

## ***Solution***      **Section 2.5 – Variation of Parameters**

### ***Exercise***

$\{y_1(x) = e^{2x}, y_2(x) = e^{-3x}\}$  is a fundamental set of solutions of  $y'' + y' - 6y = 3e^{2x}$ .

Find a particular solution of the equation?

### **Solution**

$$W = \begin{vmatrix} e^{2x} & e^{-3x} \\ 2e^{2x} & -3e^{-3x} \end{vmatrix} = -3e^{-x} - 2e^{-x} = \underline{-5e^{-x} \neq 0}$$

$$v_1(x) = - \int \frac{e^{-3x}(3e^{2x})}{-5e^{-x}} dx = \frac{3}{5} \int dx = \underline{\frac{3}{5}x}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$v_2(x) = \int \frac{e^{2x}(3e^{2x})}{-5e^{-x}} dx = -\frac{3}{5} \int e^{5x} dx = \underline{-\frac{3}{25}e^{5x}}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

The particular solution:

$$\begin{aligned} y_p &= v_1 y_1 + v_2 y_2 \\ &= \frac{3}{5} x e^{2x} - \frac{3}{25} e^{-3x} e^{5x} \\ &= \frac{3}{5} x e^{2x} - \frac{3}{25} e^{2x} \end{aligned}$$

The general solution:

$$\begin{aligned} y(x) &= C_1 e^{2x} + C_2 e^{-3x} + \frac{3}{5} x e^{2x} - \frac{3}{25} e^{2x} \\ &= \left(C_1 - \frac{3}{25}\right) e^{2x} + C_2 e^{-3x} + \frac{3}{5} x e^{2x} \\ &= \underline{C_3 e^{2x} + C_2 e^{-3x} + \frac{3}{5} x e^{2x}} \end{aligned}$$

### ***Exercise***

Find a particular solution to:  $y'' - y = t + 3$

### **Solution**

The homogeneous equation for the differential equation  $y'' - y = 0$

$$\begin{aligned} \lambda^2 - 1 &= 0 && \text{Solve for } \lambda \\ \lambda_1 &= -1 && \lambda_2 = 1 \end{aligned}$$

Therefore;  $y_1 = e^{-t}$  and  $y_2 = e^t$

$$W = \begin{vmatrix} e^{-t} & e^t \\ -e^{-t} & e^t \end{vmatrix} = 1 + 1 = 2 \neq 0$$

$v_1' = \frac{-y_2}{y_1 y_2' - y_1' y_2} g(t)$ $= -\frac{e^t}{2}(t+3)$ $v_1(t) = -\frac{1}{2} \int (t+3)e^t dt \quad \begin{cases} u = t+3 & dv = e^t dt \\ du = dt & v = e^t \end{cases}$ $= -\frac{1}{2} \left[ e^t(t+3) - \int e^t dt \right]$ $= -\frac{1}{2} (te^t + 3e^t - e^t)$ $= -\frac{1}{2} (te^t + 2e^t)$ $= -\left( \frac{1}{2} te^t + e^t \right)$	$v_2' = \frac{y_1}{y_1 y_2' - y_1' y_2} g(t)$ $= \frac{e^{-t}}{2}(t+3)$ $v_2(t) = \frac{1}{2} \int (t+3)e^{-t} dt \quad \begin{cases} u = t+3 & dv = e^{-t} dt \\ du = dt & v = -e^{-t} \end{cases}$ $= \frac{1}{2} \left[ -e^{-t}(t+3) + \int e^{-t} dt \right]$ $= \frac{1}{2} (-te^{-t} - 3e^{-t} - e^{-t})$ $= -\frac{1}{2} (te^{-t} + 4e^{-t})$ $= -\frac{1}{2} te^{-t} - 2e^{-t}$
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$$y_p = v_1 y_1 + v_2 y_2$$

$$= -\left( \frac{1}{2} te^t + e^t \right) e^{-t} - \left( \frac{1}{2} te^{-t} + 2e^{-t} \right) e^t$$

$$= -\frac{1}{2} t - 1 - \frac{1}{2} t - 2$$

$$= \underline{\underline{-t-3}}$$

### Exercise

Find a particular solution to:  $y'' - 2y' + y = e^t$

### Solution

The homogeneous equation for the differential equation  $y'' - 2y' + y = 0$

$$\lambda^2 - 2\lambda + 1 = 0 \quad \text{Solve for } \lambda$$

$$\lambda_{1,2} = 1$$

Therefore;  $y_1 = e^t$  and  $y_2 = te^t$

$$W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix}$$

$$= e^{2t} + te^{2t} - te^{2t}$$

$$= e^{2t} \neq 0$$

$$\begin{aligned} v_1' &= \frac{-y_2}{y_1 y_2' - y_1' y_2} g(t) \\ &= -\frac{te^t}{e^t(e^t + te^t) - e^t \cdot te^t} e^t \\ &= -\frac{te^{2t}}{e^{2t}} \\ &= -t \end{aligned}$$

$$\begin{aligned} v_1(t) &= \int -t dt \\ &= -\frac{1}{2}t^2 \end{aligned}$$

$$\begin{aligned} v_2' &= \frac{y_1}{y_1 y_2' - y_1' y_2} g(t) \\ &= \frac{e^t}{e^{2t}} e^t \\ &= 1 \end{aligned}$$

$$\begin{aligned} v_2(t) &= \int 1 dt \\ &= t \end{aligned}$$

$$\begin{aligned} y_p &= v_1 y_1 + v_2 y_2 \\ &= -\frac{1}{2}t^2 e^t + t^2 e^t \\ &= \frac{1}{2}t^2 e^t \end{aligned}$$

### Exercise

Find a particular solution to:  $x'' - 4x' + 4x = e^{2t}$

### Solution

The homogeneous equation for the differential equation:  $x'' - 4x' + 4x = 0$

$$\lambda^2 - 4\lambda + 4 = 0 \quad \text{Solve for } \lambda$$

$$\boxed{\lambda_{1,2} = 2}$$

Therefore;  $x_1 = e^{2t}$  and  $x_2 = te^{2t}$

$$W = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t} + 2te^{2t} \end{vmatrix}$$

$$= e^{4t} + 2te^{4t} - 2te^{4t}$$

$$= e^{4t} \neq 0$$

$v_1' = \frac{-y_2}{W} g(t)$ $= -\frac{te^{2t}}{e^{4t}} e^{2t}$ $= -t$ $v_1(t) = \int -t \, dt$ $= -\frac{1}{2}t^2$	$v_2' = \frac{y_1}{W} g(t)$ $= \frac{e^{2t}}{e^{4t}} e^{2t}$ $= 1$ $v_2(t) = \int 1 \, dt = t$
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$$y_p = v_1 y_1 + v_2 y_2$$

$$= -\frac{1}{2}t^2 e^{2t} + t^2 e^{2t}$$

$$= \frac{1}{2}t^2 e^{2t}$$

### Exercise

Find a particular solution to:  $x'' + x = \tan^2 t$

### Solution

The homogeneous equation for the differential equation:  $x'' + x = 0$

$$x_1 = \cos t \quad \text{and} \quad x_2 = \sin t$$

$$W(\cos t, \sin t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}$$

$$= 1$$

$$x_p = v_1 x_1 + v_2 x_2$$

$$v_1' = \frac{-\sin t}{1} \tan^2 t$$

$$= -\sin t (\sec^2 t - 1)$$

$$= -\sin t \left( \frac{1}{\cos^2 t} - 1 \right)$$

$$= -\frac{\sin t}{\cos^2 t} + \sin t$$

$$= -\sec t \tan t + \sin t$$

$$v_1 = -\sec t - \cos t$$

$$\begin{aligned}
 v_2' &= \frac{\cos t}{1} \tan^2 t \\
 &= \cos t (\sec^2 t - 1) \\
 &= \sec t - \cos t
 \end{aligned}$$

$$v_2 = \ln|\sec t + \tan t| - \sin t$$

$$\begin{aligned}
 x_p &= (-\sec t - \cos t) \cos t + (\ln|\sec t + \tan t| - \sin t) \sin t \\
 &= -\sec t \cos t - \cos^2 t + \sin t \ln|\sec t + \tan t| - \sin^2 t \\
 &= -\sec t \frac{1}{\sec t} - (\cos^2 t + \sin^2 t) + \sin t \ln|\sec t + \tan t| \\
 &= \underline{-2 + \sin t \ln|\sec t + \tan t|}
 \end{aligned}$$

### Exercise

Find a particular solution to the given second-order differential equation  $y'' + 25y = -2 \tan(5x)$

### Solution

$$\lambda^2 + 25 = 0 \Rightarrow \lambda_{1,2} = \pm 5i$$

$$y_p = C_1 \cos 5x + C_2 \sin 5x$$

$$W = \begin{vmatrix} \cos 5x & \sin 5x \\ -5 \sin 5x & 5 \cos 5x \end{vmatrix} = 5 \cos^2 5x + 5 \sin^2 5x = \underline{5 \neq 0}$$

$$v_1(x) = - \int \frac{\sin 5x(-2 \tan 5x)}{5} dx$$

$$= \frac{2}{5} \int \frac{\sin^2 5x}{\cos 5x} dx$$

$$= \frac{2}{5} \int \frac{1 - \cos^2 5x}{\cos 5x} dx$$

$$= \frac{2}{5} \int (\sec 5x - \cos 5x) dx$$

$$= \frac{2}{5} \left[ \frac{1}{5} \ln|\tan 5x + \sec 5x| - \frac{1}{5} \sin 5x \right]$$

$$= \frac{2}{25} (\ln|\tan 5x + \sec 5x| - \sin 5x)$$

$$v_2(x) = \int \frac{\cos 5x(-2 \tan 5x)}{5} dx = -\frac{2}{5} \int \sin 5x dx = \underline{\frac{2}{25} \cos 5x}$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$= \frac{2}{25} (\ln|\tan 5x + \sec 5x| - \sin 5x) (\cos 5x) + \frac{2}{25} \cos 5x \sin 5x$$

$$= \underline{\frac{2}{25} \ln|\tan 5x + \sec 5x|}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

### Exercise

Find a particular solution to the given second-order differential equation  $y'' - 6y' + 9y = 5e^{3x}$

### Solution

$$\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0 \Rightarrow \underline{\lambda_{1,2} = 3}$$

$$y_h = (C_1 + C_2 x)e^{3x}$$

$$W = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{6x} + 3xe^{6x} - 3xe^{6x} = \underline{e^{6x} \neq 0}$$

$$v_1(x) = - \int \frac{xe^{3x}(5e^{3x})}{e^{6x}} dx = -5 \int x dx = \underline{-\frac{5}{2}x^2}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$v_2(x) = \int \frac{e^{3x}(5e^{3x})}{e^{6x}} dx = 5 \int dx = \underline{5x}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$y_p = -\frac{5}{2}x^2 e^{3x} + 5x^2 e^{3x}$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$= \underline{\frac{5}{2}x^2 e^{3x}}$$

### Exercise

Find a particular solution to the given second-order differential equation  $y'' + 4y = 2\cos 2x$

### Solution

$$\lambda^2 + 4 = 0 \Rightarrow \underline{\lambda_{1,2} = \pm 2i}$$

$$y_h = C_1 \cos 2x + C_2 \sin 2x$$

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = \underline{2 \neq 0}$$

$$v_1(x) = - \int \frac{\sin 2x(2\cos 2x)}{2} dx = -\frac{1}{2} \int \sin 4x dx = \frac{1}{8} \cos 4x$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$v_2(x) = \int \frac{\cos 2x(2\cos 2x)}{2} dx$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \int \cos^2 2x dx$$

$$= \frac{1}{2} \int (1 + \cos 4x) dx$$

$$= \frac{1}{2} \left( x + \frac{1}{4} \sin 4x \right)$$

$$\underline{y_p = \frac{1}{2} x \sin 2x}$$

$$y_p = v_1 y_1 + v_2 y_2$$

### Exercise

Find a particular solution to the given second-order differential equation  $y'' - 5y' + 6y = 4e^{2x} + 3$

### Solution

$$\lambda^2 - 5\lambda + 6 = 0 \Rightarrow \underline{\lambda_{1,2} = \frac{5 \pm \sqrt{1}}{2} = 2, 3}$$

$$y_h = C_1 e^{2x} + C_2 e^{3x}$$

$$W = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = \underline{e^{5x} \neq 0}$$

$$v_1(x) = - \int \frac{e^{3x}(4e^{2x} + 3)}{e^{5x}} dx = - \int (4 + 3e^{-2x}) dx = -4x + \frac{3}{2} e^{-2x}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$v_2(x) = \int \frac{e^{2x}(4e^{2x} + 3)}{e^{5x}} dx = \int (4e^{-x} + 3e^{-3x}) dx = -4e^{-x} - e^{-3x}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$y_p = \left( -4x + \frac{3}{2} e^{-2x} \right) e^{2x} - \left( 4e^{-x} + e^{-3x} \right) e^{3x}$$

$$= -4xe^{2x} + \frac{3}{2} - 4e^{2x} - 1$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$\underline{= -4xe^{2x} - 4e^{2x} + \frac{1}{2}}$$

### Exercise

Verify that  $y_1(t) = t$  and  $y_2(t) = t^{-3}$  are solution to the homogenous equation

$$t^2 y''(t) + 3ty'(t) - 3y(t) = 0$$

### Solution

The homogeneous equation for the differential equation:  $y'' + \frac{3}{t} y' - \frac{3}{t^2} y = 0$

For  $y_1 = t \rightarrow y'_1 = 1 \rightarrow y''_1 = 0$

$$y'' + \frac{3}{t} y' - \frac{3}{t^2} y = 0 + \frac{3}{t}(1) - \frac{3}{t^2} t$$

$$= \frac{3}{t} - \frac{3}{t}$$

$$= 0$$

$y_1(t)$  is a solution

$$\text{For } y_2 = t^{-3} \rightarrow y_1' = -3t^{-4} \rightarrow y_1'' = 12t^{-5}$$

$$\begin{aligned} y'' + \frac{3}{t}y' - \frac{3}{t^2}y &= 12t^{-5} + \frac{3}{t}(-3t^{-4}) - \frac{3}{t^2}t^{-3} \\ &= 12t^{-5} - 9t^{-5} - 3t^{-5} \\ &= 0 \end{aligned}$$

$y_2(t)$  is a solution

$$\text{Wronskian: } W(t, t^{-3}) = \begin{vmatrix} t & t^{-3} \\ 1 & -3t^{-4} \end{vmatrix} = -4t^{-3}$$

$$v_1' = -\frac{t^{-3}t^{-3}}{-4t^{-3}} = \frac{1}{4}t^{-3} \Rightarrow v_1 = \int \left(\frac{1}{4}t^{-3}\right)dt = -\frac{1}{8}t^{-2}$$

$$v_2' = -\frac{t \cdot t^{-3}}{-4t^{-3}} = -\frac{1}{4}t \Rightarrow v_2 = \int \left(-\frac{1}{4}t\right)dt = -\frac{1}{8}t^2$$

$$\begin{aligned} y_p &= v_1 y_1 + v_2 y_2 \\ &= -\frac{1}{8}t^{-2}t - \frac{1}{8}t^2t^{-3} \\ &= -\frac{1}{8}t^{-1} - \frac{1}{8}t^{-1} \\ &= -\frac{1}{4}t^{-1} \end{aligned}$$

$$\text{Thus, the general solution is: } y(t) = C_1 t + \frac{C_2}{t^3} - \frac{1}{4t}$$

### Exercise

Find the general solution  $y'' - y = \frac{1}{x}$

### Solution

$$\text{Characteristic Eqn.: } \lambda^2 - 1 = 0 \Rightarrow \lambda_{1,2} = \pm 1$$

$$\text{The homogeneous Eqn.: } y_h = C_1 e^{-x} + C_2 e^x$$

$$W = \begin{vmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{vmatrix} = 1 + 1 = 2 \neq 0$$

$$v_1(x) = -\int \frac{e^x \frac{1}{x}}{2} dx = -\frac{1}{2} \int \frac{e^x}{x} dx$$

$$v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$



$$v_2(x) = \int -\frac{e^{-x}}{2} dx = \frac{1}{2} \int \frac{e^{-x}}{x} dx$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$y_p = -\frac{1}{2} e^{-x} \int \frac{e^x}{x} dx + \frac{1}{2} e^x \int \frac{e^{-x}}{x} dx$$

$$y_p = u_1 y_1 + u_2 y_2$$

The **general** solution: 
$$\underline{y(x) = C_1 e^{-x} + C_2 e^x - \frac{1}{2} e^{-x} \int \frac{e^x}{x} dx + \frac{1}{2} e^x \int \frac{e^{-x}}{x} dx}$$

