

Section 2.2 – Techniques for Finding Derivatives

Notations for the Derivative

The derivative of $y = f(x)$ may be written in any of the following ways:

1st derivative	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}[f(x)]$	$D_x[y]$
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Constant Rule

$$\frac{d}{dx}[c] = f'(c) = 0 \quad c \text{ is constant}$$

Proof:

Let $f(x) = c$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{c - c}{\Delta x} \\ &= 0 \end{aligned}$$

$$\text{So, } \frac{d}{dx}[c] = 0$$

Example

Find the derivative

a) $f(x) = 9$

$$f' = 0$$

b) $h(t) = \pi$

$$D_t[h(t)] = 0$$

c) $y = 2^3$

$$\frac{dy}{dx} = 0$$

Power Rule

$$f(x) = x^n \Rightarrow f'(x) = nx^{n-1} \quad n \text{ is any real number}$$

Proof

$$\text{Let } f(x) = x^n$$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^n - x^n}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2}x^{n-2}(\Delta x)^2 + \dots + (\Delta x)^n - x^n}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} nx^{n-1}\Delta x + \frac{n(n-1)}{2}x^{n-2}\Delta x + \dots + (\Delta x)^{n-1} \\ &= nx^{n-1} \end{aligned}$$

Example

Find the derivative

$$a) \quad y = x^6$$

$$\begin{aligned} y' &= 6x^{6-1} \\ &= 6x^5 \end{aligned}$$

$$b) \quad y = t$$

$$\begin{aligned} y' &= t^{1-1} \\ &= t^0 \\ &= 1 \end{aligned}$$

$$c) \quad y = \frac{1}{x^3}$$

$$y = x^{-3}$$

$$y' = -3x^{-3-1}$$

$$= -3x^{-4} \quad \text{or} \quad -\frac{3}{x^4}$$

$$e) \quad D_x \left(x^{4/3} \right)$$

$$D_x \left(x^{4/3} \right) = \frac{4}{3} x^{1/3}$$

$$f) \quad y = \sqrt{z}$$

$$\frac{dy}{dz} = \frac{d}{dz} \left[z^{1/2} \right]$$

$$= \frac{1}{2} z^{1/2-1}$$

$$= \frac{1}{2} z^{-1/2}$$

$$\frac{1}{2z^{1/2}}$$

$$\frac{1}{2\sqrt{z}}$$

Constant Times a Function

If f is a differentiable function of x , and c is a real number, then

$$\text{If } f(x) = k.g(x) \quad \Rightarrow f' = k.g'$$

Example

a) If $y = 8x^4$, find $\frac{dy}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= 8(4x^3) \\ &= 32x^3\end{aligned}$$

b) If $y = -\frac{3}{4}x^{12}$, find $\frac{dy}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= -\frac{3}{4}(12x^{11}) \\ &= -9x^{11}\end{aligned}$$

c) If $D_t(-8t)$

$$D_t(-8t) = -8$$

d) If $y = \frac{6}{x}$, find $\frac{dy}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[6x^{-1}] \\ &= -6x^{-2} \\ &= -\frac{6}{x^2}\end{aligned}$$

e) $y = \frac{9}{4x^2}$

$$= \frac{9}{4}x^{-2}$$

$$\rightarrow y' = \frac{9}{4}(-2)x^{-3}$$

$$= -\frac{9}{2x^3}$$

Sum or Difference Rule

The derivative of the sum or difference of two differentiable functions is the sum or difference of their derivatives.

$$f(x) = u(x) \pm v(x) \qquad f'(x) = u'(x) \pm v'(x)$$

Example

Find the derivative of each function

a) $y = 6x^3 + 15x^2$

$$y' = 18x^2 + 30x$$

b) $p(t) = 12t^4 - 6\sqrt{t} + \frac{5}{t}$

$$p(t) = 12t^4 - 6t^{1/2} + 5t^{-1}$$

$$p' = 48t^3 - 3t^{-1/2} - 5t^{-2}$$

$$= 48t^3 - \frac{3}{t^{1/2}} - \frac{5}{t^2}$$

c) $f(x) = \frac{x^3 + 3\sqrt{x}}{x}$

$$f(x) = \frac{x^3}{x} + 3\frac{x^{1/2}}{x}$$

$$= x^2 + 3x^{-1/2}$$

$$f'(x) = 2x - \frac{3}{2}x^{-3/2}$$

$$= 2x - \frac{3}{2x^{3/2}}$$

$$= 2x - \frac{3}{2\sqrt{x^3}}$$

d) $f(x) = (4x^2 - 3x)^2$

$$= 16x^4 - 24x^3 + 9x^2$$

$$f' = 64x^3 - 72x^2 + 18x$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

Example

Find the slope of the graph of $f(x) = x^2 - 5x + 1$ at the point (2, -5)

