

### Exercise 1

$Ox$  and  $Oy$  are bisector to 2 adjacent acute angles,  $\widehat{AOB}$  and  $\widehat{BOC}$  where the difference is  $36^\circ$  and  $\widehat{AOC} = 90^\circ$ .  $Oz$  is the bisector of the angle  $\widehat{xOy}$ . Determine the angle  $\widehat{BOz}$

#### Solution

$$\widehat{BOC} - \widehat{AOB} = 36^\circ$$

$$\widehat{BOC} + \widehat{AOB} = 90^\circ$$

$$2 \widehat{BOC} = 126^\circ$$

$$\widehat{BOC} = 63^\circ$$

$$\widehat{AOB} = 27^\circ$$

$$\widehat{xOB} = \frac{1}{2} \widehat{AOB}$$

$$= \frac{27^\circ}{2}$$

$$\widehat{BOy} = \frac{63^\circ}{2}$$

$$\widehat{xOy} = \widehat{xOB} + \widehat{BOy}$$

$$= \frac{1}{2}(63^\circ + 27^\circ)$$

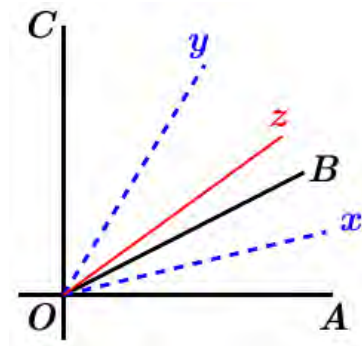
$$= 45^\circ$$

$$\widehat{xOz} = \frac{45^\circ}{2}$$

$$\widehat{BOz} = \widehat{xOz} - \widehat{xOB}$$

$$= \frac{1}{2}(45^\circ - 27^\circ)$$

$$= 9^\circ$$



## Exercise 2

$Ox$  and  $Oy$  are bisector to 2 adjacent acute angles,  $\widehat{AOB}$  and  $\widehat{BOC}$  where the difference is  $36^\circ$ .  $Oz$  is the bisector of the angle  $\widehat{xOy}$ . Determine the angle  $\widehat{BOz}$ .

### Solution

$$Ox \text{ is the bisector } \widehat{AOB} \quad (1)$$

$$Oy \text{ is the bisector } \widehat{BOC} \quad (2)$$

$$Om \text{ is the bisector } \widehat{AOC} \quad (3)$$

$$Oz \text{ is the bisector } \widehat{xOy} \quad (4)$$

$$Oy \text{ is the bisector } \widehat{BOC} \quad (5)$$

$$\widehat{BOC} - \widehat{AOB} = 36^\circ$$

$$\widehat{BOC} - \widehat{BOD} = 36^\circ$$

$$\widehat{DOC} = 36^\circ \quad |$$

$$\begin{aligned} (3) \rightarrow \widehat{AOM} &= \frac{1}{2} \widehat{AOC} \\ &= \frac{1}{2} (2\widehat{AOB} + \widehat{DOC}) \\ &= \frac{1}{2} (2\widehat{AOB} + 36^\circ) \\ &= \widehat{AOB} + 18^\circ \end{aligned}$$

$$\begin{aligned} \widehat{BOM} &= \widehat{AOM} - \widehat{AOB} \\ &= \widehat{AOB} + 18^\circ - \widehat{AOB} \\ &= 18^\circ \quad | \end{aligned}$$

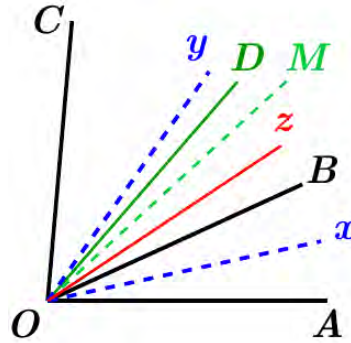
$$(1) \rightarrow \widehat{BOx} = \frac{1}{2} \widehat{AOB}$$

$$(4) \rightarrow \widehat{BOy} = \frac{1}{2} \widehat{BOC}$$

$$(1) + (4) \rightarrow \widehat{xOy} = \frac{1}{2} \widehat{AOC}$$

$$(3) \rightarrow \widehat{AOM} = \frac{1}{2} \widehat{AOC} = \widehat{xOy}$$

$$\begin{aligned} \widehat{BOz} &= \widehat{xOz} - \widehat{xOB} \\ &= \frac{1}{2} (\widehat{xOy} - \widehat{AOB}) \\ &= \frac{1}{2} (\widehat{AOM} - \widehat{AOB}) \\ &= \frac{1}{2} \widehat{BOM} \\ &= 9^\circ \quad | \end{aligned}$$



### Exercise 3

Four consecutive half-lines (segments):  $OA$ ,  $OB$ ,  $OC$ , and  $OD$  formed angles such as

$$\widehat{DOA} = \widehat{COB} = 2\widehat{AOB} \quad \text{and} \quad \widehat{COD} = 3\widehat{AOB}$$

Calculate the angles to demonstrate that the bisectors of  $\widehat{AOB}$  and  $\widehat{COD}$  are in a straight line.

#### Solution

$$\widehat{AOB} + \widehat{BOC} + \widehat{COD} + \widehat{DOA} = 360^\circ$$

$$\widehat{AOB} + 2\widehat{AOB} + 3\widehat{AOB} + 2\widehat{AOB} = 360^\circ$$

$$8\widehat{AOB} = 360^\circ$$

$$\widehat{AOB} = 45^\circ$$

$$\widehat{DOA} = \widehat{COB} = 90^\circ$$

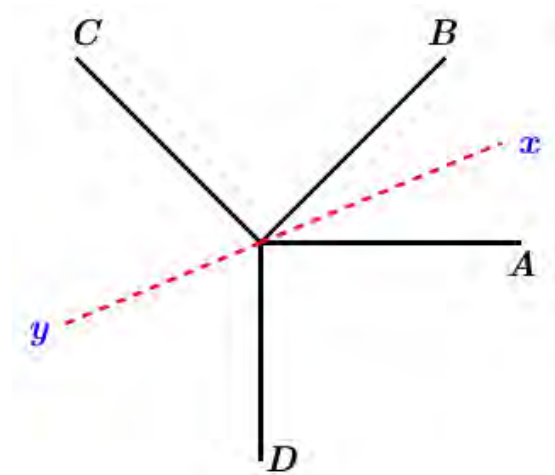
$$\widehat{COD} = 135^\circ$$

Let:

$Ox$  is the bisector  $\widehat{AOB}$

$Oy$  is the bisector  $\widehat{COD}$

$$\begin{aligned} \widehat{xOy} &= \widehat{xOB} + \widehat{BOC} + \widehat{COy} \\ &= \frac{1}{2}\widehat{AOB} + 90^\circ + \frac{1}{2}\widehat{COD} \\ &= \frac{1}{2}(45^\circ + 135^\circ) + 90^\circ \\ &= 180^\circ \end{aligned}$$



Therefore; the bisectors of  $\widehat{AOB}$  and  $\widehat{COD}$  are in a straight line

### Exercise 4

The segments  $OA$  and  $OB$  formed with  $OX$  the angles  $\alpha$  and  $\beta$ .

- a) Demonstrate that the bisector  $OC$  of the angle  $\widehat{AOB}$  made with  $OX$  an angle  $\frac{\alpha + \beta}{2}$ .
- b) Examine the cases where
  - i.  $\alpha + \beta = 90^\circ$
  - ii.  $\alpha + \beta = 180^\circ$

### Solution

**Given:**

$$\widehat{XOA} = \alpha \quad \& \quad \widehat{XOB} = \beta$$

$$\begin{aligned} \widehat{AOC} &= \frac{1}{2} \widehat{AOB} \\ &= \frac{\beta - \alpha}{2} \end{aligned}$$

$$\begin{aligned} \text{a) } \widehat{XOC} &= \widehat{XOA} + \widehat{AOC} \\ &= \alpha + \frac{\beta - \alpha}{2} \\ &= \frac{\alpha + \beta}{2} \end{aligned}$$

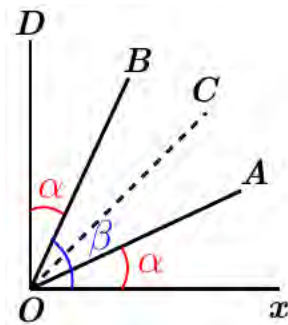
- b) **i.** If  $\alpha + \beta = 90^\circ$ , then

$$\widehat{XOC} = 45^\circ$$

Let:  $\widehat{XOD} = 90^\circ$  that implies  $OC$  is the bisector of  $\widehat{XOD}$

Since  $OC$  is the bisector of  $\widehat{AOB}$ , then

$$\begin{aligned} \widehat{BOD} &= 90^\circ - \beta & \beta &= 90^\circ - \alpha \\ &= 90^\circ - 90^\circ + \alpha \\ &= \alpha \end{aligned}$$



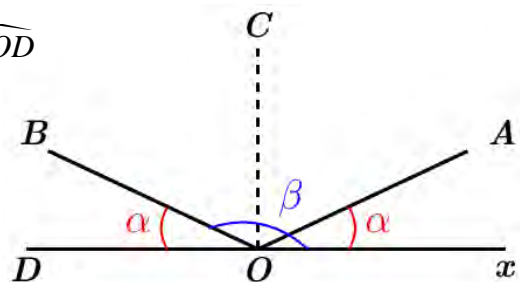
- ii.** If  $\alpha + \beta = 180^\circ$ , then

$$\widehat{XOC} = 90^\circ$$

Let:  $\widehat{XOD} = 180^\circ$  that implies  $OC$  is the bisector of  $\widehat{XOD}$

Since  $OC$  is the bisector of  $\widehat{AOB}$ , then

$$\begin{aligned} \widehat{BOD} &= 180^\circ - \beta & \beta &= 180^\circ - \alpha \\ &= 180^\circ - 180^\circ + \alpha \\ &= \alpha \end{aligned}$$



### Exercise 5

A point  $O$  takes on an infinite right  $x'Ox$  be conducted the same side half-lines  $OA$  and  $OB$ , as well as the bisectors of angles  $\widehat{xOA}$ ,  $\widehat{AOB}$ , and  $\widehat{BOx'}$ .

Calculate the angles of the figure such that the bisector of the angle  $\widehat{AOB}$  is perpendicular to  $x'Ox$  and the bisectors of the extreme angles formed an angle of  $100^\circ$ .

#### Solution

**Given:**  $\widehat{zOz'} = 100^\circ$

$$\widehat{xOC} = 90^\circ$$

$OC$  is the bisector  $\widehat{AOB}$

$$\widehat{AOC} = \widehat{COB}$$

$Oz$  is the bisector  $\widehat{xOA}$

$$\widehat{xOz} = \widehat{zOA}$$

$Oz'$  is the bisector  $\widehat{x'OB}$

$$\widehat{x'Oz'} = \widehat{z'OB}$$

$$\widehat{xOz} = \frac{180^\circ - 100^\circ}{2}$$

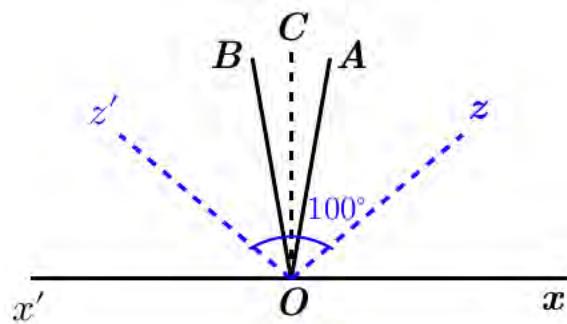
$$= 40^\circ$$

$$\widehat{AOB} = 2\widehat{AOC}$$

$$= 2(90^\circ - 2\widehat{xOz})$$

$$= 2(90^\circ - 80^\circ)$$

$$= 20^\circ$$



### Exercise 6

Four consecutive half-lines  $OA$ ,  $OB$ ,  $OC$ , and  $OD$  formed four adjacent consecutive angles which are between them like 1, 2, 3, 4.

Calculate the angles and the adjacent consecutive angles formed by their bisectors.

