Matrices

Elementary Matrix Theory

It is often desirable to use matrix notation to simplify complex mathematical expressions. The simplifying matrix notation usually makes the equations much easier to handle and manipulate. Let consider the following set of n simultaneous algebraic equations:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= y_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= y_2 \\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= y_n
\end{aligned}$$

We may simplify to $AX = Y$; A, X, and Y are defined as matrices. These three matrices are defined to be:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \\ x_n \end{pmatrix} \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ \\ y_n \end{pmatrix}$$

where the bracketed arrays are simplified of coefficients and variables.

Definition of a Matrix

A matrix is a collection of elements arranged in a rectangular or square array. Several ways of representing a matrix are as follows:

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 4 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix} = \begin{Vmatrix} 0 & 1 \\ 2 & 4 \end{Vmatrix} = \left[a_{ij} \right]_{2,2}$$

Matrix Elements

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

a_{ij} is defined as the ***element***, ***entries***, or ***components*** in the i^{th} row and the j^{th} column of the matrix (Row 1st and column last).

$$\begin{array}{rc} & \textit{Column} \\ & \begin{matrix} C_1 & C_2 & C_3 \\ \downarrow & \downarrow & \downarrow \end{matrix} \\ \begin{matrix} Row\ 1 \to R_1 \\ Row\ 2 \to R_2 \\ Row\ 3 \to R_3 \end{matrix} & \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \end{array}$$

Order of Matrix: refers to the total number of rows and columns of the matrix. In general, a matrix with n rows and m columns is termed "n x m" or "n by m".

Square Matrix: number of rows is equal to number of columns, when $m = n$,

Column Matrix: one that has one column and more than one row: mx1 matrix $m > 1$.

Row Matrix: is one that has one row and more than one column: 1xn matrix $n > 1$.

Diagonal Matrix: is a square matrix with $a_{ij} = 0$ for all $i \neq j$

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

Null Matrix: is one whose elements are equal to zero for example:

$$O=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Unit Matrix (Identity Matrix): is a diagonal matrix with all the elements on the main diagonal $(i=j)$ equal to 1. A unity matrix is often designated by either I or U.

Unity example is $I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

Null Matrix: is one whose elements are equal to zero for example:

$$O=\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}$$

Symmetric Matrix: is a square matrix that satisfies the condition $a_{ij}=a_{ji}$. For example:

$$\begin{pmatrix}6&5&1\\5&0&7\\1&7&-1\end{pmatrix};\quad\begin{bmatrix}1&-4\\-4&1\end{bmatrix}$$

Transpose of a Matrix: The transpose of a matrix A is defined as the matrix that is obtained by interchanging the corresponding rows and columns in A. Let A be an *nxm* matrix which is represented by $A=\left[a_{ij}\right]_{n,m}$.

Then the transpose of *A*, denoted by A^T *or* A' is given by $A^T=\text{transpose of } A=\left[a_{ij}\right]_{m,n}$.

For example: $A=\begin{pmatrix}3&2&1\\0&-1&5\end{pmatrix}$

$$\Rightarrow A^T=\begin{pmatrix}3&0\\2&-1\\1&5\end{pmatrix}$$

Skew-Symmetric Matrix: is a square matrix that equals its negative transpose, that is:

$$A=-A'$$

Summation or Subtraction of Matrix

Given two *m x n* matrices $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ their sum is

$$A+B=\left[a_{ij}+b_{ij}\right]$$

And their difference is

$$A-B=\left[a_{ij}-b_{ij}\right]$$

Scalar Multiplication

The scalar product of a number *k* and a matrix *A* is denoted by $\boldsymbol{kA}$.

$$kA=k\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}=\begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}$$

If $\boldsymbol{x} \in \mathbb{R}^n$ *and* $\boldsymbol{c} \in \mathbb{R}$, *then* $\boldsymbol{cx} \in \mathbb{R}^n$ is the vector whose components are c times those of x.

