Solution   *Section* 1.4 – Linear Equations

Exercise

Find the general solution of $y' - y = 3e^{t}$

Solution

$$y_h = e^{\int -dt} = e^{-t}$$

$$\int 3e^{t}e^{-t}dt = \int 3dt = 3t$$

$$y(t) = \frac{1}{e^{-t}}(3t + C)$$

$$\boxed{y(t) = 3te^{t} + Ce^{t}}$$

Exercise

Find the general solution of $y' + y = \sin t$

Solution

$$e^{\int dt} = e^{t}$$

$$\int e^{t}\sin t\ dt = e^{t}\sin t - \int e^{t}\cos t\ dt$$

$$= e^{t}\sin t - e^{t}\cos t - \int e^{t}\sin t\ dt$$

$$u = \sin t \qquad dv = e^{t}dt$$

$$du = \cos t\ dt \qquad v = e^{t}$$

$$u = \cos t \qquad dv = e^{t}dt$$

$$du = -\sin t\ dt \qquad v = e^{t}$$

$$2\int e^{t}\sin t\ dt = e^{t}\sin t - e^{t}\cos t$$

$$\int e^{t}\sin t\ dt = \frac{1}{2}e^{t}(\sin t - \cos t)$$

$$y(t) = \frac{1}{e^{t}}\left(\frac{1}{2}e^{t}(\sin t - \cos t) + C\right)$$

$$\boxed{= \frac{1}{2}\sin t - \frac{1}{2}\cos t + Ce^{-t}}$$

Exercise

Find the general solution of $y' + y = \frac{1}{1+e^t}$

Solution

$$e^{\int dt} = e^t$$

$$\int \frac{e^t}{1+e^t}\,dt = \int \frac{1}{1+e^t}\,d\left(1+e^t\right) = \ln\left(1+e^t\right)$$

$$y(t) = \frac{1}{e^t}\left(\ln\left(1+e^t\right)+C\right)$$

$$\boxed{= e^{-t}\ln\left(1+e^t\right) + Ce^{-t}}$$

Exercise

Find the general solution of $y' - y = e^{2t} - 1$

Solution

$$e^{-\int dt} = e^{-t}$$

$$\int \left(e^{2t}-1\right)e^{-t}\,dt = \int \left(e^t - e^{-t}\right)dt = e^t + e^{-t}$$

$$y(t) = \frac{1}{e^{-t}}\left(e^t + e^{-t} + C\right)$$

$$\boxed{= e^{2t} + 1 + Ce^t}$$

Exercise

Find the general solution of $y' + y = te^{-t} + 1$

Solution

$$e^{\int dt} = e^t$$

$$\int \left(te^{-t}+1\right)e^t\,dt = \int \left(t + e^t\right)dt = t + e^t$$

$$y(t) = \frac{1}{e^t}\left(t + e^t + C\right)$$

$$\boxed{= te^{-t} + 1 + Ce^{-t}}$$

Exercise

Find the general solution of $y' + y = 1 + e^{-x}\cos 2x$

Solution

$$e^{\int dx} = e^{x}$$

$$\int \left(1 + e^{-x}\cos 2x\right)e^{x}dx = \int \left(e^{x} + \cos 2x\right)dx = e^{x} + \frac{1}{2}\sin 2x$$

$$y(x) = e^{-x}\left(e^{x} + \frac{1}{2}\sin 2x + C\right)$$

$$\boxed{= e^{x} + \frac{1}{2}e^{-x}\sin 2x + Ce^{-x}}$$

Exercise

Solve the differential equation: $y' + y\cot x = \cos x$

Solution

$$e^{\int \cot x dx} = e^{\ln|\sin x|} = \sin x$$

$$\int \cos x \sin x \, dx = \int \sin x \, d(\sin x) = \frac{1}{2}\sin^2 x$$

$$y(x) = \frac{1}{\sin x}\left(\frac{1}{2}\sin^2 x + C\right)$$

$$\boxed{= \frac{1}{2}\sin x + \frac{C}{\sin x}}$$

Exercise

Solve the differential equation: $y' + y\sin t = \sin t$

Solution

$$e^{\int \sin t dt} = e^{-\cos t}$$

$$\int (\sin t)e^{-\cos t}dt = \int e^{-\cos t}d(-\cos t) = e^{-\cos t}$$

$$y(x) = \frac{1}{e^{-\cos t}}\left(e^{-\cos t} + C\right)$$

$$\boxed{= 1 + Ce^{\cos t}}$$

Exercise

Find the general solution of $y' = \cos x - y\sec x$

Solution

$$y' + (\sec x)y = \cos x$$

$$e^{\int \sec x dx} = e^{\ln|\sec x + \tan x|} = \sec x + \tan x$$

$$\int \cos x(\sec x + \tan x)dx = \int \cos x\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right)dx$$

$$= \int (1 + \sin x)dx$$

$$= x - \cos x$$

$$\boxed{y(x) = \frac{1}{\sec x + \tan x}(x - \cos x + C)}$$

Exercise

Solve the differential equation: $\quad y' + (\tan x)y = \cos^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

Solution

$$y' + (\tan x)y = \cos^2 x, \quad P(x) = \tan x, \quad Q(x) = \cos^2 x$$

$$y_h = e^{\int \tan x dx} = e^{\ln(\cos x)^{-1}} = (\cos x)^{-1} \qquad \int \tan x dx = -\ln|\cos x| = \ln(\cos x)^{-1}$$

$$\int \cos^2 x(\cos x)^{-1}dx = \int \cos x dx = \sin x$$

$$y(x) = \frac{1}{(\cos x)^{-1}}(\sin x + C)$$

$$y(x) = \cos x(\sin x + C)$$

$$\boxed{= \cos x \sin x + C\cos x}$$

Exercise

Solve the differential equation: $\quad y' + (\cot t)y = 2t\csc t$

Solution

$$e^{\int \cot t\, dt} = e^{\ln|\sin t|} = \sin t$$

$$\int 2t\csc t \sin t dt = \int 2t dt = t^2$$

$$y(t) = \frac{1}{\sin t}\left(t^2 + C\right)$$
$$\boxed{= \left(t^2 + C\right)\csc t}$$

Exercise

Solve the differential equation: $y' + (1 + \sin t)y = 0$

Solution

$$e^{\int (1+\sin t)dt} = e^{t - \cos t}$$
$$y(x) = \frac{C}{e^{t-\cos t}}$$
$$\boxed{= C\, e^{\cos t - t}}$$

Exercise

Find the general solution of $y' + \left(\frac{1}{2}\cos x\right)y = -\frac{3}{2}\cos x$

Solution

$$e^{\int \frac{1}{2}\cos x dx} = e^{\frac{1}{2}\sin x}$$
$$\int \left(-\frac{3}{2}\cos x\right)e^{\frac{1}{2}\sin x}dx = -3\int e^{\frac{1}{2}\sin x}d\left(\frac{1}{2}\sin x\right) = -3e^{\frac{1}{2}\sin x}$$
$$y(x) = e^{-\frac{1}{2}\sin x}\left(-3e^{\frac{1}{2}\sin x} + C\right)$$
$$\boxed{= -3 + Ce^{-\frac{1}{2}\sin x}}$$

Exercise

Solve the differential equation: $\frac{dy}{dx} + y = e^{3x}$

Solution

$$e^{\int dx} = e^x$$
$$\int e^x e^{3x}dx = \int e^{4x}dx = \frac{1}{4}e^{4x}$$
$$y(x) = \frac{1}{e^x}\left(\frac{1}{4}e^{4x} + C\right)$$
$$\boxed{= \frac{1}{4}e^{3x} + Ce^{-x}}$$

Exercise

Solve the differential equation: $y' - ty = t$

Solution

$$e^{\int -tdt} = e^{-\frac{1}{2}t^2}$$

$$\int te^{-\frac{1}{2}t^2}dt = -\int e^{-\frac{1}{2}t^2}d\left(-\frac{1}{2}t^2\right) = -e^{-\frac{1}{2}t^2}$$

$$y(t) = e^{\frac{1}{2}t^2}\left(e^{-\frac{1}{2}t^2} + C\right)$$

$$\boxed{= 1 + Ce^{\frac{1}{2}t^2}}$$

Exercise

Solve the differential equation: $y' = 2y + x^2 + 5$

Solution

		$\int e^{-2x}$
+	x^2+5	$-\frac{1}{2}e^{-2x}$
−	$2x$	$\frac{1}{4}e^{-2x}$
+	2	$-\frac{1}{8}e^{-2x}$

$$y' - 2y = x^2 + 5$$

$$e^{\int -2dx} = e^{-2x}$$

$$\int \left(x^2+5\right)e^{-2x}\, dx = \left(-\frac{1}{2}x^2 - \frac{5}{2} - \frac{1}{2}x - \frac{1}{4}\right)e^{-2x}$$

$$= \left(-\frac{1}{2}x^2 - \frac{1}{2}x - \frac{11}{4}\right)e^{-2x}$$

$$= -\frac{1}{4}\left(2x^2 + 2x + 11\right)e^{-2x}$$

$$y(x) = e^{2x}\left(-\frac{1}{4}\left(2x^2+2x+11\right)e^{-2x} + C\right)$$

$$\boxed{= -\frac{1}{4}\left(2x^2 + 2x + 11\right) + Ce^{2x}}$$

Exercise

Solve the differential equation: $xy' + 2y = 3$

Solution

$$y' + \frac{2}{x}y = \frac{3}{x}$$

$$e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$\int x^2 \frac{3}{x}\, dx = \int 3x\, dx = \frac{3}{2}x^2$$

$$y(x)=\frac{1}{x^2}\left(\frac{3}{2}x^2+C\right)$$

$$\boxed{=\frac{3}{2}+\frac{C}{x^2}}$$

Exercise

Find the general solution of $\frac{dy}{dt}-2y=4-t$

Solution

$$e^{\int -2dt}=e^{-2t}$$

$$\int(4-t)e^{-2t}\,dt=\int\left(4e^{-2t}-te^{-2t}\right)\,dt$$

$$=-2e^{-2t}+\frac{1}{2}te^{-2t}+\frac{1}{4}e^{-2t}$$

$$=-\frac{7}{4}e^{-2t}+\frac{1}{2}te^{-2t}$$

		$\int e^{-2t}$
+	t	$-\frac{1}{2}e^{-2t}$
−	1	$\frac{1}{4}e^{-2t}$

$$y(t)=\frac{1}{e^{-2t}}\left(-\frac{7}{4}e^{-2t}+\frac{1}{2}te^{-2t}+C\right)$$

$$\boxed{y(t)=\frac{1}{2}t-\frac{7}{4}+Ce^{2t}}$$

Exercise

Solve the differential equation: $y'+2y=1$

Solution

$$e^{\int 2dx}=e^{2x}$$

$$\int e^{2x}\,dx=\frac{1}{2}e^{2x}$$

$$y(x)=\frac{1}{e^{2x}}\left(\frac{1}{2}e^{2x}+C\right)$$

$$\boxed{=\frac{1}{2}+Ce^{-2x}}$$

Exercise

Solve the differential equation: $y'+2y=e^{-t}$

Solution

$$e^{\int 2dt} = e^{2t}$$

$$\int e^{-t}e^{2t}\, dt = \int e^{t}\, dt = e^{t}$$

$$y(x) = \frac{1}{e^{2t}}\left(e^{t} + C\right)$$

$$\underline{= e^{-t} + Ce^{-2t}}$$

Exercise

Solve the differential equation: $y' + 2y = e^{-2t}$

Solution

$$e^{\int 2dt} = e^{2t}$$

$$\int e^{-2t}e^{2t}\, dt = t$$

$$\underline{y(x) = (t + C)e^{-2t}}$$

Exercise

Find the general solution of $y' - 2y = e^{3t}$

Solution

$$e^{\int -2dt} = e^{-2t}$$

$$\int e^{3t}e^{-2t} dt = e^{t}$$

$$y(t) = e^{2t}\left(e^{t} + C\right)$$

$$\underline{= e^{3t} + Ce^{2t}}$$

Exercise

Find the general solution of $y' + 2y = e^{-x} + x + 1$

Solution

$$e^{\int 2dx} = e^{2x}$$

$$\int \left(e^{-x} + x + 1\right)e^{2x} dx = \int \left(e^{x} + (x+1)e^{2x}\right) dx$$

		$\int e^{2x}$
+	$x+1$	$\frac{1}{2}e^{2x}$
–	1	$\frac{1}{4}e^{2x}$

$$= e^{x} + \left(\frac{1}{2}x + \frac{1}{2} - \frac{1}{4}\right)e^{2x}$$

$$= e^{x} + \left(\frac{1}{2}x + \frac{1}{4}\right)e^{2x}$$

$$y(x) = e^{-2x}\left(e^{x} + \left(\frac{1}{2}x + \frac{1}{4}\right)e^{2x} + C\right)$$

$$= e^{-x} + \frac{1}{2}x + \frac{1}{4} + Ce^{-2x}$$

Exercise

Solve the differential equation: $y' + 2xy = x$

Solution

$$e^{\int 2x\,dx} = e^{x^2}$$

$$\int xe^{x^2}\,dx = \frac{1}{2}\int e^{x^2}\,d\left(x^2\right) = \frac{1}{2}e^{x^2}$$

$$y(x) = \frac{1}{e^{x^2}}\left(\frac{1}{2}e^{x^2} + C\right)$$

$$= \frac{1}{2} + Ce^{-x^2}$$

Exercise

Solve the differential equation: $y' - 2ty = t$

Solution

$$e^{\int -2t\,dt} = e^{-t^2}$$

$$\int te^{-t^2}\,dt = -\frac{1}{2}\int e^{-t^2}\,d\left(-t^2\right) = -\frac{1}{2}e^{-t^2}$$

$$y(t) = \frac{1}{e^{-t^2}}\left(-\frac{1}{2}e^{-t^2} + C\right)$$

$$= Ce^{t^2} - \frac{1}{2}$$

Exercise

Find the general solution of $y' + 2ty = 5t$

Solution

$$u(t) = e^{\int 2t\,dt} = e^{t^2}$$

$$e^{t^2} y' + 2te^{t^2} y = 5te^{t^2}$$

$$\left(e^{t^2} y\right)' = 5te^{t^2}$$

$$e^{t^2} y = \int 5te^{t^2}\,dt \qquad\qquad de^{t^2} = 2te^{t^2}\,dt$$

$$= 5\int \tfrac{1}{2} de^{t^2}$$

$$= \tfrac{5}{2} e^{t^2} + C$$

$$\boxed{y(t) = \tfrac{5}{2} + Ce^{-t^2}}$$

Exercise

Solve the differential equation: $y' - 2xy = e^{x^2}$

Solution

$$e^{\int -2x\,dx} = e^{-x^2}$$

$$\int e^{x^2} e^{-x^2}\,dx = \int dx = x$$

$$\boxed{y(x) = e^{x^2}(x + C)}$$

Exercise

Solve the differential equation: $y' + 2xy = x^3$

Solution

$$e^{\int 2x\,dx} = e^{x^2}$$

$$\int x^3 e^{x^2}\,dx = \frac{1}{2}\int x^2 e^{x^2}\,d\left(x^2\right) = \frac{1}{2}\int ue^u\,d(u)$$

$$= \frac{1}{2}\left(x^2 - 1\right)e^{x^2}$$

		$\int e^u$
+	u	e^u
−	1	e^u

$$y(x) = e^{-x^2}\left(\frac{1}{2}\left(x^2 - 1\right)e^{x^2} + C\right)$$

$$\boxed{= \frac{1}{2}\left(x^2 - 1\right) + Ce^{-x^2}}$$

Exercise

Solve the differential equation: $y' - 2y = t^2 e^{2t}$

Solution

$$e^{\int -2dt} = e^{-2t}$$

$$\int e^{-2t} t^2 e^{2t} dt = \int t^2 dt = \frac{1}{3}t^3$$

$$y(t) = \frac{1}{e^{-2t}}\left(\frac{1}{3}t^3 + C\right)$$

$$\boxed{= e^{2t}\left(\frac{1}{3}t^3 + C\right)}$$

Exercise

Find the general solution of $x' - 2\frac{x}{t+1} = (t+1)^2$

Solution

$$e^{\int -\frac{2}{t+1}dt} = e^{-2\ln(t+1)} = e^{\ln(t+1)^{-2}} = (t+1)^{-2}$$

$$\int (t+1)^2 (t+1)^{-2} dt = \int dt = t$$

$$x(t) = \frac{1}{(t+1)^{-2}}(t + C)$$

$$= (t+1)^2 (t + C)$$

$$\boxed{= t(t+1)^2 + C(t+1)^2}$$

Exercise

Find the general solution of $y' + \frac{2}{t}y = \frac{\cos t}{t^2}$

Solution

$$e^{\int \frac{2}{t}dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

$$\int \frac{\cos t}{t^2} t^2 dt = \int \cos t \ dt = \sin t$$

$$\boxed{y(t) = \frac{1}{t^2}(\sin t + C)}$$

Exercise

Solve the differential equation: $y' - 2(\cos 2t)y = 0$

Solution

$$e^{\int -2\cos 2t\, dt} = e^{-\sin 2t}$$

$$\boxed{y(x) = C\, e^{\sin 2t}}$$

Exercise

Find the general solution of $y' + 2y = \cos 3t$

Solution

$$e^{\int 2dt} = e^{2t}$$

$$\int (\cos 3t)e^{2t}dt = \left(\tfrac{1}{3}\sin 3t + \tfrac{1}{18}\cos 3t\right)e^{2t} - \tfrac{1}{36}\int (\cos 3t)e^{2t}dt$$

$$\frac{37}{36}\int (\cos 3t)e^{2t}dt = \tfrac{1}{18}(6\sin 3t + \cos 3t)e^{2t}$$

$$\int (\cos 3t)e^{2t}dt = \tfrac{2}{37}(6\sin 3t + \cos 3t)e^{2t}$$

$$y(t) = e^{-2t}\left(\tfrac{2}{37}(6\sin 3t + \cos 3t)e^{2t} + C\right)$$

$$\boxed{= \tfrac{2}{37}(6\sin 3t + \cos 3t) + Ce^{-2t}}$$

		$\int \cos 3t$
+	e^{2t}	$\frac{1}{3}\sin 3t$
−	$\frac{1}{2}e^{2t}$	$-\frac{1}{9}\cos 3t$
+	$\frac{1}{4}e^{2t}$	

