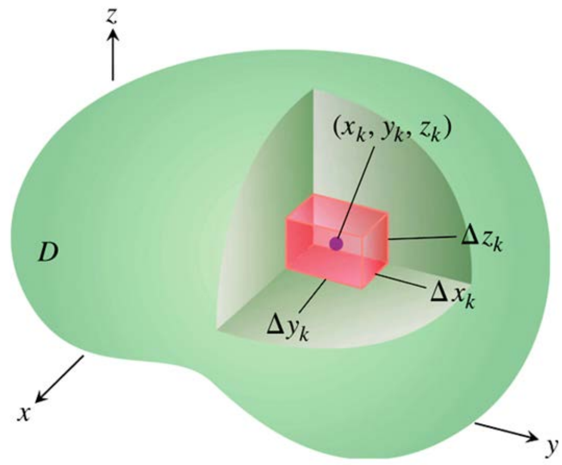
Section 3.4 – Triple Integrals

Triple Integrals

If $F(x,y,z)$ is a function defined on a closed, bounded region D in space, such a solid ball or a lump of clay, then the integral of F over D may be defined in the following way.



$$\Delta V_k = \Delta x_k \Delta y_k \Delta z_k \quad \to \quad S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\Delta V_k$$

The limit of this summation is the triple integral of F over D

$$\lim_{n\to\infty} S_n = \iiint\limits_D F(x,y,z)\,dV \quad or \quad \lim_{\|P\|\to} S_n = \iiint\limits_D F(x,y,z)\,dxdydz$$

Volume of a region in Space

Definition

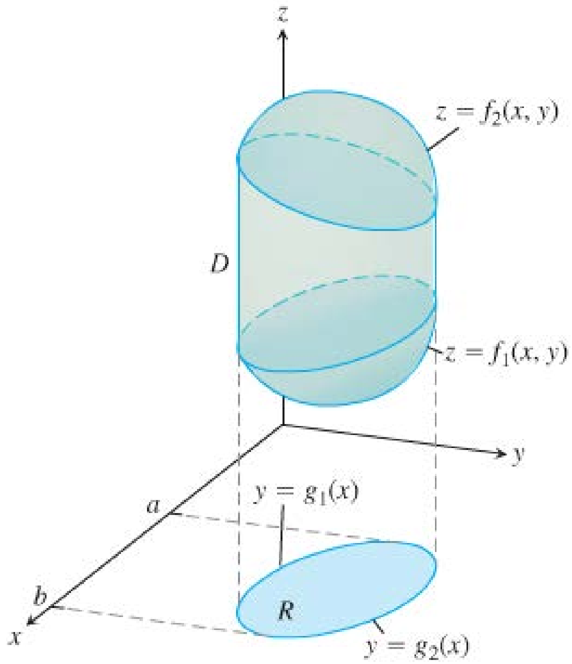
The volume of a closed, bounded region D in space is

$$V = \iiint\limits_D dV$$

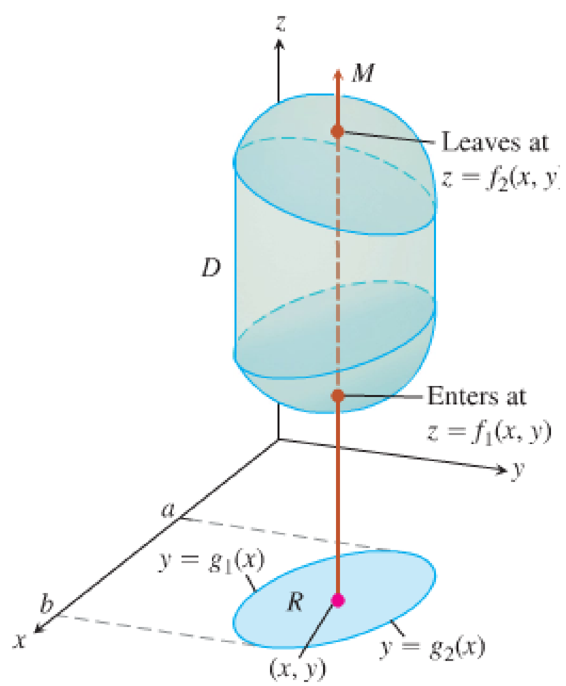
Find Limits of Integration in the Order *dz dy dx*

