Section 4.5 – Diagonalization

When $\vec{x}$ is an eigenvector, multiplication by $\boldsymbol{A}$ is just multiplication by a single number: $A\vec{x} = \lambda\vec{x}$.
The matrix $\boldsymbol{A}$ turns into a diagonal matrix $\boldsymbol{\Lambda}$ when we use the eigenvectors property.

Diagonalization

Suppose the n by n matrix $\boldsymbol{A}$ has n linearly independent eigenvectors $\vec{x}_1, \ldots, \vec{x}_n$. Put them into the column of an ***eigenvector matrix P***. Then $P^{-1}AP$ is the eigenvalue matrix $\boldsymbol{\Lambda}$:

$$P^{-1}AP = \Lambda = D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Example

The projection matrix $A = \begin{pmatrix} .5 & .5 \\ .5 & .5 \end{pmatrix}$ has $\lambda_{1,2} = 0$ *and* 1

Solution

For $\lambda_1 = 0 \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \frac{1}{2}x + \frac{1}{2}y = 0$$

$x = -y$

Therefore, $V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

For $\lambda_2 = 1 \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad -\frac{1}{2}x + \frac{1}{2}y = 0$$

$x = y$

Therefore, $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The eigenvectors are: $(-1,\ 1)$ & $(1,\ 1)$ that are the value of P.

$$P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$P^{-1} = -\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underset{P^{-1}}{\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}} \underset{A}{\begin{pmatrix} .5 & .5 \\ .5 & .5 \end{pmatrix}} \underset{P}{\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}} \underset{=}{=} \underset{D}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}$$

Definition

A square matrix A is called ***diagonalizable*** if there is an invertible matrix P such that $P^{-1}AP$ is diagonal; the matrix P is said to ***diagonalize*** A.

Theorem

Independent x from different λ - Eigenvectors $\vec{x}_1, \ldots, \vec{x}_n$ that correspond to distinct (all different) eigenvalues are linearly independent. An n by n matrix that has n different eigenvalues (no repeated λ's) must be diagonalizable.

Proof

Suppose $c_1\vec{x}_1 + c_2\vec{x}_2 = 0$ (1)

$$\begin{pmatrix} c_1\vec{x}_1 & c_2\vec{x}_2 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0$$

$$c_1\lambda_1\vec{x}_1 + c_2\lambda_2\vec{x}_2 = 0 \quad (2)$$

Multiply (1) by λ_2, that implies to

$$c_1\lambda_2\vec{x}_1 + c_2\lambda_2\vec{x}_2 = 0 \quad (3)$$

(2) − (3)

$$c_1\lambda_1\vec{x}_1 + c_2\lambda_2\vec{x}_2 - \left(c_1\lambda_2\vec{x}_1 + c_2\lambda_2\vec{x}_2\right) = 0$$

$$c_1\lambda_1\vec{x}_1 + c_2\lambda_2\vec{x}_2 - c_1\lambda_2\vec{x}_1 - c_2\lambda_2\vec{x}_2 = 0$$

$$c_1\lambda_1\vec{x}_1 - c_1\lambda_2\vec{x}_1 = 0$$

$$c_1\left(\lambda_1 - \lambda_2\right)\vec{x}_1 = 0$$

Since $\vec{x}_i \neq 0$ and λ's are different $\lambda_1 - \lambda_2 \neq 0$, we forced $\underline{c_1 = 0}|$

Similarly; Multiply (1) by λ_1, that implies to $c_1\lambda_1\vec{x}_1 + c_2\lambda_1\vec{x}_2 = 0$ (4)

(2) − (4)

$$c_1\lambda_1\vec{x}_1 + c_2\lambda_2\vec{x}_2 - c_1\lambda_1\vec{x}_1 - c_2\lambda_1\vec{x}_2 = 0$$

$$c_2\left(\lambda_2 - \lambda_1\right)\vec{x}_2 = 0 \quad \Rightarrow \underline{c_2 = 0}|$$

Therefore, $\vec{x}_1$ *and* $\vec{x}_2$ must be independent.

