $H.\left(g = 32 \ ft/s^2\right)$

c) Suppose the tank is 10 *feet* high and has radius 2 *feet* and the circular hole has radius $\frac{1}{2}$ *inch*. If the tank is initially full, how long will it take to empty?

28. A tank in the form of a right-circular cylinder cone standing on end, vertex down, is leaking water through a circular hole in its bottom.

a) Suppose the tank is 20 *feet* high and has radius 8 *inches*. Show that the differential equation governing the height h of water leaking from a tank is

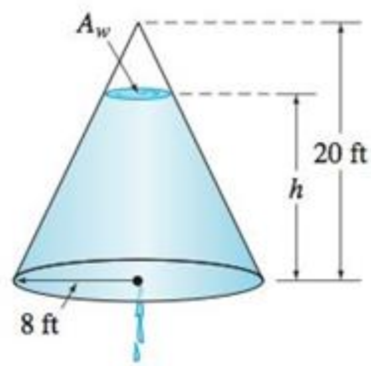
$$\frac{dh}{dt} = -\frac{5}{6h^{3/2}}$$

In this model, friction and contraction of the water at the hole were taken into account with $c = 0.6$ and $g = 32 \ ft/s^2$. If the tank is initially full, how long will it take the tank to empty?

b) Suppose the tank has a vertex angle of 60° and the circular hole has radius 2 *inches*. Determine the differential equation governing the height h of water. Use $c = 0.6$ and $g = 32$ ft/s^2.

c) If the height of the water is initially 9 *feet*, how long will it take the tank to empty?

29. Suppose that the conical tank is inverted and that water leaks out a circular hole of radius 2 *inches* in the center of its circular base. Is the time it takes to empty a full tank the same as for the tank with vertex down?



Take the friction/.contraction coefficient to be $c = 0.6$ and $g = 32$ ft/s^2

30. A differential equation for the velocity v of a falling mass m subjected to air resistance proportional to the square of the instantaneous velocity is

$$m\frac{dv}{dt} = mg - kv^2$$

Where $k > 0$ is a constant of proportionality. The positive direction is downward.

a) Solve the equation subject to the initial condition $v(0) = v_0$.

b) Use the solution in part (*a*) to determine the limiting, or terminal, velocity of the mass.

c) If the distance *s*, measured from the point where the mass was released above the ground, is related to velocity v by $\frac{ds}{dt} = v(t)$, find an explicit expression for $s(t)$ if $s(0) = 0$

31. An object is dropped from altitude y_0

a) Determine the impact velocity if the drag force is proportional to the square of velocity, with drag coefficient κ.

b) If the terminal velocity is known to −120 *mph* and the impact velocity was −90 *mph*, what was the initial altitude y_0?

32. An object is dropped from altitude y_0

a) Assume that the drag force is proportional to the velocity, with drag coefficient κ. Obtain an implicit solution relating velocity and altitude.

b) If the terminal velocity is known to −120 *mph* and the impact velocity was −90 *mph*, what was the initial altitude y_0?

33. An object of mass 3 *kg* is released from rest 500 *m* above the ground and allowed to fall under the influence of gravity. Assume the gravitational force constant, with $g = 9.81$ *m/s*^{2}, and the force due to air resistance is proportional to the velocity of the object with proportionality constant $\kappa = 3$ *N-sec/m*. Determine when the object will hit the ground.

34. A parachutist whose mass is 75 *kg* drops from helicopter hovering 4000 *m* above the ground and falls toward the earth under the influence of gravity. Assume the gravitational force is constant. Assume also that the force due to air resistance is proportional to the velocity of the parachutist, with the proportionality constant $\kappa_1 = 15$ *N-sec/m* when the chute is closed and with constant $\kappa_2 = 105$ *N-sec/m* when the chute is open. If the chute does not open until 1 *min* after the parachutist leaves the helicopter, after how many *seconds* will he reach the ground?

35. A parachutist whose mass is 75 *kg* drops from helicopter hovering 2000 *m* above the ground and falls toward the earth under the influence of gravity. Assume the gravitational force is constant. Assume also that the force due to air resistance is proportional to the velocity of the parachutist, with the proportionality constant $\kappa_1 = 30$ *N-sec/m* when the chute is closed and with constant $\kappa_2 = 90$ *N-sec/m* when the chute is open. If the chute does not open until the velocity of the parachutist reaches 20 *m/sec*, after how many seconds will he reach the ground?

36. An object of mass 5 *kg* is released from rest 1000 *m* above the ground and allowed to fall under the influence of gravity. Assume the gravitational force constant, with $g = 9.8$ *m/s*^{2}, and the force due to air resistance is proportional to the velocity of the object with proportionality constant $\kappa = 50$ *N-sec/m*. Determine when the object will hit the ground.

37. An object of mass 500 *kg* is released from rest 1000 *m* above the ground and allowed to fall under the influence of gravity. Assume the gravitational force constant, with $g = 9.8$ *m/s*^{2}, and the force due to air resistance is proportional to the velocity of the object with proportionality constant $\kappa = 50$ *N-sec/m*. Determine when the object will hit the ground.

38. A 400-*lb* object is released from rest 500 *ft* above the ground and allowed to fall under the influence of gravity. Assuming that the force in pounds due to air resistance is $-10v$, where v is the velocity of the object in *ft/s*, determine the equation of motion of the object. When will the object hit the ground?

39. An object of mass 8 *kg* is given an upward initial velocity of 20 *m/sec* and then allowed to fall under the influence of gravity. Assume that the force in Newton due to air resistance is $-16v$, where v is the velocity of the object in *m/sec*.

a) Determine the equation of motion of the object.

b) If the object is initially 100 *m* above the ground, determine when the object will hit the ground.

40. An object of mass 5 *kg* is given an downward initial velocity of 50 *m/sec* and then allowed to fall under the influence of gravity. Assume that the force in Newton due to air resistance is $-10v$, where v is the velocity of the object in *m/sec*.

a) Determine the equation of motion of the object.

