Section 4.8 – Connectivity

Paths

A path is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph.

Definitions

Let G be a graph, and let v and w be vertices in G.

A ***walk*** from v to w is a finite alternating sequence of adjacent vertices and edges of G. Thus a walk has the form

$$v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n$$

Where the v's represent vertices, the e's represents edges, $v_0 = v,\ v_n = w$ and for all $i = 1, 2, \ldots, n$, v_{i-1} *and* v_i are the endpoints of e_i. The trivial walk from v to v consists of the single vertex v.

A ***trail*** from v to w is a walk from v to w that does not contain a repeated edge.

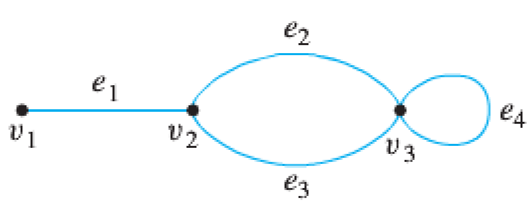
A ***closed walk*** is a walk that starts and ends at the same vertex.

A ***circuit*** is a closed walk that contains at least one edge and does not contain a repeated edge.

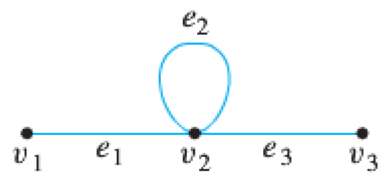
A ***simple circuit*** is a circuit that does not any other repeated vertex except first and last.

	Repeated Edge?	**Repeated Vertex**	**Starts & Ends at Same Point?**	**Must Contain at Least One Edge?**
Walk	Allowed	Allowed	Allowed	No
Trail	No	Allowed	Allowed	No
Path	No	No	No	No
Closed Walk	Allowed	Allowed	Yes	Yes
Circuit	No	Allowed	Yes	Yes
Simple Circuit	No	First & last only	Yes	Yes

Notation for Walks



The notation $e_1 e_2 e_4 e_3$ refers unambiguous to the following walk: $v_1 e_1 v_2 e_2 e_4 v_3 e_3 v_2$. On the other hand, the notation e_1 is ambiguous if used to refer to a walk. It could mean either $v_1 e_1 v_1$ or $v_2 e_1 v_1$. The notation $v_2 v_3$ is ambiguous if used to refer to a walk. It could mean $v_2 e_2 v_3$ or $v_2 e_3 v_3$. On the other hand,



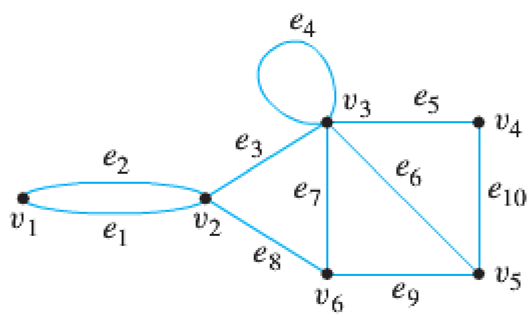
The notation $v_1v_2v_2v_3$ refers unambiguously to the walk $v_1e_1v_2e_2v_2e_3v_3$

Example

Determine which of the following walks are trails, paths, circuits, or simple circuits to the graph below.

a) $v_1e_1v_2e_3v_3e_4v_3e_5v_4$ *b)* $e_1e_3e_5e_5e_6$ *c)* $v_2v_3v_4v_5v_3v_6v_2$

d) $v_2v_3v_4v_5v_6v_2$ *e)* $v_1e_1v_2e_1v_1$ *f)* v_1



Solution

a) This walk has a repeated vertex but does not have a repeated edge, so it is a trail from v_1 *to* v_4 but not a path.

b) This is just a walk from v_1 *to* v_5. It is not a trail because it has a repeated edge.

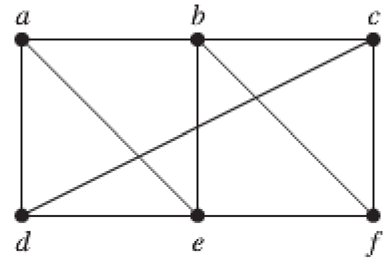
c) This walk starts and ends at v_2, contains at least one edge, and does not have a repeated edge, so it is a circuit. Since the vertex v_3 is repeated in the middle, it is not a simple circuit.

d) This walk starts and ends at v_2, contains at least one edge, and does not have a repeated edge, and does not have a repeated vertex. Thus it is a simple circuit.

e) This is just a closed walk starting and ending at v_1. It is not a circuit because edge e_1 is repeated.

f) The first vertex of this walk is the same as its last vertex, but it does not contain an edge, and so it is not a circuit, It is a closed walk from v_1 *to* v_1. (It is also a trail from v_1 *to* v_1)

Example



The given graph, a, d, c, f, e is a simple path of length 4, because $\{a, d\}$, $\{d, c\}$, $\{c, f\}$, and $\{f, e\}$ are all edges. However, d, e, c, a is not a path, because $\{e, c\}$ is not an edge.
Note that b, c, f, e, b is a circuit of length 4 because $\{b, c\}$, $\{c, f\}$, $\{f, e\}$, and $\{e, b\}$ are edges, and this path begins and ends at b. The path a, b, e, d, a, b, which is of length 5, is not simple because it contains the edge $\{a, b\}$ twice.

