Section 2.4 - Inhomogeneous Equations; the Method of Undetermined Coefficients

The second order ***nonhomogeneous*** equation is given by: $y'' + p(x)y' + q(x)y = f(x) \quad (N)$

The corresponding ***homogeneous*** equation: $y'' + p(x)y' + q(x)y = 0 \quad (H)$

Theorem

Suppose that y_p is a particular solution to the nonhomogeneous (or inhomogeneous) equation $y'' + py' + qy = f$ and that y_1 and y_2 form a fundamental set of solutions to the homogeneous equation $y'' + py' + qy = 0.$ Then the general solution to the inhomogeneous equation is given by

$$y = y_p + C_1 y_1 + C_2 y_2$$

C_1 *and* C_2 are arbitrary constants.

Theorem

Let $y = y_1(x)$ *and* $y = y_2(x)$ be ***linearly independent*** $(W(x) \neq 0)$ solutions of the reduced equation (*H*) and let $y_p(x)$ be a ***particular solution*** of (*N*). Then the general solution of (*N*) consists of the general solution of the reduced equation (*H*) ***plus*** a particular solution of (*N*):

$$y(x) = \underbrace{y_p(x)}_{\text{a Particular Solution}} + \underbrace{C_1 y_1(x) + C_2 y_2(x)}_{\text{General Solution}}$$

Forcing Term

If the forcing term *f* has a form that is replicated under differentiation, then look for a solution with the same general form as the forcing term.

Example

Find a particular solution to the equation $y'' - y' - 2y = 2e^{-2t}$

<u>Solution</u>

The forcing term $f(t) = 2e^{-2t} \Rightarrow$ the particular solution $y = ae^{-2t}$

$$y' = -2ae^{-2t}$$

$$y'' = 4ae^{-2t}$$

$$4ae^{-2t} + 2ae^{-2t} - 2ae^{-2t} = 2e^{-2t}$$

$$4ae^{-2t} = 2e^{-2t}$$

$4a = 2$

$a = \frac{1}{2}$

$$\boxed{y_p(t) = \frac{1}{2}e^{-2t}}$$

Trigonometric Forcing Term

$f(t) = A\cos\omega t + B\sin\omega t$

The general solution: $y(t) = a\cos\omega t + b\sin\omega t$

Example

Find a particular solution to the equation $y'' + 2y' - 3y = 5\sin 3t$

Solution

The particular solution: $y(t) = a\cos 3t + b\sin 3t$

$$y' = -3a\sin 3t + 3b\cos 3t$$

$$y'' = -9a\cos 3t - 9b\sin 3t$$

$$\begin{aligned} y'' + 2y' - 3y &= -9a\cos 3t - 9b\sin 3t + 2\left(-3a\sin 3t + 3b\cos 3t\right) - 3\left(a\cos 3t + b\sin 3t\right) \\ &= -9a\cos 3t - 9b\sin 3t - 6a\sin 3t + 6b\cos 3t - 3a\cos 3t - 3b\sin 3t \\ &= \left(-12a + 6b\right)\cos 3t - \left(6a + 12b\right)\sin 3t \\ &= 5\sin 3t \end{aligned}$$

$$\begin{cases} -12a + 6b = 0 \\ -\left(6a + 12b\right) = 5 \end{cases} \Rightarrow a = -\frac{1}{6},\ b = -\frac{1}{3}$$

$$\boxed{y_p(t) = -\frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t}$$

The Complex Method

Example

Find a particular solution to the equation $y'' + 2y' - 3y = 5\sin 3t$

Solution

$$5e^{3it} = 5\cos 3t + 5i\sin 3t = 5cis3t$$

$$z'' + 2z' - 3z = 5e^{3it}$$

The particular solution: $z(t) = x(t) + i\,y(t)$

$$\begin{aligned} z'' + 2z' - 3z &= (x+iy)'' + 2(x+iy)' - 3(x+iy) \\ &= (x'' + 2x' - 3x) + i(y'' + 2y' - 3y) \\ &= 5\cos 3t + i\ 5\sin 3t \end{aligned}$$