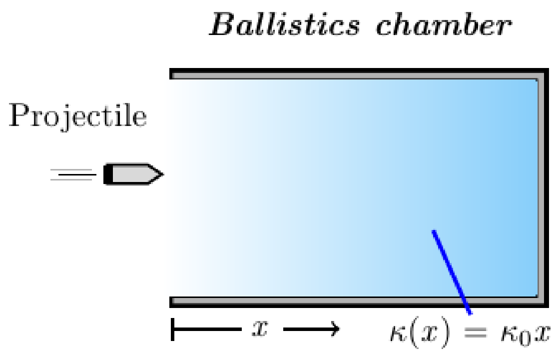
b) If the object is initially 100 *m* above the ground, determine when the object will hit the ground.

41. A shell of mass 2 *kg* is shot upward with an initial velocity of 200 *m/sec*. The magnitude of the force on the shell due to air resistance is $\frac{|v|}{20}$.

a) When will the shell reach its maximum height above the ground?

b) What is the maximum height?

42. We need to design a ballistics chamber to decelerate test projectiles fired into it. Assume the resistive force encountered by the projectile is proportional to the square of its velocity and neglect gravity.



The chamber is to be constructed so that the coefficient κ associated with this resistive force is not constant but is, in fact, a linearly increasing function of distance into the chamber:

Let $\kappa(x) = \kappa_0 x$, where κ_0 is a constant; the resistive force then has the form $\kappa(x)v^2 = \kappa_0 xv^2$.

If we use time *t* as the independent variable, Newton's law of motion leads us to the differential equation

$$m\frac{dv}{dt} + \kappa_0 xv^2 = 0 \quad with \quad v = \frac{dx}{dt}$$

a) Adopt distance x into the chamber as the new independent variable and rewrite the given differential equation as a first order equation in terms of the new independent variable.

b) Determine the value κ_0 needed if the chamber is to reduce projectile velocity to 1% of its incoming value within *d units* of distance.

43. When the velocity v of an object is very large, the magnitude of the force due to air resistance is proportional to v^2 with the force acting in opposition to the motion of the object. A shell of mass 3 *kg* is hot upward from the ground with an initial velocity of 500 *m/sec*. If the magnitude of the force due to air resistance is $(0.1)v^2$.

a) When will the shell reach its maximum height above the ground?

b) What is the maximum height?

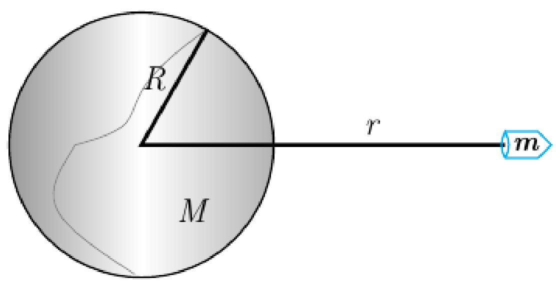
44. A sailboat has been running (on a straight course) under a light wind at 1 *m/sec*. Suddenly the wind picks up, blowing hard enough to apply a constant force of 600 *N* to the sailboat. The only other force acting on the boat is water resistance that is proportional to the velocity of the boat. If the proportionality constant for water resistance is $\kappa = 100$ *N-sec/m* and the mass of the sailboat is 50 *kg*.

a) Find the equation of motion of the sailboat.
b) What is the limiting velocity of the sailboat under this wind?
c) When the velocity of the sailboat reaches 5 *m/sec*, the boat begins to rise out of the water and plane. When this happens, the proportionality constant for the water resistance drop to $\kappa = 60$ *N-sec/m*. Find the equation of motion of the sailboat.
d) What is the limiting velocity of the sailboat under this wind as it is planning?

45. According to Newton's law of gravitation, the attractive force between two objects varies inversely as the square of the distances between them. That is, $F_g = \dfrac{GM_1M_2}{r^2}$

Where M_1 and M_2 are the masses of the objects, r is the distance between them (center to center), F_g is the attractive force, and G is the constant of proportionality.

Consider ta projectile of constant mass m being fired vertically from Earth.
Let t represent time and v the velocity of the projectile.



Earth

a) Show that the motion of the projectile, under Earth's gravitational force, is governed by the equation

$$\frac{dv}{dt} = -\frac{gR^2}{r^2},$$

Where r is the distance between the projectile and the center of Earth, R is the radius of Earth, M is the mass of Earth, and $g = \dfrac{GM}{R^2}$.

b) Use the fact the $\dfrac{dr}{dt} = v$ to obtain $v\dfrac{dv}{dr} = -\dfrac{gR^2}{r^2}$

c) If the projectile leaves Earth's surface with velocity v_0, show that $v^2 = \dfrac{2gR^2}{r} + v_0^2 - 2gR$

d) Use the result of part (*c*) to how that the velocity of the projectile remains positive if and only if $v_0^2 - 2gR > 0$. The velocity $v_e = \sqrt{2gR}$ is called the escape velocity?

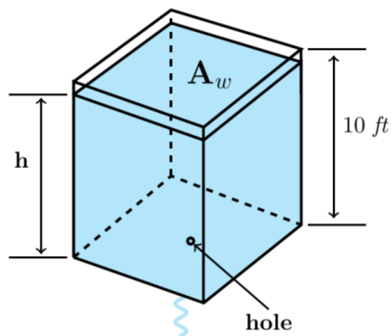
e) If $g = 9.81$ *m/sec*^{2} and $R = 6370$ *km* for Earth, what is Earth's escape velocity?

f) If the acceleration due to gravity for the Moon is $g_m = \frac{g}{6}$ and the radius of the Moon is $R_m = 1738\ km$, what is the escape velocity of the Moon?