Theorem

If $\vec{v}_1, \ldots, \vec{v}_n$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, then $\left\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\right\}$ is linearly independent set.

Theorem

If an n x n matrix $\boldsymbol{A}$ has n distinct eigenvalues, then the following are equivalent:

a) $\boldsymbol{A}$ is diagonalizable

b) $\boldsymbol{A}$ has n linearly independent eigenvectors.

Example

Given the Markov matrix $A=\begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}$

Solution

$$|A-\lambda I|=\begin{vmatrix} .8-\lambda & .3 \\ .2 & .7-\lambda \end{vmatrix}$$

$$=(.8-\lambda)(.7-\lambda)-.06$$

$$=\lambda^2-1.5\lambda+.56-.06$$

$$=\lambda^2-1.5\lambda+.5=0$$

The eigenvalues are: $\lambda_1=1, \quad \lambda_2=.5$

For $\lambda_1=1$, we have: $(A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} -.2 & .3 \\ .2 & -.3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad -.2x+.3y=0$$

$\Rightarrow 2x=3y$

Therefore, the eigenvector $V_1=\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

For $\lambda_2=.5$, we have: $(A-\lambda_2 I)V_2=0$

$$\begin{pmatrix} .3 & .3 \\ .2 & .2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad .3x+.3y=0$$

$\Rightarrow x=-y$

Therefore, the eigenvector $V_2=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

The eigenvector matrix is given by:

$$P=\begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}$$

$$P^{-1} = -\frac{1}{5}\begin{pmatrix} -1 & -1 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{3}{5} \end{pmatrix}$$

$$\overset{P}{\begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}} \overset{D}{\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}} \overset{P^{-1}}{\begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{3}{5} \end{pmatrix}} = \begin{pmatrix} 3 & \frac{1}{2} \\ 2 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{3}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{8}{10} & \frac{3}{10} \\ \frac{2}{10} & \frac{7}{10} \end{pmatrix}$$

$$= \underset{A}{\begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}}$$

Eigenvalues of AB and A + B

An eigenvalue of ***A*** times an eigenvalue of ***B*** usually does not give an eigenvalue of ***AB***.

$$\lambda_A \lambda_B \neq \lambda_{AB}$$

Commuting matrices share eigenvectors: Suppose ***A*** and ***B*** can be diagonalized. They share the eigenvector matrix ***P*** if and only if $AB = BA$.

Matrix Powers A^k

$$A^2 = PDP^{-1}PDP^{-1}$$
$$= PD^2P^{-1}$$

$$A^k = \left(PDP^{-1}\right)\cdots\left(PDP^{-1}\right)$$
$$= PD^kP^{-1}$$

The eigenvector matrix for A^k is still *S*, and the eigenvalue matrix is A^k. The eigenvectors don't change, and the eigenvalues are taken to the k^{th} power. When *A* is diagonalized, $A^k\vec{u}_0$ is easy.

Here are steps (taken from Fibonacci):

1. Find the eigenvalues of *A* and look for *n* independent eigenvectors.
2. Write $\vec{u}_0$ as a combination $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ of the eigenvectors.
3. Multiply each eigenvector $\vec{v}_i$ *by* $\left(\lambda_i\right)^k$. Then

$$\vec{u}_k = A_k\vec{u}_0$$
$$= c_1\left(\lambda_1\right)^k\vec{v}_1 + \cdots + c_n\left(\lambda_n\right)^k\vec{v}_n$$

Example

Compute A^k *where* $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

Solution

$$\det\left(A - \lambda I\right) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix}$$
$$= \left(1-\lambda\right)\left(2-\lambda\right) = 0$$

The eigenvalues are: $\boxed{\lambda_{1,2} = 1,\ 2}$

For $\lambda_1 = 1 \quad \Rightarrow \quad \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \boxed{y = 0}$$

$$\Rightarrow \quad \boxed{V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

