Section 3.3 – Properties of Division

Long Division

Divide $\left(x^3+2x^2-5x-6\right) \div (x+1)$

$$
\begin{array}{rl}
 & \overbrace{x^2+x-6}^{\textit{Quotient}} \\ \cline{2-2}
\underset{\textit{Divisor}}{x+1} \Big) & x^3+2x^2-5x-6 \longleftarrow \textit{Dividend} \\
 & \underline{x^3+x^2} \\
 & \quad x^2-5x \\
 & \quad \underline{x^2-x} \\
 & \qquad -6x-6 \\
 & \qquad \underline{-6x-6} \\
 & \qquad\qquad 0 \longleftarrow \textit{Remainder}
\end{array}
$$

$\boxed{Q(x)=x^2+x-6}$

$\boxed{R(x)=0}$

Example

Use the long division to find the quotient and the remainder: $\left(x^4-16\right)\div\left(x^2+3x+1\right)$

Solution

$$
\begin{array}{rl}
 & \qquad\qquad x^2-3x+8 \\ \cline{2-2}
x^2+3x+1 \Big) & x^4+0x^3+0x^2+0x-16 \\
 & \underline{x^4+3x^3+x^2} \\
 & -3x^3-x^2 \\
 & \underline{-3x^3-9x^2-3x} \\
 & \qquad\quad 8x^2+\ \ 3x-16 \\
 & \qquad\quad \underline{8x^2+24x+\ 8} \\
 & \qquad\qquad -21x-24
\end{array}
$$

$$\frac{x^4-16}{x^2+3x+1}=x^2-3x+8+\frac{-21x-24}{x^2+3x+1}$$

$$x^4-16=\boxed{\left(x^2+3x+1\right)\left(x^2-3x+8\right)+(-21x-24)}$$

Remainder *Theorem*

If a number c is substituted for x in the polynomial $f(x)$, then the result $f(c)$ is the remainder that would be obtained by dividing $f(x)$ by $x-c$.

That is, if $f(x)=(x-c)Q(x)+R(x)$ *then* $f(c)=R$

Example

If $f(x)=x^3-3x^2+x+5$, use the remainder theorem to find $f(2)$

Solution

$$\begin{array}{r|l} & x^2 - x - 1 \\ \hline x-2 & x^3 - 3x^2 + x + 5 \\ & \underline{x^3 - 2x^2} \\ & \quad -x^2 + x \\ & \quad \underline{-x^2 + 2x} \\ & \qquad\quad -x + 5 \\ & \qquad\quad \underline{-x + 2} \\ & \qquad\qquad\quad \boxed{3} \end{array}$$

$f(2)=3$

Factor *Theorem*

A polynomial $f(x)$ has a factor $x-c$ if and only if $f(c)=0$

Example

Show that $x-2$ is a factor of $f(x)=x^3-4x^2+3x+2$.

Solution

Since $f(2)=(2)^3-4(2)^2+3(2)=0$

From the factor theorem; $x-2$ is a factor of $f(x)$.

Synthetic Division

Use synthetic division to find the quotient and the remainder of $\left(4x^3-3x^2+x+7\right) \div (x-2)$

$$\begin{array}{c|cccc} & x^3 & x^2 & x^1 & x^0 \\ 2 & 4 & -3 & 1 & 7 \\ & & 8 & 10 & 22 \\ \hline & 4 & 5 & 11 & \boxed{29} \\ & x^2 & x^1 & x^0 & \end{array}$$

Quotient: $Q(x)=4x^2+5x+11$

Remainder: $R(x)=29$

Example

If $f(x)=3x^5-38x^3+5x^2-1$, use the synthetic division to find $f(4)$.

Solution

$$\begin{array}{c|cccccc} 4 & 3 & 0 & -38 & 5 & 0 & -1 \\ & & 12 & 48 & 40 & 180 & 720 \\ \hline & 3 & 12 & 10 & 45 & 180 & \boxed{719} \end{array}$$

$f(4)=719$

Example

Show that -11 is a zero of the polynomial $f(x)=x^3+8x^2-29x+44$

Solution

$$\begin{array}{c|cccc} -11 & 1 & 8 & -29 & 44 \\ & & -11 & 33 & -44 \\ \hline & 1 & -3 & 4 & \boxed{0} \end{array}$$

Thus, $f(-11)=0$, and -11 is a zero of f.

