Section **2.3** – **Orthogonality**

Definition

Two nonzero vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are said to be ***orthogonal*** (or ***perpendicular***) if their dot product is zero $\vec{u} \bullet \vec{v} = 0$.

We will also agree that he zero vector in $\mathbb{R}^n$ is orthogonal to every vector in $\mathbb{R}^n$. A nonempty set of vectors $\mathbb{R}^n$ is called an ***orthogonal set*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set of unit vectors is called an ***orthonormal set***.

Example

The floor of your room (extended to infinity) is a subspace V. The line where two walls meet is a subspace W (one-dimensional). Those subspaces are orthogonal. Every vector up the meeting line is perpendicular to every vector on the floor. The origin (0, 0, 0) is in the corner.

Example

Show that $\vec{u} = (-2, 3, 1, 4)$ and $\vec{v} = (1, 2, 0, -1)$ are orthogonal in $\mathbb{R}^4$

Solution

$$\vec{u} \bullet \vec{v} = (-2)(1) + (3)(2) + (1)(0) + (4)(-1)$$

$$= -2 + 6 + 0 - 4$$

$$\underline{= 0}$$

These vectors are orthogonal in $\mathbb{R}^4$

Standard Unit Vectors

$$\hat{i} \bullet \hat{j} = \hat{i} \bullet \hat{k} = \hat{j} \bullet \hat{k} = \mathbf{0}$$

Proof

$$\hat{i} \bullet \hat{j} = (1,\ 0,\ 0) \bullet (0,\ 1,\ 0)$$

$$\underline{= 0}$$

Normal

To specify slope and inclination is to use a nonzero vector $\vec{n}$, called a ***normal***, that is orthogonal to the line or plane.
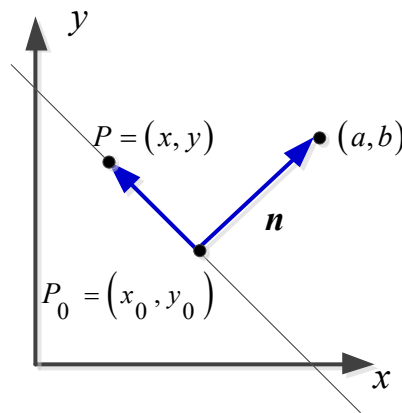
The line passes through a point $P_0\left(x_0, y_0\right)$ that has a normal $\vec{n} = (a, b)$

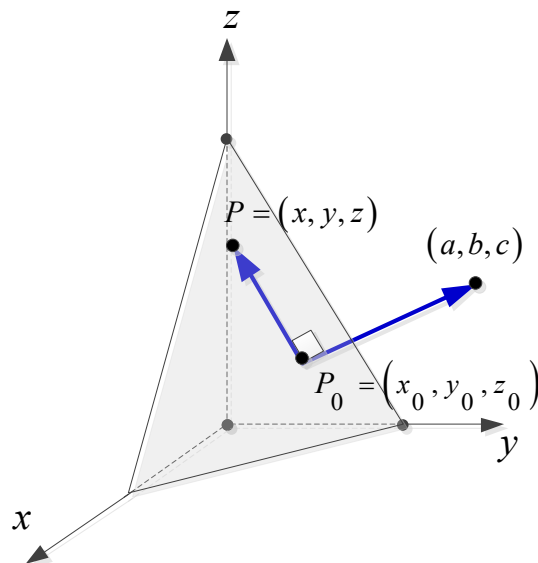The plane through $P_0\left(x_0, y_0, z_0\right)$ that has a normal $\vec{n} = (a, b, c)$.

