Solution   *Section* 2.2 – Trigonometric Integrals

Exercise

Evaluate the integral $\int \sin^5 \frac{x}{2}\,dx$

Solution

$$\sin^5\frac{x}{2} = \left(\sin^2\frac{x}{2}\right)^2 \sin\frac{x}{2}$$

$$= \left(1-\cos^2\frac{x}{2}\right)^2 \sin\frac{x}{2}$$

$$= \left(1-2\cos^2\frac{x}{2}+\cos^4\frac{x}{2}\right)\sin\frac{x}{2}$$

$$d\left(\cos\frac{x}{2}\right) = -\frac{1}{2}\sin\frac{x}{2}\,dx \quad\rightarrow\quad -2d\left(\cos\frac{x}{2}\right) = \sin\frac{x}{2}\,dx$$

$$\int \sin^5\frac{x}{2}\,dx = -2\int\left(1-2\cos^2\frac{x}{2}+\cos^4\frac{x}{2}\right)d\left(\cos\frac{x}{2}\right)$$

$$= -2\left(\cos\frac{x}{2}-\frac{2}{3}\cos^3\frac{x}{2}+\frac{1}{5}\cos^5\frac{x}{2}\right)+C$$

$$= -2\cos\frac{x}{2}+\frac{4}{3}\cos^3\frac{x}{2}-\frac{2}{5}\cos^5\frac{x}{2}+C$$

Exercise

Evaluate $\int \sin^4 6\theta\,d\theta$

Solution

$$\int \sin^4 6\theta\,d\theta = \int\left(\frac{1-\cos 12\theta}{2}\right)^2 d\theta \qquad \sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$$

$$= \frac{1}{4}\int\left(1-2\cos 12\theta+\cos^2 12\theta\right)d\theta \qquad \cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$$

$$= \frac{1}{4}\int\left(1-2\cos 12\theta+\frac{1}{2}+\frac{1}{2}\cos 24\theta\right)d\theta$$

$$= \frac{1}{4}\left(\frac{3}{2}\theta-\frac{1}{6}\sin 12\theta+\frac{1}{48}\sin 24\theta\right)+C$$

Exercise

Evaluate $\int x^2\sin^2 x\,dx$

Solution

		$\int \cos 2x$
+	x^2	$\frac{1}{2}\sin 2x$
−	$2x$	$-\frac{1}{4}\cos 2x$
+	2	$-\frac{1}{8}\sin 2x$

$$\int x^2 \sin^2 x\, dx = \frac{1}{2}\int x^2 (1-\cos 2x)\, dx$$

$$= \frac{1}{2}\int \left(x^2 - x^2\cos 2x\right) dx$$

$$= \frac{1}{2}\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 \sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x\right) + C$$

Exercise

Evaluate $\int \sin^3 3x\, dx$

Solution

$$\int \sin^3 3x\, dx = \int \sin^2 3x(\sin 3x)\, dx \qquad d(\cos 3x) = -3\sin 3x\, dx \qquad \cos^2\alpha + \sin^2\alpha = 1$$

$$= -\frac{1}{3}\int \left(1-\cos^2 3x\right) d(\cos 3x)$$

$$= -\frac{1}{3}\left(\cos 3x - \frac{1}{3}\cos^3 3x\right) + C$$

$$= \frac{1}{9}\cos^3 3x - \frac{1}{3}\cos 3x + C$$

Exercise

Evaluate the integral $\int \sin^5 x\, dx$

Solution

$$\int \sin^5 x\, dx = \int \sin^4 x\, \sin x\, dx$$

$$= -\int \left(1-\cos^2 x\right)^2 d(\cos x)$$

$$= -\int \left(1 - 2\cos^2 x + \cos^4 x\right) d(\cos x)$$

$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$$

Exercise

Evaluate the integral $\int 8\cos^4 2\pi x\, dx$

Solution

$$\int 8\cos^4 2\pi x\, dx = 8\int (\cos 2\pi x)^4\, dx \qquad \cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

$$= 8\int \left(\frac{1+\cos 4\pi x}{2}\right)^2 dx$$

$$= 2\int (1+\cos 4\pi x)^2\, dx$$

$$= 2\int \left(1 + 2\cos 4\pi x + \cos^2 4\pi x\right) dx$$

$$= 2\int dx + 4\int \cos 4\pi x dx + 2\int \cos^2 4\pi x dx$$

$$= 2x + 4\frac{1}{4\pi}\cos 4\pi x + 2\int \frac{1+\cos 8\pi x}{2}\, dx$$

$$= 2x + \frac{1}{\pi}\cos 4\pi x + \int (1+\cos 8\pi x)\, dx$$

$$= 2x + \frac{1}{\pi}\sin 4\pi x + x + \frac{1}{8\pi}\sin 8\pi x + C$$

$$= 3x + \frac{1}{\pi}\sin 4\pi x + \frac{1}{8\pi}\sin 8\pi x + C$$

Exercise

Evaluate the integral $\int x\cos^3 x\, dx$

Solution

$$\int x\cos^3 x dx = \int x\cos^2 x\cos x\, dx$$

$$= \int x\left(1-\sin^2 x\right)\cos x\, dx \qquad \cos^2\alpha + \sin^2\alpha = 1$$

$$= \int x\cos x\, dx - \int x\sin^2 x\cos x\, dx$$

$$u = x \qquad dv = \cos x dx \qquad\qquad u = x \qquad dv = \sin^2 x \cos x dx$$

$$du = dx \qquad v = \sin x \qquad\qquad du = dx \qquad v = \frac{1}{3}\sin^3 x$$

$$= x\sin x - \int \sin x\, dx - \left(\frac{1}{3}x\sin^3 x - \frac{1}{3}\int \sin^3 x\, dx\right)$$

$$= x\sin x + \cos x - \frac{1}{3}x\sin^3 x + \frac{1}{3}\int \sin^2 x \sin x\, dx$$

$$= x\sin x + \cos x - \frac{1}{3}x\sin^3 x - \frac{1}{3}\int \left(1 - \cos^2 x\right) d(\cos x)$$

$$= x\sin x + \cos x - \frac{1}{3}x\sin^3 x - \frac{1}{3}\left(\cos x - \frac{1}{3}\cos^3 x\right) + C$$

$$= x\sin x + \cos x - \frac{1}{3}x\sin^3 x - \frac{1}{3}\cos x + \frac{1}{9}\cos^3 x + C$$

$$= x\sin x + \frac{2}{3}\cos x - \frac{1}{3}x\sin^3 x + \frac{1}{9}\cos^3 x + C$$

Exercise

Evaluate the integral $\int \cos^4 x\, dx$

Solution

$$\int \cos^4 x\, dx = \frac{1}{4}\int (1 + \cos 2x)^2\, dx \qquad\qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$= \frac{1}{4}\int \left(1 + 2\cos 2x + \cos^2 2x\right) dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x + \sin 2x + \frac{1}{8}\sin 4x\right) + C$$

Exercise

Evaluate the integral $\int \cos^4 5x\, dx$

Solution

$$\int \cos^4 5x\, dx = \frac{1}{4}\int (1 + \cos 10x)^2\, dx \qquad\qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$= \frac{1}{4}\int \left(1 + 2\cos 10x + \cos^2 10x\right) dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos 10x + \frac{1}{2} + \frac{1}{2}\cos 20x\right) dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 10x + \frac{1}{2}\cos 20x\right) dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x + \frac{1}{5}\sin 10x + \frac{1}{40}\sin 20x\right) + C$$

$$= \frac{3}{8}x + \frac{1}{20}\sin 10x + \frac{1}{160}\sin 20x + C$$

Exercise

Evaluate $\displaystyle\int \cos^2 3x\, dx$

Solution

$$\int \cos^2 3x\, dx = \frac{1}{2}\int \left(1 + \cos 6x\right) dx \qquad \cos^2\alpha = \frac{1}{2}\left(1 + \cos 2\alpha\right)$$

$$= \frac{1}{2}\left(x + \frac{1}{6}\sin 6x\right) + C$$

Exercise

Evaluate $\displaystyle\int \cos^3 \frac{x}{3}\, dx$

Solution

$$\int \cos^3 \frac{x}{3}\, dx = \int \cos^2 \frac{x}{3}\cos\frac{x}{3}\, dx$$

$$= 3\int \left(1 - \sin^2 \frac{x}{3}\right)\, d\left(\sin\frac{x}{3}\right)$$

$$= 3\sin\frac{x}{3} - \sin^3\frac{x}{3} + C$$

Exercise

Evaluate the integral $\int \cos^2 4x\, dx$

Solution

$$\int \cos^2 4x\, dx = \frac{1}{2}\int (1+\cos 8x)\, dx \qquad \cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$$

$$\boxed{= \frac{1}{2}\left(x + \frac{1}{8}\sin 8x\right) + C}$$

Exercise

Evaluate the integral $\int \sqrt{1+\cos\frac{x}{2}}\, dx$

Solution

$$2\cos^2\alpha = 1+\cos 2\alpha$$

$$2\alpha = \frac{x}{2} \quad \rightarrow \quad \alpha = \frac{x}{4}$$

$$1+\cos\frac{x}{2} = 2\cos^2\frac{x}{4}$$

$$\int \sqrt{1+\cos\frac{x}{2}}\, dx = \int \sqrt{2\cos^2\frac{x}{4}}\, dx$$

$$= \sqrt{2}\int \cos\frac{x}{4}\, dx$$

$$\boxed{= 4\sqrt{2}\sin\frac{x}{4} + C}$$

Exercise

Evaluate $\int \sec^4 2x\, dx$

Solution

$$\int \sec^4 2x\, dx = \int \left(1+\tan^2 2x\right)\sec^2 2x\, dx$$

$$= \frac{1}{2}\int \left(1+\tan^2 2x\right)\, d(\tan 2x)$$

$$\boxed{= \frac{1}{2}\tan 2x + \frac{1}{6}\tan^3 2x + C}$$

Exercise

Evaluate the integral $\int 6\sec^4 x\,dx$

Solution

$$\begin{aligned}\int 6\sec^4 x\,dx &= 6\int \sec^2 x\ \sec^2 x\,dx\\ &= 6\int \sec^2 x\left(\tan^2 x+1\right)dx\\ &= 6\int \left(\tan^2 x+1\right)d\left(\tan x\right)\\ &= 6\left(\frac{1}{3}\tan^3 x+\tan x\right)+C\\ &= 2\tan^3 x+6\tan x+C\end{aligned}$$

Exercise

Evaluate $\int \sec^3 \pi x\,dx$

Solution

$u=\sec\pi x \qquad dv=\sec^2\pi x\,dx$

$du=\pi\sec\pi x\tan\pi x \qquad v=\frac{1}{\pi}\tan\pi x$

$$\begin{aligned}\int \sec^3\pi x\,dx &= \frac{1}{\pi}\sec\pi x\tan\pi x-\int \sec\pi x\tan^2\pi x\,dx\\ &= \frac{1}{\pi}\sec\pi x\tan\pi x-\int \sec\pi x\left(\sec^2\pi x-1\right)dx\end{aligned}$$

$$\int \sec^3\pi x\,dx = \frac{1}{\pi}\sec\pi x\tan\pi x-\int \sec^3\pi x\,dx+\int \sec\pi x\,dx$$

$$2\int \sec^3\pi x\,dx = \frac{1}{\pi}\sec\pi x\tan\pi x+\ln\left|\sec\pi x+\tan\pi x\right|+C_1$$

$$\int \sec^3\pi x\,dx = \frac{1}{2\pi}\sec\pi x\tan\pi x+\frac{1}{2}\ln\left|\sec\pi x+\tan\pi x\right|+C$$

Exercise

Evaluate the integral $\displaystyle\int \sec 4x\, dx$

Solution

$$\begin{aligned}\int \sec 4x\, dx &= \frac{1}{4}\int \sec 4x\, d(4x)\\ &= \frac{1}{4}\int \sec 4x \frac{\sec 4x + \tan 4x}{\sec 4x + \tan 4x} d(4x)\\ &= \frac{1}{4}\int \frac{\sec^2 4x + \sec 4x \tan 4x}{\sec 4x + \tan 4x}\, d(4x)\\ &= \frac{1}{4}\int \frac{1}{\sec 4x + \tan 4x}\, d(\sec 4x + \tan 4x)\\ &= \frac{1}{4}\ln\left|\sec 4x + \tan 4x\right| + C\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \csc^6 x\, dx$

Solution

$$\begin{aligned}\int \csc^6 x\, dx &= \int \csc^4 x\ \csc^2 x\, dx\\ &= -\int \left(\cot^2 x + 1\right)^2\, d(\cot x)\\ &= -\int \left(\cot^4 x + 2\cot^2 x + 1\right) d(\cot x)\\ &= -\frac{1}{5}\cot^5 x - \frac{2}{3}\cot^3 x - \cot x + C\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \tan^5 \frac{x}{2}\, dx$

Solution

$$\int \tan^5 \frac{x}{2}\, dx = 2\int \tan^2 \frac{x}{2}\ \tan^3 \frac{x}{2}\ d\left(\frac{x}{2}\right)$$

$$= 2\int \left(\sec^2 \frac{x}{2} - 1\right) \tan^3 \frac{x}{2}\, d\left(\frac{x}{2}\right)$$

$$= 2\int \sec^2 \frac{x}{2} \tan^3 \frac{x}{2}\, d\left(\frac{x}{2}\right) - 2\int \tan^2 \frac{x}{2} \tan \frac{x}{2}\, d\left(\frac{x}{2}\right)$$

$$= 2\int \tan^3 \frac{x}{2}\, d\left(\tan \frac{x}{2}\right) - 2\int \left(\sec^2 \frac{x}{2} - 1\right) \tan \frac{x}{2}\, d\left(\frac{x}{2}\right)$$

$$= \frac{1}{2}\tan^4 \frac{x}{2} - 2\int \sec^2 \frac{x}{2} \tan \frac{x}{2}\, d\left(\frac{x}{2}\right) + 2\int \tan \frac{x}{2}\, d\left(\frac{x}{2}\right)$$

$$= \frac{1}{2}\tan^4 \frac{x}{2} - 2\int \tan \frac{x}{2}\, d\left(\tan \frac{x}{2}\right) + 2\int \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}}\, d\left(\frac{x}{2}\right)$$

$$= \frac{1}{2}\tan^4 \frac{x}{2} - \tan^2 \frac{x}{2} - 2\int \frac{1}{\cos \frac{x}{2}}\, d\left(\cos \frac{x}{2}\right)$$

$$= \frac{1}{2}\tan^4 \frac{x}{2} - \tan^2 \frac{x}{2} - 2\ln\left|\cos \frac{x}{2}\right| + C$$

$$= \frac{1}{2}\tan^4 \frac{x}{2} - \tan^2 \frac{x}{2} + 2\ln\left|\sec \frac{x}{2}\right| + C$$