46. A 180-*lb* skydiver drops from a hot-air balloon. After 10 *sec* of free fall, a parachute is opened. The parachute immediately introduces a drag force proportional to velocity. After an additional 4 *sec*, the parachutist reaches the ground. Assume that air resistance is negligible during free fall and that the parachute is designed so that a 200-*lb* person will reach a terminal velocity of –10 *mph*.

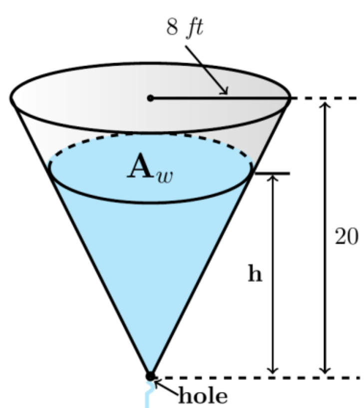
a) What is the speed of the skydiver immediately before the parachute is opened?
b) What is the parachutist's impact velocity?
c) At what altitude was the parachute opened?
d) What is the balloon's altitude?

47. Suppose water is leaking from a tank through a circular hole of area A_h at its bottom. When water leaks through a hole, friction and contraction of the stream near the hole reduce the volume of water leaving the tank per second to $cA_h\sqrt{2gh}$, where c $(0 < c < 1)$ is an empirical constant.



Determine a differential equation for the height *h* of water at time *t* for the cubical tank. The radius of the hole is 2 *in*., $g = 32$ ft/s^2, and the friction/contraction factor is $c = 0.6$

48. The right-circular tank loses water out of a circular hole at its bottom.



The radius of the hole is 2 *in*., and $g = 32$ ft/s^2, and the friction/contraction factor is $c = 0.6$.

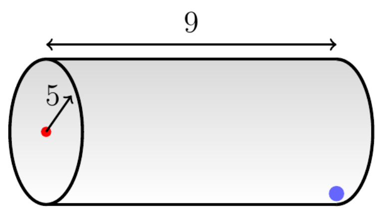
a) Determine a differential equation for the height *h* of water at time *t* for the cubical tank.
b) Find the height in function of time.

49. In meteorology, the term virga refers to falling raindrops or ice particles that evaporate before they reach the ground. Assume that a typical raindrop is spherical. Starting at some time, which we can designate as $t = 0$, the raindrop of radius r_0 falls from rest from a cloud and begins to evaporate.

a) If it is assumed that a raindrop evaporates in such a manner that its shape remains spherical, then it also makes sense to assume that the rate at which the raindrop evaporates – that is, the rate at which it loses mass – is proportional to its surface area, Show that this latter assumption implies that the rate at which the radius r of the raindrop decreases is a constant. Find $r(t)$.

b) If the positive direction is downward, construct a mathematical model for the velocity v of the falling raindrop at time $t > 0$. Ignore air resistance.

50. A horizontal cylindrical tank of length $9\ ft$, and radius $5\ ft$, is filled with oil. At $t = 0$ a plug at the lowest point of the tank is removed and a flow results.



Find y the depth of the oil in the tank at any time t while the tank is draining. The constriction coefficient is $k = \frac{1}{15}$

Section 1.7 – Direction Fields; Existence and Uniqueness

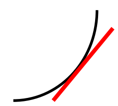
A first-order autonomous equation is an equation of the form

$$x' = f(x)$$

$$\frac{dy}{dx} = f(x, y)$$

Definition

The value $f(x, y)$ where the function f assigns to the point represent the slope of a line (line segment) call ***a lineal element***.

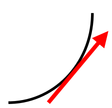


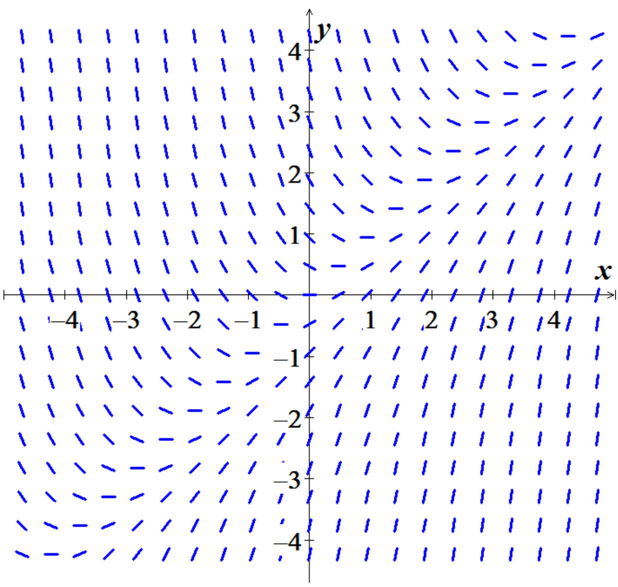
Example: Given $\frac{dy}{dx} = 0.2xy$ and consider the point $(2,\ 3)$

The slope of the lineal element is $\frac{dy}{dx} = 0.2xy = 0.2(2)(3) = 1.2$ $\quad$ (*positive sign*)

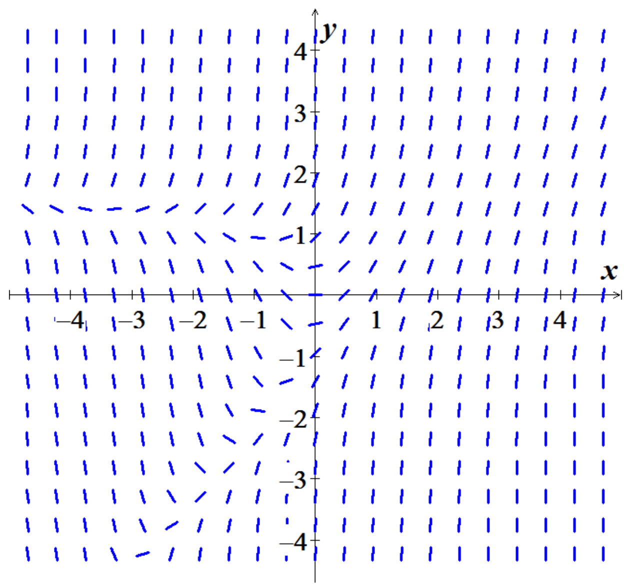
The Direction Fields

What we draw a lineal element at each point (x, y) with slope $f(x, y)$ then the collection of these lineal elements is called a ***direction field*** or a ***slope field*** of the differential equation $\frac{dy}{dx} = f(x, y)$.

