Lecture Three – Multiple Integrals

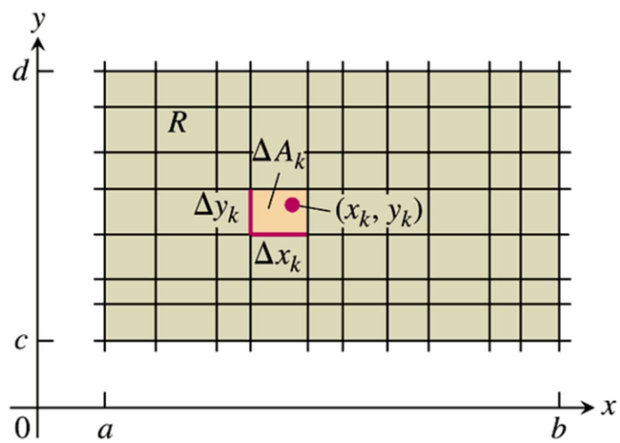
Section 3.1 – Double Integrals over Rectangular Regions

Double Integrals

Consider a function $f(x, y)$ defined on a rectangular region *R*,

$$R: \quad a \le x \le b, \quad c \le y \le d$$

A small rectangular piece of width Δx and height Δy has area $\Delta A = \Delta x \Delta y$.



To form a Riemann sum over *R*, select a point (x_k, y_k) in the kth small rectangle, multiply the value of *f* at that point by the area ΔA_k and add together the products:

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

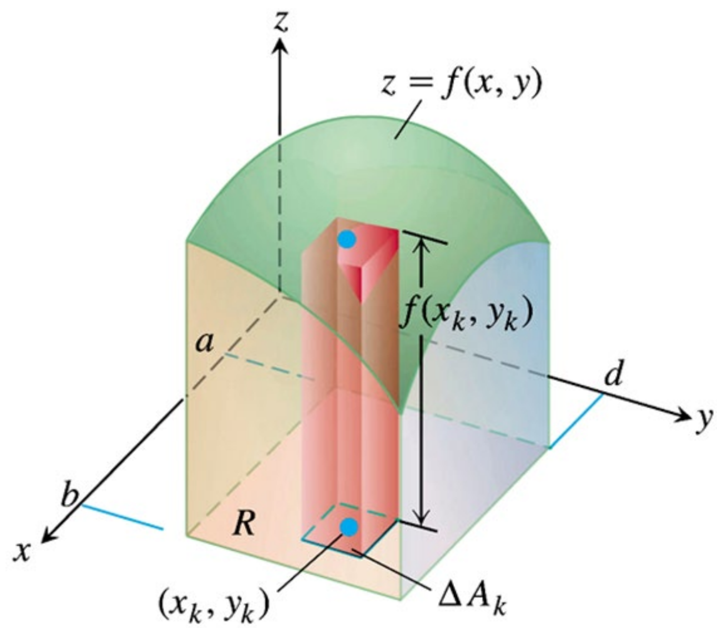
As the rectangles get narrow and short, their number *n* increases, therefore

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

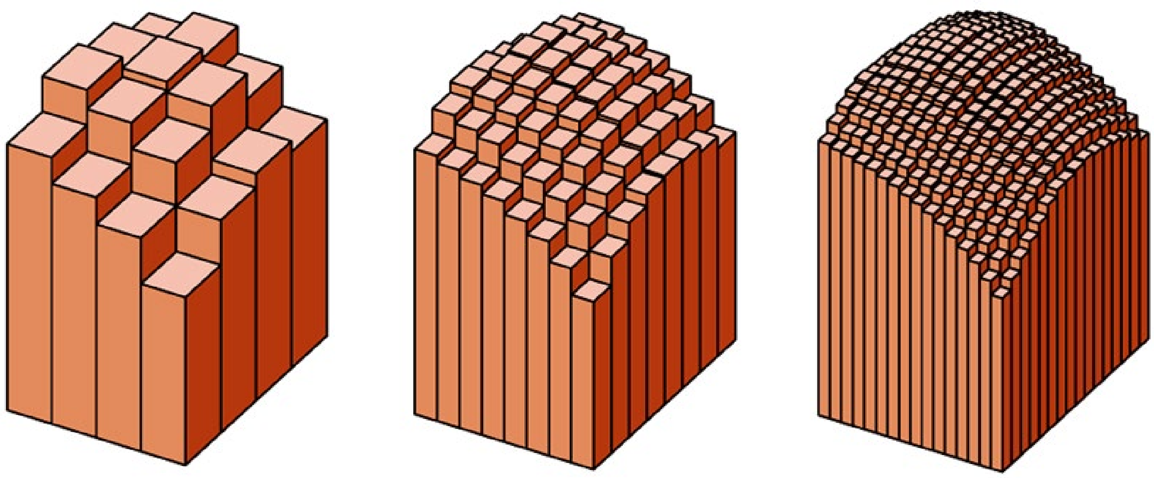
Then the function f is said to be integrable and the limit is called double integral of f over *R*,

