Lecture Three – Identities and Solving Trigonometric

Section 3.1 - Proving Identities

Reciprocal Identities

$\csc\theta = \dfrac{1}{\sin\theta}$ $\qquad \sec\theta = \dfrac{1}{\cos\theta}$ $\qquad \cot\theta = \dfrac{1}{\tan\theta}$

$\sin\theta = \dfrac{1}{\csc\theta}$ $\qquad \cos\theta = \dfrac{1}{\sec\theta}$ $\qquad \tan\theta = \dfrac{1}{\cot\theta}$

Ratio Identities

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\qquad \cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Pythagorean Identities

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos\theta = \pm\sqrt{1-\sin^2\theta}$$

$$\sin\theta = \pm\sqrt{1-\cos^2\theta}$$

$$1+\tan^2\theta = \sec^2\theta$$

$$1+\cot^2\theta = \csc^2\theta$$

Example

Write $\sec\theta\ \tan\theta$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify.

Solution

$$\sec\theta\tan\theta = \frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\cos^2\theta}$$

Example

Add $\dfrac{1}{\sin\theta} + \dfrac{1}{\cos\theta}$

Solution

$$\frac{1}{\sin\theta} + \frac{1}{\cos\theta} = \frac{\cos\theta + \sin\theta}{\sin\theta\cos\theta}$$

$$\frac{1}{\sin\theta}\frac{\cos\theta}{\cos\theta} + \frac{1}{\cos\theta}\frac{\sin\theta}{\sin\theta}$$

Example

Write: $\tan\alpha + \cot\alpha$ in terms of $\sin\alpha$ *and* $\cos\alpha$

Solution

$$\tan\alpha + \cot\alpha = \frac{\sin\alpha}{\cos\alpha} + \frac{\cos\alpha}{\sin\alpha}$$

$$= \frac{\sin\alpha}{\cos\alpha}\frac{\sin\alpha}{\sin\alpha} + \frac{\cos\alpha}{\sin\alpha}\frac{\cos\alpha}{\cos\alpha}$$

$$= \frac{\sin^2\alpha + \cos^2\alpha}{\cos\alpha\sin\alpha}$$

$$= \frac{1}{\cos\alpha\sin\alpha}$$

Example

Prove: $\tan x + \cos x = \sin x(\sec x + \cot x)$

Solution

$$\tan x + \cos x = \frac{\sin x}{\cos x} + \cos x$$

$$= \sin x\frac{1}{\cos x} + \cos x\frac{\sin x}{\sin x}$$

$$= \sin x\sec x + \sin x\frac{\cos x}{\sin x}$$

$$= \sin x(\sec x + \cot x)$$

or

$$\sin x(\sec x + \cot x) = \sin x\left(\frac{1}{\cos x} + \frac{\cos x}{\sin x}\right)$$

$$= \frac{\sin x}{\cos x} + \sin x\frac{\cos x}{\sin x}$$

$$= \tan x + \cos x$$

Example

Prove: $\cot\alpha + 1 = \csc\alpha(\cos\alpha + \sin\alpha)$

Solution

$$\begin{aligned}\csc\alpha(\cos\alpha+\sin\alpha) &= \frac{1}{\sin\alpha}(\cos\alpha+\sin\alpha)\\ &= \frac{1}{\sin\alpha}\cos\alpha + \frac{1}{\sin\alpha}\sin\alpha\\ &= \cot\alpha + 1\end{aligned}$$

Guidelines for Proving Identities

1. Work on the complicated side first (more trigonometry functions)
2. Look for trigonometry substitutions.
3. Look for algebraic operations
4. If not always change everything to sines and cosines
5. Keep an eye on the side you are not working.

Example

Prove $\dfrac{\cos^4 t - \sin^4 t}{\cos^2 t} = 1 - \tan^2 t$

Solution

$$\frac{\cos^4 t-\sin^4 t}{\cos^2 t} = \frac{(\cos^2 t-\sin^2 t)(\cos^2 t+\sin^2 t)}{\cos^2 t} \qquad a^2-b^2=(a-b)(a+b)$$

$$= \frac{(\cos^2 t-\sin^2 t)(1)}{\cos^2 t} \qquad \cos^2 t+\sin^2 t = 1$$

$$= \frac{\cos^2 t-\sin^2 t}{\cos^2 t}$$

$$= \frac{\cos^2 t}{\cos^2 t} - \frac{\sin^2 t}{\cos^2 t}$$

$$= 1-\tan^2 t$$

Example

Prove: $1+\cos\theta = \dfrac{\sin^2\theta}{1-\cos\theta}$

Solution

$$\frac{\sin^2\theta}{1-\cos\theta} = \frac{1-\cos^2\theta}{1-\cos\theta} \qquad \sin^2\theta = 1-\cos^2\theta$$

$$= \frac{(1-\cos\theta)(1+\cos\theta)}{1-\cos\theta} \qquad a^2-b^2=(a-b)(a+b)$$

$$\underline{= 1+\cos\theta \;|}$$

Example

Prove: $\tan^2\alpha\left(1+\cot^2\alpha\right) = \dfrac{1}{1-\sin^2\alpha}$

Solution

$$\tan^2\alpha\left(1+\cot^2\alpha\right) = \tan^2\alpha + \tan^2\alpha\cot^2\alpha$$

