

# Lecture Two – Functions

## Section 2.1 – Functions and Graphs

### Relations

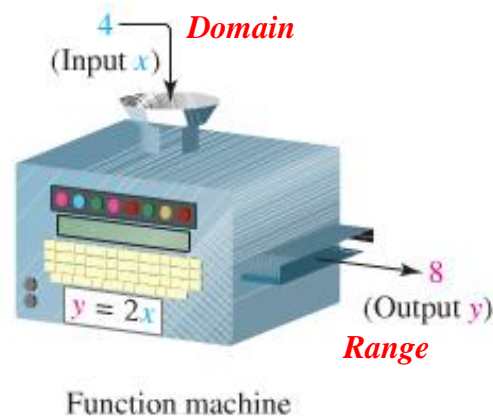
A **relation** is any set of ordered pairs. The set of all first components of ordered pairs is called the domain of the relation and the set of second components is called the range of the relation.

### Definition of a Function

A **function** is a relation between two variables such that to matches each element of a first set (called **domain**) to an element of a second set (called **range**) in such way that no element in the first set is assigned to two different elements in the second set.

The **domain** of the function is the set of all values of the independent variable for which the function is defined.

The **range** of the function is the set of all values taken on by the dependent variable.



### Example

Determine whether each relation is a function and *find the domain and the range*.

a)  $F = \{(1, 2), (-2, 4), (3, -1)\}$

Function: Yes

Domain:  $\{-2, 1, 3\}$

Range:  $\{-1, 2, 4\}$

b)  $G = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$

Function: No

Domain:  $\{1, 2\}$

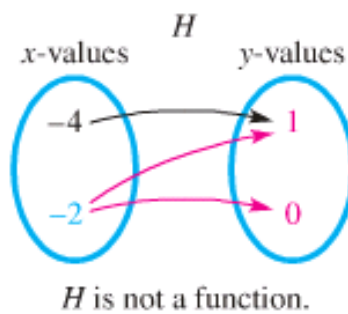
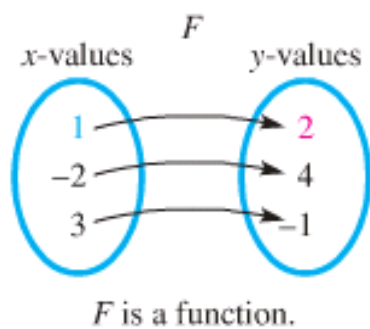
Range:  $\{1, 2, 3\}$

c)  $H = \{(-4, 1), (-2, 1), (-2, 0)\}$

Function: No

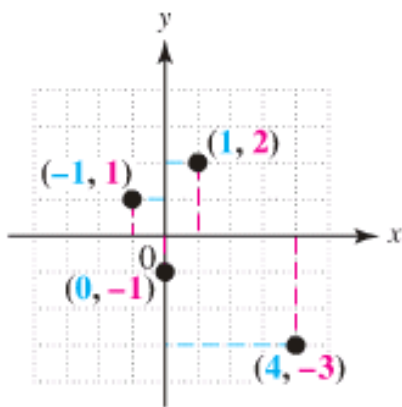
Domain:  $\{-4, -2\}$

Range:  $\{0, 1\}$



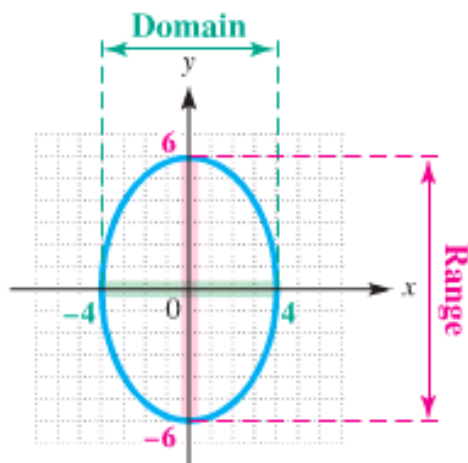
### Example

Give the domain and range of each relation



**Domain:**  $\{-1, 0, 1, 4\}$

**Range:**  $\{-3, -1, 1, 2\}$



**Domain:**  $[-4, 4]$

**Range:**  $[-6, 6]$

**Functions as Equations**       $y = -0.016x^2 + 0.93x + 8.5$

$x$ : independent

$y$ : depend on  $x$

### **Notation for Functions**

$f(x)$  read “ $f$  of  $x$ ” or “ $f$  at  $x$ ” represents the value of the function at the number  $x$ .

