Section 4.5 – Bessel's Equation and Bessel Functions

In this section we consider three special cases of ***Bessel's equation***

$$x^2y'' + xy' + \left(x^2 - \upsilon^2\right)y = 0$$

Where υ is a constant, and the solutions are called ***Bessel functions***.
The indicial equation is

$$I(r) = r(r-1) + p_0 r + q_0 = r(r-1) + r - \upsilon^2 = 0$$

$$r^2 - \upsilon^2 = 0 \quad \rightarrow \quad r = \pm\upsilon$$

We will consider the three cases $\upsilon = 0$, $\upsilon = \frac{1}{2}$, and $\upsilon = 1$ for the interval $x > 0$.

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + x\sum_{n=1}^{\infty}(n+r)a_n x^{n+r-1} + \left(x^2 - \upsilon^2\right)\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=1}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r+2} - \upsilon^2\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$x^r\left(\sum_{n=0}^{\infty}\left[(n+r)(n+r-1) + (n+r) - \upsilon^2\right]a_n x^n + \sum_{n=0}^{\infty}a_n x^{n+2}\right) = 0$$

$$(n+r)(n+r-1) + (n+r) = (n+r)(n+r-1+1) = (n+r)^2$$

$$\left(r^2 - \upsilon^2\right)a_0 + \left((1+r)^2 - \upsilon^2\right)a_1 + \underbrace{\sum_{n=2}^{\infty}\left[(n+r)^2 - \upsilon^2\right]a_n x^n}_{k=n-2} + \underbrace{\sum_{n=0}^{\infty}a_n x^{n+2}}_{k=n} = 0$$

$$\sum_{k=0}^{\infty}\left[(k+2+r)^2 - \upsilon^2\right]a_{k+2}x^{k+2} + \sum_{k=0}^{\infty}a_k x^{k+2} = 0$$

$$\sum_{k=0}^{\infty}\left[\left((k+2+r)^2 - \upsilon^2\right)a_{k+2} + a_k\right]x^{k+2} = 0$$

$$\left((k+2+r)^2 - \upsilon^2\right)a_{k+2} + a_k = 0$$

$$a_{k+2} = \frac{-a_k}{(k+2+r)^2 - \upsilon^2}$$

$$(k+2+r)^2 - \upsilon^2 = (k+2)^2 + 2r(k+2) + r^2 - \upsilon^2$$

$$= (k+2)(k+2+2r) + r^2 - \upsilon^2 \qquad r^2 - \upsilon^2 = 0$$

$$a_{k+2}=\frac{-a_k}{(k+2)(k+2+2\upsilon)}$$

We must choose $a_1=0 \quad\rightarrow\quad a_3=a_5=\cdots=0$

$$a_{2n}=-\frac{1}{2n(2n+2\upsilon)}a_{n-2}=-\frac{1}{2^2n(n+\upsilon)}a_{n-2} \qquad\qquad (2n=k+2)$$

$$a_2=-\frac{1}{2^2\cdot 1\cdot(1+\nu)}a_0$$

$$a_4(0)=-\frac{1}{2^2\cdot 2(2+\nu)}a_2=\frac{1}{2^4\cdot 1\cdot 2(1+\nu)(2+\nu)}a_0$$

$$a_6(0)=-\frac{1}{2^2\cdot 3(3+\nu)}a_4=-\frac{1}{2^6\cdot 1\cdot 2\cdot 3(1+\nu)(2+\nu)(3+\nu)}a_0$$

$$\vdots \qquad \vdots \;\; \vdots$$

$$\boxed{a_{2n}=\frac{(-1)^n}{2^{2n}n!(1+\upsilon)(2+\upsilon)\cdots(n+\upsilon)}a_0, \quad n=1,2,3,\ldots}$$

Gamma Function

$$(\upsilon+1)\cdot(\upsilon+2)\cdot\ldots\cdot(\upsilon+n)=\frac{(\upsilon+n)!}{\upsilon!}$$