$$= \tan^2\alpha + \tan^2\alpha\frac{1}{\tan^2\alpha}$$

$$= \tan^2\alpha + 1 \qquad \tan^2\alpha + 1 = \sec^2\alpha$$

$$= \sec^2\alpha$$

$$= \frac{1}{\cos^2\alpha} \qquad \cos^2\alpha = 1-\sin^2\alpha$$

$$\underline{= \frac{1}{1-\sin^2\alpha}\;|}$$

Example

Prove: $\dfrac{\sin\alpha}{1+\cos\alpha} + \dfrac{1+\cos\alpha}{\sin\alpha} = 2\csc\alpha$

Solution

$$\frac{\sin\alpha}{1+\cos\alpha} + \frac{1+\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{\sin\alpha}\cdot\frac{\sin\alpha}{1+\cos\alpha} + \frac{1+\cos\alpha}{\sin\alpha}\cdot\frac{1+\cos\alpha}{1+\cos\alpha}$$

$$= \frac{\sin^2\alpha + (1+\cos\alpha)^2}{\sin\alpha(1+\cos\alpha)}$$

$$= \frac{\sin^2\alpha + 1 + \cos^2\alpha + 2\cos\alpha}{\sin\alpha(1+\cos\alpha)}$$

$$= \frac{2+2\cos\alpha}{\sin\alpha(1+\cos\alpha)}$$

$$= \frac{2(1+\cos\alpha)}{\sin\alpha(1+\cos\alpha)}$$

$$= \frac{2}{\sin\alpha}$$

$$\boxed{= 2\csc\alpha}$$

Example

Prove $\quad \dfrac{1+\sin t}{\cos t} = \dfrac{\cos t}{1-\sin t}$

Solution

$$\frac{1+\sin t}{\cos t} = \frac{1+\sin t}{\cos t} \cdot \frac{1-\sin t}{1-\sin t}$$

$$= \frac{1-\sin^2 t}{\cos t(1-\sin t)}$$

$$= \frac{\cos^2 t}{\cos t(1-\sin t)}$$

$$\boxed{= \frac{\cos t}{1-\sin t}}$$

Example

Show that $\cot^2\theta + \cos^2\theta = \cot^2\theta\cos^2\theta$ is not an identity by finding a counterexample

Solution

$$\cot^2\frac{\pi}{4} + \cos^2\frac{\pi}{4} = \cot^2\frac{\pi}{4}\cos^2\frac{\pi}{4}$$

$$1^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1^2\left(\frac{1}{\sqrt{2}}\right)^2$$

$$1 + \frac{1}{2} = \frac{1}{2}$$

$$\frac{3}{2} \neq \frac{1}{2}$$

Exercises *Section* 3.1 – Proving Identities

(1–80) Prove the identity

1. $\cos\theta\cot\theta+\sin\theta=\csc\theta$

2. $\sec\theta\cot\theta-\sin\theta=\dfrac{\cos^2\theta}{\sin\theta}$

3. $\dfrac{\csc\theta\tan\theta}{\sec\theta}=1$

4. $(\sin\theta+\cos\theta)^2=1+2\sin\theta\cos\theta$

5. $\sin\theta(\sec\theta+\cot\theta)=\tan\theta+\cos\theta$

6. $\cos\theta(\csc\theta+\tan\theta)=\cot\theta+\sin\theta$

7. $\cot\theta+\tan\theta=\csc\theta\sec\theta$

8. $\tan x(\cos x+\cot x)=\sin x+1$

9. $\dfrac{1-\cos^4\theta}{1+\cos^2\theta}=\sin^2\theta$

10. $\dfrac{1-\sec x}{1+\sec x}=\dfrac{\cos x-1}{\cos x+1}$

11. $\dfrac{\cos x}{1+\sin x}-\dfrac{1-\sin x}{\cos x}=0$

12. $\dfrac{1+\cot^3 t}{1+\cot t}=\csc^2 t-\cot t$

13. $\tan x+\cot x=\sec x\csc x$

14. $\dfrac{\tan x-\cot x}{\sin x\cos x}=\sec^2 x-\csc^2 x$

15. $\dfrac{\sec x+\tan x}{\sec x-\tan x}=\dfrac{1+2\sin x+\sin^2 x}{\cos^2 x}$

16. $\sin^2 x-\cos^2 x=2\sin^2 x-1$

17. $\sin^4 x-\cos^4 x=\sin^2 x-\cos^2 x$

18. $\dfrac{\cos\alpha}{1+\sin\alpha}=\sec\alpha-\tan\alpha$

19. $\dfrac{\sin\alpha}{1-\sin\alpha}-\dfrac{\cos\alpha}{1-\sin\alpha}=\dfrac{1-\cot\alpha}{\csc\alpha-1}$

20. $\dfrac{\frac{1}{\tan x}+\cot x}{\frac{1}{\tan x}+\tan x}=\dfrac{2}{\sec^2 x}$

