Solution   *Section* 1.6 – Exact Differential Equations

Exercise

Solve the differential equation $\quad (2x+y)dx+(x-6y)dy=0$

Solution

$$\frac{\partial\psi}{\partial x}=M=2x+y \quad\Rightarrow\quad M_y=1$$

$$\frac{\partial\psi}{\partial y}=N=x-6y \quad\Rightarrow\quad N_x=1$$

$$\Rightarrow M_y=N_x$$

$$\frac{\partial\psi}{\partial x}=2x+y \quad\Rightarrow\quad \psi=\int(2x+y)dx=x^2+xy+h(y)$$

$$\psi_y=x+h'(y)=x-6y \quad\Rightarrow\quad h'(y)=-6y$$

$$h(y)=\int -6y\,dy=-3y^2$$

$$\psi(x,y)=\underline{x^2+xy-3y^2=C}$$

Exercise

Solve the differential equation $\quad (2x+3)dx+(2y-2)dy=0$

Solution

$$\frac{\partial\psi}{\partial x}=M=2x+3 \quad\Rightarrow\quad M_y=2$$

$$\frac{\partial\psi}{\partial y}=N=2y-2 \quad\Rightarrow\quad N_x=2$$

$$\Rightarrow M_y=N_x$$

$$\frac{\partial\psi}{\partial x}=2x+3 \quad\Rightarrow\quad \psi=\int(2x+3)dx=x^2+3x+h(y)$$

$$\psi_y=h'(y)=2y-2 \quad\Rightarrow\quad h(y)=\int(2y-2)dy$$

$$=y^2-2y+C$$

$$\psi(x,y)=\underline{x^2+3x+y^2-2y=C}$$

Exercise

Solve the differential equation $\left(1-y\sin x\right)+\left(\cos x\right)y'=0$

Solution

$$\frac{\partial\psi}{\partial x}=M=1-y\sin x \quad\Rightarrow\quad M_y=-\sin x$$

$$\frac{\partial\psi}{\partial y}=N=\cos x \quad\Rightarrow\quad N_x=-\sin x$$

$$\Rightarrow M_y=N_x$$

$$\frac{\partial\psi}{\partial x}=1-y\sin x \quad\Rightarrow\quad \psi=\int\left(1-y\sin x\right)dx=x+y\cos x+h(y)$$

$$\psi_y=\cos x+h'(y)=\cos x \quad\Rightarrow\quad h'(y)=0$$

$$h(y)=C$$

$$\psi(x,\, y)=\underline{x+y\cos x=C}$$

Exercise

Solve the differential equation $\dfrac{dy}{dx}=-\dfrac{ax+by}{bx+cy}$

Solution

$$\left(bx+cy\right)dy=-\left(ax+by\right)dx$$

$$\left(ax+by\right)dx+\left(bx+cy\right)dy=0$$

$$\frac{\partial\psi}{\partial x}=M=ax+by \quad\Rightarrow\quad M_y=b$$

$$\frac{\partial\psi}{\partial y}=N=bx+cy \quad\Rightarrow\quad N_x=b$$

$$\Rightarrow M_y=N_x$$

$$\frac{\partial\psi}{\partial x}=ax+by \quad\Rightarrow\quad \psi=\int\left(ax+by\right)dx=\frac{1}{2}ax^2+bxy+h(y)$$

$$\psi_y=bx+h'(y)=bx+cy \quad\Rightarrow\quad h'(y)=cy$$

$$h(y)=\int cydy=\frac{1}{2}cy^2$$

$$\psi(x,\, y)=\frac{1}{2}ax^2+bxy+\frac{1}{2}cy^2=D$$

$$\boxed{ax^2+2bxy+cy^2=E} \qquad (E=2D)$$

Exercise

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{3x^2 + y}{3y^2 - x}$

Solution

$$\left(3x^2 + y\right)dx - \left(3y^2 - x\right)dy = 0$$

$$\left.\begin{aligned} \frac{\partial \psi}{\partial x} &= M = 3x^2 + y \quad \Rightarrow \quad M_y = 1 \\ \frac{\partial \psi}{\partial y} &= N = -3y^2 + x \quad \Rightarrow \quad N_x = 1 \end{aligned}\right\} \Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = 3x^2 + y \quad \Rightarrow \quad \psi = \int \left(3x^2 + y\right)dx = x^3 + xy + h(y)$$

$$\psi_y = x + h'(y) = -3y^2 + x \quad \Rightarrow \quad h'(y) = -3y^2$$

$$h(y) = \int -3y^2 dy = -y^3$$

$$\psi(x, y) = \underline{x^3 + xy - y^2 = C}$$

Exercise

Solve the differential equation $2xydx + \left(x^2 - 1\right)dy = 0$

Solution

$$M(x, y) = 2xy \quad N(x, y) = x^2 - 1$$

$$\frac{\partial M}{\partial y} = 2x \quad \frac{\partial N}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int 2xy\ dx = x^2 y + h(y)$$

$$\psi_y = x^2 + h'(y) = x^2 - 1$$

$$h'(y) = -1 \quad \Rightarrow \quad h(y) = -y + C$$

$$\underline{x^2 y - y = C}$$

Exercise

Find the general solution $y' = \dfrac{x^2 + y^2}{2xy}$

Solution

Let $y = xv \quad \Rightarrow \quad y' = v + xv'$

$$v + xv' = \frac{x^2 + x^2 v^2}{2x^2 v}$$

$$xv' = \frac{1+v^2}{2v} - v$$

$$x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\frac{2v}{1-v^2}dv = \frac{dx}{x}$$

$$-\int \frac{1}{1-v^2} d\left(1-v^2\right) = \int \frac{dx}{x}$$

$$-\ln\left|1-v^2\right| = \ln|x| + \ln C$$

$$\ln\frac{1}{\left|1-v^2\right|} = \ln|\text{C}\,x|$$

$$\frac{1}{1-\left(\frac{y}{x}\right)^2} = \text{C}\,x$$

$$\frac{x^2}{x^2 - y^2} = \text{C}\,x$$

$$\frac{x}{\text{C}} = x^2 - y^2$$

$$\boxed{y^2 = x^2 - C_1 x}$$

Exercise

Find the general solution $\quad 2xyy' = x^2 + 2y^2$

Solution

Let $y = xv \quad \Rightarrow \quad y' = v + xv'$

$$2x(xv)(v + xv') = x^2 + 2x^2v^2 \qquad \textit{Divide both side by } x^2$$

$$2v^2 + 2xvv' = 1 + 2v^2$$

$$2xv\frac{dv}{dx} = 1$$

$$\int 2v\,dv = \int \frac{dx}{x}$$

$$v^2 = \ln x + C \qquad \left(v = \frac{y}{x}\right)$$

$$\frac{y^2}{x^2} = \ln x + C$$

$$\boxed{y^2 = x^2(\ln x + C)}$$

Exercise

Find the general solution $\quad xy' = y + 2\sqrt{xy}$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$$x(v + xv') = vx + 2\sqrt{x^2 v}$$

$$x(v + xv') = vx + 2x\sqrt{v} \qquad \textit{Divide both side by } x$$

$$v + xv' = v + 2\sqrt{v}$$

$$x\frac{dv}{dx} = 2\sqrt{v}$$

$$\int \frac{dv}{2\sqrt{v}} = \int \frac{dx}{x}$$

$$\sqrt{v} = \ln x + C \qquad \left(v = \frac{y}{x}\right)$$

$$\sqrt{\frac{y}{x}} = \ln x + C$$

$$\frac{y}{x} = (\ln x + C)^2$$

$$\underline{y = x(\ln x + C)^2}$$

Exercise

Find the general solution $\quad xy^2 y' = x^3 + y^3$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$$x^3 v^2 (v + xv') = x^3 + x^3 v^3 \qquad \textit{Divide both side by } x^3$$

$$v^2(v + xv') = 1 + v^3$$

$$v^3 + xv^2 v' = 1 + v^3$$

$$xv^2 \frac{dv}{dx} = 1$$

$$\int v^2 dv = \int \frac{dx}{x}$$

$$\frac{1}{3}v^3 = \ln x + C \qquad \left(v = \frac{y}{x}\right)$$

$$\frac{y^3}{x^3} = 3(\ln x + C)$$

$$\underline{y^3 = 3x^3(\ln x + C)}$$

Exercise

Find the general solution $x^2 y' = xy + x^2 e^{y/x}$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$$x^2\left(v + xv'\right) = x^2 v + x^2 e^{vx/x}$$ ***Divide both side by*** x^2

$$v + xv' = v + e^{v}$$

$$x\frac{dv}{dx} = e^{v}$$

$$\int e^{-v}dv = \int \frac{dx}{x}$$

$$e^{-v} = \ln x + \ln C \qquad \left(v = \frac{y}{x}\right)$$

$$e^{-\frac{y}{x}} = \ln Cx$$

$$-\frac{y}{x} = \ln\left(\ln Cx\right)$$

$$\boxed{y = -x\ln\left(\ln Cx\right)}$$

Exercise

Find the general solution $x^2 y' = xy + y^2$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$$x^2\left(v + xv'\right) = x^2 v + x^2 v^2$$ ***Divide both side by*** x^2

$$v + xv' = v + v^2$$

$$x\frac{dv}{dx} = v^2$$

$$\int v^{-2}dv = \int \frac{dx}{x}$$

$$-v^{-1} = \ln x + \ln C \qquad \left(v = \frac{y}{x}\right)$$

$$\frac{x}{y} = -\ln Cx$$

$$\frac{x}{y} = \ln\frac{1}{Cx}$$

$$\boxed{y(x) = \frac{x}{\ln\frac{1}{Cx}}}$$

Exercise

Find the general solution $\quad xyy' = x^2 + 3y^2$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$vx^2\left(v + xv'\right) = x^2 + 3x^2v^2$ ***Divide both side by*** x^2

$v^2 + xvv' = 1 + 3v^2$

$xv\dfrac{dv}{dx} = 1 + 2v^2$

$\dfrac{v}{1+2v^2}dv = \dfrac{dx}{x}$

$\dfrac{1}{4}\displaystyle\int \dfrac{1}{1+2v^2}d\left(1+2v^2\right) = \int \dfrac{dx}{x}$

$\dfrac{1}{4}\ln\left(1+2v^2\right) = \ln x + \ln C$

$\ln\left(1+2v^2\right) = 4\ln Cx$

$\ln\left(1 + 2\dfrac{y^2}{x^2}\right) = \ln Cx^4$

$\dfrac{x^2+2y^2}{x^2} = Cx^4$

$\boxed{x^2 + 2y^2 = Cx^6}$

Exercise

Find the general solution $\quad \left(x^2 - y^2\right)y' = 2xy$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$\left(x^2 - v^2x^2\right)\left(v + xv'\right) = 2x^2v$ ***Divide both side by*** x^2

$\left(1 - v^2\right)\left(v + xv'\right) = 2v$

$v + xv' = \dfrac{2v}{1-v^2}$

$x\dfrac{dv}{dx} = \dfrac{2v}{1-v^2} - v$

$x\dfrac{dv}{dx} = \dfrac{v^3+v}{1-v^2}$

$$\int \frac{1-v^2}{v^3+v}dv = \int \frac{dx}{x}$$

$$\frac{1-v^2}{v\left(v^2+1\right)} = \frac{A}{v} + \frac{Bv+C}{v^2+1} = \frac{(A+B)v^2+Cv+A}{v\left(v^2+1\right)}$$

$$\begin{cases} A+B=-1 \\ \quad A=1 \qquad \rightarrow B=-2 \end{cases}$$

$$\int \left(\frac{1}{v} - \frac{2v}{v^2+1}\right)dv = \int \frac{dx}{x}$$

$$\int \frac{1}{v}dv - \int \frac{1}{v^2+1}d\left(v^2+1\right) = \int \frac{dx}{x}$$

$$\ln|v| - \ln\left(v^2+1\right) = \ln|x| + \ln C$$

$$\ln \frac{|v|}{v^2+1} = \ln C\,x$$

$$\frac{y/x}{\frac{y^2}{x^2}+1} = C\,x$$

$$\frac{y}{x}\frac{x^2}{y^2+x^2} = C\,x$$

$$\underline{y(x) = C\left(y^2+x^2\right)|}$$

Exercise

Find the general solution $\qquad xyy' = y^2 + x\sqrt{4x^2+y^2}$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$$vx^2\left(v+xv'\right) = x^2v^2 + x\sqrt{4x^2+v^2x^2}$$

$$vx^2\left(v+xv'\right) = x^2v^2 + x^2\sqrt{4+v^2} \qquad \textit{Divide both side by } x^2$$

$$v^2 + xvv' = v^2 + \sqrt{4+v^2}$$

$$xv\frac{dv}{dx} = \sqrt{4+v^2}$$

$$\int \frac{v}{\sqrt{4+v^2}}dv = \int \frac{dx}{x}$$

$$\frac{1}{2}\int \frac{1}{\sqrt{4+v^2}}d\left(4+v^2\right) = \int \frac{dx}{x}$$

$$\sqrt{4+v^2} = \ln x + C$$

$$\sqrt{4+\frac{y^2}{x^2}} = \ln x + C$$

$$\frac{4x^2+y^2}{x^2} = (\ln x + C)^2$$

$$\boxed{4x^2+y^2 = x^2(\ln x + C)^2}$$

Exercise

Find the general solution $\quad xy' = y + \sqrt{x^2+y^2}$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$$x(v+xv') = xv + \sqrt{x^2+v^2x^2}$$

$$x(v+xv') = xv + x\sqrt{1+v^2}$$ *Divide both side by* x

$$v + xv' = v + \sqrt{1+v^2}$$

$$x\frac{dv}{dx} = \sqrt{1+v^2}$$

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$$

$$\ln\left|v+\sqrt{1+v^2}\right| = \ln x + \ln C$$

$$\ln\left|\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}\right| = \ln \mathrm{C} x$$

$$\frac{y}{x}+\frac{1}{x}\sqrt{x^2+y^2} = \mathrm{C}x$$

$$\boxed{y+\sqrt{x^2+y^2} = \mathrm{C}x^2}$$

Exercise

Find the general solution $\quad y^2y' + 2xy^3 = 6x$

Solution

$$y' + 2xy = 6xy^{-2}$$

Let $\quad u = y^{1+2} = y^3 \quad \Rightarrow \quad y = u^{1/3}$

$$\frac{du}{dx} = 3y^2\frac{dy}{dx} \quad \Rightarrow \quad y' = \frac{1}{3}y^{-2}u' = \frac{1}{3}u^{-2/3}u'$$

$\frac{1}{3}u^{-2/3}u' + 2xu^{1/3} = 6xu^{-2/3}$ ***Multiply both sides by*** $3u^{2/3}$

$u' + 6xu = 18x$

$$e^{\int 6xdx} = e^{3x^2}$$

$$\int 18xe^{3x^2}dx = 3\int e^{3x^2}d\left(3x^2\right) = 3e^{3x^2}$$

$$u = e^{-3x^2}\left(3e^{3x^2} + C\right)$$

$$\boxed{y^3 = 3 + Ce^{-3x^2}}$$

Exercise

Find the general solution $x^2y' + 2xy = 5y^4$

Solution

$y' + 2\frac{1}{x}y = \frac{5}{x^2}y^4$ ***Divide by*** x^2

Let $u = y^{1-4} = y^{-3} \quad \Rightarrow \quad y = u^{-1/3}$

$$\frac{du}{dx} = -3y^{-4}\frac{dy}{dx} \quad \Rightarrow \quad y' = -\frac{1}{3}y^4u' = -\frac{1}{3}u^{-4/3}u'$$

$-\frac{1}{3}u^{-4/3}u' + \frac{2}{x}u^{-1/3} = \frac{5}{x^2}u^{-4/3}$ ***Multiply both sides by*** $-3u^{4/3}$ $u = \frac{1}{x^{-6}}\left(\frac{15}{7}x^{-7} + C\right)$

$u' - \frac{6}{x}u = -\frac{15}{x^2}$

$$e^{\int -\frac{6}{x}dx} = e^{-6\ln x} = e^{\ln x^{-6}} = x^{-6}$$

$$\int x^{-6}\left(-\frac{15}{x^2}\right)dx = -15\int x^{-8}dx = \frac{15}{7}x^{-7}$$

$$y^{-3} = \frac{15 + 7Cx^7}{7x}$$

$$\boxed{y^3 = \frac{7x}{15 + 7Cx^7}}$$

Exercise

Find the general solution $2xy' + y^3e^{-2x} = 2xy$

Solution

$2xy' - 2xy = -e^{-2x}y^3$

$$y' - y = -\frac{e^{-2x}}{2x} y^3 \qquad \textit{Divide by } 2x$$

Let $\quad u = y^{1-3} = y^{-2} \quad \Rightarrow \quad y = u^{-1/2}$

$$\frac{du}{dx} = -2y^{-3}\frac{dy}{dx} \quad \Rightarrow \quad y' = -\frac{1}{2}y^3 u' = -\frac{1}{2}u^{-3/2}u'$$

$$-\frac{1}{2}u^{-3/2}u' - u^{-1/2} = -\frac{e^{-2x}}{2x}u^{-3/2} \qquad \textit{Multiply both sides by } -2u^{3/2}$$

$$u' + 2u = \frac{e^{-2x}}{x}$$

$$e^{\int 2dx} = e^{2x}$$

$$\int \frac{e^{-2x}}{x}e^{2x}dx = \int \frac{dx}{x} = \ln x$$

$$u = \frac{1}{e^{-2x}}(\ln x + C)$$

$$\frac{1}{y^2} = \frac{\ln x + C}{e^{2x}}$$

$$\boxed{y^2 = \frac{e^{2x}}{\ln x + C}}$$

Exercise

Find the general solution $\qquad y^2(xy' + y)\left(1 + x^4\right)^{1/2} = x$

Solution

$$y^2xy' + y^3 = x\left(1 + x^4\right)^{-1/2}$$

$$y' + \frac{1}{x}y = \left(1 + x^4\right)^{-1/2} y^{-2} \qquad \textit{Divide both sides by } xy^2$$

Let $\quad u = y^{1+2} = y^3 \quad \Rightarrow \quad y = u^{1/3}$

$$\frac{du}{dx} = 3y^2\frac{dy}{dx} \quad \Rightarrow \quad y' = \frac{1}{3}y^{-2}u' = \frac{1}{3}u^{-2/3}u'$$

$$\frac{1}{3}u^{-2/3}u' + \frac{1}{x}u^{1/3} = \left(1 + x^4\right)^{-1/2} u^{-2/3} \qquad \textit{Multiply both sides by } 3u^{2/3}$$

$$u' + \frac{3}{x}u = 3\left(1 + x^4\right)^{-1/2}$$

$$e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$$

$$\int 3\left(1 + x^4\right)^{-1/2} x^3 dx = \frac{3}{4}\int \left(1 + x^4\right)^{-1/2} d\left(1 + x^4\right) = \frac{3}{2}\sqrt{1 + x^4}$$

$$u = \frac{1}{x^3}\left(\frac{3}{2}\sqrt{1+x^4} + C\right)$$

$$\boxed{y^3 = \frac{1}{x^3}\left(\frac{3}{2}\sqrt{1+x^4} + C\right)}$$

Exercise

Find the general solution $\qquad 3y^2y' + y^3 = e^{-x}$

Solution

$3y' + y = e^{-x}y^{-2}$ $\qquad$ ***Divide both sides by*** y^2

Let $\quad u = y^{1+2} = y^3 \quad \Rightarrow \quad y = u^{1/3}$

$$\frac{du}{dx} = 3y^2\frac{dy}{dx} \quad \Rightarrow \quad y' = \frac{1}{3}y^{-2}u' = \frac{1}{3}u^{-2/3}u'$$

$u^{-2/3}u' + u^{1/3} = e^{-x}u^{-2/3}$ $\qquad$ ***Multiply both sides by*** $u^{2/3}$

$u' + u = e^{-x}$

$$e^{\int dx} = e^x$$

$$\int e^{-x}e^x dx = \int dx = x$$

$$u = \frac{1}{e^x}(x + C)$$

$$\boxed{y^3 = e^{-x}(x + C)}$$

Exercise

Find the general solution $\qquad 3xy^2y' = 3x^4 + y^3$

Solution

$3xy^2y' - y^3 = 3x^4$ $\qquad$ ***Divide both sides by*** xy^2

$3y' - \frac{1}{x}y = 3x^3y^{-2}$

Let $\quad u = y^{1+2} = y^3 \quad \Rightarrow \quad y = u^{1/3} \quad \Rightarrow y' = \frac{1}{3}u^{-2/3}u'$

$u^{-2/3}u' - \frac{1}{x}u^{1/3} = 3x^3u^{-2/3}$ $\qquad$ ***Multiply both sides by*** $u^{2/3}$