$$x'' + 2x' - 3x = 5\cos 3t$$

$$z(t) = ae^{3it}$$

$$z' = 3iae^{3it}$$

$$z'' = 9i^2ae^{3it} = -9ae^{3it}$$

$$\begin{aligned} z'' + 2z' - 3z &= -9ae^{3it} + 2(3i)ae^{3it} - 3ae^{3it} \\ &= -12ae^{3it} + 6iae^{3it} \\ &= -6(2-i)ae^{3it} \\ &= 5e^{3it} \end{aligned}$$

$$-6(2-i)a = 5$$

$$\begin{aligned} a &= -\frac{5}{6(2-i)}\frac{2+i}{2+i} \\ &= -\frac{5(2+i)}{6(4+1)} \\ &= -\frac{2+i}{6} \end{aligned}$$

$$\begin{aligned} z(t) &= -\tfrac{1}{6}(2+i)e^{3it} \\ &= -\tfrac{1}{6}(2+i)(\cos 3t + i\sin 3t) \\ &= -\tfrac{1}{6}\left[(2\cos 3t - \sin 3t) + i(\cos 3t + 2\sin 3t)\right] \end{aligned}$$

$$\boxed{y_p(t) = -\tfrac{1}{6}(\cos 3t + 2\sin 3t)}$$

Polynomial Forcing Term

$$f(t) = a_0 t^n + a_1 t^{n-1} + \ldots + a_{n-1} t + a_n$$

Example

Find a particular solution to the equation $y'' + 2y' - 3y = 3t + 4$

Solution

The right-hand side is a polynomial of degree 1.

The particular solution: $y(t) = at + b$

$$y' = a$$
$$y'' = 0$$

$$\begin{aligned} y'' + 2y' - 3y &= 0 + 2a - 3(at + b) \\ &= 2a - 3b - 3at \\ &= 3t + 4 \end{aligned}$$

$$\rightarrow \begin{cases} -3a = 3 \\ 2a - 3b = 4 \end{cases} \quad \Rightarrow a = -1; \quad b = -2$$

$$\boxed{y_p(t) = -t - 2}$$

Exceptional Cases

Example

Find a particular solution to the equation $y'' - y' - 2y = 3e^{-t}$

Solution

The particular solution $y = ae^{-t}$

$y'' - y' - 2y = ae^{-t} + ae^{-t} - 2ae^{-t} = 0$

The particular solution $y = ate^{-t}$ or $y = at^2 e^{-t}$

$$y' = ae^{-t} - ate^{-t} = ae^{-t}(1 - t)$$

$$y'' = -ae^{-t} - ae^{-t} + ate^{-t} = ate^{-t} - 2ae^{-t}$$

$$\begin{aligned} y'' - y' - 2y &= ate^{-t} - 2ae^{-t} - ae^{-t} + ate^{-t} - 2ate^{-t} \\ &= -3ae^{-t} \end{aligned}$$

$$-3ae^{-t} = 3e^{-t}$$
$$a = -1$$

The particular solution $\boxed{y_p = -te^{-t}}$

Summary

$f(t)$	y_p
Any Constant	A
$at+b$	$At+B$
at^2+c	At^2+Bt+C
$at^3+\cdots+b$	At^3+Bt^2+Ct+E
$\sin at \quad or \quad \cos at$	$A\cos at+B\sin at$
e^{at}	Ae^{at}
$(at+b)e^{at}$	$(At+B)e^{at}$
t^2e^{at}	$\left(At^2+Bt+C\right)e^{at}$
$e^{at}\sin bt$	$e^{at}(A\cos bt+B\sin bt)$
$t^2\sin bt$	$\left(At^2+Bt+C\right)\cos bt+\left(Et^2+Ft+G\right)\sin bt$
$te^{at}\cos bt$	$(At+B)\cos bt+(Ct+E)\sin bt$

- Check the homogeneous solution first, then start with next power for the particular solution.