Both the line and the plane are represented by the vector equation

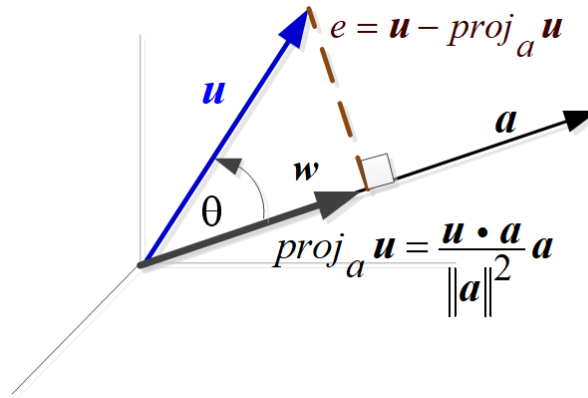$$\vec{n} \cdot \overrightarrow{P_0 P} = 0$$

The line equation: $\qquad a\left(x - x_0\right) + b\left(y - y_0\right) = 0$

The plane equation: $\qquad a\left(x - x_0\right) + b\left(y - y_0\right) + c\left(z - z_0\right) = 0$

Projections

Theorem   **Projection onto a line**

If $\vec{u}$ and $\vec{a}$ are vectors in $\mathbf{R}^n$, and if $\vec{a} \neq 0$, then $\vec{u}$ can be expressed in exactly one way in the form $\vec{u} = \vec{w} + \vec{e}$, where $\vec{w}$ is a scalar multiple of $\vec{a}$ and $\vec{e}$ is orthogonal to $\vec{a}$.

The vector $\vec{w}$ is called the ***orthogonal projection*** of $\vec{u}$ on $\vec{a}$ or sometimes ***component*** of $\vec{u}$ along $\vec{a}$. The vector $\vec{e}$ is called the vector ***component*** of $\vec{u}$ ***orthogonal*** to $\vec{a}$ (error vector and should be perpendicular to $\vec{a}$)

$$proj_{\vec{a}}\vec{u} = \frac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a} = \vec{p} \qquad \text{(\textit{vector component of } } \vec{u} \text{ \textit{ along }} \vec{a})$$

$$\vec{u} - proj_{\vec{a}}\vec{u} = \vec{u} - \frac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a} \qquad \text{(\textit{vector component of } } \vec{u} \text{ \textit{ orthogonal to }} \vec{a})$$

The length is $\left\| proj_{\vec{a}}\vec{u} \right\| = \|\vec{u}\| |\cos\theta|$

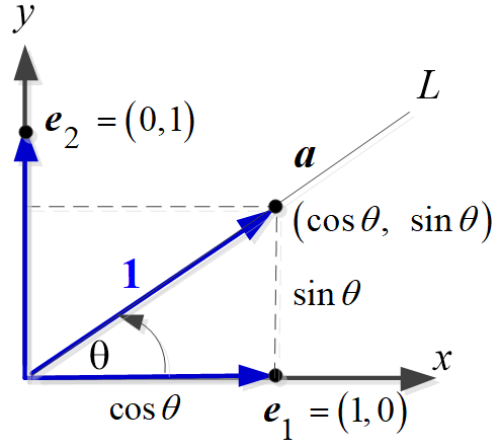$$\left\| proj_{\vec{a}}\vec{u} \right\| = \frac{|\vec{u} \bullet \vec{a}|}{\|\vec{a}\|}$$

Special case: If $\vec{u} = \vec{a}$ then $\dfrac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|^2} = 1$. The projection of $\vec{a}$ onto $\vec{a}$ is itself.

Special case: If $\vec{u}$ is perpendicular to $\vec{a}$ then $\vec{u} \bullet \vec{a} = 0$. The projection is $\vec{p} = \vec{0}$.

Example

Find the orthogonal projections of the vectors $\hat{e}_1 = (1, 0)$ and $\hat{e}_2 = (0, 1)$ on the line L that makes an angle θ with the positive x-axis in $\mathbb{R}^2$

Solution

Let $\vec{a} = (\cos\theta, \sin\theta)$ be the unit vector along the line L.

