

## Section 1.5 – Transpose, Diagonal, Triangular, and Symmetric Matrices

### Transpose

#### Definition

The transpose of a matrix  $A$  is defined as the matrix that is obtained by interchanging the corresponding rows and columns in  $A$ . Then the transpose of  $A$ , denoted by  $A^T$  or  $A'$ .

*The columns of  $A^T$  are the rows of  $A$ .*

When  $A$  is an  $m$  by  $n$  matrix, the transpose is  $n$  by  $m$ :

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \quad \text{then} \quad A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

The matrix **flips over** the main diagonal. The entry in row  $i$ , column  $j$  of  $A^T$  comes from row  $j$ , column  $i$  of the original  $A$ .

$$\left(A^T\right)_{ij} = A_{ji}$$

#### Properties of Transpose

$$a) \left(A^T\right)^T = A$$

$$b) (A + B)^T = A^T + B^T$$

$$c) (A - B)^T = A^T - B^T$$

$$d) (kA)^T = kA^T$$

$$e) (AB)^T = B^T A^T$$

*The transpose of a product of any number of matrices is the product of the transposes in the reverse order.*

#### Theorem

If  $A$  is an invertible matrix, then  $A^T$  is also invertible and

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

### ***Proof***

$$\begin{aligned}A^T \left(A^{-1}\right)^T &= \left(A^{-1}A\right)^T \\&= I^T \\&= I\end{aligned}$$

$$\begin{aligned}\left(A^{-1}\right)^T A^T &= \left(AA^{-1}\right)^T \\&= I^T \\&= I\end{aligned}$$

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$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\begin{aligned}\left(A^T\right)^{-1} &= \frac{1}{ad-bc} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \\&= \begin{bmatrix} \frac{d}{ad-bc} & -\frac{c}{ad-bc} \\ -\frac{b}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}\end{aligned}$$

## ***Trace***

### ***Definition***

If  $A$  is a square matrix, then the trace of  $A$ , denoted by  $\mathbf{tr}(A)$ , is defined to be the sum of the entries on the main diagonal of  $A$ . The trace of  $A$  is undefined if  $A$  is not a square matrix.

### ***Example***

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\mathbf{tr}(A) = a_{11} + a_{22} + a_{33}$$

## Diagonal

A square matrix in which all the entries off the main diagonal are zero is called a **diagonal matrix**. A general  $n \times n$  diagonal matrix can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

A diagonal matrix is invertible iff all of its diagonal entries are nonzero; the

$$D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}$$

Powers of diagonal matrices are:

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

## Triangular Matrices

A square matrix in which all the entries above the main diagonal are zero is called **lower diagonal triangular**.

A square matrix in which all the entries below the main diagonal are zero is called **upper diagonal triangular**.

A matrix that is either upper triangular or lower triangular is called **triangular**.

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}$$

*lower diagonal triangular*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{bmatrix}$$

*upper diagonal triangular*

### Theorem

- ✓ The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular.
- ✓ The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.
- ✓ A triangular matrix is invertible iff its diagonal entries are all nonzero.
- ✓ The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.

### Example

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

### Solution

$$A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \quad AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix} \quad BA = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 5 \end{bmatrix}$$

The goal is to describe Gaussian elimination in the most useful way by looking at them closely, which are factorizations of a matrix.

**The factors are triangular matrices.**

**The factorization that comes from elimination is  $A = LU$ .**

## Symmetric Matrices

### Definition

A square matrix  $A$  is said to be **symmetric** if  $A^T = A$ . That means a square matrix must satisfies  $a_{ij} = a_{ji}$

### Example

$$A = \begin{pmatrix} 1 & -4 \\ -4 & 1 \end{pmatrix} = A^T$$

$$A = \begin{pmatrix} 6 & 5 & 1 \\ 5 & 0 & 7 \\ 1 & 7 & -1 \end{pmatrix} = A^T$$

 The **inverse** of a symmetric matrix is also **symmetric**.

### Example

Given  $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ , show that the inverse is symmetric too?


