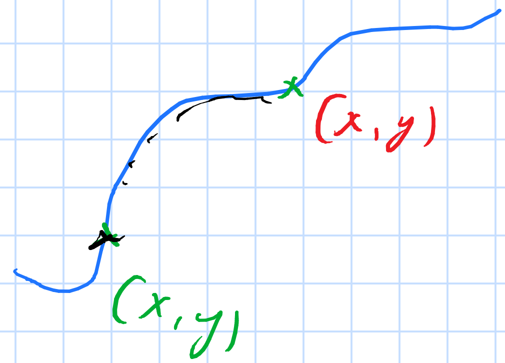
Lecture 4

4.1

$$(x, y)$$

$$(x, y) = (f(t),\ g(t))$$

$$x \xrightarrow{f} y = f(x)$$



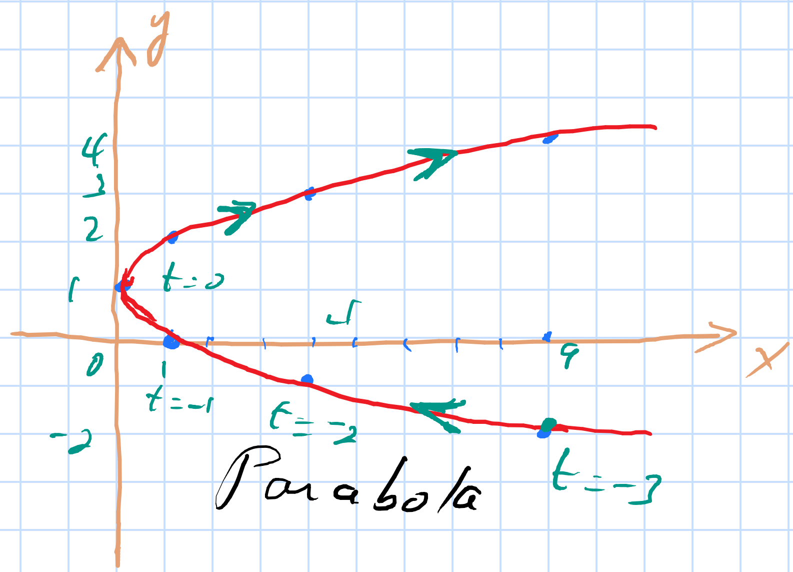
$$t \xrightarrow{g} x = g(t)$$

$$t \xrightarrow{h} y = h(t)$$

Ex

$$\begin{cases} x = t^2 & (1) \\ y = t + 1 & (2) \end{cases} \qquad -\infty < t < \infty$$

t	x	y
-3	9	-2
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3
3	9	4



(2) $\rightarrow$ $t = y - 1$

(1) $\rightarrow$ $x = (y-1)^2$

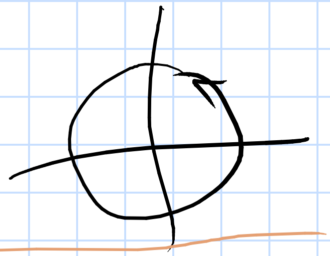
$$\boxed{x = y^2 - 2y + 1}$$

Ex $x = \cos t \quad y = \sin t \quad 0 \le t \le 2\pi$

$\cos^2 t + \sin^2 t = 1$

$x^2 + y^2 = 1$

$\therefore$ it's a circle of radius 1 & center @ Origin. (CCW)



Ex $x = a\cos t \quad y = a\sin t \quad 0 \le t \le 2\pi$

$\cos t = \frac{x}{a} \qquad \sin t = \frac{y}{a}$

$\cos^2 t + \sin^2 t = 1$

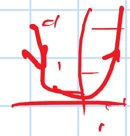
$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$

$x^2 + y^2 = a^2$

it's circle of radius a center @ origin

traces entire circle & once (CCW)

Ex $x = t \quad y = t^2 \quad -\infty \le t \le \infty$



$y = x^2$ it's a parabola with domain $\mathbb{R}$.

