Section 3.4 – Half-Angle Formulas

$\cos 2A = 2\cos^2 A - 1$

$\cos 2x = 2\cos^2 x - 1$

$\cos 2x + 1 = 2\cos^2 x$

$2\cos^2 x = \cos 2x + 1$

$\cos^2 x = \dfrac{\cos 2x + 1}{2}$ *Divide both sides by 2*

$\cos x = \pm\sqrt{\dfrac{\cos 2x + 1}{2}}$ *Replace x with* $\frac{A}{2}$

$$\Rightarrow \boxed{\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}}$$

$\cos 2A = 1 - 2\sin^2 A$

$\cos 2x = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

$\sin^2 x = \dfrac{1-\cos 2x}{2}$ *Divide both sides by 2*

$\sin x = \pm\sqrt{\dfrac{1-\cos 2x}{2}}$ *Replace x with* $\frac{A}{2}$

$$\Rightarrow \boxed{\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}}$$

Example

Find the exact value of $\cos 15°$

<u>*Solution*</u>

$$\begin{aligned}\cos 15° &= \cos\left(\tfrac{1}{2}30°\right)\\ &= \sqrt{\frac{1+\cos 30°}{2}}\\ &= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}\\ &= \sqrt{\frac{2+\sqrt{3}}{2}}\end{aligned}$$

Example

If $\cos A = \frac{3}{5}$ with $270° < A < 360°$ find $\sin\frac{A}{2},\ \cos\frac{A}{2},\ \textit{and}\ \tan\frac{A}{2}$

<u>*Solution*</u>

Since $270° < A < 360°$

$$\frac{270°}{2} < \frac{A}{2} < \frac{360°}{2}$$

$$135° < \frac{A}{2} < 180° \ \Rightarrow \frac{A}{2} \in QII$$

$$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}}$$

$$= \sqrt{\frac{1-\frac{3}{5}}{2}}$$

$$= \sqrt{\frac{5-3}{5}\frac{1}{2}}$$

$$= \sqrt{\frac{2}{5}\frac{1}{2}}$$

$$= \sqrt{\frac{1}{5}}$$

$$= \frac{1}{\sqrt{5}}$$

$$\cos\frac{A}{2} = -\sqrt{\frac{1+\cos A}{2}}$$

$$= -\sqrt{\frac{1+\frac{3}{5}}{2}}$$

$$= -\sqrt{\frac{\frac{8}{5}}{2}}$$

$$= -\sqrt{\frac{4}{5}}$$

$$= -\frac{2}{\sqrt{5}}$$

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}$$

$$= \frac{\frac{1}{\sqrt{5}}}{-\frac{2}{\sqrt{5}}}$$

$$= -\frac{1}{2}$$

Example

If $\sin A = -\frac{12}{13}$ with $180° < A < 270°$ find the six trigonometric function of $A/2$

<u>*Solution*</u>

Since $180° < A < 270°$

$$\cos A = -\sqrt{1-\sin^2 A} = -\frac{5}{13}$$

$$90° < \frac{A}{2} < 135° \qquad \Rightarrow \frac{A}{2} \in QII$$

$$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}}$$

$$= \sqrt{\frac{1-\frac{-5}{13}}{2}}$$

$$= \sqrt{\frac{13+5}{13}\frac{1}{2}}$$

$$= \sqrt{\frac{9}{13}}$$

$$= \frac{3}{\sqrt{13}}$$

$$\cos\frac{A}{2} = -\sqrt{\frac{1+\cos A}{2}}$$

$$= -\sqrt{\frac{1+\frac{-5}{13}}{2}}$$

$$= -\sqrt{\frac{\frac{8}{13}}{2}}$$

$$= -\sqrt{\frac{4}{13}}$$

$$= -\frac{2}{\sqrt{13}}$$

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}$$

$$= \frac{\frac{3}{\sqrt{13}}}{-\frac{2}{\sqrt{13}}}$$

$$= -\frac{3}{2}$$

$$\cot\frac{A}{2} = \frac{1}{\tan\frac{A}{2}}$$

$$= -\frac{2}{3}$$

$$\csc\frac{A}{2} = \frac{1}{\sin\frac{A}{2}}$$

$$= \frac{\sqrt{13}}{3}$$

$$\sec\frac{A}{2} = \frac{1}{\cos\frac{A}{2}}$$

$$= -\frac{\sqrt{13}}{2}$$

$$\boxed{\tan\frac{A}{2}=\frac{1-\cos A}{\sin A}} \qquad \boxed{\tan\frac{A}{2}=\frac{\sin A}{1+\cos A}}$$

Example

Find the exact of $\tan 15°$

<u>*Solution*</u>

$$\tan 15° = \tan\frac{30°}{2}$$

$$= \frac{1-\cos 30°}{\sin 30°}$$

$$= \frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

$$= \frac{\frac{2-\sqrt{3}}{2}}{\frac{1}{2}}$$

$$= 2-\sqrt{3}$$

Example

Prove $\sin^2\frac{x}{2} = \frac{\tan x - \sin x}{2\tan x}$

<u>*Solution*</u>

$$\sin^2\frac{x}{2} = \frac{1-\cos x}{2}$$

$$= \frac{\tan x}{\tan x}\frac{1-\cos x}{2}$$

$$= \frac{\tan x - \tan x\cos x}{2\tan x}$$

$$= \frac{\tan x - \frac{\sin x}{\cos x}\cos x}{2\tan x}$$

$$= \frac{\tan x - \sin x}{2\tan x}$$

Exercises *Section* 3.4 – Half-Angle Formulas

1. Use half-angle formulas to find the exact value of $\sin 105°$

2. Find the exact of $\tan 22.5°$

3. Given: $\cos x = \frac{2}{3}$, $\frac{3\pi}{2} < x < 2\pi$, find $\cos\frac{x}{2}$, $\sin\frac{x}{2}$, and $\tan\frac{x}{2}$

4. Prove the identity $2\csc x\cos^2\frac{x}{2} = \frac{\sin x}{1-\cos x}$

5. Prove the identity $\tan\frac{\alpha}{2} = \sin\alpha + \cos\alpha\cot\alpha - \cot\alpha$

6. Prove the following equation is an identity: $\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right) = \frac{\sin^2 x}{4}$

7. Prove the following equation is an identity: $\tan\frac{x}{2} + \cot\frac{x}{2} = 2\csc x$

8. Prove the following equation is an identity: $2\sin^2\left(\frac{x}{2}\right) = \frac{\sin^2 x}{1+\cos x}$

9. Prove the following equation is an identity: $\tan^2\left(\frac{x}{2}\right) = \frac{\sec x + \cos x - 2}{\sec x - \cos x}$

10. Prove the following equation is an identity: $\sec^2\left(\frac{x}{2}\right) = \frac{2\sec x + 2}{\sec x + 2 + \cos x}$

11. Prove the following equation is an identity: $\frac{1-\sin^2\left(\frac{x}{2}\right)}{1+\sin^2\left(\frac{x}{2}\right)} = \frac{1+\cos x}{3-\cos x}$

12. Prove the following equation is an identity: $\frac{1-\cos^2\left(\frac{x}{2}\right)}{1-\sin^2\left(\frac{x}{2}\right)} = \frac{1-\cos x}{1+\cos x}$