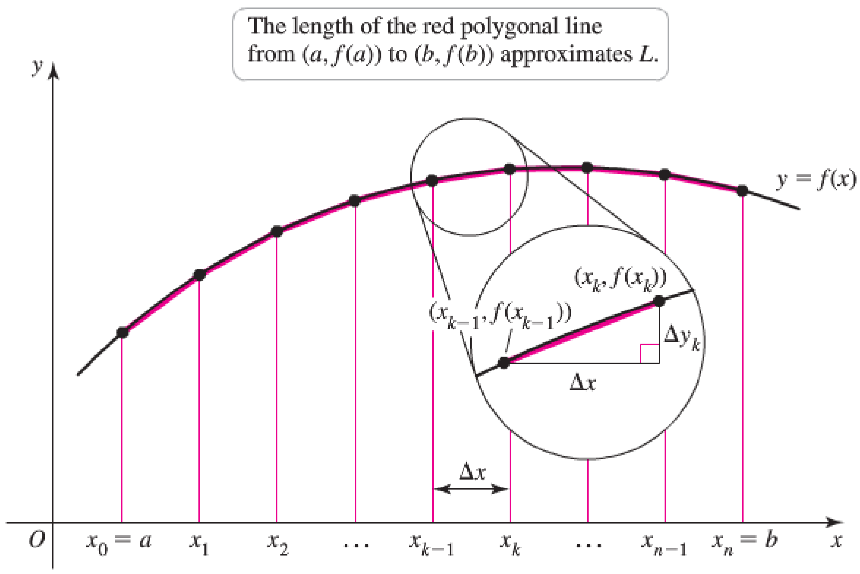
Section 1.5 – Length of Curves

Length of a curve $y = f(x)$

We assume that f has a continuous derivative at every point of $[a, b]$. Such function is called ***smooth***, and its graph is a ***smooth curve*** because it doesn't have any breaks, corners, or cusps.



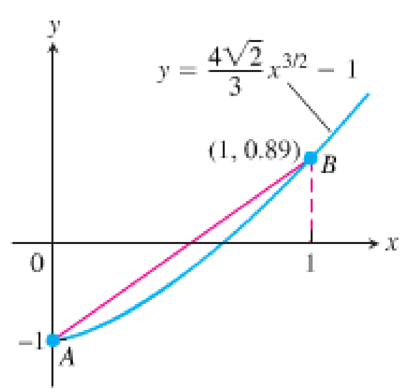
Definition

If f' is continuous on $[a, b]$, then the length (arc length) of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example

Find the length of the curve $y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \quad 0 \le x \le 1$



Solution

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3}\frac{3}{2}x^{1/2} \quad = 2\sqrt{2}x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2}x^{1/2}\right)^2 \quad = 8x$$

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^1 (1+8x)^{1/2}\ dx \qquad\qquad or\ \ u = 1+8x \quad du = 8dx \rightarrow dx = \frac{du}{8}$$

$$= \frac{1}{8}\int_0^1 (1+8x)^{1/2}\ d(1+8x)$$

$$= \frac{1}{8}\left[\frac{2}{3}(1+8x)^{3/2}\right]_0^1$$

$$= \frac{1}{12}\left[(1+8(1))^{3/2} - (1+8(0))^{3/2}\right]$$

$$= \frac{1}{12}\left[(9)^{3/2} - (1)^{3/2}\right]$$

$$= \frac{1}{12}[27-1]$$

$$= \frac{1}{12}(26)$$

$$\underline{= \frac{13}{6} \approx 2.17\ \ unit}$$

Example

Find the length of the graph of $f(x) = \frac{x^3}{12} + \frac{1}{x}, \quad 1 \le x \le 4$

Solution

$a = \frac{1}{12}, \quad m = 3, \quad b = 1, \quad n = -1$

1. $m + n = 3 - 1 = 2$ ✓

2. $abmn = \frac{1}{12}(1)(3)(-1) = -\frac{1}{4}$ ✓

$$L = \left(\frac{x^3}{12} - \frac{1}{x}\right)_1^4$$

$$= \left(\frac{4^3}{12} - \frac{1}{4}\right) - \left(\frac{1}{12} - \frac{1}{1}\right)$$

$$= \frac{72}{12}$$

$$\underline{= 6\ \ unit}$$

***Discontinuities* in $\frac{dy}{dx}$**

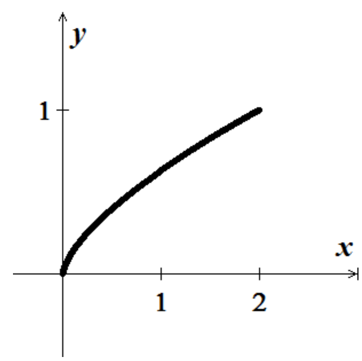
Formula for the length of $x = g(y), \quad c \le y \le d$