Ex parameterization (pt) point (a, b) having slope m

$$y = m(x-a) + b$$

let $x - a = t \Rightarrow \begin{cases} x = t + a \\ y = mt + b \end{cases} \quad t \in \mathbb{R}$

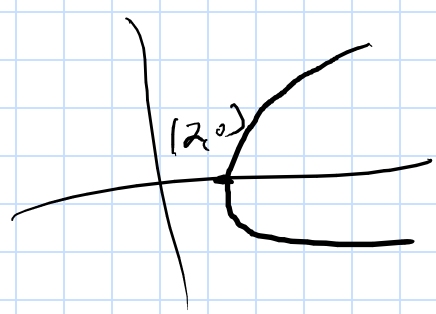
Ex $x = t + \frac{1}{t} \ge 0 \quad y = t - \frac{1}{t} \quad t > 0$

$$x + y = t + \frac{1}{t} + t - \frac{1}{t} = 2t$$

$$x = \frac{1}{2}(x+y) + \frac{2}{x+y}$$

$$t = \frac{x+y}{2}$$

$$x - y = t + \frac{1}{t} - t + \frac{1}{t} = \frac{2}{t}$$



$$(2t)\left(\frac{2}{t}\right) = 4 = (x+y)(x-y)$$

$x^2 - y^2 = 4 \Rightarrow y^2 = x^2 - 4$

$\frac{x^2}{4} - \frac{y^2}{4} = 1$ hyperbolic fctn $a = 2, b = 2$

$\boxed{x = 2} \Rightarrow x \ge 2$

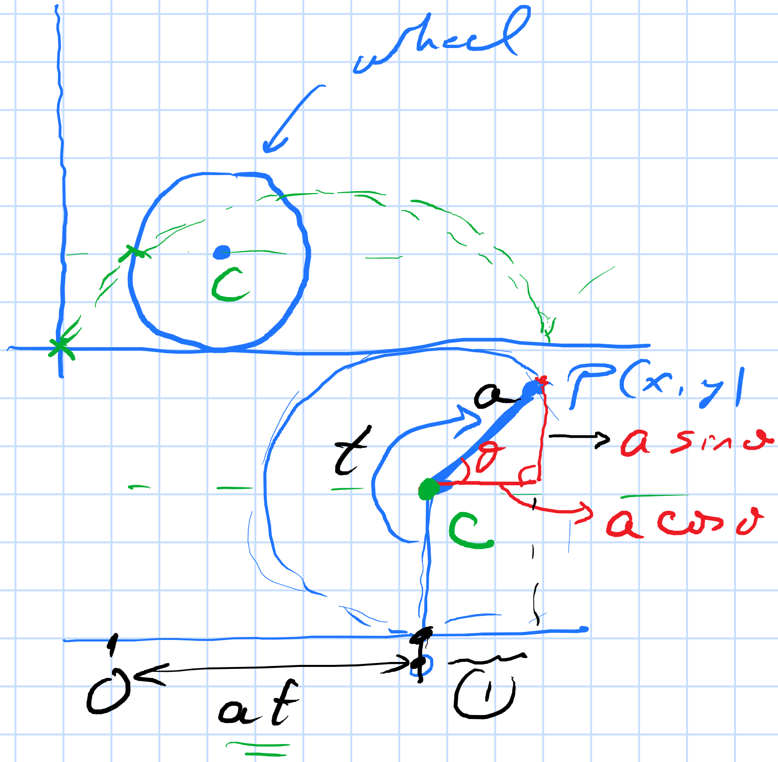
$y \in \mathbb{R}$

Cycloids

let radius $= a$

$$\boxed{s = r\theta}$$

$$s = a\,t$$



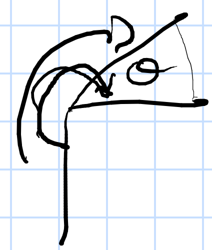
$$C\,(at, a)$$

$$\begin{cases} x = at + a\cos\theta \\ y = a + a\sin\theta \end{cases}$$

$$\begin{cases} \cos\theta = \dfrac{x - at}{a} \\ \sin\theta = \dfrac{y - a}{a} \end{cases}$$

$$t + \theta = \frac{3\pi}{2}$$

$$\theta = \frac{3\pi}{2} - t$$



$$\begin{cases} x = at + a\cos\left(\frac{3\pi}{2} - t\right) \\ y = a + a\sin\left(\frac{3\pi}{2} - t\right) \end{cases}$$

$$\cos\left(\frac{3\pi}{2} - t\right) = \cos\frac{3\pi}{2}\cos t + \sin\frac{3\pi}{2}\sin t$$
$$= -\sin t$$