To evaluate $\iiint\limits_{D} F(x,y,z)\,dV$

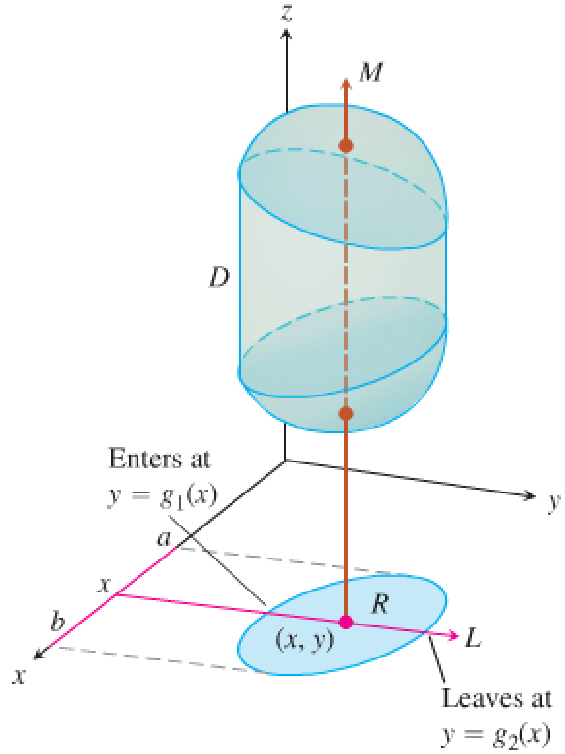
1. ***Sketch***: Sketch the region D along with its "shadow" R (vertical projection) in the xy-plane. Label the upper and lower bounding surfaces of D and R.



2. ***Find the z-limits of integration***: Draw a line M passing through (x,y) in R parallel to the z-axis. As z increases, M enters D at $z = f_1(x,y)$ and leaves at $z = f_2(x,y)$.



3. ***Find the y-limits of integration***: Draw a line L passing through (x, y) parallel to the y-axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$.



4. ***Find the x-limits of integration***: Choose x-limits that include all lines through R parallel to the y-axis $(x = a \ \ and \ \ x = b)$.

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\, dzdydx$$

Example

Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

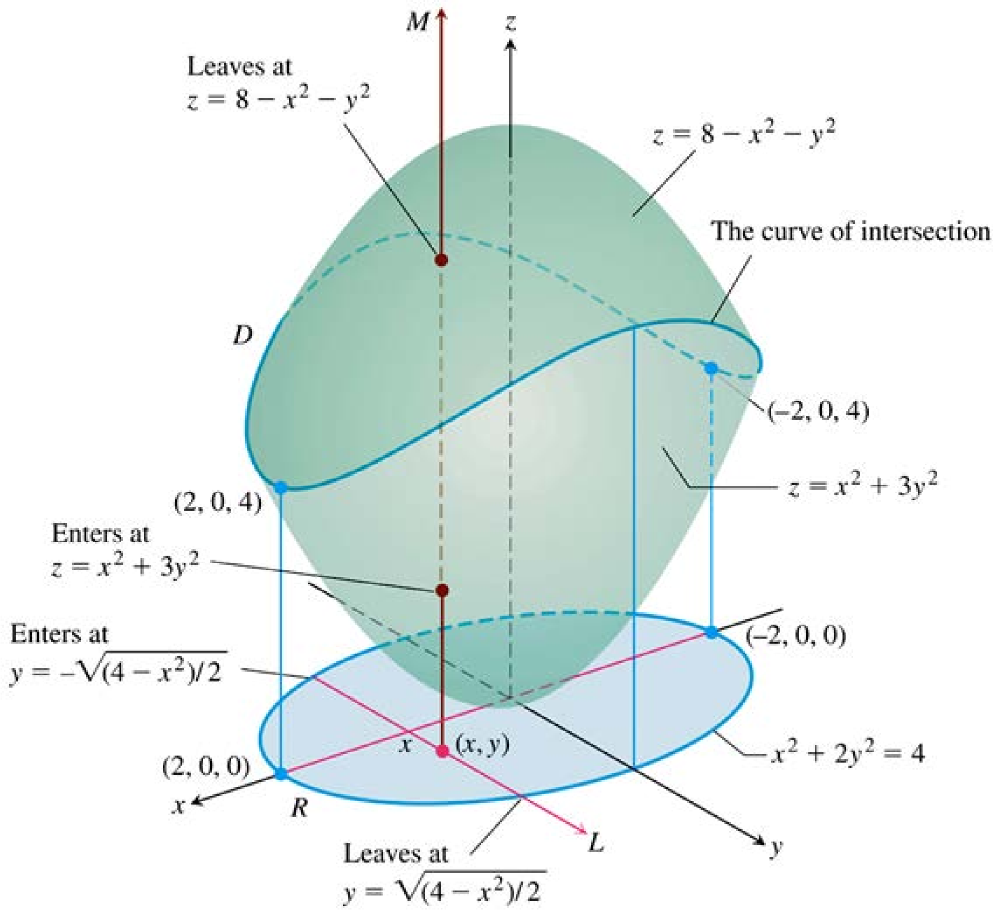
Solution

z-limits: $x^2 + 3y^2 \le z \le 8 - x^2 - y^2$

y-limits: $z = x^2 + 3y^2 = 8 - x^2 - y^2 \quad \to \quad 2x^2 + 4y^2 = 8 \ \Rightarrow \ x^2 + 2y^2 = 4$

$$y^2 = \frac{4 - x^2}{2} \quad \Rightarrow y = \pm\sqrt{\frac{4 - x^2}{2}} \quad \to -\sqrt{\frac{4 - x^2}{2}} \le y \le \sqrt{\frac{4 - x^2}{2}}$$

