Solution   *Section* 3.3 – Gram-Schmidt Process

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\mathbb{R}^m$.

$$\vec{u}_1 = (1,\ -3), \quad \vec{u}_2 = (2,\ 2)$$

Solution

$$\vec{v}_1 = \frac{\vec{u}_1}{\left\|\vec{u}_1\right\|}$$

$$= \frac{(1,\ -3)}{\sqrt{1^2 + (-3)^2}}$$

$$= \frac{(1,\ -3)}{\sqrt{10}}$$

$$= \left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right)$$

$$\vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2 \cdot \vec{u}_1\right)\vec{u}_1$$

$$= (2,\ 2) - \left[(2,\ 2).\left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right)\right]\left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right)$$

$$= (2,\ 2) - \left[\frac{2}{\sqrt{10}} - \frac{6}{\sqrt{10}}\right]\left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right)$$

$$= (2,\ 2) - \left[-\frac{4}{\sqrt{10}}\right]\left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right)$$

$$= (2,\ 2) - \left(-\frac{4}{10},\ \frac{12}{10}\right)$$

$$= (2,\ 2) - \left(-\frac{2}{5},\ \frac{6}{5}\right)$$

$$= \left(\frac{12}{5},\ \frac{4}{5}\right)$$

$$\left\|w_2\right\| = \sqrt{\left(\frac{12}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$$

$$= \sqrt{\frac{144}{25} + \frac{16}{25}}$$

$$= \sqrt{\frac{160}{25}}$$

$$= \frac{4\sqrt{10}}{5}$$

$$\vec{v}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{4\sqrt{10}}{5}\left(\frac{12}{5},\ \frac{4}{5}\right)$$

$$= \left(\frac{48\sqrt{10}}{25},\ \frac{16\sqrt{10}}{25}\right)$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\mathbb{R}^m$.

$\vec{u}_1 = (1,\ 0),\quad \vec{u}_2 = (3,\ -5)$

Solution

$$\vec{v}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|}$$

$$= \frac{(1,\ 0)}{\sqrt{1^{2}+0^{2}}}$$

$$= (1,\ 0)$$

$$\vec{w}_2 = \vec{u}_2 - \left(\vec{u}_2 \bullet \vec{v}_1\right)\vec{v}_1$$

$$= (0,\ -5)$$

$$= (3,\ -5) - \left[(3,\ -5).(1,\ 0)\right](1,\ 0)$$

$$= (3,\ -5) - [3](1,\ 0)$$

$$= (3,\ -5) - (3,\ 0)$$

$$= (0,\ -5)$$

$$\|\vec{w}_2\| = \sqrt{0^2 + (-5)^2}$$

$$= 5$$

$$\vec{v}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{1}{5}(0,\ -5)$$

$$= (0,\ -1)$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\mathbb{R}^m$.

$$\{(1,\ 1,\ 1),\ (-1,\ 1,\ 0),\ (1,\ 2,\ 1)\}$$

Solution

$$\vec{u}_1 = \frac{(1,1,1)}{\sqrt{1^2+1^2+1^2}} \qquad\qquad \vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ 1,\ 1)}{\sqrt{3}}$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

$$= (-1,\ 1,\ 0) - \left[(-1,\ 1,\ 0) \cdot \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (-1,\ 1,\ 0) - \left[-\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + 0\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (-1,\ 1,\ 0) - 0\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (-1,\ 1,\ 0)$$

$$\|\vec{w}_2\| = \sqrt{(-1)^2 + 1^2}$$

$$= \sqrt{2}$$

$$\vec{u}_2 = \frac{(-1,\ 1,\ 0)}{\sqrt{2}} \qquad\qquad \vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$\vec{v}_3 \cdot \vec{u}_1 = (1,\ 2,\ 1) \cdot \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}$$

$$= \frac{3}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}}$$

$$= \sqrt{3}$$

$$\vec{v}_3 \cdot \vec{u}_2 = (1,\ 2,\ 1) \cdot \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= -\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} + 0$$

$$= \frac{1}{\sqrt{2}}$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$$

$$= (1,\ 2,\ 1) - \sqrt{3}\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) - \sqrt{2}\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= (1,\ 2,\ 1) - (1,\ 1,\ 1) - (-1,\ 1,\ 0)$$

$$= (1,\ 0,\ 0)$$

$$\vec{u}_3 = \frac{(1,\ 0,\ 0)}{\sqrt{1}} \qquad \vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$= (1,\ 0,\ 0)$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\boldsymbol{R}^m$.

$$\{(1,\ 1,\ 1),\ (0,\ 1,\ 1),\ (0,\ 0,\ 1)\}$$

Solution

$$\vec{u}_1 = \frac{(1,\ 1,\ 1)}{\sqrt{1+1+1}} \qquad \vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ 1,\ 1)}{\sqrt{3}}$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\vec{w}_2 = (0,\ 1,\ 1) - \left[(0,\ 1,\ 1) \cdot \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) \qquad \vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

$$= (0,\ 1,\ 1) - \left[\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (0,\ 1,\ 1) - \left[\frac{2}{\sqrt{3}}\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (0,\ 1,\ 1) - \left(\frac{2}{3},\ \frac{2}{3},\ \frac{2}{3}\right)$$

$$= \left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$\|\vec{w}_2\| = \sqrt{\left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2}$$

$$= \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{1}{9}}$$

$$= \sqrt{\frac{6}{9}}$$

$$= \frac{\sqrt{6}}{3}$$

$$\vec{u}_2 = \frac{\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}{\frac{\sqrt{6}}{3}} \qquad \vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{3}{\sqrt{6}}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$= \left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$\vec{v}_3 \bullet \vec{u}_1 = (0,\ 0,\ 1) \bullet \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}}$$

$$\vec{v}_3 \bullet \vec{u}_2 = (0,\ 0,\ 1) \bullet \left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$= \frac{1}{\sqrt{6}}$$

$$\vec{w}_3 = (0,\ 0,\ 1) - \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) - \frac{1}{\sqrt{6}}\left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right) \qquad \vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \bullet \vec{u}_1\right)\vec{u}_1 - \left(\vec{v}_3 \bullet \vec{u}_2\right)\vec{u}_2$$

$$= (0,\ 0,\ 1) - \left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right) - \left(-\frac{1}{3},\ \frac{1}{6},\ \frac{1}{6}\right)$$

$$= \left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)$$

$$\vec{u}_3 = \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}} \qquad \vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$= \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\sqrt{\frac{1}{2}}}$$

$$= \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\frac{1}{\sqrt{2}}}$$

$$= \sqrt{2}\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)$$

$$= \left(0,\ -\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\mathbb{R}^m$.

$$\{(1,\ 1,\ 1),\ (0,\ 2,\ 1),\ (1,\ 0,\ 3)\}$$

Solution

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ 1,\ 1)}{\sqrt{1+1+1}}$$

$$= \frac{(1,\ 1,\ 1)}{\sqrt{3}}$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2 \cdot \vec{u}_1\right)\vec{u}_1$$

$$= (0,\ 2,\ 1) - \left[(0,\ 2,\ 1) \cdot \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (0,\ 2,\ 1) - \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (0,\ 2,\ 1) - \frac{3}{\sqrt{3}}\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (0,\ 2,\ 1) - \sqrt{3}\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (0,\ 2,\ 1) - (1,\ 1,\ 1)$$

$$= (-1,\ 1,\ 0)$$

$$\|w_2\| = \sqrt{(-1)^2 + (1)^2 + (0)^2}$$

$$= \sqrt{2}$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\left\|\vec{w}_2\right\|}$$
$$= \frac{(-1,\ 1,\ 0)}{\sqrt{2}}$$
$$\boxed{= \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)}$$

$$\vec{v}_3 \bullet \vec{u}_1 = (1,\ 0,\ 3) \bullet \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$
$$= \frac{1}{\sqrt{3}} + \frac{3}{\sqrt{3}}$$
$$\boxed{= \frac{4}{\sqrt{3}}}$$

$$\vec{v}_3 \bullet \vec{u}_2 = (1,\ 0,\ 3) \bullet \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$
$$\boxed{= -\frac{1}{\sqrt{2}}}$$

$$\vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \bullet \vec{u}_1\right)\vec{u}_1 - \left(\vec{v}_3 \bullet \vec{u}_2\right)\vec{u}_2$$
$$= (1,\ 0,\ 3) - \frac{4}{\sqrt{3}}\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) + \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$
$$= (1,\ 0,\ 3) - \left(\frac{4}{3},\ \frac{4}{3},\ \frac{4}{3}\right) + \left(-\frac{1}{2},\ \frac{1}{2},\ 0\right)$$
$$\boxed{= \left(-\frac{5}{6},\ -\frac{5}{6},\ \frac{5}{3}\right)}$$

$$\vec{u}_3 = \frac{\vec{w}_3}{\left\|\vec{w}_3\right\|}$$
$$= \frac{1}{\sqrt{\left(-\frac{5}{6}\right)^2 + \left(-\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2}}\left(-\frac{5}{6},\ -\frac{5}{6},\ \frac{5}{3}\right)$$
$$= \frac{1}{\sqrt{\frac{25}{36} + \frac{25}{36} + \frac{25}{9}}}\left(-\frac{5}{6},\ -\frac{5}{6},\ \frac{5}{3}\right)$$
$$= \frac{1}{\frac{5}{6}\sqrt{6}}\left(-\frac{5}{6},\ -\frac{5}{6},\ \frac{5}{3}\right)$$
$$\boxed{= \left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\mathbb{R}^m$.

$$\{(2,\ 2,\ 2),\ (1,\ 0,\ -1),\ (0,\ 3,\ 1)\}$$

Solution

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(2,\ 2,\ 2)}{\sqrt{2^2+2^2+2^2}}$$

$$= \frac{(2,\ 2,\ 2)}{\sqrt{12}}$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \bullet \vec{u}_1\right)\vec{u}_1 - \left(\vec{v}_3 \bullet \vec{u}_2\right)\vec{u}_2$$

$$= (1,\ 0,\ -1) - \left[(1,\ 0,\ -1) \bullet \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (1,\ 0,\ -1) - \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (0,\ 2,\ 1) - (0)\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (1,\ 0,\ -1)$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{(1,\ 0,\ -1)}{\sqrt{2}}$$

$$= \left(\frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right)$$

$$\vec{v}_3 \bullet \vec{u}_1 = (0,\ 3,\ 1) \bullet \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= \frac{3}{\sqrt{3}} + \frac{1}{\sqrt{3}}$$

$$= \frac{4}{\sqrt{3}}$$

$$\vec{v}_3 \bullet \vec{u}_2 = (0,\ 3,\ 1) \bullet \left(\frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right)$$

$$\boxed{= -\frac{1}{\sqrt{2}}}$$

$$\vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \bullet \vec{u}_1\right)\vec{u}_1 - \left(\vec{v}_3 \bullet \vec{u}_2\right)\vec{u}_2$$

$$= (0,\ 3,\ 1) - \frac{4}{\sqrt{3}}\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right)$$

$$= (0,\ 3,\ 1) - \left(\frac{4}{3},\ \frac{4}{3},\ \frac{4}{3}\right) + \left(\frac{1}{2},\ 0,\ -\frac{1}{2}\right)$$

$$\boxed{= \left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right)}$$

$$\vec{u}_3 = \frac{\vec{w}_3}{\left\|\vec{w}_3\right\|}$$

$$= \frac{1}{\sqrt{\left(-\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(-\frac{5}{6}\right)^2}}\left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right)$$

$$= \frac{1}{\sqrt{\frac{25}{36} + \frac{25}{36} + \frac{25}{9}}}\left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right)$$

$$= \frac{1}{\frac{5}{6}\sqrt{6}}\left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right)$$

$$\boxed{= \left(-\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}}\right)}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\mathbb{R}^m$.

$$\left\{(1,\ -1,\ 0),\ (0,\ 1,\ 0),\ (2,\ 3,\ 1)\right\}$$

Solution

$$\vec{u}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \frac{(1,\ -1,\ 0)}{\sqrt{2}}$$

$$\boxed{= \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)}$$

$$\vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2 \bullet \vec{u}_1\right)\vec{u}_1$$

$$= (0,\ 1,\ 0) - \left[(0,\ 1,\ 0) \bullet \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)\right]\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)$$

$$= (0,\ 1,\ 0) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)$$

$$= (0,\ 1,\ 0) + \left(\frac{1}{2},\ -\frac{1}{2},\ 0\right)$$

$$= \left(\frac{1}{2},\ \frac{1}{2},\ 0\right)$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\left\|\vec{w}_2\right\|}$$

$$= \frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}}}\left(\frac{1}{2},\ \frac{1}{2},\ 0\right)$$

$$= \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$\vec{v}_3 \bullet \vec{u}_1 = (2,\ 3,\ 1) \bullet \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)$$

$$= \frac{2}{\sqrt{2}} - \frac{3}{\sqrt{2}}$$

$$= -\frac{1}{\sqrt{2}}$$

$$\vec{v}_3 \bullet \vec{u}_2 = (2,\ 3,\ 1) \bullet \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= \frac{4}{\sqrt{2}}$$

$$= 2\sqrt{2}$$

$$\vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \bullet \vec{u}_1\right)\vec{u}_1 - \left(\vec{v}_3 \bullet \vec{u}_2\right)\vec{u}_2$$

$$= (2,\ 3,\ 1) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right) - 2\sqrt{2}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= (2,\ 3,\ 1) + \left(\frac{1}{2},\ -\frac{1}{2},\ 0\right) - (2,\ 2,\ 0)$$

$$= \left(\frac{1}{2},\ \frac{1}{2},\ 1\right)$$

$$\vec{u}_3 = \frac{\vec{w}_3}{\left\|\vec{w}_3\right\|}$$

$$= \frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}}\left(\frac{1}{2},\ \frac{1}{2},\ 1\right)$$

$$= \frac{2}{\sqrt{6}}\left(\frac{1}{2},\ \frac{1}{2},\ 1\right)$$

$$\underline{= \left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\mathbb{R}^m$.