#### Solution

$$\widehat{AOB} + \widehat{BOC} + \widehat{COD} + \widehat{DOA} = 360^\circ$$

$$\widehat{AOB} + 2\widehat{AOB} + 3\widehat{AOB} + 4\widehat{AOB} = 360^\circ$$

$$10\widehat{AOB} = 360^\circ$$

$$\widehat{AOB} = 36^\circ$$

$$\widehat{BOC} = 72^\circ$$

$$\widehat{COD} = 108^\circ$$

$$\widehat{DOA} = 144^\circ$$

$$\widehat{xOy} = \frac{1}{2}\widehat{AOB} + \frac{1}{2}\widehat{BOC}$$

$$= \frac{1}{2}36^\circ + \frac{1}{2}72^\circ$$

$$= 18^\circ + 36^\circ$$

$$= 54^\circ$$

$$\widehat{yOz} = \frac{1}{2}\widehat{BOC} + \frac{1}{2}\widehat{COD}$$

$$= \frac{1}{2}72^\circ + \frac{1}{2}108^\circ$$

$$= 36^\circ + 54^\circ$$

$$= 90^\circ$$

$$\widehat{zOw} = \frac{1}{2}\widehat{COD} + \frac{1}{2}\widehat{DOA}$$

$$= \frac{1}{2}108^\circ + \frac{1}{2}144^\circ$$

$$= 54^\circ + 72^\circ$$

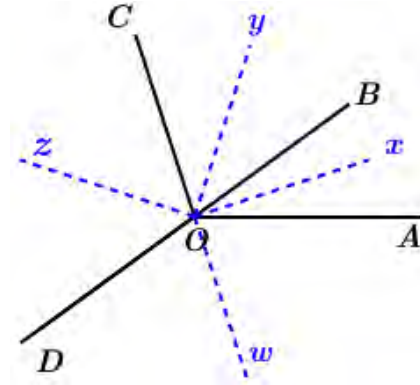
$$= 126^\circ$$

$$\widehat{wOx} = \frac{1}{2}\widehat{DOA} + \frac{1}{2}\widehat{AOB}$$

$$= \frac{1}{2}144^\circ + \frac{1}{2}36^\circ$$

$$= 72^\circ + 18^\circ$$

$$= 90^\circ$$



### Exercise 7

A point  $P$  is on the base  $BC$  of an isosceles triangle  $ABC$ . The two points  $M$  and  $N$  are the middle points of the segments  $BP$  and  $PC$ , respectively, which lead the perpendicular to the base  $BC$ ; these perpendiculars meet  $AB$  in  $E$ ,  $AC$  in  $F$ .

Demonstrate that the angle  $EPF$  is equal to  $A$ .

#### Solution

$$\widehat{BAC} = 180^\circ - \widehat{ABC} - \widehat{ACB}$$

$M$  is the middle of the segment  $BP$  and  $EM \perp$  to  $BP$ , therefore

$$EB = EP \quad \& \quad \widehat{EBP} = \widehat{EPB}$$

$N$  is the middle of the segment  $CP$  and  $FN \perp$  to  $CP$ , therefore

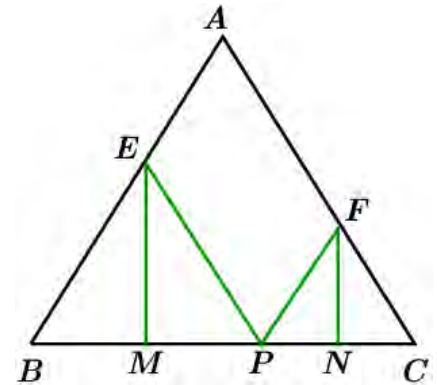
$$FP = FN \quad \& \quad \widehat{FPC} = \widehat{FCP}$$

$$\widehat{EPF} = 180^\circ - \widehat{CPF} - \widehat{BPE}$$

$$= 180^\circ - \widehat{PFC} - \widehat{PBE}$$

$$= 180^\circ - \widehat{ABC} - \widehat{ACB}$$

$$= \widehat{A} \quad \checkmark$$



### Exercise 8

Given the triangle  $ABC$  and the bisectors  $BO$  and  $CO$  of the angles of the base, where the point  $O$  is the intersection of the 2 bisectors. A line  $DOE$  passes through the point  $O$  parallel to base  $BC$ .

Prove that  $DE = DB + CE$

#### Solution

$CO$  is the bisector of  $\widehat{BCE} \Rightarrow \widehat{BCO} = \widehat{OCE}$

$OE \parallel BC \Rightarrow \widehat{COE} = \widehat{BOC}$

$\therefore \widehat{EOC} = \widehat{OCE} \rightarrow \underline{OE = EC}$

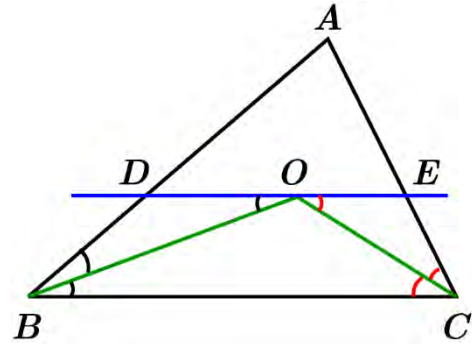
Similar;  $BO$  is the bisector of  $\widehat{DBC} \Rightarrow \widehat{DBO} = \widehat{OBC}$

$DO \parallel BC \Rightarrow \widehat{DOB} = \widehat{OBC}$

$\therefore \widehat{DOB} = \widehat{OBC} \rightarrow \underline{DO = DB}$

$DE = DO + OE$

$\underline{= DB + CE}$





## Exercise 9

A right triangle  $ABC$  at  $A$  with a height  $AH$ . We drop perpendiculars  $HE$  and  $HD$  from  $H$  to sides  $AB$  and  $AC$  respectively.

- Prove that  $DE = AH$
- Prove that  $AM$  is perpendicular to  $DE$ , where  $M$  is the middle point of  $BC$ .
- Prove that  $MN$  ( $N$  is the middle point of  $AB$ ) and the segment  $Bx$  (parallel to  $DE$ ) are intersect on  $AH$ .
- Prove that  $AM$  and  $HD$  are intersect on  $Bx$ .

### Solution

- The triangles  $AEH$  and  $ADH$  are right triangles and angle  $A$  is right angle.  
Then  $AEHD$  is a rectangle.  
Therefore,  $DE = AH$

- A middle point of a hypotenuse of a right triangle is the center of the circle of that triangle.  
Therefore,  $MC = MA = MB$

That implies to:  $\widehat{MAC} = \widehat{MCA}$

From the rectangle  $ADHE$ :  $\widehat{EAH} = \widehat{EDH}$

$$\widehat{EAH} + \widehat{HAM} + \widehat{MAC} = 90^\circ$$

$$\widehat{HAM} + \widehat{MAC} = \widehat{HAC}$$

$$\widehat{EAH} + \widehat{HAC} = 90^\circ$$

$$\widehat{EAH} + 90^\circ - \widehat{MCA} = 90^\circ$$

$$\widehat{EAH} = \widehat{MCA} = \widehat{EDH} = \widehat{MAC}$$

$$\widehat{ADE} + \widehat{EDH} = 90^\circ$$

$$\widehat{ADE} + \widehat{MAD} = 90^\circ$$

Therefore,  $AM$  is perpendicular to  $DE$ .

- $N$  is the middle point of  $AB \Rightarrow NA = NB$

$$Bx \text{ parallel to } DE \Rightarrow \widehat{ABx} = \widehat{AED} = \widehat{EDH} = \widehat{EAH}$$

Let point  $P$  the intersection of  $Bx$  and  $AH$ . Since  $\widehat{ABP} = \widehat{BAP}$ , then the triangle  $BPA$  is isosceles.  
 $PN$  is the perpendicular to  $AB$  as well  $MN$ . Which gives us that points  $M, P, N$  are on the same line.

Therefore, segment  $MN$  and  $AH$  intersect at point  $P$ .

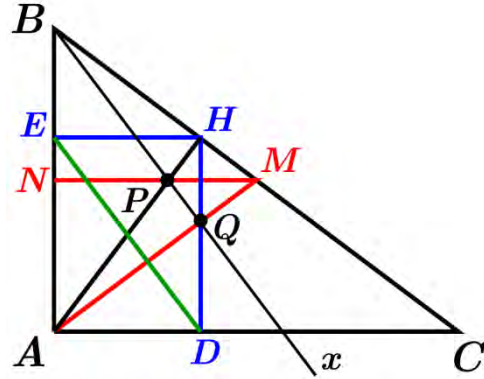
- Let Point  $Q$  be the intersection of  $AM$  and  $Bx$ .

$$\widehat{ABQ} = \widehat{BAH} \quad \& \quad \widehat{BAQ} = \widehat{ABH}$$

Then, the triangles  $BHA$  and  $BQA$  are equivalent, therefore  $AQ \perp BQ$  with hypotenuse  $AB$ .

$HQ \parallel AB$ , line  $HQ$  has to be perpendicular to  $AC$ .

$AM$  and  $HD$  are intersecting on  $Bx$  at  $Q$ .



## Exercise 10

Given an isosceles triangle  $ABC$  with a peak at  $A$ . Extend base  $BC$  the length  $CD = AB$ , then extend  $AB$  of a length  $BE = \frac{1}{2}BC$ , at the end draw a line  $EHF$ ,  $H$  is the middle point of  $BC$  and  $F$  is located on  $AD$ .

- Prove that  $\widehat{ADB} = \frac{1}{2}\widehat{ABC}$
- Prove that  $EA = HD$
- Prove that  $FA = FD = FH$
- Calculate the value of the angles  $\widehat{AFH}$  and  $\widehat{ADB}$  where  $\widehat{BAC} = 58^\circ$ .

### Solution

- Triangle  $ABC$  is isosceles, then  $\widehat{ABC} = \widehat{ACB}$

Since  $CD = AB = AC$ , then  $\widehat{CAD} = \widehat{ADC}$

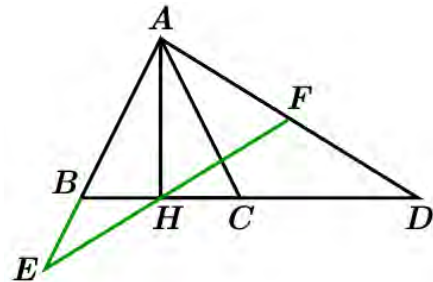
$$2\widehat{ADC} = 180^\circ - \widehat{ACD}$$

$$2\widehat{ADC} = 180^\circ - (180^\circ - \widehat{ACB})$$

$$2\widehat{ADC} = \widehat{ACB}$$

$$\widehat{ADB} = \frac{1}{2}\widehat{ACB}$$

$$= \frac{1}{2}\widehat{ABC}$$



- $BE = \frac{1}{2}BC$   $H$  the middle point of  $BC$   
 $= HC$

$$CD = AB$$

$$HC + CD = BE + AB$$

$$EA = HD \quad \checkmark$$

- $\widehat{ADH} = \frac{1}{2}\widehat{ABD}$   
 $= \frac{1}{2}(180^\circ - \widehat{HBE})$   
 $= \frac{1}{2}(180^\circ - 180^\circ + 2\widehat{BHE})$   
 $= \widehat{BHE}$

$$\Rightarrow \underline{FD = FH}$$

$$\widehat{AHF} = 90^\circ - \widehat{FHD}$$

$$= 90^\circ - \widehat{ADH} \quad (\triangle HDA)$$

$$= 90^\circ - (90^\circ - \widehat{HAF})$$

$$= \widehat{HAF}$$

$$\Rightarrow \underline{FA = FH}$$

$$FA = FD = FH \quad \checkmark$$

$$d) \widehat{BAC} = 58^\circ$$

$$\begin{aligned} \widehat{ADB} &= \frac{1}{2} \widehat{ACB} \\ &= \frac{1}{2} \left( \frac{1}{2} (180^\circ - \widehat{BAC}) \right) \\ &= \frac{1}{4} (180^\circ - 58^\circ) \\ &= \frac{122^\circ}{4} \\ &= \frac{61^\circ}{2} \quad \Bigg| \quad \underline{= 30.5^\circ} \end{aligned}$$

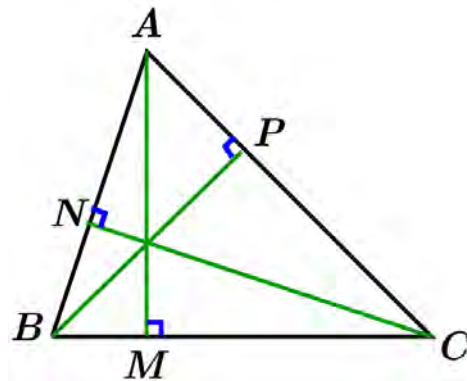
Triangle  $AFH$  is isosceles then,

$$\begin{aligned} \widehat{AFH} &= 180^\circ - \widehat{HFD} \\ &= 180^\circ - (180^\circ - 2\widehat{FDH}) \\ &= 2\widehat{FDH} \\ &= 2 \frac{61^\circ}{2} \\ &= 61^\circ \quad \Bigg| \end{aligned}$$

### Exercise 11

Demonstrate that the heights of a triangle share the angles of triangle that equal to each other.

#### Solution



Consider the 2 right triangles  $APB$  and  $ANC$ , which they have the same angle  $A$ .

Therefore,  $\widehat{ABP} = \widehat{ACN}$ .

Similar, consider the 2 right triangles  $BPC$  and  $AMC$ , which they have the same angle  $C$ .

Therefore,  $\widehat{MAC} = \widehat{CBP}$ .