If g' is continuous on [*c*, *d*], the length of the curve $x = g(y)$ from the point $A = (g(c), c)$ to the point $B = (g(d), d)$ is the value of the integral

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Example

Find the length of the curve $y = \left(\frac{x}{2}\right)^{2/3}$ from $x = 0$ to $x = 2$.



Solution

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3}\left(\frac{1}{2}\right)$$

$$= \frac{1}{3}\left(\frac{2}{x}\right)^{1/3} \qquad \boxed{x \ne 0} \ \ (\textit{CP})$$

$$y = \left(\frac{x}{2}\right)^{2/3} \ \rightarrow \ y^{3/2} = \frac{x}{2}$$ ***Raised both sides to the power 3/2***

$$x = 2y^{3/2}$$

$$\frac{dx}{dy} = 2\left(\frac{3}{2}\right)y^{1/2} = 3y^{1/2} \qquad \rightarrow \begin{cases} x = 0 & \Rightarrow y = 0 \\ x = 2 & \Rightarrow y = \left(\frac{2}{2}\right)^{2/3} = 1 \end{cases}$$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + \left(3y^{1/2}\right)^2}\, dy$$

$$= \int_0^1 \sqrt{1 + 9y}\, dy$$

$$= \int_0^1 (1 + 9y)^{1/2}\, dy$$

$$= \frac{1}{9}\frac{2}{3}(1 + 9y)^{3/2}\Big|_0^1$$

$$= \frac{2}{27}\left[(1 + 9)^{3/2} - (1 + 0)^{3/2}\right]$$

$$= \frac{2}{27}\left(10^{3/2} - 1\right)$$

$\approx 2.27 \ \textit{unit}$

Example

Find the arc length function for the curve $f(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$ on the interval $\left[1, \ \sqrt{2}\right]$

Solution

$$y = \ln\left(x + \sqrt{x^2-1}\right) \rightarrow x + \sqrt{x^2-1} = e^y$$

$$\left(\sqrt{x^2-1}\right)^2 = \left(e^y - x\right)^2$$

$$x^2 - 1 = e^{2y} - 2xe^y + x^2$$

$$2xe^y = e^{2y} + 1$$

$$x = \frac{e^{2y}+1}{2e^y}\left(\frac{e^y}{e^y}\right)$$

$$= \frac{e^y + e^{-y}}{2}$$

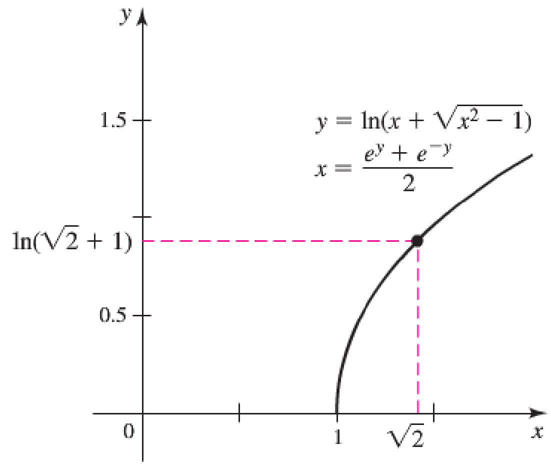
$$y = \ln\left(x + \sqrt{x^2-1}\right) \Leftrightarrow x = \frac{e^y + e^{-y}}{2} = g(y)$$

$$x = 1 \quad \rightarrow \quad y = 0$$

$$x = \sqrt{2} \quad \rightarrow \quad y = \ln\left(\sqrt{2}+1\right)$$

$$f'(x) = \frac{1 + x\left(x^2-1\right)^{-1/2}}{x + \sqrt{x^2-1}}$$

$$= \frac{\sqrt{x^2-1} + x}{x\sqrt{x^2-1} + x^2 - 1}$$



$$L = \int_0^{\ln\left(\sqrt{2}+1\right)} \sqrt{1 + g'(y)^2}\, dy$$

$$= \int_0^{\ln\left(\sqrt{2}+1\right)} \sqrt{1 + \left(\frac{e^y - e^{-y}}{2}\right)^2}\, dy$$