$$\{(3,\ 0,\ 4),\ (-1,\ 0,\ 7),\ (2,\ 9,\ 11)\}$$

Solution

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(3,\ 0,\ 4)}{\sqrt{9+16}}$$

$$\underline{= \left(\frac{3}{5},\ 0,\ \frac{4}{5}\right)}$$

$$\vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2 \bullet \vec{u}_1\right)\vec{u}_1$$

$$= (-1,\ 0,\ 7) - \left[(-1,\ 0,\ 7) \bullet \left(\frac{3}{5},\ 0,\ \frac{4}{5}\right)\right]\left(\frac{3}{5},\ 0,\ \frac{4}{5}\right)$$

$$= (-1,\ 0,\ 7) - \left(-\frac{3}{5} + \frac{28}{5}\right)\left(\frac{3}{5},\ 0,\ \frac{4}{5}\right)$$

$$= (-1,\ 0,\ 7) - 5\left(\frac{3}{5},\ 0,\ \frac{4}{5}\right)$$

$$= (-1,\ 0,\ 7) - (3,\ 0,\ 4)$$

$$\underline{= (-4,\ 0,\ 3)}$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{1}{\sqrt{16+9}}(-4,\ 0,\ 3)$$

$$\underline{= \left(-\frac{4}{5},\ 0,\ \frac{3}{5}\right)}$$

$$\vec{v}_3 \bullet \vec{u}_1 = (2,\ 9,\ 11) \bullet \left(\frac{3}{5},\ 0,\ \frac{4}{5}\right)$$

$$= \frac{6}{5} + \frac{44}{5}$$

$$\underline{= 10}$$

$$\vec{v}_3 \cdot \vec{u}_2 = (2,\ 9,\ 11) \cdot \left(-\frac{4}{5},\ 0,\ \frac{3}{5}\right)$$

$$= -\frac{8}{5} + \frac{33}{5}$$

$$\underline{= 5}$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$$

$$\vec{w}_3 = (2,\ 9,\ 11) - 10\left(\frac{3}{5},\ 0,\ \frac{4}{5}\right) - 5\left(-\frac{4}{5},\ 0,\ \frac{3}{5}\right)$$

$$\underline{= (0,\ 9,\ 0)}$$

$$\vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$= \frac{1}{9}(0,\ 9,\ 0)$$

$$\underline{= (0,\ 1,\ 0)}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\mathbb{R}^m$.

$$\{(1,\ 1,\ 1,\ 1),\ (1,\ 2,\ 1,\ 0),\ (1,\ 3,\ 0,\ 0)\}$$

Solution

$$\vec{v}_1 = \vec{u}_1$$

$$\underline{= (1,\ 1,\ 1,\ 1)}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ 1,\ 1,\ 1)}{\sqrt{4}}$$

$$\underline{= \left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2,\ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$=(1,\ 2,\ 1,\ 0)-\frac{(1,\ 2,\ 1,\ 0)\bullet(1,\ 1,\ 1,\ 1)}{\|(1,\ 1,\ 1,\ 1)\|^2}(1,\ 1,\ 1,\ 1)$$

$$=(1,\ 2,\ 1,\ 0)-\frac{1+2+1}{4}(1,\ 1,\ 1,\ 1)$$

$$=(1,\ 2,\ 1,\ 0)-\frac{4}{4}(1,\ 1,\ 1,\ 1)$$

$$=(1,\ 2,\ 1,\ 0)-(1,\ 1,\ 1,\ 1)$$

$$\boxed{=(0,\ 1,\ 0,\ -1)}$$

$$\vec{q}_2=\frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$=\frac{(0,\ 1,\ 0,\ -1)}{\sqrt{1+1}}$$

$$\boxed{=\left(0,\ \frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right)}$$

$$\vec{v}_3=\vec{u}_3-\frac{\langle\vec{u}_3,\ \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1-\frac{\langle\vec{u}_3,\ \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$=(1,\ 3,\ 0,\ 0)-\frac{(1,\ 3,\ 0,\ 0)\bullet(1,\ 1,\ 1,\ 1)}{\|(1,\ 1,\ 1,\ 1)\|^2}(1,\ 1,\ 1,\ 1)-\frac{(1,\ 3,\ 0,\ 0)\bullet(0,\ 1,\ 0,\ -1)}{\|(0,\ 1,\ 0,\ -1)\|^2}(0,\ 1,\ 0,\ -1)$$

$$=(1,\ 3,\ 0,\ 0)-\frac{4}{4}(1,\ 1,\ 1,\ 1)-\frac{3}{2}(0,\ 1,\ 0,\ -1)$$

$$=(1,\ 3,\ 0,\ 0)-(1,\ 1,\ 1,\ 1)-\left(0,\ \frac{3}{2},\ 0,\ -\frac{3}{2}\right)$$

$$\boxed{=\left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)}$$

$$\vec{q}_3=\frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$=\frac{\left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)}{\sqrt{\frac{1}{4}+1+\frac{1}{4}}}$$

$$=\frac{\left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)}{\frac{\sqrt{6}}{2}}$$

$$=\frac{2}{\sqrt{6}}\left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)$$

$$\boxed{=\left(0,\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\mathbb{R}^m$.

$$\{(0,\ 2,\ -1,\ 1),\ (0,\ 0,\ 1,\ 1),\ (-2,\ 1,\ 1,\ -1)\}$$

Solution

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(0,\ 2,\ -1,\ 1)}{\sqrt{6}}$$

$$= \left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$\vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2 \bullet \vec{u}_1\right)\vec{u}_1$$

$$= (0,\ 0,\ 1,\ 1) - \left[(0,\ 0,\ 1,\ 1) \bullet \left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)\right]\left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$= (0,\ 0,\ 1,\ 1) - \left[-\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}}\right]\left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$= (0,\ 0,\ 1,\ 1) - [0]\left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$= (0,\ 0,\ 1,\ 1)$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{(0,\ 0,\ 1,\ 1)}{\sqrt{2}}$$

$$= \left(0,\ 0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

$$\vec{v}_3 \bullet \vec{u}_1 = (-2,\ 1,\ 1,\ -1) \bullet \left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$= \frac{2}{\sqrt{6}} - \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}}$$

$$= 0$$

$$\vec{v}_3 \bullet \vec{u}_2 = (-2,\ 1,\ 1,\ -1) \bullet \left(0,\ 0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$

$$= 0$$

$$\vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \cdot \vec{u}_1\right)\vec{u}_1 - \left(\vec{v}_3 \cdot \vec{u}_2\right)\vec{u}_2$$

$$= (-2,\ 1,\ 1,\ -1) - 0 - 0$$

$$= \underline{(-2,\ 1,\ 1,\ -1)}$$

$$\vec{u}_3 = \frac{\vec{w}_3}{\left\|\vec{w}_3\right\|}$$

$$= \frac{(-2,\ 1,\ 1,\ -1)}{\sqrt{(-2)^2 + 1^2 + 1^2 + (-1)^2}}$$

$$= \frac{(-2,\ 1,\ 1,\ -1)}{\sqrt{7}}$$

$$= \underline{\left(-\frac{2}{\sqrt{7}},\ \frac{1}{\sqrt{7}},\ \frac{1}{\sqrt{7}},\ -\frac{1}{\sqrt{7}}\right)}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\mathbb{R}^m$.

$$\vec{u}_1 = (1,\ 0,\ 0),\quad \vec{u}_2 = (3,\ 7,\ -2),\quad \vec{u}_3 = (0,\ 4,\ 1)$$

Solution

$$\vec{v}_1 = \frac{\vec{u}_1}{\left\|\vec{u}_1\right\|}$$

$$= \frac{(1,\ 0,\ 0)}{\sqrt{1^2 + 0^2 + 0^2}}$$

$$= \underline{(1,\ 0,\ 0)}$$

$$\vec{w}_2 = \vec{u}_2 - \left(\vec{u}_2 \cdot \vec{v}_1\right)\vec{v}_1$$

$$= (3,\ 7,\ -2) - \left[(3,\ 7,\ -2) \cdot (1,\ 0,\ 0)\right](1,\ 0,\ 0)$$

$$= (3,7,-2) - 3(1,0,0)$$

$$= \underline{(0,\ 7,\ -2)}$$

$$\vec{v}_2 = \frac{\vec{w}_2}{\left\|\vec{w}_2\right\|}$$

$$= \frac{1}{\sqrt{53}}(0,\ 7,\ -2)$$

$$= \underline{\left(0,\ \frac{7}{\sqrt{53}},\ -\frac{2}{\sqrt{53}}\right)}$$

$$\vec{u}_3 \cdot \vec{v}_1 = (0,\ 4,\ 1) \cdot (1,\ 0,\ 0)$$

$$= 0$$

$$\vec{u}_3 \cdot \vec{v}_2 = (0,\ 4,\ 1) \cdot \left(0,\ \frac{7}{\sqrt{53}},\ -\frac{2}{\sqrt{53}}\right)$$

$$= \frac{26}{\sqrt{53}}$$

$$\vec{w}_3 = \vec{u}_3 - (\vec{u}_3 \cdot \vec{v}_1)\vec{v}_1 - (\vec{u}_3 \cdot \vec{v}_2)\vec{v}_2$$

$$= (0,\ 4,\ 1) - 0 - \frac{26}{\sqrt{53}}\left(0,\ \frac{7}{\sqrt{53}},\ -\frac{2}{\sqrt{6}}\right)$$

$$= (0,\ 4,\ 1) - 0 - \frac{26}{\sqrt{53}}\left(0,\ \frac{7}{\sqrt{53}},\ -\frac{2}{\sqrt{6}}\right)$$

$$= (0,\ 4,\ 1) - \left(0,\ \frac{182}{53},\ -\frac{52}{53}\right)$$

$$= \left(0,\ \frac{30}{53},\ \frac{105}{53}\right)$$

$$\vec{v}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$= \frac{\left(0,\ \frac{30}{53},\ \frac{105}{53}\right)}{\sqrt{\left(\frac{30}{53}\right)^2 + \left(\frac{105}{53}\right)^2}}$$

$$= \frac{53}{\sqrt{11925}}\left(0,\ \frac{30}{53},\ \frac{105}{53}\right)$$

$$= \frac{53}{15\sqrt{53}}\left(0,\ \frac{30}{53},\ \frac{105}{53}\right)$$

$$= \left(0,\ \frac{2}{\sqrt{15}},\ \frac{7}{\sqrt{15}}\right)$$

Exercise

Use the Gram-Schmidt process to find an *orthonormal* basis for the subspaces of $\mathbb{R}^m$.

$$\vec{u}_1 = (1, 1, 1, 1), \quad \vec{u}_2 = (1, 2, 4, 5), \quad \vec{u}_3 = (1, -3, -4, -2)$$

Solution

$$\vec{v}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|}$$

$$= \frac{(1, 1, 1, 1)}{\sqrt{4}}$$

$$\boxed{= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}$$

$$\vec{w}_2 = \vec{u}_2 - (\vec{u}_2 \bullet \vec{v}_1)\vec{v}_1$$

$$= (1, 2, 4, 5) - \left[(1, 2, 4, 5) \bullet \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right]\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$= (1, 2, 4, 5) - 6\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$= (1, 2, 4, 5) - (3, 3, 3, 3)$$

$$\boxed{= (-2, -1, 1, 2)}$$

$$\vec{v}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{1}{\sqrt{10}}(-2, -1, 1, 2)$$

$$\boxed{= \left(-\frac{2}{\sqrt{10}}, -\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right)}$$

$$\vec{u}_3 \bullet \vec{v}_1 = (1, -3, -4, -2) \bullet \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$= \frac{1-3-4-2}{2}$$

$$\boxed{= -4}$$

$$\vec{u}_3 \bullet \vec{v}_2 = (1, -3, -4, -2) \bullet \left(-\frac{2}{\sqrt{10}}, -\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right)$$

$$= \frac{-2+3-4-4}{\sqrt{10}}$$

$$\boxed{= -\frac{7}{\sqrt{10}}}$$

$$\vec{w}_3 = \vec{u}_3 - (\vec{u}_3 \bullet \vec{v}_1)\vec{v}_1 - (\vec{u}_3 \bullet \vec{v}_2)\vec{v}_2$$

$$=(1,\ -3,\ -4,\ -2)+4\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)+\frac{7}{\sqrt{10}}\left(-\frac{2}{\sqrt{10}},\ -\frac{1}{\sqrt{10}},\ \frac{1}{\sqrt{10}},\ \frac{2}{\sqrt{10}}\right)$$

$$=(1,\ -3,\ -4,\ -2)+(2,\ 2,\ 2,\ 2)+\left(-\frac{7}{5},\ -\frac{7}{10},\ \frac{7}{10},\ \frac{7}{5}\right)$$

$$=\left(\frac{8}{5},\ -\frac{17}{10},\ -\frac{27}{10},\ \frac{7}{5}\right)$$

$$\vec{v}_3=\frac{\vec{w}_3}{\left\|\vec{w}_3\right\|}$$

$$=\frac{1}{\sqrt{\frac{64}{25}+\frac{289}{100}+\frac{289}{100}+\frac{49}{25}}}\left(\frac{8}{5},\ -\frac{17}{10},\ -\frac{27}{10},\ \frac{7}{5}\right)$$

$$=\frac{1}{\sqrt{\frac{1030}{100}}}\left(\frac{8}{5},\ -\frac{17}{10},\ -\frac{27}{10},\ \frac{7}{5}\right)$$

$$=\left(\frac{16}{\sqrt{1030}},\ -\frac{17}{\sqrt{1030}},\ -\frac{27}{\sqrt{1030}},\ \frac{14}{\sqrt{1030}}\right)$$

Exercise

Use the Gram-Schmidt process to find an *orthonormal* basis for the subspaces of $\mathbb{R}^m$.

$$\vec{u}_1=(1,\ 1,\ 1,\ 1),\quad \vec{u}_2=(1,\ 1,\ 2,\ 4),\quad \vec{u}_3=(1,\ 2,\ -4,\ -3)$$

