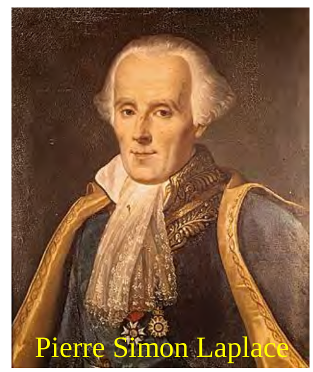
Lecture Three – Laplace and Linear Systems

Section 3.1 – Definition of the Laplace Transform

3.1-1 *Definition*

In mathematics, the ***Laplace Transform*** is an integral transform named after its discoverer Pierre-Simon Laplace. Laplace transforms are restricted to functions of ***t*** and transform a function of a complex variable ***s***. The Laplace operator will transform a linear differential equation with constant coefficients into an algebraic equation in the transformed function.



Pierre Simon Laplace
(1749 – 1827)

Suppose $f(t)$ is a function of *t* defined for $t > 0$.

Then, the ***Laplace transform*** of f is given by the formula:

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

The integral of the Laplace transform is an improper integral because the upper limit is ∞.

$$\begin{aligned} F(s) &= \int_0^{\infty} f(t)e^{-st}dt \\ &= \lim_{T\to\infty}\int_0^{T} f(t)e^{-st}dt \end{aligned}$$

The domain of F is the set of real number s for which the improper integral converges.

Example 1

Use Definition of Laplace transform to find the Laplace transform of $f(t) = e^{at}$.

Solution

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{at}e^{-st}dt \\ &= \int_0^{\infty} e^{-(s-a)t}dt \end{aligned}$$

$$F(s) = \lim_{T\to\infty} \int_0^T e^{-(s-a)t} dt$$

$$= \lim_{T\to\infty} \left. \frac{-e^{-(s-a)t}}{s-a} \right|_0^T \qquad e^{-(s-a)0} = 1$$

$$= \lim_{T\to\infty} \left(\frac{-e^{-(s-a)T}}{s-a} + \frac{1}{s-a} \right) \qquad e^{-(s-a)\infty} = \frac{1}{e^{\infty}} = 0$$

$$= \frac{1}{s-a}$$

$$\boxed{\mathcal{L}\left(e^{at}\right)(s) = \frac{1}{s-a}} \qquad \textit{for } s > a$$

Example 2

Find the Laplace transform of $f(t) = t$, using the definition.

Solution

$$F(s) = \int_0^{\infty} te^{-st} dt$$

$$\int_0^{\infty} te^{-st} dt = -\frac{1}{s} te^{-st} - \frac{1}{s^2} e^{-st} \Big|_0^{\infty}$$

$$= 0 - 0 + 0 + \frac{1}{s^2} \qquad \lim_{T\to\infty}\left(e^{-sT}\right) = 0$$

$$\boxed{= \frac{1}{s^2}}$$

		$\int e^{-st}\ dt$
+	t	$-\frac{1}{s}e^{-st}$
−	1	$\frac{1}{s^2}e^{-st}$

$$\boxed{\mathcal{L}(t)(s) = \frac{1}{s^2}}$$

Laplace transform to any power t^n

$$\pounds\left(t^n\right)(s) = \frac{n!}{s^{n+1}}$$

***Example* 3**

Use Definition of Laplace transform to find the Laplace transform of: $f(t) = \sin at$

Solution

$$F(s) = \int_0^{\infty} e^{-st} \sin at \; dt$$

		$\int \sin at \; dt$
+	e^{-st}	$-\frac{1}{a}\cos at$
−	$-se^{-st}$	$-\frac{1}{a^2}\sin at$
+	$s^2 e^{-st}$	

$$\int e^{-st} \sin at \; dt = -\frac{1}{a} e^{-st} \cos at - \frac{s}{a^2} e^{-st} \sin at - \frac{s^2}{a^2} \int e^{-st} \sin at \; dt$$

$$\int e^{-st} \sin at \; dt + \frac{s^2}{a^2} \int e^{-st} \sin at \; dt = -\frac{1}{a} e^{-st} \cos at - \frac{s}{a^2} e^{-st} \sin at$$

$$\frac{a^2+s^2}{a^2} \int e^{-st} \sin at \; dt = -\frac{1}{a} e^{-st} \cos at - \frac{s}{a^2} e^{-st} \sin at$$

$$\int e^{-st} \sin at \; dt = -\frac{ae^{-st}}{a^2+s^2} \cos at - \frac{se^{-st}}{a^2+s^2} \sin at$$

$$F(s) = \lim_{T\to\infty} \int_0^{T} e^{-st} \sin at \; dt$$

$$= \lim_{T\to\infty} \left(\left(-\frac{ae^{-sT}}{a^2+s^2} \cos aT - \frac{se^{-sT}}{a^2+s^2} \sin aT \right) - \left(-\frac{ae^{-s(0)}}{a^2+s^2} \cos a(0) - \frac{se^{-s(0)}}{a^2+s^2} \sin a(0) \right) \right)$$

$$= \lim_{T\to\infty}\left(\left(-\frac{ae^{-sT}}{a^2+s^2}\cos aT - \frac{se^{-sT}}{a^2+s^2}\sin aT\right) - \left(-\frac{a}{a^2+s^2}\right)\right)$$

$$= \lim_{T\to\infty}\left(-\frac{ae^{-sT}}{a^2+s^2}\cos aT - \frac{se^{-sT}}{a^2+s^2}\sin aT\right) + \frac{a}{a^2+s^2} \qquad \lim_{T\to\infty}\left(e^{-sT}\right) = 0$$

$$= \frac{a}{a^2+s^2}$$

$$\mathcal{L}\left(\sin at\right)(s) = \frac{a}{a^2+s^2}$$

Exercises *Section* 3.1 - The Definition of the Laplace Transform

(1 – 22) Use Definition of Laplace transform to find the Laplace transform of:

1. $f(t)=3$
2. $f(t)=t$
3. $f(t)=t^2$
4. $f(t)=e^{6t}$
5. $f(t)=e^{-2t}$
6. $f(t)=te^{-3t}$
7. $f(t)=te^{3t}$
8. $f(t)=e^{2t}\cos 3t$
9. $f(t)=\sin 3t$
10. $f(t)=\sin 2t$
11. $f(t)=\cos 2t$
12. $f(t)=\cos bt$
13. $f(t)=e^{t+7}$
14. $f(t)=e^{-2t-5}$
15. $f(t)=te^{4t}$
16. $f(t)=t^2e^{-2t}$
17. $f(t)=e^{-t}\sin t$
18. $f(t)=e^{2t}\cos 3t$
19. $f(t)=e^{-t}\sin 2t$
20. $f(t)=t\sin t$
21. $f(t)=t\cos t$
22. $f(t)=2t^4$

23. Use Definition of Laplace transform to show the Laplace transform of $f(t)=\cos\omega t$ is $F(s)=\dfrac{s}{s^2+\omega^2}$

Section 3.2 – Basic Properties of the Laplace Transform

3.2-1 The Laplace Transform of Derivatives

To find the Laplace Transform of a derivative, integrate the expression for the definition of the Laplace Transform by parts. Some useful properties that will help us derive Laplace Transform. These properties, will let us use Laplace Transform to solve differential equations and even to do higher order systems with initial conditions.

3.2-2 *Proposition*

Suppose y is a piecewise differentiable function of exponential order. Suppose also that y' is of the exponential order.

$$\begin{aligned}\pounds(y')(s) &= s.\pounds(y)(s) - y(0)\\ &= sY(s) - y(0)\end{aligned}$$

Proof

$$\pounds(y')(s) = \int_0^\infty y'(t)e^{-st}dt$$

$$= \lim_{T\to\infty} \int_0^T y'(t)e^{-st}dt$$

$$u = e^{-st} \qquad v = \int y'(t)\,dt$$

$$du = -se^{-st}dt \qquad v = y(t)$$

$$= \lim_{T\to\infty} \left[e^{-st}y(t)\ \Big|_{t=0}^{T} + s\int_0^T y(t)e^{-st}dt \right]$$

$$= \lim_{T\to\infty} e^{-st}y(t) - y(0) + s.\pounds(y)(s)$$

Let: $|y(t)| \le Ce^{at}$

$e^{-sT}|y(T)| \le Ce^{aT}e^{-sT}$

$e^{-sT}|y(T)| \le Ce^{-(s-a)T}$; which converges to 0 for $s > a$ as $T \to \infty$. Therefore,

$$\pounds(y')(s) = s.\pounds(y)(s) - y(0)$$

3.2-3 *Proposition*

$$\pounds(y'')(s) = s^2.\pounds(y)(s) - sy(0) - y'(0)$$
$$= s^2.Y(s) - sy(0) - y'(0)$$

3.2-4 *Proposition*

$$\pounds\left(y^{(k)}\right)(s) = s^k.\pounds(y)(s) - s^{k-1}y(0) - \cdots - sy^{(k-2)}(0) - y^{(k-1)}(0)$$
$$= s^k.Y(s) - s^{k-1}y(0) - \cdots - sy^{(k-2)}(0) - y^{(k-1)}(0)$$

3.2-5 Laplace Transform Linear

$$\pounds[\alpha f(t) + \beta.g(t)](s) = \alpha.\pounds[f(t)](s) + \beta.\pounds[g(t)](s)$$

Example 1

Find the Laplace transform of $f(t) = 3\sin 2t - 4t + 5e^{3t}$

Solution

$$\pounds\left[3\sin 2t - 4t + 5e^{3t}\right](s) = 3\pounds[\sin 2t](s) - 4\pounds[t](s) + 5\pounds\left[e^{3t}\right](s)$$
$$= 3\left(\frac{2}{4+s^2}\right) - 4\left(\frac{1}{s^2}\right) + 5\left(\frac{1}{s-3}\right)$$
$$= \frac{6}{4+s^2} - \frac{4}{s^2} + \frac{5}{s-3}$$

Example 2

Transform the initial value problem into an algebraic equation involving $\pounds(y)$. Solve the resulting equation for the Laplace transform of y.

$$y'' - y = e^{2t} \quad with \quad y(0) = 0 \;\; and \;\; y'(0) = 1$$

Solution

For the right-hand side

$$\pounds\left(e^{2t}\right)(s) = \frac{1}{s-2}$$

$$\pounds\{y''-y\}(s)=\pounds\{y''\}(s)-\pounds\{y\}(s)$$
$$=s^2.\pounds(y)(s)-sy(0)-y'(0)-\pounds(y)(s)$$
$$=s^2.Y(s)-sy(0)-y'(0)-Y(s) \qquad y(0)=0 \ \ and \ \ y'(0)=1$$
$$=s^2.Y(s)-1-Y(s)$$

$$s^2.Y(s)-Y(s)-1=\frac{1}{s-2}$$
$$Y(s)\left(s^2-1\right)=\frac{1}{s-2}+1$$
$$Y(s)=\frac{1}{s^2-1}\left(\frac{1}{s-2}+1\right)$$
$$=\frac{1}{(s-1)(s+1)}\left(\frac{s-1}{s-2}\right)$$
$$=\frac{1}{(s-2)(s+1)}$$

3.2-6 Laplace Transform of the Product of an Exponential with a Function

The result is a translation in the Laplace transform

$$\pounds\left(e^{ct}f(t)\right)(s)=F(s-c)$$

Example 3

Compute the Laplace transform of the function $g(t)=e^{2t}\sin 3t$

Solution

Let $f(t)=\sin 3t \xrightarrow{\pounds} F(s)=\frac{3}{s^2+9}$

With $c=2$

$$\pounds\left(e^{2t}f(t)\right)(s)=F(s-c)$$
$$=\frac{3}{(s-2)^2+9}$$
$$=\frac{3}{s^2-4s+13}$$

3.2-7 *Proposition*: Derivative of a Laplace Transform

$$\pounds(s) = -F'(s)$$

$$\pounds\left[t^n . f(t)\right](s) = (-1)^n F^{(n)}(s)$$

Example 4

Compute the Laplace transform of $t^2 e^{3t}$

Solution

$$f(t) = e^{3t} \xrightarrow{\pounds} F(s) = \frac{1}{s-3}$$

$$F'(s) = \frac{-1}{(s-3)^2}$$

$$F''(s) = \frac{2}{(s-3)^3}$$

$$\pounds\left[t^2 e^{3t}\right](s) = (-1)^2 F''(s)$$

$$\boxed{= \frac{2}{(s-3)^3}}$$

Exercises *Section* 3.2 - Basic Properties of the Laplace Transform

(1 – 3) Find the Laplace transform and defined the time domain of

1. $y(t)=t^2+4t+5$

2. $y(t)=-2\cos t+4\sin 3t$

3. $y(t)=2\sin 3t+3\cos 5t$

(4 – 7) Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of y.

4. $y'-5y=e^{-2t}, \quad with \quad y(0)=1$

5. $y'-4y=\cos 2t, \quad with \quad y(0)=-2$

6. $y''+2y'+2y=\cos 2t; \quad with \quad y(0)=1 \;\; and \;\; y'(0)=0$

7. $y''+3y'+5y=t+e^{-t}; \quad with \quad y(0)=-1 \;\; and \;\; y'(0)=0$

(8 – 59) Find the Laplace transform of $\mathcal{L}\{f(t)\}$

8. $f(t)=2t^4$

9. $f(t)=t^5$

10. $f(t)=4t-10$

11. $f(t)=7t+3$

12. $f(t)=3t^4-2t^2+1$

13. $f(t)=(t+1)^3$

14. $f(t)=(2t-1)^3$

15. $f(t)=(t-1)^4$

16. $f(t)=t^2+6t-3$

17. $f(t)=-4t^2+16t+9$

18. $f(t)=3t^2-e^{2t}$

19. $f(t)=t^2-e^{-9t}+9$

20. $f(t)=6e^{-3t}-t^2+2t-8$

21. $f(t)=5-e^{2t}+6t^2$

22. $f(t)=t^2e^{2t}$

23. $f(t)=e^{-2t}(2t+3)$

24. $f(t)=e^{-t}\left(t^2+3t+4\right)$

25. $f(t)=1+e^{4t}$

26. $f(t)=e^{2t}\cos 2t$

27. $f(t)=t^3-te^t+e^{4t}\cos t$

28. $f(t)=t^2-3t-2e^{-t}\sin 3t$

29. $f(t)=\sin^2 t$

30. $f(t)=e^{7t}\sin^2 t$

31. $f(t)=t\sin^2 t$

32. $f(t)=\cos^3 t$

33. $f(t)=te^{-t}\sin 2t$

34. $f(t)=te^{2t}\cos 5t$

35. $f(t)=t^2+e^t\sin 2t$

36. $f(t)=e^{-t}\cos 3t+e^{6t}-1$

37. $f(t)=e^{-2t}\sin 2t+t^2e^{3t}$

38. $f(t)=2t^2e^{-2t}-t+\cos 4t$

39. $f(t)=t\sin 3t$

40. $f(t)=t^2\cos 2t$

41. $f(t)=\left(1+e^{-t}\right)^2$

42. $f(t)=\left(1+e^{2t}\right)^2$

43. $f(t)=\left(e^t-e^{-t}\right)^2$

44. $f(t)=4t^2-5\sin 3t$

45. $f(t)=\cos 5t+\sin 2t$

46. $f(t)=e^{3t}\sin 6t-t^3+e^t$

47. $f(t)=t^4+t^2-t+\sin\sqrt{2}t$

48. $f(t)=t^4e^{5t}-e^t\cos\sqrt{7}t$

49. $f(t)=e^{-2t}\cos\sqrt{3}t-t^2e^{-2t}$

50. $f(t)=6e^{-5t}+e^{3t}+5t^3-9$

51. $f(t)=4\cos 4t-9\sin 4t+2\cos 10t$

52. $f(t)=3\sinh 2t+3\sin 2t$

53. $f(t)=e^{3t}+\cos 6t-e^{3t}\cos 6t$

54. $f(t)=t\cosh 3t$

55. $f(t)=t^2\sin 2t$

56. $f(t)=\sinh kt$

57. $f(t)=\cosh kt$

58. $f(t)=e^t\sinh kt$

59. $f(t)=e^{-t}\cosh kt$

(**60 – 66**) Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of y.

60. $y'+2y=t\sin t, \quad with \quad y(0)=1$

61. $y'+2y=t^2e^{-2t}, \quad with \quad y(0)=0$

62. $y''+y'+2y=e^{-t}\cos 2t, \quad with \quad y(0)=1 \ \ and \quad y'(0)=-1$

63. $y'-5y=e^{-2t}, \quad with \quad y(0)=1$

64. $y'-4y=\cos 2t, \quad with \quad y(0)=-2$

65. $y''+2y'+2y=\cos 2t; \quad with \quad y(0)=1 \ \ and \quad y'(0)=0$

66. $y''+3y'+5y=t+e^{-t}; \quad with \quad y(0)=-1 \ \ and \quad y'(0)=0$

Section 3.3 – Inverse Laplace Transform

The Laplace transform is invertible on a large class of functions. The inverse Laplace transform takes a function of a complex variable *s* and yields a function of a real variable *t* (time).

3.3-1 *Definition*

If f is a continuous function of exponential order and $\mathcal{L}(f)(s) = F(s)$, then we call f the inverse Laplace transform of *F*,

$$f(t) = \mathcal{L}^{-1}(F(s))$$

$$F(s) = \mathcal{L}(f(t)) \quad \Leftrightarrow \quad f(t) = \mathcal{L}^{-1}(F(s))$$

$$f(t) \underset{\textit{Inverse Laplace tansform} - \mathcal{L}^{-1}}{\overset{\textit{Laplace tansform} - \mathcal{L}}{\rightleftarrows}} F(S)$$

Note: Inverse transforms are not unique. If f_1 and f_2 are identical except at a discrete set of points, then $\mathcal{L}(f_1(t)) = \mathcal{L}(f_2(t))$. However, there is at most one continuous function f satisfying $\mathcal{L}\{f(t)\} = F(s)$

3.3-2 Laplace Transform Linear

3.3-3 *Proposition*

$$\begin{aligned}\mathcal{L}^{-1}[aF(s) + bG(s)] &= a.\mathcal{L}^{-1}(F(s)) + b.\mathcal{L}^{-1}(G(s)) \\ &= af(t) + bg(t)\end{aligned}$$

Example 1

Compute the inverse Laplace transform of $F(s)=\frac{1}{s-2}-\frac{16}{s^2+4}$

Solution

$$\pounds^{-1}\left\{\frac{1}{s-2}\right\}=e^{2t}$$

$$\pounds^{-1}\left\{\frac{2}{s^2+4}\right\}=\sin 2t$$

$$\pounds^{-1}\left\{\frac{1}{s-2}-8\frac{2}{s^2+4}\right\}\ =e^{2t}-8\sin 2t$$

Example 2

Compute the inverse Laplace transform of $F(s)=\frac{1}{s^2-2s-3}$; $s>3$

Solution

$$\frac{1}{s^2-2s-3}=\frac{A}{s-3}+\frac{B}{s+1}$$

$$1=As+A+Bs-3B$$

$$\begin{matrix} s \\ s^0 \end{matrix}\ \begin{cases} A+B=0 \\ A-3B=1 \end{cases}$$

$$\Delta=\begin{vmatrix}1 & 1\\ 1 & -3\end{vmatrix}=-4 \qquad \Delta_A=\begin{vmatrix}0 & 1\\ 1 & -3\end{vmatrix}=-1$$

$$A=\frac{1}{4}\quad B=-\frac{1}{4}$$

$$\frac{1}{s^2-2s-3}=\frac{1}{4}\left(\frac{1}{s-3}-\frac{1}{s+1}\right)$$

$$\pounds^{-1}\{F(s)\}=\frac{1}{4}\pounds^{-1}\left\{\frac{1}{s-3}-\frac{1}{s+1}\right\}$$

$$=\frac{1}{4}\left(e^{3t}-e^{-t}\right)$$

Example **3**

Compute the inverse Laplace transform of $F(s)=\dfrac{1}{s^2+4s+13}$

Solution

$$s^2+4s+13=s^2+4s+4+9$$

$$=(s+2)^2+3^2$$

$$f(t)=\mathcal{L}^{-1}\left\{\frac{1}{3}\frac{3}{(s+2)^2+3^2}\right\}$$

$$\boxed{=\frac{1}{3}e^{-2t}\sin 3t}$$

Example **4**

Find the inverse Laplace transform of $F(s)=\dfrac{2s^2+s+13}{(s-1)\left((s+1)^2+4\right)}$

Solution

$$\frac{2s^2+s+13}{(s-1)\left((s+1)^2+4\right)}=\frac{A}{(s-1)}+\frac{Bs+C}{(s+1)^2+4}$$

$$2s^2+s+13=As^2+2As+5A+Bs^2+(C-B)s-C$$

$$\begin{array}{c} s^2 \\ s \\ s^0 \end{array}\begin{cases} A+B=2 \\ 2A-B+C=1 \\ 5A-C=13 \end{cases}$$

$$\Delta=\begin{vmatrix} 1 & 1 & 0 \\ 2 & -1 & 1 \\ 5 & 0 & -1 \end{vmatrix}=8 \qquad \Delta_A=\begin{vmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \\ 13 & 0 & -1 \end{vmatrix}=16$$

$$\boxed{A=\frac{16}{8}=2}$$

$$\begin{cases} B=2-2=0 \\ C=5(2)-13=-3 \end{cases}$$

$$F(s)=\frac{2}{(s-1)}-\frac{3}{(s+1)^2+4}$$

$$f(t) = £^{-1}\left\{\frac{2}{(s-1)} - \frac{3}{(s+1)^2+4}\right\}$$

$$= 2£^{-1}\left\{\frac{1}{(s-1)}\right\} - 3\frac{1}{2}£^{-1}\left\{\frac{2}{(s+1)^2+4}\right\}$$

$$= 2e^{t} - \frac{3}{2}e^{-t}\sin 2t$$

Exercises *Section* 3.3 - Inverse Laplace Transform

(1 – 80) Find the ***inverse Laplace*** transform of

1. $Y(s)=\frac{1}{3s+2}$

2. $Y(s)=\frac{2}{3-5s}$

3. $Y(s)=\frac{1}{s^2+4}$

4. $Y(s)=\frac{3}{s^2}$

5. $Y(s)=\frac{3s+2}{s^2+25}$

6. $Y(s)=\frac{2-5s}{s^2+9}$

7. $Y(s)=\frac{5}{(s+2)^3}$

8. $Y(s)=\frac{1}{(s-1)^6}$

9. $Y(s)=\frac{4(s-1)}{(s-1)^2+4}$

10. $Y(s)=\frac{2s-3}{(s-1)^2+5}$

11. $Y(s)=\frac{2s-1}{(s+1)(s-2)}$

12. $Y(s)=\frac{2s-2}{(s-4)(s+2)}$

13. $Y(s)=\frac{7s^2+3s+16}{(s+1)\left(s^2+4\right)}$

14. $Y(s)=\frac{1}{(s+2)^2\left(s^2+9\right)}$

15. $Y(s)=\frac{s}{(s+2)^2\left(s^2+9\right)}$

16. $Y(s)=\frac{1}{(s+1)^2\left(s^2-4\right)}$

17. $Y(s)=\frac{7s^2+20s+53}{(s-1)\left(s^2+2s+5\right)}$

18. $F(s)=\frac{1}{s^3}$

19. $F(s)=\frac{1}{s^4}$

20. $F(s)=\frac{1}{s^2}-\frac{48}{s^5}$

21. $F(s)=\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s-2}$

22. $F(s)=\frac{4}{s}+\frac{4}{s^5}+\frac{1}{s-8}$

23. $F(s)=\frac{1}{4s+1}$

24. $F(s)=\frac{1}{5s-2}$

25. $F(s)=\frac{s+1}{s^2+2}$

26. $F(s)=\frac{2s-6}{s^2+9}$

27. $F(s)=\frac{10s}{s^2+16}$

28. $F(s)=\left(\frac{2}{s}-\frac{1}{s^3}\right)^2$

29. $F(s)=\frac{(s+1)^3}{s^4}$

30. $F(s)=\frac{(s+2)^2}{s^3}$

31. $F(s)=\frac{1}{s^4-9}$

32. $F(s)=\frac{1}{s^3+5s}$

33. $F(s)=\frac{5}{s^2+36}$

34. $F(s)=\frac{10s}{s^2+16}$

35. $F(s)=\frac{4s}{4s^2+1}$

36. $F(s)=\dfrac{1}{4s^2+1}$

37. $F(s)=\dfrac{1}{s^2+3s}$

38. $F(s)=\dfrac{s+1}{s^2-4s}$

39. $F(s)=\dfrac{1}{s^3+5s}$

40. $F(s)=\dfrac{3}{s^2+9}$

41. $F(s)=\dfrac{2}{s^2+4}$

42. $F(s)=\dfrac{3}{(2s+5)^3}$

43. $F(s)=\dfrac{6}{(s-1)^4}$

44. $F(s)=\dfrac{5}{(s+2)^4}$

45. $F(s)=\dfrac{s-1}{s^2-2s+5}$

46. $F(s)=\dfrac{3s+2}{s^2+2s+10}$

47. $F(s)=\dfrac{s}{s^2+2s-3}$

48. $F(s)=\dfrac{1}{s^2+2s-20}$

49. $F(s)=\dfrac{s+1}{s^2+2s+10}$

50. $F(s)=\dfrac{1}{s^2+4s+8}$

51. $F(s)=\dfrac{2s+16}{s^2+4s+13}$

52. $F(s)=\dfrac{2s+16}{s^2+4s+13}$

53. $F(s)=\dfrac{s-1}{2s^2+s+6}$

54. $F(s)=\dfrac{s^2+1}{s^3-2s^2-8s}$

55. $F(s)=\dfrac{6s+3}{s^4+5s^2+4}$

56. $F(s)=\dfrac{s-3}{\left(s-\sqrt{3}\right)\left(s+\sqrt{3}\right)}$

57. $F(s)=\dfrac{1}{\left(s^2+1\right)\left(s^2+4\right)}$

58. $F(s)=\dfrac{2s-4}{\left(s^2+s\right)\left(s^2+1\right)}$

59. $F(s)=\dfrac{s}{(s+2)\left(s^2+4\right)}$

60. $F(s)=\dfrac{s^2+1}{s(s-1)(s+1)(s-2)}$

61. $F(s)=\dfrac{s}{(s-2)(s-3)(s-6)}$

62. $F(s)=\dfrac{7s-1}{(s+1)(s+2)(s-3)}$

63. $F(s)=\dfrac{s^2+9s+2}{(s-1)^2(s+3)}$

64. $F(s)=\dfrac{2s^2+10s}{\left(s^2-2s+5\right)(s+1)}$

65. $F(s)=\dfrac{s^2-26s-47}{(s-1)(s+2)(s+5)}$

66. $F(s)=\dfrac{-s-7}{(s-1)(s+2)}$

67. $F(s)=\dfrac{-8s^2-5s+9}{\left(s^2-3s+2\right)(s+1)}$

68. $F(s)=\dfrac{-2s^2+8s-14}{(s+1)\left(s^2-2s+5\right)}$

69. $F(s)=\dfrac{-5s-36}{(s+2)\left(s^2+9\right)}$

70. $F(s)=\dfrac{3s^2+5s+3}{s^4+s^3}$

71. $F(s)=\dfrac{7s^3-2s^2-3s+6}{s^3(s-2)}$

72. $F(s)=\dfrac{7s^2-41s+84}{(s-1)\left(s^2-4s+13\right)}$

73. $F(s)=\dfrac{6s-5}{s^2+7}$

74. $F(s)=\dfrac{1-3s}{s^2+8s+21}$

75. $F(s)=\dfrac{3s-2}{2s^2-6s-2}$

76. $F(s)=\dfrac{s+7}{s^2-3s-10}$

77. $F(s)=\dfrac{86s-78}{(s+3)(s-4)(5s-1)}$

78. $F(s)=\dfrac{2-5s}{(s-6)\left(s^2+11\right)}$

79. $F(s)=\dfrac{25}{s^3\left(s^2+4s+5\right)}$

80. $F(s)=\dfrac{5e^{-6s}-3e^{-11s}}{(s+2)\left(s^2+9\right)}$

Section 3.4 – Using Laplace Transform to Solve Differential Equations

Both the *Laplace* transform and the *inverse Laplace* transform come into play in solving some simple ordinary differential equations with initial conditions.

