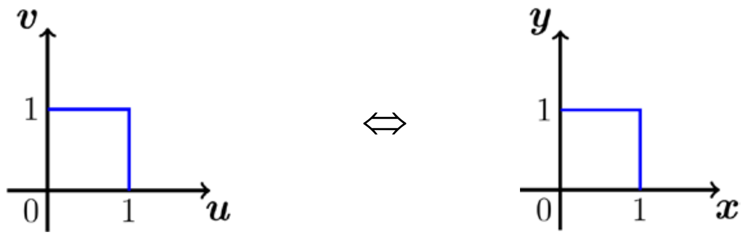
Solution *Section* 3.7 – Change of Variables in Multiple Integrals

Exercise

Let $S = \{0 \le u \le 1,\ 0 \le v \le 1\}$ be a unit square in the *uv*-plane. Find the image of *S* in the *xy*-plane under the following transformations. $T: \quad x = v,\ y = u$

Solution

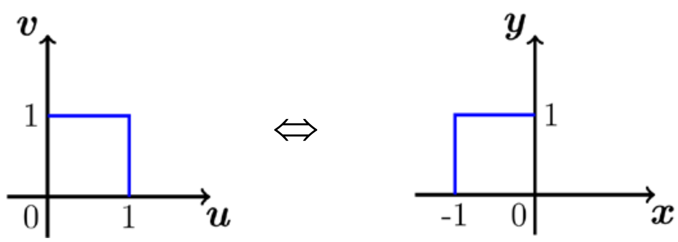


The transformation just switches the coordinates. Image *xy* is unit square.

Exercise

Let $S = \{0 \le u \le 1,\ 0 \le v \le 1\}$ be a unit square in the *uv*-plane. Find the image of *S* in the *xy*-plane under the following transformations. $T: \quad x = -v,\ y = u$

Solution



$T: \quad x = -v,\ y = u$

$$T = \{(x,\ y):\ \ 0 \le -x \le 1 \quad 0 \le y \le 1\}$$

$$= \{(x,\ y):\ \ -1 \le x \le 0 \quad 0 \le y \le 1\}$$

T is a unit square in *Q*II with one vertex at origin.

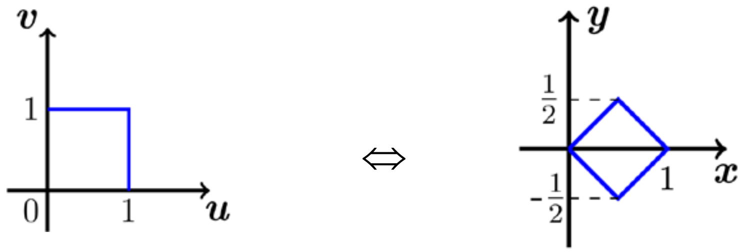
Exercise

Let $S = \{0 \le u \le 1,\ 0 \le v \le 1\}$ be a unit square in the *uv*-plane. Find the image of *S* in the *xy*-plane under the following transformations. $T: \quad x = \frac{u+v}{2},\ \ y = \frac{u-v}{2}$

Solution

$T: \quad x = \frac{u+v}{2},\ \ y = \frac{u-v}{2}$

$(u,\ v)$	$(x,\ y)$
$(0,\ 0) \to (1,\ 0)$	$(0,\ 0) \to \left(\frac{1}{2},\ \frac{1}{2}\right)$
$(1,\ 0) \to (1,\ 1)$	$\left(\frac{1}{2},\ \frac{1}{2}\right) \to (1,\ 0)$
$(1,\ 1) \to (0,\ 1)$	$(1,\ 0) \to \left(\frac{1}{2},\ -\frac{1}{2}\right)$



Diamond shape

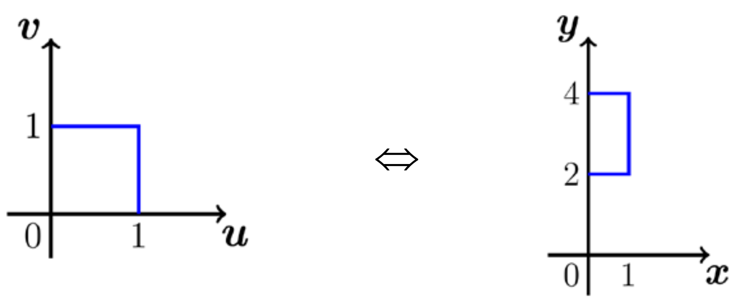
Exercise

Let $S = \{0 \le u \le 1,\ 0 \le v \le 1\}$ be a unit square in the *uv*-plane. Find the image of *S* in the *xy*-plane under the following transformations. $T: \quad x = u, \quad y = 2v + 2$

Solution

$T: \quad x = u, \quad y = 2v + 2$

$(u,\ v)$	$(x,\ y)$
$(0,\ 0) \to (1,\ 0)$	$(0,\ 2) \to (1,\ 2)$
$(1,\ 0) \to (1,\ 1)$	$(1,\ 2) \to (1,\ 4)$
$(1,\ 1) \to (0,\ 1)$	$(1,\ 4) \to (0,\ 4)$



Exercise

a) Solve the system $u = x - y, \quad v = 2x + y$ for *x* and *y* in terms of *u* and *v*. Then find the value of the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$

b) Find the image under the transformation $u = x - y, \ \ v = 2x + y$ of the triangular region with vertices (0, 0), (1, 1), and (1, −2) in the *xy*-plane. Sketch the transformed region in the *uv*-plane.

Solution

a) $\begin{matrix} u = x - y \\ v = 2x + y \end{matrix} \quad \rightarrow \quad \begin{cases} x = \frac{1}{3}u + \frac{1}{3}v \\ y = -\frac{2}{3}u + \frac{1}{3}v \end{cases}$

$$\begin{aligned} \frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix} \\ &= \frac{1}{9} + \frac{2}{9} \\ &= \frac{1}{3} \end{aligned}$$

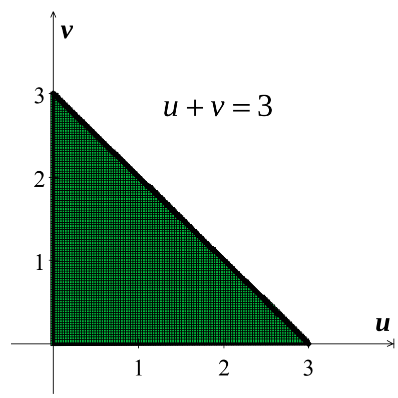
b) From (0, 0) to (1, 1) $\Rightarrow \ y = x \ \rightarrow u = x - y = 0$

From (0, 0) to (1, −2) $\Rightarrow \ y = -2x \ \rightarrow u = 2x + y = 0$

From (1, 1) to (1, −2) $\Rightarrow \ x = 1$

$$\rightarrow x = \frac{1}{3}u + \frac{1}{3}v = 1$$

$$u + v = 3$$



OR: $(0,\ 0) \ \rightarrow \ \begin{cases} u = 0 \\ v = 0 \end{cases}$

$(1,\ 1) \ \rightarrow \ \begin{cases} u = 0 \\ v = 3 \end{cases}$

$(1,\ -2) \ \rightarrow \ \begin{cases} u = 3 \\ v = 0 \end{cases}$

Exercise

Let *R* be the region in the first quadrant of the *xy*-plane bounded by the hyperbolas $xy = 1, \ xy = 9$ and the lines $y = x, \ y = 4x$. Use the transformation $x = \frac{u}{v}, \ y = uv$ with $u > 0, \ \textit{and} \ \ v > 0$ to rewrite

$$\iint\limits_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy} \right) dxdy$$

As an integral over an appropriate region G in the uv-plane. Then evaluate the uv-integral over G.

Solution

$$x = \frac{u}{v} \qquad \to \qquad u = xv \qquad \begin{cases} \frac{y}{x} = v^2 \\ xy = u^2 \end{cases}$$
$$y = uv \qquad\qquad y = xv^2$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{u}{v} + \frac{u}{v} = \boxed{2\frac{u}{v}}$$

$$xy = 1 = u^2 \quad \to \quad \begin{cases} u = 1 \\ u = 3 \end{cases} \qquad\qquad y = x \Rightarrow \frac{y}{x} = 1 = v^2 \quad \to \quad \begin{cases} v = 1 \\ v = 2 \end{cases}$$
$$xy = 9 = u^2 \qquad\qquad y = 4x \Rightarrow \frac{y}{x} = 4 = v^2$$

$$\iint\limits_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dxdy = \int_1^3 \int_1^2 (v+u)\frac{2u}{v} dvdu$$
$$= 2\int_1^3 \int_1^2 \left(u + \frac{u^2}{v}\right) dvdu$$
$$= 2\int_1^3 \left[uv + u^2 \ln v\right]_1^2 du$$
$$= 2\int_1^3 \left(2u + u^2 \ln 2 - u\right) du$$
$$= 2\int_1^3 \left(u + u^2 \ln 2\right) du$$
$$= 2\left[\frac{1}{2}u^2 + \frac{1}{3}u^3 \ln 2\right]_1^3$$
$$= 2\left(\frac{9}{2} + 9\ln 2 - \frac{1}{2} - \frac{1}{3}\ln 2\right)$$
$$\boxed{= 8 + \frac{52}{3}\ln 2}$$

Exercise

The area πab of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be found by integrating the function $f(x,y) = 1$ over the region bounded by the ellipse in the xy-plane. Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate the integral is to use the transformation $x = au,\ y = bv$ and evaluate the transformed integral over the disk G: $u^2 + v^2 \le 1$ in the uv-plane. Find the area this way.

Solution

$x = au,\ y = bv$

$$\frac{\partial(x,y)}{\partial(u,v)} = J(u,\ v) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = \underline{ab}$$

$u^2 + v^2 \le 1 \quad \rightarrow \quad -1 \le u \le 1$

$u^2 + v^2 \le 1 \quad \rightarrow \quad v^2 \le 1 - u^2 \Rightarrow \quad -\sqrt{1-u^2} \le v \le \sqrt{1-u^2}$

$$\iint_R dxdy = \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} ab\ dvdu$$

$$= ab\int_{-1}^{1}\left(\sqrt{1-u^2} + \sqrt{1-u^2}\right)du$$

$$= 2ab\int_{-1}^{1}\left(1-u^2\right)^{1/2} du \qquad\qquad \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$= 2ab\left[\frac{u}{2}\sqrt{1-u^2} + \frac{1}{2}\sin^{-1}u\right]_{-1}^{1}$$

$$= 2ab\left[\frac{1}{2}\sin^{-1}1 - \left(\frac{1}{2}\sin^{-1}(-1)\right)\right]$$

$$= 2ab\left[\frac{1}{2}\frac{\pi}{2} - \left(-\frac{1}{2}\frac{\pi}{2}\right)\right]$$

$$= 2ab\left(\frac{\pi}{2}\right)$$

$$\underline{= ab\pi}$$

Exercise

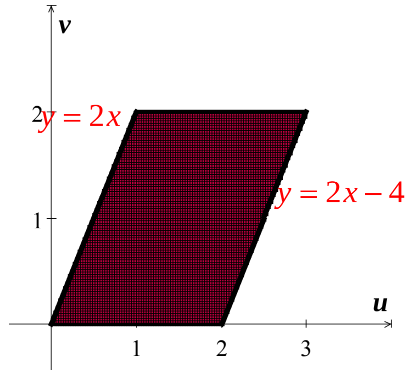
Use the transformation $x = u + \frac{1}{2}v,\ \ y = v$ to evaluate the integral

$$\int_0^2\int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2}\,dxdy$$

By first writing it as an integral over a region G in the uv-plane.