$u' - \frac{1}{x}u = 3x^3$

$$e^{\int \frac{-1}{x}dx} = e^{-\ln x} = x^{-1}$$

$$\int 3x^3x^{-1}dx = \int 3x^2dx = x^3$$

$$u = x\left(x^3 + C\right)$$

$$y^3 = x^4 + Cx$$

$$\boxed{y = \sqrt[3]{x^4 + Cx}}$$

Exercise

Find the general solution $\qquad xe^y y' = 2\left(e^y + x^3 e^{2x}\right)$

Solution

Let $u = e^y \quad \Rightarrow \quad y = \ln u \quad \Rightarrow \; y' = \dfrac{u'}{u}$

$$xu\frac{1}{u}u' = 2u + 2x^3 e^{2x}$$

$$u' - \frac{2}{x}u = 2x^2 e^{2x}$$

$$e^{\int \frac{-2}{x}dx} = e^{-2\ln x} = x^{-2}$$

$$\int 2x^2 e^{2x} x^{-2} dx = 2\int e^{2x} dx = e^{2x}$$

$$u = x^2\left(e^{2x} + C\right)$$

$$e^y = x^2 e^{2x} + Cx^2$$

$$\boxed{y = \ln\left(x^2 e^{2x} + Cx^2\right)}$$

Exercise

Find the general solution $\qquad \left(2x\sin y\cos y\right)y' = 4x^2 + \sin^2 y$

Solution

Let $u = \sin y \quad \Rightarrow \quad u' = \left(\cos y\right)y'$

$$2xuu' = 4x^2 + u^2$$

$$u' = 2x\frac{1}{u} + \frac{1}{2x}u$$

$$u' - \frac{1}{2x}u = 2xu^{-1}$$

Let $\quad v = u^{1+1} = u^2 \quad \Rightarrow \quad u = v^{1/2}$

$$v' = 2uu' \quad \Rightarrow \quad u' = \frac{1}{2}u^{-1}v' = \frac{1}{2}v^{-1/2}v'$$

$$\frac{1}{2}v^{-1/2}v' - \frac{1}{2x}v^{1/2} = 2xv^{-1/2}$$ ***Multiply both sides by*** $2v^{1/2}$

$$v' - \frac{1}{x}v = 4x$$

$$e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

$$\int x^{-1}(4x)dx = \int 4dx = 4x$$

$$v = x(4x + C)$$

$$u^2 = 4x^2 + Cx$$

$$\boxed{\sin^2 y = 4x^2 + Cx}$$

Exercise

Find the general solution $\left(x + e^y\right)y' = xe^{-y} - 1$

Solution

Let $u = e^y \quad \Rightarrow \quad y = \ln u \quad \Rightarrow \; y' = \frac{u'}{u}$

$$(x+u)\frac{u'}{u} = xu^{-1} - 1$$

$$(x+u)u' = x - u$$

Let $u = vx \quad \Rightarrow \quad u' = v + xv'$

$$(x + vx)(v + xv') = x - vx$$

$$x(1+v)(v + xv') = x(1-v)$$

$$(1+v)(v+xv') = 1 - v$$

$$v + v^2 + x(1+v)v' = 1 - v$$

$$x(1+v)\frac{dv}{dx} = 1 - 2v - v^2$$

$$\int \frac{1+v}{1-2v-v^2}dv = \int \frac{dx}{x}$$

$$-\frac{1}{2}\ln\left|1-2v-v^2\right| = \ln x + \ln C$$

$$\ln\left|1-2v-v^2\right| = -2\ln Cx \qquad\qquad v = \frac{u}{x} = \frac{e^y}{x}$$

$$\ln\left|1 - 2\frac{e^y}{x} - \frac{e^{2y}}{x^2}\right| = \ln(Cx)^{-2}$$

$$\frac{x^2 - 2xe^y - e^{2y}}{x^2} = \frac{1}{(Cx)^2}$$

$$\boxed{x^2 - 2xe^y - e^{2y} = C_1}$$

Exercise

Find the general solution $\left(x^2+y^2\right)dx+\left(x^2-xy\right)dy=0$

Solution

$$M(x,\ y)=x^2+y^2 \quad N(x,\ y)=x^2-xy$$

$$\frac{\partial M}{\partial y}=2y \quad \frac{\partial N}{\partial x}=2x-y \quad \Rightarrow \quad \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

$$\frac{M_y-N_x}{N}=\frac{2y-2x+y}{x^2+xy} \times$$

Let $y=ux \Rightarrow dy=udx+xdu$

$$-\frac{x^2+y^2}{x^2-xy}dx=udx+xdu$$

$$\left(x^2+y^2\right)dx+\left(x^2-xy\right)(udx+xdu)=0$$

$$\left(x^2+y^2\right)dx+ux(x-y)dx+x^2(x-y)du=0$$

$$\left(x^2+u^2x^2\right)dx+ux(x-ux)dx+x^2(x-ux)du=0$$

$$x^2(1+u)dx+x^3(1-u)du=0$$

$$(1+u)dx=x(u-1)du$$

$$\frac{dx}{x}=\frac{u-1}{u+1}du$$

$$\int\frac{dx}{x}=\int\left(1-\frac{2}{u+1}\right)du$$

$$\ln|x|=u-2\ln|u+1|+\ln C$$

$$\ln|x|+\ln\left(\frac{y}{x}+1\right)^2-\ln C=\frac{y}{x}$$

$$\ln\frac{x}{C}\left(\frac{(x+y)^2}{x^2}\right)=\frac{y}{x}$$

$$\frac{(x+y)^2}{Cx}=e^{\frac{y}{x}}$$

$$\boxed{(x+y)^2=Cxe^{\frac{y}{x}}}$$

Exercise

Find the general solution $x\frac{dy}{dx}+y=x^2y^2$

Solution

$$y' + \frac{1}{x}y = xy^2$$

Let $u = y^{1-2} = y^{-1} \quad \Rightarrow \quad y = \frac{1}{u}$

$$\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx} \quad \Rightarrow \quad y' = -y^2 u' = -\frac{1}{u^2}u'$$

$$-\frac{1}{u^2}u' + \frac{1}{x}\frac{1}{u} = x\frac{1}{u^2}$$

$$u' - \frac{1}{x}u = -x \qquad \left(\times - u^2\right)$$

$$e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

$$\int -xx^{-1}dx = -x$$

$$u = x(-x + C)$$

$$\frac{1}{y} = -x^2 + Cx$$

$$\boxed{y(x) = \frac{1}{-x^2 + Cx}}$$

Exercise

Solve the differential equation $\left(3x^2 - 2xy + 2\right) + \left(6y^2 - x^2 + 3\right)y' = 0$

Solution

$$\frac{\partial \psi}{\partial x} = M = 3x^2 - 2xy + 2 \quad \Rightarrow \quad M_y = -2x$$

$$\frac{\partial \psi}{\partial y} = N = 6y^2 - x^2 + 3 \quad \Rightarrow \quad N_x = -2x$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = 3x^2 - 2xy + 2$$

$$\psi = \int \left(3x^2 - 2xy + 2\right)dx = x^3 - x^2 y + 2x + h(y)$$

$$\psi_y = -x^2 + h'(y) = 6y^2 - x^2 + 3 \quad \Rightarrow \quad h'(y) = 6y^2 + 3$$

$$h(y) = \int \left(6y^2 + 3\right)dy = 2y^3 + 3y$$

$$\boxed{x^3 - x^2 y + 2x + 2y^3 + 3y = C}$$

Exercise

Solve the differential equation $\left(e^x \sin y - 2y\sin x\right)dx + \left(e^x \cos y + 2\cos x\right)dy = 0$

Solution

$$\frac{\partial\psi}{\partial x} = M = e^x \sin y - 2y\sin x \quad \Rightarrow \quad M_y = e^x \cos y - 2\sin x$$

$$\frac{\partial\psi}{\partial y} = N = e^x \cos y + 2\cos x \quad \Rightarrow \quad N_x = e^x \cos y - 2\text{sinx}$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial\psi}{\partial x} = e^x \sin y - 2y\sin x \quad \Rightarrow \quad \psi = \int\left(e^x \sin y - 2y\sin x\right)dx = e^x \sin y + 2y\cos x + h(y)$$

$$\psi_y = e^x \cos y + 2\cos x + h'(y) = e^x \cos y + 2\cos x \quad \Rightarrow \quad h'(y) = 0 \qquad \rightarrow \ \underline{h(y) = C}$$

$$\underline{e^x \sin y + 2y\cos x = C}$$

Exercise

Solve the differential equation $\left(\frac{y}{x} + 6x\right)dx + (\ln x - 2)dy = 0, \qquad x > 0$

Solution

$$\frac{\partial\psi}{\partial x} = M = \frac{y}{x} + 6x \quad \Rightarrow \quad M_y = \frac{1}{x}$$

$$\frac{\partial\psi}{\partial y} = N = \ln x - 2 \quad \Rightarrow \quad N_x = \frac{1}{x} \qquad \Rightarrow M_y = N_x$$

$$\frac{\partial\psi}{\partial x} = e^x \sin y - 2y\sin x \quad \Rightarrow \quad \psi = \int\left(\frac{y}{x} + 6x\right)dx = y\ln x + 3x^2 + h(y)$$

$$\psi_y = \ln x + h'(y) = \ln x - 2 \quad \Rightarrow \quad h'(y) = -2$$

$$h(y) = \int -2dy = -2y$$

$$\underline{y\ln x + 3x^2 - 2y = C}$$

Exercise

Solve the differential equation $\frac{xdx}{\left(x^2+y^2\right)^{3/2}} + \frac{ydy}{\left(x^2+y^2\right)^{3/2}} = 0$

Solution

Multiply both side by $\left(x^2+y^2\right)^{3/2}$ since $x^2 + y^2 \neq 0 \quad \Rightarrow \quad xdx + ydy = 0$

$$\frac{\partial\psi}{\partial x} = M = x \quad \Rightarrow \quad M_y = 0 \qquad \frac{\partial\psi}{\partial y} = N = y \quad \Rightarrow \quad N_x = 0 \quad \Rightarrow M_y = N_x$$

$$\frac{\partial\psi}{\partial x} = x \quad \Rightarrow \quad \psi = \int x\,dx = \frac{1}{2}x^2 + h(y)$$

$$\psi_y = h'(y) = y \quad \Rightarrow \quad h(y) = \int y\,dy = \frac{1}{2}y^2$$

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 = C_1$$

$$\boxed{x^2 + y^2 = C}$$

Exercise

Solve the differential equation $\left(e^{2y} - y\cos xy\right)dx + \left(2xe^{2y} - x\cos xy + 2y\right)dy = 0$

Solution

$$M(x,\ y) = e^{2y} - y\cos xy \qquad N(x,\ y) = 2xe^{2y} - x\cos xy + 2y$$

$$\frac{\partial M}{\partial y} = 2e^{2y} - \cos x + xy\sin xy \qquad \frac{\partial N}{\partial x} = 2e^{2y} - \cos x + xy\sin xy \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int\left(e^{2y} - y\cos xy\right)\,dx = xe^{2y} - \sin xy + h(y)$$

$$\psi_y = 2xe^{2y} - x\cos xy + h'(y) = 2xe^{2y} - x\cos xy + 2y$$

$$h'(y) = 2y \quad \Rightarrow \quad h(y) = y^2 + C$$

$$\psi = \boxed{xe^{2y} - \sin xy + y^2 = C}$$

Exercise

Solve the differential equation $\left(3x^2 - 2xy + 2\right) + \left(6y^2 - x^2 + 3\right)y' = 0$

Solution

$$\frac{\partial\psi}{\partial x} = M = 3x^2 - 2xy + 2 \quad \Rightarrow \quad M_y = -2x$$

$$\frac{\partial\psi}{\partial y} = N = 6y^2 - x^2 + 3 \quad \Rightarrow \quad N_x = -2x$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial\psi}{\partial x} = 3x^2 - 2xy + 2 \quad \Rightarrow \quad \psi = \int\left(3x^2 - 2xy + 2\right)dx = x^3 - x^2y + 2x + h(y)$$

$$\psi_y = -x^2 + h'(y) = 6y^2 - x^2 + 3 \quad \Rightarrow \quad h'(y) = 6y^2 + 3$$

$$h(y)=\int\left(6y^2+3\right)dy=2y^3+3y$$

$$\boxed{x^3-x^2y+2x+2y^3+3y=C}$$

Exercise

Solve the differential equation $\left(e^x\sin y-2y\sin x\right)dx+\left(e^x\cos y+2\cos x\right)dy=0$

Solution

$$\frac{\partial\psi}{\partial x}=M=e^x\sin y-2y\sin x \quad\Rightarrow\quad M_y=e^x\cos y-2\sin x$$

$$\frac{\partial\psi}{\partial y}=N=e^x\cos y+2\cos x \quad\Rightarrow\quad N_x=e^x\cos y-2\text{sinx}$$

$$\Rightarrow M_y=N_x$$

$$\frac{\partial\psi}{\partial x}=e^x\sin y-2y\sin x$$

$$\psi=\int\left(e^x\sin y-2y\sin x\right)dx$$

$$=e^x\sin y+2y\cos x+h(y)$$

$$\psi_y=e^x\cos y+2\cos x+h'(y)=e^x\cos y+2\cos x$$

$$h'(y)=0 \quad\to\quad \boxed{h(y)=C}$$

$$\boxed{e^x\sin y+2y\cos x=C}$$

Exercise

Solve the differential equation $\left(\frac{y}{x}+6x\right)dx+\left(\ln x-2\right)dy=0, \qquad x>0$

Solution

$$\frac{\partial\psi}{\partial x}=M=\frac{y}{x}+6x \quad\Rightarrow\quad M_y=\frac{1}{x} \qquad \frac{\partial\psi}{\partial y}=N=\ln x-2 \quad\Rightarrow\quad N_x=\frac{1}{x} \qquad \Rightarrow M_y=N_x$$

$$\frac{\partial\psi}{\partial x}=e^x\sin y-2y\sin x \quad\Rightarrow\quad \psi=\int\left(\frac{y}{x}+6x\right)dx=y\ln x+3x^2+h(y)$$

$$\psi_y=\ln x+h'(y)=\ln x-2 \quad\Rightarrow\quad h'(y)=-2$$

$$h(y)=\int -2dy=-2y$$

$$\boxed{y\ln x+3x^2-2y=C}$$

Exercise

Solve the differential equation $$\frac{xdx}{\left(x^2+y^2\right)^{3/2}}+\frac{ydy}{\left(x^2+y^2\right)^{3/2}}=0$$

Solution

Multiply both side by $\left(x^2+y^2\right)^{3/2}$ since $x^2+y^2\neq 0 \Rightarrow xdx+ydy=0$

$$\frac{\partial\psi}{\partial x}=M=x \Rightarrow M_y=0$$

$$\frac{\partial\psi}{\partial y}=N=y \Rightarrow N_x=0 \qquad \Rightarrow M_y=N_x$$

$$\frac{\partial\psi}{\partial x}=x \Rightarrow \psi=\int xdx=\frac{1}{2}x^2+h(y)$$

$$\psi_y=h'(y)=y \Rightarrow h(y)=\int ydy=\frac{1}{2}y^2$$

$$\frac{1}{2}x^2+\frac{1}{2}y^2=C_1$$

$$\boxed{x^2+y^2=C}$$

Exercise

Find the general solution $$\left(e^{2y}-y\cos xy\right)dx+\left(2xe^{2y}-x\cos xy+2y\right)dy=0$$

Solution

$$M(x,\ y)=e^{2y}-y\cos xy \qquad N(x,\ y)=2xe^{2y}-x\cos xy+2y$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y}&=2e^{2y}-\cos x+xy\sin xy\\ \frac{\partial N}{\partial x}&=2e^{2y}-\cos x+xy\sin xy\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\psi=\int\left(e^{2y}-y\cos xy\right)\,dx=xe^{2y}-\sin xy+h(y)$$

$$\psi_y=2xe^{2y}-x\cos xy+h'(y)=2xe^{2y}-x\cos xy+2y$$

$$h'(y)=2y \Rightarrow h(y)=y^2$$

$$\boxed{xe^{2y}-\sin xy+y^2=C}$$

Exercise

Find the general solution $(2x-1)dx+(3y+7)dy=0$

Solution

$M(x, y) = 2x - 1 \quad N(x, y) = 3y + 7$

$$\left.\begin{array}{l}\frac{\partial M}{\partial y} = 0 \\ \frac{\partial N}{\partial x} = 0\end{array}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (2x - 1)\ dx = x^2 - x + h(y)$$

$$\psi_y = h'(y) = 3y + 7 \qquad \Rightarrow \quad h(y) = \frac{3}{2}y^2 + 7$$

$$\boxed{x^2 - x + \frac{3}{2}y^2 + 7 = C}$$

Exercise

Find the general solution $\quad (5x + 4y)dx + \left(4x - 8y^3\right)dy = 0$

Solution

$M(x, y) = 5x + 4y \quad N(x, y) = 4x - 8y^3$

$$\left.\begin{array}{l}\frac{\partial M}{\partial y} = 4 \\ \frac{\partial N}{\partial x} = 4\end{array}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (5x + 4y)\ dx = \frac{5}{2}x^2 + 4xy + h(y)$$

$$\psi_y = 4x + h'(y) = 4x - 8y^3 \quad \to h'(y) = -8y^3 \quad \Rightarrow \quad h(y) = -2y^4$$

$$\boxed{\frac{5}{2}x^2 + 4xy - 2y^4 = C}$$

Exercise

Find the general solution $\quad (\sin y - y\sin x)dx + (\cos x + x\cos y - y)dy = 0$

Solution

$M(x, y) = \sin y - y\sin x \quad N(x, y) = \cos x + x\cos y - y$

$$\left.\begin{array}{l}\frac{\partial M}{\partial y} = \cos y - \sin x \\ \frac{\partial N}{\partial x} = -\sin x + \cos y\end{array}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (\sin y - y\sin x)\ dx = x\sin y + y\cos x + h(y)$$

$$\psi_y = x\cos y + \cos x + h'(y) = x\cos y + \cos x - y$$

$$\rightarrow h'(y) = -y \quad \Rightarrow \quad h(y) = -\frac{1}{2}y^2$$

$$\boxed{x\sin y + y\cos x - \frac{1}{2}y^2 = C}$$

Exercise

Find the general solution $\quad \left(2xy^2 - 3\right)dx + \left(2x^2y + 4\right)dy = 0$

Solution

$$M(x,\ y) = 2xy^2 - 3 \quad N(x,\ y) = 2x^2 + 4$$

$$\left.\begin{array}{l} \dfrac{\partial M}{\partial y} = 4xy \\ \dfrac{\partial N}{\partial x} = 4xy \end{array}\right. \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(2xy^2 - 3\right)\ dx = x^2y^2 - 3x + h(y)$$

$$\psi_y = 2x^2y + h'(y) = 2x^2y + 4 \qquad \rightarrow h'(y) = 4 \quad \Rightarrow \quad h(y) = 4y$$

$$\boxed{x^2y^2 - 3x + 4y = C}$$

Exercise

Find the general solution $\quad \left(1 + \ln x + \frac{y}{x}\right)dx - (1 - \ln x)dy = 0$

Solution

$$M(x,\ y) = 1 + \ln x + \frac{y}{x} \quad N(x,\ y) = -1 + \ln x$$

$$\left.\begin{array}{l} \dfrac{\partial M}{\partial y} = \dfrac{1}{x} \\ \dfrac{\partial N}{\partial x} = \dfrac{1}{x} \end{array}\right. \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(1 + \ln x + \frac{y}{x}\right)\ dx$$

$$= x + x\ln x - x + y\ln x + h(y)$$

$$= x\ln x + y\ln x + h(y)$$

$$\psi_y = \ln x + h'(y) = -1 + \ln x$$

$$\rightarrow h'(y) = -1 \quad \Rightarrow \quad h(y) = -y$$

$$\boxed{x\ln x + y\ln x - y = C}$$

Exercise

Find the general solution $\left(x - y^3 + y^2 \sin x\right)dx - \left(3xy^2 + 2y\cos x\right)dy = 0$

Solution

$$M(x, y) = x - y^3 + y^2 \sin x \quad N(x, y) = -3xy^2 - 2y\cos x$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= -3y^2 + 2y\sin x \\ \frac{\partial N}{\partial x} &= -3y^2 + 2y\sin x \end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(x - y^3 + y^2 \sin x\right) dx = \frac{1}{2}x^2 - xy^3 - y^2 \cos x + h(y)$$

$$\psi_y = -3xy^2 - 2y\cos x + h'(y) = -3xy^2 - 2y\cos x$$

$$\rightarrow h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$\boxed{\frac{1}{2}x^2 - xy^3 - y^2 \cos x = C}$$

Exercise

Find the general solution $\left(x^3 + y^3\right)dx + 3xy^2 dy = 0$

Solution

$$M(x, y) = x^3 + y^3 \quad N(x, y) = 3xy^2$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 3y^2 \\ \frac{\partial N}{\partial x} &= 3y^2 \end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(x^3 + y^3\right) dx = \frac{1}{4}x^4 + xy^3 + h(y)$$

$$\psi_y = 3xy^2 + h'(y) = 3xy^2$$

$$\rightarrow h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$\boxed{\frac{1}{4}x^4 + xy^3 = C}$$

Exercise

Find the general solution $\left(3x^2 y + e^y\right)dx + \left(x^3 + xe^y - 2y\right)dy = 0$