## Exercise

Find the general solution  $y'' - y = \sinh 2x$

### Solution

Characteristic Eqn.:  $\lambda^2 - 1 = 0 \rightarrow \underline{\lambda_{1,2} = \pm 1}$

$$\underline{y_h = C_1 e^{-x} + C_2 e^x}$$

$$W = \begin{vmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{vmatrix} = 1 + 1 = 2 \neq 0$$

$$\begin{aligned} v_1(x) &= - \int \frac{e^x \sinh 2x}{2} dx \\ &= -\frac{1}{4} \int e^x (e^{2x} - e^{-2x}) dx \\ &= -\frac{1}{4} \int (e^{3x} - e^{-x}) dx \\ &= -\frac{1}{4} \left( \frac{1}{3} e^{3x} + e^{-x} \right) \end{aligned}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned} v_2(x) &= \int \frac{e^{-x} \sinh 2x}{2} dx = \frac{1}{4} \int e^{-x} (e^{2x} - e^{-2x}) dx \\ &= \frac{1}{4} \int (e^x - e^{-3x}) dx \\ &= \frac{1}{4} \left( e^x + \frac{1}{3} e^{-3x} \right) \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$\begin{aligned} y_p &= \left( -\frac{1}{12} e^{3x} - \frac{1}{4} e^{-x} \right) e^{-x} + \left( \frac{1}{4} e^x + \frac{1}{12} e^{-3x} \right) e^x \\ &= -\frac{1}{12} e^{2x} - \frac{1}{4} e^{-2x} + \frac{1}{4} e^{2x} + \frac{1}{12} e^{-x} \\ &= -\frac{1}{4} e^{-2x} + \frac{1}{6} e^{2x} + \frac{1}{12} e^{-x} \end{aligned}$$

$$y_p = u_1 y_1 + u_2 y_2$$

The **general** solution: 
$$\underline{y(x) = C_1 e^{-x} + C_2 e^x - \frac{1}{4} e^{-2x} + \frac{1}{6} e^{2x} + \frac{1}{12} e^{-x}}$$

### Exercise

Find the general solution  $y'' - y = x$

### Solution

Characteristic Eqn.:  $\lambda^2 - 1 = 0 \rightarrow \lambda_{1,2} = \pm 1$

$$y_h = C_1 e^{-x} + C_2 e^x$$

$$W = \begin{vmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{vmatrix} = 1 + 1 = 2 \neq 0$$

$$v_1(x) = - \int \frac{e^x x}{2} dx$$

$$= - \frac{1}{2} (x-1) e^x$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

		$\int e^x$
+	$x$	$e^x$
-	1	$e^x$

$$v_2(x) = \int \frac{e^{-x} x}{2} dx$$

$$= - \frac{1}{2} (x+1) e^{-x}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

		$\int e^{-x}$
+	$x$	$-e^{-x}$
-	1	$e^{-x}$

$$y_p = - \frac{1}{2} (x-1) e^x e^{-x} - \frac{1}{2} (x+1) e^x e^{-x}$$

$$= -x$$

$$y_p = u_1 y_1 + u_2 y_2$$

The general solution:  $y(x) = C_1 e^{-x} + C_2 e^x - x$

### Exercise

Find the general solution  $y'' - y = \cosh x$

### Solution

Characteristic Eqn.:  $\lambda^2 - 1 = 0 \rightarrow \lambda_{1,2} = \pm 1$

$$y_h = A_1 e^{-x} + A_2 e^x$$

$$W = \begin{vmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{vmatrix} = 1 + 1 = 2 \neq 0$$

$$v_1(x) = - \int \frac{e^x \cosh x}{2} dx$$

$$= - \frac{1}{2} \int e^x \frac{e^x + e^{-x}}{2} dx$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned}
&= -\frac{1}{4} \int (e^{2x} + 1) dx \\
&= -\frac{1}{4} \left( \frac{1}{2} e^{2x} + x \right) \\
&= -\frac{1}{8} e^{2x} - \frac{1}{4} x \Big|
\end{aligned}$$

$$\begin{aligned}
v_2(x) &= \int \frac{e^{-x} \cosh x}{2} dx \\
&= \frac{1}{4} \int e^{-x} (e^x + e^{-x}) dx \\
&= \frac{1}{4} \int (1 + e^{-2x}) dx \\
&= \frac{1}{4} x - \frac{1}{8} e^{-2x} \Big|
\end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$\begin{aligned}
y_p &= \left( -\frac{1}{8} e^{2x} - \frac{1}{4} x \right) e^{-x} + \left( \frac{1}{4} x - \frac{1}{8} e^{-2x} \right) e^x \\
&= -\frac{1}{8} e^x - \frac{1}{4} x e^{-x} + \frac{1}{4} x e^x - \frac{1}{8} e^{-x} \\
&= -\frac{1}{8} e^x - \frac{1}{8} e^{-x} + \frac{1}{2} x \left( \frac{e^x - e^{-x}}{2} \right) \\
&= -\frac{1}{8} e^x - \frac{1}{8} e^{-x} + \frac{1}{2} x \sinh x \Big|
\end{aligned}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$\begin{aligned}
y(x) &= A_1 e^{-x} + A_2 e^x - \frac{1}{8} e^x - \frac{1}{8} e^{-x} + \frac{1}{2} x \sinh x \\
&= \left( A_1 - \frac{1}{8} \right) e^{-x} + \left( A_2 - \frac{1}{8} \right) e^x + \frac{1}{2} x \sinh x
\end{aligned}$$

The **general** solution:  $y(x) = C_1 e^{-x} + C_2 e^x + \frac{1}{2} x \sinh x \Big|$

**Or**  $y(x) = C_1 e^{-x} + C_2 e^x - \frac{1}{4} x e^{-x} + \frac{1}{4} x e^x \Big|$

## Exercise

Find the general solution  $y'' + y = \sin x$

### Solution

Characteristic Eqn.:  $\lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$

The homogeneous Eqn.:  $y_h = C_1 \cos x + C_2 \sin x \Big|$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0 \Big|$$

$$\begin{aligned} v_1(x) &= - \int \frac{(\sin x)(\sin x)}{1} dx \\ &= - \int (\sin^2 x) dx \\ &= - \frac{x}{2} + \frac{1}{4} \sin 2x \end{aligned}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned} v_2(x) &= \int \frac{(\cos x)(\sin x)}{1} dx \\ &= \int \sin x d(\sin x) \\ &= \frac{1}{2} \sin^2 x \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$\begin{aligned} y_p &= \left( -\frac{x}{2} + \frac{1}{4} \sin 2x \right) \cos x + \frac{1}{2} \sin^2 x \sin x \\ &= -\frac{1}{2} x \cos x + \frac{1}{4} \sin 2x \cos x + \frac{1}{2} \sin^3 x \\ &= -\frac{1}{2} x \cos x + \frac{1}{2} \sin x \cos^2 x + \frac{1}{2} \sin^3 x \\ &= -\frac{1}{2} x \cos x + \frac{1}{2} \sin x (\cos^2 x + \sin^2 x) \\ &= -\frac{1}{2} x \cos x + \frac{1}{2} \sin x \end{aligned}$$

$$y_p = u_1 y_1 + u_2 y_2$$

The **general** solution: 
$$y(x) = C_1 \cos x + C_2 \sin x - \frac{1}{2} x \cos x + \frac{1}{2} \sin x$$

## Exercise

Find the general solution  $y'' - y = e^x$

### Solution

Characteristic Eqn.:  $\lambda^2 - 1 = 0 \rightarrow \lambda_{1,2} = \pm 1$

$$y_h = C_1 e^{-x} + C_2 e^x$$

$$W = \begin{vmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{vmatrix} = 1 + 1 = 2 \neq 0$$

$$\begin{aligned} v_1(x) &= - \int \frac{e^x e^x}{2} dx \\ &= - \frac{1}{4} e^{2x} \end{aligned}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$v_2(x) = \int \frac{e^{-x} e^x}{2} dx$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \frac{1}{2}x$$

$$y_p = -\frac{1}{4}e^{2x}e^{-x} + \frac{1}{2}xe^x$$

$$= -\frac{1}{4}e^x + \frac{1}{2}xe^x$$

$$y_p = u_1 y_1 + u_2 y_2$$

The **general** solution:  $y(x) = C_1 e^{-x} + C_2 e^x - \frac{1}{4}e^x + \frac{1}{2}xe^x$

### Exercise

Find the general solution  $y'' + y = \sec x$

### Solution

Characteristic Eqn.:  $\lambda^2 + 1 = 0 \rightarrow \lambda_{1,2} = \pm i$

$$y_h = C_1 \cos x + C_2 \sin x$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0$$

$$v_1(x) = -\int \sin x \sec x \, dx$$

$$= -\int \tan x \, dx$$

$$= -\ln|\sec x|$$

$$v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$v_2(x) = \int \cos x \sec x \, dx$$

$$= x$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$y_p = -\cos x \ln|\sec x| + x \sin x$$

$$y_p = u_1 y_1 + u_2 y_2$$

The **general** solution:

$$y(x) = C_1 \cos x + C_2 \sin x - \cos x \ln|\sec x| + x \sin x$$

### Exercise

Find the general solution  $y'' + y = \tan x$

### Solution

Characteristic Eqn.:  $\lambda^2 + 1 = 0 \rightarrow \lambda_{1,2} = \pm i$

$$y_h = C_1 \cos x + C_2 \sin x$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = \underline{1 \neq 0}$$

$$\begin{aligned} v_1(x) &= - \int \sin x \tan x \, dx \\ &= - \int \frac{\sin^2 x}{\cos x} \, dx \\ &= - \int \frac{1 - \cos^2 x}{\cos x} \, dx \\ &= - \int (\sec x - \cos x) \, dx \\ &= -\ln|\sec x + \tan x| + \sin x \end{aligned}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} \, dx$$

$$\begin{aligned} v_2(x) &= \int \cos x \tan x \, dx \\ &= \int \sin x \, dx \\ &= -\cos x \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} \, dx$$

$$\begin{aligned} y_p &= (-\ln|\sec x + \tan x| + \sin x) \cos x - \cos x \sin x \\ &= -(\cos x) \ln|\sec x + \tan x| \end{aligned}$$

$$y_p = u_1 y_1 + u_2 y_2$$

The **general** solution:

$$\underline{y(x) = C_1 \cos x + C_2 \sin x - (\cos x) \ln|\sec x + \tan x|}$$

## Exercise

Find the general solution  $y'' + y = \sin x$

### Solution

$$\text{Characteristic Eqn.: } \lambda^2 + 1 = 0 \rightarrow \underline{\lambda_{1,2} = \pm i}$$

$$\underline{y_h = C_1 \cos x + C_2 \sin x}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = \underline{1 \neq 0}$$

$$\begin{aligned} v_1(x) &= - \int \sin^2 x \, dx \\ &= -\frac{1}{2} \int (1 - \cos 2x) \, dx \end{aligned}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} \, dx$$

$$= -\frac{1}{2} \left( x - \frac{1}{2} \sin 2x \right) \Big|$$

$$\begin{aligned} v_2(x) &= \int \cos x \sin x \, dx \\ &= \frac{1}{2} \int \sin 2x \, dx \\ &= -\frac{1}{4} \cos 2x \Big| \end{aligned}$$

$$\begin{aligned} y_p &= \left( -\frac{1}{2}x + \frac{1}{4} \sin 2x \right) \cos x - \frac{1}{4} \cos 2x \sin x \\ &= -\frac{1}{2}x \cos x \Big| \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$y_p = u_1 y_1 + u_2 y_2$$

The **general** solution:  $y(x) = C_1 \cos x + C_2 \sin x - \frac{1}{2}x \cos x \Big|$

### Exercise

Find the general solution  $y'' + y = \csc x$

### Solution

Characteristic Eqn.:  $\lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$

$$y_h = C_1 \cos x + C_2 \sin x \Big|$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0 \Big|$$

$$\begin{aligned} v_1(x) &= - \int \frac{\sin x \csc x}{1} dx \\ &= - \int dx \\ &= -x \Big| \end{aligned}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned} v_2(x) &= \int \cos x \csc x \, dx \\ &= \int \cot x \, dx \\ &= \ln |\sin x| \Big| \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$y_p = -x \cos x + \sin x \ln |\sin x| \Big|$$

$$y_p = u_1 y_1 + u_2 y_2$$

The **general** solution:

$$y(x) = C_1 \cos x + C_2 \sin x - x \cos x + \sin x \ln |\sin x| \Big|$$

### Exercise

Find the general solution  $y'' + y = \cos^2 x$

### Solution

Characteristic Eqn.:  $\lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$

$$\underline{y_h = C_1 \cos x + C_2 \sin x}$$

$$\begin{aligned} W &= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \\ &= \cos^2 x + \sin^2 x \\ &= \underline{1 \neq 0} \end{aligned}$$

$$\begin{aligned} v_1(x) &= - \int \frac{\sin x \cos^2 x}{1} dx \\ &= \int \cos^2 x d(\cos x) \\ &= \underline{\frac{1}{3} \cos^3 x} \end{aligned}$$

$$\begin{aligned} v_2(x) &= \int \cos x \cos^2 x dx \\ &= \int \cos^3 x dx \\ &= \int \cos^2 x d(\sin x) \\ &= \int (1 - \sin^2 x) d(\sin x) \\ &= \underline{\sin x - \frac{1}{3} \sin^3 x} \end{aligned}$$

$$\begin{aligned} y_p &= \frac{1}{3} \cos^4 x + \sin^2 x - \frac{1}{3} \sin^4 x \\ &= \frac{1}{3} (\cos^4 x - \sin^4 x) + \sin^2 x \\ &= \frac{1}{3} (\cos^2 x - \sin^2 x) (\cos^2 x + \sin^2 x) + \sin^2 x \\ &= \underline{\frac{1}{3} \cos 2x + \sin^2 x} \end{aligned}$$

The **general** solution:

$$\underline{y(x) = C_1 \cos x + C_2 \sin x + \frac{1}{3} \cos 2x + \sin^2 x}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$y_p = u_1 y_1 + u_2 y_2$$



### Exercise

Find the general solution to the given differential equation.  $y'' + y = \csc^2 x$

### Solution

$$\text{Characteristic Eqn.: } \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$$

$$y_h = C_1 \cos x + C_2 \sin x$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0$$

$$\begin{aligned} v_1(x) &= - \int \sin x \csc^2 x \, dx \\ &= - \int \frac{1}{\sin x} \, dx \\ &= - \ln |\csc x - \cot x| \end{aligned}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} \, dx$$

$$\begin{aligned} v_2(x) &= \int \cos x \csc^2 x \, dx \\ &= \int \frac{1}{\sin^2 x} d(\sin x) \\ &= - \frac{1}{\sin x} \\ &= - \csc x \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} \, dx$$

$$\begin{aligned} y_p &= -\cos x \ln |\csc x - \cot x| - \csc x \sin x \\ &= -\cos x \ln |\csc x - \cot x| - 1 \end{aligned}$$

$$y_p = v_1 y_1 + v_2 y_2$$

The **general** solution:

$$y(x) = C_1 \cos x + C_2 \sin x - \cos x \ln |\csc x - \cot x| - 1$$

### Exercise

Find the general solution to the given differential equation.  $y'' + y = \sec^2 x$

### Solution

$$\text{Characteristic Eqn.: } \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$$

$$y_h = C_1 \cos x + C_2 \sin x$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0$$

$$\begin{aligned}
 v_1(x) &= -\int \sin x \sec^2 x \, dx \\
 &= \int \frac{1}{\cos^2 x} d(\cos x) \\
 &= -\frac{1}{\cos x} \\
 &= \underline{-\sec x}
 \end{aligned}$$

$$v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned}
 v_2(x) &= \int \cos x \sec^2 x \, dx \\
 &= \int \sec x \, dx \\
 &= \underline{\ln|\sec x + \tan x|}
 \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$\begin{aligned}
 y_p &= \cos x(-\sec x) + \sin x \ln|\sec x + \tan x| \\
 &= \underline{-1 + \sin x \ln|\sec x + \tan x|}
 \end{aligned}$$

$$y_p = v_1 y_1 + v_2 y_2$$

The **general** solution:

$$y(x) = \underline{C_1 \cos x + C_2 \sin x - \cos x(-\sec x) + \sin x \ln|\sec x + \tan x|}$$

## Exercise

Find the general solution to the given differential equation.