21. $\dfrac{\cot^2\theta+3\cot\theta-4}{\cot\theta+4}=\cot\theta-1$

22. $\dfrac{\sin\theta}{1+\cos\theta}=\dfrac{1-\cos\theta}{\sin\theta}$

23. $\tan x(\csc x-\sin x)=\cos x$

24. $\sin x(\tan x\cos x-\cot x\cos x)=1-2\cos^2 x$

25. $(1+\tan x)^2+(\tan x-1)^2=2\sec^2 x$

26. $\sec x+\tan x=\dfrac{\cos x}{1-\sin x}$

27. $\dfrac{\tan x-1}{\tan x+1}=\dfrac{1-\cot x}{1+\cot x}$

28. $7\csc^2 x-5\cot^2 x=2\csc^2 x+5$

29. $1-\dfrac{\cos^2 x}{1-\sin x}=-\sin x$

30. $\dfrac{1-\cos x}{1+\cos x}=\dfrac{\sec x-1}{\sec x+1}$

31. $\dfrac{\sec x-1}{\tan x}=\dfrac{\tan x}{\sec x+1}$

32. $\dfrac{\cos x}{\cos x-\sin x}=\dfrac{1}{1-\tan x}$

33. $(\sec x+\tan x)^2=\dfrac{1+\sin x}{1-\sin x}$

34. $\dfrac{\cos x}{1+\tan x}-\dfrac{\sin x}{1+\cot x}=\cos x-\sin x$

35. $\dfrac{\cot x+\csc x-1}{\cot x-\csc x+1}=\csc x+\cot x$

36. $\dfrac{\tan x+\cot x}{\tan x-\cot x}=\dfrac{1}{\sin^2 x-\cos^2 x}$

37. $\dfrac{1-\cot^2 x}{1+\cot^2 x}+1=2\sin^2 x$

38. $\dfrac{1+\cos x}{1-\cos x}-\dfrac{1-\cos x}{1+\cos x}=4\cot x\csc x$

39. $\dfrac{\sin^3 x-\cos^3 x}{\sin x-\cos x}=1+\sin x\cos x$

40. $1+\sec^2 x\sin^2 x=\sec^2 x$

41. $\dfrac{1+\csc x}{\sec x}=\cos x+\cot x$

42. $\tan^2 x=\sec^2 x-\sin^2 x-\cos^2 x$

43. $\dfrac{\sin x}{1-\cos x}+\dfrac{\sin x}{1+\cos x}=2\csc x$

44. $\dfrac{\tan x+\sec x}{\sec x}-\dfrac{\tan x+\sec x}{\tan x}=-\cos x\cot x$

45. $\cos^2(\alpha-\beta)-\cos^2(\alpha+\beta)=\sin^2(\alpha+\beta)-\sin^2(\alpha-\beta)$

