

## ***SOLUTION***

### ***Section 3.4 – Comparison Tests***

#### ***Exercise***

Use the *Comparison Test* to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{1}{n^2 + 30}$

#### **Solution**

Comparing with  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ , which is a convergent  $p$ -series, since  $p = 2 > 1$ .

Both series have nonnegative terms for  $n \geq 1$

$$n^2 \leq n^2 + 30$$

$$\frac{1}{n^2} \geq \frac{1}{n^2 + 30}$$

Then, by Comparison Test,  $\sum_{n=1}^{\infty} \frac{1}{n^2 + 30}$  ***converges***.

#### ***Exercise***

Use the *Comparison Test* to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{n-1}{n^4 + 2}$

#### **Solution**

Comparing with  $\sum_{n=1}^{\infty} \frac{1}{n^3}$ , which is a convergent  $p$ -series, since  $p = 3 > 1$ .

Both series have nonnegative terms for  $n \geq 1$

$$n^4 \leq n^4 + 2 \Rightarrow \frac{1}{n^4} \geq \frac{1}{n^4 + 2}$$

$$\frac{n}{n^4} \geq \frac{n}{n^4 + 2} \geq \frac{n-1}{n^4 + 2}$$

$$\frac{1}{n^3} \geq \frac{n}{n^4 + 2} \geq \frac{n-1}{n^4 + 2}$$

Then, by *Comparison Test*, the given series ***converges***.

### Exercise

Use the *Comparison Test* to determine if the series converges or diverges.  $\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$

### Solution

Comparing with  $\sum_{n=1}^{\infty} \frac{1}{n}$ , which is a divergent  $p$ -series, since  $p = 1 \leq 1$ .

Both series have nonnegative terms for  $n \geq 2$

$$n^2 - n \leq n^2 \Rightarrow \frac{1}{n^2 - n} \geq \frac{1}{n^2}$$
$$\frac{n}{n^2 - n} \geq \frac{n}{n^2} = \frac{1}{n}$$

Then, by *Comparison Test*, the given series **diverges**.

### Exercise

Use the *Comparison Test* to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$

### Solution

Comparing with  $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ , which is a convergent  $p$ -series, since  $p = \frac{3}{2} > 1$ .

Both series have nonnegative terms for  $n \geq 1$

$$0 \leq \cos^2 n \leq 1$$

$$0 \leq \frac{\cos^2 n}{n^{3/2}} \leq \frac{1}{n^{3/2}}$$

Then, by *Comparison Test*, the given series **converges**.

### Exercise

Use the *Comparison Test* to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$

### Solution

Comparing with  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ , which is a divergent  $p$ -series, since  $p = \frac{1}{2} < 1$ .

Both series have nonnegative terms for  $n \geq 1$

$$\sqrt{n} \geq 1 \Rightarrow 2\sqrt{n} \geq 2$$

$$2\sqrt{n} + 1 \geq 3$$

Then, by *Comparison Test*, the given series **diverges**.

### Exercise

Use the *Comparison Test* to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$

### Solution

Comparing with  $\sum_{n=1}^{\infty} \frac{1}{2^n}$ , which is a convergent series.

$$\text{Then } 0 < \frac{1}{2^n + 1} < \frac{1}{2^n}$$

Therefore, the given series **converges** by *comparison Test*.

### Exercise

Use the *Comparison Test* to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{3n+1}{n^3 + 1}$

### Solution

$$\frac{3n+1}{n^3 + 1} \xrightarrow{n \rightarrow \infty} \frac{3}{n^2}$$

$$\frac{3n+1}{n^3 + 1} = \frac{3n}{n^3 + 1} + \frac{1}{n^3 + 1}$$

$$< \frac{3n}{n^3} + \frac{1}{n^3}$$

$$< \frac{3}{n^2} + \frac{1}{n^2}$$

$$= \frac{4}{n^2}$$

Therefore, by *Comparison Test*, the given series **converges**.

### Exercise

Use the *Comparison Test* to determine if the series converges or diverges.  $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

### Solution

$$< \ln n < n$$

$$\frac{1}{\ln n} > \frac{1}{n}$$

Since  $\sum_{n=2}^{\infty} \frac{1}{n}$  **diverges** to infinity (it is a harmonic series),

Therefore; by *Comparison Test* the series  $\sum_{n=2}^{\infty} \frac{1}{\ln n}$  **diverges**.