$$\iint_R f(x, y)\, dA \quad \text{or} \quad \iint_R f(x, y)\, dxdy$$

Double Integrals as Volumes



$$Volume = \lim_{n\to\infty} S_n = \iint_R f(x,y)\,dA, \ \ where\ \Delta A_k \to 0 \ as\ n \to \infty$$



$n = 16$ $\qquad$ $n = 64$ $\qquad$ $n = 256$

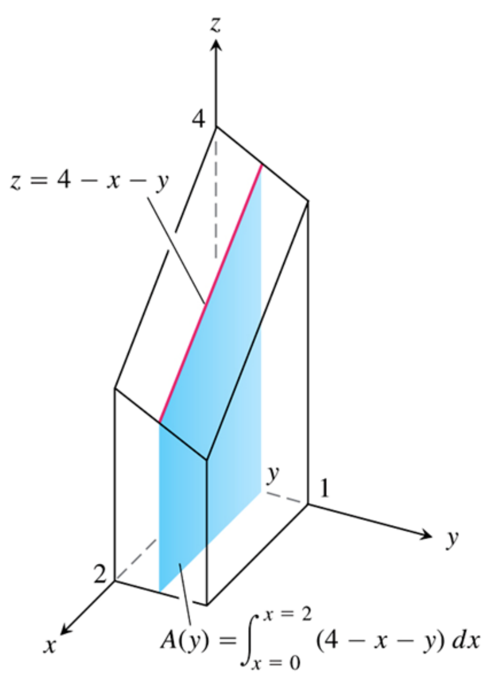
As n increases, the ***Riemann sum*** approximations approach the total volume of the solid

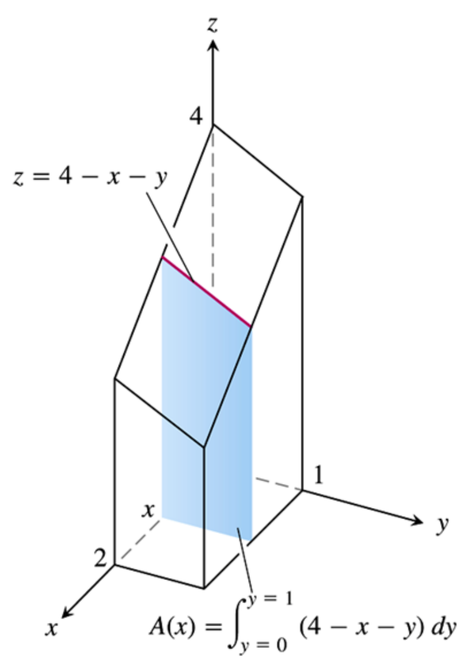
Example

Calculate the volume under the plane $z = 4 - x - y$ over the rectangular region $R: \quad 0 \le x \le 2, \quad 0 \le y \le 1$ in the xy-plane.