46. $\tan x\csc x-\sec^2 x\cos x=0$

47. $(1+\tan x)^2-2\tan x=\dfrac{1}{(1-\sin x)(1+\sin x)}$

48. $\dfrac{3\csc^2 x-5\csc x-28}{\csc x-4}=\dfrac{3}{\sin x}+7$

49. $\left(\sec^2 x-1\right)\left(\sec^2 x+1\right)=\tan^4 x+2\tan^2 x$

50. $\dfrac{\csc x}{\cot x}-\dfrac{\cot x}{\csc x}=\dfrac{\sin x}{\cot x}$

51. $\dfrac{1-\cos^2 x}{1+\cos x}=\dfrac{\sec x-1}{\sec x}$

52. $\dfrac{\cos x}{1+\cos x}=\dfrac{\sec x-1}{\tan^2 x}$

53. $\dfrac{1-2\sin^2 x}{1+2\sin x\cos x}=\dfrac{\cos x-\sin x}{\cos x+\sin x}$

54. $(\cos x-\sin x)^2+(\cos x+\sin x)^2=2$

55. $\dfrac{\sin x}{1+\cos x}+\dfrac{1+\cos x}{\sin x}=2\csc x$

56. $\dfrac{\sin x+\tan x}{\cot x+\csc x}=\sin x\tan x$

57. $\csc^2 x\sec^2 x=\sec^2 x+\csc^2 x$

58. $\cos^2 x+1=2\cos^2 x+\sin^2 x$

59. $1-\dfrac{\cos^2 x}{1+\sin x}=\sin x$

60. $\cot^2 x=(\csc x-1)(\csc x+1)$

61. $\dfrac{\csc x+\cot x}{\tan x+\sin x}=\csc x\cot x$

62. $\dfrac{1-\sec x}{\tan x}+\dfrac{\tan x}{1-\sec x}=-2\csc x$

63. $\csc x-\sin x=\cos x\cot x$

64. $\cot^3 x=\cot x\left(\csc^2 x-1\right)$

65. $\dfrac{\cot^2 x}{\csc x-1}=\dfrac{1+\sin x}{\sin x}$

66. $\cot^2 x+\csc^2 x=2\csc^2 x-1$

67. $\dfrac{\cot^2 x}{1+\csc x}=\csc x-1$

68. $\sec^4 x-\tan^4 x=\sec^2 x+\tan^2 x$

69. $\dfrac{\cos x}{1+\sin x}+\dfrac{1+\sin x}{\cos x}=2\sec x$

70. $\dfrac{\sin x+\cos x}{\sin x-\cos x}=\dfrac{1+2\sin x\cos x}{2\sin^2 x-1}$

71. $\dfrac{\csc x-1}{\csc x+1}=\dfrac{\cot^2 x}{\csc^2 x+2\csc x+1}$

72. $\csc^4 x-\cot^4 x=\csc^2 x+\cot^2 x$

73. $\tan\left(\frac{\pi}{4}+x\right)=\cot\left(\frac{\pi}{4}-x\right)$

74. $\dfrac{\sin\theta}{1+\sin\theta}-\dfrac{\sin\theta}{1-\sin\theta}=-2\tan^2\theta$

75. $\csc^2 x-\cos^2 x\csc^2 x=1$

76. $1-2\sin^2 x=2\cos^2 x-1$

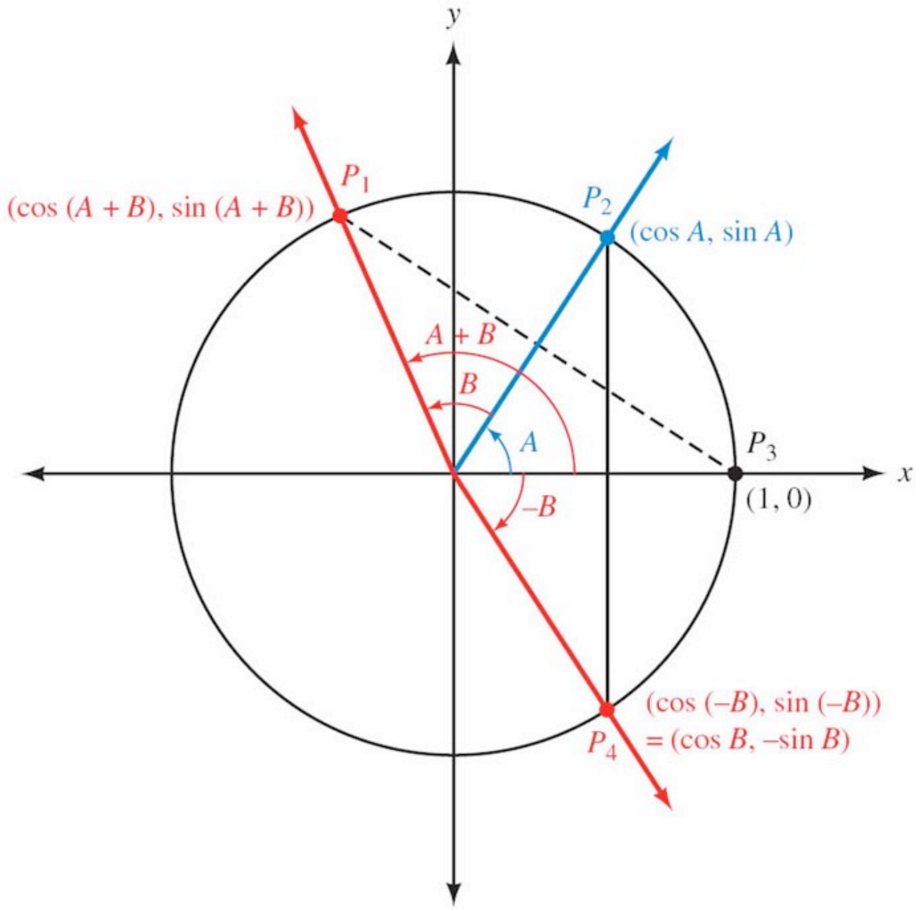
77. $\csc^2 x-\cos x\sec x=\cot^2 x$

78. $(\sec x-\tan x)(\sec x+\tan x)=1$

79. $\left(1+\tan^2 x\right)\left(1-\sin^2 x\right)=1$

80. $10\csc^2 x-6\cot^2 x=4\csc^2 x+6$

Section 3.2 – Sum and Difference Formulas



$$P_1P_3 = P_2P_4$$

$$\left(P_1P_3\right)^2 = \left(P_2P_4\right)^2$$

Distance between points

$$\left[\cos(A+B)-1\right]^2 + \left[\sin(A+B)-0\right]^2 = \left(\cos A - \cos B\right)^2 + \left(\sin A + \sin B\right)^2$$

$$\cos^2(A+B) - 2\cos(A+B) + 1 + \sin^2(A+B) = \left(\cos A - \cos B\right)^2 + \left(\sin A + \sin B\right)^2$$

$$1 - 2\cos(A+B) + 1 = \cos^2 A - 2\cos B\cos A + \cos^2 B + \sin^2 A + 2\sin B\sin A + \sin^2 B$$

$$2 - 2\cos(A+B) = \cos^2 A + \sin^2 A + \cos^2 B + \sin^2 B - 2\cos B\cos A + 2\sin B\sin A$$

$$2 - 2\cos(A+B) = 1 + 1 - 2\cos B\cos A + 2\sin B\sin A$$

$$2 - 2\cos(A+B) = 2 - 2\cos B\cos A + 2\sin B\sin A$$

$$-2\cos(A+B) = -2\cos B\cos A + 2\sin B\sin A$$

$$\cos(A+B) = \cos B\cos A - \sin B\sin A$$

$$\boxed{\begin{aligned}\cos(A+B)&=\cos A\cos B-\sin A\sin B\\ \cos(A-B)&=\cos A\cos B+\sin A\sin B\\ \sin(A+B)&=\sin A\cos B+\cos A\sin B\\ \sin(A-B)&=\sin A\cos B-\cos A\sin B\end{aligned}}$$