### Exercise

Use the *Comparison Test* to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

### Solution

$$2n-1 < 2n$$

$$\frac{1}{2n-1} > \frac{1}{2n} \quad \text{for } n \geq 1$$

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} > \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$$

By the *p-series* the series  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges  $p = 1$

Therefore; by *Comparison Test*, the given series **diverges**.

### Exercise

Use the *Comparison Test* to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$

### Solution

$$3n^2 + 2 > 3n^2 \Rightarrow \frac{1}{3n^2+2} < \frac{1}{3n^2}$$

By the *p-series* the series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges  $p > 1$

Therefore; by *Comparison Test*, the given series *converges*.

### Exercise

Use the *Comparison Test* to determine if the series converges or diverges.  $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

#### Solution

$$\begin{aligned} \sqrt{n}-1 &< \sqrt{n} \\ \frac{1}{\sqrt{n}-1} &> \frac{1}{\sqrt{n}} \quad \text{for } n \geq 2 \end{aligned}$$

By the *p-series* the series  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$  diverges  $p = \frac{1}{2} < 1$

Therefore; by *Comparison Test*, the given series *diverges*.

### Exercise

Use the *Comparison Test* to determine if the series converges or diverges.  $\sum_{n=0}^{\infty} \frac{4^n}{5^n+3}$

#### Solution

$$\frac{4^n}{5^n+3} < \frac{4^n}{5^n} = \left(\frac{4}{5}\right)^n$$

By the geometric series:  $r = \frac{4}{5} < 1$  converges

Therefore; by *Comparison Test*, the given series *converges*.

### Exercise

Use the *Comparison Test* to determine if the series converges or diverges.  $\sum_{n=2}^{\infty} \frac{\ln n}{n+1}$

#### Solution

$$\frac{\ln n}{n+1} > \frac{1}{n+1} \quad (\text{and by integral test})$$

The given series *converges* by *Comparison Test* with the divergent series.

### Exercise

Use the *Comparison Test* to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

### Solution

$$\frac{1}{\sqrt{n^3+1}} < \frac{1}{\sqrt{n^3}}$$

By the *p-series* the series  $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$  converges  $p = \frac{3}{2} > 1$

Therefore; by *Comparison Test*, the given series **converges**.

### Exercise

Use the *Comparison Test* to determine if the series converges or diverges.  $\sum_{n=0}^{\infty} \frac{1}{n!}$

### Solution

$$\frac{1}{n^2} > \frac{1}{n!} \quad \text{For } n > 3$$

By the *p-series* the series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges  $p = 2 > 1$

Therefore; by *Comparison Test*, the given series **converges**.

### Exercise

Use the *Comparison Test* to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1}$

### Solution

$$\frac{1}{4\sqrt[3]{n}-1} > \frac{1}{4\sqrt[3]{n}}$$

By the *p-series* the series  $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$  diverges  $p = \frac{1}{3} < 1$

Therefore; by *Comparison Test*, the given series **diverges**.

### Exercise

Use the *Comparison Test* to determine if the series converges or diverges.  $\sum_{n=0}^{\infty} e^{-n^2}$

#### Solution

$$\frac{1}{e^{n^2}} \leq \frac{1}{e^n}$$

Geometric series:  $r = \frac{1}{e} < 1$  converges

Therefore; by *Comparison Test*, the given series **converges**.

### Exercise

Use the *Comparison Test* to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{3^n}{2^n - 1}$

#### Solution

$$\frac{3^n}{2^n - 1} > \left(\frac{3}{2}\right)^n$$

Geometric series:  $r = \frac{3}{2} > 1$  diverges

Therefore; by *Comparison Test*, the given series **diverges**.

### Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{n-2}{n^3 - n^2 + 3}$

#### Solution

Comparing with  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ , which is a convergent  $p$ -series, since  $p = 2 > 1$ .

$$a_n = \frac{n-2}{n^3 - n^2 + 3} \Rightarrow b_n = \frac{1}{n^2}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n-2}{n^3 - n^2 + 3} \cdot \frac{n^2}{1}$$

$$= \lim_{n \rightarrow \infty} \frac{n^3 - 2n^2}{n^3 - n^2 + 3}$$

$$= 1 > 0$$

or *L'Hopital Rule*

Therefore; by *Limit Comparison Test*, the given series **converges**.

### Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges.  $\sum_{n=2}^{\infty} \frac{n(n+1)}{(n^2+1)(n-1)}$

### Solution

Comparing with  $\sum_{n=2}^{\infty} \frac{1}{n}$ , which is a divergent  $p$ -series, since  $p = 1 \leq 1$ .

$$a_n = \frac{n(n+1)}{(n^2+1)(n-1)} \Rightarrow b_n = \frac{n^2}{n^3} = \frac{1}{n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{n(n+1)}{(n^2+1)(n-1)} \cdot \frac{n}{1} \\ &= \lim_{n \rightarrow \infty} \frac{n^3 + n^2}{n^3 - n^2 + n - 1} \\ &= 1 > 0 \end{aligned}$$

Therefore; by *Limit Comparison Test*, the given series **diverges**.

### Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{2^n}{3+4^n}$

### Solution

Comparing with  $\sum_{n=1}^{\infty} \frac{1}{2^n}$ , which is a convergent geometric, since  $|r| = \frac{1}{2} < 1$ .

$$a_n = \frac{2^n}{3+4^n} \Rightarrow b_n = \frac{1}{2^n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{2^n}{3+4^n} \cdot \frac{2^n}{1} \\ &= \lim_{n \rightarrow \infty} \frac{2^{2n}}{3+4^n} \\ &= \lim_{n \rightarrow \infty} \frac{4^n}{3+4^n} \\ &= \lim_{n \rightarrow \infty} \frac{4^n \ln 4}{4^n \ln 4} \end{aligned}$$



$$= 1 > 0$$

Therefore; by *Limit Comparison Test*, the given series **converges**.

### Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}4^n}$

### Solution

Comparing with  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ , which is a divergent  $p$ -series, since  $p = \frac{1}{2} < 1$ .

$$a_n = \frac{5^n}{\sqrt{n}4^n} \Rightarrow b_n = \frac{1}{\sqrt{n}}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{5^n}{\sqrt{n}4^n} \cdot \frac{\sqrt{n}}{1} \\ &= \lim_{n \rightarrow \infty} \frac{5^n}{4^n} \\ &= \lim_{n \rightarrow \infty} \left(\frac{5}{4}\right)^n \\ &= \infty \end{aligned}$$

Therefore; by *Limit Comparison Test*, the given series **diverges**.

### Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4}\right)^n$

### Solution

Comparing with  $\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$ , which is a convergent geometric, since  $|r| = \frac{2}{5} < 1$ .

$$a_n = \left(\frac{2n+3}{5n+4}\right)^n \Rightarrow b_n = \left(\frac{2}{5}\right)^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \left(\frac{2n+3}{5n+4}\right)^n \cdot \left(\frac{5}{2}\right)^n \\ &= \lim_{n \rightarrow \infty} \left(\frac{10n+15}{10n+8}\right)^n \\ &= \lim_{n \rightarrow \infty} \left(\frac{10n}{10n}\right)^n \end{aligned}$$

$$= \lim_{n \rightarrow \infty} 1^n$$

$$= 1 > 0$$

Therefore; by *Limit Comparison Test*, the given series **converges**.

### Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges.  $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

### Solution

Comparing with  $\sum_{n=2}^{\infty} \frac{1}{n}$ , which is a divergent  $p$ -series, since  $p = 1 \leq 1$ .

$$a_n = \frac{1}{\ln n} \Rightarrow b_n = \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1}{\ln n} \cdot \frac{n}{1}$$

$$= \lim_{n \rightarrow \infty} \frac{n}{\ln n}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{1/n}$$

$$= \lim_{n \rightarrow \infty} n$$

$$= \infty$$

Therefore; by *Limit Comparison Test*, the given series **diverges**.

### Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{1}{1 + \sqrt{n}}$

### Solution

$$\text{Let } b_n = \frac{1}{\sqrt{n}}$$

$$\lim_{n \rightarrow \infty} \frac{1}{1 + \sqrt{n}} \bigg/ \frac{1}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{1 + \sqrt{n}}$$

$$= 1$$

Since the  $p$ -series diverges to infinity  $\left(p = \frac{1}{2}\right)$

Therefore; by *Limit Comparison Test*, the given series **diverges**.

### Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{n+5}{n^3-2n+3}$

#### Solution

$$\text{Let } b_n = \frac{1}{n^2}$$

$$\lim_{n \rightarrow \infty} \frac{n+5}{n^3-2n+3} \bigg/ \frac{1}{n^2} = \lim_{n \rightarrow \infty} \frac{n^3+5n^2}{n^3-2n+3} \\ = 1 < \infty$$

Since the *p-series* converges ( $p = 2$ )

Therefore; by *Limit Comparison Test*, the given series **converges**.

### Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

#### Solution

$$b_n = \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} \frac{n}{n^2+1} \cdot \frac{n}{1} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2+1} \\ = 1 < \infty$$

Since the *p-series* diverges to infinity ( $p = 1$ )

Therefore; by *Limit Comparison Test*, the given series **diverges**.

### Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{5}{4^n+1}$

#### Solution

$$b_n = \frac{1}{4^n} = \left(\frac{1}{4}\right)^n$$

$$\lim_{n \rightarrow \infty} \frac{5}{4^n+1} \cdot \frac{4^n}{1} = 5$$

By geometric series  $\left(r = \frac{1}{4} < 1\right)$  converges

Therefore; by *Limit Comparison Test*, the given series **converges**.

### Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges.  $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2 + 1}}$

### Solution

$$b_n = \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n^2 + 1}} \cdot \frac{n}{1} = 1 < \infty$$

Since the *p-series* diverges to infinity ( $p = 1$ )

Therefore; by *Limit Comparison Test*, the given series **diverges**.

### Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{2^n + 1}{5^n + 1}$

### Solution

$$b_n = \left(\frac{2}{5}\right)^n$$

$$\lim_{n \rightarrow \infty} \frac{2^n + 1}{5^n + 1} \cdot \frac{5^n}{2^n} = 1$$

By geometric series  $\left(r = \frac{2}{5} < 1\right)$  converges

Therefore; by *Limit Comparison Test*, the given series **converges**.

### Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{2n^2 - 1}{3n^5 + 2n + 1}$

### Solution

$$\text{Let } b_n = \frac{1}{n^3}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{2n^2 - 1}{3n^5 + 2n + 1} \bigg/ \frac{1}{n^3} &= \lim_{n \rightarrow \infty} \frac{2n^5 - n^3}{3n^5 + 2n + 1} \\ &= \frac{2}{3} < \infty \end{aligned}$$

Since the *p-series* converges ( $p = 3$ )

Therefore; by *Limit Comparison Test*, the given series **converges**.

### Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{1}{n^2(n+3)}$

### Solution

Let  $b_n = \frac{1}{n^3}$  By the *p-series* converges ( $p = 3$ )

$$\lim_{n \rightarrow \infty} \frac{1}{n^2(n+3)} \bigg/ \frac{1}{n^3} = \lim_{n \rightarrow \infty} \frac{n^3}{n^2(n+3)} \\ = 1 < \infty$$

Therefore; by *Limit Comparison Test*, the given series **converges**.

### Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges.  $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$

### Solution

Let  $b_n = \frac{1}{n^2}$  By the *p-series* converges ( $p = 2$ ),

$$\lim_{n \rightarrow \infty} \frac{1}{n\sqrt{n^2+1}} \cdot \frac{n^2}{1} = 1 < \infty$$

Therefore; by *Limit Comparison Test*, the given series **converges**.

