Section 3.9 – Eigenvalues and Eigenvectors

In many problems in science and mathematics, linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ come from steady state problems. Eigenvalues have their greatest importance in dynamic problems. The solution of $Ax = \lambda x$ *or* $\frac{dx}{dt} = Ax$ (is changing with time) has nonzero solutions. (***All matrices are square***)

Definition

Suppose A is an n x n matrix and

$$\boldsymbol{\lambda x} = \boldsymbol{Ax}$$

The values of λ are called eigenvalues of the matrix $\boldsymbol{A}$ and the nonzero vectors $\boldsymbol{x}$ in $\boldsymbol{R}^n$ are called the eigenvectors corresponding to that eigenvalue (λ).

- One of the meanings of the word "***eigen***" in German is "***proper***"; eigenvalues are also called ***proper values, characteristic values,*** or ***latent roots***.

Example

The vector $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 3$ since

$$Ax = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3x$$

Eigenvalues and eigenvectors have a useful geometric interpretation in $\boldsymbol{R}^2$ and $\boldsymbol{R}^3$.

The equation for the *eigenvalues*

Let's rewrite the equation $Ax = \lambda x$.

$$Ax - \lambda x = 0 \qquad\qquad \lambda: \text{ are the eigenvalues and not a vector}$$

$$Ax - \lambda I x = 0$$

$$(A - \lambda I)x = 0$$

The matrix $A - \lambda I$ times the eigenvectors $\boldsymbol{x}$ is the zero vector. The eigenvectors makes up the nullspace of $A - \lambda I$.

Definition

The number λ is an eigenvalue of $\boldsymbol{A}$ if and only if $A - \lambda I$ is singular:

$$\boxed{\det(A - \lambda I) = 0}$$

This is called ***characteristic equation*** of $\boldsymbol{A}$; the scalars satisfying this equation are the eigenvalues of $\boldsymbol{A}$. when expanding the determinant $\det(A - \lambda I)$ is a polynomial in λ called the ***characteristic polynomial*** of $\boldsymbol{A}$.

Example

Find the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$

Solution

$$\begin{aligned}
\det(A - \lambda I) &= \det\left(\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{bmatrix}\right) \\
&= (3-\lambda)(-\lambda) + 2 \\
&= \lambda^2 - 3\lambda + 2
\end{aligned}$$

The characteristic equation of $\boldsymbol{A}$ is:

$\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \boxed{\lambda_1 = 1}$ $\boxed{\lambda_2 = 2}$; these are the eigenvalues of $\boldsymbol{A}$.

Theorem

If $\boldsymbol{A}$ is an n x n triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of $\boldsymbol{A}$ are the entries on the main diagonal of $\boldsymbol{A}$.

Example

Find the eigenvalues of the lower triangular matrix

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{3}{2} & 0 \\ 5 & -8 & -\frac{1}{4} \end{pmatrix}$$

Solution

The eigenvalues are: $\lambda = \frac{1}{2}, \quad \lambda = \frac{3}{2}, \quad \textit{and} \quad \lambda = -\frac{1}{4}$

Theorem

If $\boldsymbol{A}$ is an n x n matrix, the following are equivalent.

a) λ is an eigenvalue of $\boldsymbol{A}$.

b) The system of equations $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ has nontrivial solutions.

c) There is a nonzero vector $\boldsymbol{x}$ in $\boldsymbol{R}^n$ such that $Ax = \lambda x$.

d) λ is a real solution of the characteristic equation $\det(A - \lambda I) = 0$

Eigenvectors

To find the eigenvector $\boldsymbol{x}$, for each eigenvalue λ solve $(A-\lambda I)x=0$ *or* $Ax=\lambda x$

From the eigenvalues, the eigenvectors, in the form $\boldsymbol{V}_i=\begin{pmatrix} x_i \\ y_i \end{pmatrix}$, of the system can be determined by letting:

$$\left(A-\lambda_1 I\right)V_1=0 \qquad \text{and} \left(A-\lambda_2 I\right)V_2=0$$

Example

Find the eigenvalues and the eigenvectors of the matrix $A=\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

Solution

$$\det(A-\lambda I)=\det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix}$$

$$=(1-\lambda)(4-\lambda)-4$$

$$=\lambda^2-5\lambda+4-4$$

$$=\lambda^2-5\lambda$$

$$=\lambda(\lambda-5)=\mathbf{0}$$

The eigenvalues of $\boldsymbol{A}$ are: $\lambda_1=0 \quad \lambda_2=5$

For $\lambda_1=0$, we have:

$$\left(A-\lambda_1 I\right)V_1=0$$

$$\begin{pmatrix} 1-0 & 2 \\ 2 & 4-0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{cases} x+2y=0 \\ 2x+4y=0 \end{cases} \Rightarrow x=-2y$$

