Solution   Section 1.7 – Physical Applications

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = 1 + \sin x; \quad 0 \le x \le \pi$

Solution

$$m = \int_0^{\pi} (1 + \sin x)\,dx \qquad\qquad m = \int_a^b \rho(x)\,dx$$

$$= x - \cos x \Big|_0^{\pi}$$

$$= \pi + 1 + 1$$

$$= \pi + 2$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = 1 + x^3; \quad 0 \le x \le 1$

Solution

$$m = \int_0^1 \left(1 + x^3\right) dx \qquad\qquad m = \int_a^b \rho(x)\,dx$$

$$= x + \frac{1}{4}x^4 \Big|_0^1$$

$$= 1 + \frac{1}{4}$$

$$= \frac{5}{4}$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = 2 - \frac{x}{2}; \quad 0 \le x \le 2$

Solution

$$m = \int_0^2 \left(2 - \frac{x}{2}\right) dx \qquad\qquad m = \int_a^b \rho(x)\,dx$$

$$= 2x - \frac{1}{4}x^2 \Big|_0^2$$

$$= 4 - 1$$

$$= 3$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = 5e^{-2x}; \quad 0 \le x \le 4$

Solution

$$m = \int_0^4 5e^{-2x} dx \qquad\qquad m = \int_a^b \rho(x)\,dx$$

$$= -\frac{5}{2}e^{-2x}\Big|_0^4$$

$$= -\frac{5}{2}\left(e^{-8} - 1\right)$$

$$= \frac{5}{2}\left(1 - e^{-8}\right)$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = x\sqrt{2 - x^2}; \quad 0 \le x \le 1$

Solution

$$m = \int_0^1 x\sqrt{2 - x^2}\; dx \qquad\qquad m = \int_a^b \rho(x)\,dx$$

$$= -\frac{1}{2}\int_0^1 \left(2 - x^2\right)^{1/2}\; d\left(2 - x^2\right)$$

$$= -\frac{1}{3}\left(2 - x^2\right)^{3/2}\Big|_0^1$$

$$= -\frac{1}{3}\left(1 - 2\sqrt{2}\right)$$

$$= \frac{1}{3}\left(2\sqrt{2} - 1\right)$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = \begin{cases} 1 & if \;\; 0 \le x \le 2 \\ 2 & if \;\; 2 < x \le 3 \end{cases}$

Solution

$$m = \int_0^2 1\; dx + \int_2^3 2\; dx \qquad\qquad m = \int_a^b \rho(x)\,dx$$

$$= x\Big|_0^2 + (2x)\Big|_2^3$$

$$= 2 + (6 - 4)$$

$$= 4$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x)=\begin{cases} 1 & if\ \ 0\le x\le 2 \\ 1+x & if\ \ 2< x\le 4 \end{cases}$

Solution

$$m=\int_0^2 1\,dx+\int_2^4 (1+x)\ dx \qquad\qquad m=\int_a^b \rho(x)dx$$

$$=x\Big|_0^2+\left(x+\frac{1}{2}x^2\right)\Big|_2^4$$

$$=2+(4+8-2-2)$$

$$\underline{=10}$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x)=\begin{cases} x^2 & if\ \ 0\le x\le 1 \\ x(2-x) & if\ \ 1< x\le 2 \end{cases}$

Solution

$$m=\int_0^1 x^2\ dx+\int_1^2 \left(2x-x^2\right)\ dx \qquad\qquad m=\int_a^b \rho(x)dx$$

$$=\frac{1}{3}x^3\Big|_0^1+\left(x^2-\frac{1}{3}x^3\right)\Big|_1^2$$

$$=\frac{1}{3}+4-\frac{8}{3}-1+\frac{1}{3}$$

$$\underline{=1}$$

Exercise

A heavy-duty shock absorber is compressed 2 *cm* from its equilibrium position by a mass of 500 *kg*. How much work is required tocompress the shock absorber 4 *cm* from its equilibrium position? (A mass of 500 *kg* exerts a force (in newtons) of 500 *g*)

Solution

Given: $\quad F(0.02)=500 \quad g=9.8\ m/s^2$

$$F(0.02)=0.02k=500\times 9.8 \qquad\qquad F(x)=kx=mg$$

$$k=\frac{4900}{0.02}\ \underline{=245{,}000}$$

$$W=\int_0^{.04} 245{,}000x\ dx \qquad\qquad W=\int_a^b F(x)dx$$

$$=122{,}500x^2\Big|_0^{0.04}$$

$$\underline{=195\ \ J}$$

Exercise

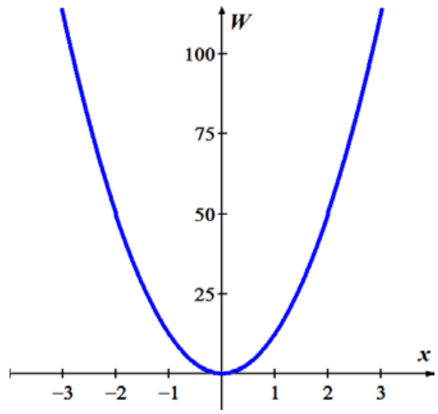
A spring has a restoring force given by $F(x)=25x$. Let $W(x)$ be the work required to stretch the spring from its equilibrium position $(x=0)$ to a variable distance *x*. Graph the work function. Compare the work required to stretch the spring *x* units from equilibrium to the work required to compress the spring *x* units from equilibrium.

Solution

$$W=\int_0^x 25t\ dt \qquad\qquad W=\int_a^b F(x)dx$$

$$=\frac{25}{2}t^2\Big|_0^x$$

$$=\frac{25}{2}x^2$$



Since $W(x)$ is an even function.

So that $W(-x)=W(x)$, and thus the work is the same to compress or stretch the spring a given distance from its equilibrium position,

Exercise

A swimming pool has the shape of a box with a base that measures 25 *m* by 15 *m* and a depth of 2.5 *m*. How much work is required to pump the water out of the pool when it is full?

Solution

$$W=\int_0^{2.5}\rho g A(y)(2.5-y)dy$$

$$=\int_0^{2.5}(1000)(9.8)(25\times 15)(2.5-y)dy$$

$$=3,675,000\left(2.5y-\frac{1}{2}y^2\right)\Big|_0^{2.5}$$

$$=3,675,000\left(6.25-\frac{6.25}{2}\right)$$

$$=11,484,375\ \ J$$

Exercise

It took 1800 J of work to stretch a spring from its natural length of 2 m to a length of 5 m. Find the spring's force constant

Solution

$$W = \int_0^3 F(x)\,dx$$

$$1800 = \int_0^3 kx\,dx$$

$$1800 = \frac{1}{2}kx^2 \Big|_0^3$$

$$1800 = \frac{1}{2}k(9-0)$$

$$1800 = \frac{9}{2}k$$

$$1800\left(\frac{2}{9}\right) = k$$

$$\boxed{k = 400 \ \ N/m}$$

Exercise

How much work is required to move am object from $x=1$ to $x=5$ (measired in meters) in the presence of a constant force at 5 N acting along the x-$axis$.