$$\|\vec{a}\| = \sqrt{\cos^2\theta + \sin^2\theta}$$
$$\underline{= 1}$$

$$\hat{e}_1 \cdot \vec{a} = (1,0)(\cos\theta, \sin\theta)$$
$$= (1)\cos\theta + (0)\sin\theta$$
$$\underline{= \cos\theta}$$

$$proj_{\vec{a}}\hat{e}_1 = \frac{\hat{e}_1 \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a}$$
$$= \frac{\cos\theta}{1}(\cos\theta, \sin\theta)$$
$$= \left(\cos^2\theta, \cos\theta\sin\theta\right)$$

$$proj_{\vec{a}}\hat{e}_2 = \frac{\hat{e}_2 \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a}$$
$$= \frac{(0, 1)(\cos\theta, \sin\theta)}{1}(\cos\theta, \sin\theta)$$
$$= \sin\theta(\cos\theta, \sin\theta)$$
$$= \left(\sin\theta\cos\theta, \sin^2\theta\right)$$

Example

Let $\vec{u} = (2, -1, 3)$ and $\vec{a} = (4, -1, 2)$. Find the vector component of $\vec{u}$ along $\vec{a}$ and the vector component of $\vec{u}$ orthogonal to $\vec{a}$.

Solution

$$proj_{\vec{a}}\vec{u} = \frac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a}$$

$$= \frac{(2, -1, 3) \bullet (4, -1, 2)}{\left(\sqrt{4^2 + (-1)^2 + 2^2}\right)^2}(4, -1, 2)$$

$$= \frac{8 + 1 + 6}{21}(4, -1, 2)$$

$$= \frac{15}{21}(4, -1, 2)$$

$$= \frac{5}{7}(4, -1, 2)$$

$$= \left(\frac{20}{7}, -\frac{5}{7}, \frac{10}{7}\right)$$

The vector component of $\vec{u}$ orthogonal to $\vec{a}$ is

$$\vec{u} - proj_{\vec{a}}\vec{u} = (2, -1, 3) - \left(\frac{20}{7}, -\frac{5}{7}, \frac{10}{7}\right)$$

$$= \left(-\frac{6}{7}, -\frac{2}{7}, \frac{11}{7}\right)$$

Theorem of *Pythagoras* in $\mathbb{R}^n$

If $\vec{u}$ and $\vec{v}$ are orthogonal vectors in $\mathbb{R}^n$ with the Euclidean inner product, then

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$

Proof

Since $\vec{u}$ and $\vec{v}$ are orthogonal, then $\vec{u} \bullet \vec{v} = 0$

$$\|\vec{u} + \vec{v}\|^2 = (\vec{u} + \vec{v}) \bullet (\vec{u} + \vec{v})$$

$$= \|\vec{u}\|^2 + 2(\vec{u} \bullet \vec{v}) + \|\vec{v}\|^2$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2$$

Distance

Theorem

In $\mathbb{R}^2$ the distance D between the point $P_0 = (x_0, y_0)$ and the line $ax + by + c = 0$ is

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

In $\mathbb{R}^3$ the distance D between the point $P_0 = (x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Exercises   Section 2.3 – Orthogonality

1.   Determine whether $\vec{u}$ and $\vec{v}$ are orthogonal

   a) $\vec{u} = (-6, \ -2), \quad \vec{v} = (5, \ -7)$ c) $\vec{u} = (1, \ -5, \ 4), \quad \vec{v} = (3, \ 3, \ 3)$

   b) $\vec{u} = (6, \ 1, \ 4), \quad \vec{v} = (2, \ 0, \ -3)$ d) $\vec{u} = (-2, \ 2, \ 3), \quad \vec{v} = (1, \ 7, \ -4)$

2.   Determine whether the vectors form an orthogonal set

   a) $\vec{v}_1 = (2, \ 3), \quad \vec{v}_2 = (3, \ 2)$

   b) $\vec{v}_1 = (1, \ -2), \quad \vec{v}_2 = (-2, \ 1)$

   c) $\vec{u} = (-4, \ 6, \ -10, \ 1) \quad \vec{v} = (2, \ 1, \ -2, \ 9)$

   d) $\vec{u} = (a, \ b) \quad \vec{v} = (-b, \ a)$

   e) $\vec{v}_1 = (-2, \ 1, \ 1), \quad \vec{v}_2 = (1, \ 0, \ 2), \quad \vec{v}_3 = (-2, \ -5, \ 1)$

   f) $\vec{v}_1 = (1, \ 0, \ 1), \quad \vec{v}_2 = (1, \ 1, \ 1), \quad \vec{v}_3 = (-1, \ 0, \ 1)$

   g) $\vec{v}_1 = (2, \ -2, \ 1), \quad \vec{v}_2 = (2, \ 1, \ -2), \quad \vec{v}_3 = (1, \ 2, \ 2)$