$$y'' + y = \sec x \tan x$$

## Solution

$$\text{Characteristic Eqn.: } \lambda^2 + 1 = 0 \Rightarrow \underline{\lambda_{1,2} = \pm i}$$

$$y_h = \underline{C_1 \cos x + C_2 \sin x}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = \underline{1 \neq 0}$$

$$\begin{aligned}
 v_1(x) &= -\int \sin x \sec x \tan x \, dx \\
 &= -\int \tan^2 x \, dx \\
 &= \int (1 - \sec^2 x) \, dx \\
 &= \underline{x - \tan x}
 \end{aligned}$$

$$v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$v_2(x) = \int \cos x \sec x \tan x \, dx$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \int \tan x \, dx$$

$$= \ln |\sec x|$$

$$y_p = \cos x (x - \tan x) + \sin x \ln |\sec x|$$

$$= x \cos x - \sin x + \sin x \ln |\sec x|$$

$$y_p = v_1 y_1 + v_2 y_2$$

The **general** solution:

$$y(x) = C_1 \cos x + C_3 \sin x + x \cos x + \sin x \ln |\sec x|$$

### Exercise

Find the general solution to the given differential equation.  $y'' + y' = x$

### Solution

$$\text{Characteristic Eqn.: } \lambda^2 + \lambda = 0 \Rightarrow \lambda_{1,2} = 0, -1$$

$$y_h = C_1 + C_2 e^{-x}$$

$$W = \begin{vmatrix} 1 & e^{-x} \\ 0 & -e^{-x} \end{vmatrix} = -e^{-x} \neq 0$$

$$v_1(x) = \int \frac{e^{-x} x}{e^{-x}} dx$$

$$= \int x \, dx$$

$$= \frac{1}{2} x^2$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$v_2(x) = \int \frac{x}{e^{-x}} dx = \frac{1}{2} x^2$$

$$= \int x e^x \, dx$$

$$= (x-1) e^x$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$y_p = \frac{1}{2} x^2 + x - 1$$

$$y_p = v_1 y_1 + v_2 y_2$$

The **general** solution:

$$y(x) = C_1 + C_2 e^{-x} + \frac{1}{2} x^2 + x - 1$$

$$= C_3 + C_2 e^{-x} + \frac{1}{2} x^2 + x$$

### Exercise

Find the general solution to the given differential equation.

$$y'' - y' = e^x \cos x$$

### Solution

Characteristic Eqn.:  $\lambda^2 - \lambda = 0 \Rightarrow \lambda_{1,2} = 0, 1$

$$y_h = C_1 + C_2 e^x$$

$$W = \begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix} = e^x \neq 0$$

		$\int \cos x$
+	$x$	$\sin x$
-	1	$-\cos x$

$$v_1(x) = - \int \frac{e^x e^x \cos x}{e^x} dx$$

$$= - \int e^x \cos x dx$$

$$\int e^x \cos x dx = (\sin x + \cos x) e^x - \int e^x \cos x dx$$

$$2 \int e^x \cos x dx = (\sin x + \cos x) e^x$$

$$= -\frac{1}{2} (\sin x + \cos x) e^x$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

		$\int \cos x$
+	$e^x$	$\sin x$
-	$e^x$	$-\cos x$
+	$e^x$	$-\int \cos x$

$$v_2(x) = \int \frac{e^x \cos x}{e^x} dx$$

$$= \int \cos x dx$$

$$= \sin x$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$y_p = -\frac{1}{2} (\sin x + \cos x) e^x + \sin x e^x$$

$$= \cos x e^x + \frac{1}{2} \sin x e^x$$

$$y_p = v_1 y_1 + v_2 y_2$$

The **general** solution:

$$y(x) = C_1 + C_2 e^x + \cos x e^x + \frac{1}{2} \sin x e^x$$

### Exercise

Find the general solution to the given differential equation.

$$y'' + y' - 2y = x e^x$$

### Solution

Characteristic Eqn.:  $\lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda_{1,2} = -2, 1$

$$y_h = C_1 e^{-2x} + C_2 e^x$$

$$W = \begin{vmatrix} e^{-2x} & e^x \\ -2e^{-2x} & e^x \end{vmatrix} = 3e^{-x} \neq 0$$

$$\begin{aligned} v_1(x) &= - \int \frac{xe^xe^x}{3e^{-x}} dx \\ &= -\frac{1}{3} \int xe^{3x} dx \\ &= -\frac{1}{3} \left( \frac{1}{3}x - \frac{1}{9} \right) e^{3x} \end{aligned}$$

		$\int e^{3x}$
+	$x$	$\frac{1}{3}e^{3x}$
-	$1$	$\frac{1}{9}e^{3x}$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned} v_2(x) &= \int \frac{xe^xe^{-2x}}{3e^{-x}} dx \\ &= \frac{1}{3} \int x dx \\ &= \frac{1}{3} x^2 \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$\begin{aligned} y_p &= e^{-2x} \left( \frac{1}{27} - \frac{1}{9}x \right) e^{3x} + \frac{1}{3} x^2 e^x \\ &= \left( \frac{1}{27} - \frac{1}{9}x + \frac{1}{3}x^2 \right) e^x \end{aligned}$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$\begin{aligned} y(x) &= C_1 e^{-2x} + C_2 e^x + \frac{1}{27} e^x - \frac{1}{9} x e^x + \frac{1}{3} x^2 e^x \\ &= \underline{C_1 e^{-2x} + C_3 e^x - \frac{1}{9} x e^x + \frac{1}{3} x^2 e^x} \end{aligned}$$

## Exercise

Find the general solution  $y'' + y' - 2y = e^{3x}$

## Solution

$$\text{Characteristic Eqn.: } \lambda^2 + \lambda - 2 = 0 \Rightarrow \underline{\lambda_{1,2} = 1, -2}$$

$$y_h = \underline{C_1 e^{-2x} + C_2 e^x}$$

$$W = \begin{vmatrix} e^{-2x} & e^x \\ -2e^{-2x} & e^x \end{vmatrix} = 3e^{-x} \neq 0$$

$$\begin{aligned} v_1(x) &= - \int \frac{e^x e^{3x}}{3e^{-x}} dx \\ &= -\frac{1}{3} \int e^{5x} dx \end{aligned}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$\left. = -\frac{1}{15}e^{5x} \right|$$

$$\begin{aligned} v_2(x) &= \int \frac{e^{-2x}e^{3x}}{3e^{-x}} dx \\ &= \frac{1}{3} \int e^{2x} dx \\ &= \frac{1}{6}e^{2x} \end{aligned}$$

$$\begin{aligned} y_p &= -\frac{1}{15}e^{-2x}e^{5x} + \frac{1}{6}e^xe^{2x} \\ &= -\frac{1}{15}e^{3x} + \frac{1}{6}e^{3x} \\ &= \frac{1}{10}e^{3x} \end{aligned}$$

$$\underline{y(x) = C_1e^{-2x} + C_2e^x + \frac{1}{10}e^{3x}}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$y_p = v_1 y_1 + v_2 y_2$$

## Exercise

Find the general solution to the given differential equation.  $y'' + 2y' + y = e^{-x} \ln x$

## Solution

$$\text{Characteristic Eqn.: } \lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda_{1,2} = -1$$

$$\underline{y_h = C_1 e^{-x} + C_2 x e^{-x}}$$

$$W = \begin{vmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & e^{-x} - x e^{-x} \end{vmatrix} = e^{-2x} - x e^{-2x} + x e^{-2x} = e^{-2x} \neq 0$$

$$\begin{aligned} v_1(x) &= - \int \frac{x e^{-x}}{e^{-2x}} (e^{-x} \ln x) dx \\ &= - \int (x \ln x) dx \\ &= \frac{1}{4}x^2 - \frac{1}{2}x^2 \ln x \end{aligned}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$u = \ln x \quad dv = x$$

$$du = \frac{1}{x} dx \quad v = \int x dx = \frac{1}{2}x^2$$

$$\int x \ln x dx = \frac{1}{2}x^2 \ln x - \frac{1}{2} \int x dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2$$

$$v_2(x) = \int \frac{e^{-x}}{e^{-2x}} (e^{-x} \ln x) dx = \int (\ln x) dx = x \ln x - x$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$u = \ln x \quad dv = 1$$

$$du = \frac{1}{x} dx \quad v = \int dx = x \rightarrow \int \ln x dx = x \ln x - x$$

$$\begin{aligned}
 y_p &= e^{-x} \left( \frac{1}{4}x^2 - \frac{1}{2}x^2 \ln x \right) + xe^{-x}(x \ln x - x) \\
 &= \frac{1}{4}x^2 e^{-x} - \frac{1}{2}x^2 e^{-x} \ln x + x^2 e^{-x} \ln x - x^2 e^{-x} \\
 &= \frac{1}{2}x^2 e^{-x} \ln x - \frac{3}{4}x^2 e^{-x} \Big|
 \end{aligned}$$

$$y_p = u_1 y_1 + u_2 y_2$$

The **general** solution:

$$\underline{y(x) = C_1 e^{-x} + C_2 x e^{-x} + \frac{1}{2}x^2 e^{-x} \ln x - \frac{3}{4}x^2 e^{-x}}$$

## Exercise

Find the general solution to the given differential equation.

$$y'' - 2y' + y = \frac{e^x}{1+x^2}$$

## Solution

$$\text{Characteristic Eqn.: } \lambda^2 - 2\lambda + 1 = 0 \Rightarrow \underline{\lambda_{1,2} = 1}$$

$$\underline{y_h = C_1 e^x + C_2 x e^x}$$

$$\begin{aligned}
 W &= \begin{vmatrix} e^x & x e^x \\ e^x & e^x + x e^x \end{vmatrix} \\
 &= e^{2x} + x e^{2x} - x e^{2x} \\
 &= e^{2x} \neq 0 \Big|
 \end{aligned}$$

$$\begin{aligned}
 v_1(x) &= - \int \frac{x e^x}{e^{2x}} \left( \frac{e^x}{1+x^2} \right) dx = - \int \frac{x}{1+x^2} dx \\
 &= - \frac{1}{2} \int \frac{1}{1+x^2} d(1+x^2) \\
 &= - \frac{1}{2} \ln(1+x^2) \Big|
 \end{aligned}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned}
 v_2(x) &= \int \frac{e^x}{e^{2x}} \left( \frac{e^x}{1+x^2} \right) dx \\
 &= \int \frac{dx}{1+x^2} \\
 &= \tan^{-1} x \Big|
 \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$\underline{y_p = -\frac{1}{2}e^x \ln(1+x^2) + x e^x \tan^{-1} x}$$

$$y_p = u_1 y_1 + u_2 y_2$$

The **general** solution:  $\underline{y(x) = C_1 e^x + C_2 x e^x - \frac{1}{2}e^x \ln(1+x^2) + x e^x \tan^{-1} x}$

### Exercise

Find the general solution  $y'' + 2y' + y = e^{-x}$

### Solution

$$\text{Characteristic Eqn.: } \lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda_{1,2} = -1$$

$$\text{The homogeneous Eqn.: } y_h = C_1 e^{-x} + C_2 x e^{-x}$$

$$W = \begin{vmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix} = e^{-2x} \neq 0$$

$$\begin{aligned} v_1(x) &= - \int \frac{x e^{-x} e^{-x}}{e^{-2x}} dx \\ &= - \int x dx \\ &= -\frac{1}{2} x^2 \end{aligned}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned} v_2(x) &= \int \frac{e^{-x} e^{-x}}{e^{-2x}} dx \\ &= \int dx \\ &= x \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$y_p = -\frac{1}{2} x^2 e^{-x} + x^2 e^{-x} = \frac{1}{2} x^2 e^{-x}$$

$$y_p = u_1 y_1 + u_2 y_2$$

The **general** solution:

$$y(x) = \left( C_1 + C_2 x + \frac{1}{2} x^2 \right) e^{-x}$$

### Exercise

Find the general solution  $y'' - 2y' - 8y = 3e^{-2x}$

### Solution

$$\text{Characteristic Eqn.: } \lambda^2 - 2\lambda - 8 = 0 \Rightarrow \lambda_{1,2} = -2, 4$$

$$\text{The homogeneous Eqn.: } y_h = C_1 e^{-2x} + C_2 e^{4x}$$

$$\begin{aligned} W &= \begin{vmatrix} e^{-2x} & e^{4x} \\ -2e^{-2x} & 4e^{4x} \end{vmatrix} \\ &= 4e^{2x} + 2e^{2x} \end{aligned}$$



$$= 6e^{2x} \neq 0$$

$$\begin{aligned} v_1(x) &= - \int \frac{3e^{2x}}{6e^{2x}} dx \\ &= -\frac{1}{2} \int dx \\ &= -\frac{1}{2}x \end{aligned}$$

$$\begin{aligned} v_2(x) &= \int \frac{3e^{-4x}}{6e^{2x}} dx \\ &= \frac{1}{2} \int e^{-6x} dx \\ &= -\frac{1}{12}e^{-6x} \end{aligned}$$

$$\begin{aligned} y_p &= -\frac{1}{2}xe^{-2x} - \frac{1}{12}e^{-6x}e^{4x} \\ &= -\frac{1}{2}xe^{-2x} - \frac{1}{12}e^{-2x} \end{aligned}$$

$$\begin{aligned} y(x) &= C_1e^{-2x} + C_2e^{4x} - \frac{1}{2}xe^{-2x} - \frac{1}{12}e^{-2x} \\ &= \left(C_1 - \frac{1}{12} - \frac{1}{2}x\right)e^{-2x} + C_2e^{4x} \\ &= \left(A_1 - \frac{x}{2}\right)e^{-2x} + C_2e^{4x} \end{aligned}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$y_p = v_1 y_1 + v_2 y_2$$

## Exercise

Find the general solution to the given differential equation.  $y'' + 3y' + 2y = \sin e^x$

### Solution

$$\text{Characteristic Eqn.: } \lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda_1 = -2, \lambda_2 = -1$$

$$\text{The homogeneous Eqn.: } y_h = C_1e^{-2x} + C_2e^{-x}$$

$$W = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix} = -e^{-3x} + 2e^{-3x} = e^{-3x} \neq 0$$

$$v_1(x) = - \int \frac{e^{-x} \sin e^x}{e^{-3x}} dx = - \int (e^{2x} \sin e^x) dx$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned}
 u &= e^x & dv &= e^x \sin e^x \\
 du &= e^x dx & v &= \int \sin e^x d(e^x) = -\cos e^x \\
 \int (e^{2x} \sin e^x) dx &= -e^x \cos e^x + \int e^x \cos e^x dx \\
 &= -e^x \cos e^x + \int \cos e^x d(e^x)
 \end{aligned}$$

$$= -e^x \cos e^x + \sin e^x \Big|$$

$$\begin{aligned}
 v_2(x) &= \int \frac{e^{-2x} \sin e^x}{e^{-3x}} dx \\
 &= \int e^x \sin e^x dx \\
 &= \int \sin e^x d(e^x) \\
 &= -\cos e^x \Big|
 \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$\begin{aligned}
 y_p &= e^{-2x} (e^x \cos e^x - \sin e^x) + e^{-x} (-\cos e^x) \\
 &= e^{-x} \cos e^x - e^{-2x} \sin e^x - e^{-x} \cos e^x \\
 &= -e^{-2x} \sin e^x \Big|
 \end{aligned}$$

$$y_p = u_1 y_1 + u_2 y_2$$

The **general** solution:

$$\underline{y(x) = C_1 e^{-2x} + C_2 e^{-x} - e^{-2x} \sin e^x \Big|}$$

## Exercise

Find the general solution  $y'' + 3y' + 2y = 4e^x$

### Solution

$$\text{Characteristic Eqn.: } \lambda^2 + 3\lambda + 2 = 0 \Rightarrow \underline{\lambda_{1,2} = -2, -1 \Big|}$$

$$\text{The homogeneous Eqn.: } \underline{y_h = C_1 e^{-2x} + C_2 e^{-x} \Big|}$$

$$W = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix} = -e^{-3x} + 2e^{-3x} = \underline{e^{-3x} \neq 0 \Big|}$$

$$v_1(x) = - \int \frac{e^{-x} (4e^x)}{e^{-3x}} dx$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned}
&= -4 \int e^{3x} dx \\
&= -\frac{4}{3} e^{3x} \Big| \\
v_2(x) &= \int \frac{e^{-2x} 4e^x}{e^{-3x}} dx \\
&= 4 \int e^{2x} dx \\
&= 2e^{2x} \Big|
\end{aligned}
\qquad
\begin{aligned}
v_2(x) &= \int \frac{y_1 g(x)}{W} dx
\end{aligned}$$

The *particular* solution:  $y_p = -\frac{4}{3} e^{3x} e^{-2x} + 2e^{2x} e^{-x} = \frac{2}{3} e^x \Big| \qquad y_p = v_1 y_1 + v_2 y_2$