Solution

$$W = \int_1^5 5\,dx$$

$$= 5(5-1)$$

$$= 20 \ \ J$$

Exercise

How much work is required to move am object from $x=0$ to $x=3$ (measired in meters) with a force (in N) is given by $F(x) = \frac{2}{x^2}$ acting along the x-$axis$.

Solution

$$W = \int_1^3 \frac{2}{x^2}\,dx$$

$$= -2\frac{1}{x}\Big|_1^3$$

$$= -2\left(\frac{1}{3} - 1\right)$$

$$= \frac{4}{3}\ J$$

Exercise

A spring on a horizontal surface can be stretched and held 0.5 m from its equilibrium position with a force of 50 N.

a) How much work is done in stretching the spring 1.5 m from its equilibrium position?

b) How much work is done in compressing the spring 0.5 m from its equilibrium position?

Solution

a) $f(x) = kx \quad \to \quad f(0.5) = 50 = 0.5k \ \Rightarrow \ \underline{k = 100}$

$$W = \int_0^{1.5} 100x\ dx$$

$$= 50x^2 \Big|_0^{1.5}$$

$$= 112.5\ J$$

b) $$W = \int_0^{-.5} 100x\ dx$$

$$= 50x^2 \Big|_0^{-0.5}$$

$$= 12.5\ J$$

Exercise

Suppose a force of 10 N is required to stretch a spring 0.1 m from its equilibrium position and hold it in that position.

a) Assuming that the spring obeys Hooke's law, find the spring constant k.

b) How much work is needed to ***compress*** the spring 0.5 m from its equilibrium position?

c) How much work is needed to ***stretch*** the spring 0.25 m from its equilibrium position?

d) How much additional work is required to stretch the spring 0.25 m if it has already been stretched 0.1 m from its equilibrium position?

Solution

a) $F(0.1) = k(0.1) = 10$

$k = \frac{10}{0.1} = 100\ N/m$ Therfore, Hooke's law for this spring: $F(x) = 100x$

b) Work is needed to ***compress*** the spring

$$W = \int_0^{-0.5} 100x\, dx$$

$$= 50x^2 \Big|_0^{-0.5}$$

$$= 50(-0.5)^2$$

$$\underline{= 12.5 \;\; J}$$

c) Work is needed to ***stretch*** the spring

$$W = \int_0^{0.25} 100x\, dx$$

$$= 50x^2 \Big|_0^{0.25}$$

$$= 50(.25)^2$$

$$\underline{= 3.125 \;\; J}$$

d) Work is required to ***stretch*** the spring

$$W = \int_{0.1}^{0.35} 100x\, dx$$

$$= 50x^2 \Big|_{0.1}^{0.35}$$

$$= 50\left(0.35^2 - 0.1^2\right)$$

$$\underline{= 5.625 \;\; J}$$

Exercise

A force of 200 N will stretch a garage door spring 0.8-*m* beyond its unstressed length.

a) How far will a 300-N-force stretch the spring?

b) How much work does it take to stretch the spring this far?

Solution

$$k = \frac{F}{x} = \frac{200}{0.8} \underline{= 250 \;\; N/m}$$

a) $300 = 250x \quad \rightarrow \quad \underline{x = 1.2 \; m}$

b) $W = \int_0^{1.2} 250x\, dx$

$$= 125x^2 \Big|_0^{1.2}$$

$$\underline{= 180 \;\; J \;\; (N-m)}$$

Exercise

A spring has a natural length of 10 *in*. An 800-*lb* force stretches the spring to 14 *in*.

a) Find the force constant.

b) How much work is done in stretching the spring from 10 *in* to 12 *in*?

c) How far beyond its natural length will a 1600-*lb* force stretch the spring?

Solution

a) $k = \frac{F}{x} = \frac{800}{14-10} = \frac{800}{4}$

$k \underline{= 200 \ \text{lb}/in}$

b) $\Delta x = 12 - 10 = 2 \ in$

$$W = k\int_0^2 x dx$$

$$= 200 \frac{1}{2} x^2 \Big|_0^2$$

$$= 100(4-0)$$

$$= 400 \ \ in.lb$$

$$= 400 \frac{1 \ ft}{12 \ in} \ \ in.lb$$

$$\underline{= 33.3 \ \ ft \cdot lb}$$

c) $F = 200x$

$1600 = 200x$

$\frac{1600}{200} = x$

$x \underline{= 8 \ \ in}$

Exercise

It takes a force of 21,714 *lb* to compress a coil spring assembly on a Transit Authority subway car from its free height of 8 *in*. to its fully compressed height of 5 *in*.

a) What is the assembly's force constant?

b) How much work does it take to compress the assembly the first half inch? The second half inch? Answer to the nearest *in-lb*.

Solution

a) $F = kx$

$21714 = k(8-5)$

$21714 = 3k$

$k \underline{= 7238 \ \ \text{lb}/in}$

b) $W = k\int_0^{0.5} x\,dx$

$$= 7238\left[\frac{1}{2}x^2\right]_0^{0.5}$$

$$= 7238\left[\frac{1}{2}(0.5)^2 - 0\right]$$

$$= 905 \quad in \cdot lb$$

$$W = 7238\int_{0.5}^{1} x\,dx$$

$$= 7238\left[\frac{1}{2}x^2\right]_{0.5}^{1}$$

$$= 3619\left[1^2 - 0.5^2\right]_{0.5}^{1}$$

$$= 2714 \quad in \cdot lb$$

Exercise

A mountain climber is about to haul up a 50 *m* length of hanging rope. How much work will it take if the rope weighs 0.624 *N/m*?

Solution

$$W = 0.624\int_0^{50} x\,dx$$

$$= 0.624\left[\frac{1}{2}x^2\right]_0^{50}$$

$$= \frac{0.624}{2}\left[(50)^2 - 0\right]$$

$$= 780 \quad J$$

Exercise

A bag of sand originally weighing 144 *lb* was lifted a constant rate. As it rose, sand also leaked out at a constant rate. The sand was half gone by the time the bag had been lifted to 18 *ft*. How much work was done lifting the sand this far? (Neglect the weight of the bag and lifting equipment.)