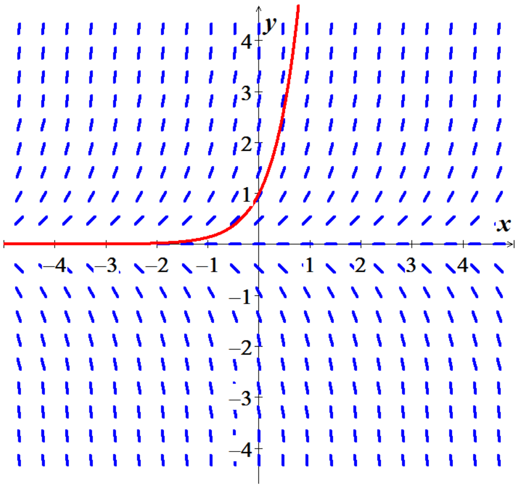
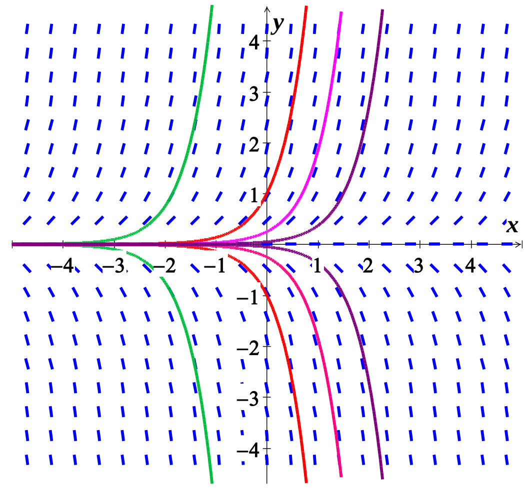




$y' = x - y$



$y' = y^2 - xy + 2x$

$y' = 2y, \quad with\ y(0) = 1 \quad \Rightarrow \quad y = e^{2x}$

Example

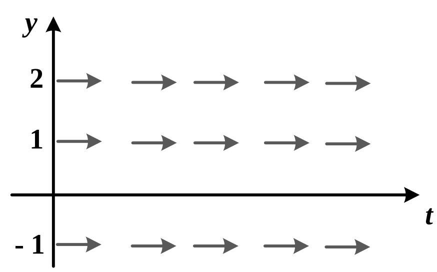
Sketch the direction field for the following differential equation. Sketch the set of integral curves for this differential equation, how the solutions behave as $t \to \infty$ and if this behavior depends on the value of $y(0)$ describe this dependency

$$y' = \left(y^2 - y - 2\right)(1-y)^2$$

Solution

$$y' = 0 \quad \Rightarrow \quad \left(y^2 - y - 2\right)(1-y)^2 = 0$$

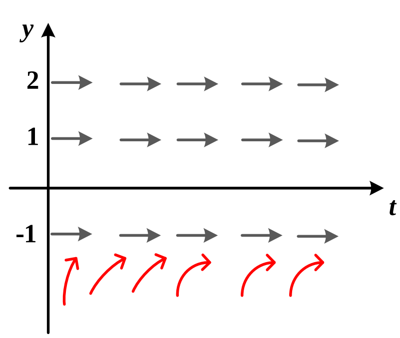
$\boxed{y = \pm 1,\ 2}$ *Slope of the tangent lines*



This divided into 4 regions.

For $y < -1$, assume $y = -2 \ \Rightarrow y' = \left(4^2 + 2 - 2\right)(1+2)^2 = 36 > \mathbf{0}$ $(\nearrow)$

$y = -1$, the slopes will flatten out while staying positive

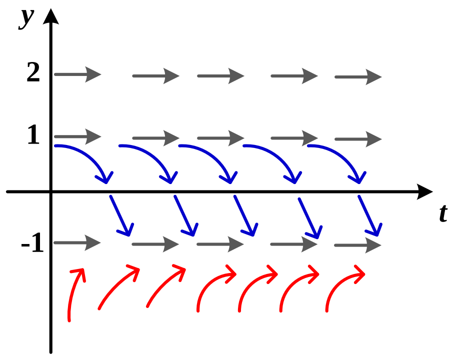


For $-1 < y < 1$, assume $y = 0 \Rightarrow y' = (-2)(1)^2 = -2 < \mathbf{0}$ $(\searrow)$

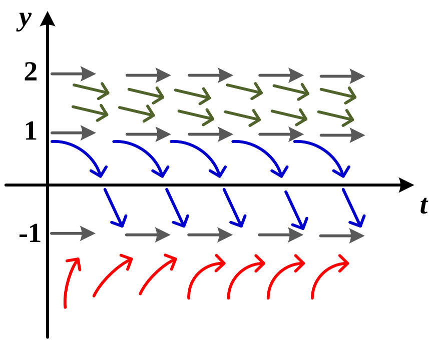
Therefore, tangent lines in this region will have negative slopes and apparently not very steep.