### Solution

$$A^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

### Theorem

If  $A$  and  $B$  are symmetric matrices with the same size, and if  $k$  is any scalar, then:

- a)  $A^T$  is symmetric
- b)  $A + B$  and  $A - B$  are symmetric.
- c)  $kA$  is symmetric

 If  $A$  is an invertible symmetric matrix, then  $A^{-1}$  is **symmetric**.

### Proof

Assume that  $A$  is symmetric and invertible then  $A = A^T$

$$\left(A^{-1}\right)^T = \left(A^T\right)^{-1} = A^{-1}$$

Which proves that  $A^{-1}$  is **symmetric**

✚ Multiplying  $M$  by  $M^T$  gives a symmetric matrix.

### **Proof**

The entry  $(i, j)$  of  $M^T M$ , it is the dot product of **row**  $i$  of  $M^T$  (column  $i$  of  $M$ ) with column  $j$  of  $M$ .

The  $(i, j)$  entry is the same dot product, column  $j$  with column  $i$ . so  $M^T M$  is symmetric.

The matrix  $M.M^T$  is also symmetric and  $M^T M$  is a different matrix from  $M.M^T$ .

✚ If  $A$  is an invertible symmetric matrix, then  $AA^T$  and  $A^T A$  are also invertible.

✚ Matrix  $A$  is symmetric across its main diagonal. So is  $A^{-1}$

✚ Matrix  $A$  is tridiagonal (only three nonzero diagonals). But  $A^{-1}$  is a full matrix with no zeros.  
(another reason we don't compute  $A^{-1}$ )

### **Example**

Given  $M = \begin{bmatrix} 1 & 2 \end{bmatrix}$  and  $M^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ . Find  $M^T M$  and  $M.M^T$

### **Solution**

$$\begin{aligned} M^T M &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} MM^T &= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 5 \end{bmatrix} \end{aligned}$$

### ***Symmetric in LDU***

When elimination is applied to a symmetric matrix,  $A^T = A$  is an advantage.

$$\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$
$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}_D \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_U$$

🎨 If  $A = A^T$  can be factored into  $LDU$  with no row exchanges, then  $U = L^T$ . The ***symmetric factorization of a symmetric matrix is***  $A = LDL^T$

## Exercises      Section 1.5 – Transpose, Diagonal, Triangular, and Symmetric Matrices

1. Solve  $Lc = b$  to find  $c$ . Then solve  $Ux = c$  to find  $x$ . What was  $A$ ?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

2. Find  $L$  and  $U$  for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on  $a, b, c, d$  to get  $A = LU$  with four pivots

3. Determine whether the given matrix is invertible

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

4. Find  $A^2$ ,  $A^{-2}$ , and  $A^{-k}$  by inspection

$$\begin{array}{lll} \text{a) } A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} & \text{b) } A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} & \text{c) } A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \end{array}$$

5. Decide whether the given matrix is symmetric

$$\begin{array}{lll} \text{a) } \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} & \text{b) } \begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{bmatrix} & \text{c) } \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \end{array}$$

6. Find all values of the unknown constant(s) in order for  $A$  to be symmetric

$$A = \begin{bmatrix} 2 & a - 2b + 2c & 2a + b + c \\ 3 & 5 & a + c \\ 0 & -2 & 7 \end{bmatrix}$$



