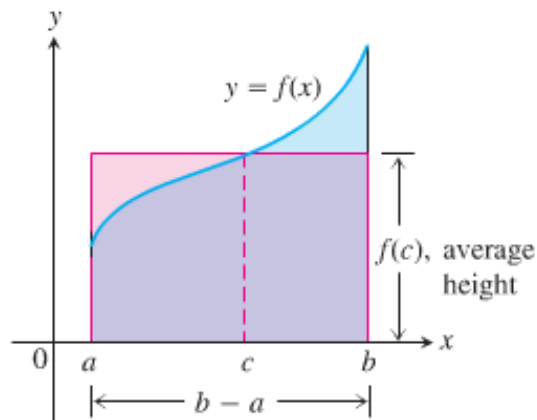


Section 4.4 – Fundamental Theorem of Calculus

Mean Value Theorem for Definite Integrals

If f is continuous on $[a, b]$, then some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx$$



Theorem – The Fundamental Theorem of Calculus, P-1

If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t) dt$ is continuous on $[a, b]$, and differentiable on (a, b)

and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x)$$

Theorem – The Fundamental Theorem of Calculus, P-2

If f is continuous at every point in $[a, b]$, then F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

Example

$$\begin{aligned} a) \quad \int_0^{\pi} \cos x \, dx &= \sin x \Big|_0^{\pi} \\ &= \sin \pi - \sin 0 \\ &= \underline{0} \end{aligned}$$

$$\begin{aligned} b) \quad \int_{-\frac{\pi}{4}}^0 \sec x \tan x \, dx &= \sec x \Big|_{-\frac{\pi}{4}}^0 \\ &= \sec 0 - \sec \left(-\frac{\pi}{4} \right) \\ &= \underline{1 - \sqrt{2}} \end{aligned}$$

$$\begin{aligned} c) \quad \int_1^4 \left(\frac{3}{2} \sqrt{x} - \frac{4}{x^2} \right) dx &= \left[x^{3/2} + \frac{4}{x} \right]_1^4 \\ &= \left((4)^{3/2} + \frac{4}{4} \right) - \left((1)^{3/2} + \frac{4}{1} \right) \\ &= (9) - (5) \\ &= \underline{4} \end{aligned}$$

Theorem – The Net Change Theorem

The net change in a function $F(x)$ over an interval $a \leq x \leq b$ is the integral of its rate of change:

$$F(\textcolor{red}{b}) - F(\textcolor{blue}{a}) = \int_{\textcolor{blue}{a}}^{\textcolor{red}{b}} F'(x) dx$$

Example

Consider the analysis of a heavy rock blown straight up from the ground by a dynamite blast. The velocity of the rock at any time t during its motion was given as $v(t) = 160 - 32t$ ft/sec

- a) Find the displacement of the rock during the time period $0 \leq t \leq 8$
- b) Find the total distance traveled during this time period.

Solution

$$\begin{aligned} \text{a) displacement: } s(t) &= \int_0^8 v(t) dt \\ &= \int_0^8 (160 - 32t) dt \\ &= \left[160t - 16t^2 \right]_0^{\textcolor{red}{8}} \\ &= \left(160(\textcolor{red}{8}) - 16(\textcolor{red}{8})^2 \right) - \left(160(\textcolor{blue}{0}) - 16(\textcolor{blue}{0})^2 \right) \\ &= \underline{256} \end{aligned}$$

The height of the rock is 256 ft above the ground 8 sec after the explosion.