Solution

The weight of sands decreases by $\frac{1}{2}144 = 72\ lb$ over the 18 ft. at rate $\frac{72}{18} = 4\ lb/\ ft$

$$F(x) = 144 - 4x$$

$$W = \int_0^{18} (144 - 4x)\,dx$$

$$= \left[144x - 2x^2\right]_0^{18}$$

$$= 144(18) - 2(18)^2 - (0)$$

$$= 1944\ \ ft \cdot lb$$

Exercise

An electric elevator with a motor at the top has a multistrand cable weighing 4.5 *lb/ft*. When the car is at the first floor, 180 *feet* of cable are paid out, and effectively 0 *foot* are out when the car is at the top floor. How much work does the motor do just lifting the cable when it takes the car from the first floor to the top?

Solution

$$F(x) = k\Delta x = 4.5(180 - x)$$

$$W = \int_0^{180} 4.5(180 - x)\,dx$$

$$= 4.5\left[180x - \tfrac{1}{2}x^2\right]_0^{180}$$

$$= 4.5\left[180(180) - \tfrac{1}{2}(180)^2 - 0\right]$$

$$= 72{,}900\ \ ft \cdot lb$$

Exercise

The rectangular cistern (storage tank for rainwater) shown has its top 10 *ft* below ground level. The cistern, currently full, is to be emptied for inspection by pumping its contents to ground level. Assume that the water weighs $62.4\ lb/ft^3$

a) How much work will it take to empty the cistern?

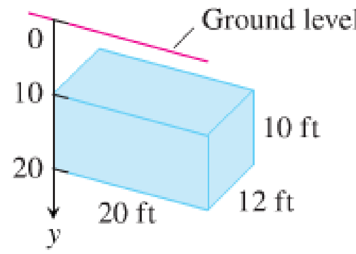
b) How long will it take a 1/-*hp* pump, rated at 275 *ft-lb/sec*, to pump the tank dry?

c) How long will it take the pump in part (*b*) to empty the tank hallway? (It will be less than half the time required to empty the tank completely)

d) What are the answers to parts (*a*) through (*c*) in a location where water weighs 62.6 lb/ft^3? 62.59 lb/ft^3?

Solution

a) $\Delta V = (20)(12)\Delta y = 240\Delta y$



$$F = 62.4(\Delta V)$$
$$= (62.4)240\Delta y$$
$$= 14976\Delta y$$

$$\Delta W = \ force \times distance$$
$$= 14976\ \Delta y \times y$$

$$W = 14976\int_{10}^{20} y dy$$
$$= 14976\left[\frac{1}{2}y^2\right]_{10}^{20}$$
$$= \frac{14976}{2}\left(20^2 - 10^2\right)$$
$$\underline{= 2,246,400\ \ ft \cdot lb}$$

b) $t = \dfrac{W}{275\frac{ft.lb}{\sec}}$

$$= \frac{2,246,400\ ft.lb}{275}\frac{\sec}{ft.lb}$$
$$\approx 8,168.73\ sec\frac{1\ \ hr}{3600\ sec}$$
$$\underline{\approx 2.27\ \ hrs} \quad 2\ hrs\ \ \&\ \ 16.1\min$$

c) $W = 14976\displaystyle\int_{10}^{15} y dy$

$$= 14976\left[\frac{1}{2}y^2\right]_{10}^{15}$$
$$= \frac{14976}{2}\left(15^2 - 10^2\right)$$
$$\underline{= 936,000\ \ ft \cdot lb}$$

$$t = \frac{W}{275\frac{ft.lb}{\sec}}$$
$$= \frac{936,000\ ft.lb}{275}\frac{\sec}{ft.lb}$$
$$\approx 3403.64\ sec\frac{1\ \ \min}{60\ sec}$$
$$\underline{\approx 56.7\ \ \min}$$

d) ***Water weighs*** $62.26\ lb\,/\ ft^3$

$$W = (62.26)(240)(150)$$

$$\boxed{= 2,214,360\ \ ft \cdot lb}$$

$$t = \frac{W}{275 \frac{ft.lb}{\sec}}$$

$$= \frac{2,241,360\ ft.lb}{275} \frac{\sec}{ft.lb}$$

$$\approx 8,150.4\ sec \frac{1\ \ hr}{3600\ sec}$$

$$\boxed{\approx 2.264\ \ hrs} \quad 2\ hrs\ \ \&\ \ 15.8 \min$$

$$W = (62.26)(240)\left(\frac{150}{2}\right)$$

$$\boxed{= 933,900\ \ ft \cdot lb}$$

$$t = \frac{W}{275 \frac{ft.lb}{\sec}}$$

$$= \frac{933,900\ ft.lb}{275} \frac{\sec}{ft.lb}$$

$$\approx 3396\ sec \frac{1\ \ \min}{60\ sec}$$

$$\boxed{\approx 56.6\ \ \min}$$

Water weighs $62.59\ lb\ /\ ft^3$

$$W = (62.59)(240)(150)$$

$$\boxed{= 2,253,240\ \ ft \cdot lb}$$

$$t = \frac{W}{275 \frac{ft.lb}{\sec}}$$

$$= \frac{2,253,240\ ft.lb}{275} \frac{\sec}{ft.lb}$$

$$\approx 8,193.60\ sec \frac{1\ \ hr}{3600\ sec}$$

$$\boxed{\approx 2.276\ \ hrs} \quad 2\ hrs\ \ \&\ \ 16.56 \min$$

$$W = (62.59)(240)\left(\frac{150}{2}\right) \quad \boxed{= 938,850\ \ ft \cdot lb}$$

$$t = \frac{W}{275 \frac{ft.lb}{\sec}}$$

$$= \frac{938,850\ ft.lb}{275} \frac{\sec}{ft.lb}$$

$$\approx 3414\ sec \frac{1\ \ \min}{60\ sec}$$

$$\boxed{\approx 56.9\ \ \min}$$

Exercise

When a particle of mass m is at $(x, 0)$, it is attracted toward the origin with a force whose magnitude is $\frac{k}{x^2}$. If the particle starts from rest at $x = b$ and is acted on by no other forces, find the work done on it by the time reaches $x = a$, $0 < a < b$.

Solution

$$F(x) = -\frac{k}{x^2}$$

$$W = \int_a^b -\frac{k}{x^2}\,dx$$

$$= k\int_a^b -\frac{1}{x^2}\,dx \qquad \int \frac{1}{x^2}\,dx = -\frac{1}{x}$$

$$= k\left[\frac{1}{x}\right]_a^b$$

$$= k\left(\frac{1}{b} - \frac{1}{a}\right)$$

$$= \frac{k(a-b)}{ab}$$

Exercise

The strength of Earth's gravitation field varies with the distance r from Earth's center, and the magnitude of the gravitational force experienced by a satellite of mass m during and after launch is

$$F(r) = \frac{mMG}{r^2}$$

Here, $M = 5.975 \times 10^{24}\ kg$ is Earth's mass, $G = 6.6720 \times 10^{-11}\ N \cdot m^2 kg^{-2}$ is the universal gravitational constant, and r is measured in meters. The work it takes to lift a 1000-*kg* satellite from Earth's surface to a circular orbit 35,780 *km* above Earth's center is therefore given by the integral

$$W = \int_{6,370,000}^{35,780,000} \frac{1000MG}{r^2}\,dr \quad \text{joules}$$

Evaluate the integral. The lower limit of integration is Earth's radius in meters at the launch site. (This calculation does not take into account energy spend lifting the launch vehicle or energy spent bringing the satellite to orbit velocity.)