7. Find a diagonal matrix  $A$  that satisfies the given condition  $A^{-2} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
8. Let  $A$  be an  $n \times n$  symmetric matrix
- Show that  $A^2$  is symmetric
  - Show that  $2A^2 - 3A + I$  is symmetric
9. Prove if  $A^T A = A$ , then  $A$  is symmetric and  $A = A^2$
10. A square matrix  $A$  is called **skew-symmetric** if  $A^T = -A$ . Prove
- If  $A$  is an invertible skew-symmetric matrix, then  $A^{-1}$  is skew-symmetric.
  - If  $A$  and  $B$  are skew-symmetric matrices, then so are  $A^T$ ,  $A + B$ ,  $A - B$ , and  $kA$  for any scalar  $k$ .
  - Every square matrix  $A$  can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.
- [Hint : Note the identity  $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$ ]
11. Suppose  $R$  is rectangular ( $m$  by  $n$ ) and  $A$  is symmetric ( $m$  by  $m$ )
- Transpose  $R^T A R$  to show its symmetric
  - Show why  $R^T R$  has no negative numbers on its diagonal.
12. If  $L$  is a lower-triangular matrix, then  $(L^{-1})^T$  is \_\_\_\_\_ Triangular
13. True or False
- The block matrix  $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$  is automatically symmetric
  - If  $A$  and  $B$  are symmetric then their product is symmetric
  - If  $A$  is not symmetric then  $A^{-1}$  is not symmetric
  - When  $A$ ,  $B$ ,  $C$  are symmetric, the transpose of  $ABC$  is  $CBA$ .
  - The transpose of a diagonal matrix is a diagonal.
  - The transpose of an upper triangular matrix is an upper triangular matrix.
  - The sum of an upper triangular matrix and a lower triangular matrix is a diagonal matrix.
  - All entries of a symmetric matrix are determined by the entries occurring on and above the main diagonal.
  - All entries of an upper triangular matrix are determined by the entries occurring on and above the main diagonal.

- j) The inverse of an invertible lower triangular matrix is an upper triangular matrix.
- k) A diagonal matrix is invertible if and only if all of its diagonal entries are positive.
- l) The sum of a diagonal matrix and a lower triangular matrix is a lower triangular matrix.
- m) A matrix that is both symmetric and upper triangular must be a diagonal matrix.
- n) If  $A$  and  $B$  are  $n \times n$  matrices such that  $A + B$  is symmetric, then  $A$  and  $B$  are symmetric.
- o) If  $A$  and  $B$  are  $n \times n$  matrices such that  $A + B$  is upper triangular, then  $A$  and  $B$  are upper triangular.
- p) If  $A^2$  is a symmetric matrix, then  $A$  is a symmetric matrix.
- q) If  $kA$  is a symmetric matrix for some  $k \neq 0$ , then  $A$  is a symmetric matrix.

14. Find 2 by 2 symmetric matrices  $A = A^T$  with these properties

- a)  $A$  is not invertible
- b)  $A$  is invertible but cannot be factored into  $LU$  (row exchanges needed)
- c)  $A$  can be factored into  $LDL^T$  but not into  $LL^T$  (because of negative  $D$ )

15. A group of matrices includes  $AB$  and  $A^{-1}$  if it includes  $A$  and  $B$ . “Products and inverses stay in the group.” Which of these sets are groups?

Lower triangular matrices  $L$  with 1's on the diagonal, symmetric matrices  $S$ , positive matrices  $M$ , diagonal invertible matrices  $D$ , permutation matrices  $P$ , matrices with  $Q^T = Q^{-1}$ . ***Invent two more matrix groups.***

16. Write  $A = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$  as the product  $EH$  of an elementary row operation matrix  $E$  and a symmetric matrix  $H$ .

17. When is the product of two symmetric matrices symmetric? Explain your answer.

18. Express  $\left((AB)^{-1}\right)^T$  in terms of  $\left(A^{-1}\right)^T$  and  $\left(B^{-1}\right)^T$

19. Find the transpose of the given matrix:  $\begin{bmatrix} 8 & -1 \\ 3 & 5 \\ -2 & 5 \\ 1 & 2 \\ -3 & -5 \end{bmatrix}$

20. Show that if  $A$  is symmetric and invertible, then  $A^{-1}$  is also symmetric.

21. Prove that  $(AB)^T = B^T A^T$

22. For the given matrix, compute  $A^T$ ,  $(A^T)^{-1}$ ,  $A^{-1}$ , and  $(A^{-1})^T$ , then compare  $(A^T)^{-1}$  and  $(A^{-1})^T$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

23. Show that a  $2 \times 2$  lower triangular matrix is invertible if and only if  $a_{11}a_{22} \neq 0$  and in this case the inverse is also lower triangular.
24. Let  $A$  be any  $2 \times 2$  diagonal matrix. Give a necessary and sufficient condition on the diagonal entries so that  $A$  has an inverse. Compute the inverse of any such matrix.