x-limits: $x^2 + 2y^2 = 4 \quad (y = 0) \quad \to x = \pm 2$



$$V=\int_{-2}^{2}\int_{-\sqrt{\left(4-x^2\right)/2}}^{\sqrt{\left(4-x^2\right)/2}}\int_{x^2+3y^2}^{8-x^2-y^2} dzdydx$$

$$=\int_{-2}^{2}\int_{-\sqrt{\left(4-x^2\right)/2}}^{\sqrt{\left(4-x^2\right)/2}} \left[z\right]_{x^2+3y^2}^{8-x^2-y^2} dydx$$

$$=\int_{-2}^{2}\int_{-\sqrt{\left(4-x^2\right)/2}}^{\sqrt{\left(4-x^2\right)/2}} \left(8-x^2-y^2-x^2-3y^2\right)dydx$$

$$=\int_{-2}^{2}\left[\left(8-2x^2\right)y-\frac{4}{3}y^3\right]_{-\sqrt{\left(4-x^2\right)/2}}^{\sqrt{\left(4-x^2\right)/2}} dx$$

$$=\int_{-2}^{2}\left[\left(8-2x^2\right)\sqrt{\frac{4-x^2}{2}}-\frac{4}{3}\left(\frac{4-x^2}{2}\right)^{3/2}+\left(8-2x^2\right)\sqrt{\frac{4-x^2}{2}}-\frac{4}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right]dx$$

$$=\int_{-2}^{2}\left[2\left(8-2x^2\right)\sqrt{\frac{4-x^2}{2}}-\frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right]dx$$

$$= \int_{-2}^{2} \left[2\left(\frac{2}{2}\right)(2)\left(4-x^2\right)\sqrt{\frac{4-x^2}{2}} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2} \right] dx$$

$$= \int_{-2}^{2} \left[8\left(\frac{4-x^2}{2}\right)\left(\frac{4-x^2}{2}\right)^{1/2} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2} \right] dx$$

$$= \int_{-2}^{2} \left[8\left(\frac{4-x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2} \right] dx$$

$$= \int_{-2}^{2} \left[\frac{16}{3}\left(\frac{4-x^2}{2}\right)^{3/2} \right] dx$$

$$= \frac{16}{3(2)^{3/2}} \int_{-2}^{2} \left(4-x^2\right)^{3/2} dx \qquad \frac{16}{3(2)^{3/2}}\frac{2^{1/2}}{2^{1/2}} = \frac{16\sqrt{2}}{3\cdot 4} = \frac{4\sqrt{2}}{3}$$

$$x = 2\sin u \quad dx = 2\cos u\, du \quad \left(4 - x^2 = 4 - 4\sin^2 u = 4\cos^2 u\right)$$

$$\begin{cases} x = 2 & \rightarrow u = \sin^{-1}\frac{x}{2} = \sin^{-1} 1 = \frac{\pi}{2} \\ x = -2 & \rightarrow u = \sin^{-1}(-1) = -\frac{\pi}{2} \end{cases}$$

$$= \frac{4\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(4\cos^2 u\right)^{3/2} \left(2\cos u\ du\right)$$

$$= \frac{4\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} 16(\cos u)^3 (\cos u)\, du$$

$$= \frac{64\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \cos^4 u\ du$$

$$= \frac{64\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(\frac{1+\cos 2u}{2}\right)^2 du$$

$$= \frac{16\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(1 + 2\cos 2u + \cos^2 2u\right) du$$

$$= \frac{16\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(1 + 2\cos 2u + \frac{1}{2} + \frac{1}{2}\cos 4u\right) du$$

$$= \frac{16\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(\frac{3}{2} + 2\cos 2u + \frac{1}{2}\cos 4u\right) du$$

$$= \frac{16\sqrt{2}}{3} \left[\frac{3}{2}u + \sin 2u + \frac{1}{8}\sin 4u\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{16\sqrt{2}}{3}\left[\frac{3\pi}{4} + \sin\pi + \frac{1}{8}\sin 2\pi - \left(-\frac{3\pi}{4} - \sin\pi - \frac{1}{8}\sin 2\pi\right)\right]$$

$$= \frac{16\sqrt{2}}{3}\left(\frac{3\pi}{2}\right)$$

$$= 8\pi\sqrt{2} \quad unit^3$$

Example

Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron D with vertices (0, 0, 0), (1, 1, 0), (0, 1, 0), and (0, 1, 1). Use the order of integration $dydzdx$.

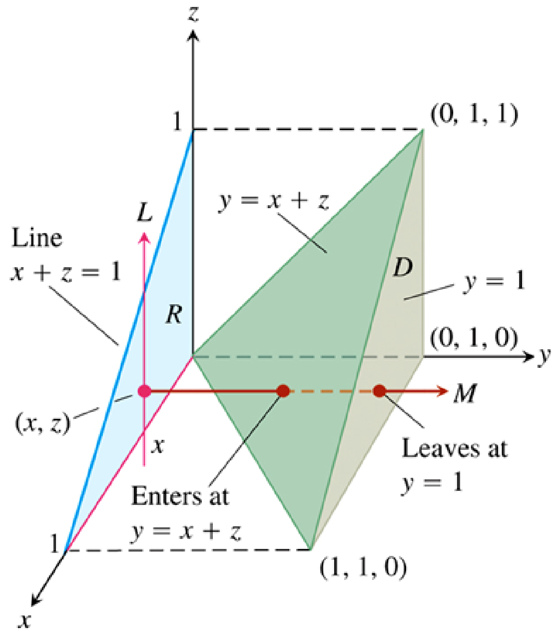
Solution

From the sketch, the upper (right-hand) bounding surface of D lies in the plane $y = 1$.