Exercise

Evaluate the integral $\int \tan^5 5\, dx$

Solution

$$\int \tan^5 x\, dx = \int \tan^2 x \tan^3 x\, dx$$

$$= \int \left(\sec^2 x - 1\right) \tan^3 x\, dx$$

$$= \int \sec^2 x \tan^3 x\, dx - \int \tan^2 x \tan x\, dx$$

$$= \int \tan^3 x\, d(\tan x) - \int \left(\sec^2 x - 1\right) \tan x\, dx$$

$$= \frac{1}{4}\tan^4 x - \int \sec^2 x \tan x\, dx + \int \tan x\, dx$$

$$= \frac{1}{4}\tan^4 x - \int \tan x\, d(\tan x) + \int \frac{\sin x}{\cos x}\, dx$$

$$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \int \frac{1}{\cos x}\, d(\cos x)$$

$$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C$$

$$\underline{= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln|\sec x| + C}$$

Exercise

Evaluate $\displaystyle\int \tan^6 3x\, dx$

Solution

$$\int \tan^6 3x\, dx = \int \left(\sec^2 3x - 1\right)\tan^4 3x\, dx$$

$$= \frac{1}{3}\int \tan^4 3x\, d(\tan 3x) - \int \left(\sec^2 3x - 1\right)\tan^2 3x\, dx$$

$$= \frac{1}{15}\tan^5 3x - \int \sec^2 3x \tan^2 3x\, dx + \int \tan^2 3x\, dx$$

$$= \frac{1}{15}\tan^5 3x - \frac{1}{3}\int \tan^2 3x\, d(\tan 3x) + \int \left(\sec^2 3x - 1\right) dx$$

$$= \frac{1}{15}\tan^5 3x - \frac{1}{9}\tan^3 3x + \frac{1}{3}\int d(\tan 3x) - \int dx$$

$$\underline{= \frac{1}{15}\tan^5 3x - \frac{1}{9}\tan^3 3x + \frac{1}{3}\tan 3x - x + C}$$

Exercise

Evaluate the integral $\displaystyle\int 20\tan^6 x\, dx$

Solution

$$\int 20\tan^6 x\, dx = 20\int \left(\sec^2 x - 1\right)\tan^4 x\, dx$$

$$= 20\int \tan^4 x\, d(\tan x) - 20\int \left(\sec^2 x - 1\right)\tan^2 x\, dx$$

$$= 4\tan^5 x - 20\int \sec^2 x \tan^2 x\, dx + 20\int \tan^2 x\, dx$$

$$= 4\tan^5 x - 20\int \tan^2 x\, d(\tan x) + 20\int \left(\sec^2 x - 1\right) dx$$

$$= 4\tan^5 x - \frac{20}{3}\tan^3 x + 20\tan x - 20x + C$$

Exercise

Evaluate the integral $\int \tan^4 x \, dx$

Solution

$$\int \tan^4 x \, dx = \int \left(\tan^2 x\right)\left(\tan^2 x\right) dx$$

$$= \int \tan^2 x \left(\sec^2 x - 1\right) dx$$

$$= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx$$

$$= \int \tan^2 x \, d(\tan x) - \int \left(\sec^2 x - 1\right) dx$$

$$= \frac{1}{3}\tan^3 x - \tan x + x + C$$

Exercise

Evaluate the integral $\int \tan^3 \theta \, d\theta$

Solution

$$\int \tan^3 \theta \, d\theta = \int \tan^2 \theta \ \tan\theta \, d\theta$$

$$= \int \left(\sec^2 \theta - 1\right) \tan\theta \, d\theta$$

$$= \int \sec^2 \theta \ \tan\theta \, d\theta - \int \tan\theta \, d\theta$$

$$= \int \tan\theta \, d(\tan\theta) - \ln\left|\sec\theta\right|$$

$$= \frac{1}{2}\tan^2 \theta - \ln\left|\sec\theta\right| + C$$

Exercise

Evaluate the integral $\displaystyle\int \tan^3 4x\, dx$

Solution

$$\begin{aligned}\int \tan^3 4x\, dx &= \int \tan^2 4x\ \tan 4x\, dx \\ &= \int \left(\sec^2 4x - 1\right)\ \tan 4x\, dx \\ &= \int \sec^2 4x\ \tan 4x\, dx - \int \tan 4x\, dx \\ &= \frac{1}{4}\int \tan 4x\, d\left(\tan 4x\right) - \int \frac{\sin 4x}{\cos 4x}\, dx \\ &= \frac{1}{8}\tan^2 4x + \frac{1}{4}\int \frac{1}{\cos 4x}\, d\left(\cos 4x\right) \\ &= \frac{1}{8}\tan^2 4x + \frac{1}{4}\ln\left|\cos 4x\right| + C\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \cot^4 x\, dx$

Solution

$$\begin{aligned}\int \cot^4 x\, dx &= \int \cot^2 x\left(\csc^2 x - 1\right)\, dx \\ &= \int \cot^2 x\ \csc^2 x\, dx - \int \cot^2 x\, dx \\ &= -\int \cot^2 x\, d\left(\cot x\right) - \int \left(\csc^2 x - 1\right)\, dx \\ &= -\frac{1}{3}\cot^3 x - \cot x + x + C\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \cot^5 3x\, dx$

Solution

$$\int \cot^5 3x\, dx = \int \cot^3 3x \left(\csc^2 3x - 1\right)\, dx$$

$$= \int \cot^3 3x\ \csc^2 3x\, dx - \int \cot^3 3x\, dx$$

$$= -\frac{1}{3}\int \cot^3 3x\, d\left(\cot 3x\right) - \int \cot 3x \left(\csc^2 3x - 1\right)\, dx$$

$$= -\frac{1}{12}\cot^4 3x - \int \cot 3x \csc^2 3x\, dx + \int \cot 3x\, dx$$

$$= -\frac{1}{12}\cot^4 3x + \frac{1}{3}\int \cot 3x\, d\left(\cot 3x\right) + \int \frac{\cos 3x}{\sin 3x}\, dx$$

$$= -\frac{1}{12}\cot^4 3x + \frac{1}{6}\cot^2 3x + \frac{1}{3}\int \frac{1}{\sin 3x}\, d\left(\sin 3x\right)$$

$$= -\frac{1}{12}\cot^4 x + \frac{1}{6}\cot^2 3x + \frac{1}{3}\ln\left|\sin 3x\right| + C$$

Exercise

Evaluate the integral $\int \sin^2 x\ \cos^2 x\, dx$

Solution

$$\int \sin^2 x\ \cos^2 x\, dx = \frac{1}{4}\int (1 - \cos 2x)(1 + \cos 2x)\, dx$$

$$= \frac{1}{4}\int \left(1 - \cos^2 2x\right) dx$$

$$= \frac{1}{4}\int \left(1 - \frac{1}{2} - \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\left(\frac{1}{2}x - \frac{1}{8}\sin 4x\right) + C$$

$$= \frac{1}{8}x - \frac{1}{32}\sin 4x + C$$

Exercise

Evaluate the integral $\displaystyle\int \sin^2 x\ \cos^3 x\, dx$

Solution

$$\begin{aligned}\int \sin^2 x\ \cos^3 x\, dx &= \int \sin^2 x\ \cos^2 x\ \cos x\, dx \\ &= \int \sin^2 x\left(1-\sin^2 x\right) d(\sin x) \\ &= \int \left(\sin^2 x-\sin^4 x\right) d(\sin x) \\ &= \frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+C\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \sin^2 x\ \cos^5 x\, dx$

Solution

$$\begin{aligned}\int \sin^2 x\ \cos^5 x\, dx &= \int \sin^2 x\ \cos^4 x\ \cos x\, dx \\ &= \int \sin^2 x\left(1-\sin^2 x\right)^2 d(\sin x) \\ &= \int \sin^2 x\left(1-2\sin^2 x+\sin^4 x\right) d(\sin x) \\ &= \int \left(\sin^2 x-2\sin^4 x+\sin^6 x\right) d(\sin x) \\ &= \frac{1}{3}\sin^3 x-\frac{2}{5}\sin^5 x+\frac{1}{7}\sin^7 x+C\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \sin^3 x\ \cos^5 x\, dx$

Solution

$$\int \sin^3 x\ \cos^5 x\, dx = \int \sin^2 x\ \cos^5 x\ \sin x\, dx$$

$$= -\int \left(1-\cos^2 x\right) \cos^5 x \, d\left(\cos x\right)$$

$$= -\int \left(\cos^5 x - \cos^7 x\right) \, d\left(\cos x\right)$$

$$= -\frac{1}{6}\cos^6 x + \frac{1}{8}\cos^8 x + C$$

Exercise

Evaluate the integral $\int \sin^3 x \cos^4 x \, dx$

Solution

$$\int \sin^3 x \cos^4 x \, dx = \int \sin^2 x \sin x \cos^4 x \, dx$$

$$= -\int \left(1-\cos^2 x\right)\cos^4 x \, d\left(\cos x\right)$$

$$= -\int \left(\cos^4 x - \cos^6 x\right) \, d\left(\cos x\right)$$

$$= -\left(\frac{1}{5}\cos^5 x - \frac{1}{7}\cos^7 x\right) + C$$

$$= \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C$$

Exercise

Evaluate the integral $\int \sin^3 2x \cos^4 x \, dx$

Solution

$$\int \sin^3 2x \cos^4 x \, dx = \int \left(2\sin x \cos x\right)^3 \cos^4 x \, dx$$

$$= 8\int \sin^3 x \cos^7 x \, dx$$

$$= -8\int \sin^2 x \cos^7 x \, d\left(\cos x\right)$$

$$= -8\int \left(1-\cos^2 x\right)\cos^7 x \, d\left(\cos x\right)$$

$$= -8\int \left(\cos^7 x - \cos^9 x\right)\, d(\cos x)$$

$$= -8\left(\frac{1}{8}\cos^8 x - \frac{1}{10}\cos^{10} x\right) + C$$

$$\boxed{= -\cos^8 x + \frac{4}{5}\cos^{10} x + C}$$

Exercise

Evaluate the integral $\int \sin^3 2x \ \cos^3 2x\, dx$

Solution

Exercise

Evaluate $\int \sin^4 x \cos^2 x\, dx$

Solution

$$\int \sin^4 x \cos^2 x\ dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 \left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{8}\int \left(1 - 2\cos 2x + \cos^2 2x\right)(1+\cos 2x)\, dx$$

$$= \frac{1}{8}\int \left(1 - \cos 2x - \cos^2 2x + \cos^3 2x\right) dx$$

$$= \frac{1}{8}\int \left(1 - \cos 2x - \frac{1}{2} - \frac{1}{2}\cos 4x\right) dx + \frac{1}{8}\int \cos^2 2x \cos 2x\ dx$$

$$= \frac{1}{8}\int \left(\frac{1}{2} - \cos 2x - \frac{1}{2}\cos 4x\right) dx + \frac{1}{16}\int \left(1 - \sin^2 2x\right)\ d(\sin 2x)$$

$$= \frac{1}{8}\left(\frac{1}{2}x - \frac{1}{2}\sin 2x - \frac{1}{4}\sin 4x\right) + \frac{1}{16}\sin 2x - \frac{1}{48}\sin^3 2x + C$$

$$\boxed{= \frac{1}{16}x - \frac{1}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C}$$

Exercise

Evaluate the integral $\int \sin^4 x \cos^3 x\, dx$

Solution

$$\int \sin^4 x \cos^3 x \, dx = \int \sin^4 x \left(1-\sin^2 x\right) \, d(\sin x)$$

$$= \int \left(\sin^4 x - \sin^6 x\right) \, d(\sin x)$$

$$\boxed{= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C}$$

Exercise

Evaluate the integral $\int \sin^4 x \cos^4 x \, dx$

Solution

$$\int \sin^4 x \cos^4 x \, dx = \int \left(\sin^2 x\right)^2 \left(\cos^2 x\right)^2 \, dx$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 \left(\frac{1+\cos 2x}{2}\right)^2 \, dx$$

$$= \frac{1}{16}\int \left(1-\cos^2 2x\right)^2 \, dx$$

$$= \frac{1}{16}\int \left(1-2\cos^2 2x+\cos^4 2x\right) \, dx$$

$$= \frac{1}{16}\int \left(1-1-\cos 4x+\left(\frac{1+\cos 4x}{2}\right)^2\right) \, dx$$

$$= \frac{1}{16}\int \left(-\cos 4x+\frac{1}{4}\left(1+2\cos 4x+\cos^2 4x\right)\right) \, dx$$

$$= \frac{1}{64}\int \left(-4\cos 4x+1+2\cos 4x+\cos^2 4x\right) \, dx$$

$$= \frac{1}{64}\int \left(1-2\cos 4x+\frac{1}{2}+\frac{1}{2}\cos 8x\right) \, dx$$

$$= \frac{1}{128}\int \left(3-4\cos 4x+\cos 8x\right) \, dx$$

$$\boxed{= \frac{1}{128}\left(3x-\sin 4x+\frac{1}{8}\sin 8x\right)+C}$$

Exercise

Evaluate the integral $\displaystyle\int \sin^4 x\ \cos^5 x\ dx$

Solution

$$\begin{aligned}
\int \sin^4 x\ \cos^5 x\ dx &= \int \sin^4 x\ \cos^4 x\ \cos x\ dx \\
&= \int \sin^4 x\ \left(1-\sin^2 x\right)^2\ d(\sin x) \\
&= \int \sin^4 x\ \left(1-2\sin^2 x+\sin^4 x\right)\ d(\sin x) \\
&= \int \left(\sin^4 x-2\sin^6 x+\sin^8 x\right)\ d(\sin x) \\
&= \frac{1}{5}\sin^5 x-\frac{2}{7}\sin^7 x+\frac{1}{9}\sin^9 x+C
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \sin^5 x\ \cos^5 x\ dx$

Solution

$$\begin{aligned}
\int \sin^5 x\ \cos^5 x\ dx &= \int \sin^5 x\ \cos^4 x\,(\cos x)\ dx \\
&= \int \sin^5 x\ \left(1-\sin^2 x\right)^2\ d(\sin x) \\
&= \int \sin^5 x\ \left(1-2\sin^2 x+\sin^4 x\right)\ d(\sin x) \\
&= \int \left(\sin^5 x-2\sin^7 x+\sin^9 x\right)\ d(\sin x) \\
&= \frac{1}{6}\sin^6 x-\frac{1}{4}\sin^8 x+\frac{1}{10}\sin^{10} x+C
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \sin^5 x\ \cos^{-2} x\ dx$

