Solution Section 2.6 – Linear Independence

Exercise

State the following statements as true or false

a) If S is a linearly dependent set, then each vector in S is a linear combination of other vectors in S.

b) Any set containing the zero vector is linearly dependent.

c) The empty set is linearly dependent.

d) Subsets of linearly dependent sets are linearly dependent.

e) Subsets of linearly independent sets are linearly independent.

f) If $a_1\vec{x}_1 + a_2\vec{x}_2 + \ \dots + a_n\vec{x}_n = \vec{0}$ and $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ are linearly independent, the null the scalars a_i are zero

Solution

a) False

b) True

c) False

d) False

e) True

f) True

Exercise

Given three independent vectors $\vec{w}_1, \vec{w}_2, \vec{w}_3$. Take combinations of those vectors to produce $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Write the combinations in a matrix form as $V = WM$.

$$\begin{aligned} \vec{v}_1 &= \vec{w}_1 + \ \vec{w}_2 \\ \vec{v}_2 &= \vec{w}_1 + 2\vec{w}_2 + \vec{w}_3 \\ \vec{v}_1 &= \qquad\ \ \vec{w}_2 + c\vec{w}_3 \end{aligned}$$

which is $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & c \end{bmatrix}$

What is the test on a matrix **V** to see if its columns are linearly independent?

If $c \ne 1$ show that $v_1, \vec{v}_2, \vec{v}_3$ are linearly independent.

If $c = 1$ show that $v's$ are linearly *dependent*.

Solution

The nullspace of **V** must contain only the *zero* vector. Then $\vec{x}=(0,\ 0,\ 0)$ is the only combination of the columns that gives $\mathbf{V}\vec{x}$ = zero vector.

$$M=\begin{bmatrix}1&1&0\\1&2&1\\0&1&c\end{bmatrix}\quad R_2-R_1$$

$$\begin{bmatrix}1&1&0\\0&1&1\\0&1&c\end{bmatrix}\quad \begin{matrix}R_1-R_2\\ \\ R_3-R_2\end{matrix}$$

$$\begin{bmatrix}1&0&-1\\0&1&1\\0&0&\boxed{c-1}\end{bmatrix}$$

If $c\neq 1$, then the matrix M is invertible. So if x is any nonzero vector we know that Mx is nonzero. Since ***w***'s are given as independent and $WM\vec{x}$ is nonzero. Since $V=WM$, this says that x is not in the nullspace of **V**. therefore; $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$ are independent.

If $c=1$, that implies

$$\begin{cases}\vec{v}_1=\vec{w}_1+\ \vec{w}_2\\ \vec{v}_2=\vec{w}_1+\vec{w}_2+\vec{w}_2+\vec{w}_3\\ \vec{v}_3=\qquad\ \vec{w}_2+\vec{w}_3\end{cases}\quad\Rightarrow\quad \begin{matrix}\vec{v}_1=\vec{w}_1+\ \vec{w}_2\\ \boxed{\vec{v}_2=\vec{v}_1+\vec{v}_3}\\ \vec{v}_3=\qquad\ \vec{w}_2+\vec{w}_3\end{matrix}$$

$-v_1+v_2-v_3=0$, which means that $v's$ are linearly *dependent*.

The other way, the vector $x=(1,\ -1,\ 1)$ is in that nullspace, and $M\vec{x}=0$. Then certainly $WM\vec{x}=0$ which is the same as $V\vec{x}=0$. So, the $v's$ are dependent.

Exercise

Find the largest possible number of independent vectors among

$$\vec{v}_1=\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}\quad \vec{v}_2=\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}\quad \vec{v}_3=\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\quad \vec{v}_4=\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\quad \vec{v}_5=\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}\quad \vec{v}_6=\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$$

Solution

Since $\vec{v}_4=\vec{v}_2-\vec{v}_1,\ \ \vec{v}_5=\vec{v}_3-\vec{v}_1,\ \ and\ \ \vec{v}_6=\vec{v}_3-\vec{v}_2$, there are at most three independent vectors among these: furthermore, applying row reduction to the matrix $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$ gives three pivots, showing that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent.

Exercise

Show that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent but $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are dependent:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \vec{v}_4 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

Solve either $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ *or* $A\vec{x} = \vec{0}$. The v's go in the columns of ***A***.

Solution

$$\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

This matrix has 3 pivots with rank of 3 equals to rows that implies the $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent.

$\vec{v}_4 = \vec{v}_1 + \vec{v}_2 - 4\vec{v}_3$ *or* $\vec{v}_1 + \vec{v}_2 - 4\vec{v}_3 - \vec{v}_4 = \vec{0}$

That shows that $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are dependent.

Exercise

Decide the dependence or independence of

a) The vectors (1, 3, 2), (2, 1, 3), and (3, 2, 1).

b) The vectors $(1, -3, 2)$, $(2, 1, -3)$, and $(-3, 2, 1)$.

Solution

a) These are *linearly independent*.

$x_1(1, 3, 2) + x_2(2, 1, 3) + x_3(3, 2, 1) = (0, 0, 0)$ only if $x_1 = x_2 = x_3 = 0$

b) These are *linearly dependent*:

$1(1, -3, 2) + 1(2, 1, -3) + 1(-3, 2, 1) = (0, 0, 0)$

Exercise

Find two independent vectors on the plane $x+2y-3z-t=0$ in $\mathbb{R}^4$. Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

Solution

This plane is the nullspace of the matrix $A=\begin{pmatrix}1 & 2 & -3 & -1\end{pmatrix}$

$x_1+2x_2-3x_3-x_4=0$

The pivot is 1st column, and the rest are 3 variables.

If $x_2=-1 \quad x_3=x_4=0 \Rightarrow x_1=2$. The vector is $(2,\ -1,\ 0,\ 0)$

If $x_3=1 \quad x_1=x_4=0 \Rightarrow x_1=3$. The vector is $(3,\ 0,\ 1,\ 0)$

If $x_4=1 \quad x_1=x_3=0 \Rightarrow x_1=1$. The vector is $(1,\ 0,\ 0,\ 1)$

The 3 vectors $(2,\ -1,\ 0,\ 0)$, $(3,\ 0,\ 1,\ 0)$, $(1,\ 0,\ 0,\ 1)$ are linearly independent.

We can't find 4 independent vectors because the nullspace only has dimension 3 (have 3 variables).