3.4-1 *Homogeneous* Equations

Example 1

Use Laplace transform to find the solution to the initial value problem

$$y'' - 2y' - 3y = 0 \qquad y(0) = 1 \quad \textit{and} \quad y'(0) = 0$$

Solution

$$\mathcal{L}\left(y'' - 2y' - 3y\right) = s^2Y(s) - sy(0) - y'(0) - 2\left(sY(s) - y(0)\right) - 3Y(s)$$

$$= \left(s^2 - 2s - 3\right)Y(s) - s + 2$$

$$= 0$$

$$Y(s) = \frac{s-2}{s^2 - 2s - 3}$$

$$= \frac{A}{s-3} + \frac{B}{s+1}$$

$$= \frac{(A+B)s + A - 3B}{(s-3)(s+1)}$$

$$\begin{cases} A + B = 1 \\ A - 3B = -2 \end{cases} \Rightarrow A = \frac{1}{4}, \quad B = \frac{3}{4}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = -4 \qquad \Delta_A = \begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix} = -1 \quad \Delta_B = \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -3$$

$$A = \frac{1}{4} \quad B = \frac{3}{4}$$

$$= \frac{1}{4}\frac{1}{s-3} + \frac{3}{4}\frac{1}{s+1}$$

$$y(t) = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \frac{1}{4}e^{3t} + \frac{3}{4}e^{-t}$$

3.4-2 *Inhomogeneous* Equations

Example 2

Use Laplace transform to find the solution to the initial value problem

$$y'' + 2y' + 2y = \cos 2t \qquad y(0) = 0 \quad \text{and} \quad y'(0) = 1$$

Solution

$$\mathcal{L}\left(y'' + 2y' + 2y\right) = s^2 Y(s) - sy(0) - y'(0) + 2\left(sY(s) - y(0)\right) + 2Y(s)$$

$$= \left(s^2 + 2s + 2\right)Y(s) - 1$$

$$\mathcal{L}\left(\cos 2t\right) = \frac{s}{s^2 + 4}$$

$$\left(s^2 + 2s + 2\right)Y(s) - 1 = \frac{s}{s^2 + 4}$$

$$\left(s^2 + 2s + 2\right)Y(s) = \frac{s}{s^2 + 4} + 1$$

$$= \frac{s^2 + s + 4}{s^2 + 4}$$

$$Y(s) = \frac{s^2 + s + 4}{\left(s^2 + 4\right)\left(s^2 + 2s + 2\right)}$$

$$= \frac{As + B}{\left(s^2 + 2s + 2\right)} + \frac{Cs + D}{\left(s^2 + 4\right)}$$

$$= \frac{(A + C)s^3 + \left(2C + B + D\right)s^2 + \left(4A + 2C + 2D\right)s + 4B + 2D}{\left(s^2 + 2s + 2\right)\left(s^2 + 4\right)}$$

$$\begin{cases} A + C = 0 & \rightarrow \quad C = -A \\ B + 2C + D = 1 & (1) \\ 4A + 2C + 2D = 1 & (2) \\ 4B + 2D = 4 & \rightarrow \quad D = 2 - 2B \end{cases}$$

$$\begin{cases} (1) \rightarrow & -2A - B = -1 \\ (2) \rightarrow & \underline{2A - 4B = -3} \end{cases}$$

$$-5B = -4$$

$$\boxed{B = \frac{4}{5}, \quad A = \frac{1}{10}, \quad C = -\frac{1}{10}, \quad D = \frac{2}{5}}$$

$$= \frac{1}{10}\frac{s + 8}{\left((s + 1)^2 + 1\right)} - \frac{1}{10}\frac{s - 4}{\left(s^2 + 4\right)}$$

$$= \frac{1}{10}\frac{s+1+7}{\left((s+1)^2+1\right)} - \frac{1}{10}\frac{s-4}{\left(s^2+4\right)}$$

$$= \frac{1}{10}\frac{s+1}{\left((s+1)^2+1\right)} + \frac{7}{10}\frac{1}{\left((s+1)^2+1\right)} - \frac{1}{10}\frac{s}{\left(s^2+4\right)} + \frac{1}{10}\frac{4}{\left(s^2+4\right)}$$

$$y(t) = \pounds^{-1}\left\{\frac{1}{10}\frac{s+1}{\left((s+1)^2+1\right)} + \frac{7}{10}\frac{1}{\left((s+1)^2+1\right)} - \frac{1}{10}\frac{s}{\left(s^2+4\right)} + \frac{1}{10}\frac{4}{\left(s^2+4\right)}\right\}$$

$$= \frac{1}{10}e^{-t}\cos t + \frac{7}{10}e^{-t}\sin t - \frac{1}{10}\cos 2t + \frac{1}{10}2\sin 2t$$

$$= \frac{1}{10}\left(e^{-t}\left(\cos t + 7\sin t\right) + 2\sin 2t - \cos 2t\right)$$

3.4-3 *Higher-Order* Equations

Example 3

Find the solution to the initial value problem

$$y^{(4)} - y = 0 \qquad y(0) = 0,\ y'(0) = 1,\ y''(0) = 0,\ and\ y'''(0) = 0$$

Solution

$$\pounds\left\{y^{(4)} - y\right\} = 0$$

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - Y(s) = 0$$

$$\left(s^4 - 1\right)Y(s) - s^2 = 0$$

$$\left(s^4 - 1\right)Y(s) = s^2$$

$$Y(s) = \frac{s^2}{s^4 - 1}$$

$$= \frac{s^2}{(s-1)(s+1)\left(s^2+1\right)}$$

$$= \frac{A}{(s-1)} + \frac{B}{(s+1)} + \frac{Cs+D}{\left(s^2+1\right)}$$

$$= \frac{As^3+As^2+As+A+Bs^3-Bs^2+Bs-B+Cs^3-Cs+Ds^2-D}{(s-1)(s+1)\left(s^2+1\right)}$$

$$=\frac{(A+B+C)s^3+(A-B+D)s^2+(A+B-C)s+A-B-D}{(s-1)(s+1)\left(s^2+1\right)}$$

$$\begin{cases} s^3 & A+B+C=0 \\ s^2 & A-B+D=1 \\ s^1 & A+B-C=0 \\ s^0 & A-B-D=0 \end{cases}$$

$$\begin{cases} A+B+C=0 \\ A+B-C=0 \end{cases} \rightarrow \quad 2A+2B=0$$

$$\begin{cases} A-B+D=1 \\ A-B-D=0 \end{cases} \rightarrow \quad 2A-2B=1$$

$$\begin{array}{c} 2A+2B=1 \\ \underline{2A-2B=0} \\ 4A=1 \end{array}$$

$$\boxed{A=\tfrac{1}{4}}$$

$$B=-A=\boxed{-\tfrac{1}{4}}$$

$$\boxed{C=0}$$

$$D=A-B=\boxed{\tfrac{1}{2}}$$

$$Y(s)=\frac{1}{4}\frac{1}{s-1}-\frac{1}{4}\frac{1}{s+1}+\frac{1}{2}\frac{1}{s^2+1}$$

$$y(t)=\pounds^{-1}\left\{\frac{1}{4}\frac{1}{s-1}-\frac{1}{4}\frac{1}{s+1}+\frac{1}{2}\frac{1}{s^2+1}\right\}$$

$$\boxed{y(t)=\tfrac{1}{4}e^{t}-\tfrac{1}{4}e^{-t}+\tfrac{1}{2}\sin t}$$

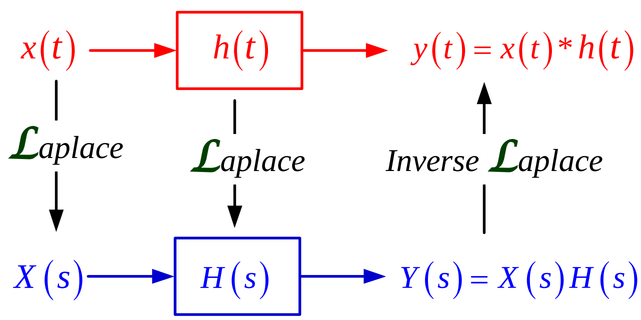
3.4-4 Electrical Circuit

Similar to the application of phasor transform to solve the steady state *AC* circuits, Laplace transform can be used to transform the time domain circuits into *s*–domain circuits to simplify the solution of integral differential equations to the manipulation of a set of algebraic equations.

The elegance of using the Laplace transform in circuit analysis lies in the automatic inclusion of the initial conditions in the transformation process, thus providing a complete (***transient*** and ***steady state***) solution.

3.4-5 Transfer Function

If we have a circuit with impulse-response $h(t)$ in the time domain, with input $x(t)$ and output $y(t)$, we can find the ***Transfer Function*** of the circuit, in the Laplace domain, by transforming all three elements: The ***transfer function*** $H(s)$ and ***input function*** $X(s)$, then $Y(s) = H(s)X(s)$ is the ***output function***.



$$
\begin{aligned}
H(s) &= \mathcal{L}\{h(t)\} \\
&= \frac{Y(s)}{X(s)}
\end{aligned}
$$

Transfer functions are powerful tools for analyzing circuits.

3.4-6 Convolution Integral

The output of a system from the input and the impulse response by using the convolution operation. We can calculate the output using the convolution operation:

$$y(t) = x(t) * h(t)$$

Where the asterisk $*$ operator denotes the ***convolution operation***.

If the input is $\delta(t)$, then $X(s)=1$ and $Y(s)=H(s)$. Hence, the physical meaning of $H(s)$ is in fact the Laplace transform of the impulse response of the corresponding circuit.

3.4-7 Circuit Element Models

The method used as follow:

1. Write the differential equation model.
2. Use Laplace transform to convert the model to *s*–domain.
3. Solve the ODE in *s*–domain.
4. Convert to *t*–time domain using the inverse Laplace transform.

Another approach, more direct approach is as follow:

1. Develop *s*–domain models for the circuit elements.
2. Draw the Laplace equivalent circuit keeping the interconnections and replacing the elements by their *s*–domain models.
3. Analyze the Laplace equivalent circuit.
4. Convert to *t*–time domain using the inverse Laplace transform.

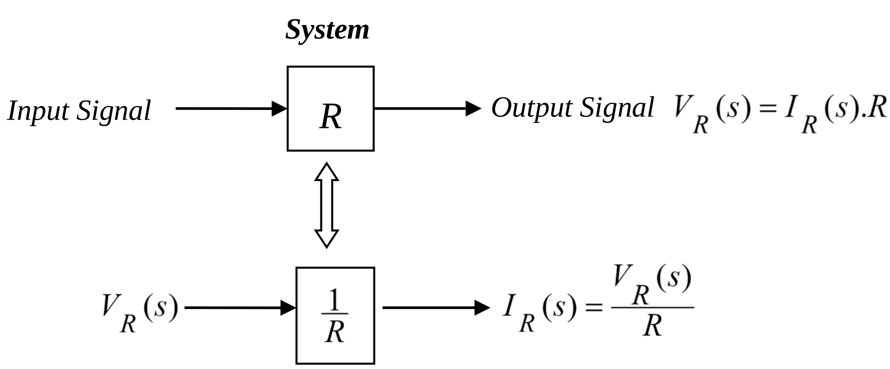
3.4-8 *Resistor*

Laplace Transform

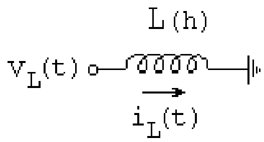
$$I_R(s).R = V_R(s)$$

$$V_R = RI$$

The block diagram:



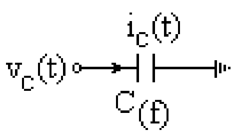
3.4-9 *Inductor*

	*Laplace Transform* $v_L(t) = L\dfrac{di_L(t)}{dt} \overset{\pounds}{\longleftrightarrow} V_L(s) = L\left(sI_L(s) - I_L(0)\right)$

The block diagram:

$$I_L(s) \longrightarrow \boxed{Ls} \longrightarrow V_L(s) = sL\,I_L(s)$$

3.4-10 *Capacitance*



$v_c(0) = E_0(0)$

Laplace Transform

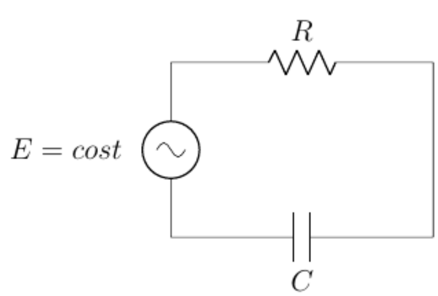
$$i_c(t) = C\frac{dV_c(t)}{dt} \overset{\pounds}{\longleftrightarrow} I_c(s) = Cs.V_c(s)$$

$$V_C(s) \longrightarrow \boxed{Cs} \longrightarrow I_C(s) \quad \Longleftrightarrow \quad I_C(s) \longrightarrow \boxed{\frac{1}{Cs}} \longrightarrow V_C(s)$$

Example 4

Suppose the electrical circuit has a resistor of $R = 2\Omega$ and a capacitor of $C = \frac{1}{5}F$. Assume the voltage source is $E = \cos t\ (V)$. If the initial current is 0 *A*, find the resulting current.

Solution



$$R\frac{dQ}{dt} + \frac{1}{C}Q = E$$

$$2Q' + 5Q = \cos t$$

$$\pounds\left(2Q' + 5Q\right) = \pounds\left(\cos t\right)$$

$$2sQ(s) - 2Q(0) + 5Q(s) = \frac{s}{s^2 + 1}$$

$$\left(2s + 5\right)Q(s) = \frac{s}{s^2 + 1}$$

$$2\left(s+\tfrac{5}{2}\right)Q(s)=\frac{s}{s^2+1}$$

$$Q(s)=\frac{1}{2}\frac{s}{\left(s+\frac{5}{2}\right)\left(s^2+1\right)}$$

$$\frac{s}{\left(s+\frac{5}{2}\right)\left(s^2+1\right)}=\frac{A}{s+\frac{5}{2}}+\frac{Bs+C}{s^2+1}$$

$$s=As^2+A+Bs^2+\tfrac{5}{2}Bs+Cs+\tfrac{5}{2}C$$

$$s=(A+B)s^2+\left(\tfrac{5}{2}B+C\right)s+A+\tfrac{5}{2}C$$

$$\begin{cases} A+B=0 & \rightarrow A=-B \\ \frac{5}{2}B+C=1 & \rightarrow C=1-\frac{5}{2}B \\ A+\frac{5}{2}C=0 & (1) \end{cases}$$

$$(1)\quad \rightarrow \quad -B+\frac{5}{2}-\frac{25}{4}B=0$$

$$\frac{29}{4}B=\frac{5}{2}$$

$$\boxed{B=\frac{10}{29}}$$

$$A=-B=\boxed{-\frac{10}{29}}$$

$$C=1-\frac{25}{29}=\boxed{\frac{4}{29}}$$

$$Q(s)=\frac{1}{2}\left(-\frac{10}{29}\frac{1}{s+\frac{5}{2}}+\frac{10}{29}\frac{s}{s^2+1}+\frac{4}{29}\frac{1}{s^2+1}\right)$$

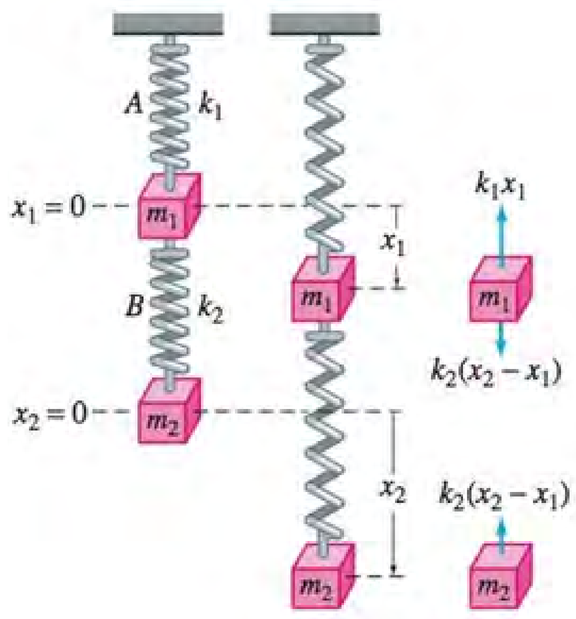
$$=\frac{1}{29}\left(-5\frac{1}{s+\frac{5}{2}}+5\frac{s}{s^2+1}+2\frac{1}{s^2+1}\right)$$

$$q(t)=\frac{1}{29}\left(-5\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{5}{2}}\right\}+5\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}+2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\right)$$

$$\boxed{=\frac{1}{29}\left(-5e^{-5t/2}+5\cos t+2\sin t\right)}$$

3.4-11 Springs-Masses

Two masses m_1 and m_2 are connected to two springs A and B of negligible mass having spring constants k_1 and k_2, respectively.



Let $x_1(t)$ and $x_2(t)$ denote the vertical displacements of the masses from their equilibrium positions. When the system is in motion, spring B is subject to both an elongation and a compression; hence its net elongation is $x_2 - x_1$. Therefore, it follows from Hooke's Law that springs A and B exert forces $-k_1x_1$ and $k_2(x_2 - x_1)$, respectively, on m_1.

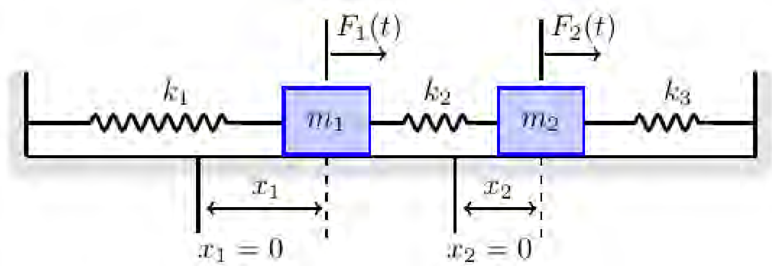
If no external force is impressed on the system and if no damping force is present, then the net force on m_1 is $-k_1x_1 + k_2(x_2 - x_1)$.

By Newton's second law we can write

$$\begin{cases} m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1) \\ m_2 x_2'' = -k_2(x_2 - x_1) \end{cases}$$

Example 5

Write down the system of differential equations for the spring and mass system as shown below. Both masses move to the right of their equilibrium points. The mass m_1 moves farther than m_2.



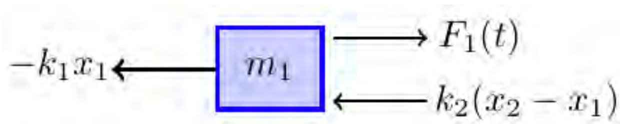
Solution

Let $x_1(t)$ and $x_2(t)$ denote the horizontal displacements of the masses from their equilibrium positions.

If both masses move the same amount in the same direction, then the middle spring will not have changed length and it will result $x_2 - x_1 = 0$.

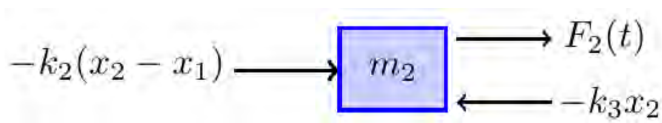
If both masses move in the positive direction then if m_1 moves more than m_2 then $x_2 - x_1 < 0$, the spring will be compressed. If m_2 moves more than m_1 then $x_2 - x_1 > 0$ (stretched).

At mass m_1 with $x_1 > 0$:



$$m_1x_1'' = -k_1x_1 + k_2(x_2 - x_1) + F_1(t)$$

At mass m_2:



$$m_2x_2'' = -k_3x_2 - k_2(x_2 - x_1) + F_2(t)$$

Exercises *Section* 3.4 - Using Laplace Transform to Solve Differential Equations

(1 – 74) Solve using the Laplace transform:

1. $y' + y = te^t, \quad y(0) = -2$

2. $y' - y = 2\cos 5t, \quad y(0) = 0$

3. $y' - y = 1 + te^t, \quad y(0) = 0$

4. $y' + 3y = e^{2t}, \quad y(0) = -1$

5. $y' + 4y = \cos t, \quad y(0) = 0$

6. $y' + 4y = e^{-4t}, \quad y(0) = 2$

7. $y' - 4y = t^2 e^{-2t}, \quad y(0) = 1$

8. $y' + 9y = e^{-t}, \quad y(0) = 0$

9. $y' + 16y = \sin 3t, \quad y(0) = 1$

10. $y'' - y = e^{2t}; \quad y(0) = 0, \quad y'(0) = 1$

11. $y'' - y = 2t; \quad y(0) = 0, \quad y'(0) = -1$

12. $y'' - y = t - 2\ ; \quad y(2) = 3, \quad y'(2) = 0$

13. $y'' + y = t\ ; \quad y(\pi) = y'(\pi) = 0$

14. $y'' - 2y' + 5y = -8e^{\pi - t}\ ; \quad y(\pi) = 2, \quad y'(\pi) = 12$

15. $y'' + y = t^2 + 2\ ; \quad y(0) = 1, \quad y'(0) = -1$

16. $y'' + y = \sqrt{2}\sin\sqrt{2}t; \quad y(0) = 10, \quad y'(0) = 0$

17. $y'' + y = -2\cos 2t\ ; \quad y(0) = 1, \quad y'(0) = -1$

18. $y'' - y' = e^t \cos t\ ; \quad y(0) = 0, \quad y'(0) = 0$

19. $y'' + y' - y = t^3\ ; \quad y(0) = 1, \quad y'(0) = 0$

20. $y'' - y' - 2y = 4t^2, \quad y(0) = 1, \quad y'(0) = 4$

21. $y'' - y' - 2y = e^{2t}\ ; \quad y(0) = -1, \quad y'(0) = 0$

22. $y'' - y' - 2y = 0, \quad y(0) = -2, \quad y'(0) = 5$

23. $y'' - y' - 2y = -8\cos t - 2\sin t\ ; \quad y\left(\frac{\pi}{2}\right) = 1, \quad y'\left(\frac{\pi}{2}\right) = 0$

24. $x'' - x' - 6x = 0\ ; \quad x(0) = 2, \quad x'(0) = -1$

25. $y'' + 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = 1$

26. $y'' + 2y' + y = t\ , \quad y(0) = -3, \quad y(1) = -1$

27. $y'' - 2y' - y = e^{2t} - e^t\ ; \quad y(0) = 1, \quad y'(0) = 3$

28. $y'' - 2y' + y = 6t - 2\ ; \quad y(-1) = 3, \quad y'(-1) = 7$

29. $y'' - 2y' + y = \cos t - \sin t\ ; \quad y(0) = 1, \quad y'(0) = 3$

30. $y'' - 2y' + 5y = 0, \quad y(0) = 2, \quad y'(0) = 4$

31. $y'' - 2y' + 5y = 1 + t, \quad y(0) = 0, \quad y'(0) = 0$

32. $y'' + 3y' = -3t \;; \quad y(0) = -1, \quad y'(0) = 1$

33. $y'' + 3y' = t^3 \;; \quad y(0) = 0, \quad y'(0) = 0$

34. $y'' - 3y' + 2y = e^{-t} \;, \quad y(1) = 0, \quad y'(1) = 0$

35. $y'' - 3y' + 2y = \cos t \;; \quad y(0) = 0, \quad y'(0) = -1$

36. $y'' - 4y' + 4y = t^3 e^{2t}, \quad y(0) = 0, \quad y'(0) = 0$

37. $y'' - 4y' + 4y = t^3, \quad y(0) = 1, \quad y'(0) = 0$

38. $y'' - 4y = e^{-t} \;; \quad y(0) = -1, \quad y'(0) = 0$

39. $y'' - 4y' = 6e^{3t} - 3e^{-t}; \quad y(0) = 1, \quad y'(0) = -1$

40. $x'' + 4x' + 4x = t^2; \quad x(0) = x'(0) = 0$

41. $y'' + 4y = 4t^2 - 4t + 10 \;; \quad y(0) = 0, \quad y'(0) = 3$

42. $y'' - 4y = 4t - 8e^{-2t}; \quad y(0) = 0, \quad y'(0) = 5$

43. $y'' + 4y' = \cos(t - 3) + 4t \;, \quad y(3) = 0, \quad y'(3) = 7$

44. $y'' + 4y' + 8y = \sin t \;, \quad y(0) = 1, \quad y'(0) = 0$

45. $y'' + 5y' - y = e^t - 1 \;; \quad y(0) = 1, \quad y'(0) = 1$

46. $y'' + 5y' - 6y = 21e^{t-1} \quad y(1) = -1, \quad y'(1) = 9$

47. $y'' + 5y' + 4y = 0; \quad y(0) = 1, \quad y'(0) = 0$

48. $y'' + 6y = t^2 - 1 \;; \quad y(0) = 0, \quad y'(0) = -1$

49. $y'' - 6y' + 9y = t, \quad y(0) = 0, \quad y'(0) = 1$

50. $y'' - 6y' + 13y = 0, \quad y(0) = 0, \quad y'(0) = -3$

51. $y'' - 6y' + 15y = 2\sin 3t \;, \quad y(0) = -1, \quad y'(0) = -4$

52. $y'' + 6y' + 9y = 0, \quad y(0) = -1, \quad y'(0) = 6$

53. $y'' + 6y' + 5y = 12e^t \;, \quad y(0) = -1, \quad y'(0) = 7$

54. $y'' - 7y' + 10y = 9\cos t + 7\sin t \;; \quad y(0) = 5, \quad y'(0) = -4$

55. $y'' + 8y' + 25y = 0 \;, \quad y(\pi) = 0, \quad y'(\pi) = 6$

56. $y'' + 9y = 2\sin 2t; \quad y(0) = 0, \quad y'(0) = -1$

57. $y'' + 9y = 3\sin 2t; \quad y(0) = 0, \quad y'(0) = -1$

58. $y'' - 10y' + 9y = 5t \;; \quad y(0) = -1, \quad y'(0) = 2$

59. $y'' + 16y = 2\sin 4t \;; \quad y(0) = -\frac{1}{2}, \quad y'(0) = 0$

60. $2y'' + 3y' - 2y = te^{-2t}, \quad y(0) = 0, \quad y'(0) = -2$

61. $2y'' + 20y' + 51y = 0, \quad y(0) = 2, \quad y'(0) = 0$

62. $y^{(3)} + y' = e^t \ , \quad y(0) = y'(0) = y''(0) = 0$

63. $2y^{(3)} + 3y'' - 3y' - 2y = e^{-t}; \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1$

64. $y^{(3)} + 2y'' - y' - 2y = \sin 3t; \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1$

65. $y^{(3)} - y'' + y' - y = 0 \ ; \quad y(0) = 1, \quad y'(0) = 1, \quad y''(0) = 3$

66. $y^{(3)} + 4y'' + y' - 6y = -12 \ ; \quad y(0) = 1, \quad y'(0) = 4, \quad y''(0) = -2$

67. $y^{(3)} + 3y'' + 3y' + y = 0 \ ; \quad y(0) = -4, \quad y'(0) = 4, \quad y''(0) = -2$

68. $y^{(3)} - 3y'' + 3y' - y = t^2 e^t \ , \quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 3$

69. $y^{(3)} + y'' + 3y' - 5y = 16e^{-t} \ ; \quad y(0) = 0, \quad y'(0) = 2, \quad y''(0) = -4$

70. $y''' + 4y'' + 5y' + 2y = 10\cos t \ , \quad y(0) = y'(0) = 0, \quad y''(0) = 3$

71. $y^{(4)} + 2y'' + y = 4te^t; \quad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0$

72. $y^{(4)} - y = 0 \ ; \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0, \quad y^{(3)}(0) = 0$

73. $y^{(4)} - 4y = 0 \ ; \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2, \quad y^{(3)}(0) = 0$

74. $y^{(4)} - 4y^{(3)} + 6y'' - 4y' + y = 0 \ ; \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y^{(3)}(0) = 1$

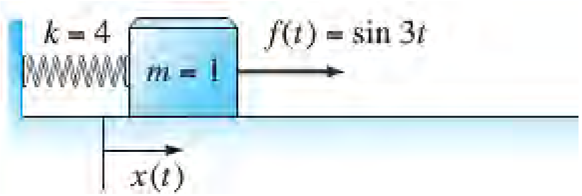
75. Given: $y'' - 4y' + 3y = 0, \quad y(0) = 1 \quad y'(0) = -1$

a) Show that the general solution is: $y(t) = C_1 e^{3t} + C_2 e^t$ and find C_1 *and* C_2

b) Use Laplace transform to solve the system

76. Solve the initial value problem $x'' + 4x = \sin 3t; \quad x(0) = x'(0) = 0.$

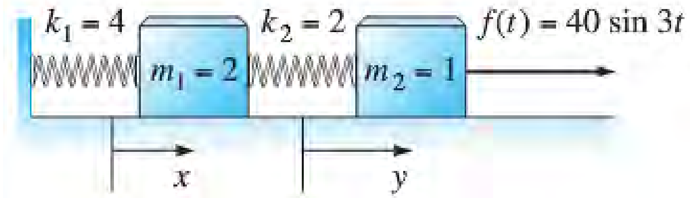
Such problem arises in the motion of a mass-and-spring system with external force as shown below.



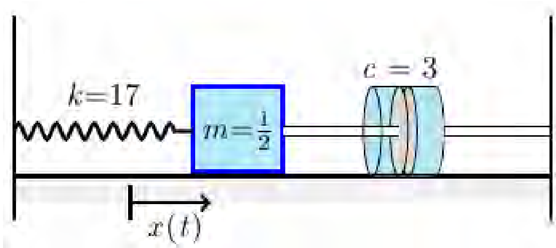
77. Solve the system $\begin{cases} 2x'' = -6x + 2y \\ y'' = 2x - 2y + 40\sin 3t \end{cases}$

Subject to the initial conditions $x(0) = x'(0) = y(0) = y'(0) = 0$

Thus the force $f(t) = 40\sin 3t$ is applied to the second mass as shown below, beginning at time $t = 0$ when the system is at rest in its equilibrium position.



78. Consider a mass-spring system with $m = \frac{1}{2}$, $k = 17$, *and* $c = 3$.



Let $x(t)$ be the displacement of the mass m from its equilibrium position. If the mass is set in motion with $x(0) = 3$ *and* $x'(0) = 1$, find $x(t)$ for the resulting damped free oscillations.

79. A $4-lb$ weight stretches a spring 2 *feet*. The weight is released from rest 18 *inches* above the equilibrium position, and the resulting motion takes place in a medium offering a damping force numerically equal to $\frac{7}{8}$ times the instantaneous velocity. Use the Laplace transform to find the equation of motion $x(t)$.

80. Consider a mass-spring-dashpot system with $m = \frac{1}{2}$, $k = 17$, $c = 3$, *and* $f(t) = 15\sin 2t$ with initial conditions $x(0) = x'(0) = 0$. Let $x(t)$ be the displacement of the mass m from its equilibrium position. Find the resulting transient motion and steady periodic motion of the mass.

81. A 8-*kg* mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by 3 *N-sec/m* and the spring constant is 40 *N/m*. If the mass is driven by an external force equal to $f(t) = 2\sin 2t \cos 2t$ *N*. Find the solution.

82. A 2-*kg* mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by $c = 8$ *kg/sec* and the spring constant is $k = 80$ *N/m*. At time $t = 0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t) = 20e^{-t}$ *N*. (*t* in *seconds*). Find the equation of motion.

83. A 10-*kg* mass is attached to a spring having a spring constant of 140 *N/m*. The mass is started in motion initially from the equilibrium position with an initial velocity 1 *m/sec* in the upward direction and with an applied external force $F(t) = 5\sin t$. If the force due to air resistance is $-90y'$ *N*. Find the equation motion of the mass.

84. A 128-*lb* weight is attached to a spring having a spring constant of 64 *lb/ft*. The weight is started in motion initially by displacing it 6 *in* above the equilibrium position with no initial velocity and with an applied external force $F(t) = 8\sin 4t$. Assume no air resistance. Find the equation motion of the mass.

85. Find the motion of a damped mass-and-spring system with $m = 1$, $c = 2$, and $k = 26$ under the influence of an external force $F(t) = 82\cos 4t$ with $x(0) = 6$ and $x'(0) = 0$.

86. A spring with a mass of 2-*kg* has natural length 0.5 *m*. A force of 25.6 *N* is required to maintain it stretched to a length of 0.7 *m*. The spring is immersed in a fluid with damping constant $c = 40$. If the spring is started from the equilibrium position and is given a push to start it with initial velocity 0.6 *m/s*. Find the position of the mass at any time *t*.

87. A spring with a mass of 3-*kg* is held stretched 0.6 *m* beyond its natural length by a force of 20 *N*. If the spring begins at its equilibrium and with initial velocity 1.2 *m/s*. Find the position of the mass.

88. A spring with a mass of 2-*kg* is held stretched 0.5 *m*, has damping constant 14, and a force of 6 *N*. If the spring is stretched 1 *m* beyond at its equilibrium and with no initial velocity. Find the position of the mass at any time *t*.

89. Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit when $L = 0.25$ *H*, $R = 10\ \Omega$, $C = 0.001$ *F*, $E(t) = 0$, $q(0) = q_0$ *C*, and $i(0) = 0$.

90. Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit at $t = 0.01$ *sec* when $L = 0.05$ *h*, $R = 2\ \Omega$, $C = 0.01$ *f*, $E(t) = 0$, $q(0) = 5$ *C*, and $i(0) = 0$ *A*.

91. Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit when $L=\frac{5}{3}\ h$, $R=10\ \Omega$, $C=\frac{1}{30}\ f$, $E(t)=0$, $q(0)=4\ C$, and $i(0)=0\ A$.