Exercise

Find the general solution of $y' - 3y = 5$

Solution

$$u(t) = e^{-\int 3dt} = e^{-3t}$$

$$e^{-3t}y' - 3e^{-3t}y = 5e^{-3t}$$

$$\left(e^{-3t}y\right)' = 5e^{-3t}$$

$$e^{-3t}y = \int 5e^{-3t}dt$$

$$e^{-3t}y = -\tfrac{5}{3}e^{-3t} + C$$

$$\boxed{y(t) = -\tfrac{5}{3} + Ce^{3t}}$$

Exercise

Solve the differential equation: $y' + 3y = 2xe^{-3x}$

Solution

$$e^{\int 3dx} = e^{3x}$$

$$\int 2xe^{-3x}e^{3x}\,dx = \int 2x\,dx = x^2$$

$$\boxed{y(x) = \frac{1}{e^{3x}}\left(x^2 + C\right)}$$

Exercise

Find the general solution of $y' + 3t^2y = t^2$

Solution

$$e^{\int 3t^2 dt} = e^{t^3}$$

$$\int t^2 e^{t^3}\,dt = \frac{1}{3}\int e^{t^3}\,d\left(t^3\right) = \frac{1}{3}e^{t^3}$$

$$y(t) = \frac{1}{e^{t^3}}\left(\frac{1}{3}e^{t^3} + C\right)$$

$$= \boxed{\frac{1}{3} + Ce^{-t^3}}$$

Exercise

Solve the differential equation: $y' + 3x^2y = x^2$

Solution

$$e^{\int 3x^2 dx} = e^{x^3}$$

$$\int x^2 e^{x^3} dx = \frac{1}{3}\int e^{x^3} d\left(e^{x^3}\right) = \frac{1}{3}e^{x^3}$$

$$y(x) = \frac{1}{e^{x^3}}\left(\frac{1}{3}e^{x^3} + C\right)$$

$$\boxed{= \frac{1}{3} + Ce^{-x^3}}$$

Exercise

Find the general solution of $y' + \frac{3}{t}y = \frac{\sin t}{t^3}, \quad (t \neq 0)$

Solution

$$e^{\int \frac{3}{t}dt} = e^{3\ln t} = e^{\ln t^3} = t^3$$

$$\int \frac{\sin t}{t^3} t^3 \, dt = \int \sin t \, dt = -\cos t$$

$$y(t) = \frac{1}{t^3}(-\cos t + C)$$

$$= \frac{C}{t^3} - \frac{\cos t}{t^3}$$

Exercise

Find the general solution of $y' + \frac{3}{x}y = 1 + \frac{1}{x}$

Solution

$$e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$$

$$\int \left(1 + \frac{1}{x}\right)x^3 dx = \int \left(x^3 + x^2\right)dx = \frac{1}{4}x^4 + \frac{1}{3}x^3$$

$$y(x) = \frac{1}{x^3}\left(\frac{1}{4}x^4 + \frac{1}{3}x^3 + C\right)$$

$$= \frac{1}{4}x + \frac{1}{3}x + \frac{C}{x^3}$$

Exercise

Find the general solution of $y' + \frac{3}{2}y = \frac{1}{2}e^x$

Solution

$$e^{\int \frac{3}{2}dx} = e^{3x/2}$$

$$\int \left(\frac{1}{2}e^x\right)e^{3x/2}dx = \frac{1}{2}\int e^{5x/2}dx = \frac{1}{5}e^{5x/2}$$

$$y(x) = e^{-3x/2}\left(\frac{1}{5}e^{5x/2} + C\right)$$

$$= \frac{1}{5}e^x + Ce^{-3x/2}$$

Exercise

Find the general solution of $y' + 5y = t + 1$

Solution

$$e^{\int 5dt} = e^{5t}$$

$$\int (t+1)e^{5t}dt = \left(\frac{1}{5}t + \frac{1}{5} + \frac{1}{25}\right)e^{5t} = \frac{1}{5}\left(t + \frac{6}{5}\right)e^{5t}$$

		$\int e^{5t}$
+	$t+1$	$\frac{1}{5}e^{5t}$
−	1	$\frac{1}{25}e^{5t}$

$$y(t) = \frac{1}{e^{5t}}\left(\frac{1}{5}\left(t + \frac{6}{5}\right)e^{5t} + C\right)$$

$$= \frac{1}{5}\left(t + \frac{6}{5}\right) + Ce^{-5t}$$

Exercise

Solve the differential equation: $xy' - y = x^2 \sin x$

Solution

$$y' - \frac{1}{x}y = x\sin x$$

$$e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\int x^2 \sin x \, dx = -x^2\cos x + 2x\sin x + 2\cos x$$

		$\int \sin x$
+	x^2	$-\cos x$
−	$2x$	$-\sin x$
+	2	$\cos x$

$$y(x) = \frac{1}{x}\left(-x^2\cos x + 2x\sin x + 2\cos x + C\right)$$

$$= -x\cos x + 2\sin x + \frac{2}{x}\cos x + \frac{C}{x}$$

Exercise

Solve the differential equation: $x\frac{dy}{dx} + y = e^x, \quad x > 0$

Solution

$$y' + \frac{1}{x}y = \frac{e^x}{x}$$

$$e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\int \frac{e^x}{x}e^{\int \frac{1}{x}dx}dx = \int x\frac{e^x}{x}dx = \int e^x dx = e^x$$

$$y(x) = \frac{1}{x}\left(e^x + C\right), \quad x > 0$$

Exercise

Solve the differential equation: $x\dfrac{dy}{dx}+2y=1-\dfrac{1}{x}, \quad x>0$

Solution

$$\frac{dy}{dx}+\frac{2}{x}y=\frac{1}{x}-\frac{1}{x^2}$$

$$e^{\int \frac{2}{x}dx}=e^{2\ln x}=e^{\ln x^2}=x^2$$

$$\int\left(\frac{1}{x}-\frac{1}{x^2}\right)x^2dx=\int(x-1)\,dx=\frac{1}{2}x^2-x$$

$$y(x)=\frac{1}{x^2}\left(\frac{1}{2}x^2-x+C\right)$$

$$=\frac{1}{2}-\frac{1}{x}+\frac{C}{x^2}, \quad x>0$$

Exercise

Find the general solution of $y\dfrac{dx}{dy}+2x=5y^3$

Solution

$$x'+\frac{2}{y}x=5y^2$$

$$e^{\int \frac{2}{y}dy}=e^{2\ln y}=y^2$$

$$\int 5y^2y^2dy=5\int y^4dx=y^5$$

$$x(y)=\frac{1}{y^2}\left(y^5+C\right)$$

$$=y^3+\frac{C}{y^2}$$

Exercise

Find the general solution of $ty'+y=\cos t$

Solution

$$y'+\frac{1}{t}y=\frac{\cos t}{t}$$

$$e^{\int \frac{1}{t}dt}=e^{\ln t}=t$$

$$\int t\frac{\cos t}{t}dt = \int \cos t\ dt = \sin t$$

$$\boxed{y(t) = \frac{1}{t}(\sin t + C)}$$

Exercise

Solve the differential equation: $xy' + 2y = x^2$

Solution

$$y' + \frac{2}{x}y = x$$

$$e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$\int x^3\ dx = \frac{1}{4}x^4$$

$$y(x) = \frac{1}{x^2}\left(\frac{1}{4}x^4 + C\right)$$

$$\boxed{= \frac{1}{4}x^2 + \frac{C}{x^2}}$$

Exercise

Solve the differential equation: $xy' = 2y + x^3\cos x$

Solution

$$y' - \frac{2}{x}y = x^2\cos x$$

$$e^{\int \frac{-2}{x}dx} = e^{-2\ln x} = x^{-2}$$

$$\int x^{-2}x^2\cos x\ dx = \int \cos x\ dx = \sin x$$

$$\boxed{y(x) = x^2(\sin x + C)}$$

Exercise

Find the general solution of $xy' + 2y = x^{-3}$

Solution

$$y' + \frac{2}{x}y = x^{-4}$$

$$e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$\int x^{-4}x^2 dx = \int x^{-2}dx = -\frac{1}{x}$$

$$y(x) = \frac{1}{x^2}\left(-\frac{1}{x} + C\right)$$

$$\underline{= \frac{C}{x^2} - \frac{1}{x^3}}|$$

Exercise

Find the general solution of $ty' + 2y = t^2$

Solution

$$y' + \frac{2}{t}y = t$$

$$e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2$$

$$\int t^2(t)dt = \int t^3 dt = \frac{1}{4}t^4$$

$$y(t) = \frac{1}{t^2}\left(\frac{1}{4}t^4 + C\right)$$

$$\underline{= \frac{1}{4}t^2 + \frac{C}{t^2}}|$$

Exercise

Find the general solution of $xy' + 2\left(y + x^2\right) = \frac{\sin x}{x}$

Solution

$$y' + \frac{2}{x}y = \frac{\sin x}{x^2} - 2x$$

$$e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$\int \left(\frac{\sin x}{x^2} - 2x\right)x^2 dx = \int \left(\sin x - 2x^3\right)dx = -\cos x - \frac{1}{2}x^4$$

$$y(x) = \frac{1}{x^2}\left(-\cos x - \frac{1}{2}x^4 + C\right)$$

$$\underline{= -\frac{\cos x}{x^2} - \frac{1}{2}x^2 + \frac{C}{x^2}}|$$

Exercise

Solve the differential equation: $xy' + 4y = x^3 - x$

Solution

$$y' + \frac{4}{x}y = x^2 - 1$$

$$e^{\int \frac{4}{x}dx} = e^{4\ln x} = x^4$$

$$\int x^4\left(x^2-1\right)dx = \int \left(x^6 - x^4\right)dx = \frac{1}{7}x^7 - \frac{1}{5}x^5$$

$$y(x) = \frac{1}{x^4}\left(\frac{1}{7}x^7 - \frac{1}{5}x^5 + C\right)$$

$$\boxed{= \frac{1}{7}x^3 - \frac{1}{5}x + Cx^{-4}}$$

Exercise

Solve the differential equation: $xy' + (x+1)y = e^{-x}\sin 2x$

Solution

$$y' + \left(1 + \frac{1}{x}\right)y = \frac{1}{x}e^{-x}\sin 2x$$

$$e^{\int \left(1+\frac{1}{x}\right)dx} = e^{x+\ln x} = e^x e^{\ln x} = xe^x$$

$$\int xe^x \frac{1}{x}e^{-x}\sin 2x\; dx = \int \sin 2x\; dx = -\frac{1}{2}\cos 2x$$

$$\boxed{y(x) = \frac{1}{xe^x}\left(\frac{1}{2}\cos 2x + C\right)}$$

Exercise

Solve the differential equation: $xy' + (3x+1)y = e^{3x}$

Solution

$$y' + \left(3 + \frac{1}{x}\right)y = \frac{e^{3x}}{x}$$

$$e^{\int \left(3+\frac{1}{x}\right)dx} = e^{3x+\ln x} = xe^{3x}$$

$$\int xe^{3x}\frac{e^{3x}}{x}\; dx = \int e^{3x}\; dx = \frac{1}{3}e^{3x}$$

$$y(x) = \frac{1}{xe^{3x}}\left(e^{3x} + C\right)$$

$$\boxed{= \frac{1}{x} + Ce^{3x}} \quad x > 0$$

Exercise

Solve the differential equation: $xy' + (2x-3)y = 4x^4$

Solution

$$y' + \left(2 - \frac{3}{x}\right)y = 4x^3$$

$$e^{\int \left(2-\frac{3}{x}\right)dx} = e^{2x-3\ln x} = e^{2x}e^{-3\ln x} = x^{-3}e^{2x}$$

$$\int 4x^3 x^{-3}e^{2x}\ dx = 4\int e^{2x}dx = 2e^{2x}$$

$$y(x) = \frac{1}{x^{-3}e^{2x}}\left(2e^{2x} + C\right)$$

$$\boxed{= 2x^3 + Cx^3e^{-2x}}$$

Exercise

Solve the differential equation: $2xy'' - 3y = 9x^3$

Solution

$$y'' - \frac{3}{2x}y = \frac{9}{2}x^2$$

$$e^{\int \frac{-3}{2x}dx} = e^{\frac{-3}{2}\ln x} = x^{-3/2}$$

$$\int \frac{9}{2}x^2x^{-3/2}\ dx = \frac{9}{2}\int x^{1/2}\ dx = 3x^{3/2}$$

$$y(x) = x^{3/2}\left(3x^{3/2} + C\right)$$

$$\boxed{= 3 + Cx^{3/2}}$$

Exercise

Solve the differential equation: $2y' + 3y = e^{-t}$

Solution

$$y' + \frac{3}{2}y = \frac{1}{2}e^{-t}$$

$$e^{\int \frac{3}{2}dt} = e^{3t/2}$$

$$\int e^{3t/2}e^{-t}dt = \int e^{t/2}dt = 2e^{t/2}$$

$$y(t) = \frac{1}{e^{3t/2}}\left(2e^{t/2} + C\right)$$

$$\boxed{= 2e^{-t} + Ce^{-3t/2}}$$

Exercise

Solve the differential equation: $2y' + 2ty = t$

Solution

$$y' + ty = \frac{1}{2}t$$

$$e^{\int tdt} = e^{t^2/2}$$

$$\int te^{t^2/2}dt = \int e^{t^2/2}d\left(\frac{1}{2}t^2\right) = e^{t^2/2}$$

$$y(t) = \frac{1}{e^{t^2/2}}\left(e^{t^2/2} + C\right)$$

$$\boxed{= 1 + Ce^{t^2/2}}$$

Exercise

Solve the differential equation: $3xy' + y = 10\sqrt{x}$

Solution

$$y' + \frac{1}{3x}y = \frac{10}{3}x^{-1/2}$$

$$e^{\int \frac{1}{3x}dx} = e^{\frac{1}{3}\ln x} = x^{1/3}$$

$$\int \frac{10}{3}x^{-1/2}x^{1/3}\,dx = \frac{10}{3}\int x^{-1/6}\,dx = 4x^{5/6}$$

$$y(x) = x^{-1/3}\left(4x^{5/6} + C\right)$$

$$\boxed{= 4x^{1/2} + Cx^{-1/3}}$$

Exercise

Solve the differential equation: $3xy' + y = 12x$

Solution

$$y' + \frac{1}{3x}y = 4$$

$$e^{\int \frac{1}{3x}dx} = e^{\frac{1}{3}\ln x} = x^{1/3}$$

$$\int 4x^{1/3}\, dx = 3x^{4/3}$$

$$y(x) = x^{-1/3}\left(3x^{4/3} + C\right)$$

$$\boxed{= 3x + Cx^{-1/3}}$$

Exercise

Solve the differential equation: $x^2y' + xy = 1$

Solution

$$y' + \frac{1}{x}y = \frac{1}{x^2}$$

$$e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\int x\frac{1}{x^2}dx = \int \frac{1}{x}dx = \ln x$$

$$\boxed{y(x) = \frac{1}{x}(\ln x + C)}$$

Exercise

Solve the differential equation: $x^2y' + x(x+2)y = e^x$

Solution

$$y' + \left(1 + \frac{2}{x}\right)y = \frac{e^x}{x^2}$$

$$e^{\int \left(1+\frac{2}{x}\right)dx} = e^{x+2\ln x} = e^x e^{\ln x^2} = x^2e^x$$

$$\int x^2e^x\frac{e^x}{x^2}dx = \int e^{2x}dx = \frac{1}{2}e^{2x}$$

$$y(x) = \frac{1}{x^2e^x}\left(\frac{1}{2}e^{2x} + C\right)$$

$$\boxed{= \frac{1}{x^2}\left(\frac{1}{2}e^x + Ce^{-x}\right)}$$

Exercise

Find the general solution of $y^2 + (y')^2 = 1$

Solution

$$\left(\frac{dy}{dx}\right)^2 = 1 - y^2$$

$$\frac{dy}{dx} = \pm\sqrt{1-y^2}$$

$$\frac{dy}{\sqrt{1-y^2}} = \pm dx$$

$$\int \frac{dy}{\sqrt{1-y^2}} = \pm\int dx$$

$$\sin^{-1} y = \pm(x+c)$$

$$y = \sin(\pm(x+c))$$

$$y(x) = \pm\sin(x+c)$$

Exercise

Solve the differential equation: $(1+x)y' + y = \sqrt{x}$

Solution

$$\frac{dy}{dx} + \frac{1}{1+x}y = \frac{\sqrt{x}}{1+x}$$

$$e^{\int \frac{1}{1+x}dx} = e^{\ln(1+x)} = 1+x$$

$$\int \frac{\sqrt{x}}{1+x}(1+x)\,dx = \int x^{1/2}dx = \frac{2}{3}x^{3/2}$$

$$y(x) = \frac{1}{1+x}\left(\frac{2}{3}x^{3/2} + C\right)$$

$$= \frac{2x^{3/2}}{3(1+x)} + \frac{C}{1+x}$$

Exercise

Find the general solution of $(1+x)y' + y = \cos x$

Solution

$$y' + \frac{1}{1+x} y = \frac{\cos x}{1+x}$$

$$y_h = e^{\int \frac{1}{1+x}dx} = e^{\ln(1+x)} = 1+x$$

$$\int \frac{\cos x}{1+x}(1+x)\,dx = \int \cos x\,dx = \sin x$$

$$y(x) = \frac{1}{x+1}(\sin x + C)$$

$$\boxed{y(x) = \frac{\sin x + C}{x+1}}$$

Exercise

Solve the differential equation: $(x+1)y' + (x+2)y = 2xe^{-x}$

Solution

$$y' + \frac{x+2}{x+1} y = \frac{2xe^{-x}}{x+1} \qquad \frac{x+2}{x+1} = 1 + \frac{1}{x+1}$$

$$e^{\int\left(1+\frac{1}{x+1}\right)dx} = e^{x+\ln(x+1)} = (x+1)e^{x}$$

$$\int (x+1)e^{x}\,\frac{2xe^{-x}}{x+1}\,dx = \int 2x\,dx = x^2$$

$$\boxed{y(x) = \frac{e^{-x}}{x+1}\left(x^2 + C\right)}$$

Exercise

Solve the differential equation: $(x+1)y' - xy = x + x^2$

Solution

$$y' - \frac{x}{x+1} y = \frac{x(x+1)}{x+1} = x$$

$$e^{\int\left(\frac{-x}{x+1}\right)dx} = e^{\int\left(\frac{1}{x+1}-1\right)dx} = e^{\ln|x+1|-x} = e^{\ln(x+1)}e^{-x} = (x+1)e^{-x}$$

$$\int x(x+1)e^{-x}\,dx = \left(-x^2 - x - 2x - 1 - 2\right)e^{-x}$$

$$= \left(-x^2 - 3x - 3\right)e^{-x}$$

		$\int e^{-x}$
+	$x^2 + x$	$-e^{-x}$
−	$2x+1$	e^{-x}
+	2	$-e^{-x}$

$$y(x) = \frac{e^{x}}{x+1}\left(-\left(x^2 + 3x + 3\right)e^{-x} + C\right)$$

$$\boxed{= \frac{x^2+3x+3}{x+1} + \frac{Ce^{x}}{x+1}} \qquad x > -1$$

Exercise

Find the general solution of $\left(1+x^3\right)y' = 3x^2y + x^2 + x^5$

Solution

$$y' - \frac{3x^2}{1+x^3}y = \frac{x^2\left(1+x^3\right)}{1+x^3} = x^2$$

$$e^{\int -\frac{3x^2}{1+x^3}dx} = e^{-\int \frac{d\left(1+x^3\right)}{1+x^3}} = e^{-\ln\left(1+x^3\right)} = e^{\ln\left(1+x^3\right)^{-1}} = \frac{1}{1+x^3}$$

$$\int \frac{x^2}{1+x^3}dx = \frac{1}{3}\int \frac{d\left(1+x^3\right)}{1+x^3} = \frac{1}{3}\ln\left(1+x^3\right)$$

$$y(x) = \left(1+x^3\right)\left(\frac{1}{3}\ln\left(1+x^3\right) + C\right)$$

$$\boxed{= \frac{1}{3}\left(1+x^3\right)\ln\left(1+x^3\right) + C\left(1+x^3\right)}$$

Exercise

Solve the differential equation: $\quad (t+1)\frac{ds}{dt} + 2s = 3(t+1) + \frac{1}{(t+1)^2}, \quad t > -1$