Multiplication of Matrices

Let A be an $m \times n$ matrix and let B be an $n \times k$ matrix. To find the element in the i^{th} row and j^{th} column of the product matrix AB, multiply each element in the i^{th} row of A by the corresponding element in the j^{th} column of B, and then add these products. The product matrix AB is an $m \times k$ matrix.

*Matrix **A***   *Matrix B*

$m \times n$ $n \times k$

Inner *must be equal*

Outer: Order of AB is $m \times k$

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$a_{11} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & \_ \\ \_ & \_ \end{bmatrix}$$

$$a_{12} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & af+bh \\ \_ & \_ \end{bmatrix}$$

$$a_{21} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ ce+dg & \_ \end{bmatrix}$$

$$a_{22} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ \_ & cf+dh \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

$$C = AB = \left| a_{ij} \right|_{m,n} \left| b_{ij} \right|_{n,k}$$

$$= \left| c_{ij} \right|_{m,k}$$

$$c_{ij} = \sum_{n=1}^{k} a_{in} b_{nj} \qquad \text{for } i = 1, 2, \ldots, k \qquad \text{and } j = 1, 2, \ldots, k$$

Definition

We define the product $A\boldsymbol{x}$ of a matrix A and the vector $\boldsymbol{x}$ to be the linear combination of the column vectors of A with coefficients from the vector $\boldsymbol{x}$

$$A\boldsymbol{x} = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

Example: Given the matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}$$

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} \end{bmatrix}$$

Example

$$AB = \begin{bmatrix} 1 & -3 \\ 7 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1)+(-3)3 & 1(0)+(-3)1 & 1(-1)+(-3)4 & 1(2)+(-3)(-1) \\ 7(1)+2(3) & 7(0)+2(1) & 7(-1)+2(4) & 7(2)+2(-1) \end{bmatrix}$$

$$= \begin{bmatrix} -8 & -3 & -13 & 5 \\ 13 & 2 & 1 & 12 \end{bmatrix}$$

Determinant of a Matrix: with each square matrix a determinant having the same elements and order. The determinant of a square matrix A is designated by:

$$Det\ A = \Delta_A = D_A = |A|$$

Determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is define as $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Consider the matrix: $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & 0 \end{bmatrix}$

Then: $|A| = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & 0 \end{vmatrix}$

$$= 1\begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix}$$

$$= 1(0-2) - 0(0-2) + (-1)(0-3)$$

$$= 1$$

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$-a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Geometric Interpretation of the Determinant

Let A be the matrix of the vectors v_1 and v_2

$$A = \left[v_1,\ v_2 \right]$$

The vectors v_1 and v_2 determine a parallelogram where the area is equal to $|A|$

In three-dimensions $A = \left[v_1,\ v_2,\ v_3 \right]$, the vectors v_1, v_2 and v_3 determine a cube where the volume is equal to $|A|$.

Singular Matrix: is said to be singular if the value of its determinant is zero. Where a matrix is singular, it usually means that not all the rows or not all the columns of the matrix are independent of each other. Let consider the following set of equations:

$$
\begin{aligned}
2x_1 - 3x_2 + x_3 &= 0 \\
-1x_1 + x_2 + x_3 &= 0 \\
-x_1 - 2x_2 + 2x_3 &= 0
\end{aligned}
$$

The third equation is equal to the sum of the first two equations. Therefore these three equations are not completely independent. In matrix form, these equations may be represented by $AX = 0$

Where $A = \begin{bmatrix} 2 & -3 & 1 \\ -1 & 1 & 1 \\ -1 & -2 & 2 \end{bmatrix}$ & $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Determinant of A:

$$|A| = \begin{vmatrix} 2 & -3 & 1 \\ -1 & 1 & 1 \\ -1 & -2 & 2 \end{vmatrix} = 0$$

$\Rightarrow$ Therefore the matrix is ***singular***.

The $n \times n$ matrix A is said to be ***nonsingular*** if we can solve the system $A\mathbf{x} = \boldsymbol{b}$ for any choice of the vector $\boldsymbol{b}$ *in* $\mathbb{R}^n$. Otherwise the matrix is ***singular***.

