Section 1.5 – Maximization Problems *with constraints of the form* $\le$

Example

A farmer has 100 acres of available land on which he wishes to plant a mixture of potatoes, corn, and cabbage. It costs him \$400 to produce an acre of potatoes, \$160 to produce an acre of corn, and \$280 to produce an acre of cabbage. He has a maximum of \$20,000 to spend. He makes a profit of \$120 per acre of potatoes, \$40 per acre of corn, and \$60 per acre of cabbage. How many acres of each crop should he plant to maximize his profit?

Solution

	Potatoes	***Corn***	***Cabbage***		***Total***
Number of acres	x_1	x_2	x_3	$\le$	100
Cost (per acre)	\$400	\$160	\$280	$\le$	\$20,000
Profit (per acre)	\$120	\$40	\$60		

$$\begin{array}{lll} \textit{Maximize} & P = 120x_1 + 40x_2 + 60x_3 & (1) \\ \textit{subject to} & x_1 + x_2 + x_3 \le 100 & (2) \\ & 400x_1 + 160x_2 + 280x_3 \le 20,000 & (3) \\ \textit{with} & x_1, x_2, x_3 \ge 0 & \end{array}$$

Divide **(3)** by 40: $\frac{1}{40}(3) \Rightarrow 10x_1 + 4x_2 + 7x_3 \le 500$

$$\begin{array}{lr} P - 120x_1 - 40x_2 - 60x_3 = 0 & (1) \\ x_1 + x_2 + x_3 + s_1 \qquad = 100 & (2) \\ 10x_1 + 4x_2 + 7x_3 \qquad + s_2 = 500 & (3) \\ x_1, x_2, x_3, s_1, s_2 \ge 0 & \end{array}$$

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & P \end{array}$$
$$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 100 \\ 10 & 4 & 7 & 0 & 1 & 0 & 500 \\ \hline -120 & -40 & -60 & 0 & 0 & 1 & 0 \end{array}\right]$$

So the ***basic feasible solution*** at this point is:

$x_1 = 0, \ x_2 = 0, \ x_3 = 0, \ s_1 = 100, \ s_2 = 500, \ P = 0$

First Pivot

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & P \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 100 \\ 10 & 4 & 7 & 0 & 1 & 0 & 500 \\ \hline -120 & -40 & -60 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad \begin{array}{l} \frac{100}{1} = 100 \\ \frac{500}{10} = 50 \leftarrow \\ \end{array}$$

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & P \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 100 \\ 10 & 4 & 7 & 0 & 1 & 0 & 500 \\ \hline -120 & -40 & -60 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad \begin{array}{l} \\ \frac{1}{10} R_2 \\ \end{array}$$

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & P \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 100 \\ 1 & .4 & .7 & 0 & .1 & 0 & 50 \\ \hline -120 & -40 & -60 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 120R_2 \end{array}$$

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & P \end{array} \\ \left[\begin{array}{cccccc|c} 0 & .6 & .3 & 1 & -.1 & 0 & 50 \\ 1 & .4 & .7 & 0 & 1 & 0 & 50 \\ \hline 0 & 8 & 24 & 0 & 12 & 1 & 6000 \end{array}\right] \end{array}$$

So the ***basic feasible solution*** at this point is:

$$x_1 = 50, \; x_2 = 0, \; x_3 = 0, \; s_1 = 50, \; s_2 = 0, \; P = 6000$$

The farmer will make a maximum of $6,000 by planning 50 acres of potatoes, no acres of corn, and no acres of cabbage.

Simplex Method

1. Determine the objective constraints
2. Write all the necessary constraints
3. Convert each constraint into an equation by adding a slack variable in each.
4. Set up the initial simplex tableau.
5. Locate the most negative indicator. If there are two such indicators, choose the one farther to the left.
6. Form the necessary quotients to find the pivot. Disregard any quotients with 0 or a negative number in the denominator. The smallest nonnegative quotient gives the location of the pivot. If all quotients must be disregarded, no maximum solution exists. If two quotients are both equal and smallest, choose the pivot in the row nearest the top of the matrix.
7. Use the row operations to change all other numbers in the pivot column to zero by adding a suitable multiple of the pivot row to a positive multiple of each row.
8. If the indicators are all positive or 0, this is the final tableau.
9. Read the solution from the final tableau

Example

Solve the simplex method:

$$\begin{aligned} &\textit{Maximize}: && P = 25x_1 + 30x_2 \\ &\textit{subject to}: && x_1 + x_2 \le 65 \\ & && 4x_1 + 5x_2 \le 300 \\ &\textit{with} && x_1, x_2 \ge 0 \end{aligned}$$

The initial tableau

$$\begin{array}{c} \begin{matrix} x_1 & \quad x_2 & \quad s_1 & s_2 & P \end{matrix} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 65 \\ 4 & 5 & 0 & 1 & 0 & 300 \\ \hline -25 & -30 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad \begin{array}{l} \frac{65}{1}=65 \\ \frac{300}{5}=60 \end{array}$$