Exercise

Determine whether the vectors are linearly dependent or linearly independent in $\mathbb{R}^3$

a) (4, –1, 2), (–4, 10, 2)

b) (8, –1, 3), (4, 0, 1)

c) (–3, 0, 4), (5, –1, 2), (1, 1, 3)

d) (–2, 0, 1), (3, 2, 5), (6, –1, 1), (7, 0, –2)

Solution

a) The vector equation *a* (4, –1, 2) + *b* (–4, 10, 2) = (0, 0 ,0)

$$\left[\begin{array}{cc|c} 4 & -4 & 0 \\ -1 & 10 & 0 \\ 2 & 2 & 0 \end{array}\right] \quad \tfrac{1}{4}R_1$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 10 & 0 \\ 2 & 2 & 0 \end{array}\right] \quad \begin{array}{l} \\ R_2+R_1 \\ R_3-2R_1 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad 9R_1+R_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \\ \frac{1}{9}R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore, the system has only the trivial solution $a = b = 0$.
We conclude that the given set of vectors is linearly independent.

b) $A\,(8, -1, 3) + b\,(4, 0, 1) = (0, 0\ ,0)$

$$\left[\begin{array}{cc|c} 8 & 4 & 0 \\ -1 & 0 & 0 \\ 3 & 1 & 0 \end{array}\right] \quad \begin{array}{l} \frac{1}{8}R_1 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ -1 & 0 & 0 \\ 3 & 1 & 0 \end{array}\right] \quad \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 \end{array}\right] \quad \begin{array}{l} R_1 - R_2 \\ \\ R_3 + R_2 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \\ 2R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore, the system has only one trivial solution $a = b = 0$.
We conclude that the given set of vectors is linearly independent

c) The vector equation:

$a\,(-3, 0, 4) + b\,(5, -1, 2) + \ c(1, 1, 3) = (0, 0\ ,0)$

$$\left[\begin{array}{ccc|c} -3 & 5 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 4 & 2 & 3 & 0 \end{array}\right] \quad \begin{array}{l} \\ \\ 3R_3 + 4R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} -3 & 5 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 26 & 13 & 0 \end{array}\right] \quad -R_2$$

$$\left[\begin{array}{ccc|c} -3 & 5 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 26 & 13 & 0 \end{array}\right] \quad \begin{array}{l} R_1 - 5R_2 \\ \\ R_3 - 26R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} -3 & 0 & 6 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 39 & 0 \end{array}\right] \quad \tfrac{1}{39}R_3$$

$$\left[\begin{array}{ccc|c} -3 & 0 & 6 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R_1 - 6R_3 \\ R_2 + R_3 \\ \end{array}$$

$$\left[\begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad -\tfrac{1}{3}R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore, the system has only the trivial solution $a = b = c = 0$.
We conclude that the given set of vectors is linearly independent.

d) The vector equation:

$a\,(-2, 0, 1) + b\,(3, 2, 5) + c\,(6, -1, 1) + d\,(7, 0, -2) = (0, 0, 0)$

$$\left[\begin{array}{cccc|c} -2 & 3 & 6 & 7 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 1 & 5 & 1 & -2 & 0 \end{array}\right] \quad \begin{array}{l} \\ \\ 2R_3 + R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} -2 & 3 & 6 & 7 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 13 & 8 & 3 & 0 \end{array}\right] \quad \begin{array}{l} 2R_1 - 3R_2 \\ \\ 2R_3 - 13R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} -4 & 0 & 15 & 14 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 29 & 6 & 0 \end{array}\right] \quad \begin{array}{l} 29R_1 - 15R_3 \\ 29R_2 + R_3 \\ \end{array}$$

$$\left[\begin{array}{cccc|c} -116 & 0 & 0 & 316 & 0 \\ 0 & 58 & 0 & 6 & 0 \\ 0 & 0 & 29 & 6 & 0 \end{array}\right] \quad \begin{array}{l} -\frac{1}{4}R_1 \\ \frac{1}{58}R_2 \\ \frac{1}{29}R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{79}{29} & 0 \\ 0 & 1 & 0 & \frac{3}{29} & 0 \\ 0 & 0 & 1 & \frac{6}{29} & 0 \end{array}\right]$$

Therefore, the system has nontrivial solutions $a = \frac{79}{29}t, \quad b = -\frac{3}{29}t, \quad c = -\frac{6}{29}t, \quad d = t$

We conclude that the given set of vectors is linearly dependent.

Exercise

Determine whether the vectors are linearly dependent or linearly independent in $\mathbb{R}^4$

a) $\{(3,\ 8,\ 7,\ -3),\ (1,\ 5,\ 3,\ -1),\ (2,\ -1,\ 2,\ 6),\ (1,\ 4,\ 0,\ 3)\}$

b) $\{(0,\ 0,\ 2,\ 2),\ (3,\ 3,\ 0,\ 0),\ (1,\ 1,\ 0,-1)\}$

c) $\{(0,\ 3,\ -3,\ -6),\ (-2,\ 0,\ 0,\ -6),\ (0,\ -4,\ -2,\ -2),\ (0,\ -8,\ 4,\ -4)\}$

d) $\{(3,\ 0,\ -3,\ 6),\ (0,\ 2,\ 3,\ 1),\ (0,\ -2,\ -2,\ 0),\ (-2,\ 1,\ 2,\ 1)\}$

e) $\{(1,\ 3,\ -4,\ 2),\ (2,\ 2,\ -4,\ 0),\ (2,\ 3,\ 2,\ -4),\ (-1,\ 0,\ 1,\ 0)\}$

f) $\{(1,\ 3,\ -4,\ 2),\ (2,\ 2,\ -4,\ 0),\ (1,\ -3,\ 2,\ -4),\ (-1,\ 0,\ 1,\ 0)\}$

g) $\{(1,\ 0,\ 0,\ -1),\ (0,\ 1,\ 0,\ -1),\ (0,\ 1,\ 0,\ -1),\ (0,\ 0,\ 0,\ 1)\}$

Solution

a) $\det\begin{pmatrix} 3 & 1 & 2 & 1 \\ 8 & 5 & -1 & 4 \\ 7 & 3 & 2 & 0 \\ -3 & -1 & 6 & 3 \end{pmatrix} \underline{= 128 \neq 0}$

The system has only the trivial solution and the vectors are *linearly independent*.

b) $k_1(0,0,2,2) + k_2(3,3,0,0) + k_3(1,1,0,-1) = (0,0,0,0)$

$$\left[\begin{array}{ccc|c} 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 \end{array}\right]$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{bmatrix} \quad \begin{matrix} \\ R_2 - R_1 \\ \\ R_4 - R_3 \end{matrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \\ R_2 + R_3 \\ \\ \\ \end{matrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \\ \\ R_3 - R_2 \\ \\ \end{matrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \frac{1}{2}R_3 \\ \frac{1}{3}R_2 \\ \\ \\ \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 = k_2 = k_3 = 0$

