Section 4.2 - Law of Cosines

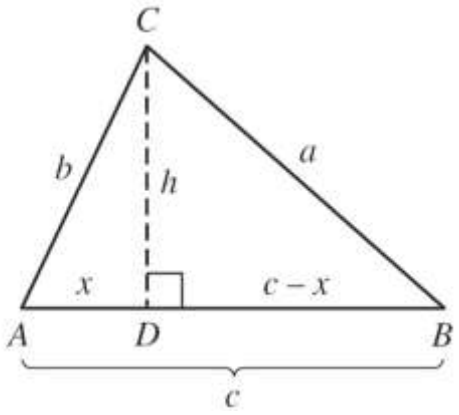
Law of Cosines (*SAS*)

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Derivation



$$a^2 = (c-x)^2 + h^2$$
$$= c^2 - 2cx + x^2 + h^2 \qquad \textbf{(1)}$$

$$b^2 = x^2 + h^2 \qquad \textbf{(2)}$$

From **(2)**:

(1) $a^2 = c^2 - 2cx + b^2$

$$a^2 = c^2 + b^2 - 2cx \qquad \textbf{(3)}$$

$$\cos A = \frac{x}{b}$$

$$b\cos A = x$$

(3) $\Rightarrow a^2 = c^2 + b^2 - 2cb\cos A$

Example

Find the missing parts in triangle ABC if $A = 60°$, $b = 20$ in, and $c = 30$ in.

Solution

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ &= 20^2 + 30^2 - 2(20)(30)\cos 60° \\ &= 700 \end{aligned}$$

$$a \approx 26$$

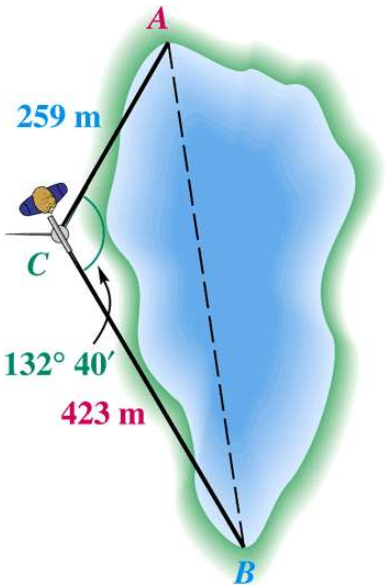
$$\begin{aligned} \sin B &= \frac{b\sin A}{a} \\ &= \frac{20\sin 60°}{26} \\ &= 0.6662 \end{aligned}$$

$$\begin{aligned} B &= \sin^{-1}(0.6662) \\ &= 42° \end{aligned}$$

$$\begin{aligned} C &= 180° - A - B \\ &= 180° - 60° - 42° \\ &= 78° \end{aligned}$$

Example

A surveyor wishes to find the distance between two inaccessible points A and B on opposite sides of a lake. While standing at point C, she finds that AC = 259 m, BC = 423 m, and angle ACB = 132°40′. Find the distance AB.



Solution

$$\begin{aligned} AB^2 &= AC^2 + BC^2 - 2(AC)(BC)\cos C \\ &= 259^2 + 423^2 - 2(259)(423)\cos(132°40') \\ &= 394510 \end{aligned}$$

$$\boxed{AB \approx 628}$$

Law of Cosines (***SSS***) - Three Sides

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Example

Solve triangle *ABC* if *a* = 34 km, *b* = 20 km, and c = 18 km

Solution

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$= \frac{20^2 + 18^2 - 34^2}{2(20)(18)}$$
$$= -0.6$$

$$A = \cos^{-1}(-0.6)$$
$$= 127°$$

OR

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$
$$= \frac{34^2 + 20^2 - 18^2}{2(34)(20)}$$
$$= 0.91$$

$$C = \cos^{-1}(0.91)$$
$$= 25°$$

$$\sin C = \frac{c \sin A}{a}$$
$$= \frac{18 \sin 127°}{34}$$
$$= 0.4228$$

$$C = \sin^{-1}(0.4228)$$
$$= 25°$$

$$B = 180° - A - C$$
$$= 180° - 127° - 25°$$
$$= 28°$$

Example

A plane is flying with an airspeed of 185 miles per hour with heading 120°. The wind currents are running at a constant 32 miles per hour at 165° clockwise from due north. Find the true course and ground speed of the plane.