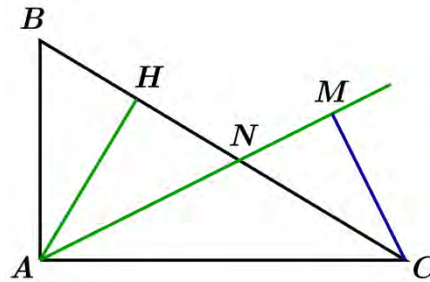
Similar, consider the 2 right triangles  $BNC$  and  $AMB$ , which they have the same angle  $B$ .

Therefore,  $\widehat{BCN} = \widehat{BAM}$ .

## Exercise 12

A right triangle  $ABC$  at  $A$  where  $AB < AC$ , drop a perpendicular  $AH$  from  $A$  to the hypotenuse  $BC$  where  $HN = HB$ . From  $C$  drops a perpendicular  $CM$  at  $AN$ . Demonstrate that  $BC$  is the bisector of the angle  $\widehat{ACM}$ .

### Solution



Consider the 2 right triangles  $ABC$  and  $ABH$  with a common angle  $B$ , then

$$\widehat{BAH} = \widehat{ACB}$$

Given:  $HN = HB$ , then  $\widehat{HAN} = \widehat{BAH} = \widehat{ACB}$

$$\begin{aligned}\widehat{NAC} &= 90^\circ - \widehat{HAB} - \widehat{HAN} \\ &= 90^\circ - 2\widehat{HCA}\end{aligned}$$

Consider the 2 right triangles  $AHN$  and  $CMC$ , where  $\widehat{HNA} = \widehat{MNC}$

Therefore,  $\widehat{HAN} = \widehat{NCM}$

Since  $\widehat{HAN} = \widehat{ACB}$

Then  $\widehat{ACB} = \widehat{MCB}$

Therefore,  $BC$  is the bisector of the angle  $\widehat{ACM}$

### Exercise 13

On the sides of an angle that it takes the length  $OA$  and  $OB$ , so that  $OA + OB = \ell$  (is given) and construct a parallelogram  $OABC$ . What is the place of the summit  $C$  of parallelogram?

#### Solution

Let segment  $OE$  extension of segment  $OA$  such that  $OE = \ell$

Let segment  $OF$  extension of segment  $OB$  such that  $OF = \ell$

Then, the triangle  $OEF$  is isosceles.

$$\widehat{OEF} = \widehat{OFE} = 90^\circ - \frac{1}{2}\widehat{EOF}$$

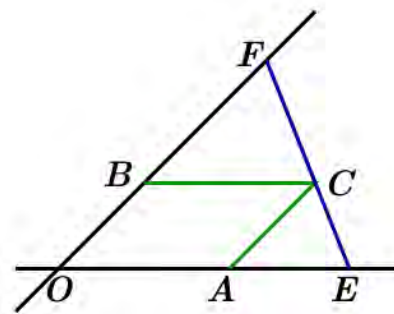
$$OA + OB = \ell$$

$$\begin{cases} OA + AC = \ell \\ OA + AE = \ell \end{cases} \Rightarrow AC = AE$$

$$\begin{cases} OB + BC = \ell \\ OB + BF = \ell \end{cases} \Rightarrow BC = BF$$

$$\widehat{OEF} = \widehat{OFE} = \widehat{FCB} = \widehat{ACE}$$

Therefore, the point  $C$ ,  $E$ , and  $F$  are aligned.



### Exercise 14

Demonstrate that the sum of distances from a point  $M$  on the base  $BC$  of an isosceles triangle  $ABC$  to the sides equal a constant.

**Solution**

The shortest distance from a point to a line is the perpendicular from that point to the line.

Therefore, let:

$$MP \perp AB$$

$$MQ \perp AC$$

Let  $BH \perp AC$  (Shortest distance from  $B$  to side  $AC$ .)

Let  $D$  be the point of intersection  $ME$  and  $BH$ .

Let  $ME \parallel AC$

Where the point  $E$  is the intersection of the lines  $MD$  and  $AB$ .

Since  $MD \parallel AC$  then  $\widehat{DMB} = \widehat{ACB}$

Since triangle  $ABC$  is an isosceles triangle.

$$\widehat{DMB} = \widehat{ACB} = \widehat{PBM}$$

The right triangles  $BPM$  and  $BDM$  at  $P$  &  $D$  and have the same hypotenuse, then

$$\Rightarrow |MP| = |BD|$$

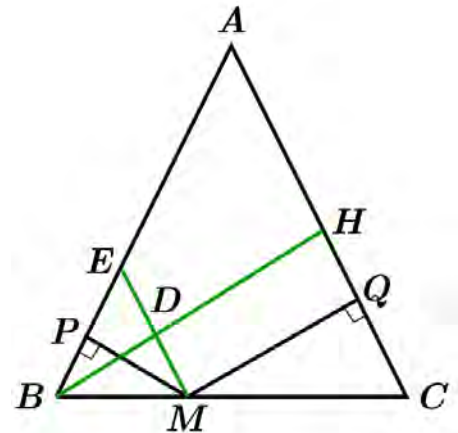
$$MD \parallel HQ \text{ and } DH \text{ \& } MQ \perp HQ$$

$$\Rightarrow |MQ| = |DH|$$

$$|MP| + |MQ| = |BD| + |DH|$$

$$= |BH|$$

$$= \text{constant}$$



Therefore, the sum of distances from a point  $M$  on the base  $BC$  of an isosceles triangle  $ABC$  to the sides equal a constant.

## Exercise 15

Demonstrate that the difference of distances from a point  $M$  taken on the extension of the base  $BC$  of an isosceles triangle  $ABC$  to the sides equal a constant.

### Solution

The shortest distance from a point to a line is the perpendicular from that point to the line.

Therefore, let:

$$MP \perp AB$$

$$MQ \perp AC$$

Let  $BH \perp AC$  (Shortest distance from  $B$  to side  $AC$ .)

Let  $D$  be the point of intersection  $ME$  and  $BH$ .

Let  $ME \parallel AC$

Where the point  $E$  is the intersection of the extensions of the lines  $MD$  and  $AB$ .

Since  $MD \parallel AC$  then  $\widehat{DMB} = \widehat{ACB}$

Since triangle  $ABC$  is an isosceles triangle.

$$\widehat{DMB} = \widehat{ACB} = \widehat{PBM}$$

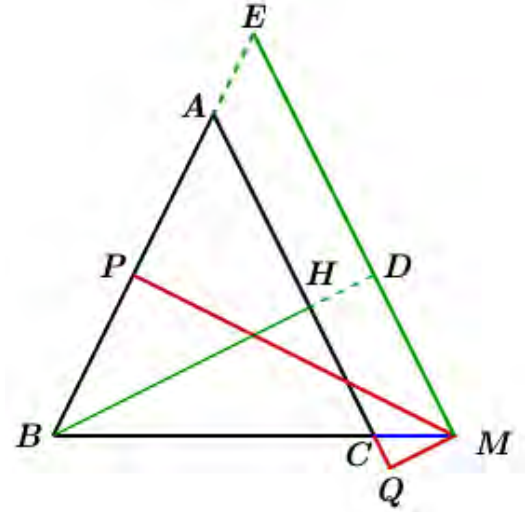
The right triangles  $BPM$  and  $BDM$  at  $P$  &  $D$  and have the same hypotenuse, then

$$\Rightarrow |MP| = |BD|$$

$$MD \parallel HQ \text{ and } DH \text{ \& } MQ \perp HQ$$

$$\Rightarrow |MQ| = |DH|$$

$$\begin{aligned} |MP| - |MQ| &= |BD| - |DH| \\ &= |BH| \\ &= \text{constant} \end{aligned}$$



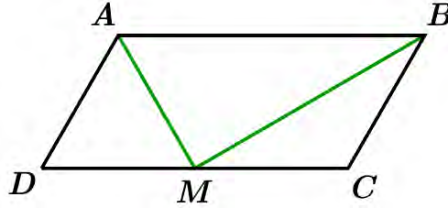
Therefore, the difference of distances from a point  $M$  taken on the extension of the base  $BC$  of an isosceles triangle  $ABC$  to the sides equal a constant.



### Exercise 16

Consider a parallelogram  $ABCD$  in which  $CD = 2AD$ . In the joint  $A$  and  $B$  the middle  $M$  of  $BC$ .  
Prove that the angle  $\widehat{AMB}$  is a right angle.

#### Solution



Since the point  $M$  is the middle of side  $BC$ , then

$$MD = MC = \frac{1}{2}CD$$

$$\Rightarrow MD = AD = BC$$

Therefore, the triangles  $ADM$  and  $BCM$  are isosceles at  $D$  and  $C$  respectively.

Which implies that  $MA = MB$

Let  $O$  be the middle point of the side  $AB$ , and  $OA = OB = AD$

$O$  and  $M$  are middle of the parallelogram  $ABCD$ , that implies

$$OM = BC = AD$$

$$\Rightarrow OA = OB = OM$$

The triangle  $MAB$  inscribed in a circle with center at  $O$  and diameter  $AB$ , that will imply that is a right triangle at the point  $M$ .

**Or**

$$\widehat{AMD} = \frac{1}{2}(180^\circ - \widehat{MDA})$$

$$\widehat{BMC} = \frac{1}{2}(180^\circ - \widehat{MCB})$$

$$\widehat{ADM} + \widehat{MCB} = 180^\circ$$

$$\widehat{DMA} + \widehat{AMB} + \widehat{BMC} = 180^\circ$$

$$\widehat{AMB} = 180^\circ - (\widehat{BMC} + \widehat{DMA})$$

$$= 180^\circ - \left(90^\circ - \frac{1}{2}\widehat{MDA} + 90^\circ - \frac{1}{2}\widehat{MCB}\right)$$

$$= \frac{1}{2}(\widehat{MDA} + \widehat{MCB})$$

$$= \frac{1}{2}(180^\circ)$$

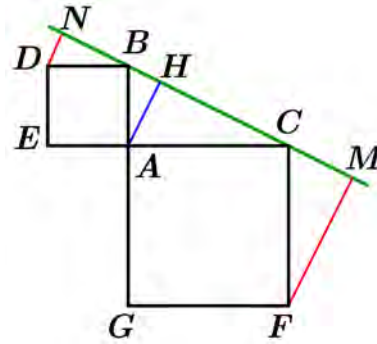
$$= 90^\circ \quad |$$

### Exercise 17

From the sides  $AB$  and  $AC$  of a right triangle  $ABC$  at  $A$ , draw two squares  $ABDE$  and  $ACFG$ . Then lead  $DN$  and  $FM$  perpendicular to the line  $BC$ .

- Prove that  $DN + FM = BC$
- Prove that the points  $D, A, F$  on a straight line.
- Prove that the lines  $DE$  and  $FG$  contribute on the extension of the height  $AH$ .

### Solution



- a) Let consider the 2 right triangles  $DNB$  &  $BHA$  at points  $N$  &  $H$  respectively, with  $DB = AB$ . Then

$$\begin{aligned}\widehat{HAB} &= 90^\circ - \widehat{ABH} \\ &= 90^\circ - (90^\circ - \widehat{NBD}) \\ &= \widehat{NBD} \\ \Rightarrow \widehat{BDN} &= \widehat{ABH}\end{aligned}$$

$\therefore$  The 2 triangles are equals, which implies that  $\underline{DN = BH}$

Similar, for the 2 right triangles  $CMF$  &  $AHC$  at points  $M$  &  $H$  respectively, with  $AC = CF$ . Then

$$\begin{aligned}\widehat{HAC} &= 90^\circ - \widehat{ACH} \\ &= 90^\circ - (90^\circ - \widehat{MCF}) \\ &= \widehat{MCF} \\ \Rightarrow \widehat{ACH} &= \widehat{CFM}\end{aligned}$$

$\therefore$  The 2 triangles are equals, which implies that  $\underline{FM = HC}$

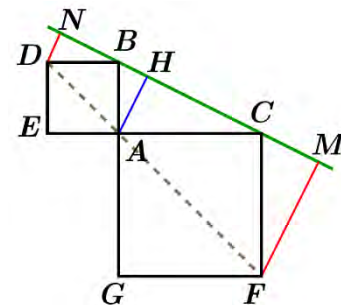
$$\begin{aligned}DN + FM &= BH + HC \\ &= \underline{BC} \quad \checkmark\end{aligned}$$

- b) Since  $ABDE$  is a square, then  $\widehat{BAD} = 45^\circ$

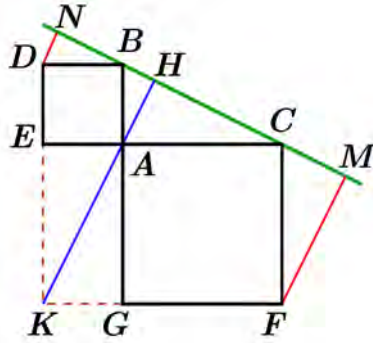
And  $ACFG$  is a square, then  $\widehat{CAF} = 45^\circ$

$$\begin{aligned}\widehat{DAF} &= \widehat{DAB} + \widehat{BAC} + \widehat{CAF} \\ &= 45^\circ + 90^\circ + 45^\circ \\ &= \underline{180^\circ}\end{aligned}$$

$\therefore$  The points  $D, A,$  &  $F$  are on a straight line.