The lower (left-hand) bounding surface lies in the plane $y = x + z$.

The upper boundary of R is the line $z = 1 - x$.

The lower boundary is the line $z = 0$.



y-limits: The line through (x, z) in R parallel to the y-axis enters D at $y = x + z$ and leaves at $y = 1$.

z-limits: The line through (x, z) in R parallel to the z-axis enters R at $z = 0$ and leaves at $z = 1 - x$.

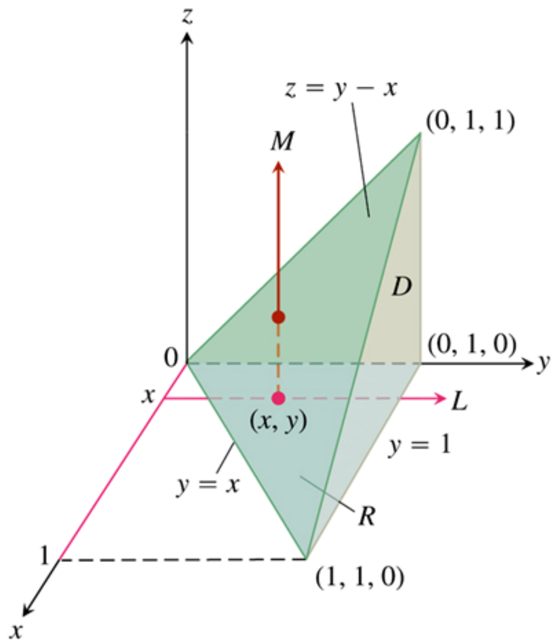
x-limits: $0 \le x \le 1$

$$\int_0^1 \int_0^{1-x} \int_{x+z}^{1} F(x, y, z)\, dydzdx$$

Example

Integrate $F(x, y, z) = 1$ over the tetrahedron D in the previous example in the order $dz\ dy\ dx$, and then integrate in the order $dy\ dz\ dx$.

Solution



z-limits of integration: A line M parallel to the z-axis through a typical point (x, y) in the xy-plane "shadow" enters the tetrahedron at $z = 0$ and exists through the upper plane where $z = y - x$. $0 \le z \le y - x$

Line is given by: $ax + by + cz = 0$ passes through the 2 points:

$(1,1,0) \to a + b = 0 \quad \Rightarrow \quad a = -b$

and $(0,1,1) \to b + c = 0 \quad \Rightarrow \quad c = -b$

$\to -bx + by - bz = 0$

$-x + y - z = 0 \quad \Rightarrow \quad z = y - x$

y-limits of integration: On the xy-plane, where $z = 0$, the sloped side of the tetrahedron crosses the plane along the line $y = x$. A line L through (x, y) parallel to the y-axis enters the shadow in the xy-plane at $y = x$ and exists at $y = 1$. $x \le y \le 1$

x-limits of integration: A line L parallel to the y-axis through a typical point (x, y) in the xy-plane sweeps out the shadow, where $0 \le x \le 1$ at the point $(1,1,0)$

The integral is: $\displaystyle\int_0^1 \int_x^1 \int_0^{y-x} F(x, y, z)\, dz\, dy\, dx$

$$V = \int_0^1 \int_x^1 \int_0^{y-x} dzdydx$$

$$= \int_0^1 \int_x^1 [z]_0^{y-x} dydx$$

$$= \int_0^1 \int_x^1 (y-x) dydx$$

$$= \int_0^1 \left[\tfrac{1}{2} y^2 - xy\right]_x^1 dx$$

$$= \int_0^1 \left[\tfrac{1}{2} - x - \left(\tfrac{1}{2} x^2 - x^2\right)\right] dx$$

$$= \int_0^1 \left(\tfrac{1}{2} - x + \tfrac{1}{2} x^2\right) dx$$

$$= \left[\frac{1}{2} x - \frac{1}{2} x^2 + \frac{1}{6} x^3\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{6}$$

$$= \frac{1}{6} \quad unit^3$$

$$V = \int_0^1 \int_0^{1-x} \int_{x+z}^1 dydzdx$$

$$= \int_0^1 \int_0^{1-x} [y]_{x+z}^1 dzdx$$

$$= \int_0^1 \int_0^{1-x} (1-x-z) dzdx$$

$$= \int_0^1 \left[z - xz - \tfrac{1}{2} z^2\right]_0^{1-x} dx$$

$$= \int_0^1 \left(1 - x - x(1-x) - \tfrac{1}{2}(1-x)^2\right) dx$$

$$= \int_0^1 \left((1-x)(1-x) - \tfrac{1}{2}(1-x)^2\right) dx$$

$$= \int_0^1 \left((1-x)^2 - \frac{1}{2}(1-x)^2 \right) dx$$

$$= \int_0^1 \frac{1}{2}(1-x)^2 \, dx$$

$$= -\frac{1}{6}(1-x)^3 \Big|_0^1$$

$$\boxed{= \frac{1}{6} \quad unit^3}$$

Average Value of a Function in Space

The average value of a function F over a region D in space is defined by the formula

$$\textbf{\textit{Average value}} \text{ of } F \text{ over } D = \frac{1}{\text{volume of } D} \iiint\limits_D F\,dV$$