Example

Find the exact value for $\cos 75°$

Solution

$$\begin{aligned}\cos 75° &= \cos(45°+30°)\\ &= \cos 45°\cos 30° - \sin 45°\sin 30°\\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\frac{1}{2}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Example

Show that $\cos(x+2\pi)=\cos x$

Solution

$$\begin{aligned}\cos(x+2\pi) &= \cos x\cos 2\pi - \sin x\sin 2\pi\\ &= \cos x\cdot(1)-\sin x\cdot(0)\\ &= \cos x\end{aligned}$$

Example

Simplify: $\cos 3x\cos 2x - \sin 3x\sin 2x$

Solution

$$\begin{aligned}\cos 3x\cos 2x - \sin 3x\sin 2x &= \cos(3x+2x)\\ &= \cos 5x\end{aligned}$$

Example

Show that $\cos(90° - A) = \sin A$

Solution

$$\begin{aligned}\cos(90° - A) &= \cos 90° \cos A + \sin 90° \sin A \\ &= 0 \cdot \cos A + 1 \cdot \sin A \\ &= \sin A\end{aligned}$$

Example

Find the exact value of $\sin\frac{\pi}{12}$

Solution

$$\begin{aligned}\sin\frac{\pi}{12} &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ &= \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4} \\ &= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} - \frac{1}{2}\frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

Example

Find the exact value of $\cos 15°$

Solution

$$\begin{aligned}\cos 15° &= \cos(45° - 30°) \\ &= \cos(45°)\cos(30°) + \sin(45°)\sin(30°) \\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} \\ &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} \\ &= \frac{\sqrt{6} + \sqrt{2}}{4}\end{aligned}$$

Example

If $\sin A = \frac{3}{5}$ with A in QI, and $\cos B = -\frac{5}{13}$ with B in QIII, find $\sin(A+B)$, $\cos(A+B)$, and $\tan(A+B)$

Solution

$\sin A = \frac{3}{5} \rightarrow A \in QI$

$\cos A = \frac{4}{5}$

$\cos B = -\frac{5}{13} \rightarrow B \in QIII$

$\sin B = -\frac{12}{13}$

$$\begin{aligned}\sin(A+B) &= \sin A \cos B + \cos A \sin B \\ &= \frac{3}{5}\left(-\frac{5}{13}\right) + \frac{4}{5}\left(-\frac{12}{13}\right) \\ &= -\frac{15}{65} - \frac{48}{65} \\ &= -\frac{63}{65}\end{aligned}$$

$$\begin{aligned}\cos(A+B) &= \cos A \cos B - \sin A \sin B \\ &= \frac{4}{5}\left(-\frac{5}{13}\right) - \frac{3}{5}\left(-\frac{12}{13}\right) \\ &= -\frac{20}{65} + \frac{36}{65} \\ &= \frac{16}{65}\end{aligned}$$

$$\begin{aligned}\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} \\ &= -\frac{63}{16}\end{aligned}$$

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}$$

$$=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

$$=\frac{\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B-\sin A\sin B}{\cos A\cos B}}$$

$$=\frac{\frac{\sin A\cos B}{\cos A\cos B}+\frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B}-\frac{\sin A\sin B}{\cos A\cos B}}$$

$$=\frac{\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}}{1-\frac{\sin A}{\cos A}\frac{\sin B}{\cos B}}$$

$$=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$\boxed{\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}}$$

$$\boxed{\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}}$$

Example

If $\sin A=\frac{3}{5}$ with A in Q*I*, and $\cos B=-\frac{5}{13}$ with B in Q*III*, find $\tan(A+B)$

Solution

$$\tan A=\frac{3/5}{4/5}$$

$$\tan A=\frac{\sin A}{\cos A}$$

$$=\frac{3}{4}$$

$$\tan B=\frac{-12/13}{-5/13}$$

$$\tan B=\frac{\sin B}{\cos B}$$

$$=\frac{12}{5}$$

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$=\frac{\frac{3}{4}+\frac{12}{5}}{1-\frac{3}{4}\frac{12}{5}}$$

$$=\frac{\frac{15}{20}+\frac{48}{20}}{1-\frac{36}{20}}$$

$$=\frac{\frac{63}{20}}{-\frac{16}{20}}$$

$$\boxed{=-\frac{63}{16}}$$

Example

Establish the identity: $\dfrac{\cos(x-y)}{\sin x \sin y}=\cot x\cot y+1$

Solution

$$\frac{\cos(x-y)}{\sin x \sin y}=\frac{\cos x\cos y+\sin x\sin y}{\sin x \sin y}$$