Solution

$$M(x, y) = 3x^2 y + e^y \quad N(x, y) = x^3 + xe^y - 2y$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 3x^2 + e^y \\ \frac{\partial N}{\partial x} &= 3x^2 + e^y\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(3x^2 y + e^y\right) dx = x^3 y + xe^y + h(y)$$

$$\psi_y = x^3 + xe^y + h'(y) = x^3 + xe^y - 2y \quad \rightarrow h'(y) = -2y \quad \Rightarrow \quad h(y) = -y^2$$

$$\underline{x^3 y + xe^y - y^2 = C}$$

Exercise

Find the general solution $\quad xdy + \left(y - 2xe^x - 6x^2\right)dx = 0$

Solution

$$M(x,\, y) = y - 2xe^x - 6x^2 \quad N(x,\, y) = x$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 1 \\ \frac{\partial N}{\partial x} &= 1\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(y - 2xe^x - 6x^2\right) dx$$

$$= xy - (2x - 2)e^x - 2x^3 + h(y)$$

$$\psi_y = x + h'(y) = x$$

$$\rightarrow h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$\underline{xy - 2xe^x + 2e^x - 2x^3 = C}$$

Exercise

Find the general solution $\quad \left(1 - \frac{3}{y} + x\right)dy + \left(y - \frac{3}{x} + 1\right)dx = 0$

Solution

$$M(x,\, y) = y - \frac{3}{x} + 1 \quad N(x,\, y) = 1 - \frac{3}{y} + x$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 1 \\ \frac{\partial N}{\partial x} &= 1\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(y - \frac{3}{x} + 1\right) dx = xy - 3\ln|x| + x + h(y)$$

$$\psi_y = x + h'(y) = 1 - \frac{3}{y} + x$$

$$\rightarrow h'(y) = 1 - \frac{3}{y} \quad \Rightarrow \quad h(y) = y - 3\ln|y|$$

$$xy - 3\ln|x| + x + y - 3\ln|y| = C$$

$$\boxed{xy + x + y - 3\ln|xy| = C}$$

Exercise

Find the general solution $\left(x^2y^3 - \dfrac{1}{1+9x^2}\right)\dfrac{dx}{dy} + x^3y^2 = 0$

Solution

$$\left(x^2y^3 - \frac{1}{1+9x^2}\right)dx + \left(x^3y^2\right)dy = 0$$

$$M(x, y) = x^2y^3 - \frac{1}{1+9x^2} \qquad N(x, y) = x^3y^2$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 3x^2y^2 \\ \frac{\partial N}{\partial x} &= 3x^2y^2 \end{aligned}\right. \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(x^2y^3 - \frac{1}{1+9x^2}\right) dx$$

$$= \frac{1}{3}x^3y^3 - \frac{1}{3}\arctan(3x) + h(y)$$

$$\psi_y = x^3y^2 + h'(y) = x^3y^2$$

$$\rightarrow h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$\frac{1}{3}x^3y^3 - \frac{1}{3}\arctan(3x) = C_1$$

$$\boxed{x^3y^3 - \arctan(3x) = C}$$

Exercise

Find the general solution $(5y - 2x)y' - 2y = 0$

Solution

$$M(x, y) = -2y \quad N(x, y) = 5y - 2x$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= -2 \\ \frac{\partial N}{\partial x} &= -2\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (-2y)\, dx = -2xy + h(y)$$

$$\psi_y = -2x + h'(y) = 5y - 2x$$

$$\rightarrow h'(y) = 5y \quad \Rightarrow \quad h(y) = \frac{5}{2}y^2$$

$$\boxed{-2xy + \frac{5}{2}y^2 = C}$$

Exercise

Find the general solution $\quad (x - y)dx - xdy = 0$

Solution

$$M(x, y) = x - y \quad N(x, y) = -x$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= -1 \\ \frac{\partial N}{\partial x} &= -1\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (x - y)\, dx$$

$$= \frac{1}{2}x^2 - xy + h(y)$$

$$\psi_y = -x + h'(y) = -x$$

$$\rightarrow h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$\boxed{\frac{1}{2}x^2 - xy = C}$$

Exercise

Find the general solution $\quad (x + y)dx + xdy = 0$

Solution

$$M(x, y) = x + y \quad N(x, y) = x$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 1 \\ \frac{\partial N}{\partial x} &= 1\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (x+y)\, dx$$
$$= \frac{1}{2}x^2 + xy + h(y)$$
$$\psi_y = x + h'(y) = x$$
$$\to h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$
$$\frac{1}{2}x^2 + xy = C$$

Exercise

Find the general solution $\quad \dfrac{dy}{dx} = -\dfrac{2xy^2+1}{2x^2y}$

Solution

$$2x^2ydy = -\left(2xy^2+1\right)dx$$
$$\left(2xy^2+1\right)dx + 2x^2ydy = 0$$
$$M(x, y) = 2xy^2+1 \quad N(x, y) = 2x^2y$$
$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 4xy\\ \frac{\partial N}{\partial x} &= 4xy\end{aligned}\right. \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
$$\psi = \int \left(2xy^2+1\right) dx$$
$$= x^2y^2 + x + h(y)$$
$$\psi_y = 2x^2y + h'(y) = 2x^2y$$
$$\to h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$
$$x^2y^2 + x = C$$

Exercise

Find the general solution $\quad \left(1+e^xy+xe^xy\right)dx + \left(xe^x+2\right)dy = 0$

Solution

$$M(x, y) = 1+e^xy+xe^xy \quad N(x, y) = xe^x+2$$
$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= e^x + xe^x\\ \frac{\partial N}{\partial x} &= e^x + xe^x\end{aligned}\right. \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(1 + e^x y + xe^x y\right)\, dx$$

$$= x + e^x y + (x-1)e^x y + h(y)$$

$$= x + xe^x y + h(y)$$

$$\psi_y = xe^x + h'(y) = xe^x + 2$$

$$\to h'(y) = 2 \quad \Rightarrow \quad h(y) = 2y + C$$

$$\boxed{x + xe^x y + 2y + C = 0}$$

Exercise

Find the general solution $\left(2xy^3 + 1\right)dx + \left(3x^2y^2 - \frac{1}{y}\right)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}\left(2xy^3 + 1\right) = 6xy^2$$

$$N_x = \frac{\partial}{\partial x}\left(3x^2y^2 - \frac{1}{y}\right) = 6xy^2$$

$$\Rightarrow M_y = N_x$$

$$\psi = \int \left(2xy^3 + 1\right)\, dx$$

$$= x^2y^3 + x + h(y)$$

$$\psi_y = 3x^2y^2 + h'(y)$$

$$= 3x^2y^2 - \frac{1}{y}$$

$$\to h'(y) = -\frac{1}{y} \quad \Rightarrow \quad h(y) = -\ln|y| + C$$

$$\boxed{x^2y^3 + x - \ln|y| = C}$$

Exercise

Find the general solution $(2x + y)dx + (x - 2y)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}(2x + y) = 1$$

$$N_x = \frac{\partial}{\partial x}(x - 2y) = 1$$

$$\Rightarrow M_y = N_x$$

$$\psi = \int (2x + y)\,dx$$

$$= x^2 + xy + h(y)$$

$$\psi_y = x + h'(y) = x - 2y$$

$$\rightarrow h'(y) = -2y \quad \Rightarrow \quad h(y) = -y^2$$

$$\boxed{x^2 + xy - y^2 = C}$$

Exercise

Find the general solution $\quad e^x(y - x)dx + (1 + e^x)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}\left(e^x(y - x)\right) = e^x$$

$$N_x = \frac{\partial}{\partial x}\left(1 + e^x\right) = e^x$$

$$\Rightarrow M_y = N_x$$

$$\psi = \int \left(ye^x - xe^x\right)\,dx$$

$$= ye^x - (x - 1)e^x + h(y)$$

$$\psi_y = e^x + h'(y) = 1 + e^x$$

$$\rightarrow h'(y) = 1 \quad \Rightarrow \quad h(y) = y$$

$$\boxed{ye^x - (x - 1)e^x + y = C}$$

$$\boxed{y(x) = \frac{(x - 1)e^x + C}{1 + e^x}}$$

Exercise

Find the general solution $\quad \left(ye^{xy} - \frac{1}{y}\right)dx + \left(xe^{xy} + \frac{x}{y^2}\right)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}\left(ye^{xy} - \frac{1}{y}\right) = e^{xy} + xye^{xy} + \frac{1}{y^2}$$

$$N_x = \frac{\partial}{\partial x}\left(xe^{xy} + \frac{x}{y^2}\right) = e^{xy} + xye^{xy} + \frac{1}{y^2}$$

$$\Rightarrow M_y = N_x$$

$$\psi = \int \left(ye^{xy} - \frac{1}{y} \right) dx = e^{xy} - \frac{x}{y} + h(y)$$

$$\psi_y = xe^{xy} + \frac{x}{y^2} + h'(y) = xe^{xy} + \frac{x}{y^2}$$

$$\rightarrow h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$\boxed{e^{xy} - \frac{x}{y} = C}$$

Exercise

Find the general solution $\quad (\tan x - \sin x \sin y)dx + (\cos x \cos y)dy = 0$

Solution

$$M(x, y) = \tan x - \sin x \sin y \quad N(x, y) = \cos x \cos y$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= -\sin x \cos y \\ \frac{\partial N}{\partial x} &= -\sin x \cos y \end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (\tan x - \sin x \sin y)\, dx = \ln|\sec x| + \cos x \sin y + h(y)$$

$$\psi_y = \cos x \cos y + h'(y) = \cos x \cos y \quad \rightarrow h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$\boxed{\ln|\sec x| + \cos x \sin y = C}$$

Exercise

Find the general solution $\quad \left(2x^3 - xy^2 - 2y + 3\right)dx - \left(x^2 y + 2x\right)dy = 0$

Solution

$$M(x, y) = 2x^3 - xy^2 - 2y + 3 \quad N(x, y) = -x^2 y - 2x$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= -2xy - 2 \\ \frac{\partial N}{\partial x} &= -2xy - 2 \end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(2x^3 - xy^2 - 2y + 3\right) dx = \frac{1}{2}x^4 - \frac{1}{2}x^2 y^2 - 2xy + 3x + h(y)$$

$$\psi_y = -x^2 y - 2x + h'(y) = -x^2 y - 2x \quad \rightarrow h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$\boxed{\frac{1}{2}x^4 - \frac{1}{2}x^2 y^2 - 2xy + 3x = C}$$

Exercise

Find the general solution $\quad (x+\sin y)dx+(x\cos y-2y)dy=0$

Solution

$$M(x,\ y)=x+\sin y \quad N(x,\ y)=x\cos y-2y$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y}&=\cos y\\ \frac{\partial N}{\partial x}&=\cos y\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\psi=\int (x+\sin y)\ dx=\frac{1}{2}x^2+x\sin y+h(y)$$

$$\psi_y=x\cos y+h'(y)=x\cos y-2y$$

$$\rightarrow h'(y)=-2y \quad \Rightarrow \quad h(y)=-y^2$$

$$\boxed{\frac{1}{2}x^2+x\sin y-y^2=C}$$

Exercise

Find the general solution $\quad \left(x+\frac{1}{\sqrt{y^2-x^2}}\right)dx+\left(1-\frac{x}{y\sqrt{y^2-x^2}}\right)dy=0$

Solution

$$M(x,\ y)=x+\frac{1}{\sqrt{y^2-x^2}} \quad N(x,\ y)=1-\frac{x}{y\sqrt{y^2-x^2}}$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y}&=-\frac{1}{2}(2y)\frac{1}{\sqrt{y^2-x^2}}=-\frac{y}{\sqrt{y^2-x^2}}\\ \frac{\partial N}{\partial x}&=-\frac{1}{y}\frac{1}{\sqrt{y^2-x^2}}\left(y^2-x^2+x^2\right)=-\frac{y}{\sqrt{y^2-x^2}}\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\psi=\int\left(x+\frac{1}{\sqrt{y^2-x^2}}\right)dx=\frac{1}{2}x^2+\sin^{-1}\frac{x}{y}+h(y) \qquad \int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}$$

$$\psi_y=-\frac{1}{y^2\sqrt{1-\left(\frac{x}{y}\right)^2}}+h'(y)=-\frac{1}{y\sqrt{y^2-x^2}}+h'(y)=1-\frac{1}{y\sqrt{y^2-x^2}} \qquad \frac{d}{dx}(\arcsin u)=\frac{u'}{\sqrt{1-u^2}}$$

$$\rightarrow h'(y)=1 \quad \Rightarrow \quad h(y)=y$$

$$\boxed{\frac{1}{2}x^2+\sin^{-1}\frac{x}{y}+y=C}$$

Exercise

Find the general solution $\left(2x+y^2-\cos(x+y)\right)dx+\left(2xy-\cos(x+y)-e^y\right)dy=0$

<u>***Solution***</u>

$$M_y=\frac{\partial}{\partial y}\left(2x+y^2-\cos(x+y)\right)=2y+\sin(x+y)$$
$$N_x=\frac{\partial}{\partial x}\left(2xy-\cos(x+y)-e^y\right)=2y+\sin(x+y)$$
$$\Rightarrow M_y=N_x$$

$$\psi=\int\left(2x+y^2-\cos(x+y)\right)\,dx=x^2+xy^2-\sin(x+y)+h(y)$$

$$\psi_y=2xy-\cos(x+y)+h'(y)=2xy-\cos(x+y)-e^y$$

$$\to h'(y)=-e^y \quad\Rightarrow\quad h(y)=-e^y$$

$$\boxed{x^2+xy^2-\sin(x+y)-e^y=C}$$

Exercise

Find the general solution $\left(\dfrac{2}{\sqrt{1-x^2}}+y\cos(xy)\right)dx+\left(x\cos(xy)-y^{-1/3}\right)dy=0$

<u>***Solution***</u>

$$M_y=\frac{\partial}{\partial y}\left(\frac{2}{\sqrt{1-x^2}}+y\cos(xy)\right)=\cos(xy)-xy\sin(xy)$$
$$N_x=\frac{\partial}{\partial x}\left(x\cos(xy)-y^{-1/3}\right)=\cos(xy)-xy\sin(xy)$$
$$\Rightarrow M_y=N_x$$

$$\psi=\int\left(x\cos(xy)-y^{-1/3}\right)\,dy=\sin(xy)-\frac{3}{2}y^{2/3}+h(x)$$

$$\psi_x=y\cos(xy)+h'(x)=\frac{2}{\sqrt{1-x^2}}+y\cos(xy)$$

$$\to h'(x)=\frac{2}{\sqrt{1-x^2}} \quad\Rightarrow\quad h(x)=2\arcsin x$$

$$\boxed{\sin(xy)-\frac{3}{2}y^{2/3}+2\arcsin x=C}$$

Exercise

Find the general solution $\left(2x+y\cos(xy)\right)dx+\left(x\cos(xy)-2y\right)dy=0$

<u>***Solution***</u>

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(2x + y\cos(xy)\right) = \cos(xy) - xy\sin(xy) \\ N_x &= \frac{\partial}{\partial x}\left(x\cos(xy) - 2y\right) = \cos(xy) - xy\sin(xy) \end{aligned}\right\} \Rightarrow M_y = N_x$$

$$\psi = \int \left(2x + y\cos(xy)\right)\, dy = x^2 + \sin(xy) + h(y)$$

$$\psi_y = x\cos(xy) + h'(y) = x\cos(xy) - 2y$$

$$\rightarrow h'(y) = -2y \quad \Rightarrow \quad h(y) = -y^2$$

$$x^2 + \sin(xy) - y^2 = C$$

Exercise

Find the general solution $\left(e^x \sin y - 3x^2\right)dx + \left(e^x \cos y + \frac{1}{3}y^{-2/3}\right)dy = 0$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(e^x \sin y - 3x^2\right) = e^x \cos y \\ N_x &= \frac{\partial}{\partial x}\left(e^x \cos y + \frac{1}{3}y^{-2/3}\right) = e^x \cos y \end{aligned}\right\} \Rightarrow M_y = N_x$$

$$\psi = \int \left(e^x \sin y - 3x^2\right)\, dy = e^x \sin y - x^3 + h(y)$$

$$\psi_y = e^x \cos y + h'(y) = e^x \cos y + \frac{1}{3}y^{-2/3}$$

$$\rightarrow h'(y) = \frac{1}{3}y^{-2/3} \quad \Rightarrow \quad h(y) = y^{1/3}$$

$$e^x \sin y - x^3 + y^{1/3} = C$$

Exercise

Find the general solution $\left(2y\sin x\cos x - y + 2y^2 e^{xy^2}\right)dx = \left(x - \sin^2 x - 4xye^{xy^2}\right)dy$

Solution

$$\left(2y\sin x\cos x - y + 2y^2 e^{xy^2}\right)dx - \left(x - \sin^2 x - 4xye^{xy^2}\right)dy = 0$$

$$M(x,\ y) = 2y\sin x\cos x - y + 2y^2 e^{xy^2} \qquad N(x,\ y) = -x + \sin^2 x + 4xye^{xy^2}$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 2\sin x\cos x - 1 + 4ye^{xy^2} + 4xy^3 e^{xy^2} \\ \frac{\partial N}{\partial x} &= -1 + 2\sin x\cos x + 4ye^{xy^2} + 4xy^3 e^{xy^2} \end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(2y\sin x\cos x - y + 2y^2 e^{xy^2}\right) dx$$

$$= \int (2y\sin x)\ d(\sin x) - xy + 2e^{xy^2}$$

$$= y\sin^2 x - xy + 2e^{xy^2}$$

$$\psi_y = \sin^2 x - x + 4xye^{xy^2} + h'(y) = -x + \sin^2 x + 4xye^{xy^2}$$

$$\to h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$\boxed{y\sin^2 x - xy + 2e^{xy^2} = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$x^2y^3 + x\left(1+y^2\right)y' = 0, \qquad \mu(x,\ y) = \frac{1}{xy^3}$$

Solution

$$M_y = \frac{\partial}{\partial y}\left(x^2y^3\right) = 3x^2y^2$$

$$N_x = \frac{\partial}{\partial x}\left(x + xy^2\right) = 1 + y^2$$

$$\Rightarrow M_y \neq N_x$$

$$x^2y^3\left(\frac{1}{xy^3}\right) + x\left(1+y^2\right)\left(\frac{1}{xy^3}\right)y' = 0$$

$$x + \left(\frac{1+y^2}{y^3}\right)\frac{dy}{dx} = 0 \quad \Rightarrow \quad \left(\frac{1+y^2}{y^3}\right)dy = -xdx$$

$$\int \left(y^{-3} + \frac{1}{y}\right)dy = -\int xdx$$

$$-\frac{1}{2}y^{-2} + \ln|y| = -\frac{1}{2}x^2 + C_0$$

$$\boxed{x^2 - y^{-2} + \ln|y| = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$y^2 - xy + \left(x^2\right)y' = 0, \qquad \mu(x,\ y) = \frac{1}{xy^2}$$

Solution

$$M_y = \frac{\partial}{\partial y}\left(y^2 - xy\right) = 2y - x \qquad N_x = \frac{\partial}{\partial x}\left(x^2\right) = 2x \quad \Rightarrow M_y \neq N_x$$

$$\left(y^2 - xy\right)\left(\frac{1}{xy^2}\right) + \left(x^2\right)\left(\frac{1}{xy^2}\right)y' = 0$$

$$\left(\frac{1}{x} - \frac{1}{y}\right) + \left(\frac{x}{y^2}\right)y' = 0$$

$$M_y = \frac{\partial}{\partial y}\left(\frac{1}{x} - \frac{1}{y}\right) = \frac{1}{y^2}$$

$$N_x = \frac{\partial}{\partial x}\left(\frac{x}{y^2}\right) = \frac{1}{y^2}$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = \frac{1}{x} - \frac{1}{y} \quad \Rightarrow \quad \psi = \int \left(\frac{1}{x} - \frac{1}{y}\right)dx = \ln|x| - \frac{x}{y} + h(y)$$

$$\psi_y = \frac{x}{y^2} + h'(y) = \frac{x}{y^2}$$

$$\Rightarrow \quad h'(y) = 0 \quad \to h(y) = C$$

$$\boxed{\ln|x| - \frac{x}{y} = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$x^2y^3 - y + x\left(1 + x^2y^2\right)y' = 0, \qquad \mu(x, y) = \frac{1}{xy}$$

Solution

$$M_y = \frac{\partial}{\partial y}\left(x^2y^3 - y\right) = 3y^2 - 1 \qquad N_x = \frac{\partial}{\partial x}\left(x + x^3y^2\right) = 1 + 3x^2y^2 \quad \Rightarrow M_y \neq N_x$$

$$\left(x^2y^3 - y\right)\left(\frac{1}{xy}\right) + x\left(1 + x^2y^2\right)\left(\frac{1}{xy}\right)y' = 0$$

$$\left(xy^2 - \frac{1}{x}\right) + \left(\frac{1}{y} + x^2y\right)y' = 0$$

$$M_y = \frac{\partial}{\partial y}\left(xy^2 - \frac{1}{x}\right) = 2xy$$

$$N_x = \frac{\partial}{\partial x}\left(\frac{1}{y} + x^2y\right) = 2xy$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = xy^2 - \frac{1}{x} \quad \Rightarrow \quad \psi = \int \left(xy^2 - \frac{1}{x}\right)dx = \frac{1}{2}x^2y^2 - \ln|x| + h(y)$$

$$\psi_y = x^2 y + h'(y) = \frac{1}{y} + x^2 y$$

$$\Rightarrow \quad h'(y) = \frac{1}{y} \quad \to \quad h(y) = \ln|y|$$

$$\boxed{\frac{1}{2}x^2 y^2 - \ln x + \ln y = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$\left(\frac{\sin y}{y} - 2e^{-x}\sin x\right)dx + \left(\frac{\cos y + 2e^{-x}\cos x}{y}\right)dy = 0, \qquad \mu(x, y) = ye^x$$