The gamma function is defined by $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt \quad for\ x>0$

$$\Gamma(x)=\frac{\Gamma(x+1)}{x}=\frac{\Gamma(x+n)}{x\cdot(x+1)\cdot\ldots\cdot(x+n-1)}$$

$$x!=\Gamma(x+1)$$

$$(\upsilon+n)!=\Gamma(\upsilon+n+1)$$

$$a_{2n}=\frac{(-1)^n}{2^{2n+\upsilon}n!\Gamma(1+\upsilon+n)}, \quad n=0,1,2,3,\ldots$$

The series solution is denoted by $J_\upsilon(x)$: $\qquad J_\upsilon(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$

For $r_2=-\upsilon$, then $\qquad J_{-\upsilon}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1-\upsilon+n)}\left(\frac{x}{2}\right)^{2n-\upsilon}$

The functions $J_\upsilon(x)$ and $J_{-\upsilon}(x)$ are called the ***Bessel function of the first kind*** of order υ and $-\upsilon$.

Bessel Equation of Order Zero

In this case $\upsilon = 0$, that implies to Bessel's equation: $x^2 y'' + xy' + x^2 y = 0$

The roots of the indicial equation are equal: $r_1 = r_2 = 0$

Hence, $y_1(x) = a_0 \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} \right]$

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+n)} \left(\frac{x}{2}\right)^{2n}$$

$$y_2(x) = J_0(x)\ln x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H(n)}{(n!)^2} \left(\frac{x}{2}\right)^{2n} \qquad H(n) = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{n}$$

$$Y_0(x) = \frac{2}{\pi}\left[y_2(x) + (\gamma - \ln 2) J_0(x) \right]$$

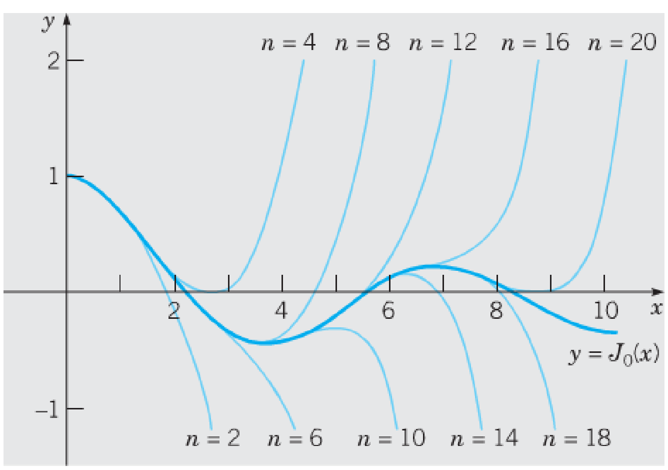
$$= \frac{2}{\pi}\left[\left(\ln\frac{x}{2} + \gamma\right) J_0(x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H(n)}{(n!)^2} \left(\frac{x}{2}\right)^{2n} \right]$$

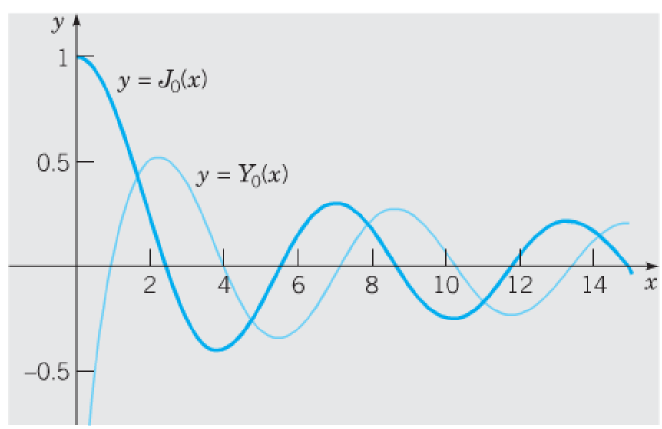
Where γ is ***Euler's constant***, defined by

$$\gamma = \lim_{n\to\infty} \left[H(n) - \ln n \right]$$

$$= \lim_{x\to\infty} \left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n \right]$$

$$= 0.5772156\cdots$$





Bessel Equation of Order *One-Half*

In this case $\upsilon = \frac{1}{2}$, that implies to Bessel's equation: $x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$