$$= \frac{1}{2}\int_0^{\ln\left(\sqrt{2}+1\right)} \sqrt{4 + e^{2y} - 2 + e^{-2y}}\, dy$$

$$= \frac{1}{2}\int_0^{\ln\left(\sqrt{2}+1\right)} \sqrt{e^{2y} + 2 + e^{-2y}}\, dy$$

$$= \frac{1}{2}\int_0^{\ln\left(\sqrt{2}+1\right)} \sqrt{\left(e^y + e^{-y}\right)^2}\, dy$$

$$= \frac{1}{2}\int_0^{\ln\left(\sqrt{2}+1\right)} \left(e^y + e^{-y}\right) dy$$

OR

$a = \frac{1}{2}, \quad m = 1, \quad b = \frac{1}{2}, \quad n = -1$

1. $m = -n$ ✓

2. $abmn = -\frac{1}{4}$ ✓

$$L = \frac{1}{2}\left(e^y - e^{-y}\right)\Big|_0^{\ln\left(\sqrt{2}+1\right)}$$

$$= \frac{1}{2}\left(\sqrt{2} + 1 - \frac{1}{\sqrt{2}+1} - 1 + 1\right)$$

$$= \frac{1}{2}\left(\frac{3 + 2\sqrt{2} - 1}{\sqrt{2}+1}\right)$$

$$= \frac{1}{2}\left(\frac{2 + 2\sqrt{2}}{\sqrt{2}+1}\right)$$

$$= 1 \ \text{unit}$$

$$=\frac{1}{2}\left(e^{y}-e^{-y}\right)\Big|_{0}^{\ln\left(\sqrt{2}+1\right)}$$

$$=\frac{1}{2}\left(\sqrt{2}+1-\frac{1}{\sqrt{2}+1}-1+1\right)$$

$$=\frac{1}{2}\left(\frac{3+2\sqrt{2}-1}{\sqrt{2}+1}\right)$$

$$=\frac{1}{2}\left(\frac{2+2\sqrt{2}}{\sqrt{2}+1}\right)$$

$$=1\ \ unit$$

The differential Formula for Arc length

If $y=f(x)$ and if f' is continuous on $[a, b]$, then by the Fundamental Theorem of Calculus, we can define a new function

$$s(x)=\int_{a}^{x}\sqrt{1+\left[f'(t)\right]^{2}}\ dt$$

$$\frac{ds}{dx}=\sqrt{1+\left[f'(t)\right]^{2}}=\sqrt{1+\left(\frac{dy}{dx}\right)^{2}}\quad\Rightarrow\quad ds=\sqrt{1+\left(\frac{dy}{dx}\right)^{2}}dx$$

$$=\sqrt{(dx)^{2}+(dx)^{2}\frac{(dy)^{2}}{(dx)^{2}}}$$

$$\boxed{ds=\sqrt{dx^{2}+dy^{2}}}$$

Example

Find the arc length function for the curve $f(x)=\frac{x^{3}}{12}+\frac{1}{x}$ taking $A=\left(1,\ \frac{13}{12}\right)$ as the starting point

Solution

$$1+\left[f'(x)\right]^{2}=\left(\frac{x^{2}}{4}+\frac{1}{x^{2}}\right)^{2}$$

$$s(x)=\int_{1}^{x}\sqrt{1+\left[f'(t)\right]^{2}}\ dt$$

$$= \int_1^x \left(\frac{t^2}{4} + \frac{1}{t^2} \right) dt$$

$$= \left(\frac{t^3}{12} - \frac{1}{t} \right)_1^x$$

$$= \left(\frac{x^3}{12} - \frac{1}{x} \right) - \left(\frac{1}{12} - 1 \right)$$