The **general** solution:

$$y(x) = \underline{C_1 e^{-2x} + C_2 e^{-x} + \frac{2}{3} e^x}$$

### Exercise

Find the general solution  $y'' + 3y' + 2y = \frac{1}{1+e^x}$

### Solution

Characteristic Eqn.:  $\lambda^2 + 3\lambda + 2 = 0 \Rightarrow \underline{\lambda_{1,2} = -2, -1}$

$$y_h = \underline{C_1 e^{-2x} + C_2 e^{-x}}$$

$$\begin{aligned}
W &= \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix} \\
&= -e^{-3x} + 2e^{-3x} \\
&= \underline{e^{-3x} \neq 0}
\end{aligned}$$

$$\begin{aligned}
v_1(x) &= - \int \frac{e^{-x}}{e^{-3x}} \frac{1}{1+e^x} dx \\
&= - \int \frac{e^{2x}}{1+e^x} dx \\
&= - \int \left( \frac{e^x}{1+e^x} - e^x \right) dx \\
&= \int e^x dx - \int \frac{e^x}{1+e^x} dx \\
&= \int e^x dx - \int \frac{1}{1+e^x} d(1+e^x)
\end{aligned}
\qquad
v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$\underline{= e^x - \ln(1 + e^x) \Big|}$$

$$\begin{aligned} v_2(x) &= \int \frac{e^{-2x}}{e^{-3x}} \frac{1}{1+e^x} dx \\ &= \int \frac{e^x}{1+e^x} dx \\ &= \int \frac{1}{1+e^x} d(1+e^x) \\ &= \ln(1+e^x) \Big| \end{aligned}$$

$$\begin{aligned} y_p &= e^{-2x} \left( e^x - \ln(1+e^x) \right) + e^{-x} \ln(1+e^x) \\ &= e^{-x} \Big| \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$y_p = v_1 y_1 + v_2 y_2$$

The **general** solution:

$$y(x) = \underline{C_1 e^{-2x} + C_2 e^{-x} + e^{-x} \Big|}$$

## Exercise

Find the general solution to the given differential equation.

$$y'' - 4y = \sinh 2x$$

## Solution

$$\text{Characteristic Eqn.: } \lambda^2 - 4 = 0 \Rightarrow \lambda_{1,2} = -2, 2$$

$$\text{The homogeneous Eqn.: } \underline{y_h = C_1 e^{-2x} + C_2 e^{2x} \Big|}$$

$$W = \begin{vmatrix} e^{-2x} & e^{2x} \\ -2e^{-2x} & 2e^{2x} \end{vmatrix} = 2 + 2 = 4 \neq 0$$

$$\begin{aligned} v_1(x) &= -\frac{1}{4} \int e^{2x} \sinh 2x \, dx \\ &= -\frac{1}{8} \int e^{2x} (e^{2x} - e^{-2x}) \, dx \\ &= -\frac{1}{8} \int (e^{4x} - 1) \, dx \\ &= -\frac{1}{8} \left( \frac{1}{4} e^{4x} - x \right) \Big| \end{aligned}$$

$$v_2(x) = \frac{1}{4} \int e^{-2x} \sinh 2x \, dx$$

$$v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$\begin{aligned}
&= -\frac{1}{8} \int e^{-2x} (e^{2x} - e^{-2x}) dx \\
&= \frac{1}{8} \int (1 - e^{4x}) dx \\
&= \frac{1}{8} \left( x - \frac{1}{4} e^{4x} \right)
\end{aligned}$$

$$\begin{aligned}
y_p &= \left( \frac{x}{8} - \frac{1}{32} e^{4x} \right) e^{-2x} + \left( \frac{x}{8} - \frac{1}{32} e^{4x} \right) e^{2x} \\
&= \left( \frac{x}{4} - \frac{1}{16} e^{4x} \right) \left( \frac{e^{-2x} + e^{2x}}{2} \right) \\
&= \left( \frac{x}{4} - \frac{1}{16} e^{4x} \right) \cosh 2x
\end{aligned}$$

$$y(x) = C_1 e^{-2x} + C_2 e^{4x} + \left( \frac{x}{4} - \frac{1}{16} e^{4x} \right) \cosh 2x$$

$$y_p = v_1 y_1 + v_2 y_2$$

### Exercise

Find the general solution  $y'' + 4y = \sec 2x$

### Solution

$$\text{Characteristic Eqn.: } \lambda^2 + 4 = 0 \Rightarrow \lambda_{1,2} = \pm 2i$$

$$y_h = C_1 \cos 2x + C_2 \sin 2x$$

$$\begin{aligned}
W &= \begin{vmatrix} \cos 2x & \sin 2x \\ -2 \sin 2x & 2 \cos 2x \end{vmatrix} \\
&= 2 \cos^2 2x + 2 \sin^2 2x \\
&= 2 \neq 0
\end{aligned}$$

$$\begin{aligned}
v_1(x) &= -\frac{1}{2} \int \sin 2x \sec 2x \, dx \\
&= -\frac{1}{2} \int \tan 2x \, dx \\
&= -\frac{1}{4} \ln |\sec 2x|
\end{aligned}$$

$$\begin{aligned}
v_2(x) &= \frac{1}{2} \int \cos 2x \sec 2x \, dx \\
&= \frac{1}{2} \int dx \\
&= \frac{1}{2} x
\end{aligned}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$y_p = -\frac{1}{4} \ln|\sec x| \cos 2x + \frac{1}{2} x \sin 2x$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$y(x) = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{4} \ln|\sec x| \cos 2x + \frac{1}{2} x \sin 2x$$

### Exercise

Find the general solution to the given differential equation.  $y'' + 4y = \cos 3x$

### Solution

$$\text{Characteristic Eqn.: } \lambda^2 + 4 = 0 \Rightarrow \lambda_{1,2} = \pm 2i$$

$$\text{The homogeneous Eqn.: } y_h = C_1 \cos 2x + C_2 \sin 2x$$

$$\begin{aligned} W &= \begin{vmatrix} \cos 2x & \sin 2x \\ -2 \sin 2x & 2 \cos 2x \end{vmatrix} \\ &= 2 \cos^2 2x + 2 \sin^2 2x \\ &= 2 \neq 0 \end{aligned}$$

$$\begin{aligned} v_1(x) &= -\frac{1}{2} \int \sin 2x \cos 3x \, dx \\ &= -\frac{1}{4} \int (\sin 5x - \sin x) \, dx \\ &= -\frac{1}{4} \left( -\frac{1}{5} \cos 5x + \cos x \right) \\ &= \frac{1}{20} \cos 5x - \frac{1}{4} \cos x \end{aligned}$$

$$v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$\cos \alpha \sin \beta = \frac{1}{2} [\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

$$\begin{aligned} v_2(x) &= \frac{1}{2} \int \cos 2x \cos 3x \, dx \\ &= \frac{1}{4} \int (\cos 5x + \cos x) \, dx \\ &= \frac{1}{20} \sin 5x + \frac{1}{4} \sin x \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

$$\begin{aligned} y_p &= \left( \frac{1}{20} \cos 5x - \frac{1}{4} \cos x \right) \cos 2x + \left( \frac{1}{20} \sin 5x + \frac{1}{4} \sin x \right) \sin 2x \\ &= \frac{1}{20} \cos 5x \cos 2x - \frac{1}{4} \cos x \cos 2x + \frac{1}{20} \sin 5x \sin 2x + \frac{1}{4} \sin x \sin 2x \\ &= \frac{1}{40} \cos 5x + \frac{1}{40} \cos 3x - \frac{1}{8} \cos 3x - \frac{1}{8} \cos x + \frac{1}{40} \cos 3x - \frac{1}{40} \cos 2x + \frac{1}{8} \cos x - \frac{1}{8} \cos 3x \\ &= \frac{1}{40} \cos 5x - \frac{1}{5} \cos 3x - \frac{1}{40} \cos 2x \end{aligned}$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$y(x) = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{40} \cos 5x - \frac{1}{5} \cos 3x - \frac{1}{40} \cos 2x$$

$$= \underline{A_1 \cos 2x + C_2 \sin 2x + \frac{1}{40} \cos 5x - \frac{1}{5} \cos 3x}$$

### Exercise

Find the general solution to the given differential equation.  $y'' + 4y = \sin^2 x$

### Solution

Characteristic Eqn.:  $\lambda^2 + 4 = 0 \Rightarrow \lambda_{1,2} = \pm 2i$

The homogeneous Eqn.:  $\underline{y_h = C_1 \cos 2x + C_2 \sin 2x}$

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x$$

$$= \underline{2 \neq 0}$$

$$v_1(x) = -\frac{1}{2} \int \sin 2x \sin^2 x \, dx$$

$$= -\int \sin x \cos x \sin^2 x \, dx$$

$$= -\int \sin^3 x \, d(\sin x)$$

$$= \underline{-\frac{1}{4} \sin^4 x}$$

$$v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$v_2(x) = \frac{1}{2} \int \cos 2x \sin^2 x \, dx$$

$$= \frac{1}{2} \int (1 - 2\sin^2 x) \sin^2 x \, dx$$

$$= \frac{1}{2} \int (\sin^2 x - 2\sin^4 x) \, dx$$

$$= \frac{1}{2} \int \left( \frac{1}{2} - \frac{1}{2} \cos 2x - 2 \left( \frac{1 - \cos 2x}{2} \right)^2 \right) dx$$

$$= \frac{1}{4} \int (1 - \cos 2x - 1 + 2\cos 2x - \cos^2 2x) \, dx$$

$$= \frac{1}{4} \int \left( \cos 2x - \frac{1}{2} - \frac{1}{2} \cos 4x \right) dx$$

$$= \underline{\frac{1}{8} \sin 2x - \frac{x}{8} - \frac{1}{32} \sin 4x}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$y_p = -\frac{1}{4}\sin^4 x \cos x + \left(\frac{1}{8}\sin 2x - \frac{x}{8} - \frac{1}{32}\sin 4x\right)\sin x \quad y_p = v_1 y_1 + v_2 y_2$$

$$y(x) = \underline{C_1 \cos 2x + C_2 \sin 2x - \frac{1}{4}\sin^4 x \cos x + \left(\frac{1}{8}\sin 2x - \frac{x}{8} - \frac{1}{32}\sin 4x\right)\sin x}$$

### Exercise

Find the general solution to the given differential equation.  $y'' - 4y = \frac{e^x}{x}$

### Solution

$$\lambda^2 - 4 = 0 \Rightarrow \lambda_{1,2} = \pm 2$$

$$y_h = \underline{C_1 e^{-2x} + C_2 e^{2x}}$$

$$W = \begin{vmatrix} e^{-2x} & e^{2x} \\ -2e^{-2x} & 2e^{2x} \end{vmatrix} = 2 + 2 = 4 \neq 0$$

$$\begin{aligned} v_1(x) &= -\int \frac{1}{4} e^{2x} \frac{e^{2x}}{x} dx = -\frac{1}{4} \int \frac{e^{4x}}{x} \\ &= -\frac{1}{4} \int \frac{e^{4x}}{x} \end{aligned}$$

$$v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned} v_2(x) &= \frac{1}{4} \int e^{-2x} \frac{e^{2x}}{x} dx \\ &= \frac{1}{4} \int \frac{1}{x} dx \\ &= \underline{\frac{1}{4} \ln|x|} \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$y_p = \underline{-\frac{1}{4} e^{-2x} \int \frac{e^{4x}}{x} dx + \frac{1}{4} e^{2x} \ln|x|}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y(x) = \underline{C_1 e^{-2x} + C_2 e^{2x} + \frac{1}{4} e^{2x} \ln|x| - \frac{1}{4} e^{-2x} \int \frac{e^{4x}}{x} dx}$$

### Exercise

Find the general solution to the given differential equation.  $y'' - 4y = xe^x$

### Solution

$$\lambda^2 - 4 = 0 \Rightarrow \lambda_{1,2} = \pm 2$$

$$y_h = \underline{C_1 e^{-2x} + C_2 e^{2x}}$$



$$W = \begin{vmatrix} e^{-2x} & e^{2x} \\ -2e^{-2x} & 2e^{2x} \end{vmatrix} = 2 + 2 = 4 \neq 0$$

$$\begin{aligned} v_1(x) &= -\frac{1}{4} \int e^{2x} (xe^x) dx \\ &= -\frac{1}{4} \int xe^{3x} dx \\ &= -\frac{1}{4} \left( \frac{1}{3}x - \frac{1}{9} \right) e^{3x} \end{aligned}$$

$$v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned} v_2(x) &= \frac{1}{4} \int e^{-2x} (xe^x) dx \\ &= \frac{1}{4} \int xe^{-x} dx \\ &= \frac{1}{4} (-x - 1) e^{-x} \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$y_p = \frac{1}{36} (1 - 3x) e^x - \frac{1}{4} (x + 1) e^{-x}$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$y(x) = C_1 e^{-2x} + C_2 x e^{2x} + \frac{1}{36} (1 - 3x) e^x - \frac{1}{4} (x + 1) e^{-x}$$

## Exercise

Find the general solution  $y'' + 4y = \sin^2 2t$

## Solution

$$\text{Characteristic Eqn.: } \lambda^2 + 4 = 0 \Rightarrow \lambda_{1,2} = \pm 2i$$

$$y_h = C_1 \cos 2t + C_2 \sin 2t$$

$$\begin{aligned} W &= \begin{vmatrix} \cos 2t & \sin 2t \\ -2 \sin 2t & 2 \cos 2t \end{vmatrix} \\ &= 2 \cos^2 t + 2 \sin^2 t \\ &= 2 \neq 0 \end{aligned}$$

$$\begin{aligned} v_1(t) &= -\int \frac{\sin 2t}{2} \sin^2 2t dt \\ &= \frac{1}{4} \int (1 - \cos^2 2t) d(\cos 2t) \\ &= \frac{1}{4} \left( \cos 2t - \frac{1}{3} \cos^3 2t \right) \end{aligned}$$

$$v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned}
 v_2(t) &= \int \frac{\cos 2t}{2} \sin^2 2t \, dt \\
 &= \frac{1}{4} \int \sin^2 2t \, d(\sin 2t) \\
 &= \frac{1}{12} \sin^3 2t \Big|
 \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$\begin{aligned}
 y_p &= \left( \frac{1}{4} \cos 2t - \frac{1}{12} \cos^3 2t \right) \cos 2t + \frac{1}{12} \sin^4 2t & y_p &= u_1 y_1 + u_2 y_2 \\
 &= \frac{1}{4} \cos^2 2t - \frac{1}{12} \cos^4 2t + \frac{1}{12} \sin^4 2t \\
 &= \frac{1}{4} \cos^2 2t + \frac{1}{12} (\sin^4 2t - \cos^4 2t) \\
 &= \frac{1}{4} \cos^2 2t + \frac{1}{12} (\sin^2 2t - \cos^2 2t) (\sin^2 2t + \cos^2 2t) \\
 &= \frac{1}{4} \cos^2 2t + \frac{1}{12} \sin^2 2t - \frac{1}{12} \cos^2 2t \\
 &= \frac{1}{6} \cos^2 2t + \frac{1}{12} \sin^2 2t \Big|
 \end{aligned}$$

The **general** solution:

$$y(t) = C_1 \cos 2t + C_2 \sin 2t + \frac{1}{6} \cos^2 2t + \frac{1}{12} \sin^2 2t \Big|$$

### Exercise

Find the general solution to the given differential equation.

$$y'' - 4y' + 4y = 2e^{2x}$$

### Solution

$$\lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda_{1,2} = 2 \Big|$$

$$y_h = C_1 e^{2x} + C_2 x e^{2x} \Big|$$

$$\begin{aligned}
 W &= \begin{vmatrix} e^{2x} & x e^{2x} \\ 2e^{2x} & e^{2x} + 2x e^{2x} \end{vmatrix} \\
 &= e^{4x} + 2x e^{4x} - 2x e^{4x} \\
 &= e^{4x} \neq 0 \Big|
 \end{aligned}$$

$$\begin{aligned}
 v_1(x) &= - \int \frac{2x e^{4x}}{e^{4x}} dx \\
 &= -2 \int x dx \\
 &= -x^2 \Big|
 \end{aligned}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned}
 v_2(x) &= \int \frac{2e^{4x}}{e^{4x}} dx \\
 &= 2 \int dx \\
 &= 2x
 \end{aligned}$$

$$\begin{aligned}
 y_p &= -x^2 e^{2x} + 2x^2 e^{2x} = x^2 e^{2x} \\
 y(x) &= C_1 e^{2x} + C_2 x e^{2x} + x^2 e^{2x}
 \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$y_p = v_1 y_1 + v_2 y_2$$