### Exercise

Use any method to determine if the series converges or diverges  $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n} + \sqrt[3]{n}}$

### Solution

Comparing with  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ , which is a divergent *p-series*, since  $p = \frac{1}{2} < 1$ .

$$a_n = \frac{1}{2\sqrt{n} + \sqrt[3]{n}} \Rightarrow b_n = \frac{1}{\sqrt{n}}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1}{2\sqrt{n} + \sqrt[3]{n}} \cdot \frac{\sqrt{n}}{1} \\ = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{2\sqrt{n} + \sqrt[3]{n}}$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \frac{1}{2 + n^{1/3 - 1/2}} \\
&= \lim_{n \rightarrow \infty} \frac{1}{2 + n^{-1/6}} \\
&= \frac{1}{2} > 0
\end{aligned}$$

Then, by Comparison Test, the given series *diverges*.

### Exercise

Use any method to determine if the series converges or diverges  $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$

### Solution

Comparing with  $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ , which is a convergent geometric, since  $|r| = \frac{1}{2} < 1$ .

By the Direct Comparison Test:  $\frac{\sin^2 n}{2^n} \leq \frac{1}{2^n}$

The given series *converges*.

### Exercise

Use any method to determine if the series converges or diverges  $\sum_{n=1}^{\infty} \frac{n+1}{n^2 \sqrt{n}}$

### Solution

Comparing with  $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ , which is a convergent  $p$ -series, since  $p = \frac{3}{2} > 1$ .

$$a_n = \frac{n+1}{n^2 \sqrt{n}} \Rightarrow b_n = \frac{1}{n^{3/2}}$$

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{n+1}{n^2 \sqrt{n}} \cdot \frac{n^{3/2}}{1} \\
&= \lim_{n \rightarrow \infty} \frac{n+1}{n} \\
&= 1 > 0
\end{aligned}$$

Then, by Comparison Test, the given series *converges*.

### Exercise

Use any method to determine if the series converges or diverges  $\sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$

### Solution

Comparing with  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ , which is a convergent  $p$ -series, since  $p = 2 > 1$ .

$$a_n = \frac{10n+1}{n(n+1)(n+2)} \Rightarrow b_n = \frac{1}{n^2}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{10n+1}{n(n+1)(n+2)} \cdot \frac{n^2}{1} \\ &= \lim_{n \rightarrow \infty} \frac{10n^3 + n^2}{n(n^2 + 3n + 2)} \\ &= \lim_{n \rightarrow \infty} \frac{10n^3 + n^2}{n^3 + 3n^2 + 2} \\ &= 10 > 0 \end{aligned}$$

Then, by Comparison Test, the given series **converges**.

### Exercise

Use any method to determine if the series converges or diverges  $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$

### Solution

By the Direct Comparison Test:  $\left(\frac{n}{3n+1}\right)^n < \left(\frac{n}{3n}\right)^n = \left(\frac{1}{3}\right)^n$

$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ , which is a convergent geometric, since  $|r| = \frac{1}{3} < 1$ .

Therefore, the given series **converges**.

### Exercise

Use any method to determine if the series converges or diverges  $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$

### Solution

Comparing with  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ , which is a convergent  $p$ -series, since  $p = 2 > 1$ .

$$a_n = \frac{(\ln n)^2}{n^3} \Rightarrow b_n = \frac{1}{n^2}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{(\ln n)^2}{n^3} \cdot \frac{n^2}{1} \\ &= \lim_{n \rightarrow \infty} \frac{(\ln n)^2}{n} \\ &= \lim_{n \rightarrow \infty} \frac{2 \ln n \left( \frac{1}{n} \right)}{1} \\ &= 2 \lim_{n \rightarrow \infty} \frac{\ln n}{n} \\ &= 0 \end{aligned}$$

Then, by Comparison Test, the given series **converges**.

### Exercise

Use any method to determine if the series converges or diverges

$$\sum_{n=1}^{\infty} \frac{1 + \sin n}{n^2}$$

### Solution

$$\text{Let } b_n = \frac{1}{n^2}$$

$$\lim_{n \rightarrow \infty} \frac{1 + \sin n}{n^2} \bigg/ \frac{1}{n^2} = \lim_{n \rightarrow \infty} (1 + \sin n) \text{ which does not exist.}$$

$$\frac{1 + \sin n}{n^2} \leq \frac{2}{n^2}, \text{ then the given series converges by comparison test}$$

### Exercise

Use any method to determine if the series converges or diverges

$$\sum_{n=1}^{\infty} \frac{1}{2 + 3^n}$$

### Solution

$$a_n = \frac{1}{2 + 3^n} < \frac{1}{3^n} = b_n$$

So, by the Direct Comparison Test, the series converges.