  If we use the same power, we can add the constants which result with another constant. Therefore, we do not need to use the same power or the same terms as the homogeneous solution.

Example

$y''-y'-12y=e^{4x}$	$y_h=C_1e^{-3x}+C_2e^{4x}$	$y_P=Axe^{4x}$
$y''-2y'=12x-10$	$y_h=C_1+C_2e^{2x}$	$y_P=Ax^2+Bx$
$y''+2y'+y=x^2e^{-x}$	$y_h=\left(C_1+C_2x\right)e^{-x}$	$y_p=\left(Ax^4+Bx^3+Cx^2\right)e^{-x}$
$y''+6y'+13y=e^{-3x}\cos 2x$	$y_h=e^{-3x}\left(C_1\cos 2x+C_2\sin 2x\right)$	$y_p=e^{-3x}(Ax\cos 2x+Bx\sin 2x)$

Exercises *Section* 2.4 - Inhomogeneous Equations; the Method of Undetermined Coefficients

1. Show that the 3 solutions $y_1 = x, \quad y_2 = x\ln x, \quad y_3 = x^2$ of the 3rd order equation $x^3y''' - x^2y'' + 2xy' - 2y = 0$ are linearly independent on an open interval $x > 0$. Then find a particular solution that satisfies the initial conditions $y(1) = 3, \quad y'(1) = 2, \quad y''(1) = 1$