Some Operations of a Matrix Transpose:

$(A')' = A$

$(kA)' = kA'$; where k is a scalar

$(A+B)' = A' + B'$

$(AB)' = B'A'$

Adjoint of a Matrix: Let A be a square matrix of order n. The Adjoint matrix of A denoted by $Adj(A)$ is defined as $Adj(A) = [ij\ cofactor\ of\ DetA]'_{n,n}$; where the ij cofactor of the determinant of A is the determinant obtained by omitting the i^{th} row and the j^{th} column of $|A|$ and then multiplying by $(1)^{i+j}$.

Example

Determine the Adjoint matrix of: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

The determinant of A is $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$\Rightarrow adjA = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$Adj(A) = \begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} & -\left(a_{12}a_{33} - a_{13}a_{32}\right) & a_{12}a_{23} - a_{13}a_{22} \\ -\left(a_{21}a_{33} - a_{23}a_{31}\right) & a_{11}a_{33} - a_{13}a_{31} & -\left(a_{11}a_{23} - a_{13}a_{21}\right) \\ a_{21}a_{32} - a_{22}a_{31} & -\left(a_{11}a_{32} - a_{12}a_{31}\right) & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix}$$

Inverse of a matrix

The $n \times n$ matrix A is invertible if there is an $n \times n$ matrix B such that $AB = I$ and $BA = I$.
A matrix B with the property is called an inverse of A.

$$A \text{ is } A^{-1} = \frac{adjA}{|A|}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \qquad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A^{-1} doesn't exist

To find inverse matrix using Gauss-Jordan method:

$$\left[A \middle| I\right] \rightarrow \left[I \middle| A^{-1}\right] \qquad \textit{where } A^{-1} \textit{ read as } "A \textit{ inverse}"$$

Properties of the Inverse Matrix

- ✓ $AA^{-1} = A^{-1}A = I$
- ✓ $\left(A^{-1}\right)^{-1} = A$
- ✓ $(AB)^{-1} = B^{-1}A^{-1}$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] R_2 + R_1$$

$$\begin{array}{cccccc} -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 \\ \hline 0 & 2 & 5 & 1 & 1 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] R_3 - R_1$$

$$\begin{array}{cccccc} 1 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & -1 & 0 & 0 \\ \hline 0 & -1 & -2 & -1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right] \frac{1}{2} R_2$$

$$\begin{array}{cccccc} 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right] R_3 + R_2$$

$$\begin{array}{cccccc} 0 & -1 & -2 & -1 & 0 & 1 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ \hline 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}\right] 2R_3$$

$$\begin{array}{cccccc} 0 & 0 & 1 & -1 & 1 & 2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right] R_2 - \frac{5}{2} R_3$$

$$\begin{array}{cccccc} 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{5}{2} & \frac{5}{2} & -\frac{5}{2} & -5 \\ \hline 0 & 1 & 0 & 3 & -2 & -5 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right] R_1 - 2R_3$$

$$\begin{array}{cccccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 2 & -2 & -4 \\ \hline 1 & 0 & 0 & 3 & -2 & -4 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 1 & 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right] \quad \Rightarrow \quad A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

Properties of Matrix

Addition and Scalar Multiplication

$A+B=B+A$	*Commutative Property of Addition*
$A+(B+C)=(A+B)+C$	*Associative Property of Addition*
$(kl)A=k(lA)$	*Associative Property of Scalar Multiplication*
$k(A+B)=kA+kB$	*Distributive Property*
$(k+l)A=kA+lA$	*Distributive Property*
$A+0=0+A=A$	*Additive Identity Property*
$A+(-A)=(-A)+A=0$	*Additive Inverse Property*

Multiplication

$A(BC)=(AB)C$	*Associative Property of Multiplication*
$A(B+C)=AB+AC$	*Distributive Property*
$(B+C)A=BA+CA$	*Distributive Property*