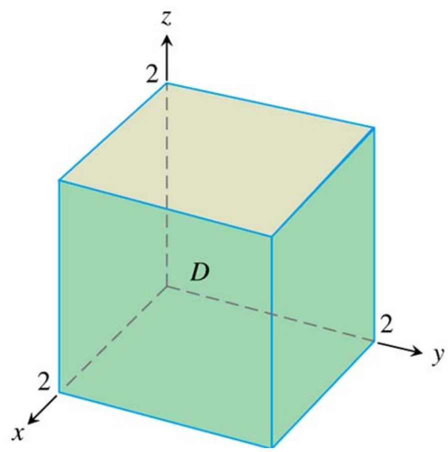
Example

Find the average of $F(x, y, z) = xyz$ throughout the cubical region D bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$ in the first octant.

Solution

$$Volume = 2 \cdot 2 \cdot 2 = 8$$

The value of the integral of F over the cube is



$$V = \int_0^2 \int_0^2 \int_0^2 xyz\,dx\,dy\,dz$$

$$= \int_0^2 z\,dz \int_0^2 y\,dy \int_0^2 x\,dx$$

$$= \left[\frac{1}{2}z^2\right]_0^2 \left[\frac{1}{2}y^2\right]_0^2 \left[\frac{1}{2}x^2\right]_0^2$$

$$= \frac{1}{8}(4)(4)(4)$$

$$= 8 \quad unit^3$$

$$\textbf{\textit{Average value}} \text{ of } xyz \text{ over } cube = \frac{1}{\text{volume of } D} \iiint\limits_{cube} xyz\,dV$$

$$= \left(\frac{1}{8}\right)(8)$$

$$= 1$$

Exercises   *Section* 3.4 – Triple Integrals

Evaluate the integral

1. $\displaystyle\int_0^1\int_0^1\int_0^1 \left(x^2+y^2+z^2\right)dzdydx$

2. $\displaystyle\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dzdxdy$

3. $\displaystyle\int_0^{\pi/6}\int_0^{1}\int_{-2}^{3} y\sin z\ dxdydz$

4. $\displaystyle\int_{-1}^1\int_0^1\int_0^2 (x+y+z)\,dydxdz$

5. $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2}} dzdydx$

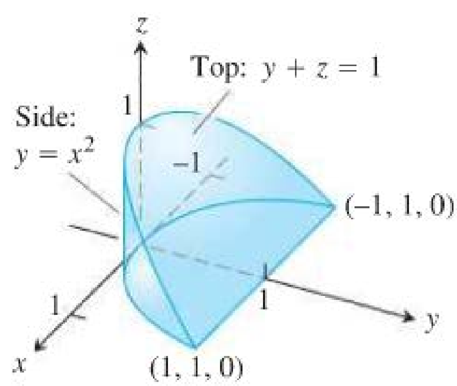
6. $\displaystyle\int_0^1\int_0^{1-x^2}\int_0^{4-x^2-y} xdzdydx$

7. $\displaystyle\int_0^\pi\int_0^\pi\int_0^\pi \cos(u+v+w)\,dudvdw$

8. $\displaystyle\int_0^{\pi/4}\int_0^{\ln\sec v}\int_{-\infty}^{2t} e^x dxdtdv$

9. Here is the region of integration of the integral

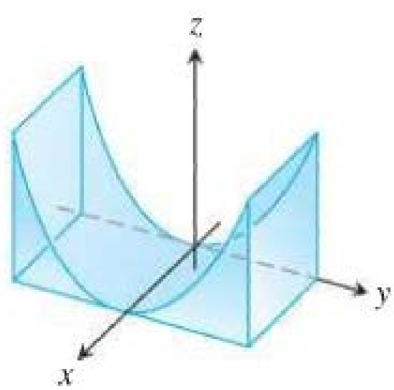
$$\int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} dzdydx$$