The system has only the trivial solution and the vectors are *linearly independent*.

c) $\det\begin{pmatrix} 0 & -2 & 0 & 0 \\ 3 & 0 & -4 & -8 \\ -3 & 0 & -2 & 4 \\ -6 & -6 & -2 & -4 \end{pmatrix} = 480 \neq 0$

The system has only the trivial solution and the vectors are *linearly independent*.

d) $a(3,\ 0,\ -3,\ 6) + b(0,\ 2,\ 3,\ 1) + c(0,\ -2,\ -2,\ 0) + d(-2,\ 1,\ 2,\ 1) = (0,\ 0,\ 0,\ 0)$

$$\left[\begin{array}{cccc|c} 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & 1 & 0 \\ -3 & 3 & -2 & 2 & 0 \\ 6 & 1 & 0 & 1 & 0 \end{array}\right] \quad \begin{matrix} \\ \\ R_3 + R_1 \\ R_4 - 2R_1 \end{matrix}$$

$$\left[\begin{array}{cccc|c} 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & 1 & 0 \\ 0 & 3 & -2 & 0 & 0 \\ 0 & 1 & 0 & 5 & 0 \end{array}\right] \begin{array}{l} \\ \\ 2R_3 - 3R_2 \\ 2R_4 - R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & 1 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 0 & 2 & 12 & 0 \end{array}\right] \begin{array}{l} \\ R_2 + R_3 \\ \\ R_4 - R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 9 & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ \frac{1}{9}R_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 + 2R_4 \\ R_2 + 2R_4 \\ R_3 + 3R_4 \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \frac{1}{3}R_1 \\ \frac{1}{2}R_2 \\ \frac{1}{2}R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore, the system has only one trivial solution $a = b = c = d = 0$.
The given set of vectors is *linearly independent*

e) $\{(1, 3, -4, 2), (2, 2, -4, 0), (2, 3, 2, -4), (-1, 0, 1, 0)\}$

$$\begin{vmatrix} 1 & 2 & 2 & -1 \\ 3 & 2 & 3 & 0 \\ -4 & -4 & 2 & 1 \\ 2 & 0 & -4 & 0 \end{vmatrix} = 28 \neq 0$$

The system has only the trivial solution and the vectors are *linearly independent*.

f) $\{(1,\ 3,\ -4,\ 2),\ (2,\ 2,\ -4,\ 0),\ (1,\ -3,\ 2,\ -4),\ (-1,\ 0,\ 1,\ 0)\}$

$$\begin{vmatrix} 1 & 2 & 1 & -1 \\ 3 & 2 & -3 & 0 \\ -4 & -4 & 2 & 1 \\ 2 & 0 & -4 & 0 \end{vmatrix} \underline{=0}$$

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 3 & 2 & -3 & 0 \\ -4 & -4 & 2 & 1 \\ 2 & 0 & -4 & 0 \end{pmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 + 4R_1 \\ R_4 - 2R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -4 & -6 & 3 \\ 0 & 4 & 6 & -3 \\ 0 & -4 & -6 & 2 \end{pmatrix} \begin{matrix} 2R_1 + R_2 \\ \\ R_3 + R_2 \\ R_4 - R_2 \end{matrix}$$

$$\begin{pmatrix} 2 & 0 & -4 & 1 \\ 0 & -4 & -6 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & -4 & 1 \\ 0 & -4 & -6 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} R_1 - R_3 \\ R_2 - 3R_3 \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} 2 & 0 & -4 & 0 \\ 0 & -4 & -6 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \frac{1}{2}R_1 \\ -\frac{1}{4}R_2 \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underline{x_4 = 0, \quad x_2 = -\frac{3}{2}x_3, \quad x_1 = 2x_3}$$

$\therefore$ The set is *linearly independent*.

g) $\{(1,\ 0,\ 0,\ -1),\ (0,\ 1,\ 0,\ -1),\ (0,\ 1,\ 0,\ -1),\ (0,\ 0,\ 0,\ 1)\}$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 \end{vmatrix} \underline{= 0}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad R_3 + R_1$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad R_3 + R_2$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} x_1 &= 0 \\ x_2 &= -x_3 \\ x_4 &= 0 \end{aligned}$$

$\therefore$ The set is *linearly independent*.

Exercise

a) Show that the three vectors $\vec{v}_1 = (1,\ 2,\ 3,\ 4)$ $\vec{v}_2 = (0,\ 1,\ 0,\ -1)$ $\vec{v}_3 = (1,\ 3,\ 3,\ 3)$ form a linearly dependent set in $\mathbb{R}^4$.

b) Express each vector in part (*a*) as a linear combination of the other two.

Solution

a) The vector equation:

$$k_1(1,\ 2,\ 3,\ 4) + k_2(0,\ 1,\ 0,\ -1) + k_3(1,\ 3,\ 3,\ 3) = (0,\ 0,\ 0,\ 0)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 \\ 3 & 0 & 3 & 0 \\ 4 & -1 & 3 & 0 \end{array}\right] \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 4R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 4R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \quad \begin{array}{l} \\ \\ \\ R_4 + R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The solution: $k_1 = -t, \ k_2 = -t, \ k_3 = t$

Since the system has nontrivial solutions, the given set of vectors is *linearly dependent*.

b) Since $k_1 = -t, \ k_2 = -t, \ k_3 = t$ and if we let $t = 1$, then $-\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = 0$

$$\vec{v}_1 = -\vec{v}_2 + \vec{v}_3, \quad \vec{v}_2 = -\vec{v}_1 + \vec{v}_3, \quad \vec{v}_3 = \vec{v}_1 + \vec{v}_2$$

Exercise

For which real values of λ do the following vectors form a linearly dependent set in $\boldsymbol{R}^3$

$$\vec{v}_1 = \left(\lambda, \ -\tfrac{1}{2}, \ -\tfrac{1}{2}\right) \quad \vec{v}_2 = \left(-\tfrac{1}{2}, \ \lambda, \ -\tfrac{1}{2}\right) \quad \vec{v}_3 = \left(-\tfrac{1}{2}, \ -\tfrac{1}{2}, \ \lambda\right)$$

Solution

$$k_1\left(\lambda, \ -\tfrac{1}{2}, \ -\tfrac{1}{2}\right) + k_2\left(-\tfrac{1}{2}, \ \lambda, \ -\tfrac{1}{2}\right) + k_3\left(-\tfrac{1}{2}, \ -\tfrac{1}{2}, \ \lambda\right) = (0, \ 0, \ 0, \ 0)$$

$$\det\begin{pmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{pmatrix} = \tfrac{1}{4}\left(4\lambda^3 - 3\lambda - 1\right)$$

$4\lambda^3 - 3\lambda - 1 = 0$

For $\lambda = 1 \quad \lambda = -\frac{1}{2}$, the determinant is zero and the vectors form a *linearly dependent* set.