92. Find the current $i(t)$ in an *LRC*–series circuit when $L=1\ h$, $R=20\ \Omega$, $C=0.005\ f$, $E(t)=150\ V$, $q(0)=0\ C$, and $i(0)=0\ A$.

(93 – 96) A resistor $R=20\ \Omega$ and a capacitor of $C=0.1\ F$ are joined in series with an electronic force (*emf*) $E=E(t)$ and no charge on the capacitor at $t=0$. Find the ensuing charge on the capacitor at time *t* for the given:

93. $E(t)=100\sin 2t$

94. $E(t)=100e^{-0.1t}$

95. $E(t)=100\left(1-e^{-0.1t}\right)$

96. $E(t)=100\cos 3t$

(97 – 99) An inductor $(L=1\ H)$ and a resistor $(R=0.1\ \Omega)$ are joined in series with an electronic force (*emf*) $E=E(t)$ and no charge on the capacitor at $t=0$. Find the ensuing current in the current at time *t* for the given:

97. $E(t)=10-2t$

98. $E(t)=4\cos 3t$

99. $E(t)=4\sin 2\pi t$

100. Solve the general initial value problem modeling the *RC* circuit

$$R\frac{dQ}{dt}+\frac{1}{C}Q=E, \quad Q(0)=0$$

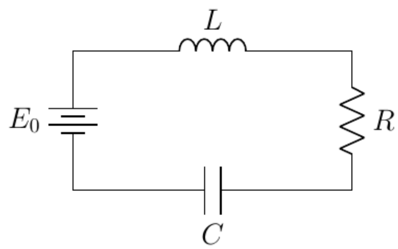
Where E is a constant source of *emf*

101. Solve the general initial value problem modeling the *LR* circuit

$$L\frac{dI}{dt}+RI=E, \quad I(0)=I_0$$

Where E is a constant source of *emf*

102. Consider a battery of constant voltage E_0 that charges the capacitor. $L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=E(t)$



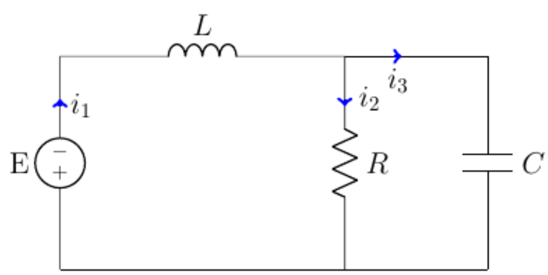
Divide the given equation by L and define $2\lambda=\frac{R}{L}$ and $\omega^2=\frac{1}{LC}$.

a) Use the Laplace transform to show that the solution $q(t)$ of $q'' + 2\lambda q' + \omega^2 q = \dfrac{E_0}{L}$ subject to $q(0) = 0$, $i(0) = 0$ is

$$q(t) = \begin{cases} E_0 C\left[1 - e^{-\lambda t}\left(\cosh\sqrt{\lambda^2 - \omega^2}\,t + \dfrac{\lambda}{\sqrt{\lambda^2 - \omega^2}}\sinh\sqrt{\lambda^2 - \omega^2}\,t\right)\right] & \lambda > \omega \\ E_0 C\left[1 - e^{-\lambda t}(1 + \lambda t)\right] & \lambda = \omega \\ E_0 C\left[1 - e^{-\lambda t}\left(\cos\sqrt{\omega^2 - \lambda^2}\,t + \dfrac{\lambda}{\sqrt{\omega^2 - \lambda^2}}\sin\sqrt{\omega^2 - \lambda^2}\,t\right)\right] & \lambda < \omega \end{cases}$$

b) Use the Laplace transform to find the charge $q(t)$ in an *RC* series when $q(0) = 0$ and $E(t) = E_0 e^{-kt}$, $k > 0$. Consider *two* cases: $k \neq \dfrac{1}{RC}$ and $k = \dfrac{1}{RC}$

103. Solve the system under the conditions $E(t) = 60\ V$, $L = 1\ h$, $R = 50\ \Omega$, $C = 10^{-4}\ f$, and the currents i_1 and i_2 are initially zero.



104. Solve

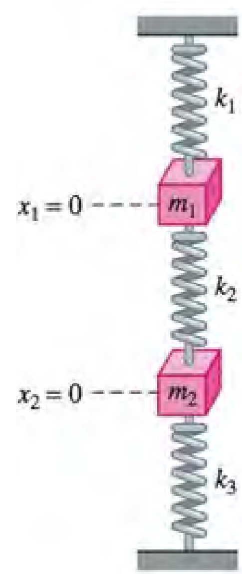
$$\begin{aligned} x_1'' + 10x_1 \quad - 4x_2 &= 0 \\ -4x_1 + x_2'' + 4x_2 &= 0 \end{aligned}$$

Subject to $x_1(0) = 0,\ x_1'(0) = 1,\ x_2(0) = 0,\ x_2'(0) = -1$

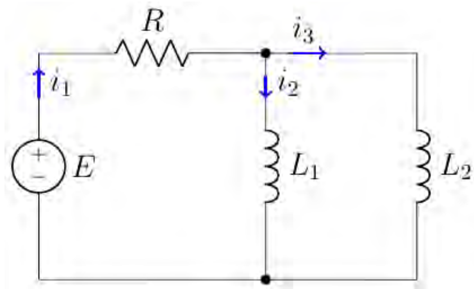
105. Derive the system of differential equations describing the straight-line vertical motion of the coupled springs. Use the Laplace transform to solve the system when

$k_1 = 1,\ k_2 = 1,\ k_3 = 1,\ m_1 = 1,\ m_2 = 1$ and

$x_1(0) = 0,\ x_1'(0) = -1,\ x_2(0) = 0,\ x_2'(0) = 1$

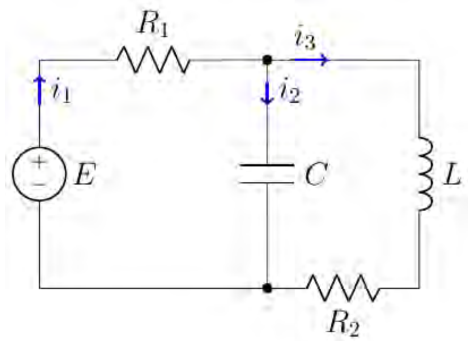


106. Solve the currents $i_1(t)$, $i_2(t)$ and $i_3(t)$ in the given electrical network.



Given $R = 5\,\Omega$ $\quad L_1 = 0.01\,h,\quad L_2 = 0.0125\,h,\quad E = 100\,V$ and $i_2(0) = 0 \quad i_3(0) = 0$

107. Find the charge on the capacitor $q(t)$ and the current $i_3(t)$ in the given electrical network.

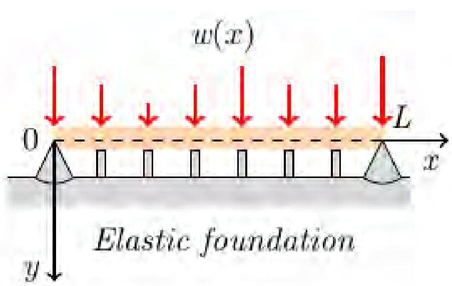


Given: $\quad R_1 = 1\,\Omega,\quad R_2 = 1\,\Omega,\quad L = 1\,h,\quad C = 1\,f\quad \&\ q(0) = 0,\quad i_3(0) = 0$

$$E(t) = \begin{cases} 0, & 0 < t < 1 \\ 50e^{-t}, & t \ge 1 \end{cases}$$

108. When a uniform beam is supported by an elastic foundation, the differential equation for its deflection $y(x)$ is

$$EI\frac{d^4y}{dx^4} + ky = w(x)$$



Where k is the modulus of the foundation and $-ky$ is the restoring force of the foundation that acts in the direction opposite to that of the load $w(x)$. For algebraic convenience suppose that the differential equation is written as

$$\frac{d^4y}{dx^4}+4a^4y=\frac{w(x)}{EI}$$

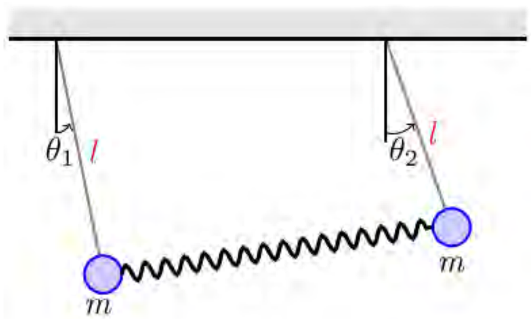
Where $a=\left(\frac{k}{4EI}\right)^{1/4}$. Assume $L=\pi$ and $a=1$. Find the deflection $y(x)$ of a beam that is supported on an elastic foundation when

a) The beam is simply supported at both ends and a constant load w_0 is uniformly distributed along its length,

b) The bean is embedded at both ends and $w(x)$ is concentrated load w_0 applied at $x=\frac{\pi}{2}$

109. Suppose two identical pendulums are coupled by means of a spring with constant *k*. when the displacement angles $\theta_1(t)$ and $\theta_2(t)$ are small, the system of linear differential equations describing the motion is

$$\begin{cases}\theta_1''+\frac{g}{l}\theta_1=-\frac{k}{m}(\theta_1-\theta_2)\\ \theta_2''+\frac{g}{l}\theta_2=\frac{k}{m}(\theta_1-\theta_2)\end{cases}$$



a) Use Laplace transform to solve the system when

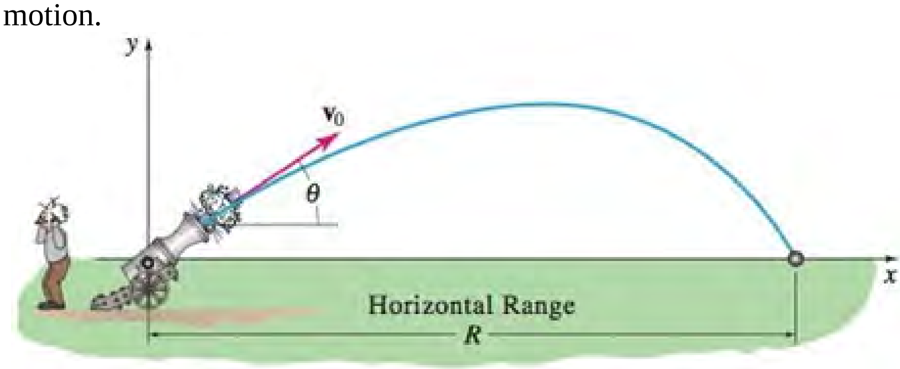
$$\theta_1'(0)=0\quad \theta_1(0)=\theta_0\quad \theta_2'(0)=0\quad \theta_2(0)=\psi_0$$

Where θ_0 and ψ_0 constants. Let $\omega^2=\frac{g}{l}$, $K=\frac{k}{m}$

b) Use the solution in part (*a*) to discuss the motion of the coupled pendulums in the special case when the initial conditions are $\theta_1'(0)=0,\ \ \theta_1(0)=\theta_0,\ \ \theta_2'(0)=\theta_0,\ \ \theta_2(0)=0$

c) Use the solution in part (*a*) to discuss the motion of the coupled pendulums in the special case when the initial conditions are $\theta_1'(0)=0,\ \ \theta_1(0)=\theta_0,\ \ \theta_2'(0)=-\theta_0,\ \ \theta_2(0)=0$

110. A projectile, such as the canon ball, has weight $w=mg$ and initial velocity $\boldsymbol{v}_0$ that is tangent to its path of motion.



If air resistance and all other forces except its weight are ignored, that motion of the projectile is describe by the system of linear differential equations:

$$\begin{cases} m\dfrac{d^2x}{dt^2}=0 \\ m\dfrac{d^2y}{dt^2}=-mg \end{cases}$$

a) Use Laplace transform to solve the system when

$$x(0)=0 \quad x'(0)=v_0\cos\theta \quad y(0)=0 \quad y'(0)=v_0\sin\theta$$

Where $v_0=|v|$ is constant and θ is the constant angle of elevation.

The solutions $x(t)$ and $y(t)$ are parametric equations of the trajectory of the projectile.

b) Use $x(t)$ in part (*a*) to eliminate the parameter *t* in $y(t)$. Use the resulting equation for *y* to show that the horizontal range *R* of the projectile is given by

$$R=\frac{v_0^2}{g}\sin 2\theta$$

c) From the formula in part (*b*), we see that *R* is a maximum when $\sin 2\theta=1$ or when $\theta=\frac{\pi}{4}$. Show that the same range – less than the maximum– can be obtained by firing the gun at either of two complementary angles θ and $\frac{\pi}{2}-\theta$. The only difference is that the smaller angle results in a low trajectory whereas the larger angle fives a high trajectory.

d) Suppose $g=32$ *ft/s*^{2}, $\theta=30°$, and $v_0=300$ *ft/s*. Use part (*b*) to find the horizontal range of the projectile.

e) Find the time when the projectile hits the ground.

f) Use the parametric equations $x(t)$ and $y(t)$ in part (*a*) along with the numerical data in part (*d*) to plot the ballistic curve of the projectile.

g) Repeat with $\theta=52°$ and $v_0=300$ *ft/s*.

h) Superimpose both curves part (*f*) & (*g*) on the same coordinate system.

Section 3.5 – Introduction & Basic Theory of Linear Systems

A system of differential equations is a set of one or more equations, involving one or more differential equations.

There are several physical problems that involve number of separate elements such as an example: electrical networks, mechanical, and more other fields.

3.5-1 Example of *Predator-Prey* Systems (*Ecology*)

The dynamical of biological growth of populations is a branch of ecology.

The growth rate of species is depending on their population. The rate is increased by birth and food supply causes the species to live, and decreased by death, overcrowded, etc.….

Consider two species that exist together and interact as an example such as wolves and deer, shark and food fish, etc... Vito Volterra, an Italian mathematician, formulated a predator-prey system model.

Let the *prey* population denoted by $F(t)$.

Let the *predator* population denoted by $G(t)$.

For each $F(t)$ and $G(t)$, we have a reproductive rate which denoted by r_F and r_G for prey and predator respectively.

Therefore, that will imply to:

$$\begin{cases} F' = r_F F(t) \\ G' = r_G G(t) \end{cases}$$

Let assume there is absence of predator, by using *Malthusian* model, the prey population will be given by

$$G = 0 \qquad \Rightarrow R_F = a > 0$$

When there are predator activities, then, and the decrease in the reproductive rate would also be proportional to $G(t)$.

$$R_F = a - bG \qquad a,\ b > 0$$

In the absence of prey, by using *Malthusian* model, the predator population will be given by

$$F = 0$$

$$\rightarrow \quad R_G = -c < 0$$

The presence of the prey would decrease in the reproductive rate would be proportional to the size of the prey population.

$$R_G = -c + d\ F \qquad c,\ d > 0$$

That will give us a system of:

$$\begin{cases} F' = (a - bG)F \\ G' = (-c + d\ F)G \end{cases}$$

This model is ***nonlinear*** because the right-side contains the product FG.

It is ***autonomous*** because the right-side doesn't depend explicitly on the independent variable.

3.5-2 *Summary of Predator-Prey*

The Predator-Prey or ***Lotka–Volterra*** system is given by:

$$\begin{cases} \dot{x} = -ax + bxy \\ \dot{y} = cy - dxy \end{cases}$$

Where *x* is the predator, their prey is '*y*', and the coefficient *a*, *b*, *c*, and *d* are positive real numbers and they are defined as follow:

a: is the natural decay.

ax: is a rate term, which shows that without prey to eat, the predator population diminishes.

c: is the natural growth coefficient.

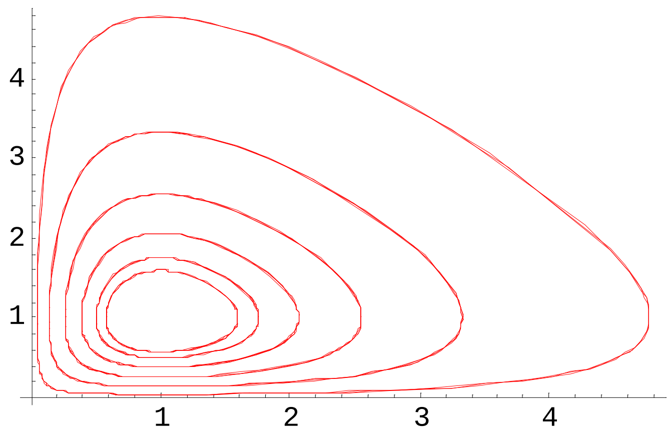
cy: is a rate term, where the prey population increases.

b** and **c: predator efficiency in converting food into fertility and the probability that are predator-prey encounter removes of the prey.

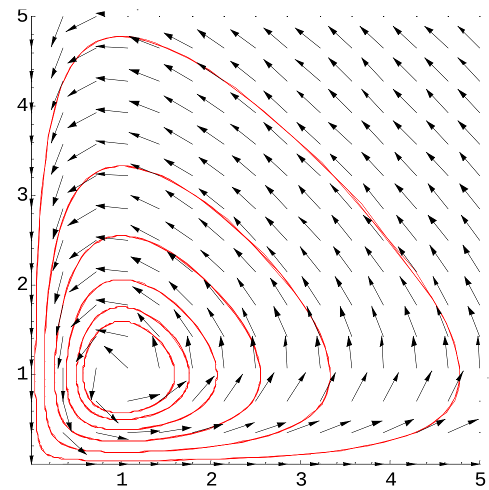
bxy** and **cxy: Food promotes the predator population's growth rate, while serving as food diminishes the prey populations' growth rate.

The predator-prey system or model is based on the population Law of Mass Action.
The Law of Mass Action is defined as:

"*The rate of change of one population due to interaction of another is proportional to the product of the two populations*."



(*a*)



(*b*) Flow direction.

$$\begin{cases} \dot{x} = x - xy \\ \dot{y} = -y + xy \end{cases}$$

Where $x(t)$ is the prey, $y(t)$ is the predator.

3.5-3 *Definition*

A linear system of differential equations is any set of differential equations having the following ***standard form***:

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + f_1(t) \\ x_2' &= a_{21}(t)x_1 + \cdots + a_{2n}(t)x_n + f_2(t) \\ &\vdots \qquad\qquad \vdots \\ x_n' &= a_{n1}(t)x_1 + \cdots + a_{nn}(t)x_n + f_n(t) \end{aligned}$$

Where $x_1, \cdots, \textit{and}\ x_n$ are the unknown functions. The ***coefficients*** $a_{ij}(t)$ and $f_i(t)$ are known functions of the independent variable $t \in (a,\ b)$ is an interval in **R**.

If all of the $f_i(t) = 0$, the system said to be ***homogeneous***. Otherwise it is ***inhomogeneous***.

The inhomogeneous part is sometimes called the ***forcing term***.

3.5-4 Matrix Notation for Linear Systems

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

In simple form, we can rewrite:

$$x'(t) = A(t)x(t) + f(t)$$

$$x' = Ax + f$$

Example 1

Given the linear system $\begin{cases} x_1' = x_1 + 2x_2 \\ x_2' = 2x_1 + x_2 \end{cases}$

Write in the form $x' = Ax + f$

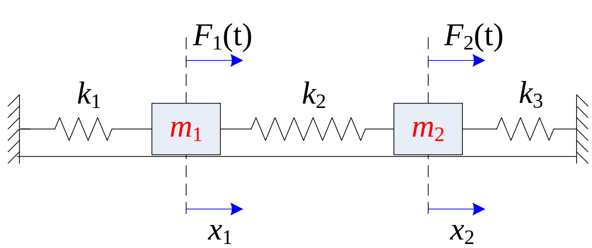
Solution

$$x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

3.5-5 Example of a *Spring-Mass* (*mechanical*)

Two masses move on a frictionless surface under the influence of external forces $F_1(t)$ and $F_2(t)$, and they are also constraint by the three springs whose constant are $k_1, k_2,$ *and* k_3



Let examine the forces acting on m_1

The ***first*** spring exerts a force of: $f_1 = -k_1 x_1$ $\qquad k_1$ ***is the spring constant.***

The ***second*** spring exerts a force of: $f_2 = k_2\left(x_2 - x_1\right)$

By Newton's second law:

$$m_1 \frac{d^2x_1}{dt^2} = \sum forces$$

$$= f_1 + f_2 + F_1(t)$$

$$= -k_1x_1 + k_2\left(x_2 - x_1\right) + F_1(t)$$

$$= -\left(k_1 + k_2\right)x_1 + k_2x_2 + F_1(t)$$

The forces acting on m_2

The ***second*** spring exerts a force of: $-f_2 = -k_2\left(x_2 - x_1\right)$

The ***third*** spring exerts a force of: $f_3 = -k_3x_2$

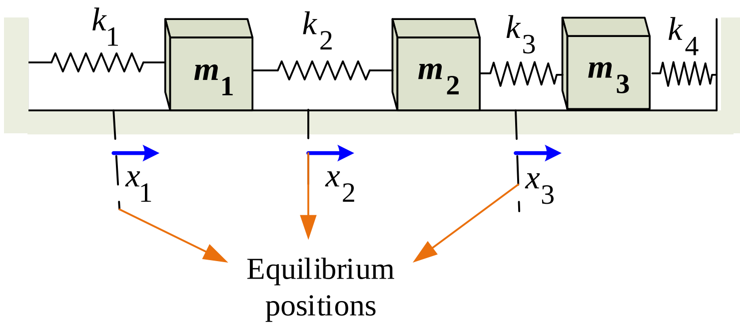
$$m_2 \frac{d^2x_2}{dt^2} = -k_2\left(x_2 - x_1\right) - k_3x_2 + F_2(t)$$

$$= k_2x_1 - \left(k_2 + k_3\right)x_2 + F_2(t)$$

Therefore;

$$\begin{cases} m_1x_1'' = -\left(k_1 + k_2\right)x_1 + k_2x_2 + F_1(t) \\ m_2x_2'' = k_2x_1 - \left(k_2 + k_3\right)x_2 + F_2(t) \end{cases}$$

Example 2



Consider the mass-and–spring systems, as shown above. Three masses connected to each other and to two walls by 4 indicated springs. Assume the masses slide without friction and each spring obeys Hooke's Law $(F = -kx)$.

By applying Newton's law $F = ma$ to the 3-masses:

$$
\begin{aligned}
m_1 x_1'' &= -k_1 x_1 \qquad\qquad + k_2 (x_2 - x_1) \\
m_2 x_2'' &= -k_2 (x_2 - x_1) + k_3 (x_3 - x_2) \\
m_3 x_3'' &= -k_3 (x_3 - x_2) - k_4 x_3
\end{aligned}
$$

$$
\begin{cases}
x_1'' = -\dfrac{k_1 + k_2}{m_1} x_1 + \dfrac{k_2}{m_1} x_2 \\[2ex]
x_2'' = \dfrac{k_2}{m_2} x_1 - \dfrac{k_2 + k_3}{m_2} x_2 + \dfrac{k_3}{m_2} x_3 \\[2ex]
x_3'' = \qquad \dfrac{k_3}{m_3} x_2 - \dfrac{k_3 + k_4}{m_3} x_3
\end{cases}
$$

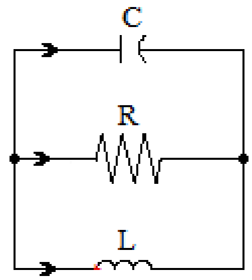
The displacement vector: $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

The mass matrix $M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}$

The stiffness matrix $K = \begin{pmatrix} -k_1 - k_2 & k_2 & 0 \\ k_2 & -k_2 - k_3 & k_3 \\ 0 & k_3 & -k_3 - k_4 \end{pmatrix}$

3.5-6 Example of a *parallel* LRC circuit

Consider the parallel LRC circuit as shown below



Let V be the voltage drop across the capacitor and I current through the inductor.

The current is described by the ***equation***: $\frac{dI}{dt}=\frac{V}{L}$

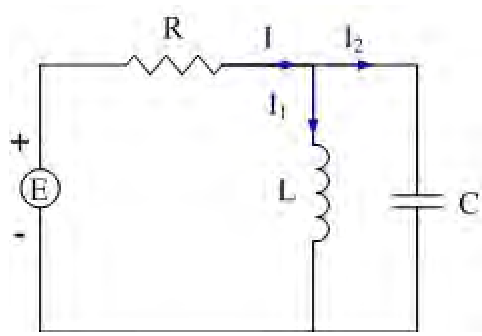
The voltage is described by the ***equation***: $\frac{dV}{dt}=-\frac{1}{C}-\frac{V}{RC}$

Therefore, we can rewrite the equation as ***system equations***

$$\begin{cases} I' = \frac{1}{L}V \\ V' = -\frac{1}{C} - \frac{1}{RC}V \end{cases}$$

Example 3

Find a first-order system the models the circuit below



Solution

Using Kirchhoff's current law: $I = I_1 + I_2$

Kirchhoff's voltage law applied to the loop containing the source and the inductor:

$$E = RI + LI_1'$$

$$= R\left(I_1 + I_2\right) + LI_1'$$

$$LI_1' = E - R\left(I_1 + I_2\right)$$

$$I_1' = \frac{1}{L}\left[E - R\left(I_1 + I_2\right)\right]$$

Kirchhoff's voltage law applied to the loop containing the source, resistor and the capacitor:

$$E = RI + \frac{1}{C}Q$$

Differentiate the equation:

$$\begin{aligned} E' &= RI' + \frac{1}{C}Q' \qquad\qquad Q' = I_2 \\ &= R\left(I_1' + I_2'\right) + \frac{1}{C}I_2 \\ &= RI_1' + RI_2' + \frac{1}{C}I_2 \end{aligned}$$

$$\begin{aligned} RI_2' &= E' - RI_1' - \frac{1}{C}I_2 \\ &= E' - R\frac{1}{L}\left[E - R\left(I_1 + I_2\right)\right] - \frac{1}{C}I_2 \end{aligned}$$

$$\begin{aligned} I_2' &= \frac{1}{R}E' - \frac{1}{L}\left[E - R\left(I_1 + I_2\right)\right] - \frac{1}{RC}I_2 \\ &= \frac{1}{R}E' - \frac{1}{L}E + \frac{R}{L}I_1 + \frac{R}{L}I_2 - \frac{1}{RC}I_2 \\ &= \frac{R}{L}I_1 + \left(\frac{R}{L} - \frac{1}{RC}\right)I_2 + \frac{E'}{R} - \frac{E}{L} \end{aligned}$$

$$\begin{cases} I_1' = -\frac{R}{L}I_1 - \frac{R}{L}I_2 + \frac{E}{L} & (1) \\ I_2' = \frac{R}{L}I_1 + \left(\frac{R}{L} - \frac{1}{RC}\right)I_2 + \frac{E'}{R} - \frac{E}{L} & (2) \end{cases}$$

Properties of Linear Systems

3.5-7 *Properties of Homogeneous Systems*

3.5-8 *Theorem*

Suppose x_1 and x_2 are solution to the homogeneous linear system

$$x' = Ax$$

If C_1 and C_2 are any constants, then $x = C_1 x_1 + C_2 x_2$ is also a solution

3.5-9 *Theorem*

Suppose $x_1, x_2, \ldots,\ and\ x_n$ are solution to the homogeneous linear system

If $C_1, C_2, \ldots,\ and\ C_n$ are any constants, then

$$x(t) = C_1 x_1(t) + C_2 x_2(t) + \cdots + C_n x_n(t)$$

is also a solution to $x' = Ax$

3.5-10 *Linearly Independence and Dependence*

3.5-11 Proposition

Suppose $y_1, y_2, \ldots,\ and\ y_n$ are solution to the n-dimensional system $y' = Ay$ defined on the interval $I = (\alpha,\ \beta)$.

1. If the vectors $y_1(0), y_2(0), \ldots,\ and\ y_n(0)$ are linearly dependent for some $t_0 \in I$, then there are constants $C_1, C_2, \ldots,\ and\ C_n$ not all zero, such that $C_1 y_1(t) + C_2 y_2(t) + \cdots + C_n y_n(t) = 0$ for all $t \in I$. In particular, $y_1(t), y_2(t), \ldots,\ and\ y_n(t)$ are linearly dependent for all $t \in I$.

2. If for some $t_0 \in I$ the vectors $y_1(0), y_2(0), \ldots,\ and\ y_n(0)$ are linearly independent, then $y_1(t), y_2(t), \ldots,\ and\ y_n(t)$ are linearly independent for all $t \in I$.

3.5-12 *Definition*

A set of ***n*** solutions to the linear system $x' = Ax$ is linearly independent if it is linearly independent for any one value of t.

Example 4

Given $x_1(t)=\begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$ and $x_2(t)=\begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}$ are solutions to the homogeneous system

$$x'(t)=\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}x(t)$$

Show that all solutions to this system can be expressed as linear combination of x_1 and x_2

Solution

$$x_1(t)=\begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}=e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$x_2(t)=\begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}=e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x(t)=C_1x_1(t)+C_2x_2(t)$$

$$\begin{aligned} x(t=0)&=C_1x_1(0)+C_2x_2(0) \\ &=C_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}+C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &=\begin{pmatrix} C_1+C_2 \\ -C_1+C_2 \end{pmatrix} \\ &=\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}=\left[x_1(0),\ x_2(0)\right]$$

$$\det=\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}=2\neq 0$$

$\Rightarrow$ The matrix is nonsingular and $x_1(0)$, $x_2(0)$ are linearly independent

$$x(0)=\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

Example 5

Consider the system of homogeneous equations

$$x'(t) = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix} x(t)$$

We can show that

$$x_1(t) = \begin{pmatrix} e^t \cos t \\ e^t(\cos t - \sin t) \end{pmatrix} \qquad x_2(t) = \begin{pmatrix} e^t \sin t \\ e^t(\cos t + \sin t) \end{pmatrix}$$

are solutions the given system

$$x_1(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x(t) = C_1 x_1(t) + C_2 x_2(t)$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

$$\det = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \neq 0$$

$\Rightarrow$ $x_1(0)$ and $x_2(0)$ are linearly independent

Exercises *Section* 3.5 – Introduction & Basic Theory of Linear Systems

(1 – 3) For the linear systems, which are homogeneous? Which are inhomogeneous?