$$=\frac{\cos x\cos y}{\sin x \sin y}+\frac{\sin x\sin y}{\sin x \sin y}$$

$$\boxed{=\cot x\cot y+1} \quad \checkmark$$

Example

Establish the identity: $\cot(x+y)=\dfrac{\cot x\cot y-1}{\cot x+\cot y}$

Solution

$$\cot(x+y)=\frac{\cos(x+y)}{\sin(x+y)}$$

$$=\frac{\cos x\cos y-\sin x\sin y}{\sin x\cos y+\cos x\sin y}$$

$$=\frac{\frac{\cos x\cos y}{\sin x\sin y}-\frac{\sin x\sin y}{\sin x\sin y}}{\frac{\sin x\cos y}{\sin x\sin y}+\frac{\cos x\sin y}{\sin x\sin y}}$$

$$\boxed{=\frac{\cot x\cot y-1}{\cot x+\cot y}} \quad \checkmark$$

Example

Establish the identity: $\sec(x-y)=\dfrac{\cos x\cos y-\sin x\sin y}{\cos^2 x-\sin^2 y}$

Solution

$$\sec(x-y)=\frac{1}{\cos(x-y)}\frac{\cos(x+y)}{\cos(x+y)}$$

$$=\frac{\cos x\cos y-\sin x\sin y}{(\cos x\cos y+\sin x\sin y)(\cos x\cos y-\sin x\sin y)}$$

$$=\frac{\cos x\cos y-\sin x\sin y}{\cos^2 x\cos^2 y-\sin^2 x\sin^2 y}$$

$$=\frac{\cos x\cos y-\sin x\sin y}{\cos^2 x\cos^2 y-\sin^2 x\sin^2 y}$$

$$=\frac{\cos x\cos y-\sin x\sin y}{\cos^2 x\left(1-\sin^2 y\right)-\left(1-\cos^2 x\right)\sin^2 y}$$

$$=\frac{\cos x\cos y-\sin x\sin y}{\cos^2 x-\cos^2 x\sin^2 y-\sin^2 y+\cos^2 x\sin^2 y}$$

$$=\frac{\cos x\cos y-\sin x\sin y}{\cos^2 x-\sin^2 y}\quad \checkmark$$

Exercises *Section* 3.2 – Sum and Difference Formulas

1. Write the expression as a single trigonometric function $\sin 8x \cos x - \cos 8x \sin x$

2. Show that $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$

3. If $\sin A = \frac{4}{5}\ (A \in QII)$, and $\cos B = -\frac{5}{13}\ (B \in QIII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

4. If $\sin A = \frac{3}{5}\ (A \in QII)$, and $\cos B = -\frac{12}{13}\ (B \in QIII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

5. If $\sin A = \frac{1}{\sqrt{5}}\ (A \in QI)$, and $\tan B = \frac{3}{4}\ (B \in QI)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

6. If $\sin A = \frac{3}{5}\ (A \in QII)$, and $\cos B = \frac{12}{13}\ (B \in QIV)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

7. If $\sin A = \frac{7}{25}\ (A \in QII)$, and $\cos B = -\frac{8}{17}\ (B \in QIII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

8. If $\cos A = -\frac{4}{5}\ (A \in QII)$, and $\sin B = \frac{24}{25}\ (B \in QII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

9. If $\cos A = \frac{15}{17}\ (A \in QI)$, and $\cos B = -\frac{12}{13}\ (B \in QII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

10. If $\sin A = -\frac{3}{5}\ (A \in QIV)$, and $\sin B = \frac{7}{25}\ (B \in QII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

11. If $\sec A = \sqrt{5}$ with A in Q*I*, and $\sec B = \sqrt{10}$ with B in Q*I*, find $\sec(A+B)$

(12–30) Prove the identity

12. $\dfrac{\sin(A-B)}{\cos A\cos B}=\tan A-\tan B$

13. $\sec(A+B)=\dfrac{\cos(A-B)}{\cos^2 A-\sin^2 B}$

14. $\dfrac{\cos 4\alpha}{\sin\alpha}-\dfrac{\sin 4\alpha}{\cos\alpha}=\dfrac{\cos 5\alpha}{\sin\alpha\cos\alpha}$

15. $\dfrac{\cos(x+y)}{\cos(x-y)}=\dfrac{\cot y-\tan x}{\cot y+\tan x}$

16. $\dfrac{\sin(x+y)}{\sin(x-y)}=\dfrac{\cot y+\cot x}{\cot y-\cot x}$

17. $\dfrac{\sin(x-y)}{\sin x\cos y}=1-\cot x\tan y$

18. $\dfrac{\sin(x-y)}{\sin x\sin y}=\cot y-\cot x$

19. $\dfrac{\cos(x+y)}{\cos x\sin y}=\cot y-\tan x$

20. $\dfrac{\sin(x+y)}{\cos(x-y)}=\dfrac{1+\cot x\tan y}{\cot x+\tan y}$

21. $\sin\left(\frac{\pi}{4}+x\right)+\sin\left(\frac{\pi}{4}-x\right)=\sqrt{2}\cos x$