Find the particular solution for the given differential equation

2. $y'' + 3y' + 2y = 4e^{-3t}$

3. $y'' + 6y' + 8y = -3e^{-t}$

4. $y'' + 2y' + 5y = 12e^{-t}$

5. $y'' + 3y' - 18y = 18e^{2t}$

6. $y'' + 4y = \cos 3t$

7. $y'' + 7y' + 6y = 3\sin 2t$

8. $y'' + 5y' + 4y = 2 + 3t$

9. $y'' + 6y' + 8y = 2t - 3$

10. $y'' + 3y' + 4y = t^3$

11. $y'' + 2y' + 2y = 2 + \cos 2t$

12. $y'' - y = t - e^{-t}$

13. $y'' - 2y' + y = 10e^{-2t}\cos t$

14. $y''' - 4y'' + 4y' = 5t^2 - 6t + 4t^2e^t + 3e^{5t}$

Use the ***complex method*** to find the particular solution for

15. $y'' + 4y' + 3y = \cos 2t + 3\sin 2t$

16. $y'' + 4y = \cos 3t$

Find the general solution for the given differential equation

17. $y'' + y = 2\cos x$

18. $y'' + y = \cos 3x$

19. $y'' + y = 2x\sin x$

20. $y'' - y = x^2e^x + 5$

21. $y'' - y' = -3$

22. $y'' - y' = 2\sin x$

23. $y'' - y' = \sin x$

24. $y'' - y' = -8x + 3$

25. $y'' + y = 2x + 3e^x$

26. $y'' - y = x^2 + e^x$

27. $y'' + y' = 10x^4 + 2$

28. $y'' - y' = 5e^x - \sin 2x$

29. $y'' + y = x\cos x - \cos x$

30. $y'' + y = e^x \sin x$

31. $y'' - y' - 2y = 20\cos x$

32. $y'' - y' + \frac{1}{4}y = 3 + e^{x/2}$

33. $y'' + y' + \frac{1}{4}y = e^x(\sin 3x - \cos 3x)$

34. $y'' - y' - 2y = e^{3x}$

35. $y'' - y' - 6y = 20e^{-2x}$

36. $y'' + y' - 6y = 2x$

37. $y'' - y' - 6y = e^{-x} - 7\cos x$

38. $y'' + y' + 8y = x\cos 3x + (10x^2 + 21x + 9)\sin 3x$

39. $y'' - y' - 12y = e^{4x}$

40. $y'' + 2y' = 2x + 5 - e^{-2x}$

41. $y'' - 2y' = 12x - 10$

42. $y'' + 2y' + y = \sin x + 3\cos 2x$

43. $y'' - 2y' + y = 6e^x$

44. $y'' + 2y' + y = x^2$

45. $y'' + 2y' + y = x^2e^{-x}$

46. $y'' - 2y' + y = x^3 + 4x$

47. $y'' + 2y' + y = 6\sin 2x$

48. $y'' - 2y' + y = \left(x^2 - 1\right)e^{2x} + (3x+4)e^x$

49. $y'' + 2y' + 2y = 5e^{6x}$

50. $y'' + 2y' + 2y = x^3$

51. $y'' + 2y' + 2y = \cos x + e^{-x}$

52. $y'' - 2y' + 2y = e^x \sin x$

53. $y'' - 2y' + 2y = e^{2x}(\cos x - 3\sin x)$

54. $y'' - 2y' - 3y = 1 - x^2$

55. $y'' - 2y' - 3y = 4e^x - 9$

56. $y'' - 2y' - 3y = 2e^{-x}\cos x + x^2 + xe^{3x}$

57. $y'' - 2y' + 5y = 25x^2 + 12$

58. $y'' - 2y' + 5y = e^x \cos 2x$

59. $y'' - 2y' + 5y = e^x \sin x$

60. $y'' + 2y' - 24y = 16 - (x+2)e^{4x}$

61. $y'' + 3y = -48x^2e^{3x}$

62. $y'' - 3y' = e^{3x} - 12x$

63. $y'' + 3y' = 4x - 5$

64. $y'' - 3y' = 8e^{3x} + 4\sin x$

65. $y'' + 3y' + 2y = 6$

66. $y'' + 3y' + 2y = 4x^2$

67. $y'' - 3y' + 2y = 5e^x$

68. $y'' - 3y' + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$

69. $y'' - 3y' + 2y = 14\sin 2x - 18\cos 2x$

70. $y'' + 3y' + 2y = e^{-x} + e^{-2x} - x$

71. $y''' - 3y'' + 3y' - y = 3e^x$

72. $y'' - 3y' - 10y = -3$

73. $y'' - 3y' - 10y = 2x - 3$

74. $y'' + 3y' - 10y = 6e^{4x}$

75. $y'' + 3y' - 10y = x\left(e^x + 1\right)$

76. $y'' - 4y = 4x^2$

77. $y'' + 4y = 3x^3$

78. $y'' + 4y = 3\sin x$