Solution

$$M_y = \frac{\partial}{\partial y}\left(\frac{\sin y}{y} - 2e^{-x}\sin x\right) = \frac{y\cos y - \sin y}{y^2}$$

$$N_x = \frac{\partial}{\partial x}\left(\frac{\cos y + 2e^{-x}\cos x}{y}\right) = \frac{1}{y}\left(-2e^{-x}\cos x - 2e^{-x}\sin x\right)$$

$$\Rightarrow \; M_y \neq N_x$$

$$\left(ye^x\right)\left(\frac{\sin y}{y} - 2e^{-x}\sin x\right)dx + \left(ye^x\right)\left(\frac{\cos y + 2e^{-x}\cos x}{y}\right)dy = 0$$

$$\left(e^x\sin y - 2y\sin x\right)dx + \left(e^x\cos y + 2\cos x\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(e^x\sin y - 2y\sin x\right) = e^x\cos y - 2\sin x$$

$$N_x = \frac{\partial}{\partial x}\left(e^x\cos y + 2\cos x\right) = e^x\cos y - 2\sin x$$

$$\Rightarrow \; M_y = N_x$$

$$\psi = \int\left(e^x\sin y - 2y\sin x\right)dx$$

$$= e^x\sin y + 2y\cos x + h(y)$$

$$\psi_y = e^x\cos y + 2\cos x + h'(y)$$

$$= e^x\cos y + 2\cos x$$

$$\Rightarrow \quad h'(y) = 0 \quad \to \quad h(y) = C$$

$$\boxed{e^x\sin y + 2y\cos x = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$(x+2)\sin y\,dx + x\cos y\,dy = 0, \qquad \mu(x,y) = xe^{x}$$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\big((x+2)\sin y\big) = (x+2)\cos y \\ N_x &= \frac{\partial}{\partial x}(x\cos y) = -x\sin y \end{aligned}\right\} \Rightarrow M_y \neq N_x$$

$$\left(xe^{x}\right)(x+2)\sin y\,dx + \left(xe^{x}\right)x\cos y\,dy = 0$$

$$\left(x^2+2x\right)e^{x}\sin y\,dx + x^2e^{x}\cos y\,dy = 0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(\left(x^2+2x\right)e^{x}\sin y\right) = \left(x^2+2x\right)e^{x}\cos y \\ N_x &= \frac{\partial}{\partial x}\left(x^2e^{x}\cos y\right) = \left(2xe^{x}+x^2\right)e^{x}\cos y \end{aligned}\right\} \Rightarrow M_y = N_x$$

$$\psi = \int\left(x^2e^{x}\cos y\right)dy = x^2e^{x}\sin y + h(x)$$

$$\psi_x = \left(x^2+2x\right)e^{x}\sin y + h'(x) = \left(x^2+2x\right)e^{x}\sin y$$

$$\Rightarrow \quad h'(x) = 0 \;\to\; h(x) = C$$

$$\boxed{x^2e^{x}\sin y = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$\left(x^2+y^2-x\right)dx - y\,dy = 0, \qquad \mu(x,y) = \frac{1}{x^2+y^2}$$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(x^2+y^2-x\right) = 2y \\ N_x &= \frac{\partial}{\partial x}(-y) = 0 \end{aligned}\right\} \Rightarrow M_y \neq N_x$$

$$\frac{1}{x^2+y^2}\left(x^2+y^2-x\right)dx - \frac{y}{x^2+y^2}dy = 0$$

$$\left(1-\frac{x}{x^2+y^2}\right)dx - \frac{y}{x^2+y^2}dy = 0$$

$$M_y = \left(1 - \frac{x}{x^2 + y^2}\right) = \frac{2xy}{\left(x^2 + y^2\right)^2}$$

$$\Rightarrow M_y = N_x$$

$$N_x = \left(\frac{-y}{x^2 + y^2}\right) = \frac{2xy}{\left(x^2 + y^2\right)^2}$$

$$\frac{d\psi}{dx} = 1 - \frac{x}{x^2 + y^2}$$

$$\psi = \int \left(1 - \frac{x}{x^2 + y^2}\right) dx$$

$$= \int dx - \frac{1}{2} \int \frac{1}{x^2 + y^2} d\left(x^2 + y^2\right)$$

$$= x - \frac{1}{2} \ln\left(x^2 + y^2\right) + h(y)$$

$$\psi_y = -\frac{y}{x^2 + y^2} + h'(y)$$

$$= -\frac{y}{x^2 + y^2}$$

$$h'(y) = 0 \quad \rightarrow \quad h(y) = C$$

$$\boxed{x - \frac{1}{2} \ln\left(x^2 + y^2\right) = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation $(2y - 6x)dx + \left(3x - 4x^2 y^{-1}\right)dy = 0, \qquad \mu(x, y) = xy^2$

Solution

$$M_y = \frac{\partial}{\partial y}(2y - 6x) = 2$$

$$\Rightarrow M_y \neq N_x$$

$$N_x = \frac{\partial}{\partial x}\left(3x - 4x^2 y^{-1}\right) = 3 - \frac{8x}{y}$$

$$xy^2 (2y - 6x)dx + xy^2 \left(3x - 4x^2 y^{-1}\right)dy = 0$$

$$\left(2xy^3 - 6x^2 y^2\right)dx + \left(3x^2 y^2 - 4x^3 y\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(2xy^3 - 6x^2 y^2\right) = 6xy^2 - 12x^2 y$$

$$\Rightarrow M_y = N_x$$

$$N_x = \frac{\partial}{\partial x}\left(3x^2 y^2 - 4x^3 y\right) = 6xy^2 - 12x^2 y$$

$$\psi = \int \left(2xy^3 - 6x^2y^2\right) dx$$

$$= x^2y^3 - 2x^3y^2 + h(y)$$

$$\psi_y = 3x^2y^2 - 4x^3y + h'(y)$$

$$= 3x^2y^2 - 4x^3y$$

$$h'(y) = 0 \quad \to \quad h(y) = C$$

$$x^2y^3 - 2x^3y^2 = C$$

Exercise

Find the general solution of the homogenous equation $\left(x^2 + y^2\right)dx - 2xydy = 0$

Solution

$$\left.\begin{array}{l} M_y = \frac{\partial}{\partial y}\left(x^2 + y^2\right) = 2y \\ N_x = \frac{\partial}{\partial x}(-2xy) = -2y \end{array}\right\} \Rightarrow M_y \neq N_x$$

$$\frac{M_y - N_x}{N} = \frac{2y + 2y}{-2xy} = -\frac{4y}{2xy} = -\frac{2}{x}$$

$$\frac{d\mu}{dx} = -\mu\frac{2}{x} \quad \Rightarrow \quad \int \frac{d\mu}{\mu} = -2\int \frac{dx}{x}$$

$$\ln \mu = -2\ln x$$

$$\ln \mu = \ln x^{-2} \quad \Rightarrow \quad \mu = \frac{1}{x^2}$$

$$\frac{1}{x^2}\left(x^2 + y^2\right)dx - \frac{1}{x^2}2xydy = 0 \qquad \Rightarrow \quad \left(1 + \frac{y^2}{x^2}\right)dx - \frac{2y}{x}dy = 0$$

$$\left.\begin{array}{l} M_y = \frac{\partial}{\partial y}\left(1 + \frac{y^2}{x^2}\right) = \frac{2y}{x^2} \\ N_x = \frac{\partial}{\partial x}\left(-\frac{2y}{x}\right) = \frac{2y}{x^2} \end{array}\right\} \Rightarrow M_y = N_x$$

$$\psi = \int \left(1 + \frac{y^2}{x^2}\right)dx$$

$$= x - \frac{y^2}{x} + h(y)$$

$$\psi_y = -\frac{2y}{x} + h'(y) = -\frac{2y}{x}$$

$$h'(y)=0 \quad \Rightarrow \quad h(y)=C$$

$$x-\frac{y^2}{x}=C \qquad\qquad \textit{multiply by } x$$

$$\boxed{x^2-y^2=Cx}$$

Exercise

Find the general solution of the homogenous equation $(x+y)dx+(y-x)dy=0$

Solution

$$(x+y)dx=-(y-x)dy$$

$$\frac{dy}{dx}=\frac{x+y}{x-y}$$

$$=\frac{\frac{x+y}{x}}{\frac{x-y}{x}}$$

$$=\frac{1+\frac{y}{x}}{1-\frac{y}{x}}$$

$$\frac{dy}{dx}=\frac{1+v}{1-v}=x\frac{dv}{dx}+v$$

$$x\frac{dv}{dx}=\frac{1+v}{1-v}-v=\frac{1+v^2}{1-v}$$

$$\frac{1-v}{1+v^2}dv=\frac{dx}{x}$$

$$\int\frac{dx}{x}=\int\frac{1}{1+v^2}dv-\int\frac{v}{1+v^2}dv \qquad\qquad \int\frac{1}{a^2+x^2}dx=\arctan\frac{x}{a}$$

$$\ln x=\arctan v-\frac{1}{2}\int\frac{1}{1+v^2}d\left(1+v^2\right)$$

$$\ln x+C=\arctan v-\frac{1}{2}\ln\left(1+v^2\right)$$

$$\ln x+C=\arctan\frac{y}{x}-\frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right)$$

$$\boxed{\arctan\frac{y}{x}-\frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right)-\ln x=C}$$

Exercise

Find the general solution of the homogenous equation $\frac{dy}{dx}=\frac{y\left(x^2+y^2\right)}{xy^2-2x^3}$

Solution

$$\frac{dy}{dx}=\frac{y}{x}\frac{\frac{x^2+y^2}{x^2}}{\frac{y^2-2x^2}{x^2}}$$

$$=\frac{y}{x}\frac{1+\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)^2-2}$$

$$=v\frac{1+v^2}{v^2-2}=x\frac{dv}{dx}+v$$

$$\frac{v+v^3}{v^2-2}-v=x\frac{dv}{dx}$$

$$x\frac{dv}{dx}=\frac{v+v^3-v^3+2v}{v^2-2}=\frac{3v}{v^2-2}$$

$$\int\frac{dx}{x}=\int\frac{v^2-2}{3v}dv=\frac{1}{3}\int\left(v-\frac{2}{v}\right)dv$$

$$3\ln x+C=\frac{1}{2}v^2-2\ln v$$

$$3\ln x+C=\frac{1}{2}\frac{y^2}{x^2}-2\ln\frac{y}{x}$$

$$3\ln x+C=\frac{1}{2}\frac{y^2}{x^2}-2\left(\ln y-\ln x\right)$$

$$6\ln x+C=\frac{y^2}{x^2}-4\ln y+4\ln x$$

$$\boxed{\frac{y^2}{x^2}-4\ln y-2\ln x=C}$$

Exercise

Find an integrating factor and solve the given equation $\left(3x^2y+2xy+y^3\right)dx+\left(x^2+y^2\right)dy=0$

Solution

$$\left.\begin{aligned}M_y&=\frac{\partial}{\partial y}\left(3x^2y+2xy+y^3\right)=3x^2+2x+3y^2\\N_x&=\frac{\partial}{\partial x}\left(x^2+y^2\right)=2x\end{aligned}\right\}\Rightarrow M_y\neq N_x$$

$$\frac{M_y - N_x}{N} = \frac{3x^2 + 2x + 3y^2 - 2x}{x^2 + y^2} = 3$$

$$\frac{d\mu}{dx} = 3\mu \quad \Rightarrow \quad \int \frac{d\mu}{\mu} = 3\int dx$$

$$\ln \mu = 3x \quad \Rightarrow \quad \mu = e^{3x}$$

$$e^{3x}\left(3x^2 y + 2xy + y^3\right)dx + e^{3x}\left(x^2 + y^2\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left[e^{3x}\left(3x^2 y + 2xy + y^3\right)\right] = e^{3x}\left(3x^2 + 2x + 3y^2\right)$$

$$N_x = \frac{\partial}{\partial x} e^{3x}\left(x^2 + y^2\right) = 3e^{3x}\left(x^2 + y^2\right) + 2xe^{3x} = e^{3x}\left(3x^2 + 3y^2 + 2x\right)$$

$$\Rightarrow M_y = N_x$$

$$\psi = \int \left(e^{3x}\left(3x^2 y + 2xy + y^3\right)\right)dx = +h(y)$$

$$= e^{3x}\left(x^2 y + \tfrac{2}{3}xy + \tfrac{1}{3}y^3 - \tfrac{2}{3}xy - \tfrac{2}{9}y + \tfrac{2}{9}y\right) + h(y)$$

$$= e^{3x}\left(x^2 y + \tfrac{1}{3}y^3\right) + h(y)$$

	$\int e^{3x}$
$3x^2 y + 2xy + y^3$	$\frac{1}{3}e^{3x}$
$6xy + 2y$	$\frac{1}{9}e^{3x}$
$6y$	$\frac{1}{27}e^{3x}$

$$\psi_y = e^{3x}\left(x^2 + y^2\right) + h'(y) = e^{3x}\left(x^2 + y^2\right)$$

$$h'(y) = 0 \Rightarrow h(y) = C$$

$$\boxed{e^{3x}\left(x^2 y + \tfrac{1}{3}y^3\right) = C}$$

Exercise

Find an integrating factor and solve the given equation $\quad dx + \left(\frac{x}{y} - \sin y\right)dy = 0$

Solution

$ydx + (x - y\sin y)dy = 0$ *Multiply by **y** both sides*

$$M_y = \frac{\partial}{\partial y}(y) = 1; \quad N_x = \frac{\partial}{\partial x}(x - y\sin y) = 1; \quad M_y = N_x$$

$$\frac{\partial\psi}{\partial x} = 2x + y^2 \quad \Rightarrow \quad \psi = \int ydx = xy + h(y)$$

$$\psi_y = x + h'(y) = x - y\sin y \quad \Rightarrow \quad h'(y) = -y\sin y$$

$$h(y) = -\int y\sin y dy = y\cos y - \sin y$$

	$\int -\sin y$
y	$\cos y$
1	$\sin y$

$$\boxed{xy + y\cos y - \sin y = C}$$

Exercise

Find an integrating factor and solve the given equation $e^x dx + \left(e^x \cot y + 2y \csc y\right) dy = 0$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(e^x\right) = 0 \\ N_x &= \frac{\partial}{\partial x}\left(e^x \cot y + 2y \csc y\right) = e^x \end{aligned}\right\} \Rightarrow M_y \neq N_x$$

$$e^x dx + \left(e^x \frac{\cos y}{\sin y} + 2y \frac{1}{\sin y}\right) dy = 0 \qquad \textit{Multiply by } \textbf{\textit{siny}} \textit{ both sides}$$

$$(\sin y) e^x dx + (\sin y)\left(e^x \frac{\cos y}{\sin y} + 2y \frac{1}{\sin y}\right) dy = 0$$

$$e^x \sin y dx + \left(e^x \cos y + 2y\right) dy = 0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(e^x \sin y\right) = e^x \cos y \\ N_x &= \frac{\partial}{\partial x}\left(e^x \cos y + 2y\right) = e^x \cos y \end{aligned}\right\} \Rightarrow M_y = N_x$$

$$\psi = \int \left(e^x \sin y\right) dx = e^x \sin y + h(y)$$

$$\psi_y = e^x \cos y + h'(y) = e^x \cos y + 2y \quad \rightarrow h'(y) = 2y \ \Rightarrow\ h(y) = y^2$$

$$\psi(x, y) = e^x \sin y + y^2 = C$$

$$\boxed{e^x \sin y + y^2 = C}$$

Exercise

Find an integrating factor and solve the given equation $\left(3x + \frac{6}{y}\right) dx + \left(\frac{x^2}{y} + 3\frac{y}{x}\right) dy = 0$

Solution

$$xy\left(3x + \frac{6}{y}\right) dx + xy\left(\frac{x^2}{y} + 3\frac{y}{x}\right) dy = 0$$

$$\left(3x^2 y + 6x\right) dx + \left(x^3 + 3y^2\right) dy = 0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(3x^2 y + 6x\right) = 3x^2 \\ N_x &= \frac{\partial}{\partial x}\left(x^3 + 3y^2\right) = 3x^2 \end{aligned}\right\} \Rightarrow M_y = N_x$$

$$\psi = \int \left(3x^2 y + 6x\right) dx = x^3 y + 3x^2 + h(y)$$

$$\psi_y = x^3 + h'(y) = x^3 + 3y^2$$

$$h'(y) = 3y^2 \quad \Rightarrow \quad h(y) = y^3$$

$$\boxed{x^3 y + 3x^2 + y^3 = C}$$

Exercise

Find an integrating factor and solve the given equation $\left(x + 3x^3 \sin y\right)dx + \left(x^4 \cos y\right)dy = 0$

Solution

$$\left.\begin{array}{l} M_y = \frac{\partial}{\partial y}\left(x + 3x^3 \sin y\right) = 3x^3 \cos y \\ N_x = \frac{\partial}{\partial x}\left(x^4 \cos y\right) = 4x^3 \cos y \end{array}\right\} \Rightarrow M_y \neq N_x$$

$$\frac{M_y - N_x}{N} = \frac{3x^3 \cos y - 4x^3 \cos y}{x^4 \cos y} = -\frac{1}{x}$$

$$\mu = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \boxed{\frac{1}{x}} \qquad\qquad \mu = e^{\int \frac{M_y - N_x}{N}dx}$$

$$\frac{1}{x}\left(x + 3x^3 \sin y\right)dx + \frac{1}{x}\left(x^4 \cos y\right)dy = 0$$

$$\left(1 + 3x^2 \sin y\right)dx + \left(x^3 \cos y\right)dy = 0$$

$$\left.\begin{array}{l} M_y = \frac{\partial}{\partial y}\left(1 + 3x^2 \sin y\right) = 3x^2 \cos y \\ N_x = \frac{\partial}{\partial x}\left(x^3 \cos y\right) = 3x^2 \cos y \end{array}\right\} \Rightarrow M_y = N_x$$

$$\psi = \int \left(1 + 3x^2 \sin y\right)dx = x + x^3 \sin y + h(y)$$

$$\psi_y = -x^3 \cos y + h'(y) = -x^3 \cos y$$

$$h'(y) = 0 \Rightarrow h(y) = C$$

$$\boxed{x + x^3 \sin y = C}$$

Exercise

Find an integrating factor and solve the given equation $\left(2x^2 + y\right)dx + \left(x^2 y - x\right)dy = 0$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(2x^2+y\right)=1 \\ N_x &= \frac{\partial}{\partial x}\left(x^2y-x\right)=2xy-1 \end{aligned}\right\} \Rightarrow M_y \neq N_x$$

$$\frac{M_y-N_x}{N}=\frac{1-2xy+1}{x^2y-x}=\frac{2(1-xy)}{x(xy-1)}=-\frac{2}{x}$$

$$\mu=e^{-\int \frac{2}{x}dx}=e^{-2\ln x}\ \underline{=x^{-2}|} \qquad \mu=e^{\int \frac{M_y-N_x}{N}dx}$$

$$x^{-2}\left(2x^2+y\right)dx+x^{-2}\left(x^2y-x\right)dy=0$$

$$\left(2+\frac{y}{x^2}\right)dx+\left(y-\frac{1}{x}\right)dy=0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(2+\frac{y}{x^2}\right)=\frac{1}{x^2} \\ N_x &= \frac{\partial}{\partial x}\left(y-\frac{1}{x}\right)=\frac{1}{x^2} \end{aligned}\right\} \Rightarrow M_y = N_x$$

$$\psi=\int\left(2+\frac{y}{x^2}\right)dx$$

$$=2x-\frac{y}{x}+h(y)$$

$$\psi_y=-\frac{1}{x}+h'(y)=y-\frac{1}{x}$$

$$h'(y)=y \Rightarrow h(y)=\frac{1}{2}y^2$$

$$\boxed{2x-\frac{y}{x}+\frac{1}{2}y^2=C}$$

Exercise

Find an integrating factor and solve the given equation $\left(3x^2+y\right)dx+\left(x^2y-x\right)dy=0$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(3x^2+y\right)=1 \\ N_x &= \frac{\partial}{\partial x}\left(x^2y-x\right)=2xy-1 \end{aligned}\right\} \Rightarrow M_y \neq N_x$$

$$\frac{M_y-N_x}{N}=\frac{1-2xy+1}{x^2y-x}$$

$$=\frac{2(1-xy)}{x(xy-1)}$$

$$=-\frac{2}{x}$$

$$e^{-\int \frac{2}{x}dx}=e^{-2\ln x}\ \underline{=x^{-2}}|$$

$$x^{-2}\left(3x^2+y\right)dx+x^{-2}\left(x^2y-x\right)dy=0$$

$$\left(3+\frac{y}{x^2}\right)dx+\left(y-\frac{1}{x}\right)dy=0$$

$$\begin{aligned} M_y&=\frac{\partial}{\partial y}\left(3+\frac{y}{x^2}\right)=\frac{1}{x^2}\\ N_x&=\frac{\partial}{\partial x}\left(y-\frac{1}{x}\right)=\frac{1}{x^2}\end{aligned}\quad \Rightarrow M_y=N_x$$

$$\psi=\int\left(3+\frac{y}{x^2}\right)dx$$

$$=3x-\frac{y}{x}+h(y)$$

$$\psi_y=-\frac{1}{x}+h'(y)$$

$$=y-\frac{1}{x}$$

$$h'(y)=y\ \Rightarrow\ h(y)=\frac{1}{2}y^2$$

$$\underline{3x-\frac{y}{x}+\frac{1}{2}y^2=C}|$$

Exercise

Find an integrating factor and solve the given equation $\left(2y^2+2y+4x^2\right)dx+(2xy+x)dy=0$