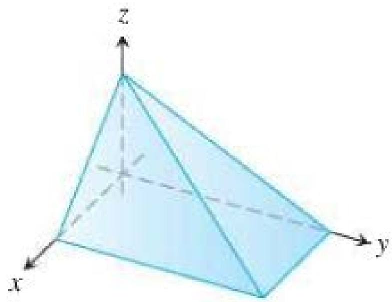
a) *dydzdx* ***b***) *dydxdz* ***c***) *dxdydz*

d) *dxdzdy* ***e***) *dzdxdy*

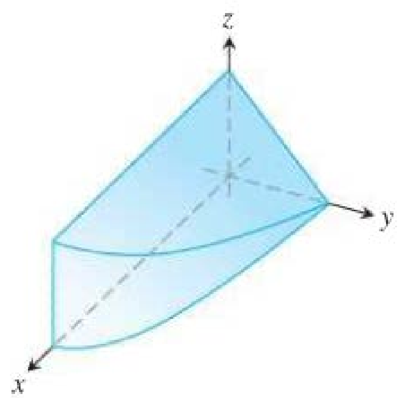
10. Find the volumes of the region between the cylinder $z=y^2$ and the *xy*-plane that is bounded by the planes $x=0,\quad x=1,\quad y=-1,\quad y=1$



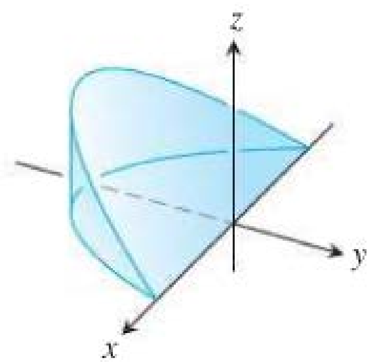
11. Find the volumes of the region in the first octant bounded by the coordinate planes and the planes $x+z=1, \quad y+2z=2$



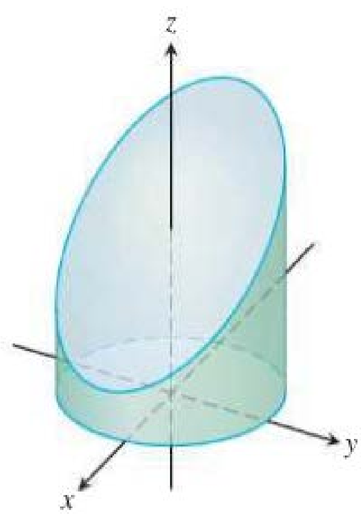
12. Find the volumes of the region in the first octant bounded by the coordinate planes and the plane $y+z=2$, and the cylinder $x=4-y^2$



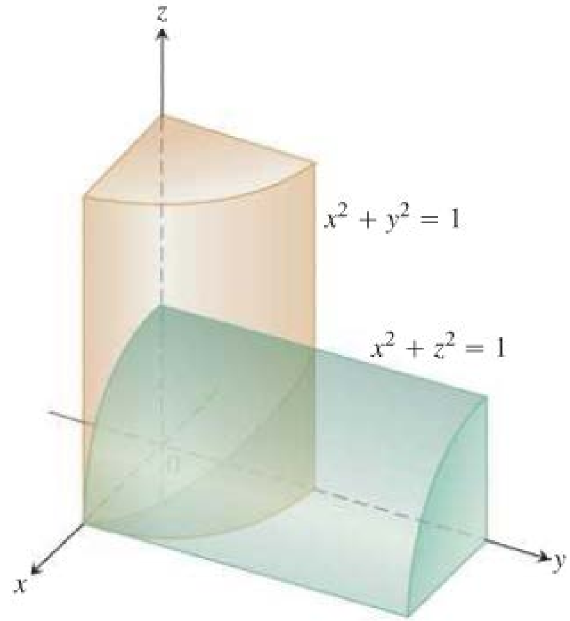
13. Find the volumes of the wedge cut from the cylinder $x^2+y^2=1$ by the planes $z=-y, \quad z=0$



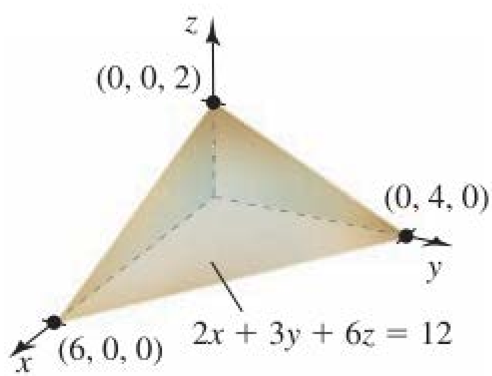
14. Find the volumes of the region cut from the cylinder $x^2+y^2=4$ by the plane $z=0$ and the plane $x+z=3$



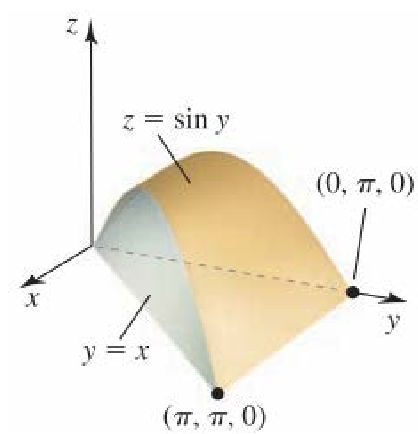
15. Find the volumes of the region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$, one-eighth of which is shown below