Solution

$$\frac{\partial(x,y)}{\partial(u,v)}=J(u,\ v)=\begin{vmatrix}1 & \frac{1}{2}\\ 0 & 1\end{vmatrix}=\underline{1}$$



$$\begin{aligned} x &= u+\tfrac{1}{2}v \\ y &= v \end{aligned} \quad \to \quad \begin{aligned} u &= x-\tfrac{1}{2}y \\ v &= y \end{aligned}$$

$$x=\frac{y}{2}\to y=2x$$

$$x=\frac{y+4}{2}\to y=2x-4$$

$$0\le x\le 2$$

$x=\frac{y}{2}$	$u=x-\frac{y}{2}=\frac{y}{2}-\frac{y}{2}=0$	$u=0$
$x=\frac{y}{2}+2$	$u=x-\frac{y}{2}=\frac{y}{2}+2-\frac{y}{2}=2$	$u=2$
$y=0$	$v=0$	$v=0$
$y=2$	$v=2$	$v=2$

$$\int_0^2\int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2}dxdy=\int_0^2\int_0^2 v^3(2u)e^{4u^2}dudv \qquad d\left(4u^2\right)=8udu$$

$$=\frac{1}{4}\int_0^2\int_0^2 v^3e^{4u^2}d\left(4u^2\right)dv$$

$$=\frac{1}{4}\int_0^2 v^3\left[e^{4u^2}\right]_0^2dv$$

$$=\frac{1}{4}\left(e^{16}-1\right)\int_0^2 v^3dv$$

$$=\frac{1}{4}\left(e^{16}-1\right)\left[\frac{1}{4}v^4\right]_0^2$$

$$=\underline{e^{16}-1}$$

Exercise

Use the transformation $x=\frac{u}{v},\ \ y=uv$ to evaluate the integral

$$\int_1^2\int_{1/y}^{y}\left(x^2+y^2\right)dxdy+\int_2^4\int_{y/4}^{4/y}\left(x^2+y^2\right)dxdy$$

Solution

$$x = \frac{u}{v} \qquad u = xv \qquad \begin{cases} \frac{y}{x} = v^2 \\ xy = u^2 \end{cases}$$

$$y = uv \quad \rightarrow \quad y = xv^2$$

$$J(u,v) = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{u}{v} + \frac{u}{v} = \underline{\frac{2u}{v}}$$

$x = y$	$\frac{y}{x} = 1 = v^2$	$v = 1$
$x = \frac{1}{y}$	$xy = 1 = u^2$	$u = 1$
$x = \frac{4}{y}$	$xy = 4 = u^2$	$u = 2$
$x = \frac{y}{4}$	$\frac{y}{x} = 4 = v^2$	$v = 2$

$$\int_1^2 \int_{1/y}^{y} \left(x^2 + y^2\right) dxdy + \int_2^4 \int_{y/4}^{4/y} \left(x^2 + y^2\right) dxdy = \int_1^2 \int_1^2 \left(\frac{u^2}{v^2} + u^2 v^2\right)\left(\frac{2u}{v}\right) dudv$$

$$= 2\int_1^2 \int_1^2 \left(\frac{u^3}{v^3} + u^3 v\right) dudv$$

$$= 2\int_1^2 \left(\frac{1}{v^3} + v\right)\left[\frac{1}{4}u^4\right]_1^2 dv$$

$$= \frac{1}{2}(16-1)\int_1^2 \left(v^{-3} + v\right) dv$$

$$= \frac{15}{2}\left[-\frac{1}{2}v^{-2} + \frac{1}{2}v^2\right]_1^2$$

$$= \frac{15}{4}\left[-\frac{1}{4} + 4 - (-1+1)\right]$$

$$= \frac{15}{4}\left(\frac{15}{4}\right)$$

$$= \underline{\frac{225}{16}}$$

Exercise

Find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ of the transformation

a) $x = u\cos v, \quad y = u\sin v$

b) $x = u\sin v, \quad y = u\cos v$

Solution

a) $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix}$

$= u\cos^2 v + u\sin^2 v$

$\underline{= u}\,\big|$

b) $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sin v & u\cos v \\ \cos v & -u\sin v \end{vmatrix}$

$= -u\sin^2 v - u\cos^2 v$

$\underline{= -u}\,\big|$

Exercise

Find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ of the transformation

a) $x = u\cos v, \quad y = u\sin v, \quad z = w$

b) $x = 2u - 1, \quad y = 3v - 4, \quad z = \frac{1}{2}(w-4)$

Solution

a) $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \cos v & -u\sin v & 0 \\ \sin v & u\cos v & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= u\cos^2 v + u\sin^2 v$

$\underline{= u}\,\big|$

b) $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{vmatrix} \underline{= 3}\,\big|$

Exercise

Evaluate the appropriate determinant to show that the Jacobian of the transformation from Cartesian $\rho\phi\theta$–space to Cartesian *xyz*–space is $\rho^2 \sin\phi$

Solution

$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \begin{matrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta \\ \cos\phi & -\rho\sin\phi \end{matrix}$$

$$= \rho^2 \cos^2\phi \sin\phi \cos^2\theta + \rho^2 \sin^3\phi \sin^2\theta + \rho^2 \sin\phi \cos^2\phi \sin^2\theta + \rho^2 \sin^3\phi \cos^2\theta$$

$$= \rho^2 \cos^2\phi \sin\phi \left(\cos^2\theta + \sin^2\theta\right) + \rho^2 \sin^3\phi \left(\sin^2\theta + \cos^2\theta\right)$$

$$= \rho^2 \cos^2\phi \sin\phi + \rho^2 \sin^3\phi$$

$$= \rho^2 \sin\phi \left(\cos^2\phi + \sin^2\phi\right)$$

$$= \rho^2 \sin\phi$$

Exercise

How can substitutions in single definite integrals be viewed as transformations of regions? What is the Jacobian in such a case? Illustrate with an example.

Solution

Let $u = g(x) \quad \Rightarrow \quad J(x) = \frac{du}{dx} = g'(x)$

$$\int_a^b f(u)\,du = \int_{g(a)}^{g(b)} f\left(g(x)\right) g'(x)\,dx$$

$g'(x)$ represents the Jacobian of the transformation $u = g(x) \quad or \quad x = g^{-1}(u)$

Exercise

Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

(***Hint***: Let $x = au,\ y = bv,\ and\ z = cw$. Then find the volume of an appropriate region in *uvw*-space)

Solution

$$J(u,v,w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \Rightarrow \frac{(au)^2}{a^2} + \frac{(bv)^2}{b^2} + \frac{(cw)^2}{c^2} \le 1 \quad \rightarrow \quad u^2 + v^2 + w^2 \le 1$$

$$u^2 + v^2 + w^2 \le 1 \quad \Rightarrow \quad V = \frac{4\pi}{3} = \iiint_G dudvdw$$

$$V = \iiint_R dxdydz$$

$$= \iiint\limits_G abc\ dudvdw$$

$$= abc \iiint\limits_G dudvdw$$

$$\boxed{= \frac{4\pi abc}{3}}$$

Exercise

Use the transformation $x = u^2 - v^2,\ \ y = 2uv$ to evaluate the integral

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2}\ dydx$$

(Hint: Show that the image of the triangular region G with vertices (0, 0), (1, 0), (1, 1) in the uv-plane is the region of integration R in the xy-plane defined by the limits of integration.)

Solution

$x = u^2 - v^2,\ \ y = 2uv$

$$J(u,v) = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} \qquad\qquad J(u,\ v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= 4u^2 + 4v^2$$

$$\boxed{= 4\left(u^2 + v^2\right)}$$

$y = 2\sqrt{1-x}$	$2uv = 2\sqrt{1-u^2+v^2} \rightarrow u^2v^2 = 1-u^2+v^2$ $u^2v^2 + u^2 = 1+v^2 \Rightarrow u^2\left(v^2+1\right) = 1+v^2$	$u = \pm 1$
$y = 0$	$2uv = 0$	$u = 0,\ v = 0$
$x = 0$	$u^2 - v^2 = 0$	$u = \pm v$

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2}\ dydx = \int_0^1 \int_0^u \sqrt{\left(u^2 - v^2\right)^2 + (2uv)^2} \cdot 4\left(u^2 + v^2\right) dvdu$$

$$= 4\int_0^1 \int_0^u \sqrt{u^4 + v^4 - 2u^2v^2 + 4u^2v^2} \cdot \left(u^2 + v^2\right) dvdu$$

$$= 4\int_0^1\int_0^u \sqrt{u^4+v^4+2u^2v^2}\cdot\left(u^2+v^2\right)dvdu$$

$$= 4\int_0^1\int_0^u \sqrt{\left(u^2+v^2\right)^2}\cdot\left(u^2+v^2\right)dvdu$$

$$= 4\int_0^1\int_0^u \left(u^2+v^2\right)^2 dvdu$$

$$= 4\int_0^1\int_0^u \left(u^4+v^4+2u^2v^2\right)dvdu$$

$$= 4\int_0^1 \left[u^4v+\frac{1}{5}v^5+\frac{2}{3}u^2v^3\right]_0^u du$$

$$= 4\int_0^1 \left(u^5+\frac{1}{5}u^5+\frac{2}{3}u^5\right)du$$

$$= \frac{112}{15}\int_0^1 u^5 du$$

$$= \frac{112}{15}\left[\frac{1}{6}u^6\right]_0^1$$

$$= \frac{56}{45}$$

Exercise

Evaluate $\iint_R y^4 dA$; R is the region bounded by the hyperbolas $xy=1$ and $xy=4$ and the lines $\frac{y}{x}=1$, and $\frac{y}{x}=3$

Solution

Let $\begin{cases} u = xy & \rightarrow x = \frac{u}{y} \\ v = \frac{y}{x} & \rightarrow x = \frac{y}{v} \end{cases}$

$x = \frac{y}{v} = \frac{u}{y} \quad \rightarrow \quad y^2 = uv \quad \Rightarrow \quad y = \sqrt{uv}$