### **Example**

Let  $f(x) = -x^2 + 5x - 3$

a)  $f(2)$

$$f(x) = -x^2 + 5x - 3$$

$$f(\text{---}) = -(\text{---})^2 + 5(\text{---}) - 3$$

$$f(2) = -(2)^2 + 5(2) - 3$$

$$= 3$$

b)  $f(q)$

$$f(q) = -(q)^2 + 5(q) - 3$$

$$= -q^2 + 5q - 3$$

### **Example**

If  $f(x) = x^2 - 2x + 7$ , evaluate each of the following:

a)  $f(-5)$

b)  $f(x+4)$

### **Solution**

a)  $f(-5) = ?$

$$f(\text{---}) = (\text{---})^2 - 2(\text{---}) + 7$$

$$f(-5) = (-5)^2 - 2(-5) + 7$$

$$= 25 + 10 + 7$$

$$= 42$$

b)  $f(x+4) = ?$

$$f(\text{---}) = (\text{---})^2 - 2(\text{---}) + 7$$

$$f(x+4) = (x+4)^2 - 2(x+4) + 7$$

$$= x^2 + 2(4)x + 4^2 - 2x - 8 + 7$$

$$= x^2 + 8x + 16 - 2x - 1$$

$$= \underline{x^2 + 6x + 15}$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

### Example

Let  $g(x) = 2x + 3$ , find  $g(a+1)$

### Solution

$$g(x) = 2x + 3$$

$$g(a+1) = 2(a+1) + 3$$

$$= 2a + 2 + 3$$

$$= \underline{2a + 5}$$

### Example

Given:  $f(x) = 2x^2 - x + 3$ , find the following.

a)  $f(0)$

b)  $f(-7)$

c)  $f(5a)$

### Solution

$$\begin{aligned} \text{a) } f(x=0) &= 2(0)^2 - (0) + 3 \\ &= \underline{3} \end{aligned}$$

$$\begin{aligned} \text{b) } f(-7) &= 2(-7)^2 - (-7) + 3 \\ &= \underline{108} \end{aligned}$$

$$\begin{aligned} \text{c) } f(5a) &= 2(5a)^2 - (5a) + 3 \\ &= \underline{50a^2 - 5a + 3} \end{aligned}$$

## Increasing and Decreasing Functions

- ✚ A function *rises from left to right* ( $x$ -coordinate), the function  $f$  is said to be **increasing** on an open interval  $I(a, b)$  ( $x$ -coordinate)

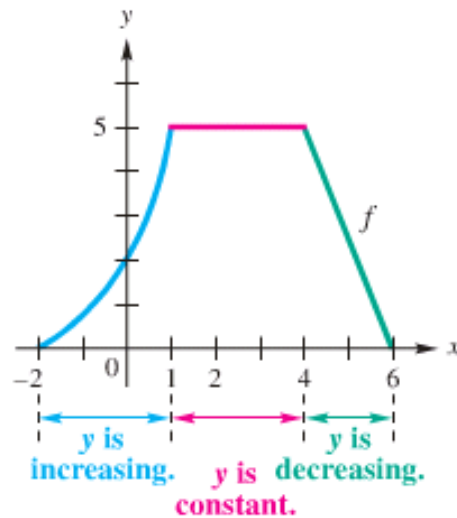
$$a < b \Rightarrow f(a) < f(b)$$

- ✚ A function  $f$  is said to be **decreasing** on an open interval  $I$

$$a < b \Rightarrow f(a) > f(b)$$

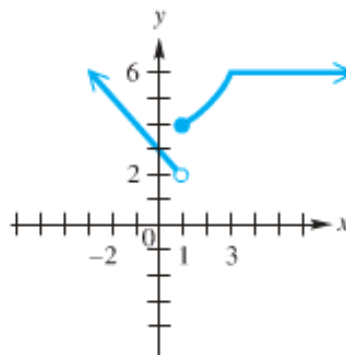
- ✚ A function  $f$  is said to be **constant** on an open interval  $I$

$$a < b \Rightarrow f(a) = f(b)$$



### Example

Determine the intervals over which the function is increasing, decreasing, or constant



Increasing:  $[1, 3]$

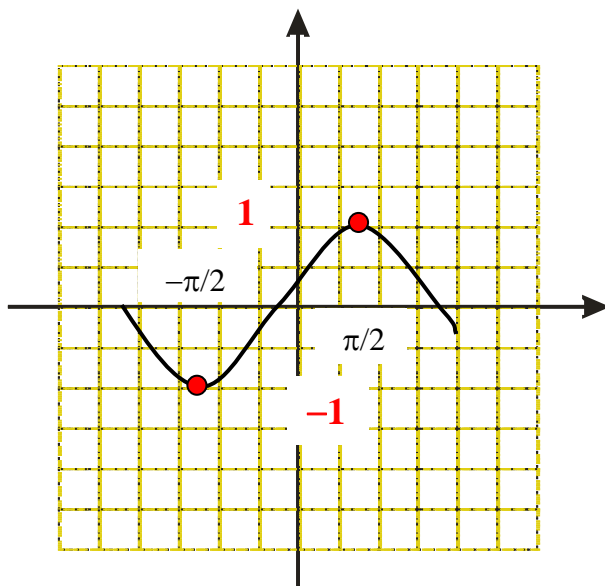
Decreasing:  $(-\infty, 1)$

Constant:  $[3, \infty)$

## Relative *Maxima* (um) and *Minima* (um)

$f(a)$  is a relative maximum if there exists an open interval  $I$  about  $a$  such that  $f(a) > f(x)$ , for all  $x$  in  $I$ .

$f(a)$  is a relative minimum if there exists an open interval  $I$  about  $a$  such that  $f(a) < f(x)$ , for all  $x$  in  $I$ .