Solution

$$f'(x) = 2x - 5$$

$$\begin{aligned}\text{Slope} &= f'(2) \\ &= 2(2) - 5 \\ &= -1\end{aligned}$$

Example

Researchers have determined that the daily energy requirements of female beagles who are at least 1 year old change with respect to age according to the function

$$E(t) = 753t^{-0.1321}$$

where $E(t)$ is the daily energy requirements (in $\text{kJ} / \text{W}^{0.67}$) for a dog that is t years old.

a) Find $E'(t)$

$$\begin{aligned}E' &= 753(-0.1321)t^{-0.1321-1} \\ &= -99.4713t^{-1.1321}\end{aligned}$$

b) Determine the rate of change of the daily energy requirements of a 2-year old female beagle

$$\begin{aligned}E'(\mathbf{2}) &= -99.4713(\mathbf{2})^{-1.1321} \\ &= -45.4 \text{ kJ} / \text{W}^{0.67}\end{aligned}$$

The daily energy requirements of a 2-year old female beagle are decreasing at the rate

Example

From 1998 through 2005, the revenue per share R (in dollars) for McDonald's Corporation can be modeled by

$$R = 0.0598t^2 - 0.379t + 8.44 \qquad 8 \leq t \leq 15$$

Where t represents the year, with $t = 8$ corresponding to 1998. At what rate was McDonald's revenue per share changing in 2003?

Solution

$$2003 \Rightarrow t = 13$$

$$R' = 0.1196t - 0.379$$

$$\begin{aligned}\rightarrow R' &= 0.1196(13) - 0.379 \\ &= 1.1758\end{aligned}$$

Marginal Analysis

$$\text{Profit} = P \quad \text{Revenue} = R \quad \text{Cost} = C \quad P = R - C$$

The derivatives of these quantities are called ***Marginal***

$$\frac{dP}{dx} = \text{Marginal Profit}$$

$$\frac{dR}{dx} = \text{Marginal Revenue}$$

$$\frac{dC}{dx} = \text{Marginal Cost}$$

Marginal Cost

Example

Suppose that the total cost in hundreds of dollars to produce x thousand barrels of a beverage is given by

$$C(x) = 4x^2 + 100x + 500$$

Find the marginal cost for the following values of x .

a) $x = 5$

Solution

$$C' = 8x + 100$$

$$C'(5) = 8(5) + 100$$

$$= 140$$

After 5 thousand barrels, the cost will be 140 (hundred dollars) or \$14,000.00

b) $x = 30$

Solution

$$C'(30) = 8(30) + 100$$

$$= 340$$

After 30 thousand barrels, the cost will be \$34,000.00

Demand Functions

The numbers of unit q that are willing to purchase at a given price per unit p

$$p = f(q)$$

Total Revenue R

Related to the price per unit and the quantity demanded (or sold): $R(q) = q \cdot p$

Example

The demand function for a certain product is given by $p = \frac{50,000 - q}{25,000}$

Find the marginal revenue when $q = 10,000$ units and p is in dollars.

Solution

$$\begin{aligned} R &= q \cdot p \\ &= q \left(\frac{50,000 - q}{25,000} \right) \\ &= \frac{50,000q - q^2}{25,000} \\ &= \frac{50,000q}{25,000} - \frac{q^2}{25,000} \\ &= 2q - \frac{1}{25,000}q^2 \\ R' &= 2 - \frac{2}{25,000}q \\ R'(\textcolor{red}{10000}) &= 2 - \frac{2}{25,000}(\textcolor{red}{10000}) \\ &= \textcolor{blue}{1.2} \end{aligned}$$

When $q = 10,000$ units; the marginal revenue is \$1.20 per unit.