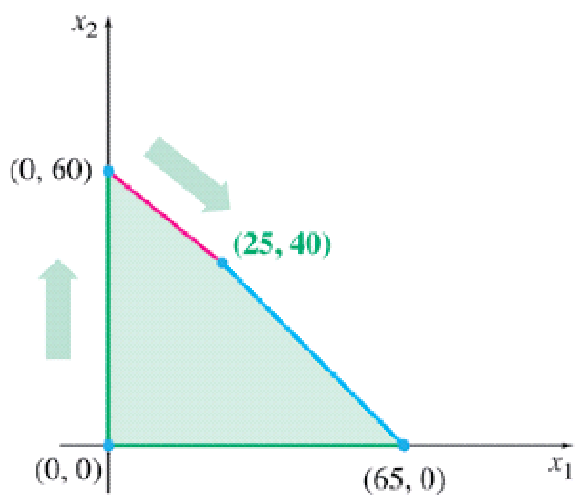
$$\begin{array}{c} \begin{matrix} x_1 & \quad x_2 & \quad s_1 & s_2 & P \end{matrix} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 65 \\ 4 & 5 & 0 & 1 & 0 & 300 \\ \hline -25 & -30 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad \frac{1}{5}R_2$$

$$\begin{array}{c} \begin{matrix} x_1 & \quad x_2 & \quad s_1 & s_2 & P \end{matrix} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 65 \\ .8 & 1 & 0 & .2 & 0 & 60 \\ \hline -25 & -30 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 30R_2 \end{array}$$

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & P \end{matrix} \\ \left[\begin{array}{ccccc|c} .2 & 0 & 1 & -.2 & 0 & 5 \\ .8 & 1 & 0 & .2 & 0 & 60 \\ \hline -1 & 0 & 0 & 6 & 1 & 1800 \end{array}\right] \end{array} \quad \frac{1}{.2}R_1$$

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & P \end{matrix} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 5 & -1 & 0 & 25 \\ .8 & 1 & 0 & .2 & 0 & 60 \\ \hline -1 & 0 & 0 & 6 & 1 & 1800 \end{array}\right] \end{array} \quad \begin{array}{l} \\ R_2 - .8R_1 \\ R_3 + R_1 \end{array}$$

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & P \end{matrix} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 5 & -1 & 0 & 25 \\ 0 & 1 & -4 & 1 & 0 & 40 \\ \hline 0 & 0 & 5 & 5 & 1 & 1825 \end{array}\right] \end{array}$$



The solution is $x_1 = 25$ and $x_2 = 40$ with $P = 1825$

Example

Find the pivot for the following initial simplex tableau

$$\begin{array}{cccccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & z & \\
\end{array}$$
$$\left[\begin{array}{cccccc|c}
1 & -2 & 1 & 0 & 0 & 0 & 100 \\
3 & 4 & 0 & 1 & 0 & 0 & 200 \\
5 & 0 & 0 & 0 & 1 & 0 & 150 \\
\hline
-10 & -25 & 0 & 0 & 0 & 1 & 0
\end{array}\right]$$

The most negative indicator is -25, the pivot column is 2

Pivot row: $\frac{100}{-2}$, $\frac{200}{4}$, $\frac{150}{0}$

Variables have to be nonnegative, and no zero in denominator, the only usable quotient is $\frac{200}{4}$

The pivot is (4) column 2 , row 2

Exercises Section 1.5 – Maximization Problems *with constraints of the form* $\le$

1. Solve the simplex method:

Maximize: $P = 50x_1 + 80x_2$

Subject to
$$\begin{aligned} x_1 + 2x_2 &\le 32 \\ 3x_1 + 4x_2 &\le 84 \\ x_1, x_2 &\ge 0 \end{aligned}$$

2. Solve the simplex method:

Maximize: $P = 2x_1 + 3x_2$

Subject to:
$$\begin{cases} -3x_1 + 4x_2 \le 12 \\ x_2 \le 2 \\ x_1, x_2 \ge 0 \end{cases}$$

3. Solve the simplex method:

Maximize: $P = 2x_1 + x_2$

Subject to:
$$\begin{cases} 5x_1 + x_2 \le 9 \\ x_1 + x_2 \le 5 \\ x_1, x_2 \ge 0 \end{cases}$$

4. The initial tableau of a linear programing is given. Use the simplex method to solve it.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 2 & 4 & 1 & 0 & 0 & 8 \\ 5 & 8 & 1 & 0 & 1 & 0 & 10 \\ \hline -3 & -24 & 1 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

5. Carrie is working to raise money for the homeless by sending information letters and making follow-up calls to local labor organizations and church groups. She discovers that each church group requires 2 hours of letter writing and 1 hour of follow-up, while for each labor union she needs 2 hours of letter writing and 3 hours of follow-up. Carrie can raise $100 from each church group and $200 from each union local, and she has a maximum of 16 hours of letter-writing time and a maximum of 12 hours of follow-up time available per month. Determine the most profitable mixture of groups she should contact and the most money she can raise in a month.