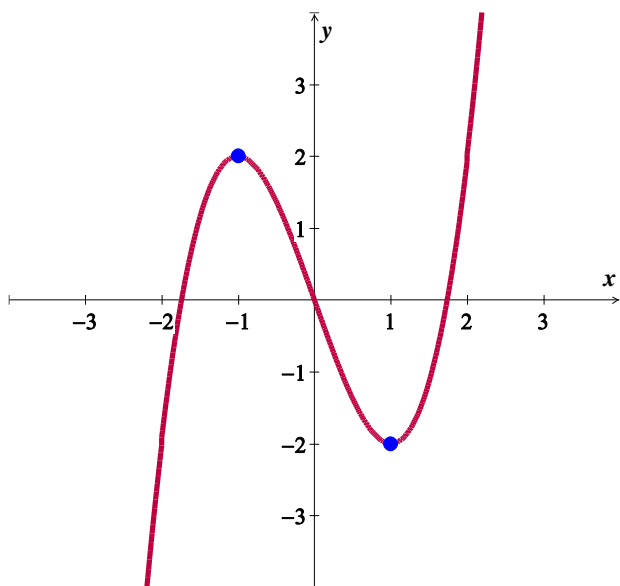


The relative minimum value of the function is  $-1$  @  $x = -\pi/2$

The relative maximum value of the function is  $1$  @  $x = \pi/2$

## Example

State the intervals on which the given function  $f(x) = x^3 - 3x$  is increasing, decreasing, or constant, and determine the extreme values



**Increasing**  $(-\infty, -1), (1, \infty)$

**Decreasing**  $(-1, 1)$

**RMIN**  $(1, -2)$

**RMAX**  $(-1, 2)$

## Piecewise-Defined Functions

Function are sometimes described by more than one expression, we call such functions *piecewise-defined functions*.

### Example

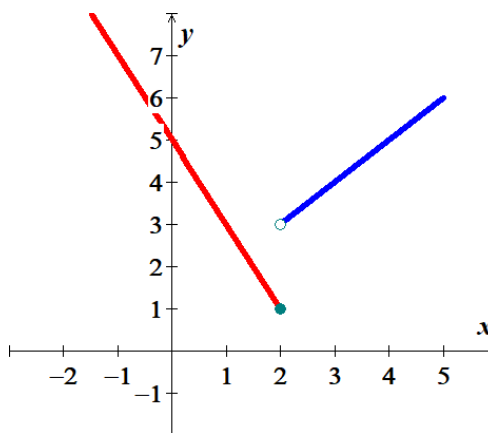
Graph function

$$f(x) = \begin{cases} -2x+5 & \text{if } x \leq 2 \\ x+1 & \text{if } x > 2 \end{cases}$$

Find:  $f(2) = -2(\textcolor{red}{2}) + 5 = \textcolor{blue}{1}$

$$f(0) = -2(\textcolor{red}{0}) + 5 = \textcolor{blue}{5}$$

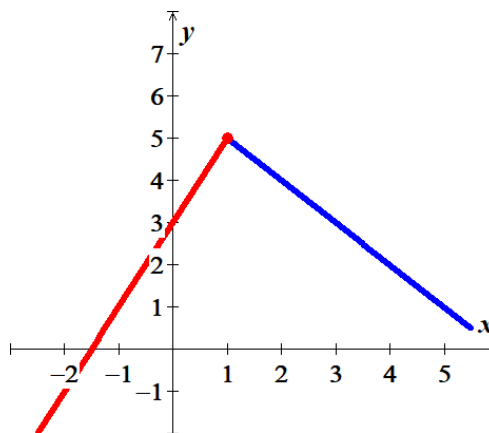
$$f(4) = \textcolor{red}{4} + 1 = \textcolor{blue}{5}$$



### Example

Graph function

$$f(x) = \begin{cases} 2x+3 & \text{if } x \leq 1 \\ -x+6 & \text{if } x > 1 \end{cases}$$



### Example

$$C(t) = \begin{cases} 20 & \text{if } 0 \leq t \leq 60 \\ 20 + 0.40(t - 60) & \text{if } t > 60 \end{cases}$$

Find  $C(40)$ ,  $C(80)$ , and  $C(60)$

### Solution

a)  $C(40) = 20$

b)  $C(80) = 20 + 0.40(80 - 60) = 28$

c)  $C(60) = 20$

## **Exercise**      **Section 2.1 – Functions and Graphs**

(1 – 7) Determine whether each relation is a function and *find the domain and the range*.