Solution

$$\frac{ds}{dt} + \frac{2}{t+1}s = 3 + \frac{1}{(t+1)^3}$$

$$e^{\int \frac{2}{t+1}dt} = e^{2\ln(t+1)} = e^{\ln(t+1)^2} = (t+1)^2$$

$$\int \left(3 + \frac{1}{(t+1)^3}\right)(t+1)^2\,dt = \int \left(3(t+1)^2 + \frac{1}{t+1}\right)dt \qquad d(t+1) = dt$$

$$= 3\int (t+1)^2\,d(t+1) + \int \frac{1}{t+1}d(t+1)$$

$$= (t+1)^3 + \ln(t+1)$$

$$s(t) = \frac{1}{(t+1)^2}\left((t+1)^3 + \ln(t+1) + C\right)$$

$$\boxed{= t+1 + \frac{\ln(t+1)}{(t+1)^2} + \frac{C}{(t+1)^2}, \quad t > -1}$$

Exercise

Solve the differential equation: $(x+2)^2 y' = 5 - 8y - 4xy$

Solution

$$y' + \frac{4}{x+2} y = 5(x+2)^{-2}$$

$$e^{\int \left(\frac{4}{x+2}\right) dx} = e^{4\ln(x+2)} = (x+2)^4$$

$$\int 5(x+2)^{-2}(x+2)^4 \ dx = 5\int (x+2)^2 \ d(x+2) = \frac{5}{3}(x+2)^3$$

$$y(x) = (x+2)^{-4}\left(\frac{5}{3}(x+2)^3 + C\right)$$

$$\boxed{= \frac{5}{3}(x+2)^{-1} + C(x+2)^{-4}}$$

Exercise

Solve the differential equation: $\left(x^2 - 1\right) y' + 2y = (x+1)^2$

Solution

$$y' + \frac{2}{(x-1)(x+1)} y = \frac{(x+1)^2}{(x-1)(x+1)} = \frac{x+1}{x-1}$$

$$e^{\int \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx} = e^{\ln|x-1| - \ln|x+1|} = e^{\ln|x-1|} e^{-\ln|x+1|} = \frac{x-1}{x+1}$$

$$\frac{2}{(x-1)(x+1)} = \frac{1}{x-1} - \frac{1}{x+1}$$

$$\int \frac{x-1}{x+1}\frac{x+1}{x-1} \ dx = \int \ dx = x$$

$$\boxed{y(x) = \frac{x+1}{x-1}(x+C)} \quad -1 < x < 1$$

Exercise

Solve the differential equation: $\left(x^2 + 4\right) y' + 2xy = x^2\left(x^2 + 4\right)$

Solution

$$y' + \frac{2x}{x^2 + 4} y = x^2$$

$$e^{\int \frac{2x}{x^2+4} dx} = e^{\ln\left(x^2+4\right)} = x^2 + 4$$

$$\int x^2\left(x^2 + 4\right) \ dx = \int \left(x^4 + 4x^2\right) \ dx = \frac{1}{5}x^5 + \frac{4}{3}x^3$$

$$\boxed{y(x) = \frac{1}{x^2 + 4}\left(\frac{1}{5}x^5 + \frac{4}{3}x^3 + C\right)}$$

Exercise

Find the general solution of $\left(1+e^t\right)y'+e^t y=0$

Solution

$$y'+\frac{e^t}{1+e^t}y=0$$

$P(t)=\frac{e^t}{1+e^t}, \quad Q(t)=0$ $e^{\int \frac{e^t}{1+e^t}dt}=e^{\int \frac{1}{1+e^t}d\left(1+e^t\right)}=e^{\ln\left(1+e^t\right)}=1+e^t$ $y(t)=\frac{1}{1+e^t}(0+c)$ $=\boxed{\frac{c}{1+e^t}}$	$\frac{dy}{dt}=-\frac{e^t}{1+e^t}y$ $\frac{dy}{y}=-\frac{e^t}{1+e^t}dt$ $\int\frac{dy}{y}=-\int\frac{1}{1+e^t}d\left(1+e^t\right)$ $\ln y=-\ln\left(1+e^t\right)+C$ $\ln y=\ln\left(\frac{1}{1+e^t}\right)+\ln c$ $\ln y=\ln\left(\frac{c}{1+e^t}\right)$ $\boxed{y(t)=\frac{c}{1+e^t}}$

Exercise

Find the general solution of $\left(t^2+9\right)y'+ty=0$

Solution

$$y'+\frac{t}{t^2+9}y=0$$

$$e^{\int \frac{t}{t^2+9}dt}=e^{\frac{1}{2}\int \frac{1}{t^2+9}d\left(t^2+9\right)}$$

$$d\left(t^2+9\right)=2tdt \Rightarrow \frac{1}{2}d\left(t^2+9\right)=tdt$$

$$=e^{\frac{1}{2}\ln\left(t^2+9\right)}=e^{\ln\sqrt{t^2+9}}=\sqrt{t^2+9}$$

$$y(t)=\frac{1}{\sqrt{t^2+9}}(0+c)$$

$$=\boxed{\frac{c}{\sqrt{t^2+9}}}$$

Exercise

Solve the differential equation: $e^{2x}y' + 2e^{2x}y = 2x$

Solution

$$y' + 2y = 2xe^{-2x}$$

$$e^{\int 2dx} = e^{2x}$$

$$\int 2xe^{-2x}\left(e^{2x}\right)dx = 2\int xdx = x^2$$

$$\underline{y(x)} = \frac{1}{e^{2x}}\left(x^2 + C\right)$$

$$\boxed{= x^2e^{-2x} + Ce^{-2x}}$$

Exercise

Solve the differential equation: $\tan\theta\frac{dr}{d\theta} + r = \sin^2\theta, \quad 0 < \theta < \frac{\pi}{2}$

Solution

$$\frac{dr}{d\theta} + \frac{1}{\tan\theta}r = \frac{\sin^2\theta}{\tan\theta}$$

$$\frac{dr}{d\theta} + \frac{1}{\tan\theta}r = \sin^2\theta\frac{\cos\theta}{\sin\theta}$$

$$\frac{dr}{d\theta} + \frac{1}{\tan\theta}r = \sin\theta\cos\theta, \quad P(\theta) = \frac{1}{\tan\theta} = \cot\theta \quad Q(\theta) = \sin\theta\cos\theta$$

$$e^{\int\cot\theta d\theta} = e^{\ln|\sin\theta|} = \sin\theta, \quad 0 < \theta < \frac{\pi}{2}$$

$$\int(\sin\theta\cos\theta)(\sin\theta)d\theta = \int\left(\sin^2\theta\cos\theta\right)d\theta \qquad d(\sin\theta) = \cos\theta d\theta$$

$$= \int\sin^2\theta d(\sin\theta)$$

$$= \frac{1}{3}\sin^3\theta$$

$$\underline{r(\theta)} = \frac{1}{\sin\theta}\left(\frac{1}{3}\sin^3\theta + C\right)$$

$$\boxed{= \frac{1}{3}\sin^2\theta + \frac{C}{\sin\theta}}$$

Exercise

Find the general solution of $(\cos t)y' + (\sin t)y = 1$

Solution

$$y' + (\tan t)y = \frac{1}{\cos t}$$

$$e^{\int \tan t dt} = e^{-\ln|\cos t|} = e^{\ln\frac{1}{|\cos t|}} = \frac{1}{|\cos t|} = \sec t$$

$$\int \sec^2 t\ dt = \tan t$$

$$y(t) = \frac{1}{\sec t}(\tan t + C)$$

$$= \cos t\left(\frac{\sin t}{\cos t} + C\right)$$

$$\underline{= \sin t + C\cos t}$$

Exercise

Solve the differential equation: $\cos x\frac{dy}{dx} + (\sin x)y = 1$

Solution

$$y' + (\tan x)y = \sec x$$

$$e^{\int(\tan x)dx} = e^{\ln|\sec x|} = \sec x$$

$$\int \sec^2 x\ dx = \tan x$$

$$y(x) = \cos x(\tan x + C)$$

$$\underline{= \sin x + C\cos x}$$

Exercise

Solve the differential equation: $\cos^2 x\sin x\frac{dy}{dx} + \left(\cos^3 x\right)y = 1$

Solution

$$y' + (\cot x)y = \frac{1}{\cos^2 x\sin x}$$

$$e^{\int(\cot x)dx} = e^{\ln|\sin x|} = \sin x$$

$$\int \frac{\sin x}{\cos^2 x\sin x}\ dx = \int \sec^2 x\ dx = \tan x$$

$$y(x) = \frac{1}{\sin x}(\tan x + C)$$

$$\boxed{= \sec x + \frac{C}{\sin x}}$$

Exercise

Solve the differential equation: $\frac{dr}{d\theta} + r\sec\theta = \cos\theta$

Solution

$$e^{\int(\sec\theta)d\theta} = e^{\ln|\sec\theta+\tan\theta|} = \sec\theta + \tan\theta$$

$$\int \cos\theta(\sec\theta + \tan\theta)\, d\theta = \int (1+\sin\theta)\, d\theta = \theta - \cos\theta$$

$$\boxed{r(\theta) = \frac{1}{\sec\theta + \tan\theta}(\theta - \cos\theta + C)}$$

Exercise

Find the general solution of $\frac{dr}{d\theta} + r\tan\theta = \sec\theta$

Solution

$$e^{\int \tan\theta d\theta} = e^{\ln|\sec\theta|} = \sec\theta$$

$$\int \sec^2\theta d\theta = \tan\theta$$

$$r(\theta) = \frac{1}{\sec\theta}(\tan\theta + C)$$

$$\boxed{= \sin\theta + C\cos\theta}$$

Exercise

Solve the differential equation: $\frac{dP}{dt} + 2tP = P + 4t - 2$

Solution

$$P' + (2t-1)P = 4t - 2$$

$$e^{\int(2t-1)dt} = e^{t^2-t}$$

$$\int e^{t^2-t}(4t-2)\, dt = 2\int e^{t^2-t}\, d\left(t^2-t\right) = 2e^{t^2-t}$$

$$P(t) = \frac{1}{e^{t^2-t}}\left(2e^{t^2-t} + C\right)$$

$$\boxed{= 2 + Ce^{t-t^2}}$$

Exercise

Solve the differential equation: $ydx - 4\left(x + y^6\right)dy = 0$

Solution

$$y\frac{dx}{dy} - 4x - 4y^6 = 0$$

$$\frac{dx}{dy} - \frac{4}{y}x = 4y^5$$

$$e^{\int -\frac{4}{y}dy} = e^{-4\ln y} = e^{\ln y^{-4}} = y^{-4}$$

$$\int 4y^5 y^{-4}\ dx = 4\int y\ dy = 2y^2$$

$$x(y) = y^4\left(2y^2 + C\right)$$

$$\boxed{= 2y^6 + Cy^4}$$

Exercise

Solve the differential equation: $ydx = \left(ye^y - 2x\right)dy$

Solution

$$y\frac{dx}{dy} = ye^y - 2x \quad \rightarrow \quad \frac{dx}{dy} + \frac{2}{y}x = e^y$$

$$e^{\int \frac{2}{y}dy} = e^{2\ln y} = e^{\ln y^2} = y^2$$

$$\int y^2 e^y\ dx = \left(y^2 - 2y + 2\right)e^y$$

$$\boxed{x(y) = \frac{1}{y^2}\left(\left(y^2 - 2y + 2\right)e^y + C\right)}$$

		$\int e^y$
+	y^2	e^y
–	$2y$	e^y
+	2	e^y

Exercise

Find the general solution of $(x + y + 1)dx - dy = 0$

Solution

$$\frac{dy}{dx} = x + y + 1$$

$$y' - y = x + 1$$

$$e^{\int -dx} = e^{-x}$$

$$\int (x+1)e^{-x}dx = (-x-2)e^{-x}$$

		$\int e^{-x}$
+	$x+1$	$-e^{-x}$
−	1	e^{-x}

$$y(x) = e^{x}\left((-x-2)e^{-x} + C\right)$$

$$\underline{= -x - 2 + Ce^{x}}$$

Exercise

Find the general solution of $\dfrac{dy}{dx} = x^2 e^{-4x} - 4y$

Solution

$$y' + 4y = x^2 e^{-4x}$$

$$e^{\int 4dx} = e^{4x}$$

$$\int x^2 e^{-4x} e^{4x} dx = \int x^2 dx = \frac{1}{3}x^3$$

$$\underline{y(x) = e^{-4x}\left(\frac{1}{3}x^3 + C\right)}$$

Exercise

Find the general solution of $\left(x^2+1\right)y' + xy - x = 0$

Solution

$$y' + \frac{x}{x^2+1}y = \frac{x}{x^2+1}$$

$$e^{\int \frac{x}{x^2+1}dx} = e^{\frac{1}{2}\ln\left(x^2+1\right)} = \left(x^2+1\right)^{1/2}$$

$$\int \frac{x}{x^2+1}\left(x^2+1\right)^{1/2} dx = \frac{1}{2}\int \left(x^2+1\right)^{-1/2} d\left(x^2+1\right) = \sqrt{x^2+1}$$

$$y(x) = \frac{1}{\sqrt{x^2+1}}\left(\sqrt{x^2+1} + C\right)$$

$$\underline{= 1 + \frac{C}{\sqrt{x^2+1}}}$$

Exercise

Find the general solution of $\frac{dx}{dt} = 9.8 - 0.196x$

Solution

$$x' + 0.196x = 9.8$$

$$e^{\int .196dx} = e^{0.196t}$$

$$\int 9.8e^{0.196t}dt = 50e^{0.196t}$$

$$x(t) = \frac{1}{e^{0.196t}}\left(50e^{0.196t} + C\right)$$

$$= 50 + Ce^{-0.196t}$$

Exercise

Find the general solution of $\frac{di}{dt} + 500i = 10\sin\omega t$

Solution

$$e^{\int 500dt} = e^{500t}$$

$$\int 10(\sin\omega t)e^{500t}dt$$

		$\int \sin\omega t$
+	e^{500t}	$-\frac{1}{\omega}\cos\omega t$
−	$500e^{500t}$	$-\frac{1}{\omega^2}\sin\omega t$
+	$25\times 10^4 e^{500t}$	$-\int \frac{1}{\omega^2}\sin\omega t$

$$\int (\sin\omega t)e^{500t}dt = \left(-\frac{1}{\omega}\cos\omega t + \frac{500}{\omega^2}\sin\omega t\right)e^{500t} - \frac{25\times 10^4}{\omega^2}\int (\sin\omega t)e^{500t}dt$$

$$\left(\frac{\omega^2 + 25\times 10^4}{\omega^2}\right)\int (\sin\omega t)e^{500t}dt = \frac{1}{\omega^2}(-\omega\cos\omega t + 500\sin\omega t)e^{500t}$$

$$\int 10(\sin\omega t)e^{500t}dt = \frac{10}{\omega^2 + 25\times 10^4}(-\omega\cos\omega t + 500\sin\omega t)e^{500t}$$

$$i(t) = e^{-500t}\left(\frac{10}{\omega^2 + 25\times 10^4}(-\omega\cos\omega t + 500\sin\omega t)e^{500t} + C\right)$$

$$= \frac{10}{\omega^2 + 25\times 10^4}(-\omega\cos\omega t + 500\sin\omega t) + Ce^{-500t}$$

Exercise

Find the general solution of $2\frac{dQ}{dt}+100Q=10\sin 60t$

		$\int \sin 60t$
+	e^{50t}	$-\frac{1}{60}\cos 60t$
–	$50e^{50t}$	$-\frac{1}{3600}\sin 60t$
+	$2500e^{50t}$	$-\frac{1}{3600}\int \sin 60t$

Solution

$$2\frac{dQ}{dt}+\frac{1}{.01}Q=10\sin 60t \quad \rightarrow \quad \frac{dQ}{dt}+50Q=5\sin 60t$$

$$e^{\int 50dt}=e^{50t}$$

$$5\int e^{50t}(\sin 60t)\,dt=$$

$$\int e^{50t}(\sin 60t)\,dt=\left(-\frac{1}{60}\cos 60t+\frac{1}{72}\sin 60t\right)e^{50t}-\frac{25}{36}\int e^{50t}(\sin 60t)\,dt$$

$$\frac{61}{36}\int e^{50t}(\sin 60t)\,dt=\left(-\frac{1}{60}\cos 60t+\frac{1}{72}\sin 60t\right)e^{50t}$$

$$\int e^{50t}(\sin 60t)\,dt=\frac{36}{21,960}(-6\cos 60t+5\sin 60t)e^{50t}$$

$$5\int e^{50t}(\sin 60t)\,dt=\frac{1}{122}(-6\cos 60t+5\sin 60t)e^{50t}$$

$$Q(t)=\frac{1}{e^{50t}}\left(\frac{1}{122}(-6\cos 60t+5\sin 60t)e^{50t}+C\right)$$

$$=\frac{1}{122}(-6\cos 60t+5\sin 60t)+Ce^{-50t}$$

$$Q(0)=-\frac{6}{122}+C=0 \Rightarrow \boxed{C=\frac{3}{61}}$$

$$\boxed{Q(t)=\frac{1}{122}\left(-5\cos 60t+6\sin 60t+6e^{-50t}\right)}$$

Exercise

Find the general solution of $y'-3y=4$; $y(0)=2$

Solution

$$e^{-\int 3dt}=e^{-3t}$$

$$\int 4e^{-3t}dt=-\frac{4}{3}e^{-3t}$$

$$y(t)=e^{3t}\left(-\frac{4}{3}e^{-3t}+C\right)$$

$$\boxed{=-\frac{4}{3}+Ce^{3t}}$$

$$y(0)=-\frac{4}{3}+Ce^{3(0)}$$

$$2 = -\frac{4}{3} + C \qquad C = \frac{4}{3} + 2 = \underline{\frac{10}{3}}$$

$$\boxed{y(t) = -\frac{4}{3} + \frac{10}{3} e^{3t}}$$

Exercise

Find the general solution of $y' = y + 2xe^{2x}$; $y(0) = 3$

Solution

$$y' - y = 2xe^{2x}$$

$$e^{\int -1 dx} = e^{-x}$$

$$\int 2xe^{2x}\left(e^{-x}\right)dx = 2\int xe^{x}dx = 2\left(xe^{x} - e^{x}\right)$$

$$y(x) = \frac{1}{e^{-x}}\left(2xe^{x} - 2e^{x} + C\right)$$

$$= e^{x}\left(2xe^{x} - 2e^{x} + C\right)$$

$$= 2xe^{2x} - 2e^{2x} + Ce^{x}$$

$$y(x = 0) = 2(0)e^{2(0)} - 2e^{2(0)} + Ce^{(0)}$$

$$3 = -2 + C \qquad \to \boxed{C = 5}$$

$$\boxed{y(x) = 2xe^{2x} - 2e^{2x} + 5e^{x}}$$

Exercise

Find the general solution of $\left(x^2 + 1\right)y' + 3xy = 6x$; $y(0) = -1$

Solution

$$y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}$$

$$e^{\int \frac{3x}{x^2+1}dx} = e^{\frac{3}{2}\ln\left(x^2+1\right)} = e^{\ln\left(x^2+1\right)^{\frac{3}{2}}} = \underline{\left(x^2+1\right)^{\frac{3}{2}}}$$

$$\int \left(x^2+1\right)^{\frac{3}{2}} \frac{6x}{x^2+1}dx = 3\int \left(x^2+1\right)^{\frac{1}{2}} d\left(x^2+1\right) = 2\left(x^2+1\right)^{\frac{3}{2}}$$

$$y(x) = 2 + C\left(x^2+1\right)^{-\frac{3}{2}}$$

$$y(0) = 2 + C\left((0)^2+1\right)^{-\frac{3}{2}}$$

$$-1 = 2 + C(1)^{-\frac{3}{2}} \quad \to \underline{C = -3}$$

$$\boxed{y(x) = 2 - 3\left(x^2 + 1\right)^{-\frac{3}{2}}}$$

Exercise

Solve the initial value problem: $\quad t\dfrac{dy}{dt} + 2y = t^3, \quad t > 0, \quad y(2) = 1$