1. $\begin{cases} x_1' = -2x_1 + x_1x_2 \\ x_2' = -3x_1 - x_2 \end{cases}$ **2.** $\begin{cases} x_1' = -x_2 \\ x_2' = \sin x_1 \end{cases}$ **3.** $\begin{cases} x_1' = x_1 + (\sin t)x_2 \\ x_2' = 2tx_1 - x_2 \end{cases}$

(4 – 5) Write the given system of equations in matrix-form then show that the given vector is a solution to the system

4. $\begin{cases} x_1' = -3x_1 + x_2 \\ x_2' = -2x_1 \end{cases}$ $\quad v = \left(-e^{-2t} + e^{-t}, -e^{-2t} + 2e^{-t}\right)^T$

5. $\begin{cases} x_1' = -x_1 + 4x_2 \\ x_2' = \quad 3x_2 \end{cases}$ $\quad v = \left(e^{3t} - e^{-t}, e^{3t}\right)^T$

(6 – 8) Verify by substitution that $x_1(t)$ and $x_2(t)$ are solutions of the given homogenous equation. Show also that the solutions $x_1(t)$ and $x_2(t)$ are linearly independent. Find the solution of the given homogeneous equation with the initial condition $x(0) = x_0$

6. $\begin{cases} x_1(t) = \begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix}, & x_2(t) = \begin{pmatrix} -e^{-2t} \\ e^{-2t} \end{pmatrix} \\ x' = \begin{pmatrix} -6 & -4 \\ 8 & 6 \end{pmatrix} x & x(0) = \begin{pmatrix} -5 \\ 8 \end{pmatrix} \end{cases}$

7. $\begin{cases} x_1(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}, & x_2(t) = \begin{pmatrix} e^{2t}(t+2) \\ e^{2t}(t+1) \end{pmatrix} \\ x' = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} x & x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$

8. $\begin{cases} x_1(t) = \begin{pmatrix} \frac{1}{2}\cos t - \frac{1}{2}\sin t \\ \cos t \end{pmatrix}, & x_2(t) = \begin{pmatrix} \frac{1}{2}\cos t + \frac{1}{2}\sin t \\ \sin t \end{pmatrix} \\ x' = \begin{pmatrix} 3 & -1 \\ 2 & -1 \end{pmatrix} x & x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$

(9 – 13) Rewrite the given equation into a system in normal form with initial value.

9. $y^{(4)} - y^{(3)} + 7y = \cos t$; $\quad y(0) = y'(0) = 1, \quad y''(0) = 0, \quad y^{(3)}(0) = 2$

10. $y^{(4)} + 3y'' - (\sin t)y' + 8y = t^2$, $\quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 3, \quad y'''(0) = 4$

11. $y^{(6)} - (y')^3 = e^{2t} - \sin y$; $\quad y(0) = y'(0) = y''(0) = y^{(3)}(0) = y^{(4)}(0) = y^{(5)}(0) = 0$

12. $\begin{cases} 3x'' = -5x + 2y \\ 4y'' = 6x - 2y \end{cases} \quad \begin{cases} x(0) = -1, & x'(0) = 0 \\ y(0) = 1, & y'(0) = 2 \end{cases}$

13. $\begin{cases} x''' - y = t \\ 2x'' + 5y'' - 2y = 1 \end{cases} \quad \begin{cases} x(0) = x'(0) = x''(0) = 4 \\ y(0) = y'(0) = 1 \end{cases}$

(14 – 21) Transform the given differential equation or system into an equivalent system of 1st-order differential equation

14. $x'' + 3x' + 7x = t^2$

15. $x^{(4)} + 6x'' - 3x' + x = \cos 3t$

16. $t^2x'' + tx' + (t^2 - 1)x = 0$

17. $t^3x^{(3)} - 2t^2x'' + 3tx' + 5x = \ln t$

18. $x'' - 5x + 4y = 0, \quad y'' + 4x - 5y = 0$

19. $x'' - 3x' + 4x - 2y = 0, \quad y'' + 2y' - 3x + y = \cos t$

20. $x'' = 3x - y + 2z, \quad y'' = x + y - 4z, \quad z'' = 5x - y - z$

21. $x'' = (1 - y)x, \quad y'' = (1 - x)y$

22. Prove that the general solution of

$$X' = \begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} X$$

On the interval $(-\infty, \infty)$ is

$$X = C_1 \begin{pmatrix} 6 \\ -1 \\ -5 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} e^{-2t} + C_3 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} e^{3t}$$

23. Prove that the general solution of

$$X' = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} X + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^2 + \begin{pmatrix} 4 \\ -6 \end{pmatrix} t + \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

On the interval $(-\infty, \infty)$ is

$$X = C_1 \begin{pmatrix} 1 \\ -1 - \sqrt{2} \end{pmatrix} e^{\sqrt{2}t} + C_2 \begin{pmatrix} 1 \\ -1 + \sqrt{2} \end{pmatrix} e^{-\sqrt{2}t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^2 + \begin{pmatrix} -2 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(24 – 30) For the systems below:

a) Verify that the given vectors are solutions of the given system.
b) Use the Wronskian to show that they are linearly independent.
c) Write the general solution of the system.
d) Find the particular solution that satisfies the given initial conditions

24. $x' = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} x; \quad \vec{x}_1 = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}$

25. $\mathbf{x}' = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} \mathbf{x}; \quad \vec{x}_1 = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 0 \\ x_2(0) = 5 \end{cases}$

26. $\mathbf{x}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \mathbf{x}; \quad \vec{x}_1 = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = e^{-2t} \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \quad \begin{cases} x_1(0) = 5 \\ x_2(0) = -3 \end{cases}$

27. $\mathbf{x}' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \mathbf{x}; \quad \vec{x}_1 = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 8 \\ x_2(0) = 0 \end{cases}$

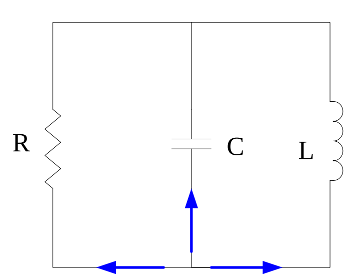
28. $\mathbf{x}' = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} \mathbf{x}; \quad \vec{x}_1 = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} -2e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 0 \\ x_2(0) = 0 \\ x_3(0) = 4 \end{cases}$

29. $\mathbf{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x}; \quad \vec{x}_1 = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 10 \\ x_2(0) = 12 \\ x_3(0) = -1 \end{cases}$

30. $\mathbf{x}' = \begin{bmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{bmatrix} \mathbf{x}; \quad \vec{x}_1 = \begin{bmatrix} e^{-t} \\ 0 \\ 0 \\ e^{-t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 0 \\ 0 \\ e^{-t} \\ 0 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 0 \\ e^t \\ 0 \\ -2e^t \end{bmatrix}, \quad \vec{x}_4 = \begin{bmatrix} e^t \\ 0 \\ 3e^t \\ 0 \end{bmatrix}, \quad \begin{cases} x_1(0) = 1 \\ x_2(0) = 3 \\ x_3(0) = 4 \\ x_4(0) = 7 \end{cases}$

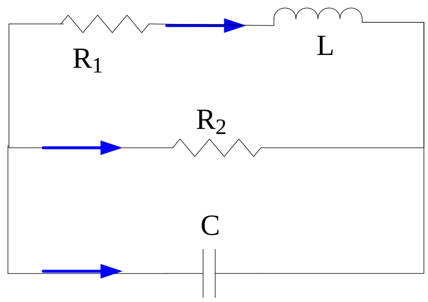
31. Consider the *RLC* parallel circuit below. Let *V* represent the voltage drop across the capacitor and *I* represent the current across the inductor.

Show that: $\begin{cases} V' = -\dfrac{V}{RC} - \dfrac{1}{C} \\ I' = \dfrac{V}{L} \end{cases}$

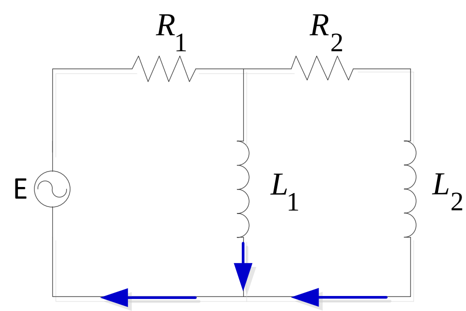


32. Consider the *RLC* parallel circuit below. Let V represent the voltage drop across the capacitor and I represent the current across the inductor.

Show that: $$\begin{cases} CV' = -I - \dfrac{V}{R_2} \\ LI' = -R_1 I + V \end{cases}$$



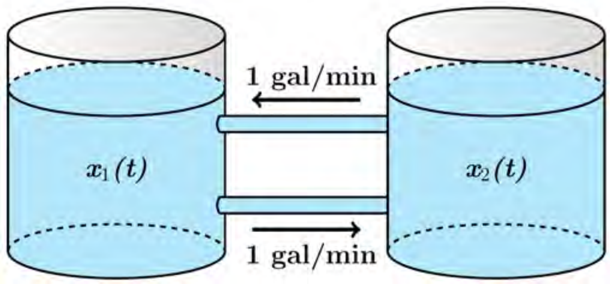
33. Consider the circuit below.



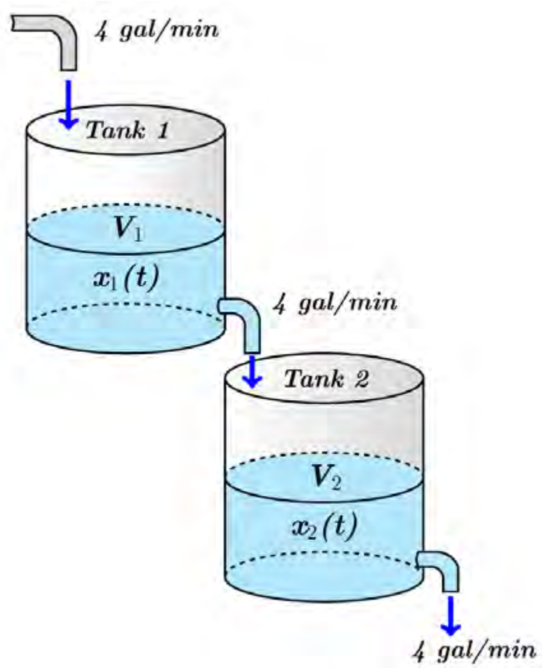
Let I_1 and I_2 represent the current flow across the indicators L_1 and L_2 respectively. Show that the circuit is modeled by the system

$$\begin{cases} L_1 I_1' = -R_1 I_1 - R_1 I_2 + E \\ L_2 I_2' = -R_1 I_1 - \left(R_1 + R_2\right) I_2 + E \end{cases}$$

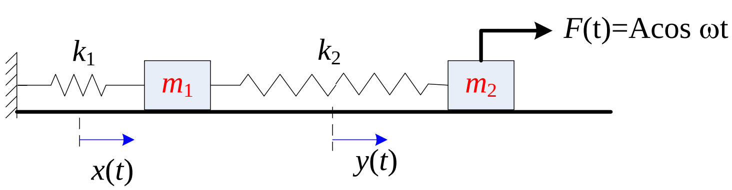
34. Two tanks are connected by two pipes. Each tank contains 500 *gallons* of a salt solution. Through on pipe solution is pumped from the first tank to the second at 1 *gal/min*. Through the other pipe, solution is pumped at the same rate from the second to the first tank. Show the salt content in each tank varies with time.



35. Each tank contains 100 *gallons* of a salt solution. Pure water flows into the upper tank at a rate of 4 *gal/min*. Salt solution drains from the upper tank into the lower tank at a rate of 4 *gal/min*. Finally, salt solution drains from the lower tank at a rate of 4 *gal/min*, effectively keeping the volume of solution in each tank at a constant 100 *gal*. If the initial salt content of the upper and lower tanks is 10 and 20 pounds, respectively. Set up an initial value problem that models the amount of salt in each tank over time (do not solve). Write the model in matrix-vector form. Is the system homogeneous or inhomogeneous?

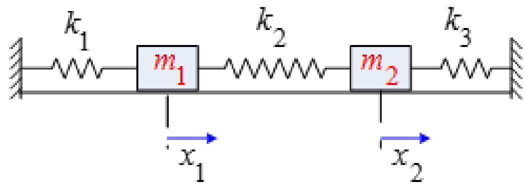


36. Two masses on a frictionless tabletop are connected with a spring having spring constant k_2. The first mass is connected to a vertical support with a spring having spring constant k_1. The second mass is shaken harmonically via a force equaling $F = A\cos\omega t$. Let $x(t)$ and $y(t)$ measure the displacements of the masses m_1 and m_2, respectively, from their equilibrium positions as a function of time. If both masses start from rest at their equilibrium positions at time $t = 0$.



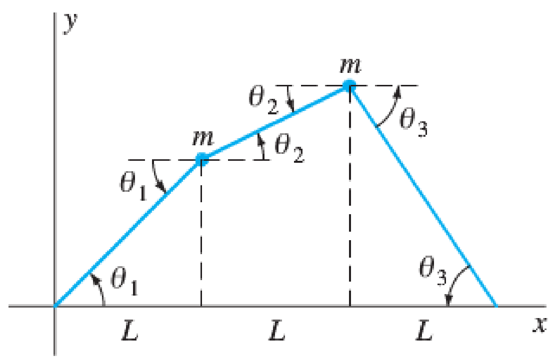
Set up an initial value problem that models the position of the masses over time (do not solve). Write the model in matrix-vector form. Is the system homogeneous or inhomogeneous?

37. Derive the equations
$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2 \end{cases}$$



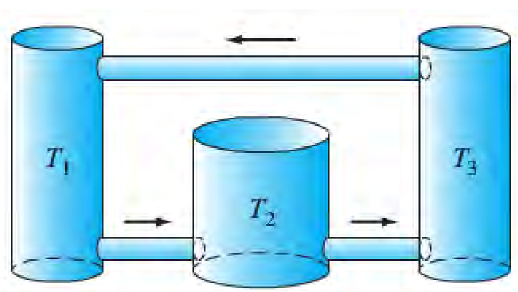
For the displacements (from equilibrium) of the 2 masses.

38. Two particles each of mass m are attached to a string under (constant) tension T. Assume that the particles oscillate vertically (that is, parallel to the y-axis) with amplitudes so small that the sines of the angles shown are accurately approximated by their tangents. Show that the displacement y_1 and y_2 satisfy the equations



$$\begin{cases} k y_1'' = -2y_1 + y_2 \\ k y_2'' = y_1 - 2y_2 \end{cases} \qquad \text{where } k = \frac{mL}{T}$$

39. There 100-gal fermentation vats are connected, and the mixtures in each tank are kept uniform by stirring. Denote by $x_i(t)$ the amount (in pounds) of alcohol in tank T_i at time t (i = 1, 2, 3). Suppose that the mixture circulates between the tanks at the rate of 10 gal/min. Derive the equations



$$\begin{cases} 10x_1' = -x_1 \qquad + x_3 \\ 10x_2' = x_1 - x_2 \\ 10x_3' = \qquad x_2 - x_3 \end{cases}$$

40. Suppose that a particle with mass m and electrical charge q moves in the xy-plane under the influence of the magnetic field $\vec{B} = B\hat{k}$ (thus a uniform field parallel to the z-axis), so the force on the particle is $\vec{F} = q\vec{v} \times \vec{B}$ if its velocity is $\vec{v}$. Show that the equations of motion of the particle are

$$mx'' = +qBy', \quad my'' = -qBx'$$

Section 3.6 – Planar Systems – *Distinct, Complex, and Repeated Eigenvalues – Eigenvectors*

Consider the system equation: $x' = Ax$

Where A is a matrix with constant entries $\left[a_{ij}\right]$

The 1st-order homogeneous equation can be written as $x' = ax$

The solution to this system is given by: $x = Ce^{at}$

We can rewrite the solution in form of vector: $x = ve^{\lambda t}$

The first derivative of the solution: $x' = \lambda ve^{\lambda t}$

$$x' = Ax$$
$$\lambda ve^{\lambda t} = Ave^{\lambda t}$$
$$\lambda v = Av$$

3.6-1 *Definition*

Suppose A is an nxn matrix and $Av = \lambda v$

The values of λ are called eigenvalues of the matrix $\boldsymbol{A}$ and the nonzero vectors $\boldsymbol{v}$ are called the eigenvectors corresponding to that eigenvalue.

3.6-2 *Eigenvalues*

Let's change the form of the system to a general matrix form and is defined by the form: $X' = AX(t)$
Where A is a square matrix (n x n)

The behavior of a system can be determined from equilibrium point(s) by finding the eigenvalues and the eigenvectors of the system.
Therefore; the equation can be rewritten into the form:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

Let's rewrite the equation $Av = \lambda v$.

$Av - \lambda v = 0$ $\qquad\qquad$ λ: are the eigenvalues and not a vector

$Av - \lambda Iv = 0$

$(A - \lambda I)v = 0$

$$\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} a_{11} - \lambda_1 & a_{12} \\ a_{21} & a_{22} - \lambda_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

Since v is a nonzero vector that implies that the matrix $A - \lambda I$ has a nontrivial null space.

This exists if and only if (*iff*):

$$\det\left(A - \lambda I\right) = 0$$

Therefore, the eigenvalues (λ's) are the roots which can be determined by solving the determinant:

$$\begin{vmatrix} a_{11} - \lambda_1 & a_{12} \\ a_{21} & a_{22} - \lambda_2 \end{vmatrix} = 0$$

$$\left(a_{11} - \lambda_1\right)\left(a_{22} - \lambda_2\right) - a_{12}a_{21} = 0$$

Example 1

Find the eigenvalues of the matrix $A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$

Solution

$$\left|A - \lambda I\right| = 0$$

$$\begin{vmatrix} -4 - \lambda & 6 \\ -3 & 5 - \lambda \end{vmatrix} = (-4 - \lambda)(5 - \lambda) + 18$$

$$= -20 + 4\lambda - 5\lambda + \lambda^2 + 18$$

$$= \lambda^2 - \lambda - 2$$

The characteristic polynomial is: $\lambda^2 - \lambda - 2 = 0$

Thus, the eigenvalues of $\boldsymbol{A}$ are 2 and –1.

3.6-3 *Eigenvectors*

From the eigenvalues, the eigenvectors, in the form $V_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$, of the system can be determined by letting:

$$\left(A-\lambda_1 I\right)V_1 = 0 \qquad \text{and} \left(A-\lambda_2 I\right)V_2 = 0$$

The general solution can be written as: $\quad x(t) = V_i e^{\lambda t}$

Example 2

Find the eigenvectors of the matrix $\qquad A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$

Solution

The eigenvalues of $\boldsymbol{A}$ are 2 and -1.

For $\lambda = 2$, we have $\quad \left(A-\lambda_1 I\right)V_1 = 0$

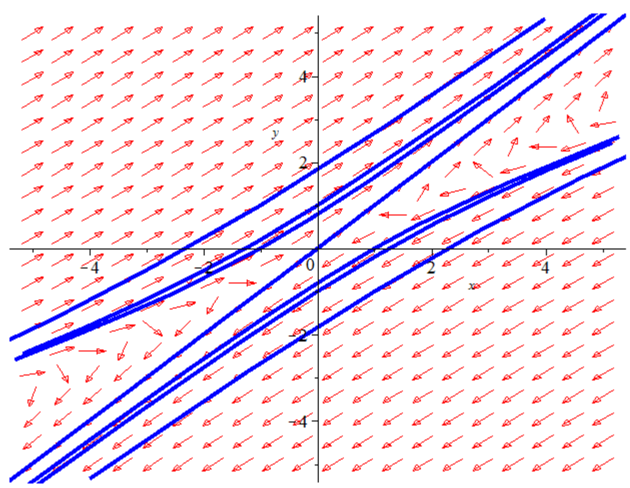
$$\begin{pmatrix} -4-2 & 6 \\ -3 & 5-2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -6 & 6 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -6x+6y=0 \\ -3x+3y=0 \end{cases} \Rightarrow x = y$$

If $x = c \quad \Rightarrow y = c$

$$V_1 = c\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



For $\lambda = -1$, we have $\quad \left(A-\lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -3 & 6 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

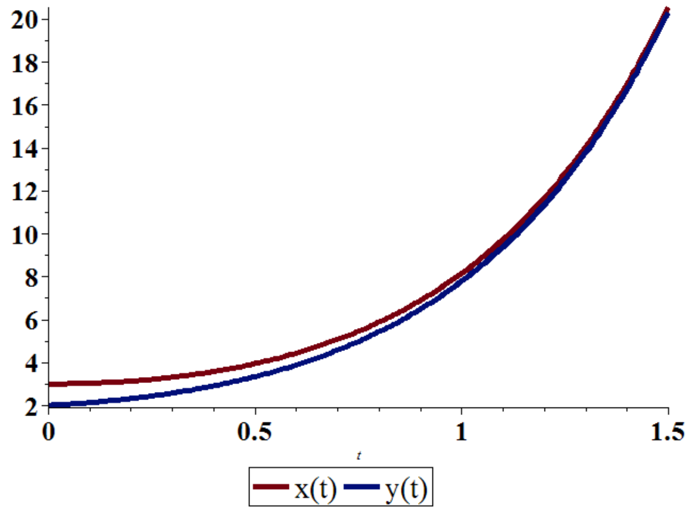
$$\begin{cases} -3x+6y=0 \\ -3x+6y=0 \end{cases} \Rightarrow x = 2y$$

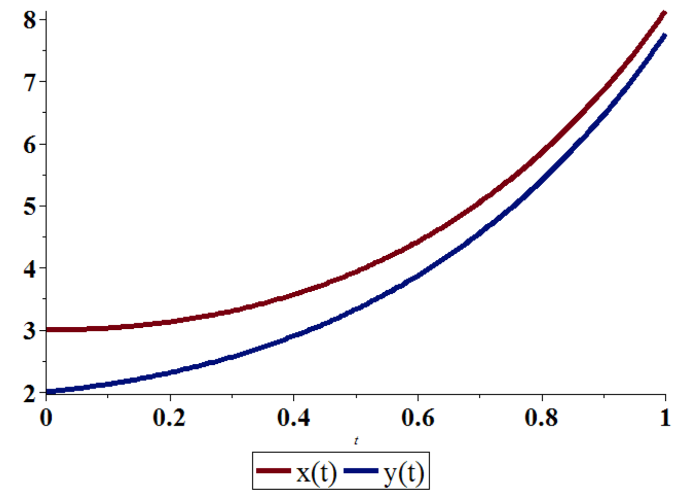
If $y = c \quad \Rightarrow x = 2c$

$$V_2 = c\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x(t) = V_1 e^{\lambda_1 t} + V_2 e^{\lambda_2 t}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}$$





3.6-4 Planar Systems

2-dimension linear systems are also called planar systems, we will enable to solve the system

$$y' = Ay$$

$$where \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix}$$

$$= \lambda^2 - \left(a_{11} + a_{22}\right)\lambda + a_{11}a_{22} - a_{12}a_{21}$$

$$D = \det(A) = a_{11}a_{22} - a_{12}a_{21}$$

$$T = tr(A) = a_{11} + a_{22} \qquad tr(A): trace$$

$$\lambda^2 - T\lambda + D = 0$$

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

3.6-5 *Summary*

In general, a polynomial of degree n has n roots. Each root λ is an eigenvalue, and for each we can find an eigenvector v. From these, we can form the solution $y(t) = ve^{\lambda t}$.

However, the numbers of the eigenvalue solutions are as follow:

1. Two Distinct real roots $\left(T^2 - 4D > 0\right)$
2. Two complex conjugate roots $\left(T^2 - 4D < 0\right)$
3. One real Repeated root $\left(T^2 - 4D = 0\right)$

3.6-6 Proposition

Suppose λ_1 and λ_2 are eigenvalues of an $n \times n$ matrix A. Suppose $V_1 \neq 0$ is an eigenvector for λ_1 and $V_2 \neq 0$ is an eigenvector for λ_2. If $\lambda_1 \neq \lambda_2$ then V_1 and V_2 are linearly independent.

3.6-7 Distinct Real Eigenvalues

If $T^2 - 4D > 0$, then the solutions of the characteristic equation are:

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2} \quad \textit{and} \quad \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}$$

Then $\lambda_1 < \lambda_2$, and both are real eigenvalues of *A*.

Let V_1 and V_2 be the associated eigenvectors. Then we have two exponential solutions:

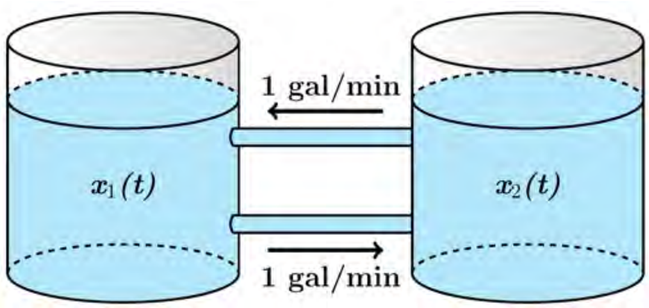
$$y_1(t) = V_1 e^{\lambda_1 t} \quad \textit{and} \quad y_2(t) = V_2 e^{\lambda_2 t}$$

The general solution is:

$$\begin{aligned} y(t) &= C_1 y_1(t) + C_2 y_2(t) \\ &= C_1 V_1 e^{\lambda_1 t} + C_2 V_2 e^{\lambda_2 t} \end{aligned}$$

Example 3

Two tanks are connected by two pipes. Each tank contains 500 *gallons* of a salt solution. Through on pipe solution is pumped from the first tank to the second at 1 *gal/min*. Through the other pipe, solution is pumped at the same rate from the second to the first tank. Suppose that at time $t = 0$ there is no salt in the tank on the right and 100 *lb*. in the tank on the left. Show the salt content in each tank varies with time.



Solution

$$\textit{Rate in} = 1 gal/min \frac{x_2}{500} lb/gal = \frac{x_2}{500} lb/min$$

$$\textit{Rate out} = 1 gal/min \frac{x_1}{500} lb/gal = \frac{x_1}{500} lb/min$$

$$\begin{aligned} \frac{dx_1}{dt} &= \textit{Rate in} - \textit{Rate out} = \frac{x_2}{500} - \frac{x_1}{500} \\ &= -\frac{x_1}{500} + \frac{x_2}{500} \end{aligned}$$

$$\frac{dx_2}{dt}=\frac{x_1}{500}-\frac{x_2}{500}$$

$$\begin{cases} x_1' = -\frac{x_1}{500}+\frac{x_2}{500} \\ x_2' = \frac{x_1}{500}-\frac{x_2}{500} \end{cases}$$

The system is: $x' = Ax(t)$

Where $A=\begin{pmatrix} -\frac{1}{500} & \frac{1}{500} \\ \frac{1}{500} & -\frac{1}{500} \end{pmatrix}$

$$\det\left(A-\lambda I\right)=\begin{vmatrix} -\frac{1}{500}-\lambda & \frac{1}{500} \\ \frac{1}{500} & -\frac{1}{500}-\lambda \end{vmatrix}$$

$$=\left(-\frac{1}{500}-\lambda\right)\left(-\frac{1}{500}-\lambda\right)-\frac{1}{500}\frac{1}{500}$$

$$=\frac{1}{500^2}+\frac{2}{500}\lambda+\lambda^2-\frac{1}{500^2}$$

$$=\lambda^2+\frac{1}{250}\lambda$$

$$=\lambda\left(\lambda+\frac{1}{250}\right)$$

The eigenvalues are: $\lambda_1=-\frac{1}{250} \qquad \lambda_2=0$

For $\lambda_1=-\frac{1}{250}$, we have

$$\left(A-\lambda_1 I\right)V_1=0$$

$$\begin{pmatrix} -\frac{1}{500}+\frac{1}{250} & \frac{1}{500} \\ \frac{1}{500} & -\frac{1}{500}+\frac{1}{250} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{500} & \frac{1}{500} \\ \frac{1}{500} & \frac{1}{500} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \frac{1}{500}x+\frac{1}{500}y=0 \\ \frac{1}{500}x+\frac{1}{500}y=0 \end{cases} \Rightarrow \begin{matrix} x+y=0 \\ x+y=0 \end{matrix} \Rightarrow x=-y$$

$$V_1=\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \Rightarrow \quad \underline{x_1(t)=\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t/250}}$$

For $\lambda_2 = 0$, we have

$$\left(A - \lambda_2 I\right)V_2 = 0$$

$$\begin{pmatrix} -\frac{1}{500} & \frac{1}{500} \\ \frac{1}{500} & -\frac{1}{500} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -\frac{1}{500}x + \frac{1}{500}y = 0 \\ \frac{1}{500}x - \frac{1}{500}y = 0 \end{cases} \Rightarrow \begin{matrix} -x + y = 0 \\ x - y = 0 \end{matrix} \Rightarrow x = y$$

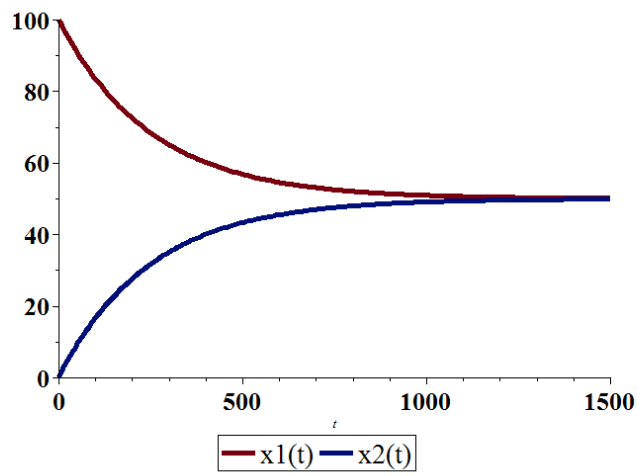
$$V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{0t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The general solution:

$$x(t) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t/250} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} C_1 e^{-t/250} + C_2 \\ -C_1 e^{-t/250} + C_2 \end{pmatrix}$$



$$x(0) = \begin{pmatrix} C_1 e^{-0/250} + C_2 \\ -C_1 e^{-0/250} + C_2 \end{pmatrix}$$

$$\begin{pmatrix} 100 \\ 0 \end{pmatrix} = \begin{pmatrix} C_1 + C_2 \\ -C_1 + C_2 \end{pmatrix}$$

$$\rightarrow \begin{cases} C_1 + C_2 = 100 \\ -C_1 + C_2 = 0 \end{cases} \Rightarrow \quad C_1 = 50 \quad C_2 = 50$$

$$x(t) = \begin{pmatrix} 50 + 50e^{-t/250} \\ 50 - 50e^{-t/250} \end{pmatrix}$$

3.6-8 Complex Eigenvalues

If $T^2 - 4D < 0$, then the solutions of the characteristic equation are the complex conjugate:

$$\lambda_1 = \frac{T + i\sqrt{4D - T^2}}{2} \quad \text{and} \quad \lambda_2 = \frac{T - i\sqrt{4D - T^2}}{2}$$

Example 4

Find the eigenvalues and eigenvectors for the matrix $A = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}$

Solution

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -\lambda & 1 \\ -2 & 2-\lambda \end{vmatrix} \\ &= -\lambda(2-\lambda) + 2 \\ &= \lambda^2 - 2\lambda + 2 \\ &= 0 \end{aligned}$$

$$\lambda^2 - 2\lambda + 2 = 0 \quad \Rightarrow \lambda = 1 + i \quad \bar{\lambda} = 1 - i$$

For $\lambda = 1 + i$; $\quad (A - \lambda I)\begin{pmatrix} x \\ y \end{pmatrix} = 0$

$$\begin{pmatrix} -1-i & 1 \\ -2 & 2-i-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} -1-i & 1 \\ -2 & 1-i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\rightarrow \begin{cases} -(1+i)x + y = 0 \\ -2x + (1-i)y = 0 \end{cases}$$

$$\rightarrow \ (1+i)x = y$$

$$V_1 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} \quad \Rightarrow \quad x_1(t) = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} e^{(1+i)t}$$

For $\bar{\lambda} = 1 - i$; $\quad V_2 = \bar{V}_1 \quad \Rightarrow \quad x_2(t) = V_2\, e^{\bar{\lambda} t}$

$$V_2 = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} \quad \Rightarrow \quad x_2(t) = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^{(1-i)t}$$

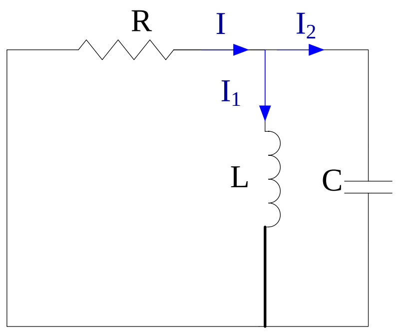
3.6-9 *Theorem*

Suppose that A is a 2×2 matrix with complex conjugate eigenvalues λ and $\bar{\lambda}$. Suppose that V is an eigenvector associated with λ. Then the general solution to the system $x' = Ax$ is

$$x(t) = C_1V_1e^{\lambda t} + C_2V_2e^{\bar{\lambda} t}$$

Example 5

Find the current I_1 and I_2 for the circuit below, where $R = 1\,\Omega$, $L = 1\ henry$, and $C = \frac{5}{4}\ farad$. Assume that $I_1(0) = 5A$ and $I_2(0) = 1A$.