- c) Let the point  $K$  be the intersection of the extension of the sides  $DE$  and  $FG$ .  
Which will result of  $GKEA$  is a rectangle with  $AE = GK$  &  $EK = AG$



Consider the 2 right triangles  $BAC$  &  $KGA$  at points  $A$  &  $G$  respectively with  $AE = AB = GK$   
 $\widehat{ACB} = \widehat{GAK} = \widehat{ACH}$

From the right triangle  $AHC$ :

$$\widehat{HAC} + \widehat{ACH} = 90^\circ$$

$$\rightarrow \widehat{HAC} + \widehat{KAG} = 90^\circ$$

$$\begin{aligned} \widehat{HAC} + \widehat{CAG} + \widehat{KAG} &= (\widehat{HAC} + \widehat{KAG}) + \widehat{CAG} \\ &= 90^\circ + 90^\circ \\ &= 180^\circ \end{aligned}$$

$\therefore$  The points  $K$ ,  $A$ , &  $H$  are on a straight line.

### Exercise 18

Given a diamond  $ABCD$ ; the peak  $B$  and  $D$ , the same the perpendiculars  $BM, BN, DP, DQ$  on opposite sides. These perpendiculars are intersected at  $E$  and  $F$ .

Demonstrate that the angles of the quadrilateral  $BFDE$  are equals to the diamond and which is a diamond itself.

#### Solution

From the right triangles  $BPD$  &  $BMD$ , that implies  $\widehat{MBD} = \widehat{PDB}$

$$\Rightarrow \widehat{EBD} = \widehat{EDB}$$

Similar, from the right triangles  $BND$  &  $BQD$ , that implies  $\widehat{NBD} = \widehat{QDB}$

$$\Rightarrow \widehat{FBD} = \widehat{FDB}$$

$$\widehat{EBD} + \widehat{DBF} = \widehat{EDB} + \widehat{BDF}$$

$$\widehat{EBF} = \widehat{EDF}$$

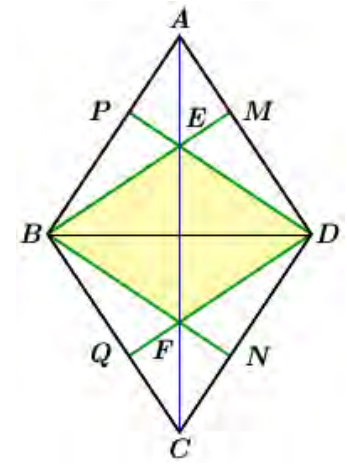
Since,  $AC \perp BD$ , then  $EF \perp BD$

The 2 triangles  $EBF$  &  $EDF$  have  $EF$  as a common side and  $\widehat{EBF} = \widehat{EDF}$ , then

$$\widehat{BEF} = \widehat{DEF} = \widehat{BFE} = \widehat{DFE}$$

$$\widehat{BED} = \widehat{BFD}$$

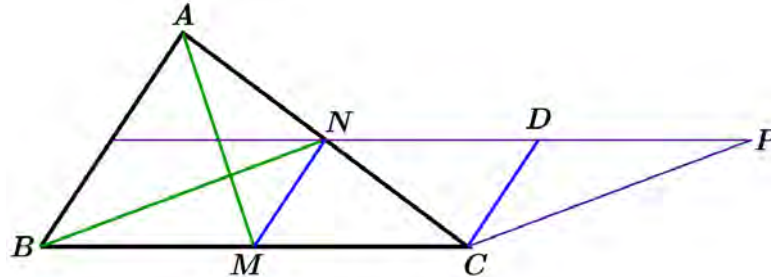
Therefore, the angles of the quadrilateral  $BFDE$  are equals to the diamond and which is a diamond itself.



### Exercise 19

In a triangle  $ABC$ , we trace the median  $AM$  and  $BN$  and from  $N$  a parallel to  $BC$ , from  $C$  a parallel to  $BN$ ; that the two sides intersect at a point  $P$ . Let  $D$  be the middle point of the segment  $PN$ . Prove that  $CD$  is parallel to  $MN$ .

#### Solution



Since the points  $M$  &  $N$  are middle of the sides  $BC$  &  $AC$  of the triangle  $ABC$ , then  
 $MN \parallel AB$

Given:  $NP \parallel MC$

$BN \parallel CP$

Since  $M$  &  $D$  are the middle points of the segments  $BC$  and  $NP$  respectively, then  
 $BN \parallel CP \parallel MD$

Therefore,  $BNPC$  is a parallelogram, and  $MC = ND$ .

Since  $MC = ND$  &  $MN = CD$

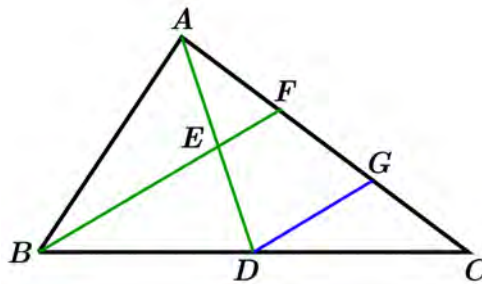
Therefore,  $MCDN$  is a parallelogram which implies to  $CD$  parallel to  $MN$

### Exercise 20

The median  $AD$  of a given triangle  $ABC$  to the side  $BC$ . The same the median  $BE$  to the side  $AC$  which intersect  $AC$  at a point  $F$ .

Prove that where  $AF = \frac{1}{3} AC$

#### Solution



Let  $DG$  be parallel to segment  $BEF$ .

**Given:**  $E$  is the middle point of the segment  $AD \Rightarrow AE = ED$

$D$  is the middle point of the segment  $BC \Rightarrow BD = DC$

Since  $EF \parallel DG$ , and  $AE = ED$ , that implies  $AF = FG$

Consider the triangles  $CDG$  and  $CBF$ :

$EF \parallel DG$ , and  $CD = DB$ , that implies  $GC = FG$

That will imply to:  $AF = FG = GC$

$$AC = AF + FG + GC$$

$$= 3AF$$

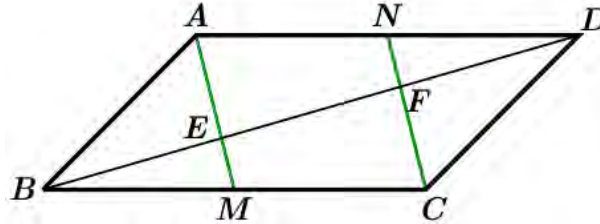
Therefore;  $\underline{AF = \frac{1}{3} AC}$

### Exercise 21

In a parallelogram  $ABCD$ , from the points peak  $A$  and  $C$  joint the middle of opposite sides at  $M$  and  $N$  respectively.

Prove that the diagonal  $BD$  is divided in three equal parts.

#### Solution



$M$  is the middle point of the segment  $BC$ , then  $BM = CM$

$N$  is the middle point of the segment  $AD$ , then  $NA = ND$

From these, implies that  $AM \parallel CN$ .

From the triangles  $BEM$  &  $BCF$ , and since  $ME \parallel CF$

It will give us that  $BE = EF$

From the triangles  $DFN$  &  $DEA$ , and since  $AE \parallel FN$

It will give us that  $\Rightarrow DF = EF$

Therefore,  $BE = EF = DF$

$$BD = BE + EF + FD$$

$$= 3BE$$

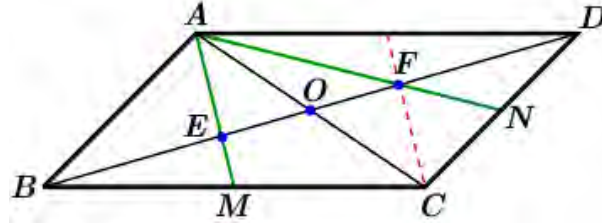
Therefore, the diagonal  $BD$  is divided in three equal parts

### Exercise 22

In a parallelogram  $ABCD$ , from the point peak  $A$ , extend to the middle of sides  $BC$  and  $DC$  at  $M$  and  $N$  respectively.

Prove that the diagonal  $BD$  is divided in three equal parts.

#### Solution



Let a point  $E$  be the intersection of the segments  $AM$  &  $BD$ .

Let a point  $F$  be the intersection of the segments  $AN$  &  $BD$ .

Let  $O$  be the intersection of both diagonal  $AC$  &  $BD$ .

From the triangles  $BEM$  &  $BCF$ , and since  $ME \parallel CF$

$$\Rightarrow BE = EF$$

Similar,  $\Rightarrow DF = EF$

$$BO = OF \rightarrow OE = OF$$

$$BO = BE + EO$$

$$= BE + \frac{1}{2}BE$$

$$= \frac{3}{2}BE$$

$$BE = \frac{2}{3}BO$$

$$= \frac{2}{3}\left(\frac{1}{2}BD\right)$$

$$= \frac{1}{3}BD$$

$$DF = \frac{2}{3}DO$$

$$= \frac{2}{3}\left(\frac{1}{2}BD\right)$$

$$= \frac{1}{3}BD$$

Therefore, the diagonal  $BD$  is divided in three equal parts



### Exercise 23

Consider a triangle  $ABC$  with a bisector  $AF$  of the angle  $A$ . by  $F$ , we lead  $FE$  parallel to  $AB$ , and by  $E$  we lead  $ED$  parallel to  $BC$ .

Prove that  $AE = BD$

#### Solution

**Given:**  $\widehat{EAF} = \widehat{FAB}$

Since  $FE \parallel AB$ , then

$$\widehat{FEC} = \widehat{BAE} = 2\widehat{EAF}$$

$$\begin{aligned}\widehat{AEF} &= 180^\circ - \widehat{FEC} \\ &= 180^\circ - 2\widehat{EAF}\end{aligned}$$

Consider the triangle  $AEF$ :

$$\begin{aligned}\widehat{EAF} + \widehat{EFA} + \widehat{AEF} &= 180^\circ \\ \widehat{EAF} + \widehat{EFA} + 180^\circ - 2\widehat{EAF} &= 180^\circ \\ \widehat{EFA} - \widehat{EAF} &= 0^\circ \\ \widehat{EFA} &= \widehat{EAF}\end{aligned}$$

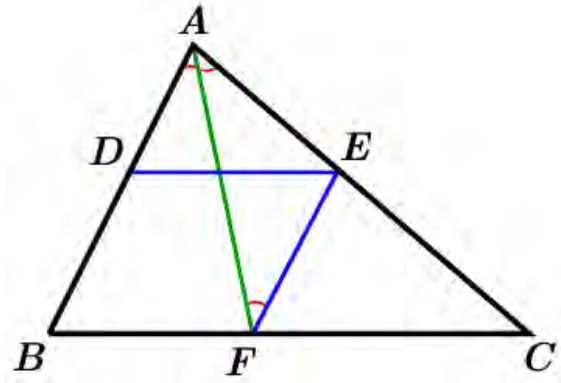
$\therefore$  Triangle  $AEF$  is isosceles

$$\Rightarrow \underline{AE = EF}$$

**Given**  $DE \parallel BF$  &  $FE \parallel DB$

$FEDB$  is a parallelogram.

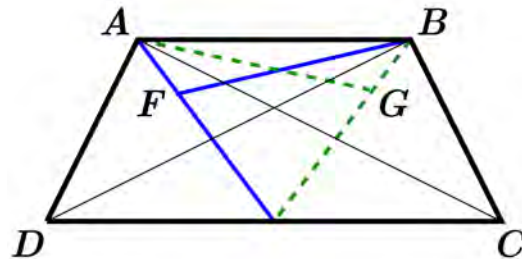
Then,  $\underline{EF = DB = AE}$



### Exercise 24

Given an isosceles trapezoid  $ABCD$  ( $AD = BC$ ) with diagonals  $AC$  and  $BD$ . The bisector of angles  $\widehat{DAB}$  and  $\widehat{DBA}$  intersect in  $F$ , and the bisector of angles  $\widehat{CBA}$  and  $\widehat{CAB}$  intersect in  $G$ . Demonstrate that  $FG$  is parallel to  $AB$ .