79. $y'' + 4y = 3\sin 2x$

80. $y'' + 4y = 4\cos x + 3\sin x - 8$

81. $y'' - 4y = \left(x^2 - 3\right)\sin 2x$

82. $y'' + 4y' + 4y = 2x + 6$

83. $y'' + 4y' + 5y = 5x + e^{-x}$

84. $y'' + 4y' + 5y = 2e^{-2x} + \cos x$

85. $y'' + 5y' = 15x^2$

86. $y'' - 5y' = 2x^3 - 4x^2 - x + 6$

87. $y'' + 6y' + 8y = 3e^{-2x} + 2x$

88. $y'' - 6y' + 9y = e^{3x}$

89. $y'' + 6y' + 9y = -xe^{4x}$

90. $y'' + 6y' + 13y = e^{-3x}\cos 2x$

91. $y'' - 7y' = -3$

92. $y'' + 7y' = 42x^2 + 5x + 1$

93. $y'' + 8y = 5x + 2e^{-x}$

94. $y'' - 8y' + 20y = 100x^2 - 26xe^x$

95. $y'' - 9y = 54$

96. $y'' + 9y = x^2\cos 3x + 4\sin x$

97. $y'' + 10y' + 25y = 14e^{-5x}$

98. $y'' - 10y' + 25y = 30x + 3$

99. $y'' - 16y = 2e^{4x}$

100. $y'' + 25y = 6\sin x$

101. $y'' + 25y = 20\sin 5x$

102. $\frac{1}{4}y'' + y' + y = x^2 - 2x$

103. $2y'' - 5y' + 2y = -6e^{x/2}$

104. $2y'' - 7y' + 5y = -29$

105. $4y'' + 9y = 15$

106. $4y'' - 4y' - 3y = \cos 2x$

107. $9y'' - 6y' + y = 9xe^{x/3}$

108. $y^{(3)}+y''=8x^2$

109. $y^{(3)}-y''-4y'+4y=5-e^x+e^{2x}$

110. $y^{(3)}+y''=3e^x+4x^2$

111. $y^{(3)}+2y''+y'=10$

112. $y^{(3)}-2y''-4y'+8y=6xe^{2x}$

113. $y^{(3)}-3y''+3y'-y=x-4e^x$

114. $y^{(3)}-4y''+y'+6y=4\sin 2x$

115. $y^{(3)}-3y''+3y'-y=e^x-x+16$

116. $y^{(3)}-6y''=3-\cos x$

117. $y^{(3)}-6y''+11y'-6y=2xe^{-x}$

118. $y^{(3)}+8y''=-6x^2+9x+2$

119. $y^{(4)}+y''=3x^2+4\sin x-2\cos x$

120. $y^{(4)}+2y''+y=(x-2)^2$

121. $y^{(4)}-y''=4x+2xe^{-x}$

122. $\left(D^2+D-2\right)y=2x-40\cos 2x$

123. $\left(D^2-3D+2\right)y=2\sin x$

124. $(D-2)^3\left(D^2+9\right)y=x^2e^{2x}+x\sin 3x$

Find the general solution that satisfy the given initial conditions

125. $y''+y=\cos x;\quad y(0)=1,\quad y'(0)=-1$

126. $y''+y'=x\ ;\quad y(1)=0,\quad y'(1)=1$

127. $y''+y'=-x\ ;\quad y(0)=1,\quad y'(0)=0$

128. $y''+y=8\cos 2t-4\sin t\quad y\left(\frac{\pi}{2}\right)=-1,\quad y'\left(\frac{\pi}{2}\right)=0$

129. $y''-y'-2y=4x^2;\quad y(0)=1,\quad y'(0)=4$

130. $y''-y'-2y=e^{3x};\quad y(1)=2,\quad y'(1)=1$

131. $y''-y'-2y=e^{3x};\quad y(0)=1,\quad y'(0)=2$

132. $y''-y'-2y=e^{3x};\quad y(0)=2,\quad y'(0)=1$

133. $y''+2y'+y=2\cos t\ ;\quad y(0)=3,\quad y'(0)=0$

134. $y''-2y'+y=t^3;\quad y(0)=1,\quad y'(0)=0$

135. $y''-2y'+y=-3-x+x^2\ ;\quad y(0)=-2,\quad y'(0)=1$

136. $y''-2y'+2y=x+1;\quad y(0)=3,\quad y'(0)=0$

137. $y''+2y'+2y=\sin 3x;\quad y(0)=2,\quad y'(0)=0$

138. $y''+2y'+2y=2\cos 2t;\quad y(0)=-2,\quad y'(0)=0$

139. $y''-2y'-3y=2e^x-10\sin x\ ;\quad y(0)=2,\quad y'(0)=4$

140. $y''+2y'+10y=4+26x+6x^2+10x^3\ ;\quad y(0)=2,\quad y'(0)=9$

141. $y''-2y'+10y=6\cos 3t-\sin 3t\ ;\quad y(0)=2,\quad y'(0)=-8$

142. $y''+3y'+2y=e^x;\quad y(0)=0,\quad y'(0)=3$

143. $y''-3y'+2y=3e^{-x}-10\cos 3x\quad y(0)=1\quad y'(0)=2$

144. $y'' + 4y = -2$; $y\left(\frac{\pi}{8}\right) = \frac{1}{2}$, $y'\left(\frac{\pi}{8}\right) = 2$