Connectedness

Definition

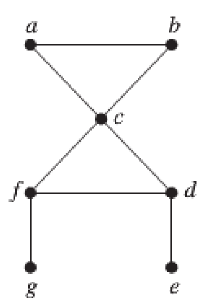
Let G be a graph. Two vertices v and w of G are ***connected*** if, and only if, there is a walk from v to w. The graph G is connected if, and only if, given any two vertices v and w in G, there is a walk from v to w.
Symbolically,

$$G \text{ is connected } \Leftrightarrow \forall \text{ vertices } v, w \in V(G), \exists \text{ a walk from } v \text{ to } w.$$

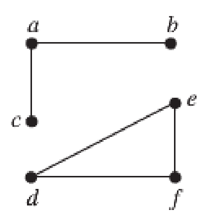
Definition

An undirected graph is called ***connected*** if there is a path between every pair of distinct vertices of the graph. An undirected graph that is not *connected* is called ***disconnected***. We say that we *disconnect* a graph when we remove vertices or edges, or both, to produce a disconnected subgraph.

Example



G_1



G_2

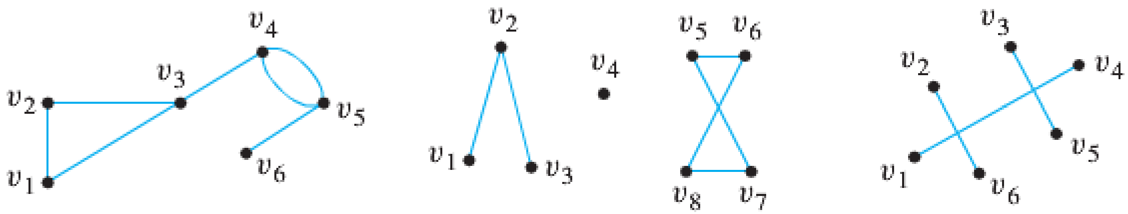
The graph G_1 is connected, because for every pair of distinct vertices there is a path between them.

However, the graph G_2 is not connected. For instance, there is no path in G_2 between vertices a and b.

Connected and Disconnected Graphs

Example

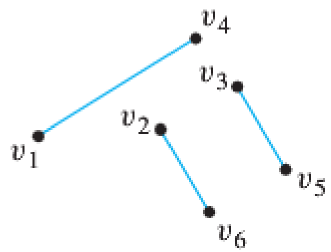
Which of the following graphs are connected?



(*a*) **(*b*)** **(*c*)**

Solution

The graph represented in (*a*) is connected, whereas those of (*b*) and (*c*) are not.
To understand why (*c*) is not connected, two edges may cross at a point that is not a vertex.
Thus the graph in (*c*) can be drawn as follows:



Theorem

There is a simple path between every pair distinct vertices of a connected undirected graph.

Proof

Let *u* and *v* be two distinct vertices of the connected undirected graph $G = (V, E)$. Because *G* is connected, there is at least one path between *u* and *v*. Let $x_0, x_1, \ldots, x_n$ where $x_0 = u$ and $x_n = v$, be the vertex sequence of a path of least length. This path of least length is simple. To see this, suppose it is not simple. Then $x_i = x_j$ for some *i* and *j* with $0 \le i < j$. This means that there is a path from *u* to *v* of shorter length with vertex sequence $x_0, x_1, \ldots, x_{i-1}\ x_j, \ldots, x_n$ obtained by deleting the edges corresponding to the vertex sequence $x_i, \ldots, x_{j-1}$

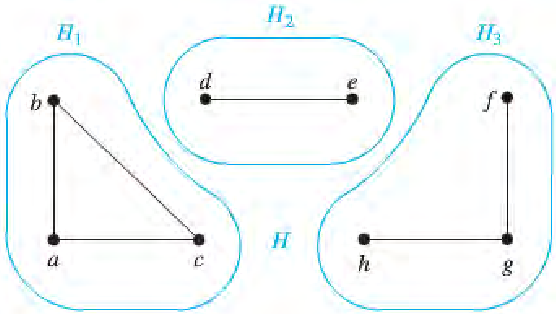
Definition

A graph *H* is a connected component of a graph *G* if, and only if,

- *H* is subgraph of *G*;
- *H* is connected; and
- No connected subgraph of *G* has *H* as a subgraph and contains vertices or edges that are not in *H*.

Example

What are the connected components of the graph H shown below?

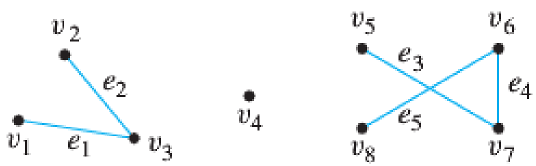


Solution

The graph H is the union of the three disjoint connected subgraphs $H_1, H_2, \textit{and } H_3$. These three subgraphs are the connected components of H.