Exercise

Show that if $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a linearly independent set of vectors, then so is every nonempty subset of S.

Solution

Let $\{\vec{v}_a, \vec{v}_b, \ldots, \vec{v}_r\}$ be a nonempty subset of *S*.

If this set is linearly dependent, then there would be a nonzero solution $(k_a, k_b, \ldots, k_r)$ to $k_a\vec{v}_a + k_b\vec{v}_b + \ldots + k_r\ \vec{v}_r = 0$. This can be expanded to a nonzero solution of $k_1\vec{v}_1 + k_2\vec{v}_2 + \ldots + k_n\ \vec{v}_n = 0$ by taking all other coefficients as 0. This contradicts the linear independence of *S*, so the subset must be *linearly independent*.

Exercise

Show that if $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ is a linearly dependent set of vectors in a vector space *V*, and if $\vec{v}_{r+1}, \ldots, \vec{v}_n$ are vectors in *V* that are not in *S*, then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r, \vec{v}_{r+1}, \ldots, \vec{v}_n\}$ is also linearly dependent.

Solution

If *S* is linearly dependent, then there is a nonzero solution $(k_1, k_2, \ldots, k_r)$ to

$k_1\vec{v}_1 + k_2\vec{v}_2 + \ldots + k_r\vec{v}_r = 0$.

Thus $(k_1, k_2, \ldots, k_r, 0, 0, \ldots, 0)$ is a nonzero solution to

$k_1\vec{v}_1 + k_2\vec{v}_2 + \ldots + k_r\ \vec{v}_r + k_{r+1}\ \vec{v}_{r+1} \ldots + k_n\ \vec{v}_n = 0$

So, the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r, \vec{v}_{r+1}, \ldots, \vec{v}_n\}$ is *linearly dependent*.

Exercise

Show that $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent and $\vec{v}_3$ does not lie in span $\{\vec{v}_1, \vec{v}_2\}$, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent.

Solution

If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly dependent, there exist a nonzero solution to $k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = 0$ with $k_3 \neq 0$ (since $\vec{v}_1$ and $\vec{v}_2$ are linearly independent).

$k_3\vec{v}_3 = -k_1\vec{v}_1 - k_2\vec{v}_2 \quad \Rightarrow \quad \vec{v}_3 = -\frac{k_1}{k_3}\vec{v}_1 - \frac{k_2}{k_3}\vec{v}_2$ which contradicts that $\vec{v}_3$ is not in span $\{\vec{v}_1, \vec{v}_2\}$. Thus $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent.

Exercise

By using the appropriate identities, where required, determine $F(-\infty, \infty)$ are linearly dependent.

a) $6,\ 3\sin^2 x,\ 2\cos^2 x$

b) $x,\ \cos x$

c) $1,\ \sin x,\ \sin 2x$

d) $(3-x)^2,\ x^2-6x,\ 5$

e) $\cos 2x,\ \sin^2 x,\ \cos^2 x$

Solution

a) From the identity $\sin^2 x + \cos^2 x = 1$

$$(-1)(6) + (2)\left(3\sin^2 x\right) + (3)\left(2\cos^2 x\right) = -6 + 6\left(\sin^2 x + \cos^2 x\right)$$
$$= 0$$

Therefore, the set is *linearly dependent*.

b) $ax + b\cos x = 0$

$x = 0 \quad \Rightarrow \quad b = 0$

$x = \frac{\pi}{2} \quad \Rightarrow \quad a = 0$

Therefore, the set is *linearly independent*.

c) $a(1) + b\sin x + c\sin 2x = 0$

$x = 0 \quad \Rightarrow \quad a = 0$

$x = \frac{\pi}{2} \quad \Rightarrow \quad b = 0$

$x = \frac{\pi}{4} \quad \Rightarrow \quad c = 0$

Therefore, the set is *linearly independent*.

d) $(3-x)^2 = 9-6x+x^2$

$$(3-x)^2 - \left(9-6x+x^2\right) = 0$$

$$(3-x)^2 - \left(x^2-6x\right) - 9 = 0$$

$$(1)(3-x)^2 + (-1)\left(x^2-6x\right) + \left(-\tfrac{9}{5}\right)5 = 0$$

Therefore, the set is linearly dependent.

e) By using the double angle:

$\cos 2x = \cos^2 x - \sin^2 x$ are *linearly dependent*.

Exercise

$f_1(x) = \sin x,\quad f_2(x) = \cos x$ are linearly independent in $F(-\infty, \infty)$ because neither function is a scalar multiple of the other. Confirm the linear independence using Wroński's test.

Solution

The Wronskian: $W(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$

$$= -\sin^2 x - \cos^2 x$$

$$= -\left(\sin^2 x + \cos^2 x\right)$$

$$\underline{= -1 \neq 0}$$

$\sin x$ *and* $\cos x$ are *linearly independent*

Exercise

Show $f_1(x) = e^x,\quad f_2(x) = xe^x \quad f_3(x) = x^2e^x$ are linearly independent in $F(-\infty, \infty)$

Solution

$$W = \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & e^x + xe^x & 2xe^x + x^2e^x \\ e^x & 2e^x + xe^x & 2e^x + 4xe^x + x^2e^x \end{vmatrix} \qquad \begin{matrix} e^x \\ \textit{factor}\ \ e^x \\ e^x \end{matrix}$$

$$= e^{3x}\begin{vmatrix} 1 & x & x^2 \\ 1 & 1+x & 2x+x^2 \\ 1 & 2+x & 2+4x+x^2 \end{vmatrix}$$

$$= e^{3x}\left[(1+x)\left(2+4x+x^2\right)+2x^2+x^3+2x^2+x^3-x^2-x^3-\left(2x+x^2\right)(2+x)-2x-4x^2-x^3\right]$$

$$= e^{3x}\left[2+4x+x^2+2x+4x^2+x^3-4x-2x^2-2x^2-x^3-2x-x^2\right]$$

$$\underline{= 2e^{3x} \neq 0}$$

$\left\{e^x,\ xe^x,\ x^2e^x\right\}$ are *linearly independent*

Exercise

Use the Wronskian to show that $f_1(x)=\sin x,\quad f_2(x)=\cos x,\quad f_3(x)=x\cos x$ span a three-dimensional subspace of $F(-\infty,\ \infty)$

Solution

The Wronskian: $W(x)=\begin{vmatrix} \sin x & \cos x & x\cos x \\ \cos x & -\sin x & \cos x - x\sin x \\ -\sin x & -\cos x & -2\sin x - x\cos x \end{vmatrix}$

$$\begin{aligned} &= 2\sin^3 x + x\sin^2 x\cos x - \sin x\cos^2 x + x\sin^2 x\cos x - x\cos^3 x \\ &\quad - x\sin^2 x\cos x + \sin x\cos^2 x - x\sin^2 x\cos x + 2\sin x\cos^2 x + x\cos^3 x \\ &= 2\sin^3 x + 2\sin x\cos^2 x \\ &= 2\sin x\left(\sin^2 x + \cos^2 x\right) \\ &= \underline{2\sin x} \end{aligned}$$

Since $\sin x \neq 0$ for all real x values, the vectors are *linearly independent.*

Exercise

Show by inspection that the vectors are linearly dependent.