$x = \frac{\sqrt{uv}}{v} = \sqrt{\frac{u}{v}}$

$x = u^{1/2}v^{-1/2} \qquad y = u^{1/2}v^{1/2}$

$$J(u,\ v) = \begin{vmatrix} \frac{1}{2}u^{-1/2}v^{-1/2} & -\frac{1}{2}u^{1/2}v^{-3/2} \\ \frac{1}{2}u^{-1/2}v^{1/2} & \frac{1}{2}u^{1/2}v^{-1/2} \end{vmatrix} \qquad J(u,\ v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \frac{1}{4}v^{-1} + \frac{1}{4}v^{-1}$$

$$= \frac{1}{2v}$$

$xy = 1$	$u = xy = 1$	$u = 1$
$xy = 4$	$u = xy = 4$	$u = 4$
$\frac{y}{x} = 1$	$v = \frac{y}{x} = 1$	$v = 1$
$\frac{y}{x} = 3$	$v = \frac{y}{x} = 3$	$v = 3$

$$\iint_R y^4 dA = \int_1^4 \int_1^3 \frac{1}{2v}\left(\sqrt{uv}\right)^4 dvdu \qquad \iint_R f(x,y)\,dxdy = \iint_G f(g(u,v), h(u,v))\left|J(u,v)\right| dudv$$

$$= \frac{1}{2}\int_1^4 u^2\ du \int_1^3 v\ dv$$

$$= \frac{1}{12}\ u^3 \Big|_1^4\ v^2 \Big|_1^3$$

$$= \frac{1}{12}(64-1)(9-1)$$

$$= \frac{504}{12}$$

$$= 42$$

Exercise

Evaluate $\iint_R \left(y^2 + xy - 2x^2\right) dA$; R is the region bounded by the lines $y = x$, $y = x - 3$, $y = -2x + 3$, and $y = -2x - 3$

Solution

$$\begin{cases} y - x = 0 & y - x = -3 \\ y + 2x = \pm 3 \end{cases}$$

Let $\begin{cases} u = y - x \\ v = y + 2x \end{cases}$

$$y - x = u$$
$$\frac{y + 2x = v}{x = \frac{1}{3}(v - u)} \quad \to y = \frac{1}{3}(v + 2u)$$

$$J(u,\ v) = \begin{vmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{vmatrix} \qquad\qquad J(u,\ v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$
$$= -\frac{1}{3}$$

$y = x$	$\frac{1}{3}(v + 2u) = \frac{1}{3}(v - u) \to 2u = -u$	$u = 0$
$y = x - 3$	$\frac{1}{3}(v + 2u) = \frac{1}{3}(v - u) - 3 \ \to 2u = -u - 9$	$u = -3$
$y = -2x + 3$	$\frac{1}{3}(v + 2u) = -\frac{2}{3}(v - u) + 3 \ \to \ v = -2v + 9$	$v = 3$
$y = -2x - 3$	$\frac{1}{3}(v + 2u) = -\frac{2}{3}(v - u) - 3 \ \to \ v = -2v - 9$	$v = -3$

$$y^2 + xy - 2x^2 = \frac{1}{9}(v + 2u)^2 + \frac{1}{9}(v + 2u)(v - u) - \frac{2}{9}(v - u)^2$$
$$= \frac{1}{9}\left(v^2 + 4uv + 4u^2 + v^2 + uv - 2u^2 - 2v^2 + 4uv - 2u^2\right)$$
$$= uv$$

$$\iint_R \left(y^2 + xy - 2x^2\right) dA = \int_{-3}^{0}\int_{-3}^{3}\left(-\frac{1}{3}\right) uv\ dvdu$$
$$= -\frac{1}{3}\int_{-3}^{0} u\ du \int_{-3}^{3} v\ dv$$
$$= -\frac{1}{12} u^2 \Big|_{-3}^{0} \quad v^2 \Big|_{-3}^{3}$$
$$= -\frac{1}{12}(-9)(9 - 9)$$
$$= 0$$

Exercise

Evaluate $\iint\limits_D x\ dV$; R is bounded by the planes $y - 2x = 0,\ y - 2x = 1,\ z - 3y = 0,\ z - 3y = 1,$ $z - 4x = 0$ and $z - 4x = 3$

Solution

Let $\begin{cases} u = y-2x & \rightarrow & 0 \le u \le 1 \\ v = z-3y & \rightarrow & 0 \le v \le 1 \\ w = z-4x & \rightarrow & 0 \le w \le 3 \end{cases}$

$$\begin{cases} u = y-2x & \rightarrow & y = u+2x \\ w = z-4x & \rightarrow & z = w+4x \end{cases}$$

$$v = z-3y$$

$$= w+4x-3u-6x$$

$$2x = w-3u-v$$

$$\begin{cases} x = -\frac{3}{2}u - \frac{1}{2}v + \frac{1}{2}w \\ y = -2u - v + w \\ z = -6u - 2v + 3w \end{cases}$$

$$J(u,\ v,\ w) = \begin{vmatrix} -2 & -1 & 1 \\ -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ -6 & -2 & 3 \end{vmatrix}$$

$$\boxed{= \frac{1}{2}}$$

$$\iint\limits_D x\ dV = \int_0^3 \int_0^1 \int_0^1 \left(\frac{1}{2}\right)\frac{1}{2}(-3u - v + w)\,dudvdw$$

$$= \frac{1}{4}\int_0^3 \int_0^1 \left(-\frac{3}{2}u^2 - vu + wu\right)\Big|_0^1\ dvdw$$

$$= \frac{1}{4}\int_0^3 \int_0^1 \left(-\frac{3}{2} - v + w\right)dvdw$$

$$= \frac{1}{4}\int_0^3 \left(-\frac{3}{2}v - \frac{1}{2}v^2 + wv\right)\Big|_0^1\ dw$$

$$= \frac{1}{4}\int_0^3 \left(-\frac{3}{2} - \frac{1}{2} + w\right)dw$$

$$= \frac{1}{4}\left(-2w + \frac{1}{2}w^2\right)\Big|_0^3$$

$$= \frac{1}{4}\left(-6 + \frac{9}{2}\right)$$

$$\boxed{= -\frac{3}{8}}$$

Exercise

Let R be the region bounded by the lines $x+y=1;\quad x+y=4;\quad x-2y=0;\quad x-2y=-4$

Evaluate the integral $\displaystyle\iint_R 3xy\,dA$

Solution

Let $\begin{cases} u = x+y \\ v = x-2y \end{cases}$

$$\overline{u - v = 3y} \quad \to \quad \boxed{y = \frac{1}{3}(u-v)}$$

$$\begin{aligned} x &= u - y \\ &= u - \frac{1}{3}u + \frac{1}{3}v \\ &= \frac{2}{3}u + \frac{1}{3}v \end{aligned}$$

$$J(u,\ v) = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} \qquad\qquad J(u,\ v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= -\frac{1}{3}$$

$x+y=1$	$\frac{2}{3}u+\frac{1}{3}v+\frac{1}{3}u-\frac{1}{3}v=1$	$u=1$
$x+y=4$	$\frac{2}{3}u+\frac{1}{3}v+\frac{1}{3}u-\frac{1}{3}v=4$	$u=4$
$x-2y=0$	$\frac{2}{3}u+\frac{1}{3}v-\frac{2}{3}u+\frac{2}{3}v=0$	$v=0$
$x-2y=-4$	$\frac{2}{3}u+\frac{1}{3}v-\frac{2}{3}u+\frac{2}{3}v=-4$	$v=-4$

$$\begin{aligned} \iint_R 3xy\,dA &= \int_1^4 \int_{-4}^0 3\frac{1}{3}(2u+v)\frac{1}{3}(u-v)\left|-\frac{1}{3}\right| dv\,du \\ &= \frac{1}{9}\int_1^4 \int_{-4}^0 \left(2u^2 - uv - v^2\right) dv\,du \\ &= \frac{1}{9}\int_1^4 \left(2u^2v - \frac{1}{2}uv^2 - \frac{1}{3}v^3\right)\Big|_{-4}^0\ du \\ &= \frac{1}{9}\int_1^4 \left(8u^2 + 8u - \frac{64}{3}\right) du \\ &= \frac{8}{9}\left(\frac{1}{3}u^3 + \frac{1}{2}u^2 - \frac{8}{3}u\right)\Big|_1^4 \end{aligned}$$

$$= \frac{8}{9}\left(\frac{64}{3} + 8 - \frac{32}{3} - \frac{1}{3} - \frac{1}{2} + \frac{8}{3}\right)$$

$$= \frac{8}{9}\left(\frac{39}{3} + \frac{15}{2}\right)$$

$$= \frac{8}{9}\left(\frac{123}{6}\right)$$

$$= \frac{164}{9}$$

Exercise

Let *R* be the region bounded by the square with vertices (0, 1), (1, 2), (2, 1), & (1, 0).

Evaluate the integral $\iint\limits_R (x+y)^2 \sin^2(x-y)\,dA$

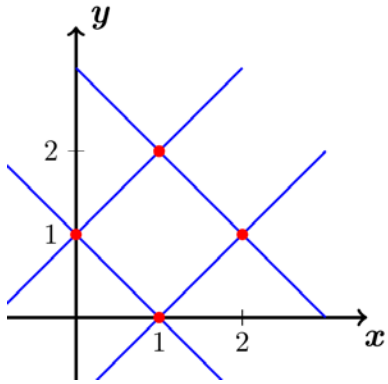
Solution

$(0,\ 1)\ \&\ (1,\ 2)\ \rightarrow m = \frac{2-1}{1-0} = 1 \ \Rightarrow \ y = x+1$

$(0,\ 1)\ \&\ (1,\ 0)\ \rightarrow m = \frac{0-1}{1-0} = -1 \ \Rightarrow \ y = -x+1$

$(2,\ 1)\ \&\ (1,\ 0)\ \rightarrow m = 1 \ \Rightarrow \ y = x-1$

$(2,\ 1)\ \&\ (1,\ 2)\ \rightarrow m = -1 \ \Rightarrow \ y = -x+3$



$$\begin{cases} y - x = 1 & y + x = 1 \\ y - x = -1 & y + x = 3 \end{cases}$$

Let $\begin{cases} u = y - x \\ v = y + x \end{cases}$

$$\overline{u + v = 2y} \ \rightarrow \ y = \frac{1}{2}u + \frac{1}{2}v$$

$$x = y - u$$
$$= -\frac{1}{2}u + \frac{1}{2}v$$

$$J(u,\ v) = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} \qquad J(u,\ v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= -\frac{1}{2}$$

$y - x = 1$	$\frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}u - \frac{1}{2}v = 1$	$u = 1$
$y - x = -1$	$\frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}u - \frac{1}{2}v = -1$	$u = -1$
$x + y = 1$	$\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}u + \frac{1}{2}v = 1$	$v = 1$