Example

Find all connected components of the following graph G.



Solution

G has three connected components: $H_1, H_2, \textit{and } H_3$ with vertex sets $V_1, V_2, \textit{and } V_3$ and edges $E_1, E_2, \textit{and } E_3$, where

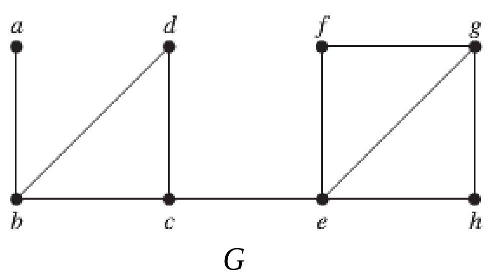
$$V_1 = \{v_1, v_2, v_3\} \qquad E_1 = \{e_1, e_2\}$$

$$V_2 = \{v_4\} \qquad E_2 = \varnothing$$

$$V_3 = \{v_5, v_6, v_7, v_8\} \qquad E_3 = \{e_3, e_4, e_5\}$$

Example

Find the cut vertices and cut edges in the graph G.



Solution

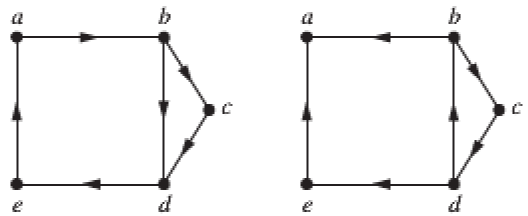
The cut vertices of G are b, c, and e. The removal of one of these vertices (and its adjacent edges) disconnects the graph. The cut edges are $\{a, b\}$ and $\{c, e\}$. Removing either one of these edges disconnects G.

Definition

A directed graph is weakly connected of there is a path between every two vertices in the underlying undirected graph.

Example

Are the directed graphs G and H shown below strongly connected? Are they weakly connected?



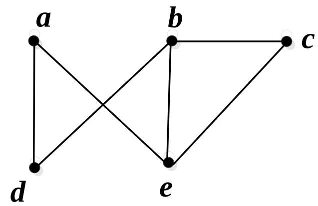
G H

Solution

G is strongly connected because there is a path between any two vertices in this directed graph. Hence, G is also weakly connected.
The graph H is not strongly connected. There is no directed path from a to b in this graph. However, H is weakly connected, because there is a path between any 2 vertices in the underlying undirected graph of H.

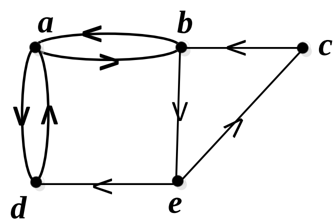
Exercises *Section* 4.8 – Connectivity

1. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? Which are the lengths of those that are paths?



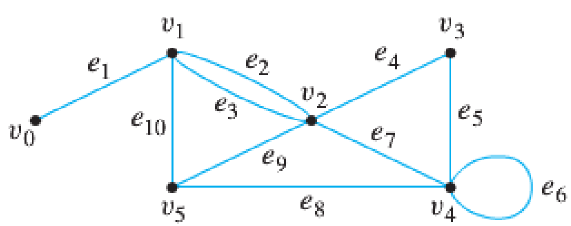
a) *a, e, b, c, b* ***b)*** *a, e, a, d, b, c, a* ***c)*** *e, b, a, d, b, e* ***d)*** *c, b , d, a, e, c*

2. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? Which are the lengths of those that are paths?



a) *a, b, e, c, b* ***b)*** *a, d, a, d, a* ***c)*** *a, d, b, e, a* ***d)*** *a, b , e, c, b, d, a*

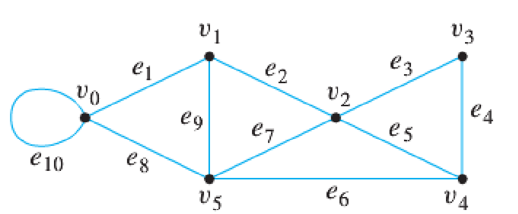
3. Determine whether of the following walks are trails, paths, circuits, or simple circuits or just walk to the graph below.



a) $v_0e_1v_1e_{10}v_5e_9v_2e_2v_1$ *b)* $v_4e_7v_2e_9v_5e_{10}v_1e_3v_2e_9v_5$ *c)* v_2

d) $v_5v_2v_3v_4v_4v_5$ *e)* $v_2v_3v_4v_5v_2v_4v_3v_2$ *f)* $e_5e_8e_{10}e_3$

4. Determine whether of the following walks are trails, paths, circuits, or simple circuits or just walk to the graph below.



a) $v_1e_2v_2e_3v_3e_4v_4e_5v_2e_2v_1e_1v_0$ *b)* $v_2v_3v_4v_5v_2$ *c)* $v_4v_2v_3v_4v_5v_2v_4$

d) $v_2v_1v_5v_2v_3v_4v_2$ *e)* $v_0v_5v_2v_3v_4v_2v_1$ *f)* $v_5v_4v_2v_1$