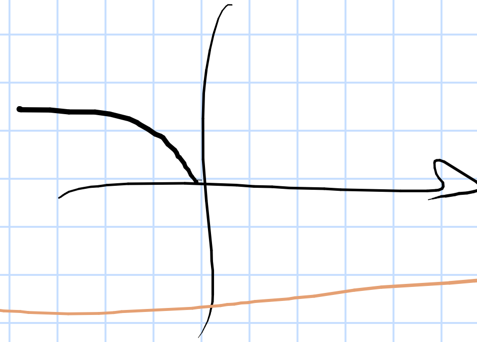
$$\sin\left(\frac{3\pi}{2} - t\right) = \sin\frac{3\pi}{2}\cos t - \cos\frac{3\pi}{2}\sin t$$
$$= -\cos t$$

$$\begin{cases} x = at - a\sin t = a(t - \sin t) \\ y = a - a\cos t = a(1 - \cos t) \end{cases}$$

2/ $x = -\sqrt{t}$ $\quad y = t$ $\quad t \geq 0$

$x \leq 0$ $\quad y \geq 0$

$$x = -\sqrt{t}$$
$$= -\sqrt{y}$$



#16 $x = e^{2t}$ ① $\quad y = e^t + 1$ ② $\quad 0 \leq t \leq 25$

$x = (e^t)^2$

② $\to e^t = y - 1$

$$x = (y-1)^2$$
$$= y^2 - 2y + 1 \quad \text{parabola.}$$

$t = 0 \to y = 2$

$t = 25 \to y = e^{25} + 1$

domain: $[2, e^{25}+1]$

#19 $\begin{cases} x = 1 - 3\sin 4\pi t & ① \\ y = 2 + 3\cos 4\pi t & ② \end{cases}$ $\quad 0 \leq t \leq \frac{1}{2}$

① $\sin 4\pi t = \dfrac{1-x}{3}$

② $\cos 4\pi t = \dfrac{y-2}{3}$

$$(\sin 4\pi t)^2 + \cos^2 4\pi t = 1$$
$$\frac{(1-x)^2}{9} + \frac{(y-2)^2}{9} = 1$$

$$(x-1)^2+(y-2)^2=9$$

circle of radius 3 & center $(1,2)$

$0 \le t \le \frac{1}{2}$ $\quad \begin{cases} x = 1-3\sin 4\pi t \\ y = 2+3\cos 4\pi t \end{cases}$

$t=0 \Rightarrow x=1, \ y=5$

$t=\frac{1}{2} \Rightarrow x=1 \ \ y=5$

same point

an entire circle (ccw)

$x = \ln t \qquad y = 8 \ln t^2 \qquad 1 \le t \le e^2$

$$y = 8\ln(t^2)$$
$$= 16 \ln t$$
$$= 16x$$

$t=1 \to x=0$

$t=e^2 \to x=2$

$0 \le x \le 2$

Sec 4.2 Calculus

$$\frac{dy}{dx} \quad \begin{cases} (1) & y \,\&\, x \quad \text{cartesian} \\ (2) & \end{cases}$$

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = y' \\[2ex] \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{dy'}{dx} = \dfrac{dy'/dt}{dx/dt} \end{cases}$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

Ex Tangent $\begin{cases} x = \sec t \\ y = \tan t \end{cases}$ $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

$t = \frac{\pi}{4} \to (\sqrt{2}, 1)$

$$m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sec^2 t}{\sec t \tan t} = \left.\frac{\sec t}{\tan t}\right|_{t=\frac{\pi}{4}} = \sqrt{2}$$

$$y = \sqrt{2}(x - \sqrt{2}) + 1$$
$$\boxed{= \sqrt{2}x - 1}$$

$\sec^2 t = \tan^2 t + 1$

$x^2 = y^2 + 1$

$2x = 2yy'$

$y' = \frac{x}{y}$

Ex $\quad x = t - t^2 \qquad y = t - t^3$

$$\frac{d^2y}{dx^2} \,? = \frac{dy'/dt}{dx/dt} \leftarrow$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{1-3t^2}{1-2t}$$

$$\frac{dy'}{dt} = \left(\frac{-3t^2+1}{-2t+1}\right)'$$

$$\begin{array}{ccc} -3 & 0 & 1 \\ 0 & -2 & 1 \end{array}$$

$$= \frac{6t^2-6t+2}{(-2t+1)^2}$$

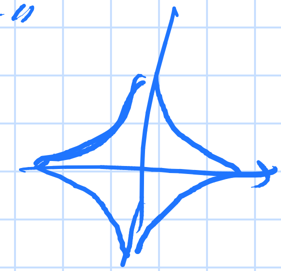
$$\frac{dx}{dt} = 1-2t$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$= \boxed{\frac{6t^2-6t+2}{(-2t+1)^3}}$$