1.  $\{(1, 2), (3, 4), (5, 6), (5, 8)\}$
2.  $\{(1, 2), (3, 4), (6, 5), (8, 5)\}$
3.  $\{(9, -5), (9, 5), (2, 4)\}$
4.  $\{(-2, 5), (5, 7), (0, 1), (4, -2)\}$
5.  $\{(-5, 3), (0, 3), (6, 3)\}$
6.  $\{(1, 2), (3, 4), (6, 5), (8, 5), (1, 5)\}$
7.  $\{(-1, 3), (3, 4), (6, 5), (8, 5), (1, 5)\}$
8. Let  $f(x) = -3x + 4$ , find  $f(0)$ ,  $f(-1)$ ,  $f(h)$ , and  $f(a-1)$
9. Let  $g(x) = -x^2 + 4x - 1$ , find  $g(-x)$ ,  $g(2)$ , and  $g(-2)$
10. Let  $f(x) = -3x + 4$ , find  $f(a+4)$
11. Given:  $f(x) = 2|x| + 3x$ , find  $f(2-h)$ .
12. Given:  $g(x) = \frac{x-4}{x+3}$ , find  $g(x+h)$
13. Given:  $g(x) = \frac{x}{\sqrt{1-x^2}}$ , find  $g(0)$  and  $g(-1)$
14. Given that  $g(x) = 2x^2 + 2x + 3$ . Find  $g(p+3)$
15. If  $f(x) = x^2 - 2x + 7$ , evaluate each of the following:  $f(-5)$ ,  $f(x+4)$ ,  $f(-x)$
16. Find  $g(0)$ ,  $g(-4)$ ,  $g(7)$ , and  $g\left(\frac{3}{2}\right)$  for  $g(x) = \frac{x}{\sqrt{16-x^2}}$
17. For  $f(x) = 3x - 4$ , determine
  - a)  $f(0)$
  - b)  $f\left(\frac{5}{3}\right)$
  - c)  $f(-2a)$
  - d)  $f(x+h)$
18. For  $f(x) = 3x^2 + 3x - 1$ , determine
  - a)  $f(0)$
  - b)  $f(x+h)$
  - c)  $f(2)$
  - d)  $f(h)$



19. For  $f(x) = 2x^2 - 4$ , determine

a)  $f(0)$                       b)  $f(x+h)$                       c)  $f(2)$                       d)  $f(2) - f(-3)$

20. For  $f(x) = 3x^2 + 4x - 2$ , determine

a)  $f(0)$                       b)  $f(x+h)$                       c)  $f(3)$                       d)  $f(-5)$

21. For  $f(x) = -x^3 - x^2 - x + 10$ , determine

a)  $f(0)$                       b)  $f(-1)$                       c)  $f(2)$                       d)  $f(1) - f(-2)$

22. For  $\frac{1}{10}x^{10} - \frac{1}{2}x^6 + \frac{2}{3}x^3 - 10x$ , determine

a)  $f(2) - f(-2)$                       b)  $f(1) - f(-1)$                       c)  $f(2) - f(0)$

23. For  $f(x) = 3x^4 + x^2 - 4$ , determine

a)  $f(2) - f(-2)$                       b)  $f(1) - f(-1)$                       c)  $f(2) - f(0)$

24. For  $f(x) = -\frac{2}{3}x^3 + 4x$ , determine

a)  $f(2) - f(-2)$                       b)  $f(1) - f(-1)$                       c)  $f(2) - f(0)$

25. For  $f(x) = \frac{2x-3}{x-4}$ , determine

a)  $f(0)$                       b)  $f(3)$                       c)  $f(x+h)$                       d)  $f(-4)$

26. For  $f(x) = \frac{3x-1}{x-5}$ , determine

a)  $f(0)$                       b)  $f(3)$                       c)  $f(x+h)$                       d)  $f(-5)$

27. 
$$f(x) = \begin{cases} 2+x & \text{if } x < -4 \\ -x & \text{if } -4 \leq x \leq 2 \\ 3x & \text{if } x > 2 \end{cases}$$
 Find:  $f(-5)$ ,  $f(-1)$ ,  $f(0)$ , and  $f(3)$

28. 
$$f(x) = \begin{cases} -2x & \text{if } x < -3 \\ 3x-1 & \text{if } -3 \leq x \leq 2 \\ -4x & \text{if } x > 2 \end{cases}$$
 Find:  $f(-5)$ ,  $f(-1)$ ,  $f(0)$ , and  $f(3)$

29.  $f(x) = \begin{cases} x^3 + 3 & \text{if } -2 \leq x \leq 0 \\ x + 3 & \text{if } 0 < x < 1 \\ 4 + x - x^2 & \text{if } 1 \leq x \leq 3 \end{cases}$  Find:  $f(-5)$ ,  $f(-1)$ ,  $f(0)$ , and  $f(3)$

30.  $h(x) = \begin{cases} \frac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$  Find:  $h(5)$ ,  $h(0)$ , and  $h(3)$

31.  $f(x) = \begin{cases} 3x + 5 & \text{if } x < 0 \\ 4x + 7 & \text{if } x \geq 0 \end{cases}$  Find

a)  $f(0)$       b)  $f(-2)$       c)  $f(1)$       d)  $f(3) + f(-3)$       e) Graph  $f(x)$

32.  $f(x) = \begin{cases} 6x - 1 & \text{if } x < 0 \\ 7x + 3 & \text{if } x \geq 0 \end{cases}$  Find

a)  $f(0)$       b)  $f(-1)$       c)  $f(4)$       d)  $f(2) + f(-2)$       e) Graph  $f(x)$