22. $\cos(A+B)+\cos(A-B)=2\cos A\cos B$

23. $\sin(x-y)-\sin(y-x)=2\sin x\cos y-2\cos x\sin y$

24. $\cos(x-y)+\cos(y-x)=2\cos x\cos y+2\sin x\sin y$

25. $\tan(x+y)\tan(x-y)=\dfrac{\tan^2 x-\tan^2 y}{1-\tan^2 x\tan^2 y}$

26. $\dfrac{\cos(\alpha-\beta)}{\sin(\alpha+\beta)}=\dfrac{1+\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$

27. $\sec(x+y)=\dfrac{\cos x\cos y+\sin x\sin y}{\cos^2 x-\sin^2 y}$

28. $\csc(x-y)=\dfrac{\sin x\cos y+\cos x\sin y}{\sin^2 x-\sin^2 y}$

29. $\tan(x+y)+\tan(x-y)=\dfrac{2\tan x}{\cos^2 y\left(1-\tan^2 x\tan^2 y\right)}$

30. $\dfrac{\cos(x-y)}{\cos(x+y)}=\dfrac{1+\tan x\tan y}{1-\tan x\tan y}$

31. Common household current is called ***alternating current*** because the current alternates direction within the wires. The voltage V in a typical 115-*volt* outlet can be expressed by the function $V(t)=163\sin\omega t$ where ω is the angular speed (in *radians* per *second*) of the rotating generator at the electrical plant, and t is time measured in seconds.

a) It is essential for electric generators to rotate at precisely 60 cycles per second so household appliances and computers will function properly. Determine ω for these electric generators.

b) Determine a value of ϕ so that the graph of $V(t)=163\cos(\omega t-\phi)$ is the same as the graph of $V(t)=163\sin\omega t$

Section 3.3 – Double-angle and Half-Angle Formulas

$$\begin{aligned} \sin 2A &= \sin(A+A) \\ &= \sin A\cos A + \cos A\sin A \\ &= \boxed{2\sin A\cos A} \end{aligned}$$

$$\sin 2A \neq 2\sin A$$

$$\begin{aligned} \cos 2A &= \cos(A+A) \\ &= \cos A\cos A - \sin A\sin A \\ &= \boxed{\cos^2 A - \sin^2 A} \end{aligned}$$

$$\begin{aligned} \cos 2A &= \cos^2 A - \sin^2 A \\ &= \cos^2 A - \left(1-\cos^2 A\right) \\ &= \cos^2 A - 1 + \cos^2 A \\ &= \boxed{2\cos^2 A - 1} \end{aligned}$$

$$\begin{aligned} \cos 2A &= \cos^2 A - \sin^2 A \\ &= \left(1-\sin^2 A\right) - \sin^2 A \\ &= 1 - \sin^2 A - \sin^2 A \\ &= \boxed{1 - 2\sin^2 A} \end{aligned}$$

Example

If $\sin A = \frac{3}{5}$ with A in QII, find $\sin 2A$

Solution

$$\cos A = \boxed{-\frac{4}{5}}$$

$$\begin{aligned} \sin 2A &= 2\sin A\cos A \\ &= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) \\ &= \boxed{-\frac{24}{25}} \end{aligned}$$

Example

Prove $\quad (\sin\theta + \cos\theta)^2 = 1 + \sin 2\theta$

Solution

$$(\sin\theta + \cos\theta)^2 = \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta$$

$$= \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta$$

$$= 1 + 2\sin\theta\cos\theta$$

$$\underline{= 1 + \sin 2\theta}\,\rfloor$$

Example

Prove $\quad \sin 2x = \dfrac{2\cot x}{1+\cot^2 x}$

Solution

$$\frac{2\cot x}{1+\cot^2 x} = \frac{2\frac{\cos x}{\sin x}}{1+\frac{\cos^2 x}{\sin^2 x}}$$

$$= \frac{2\frac{\cos x}{\sin x}}{\frac{\sin^2 x+\cos^2 x}{\sin^2 x}}$$

$$= 2\frac{\cos x}{\sin x}\frac{\sin^2 x}{\sin^2 x+\cos^2 x}$$

$$= 2\frac{\cos x}{1}\frac{\sin x}{1}$$

$$= 2\cos x\sin x$$

$$\underline{= \sin 2x}\,\rfloor$$

Example

Prove $\quad \cos 4x = 8\cos^4 x - 8\cos^2 x + 1$

Solution

$$\cos 4x = \cos(2.2x)$$

$$= 2\cos^2 2x - 1$$

$$= 2(\cos 2x)^2 - 1$$

$$= 2\left(2\cos^2 x - 1\right)^2 - 1$$

$$= 2\left(4\cos^4 x - 4\cos^2 x + 1\right) - 1$$

$$= 8\cos^4 x - 8\cos^2 x + 2 - 1$$

$$\underline{= 8\cos^4 x - 8\cos^2 x + 1}$$

$$\tan 2A = \tan(A + A)$$

$$= \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$\boxed{\tan 2A = \frac{2\tan A}{1-\tan^2 A}}$$