### Exercise

Use any method to determine if the series converges or diverges

$$\sum_{n=1}^{\infty} \frac{1}{2 + \sqrt{n}}$$

### Solution

$$2 + \sqrt{n} < n \Rightarrow \frac{1}{2 + \sqrt{n}} \geq \frac{1}{n}$$

By the *p-series* the series  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges  $p = 1$

The given series **diverges** by Comparison Test using *p-series*.

### Exercise

Use any method to determine if the series converges or diverges

$$\sum_{n=1}^{\infty} \frac{1}{an + b}$$

### Solution

$$an + b < n \Rightarrow \frac{1}{an + b} \geq \frac{1}{n} \quad a, b > 0$$

By the *p-series* the series  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges  $p = 1$

The given series **diverges** by Comparison Test using *p-series*.

$$\lim_{n \rightarrow \infty} \frac{1}{an + b} \cdot \frac{n}{1} = \frac{1}{a} > 0$$

The given series **diverges** by Limit Comparison Test

### Exercise

Use any method to determine if the series converges or diverges

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}$$

### Solution

Let  $b_n = \frac{1}{n^{3/2}}$  By the *p-series* converges  $\left(p = \frac{3}{2} > 1\right)$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n^2 + 1} \cdot \frac{n^{3/2}}{1} = 1$$

Therefore, the given series **converges** by the limit comparison test.

### Exercise

Use any method to determine if the series converges or diverges

$$\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$$

### Solution

$$\frac{\sqrt[3]{n}}{n} = \frac{1}{n^{2/3}}$$

By the *p-series* the series  $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$  diverges  $p = \frac{2}{3} < 1$

The given series *diverges* by comparison test using *p-series*.

### Exercise

Use any method to determine if the series converges or diverges

$$\sum_{n=0}^{\infty} 5\left(-\frac{4}{3}\right)^n$$

### Solution

$$|r| = \frac{4}{3} > 1$$

The given series *diverges* by Geometric series

### Exercise

Use any method to determine if the series converges or diverges

$$\sum_{n=1}^{\infty} \frac{1}{5^n + 1}$$

### Solution

$$\frac{1}{5^n + 1} < \left(\frac{1}{5}\right)^n$$

The given series converges by a Direct Comparison with the convergent geometric series  $\left(r = \frac{1}{5} < 1\right)$

### Exercise

Use any method to determine if the series converges or diverges

$$\sum_{n=2}^{\infty} \frac{1}{n^3 - 8}$$

### Solution

Let  $b_n = \frac{1}{n^3}$  By the *p-series* converges ( $p = 3$ )

$$\lim_{n \rightarrow \infty} \frac{1}{n^3 - 8} \cdot \frac{n^3}{1} = 1$$

Therefore, the given series *converges* by the limit comparison test with *p-series*.

### Exercise

Use any method to determine if the series converges or diverges

$$\sum_{n=1}^{\infty} \frac{2n}{3n-2}$$

### Solution

$$\lim_{n \rightarrow \infty} \frac{2n}{3n-2} = \frac{2}{3} \neq 0$$

The given series *diverges* by the Limit.

### Exercise

Use any method to determine if the series converges or diverges

$$\sum_{n=1}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+2} \right)$$

### Solution

$$\begin{aligned} \sum_{n=1}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+2} \right) &= \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \left( \frac{1}{4} - \frac{1}{5} \right) + \dots \\ &= \frac{1}{2} \end{aligned}$$

The given series *converges* by telescoping series.

### Exercise

Use any method to determine if the series converges or diverges

$$\sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^2}$$

### Solution

$$\begin{aligned} \int_1^{\infty} \frac{x}{(x^2 + 1)^2} dx &= \frac{1}{2} \int_1^{\infty} \frac{1}{(x^2 + 1)^2} d(x^2 + 1) \\ &= -\frac{1}{2} \frac{1}{x^2 + 1} \Big|_1^{\infty} \\ &= -\frac{1}{2} \left( 0 - \frac{1}{2} \right) \end{aligned}$$

$$\left| = \frac{1}{4} \right|$$

The given series **converges** by the *Integral Test*.