33.  $f(x) = \begin{cases} 2x + 1 & \text{if } x \leq 1 \\ 3x - 2 & \text{if } x > 1 \end{cases}$  Find

a)  $f(0)$       b)  $f(2)$       c)  $f(-2)$       d)  $f(1) + f(-1)$       e) Graph  $f(x)$

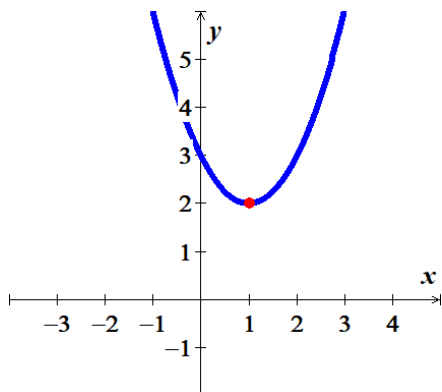
34. Graph the piecewise function defined by  $f(x) = \begin{cases} 3 & \text{if } x \leq -1 \\ x - 2 & \text{if } x > -1 \end{cases}$

35. Sketch the graph  $f(x) = \begin{cases} x + 2 & \text{if } x \leq -1 \\ x^3 & \text{if } -1 < x < 1 \\ -x + 3 & \text{if } x \geq 1 \end{cases}$

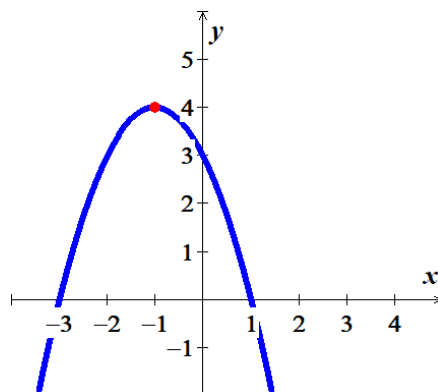
36. Sketch the graph  $f(x) = \begin{cases} x - 3 & \text{if } x \leq -2 \\ -x^2 & \text{if } -2 < x < 1 \\ -x + 4 & \text{if } x \geq 1 \end{cases}$

(37 – 42) Determine any *relative maximum* or *minimum* of the function, determine the intervals on which the function *increasing* or *decreasing*, and then find the *domain* and the *range*.