### Exercise

Find the general solution  $y'' - 4y' + 4y = (x+1)e^{2x}$

### Solution

$$\text{Characteristic Eqn.: } \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0 \Rightarrow \lambda_{1,2} = 2$$

$$\text{The homogeneous Eqn.: } y_h = C_1 e^{2x} + C_2 x e^{2x}$$

$$\begin{aligned}
 W &= \begin{vmatrix} e^{2x} & x e^{2x} \\ 2e^{2x} & e^{2x} + 2x e^{2x} \end{vmatrix} \\
 &= e^{4x} + 2x e^{4x} - 2x e^{4x} \\
 &= e^{4x} \neq 0
 \end{aligned}$$

$$u'_1 = -\frac{x e^{2x} (x+1) e^{2x}}{e^{4x}} = -x^2 - x$$

$$u_1 = \int (-x^2 - x) dx = -\frac{1}{3}x^3 - \frac{1}{2}x^2$$

$$u'_2 = \frac{e^{2x} (x+1) e^{2x}}{e^{4x}} = x+1$$

$$u_2 = \int (x+1) dx = \frac{1}{2}x^2 + x$$

$$\begin{aligned}
 y_p &= u_1 y_1 + u_2 y_2 = \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right) e^{2x} + \left(\frac{1}{2}x^2 + x\right) x e^{2x} \\
 &= \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{2}x^3 + x^2\right) e^{2x} \\
 &= \left(\frac{1}{6}x^3 + \frac{1}{2}x^2\right) e^{2x}
 \end{aligned}$$

$$y(x) = C_1 e^{2x} + C_2 x e^{2x} + \left(\frac{1}{6}x^3 + \frac{1}{2}x^2\right) e^{2x}$$

$$u'_1 = -\frac{y_2 g(t)}{W}$$

$$u'_2 = \frac{y_1 g(t)}{W}$$

## Exercise

Find the general solution to the given differential equation.  $y'' + 4y' + 5y = 10$

## Solution

$$\lambda^2 + 4\lambda + 5 = 0 \Rightarrow \lambda_{1,2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

$$y_h = e^{-2x} (C_1 \cos x + C_2 \sin x)$$

$$W = \begin{vmatrix} e^{-2x} \cos x & e^{-2x} \sin x \\ -2e^{-2x} \cos x - e^{-2x} \sin x & -2e^{-2x} \sin x + e^{-2x} \cos x \end{vmatrix}$$

$$= -2e^{-4x} \cos x \sin x + e^{-4x} \cos^2 x + 2e^{-4x} \cos x \sin x + e^{-4x} \sin^2 x$$

$$= e^{-4x} \neq 0$$

$$v_1(x) = - \int \frac{10e^{-2x} \sin x}{e^{-4x}} dx$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$= -10 \int e^{2x} \sin x dx$$

$$\int e^{2x} \sin x dx = -\cos x e^{2x} + 2 \sin x e^{2x} - 4 \int e^{2x} \sin x dx$$

$$\int e^{2x} \sin x dx = \frac{1}{5} (-\cos x + 2 \sin x) e^{2x}$$

$$= (-10) \frac{1}{5} (-\cos x + 2 \sin x) e^{2x}$$

$$= (2 \cos x - 4 \sin x) e^{2x}$$

$$v_2(x) = \int \frac{10e^{-2x} \cos x}{e^{-4x}} dx$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= 10 \int e^{2x} \cos x dx$$

$$\int e^{2x} \cos x dx = \sin x e^{2x} + 2 \cos x e^{2x} - 4 \int e^{2x} \cos x dx$$

$$\int e^{2x} \cos x dx = \frac{1}{5} (\sin x + 2 \cos x) e^{2x}$$

$$= (10) \frac{1}{5} (\sin x + 2 \cos x) e^{2x}$$

$$= (2 \sin x + 4 \cos x) e^{2x}$$

$$y_p = (2 \cos x - 4 \sin x) e^{2x} (\cos x) e^{-2x} + (2 \sin x + 4 \cos x) e^{2x} (\sin x) e^{-2x}$$

$$y_p = v_1 y_1 + v_2 y_2$$

		$\int \sin x$
+	$e^{2x}$	$-\cos x$
-	$2e^{2x}$	$-\sin x$
+	$4e^{2x}$	

		$\int \cos x$
+	$e^{2x}$	$\sin x$
-	$2e^{2x}$	$-\cos x$
+	$4e^{2x}$	

$$\begin{aligned}
 &= 2 \cos^2 x - 4 \sin x \cos x + 2 \sin^2 x + 4 \cos x \sin x \\
 &= 2 \\
 \underline{y(x) = e^{-2x} (C_1 \cos x + C_2 \sin x) + 2}
 \end{aligned}$$

### Exercise

Find the general solution to the given differential equation.

$$y'' - 9y = \frac{9x}{e^{3x}}$$

### Solution

$$\lambda^2 - 9 = 0 \Rightarrow \lambda_{1,2} = \pm 3$$

$$\underline{y_h = C_1 e^{-3x} + C_2 e^{3x}}$$

$$W = \begin{vmatrix} e^{-3x} & e^{3x} \\ -3e^{-3x} & 3e^{3x} \end{vmatrix} = 3 + 3 = 6 \neq 0$$

$$\begin{aligned}
 v_1(x) &= - \int \frac{e^{3x}}{6} \frac{9x}{e^{3x}} dx \\
 &= - \frac{3}{2} \int x dx \\
 &= - \frac{3}{4} x^2
 \end{aligned}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned}
 v_2(x) &= \int \frac{e^{-3x}}{6} \frac{9x}{e^{3x}} dx \\
 &= \frac{3}{2} \int x e^{-6x} dx \\
 &= \frac{3}{2} \left( -\frac{1}{6} x - \frac{1}{36} \right) e^{-6x} \\
 &= \left( -\frac{1}{4} x - \frac{1}{24} \right) e^{-6x}
 \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$\begin{aligned}
 y_p &= -\frac{3}{4} x^2 e^{-3x} - \left( \frac{1}{4} x + \frac{1}{24} \right) e^{-6x} e^{3x} \\
 &= -\left( \frac{3}{4} x^2 + \frac{1}{4} x + \frac{1}{24} \right) e^{-3x}
 \end{aligned}$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$\begin{aligned}
 y(x) &= C_1 e^{-3x} + C_2 e^{3x} - \left( \frac{3}{4} x^2 + \frac{1}{4} x + \frac{1}{24} \right) e^{-3x} \\
 &= C_2 e^{3x} - \left( \frac{3}{4} x^2 + \frac{1}{4} x + C_3 \right) e^{-3x}
 \end{aligned}$$

### Exercise

Find the general solution  $y'' + 9y = \csc 3x$

#### Solution

$$\text{Characteristic Eqn.: } \lambda^2 + 9 = 0 \Rightarrow \lambda_{1,2} = \pm 3i$$

$$\text{The homogeneous Eqn.: } \underline{y_h = C_1 \cos 3x + C_2 \sin 3x}$$

$$\begin{aligned} W &= \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} \\ &= 3\cos^2 3x + 3\sin^2 3x \\ &= 3 \neq 0 \end{aligned}$$

$$u'_1 = -\frac{(\sin 3x)(\csc 3x)}{3} = -\frac{1}{3}$$

$$u'_1 = -\frac{y_2 g(t)}{W}$$

$$u_1 = \int \left(-\frac{1}{3}\right) dx = -\frac{1}{3}x$$

$$u'_2 = \frac{(\cos 3x)(\csc 3x)}{3} = \frac{1}{3} \frac{\cos 3x}{\sin 3x}$$

$$u'_2 = \frac{y_1 g(t)}{W}$$

$$\begin{aligned} u_2 &= \int \left(\frac{1}{3} \frac{\cos 3x}{\sin 3x}\right) dx \\ &= \frac{1}{9} \int \frac{1}{\sin 3x} d(\sin 3x) \\ &= \frac{1}{9} \ln |\sin 3x| \end{aligned}$$

$$\underline{y_p = -\frac{1}{3}x \cos 3x + \frac{1}{9} \sin 3x \ln |\sin 3x|}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$\underline{y(x) = C_1 \cos 3x + C_2 \sin 3x - \frac{1}{3}x \cos 3x + \frac{1}{9} \sin 3x \ln |\sin 3x|}$$

### Exercise

Find the general solution to the given differential equation.  $y'' + 9y = 3 \tan 3t$

#### Solution

$$\text{Characteristic Eqn.: } \lambda^2 + 9 = 0 \Rightarrow \lambda_{1,2} = \pm 3i$$

$$\text{The homogeneous Eqn.: } \underline{y_h = C_1 \cos 3t + C_2 \sin 3t}$$

$$\begin{aligned} W &= \begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix} \\ &= 3\cos^2 3t + 3\sin^2 3t \\ &= 3 \neq 0 \end{aligned}$$

$$\begin{aligned}
 v_1(t) &= -\int \frac{\sin 3t(3 \tan 3t)}{3} dt \\
 &= -\int \frac{\sin^2 3t}{\cos 3t} dt \\
 &= -\int \frac{1-\cos^2 3t}{\cos 3t} dt \\
 &= -\int (\sec 3t - \cos 3t) dt \\
 &= -\frac{1}{3} \ln |\sec 3t + \tan 3t| + \frac{1}{3} \sin 3t
 \end{aligned}$$

$$v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned}
 v_2(t) &= \int \frac{\cos 3t(3 \tan 3t)}{3} dt \\
 &= \int \sin 3t dt \\
 &= -\frac{1}{3} \cos 3t
 \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$\begin{aligned}
 y_p &= -\frac{1}{3} \cos 3t \ln |\sec 3t + \tan 3t| + \frac{1}{3} \cos 3t \sin 3t - \frac{1}{3} \cos 3t \sin 3t \\
 &= -\frac{1}{3} (\cos 3t) \ln |\sec 3t + \tan 3t|
 \end{aligned}$$

$$y_p = u_1 y_1 + u_2 y_2$$

The **general** solution:

$$y(t) = C_1 \cos 3t + C_2 \sin 3t - \frac{1}{3} (\cos 3t) \ln |\sec 3t + \tan 3t|$$

## Exercise

Find the general solution to the given differential equation.

$$y'' + 9y = \sin 3x$$

## Solution

$$\text{Characteristic Eqn.: } \lambda^2 + 9 = 0 \Rightarrow \lambda_{1,2} = \pm 3i$$

$$\text{The homogeneous Eqn.: } y_h = C_1 \cos 3x + C_2 \sin 3x$$

$$\begin{aligned}
 W &= \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} \\
 &= 3\cos^2 3x + 3\sin^2 3x \\
 &= 3 \neq 0
 \end{aligned}$$

$$\begin{aligned}
 v_1(x) &= -\frac{1}{3} \int \sin^2 3x dx \\
 &= -\frac{1}{6} \int (1 - \cos 6x) dx
 \end{aligned}$$

$$v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= -\frac{1}{6} \left( x - \frac{1}{6} \sin 6x \right) \Big|$$

$$\begin{aligned} v_2(x) &= \frac{1}{3} \int \cos 3x \sin 3x \, dx \\ &= \frac{1}{9} \int \sin 3x \, d(\sin 3x) \\ &= \frac{1}{18} \sin^2 3x \Big| \end{aligned}$$

$$y_p = \left( \frac{1}{6}x - \frac{1}{36} \sin 6x \right) \cos 3x + \frac{1}{18} \sin^3 3x \Big|$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$y(x) = \underline{C_1 \cos 3x + C_2 \sin 3x + \left( \frac{1}{6}x - \frac{1}{36} \sin 6x \right) \cos 3x + \frac{1}{18} \sin^3 3x \Big|}$$

### Exercise

Find the general solution to the given differential equation.  $y'' + 9y = \sec 3x$

### Solution

$$\text{Characteristic Eqn.: } \lambda^2 + 9 = 0 \Rightarrow \underline{\lambda_{1,2} = \pm 3i}$$

$$\text{The homogeneous Eqn.: } \underline{y_h = C_1 \cos 3x + C_2 \sin 3x}$$

$$\begin{aligned} W &= \begin{vmatrix} \cos 3x & \sin 3x \\ -3 \sin 3x & 3 \cos 3x \end{vmatrix} \\ &= 3 \cos^2 3x + 3 \sin^2 3x \\ &= \underline{3 \neq 0} \end{aligned}$$

$$\begin{aligned} v_1(x) &= -\frac{1}{3} \int \sin 3x \sec 3x \, dx \\ &= \frac{1}{9} \int \frac{1}{\cos 3x} d(\cos 3x) \\ &= \frac{1}{9} \ln |\cos 3x| \Big| \end{aligned}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned} v_2(x) &= \frac{1}{3} \int \cos 3x \sec 3x \, dx \\ &= \frac{1}{3} \int dx \\ &= \underline{\frac{x}{3}} \Big| \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$y_p = \underline{\frac{1}{9} \cos 3x \ln |\cos 3x| + \frac{x}{3} \sin 3x \Big|}$$

$$y_p = v_1 y_1 + v_2 y_2$$



The **general** solution:

$$y(x) = \underline{C_1 \cos 3x + C_2 \sin 3x + \frac{1}{9} \cos 3x \ln |\cos 3x| + \frac{x}{3} \sin 3x}$$

### Exercise

Find the general solution to the given differential equation.  $y'' + 9y = 2 \sec 3x$

### Solution

$$\lambda^2 + 9 = 0 \Rightarrow \lambda_{1,2} = \pm 3i$$

$$y_h = \underline{C_1 \cos 3x + C_2 \sin 3x}$$

$$\begin{aligned} W &= \begin{vmatrix} \cos 3x & \sin 3x \\ -3 \sin 3x & 3 \cos 3x \end{vmatrix} \\ &= 3 \cos^2 3x + 3 \sin^2 3x \\ &= \underline{3 \neq 0} \end{aligned}$$

$$\begin{aligned} v_1(x) &= -\frac{2}{3} \int \sin 3x \sec 3x \, dx \\ &= \frac{2}{9} \int \frac{1}{\cos 3x} d(\cos 3x) \\ &= \underline{\frac{2}{9} \ln |\cos 3x|} \end{aligned}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned} v_2(x) &= \frac{2}{3} \int \cos 3x \sec 3x \, dx \\ &= \frac{2}{3} \int dx \\ &= \underline{\frac{2}{3} x} \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$y_p = \underline{\frac{2}{9} \cos 3x \ln |\cos 3x| + \frac{2}{3} x \sin 3x}$$

$$y_p = v_1 y_1 + v_2 y_2$$

The **general** solution:

$$y(x) = \underline{C_1 \cos 3x + C_2 \sin 3x + \frac{2}{9} \cos 3x \ln |\cos 3x| + \frac{2}{3} x \sin 3x}$$

### Exercise

Find the general solution to the given differential equation.  $4y'' + 36y = \csc 3x$

### Solution

$$\text{Characteristic Eqn.: } 4\lambda^2 + 36 = 0 \Rightarrow \lambda_{1,2} = \pm 3i$$

$$\text{The homogeneous Eqn.: } y_h = C_1 \cos 3x + C_2 \sin 3x$$

$$\begin{aligned} W &= \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} \\ &= 3\cos^2 3x + 3\sin^2 3x \\ &= 3 \neq 0 \end{aligned}$$

$$y'' + 9y = \frac{1}{4} \csc 3x$$

$$\begin{aligned} v_1(x) &= -\frac{1}{12} \int \sin 3x \csc 3x \, dx \\ &= -\frac{1}{12} \int dx \\ &= -\frac{1}{12} x \end{aligned}$$

$$v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned} v_2(x) &= \frac{1}{12} \int \cos 3x \csc 3x \, dx \\ &= \frac{1}{12} \int \cot 3x \, dx \\ &= \frac{1}{36} \ln |\sin 3x| \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$y_p = -\frac{1}{12} x \cos 3x + \frac{1}{36} \sin 3x \ln |\sin 3x|$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$y(x) = C_1 \cos 3x + C_2 \sin 3x - \frac{1}{12} x \cos 3x + \frac{1}{36} \sin 3x \ln |\sin 3x|$$

### Exercise

Find the general solution  $(D^2 + 5D + 6)y = x^2 + 2x$

### Solution

$$\text{Characteristic Eqn.: } \lambda^2 + 5\lambda + 6 = 0 \Rightarrow \lambda_{1,2} = -3, -2$$

$$y_h = C_1 e^{-3x} + C_2 e^{-2x}$$

$$\begin{aligned}
 W &= \begin{vmatrix} e^{-3x} & e^{-2x} \\ -3e^{-3x} & -2e^{-2x} \end{vmatrix} \\
 &= -2e^{-5x} + 3e^{-5x} \\
 &= e^{-5x} \neq 0
 \end{aligned}$$

$$\begin{aligned}
 v_1(x) &= - \int \frac{(x^2 + 2x)e^{-2x}}{e^{-5x}} dx \\
 &= - \int (x^2 + 2x)e^{3x} dx \\
 &= - \left( \frac{1}{3}x^2 + \frac{2}{3}x - \frac{2}{9}x - \frac{2}{9} + \frac{2}{27} \right) e^{3x} \\
 &= - \left( \frac{1}{3}x^2 + \frac{4}{9}x - \frac{4}{27} \right) e^{3x}
 \end{aligned}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

		$\int e^{3x}$
+	$x^2 + 2x$	$\frac{1}{3}e^{3x}$
-	$2x + 2$	$\frac{1}{9}e^{3x}$
+	2	$\frac{1}{27}e^{3x}$

$$\begin{aligned}
 v_2(x) &= \int \frac{(x^2 + 2x)e^{-3x}}{e^{-5x}} dx \\
 &= \int (x^2 + 2x)e^{2x} dx \\
 &= - \left( \frac{1}{2}x^2 + x - \frac{1}{2}x - \frac{1}{2} + \frac{1}{4} \right) e^{2x} \\
 &= \left( \frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{4} \right) e^{2x}
 \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