Solution

$$W = 1000MG\int_{6,370,000}^{35,780,000} \frac{dr}{r^2}$$

$$= 1000MG\left[-\frac{1}{r}\right]_{6,370,000}^{35,780,000}$$

$$= 1000\left(5.975\times 10^{24}\right)\left(6.6720\times 10^{-11}\right)\left(\frac{1}{6,370,000} - \frac{1}{35,780,000}\right)$$

$$\approx 5.144\times 10^{10}\ J$$

Exercise

You drove an 800-*gal* truck from the base of Mt. Washington to the summit and discovered on arrival that the tank was only half full. You started with a full tank, climbed at a steady rate, and accomplished the 4750-*ft* elevation change in 50 *minutes*.

Assuming that the water leaked out at a steady rate, how much work was spent in carrying the water to the top? Do not count the work done in getting yourself and the truck there. Water weighs 8- *lb./gal*.

Solution

The force required to lift the water is equal to the water's weight which varies $8(800)$ *lbs*. to $8(400)$ *lbs*. over the 4750 *ft* change in elevation. Since it looses half of the water when the truck reaches its destination, it would loose all of the water if it went twice the distance. When the truck is *x* feet of the base of Mt. Washington, the water's weight is the following proportion.

$$F(x) = 8(800)\left(\frac{2(4750)-x}{2(4750)}\right) = 6400\left(1-\frac{x}{9500}\right)$$

$$W = 6400\int_0^{4750}\left(1-\frac{x}{9500}\right)dx$$

$$= 6400\left(x-\frac{x^2}{19000}\right)\Big|_0^{4750}$$

$$= 22,800,000\ \ ft-lbs$$

Exercise

A cylindrical water tank has height 8 *m* and radius 2 *m*

a) If the tank is full of water, how much work is required to pump the water to the level of the top of the tank and out of the tank?

b) Is it true it takes half as much work to pump the water out of the tank when it is half full as when it is full? Explain.

Solution

a) $W = \rho g\int_0^a (a-y)w(y)dy$

$$= 1,000(9.8)\int_0^8 (8-y)\left(2^2\pi\right)dy$$
$$= 39,200\pi\left(8y - \frac{1}{2}y^2\right)\Big|_0^8$$
$$= 39,200\pi(64-32)$$
$$= 125,400\pi$$
$$\boxed{\approx 3.941\times 10^6\ J}$$

b) The work done pumping the water from a half-full tank

$$W = 9,800(4\pi)\int_4^8 (8-y)\,dy$$
$$= 39,200\pi\left(8y - \frac{1}{2}y^2\right)\Big|_4^8$$
$$= 39,200\pi(32-32+8)$$
$$\boxed{\approx 985203\ J}$$

To empty a half-full tank, the work is

$$W = 39,200\pi\left(8y - \frac{1}{2}y^2\right)\Big|_0^4$$
$$= 39,200\pi(32-16)$$
$$\boxed{\approx 2.9556\times 10^6\ J}$$

NO, it is not true.

Exercise

A water tank is shaped like an inverted cone with height 6 *m* and base radius 1.5 *m*.

a) If the tank is full of water, how much work is required to pump the water to the level of the top of the tank and out of the tank?

b) Is it true it takes half as much work to pump the water out of the tank when it is half full as when it is full? Explain.

Solution

$$\frac{x}{1.5} = \frac{y}{6} \quad\to\quad x = \frac{y}{4}$$
$$Area = \pi x^2 = \frac{\pi}{16}y^2$$

a) $W = \rho g\int_0^a A(y)(a-y)\,dy$

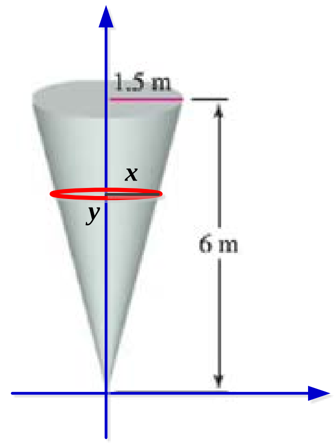
$$= 9{,}800 \int_0^6 \frac{\pi}{16} y^2 (6 - y)\, dy$$

$$= 612.5\pi \int_0^6 \left(6y^2 - y^3\right) dy$$

$$= 612.5\pi \left(2y^3 - \frac{1}{4} y^4\right)\Big|_0^6$$

$$= 612.5\pi (432 - 324)$$

$$\underline{= 66{,}150\pi\ J}$$



b) $W = 9{,}800 \int_0^3 \frac{\pi}{16} y^2 (6 - y)\, dy$

$$= 612.5\pi \left(2y^3 - \frac{1}{4} y^4\right)\Big|_0^3$$

$$= 612.5\pi (54 - 20.25)$$

$$\underline{\approx 20{,}672\pi\ J}$$

The work done is less than half the half amount from part (*a*).

It is not true, while the water must be raised further than water in the top half, due to the shape of the tank, there is far less water in the bottom half than in the top.

Exercise

A spherical water tank with an inner radius of 8 *m* has its lowest point 2 *m* above the ground. It is filled by a pipe that feed the tank at its lowest point.

a) Neglecting the volume of the inflow pipe, how much work is required to fill the tank if it is initially empty?

b) Now assume that the inflow pipe feeds the tank at the top of the tank. Neglecting the volume of the inflow pipe, how much work is required to fill the tank if it is initially empty?

Solution

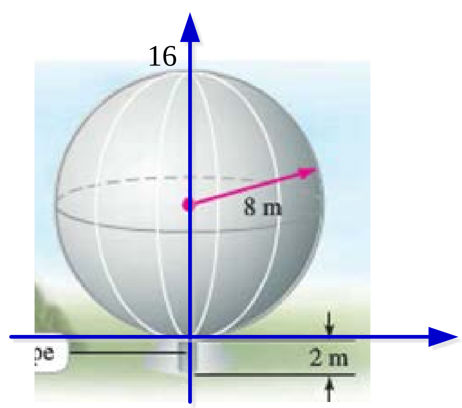
a) Equation of the tank: $x^2 + (y - 8)^2 = 64 \quad \to \quad x^2 = 16y - y^2$

