

Solution **Section 4.2 – Area under Curves**

Exercise

Use finite approximations to estimate the area under the graph of the function using

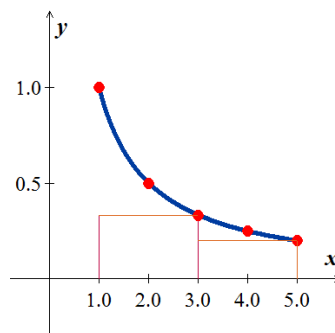
$$f(x) = \frac{1}{x} \quad \text{between } x=1 \quad \text{and} \quad x=5$$

- a) A lower sum with two rectangles of equal width
- b) A lower sum with four rectangles of equal width
- c) An upper sum with two rectangles of equal width
- d) An upper sum with four rectangles of equal width

Solution

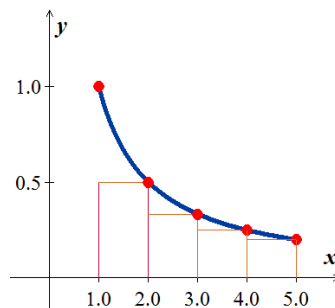
- a) Using 2 lower rectangles: $\Delta x = \frac{5-1}{2} = 2$

$$\begin{aligned} A &\approx \Delta x \left(f(x_1) + f(x_2) \right) \\ &\approx 2 \cdot \left(f(3) + f(5) \right) \\ &\approx 2 \cdot \left(\frac{1}{3} + \frac{1}{5} \right) \\ &\approx \frac{16}{15} \end{aligned}$$



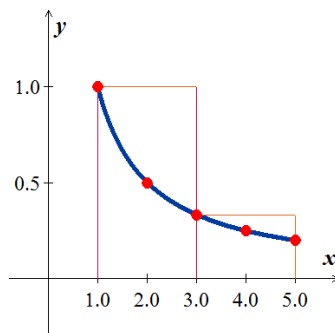
- b) Using 4 lower rectangles: $\Delta x = \frac{5-1}{4} = 1$

$$\begin{aligned} A &\approx 1 \cdot \left(f(2) + f(3) + f(4) + f(5) \right) \\ &\approx 1 \cdot \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \right) \\ &\approx \frac{77}{60} \end{aligned}$$



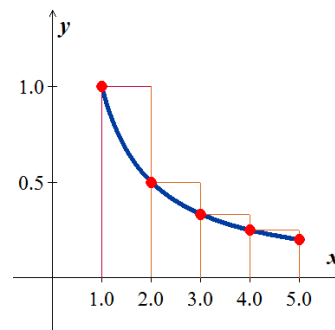
- c) Using 2 upper rectangles: $\Delta x = \frac{5-1}{2} = 2$

$$\begin{aligned} A &\approx 2 \cdot \left(f(1) + f(3) \right) \\ &\approx 2 \cdot \left(1 + \frac{1}{3} \right) \\ &\approx \frac{8}{3} \end{aligned}$$



- d) Using 4 upper rectangles: $\Delta x = \frac{5-1}{4} = 1$

$$\begin{aligned} A &\approx 1 \cdot \left(f(1) + f(2) + f(3) + f(4) \right) \\ &\approx 1 \cdot \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \right) \\ &\approx \frac{25}{12} \end{aligned}$$



Exercise

Use finite approximations to estimate the area under the graph of the function using

$$f(x) = 4 - x^2 \quad \text{between } x = -2 \quad \text{and} \quad x = 2$$

- a) A lower sum with two rectangles of equal width
- b) A lower sum with four rectangles of equal width
- c) An upper sum with two rectangles of equal width
- d) An upper sum with four rectangles of equal width

Solution

- a) Using 2 lower rectangles:

$$\Delta x = \frac{2 - (-2)}{2} = 2$$

$$\begin{aligned} A &\approx \Delta x (f(x_1) + f(x_2)) \\ &\approx 2 \cdot (f(-2) + f(2)) \\ &\approx 2 \cdot \left[(4 - (-2)^2) + (4 - 2^2) \right] \\ &= 0 \end{aligned}$$

- b) Using 4 lower rectangles:

$$\Delta x = \frac{2 - (-2)}{4} = 1$$

$$\begin{aligned} A &\approx 1 \cdot (f(-2) + f(-1) + f(1) + f(2)) \\ &\approx 1 \cdot (0 + 3 + 3 + 0) \\ &= 6 \end{aligned}$$

