Surface

***Surface* of a curve** $y = f(x)$ is given by the formula:

$$S = 2\pi \int_a^b f(x)\sqrt{1+\left(f'(x)\right)^2}\, dx$$

If $f(x) = ax^m + bx^n$, then

$$\sqrt{1+\left(f'(x)\right)^2} = \overline{f'(x)}$$

$\overline{f'(x)}$: is the conjugate of $f'(x)$

Iff $f(x)$ satisfies these 2 conditions:

1. $m+n=2$
2. $abmn = -\frac{1}{4}$

Proof

$$f'(x) = max^{m-1} + nbx^{n-1}$$

$$1+\left(f'\right)^2 = 1+\left(max^{m-1}+nbx^{n-1}\right)^2$$

$$= 1+m^2a^2x^{2m-2} + 2abmnx^{m+n-2} + n^2b^2x^{2n-2}$$

We need to combined to a perfect square $a^2 - 2ab + b^2 = (a-b)^2$

- If $x^{m+n-2} = 1 = x^0 \rightarrow \boxed{m+n=2}$

$$= m^2a^2x^{2m-2} + (1+2abmn) + n^2b^2x^{2n-2}$$ $a^2 - 2ab + b^2 = (a-b)^2$

- Let $1+2abmn = -2abmn \rightarrow \boxed{abmn = -\frac{1}{4}}$

$$= m^2a^2x^{2m-2} - 2abmn + n^2b^2x^{2n-2}$$ $x^{2(m+n-2)} = 1$

$$= \left(max^{m-1} - nbx^{n-1}\right)^2$$

$$\sqrt{\left(max^{m-1}-nbx^{n-1}\right)^2} = max^{m-1} - nbx^{n-1}$$ ✓

$$f'(x) = max^{m-1} + nbx^{n-1} \Rightarrow \sqrt{1+\left(f'(x)\right)^2} = max^{m-1} - nbx^{n-1} = \overline{f'(x)}$$

Example

Find the surface of the graph of $f(x)=\frac{x^3}{12}+\frac{1}{x}, \quad 1\le x\le 4$

Solution

$$f'(x)=\frac{x^2}{4}-\frac{1}{x^2}$$

$$\begin{aligned}1+\left[f'(x)\right]^2 &= 1+\left(\frac{x^2}{4}-\frac{1}{x^2}\right)^2\\ &= 1+\frac{x^4}{16}-\frac{1}{2}+\frac{1}{x^4}\\ &= \frac{x^4}{16}+\frac{1}{2}+\frac{1}{x^4}\\ &= \left(\frac{x^2}{4}+\frac{1}{x^2}\right)^2\end{aligned}$$

$a=\frac{1}{12}, \quad m=3, \quad b=1, \quad n=-1$

1. $m+n=3-1=2$ ✓

2. $abmn=\frac{1}{12}(1)(3)(-1)=-\frac{1}{4}$ ✓

$$\begin{aligned}S &= 2\pi\int_1^4 f(x)\sqrt{1+\left(f'(x)\right)^2}\,dx\\ &= 2\pi\int_1^4\left(\frac{x^3}{12}+\frac{1}{x}\right)\sqrt{\left(\frac{x^2}{4}+\frac{1}{x^2}\right)^2}\,dx\\ &= 2\pi\int_1^4\left(\frac{x^3}{12}+\frac{1}{x}\right)\left(\frac{x^2}{4}+\frac{1}{x^2}\right)dx\\ &= 2\pi\int_1^4\left(\frac{1}{48}x^5+\frac{1}{12}x+\frac{1}{4}x+x^{-3}\right)dx\\ &= 2\pi\left(\frac{1}{288}x^6+\frac{1}{6}x^2-\frac{1}{2x^2}\right)\Big|_1^4\\ &= \pi\left(\frac{256}{9}+\frac{16}{3}-\frac{1}{16}-\frac{1}{144}-\frac{1}{3}+1\right)\end{aligned}$$

$$S=2\pi\int_1^4\left(\frac{x^3}{12}+\frac{1}{x}\right)\left(\frac{x^2}{4}+\frac{1}{x^2}\right)dx$$

$$\boxed{=\frac{275}{8}\pi \quad unit^2}$$

If $f(x) = ae^{mx} + be^{nx}$, then

$$S = 2\pi \int_a^b f(x)\sqrt{1+\left(f'(x)\right)^2}\ dx = 2\pi \int_a^b f(x)\,\overline{f'(x)}\ dx$$

Iff $f(x)$ satisfies these 2 conditions:

1. $m = -n$
2. $abmn = -\frac{1}{4}$

Proof

$$f'(x) = ame^{mx} + bne^{nx}$$

$$1+\left(f'\right)^2 = 1+\left(ame^{mx} + bne^{nx}\right)^2$$

$$= 1 + m^2a^2e^{2mx} + 2abmne^{(m+n)x} + n^2b^2e^{2nx}$$

- If $e^{(m+n)x} = 1 = e^{(x=0)} \quad \rightarrow \quad \boxed{m-n}$

$$= m^2a^2x^{2m-2} + (1+2abmn) + n^2b^2x^{2n-2} \qquad a^2 - 2ab + b^2 = (a-b)^2$$

- Let $1 + 2abmn = -2abmn \quad \rightarrow \quad \boxed{abmn = -\frac{1}{4}}$

$$= m^2a^2e^{2mx} - 2abmne^{(m+n)x} + n^2b^2e^{2nx} \qquad x^{2(m+n-2)} = 1$$

$$= \left(ame^{mx} - bne^{nx}\right)^2$$

$$\sqrt{1+\left(f'\right)^2} = \sqrt{\left(ame^{mx} - bne^{nx}\right)^2}$$

$$\sqrt{1+f'(x)} = \overline{f'(x)} \qquad \checkmark$$