$$A(y) = \pi x^2 = \pi\left(16y - y^2\right)$$

$$W = \rho g \pi \int_0^{16} \left(16y - y^2\right) dy$$

$$= 9800\pi \left(8y^2 - \frac{1}{3} y^3\right)\Big|_0^{16}$$

$$= 9800\pi \left(16^2\right)\left(8 - \frac{16}{3}\right)$$



$\approx 2.102\times 10^{8}\ J$

b) The toal weight of the water lifted up for 18 *m* is

$$W = \frac{4\pi}{3}R^3\rho g h$$

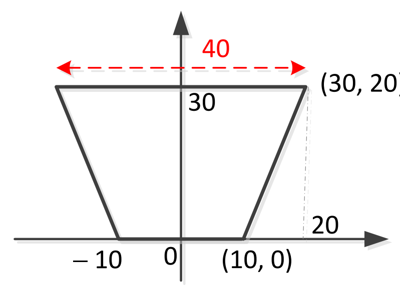
$$= \frac{4\pi}{3}8^3(9800)(18)$$

$$\approx 3.783\times 10^{8}\ J$$

Exercise

A large vertical dam in the shape of a symmetric trapezoid has a height of 30 *m*. a width of 20 *m*. at its base, and a width of 40 *m*. at the top. What is the total force on the face of the dam when the reservoir is full? $\left(\rho = 1000\frac{kg}{m^3},\ g = 9.8\frac{m}{s^2}\right)$

Solution

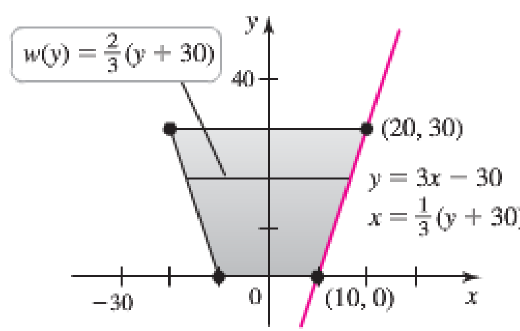


$$y - 0 = \frac{30}{10}(x-10)$$

$$y = 3x - 30 \quad \rightarrow \quad x = \frac{1}{3}(y+30)$$

Depth: $0 \le y \le 30$

Width: $w(y) = 2x = \frac{2}{3}(y+30)$



$$F = \int_0^a \rho g(a-y)w(y)\,dy$$

$$= \int_0^{30}\left(10^3\right)(9.8)(30-y)\frac{2}{3}(y+30)\,dy$$

$$= \frac{19600}{3}\int_0^{30}\left(900-y^2\right)dy$$

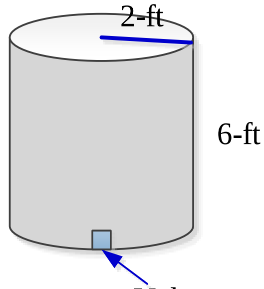
$$= \frac{19600}{3}\left(900y - \frac{1}{3}y^3\right)\Big|_0^{30}$$

$$= \frac{19600}{3}(27000-9000)$$

$$= 1.176\times 10^{8}\ kg$$

Exercise

Pumping water from a lake 15-*feet* below the bottom of the tank can fill the cylindrical tank shown here.



There are two ways to go about it. One is to pump the water through a hose attached to a valve in the bottom of the tank. The other is to attach the hose to the rim of the tank and let the water pour in. Which way will be faster? Give reason for answer

Solution

The water is being pumped from a lake that is 15-*feet* below the tank. That is the distance it takes to get the water from the lake to the valve, but this is not the total distance that the water is moved. We are forcing the water up into the tank, so the water travels a distance of y in the tank.

The total distance the water travels is 15 + y.

The cross-section is a circle: $A(y) = \pi r^2 = 4\pi$

$$W = 62.4\int_0^6 (4\pi)(15+y)\,dy$$

$$= 249.6\pi\left(15y + \frac{1}{2}y^2\right)\Big|_0^6$$

$$= 249.6\pi(90+18)$$

$$\approx 84,687.3 \ \ ft-lbs$$

Now we are pumping the water to the top of the tank and letting it pour in.
Therefore, the distance that the water is pumped is $15+6=21\ ft$

$$W = 62.4\int_0^6 21(4\pi)\,dy$$

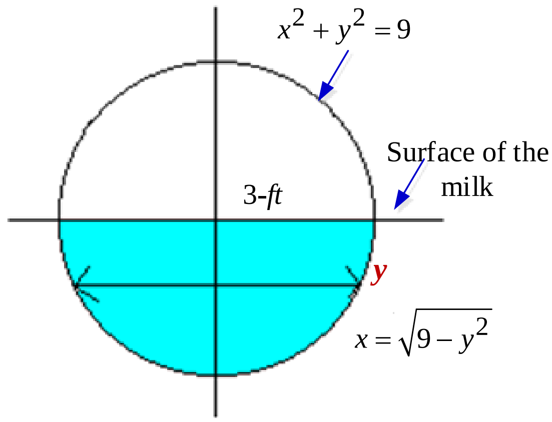
$$= 16,466.97(y)\Big|_0^6$$

$$\approx 98,801.83 \ \ ft-lbs$$

Exercise

A tank truck hauls milk in a 6-*feet* diameter horizontal right circular cylindrical tank. How must force does the milk exert on each end of the tank when the tank is half full?

Solution



Diameter = 6 $\rightarrow r=3$

Circular cylinder: $x^2+y^2=9$

$$L(y)=2x=2\sqrt{9-y^2}$$

Weight density of milk is $64.5\ \ lbs/\ ft^3$

$$F=64.5\int_{-3}^{0}2\sqrt{9-y^2}\,(0-y)\,dy$$

$$=64.5\int_{-3}^{0}\left(9-y^2\right)^{1/2}\ d\left(9-y^2\right)$$

$$=64.5\left(\tfrac{2}{3}\right)\left(9-y^2\right)^{3/2}\Big|_{-3}^{0}$$

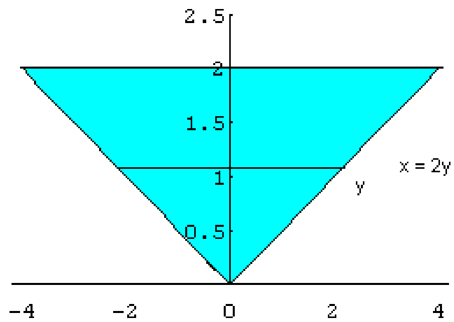
$$=43(27)$$

$$\underline{=1{,}161\ \ lbs}$$

Exercise

The vertical triangular plate shown here is the end plate of a trough full of water. What is the fluid force against the plate?