### Solution



Consider the 2 triangles  $ABD$  &  $ABC$ :

- Both has the  $AB$  as common
- $AD = BC$

That implies to:  $\widehat{ABD} = \widehat{CAB}$

Since  $BF$  is the bisector of the angle  $\widehat{ABD}$

$$\widehat{ABF} = \widehat{FBD}$$

$$\begin{aligned} \Rightarrow \widehat{ABF} &= \frac{1}{2} \widehat{ABD} \\ &= \frac{1}{2} \widehat{CAB} \\ &= \frac{1}{2} (2\widehat{BAG}) \\ &= \widehat{BAG} \end{aligned}$$

$$\widehat{ABF} = \widehat{BAG} \quad |$$

From the 2 triangles  $AFB$  &  $AGB$

- Both has the  $AB$  as common
- $\widehat{ABF} = \widehat{BAG}$

$$\underline{FG \parallel AB} \quad |$$

### Exercise 25

Let  $M$  and  $N$  be the middle points of the bases  $AB$  and  $CD$  of a trapezoid  $ABCD$ .

Let  $P$  and  $Q$  be the middle points of the diagonals  $AC$  and  $BD$  respectively.

Demonstrate that the angles  $\widehat{M}$  and  $\widehat{N}$  of quadrilateral  $MNPQ$  are equals to the angle formed by extending the sides not parallel to  $BC$  and  $AD$ , where intersect at point  $E$ .

#### Solution

Since  $N$  is the mid-point of the side  $DC$ , and  
 $P$  is the mid-point of the side  $AC$ , then

$$\Rightarrow NP \parallel AD$$

Since  $M$  is the mid-point of the side  $AB$ , and  
 $Q$  is the mid-point of the side  $DB$ , then

$$\Rightarrow QM \parallel AD$$

$$\therefore NP \parallel QM \parallel AE$$

Since  $N$  is the mid-point of the side  $DC$ , and  
 $Q$  is the mid-point of the side  $DB$ , then

$$\Rightarrow NQ \parallel CB$$

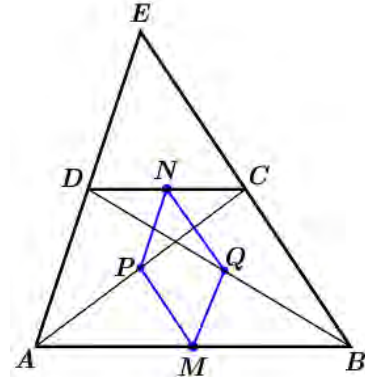
Since  $M$  is the mid-point of the side  $AB$ , and  
 $P$  is the mid-point of the side  $AC$ , then

$$\Rightarrow MP \parallel CB$$

$$\therefore NQ \parallel MP \parallel BE$$

$$\rightarrow \begin{cases} NP \parallel QM \\ NQ \parallel PM \end{cases}$$

$$\therefore \underline{\widehat{PMQ} = \widehat{PNQ}}$$



## Exercise 26

In a triangle  $ABC$ , the medians segment  $BM$  and  $CN$  intersect in right angles and the measurement are 3 and 6 units respectively.

1. Construct a geometrical to the triangle  $ABC$ .
2. In the trace of third median  $AP$  which leads  $MN$  extension such the distance  $MD = MN$ , which lead to the segments  $AD$  and  $PD$ . Calculate  $AD$  and  $DP$ .
3. What is the nature of the triangle  $APD$  ?

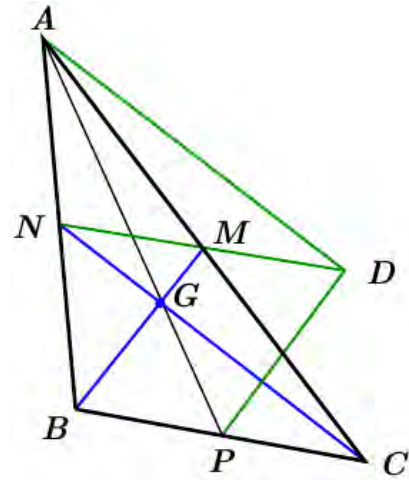
### Solution

1. Since  $M$  and  $N$  are the middle point of the side's  $AC$  &  $AB$ , then

$$\begin{aligned} BG &= \frac{2}{3} BM \\ &= \frac{2}{3}(3) \\ &= 2 \text{ units} \end{aligned}$$

Similar,

$$\begin{aligned} CG &= \frac{2}{3} CN \\ &= \frac{2}{3}(6) \\ &= 4 \text{ units} \end{aligned}$$



Wish, we lead to:  $GM = 1$  &  $GN = 2$

We can construct 2 perpendicular lines intersect at a point  $G$ , then we use to measure the distance from the point  $G$  to get the points  $B$ ,  $C$ ,  $M$ , &  $N$ .

By extending the segment  $BN$  and  $CM$  with equal distance and which it will intersect at point  $A$ .

2. Since  $ND \parallel BC$  &  $MD = MN$

The parallelogram  $BPDM$ ,  $BP = MD = MN$

Then  $PD = MB = 3 \text{ units}$

$AD \parallel CN$  and  $M$  is the intersection of the diagonals of the parallelogram  $ADCN$ , then

$$AD = CN = 6 \text{ units}$$

3.  $PD \parallel BN$  &  $MB \perp CN$ , then  $PD \perp CN$

$AD \parallel CN$  &  $PD \perp CN$ , then  $AD \perp PD$

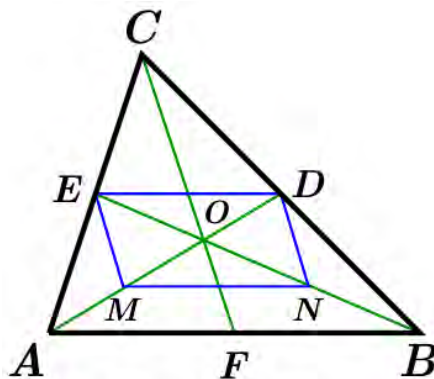
Therefore, the triangle  $ADP$  is right triangle at point  $D$ .

### Exercise 27

Inside the triangle  $ABC$ , the median  $AD$ ,  $BE$ , and  $CF$  intersect at a point  $O$ . We take  $M$  the middle point of the segment  $OA$ ,  $N$  the middle point of segment  $OB$ .

Show that  $DEMN$  is a parallelogram.

### Solution



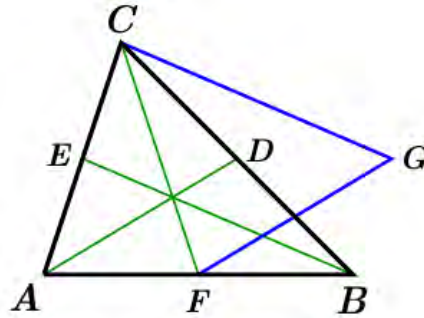
$DE$  &  $MN$  are parallel to  $AB$  and equals to  $\frac{1}{2}|AB|$

That implies to  $ME \parallel DN$ .

Therefore,  $DEMN$  is a parallelogram

### Exercise 28

Inside the triangle  $ABC$ , the median  $AD$ ,  $BE$ , and  $CF$  intersect at a point  $O$ . From the point  $F$ , draw  $FG$  parallel to  $AD$  and are equals, then joint  $A$  to  $G$ .



Show that  $CG = BE$ .

#### Solution

**Given:**  $FG \parallel AD$  &  $FG = AD$

Then, the quadrilateral  $AFGD$  is a parallelogram which it results to  $DG \parallel AF$  &  $DG = AF$ .

$\rightarrow DG \parallel BF$

Since  $F$  is the mid-point of the side  $AB$ , then  $AF = DG = FB$ .

Then, the quadrilateral  $BFDG$  is a parallelogram which it results to  $FD \parallel BG$  &  $DF = GB$ .

So,  $FD \parallel BG \parallel CE$

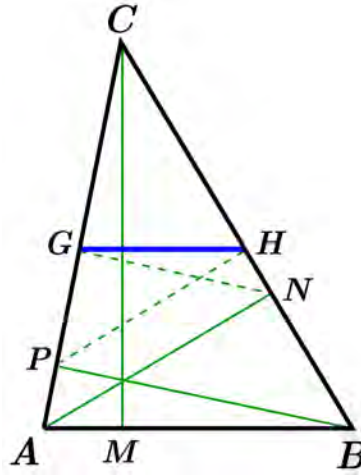
Given that  $D$  &  $F$  are midpoints, then  $DF = \frac{1}{2}AC = CE$

And  $CE \parallel BG$  &  $DF = CE$ , then  $BGCE$  is a parallelogram.

Therefore,  $CG = BE$

### Exercise 29

The height of a triangle  $ABC$  (each of the three lines passing through a vertex of the triangle and perpendicular to the opposite side to this vertex) are  $AN$ ,  $BP$ ,  $CM$ .



From  $P$ , let  $PH$  perpendicular to  $BC$ , same from  $N$ , let  $NG$  perpendicular to  $AC$ . Show that  $GH$  is parallel to  $AB$ .

#### Solution

Let the point  $O$  be the middle of the segment  $AB$ .

Then  $O$  is the center of the 2 triangles  $ANB$  &  $APB$ .

The triangle  $OBN$  is isosceles, implies to  $\widehat{ONB} = \widehat{OBN}$

The triangle  $OPA$  is isosceles, implies to  $\widehat{OPA} = \widehat{OAP}$

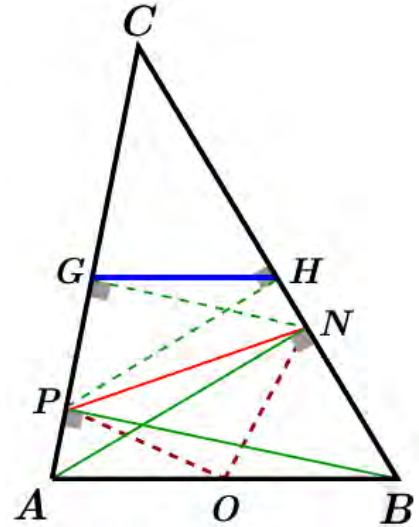
$$\begin{aligned}\widehat{PON} &= 180^\circ - (\widehat{NOB} + \widehat{POA}) \\ &= 180^\circ - (180^\circ - 2\widehat{NBO} + 180^\circ - 2\widehat{OAP}) \\ &= 2\widehat{B} + 2\widehat{A} - 180^\circ\end{aligned}$$

Consider the triangle  $PON$  with  $OP = ON$ , then

$$\begin{aligned}\widehat{OPN} &= \widehat{ONP} \\ \widehat{OPN} &= \frac{1}{2}(180^\circ - \widehat{PON}) \\ &= \frac{1}{2}(180^\circ - 2\widehat{B} - 2\widehat{A} + 180^\circ) \\ &= 180^\circ - \widehat{B} - \widehat{A}\end{aligned}$$

$$\begin{aligned}\widehat{APN} &= \widehat{APO} + \widehat{OPN} \\ &= \widehat{A} + 180^\circ - \widehat{B} - \widehat{A} \\ &= 180^\circ - \widehat{B}\end{aligned}$$

$$\begin{aligned}\widehat{CPN} &= 180^\circ - \widehat{APN} \\ &= 180^\circ - 180^\circ + \widehat{B} \\ &= \widehat{B}\end{aligned}$$



From the 2 right triangles  $CHP$  &  $CGN$

$$\widehat{HPC} = \widehat{GNC}$$

$$\begin{aligned}\widehat{GHN} &= 180^\circ - \widehat{HGN} - \widehat{HNG} \\ &= 180^\circ - \widehat{HGN} - \widehat{CPH}\end{aligned}$$

$$180^\circ - \widehat{GHC} = 180^\circ - \widehat{HGN} - \widehat{CPH}$$

$$\widehat{GHC} = \widehat{HGN} + \widehat{CPH} \quad |$$

Let  $Q$  be the middle point of the segment  $PN$ .

Since  $PGN$  &  $PHN$  are right triangle with the same hypothesis.