Solution

$$\int \sin^5 x \ \cos^{-2} x \ dx = \int \sin^4 x \ \cos^{-2} x \ \sin x \ dx$$

$$= -\int \left(1-\cos^2 x\right)^2 \cos^{-2} x \ d\left(\cos x\right)$$

$$= -\int \left(1-2\cos^2 x+\cos^4 x\right)\cos^{-2} x \ d\left(\cos x\right)$$

$$= -\int \left(\cos^{-2} x-2+\cos^2 x\right) \ d\left(\cos x\right)$$

$$= \cos^{-1} x+2\cos x-\frac{1}{3}\cos^3 x+C$$

$$= \sec x+2\cos x-\frac{1}{3}\cos^3 x+C$$

Exercise

Evaluate the integral $\int \sin 3x \cos^6 3x \ dx$

Solution

$$\int \sin 3x \cos^6 3x \ dx = -\frac{1}{3}\int \cos^6 3x \ d\left(\cos 3x\right)$$

$$= -\frac{1}{21}\cos^7 3x+C$$

Exercise

Evaluate the integral $\int \sin^4 2x \cos 2x dx$

Solution

$$d\left(\sin 2x\right) = 2\cos 2x dx$$

$$\frac{1}{2}d\left(\sin 2x\right) = \cos 2x dx$$

$$\int \sin^4 2x \cos 2x dx = \frac{1}{2}\int \sin^4 2x \ d\left(\sin 2x\right)$$

$$= \frac{1}{10}\sin^5 2x+C$$

Exercise

Evaluate the integral $\int \cos^3 2x \sin^5 2x \, dx$

Solution

$$\int \cos^3 2x \sin^5 2x \, dx = \int \left(\cos^2 2x\right) \cos 2x \sin^5 2x \, dx \qquad d(\sin 2x) = 2\cos 2x \, dx$$

$$= \int \left(1 - \sin^2 2x\right) \sin^5 2x \, \left(\tfrac{1}{2} d \sin 2x\right)$$

$$= \frac{1}{2} \int \left(\sin^5 2x - \sin^7 2x\right) \, (d \sin 2x)$$

$$= \frac{1}{2}\left(\frac{1}{6}\sin^6 2x - \frac{1}{8}\sin^8 2x\right) + C$$

$$\boxed{= \frac{1}{12}\sin^6 2x - \frac{1}{16}\sin^8 2x + C}$$

Exercise

Evaluate the integral $\int 16 \sin^2 x \cos^2 x dx$

Solution

$$\int 16 \sin^2 x \cos^2 x dx = 16 \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx \qquad \cos^2 \alpha = \frac{1+\cos 2\alpha}{2} \quad \sin^2 \alpha = \frac{1-\cos 2\alpha}{2}$$

$$= 4 \int \left(1 - \cos^2 2x\right) dx$$

$$= 4 \int \left(1 - \frac{1+\cos 4x}{2}\right) dx$$

$$= 4 \int \frac{1-\cos 4x}{2} dx$$

$$= 2\left(x - \frac{1}{4}\sin 4x\right) + C$$

$$= 2x - \frac{1}{2}\left(2\sin 2x \cos 2x\right) + C$$

$$= 2x - \left(2\sin x \cos x\right)\left(2\cos^2 x - 1\right) + C$$

$$\boxed{= 2x - 4\sin x \cos^3 x + 2\sin x \cos x + C}$$

Exercise

Evaluate the integral $\int \sin 2x \cos 3x \, dx$

Solution

$$\int \sin 2x \cos 3x \, dx = \frac{1}{2}\int \left(\sin 5x + \sin(-x)\right) dx \qquad \sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

$$= \frac{1}{2}\int \left(\sin 5x - \sin x\right) dx$$

$$= \frac{1}{2}\left(-\frac{1}{5}\cos 5x + \cos x\right) + C$$

$$= \frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C$$

Exercise

Evaluate the integral $\int \sin^2\theta \cos 3\theta \, d\theta$

Solution

$$\int \sin^2\theta \cos 3\theta \, d\theta = \int \frac{1-\cos 2\theta}{2}\cos 3\theta \, d\theta \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \frac{1}{2}\int \left(\cos 3\theta - \cos 2\theta \cos 3\theta\right) d\theta$$

$$= \frac{1}{2}\int \cos 3\theta \, d\theta - \frac{1}{2}\int \cos 2\theta \cos 3\theta \, d\theta$$

$$= \frac{1}{6}\sin 3\theta - \frac{1}{2}\int \frac{1}{2}\left(\cos(5\theta) + \cos(-\theta)\right) d\theta \qquad \cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$$

$$= \frac{1}{6}\sin 3\theta - \frac{1}{4}\left(\frac{1}{5}\sin 5\theta + \sin\theta\right) + C$$

$$= \frac{1}{6}\sin 3\theta - \frac{1}{20}\sin 5\theta - \frac{1}{4}\sin\theta + C$$

Exercise

Evaluate the integral $\int \cos^3\theta \sin 2\theta \, d\theta$

Solution

$$\int \cos^3\theta \sin 2\theta \, d\theta = \int \cos^3\theta \left(2\sin\theta\cos\theta\right) d\theta \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= -2\int \cos^4\theta \; d(\cos\theta) \qquad\qquad d(\cos\theta) = -\sin\theta d\theta$$

$$\boxed{= -\frac{2}{5}\cos^5\theta + C}$$

Exercise

Evaluate the integral $\displaystyle\int \sin^{-3/2}x \; \cos^3 x \; dx$

Solution

$$\int \sin^{-3/2}x \; \cos^3 x \; dx = \int \sin^{-3/2}x \; \cos^2 x \; \cos x \; dx$$

$$= \int \sin^{-3/2}x \left(1-\sin^2 x\right) \; d(\sin x)$$

$$= \int \left(\sin^{-3/2}x - \sin^{1/2}x\right) \; d(\sin x)$$

$$\boxed{= -2\sin^{-1/2}x - \frac{2}{3}\sin^{3/2}x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \sin^3 x \; \cos^{3/2}x \; dx$

Solution

$$\int \sin^3 x \; \cos^{3/2}x \; dx = \int \sin^2 x \; \cos^{3/2}x \; \sin x \; dx$$

$$= -\int \left(1-\cos^2 x\right) \; \cos^{3/2}x \; d(\cos x)$$

$$= \int \left(-\cos^{3/2}x + \cos^{7/2}x\right) \; d(\cos x)$$

$$\boxed{= \frac{2}{9}\cos^{9/2}x - \frac{2}{5}\cos^{5/2} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \sin\theta \sin 2\theta \sin 3\theta \; d\theta$

Solution

$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$

$$\int \sin\theta\sin 2\theta\sin 3\theta\, d\theta = \int \frac{1}{2}\left(\cos(1-2)\theta - \cos(1+2)\theta\right) sin3\theta\, d\theta$$

$$= \frac{1}{2}\int \left(\cos(-\theta)-\cos(3\theta)\right)\sin 3\theta\, d\theta$$

$$= \frac{1}{2}\int \cos\theta\ \sin 3\theta\, d\theta - \frac{1}{2}\int \cos 3\theta\ \sin 3\theta\, d\theta$$

$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$

$$= \frac{1}{4}\int (\sin 4\theta + \sin 2\theta)\ d\theta - \frac{1}{4}\int \left(\sin 6\theta + \sin(0)\right)\ d\theta$$

$$= \frac{1}{4}\left(-\frac{1}{4}\cos 4\theta - \frac{1}{2}\cos 2\theta\right) + \frac{1}{24}\cos 6\theta + C$$

$$= -\frac{1}{16}\cos 4\theta - \frac{1}{8}\cos 2\theta + \frac{1}{24}\cos 6\theta + C$$

Exercise

Evaluate the integral $\int \sin 3x\ \cos 6x\ dx$

Solution

$$\int \sin 3x\ \cos 6x\ dx = \frac{1}{2}\int \left(\sin 9x + \sin(-3x)\right) dx$$

$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$

$$= \frac{1}{2}\int (\sin 10x - \sin 3x)\, dx$$

$$= \frac{1}{2}\left(-\frac{1}{10}\cos 10x + \frac{1}{3}\cos 3x\right) + C$$

$$= \frac{1}{6}\cos 3x - \frac{1}{20}\cos 10x + C$$

Exercise

Evaluate the integral $\int \sin 3x \cos 7x\ dx$

Solution

$$\int \sin 3x \cos 7x\ dx = \frac{1}{2}\int \left(\sin 10x + \sin(-4x)\right) dx$$

$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$

$$=\frac{1}{2}\int(\sin 10x-\sin 4x)\,dx$$

$$=\frac{1}{2}\left(-\frac{1}{10}\cos 10x+\frac{1}{4}\cos 4x\right)+C$$

$$\boxed{=\frac{1}{8}\cos 4x-\frac{1}{20}\cos 10x+C}$$

Exercise

Evaluate the integral $\displaystyle\int \sin 5x\cos 4x\ dx$

Solution

$$\int \sin 5x\cos 4x\ dx=\frac{1}{2}\int(\sin x+\sin 9x)\,dx \qquad \sin\alpha\cos\beta=\tfrac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

$$=\frac{1}{2}\left(-\cos x-\frac{1}{9}\cos x9x\right)+C$$

$$\boxed{=\frac{1}{2}-\cos x-\frac{1}{18}\cos x9x+C}$$

Exercise

Evaluate the integral $\displaystyle\int \cos 2\theta\ \cos 6\theta\ d\theta$

Solution

$$\int \cos 2\theta\ \cos 6\theta\ d\theta=\frac{1}{2}\int\left(\cos 8\theta+\cos(-4\theta)\right)\ d\theta \qquad \cos\alpha\cos\beta=\tfrac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$$

$$=\frac{1}{2}\int(\cos 8\theta+\cos 4\theta)\ d\theta$$

$$=\frac{1}{2}\left(\frac{1}{8}\sin 8\theta+\frac{1}{4}\sin 4\theta\right)+C$$

$$\boxed{=\frac{1}{16}\sin 8\theta+\frac{1}{8}\sin 4\theta+C}$$

Exercise

Evaluate the integral $\displaystyle\int \cos 5\theta\ \cos 3\theta\ d\theta$

Solution

$$\int \cos 5\theta \ \cos 3\theta \ d\theta = \frac{1}{2}\int (\cos 8\theta + \cos 2\theta) \ d\theta \qquad \cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

$$= \frac{1}{2}\left(\frac{1}{8}\sin 8\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$\boxed{= \frac{1}{16}\sin 8\theta + \frac{1}{4}\sin 2\theta + C}$$

Exercise

Evaluate the integral $\int \sin 2\theta \ \cos 4\theta \ d\theta$

Solution

$$\int \sin 2\theta \ \cos 4\theta \ d\theta = \frac{1}{2}\int (\sin 6\theta + \sin(-2\theta)) \ d\theta \qquad \sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$= \frac{1}{2}\int (\sin 6\theta - \sin 2\theta) \ d\theta$$

$$= \frac{1}{2}\left(-\frac{1}{6}\cos 6\theta + \frac{1}{2}\cos 2\theta\right) + C$$

$$\boxed{= \frac{1}{4}\cos 4\theta - \frac{1}{12}\cos 6\theta + C}$$

Exercise

Evaluate the integral $\int \sin(-7\theta) \ \cos 6\theta \ d\theta$

Solution

$$\int \sin(-7\theta) \ \cos 6\theta \ d\theta = -\int \sin 7\theta \ \cos 6\theta \ d\theta$$

$$= -\frac{1}{2}\int (\sin 13\theta + \sin\theta) \ d\theta \qquad \sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$= -\frac{1}{2}\left(-\frac{1}{13}\cos 13\theta - \cos\theta\right) + C$$

$$\boxed{= \frac{1}{26}\cos 13\theta + \frac{1}{2}\cos\theta + C}$$

Exercise

Evaluate the integral $\int \sin\theta\ \sin 3\theta\ d\theta$

Solution

$$\int \sin\theta\ \sin 3\theta\ d\theta = \frac{1}{2}\int \left(\cos(-2\theta) - \cos 4\theta\right)\ d\theta \qquad \sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$= \frac{1}{2}\int \left(\cos 2\theta - \cos 4\theta\right)\ d\theta$$

$$= \frac{1}{2}\left(\frac{1}{2}\sin 2\theta - \frac{1}{4}\sin 4\theta\right) + C$$

$$= \frac{1}{4}\sin 2\theta - \frac{1}{8}\sin 4\theta + C$$

Exercise

Evaluate the integral $\int \sin 5\theta\ \sin 4\theta\ d\theta$

Solution

$$\int \sin 5\theta\ \sin 4\theta\ d\theta = \frac{1}{2}\int \left(\cos\theta - \cos 9\theta\right)\ d\theta \qquad \sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$= \frac{1}{2}\left(\sin\theta - \frac{1}{9}\sin 9\theta\right) + C$$

$$= \frac{1}{2}\sin\theta - \frac{1}{18}\sin 9\theta + C$$

Exercise

Evaluate the integral $\int \sin x \cos^5 x\ dx$

Solution

$$\int \sin x\cos^5 x\ dx = -\int \cos^5 x\ d(\cos x)$$

$$= -\frac{1}{6}\cos^6 x + C$$

Exercise

Evaluate the integral $\quad \displaystyle\int \sin^7 2x \ \cos 2x \ dx$

Solution

$$\int \sin^7 2x \ \cos 2x \ dx = \frac{1}{2}\int \sin^7 2x \ d(\sin 2x)$$

$$= \frac{1}{16}\sin^8 2x + C$$

Exercise

Evaluate the integral $\quad \displaystyle\int \sin^3 2x \ \sqrt{\cos 2x} \ dx$

Solution

$$\int \sin^3 2x \ \sqrt{\cos 2x} \ dx = -\frac{1}{2}\int \left(1-\cos^2 2x\right)(\cos 2x)^{1/2} \ d(\cos 2x)$$

$$= -\frac{1}{2}\int \left((\cos 2x)^{1/2} - (\cos 2x)^{5/2}\right) \ d(\cos 2x)$$

$$= -\frac{1}{2}\left(\frac{2}{3}(\cos 2x)^{3/2} - \frac{2}{7}(\cos 2x)^{7/2}\right) + C$$

$$= \frac{1}{7}(\cos 2x)^{7/2} - \frac{1}{3}(\cos 2x)^{3/2} + C$$

Exercise

Evaluate $\quad \displaystyle\int \sin^3 x \cos^2 x \ dx$

Solution

$$\int \sin^3 x \cos^2 x \ dx = \int \sin^2 x \cos^2 x \ \sin x dx \qquad d(\cos x) = -\sin x \ dx$$

$$= -\int \left(1-\cos^2 x\right)\cos^2 x \ d(\cos x)$$

$$= \int \left(\cos^4 x - \cos^2 x\right) \ d(\cos x)$$

$$= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\cos^3\theta}{\sqrt{\sin\theta}}\,d\theta$

Solution

$$\begin{aligned}
\int \frac{\cos^3\theta}{\sqrt{\sin\theta}}\,d\theta &= \int (\sin\theta)^{-1/2}\cos^2\theta\cos\theta\,d\theta \\
&= \int (\sin\theta)^{-1/2}\left(1-\sin^2\theta\right)\,d(\sin\theta) \\
&= \int \left(\sin^{-1/2}\theta-\sin^{3/2}\theta\right)\,d(\sin\theta) \\
&= 2\sqrt{\sin\theta}-\frac{2}{5}\sin^{5/2}\theta+C
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\cos^5\theta}{\sqrt{\sin\theta}}\,d\theta$

Solution

$$\begin{aligned}
\int \frac{\cos^5\theta}{\sqrt{\sin\theta}}\,d\theta &= \int (\sin\theta)^{-1/2}\left(1-\sin^2\theta\right)^2\,d(\sin\theta) \\
&= \int (\sin\theta)^{-1/2}\left(1-2\sin^2\theta+\sin^4\theta\right)\,d(\sin\theta) \\
&= \int \left((\sin\theta)^{-1/2}-2(\sin\theta)^{3/2}+(\sin\theta)^{7/2}\right)\,d(\sin\theta) \\
&= 2(\sin\theta)^{1/2}-\frac{1}{5}(\sin\theta)^{5/2}+\frac{2}{9}(\sin\theta)^{9/2}+C
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\cos^2 x}{\sin^5 x}\,dx$

