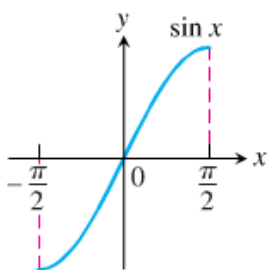


Section 2.9 – Derivatives of Inverse Trigonometric Functions

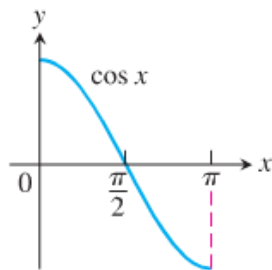
Defining the inverses

The six basic trigonometric functions are not one-to-one.



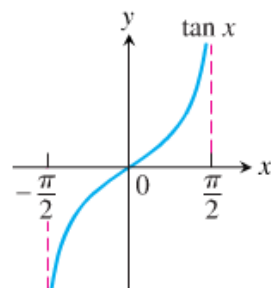
$$y = \sin x$$

Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



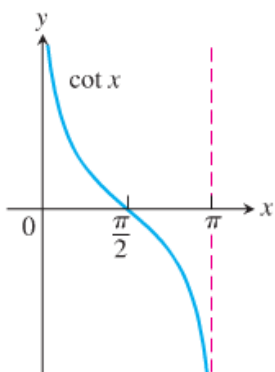
$$y = \cos x$$

Domain: $[0, \pi]$
Range: $[-1, 1]$



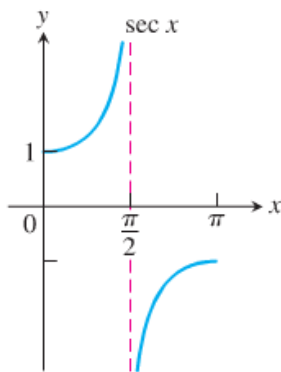
$$y = \tan x$$

Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



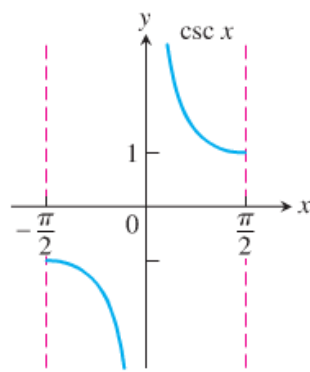
$$y = \cot x$$

Domain: $(0, \pi)$
Range: $(-\infty, \infty)$



$$y = \sec x$$

Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$



$$y = \csc x$$

Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

Since these restricted functions are now one-to-one, they have inverses, which we denoted by

$$y = \sin^{-1} x \quad \text{or} \quad y = \arcsin x$$

$$y = \cos^{-1} x \quad \text{or} \quad y = \arccos x$$

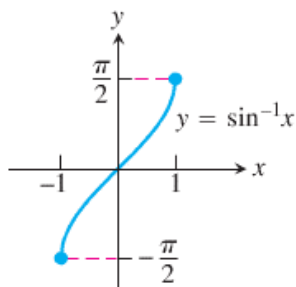
$$y = \tan^{-1} x \quad \text{or} \quad y = \arctan x$$

$$y = \cot^{-1} x \quad \text{or} \quad y = \operatorname{arccot} x$$

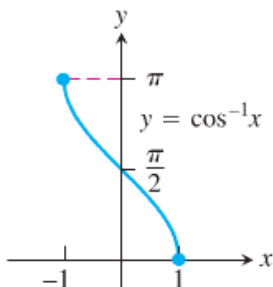
$$y = \sec^{-1} x \quad \text{or} \quad y = \operatorname{arcsec} x$$

$$y = \csc^{-1} x \quad \text{or} \quad y = \operatorname{arccsc} x$$

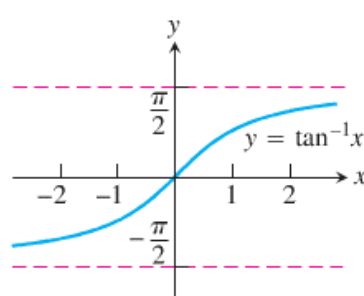
Domain: $-1 \leq x \leq 1$
 Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$



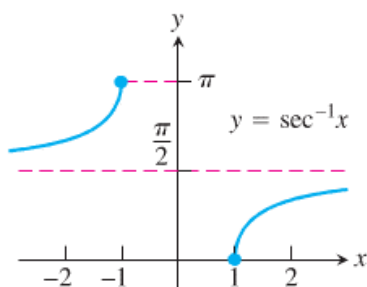
Domain: $-1 \leq x \leq 1$
 Range: $0 \leq y \leq \pi$



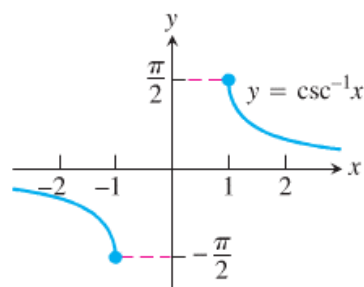
Domain: $-\infty < x < \infty$
 Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



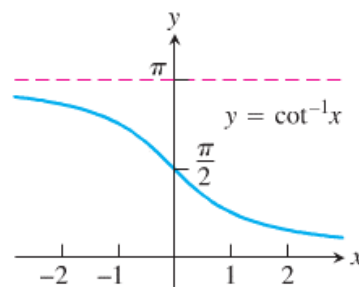
Domain: $x \leq -1$ or $x \geq 1$
 Range: $0 \leq y \leq \pi, y \neq \frac{\pi}{2}$