Then, the triangles  $HQN$  &  $GQN$  are isosceles.

$$\widehat{H} = \widehat{N} \quad \& \quad \widehat{G} = \widehat{P}$$

$$\begin{aligned}\widehat{GQH} &= 180^\circ - (180^\circ - 2\widehat{P} + 180^\circ - 2\widehat{N}) \\ &= 2\widehat{P} + 2\widehat{N} - 180^\circ\end{aligned}$$

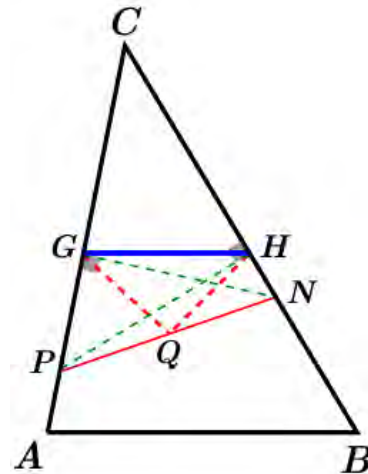
$$\text{Since } QG = QH \Rightarrow \widehat{QGH} = \widehat{QHG}$$

$$\begin{aligned}\widehat{QGH} &= \frac{1}{2}(180^\circ - \widehat{GQH}) \\ &= \frac{1}{2}(180^\circ - 2\widehat{P} - 2\widehat{N} + 180^\circ) \\ &= 180^\circ - \widehat{P} - \widehat{N}\end{aligned}$$

$$\begin{aligned}\widehat{HGN} &= \widehat{QGH} - \widehat{QGN} \\ &= 180^\circ - \widehat{P} - \widehat{N} - 90^\circ + \widehat{QGP} \\ &= 90^\circ - \widehat{P} - \widehat{N} + \widehat{P} \\ &= 90^\circ - \widehat{N} \\ &= \widehat{NPH}\end{aligned}$$

$$\begin{aligned}\widehat{CHG} &= \widehat{HGN} + \widehat{CPH} \\ &= \widehat{NPH} + \widehat{CPH} \\ &= \widehat{NPC} \\ &= \widehat{B} \quad | \end{aligned}$$

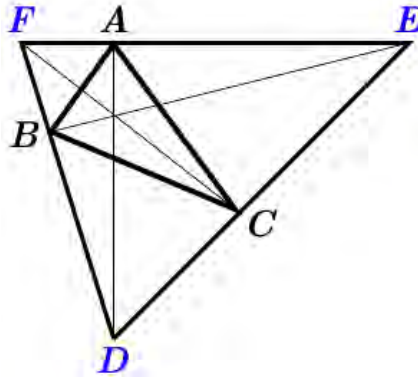
Therefore,  $GH \parallel AB$





### Exercise 30

From the top of a triangle, we lead the external bisectors of angles such that formed an outside triangle such that the top of the first are the feet of the second heights.



### Solution

Let the triangle  $DEF$  where  $DA$ ,  $BE$ , and  $FC$  are heights (perpendicular to sides).

Let the point  $M$  be the middle points of the same hypotenuse of the 2 right triangles  $FAD$  &  $FCD$ . Then, the 2 triangles inscribed the same circle with the center at point  $M$ .

$$MF = MA = MC = MD$$

$$\widehat{MFA} = \widehat{MAF} \quad \& \quad \widehat{MCD} = \widehat{MDC}$$

Therefore, the triangle  $AMC$  is isosceles.

$$MA = MC \quad \& \quad \widehat{MAC} = \widehat{ACM}$$

$$\begin{aligned} \widehat{AMC} &= 180^\circ - (\widehat{FMA} + \widehat{CMD}) \\ &= 180^\circ - (180^\circ - 2\widehat{F} + 180^\circ - 2\widehat{D}) \\ &= \underline{2\widehat{F} + 2\widehat{D} - 180^\circ} \end{aligned}$$

$$\begin{aligned} \widehat{ACM} &= \frac{1}{2}(180^\circ - \widehat{AMC}) \\ &= \frac{1}{2}(180^\circ + 180^\circ - 2\widehat{F} - 2\widehat{D}) \\ &= \underline{180^\circ - \widehat{F} - \widehat{D}} \end{aligned}$$

$$\begin{aligned} \widehat{DCA} &= \widehat{DCM} + \widehat{MCA} \\ &= \widehat{D} + 180^\circ - \widehat{F} - \widehat{D} \end{aligned}$$

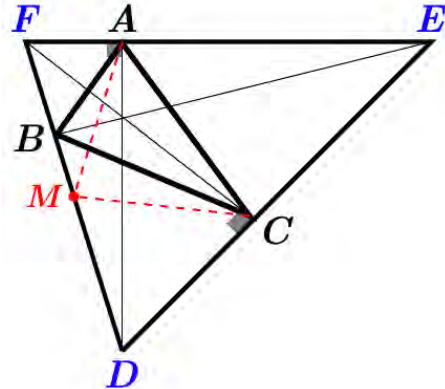
$$180^\circ - \widehat{ACE} = 180^\circ - \widehat{F}$$

$$\underline{\widehat{ACE} = \widehat{F}}$$

Similar,

Let the point  $N$  be the middle points of the same hypotenuse of the 2 right triangles  $FCE$  &  $FBE$ . Then, the 2 triangles inscribed the same circle with the center at point  $N$ .

$$NF = NB = NC = NE$$



$$\widehat{NBF} = \widehat{BFN} \quad \& \quad \widehat{NEC} = \widehat{NCE}$$

Therefore, the triangle  $NBC$  is isosceles.

$$MA = MC \quad \& \quad \widehat{MAC} = \widehat{ACM}$$

$$\begin{aligned} \widehat{BNC} &= 180^\circ - (\widehat{FNB} + \widehat{CNE}) \\ &= 180^\circ - (180^\circ - 2\widehat{F} + 180^\circ - 2\widehat{E}) \\ &= \underline{2\widehat{F} + 2\widehat{E} - 180^\circ} \end{aligned}$$

$$\begin{aligned} \widehat{BCN} &= \frac{1}{2}(180^\circ - \widehat{BNC}) \\ &= \frac{1}{2}(180^\circ + 180^\circ - 2\widehat{F} - 2\widehat{E}) \\ &= \underline{180^\circ - \widehat{F} - \widehat{E}} \end{aligned}$$

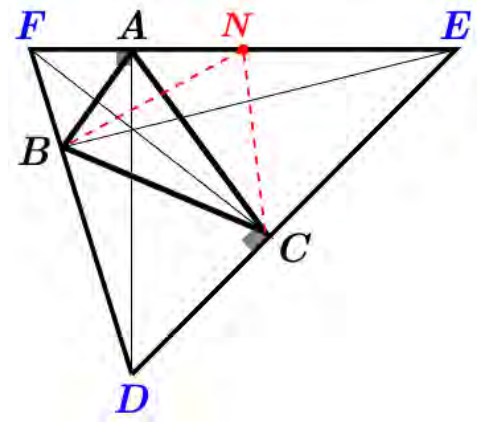
$$\begin{aligned} \widehat{BCE} &= \widehat{BCN} + \widehat{NCE} \\ &= \widehat{E} + 180^\circ - \widehat{F} - \widehat{E} \end{aligned}$$

$$180^\circ - \widehat{BCD} = 180^\circ - \widehat{F}$$

$$\underline{\widehat{BCD} = \widehat{F}} \quad |$$

Then,  $\underline{\widehat{ACE} = \widehat{F} = \widehat{BCD}}$

Therefore,  $CF$  is the interior bisector of  $\widehat{BCA}$  and  $DCE$  is the exterior bisector.



$$\begin{aligned} \widehat{MAC} &= \frac{1}{2}(180^\circ - \widehat{AMC}) \\ &= \frac{1}{2}(180^\circ + 180^\circ - 2\widehat{F} - 2\widehat{D}) \\ &= \underline{180^\circ - \widehat{F} - \widehat{D}} \end{aligned}$$

$$\begin{aligned} \widehat{FAC} &= \widehat{FAM} + \widehat{MAC} \\ &= \widehat{F} + 180^\circ - \widehat{F} - \widehat{D} \end{aligned}$$

$$180^\circ - \widehat{CAE} = 180^\circ - \widehat{D}$$

$$\underline{\widehat{CAE} = \widehat{D}} \quad |$$

Let the point  $P$  be the middle points of the same hypotenuse of the 2 right triangles  $DAE$  &  $BDE$ .  
Then, the 2 triangles inscribed the same circle with the center at point  $P$ .

$$PE = PA = PB = PD$$

$$\widehat{PAE} = \widehat{PAE} \quad \& \quad \widehat{PBD} = \widehat{PDB}$$

Therefore, the triangle  $APB$  is isosceles.

$$PA = PB \quad \& \quad \widehat{PAB} = \widehat{PBA}$$

$$\begin{aligned}\widehat{APB} &= 180^\circ - (\widehat{DPB} + \widehat{APE}) \\ &= 180^\circ - (180^\circ - 2\hat{D} + 180^\circ - 2\hat{E}) \\ &= \underline{2\hat{D} + 2\hat{E} - 180^\circ} \quad | \end{aligned}$$

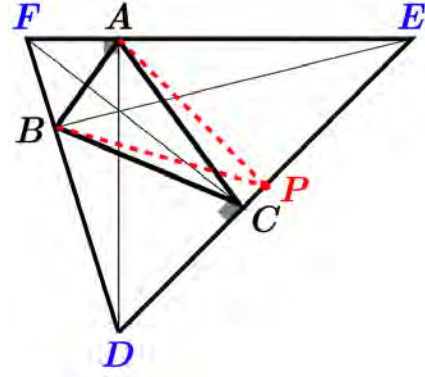
$$\begin{aligned}\widehat{PAB} &= \frac{1}{2}(180^\circ - \widehat{APB}) \\ &= \frac{1}{2}(180^\circ + 180^\circ - 2\hat{D} - 2\hat{E}) \\ &= \underline{180^\circ - \hat{D} - \hat{E}} \quad | \end{aligned}$$

$$\begin{aligned}\widehat{BAE} &= \widehat{PAE} + \widehat{PAB} \\ &= \hat{E} + 180^\circ - \hat{D} - \hat{E} \\ 180^\circ - \widehat{FAB} &= 180^\circ - \hat{D} \end{aligned}$$

$$\underline{\widehat{FAB} = \hat{D}} \quad |$$

Then,  $\widehat{FAB} = \hat{D} = \widehat{CAE}$

Therefore,  $AD$  is the interior bisector of  $\widehat{BAC}$  and  $FAE$  is the exterior bisector.



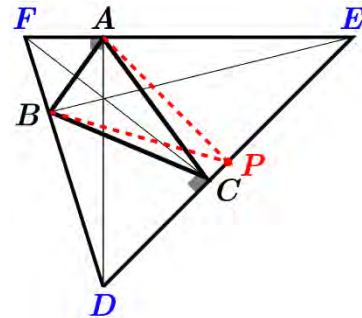
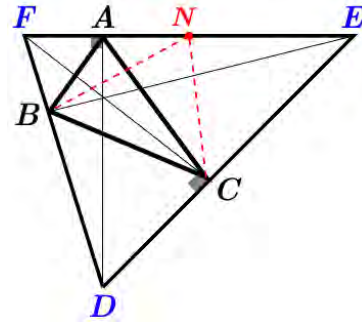
$$\begin{aligned}\widehat{NBC} &= \frac{1}{2}(180^\circ - \widehat{BNC}) \\ &= \frac{1}{2}(180^\circ + 180^\circ - 2\hat{F} - 2\hat{E}) \\ &= \underline{180^\circ - \hat{F} - \hat{E}} \quad | \end{aligned}$$

$$\begin{aligned}\widehat{CBF} &= \widehat{CBN} + \widehat{NBF} \\ &= 180^\circ - \hat{F} - \hat{E} + \hat{F} \\ 180^\circ - \widehat{CBD} &= 180^\circ - \hat{E} \end{aligned}$$

$$\underline{\widehat{CBD} = \hat{E}} \quad |$$

$$\begin{aligned}\widehat{PBA} &= \frac{1}{2}(180^\circ - \widehat{APB}) \\ &= \frac{1}{2}(180^\circ + 180^\circ - 2\hat{D} - 2\hat{E}) \\ &= \underline{180^\circ - \hat{D} - \hat{E}} \quad | \end{aligned}$$

$$\begin{aligned}\widehat{ABD} &= \widehat{DBP} + \widehat{PBA} \\ &= \hat{D} + 180^\circ - \hat{D} - \hat{E} \\ 180^\circ - \widehat{ABF} &= 180^\circ - \hat{E} \end{aligned}$$



$$\widehat{ABF} = \widehat{E}$$

Then,  $\widehat{CBD} = \widehat{E} = \widehat{ABF}$

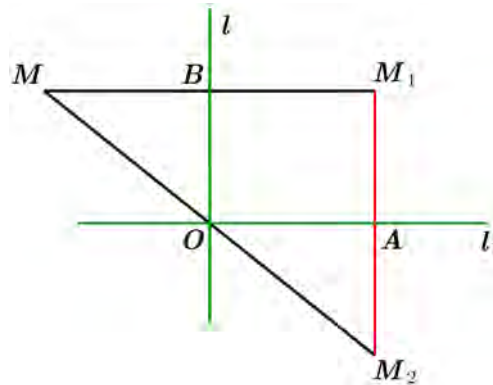
Therefore,  $BE$  is the interior bisector of  $\widehat{ABC}$  and  $DBF$  is the exterior bisector.

### Exercise 31

Consider a point  $O$  on a vertical line  $\ell$ , a point  $M$  outside the line  $\ell$ . We take the symmetries  $M_1$  and  $M_2$  from  $M$  across the line  $\ell$  and the point  $O$ , respectively.

Demonstrate that the points  $M_1$  and  $M_2$  are symmetries with regard to a line perpendicular to the line  $\ell$  passing through the point  $O$ .

#### Solution



Since  $M_1$  is the symmetry of  $M$  across the line  $\ell$ , let  $B$  the middle point of the segment  $MM_1$ .

That implies to:  $BM = BM_1$ .

Similarly, the point  $O$  is the middle point of the segment  $MM_2$ .