Solution

$$\alpha = 180° - 120°$$
$$= 60°$$

$$\theta = 360° - 165° - \alpha$$
$$= 360° - 165° - 60°$$
$$= 135°$$

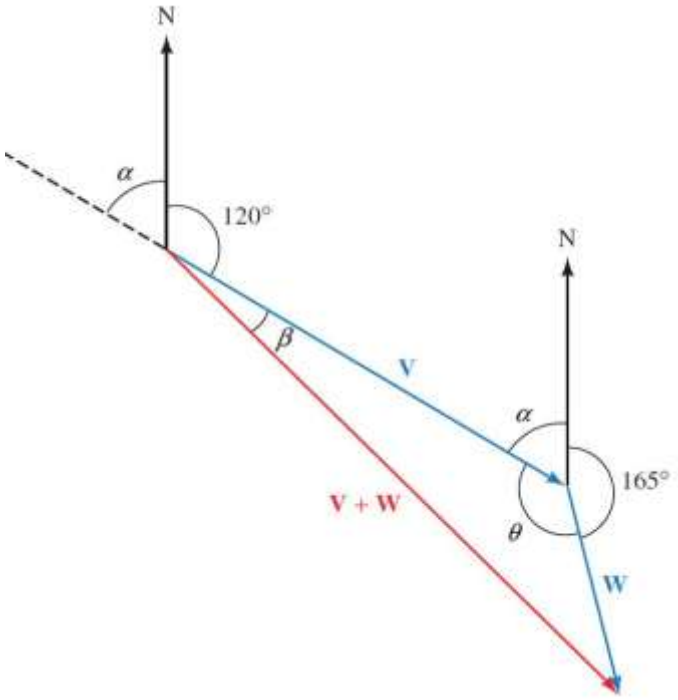
$$|V+W|^2 = |V|^2 + |W|^2 - 2|V| \cdot |W| \cos\theta$$
$$= 185^2 + 32^2 - 2(185)(32)\cos 135°$$
$$= 43{,}621$$

$$|V+W| = 210\ mph$$

$$\frac{\sin\beta}{32} = \frac{\sin\theta}{210}$$

$$\sin\beta = \frac{32\sin 135°}{210}$$
$$= 0.1077$$

$$\beta = \sin^{-1}(0.1077)$$
$$= 6°$$



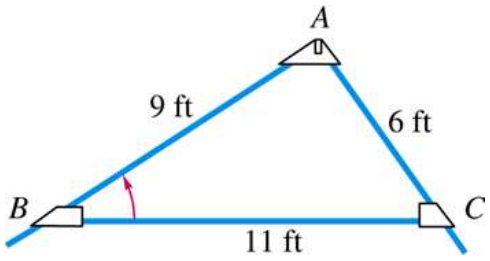
The true course is:

$120° + \beta = 120° + 6° = 126°$.

The speed of the plane with respect to the ground is 210 mph.

Example

Find the measure of angle B in the figure of a roof truss.