Solution

$$\vec{v}_1=\frac{\vec{u}_1}{\left\|\vec{u}_1\right\|}$$

$$=\frac{(1,\ 1,\ 1,\ 1)}{\sqrt{4}}$$

$$=\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$\vec{w}_2=\vec{u}_2-\left(\vec{u}_2\bullet\vec{v}_1\right)\vec{v}_1$$

$$=(1,\ 1,\ 2,\ 4)-\left[(1,\ 1,\ 2,\ 4)\bullet\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)\right]\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$=(1,\ 1,\ 2,\ 4)-4\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$=(1,\ 1,\ 2,\ 4)-(2,\ 2,\ 2,\ 2)$$

$$=(-1,\ -1,\ 0,\ 2)$$

$$\vec{v}_2 = \frac{\vec{w}_2}{\left\|\vec{w}_2\right\|}$$

$$= \frac{1}{\sqrt{1+1+4}}(-1,\ -1,\ 0,\ 2)$$

$$\underline{= \left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0,\ \frac{2}{\sqrt{6}}\right)}$$

$$\vec{u}_3 \bullet \vec{v}_1 = (1,\ 2,\ -4,\ -3) \bullet \left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$= \frac{1+2-4-3}{2}$$

$$\underline{= -2}$$

$$\vec{u}_3 \bullet \vec{v}_2 = (1,\ 2,\ -4,\ -3) \bullet \left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0,\ \frac{2}{\sqrt{6}}\right)$$

$$= \frac{-1-2-6}{\sqrt{6}}$$

$$\underline{= -\frac{9}{\sqrt{6}}}$$

$$\vec{w}_3 = \vec{u}_3 - (\vec{u}_3 \bullet \vec{v}_1)\vec{v}_1 - (\vec{u}_3 \bullet \vec{v}_2)\vec{v}_2$$

$$= (1,\ 2,\ -4,\ -3) + 2\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right) + \frac{9}{\sqrt{6}}\left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0,\ \frac{2}{\sqrt{6}}\right)$$

$$= (1,\ 2,\ -4,\ -3) + (1,\ 1,\ 1,\ 1) + \left(-\frac{3}{2},\ -\frac{3}{2},\ 0,\ 3\right)$$

$$\underline{= \left(\frac{1}{2},\ \frac{3}{2},\ -3,\ 1\right)}$$

$$\vec{v}_3 = \frac{\vec{w}_3}{\left\|\vec{w}_3\right\|}$$

$$= \frac{1}{\sqrt{\frac{1}{4}+\frac{9}{4}+9+1}}\left(\frac{1}{2},\ \frac{3}{2},\ -3,\ 1\right)$$

$$= \frac{2}{5\sqrt{2}}\left(\frac{1}{2},\ \frac{3}{2},\ -3,\ 1\right)$$

$$\underline{= \left(\frac{1}{5\sqrt{2}},\ \frac{3}{5\sqrt{2}},\ -\frac{6}{5\sqrt{2}},\ \frac{2}{5\sqrt{2}}\right)}$$

Exercise

Use the Gram-Schmidt process to find an *orthonormal* basis for the subspaces of $\mathbb{R}^m$.

$$\vec{u}_1 = (0,\ 2,\ 1,\ 0);\quad \vec{u}_2 = (1,\ -1,\ 0,\ 0);\quad \vec{u}_3 = (1,\ 2,\ 0,\ -1);\quad \vec{u}_4 = (1,\ 0,\ 0,\ 1)$$

Solution

$$\vec{v}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|}$$

$$= \frac{(0,\ 2,\ 1,\ 0)}{\sqrt{5}}$$

$$= \left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)$$

$$\vec{w}_2 = \vec{u}_2 - (\vec{u}_2 \bullet \vec{v}_1)\vec{v}_1$$

$$= (1, -1,\ 0,\ 0) - \left[(1, -1,\ 0,\ 0) \bullet \left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)\right]\left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)$$

$$= (1, -1,\ 0,\ 0) - \left(-\frac{2}{\sqrt{5}}\right)\left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right).$$

$$= \left(1,\ -\frac{1}{5},\ \frac{2}{5},\ 0\right)$$

$$\vec{v}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{1}{\sqrt{1+\frac{1}{25}+\frac{4}{25}+0}}\left(1,\ -\frac{1}{5},\ \frac{2}{5},\ 0\right)$$

$$= \frac{5}{\sqrt{30}}\left(1, -\frac{1}{5},\ \frac{2}{5},\ 0\right)$$

$$= \left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right)$$

$$u_3 \bullet v_1 = (1,\ 2,\ 0,\ -1) \bullet \left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)$$

$$= \frac{4}{\sqrt{5}}$$

$$u_3 \bullet v_2 = (1,\ 2,\ 0,\ -1) \bullet \left(\frac{5}{\sqrt{30}},\ -\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right)$$

$$= \frac{5}{\sqrt{30}} - \frac{2}{\sqrt{30}}$$

$$=\boxed{\frac{3}{\sqrt{30}}}$$

$$\vec{w}_3=\vec{u}_3-(\vec{u}_3\cdot\vec{v}_1)\vec{v}_1-(\vec{u}_3\cdot\vec{v}_2)\vec{v}_2$$

$$=(1,\ 2,\ 0,\ -1)-\left(\frac{4}{\sqrt{5}}\right)\left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)-\left(\frac{3}{\sqrt{30}}\right)\left(\frac{5}{\sqrt{30}},\ -\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right)$$

$$=(1,\ 2,\ 0,\ -1)-\left(0,\ \frac{8}{5},\ \frac{4}{5},\ 0\right)-\left(\frac{1}{2},\ -\frac{1}{10},\ \frac{1}{5},\ 0\right)$$

$$=\boxed{\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)}$$

$$\vec{v}_3=\frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$=\frac{\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)}{\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+(-1)^2+(-1)^2}}$$

$$=\frac{1}{\sqrt{\frac{5}{2}}}\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)$$

$$=\frac{\sqrt{2}}{\sqrt{5}}\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)=\frac{2}{\sqrt{10}}\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)$$

$$=\boxed{\left(\frac{1}{\sqrt{10}},\ \frac{1}{\sqrt{10}},\ -\frac{2}{\sqrt{10}},\ -\frac{2}{\sqrt{10}}\right)}$$

$$u_4\cdot v_1=(1,\ 0,\ 0,\ 1)\cdot\left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)$$

$$=\boxed{0}$$

$$u_4\cdot v_2=(1,\ 0,\ 0,\ 1)\cdot\left(\frac{5}{\sqrt{30}},\ -\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right)$$

$$=\boxed{\frac{5}{\sqrt{30}}}$$

$$u_4\cdot v_3=(1,\ 0,\ 0,\ 1)\cdot\left(\frac{1}{\sqrt{10}},\ \frac{1}{\sqrt{10}},\ -\frac{2}{\sqrt{10}},\ -\frac{2}{\sqrt{10}}\right)$$

$$=\boxed{-\frac{1}{\sqrt{10}}}$$

$$\vec{w}_4=\vec{u}_4-(\vec{u}_4\cdot\vec{v}_1)\vec{v}_1-(\vec{u}_4\cdot\vec{v}_2)\vec{v}_2-(\vec{u}_4\cdot\vec{v}_3)\vec{v}_3$$

$$=(1,\ 2,\ 0,\ -1)-(0)-\left(\frac{5}{\sqrt{30}}\right)\left(\frac{5}{\sqrt{30}},\ -\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right)+\left(\frac{1}{\sqrt{10}}\right)\left(\frac{1}{\sqrt{10}},\ \frac{1}{\sqrt{10}},\ -\frac{2}{\sqrt{10}},\ -\frac{2}{\sqrt{10}}\right)$$

$$= (1,\ 2,\ 0,\ -1) - \left(\frac{5}{6},\ -\frac{1}{6},\ \frac{1}{3},\ 0\right) + \left(\frac{1}{10},\ \frac{1}{10},\ -\frac{1}{5},\ -\frac{1}{5}\right)$$

$$\boxed{= \left(\frac{4}{15},\ \frac{4}{15},\ -\frac{8}{15},\ \frac{4}{5}\right)}$$

$$\vec{v}_4 = \frac{\vec{w}_4}{\|\vec{w}_4\|}$$

$$= \frac{\left(\frac{4}{15},\ \frac{4}{15},\ -\frac{8}{15},\ \frac{4}{5}\right)}{\sqrt{\left(\frac{4}{15}\right)^2 + \left(\frac{4}{15}\right)^2 + \left(-\frac{8}{15}\right)^2 + \left(\frac{4}{5}\right)^2}}$$

$$= \frac{1}{\sqrt{\frac{240}{225}}}\left(\frac{4}{15},\ \frac{4}{15},\ -\frac{8}{15},\ \frac{4}{5}\right)$$

$$= \frac{1}{\frac{4}{\sqrt{15}}}\left(\frac{4}{15},\ \frac{4}{15},\ -\frac{8}{15},\ \frac{4}{5}\right)$$

$$= \frac{15}{4\sqrt{15}}\left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right)$$

$$\boxed{= \left(\frac{1}{\sqrt{15}},\ \frac{1}{\sqrt{15}},\ -\frac{2}{\sqrt{15}},\ \frac{3}{\sqrt{15}}\right)}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\{(1,\ 1,\ 0),\ (1,\ 2,\ 0),\ (0,\ 1,\ 2)\}$$

Solution

$$\boxed{\vec{v}_1 = (1,\ 1,\ 0)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (0,\ 2,\ 1) - \frac{(1,\ 2,\ 0) \bullet (1,\ 1,\ 0)}{1+1+0}(1,\ 1,\ 0)$$

$$= (0,\ 2,\ 1) - \frac{3}{2}(1,\ 1,\ 0)$$

$$\boxed{= \left(-\frac{3}{2},\ \frac{1}{2},\ 1\right)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(0,\ 1,\ 2) \bullet (1,\ 1,\ 0)}{2}(1,\ 1,\ 0)$$

$$=\left(\frac{1}{2},\ \frac{1}{2},\ 0\right)$$

$$\frac{\langle\vec{u}_3,\vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2=\frac{1}{\frac{9}{4}+\frac{1}{4}+1}(0,\ 1,\ 2)\bullet\left(-\frac{3}{2},\ \frac{1}{2},\ 1\right)\left(-\frac{3}{2},\ \frac{1}{2},\ 1\right)$$

$$=\frac{2}{7}\frac{5}{2}\left(-\frac{3}{2},\ \frac{1}{2},\ 1\right)$$

$$=\left(-\frac{15}{14},\ \frac{5}{14},\ \frac{5}{7}\right)$$

$$\vec{v}_3=\vec{u}_3-\frac{\langle\vec{u}_3,\vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1-\frac{\langle\vec{u}_3,\vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$=(0,\ 1,\ 2)-\left(\frac{1}{2},\ \frac{1}{2},\ 0\right)-\left(-\frac{15}{14},\ \frac{5}{14},\ \frac{5}{7}\right)$$

$$=\left(\frac{4}{7},\ \frac{1}{7},\ \frac{9}{7}\right)$$

$$\vec{q}_1=\frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$=\frac{1}{\sqrt{2}}(1,\ 1,\ 0)$$

$$=\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$\vec{q}_2=\frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$=\frac{1}{\sqrt{\frac{9}{4}+\frac{1}{4}+1}}\left(-\frac{3}{2},\ \frac{1}{2},\ 1\right)$$

$$=\frac{2}{\sqrt{14}}\left(-\frac{3}{2},\ \frac{1}{2},\ 1\right)$$

$$=\left(-\frac{2}{\sqrt{14}},\ \frac{1}{\sqrt{14}},\ \frac{2}{\sqrt{14}}\right)$$

$$\vec{q}_3=\frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$=\frac{1}{\sqrt{\frac{16}{49}+\frac{1}{49}+\frac{81}{49}}}\left(\frac{4}{7},\ \frac{1}{7},\ \frac{9}{7}\right)$$

$$=\frac{7}{\sqrt{98}}\left(\frac{4}{7},\ \frac{1}{7},\ \frac{9}{7}\right)$$

$$= \frac{7}{7\sqrt{2}}\left(\frac{4}{7},\ \frac{1}{7},\ \frac{9}{7}\right)$$

$$= \left(\frac{4}{7\sqrt{2}},\ \frac{1}{7\sqrt{2}},\ \frac{9}{7\sqrt{2}}\right)$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\{(1,\ -2,\ 2),\ (2,\ 2,\ 1),\ (2,\ -1,\ -2)\}$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (1,\ -2,\ 2)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (2,\ 2,\ 1) - \frac{(2,\ 2,\ 1)\bullet(1,\ -2,\ 2)}{9}(1,\ -2,\ 2)$$

$$= (2,\ 2,\ 1) - \frac{0}{9}(1,\ -2,\ 2)$$

$$= (2,\ 2,\ 1)$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(2,\ -1,\ -2)\bullet(1,\ -2,\ 2)}{9}(1,\ -2,\ 2)$$

$$= \frac{0}{9}(1,\ -2,\ 2)$$

$$= (0,\ 0,\ 0)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{1}{9}\left[(2,\ -1,\ -2)\bullet(2,\ 2,\ 1)\right](2,\ 2,\ 1)$$

$$= (0,\ 0,\ 0)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (2,\ -1,\ -2) - (0,\ 0,\ 0) - (0,\ 0,\ 0)$$

$$= (2,\ -1,\ -2)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$
$$= \frac{1}{3}(1,\ -2,\ 2)$$
$$= \left(\frac{1}{3},\ -\frac{2}{3},\ \frac{2}{3}\right)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$
$$= \frac{1}{3}(2,\ 2,\ 1)$$
$$= \left(\frac{2}{3},\ \frac{2}{3},\ \frac{1}{3}\right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$
$$= \frac{1}{3}(2,\ -1,\ -2)$$
$$= \left(\frac{2}{3},\ -\frac{1}{3},\ -\frac{2}{3}\right)$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\{(1,\ 0,\ 0),\ (1,\ 1,\ 1),\ (1,\ 1,\ -1)\}$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (1,\ 0,\ 0)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$
$$= (1,\ 1,\ 1) - \frac{(1,\ 1,\ 1) \bullet (1,\ 0,\ 0)}{1}(1,\ 0,\ 0)$$
$$= (1,\ 1,\ 1) - (1,\ 0,\ 0)$$
$$= (0,\ 1,\ 1)$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(1,\ 1,\ -1) \bullet (1,\ 0,\ 0)}{1}(1,\ 0,\ 0)$$
$$= (1,\ 0,\ 0)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{1}{1}\left[(1,\ 1,\ -1) \bullet (0,\ 1,\ 1)\right](0,\ 1,\ 1)$$

$$= 0(0,\ 1,\ 1)$$

$$\underline{= (0,\ 0,\ 0)\ |}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (1,\ 1,\ -1) - (1,\ 0,\ 0) - (0,\ 0,\ 0)$$

$$\underline{= (0,\ 1,\ -1)\ |}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{1}(1,\ 0,\ 0)$$

$$\underline{= (1,\ 0,\ 0)\ |}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{\sqrt{2}}(0,\ 1,\ 1)$$

$$\underline{= \left(0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)\ |}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\sqrt{2}}(0,\ 1,\ -1)$$

$$\underline{= \left(0,\ \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right)\ |}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\{(4,\ -3,\ 0),\ (1,\ 2,\ 0),\ (0,\ 0,\ 4)\}$$