Solution

$$\begin{aligned} M_y&=\frac{\partial}{\partial y}\left(2y^2+2y+4x^2\right)=4y+2\\ N_x&=\frac{\partial}{\partial x}(2xy+x)=2y+1\end{aligned}\quad \Rightarrow M_y\neq N_x$$

$$\frac{M_y-N_x}{N}=\frac{4y+2-2y-1}{x(2y+1)}=\frac{2y+1}{x(2y+1)}=\frac{1}{x}$$

$$\mu=e^{\int \frac{1}{x}dx}=e^{\ln x}\ \underline{=x}| \qquad \mu=e^{\int \frac{M_y-N_x}{N}dx}$$

$$x\left(2y^2+2y+4x^2\right)dx+x\left(2xy+x\right)dy=0$$

$$\left(2xy^2+2xy+4x^3\right)dx+\left(2x^2y+x^2\right)dy=0$$

$$\left.\begin{array}{l} M_y=\frac{\partial}{\partial y}\left(2xy^2+2xy+4x^3\right)=4xy+2x \\ N_x=\frac{\partial}{\partial x}\left(2x^2y+x^2\right)=4xy+2x \end{array}\right. \Rightarrow M_y=N_x$$

$$\psi=\int\left(2xy^2+2xy+4x^3\right)dx$$

$$=x^2y^2+x^2y+x^4+h(y)$$

$$\psi_y=2x^2y+x^2+h'(y)=2x^2y+x^2$$

$$h'(y)=0 \Rightarrow h(y)=C$$

$$\boxed{x^2y^2+x^2y+x^4=C}$$

Exercise

Find an integrating factor and solve the given equation $\quad \left(x^4-x+y\right)dx-xdy=0$

Solution

$$\left.\begin{array}{l} M_y=\frac{\partial}{\partial y}\left(x^4-x+y\right)=1 \\ N_x=\frac{\partial}{\partial x}(-x)=-1 \end{array}\right. \Rightarrow M_y\neq N_x$$

$$\frac{M_y-N_x}{N}=-\frac{2}{x}$$

$$\mu=e^{-\int\frac{2}{x}dx}=e^{-2\ln x}\ \boxed{=x^{-2}} \qquad \mu=e^{\int\frac{M_y-N_x}{N}dx}$$

$$x^{-2}\left(x^4-x+y\right)dx-x^{-2}xdy=0$$

$$\left(x^2-x^{-1}+yx^{-2}\right)dx-x^{-1}dy=0$$

$$\left.\begin{array}{l} M_y=\frac{\partial}{\partial y}\left(x^2-x^{-1}+yx^{-2}\right)=x^{-2} \\ N_x=\frac{\partial}{\partial x}\left(-x^{-1}\right)=x^{-2} \end{array}\right. \Rightarrow M_y=N_x$$

$$\psi=\int\left(x^2-x^{-1}+yx^{-2}\right)dx$$

$$=\frac{1}{3}x^3-\ln|x|-\frac{y}{x}+h(y)$$

$$\psi_y=-\frac{1}{x}+h'(y)$$

$$=-\frac{1}{x}$$

$$h'(y)=0 \;\Rightarrow\; h(y)=C$$

$$\boxed{\frac{1}{3}x^3-\ln|x|-\frac{y}{x}=C}$$

Exercise

Find an integrating factor and solve the given equation $(2xy)dx+\left(y^2-3x^2\right)dy=0$

Solution

$$M_y=\frac{\partial}{\partial y}(2xy)=2x$$

$$N_x=\frac{\partial}{\partial x}\left(y^2-3x^2\right)=-6x \qquad \Rightarrow M_y\neq N_x$$

$$\frac{N_x-M_y}{M}=\frac{-8x}{2xy}=-\frac{4}{y}$$

$$\mu=e^{-\int\frac{4}{y}dy}=e^{-4\ln y}=\boxed{y^{-4}} \qquad \mu=e^{\int\frac{N_x-M_y}{M}dy}$$

$$y^{-4}(2xy)dx+y^{-4}\left(y^2-3x^2\right)dy=0$$

$$\left(2xy^{-3}\right)dx+\left(y^{-2}-3x^2y^{-4}\right)dy=0$$

$$M_y=\frac{\partial}{\partial y}\left(2xy^{-3}\right)=-6xy^{-4}$$

$$N_x=\frac{\partial}{\partial x}\left(y^{-2}-3x^2y^{-4}\right)=-6xy^{-4} \qquad \Rightarrow M_y=N_x$$

$$\psi=\int\left(2xy^{-3}\right)dx=x^2y^{-3}+h(y)$$

$$\psi_y=-3x^2y^{-4}+h'(y)=y^{-2}-3x^2y^{-4}$$

$$h'(y)=\frac{1}{y^2} \;\Rightarrow\; h(y)=-\frac{1}{y}$$

$$\boxed{x^2y^{-3}-\frac{1}{y}=C}$$

Exercise

Solve the given initial-value problem $\quad \frac{dy}{dx}=\frac{xy^2-\cos x\sin x}{y\left(1-x^2\right)}, \quad y(0)=2$

Solution

$$\left(xy^2-\cos x\sin x\right)dx-y\left(1-x^2\right)dy=0$$

$$M_y=\frac{\partial}{\partial y}\left(xy^2-\cos x\sin x\right)=2xy \qquad N_x=\frac{\partial}{\partial x}\left(-y+yx^2\right)=2xy \qquad \Rightarrow M_y=N_x$$

$$\begin{aligned}\psi &= \int\left(xy^2-\cos x\sin x\right)dx \\ &= \int\left(xy^2-\frac{1}{2}\sin 2x\right)dx \\ &= \frac{1}{2}x^2y^2+\frac{1}{4}\cos 2x+h(y)\end{aligned}$$

$$\psi_y=x^2y+h'(y)=-y+yx^2$$

$$h'(y)=-y \quad \Rightarrow \quad h(y)=-\frac{1}{2}y^2$$

$$\psi=\frac{1}{2}x^2y^2+\frac{1}{4}\cos 2x-\frac{1}{2}y^2=C$$

$$y(0)=2 \quad \Rightarrow \quad \frac{1}{4}-\frac{1}{2}(4)=C\to \boxed{C=-\frac{7}{4}}$$

$$\frac{1}{2}x^2y^2+\frac{1}{4}\cos 2x-\frac{1}{2}y^2=-\frac{7}{4}$$

$$\boxed{2x^2y^2+\cos 2x-2y^2=-7} \qquad 2x^2y^2+2\cos^2 x-1-2y^2=-7$$

$$x^2y^2+\cos^2 x-y^2=-3$$

Exercise

Solve the given initial-value problem $\quad (x+y)^2\,dx+\left(2xy+x^2-1\right)dy, \quad y(1)=1$

Solution

$$\left.\begin{aligned}M_y &= \frac{\partial}{\partial y}(x+y)^2=2(x+y) \\ N_x &= \frac{\partial}{\partial x}\left(2xy+x^2-1\right)=2y+2x\end{aligned}\right\} \Rightarrow M_y=N_x$$

$$\begin{aligned}\psi &= \int\left(x^2+2xy+y^2\right)dx \\ &= \frac{1}{3}x^3+x^2y+xy^2+h(y)\end{aligned}$$

$$\psi_y = x^2 + 2xy + h'(y)$$
$$= 2xy + x^2 - 1$$
$$h'(y) = -1 \quad \Rightarrow \quad h(y) = -y$$
$$\psi = \frac{1}{3}x^3 + x^2y + xy^2 - y = C$$
$$y(1) = 1 \quad \Rightarrow \quad \frac{1}{3} + 1 + 1 - 1 = C \ \rightarrow \ \boxed{C = \frac{4}{3}}$$
$$\boxed{\frac{1}{3}x^3 + x^2y + xy^2 - y = \frac{4}{3}}$$

Exercise

Solve the given initial-value problem $\left(e^x + y\right)dx + \left(2 + x + ye^y\right)dy, \quad y(0) = 1$

Solution

$$M_y = \frac{\partial}{\partial y}\left(e^x + y\right) = 1$$
$$N_x = \frac{\partial}{\partial x}\left(2 + x + ye^y\right) = 1 \quad \Rightarrow \boxed{M_y = N_x}$$
$$\psi = \int \left(e^x + y\right)dx$$
$$= e^x + xy + h(y)$$
$$\psi_y = x + h'(y)$$
$$= 2 + x + ye^y$$
$$h'(y) = 2 + ye^y$$
$$h(y) = 2y + e^y(y-1)$$
$$e^x + xy + 2y + e^y(y-1) = C$$
$$y(0) = 1 \quad \Rightarrow \quad 1 + 2 = C \ \rightarrow \ \boxed{C = 3}$$
$$\boxed{e^x + xy + 2y + e^y(y-1) = 3}$$

Exercise

Solve the given initial-value problem $(2x - y)dx + (2y - x)dy, \quad y(1) = 3$

Solution

$$\begin{cases} M_y = \frac{\partial}{\partial y}(2x-y) = -1 \\ N_x = \frac{\partial}{\partial x}(2y-x) = -1 \end{cases} \Rightarrow \underline{M_y = N_x}|$$

$$\psi = \int (2x-y)\,dx$$

$$= x^2 - xy + h(y)$$

$$\psi_y = -x + h'(y) = 2y - x$$

$$h'(y) = 2y \quad \to h(y) = y^2$$

$$x^2 - xy + y^2 = C$$

$$y(1) = 3 \quad \Rightarrow \quad 1 - 3 + 9 = C \quad \to \quad \underline{C = 7}|$$

$$\boxed{x^2 - xy + y^2 = 7}$$

$$y^2 - xy + x^2 - 7 = 0 \quad \to \quad y = \frac{x \pm \sqrt{x^2 - 4x^2 + 28}}{2} = \frac{x \pm \sqrt{-3x^2 + 28}}{2}$$

$$since\ \ y(1) = 3 \quad \to \quad y = \frac{1}{2}\left(1 \pm \sqrt{-3(1)^2 + 28}\right) = \frac{1}{2}(1 \pm 5) \begin{cases} \frac{1+5}{2} = 3 \\ \cancel{\frac{1-5}{2} = -2} \end{cases}$$

$$\underline{y(x) = \frac{x + \sqrt{-3x^2 + 28}}{2}}| \qquad |x| < \sqrt{\frac{28}{3}}$$

Exercise

Solve the given initial-value problem $\left(9x^2 + y - 1\right)dx - (4y - x)dy, \quad y(1) = 0$

Solution

$$M_y = \frac{\partial}{\partial y}\left(9x^2 + y - 1\right) = 1$$

$$N_x = \frac{\partial}{\partial x}(x - 4y) = 1 \qquad \Rightarrow \underline{M_y = N_x}|$$

$$\psi = \int \left(9x^2 + y - 1\right)dx$$

$$= 3x^3 + xy - x + h(y)$$

$$\psi_y = x + h'(y) = x - 4y$$

$$h'(y) = -4y \quad \to h(y) = -2y^2$$

$$3x^3 + xy - x - 2y^2 = C$$

$$y(1) = 0 \quad \Rightarrow \quad 3 - 1 = C \quad \to \quad \underline{C = 2}|$$

$$\boxed{-2y^2 + xy + 3x^3 - x = 2}$$

$$-2y^2 + xy + 3x^3 - x - 2 = 0 \quad \to \quad y = \frac{-x \pm \sqrt{x^2 + 24x^3 - 8x - 16}}{-4}$$

$$since \;\; y(1) = 0 \quad \to \quad y = -\frac{1}{4}\left(-1 \pm \sqrt{1 + 24 - 8 - 16}\right) = -\frac{1}{4}(-1 \pm 1) \begin{cases} -\frac{-1+1}{4} = 0 \\ \cancel{\frac{-1-1}{-4} = \frac{1}{2}} \end{cases}$$

$$\boxed{y(x) = \frac{x - \sqrt{x^2 + 24x^3 - 8x - 16}}{4}}$$

Exercise

Solve $\qquad \dfrac{dy}{dx} = \dfrac{xy^2 - \cos x \sin x}{y\left(1 - x^2\right)}, \quad y(0) = 2$

Solution

$$\left(xy^2 - \cos x \sin x\right)dx - y\left(1 - x^2\right)dy = 0$$

$$\left(xy^2 - \cos x \sin x\right)dx + y\left(x^2 - 1\right)dy = 0$$

$$M(x, y) = xy^2 - \cos x \sin x \qquad N(x, y) = y\left(x^2 - 1\right)$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 2xy \\ \frac{\partial N}{\partial x} &= 2xy \end{aligned}\right. \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\begin{aligned} \psi &= \int \left(xy^2 - \cos x \sin x\right) dx \\ &= \int xy^2 \, dx - \int \sin x \, d(\sin x) \\ &= \frac{1}{2}x^2y^2 - \frac{1}{2}\sin^2 x + h(y) \end{aligned}$$

$$\psi_y = x^2y + h'(y) = x^2y - y$$

$$h'(y) = -y \quad \Rightarrow \quad h(y) = -\frac{1}{2}y^2$$

$$\boxed{\frac{1}{2}x^2y^2 - \frac{1}{2}\sin^2 x - \frac{1}{2}y^2 = C}$$

$$y(0) = 2 \quad \to \quad \boxed{-2 = C}$$

$$\boxed{\frac{1}{2}x^2y^2 - \frac{1}{2}\sin^2 x - \frac{1}{2}y^2 = -2}$$

Exercise

Solve the given initial-value problem $\frac{dy}{dx}=(-2x+y)^2-7, \quad y(0)=0$

Solution

Let $u=-2x+y \quad \Rightarrow \quad \frac{du}{dx}=-2+\frac{dy}{dx}$

$$\frac{du}{dx}+2=u^2-7$$

$$\frac{du}{dx}=u^2-9$$

$$\int \frac{du}{u^2-9}=\int dx$$

$$\frac{1}{u^2-9}=\frac{A}{u-3}+\frac{B}{u+3}$$

$$Au+3A+Bu-3B=0 \quad \begin{cases} A+B=0 \\ 3A-3B=1 \end{cases} \rightarrow A=\frac{1}{6}, \quad B=-\frac{1}{6}$$

$$\frac{1}{6}\int\left(\frac{1}{u-3}-\frac{1}{u+3}\right)du=\int dx$$

$$\frac{1}{6}\left(\ln|u-3|-\ln|u+3|\right)=x+C$$

$$\ln\left|\frac{u-3}{u+3}\right|=6x+C$$

$$\frac{u-3}{u+3}=e^{6x+C}=Ae^{6x}$$

$$u-3=Aue^{6x}+3Ae^{6x}$$

$$u=\frac{3+3Ae^{6x}}{1-Ae^{6x}}=-2x+y$$

$$y=2x+\frac{3+3Ae^{6x}}{1-Ae^{6x}} \qquad y(0)=0$$

$$0=\frac{3+3A}{1-A} \rightarrow \underline{A=-1}$$

$$\boxed{y(x)=2x+\frac{3\left(1-e^{6x}\right)}{1+e^{6x}}}$$

Exercise

Solve the given initial-value problem $(2y-x)y'-y+2x=0, \quad y(1)=0$

Solution

$$M_y=\frac{\partial}{\partial y}(2y-x)=2$$

$$N_x=\frac{\partial}{\partial x}(-y+2x)=2$$

$$\Rightarrow \underline{M_y=N_x}$$

$$\psi = \int(-y+2x)\,dx = x^2 - xy + h(y)$$

$$\psi_y = -x + h'(y) = 2y - x \quad \Rightarrow h'(y) = 2y \quad \to h(y) = y^2$$

$$x^2 - xy + y^2 = C$$

$$y(1) = 0 \quad \to 1 = C$$

$$x^2 - xy + y^2 = 1$$

$$y^2 - xy + x^2 - 1 = 0$$

$$y = \frac{x \pm \sqrt{x^2 - 4x^2 + 4}}{2} = \frac{x \pm \sqrt{4 - 3x^2}}{2}$$

Since $y(1) = 0 \quad \to \boxed{y(x) = \frac{x - \sqrt{4 - 3x^2}}{2}}$

Exercise

Solve the given initial-value problem $\left(x + y^3\right)y' + y + x^3 = 0, \quad y(0) = -2$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(y + x^3\right) = 1 \\ N_x &= \frac{\partial}{\partial x}\left(x + y^3\right) = 1 \end{aligned}\right\} \Rightarrow \boxed{M_y = N_x}$$

$$\psi = \int\left(y + x^3\right)dx = xy + \frac{1}{4}x^4 + h(y)$$

$$\psi_y = x + h'(y) = x + y^3$$

$$h'(y) = y^3 \quad \to h(y) = \frac{1}{4}y^4$$

$$xy + \frac{1}{4}x^4 + \frac{1}{4}y^4 = C$$

$$y(0) = -2 \quad \to \boxed{4 = C}$$

$$\boxed{xy + \frac{1}{4}x^4 + \frac{1}{4}y^4 = 4}$$

Exercise

Solve the given initial-value problem $y' = \left(3x^2+1\right)\left(y^2+1\right), \quad y(0)=1$

Solution

$$\frac{1}{y^2+1}y' - \left(3x^2+1\right) = 0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(-3x^2-1\right) = 0 \\ N_x &= \frac{\partial}{\partial x}\left(\frac{1}{y^2+1}\right) = 0 \end{aligned}\right\} \Rightarrow M_y = N_x$$

$$\psi = \int\left(-3x^2-1\right)dx = -x^3 - x + h(y)$$

$$\psi_y = h'(y) = \frac{1}{y^2+1} \quad \to h(y) = \tan^{-1} y$$

$$-x^3 - x + \tan^{-1} y = C$$

$$y(0)=1 \quad \to \tan^{-1} 1 = C \;\Rightarrow\; \boxed{C = \frac{\pi}{4}}$$

$$-x^3 - x + \tan^{-1} y = \frac{\pi}{4}$$

$$\tan^{-1} y = x^3 + x + \frac{\pi}{4}$$

$$\boxed{y(x) = \tan\left(x^3 + x + \frac{\pi}{4}\right)}$$

Exercise

Solve the given initial-value problem $\left(y^3+\cos t\right)y' = 2 + y\sin t, \quad y(0)=-1$

Solution

$$\left(y^3+\cos t\right)y' - (2 + y\sin t) = 0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}(-2 - y\sin t) = -\sin t \\ N_t &= \frac{\partial}{\partial t}\left(y^3+\cos t\right) = -\sin t \end{aligned}\right\} \Rightarrow \boxed{M_y = N_t}$$

$$\begin{aligned} \psi &= \int(-2 - y\sin t)\,dt \\ &= -2t + y\cos t + h(y) \end{aligned}$$

$$\psi_y = \cos t + h'(y) = y^3 + \cos t$$

$$h'(y) = y^3 \quad \to h(y) = \frac{1}{4}y^4$$

$$-2t + y\cos t + \frac{1}{4}y^4 = C$$

$$y(0) = -1 \quad \rightarrow -1 + \frac{1}{4} = C \;\Rightarrow\; \boxed{C = -\frac{3}{4}}$$

$$\boxed{-2t + y\cos t + \frac{1}{4}y^4 = -\frac{3}{4}}$$

Exercise

Solve the given initial-value problem $\left(y^3 - t^3\right)y' = 3t^2y + 1, \quad y(-2) = -1$