Solution

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$= \frac{11^2 + 9^2 - 6^2}{2(11)(9)}$$

$$B = \cos^{-1}\left(\frac{11^2 + 9^2 - 6^2}{2(11)(9)}\right)$$

$$\approx 33°$$

Heron's Area Formula (*SSS*)

If a triangle has sides of lengths a, b, and c, with semi-perimeter

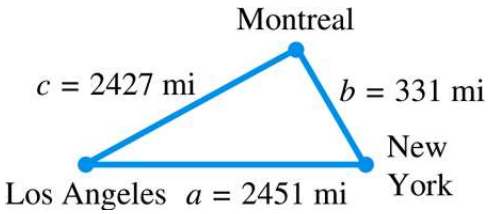
$$s = \frac{1}{2}(a+b+c)$$

Then the area of the triangle is:

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

Example

The distance "as the crow flies" from Los Angeles to New York is 2451 miles, from New York to Montreal is 331 miles, and from Montreal to Los Angeles is 2427 miles. What is the area of the triangular region having these three cities as vertices? (Ignore the curvature of Earth.)



Solution

The semiperimeter s is:

$$s = \frac{1}{2}(a+b+c)$$
$$= \frac{1}{2}(2451+331+2427)$$
$$= 2604.5$$

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{2604.5(2604.5-2451)(2604.5-331)(2604.5-2427)}$$
$$\approx 401,700\ mi^2$$

Exercises *Section* 4.2 - Law of Cosines

1. If a = 13 yd., b = 14 yd., and c = 15 yd., find the largest angle.

2. Solve triangle ABC if b = 63.4 km, and c = 75.2 km, A = 124° 40′

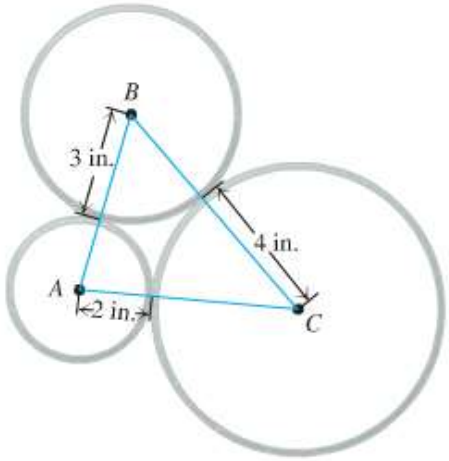
3. Solve triangle ABC if $A = 42.3°,\ b = 12.9m,\ and\ c = 15.4m$

4. Solve triangle ABC if a = 832 ft., b = 623 ft., and c = 345 ft.

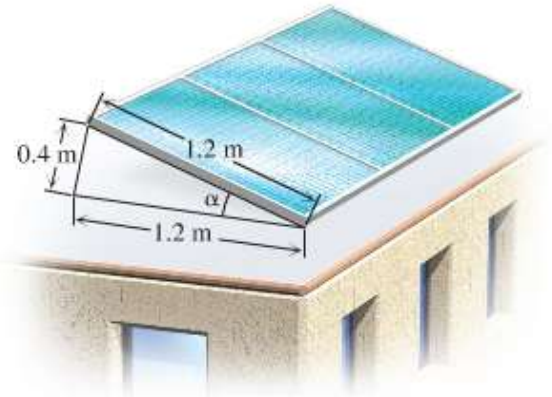
5. Solve triangle ABC if $a = 9.47\,ft,\ b = 15.9\,ft,\ and\ c = 21.1\,ft$

6. The diagonals of a parallelogram are 24.2 cm and 35.4 cm and intersect at an angle of 65.5°. Find the length of the shorter side of the parallelogram

7. An engineer wants to position three pipes at the vertices of a triangle. If the pipes A, B, and C have radii 2 in, 3 in, and 4 in, respectively, then what are the measures of the angles of the triangle ABC?