Solution

$$\begin{aligned}
\int \frac{\cos^2 x}{\sin^5 x}\,dx &= \int \frac{\cos^2 x}{\sin^2 x}\frac{1}{\sin^3 x}\,dx \\
&= \int \cot^2 x\ \csc^3 x\,dx
\end{aligned}$$

$$= \int \left(\csc^2 x - 1\right) \csc^3 x \, dx$$

$$= \int \csc^5 x \, dx - \int \csc^3 x \, dx$$

$$u = \csc^3 x \qquad dv = \csc^2 x dx$$

$$du = -3\csc^3 x \cot x dx \qquad v = -\cot x$$

$$\int \csc^5 x dx = -\csc^3 x \cot x - 3\int \csc^3 x \cot^2 x \, dx$$

$$= -\csc^3 x \cot x - 3\int \csc^3 x \left(\csc^2 x - 1\right) dx$$

$$= -\csc^3 x \cot x - 3\int \csc^5 x \, dx + 3\int \csc^3 x \, dx$$

$$5\int \csc^5 x \, dx = -\csc^3 x \cot x + 3\int \csc^3 x \, dx$$

$$\int \csc^5 x \, dx = -\frac{1}{5}\csc^3 x \cot x + \frac{3}{5}\int \csc^3 x \, dx$$

$$u = \csc x \qquad dv = \csc^2 x dx$$

$$du = -\csc x \cot x dx \qquad v = -\cot x$$

$$\int \csc^3 x dx = -\csc x \cot x - \int \csc x \cot^2 x \, dx$$

$$= -\csc x \cot x - \int \csc x\left(\csc^2 x - 1\right) dx$$

$$= -\csc x \cot x - \int \csc^3 x \, dx + \int \csc x \frac{\csc x + \cot x}{\csc x + \cot x} \, dx$$

$$2\int \csc^3 x \, dx = -\csc x \cot x + \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} \, dx$$

$$2\int \csc^3 x \, dx = -\csc x \cot x - \int \frac{1}{\csc x + \cot x} \, d\left(\csc x + \cot x\right)$$

$$\int \csc^3 x \, dx = -\frac{1}{2}\csc x \cot x - \ln\left|\csc x + \cot x\right|$$

$$\int \frac{\cos^2 x}{\sin^5 x} dx = -\frac{1}{5}\csc^3 x \cot x + \frac{3}{5}\left(-\frac{1}{2}\csc x \cot x - \ln\left|\csc x + \cot x\right|\right) + \frac{1}{2}\csc x \cot x + \ln\left|\csc x + \cot x\right|$$

$$= -\frac{1}{5}\csc^3 x \cot x - \frac{3}{10}\csc x \cot x - \frac{3}{5}\ln\left|\csc x + \cot x\right| + \frac{1}{2}\csc x \cot x + \ln\left|\csc x + \cot x\right|$$

$$= \frac{1}{5}\csc x \cot x\left(1 - \csc^2 x\right) + + \frac{2}{5}\ln\left|\csc x + \cot x\right| + C$$

$$\boxed{= \frac{1}{5}\csc x \cot^3 x + \frac{2}{5}\ln\left|\csc x + \cot x\right| + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sin^3 x}{\cos^4 x}\,dx$

Solution

$$\int \frac{\sin^3 x}{\cos^4 x}\,dx = \int \frac{\sin^2 x \sin x}{\cos^4 x}\,dx \qquad \cos^2\alpha + \sin^2\alpha = 1$$

$$= \int \frac{\left(1-\cos^2 x\right)\sin x}{\cos^4 x}\,dx$$

$$= -\int \left(\frac{1}{\cos^4 x} - \frac{\cos^2 x}{\cos^4 x}\right) d\left(\cos x\right)$$

$$= -\int \left(\cos^{-4} x - \cos^{-2} x\right) d\left(\cos x\right)$$

$$= -\left(-\frac{1}{3}\cos^{-3} x + \cos^{-1} x\right) + C$$

$$= \frac{1}{3}\frac{1}{\cos^3 x} - \frac{1}{\cos x} + C$$

$$\boxed{= \frac{1}{3}\csc^3 x - \csc x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sin^4 x}{\cos^6 x}\,dx$

Solution

$$\int \frac{\sin^4 x}{\cos^6 x}\,dx = \int \frac{\sin^4 x}{\cos^4 x}\cdot\frac{1}{\cos^2 x}\,dx$$

$$= \int \tan^4 x\ \sec^2 x\,dx$$

$$= \int \tan^4 x\ \ d\left(\tan x\right)$$

$$\boxed{= \frac{1}{5}\tan^5 x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{2\cos x + 3\sin x}{\sin^3 x}\, dx$

Solution

$$\int \frac{2\cos x + 3\sin x}{\sin^3 x}\, dx = 2\int \frac{\cos x}{\sin^3 x}\, dx + 3\int \frac{\sin x}{\sin^3 x}\, dx$$

$$= 2\int \sin^{-3} x\, d(\sin x) + 3\int \frac{1}{\sin^2 x}\, dx$$

$$= -\sin^{-2} x + 3\int \csc^2 x\, dx$$

$$= -\csc^2 x - 3\cot x + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{2 + \sin x + 2\cos x}{1 + \cos x}\, dx$

Solution

$$\int \frac{2 + \sin x + 2\cos x}{1 + \cos x}\, dx = \int \frac{2(1 + \cos x)}{1 + \cos x}\, dx + \int \frac{\sin x}{1 + \cos x}\, dx$$

$$= 2\int dx - \int \frac{1}{1 + \cos x}\, d(1 + \cos x)$$

$$= 2x - \ln|1 + \cos x| + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{1 - \cos x}$

Solution

$$\int \frac{dx}{1 - \cos x} = \int \frac{1}{1 - \cos x} \cdot \frac{1 + \cos x}{1 + \cos x}\, dx$$

$$= \int \frac{1 + \cos x}{1 - \cos^2 x}\, dx$$

$$= \int \frac{1 + \cos x}{\sin^2 x}\, dx$$

$$= \int \frac{1}{\sin^2 x}\, dx + \int \frac{\cos x}{\sin^2 x}\, dx$$

$$= \int \csc^2 x\, dx + \int \frac{1}{\sin^2 x}\, d(\sin x)$$

$$= -\cot x - \frac{1}{\sin x} + C$$

$$\underline{= -\cot x - \csc x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{1-\sin x}$

Solution

$$\int \frac{dx}{1-\sin x} = \int \frac{1}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x}\, dx$$

$$= \int \frac{1+\sin x}{1-\sin^2 x}\, dx$$

$$= \int \frac{1+\sin x}{\cos^2 x}\, dx$$

$$= \int \frac{1}{\cos^2 x}\, dx + \int \frac{\sin x}{\cos^2 x}\, dx$$

$$= \int \sec^2 x\, dx - \int \frac{1}{\cos^2 x}\, d(\cos x)$$

$$= \tan x + \frac{1}{\cos x} + C$$

$$\underline{= \tan x + \sec x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sin\theta\cos\theta}{2-\cos\theta}\, d\theta$

Solution

$$\int \frac{\sin\theta\cos\theta}{2-\cos\theta}\, d\theta = \int \frac{\sin\theta(\cos\theta + 2 - 2)}{2-\cos\theta}\, d\theta$$

$$= \int \frac{\sin\theta(\cos\theta - 2) + 2\sin\theta}{2-\cos\theta}\, d\theta$$

$$= -\int \frac{\sin\theta\left(2-\cos\theta\right)}{2-\cos\theta}\, d\theta + \int \frac{2\sin\theta}{2-\cos\theta}\, d\theta$$

$$= -\int \sin\theta\, d\theta + \int \frac{2}{2-\cos\theta}\, d\left(2-\cos\theta\right)$$

$$\boxed{= \cos\theta + \ln\left|2-\cos\theta\right| + C}$$

Exercise

Evaluate the integral $\displaystyle\int \tan^3 x\ \sec^3 x\, dx$

Solution

$$\int \tan^3 x\ \sec^3 x\, dx = \int \tan^2 x\ \sec^2 x\ \left(\tan x\ \sec x\right)\, dx$$

$$= \int \left(\sec^2 x - 1\right)\ \sec^2 x\, d\left(\sec x\right)$$

$$= \int \left(\sec^4 x - \sec^2 x\right)\, d\left(\sec x\right)$$

$$\boxed{= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \sec x \tan^2 x\, dx$

Solution

$$\int \sec x \tan^2 x\, dx = \int \sec x \tan x \tan x\, dx$$

$u = \tan x \qquad dv = \sec x \tan x dx$

$du = \sec^2 x dx \qquad v = \sec x$

$$\int \sec x \tan^2 x\, dx = \tan x \sec x - \int \sec x \sec^2 x\, dx$$

$$= \tan x \sec x - \int \sec x\left(1+\tan^2 x\right)\, dx$$

$$= \tan x \sec x - \left[\int \sec x\, dx + \int \sec x \tan^2 x\, dx\right]$$

$$= \tan x \sec x - \ln\left|\sec x + \tan x\right| - \int \sec x \tan^2 x \, dx$$

$$\int \sec x \tan^2 x \, dx + \int \sec x \tan^2 x \, dx = \tan x \sec x - \ln\left|\sec x + \tan x\right|$$

$$2\int \sec x \tan^2 x \, dx = \tan x \sec x - \ln\left|\sec x + \tan x\right|$$

$$\int \sec x \tan^2 x \, dx = \boxed{\frac{1}{2}\tan x \sec x - \frac{1}{2}\ln\left|\sec x + \tan x\right| + C}$$

Exercise

Evaluate the integral $\int \sec^2 x \tan^2 x \, dx$

Solution

$$\int \sec^2 x \tan^2 x dx = \int \tan^2 x \, d(\tan x) \qquad d(\tan x) = \sec^2 x dx$$

$$= \boxed{\frac{1}{3}\tan^3 x + C}$$

Exercise

Evaluate the integral $\int \sec^4 x \tan^2 x \, dx$

Solution

$$\int \sec^4 x \tan^2 x dx = \int \sec^2 x \sec^2 x \tan^2 x \, dx \qquad d(\tan x) = \sec^2 x dx \quad \sec^2 x = 1 + \tan^2 x$$

$$= \int \left(1 + \tan^2 x\right)\tan^2 x \, d(\tan x)$$

$$= \int \left(\tan^2 x + \tan^4 x\right) d(\tan x)$$

$$= \boxed{\frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C}$$

Exercise

Evaluate the integral $\int \sec^6 4x \ \tan 4x \ dx$

Solution

$$\int \sec^6 4x \ \tan 4x \ dx = \frac{1}{4}\int \sec^5 4x \ (\sec 4x \ \tan 4x) \ d(4x)$$

$$= \frac{1}{4}\int \sec^5 4x \ d(\sec 4x)$$

$$\boxed{= \frac{1}{24}\sec^6 4x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \sec^2 \frac{x}{2} \ \tan\frac{x}{2} \ dx$

Solution

$$\int \sec^2 \frac{x}{2} \ \tan\frac{x}{2} \ dx = 2\int \sec\frac{x}{2}\left(\sec\frac{x}{2}\tan\frac{x}{2}\right) \ d\left(\frac{x}{2}\right)$$

$$= 2\int \sec\frac{x}{2} \ d\left(\sec\frac{x}{2}\right)$$

$$\boxed{= \sec^2\frac{x}{2} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \tan^3 2x \ \sec^3 2x \ dx$

Solution

$$\int \tan^3 2x \ \sec^3 2x \ dx = \frac{1}{2}\int \left(\tan^2 2x\right) \ \sec^2 2x \ (\sec 2x \tan 2x) \ d(2x)$$

$$= \frac{1}{2}\int \left(\sec^2 2x - 1\right) \ \sec^2 2x \ d(\sec 2x)$$

$$= \frac{1}{2}\int \left(\sec^4 2x - \sec^2 2x\right) \ d(\sec 2x)$$

$$= \frac{1}{2}\left(\frac{1}{5}\sec^5 2x - \frac{1}{3}\sec^3 2x\right) + C$$

$$\boxed{= \frac{1}{10}\sec^5 2x - \frac{1}{6}\sec^3 2x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \tan^5 2x\ \sec^4 2x\ dx$

Solution

$$\begin{aligned}\int \tan^5 2x\ \sec^4 2x\ dx &= \frac{1}{2}\int \tan^5 2x\ \sec^2 2x\ \sec^2 2x\ d(2x)\\ &= \frac{1}{2}\int \tan^5 2x\ \left(\tan^2 2x + 1\right)\ d(\tan 2x)\\ &= \frac{1}{2}\int \left(\tan^7 2x + \tan^5 2x\right)\ d(\tan 2x)\\ &= \frac{1}{2}\left(\frac{1}{8}\tan^8 2x + \frac{1}{6}\tan^6 2x\right) + C\\ &= \frac{1}{16}\tan^8 2x + \frac{1}{12}\tan^6 2x + C\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \tan^3 x\ \sec^5 x\ dx$

Solution

$$\begin{aligned}\int \tan^3 x\ \sec^5 x\ dx &= \int \tan^4 x\ \sec^4 x\ (\sec x \tan x)\, dx\\ &= \int \left(\sec^2 x - 1\right)^2 \sec^4 x\ d(\sec x)\\ &= \int \left(\sec^4 x - 2\sec^2 x + 1\right)\sec^4 x\ d(\sec x)\\ &= \int \left(\sec^8 x - 2\sec^6 x + \sec^4 x\right)\ d(\sec x)\\ &= \frac{1}{9}\sec^9 x - \frac{2}{7}\sec^7 x + \frac{1}{5}\sec^5 x + C\end{aligned}$$

Exercise

Evaluate $\displaystyle\int \tan^3 x\ \sec^4 x\ dx$

Solution

$$\int \tan^3 x \sec^4 x\, dx = \int \tan^3 x\left(1+\tan^2 x\right)\sec^2 x\, dx \qquad \sec^2 x = 1+\tan^2 x$$

$$= \int \left(\tan^3 x + \tan^5 x\right)\, d\left(\tan x\right) \qquad d\left(\tan x\right) = \sec^2 x dx$$

$$\boxed{= \frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x + C}$$

Exercise

Evaluate the integral $\int \tan^5 \theta\, \sec^4 \theta\, d\theta$

Solution

$$\int \tan^5 \theta\, \sec^4 \theta\, d\theta = \int \tan^5 \theta\, \sec^2 \theta\, \sec^2 \theta\, d\theta$$

$$= \int \tan^5 \theta\, \left(1+\tan^2 \theta\right)\, d\left(\tan \theta\right)$$

$$= \int \left(\tan^5 \theta + \tan^7 \theta\right)\, d\left(\tan \theta\right)$$

$$\boxed{= \frac{1}{6}\tan^6 \theta + \frac{1}{8}\tan^8 \theta + C}$$

Exercise

Evaluate the integral $\int \tan^5 \theta\, \sec^7 \theta\, d\theta$

Solution

$$\int \tan^5 \theta\, \sec^7 \theta\, d\theta = \int \tan^4 \theta\, \sec^6 \theta\left(\tan \theta\, \sec \theta\right)\, d\theta$$

$$= \int \left(\sec^2 \theta - 1\right)^2 \sec^6 \theta\, d\left(\sec \theta\right)$$

$$= \int \left(\sec^4 \theta - 2\sec^2 \theta + 1\right)\sec^6 \theta\, d\left(\sec \theta\right)$$

$$= \int \left(\sec^{10} \theta - 2\sec^8 \theta + \sec^6 \theta\right)\, d\left(\sec \theta\right)$$

$$\boxed{= \frac{1}{11}\sec^{11} \theta - \frac{2}{9}\sec^9 \theta + \frac{1}{7}\sec^7 \theta + C}$$