3.   Find a unit vector that is orthogonal to both $\vec{u} = (1, 0, 1)$ and $\vec{v} = (0, 1, 1)$

4.   a) Show that $\vec{v} = (a, \ b)$ and $\vec{w} = (-b, \ a)$ are orthogonal vectors.
   b) Use the result to find two vectors that are orthogonal to $\vec{v} = (2, -3)$.
   c) Find two unit vectors that are orthogonal to $(-3, 4)$

5.   Find the vector component of $\vec{u}$ along $\vec{a}$ and the vector component of $\vec{u}$ orthogonal to $\vec{a}$.

   a) $\vec{u} = (6, \ 2), \quad \vec{a} = (3, \ -9)$ d) $\vec{u} = (1, \ 1, \ 1), \quad \vec{a} = (0, \ 2, \ -1)$

   b) $\vec{u} = (3, \ 1, \ -7), \quad \vec{a} = (1, \ 0, \ 5)$ e) $\vec{u} = (2, \ 1, \ 1, \ 2), \quad \vec{a} = (4, \ -4, \ 2, \ -2)$

   c) $\vec{u} = (1, \ 0, \ 0), \quad \vec{a} = (4, \ 3, \ 8)$ f) $\vec{u} = (5, \ 0, \ -3, \ 7), \quad \vec{a} = (2, \ 1, \ -1, -1)$

6.   Project the vector $\vec{v}$ onto the line through $\vec{a}$, check that $\vec{e} = \vec{u} - proj_{\vec{a}}\vec{u}$ is perpendicular to $\vec{a}$:

   a) $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\vec{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ b) $\vec{v} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ and $\vec{a} = \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix}$ c) $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

7.   Find the projection matrix $proj_{\vec{a}}\vec{u} = \dfrac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a}$ onto the line through $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

(8 – 9)   Draw the projection of $\vec{b}$ onto $\vec{a}$ and also compute it from $proj_{\vec{a}}\vec{u} = \dfrac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a}$

8. $\vec{b} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **9.** $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

10.   Show that if $\vec{v}$ is orthogonal to both $\vec{w}_1$ and $\vec{w}_2$, then v is orthogonal to $k_1\vec{w}_1 + k_2\vec{w}_2$ for all scalars k_1 and k_2.

11.   *a)* Project the vector $\vec{v} = (3, 4, 4)$ onto the line through $\vec{a} = (2, 2, 1)$ and then onto the plane that also contains $\vec{a}* = (1, 0, 0)$.

     b)   Check that the first error vector $\vec{v} - \vec{p}$ is perpendicular to $\vec{a}$, and the second error vector $\vec{v} - \vec{p}*$ is also perpendicular to $\vec{a}*$.

12.   Compute the projection matrices $\vec{a}\vec{a}^T / \vec{a}^T\vec{a}$ onto the lines through $\vec{a}_1 = (-1, 2, 2)$ and $\vec{a}_2 = (2, 2, -1)$. Multiply those projection matrices and explain why their product P_1P_2 is what it is. Project $\vec{v} = (1, 0, 0)$ onto the lines $\vec{a}_1$, $\vec{a}_2$, and also onto $\vec{a}_3 = (2, -1, 2)$. Add up the three projections $p_1 + p_2 + p_3$.

13.   If $P^2 = P$ show that $(I - P)^2 = I - P$. When P projects onto the column space of A, $I - P$ projects onto the _____.