$y = .9 \quad \Rightarrow \quad y' = -.0209$

$y = -.9 \quad \Rightarrow \quad y' = -1.0469$ (Steeper)

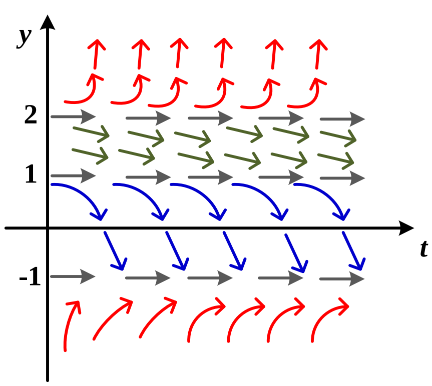


For $1 < y < 2$, assume $y = 1.5 \Rightarrow y' = \left(1.5^2 - 1.5 - 2\right)(-.5)^2 = -0.3125 < \mathbf{0}$ $(\searrow)$ Not to steep

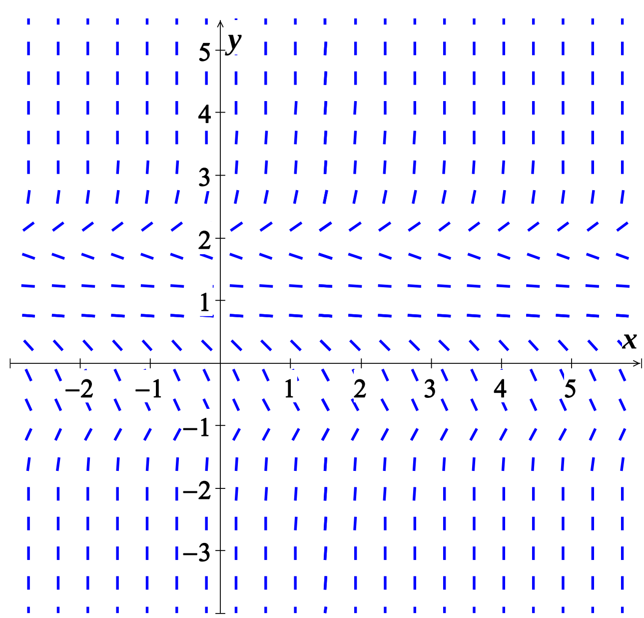


For $y > 2$, assume $y = 3 \Rightarrow y' = (4)(-2)^2 = 16 > \mathbf{0}$ $(\nearrow)$

Start out fairly flat neary = 2, then will get fairly steep.



Value of $y(0)$	$t \to \infty$
$y(0) < -1$	$y \to -1$
$-1 \le y(0) < 2$	$y \to 1$
$y(0) = 2$	$y \to 2$
$y(0) > 2$	$y \to \infty$



The questions of ***existence and uniqueness***

- When can we be sure that a solution exists**?**
- How many different solutions are there

Existence of Solutions

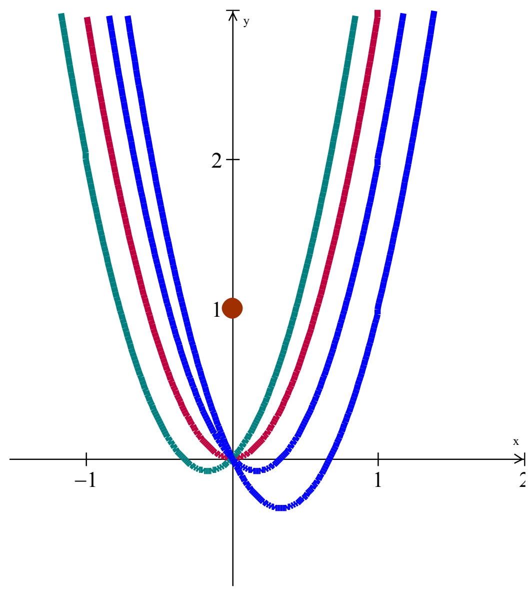
The fundamental questions in a course on differential equations are:

- Does the given initial-value problem (*IVP*) have a solution? Do solutions to the problem exists?
- If a solution does exist, is it unique? Is there exactly one solution to the problem or is there more than one solution?

Example

Consider the initial value problem: $tx' = x + 3t^2$ with $x(0) = 1$

Solution



$$x' = \frac{1}{t}x + 3t$$

$$x' = \frac{1}{t}x + 3t \qquad t \neq 0$$

There is ***no solution*** to the given initial value

$$u(t) = e^{-\int \frac{1}{t}dt}$$

$$= e^{-\ln t}$$

$$= \frac{1}{t}$$

$$\left[\frac{x}{t}\right]' = 3$$

$$\frac{x}{t} = \int 3dt$$

$$= 3t + C$$

$$\boxed{x(t) = 3t^2 + Ct}$$

Theorem: Existence of Solutions

Suppose the function $f(t,x)$ is defined and continuous on the rectangle ***R*** in the *tx*-plane. Then given any point $\left(t_0,\ x_0\right) \in \boldsymbol{R}$, the initial value problem

$$x' = f(t,x) \quad \textit{and} \quad x(t_0) = x_0$$

has a solution $x(t)$ defined in an interval containing x_0. Furthermore, the solution will be defined at least until the solution curve $t \to \left(t,\ x(t)\right)$ leaves the rectangle ***R***.

Interval of Existence of a Solution

Example

Consider the initial value problem $x' = 1 + x^2$ with $x(0) = 0$. Find the solution and its interval of existence.

Solution

The right-hand side is $f(t,x) = 1 + x^2$ which is continuous on the entire *tx*-plane.

The solution to the initial value problem is:

$$\frac{dx}{dt} = 1 + x^2$$

$$\frac{dx}{1+x^2} = dt$$

$$\int \frac{dx}{1+x^2} = \int dt$$

$$\tan^{-1} x = t$$

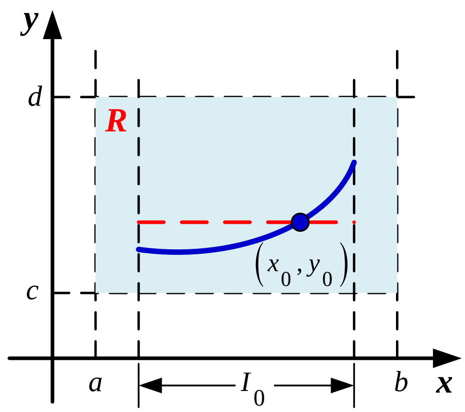
$$x(t) = \tan t$$

$x(t)$ is discontinuous at $t = \pm\frac{\pi}{2}$. Hence the solution to the initial value problem is defined only for $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