For $\lambda_2 = 2 \quad \Rightarrow \quad \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x = y$$

$$\Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The eigenvector matrix is given by:

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \Rightarrow \quad D^k = \begin{pmatrix} 1^k & 0 \\ 0 & 2^k \end{pmatrix}$$

$$\begin{aligned} A^k &= PD^kP^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^k \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2^k \\ 0 & 2^k \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{pmatrix} \end{aligned}$$

Similar Matrices

Definition

If A and B are square matrices, then we say that ***B* is similar to *A*** if there exists an invertible matrix P such that $B = P^{-1}AP \quad or \quad A = PBP^{-1}$

- Similar matrices B and $M^{-1}AM$ have the same eigenvalues. If $\vec{x}$ is an eigenvector of A then $M^{-1}\vec{x}$ is an eigenvector of $B = M^{-1}AM$.

Proof

Since $B = M^{-1}AM \Rightarrow A = MBM^{-1}$

Suppose $A\vec{x} = \lambda\vec{x}$:

$$MBM^{-1}\vec{x} = \lambda\vec{x}$$
$$BM^{-1}\vec{x} = \lambda M^{-1}\vec{x}$$

The eigenvalue of B is the same λ. The eigenvector is now $M^{-1}x$

Example

The projection $A = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ is similar to $D = P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Choose $M = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$; the similar matrix $M^{-1}AM \;\; is \;\; \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Also choose $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$; the similar matrix $M^{-1}AM \;\; is \;\; \begin{bmatrix} .5 & -.5 \\ -.5 & .5 \end{bmatrix}$

These matrices $M^{-1}AM$ all have the same eigenvalues 1 and 0.

Every 2 by 2 matrix with those eigenvalues is similar to A.

The eigenvectors change with M.

Example

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is similar to every matrix $B = \begin{bmatrix} cd & d^2 \\ -c^2 & -cd \end{bmatrix}$ except $B = 0$.

These matrices B all have zero determinant (like A). T
hey all have rank one (like A). Their trace is $cd - cd = 0$.

Their eigenvalues are 0 and 0 (like A).

Choose $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with* $ad - cd = 1$ and $B = M^{-1}AM$

Connections between similar matrices A and B:

Not Changed	***Changed***
Eigenvalues	Eigenvectors
Trace and determinant	Nullspace
Rank	Column space
Number of independent	Row space
eigenvectors	Left nullspace
Jordan form	Singular values

Example

Jordan matrix J has triple eigenvalues 5, 5, 5. Its only eigenvectors are multiples of (1, 0, 0). Algebraic multiplicity 3, geometric multiplicity 1:

$$\text{If } J = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \quad \textit{then} \quad J - 5I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ has rank 2.}$$

Every similar matrix $B = M^{-1}JM$ has the same triple eigenvalues 5, 5, 5. Also $B - 5I$ must have the same rank 2. Its nullspace has dimension 3 − 2 = 1. So each similar matrix B also has only one independent eigenvector.

The transpose matrix J^T has the same eigenvalues 5, 5, 5, and $J^T - 5I$ has the same rank 2. ***Jordan's theory says that*** J^T ***is similar to J***. The matrix that produces the similarity happens to be the reserve identity M:

$$J^T = M^{-1}JM \quad \textit{is} \quad \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

There is one line of eigenvectors $(x_1, 0, 0)$ for J and another line $(0, 0, x_3)$ for J^T.

Fibonacci Numbers

Every new Fibonacci number is the sum of the two previous F's.

The ***sequence*** 0, 1, 1, 2, 3, 5, 8, 13, …. comes from $F_{k+2} = F_{k+1} + F_k$

Problem

Find the Fibonacci number F_{100}

We can apply the rule one step at a time, or just use Linear algebra.

Let consider the matrix equation: $u_{k+1} = Au_k$. Fibonacci rule gave us a two-step rule for scalars.