$\vec{v}_1(4,\ -1,\ 3),\quad \vec{v}_2(2,\ 3,\ -1),\quad \vec{v}_3(-1,\ 2,\ -1),\quad \vec{v}_4(5,\ 2,\ 3),\quad in\ \mathbb{R}^3$

Solution

$$\begin{bmatrix} 4 & 2 & -1 & 5 \\ -1 & 3 & 2 & 2 \\ 3 & -1 & -1 & 3 \end{bmatrix} \qquad \begin{matrix} \\ 4R_2 + R_1 \\ 4R_3 - 3R_1 \end{matrix}$$

$$\begin{bmatrix} 4 & 2 & -1 & 5 \\ 0 & 14 & 7 & 13 \\ 0 & -10 & -1 & -3 \end{bmatrix} \quad \begin{matrix} 7R_1 - R_2 \\ \\ 14R_3 + 10R_2 \end{matrix}$$

$$\begin{bmatrix} 28 & 0 & -14 & 22 \\ 0 & 14 & 7 & 13 \\ 0 & 0 & 56 & 88 \end{bmatrix} \quad \begin{matrix} 4R_1 + R_3 \\ 8R_2 - R_3 \\ \\ \end{matrix}$$

$$\begin{bmatrix} 112 & 0 & 0 & 176 \\ 0 & 112 & 0 & 16 \\ 0 & 0 & 56 & 88 \end{bmatrix} \quad \begin{matrix} \frac{1}{112}R_1 \\ \frac{1}{112}R_2 \\ \frac{1}{56}R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{11}{7} \\ 0 & 1 & 0 & \frac{1}{7} \\ 0 & 0 & 1 & \frac{11}{7} \end{bmatrix} \quad \begin{matrix} \vec{v}_1 = -\frac{11}{7}\vec{v}_4 \\ \vec{v}_2 = -\frac{1}{7}\vec{v}_4 \\ \vec{v}_3 = -\frac{11}{7}\vec{v}_4 \end{matrix}$$

$$-\frac{11}{7}\vec{v}_1 - \frac{1}{7}\vec{v}_2 - \frac{11}{7}\vec{v}_2 + \vec{v}_4 = 0$$

$$7\vec{v}_4 = 11\vec{v}_1 + \vec{v}_2 + 11\vec{v}_2$$

Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$(2,\ -1,\ 3),\quad (3,\ 4,\ 1),\quad (2,\ -3,\ 4),\quad in\ \mathbb{R}^3$$

Solution

$$a(2,\ -1,\ 3) + b(3,\ 4,\ 1) + c(2,\ -3,\ 4) = (0,\ 0,\ 0)$$

$$\begin{bmatrix} 2 & 3 & 2 & 0 \\ -1 & 4 & -3 & 0 \\ 3 & 1 & 4 & 0 \end{bmatrix} \quad \begin{matrix} \\ 2R_2 + R_1 \\ 2R_3 - 3R_1 \end{matrix}$$

$$\begin{bmatrix} 2 & 3 & 2 & 0 \\ 0 & 11 & -4 & 0 \\ 0 & -7 & 2 & 0 \end{bmatrix} \quad \begin{matrix} 11R_1 - 3R_2 \\ \\ 11R_3 + 7R_2 \end{matrix}$$

$$\begin{bmatrix} 22 & 0 & 34 & 0 \\ 0 & 11 & -4 & 0 \\ 0 & 0 & -6 & 0 \end{bmatrix} \quad -\frac{1}{6}R_3$$

$$\begin{bmatrix} 22 & 0 & 34 & 0 \\ 0 & 11 & -4 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{matrix} R_1 - 34R_3 \\ R_2 + 4R_3 \end{matrix}$$

$$\begin{bmatrix} 22 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{matrix} \frac{1}{22}R_1 \\ \frac{1}{11}R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The system has only he trivial solution $a = b = c = 0$.

$$\begin{vmatrix} 2 & 3 & 2 \\ -1 & 4 & -3 \\ 3 & 1 & 4 \end{vmatrix} = 32 - 27 - 2 - 24 + 6 + 12 \ \neq 0$$

The system has only the trivial solution and the vectors are *linearly independent*

Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$(1,\ 0,\ 0,\ 0),\quad (1,\ 1,\ 0,\ 0),\quad (1,\ 1,\ 1,\ 0),\quad (1,\ 1,\ 1,\ 1),\quad \text{in } \mathbb{R}^4$$

Solution

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

The system has only the trivial solution and the vectors are *linearly independent*

Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$A_1\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix},\quad A_2\begin{bmatrix}-2 & 4\\ 1 & 0\end{bmatrix},\quad A_3\begin{bmatrix}3 & -1\\ 2 & 0\end{bmatrix},\quad in\ M_{22}$$

Solution

$$\begin{bmatrix}1 & -2 & 3 & 0\\ 2 & 4 & -1 & 0\\ 3 & 1 & 2 & 0\end{bmatrix}\quad \begin{matrix} \\ R_2-2R_1\\ R_3-3R_1\end{matrix}$$

$$\begin{bmatrix}1 & -2 & 3 & 0\\ 0 & 8 & -7 & 0\\ 0 & 7 & -7 & 0\end{bmatrix}\quad \begin{matrix}4R_1+R_2\\ \\ 8R_3-7R_2\end{matrix}$$

$$\begin{bmatrix}4 & 0 & 5 & 0\\ 0 & 8 & -7 & 0\\ 0 & 0 & -7 & 0\end{bmatrix}\quad \begin{matrix} \\ \\ -\frac{1}{7}R_3\end{matrix}$$

$$\begin{bmatrix}4 & 0 & 5 & 0\\ 0 & 8 & -7 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\quad \begin{matrix}R_1-5R_3\\ R_2+7R_3\\ \\ \end{matrix}$$

$$\begin{bmatrix}4 & 0 & 0 & 0\\ 0 & 8 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\quad \begin{matrix}\frac{1}{4}R_1\\ \frac{1}{8}R_2\\ \\ \end{matrix}$$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$$