$$\text{b) } v(t) = 160 - 32t = 0 \rightarrow \boxed{t = 5 \text{ sec}}$$

The velocity is positive over the time $[0, 5]$ and negative over $[5, 8]$

$$\begin{aligned} \int_0^8 |v(t)| dt &= \int_0^5 |v(t)| dt + \int_5^8 |v(t)| dt \\ &= \int_0^5 (160 - 32t) dt - \int_5^8 (160 - 32t) dt \\ &= \left[160t - 16t^2 \right]_0^5 - \left[160t - 16t^2 \right]_5^8 \\ &= \left[\left(160(\textcolor{red}{5}) - 16(\textcolor{red}{5})^2 \right) - \left(160(\textcolor{blue}{0}) - 16(\textcolor{blue}{0})^2 \right) \right] \\ &\quad - \left[\left(160(\textcolor{red}{8}) - 16(\textcolor{red}{8})^2 \right) - \left(160(\textcolor{blue}{5}) - 16(\textcolor{blue}{5})^2 \right) \right] \\ &= 400 - (-144) \\ &= \underline{544} \end{aligned}$$

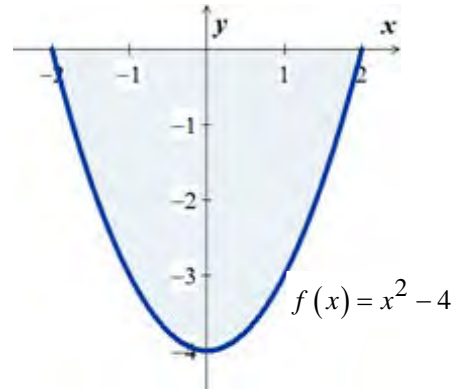
Example

Shows the graph of $f(x) = x^2 - 4$ and its mirror image $g(x) = 4 - x^2$ are reflected across the x -axis. For each function, compute

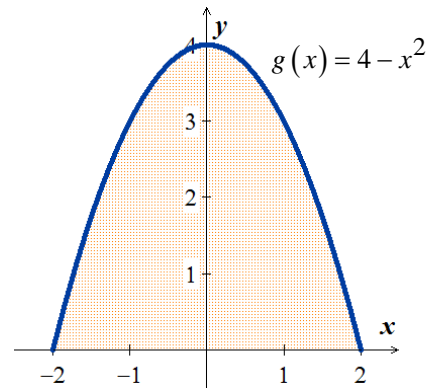
- a) The definite integral over the interval $[-2, 2]$
- b) The area between the graph and the x -axis over $[-2, 2]$

Solution

$$\begin{aligned} a) \quad \int_{-2}^2 f(x) dx &= \int_{-2}^2 (x^2 - 4) dx \\ &= \left[\frac{x^3}{3} - 4x \right]_{-2}^2 \\ &= \left[\frac{(2)^3}{3} - 4(2) \right] - \left[\frac{(-2)^3}{3} - 4(-2) \right] \\ &= \left(\frac{8}{3} - 8 \right) - \left(-\frac{8}{3} + 8 \right) \\ &= -\frac{32}{3} \end{aligned}$$



$$\begin{aligned} \int_{-2}^2 g(x) dx &= \int_{-2}^2 (4 - x^2) dx \\ &= \left[4x - \frac{x^3}{3} \right]_{-2}^2 \\ &= \left[4(2) - \frac{(2)^3}{3} \right] - \left[4(-2) - \frac{(-2)^3}{3} \right] \\ &= \frac{32}{3} \end{aligned}$$



- b) In both cases, the area between the curve and the x -axis over $[-2, 2]$ is $\frac{32}{3}$ units.

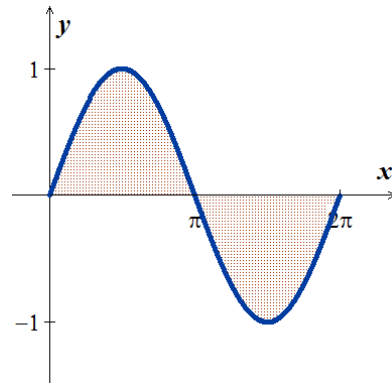
Example

Shows the graph of $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$. Compute

- a) The definite integral of $f(x)$ over $[0, 2\pi]$
- b) The area between the graph and the x -axis over $[0, 2\pi]$

Solution

$$\begin{aligned} a) \int_0^{2\pi} \sin x dx &= -\cos x \Big|_0^{2\pi} \\ &= -(\cos 2\pi - \cos 0) \\ &= -(1 - 1) \\ &= \underline{0} \end{aligned}$$