Solution

$$
\begin{aligned}
Volume &= \int_{x=0}^{x=2} A(x)\,dx \\
&= \int_{x=0}^{x=2} \int_{y=0}^{y=1} (4 - x - y)\,dy\;dx \\
&= \int_{x=0}^{x=2} \left[4y - xy - \frac{1}{2}y^2 \right]_{y=0}^{y=1} dx \\
&= \int_{x=0}^{x=2} \left(4 - x - \frac{1}{2} \right) dx \\
&= \int_{x=0}^{x=2} \left(\frac{7}{2} - x \right) dx \\
&= \left[\frac{7}{2}x - \frac{1}{2}x^2 \right]_0^2 \\
&= 7 - 2 \\
&= 5 \quad unit^3
\end{aligned}
$$





$$
Volume = \int_0^1 \int_0^2 (4 - x - y)\,dx\,dy
$$

Theorem – Fubini's Theorem

If $f(x,y)$ is continuous throughout the rectangular region R: $a \le x \le b,\ c \le y \le d$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dxdy = \int_a^b \int_c^d f(x,y)\,dydx$$

Example

Calculate $\displaystyle\iint_R f(x,y)\,dA$ for $f(x,y) = 100 - 6x^2y$ *and* $R:\ 0 \le x \le 2,\ \ -1 \le y \le 1$

Solution

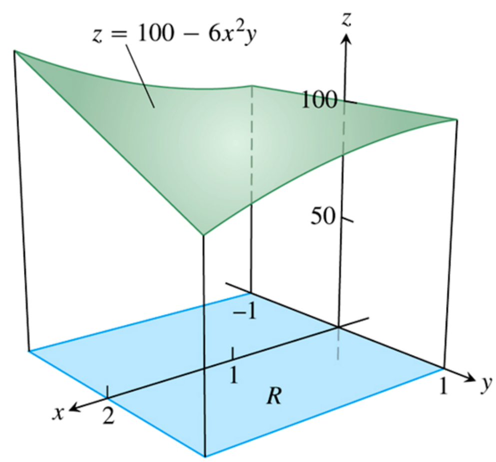
$$\int_{-1}^{1}\int_{0}^{2}\left(100 - 6x^2y\right)dxdy = \int_{-1}^{1}\left(100x - 2x^3y\ \Big|_0^2\ dy\right.$$

$$= \int_{-1}^{1}(200 - 16y)\,dy$$

$$= 200y - 8y^2\ \Big|_{-1}^{1}$$

$$= 200 - 8 - (-200 - 8)$$

$$= 400$$



Example

Evaluate $\displaystyle\int\int e^{4x}y^3dydx$

Solution

$$\int\int e^{4x}y^3dydx = \int e^{4x}dx\int y^3dy$$

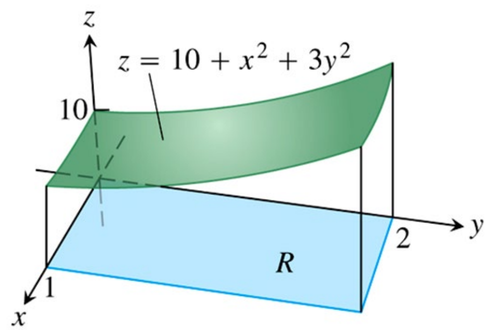
$$= \frac{1}{4}e^{4x}\left(\frac{1}{4}y^4\right) + C$$

$$= \frac{1}{16}y^4e^{4x} + C$$

Example

Find the volume of the region bounded above the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below the rectangle $R:\ 0 \le x \le 1, \quad 0 \le y \le 2$

Solution



$$Volume = \int_0^1 \int_0^2 \left(10 + x^2 + 3y^2\right) dydx$$

$$= \int_0^1 \left(10y + yx^2 + y^3 \Big|_0^2\right. dx$$

$$= \int_0^1 \left(2x^2 + 28\right) dx$$

$$= \frac{2}{3}x^3 + 28x \Big|_0^1$$

$$= \frac{2}{3} + 28$$

$$\boxed{= \frac{86}{3} \ \ unit^3}$$

Example

Evaluate $\displaystyle\int_0^1 \int_y^1 ye^{-x^3} dxdy$

Solution

$$y \le x \le 1 \quad \rightarrow \quad \begin{cases} y = x \\ x = 1 \end{cases}$$