The interval: $\boxed{\left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)}$

Theorem: Existence of a Unique Solution

Let R be a rectangular region in the *xy*-plane defined by $a \le x \le b, \ \ c \le y \le d$ that contains the point $\left(x_0, y_0\right)$ in its interior. If $f\left(x, y\right)$ and $\dfrac{\partial f}{\partial y}$ are continuous on R, then there exists some interval $I_0 : \left(x_0 - h, \ x_0 + h\right), \ \ h > 0$, contained in [$a$, b], and a unique function $y\left(x\right)$, defined on I_0 that is a solution of the initial-value problem (*IVP*)



Mathematics & Theorems

Any theorem is a logical statement which has hypotheses (when it's true) and conclusions (true)

The Hypotheses of the Uniqueness of Solutions Theorem

1. The equation is in normal form $y' = f(t, y)$
2. The right-hand side $f(t, y)$ and its derivative $\dfrac{\partial f}{\partial y}$ are both continuous in the rectangle ***R***.
3. The initial point $\left(t_0, y_0\right)$ is in the rectangle ***R***.

For the uniqueness theorem the conclusions are as follows:

1- There is one and only one solution to the initial value problem.
2- The solution exists until the solution curve $t \to \left(t, \ y(t)\right)$ leaves the rectangle ***R***.

Example

Consider the initial value problem $tx' = x + 3t^2$. Is there a solution to this equation with initial condition $x(1) = 2$? If so, is the solution unique?

Solution

$$x' = \frac{x}{t} + 3t$$

The right-hand side: $f(t,x) = \frac{x}{t} + 3t$ is continuous except where $t = 0$.

We can take ***R*** to be any rectangle which contains the point $(1,\ 2)$ to avoid $t = 0$, we can choose

$\frac{1}{2} < t < 2$ and $0 < x < 4$

Then f is continuous everywhere in ***R*** $\Rightarrow$ hypotheses of the existence theorem are satisfied.

Since $\frac{\partial f}{\partial x} = \frac{1}{t}$ is also continuous in ***R***.

There is only one solution.

It is important to determine and prove a theorem concerning the existence and uniqueness of solutions of an O.D.E.

- Are the $x_{1,2} = \dfrac{1 \pm \sqrt{1-4\mu}}{2}$ solutions to $P_+ = \dfrac{1+\sqrt{1-4\mu}}{2}$ exist?

  $P_- = \dfrac{1-\sqrt{1-4\mu}}{2}$

  $\Rightarrow$ Solutions exist for the system.

- ***Uniqueness***: Assume $\mu > \frac{1}{4}$ is another solution. We want to prove $f_\mu(x)$ is actually $f_\mu(x)$ i.e.

  $\mu > \frac{1}{4} \qquad f_\mu(x)$

  $\mu > \frac{1}{4} \qquad f'_\mu$

  So that, $\frac{d}{dx}\left[f_\mu(x) \right] = 2x + \mu$, then multiply both sides by $f'_\mu(P_+) = 1 + \sqrt{1-4\mu}$ to obtain:

  $f'_\mu(P_-) = 1 - \sqrt{1-4\mu}$

Exercises *Section* 1.7 - Direction Fields; Existence and Uniqueness of Solutions

Which of the initial value problems are guaranteed a unique solution

1. $y' = 4 + y^2, \quad y(0) = 1$
2. $y' = \sqrt{y}, \quad y(4) = 0$
3. $y' = t\tan^{-1} y, \quad y(0) = 2$
4. $\omega' = \omega \sin \omega + s, \quad \omega(0) = -1$
5. $x' = \dfrac{t}{x+1}, \quad x(0) = 0$
6. $y' = \dfrac{1}{x} y + 2, \quad y(0) = 1$
7. $y' = e^t y - y^3, \quad y(0) = 0$
8. $y' = ty^2 - \dfrac{1}{3y+t}, \quad y(0) = 1$
9. $y' = xy, \quad y(0) = 1$
10. $y' = -\dfrac{t^2}{1-y^2}, \quad y(-1) = \frac{1}{2}$
11. $y' = \dfrac{y}{\sin t}, \quad y\left(\frac{\pi}{2}\right) = 1$
12. $y' = \sqrt{1-y^2}, \quad y(0) = 1$

13. Show that $y(t) = 0$ and $y(t) = t^3$ are both solutions of the initial value problem $y' = 3y^{2/3}$, where $y(0) = 0$. Explain why this fact doesn't contradict Theorem

14. Use a numerical solver to sketch the solution of the given initial value problem

$$\frac{dy}{dt} = \frac{t}{y+1}, \qquad y(2) = 0$$

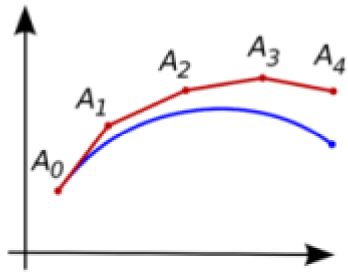
a) Where does your solver experience difficulty? why? Use the image of your solution to estimate the interval of existence.

b) Find an explicit solution; then use your formula to determine the interval of existence. How does it compare with the approximation found in part (*a*).

Section 1.8 – Numerical Methods

Euler's method named after ***Leonhard Euler*** is an example of a ***fixed-step*** solver.

Euler's method is a first-order numerical procedure for solving ordinary differential equations (***ODE***s) with a given initial value. It is the most basic kind of explicit method for numerical integration of ordinary differential equations.