- b) The area between the graph and the axis is obtained by adding the absolute values

$$\begin{aligned} \text{Area} &= \left| \int_0^{\pi} \sin x dx \right| + \left| \int_{\pi}^{2\pi} \sin x dx \right| \\ &= \left| -\cos x \right|_0^{\pi} + \left| -\cos x \right|_{\pi}^{2\pi} \\ &= |-(\cos \pi - \cos 0)| + |-(\cos 2\pi - \cos \pi)| \\ &= | -(-1 - 1) | + | -(1 - (-1)) | \\ &= |2| + |-2| \\ &= \underline{4} \end{aligned}$$

Summary

To find the area between the graph of $y = f(x)$ and the x -axis over the interval $[a, b]$:

1. Subdivide $[a, b]$ at the zeros of f .
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

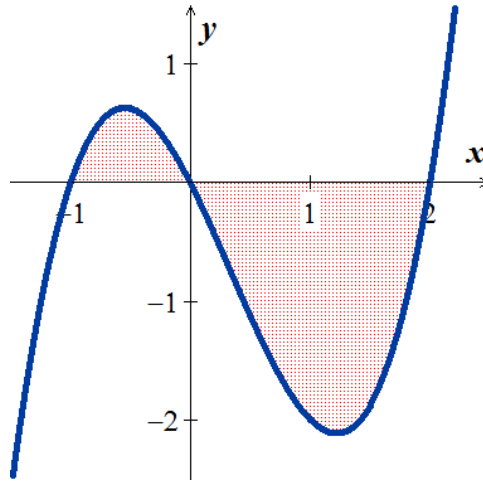
Example

Find the area of the region between the x -axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \leq x \leq 2$

Solution

The zeros of: $f(x) = x^3 - x^2 - 2x = 0$

$$x(x^2 - x - 2) = 0 \Rightarrow x = 0, -1, 2$$



$$\int_{-1}^0 (x^3 - x^2 - 2x) dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_{-1}^0$$

$$= \left[0 - \left(\frac{(-1)^4}{4} - \frac{(-1)^3}{3} - (-1)^2 \right) \right]$$

$$= -\left(\frac{1}{4} + \frac{1}{3} - 1 \right)$$

$$= \frac{5}{12}$$

$$\int_0^2 (x^3 - x^2 - 2x) dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_0^2$$

$$= \left[\left(\frac{(2)^4}{4} - \frac{(2)^3}{3} - (2)^2 \right) - 0 \right]$$

$$= \left(4 - \frac{8}{3} - 4 \right)$$

$$= -\frac{8}{3}$$

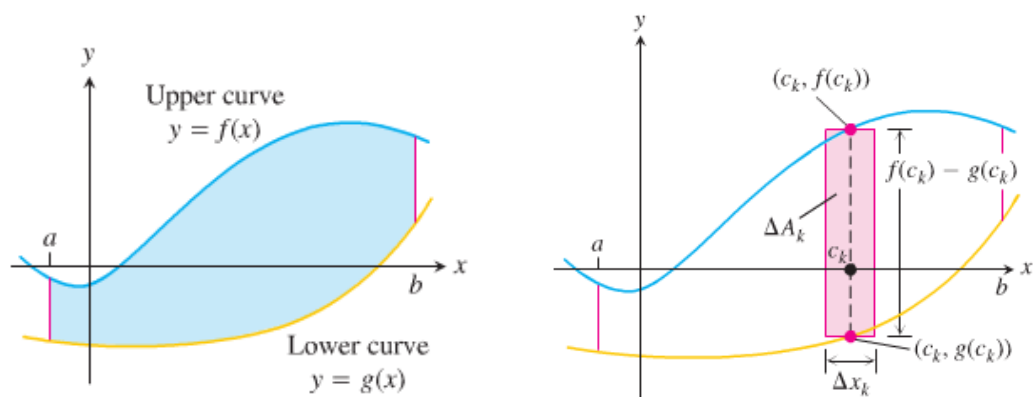
$$\text{Area} = \left| \int_{-1}^0 (x^3 - x^2 - 2x) dx \right| + \left| \int_0^2 (x^3 - x^2 - 2x) dx \right|$$