$x+y=3$	$\frac{1}{2}u+\frac{1}{2}v-\frac{1}{2}u+\frac{1}{2}v=3$	$v=3$

$$\iint\limits_R (x+y)^2\sin^2(x-y)\,dA=\int_{-1}^{1}\int_{1}^{3}\left(-\frac{1}{2}u+\frac{1}{2}v+\frac{1}{2}u+\frac{1}{2}v\right)^2\sin^2\left(-\frac{1}{2}u+\frac{1}{2}v-\frac{1}{2}u-\frac{1}{2}v\right)dvdu$$

$$=\int_{-1}^{1}\int_{1}^{3}v^2\sin^2(-u)\,dvdu$$

$$=\int_{-1}^{1}\frac{1}{2}(1-\cos 2u)\,du\quad\left(\frac{1}{3}v^3\right)\Big|_1^3$$

$$=\frac{1}{6}(27-1)\quad\left(u-\frac{1}{2}\sin 2u\right)\Big|_{-1}^{1}$$

$$=\frac{13}{3}\left(1-\frac{1}{2}\sin 2+1-\frac{1}{2}\sin 2\right)$$

$$=\frac{13}{3}(2-\sin 2)$$

Exercise

Evaluate $\iiint\limits_D yzdV$ D is bounded by the planes: $x+2y=1,\ \ x+2y=2,\ x-z=0,\ x-z=2,$ $2y-z=0,$ and $2y-z=3$

Solution

Let $\begin{cases} u=x+2y \\ v=x-z \\ w=2y-s \end{cases} \rightarrow v-w=x-2y$

$$\begin{cases} u=x+2y \\ v-w=x-2y \end{cases} \rightarrow\ u+v-w=2x$$

$$2y=\frac{1}{2}u+\frac{1}{2}v-\frac{1}{2}w-v+u$$

$$z=\frac{1}{2}u+\frac{1}{2}v-\frac{1}{2}w-v$$

$$\begin{cases} x=\frac{1}{2}u+\frac{1}{2}v-\frac{1}{2}w \\ y=\frac{1}{4}u-\frac{1}{4}v+\frac{1}{4}w \\ z=\frac{1}{2}u-\frac{1}{2}v+\frac{1}{2}w \end{cases}$$

$$J(u,\ v,\ w)=\begin{vmatrix}\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4}\\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}\end{vmatrix}$$

$$=\frac{1}{16}+\frac{1}{16}+\frac{1}{16}-\frac{1}{16}+\frac{1}{16}+\frac{1}{16}$$

$$=\frac{1}{4}$$

$x+2y=1$	$\frac{1}{2}u+\frac{1}{2}v-\frac{1}{2}w+\frac{1}{2}u-\frac{1}{2}v+\frac{1}{2}w=1$	$u=1$
$x+2y=2$	$\frac{1}{2}u+\frac{1}{2}v-\frac{1}{2}w+\frac{1}{2}u-\frac{1}{2}v+\frac{1}{2}w=2$	$u=2$
$x-z=0$	$\frac{1}{2}u+\frac{1}{2}v-\frac{1}{2}w-\frac{1}{2}u+\frac{1}{2}v+\frac{1}{2}w=0$	$v=0$
$x-z=2$	$\frac{1}{2}u+\frac{1}{2}v-\frac{1}{2}w-\frac{1}{2}u+\frac{1}{2}v+\frac{1}{2}w=2$	$v=2$
$2y-z=0$	$\frac{1}{2}u-\frac{1}{2}v+\frac{1}{2}w-\frac{1}{2}u+\frac{1}{2}v+\frac{1}{2}w=0$	$w=0$
$2y-z=3$	$\frac{1}{2}u-\frac{1}{2}v+\frac{1}{2}w-\frac{1}{2}u+\frac{1}{2}v+\frac{1}{2}w=3$	$w=3$

$$\iiint_D yz\,dV=\int_1^2\int_0^2\int_0^3\frac{1}{8}(u-v+w)(u-v-w)\frac{1}{4}\ dwdvdu$$

$$=\frac{1}{32}\int_1^2\int_0^2\int_0^3\left((u-v)^2+w^2\right)\ dwdvdu$$

$$=\frac{1}{32}\int_1^2\int_0^2\left(\left(u^2-2uv+v^2\right)w+\frac{1}{3}w^3\right)\Big|_0^3\ dvdu$$

$$=\frac{1}{32}\int_1^2\int_0^2\left(3u^2-6uv+3v^2+9\right)\ dvdu$$

$$=\frac{1}{32}\int_1^2\left(3u^2v-3uv^2+v^3+9v\right)\Big|_0^2\ du$$

$$=\frac{1}{32}\int_1^2\left(6u^2-12u+8+18\right)\ du$$

$$=\frac{1}{32}\int_1^2\left(6u^2-12u+26\right)\ du$$

$$= \frac{1}{32}\left(2u^3 - 6u^2 + 26u\right)\Big|_1^2$$

$$= \frac{1}{16}(8 - 12 + 26 - 1 + 3 - 13)$$

$$\boxed{= \frac{11}{16}}$$

Exercise

Evaluate $\iint\limits_R xy\ dA$; R is the square with vertices $(0,\ 0)$, $(1,\ 1)$, $(2,\ 0)$, and $(1,\ -1)$

Solution

$(0,\ 0)\ \&\ (1,\ 1) \to y = x \quad \Rightarrow y - x = 0$

$(0,\ 0)\ \&\ (2,\ 0) \to y = 0$

$(0,\ 0)\ \&\ (1,\ -1) \to y = -x \quad \Rightarrow y + x = 0$

$(1,\ 1)\ \&\ (2,\ 0) \to y = -x + 2 \quad \Rightarrow y + x = 2$

$(1,\ 1)\ \&\ (1,\ -1) \to x = 1$

$(2,\ 0)\ \&\ (1,\ -1) \to y = x - 2 \quad \Rightarrow y - x = -2$

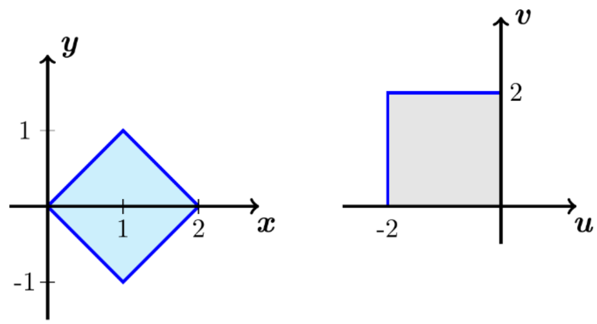
Let $\begin{cases} u = y - x \\ v = y + x \end{cases}$

$\overline{u + v = 2y} \quad \to \quad \boxed{y = \frac{1}{2}(u + v)}$

$x = v - y$

$\quad = v - \frac{1}{2}u - \frac{1}{2}v$

$\quad \boxed{= \frac{1}{2}v - \frac{1}{2}u}$



$$\begin{cases} x = -\frac{1}{2}u + \frac{1}{2}v \\ y = \frac{1}{2}u + \frac{1}{2}v \end{cases}$$

$y - x = 0$	$\frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}u - \frac{1}{2}v = 0$	$u = 0$
$y - x = -2$	$\frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}u - \frac{1}{2}v = -2$	$u = -2$
$y + x = 0$	$\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}u + \frac{1}{2}v = 0$	$v = 0$
$y + x = 2$	$\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}u + \frac{1}{2}v = 2$	$v = 2$

$$J(u,\ v) = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} \qquad\qquad J(u,\ v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= -\frac{1}{2}$$

$$\iint\limits_R xy\ dA = \int_0^2 \int_{-2}^2 \left(-\frac{1}{2}u + \frac{1}{2}v\right)\left(\frac{1}{2}u + \frac{1}{2}v\right)\left|-\frac{1}{2}\right| dudv$$

$$= \frac{1}{8}\int_0^2 \int_{-2}^2 (v-u)(v-u)\,dudv$$

$$= \frac{1}{8}\int_0^2 \int_{-2}^2 \left(v^2 - u^2\right) dudv$$

$$= \frac{1}{8}\int_0^2 \left(v^2u - \frac{1}{3}u^3\right)\Big|_{-2}^{2}\ dv$$

$$= \frac{1}{8}\int_0^2 \left(2v^2 - \frac{8}{3} + 2v^2 - \frac{8}{3}\right) dv$$

$$= \frac{1}{8}\int_0^2 \left(4v^2 - \frac{16}{3}\right) dv$$

$$= \frac{1}{2}\left(\frac{1}{3}v^3 - \frac{4}{3}v\right)\Big|_0^2$$

$$= \frac{1}{6}(8-8)$$

$$= 0$$

Exercise

Evaluate $\iint\limits_R x^2y\ dA$; $R = \{(x,\ y):\ \ 0 \le x \le 2,\quad x \le y \le x+4\}$

Solution

$y = x \quad \to \quad y - x = 0$

$y = x + 4 \quad \to \quad y - x = 4$

Let $\begin{cases} u = x \\ v = y - x \end{cases} \qquad y = u + v$

$$\begin{cases} x = u \\ y = u + v \end{cases}$$

$$J(u,\ v) = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \qquad\qquad J(u,\ v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= 1$$

$x = 0$		$u = 0$
$x = 2$		$u = 2$
$y - x = 0$	$u + v - u = 0$	$v = 0$
$y - x = 4$	$u + v - u = 4$	$v = 4$

$$\iint_R x^2 y\ dA = \int_0^4 \int_0^2 u^2 (u + v)|1|\ dudv$$

$$= \int_0^4 \int_0^2 \left(u^3 + vu^2\right) dudv$$

$$= \int_0^4 \left(\frac{1}{4}u^4 + \frac{1}{3}vu^3\right)\Big|_0^2\ dv$$

$$= \int_0^4 \left(4 + \frac{8}{3}v\right) dv$$

$$= \left(4v + \frac{4}{3}v^2\right)\Big|_0^4$$

$$= 16 + \frac{64}{3}$$

$$= \frac{112}{3}$$

Exercise

Evaluate $\displaystyle\iint_R x^2 \sqrt{x + 2y}\ dA$; $R = \left\{(x,\ y):\ \ 0 \le x \le 2,\ \ -\frac{x}{2} \le y \le 1 - x\right\}$

Solution

$$y = -\frac{1}{2}x \ \rightarrow\ 2y + x = 0$$

$$y = 1 - x \ \rightarrow\ y + x = 1$$

Let $\begin{cases} u = \frac{1}{2}x \\ v = y + x \end{cases}$

$\begin{cases} x = 2u \\ y = -2u + v \end{cases}$

$$J(u,\ v) = \begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} \qquad\qquad J(u,\ v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= 2$$

$x = 0$	$2u = 0$	$u = 0$
$x = 2$	$2u = 2$	$u = 1$
$y + x = 1$	$v = 1$	$v = 1$
$y = -\frac{1}{2}x$	$-2u + v = -u$	$v = u$

$$\iint_R x^2\sqrt{x+2y}\ dA = \int_0^1 \int_u^1 4u^2\sqrt{2u-4u+2v}\,|2|\ dvdu$$

$$= 4\int_0^1 \int_u^1 u^2(-2u+2v)^{1/2}\ d(-2u+2v)\,du$$

$$= \frac{8}{3}\int_0^1 u^2(-2u+2v)^{3/2}\Big|_u^1\ du$$

$$= \frac{8}{3}\int_0^1 u^2\left((-2u+2)^{3/2} - 0\right)du$$

$$= \frac{16\sqrt{2}}{3}\int_0^1 u^2(1-u)^{3/2}\ du$$

Let $w = 1-u \ \rightarrow\ dw = -du$

$u = 1-w$

$$= -\frac{16\sqrt{2}}{3}\int_0^1 (1-w)^2\ w^{3/2}dw$$

$$= -\frac{16\sqrt{2}}{3}\int_0^1 \left(1-2w+w^2\right)w^{3/2}dw$$

$$= -\frac{16\sqrt{2}}{3}\int_0^1 \left(w^{3/2} - 2w^{5/2} + w^{7/2}\right)dw$$

$$= -\frac{16\sqrt{2}}{3}\left(\frac{2}{5}(1-u)^{5/2} - \frac{4}{7}(1-u)^{7/2} + \frac{2}{9}(1-u)^{9/2}\right)\Bigg|_0^1$$

$$= -\frac{16\sqrt{2}}{3}\left(-\frac{2}{5} + \frac{4}{7} - \frac{2}{9}\right)$$

$$= -\frac{16\sqrt{2}}{3}\left(\frac{-126+184-70}{315}\right)$$

$$= -\frac{16\sqrt{2}}{3}\left(-\frac{16}{315}\right)$$

$$= \frac{256\sqrt{2}}{945}$$

Exercise

Evaluate $\iint\limits_R xy\ dA$; where R is bounded by the ellipse $9x^2 + 4y^2 = 36$.