Solution



$$L(y)=2x=4y$$

$$F=62.4\int_{0}^{2}(2-y)\cdot(4y)\,dy$$

$$=249.6\int_{0}^{2}\left(2y-y^2\right)dy$$

$$=249.6\left(y^2-\tfrac{1}{3}y^3\right)\Big|_{0}^{2}$$

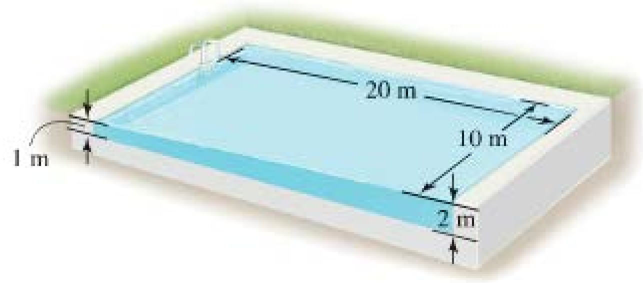
$$=249.6\left(4-\tfrac{8}{3}\right)$$

$$\underline{=332.8\ \ lbs}$$

Exercise

A swimming pool is 20 *m* leg long and 10 *m* wide, with a bottom that slopes uniformly from a depth of 1 *m* at one end to a depth of 2 *m* at the other end.
Assuming the pool is full, how much work is required to pump the water to a level 0.2 *m* above the top of the pool?

Solution



Depth: $(2+.2)-y=2.2-y$

From 0–1 *m*:

$y=\frac{1}{20}(10-x) \quad \rightarrow \quad 10-x=2y$

$$A(y)=10(20y)=200y$$

From 1–2 *m*: $A(y)=10(20)=200$

$$W=\rho g\int_0^1 200y(2.2-y)dy+\rho g\int_1^2 200(2.2-y)dy$$

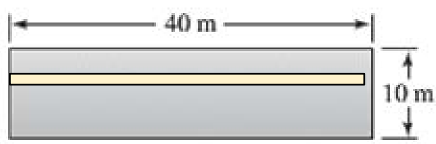
$$=(200\rho g)\left\{\left(1.1y^2-\frac{1}{3}y^3\right)\Big|_0^1+\left(2.2y-\frac{1}{2}y^2\right)\Big|_1^2\right\}$$

$$=\left(1.96\times10^6\right)\left(1.1-\frac{1}{3}+4.4-2-2.2+\frac{1}{2}\right)$$

$$\approx 2.875\times10^6\ J$$

Exercise

Find the total force on the face of the given dam



Solution

Freshwater Weight density: $\rho=62.4\ lb/ft^3=10^3\ kg/m^3$

$$F=\int_0^{10}\left(10^3\right)(9.8)(10-y)(40)dy \qquad F=\int_0^a \rho g(a-y)w(y)dy$$

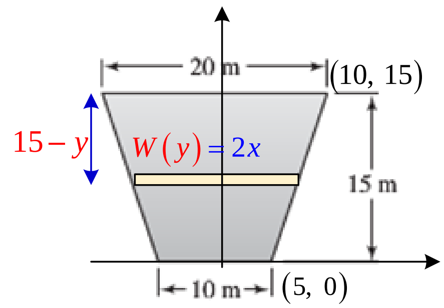
$$=392\times10^3\left(10y-\frac{1}{2}y^2\right)\Big|_0^{10}$$

$$=392\times10^3(100-50)$$

$$=196\times10^5\ N$$

Exercise

Find the total force on the face of the given dam



Solution

Freshwater Weight density: $\rho = 10^3\ kg/m^3$

$$y = \frac{15-0}{10-5}(x-5) = 3x-15$$

$$x = \frac{1}{3}(y+15) \Rightarrow \boxed{2x = \frac{2}{3}(y+15)}$$

$$F = \int_0^{15}\left(10^3\right)(9.8)(15-y)\frac{2}{3}(y+15)\,dy \qquad F = \int_0^a \rho g(a-y)w(y)\,dy$$

$$= \frac{19.6}{3}\times 10^3\int_0^{15}\left(225-y^2\right)dy$$

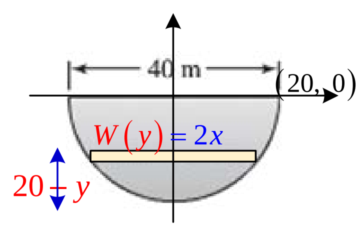
$$= \frac{19.6}{3}\times 10^3\left(225y-\frac{1}{3}y^3\right)\Big|_0^{15}$$

$$= \frac{19.6}{3}\times 10^3\left(15^5-\frac{1}{3}15^3\right)$$

$$\boxed{= 1.47\times 10^7\ N}$$

Exercise

Find the total force on the face of the given dam



Solution

Freshwater Weight density: $\rho = 10^3\ kg/m^3$

$$x^2 + (y-20)^2 = 20^2$$

$$x = \sqrt{400-\left(y^2-40y+400\right)} \quad \to \quad 2x = 2\sqrt{40y-y^2}$$

$$F = \int_0^{20}\left(10^3\right)(9.8)(20-y)(2)\sqrt{40y-y^2}\,dy \qquad F = \int_0^a \rho g(a-y)w(y)\,dy$$

$$= 9.8\times 10^3\int_0^{20}\left(40y-y^2\right)^{1/2} d\left(40y-y^2\right).$$

$$= \frac{19.6}{3}\times 10^3\left(40y-y^2\right)^{3/2}\Big|_0^{20}$$

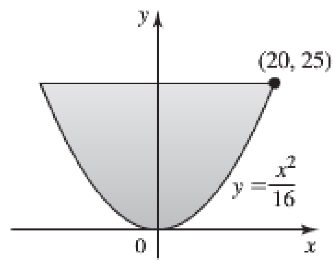
$$= \frac{19.6}{3}\times 10^3(800-400)^{3/2}$$

$$= \frac{19.6}{3}\times 10^3(20)^3$$

$$\boxed{= 5.227\times 10^7\ N}$$

Exercise

Find the total force on the face of the given dam



Solution

Freshwater Weight density: $\rho = 10^3\ kg/m^3$

$x^2 = 16y \to x = 4\sqrt{y} \quad \Rightarrow \quad \underline{2x = 8\sqrt{y}}$

$$F = \int_0^{25} \left(10^3\right)(9.8)(25-y)(8)\sqrt{y}\ dy \qquad\qquad F = \int_0^a \rho g(a-y)w(y)dy$$

$$= 78.4\times 10^3 \int_0^{25} \left(25y^{1/2} - y^{3/2}\right)dy$$

$$= 78.4\times 10^3 \left(\frac{50}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right)\Big|_0^{25}$$

$$= 78.4\times 10^3 \left(\frac{2}{3}5^5 - \frac{2}{5}5^5\right)$$

$$= 78.4\times 5^5 \times 10^3 \left(\frac{4}{15}\right)$$

$$\underline{= 6.533\times 10^7\ \ N}$$

Exercise

A large building shaped like a box is 50 *m* high with a face that is 80 *m* wide. A strong wind blows directly at the face of the building, exerting a pressure of 150 N/m^2 at the ground and increasing with height according to $P(y) = 150 + 2y$, where *y* is the height above the ground. Calculate the total force on the building, which is a meaure of the resistance that must be included in the design of the building.