$$= \frac{5}{12} + \left| -\frac{8}{3} \right|$$

$$= \frac{5}{12} + \frac{8}{3}$$

$$= \frac{37}{12}$$

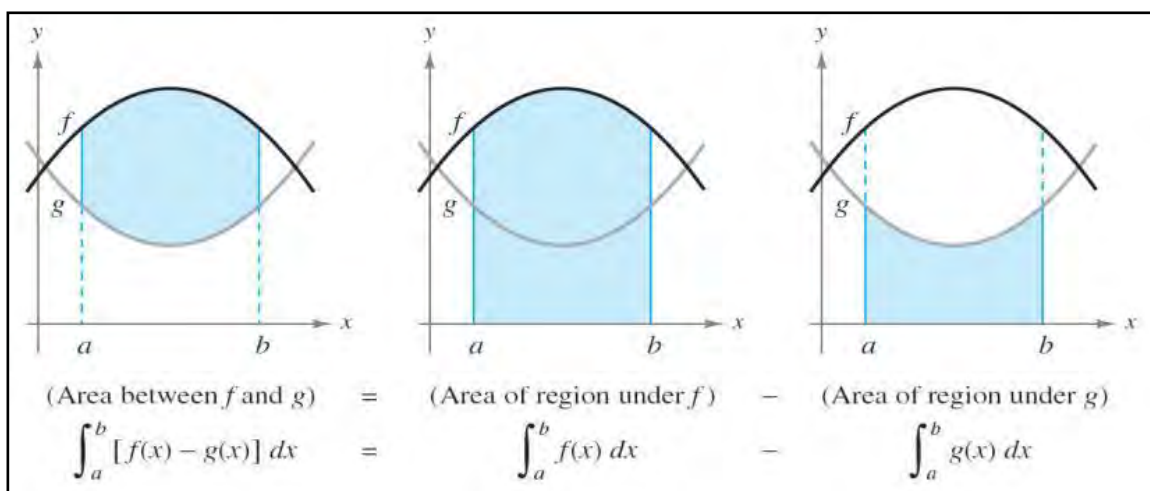
Areas between Curves



Definition

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves** $y = f(x)$ and $y = g(x)$ **from a to b** is:

$$A = \int_a^b [f(x) - g(x)] dx$$



Example

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

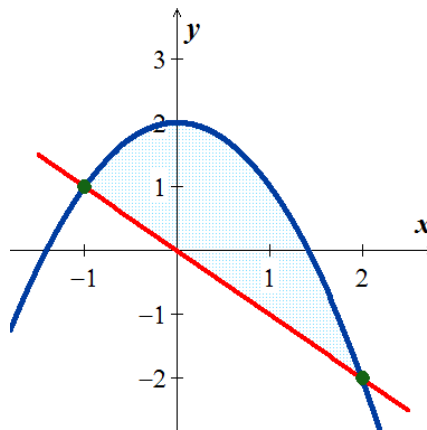
Solution

The limits of integrations are found by letting:

$$2 - x^2 = -x \quad \Rightarrow \quad x^2 - x - 2 = 0 \quad \rightarrow \quad \underline{x = -1, 2}$$

$$A = \int_{-1}^2 [f(x) - g(x)] dx$$

$$\begin{aligned}
&= \int_{-1}^2 \left[2 - x^2 - (-x) \right] dx \\
&= \int_{-1}^2 (2 - x^2 + x) dx \\
&= \left[2x - \frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^2 \\
&= \left(4 - \frac{8}{3} + \frac{4}{2} \right) - \left(-2 + \frac{1}{3} + \frac{1}{2} \right) \\
&= \frac{9}{2}
\end{aligned}$$