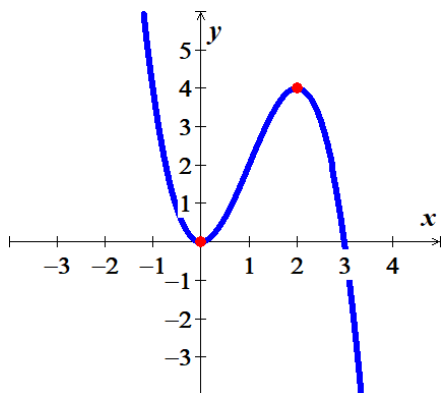
37.  $f(x) = x^2 - 2x + 3$



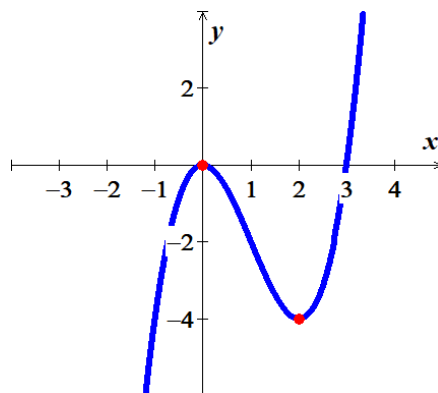
38.  $f(x) = -x^2 - 2x + 3$



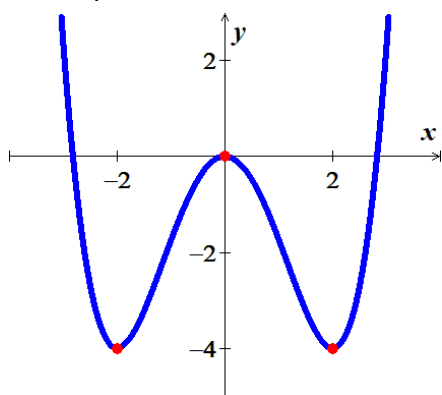
39.  $f(x) = -x^3 + 3x^2$



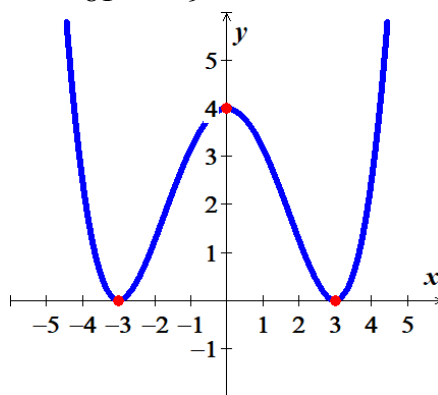
40.  $f(x) = x^3 - 3x^2$



41.  $f(x) = \frac{1}{4}x^4 - 2x^2$



42.  $f(x) = \frac{4}{81}x^4 - \frac{8}{9}x^2 + 4$

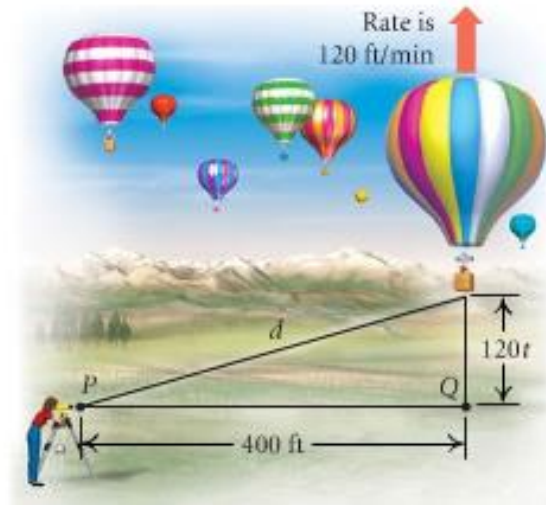


43. The elevation  $H$ , in *meters*, above sea level at which the boiling point of water is in  $t$  *degrees Celsius* is given by the function

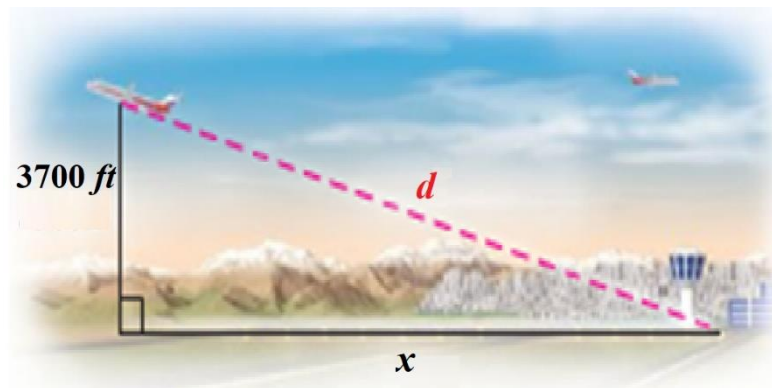
$$H(t) = 1000(100 - t) + 580(100 - t)^2$$

At what elevation is the boiling point  $99.5^\circ$ .

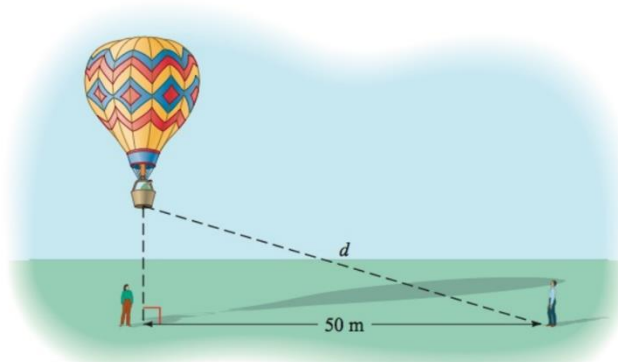
44. A hot-air balloon rises straight up from the ground at a rate of  $120 \text{ ft./min.}$  The balloon is tracked from a rangefinder on the ground at point  $P$ , which is  $400 \text{ feet.}$  from the release point  $Q$  of the balloon. Let  $d$  be the distance from the balloon to the rangefinder and  $t$  – the time, in *minutes*, since the balloon was released. Express  $d$  as a function of  $t$ .



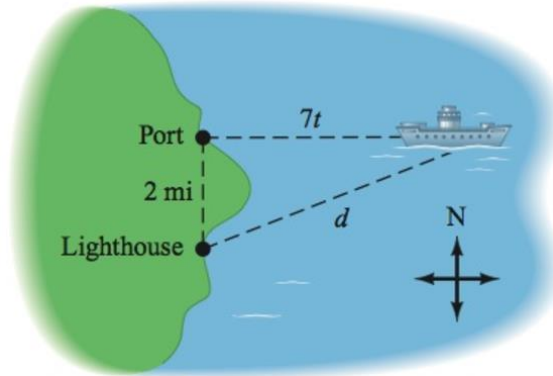
45. An airplane is flying at an altitude of  $3700 \text{ feet.}$  The slanted distance directly to the airport is  $d \text{ feet.}$  Express the horizontal distance  $x$  as a function of  $d$ .



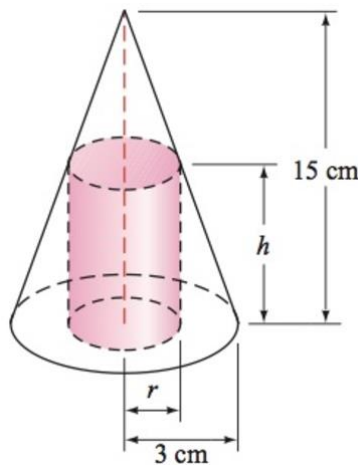
46. For the first minute of flight, a hot air balloon rises vertically at a rate of  $3 \text{ m/sec.}$  If  $t$  is the time in *seconds* that the balloon has been airborne, write the distance  $d$  between the balloon and a point on the ground  $50 \text{ meters}$  from the point to lift off as a function of  $t$ .