- c) Using 2 upper rectangles:

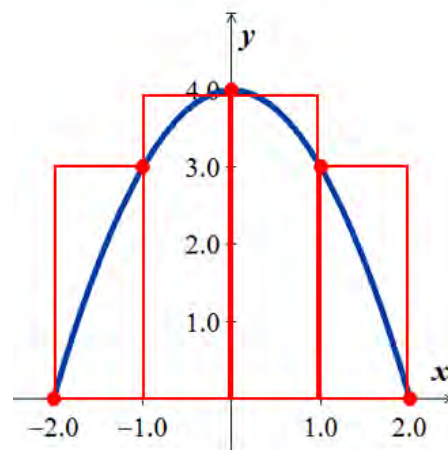
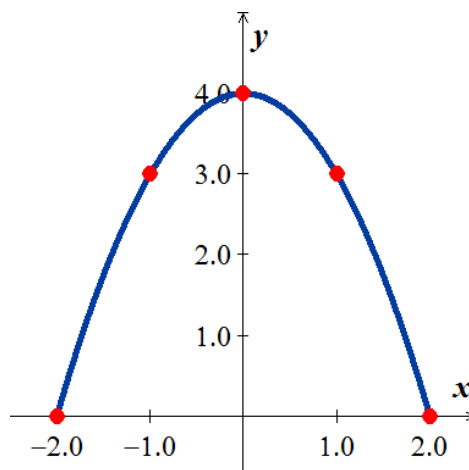
$$\Delta x = \frac{2 - (-2)}{2} = 2$$

$$\begin{aligned} A &\approx 2 \cdot (f(0) + f(0)) \\ &\approx 2 \cdot (4 + 4) \\ &= 16 \end{aligned}$$

- d) Using 4 upper rectangles:

$$\Delta x = \frac{2 - (-2)}{4} = 1$$

$$\begin{aligned} A &\approx 1 \cdot (f(-1) + f(0) + f(1) + f(2)) \\ &\approx 1 \cdot (3 + 4 + 4 + 3) \\ &= 14 \end{aligned}$$



Exercise

Use finite approximations to estimate the average value of f on the given interval by partitioning the interval into four subintervals of equal length and evaluating f at the subinterval midpoints.

$$f(t) = \frac{1}{2} + \sin^2 \pi t \quad \text{on } [0, 2]$$

Solution

$$\Delta x = \frac{2-0}{4} = 0.5$$

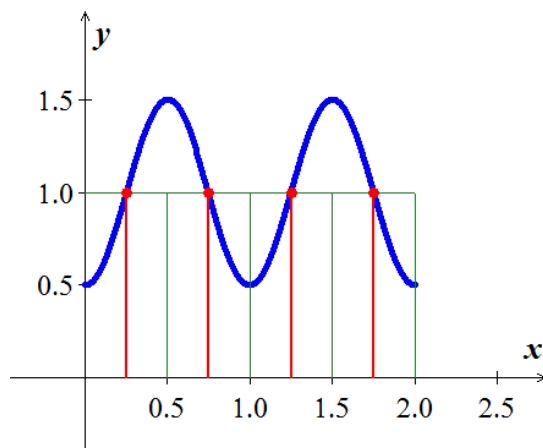
$$f(t = .25) = \frac{1}{2} + \sin^2 (.25\pi) = 1$$

$$f(t = .75) = \frac{1}{2} + \sin^2 (.75\pi) = 1$$

$$f(t = 1.25) = \frac{1}{2} + \sin^2 (1.25\pi) = 1$$

$$f(t = 1.75) = \frac{1}{2} + \sin^2 (1.75\pi) = 1$$

$$\begin{aligned} A &\approx .5 \cdot (f(.25) + f(.75) + f(1.25) + f(1.75)) \\ &= .5(1 + 1 + 1 + 1) \\ &= \underline{2} \end{aligned}$$



$$\begin{aligned} \text{Average value} &\approx \frac{\text{Area}}{\text{Length } [0, 2]} \\ &= \frac{2}{2} \\ &= \underline{1} \end{aligned}$$

Exercise

Use finite approximations to estimate the average value of f on the given interval by partitioning the interval into four subintervals of equal length and evaluating f at the subinterval midpoints.