The vectors are *linearly independent*

Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{\begin{pmatrix}1 & -3 & 2\\ -4 & 0 & 5\end{pmatrix}, \begin{pmatrix}-3 & 7 & 4\\ 6 & -2 & -7\end{pmatrix}, \begin{pmatrix}-2 & 3 & 11\\ -1 & -3 & 2\end{pmatrix}\right\} \text{ in } M_{2\times 3}(\mathbb{R})$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ -3 & 7 & 3 & 0 \\ 2 & 4 & 11 & 0 \\ -4 & 6 & -1 & 0 \\ 0 & -2 & -3 & 0 \\ 5 & -7 & 2 & 0 \end{array}\right] \quad \begin{array}{l} \\ R_2 + 3R_1 \\ R_3 - 2R_1 \\ R_4 + 4R_1 \\ \\ R_6 - 5R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & 10 & 15 & 0 \\ 0 & -6 & -9 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & 8 & 12 & 0 \end{array}\right] \quad \begin{array}{l} \\ \\ R_3 + 5R_2 \\ R_4 - 3R_2 \\ R_5 - R_2 \\ R_6 + 4R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \to\ a_1 = 3a_2 + 2a_3 \\ \to\ 2a_2 = -3a_3 \\ \\ \\ \\ \\ \end{array}$$

$$\underline{a_2 = -\frac{3}{2}a_3}$$

$$\begin{aligned} a_1 &= -\frac{9}{2}a_3 + 2a_3 \\ &= \underline{-\frac{5}{2}a_3} \end{aligned}$$

It is *linearly dependent*.

$if\ a_3 = -2 \quad a_2 = 3 \quad a_1 = 5$

$$5\begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix} + 3\begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix} - 2\begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix}\right\} \text{ in } M_{2\times 2}(\mathbb{R})$$

Solution

$$\begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} = -2\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}$$

$$2\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} + \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\therefore$ *Linearly dependent*

Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{\begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix}\right\} \text{ in } M_{2\times 2}(\mathbb{R})$$

Solution

$$\begin{bmatrix} 1 & -1 \\ -2 & 1 \\ -1 & 2 \\ 4 & -4 \end{bmatrix} \quad \begin{array}{l} \\ R_2 + 2R_1 \\ R_3 + R_1 \\ R_4 - 4R_1 \end{array}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$a_2 = 0 \quad \to \ a_1 = a_2 = 0$

$\therefore$ *Linearly independent*

Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -4 & 4 \end{pmatrix}\right\} \quad in \quad M_{2\times 2}(\mathbb{R})$$

Solution

$$\begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & 1 \\ -2 & 1 & 1 & -4 \\ 1 & 1 & 0 & 4 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 1 \\ 1 & 1 & -4 \\ 1 & 0 & 4 \end{vmatrix} - \begin{vmatrix} 0 & -1 & 1 \\ -2 & 1 & -4 \\ 1 & 1 & 4 \end{vmatrix} - 2\begin{vmatrix} 0 & -1 & 2 \\ -2 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= -21 + 7 + 14$$

$$= 0$$

$\therefore$ *Linearly dependent*

Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & -2 \end{pmatrix}\right\} \quad in \quad M_{2\times 2}(\mathbb{R})$$

Solution

$$W = \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & 1 \\ -2 & 1 & 1 & 2 \\ 1 & 1 & 0 & -2 \end{vmatrix} = 24 \neq 0$$

$\therefore$ *Linearly independent*

Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{e^x, \ \ln x\right\} \quad in \quad \mathbb{R}$$

Solution

$$W = \begin{vmatrix} e^x & \ln x \\ e^x & \frac{1}{x} \end{vmatrix}$$

$$= e^x\left(\frac{1}{x} - \ln x\right) \neq 0$$

$\therefore$ *Linearly independent*

Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$\left\{x,\ \frac{1}{x}\right\}$ *in* $\mathbb{R}$

Solution

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

$$= -\frac{1}{x} - \frac{1}{x}$$

$$= -\frac{2}{x} \neq 0$$

$\therefore$ *Linearly independent*

Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$\{1+x,\ 1-x\}$ *in* $P_2(\mathbb{R})$

Solution

$$W = \begin{vmatrix} 1+x & 1-x \\ 1 & -1 \end{vmatrix}$$

$$= -1 - x - 1 + x$$

$$= -2 \neq 0$$

$\therefore$ *Linearly independent*

Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$\{9x^2 - x + 3,\ 3x^2 - 6x + 5,\ -5x^2 + x + 1\}$ *in* $P_3(\mathbb{R})$

Solution

$$W=\begin{vmatrix} 9 & -1 & 3 \\ 3 & -6 & 5 \\ -5 & 1 & 1 \end{vmatrix} \quad \underline{=-152 \neq 0}$$

$\therefore$ *Linearly independent*

$$W=\begin{vmatrix} 9x^2-x+3 & 3x^2-6x+5 & -5x^2+x+1 \\ 18x-1 & 6x-6 & -10x+1 \\ 18 & 6 & -10 \end{vmatrix}$$

$$=(-60x-60)\left(9x^2-x+3\right)+(-180x+18)\left(3x^2-6x+5\right)+(108x-6)\left(-5x^2+x+1\right)$$

$$-(108x-108)\left(-5x^2+x+1\right)-(-60x+6)\left(9x^2-x+3\right)-(-180x+10)\left(3x^2-6x+5\right)$$

$$\begin{array}{ll} x^3 & -540-540-540+540+540+540 \\ x^2 & -540+60+54+1080+30+108+540-108-54-60-30-1080 \\ x^1 & 60-180-900-6+108+108-108+6+180+60+900 \\ x^0 & -180+90-6+108-18-50 \end{array}$$

$$\underline{=228x-56 \quad \neq 0}$$

Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{-x^2,\ 1+4x^2\right\} \quad in \quad P_3(\mathbb{R})$$

Solution

$$W=\begin{vmatrix} -x^2 & 4x^2+1 \\ -2x & 8x \end{vmatrix}$$

$$\underline{=2x \neq 0}$$

$\therefore$ *Linearly independent*

Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{7x^2+x+2,\ 2x^2-x+3,\ -3x^2+4\right\} \quad in \quad P_3(\mathbb{R})$$

Solution

$$W = \begin{vmatrix} 7 & 1 & 2 \\ 2 & -1 & 3 \\ -3 & 0 & 4 \end{vmatrix}$$

$$\underline{= -51 \neq 0}$$

$\therefore$ *Linearly independent*

Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{3x^2+3x+8,\ \ 2x^2+x,\ \ 2x^2+2x+2,\ \ 5x^2-2x+8\right\} \quad in \quad P_3(\mathbb{R})$$