$0 \le y \le 1$

$0 \le y \le x \quad \& \quad 0 \le x \le 1$

$$\int_0^1 \int_y^1 ye^{-x^3} dxdy = \int_0^1 \int_0^x ye^{-x^3} dydx$$

$$= \frac{1}{2}\int_0^1 e^{-x^3} y^2 \Big|_0^x dx$$

$$=\frac{1}{2}\int_0^1 x^2 e^{-x^3}\ dx$$

$$=-\frac{1}{6}\int_0^1 e^{-x^3}\ d\left(-x^3\right)$$

$$=-\frac{1}{6}e^{-x^3}\ \Big|_0^1$$

$$=-\frac{1}{6}\left(e^{-1}-1\right)$$

$$=\frac{1}{6}\left(1-\frac{1}{e}\right)$$

$$\boxed{=\frac{e-1}{6e}}$$

Exercises *Section* 3.1 – Double Integrals over Rectangular Regions

(1 – 18) Evaluate the iterated integral

1. $\displaystyle\int_1^2\int_0^4 2xy\ dydx$

2. $\displaystyle\int_0^2\int_{-1}^1 (x-y)\ dydx$

3. $\displaystyle\int_0^1\int_0^1\left(1-\frac{x^2+y^2}{2}\right)dxdy$

4. $\displaystyle\int_0^3\int_{-2}^0\left(x^2y-2xy\right)dydx$

5. $\displaystyle\int_0^1\int_0^1\frac{y}{1+xy}dxdy$

6. $\displaystyle\int_0^{\ln 2}\int_1^{\ln 5} e^{2x+y}dydx$

7. $\displaystyle\int_0^1\int_1^2 xye^x dydx$

8. $\displaystyle\int_\pi^{2\pi}\int_0^\pi (\sin x+\cos y)dxdy$

9. $\displaystyle\int_1^2\int_1^4\frac{xy}{\left(x^2+y^2\right)^2}dxdy$

10. $\displaystyle\int_1^3\int_1^{e^x}\frac{x}{y}\ dydx$

11. $\displaystyle\int_1^2\int_0^{\ln x} x^3e^y\, dydx$

12. $\displaystyle\int_1^{10}\int_0^{1/y} ye^{xy}dxdy$

13. $\displaystyle\int_0^1\int_{\sqrt{y}}^{2-\sqrt{y}} xy\ dxdy$

14. $\displaystyle\int_0^1\int_{x^2}^{x}\sqrt{x}\ dydx$

15. $\displaystyle\int_0^{3/2}\int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} ydxdy$

16. $\displaystyle\int_0^2\int_0^{4-x^2} 2x\ dydx$

17. $\displaystyle\int_0^1\int_{2y}^2 4\cos\left(x^2\right)\ dxdy$

18. $\displaystyle\int_0^1\int_{\sqrt[3]{y}}^1\frac{2\pi\sin\pi x^2}{x^2}\ dxdy$

(19 – 26) Evaluate the double integral over the given region *R*.

19. $\displaystyle\iint_R\left(6y^2-2x\right)dA \quad R:\ 0\le x\le 1,\quad 0\le y\le 2$

20. $\displaystyle\iint_R\left(\frac{\sqrt{x}}{y^2}\right)dA \quad R:\ 0\le x\le 4,\quad 1\le y\le 2$

21. $\displaystyle\iint_R y\sin(x+y)\,dA \quad R:\ -\pi \le x \le 0,\quad 0 \le y \le \pi$

22. $\displaystyle\iint_R e^{x-y}\,dA \quad R:\ 0 \le x \le \ln 2,\quad 0 \le y \le \ln 2$

23. $\displaystyle\iint_R \frac{y}{x^2y^2+1}\,dA \quad R:\ 0 \le x \le 1,\quad 0 \le y \le 1$

24. $\displaystyle\iint_R x^{-1/2}e^{y}\,dA$; R is the region bounded by $x=1$, $x=4$, $y=\sqrt{x}$, and $y=0$