Let $\vec{u}_k = \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix}$, the rule $\begin{matrix} F_{k+2} = F_{k+1} + F_k \\ F_{k+1} = F_{k+1} \end{matrix}$ becomes $\vec{u}_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \vec{u}_k$.

Every step multiplies by $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, after 100 steps we reach $\vec{u}_{100} = \begin{pmatrix} F_{101} \\ F_{100} \end{pmatrix}$

$$\vec{u}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \dots \quad \vec{u}_{100} = \begin{pmatrix} F_{101} \\ F_{100} \end{pmatrix}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} \\ &= -\lambda(1-\lambda) - 1 \\ &= \lambda^2 - \lambda - 1 \end{aligned}$$

The characteristic equation is $\lambda^2 - \lambda - 1 = 0$ and the solutions are

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618 \quad \textit{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -.618$$

For $\lambda_1 \quad \Rightarrow \quad (A - \lambda_1 I)\vec{v}_1 = 0$

$$\begin{pmatrix} 1-\lambda_1 & 1 \\ 1 & -\lambda_1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x_1 - \lambda_1 y_1 = 0$$

$x_1 = \lambda_1 y_1$

$$\Rightarrow \boxed{\vec{v}_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}}$$

For $\lambda_2 \quad \Rightarrow \quad (A - \lambda_2 I)\vec{v}_2 = 0$

$$\begin{pmatrix} 1-\lambda_2 & 1 \\ 1 & -\lambda_2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x_1 - \lambda_2 y_1 = 0$$

$$x_1 = \lambda_2 y_1$$

$$\Rightarrow \vec{v}_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$$

The eigenvector matrix is given by:

$$S = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$$

The combination of these eigenvectors that give $\vec{u}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\lambda_1 - \lambda_2}\left(\begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} - \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}\right)$$

$$= \frac{\vec{v}_1 - \vec{v}_2}{\lambda_1 - \lambda_2}$$

$$\vec{u}_{100} = \frac{(\lambda_1)^{100}\,\vec{v}_1 - (\lambda_2)^{100}\,\vec{v}_2}{\lambda_1 - \lambda_2}$$

$$F_{100} = \frac{1}{\lambda_1 - \lambda_2}\left[(\lambda_1)^{100} - (\lambda_2)^{100}\right]$$

$$= \frac{1}{\frac{1+\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{100} - \left(\frac{1+\sqrt{5}}{2}\right)^{100}\right]$$

$$\approx 2.54 \times 10^{20}$$

The *Jordan* Form

For every A, we want to choose M so that $M^{-1}AM$ is as nearly diagonal as possible. When A has a full set of n eigenvectors, they go into the columns of M. Then $M = P$. The matrix $P^{-1}AP$ is diagonal.

If A has $\boldsymbol{s}$ independent eigenvectors, it is similar to a matrix J that has $\boldsymbol{s}$ Jordan blocks on its diagonal. There is a matrix M such that

$$M^{-1}AM = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix} = J$$

Each block in J has one eigenvalue λ_i, one eigenvector, and 1's above the diagonal:

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

A is similar to B if they share the same Jordan form J – not otherwise.

Exercises Section 4.5 – Diagonalization

1. The Lucas numbers are like Fibonacci numbers except they start with $L_1 = 1$ *and* $L_2 = 3$. Following the rule $L_{k+2} = L_{k+1} + L_k$. The next Lucas numbers are 4, 7, 11, 18. Show that the Lucas number $L_{100} = \lambda_1^{100} + \lambda_2^{100}$.

2. Find all eigenvector matrices *S* that diagonalize *A* (rank 1) to give $S^{-1}AS = \Lambda$:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

What is A^n? Which matrices *B* commute with *A* (so that $AB = BA$)

(3 – 6) Determine whether the matrix is diagonalizable

3. $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$

4. $\begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$

5. $\begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$

6. $\begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

(7 – 26) Determine if the matrices are diagonalizable. If so, find a matrix ***P*** that diagonalizes *A* and determine $P^{-1}AP$.