6. The Texas Poker Company assembles three different poker sets. Each Royal Flush poker set contains 1000 poker chips, 4 decks of cards, 10 dice, and 2 dealer buttons. Each Deluxe Diamond poker set contains 600 poker chips, 2 decks of cards, 5 dice, and one dealer button. The full House poker set contains 300 poker chips, 2 decks of cards, 5 dice, and one dealer button. The Texas Poker Company has 2,800,000 poker chips, 10,000 decks of cards, 25,000 dice, and 6000 dealer buttons in stock. They earn a profit of $38 for each Royal Flush poker set, $22 for each Deluxe Diamond poker set, and $12 for each Full House poker set.

a) How many of each type of poker set should they assemble to maximize profit? What is the maximum profit?

b) Find the values of any nonzero slack variables and describe what they tell you about any unused components.

7. The Muro Manufacturing Company makes two kinds of plasma screen TV sets. It produces the Flexscan set that sells for $350 profit and the Panoramic I that sells for $500 profit. On the assembly line, the Flexscan requires 5 hours, and the Panoramic I takes 7 hours. The cabinet shop spends 1 hour on the cabinet for the Flexscan and 2 hours on the cabinet for the Panoramic I. Both sets require 4 hours for testing and packing. On a particular production run, the Muro Company has available 3600 work-hours on the assembly line, 900 work-hours in the cabinet shop, and 2600 work-hours in the testing and packing department.

a) How many sets of each type should it produce to make a maximum profit? What is the maximum profit?

b) Find the values of any nonzero slack variables and describe what they tell you about unused time.

8. A baker has 150 units of flour, 90 of sugar, and 150 of raisins. A loaf of raisin bread requires 1 unit of flour, 1 of sugar, and 2 of raisins, while a raisin cake needs 5, 2, and 1 units, respectively.

a) If raisin bread sells for $1.75 a loaf and raisin cake for $4.00 each, how many of each should baked so that gross income is maximized?

b) What is the maximum gross income?

c) Does it require all of the available units of flour, sugar, and raisins to produce the number the maximum profit? If not, how much of each ingredient is left over? Compare any leftover to the value of the relevant slack variable.

9. A farmer owns a 100 acre farm and plans to plant at most three crops. The seed for crops A, B, and C costs $24, $40, and $30 per acre, respectively. A maximum of $3,600 can be spent on seed. Crops A, B, and C require 1, 2, and 2 workdays per acre, respectively, and three are a maximum of 160 workdays available. If the farmer can make a profit of $140 per acre on crop A, $200 per acre on crop B, and $160 per acre on crop C, how many acres of each crop that should be planted to maximize the profit?

10. A candy company makes three types of candy, solid, fruit, and cream filled, and packages these candies in three different assortments. A box of assortment I contains 4 solid, 4 fruit, and 12 cream and sells for $9.40. A box of assortment II contains 12 solid, 4 fruit, and 4 cream and sells for $7.60. A box of assortment III contains 8 solid, 8 fruit, and 8 cream and sells for $11.00. The manufacturing costs per piece of candy are $0.20 for solid, $0.25 for fruit, and $0.30 for cream. The company can manufacture up to 4800 solid, 4000 fruit, and 5600 cream candies weekly. How many boxes of each type should the company produce in order to maximize profit? What is their maximum profit?

11. A small company manufactures three different electronic components for computers. Component A requires 2 hours of fabrication and 1 hour of assembly; component B requires 3 hours of fabrication and 1 hour of assembly; and component C requires 2 hours of fabrication and 2 hours of assembly. The company has up to 1,000 labor-hours of fabrication time and 800 labor-hours of assembly time available per week. The profit on each component, A, B, and C is \$7, \$8, and \$10, respectively. How many components of each week in order to maximize its profit (assuming that all components that it manufactures can be sold)? What is the maximum profit?

12. An investor has at most \$100,000 to invest in government bonds, mutual funds, and money market funds. The average yields for government bonds, mutual funds, and money market funds are 8%, 13%,, and 15%, respectively. The investor's policy requires that the total amount invested in mutual and money market funds not exceed the amount invested in government bonds. How much should be invested in each type of investment in order to maximize the return? What is the maximum return?

13. A department store chain up to \$20,000 to spend on television advertising for a sale. All ads will be placed with one television station, where 30-second as costs \$1,000 on daytime TV and is viewed by 14,000 potential customers, \$2000 on prime-time TV and is viewed by 24,000 potential customer, and \$1,500 on late-night TV and is viewed by 18,000 potential customers. The television station will not accept a total of more than 15 ads in all three time periods. How many ads should be placed in each time period in order to maximize the number of potential customers who will see the ads? How many potential customers will see the ads?

14. A political scientist has received a grant to find a research project involving voting trends. The budget of the grant includes \$3,200 for conducting door-to-door interviews the day before an election. Undergraduate students, graduate students, and faculty members will be hired to conduct the interviews. Each undergraduate student will conduct 18 interviews and be paid \$100. Each graduate student will conduct 25 interviews and be paid \$150. Each faculty members will conduct 30 interviews and be paid \$200. Due to limited transportation facilities, no more than 20 interviews can be hired. How many undergraduate students, graduate students, and faculty members should be hired in order to maximize the number of interviews that will be conducted? What is the maximum number of interviews?