Solution

$$\underline{\vec{v}_1 = \vec{u}_1 = (4,\ -3,\ 0)\ |}$$

$$
\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 \\
&= (1,\ 2,\ 0) - \frac{(1,\ 2,\ 0)\bullet(4,\ -3,\ 0)}{25}(4,\ -3,\ 0) \\
&= (1,\ 2,\ 0) + \frac{2}{25}(4,\ -3,\ 0) \\
&= \boxed{\left(\frac{33}{25},\ \frac{44}{25},\ 0\right)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 &= \frac{(0,\ 0,\ 4)\bullet(4,\ -3,\ 0)}{25}(4,\ -3,\ 0) \\
&= \boxed{(0,\ 0,\ 0)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 &= \frac{225}{3,025}\left[(0,\ 0,\ 4)\bullet\left(\frac{33}{25},\ \frac{44}{25},\ 0\right)\right]\left(\frac{33}{25},\ \frac{44}{25},\ 0\right) \\
&= \boxed{(0,\ 0,\ 0)}
\end{aligned}
$$

$$
\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 \\
&= (0,\ 0,\ 4) - (0,\ 0,\ 0) - (0,\ 0,\ 0) \\
&= \boxed{(0,\ 0,\ 4)}
\end{aligned}
$$

$$
\begin{aligned}
\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
&= \frac{1}{\sqrt{16+9}}(4,\ -3,\ 0) \\
&= \boxed{\left(\frac{4}{5},\ -\frac{3}{5},\ 0\right)}
\end{aligned}
$$

$$
\begin{aligned}
\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \frac{25}{\sqrt{3,025}}\left(\frac{33}{25},\ \frac{44}{25},\ 0\right) \\
&= \frac{25}{55}\left(\frac{33}{25},\ \frac{44}{25},\ 0\right) \\
&= \boxed{\left(\frac{3}{5},\ \frac{4}{5},\ 0\right)}
\end{aligned}
$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{4}(0,\ 0,\ 4)$$

$$= (0,\ 0,\ 1)$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\{(0,\ 1,\ 2),\ (2,\ 0,\ 0),\ (1,\ 1,\ 1)\}$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (0,\ 1,\ 2)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (2,\ 0,\ 0) - \frac{(2,\ 0,\ 0) \bullet (0,\ 1,\ 2)}{5}(0,\ 1,\ 2)$$

$$= (2,\ 0,\ 0) + \frac{0}{5}(0,\ 1,\ 2)$$

$$= (2,\ 0,\ 0)$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(1,\ 1,\ 1) \bullet (0,\ 1,\ 2)}{5}(0,\ 1,\ 2)$$

$$= \frac{3}{5}(0,\ 1,\ 2)$$

$$= \left(0,\ \frac{3}{5},\ \frac{6}{5}\right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{1}{4}\left[(1,\ 1,\ 1) \bullet (2,\ 0,\ 0)\right](2,\ 0,\ 0)$$

$$= \frac{1}{2}(2,\ 0,\ 0)$$

$$= (1,\ 0,\ 0)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (1,\ 1,\ 1) - (1,\ 0,\ 0) - \left(0,\ \frac{3}{5},\ \frac{6}{5}\right)$$

$$\boxed{= \left(0,\ \frac{2}{5},\ -\frac{1}{5}\right)}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{5}}(0,\ 1,\ 2)$$

$$\boxed{= \left(0,\ \frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right)}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{2}(2,\ 0,\ 0)$$

$$\boxed{= (1,\ 0,\ 0)}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{5}{\sqrt{5}}\left(0,\ \frac{2}{5},\ -\frac{1}{5}\right)$$

$$\boxed{= \left(0,\ \frac{2}{\sqrt{5}},\ -\frac{1}{\sqrt{5}}\right)}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\{(0,\ 1,\ 1),\ (1,\ 1,\ 0),\ (1,\ 0,\ 1)\}$$

Solution

$$\boxed{\vec{v}_1 = \vec{u}_1 = (0,\ 1,\ 1)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (1,\ 1,\ 0) - \frac{(1,\ 1,\ 0) \bullet (0,\ 1,\ 1)}{2}(0,\ 1,\ 1)$$

$$= (1,\ 1,\ 0) - \frac{1}{2}(0,\ 1,\ 1)$$

$$\boxed{= \left(1,\ \frac{1}{2},\ -\frac{1}{2}\right)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(1,\ 0,\ 1)\cdot(0,\ 1,\ 1)}{2}(0,\ 1,\ 1)$$

$$= \left(0,\ \tfrac{1}{2},\ \tfrac{1}{2}\right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{4}{6}\left[(1,\ 0,\ 1)\cdot\left(1,\ \tfrac{1}{2},\ -\tfrac{1}{2}\right)\right]\left(1,\ \tfrac{1}{2},\ -\tfrac{1}{2}\right)$$

$$= \frac{1}{3}\left(1,\ \tfrac{1}{2},\ -\tfrac{1}{2}\right)$$

$$= \left(\tfrac{1}{3},\ \tfrac{1}{6},\ -\tfrac{1}{6}\right)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (1,\ 0,\ 1) - \left(0,\ \tfrac{1}{2},\ \tfrac{1}{2}\right) - \left(\tfrac{1}{3},\ \tfrac{1}{6},\ -\tfrac{1}{6}\right)$$

$$= \left(\tfrac{2}{3},\ -\tfrac{2}{3},\ \tfrac{2}{3}\right)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{2}}(0,\ 1,\ 1)$$

$$= \left(0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{2}{\sqrt{6}}\left(1,\ \tfrac{1}{2},\ -\tfrac{1}{2}\right)$$

$$= \left(\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}}\right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{3}{\sqrt{12}}\left(\tfrac{2}{3},\ -\tfrac{2}{3},\ \tfrac{2}{3}\right)$$

$$= \left(\frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\{(1,\ 2,\ -2),\ (1,\ 0,\ -4),\ (5,\ 2,\ 0),\ (1,\ 1,\ -1)\}$$

Solution

$$\boxed{\vec{v}_1 = \vec{u}_1 = (1,\ 2,\ -2)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (1,\ 0,\ -4) - \frac{(1,\ 0,\ -4)\bullet(1,\ 2,\ -2)}{9}(1,\ 2,\ -2)$$

$$= (1,\ 0,\ -4) - (1,\ 2,\ -2)$$

$$\boxed{= (0,\ -2,\ -2)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(5,\ 2,\ 0)\bullet(1,\ 2,\ -2)}{9}(1,\ 2,\ -2)$$

$$\boxed{= (1,\ 2,\ -2)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{1}{8}\left[(5,\ 2,\ 0)\bullet(0,\ -2,\ -2)\right](0,\ -2,\ -2)$$

$$= -\frac{1}{2}(0,\ -2,\ -2)$$

$$\boxed{= (0,\ 1,\ 1)}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (5,\ 2,\ 0) - (1,\ 2,\ -2) - (0,\ 1,\ 1)$$

$$\boxed{= (4,\ -1,\ 1)}$$

$$\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(1,\ 1,\ -1)\bullet(1,\ 2,\ -2)}{9}(1,\ 2,\ -2)$$

$$= \frac{5}{9}(1,\ 2,\ -2)$$

$$\boxed{= \left(\frac{5}{9},\ \frac{10}{9},\ -\frac{10}{9}\right)}$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{1}{8}\left[(1,\ 1,\ -1)\bullet(0,\ -2,\ -2)\right](0,\ -2,\ -2)$$

$$= \boxed{(0,\ 0,\ 0)}$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3 = \frac{1}{18}\left[(1,\ 1,\ -1)\bullet(4,\ -1,\ 1)\right](4,\ -1,\ 1)$$

$$= \frac{1}{9}(4,\ -1,\ 1)$$

$$= \boxed{\left(\frac{4}{9},\ -\frac{1}{9},\ \frac{1}{9}\right)}$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$

$$= (1,\ 1,\ -1) - \left(\frac{5}{9},\ \frac{10}{9},\ -\frac{10}{9}\right) - \left(\frac{4}{9},\ -\frac{1}{9},\ \frac{1}{9}\right)$$

$$= \boxed{(0,\ 0,\ 0)}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{3}(1,\ 2,\ -2)$$

$$= \boxed{\left(\frac{1}{3},\ \frac{2}{3},\ -\frac{2}{3}\right)}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{2\sqrt{2}}(0,\ -2,\ -2)$$

$$= \boxed{\left(0,\ -\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right)}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{3\sqrt{2}}(4,\ -1,\ 1)$$

$$= \boxed{\left(\frac{4}{3\sqrt{2}},\ -\frac{1}{3\sqrt{2}},\ \frac{1}{3\sqrt{2}}\right)}$$

$$\vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

$$= \boxed{(0,\ 0,\ 0)}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\{(-3,\ 1,\ 2),\ (1,\ 1,\ 1),\ (2,\ 0,\ -1),\ (1,\ -3,\ 2)\}$$

Solution

$$\boxed{\vec{v}_1 = \vec{u}_1 = (-3,\ 1,\ 2)}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 \\ &= (1,\ 1,\ 1) - \frac{(1,\ 1,\ 1)\bullet(-3,\ 1,\ 2)}{14}(-3,\ 1,\ 2) \\ &= (1,\ 1,\ 1) - \frac{0}{14}(1,\ 2,\ -2) \\ &= (1,\ 1,\ 1)\ \boxed{}\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 &= \frac{(2,\ 0,\ -1)\bullet(-3,\ 1,\ 2)}{14}(-3,\ 1,\ 2) \\ &= -\frac{4}{7}(-3,\ 1,\ 2) \\ &= \left(\frac{12}{7},\ -\frac{4}{7},\ -\frac{8}{7}\right)\ \boxed{}\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 &= \frac{1}{3}\left[(2,\ 0,\ -1)\bullet(1,\ 1,\ 1)\right](1,\ 1,\ 1) \\ &= \frac{1}{3}(1,\ 1,\ 1) \\ &= \left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right)\ \boxed{}\end{aligned}$$

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 \\ &= (2,\ 0,\ -1) - \left(\frac{12}{7},\ -\frac{4}{7},\ -\frac{8}{7}\right) - \left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right) \\ &= \left(-\frac{1}{21},\ \frac{5}{21},\ -\frac{4}{21}\right)\ \boxed{}\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 &= \frac{(1,\ -3,\ 2)\bullet(-3,\ 1,\ 2)}{14}(-3,\ 1,\ 2) \\ &= -\frac{1}{7}(-3,\ 1,\ 2)\end{aligned}$$

$$=\left(\frac{3}{7},\ -\frac{1}{7},\ -\frac{2}{7}\right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{(1,\ -3,\ 2)\bullet(1,\ 1,\ 1)}{3}(1,\ 1,\ 1)$$

$$=(0,\ 0,\ 0)$$

$$\frac{\langle \vec{u}_4, \vec{v}_3\rangle}{\|\vec{v}_3\|^2}\vec{v}_3 = \frac{441}{42}\left[(1,\ -3,\ 2)\bullet\left(-\frac{1}{21},\ \frac{5}{21},\ -\frac{4}{21}\right)\right]\left(-\frac{1}{21},\ \frac{5}{21},\ -\frac{4}{21}\right)$$

$$=\frac{441}{42}\left(-\frac{24}{21}\right)\left(-\frac{1}{21},\ \frac{5}{21},\ -\frac{4}{21}\right)$$

$$=\left(\frac{4}{7},\ -\frac{20}{7},\ \frac{16}{7}\right)$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3\rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$

$$=(1,\ -3,\ 2)-\left(\frac{3}{7},\ -\frac{1}{7},\ -\frac{2}{7}\right)-(0,\ 0,\ 0)-\left(\frac{4}{7},\ -\frac{20}{7},\ \frac{16}{7}\right)$$

$$=(0,\ 0,\ 0)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$=\frac{1}{\sqrt{14}}(-3,\ 1,\ 2)$$

$$=\left(-\frac{3}{\sqrt{14}},\ \frac{1}{\sqrt{14}},\ \frac{2}{\sqrt{14}}\right)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$=\frac{1}{\sqrt{3}}(1,\ 1,\ 1)$$

$$=\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$=\frac{21}{\sqrt{42}}\left(-\frac{1}{21},\ \frac{5}{21},\ -\frac{4}{21}\right)$$

$$=\left(-\frac{1}{\sqrt{42}},\ \frac{5}{\sqrt{42}},\ -\frac{4}{\sqrt{42}}\right)$$