		$\int e^{2x}$
+	$x^2 + 2x$	$\frac{1}{2}e^{2x}$
-	$2x + 2$	$\frac{1}{4}e^{2x}$
+	2	$\frac{1}{8}e^{2x}$

$$\begin{aligned}
 y_p &= - \left( \frac{1}{3}x^2 + \frac{4}{9}x - \frac{4}{27} \right) e^{3x} e^{-3x} + \left( \frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{4} \right) e^{2x} e^{-2x} \\
 &= - \frac{1}{3}x^2 - \frac{4}{9}x + \frac{4}{27} + \frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{4} \\
 &= \frac{1}{6}x^2 + \frac{1}{18}x - \frac{11}{108}
 \end{aligned}$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$y(x) = C_1 e^{-3x} + C_2 e^{-2x} + \frac{1}{6}x^2 + \frac{1}{18}x - \frac{11}{108}$$

## Exercise

Find the general solution  $(D^2 - 3D + 2)y = \frac{1}{1 + e^{-x}}$

## Solution

$$\text{Characteristic Eqn.: } \lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda_{1,2} = 1, 2$$

$$y_h = C_1 e^x + C_2 e^{2x}$$

$$\begin{aligned}
 W &= \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} \\
 &= 2e^{3x} - e^{3x} \\
 &= \underline{e^{3x} \neq 0}
 \end{aligned}$$

$$\begin{aligned}
 v_1(x) &= - \int \frac{e^{2x}}{e^{3x}} \frac{1}{1+e^{-x}} dx \\
 &= - \int \frac{e^{-x}}{1+e^{-x}} dx \\
 &= \int \frac{1}{1+e^{-x}} d(1+e^{-x}) \\
 &= \underline{\ln(1+e^{-x})}
 \end{aligned}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned}
 v_2(x) &= \int \frac{e^x}{e^{3x}} \frac{1}{1+e^{-x}} dx \\
 &= \int \frac{1}{e^{2x} + e^x} dx \\
 &= \int \frac{1}{e^x(e^x + 1)} dx \\
 &= \int \left( \frac{1}{e^x} - \frac{1}{e^x + 1} \frac{e^{-x}}{e^{-x}} \right) dx \\
 &= \int \left( e^{-x} - \frac{e^{-x}}{1+e^{-x}} \right) dx \\
 &= -e^{-x} - \int \frac{1}{1+e^{-x}} d(1+e^{-x}) \\
 &= \underline{-e^{-x} - \ln(1+e^{-x})}
 \end{aligned}$$

$$\frac{1}{e^x(e^x + 1)} = \frac{1}{e^x} - \frac{1}{e^x + 1}$$

$$\begin{aligned}
 y_p &= e^x \ln(1+e^{-x}) + (-e^{-x} - \ln(1+e^{-x}))e^{2x} \\
 &= \underline{e^x \ln(1+e^{-x}) - e^x - e^{2x} \ln(1+e^{-x})}
 \end{aligned}$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$y(x) = \underline{C_1 e^x + C_2 e^{2x} + e^x \ln(1+e^{-x}) - e^x - e^{2x} \ln(1+e^{-x})}$$

### Exercise

Find the general solution  $y''' + y' = \sec x$

### Solution

$$\text{Characteristic Eqn.: } \lambda^3 + \lambda = \lambda(\lambda^2 + 1) = 0 \rightarrow \lambda_{1,2,3} = 0, \pm i$$

$$\underline{y_h = C_1 + C_2 \cos x + C_3 \sin x}$$

$$W = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = \sin^2 x + \cos^2 x = \underline{1 \neq 0}$$

$$W_1 = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \sec x & -\cos x & -\sin x \end{vmatrix} = \cos^2 x \sec x + \sin^2 x \sec x = \underline{\sec x}$$

$$W_2 = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \sec x & -\sin x \end{vmatrix} = \underline{-1}$$

$$W_3 = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \sec x \end{vmatrix} = -\sin x \sec x = \underline{-\tan x}$$

$$u_1(x) = \int \sec x \, dx = \underline{\ln|\sec x + \tan x|}$$

$$u_1 = \int \frac{W_1}{W}$$

$$u_2(x) = -\int dx = \underline{-x}$$

$$u_2 = \int \frac{W_2}{W}$$

$$u_3(x) = \int -\tan x \, dx = \underline{\ln|\cos x|}$$

$$u_3 = \int \frac{W_3}{W}$$

$$\underline{y_p = \ln|\sec x + \tan x| - x \cos x + (\sin x) \ln|\cos x|}$$

$$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$$

$$\underline{y(x) = C_1 + C_2 \cos x + C_3 \sin x + \ln|\sec x + \tan x| - x \cos x + (\sin x) \ln|\cos x|}$$

### **Exercise**

Find the general solution  $y''' - 3y'' + 2y' = \frac{e^x}{1 + e^{-x}}$

### Solution

$$\text{Characteristic Eqn.: } \lambda^3 - 3\lambda^2 + 2\lambda = \lambda(\lambda^2 - 3\lambda + 2) = 0 \rightarrow \lambda_{1,2,3} = 0, 1, 2$$

$$\underline{y_h = C_1 + C_2 e^x + C_3 e^{2x}}$$

$$W = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix} = 4e^{3x} - 2e^{3x} = \underline{2e^{3x} \neq 0}$$

$$W_1 = \begin{vmatrix} 0 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ \frac{e^x}{1+e^{-x}} & e^x & 4e^{2x} \end{vmatrix} = \frac{2e^{4x}}{1+e^{-x}} - \frac{e^{4x}}{1+e^{-x}} = \underline{\frac{e^{4x}}{1+e^{-x}}}$$

$$W_2 = \begin{vmatrix} 1 & 0 & e^{2x} \\ 0 & 0 & 2e^{2x} \\ 0 & \frac{e^x}{1+e^{-x}} & 4e^{2x} \end{vmatrix} = \underline{-\frac{2e^{3x}}{1+e^{-x}}}$$

$$W_3 = \begin{vmatrix} 1 & e^x & 0 \\ 0 & e^x & 0 \\ 0 & e^x & \frac{e^x}{1+e^{-x}} \end{vmatrix} = \underline{\frac{e^{2x}}{1+e^{-x}}}$$

$$\begin{aligned} u_1(x) &= \frac{1}{2} \int \frac{e^x}{1+e^{-x}} \frac{e^x}{e^x} dx \\ &= \frac{1}{2} \int \frac{e^{2x}}{e^x+1} dx \\ &= \frac{1}{2} \int \left( e^x - \frac{e^x}{e^x+1} \right) dx \\ &= \frac{1}{2} \int e^x dx - \frac{1}{2} \int \frac{1}{e^x+1} d(e^x+1) \\ &= \underline{\frac{1}{2}e^x - \frac{1}{2}\ln(e^x+1)} \end{aligned}$$

$$u_1 = \int \frac{W_1}{W}$$

$$\begin{aligned} u_2(x) &= - \int \frac{1}{1+e^{-x}} \frac{e^x}{e^x} dx \\ &= - \int \frac{e^x}{e^x+1} dx \\ &= - \int \frac{1}{e^x+1} d(e^x+1) \\ &= \underline{-\ln(e^x+1)} \end{aligned}$$

$$u_2 = \int \frac{W_2}{W}$$

$$\begin{aligned}
 u_3(x) &= \int \frac{1}{2e^{3x}} \frac{e^{2x}}{1+e^{-x}} dx \\
 &= \frac{1}{2} \int \frac{e^{-x}}{1+e^{-x}} dx \\
 &= -\frac{1}{2} \int \frac{1}{1+e^{-x}} d(1+e^{-x}) \\
 &= -\frac{1}{2} \ln(1+e^{-x})
 \end{aligned}$$

$$u_3 = \int \frac{W_3}{W}$$

$$y_p = \frac{1}{2}e^x - \frac{1}{2}\ln(e^x + 1) - e^x \ln(e^x + 1) - \frac{1}{2}e^{2x} \ln(1+e^{-x}) \quad y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$$

$$y(x) = C_1 + C_2 e^x + C_3 e^{2x} + \frac{1}{2}e^x - \left(\frac{1}{2} + e^x\right) \ln(e^x + 1) - \frac{1}{2}e^{2x} \ln(1+e^{-x})$$

### Exercise

Find the general solution  $y''' - 6y'' + 11y' - 6y = e^x$

### Solution

$$\text{Characteristic Eqn.: } \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \rightarrow \lambda_1 = 1$$

$$\begin{array}{c|ccc}
 1 & 1 & -6 & 11 & -6 \\
 & & 1 & -5 & 6 \\
 \hline
 & 1 & -5 & 6 & 0
 \end{array} \rightarrow \lambda^2 - 5\lambda + 6 = 0 \Rightarrow \lambda_{2,3} = 2, 3$$

$$y_h = C_1 e^x + C_2 e^{2x} + C_3 e^{3x}$$

$$\begin{aligned}
 W &= \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} \\
 &= 18e^{6x} + 3e^{6x} + 4e^{6x} - 2e^{6x} - 12e^{6x} - 9e^{6x} \\
 &= 2e^{6x} \neq 0
 \end{aligned}$$

$$W_1 = \begin{vmatrix} 0 & e^{2x} & e^{3x} \\ 0 & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^{6x}$$

$$W_2 = \begin{vmatrix} e^x & 0 & e^{3x} \\ e^x & 0 & 3e^{3x} \\ e^x & e^x & 9e^{3x} \end{vmatrix} = -2e^{5x}$$

$$W_3 = \begin{vmatrix} e^x & e^{2x} & 0 \\ e^x & 2e^{2x} & 0 \\ e^x & 4e^{2x} & e^x \end{vmatrix} = e^{4x}$$

$$\begin{aligned} u_1(x) &= \int \frac{e^{6x}}{2e^{6x}} dx \\ &= \frac{1}{2} \int dx \\ &= \frac{1}{2}x \end{aligned}$$

$$u_1 = \int \frac{W_1}{W}$$

$$\begin{aligned} u_2(x) &= - \int \frac{2e^{5x}}{2e^{6x}} dx \\ &= - \int e^{-x} dx \\ &= e^{-x} \end{aligned}$$

$$u_2 = \int \frac{W_2}{W}$$

$$\begin{aligned} u_3(x) &= \int \frac{e^{4x}}{2e^{6x}} dx \\ &= \frac{1}{2} \int e^{-2x} dx \\ &= -\frac{1}{4}e^{-2x} \end{aligned}$$

$$u_3 = \int \frac{W_3}{W}$$

$$\begin{aligned} y_p &= \frac{1}{2}xe^x + e^{-x}e^{2x} - \frac{1}{4}e^{-2x}e^{3x} \\ &= \frac{1}{2}xe^x + e^x - \frac{1}{4}e^x \\ &= \frac{1}{2}xe^x + \frac{3}{4}e^x \end{aligned}$$

$$y_p = u_1y_1 + u_2y_2 + u_3y_3$$

$$\begin{aligned} y(x) &= C_1e^x + C_2e^{2x} + C_3e^{3x} + \frac{1}{2}xe^x + \frac{3}{4}e^x \\ &= C_4e^x + C_2e^{2x} + C_3e^{3x} + \frac{1}{2}xe^x \end{aligned}$$

$$\left(C_1 + \frac{3}{4}\right)e^x = C_4e^x$$

### Exercise

Find the general solution  $x^3y^{(3)} - 4x^2y'' + 8xy' - 8y = 4\ln x$

### Solution

Characteristic Eqn.:  $\lambda(\lambda-1)(\lambda-2) - 4\lambda(\lambda-1) + 8\lambda - 8 = 0$

$$\lambda^3 - 3\lambda^2 + 2\lambda - 4\lambda^2 + 4\lambda + 8\lambda - 8 = 0$$



$$\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$$

$$\begin{array}{c|ccc} 1 & 1 & -7 & 14 & -8 \\ & & 1 & -6 & 8 \\ \hline & 1 & -6 & 8 & \color{red}{0} \end{array} \quad \rightarrow \lambda^2 - 6\lambda + 8 = 0 \quad \lambda = \frac{6 \pm 2}{2}$$

The roots are:  $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 4$

$$y_h = C_1 x + C_2 x^2 + C_3 x^4$$

$$W = \begin{vmatrix} x & x^2 & x^4 \\ 1 & 2x & 4x^3 \\ 0 & 2 & 16x^2 \end{vmatrix}$$

$$= 32x^4 + 2x^4 - 8x^4 - 16x^4$$

$$= 6x^4 \neq 0$$

$$y^{(3)} - \frac{4}{x}y'' + \frac{8}{x^2}y' - \frac{8}{x^3}y = \frac{4\ln x}{x^3}$$

$$W_1 = \begin{vmatrix} 0 & x^2 & x^4 \\ 0 & 2x & 4x^3 \\ \frac{4\ln x}{x^3} & 2 & 16x^2 \end{vmatrix} = \frac{4\ln x}{x^3} (4x^5 - 2x^5) = 8x^2 \ln x$$

$$W_2 = \begin{vmatrix} x & 0 & x^4 \\ 1 & 0 & 4x^3 \\ 0 & \frac{4\ln x}{x^3} & 16x^2 \end{vmatrix} = -\frac{4\ln x}{x^3} (4x^4 - x^4) = -12x \ln x$$

$$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & \frac{4\ln x}{x^3} \end{vmatrix} = \frac{4\ln x}{x^3} (2x^2 - x^2) = \frac{4\ln x}{x}$$

$$u_1(x) = \int \frac{8x^2 \ln x}{6x^4} dx$$

$$= \frac{4}{3} \int \frac{\ln x}{x^2} dx$$

$$= \frac{4}{3} \left( -\frac{\ln x}{x} + \int \frac{1}{x^2} dx \right)$$

$$u_1 = \int \frac{W_1}{W}$$

$u = \ln x$	$dv = x^{-2} dx$
$du = \frac{dx}{x}$	$v = -\frac{1}{x}$

$$= \frac{4}{3} \left( -\frac{\ln x}{x} - \frac{1}{x} \right)$$

$$= -\frac{4 \ln x}{3x} - \frac{4}{3x} \Big|$$

$$u_2(x) = \int \frac{-12x \ln x}{6x^4} dx$$

$$= -2 \int \frac{\ln x}{x^3} dx$$

$$= -2 \left( -\frac{\ln x}{2x^2} + \frac{1}{2} \int x^{-3} dx \right)$$

$$= -2 \left( -\frac{\ln x}{2x^2} - \frac{1}{4x^2} \right)$$

$$= \frac{\ln x}{x^2} + \frac{1}{2x^2} \Big|$$

$$u_3(x) = \int \frac{4 \ln x}{x} \frac{1}{6x^4} dx$$

$$= \frac{2}{3} \int \frac{\ln x}{x^5} dx$$

$$= \frac{2}{3} \left( -\frac{\ln x}{4x^4} + \frac{1}{4} \int \frac{1}{x^5} dx \right)$$

$$= \frac{2}{3} \left( -\frac{\ln x}{4x^4} - \frac{1}{16x^4} \right)$$

$$= -\frac{\ln x}{6x^4} - \frac{1}{24x^4} \Big|$$

$$y_p = \left( -\frac{4 \ln x}{3x} - \frac{4}{3x} \right) x + \left( \frac{\ln x}{x^2} + \frac{1}{2x^2} \right) x^2 + \left( -\frac{\ln x}{6x^4} - \frac{1}{24x^4} \right) x^4$$

$$= -\frac{4}{3} \ln x - \frac{4}{3} + \ln x + \frac{1}{2} - \frac{\ln x}{6} - \frac{1}{24}$$

$$= -\frac{1}{2} \ln x - \frac{7}{8} \Big|$$

$$y(x) = C_1 x + C_2 x^2 + C_3 x^4 - \frac{1}{2} \ln x - \frac{7}{8} \Big|$$

$$u_2 = \int \frac{W_2}{W}$$

$u = \ln x$	$dv = x^{-3} dx$
$du = \frac{dx}{x}$	$v = -\frac{1}{2} x^{-2}$

$$u_3 = \int \frac{W_3}{W}$$

$u = \ln x$	$dv = x^{-5} dx$
$du = \frac{dx}{x}$	$v = -\frac{1}{4} x^{-4}$

$$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$$

### Exercise

Find the general solution by to **variation of parameters** with the given initial conditions.