Solution

$u^2 + v^2 = 1 \quad \rightarrow \quad v = \pm\sqrt{1-u^2}$

$-1 \le u \le 1$

$\frac{x^2}{4} + \frac{y^2}{9} = 1 \quad \rightarrow \quad \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$$\begin{cases} \frac{x}{2} = u & \rightarrow \quad x = 2u \\ \frac{y}{3} = v & \rightarrow \quad y = 3v \end{cases}$$

$$J(u,\ v) = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix}$$

$$= 6$$

$$\iint\limits_R xy\ dA = \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} (2u)(3v)|6|\ dvdu$$

$$= 18\int_{-1}^{1} uv^2 \Big|_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\ du$$

$$= 36\int_{-1}^{1} u\left(1-u^2\right)du$$

$$= 36\int_{-1}^{1}\left(u-u^3\right)du$$

$$= 36\left(\frac{1}{2}u^2-\frac{1}{4}u^4\right)\Big|_{-1}^{1}$$

$$= 36\left(\frac{1}{2}-\frac{1}{4}-\frac{1}{2}+\frac{1}{4}\right)$$

$$\underline{= 0}\,\rfloor$$

Exercise

Evaluate $= \int_0^1\int_{2u-2}^{2u}\sqrt{u+u-v}\,|1|\,dvdu$

Solution

$$x = y, \quad x = y+2$$

$$0 \le x \le 1$$

$$\begin{cases} u = x \\ v = x+y \end{cases} \rightarrow \begin{cases} x = u \\ y = -u+v \end{cases}$$

$$J(u,\ v) = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}$$

$$\underline{= 1}\,\rfloor$$

$x = 0$		$u = 0$
$x = 1$		$u = 1$
$x = y$	$u = -u+v$	$v = 2u$
$x = y+2$	$u = -u+v+2$	$v = 2u-2$

$$\int_0^1\int_y^{y+2}\sqrt{x-y}\ dxdy = \int_0^1\int_{2u-2}^{2u}\sqrt{u+u-v}\,|1|\ dvdu$$

$$= -\int_0^1\int_{2u-2}^{2u}(2u-v)^{1/2}\ d(2u-v)\,du$$

$$= -\frac{2}{3}\int_0^1 (2u-v)^{3/2}\Big|_{2u-2}^{2u}\ du$$

$$= \frac{2}{3}\int_0^1 (2)^{3/2}\ du$$

$$\underline{= \frac{4\sqrt{2}}{3}}\,\rfloor$$

Exercise

Evaluate $\iint\limits_R \sqrt{y^2-x^2}\ dA$; where R is the diamond bounded by $y-x=0,\ \ y-x=2,\ \ y+x=0$, and $y+x=2$

Solution

$$\begin{cases} u=y-x \\ v=y+x \end{cases} \quad \to \quad \begin{cases} x=-\frac{1}{2}u+\frac{1}{2}v \\ y=\frac{1}{2}u+\frac{1}{2}v \end{cases}$$

$$J(u,\ v)=\begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$=-\frac{1}{2}$$

$y-x=0$	$\frac{1}{2}u+\frac{1}{2}v+\frac{1}{2}u-\frac{1}{2}v=0$	$u=0$
$y-x=2$	$\frac{1}{2}u+\frac{1}{2}v+\frac{1}{2}u-\frac{1}{2}v=2$	$u=2$
$y+x=0$	$\frac{1}{2}u+\frac{1}{2}v-\frac{1}{2}u+\frac{1}{2}v=0$	$v=0$
$y+x=2$	$\frac{1}{2}u+\frac{1}{2}v-\frac{1}{2}u+\frac{1}{2}v=2$	$v=2$

$$\iint\limits_R \sqrt{y^2-x^2}\ dA=\int_0^2\int_0^2\left|-\frac{1}{2}\right|\sqrt{\left(\frac{1}{2}u+\frac{1}{2}v\right)^2-\left(-\frac{1}{2}u+\frac{1}{2}v\right)^2}\ dvdu$$

$$=\frac{1}{8}\int_0^2\int_0^2\sqrt{u^2+2uv+v^2-u^2+2uv-v^2}\ dvdu$$

$$=\frac{1}{4}\int_0^2\int_0^2\sqrt{uv}\ dvdu$$

$$=\frac{1}{4}\int_0^2\int_0^2\frac{1}{u}(uv)^{1/2}\ d(uv)\,du$$

$$=\frac{1}{6}\int_0^2\frac{1}{u}(uv)^{3/2}\Big|_0^2\ du$$

$$=\frac{\sqrt{2}}{3}\int_0^2\frac{1}{u}(u)^{3/2}\ du$$

$$= \frac{\sqrt{2}}{3} \int_0^2 (u)^{1/2}\, du$$

$$= \frac{2\sqrt{2}}{9} (u)^{3/2} \Big|_0^2$$

$$\underline{= \frac{16}{9}}$$

Exercise

Evaluate $\iint\limits_R \left(\frac{y-x}{y+2x+1} \right)^4 dA$; where R is the parallelogram bounded by $y - x = 1$, $y - x = 2$, $y + 2x = 0$, and $y + 2x = 4$

Solution

Let $\begin{cases} u = y - x \\ v = y + 2x \end{cases} \rightarrow \begin{cases} x = -\frac{1}{3}u + \frac{1}{3}v \\ y = \frac{2}{3}u + \frac{1}{3}v \end{cases}$

$$J(u,\ v) = \begin{vmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix}$$

$$\underline{= -1}$$

$y - x = 1$	$\frac{2}{3}u + \frac{1}{3}v + \frac{1}{3}u - \frac{1}{3}v = 1$	$u = 1$
$y - x = 2$	$\frac{2}{3}u + \frac{1}{3}v + \frac{1}{3}u - \frac{1}{3}v = 2$	$u = 2$
$y + 2x = 0$	$\frac{2}{3}u + \frac{1}{3}v - \frac{2}{3}u + \frac{2}{3}v = 0$	$v = 0$
$y + 2x = 4$	$\frac{2}{3}u + \frac{1}{3}v - \frac{2}{3}u + \frac{2}{3}v = 4$	$v = 4$

$$\iint\limits_R \left(\frac{y-x}{y+2x+1} \right)^4 dA = \int_0^4 \int_1^2 |-1| \left(\frac{\frac{1}{3}(2u + v + u - v)}{\frac{1}{3}(2u + v - 2u + 2v) + 1} \right)^4 dudv$$

$$= \int_0^4 \int_1^2 \left(\frac{u}{v+1} \right)^4 dudv$$

$$= \int_0^4 (v+1)^{-4}\, d(3v+1) \int_1^2 u^4\, du$$

$$= -\frac{1}{3}(v+1)^{-3}\Big|_0^4 \quad \left(\frac{1}{5}u^5\right)\Big|_1^2$$

$$= -\frac{1}{15}\left(\frac{1}{125}-1\right)(32)$$

$$= -\frac{32}{15}\left(-\frac{124}{125}\right)$$

$$= \boxed{\frac{3,968}{1,875}}$$

Exercise

Evaluate $\iint\limits_R e^{xy}\ dA$; where R is the region bounded by $xy=1,\ \ xy=4,\ \ \frac{y}{x}=1$, and $\frac{y}{x}=3$

Solution

Let $\begin{cases} u = xy \\ v = \frac{y}{x} \end{cases} \to \begin{array}{l} y = \frac{u}{x} \\ v = \frac{u}{x^2} \end{array}$

$$\begin{cases} x = \sqrt{\frac{u}{v}} \\ y = \sqrt{uv} \end{cases}$$

$$J(u,\ v) = \begin{vmatrix} \frac{1}{2\sqrt{uv}} & -\frac{\sqrt{u}}{v^{3/2}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{vmatrix}$$

$$= \frac{1}{4v} + \frac{1}{4v}$$

$$= \boxed{\frac{1}{2v}}$$

$xy=1$	$\sqrt{\frac{u}{v}}\sqrt{uv}=1$	$u=1$
$xy=4$	$\sqrt{\frac{u}{v}}\sqrt{uv}=4$	$u=4$
$\frac{y}{x}=1$	$\sqrt{\frac{v}{u}}\sqrt{uv}=1$	$v=1$
$\frac{y}{x}=3$	$\sqrt{\frac{v}{u}}\sqrt{uv}=3$	$v=3$

$$\iint\limits_R e^{xy}\ dA = \int_1^3\int_1^4 \left|\frac{1}{2v}\right| e^{\sqrt{\frac{u}{v}}\sqrt{uv}}\ dudv$$

$$= \frac{1}{2} \int_1^3 \frac{1}{v} dv \ \int_1^4 e^u \ du$$

$$= \frac{1}{2} \ln v \Big|_1^3 \ \ e^u \Big|_1^4$$

$$= \frac{\ln 3}{2}\left(e^4 - e\right)$$

Exercise

Evaluate $\iint\limits_R xy \ dA$; where R is the region bounded by the hyperbolas $xy = 1$, $xy = 4$, $y = 1$, and $y = 3$

Solution

Let $\begin{cases} u = xy \\ v = y \end{cases} \rightarrow \begin{aligned} x &= \frac{u}{v} \\ y &= v \end{aligned}$

$$J(u, v) = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix}$$

$$= \frac{1}{v}$$

$xy = 1$	$u = 1$
$xy = 4$	$u = 4$
$y = 1$	$v = 1$
$y = 3$	$v = 3$

$$\iint\limits_R xy \ dA = \int_1^3 \int_1^4 \left|\frac{1}{v}\right| u \ dudv$$

$$= \int_1^3 \frac{1}{v} dv \ \int_1^4 u \ du$$

$$= \ln v \Big|_1^3 \ \ \frac{1}{2} u^2 \Big|_1^4$$

$$= \frac{15}{2} \ln 3$$

Exercise

Evaluate $\iint\limits_R (x-y)\sqrt{x-2y}\ dA$; where R is the triangular region bounded by $y=0,\ x-2y=0,$ and $x-y=1$