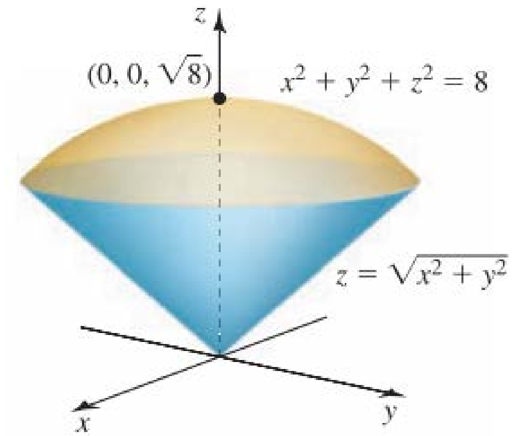
16. Find the volume of the solid in the first octant bounded by the plane $2x + 3y + 6z = 12$ and the coordinate planes



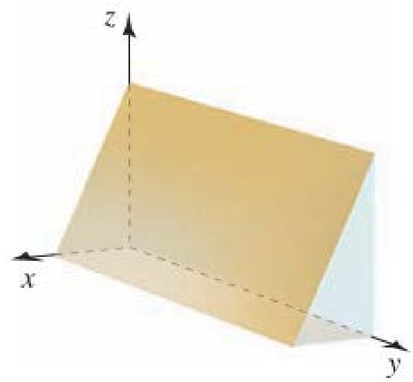
17. Find the volume of the solid in the first octant formed when the cylinder $z = \sin y$, for $0 \le y \le \pi$, is sliced by the planes $y = x$ and $x = 0$



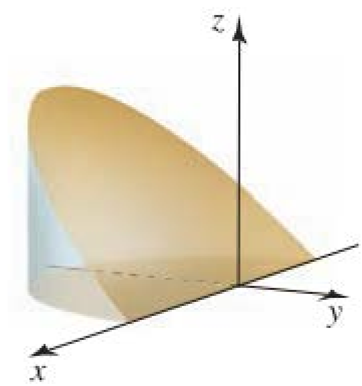
18. Find the volume of the solid bounded below by the cone $z=\sqrt{x^2+y^2}$ and bounded above the sphere $x^2+y^2+z^2=8$



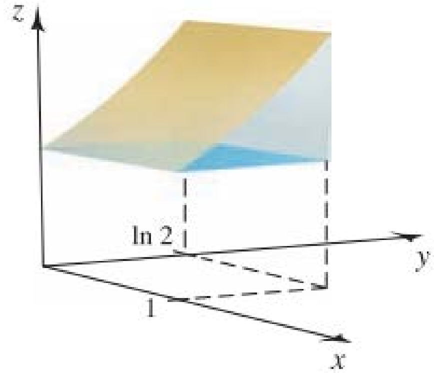
19. Find the volume of the prism in the first octant bounded below by $z=2-4x$ and $y=8$



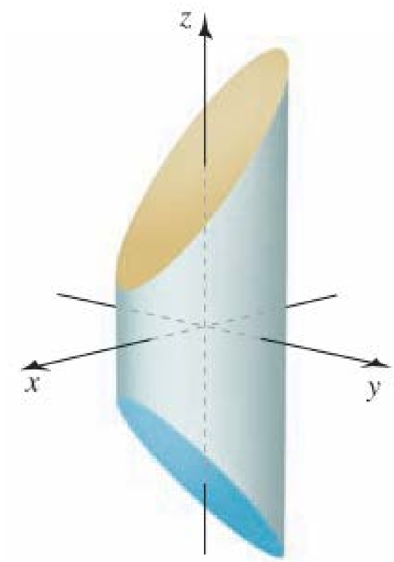
20. Find the volume of the wedge above the *xy*-plane formed when the cylinder $x^2+y^2=4$ is cut by the planes $z=0$ and $y=-z$



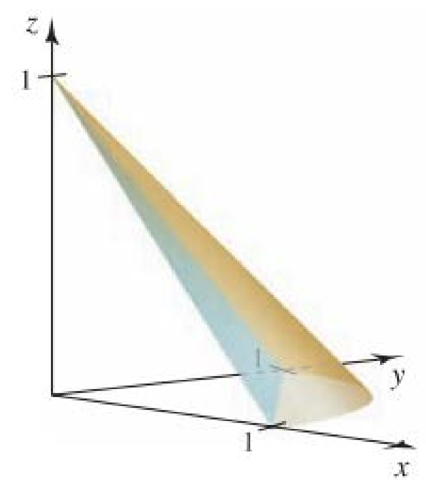
21. Find the volume of the solid bounded by the surfaces $z = e^y$ and $z = 1$ over the rectangle $\{(x, y):\ 0 \le x \le 1,\ 0 \le y \le \ln 2\}$



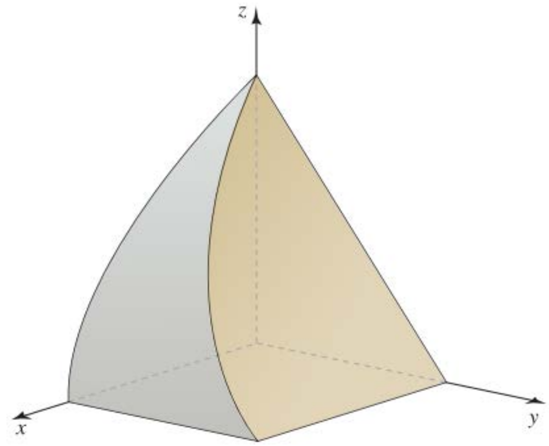
22. Find the volume of the wedge of the cylinder $x^2 + 4y^2 = 4$ created by the planes $z = 3 - x$ and $z = x - 3$