$$y'' + y = \sec t ; \quad y(0) = 1, \quad y'(0) = 2$$

### Solution

$$\lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$$

$$y_h = C_1 \cos t + C_2 \sin t$$

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1 \neq 0$$

$$\begin{aligned} v_1(t) &= -\int \frac{\sin t}{1} \sec t \, dt \\ &= -\int \tan t \, dt \\ &= \ln|\cos t| \end{aligned}$$

$$v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned} v_2(t) &= \int \frac{\cos t}{1} \sec t \, dt \\ &= \int dt \\ &= t \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$y_p = \ln|\cos t| \cos t + t \sin t$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$y(t) = C_1 \cos t + C_2 \sin t + \ln|\cos t| \cos t + t \sin t$$

$$y(0) = 1 \rightarrow C_1 = 1$$

$$y' = -C_1 \sin t + C_2 \cos t - \sin t - \ln|\cos t| \sin t + \sin t + t \cos t$$

$$y'(0) = 2 \rightarrow C_2 = 2$$

$$y(t) = \cos t + 2 \sin t + \ln|\cos t| \cos t + t \sin t$$

### Exercise

Find the general solution by to **variation of parameters** with the given initial conditions.

$$y'' + y = \sec^3 t; \quad y(0) = 1, \quad y'(0) = \frac{1}{2}$$

### Solution

$$\lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$$

$$y_h = C_1 \cos t + C_2 \sin t$$

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1 \neq 0$$

$$\begin{aligned}
 v_1(t) &= -\int \frac{\sin t}{1} \sec^3 t \, dt \\
 &= -\int \tan t \sec^2 t \, dt \\
 &= -\int \sec t \, d(\sec t) \\
 &= -\frac{1}{2} \sec^2 t \Big|
 \end{aligned}$$

$$v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned}
 v_2(t) &= \int \frac{\cos t}{1} \sec^3 t \, dt \\
 &= \int \sec^2 t \, dt \\
 &= \tan t \Big|
 \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$\begin{aligned}
 y_p &= -\frac{1}{2} \sec^2 t \cos t + \tan t \sin t \\
 &= -\frac{1}{2 \cos t} + \frac{\sin^2 t}{\cos t} \\
 &= \frac{1}{2} \frac{2 \sin^2 t - 1}{\cos t} \\
 &= -\frac{1}{2} \frac{\cos 2t}{\cos t} \Big|
 \end{aligned}$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$\begin{aligned}
 y(t) &= C_1 \cos t + C_2 \sin t - \frac{1}{2} \frac{\cos 2t}{\cos t} \Big| \\
 y(0) &= 1 \rightarrow C_1 - \frac{1}{2} = 1 \quad \underline{C_1 = \frac{3}{2}} \Big|
 \end{aligned}$$

$$\begin{aligned}
 y' &= -C_1 \sin t + C_2 \cos t - \frac{1}{2} \frac{-2 \sin 2t \cos 2t + \sin t \cos 2t}{\cos^2 t} \\
 y'(0) &= \frac{1}{2} \rightarrow \underline{C_2 = \frac{1}{2}} \Big|
 \end{aligned}$$

$$y(t) = \frac{3}{2} \cos t + \frac{1}{2} \sin t - \frac{1}{2} \frac{\cos 2t}{\cos t} \Big|$$

### Exercise

Find the general solution by to **variation of parameters** with the give n initial conditions.

$$y'' - y = t + \sin t ; \quad y(0) = 2, \quad y'(0) = 3$$

### Solution

$$\lambda^2 - 1 = 0 \Rightarrow \underline{\lambda_{1,2} = \pm 1}$$

$$\underline{y_h = C_1 e^{-t} + C_2 e^t}$$

$$W = \begin{vmatrix} e^{-t} & e^t \\ -e^{-t} & e^t \end{vmatrix} = 2 \neq 0$$

$$\begin{aligned} v_1(t) &= -\frac{1}{2} \int e^t (t + \sin t) dt \\ &= -\frac{1}{2} \int (te^t + e^t \sin t) dt \end{aligned}$$

$$\int (e^t \sin t) dt = e^t (-\cos t + \sin t) - \int (e^t \sin t) dt$$

$$2 \int (e^t \sin t) dt = e^t (\sin t - \cos t)$$

$$= -\frac{1}{2} \left[ (t-1)e^t + \frac{1}{2}e^t (\sin t - \cos t) \right]$$

$$= -\frac{1}{2} \left( t-1 + \frac{1}{2} \sin t - \frac{1}{2} \cos t \right) e^t$$

$$v_2(t) = \frac{1}{2} \int e^{-t} (t + \sin t) dt$$

$$= \frac{1}{2} \int (te^{-t} + e^{-t} \sin t) dt$$

$$\int (e^{-t} \sin t) dt = e^{-t} (-\cos t - \sin t) - \int (e^{-t} \sin t) dt$$

$$2 \int (e^{-t} \sin t) dt = -e^{-t} (\sin t + \cos t)$$

$$= \frac{1}{2} \left[ (-t-1)e^{-t} - \frac{1}{2}e^{-t} (\sin t + \cos t) \right]$$

$$= -\frac{1}{2} \left( t+1 + \frac{1}{2} \sin t + \frac{1}{2} \cos t \right) e^{-t}$$

$$y_p = -\frac{1}{2} \left( t-1 + \frac{1}{2} \sin t - \frac{1}{2} \cos t \right) e^t e^{-t} - \frac{1}{2} \left( t+1 + \frac{1}{2} \sin t + \frac{1}{2} \cos t \right) e^{-t} e^t$$

$$= -\frac{1}{2}t + \frac{1}{2} - \frac{1}{4} \sin t + \frac{1}{4} \cos t - \frac{1}{2}t - \frac{1}{2} - \frac{1}{4} \sin t - \frac{1}{4} \cos t$$

$$= -t - \frac{1}{2} \sin t$$

$$y(t) = C_1 e^{-t} + C_2 e^t - t - \frac{1}{2} \sin t$$

$$y(0) = 2 \rightarrow C_1 + C_2 = 2$$

$$y' = -C_1 e^{-t} + C_2 e^t - 1 - \frac{1}{2} \cos t$$

$$y'(0) = 3 \rightarrow -C_1 + C_2 - 1 - \frac{1}{2} = 3 \Rightarrow -C_1 + C_2 = \frac{9}{2}$$

		$\int e^t$
+	$t$	$e^t$
-	$1$	$e^t$

		$\int \sin t$
+	$e^t$	$-\cos t$
-	$e^t$	$-\sin t$
+	$e^t$	

		$\int e^{-t}$
+	$t$	$-e^{-t}$
-	$1$	$e^{-t}$

		$\int \sin t$
+	$e^{-t}$	$-\cos t$
-	$-e^{-t}$	$-\sin t$
+	$e^{-t}$	

$$y_p = v_1 y_1 + v_2 y_2$$

$$\begin{cases} C_1 + C_2 = 2 \\ -C_1 + C_2 = \frac{9}{2} \end{cases} \quad \underline{C_2 = \frac{13}{4} \quad C_1 = -\frac{5}{4}}$$

$$\underline{y(t) = -\frac{5}{4}e^{-t} + \frac{13}{4}e^t - t - \frac{1}{2}\sin t}$$

### Exercise

Find the general solution by to **variation of parameters** with the give n initial conditions.

$$y'' - 2y' + y = \frac{e^x}{x}; \quad y(1) = 0, \quad y'(1) = 0$$

### Solution

$$\lambda^2 - 2\lambda + 1 = 0 \rightarrow \underline{\lambda_{1,2} = 1}$$

$$\underline{y_h = C_1 e^x + C_2 x e^x}$$

$$W = \begin{vmatrix} e^x & x e^x \\ e^x & e^x + x e^x \end{vmatrix} = e^{2x} + x e^{2x} - x e^{2x} = \underline{e^{2x} \neq 0}$$

$$\begin{aligned} v_1(x) &= - \int \frac{x e^x}{e^{2x}} \frac{e^x}{x} dx \\ &= - \int dx \\ &= \underline{-x} \end{aligned}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned} v_2(x) &= \int \frac{e^x}{e^{2x}} \frac{e^x}{x} dx \\ &= \int \frac{dx}{x} \\ &= \underline{\ln|x|} \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$\underline{y_p = -x e^x + x e^x \ln|x|}$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$y(x) = C_1 e^x + C_2 x e^x - x e^x + x e^x \ln|x|$$

$$\underline{y(1) = 0} \rightarrow e C_1 + e C_2 - e = 0 \Rightarrow C_1 + C_2 = 1$$

$$\begin{aligned} y' &= C_1 e^x + C_2 e^x + C_2 x e^x - x e^x - e^x + e^x \ln|x| + x e^x \ln|x| + e^x \\ &= C_1 e^x + C_2 e^x + C_2 x e^x - x e^x + e^x \ln|x| + x e^x \ln|x| \end{aligned}$$

$$\underline{y'(1) = 0} \rightarrow e C_1 + 2e C_2 - e = 0 \Rightarrow C_1 + 2C_2 = 1$$

$$\begin{cases} C_1 + C_2 = 1 \\ -C_1 - 2C_2 = -1 \end{cases} \quad \underline{C_2 = 0} \quad \underline{C_1 = 1}$$

$$\underline{y(x) = e^x - xe^x + xe^x \ln|x|}$$

### Exercise

Find the general solution by to **variation of parameters** with the given initial conditions.

$$y'' + 2y' - 8y = 2e^{-2x} - e^{-x}; \quad y(0) = 1, \quad y'(0) = 0$$

### Solution

$$\lambda^2 + 2\lambda - 8 = 0 \Rightarrow \underline{\lambda_{1,2} = -4, 2}$$

$$\underline{y_h = C_1 e^{-4x} + C_2 e^{2x}}$$

$$W = \begin{vmatrix} e^{-4x} & e^{2x} \\ -4e^{-4x} & 2e^{2x} \end{vmatrix} = 2e^{-2x} + 4e^{-2x} = \underline{6e^{-2x} \neq 0}$$

$$\begin{aligned} v_1(x) &= - \int \frac{e^{2x}}{6e^{-2x}} (2e^{-2x} - e^{-x}) dx \\ &= -\frac{1}{6} \int (2e^{2x} - e^{3x}) dx \\ &= -\frac{1}{6} \left( e^{2x} - \frac{1}{3} e^{3x} \right) \end{aligned}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned} v_2(x) &= \frac{1}{6} \int \frac{e^{-4x}}{e^{-2x}} (2e^{-2x} - e^{-x}) dx \\ &= \frac{1}{6} \int (2e^{-4x} - e^{-3x}) dx \\ &= \frac{1}{6} \left( -\frac{1}{2} e^{-4x} + \frac{1}{3} e^{-3x} \right) \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$\begin{aligned} y_p &= \left( -\frac{1}{6} e^{2x} + \frac{1}{18} e^{3x} \right) e^{-4x} + \left( -\frac{1}{12} e^{-4x} + \frac{1}{18} e^{-3x} \right) e^{2x} \\ &= -\frac{1}{6} e^{-2x} + \frac{1}{18} e^{-x} - \frac{1}{12} e^{-2x} + \frac{1}{18} e^{-x} \\ &= \underline{-\frac{1}{4} e^{-2x} + \frac{1}{9} e^{-x}} \end{aligned}$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$\underline{y(x) = C_1 e^{-4x} + C_2 e^{2x} - \frac{1}{4} e^{-2x} + \frac{1}{9} e^{-x}}$$

$$\underline{y(0) = 1} \rightarrow C_1 + C_2 - \frac{1}{4} + \frac{1}{9} = 1 \Rightarrow C_1 + C_2 = \frac{41}{36}$$

$$y'(x) = -4C_1 e^{-4x} + 2C_2 e^{2x} + \frac{1}{2}e^{-2x} - \frac{1}{9}e^{-x}$$

$$y'(0) = 0 \rightarrow -4C_1 + 2C_2 + \frac{1}{2} - \frac{1}{9} = 0 \Rightarrow -4C_1 + 2C_2 = -\frac{7}{18}$$

$$\begin{cases} C_1 + C_2 = \frac{41}{36} \\ -4C_1 + 2C_2 = -\frac{7}{18} \end{cases} \rightarrow \underline{C_1 = \frac{4}{9}, C_2 = \frac{25}{36}}$$

$$\underline{y(x) = \frac{4}{9}e^{-4x} + \frac{25}{36}e^{2x} - \frac{1}{4}e^{-2x} + \frac{1}{9}e^{-x}}$$

### Exercise

Find the general solution by to **variation of parameters** with the given initial conditions.

$$y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x; \quad y(0) = 1, \quad y'(0) = 2$$

### Solution

$$\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \underline{\lambda_{1,2} = 1, 2}$$

$$\underline{y_h = C_1 e^x + C_2 e^{2x}}$$

$$W = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = \underline{e^{3x} \neq 0}$$

$$v_1(x) = - \int \frac{e^{2x}}{e^{3x}} (3e^{-x} - 10\cos 3x) dx$$

$$= - \int e^{-x} (3e^{-x} - 10\cos 3x) dx$$

$$= \int (10e^{-x} \cos 3x - 3e^{-2x}) dx$$

$$= 10 \int e^{-x} \cos 3x dx - \int 3e^{-2x} dx$$

$$\int e^{-x} \cos 3x dx = e^{-x} \left( \frac{1}{3} \sin 3x - \frac{1}{9} \cos 3x \right) - \frac{1}{9} \int e^{-x} \cos 3x dx$$

$$\frac{10}{9} \int e^{-x} \cos 3x dx = \frac{1}{9} e^{-x} (3 \sin 3x - \cos 3x)$$

$$\int e^{-x} \cos 3x dx = \frac{1}{10} e^{-x} (3 \sin 3x - \cos 3x)$$

$$\underline{= e^{-x} (3 \sin 3x - \cos 3x) + \frac{3}{2} e^{-2x}}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

		$\int \cos 3x$
+	$e^{-x}$	$\frac{1}{3} \sin 3x$
-	$-e^{-x}$	$-\frac{1}{9} \cos 3x$
+	$e^{-x}$	



$$\begin{aligned}
 v_2(x) &= \int \frac{e^x}{e^{3x}} (3e^{-x} - 10 \cos 3x) dx \\
 &= \int e^{-2x} (3e^{-x} - 10 \cos 3x) dx \\
 &= 3 \int e^{-3x} dx - 10 \int e^{-2x} \cos 3x dx
 \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$\begin{aligned}
 \int e^{-2x} \cos 3x dx &= e^{-2x} \left( \frac{1}{3} \sin 3x - \frac{2}{9} \cos 3x \right) - \frac{4}{9} \int e^{-2x} \cos 3x dx \\
 \frac{13}{9} \int e^{-2x} \cos 3x dx &= \frac{1}{9} e^{-2x} (3 \sin 3x - 2 \cos 3x) \\
 \int e^{-2x} \cos 3x dx &= \frac{1}{13} e^{-2x} (3 \sin 3x - 2 \cos 3x) \\
 &= -e^{-3x} - \frac{10}{13} e^{-2x} (3 \sin 3x - 2 \cos 3x)
 \end{aligned}$$

		$\int \cos 3x$
+	$e^{-2x}$	$\frac{1}{3} \sin 3x$
-	$-2e^{-2x}$	$-\frac{1}{9} \cos 3x$
+	$4e^{-2x}$	

$$\begin{aligned}
 y_p &= \left( e^{-x} (3 \sin 3x - \cos 3x) + \frac{3}{2} e^{-2x} \right) e^x + \left( -e^{-3x} - \frac{10}{13} e^{-2x} (3 \sin 3x - 2 \cos 3x) \right) e^{2x} \\
 &= 3 \sin 3x - \cos 3x + \frac{3}{2} e^{-x} - e^{-x} - \frac{30}{13} \sin 3x + \frac{20}{13} \cos 3x \\
 &= \frac{1}{2} e^{-x} + \frac{9}{13} \sin 3x + \frac{7}{13} \cos 3x
 \end{aligned}$$

$$y(x) = C_1 e^x + C_2 e^{2x} + \frac{1}{2} e^{-x} + \frac{9}{13} \sin 3x + \frac{7}{13} \cos 3x$$

$$y(0) = 1 \rightarrow C_1 + C_2 + \frac{1}{2} + \frac{7}{13} = 1 \Rightarrow C_1 + C_2 = -\frac{1}{26}$$

$$y'(x) = C_1 e^x + 2C_2 e^{2x} - \frac{1}{2} e^{-x} + \frac{27}{13} \cos 3x - \frac{21}{13} \sin 3x$$

$$y'(0) = 2 \rightarrow C_1 + 2C_2 - \frac{1}{2} + \frac{27}{13} = 2 \Rightarrow C_1 + 2C_2 = \frac{11}{26}$$

$$\begin{cases} C_1 + C_2 = -\frac{1}{26} \\ C_1 + 2C_2 = \frac{11}{26} \end{cases} \rightarrow \Delta = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \quad \Delta_1 = \begin{vmatrix} -\frac{1}{26} & 1 \\ \frac{11}{26} & 2 \end{vmatrix} = -\frac{1}{2} \quad \Delta_2 = \begin{vmatrix} 1 & -\frac{1}{26} \\ 1 & \frac{11}{26} \end{vmatrix} = \frac{6}{13}$$

$$C_1 = -\frac{1}{2}, C_2 = \frac{6}{13}$$

$$y(x) = -\frac{1}{2} e^x + \frac{6}{13} e^{2x} + \frac{1}{2} e^{-x} + \frac{9}{13} \sin 3x + \frac{7}{13} \cos 3x$$