Solution

$$\begin{bmatrix} 3 & 2 & 2 & 5 \\ 3 & 1 & 2 & -2 \\ 8 & 0 & 2 & 8 \end{bmatrix} \quad \begin{matrix} \\ R_2 - R_1 \\ 3R_3 - 8R_1 \end{matrix}$$

$$\begin{bmatrix} 3 & 2 & 2 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & -16 & -10 & -16 \end{bmatrix} \quad R_3 + 16R_2$$

$$\begin{bmatrix} 3 & 2 & 2 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & -10 & 96 \end{bmatrix} \quad \begin{aligned} 3a_1 &= -2a_2 - 2a_3 - 5a_4 \\ \underline{a_2} &\underline{= -7a_4} \\ \underline{a_3} &\underline{= \tfrac{48}{5}a_4} \end{aligned}$$

$$3a_1 = 14a_4 - \frac{96}{5}a_4 - 5a_4$$

$$\underline{a_1 = \frac{3}{5}a_4}$$

$\therefore$ *Linearly dependent*

Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{x^3+2x^2,\ \ -x^2+3x+1,\ \ x^3-x^2+2x-1\right\} \quad in \quad P_3(\mathbb{R})$$

Solution

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & -1 \\ 0 & 3 & 2 \\ 0 & 1 & -1 \end{bmatrix} \quad R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -3 \\ 0 & 3 & 2 \\ 0 & 1 & -1 \end{bmatrix} \quad \begin{matrix} R_3 + 3R_2 \\ R_4 + R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & -7 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\begin{cases} a_3 = 0 \\ a_1 = -3a_3 = 0 \\ a_1 = -a_3 = 0 \end{cases}$$

$\therefore$ *Linearly independent*

Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{x^3 - x,\ \ 2x^2 4,\ \ -2x^3 + 3x^2 + 2x + 6\right\} \quad in \quad P_3(\mathbb{R})$$

Solution

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & 4 & 6 \end{bmatrix} \quad R_3 + R_1$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 4 & 6 \end{bmatrix} \quad R_4 - 2R_2$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} 2a_2 = -3a_3 \\ a_1 = 2a_3 \end{cases}$$

If $a_3 = 2 \quad \to \quad a_1 = 4 \quad a_2 = -3$

$$\to \quad 4\left(x^3 - x\right) - 3\left(2x^2 + 4\right) + 2\left(-2x^3 + 3x^2 + 2x + 6\right) = 0$$

$\therefore$ *Linearly dependent*

Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{ \begin{matrix} x^4 - x^3 + 5x^2 - 8x + 6, \quad -x^4 + x^3 - 5x^2 + 5x - 3, \quad x^4 + 3x^2 - 3x + 5, \\ 2x^4 + 3x^3 + 4x^2 - x + 1, \quad x^3 - x + 2 \end{matrix} \right\} \quad in \quad P_4(\mathbb{R})$$

Solution

$$\begin{vmatrix} 1 & -1 & 1 & 2 & 0 \\ -1 & 1 & 0 & 3 & 1 \\ 5 & -5 & 3 & 4 & 0 \\ -8 & 5 & -3 & -1 & -1 \\ 6 & -3 & 5 & 1 & 2 \end{vmatrix} \underline{= -60 \neq 0}$$

$\therefore$ *Linearly dependent*

Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{ \begin{matrix} x^4 - x^3 + 5x^2 - 8x + 6, \quad -x^4 + x^3 - 5x^2 + 5x - 3, \\ x^4 + 3x^2 - 3x + 5, \quad 2x^4 + x^3 + 4x^2 + 8x \end{matrix} \right\} \quad in \quad P_4(\mathbb{R})$$

Solution

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ -1 & 1 & 0 & 1 \\ 5 & -5 & 3 & 4 \\ -8 & 5 & -3 & 8 \\ 6 & -3 & 5 & 0 \end{bmatrix} \quad \begin{matrix} \\ R_2 + R_1 \\ R_3 - 5R_1 \\ R_4 + 8R_1 \\ R_5 - 6R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & -6 \\ 0 & -3 & 5 & 24 \\ 0 & 3 & -1 & -12 \end{bmatrix} \quad R_5 + R_4$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & -6 \\ 0 & -3 & 5 & 24 \\ 0 & 0 & 4 & 12 \end{bmatrix} \quad \begin{matrix} \\ \\ R_3 + 2R_2 \\ \\ R_5 - 4R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 5 & 24 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \to a_1 = a_2 - a_3 - 2a_4 \\ \to a_3 = -3a_4 \\ \\ \to 3a_2 = 5a_3 + 24a_4 \\ \\ \end{matrix}$$

If $a_4 = -1 \quad \to a_3 = 3 \quad a_2 = -3 \quad a_1 = -4$

$$-4\left(x^4 - x^3 + 5x^2 - 8x + 6\right) - 3\left(-x^4 + x^3 - 5x^2 + 5x - 3\right)$$
$$+3\left(x^4 + 3x^2 - 3x + 5\right) - \left(2x^4 + x^3 + 4x^2 + 8x\right) = 0$$

$\therefore$ *Linearly dependent*

Exercise

Suppose that the vectors $\vec{u}_1, \vec{u}_2, \textit{and } \vec{u}_3$ are linearly dependent. Are the vectors $\vec{v}_1 = \vec{u}_1 + \vec{u}_2$, $\vec{v}_2 = \vec{u}_1 + \vec{u}_3$, and $\vec{v}_3 = \vec{u}_2 + \vec{u}_3$ also linearly dependent?

(***Hint***: Assume that $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{0}$, and see what the a_i's can be.)

Solution

Given: $\vec{u}_1, \vec{u}_2, \textit{and } \vec{u}_3$ are linearly dependent, then there are scalar $b_1, b_2, \textit{and } b_3$ such that

$$b_1\vec{u}_1 + b_2\vec{u}_2 + b_3\vec{u}_3 = 0.$$

Assume that $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{0}$

$$a_1\left(\vec{u}_1 + \vec{u}_2\right) + a_2\left(\vec{u}_1 + \vec{u}_3\right) + a_3\left(\vec{u}_2 + \vec{u}_3\right) = 0$$
$$a_1\vec{u}_1 + a_1\vec{u}_2 + a_2\vec{u}_1 + a_2\vec{u}_3 + a_3\vec{u}_2 + a_3\vec{u}_3 = 0$$

$$\left(a_1+a_2\right)\vec{u}_1+\left(a_1+a_3\right)\vec{u}_2+\left(a_2+a_3\right)\vec{u}_3=0$$

If $a_1+a_2=b_1 \quad a_1+a_3=b_2 \quad a_2+a_3=b_3$ and since $\vec{u}_1,\ \vec{u}_2,\ and\ \vec{u}_3$ are linearly dependent, therefore, $\vec{v}_1,\ \vec{v}_2,\ and\ \vec{v}_3$ are *linearly dependent*.