Example

Find the revenue function and marginal revenue for a demand function of $P = 2000 - 4x$

Solution

$$\begin{aligned} \text{Revenue} &= \text{quantity} * \text{price} \\ \text{Revenue: } R &= x \cdot P \\ &= x(2000 - 4x) \\ &= 2000x - 4x^2 \end{aligned}$$

$$\text{Marginal: } R' = 2000 - 8x$$

Marginal Profit

Example

Suppose that the function for the product $p = \frac{50,000 - x}{25,000}$ is given by

$$C(x) = 2100 + 0.25x \quad \text{where } 0 \leq x \leq 30,000$$

Find the marginal profit from the production of the following numbers of units.

a) $x = 15,000$

b) $x = 25,000$

Solution

a) $R(x) = 2x - \frac{1}{25,000}x^2$

The profit is given by: $P = R - C$

$$P = 2x - \frac{1}{25,000}x^2 - (2100 + 0.25x)$$

$$= 2x - \frac{1}{25,000}x^2 - 2100 - 0.25x$$

$$= -\frac{1}{25,000}x^2 + 1.75x - 2100$$

$$P' = -\frac{2}{25,000}x + 1.75$$

$$P'(\textcolor{red}{15000}) = -\frac{2}{25,000}(\textcolor{red}{15000}) + 1.75$$
$$= \textcolor{blue}{0.55}$$

The marginal profit is \$0.55 per unit

b) $P'(\textcolor{red}{25000}) = -\frac{2}{25,000}(\textcolor{red}{25000}) + 1.75$

$$= \textcolor{blue}{-0.25}$$

The marginal profit is $\textcolor{red}{-\$0.25}$ per unit, which will reduce the profit.

Exercises **Section 2.2 – Techniques for Finding Derivatives**

Find the derivative of

1. $f(x) = -2$

2. $y = \pi$

3. $y = \sqrt{5}$

4. $f(x) = x^4$

5. $s(t) = \frac{1}{t}$

6. $y = 4x^2$

7. $y = \frac{9}{4x^2}$

8. $y = \frac{9}{(4x)^2}$

9. $y = \sqrt{5x}$

10. $f(x) = 16x^{1/2}$

11. $y = \sqrt[3]{x}$

12. $y = \frac{t}{4}$

13. $y = \frac{0.4}{\sqrt{x^3}}$

14. $y = -\frac{2}{\sqrt[3]{x}}$

15. $y = \frac{1}{\sqrt[3]{x}}$

16. $y = \frac{x^3 - 4x}{\sqrt{x}}$

17. $f(x) = 3x^2 + 2x$

18. $f(x) = 4 + 2x^3 - 3x^{-1}$

19. $f(x) = \frac{5}{3x^2} - \frac{2}{x^4} + \frac{x^3}{9}$

20. $f(x) = \frac{3}{x^{3/5}} - \frac{6}{x^{1/2}}$

21. $f(x) = \frac{5}{x^{1/5}} - \frac{8}{x^{3/2}}$

22. $y = \frac{1.2}{\sqrt{x}} - 3.2x^{-2} + x$

23. $f(x) = x^2 - 3x - 4\sqrt{x}$
24. $f(x) = 3\sqrt[3]{x^4} - 2x^3 + 4x$
25. $f(x) = 0.05x^4 + 0.1x^3 - 1.5x^2 - 1.6x + 3$
26. $y = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$
27. $f(t) = -3t^2 + 2t - 4$
28. $g(x) = 4\sqrt[3]{x} + 2$
29. $f(x) = x(x^2 + 1)$
30. $f(x) = \frac{2x^2 - 3x + 1}{x}$
31. $f(x) = \frac{4x^3 - 3x^2 + 2x + 5}{x^2}$
32. $f(x) = \frac{-6x^3 + 3x^2 - 2x + 1}{x}$
33. Find the slope of the graph of $f(x) = x^2 - 5x + 1$ at the point $(2, -5)$
34. Find an equation of the tangent line to the graph of $f(x) = -x^2 + 3x - 2$ at the point $(2, 0)$
35. Find the slope of the graph of $f(x) = x^3$ when $x = -1, 0$, and 1 .
36. The height h (in feet) of a free-falling object at time (in seconds) is given by $h = -16t^2 + 180$. Find the average velocity of the object over each interval.
- $[0, 1]$
 - $[1, 2]$
37. Give the position function of a diver who jumps from a board 12 feet high with initial velocity 16 feet per second. Then find the diver's velocity function.
38. An analyst has found that a company's costs and revenues in dollars for one product are given by
- $$C(x) = 2x \qquad R(x) = 6x - \frac{x^2}{1000}$$

Respectively, where x is the number of items produced.

- Find the marginal cost function
- Find the marginal revenue function
- Using the fact that profit is the difference between revenue and costs, find the marginal profit function.
- What value of x makes the marginal profit is 0.
- Find the profit when the marginal profit is 0.