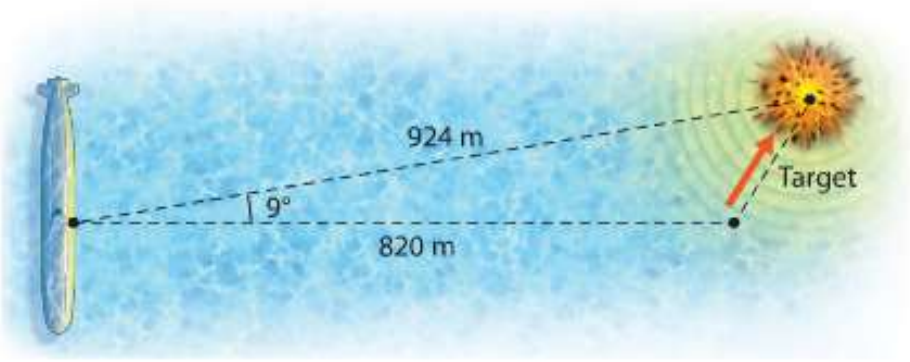


8. A solar panel with a width of 1.2 m is positioned on a flat roof. What is the angle of elevation α of the solar panel?

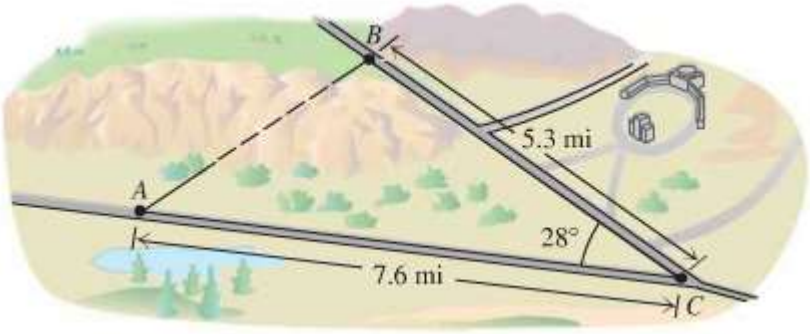


9. Andrea and Steve left the airport at the same time. Andrea flew at 180 mph on a course with bearing 80°, and Steve flew at 240 mph on a course with bearing 210°. How far apart were they after 3 hr.?

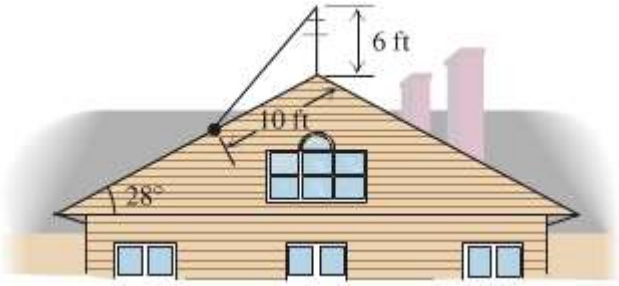
10. A submarine sights a moving target at a distance of 820 m. A torpedo is fired 9° ahead of the target, and travels 924 m in a straight line to hit the target. How far has the target moved from the time the torpedo is fired to the time of the hit?



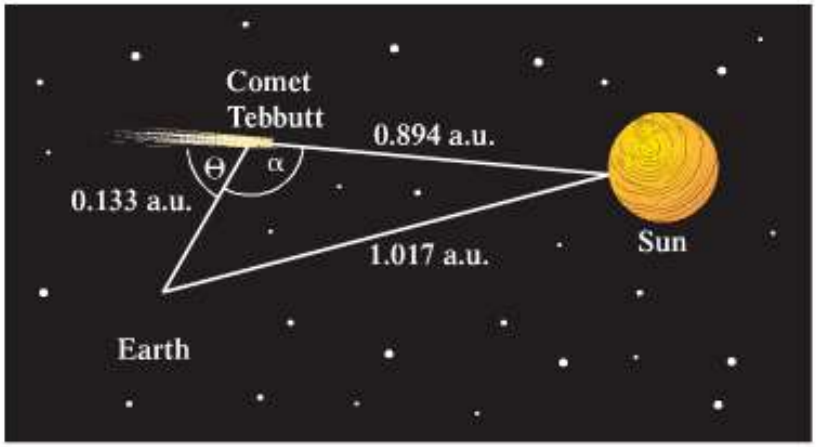
11. A tunnel is planned through a mountain to connect points A and B on two existing roads. If the angle between the roads at point C is 28°, what is the distance from point A to B? Find $\angle CBA$ and $\angle CAB$ to the nearest tenth of a degree.