Exercise

Evaluate the integral $\displaystyle\int \tan^7 \theta\ \sec^5 \theta\ d\theta$

Solution

$$\int \tan^7 \theta\ \sec^5 \theta\ d\theta = \int \tan^6 \theta\ \sec^4 \theta (\tan\theta\ \sec\theta)\ d\theta$$

$$= \int \left(\sec^2 \theta - 1\right)^3 \sec^4 \theta\ d(\sec\theta)$$

$$= \int \left(\sec^6 \theta - 3\sec^4 \theta + 3\sec^2 \theta - 1\right) \sec^4 \theta\ d(\sec\theta)$$

$$= \int \left(\sec^{10} \theta - 3\sec^8 \theta + 3\sec^6 \theta - \sec^4 \theta\right)\ d(\sec\theta)$$

$$= \frac{1}{11}\sec^{11}\theta - \frac{1}{3}\sec^9\theta + \frac{3}{7}\sec^7\theta - \frac{1}{5}\sec^5\theta + C$$

Exercise

Evaluate the integral $\displaystyle\int \sec^4 3x \tan^3 3x\ dx$

Solution

$$\int \sec^4 3x \tan^3 3x\ dx = \int \sec^2 3x \tan^3 3x\ \sec^2 3x\ dx$$

$$= \frac{1}{3}\int \left(1 + \tan^2 3x\right) \tan^3 3x\ d(\tan 3x)$$

$$= \frac{1}{3}\int \left(\tan^3 3x + \tan^5 3x\right)\ d(\tan 3x)$$

$$= \frac{1}{3}\left(\frac{1}{4}\tan^4 3x + \frac{1}{6}\tan^6 3x\right) + C$$

$$= \frac{1}{12}\tan^4 3x + \frac{1}{18}\tan^6 3x + C$$

Exercise

Evaluate $\displaystyle\int \tan^3 \frac{\pi x}{2}\ \sec^2 \frac{\pi x}{2}\ dx$

Solution

$$\int \tan^3 \frac{\pi x}{2} \sec^2 \frac{\pi x}{2} \, dx = \frac{2}{\pi} \int \tan^3 \frac{\pi x}{2} \, d\left(\tan \frac{\pi x}{2}\right)$$

$$= \frac{1}{2\pi} \tan^4 \frac{\pi x}{2} + C$$

Exercise

Evaluate the integral $\int \sec^{-2} x \tan^3 x \, dx$

Solution

$$\begin{aligned}
\int \sec^{-2} x \tan^3 x \, dx &= \int \sec^{-2} x \tan^2 x \tan x \, dx \\
&= \int \sec^{-2} x \left(\sec^2 x - 1\right) \tan x \, dx \\
&= \int \left(1 - \sec^{-2} x\right) \tan x \, dx \\
&= \int \tan x \, dx - \int \sec^{-2} x \tan x \, dx \\
&= \int \frac{\sin x}{\cos x} \, dx - \int \cos^2 x \cdot \frac{\sin x}{\cos x} \, dx \\
&= -\int \frac{1}{\cos x} \, d(\cos x) + \int \cos x d(\cos x) \\
&= -\ln|\cos x| + \frac{1}{2}\cos^2 x + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \sqrt{\tan x} \sec^4 x \, dx$

Solution

$$\begin{aligned}
\int \sqrt{\tan x} \sec^4 x \, dx &= \int (\tan x)^{1/2} \left(\tan^2 x + 1\right) \sec^2 x \, dx \\
&= \int \left(\tan^{5/2} x + \tan^{1/2} x\right) d(\tan x) \\
&= \frac{2}{7}\tan^{7/2} x + \frac{2}{3}\tan^{3/2} x + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \tan^5\theta \csc^2\theta \, d\theta$

Solution

$$\begin{aligned}
\int \tan^5\theta \csc^2\theta \, d\theta &= \int \frac{\sin^5\theta}{\cos^5\theta}\cdot\frac{1}{\sin^2\theta}\, d\theta \\
&= \int \frac{\sin^3\theta}{\cos^5\theta}\, d\theta \\
&= \int \frac{\sin^3\theta}{\cos^3\theta}\cdot\frac{1}{\cos^2\theta}\, d\theta \\
&= \int \tan^3\theta \sec^2\theta \, d\theta \\
&= \int \tan^3\theta \, d(\tan\theta) \\
&= \frac{1}{4}\tan^4\theta + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \csc^2 x \cot x \, dx$

Solution

$$\begin{aligned}
\int \csc^2 x \cot x \, dx &= -\int \cot x \, d(\cot x) \\
&= -\frac{1}{2}\cot^2 x + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \csc^{10} x \cot x \, dx$

Solution

$$\begin{aligned}
\int \csc^{10} x \cot x \, dx &= -\int \csc^9 x \, d(\csc x) \\
&= -\frac{1}{10}\csc^{10} x + C
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int (\cot 2x - \csc 2x)^2\, dx$

Solution

$$\begin{aligned}\int (\cot 2x - \csc 2x)^2\, dx &= \int \left(\cot^2 2x - 2\cot 2x \csc 2x + \csc^2 2x\right) dx \\ &= \int \cot^2 2x\, dx - \int \left(2\cot 2x \csc 2x\right) dx + \int \csc^2 2x\, dx \\ &= \int \left(\csc^2 2x - 1\right)\, dx - \int \left(\cot 2x \csc 2x\right) d(2x) - \frac{1}{2}\cot 2x \\ &= -\frac{1}{2}\cot 2x - x + \csc 2x - \frac{1}{2}\cot 2x + C \\ &= \csc 2x - \cot 2x - x + C\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \operatorname{sech}^4 x\ dx$

Solution

$$\begin{aligned}\int \operatorname{sech}^4 x\ dx &= \int \operatorname{sech}^2 x\ \operatorname{sech}^2 x\, dx \\ &= \int \operatorname{sech}^2 x\ \left(1 - \tanh^2 x\right)\, dx \\ &= \int \operatorname{sech}^2 x\ dx - \int \operatorname{sech}^2 x\ \tanh^2 x\, dx \\ &= \tanh x - \int \tanh^2 x\, d(\tanh x) \\ &= \tanh x - \frac{1}{3}\tanh^3 x + C\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \sinh^3 x\ \cosh^2 x\, dx$

Solution

$$\int \sinh^3 x\ \cosh^2 x\, dx = \int \sinh^2 x\ \cosh^2 x\ (\sinh x\ dx)$$

$$= \int \left(\cosh^2 x - 1\right)\ \cosh^2 x\ \ d\left(\cosh x\right)$$

$$= \int \left(\cosh^4 x - \cosh^2 x\right)\ d\left(\cosh x\right)$$

$$\boxed{= \frac{1}{5}\cosh^5 x - \frac{1}{3}\cosh^3 x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \operatorname{sech}^2 x\ \sinh x\ dx$

Solution

$$\int \operatorname{sech}^2 x\ \sinh x\ dx = \int \frac{d\left(\cosh x\right)}{\cosh^2 x}$$

$$= -\frac{1}{\cosh x} + C$$

$$\boxed{= -\operatorname{sech} x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\tan^3 x}{\sqrt{\sec x}}\, dx$

Solution

$$\int \frac{\tan^3 x}{\sqrt{\sec x}}\, dx = \int \frac{\tan^2 x \tan x}{\left(\sec x\right)^{1/2}} \frac{\sec x}{\sec x}\, dx$$

$1 + \tan^2 \alpha = \sec^2 \alpha$

$$= \int \left(\sec x\right)^{-3/2}\left(\sec^2 x - 1\right) d\left(\sec x\right)$$

$$= \int \left(\left(\sec x\right)^{1/2} - \left(\sec x\right)^{-3/2}\right) d\left(\sec x\right)$$

$$\boxed{= \frac{2}{3}\left(\sec x\right)^{3/2} + 2\left(\sec x\right)^{-1/2} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\tan^2 x}{\sec x}\,dx$

Solution

$$\int \frac{\tan^2 x}{\sec x}\,dx = \int \frac{\sec^2 x - 1}{\sec x}\,dx$$

$$= \int \left(\sec x - \frac{1}{\sec x}\right) dx$$

$$= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\,dx - \int \cos x\;dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx - \sin x$$

$$= \int \frac{d\left(\sec x + \tan x\right)}{\sec x + \tan x} - \sin x$$

$$= \ln\left|\sec x + \tan x\right| - \sin x + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sec x}{\tan^2 x}\,dx$

Solution

$$\int \frac{\sec x}{\tan^2 x}\,dx = \int \frac{1}{\cos x}\frac{\cos^2 x}{\sin^2 x}\,dx$$

$$= \int \frac{\cos x}{\sin^2 x}\,dx$$

$$= \int \frac{1}{\sin^2 x}\,d\left(\sin x\right)$$

$$= -\frac{1}{\sin x} + C$$

$$= -\csc x + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sec^2 x}{\tan^5 x}\,dx$

Solution

$$\int \frac{\sec^2 x}{\tan^5 x}\,dx = \int \tan^{-5} x \, d(\tan x)$$

$$= -\frac{1}{4}\tan^{-4} x + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\csc^4 x}{\cot^2 x}\,dx$

Solution

$$\int \frac{\csc^4 x}{\cot^2 x}\,dx = \int \frac{\csc^2 x\left(\cot^2 x + 1\right)}{\cot^2 x}\,dx$$

$$= -\int \frac{\cot^2 x + 1}{\cot^2 x}\,d(\cot x)$$

$$= -\int \left(1 + \frac{1}{\cot^2 x}\right) d(\cot x)$$

$$= -\cot x + \frac{1}{\cot x} + C$$

$$= -\cot x + \tan x + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sec^4(\ln x)}{x}\,dx$

Solution

$$\int \frac{\sec^4(\ln x)}{x}\,dx = \int \sec^4(\ln x)\, d(\ln x)$$

$$= \int \sec^2(\ln x)\left(\tan^2(\ln x) + 1\right) d(\ln x)$$

$$= \int \sec^2(\ln x)\tan^2(\ln x)\, d(\ln x) + \int \sec^2(\ln x)\, d(\ln x)$$

$$= \int \tan^2(\ln x)\, d(\tan(\ln x)) + \tan(\ln x)$$

$$= \frac{1}{3}\tan^3(\ln x) + \tan(\ln x) + C$$

Exercise

Evaluate the integral $\displaystyle\int e^x \sec\left(e^x+1\right)dx$

Solution

$$\int e^x \sec\left(e^x+1\right)dx = \int \sec\left(e^x+1\right)d\left(e^x+1\right)$$

$$= \int \sec\left(e^x+1\right)\frac{\sec\left(e^x+1\right)+\tan\left(e^x+1\right)}{\sec\left(e^x+1\right)+\tan\left(e^x+1\right)}d\left(e^x+1\right)$$

$$= \int \frac{\sec^2\left(e^x+1\right)+\sec\left(e^x+1\right)\tan\left(e^x+1\right)}{\sec\left(e^x+1\right)+\tan\left(e^x+1\right)}d\left(e^x+1\right)$$

$$= \int \frac{1}{\sec\left(e^x+1\right)+\tan\left(e^x+1\right)}d\left(\sec\left(e^x+1\right)+\tan\left(e^x+1\right)\right)$$

$$= \ln\left|\sec\left(e^x+1\right)+\tan\left(e^x+1\right)\right|+C$$

Exercise

Evaluate the integral $\displaystyle\int e^x \sec^3\left(e^x\right)\,dx$

Solution

$$u = \sec\left(e^x\right) \qquad dv = \sec\left(e^x\right)e^x dx$$

$$du = \sec\left(e^x\right)\tan\left(e^x\right)e^x dx \qquad v = \int \sec\left(e^x\right)d\left(e^x\right) = \tan\left(e^x\right)$$

$$\int e^x \sec^3\left(e^x\right)\,dx = \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec\left(e^x\right)\tan^2\left(e^x\right)e^x dx$$

$$= \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec\left(e^x\right)\left(\sec^2\left(e^x\right)-1\right)e^x dx$$

$$= \sec\left(e^x\right)\tan\left(e^x\right) - \int \left(\sec^3\left(e^x\right)-\sec\left(e^x\right)\right)e^x dx$$

$$= \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec^3\left(e^x\right)e^x dx + \int \sec\left(e^x\right)e^x dx \qquad d\left(e^x\right) = e^x dx$$

$$= \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec^3\left(e^x\right)e^x dx + \int \sec\left(e^x\right)d\left(e^x\right)$$

$$\int \sec^3\left(e^x\right)e^x dx = \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec^3\left(e^x\right)e^x dx + \ln\left|\sec\left(e^x\right) + \tan\left(e^x\right)\right|$$

$$2\int \sec^3\left(e^x\right)e^x dx = \sec\left(e^x\right)\tan\left(e^x\right) + \ln\left|\sec\left(e^x\right) + \tan\left(e^x\right)\right| + C$$

$$\int \sec^3\left(e^x\right)e^x dx = \underline{\tfrac{1}{2}\sec\left(e^x\right)\tan\left(e^x\right) + \tfrac{1}{2}\ln\left|\sec\left(e^x\right) + \tan\left(e^x\right)\right| + C}$$

Exercise

Evaluate the integral $\int e^x \sqrt{\tan^2 e^x + 1}\, dx$

Solution

$$\begin{aligned}
\int e^x \sqrt{\tan^2\left(e^x\right)+1}\, dx &= \int \sqrt{\tan^2\left(e^x\right)+1}\, d\left(e^x\right) \\
&= \int \sqrt{\sec^2\left(e^x\right)}\, d\left(e^x\right) \\
&= \int \sec e^x \cdot \frac{\sec e^x + \tan e^x}{\sec e^x + \tan e^x}\, d\left(e^x\right) \\
&= \int \frac{\sec^2 e^x + \sec e^x \tan e^x}{\sec e^x + \tan e^x}\, d\left(e^x\right) \\
&= \int \frac{1}{\sec e^x + \tan e^x}\, d\left(\sec e^x + \tan e^x\right) \\
&= \underline{\ln\left|\sec e^x + \tan e^x\right| + C}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\sqrt{\frac{\pi}{2}}} x\sin^3\left(x^2\right)dx$