Ex Area?

$$\begin{cases} x = \cos^3 t \\ y = \sin^3 t \end{cases} \quad 0 \le t \le 2\pi$$



$$A = \int_a^b y\,dx$$

$$\frac{dx}{dt} = \frac{d}{dt}(\cos^3 t)$$

$$= -3\sin t\cos^2 t$$

$$dx = -3\sin t\cos^2 t\,dt$$

$$= \int_0^{2\pi} \sin^3 t\,(-3\sin t\cos^2 t)\,dt$$

$$= +3\int_0^{2\pi} \sin^4 t\cos^2 t\,dt$$

$$= +3\int_0^{2\pi} \left(\overset{a-b}{\frac{1-\cos 2t}{2}}\right)^2 \left(\overset{a+b}{\frac{1+\cos 2t}{2}}\right) dt$$

$$= +\frac{3}{8}\int_0^{2\pi} (1-\cos^2 2t)(1-\cos 2t)\,dt$$

$$= +\frac{3}{8}\int_0^{2\pi} \left(1-\cos 2t-\left(\frac{1}{2}+\frac{1}{2}\cos 4t\right)+\cos^3 2t\right) dt$$

$$= +\frac{3}{8}\left[\int_0^{2\pi}\left(\frac{1}{2}-\cos 2t-\frac{1}{2}\cos 4t\right)dt + \int_0^{2\pi} \cos^2 2t\,\cos 2t\,dt\right]$$

$$A = +\frac{3}{8}\left[\frac{1}{2}t - \frac{1}{2}\sin 2t - \frac{1}{8}\sin 4t \Big|_0^{2\pi} + \frac{1}{2}\int_0^{2\pi}(1-\sin^2 2t)\, d(\sin 2t)\right]$$

$$= +\frac{3}{8}\left[\pi + \frac{1}{2}\left(\sin 2t - \frac{1}{3}\sin^3 2t\right)\Big|_0^{2\pi}\right]$$

$$= +\frac{3\pi}{8} \text{ unit}^2$$

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Ex $x = r\cos t \quad y = r\sin t \quad 0 \leq t \leq 2\pi$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-r\sin t)^2 + (r\cos t)^2}$$

$$= \sqrt{r^2\sin^2 t + r^2\cos^2 t}$$

$$= r$$

$$L = \int_0^{2\pi} r\, dt$$

$$= r t \Big|_0^{2\pi}$$

$$= \underline{2\pi r \text{ unit}}$$

$$\boxed{\int_0^a dt = a \qquad \int_0^a d\theta = a}$$

Ex $x=\cos^3 t \quad y=\sin^3 t \quad 0\le t\le 2\pi$

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(-3\cos^2 t\sin t\right)^2+\left(3\sin^2 t\cos t\right)^2$$

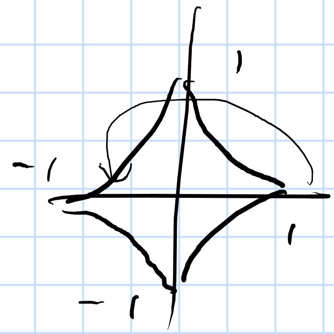
$$=9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t$$

$$=9\sin^2 t\cos^2 t\left(\cos^2 t+\sin^2 t\right)$$

$$=9\sin^2 t\cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=3\sin t\cos t$$

$$=\frac{3}{2}\sin 2t$$



$$L=4\left(\frac{3}{2}\right)\int_0^{\pi/2}\sin 2t\,dt$$

$$=-6\cos 2t\Big|_0^{\pi/2}$$

$$=-3(-1-1)$$

$$=\underline{6 \text{ unit}}$$