$$\vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

$$= (0,\ 0,\ 0)$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\{(2,\ 1,\ 1),\ (0,\ 3,\ -1),\ (3,\ -4,\ -2),\ (-1,\ -1,\ 3)\}$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (2,\ 1,\ 1)$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 \\
&= (0,\ 3,\ -1) - \frac{(0,\ 3,\ -1)\cdot(2,\ 1,\ 1)}{6}(2,\ 1,\ 1) \\
&= (0,\ 3,\ -1) - \frac{1}{3}(2,\ 1,\ 1) \\
&= \left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 &= \frac{(3,\ -4,\ -2)\cdot(2,\ 1,\ 1)}{6}(2,\ 1,\ 1) \\
&= (0,\ 0,\ 0)
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 &= \frac{9}{84}\left[(3,\ -4,\ -2)\cdot\left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right)\right]\left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right) \\
&= \frac{3}{28}(-10)\left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right) \\
&= \left(\frac{5}{7},\ -\frac{20}{7},\ \frac{10}{7}\right)
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 \\
&= (3,\ -4,\ -2) - (0,\ 0,\ 0) - \left(\frac{5}{7},\ -\frac{20}{7},\ \frac{10}{7}\right) \\
&= \left(\frac{16}{7},\ -\frac{8}{7},\ -\frac{24}{7}\right)
\end{aligned}$$

$$\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(-1,\ -1,\ 3)\cdot(2,\ 1,\ 1)}{6}(2,\ 1,\ 1)$$

$$\underline{=(0,\ 0,\ 0)}$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{9}{84}\left[(-1,\ -1,\ 3)\cdot\left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right)\right]\left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right)$$

$$= \frac{3}{28}\left(-\frac{18}{3}\right)\left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right)$$

$$\underline{=\left(\frac{3}{7},\ -\frac{12}{7},\ \frac{6}{7}\right)}$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3 = \frac{49}{896}\left[(-1,\ -1,\ 3)\cdot\left(\frac{16}{7},\ -\frac{8}{7},\ -\frac{24}{7}\right)\right]\left(\frac{16}{7},\ -\frac{8}{7},\ -\frac{24}{7}\right)$$

$$= \frac{7}{128}\left(-\frac{80}{7}\right)\left(\frac{16}{7},\ -\frac{8}{7},\ -\frac{24}{7}\right)$$

$$\underline{=\left(-\frac{10}{7},\ \frac{5}{7},\ \frac{15}{7}\right)}$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$

$$= (-1,\ -1,\ 3) - (0,\ 0,\ 0) - \left(\frac{3}{7},\ -\frac{12}{7},\ \frac{6}{7}\right) - \left(-\frac{10}{7},\ \frac{5}{7},\ \frac{15}{7}\right)$$

$$\underline{=(0,\ 0,\ 0)}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{6}}(2,\ 1,\ 1)$$

$$\underline{=\left(\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{3}{2\sqrt{21}}\left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right)$$

$$\underline{=\left(-\frac{1}{\sqrt{21}},\ \frac{4}{\sqrt{21}},\ -\frac{2}{\sqrt{21}}\right)}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{7}{8\sqrt{14}}\left(\frac{16}{7},\ -\frac{8}{7},\ -\frac{24}{7}\right)$$

$$\boxed{= \left(\frac{2}{\sqrt{14}},\ -\frac{1}{\sqrt{14}},\ -\frac{3}{\sqrt{14}}\right)}$$

$$\vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

$$\boxed{= (0,\ 0,\ 0)}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\vec{u}_1 = (1, 1, 0, -1), \quad \vec{u}_2 = (1, 3, 0, 1), \quad \vec{u}_3 = (4, 2, 2, 0)$$

Solution

$$\boxed{\vec{v}_1 = \vec{u}_1 = (1, 1, 0, -1)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (1, 3, 0, 1) - \frac{(1, 3, 0, 1) \bullet (1, 1, 0, -1)}{3}(1, 1, 0, -1)$$

$$= (1, 3, 0, 1) - (1, 1, 0, -1)$$

$$\boxed{= (0, 2, 0, 2)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(4, 2, 2, 0) \bullet (1, 1, 0, -1)}{3}(1, 1, 0, -1)$$

$$\boxed{= (2, 2, 0, -2)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{(4, 2, 2, 0) \bullet (0, 2, 0, 2)}{8}(0, 2, 0, 2)$$

$$= \frac{1}{2}(0, 2, 0, 2)$$

$$\boxed{= (0, 1, 0, 1)}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (4,\ 2,\ 2,\ 0) - (2,\ 2,\ 0,\ -2) - (0,\ 1,\ 0,\ 1)$$

$$= (2,\ -1,\ 2,\ 1)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{3}}(1,\ 1,\ 0,\ -1)$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ 0,\ -\frac{1}{\sqrt{3}}\right)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{2\sqrt{2}}(0,\ 2,\ 0,\ 2)$$

$$= \left(0,\ \frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\sqrt{10}}(2,\ -1,\ 2,\ 1)$$

$$= \left(\frac{2}{\sqrt{10}},\ -\frac{1}{\sqrt{10}},\ \frac{2}{\sqrt{10}},\ -\frac{1}{\sqrt{10}}\right)$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\vec{u}_1 = (1,\ 1,\ 1,\ 1), \quad \vec{u}_2 = (1,\ 1,\ 2,\ 4), \quad \vec{u}_3 = (1,\ 2,\ -4,\ -3)$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (1,\ 1,\ 1,\ 1)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (1,\ 1,\ 2,\ 4) - \frac{(1,\ 1,\ 2,\ 4) \bullet (1,\ 1,\ 1,\ 1)}{4}(1,\ 1,\ 1,\ 1)$$

$$= (1,\ 1,\ 2,\ 4) - 2(1,\ 1,\ 1,\ 1)$$

$$\underline{= (-1,\ -1,\ 0,\ 2)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(1,\ 2,\ -4,\ -3) \bullet (1,\ 1,\ 1,\ 1)}{4}(1,\ 1,\ 1,\ 1)$$

$$\underline{= (-1,\ -1,\ -1,\ -1)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{(1,\ 2,\ -4,\ -3) \bullet (-1,\ -1,\ 0,\ 2)}{6}(-1,\ -1,\ 0,\ 2)$$

$$= -\frac{3}{2}(-1,\ -1,\ 0,\ 2)$$

$$\underline{= \left(\frac{3}{2},\ \frac{3}{2},\ 0,\ -3\right)}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (1,\ 2,\ -4,\ -3) - (-1,\ -1,\ -1,\ -1) - \left(\frac{3}{2},\ \frac{3}{2},\ 0,\ -3\right)$$

$$\underline{= \left(\frac{1}{2},\ \frac{3}{2},\ -3,\ 1\right)}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{2}(1,\ 1,\ 1,\ 1)$$

$$\underline{= \left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{\sqrt{6}}(-1,\ -1,\ 0,\ 2)$$

$$\underline{= \left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0,\ \frac{2}{\sqrt{6}}\right)}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{2}{\sqrt{50}}\left(\frac{1}{2},\ \frac{3}{2},\ -3,\ 1\right)$$

$$= \frac{2}{5\sqrt{2}}\left(\frac{1}{2},\ \frac{3}{2},\ -3,\ 1\right)$$

$$\boxed{= \left(\frac{1}{5\sqrt{2}},\ \frac{3}{5\sqrt{2}},\ -\frac{6}{5\sqrt{2}},\ \frac{2}{5\sqrt{2}}\right)}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\{(3,\ 4,\ 0,\ 0),\ (-1,\ 1,\ 0,\ 0),\ (2,\ 1,\ 0,\ -1),\ (0,\ 1,\ 1,\ 0)\}$$

Solution

$$\boxed{\vec{v}_1 = \vec{u}_1 = (3,\ 4,\ 0,\ 0)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (-1,\ 1,\ 0,\ 0) - \frac{(-1,\ 1,\ 0,\ 0)\cdot(3,\ 4,\ 0,\ 0)}{25}(3,\ 4,\ 0,\ 0)$$

$$= (-1,\ 1,\ 0,\ 0) - \frac{1}{25}(3,\ 4,\ 0,\ 0)$$

$$\boxed{= \left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(2,\ 1,\ 0,\ -1)\cdot(3,\ 4,\ 0,\ 0)}{25}(3,\ 4,\ 0,\ 0)$$

$$= \frac{10}{25}(3,\ 4,\ 0,\ 0)$$

$$\boxed{= \left(\frac{6}{5},\ \frac{8}{5},\ 0,\ 0\right)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{625}{1,225}\left[(2,\ 1,\ 0,\ -1)\cdot\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)\right]\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)$$

$$= \frac{25}{49}\left(-\frac{35}{25}\right)\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)$$

$$\boxed{= \left(\frac{4}{5},\ -\frac{3}{5},\ 0,\ 0\right)}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (2,\ 1,\ 0,\ -1) - \left(\frac{6}{5},\ \frac{8}{5},\ 0,\ 0\right) - \left(\frac{4}{5},\ -\frac{3}{5},\ 0,\ 0\right)$$

$$= \left(0,\ 0,\ 0,\ -1\right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(0,\ 1,\ 1,\ 0) \bullet (3,\ 4,\ 0,\ 0)}{25}(3,\ 4,\ 0,\ 0)$$

$$= \frac{4}{25}(3,\ 4,\ 0,\ 0)$$

$$= \left(\frac{12}{25},\ \frac{16}{25},\ 0,\ 0\right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{25}{49}\left[(0,\ 1,\ 1,\ 0) \bullet \left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)\right]\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)$$

$$= \frac{25}{49}\left(\frac{21}{25}\right)\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)$$

$$= \left(-\frac{12}{25},\ \frac{9}{25},\ 0,\ 0\right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3 = \left[(0,\ 1,\ 1,\ 0) \bullet (0,\ 0,\ 0,\ -1)\right](0,\ 0,\ 0,\ -1)$$

$$= (0,\ 0,\ 0,\ 0)$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$

$$= (0,\ 1,\ 1,\ 0) - \left(\frac{12}{25},\ \frac{16}{25},\ 0,\ 0\right) - \left(-\frac{12}{25},\ \frac{9}{25},\ 0,\ 0\right) - (0,\ 0,\ 0,\ 0)$$

$$= (0,\ 0,\ 1,\ 0)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{5}(3,\ 4,\ 0,\ 0)$$

$$= \left(\frac{3}{5},\ \frac{4}{5},\ 0,\ 0\right)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{25}{35}\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)$$

$$= \left(-\frac{4}{5},\ \frac{3}{5},\ 0,\ 0\right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= (0,\ 0,\ 0,\ -1)$$

$$\vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

$$= (0,\ 0,\ 1,\ 0)$$

Exercise

Find the QR–decomposition of

a) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

Solution

a) Since $\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5 \neq 0$, The matrix is invertible

$$\vec{u}_1 (1,\ 2), \quad \vec{u}_2 = (-1,\ 3)$$

$$\vec{v}_1 = \vec{u}_1 = (1,\ 2)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ 2)}{\sqrt{1^2 + 2^2}}$$

$$= \frac{(1,\ 2)}{\sqrt{5}}$$

$$= \left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}} \right)$$

$$\vec{v}_2 = \vec{u}_2 - (\vec{u}_2 \cdot \vec{v}_1)\vec{v}_1$$

$$= (-1,\ 3) - \left[(-1,\ 3) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) \right] \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)$$

$$= (-1,\ 3) - \left(\frac{5}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$= (-1,\ 3) - (1,\ 2)$$

$$\underline{= (-2,\ 1)}\,\big|$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{(-2,\ 1)}{\sqrt{(-2)^2 + 1^2}}$$

$$\underline{= \left(-\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right)}\,\big|$$

$$\langle \vec{u}_1, \vec{q}_1 \rangle = (1,\ 2) \bullet \left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right)$$

$$= \frac{5}{\sqrt{5}}$$

$$\underline{= \sqrt{5}}\,\big|$$

$$\langle \vec{u}_2, \vec{q}_1 \rangle = (-1,\ 3) \bullet \left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right)$$

$$= \frac{5}{\sqrt{5}}$$

$$\underline{= \sqrt{5}}\,\big|$$

$$\langle \vec{u}_2, \vec{q}_2 \rangle = (-1,\ 3) \bullet \left(-\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right)$$

$$= \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{5}}$$

$$\underline{= \sqrt{5}}\,\big|$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}$$

The QR-decomposition of the matrix is

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}$$

b) The column vectors of are: $\vec{u}_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$$\boxed{\vec{v}_1 = \vec{u}_1 = (3, \ -4)}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(3, \ -4)}{\sqrt{9+16}}$$

$$\boxed{= \left(\frac{3}{5}, \ -\frac{4}{5}\right)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (5, \ 0) - \frac{(5, \ 0) \bullet (3, \ -4)}{25}(3, \ -4)$$

$$= (5, \ 0) - \frac{15}{25}(3, \ -4)$$

$$= (5, \ 0) - \frac{3}{5}(3, \ -4)$$

$$= (5, \ 0) - \left(\frac{9}{5}, \ -\frac{12}{5}\right)$$

$$\boxed{= \left(\frac{16}{5}, \ \frac{12}{5}\right)}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{\left(\frac{16}{5}, \ \frac{12}{5}\right)}{\sqrt{\frac{256}{25} + \frac{144}{25}}}$$

$$= \frac{1}{\sqrt{\frac{400}{25}}}\left(\frac{16}{5}, \ \frac{12}{5}\right)$$

$$= \frac{1}{\sqrt{16}}\left(\frac{16}{5}, \ \frac{12}{5}\right)$$

$$= \frac{1}{4}\left(\frac{16}{5}, \ \frac{12}{5}\right)$$

$$\boxed{= \left(\frac{4}{5}, \ \frac{3}{5}\right)}$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix}$$

$$=\begin{bmatrix} 3\left(\frac{3}{5}\right)-4\left(-\frac{4}{5}\right) & 5\left(\frac{3}{5}\right)+0\left(-\frac{4}{5}\right) \\ 0 & 5\left(\frac{4}{5}\right)-0\left(\frac{3}{5}\right) \end{bmatrix}$$

$$=\begin{bmatrix} 5 & 3 \\ 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \cdot \begin{bmatrix} 5 & 3 \\ 0 & 4 \end{bmatrix}$$

$$\boldsymbol{A} \quad = \quad \boldsymbol{Q} \qquad \boldsymbol{R}$$

c) Since the column vectors $\vec{u}_1\,(1,\ 0,\ 1),\quad \vec{u}_2 = (2,\ 1,\ 4)$ are linearly independent, so has a QR–decomposition.

$$\vec{v}_1 = \vec{u}_1 = (1,\ 0,\ 1)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ 0,\ 1)}{\sqrt{1^2+0+1^2}}$$

$$= \frac{(1,\ 0,\ 1)}{\sqrt{2}}$$

$$= \left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \langle \vec{u}_2,\ \vec{v}_1 \rangle \vec{v}_1$$

$$= (2,1,4) - \left[(2,1,4) \bullet \left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\right]\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$= (2,1,4) - \left(\frac{6}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$= (2,1,4) - (3,\ 0,\ 3)$$

$$= (-1,\ 1,\ 1)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{(-1,\ 1,\ 1)}{\sqrt{(-1)^2+1^2+1^2}}$$

$$= \left(-\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}} \right)$$

$$\langle \vec{u}_1,\ \vec{q}_1 \rangle = (1,\ 0,\ 1) \cdot \left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}} \right)$$

$$= \frac{2}{\sqrt{2}}$$

$$= \sqrt{2}$$

$$\langle \vec{u}_2,\ \vec{q}_1 \rangle = (2,\ 1,\ 4) \cdot \left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}} \right)$$

$$= \frac{2}{\sqrt{2}} + \frac{4}{\sqrt{2}}$$

$$= 3\sqrt{2}$$

$$\langle \vec{u}_2,\ \vec{q}_2 \rangle = (2,\ 1,\ 4) \cdot \left(-\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}} \right)$$

$$= -\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{4}{\sqrt{3}}$$

$$= \frac{3}{\sqrt{3}}$$

$$= \sqrt{3}$$

$$R = \begin{bmatrix} \langle \vec{u}_1,\ \vec{q}_1 \rangle & \langle \vec{u}_2,\ \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2,\ \vec{q}_2 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$$

The QR–decomposition of the matrix is

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$$

$$A \quad = \quad Q \qquad\qquad R$$

d) Since $\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{vmatrix} = -4 \neq 0$,

The matrix is invertible, so it has a QR-decomposition.