$y' = f(x, y) \qquad y(x_0) = y_0$

The setting size: $h = \dfrac{b-a}{k} > 0$; $\quad k \in \mathbb{N}$

Then, $\quad x_0 = a$

$x_1 = x_0 + h = a + h$

$x_k = x_{k-1} + h = a + kh$

Last point $\quad x_k = a + kh = b$

By the definition of the derivative:

$$y'\left(x_k\right) \approx \frac{y\left(x_{k+1}\right) - y\left(x_k\right)}{h}$$

$$y'\left(x_k\right) \approx \frac{y_{k+1} - y_k}{h} = f\left(x_k, y_k\right) : \textit{slope}$$

The tangent line at the point $\left(x_0, y(x_0)\right)$ is:

$$y_{k+1} = y_k + h.f\left(x_k, y_k\right)$$

$$y_{k+1} = y_k + \Delta x_{step} . f\left(x_k, y_k\right)$$

This method is known as *Euler's Method* with step size h.

Example

Compute the first four step in the Euler's method approximation to the solution of $y' = y - x$ with $y(1) = 1$, using the step size $h = 0.1$. Compare the result with the actual solution to the initial value problem.

Solution

$y(1) = 1 \Rightarrow x_0 = 1 \;\; and \;\; y_0 = 1$

The *first* step:

$$y_1 = y_0 + h\left(y_0 - x_0\right)$$
$$= 1 + 0.1(1-1)$$
$$= 1$$
$$x_1 = x_0 + h = 1 + 0.1 = 1.1$$

The *second* step:

$$y_2 = y_1 + h\left(y_1 - x_1\right)$$
$$= 1 + 0.1(1-1.1)$$
$$= 0.99$$
$$x_2 = x_1 + h = 1.1 + 0.1 = 1.2$$

The *third* step:

$$y_3 = y_2 + h\left(y_2 - x_2\right)$$
$$= 0.99 + 0.1(0.99 - 1.2)$$
$$= 0.969$$
$$x_3 = x_2 + h = 1.2 + 0.1 = 1.3$$

The *fourth* step:

$$y_4 = y_3 + h\left(y_3 - x_3\right)$$
$$= 0.969 + 0.1(0.969 - 1.3)$$
$$= 0.9359$$
$$x_3 = x_2 + h = 1.3 + 0.1 = 1.4$$

The exact solution to $y' = y - x$ is $y(x) = 1 + x - e^{x-1}$

x_k	y_k : Euler's	y_k - exact	*Error*
1.0	1.0	1.0	0
1.1	1.0	0.9948	-0.0052
1.2	0.990	0.9786	-0.0114
1.3	0.969	0.9501	-0.0189
1.4	0.9359	0.9082	-0.0277

Runge-Kutta Methods

Like Euler's method, the Runge-Kutta methods are fixed-step solvers.

The second-Order Runge-Kutta Method

The second-Order Runge-Kutta method is also known as the improved Euler's method.

Starting from the initial value point (x_0 , y_0), we compute two slopes:

$$s_1 = f(t_0 , y_0)$$

$$s_2 = f(t_0 + h , y_0 + hs_1)$$

$$y_1 = y_0 + h\frac{s_1 + s_2}{2}$$

But an analysis using Taylor's theorem reveals that there is an improvement in the estimate for the truncation error.
For the second-Order Runge-Kutta method, we have

$$|y(t_1) - y_1| \le Mh^3$$

The constant M depends on the function $f(t, y)$.
The second-Order Runge-Kutta method is controlled by the cube of the step size instead of the square.

Input t_0 *and* y_0

For k = 1 to N

$$s_1 = f(t_{k-1} , y_{k-1})$$

$$s_2 = f(t_{k-1} + h , y_{k-1} + hs_1)$$

$$y_k = y_{k-1} + h\frac{s_1 + s_2}{2}$$

$$t_k = t_{k-1} + h$$

Example

Compute the first four step in the second-Order Runge-Kutta method approximation to the solution of $y' = y - t$ with $y(1) = 1$, using the step size $h = 0.1$. Compare the result with the actual solution to the initial value problem.

Solution

$\Rightarrow t_0 = 1 \;\; and \;\; y_0 = 1$

The *first* step:

$$\begin{aligned} s_1 &= f\left(t_0 , y_0\right) \\ &= y_0 - t_0 \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} s_2 &= f\left(t_0 + h , y_0 + hs_1\right) \\ &= \left(y_0 + hs_1\right) - \left(t_0 + h\right) \\ &= \left(1 + .1(0)\right) - \left(1 + .1\right) \\ &= 1 - 1.1 \\ &= -0.1 \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + h\frac{s_1 + s_2}{2} \\ &= 1 + 0.1\left(\frac{0 - 0.1}{2}\right) \\ &= 0.995 \end{aligned}$$

$$\begin{aligned} t_1 &= t_0 + h \\ &= 1 + 0.1 \\ &= 1.1 \end{aligned}$$

The *second* step:

$$s_1 = y_1 - t_1 = 0.995 - 1.1 = -0.105$$

$$s_2 = \left(y_1 + hs_1\right) - \left(t_1 + h\right) = \left(0.995 + .1(-0.105)\right) - \left(1.1 + .1\right) = -.2155$$

$$y_2 = y_1 + h\frac{s_1 + s_2}{2} = .995 + 0.1\left(\frac{-.105 - .2155}{2}\right) = .978975$$

$$t_2 = t_1 + h = 1.1 + .1 = 1.2$$

The *third* step:

$$s_1 = y_2 - t_2 = 0.978975 - 1.2 = -0.221025$$

$$s_2 = \left(y_2 + hs_1\right) - \left(t_2 + h\right) = \left(0.978975 + .1(-0.221025)\right) - \left(1.2 + .1\right) = -.3431275$$

$$y_3 = y_2 + h\frac{s_1 + s_2}{2} = .978975 + 0.1\left(\frac{-.221025 - .3431275}{2}\right) = 0.9507673$$

$$t_3 = t_2 + h = 1.3$$

The *fourth* step:

$$s_1 = y_3 - t_3 = 0.9507673 - 1.3 = -0.3492327$$

$$s_2 = \left(y_3 + hs_1\right) - \left(t_3 + h\right) = \left(0.9507673 + .1(-0.3492327)\right) - \left(1.3 + .1\right) = -.48415597$$

$$y_4 = y_3 + h\frac{s_1 + s_2}{2} = .9507673 + 0.1\left(\frac{-.3492327 - .48415597}{2}\right) = 0.9090979$$

$$t_4 = t_3 + h = 1.4$$

t_k	y_k : Runge-Kutta	y_k - Exact	***Runge-Kutta Error***	***Euler's Error***
1.0	1.0	1.0	**0**	**0**
1.1	0.9950000	0.994829081	**-0.000170918**	**-0.0052**
1.2	0.9789750	0.978597241	**-0.000377758**	**-0.0114**
1.3	0.9507673	0.950141192	**-0.000626182**	**-0.0189**
1.4	0.9090979	0.908175302	**-0.000922647**	**-0.0277**