If $y=-1 \Rightarrow x=2$, therefore the eigenvector $V_1=\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Or $\begin{pmatrix} -2y \\ y \end{pmatrix}=y\begin{pmatrix} -2 \\ 1 \end{pmatrix} \Rightarrow V_1=\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ *or* $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

For $\lambda_2=5$, we have:

$$\left(A-\lambda_2 I\right)V_2 = 0$$

$$\begin{pmatrix} 1-5 & 2 \\ 2 & 4-5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{cases} -4x+2y=0 \\ 2x-y=0 \end{cases} \Rightarrow 2x = y$$

If $x = 1 \Rightarrow y = 2$, therefore the eigenvector $V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Power of a Matrix

Theorem

If k is a positive integer, λ is an eigenvalue of a matrix A, and $\boldsymbol{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\boldsymbol{x}$ is a corresponding eigenvector.

Example

Find the eigenvalues of A^7 for $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

Solution

$$\det\left(A-\lambda I\right) = \begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix} = \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

The eigenvalues of A: $\lambda = 1$ *and* $\lambda = 2$

The eigenvalues of A^7 are: $\boxed{\lambda = 1^7 = 1}$ *and* $\boxed{\lambda = 2^7 = 128}$

Theorem

A square matrix A is invertible *iff* $\lambda = 0$ is not an eigenvalue of A.

Summary

To solve the eigenvalue problem for an n by n matrix:

1. **1.** Compute the determinant of $A-\lambda I$. With λ subtracted along the diagonal, this determinant starts with λ^n or $-\lambda^n$. It is a polynomial in λ of degree n.
2. **2.** Find the roots of this polynomial, by solving $\det(A-\lambda I)=0$. The n roots are the n eigenvalues of A. They make $A-\lambda I$ singular.
3. **3.** For each eigenvalue λ, solve $(A-\lambda I)\mathbf{x}=\mathbf{0}$ ***to find an eigenvector x***.

Imaginary Eigenvalues

Example

Find the eigenvalues and the eigenvectors of the matrix $A=\begin{pmatrix}-2 & -1\\ 5 & 2\end{pmatrix}$

Solution

$$\det(A-\lambda I)=\begin{vmatrix}-2-\lambda & -1\\ 5 & 2-\lambda\end{vmatrix}$$

$$=(-2-\lambda)(2-\lambda)+5$$

$$=\lambda^2-4+5$$

$$=\lambda^2+1=0$$

$$\Rightarrow \lambda^2=-1$$

The solutions are: $\lambda=\pm i$.

$$\lambda_1=i:\left(A-\lambda_1 I\right)V_1=0$$

$$\begin{pmatrix}-2-i & -1\\ 5 & 2-i\end{pmatrix}\begin{pmatrix}x_1\\ y_1\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\to\begin{cases}(-2-i)x_1-y_1=0\\ 5x_1+(2-i)y_1=0\end{cases}\Rightarrow\begin{cases}(-2-i)x_1=y_1\\ 5x_1=-(2-i)y_1\end{cases}$$

Therefore the eigenvector $V_1=\begin{pmatrix}-1\\ 2+i\end{pmatrix}$

$$\lambda_1=-i:\left(A-\lambda_2 I\right)V_2=0$$

$$\begin{pmatrix}-2+i & -1\\ 5 & 2+i\end{pmatrix}\begin{pmatrix}x_2\\ y_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\to\begin{cases}(-2+i)x_2-y_2=0\\ 5x_2+(2+i)y_2=0\end{cases}\Rightarrow\begin{cases}(-2+i)x_2=y_2\\ 5x_2=-(2+i)y_2\end{cases}$$

Therefore the eigenvector $V_2=\begin{pmatrix}1\\ -2+i\end{pmatrix}$

Example

Find the eigenvalues and the eigenvectors of the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda^2 = -1 \quad \rightarrow \quad \lambda_1 = i \quad \lambda_2 = -i$$

The matrix $\boldsymbol{A}$ is a 90° rotation which has no real eigenvalues or eigenvectors.

No vector $\boldsymbol{Ax}$ stays in the same direction as $\boldsymbol{x}$ (except the zero vector which is useless).

If we add the eigenvalues together the result is zero which is the trace of $\boldsymbol{A}$.

$$\lambda_1 = i : \left(A - \lambda_1 I\right) V_1 = 0$$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -ix + y = 0 \\ -x - iy = 0 \end{cases} \Rightarrow x = -iy$$

Therefore the eigenvector $V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$$\lambda_2 = -i : \left(A - \lambda_2 I\right) V_2 = 0$$

$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} ix + y = 0 \\ -x + iy = 0 \end{cases} \Rightarrow x = iy$$

Therefore the eigenvector $V_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

Exercises *Section* 3.9 – Eigenvalues and Eigenvectors

1. Find the eigenvalues and eigenvectors of $A, A^2, A^{-1}, \text{ and } A+4I$:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad and \quad A^2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

Check the trace $\lambda_1 + \lambda_2$ and the determinant $\lambda_1 \lambda_2$ for A and also A^2.