Solution

Kirchhoff's current law: $I = I_1 + I_2$

Since there is no voltage: $0 = RI + LI_1' \quad \Rightarrow LI_1' = -RI = -R\left(I_1 + I_2\right)$

$$I_1' = -\frac{R}{L}I_1 - \frac{R}{L}I_2 \quad (1)$$

Kirchhoff's law applied to the large loop:

$$0 = RI + \frac{1}{C}Q \quad \Rightarrow \quad \frac{1}{C}Q + R\left(I_1 + I_2\right) = 0$$

$$\frac{1}{C}Q' + R\left(I_1' + I_2'\right) = 0$$

$$\frac{1}{C}I_2 + R\left(I_1' + I_2'\right) = 0$$

$$RI_2' = -\frac{1}{C}I_2 - RI_1'$$

$$I_2' = -\frac{1}{RC}I_2 - \left(-\frac{R}{L}I_1 - \frac{R}{L}I_2\right)$$

$$= \frac{R}{L}I_1 + \left(\frac{R}{L} - \frac{1}{RC}\right)I_2 \quad (2)$$

$$\begin{cases} I_1' = -\frac{R}{L}I_1 - \frac{R}{L}I_2 \\ I_2' = \frac{R}{L}I_1 + \left(\frac{R}{L} - \frac{1}{RC}\right)I_2 \end{cases}$$

$$\Rightarrow \begin{cases} I_1' = -I_1 - I_2 \\ I_2' = I_1 + \left(1 - \frac{1}{5/4}\right)I_2 \end{cases}$$

$$\begin{cases} I_1' = -I_1 - I_2 \\ I_2' = I_1 + \frac{1}{5}I_2 \end{cases}$$

The system can be written as: $I' = AI$, where $A = \begin{pmatrix} -1 & -1 \\ 1 & \frac{1}{5} \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & -1 \\ 1 & \frac{1}{5}-\lambda \end{vmatrix}$$

$$= (-1-\lambda)\left(\frac{1}{5}-\lambda\right) + 1 = 0$$

$$\lambda^2 + \frac{4}{5}\lambda + \frac{4}{5} = 0$$

$$5\lambda^2 + 4\lambda + 4 = 0 \qquad \lambda = \frac{-2 \pm 4i}{5}$$

For $\lambda = \frac{-2+4i}{5}$;

$$\begin{pmatrix} -1 + \frac{2}{5} - \frac{4i}{5} & -1 \\ 1 & \frac{1}{5} + \frac{2}{5} - \frac{4i}{5} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} -\frac{3}{5} - \frac{4i}{5} & -1 \\ 1 & \frac{3}{5} - \frac{4i}{5} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} -\frac{1}{5}(3+4i)x - y = 0 \\ x + \frac{1}{5}(3-4i)y = 0 \end{cases} \rightarrow x = -5 \qquad y = 3+4i$$

$$V_1 = \begin{pmatrix} -5 \\ 3+4i \end{pmatrix}$$

$$x(t) = V_1 e^{\lambda t}$$

$$= \begin{pmatrix} -5 \\ 3+4i \end{pmatrix} e^{\left(\frac{-2+4i}{5}\right)t}$$

$$= \begin{pmatrix} -5 \\ 3+4i \end{pmatrix} e^{-\frac{2}{5}t} e^{\frac{4}{5}it}$$

$$= e^{-\frac{2}{5}t} \begin{pmatrix} -5 \\ 3+4i \end{pmatrix} \left(\cos\left(\tfrac{4}{5}t\right) + \boldsymbol{i} \sin\left(\tfrac{4}{5}t\right) \right)$$

$$= e^{-\frac{2}{5}t} \begin{pmatrix} -5\cos\left(\frac{4}{5}t\right) - i5\sin\left(\frac{4}{5}t\right) \\ 3\cos\left(\frac{4}{5}t\right) - 4\sin\left(\frac{4}{5}t\right) + i\left(4\cos\left(\frac{4}{5}t\right) + 3\sin\left(\frac{4}{5}t\right)\right) \end{pmatrix}$$

$$x_1(t) = e^{-\frac{2}{5}t} \begin{pmatrix} -5\cos\left(\frac{4}{5}t\right) \\ 3\cos\left(\frac{4}{5}t\right) - 4\sin\left(\frac{4}{5}t\right) \end{pmatrix}$$

$$x_2(t) = e^{-\frac{2}{5}t} \begin{pmatrix} -5\sin\left(\frac{4}{5}t\right) \\ 4\cos\left(\frac{4}{5}t\right) + 3\sin\left(\frac{4}{5}t\right) \end{pmatrix}$$

$$I(t) = C_1 x_1 + C_2 x_2$$

$$= C_1 e^{-\frac{2}{5}t} \begin{pmatrix} -5\cos\left(\frac{4}{5}t\right) \\ 3\cos\left(\frac{4}{5}t\right) - 4\sin\left(\frac{4}{5}t\right) \end{pmatrix} + C_2 e^{-\frac{2}{5}t} \begin{pmatrix} -5\sin\left(\frac{4}{5}t\right) \\ 4\cos\left(\frac{4}{5}t\right) + 3\sin\left(\frac{4}{5}t\right) \end{pmatrix}$$

$$I(0) = C_1 e^{-\frac{2}{5}(0)} \begin{pmatrix} -5\cos\left(\frac{4}{5}(0)\right) \\ 3\cos\left(\frac{4}{5}(0)\right) - 4\sin\left(\frac{4}{5}(0)\right) \end{pmatrix} + C_2 e^{-\frac{2}{5}(0)} \begin{pmatrix} -5\sin\left(\frac{4}{5}(0)\right) \\ 4\cos\left(\frac{4}{5}(0)\right) + 3\sin\left(\frac{4}{5}(0)\right) \end{pmatrix}$$

$$\begin{pmatrix} 5 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} -5 \\ 3 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

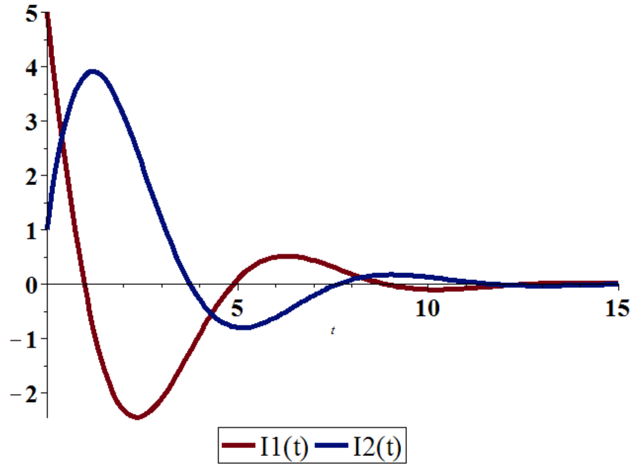
$$\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -5C_1 \\ 3C_1 + 4C_2 \end{pmatrix}$$

$$\begin{cases} -5C_1 = 5 \\ 3C_1 + 4C_2 = 1 \end{cases} \Rightarrow \underline{C_1 = -1|} \quad \underline{C_2 = 1|}$$

The general solution is:

$$I(t) = -e^{-\frac{2}{5}t}\begin{pmatrix} -5\cos\left(\frac{4}{5}t\right) \\ 3\cos\left(\frac{4}{5}t\right) - 4\sin\left(\frac{4}{5}t\right) \end{pmatrix} + e^{-\frac{2}{5}t}\begin{pmatrix} -5\sin\left(\frac{4}{5}t\right) \\ 4\cos\left(\frac{4}{5}t\right) + 3\sin\left(\frac{4}{5}t\right) \end{pmatrix}$$

$$= e^{-\frac{2}{5}t}\begin{pmatrix} 5\cos\left(\frac{4}{5}t\right) - 5\sin\left(\frac{4}{5}t\right) \\ \cos\left(\frac{4}{5}t\right) + 7\sin\left(\frac{4}{5}t\right) \end{pmatrix}$$



3.6-10 One Real Eigenvalue of *Multiplicity* 2

If $T^2 - 4D = 0 \Rightarrow T^2 = 4D$, then the solutions of the characteristic equation: $\lambda_1 = \lambda_2 = \frac{T}{2}$

$$\left.\begin{aligned} x_1(t) &= V_1 e^{\lambda t} \\ x_2(t) &= \left(V_1 t + V_2\right)e^{\lambda t} \end{aligned}\right\} \Rightarrow \quad x(t) = C_1 x_1(t) + C_2 x_2(t)$$

Example 6

Find all exponential solutions for $\quad A = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}$

Solution

$$\left|A - \lambda I\right| = \begin{vmatrix} -1-\lambda & -1 \\ 1 & -3-\lambda \end{vmatrix}$$

$$= \lambda^2 + 4\lambda + 4$$

$$= (\lambda + 2)^2 = 0$$

For $\lambda_{1,2} = -2 \qquad \to \ (A - \lambda I)^2 V_2 = 0$

$$\begin{pmatrix}1 & -1\\1 & -1\end{pmatrix}\begin{pmatrix}1 & -1\\1 & -1\end{pmatrix}V_2 = 0$$

$$\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}V_2 = 0 \qquad \Rightarrow \quad V_2 = \begin{pmatrix}1\\0\end{pmatrix}$$

$$(A+2I)V_2 = V_1$$

$$\begin{pmatrix}1 & -1\\1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = V_1 \qquad \Rightarrow \quad \begin{pmatrix}1\\1\end{pmatrix} = V_1$$

$$x_1(t) = \begin{pmatrix}1\\1\end{pmatrix}e^{-2t}$$

$$\begin{aligned} x_2(t) &= \left(V_1 t + V_2\right)e^{-2t} \\ &= \left(\begin{pmatrix}1\\1\end{pmatrix}t + \begin{pmatrix}1\\0\end{pmatrix}\right)e^{-2t} \\ &= \begin{pmatrix}t+1\\t\end{pmatrix}e^{-2t} \end{aligned}$$

$$\begin{aligned} x(t) &= C_1 x_1(t) + C_2 x_2(t) \\ &= C_1\begin{pmatrix}1\\1\end{pmatrix}e^{-2t} + C_2\begin{pmatrix}t+1\\t\end{pmatrix}e^{-2t} \end{aligned}$$

Exercises Section 3.6 – Planar Systems – *Distinct, Complex, and Repeated Eigenvalues – Eigenvectors*

(1 – 13) Find the eigenvalues and the eigenvectors for each of the matrices.

1. $A=\begin{pmatrix} 12 & 14 \\ -7 & -9 \end{pmatrix}$

2. $A=\begin{pmatrix} -4 & 1 \\ -2 & 1 \end{pmatrix}$

3. $A=\begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$

4. $A=\begin{pmatrix} -2 & 3 \\ 0 & -5 \end{pmatrix}$

5. $A=\begin{pmatrix} 6 & 10 \\ -5 & -9 \end{pmatrix}$

6. $A=\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

7. $A=\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$

8. $A=\begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

9. $A=\begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ -8 & -4 & -3 \end{pmatrix}$

10. $A=\begin{pmatrix} -1 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & -12 & 2 \end{pmatrix}$

11. $A=\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$

12. $A=\begin{pmatrix} -6 & 4 & 4 \\ -4 & 2 & 4 \\ -10 & 8 & 4 \end{pmatrix}$

13. $A=\begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(14 – 15) Find a fundamental set of solutions for the system $x' = Ax$, where A is the given matrices.

14. $A=\begin{pmatrix} 2 & 0 \\ -4 & -1 \end{pmatrix}$

15. $A=\begin{pmatrix} -1 & 0 & 0 \\ 2 & -5 & -6 \\ -2 & 3 & 4 \end{pmatrix}$

(16 – 59) Find the general solution of the system

16. $\begin{cases} x_1'(t) = x_1 + 2x_2 \\ x_2'(t) = 4x_1 + 3x_2 \end{cases}$

17. $\begin{cases} x_1'(t) = 2x_1 + 2x_2 \\ x_2'(t) = x_1 + 3x_2 \end{cases}$

18. $\begin{cases} x_1'(t) = -4x_1 + 2x_2 \\ x_2'(t) = -\frac{5}{2}x_1 + 2x_2 \end{cases}$

19. $\begin{cases} x_1'(t) = -\frac{5}{2}x_1 + 2x_2 \\ x_2'(t) = \frac{3}{4}x_1 - 2x_2 \end{cases}$

20. $\begin{cases} x_1'(t) = 3x_1 - x_2 \\ x_2'(t) = 9x_1 - 3x_2 \end{cases}$

21. $\begin{cases} x_1'(t) = -6x_1 + 5x_2 \\ x_2'(t) = -5x_1 + 4x_2 \end{cases}$

22. $\begin{cases} x_1'(t) = 6x_1 - x_2 \\ x_2'(t) = 5x_1 + 2x_2 \end{cases}$

23. $\begin{cases} x_1'(t) = x_1 + x_2 \\ x_2'(t) = -2x_1 - x_2 \end{cases}$

24. $\begin{cases} x_1'(t) = 5x_1 + x_2 \\ x_2'(t) = -2x_1 + 3x_2 \end{cases}$

25. $\begin{cases} x_1'(t) = 4x_1 + 5x_2 \\ x_2'(t) = -2x_1 + 6x_2 \end{cases}$

26. $\begin{cases} x_1'(t) = 5x_1 - 4x_2 \\ x_2'(t) = 2x_1 - x_2 \end{cases}$

27. $\begin{cases} x_1'(t) = 6x_1 - 6x_2 \\ x_2'(t) = 4x_1 - 4x_2 \end{cases}$

28. $\begin{cases} x_1'(t) = 5x_1 - 3x_2 \\ x_2'(t) = 2x_1 \end{cases}$

29. $\begin{cases} x_1'(t) = 5x_1 - 4x_2 \\ x_2'(t) = 3x_1 - 2x_2 \end{cases}$

30. $\begin{cases} x_1'(t) = 9x_1 - 8x_2 \\ x_2'(t) = 6x_1 - 5x_2 \end{cases}$

31. $\begin{cases} x_1'(t) = 10x_1 - 6x_2 \\ x_2'(t) = 12x_1 - 7x_2 \end{cases}$

32. $\begin{cases} x_1'(t) = 6x_1 - 10x_2 \\ x_2'(t) = 2x_1 - 3x_2 \end{cases}$

33. $\begin{cases} x_1'(t) = 11x_1 - 15x_2 \\ x_2'(t) = 6x_1 - 8x_2 \end{cases}$

34. $\begin{cases} x_1'(t) = 3x_1 + x_2 \\ x_2'(t) = x_1 + 3x_2 \end{cases}$

35. $\begin{cases} x_1'(t) = 4x_1 + 2x_2 \\ x_2'(t) = 2x_1 + 4x_2 \end{cases}$

36. $\begin{cases} x_1'(t) = 9x_1 + 2x_2 \\ x_2'(t) = 2x_1 + 6x_2 \end{cases}$

37. $\begin{cases} x_1'(t) = 13x_1 + 4x_2 \\ x_2'(t) = 4x_1 + 7x_2 \end{cases}$

38. $\begin{cases} x_1'(t) = 3x_1 - 2x_2 \\ x_2'(t) = 2x_1 - 2x_2 \end{cases}$

39. $\begin{cases} x_1'(t) = 2x_1 - x_2 \\ x_2'(t) = 3x_1 - 2x_2 \end{cases}$

40. $\begin{cases} x_1'(t) = 5x_1 - x_2 \\ x_2'(t) = 3x_1 - x_2 \end{cases}$

41. $\begin{cases} x_1'(t) = x_1 + x_2 \\ x_2'(t) = 4x_1 - 2x_2 \end{cases}$

42. $\begin{cases} x_1'(t) = -x_1 - 4x_2 \\ x_2'(t) = x_1 - x_2 \end{cases}$

43. $\begin{cases} x_1'(t) = 2x_1 + 3x_2 - 7 \\ x_2'(t) = -x_1 - 2x_2 + 5 \end{cases}$

44. $\begin{cases} x_1'(t) = 5x_1 + 9x_2 + 2 \\ x_2'(t) = -x_1 + 11x_2 + 6 \end{cases}$

45. $\begin{cases} y_1'(t) = 6y_1 + y_2 + 6t \\ y_2'(t) = 4y_1 + 3y_2 - 10t + 4 \end{cases}$

46. $\begin{cases} x'(t) = 5x + 3y - 2e^{-t} + 1 \\ y'(t) = -x + y + e^{-t} - 5t + 7 \end{cases}$

47. $\begin{cases} x'(t) = -3x + y + 3t \\ y'(t) = 2x - 4y + e^{-t} \end{cases}$

48. $\begin{cases} x'(t) = 2x - y + (\sin 2t)e^{2t} \\ y'(t) = 4x + 2y + (2\cos 2t)e^{2t} \end{cases}$

49. $\begin{cases} x'(t) = \qquad 2y + e^{t} \\ y'(t) = -x + 3y - e^{t} \end{cases}$

50. $\begin{cases} x'(t) = 2y + 2 \\ y'(t) = -x + 3y + e^{-3t} \end{cases}$

51. $\begin{cases} x'(t) = x + 8y + 12t \\ y'(t) = x - y + 12t \end{cases}$

52. $\begin{cases} x_1'(t) = x_1 + x_2 - x_3 \\ x_2'(t) = 2x_2 \\ x_3'(t) = x_2 - x_3 \end{cases}$

53. $\begin{cases} x_1'(t) = 2x_1 - 7x_2 \\ x_2'(t) = 5x_1 + 10x_2 + 4x_3 \\ x_3'(t) = 5x_2 + 2x_3 \end{cases}$

54. $\begin{cases} x_1'(t) = 3x_1 - x_2 - x_3 \\ x_2'(t) = x_1 + x_2 - x_3 \\ x_3'(t) = x_1 - x_2 + x_3 \end{cases}$

55. $\begin{cases} x_1'(t) = 3x_1 + 2x_2 + 4x_3 \\ x_2'(t) = 2x_1 + 2x_3 \\ x_3'(t) = 4x_1 + 2x_2 + 3x_3 \end{cases}$

56. $\begin{cases} x_1'(t) = x_1 + x_2 + x_3 \\ x_2'(t) = 2x_1 + x_2 - x_3 \\ x_3'(t) = -8x_1 - 5x_2 - 3x_3 \end{cases}$

57. $\begin{cases} x_1'(t) = x_1 - x_2 + 4x_3 \\ x_2'(t) = 3x_1 + 2x_2 - x_3 \\ x_3'(t) = 2x_1 + x_2 - x_3 \end{cases}$

58. $\begin{cases} x_1'(t) = x_1 + x_2 + e^{t} \\ x_2'(t) = x_1 + x_2 + e^{2t} \\ x_3'(t) = 3x_3 + te^{3t} \end{cases}$

59. $\begin{cases} x_1'(t) = 3x_1 - x_2 - x_3 \\ x_2'(t) = x_1 + x_2 - x_3 + t \\ x_3'(t) = x_1 - x_2 + x_3 + 2e^{t} \end{cases}$

(60 – 96) Find the general solution of the system $y' = Ay$

60. $\begin{cases} y_1'(t) = -y_1 + 6y_2 \\ y_2'(t) = -3y_1 + 8y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

61. $\begin{cases} y_1'(t) = y_1 + 2y_2 \\ y_2'(t) = -y_1 + 4y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

62. $\begin{cases} y_1'(t) = -4y_1 - 8y_2 \\ y_2'(t) = 4y_1 + 4y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

63. $\begin{cases} y_1'(t) = -y_1 - 2y_2 \\ y_2'(t) = 4y_1 + 3y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

64. $\begin{cases} y_1'(t) = 3y_1 - y_2 \\ y_2'(t) = y_1 + y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

65. $\begin{cases} y_1'(t) = -3y_1 + y_2 \\ y_2'(t) = -y_1 - y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$

66. $\begin{cases} y_1'(t) = 2y_1 + 4y_2 \\ y_2'(t) = -y_1 + 6y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

67. $\begin{cases} y_1'(t) = -8y_1 - 10y_2 \\ y_2'(t) = 5y_1 + 7y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

68. $\begin{cases} y_1'(t) = -3y_1 + 2y_2 \\ y_2'(t) = -3y_1 + 4y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

69. $\begin{cases} y_1'(t) = 3y_1 - y_2 \\ y_2'(t) = 5y_1 - 3y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

70. $\begin{cases} y_1'(t) = y_1 + 9y_2 \\ y_2'(t) = -2y_1 - 5y_2 \end{cases}$ $y(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

71. $\begin{cases} y_1'(t) = 4y_1 + y_2 \\ y_2'(t) = -2y_1 + y_2 \end{cases}$ $y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

72. $\begin{cases} y_1'(t) = 2y_1 + y_2 - e^{2t} \\ y_2'(t) = y_1 + 2y_2 \end{cases}$ $y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

73. $\begin{cases} y_1'(t) = 3y_1 - 2y_2 \\ y_2'(t) = 2y_1 - 2y_2 \end{cases}$ $y(0) = \begin{pmatrix} 3 \\ \frac{1}{2} \end{pmatrix}$

74. $\begin{cases} y_1'(t) = y_1 - 2y_2 \\ y_2'(t) = 3y_1 - 4y_2 \end{cases}$ $y(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

75. $\begin{cases} y_1'(t) = y_1 - 4y_2 \\ y_2'(t) = 4y_1 - 7y_2 \end{cases}$ $y(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

76. $\begin{cases} y_1'(t) = 3y_1 + 9y_2 \\ y_2'(t) = -y_1 - 3y_2 \end{cases}$ $y(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

77. $\begin{cases} y_1'(t) = 2y_1 + \frac{3}{2}y_2 \\ y_2'(t) = -\frac{3}{2}y_1 - y_2 \end{cases}$ $y(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

78. $\begin{cases} y_1'(t) = -5y_1 + 12y_2 \\ y_2'(t) = -2y_1 + 5y_2 \end{cases}$ $y(0) = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$

79. $\begin{cases} y_1'(t) = -4y_1 + 6y_2 \\ y_2'(t) = -3y_1 + 5y_2 \end{cases}$ $y(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

80. $\begin{cases} y_1'(t) = y_1 + 2y_2 \\ y_2'(t) = 3y_1 + 2y_2 \end{cases}$ $y(0) = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$

81. $\begin{cases} y_1'(t) = -5y_1 + y_2 \\ y_2'(t) = 4y_1 - 2y_2 \end{cases}$ $y(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

82. $\begin{cases} y_1'(t) = 3y_1 - 9y_2 \\ y_2'(t) = 4y_1 - 3y_2 \end{cases}$ $y(0) = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

83. $\begin{cases} y_1'(t) = -y_1 + \frac{3}{2}y_2 \\ y_2'(t) = -\frac{1}{6}y_1 - 2y_2 \end{cases}$ $y(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

84. $\begin{cases} y_1'(t) = 3y_1 - 3y_2 + 2 \\ y_2'(t) = -6y_1 - t \end{cases}$ $y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

85. $\begin{cases} y_1'(t) = y_1 + 2y_2 + 2t \\ y_2'(t) = 3y_1 + 2y_2 - 4t \end{cases}$ $y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

86. $\begin{cases} x_1'(t) = 3x_1 - x_2 + 4e^{2t} \\ x_2'(t) = -x_1 + 3x_2 + 4e^{4t} \end{cases}$ $X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

87. $\begin{cases} x_1'(t) = x_1 - x_2 + \frac{1}{t} \\ x_2'(t) = x_1 - x_2 + \frac{1}{t} \end{cases}$ $X(1) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

88. $\begin{cases} y_1'(t) = 3y_1 - 13y_2 \\ y_2'(t) = 5y_1 + y_2 \end{cases}$ $y(0) = \begin{pmatrix} 3 \\ -10 \end{pmatrix}$

89. $\begin{cases} y_1'(t) = 7y_1 + y_2 \\ y_2'(t) = -4y_1 + 3y_2 \end{cases}$ $y(0) = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$

90. $\begin{cases} y_1'(t) + 2y_2'(t) = 4y_1 + 5y_2 \\ 2y_1'(t) - y_2'(t) = 3y_1 \end{cases}$ $y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

91. $\begin{cases} y_1'(t) = -5y_1 + y_2 + 6e^{2t} \\ y_2'(t) = 4y_1 - 2y_2 - e^{2t} \end{cases}$ $y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

92. $\begin{cases} x_1'(t) = 3x_1 - 2x_2 - 2e^{-t} \\ x_2'(t) = x_1 \quad - 2e^{-t} \end{cases}$ $X(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

93. $\begin{cases} y_1'(t) = y_1 \\ y_2'(t) = -4y_1 + y_2 \\ y_3'(t) = 3y_1 + 6y_2 + 2y_3 \end{cases}$ $y(0) = \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix}$

94. $\begin{cases} y_1'(t) = -\frac{5}{2}y_1 + y_2 + y_3 \\ y_2'(t) = y_1 - \frac{5}{2}y_2 + y_3 \\ y_3'(t) = y_1 + y_2 - \frac{5}{2}y_3 \end{cases}$ $y(0) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

95. $\begin{cases} x_1'(t) = 3x_1 - x_2 - x_3 \\ x_2'(t) = x_1 + x_2 - x_3 + t \\ x_3'(t) = x_1 - x_2 + x_3 + 2e^t \end{cases}$ $X(0) = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

96. $\begin{cases} x_1'(t) = x_1 + x_2 \quad + e^t \\ x_2'(t) = x_1 + x_2 \quad + e^{2t} \\ x_3'(t) = \qquad 3x_3 + te^{3t} \end{cases}$ $X(0) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

(97 – 103) Find the general solution of the system

97. $x'' + x = 3$; $x(\pi) = 1, \quad x'(\pi) = 2$

98. $\begin{cases} x'' = x - y \\ y'' = x - y \end{cases}$ $\begin{cases} x(3) = 5, & x'(3) = 2 \\ y(3) = 1, & y'(3) = -1 \end{cases}$

99. $\begin{cases} x'' = x - y \\ y'' = -x + y \end{cases}$ $\begin{cases} x(0) = -1, & x'(0) = 0 \\ y(0) = 1, & y'(0) = 0 \end{cases}$

100. $\begin{cases} \dfrac{d^2x}{dt^2} = y; & x(0) = 3, \quad x'(0) = 1 \\ \dfrac{d^2y}{dt^2} = x; & y(0) = 1, \quad y'(0) = -1 \end{cases}$

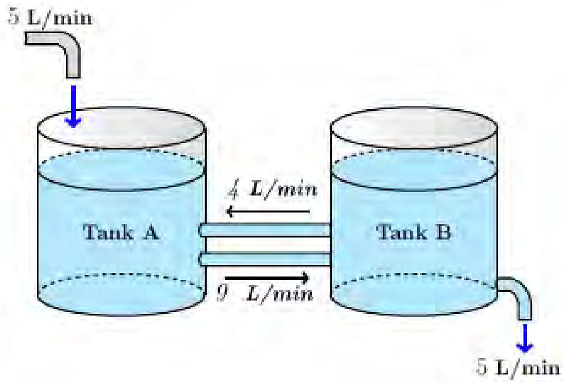
101. $\begin{cases} x'' + 5x - 2y = 0 & x(0) = x'(0) = 0 \\ y'' + 2y - 2x = 3\sin 2t & y(0) = 1, \quad y'(0) = 0 \end{cases}$

102. $\begin{cases} x'' = -2x' - 5y + 3 \\ y' = x' + 2y \end{cases}$ $\quad x(0) = 0,\ x'(0) = 0,\ y(0) = 1$

103. $\begin{cases} x'' = 2x' + 5y + 3 \\ y' = -x' - 2y \end{cases}$ $\quad x(0) = 0,\ x'(0) = 0,\ y(0) = 1$

104. Find the real and imaginary part of $z(t) = e^{2it}\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$

105. Two tanks, each containing 360 *liters* of a salt solution. Pure water pours into tank *A* at a rate of 5 *L/min*. There are two pipes connecting tank *A* to tank *B*. The first pumps salt solution from tank *B* into tank *A* at a rate of 4 *L/min*. The second pumps salt solution from tank *A* into tank *B* at a rate of 9 *L/min*. Finally, there is a drain on tank *B* from which salt solution drains at a rate of 5 *L/min*. Thus, each tank maintains a constant volume of 360 *liters* of salt solution. Initially, there are 60 *kg* of salt present in tank *A*, but tank *B* contains pure water.



a) Set up, in matrix-vector form, an initial value problem that models the salt content in each tank over time.

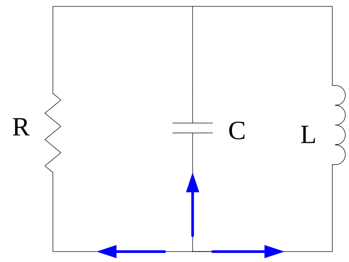
b) Find the eigenvalues and eigenvectors of the coefficient matrix in part (*a*), then find the general solution in vector form. Find the solution that satisfies the initial conditions posed in part (*a*).

c) Plot each component of your solution in part (*b*) over a period of four-time constants $\left[0,\ 4T_c\right]$. What is the eventual salt content in each tank? Give both a physical and a mathematical reason for your answer.

106. Consider the *RLC* parallel circuit below. Let *V* represent the voltage drop across the capacitor and *I* represent the current across the inductor that satisfied the system.

$$\begin{cases} V' = -\dfrac{V}{RC} - \dfrac{1}{C} \\ I' = \dfrac{V}{L} \end{cases}$$

Suppose that the resistance is $R = \frac{1}{2}\Omega$, the capacitor is $C = 1$ *farad*, and the inductance is $L = \frac{1}{2}$ *henry*. If the initial voltage across the capacitor is $V(0) = 10$ *volts* and there is no initial current across the inductor, solve the system to determine the voltage and current as a function of time.