The roots of the indicial equation are equal: $r_1 = \frac{1}{2}, \quad r_2 = -\frac{1}{2}$

$$a_{2n} = -\frac{1}{2^2 n(n+\upsilon)} a_{n-2} = -\frac{1}{2^2 n\left(n+\frac{1}{2}\right)} a_{n-2} = -\frac{1}{2n(2n+1)} a_{n-2}$$

$$a_{2n} = \frac{(-1)^n}{2^{2n} n!(1+\upsilon)(2+\upsilon)\cdots(n+\upsilon)} a_0, \quad n = 1, 2, 3, \ldots$$

Taking $a_0 = 1$, we obtain

$$y_1(x) = x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}\right] = x^{-1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x, \quad x > 0$$

For $r_2 = -\frac{1}{2}$, $a_{2n} = \frac{(-1)^n}{(2n)!} a_0, \quad a_{2n+1} = \frac{(-1)^n}{(2n+1)!} a_1, \quad n = 1, 2, \ldots$

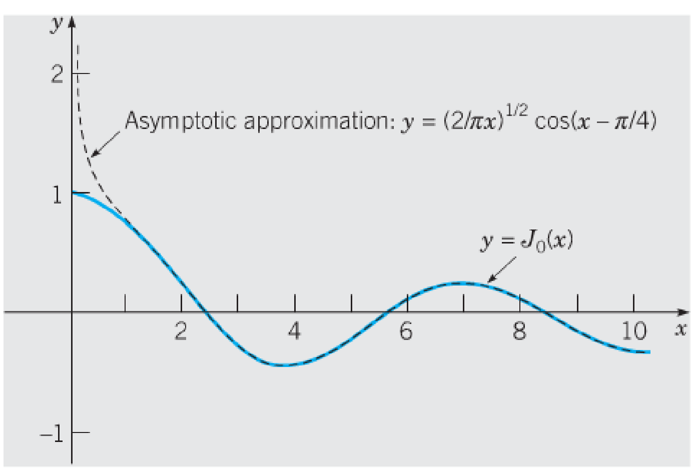
$$y_2(x) = x^{-1/2}\left[a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right]$$

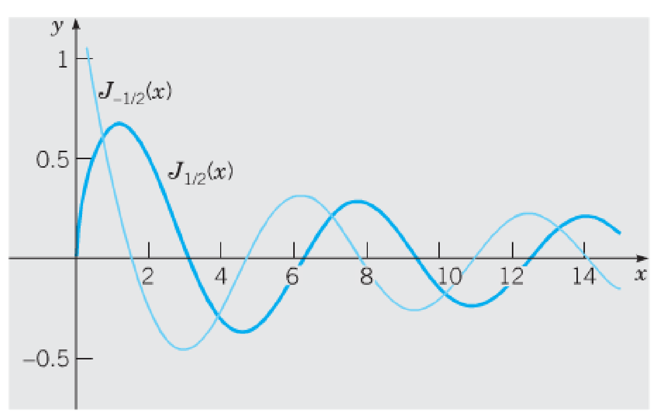
$$= a_0 \frac{\cos x}{x^{1/2}} + a_1 \frac{\sin x}{x^{1/2}}$$

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x, \quad x > 0$$

The general solution is:

$$y(x) = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$$





Bessel Equation of Order *One*

In this case $\upsilon = 1$, that implies to Bessel's equation: $x^2 y'' + xy' + \left(x^2 - 1\right) y = 0$