Example

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below the x -axis and the line $y = x - 2$

Solution

$$(y = \sqrt{x}) \cap (y = 0) \rightarrow (0, 0)$$

$$(y = \sqrt{x}) \cap (y = x - 2) \rightarrow \sqrt{x} = x - 2$$

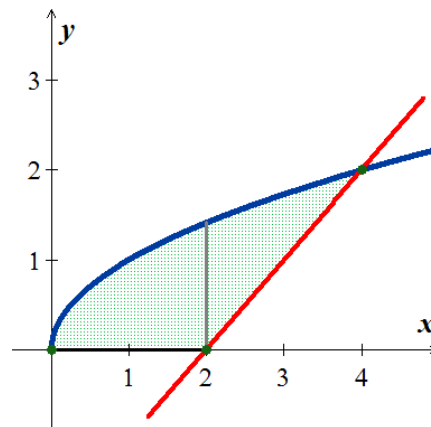
$$(\sqrt{x})^2 = (x - 2)^2$$

$$x = x^2 - 4x + 4$$

$$x^2 - 5x + 4 = 0$$

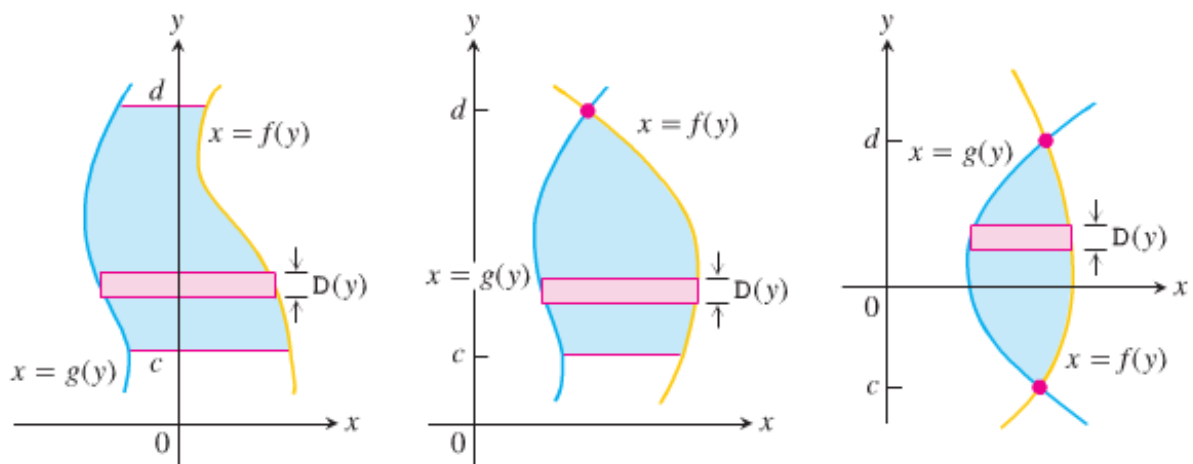
$$\rightarrow x = \text{X}, 4$$

$$(y = 0) \cap (y = x - 2) \rightarrow x = 2$$



$$\begin{aligned}
\text{Total Area} &= \int_0^2 [\sqrt{x} - 0] dx + \int_2^4 [\sqrt{x} - (x - 2)] dx \\
&= \left[\frac{2}{3} x^{3/2} \right]_0^2 + \left[\frac{2}{3} x^{3/2} - \frac{x^2}{2} + 2x \right]_2^4 \\
&= \left[\frac{2}{3} (2^{3/2}) - 0 \right] + \left(\frac{2}{3} 4^{3/2} - \frac{4^2}{2} + 2(4) \right) - \left(\frac{2}{3} 2^{3/2} - \frac{2^2}{2} + 2(2) \right) \\
&= \frac{2}{3} (2^{3/2}) + \frac{2}{3} 4^{3/2} - \frac{16}{2} + 8 - \frac{2}{3} 2^{3/2} + \frac{4}{2} - 4 \\
&= \frac{2}{3} (8) - 2 \\
&= \frac{10}{3}
\end{aligned}$$

Integration with Respect to y



$$A = \int_c^d [f(y) - g(y)] dy \quad (\text{From right hand to left hand})$$

Example

Find the area of the region by integrating with respect to y , in the first quadrant that is bounded above by $y = \sqrt{x}$ and below the x -axis and the line $y = x - 2$.