Solution

$$\begin{aligned}
\int_0^{\sqrt{\frac{\pi}{2}}} x\sin^3\left(x^2\right)dx &= \frac{1}{2}\int_0^{\sqrt{\frac{\pi}{2}}} \sin^3\left(x^2\right)d\left(x^2\right) \\
&= \frac{1}{2}\int_0^{\sqrt{\frac{\pi}{2}}} \sin^2\left(x^2\right)\ \sin\left(x^2\right)\ d\left(x^2\right)
\end{aligned}$$

$$= -\frac{1}{2}\int_0^{\sqrt{\frac{\pi}{2}}} \left(1 - \cos^2\left(x^2\right)\right)\, d\left(\cos x^2\right)$$

$$= -\frac{1}{2}\left(\cos x^2 - \frac{1}{3}\cos^3\left(x^2\right)\right) \Bigg|_0^{\sqrt{\frac{\pi}{2}}}$$

$$= -\frac{1}{2}\left(\cos\frac{\pi}{2} - \frac{1}{3}\cos^3\frac{\pi}{2} - 1 + \frac{1}{3}\right)$$

$$= -\frac{1}{2}\left(-\frac{2}{3}\right)$$

$$= \frac{1}{3}$$

Exercise

Evaluate the integral $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\, dx$

Solution

$$\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\, dx = \int_{\pi/6}^{\pi/3} (\sin x)^{-1/2}\left(1 - \sin^2 x\right)\, d(\sin x)$$

$$= \int_{\pi/6}^{\pi/3} \left((\sin x)^{-1/2} - (\sin x)^{3/2}\right)\, d(\sin x)$$

$$= 2(\sin x)^{1/2} - \frac{2}{5}(\sin x)^{5/2} \Big|_{\pi/6}^{\pi/3}$$

$$= 2\left(\frac{\sqrt{3}}{2}\right)^{1/2} - \frac{2}{5}\left(\frac{\sqrt{3}}{2}\right)^{5/2} - 2\left(\frac{1}{2}\right)^{1/2} + \frac{2}{5}\left(\frac{1}{2}\right)^{5/2}$$

$$= \sqrt[4]{3}\sqrt{2} - \frac{3}{10}\frac{\sqrt[4]{3}}{\sqrt{2}} - \sqrt{2} + \frac{\sqrt{2}}{20}$$

$$= \frac{\sqrt{2}}{20}\left(17\sqrt[4]{3} - 19\right)$$

Exercise

Evaluate the integral $\displaystyle\int_{\pi/6}^{\pi/2} \frac{dx}{\sin x}$

Solution

$$\int_{\pi/6}^{\pi/2} \frac{dx}{\sin x} = \int_{\pi/6}^{\pi/2} \csc x\,\frac{\csc x + \cot x}{\csc x + \cot x}\, dx$$

$$= \int_{\pi/6}^{\pi/2} \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\, dx$$

$$= -\int_{\pi/6}^{\pi/2} \frac{1}{\csc x + \cot x}\, d\left(\csc x + \cot x\right)$$

$$= -\ln\left|\csc x + \cot x\right| \Big|_{\pi/6}^{\pi/2}$$

$$= -\ln\left|1+0\right| + \ln\left|2+\sqrt{3}\right|$$

$$\boxed{= \ln\left(2+\sqrt{3}\right)}$$

Exercise

Evaluate the integral $\int_{\pi/6}^{\pi/3} \cot^3 \theta\, d\theta$

Solution

$$\int_{\pi/6}^{\pi/3} \cot^3 \theta\, d\theta = \int_{\pi/6}^{\pi/3} \cot\theta\left(\csc^2\theta - 1\right)\, d\theta$$

$$= \int_{\pi/6}^{\pi/3} \cot\theta \csc^2 \theta\, d\theta - \int_{\pi/6}^{\pi/3} \cot\theta\, d\theta$$

$$= -\int_{\pi/6}^{\pi/3} \cot\theta\, d\left(\cot\theta\right) - \int_{\pi/6}^{\pi/3} \frac{\cos\theta}{\sin\theta}\, d\theta$$

$$= -\frac{1}{2}\cot^2\theta - \int_{\pi/6}^{\pi/3} \frac{1}{\sin\theta}\, d\left(\sin\theta\right)$$

$$= -\frac{1}{2}\cot^2\theta - \ln\left|\sin\theta\right| \Big|_{\pi/6}^{\pi/3}$$

$$= -\frac{1}{2}\frac{1}{3} - \ln\frac{\sqrt{3}}{2} + \frac{1}{2}3 + \ln\frac{1}{2}$$

$$= \frac{3}{2} - \frac{1}{6} + \ln\frac{1}{2} - \ln\frac{\sqrt{3}}{2}$$

$$= \frac{4}{3} + \ln\frac{1}{\sqrt{3}}$$

$$\boxed{= \frac{4}{3} - \ln\sqrt{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/3} \tan^2 x\, dx$

Solution

$$\int_0^{\pi/3} \tan^2 x\, dx = \int_0^{\pi/3} \left(\sec^2 x - 1\right) dx$$

$$= \tan x - x \Big|_0^{\pi/3}$$

$$= \sqrt{3} - \frac{\pi}{3}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/4} 6\tan^3 x\, dx$

Solution

$$\int_0^{\pi/4} 6\tan^3 x\, dx = 6\int_0^{\pi/4} \tan x \tan^2 x\, dx$$

$$= 6\int_0^{\pi/4} \tan x\left(\sec^2 x - 1\right) dx$$

$$= 6\int_0^{\pi/4} \tan x \sec^2 x\, dx - 6\int_0^{\pi/4} \tan x\, dx$$

$$= 6\int_0^{\pi/4} \tan x\; d(\tan x) - 6\int_0^{\pi/4} \frac{\sin x}{\cos x}\, dx$$

$$= 3\tan^2 x \Big|_0^{\pi/4} + 6\int_0^{\pi/4} \frac{1}{\cos x}\, d(\cos x)$$

$$= 3 + 6\ln(\cos x) \Big|_0^{\pi/4}$$

$$= 3 + 6\ln\frac{\sqrt{2}}{2} - 6\ln 1$$

$$= 3 + 6\ln\frac{\sqrt{2}}{2}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/4} \tan^4 x dx$

Solution

$$\int_0^{\pi/4} \tan^4 x dx = \int_0^{\pi/4} \tan^2 x\left(\sec^2 x - 1\right) dx$$

$$= \int_0^{\pi/4} \tan^2 x\left(\sec^2 x - 1\right) dx$$

$$= \int_0^{\pi/4} \tan^2 x \sec^2 x\, dx - \int_0^{\pi/4} \tan^2 x\, dx$$

$$= \int_0^{\pi/4} \tan^2 x\, d(\tan x) - \int_0^{\pi/4} \left(\sec^2 x - 1\right) dx$$

$$= \frac{1}{3}\tan^3 x - \tan x + x \Big|_0^{\pi/4}$$

$$= \frac{1}{3} - 1 + \frac{\pi}{4}$$

$$= \frac{\pi}{4} - \frac{2}{3}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi} 8\sin^4 y \cos^2 y\, dy$

Solution

$$\int_0^{\pi} 8\sin^4 y \cos^2 y\, dy = 8\int_0^{\pi} \left(\frac{1-\cos 2y}{2}\right)^2 \left(\frac{1+\cos 2y}{2}\right) dy$$

$$= \int_0^{\pi} \left(1 - 2\cos 2y + \cos^2 2y\right)\left(1+\cos 2y\right) dy$$

$$= \int_0^{\pi} \left(1 - 2\cos 2y + \cos^2 2y + \cos 2y - 2\cos^2 2y + \cos^3 2y\right) dy$$

$$= \int_0^{\pi} \left(1 - \cos 2y - \cos^2 2y + \cos^3 2y\right) dy$$

$$= \int_0^{\pi} \left(1 - \cos 2y - \frac{1}{2} - \frac{1}{2}\cos 4y\right) dy + \int_0^{\pi} \cos^2 2y \cos 2y\, dy$$

$$= \int_0^{\pi} \left(\frac{1}{2} - \cos 2y - \frac{1}{2}\cos 4y\right) dy + \frac{1}{2}\int_0^{\pi} \left(1 - \sin^2 2y\right) d\left(\sin 2y\right)$$

$$= \left[\frac{1}{2}y - \frac{1}{2}\sin 2y - \frac{1}{8}\sin 4y + \frac{1}{2}\left(\sin 2y - \frac{1}{3}\sin^3 2y\right)\right]_0^{\pi}$$

$$= \frac{\pi}{2}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/6} 3\cos^5 3x \, dx$

Solution

$$\int_0^{\pi/6} 3\cos^5 3x \, dx = \int_0^{\pi/6} 3\left(\cos^2 3x\right)^2 \cos 3x \, dx$$

$$= \int_0^{\pi/6} \left(1 - \sin^2 3x\right)^2 d\left(\sin 3x\right)$$

$$= \int_0^{\pi/6} \left(1 - 2\sin^2 3x + \sin^4 3x\right) d\left(\sin 3x\right)$$

$$= \left[\sin 3x - \frac{2}{3}\sin^2 3x + \frac{1}{5}\sin^4 3x\right]_0^{\pi/6}$$

$$= \sin\frac{\pi}{2} - \frac{2}{3}\sin^2\frac{\pi}{2} + \frac{1}{5}\sin^4\frac{\pi}{2} - 0$$

$$= 1 - \frac{2}{3} + \frac{1}{5}$$

$$= \frac{8}{15}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta \, d\theta$

Solution

$$\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta \, d\theta = \int_0^{\pi/2} \sin^2 2\theta \left(\cos^2 2\theta\right) \cos 2\theta \, d\theta \qquad d\left(\sin 2\theta\right) = 2\cos 2\theta d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^2 2\theta \left(1 - \sin^2 2\theta\right) d\left(\sin 2\theta\right)$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(\sin^2 2\theta - \sin^4 2\theta\right)d\left(\sin 2\theta\right)$$

$$= \frac{1}{2}\left[\frac{1}{3}\sin^3 2\theta - \frac{1}{5}\sin^5 2\theta\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left(\frac{1}{3}\sin^3 \pi - \frac{1}{5}\sin^5 \pi - 0\right)$$

$$\underline{= 0}$$

Exercise

Evaluate the integral $\int_0^{2\pi}\sqrt{\frac{1-\cos x}{2}}dx$

Solution

$$\int_0^{2\pi}\sqrt{\frac{1-\cos x}{2}}dx = \int_0^{2\pi}\sin\frac{x}{2}dx \qquad \left|\sin\left(\frac{\alpha}{2}\right)\right| = \sqrt{\frac{1-\cos\alpha}{2}}$$

$$= \left[-2\cos\frac{x}{2}\right]_0^{2\pi}$$

$$= -2\left(\cos\pi - \cos 0\right)$$

$$\underline{= 2}$$

Exercise

Evaluate the integral $\int_0^{\pi}\sqrt{1-\cos^2\theta}\,d\theta$

Solution

$$\int_0^{\pi}\sqrt{1-\cos^2\theta}\,d\theta = \int_0^{\pi}\left|\sin\theta\right|d\theta$$

$$= \left[-\cos\theta\right]_0^{\pi}$$

$$= -\cos\pi + \cos 0$$

$$\underline{= 2}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/6} \sqrt{1+\sin x}\; dx$

Solution

$$\int_0^{\pi/6} \sqrt{1+\sin x}\; dx = \int_0^{\pi/6} \sqrt{1+\sin x}\,\frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}}\; dx$$

$$= \int_0^{\pi/6} \frac{\sqrt{1-\sin^2 x}}{\sqrt{1-\sin x}}\; dx \qquad\qquad \cos x = \sqrt{1-\sin^2 x}$$

$$= \int_0^{\pi/6} \frac{\cos x}{\sqrt{1-\sin x}}\; dx \qquad\qquad d\left(1-\sin x\right) = -\cos x dx$$

$$= -\int_0^{\pi/6} (1-\sin x)^{-1/2}\; d\left(1-\sin x\right)$$

$$= -2\Big[(1-\sin x)^{1/2} \Big]_0^{\pi/6}$$

$$= -2\left(\sqrt{1-\sin\frac{\pi}{6}} - 1 \right)$$

$$= -2\left(\sqrt{1-\frac{1}{2}} - 1 \right)$$

$$= -2\left(\frac{1}{\sqrt{2}} - 1 \right)$$

$$= -2\left(\frac{\sqrt{2}}{2} - 1 \right)$$

$$= 2-\sqrt{2}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2\theta - 1}\; d\theta$

Solution

$$\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2\theta - 1}\; d\theta = \int_{-\pi/3}^{\pi/3} \tan\theta\; d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \frac{\sin\theta}{\cos\theta}\; d\theta$$

$$= -\int_{-\pi/3}^{\pi/3} \frac{1}{\cos\theta}\, d(\cos\theta)$$

$$= -\ln|\cos\theta| \Big|_{-\pi/3}^{\pi/3}$$

$$= -\left(\ln\left(\cos\frac{\pi}{3}\right) - \ln\left(\cos\frac{\pi}{3}\right)\right)$$

$$\underline{= 0}$$

Exercise

Evaluate the integral $\int_0^{\pi} (1-\cos 2x)^{3/2}\, dx$

Solution

$$\int_0^{\pi} (1-\cos 2x)^{3/2}\, dx = \int_0^{\pi} \left(2\sin^2 x\right)^{3/2} dx$$

$$= 2\sqrt{2}\int_0^{\pi} \sin^3 x\, dx$$

$$= 2\sqrt{2}\int_0^{\pi} \sin^2 x\, \sin x\, dx$$

$$= -2\sqrt{2}\int_0^{\pi} \left(1-\cos^2 x\right) d(\cos x)$$

$$= -2\sqrt{2}\left(\cos x - \frac{1}{3}\cos^3 x\right) \Big|_0^{\pi}$$

$$= -2\sqrt{2}\left(-1+\frac{1}{3}-1+\frac{1}{3}\right)$$

$$= -2\sqrt{2}\left(-\frac{4}{3}\right)$$

$$\underline{= \frac{8\sqrt{2}}{3}}$$

Exercise

Evaluate the integral $\int_0^{\pi} \left(1-\cos^2 x\right)^{3/2} dx$

Solution

$$\int_0^{\pi}\left(1-\cos^2 x\right)^{3/2}dx = \int_0^{\pi}\left(\sin^2 x\right)^{3/2}dx$$

$$= \int_0^{\pi}\sin^3 x\ dx$$

$$= \int_0^{\pi}\sin^3 x\ dx \qquad\qquad \sin^2 x = 1-\cos^2 x$$

$$= \int_0^{\pi}\left(1-\cos^2 x\right)\sin x\ dx \qquad\qquad d\left(\cos x\right) = -\sin x dx$$

$$= -\int_0^{\pi}\left(1-\cos^2 x\right)d\left(\cos x\right)$$

$$= -\left(\cos x - \tfrac{1}{3}\cos^3 x\ \right|_0^{\pi}$$

$$= -\left(-1+\tfrac{1}{3}-\left(1-\tfrac{1}{3}\right)\right)$$

$$= 1-\tfrac{1}{3}+1-\tfrac{1}{3}$$

$$= 2-\tfrac{2}{3}$$

$$\boxed{= \tfrac{4}{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\pi}^{\pi}\left(1-\cos^2 x\right)^{3/2}dx$