$\vec{u}_1\ (1,\ 1,\ 0),\quad \vec{u}_2 = (2,\ 1,\ 3),\quad \vec{u}_3 = (1,\ 1,\ 1)$

$$\boxed{\vec{v}_1 = \vec{u}_1 = (1,\ 1,\ 0)}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ 1,\ 0)}{\sqrt{1^2 + 1^2 + 0}}$$

$$= \frac{(1,\ 1,\ 0)}{\sqrt{2}}$$

$$\boxed{= \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)}$$

$$\vec{v}_2 = \vec{u}_2 - (\vec{u}_2 \bullet \vec{v}_1)\vec{v}_1$$

$$= (2,\ 1,\ 3) - \left[(2,\ 1,\ 3) \bullet \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)\right]\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= (2,\ 1,\ 3) - \frac{3}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= (2,\ 1,\ 3) - \left(\frac{3}{2},\ \frac{3}{2},\ 0\right)$$

$$\boxed{= \left(\frac{1}{2},\ -\frac{1}{2},\ 3\right)}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{\left(\frac{1}{2},\ -\frac{1}{2},\ 3\right)}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + 3^2}}$$

$$= \frac{\left(\frac{1}{2},\ -\frac{1}{2},\ 3\right)}{\sqrt{\frac{19}{2}}}$$

$$= \frac{\sqrt{2}}{\sqrt{19}}\left(\frac{1}{2},\ -\frac{1}{2},\ 3\right)$$

$$\boxed{= \left(\frac{1}{\sqrt{38}},\ -\frac{1}{\sqrt{38}},\ \frac{6}{\sqrt{38}}\right)}$$

$$\vec{v}_3 = \vec{u}_3 - (\vec{u}_3 \bullet \vec{v}_1)\vec{v}_1 - (\vec{u}_3 \bullet \vec{v}_2)\vec{v}_2$$

$$= (1,1,1) - \left[(1,1,1).\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)\right]\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$$

$$-\left[(1,1,1).\left(\frac{1}{\sqrt{38}},-\frac{1}{\sqrt{38}},\frac{6}{\sqrt{38}}\right)\right]\left(\frac{1}{\sqrt{38}},-\frac{1}{\sqrt{38}},\frac{6}{\sqrt{38}}\right)$$

$$= (1,1,1) - \frac{2}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right) - \frac{6}{\sqrt{38}}\left(\frac{1}{\sqrt{38}},-\frac{1}{\sqrt{38}},\frac{6}{\sqrt{38}}\right)$$

$$= (1,1,1) - (1,1,0) - \left(\frac{3}{19},-\frac{3}{19},\frac{18}{19}\right)$$

$$= \left(-\frac{3}{19},\ \frac{3}{19},\ \frac{1}{19}\right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{\left(-\frac{3}{19},\ \frac{3}{19},\ \frac{1}{19}\right)}{\sqrt{\left(-\frac{3}{19}\right)^2 + \left(\frac{3}{19}\right)^2 + \left(\frac{1}{19}\right)^2}}$$

$$= \frac{19}{\sqrt{19}}\left(-\frac{3}{19},\ \frac{3}{19},\ \frac{1}{19}\right)$$

$$= \left(-\frac{3}{\sqrt{19}},\ \frac{3}{\sqrt{19}},\ \frac{1}{\sqrt{19}}\right)$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix}$$

$$\langle \vec{u}_1, \vec{q}_1 \rangle = (1,\ 1,\ 0) \bullet \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= \frac{2}{\sqrt{2}}$$

$$= \sqrt{2}$$

$$\langle \vec{u}_2, \vec{q}_1 \rangle = (2,\ 1,\ 3) \bullet \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= \frac{3}{\sqrt{2}}$$

$$\langle \vec{u}_2, \vec{q}_2 \rangle = (2,\ 1,\ 3) \bullet \left(\frac{1}{\sqrt{38}},-\frac{1}{\sqrt{38}},\frac{6}{\sqrt{38}}\right)$$

$$= \frac{2-1+18}{\sqrt{38}}$$

$$= \frac{19}{\sqrt{38}}$$

$$= \frac{19}{\sqrt{2}\sqrt{19}}$$

$$\boxed{= \frac{\sqrt{19}}{\sqrt{2}}}$$

$$\left\langle \vec{u}_3,\ \vec{q}_1 \right\rangle = (1,\ 1,\ 1) \bullet \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0 \right)$$

$$= \frac{2}{\sqrt{2}}$$

$$\boxed{= \sqrt{2}}$$

$$\left\langle \vec{u}_3,\ \vec{q}_2 \right\rangle = (1,\ 1,\ 1) \bullet \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}} \right)$$

$$= \frac{1-1+6}{\sqrt{38}}$$

$$= \frac{6}{\sqrt{2}\sqrt{19}} \frac{\sqrt{2}}{\sqrt{2}}$$

$$\boxed{= \frac{3\sqrt{2}}{\sqrt{19}}}$$

$$\left\langle \vec{u}_3,\ \vec{q}_3 \right\rangle = (1,\ 1,\ 1) \bullet \left(-\frac{3}{\sqrt{19}},\ \frac{3}{\sqrt{19}},\ \frac{1}{\sqrt{19}} \right)$$

$$= \frac{-3+3+1}{\sqrt{19}}$$

$$\boxed{= \frac{1}{\sqrt{19}}}$$

$$R = \begin{bmatrix} \left\langle \vec{u}_1, \vec{q}_1 \right\rangle & \left\langle \vec{u}_2, \vec{q}_1 \right\rangle & \left\langle \vec{u}_3, \vec{q}_1 \right\rangle \\ 0 & \left\langle \vec{u}_2, \vec{q}_2 \right\rangle & \left\langle \vec{u}_3, \vec{q}_2 \right\rangle \\ 0 & 0 & \left\langle \vec{u}_3, \vec{q}_3 \right\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix}$$

The QR–decomposition of the matrix is

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix}$$

$$\boldsymbol{A} \quad = \quad \boldsymbol{Q} \quad\quad \boldsymbol{R}$$

e)
$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \quad \begin{matrix} \\ R_2 + R_1 \\ R_3 - R_1 \\ R_4 + R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \quad \begin{matrix} \\ \\ \\ R_4 - R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix is linearly dependent, so ***doesn't*** have a QR–decomposition.

Exercise

Verify that the Cauchy-Schwarz inequality holds for the given vectors using the Euclidean inner product

$$\vec{u} = (0,\ -2,\ 2,\ 1), \quad \vec{v} = (-1,\ -1,\ 1,\ 1)$$

Solution

$$\langle \vec{u},\ \vec{v} \rangle = 0 - 2(-1) + 2(1) + 1(1)$$
$$= 5$$

$$\|\langle \vec{u},\ \vec{v} \rangle\| = \sqrt{5}$$

$$\|\vec{u}\|\ \|\vec{v}\| = \sqrt{0+4+4+1}\ \sqrt{1+1+1+1}$$
$$= \sqrt{9}\sqrt{4}$$
$$= 6$$

$$\sqrt{5} < 6 \quad \Rightarrow \quad \|\langle \vec{u},\ \vec{v} \rangle\| \le \|\vec{u}\|\ \|\vec{v}\|$$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1,\ 1]$ spanned by the functions, using the inner product $f_1(x) = x + 2,\quad f_2(x) = x^2 - 3x + 4$

Solution

Let $\vec{u}_1 = f_1 = x + 2,\quad \vec{u}_2 = f_2 = x^2 - 3x + 4$

$$\boxed{\vec{v}_1 = \vec{u}_1 = x + 2}$$

$$\left\langle \vec{v}_1,\ \vec{v}_1 \right\rangle = \int_{-1}^{1} (x+2)^2\, dx$$

$$= \frac{1}{3}(x+2)^3 \Big|_{-1}^{1}$$

$$= \frac{1}{3}(27 - 1)$$

$$= \boxed{\frac{26}{3}}$$

$$\left\langle \vec{u}_2,\ \vec{v}_1 \right\rangle = \int_{-1}^{1} \left(x^2 - 3x + 4\right)(x+2)\ dx$$

$$= \int_{-1}^{1} \left(x^3 - x^2 - 2x + 8\right)\ dx$$

$$= \left(\frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 + 8x\right)\Big|_{-1}^{1}$$

$$= \frac{1}{4} - \frac{1}{3} - 1 + 8 - \frac{1}{4} - \frac{1}{3} + 1 + 8$$

$$= \boxed{\frac{46}{3}}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2,\ \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1$$

$$\vec{v}_2 = x^2 - 3x + 4 - \frac{46}{3}\left(\frac{3}{26}\right)(x+2)$$

$$= x^2 - 3x + 4 - \frac{23}{13}x - \frac{46}{13}$$

$$\boxed{= x^2 - \frac{62}{13}x + \frac{6}{13}}$$

The orthogonal basis is $\left\{ x + 2,\quad x^2 - \frac{62}{13}x + \frac{6}{13} \right\}$

$$\left\langle \vec{v}_2, \vec{v}_2 \right\rangle = \int_{-1}^{1} \left(x^2 - \frac{62}{13}x + \frac{6}{13} \right)^2 dx$$

$$= \frac{1}{169} \int_{-1}^{1} \left(13x^2 - 62x + 6 \right)^2 dx$$

$$= \frac{1}{169} \int_{-1}^{1} \left(169x^4 + 3{,}844x^2 + 36 - 1{,}612x^3 + 156x^2 - 744x \right) dx$$

$$= \frac{1}{169} \left(\frac{169}{5}x^5 + \frac{4{,}000}{3}x^3 + 36x - 403x^4 - 372x^2 \right) \Bigg|_{-1}^{1}$$

$$= \frac{1}{169} \left(\frac{169}{5} + \frac{4{,}000}{3} + 36 - 403 - 372 + \frac{169}{5} + \frac{4{,}000}{3} + 36 + 403 + 372 \right)$$

$$= \frac{1}{169} \left(\frac{338}{5} + \frac{8{,}000}{3} + 72 \right)$$

$$= \boxed{\frac{3{,}238}{195}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\left\| \vec{v}_1 \right\|}$$

$$= \boxed{\frac{\sqrt{3}}{\sqrt{26}} (x + 2)}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\left\| \vec{v}_2 \right\|}$$

$$= \boxed{\sqrt{\frac{195}{3238}} \left(x^2 - \frac{62}{13}x + \frac{6}{13} \right)}$$

The orthonormal basis is $\left\{ \frac{\sqrt{3}}{\sqrt{26}} (x + 2) \, , \quad \sqrt{\frac{195}{3238}} \left(x^2 - \frac{62}{13}x + \frac{6}{13} \right) \right\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1,\ 1]$ spanned by the functions, using the inner product $f_1(x)=x, \quad f_2(x)=x^3, \quad f_3(x)=x^5$

