

Lecture Three

Section 3.1 – Inner Products

Definition

An **inner product** on a real vector space V is a function that associates a real number $\langle \vec{u}, \vec{v} \rangle$ with each pair of vectors in V in such a way that the following axioms are satisfied for all vectors \vec{u}, \vec{v} , and \vec{w} in V and all scalars k .

1. $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ *Symmetry axiom*
2. $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ *Additivity axiom*
3. $\langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle$ *Homogeneity axiom*
4. $\langle \vec{v}, \vec{v} \rangle \geq 0$ and $\langle \vec{v}, \vec{v} \rangle = 0$ iff $\vec{v} = 0$ *Positivity axiom*

A real vector space with an inner product is called a **real inner product space**.

$$\langle \vec{u}, \vec{u} \rangle = \vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

This is called the **Euclidean inner product** (or the **standard inner product**)

Definition

If V is a real inner product space, then the norm (or length) of a vector \vec{v} in V is denoted by $\|\vec{v}\|$ and is defined by

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$$

And the **distance** between two vectors is denoted by $d(\vec{u}, \vec{v})$ and is defined by

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{\langle \vec{u} - \vec{v}, \vec{u} - \vec{v} \rangle}$$

A vector of norm 1 is called a **unit vector**.

Theorem

If \vec{u} and \vec{v} are vectors in a real inner product space V , and if k is a scalar, then:

- a) $\|\vec{v}\| \geq 0$ with equality iff $\vec{v} = 0$
- b) $\|k\vec{v}\| = |k| \|\vec{v}\|$
- c) $d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$
- d) $d(\vec{u}, \vec{v}) \geq 0$ with equality iff $\vec{u} = \vec{v}$

Although the Euclidean inner product is the most important inner product on \mathbb{R}^n , there are various applications in which is desirable to modify it by weighing each term differently. More precisely, if w_1, w_2, \dots, w_n are positive real numbers, which we will call weighs, and if $\vec{u} = (u_1, u_2, \dots, u_n)$ and $\vec{v} = (v_1, v_2, \dots, v_n)$ are vectors in \mathbb{R}^n , then it can be shown that the formula

$$\langle \vec{u}, \vec{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \dots + w_n u_n v_n$$

Defines an inner product on \mathbb{R}^n that we call the **weighted Euclidean inner product** with weights w_1, w_2, \dots, w_n

Example

Let $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ be vectors in \mathbb{R}^2 , verify that the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 3u_1 v_1 + 2u_2 v_2$ satisfies the four inner product axioms.

Solution

$$\begin{aligned} \text{Axiom 1: } \langle \vec{u}, \vec{v} \rangle &= 3u_1 v_1 + 2u_2 v_2 \\ &= 3v_1 u_1 + 2v_2 u_2 \\ &= \langle \vec{v}, \vec{u} \rangle \end{aligned}$$

$$\begin{aligned} \text{Axiom 2: } \langle \vec{u} + \vec{v}, \vec{w} \rangle &= 3(u_1 + v_1)w_1 + 2(u_2 + v_2)w_2 \\ &= 3(u_1 w_1 + v_1 w_1) + 2(u_2 w_2 + v_2 w_2) \\ &= 3u_1 w_1 + 3v_1 w_1 + 2u_2 w_2 + 2v_2 w_2 \\ &= (3u_1 w_1 + 2u_2 w_2) + (3v_1 w_1 + 2v_2 w_2) \\ &= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \end{aligned}$$

$$\begin{aligned} \text{Axiom 3: } \langle k\vec{u}, \vec{v} \rangle &= 3(ku_1)v_1 + 2(ku_2)v_2 \\ &= k(3u_1 v_1 + 2u_2 v_2) \\ &= k\langle \vec{u}, \vec{v} \rangle \end{aligned}$$

$$\begin{aligned} \text{Axiom 4: } \langle \vec{v}, \vec{v} \rangle &= 3v_1 v_1 + 2v_2 v_2 \\ &= 3v_1^2 + 2v_2^2 \geq 0 \\ v_1 = v_2 = 0 &\text{ iff } \vec{v} = \vec{0} \end{aligned}$$

Exercises Section 3.1 – Inner Products

1. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on \mathbb{R}^2 , and let $\vec{u} = (1, 1)$, $\vec{v} = (3, 2)$, $\vec{w} = (0, -1)$, and $k = 3$. Compute the following.

$$\begin{array}{lll} a) \langle \vec{u}, \vec{v} \rangle & c) \langle \vec{u} + \vec{v}, \vec{w} \rangle & e) d(\vec{u}, \vec{v}) \\ b) \langle k\vec{v}, \vec{w} \rangle & d) \|\vec{v}\| & f) \|\vec{u} - k\vec{v}\| \end{array}$$

2. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on \mathbb{R}^2 , and let $\vec{u} = (1, 1)$, $\vec{v} = (3, 2)$, $\vec{w} = (0, -1)$ and $k = 3$. Compute the following for the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 2u_1v_1 + 3u_2v_2$.

$$\begin{array}{lll} a) \langle \vec{u}, \vec{v} \rangle & c) \langle \vec{u} + \vec{v}, \vec{w} \rangle & e) d(\vec{u}, \vec{v}) \\ b) \langle k\vec{v}, \vec{w} \rangle & d) \|\vec{v}\| & f) \|\vec{u} - k\vec{v}\| \end{array}$$

3. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on \mathbb{R}^2 , and let $\vec{u} = (3, -2)$, $\vec{v} = (4, 5)$, $\vec{w} = (-1, 6)$, and $k = -4$. Verify the following.

$$\begin{array}{ll} a) \langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle & d) \langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle \\ b) \langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle & e) \langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0 \\ c) \langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle & \end{array}$$

4. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on \mathbb{R}^2 , and let $\vec{u} = (3, -2)$, $\vec{v} = (4, 5)$, $\vec{w} = (-1, 6)$, and $k = -4$. Verify the following for the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 4u_1v_1 + 5u_2v_2$.

$$\begin{array}{ll} a) \langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle & d) \langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle \\ b) \langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle & e) \langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0 \\ c) \langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle & \end{array}$$

5. Let $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$. Show that the following are inner product on \mathbb{R}^2 by verifying that the inner product axioms hold. $\langle \vec{u}, \vec{v} \rangle = 3u_1v_1 + 5u_2v_2$

6. Show that the following identity holds for the vectors in any inner product space

$$\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$$

7. Show that the following identity holds for the vectors in any inner product space

$$\langle \vec{u}, \vec{v} \rangle = \frac{1}{4} \left(\|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 \right)$$

8. Prove that $\|k\vec{v}\| = |k| \|\vec{v}\|$