39. A business sells 2000 units per month at a price \$10 each. If monthly sales increases 200 units for each \$0.10 reduction in price.

40. From 1998 through 2005, the revenue per share R (in dollars) for McDonald's Corporation can be modeled by

$$R = 0.0598t^2 - 0.379t + 8.44 \quad 8 \leq t \leq 15$$

Where t represents the year, with $t = 8$ corresponding to 1998. At what rate was McDonald's revenue per share changing in 2003?

41. The cost C (in dollars) of producing x units of a product is given by $C = 3.6\sqrt{x} + 500$

- Find the additional cost when the production increases from 9 to 10 units.
- Find the marginal cost when $x = 9$
- Compare the results of parts (a) and (b)

42. The revenue R (in dollars) of renting x apartments can be modeled by $R = 2x(900 + 32x - x^2)$

- Find the additional revenue when the number of rentals is increased from 14 to 15
- Find the marginal revenue when $x = 14$
- Compare the results of parts (a) and (b)

43. The profit P (in dollars) of selling x units of calculus textbooks is given by

$$P = -0.05x^2 + 20x - 1000$$

- Find the additional profit when the sales increase from 150 to 151 units.
- Find the marginal profit when $x = 150$
- Compare the results of parts (a) and (b)

44. From 1998 through 2005, the revenue per share R (in dollars) for McDonald's Corporation can be modeled by

$$R = 0.0598t^2 - 0.379t + 8.44 \quad 8 \leq t \leq 15$$

Where t represents the year, with $t = 8$ corresponding to 1998. At what rate was McDonald's revenue per share changing in 2003?

45. The profit derived from selling x units, is given by $P = 0.0002x^3 + 10x$, find the marginal profit for a production level of 100 units. Compare this with the actual gain in profit by increasing production from 100 to 101 units.

46. The Cost of producing x hamburgers is $C = 5000 + 0.56x$, $0 \leq x \leq 50,000$ and the revenue function is given by

$$R = \frac{1}{20000} \left(60000x - x^2 \right)$$

Compare the marginal profit when 10,000 units are produced with the actual increase in profit from 10,000 units to 10,001 units

47. An object moves along the y -axis (marked in feet) so that its position at time x (in seconds) is

$$f(x) = x^3 - 6x^2 + 9x$$

- a) Find the instantaneous velocity function v .
- b) Find the velocity at $x = 2$ and $x = 5$ seconds
- c) Find the time(s) when the velocity is 0.

48. A company's total sales (in millions of dollars) t months from now are given by

$$S(t) = 0.03t^3 + 0.5t^2 + 2t + 3$$

- a) Find $S'(t)$.
- b) Find $S(5)$ and $S'(5)$ (to two decimal places). Write a brief verbal interpretation of these results.
- c) Find $S(10)$ and $S'(10)$ (to two decimal places). Write a brief verbal interpretation of these results.

49. A company's total sales (in millions of dollars) t months from now are given by

$$S(t) = 0.015t^4 + 0.4t^3 + 3.4t^2 + 10t - 3$$

- a) Find $S'(t)$.
- b) Find $S(4)$ and $S'(4)$ (to two decimal places). Write a brief verbal interpretation of these results.
- c) Find $S(8)$ and $S'(8)$ (to two decimal places). Write a brief verbal interpretation of these results.

50. A marine manufacturer will sell $N(x)$ power boats after spending $\$x$ thousand on advertising, as given by

$$N(x) = 1,000 - \frac{3,780}{x} \quad 5 \leq x \leq 30$$

- a) Find $N'(x)$.
- b) Find $N(20)$ and $N'(20)$ (to two decimal places). Write a brief verbal interpretation of these results.

51. A company manufactures and sells x transistor radios per week. If the weekly cost and revenue equations are

$$C(x) = 5,000 + 2x \quad R(x) = 10x - \frac{x^2}{1,000} \quad 0 \leq x \leq 8,000$$

Then find the approximate changes in revenue and profit if production is increased from 2,000 to 2,010 per week.

52. A company manufactures fuel tanks for cars. The total weekly cost (in dollars) of producing x tanks given by

$$C(x) = 10,000 + 90x - 0.05x^2$$

- Find the marginal cost function.
- Find the marginal cost at a production level of 500 tanks per week.
- Interpret the result of part b.
- Find the exact cost of producing the 501st item.