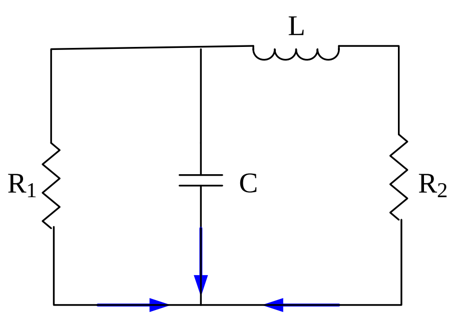


Plot the voltage and current as a function of time. Assume current flows in the directions indicated.

107. Show that the voltage V across the capacitor and the current I through the inductor satisfy the system

$$\begin{cases} I' = -\dfrac{R_1}{L} I + \dfrac{1}{L} V \\ V' = -\dfrac{1}{C} I - \dfrac{1}{R_2 C} V \end{cases}$$

Suppose that the capacitance is $C = 1$ *farad*, the inductance is $L = 1$ *henry*, the leftmost resistor has resistance $R_2 = 1\ \Omega$, and the rightmost resistor has resistance $R_1 = 5\ \Omega$.



If the initial voltage across the capacitor is 12 *volts* and the initial current through the inductor is zero, determine the voltage V across the capacitor and the current I through the inductor as functions of time. Plot the voltage and current as functions of time. Assume current flows in the directions indicated.

Section 3.7 – Phase Plane Portraits & Applications

3.7-1 Equilibrium Points

The dynamical behavior of a linear system is easier than non-linear system. We need to determine a set of points to satisfy the autonomous system $y' = 0\ \left(y' = f\left(y(t), t \right) \equiv 0 \right)$. These set of points are called ***equilibrium points***.

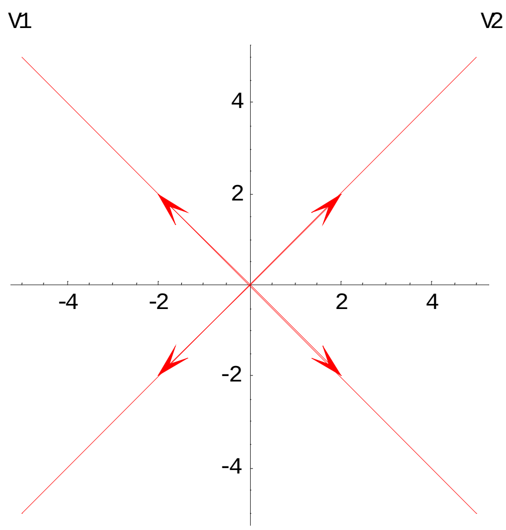
From these equilibrium points, we can determine the stability of the system.

The equilibrium point O_1 is the intersection of the eigenvectors, and we can plot those two lines by joining these points V_1O_1 and V_2O_1 together.

The general solution for the system is given by:

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 V_1 e^{\lambda_1 t} + C_2 V_2 e^{\lambda_2 t}$$

The behavior of the system or the solutions is depending on the value of λ_1 and λ_2 , and if they are real or complex values.



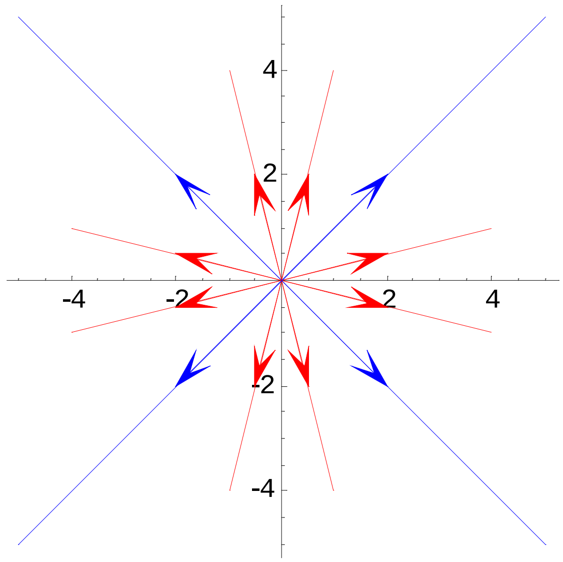
Eigenvectors V_1 and V_2 plot.

The family of all solution curves without the presence of the independent variable is **called *phase portrait***.

3.7-2 Stability of the equilibrium point condition

- An equilibrium point is *stable* if all nearby solutions stay nearby
- An equilibrium point is *asymptotically stable* if all nearby solutions not only stay nearby, but also tend the equilibrium point.

***Case* 1**: If $\lambda_1 > 0$ and $\lambda_2 > 0$ are real values.



$\lambda_1 > 0$ and $\lambda_2 > 0$ *source* or *repel* (unstable at point (0, 0))

The system is unstable and the solution as the time go by, will diverge away from the equilibrium point

Example 1

Plot the phase portrait of the behavior of $y' = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} y$

Solution

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= \lambda^2 - 3\lambda + 2 = 0$$

$$\lambda_{1,2} = 1,\ 2$$

For $\lambda_1 = 1;$ $\quad (A - I)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow x_1 = -y_1$$

$$V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

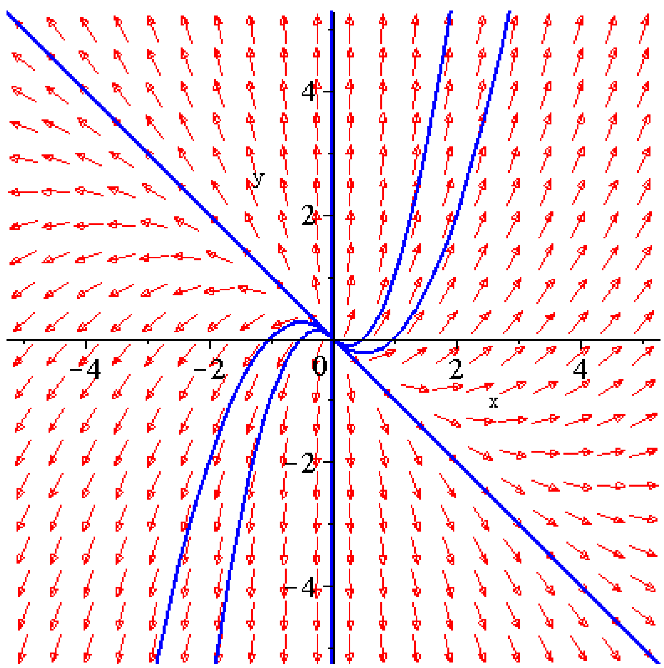
For $\lambda_2 = 2$; $\quad (A - 2I)\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = 0$

$$\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} x_2 = 0 \\ y_2 = 1 \end{cases}$$

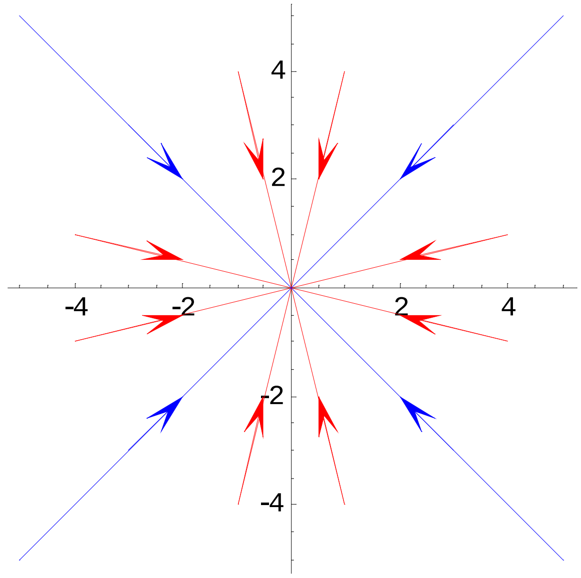
$$V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$y(t) = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}$$

$$\begin{cases} y_1(t) = -C_1 e^t \\ y_2(t) = C_1 e^t + C_2 e^{2t} \end{cases}$$



***Case* 2**: If $\lambda_1 \,\&\, \lambda_2 < 0$



$\lambda_1 \,\&\, \lambda_2 < 0$ *sink* or *attractor* ((0, 0) is asymptotically stable)

$\lambda_1 = \lambda_2 < 0$ *proper node*.

Example **2**

Plot the phase portrait of the behavior of $y' = \begin{pmatrix} -3 & -1 \\ -1 & -3 \end{pmatrix} y$

Solution

$$|A-\lambda I| = \begin{vmatrix} -3-\lambda & -1 \\ -1 & -3-\lambda \end{vmatrix}$$

$$= \lambda^2 + 6\lambda + 8 = 0$$

$$\lambda_{1,2} = -4, \ -2$$

For $\lambda_1 = -4$; $(A+4I)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow x_1 = y_1$$

$$V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

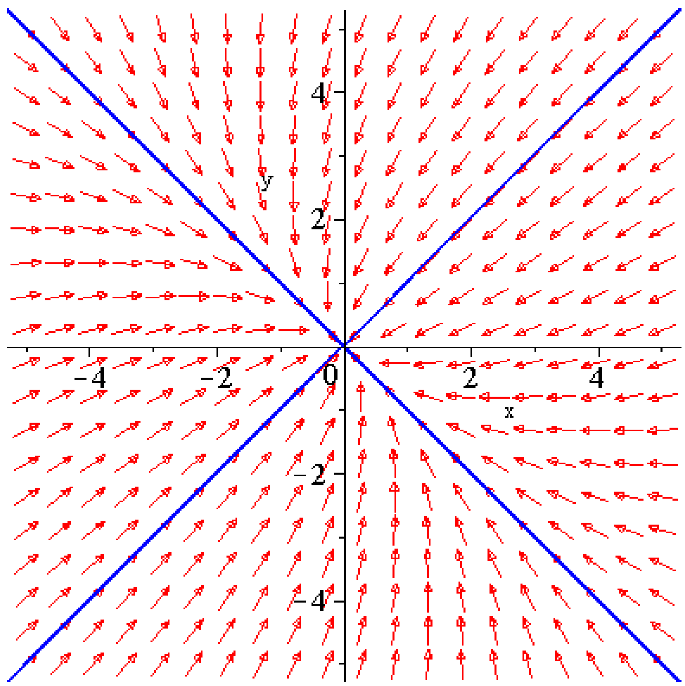
For $\lambda_1 = -2$; $(A+2I)\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = 0$

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x_2 = -y_2$$

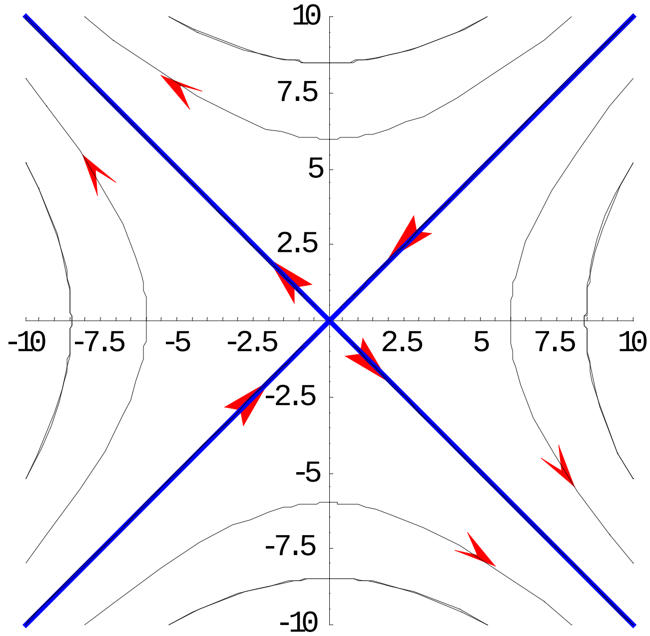
$$V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$y(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-4t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t}$$

$$\begin{cases} y_1(t) = C_1 e^{-4t} - C_2 e^{-2t} \\ y_2(t) = C_1 e^{-4t} + C_2 e^{-2t} \end{cases}$$



***Case* 3**: If $\lambda_1 > 0$ & $\lambda_2 < 0$



$\lambda_1 > 0$ & $\lambda_2 < 0$

A *saddle point*. ((0,0) is semi-stable)

Example 3

Plot the phase portrait of the behavior of $\qquad y' = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix} y$

Solution

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 4 \\ 2 & -1-\lambda \end{vmatrix}$$

$$= \lambda^2 - 9 = 0$$

$$\lambda_{1,2} = -3,\ 3$$

For $\lambda_1 = -3;\ \ (A + 3I)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$

$$\begin{pmatrix} 4 & 4 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x_1 = -y_1$$

$$V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

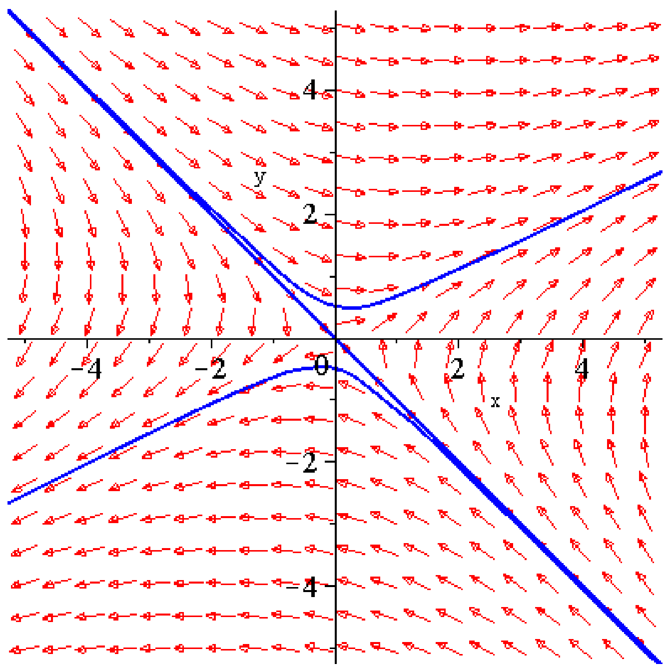
For $\lambda_1 = 3$; $\quad (A+3I)\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = 0$

$$\begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad x_2 = 2y_2$$

$$V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

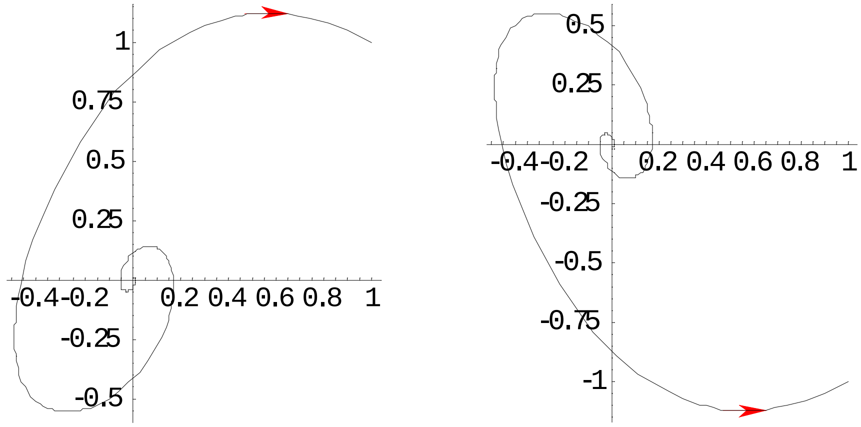
$$y(t) = C_1\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{-3t} + C_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{3t}$$

$$\begin{cases} y_1(t) = -C_1e^{-3t} + 2C_2e^{3t} \\ y_2(t) = C_1e^{-3t} + C_2e^{3t} \end{cases}$$



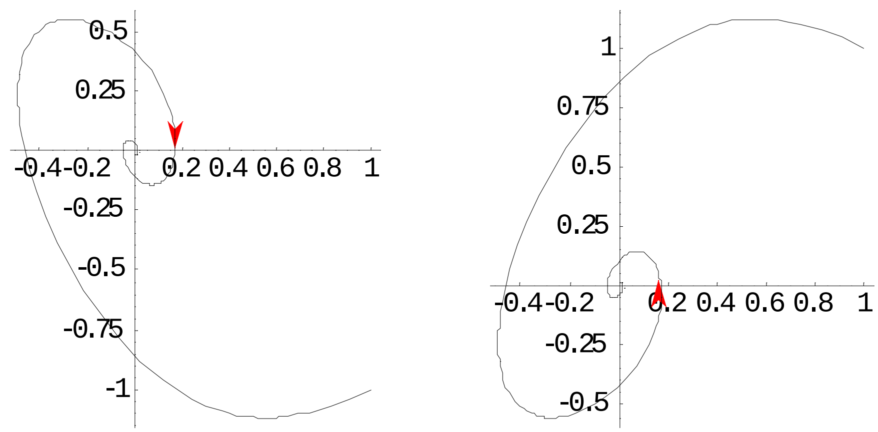
***Case* 4**: If λ_1 & λ_2 are complex values: $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$

If $a > 0$, the behavior of the system is spiral clockwise (*cw*), then otherwise is *ccw*.

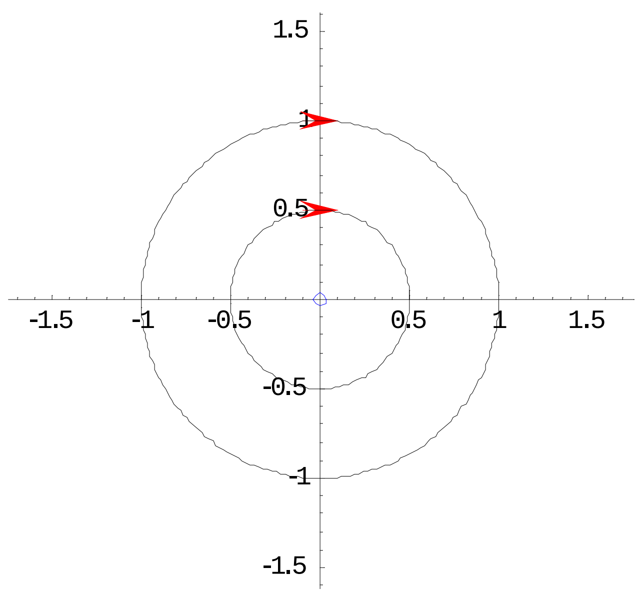


<u>*spiral out*</u>. (unstable at (0,0) point)

or



$a < 0$ <u>*spiral in*</u>. (asymptotically stable at (0,0) point)



$a = 0$ $\lambda_{1,2} = \pm\, ib$ ‘circle’ periodic solution- (0, 0) is a <u>*center*</u> stable.

Example 4

Plot the phase portrait of the behavior of $y' = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} y$

Solution

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 2 \\ -2 & -\lambda \end{vmatrix}$$

$$= \lambda^2 + 4 = 0 \quad \to \quad \underline{\lambda_{1,2} = \pm 2i}$$

For $\lambda_1 = -2i$; $\quad (A + 2iI)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$

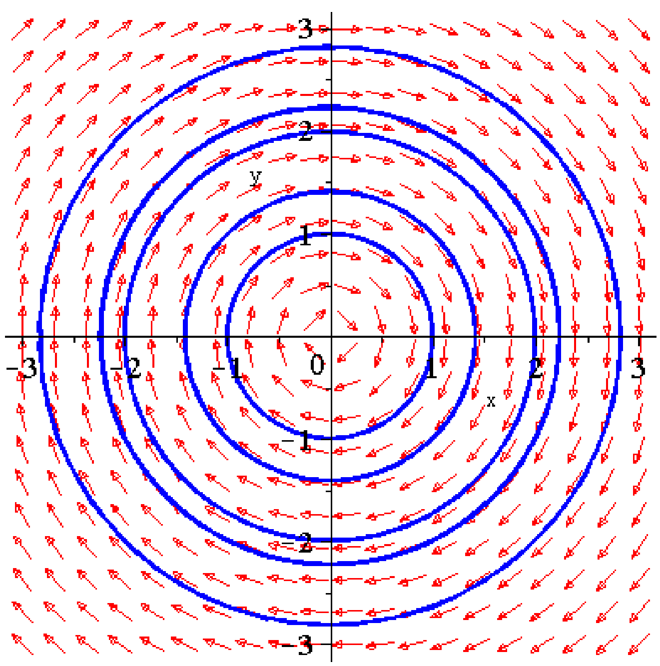
$$\begin{pmatrix} 2i & 2 \\ -2 & 2i \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad ix_1 = -y_1 \qquad \Rightarrow \quad \underline{V_1 = \begin{pmatrix} -1 \\ i \end{pmatrix}}$$

$$z(t) = \begin{pmatrix} -1 \\ i \end{pmatrix} e^{-2it}$$

$$= \left(\begin{pmatrix} -1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) (\cos 2t - i \sin 2t)$$

$$= \begin{pmatrix} -\cos 2t \\ \sin 2t \end{pmatrix} + i \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}$$

$$\underline{y(t) = C_1 \begin{pmatrix} -\cos 2t \\ \sin 2t \end{pmatrix} + C_2 \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}}$$



The equilibrium point is the center, but the solution curves are circles.

Example 5

Plot the phase portrait of the behavior of $y' = \begin{pmatrix} 4 & -10 \\ 2 & -4 \end{pmatrix} y$

Solution

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & -10 \\ 2 & -4-\lambda \end{vmatrix}$$

$$= \lambda^2 + 4 = 0 \quad \rightarrow \quad \lambda_{1,2} = \pm 2i$$

For $\lambda_1 = -2i$; $\quad (A + 2iI)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$

$$\begin{pmatrix} 4+2i & -10 \\ 2 & -4+2i \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x_1 = (2+i)y_1$$

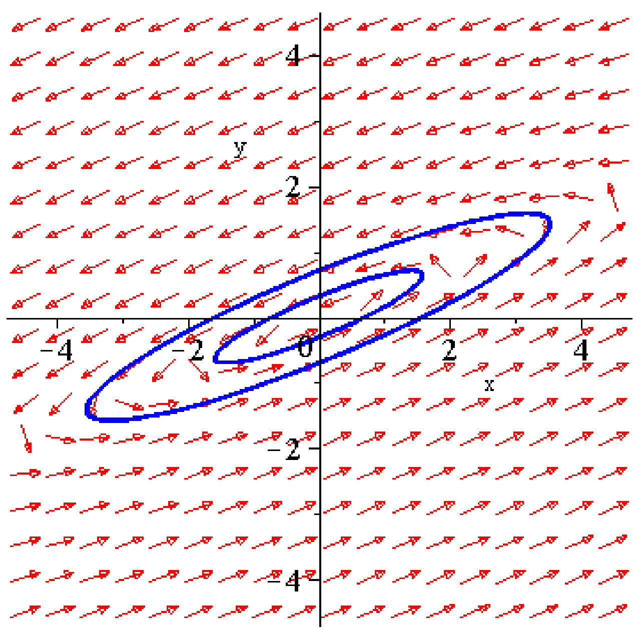
$$\Rightarrow \quad V_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

$$z(t) = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{-2it} \qquad\qquad e^{ibt} = \cos bt + i \sin bt$$

$$= \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)(\cos 2t - i \sin 2t)$$

$$= \begin{pmatrix} 2\cos 2t + \sin 2t \\ \cos 2t \end{pmatrix} + i \begin{pmatrix} -2\sin 2t + \cos 2t \\ -\sin 2t \end{pmatrix}$$

$$y(t) = C_1 \begin{pmatrix} 2\cos 2t + \sin 2t \\ \cos 2t \end{pmatrix} + C_2 \begin{pmatrix} -2\sin 2t + \cos 2t \\ -\sin 2t \end{pmatrix}$$



The equilibrium point is the center, but the solution curves are ellipses.

Example 6

Plot the phase portrait of the behavior of $\quad y' = \begin{pmatrix} 1 & -4 \\ 2 & -3 \end{pmatrix} y$

Solution

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -4 \\ 2 & -3-\lambda \end{vmatrix}$$

$$= \lambda^2 + 2\lambda + 5 = 0 \quad \rightarrow \quad \underline{\lambda_{1,2} = -1 \pm 2i}$$

For $\lambda_1 = -1 - 2i$; $\quad \begin{pmatrix} 2+2i & -4 \\ 2 & -2+2i \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x_1 = (1-i) y_1$

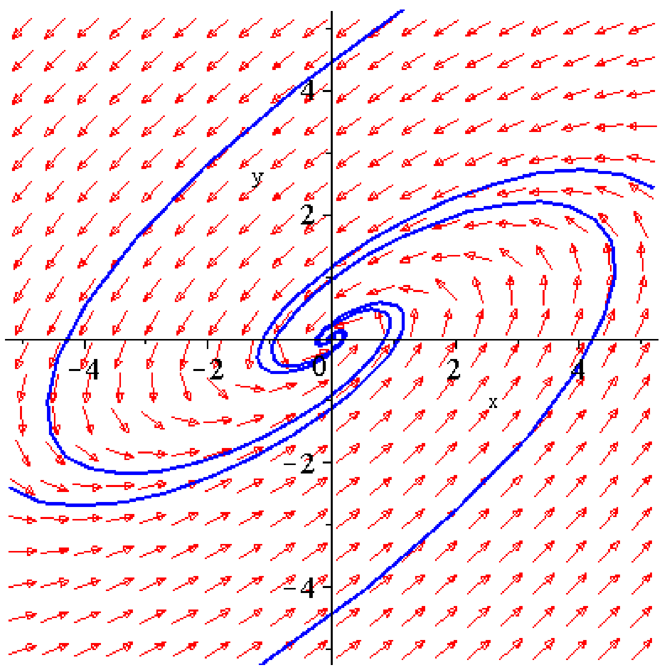
$$\Rightarrow \quad \underline{V_1 = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}}$$

$$z(t) = \begin{pmatrix} 1-i \\ 1 \end{pmatrix} e^{(-1-2i)t} \qquad\qquad e^{(a+ib)t} = (\cos bt + i \sin bt) e^{at}$$

$$= \begin{pmatrix} 1-i \\ 1 \end{pmatrix} (\cos 2t - i \sin 2t) e^{-t}$$

$$= \begin{pmatrix} \cos 2t - \sin 2t + i(\sin 2t - \cos 2t) \\ \cos 2t - i \sin 2t \end{pmatrix} e^{-t}$$

$$\underline{y(t) = C_1 \begin{pmatrix} \cos 2t - \sin 2t \\ \cos 2t \end{pmatrix} + C_2 \begin{pmatrix} \sin 2t - \cos 2t \\ -\sin 2t \end{pmatrix}}$$



The behavior of the system at the equilibrium point center is an asymptotically stable and spiral in.

3.7-3 *Stability properties of linear systems (in 2-dimensions)*

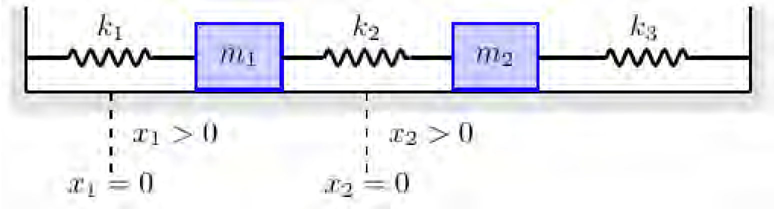
Eigenvalues	*Type of critical point*	*Stability*
$\lambda_1 > \lambda_2 > 0$	*Improper node*	*Unstable.*
$\lambda_1 < \lambda_2 < 0$	*Improper node*	*Asymptotically stable*
$\lambda_2 < 0 < \lambda_1$	*Saddle point*	*Unstable.*
$\lambda_1 = \lambda_2 > 0$	*Proper/improper node*	*Unstable*
$\lambda_1 = \lambda_2 < 0$	*Proper/improper node*	*Asymptotically stable*
$\lambda_{1,2} = a \pm ib$	*Spiral point*	
a > 0	*spiral out*	*Unstable*
a < 0	*spiral in*	*Asymptotically stable*
$\lambda_{1,2} = \pm ib$	*Center*	*Stable*

3.7-4 *Stability properties of linear systems (in 3-dimensions)*

Eigenvalues	*Type of critical point Stability*
$\lambda_1 > \lambda_2 > \lambda_3 > 0$	*3-dimensional node*
$\lambda_1 < \lambda_2 < \lambda_3 < 0$	*3-dimensional node*
$\lambda_1 < \lambda_2 < 0 < \lambda_3$	*Saddle node*
$\lambda_{1,2} \in \mathbb{R},\ \ \lambda_3 \in \nabla$	
$Re\left(\lambda_1,\ \lambda_2\right) < \lambda_3 < 0$	*Negative attractor*
$\lambda_3 < Re\left(\lambda_1,\ \lambda_2\right) < 0$	*Positive attractor*
$Re\left(\lambda_1,\ \lambda_2\right) < 0 < \lambda_3$	*Positive attractor towards* ξ_3
$\lambda_3 < 0 < Re\left(\lambda_1,\ \lambda_2\right)$	*Negative attractor towards* ξ_3
$Re\left(\lambda_1,\ \lambda_2\right) = 0,\ \ \lambda_3 > 0$	*Towards positive direction attractor*
$Re\left(\lambda_1,\ \lambda_2\right) = 0,\ \ \lambda_3 < 0$	*Towards negative direction attractor*

Example 7

Consider the spring-mass system consisting of two masses that are constraint by the three springs whose constant are $k_1,\ k_2,\ and\ k_3$. Assume there is no damping and there are ***no*** *external* forces.



Write an equivalent linear system of the first-order differential equations.