The roots of the indicial equation are equal: $r_1 = 1, \quad r_2 = -1$

$$a_{2n} = -\frac{1}{2^2 n(n+1)} a_{n-2}$$

$$a_{2n} = \frac{(-1)^n}{2^{2n} n!(n+1)!} a_0, \quad n = 1, 2, 3, \ldots$$

Taking $a_0 = \frac{1}{2}$, we obtain

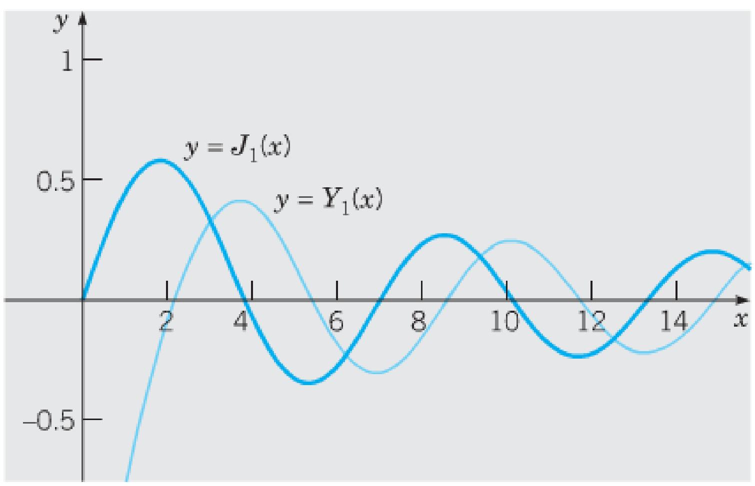
$$J_1(x) = \frac{x}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n+1)!n!}$$

$$y_2(x) = -J_1(x) \ln x + \frac{1}{x}\left[1 - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(H_n + H_{n-1}\right)}{2^{2n} n!(n-1)!} x^{2n}\right]$$

$$Y_1(x) = \frac{2}{\pi}\left[-y_2(x) + (\gamma - \ln 2) J_1(x)\right]$$

The general solution is:

$$y(x) = c_1 J_1(x) + c_2 Y_1(x)$$



Applications of *Bessel* Functions

The importance of Bessel functions stems not only from the frequent appearance of Bessel's equation in applications, but also from the fact that the solutions of many other second-order linear differential equations can be expressed in terms of Bessel functions.

The Bessel's equation is given by: $z^2 \frac{d^2w}{dz^2} + z\frac{dw}{dz} + \left(z^2 - \upsilon^2\right)w = 0$

Let $w = x^{-\alpha} y, \quad z = kx^{\beta}$

$$z = kx^{\beta} \quad \to x = \left(\frac{z}{k}\right)^{1/\beta}$$

$$\begin{aligned}
\frac{dw}{dz} &= \frac{dw}{dx}\frac{dx}{dz} \\
&= \frac{d}{dx}\left(x^{-\alpha} y\right)\frac{d}{dz}\left(\left(\frac{z}{k}\right)^{1/\beta}\right) \\
&= \left(-\alpha x^{-\alpha-1} y + x^{-\alpha}\frac{dy}{dx}\right)\left(\frac{1}{k\beta}\left(\frac{z}{k}\right)^{1/\beta - 1}\right) \\
&= \frac{1}{k\beta}\left(-\alpha x^{-\alpha-1} y + x^{-\alpha}\frac{dy}{dx}\right)\left(x^{\beta}\right)^{1/\beta-1} \\
&= \frac{1}{k\beta}\left(-\alpha x^{-\alpha-1} y + x^{-\alpha}\frac{dy}{dx}\right)x^{1-\beta} \\
&= \underline{\frac{1}{k\beta}\left(-\alpha x^{-\alpha-\beta} y + x^{1-\alpha-\beta}\frac{dy}{dx}\right)}
\end{aligned}$$

$$\begin{aligned}
\frac{d^2w}{dz^2} &= \frac{d}{dx}\left(\frac{dw}{dz}\right)\frac{dx}{dz} \\
&= \frac{1}{k\beta}\frac{d}{dx}\left(-\alpha x^{-\alpha-\beta} y + x^{1-\alpha-\beta}\frac{dy}{dx}\right)\frac{d}{dz}\left(\left(\frac{z}{k}\right)^{1/\beta}\right) \\
&= \frac{1}{k\beta}\left(\left(\alpha^2 + \alpha\beta\right)x^{-\alpha-\beta-1} y - \alpha x^{-\alpha-\beta}\frac{dy}{dx} + (1-\alpha-\beta)x^{-\alpha-\beta}\frac{dy}{dx} + x^{1-\alpha-\beta}\frac{d^2y}{dx^2}\right)\left(\frac{1}{k\beta}x^{1-\beta}\right) \\
&= \frac{1}{k^2\beta^2}\left(\left(\alpha^2 + \alpha\beta\right)x^{-\alpha-2\beta} y + \left((1-\alpha-\beta)x^{1-\alpha-2\beta} - \alpha x^{1-\alpha-2\beta}\right)\frac{dy}{dx} + x^{2-\alpha-2\beta}\frac{d^2y}{dx^2}\right) \\
&= \underline{\frac{1}{k^2\beta^2}\left(\left(\alpha^2 + \alpha\beta\right)x^{-\alpha-2\beta} y + (1-2\alpha-\beta)x^{1-\alpha-2\beta}\frac{dy}{dx} + x^{2-\alpha-2\beta}\frac{d^2y}{dx^2}\right)}
\end{aligned}$$