That implies to:  $OM = OM_2$ .

Let  $A$  be the point intersection of the segment  $M_1M_2$  and line  $\ell_1$ .

Since  $MM_1 \perp \ell$  and  $\ell \perp \ell_1 \Rightarrow MM_1 \parallel \ell_1$

From the right triangle  $MM_1M_2$  Since  $O$  is the middle of  $MM_2$  and  $OA \perp MM_1$ .

Therefore, the point  $A$  the middle point of the segment  $M_1M_2$

### Exercise 32

In a quadrilateral  $ABCD$  (Kite), the sides  $AB = AD$ ,  $\angle A = 135^\circ$  and  $\angle B = \angle D = 90^\circ$ .

1. Prove the symmetry in the figure.
2. Prove that the middles of the sides are the top of rectangle.
3. Prove there exists an interior of the given quadrilateral a point equidistant of 4 sides; determine these points.
4. On the same exterior bisector of angles  $A, B, C, D$ ; they formed a quadrilateral

### Solution

$$\begin{aligned} 1. \quad C &= 180^\circ - 135^\circ \\ &= 45^\circ \end{aligned}$$

Since  $AB = AD$ , then the side  $AC$  is the bisector of  $\widehat{BAD}$ .

The 2 right triangles  $ABC$  &  $ADC$  are rights on  $B$  and  $D$  respectively, with same hypotenuse  $AC$ , therefore the figure is symmetric about the side  $AC$ .

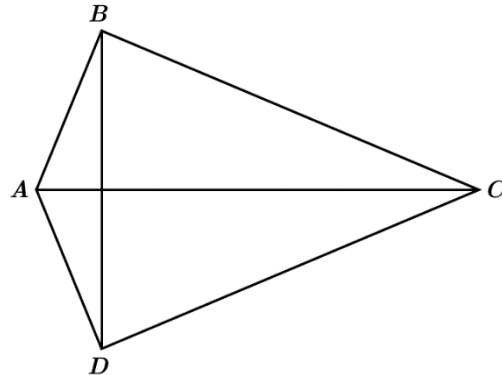
$$\begin{aligned} \widehat{BAC} &= \frac{135^\circ}{2} \\ &= 67.5^\circ \end{aligned}$$

$$\begin{aligned} \widehat{BCA} &= \frac{45^\circ}{2} \\ &= 22.5^\circ \end{aligned}$$

$$\begin{aligned} \widehat{ABD} &= 90^\circ - 67.5^\circ \\ &= 22.5^\circ \end{aligned}$$

$$\widehat{ADB} = 22.5^\circ$$

$$\begin{aligned} \widehat{DBC} &= 90^\circ - 22.5^\circ \\ &= 67.5^\circ \end{aligned}$$



2. Since  $AB = AD$ , then the triangle  $BAD$  is isosceles.

Let  $M$  and  $Q$  the middle points of the sides  $AB$  and  $AD$  respectively.

Therefore, the segment  $MQ \parallel BD$ .

Similar, the triangle  $BCD$  is isosceles, since  $CB = CD$ .

Let  $N$  and  $P$  the middle points of the sides  $CA$  and  $CD$  respectively.

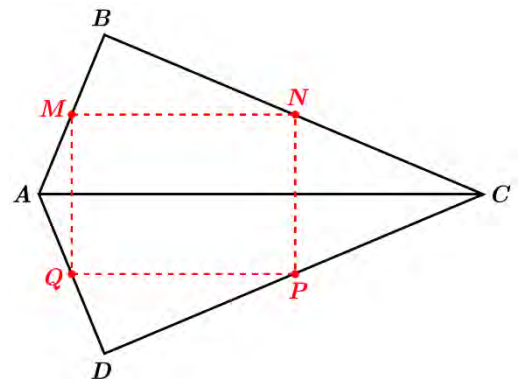
Therefore, the segment  $NP \parallel BD$ .

The points  $M$  and  $N$  are the middle points of the sides  $AB$  and  $CB$  respectively in the right triangle  $ABC$ .

Therefore, the segment  $MN \parallel AC$ .

The points  $P$  and  $Q$  are the middle points of the sides  $DC$  and  $DA$  respectively in the right triangle  $ADC$ .

Therefore, the segment  $PQ \parallel AC$ .



$$PQ \parallel MN \parallel AC \quad \& \quad MQ \parallel NP \parallel BD$$

That implies to  $MQ \parallel NP$  and  $MN \parallel PQ$ , which implies that  $MNPQ$  is a parallelogram.

From the quadrilateral  $ABCD$ ,  $AC \perp BD$

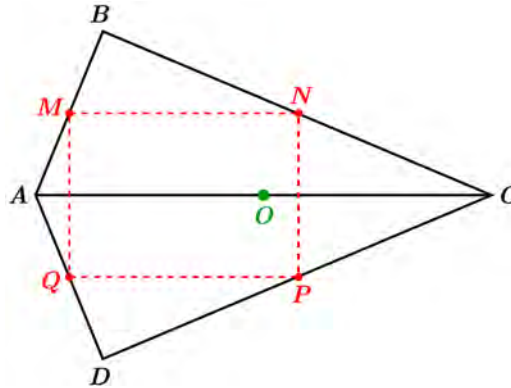
Therefore,  $MNPQ$  is rectangular.

3. Let the point  $O$  be the middle point of the segment  $AC$ .

Given that the 2 right triangles  $ABC$  &  $ADC$  have the same hypothesis segment  $AC$ .

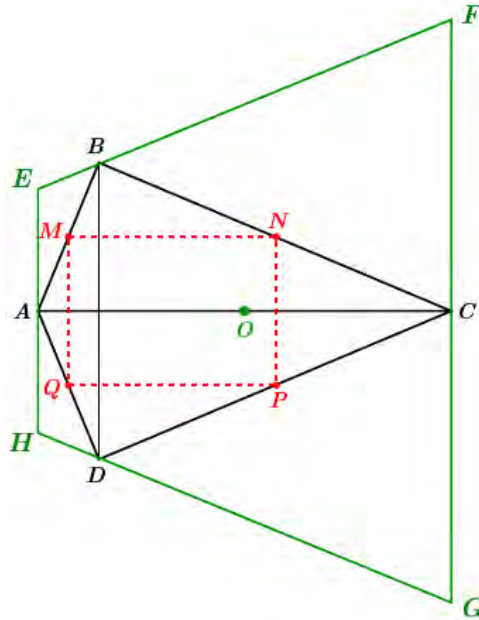
The 4 point are inscribed in a circle with diameter  $AC$  where the radius is:

$$OA = OB = OC = OD$$



Therefore, the point  $O$  is an interior of the given quadrilateral which is a point equidistant of 4 sides.

4. Let two segments pass through the points  $A$  and  $C$  perpendicular to the segment  $AC$ , which are parallel to segment  $BD$ .



Let the segment  $EF$  passes through the point  $B$  such that:

$$\begin{aligned} \widehat{EBA} &= \widehat{FBC} \\ &= \frac{90^\circ}{2} \\ &= 45^\circ \end{aligned}$$

$$\begin{aligned}\widehat{EAB} &= 90^\circ - \widehat{BAC} \\ &= 90^\circ - 67.5^\circ \\ &= \underline{22.5^\circ}\end{aligned}$$

$$\begin{aligned}\widehat{AEB} &= 180^\circ - \widehat{EBA} - \widehat{BAE} \\ &= 180^\circ - 45^\circ - 22.5^\circ \\ &= \underline{112.5^\circ}\end{aligned}$$

$$\begin{aligned}\widehat{BCF} &= 90^\circ - \widehat{ACB} \\ &= 90^\circ - 22.5^\circ \\ &= \underline{67.5^\circ}\end{aligned}$$

$$\begin{aligned}\widehat{BFC} &= 180^\circ - \widehat{FBC} - \widehat{BCF} \\ &= 180^\circ - 45^\circ - 67.5^\circ \\ &= \underline{67.5^\circ}\end{aligned}$$



### Exercise 33

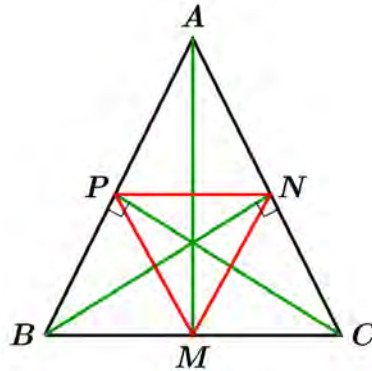
Given an isosceles triangle  $ABC$  with a peak at  $A$  with angle of  $47^\circ$ .

The height of a triangle  $ABC$  of each of the three lines passing through a vertex of the triangle and perpendicular to the opposite side to this vertex, are  $AM$ ,  $BN$ ,  $CP$ .

What is the angle of the peak of the isosceles triangle  $MNP$  ?

### Solution

The triangle  $ABC$  is isosceles triangle at  $A$  then the height  $AM$  is the bisector. So, the point  $M$  is the middle point of the side  $BC$ .



Since the 2 right triangles  $BPC$  &  $BNC$  have the same hypothesis segment  $BC$ .

There are inscribed in a circle with diameter  $BC$  with the center at point  $M$ .

Therefore, the triangle  $MNP$  is isosceles with a peak at the point  $M$ .

$$C = \widehat{MNC} = B = \widehat{BPM}$$

$$\begin{cases} \widehat{BMP} = 180^\circ - 2B \\ \widehat{CMN} = 180^\circ - 2C \end{cases}$$

$$\begin{aligned} B &= \frac{180^\circ - 47^\circ}{2} \\ &= \frac{133^\circ}{2} \end{aligned}$$

$$\begin{aligned} \widehat{PMN} &= 180^\circ - \widehat{BMP} - \widehat{CMN} \\ &= 180^\circ - 180^\circ + 2B - 180^\circ + 2B \\ &= 4B - 180^\circ \\ &= 4\left(\frac{133^\circ}{2}\right) - 180^\circ \\ &= 266^\circ - 180^\circ \\ &= 86^\circ \end{aligned}$$

### Exercise 34

The angles  $A$ ,  $B$ , and  $C$  of a triangle are:

$$A = 68^\circ \quad B = 62^\circ \quad C = 50^\circ$$

Let the point  $H$  be the intersection point of the heights of the triangle  $ABC$ .

The bisectors inside the triangle  $BHC$  intersect at a point  $O$ .

Find the angles:  $\widehat{BOH}$ ,  $\widehat{HOC}$ , and  $\widehat{COB}$

#### Solution

Triangle  $BNC$ :

$$\begin{aligned}\widehat{NBC} &= 90^\circ - C \\ &= 90^\circ - 50^\circ \\ &= 40^\circ\end{aligned}$$

Triangle  $BPC$ :

$$\begin{aligned}\widehat{PCB} &= 90^\circ - B \\ &= 90^\circ - 62^\circ \\ &= 28^\circ\end{aligned}$$

Triangle  $BHC$ :

$$\begin{aligned}\widehat{BHC} &= 180^\circ - C - \widehat{BPC} \\ &= 180^\circ - 40^\circ - 28^\circ \\ &= 112^\circ\end{aligned}$$

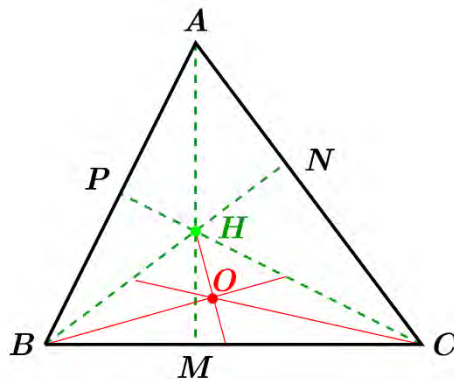
$$\begin{aligned}\widehat{OCB} &= \frac{1}{2}\widehat{BCP} \\ &= \frac{1}{2}(28^\circ) \\ &= 14^\circ\end{aligned}$$

$$\begin{aligned}\widehat{OBC} &= \frac{1}{2}\widehat{NBC} \\ &= \frac{1}{2}(40^\circ) \\ &= 20^\circ\end{aligned}$$

Triangle  $BOC$ :

$$\begin{aligned}\widehat{COB} &= 180^\circ - \widehat{OCB} - \widehat{OBC} \\ &= 180^\circ - 14^\circ - 20^\circ \\ &= 146^\circ\end{aligned}$$

$$\begin{aligned}\widehat{BHO} &= \frac{1}{2}\widehat{BHC} \\ &= \frac{1}{2}(112^\circ)\end{aligned}$$



$$\underline{= 56^\circ}$$

Triangle  $BHO$ :

$$\begin{aligned}\widehat{HOB} &= 180^\circ - \widehat{OBH} - \widehat{OHB} \\ &= 180^\circ - 20^\circ - 56^\circ \\ &\underline{= 104^\circ}\end{aligned}$$

### Exercise 35

On the sides of an angle  $A$ , where the longest of  $AB = AB'$  is  $AC = AC'$

- Prove that  $BC' = B'C$
- Let a point  $I$  be the intersection of the sides  $BC'$  and  $B'C$ . Prove that  $IB = IB'$  and  $IC = IC'$ .
- Prove that the point  $I$  is located on the bisector of angle  $A$ .