Solution

$$\left(y^3 - t^3\right)y' - \left(3t^2y + 1\right) = 0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(-3t^2y - 1\right) = -3t^2 \\ N_t &= \frac{\partial}{\partial t}\left(y^3 - t^3\right) = -3t^2 \end{aligned}\right. \quad \Rightarrow \boxed{M_y = N_t}$$

$$\psi = \int \left(-3t^2y - 1\right)dt$$

$$= -t^3y - t + h(y)$$

$$\psi_y = -t^3 + h'(y)$$

$$= y^3 - t^3$$

$$h'(y) = y^3 \quad \rightarrow h(y) = \frac{1}{4}y^4$$

$$-t^3y - t + \frac{1}{4}y^4 = C$$

$$y(-2) = -1 \quad -8 + 2 + \frac{1}{4} = C \quad \Rightarrow \boxed{C = -\frac{23}{4}}$$

$$\boxed{-t^3y - t + \frac{1}{4}y^4 = -\frac{23}{4}}$$

Exercise

Solve the given initial-value problem $\left(e^{2y} + t^2y\right)y' + ty^2 + \cos t = 0, \quad y\left(\frac{\pi}{2}\right) = 0$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(ty^2 + \cos t\right) = 2yt \\ N_t &= \frac{\partial}{\partial t}\left(e^{2y} + t^2y\right) = 2ty \end{aligned}\right. \quad \Rightarrow \boxed{M_y = N_t}$$

$$\psi = \int \left(ty^2 + \cos t\right) dt$$

$$= \frac{1}{2}t^2y^2 + \sin t + h(y)$$

$$\psi_y = t^2y + h'(y)$$

$$= t^2y + e^{2y}$$

$$h'(y) = e^{2y} \quad \rightarrow h(y) = \frac{1}{2}e^{2y}$$

$$\frac{1}{2}t^2y^2 + \sin t + \frac{1}{2}e^{2y} = C$$

$$y\left(\frac{\pi}{2}\right) = 0 \quad 1 + \frac{1}{2} = C \Rightarrow \boxed{C = \frac{3}{2}}$$

$$\boxed{\frac{1}{2}t^2y^2 + \sin t + \frac{1}{2}e^{2y} = \frac{3}{2}}$$

Exercise

Solve the given initial-value problem $\quad y' = -\dfrac{y\cos(ty) + 1}{t\cos(ty) + 2ye^{y^2}}, \quad y(\pi) = 0$

Solution

$$\left(t\cos(ty) + 2ye^{y^2}\right)y' + \left(y\cos(ty) + 1\right) = 0$$

$$M_y = \frac{\partial}{\partial y}\left(y\cos ty + 1\right) = \cos ty - ty\sin ty$$

$$N_t = \frac{\partial}{\partial t}\left(t\cos ty + 2ye^{y^2}\right) = \cos ty - ty\sin ty \qquad \Rightarrow \boxed{M_y = N_t}$$

$$\psi = \int \left(y\cos ty + 1\right) dt$$

$$= \sin ty + t + h(y)$$

$$\psi_y = t\cos ty + h'(y)$$

$$= t\cos ty + 2ye^{y^2}$$

$$h'(y) = 2ye^{y^2} \quad \rightarrow h(y) = e^{y^2}$$

$$\sin ty + t + e^{y^2} = C$$

$$y(\pi) = 0 \quad \Rightarrow \boxed{C = -\pi - 1}$$

$$\boxed{\sin ty + t + e^{y^2} = \pi + 1}$$

Exercise

Solve the given initial-value problem $\left(2ty+\dfrac{1}{y}\right)y'+y^2=1, \quad y(1)=1$

Solution

$$\left(2ty+\frac{1}{y}\right)y'+y^2-1=0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(y^2-1\right)=2y \\ N_t &= \frac{\partial}{\partial t}\left(2ty+\frac{1}{y}\right)=2y \end{aligned}\right\} \Rightarrow \boxed{M_y=N_t}$$

$$\psi=\int\left(y^2-1\right)dt$$

$$=ty^2-t+h(y)$$

$$\psi_y=t\cos ty+h'(y)$$

$$=2ty+h'(y)$$

$$=2ty+\frac{1}{y}$$

$$h'(y)=\frac{1}{y} \quad \to h(y)=\ln y$$

$$ty^2-t+\ln|y|=C$$

$$y(1)=1 \quad \Rightarrow \boxed{C=0}$$

$$\boxed{ty^2-t+\ln|y|=0}$$

Exercise

Solve the given initial-value problem $\left(ye^x+1\right)dx+\left(e^x-1\right)dy=0 \quad y(1)=1$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(ye^x+1\right)=e^x \\ N_x &= \frac{\partial}{\partial x}\left(e^x-1\right)=e^x \end{aligned}\right\} \Rightarrow M_y=N_x$$

$$\psi=\int\left(ye^x+1\right)dx$$

$$=ye^x+x+h(y)$$

$$\psi_y=e^x+h'(y)=e^x-1$$

$$\to h'(y)=-1 \Rightarrow h(y)=-y$$

$$\boxed{ye^x + x - y = C}$$

$y(1) = 1 \quad \rightarrow \quad \boxed{C = e}$

$$\boxed{ye^x + x - y = e}$$

Exercise

Solve the given initial-value problem $\quad 2xy^2 + 4 = 2\left(3 - x^2 y\right)y' \quad y(-1) = 8$

Solution

$$2xy^2 + 4 - 2\left(3 - x^2 y\right)y' = 0$$

$$\left.\begin{aligned} M &= 2xy^2 + 4 \quad \Rightarrow \quad M_y = 4xy \\ N &= -6 + 2x^2 y \quad \Rightarrow \quad N_x = 4xy \end{aligned}\right\} \rightarrow \quad M_y = N_x$$

$$\psi = \int \left(2xy^2 + 4\right) dx \qquad\qquad \psi = \int M dx$$

$$= x^2 y^2 + 4x + h(y)$$

$$\psi_y = 2x^2 y + h'(y)$$

$$= 2x^2 y - 6$$

$$h'(y) = -6 \quad \rightarrow \quad h(y) = -6y$$

$$\boxed{x^2 y^2 + 4x - 6y = C}$$

$y(-1) = 8 \quad \rightarrow \quad 64 - 4 - 48 = C \quad \boxed{C = 12}$

$$\boxed{x^2 y^2 + 4x - 6y = 12}$$

Exercise

Solve the given initial-value problem $\quad y' + \frac{4}{x}y = x^3 y^2 \quad y(2) = -1$

Solution

Let $u = y^{1-2} = y^{-1} \quad \Rightarrow \quad y = \frac{1}{u}$

$$\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx} \quad \Rightarrow \quad u' = -\frac{1}{y^2}y'$$

$$y' = -y^2 u' = -\frac{1}{u^2}u'$$

$$y' + \frac{4}{x}y = x^3 y^2$$

$$-\frac{1}{u^2}u' + \frac{4}{x}\frac{1}{u} = x^3\frac{1}{u^2}$$

$$u' - \frac{4}{x}u = -x^3$$

$$e^{\int -\frac{4}{x}dx} = e^{-4\ln x} = x^{-4}$$

$$\int -x^3 x^{-4} dx = -\int \frac{1}{x}dx = -\ln x$$

$$u = x^4(-\ln x + C)$$

$$y = \frac{1}{x^4(C - \ln x)} \qquad y = \frac{1}{u}$$

$$y(2) = -1 \quad \to -1 = \frac{1}{16(C - \ln 2)} \qquad \boxed{C = \ln 2 - \frac{1}{16}}$$

$$\boxed{y(x) = \frac{1}{x^4\left(\ln 2 - \frac{1}{16} - \ln x\right)}}$$

Exercise

Solve the given initial-value problem $\quad y' = 5y + e^{-2x}y^{-2} \quad y(0) = 2$

Solution

$$y^2y' - 5y^3 = e^{-2x}$$

Let $u = y^3 \quad \Rightarrow \quad y = u^{1/3}$

$$u' = 3y^2y' \quad \Rightarrow \quad y' = \frac{1}{3}u^{-2/3}u'$$

$$\frac{1}{3}u' - 5u = e^{-2x}$$

$$u' - 15u = 3e^{-2x}$$

$$e^{\int -15dx} = e^{-15x}$$

$$\int 3e^{-2x}e^{-15x}dx = 3\int e^{-17x}dx = -\frac{3}{17}e^{-17x}$$

$$u = e^{15x}\left(-\frac{3}{17}e^{-17x} + C\right) \qquad u = y^3$$

$$y^3 = e^{15x}C - \frac{3}{17}e^{-2x}$$

$$y(0) = 2 \quad \to \quad 8 = C - \frac{3}{17} \qquad \boxed{C = \frac{139}{17}}$$

$$\boxed{y(x) = \left(\frac{139e^{15x} - 3e^{-2x}}{17}\right)^{1/3}}$$

Exercise

Solve the given initial-value problem $6y' - 2y = xy^4 \quad y(0) = -2$

Solution

$$6y^{-4}y' - 2y^{-3} = x$$

Let $u = y^{-3} \quad \Rightarrow \quad y = u^{-1/3}$

$$y' = -\frac{1}{3}u^{-4/3}u'$$

$$6u^{4/3}\left(-\frac{1}{3}u^{-4/3}\right)u' - 2u = x$$

$$-2u' - 2u = x$$

$$u' + u = -\frac{1}{2}x$$

$$e^{\int dx} = e^x$$

$$-\int \frac{1}{2}xe^x dx = -\frac{1}{2}(x-1)e^x$$

$$u = \frac{1}{e^x}\left(-\frac{1}{2}(x-1)e^x + C\right) \qquad u = y^{-3}$$

$$y^{-3} = \frac{C}{e^x} - \frac{1}{2}(x-1)$$

$$y(0) = -2 \quad \to \quad -\frac{1}{8} = C + \frac{1}{2} \quad \boxed{C = -\frac{5}{8}}$$

$$\frac{1}{y^3} = -\frac{5}{8}e^{-x} - \frac{1}{2}(x-1)$$

$$= -\frac{5e^{-x} - 4x + 4}{8}$$

$$\boxed{y(x) = -\frac{2}{\left(5e^{-x} - 4x + 4\right)^{1/3}}}$$

Exercise

Solve the given initial-value problem $y' + \frac{y}{x} - \sqrt{y} = 0 \quad y(1) = 0$

Solution

$$y' + \frac{1}{x}y = y^{1/2}$$

$$y^{-1/2}y' + \frac{1}{x}y^{1/2} = 1$$

Let $u = y^{1/2} \quad \Rightarrow \quad y = u^2$

$$y' = 2u\,u'$$

$$u^{-1}(2uu') + \frac{1}{x}u = 1$$

$$2u' + \frac{1}{x}u = 1$$

$$u' + \frac{1}{2x}u = \frac{1}{2}$$

$$e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\ln x} = x^{1/2}$$

$$\int \frac{1}{2}x^{1/2}dx = \frac{1}{3}x^{3/2}$$

$$u = \frac{1}{x^{1/2}}\left(\frac{1}{3}x^{3/2} + C\right)$$

$$y^{1/2} = \frac{1}{3}x + Cx^{-1/2}$$

$$y(1) = 0 \quad \to \quad 0 = C + \frac{1}{3} \quad \boxed{C = -\frac{1}{3}}$$

$$\boxed{y(x) = \left(\frac{1}{3}x + \frac{1}{3}x^{-1/2}\right)^2}$$

Exercise

Solve the given initial-value problem $\quad xyy' + 4x^2 + y^2 = 0 \quad y(2) = -7$

Solution

$$xyy' = -4x^2 - y^2$$

$$\frac{y}{x}y' = -4 - \left(\frac{y}{x}\right)^2$$

Let $\frac{y}{x} = v \quad \to y = xv \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v\left(v + x\frac{dv}{dx}\right) = -4 - v^2$$

$$vx\frac{dv}{dx} = -4 - 2v^2$$

$$-\frac{v}{4 + 2v^2}dv = \frac{dx}{x}$$

$$-\frac{1}{4}\int \frac{1}{4 + 2v^2}d\left(4 + 2v^2\right) = \int \frac{1}{x}dx$$

$$-\frac{1}{4}\ln\left(4 + 2v^2\right) = \ln x + \ln C_1$$

$$\ln\left(4 + 2v^2\right)^{-\frac{1}{4}} = \ln C_1 x$$

$$\left(4 + 2v^2\right)^{-\frac{1}{4}} = C_1 x$$

$$4+2\left(\frac{y}{x}\right)^2 = Cx^{-4}$$

$$\frac{2y^2}{x^2} = \frac{C}{x^4} - 4$$

$$y^2 = \frac{1}{2}\frac{C-4x^4}{x^2}$$

$$y(2) = -7 \quad \to 49 = \frac{1}{2}\frac{C-64}{4} \quad \to \quad \underline{C = 456|}$$

$$y^2 = \frac{1}{2}\frac{456-4x^4}{x^2}$$

$$y^2 = \frac{228-2x^4}{x^2}$$

$$y = \pm\frac{\sqrt{228-2x^4}}{x}$$

Since the given initial $y(2) = -7$

$$\boxed{y = -\frac{\sqrt{228-2x^4}}{x}}$$

Exercise

Solve the given initial-value problem $\quad xy' = y(\ln x - \ln y) \quad y(1) = 4$

Solution

$$y' = \frac{y}{x}\ln\frac{x}{y}$$

Let $\quad \frac{y}{x} = v \quad \to y = xv \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + xv' = v\ln\frac{1}{v}$$

$$x\frac{dv}{dx} = -v(1+\ln v)$$

$$\int\frac{dv}{v(1+\ln v)} = -\int\frac{dx}{x}$$

$$\int\frac{d(1+\ln v)}{1+\ln v} = -\int\frac{dx}{x}$$

$$\ln(1+\ln v) = -\ln x + \ln C$$

$$\ln\left(1+\ln\frac{y}{x}\right) = \ln\frac{C}{x}$$

$$1+\ln\frac{y}{x} = \frac{C}{x}$$

$$y(1) = 4 \quad \to \quad \underline{C = 1+\ln 4|}$$

$$\ln\frac{y}{x} = \frac{1+\ln 4}{x} - 1$$

$$\boxed{y(x) = xe^{\frac{1+\ln 4}{x} - 1}}$$

Exercise

Solve the given initial-value problem $\quad y' - (4x - y + 1)^2 = 0 \quad y(0) = 2$

Solution

Let $\quad v = 4x - y \quad \Rightarrow \quad v' = 4 - y'$

$$y' = 4 - v'$$

$$4 - v' - (v+1)^2 = 0$$

$$v' = 4 - (v+1)^2$$

$$= -v^2 - 2v + 3$$

$$\int \frac{dv}{v^2 + 2v - 3} = -\int dx$$

$$\int \frac{dv}{(v-1)(v+3)} = -\int dx$$

$$\frac{1}{(v-1)(v+3)} = \frac{A}{v-1} + \frac{B}{v+3}$$

$$Av + 3A + Bv - B = 1$$

$$\begin{cases} A + B = 0 \\ 3A - B = 1 \end{cases} \rightarrow \boxed{A = \frac{1}{4},\ B = -\frac{1}{4}}$$

$$\frac{1}{4}\int \left(\frac{1}{v-1} - \frac{1}{v+3}\right) = -\int dx$$

$$\frac{1}{4}\left(\ln(v-1) - \ln(v+3)\right) = -x + C_1$$

$$\ln\frac{v-1}{v+3} = C - 4x$$

$$\ln\left|\frac{4x - y - 1}{4x - y + 3}\right| = C - 4x$$

$$y(0) = 2 \quad \rightarrow \quad \boxed{C = \ln 3}$$

$$\frac{4x - y - 1}{4x - y + 3} = e^{\ln 3 - 4x}$$

$$4x - y - 1 = 3(4x - y + 3)e^{-4x}$$

$$(4x - y - 1)e^{4x} = 12x - 3y + 9$$

$$3y - ye^{4x} = 12x + 9 + (1 - 4x)e^{4x}$$

$$\boxed{y(x) = \frac{12x + 9 + (1 - 4x)e^{4x}}{3 - e^{4x}}}$$

Exercise

Solve the given initial-value problem $\left(e^{t+y}+2y\right)y'+\left(e^{t+y}+3t^2\right)=0,\quad y(0)=0$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(e^{t+y}+3t^2\right)=e^{t+y} \\ N_t &= \frac{\partial}{\partial t}\left(e^{t+y}+2y\right)=e^{t+y} \end{aligned}\right\} \Rightarrow \underline{M_y=N_t}$$

$$\begin{aligned}\psi &= \int\left(e^{t+y}+3t^2\right)dt \\ &= e^{t+y}+t^3+h(y)\end{aligned}$$

$$\psi_y = e^{t+y}+h'(y)=e^{t+y}+2y$$

$$h'(y)=2y \quad \rightarrow h(y)=y^2$$

$$e^{t+y}+t^3+y^2=C$$

$$y(0)=0 \quad \Rightarrow \quad \underline{C=1}$$

$$\underline{e^{t+y}+t^3+y^2=1}$$

Exercise

Solve the given initial-value problem $(4y+2x-5)dx+(6y+4x-1)dy,\quad y(-1)=2$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}(4y+2x-5)=4 \\ N_x &= \frac{\partial}{\partial x}(6y+4x-1)=4 \end{aligned}\right\} \Rightarrow \underline{M_y=N_x}$$

$$\begin{aligned}\psi &= \int(4y+2x-5)dx \\ &= 4xy+x^2-5x+h(y)\end{aligned}$$

$$\begin{aligned}\psi_y &= 4x+h'(y) \\ &= 6y+4x-1\end{aligned}$$

$$\Rightarrow h'(y)=6y-1 \quad \rightarrow h(y)=3y^2-y$$

$$\psi=4xy+x^2-5x+3y^2-y=C$$

$$y(-1)=2 \quad \Rightarrow \quad 4(-1)(2)+1+5+12-2=C \ \rightarrow \ \underline{C=8}$$

$$\underline{4xy+x^2-5x+3y^2-y=8}$$

Exercise

Solve the given initial-value problem $\left(ye^{xy}-\frac{1}{y}\right)dx+\left(xe^{xy}+\frac{x}{y^2}\right)dy=0 \quad y(1)=1$

Solution

$$M_y=\frac{\partial}{\partial y}\left(ye^{xy}-\frac{1}{y}\right)=(1+xy)e^{xy}+\frac{1}{y^2}$$

$$N_x=\frac{\partial}{\partial x}\left(xe^{xy}+\frac{x}{y^2}\right)=(1+xy)e^{xy}+\frac{1}{y^2}$$

$$\Rightarrow M_y=N_x$$

$$\psi=\int\left(ye^{xy}-\frac{1}{y}\right)dx=e^{xy}-\frac{x}{y}+h(y)$$

$$\psi_y=xe^{xy}+\frac{x}{y^2}+h'(y)=xe^{xy}+\frac{x}{y^2}$$

$$\to h'(y)=0 \Rightarrow h(y)=C$$

$$\boxed{e^{xy}-\frac{x}{y}=C}$$

$$y(1)=1 \quad \to \underline{C=e-1}$$

$$\boxed{e^{xy}-\frac{x}{y}=e-1}$$

Exercise

Solve the given initial-value problem $(2y\ln t-t\sin y)y'+\frac{1}{t}y^2+\cos y=0, \quad y(2)=0$

Solution

$$M_y=\frac{\partial}{\partial y}\left(\frac{1}{t}y^2+\cos y\right)=\frac{2y}{t}-\sin y$$

$$N_t=\frac{\partial}{\partial t}(2y\ln t-t\sin y)=\frac{2y}{t}-\sin y$$

$$\Rightarrow \boxed{M_y=N_t}$$

$$\psi=\int\left(\frac{1}{t}y^2+\cos y\right)dt$$

$$=y^2\ln|t|+t\cos y+h(y)$$

$$\psi_y=2y\ln t-t\sin y+h'(y)=2y\ln t-t\sin y$$

$$\Rightarrow h'(y)=0 \qquad \to h(y)=C$$

$$y^2\ln|t|+t\cos y+C=0$$

$$y(2)=0 \quad \Rightarrow \underline{C=-2}$$

$$\boxed{y^2\ln|t|+t\cos y-2=0}$$

Exercise

Solve the given initial-value problem $\quad (\tan y - 2)dx + \left(x\sec^2 y + \frac{1}{y}\right)dy = 0 \quad y(0) = 1$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}(\tan y - 2) = \sec^2 y \\ N_x &= \frac{\partial}{\partial x}\left(x\sec^2 y + \frac{1}{y}\right) = \sec^2 y \end{aligned}\right. \quad \Rightarrow M_y = N_x$$

$$\begin{aligned} \psi &= \int (\tan y - 2)dx \\ &= x\tan y - 2x + h(y) \end{aligned}$$

$$\begin{aligned} \psi_y &= x\sec^2 y + h'(y) \\ &= x\sec^2 y + \frac{1}{y} \end{aligned}$$

$$\to h'(y) = \frac{1}{y} \quad \Rightarrow \quad h(y) = \ln|y|$$

$$\underline{x\tan y - 2x + \ln|y| = C|}$$

$$y(0) = 1 \quad \to \quad \underline{C = 0|}$$

$$\underline{x\tan y - 2x + \ln y = 0|}$$

Exercise

Solve the given initial-value problem $\quad 2xy - 9x^2 + \left(2y + x^2 + 1\right)\frac{dy}{dx} = 0 \quad y(0) = -3$

Solution

$$\left.\begin{aligned} M &= 2xy - 9x^2 \quad \Rightarrow \quad M_y = 2x \\ N &= 2y + x^2 + 1 \quad \Rightarrow \quad N_x = 2x \end{aligned}\right. \quad \to \quad M_y = N_x$$

$$\psi = \int \left(2xy - 9x^2\right)\, dx \qquad\qquad \psi = \int M dx$$

$$= x^2 y - 3x^3 + h(y)$$

$$\begin{aligned} \psi_y &= x^2 + h'(y) \\ &= 2y + x^2 + 1 \end{aligned}$$

$$h'(y) = 2y + 1 \quad \to \quad h(y) = y^2 + y$$

$$\underline{x^2 y - 3x^3 + y^2 + y = C|}$$

$y(0) = -3 \quad \rightarrow \; 9 - 3 = C \quad \boxed{C = 6}$

$$\boxed{x^2 y - 3x^3 + y^2 + y = 6}$$

Exercise

Solve the given initial-value problem $\quad \dfrac{2t}{t^2+1} y - 2t + \left(2 - \ln\left(t^2+1\right)\right)\dfrac{dy}{dt} = 0 \quad y(5) = 0$