7. $A = \begin{pmatrix} -14 & 12 \\ -20 & 17 \end{pmatrix}$

8. $A = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}$

9. $A = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}$

10. $A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$

11. $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$

12. $A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 3 & 2 \end{pmatrix}$

13. $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$

14. $A = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{pmatrix}$

15. $A=\begin{pmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

16. $A=\begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$

17. $A=\begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & -2 \\ 1 & 3 & 1 \end{pmatrix}$

18. $A=\begin{pmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{pmatrix}$

19. $A=\begin{pmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{pmatrix}$

20. $A=\begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

21. $A=\begin{pmatrix} 3 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

22. $A=\begin{pmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{pmatrix}$

23. $A=\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}$

24. $A=\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

25. $A=\begin{pmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

26. $A=\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}$

27. The 4 by 4 triangular Pascal matrix P_L and its inverse (alternating diagonals) are

$$P_L=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \quad \textit{and} \quad P_L^{-1}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

Check that P_L and P_L^{-1} have the same eigenvalues. Find a diagonal matrix D with alternating signs that gives $P_L^{-1}=D^{-1}P_L D$, so P_L is similar to P_L^{-1}. Show that $P_L D$ with columns of alternating signs is its own inverse.

Since P_L and P_L^{-1} are similar they have the same Jordan form J. Find J by checking the number of independent eigenvectors of P_L with $\lambda = 1$.

28. These Jordan matrices have eigenvalues 0, 0, 0, 0. They have two eigenvectors (one from each block). But the block sizes don't match and they are not similar:

$$J = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad and \quad K = \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

For any matrix M compare JM with MK. If they are equal show that M is not invertible. Then $M^{-1}JM = K$ is Impossible; J is not similar to K.

29. If $\boldsymbol{x}$ is in the nullspace of A show that $M^{-1}\boldsymbol{x}$ is in the nullspace of $M^{-1}AM$.
The nullspaces of A and $M^{-1}AM$ have the same (vectors) (basis) (dimension)

30. Prove that A^T is always similar to A (λ's are the same):

a) For one Jordan block J_i, find M_i so that $M_i^{-1}J_iM_i = J_i^T$.

b) For any J with blocks J_i, build M_0 from blocks so that $M_0^{-1}JM_0 = J^T$.

c) For any $A = MJM^{-1}$: Show that A^T is similar to J^T and so to J and so to A.

31. Why are these statements all true?

a) If A is similar to B then A^2 is similar to B^2.

b) A^2 and B^2 can be similar when A and B are not similar.

c) $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ is similar to $\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}$

d) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ is not similar to $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

e) If we exchange rows 1 and 2 of A, and then exchange columns 1 and 2 the eigenvalues stay the same. In this case M=?

32. If an n x n matrix A has all eigenvalues $\lambda = 0$ prove that A^n is the zero matrix.

33. If A is similar to A^{-1}, must all the eigenvalues equal to 1 or –1?.

(34 – 42) Determine whether the *two matrices* are similar matrices

34. $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}$

35. $A = \begin{pmatrix} 4 & -1 \\ 2 & 4 \end{pmatrix}$ $B = \begin{pmatrix} 4 & 1 \\ 2 & 4 \end{pmatrix}$

36. $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 2 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

37. $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}1&2&3\\0&4&5\\0&0&6\end{pmatrix}$

38. $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}4&0&0\\7&1&0\\8&9&6\end{pmatrix}$

39. $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}4&7&0\\0&1&0\\8&9&6\end{pmatrix}$

40. $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}1&2&0\\3&4&5\\0&7&6\end{pmatrix}$

41. $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}1&2&0\\3&4&0\\0&5&6\end{pmatrix}$

42. $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}1&0&0\\2&5&1\\3&1&5\end{pmatrix}$

43. Prove that two similar matrices have the same determinant. Explain geometrically why this is reasonable.

44. Prove that two similar matrices have the same characteristic polynomial and thus the same eigenvalues. Explain geometrically why this is reasonable.