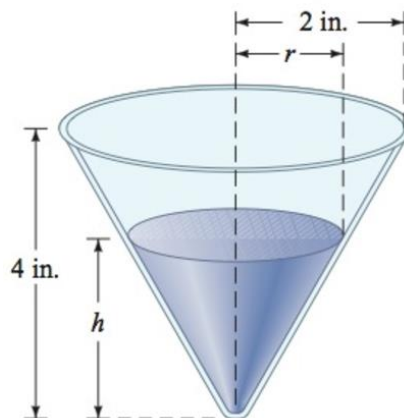
47. A light house is 2 miles south of a port. A ship leaves port and sails east at a rate of 7 miles per hour. Express the distance  $d$  between the ship and the lighthouse as a function of time, given that the ship has been sailing for  $t$  hours.



48. A cone has an altitude of 15 cm and a radius of 3 cm. A right circular cylinder of radius  $r$  and height  $h$  is inscribed in the cone. Use similar triangles to write  $h$  as a function of  $r$ .

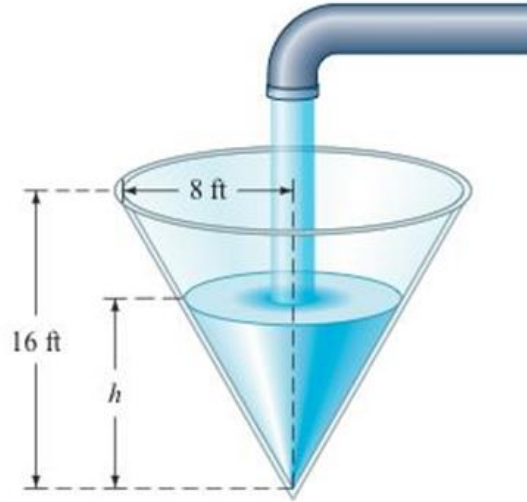


49. Water is flowing into a conical drinking cup with an altitude of 4 inches and a radius of 2 inches.

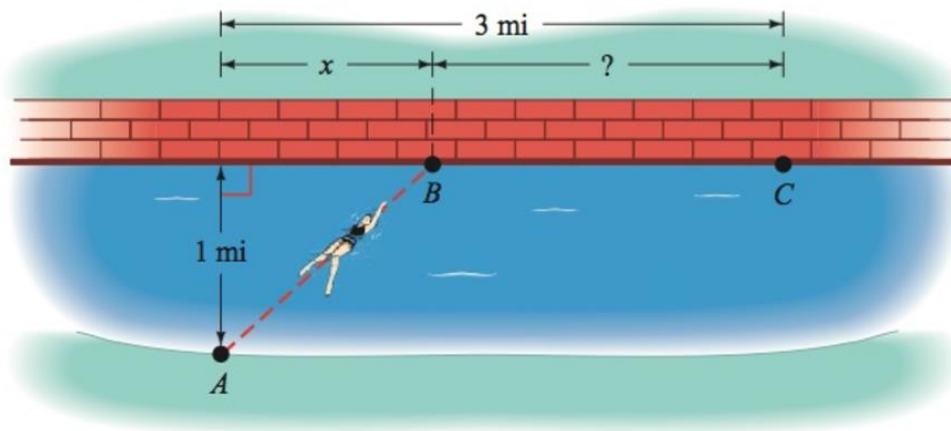


- Write the radius  $r$  of the surface of the water as a function of its depth  $h$ .
- Write the volume  $V$  of the water as a function of its depth  $h$ .

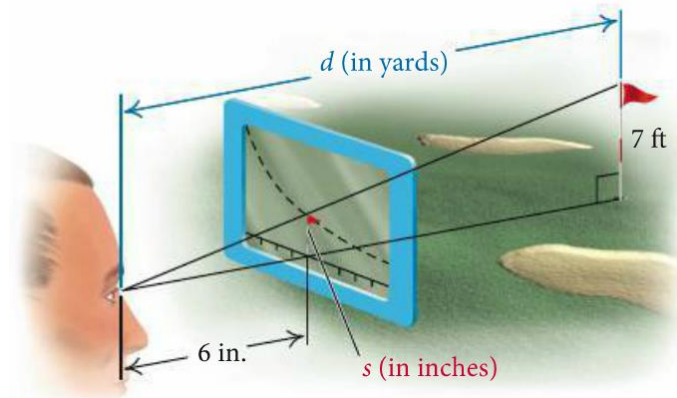
50. A water tank has the shape of a right circular cone with height 16 feet and radius 8 feet. Water is running into the tank so that the radius  $r$  (in feet) of the surface of the water is given by  $r = 1.5t$ , where  $t$  is the time (in minutes) that the water has been running.