Solution

$$\left.\begin{aligned} M &= \frac{2t}{t^2+1} y - 2t \quad \Rightarrow \quad M_y = \frac{2t}{t^2+1} \\ N &= 2 - \ln\left(t^2+1\right) \quad \Rightarrow \quad N_t = \frac{2t}{t^2+1} \end{aligned}\right\} \rightarrow \; M_y = N_t$$

$$\psi = \int \left(\frac{2t}{t^2+1} y - 2t\right) dt \qquad\qquad \psi = \int M dt$$

$$= y \int \frac{1}{t^2+1} d\left(t^2+1\right) - t^2$$

$$= \ln\left(t^2+1\right) y - t^2 + h(y)$$

$$\psi_y = \ln\left(t^2+1\right) + h'(y)$$

$$= 2 - \ln\left(t^2+1\right)$$

$$h'(y) = -2 \quad \rightarrow \quad h(y) = -2y$$

$$\ln\left(t^2+1\right) y - t^2 - 2y = C$$

$y(5) = 0 \quad \rightarrow \quad \boxed{C = -25}$

$$\left(\ln\left(t^2+1\right) - 2\right) y - t^2 = -25$$

$$\boxed{y(t) = \frac{t^2 - 25}{\ln\left(t^2+1\right) - 2}}$$

Exercise

Solve the given initial-value problem $\quad 3y^3 e^{3xy} - 1 + \left(2ye^{3xy} + 3xy^2 e^{3xy}\right) y' = 0 \quad y(0) = 1$

Solution

$$M = 3y^3 e^{3xy} - 1 \qquad \Rightarrow \quad M_y = 9y^2 e^{3xy} + 9xy^3 e^{3xy}$$

$$N = 2ye^{3xy} + 3xy^2 e^{3xy} \quad \Rightarrow \quad N_x = 6y^2 e^{3xy} + 3y^2 e^{3xy} + 9xy^3 e^{3xy}$$

$$\rightarrow \quad M_y = N_x$$

$$\psi = \int \left(3y^3 e^{3xy} - 1\right)\, dx \qquad\qquad \psi = \int M dx$$

$$= y^2 e^{3xy} - x + h(y)$$

$$\psi_y = 2ye^{3xy} + 3xy^2 e^{3xy} + h'(y)$$

$$= 2ye^{3xy} + 3xy^2 e^{3xy}$$

$$\Rightarrow h'(y) = 0 \quad \rightarrow \quad h(y) = C$$

$$y^2 e^{3xy} - x = C$$

$$y(0) = 1 \quad \rightarrow \quad \underline{C = 1}$$

$$\boxed{y^2 e^{3xy} - x = 1}$$

Exercise

Solve the given initial-value problem $2xydx + \left(1 + x^2\right)dy = 0$; $y(2) = -5$

Solution

$$M = 2xy \quad \Rightarrow \quad M_y = 2x$$

$$N = 1 + x^2 \quad \Rightarrow \quad N_x = 2x \qquad \rightarrow \quad M_y = N_x$$

$$\psi = \int (2xy)\, dx = x^2 y + h(y)$$

$$\psi_y = x^2 + h'(y) = 1 + x^2$$

$$\Rightarrow h'(y) = 1 \quad \rightarrow \quad h(y) = y$$

$$x^2 y + y = C$$

$$y\left(x^2 + 1\right) = C$$

$$y(2) = -5 \quad \rightarrow -20 - 5 = C \quad \Rightarrow \underline{C = -25}$$

$$\boxed{y(x) = -\frac{25}{1 + x^2}}$$

Exercise

Solve the given initial-value problem $\dfrac{dy}{dx} = -\dfrac{2x\cos y + 3x^2 y}{x^3 - x^2 \sin y - y}$; $y(0) = 2$

Solution

$$\left(x^3 - x^2 \sin y - y\right)dy = -\left(2x\cos y + 3x^2 y\right)dx$$

$$\left(2x\cos y + 3x^2 y\right)dx + \left(x^3 - x^2 \sin y - y\right)dy = 0$$

$$\begin{aligned} M &= 2x\cos y + 3x^2 y \quad \Rightarrow \quad M_y = -2x\sin y + 3x^2 \\ N &= x^3 - x^2 \sin y - y \quad \Rightarrow \quad N_x = 3x^2 - 2x\sin y \end{aligned} \quad \to \quad M_y = N_x$$

$$\psi = \int \left(2x\cos y + 3x^2 y\right) dx = x^2 \cos y + x^3 y + h(y)$$

$$\psi_y = -x^2 \sin y + x^3 + h'(y) = x^3 - x^2 \sin y - y$$

$$\Rightarrow h'(y) = -y \quad \to \quad h(y) = -\frac{1}{2}y^2$$

$$x^2 \cos y + x^3 y - \frac{1}{2}y^2 = C$$

$$y(0) = 2 \quad \to \quad \underline{-2 = C}$$

$$\boxed{x^2 \cos y + x^3 y - \frac{1}{2}y^2 = -2}$$

Exercise

Find an integrating factor of the form $x^n y^m$ and solve the equation

$$\left(2y^2 - 6xy\right)dx + \left(3xy - 4x^2\right)dy = 0$$

Solution

$$x^n y^m \left(2y^2 - 6xy\right)dx + x^n y^m \left(3xy - 4x^2\right)dy = 0$$

$$\left(2x^n y^{m+2} - 6x^{n+1} y^{m+1}\right)dx + \left(3x^{n+1} y^{m+1} - 4x^{n+2} y^m\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(2x^n y^{m+2} - 6x^{n+1} y^{m+1}\right) = 2(m+2)x^n y^{m+1} - 6(m+1)x^{n+1} y^m$$

$$N_x = \frac{\partial}{\partial x}\left(3x^{n+1} y^{m+1} - 4x^{n+2} y^m\right) = 3(n+1)x^n y^{m+1} - 4(n+2)x^{n+1} y^m$$

For the equation to be exact, then

$$2(m+2)x^n y^{m+1} - 6(m+1)x^{n+1} y^m = 3(n+1)x^n y^{m+1} - 4(n+2)x^{n+1} y^m$$

$$2(m+2)y - 6(m+1)x = 3(n+1)y - 4(n+2)x$$

$$\begin{cases} 2m + 4 = 3n + 3 \\ 3m + 3 = 2n + 4 \end{cases} \quad \to \quad \begin{cases} 2m - 3n = -1 \\ 3m - 2n = 1 \end{cases} \quad \Rightarrow \quad \underline{m = 1, \ \ n = 1}$$

$$xy\left(2y^2 - 6xy\right)dx + xy\left(3xy - 4x^2\right)dy = 0$$

$$\left(2xy^3 - 6x^2y^2\right)dx + \left(3x^2y^2 - 4x^3y\right)dy = 0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(2xy^3 - 6x^2y^2\right) = 6xy^2 - 12x^2y \\ N_x &= \frac{\partial}{\partial x}\left(3x^2y^2 - 4x^3y\right) = 6xy^2 - 12x^2y \end{aligned}\right. \Rightarrow M_y = N_x$$

$$\psi = \int \left(2xy^3 - 6x^2y^2\right)dx$$

$$= x^2y^3 - 2x^3y^2 + h(y)$$

$$\psi_y = 3x^2y^2 - 4x^3y + h'(y)$$

$$= 3x^2y^2 - 4x^3y$$

$$h'(y) = 0 \Rightarrow h(y) = C$$

$$\underline{x^2y^3 - 2x^3y^2 = C}|$$

Exercise

Find an integrating factor of the form $x^n y^m$ and solve the equation

$$(12 + 5xy)dx + \left(6xy^{-1} + 3x^2\right)dy = 0$$

Solution

$$x^n y^m (12 + 5xy)dx + x^n y^m \left(6xy^{-1} + 3x^2\right)dy = 0$$

$$\left(12x^n y^m + 5x^{n+1}y^{m+1}\right)dx + \left(6x^{n+1}y^{m-1} + 3x^{n+2}y^m\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(12x^n y^m + 5x^{n+1}y^{m+1}\right) = 12mx^n y^{m-1} + 5(m+1)x^{n+1}y^m$$

$$N_x = \frac{\partial}{\partial x}\left(6x^{n+1}y^{m-1} + 3x^{n+2}y^m\right) = 6(n+1)x^n y^{m-1} + 3(n+2)x^{n+1}y^m$$

For the equation to be exact, then

$$12mx^n y^{m-1} + 5(m+1)x^{n+1}y^m = 6(n+1)x^n y^{m-1} + 3(n+2)x^{n+1}y^m$$

$$12m + 5(m+1)xy = 6(n+1) + 3(n+2)xy$$

$$\begin{cases} 12m = 6n + 6 \\ 5m + 5 = 3n + 6 \end{cases} \rightarrow \begin{cases} 2m - n = 1 \\ 5m - 6n = 1 \end{cases} \Rightarrow \underline{n = 3, \ m = 2}|$$

$$x^3y^2(12 + 5xy)dx + x^3y^2\left(6xy^{-1} + 3x^2\right)dy = 0$$

$$\left(12x^3y^2 + 5x^4y^3\right)dx + \left(6x^4y + 3x^5y^2\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(12x^3y^2 + 5x^4y^3\right) = 24x^3y + 15x^4y^2$$
$$N_x = \frac{\partial}{\partial x}\left(6x^4y + 3x^5y^2\right) = 24x^3y + 15x^4y^2 \quad \Rightarrow \quad M_y = N_x$$

$$\psi = \int \left(12x^3y^2 + 5x^4y^3\right)dx$$
$$= 3x^4y^2 + x^5y^3 + h(y)$$
$$\psi_y = 6x^4y + 3x^5y^2 + h'(y)$$
$$= 6x^4y + 3x^5y^2$$
$$h'(y) = 0 \;\Rightarrow\; h(y) = C$$
$$\boxed{3x^4y^2 + x^5y^3 = C}$$

Exercise

Find an integrating factor of the form $x^n y^m$ and solve the equation

$$\left(3y + 4xy^2\right)dx + \left(2x + 3x^2y\right)dy = 0$$

Solution

$$x^ny^m\left(3y + 4xy^2\right)dx + x^ny^m\left(2x + 3x^2y\right)dy = 0$$
$$\left(3x^ny^{m+1} + 4x^{n+1}y^{m+2}\right)dx + \left(2x^{n+1}y^m + 3x^{n+2}y^{m+1}\right)dy = 0$$
$$M = 3x^ny^{m+1} + 4x^{n+1}y^{m+2} \;;\quad N = 2x^{n+1}y^m + 3x^{n+2}y^{m+1}$$
$$M_y = \frac{\partial}{\partial y}\left(3x^ny^{m+1} + 4x^{n+1}y^{m+2}\right) = 3(m+1)x^ny^m + 4(m+2)x^{n+1}y^{m+1}$$
$$N_x = \frac{\partial}{\partial x}\left(2x^{n+1}y^m + 3x^{n+2}y^{m+1}\right) = 2(n+1)x^ny^m + 3(n+2)x^{n+1}y^{m+1}$$

For the equation to be exact, then

$$3(m+1)x^ny^m + 4(m+2)x^{n+1}y^{m+1} = 2(n+1)x^ny^m + 3(n+2)x^{n+1}y^{m+1}$$
$$3(m+1) + 4(m+2)xy = 2(n+1) + 3(n+2)xy$$
$$\begin{cases} 3m + 3 = 2n + 2 \\ 4m + 8 = 3n + 6 \end{cases} \rightarrow \begin{cases} 3m - 2n = -1 \\ 4m - 3n = -2 \end{cases}$$
$$\underline{m} = \frac{\begin{vmatrix} -1 & -2 \\ -2 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 4 & -3 \end{vmatrix}} = \frac{-1}{-1} \underline{= 1} \qquad \underline{n} = \frac{\begin{vmatrix} 3 & -1 \\ 4 & -2 \end{vmatrix}}{-1} \underline{= 2}$$

$$x^2y\left(3y+4xy^2\right)dx+x^2y\left(2x+3x^2y\right)dy=0$$

$$\left(3x^2y^2+4x^3y^3\right)dx+\left(2x^3y+3x^4y^2\right)dy=0$$

$$\left.\begin{array}{l} M_y=\frac{\partial}{\partial y}\left(3x^2y^2+4x^3y^3\right)=6x^2y+12x^3y^2 \\ N_x=\frac{\partial}{\partial x}\left(2x^3y+3x^4y^2\right)=6x^2y+12x^3y^2 \end{array}\right. \Rightarrow M_y=N_x$$

$$\psi=\int\left(3x^2y^2+4x^3y^3\right)dx$$

$$=x^3y^2+x^4y^3+h(y)$$

$$\psi_y=2x^3y+3x^4y^2+h'(y)=2x^3y+3x^4y^2$$

$$h'(y)=0 \Rightarrow h(y)=C$$

$$\boxed{x^3y^2+x^4y^3=C}$$

Exercise

Find the general solution by using either Bernoulli $\frac{dy}{dx}-5y=-\frac{5}{2}xy^3$

Solution

$$y^3 \Rightarrow n=3$$

Let $u=y^{1-3}=y^{-2}=\frac{1}{y^2} \Rightarrow y=\frac{1}{\sqrt{u}}$

$$\frac{du}{dx}=-2y^{-3}\frac{dy}{dx} \Rightarrow -\frac{1}{2}y^3\frac{du}{dx}=\frac{dy}{dx}$$

$$\frac{dy}{dx}-5y=-\frac{5}{2}xy^3$$

$$-\frac{1}{2}u^{-3/2}\frac{du}{dx}-5u^{-1/2}=-\frac{5}{2}xu^{-3/2} \qquad \times\ -2u^{3/2}$$

$$\frac{du}{dx}+10u=5x$$

		$\int e^{10x}$
+	$5x$	$\frac{1}{10}e^{10x}$
−	5	$\frac{1}{100}e^{10x}$

$$e^{\int 10dx}=e^{10x}$$

$$\int 5xe^{10x}dx=\left(\frac{1}{2}x-\frac{1}{20}\right)e^{10x}$$

$$u=\frac{1}{e^{10x}}\left(\left(\frac{1}{2}x-\frac{1}{20}\right)e^{10x}+C\right)$$

$$\boxed{\frac{1}{y^2}=\frac{1}{2}x-\frac{1}{20}+Ce^{-10x}}$$

Exercise

Find the general solution by using either Bernoulli $\frac{dy}{dx}+\frac{y}{x}=x^2y^2$

Solution

$y^2 \Rightarrow n=2$

Let $u=y^{1-2}=\frac{1}{y} \Rightarrow y=\frac{1}{u}$

$$\frac{du}{dx}=-\frac{1}{y^2}\frac{dy}{dx} \Rightarrow -\frac{1}{u^2}\frac{du}{dx}=\frac{dy}{dx}$$

$$\frac{dy}{dx}+\frac{y}{x}=x^2y^2$$

$$-\frac{1}{u^2}\frac{du}{dx}+\frac{1}{xu}=x^2\frac{1}{u^2} \qquad \times\ -u^2$$

$$\frac{du}{dx}-\frac{1}{x}u=-x^2$$

$$e^{\int -\frac{1}{x}dx}=e^{-\ln|x|}=\frac{1}{x}$$

$$\int x^2\frac{1}{x}dx=\frac{1}{2}x^2$$

$$u=x\left(\frac{1}{2}x^2+C_1\right)$$

$$\frac{1}{y}=\frac{1}{2}x^3+C_1x$$

$$\boxed{y(x)=\frac{2}{x^3+Cx}}$$

Exercise

Find the general solution by using either Bernoulli $\frac{dy}{dx}-y=e^{2x}y^3$

Solution

$y^3 \Rightarrow n=3$

Let $u=y^{1-3}=y^{-2}=\frac{1}{y^2} \Rightarrow y=\frac{1}{\sqrt{u}}=u^{-1/2}$

$$\frac{du}{dx}=-2y^{-3}\frac{dy}{dx} \Rightarrow -\frac{1}{2}y^3\frac{du}{dx}=\frac{dy}{dx}$$

$$\frac{dy}{dx}-y=e^{2x}y^3$$

$$-\frac{1}{2}u^{-3/2}\frac{du}{dx}-u^{-1/2}=e^{2x}u^{-3/2} \qquad \times\ -2u^{3/2}$$

$$\frac{du}{dx}+2u=-2e^{2x}$$

$$e^{\int 2dx} = e^{2x}$$

$$\int -2e^{2x}e^{2x}dx = -\frac{1}{2}e^{4x}$$

$$u = e^{-2x}\left(-\frac{1}{2}e^{4x} + C\right)$$

$$\boxed{y^{-2} = -\frac{1}{2}e^{2x} + Ce^{-2x}} \qquad y = \pm\sqrt{\frac{-2}{e^{2x} + Ce^{-2x}}}$$

Exercise

Find the general solution by using either Bernoulli $\frac{dy}{dx} + \frac{y}{x-2} = 5(x-2)y^{1/2}$

Solution

$$y^{1/2} \Rightarrow n = \frac{1}{2}$$

Let $u = y^{1-\frac{1}{2}} = y^{1/2} \Rightarrow y = u^2 \qquad \rightarrow \frac{dy}{dx} = 2u\frac{du}{dx}$

$$\frac{dy}{dx} + \frac{1}{x-2}y = 5(x-2)y^{1/2}$$

$$2u\frac{du}{dx} + \frac{1}{x-2}u^2 = 5(x-2)u \qquad \times \frac{1}{2u}$$

$$\frac{du}{dx} + \frac{1}{2}\frac{1}{x-2}u = \frac{5}{2}(x-2)$$

$$e^{\frac{1}{2}\int \frac{1}{x-2}dx} = e^{\frac{1}{2}\ln|x-2|} = \sqrt{x-2}$$

$$\int \frac{1}{2}(x-2)\sqrt{x-2}\; dx = \int \frac{1}{2}(x-2)^{3/2}\; dx = \frac{1}{5}(x-2)^{5/2}$$

$$u = \frac{1}{\sqrt{x-2}}\left(\frac{1}{5}(x-2)^{5/2} + C\right)$$

$$y^{1/2} = \frac{1}{5}(x-2)^{3/2} + \frac{C}{\sqrt{x-2}}$$

$$\boxed{y(x) = \left(\frac{1}{5}(x-2)^{3/2} + \frac{C}{\sqrt{x-2}}\right)^2}$$

Exercise

Find the general solution by using either Bernoulli $\frac{dy}{dx} + y = e^x y^{-2}$

Solution

$$y^{-2} \Rightarrow n = -2$$

Let $\quad u = y^{1+2} = y^3 \quad \Rightarrow \quad y = u^{1/3}$

$$\to \frac{dy}{dx} = \frac{1}{3}u^{-2/3}\frac{du}{dx}$$

$$\frac{dy}{dx} + y = e^x y^{-2}$$

$$\frac{1}{3}u^{-2/3}\frac{du}{dx} + u^{1/3} = e^x u^{-2/3} \qquad \times\ 3u^{2/3}$$

$$\frac{du}{dx} + 3u = 3e^x$$

$$e^{\int 3dx} = e^{3x}$$

$$\int 3e^x e^{3x}\ dx = \frac{3}{4}e^{4x}$$

$$u = e^{-3x}\left(\frac{3}{4}e^{4x} + C\right)$$

$$\boxed{y^3 = \frac{3}{4}e^x + Ce^{-3x}}$$

Exercise

Find the general solution by using either Bernoulli $\quad \frac{dy}{dx} + y^3 x + y = 0$

Solution

$$\frac{dy}{dx} + y = -xy^3$$

$$y^3 \ \Rightarrow\ n = 3$$

Let $\quad u = y^{1-3} = y^{-2} = \frac{1}{y^2} \quad \Rightarrow \quad y = \frac{1}{\sqrt{u}} = u^{-1/2}$

$$\frac{dy}{dx} = -\frac{1}{2}u^{-3/2}\frac{du}{dx}$$

$$-\frac{1}{2}u^{-3/2}\frac{du}{dx} + u^{-1/2} = -xu^{-3/2} \qquad \times\ -2u^{3/2}$$

$$\frac{du}{dx} - 2u = 2x$$

$$e^{\int -2dx} = e^{-2x}$$

$$\int 2xe^{-2x}dx = \left(-x - \frac{1}{2}\right)e^{-2x}$$

		$\int e^{-2x}$
+	$2x$	$-\frac{1}{2}e^{-2x}$
–	2	$\frac{1}{4}e^{-2x}$

$$u = e^{2x}\left(\left(-x - \frac{1}{2}\right)e^{-2x} + C\right)$$

$$\boxed{\frac{1}{y^2} = -x - \frac{1}{2} + Ce^{2x}} \qquad y(x) = \pm\frac{1}{\sqrt{Ce^{2x} - x - \frac{1}{2}}}$$

Exercise

Find the general solution by using homogeneous equations. $\left(xy+y^2\right)dx-x^2dy=0$