Solution

Let $\begin{cases} u = x-2y \\ v = y \end{cases} \rightarrow \begin{aligned} x &= u+2v \\ y &= v \end{aligned}$

$$J(u,\ v) = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$$

$$\underline{=1}$$

$y=0$		$v=0$
$x-2y=0$		$u=0$
$x-y=1$	$u+2v-v=1$	$v=1-u$

$$\iint\limits_R (x-y)\sqrt{x-2y}\ dA = \int_0^1 \int_0^{1-u} (u+v)\sqrt{u}\ dvdu$$

$$= \int_0^1 \int_0^{1-u} \left(u^{3/2} + vu^{1/2}\right) dvdu$$

$$= \int_0^1 \left(u^{3/2}v + \frac{1}{2}u^{1/2}v^2\right)\Big|_0^{1-u} du$$

$$= \int_0^1 \left(u^{3/2}(1-u) + \frac{1}{2}u^{1/2}\left(1-2u+u^2\right)\right) du$$

$$= \int_0^1 \left(u^{3/2} - u^{5/2} + \frac{1}{2}u^{1/2} - u^{3/2} + \frac{1}{2}u^{5/2}\right) du$$

$$= \int_0^1 \left(\frac{1}{2}u^{1/2} - \frac{1}{2}u^{5/2}\right) du$$

$$= \frac{1}{3}u^{3/2} - \frac{1}{7}u^{7/2}\Big|_0^1$$

$$= \frac{1}{3} - \frac{1}{7}$$

$$\underline{= \frac{4}{21}}$$

Exercise

Evaluate $\iiint_D xy\ dV$: D is bounded by the planes: $y-x=0,\ \ y-x=2,\ \ z-y=0,\ \ z-y=2,$

$z=0,$ and $z=3$

Solution

Let: $\begin{cases} u=y-x \\ v=z-y \\ w=z \end{cases} \quad \to \begin{cases} x=-u-v+w \\ y=-v+w \end{cases}$

$$J(u,\ v,\ w)=\begin{vmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\underline{=1}$$

$y-x=0$	$u=0$
$y-x=2$	$u=2$
$z-y=0$	$v=0$
$z-y=2$	$v=2$
$z=0$	$w=0$
$z=3$	$w=3$

$$\iiint_D xydV = \int_0^3\int_0^2\int_0^2 (w-u-v)(w-v)\ dudvdw$$

$$=\int_0^3\int_0^2\int_0^2 \left(w^2-2vw-uw+uv+v^2\right)\ dudvdw$$

$$=\int_0^3\int_0^2 \left(w^2u-2vwu-\frac{1}{2}wu^2+\frac{1}{2}vu^2+v^2u\right)\Big|_0^2\ dvdw$$

$$=\int_0^3\int_0^2 \left(2w^2-4vw-2w+2v+2v^2\right)dvdw$$

$$=\int_0^3 \left(2w^2v-2wv^2-2wv+v^2+\frac{2}{3}v^3\right)\Big|_0^2\ dw$$

$$=\int_0^3 \left(4w^2-12w+\frac{28}{3}\right)dw$$

$$=\left(\frac{4}{3}w^3-6w^2+\frac{28}{3}w\right)\Big|_0^3$$

$$=36-54+28$$

$$\underline{=10}$$

Exercise

Evaluate $\iiint_D dV$: D is bounded by the planes: $y-2x=0,\ \ y-2x=1,\ \ z-3y=0,\ \ z-3y=1,$

$z-4x=0,$ and $z-4x=3$

Solution

Let: $\begin{cases} u=y-2x \\ v=z-3y \\ w=z-4x \end{cases} \rightarrow \begin{cases} x=\frac{1}{2}y-\frac{1}{2}u \\ z=v+3y \\ \end{cases}$

$w=v+3y-2y+2u$

$\begin{cases} y=-2u-v+w \\ x=-\frac{3}{2}u-\frac{1}{2}v+\frac{1}{2}w \\ z=-6u-2v+3w \end{cases}$

$$J(u,\ v,\ w)=\begin{vmatrix} -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ -2 & -1 & 1 \\ -6 & -2 & 3 \end{vmatrix}$$

$$=\frac{9}{2}+3+2-3-3-3$$

$$=\frac{1}{2}$$

$y-2x=0$	$u=0$
$y-2x=1$	$u=1$
$z-3y=0$	$v=0$
$z-3y=1$	$v=1$
$z-x=0$	$w=0$
$z-4x=3$	$w=3$

$$\iiint_D dV=\int_0^3 dw\ \int_0^1 dv\ \int_0^1 \frac{1}{2}\ du$$

$$=\frac{1}{2}(3)(1)(1)$$

$$=\frac{3}{2}$$

Exercise

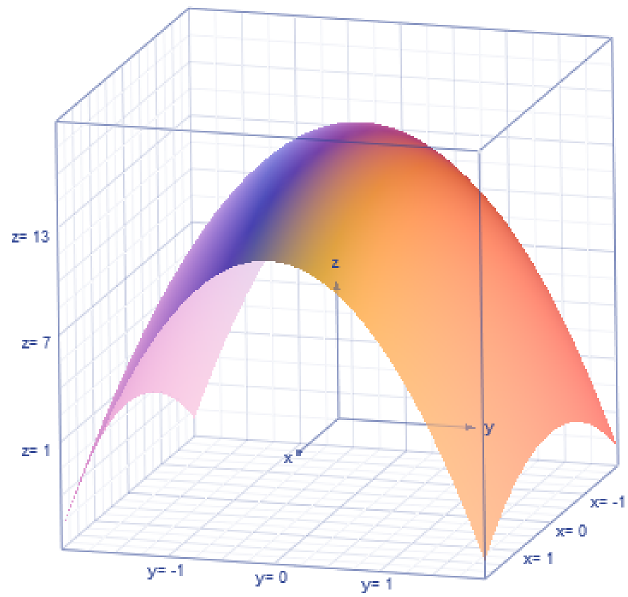
Evaluate $\iiint_D z\,dV$: D is bounded by the paraboloid $z = 16 - x^2 - 4y^2$ and the xy-plane.

Solution

$$z = 16 - x^2 - 4y^2 = 0$$

$$x^2 + 4y^2 = 16 \quad \rightarrow \quad \left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

Let: $\begin{cases} u = \frac{x}{4} \\ v = \frac{y}{2} \\ w = z \end{cases} \quad \begin{cases} x = 4u \\ y = 2v \\ z = w \end{cases}$



$$u^2 + v^2 = 1 \quad \rightarrow \quad v = \pm\sqrt{1-u^2}$$

$$\underline{-1 \le u \le 1\,|}$$

$$w = z = 16 - 16u^2 - 16v^2$$

$$J(u,\ v,\ w) = \begin{vmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\underline{= 8\,|}$$

$$\iiint_D z\,dV = \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\int_0^{16-16u^2-16v^2} 8w\,dw\,dv\,du$$

$$= 4\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} w^2 \Big|_0^{16-16u^2-16v^2}\,dv\,du$$

$$= 4\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} 16^2\left(1-\left(u^2+v^2\right)\right)^2\,dv\,du$$

$$= 1{,}024\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\left(1 - 2\left(u^2+v^2\right) + \left(u^2+v^2\right)^2\right)dv\,du$$

$$= 1{,}024\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\left(1 - 2u^2 - 2v^2 + u^4 + 2u^2v^2 + v^4\right)dv\,du$$

$$=1,024\int_{-1}^{1}\left(v-2u^2v-\frac{2}{3}v^3+u^4v+\frac{2}{3}u^2v^3+\frac{1}{5}v^5\right)\Bigg|_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\ du$$

$$=1,024\int_{-1}^{1}\left(\left(1-2u^2+u^4\right)v-\frac{2}{3}\left(1-u^2\right)v^3+\frac{1}{5}v^5\right)\Bigg|_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\ du$$

$$=2,048\int_{-1}^{1}\left(\left(1-u^2\right)^2\left(1-u^2\right)^{1/2}-\frac{2}{3}\left(1-u^2\right)^{5/2}+\frac{1}{5}\left(1-u^2\right)^{5/2}\right)du$$

$$=2,048\left(\frac{8}{15}\right)\int_{-1}^{1}\left(1-u^2\right)^{5/2}du$$

$$u=\sin t\quad\rightarrow\quad du=\cos t\,dt$$

$$\begin{cases} u=1=\sin t & \rightarrow t=\frac{\pi}{2} \\ u=-1=\sin t & \rightarrow t=-\frac{\pi}{2}\end{cases}$$

$$=2,048\left(\frac{8}{15}\right)\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos^5 t\cos t\ dt$$

$$=2,048\left(\frac{8}{15}\right)\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos^6 t\ dt$$

$$=\frac{2,048}{15}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\left(1+\cos 2t\right)^3\ dt$$

$$=\frac{2,048}{15}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\left(1+3\cos 2t+3\cos^2 2t+\cos^3 2t\right)\ dt$$

$$=\frac{2,048}{15}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\left(\frac{5}{2}+3\cos 2t+\frac{3}{2}\cos 4t+\cos^2 2t\left(\sin 2t\right)\right)\ dt$$

$$\int\cos^2 2t\left(\sin 2t\right)dt=-\frac{1}{2}\int\cos^2 2t\ d\left(\cos 2t\right)$$

$$=-\frac{1}{6}\cos^3 2t$$

$$=\frac{2,048}{15}\left(\frac{5}{2}t+\frac{3}{2}\sin 2t+\frac{3}{8}\sin 4t-\frac{1}{6}\cos^3 2t\right)\Bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= \frac{2,048}{15}\left(\frac{15\pi}{4} + \frac{1}{6} - \frac{5\pi}{4} - \frac{1}{6}\right)$$

$$= \frac{2,048}{15}\left(\frac{5\pi}{2}\right)$$

$$= \frac{1,024\pi}{3}$$

Exercise

Evaluate $\iiint_D dV : D$ is bounded by the upper half of the ellipsoid $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$ and the *xy*-plane.