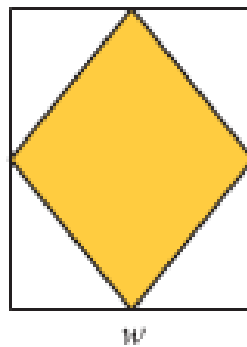
- a) The area  $A$  of the surface of the water is  $A = \pi r^2$ . Find  $A(t)$  and use it to determine the area of the surface of the water when  $t = 2$  minutes.
- b) The volume  $V$  of the water is given by  $V = \frac{1}{3}\pi r^2 h$ . Find  $V(t)$  and use it to determine the volume of the water when  $t = 3$  minutes.
51. An athlete swims from point  $A$  to point  $B$  at a rate of 2 miles per hour and runs from point  $B$  to point  $C$  at a rate of 8 miles per hour. Use the dimensions in the figure to write the time  $t$  required to reach point  $C$  as a function of  $x$ .



52. A device used in golf to estimate the distance  $d$ , in yards, to a hole measures the size  $s$ , in inches, that the 7-foot pin appears to be in a viewfinder. Express the distance  $d$  as a function of  $s$ .



53. A rhombus is inscribed in a rectangle that is  $w$  meters wide with a perimeter of 40 m. Each vertex of the rhombus is a midpoint of a side of the rectangle. Express the area of the rhombus as a function of the rectangle's width.

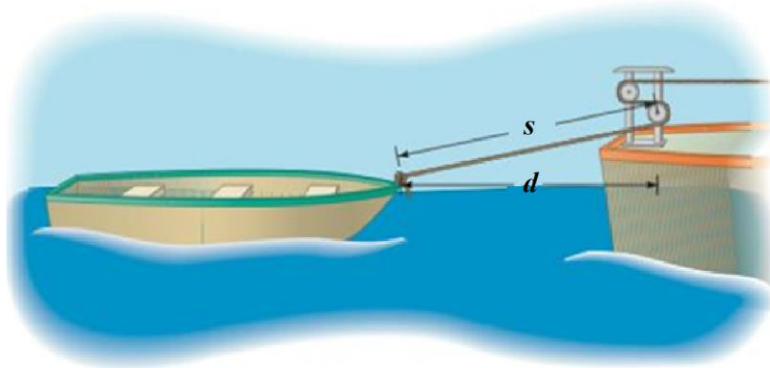


54. The surface area  $S$  of a right circular cylinder is given by the formula  $S = 2\pi rh + 2\pi r^2$ . if the height is twice the radius, find each of the following.

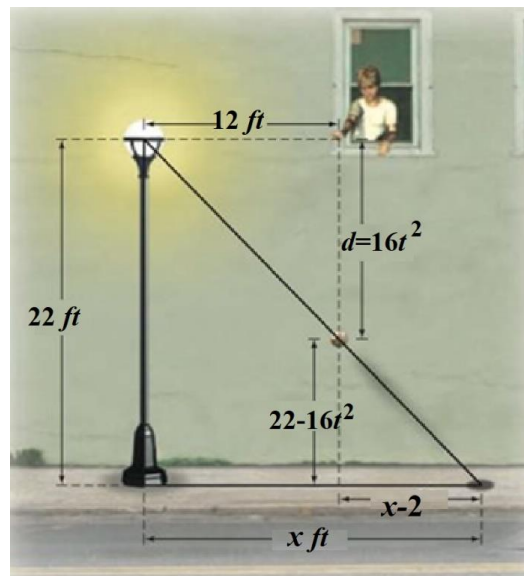


- a) A function  $S(r)$  for the surface area as a function of  $r$ .  
 b) A function  $S(h)$  for the surface area as a function of  $h$ .

55. A boat is towed by a rope that runs through a pulley that is 4 feet above the point where the rope is tied to the boat. The length (in feet) of the rope from the boat to the pulley is given by  $s = 48 - t$ , where  $t$  is the time in seconds that the boat has been in tow. The horizontal distance from the pulley to the boat is  $d$ .

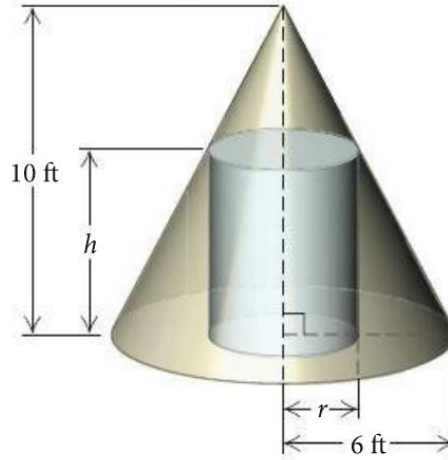


- a) Find  $d(t)$   
 b) Evaluate  $s(35)$  and  $d(35)$
56. The light from a lamppost casts a shadow from a ball that was dropped from a height of 22 feet above the ground. The distance  $d$ , in feet, the ball has dropped  $t$  seconds after it is released is given by  $d(t) = 16t^2$ . Find the distance  $x$ , in feet, of the shadow from the base of the lamppost as a function of time  $t$ .





57. A right circular cylinder of height  $h$  and a radius  $r$  is inscribed in a right circular cone with a height of 10 feet and a base with radius 6 feet.



- Express the height  $h$  of the cylinder as a function of  $r$ .
- Express the volume  $V$  of the cylinder as a function of  $r$ .
- Express the volume  $V$  of the cylinder as a function of  $h$ .