$$= \frac{x^3}{12} - \frac{1}{x} + \frac{11}{12}$$

$$s(4) = \frac{4^3}{12} - \frac{1}{4} + \frac{11}{12}$$

$$= 6 \ \ unit$$

Exercises *Section* 1.5 – Length of Curves

Find the length of the curve of

1. $y=\frac{1}{3}\left(x^2+2\right)^{3/2}$ from $x=0$ to $x=3$

2. $y=(x)^{3/2}$ from $x=0$ to $x=4$

3. $x=\frac{y^{3/2}}{3}-y^{1/2}$ from $y=1$ to $y=9$

4. $x=\frac{y^3}{6}+\frac{1}{2y}$ from $y=2$ to $y=3$

5. $f(x)=x^3+\frac{1}{12x}$ *for* $\frac{1}{2}\le x\le 2$

6. $f(x)=\frac{1}{5}x^5+\frac{1}{12x^3}$ $1\le x\le 2$

7. $y=\frac{1}{3}x^{1/2}-x^{3/2},\quad 0\le x\le \frac{1}{3}$

8. $y=\frac{1}{3}x^3+\frac{1}{4x},\quad 1\le x\le 2$

9. $y=2e^x+\frac{1}{8}e^{-x}$ $0\le x\le \ln 2$

10. $y=e^{2x}+\frac{1}{16}e^{-2x}$ $0\le x\le \ln 3$

11. $y=\ln(\cos x)$ $0\le x\le \frac{\pi}{4}$

12. $f(y)=2e^{\sqrt{2}y}+\frac{1}{16}e^{-\sqrt{2}y}$ $0\le y\le \frac{\ln 2}{\sqrt{2}}$

13. $y=\frac{x^3}{3}+x^2+x+1+\frac{1}{4x+4}$ $0\le x\le 2$

14. $y=\frac{x^3}{3}+x^2+x+1+\frac{1}{4x+4}$ $0\le x\le 4$

15. $y=\ln\left(e^x-1\right)-\ln\left(e^x+1\right)$ $\ln 2\le x\le \ln 3$

16. $f(x)=\frac{2}{3}x^{3/2}-\frac{1}{2}x^{1/2}$ $1\le x\le 4$

17. $f(x)=x^3+\frac{1}{12x}$ $1\le x\le 4$

18. $f(x)=\frac{1}{8}x^4+\frac{1}{4x^2}$ $1\le x\le 10$

19. $f(x)=\frac{1}{4}x^4+\frac{1}{8x^2}$ $3\le x\le 8$

20. $f(x)=\frac{1}{10}x^5+\frac{1}{6x^3}$ $1\le x\le 7$

21. $f(x)=\frac{3}{10}x^{1/3}-\frac{3}{2}x^{5/3}$ $0\le x\le 12$

22. $f(x)=x^{1/2}-\frac{1}{3}x^{3/2}$ $2\le x\le 9$

23. $y=x^{1/2}-\frac{1}{3}x^{3/2}$ $1\le x\le 4$

24. $x=y^{2/3},\quad 1\le y\le 8$

25. $y=2x+4$ $-2\le x\le 2$

26. $y=\frac{x^3}{6}+\frac{1}{2x}$ $x\in[1,\ 2]$

27. $f(x)=x^{1/2}-\frac{1}{3}x^{3/2}$ $1\le x\le 3$

28. $y=\frac{3}{4}x^{4/3}-\frac{3}{8}x^{2/3}+5,\quad 1\le x\le 8$

29. $y=\ln x-\frac{1}{8}x^2;\quad 1\le x\le 2$

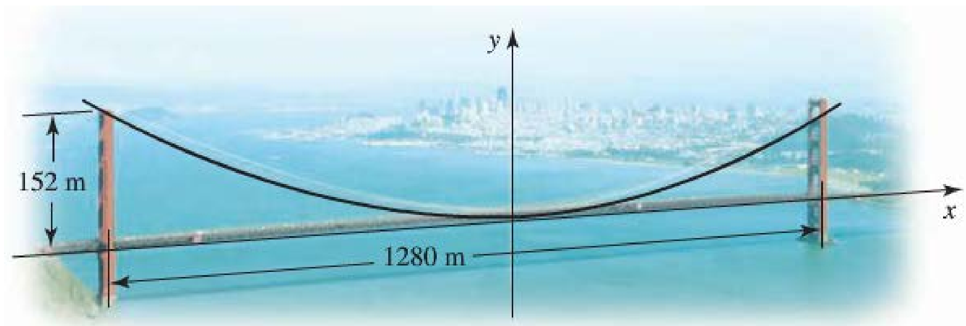
30. $y=\frac{1}{2}x^2-\frac{1}{4}\ln x;\quad 1\le x\le 3$