Solution

Let $\vec{u}_1=f_1=x, \quad \vec{u}_2=f_2=x^3, \quad \vec{u}_3=f_3=x^5$

$$\boxed{\vec{v}_1=\vec{u}_1=x}$$

$$\left\langle \vec{v}_1,\ \vec{v}_1\right\rangle=\int_{-1}^{1}x^2dx$$

$$=\frac{1}{3}x^3\Big|_{-1}^{1}$$

$$=\boxed{\frac{2}{3}}$$

$$\left\langle \vec{u}_2,\ \vec{v}_1\right\rangle=\int_{-1}^{1}x^4\ dx$$

$$=\frac{1}{5}x^5\Big|_{-1}^{1}$$

$$=\boxed{\frac{2}{5}}$$

$$\vec{v}_2=\vec{u}_2-\frac{\left\langle \vec{u}_2,\ \vec{v}_1\right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$=x^3-\frac{2}{5}\left(\frac{3}{2}\right)(x)$$

$$\boxed{=x^3-\frac{3}{5}x}$$

$$\left\langle \vec{v}_2,\ \vec{v}_2\right\rangle=\int_{-1}^{1}\left(x^3-\frac{3}{5}x\right)^2\ dx$$

$$=\int_{-1}^{1}\left(x^6-\frac{6}{5}x^4+\frac{9}{25}x^2\right)\ dx$$

$$=\left(\frac{1}{7}x^7-\frac{6}{25}x^5+\frac{3}{25}x^3\right)\Big|_{-1}^{1}$$

$$=2\left(\frac{1}{7}-\frac{6}{25}+\frac{3}{25}\right)$$

$$= \boxed{\frac{8}{175}}$$

$$\left\langle \vec{u}_3, \vec{v}_1 \right\rangle = \int_{-1}^{1} x^6 \, dx$$

$$= \frac{1}{7} x^7 \Big|_{-1}^{1}$$

$$= \boxed{\frac{2}{7}}$$

$$\left\langle \vec{u}_3, \vec{v}_2 \right\rangle = \int_{-1}^{1} x^5 \left(x^3 - \frac{3}{5} x \right) dx$$

$$= \int_{-1}^{1} \left(x^8 - \frac{3}{5} x^6 \right) dx$$

$$= \left(\frac{1}{9} x^9 - \frac{3}{35} x^7 \right) \Big|_{-1}^{1}$$

$$= 2 \left(\frac{1}{9} - \frac{3}{35} \right)$$

$$= \boxed{\frac{16}{315}}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\left\langle \vec{u}_3, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1 - \frac{\left\langle \vec{u}_3, \vec{v}_2 \right\rangle}{\left\| \vec{v}_2 \right\|^2} \vec{v}_2$$

$$= x^5 - \frac{16}{315} \left(\frac{175}{8} \right) \left(x^3 - \frac{3}{5} x \right) - \frac{2}{7} \left(\frac{3}{2} \right) x$$

$$= x^5 - \frac{70}{63} \left(x^3 - \frac{3}{5} x \right) - \frac{3}{7} x$$

$$= x^5 - \frac{70}{63} x^3 + \frac{14}{21} x - \frac{3}{7} x$$

$$= \boxed{x^5 - \frac{70}{63} x^3 + \frac{5}{21} x}$$

The orthogonal basis is $\left\{ x, \ x^3 - \frac{3}{5} x \ , \ x^5 - \frac{70}{63} x^3 + \frac{5}{21} x \ \right\}$

$$\left\langle \vec{v}_3, \vec{v}_3 \right\rangle = \int_{-1}^{1} \left(x^5 - \frac{70}{63} x^3 + \frac{5}{21} x \right)^2 dx$$

$$= \int_{-1}^{1} \frac{1}{3{,}969} \left(63x^5 - 70x^3 + 15x \right)^2 dx$$

$$= \frac{1}{3{,}969}\int_{-1}^{1}\left(3{,}969x^{10} - 8{,}820x^{8} + 1{,}890x^{6} - 2{,}100x^{4} + 4{,}900x^{6} + 225x^{2}\right)dx$$

$$= \frac{1}{3{,}969}\left(\frac{3{,}969}{11}x^{11} - 980x^{9} + 970x^{7} - 420x^{5} + 75x^{3}\right)\Big|_{-1}^{1}$$

$$= \frac{2}{3{,}969}\left(\frac{3{,}969}{11} - 980 + 970 - 420 + 75\right)$$

$$= \frac{2}{3{,}969}\left(\frac{3{,}969}{11} - 355\right)$$

$$= \frac{2}{3{,}969}\left(\frac{64}{11}\right)$$

$$\boxed{= \frac{128}{43{,}659}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{x}{\sqrt{2/3}}$$

$$\boxed{= \frac{\sqrt{3}}{\sqrt{2}}x}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \sqrt{\frac{175}{8}}\left(x^3 - \frac{3}{5}x\right)$$

$$\boxed{= \frac{5\sqrt{7}}{2\sqrt{2}}\left(x^3 - \frac{3}{5}x\right)}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \sqrt{\frac{43{,}659}{128}}\left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x\right)$$

$$\boxed{= \frac{63\sqrt{11}}{8\sqrt{2}}\left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x\right)}$$

The orthonormal basis is $\left\{\frac{\sqrt{3}}{\sqrt{2}}x,\ \ \frac{5}{2}\sqrt{\frac{7}{2}}\left(x^3 - \frac{3}{5}x\right),\ \ \frac{63}{8}\sqrt{\frac{11}{2}}\left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x\right)\right\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1,\ 1]$ spanned by the functions, using the inner product $\quad f_1(x)=1, \quad f_2(x)=x, \quad f_3(x)=\frac{1}{2}\left(3x^2-1\right)$

Solution

Let $\vec{u}_1 = f_1 = 1, \quad \vec{u}_2 = f_2 = x, \quad \vec{u}_3 = f_3 = \frac{3}{2}x^2 - \frac{1}{2}$

$$\boxed{\vec{v}_1 = \vec{u}_1 = 1}$$

$$\left\langle \vec{v}_1, \vec{v}_1 \right\rangle = \int_{-1}^{1} 1\, dx$$

$$= x \Big|_{-1}^{1}$$

$$\boxed{= 2}$$

$$\left\langle \vec{u}_2, \vec{v}_1 \right\rangle = \int_{-1}^{1} x\, dx$$

$$= \frac{1}{2}x^2 \Big|_{-1}^{1}$$

$$\boxed{= 0}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2}\vec{v}_1$$

$$= x - \frac{0}{2}(1)$$

$$\boxed{= x}$$

$$\left\langle \vec{v}_2, \vec{v}_2 \right\rangle = \int_{-1}^{1} x^2\, dx$$

$$= \frac{1}{3}x^3 \Big|_{-1}^{1}$$

$$\boxed{= \frac{2}{3}}$$

$$\left\langle \vec{u}_3, \vec{v}_1 \right\rangle = \frac{1}{2}\int_{-1}^{1} \left(3x^2 - 1\right) dx$$

$$= \frac{1}{2}\left(x^3 - x\right) \Big|_{-1}^{1}$$

$$\boxed{= 0}$$

$$\left\langle \vec{u}_3, \vec{v}_2 \right\rangle = \frac{1}{2}\int_{-1}^{1} x\left(3x^2 - 1\right) dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(3x^3 - x\right) dx$$

$$= \frac{1}{2}\left(\frac{3}{4}x^4 - \frac{1}{2}x^2\right)\Big|_{-1}^{1}$$

$$\boxed{= 0}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\left\langle \vec{u}_3, \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1 - \frac{\left\langle \vec{u}_3, \vec{v}_2 \right\rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2$$

$$= \frac{3}{2}x^2 - \frac{1}{2} - \frac{0}{1}(1) - \frac{0}{\frac{2}{3}}(x)$$

$$\boxed{= \frac{1}{2}\left(3x^2 - 1\right)}$$

The orthogonal basis is $\left\{1,\ x\ ,\ \frac{1}{2}\left(3x^2 - 1\right)\right\}$

$$\left\langle \vec{v}_3, \vec{v}_3 \right\rangle = \frac{1}{4}\int_{-1}^{1} \left(3x^2 - 1\right)^2 dx$$

$$= \frac{1}{4}\int_{-1}^{1} \left(9x^4 - 6x^2 + 1\right) dx$$

$$= \frac{1}{4}\left(\frac{9}{5}x^5 - 2x^3 + x\right)\Big|_{-1}^{1}$$

$$= \frac{1}{2}\left(\frac{9}{5} - 2 + 1\right)$$

$$\boxed{= \frac{2}{5}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$\boxed{= \frac{1}{\sqrt{2}}}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$\boxed{= \sqrt{\frac{3}{2}}\ x}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\sqrt{\frac{2}{5}}}\frac{1}{2}\left(3x^2 - 1\right)$$

$$\boxed{= \frac{1}{2}\sqrt{\frac{5}{2}}\left(3x^2 - 1\right)}$$

The orthonormal basis is $\left\{\frac{1}{\sqrt{2}},\ \sqrt{\frac{3}{2}}x\ ,\ \frac{1}{2}\sqrt{\frac{5}{2}}\left(3x^2 - 1\right)\right\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1,\ 1]$ spanned by the functions, using the inner product $\quad f_1(x) = 1, \quad f_2(x) = \sin \pi x, \quad f_3(x) = \cos \pi x$

Solution

Let $\quad \vec{u}_1 = f_1 = 1, \quad \vec{u}_2 = f_2 = \sin \pi x, \quad \vec{u}_3 = f_3 = \cos \pi x$

$$\boxed{\vec{v}_1 = \vec{u}_1 = 1}$$

$$\left\langle \vec{v}_1,\ \vec{v}_1 \right\rangle = \int_{-1}^{1} 1dx$$

$$= x\Big|_{-1}^{1}$$

$$\boxed{= 2}$$

$$\left\langle \vec{u}_2,\ \vec{v}_1 \right\rangle = \int_{-1}^{1} \sin \pi x\ dx$$

$$= -\frac{1}{\pi}\cos \pi x\Big|_{-1}^{1}$$

$$\boxed{= 0}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2,\ \vec{v}_1 \right\rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$\boxed{= \sin \pi x}$$

$$\left\langle \vec{v}_2,\ \vec{v}_2 \right\rangle = \int_{-1}^{1} \sin^2 \pi x\ dx$$

$$=\frac{1}{2}\int_{-1}^{1}(1-\cos 2\pi x)\,dx$$

$$=\frac{1}{2}\left(x-\frac{1}{2\pi}\sin 2\pi x\right)\Big|_{-1}^{1}$$

$$\underline{=1}$$

$$\langle \vec{u}_3,\ \vec{v}_1\rangle=\int_{-1}^{1}\cos\pi x\,dx$$

$$=\frac{1}{\pi}\sin\pi x\Big|_{-1}^{1}$$

$$\underline{=0}$$

$$\langle \vec{u}_3,\ \vec{v}_2\rangle=\int_{-1}^{1}\cos\pi x\sin\pi x\,dx$$

$$=\frac{1}{2}\int_{-1}^{1}\sin 2\pi x\,dx$$

$$=-\frac{1}{4\pi}\cos 2\pi x\ \Big|_{-1}^{1}$$

$$\underline{=0}$$

$$\vec{v}_3=\vec{u}_3-\frac{\langle \vec{u}_3,\ \vec{v}_1\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1-\frac{\langle \vec{u}_3,\ \vec{v}_2\rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2$$

$$=\cos\pi x-\frac{0}{}-\frac{0}{}$$

$$\underline{=\cos\pi x}$$

The orthogonal basis is $\left\{1,\ \ \sin\pi x-\frac{1}{\pi}\ ,\ \ \cos\pi x\right\}$

$$\langle \vec{v}_3,\ \vec{v}_3\rangle=\int_{-1}^{1}\cos^2\pi x\,dx$$

$$=\frac{1}{2}\int_{-1}^{1}(1+\cos 2\pi x)\,dx$$

$$=\frac{1}{2}\left(x+\frac{1}{2\pi}\sin 2\pi x\right)\ \Big|_{-1}^{1}$$

$$\underline{=1}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \frac{1}{\sqrt{2}}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \sin \pi x$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$= \cos \pi x$$

The orthonormal basis is $\left\{\frac{1}{\sqrt{2}},\ \sin \pi x\ ,\ \cos \pi x\right\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1,\ 1]$ spanned by the functions, using the inner product $f_1(x) = \sin \pi x, \quad f_2(x) = \sin 2\pi x, \quad f_3(x) = \sin 3\pi x$

Solution

Let $\vec{u}_1 = f_1 = \sin \pi x, \quad \vec{u}_2 = f_2 = \sin 2\pi x, \quad \vec{u}_3 = f_3 = \sin 3\pi x$

$$\vec{v}_1 = \vec{u}_1 = \sin \pi x$$

$$\left\langle \vec{v}_1,\ \vec{v}_1 \right\rangle = \int_{-1}^{1} \sin^2 \pi x\ dx$$

$$= \frac{1}{2}\int_{-1}^{1} (1 - \cos 2\pi x)\ dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2\pi}\sin 2\pi x\right)\Big|_{-1}^{1}$$

$$= 1$$

$$\left\langle \vec{u}_2,\ \vec{v}_1 \right\rangle = \int_{-1}^{1} \sin \pi x \sin 2\pi x\ dx \qquad \sin a \sin b = \frac{1}{2}\left[\cos(a+b) - \cos(a-b)\right]$$

$$= \frac{1}{2}\int_{-1}^{1} \left(\cos 3\pi x - \cos(-\pi x)\right)\ dx$$

$$= \frac{1}{2}\int_{-1}^{1} (\cos 3\pi x - \cos \pi x)\ dx$$

$$= \frac{1}{2}\left(\frac{1}{3\pi}\sin 3\pi x - \frac{1}{\pi}\sin \pi x\right)\Big|_{-1}^{1}$$

$$\underline{= 0 \,|}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$\underline{= \sin 2\pi x \,|}$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^{1} \sin^2 2\pi x\ dx$$

$$= \frac{1}{2}\int_{-1}^{1} (1 - \cos 4\pi x)\ dx$$

$$= \frac{1}{2}\left(x - \frac{1}{4\pi}\sin 4\pi x\right)\Big|_{-1}^{1}$$

$$\underline{= 1 \,|}$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_{-1}^{1} \sin \pi x \sin 3\pi x\ dx \qquad \sin a \sin b = \frac{1}{2}\left[\cos(a+b) - \cos(a-b)\right]$$

$$= \frac{1}{2}\int_{-1}^{1} (\cos 4\pi x - \cos(-2\pi x))\ dx$$

$$= \frac{1}{2}\int_{-1}^{1} (\cos 4\pi x - \cos 2\pi x)\ dx$$

$$= \frac{1}{2}\left(\frac{1}{4\pi}\sin 4\pi x - \frac{1}{2\pi}\sin 2\pi x\right)\Big|_{-1}^{1}$$

$$\underline{= 0 \,|}$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^{1} \sin 3\pi x \sin 2\pi x\ dx \qquad \sin a \sin b = \frac{1}{2}\left[\cos(a+b) - \cos(a-b)\right]$$

$$= \frac{1}{2}\int_{-1}^{1} (\cos 5\pi x - \cos \pi x)\ dx$$

$$= \frac{1}{2}\left(\frac{1}{5\pi}\sin 5\pi x - \frac{1}{\pi}\sin \pi x\right)\Big|_{-1}^{1}$$

$$\underline{= 0 \,|}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= \sin 3\pi x$$