### Exercise

Use any method to determine if the series converges or diverges  $\sum_{n=1}^{\infty} \frac{n2^n}{4n^3 + 1}$

### Solution

$$b_n = \frac{n^2}{2^n}$$

$$a_n = \frac{n2^n}{4n^3 + 1}$$

$$\lim_{k \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n2^n}{4n^3 + 1} \cdot \frac{n^2}{2^n}$$

$$\left| = \frac{1}{4} \right|$$

Therefore; by the *Limit Comparison Test*, the given series **converges**.

### Exercise

Use any method to determine if the series converges or diverges  $\sum_{k=1}^{\infty} \frac{|\sin k|}{k^2}$

### Solution

$$|\sin k| \leq 1$$

$$\frac{|\sin k|}{k^2} \leq \frac{1}{k^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \text{ converges by } \mathbf{p}\text{-series } (p = 2 > 1)$$

Therefore; by the *Comparison Test*, the given series **converges**.

### Exercise

Use any method to determine if the series converges or diverges  $\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2}$

### Solution

$$0 \leq \sin^2 k \leq 1$$

$$0 \leq \frac{\sin^2 k}{k^2} \leq \frac{1}{k^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \text{ converges by } \mathbf{p}\text{-series } (p = 2 > 1)$$

Therefore; by the *Comparison Test*, the given series **converges**.

### Exercise

Use any method to determine if the series converges or diverges  $\sum_{k=1}^{\infty} \sin^2 \frac{1}{k}$

### Solution

$$\text{Let } b_k = \sum_{k=1}^{\infty} \frac{1}{k^2} \text{ converges by } \mathbf{p}\text{-series } (p = 2 > 1)$$

$$a_k = \sin^2 \frac{1}{k}$$

$$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = \lim_{k \rightarrow \infty} \frac{\sin^2 \frac{1}{k}}{\frac{1}{k^2}} = \frac{0}{0}$$

$$= \lim_{k \rightarrow \infty} \left( \frac{\sin \frac{1}{k}}{\frac{1}{k}} \right)^2$$

$$\text{Let } x = \frac{1}{k} \rightarrow 0$$

$$= \lim_{x \rightarrow 0} \left( \frac{\sin x}{x} \right)^2$$

$$= 1$$

Therefore; by the *Limit Comparison Test*, the given series **converges**.

### Exercise

Use any method to determine if the series converges or diverges  $\sum_{k=1}^{\infty} \sin \frac{1}{k}$

### Solution

$$\text{Let } b_k = \sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges by } \mathbf{p}\text{-series } (p = 1 \leq 1)$$

$$a_k = \sin \frac{1}{k}$$

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{a_k}{b_k} &= \lim_{k \rightarrow \infty} \frac{\sin \frac{1}{k}}{\frac{1}{k}} = \frac{0}{0} && \text{Let } x = \frac{1}{k} \rightarrow 0 \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \\ &= 1 \end{aligned}$$

Therefore; by the *Limit Comparison Test*, the given series **diverges**.

### Exercise

Use any method to determine if the series converges or diverges  $\sum_{k=1}^{\infty} \frac{1}{k} \sin \frac{1}{k}$

### Solution

Let  $b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$  converges by **p-series** ( $p = 2 > 1$ )

$$a_k = \frac{1}{k} \sin \frac{1}{k}$$

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{a_k}{b_k} &= \lim_{k \rightarrow \infty} \frac{\frac{1}{k} \sin \frac{1}{k}}{\frac{1}{k^2}} \\ &= \lim_{k \rightarrow \infty} \frac{\sin \frac{1}{k}}{\frac{1}{k}} && \text{Let } x = \frac{1}{k} \rightarrow 0 \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \\ &= 1 \end{aligned}$$

Therefore; by the *Limit Comparison Test*, the given series **converges**.