Solution

$$\frac{dy}{dx}=\frac{xy+y^2}{x^2}\quad=\frac{y}{x}+\left(\frac{y}{x}\right)^2$$

Let $v=\frac{y}{x}\rightarrow y=xv \quad\Rightarrow\quad y'=v+xv'$

$$v+xv'=v+v^2$$

$$x\frac{dv}{dx}=v^2$$

$$\int\frac{dv}{v^2}=\int\frac{dx}{x}$$

$$-\frac{1}{v}=\ln|x|+C$$

$$-\frac{x}{y}=\ln|x|+C$$

$$\boxed{y(x)=-\frac{x}{\ln|x|+C}}$$

Exercise

Find the general solution by using homogeneous equations. $\left(x^2+y^2\right)dx+2xydy=0$

Solution

$$\frac{dy}{dx}=-\frac{x^2+y^2}{2xy}$$

$$=-\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right)$$

Let $v=\frac{y}{x}\rightarrow y=xv \quad\Rightarrow\quad y'=v+xv'$

$$v+xv'=-\frac{1}{2}v-\frac{1}{2v}$$

$$xv'=-\frac{3}{2}v-\frac{1}{2v}$$

$$x\frac{dv}{dx}=-\left(\frac{3v^2+1}{2v}\right)$$

$$\int\frac{2v}{3v^2+1}dv=-\int\frac{1}{x}dx$$

$$\frac{1}{3}\ln\left(3v^2+1\right)=-\ln|x|+\ln C_1$$

$$\ln\left(3v^2+1\right)=-3\ln\left|C_1 x\right|$$

$$3v^2+1=\frac{C}{|x|^3}$$

$$3\frac{y^2}{x^2}=\frac{C}{|x|^3}-1$$

$$3xy^2=C-x^3$$

$$\boxed{3xy^2+x^3=C}$$

Exercise

Find the general solution by using homogeneous equations. $\left(y^2-xy\right)dx+x^2dy=0$

Solution

$$\frac{dy}{dx}=\frac{xy-y^2}{x^2}$$

$$=\frac{y}{x}-\left(\frac{y}{x}\right)^2$$

Let $v=\frac{y}{x}\to y=xv \quad\Rightarrow\quad y'=v+xv'$

$$v+xv'=v-v^2$$

$$x\frac{dv}{dx}=-v^2$$

$$-\int\frac{1}{v^2}dv=\int\frac{1}{x}dx$$

$$\frac{1}{v}=\ln|x|+C$$

$$\frac{x}{y}=\ln|x|+C$$

$$\boxed{y(x)=\frac{x}{\ln|x|+C}}$$

Exercise

Find the general solution by using homogeneous equations. $\frac{dy}{d\theta}=\frac{1}{\theta}\left(\theta\sec\left(\frac{y}{\theta}\right)+y\right)$

Solution

$$\frac{dy}{d\theta}=\sec\left(\frac{y}{\theta}\right)+\frac{y}{\theta}$$

Let $v=\frac{y}{\theta}\to y=\theta v \quad\Rightarrow\quad y'=v+\theta v'$

$$v+\theta v' = \sec v + v$$

$$\int \cos v\ dv = \int \frac{d\theta}{\theta}$$

$$\sin v = \ln|\theta| + C$$

$$\frac{y}{\theta} = \arcsin\left(\ln|\theta| + C\right)$$

$$\underline{y(\theta) = \theta \arcsin\left(\ln|\theta| + C\right)}$$

Exercise

Find the general solution by using homogeneous equations. $\dfrac{dy}{dx} = \dfrac{y(\ln y - \ln x + 1)}{x}$

Solution

$$\frac{dy}{dx} = \frac{y}{x}\left(\ln\frac{y}{x} + 1\right)$$

Let $v = \dfrac{y}{x} \to y = xv \quad \Rightarrow \quad y' = v + xv'$

$$v + xv' = v\ln|v| + v$$

$$x\frac{dv}{dx} = v\ln|v|$$

$$\frac{dv}{v\ln|v|} = \frac{dx}{x}$$

$$\int \frac{1}{\ln|v|}\, d(\ln v) = \int \frac{dx}{x}$$

$$\ln|\ln v| = \ln|x| + \ln C$$

$$\ln|\ln v| = \ln|Cx|$$

$$\ln v = Cx$$

$$v = e^{Cx} = \frac{y}{x}$$

$$\underline{y(x) = xe^{Cx}}$$

Exercise

Find the general solution by using Equation with Linear Coefficients

$$(-3x + y - 1)dx + (x + y + 3)dy = 0$$

Solution

$$\begin{cases} a_1 b_2 = (-3)(1) = -3 \\ \quad a_2 b_1 = (1)(1) = 1 \end{cases} \quad \to \quad a_1 b_2 \neq a_2 b_1$$

$$x = u + h \quad y = v + k$$

$$\begin{cases} -3h+k-1=0 \\ h+k+3=0 \end{cases} \rightarrow \begin{cases} -3h+k=1 \\ h+k=-3 \end{cases} \rightarrow \underline{h=-1,\ k=-2}|$$

$$\begin{cases} x=u+h=u-1 \\ y=v+k=v-2 \end{cases}$$

$$(-3u+3+v-2-1)du+(u-1+v-2+3)dv=0$$

$$(-3u+v)du+(u+v)dv=0 \quad \rightarrow \quad \frac{dv}{du}=\frac{3-\frac{v}{u}}{1+\frac{v}{u}}$$

Let $w=\frac{v}{u} \quad \rightarrow \ v=uw \quad \rightarrow \frac{dv}{du}=w+u\frac{dw}{du}$

$$w+u\frac{dw}{du}=\frac{3-w}{1+w}$$

$$u\frac{dw}{du}=\frac{3-w}{1+w}-w$$

$$u\frac{dw}{du}=\frac{3-2w-w^2}{1+w}$$

$$\frac{w+1}{w^2+2w-3}dw=-\frac{du}{u}$$

$$\frac{1}{2}\int\frac{1}{w^2+2w-3}d\left(w^2+2w-3\right)=-\int\frac{du}{u}$$

$$\frac{1}{2}\ln\left|w^2+2w-3\right|=-\ln u+\ln C_1$$

$$\ln\sqrt{w^2+2w-3}=\ln C_1\frac{1}{u}$$

$$\sqrt{w^2+2w-3}=C_1\frac{1}{u}$$

$$w^2+2w-3=C\frac{1}{u^2}$$

$$\frac{v^2}{u^2}+2\left(\frac{v}{u}\right)-3=C\frac{1}{u^2}$$

$$v^2+2uv-3u^2=C \qquad x=u-1 \quad y=v-2$$

$$\underline{(y+2)^2+2(x+1)(y+2)-3(x+1)^2=C}|$$

Exercise

Find the general solution by using Equation with Linear Coefficients

$$(x+y-1)dx+(y-x-5)dy=0$$

Solution

$$\begin{cases} a_1b_2=(1)(-1)=-1 \\ a_2b_1=(1)(1)=1 \end{cases} \rightarrow \quad a_1b_2\neq a_2b_1$$

$$x = u + h \quad y = v + k$$

$$\begin{cases} h + k - 1 = 0 \\ -h + k - 5 = 0 \end{cases} \rightarrow \begin{cases} h + k = 1 \\ -h + k = 5 \end{cases} \rightarrow \underline{h = -2,\ k = 3}$$

$$\begin{cases} x = u + h = u - 2 \\ y = v + k = v + 3 \end{cases}$$

$$(u - 2 + v + 3 - 1)du + (v + 3 - u + 2 - 5)dv = 0$$

$$(u + v)du + (v - u)dv = 0 \quad \rightarrow \quad \frac{dv}{du} = \frac{u + v}{u - v} = \frac{1 + \frac{v}{u}}{1 - \frac{v}{u}}$$

Let $w = \frac{v}{u} \quad \rightarrow v = uw \quad \rightarrow \frac{dv}{du} = w + u\frac{dw}{du}$

$$w + u\frac{dw}{du} = \frac{1 + w}{1 - w}$$

$$u\frac{dw}{du} = \frac{1 + w}{1 - w} - w$$

$$u\frac{dw}{du} = \frac{1 + w^2}{1 - w}$$

$$\frac{1 - w}{w^2 + 1}dw = \frac{du}{u}$$

$$\int \frac{1}{w^2 + 1}dw - \int \frac{w}{w^2 + 1}dw = \int \frac{du}{u}$$

$$\arctan\frac{v}{u} - \frac{1}{2}\ln\left(\frac{v^2}{u^2} + 1\right) = \ln u + C_1$$

$$2\arctan\frac{v}{u} - \ln\left(\frac{v^2 + u^2}{u^2}\right) = 2\ln u + 2C_1$$

$$2\arctan\frac{v}{u} - \ln\left(\frac{v^2 + u^2}{u^2}\right) - \ln u^2 = C$$

$$2\arctan\frac{v}{u} - \ln\left(v^2 + u^2\right) = C \qquad u = x + 2 \quad v = y - 3$$

$$\underline{2\arctan\frac{y - 3}{x + 2} - \ln\left((x + 2)^2 + (y - 3)^2\right) = C}$$

Exercise

Find the general solution by using Equation with Linear Coefficients

$$(2x + y + 4)dx + (x - 2y - 2)dy = 0$$

Solution

$$\begin{cases} a_1b_2 = (2)(-2) = -4 \\ a_2b_1 = (1)(1) = 1 \end{cases} \rightarrow \quad a_1b_2 \neq a_2b_1$$

$x = u + h \quad y = v + k$

$$\begin{cases} 2h + k + 4 = 0 \\ h - 2k - 2 = 0 \end{cases} \to \begin{cases} 2h + k = -4 \\ h - 2k = 2 \end{cases} \to \quad \underline{h = -\frac{6}{5},\ k = -\frac{8}{5}}$$

$$\begin{cases} x = u + h = u - \frac{6}{5} \\ y = v + k = v - \frac{8}{5} \end{cases}$$

$$\left(2u - \frac{12}{5} + v - \frac{8}{5} + 4\right)du + \left(u - \frac{6}{5} - 2v + \frac{16}{5} - 2\right)dv = 0$$

$$(2u + v)du + (u - 2v)dv = 0 \quad \to \quad \frac{dv}{du} = \frac{2u + v}{2v - u} = \frac{2 + \frac{v}{u}}{2\frac{v}{u} - 1}$$

Let $w = \frac{v}{u} \quad \to \ v = uw \quad \to \frac{dv}{du} = w + u\frac{dw}{du}$

$$w + u\frac{dw}{du} = \frac{2 + w}{2w - 1}$$

$$u\frac{dw}{du} = \frac{2 + w}{2w - 1} - w$$

$$u\frac{dw}{du} = \frac{2 + 2w - 2w^2}{2w - 1}$$

$$\frac{2w - 1}{2 + 2w - 2w^2}dw = \frac{du}{u}$$

$$-\int \frac{1}{w^2 - w - 1}d\left(w^2 - w - 1\right) = 2\int \frac{du}{u}$$

$$-\ln\left|w^2 - w - 1\right| = 2\ln|u| + C_1$$

$$\ln\left|w^2 - w - 1\right| + \ln u^2 = C_1$$

$$\ln\left|u^2\left(\frac{v^2}{u^2} - \frac{v}{u} - 1\right)\right| = \ln C$$

$$v^2 - uv - u^2 = C \qquad u = x + \frac{6}{5} \quad v = y + \frac{8}{5}$$

$$\underline{\left(x + \frac{8}{5}\right)^2 - \left(x + \frac{8}{5}\right)\left(x + \frac{6}{5}\right) - \left(x + \frac{6}{5}\right)^2 = C} \qquad (5x + 8)^2 - (5x + 8)(5x + 6) - (5x + 6)^2 = C_2$$

Exercise

Find the general solution by using Equation with Linear Coefficients

$$(2x - y)dx + (4x + y - 3)dy = 0$$

Solution

$$\begin{cases} a_1 b_2 = (2)(1) = 2 \\ a_2 b_1 = (-1)(4) = -4 \end{cases} \rightarrow \quad a_1 b_2 \neq a_2 b_1$$

$x = u + h \quad y = v + k$

$$\begin{cases} 2h - k = 0 \\ 4h + k = 3 \end{cases} \rightarrow \quad \underline{h = \tfrac{1}{2},\ k = 1|}$$

$$\begin{cases} x = u + h = u + \frac{1}{2} \\ y = v + k = v + 1 \end{cases}$$

$$(2u + 1 - v - 1)du + (4u + 2 + v + 1 - 3)dv = 0$$

$$(2u - v)du + (4u + v)dv = 0 \quad \rightarrow \quad \frac{dv}{du} = \frac{v - 2u}{4u + v} = \frac{\frac{v}{u} - 2}{4 + \frac{v}{u}}$$

Let $w = \frac{v}{u} \quad \rightarrow \ v = uw \quad \rightarrow \frac{dv}{du} = w + u\frac{dw}{du}$

$$w + u\frac{dw}{du} = \frac{w - 2}{4 + w}$$

$$u\frac{dw}{du} = \frac{w - 2}{4 + w} - w$$

$$u\frac{dw}{du} = \frac{-w^2 - 3w - 2}{4 + w}$$

$$\frac{w + 4}{w^2 + 3w + 2}dw = -\frac{du}{u}$$

$$\frac{w + 4}{w^2 + 3w + 2} = \frac{A}{w + 1} + \frac{B}{w + 2} \quad \rightarrow (A + B)w + 2A + B = w + 4$$

$$\begin{cases} A + B = 1 \\ 2A + B = 4 \end{cases} \rightarrow \quad A = 3, \quad B = -2$$

$$\int \frac{3}{w + 1}dw - \int \frac{2}{w + 2}dw = -\int \frac{du}{u}$$

$$3\ln|w + 1| - 2\ln|w + 2| = -\ln|u| + \ln C$$

$$\ln|w + 1|^3 - \ln(w + 2)^2 + \ln|u| = \ln C$$

$$\ln \frac{u(w + 1)^3}{(w + 2)^2} = \ln C$$

$$\frac{u(w + 1)^3}{(w + 2)^2} = C$$

$$u\left(\frac{v}{u} + 1\right)^3 = C\left(\frac{v}{u} + 2\right)^2 \qquad u = x - \frac{1}{2} \quad v = y - 1$$

$$\frac{1}{u^2}(v + u)^3 = C\frac{1}{u^2}(v + 2u)^2$$

$$\left(x-\tfrac{1}{2}+y-1\right)^3=C\left(y-1+2x-1\right)^2$$

$$\left(x-\tfrac{3}{2}+y\right)^3=C\left(y+2x-2\right)^2$$

$$\tfrac{1}{8}\left(2x-3+2y\right)^3=C\left(y+2x-2\right)^2 \qquad C_1=8C$$

$$\boxed{\left(2x-3+2y\right)^3=C_1\left(y+2x-2\right)^2}$$

Exercise

Prove that $Mdx+Ndy=0$ has an integrating factor that depends only on the sum $x+y$ if and only if the expression

$$\frac{N_x-M_y}{M-N} \text{ depends only on } x+y$$

Use the prove to solve the equation $(3+y+xy)dx+(3+x+xy)dy=0$

Solution

An equation $Mdx+Ndy=0$ has an integrating factor $\mu(x+y)$ iff $\mu(x+y)Mdx+\mu(x+y)Ndy=0$

For the equation to be exact, then

$$\frac{\partial}{\partial y}\left[\mu(x+y)M(x,y)\right]=\frac{\partial}{\partial x}\left[\mu(x+y)N(x,y)\right]$$

$$\mu'(x+y)M+\mu(x+y)M_y=\mu'(x+y)N+\mu(x+y)N_x$$

$$\mu'(x+y)(M-N)=\mu(x+y)\left(N_x-M_y\right)$$

$$\frac{\mu'(x+y)}{\mu(x+y)}=\frac{N_x-M_y}{M-N}$$

$$\int\frac{\mu'(x+y)}{\mu(x+y)}=\int\frac{N_x-M_y}{M-N}d(x+y)$$

$$\ln\left|\mu(x+y)\right|=\int\frac{N_x-M_y}{M-N}d(x+y)$$

$$\mu(x+y)=\pm e^{\int\frac{N_x-M_y}{M-N}d(x+y)}$$

$$(3+y+xy)dx+(3+x+xy)dy=0$$

$$M_y=\frac{\partial}{\partial y}(3+y+xy)=1+x$$

$$N_x=\frac{\partial}{\partial x}(3+x+xy)=1+y$$

$$\frac{N_x-M_y}{M-N}=\frac{1+y-(1+x)}{3+y+xy-(3+x+xy)}$$

$$= \frac{y-x}{y-x}$$

$$\underline{= 1|}$$

$$\mu(x+y) = e^{\int d(x+y)} = e^{x+y}$$

$$e^{x+y}(3+y+xy)dx + e^{x+y}(3+x+xy)dy = 0$$

$$M_y = \frac{\partial}{\partial y}(3+y+xy)e^{x+y} = (1+x)e^{x+y} + (3+y+xy)e^{x+y} = (4+x+y+xy)e^{x+y}$$

$$N_x = \frac{\partial}{\partial x}(3+x+xy)e^{x+y} = (1+y)e^{x+y} + (3+x+xy)e^{x+y} = (4+x+y+xy)e^{x+y}$$

$$\Rightarrow \; M_y = N_x$$

$$\psi = \int (3+y+xy)e^{x+y}dx = (3+y+y(x-1))e^{x+y} + h(y)$$

$$= (3+xy)e^{x+y} + h(y)$$

$$\psi_y = (3+xy+x)e^{x+y} + h'(y) = (3+xy+x)e^{x+y}$$

$$h'(y) = 0 \;\Rightarrow\; h(y) = C$$

$$\underline{(3+xy)e^{x+y} = C|}$$

Exercise

A portion of a uniform chain of length 8 *feet* is loosely coiled around a peg at the edge of a high horizontal platform, and the remaining portion of the chain hangs at rest over the edge of the platform.

Suppose the length of the overhanging chain is 3 *feet*, that the chain weighs 2 2 *lb/ft* , and that the positive direction is downward. Starting at $t = 0$ seconds, the weight of the overhanging portion causes the chain on the table to uncoil smoothly and to fall to the floor. If $x(t)$ denotes the length of the chain overhanging the table at time $t > 0$, then $v = \frac{dx}{dt}$ is its velocity. When all resistive forces are ignored, it can be shown that a mathematical model relating v to x is given by

$$xv\frac{dv}{dx} + v^2 = 32x$$

a) Rewrite this model in differential form and solve the *DE* for v in terms of x by finding am appropriate integrating factor. Find an explicit solution $v(x)$.

b) Determine the velocity with which the chain leaves the platform.

Solution

a) $\left(v^2 - 32x\right)dx + xvdv = 0$

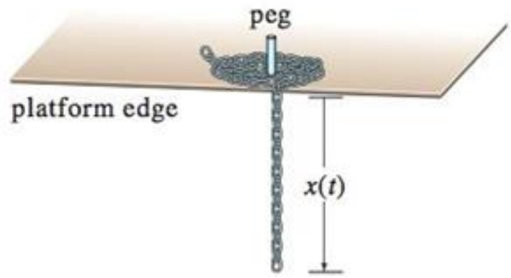
$$M(x,\ v) = v^2 - 32x \qquad N(x,\ v) = xv$$

$$\frac{\partial M}{\partial v} = 2v \qquad \frac{\partial N}{\partial x} = v \qquad \Rightarrow \quad \frac{\partial M}{\partial v} \neq \frac{\partial N}{\partial x}$$

$$\frac{M_v - N_x}{N} = \frac{2v - v}{xv} = \frac{1}{x}$$

$$\frac{d\mu}{dx} = \frac{\mu}{x} \quad \Rightarrow \quad \int \frac{d\mu}{\mu} = \int \frac{dx}{x}$$

$$\ln \mu = \ln x \ \rightarrow \ \underline{\mu(x) = x}$$



$$x\left(v^2 - 32x\right)dx + x^2 v\,dv = 0$$

$$M_v = \frac{\partial}{\partial v}\left(v^2 x - 32x^2\right) = 2vx \qquad N_x = \frac{\partial}{\partial x}\left(vx^2\right) = 2vx \qquad \Rightarrow M_y = N_x$$

$$\psi = \int \left(v^2 x - 32x^2\right)dx = \frac{1}{2}v^2x^2 - \frac{32}{3}x^3 + h(v)$$

$$\psi_v = vx^2 + h'(v) = vx^2 \quad \Rightarrow \quad h'(v) = 0 \qquad \Rightarrow h(y) = C$$

$$\frac{1}{2}v^2x^2 - \frac{32}{3}x^3 = C$$

Given: $v = 0 \ \ x = 3 \qquad \Rightarrow \underline{-288 = C}$

$$\frac{1}{2}v^2x^2 - \frac{32}{3}x^3 = -288$$

$$3v^2x^2 - 64x^3 = -1728$$

$$3v^2x^2 = 64\left(x^3 - 27\right)$$

$$v^2 = 64\frac{x^3 - 27}{3x^2}$$

$$\underline{v(x) = \frac{8}{x}\sqrt{\frac{x^3}{3} - 9}}$$

b) The chain leaves the platform when $x = 8$

$$v(8) = \sqrt{\frac{512}{3} - 9} = \underline{\sqrt{\frac{485}{3}}\ ft/s} \qquad \underline{\approx 12.7\ ft/s}$$