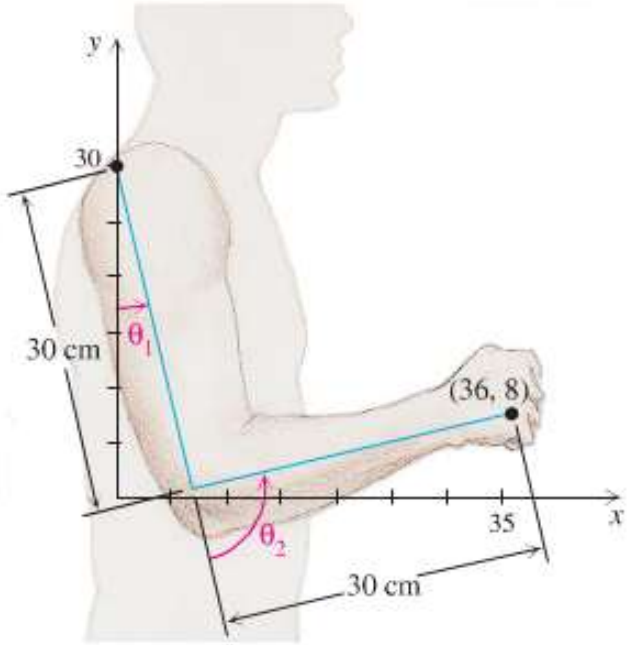
12. A 6-ft antenna is installed at the top of a roof. A guy wire is to be attached to the top of the antenna and to a point 10 ft down the roof. If the angle of elevation of the roof is 28°, then what length guy wire is needed?



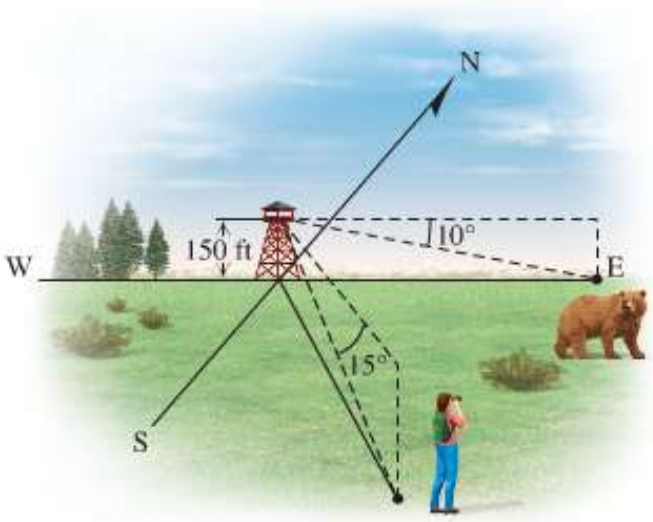
13. On June 30, 1861, Comet Tebutt, one of the greatest comets, was visible even before sunset. One of the factors that causes a comet to be extra bright is a small scattering angle θ. When Comet Tebutt was at its brightest, it was 0.133 a.u. from the earth, 0.894 a.u. from the sun, and the earth was 1.017 a.u. from the sun. Find the phase angle α and the scattering angle θ for Comet Tebutt on June 30, 1861. (One astronomical unit (a.u) is the average distance between the earth and the sub.)



14. A human arm consists of an upper arm of 30 cm and a lower arm of 30 cm. To move the hand to the point (36, 8), the human brain chooses angle θ_1 *and* θ_2 to the nearest tenth of a degree.



15. A forest ranger is 150 ft above the ground in a fire tower when she spots an angry grizzly bear east of the tower with an angle of depression of 10°. Southeast of the tower she spots a hiker with an angle of depression of 15°. Find the distance between the hiker and the angry bear.


N
W
E
S
150 ft
10°
15°