$$f(t) = 1 - \left(\cos \frac{\pi t}{4}\right)^4 \quad \text{on } [0, 4]$$

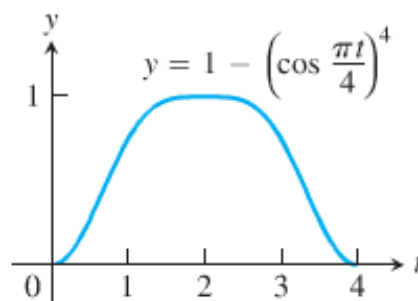
Solution

$$\Delta x = \frac{4-0}{4} = 1$$

$$f(t = 0.5) = 1 - \left(\cos \frac{0.5\pi}{4}\right)^4 = 0.27145$$

$$f(t = 1.5) = 1 - \left(\cos \frac{1.5\pi}{4}\right)^4 = 0.97855$$

$$f(t = 2.5) = 1 - \left(\cos \frac{2.5\pi}{4}\right)^4 = 0.97855$$



$$f(t = 3.5) = 1 - \left(\cos \frac{3.5\pi}{4}\right)^4 = 0.27145$$

$$\begin{aligned} A &\approx 1 \cdot (f(.5) + f(1.5) + f(2.5) + f(3.5)) \\ &= 1(0.27145 + 0.97855 + 0.97855 + 0.27145) \\ &= 2.5 \end{aligned}$$

$$\begin{aligned} \text{Average value} &\approx \frac{\text{Area}}{\text{Length } [0, 2]} \\ &= \frac{2.5}{4} \\ &= 0.625 \end{aligned}$$

Exercise

Write the sums without sigma notation. Then evaluate:

$$\sum_{k=1}^2 \frac{6k}{k+1}$$

Solution

$$\begin{aligned} \sum_{k=1}^2 \frac{6k}{k+1} &= \frac{6}{2} + \frac{12}{3} \\ &= 7 \end{aligned}$$

Exercise

Write the sums without sigma notation. Then evaluate:

$$\sum_{k=1}^3 \frac{k-1}{k}$$

Solution

$$\begin{aligned} \sum_{k=1}^3 \frac{k-1}{k} &= \frac{0}{1} + \frac{1}{2} + \frac{2}{3} \\ &= \frac{7}{6} \end{aligned}$$

Exercise

Write the sums without sigma notation. Then evaluate:

$$\sum_{k=1}^5 \sin k\pi$$

Solution

$$\begin{aligned}\sum_{k=1}^5 \sin k\pi &= \sin \pi + \sin 2\pi + \sin 3\pi + \sin 4\pi + \sin 5\pi \\ &= 0 + 0 + 0 + 0 \\ &= 0\end{aligned}$$

Exercise

Write the sums without sigma notation. Then evaluate: $\sum_{k=1}^4 (-1)^k \cos k\pi$

Solution

$$\begin{aligned}\sum_{k=1}^4 (-1)^k \cos k\pi &= -\cos \pi + \cos 2\pi - \cos 3\pi + \cos 4\pi \\ &= -(-1) + 1 - (-1) + 1 \\ &= 4\end{aligned}$$

Exercise

Write the following expression $1 + 2 + 4 + 8 + 16 + 32$ in sigma notation

Solution

$$\begin{aligned}1 + 2 + 4 + 8 + 16 + 32 &= \sum_{k=1}^6 2^{k-1} \\ 1 + 2 + 4 + 8 + 16 + 32 &= \sum_{k=0}^5 2^k\end{aligned}$$

Exercise

Write the following expression $1 - 2 + 4 - 8 + 16 - 32$ in sigma notation

Solution

$$1 - 2 + 4 - 8 + 16 - 32 = \sum_{k=1}^6 (-2)^{k-1}$$

$$1 - 2 + 4 - 8 + 16 - 32 = \sum_{k=0}^5 (-1)^k 2^k$$

Exercise

Write the following expression $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ in sigma notation

Solution

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} = \sum_{k=1}^4 \frac{1}{2^k}$$

Exercise

Write the following expression $-\frac{1}{5} + \frac{2}{5} - \frac{3}{5} + \frac{4}{5} - \frac{5}{5}$ in sigma notation

Solution

$$-\frac{1}{5} + \frac{2}{5} - \frac{3}{5} + \frac{4}{5} - \frac{5}{5} = \sum_{k=1}^5 (-1)^k \frac{k}{5}$$

Exercise

Suppose that $\sum_{k=1}^n a_k = -5$ and $\sum_{k=1}^n b_k = 6$. Find the value of $\sum_{k=1}^n (b_k - 2a_k)$