Solution

$$y = \sqrt{x} \rightarrow x = y^2$$

$$y = x - 2 \rightarrow x = y + 2$$

$$(x = y^2) \cap (y = 0) \rightarrow (0, 0)$$

$$(x = y^2) \cap (x = y + 2) \rightarrow y^2 = y + 2$$

$$y^2 - y - 2 = 0 \rightarrow y = -1, 2$$

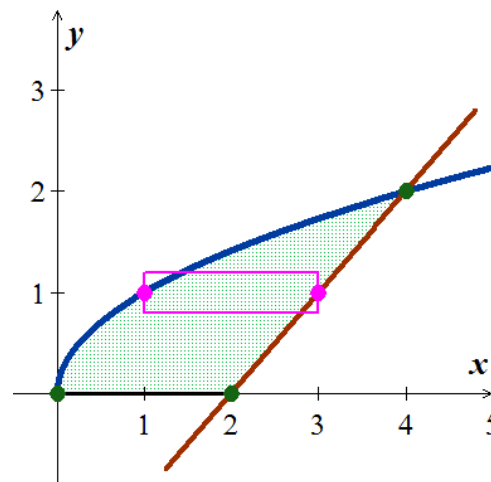
$$(y = 0) \cap (x = y + 2) \rightarrow y = 0$$

$$A = \int_0^2 [y + 2 - y^2] dy$$

$$= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_0^2$$

$$= \frac{2^2}{2} + 2(2) - \frac{2^3}{3} - 0$$

$$= \frac{10}{3}$$



Exercises Section 4.4 – Fundamental Theorem of Calculus

Evaluate the integrals

1. $\int_0^3 (2x+1)dx$
2. $\int_0^2 x(x-3)dx$
3. $\int_0^4 \left(3x - \frac{x^3}{4}\right)dx$
4. $\int_{-2}^2 (x^3 - 2x + 3)dx$
5. $\int_0^1 (x^2 + \sqrt{x})dx$
6. $\int_0^{\pi/3} 4\sec u \tan u \, du$
7. $\int_{\pi/4}^{3\pi/4} \csc \theta \cot \theta d\theta$
8. $\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right)dt$
9. $\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} dy$
10. $\int_1^8 \frac{(x^{1/3} + 1)(2 - x^{2/3})}{x^{1/3}} dx$
11. $\int_{\pi/2}^{\pi} \frac{\sin 2x}{2\sin x} dx$
12. $\int_0^{\pi/3} (\cos x + \sec x)^2 dx$
13. $\int_0^{\pi} \frac{1}{2}(\cos x + |\cos x|)dx$
14. $\int_0^1 2x(4 - x^2)dx$
15. $\int_0^4 (8 - 2x)dx$
16. $\int_0^4 \frac{1}{\sqrt{16 - x^2}} dx$
17. $\int_{-4}^2 (2x + 4)dx$
18. $\int_0^2 (1 - x)dx$
19. $\int_0^2 (x^2 - 2)dx$
20. $\int_0^{\pi/2} \cos x \, dx$
21. $\int_1^7 \frac{dx}{x}$
22. $\int_4^9 3\sqrt{x} \, dx$
23. $\int_{-2}^3 (x^2 - x - 6)dx$
24. $\int_0^1 (1 - \sqrt{x})dx$
25. $\int_0^{\pi/4} 2\cos x \, dx$
26. $\int_{-\pi/4}^{7\pi/4} (\sin x + \cos x)dx$
27. $\int_0^{\ln 8} e^x dx$
28. $\int_1^4 \left(\frac{x-1}{x}\right)dx$
29. $\int_{-2}^{-1} \left(3e^{3x} + \frac{2}{x}\right)dx$
30. $\int_0^2 \frac{dx}{x^2 + 4}$