Fourth-Order Runge-Kutta Method

This method is the most commonly used solution algorithm. For most equations and systems it is suitably fast and accurate.

Starting from the initial value point $\left(t_0 \ , y_0\right)$, we compute two slopes:

$$s_1 = f\left(t_0 \ , y_0\right)$$

$$s_2 = f\left(t_0 + \frac{h}{2} \ , y_0 + \frac{h}{2} s_1\right)$$

$$s_3 = f\left(t_0 + \frac{h}{2} \ , y_0 + \frac{h}{2} s_2\right)$$

$$s_4 = f\left(t_0 + h \ , y_0 + h s_3\right)$$

$$y_1 = y_0 + h \frac{s_1 + 2s_2 + 2s_3 + s_4}{6}$$

Example

Compute the first four step in the second-Order Runge-Kutta method approximation to the solution of $y' = y - t$ with $y(1) = 1$, using the step size $h = 0.1$. Compare the result with the actual solution to the initial value problem.

Solution

$$\Rightarrow t_0 = 1 \ \ and \ \ y_0 = 1$$

The *first* step:

$$s_1 = f\left(t_0 \ , y_0\right)$$
$$= 1 - 1$$
$$= 0$$

$$s_2 = f\left(t_0 + \frac{h}{2} \ , y_0 + \frac{h}{2} s_1\right)$$
$$= f\left(1.05 \ , 1\right)$$
$$= 1 - 1.05$$
$$= -0.05$$

$$s_3 = f\left(t_0 + \frac{h}{2} \ , y_0 + \frac{h}{2} s_2\right)$$
$$= f\left(1.05 \ , .9975\right)$$
$$= .9975 - 1.05$$
$$= -0.0525$$

$$s_4 = f\left(t_0 + h \ , y_0 + h s_3\right)$$

$= f(1.1, .99475)$

$= .99475 - 1.1$

$= -0.10525$

$$y_1 = y_0 + h\frac{s_1 + 2s_2 + 2s_3 + s_4}{6}$$

$$= 1 + 0.1\left(\frac{0 + 2(-.05) + 2(-.0525) + (-.10525)}{6}\right)$$

$$= 0.99482916667$$

$$t_1 = t_0 + h$$

$$= 1.1$$

t_k	y_k : Runge-Kutta	y_k - Exact	***Runge-Kutta Error***
1.0	1.0	1.0	**0**
1.1	0.994829167	0.994829081	**-0.000000086**
1.2	0.978597429	0.978597241	**0.000000295**
1.3	0.950141502	0.950141192	**-0.000000310**
1.4	0.908175759	0.908175302	**-0.000000457**

Exercises *Section* 1.8 – Numerical Methods

Calculate the first five iterations of Euler's method with step $h = 0.1$ of

1. $y' = ty \quad y(0) = 1$

2. $z' = x - 2z \quad z(0) = 1$

3. $z' = 5 - z \quad z(0) = 0$

4. Given: $y' + 2xy = x \quad y(0) = 8$

a) Use a computer and Euler's method to calculate three separate approximate solutions on the interval $[0,\ 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

5. Given: $y' + 2y = 2 - e^{-4t} \quad y(0) = 1$

a) Solve the differential equation

b) Use Euler's method and Runge-Kutta methods to calculate three separate approximate solutions on the interval $[0,\ 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$. Plot the exact solution and approximate solutions as discrete points.

6. Given: $z' - 2z = xe^{2x} \quad z(0) = 1$

a) Use a computer and Euler's method to calculate three separate approximate solutions on the interval $[0,\ 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a third with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

7. Consider the initial value problem $y' = 12y(4 - y) \quad y(0) = 1$

Use Euler's method with step size $h = 0.04$ to sketch solution on the interval $[0,\ 2]$

8. You've seen that the error in Euler's method varies directly as the first power of the step size $\left(i.e\ \ E_h \approx \lambda h\right)$. This makes Euler's method an order to halve the error? How does this affect the number of required iterations?

9. Use Euler's method to provide an approximate solution over the given time interval using the given steps sizes. Provide a plot of ***v*** versus ***y*** for each step size

$$y'' + 4y = 0, \quad y(0) = 4, \quad y'(0) = 0, \quad [0,\ 2\pi]; \quad h = 0.1,\ 0.01,\ 0.001$$

10. Given $z' + z = \cos x \quad z(0) = 1$

a) Use a computer and Runge-Kutta method to calculate three separate approximate solutions on the interval $[0,\ 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

11. Given $x' = \dfrac{t}{x} \quad x(0) = 1$

a) Use a computer and Runge-Kutta method to calculate three separate approximate solutions on the interval $[0,\ 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

12. Consider the initial value problem $y' = \dfrac{t}{y^2} \quad y(0) = 1$

Use Runge-Kutta method with step size $h = 0.04$ to sketch solution on the interval $[0,\ 2]$

13. Consider the initial value problem $y' - y = -\frac{1}{2}e^{t/2}\sin 5t + 5e^{t/2}\cos 5t \quad y(0) = 0$

Use Runge-Kutta method with step size $h = 0.05$ to sketch solution on the interval $[0,\ 5]$