The Rational Zeros *Theorem*

If the polynomial $f(x)=a_n x^n + a_{n-1}x^{n-1}+\ldots+a_1 x+a_0$ *has integer* coefficients and if $\frac{c}{d}$ is a rational zero of $f(x)$ such that ***c*** and ***d*** have no common prime factor, then

1. The numerator c of the zero is a factor of the constant term a_0
2. The denominator d of the zero is a factor of the leading coefficient a_n

$$\text{possible rational zeros} = \frac{\text{factors of the constant term } a_0}{\text{factors of the leading coefficient } a_n} = \frac{\text{possibilities for } a_0}{\text{possibilities for } a_n}$$

Example

Find all rational solutions of the equation: $3x^4+14x^3+14x^2-8x-8=0$

Solution

possibilities for a_0	$\pm1,\ \pm2,\ \pm4,\ \pm8$
possibilities for a_n	$\pm1,\ \pm3$
possibilities for c/d	$\pm1,\ \pm2,\ \pm4,\ \pm8,\ \pm\frac{1}{3},\ \pm\frac{2}{3},\ \pm\frac{4}{3},\ \pm\frac{8}{3}$

Using the calculator, the result will show that -2 is a zero.

$$\begin{array}{r|rrrrr} -2 & 3 & 14 & 14 & -8 & -8 \\ & & -6 & -16 & 4 & 8 \\ \hline & 3 & 8 & -2 & -4 & \boxed{0} \end{array}$$

We have the factorization of: $(x+2)\left(3x^3+8x^2-2x-4\right)=0$

For $3x^3+8x^2-2x-4 \quad \Rightarrow \quad \frac{c}{d}=\frac{\pm1,\ \pm2,\ \pm4}{\pm1,\ \pm3}$

$x=-\frac{2}{3}$ is another solution.

$$\begin{array}{r|rrrr} -\frac{2}{3} & 3 & 8 & -2 & -4 \\ & & -2 & -4 & 4 \\ \hline & 3 & 6 & -6 & \boxed{0} \end{array}$$

We have the factorization of: $(x+2)\left(x+\frac{2}{3}\right)\left(3x^2+6x-6\right)=0$

By applying quadratic formula to solve: $3x^2+6x-6=0 \Rightarrow x=-1\pm\sqrt{3}$

Hence, the polynomial has two rational roots $x=-2$ *and* $-\frac{2}{3}$ and two irrational roots $x=-1\pm\sqrt{3}$.

Exercises *Section* 3.3 – Properties of Division

1. Find the quotient and remainder if $f(x)$ is divided by $p(x)$:

$f(x)=2x^4-x^3+7x-12; \quad p(x)=x^2-3$

2. Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=3x^3+2x-4; \quad p(x)=2x^2+1$

3. Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=7x+2; \quad p(x)=2x^2-x-4$

4. Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=9x+4; \quad p(x)=2x-5$

5. Use the remainder theorem to find $f(c)$: $f(x)=x^4-6x^2+4x-8; \quad c=-3$

6. Use the remainder theorem to find $f(c)$: $f(x)=x^4+3x^2-12; \quad c=-2$

7. Use the factor theorem to show that $x-c$ is a factor of $f(x)$: $f(x)=x^3+x^2-2x+12; \quad c=-3$

8. Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $2x^3-3x^2+4x-5; \quad x-2$

9. Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $5x^3-6x^2+15; \quad x-4$

10. Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $9x^3-6x^2+3x-4; \quad x-\frac{1}{3}$

11. Use the synthetic division to find $f(c)$: $f(x)=2x^3+3x^2-4x+4; \quad c=3$

12. Use the synthetic division to find $f(c)$: $f(x)=8x^5-3x^2+7; \quad c=\frac{1}{2}$

13. Use the synthetic division to find $f(c)$: $f(x)=x^3-3x^2-8; \quad c=1+\sqrt{2}$

14. Use the synthetic division to show that c is a zero of $f(x)$:

$f(x)=3x^4+8x^3-2x^2-10x+4; \quad c=-2$

15. Use the synthetic division to show that c is a zero of $f(x)$:

$f(x)=27x^4-9x^3+3x^2+6x+1; \quad c=-\frac{1}{3}$

16. Find all values of k such that $f(x)$ is divisible by the given linear polynomial:

$f(x)=kx^3+x^2+k^2x+3k^2+11; \quad x+2$

17. Find all solutions of the equation: $x^3-x^2-10x-8=0$

18. Find all solutions of the equation: $x^3+x^2-14x-24=0$

19. Find all solutions of the equation: $2x^3 - 3x^2 - 17x + 30 = 0$

20. Find all solutions of the equation: $12x^3 + 8x^2 - 3x - 2 = 0$

21. Find all solutions of the equation: $x^4 + 3x^3 - 30x^2 - 6x + 56 = 0$

22. Find all solutions of the equation: $3x^5 - 10x^4 - 6x^3 + 24x^2 + 11x - 6 = 0$

23. Find all solutions of the equation: $6x^5 + 19x^4 + x^3 - 6x^2 = 0$