Solution

$$F = \int_0^{50} (150 + 2y)(80)\ dy \qquad\qquad F = \int_0^a P(y)w(y)dy$$

$$= 80\left(150y + y^2\right)\Big|_0^{50}$$

$$= 80(7500 + 2500)$$

$$\underline{= 8\times 10^5\ N}$$

Exercise

Adiving pool that is 4 *m* deep full of water has a viewing window on one of its vertical walls. Find the force of the a square window, 0.5 *m* on side, with the lower edge of the window on the bottom of the pool.

Solution

Freshwater Weight density: $\rho = 10^3\ kg/m^3$

$$F = \int_0^{0.5} 10^3 (9.8)(4-y)(0.5)\ dy \qquad\qquad F = \int_0^a \rho g(a-y)w(y)dy$$

$$= 4.9\times 10^3\left(4y - \frac{1}{2}y^2\right)\Big|_0^{0.5}$$

$$= 4.9\times 10^3\left(2 - \frac{1}{8}\right)$$

$$= 4.9\times 10^3\left(\frac{15}{8}\right)$$

$$= 9187.5\ N$$

Exercise

Adiving pool that is 4 *m* deep full of water has a viewing window on one of its vertical walls. Find the force of the a square window, 0.5 *m* on side, with the lower edge of the window 1 *m* from the bottom of the pool.

Solution

Freshwater Weight density: $\rho = 10^3\ kg/m^3$

$$F = \int_1^{1.5} 10^3 (9.8)(4-y)(0.5)\ dy \qquad\qquad F = \int_0^a \rho g(a-y)w(y)dy$$

$$= 4.9\times 10^3\left(4y - \frac{1}{2}y^2\right)\Big|_1^{1.5}$$

$$= 4.9\times 10^3\left(6 - \frac{9}{8} - 4 + \frac{1}{2}\right)$$

$$= 4.9\times 10^3\left(2 - \frac{5}{8}\right)$$

$$= 4.9\times 10^3\left(\frac{11}{8}\right)$$

$$= 6737.5\ N$$

Exercise

Adiving pool that is 4 *m* deep full of water has a viewing window on one of its vertical walls. Find the force of the a circle window, with a radius of 0.5 *m*, tangent to the bottom of the pool.

Solution

Equation of the circle: $x^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4}$

$$x^2=\frac{1}{4}-y^2+y-\frac{1}{4}=y-y^2$$

$$x=\sqrt{y-y^2}\quad\Rightarrow\quad 2x=2\sqrt{y-y^2}$$

$$F=\int_0^1 10^3(9.8)(4-y)\left(2\sqrt{y-y^2}\right)dy \qquad F=\int_0^a \rho g(a-y)w(y)dy$$

$$=19.6\times10^3\int_0^1\left(\frac{7}{2}+\frac{1}{2}-y\right)\sqrt{y-y^2}\,dy \qquad d\left(y-y^2\right)=1-2y=2\left(\frac{1}{2}-y\right)$$

$$=19.6\times10^3\int_0^1\frac{7}{2}\sqrt{y-y^2}\,dy+19.6\times10^3\int_0^1\left(\frac{1}{2}-y\right)\sqrt{y-y^2}\,dy$$

$$=7\times9.8\times10^3\int_0^1\sqrt{y-y^2}\,dy+39.2\times10^3\int_0^1\sqrt{y-y^2}\,d\left(y-y^2\right) \qquad \textit{Area of semicircle}:\ \frac{1}{2}\pi r^2$$

$$=7\times9.8\times10^3\left(\frac{1}{2}\frac{\pi}{4}\right)+39.2\times10^3\left(\frac{1}{2}y^2-\frac{1}{3}y^3\right)\Big|_0^1$$

$$=\frac{7\pi}{8}\times9.8\times10^3+39.2\times10^3\left(\frac{1}{2}-\frac{1}{3}\right)$$

$$=\frac{7\pi}{8}\times9.8\times10^3+39.2\times10^3\left(\frac{1}{6}\right)$$

$$\approx 2.694\times10^4\ N$$

Exercise

A rigid body with a mass of 2 *kg* moves along a line due to a force that produces a position function $x(t)=4t^2$, where *x* is measured in *meters* and *t* is measured in *seconds*. Find the work done during the first 5 *sec*. in two ways.

a) Note that $x''(t)=8$; then use Newton's second law, $\left(F=ma=mx''(t)\right)$ to evaluate the work integral $W=\int_{x_0}^{x_f}F(x)dx$, where x_0 and x_f are the initial and final positions, repectively.

b) Change variables in the work integral and integrate with respect to *t*.

Solution

a) $W = \int_{x_0}^{x_f} mx''dx$

$= \int_0^{x(5)} 2\times 8dx \qquad x(5)=4(5)^2=100$

$= 16x \Big|_0^{100}$

$\underline{= 1600 \;\; J}$

b) $W = \int_{x_0}^{x_f} mx''dx$

$= \int_0^5 2\times 8\frac{dx}{dt}dt$

$= 16\int_0^5 (8t)dt$

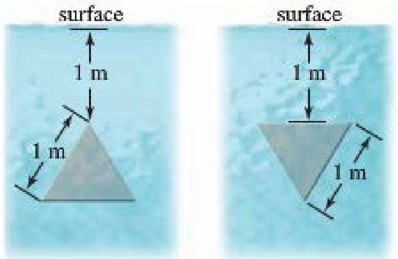
$= 64t^2 \Big|_0^5$

$\underline{= 1600 \;\; J}$

Exercise

A plate shaped like an equilateral triangle 1 *m* on a side is placed on a vetical wall 1 *m* below the surface of a pool filled with water. On which plate in the figure is the force is greater

Solution

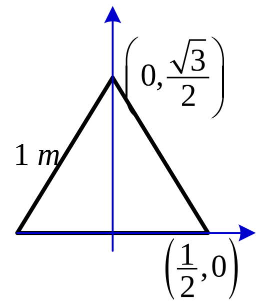


The left picture has more force than the right, because of the bottom part is wider side in the pool.