14.   What linear combination of $(1, 2, -1)$ and $(1, 0, 1)$ is closest to $\vec{v} = (2, 1, 1)$?

15.   Show that $\vec{u} - \vec{v}$ is orthogonal to $\vec{u} + \vec{v}$ if and only if $\|\vec{u}\| = \|\vec{v}\|$

16.   Given $\vec{u} = (3, -1, 2)$ $\vec{v} = (4, -1, 5)$ and $\vec{w} = (8, -7, -6)$

     a)   Find $3\vec{v} - 4(5\vec{u} - 6\vec{w})$

     b)   Find $\vec{u} \cdot \vec{v}$ and then the angle θ between $\vec{u}$ and $\vec{v}$.

17.   Given: $\vec{u} = (3, 1, 3)$ $\vec{v} = (4, 1, -2)$

     a)   Compute the projection $\vec{w}$ of $\vec{u}$ on $\vec{v}$

     b)   Find $\vec{p} = \vec{u} - \vec{v}$ and show that $\vec{p}$ is perpendicular to $\vec{v}$.

18.   *a)* Show that $\vec{v} = (a, b)$ and $\vec{w} = (-b, a)$ are orthogonal vectors

     b) Use the result in part (*a*) to find two vectors that are orthogonal to $\vec{v} = (2, -3)$

     c) Find two unit vectors that are orthogonal to $(-3, 4)$

19. Show that A (3, 0, 2), B (4, 3, 0), and C (8, 1, −1) are vertices of a right triangle. At which vertex is the right angle?

20. Establish the identity: $\vec{u} \cdot \vec{v} = \dfrac{1}{4}\|\vec{u} + \vec{v}\|^2 - \dfrac{1}{4}\|\vec{u} - \vec{v}\|^2$

21. Find the Euclidean inner product $\vec{u} \cdot \vec{v}$: $\vec{u} = (-1,\ 1,\ 0,\ 4,\ -3)$ $\vec{v} = (-2,\ -2,\ 0,\ 2,\ -1)$

22. Find the Euclidean distance between $\vec{u}$ and $\vec{v}$: $\vec{u} = (3,\ -3,\ -2,\ 0,\ -3)$ $\vec{v} = (-4,\ 1,\ -1,\ 5,\ 0)$

(*Exercises* **22 − 26**) Find

    a) $\vec{v} \cdot \vec{u},\ |\vec{v}|,\ |\vec{u}|$

    b) The cosine of the angle between $\vec{v}$ and $\vec{u}$
    c) The scalar component of $\vec{u}$ in the direction of $\vec{v}$
    d) The vector $proj_{\vec{v}}\vec{u}$

23. $\vec{v} = 2\hat{i} - 4\hat{j} + \sqrt{5}\,\hat{k},\quad \vec{u} = -2\hat{i} + 4\hat{j} - \sqrt{5}\,\hat{k}$

24. $\vec{v} = \dfrac{3}{5}\hat{i} + \dfrac{4}{5}\hat{k},\quad \vec{u} = 5\hat{i} + 12\hat{j}$

25. $\vec{v} = 2\hat{i} + 10\hat{j} - 11\hat{k},\quad \vec{u} = 2\hat{i} + 2\hat{j} + \hat{k}$

26. $\vec{v} = 5\hat{i} + \hat{j},\quad \vec{u} = 2\hat{i} + \sqrt{17}\,\hat{j}$

27. $\vec{v} = \left(\dfrac{1}{\sqrt{2}},\ \dfrac{1}{\sqrt{3}}\right),\quad \vec{u} = \left(\dfrac{1}{\sqrt{2}},\ -\dfrac{1}{\sqrt{3}}\right)$

28. Suppose Ted weighs 180 *lb.* and he is sitting on an inclined plane that drops 3 *units* for every 4 horizontal units. The gravitational force vector is $\vec{F}_g = \begin{pmatrix} 0 \\ -180 \end{pmatrix}$.

    a) Find the force pushing Ted down the slope.
    b) Find the force acting to hold Ted against the slope

29. Prove that is two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$ are orthogonal to nonzero vector $\vec{w}$ in $\mathbb{R}^2$, then $\vec{u}$ and $\vec{v}$ are scalar multiples of each other.