Solution

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 + z^2 = 1$$

Let: $\begin{cases} u = \frac{x}{3} \\ v = \frac{y}{2} \\ w = z \end{cases} \qquad \begin{cases} x = 3u \\ y = 2v \\ z = w \end{cases}$

$$J(u,\ v,\ w) = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 6$$

$$u^2 + v^2 + w^2 = 1$$

$$-\sqrt{1 - v^2 - w^2} \le u \le \sqrt{1 - v^2 - w^2}$$

$$-\sqrt{1 - w^2} \le v \le \sqrt{1 - w^2} \quad \& \quad 0 \le w \le 1$$

$$\iiint_D dV = \int_0^1 \int_{-\sqrt{1-w^2}}^{\sqrt{1-w^2}} \int_{-\sqrt{1-v^2-w^2}}^{\sqrt{1-v^2-w^2}} (6)\ dudvdw$$

$$= 6\int_0^1 \int_{-\sqrt{1-w^2}}^{\sqrt{1-w^2}} u \Big|_{-\sqrt{1-v^2-w^2}}^{\sqrt{1-v^2-w^2}} dvdw$$

$$= 12\int_0^1 \int_{-\sqrt{1-w^2}}^{\sqrt{1-w^2}} \sqrt{1 - \left(v^2 + w^2\right)}\ dvdw$$

$$v^2 + w^2 = r^2 \quad \rightarrow \quad 0 \le r \le 1 \quad 0 \le \theta \le \pi$$

$$= 12 \int_0^{\pi} \int_0^1 \sqrt{1-r^2} \; r \; dr d\theta$$

$$= -6 \int_0^{\pi} d\theta \;\; \int_0^1 \left(1-r^2\right)^{1/2} d\left(1-r^2\right)$$

$$= -6\pi \frac{2}{3} \left(1-r^2\right)^{3/2} \Big|_0^1$$

$$= 4\pi$$

Exercise

Evaluate $\iiint_D xz \; dV : D$ is bounded by the planes: $y = x, \;\; y = x + 2, \;\; x - z = 0, \;\; z = x + 3, \;\; z = 0,$ and $z = 4$

Solution

$y - x = 0 \quad y - x = 2$

$z - x = 0 \quad z - x = 3$

Let: $\begin{cases} u = y - x \\ v = z - x \\ w = z \end{cases} \quad \rightarrow \begin{cases} y = u - v + w \\ x = -v + w \end{cases}$

$$J(u, \; v, \; w) = \begin{vmatrix} 0 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1$$

$y = x$	$u - v + w = w - v$	$u = 0$
$y = x + 2$	$u - v + w = w - v + 2$	$u = 2$
$x - z = 0$	$w - v - w = 0$	$v = 0$
$z = x + 3$	$w + v - w = 3$	$v = 3$
$z = 0$		$w = 0$
$z = 4$		$w = 4$

$$\iiint_D xzdV = \int_0^2 \int_0^3 \int_0^4 (w - v)(w) \; dwdvdu$$

$$= \int_0^2 du \;\; \int_0^3 \int_0^4 \left(w^2 - vw\right) \; dwdv$$

$$= 2\int_0^3 \left(\frac{1}{3}w^3 - \frac{1}{2}vw^2\right)\Big|_0^4 \, dv$$

$$= 2\int_0^3 \left(\frac{64}{3} - 8v\right) dv$$

$$= 2\left(\frac{64}{3}v - 4v^2\right)\Big|_0^3$$

$$= 18(64 - 36)$$

$$\underline{= 56}$$

Exercise

Let R be the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$ are real numbers.

Find the area of R.

Solution

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Let: $\begin{cases} u = \frac{x}{a} & \rightarrow \ x = au \\ v = \frac{y}{b} & \rightarrow \ y = bv \end{cases}$

$$J(u,\ v) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}$$

$$\underline{= ab}$$

$u^2 + v^2 = 1 \quad \rightarrow \quad u = \sqrt{1 - v^2}$

$-\sqrt{1 - v^2} \le u \le \sqrt{1 - v^2} \quad \& \quad -1 \le v \le 1$

Since, $u^2 + v^2 = 1$ is a unit circle, then the area $\pi r^2 = \pi$. Therefore, the area of the ellipse is $ab\pi$

$$\iint_R dA = \int_{-1}^{1} \int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} ab \; dudv$$

$$= ab\int_{-1}^{1} u \Big|_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} \, dv$$

$$= 2ab\int_{-1}^{1} \sqrt{1 - v^2} \; dv$$

$$v = \sin t \quad \rightarrow \quad dv = \cos t dt$$

$$\begin{cases} v = 1 = \sin t & \rightarrow t = \frac{\pi}{2} \\ v = -1 = \sin t & \rightarrow t = -\frac{\pi}{2} \end{cases}$$

$$= 2ab \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t \ dt$$

$$= ab \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos 2t) \ dt$$

$$= ab\left(t + \frac{1}{2}\sin 2t\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= ab\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$\underline{ab\pi}$$

Exercise

Let R be the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$ are real numbers.

Evaluates $\iint_R |xy| \, dA$

Solution

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Let: $\begin{cases} u = \frac{x}{a} & \rightarrow x = au \\ v = \frac{y}{b} & \rightarrow y = bv \end{cases}$

$$J(u, v) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}$$

$$\underline{= ab}$$

$$u^2 + v^2 = 1 \quad \rightarrow \quad u = \sqrt{1 - v^2}$$

$$-\sqrt{1 - v^2} \le u \le \sqrt{1 - v^2} \quad \& \quad -1 \le v \le 1$$

$$\iint_R |xy| \, dA = \int_{-1}^{1} \int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} |abuv| \, ab \ dudv$$

$$= 4a^2b^2 \int_0^1 \int_0^{\sqrt{1-v^2}} uv\ dudv$$

$$= 2a^2b^2 \int_0^1 vu^2 \Big|_0^{\sqrt{1-v^2}} dv$$

$$= 2a^2b^2 \int_0^1 v\left(1-v^2\right) dv$$

$$= 2a^2b^2 \int_0^1 \left(v-v^3\right) dv$$

$$= 2a^2b^2 \left(\frac{1}{2}v^2 - \frac{1}{4}v^4\right)\Big|_0^1$$

$$= 2a^2b^2 \left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \frac{1}{2}a^2b^2$$

Exercise

Let R be the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$ are real numbers.

Find the center of mass of the upper half of R $(y \ge 0)$ assuming it has a constant density.

Solution

Let: $\begin{cases} u = \frac{x}{a} & \to\ x = au \\ v = \frac{y}{b} & \to\ y = bv \end{cases}$

$$J(u,\ v) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}$$

$$= ab$$

$u^2 + v^2 = 1 \quad \to \quad v = \sqrt{1-u^2}$

$0 \le v \le \sqrt{1-u^2} \quad \& \quad -1 \le u \le 1$

Since, $u^2 + v^2 = 1$ is a unit circle, then the area $\pi r^2 = \pi$. Therefore, the area of the ellipse is $ab\pi$

Mass of the upper half is given by:

$$m = \frac{1}{2}\pi ab$$

By symmetry, $\overline{x} = 0$

$$\bar{y} = \frac{2}{\pi ab}\int_{-1}^{1}\int_{0}^{\sqrt{1-u^2}} aby\ dvdu$$

$$= \frac{2}{\pi}\int_{-1}^{1}\int_{0}^{\sqrt{1-u^2}} bv\ dvdu$$

$$= \frac{b}{\pi}\int_{-1}^{1} v^2 \Big|_{0}^{\sqrt{1-u^2}}\ du$$

$$= \frac{b}{\pi}\int_{-1}^{1}\left(1-u^2\right)du$$

$$= \frac{b}{\pi}\left(u-\frac{1}{3}u^3\right)\Big|_{-1}^{1}$$

$$= \frac{2b}{\pi}\left(1-\frac{1}{3}\right)$$

$$= \frac{4b}{3\pi}$$

$\therefore$ the center of mass of the upper half of R is $\left(0,\ \ \frac{4b}{3\pi}\right)$

Exercise

Let R be the region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>0$ and $b>0$ are real numbers.

Find the average square of the distance between points of R and the origin.

Solution

Let: $\begin{cases} u=\frac{x}{a} & \rightarrow\ x=au \\ v=\frac{y}{b} & \rightarrow\ y=bv \end{cases}$

The distance between points of R and the origin is $d=\sqrt{x^2+y^2}$

$$d=\sqrt{a^2u^2+b^2v^2}$$

$$J(u,\ v)=\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}$$

$$= ab$$

$$u^2+v^2=1 \quad \rightarrow \quad v=\sqrt{1-u^2}$$

$$0\le v\le\sqrt{1-u^2} \quad \& \quad -1\le u\le 1$$

Average square of the distance is:

$$avg = \frac{1}{\pi ab}\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} ab\left(a^2u^2 + b^2v^2\right)\, dvdu$$

$$= \frac{1}{\pi}\int_{-1}^{1}\left(a^2u^2v + \frac{1}{3}b^2v^3\right)\Big|_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\, du$$

$$= \frac{2}{\pi}\int_{-1}^{1}\left(a^2u^2\left(1-u^2\right)^{1/2} + \frac{1}{3}b^2\left(1-u^2\right)^{3/2}\right)du$$

$$u = \sin t \quad \rightarrow \quad du = \cos t dt$$

$$\begin{cases} u = 1 = \sin t & \rightarrow t = \frac{\pi}{2} \\ u = -1 = \sin t & \rightarrow t = -\frac{\pi}{2} \end{cases}$$

$$\int u^2\left(1-u^2\right)^{1/2}du = \int \sin^2 t\cos t\cos t\; dt$$

$$= \frac{1}{4}\int (1-\cos 2t)(1+\cos 2t)\,dt$$

$$= \frac{1}{4}\int \left(1-\cos^2 2t\right)dt$$

$$= \frac{1}{4}\int \left(\frac{1}{2}-\frac{1}{2}\cos 4t\right)dt$$

$$= \frac{1}{8}\left(t-\frac{1}{4}\sin 4t\right)$$

$$\int \left(1-u^2\right)^{3/2}du = \int \cos^4 t\; dt$$

$$= \frac{1}{4}\int (1+\cos 2t)^2\; dt$$

$$= \frac{1}{4}\int \left(1+2\cos 2t+\cos^2 2t\right)dt$$

$$= \frac{1}{4}\int \left(\frac{3}{2}+2\cos 2t+\frac{1}{2}\cos 4t\right)dt$$

$$= \frac{1}{4}\left(\frac{3}{2}t+\sin 2t+\frac{1}{8}\sin 4t\right)$$

$$= \frac{2}{\pi}\left(\frac{a^2}{8}\left(t-\frac{1}{4}\sin 4t\right)+\frac{b^2}{12}\left(\frac{3}{2}t+\sin 2t+\frac{1}{8}\sin 4t\right)\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{4}{\pi}\left(\frac{a^2}{8}\frac{\pi}{2} + \frac{b^2}{8}\frac{\pi}{2}\right)$$

$$= \frac{a^2 + b^2}{4}$$

Exercise

Let R be the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$ are real numbers.

Find the average distance between points in the upper half of R and the x-axis.

Solution

Let: $\begin{cases} u = \frac{x}{a} & \rightarrow \ x = au \\ v = \frac{y}{b} & \rightarrow \ y = bv \end{cases}$

The distance between points in the upper half of R and the x-axis is $d = y = bv$

$$J(u,\ v) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}$$

$$= ab$$

$$u^2 + v^2 = 1 \quad \rightarrow \quad v = \sqrt{1-u^2}$$

$$0 \le v \le \sqrt{1-u^2} \quad \& \quad -1 \le u \le 1$$

$$avg = \frac{2}{\pi ab}\int_{-1}^{1}\int_{0}^{\sqrt{1-u^2}} ab^2 v \ dvdu$$

$$= \frac{b}{\pi}\int_{-1}^{1} v^2 \Big|_0^{\sqrt{1-u^2}} \ du$$

$$= \frac{b}{\pi}\int_{-1}^{1} \left(1-u^2\right)du$$

$$= \frac{2b}{\pi}\left(u - \frac{1}{3}u^3\right)\Big|_0^1$$

$$= \frac{2b}{\pi}\left(1 - \frac{1}{3}\right)$$

$$= \frac{4b}{3\pi}$$

Exercise

Let D be the region bounded by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, where $a>0$, $b>0$ and $c>0$ are real numbers. Find the Volume of D.