The orthogonal basis is $\{\sin \pi x, \quad \sin 2\pi x\ , \quad \sin 3\pi x\}$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_{-1}^{1} \sin^2 3\pi x\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} (1 - \cos 6\pi x)\, dx$$

$$= \frac{1}{2}\left(x - \frac{1}{6\pi}\sin 6\pi x\right) \Big|_{-1}^{1}$$

$$= 1$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \sin \pi x$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \sin 2\pi x$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \sin 3\pi x$$

The orthonormal basis is $\{\sin \pi x, \quad \sin 2\pi x\ , \quad \sin 3\pi x\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $\quad f_1(x) = \cos \pi x, \quad f_2(x) = \cos 2\pi x, \quad f_3(x) = \cos 3\pi x$

Solution

Let $\quad \vec{u}_1 = f_1 = \cos \pi x, \quad \vec{u}_2 = f_2 = \cos 2\pi x, \quad \vec{u}_3 = f_3 = \cos 3\pi x$

$$\vec{v}_1 = \vec{u}_1 = \cos \pi x$$

$$\left\langle \vec{v}_1, \vec{v}_1 \right\rangle = \int_{-1}^{1} \cos^2 \pi x \, dx$$

$$= \frac{1}{2}\int_{-1}^{1} (1+\cos 2\pi x) \, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{2\pi}\sin 2\pi x \right) \Big|_{-1}^{1}$$

$$\underline{= 1}$$

$$\left\langle \vec{u}_2, \vec{v}_1 \right\rangle = \int_{-1}^{1} \cos 2\pi x \cos \pi x \, dx$$

$$\cos a \cos b = \frac{1}{2}\left[\cos(a+b) + \cos(a-b) \right]$$

$$= \frac{1}{2}\int_{-1}^{1} (\cos 3\pi x + \cos \pi x) \, dx$$

$$= \frac{1}{2}\left(\frac{1}{3\pi}\sin 3\pi x + \frac{1}{\pi}\sin \pi x \right) \Big|_{-1}^{1}$$

$$\underline{= 0}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2}\vec{v}_1$$

$$\underline{= \cos 2\pi x}$$

$$\left\langle \vec{v}_2, \vec{v}_2 \right\rangle = \int_{-1}^{1} \cos^2 2\pi x \, dx$$

$$= \frac{1}{2}\int_{-1}^{1} (1+\cos 4\pi x) \, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{4\pi}\sin 4\pi x \right) \Big|_{-1}^{1}$$

$$\underline{= 1}$$

$$\left\langle \vec{u}_3, \vec{v}_1 \right\rangle = \int_{-1}^{1} \cos 3\pi x \cos \pi x \, dx$$

$$\cos a \cos b = \frac{1}{2}\left[\cos(a+b) + \cos(a-b) \right]$$

$$= \frac{1}{2}\int_{-1}^{1} (\cos 4\pi x + \cos 2\pi x) \, dx$$

$$= \frac{1}{2}\left(\frac{1}{4\pi}\sin 4\pi x + \frac{1}{2\pi}\sin 2\pi x \right) \Big|_{-1}^{1}$$

$$\underline{=0}$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^{1} \cos 3\pi x \cos 2\pi x \, dx \qquad \cos a \cos b = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right]$$

$$= \frac{1}{2}\int_{-1}^{1} (\cos 5\pi x + \cos \pi x) \, dx$$

$$= \frac{1}{2}\left(\frac{1}{5\pi}\sin 5\pi x + \frac{1}{\pi}\sin \pi x\right) \Big|_{-1}^{1}$$

$$\underline{=0}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$\underline{= \cos 3\pi x}$$

The orthogonal basis is $\{\cos \pi x, \quad \cos 2\pi x\,, \quad \cos 3\pi x\}$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_{-1}^{1} \cos^2 3\pi x \, dx$$

$$= \frac{1}{2}\int_{-1}^{1} (1 + \cos 6\pi x) \, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{6\pi}\sin 6\pi x\right) \Big|_{-1}^{1}$$

$$\underline{=1}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\underline{= \cos \pi x}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$\underline{= \cos 2\pi x}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$\underline{= \cos 3\pi x}$$

The orthonormal basis is $\{\cos \pi x, \quad \cos 2\pi x\,, \quad \cos 3\pi x\}$

Exercise

For $\mathbb{P}_3[x]$, define the inner product over $\mathbb{R}$ as

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\, dx$$

a) If $f(x)=1$ is a unit vector in $\mathbb{P}_3[x]$?

b) Find an orthonormal basis for the subspace spanned by $x \text{ and } x^2$.

c) Complete the basis in part (*b*) to an orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product.

d) Is

$$[f, g] = \int_{0}^{1} f(x)g(x)\, dx$$

Also, an inner product for $\mathbb{P}_3[x]$

e) Find a pair of vectors $\vec{v}$ and $\vec{w}$ such that

$$\langle \vec{v}, \vec{w}\rangle = 0 \quad but \quad [\vec{v}, \vec{w}] \neq 0$$

f) Is the basis found in part (*c*) are orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product in part (*d*)?

Solution

a) $f(x)=1$

$$\langle f, f\rangle = \int_{-1}^{1} f(x) f(x)\, dx$$

$$= \int_{-1}^{1} dx$$

$$= x \Big|_{-1}^{1}$$

$$= 1+1$$

$$\underline{= 2 \neq 1}$$

Therefore, when $f(x)=1$ is ***not*** a unit vector in $\mathbb{P}_3[x]$

b) Let $\vec{u}_1 = f = x, \quad \vec{u}_2 = g = x^2$

$$\underline{\vec{v}_1 = \vec{u}_1 = x}$$

$$\left\langle \vec{v}_1, \vec{v}_1 \right\rangle = \int_{-1}^{1} x^2\, dx$$

$$= \frac{1}{3}x^3 \Big|_{-1}^{1}$$

$$= \frac{1}{3}(1+1)$$

$$= \frac{2}{3}$$

$$\left\langle \vec{u}_2, \vec{v}_1 \right\rangle = \int_{-1}^{1} x^2(x)\, dx$$

$$= \int_{-1}^{1} x^3\, dx$$

$$= \frac{1}{4}x^4 \Big|_{-1}^{1}$$

$$= \frac{1}{4}(1-1)$$

$$= 0$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2}\vec{v}_1$$

$$= x^2 - \frac{0}{*}(\ \ )$$

$$= x^2$$

$$\left\langle \vec{v}_2, \vec{v}_2 \right\rangle = \int_{-1}^{1} x^4\, dx$$

$$= \frac{1}{5}x^5 \Big|_{-1}^{1}$$

$$= \frac{1}{5}(1+1)$$

$$= \frac{2}{5}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\left\| \vec{v}_1 \right\|}$$

$$= \frac{x}{\sqrt{\frac{2}{3}}}$$

$$= \sqrt{\frac{3}{2}}\, x$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{x^2}{\sqrt{\frac{2}{5}}}$$

$$= \sqrt{\frac{5}{2}}\, x^2$$

The orthonormal basis is $\left\{ \sqrt{\frac{3}{2}}\, x,\ \sqrt{\frac{5}{2}}\, x^2 \right\}$

c) Since $\vec{u}_1 = x, \quad \vec{u}_2 = x^2$ in $\mathbb{P}_3[x]$

Then, let $\vec{u}_3 = 1$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_{-1}^{1} (1)(x)\, dx$$

$$= \int_{-1}^{1} x\, dx$$

$$= \frac{1}{2}x^2 \Big|_{-1}^{1}$$

$$= \frac{1}{2}(1-1)$$

$$= 0$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^{1} (1)\left(x^2\right) dx$$

$$= \int_{-1}^{1} x^2\, dx$$

$$= \frac{1}{3}x^3 \Big|_{-1}^{1}$$

$$= \frac{1}{3}(1+1)$$

$$= \frac{2}{3}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$=1-\frac{0}{\frac{2}{3}}(x)-\frac{\frac{2}{3}}{\frac{2}{5}}\left(x^2\right)$$

$$\boxed{=1-\frac{5}{3}x^2}$$

$$\left\langle \vec{v}_3, \vec{v}_3 \right\rangle = \int_{-1}^{1}\left(1-\frac{5}{3}x^2\right)^2\,dx$$

$$= \int_{-1}^{1}\left(1-\frac{10}{3}x^2+\frac{25}{9}x^4\right)dx$$

$$= \left(x-\frac{10}{9}x^3+\frac{5}{9}x^5\right)\Big|_{-1}^{1}$$

$$= 2\left(1-\frac{10}{9}+\frac{5}{9}\right)$$

$$= 2\left(\frac{9-5}{9}\right)$$

$$\boxed{=\frac{8}{9}}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$=\left(\sqrt{\frac{9}{8}}\right)\left(1-\frac{5}{3}x^2\right)$$

$$=\frac{3}{2\sqrt{2}}\left(1-\frac{5}{3}x^2\right)$$

$$\boxed{=\frac{3}{2\sqrt{2}}-\frac{5}{2\sqrt{2}}\,x^2}$$

The orientation basis is $\left\{ \sqrt{\frac{3}{2}}\,x,\ \ \sqrt{\frac{5}{2}}\,x^2,\ \ \frac{3}{2\sqrt{2}}-\frac{5}{2\sqrt{2}}\,x^2 \right\}$

d) $[f, g] = \int_0^1 f(x)g(x)\,dx$

Let $\vec{u}_1 = 1,\quad \vec{u}_2 = x,\quad \vec{u}_3 = x^2$

$\boxed{\vec{v}_1 = \vec{u}_1 = 1}$

$$\left\langle \vec{v}_1, \vec{v}_1 \right\rangle = \int_0^1 1\,dx$$

$$= x \Big|_0^1$$

$$\boxed{= 1}$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_0^1 x(1)\, dx$$

$$= \int_0^1 x\, dx$$

$$= \frac{1}{2}x^2 \Big|_0^1$$

$$\boxed{= \frac{1}{2}}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= x - \frac{\frac{1}{2}}{1}(1)$$

$$\boxed{= x - \frac{1}{2}}$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_0^1 \left(x - \frac{1}{2}\right)^2\, dx$$

$$= \int_0^1 \left(x - \frac{1}{2}\right)^2\, d\left(x - \frac{1}{2}\right)$$

$$= \frac{1}{3}\left(x - \frac{1}{2}\right)^3 \Big|_0^1$$

$$= \frac{1}{3}\left(\left(\frac{1}{2}\right)^3 - \left(-\frac{1}{2}\right)^3\right)$$

$$= \frac{1}{3}\left(\frac{1}{8} + \frac{1}{8}\right)$$

$$\boxed{= \frac{1}{12}}$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_0^1 \left(x^2\right)(1)\, dx$$

$$= \int_0^1 x^2\, dx$$

$$= \frac{1}{3}x^3 \Big|_0^1$$

$$\boxed{= \frac{1}{3}}$$

$$\left\langle \vec{u}_3, \vec{v}_2 \right\rangle = \int_0^1 \left(x^2\right)\left(x - \frac{1}{2}\right) dx$$

$$= \int_0^1 \left(x^3 - \frac{1}{2}x^2\right) dx$$

$$= \left(\frac{1}{4}x^4 - \frac{1}{6}x^3\right) \Big|_0^1$$

$$= \frac{1}{4} - \frac{1}{6}$$

$$\boxed{= \frac{1}{12}}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\left\langle \vec{u}_3, \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1 - \frac{\left\langle \vec{u}_3, \vec{v}_2 \right\rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2$$

$$= x^2 - \frac{\frac{1}{3}}{1}(1) - \frac{\frac{1}{12}}{\frac{1}{12}}\left(x - \frac{1}{2}\right)$$

$$= x^2 - \frac{1}{3} - x + \frac{1}{2}$$

$$\boxed{= x^2 - x + \frac{1}{6}}$$

$$\left\langle \vec{v}_3, \vec{v}_3 \right\rangle = \int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 dx$$

$$= \int_0^1 \left(x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36}\right) dx$$

$$= \left(\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{4}{9}x^3 - \frac{1}{6}x^2 + \frac{1}{36}x\right) \Big|_0^1$$

$$= \frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36}$$

$$= \frac{2-5}{10} + \frac{16-6+1}{36}$$

$$= -\frac{3}{10} + \frac{11}{36}$$

$$= \frac{-108+110}{360}$$

$$= \frac{1}{180}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= 1$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{x - \frac{1}{2}}{\sqrt{\frac{1}{12}}}$$

$$= 2\sqrt{3}\left(x - \frac{1}{2}\right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \left(\sqrt{180}\right)\left(x^2 - x + \frac{1}{6}\right)$$

$$= \left(6\sqrt{5}\right)\left(x^2 - x + \frac{1}{6}\right)$$

$$= 6\sqrt{5}\,x^2 - 6\sqrt{5}\,x + \sqrt{5}$$

The orthonormal basis is $\left\{ 1,\ 2\sqrt{3}\left(x - \frac{1}{2}\right),\ \sqrt{5}\left(6x^2 - 6x + 1\right) \right\}$

Therefore, $[f,\ g] = \int_0^1 f(x)\,g(x)\,dx$ is an inner product for $\mathbb{P}_3[x]$

e) Let assume: $\vec{v} = 1$ *and* $\vec{w} = x$

$$\langle \vec{v},\ \vec{w} \rangle = \int_{-1}^{1} 1(x)\,dx$$

$$= \int_{-1}^{1} x\,dx$$

$$= \frac{1}{2}x^2 \Big|_{-1}^{1}$$

$$= \frac{1}{2}(1 - 1)$$

$$= 0 \quad \checkmark$$

$$[\vec{v},\ \vec{w}] = \int_0^1 1(x)\,dx$$

$$= \frac{1}{2}x^2 \Big|_0^1$$

$$\boxed{= \frac{1}{2} \neq 0} \quad \checkmark$$

f) The orthonormal basis in part (c) $\left\{ \sqrt{\frac{3}{2}}\,x,\ \sqrt{\frac{5}{2}}\,x^2,\ \frac{3}{2\sqrt{2}} - \frac{5}{2\sqrt{2}}\,x^2 \right\}$ are ***not*** the same as

the orthonormal basis in part (d) $\left\{ 1,\ 2\sqrt{3}\left(x - \frac{1}{2}\right),\ \sqrt{5}\left(6x^2 - 6x + 1\right) \right\}$