Solution

$$\frac{dy}{dt} + \frac{2}{t}y = t^2, \quad P(t) = \frac{2}{t}, \quad Q(t) = t^2$$

$$e^{\int \frac{2}{t}dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

$$\int t^2 t^2 dt = \int t^4 dt = \frac{1}{5}t^5$$

$$y(t) = \frac{1}{t^2}\left(\frac{1}{5}t^5 + C\right) = \frac{1}{5}t^3 + \frac{C}{t^2}$$

$$y(2) = \frac{1}{5}2^3 + \frac{C}{2^2}$$

$$1 = \frac{8}{5} + \frac{C}{4} \quad \to \frac{C}{4} = 1 - \frac{8}{5} = -\frac{3}{5} \Rightarrow \boxed{C = -\frac{12}{5}}$$

$$\boxed{y(t) = \frac{1}{5}t^3 - \frac{12}{5t^2}}$$

Exercise

Solve the initial value problem: $\quad \theta\dfrac{dy}{d\theta} + y = \sin\theta, \quad \theta > 0, \quad y\left(\frac{\pi}{2}\right) = 1$

Solution

$$\frac{dy}{d\theta} + \frac{1}{\theta}y = \frac{\sin\theta}{\theta}, \quad P(\theta) = \frac{1}{\theta}, \quad Q(\theta) = \frac{\sin\theta}{\theta}$$

$$e^{\int \frac{1}{\theta}d\theta} = e^{\ln|\theta|} = \theta \quad (>0)$$

$$\int \frac{\sin\theta}{\theta}\theta d\theta = \int \sin\theta d\theta = -\cos\theta$$

$$y(\theta) = \frac{1}{\theta}(-\cos\theta + C)$$

$$y\left(\frac{\pi}{2}\right) = 1 \quad \to 1 = \frac{2}{\pi}(0 + C) \quad \Rightarrow \underline{C = \frac{\pi}{2}}$$

$$\boxed{y(\theta) = -\frac{\cos\theta}{\theta} + \frac{\pi}{2\theta}}$$

Exercise

Solve the initial value problem: $\dfrac{dy}{dx}+xy=x, \quad y(0)=-6$

Solution

$$\frac{dy}{dx}+xy=x$$

$$e^{\int x\,dx}=e^{x^2/2}$$

$$\int xe^{x^2/2}dx=\int e^{x^2/2}d\left(\frac{x^2}{2}\right)=e^{x^2/2} \qquad\qquad d\left(\frac{x^2}{2}\right)=x\,dx$$

$$y(x)=\frac{1}{e^{x^2/2}}\left(e^{x^2/2}+C\right)$$

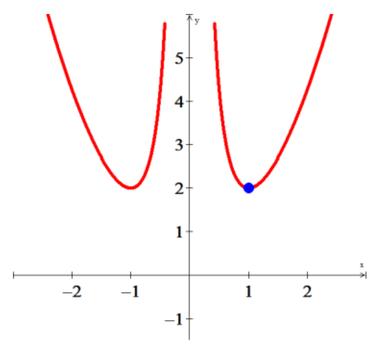
$$y(0)=-6 \quad -6=1+C \quad \to \underline{C=-7}$$

$$y(x)=\frac{1}{e^{x^2/2}}\left(e^{x^2/2}-7\right)$$

$$\underline{=1-\frac{7}{e^{x^2/2}}}$$

Exercise

Solve the initial value problem: $ty'+2y=4t^2, \quad y(1)=2$



Solution

$$y'+\frac{2}{t}y=4t$$

$$e^{\int\frac{2}{t}dt}=e^{2\ln|t|}=e^{\ln t^2}=t^2$$

1st method

$$\int t^2(4t)dt=4\int t^3dt=t^4$$

$$y(t)=\frac{1}{t^2}\left(t^4+C\right) \qquad y=\frac{1}{e^{\int P dx}}\left(\int Q.e^{\int P dx}dx+C\right)$$

2nd method

$$t^2y'+t^2\frac{2}{t}y=4t\left(t^2\right)$$

$$t^2y'+2ty=4t^3$$

$$\left(t^2y\right)'=4t^3$$

$$t^2y=t^4+C$$

$$y(t)=\frac{1}{t^2}\left(t^4+C\right)$$

$$y(1)=\frac{1}{1^2}\left(1^4+C\right)$$

$$2=1+C \to \underline{C=1}$$

$$y(t)=\frac{1}{t^2}\left(t^4+1\right)$$

$$\boxed{y(t)=t^2+\frac{1}{t^2}}$$

Exercise

Find the solution of the initial value problem $\left(1+t^2\right)y'+4ty=\left(1+t^2\right)^{-2}, \quad y(1)=0$

Solution

$$y'+\frac{4t}{1+t^2}y=\frac{\left(1+t^2\right)^{-2}}{1+t^2}$$

$$y'+\frac{4t}{1+t^2}y=\left(1+t^2\right)^{-3}$$

$$e^{\int\frac{4t}{1+t^2}dt}=e^{2\int\frac{d\left(1+t^2\right)}{1+t^2}}=e^{2\ln\left(1+t^2\right)}=e^{\ln\left(1+t^2\right)^2}=\left(1+t^2\right)^2$$

$$\int\left(1+t^2\right)^2\left(1+t^2\right)^{-3}dt=\int\frac{dt}{1+t^2}=\boxed{\tan^{-1}|t|}$$

$$y(t)=\frac{1}{\left(1+t^2\right)^2}\left(\tan^{-1}|t|+C\right)$$

Given $y(1)=0$, then $\quad 0=\frac{1}{\left(1+1^2\right)^2}\left(\tan^{-1}|1|+C\right)$

$$0=\frac{\pi}{4}+C \quad\Rightarrow\quad C=-\frac{\pi}{4}$$

$$\boxed{y(t)=\frac{1}{\left(1+t^2\right)^2}\left(\tan^{-1}|t|-\frac{\pi}{4}\right)}$$

Exercise

Solve the initial value problem: $\quad y'=x+5y, \quad y(0)=3$

Solution

$$y'-5y=x$$

$$e^{\int-5dx}=\boxed{e^{-5x}}$$

		$\int e^{-5x}$
+	x	$-\frac{1}{5}e^{-5x}$
−	1	$\frac{1}{25}e^{-5x}$

$$\int xe^{-5x}dx=\left(-\frac{1}{5}x-\frac{1}{25}\right)e^{-5x}$$

$$y(x) = e^{5x}\left(\left(-\frac{1}{5}x - \frac{1}{25}\right)e^{-5x} + C\right)$$

$$= -\frac{1}{5}x - \frac{1}{25} + Ce^{5x} \qquad y(0) = 3$$

$$3 = -\frac{1}{25} + C \quad \rightarrow \quad C = \frac{76}{25}$$

$$y(x) = -\frac{1}{5}x - \frac{1}{25} + \frac{76}{25}e^{5x}$$

Exercise

Solve the initial value problem: $\quad y' = 2x - 3y, \quad y(0) = \frac{1}{3}$

Solution

$$y' + 3y = 2x$$

$$e^{\int 3dx} = e^{3x}$$

$$\int 2xe^{3x}dx = \left(\frac{2}{3}x - \frac{2}{9}\right)e^{3x}$$

		$\int e^{3x}$
+	$2x$	$\frac{1}{3}e^{3x}$
−	2	$\frac{1}{9}e^{3x}$

$$y(x) = e^{-3x}\left(\left(\frac{2}{3}x - \frac{2}{9}\right)e^{3x} + C\right)$$

$$= \frac{2}{3}x - \frac{2}{9} + Ce^{-3x} \qquad y(0) = \frac{1}{3}$$

$$\frac{1}{3} = -\frac{2}{9} + C \quad \rightarrow \quad C = \frac{5}{9}$$

$$y(x) = \frac{2}{3}x - \frac{2}{9} + \frac{5}{9}e^{-3x}$$

Exercise

Solve the initial value problem: $\quad xy' + y = e^x, \quad y(1) = 2$

Solution

$$y' + \frac{1}{x}y = \frac{e^x}{x}$$

$$e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\int x\frac{e^x}{x}dx = \int e^x dx = e^x$$

$$y(x) = \frac{1}{x}\left(e^x + C\right)$$

$$y(1) = 2 \quad 2 = e + C \quad \rightarrow \quad C = 2 - e$$

$$y(x) = \frac{1}{x}\left(e^x + 2 - e\right)$$

Exercise

Solve the initial value problem: $y\frac{dx}{dy} - x = 2y^2, \quad y(1) = 5$

Solution

$$\frac{dx}{dy} - \frac{1}{y}x = 2y$$

$$e^{\int -\frac{1}{y}dy} = e^{-\ln y} = e^{\ln y^{-1}} = y^{-1}$$

$$\int 2yy^{-1}\,dx = 2\int dy = 2y$$

$$x(y) = y(2y + C)$$

$$y(1) = 5 \quad \rightarrow 1 = 5(10 + C) \quad \Rightarrow \quad C = -\frac{49}{5}$$

$$x(y) = 2y^2 - \frac{49}{5}y$$

Exercise

Solve the initial value problem: $xy' + y = 4x + 1, \quad y(1) = 8$

Solution

$$y' + \frac{1}{x}y = \frac{4x+1}{x}$$

$$e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\int x\frac{4x+1}{x}dx = \int (4x+1)dx = 2x^2 + x$$

$$y(x) = \frac{1}{x}\left(2x^2 + x + C\right) \qquad y(1) = 8$$

$$8 = 3 + C \quad \rightarrow \quad C = 5$$

$$y(x) = 2x + 1 + \frac{5}{x}$$

Exercise

Solve the initial value problem: $y' + 4xy = x^3e^{x^2}, \quad y(0) = -1$

Solution

$$e^{\int 4xdx} = e^{2x^2}$$

$$\int x^3e^{x^2}e^{2x^2}\,dx = \int x^3e^{3x^2}\,dx = \frac{1}{6}\int x^2e^{3x^2}\,d\left(3x^2\right)$$

	$u = 3x^2$	$\int e^u$
+	u	e^u
−	1	e^u

$$= \frac{1}{18}\int ue^u d(u)$$

$$= \frac{1}{18}\left(3x^2 - 1\right)e^{3x^2}$$

$$y(x) = \frac{1}{e^{2x^2}}\left(\frac{1}{18}\left(3x^2 - 1\right)e^{3x^2} + C\right)$$

$$y(0) = -1 \qquad -1 = -\frac{1}{18} + C \quad \to \quad \underline{C = -\frac{17}{18}}$$

$$\underline{y(x) = \frac{1}{18}\left(3x^2 - 1\right)e^{x^2} - \frac{17}{18}e^{2x^2}}$$

Exercise

Solve the initial value problem: $\quad (x+1)y' + y = \ln x, \quad y(1) = 10$

Solution

$$y' + \frac{1}{x+1}y = \frac{\ln x}{x+1}$$

$$e^{\int \frac{dx}{x+1}} = e^{\ln(x+1)} \; \underline{= x+1}$$

$$\int \frac{\ln x}{x+1}(x+1)\,dx = \int \ln x\,dx = x\ln x - x$$

$$y(x) = \frac{1}{x+1}(x\ln x - x + C) \qquad y(1) = 10$$

$$10 = \frac{1}{2}(-1 + C) \; \to \; \underline{C = 21}$$

$$\underline{y(x) = \frac{1}{x+1}(x\ln x - x + 21)}$$

Exercise

Solve the initial value problem: $\quad y' - (\sin x)y = 2\sin x, \quad y\left(\frac{\pi}{2}\right) = 1$

Solution

$$e^{\int -\sin x dx} \; \underline{= e^{\cos x}}$$

$$\int 2\sin x e^{\cos x} dx = -2\int e^{\cos x}\, d(\cos x) = -2e^{\cos x}$$

$$y(x) = \frac{1}{e^{\cos x}}\left(-2e^{\cos x} + C\right) \qquad y\left(\frac{\pi}{2}\right) = 1$$

$$1 = -2 + C \quad \to \; \underline{C = 3}$$

$$\underline{y(x) = -2 + \frac{3}{e^{\cos x}}}$$

Exercise

Solve the initial value problem: $L\frac{di}{dt} + RI = E, \quad i(0) = i_0$

Solution

$$e^{\int R dt} = e^{Rt}$$

$$\int Ee^{Rt} dt = \frac{E}{R}e^{Rt}$$

$$I(t) = \frac{1}{e^{Rt}}\left(\frac{E}{R}e^{Rt} + C\right) \qquad i(0) = i_0$$

$$i_0 = \frac{E}{R} + C \rightarrow C = i_0 - \frac{E}{R}$$

$$\boxed{I(t) = \frac{E}{R} + \left(i_0 - \frac{E}{R}\right)e^{-Rt}}$$

Exercise

Solve the initial value problem: $\frac{dT}{dt} = k\left(T - T_m\right) \quad T(0) = T_0$

Solution

$$\frac{dT}{dt} - kT = -kT_m$$

$$e^{\int -k dt} = e^{-kt}$$

$$\int -kT_m e^{-kt} dt = T_m e^{-kt}$$

$$T(t) = \frac{1}{e^{-kt}}\left(T_m e^{-kt} + C\right) \qquad T(0) = T_0$$

$$T_0 = T_m + C \rightarrow C = T_0 - T_m$$

$$\boxed{T(t) = T_m + \left(T_0 - T_m\right)e^{kt}}$$

Exercise

Solve the initial value problem: $y' + y = 2, \quad y(0) = 0$

Solution

$$e^{\int dx} = e^x$$

$$\int 2e^x dx = 2e^x$$

$$y(x)=\frac{1}{e^x}\left(2e^x+C\right)$$

$$\boxed{=2+Ce^{-x}}$$

$$y(0)=0 \quad \to 0=2+C \quad \Rightarrow \quad \boxed{C=-2}$$

$$\boxed{y(x)=2-2e^{-x}}$$

Exercise

Solve the initial value problem: $\quad y'-2y=3e^{2x}, \quad y(0)=0$

Solution

$$e^{\int -2dx}=e^{-2x}$$

$$\int 3e^{2x}e^{-2x}dx=3x$$

$$y(x)=\boxed{e^{2x}(3x+C)}$$

$$y(0)=0 \quad \to \boxed{0=C}$$

$$\boxed{y(x)=3xe^{2x}}$$

Exercise

Solve the initial value problem: $\quad xy'+2y=3x, \quad y(1)=5$

Solution

$$y'+\frac{2}{x}y=3$$

$$e^{\int \frac{2}{x}dx}=e^{2\ln x}=x^2$$

$$\int 3x^2\, dx=x^3$$

$$y(x)=\frac{1}{x^2}\left(x^3+C\right)$$

$$\boxed{=x+\frac{C}{x^2}}$$

$$y(1)=5 \quad \to 5=1+C \quad \Rightarrow \quad \boxed{C=4}$$

$$\boxed{y(x)=x+\frac{4}{x^2}}$$

Exercise

Solve the initial value problem: $xy' + 5y = 7x^2, \quad y(2) = 5$

Solution

$$y' + \frac{5}{x}y = 7x$$

$$e^{\int \frac{5}{x}dx} = e^{5\ln x} = x^5$$

$$\int 7x^2 x^5\, dx = \frac{7}{8}x^8$$

$$y(x) = \frac{1}{x^5}\left(\frac{7}{8}x^8 + C\right)$$

$$= \frac{7}{8}x^3 + \frac{C}{x^5}$$

$$y(2) = 5 \quad \to 5 = 7 + \frac{1}{32}C \quad \Rightarrow \quad C = -64$$

$$y(x) = \frac{7}{8}x^3 - \frac{64}{x^5}$$

Exercise

Solve the initial value problem: $xy' - y = x, \quad y(1) = 7$

Solution

$$y' - \frac{1}{x}y = 1$$

$$e^{\int \frac{-1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$\int \frac{1}{x}\, dx = \ln x$$

$$y(x) = x(\ln x + C)$$

$$= x\ln x + Cx$$

$$y(1) = 7 \quad \to 7 = C$$

$$y(x) = x\ln x + 7x$$

Exercise

Solve the initial value problem: $xy' + y = 3xy, \quad y(1) = 0$

Solution

$$xy' + (1-3x)y = 0$$

$$y' + \left(\frac{1}{x} - 3\right)y = 0$$

$$e^{\int\left(\frac{1}{x}-3\right)dx} = e^{\ln x - 3x} = e^{\ln x}e^{-3x} = xe^{-3x}$$

$$y(x) = \frac{1}{xe^{-3x}}C$$

$$= \boxed{\frac{Ce^{3x}}{x}}$$

$$y(1) = 0 \quad \to \underline{0 = C}$$

$$\boxed{y(x) = 0}$$

Exercise

Solve the initial value problem: $xy' + 3y = 2x^5, \quad y(2) = 1$

Solution

$$y' + \frac{3}{x}y = 2x^4$$

$$e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$$

$$\int 2x^4x^3\, dx = \frac{1}{4}x^8$$

$$y(x) = \frac{1}{x^3}\left(\frac{1}{4}x^8 + C\right)$$

$$= \boxed{\frac{1}{4}x^5 + Cx^{-3}}$$

$$y(2) = 1 \quad \to 1 = 8 + \frac{C}{8} \quad \Rightarrow \underline{C = -56}$$

$$\boxed{y(x) = \frac{1}{4}x^5 - 56x^{-3}}$$

Exercise

Solve the initial value problem: $y' + y = e^x, \quad y(0) = 1$

Solution

$$e^{\int dx} = e^x$$

$$\int e^x e^x\, dx = \int e^{2x}\, dx = \frac{1}{2}e^{2x}$$

$$y(x) = \frac{1}{e^x}\left(\frac{1}{2}e^{2x} + C\right)$$

$$= \frac{1}{2}e^x + Ce^{-x}$$

$$y(0) = 1 \quad \to 1 = \frac{1}{2} + C \quad \Rightarrow C = \frac{1}{2}$$

$$y(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$$

Exercise

Solve the initial value problem: $\quad xy' - 3y = x^3, \quad y(1) = 10$

Solution

$$y' - \frac{3}{x}y = x^2$$

$$e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = x^{-3}$$

$$\int x^{-3}x^3\, dx = \int dx = x$$

$$y(x) = x^3(x + C)$$

$$= x^4 + Cx^3$$

$$y(1) = 10 \quad \to 10 = 1 + C \quad \Rightarrow C = 9$$

$$y(x) = x^4 + 9x^3$$

Exercise

Solve the initial value problem: $\quad y' + 2xy = x, \quad y(0) = -2$

Solution

$$e^{\int 2x dx} = e^{x^2}$$

$$\int xe^{x^2}\, dx = \frac{1}{2}e^{x^2}$$

$$y(x) = e^{-x^2}\left(\frac{1}{2}e^{x^2} + C\right)$$

$$= \frac{1}{2} + Ce^{-x^2}$$

$$y(0) = -2 \quad \to -2 = \frac{1}{2} + C \quad \Rightarrow C = -\frac{5}{2}$$

$$\boxed{y(x) = \frac{1}{2} - \frac{5}{2}e^{-x^2}}$$

Exercise

Solve the initial value problem: $y' = (1-y)\cos x, \quad y(\pi) = 2$

Solution

$$y' + (\cos x)y = \cos x$$

$$e^{\int \cos x dx} = e^{\sin x}$$

$$\int \cos x e^{\sin x}\, dx = e^{\sin x}$$

$$y(x) = \frac{1}{e^{\sin x}}\left(e^{\sin x} + C\right)$$

$$\underline{= 1 + Ce^{-\sin x}}$$

$$y(\pi) = 2 \quad \to \quad 2 = 1 + C \quad \Rightarrow \underline{C = 1}$$

$$\boxed{y(x) = 1 + e^{-\sin x}}$$

Exercise

Solve the initial value problem: $(1+x)y' + y = \cos x, \quad y(0) = 1$

Solution

$$y' + \frac{1}{x+1}y = \frac{\cos x}{1+x}$$

$$e^{\int \frac{1}{1+x}dx} = e^{\ln(1+x)} = 1 + x$$

$$\int \frac{\cos x}{1+x}(1+x)\, dx = \sin x$$

$$y(x) = \frac{1}{1+x}(\sin x + C)$$

$$y(0) = 1 \quad \Rightarrow \underline{C = 1}$$

$$\boxed{y(x) = \frac{1}{1+x}(\sin x + 1)}$$

Exercise

Solve the initial value problem: $y' = 1 + x + y + xy, \quad y(0) = 0$

Solution

$$y' - (1+x)y = 1 + x$$

$$e^{-\int(1+x)dx} = e^{-x-\frac{1}{2}x^2}$$

$$\int(1+x)e^{-\left(x+x^2/2\right)}\,dx = -e^{-\left(x+x^2/2\right)}$$

$$y(x) = e^{x+\frac{1}{2}x^2}\left(-e^{-\left(x+\frac{1}{2}x^2\right)} + C\right)$$