$$z^2 \frac{d^2w}{dz^2} + z\frac{dw}{dz} + \left(z^2 - \upsilon^2\right)w = 0$$

$$k^2x^{2\beta}\frac{1}{k^2\beta^2}\left(\left(\alpha^2+\alpha\beta\right)x^{-\alpha-2\beta}y+(1-2\alpha-\beta)x^{1-\alpha-2\beta}\frac{dy}{dx}+x^{2-\alpha-2\beta}\frac{d^2y}{dx^2}\right)$$
$$+kx^{\beta}\frac{1}{k\beta}\left(-\alpha x^{-\alpha-\beta}y+x^{1-\alpha-\beta}\frac{dy}{dx}\right)+\left(k^2x^{2\beta}-\upsilon^2\right)x^{-\alpha}y=0$$

$$\frac{1}{\beta^2}\left(\left(\alpha^2+\alpha\beta\right)x^{-\alpha}y+(1-2\alpha-\beta)x^{1-\alpha}\frac{dy}{dx}+x^{2-\alpha}\frac{d^2y}{dx^2}\right)+\frac{1}{\beta}\left(-\alpha x^{-\alpha}y+x^{1-\alpha}\frac{dy}{dx}\right)$$
$$+\left(k^2x^{2\beta}-\upsilon^2\right)x^{-\alpha}y=0$$

$$\left(\alpha^2+\alpha\beta\right)x^{-\alpha}y+(1-2\alpha-\beta)x^{1-\alpha}\frac{dy}{dx}+x^{2-\alpha}\frac{d^2y}{dx^2}-\alpha\beta x^{-\alpha}y+\beta x^{1-\alpha}\frac{dy}{dx}$$
$$+\left(k^2\beta^2x^{2\beta}-\beta^2\upsilon^2\right)x^{-\alpha}y=0$$

$$x^2x^{-\alpha}\frac{d^2y}{dx^2}+(1-2\alpha-\beta+\beta)xx^{-\alpha}\frac{dy}{dx}+\left(\alpha^2+\alpha\beta+k^2\beta^2x^{2\beta}-\beta^2\upsilon^2-\alpha\beta\right)x^{-\alpha}y=0$$

Then substitute into the Bessel's equation:

$$x^2\frac{d^2y}{dx^2}+(1-2\alpha)x\frac{dy}{dx}+\left(\alpha^2-\beta^2\upsilon^2+k^2\beta^2x^{2\beta}\right)y=0$$

That is equivalent to:

$$x^2\frac{d^2y}{dx^2}+Ax\frac{dy}{dx}+\left(B+Cx^p\right)y=0$$

Where $A=1-2\alpha,\quad B=\alpha^2-\beta^2\upsilon^2,\quad C=\beta^2k^2,\quad p=2\beta$

$$\to\upsilon^2=\frac{\alpha^2-B}{\beta^2}$$

$$\alpha=\frac{1-A}{2},\quad \beta=\frac{p}{2},\quad k=\frac{2\sqrt{C}}{p},\quad \upsilon=\frac{\sqrt{(1-A)^2-4B}}{p}\qquad \left((1-A)^2-4B>0\right)$$

Which follows that the general solution is:

$$y(x)=x^{\alpha}w(z)==x^{\alpha}w\left(kx^{\beta}\right)$$

Where

$$w(z)=c_1J_{\upsilon}(z)+c_2Y_{-\upsilon}(z)$$

Theorem: Solutions in Bessel Functions

If $C>0,\ p\neq 0$, and $(1-A)^2\geq 4B$, then the general solution (for $x>0$)

$$y(x)=x^{\alpha}\left[c_1J_{\upsilon}\left(kx^{\beta}\right)+c_2J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

Where α,β,k, and υ are given. If υ is an integer, then $J_{-\upsilon}$ is to be replaced by Y_{υ}.