53. A company's market research department recommends the manufacture and marketing of a new headphone set for MP3 players. After suitable test marketing, the research department presents the following *price-demand* equation:

$$x = 10,000 - 1,000p \rightarrow p = 10 - 0.001x$$

Where x is the number of headphones that retailers are likely to buy at $\$p$ per set.

The financial department provides the cost function

$$C(x) = 7,000 + 2x$$

Where $\$7,000$ is the estimate of fixed costs (tooling and overhead) and $\$2$ is the estimate of variable costs per headphone set (materials, labor, marketing, transportation, storage, etc.).

- Find the domain of the function defined by the price demand function.
- Find and interpret the marginal cost function $C'(x)$.
- Find the revenue function as a function of x and find its domain.
- Find the marginal revenue at $x = 2,000$, $5,000$, and $7,000$. Interpret these results.
- Graph the cost function and the revenue function in the same coordinate system, Find the intersection points of these two graphs and interpret the results.
- Find the profit function and its domain and sketch the graph of the function.
- Find the marginal profit at $x = 1,000$, $4,000$, and $6,000$. Interpret these results.

54. A small machine shop manufactures drill bits used in the petroleum industry. The manager estimates that the total daily cost (in dollars) of producing x bits is

$$C(x) = 1,000 + 25x - 0.1x^2$$

- Find $\bar{C}(x)$ and $\bar{C}'(x)$
- Find $\bar{C}(10)$ and $\bar{C}'(10)$. Interpret these quantities.
- Use the results in part (b) to estimate the average cost per bit at a production level of 11 bits per day.

55. The total profit (in dollars) from the sale of x calendars is

$$P(x) = 22x - 0.2x^2 - 400 \quad 0 \leq x \leq 100$$

- Find the exact profit from the sale of the 41st calendar.
- Use the marginal profit to approximate the profit from the sale of the 41st calendar.

56. The total profit (in dollars) from the sale of x cameras is

$$P(x) = 12x - 0.02x^2 - 1,000 \quad 0 \leq x \leq 600$$

Evaluate the marginal profit at the given values of x , and interpret the results.

- a) $x = 200$.
- b) $x = 350$.

57. The total profit (in dollars) from the sale of x gas grills is

$$P(x) = 20x - 0.02x^2 - 320 \quad 0 \leq x \leq 1,000$$

- a) Find the average profit per grill if 40 grills are produced.
- b) Find the marginal average profit at a production level of 40 grills and interpret the results.
- c) Use the results from parts (a) and (b) to estimate the average profit per grill if 41 grills are produced.

58. The price p (in dollars) and the demand x for a particular steam iron are related by the equation

$$x = 1,000 - 20p$$

- a) Express the price p in terms of the demand x , and find the domain of this function.
- b) Find the revenue $R(x)$ from the sale of x steam irons. What is the domain of R ?
- c) Find the marginal revenue at a production level of 400 steam irons and interpret the results.
- d) Find the marginal revenue at a production level of 650 steam irons and interpret the results.

59. The price-demand equation and the cost function for the production of TVs are given respectively, by

$$x = 9,000 - 30p \quad \text{and} \quad C(x) = 150,000 + 30x$$

Where x is the number of TVs that can be sold at a price of $\$p$ per TV and $C(x)$ is the total cost (in dollars) of producing x TVs.

- a) Express the price p as a function of the demand x , and find the domain of this function.
- b) Find the marginal cost.
- c) Find the revenue function and state its domain.
- d) Find the marginal revenue.
- e) Find $R'(3,000)$ and $R'(6,000)$ and interpret these quantities.
- f) Graph the cost function and the revenue function on the same coordinate system for $0 \leq x \leq 9,000$. Find the break-even points and indicate regions of loss and profit.
- g) Find the profit function in terms of x .
- h) Find the marginal profit.
- i) Find $P'(1,500)$ and $P'(4,500)$ and interpret these quantities

- 60.** The total cost and the total revenue (in dollars) for the production and sale of x hair dryers are given, respectively, by

$$C(x) = 5x + 2,340 \quad \text{and} \quad R(x) = 40x - 0.1x^2 \quad 0 \leq x \leq 400$$

- a) Find the value of x where the graph of $R(x)$ has a horizontal tangent line.
- b) Find the profit function $P(x)$.
- c) Find the value of x where the graph of $P(x)$ has a horizontal tangent line.
- d) Graph $C(x)$, $R(x)$, and $P(x)$ on the same coordinate system for $0 \leq x \leq 400$. Find the break-even points. Find the x intercept of the graph of $P(x)$.