$$= -1 + Ce^{x+\frac{1}{2}x^2}$$

$$y(0) = 0 \quad \to \quad 0 = -1 + C \quad \Rightarrow \underline{C = 1}$$

$$\underline{y(x) = -1 + e^{x+\frac{1}{2}x^2}}$$

Exercise

Solve the initial value problem: $xy' = 3y + x^4\cos x, \quad y(2\pi) = 0$

Solution

$$y' - \frac{3}{x}y = x^3\cos x$$

$$e^{-\int\frac{3}{x}dx} = e^{-3\ln x} = x^{-3}$$

$$\int x^{-3}x^3\cos x\,dx = \int\cos x\,dx = \sin x$$

$$y(x) = x^3(\sin x + C)$$

$$y(2\pi) = 0 \quad \to \quad \underline{0 = C}$$

$$\underline{y(x) = x^3\sin x}$$

Exercise

Solve the initial value problem: $y' = 2xy + 3x^2e^{x^2}, \quad y(0) = 5$

Solution

$$y' - 2xy = 3x^2e^{x^2}$$

$$e^{-\int 2x\,dx} = e^{-x^2}$$

$$\int 3x^2e^{x^2}e^{-x^2}\,dx = \int 3x^2\,dx = x^3$$

$$y(x) = e^{x^2}\left(x^3 + C\right)$$

$$y(0)=5 \quad \rightarrow \quad \underline{5=C}$$

$$\boxed{y(x)=e^{x^2}\left(x^3+5\right)}$$

Exercise

Solve the initial value problem: $\quad \left(x^2+4\right)y'+3xy=x, \quad y(0)=1$

Solution

$$y'+\frac{3x}{x^2+4}y=\frac{x}{x^2+4}$$

$$e^{\int\frac{3x}{x^2+4}dx}=e^{\frac{3}{2}\int\frac{1}{x^2+4}d\left(x^2+4\right)}=e^{\frac{3}{2}\ln\left(x^2+4\right)}=\left(x^2+4\right)^{3/2}$$

$$\int\frac{x}{x^2+4}\left(x^2+4\right)^{3/2}\,dx=\frac{1}{2}\int\left(x^2+4\right)^{1/2}\,d\left(x^2+4\right)=\frac{1}{3}\left(x^2+4\right)^{3/2}$$

$$y(x)=\left(x^2+4\right)^{-3/2}\left(\frac{1}{3}\left(x^2+4\right)^{3/2}+C\right)$$

$$=\frac{1}{3}+C\left(x^2+4\right)^{-3/2}$$

$$y(0)=1 \quad \rightarrow \quad 1=\frac{1}{3}+\frac{1}{8}C \quad \Rightarrow \underline{C=\frac{16}{3}}$$

$$\boxed{y(x)=\frac{1}{3}+\frac{16}{3}\left(x^2+4\right)^{-3/2}}$$

Exercise

Solve the initial value problem: $\quad \left(x^2+1\right)y'+3x^3y=6xe^{-3x^2/2}, \quad y(0)=1$

Solution

$$y'+\frac{3x^3}{x^2+1}y=\frac{6xe^{-3x^2/2}}{x^2+1}$$

$$y'+\left(3x-\frac{3x}{x^2+1}\right)y=\frac{6xe^{-3x^2/2}}{x^2+1}$$

$$e^{\int\left(3x-\frac{3x}{x^2+1}\right)dx}=e^{\frac{3}{2}x^2-\frac{3}{2}\ln\left(x^2+1\right)}$$

$$=e^{\frac{3}{2}x^2}e^{\ln\left(x^2+1\right)^{-3/2}}$$

$$=e^{\frac{3}{2}x^2}\left(x^2+1\right)^{-3/2}$$

$$\int \frac{6xe^{-3x^2/2}}{x^2+1} e^{\frac{3}{2}x^2}\left(x^2+1\right)^{-3/2} dx = 3\int \left(x^2+1\right)^{-5/2} d\left(x^2+1\right)$$

$$= -2\left(x^2+1\right)^{-3/2}$$

$$y(x) = e^{-3x^2/2}\left(x^2+1\right)^{3/2}\left(-2\left(x^2+1\right)^{-3/2} + C\right)$$

$$= e^{-3x^2/2}\left(-2 + C\left(x^2+1\right)^{3/2}\right)$$

$$y(0) = 1 \quad \to \ 1 = -2 + C \quad \Rightarrow \underline{C = 3}$$

$$\underline{y(x) = e^{-3x^2/2}\left(-2 + 3\left(x^2+1\right)^{3/2}\right)}$$

Exercise

Solve the initial value problem: $\quad y' - 2y = e^{3x}; \quad y(0) = 3$

Solution

$$e^{\int -2dx} = e^{-2x}$$

$$\int e^{3x}e^{-2x}dx = \int e^{x}dx = e^{x}$$

$$y(x) = e^{2x}\left(e^{x} + C\right)$$

$$\underline{= e^{3x} + Ce^{2x}}$$

$$y(0) = 1 \quad \to 1 = 1 + C \ \Rightarrow \underline{C = 0}$$

$$\underline{y(x) = e^{3x}}$$

Exercise

Solve the initial value problem: $\quad y' - 3y = 6; \quad y(0) = 1$

Solution

$$e^{\int -3dx} = e^{-3x}$$

$$\int 6e^{-3x}dx = -2e^{-3x}$$

$$y(x) = e^{3x}\left(-2e^{-3x} + C\right)$$

$$\underline{= -2 + Ce^{3x}}$$

$$y(0)=1 \quad \to 1=-2+C \quad \Rightarrow \boxed{C=3}$$

$$\boxed{y(x)=-2+3e^{3x}}$$

Exercise

Solve the initial value problem: $\quad 2y'+3y=e^{x}; \quad y(0)=0$

Solution

$$y'+\frac{3}{2}y=\frac{1}{2}e^{x}$$

$$e^{\int \frac{3}{2}dx}=e^{3x/2}$$

$$\int e^{x}e^{3x/2}dx=\int e^{5x/2}dx=\frac{2}{5}e^{5x/2}$$

$$y(x)=e^{-3x/2}\left(\frac{2}{5}e^{5x/2}+C\right)$$

$$\boxed{=\frac{2}{5}e^{x}+Ce^{-3x/2}}$$

$$y(0)=0 \quad \to \quad 0=\frac{2}{5}+C \quad \Rightarrow \boxed{C=-\frac{2}{5}}$$

$$\boxed{y(x)=\frac{2}{5}e^{x}-\frac{2}{5}e^{-3x/2}}$$

Exercise

Solve the initial value problem: $\quad y'+y=1+e^{-x}\cos 2x; \quad y\left(\frac{\pi}{2}\right)=0$

Solution

$$e^{\int dx}=e^{x}$$

$$\int e^{x}\left(1+e^{-x}\cos 2x\right)dx=\int\left(e^{x}+\cos 2x\right)dx=e^{x}+\frac{1}{2}\sin 2x$$

$$y(x)=e^{-x}\left(e^{x}+\frac{1}{2}\sin 2x+C\right)$$

$$\boxed{=1+\frac{1}{2}e^{-x}\sin 2x+Ce^{-x}}$$

$$y\left(\frac{\pi}{2}\right)=0 \quad \to \quad 0=1+Ce^{-\pi/2} \quad \Rightarrow \boxed{C=-e^{\pi/2}}$$

$$\boxed{y(x)=1+\frac{1}{2}e^{-x}\sin 2x-e^{-x+\pi/2}}$$

Exercise

Solve the initial value problem: $2y' + (\cos x)y = -3\cos x; \quad y(0) = -4$

Solution

$$y' + \left(\frac{1}{2}\cos x\right)y = -\frac{3}{2}\cos x$$

$$e^{\frac{1}{2}\int \cos x\, dx} = e^{\frac{1}{2}\sin x}$$

$$\int e^{\frac{1}{2}\sin x}(-3\cos x)\,dx = -6\int e^{\frac{1}{2}\sin x}\,d\left(\frac{1}{2}\sin x\right) = -6e^{\frac{1}{2}\sin x}$$

$$y(x) = e^{-\frac{1}{2}\sin x}\left(-6e^{\frac{1}{2}\sin x} + C\right)$$

$$= -6 + Ce^{-\frac{1}{2}\sin x}$$

$$y(0) = -4 \quad \to \quad -4 = -6 + C \quad \Rightarrow \underline{C = 2}$$

$$\boxed{y(x) = -6 + 2e^{-\frac{1}{2}\sin x}}$$

Exercise

Solve the initial value problem: $y' + 2y = e^{-x} + x + 1; \quad y(-1) = e$

Solution

$$e^{\int 2dx} = e^{2x}$$

$$\int \left(e^{-x} + x + 1\right)e^{2x}dx = \int \left(e^{x} + (x+1)e^{2x}\right)dx$$

$$= e^{x} + \left(\frac{1}{2}x + \frac{1}{4}\right)e^{2x}$$

		$\int e^{2x}$
+	$x+1$	$\frac{1}{2}e^{2x}$
−	1	$\frac{1}{4}e^{2x}$

$$y(x) = e^{-2x}\left(e^{x} + \left(\frac{1}{2}x + \frac{1}{4}\right)e^{2x} + C\right)$$

$$= e^{-x} + \frac{1}{2}x + \frac{1}{4} + Ce^{-2x}$$

$$y(-1) = e \quad \to \quad e = e - \frac{1}{2} + \frac{1}{4} + Ce^{2} \quad \Rightarrow C = \frac{1}{4}e^{-2}$$

$$\boxed{y(x) = e^{-x} + \frac{1}{2}x + \frac{1}{4} + \frac{1}{4}e^{-2x-2}}$$

Exercise

Solve the initial value problem: $y' + \frac{y}{x} = xe^{-x}; \quad y(1) = e - 1$

Solution

$$e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\int x^2 e^{-x} dx = \left(-x^2 - 2x - 2\right)e^{-x}$$

		$\int e^{-x}$
+	x^2	$-e^{-x}$
−	$2x$	e^{-x}
+	2	$-e^{-x}$

$$y(x) = \frac{1}{x}\left(-\left(x^2 + 2x + 2\right)e^{-x} + C\right)$$

$$y(1) = e - 1 \quad \to e - 1 = -5e^{-1} + C \quad \Rightarrow \boxed{C = 5e^{-1} + e - 1}$$

$$\boxed{y(x) = \frac{1}{x}\left(-\left(x^2 + 2x + 2\right)e^{-x} + 5e^{-1} + e - 1\right)}$$

Exercise

Solve the initial value problem: $y' + 4y = e^{-x}; \quad y(1) = \frac{4}{3}$

Solution

$$e^{\int 4dx} = e^{4x}$$

$$\int e^{-x}e^{4x}dx = \int e^{3x}dx = \frac{1}{3}e^{3x}$$

$$y(x) = e^{-4x}\left(\frac{1}{3}e^{3x} + C\right)$$

$$= \frac{1}{3}e^{-x} + Ce^{-4x}$$

$$y(1) = \frac{4}{3} \quad \to \quad \frac{4}{3} = \frac{1}{3}e^{-1} + Ce^{-4} \quad \Rightarrow \boxed{C = \frac{1}{3}\left(4e^4 - e^3\right)}$$

$$\boxed{y(x) = \frac{1}{3}e^{-x} + \frac{1}{3}\left(4e^4 - e^3\right)e^{-4x}}$$

Exercise

Solve the initial value problem: $x^2 y' + 3xy = x^4 \ln x + 1; \quad y(1) = 0$

Solution

$$y' + \frac{3}{x}y = x^2 \ln x + \frac{1}{x^2}$$

$$e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$$

$$\int\left(x^2\ln x+\frac{1}{x^2}\right)x^3\,dx=\int\left(x^5\ln x+x\right)dx \qquad u=\ln x \quad dv=x^5 \qquad du=\frac{1}{x} \quad v=\frac{1}{6}x^6$$

$$=\frac{1}{6}x^6\ln x-\frac{1}{6}\int x^5dx+\frac{1}{2}x^2$$

$$=\frac{1}{6}x^6\ln x-\frac{1}{36}x^6+\frac{1}{2}x^2$$

$$y(x)=\frac{1}{x^3}\left(\frac{1}{6}x^6\ln x-\frac{1}{36}x^6+\frac{1}{2}x^2+C\right)$$

$$=\frac{1}{6}x^3\ln x-\frac{1}{36}x^3+\frac{1}{2x}+\frac{C}{x^3}$$

$$y(1)=0 \quad \to 0=-\frac{1}{36}+\frac{1}{2}+C\Rightarrow C=-\frac{17}{36}$$

$$\boxed{y(x)=\frac{1}{6}x^3\ln x-\frac{1}{36}x^3+\frac{1}{2x}-\frac{17}{36x^3}}$$

Exercise

Find the solution of the initial value problem $\qquad y'+\frac{3}{x}y=3x-2 \quad y(1)=1$

Solution

$$e^{\int\frac{3}{x}dx}=e^{3\ln x}=x^3$$

$$\int(3x-2)x^3dx=\int\left(3x^4-2x^3\right)dx=\frac{3}{5}x^5-\frac{1}{2}x^4$$

$$y(x)=\frac{1}{x^3}\left(\frac{3}{5}x^5-\frac{1}{2}x^4+C\right)$$

$$\boxed{=\frac{3}{5}x^2-\frac{1}{2}x+\frac{C}{x^3}}$$

$$y(1)=1 \quad \to 1=\frac{3}{5}-\frac{1}{2}+C \quad \Rightarrow \boxed{C=\frac{9}{10}}$$

$$\boxed{y(x)=\frac{3}{5}x^2-\frac{1}{2}x+\frac{9}{10}x^{-3}}$$

Exercise

Find the solution of the initial value problem $\qquad (\cos x)y'+y\sin x=2x\cos^2 x \quad y\left(\frac{\pi}{4}\right)=\frac{-15\sqrt{2}\pi^2}{32}$

Solution

$$y'+(\tan x)y=2x\cos x$$

$$e^{\int\tan x\,dx}=e^{\ln\sec x}=\sec x$$

$$\int 2x\cos x(\sec x)\,dx = \int 2x\,dx = x^2$$

$$y(x) = \frac{1}{\sec x}\left(x^2 + C\right)$$

$$= \cos x\left(x^2 + C\right)$$

$$y\left(\frac{\pi}{4}\right) = \frac{-15\sqrt{2}\pi^2}{32} \quad \rightarrow \quad \frac{-15\sqrt{2}\pi^2}{32} = \frac{\sqrt{2}}{2}\left(\frac{\pi^2}{16} + C\right)$$

$$\Rightarrow C = \frac{-15\sqrt{2}\pi^2}{32} - \frac{\sqrt{2}\pi^2}{32} = \boxed{-\frac{\sqrt{2}\pi^2}{2}}$$

$$\boxed{y(x) = \cos x\left(x^2 - \frac{\sqrt{2}\pi^2}{2}\right)}$$

Exercise

Find the solution of the initial value problem $(\cos x)y' + (\sin x)y = 2\cos^3 x\sin x - 1 \quad y\left(\frac{\pi}{4}\right) = 3\sqrt{2}$

Solution

$$y' + (\tan x)y = 2\cos^2 x\sin x - \sec x$$

$$e^{\int \tan x dx} = e^{\ln|\sec x|} = \sec x$$

$$\int\left(2\cos^2 x\sin x - \sec x\right)\sec x\,dx = \int\left(2\cos x\sin x - \sec^2 x\right)dx$$

$$= \int\left(\sin 2x - \sec^2 x\right)dx$$

$$= -\frac{1}{2}\cos 2x - \tan x$$

$$y(x) = \frac{1}{\sec x}\left(-\frac{1}{2}\cos 2x - \tan x + C\right)$$

$$y(x) = -\frac{1}{2}\cos 2x\cos x - \sin x + C\cos x \qquad y\left(\frac{\pi}{4}\right) = 3\sqrt{2}$$

$$3\sqrt{2} = -\frac{\sqrt{2}}{2} + C\frac{\sqrt{2}}{2} \quad \rightarrow \quad \boxed{C = 7}$$

$$\boxed{y(x) = -\frac{1}{2}\cos 2x\cos x - \sin x + 7\cos x}$$

Exercise

Find the solution of the initial value problem $t\,y' + 2y = t^2 - t + 1 \quad y(1) = \frac{1}{2}$

Solution

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

$$e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = t^2$$

$$\int \left(t - 1 + \frac{1}{t}\right)t^2 dt = \int \left(t^3 - t^2 + t\right)dt = \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2$$

$$y(t) = \frac{1}{t^2}\left(\frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + C\right)$$

$$= \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{C}{t^2} \qquad y(1) = \frac{1}{2}$$

$$\frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C \quad \to \quad C = \frac{1}{12}$$

$$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{2t^2}$$

Exercise

Find the solution of the initial value problem $\quad t\,y' - 2y = t^5 \sin 2t - t^3 + 4t^4 \quad y(\pi) = \frac{3}{2}\pi^4$

Solution

$$y' - \frac{2}{t}y = t^4 \sin 2t - t^2 + 4t^3$$

$$e^{\int -\frac{2}{t}dt} = e^{-2\ln|t|} = t^{-2}$$

$$\int \left(t^4 \sin 2t - t^2 + 4t^3\right)\frac{1}{t^2}dt = \int \left(t^2 \sin 2t - 1 + 4t\right)dt$$