$x^2 y'' + xy' + \left(x^2 - \upsilon^2\right) y = 0$	
$\upsilon = 0$	$Y_0(x) = \frac{2}{\pi}\left[y_2(x) + (\gamma - \ln 2) J_0(x)\right] = \frac{2}{\pi}\left[\left(\ln\frac{x}{2} + \gamma\right) J_0(x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H(n)}{(n!)^2}\left(\frac{x}{2}\right)^{2n}\right]$
$\upsilon = \frac{1}{2}$	$y(x) = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x) = c_1 \left(\frac{2}{\pi x}\right)^{1/2} \sin x + c_2 \left(\frac{2}{\pi x}\right)^{1/2} \cos x$
$\upsilon = 1$	$y(x) = c_1 J_1(x) + c_2 Y_1(x) = c_1 \frac{x}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n+1)!n!} + c_2 \frac{2}{\pi}\left[-y_2(x) + (\gamma - \ln 2) J_1(x)\right]$

Zeros of J_0, J_1, Y_0 *and* Y_1			
$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$
2.4048	0.0000	0.8936	2.1971
5.5201	3.8317	3.9577	5.4297
8.6537	7.156	7.0861	8.5960
11.7915	10.1735	10.2223	11.7492
14.9309	13.3237	13.3611	14.8974

Numerical Values J_0, J_1, Y_0 *and* Y_1				
x	$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$
0	1.0000	0.00	–	–
1	0.7652	0.4401	0.0883	–0.7812
2	0.2239	0.5767	0.5104	–0.1070
3	–0.2601	0.3391	0.3769	0.3247
4	–0.3971	–0.0660	–0.0169	0.3979
5	–0.1776	–0.3276	–0.3085	0.1479
6	0.1506	–0.2767	–0.2882	–0.1750
7	0.3001	–0.0047	–0.0259	–0.3027
8	0.1717	0.2346	0.2235	–0.1581
9	–0.0903	0.2453	0.2499	0.143
10	–0.2459	0.0435	0.0557	0.2490
11	–0.1712	–0.1768	–0.1688	0.1637
12	0.0477	–0.2234	–0.2252	–0.0571
13	0.2069	–0.0703	–0.0782	–0.2101
14	0.1711	0.1334	0.1272	–0.1666
15	–0.0142	0.2051	0.2055	0.0211

Exercises Section 4.5 – Bessel's Equation and Bessel Functions

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2y'' + xy' + \left(x^2 - \upsilon^2\right)y = 0$

1. $x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0$
2. $x^2y'' + xy' + \left(x^2 - 1\right)y = 0$
3. $4x^2y'' + 4xy' + \left(4x^2 - 25\right)y = 0$
4. $16x^2y'' + 16xy' + \left(16x^2 - 1\right)y = 0$
5. $xy'' + y' + xy = 0$
6. $xy'' + y' + \left(x - \frac{4}{x}\right)y = 0$

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2y'' + xy' + \left(\alpha^2x^2 - \upsilon^2\right)y = 0$

7. $x^2y'' + xy' + \left(9x^2 - 4\right)y = 0$
8. $x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right)y = 0$
9. $x^2y'' + xy' + \left(25x^2 - \frac{4}{9}\right)y = 0$
10. $x^2y'' + xy' + \left(2x^2 - 64\right)y = 0$

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

11. $4x^2y'' + 8xy' + \left(x^4 - 3\right)y = 0$
12. $y'' + 9xy = 0$
13. $xy'' + (x - 3)y = 0$
14. $xy'' + \left(4x^3 - 1\right)y = 0$
15. $x^2y'' + xy' - \left(\frac{1}{4} + x^2\right)y = 0$
16. $xy'' + (2x + 1)y' + (2x + 1)y = 0$
17. $xy'' - y' - xy = 0$
18. $x^4y'' + a^2y = 0$
19. $y'' - x^2y = 0$
20. $x^2y'' - xy' + \left(1 + x^2\right)y = 0$
21. $xy'' + 3y' + xy = 0$
22. $xy'' - y' + 36x^3y = 0$
23. $x^2y'' - 5xy' + (8 + x)y = 0$
24. $36x^2y'' + 60xy' + \left(9x^3 - 5\right)y = 0$
25. $16x^2y'' + 24xy' + \left(1 + 144x^3\right)y = 0$
26. $x^2y'' + 3xy' + \left(1 + x^2\right)y = 0$
27. $4x^2y'' - 12xy' + (15 + 16x)y = 0$
28. $16x^2y'' - \left(5 - 144x^3\right)y = 0$
29. $2x^2y'' + 3xy' - \left(28 - 2x^5\right)y = 0$
30. $y'' + x^4y = 0$
31. $y'' + 4x^3y = 0$

