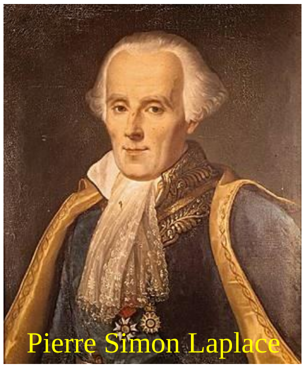
The Laplace Transform

Section 2.7 – Definition of Laplace Transform

Definition

Suppose $f(t)$ is a function of t defined for $0 < t < \infty$. The ***Laplace transform*** of f is the function

$$\mathcal{L}\left[(f)(s)\right] = F(s) = \int_0^\infty f(t)e^{-st}dt$$



(1749 – 1827)

The integral of the Laplace transform is an improper integral because the upper limit is ∞.

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \lim_{T\to\infty}\int_0^T f(t)e^{-st}dt$$

The domain of F is the set of real number s for which the improper integral converges.

Example

Assume $f(t) = e^{at}$

Solution

$$F(s) = \int_0^\infty e^{at}e^{-st}dt$$

$$= \int_0^\infty e^{-(s-a)t}dt$$

$$F(s) = \lim_{T\to\infty}\int_0^T e^{-(s-a)t}dt$$

$$= \lim_{T\to\infty} \left.\frac{-e^{-(s-a)t}}{s-a}\right|_0^T \qquad e^{-(s-a)0} = 1$$

$$= \lim_{T\to\infty}\left(\frac{-e^{-(s-a)T}}{s-a} + \frac{1}{s-a}\right) \qquad e^{-(s-a)\infty} = \frac{1}{e^\infty} = 0$$

$$= \frac{1}{s-a}$$

$$\mathcal{L}\left(e^{at}\right)(s) = F(s) = \boxed{\frac{1}{s-a}} \qquad \text{for } s > a$$

Example

Assume $f(t) = t$

Solution

$$F(s) = \int_0^{\infty} te^{-st} dt$$

$$u = t \qquad dv = \int e^{-st} dt$$

$$du = dt \qquad v = -\frac{1}{s} e^{-st}$$

$$\int te^{-st} dt = -\frac{1}{s} te^{-st} - \int \left(-\frac{1}{s}\right) e^{-st} dt$$

$$= -\frac{1}{s} te^{-st} + \frac{1}{s} \int e^{-st} dt$$

$$= -\frac{1}{s} te^{-st} + \frac{1}{s}\left(-\frac{1}{s}\right) e^{-st}$$

$$= -\frac{1}{s} te^{-st} - \frac{1}{s^2} e^{-st}$$

$$F(s) = \lim_{T \to \infty} \left(-\frac{1}{s} te^{-st} - \frac{1}{s^2} e^{-st} \right)_{t=0}^{T}$$

$$= \lim_{T \to \infty} \left(-\frac{1}{s} Te^{-sT} - \frac{1}{s^2} e^{-sT} + \frac{1}{s^2} \right) \qquad \lim_{T \to \infty} \left(e^{-sT} \right) = 0$$

$$= \boxed{\frac{1}{s^2}}$$

Laplace transform to any power t^n

$$\mathcal{L}\left(t^n\right)(s) = \frac{n!}{s^{n+1}}$$

Example

Assume $f(t) = \sin at$

Solution

$$F(s) = \int_0^\infty e^{-st} \sin at \; dt$$

$$u = e^{-st} \qquad dv = \int \sin at \; dt$$

$$du = -se^{-st} dt \qquad v = -\frac{1}{a}\cos at$$

$$\int e^{-st} \sin at dt = -\frac{1}{a}e^{-st}\cos at - \int \left(-\frac{1}{a}\cos at\right)\left(-se^{-st}\right)dt$$

$$= -\frac{1}{a}e^{-st}\cos at - \frac{s}{a}\int \left(e^{-st}\cos at\right)dt$$

$$\int \left(e^{-st}\cos at\right)dt \qquad u = e^{-st} \qquad dv = \int \cos at \; dt$$