25. $\displaystyle\iint_R \left(x^2+y^2\right)dA$; R is the region $\{(x,y):\ 0 \le x \le 2,\ 0 \le y \le x\}$

26. $\displaystyle\iint_R \frac{2y}{\sqrt{x^4+1}}\,dA$; *R is the region bounded by* $x=1,\quad x=2,\quad y=x^{3/2},\quad y=0$

27. Integrate $f(x,y)=\dfrac{1}{xy}$ over the ***square*** $1 \le x \le 2,\quad 1 \le y \le 2$

28. Integrate $f(x,y)=y\cos xy$ over the ***rectangle*** $0 \le x \le \pi,\quad 0 \le y \le 1$

29. Find the volume of the region bounded above the paraboloid $z=x^2+y^2$ and below by the square $R:\ -1 \le x \le 1,\quad -1 \le y \le 1$

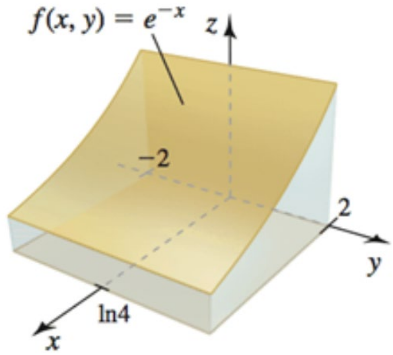
30. Find the volume of the region bounded above the plane $z=\dfrac{y}{2}$ and below by the rectangle $R:\ 0 \le x \le 4,\quad 0 \le y \le 2$

31. Find the volume of the region bounded above the surface $z=4-y^2$ and below by the rectangle $R:\ 0 \le x \le 1,\quad 0 \le y \le 2$

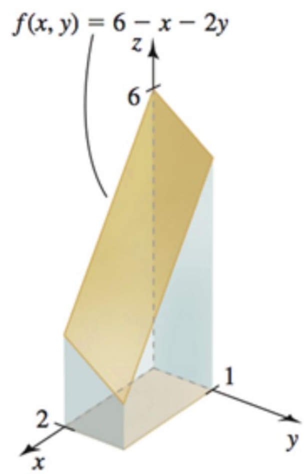
32. Find the volume of the region bounded above the ellipitical paraboloid $z=16-x^2-y^2$ and below by the square $R:\ 0 \le x \le 2,\quad 0 \le y \le 2$

33. Evaluate $\displaystyle\int_0^{1/2}\left(\sin^{-1}[2x]-\sin^{-1}x\right)dx$ by converting it to a double integral.

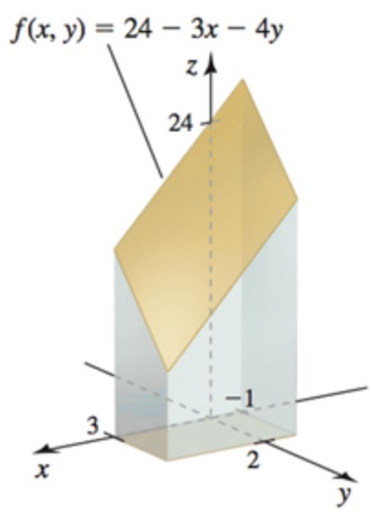
34. Find the volume of the solid beneath the cylinder $f(x, y) = e^{-x}$ and above the region $R = \{(x, y):\ 0 \le x \le \ln 4,\ \ -2 \le y \le 2\}$



35. Find the volume of the solid beneath the plane $f(x, y) = 6 - x - 2y$ and above the region $R = \{(x, y):\ 0 \le x \le 2,\ \ 0 \le y \le 1\}$



36. Find the volume of the solid beneath the plane $f(x, y) = 24 - 3x - 4y$ and above the region $R = \{(x, y):\ -1 \le x \le 3,\ \ 0 \le y \le 2\}$



37. Find the volume of the solid beneath the paraboloid $f(x, y) = 12 - x^2 - 2y^2$ and above the region $R = \{(x, y): \; 1 \le x \le 2, \; 0 \le y \le 1\}$