Domain: $x \leq -1$ or $x \geq 1$
 Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$

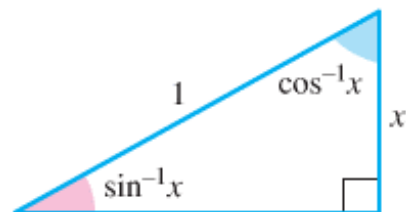


Domain: $-\infty < x < \infty$
 Range: $0 < y < \pi$



Definitions

- ✓ $y = \sin^{-1} x$ is the number in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for which $\sin y = x$
- ✓ $y = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$
- ✓ $y = \tan^{-1} x$ is the number in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which $\tan y = x$
- ✓ $y = \cot^{-1} x$ is the number in $(0, \pi)$ for which $\cot y = x$



Example $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

Inverse Function – Inverse Cofunction Identities

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$

$$\cot^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{x}\right)$$

$$\csc^{-1} x = \frac{\pi}{2} - \sec^{-1} x$$

$$\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{x}\right)$$

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

$$\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1}(x)$$

Derivative of $y = \sin^{-1} u$

$$\begin{aligned}
 (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\
 &= \frac{1}{\cos(\sin^{-1} x)} \\
 &= \frac{1}{\sqrt{1-\sin^2(\sin^{-1} x)}} \\
 &= \frac{1}{\sqrt{1-x^2}}
 \end{aligned}$$

$$\boxed{\frac{d}{dx} \sin^{-1} u = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad |u| < 1}$$

$$\boxed{\frac{d}{dx} \sin^{-1} u = \frac{u'}{\sqrt{1-u^2}}}$$

$\frac{d}{dx} \sin^{-1} u = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad u < 1$	$(\sin^{-1} u)' = \frac{u'}{\sqrt{1-u^2}}$
$\frac{d}{dx} \cos^{-1} u = -\frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad u < 1$	$(\cos^{-1} u)' = -\frac{u'}{\sqrt{1-u^2}}$
$\frac{d}{dx} \tan^{-1} u = \frac{1}{1+u^2} \frac{du}{dx}$	$(\tan^{-1} u)' = \frac{u'}{1+u^2}$
$\frac{d}{dx} \cot^{-1} u = -\frac{1}{1+u^2} \frac{du}{dx}$	$(\cot^{-1} u)' = -\frac{u'}{1+u^2}$
$\frac{d}{dx} \sec^{-1} u = \frac{1}{ u \sqrt{u^2-1}} \frac{du}{dx}, \quad u > 1$	$(\sec^{-1} u)' = \frac{u'}{ u \sqrt{u^2-1}}$
$\frac{d}{dx} \csc^{-1} u = -\frac{1}{ u \sqrt{u^2-1}} \frac{du}{dx}, \quad u > 1$	$(\csc^{-1} u)' = -\frac{u'}{ u \sqrt{u^2-1}}$

Example

Find the derivative of $\frac{d}{dx}(\sin^{-1} x^2)$

Solution

$$\begin{aligned}\frac{d}{dx}(\sin^{-1} x^2) &= \frac{1}{\sqrt{1-(x^2)^2}} \cdot \frac{d}{dx}(x^2) \\ &= \frac{2x}{\sqrt{1-x^4}}\end{aligned}$$

Example

Find the derivative of $\frac{d}{dx}(\sec^{-1} 5x^4)$

Solution

$$\begin{aligned}\frac{d}{dx}(\sec^{-1} 5x^4) &= \frac{(5x^4)'}{5x^4 \sqrt{(5x^4)^2 - 1}} \\ &= \frac{20x^3}{5x^4 \sqrt{25x^8 - 1}} \\ &= \frac{4}{x \sqrt{25x^8 - 1}}\end{aligned}$$

$$5x^4 - 1 > 0$$

Exercises Section 2.9 – Derivatives of Inverse Trigonometric Functions

(1 – 2) Find the value of

1. $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

2. $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

(3 – 5) Find the limit

3. $\lim_{x \rightarrow -1^+} \cos^{-1} x$

4. $\lim_{x \rightarrow -\infty} \tan^{-1} x$

5. $\lim_{x \rightarrow \infty} \csc^{-1} x$

(6 – 17) Find the derivative

6. $y = \cos^{-1}\left(\frac{1}{x}\right)$

10. $y = \ln\left(\tan^{-1} x\right)$

14. $y = \ln\left(x^2 + 4\right) - x \tan^{-1}\left(\frac{x}{2}\right)$

7. $y = \sin^{-1} \sqrt{2}t$

11. $y = \tan^{-1}(\ln x)$

15. $f(x) = \sin^{-1} \frac{1}{x}$

8. $y = \sec^{-1}(5s)$

12. $y = \csc^{-1}(e^t)$

16. $\left. \frac{d}{dx} \left(x \sec^{-1} x \right) \right|_{x=\frac{2}{\sqrt{3}}}$

9. $y = \cot^{-1} \sqrt{t-1}$

13. $y = x\sqrt{1-x^2} + \cos^{-1} x$

17. $\left. \frac{d}{dx} \left(\tan^{-1} e^{-x} \right) \right|_{x=0}$

18. Find the angle α