Solution

$$\int_{-\pi}^{\pi}\left(1-\cos^2 x\right)^{3/2}dx = \int_{-\pi}^{\pi}\left(\sin^2 x\right)^{3/2}dx$$

$$= \int_{-\pi}^{\pi}\left|\sin^3 x\right|dx$$

$$= -\int_{-\pi}^{0}\sin^3 x\ dx + \int_0^{\pi}\sin^3 x\ dx \qquad\qquad \sin^2 x = 1-\cos^2 x$$

$$= -\int_{-\pi}^{0}\left(1-\cos^2 x\right)\sin x\ dx + \int_0^{\pi}\left(1-\cos^2 x\right)\sin x\ dx \qquad d\left(\cos x\right) = -\sin x dx$$

$$= \int_{-\pi}^{0} \left(1-\cos^2 x\right) d\left(\cos x\right) - \int_{0}^{\pi} \left(1-\cos^2 x\right) d\left(\cos x\right)$$

$$= \left(\cos x - \frac{1}{3}\cos^3 x\right. \Big|_{-\pi}^{0} - \left(\cos x - \frac{1}{3}\cos^3 x\right. \Big|_{0}^{\pi}$$

$$= \left(1-\frac{1}{3}-\left(-1+\frac{1}{3}\right)\right) - \left(-1+\frac{1}{3}-\left(1-\frac{1}{3}\right)\right)$$

$$= 1-\frac{1}{3}+1-\frac{1}{3}+1-\frac{1}{3}+1-\frac{1}{3}$$

$$= 4-\frac{4}{3}$$

$$= \frac{8}{3}$$

Exercise

Evaluate the integral $\displaystyle\int_{\pi/4}^{\pi/2} \csc^4 \theta d\theta$

Solution

$$\int_{\pi/4}^{\pi/2} \csc^4 \theta d\theta = \int_{\pi/4}^{\pi/2} \left(1+\cot^2 \theta\right)\csc^2 \theta d\theta \qquad \csc^2 \theta = 1+\cot^2 \theta$$

$$= \int_{\pi/4}^{\pi/2} \csc^2 \theta d\theta + \int_{\pi/4}^{\pi/2} \cot^2 \theta \csc^2 \theta d\theta \qquad d\left(\cot \theta\right) = -\csc^2 \theta d\theta$$

$$= \int_{\pi/4}^{\pi/2} \csc^2 \theta d\theta - \int_{\pi/4}^{\pi/2} \cot^2 \theta\, d\left(\cot \theta\right)$$

$$= \left(-\cot \theta - \frac{1}{3}\cot^3 \theta\right. \Big|_{\pi/4}^{\pi/2}$$

$$= -\left(\cot\frac{\pi}{2} + \frac{1}{3}\cot^3\frac{\pi}{2} - \cot\frac{\pi}{4} - \frac{1}{3}\cot^3\frac{\pi}{4}\right)$$

$$= -\left(0+\frac{1}{3}(0)-1-\frac{1}{3}\right)$$

$$= \frac{4}{3}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\pi}^{\pi} \sin 3x \sin 3x \, dx$

Solution

$$\int_{-\pi}^{\pi} \sin 3x \sin 3x \, dx = \frac{1}{2}\int_{-\pi}^{\pi} (\cos 0 - \cos 6x) \, dx \qquad \sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} (1 - \cos 6x) \, dx$$

$$= \frac{1}{2}\left[x - \frac{1}{6}\sin 6x\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left(\pi - \frac{1}{6}\sin 6\pi - \left(-\pi - \frac{1}{6}\sin(-6\pi)\right)\right)$$

$$= \frac{1}{2}(\pi + \pi)$$

$$= \pi$$

Exercise

Evaluate the integral $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x \cos 7x \, dx$

Solution

$$\int_{-\pi/2}^{\pi/2} \cos x \cos 7x dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (\cos 8x + \cos(-6x)) \, dx \qquad \cos\alpha\cos\beta = \tfrac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} (\cos 8x + \cos 6x) \, dx$$

$$= \frac{1}{2}\left(\frac{1}{6}\sin 6x + \frac{1}{8}\sin 8x \right|_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2}\left(\frac{1}{6}\sin(3\pi) + \frac{1}{8}\sin(4\pi) - \frac{1}{6}\sin(-3\pi) - \frac{1}{8}\sin(-4\pi)\right)$$

$$= 0$$

Exercise

Evaluate the integral $\displaystyle\int_{0}^{\pi/4} \cos^5 2x \, \sin^2 2x \, dx$

Solution

$$\int_0^{\pi/4} \cos^5 2x \ \sin^2 2x \, dx = \int_0^{\pi/4} \cos^4 2x \ \cos 2x \ \sin^2 2x \, dx$$

$$= \frac{1}{2}\int_0^{\pi/4} \left(1-\sin^2 2x\right)^2 \ \sin^2 2x \ d\left(\sin 2x\right)$$

$$= \frac{1}{2}\int_0^{\pi/4} \left(1-2\sin^2 2x+\sin^4 2x\right) \ \sin^2 2x \ d\left(\sin 2x\right)$$

$$= \frac{1}{2}\int_0^{\pi/4} \left(\sin^2 2x-2\sin^4 2x+\sin^6 2x\right) \ d\left(\sin 2x\right)$$

$$= \frac{1}{2}\left(\frac{1}{3}\sin^3 2x-\frac{2}{5}\sin^5 2x+\frac{1}{7}\sin^7 2x\right) \Big|_0^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{1}{3}\sin^3 \frac{\pi}{2}-\frac{2}{5}\sin^5 \frac{\pi}{2}+\frac{1}{7}\sin^7 \frac{\pi}{2}-0\right)$$

$$= \frac{1}{2}\left(\frac{1}{3}-\frac{2}{5}+\frac{1}{7}\right)$$

$$= \frac{1}{2}\left(\frac{35-42+25}{105}\right)$$

$$= \frac{4}{105}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/6} \sin^5 x \, dx$

Solution

$$\int_0^{\pi/6} \sin^5 x \, dx = \int_0^{\pi/6} \sin^4 x \sin x \, dx$$

$$= -\int_0^{\pi/6} \left(1-\cos^2 x\right)^2 \ d\left(\cos x\right)$$

$$= -\int_0^{\pi/6} \left(1-2\cos^2 x+\cos^4 x\right) \ d\left(\cos x\right)$$

$$= -\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x \ \Big|_0^{\pi/6}$$

$$= -\frac{\sqrt{3}}{2}+\frac{2}{3}\frac{3\sqrt{3}}{8}-\frac{1}{5}\frac{9\sqrt{3}}{32}+1-\frac{2}{3}+\frac{1}{5}$$

$$= -\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4} - \frac{9\sqrt{3}}{160} + \frac{15-10+3}{15}$$

$$= \left(\frac{-80+40-9}{160}\right)\sqrt{3} + \frac{8}{15}$$

$$= -\frac{49\sqrt{3}}{160} + \frac{8}{15}$$

$$= \frac{256-147\sqrt{3}}{480}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\pi}^{\pi} \sin^2 x \, dx$

Solution

$$\int_{-\pi}^{\pi} \sin^2 x \, dx = 2\int_{0}^{\pi} \sin^2 x \, dx$$

$$= 2\int_{0}^{\pi} \frac{1}{2}(1-\cos 2x) \, dx$$

$$= x - \frac{1}{2}\sin 2x \Big|_{0}^{\pi}$$

$$= \pi$$

Exercise

Evaluate the integral $\displaystyle\int_{-\pi/2}^{\pi/2} \left(\sin^2 x + 1\right) dx$

Solution

$$\int_{-\pi/2}^{\pi/2} \left(\sin^2 x + 1\right) dx = \int_{-\pi/2}^{\pi/2} \left(\frac{1}{2} - \frac{1}{2}\cos 2x + 1\right) dx$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{3}{2} - \frac{1}{2}\cos 2x\right) dx$$

$$= \frac{3}{2}x - \frac{1}{4}\sin 2x \Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{3\pi}{4} + \frac{3\pi}{4}$$

$$= \frac{3\pi}{2}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/3} \sec^{3/2} x \ \tan x \, dx$

Solution

$$\begin{aligned}\int_0^{\pi/3} \sec^{3/2} x \ \tan x \, dx &= \int_0^{\pi/3} \sec^{1/2} x \ (\sec x \tan x) \ dx \\ &= \int_0^{\pi/3} \sec^{1/2} x \, d(\sec x) \\ &= \frac{2}{3}\sec^{3/2} x \ \Big|_0^{\pi/3} \\ &= \frac{2}{3}\left((2)^{3/2} - 1\right) \\ &= \frac{2}{3}\left(2\sqrt{2} - 1\right)\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1+\sin x} dx$

Solution

$$\begin{aligned}\int_0^{\pi/2} \frac{\cos x}{1+\sin x} dx &= \int_0^{\pi/2} \frac{1}{1+\sin x} d(1+\sin x) \\ &= \ln|1+\sin x| \ \Big|_0^{\pi/2} \\ &= \ln 2 - \ln 1 \\ &= \ln 2\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int_{\pi/6}^{\pi/3} \sin 6x \ \cos 4x \ dx$

Solution

$$\int_{\pi/6}^{\pi/3} \sin 6x \ \cos 4x \ dx = \frac{1}{2}\int_{\pi/6}^{\pi/3} (\sin 10x + \sin 2x) \ dx \qquad \sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$=\frac{1}{2}\left(-\frac{1}{10}\cos 10x-\frac{1}{2}\cos 2x\right|_{\pi/6}^{\pi/3}$$

$$=-\frac{1}{2}\left(\frac{1}{10}\cos\frac{10\pi}{3}+\frac{1}{2}\cos\frac{2\pi}{3}-\frac{1}{10}\cos\frac{5\pi}{3}-\frac{1}{2}\cos\frac{\pi}{3}\right)$$

$$=-\frac{1}{2}\left(-\frac{1}{20}-\frac{1}{4}-\frac{1}{20}-\frac{1}{4}\right)$$

$$=\frac{1}{2}\left(\frac{1}{10}+\frac{1}{2}\right)$$

$$=\frac{6}{20}$$

$$=\frac{3}{10}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\pi/2}^{\pi/2} 3\cos^3 x\,dx$

Solution

$$\int_{-\pi/2}^{\pi/2} 3\cos^3 x\,dx = 3\int_{-\pi/2}^{\pi/2}\cos^2 x\,\cos x\,dx$$

$$=3\int_{-\pi/2}^{\pi/2}\left(1-\sin^2 x\right)\,d(\sin x)$$

$$=3\left(\sin x-\frac{1}{3}\sin^3 x\right|_{-\pi/2}^{\pi/2}$$

$$=3\left(1-\frac{1}{3}+1-\frac{1}{3}\right)$$

$$=3\left(2-\frac{2}{3}\right)$$

$$=4$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi}\sec^2 x\,dx$

Solution

$$\int_0^{\pi}\sec^2 x\,dx=\tan x\Big|_0^{\pi}$$

$$=\tan\pi-\tan 0$$

$$=0$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\ln(\sqrt{3}+2)} \frac{\cosh x}{\sqrt{4-\sinh^2 x}}\,dx$

Solution

$$\int_0^{\ln(\sqrt{3}+2)} \frac{\cosh x}{\sqrt{4-\sinh^2 x}}\,dx = \int_0^{\ln(\sqrt{3}+2)} \frac{1}{\sqrt{4-\sinh^2 x}}\,d(\sinh x)$$

$$= \sin^{-1}\left(\tfrac{1}{2}\sinh x\right)\Big|_0^{\ln(\sqrt{3}+2)}$$

$$= \sin^{-1}\left(\tfrac{1}{2}\sinh\left(\ln\left(\sqrt{3}+2\right)\right)\right) - \sin^{-1}\left(\tfrac{1}{2}\sinh 0\right)$$

$$= \sin^{-1}\left(\frac{1}{4}\left(e^{\ln(\sqrt{3}+2)} - e^{-\ln(\sqrt{3}+2)}\right)\right)$$

$$= \sin^{-1}\left(\frac{1}{4}\left(\sqrt{3}+2-\frac{1}{\sqrt{3}+2}\right)\right)$$

$$= \sin^{-1}\left(\frac{1}{4}\left(\frac{3+4\sqrt{3}+4-1}{\sqrt{3}+2}\right)\right)$$

$$= \sin^{-1}\left(\frac{1}{4}\left(\frac{6+4\sqrt{3}}{\sqrt{3}+2}\,\frac{\sqrt{3}-2}{\sqrt{3}-2}\right)\right)$$

$$= \sin^{-1}\left(\frac{1}{2}\left(\frac{-\sqrt{3}}{3-4}\right)\right)$$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{3}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/2} \cos^4 x\,dx$

Solution

$$\int_0^{\pi/2} \cos^4 x\,dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{\pi}{2}\right)$$

$$\int_0^{\pi/2} \cos^n x\,dx = \left(\tfrac{1}{2}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{5}{6}\right)\cdots\left(\tfrac{n-1}{n}\right)\left(\tfrac{\pi}{2}\right)$$

$$= \frac{3\pi}{16}$$

Exercise

Evaluate $\displaystyle\int_0^{\pi/2} \cos^{10} x\, dx$

Solution

$$\int_0^{\pi/2} \cos^{10}\theta\, d\theta = \int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^5 d\theta$$

$$= \frac{1}{32}\int_0^{\pi/2} \left(1+5\cos 2\theta+10\cos^2 2\theta+10\cos^3 2\theta+5\cos^4 2\theta+\cos^5 2\theta\right) d\theta$$

$$= \frac{1}{32}\int_0^{\pi/2} \left(\begin{array}{l} 1+5\cos 2\theta+5+5\cos 4\theta+\frac{5}{4}(1+\cos 4\theta)^2 \\ \quad +\left(10+\cos^2 2\theta\right)\cos^3 2\theta \end{array}\right) d\theta$$

$$= \frac{1}{32}\int_0^{\pi/2} \left(6+5\cos 2\theta+5\cos 4\theta+\frac{5}{4}\left(1+2\cos 4\theta+\cos^2 4\theta\right)\right) d\theta$$

$$+\frac{5}{16}\int_0^{\pi/2} \cos^3 2\theta\, d\theta+\frac{1}{32}\int_0^{\pi/2} \cos^5 2\theta\, d\theta$$

$$= \frac{1}{32}\int_0^{\pi/2} \left(\frac{63}{8}+5\cos 2\theta+\frac{11}{2}\cos 4\theta+\frac{5}{8}\cos 8\theta\right) d\theta$$

$$+\frac{5}{32}\int_0^{\pi/2} \left(1-\sin^2 2\theta\right) d(\sin 2\theta)+\frac{1}{64}\int_0^{\pi/2} \left(1-\sin^2 2\theta\right)^2 d(\sin 2\theta)$$

$$= \left[\frac{1}{32}\left(\frac{63}{8}\theta+\frac{5}{2}\sin 2\theta+\frac{11}{8}\sin 4\theta+\frac{5}{64}\sin 8\theta\right)+\frac{5}{32}\left(\sin 2\theta-\frac{1}{3}\sin^3 2\theta\right)\right]_0^{\pi/2}$$