Find the total area between the region between the given graph and the x -axis

31. $y = -x^2 - 2x, \quad -3 \leq x \leq 2$
32. $y = x^3 - 3x^2 + 2x, \quad 0 \leq x \leq 2$
33. $y = x^{1/3} - x, \quad -1 \leq x \leq 8$
34. $f(x) = x^2 + 1, \quad 2 \leq x \leq 3$
35. Find the area of the region between the graph of $y = 4x - 8$ and the x -axis, for $-4 \leq x \leq 8$
36. Find the area of the region between the graph of $y = -3x$ and the x -axis, for $-2 \leq x \leq 2$

37. Find the area of the region between the graph of $y = 3x + 6$ and the x -axis, for $0 \leq x \leq 6$
38. Find the area of the region between the graph of $y = 1 - |x|$ and the x -axis, for $-2 \leq x \leq 2$
39. Find the area of the region above the x -axis bounded by $y = 4 - x^2$
40. Find the area of the region above the x -axis bounded by $y = x^4 - 16$
41. Find the area of the region between the graph of $y = 6\cos x$ and the x -axis, for $-\frac{\pi}{2} \leq x \leq \pi$
42. Find the area of the region between the graph of $f(x) = \frac{1}{x}$ and the x -axis, for $-2 \leq x \leq -1$
43. Find the area of the region bounded by the graph of $f(x) = x^2 - 4x + 3$ x -axis on $0 \leq x \leq 3$
44. Find the area of the region bounded by the graph of $f(x) = x^2 + 4x + 3$ x -axis on $-3 \leq x \leq 0$
45. Find the area of the region bounded by the graph of $f(x) = x^2 - 3x + 2$ x -axis on $0 \leq x \leq 2$
46. Find the area of the region bounded by the graph of $f(x) = x^2 + 3x + 2$ x -axis on $-2 \leq x \leq 0$
47. Find the area of the region bounded by the graph of $f(x) = 2x^2 - 4x + 2$ x -axis on $0 \leq x \leq 2$
48. Find the area of the region bounded by the graph of $f(x) = 2x^2 + 4x + 2$ x -axis on $-1 \leq x \leq 1$
49. Find the area of the region bounded by the graphs of $x = y^2 - y$ and $x = 2y^2 - 2y - 6$
50. Find the area of the region bounded by the graphs of $y = x^2 - 4$ & $y = -x^2 - 2x$

Compute the area of the region bounded by the graph of f and the x -axis on the given interval.

51. $f(x) = \frac{1}{x^2 + 1}$ on $[-1, \sqrt{3}]$

52. $f(x) = 2\sin \frac{x}{4}$ on $[0, 2\pi]$

53. Archimedes, inventor, military engineer, physicist, and the greatest mathematician of classical times in the Western world, discovered that the area under a parabolic arch is two-thirds the base times the height. Sketch the parabolic arch $y = h - \left(\frac{4h}{b^2}\right)x^2$ $-\frac{b}{2} \leq x \leq \frac{b}{2}$, assuming that h and b are positive.

Then use calculus to find the area of the region enclosed between the arch and the x -axis

54. Suppose that a company's marginal revenue from the manufacture and sale of eggbeaters is

$$\frac{dr}{dx} = 2 - \frac{2}{(x+1)^2}$$

Where r is measured in thousands of dollars and x in thousands of units. How much money should the company expect from a production run of $x = 3$ thousand eggbeaters? To find out, integrate the marginal revenue from $x = 0$ to $x = 3$.

55. The height H (ft) of a palm tree after growing for t years is given by

$$H = \sqrt{t+1} + 5t^{1/3} \quad \text{for } 0 \leq t \leq 8$$

- a) Find the tree's height when $t = 0$, $t = 4$, and $t = 8$.
- b) Find the tree's average height for $0 \leq t \leq 8$