32. Find a Frobenius solution of Bessel's equation of order ***zero*** $x^2y'' + xy' + x^2y = 0$

33. Derive the formula $\quad xJ'_{\upsilon}(x) = \upsilon J_{\upsilon}(x) - xJ_{\upsilon+1}(x)$

34. Derive the formula $x J'_{\upsilon}(x) = -\upsilon J_{\upsilon}(x) + x J_{\upsilon-1}(x)$

35. Derive the formula $2\upsilon J'_{\upsilon}(x) = x J_{\upsilon+1}(x) + x J_{\upsilon-1}(x)$

36. Prove that $\frac{d}{dx}\left[x^{\upsilon+1} J_{\upsilon+1}(x)\right] = x^{\upsilon+1} J_{\upsilon}(x)$

37. Show that $y = \sqrt{x}\, J_{3/2}(x)$ is a solution of $x^2 y'' + \left(x^2 - 2\right) y = 0$

38. Show that $4J''_{\upsilon}(x) = J_{\upsilon-2}(x) - 2J_{\upsilon}(x) + J_{\upsilon+2}(x)$

39. Show that $y = x^{1/2} w\left(\frac{2}{3}\alpha x^{3/2}\right)$ is a solution of Airy's differential equation $y'' + \alpha^2 xy = 0$, $x > 0$, whenever w is a solution of Bessel's equation of order $\frac{2}{3}$, that is, $t^2 w'' + tw' + \left(t^2 - \frac{1}{9}\right) w = 0$, $t > 0$.

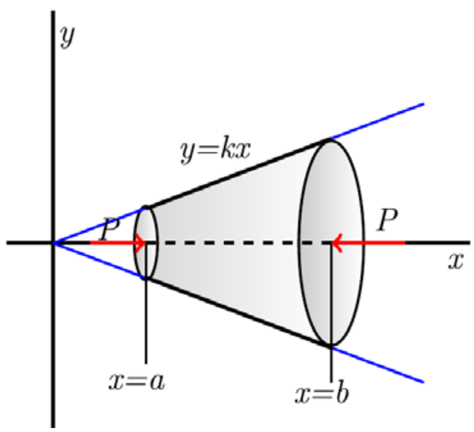
[*Hint*: After differentiating, substituting, and simplifying, then let $t = \frac{2}{3}\alpha x^{3/2}$].

40. Use the relation $\Gamma(x+1) = x\Gamma(x)$ and if υ is nonnegative integer, then show that

$$J_{\upsilon}(x) = \frac{1}{\Gamma(\upsilon+1)}\left(\frac{x}{2}\right)^{\upsilon}\left[1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(\upsilon+1)(\upsilon+2)\cdots(\upsilon+n)}\left(\frac{x}{2}\right)^{2n}\right]$$

41. A linearly tapered rod with circular cross section, subject to an axial force P of compression. Its deflection curve $y = y(x)$ satisfies the endpoint value problem

$$EIy'' + Py = 0\,; \quad y(a) = y(b) = 0 \quad (1)$$



Here, however, the moment of inertia $I = I(x)$ of the cross section at x is given by

$$I(x) = \frac{1}{4}\pi(kx)^4 = I_0\left(\frac{x}{b}\right)^4 \quad (2)$$

Where $I_0 = I(b)$, the value of I at $x = b$. Substitution of $I(x)$ in the differential equation (1) yields to the eigenvalue problem

$$x^4 y'' + \lambda y = 0\,; \quad y(a) = y(b) = 0 \quad (3)$$

Where $\lambda = \mu^2 = \dfrac{Pb^4}{EI_0}$

a) Show that the general solution of $x^4 y'' + \mu^2 y = 0$ is $y(x) = x\left(A\cos\dfrac{\mu}{x} + B\sin\dfrac{\mu}{x} \right)$