$$+\frac{1}{64}\int_0^{\pi/2} \left(1-2\sin^2 2\theta+\sin^4 2\theta\right) d(\sin 2\theta)$$

$$= \frac{1}{32}\left(\frac{63\pi}{16}\right)+\frac{1}{64}\left(\sin 2\theta-\frac{2}{3}\sin^3 2\theta+\frac{1}{5}\sin^5 2\theta\right)\Big|_0^{\pi/2}$$

$$\boxed{= \frac{63\,\pi}{512}}$$

Or

$$\int_0^{\pi/2} \cos^{10} x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\left(\frac{9}{10}\right)\left(\frac{\pi}{2}\right)$$

$$\int_0^{\pi/2} \cos^n x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

$$= \boxed{\frac{63\pi}{512}}$$

Exercise

Evaluate $\int_0^{\pi/2} \cos^7 x\, dx$

Solution

$$\int_0^{\pi/2} \cos^7 x\, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)$$

$$\int_0^{\pi/2} \cos^n \theta\, d\theta = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right)$$

$$= \boxed{\frac{16}{35}}$$

Or

$$\int_0^{\pi/2} \cos^7 x\, dx = \int_0^{\pi/2} \left(\cos^2 x\right)^3\, d(\sin x)$$

$$= \int_0^{\pi/2} \left(1 - \sin^2 x\right)^3\, d(\sin x)$$

$$= \int_0^{\pi/2} \left(1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x\right)\, d(\sin x)$$

$$= \left(\sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x\right)_0^{\pi/2}$$

$$= \frac{3}{5} - \frac{1}{7}$$

$$= \boxed{\frac{16}{37}}$$

Exercise

Evaluate $\int_0^{\pi/2} \cos^9 x\, dx$

Solution

$$\int_0^{\pi/2} \cos^9 x\, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\left(\frac{8}{9}\right)$$

$$\int_0^{\pi/2} \cos^n \theta\, d\theta = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right)$$

$$= \boxed{\frac{128}{315}}$$

Or

$$\int_0^{\pi/2} \cos^9\theta\, d\theta = \int_0^{\pi/2} \left(1-\sin^2 x\right)^4\, d(\sin x)$$

$$= \int_0^{\pi/2} \left(1-4\sin^2 x+6\sin^4 x-4\sin^6 x+\sin^8 x\right)\, d(\sin x)$$

$$= \left(\sin x-\frac{4}{3}\sin^3 x+\frac{6}{5}\sin^5 x-\frac{4}{7}\sin^7 x+\frac{1}{9}\sin^9 x\right)_0^{\pi/2}$$

$$= 1-\frac{4}{3}+\frac{6}{5}-\frac{4}{7}+\frac{1}{9}$$

$$= \frac{128}{315}$$

Exercise

Evaluate $\displaystyle\int_0^{\pi/2} \sin^5 x\, dx$

Solution

$$\int_0^{\pi/2} \sin^5 x\, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)$$

$$\int_0^{\pi/2} \cos^n\theta\, d\theta = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right)$$

$$= \frac{8}{15}$$

Or

$$\int_0^{\pi/2} \sin^5 x\, dx = \int_0^{\pi/2} \left(1-\cos^2 x\right)^2\, d(\cos x)$$

$$= \int_0^{\pi/2} \left(1-2\cos^2 x+\cos^4 x\right)\, d(\cos x)$$

$$= \left(\cos x-\frac{2}{3}\cos^3 x+\frac{1}{5}\cos^5 x\right)_0^{\pi/2}$$

$$= -1+\frac{2}{3}-\frac{1}{5}$$

$$= -\frac{8}{15}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/2} \sin^6 x\, dx$

Solution

$$\int_0^{\pi/2} \sin^6 x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{\pi}{2}\right)$$

$$\int_0^{\pi/2} \sin^n x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

$$= \frac{5\pi}{32}$$

Exercise

Evaluate the integral $\int_0^{\pi/2} \sin^8 x\, dx$

Solution

$$\int_0^{\pi/2} \sin^8 x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\left(\frac{\pi}{2}\right)$$

$$\int_0^{\pi/2} \sin^n x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

$$= \frac{35\pi}{256}$$

Exercise

Evaluate the integral $\int_0^{\pi/2} \tan^2 \frac{x}{2}\, dx$

Solution

$$\int_0^{\pi/2} \tan^2 \frac{x}{2}\, dx = \int_0^{\pi/2} \left(\sec^2 \frac{x}{2} - 1\right) dx$$

$$= 2\tan\frac{x}{2} - x \Big|_0^{\frac{\pi}{2}}$$

$$= 2\tan\frac{\pi}{4} - \frac{\pi}{2}$$

$$= 2 - \frac{\pi}{2}$$

Exercise

Find the area of the region bounded by the graphs of $y = \tan x$ and $y = \sec x$ on the interval $\left[0, \frac{\pi}{4}\right]$

Solution

$$A = \int_0^{\pi/4} (\sec x - \tan x)\, dx$$

$$= \ln\left|\sec x + \tan x\right| + \ln\left|\cos x\right| \Bigg|_0^{\pi/4}$$

$$= \ln\left(\sqrt{2}+1\right) + \ln\frac{\sqrt{2}}{2} - 0$$

$$= \ln\left(\frac{\sqrt{2}}{2}\left(\sqrt{2}+1\right)\right)$$

$$= \ln\left(1+\frac{\sqrt{2}}{2}\right)$$

Exercise

Find the area of the region bounded by the graphs of the equations $y = \sin x, \quad y = \sin^3 x, \quad x = 0, \quad x = \frac{\pi}{2}$

Solution

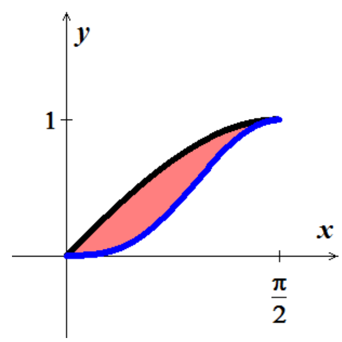
$$A = \int_0^{\pi/2} \left(\sin x - \sin^3 x\right) dx$$

$$= \int_0^{\pi/2} \sin x \, dx - \int_0^{\pi/2} \sin^3 x \, dx$$

$$= -\cos x \Big|_0^{\pi/2} - \frac{2}{3}$$

$$= 1 - \frac{2}{3}$$

$$= \frac{1}{3}$$



Exercise

Find the area of the region bounded by the graphs of the equations $y = \sin^2 \pi x, \quad y = 0, \quad x = 0, \quad x = 1$

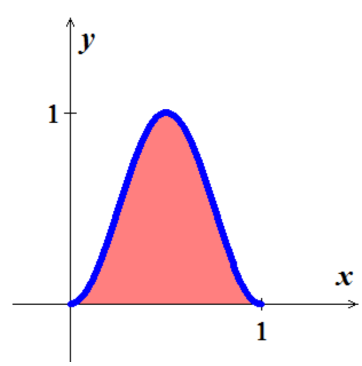
Solution

$$A = \int_0^1 \sin^2 \pi x \, dx$$

$$= \frac{1}{2}\int_0^1 \left(1 + \cos 2\pi x\right) dx$$

$$= \frac{1}{2}\left(x + \frac{1}{2\pi}\sin 2\pi x\right)\Bigg|_0^1$$

$$= \frac{1}{2}$$



Exercise

Find the area of the region bounded by the graphs of the equations

$$y = \cos^2 x, \quad y = \sin^2 x, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$$

Solution

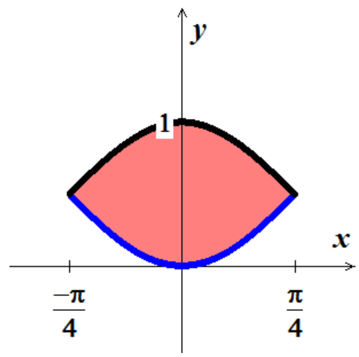
$$A = \int_{-\pi/4}^{\pi/4} \left(\cos^2 x - \sin^2 x\right) dx$$

$$= \int_{-\pi/4}^{\pi/4} \cos 2x \, dx$$

$$= \frac{1}{2}\sin 2x \Big|_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{2}(1+1)$$

$$= 1$$



Exercise

Find the area of the region bounded by the graphs of the equations

$$y = \cos^2 x, \quad y = \sin x \cos x, \quad x = -\frac{\pi}{2}, \quad x = \frac{\pi}{4}$$

Solution

$$A = \int_{-\pi/2}^{\pi/4} \left(\cos^2 x - \sin x \cos x\right) dx$$

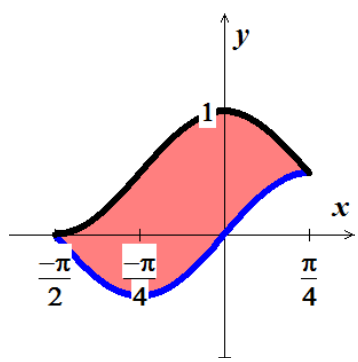
$$= \int_{-\pi/2}^{\pi/4} \left(\frac{1}{2} + \frac{1}{2}\cos 2x - \frac{1}{2}\sin 2x\right) dx$$

$$= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x + \frac{1}{2}\cos 2x\right)\Big|_{-\pi/2}^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2} + \frac{\pi}{2} + \frac{1}{2}\right)$$

$$= \frac{1}{2}\left(\frac{3\pi}{4} + 1\right)$$

$$= \frac{3\pi}{8} + \frac{1}{2}$$



Exercise

Find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the *x-axis* $y = \tan x, \quad y = 0, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$

Solution

Disks Method:

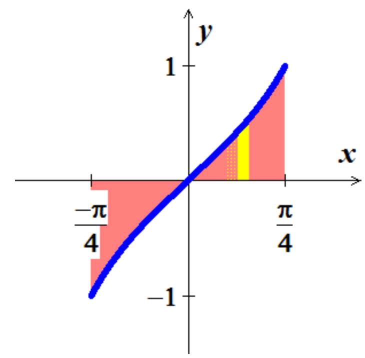
$$V = 2\pi \int_0^{\pi/4} \tan^2 x \, dx$$

$$= 2\pi \int_0^{\pi/4} \left(\sec^2 x - 1\right) dx$$

$$= 2\pi \left(\tan x - x\right) \Big|_0^{\pi/4}$$

$$= 2\pi \left(1 - \frac{\pi}{4}\right)$$

$$= 2\pi - \frac{1}{2}$$



Exercise

Find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the *x-axis* $y = \cos \frac{x}{2}, \quad y = \sin \frac{x}{2}, \quad x = 0, \quad x = \frac{\pi}{2}$

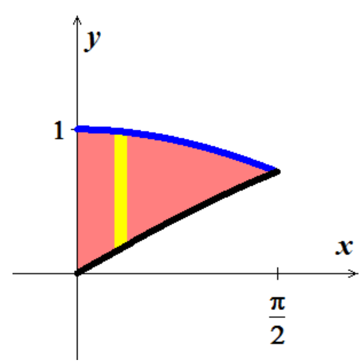
Solution

$$V = \pi \int_0^{\pi/2} \left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}\right) dx \qquad \cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$= \pi \int_0^{\pi/2} \cos x \, dx$$

$$= \pi \sin x \Big|_0^{\pi/2}$$

$$= \pi$$

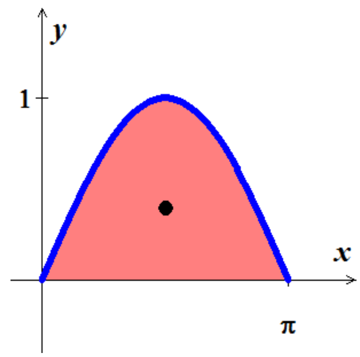


Exercise

Find the ***volume*** of the solid generated by revolving the region bounded by the graphs of the equations about the *x-axis*, then find the ***centroid*** of the region

$$y = \sin x, \quad y = 0, \quad x = 0, \quad x = \pi$$

Solution



$$V = \pi \int_0^{\pi} \sin^2 x \, dx$$

$$= \frac{\pi}{2} \int_0^{\pi} (1 - \cos 2x) \, dx$$

$$= \frac{\pi}{2} \left(x - \frac{1}{2} \sin 2x \right) \Big|_0^{\pi}$$

$$= \frac{\pi^2}{2}$$

$$A = \int_0^{\pi} \sin x \, dx$$

$$= -\cos x \Big|_0^{\pi}$$

$$= -(-1 - 1)$$

$$= 2$$

$$\bar{x} = \frac{1}{A} \int_0^{\pi} x \sin x \, dx$$

		$\int \sin x$
+	x	$-\cos x$
−	1	$-\sin x$

$$= \frac{1}{2} (-x \cos x + \sin x) \Big|_0^{\pi}$$

$$= \frac{\pi}{2}$$

$$\bar{y} = \frac{1}{2A} \int_0^{\pi} \sin^2 x \, dx$$

$$= \frac{1}{8} \int_0^{\pi} (1 - \cos 2x) \, dx$$

$$= \frac{1}{8} \left(x - \frac{1}{2} \sin 2x \right) \Big|_0^{\pi}$$

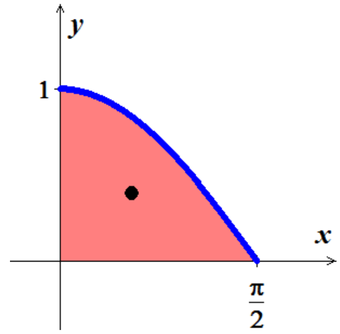
$$= \frac{\pi}{8}$$

$$(\bar{x}, \bar{y}) = \left(\frac{1}{2}, \frac{\pi}{8} \right)$$

Exercise

Find the ***volume*** of the solid generated by revolving the region bounded by the graphs of the equations about the *x-axis*, then find the ***centroid*** of the region

$$y = \cos x, \quad y = 0, \quad x = 0, \quad x = \frac{\pi}{2}$$



Solution

$$V = \pi \int_0^{\pi/2} \cos^2 x \, dx$$

$$= \frac{\pi}{2} \int_0^{\pi/2} (1 + \cos 2x) \, dx$$

$$= \frac{\pi}{2} \left(x + \frac{1}{2} \sin 2x \right) \Big|_0^{\pi/2}$$

$$= \frac{\pi^2}{4}$$

$$A = \int_0^{\pi/2} \cos x \, dx$$

$$= \sin x \Big|_0^{\pi/2}$$

$$= 1$$

$$\bar{x} = \frac{1}{A} \int_0^{\pi/2} x \cos x \, dx$$

		$\int \cos x$
+	x	$\sin x$
−	1	$-\cos x$

$$= (x \sin x + \cos x) \Big|_0^{\pi/2}$$

$$= \frac{\pi}{2} - 1$$

$$\bar{y} = \frac{1}{2A} \int_0^{\pi/2} \cos^2 x \, dx$$

$$= \frac{1}{4} \int_0^{\pi/2} (1 + \cos 2x) \, dx$$

$$= \frac{1}{4} \left(x + \frac{1}{2} \sin 2x \right) \Big|_0^{\pi/2}$$

$$= \frac{\pi}{8}$$

$$(\bar{x}, \bar{y}) = \left(\frac{\pi - 2}{2}, \ \frac{\pi}{8} \right)$$