#### Solution

- a) From the 2 triangles  $AB'C$  and  $ABC'$ , they have common:

- ✓ Common angle  $A$ .
- ✓  $AB = AB'$
- ✓  $AC = AC'$

The 2 triangles are equivalent.

Therefore, that implies to  $BC' = B'C$ .

- b) From the 2 triangles  $IBC$  and  $IB'C'$ .

- ✓  $BC = B'C'$
- ✓  $\widehat{BIC} = \widehat{B'IC'}$
- ✓  $C = C'$

That implies to the 2 triangles  $IBC$  and  $IB'C'$  are equivalent

Therefore,  $IB = IB'$  and  $IC = IC'$

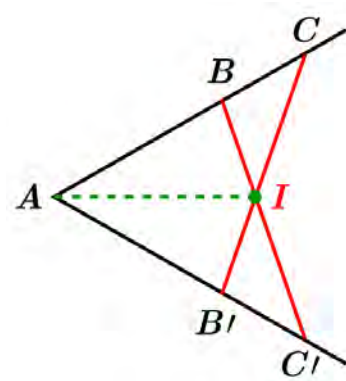
- c) Consider the 2 triangles  $ABI$  and  $AB'I$

- ✓  $AB = AB'$
- ✓  $IB = IB'$
- ✓ Common side  $AI$

That implies to

$$\widehat{IAB} = \widehat{IAB'}$$

Therefore,  $AI$  is the **bisector** of angle  $A$ .



### Exercise 36

Given two adjacent angles of  $60^\circ$ ,  $\widehat{AOB}$  and  $\widehat{BOC}$ .

A point  $Z$  is located inside the angle  $\widehat{BOC}$ , where leads the perpendicular  $ZP$ ,  $ZT$ ,  $ZS$  on  $OC$ ,  $OA$ , and  $OB$ .

Prove that  $ZT = ZS + ZP$

#### Solution

Let the segment  $DE$  passing through the point  $Z$  and parallel to  $OA$ .

So that,

$$\widehat{CDE} = 120^\circ$$

$$\begin{aligned}\widehat{ODE} &= 180^\circ - 120^\circ \\ &= 60^\circ\end{aligned}$$

And it is given that

$$\widehat{DOE} = 60^\circ$$

Which implies that the triangle  $ODE$  is equilateral.

Let  $OF$  perpendicular to the side  $OE$ .

And since all the heights in the equilateral  $ODE$  triangle are equals ( $OO' = DF$ ).

Since side  $ZT$  is perpendicular to the line  $OA$ , then the points  $OO'ZT$  is a rectangle with  $OO' = ZT$

That implies to  $DF = ZT$

Let the point  $G$  on the segment  $DF$  such that  $ZG$  perpendicular to  $DF$  and  $ZS = GF$  from the rectangle  $ZSFG$ .

From the right triangle  $DGZ$  at  $G$ .

$$\widehat{GDZ} = 30^\circ \text{ (from the equilateral } ODE \text{ triangle).}$$

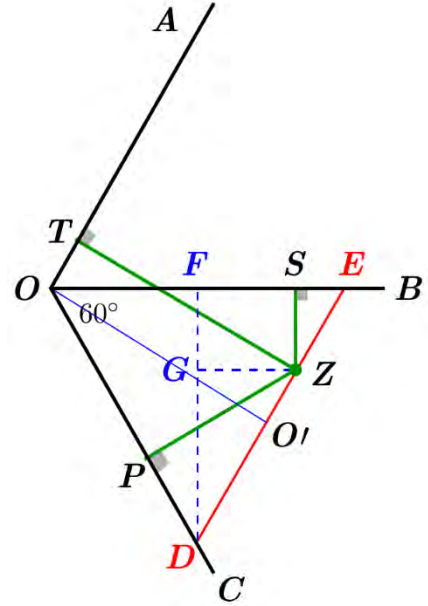
$$\Rightarrow \widehat{GZD} = 60^\circ$$

Consider the 2 right triangles  $DPZ$  and  $DGZ$ , they have

- ✓  $ZD$  common hypotenuse
- ✓  $\widehat{P} = \widehat{G} = 90^\circ$
- ✓  $\widehat{PDZ} = \widehat{GZD} = 60^\circ$

That imply to  $\underline{GD = ZP}$

$$\begin{aligned}ZT &= \overline{DF} \\ &= \overline{DG} + \overline{GF} \\ &= \underline{ZP + ZS}\end{aligned}$$



### Exercise 37

Let two perpendicular lines,  $x'ox$  &  $y'oy$ . Consider a point  $P$  in  $xoy$  plane.

Let  $PA$  be perpendicular to  $ox$  and  $PB$  be perpendicular to  $oy$ .

From the point  $P$ , a secant line  $PZ$  intersect  $x'ox$  &  $y'oy$ , a point  $Q$  on  $x'x$  and  $R$  on  $y'y$  such that  $QS = PR$ .

Let  $SD$  and  $SC$  be perpendicular to  $x'ox$  &  $y'oy$  respectively.

- Prove that  $BD$  and  $AC$  are parallel
- Prove that  $\overline{PS} = \overline{BD} + \overline{AC}$
- Prove that  $\overline{QR} = \overline{BD} - \overline{AC}$

#### Solution

- Consider the 3 right triangles  $PBR$ ,  $AOC$  &  $QDS$ :

- ✓  $BP \parallel DQ \parallel OA$
- ✓  $BR \parallel DS \parallel OC$
- ✓  $BP = OA$
- ✓  $DS = OC$
- ✓  $\widehat{RBP} = \widehat{QDS} = \widehat{AOC} = 90^\circ$
- ✓  $PR = QS$  (hypotenuse)

The 3 triangles are equals.

Therefore,  $AC \parallel PR \parallel QS \Rightarrow AC \parallel PS$

Since,  $BP = DQ$  then  $BD \parallel PQ$  from the parallelogram  $PBDQ$ .

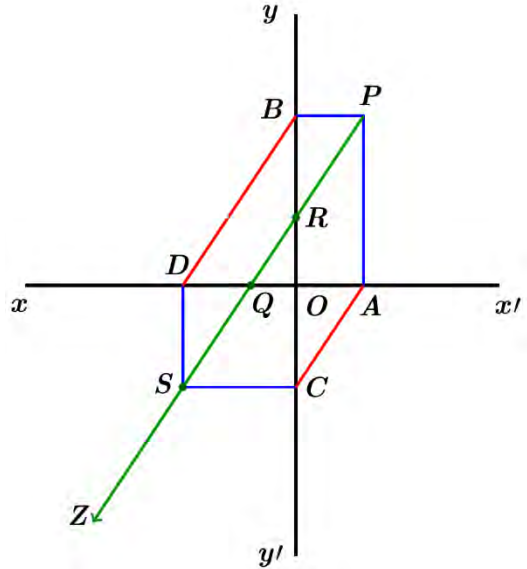
$BD \parallel PQ \parallel AC$

Therefore,  $BD$  and  $AC$  are *parallel*.

- $\overline{BP} = \overline{DQ}$

$$\begin{aligned}\overline{PS} &= \overline{PQ} + \overline{QS} \\ &= \overline{AC} + \overline{BD}\end{aligned}$$

- $\overline{QR} = \overline{PQ} - \overline{PR}$   
 $= \overline{BD} - \overline{AC}$



### Exercise 38

Given a right triangle  $BAC$  at  $A$ , the bisector  $AD$  of angle  $A$ , and a perpendicular line  $DE$ , where the point  $E$  intersects side  $AC$ .

Prove that  $DB = DE$

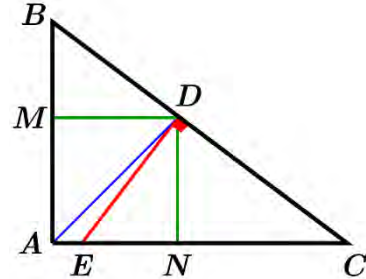
#### Solution

Let a point  $M$  be the projection of the point  $D$  onto  $AB$ .

Let a point  $N$  be the projection of the point  $D$  onto  $AC$ .

Consider the 2 right triangles  $AMD$  and  $AND$ :

- ✓  $\widehat{MAD} = \widehat{NAD}$
- ✓  $\widehat{M} = \widehat{N} = 90^\circ$
- ✓  $DA$  hypotenuse



Therefore, the 2 triangles  $AMD$  and  $AND$  are equals

$$\begin{cases} DM = DN \\ AM = AN \end{cases}$$

And  $DN \parallel AB$

$$\widehat{NDC} = \widehat{ABD} = \widehat{B}$$

$$\begin{aligned} \widehat{EDN} &= 90^\circ - \widehat{NDC} \\ &= 90^\circ - \widehat{B} \end{aligned}$$

$$\begin{aligned} \widehat{MDB} &= 90^\circ - \widehat{B} \\ &= \widehat{EDN} \end{aligned}$$

Consider the 2 right triangles  $BMD$  and  $DNE$ :

- ✓  $\widehat{B} = \widehat{DEN}$
- ✓  $\widehat{M} = \widehat{N} = 90^\circ$
- ✓  $MD = ND$

$$\Rightarrow \widehat{MDB} = \widehat{EDN}$$

The 2 triangles  $BMD$  and  $DNE$  are equals

Therefore,  $DB = DE$

### Exercise 39

Consider a square  $ABCD$ , on the side  $BC$ , outside the square, consider an equilateral triangle  $BCE$ . Consider inside the square, on the side  $AB$ , an equilateral triangle  $ABF$ .

Show that the points  $D$ ,  $E$ , and  $F$  are on the same line.

#### Solution

Given the square  $ABCD$  and equilateral triangles  $ABF$  and  $BCE$ . Therefore,

$$AD = AF = BF = BC = BE$$

$$\widehat{AFB} = \widehat{FAB} = \widehat{CBE} = \widehat{BEC} = 60^\circ$$

Since  $AD = AF$

Then, the triangle  $ADF$  is isosceles:

$$\widehat{ADF} = \widehat{AFD}$$

$$\begin{aligned}\widehat{DAF} &= 90^\circ - \widehat{FAB} \\ &= 90^\circ - 60^\circ \\ &= 30^\circ \quad | \end{aligned}$$

$$\widehat{ADF} + \widehat{AFD} + \widehat{DAF} = 180^\circ$$

$$\begin{aligned}2\widehat{AFD} &= 180^\circ - \widehat{DAF} \\ &= 180^\circ - 30^\circ \\ &= 150^\circ \quad | \end{aligned}$$

$$\widehat{AFD} = \frac{1}{2}(150^\circ)$$

$$\widehat{AFD} = 75^\circ \quad |$$

From the equilateral triangle  $ABF$ :

$$\widehat{AFB} = 60^\circ \quad |$$

$$\begin{aligned}\widehat{FBC} &= 90^\circ - \widehat{ABF} \\ &= 90^\circ - 60^\circ \\ &= 30^\circ \quad | \end{aligned}$$

From the equilateral triangle  $BCE$ :

$$\widehat{CBE} = 60^\circ \quad |$$

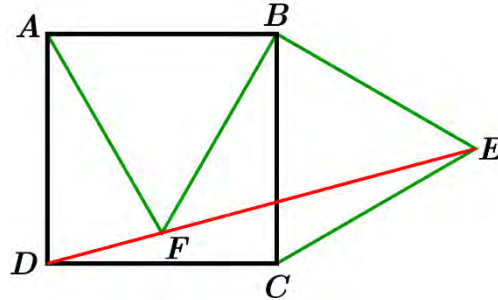
$$\begin{aligned}\widehat{FBE} &= \widehat{FBC} + \widehat{CBE} \\ &= 30^\circ + 60^\circ \\ &= 90^\circ \quad | \end{aligned}$$

Since  $BF = BE$

Then, the triangle  $EBF$  is isosceles

$$\widehat{BFE} + \widehat{BEF} + \widehat{EBF} = 180^\circ$$

$$\begin{aligned}2\widehat{BFE} &= 180^\circ - \widehat{EBF} \\ &= 180^\circ - 90^\circ \end{aligned}$$





$$\underline{= 90^\circ \mid}$$

$$\widehat{BFE} = \frac{1}{2}(90^\circ)$$

$$\underline{\widehat{BFE} = 45^\circ \mid}$$

$$\widehat{AFD} + \widehat{AFB} + \widehat{BFE} = 75^\circ + 90^\circ + 45^\circ$$

$$\underline{= 180^\circ \mid}$$

The points  $D$ ,  $F$ , and  $E$  are on the same line.

## *Exercise 4*

### *Solution*