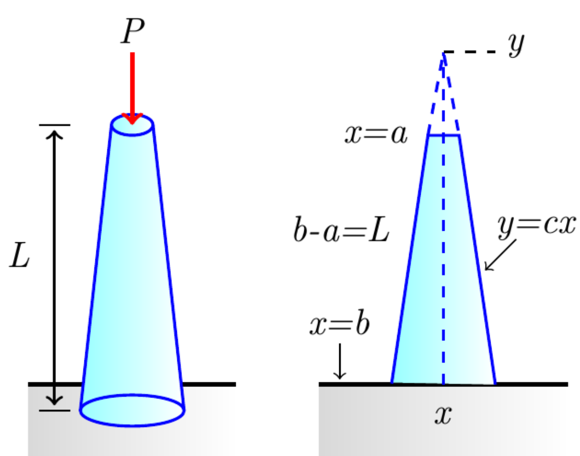
b) Conclude that the *n*th eigenvalue is given by $\mu_n = n\pi \dfrac{ab}{L}$, where $L = b - a$ is the length of the rod, and hence that the *n*th buckling force is

$$P_n = \frac{n^2\pi^2}{L^2}\left(\frac{a}{b}\right)^2 EI_0$$

42. When a constant vertical compressive force or load *P* was applied to a thin column of uniform cross section, the deflection $y(x)$ was a solution of the boundary-value problem

$$EI\frac{d^2y}{dy^2} + Py = 0\ ; \quad y(0) = 0, \quad y(L) = 0$$

The assumption here is that the column is hinged at both ends. The column will buckle or deflect only when the compression force is a critical load P_n



a) Let assume that the column is of length *L*, is hinged at both ends, has circular cross sections, and is tapered. If the column, a truncated cone, has a linear taper $y = cx$ in cross section, the moment of inertia of a cross section with respect to an axis perpendicular to the $xy - plane$ is $I = \frac{1}{4}\pi r^4$, where $r = y$ and $y = cx$. Hence, we can write $I(x) = I_0\,(x\ b)^4$, where $I_0 = I(b) = \frac{1}{4}\pi(cb)^4$. Substituting $I(x)$ into the differential equation, we see that the deflection in this case is determine from the *BVP*?

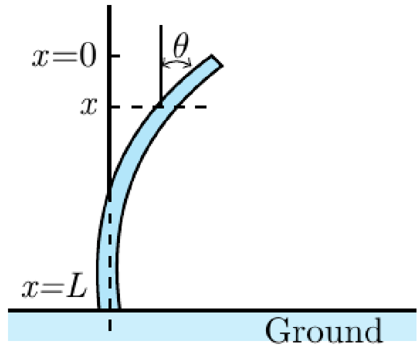
$$x^4\frac{d^2y}{dx^2} + \lambda y = 0\ ; \quad y(a) = 0, \quad y(b) = 0$$

Where $\lambda = Pb^4 EI_0$

Find the critical loads P_n for the tapered column. Use an appropriate identity to express the buckling modes $y_n(x)$ as a single function.

b) Plot the graph of the first buckling mode $y_1(x)$ corresponding to the Euler load P_1 when $b = 11$ and $a = 1$

43. For a practical application, we now consider the problem of determining when a uniform vertical column will buckle under its own weight (after, perhaps, being nudged laterally just a bit by a passing breeze). We take $x = 0$ at the free top end of the column and $x = L > 0$ at its bottom; we assume that the bottom is rigidly imbedded in the ground, perhaps in concrete.



Denote the angular deflection of the column at the point *x* by $\theta(x)$. From the theory of elasticity it follows that

$$EI\frac{d^2\theta}{dx^2} + g\rho x\theta = 0$$

Where *E* is the Young's modulus of the material of the column,

I is its cross-sectional moment of inertia

ρ is the linear density of the column

g is gravitational acceleration.

For physical reasons – no bending at the free top of the column and no deflection at its imbedded bottom – the boundary conditions are $\theta'(0) = 0, \quad \theta(L) = 0$

Determine the general equation of the length *L*.