Solution

$$m_1 x_1'' = -\left(k_1 + k_2\right)x_1 + k_2 x_2$$

$$m_2 x_2'' = k_2 x_1 - \left(k_2 + k_3\right)x_2$$

$$\begin{cases} x_1'' = -\dfrac{k_1+k_2}{m_1}x_1 + \dfrac{k_2}{m_1}x_2 \\ x_2'' = \dfrac{k_2}{m_2}x_1 - \dfrac{k_2+k_3}{m_2}x_2 \end{cases}$$

To write an equivalent first-order system, let $Y = \left(y_1,\ y_2,\ y_3,\ y_4\right)^T$

Where $y_1(t) = x_1(t) \qquad y_2(t) = x_1'(t) \qquad y_3(t) = x_2(t) \qquad y_4(t) = x_2'(t)$

$$\begin{cases} y_2' = x_1'' = -\dfrac{k_1+k_2}{m_1}y_1 + \dfrac{k_2}{m_1}y_3 \\ y_4' = x_2'' = \dfrac{k_2}{m_2}y_1 - \dfrac{k_2+k_3}{m_2}y_3 \end{cases}$$

$$\to \begin{cases} y_1' = y_2 \\ y_2' = -\dfrac{k_1+k_2}{m_1}y_1 + \dfrac{k_2}{m_1}y_3 \\ y_3' = y_4 \\ y_4' = \dfrac{k_2}{m_2}y_1 - \dfrac{k_2+k_3}{m_2}y_3 \end{cases}$$

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2+k_3}{m_2} & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2+k_3}{m_2} & 0 \end{pmatrix}$$

$$|A-\mu I| = \begin{vmatrix} -\mu & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\mu & \frac{k_2}{m_1} & 0 \\ 0 & 0 & -\mu & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2+k_3}{m_2} & -\mu \end{vmatrix}$$

$$= \mu^4 + \frac{k_2+k_3}{m_2}\mu^2 + \frac{k_1+k_2}{m_1}\mu^2 - \frac{k_2^2}{m_1 m_2} + \frac{(k_1+k_2)(k_2+k_3)}{m_1 m_2}$$

$$= \mu^4 + \frac{(k_2+k_3)m_1 + (k_1+k_2)m_2}{m_1 m_2}\mu^2 + \frac{k_1k_2+k_1k_3+k_2k_3}{m_1 m_2} = 0$$

$$\mu^2 = -\frac{(k_2+k_3)m_1 + (k_1+k_2)m_2}{2m_1 m_2} \pm \frac{1}{2}\sqrt{\left(\frac{(k_2+k_3)m_1 + (k_1+k_2)m_2}{m_1 m_2}\right)^2 - 4\frac{k_1k_2+k_1k_3+k_2k_3}{m_1 m_2}}$$

$$\mu^2 = -\frac{(k_2+k_3)m_1 + (k_1+k_2)m_2}{2m_1 m_2} - \frac{1}{2}\sqrt{\left(\frac{(k_2+k_3)m_1 + (k_1+k_2)m_2}{m_1 m_2}\right)^2 - 4\frac{k_1k_2+k_1k_3+k_2k_3}{m_1 m_2}} < 0$$

$$\sqrt{\left(\frac{(k_2+k_3)m_1 + (k_1+k_2)m_2}{m_1 m_2}\right)^2 - 4\frac{k_1k_2+k_1k_3+k_2k_3}{m_1 m_2}} < \sqrt{\left(\frac{(k_2+k_3)m_1 + (k_1+k_2)m_2}{m_1 m_2}\right)^2}$$

$$= \frac{(k_2+k_3)m_1 + (k_1+k_2)m_2}{m_1 m_2}$$

$$\therefore\ \mu^2_{1,\ 2} < 0 \ \rightarrow\ \lambda_{1,\ 2,\ 3,\ 4} \in \mathbb{C}$$

Therefore, the eigenvalues μ^2 are negative real numbers which typical for mechanical system. If

$$\mu^2 = \lambda = -\omega^2 < 0$$

- Another approach to solve second-order system without substitute to first-order system, we can let:

$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$ a nonsingular mass matrix and

$K = \begin{pmatrix} -\dfrac{k_1+k_2}{m_1} & \dfrac{k_2}{m_1} \\ \dfrac{k_2}{m_2} & -\dfrac{k_2+k_3}{m_2} \end{pmatrix}$ stiffness matrix, then

$$M\,\vec{x}'' = K\,\vec{x}$$

$$M^{-1}M\vec{x}'' = M^{-1}K\vec{x}$$

$$\vec{x}'' = A\vec{x} \quad where \quad A = M^{-1}K$$

$$M^{-1}K = \frac{1}{m_1 m_2}\begin{pmatrix} m_2 & 0 \\ 0 & m_1 \end{pmatrix}\begin{pmatrix} -\dfrac{k_1+k_2}{m_1} & \dfrac{k_2}{m_1} \\ \dfrac{k_2}{m_2} & -\dfrac{k_2+k_3}{m_2} \end{pmatrix}$$

$$A = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} -\dfrac{k_1+k_2}{m_1} & \dfrac{k_2}{m_1} \\ \dfrac{k_2}{m_2} & -\dfrac{k_2+k_3}{m_2} \end{pmatrix}$$

$$= \begin{pmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_2 - k_3 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -k_1 - k_2 - \lambda & k_2 \\ k_2 & -k_2 - k_3 - \lambda \end{vmatrix}$$

$$= \lambda^2 + \left(k_1 + 2k_2 + k_3\right)\lambda + k_1k_2 + k_1k_3 + k_2k_3 = 0$$

$$\lambda = -\left(k_1 + 2k_2 + k_3\right) \pm \frac{1}{2}\sqrt{\left(k_1 + 2k_2 + k_3\right)^2 - 4\left(k_1k_2 + k_1k_3 + k_2k_3\right)}$$

$$\left(k_1 + 2k_2 + k_3\right)^2 - 4\left(k_1k_2 + k_1k_3 + k_2k_3\right) < \left(k_1 + 2k_2 + k_3\right)^2$$

That implies: $\lambda_{1,\,2} < 0$

To solve the system $\vec{x}(t) = \vec{v}e^{\alpha t}$

$\vec{x}' = \alpha\vec{v}e^{\alpha t}$

$\vec{x}'' = \alpha^2\vec{v}e^{\alpha t} = A\vec{v}e^{\alpha t}$

That implies $A\vec{v} = \alpha^2\vec{v}$

Therefore, $\vec{x}(t) = \vec{v}e^{\alpha t}$ is a solution of $\vec{x}'' = \boldsymbol{A}\vec{x}$ if and only if $\alpha^2 = \lambda$, an eigenvalue of the matrix A, and $\vec{v}$ is an associated eigenvector.

$\alpha^2 = \lambda = -\omega^2 < 0$

Then $\alpha = \pm\omega i$, which in this case the solution is given by

$$\vec{x}(t) = \vec{v}e^{i\omega t} = \vec{v}\left(\cos\omega t + i\sin\omega t\right)$$

3.7-5 *Theorem* Second-Order Homogeneous Linear Systems

If a matrix A $(n\times n)$ has distinct negative eigenvalues $-\omega_1^2,\ -\omega_2^2,\ \ldots,\ -\omega_n^2$ with associated real eigenvectors $\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_n$, then a general solution of the form $\boldsymbol{x}'' = A\boldsymbol{x}$ is given by

$$\vec{x}(t) = \sum_{i=1}^{n}\left(a_i\cos\omega_i t + b_i\sin\omega_i t\right)\vec{v}_i$$

With a_i and b_i arbitrary constants.

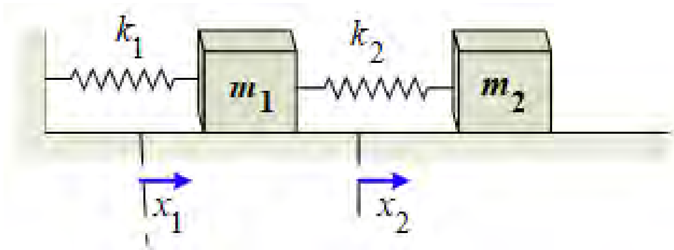
Where $\lambda = -\omega_i^2$

In the special case of non-repeated zero eigenvalue λ_0 with associated $\boldsymbol{v}_0$

$$\vec{x}_0(t) = \left(a_0 + b_0 t\right)\vec{v}_0$$

Example 8

Consider the mass-and-spring system.



Where $m_1 = 2, \quad m_2 = 1, \quad k_1 = 100, \quad k_2 = 50$ and $M\,\vec{x}'' = K\,\vec{x}$

Solution

$$\begin{cases} m_1 x_1'' = -k_1 x_1 + k_2\left(x_2 - x_1\right) \\ m_2 x_2'' = -k_2\left(x_2 - x_1\right) \end{cases}$$

$$\begin{cases} m_1 x_1'' = \left(-k_1 - k_2\right)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - k_2 x_2 \end{cases}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} x'' = \begin{pmatrix} -150 & 50 \\ 50 & -50 \end{pmatrix} \vec{x}$$

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad M^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$$

$$x'' = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -150 & 50 \\ 50 & -50 \end{pmatrix} \vec{x} \qquad M^{-1} M \vec{x}'' = M^{-1} K \vec{x}$$

$$= \begin{pmatrix} -75 & 25 \\ 50 & -50 \end{pmatrix} \vec{x} \qquad \vec{x}'' = \boldsymbol{A}\vec{x}$$

$$A = \begin{pmatrix} -75 & 25 \\ 50 & -50 \end{pmatrix}$$

$$\left|A - \lambda I\right| = \begin{vmatrix} -75 - \lambda & 25 \\ 50 & -50 - \lambda \end{vmatrix}$$

$$= \left(-75 - \lambda\right)\left(-50 - \lambda\right) - 1250$$

$$= \lambda^2 + 125\lambda + 2500 = 0$$

The eigenvalues are: $\lambda_1 = -100 = -\omega_1^2, \quad \lambda_2 = -25 = -\omega_2^2$

By the theorem, the natural frequencies: $\omega_1 = 10 \quad \textit{and} \quad \omega_2 = 5$

For $\lambda_1 = -100 \quad \Rightarrow \quad (A+100I)V_1 = 0$

$$\begin{pmatrix} 25 & 25 \\ 50 & 50 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad a = -b$$

$$\to V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\lambda_2 = -25 \quad \Rightarrow \quad (A+25I)V_2 = 0$

$$\begin{pmatrix} -50 & 25 \\ 50 & -25 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad 2a = b$$

$$\to V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The free oscillation of the mass-and-spring system, follows by:

$$\vec{x}(t) = \left(a_1 \cos 10t + b_1 \sin 10t\right)V_1 + \left(a_2 \cos 5t + b_2 \sin 5t\right)V_2$$

The natural mode:

$$\vec{x}_1(t) = \left(a_1 \cos 10t + b_1 \sin 10t\right)V_1$$

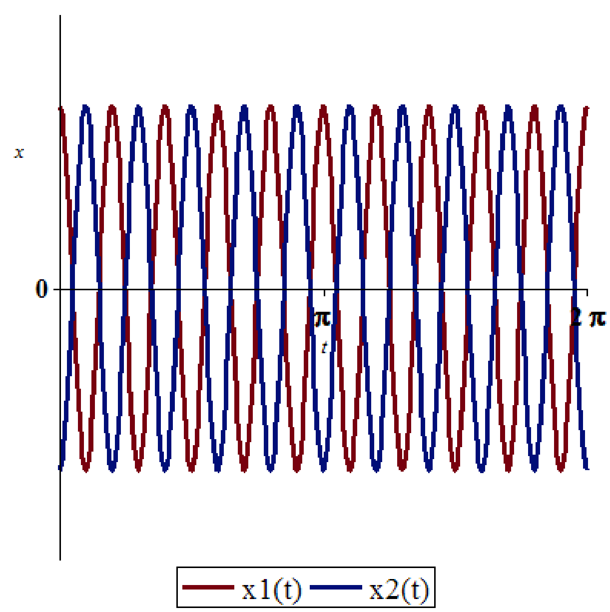
$$= c_1 \cos\left(10t - \alpha_1\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Where $c_1 = \sqrt{a_1^2 + b_1^2}$;

$$\cos\alpha_1 = \frac{a_1}{c_1} \qquad \sin\alpha_1 = \frac{b_1}{c_1}$$

Which has the scalar equations:

$$\begin{cases} x_1(t) = c_1 \cos\left(10t - \alpha_1\right) \\ x_2(t) = -c_1 \cos\left(10t - \alpha_1\right) \end{cases}$$



The ***second*** part:

$$\vec{x}_2(t) = \left(a_2 \cos 5t + b_2 \sin 5t\right)V_2$$

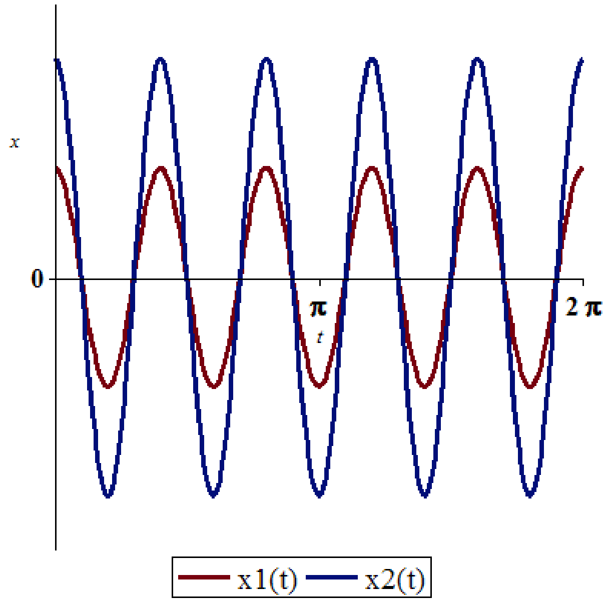
$$= c_2 \cos\left(5t - \alpha_2\right)\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Where $c_2 = \sqrt{a_2^2 + b_2^2}$;

$$\cos\alpha_2 = \frac{a_2}{c_2} \qquad \sin\alpha_2 = \frac{b_2}{c_2}$$

Which has the scalar equations:

$$\begin{cases} x_1(t) = c_2 \cos\left(5t - \alpha_2\right) \\ x_2(t) = 2c_2 \cos\left(5t - \alpha_2\right) \end{cases}$$



Exercises *Section* 3.7 – Phase Plane Portraits & Applications

(**1 – 4**) Sketch a rough approximation of a solution in each region determined by the half-line solutions. Use arrows to indicate the direction of motion on all solutions. Determine the behavior of the equilibrium point and the stability.

1. $y(t)=C_1e^{-t}\begin{pmatrix}2\\1\end{pmatrix}+C_2e^{-2t}\begin{pmatrix}-1\\1\end{pmatrix}$

2. $y(t)=C_1e^{t}\begin{pmatrix}-1\\-2\end{pmatrix}+C_2e^{2t}\begin{pmatrix}3\\-1\end{pmatrix}$

3. $y(t)=C_1e^{t}\begin{pmatrix}1\\1\end{pmatrix}+C_2e^{-2t}\begin{pmatrix}1\\-1\end{pmatrix}$

4. $y(t)=C_1e^{-t}\begin{pmatrix}-5\\2\end{pmatrix}+C_2e^{2t}\begin{pmatrix}-1\\4\end{pmatrix}$

(**5 – 6**) Sketch a rough approximation of the given system. Use arrows to indicate the direction of motion on all solutions. Determine the behavior of the equilibrium point and the stability.

5. $y'=\begin{pmatrix}-3 & \sqrt{2}\\ \sqrt{2} & -2\end{pmatrix}y$

6. $y'=\begin{pmatrix}1 & 1\\ 4 & 1\end{pmatrix}y$

(**7 – 11**) Calculate the eigenvalues to determine the behavior of the system whether the equilibrium point at the origin is the center, a spiral sink or a source. Calculate and sketch the vector generated by the right-hand side of the system at the point (1, 0). Use this to help sketch the solution trajectory for the system passing through the point (1, 0).

7. $y'=\begin{pmatrix}0 & 3\\ -3 & 0\end{pmatrix}y$

8. $y'=\begin{pmatrix}-2 & 2\\ -1 & 0\end{pmatrix}y$

9. $y'=\begin{pmatrix}7 & -10\\ 4 & -5\end{pmatrix}y$

10. $y'=\begin{pmatrix}-4 & 8\\ -4 & 4\end{pmatrix}y$

11. $y'=\begin{pmatrix}-3 & 2\\ -4 & 1\end{pmatrix}y$

12. For the given system $y'=\begin{pmatrix}-1 & 6\\ -3 & 8\end{pmatrix}y$

a) Sketch a rough approximation of the given system. Use arrows to indicate the direction of motion on all solutions. Determine the behavior of the equilibrium point and the stability.

b) Find the solution of the initial-value problem $y(0)=(0,\ 1)^T$

(**13 – 27**) Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system.

13. $x_1'=x_1+2x_2,\quad x_2'=2x_1+x_2$

14. $x_1'=2x_1+3x_2,\quad x_2'=2x_1+x_2$

15. $x_1'=6x_1-7x_2,\quad x_2'=x_1-2x_2$

16. $x_1' = -3x_1 + 4x_2,\quad x_2' = 6x_1 - 5x_2$

17. $x_1' = x_1 - 5x_2,\quad x_2' = x_1 - x_2$

18. $x_1' = -3x_1 - 2x_2,\quad x_2' = 9x_1 + 3x_2$

19. $x_1' = x_1 - 5x_2,\quad x_2' = x_1 + 3x_2$

20. $x_1' = 5x_1 - 9x_2,\quad x_2' = 2x_1 - x_2$

21. $x_1' = 3x_1 + 4x_2,\quad x_2' = 3x_1 + 2x_2;\quad x_1(0) = x_2(0) = 1$

22. $x_1' = 9x_1 + 5x_2,\quad x_2' = -6x_1 - 2x_2;\quad x_1(0) = 1,\ x_2(0) = 0$

23. $x_1' = 2x_1 - 5x_2,\quad x_2' = 4x_1 - 2x_2;\quad x_1(0) = 2,\ x_2(0) = 3$

24. $x_1' = x_1 - 2x_2,\quad x_2' = 2x_1 + x_2;\quad x_1(0) = 0,\ x_2(0) = 4$

25. $x_1' = x_1 - 2x_2,\quad x_2' = 3x_1 - 4x_2;\quad x_1(0) = -1,\ x_2(0) = 2$

26. $x_1' = -0.5x_1 + 2x_2,\quad x_2' = -2x_1 - 0.5x_2;\quad x_1(0) = -2,\ x_2(0) = 2$

27. $x_1' = 1.25x_1 + 0.75x_2,\quad x_2' = 0.75x_1 + 1.25x_2;\quad x_1(0) = -2,\ x_2(0) = 1$

(28 – 33) Find the general solution of the given system.

28. $x_1' = 4x_1 + x_2 + 4x_3,\quad x_2' = x_1 + 7x_2 + x_3,\quad x_3' = 4x_1 + x_2 + 4x_3$

29. $x_1' = x_1 + 2x_2 + 2x_3,\quad x_2' = 2x_1 + 7x_2 + x_3,\quad x_3' = 2x_1 + x_2 + 7x_3$

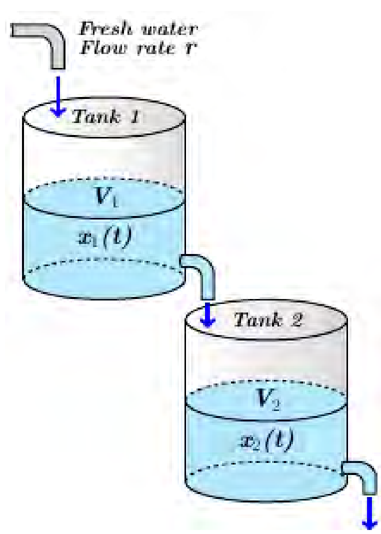
30. $x_1' = 4x_1 + x_2 + x_3,\quad x_2' = x_1 + 4x_2 + x_3,\quad x_3' = x_1 + x_2 + 4x_3$

31. $x_1' = 5x_1 + x_2 + 3x_3,\quad x_2' = x_1 + 7x_2 + x_3,\quad x_3' = 3x_1 + x_2 + 5x_3$

32. $x_1' = 5x_1 - 6x_3,\quad x_2' = 2x_1 - x_2 - 2x_3,\quad x_3' = 4x_1 - 2x_2 - 4x_3$

33. $x_1' = 3x_1 + 2x_2 + 2x_3,\quad x_2' = -5x_1 - 4x_2 - 2x_3,\quad x_3' = 5x_1 + 5x_2 + 3x_3$

(34 – 37) Find the amount $x_1(t),\ x_2(t)$ of salt in each tank at time $t \ge 0$, with $x_1(0) = 15\ lb \quad x_2(0) = 0$. If



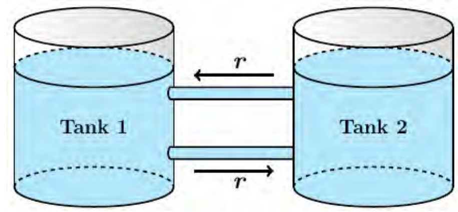
34. $V_1 = 50\ gal,\quad V_2 = 25\ gal,\quad r = 10\ gal/\min$

35. $V_1 = 25\ gal,\quad V_2 = 40\ gal,\quad r = 10\ gal/\min$

36. $V_1 = 50\ gal,\quad V_2 = 25\ gal,\quad r = 5\ gal/\min$

37. $V_1 = 25\ gal,\quad V_2 = 40\ gal,\quad r = 5\ gal/min$

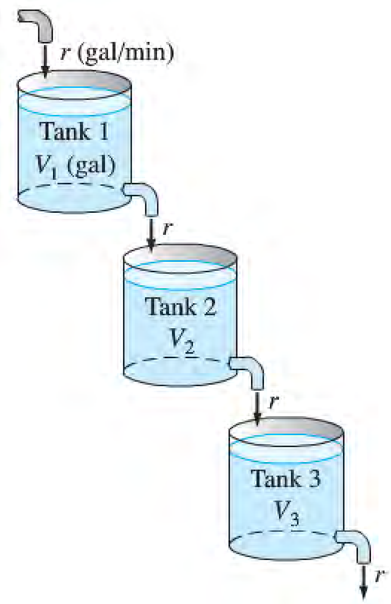
(**38 – 39**) Find the amount $x_1(t),\ x_2(t)$ of salt in each tank at time $t \ge 0$, with $x_1(0) = 15\ lb \quad x_2(0) = 0$. If



38. $V_1 = 50\ gal,\quad V_2 = 25\ gal,\quad r = 10\ gal/\min$

39. $V_1 = 25\ gal,\quad V_2 = 40\ gal,\quad r = 10\ gal/\min$

(**40 – 43**) Find the amount $x_1(t),\ x_2(t),\ x_3(t)$ of salt in each tank at time $t \ge 0$, if



40. $V_1 = 30\ gal,\quad V_2 = 15\ gal,\quad V_3 = 10\ gal,\quad r = 30\ gal/\min$

$x_1(0) = 27\ lb \quad x_2(0) = x_3(0) = 0$

41. $V_1 = 20\ gal,\quad V_2 = 30\ gal,\quad V_3 = 60\ gal,\quad r = 60\ gal/\min$

$x_1(0) = 45\ lb \quad x_2(0) = x_3(0) = 0$

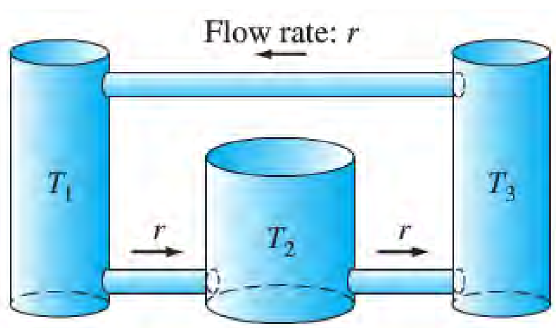
42. $V_1 = 15\ gal,\quad V_2 = 10\ gal,\quad V_3 = 30\ gal,\quad r = 60\ gal/\min$

$x_1(0) = 45\ lb \quad x_2(0) = x_3(0) = 0$

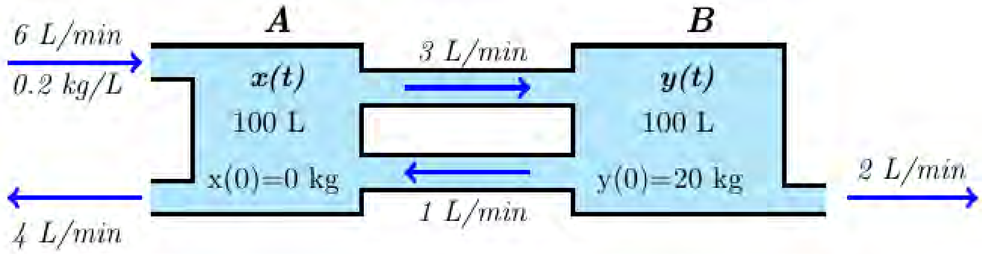
43. $V_1 = 20\ gal,\quad V_2 = 40\ gal,\quad V_3 = 50\ gal,\quad r = 10\ gal/\min$

$x_1(0) = 15 \quad x_2(0) = x_3(0) = 0$

44. If $V_1 = 50\ gal,\quad V_2 = 25\ gal,\quad V_3 = 50\ gal,\quad r = 10\ gal/\min$, find the amount $x_1(t),\ x_2(t),\ x_3(t)$ of salt in each tank at time $t \ge 0$

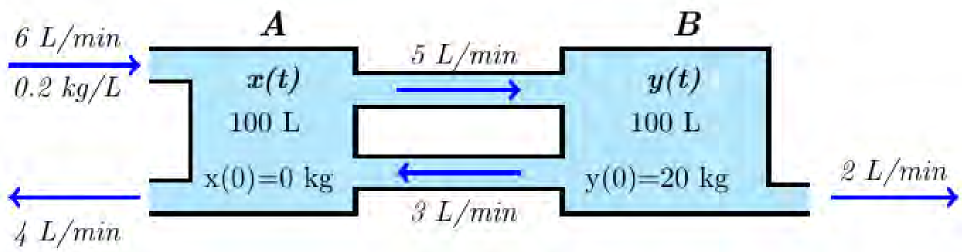


45. Two large tanks, each holding 100 *L* of liquid, are interconnected by pipes, with the liquid following from tank *A* into tank *B* at a rate of 3 *L/min* and from *B* to into *A* at a rate of 1 *L/min*.



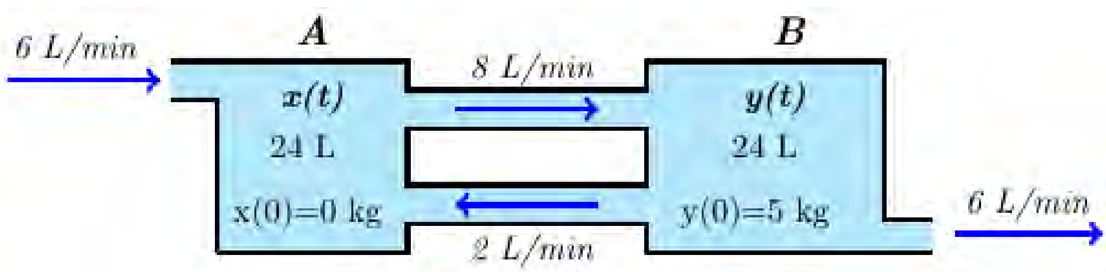
The liquid inside each tank is kept well stirred. A brine solution with a concentration of 0.2 *kg/L* of salt flows into tank *A* at a rate of 6 *L/min*. The diluted solution flows out the system from tank *A* at 4 *L/min* and from tank *B* at 2 *L/min*. If, initially, tank *A* contains pure water and tank *B* contains 20 *kg* of salt, determine the mass of salt in each tank at time $t \geq 0$.

46. Two large tanks, each holding 100 *L* of liquid, are interconnected by pipes, with the liquid following from tank *A* into tank *B* at a rate of 5 *L/min* and from *B* to into *A* at a rate of 3 *L/min*.



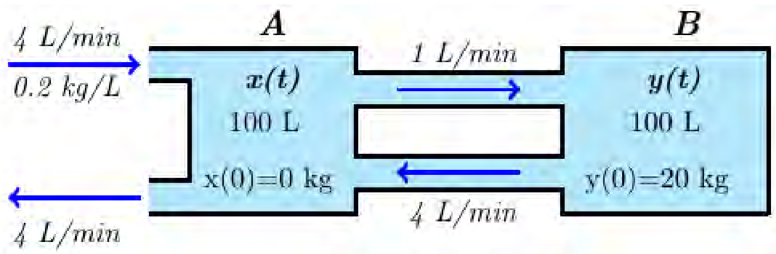
The liquid inside each tank is kept well stirred. A brine solution with a concentration of 0.2 *kg/L* of salt flows into tank *A* at a rate of 6 *L/min*. The diluted solution flows out the system from tank *A* at 4 *L/min* and from tank *B* at 2 *L/min*. If, initially, tank *A* contains pure water and tank *B* contains 20 *kg* of salt, determine the mass of salt in each tank at time $t \geq 0$.

47. Two large tanks, each holding 24 *L* of liquid, are interconnected by pipes, with the liquid following from tank *A* into tank *B* at a rate of 8 *L/min* and from *B* to into *A* at a rate of 2 *L/min*.



The liquid inside each tank is kept well stirred. A brine solution flows into tank *A* at a rate of 6 *L/min*. The diluted solution flows out the system from tank *B* at 6 *L/min*. If, initially, tank *A* contains pure water and tank *B* contains 5 *kg* of salt, determine the mass of salt in each tank at time $t \geq 0$.

48. Two large tanks, each holding 100 *L* of liquid, are interconnected by pipes, with the liquid following from tank *A* into tank *B* at a rate of 1 *L/min* and from *B* to into *A* at a rate of 4 *L/min*.

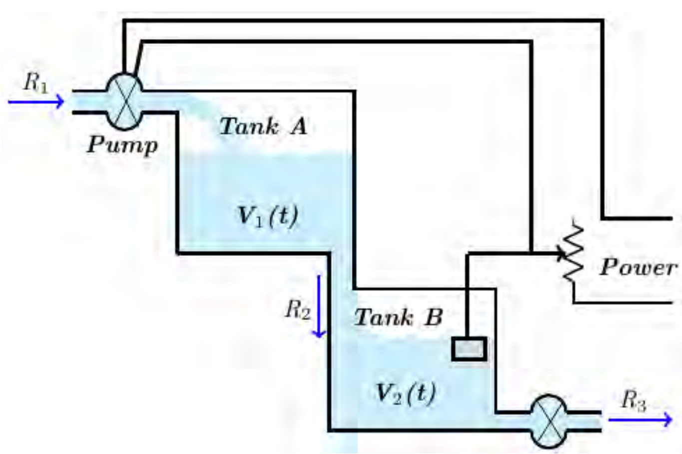


The liquid inside each tank is kept well stirred. A brine solution flows into tank *A* at a rate of 4 *L/min*. The diluted solution flows out the system from tank *A* at 4 *L/min*. If, initially, tank *A* contains pure water and tank *B* contains 20 *kg* of salt, determine the mass of salt in each tank at time $t \geq 0$.

49. Two 1,000-liter tanks are with salt water. Tank ***A*** contains 800 *liters* of water initially containing 20 *grams* of salt dissolved in it and Tank ***B*** contains 1,000 *liters* of water initially containing 80 *grams* of salt dissolved in it. Salt water with a concentration of $\frac{1}{2}$ *g/L* of salt enters Tank ***A*** at a rate of 4 *L/hr*. Fresh water enters Tank ***B*** at a rate of 7 *L/hr*. Through a connecting pipe water flows from Tank ***B*** into Tank ***A*** at a rate of 10 *L/hr*. Through a different connecting pipe 14 *L/hr* flows out of Tank ***A*** and 11 *L/hr* are drained out of the pipe (and hence out of the system completely) and only 3 *L/hr* flows back into Tank ***B***.

Find the amount of salt in each tank at any time.

50. Many physical and biological systems involve time delays. A pure time delay has its output the same as its input but shifted in time. A more common type of delay is pooling delay. Here the level of fluid in tank *B* determines the rate at which fluid enters tank *A*. Suppose this rate is given by $R_1(t) = \alpha\left(V - V_2(t)\right)$, where α and *V* are positive constants and $V_2(t)$ is the volume of fluid in tank *B* at time *t*.