		$\int \sin 2t$
+	t^2	$-\frac{1}{2}\cos 2t$
–	$2t$	$-\frac{1}{4}\sin 2t$
+	2	$\frac{1}{8}\cos 2t$

$$= -\frac{1}{2}t^2 \cos 2t + \frac{1}{2}t \sin 2t + \frac{1}{4}\cos 2t - t + 2t^2$$

$$y(t) = t^2\left(-\frac{1}{2}t^2 \cos 2t + \frac{1}{2}t \sin 2t + \frac{1}{4}\cos 2t - t + 2t^2 + C\right)$$

$$= -\frac{1}{2}t^4 \cos 2t + \frac{1}{2}t^3 \sin 2t + \frac{1}{4}t^2 \cos 2t - t^3 + 2t^4 + Ct^2 \qquad y(\pi) = \frac{3}{2}\pi^4$$

$$\frac{3}{2}\pi^4 = -\frac{1}{2}\pi^4 + \frac{1}{4}\pi^2 - \pi^3 + 2\pi^4 + \pi^2 C$$

$$C = \pi - \frac{1}{4}$$

$$y(t) = -\frac{1}{2}t^4 \cos 2t + \frac{1}{2}t^3 \sin 2t + \frac{1}{4}t^2 \cos 2t - t^3 + 2t^4 + \left(\pi - \frac{1}{4}\right)t^2$$

Exercise

Find the solution of the initial value problem $\quad 2y' - y = 4\sin 3t \quad y(0) = y_0$

Solution

$$y' - \frac{1}{2}y = 2\sin 3t$$

$$e^{\int -\frac{1}{2}dt} = e^{-\frac{t}{2}}$$

$$\int 2e^{-t/2}\sin 3t\ dt = e^{-t/2}\left(-\frac{2}{3}\cos 3t - \frac{1}{9}\sin 3t\right) - \frac{1}{18}\int e^{-t/2}\sin 3t\ dt$$

$$\frac{37}{18}\frac{1}{2}\int 2e^{-t/2}\sin 3t\ dt = e^{-t/2}\left(-\frac{2}{3}\cos 3t - \frac{1}{9}\sin 3t\right)$$

$$\int 2e^{-t/2}\sin 3t\ dt = \frac{36}{37}e^{-t/2}\left(-\frac{2}{3}\cos 3t - \frac{1}{9}\sin 3t\right)$$

$$= \left(-\frac{24}{37}\cos 3t - \frac{4}{37}\sin 3t\right)e^{-t/2}$$

		$\int \sin 3t$
+	$2e^{-t/2}$	$-\frac{1}{3}\cos 3t$
−	$-e^{-t/2}$	$-\frac{1}{9}\sin 3t$
+	$\frac{1}{2}e^{-t/2}$	

$$y(t) = e^{t/2}\left(\left(-\frac{24}{37}\cos 3t - \frac{4}{37}\sin 3t\right)e^{-t/2} + C\right)$$

$$= -\frac{24}{37}\cos 3t - \frac{4}{37}\sin 3t + Ce^{t/2} \qquad y(0) = y_0$$

$$y_0 = -\frac{24}{37} + C \quad \to \quad C = y_0 + \frac{24}{37}$$

$$y(t) = -\frac{24}{37}\cos 3t - \frac{4}{37}\sin 3t + \left(y_0 + \frac{27}{37}\right)e^{t/2}$$

Exercise

Find the solution of the initial value problem $\quad y' + 2y = 2 - e^{-4t} \quad y(0) = 1$

Solution

$$e^{\int 2dt} = e^{2t}$$

$$\int \left(2 - e^{-4t}\right)e^{2t}dt = \int \left(2e^{2t} - e^{-2t}\right)dt = e^{2t} + \frac{1}{2}e^{-2t}$$

$$y(t) = \frac{1}{e^{2t}}\left(e^{2t} + \frac{1}{2}e^{-2t} + C\right)$$

$$= 1 + \frac{1}{2}e^{-4t} + Ce^{-2t}$$

$$y(0) = 1 \quad \to 1 = 1 + \frac{1}{2} + C \quad \Rightarrow \quad C = -\frac{1}{2}$$

$$y(t) = 1 + \frac{1}{2}e^{-4t} - \frac{1}{2}e^{-2t}$$

Exercise

Find the solution of the initial value problem $y' - y = -\frac{1}{2}e^{t/2}\sin 5t + 5e^{t/2}\cos 5t \quad y(0) = 0$

Solution

$$e^{\int -dt} = e^{-t}$$

$$\int \left(-\frac{1}{2}e^{t/2}\sin 5t + 5e^{t/2}\cos 5t\right)e^{-t}dt = -\frac{1}{2}\int \left(e^{-t/2}\sin 5t\right)dt + 5\int \left(e^{-t/2}\cos 5t\right)dt$$

		$\int \sin 5t$
+	$e^{-t/2}$	$-\frac{1}{5}\cos 5t$
–	$-\frac{1}{2}e^{-t/2}$	$-\frac{1}{25}\sin 5t$
+	$\frac{1}{4}e^{-t/2}$	

		$\int \cos 5t$
+	$e^{-t/2}$	$\frac{1}{5}\sin 5t$
–	$-\frac{1}{2}e^{-t/2}$	$-\frac{1}{25}\cos 5t$
+	$\frac{1}{4}e^{-t/2}$	

$$\int \left(e^{-t/2}\sin 5t\right)dt = \left(-\frac{1}{5}\cos 5t - \frac{1}{50}\sin 5t\right)e^{-t/2} - \frac{1}{100}\int \left(e^{-t/2}\sin 5t\right)dt$$

$$\frac{101}{100}\int \left(e^{-t/2}\sin 5t\right)dt = -\frac{1}{50}(10\cos 5t + \sin 5t)e^{-t/2}$$

$$\int \left(e^{-t/2}\sin 5t\right)dt = -\frac{2}{101}(10\cos 5t + \sin 5t)e^{-t/2}$$

$$\int \left(e^{-t/2}\cos 5t\right)dt = e^{-t/2}\left(\frac{1}{5}\sin 5t - \frac{1}{50}\cos 5t\right) - \frac{1}{100}\int \left(e^{-t/2}\cos 5t\right)dt$$

$$\frac{101}{100}\int \left(e^{-t/2}\cos 5t\right)dt = \frac{1}{50}e^{-t/2}(10\sin 5t - \cos 5t)$$

$$\int \left(e^{-t/2}\cos 5t\right)dt = \frac{2}{101}e^{-t/2}(10\sin 5t - \cos 5t)$$

$$\int \left(-\frac{1}{2}e^{t/2}\sin 5t + 5e^{t/2}\cos 5t\right)e^{-t}dt = \left(\frac{10}{101}\cos 5t + \frac{1}{101}\sin 5t + \frac{100}{101}\sin 5t - \frac{10}{101}\cos 5t\right)e^{-t/2}$$

$$= e^{-t/2}\sin 5t$$

$$y(t) = e^{t}\left(e^{-t/2}\sin 5t + C\right)$$

$$= e^{t/2}\sin 5t + Ce^{t}$$

$$y(0) = 0 \quad \rightarrow \quad \underline{C = 0}$$

$$\boxed{y(t) = e^{t/2}\sin 5t}$$

Exercise

Find the solution of the initial value problem $y' + 2y = 3\ ;\quad y(0) = -1$

Solution

$$e^{\int 2dt} = e^{2t}$$

$$\int 3e^{2t}dt = \frac{3}{2}e^{2t}$$

$$y(t) = \frac{1}{e^{2t}}\left(\frac{3}{2}e^{2t} + C\right)$$

$$= \frac{3}{2} + Ce^{-2t}$$

$$y(0) = -1 \quad \rightarrow \quad \frac{3}{2} + C = -1 \quad \Rightarrow C = -\frac{5}{2}$$

$$\boxed{y(t) = \frac{3}{2} - \frac{5}{2}e^{-2t}}$$

Exercise

Find the solution of the initial value problem $y' + (\cos t)y = \cos t\ ;\quad y(\pi) = 2$

Solution

$$e^{\int \cos t\ dt} = e^{\sin t}$$

$$\int (\cos t)e^{\sin t}dt = \int e^{\sin t}d(\sin t) = e^{\sin t}$$

$$y(t) = \frac{1}{e^{\sin t}}\left(e^{\sin t} + C\right)$$

$$= 1 + Ce^{-\sin t}$$

$$y(\pi) = 2 \quad \rightarrow \quad 1 + C = 2 \quad \Rightarrow C = 1$$

$$\boxed{y(t) = 1 + e^{-\sin t}}$$

Exercise

Find the solution of the initial value problem $y' + 2ty = 2t\ ;\quad y(0) = 1$

Solution

$$e^{\int 2tdt} = e^{t^2}$$

$$\int (2t)e^{t^2}dt = \int e^{t^2}d\left(t^2\right) = e^{t^2}$$

$$y(t) = \frac{1}{e^{t^2}}\left(e^{t^2} + C\right)$$

$$= 1 + Ce^{-t^2}$$

$$y(0) = 1 \quad \rightarrow \; 1 + C = 1 \quad \Rightarrow \underline{C = 0}$$

$$y(t) = 1$$

Exercise

Find the solution of the initial value problem $\quad y' + y = \frac{e^{-t}}{t^2} \; ; \quad y(1) = 0$

Solution

$$e^{\int dt} = e^t$$

$$\int \left(e^t\right)\frac{e^{-t}}{t^2}dt = \int t^{-2}dt = -\frac{1}{t}$$

$$y(t) = \frac{1}{e^t}\left(-\frac{1}{t} + C\right)$$

$$y(1) = 0 \quad \rightarrow \; \frac{1}{e}(-1 + C) = 0 \quad \Rightarrow \underline{C = 1}$$

$$y(t) = \frac{1}{e^t}\left(-\frac{1}{t} + 1\right)$$

Exercise

Find the solution of the initial value problem $\quad ty' + 2y = \sin t \; ; \quad y(\pi) = \frac{1}{\pi}$

Solution

$$y' + \frac{2}{t}y = \frac{\sin t}{t}$$

$$e^{\int \frac{2}{t}dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

$$\int \left(t^2\right)\frac{\sin t}{t}dt = \int (t\sin t)dt = -t\cos t + \sin t$$

		$\int \sin t$
+	t	$-\cos t$
−	1	$-\sin t$

$$y(t) = \frac{1}{t^2}(\sin t - t\cos t + C)$$

$$y(\pi) = \frac{1}{\pi} \quad \rightarrow \; \frac{1}{\pi^2}(\pi + C) = \frac{1}{\pi} \quad \Rightarrow \underline{C = 0}$$

$$\boxed{y(t)=\frac{1}{t^2}(\sin t - t\cos t)}$$

Exercise

Find the solution of the initial value problem $\quad y' + \frac{2}{t}y = \frac{\cos t}{t^2}\ ;\quad y(\pi)=0$

Solution

$$e^{\int \frac{2}{t}dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

$$\int \left(t^2\right)\frac{\cos t}{t^2}dt = \int (\cos t)\,dt = \sin t$$

$$\boxed{y(t)=\frac{1}{t^2}(\sin t + C)}$$

$$y(\pi)=0 \quad \to \quad \frac{1}{\pi^2}(C)=0 \quad \Rightarrow \boxed{C=0}$$

$$\boxed{y(t)=\frac{\sin t}{t^2}}$$

Exercise

Find the solution of the initial value problem $\quad (\sin t)y' + (\cos t)y = 0\ ;\quad y\left(\frac{3\pi}{4}\right)=2$

Solution

$$y' + (\cot t)y = 0$$

$$e^{\int (\cot t)dt} = e^{\ln(\sin t)} = \sin t$$

$$\boxed{y(t)=\frac{C}{\sin t}}$$

$$y\left(\frac{3\pi}{4}\right)=2 \quad \to \quad C\left(\sqrt{2}\right)=2 \quad \Rightarrow \boxed{C=\sqrt{2}}$$

$$\boxed{y(t)=\sqrt{2}\csc t}$$

Exercise

Find the solution of the initial value problem $\quad y' + 3t^2 y = t^2\ ;\quad y(0)=2$

Solution

$$e^{\int 3t^2 dt} = e^{t^3}$$

$$\int (t^2) e^{t^3} dt = \frac{1}{3}\int e^{t^3} d(t^3) = \frac{1}{3}e^{t^3}$$

$$y(t) = \frac{1}{e^{t^3}}\left(\frac{1}{3}e^{t^3} + C\right)$$

$$= \frac{1}{3} + Ce^{-t^3}$$

$$y(0) = 2 \quad \rightarrow \quad \frac{1}{3} + C = 2 \quad \Rightarrow C = \frac{5}{3}$$

$$y(t) = \frac{1}{3} + \frac{5}{3}e^{-t^3}$$

Exercise

Find the solution of the initial value problem $\quad ty' + y = t\sin t \;; \quad y(\pi) = -1$

Solution

$$y' + \frac{1}{t}y = \sin t$$

$$e^{\int \frac{1}{t}dt} = e^{\ln t} = t$$

$$\int t\sin t \; dt = \sin t - t\cos t$$

		$\int \sin t$
+	t	$-\cos t$
–	1	$-\sin t$

$$y(t) = \frac{1}{t}(\sin t - t\cos t + C)$$

$$y(\pi) = -1 \quad \rightarrow \quad \frac{1}{\pi}(\pi + C) = -1 \quad \Rightarrow C = -2\pi$$

$$y(t) = \frac{1}{t}(\sin t - t\cos t - 2\pi)$$

Exercise

Find the solution of the initial value problem $\quad y' + y = \sin t \;; \quad y(\pi) = 1$

Solution

$$e^{\int dt} = e^t$$

$$\int e^t \sin t \; dt = e^t(-\cos t + \sin t) - \int e^t \sin t \; dt$$

$$2\int e^t \sin t \; dt = e^t(-\cos t + \sin t)$$

$$\int e^t \sin t \; dt = \frac{1}{2}e^t(-\cos t + \sin t)$$

		$\int \sin t$
+	e^t	$-\cos t$
–	e^t	$-\sin t$
+	e^t	

$$y(t) = \frac{1}{e^t}\left(\frac{1}{2}e^t(\sin t - \cos t) + C\right)$$

$$= \frac{1}{2}(\sin t - \cos t) + Ce^{-t}$$

$$y(\pi) = 1 \quad \to \quad \frac{1}{2} + Ce^{-\pi} = 1 \quad \Rightarrow C = \frac{1}{2}e^{\pi}$$

$$y(t) = \frac{1}{2}(\sin t - \cos t) + \frac{1}{2}e^{\pi}e^{-t}$$

Exercise

Find the solution of the initial value problem $\quad y' + y = \cos 2t\ ;\quad y(0) = 5$

Solution

$$e^{\int dt} = e^t$$

$$\int e^t \cos 2t\ dt = e^t\left(\frac{1}{2}\sin 2t + \frac{1}{4}\cos 2t\right) - \frac{1}{4}\int e^t \cos 2t\ dt$$

		$\int \cos 2t$
+	e^t	$\frac{1}{2}\sin 2t$
−	e^t	$-\frac{1}{4}\cos 2t$
+	e^t	

$$\left(1 + \frac{1}{4}\right)\int e^t \cos 2t\ dt = e^t\left(\frac{1}{2}\sin 2t + \frac{1}{4}\cos 2t\right)$$

$$\frac{5}{4}\int e^t \cos 2t\ dt = \frac{1}{4}e^t(2\sin 2t + \cos 2t)$$

$$\int e^t \cos 2t\ dt = \frac{1}{5}e^t(2\sin 2t + \cos 2t)$$

$$y(t) = \frac{1}{e^t}\left(\frac{1}{5}e^t(2\sin 2t + \cos 2t) + C\right)$$

$$= \frac{1}{5}(2\sin 2t + \cos 2t) + Ce^{-t}$$

$$y(0) = 5 \quad \to \quad \frac{1}{5} + C = 5 \quad \Rightarrow C = \frac{24}{5}$$

$$y(t) = \frac{1}{5}(2\sin 2t + \cos 2t) + \frac{25}{5}e^{-t}$$

Exercise

Find the solution of the initial value problem $\quad y' + 3y = \cos 2t\ ;\quad y(0) = -1$

Solution

$$e^{\int 3dt} = e^{3t}$$

$$\int e^{3t} \cos 2t\ dt = e^{3t}\left(\frac{1}{2}\sin 2t + \frac{3}{4}\cos 2t\right) - \frac{9}{4}\int e^{3t} \cos 2t\ dt$$

$$\left(1+\frac{9}{4}\right)\int e^t\cos 2t\ dt = e^t\left(\frac{1}{2}\sin 2t+\frac{3}{4}\cos 2t\right)$$

$$\frac{13}{4}\int e^t\cos 2t\ dt = \frac{1}{4}e^t\left(2\sin 2t+3\cos 2t\right)$$

$$\int e^t\cos 2t\ dt = \frac{1}{13}e^t\left(2\sin 2t+3\cos 2t\right)$$

		$\int \cos 2t$
+	e^{3t}	$\frac{1}{2}\sin 2t$
−	$3e^{3t}$	$-\frac{1}{4}\cos 2t$
+	$9e^{3t}$	

$$y(t)=\frac{1}{e^{3t}}\left(\frac{1}{13}e^{3t}\left(2\sin 2t+3\cos 2t\right)+C\right)$$

$$=\frac{1}{13}\left(2\sin 2t+3\cos 2t\right)+Ce^{-3t}$$

$$y(0)=-1 \quad\rightarrow\quad \frac{3}{13}+C=-1 \quad\Rightarrow C=-\frac{16}{13}$$

$$y(t)=\frac{1}{13}\left(2\sin 2t+3\cos 2t\right)-\frac{16}{13}e^{-3t}$$

Exercise

Find the solution of the initial value problem $\quad y'-2y=7e^{2t}\ ;\quad y(0)=3$

Solution

$$e^{\int -2dt}=e^{-2t}$$

$$\int 7e^{2t}e^{-2t}\ dt=7\int dt=7t$$

$$y(t)=\frac{1}{e^{-2t}}(7t+C)$$

$$=e^{2t}(7t+C)$$

$$y(0)=3 \quad\rightarrow\quad C=3$$

$$y(t)=e^{2t}(7t+3)$$

Exercise

Find the solution of the initial value problem $\quad y'-2y=3e^{-2t}\ ;\quad y(0)=10$

Solution

$$e^{\int -2dt}=e^{-2t}$$

$$\int 3e^{-2t}e^{-2t}\ dt=3\int e^{-4t}\ dt=-\frac{3}{4}e^{-4t}$$

$$y(t) = \frac{1}{e^{-2t}}(7t + C)$$

$$= \boxed{e^{2t}(7t + C)}$$

$$y(0) = 3 \quad \rightarrow \quad \boxed{C = 3}$$

$$\boxed{y(t) = e^{2t}(7t + 3)}$$

Exercise

Find the solution of the initial value problem $\quad y' + 2y = t^2 + 2t + 1 + e^{4t} \;\; ; \quad y(0) = 0$