45. Suppose that A is a matrix. Suppose that the linear transformation associated to A has two linearly independent eigenvectors. Prove that A is similar to a diagonal matrix.

46. Prove that if A is a 2×2 matrix that has two distinct eigenvalues, then A is similar to a diagonal matrix.

47. Suppose that the characteristic polynomial of a matrix has a double root. Is it true that the matrix has two linearly independent eigenvectors? Consider the example $\begin{pmatrix}0&0\\1&0\end{pmatrix}$. Is it true that matrices with equal characteristic polynomial are necessarily similar?

48. Show that the given matrix is not diagonalizable. $\begin{bmatrix}1&1\\-1&-1\end{bmatrix}$

49. Determine if the given matrix is diagonalizable. If, so, find matrices S and $\Lambda(D)$ such that the given matrix equals $S\Lambda S^{-1}$

a) $\begin{bmatrix}3&3\\4&2\end{bmatrix}$

b) $\begin{bmatrix}1&0&1\\0&2&0\\-1&0&-1\end{bmatrix}$

50. A is a 5×5 matrix with *two* eigenvalues. One eigenspace is *three*–dimensional, and the other eigenspace is *two*–dimensional. Is A diagonalizable? Why?

51. A is a 3×3 matrix with *two* eigenvalues. Each eigenspace is *one*–dimensional. Is A diagonalizable? Why?

52. A is a 4×4 matrix with *three* eigenvalues. One eigenspace is *one*–dimensional, and one of the other eigenspace is *two*–dimensional. Is it possible that A is *not* diagonalizable? Justify your answer?

53. A is a 7×7 matrix with *three* eigenvalues. One eigenspace is *two*–dimensional, and one of the other eigenspace is *three*–dimensional. Is it possible that A is *not* diagonalizable? Justify your answer?

54. Show that if A is diagonalizable and invertible, then so is A^{-1}.

55. Show that if A has n linearly independent eigenvectors, then so does A^T.

56. A factorization $A = PDP^{-1}$ is not unique. Demonstrate this for the matrix $A = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix}$ with $D_1 = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$, find a matrix P_1 such that $A = P_1 D_1 P_1^{-1}$.

57. Construct a nonzero 2×2 matrix that is invertible but not diagonalizable.

58. Construct a nonzero 2×2 matrix that is diagonalizable but not invertible.

59. What are the matrices that are similar to themselves only?

60. For any scalars a, b, and c, show that

$$A = \begin{pmatrix} b & c & a \\ c & a & b \\ a & b & c \end{pmatrix}, \quad B = \begin{pmatrix} c & a & b \\ a & b & c \\ b & c & a \end{pmatrix}, \quad C = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$$

are similar.

Moreover, if $BC = CB$, then A has two zero eigenvalues.

(61 – 64) For positive integer $k \geq 2$, compute

61. $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}^k$

62. $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^k$

63. $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^k$

64. $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^k$

65. Let $A=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Show that A^k is similar to A fro every positive integer k. It is true more generally for any matrix with all eigenvalues equal to 1.

66. Can a matrix be similar to two different diagonal matrices?

67. Prove that if A is diagonalizable, then A^T is diagonalizable.

68. Prove that if the eigenvalues of a diagonalizable matrix A are all ± 1, then the matrix is equal to its inverse.

69. Prove that if A is diagonalizable with n real eigenvalues $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$, then $|A| = \lambda_1 \cdot \lambda_2 \cdot\ \ldots \cdot \lambda_n$

70. If x is a real number, then we can define e^x by the series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ \cdots$$

In similar way, If X is a square matrix, then we can define e^X by the series

$$e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \frac{1}{4!}X^4 + \ \cdots$$

Evaluate e^X, where X is the indicated square matrix.

a) $X=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

b) $X=\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

c) $X=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

d) $X=\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$