Exercise

Show that the set $F=\left\{1+t,\ t^2,\ t-2\right\}$ is a linearly independent subset of P_2.

Solution

$$W=\begin{vmatrix}1+t & t^2 & t-2\\ 1 & 2t & 1\\ 0 & 2 & 0\end{vmatrix}$$

$$=2t-4-2-2t$$

$$\underline{=-6\ \neq 0} \qquad \therefore \textit{Linearly Independent.}$$

$$\exists\, c_1,\ c_2,\ c_3\ \textit{constants}\ \ni\ 0=c_1(1+t)+c_2t^2+c_3(t-2)$$

$$\Rightarrow\begin{cases} t^0 & c_1-2c_3=0\\ t & c_1+c_3=0 \qquad \rightarrow \underline{c_1=c_3=0}\\ t^2 & \underline{c_2=0}\end{cases}$$

Since the only solution to this system is the trivial one. F is Linearly Independent subset of P_2

Exercise

Suppose that A is linearly dependent set of vectors and B is any set containing A. Show that B must be linearly dependent.

Solution

If A is linearly dependent, then there are vectors $\vec{x}_1,\ \vec{x}_2,\ \ldots,\ \vec{x}_n$ in A and $\mathbb{R}$, $c_1,\ c_2,\ldots\ c_n$ with all not $c_i\neq 0$ and $c_1\vec{x}_1+c_2\vec{x}_2+\ldots+c_n\vec{x}_n=\vec{0}$

If B any set that contains A, then this same relation holds in B set.

B is also dependent.

Exercise

Show that $\{\sin t,\ \sin 2t,\ \cos t\}$ is a linearly independent, subset of $C[0,\ 1]$. Does it span $C[0,\ 1]$

Solution

$$W = \begin{vmatrix} \sin t & \sin 2t & \cos t \\ \cos t & 2\cos 2t & -\sin t \\ -\sin t & -4\sin 2t & -\cos t \end{vmatrix}$$

$$= -2\sin t\cos t\cos 2t + \sin^2 t\sin 2t - 4\cos^2 t\sin 2t + 2\sin t\cos t\cos 2t - 4\sin^2 t\sin 2t + \cos^2 t\sin 2t$$

$$= \sin 2t - 4\sin 2t$$

$$= -3\sin 2t \neq 0$$

$\therefore$ *Linearly Independent.*

$a\sin t + b\sin 2t + d\cos t = 0$

$$\text{If}\begin{cases} t = 0 & \rightarrow & d = 0 \\ t = \frac{\pi}{2} & \rightarrow & a = 0 \\ t = \frac{\pi}{4} & \rightarrow & b = 0 \end{cases}$$

Since all the polynomials are in $C[0,\ 1]$ and there is no other way that we can write them as linear combinations of $\sin t,\ \sin 2t,$ *and* $\cos t$.

The set can't possible span $C[0,\ 1]$

Exercise

Show that the set $\{\sin(t+a),\ \sin(t+b),\ \sin(t+c)\}$ is linearly dependent on $C[0,\ 1]$.

Solution

$$W = \begin{vmatrix} \sin(t+a) & \sin(t+b) & \sin(t+c) \\ \cos(t+a) & \cos(t+b) & \cos(t+c) \\ -\sin(t+a) & -\sin(t+b) & -\sin(t+c) \end{vmatrix}$$

$$= -\sin(t+a)\cos(t+b)\sin(t+c) - \sin(t+a)\cos(t+c)\sin(t+b) - \sin(t+b)\cos(t+a)\sin(t+c)$$
$$+ \sin(t+a)\cos(t+b)\sin(t+c) + \sin(t+a)\cos(t+c)\sin(t+b) + \sin(t+b)\cos(t+a)\sin(t+c)$$

$$= 0$$

$\therefore$ The set is linearly dependent on $C[0,\ 1]$

$k_1\sin(t+a) + k_2\sin(t+b) + k_3\sin(t+c) = 0$

$$\text{If } \begin{cases} t=-a & \to & k_2+k_3=0 \\ t=-b & \to & k_1+k_3=0 \\ t=-c & \to & k_1+k_2=0 \end{cases}$$

$t=0 \quad \to \quad k_1 \sin a + k_2 \sin b + k_3 \sin c = 0$

$t=\frac{\pi}{2} \quad \to \quad k_1 \cos a + k_2 \cos b + k_3 \cos c = 0$

$t=\pi \quad \to \quad -\left(k_1 \sin a + k_2 \sin b + k_3 \sin c\right) = 0$

Exercise

Show that if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent and $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta$ are linearly dependent, then β can be uniquely expressed as a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$

Solution

Since, $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent, then

$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = 0$ when all $a_i = 0$.

Let assume that:

$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = b\beta$

$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n - b\beta = 0$

If $b=0$, then $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = 0$ and β doesn't exist.

If $b \neq 0$, and $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta$ are linearly dependent, then

$$\beta = \frac{a_1}{b}\alpha_1 + \frac{a_2}{b}\alpha_2 + \ldots + \frac{a_n}{b}\alpha_n$$
$$= c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n$$

If $\frac{a_1}{b} = c_1 \quad \Rightarrow \quad \frac{a_1}{b} - c_1 = 0$

$$\left(\frac{a_1}{b} - c_1\right)\alpha_1 + \left(\frac{a_2}{b} - c_2\right)\alpha_2 + \ldots + \left(\frac{a_n}{b} - c_n\right)\alpha_n = 0$$

Then $\frac{a_i}{b} - c_i = 0 \quad (1 \le i \le n)$ since $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent and contradict that $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta$ are linearly dependent.

Therefore, β can be uniquely expressed as a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$ in the form

$\beta = c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n$

Exercise

Show that if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly dependent with $\left(\alpha_1 \neq 0\right)$ if and only if there exists an integer k $\left(1 < k \leq n\right)$, such that α_k is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$

Solution

Since, $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly dependent, then if

$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = 0$

then there exists an $\alpha_k \neq 0$ $\left(1 < k \leq n\right)$

If we let $i > k$ where $a_i = 0$, then $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_{k-1}\alpha_{k-1} - a_k\alpha_k = 0$

$a_k\alpha_k = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_{k-1}\alpha_{k-1}$