145. $y'' + 4y = 2x$; $y(0) = 1$, $y'(0) = 2$

146. $y'' + 4y = \sin^2 2t$; $x\left(\frac{\pi}{8}\right) = 0$ $x'\left(\frac{\pi}{8}\right) = 0$

147. $y'' - 4y' + 8y = x^3$; $y(0) = 2$, $y'(0) = 4$

148. $y'' + 4y' + 4y = (3 + x)e^{-2x}$; $y(0) = 2$, $y'(0) = 5$

149. $y'' + 4y' + 4y = 4 - t$; $y(0) = -1$, $y'(0) = 0$

150. $y'' - 4y' + 4y = e^x$; $y(0) = 2$, $y'(0) = 0$

151. $y'' - 4y' - 5y = 4e^{-2t}$; $y(0) = 0$, $y'(0) = -1$

152. $y'' + 4y' + 5y = 35e^{-4x}$; $y(0) = -3$, $y'(0) = 1$

153. $y'' + 4y' + 8y = \sin t$; $y(0) = 1$, $y'(0) = 0$

154. $y'' - 4y' - 12y = 3e^{5t}$; $y(0) = \frac{18}{7}$, $y'(0) = -\frac{1}{7}$

155. $y'' - 4y' - 12y = \sin 2t$; $y(0) = 0$, $y'(0) = 0$

156. $y'' - 5y' = t - 2$; $y(0) = 0$, $y'(0) = 2$

157. $y'' + 5y' - 6y = 10e^{2x}$; $y(0) = 1$, $y'(0) = 1$

158. $y'' + 6y' + 10y = 22 + 20x$; $y(0) = 2$, $y'(0) = -2$

159. $y'' + 7y' + 12y = -2\cos 2x + 36\sin 2x$; $y(0) = -3$, $y'(0) = 3$

160. $y'' + 8y' + 7y = 10e^{-2x}$; $y(0) = -2$, $y'(0) = 10$

161. $y'' + 9y = \sin 2x$; $y(0) = 1$, $y'(0) = 0$

162. $y'' - 64y = 16$; $y(0) = 1$, $y'(0) = 0$

163. $2y'' + 3y' - 2y = 14x^2 - 4x + 11$; $y(0) = 0$, $y'(0) = 0$

164. $5y'' + y' = -6x$; $y(0) = 0$, $y'(0) = -10$

165. $x'' + 9x = 10\cos 2t$; $x(0) = x'(0) = 0$

166. $x'' + 4x = 5\sin 3t$; $x(0) = x'(0) = 0$

167. $x'' + 100x = 225\cos 5t + 300\sin 5t$; $x(0) = 375$, $x'(0) = 0$

168. $x'' + 25x = 90\cos 4t$; $x(0) = 0$, $x'(0) = 90$

169. $y^{(3)} - y' = 4e^{-x} + 3e^{2x}$; $y(0) = 0$, $y'(0) = -1$, $y''(0) = 2$

170. $y^{(3)} + y'' = x + e^{-x}$; $y(0) = 1$, $y'(0) = 0$, $y''(0) = 1$

171. $y^{(3)} - 2y'' + y' = 1 + xe^x$; $y(0) = y'(0) = 0$, $y''(0) = 1$

172. $y^{(4)} - 4y'' = x^2$; $y(0) = y'(0) = 1$, $y''(0) = y^{(3)}(0) = -1$

173. $y^{(4)} - y = 5$; $y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0$

174. $y^{(4)} - y''' = x + e^{x};$ $\quad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0$

175. If k and b are positive constants, then find the general solution of $\quad y'' + k^{2}y = \sin bx$