### Exercise

Find the general solution by to *variation of parameters* with the give n initial conditions.

$$y'' + 4y = \sin^2 2t ; \quad y\left(\frac{\pi}{8}\right) = 0, \quad y'\left(\frac{\pi}{8}\right) = 0$$

### Solution

$$\lambda^2 + 4 = 0 \rightarrow \lambda_{1,2} = \pm 2i$$

$$y_h = C_1 \cos 2t + C_2 \sin 2t$$

$$W = \begin{vmatrix} \cos 2t & \sin 2t \\ -2 \sin 2t & 2 \cos 2t \end{vmatrix} = 2 \cos^2 2t + 2 \sin^2 2t = 2 \neq 0$$

$$\begin{aligned} v_1(t) &= - \int \frac{\sin 2t \sin^2 2t}{2} dt \\ &= \frac{1}{4} \int (1 - \cos^2 2t) d(\cos 2t) \\ &= \frac{1}{4} \left( \cos 2t - \frac{1}{3} \cos^3 2t \right) \end{aligned}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned} v_2(t) &= \frac{1}{2} \int \cos 2t \sin^2 2t dt \\ &= \frac{1}{4} \int \sin^2 2t d(\sin 2t) \\ &= \frac{1}{12} \sin^3 2t \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$\begin{aligned} y_p &= \frac{1}{4} \left( \cos 2t - \frac{1}{3} \cos^3 2t \right) \cos 2t + \frac{1}{12} \sin^3 2t (\sin 2t) \\ &= \frac{1}{4} \cos^2 2t - \frac{1}{12} \cos^4 2t + \frac{1}{12} \sin^4 2t \\ &= \frac{1}{4} \cos^2 2t - \frac{1}{12} (\cos^4 2t - \sin^4 2t) \\ &= \frac{1}{4} \cos^2 2t - \frac{1}{12} (\cos^2 2t - \sin^2 2t) (\cos^2 2t + \sin^2 2t) \\ &= \frac{1}{8} (1 + \cos 4t) - \frac{1}{12} \cos 4t \\ &= \frac{1}{8} + \frac{1}{24} \cos 4t \end{aligned} \quad y_p = v_1 y_1 + v_2 y_2$$

$$y(t) = C_1 \cos 2t + C_2 \sin 2t + \frac{1}{8} + \frac{1}{24} \cos 4t$$

$$y\left(\frac{\pi}{8}\right) = 0 \rightarrow \frac{\sqrt{2}}{2} C_1 + \frac{\sqrt{2}}{2} C_2 + \frac{1}{8} = 0 \Rightarrow \sqrt{2} C_1 + \sqrt{2} C_2 = -\frac{1}{4}$$

$$y' = -2C_1 \sin 2t + 2C_2 \cos 2t - \frac{1}{6} \sin 4t$$

$$y'\left(\frac{\pi}{8}\right)=0 \rightarrow -\sqrt{2}C_1 + \sqrt{2}C_2 = \frac{1}{6}$$

$$\begin{cases} \sqrt{2}C_1 + \sqrt{2}C_2 = -\frac{1}{4} \\ -\sqrt{2}C_1 + \sqrt{2}C_2 = \frac{1}{6} \end{cases} \quad \underline{C_2 = -\frac{1}{24\sqrt{2}}} \quad \sqrt{2}C_1 = -\frac{1}{4} + \frac{1}{24} \Rightarrow \underline{C_1 = -\frac{5}{24\sqrt{2}}}$$

$$\underline{y(t) = -\frac{5\sqrt{2}}{48}\cos 2t - \frac{\sqrt{2}}{48}\sin 2t + \frac{1}{8} + \frac{1}{24}\cos 4t}$$

### Exercise

Find the general solution by to **variation of parameters** with the give n initial conditions.

$$y'' + 4y = \sin^2 2t ; \quad y(0) = 0, \quad y'(0) = 0$$

### Solution

$$\lambda^2 + 4 = 0 \rightarrow \underline{\lambda_{1,2} = \pm 2i}$$

$$\underline{y_h = C_1 \cos 2t + C_2 \sin 2t}$$

$$W = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2\cos^2 2t + 2\sin^2 2t = \underline{2 \neq 0}$$

$$\begin{aligned} v_1(t) &= -\int \frac{\sin 2t \sin^2 2t}{2} dt \\ &= \frac{1}{4} \int (1 - \cos^2 2t) d(\cos 2t) \\ &= \underline{\frac{1}{4} \left( \cos 2t - \frac{1}{3} \cos^3 2t \right)} \end{aligned}$$

$$v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned} v_2(t) &= \frac{1}{2} \int \cos 2t \sin^2 2t dt \\ &= \frac{1}{4} \int \sin^2 2t d(\sin 2t) \\ &= \underline{\frac{1}{12} \sin^3 2t} \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$\begin{aligned} y_p &= \frac{1}{4} \left( \cos 2t - \frac{1}{3} \cos^3 2t \right) \cos 2t + \frac{1}{12} \sin^3 2t (\sin 2t) \\ &= \frac{1}{4} \cos^2 2t - \frac{1}{12} \cos^4 2t + \frac{1}{12} \sin^4 2t \\ &= \frac{1}{4} \cos^2 2t - \frac{1}{12} (\cos^4 2t - \sin^4 2t) \\ &= \frac{1}{4} \cos^2 2t - \frac{1}{12} (\cos^2 2t - \sin^2 2t) (\cos^2 2t + \sin^2 2t) \\ &= \frac{1}{8} (1 + \cos 4t) - \frac{1}{12} \cos 4t \end{aligned}$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$= \frac{1}{8} + \frac{1}{24} \cos 4t \quad |$$

$$y(t) = C_1 \cos 2t + C_2 \sin 2t + \frac{1}{8} + \frac{1}{24} \cos 4t$$

$$y(0) = 0 \rightarrow C_1 + \frac{1}{8} + \frac{1}{24} = 0 \Rightarrow \underline{C_1 = -\frac{1}{6}} \quad |$$

$$y' = -2C_1 \sin 2t + 2C_2 \cos 2t - \frac{1}{6} \sin 4t$$

$$y'(0) = 0 \rightarrow \underline{C_2 = 0} \quad |$$

$$\underline{y(t) = -\frac{1}{6} \cos 2t + \frac{1}{8} + \frac{1}{24} \cos 4t} \quad |$$

### Exercise

Find the general solution by to **variation of parameters** with the given initial conditions.

$$y'' - 4y' + 4y = (12x^2 - 6x)e^{2x}; \quad y(0) = 1, \quad y'(0) = 0$$

### Solution

$$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0 \Rightarrow \underline{\lambda_{1,2} = 2} \quad |$$

$$\underline{y_h = C_1 e^{2x} + C_2 x e^{2x}} \quad |$$

$$W = \begin{vmatrix} e^{2x} & x e^{2x} \\ 2e^{2x} & (1+2x)e^{2x} \end{vmatrix} = e^{4x} + 2x e^{4x} - 2x e^{4x} = \underline{e^{4x} \neq 0} \quad |$$

$$\begin{aligned} v_1(x) &= - \int \frac{x e^{2x}}{e^{4x}} (12x^2 - 6x) e^{2x} dx \\ &= - \int (12x^3 - 6x^2) dx \\ &= \underline{2x^3 - 3x^4} \quad | \end{aligned}$$

$$v_1(x) = - \int \frac{y_2 g(x)}{W} dx$$

$$\begin{aligned} v_2(x) &= \int \frac{e^{2x}}{e^{4x}} (12x^2 - 6x) e^{2x} dx \\ &= \int (12x^2 - 6x) dx \\ &= \underline{4x^3 - 3x^2} \quad | \end{aligned}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$\begin{aligned} y_p &= (2x^3 - 3x^4) e^{2x} + (4x^3 - 3x^2) x e^{2x} \\ &= (2x^3 - 3x^4 + 4x^4 - 3x^3) e^{2x} \end{aligned}$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$= \left( x^4 - x^3 \right) e^{2x} \Big|$$

$$y(x) = \left( C_1 + C_2 x - x^3 + x^4 \right) e^{2x}$$

$$y(0) = 1 \rightarrow \underline{C_1 = 1}$$

$$y'(x) = \left( C_2 - 3x^2 + 4x^3 + 2C_1 + 2C_2 x - 2x^3 + 2x^4 \right) e^{2x}$$

$$y'(0) = 0 \rightarrow C_2 + 2C_1 = 0 \Rightarrow \underline{C_2 = -2}$$

$$y(x) = \left( 1 - 2x - x^3 + x^4 \right) e^{2x} \Big|$$

### Exercise

Find the general solution by to **variation of parameters** with the given initial conditions.

$$2y'' + y' - y = x + 1; \quad y(0) = 1, \quad y'(0) = 0$$

### Solution

$$2\lambda^2 + \lambda - 1 = 0 \Rightarrow \underline{\lambda_{1,2} = -1, \frac{1}{2}}$$

$$y_h = C_1 e^{-x} + C_2 e^{x/2} \Big|$$

$$W = \begin{vmatrix} e^{-x} & e^{x/2} \\ -e^{-x} & \frac{1}{2}e^{x/2} \end{vmatrix} = \frac{1}{2}e^{-x/2} + e^{-x/2} = \underline{\frac{3}{2}e^{-x/2} \neq 0}$$

$$y'' + \frac{1}{2}y' - \frac{1}{2}y = \frac{1}{2}(x+1)$$

$$\begin{aligned} v_1(x) &= - \int \frac{2}{3} \frac{e^{x/2}}{e^{-x/2}} \frac{1}{2}(x+1) dx & v_1(x) &= - \int \frac{y_2 g(x)}{W} dx \\ &= - \frac{1}{3} \int (x+1) e^x dx \\ &= \underline{-\frac{1}{3} x e^x} \end{aligned}$$

		$\int e^x$
+	$x+1$	$e^x$
-	1	$e^x$

$$\begin{aligned} v_2(x) &= \frac{1}{3} \int \frac{e^{-x}}{e^{-x/2}} (x+1) dx & v_2(x) &= \int \frac{y_1 g(x)}{W} dx \\ &= \frac{1}{3} \int (x+1) e^{-x/2} dx \\ &= \underline{\frac{1}{3}(-2x-6)e^{-x/2}} \end{aligned}$$

		$\int e^{-x/2}$
+	$x+1$	$-2e^{-x/2}$
-	1	$4e^{-x/2}$

$$y_p = -\frac{1}{3} x e^x e^{-x} - \frac{2}{3} (x+3) e^{-x/2} e^{x/2}$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$= -\frac{1}{3}x - \frac{2}{3}x - 2$$

$$= -x - 2$$

$$y(x) = C_1 e^{-x} + C_2 e^{x/2} - x - 2$$

$$y(0) = 1 \rightarrow C_1 + C_2 = 3$$

$$y'(x) = -C_1 e^{-x} + \frac{1}{2}C_2 e^{x/2} - 1$$

$$y'(0) = 0 \rightarrow -C_1 + \frac{1}{2}C_2 - 1 = 0$$

$$-2C_1 + C_2 = 2$$

$$\begin{cases} C_1 + C_2 = 3 \\ -2C_1 + C_2 = 2 \end{cases} \rightarrow C_1 = \frac{1}{3}, C_2 = \frac{8}{3}$$

$$y(x) = \frac{1}{3}e^{-x} + \frac{8}{3}e^{x/2} - x - 2$$

### Exercise

Find the general solution by to **variation of parameters** with the given initial conditions.

$$4y'' - y = xe^{x/2}; \quad y(0) = 1, \quad y'(0) = 0$$

### Solution

$$4\lambda^2 - 1 = 0 \Rightarrow \lambda_{1,2} = \pm \frac{1}{2}$$

$$y_h = C_1 e^{-x/2} + C_2 e^{x/2}$$

$$W = \begin{vmatrix} e^{-x/2} & e^{x/2} \\ -\frac{1}{2}e^{-x/2} & \frac{1}{2}e^{x/2} \end{vmatrix} = \frac{1}{2} + \frac{1}{2} = 1 \neq 0$$

$$y'' - \frac{1}{4}y = \frac{1}{4}xe^{x/2}$$

$$v_1(x) = -\int \frac{1}{4}xe^{x/2}e^{x/2} dx$$

$$= -\frac{1}{4} \int xe^x dx$$

$$= -\frac{1}{4}(x-1)e^x$$

$$v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$v_2(x) = \int \frac{1}{4}xe^{x/2}e^{-x/2} dx$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \frac{1}{4} \int x \, dx$$

$$= \frac{1}{8} x^2 \Big|$$

$$y_p = -\frac{1}{4}(x-1)e^x e^{-x/2} + \frac{1}{8}x^2 e^{x/2}$$

$$= \left( \frac{1}{8}x^2 - \frac{1}{4}x + \frac{1}{4} \right) e^{x/2} \Big|$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$y(x) = C_1 e^{-x/2} + C_2 e^{x/2} + \left( \frac{1}{8}x^2 - \frac{1}{4}x + \frac{1}{4} \right) e^{x/2}$$

$$= C_1 e^{-x/2} + \left( \frac{1}{8}x^2 - \frac{1}{4}x + C_3 \right) e^{x/2}$$

$$y(0) = 1 \rightarrow C_1 + C_3 = 1 \Big|$$

$$y' = -\frac{1}{2}C_1 e^{-x/2} + \left( \frac{1}{4}x - \frac{1}{4} + \frac{1}{16}x^2 - \frac{1}{8}x + \frac{1}{2}C_3 \right) e^{x/2}$$

$$y'(0) = 0 \rightarrow -\frac{1}{2}C_1 - \frac{1}{4} + \frac{1}{2}C_3 = 0$$

$$-2C_1 + 2C_3 = 1 \Big|$$

$$\begin{cases} C_1 + C_3 = 1 \\ -2C_1 + 2C_3 = 1 \end{cases} \rightarrow C_3 = \frac{3}{4}, C_1 = \frac{1}{4} \Big|$$

$$y(x) = \frac{1}{4}e^{-x/2} + \left( \frac{1}{8}x^2 - \frac{1}{4}x + \frac{3}{4} \right) e^{x/2} \Big|$$

## Exercise

Find the general solution  $t^2 y'' - ty' + y = t$  ;  $y(1) = 1, \quad y'(1) = 4$

## Solution

Characteristic Eqn.:  $\lambda(\lambda-1) - \lambda + 1 = 0$

$$\lambda^2 - 2\lambda + 1 = 0$$

The roots are:  $\lambda_{1,2} = 1 \Big|$

$$y_h = (C_1 + C_2 \ln t) e^{\ln t}$$

$$= (C_1 + C_2 \ln t) t$$

$$= C_1 t + C_2 t \ln t \Big|$$

$$W = \begin{vmatrix} t & t \ln t \\ 1 & 1 + \ln t \end{vmatrix}$$

$$= t + t \ln t - t \ln t$$

$$= t \neq 0$$

$$y'' - \frac{1}{t} y' - \frac{1}{t^2} y = \frac{1}{t}$$

$$W_1 = \begin{vmatrix} 0 & t \ln t \\ \frac{1}{t} & 1 + \ln t \end{vmatrix} = -\ln t$$

$$W_2 = \begin{vmatrix} t & 0 \\ 1 & \frac{1}{t} \end{vmatrix} = 1$$

$$\begin{aligned} u_1(t) &= \int \frac{-\ln t}{t} dt \\ &= -\int \ln t \, d(\ln t) \\ &= -\frac{1}{2}(\ln t)^2 \end{aligned}$$

$$u_1 = \int \frac{W_1}{W}$$

$$\begin{aligned} u_2(t) &= \int \frac{1}{t} dt \\ &= \ln t \end{aligned}$$

$$u_2 = \int \frac{W_2}{W}$$

$$\begin{aligned} y_p &= -\frac{1}{2}(\ln t)^2 t + (\ln t)t \ln t \\ &= \frac{1}{2}t(\ln t)^2 \end{aligned}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y(x) = C_1 t + C_2 t \ln t + \frac{1}{2} t (\ln t)^2$$

$$y(1) = 1 \rightarrow C_1 = 1$$

$$y'(t) = C_1 + C_2 (1 + \ln t) + \frac{1}{2} ((\ln t)^2 + 2 \ln t)$$

$$y'(1) = 4 \rightarrow C_1 + C_2 = 4 \quad C_2 = 3$$

$$y(x) = t + 3t \ln t + \frac{1}{2} t (\ln t)^2$$