2. Show directly that the given vectors are eigenvectors of the given matrix. What are the corresponding eigenvalues

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$$

3. For which real numbers c does this matrix A have

$$A = \begin{pmatrix} 2 & -c \\ -1 & 2 \end{pmatrix}$$

a) Two real eigenvalues and eigenvectors.

b) A repeated eigenvalue with only one eigenvector

c) Two complex eigenvalues and eigenvectors.

4. Find the eigenvalues of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{AB}$, and $\boldsymbol{BA}$:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad AB = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad BA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

a) The eigenvalues of $\boldsymbol{AB}$ (are equal to) (are not equal to) eigenvalues of $\boldsymbol{A}$ times eigenvalues of $\boldsymbol{B}$.

b) The eigenvalues of $\boldsymbol{AB}$ (are equal to) (are not equal to) eigenvalues of $\boldsymbol{BA}$.

5. When $a+b=c+d$ show that (1, 1) is an eigenvector and find both eigenvalues of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

6. The eigenvalues of A equal to the eigenvalues of A^T. This is because $\det(A-\lambda I)$ equals $\det\left(A^T - \lambda I\right)$. That is true because _____. Show by an example that the eigenvectors of A and A^T are not the same.

7. Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$. Compute the eigenvalues and eigenvectors of A.

8. Let $A=\begin{bmatrix}2 & -2 & 3\\ 1 & 1 & 1\\ 1 & 3 & -1\end{bmatrix}$

a) What is the characteristic polynomial for A (i.e. compute $\det(A-\lambda I)$?

b) Verify that 1 is an eigenvalue of A. What is a corresponding eigenvector?

c) What are the other eigenvalues of A?

9. For the following matrices:

$a)\ \begin{pmatrix}3 & 0\\ 8 & -1\end{pmatrix}$ $\quad b)\ \begin{pmatrix}10 & -9\\ 4 & -2\end{pmatrix}$ $\quad c)\ \begin{pmatrix}0 & 3\\ 4 & 0\end{pmatrix}$

$d)\ \begin{bmatrix}-2 & -7\\ 1 & 2\end{bmatrix}$ $\quad e)\ \begin{pmatrix}4 & 0 & 1\\ -2 & 1 & 0\\ -2 & 0 & 1\end{pmatrix}$ $\quad f)\ \begin{pmatrix}3 & 0 & -5\\ \frac{1}{5} & -1 & 0\\ 1 & 1 & -2\end{pmatrix}$

$g)\ \begin{pmatrix}-2 & 0 & 1\\ -6 & -2 & 0\\ 19 & 5 & -4\end{pmatrix}$ $\quad h)\ \begin{pmatrix}0 & 0 & 2 & 0\\ 1 & 0 & 1 & 0\\ 0 & 1 & -2 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$ $\quad i)\ \begin{pmatrix}10 & -9 & 0 & 0\\ 4 & -2 & 0 & 0\\ 0 & 0 & -2 & -7\\ 0 & 0 & 1 & 2\end{pmatrix}$

$j)\ \begin{bmatrix}-1 & 0 & 1\\ -1 & 3 & 0\\ -4 & 13 & -1\end{bmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

10. Find the eigenvalues of A^9 for $A=\begin{pmatrix}1 & 3 & 7 & 11\\ 0 & \frac{1}{2} & 3 & 8\\ 0 & 0 & 0 & 4\\ 0 & 0 & 0 & 2\end{pmatrix}$

11. Find the eigenvalues of the matrices

$$A=\begin{pmatrix}0.7 & 0.4\\ 0.3 & 0.6\end{pmatrix},\quad A^2=\begin{pmatrix}0.61 & 0.52\\ 0.39 & 0.48\end{pmatrix},\quad A^\infty=\begin{pmatrix}0.57143 & 0.57143\\ 0.42857 & 0.42857\end{pmatrix},\quad \textit{and}\ \ B=\begin{pmatrix}0.3 & 0.6\\ 0.7 & 0.4\end{pmatrix}$$

12. Given the matrix $\begin{bmatrix} -1 & -3 \\ -3 & 7 \end{bmatrix}$

a) Find the characteristic polynomial.
b) Find the eigenvalues
c) Find the bases for its eigenspaces
d) Graph the eigenspaces
e) Verify directly that $A\boldsymbol{v} = \lambda\boldsymbol{v}$, for all associated eigenvectors and eigenvalues.

13. Given the matrix $\begin{bmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{bmatrix}$

a) Find the characteristic polynomial.
b) Find the eigenvalues
c) Find the bases for its eigenspaces
d) Graph the eigenspaces
e) Verify directly that $A\boldsymbol{v} = \lambda\boldsymbol{v}$, for all associated eigenvectors and eigenvalues.

14. Given: $A = \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}$. Compute A^{11}