For the left side plate:

$$y = h = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}$$

Line segement: $y = \dfrac{0-\frac{\sqrt{3}}{2}}{\frac{1}{2}-0}\left(x-\frac{1}{2}\right) = -\sqrt{3}\left(x-\frac{1}{2}\right)$



$$x = \frac{1}{2} - \frac{1}{\sqrt{3}}y \;\rightarrow\; \underline{2x = 1 - \frac{2}{\sqrt{3}}y}$$

$$F = \int_0^{\sqrt{3}/2} \rho g\left(1+\frac{\sqrt{3}}{2}-y\right)\left(1-\frac{2}{\sqrt{3}}y\right)dy$$

$$= \rho g \int_0^{\sqrt{3}/2} \left(1 + \frac{\sqrt{3}}{2} - y - \frac{2\sqrt{3}}{3} y - y + \frac{2\sqrt{3}}{3} y^2\right) dy$$

$$= \rho g \int_0^{\sqrt{3}/2} \left(1 + \frac{\sqrt{3}}{2} - \frac{2}{3}\left(3 + \sqrt{3}\right) y + \frac{2\sqrt{3}}{3} y^2\right) dy$$

$$= \rho g \left(\left(1 + \frac{\sqrt{3}}{2}\right) y - \frac{1}{3}\left(3 + \sqrt{3}\right) y^2 + \frac{2\sqrt{3}}{9} y^3 \right) \Big|_0^{\sqrt{3}/2}$$

$$= 9{,}800 \left(\frac{\sqrt{3}}{2} + \frac{3}{4} - \frac{3}{4} - \frac{\sqrt{3}}{4} + \frac{1}{4} \right) \qquad \rho = 1000 \ \frac{kg}{m^3} \quad g = 9.8 \ m/s^2$$

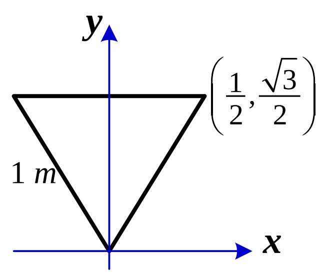
$$\underline{= 2{,}450\left(1 + \sqrt{3}\right) \ N}$$

For the right side plate:

Line segement: $y = \dfrac{\frac{\sqrt{3}}{2} - 0}{\frac{1}{2} - 0}(x - 0) = \sqrt{3}x$

$$x = \frac{1}{\sqrt{3}} y \ \rightarrow \ \underline{2x = \frac{2}{\sqrt{3}} y}$$

$$F = \int_0^{\sqrt{3}/2} \rho g \left(1 + \frac{\sqrt{3}}{2} - y\right)\left(\frac{2}{\sqrt{3}} y\right) dy$$



$$= \rho g \int_0^{\sqrt{3}/2} \left(\frac{2\sqrt{3}}{3} y + y - \frac{2\sqrt{3}}{3} y^2 \right) dy$$

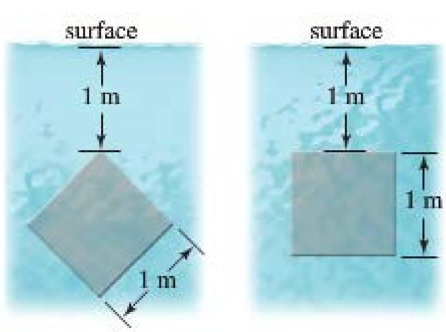
$$= \rho g \left(\frac{\sqrt{3}}{3} y^2 + \frac{1}{2} y^2 - \frac{2\sqrt{3}}{9} y^3 \right) \Big|_0^{\sqrt{3}/2}$$

$$= 9{,}800 \left(\frac{\sqrt{3}}{4} + \frac{3}{8} - \frac{1}{4} \right)$$

$$\underline{= 2{,}450\left(1 + 2\sqrt{3}\right) \ N} \qquad \underline{\approx 10{,}937 \ N}$$

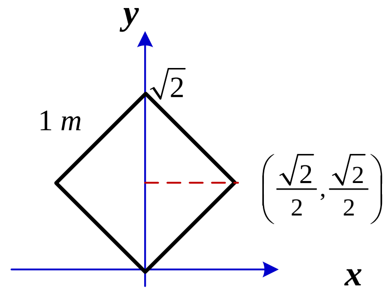
Exercise

A square plate 1 *m* on a side is placed on a vetical wall 1 *m* below the surface of a pool filled with water.



On which plate in the figure is the force is greater

Solution



For the plate on left side :

$$2x^2 = 1 \;\rightarrow\; x = \frac{\sqrt{2}}{2}$$

Line segment OA: $\underline{y = x}| \;\rightarrow 2x = 2y$

Line segment AB: $\lfloor y = \dfrac{\sqrt{2} - \frac{\sqrt{2}}{2}}{0 - \frac{\sqrt{2}}{2}}x + \sqrt{2} \;\underline{= -x + \sqrt{2}}| \;\rightarrow\; x = \sqrt{2} - y$

$$F = \rho g \int_0^{\sqrt{2}/2} \left(1 + \sqrt{2} - y\right)(2y)\,dy + \rho g \int_{\sqrt{2}/2}^{\sqrt{2}} \left(1 + \sqrt{2} - y\right)(2)\left(\sqrt{2} - y\right)dy$$

$$= 2\rho g \int_0^{\sqrt{2}/2} \left(y + \sqrt{2}y - y^2\right)dy + 2\rho g \int_{\sqrt{2}/2}^{\sqrt{2}} \left(\sqrt{2} + 2 - y - 2\sqrt{2}y + y^2\right)dy$$

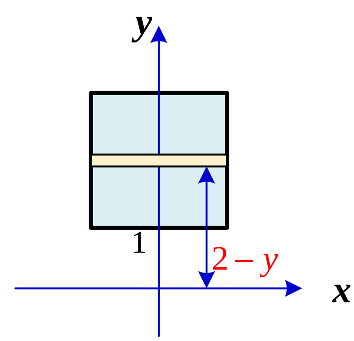
$$= 2\rho g \left(\frac{1}{2}y^2 + \frac{\sqrt{2}}{2}y^2 - \frac{1}{3}y^3\right)\Bigg|_0^{\sqrt{2}/2} + 2\rho g \left(\sqrt{2}y + 2y - \frac{1}{2}y^2 - \sqrt{2}y^2 + \frac{1}{3}y^3\right)\Bigg|_{\sqrt{2}/2}^{\sqrt{2}}$$

$$= 2\rho g \left(\frac{1}{4} + \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{12} + 2 + 2\sqrt{2} - 2 - 2\sqrt{2} + \frac{2\sqrt{2}}{3} - 1 - \sqrt{2} + \frac{1}{4} + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{12}\right)$$

$$= 2\rho g \left(\frac{1}{2} + \frac{\sqrt{2}}{4}\right) \qquad \rho = 1000 \;\frac{kg}{m^3} \quad g = 9.8 \;\; m/s^2$$

$$\underline{= 9,800\left(1 + \frac{\sqrt{2}}{2}\right)\; N}| \qquad \underline{\approx 16,730 \;\; N}|$$

For the plate on right side:



$$F = \rho g \int_0^1 (2 - y)(1)\,dy$$

$$= \rho g \left(2y - \frac{1}{2}y^2\right)\Big|_0^1$$

$$= 9,800\left(\frac{3}{2}\right)$$

$$\underline{= 14,700 \; N}|$$