Solution

$$e^{\int 2dt} = e^{2t}$$

$$\int e^{2t}\left(t^2 + 2t + 1 + e^{4t}\right)\,dt = \int \left(t^2 + 2t + 1\right)e^{2t}\,dt + \int e^{6t}\,dt$$

$$= \left(\frac{1}{2}t^2 + t + \frac{1}{2} - \frac{1}{2}t - \frac{1}{2} + \frac{1}{4}\right)e^{2t} + \frac{1}{6}e^{6t}$$

$$y(t) = \frac{1}{e^{2t}}\left(\left(\frac{1}{2}t^2 + \frac{1}{2}t + \frac{1}{4}\right)e^{2t} + \frac{1}{6}e^{6t} + C\right)$$

$$= \boxed{\left(\frac{1}{2}t^2 + \frac{1}{2}t + \frac{1}{4}\right)e^{2t} + \frac{1}{6}e^{4t} + Ce^{-2t}}$$

$$y(0) = 0 \quad \rightarrow \quad \frac{1}{4} + \frac{1}{6} + C = 0 \quad \Rightarrow \boxed{C = -\frac{5}{12}}$$

$$\boxed{y(t) = \left(\frac{1}{2}t^2 + \frac{1}{2}t + \frac{1}{4}\right)e^{2t} + \frac{1}{6}e^{4t} - \frac{5}{12}e^{-2t}}$$

		$\int e^{2t}$
+	$t^2 + 2t + 1$	$\frac{1}{2}e^{2t}$
−	$2t + 2$	$\frac{1}{4}e^{2t}$
+	2	$\frac{1}{8}e^{2t}$

Exercise

Find the solution of the initial value problem $\quad y' - 3y = 2t - e^{4t} \;\; ; \quad y(0) = 0$

Solution

$$e^{\int -3dt} = e^{-3t}$$

$$\int e^{-3t}\left(2t - e^{4t}\right)\,dt = \int 2te^{-3t}\,dt - \int e^{t}\,dt$$

$$= \left(-\frac{2}{3}t - \frac{2}{9}\right)e^{-3t} - e^{t}$$

		$\int e^{-3t}$
+	$2t$	$-\frac{1}{3}e^{-3t}$
−	2	$\frac{1}{9}e^{-3t}$

$$y(t) = \frac{1}{e^{-3t}}\left(\left(-\frac{2}{3}t - \frac{2}{9}\right)e^{-3t} - e^{t} + C\right)$$

$$=-\frac{2}{3}t-\frac{2}{9}-e^{4t}+Ce^{3t}$$

$$y(0)=0 \quad \rightarrow \quad -\frac{2}{9}-1+C=0 \quad \Rightarrow C=\frac{11}{9}$$

$$y(t)=-\frac{2}{3}t-\frac{2}{9}-e^{4t}+\frac{11}{9}e^{3t}$$

Exercise

Find the solution of the initial value problem $y'+y=t^3+\sin 3t \;;\; y(0)=0$

Solution

$$e^{\int dt}=e^t$$

$$\int e^t\left(t^3+\sin 3t\right)\,dt=\int t^3e^t\,dt+\int e^t\sin 3t\,dt$$

$$\int e^t\sin 3t\,dt=e^t\left(-\frac{1}{3}\cos 3t+\frac{1}{9}\sin 3t\right)-\frac{1}{9}\int e^t\sin 3t\,dt$$

$$\left(1+\frac{1}{9}\right)\int e^t\sin 3t\,dt=\frac{1}{9}e^t\left(\sin 3t-3\cos 3t\right)$$

$$\frac{10}{9}\int e^t\sin 3t\,dt=\frac{1}{9}e^t\left(\sin 3t-3\cos 3t\right)$$

$$\int e^t\sin 3t\,dt=\frac{1}{10}e^t\left(\sin 3t-3\cos 3t\right)$$

		$\int \sin 3t$
+	e^t	$-\frac{1}{3}\cos 3t$
−	e^t	$-\frac{1}{9}\sin 3t$
+	e^t	

		$\int e^t$
+	t^3	e^t
−	$3t^2$	e^t
+	$6t$	e^t
−	6	e^t

$$\int e^t\left(t^3+\sin 3t\right)\,dt=\int t^3e^t\,dt+\int e^t\sin 3t\,dt$$

$$=\left(t^3-3t^2+6t-6\right)e^t+\frac{1}{10}e^t\left(\sin 3t-3\cos 3t\right)$$

$$=\left(t^3-3t^2+6t-6+\frac{1}{10}\sin 3t-\frac{3}{10}\cos 3t\right)e^t$$

$$y(t)=\frac{1}{e^t}\left(\left(t^3-3t^2+6t-6+\frac{1}{10}\sin 3t-\frac{3}{10}\cos 3t\right)e^t+C\right)$$

$$=t^3-3t^2+6t-6+\frac{1}{10}\sin 3t-\frac{3}{10}\cos 3t+Ce^{-t}$$

$$y(0)=0 \quad \rightarrow \quad -6-\frac{3}{10}+C=0 \quad \Rightarrow C=\frac{63}{10}$$

$$y(t)=t^3-3t^2+6t-6+\frac{1}{10}\sin 3t-\frac{3}{10}\cos 3t+\frac{63}{10}e^{-t}$$

Exercise

Find the solution of the initial value problem $\quad y' + 2y = \cos 2t + 3\sin 2t + e^{-t} \;\; ; \quad y(0) = 0$

Solution

$$e^{\int 2dt} = e^{2t}$$

$$\int e^{2t}\left(\cos 2t + 3\sin 2t + e^{-t}\right)\,dt = \int e^{2t}\cos 2t\,dt + 3\int e^{2t}\sin 2t\,dt + \int e^{t}dt$$

$$\int e^{2t}\cos 2t\,dt = e^{2t}\left(\tfrac{1}{2}\sin 2t + \tfrac{1}{2}\cos 2t\right) - \int e^{2t}\cos 2t\,dt$$

		$\int \cos 2t$
+	e^{2t}	$\frac{1}{2}\sin 2t$
−	$2e^{2t}$	$-\frac{1}{4}\cos 2t$
+	$4e^{2t}$	

$$2\int e^{2t}\cos 2t\,dt = \tfrac{1}{2}e^{2t}\left(\sin 2t + \cos 2t\right)$$

$$\int e^{2t}\cos 2t\,dt = \tfrac{1}{4}e^{2t}\left(\sin 2t + \cos 2t\right)$$

$$\int e^{2t}\sin 2t\,dt = e^{2t}\left(-\tfrac{1}{2}\cos 2t + \tfrac{1}{2}\sin 2t\right) - \int e^{2t}\sin 2t\,dt$$

		$\int \sin 2t$
+	e^{2t}	$-\frac{1}{2}\cos 2t$
−	$2e^{2t}$	$-\frac{1}{4}\sin 2t$
+	$4e^{2t}$	

$$2\int e^{2t}\sin 2t\,dt = \tfrac{1}{2}e^{2t}\left(\sin 2t - \cos 2t\right)$$

$$\int e^{2t}\sin 2t\,dt = \tfrac{1}{4}e^{2t}\left(\sin 2t - \cos 2t\right)$$

$$\int e^{2t}\left(\cos 2t + 3\sin 2t + e^{-t}\right)\,dt = \int e^{2t}\cos 2t\,dt + 3\int e^{2t}\sin 2t\,dt + \int e^{t}dt$$

$$= \tfrac{1}{4}e^{2t}\left(\sin 2t + \cos 2t\right) + \tfrac{3}{4}e^{2t}\left(\sin 2t - \cos 2t\right) + e^{t}$$

$$= \tfrac{1}{4}e^{2t}\left(\sin 2t + \cos 2t + 3\sin 2t - 3\cos 2t\right) + e^{t}$$

$$= \tfrac{1}{4}e^{2t}\left(4\sin 2t - 2\cos 2t\right) + e^{t}$$

$$= e^{2t}\left(\sin 2t - \tfrac{1}{2}\cos 2t\right) + e^{t}$$

$$y(t) = \frac{1}{e^{2t}}\left(e^{2t}\left(\sin 2t - \tfrac{1}{2}\cos 2t\right) + e^{t} + C\right)$$

$$= \sin 2t - \tfrac{1}{2}\cos 2t + e^{-t} + Ce^{-2t}$$

$$y(0) = 0 \;\;\to\;\; -\tfrac{1}{2} + 1 + C = 0 \;\;\Rightarrow C = -\tfrac{1}{2}$$

$$y(t) = \sin 2t - \tfrac{1}{2}\cos 2t + e^{-t} - \tfrac{1}{2}e^{-2t}$$

Exercise

Find the solution of the initial value problem $y' + y = e^{3t}$; $y(0) = y_0$

Solution

$$e^{\int dt} = e^t$$

$$\int e^t e^{3t}\, dt = \int e^{4t} dt = \frac{1}{4} e^{4t}$$

$$y(t) = \frac{1}{e^t}\left(\frac{1}{4}e^{4t} + C\right)$$

$$= \frac{1}{4}e^{3t} + Ce^{-t}$$

$$y(0) = y_0 \quad \to \quad \frac{1}{4} + C = y_0 \quad \Rightarrow C = y_0 - \frac{1}{4}$$

$$y(t) = \frac{1}{4}e^{3t} + \left(y_0 - \frac{1}{4}\right)e^{-t}$$

Exercise

Find the solution of the initial value problem $t^2 y' - ty = 1$; $y(1) = y_0$

Solution

$$y' - \frac{1}{t} y = \frac{1}{t^2}$$

$$e^{\int -\frac{1}{t} dt} = e^{-\ln t} = \frac{1}{t}$$

$$\int \frac{1}{t}\frac{1}{t^2}\, dt = \int t^{-3} dt = -\frac{1}{2} t^{-2}$$

$$y(t) = t\left(-\frac{1}{2t^2} + C\right)$$

$$= -\frac{1}{2t} + Ct$$

$$y(1) = y_0 \quad \to \quad -\frac{1}{2} + C = y_0 \quad \Rightarrow C = y_0 + \frac{1}{2}$$

$$y(t) = -\frac{1}{2t} + \left(y_0 + \frac{1}{2}\right)t$$

Exercise

Find the solution of the initial value problem $y' + ay = e^{at}$; $y(0) = y_0$, $a \neq 0$

Solution

$$e^{\int a\,dt} = e^{at}$$

$$\int e^{at}e^{at}\, dt = \int e^{2at}dt = \frac{1}{2a}e^{2at}$$

$$y(t) = \frac{1}{e^{at}}\left(\frac{1}{2a}e^{2at} + C\right)$$

$$= \frac{1}{2a}e^{at} + Ce^{-at}$$

$$y(0) = y_0 \quad \rightarrow \quad \frac{1}{2a} + C = y_0 \quad \Rightarrow C = y_0 - \frac{1}{2a}$$

$$y(t) = \frac{1}{2a}e^{at} + \left(y_0 - \frac{1}{2a}\right)e^{-at}$$

Exercise

Find the solution of the initial value problem $3y' + 12y = 4$; $y(0) = y_0$

Solution

$$y' + 4y = \frac{4}{3}$$

$$e^{\int 4dt} = e^{4t}$$

$$\int \frac{4}{3}e^{4t}\, dt = \frac{1}{3}e^{4t}$$

$$y(t) = \frac{1}{e^{4t}}\left(\frac{1}{3}e^{4t} + C\right)$$

$$= \frac{1}{3} + Ce^{-4t}$$

$$y(0) = y_0 \quad \rightarrow \quad \frac{1}{3} + C = y_0 \quad \Rightarrow C = y_0 - \frac{1}{3}$$

$$y(t) = \frac{1}{3} + \left(y_0 - \frac{1}{3}\right)e^{-4t}$$

Exercise

Find a solution to the initial value problem that is continuous on the given interval $[a,\ b]$

$$y' + \frac{1}{x}y = f(x), \quad y(1) = 1 \qquad f(x) = \begin{cases} 3x, & 1 \le x \le 2 \\ 0, & 2 < x \le 3 \end{cases} \qquad [a,\ b] = [1,\ 3]$$

Solution

For $1 \le x \le 2$:

$$y' + \frac{1}{x}y = 3x, \quad y(1) = 1$$

$$e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\int 3x^2 dx = x^3$$

$$y(x) = \frac{1}{x}\left(x^3 + C\right)$$

$$= x^2 + \frac{C}{x}$$

$$y(1) = 1 \quad \to \ 1 = 1 + C \ \Rightarrow C = 0$$

$$y(x) = x^2$$

For $2 \le x \le 3$:

$$y' + \frac{1}{x}y = 0 \qquad x = 2 \ \Rightarrow \ y = 4$$

$$y(x) = \frac{C}{x}$$

$$y(2) = 4 \quad \to \ 4 = \frac{C}{2} \ \Rightarrow \ C = 8$$

$$y(x) = \frac{8}{x}$$

Exercise

Find a solution to the initial value problem that is continuous on the given interval $[a,\ b]$

$$y' + (\sin x)y = f(x), \quad y(0) = 3 \qquad f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ -\sin x, & \pi < x \le 2\pi \end{cases} \qquad [a,\ b] = [0,\ 2\pi]$$

Solution

For $0 \le x \le \pi$:

$$y' + (\sin x)y = \sin x, \quad y(0) = 3$$

$$e^{\int \sin x dx} = e^{-\cos x}$$

$$\int \sin x\ e^{-\cos x} dx = \int e^{-\cos x} d(-\cos x) = e^{-\cos x}$$

$$y(x) = e^{\cos x}\left(e^{-\cos x} + C\right)$$

$$= 1 + Ce^{\cos x}$$

$$y(0) = 3 \quad \rightarrow \quad 3 = 1 + Ce \quad \Rightarrow \boxed{C = 2e^{-1}}$$

$$\boxed{y(x) = 1 + 2e^{\cos x - 1}} \qquad y(\pi) = 1 + 2e^{-2}$$

For $\pi \le x \le 2\pi$:

$$y' + (\sin x)y = -\sin x \qquad y(\pi) = 1 + 2e^{-2}$$

$$e^{\int \sin x dx} = e^{-\cos x}$$

$$\int -\sin x \ e^{-\cos x} dx = \int -e^{-\cos x} d(-\cos x) = -e^{-\cos x}$$

$$y(x) = e^{\cos x}\left(-e^{-\cos x} + C\right)$$

$$= -1 + Ce^{\cos x}$$

$$y(\pi) = 1 + 2e^{-2} \quad \rightarrow \quad 1 + 2e^{-2} = -1 + Ce^{-1}$$

$$Ce^{-1} = 2 + 2e^{-2} \quad \Rightarrow \quad C = 2e + 2e^{-1}$$

$$\boxed{y(x) = -1 + \left(2e + 2e^{-1}\right)e^{\cos x - 1}}$$

Exercise

Find a solution to the initial value problem that is continuous on the given interval $[a, b]$

$$y' + p(t)y = 2, \quad y(0) = 1 \qquad p(t) = \begin{cases} 0, & 0 \le t \le 1 \\ \frac{1}{t}, & 1 < t \le 2 \end{cases} \qquad [a, b] = [0, 2]$$

Solution

For $0 \le t \le 1$:

$$y' = 2, \quad y(0) = 1$$

$$\int dy = \int 2dt$$

$$y(t) = 2t + C$$

$$y(0) = 1 \quad \Rightarrow \boxed{C = 1}$$

$$\boxed{y(t) = 2t + 1} \qquad t = 1 \ \Rightarrow \ y = 3$$

For $1 \le t \le 2$:

$$y' + \frac{1}{t}y = 2, \quad y(1) = 3$$

$$e^{\int \frac{1}{t}dt} = e^{\ln t} = t$$

$$\int 2t\ dt = t^2$$

$$y(t) = \frac{1}{t}\left(t^2 + C\right)$$

$$y(1) = 3 \quad \to \ 3 = 1 + C \ \Rightarrow \ \underline{C = 2|}$$

$$\underline{y(t) = t + \frac{2}{t}|}$$

Exercise

Find a solution to the initial value problem that is continuous on the given interval $[a,\ b]$

$$y' + p(t)y = 0, \quad y(0) = 3 \qquad p(t) = \begin{cases} 2t - 1, & 0 \le t \le 1 \\ 0, & 1 < t \le 3 \\ -\frac{1}{t}, & 3 < t \le 4 \end{cases} \qquad [a,\ b] = [0,\ 4]$$

Solution

For $0 \le t \le 1$:

$$y' + (2t - 1)y = 0, \quad y(0) = 3$$

$$e^{\int (2t-1)dt} = e^{t^2 - t}$$

$$y(t) = Ce^{-t^2 + t}$$

$$y(0) = 3 \quad \Rightarrow \underline{C = 3|}$$

$$\underline{y(t) = 3e^{t - t^2}|} \qquad t = 1 \ \Rightarrow \ y = 3$$

For $1 \le t \le 3$:

$$y' = 0, \quad y(1) = 3$$

$$y(t) = C$$

$$y(1) = 3 \quad \Rightarrow \ \underline{C = 3|}$$

$$\underline{y(t) = 3|} \qquad t = 3 \ \Rightarrow \ y = 3$$

For $3 \le t \le 4$:

$$y' - \frac{1}{t}y = 0, \quad y(3) = 3$$

$$e^{\int \frac{-1}{t}dt} = e^{-\ln t} = \frac{1}{t}$$

$$y(t) = Ct$$

$$y(3)=3 \quad \Rightarrow \underline{C=1}$$

$$\underline{y(t)=t}$$

Exercise

Solve $xy'+2y=\sin x$ *for* y' $\qquad y\left(\frac{\pi}{2}\right)=0$

Solution

$$xy'+2y=\sin x$$

$$y'+\frac{2}{x}y=\frac{\sin x}{x} \qquad x\neq 0$$

$$e^{\int\frac{2}{x}dx}=e^{2\ln|x|}=e^{\ln x^2}=x^2$$

$$\int x^2\frac{\sin x}{x}dx=\int x\sin x\,dx$$

$$=-x\cos x+\sin x$$

		$\int \sin x$
+	x	$-\cos x$
–	1	$-\sin x$

$$y(x)=\frac{1}{x^2}\left(-x\cos x+\sin x+C\right)$$

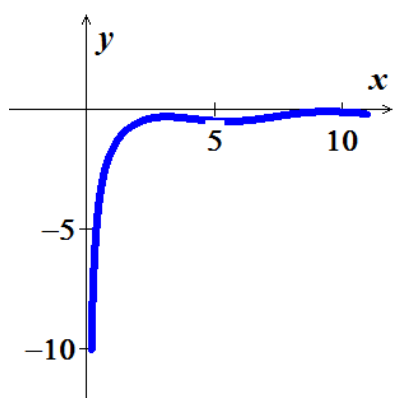
$$\underline{-\frac{1}{x}\cos x+\frac{1}{x^2}\sin x+\frac{C}{x^2}}$$

$$y\left(\frac{\pi}{2}\right)=-\frac{1}{\left(\frac{\pi}{2}\right)}\cos\left(\frac{\pi}{2}\right)+\frac{1}{\left(\frac{\pi}{2}\right)^2}\sin\left(\frac{\pi}{2}\right)+\frac{C}{\left(\frac{\pi}{2}\right)^2}$$

$$0=\frac{4}{\pi^2}+\frac{4}{\pi^2}C$$

$$\frac{4}{\pi^2}C=-\frac{4}{\pi^2} \quad\rightarrow\quad \underline{C=-1}$$

$$\underline{y(x)=-\frac{1}{x}\cos x+\frac{1}{x^2}\sin x-\frac{1}{x^2}} \qquad x\neq 0$$