31. Find the length of the curve $y=\int_{-2}^{x}\sqrt{2t^4-1}\,dt$ $-2\le x\le -1$

32. Find the length of the curve $x=\int_{0}^{y}\sqrt{\sec^4 t-1}\,dt$ $-\frac{\pi}{4}\le y\le \frac{\pi}{4}$

33. Find the length of the curve $y = 3 - 2x \quad 0 \le x \le 2$. Check your answer by finding the length of the segment as the hypotenuse of a right triangle.

34. The profile of the cables on a suspension bridge may be modeled by a parabola. The central span of the Golden Gate Bridge is 1280 *m* long and 152 *m* high. The parabola $y = 0.00037x^2$ gives a good fit to the shape of the cables, where $|x| \le 640$, and *x* and *y* are measured in meters. Approximate the length of the cables that stretch between the tops of the two towers.

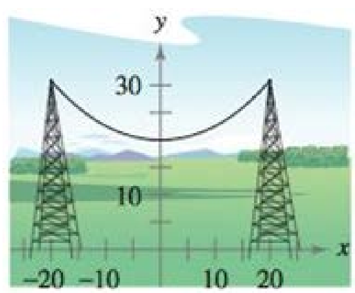


35. Find a curve through the origin in the *xy*-plane whose length from *x* = 0 to *x* = 1 is

$$L = \int_0^1 \sqrt{1 + \tfrac{1}{4}e^x}\, dx$$

36. Confirm that the circumference of a circle of radius *a* is $2\pi a$

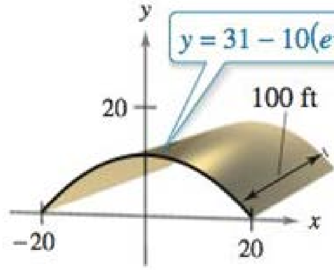
37. Electrical wires suspended between two towers form a caternary modeled by the equation



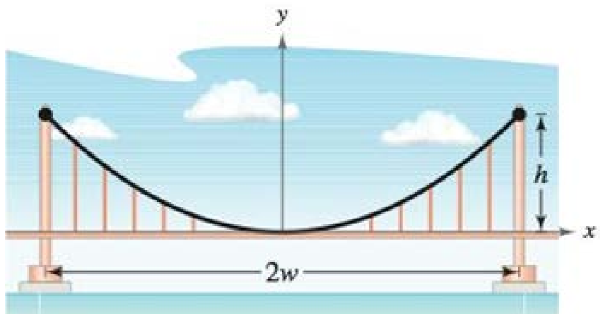
$$y = 20\cosh\frac{x}{20}, \quad -20 \le x \le 20$$

Where *x* and *y* are measured in meters. The towers are 40 meters apart. Find the length of the suspended cable.

38. A barn is 100 feet long and 40 feet wide. A cross section of the roof is the inverted caternary $y = 31 - 10\left(e^{x/20} + e^{-x/20}\right)$. Find the number of square feet of roofing on the barn.



39. A cable for a suspension bridge has the shape of a parabola with equation $y = kx^2$. Let h represent the height of the cable from it lowest point to its highest point and let $2w$ represent the total span of the bridge.



Show that the length C of the cable is given by $C = 2\int_0^w \sqrt{1 + \frac{4h^2}{w^4}x^2}\, dx$

40. Find the total length of the graph of the astroid $x^{2/3} + y^{2/3} = 4$

41. Find the arc length from $(0,\ 3)$ clockwise to $(2,\ \sqrt{5})$ along the circle $x^2 + y^2 = 9$

42. Find the arc length from $(-3,\ 4)$ clockwise to $(4,\ 3)$ along the circle $x^2 + y^2 = 25$. Show that the result is one-fourth the circumference of the circle.

43. $y = \ln x$ between $x = 1$ and $x = b > 1$ that

$$\int \frac{\sqrt{x^2 + a^2}}{x} dx = \sqrt{x^2 + a^2} - a \ln\left(\frac{a + \sqrt{x^2 + a^2}}{x}\right) + C$$

Use any means to approximate the value of b for which the curve has length 2.