Solution

$$\begin{aligned} \sum_{k=1}^n (b_k - 2a_k) &= \sum_{k=1}^n b_k - 2 \sum_{k=1}^n a_k \\ &= 6 - 2(-5) \\ &= 16 \end{aligned}$$

Exercise

Evaluate the sums $\sum_{k=1}^{10} k^3$

Solution

$$\begin{aligned}\sum_{k=1}^{10} k^3 &= \left(\frac{10(10+1)}{2} \right)^2 \\ &= 55^2 \\ &= \underline{3,025} \quad | \end{aligned}$$

Exercise

Evaluate the sums $\sum_{k=1}^7 (-2k)$

Solution

$$\begin{aligned}\sum_{k=1}^7 (-2k) &= -2 \sum_{k=1}^7 k \\ &= -2 \left(\frac{7(7+1)}{2} \right) \\ &= \underline{-56} \quad | \end{aligned}$$

Exercise

Evaluate the sums $\sum_{k=1}^5 \frac{\pi k}{15}$

Solution

$$\begin{aligned}\sum_{k=1}^5 \frac{\pi k}{15} &= \frac{\pi}{15} \sum_{k=1}^5 k \\ &= \frac{\pi}{15} \left(\frac{5(5+1)}{2} \right) \\ &= \underline{\pi} \quad | \end{aligned}$$

Exercise

Evaluate the sums $\sum_{k=1}^5 k(3k+5)$

Solution

$$\begin{aligned}
 \sum_{k=1}^5 k(3k+5) &= \sum_{k=1}^5 (3k^2 + 5k) \\
 &= 3 \sum_{k=1}^5 k^2 + 5 \sum_{k=1}^5 k \\
 &= 3 \left(\frac{5(5+1)(2(5)+1)}{6} \right) + 5 \frac{5(5+1)}{2} \\
 &= 240
 \end{aligned}$$

Exercise

Evaluate the sums $\sum_{k=1}^5 \frac{k^3}{225} + \left(\sum_{k=1}^5 k \right)^3$

Solution

$$\begin{aligned}
 \sum_{k=1}^5 \frac{k^3}{225} + \left(\sum_{k=1}^5 k \right)^3 &= \frac{1}{225} \sum_{k=1}^5 k^3 + \left(\sum_{k=1}^5 k \right)^3 \\
 &= \frac{1}{225} \left(\frac{5(5+1)}{2} \right)^2 + \left(\frac{5(5+1)}{2} \right)^3 \\
 &= \frac{1}{225} \left(\frac{900}{4} \right) + \frac{27,000}{8} \\
 &= 1 + 3,375 \\
 &= 3,376
 \end{aligned}$$

Exercise

Evaluate the sums $\sum_{k=1}^{500} 7$

Solution

$$\sum_{k=1}^{500} 7 = 7(500)$$

$$\underline{= 3,500}$$

Exercise

Evaluate the sums $\sum_{k=18}^{71} k(k-1)$

Solution

$$\text{Let } n = (k-18)+1$$

$$= k-17$$

$$\begin{cases} k=18 & \rightarrow n=1 \\ k=71 & \rightarrow n=54 \end{cases}$$

$$\Rightarrow \underline{k = n+17}$$

$$\begin{aligned} \sum_{k=18}^{71} k(k-1) &= \sum_{n=1}^{54} (n+17)(n+17-1) \\ &= \sum_{n=1}^{54} (n+17)(n+16) \\ &= \sum_{n=1}^{54} (n^2 + 33n + 272) \\ &= \sum_{n=1}^{54} n^2 + 33 \sum_{n=1}^{54} n + \sum_{n=1}^{54} 272 \\ &= \frac{54(54+1)(54(2)+1)}{6} + 33 \cdot \frac{54(54+1)}{6} + 272(54) \\ &\underline{= 117,648} \end{aligned}$$

Exercise

Evaluate the sums $\sum_{k=1}^n \left(\frac{1}{n} + 2n \right)$

Solution

$$\sum_{k=1}^n \left(\frac{1}{n} + 2n \right) = n \cdot \left(\frac{1}{n} + 2n \right) \\ = 1 + 2n^2$$

Exercise

Graph the function $f(x) = x^2 - 1$ over the given interval $[0, 2]$. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum

$$\sum_{k=1}^4 f(c_k) \Delta x_k, \text{ given } c_k \text{ is the}$$

- a) Left-hand endpoint
- b) Right-hand endpoint
- c) Midpoint of k^{th} subinterval.

Solution