### Exercise

Use any method to determine if the series converges or diverges  $\sum_{k=1}^{\infty} \frac{1}{k^2} \sin \frac{1}{k}$

### Solution

$$-1 \leq \sin \frac{1}{k} \leq 1$$

$$-\frac{1}{k^2} \leq \frac{\sin \frac{1}{k}}{k^2} \leq \frac{1}{k^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \text{ converges by } \mathbf{p}\text{-series } (p = 2 > 1)$$

Therefore; by the *Comparison Test*, the given series **converges**.

### Exercise

Use any method to determine if the series converges or diverges  $\sum_{k=1}^{\infty} (-1)^k k \sin \frac{1}{k}$

### Solution

$$-1 \leq \sin \frac{1}{k} \leq 1$$

$$-k \leq k \sin \frac{1}{k} \leq k$$

$$\lim_{k \rightarrow \infty} k = \infty$$

Therefore; by the *Comparison Test*, the given series **diverges**.

### Exercise

Use any method to determine if the series converges or diverges  $\sum_{k=1}^{\infty} \frac{1}{k^2} \sin \frac{\pi}{2k}$

### Solution

$$-1 \leq \sin \frac{\pi}{2k} \leq 1$$

$$-\frac{1}{k^2} \leq \frac{1}{k^2} \sin \frac{\pi}{2k} \leq \frac{1}{k^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \text{ converges by } \mathbf{p}\text{-series } (p = 2 > 1)$$

Therefore; by the *Comparison Test*, the given series **converges**.

### Exercise

Use any method to determine if the series converges or diverges  $\sum_{k=1}^{\infty} \tan \frac{1}{k}$

### Solution

Let  $b_k = \sum_{k=1}^{\infty} \frac{1}{k}$  diverges by **p-series** ( $p = 1 \leq 1$ )

$$a_k = \tan \frac{1}{k}$$

$$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = \lim_{k \rightarrow \infty} \frac{\tan \frac{1}{k}}{\frac{1}{k}}$$

$$= \lim_{x \rightarrow 0} \frac{\tan x}{x} \quad \text{Let } x = \frac{1}{k} \rightarrow 0$$

$$= 1$$

Therefore; by the *Limit Comparison Test*, the given series **diverges**.

### Exercise

Use any method to determine if the series converges or diverges  $\sum_{k=1}^{\infty} (-1)^k \tan^{-1} k$

### Solution

$$\lim_{k \rightarrow \infty} \tan^{-1} k = \tan^{-1} \infty$$

$$= \frac{\pi}{2} \neq 0$$

Therefore; by the *Divergence Test*, the given series **diverges**.

### Exercise

Use any method to determine if the series converges or diverges  $\sum_{n=1}^{\infty} \frac{\cos n}{n^3}$

### Solution

$$|\cos n| \leq 1$$

$$\frac{|\cos n|}{n^3} \leq \frac{1}{n^3}$$

$$\sum \frac{1}{n^3} \text{ converges by } \mathbf{p}\text{-series} \quad (p = 3 > 1)$$



Therefore; by the *Comparison Test*, the given series **converges** (absolutely)

### Exercise

Use any method to determine if the series converges or diverges

$$\sum_{k=2}^{\infty} \frac{k}{\ln k}$$

### Solution

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{k}{\ln k} &= \frac{\infty}{\infty} \\ &= \lim_{k \rightarrow \infty} \frac{1}{\frac{1}{k}} \\ &= \lim_{k \rightarrow \infty} k \\ &= \infty \end{aligned}$$

Therefore; by the *Divergence Test*, the given series **diverges**.

### Exercise

Use any method to determine if the series converges or diverges

$$\frac{1}{1+\sqrt{1}} + \frac{1}{1+\sqrt{2}} + \frac{1}{1+\sqrt{3}} + \dots$$

### Solution

$$\frac{1}{1+\sqrt{1}} + \frac{1}{1+\sqrt{2}} + \frac{1}{1+\sqrt{3}} + \dots = \sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$$

$$\sqrt{n} < n$$

$$1 + \sqrt{n} < n$$

$$\frac{1}{1+\sqrt{n}} > \frac{1}{n}$$

$$\sum \frac{1}{n} \text{ diverges by } \mathbf{p}\text{-series } (p = 1 \leq 1)$$

Therefore; by the *Comparison Test*, the given series **diverges**.