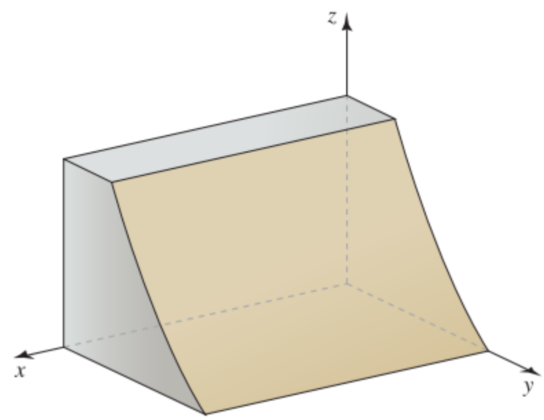
23. Find the volume of the solid in the first octant bounded by the cone $z = 1 - \sqrt{x^2 + y^2}$ and the plane $x + y + z = 1$



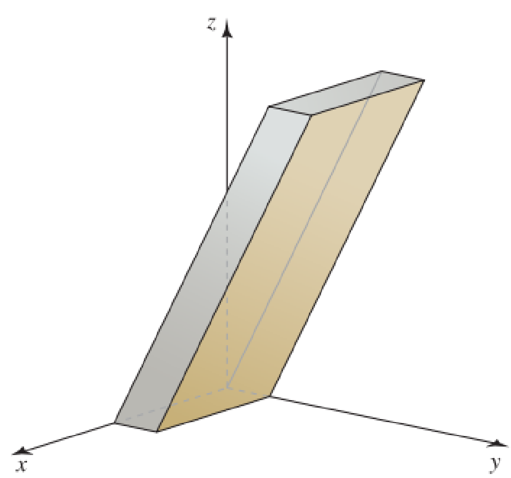
24. Find the volume of the solid bounded by $x = 0$, $x = 1 - z^2$, $y = 0$, $z = 0$, and $z = 1 - y$



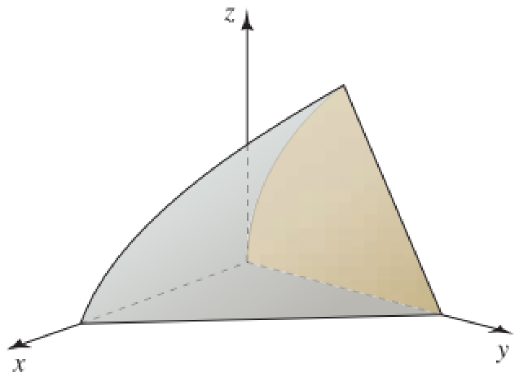
25. Find the volume of the solid bounded by $x = 0$, $x = 2$, $y = 0$, $y = e^{-z}$, $z = 0$, and $z = 1$



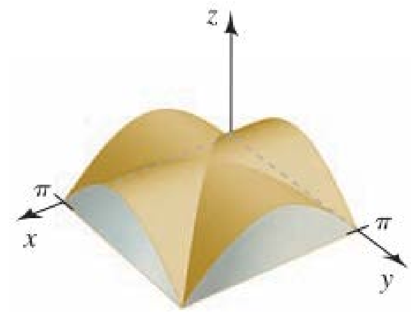
26. Find the volume of the solid bounded by $x = 0$, $x = 2$, $y = z$, $y = z + 1$, $z = 0$, and $z = 4$



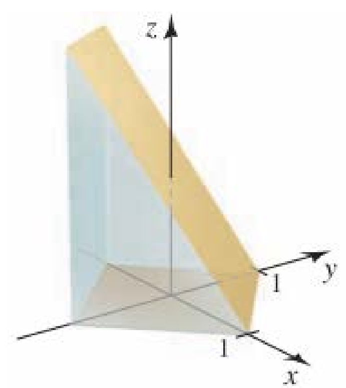
27. Find the volume of the solid bounded by $x = 0$, $y = z^2$, $z = 0$, and $z = 2 - x - y$



28. Find the volume of the solid common to the cylinders $z = \sin x$ and $z = \sin y$ over the square $R = \{(x, y):\ 0 \le x \le \pi,\ 0 \le y \le \pi\}$



29. Find the volume of the wedge of the square column $|x| + |y| = 1$ created by the planes $z = 0$ and $x + y + z = 1$



30. Find the volume of a right circular cone with height h and base radius r.

31. Find the volume of a tetrahedron whose vertices are located at $(0,\ 0,\ 0)$, $(a,\ 0,\ 0)$, $(0,\ b,\ 0)$, and $(0,\ 0,\ c)$

32. Find the volume of a truncated cone of height h whose ends have radii r and R.