Exercise

Find the solution of the initial value problem. Discuss the interval of existence and provide a sketch of your solution $(2x+3)y'=y+(2x+3)^{1/2}$; $\qquad y(-1)=0$

Solution

$$(2x+3)y'-y=(2x+3)^{1/2}$$

$$y'-\frac{1}{2x+3}y=(2x+3)^{-1/2}$$

$$e^{\int\frac{-1}{2x+3}dx}=e^{-\frac{1}{2}\int\frac{1}{2x+3}d(2x+3)}=e^{-\frac{1}{2}\ln(2x+3)}=e^{\ln(2x+3)^{-1/2}}=|2x+3|^{-1/2}$$

$$\int (2x+3)^{-1/2}(2x+3)^{-1/2}\,dx = \int (2x+3)^{-1}\,dx$$

$$= \frac{1}{2}\int \frac{d(2x+3)}{2x+3} \qquad d(2x+3) = 2dx$$

$$= \frac{1}{2}\ln|2x+3|$$

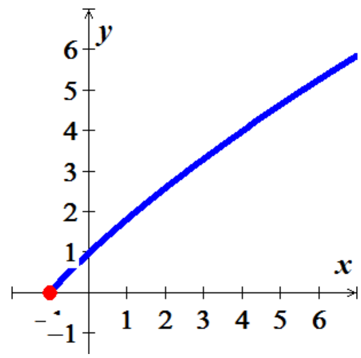
$$y(x) = \frac{1}{(2x+3)^{-1/2}}\left(\frac{1}{2}\ln(2x+3) + C\right)$$

$$= \frac{1}{2}(2x+3)^{1/2}\ln(2x+3) + C(2x+3)^{1/2}$$

$$0 = \frac{1}{2}(2(-1)+3)^{1/2}\ln(2(-1)+3) + C(2(-1)+3)^{1/2}$$

$$0 = \frac{1}{2}(1)^{1/2}\ln(1) + C(1)^{1/2} \quad \to \quad \underline{C=0}$$

$$\boxed{y(x) = \frac{1}{2}(2x+3)^{1/2}\ln(2x+3)}$$



Exercise

Find the general solution of the given differential equation. Then find the particular solution satisfying the given initial condition of $y' - 3y = 4, \quad y(0) = 2$

Solution

$$e^{\int -3dx} = e^{-3x}$$

$$\int 4e^{-3x}dx = -\frac{4}{3}e^{-3x}$$

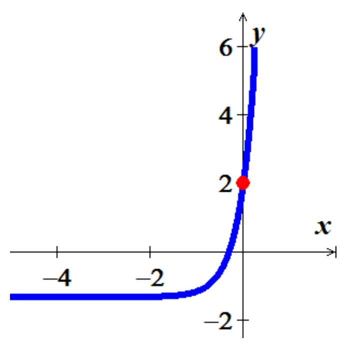
$$y(x) = \frac{1}{e^{-3x}}\left(-\frac{4}{3}e^{-3x} + C\right)$$

$$= -\frac{4}{3} + Ce^{3x}$$

$$y(0) = 2 \quad \to 2 = -\frac{4}{3} + Ce^{0} \quad \Rightarrow \quad \underline{C = \frac{10}{3}}$$

$$y(x) = -\frac{4}{3} + \frac{10}{3}e^{3x}$$

$$= \boxed{\frac{10e^{3x} - 4}{3}}$$



Exercise

Find the general solution of the given differential equation. Then find the particular solution satisfying the given initial condition of

$$y' + \frac{1}{2}y = t, \quad y(0) = 1$$

Solution

$$e^{\int \frac{1}{2}dt} = e^{t/2}$$

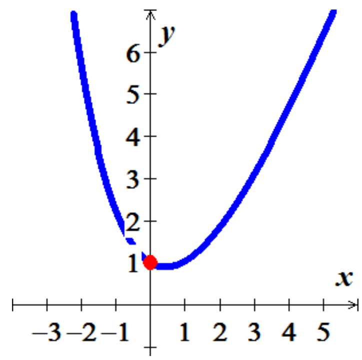
$$\left(e^{t/2}y\right)' = te^{t/2}$$

$$e^{t/2}y = \int te^{t/2}dt \qquad\qquad \int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1)$$

$$= \frac{e^{t/2}}{\left(\frac{1}{2}\right)^2}\left(\frac{t}{2}-1\right) + C$$

$$= 4e^{t/2}\left(\frac{t}{2}-1\right) + C$$

$$= (2t-4)e^{t/2} + C$$



$$y(t) = (2t-4) + Ce^{-t/2}$$

$$y(0) = 1 \quad \to 1 = -4 + C \quad \Rightarrow \underline{C = 5}$$

$$\boxed{y(t) = 2t - 4 + 5e^{-t/2}}$$

Exercise

Find the general solution of the given differential equation. Then find the particular solution satisfying the given initial condition of

$$y' + y = e^t, \quad y(0) = 1$$

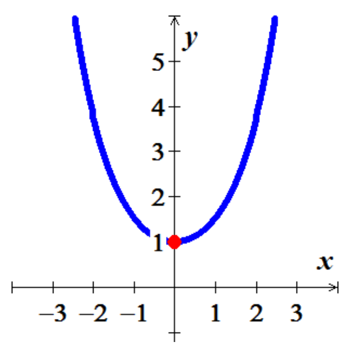
Solution

$$e^{\int dt} = e^t$$

$$\int e^t e^t dt = \int e^{2t}dt = \frac{1}{2}e^{2t}$$

$$y(t) = \frac{1}{e^t}\left(\frac{1}{2}e^{2t} + C\right)$$

$$\boxed{= \frac{1}{2}e^t + Ce^{-t}}$$



$$y(0) = 1 \quad \to 1 = \frac{1}{2} + C \quad \Rightarrow \underline{C = \frac{1}{2}}$$

$$\boxed{y(t) = \frac{1}{2}\left(e^t + e^{-t}\right)}$$

Exercise

The following system of differential equations is encountered in the study of the decay of a special type of radioactive series of elements

$$\frac{dx}{dt} = -\lambda_1 x$$

$$\frac{dy}{dt} = \lambda_1 x - \lambda_2 y$$

Where λ_1 *and* λ_2 are constants.

Discuss how to solve this system subject to $x(0) = x_0,\ y(0) = y_0$

Solution

$$\int \frac{dx}{x} = -\lambda_1 \int dt$$

$$\ln x = -\lambda_1 t + C$$

$$x(t) = e^{-\lambda_1 t + C} = Ae^{-\lambda_1 t}$$

$$x(0) = A = x_0$$

$$\boxed{x(t) = x_0 e^{-\lambda_1 t}}$$

$$\frac{dy}{dt} = x_0 \lambda_1 e^{-\lambda_1 t} - \lambda_2 y$$

$$y' + \lambda_2 y = x_0 \lambda_1 e^{-\lambda_1 t}$$

$$e^{\int \lambda_2 dt} = e^{\lambda_2 t}$$

$$x_0 \lambda_1 \int e^{-\lambda_1 t} e^{\lambda_2 t} dt = x_0 \lambda_1 \int e^{(\lambda_2 - \lambda_1)t} dt$$

$$= \frac{x_0 \lambda_1}{\lambda_2 - \lambda_1} e^{(\lambda_2 - \lambda_1)t}$$

$$y(t) = \frac{1}{e^{\lambda_2 t}} \left(\frac{x_0 \lambda_1}{\lambda_2 - \lambda_1} e^{(\lambda_2 - \lambda_1)t} + C \right)$$

$$= \frac{x_0 \lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + Ce^{-\lambda_2 t}$$

$$y(0) = \frac{x_0 \lambda_1}{\lambda_2 - \lambda_1} + C = y_0$$

$$C = y_0 - \frac{x_0 \lambda_1}{\lambda_2 - \lambda_1} = \frac{\lambda_2 y_0 - \lambda_1 y_0 - x_0 \lambda_1}{\lambda_2 - \lambda_1}$$

$$y(t) = \boxed{\frac{x_0\lambda_1}{\lambda_2 - \lambda_1}e^{-\lambda_1 t} + \frac{\lambda_2 y_0 - \lambda_1 y_0 - x_0\lambda_1}{\lambda_2 - \lambda_1}e^{-\lambda_2 t}}$$

Exercise

Let $P(t)$ be the performance level of someone learning a skill as a function of the training time *t*. The graph of *P* is called a ***learning curve***. We proposed the differential equation

$$\frac{dP}{dt} = k(M - P(t))$$

As a reasonable model for learning, where *k* is a positive constant. Solve it as a linear differential equation and use your solution to graph the learning curve.

Solution

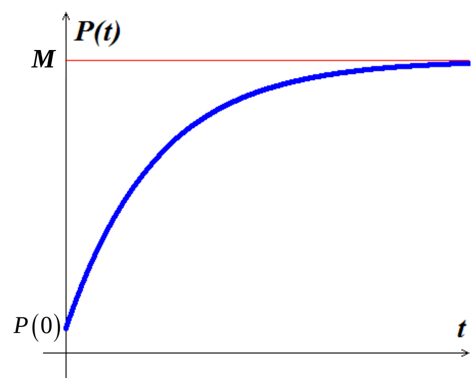
$$\frac{dP}{dt} + kP = kM$$

$$e^{\int k\,dt} = e^{kt}$$

$$\int kMe^{kt}\,dt = Me^{kt}$$

$$P(t) = \frac{1}{e^{kt}}\left(Me^{kt} + C\right)$$

$$\underline{= M + Ce^{-kt}|} \quad k > 0$$



Exercise

A differential equation describing the velocity *v* of a falling mass subject to air resistance proportional to the instantaneous velocity is

$$m\frac{dv}{dt} = mg - kv$$

Where $k > 0$ is a constant of proportionality. The positive direction is downward.

a) Solve the equation subject to the initial condition $v(0) = v_0$

b) Use the solution in part (*a*) to determine the limiting, or terminal, velocity of the mass.

c) If the distance *s*, measured from the point where the mass was released above ground, is related to velocity *v* by $\frac{ds}{dt} = v(t)$, find an explicit expression for $s(t)$ if $s(0) = 0$

Solution

a) $\frac{dv}{dt} + \frac{k}{m}v = g$

$$e^{\int \frac{k}{m}dt} = e^{kt/m}$$

$$\int ge^{kt/m}\,dt = \frac{mg}{k}e^{kt/m}$$

$$v(t) = e^{-kt/m}\left(\frac{mg}{k}e^{kt/m} + C\right)$$

$$= \frac{mg}{k} + Ce^{-kt/m}$$

$$v(0) = v_0 \quad \Rightarrow v_0 = \frac{mg}{k} + C \quad \boxed{C = v_0 - \frac{mg}{k}}$$

$$\boxed{v(t) = \frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right)e^{-kt/m}}$$

b) $\lim\limits_{t\to\infty} v(t) = \lim\limits_{t\to\infty}\left(\frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right)e^{-kt/m}\right) \boxed{= \frac{mg}{k}}$

c) $\frac{ds}{dt} = v(t)$

$$s(t) = \int\left(\frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right)e^{-kt/m}\right)dt$$

$$= \frac{mg}{k}t - \frac{m}{k}\left(v_0 - \frac{mg}{k}\right)e^{-kt/m} + C$$

$$s(0) = 0 \quad \Rightarrow \quad 0 = -\frac{m}{k}\left(v_0 - \frac{mg}{k}\right) + C$$

$$\rightarrow \boxed{C = \frac{m}{k}\left(v_0 - \frac{mg}{k}\right)}$$

$$\boxed{s(t) = \frac{mg}{k}t - \frac{m}{k}\left(v_0 - \frac{mg}{k}\right)e^{-kt/m} + \frac{m}{k}\left(v_0 - \frac{mg}{k}\right)}$$

Exercise

As a raindrop falls, it evaporates while retaining its spherical shape. If we make the further assumptions that the rate at which the raindrop evaporates is proportional to its surface and that air resistance is negligible, then a model for the velocity $v(t)$ of the raindrop is

$$\frac{dv}{dt} + \frac{3(k/\rho)}{\frac{k}{\rho}t + r_0}v = g$$

Here ρ is the density of water, r_0 is the radius of the raindrop at $t = 0$, $k < 0$ is the constant of proportionality, and downward direction is taken to be the positive direction.

a) Solve for $v(t)$ if the raindrop falls from rest.

b) Show that the radius of the raindrop at time *t* is $r(t) = \frac{k}{\rho}t + r_0$.

c) If $r_0 = 0.01\ ft$ and $r = 0.007\ ft$ 10 *seconds* after the raindrop falls from a cloud, determine the time at which the raindrop has evaporated completely.

Solution

a) $$e^{3\frac{k}{\rho}\int\frac{dt}{\frac{k}{\rho}t+r_0}} = e^{3\ln\left(\frac{k}{\rho}t+r_0\right)} = \left(\frac{k}{\rho}t+r_0\right)^3$$

$$\int g\left(\frac{k}{\rho}t+r_0\right)^3 dt = \frac{\rho g}{k}\int\left(\frac{k}{\rho}t+r_0\right)^3 d\left(\frac{k}{\rho}t+r_0\right) = \frac{\rho g}{4k}\left(\frac{k}{\rho}t+r_0\right)^4$$

$$v(t) = \left(\frac{k}{\rho}t+r_0\right)^{-3}\left(\frac{\rho g}{4k}\left(\frac{k}{\rho}t+r_0\right)^4 + C\right)$$

$$= \frac{\rho g}{4k}\left(\frac{k}{\rho}t+r_0\right) + C\left(\frac{kt+\rho r_0}{\rho}\right)^{-3}$$

$$v(0)=0 \quad \Rightarrow 0 = \frac{\rho g r_0}{4k} + C(r_0)^{-3} \quad \to \boxed{C = -\frac{\rho g r_0^4}{4k}}$$

$$\boxed{v(t) = \frac{\rho g}{4k}\left(\frac{k}{\rho}t+r_0\right) - \frac{\rho g r_0^4}{4k}\left(\frac{\rho}{kt+\rho r_0}\right)^3}$$

b) $$\frac{dr}{dt} = \frac{k}{\rho}$$

$$\int dr = \int \frac{k}{\rho}dt$$

$$r(t) = \frac{k}{\rho}t + C$$

$$r(0) = r_0 \quad \Rightarrow \boxed{r_0 = C}$$

$$\boxed{r(t) = \frac{k}{\rho}t + r_0}$$

c) ***Given***: $r_0 = 0.01\ ft$ and $r = 0.007\ ft$

$$r(t=10) = \frac{k}{\rho}(10) + .01 = .007$$

$$10\frac{k}{\rho} = -.003$$

$$\boxed{\frac{k}{\rho} = -.0003}$$

$$r(t) = -.0003t + .01 = 0$$

$$t = \frac{.01}{.0003} \boxed{\approx 33.3\ sec}$$

Exercise

A model that describes the population of a fishery in which harvesting takes place at a constant rate is given by

$$\frac{dP}{dt} = kP - h$$

Where k and h are positive constants.

a) Solve $P(t)$ given the initial value $P(0) = P_0$

b) Describe the behavior of the population $P(t)$ for increasing time in three cases $P_0 > \frac{h}{k}$, $P_0 = \frac{h}{k}$, and $P_0 < \frac{h}{k}$

c) Use the results from part (*b*) to determine whether the fish population will ever go extinct in finite time, that is, whether there exists a time $T > 0$ such that $P(T) = 0$. If the population goes extinct then find *T*.

Solution

a) $\frac{dP}{dt} - kP = -h$

$$e^{\int -kdt} = e^{-kt}$$

$$\int -he^{-kt}dt = \frac{h}{k}e^{-kt}$$

$$P(t) = e^{kt}\left(\frac{h}{k}e^{-kt} + C\right)$$

$$= \frac{h}{k} + Ce^{kt}$$

$$P(0) = P_0 \quad \to P_0 = \frac{h}{k} + C \quad C = P_0 - \frac{h}{k}$$

$$P(t) = \frac{h}{k} + \left(P_0 - \frac{h}{k}\right)e^{kt}$$

b) For $P_0 > \frac{h}{k} \quad \Rightarrow \quad P_0 - \frac{h}{k} > 0$

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty}\left(\frac{h}{k} + \left(P_0 - \frac{h}{k}\right)e^{kt}\right) = \infty$$

$P(t)$ increases as time increases

For $P_0 = \frac{h}{k} \quad \Rightarrow \quad P_0 - \frac{h}{k} = 0$

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty}\left(\frac{h}{k}\right) = \frac{h}{k}$$

$P(t)$ remains constant as time increases

For $P_0 < \frac{h}{k} \quad \Rightarrow \quad P_0 - \frac{h}{k} < 0$

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty}\left(\frac{h}{k} + \left(P_0 - \frac{h}{k}\right)e^{kt}\right) = -\infty$$

$P(t)$ decreases as time increases.

c) $P(T) = \frac{h}{k} + \left(P_0 - \frac{h}{k}\right)e^{kT} = 0$

$$\left(\frac{kP_0 - h}{k}\right)e^{kT} = -\frac{h}{k}$$

$$e^{kT} = -\frac{h}{kP_0 - h}$$

$$kT = \ln\left(\frac{h}{h - kP_0}\right)$$

$$\boxed{T = \frac{1}{k}\ln\left(\frac{h}{h - kP_0}\right)}$$

Exercise

A certain body weighing $45\ lb$, is heated to a temperature of $300°$. Then at $t = 0$ it is plunged into 100 lb of water at a temperature of $50°$. Given that the specific heat of the body is $\frac{1}{9}$, find the formula for the temperature T of the body during its cooling.

Solution

$H = mc\Delta t$ $\qquad$ c: specific heat

Let T_w : temperature of the water

The amount of heat lost by the body is: $\quad 45\left(\frac{1}{9}\right)(300 - T) = 5(300 - T)$

The amount of heat gained by the body is: $\quad 100(1)\left(T_w - 50\right) = 100\left(T_w - 50\right)$

The 2 amounts are the same

$$5(300 - T) = 100\left(T_w - 50\right)$$

$$300 - T = 20T_w - 1000$$

$$\boxed{T_w = \frac{1}{20}(1300 - T)}$$

Using Newton's law of cooling: $-\frac{dT}{dt} = k\left(T - T_w\right)$

$$\begin{aligned}\frac{dT}{dt} &= -k\left(T - \frac{1}{20}(1300 - T)\right)\\ &= -k\left(T - 65 + \frac{1}{20}T\right)\\ &= -\frac{21}{20}kT + 65k\end{aligned}$$

$$\frac{dT}{dt} + \frac{21}{20}kT = 65k$$

$$e^{\int \frac{21}{20}k\,dt} = e^{\frac{21}{20}kt}$$

$$\int (65k)e^{\frac{21}{20}kt}dt = \frac{1,300}{21}e^{\frac{21}{20}kt}$$

$$T(t) = \frac{1}{e^{\frac{21}{20}kt}}\left(\frac{1,300}{21}e^{\frac{21}{20}kt} + C\right)$$

$$\boxed{= \frac{1,300}{21} + Ce^{-\frac{21}{20}kt}}$$

$$T(0) = 300 \quad \rightarrow 300 = \frac{1300}{21} + C \quad \Rightarrow \boxed{C = \frac{5,000}{21}}$$

$$\boxed{T(t) = \frac{1,300}{21} + \frac{5,000}{21}e^{-\frac{21}{20}kt}}$$