Solution

Let: $\begin{cases} u=\frac{x}{a} & \rightarrow\ x=au \\ v=\frac{y}{b} & \rightarrow\ y=bv \\ w=\frac{z}{c} & \rightarrow\ z=cw \end{cases}$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \rightarrow \quad u^2+v^2+w^2=1$$

$$J(u,\ v)=\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

$$\underline{=abc\ |}$$

Since, $u^2+v^2+w^2=1$ is a unit sphere, then the volume $\frac{4\pi r^2}{3}=\frac{4\pi}{3}$. Therefore, the volume of the ellipsoid is $\frac{4}{3}abc\pi$

Exercise

Let D be the region bounded by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, where $a>0$, $b>0$ and $c>0$ are real numbers. Evaluates $\iint_V |xyz|\,dV$

Solution

Let: $\begin{cases} u=\frac{x}{a} & \rightarrow\ x=au \\ v=\frac{y}{b} & \rightarrow\ y=bv \\ w=\frac{z}{c} & \rightarrow\ z=cw \end{cases}$

$$J(u,\ v)=\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

$$\underline{=abc\ |}$$

$$u^2+v^2+w^2=1 \quad \rightarrow \quad w=\sqrt{1-u^2-v^2}$$

$$-\sqrt{1-u^2-v^2} \le w \le \sqrt{1-u^2-v^2}$$

$$-\sqrt{1-u^2} \le v \le \sqrt{1-u^2} \quad \& \quad -1 \le u \le 1$$

$$\iint_R |xyz|\, dA = \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\int_{-\sqrt{1-u^2-v^2}}^{\sqrt{1-u^2-v^2}} (aubvcw)\,abc \;\; dwdvdu$$

$$= 4a^2b^2c^2 \int_0^1\int_0^{\sqrt{1-u^2}} uvw^2 \Big|_0^{\sqrt{1-u^2-v^2}} \; dvdu$$

$$= 4a^2b^2c^2 \int_0^1\int_0^{\sqrt{1-u^2}} \left(uv-u^3v-uv^3\right) dvdu$$

$$= 4a^2b^2c^2 \int_0^1 \left(\frac{1}{2}\left(u-u^3\right)v^2-\frac{1}{4}uv^4\right)\Big|_0^{\sqrt{1-u^2}} \; du$$

$$= a^2b^2c^2 \int_0^1 \left(2u\left(1-u^2\right)\left(1-u^2\right)-u\left(1-u^2\right)^2\right) du$$

$$= a^2b^2c^2 \int_0^1 \left(1-u^2\right)^2 (2u-u)\, du$$

$$= a^2b^2c^2 \int_0^1 u\left(1-u^2\right)^2 du$$

$$= -\frac{1}{2}a^2b^2c^2 \int_0^1 \left(1-u^2\right)^2 d\left(1-u^2\right)$$

$$= -\frac{1}{6}a^2b^2c^2 \left(1-u^2\right)^3 \Big|_0^1$$

$$= \frac{1}{6}a^2b^2c^2$$

Exercise

Let D be the region bounded by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, where $a>0$, $b>0$ and $c>0$ are real numbers. Find the center of mass of the upper half of D $(z\geq 0)$ assuming it has a constant density.

Solution

Let: $\begin{cases} u=\frac{x}{a} & \rightarrow\ x=au \\ v=\frac{y}{b} & \rightarrow\ y=bv \\ w=\frac{z}{c} & \rightarrow\ z=cw \end{cases}$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad\rightarrow\quad u^2+v^2+w^2=1$$

$$J(u,\ v)=\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

$$=abc$$

Since, $u^2+v^2+w^2=1$ is a unit sphere, then the volume $\frac{4\pi r^2}{3}=\frac{4\pi}{3}$. Therefore, the volume of the ellipsoid is $\frac{4}{3}abc\pi$

$m=\frac{1}{2}\frac{4}{3}abc\pi$ (upper half of D)

$$m=\frac{2}{3}abc\pi$$

By symmetry $\overline{x}=\overline{y}=0$

$$\overline{z}=\frac{3}{2\pi abc}\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\int_{0}^{\sqrt{1-u^2-v^2}}(abc)(cw)\ dwdvdu$$

$$=\frac{3c}{4\pi}\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} w^2\Big|_0^{\sqrt{1-u^2-v^2}}\ dvdu$$

$$=\frac{3c}{4\pi}\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\left(1-u^2-v^2\right)dvdu$$

$$=\frac{3c}{2\pi}\int_{-1}^{1}\left(\left(1-u^2\right)v-\frac{1}{3}v^3\right)\Big|_0^{\sqrt{1-u^2}}\ du$$

$$= \frac{3c}{2\pi} \int_{-1}^{1} \left(\left(1-u^2\right)^{3/2} - \frac{1}{3}\left(1-u^2\right)^{3/2} \right) du$$

$$= \frac{c}{\pi} \int_{-1}^{1} \left(1-u^2\right)^{3/2} du$$

$$u = \sin t \quad \to \quad du = \cos t dt$$

$$\begin{cases} u = 1 = \sin t & \to t = \frac{\pi}{2} \\ u = -1 = \sin t & \to t = -\frac{\pi}{2} \end{cases}$$

$$\begin{aligned} \int \left(1-u^2\right)^{3/2} du &= \int \cos^4 t \; dt \\ &= \frac{1}{4} \int (1+\cos 2t)^2 \, dt \\ &= \frac{1}{4} \int \left(1 + 2\cos 2t + \cos^2 2t\right) dt \\ &= \frac{1}{4} \int \left(\frac{3}{2} + 2\cos 2t + \frac{1}{2}\cos 4t\right) dt \\ &= \frac{1}{4}\left(\frac{3}{2}t + \sin 2t + \frac{1}{8}\sin 4t\right) \end{aligned}$$

$$= \frac{c}{4\pi} \left(\frac{3}{2}t + \sin 2t + \frac{1}{8}\sin 4t\right) \Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{c}{4\pi} \left(2 \frac{3}{2} \frac{\pi}{2}\right)$$

$$= \frac{3c}{8}$$

Exercise

Let D be the region bounded by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, where $a > 0$, $b > 0$ and $c > 0$ are real numbers. Find the average square of the distance between points of D and the origin.

Solution

Let: $\begin{cases} u = \frac{x}{a} & \to x = au \\ v = \frac{y}{b} & \to y = bv \\ w = \frac{z}{c} & \to z = cw \end{cases}$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \to \quad u^2+v^2+w^2=1$$

$$J(u,\ v)=\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

$$\boxed{= abc}$$

The distance between a point on D and the origin is

$$d=\sqrt{x^2+y^2+z^2}$$

$$=\sqrt{a^2u^2+b^2v^2+c^2w^2}$$

Since, $u^2+v^2+w^2=1$ is a unit sphere, then the volume $\frac{4\pi r^2}{3}=\frac{4\pi}{3}$. Therefore, the volume of the ellipsoid is $\frac{4}{3}abc\pi$

$m=\frac{1}{2}\frac{4}{3}abc\pi$ (upper half of D)

$$\boxed{m=\frac{2}{3}abc\pi}$$

$$avg=\frac{1}{2}\frac{3}{2\pi abc}\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\int_{-\sqrt{1-u^2-v^2}}^{\sqrt{1-u^2-v^2}}\left(a^2u^2+b^2v^2+c^2w^2\right)(abc)\ dwdvdu$$

$$=\frac{3}{4\pi}\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\int_{-\sqrt{1-u^2-v^2}}^{\sqrt{1-u^2-v^2}}\left(a^2u^2+b^2v^2+c^2w^2\right)\ dwdvdu$$

Let $u=r\cos\theta \quad v=r\sin\theta$

$$avg=\frac{3}{4\pi}\int_0^{2\pi}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}}\left(a^2r^2\cos^2\theta+b^2r^2\sin^2\theta+c^2z^2\right)\ r\ dzdrd\theta$$

$$=\frac{3}{2\pi}\int_0^{2\pi}\int_0^1\left(\left(a^2\cos^2\theta+b^2\sin^2\theta\right)r^3z+\frac{1}{3}rc^2z^3\right)\Bigg|_0^{\sqrt{1-r^2}}\ drd\theta$$

$$=\frac{3}{2\pi}\int_0^{2\pi}\int_0^1\left(\left(a^2\cos^2\theta+b^2\sin^2\theta\right)r^3\left(1-r^2\right)^{1/2}+\frac{1}{3}rc^2\left(1-r^2\right)^{3/2}\right)drd\theta$$

$$r=\sin t \quad \to \quad dr=\cos t\,dt$$

$$\begin{cases} r=1=\sin t & \to t=\frac{\pi}{2} \\ r=0=\sin t & \to t=0 \end{cases}$$

$$\int_0^1 r^3\left(1-r^2\right)^{1/2} dr = \int_0^{\frac{\pi}{2}} \sin^3 t \cos^2 t dt$$

$$= -\int_0^{\frac{\pi}{2}} \left(1-\cos^2 t\right)\cos^2 t \; d\left(\cos t\right)$$

$$= -\int_0^{\frac{\pi}{2}} \left(\cos^2 t - \cos^4 t\right) d\left(\cos t\right)$$

$$= \frac{1}{5}\cos^5 t - \frac{1}{3}\cos^3 t \Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{5}+\frac{1}{3}$$

$$\boxed{= \frac{2}{15}}$$

$$= \frac{1}{5\pi}\int_0^{2\pi}\left(a^2\cos^2\theta + b^2\sin^2\theta\right)d\theta - \frac{c^2}{4\pi}\int_0^{2\pi} d\theta \int_0^1 \left(1-r^2\right)^{3/2} d\left(1-r^2\right)$$

$$= \frac{1}{10\pi}\int_0^{2\pi}\left(a^2\left(1+\cos 2\theta\right) + b^2\left(1-\cos 2\theta\right)\right)d\theta - \frac{c^2}{5}\left(1-r^2\right)^{5/2}\Big|_0^1$$

$$= \frac{1}{10\pi}\left(a^2\left(\theta+\frac{1}{2}\sin 2\theta\right) + b^2\left(\theta - \frac{1}{2}\sin 2\theta\right)\right)\Big|_0^{2\pi} + \frac{c^2}{5}$$

$$= \frac{1}{10\pi}\left(2\pi a^2 + 2\pi b^2\right) + \frac{c^2}{5}$$

$$\boxed{= \frac{a^2+b^2+c^2}{5}}$$