$$du = -se^{-st}dt \qquad v = \frac{1}{a}\sin at$$

$$\int e^{-st}\sin at \; dt = -\frac{1}{a}e^{-st}\cos at - \frac{s}{a}\left[\frac{1}{a}e^{-st}\sin at - \frac{1}{a}\int\left(-se^{-st}\right)(\sin at)dt\right]$$

$$\int e^{-st}\sin at \; dt = -\frac{1}{a}e^{-st}\cos at - \frac{s}{a^2}e^{-st}\sin at - \frac{s^2}{a^2}\int e^{-st}\sin at \; dt$$

$$\int e^{-st}\sin at \; dt + \frac{s^2}{a^2}\int e^{-st}\sin at \; dt = -\frac{1}{a}e^{-st}\cos at - \frac{s}{a^2}e^{-st}\sin at$$

$$\frac{a^2+s^2}{a^2}\int e^{-st}\sin at \; dt = -\frac{1}{a}e^{-st}\cos at - \frac{s}{a^2}e^{-st}\sin at$$

$$\int e^{-st}\sin at \; dt = -\frac{ae^{-st}}{a^2+s^2}\cos at - \frac{se^{-st}}{a^2+s^2}\sin at$$

$$F(s) = \lim_{T\to\infty}\int_0^T e^{-st}\sin at \; dt$$

$$= \lim_{T\to\infty}\left(\left(-\frac{ae^{-sT}}{a^2+s^2}\cos aT - \frac{se^{-sT}}{a^2+s^2}\sin aT\right) - \left(-\frac{ae^{-s(0)}}{a^2+s^2}\cos a(0) - \frac{se^{-s(0)}}{a^2+s^2}\sin a(0)\right)\right)$$

$$= \lim_{T\to\infty}\left(\left(-\frac{ae^{-sT}}{a^2+s^2}\cos aT - \frac{se^{-sT}}{a^2+s^2}\sin aT\right) - \left(-\frac{a}{a^2+s^2}\right)\right)$$

$$= \lim_{T\to\infty}\left(-\frac{ae^{-sT}}{a^2+s^2}\cos aT - \frac{se^{-sT}}{a^2+s^2}\sin aT\right) + \frac{a}{a^2+s^2} \qquad \lim_{T\to\infty}\left(e^{-sT}\right) = 0$$

$$= \frac{a}{a^2+s^2}$$

Definition

f is of exponential order λ if there exists a positive number M and a nonnegative number A such that

$$|f(x)| \le Me^{\lambda x} \quad on \quad [A, \infty)$$

Theorem

Let f be a continuous function on $[0, \infty)$. If f is of exponential order λ, then the Laplace transform $\mathcal{L}[f(x)] = F(s)$ exists for $s > \lambda$.

Exercises *Section* 2.7 – Definition of Laplace Transform

Use Definition of Laplace transform to find the Laplace transform of:

1. $f(t) = 3$

2. $f(t) = t$

3. $f(t) = t^2$

4. $f(t) = e^{6t}$

5. $f(t) = e^{-2t}$

6. $f(t) = te^{-3t}$

7. $f(t) = te^{3t}$

8. $f(t) = e^{2t}\cos 3t$

9. $f(t) = \sin 3t$

10. $f(t) = \sin 2t$

11. $f(t) = \cos 2t$

12. $f(t) = \cos bt$

13. $f(t) = e^{t+7}$

14. $f(t) = e^{-2t-5}$

15. $f(t) = te^{4t}$

16. $f(t) = t^2 e^{-2t}$

17. $f(t) = e^{-t}\sin t$

18. $f(t) = e^{2t}\cos 3t$

19. $f(t) = e^{-t}\sin 2t$

20. $f(t) = t\sin t$

21. $f(t) = t\cos t$

22. $f(t) = 2t^4$

Use Definition of Laplace transform to show the Laplace transform of

23. $f(t) = \cos \omega t$ is $F(s) = \dfrac{s}{s^2 + \omega^2}$