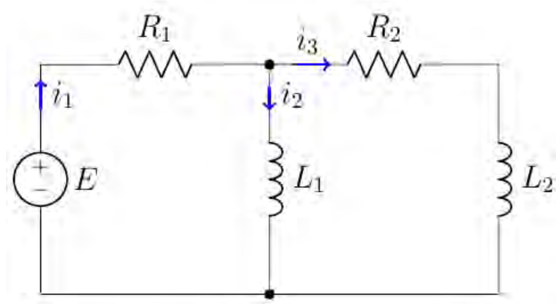
a) If the outflow rate R_3 from tank *B* is constant and the flow rate R_2 from tank *A* into tank *B* is $R_2(t) = KV_1(t)$ is the volume of fluid in tank *A* at time *t*, then show that this feedback system is governed by the system

$$\begin{cases} \dfrac{dV_1}{dt} = \alpha\left(V - V_2(t)\right) - KV_1(t) \\ \dfrac{dV_2}{dt} = KV_1(t) - R_3 \end{cases}$$

b) Find a general solution for the system in part (*a*) when $\alpha = 5\ min^{-1}$, $V = 20\ L$, $K = 2\ min^{-1}$, and $R_3 = 10\ L/min$.

c) Using the general solution obtained in part (*b*), what can be said about the volume of fluid in each of the tanks as $t \to +\infty$?

51. The electrical network shown below

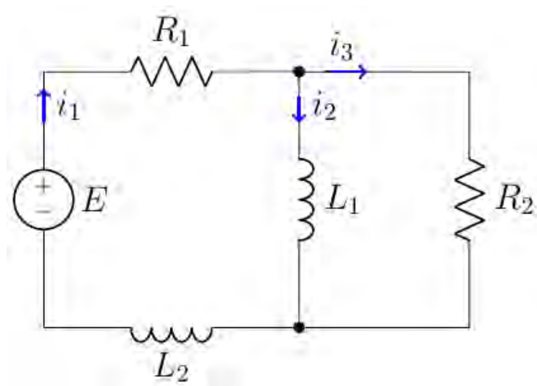


a) Find the system equations for the currents $i_2(t)$ and $i_3(t)$

b) Solve the system for the given: $R_1 = 2\ \Omega$, $R_2 = 3\ \Omega$, $L_1 = 1\ h$, $L_2 = 1\ h$, $E = 60\ V$, with the initial values $i_2(0) = 0$ & $i_3(0) = 0$

c) Determine the current $i_1(t)$

52. The electrical network shown below

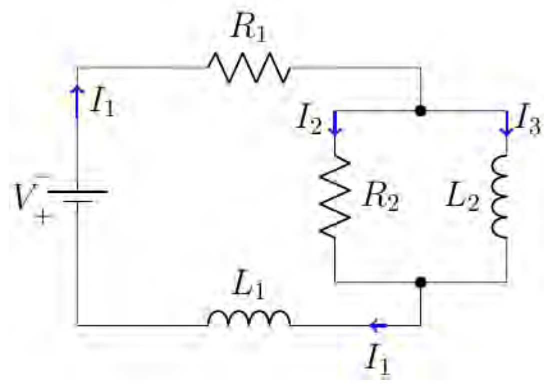


a) Find the system equations for the currents $i_1(t)$ and $i_2(t)$

b) Solve the system for the given: $R_1 = 8\ \Omega$, $R_2 = 3\ \Omega$, $L_1 = 1\ h$, $L_2 = 1\ h$, $E = 100\sin t\ V$, with the initial values $i_1(0) = 0$ & $i_2(0) = 0$

c) Determine the current $i_3(t)$

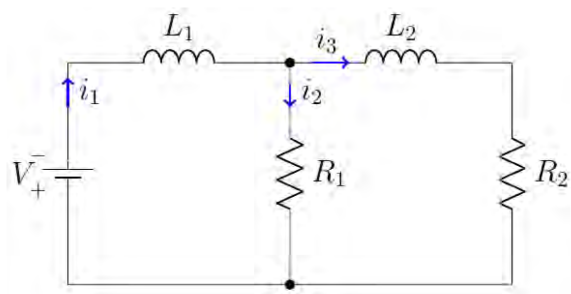
(53 – 54) Find a system of differential equations and solve for the currents in the given network, with initial values: $I_1(0) = I_2(0) = I_3(0) = 0$



53. $R_1 = 2\ \Omega,\quad R_2 = 1\ \Omega,\quad L_1 = 0.2\ H,\quad L_2 = 0.1\ H,\quad V = 6\ V$

54. $R_1 = 2\ \Omega,\quad R_2 = 1\ \Omega,\quad L_1 = 0.1\ H,\quad L_2 = 0.2\ H,\quad V = 6\ V$

(55 – 56) Find a system of differential equations and solve for the currents in the given network with initial values: $i_1(0) = i_2(0) = i_3(0) = 0$

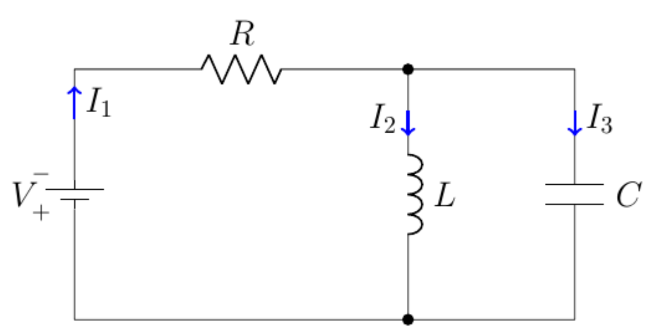


55. $R_1 = 10\Omega,\quad R_2 = 20\ \Omega,\quad L_1 = 0.005\ H,\quad L_2 = 0.01\ H,\quad V = 50\ V$

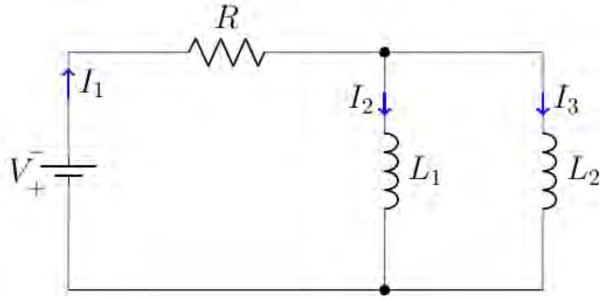
56. $R_1 = 10\ \Omega,\quad R_2 = 40\ \Omega,\quad L_1 = 10\ H,\quad L_2 = 20\ H,\quad V = 20\ V$

57. Find a system of differential equations and determine the charge on the capacitor and the currents in the given network with initial values: $I_1(0) = I_2(0) = I_3(0) = 0$

$R = 20\ \Omega,\quad L = 1\ H,\quad C = \frac{1}{160}\ F,\quad V = 5\ V,\quad q(0) = 2\ C$



(58 – 59) Find a system of differential equations and solve for the currents in the given network with initial values: $I_1(0) = I_2(0) = I_3(0) = 0$

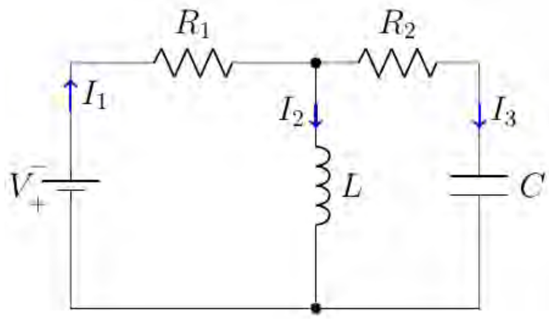


58. $R = 10\ \Omega,\quad L_1 = 0.02\ H,\quad L_2 = 0.025\ H,\quad V = 10\ V$

59. $R = 10\ \Omega,\quad L_1 = 2\ H,\quad L_2 = 25\ H,\quad V = 20\ V$

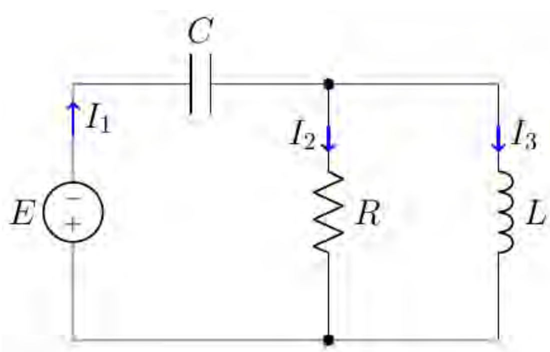
60. Find a system of differential equations and solve for the currents in the given network with initial values: $I_1(0) = I_2(0) = I_3(0) = 0$

$R_1 = 10\ \Omega,\quad R_2 = 5\ \Omega,\quad L = 20\ H,\quad C = \frac{1}{30}\ F,\quad V = 10\ V$



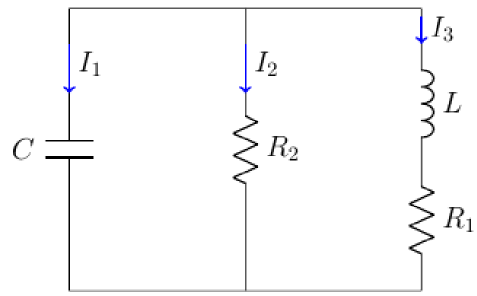
61. Find a system of differential equations and solve for the currents in the given network with initial values: $I_1(0) = I_2(0) = I_3(0) = 0$

$R = 1\ \Omega,\quad L = 0.5\ H,\quad C = 0.5\ F,\quad E = \cos 3t\ V$

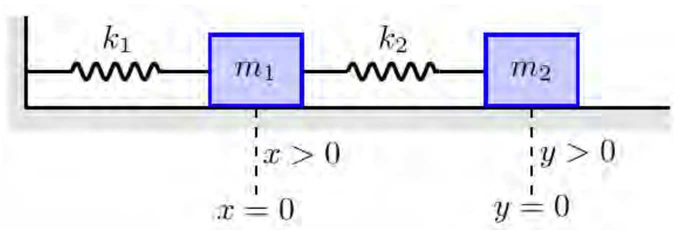


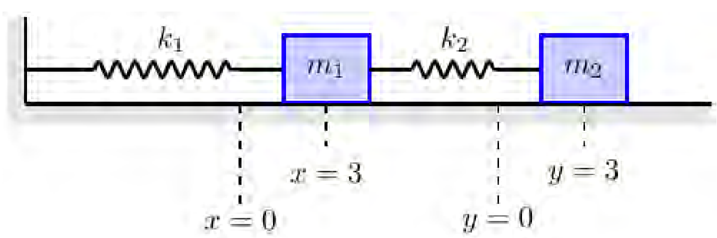
62. Derive three equations for the unknown currents $I_1, I_2,$ *and* I_3 with thegiven values of the given electric circuit shown below, then find the general solution

$R_1 = R_2 = 1\,\Omega$, $C = 1\,F$, and $L = 1\,H$.

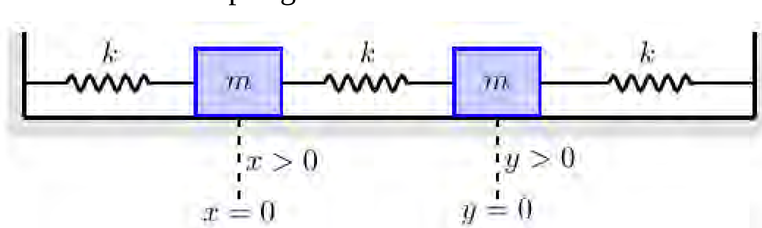


63. On a smooth horizontal surface $m_1 = 2\ kg$ is attached to a fixed wall by a spring with spring constant $k_1 = 4\ N/m$. Another mass $m_2 = 1\ kg$ is attached to the first object by a spring with spring constant $k_2 = 2\ N/m$. The object is aligned horizontally so that the springs are their natural lengths. If both objects are displaced 3 m to the right of their equilibrium positions and then released, what are the equations of motion for the two objects?





64. Three identical springs with spring constant k and two identical masses m are attached in a straight line with the ends of the outside springs fixed.



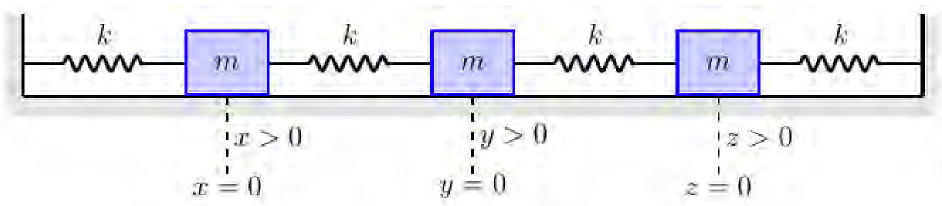
a) Determine and interpret the normal modes of the system.

b) Given the values $m = 2$ *kg*, and $k = 2$ *N/m* with initial value $x(0) = 1, \quad x'(0) = 0, \quad y(0) = 1, \quad y'(0) = 0$. what are the equations of motion for the two objects?

c) Given the values $m = 2$ *kg*, and $k = 2$ *N/m* with initial value $x(0) = 1, \quad x'(0) = 0, \quad y(0) = -1, \quad y'(0) = 0$. what are the equations of motion for the two objects?

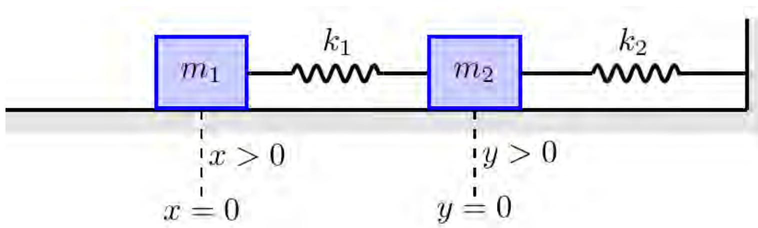
d) Given the values $m = 2$ *kg*, and $k = 2$ *N/m* with initial value $x(0) = 1, \quad x'(0) = 0, \quad y(0) = 2, \quad y'(0) = 0$. what are the equations of motion for the two objects?

65. Four springs with the same spring constant and three equal masses are attached in a straight line on a horizontal frictionless surface.



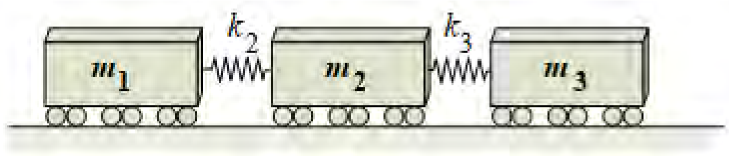
a) What are the equations of motion for the three objects?
b) Determine the normal frequencies for the system, describe the three normal modes of vibration.

66. Two springs and two masses are attached in a straight line on a horizontal frictionless surface. The system is set in motion by holding the mass m_2 at it equilibrium position and pulling the mass m_1 to the left of its equilibrium position a distance 1 *m* and them releasing both masses.



a) Express Newton's law for the system and determine the equations of motion for the two masses if $m_1 = 1$ *kg*, $m_2 = 2$ *kg*, $k_1 = 4$ *N/m*, and $k_2 = \frac{10}{3}$ *N/m*
b) Express Newton's law for the system and determine the equations of motion for the two masses if $m_1 = 1$ *kg*, $m_2 = 1$ *kg*, $k_1 = 3$ *N/m*, and $k_2 = 2$ *N/m*

67. Three railway cars are connected by buffer springs that react when is compressed but disengage instead of stretching.

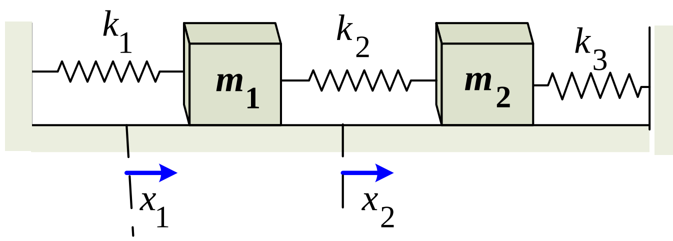


Given that $k_2 = k_3 = k = 3000\ lb/ft$ and $m_1 = m_3 = 750\ lbs$ *and* $m_2 = 500\ lbs$

Suppose that the leftmost car is moving to the right with velocity v_0 and at time $t = 0$ strikes the other 2 cars. The corresponding initial conditions are:

$$x_1(0) = x_2(0) = x_3(0) = 0$$
$$x_1'(0) = v_0 \quad x_2'(0) = x_3'(0) = 0$$

(68- 71) Consider the mass-and-spring system shown below and with the given masses and spring constants values.



Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

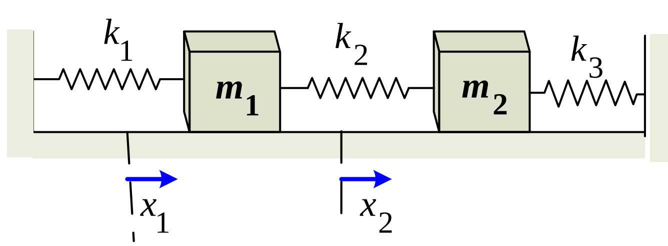
68. $m_1 = m_2 = 1; \quad k_1 = 0,\ k_2 = 2,\ k_3 = 0 \ \ (no\ walls)$

69. $m_1 = m_2 = 1; \quad k_1 = 1,\ k_2 = 2,\ k_3 = 1$

70. $m_1 = m_2 = 1; \quad k_1 = 2,\ k_2 = 1,\ k_3 = 2$

71. $m_1 = 1,\ m_2 = 2; \quad k_1 = 2,\ \ k_2 = k_3 = 4$

(72- 74) Consider the mass-and-spring system shown below and with the given masses and spring constants values.



The mass-and-spring system is set in motion from rest $x_1'(0) = x_2'(0) = 0$ in its equilibrium position $x_1(0) = x_2(0) = 0$.

Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

For the given external forces $F_1(t)$ and $F_2(t)$ acting on the masses m_1 and m_2, respectively. Find the resulting motion of the system and describe it as a superposition of oscillations at three different frequencies.

72. $m_1 = m_2 = 1;\quad k_1 = 1,\ k_2 = 4,\ k_3 = 1\quad F_1(t) = 96\cos 5t,\ F_2(t) = 0$

73. $m_1 = 1,\ m_2 = 2;\quad k_1 = 1,\ k_2 = k_3 = 2;\quad F_1(t) = 0,\ F_2(t) = 120\cos 3t$

74. $m_1 = m_2 = 1;\quad k_1 = 4,\ k_2 = 6,\ k_3 = 4;\quad F_1(t) = 30\cos t,\ F_2(t) = 60\cos t$

75. Consider a mass-and-spring system containing two masses $m_1 = m_2 = 1$ whose displacement functions $x(t)$ and $y(t)$ satisfy the differential equations

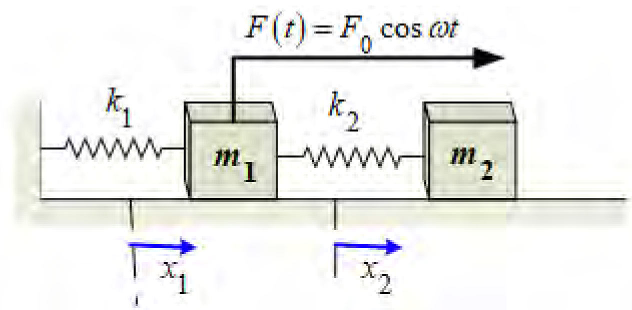
$$\begin{aligned} x'' &= -40x + 8y \\ y'' &= 12x - 60y \end{aligned}$$

a) Describe the two fundamental modes of free oscillation of the system.
b) Assume that the two masses start in motion with the initial conditions

$$x(0) = 19,\ x'(0) = 12 \quad \textit{and} \quad y(0) = 3,\ y'(0) = 6$$

And are acted on by the same force, $F_1(t) = F_2(t) = -195\cos 7t$. Describe the resulting motion as a superposition of oscillations at three different frequencies.

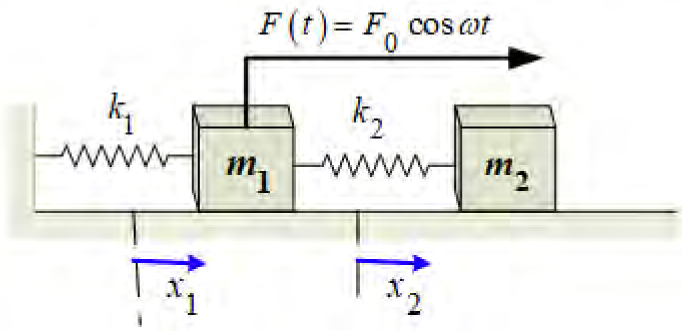
76. Consider a mass-and-spring system shown below. Assume that $m_1 = 1;\quad k_1 = 50;\quad F_0 = 5$ in *mks* units, and that $\omega = 10$. Then find m_2 so that in the resulting steady periodic oscillations, the mass m_1 will remain at rest (!).



Thus, the effect of the second mass-and-spring pair will be to neutralize the effect of the force on the first mass. This is an example of a dynamic damper. It has an electrical analogy that some cable companies use to prevent your reception of certain cable channels.

77. Consider a mass-and-spring system shown below. Assume that

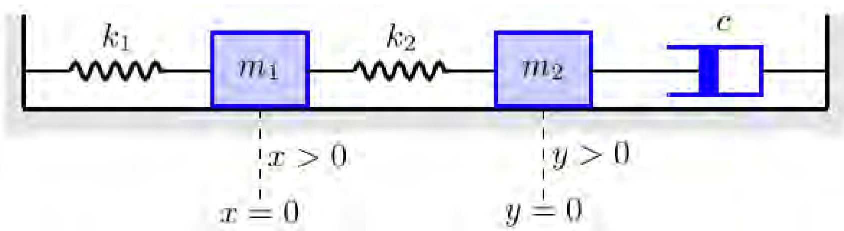
$$m_1 = 2,\ m_2 = \frac{1}{2};\quad k_1 = 75,\ k_2 = 25;\quad F_0 = 100 \quad \textit{and}\ \omega = 10\ \text{(in } \boldsymbol{mks}\ \textit{units}\text{)}.$$



Find the solution of the system $M\,\vec{x}'' = K\,\vec{x} + F$ that satisfies the initial conditions

$\vec{x}(0) = \vec{x}'(0) = \mathbf{0}$

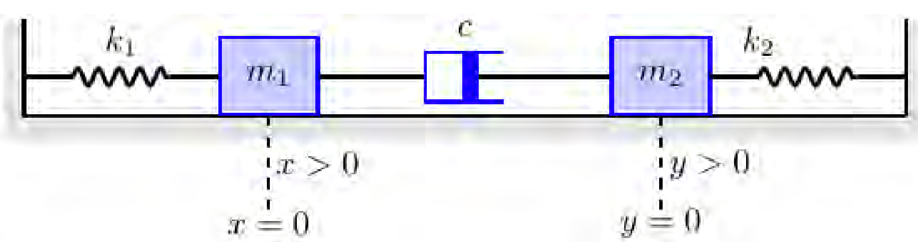
78. Two springs, two masses, and a dashpot are attached in a straight line on a horizontal frictionless surface. The dashpot damping force on mass m_2, given by $F = -cy'$



.

Derive the system equation of differential equations for the displacements *x* and *y*.

79. Two springs, two masses, and a dashpot are attached in a straight line on a horizontal frictionless surface. The system is set in motion by holding the mass m_2 at equilibrium position and pushing the mass m_1 to the left of its equilibrium position a distance 2 *m* and then releasing both masses.



If $m_1 = m_2 = 1\ kg$ and $k_1 = k_2 = 1\ N/m$, and $c = 1$ *N-sec*

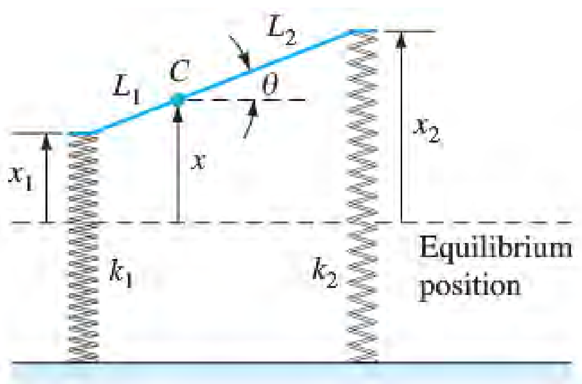
Determine the equations of motion for the two masses

80. A car with two axles and with separate front and rear suspension systems.

We assume that the car body acts as would a solid bar of mass m and length $L = L_1 + L_2$. It has moment of inertia I about its center of mass C, which is at distance L_1 from the front of the car. The car has front and back suspension springs with Hooke's constants k_1 and k_2, respectively. When the car is in motion, let $x(t)$ denote the vertical displacement of the center of mass of the car from equilibrium; let $\theta(t)$ denote its angular displacement (in radians) from the horizontal. Then Newton's laws of motion for linear and angular acceleration can be used to derive the equations.

$$mx'' = -(k_1 + k_2)x + (k_1 L_1 - k_2 L_2)\theta$$

$$I\theta'' = (k_1 L_1 - k_2 L_2)x - \left(k_1 L_1^2 + k_2 L_2^2\right)\theta$$



Suppose that $m = 75$ *slugs* (the car weighs 2400 *lb.*), $L_1 = 7\, ft$, $L_2 = 3\, ft$ (it's a rear engine car), $k_1 = k_2 = 2000$ lb/ft, and $I = 1000$ $ft.lb.s^2$.

a) Find the two natural frequencies ω_1 and ω_2 of the car.

b) Now suppose that the car is driven at a speed of v $ft/\sec$ along a washboard surface shaped like a sine curve with a wavelength of 40 *ft*. The result is a periodic force on the car with frequency $\omega = \frac{2\pi}{40}v = \frac{\pi}{20}v$. Resonance occurs when $\omega = \omega_1$ *or* $\omega = \omega_2$. Find the corresponding two critical speeds of the car (in *ft/sec*)

(81 – 83) The system is taken as a model for an undamped car with the given parameters in *fps* units.

a) Find the two natural frequencies ω_1 and ω_2 of the car (in hertz).

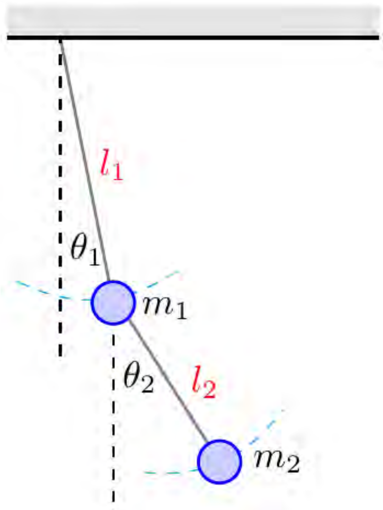
b) Assume that his car is driven along a sinusoidal washboard surface with a wavelength of 40 *ft*. The result is a periodic force on the car with frequency $\omega = \frac{2\pi}{40}v = \frac{\pi}{20}v$. Resonance occurs when $\omega = \omega_1$ *or* $\omega = \omega_2$. Find the corresponding two critical speeds of the car (in *ft/sec*)

81. $m = 100;\quad I = 800;\quad L_1 = L_2 = 5;\quad k_1 = k_2 = 2000$

82. $m=100;\quad I=1000;\quad L_1=6,\ L_2=4;\quad k_1=k_2=2000$

83. $m=100;\quad I=800;\quad L_1=L_2=5;\quad k_1=1000,\ k_2=2000$

84. A double pendulum swinging in a vertical plane under the influence of gravity satisfies the system



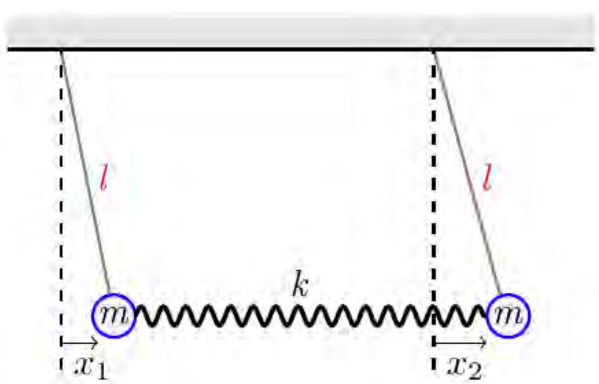
$$\begin{cases}(m_1+m_2)\ell_1^2\theta_1''+m_2\ell_1\ell_2\theta_2''+(m_1+m_2)\ell_1 g\theta_1=0\\ m_2\ell_2^2\theta_2''+m_2\ell_1\ell_2\theta_1''+m_2\ell_2 g\theta_2=0\end{cases}$$

Where θ_1 and θ_2 are small angles.

Solve the system when $m_1=3\ kg,\quad m_2=2\ kg,\quad \ell_1=\ell_2=5\ m$

$$\theta_1(0)=\frac{\pi}{6},\quad \theta_2(0)=0,\quad \theta_1'(0)=\theta_2'(0)=0$$

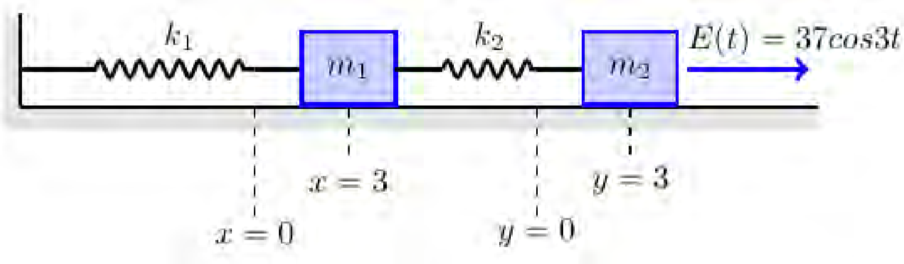
85. The motion of a pair of identical pendulums coupled by a spring is modeled by the system



$$\begin{cases}mx_1''=-\frac{mg}{\ell}x_1-k(x_1-x_2)\\ mx_2''=-\frac{mg}{\ell}x_2+k(x_1-x_2)\end{cases}$$

For small displacements. Determine the two normal frequencies for the system.

86. On a smooth horizontal surface $m_1 = 2\ kg$ is attached to a fixed wall by a spring with spring constant $k_1 = 4\ N/m$. Another mass $m_2 = 1\ kg$ is attached to the first object by a spring with spring constant $k_2 = 2\ N/m$. The objects are aligned horizontally so that the springs are their natural lengths.



Suppose an external force $E(t) = 37\cos 3t$ is applied to the second object of mass 1 *kg*.

a) Find the general solution

b) Show that $x(t)$ satisfies the equation $x^{(4)}(t) + 5x''(t) + 4x(t) = 37\cos 3t$

c) Find a general solution $x(t)$ to equation in part (*b*).

d) Substitute $x(t)$ to obtain a formula for $y(t)$

e) If both masses are displaced 2 *m* to the right of their equilibrium positions and then released, find the displacement functions $x(t)$ and $y(t)$