Example

Simplify $\dfrac{2\tan 15^\circ}{1 - \tan^2 15^\circ}$

Solution

$$\frac{2\tan 15^\circ}{1 - \tan^2 15^\circ} = \tan(2\cdot 15^\circ)$$

$$= \tan(30^\circ)$$

$$\underline{= \frac{1}{\sqrt{3}}}$$

Example

Prove $\tan\theta = \dfrac{1 - \cos 2\theta}{\sin 2\theta}$

Solution

$$\frac{1 - \cos 2\theta}{\sin 2\theta} = \frac{1 - (1 - 2\sin^2\theta)}{2\sin\theta\cos\theta}$$

$$= \frac{1 - 1 + 2\sin^2\theta}{2\sin\theta\cos\theta}$$

$$= \frac{2\sin^2\theta}{2\sin\theta\cos\theta}$$

$$= \frac{\sin\theta}{\cos\theta}$$

$$\underline{= \tan\theta}$$

Example

Given $\cos\theta = \frac{3}{5}$ and $\sin\theta < 0$, find $\sin 2\theta,\ \cos 2\theta,\ \textit{and}\ \tan 2\theta$

Solution

$$\sin\theta = -\frac{4}{5}$$

$$\begin{aligned} \sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\left(-\frac{4}{5}\right)\left(\frac{3}{5}\right) \\ &= -\frac{24}{25} \end{aligned}$$

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= \left(\frac{3}{5}\right)^2 - \left(-\frac{4}{5}\right)^2 \\ &= \frac{9}{25} - \frac{16}{25} \\ &= -\frac{7}{25} \end{aligned}$$

$$\begin{aligned} \tan 2\theta &= \frac{\sin 2\theta}{\cos 2\theta} \\ &= \frac{24}{7} \end{aligned}$$

Half-Angle Formulas

$\cos 2A = 2\cos^2 A - 1$

$\cos 2x = 2\cos^2 x - 1$

$\cos 2x + 1 = 2\cos^2 x$

$2\cos^2 x = \cos 2x + 1$

$\cos^2 x = \dfrac{\cos 2x + 1}{2}$ *Divide both sides by 2*

$\cos x = \pm\sqrt{\dfrac{\cos 2x + 1}{2}}$ *Replace x with* $\frac{A}{2}$

$$\Rightarrow \boxed{\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}}$$

$\cos 2A = 1 - 2\sin^2 A$

$\cos 2x = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

$\sin^2 x = \dfrac{1 - \cos 2x}{2}$ *Divide both sides by 2*

$\sin x = \pm\sqrt{\dfrac{1 - \cos 2x}{2}}$ *Replace x with* $\frac{A}{2}$

$$\Rightarrow \boxed{\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}}$$

Example

If $\sin A = -\frac{12}{13}$ with $180° < A < 270°$ find the six trigonometric function of $A/2$

Solution

Since $180° < A < 270°$

$$\cos A = -\sqrt{1 - \sin^2 A}$$
$$= -\frac{5}{13}$$

$90° < \frac{A}{2} < 135°$ $\Rightarrow \frac{A}{2} \in QII$

$$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}}$$

$$= \sqrt{\frac{1}{2}\left(1-\frac{-5}{13}\right)}$$

$$= \sqrt{\frac{1}{2}\left(\frac{13+5}{13}\right)}$$

$$= \sqrt{\frac{9}{13}}$$

$$= \frac{3}{\sqrt{13}}$$

$$\cos\frac{A}{2} = -\sqrt{\frac{1+\cos A}{2}}$$

$$= -\sqrt{\frac{1}{2}\left(1+\frac{-5}{13}\right)}$$

$$= -\sqrt{\frac{1}{2}\frac{8}{13}}$$

$$= -\sqrt{\frac{4}{13}}$$

$$= -\frac{2}{\sqrt{13}}$$

$$\tan\frac{A}{2} = \frac{\frac{3}{\sqrt{13}}}{-\frac{2}{\sqrt{13}}} \qquad \tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}$$

$$= -\frac{3}{2}$$

$$\csc\frac{A}{2} = \frac{\sqrt{13}}{3} \qquad \csc\frac{A}{2} = \frac{1}{\sin\frac{A}{2}}$$

$$\sec\frac{A}{2} = -\frac{\sqrt{13}}{2} \qquad \sec\frac{A}{2} = \frac{1}{\cos\frac{A}{2}}$$

$$\cot\frac{A}{2} = -\frac{2}{3} \qquad \cot\frac{A}{2} = \frac{1}{\tan\frac{A}{2}}$$

$$\boxed{\tan\frac{A}{2}=\frac{1-\cos A}{\sin A}}\qquad\qquad \boxed{\tan\frac{A}{2}=\frac{\sin A}{1+\cos A}}$$

Example

Find the exact of $\tan 15°$

Solution

$$\begin{aligned}\tan 15° &= \tan\frac{30°}{2}\\ &= \frac{1-\cos 30°}{\sin 30°}\\ &= \frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}}\\ &= \frac{\frac{2-\sqrt{3}}{2}}{\frac{1}{2}}\\ &= 2-\sqrt{3}\end{aligned}$$

Example

Prove $\quad \sin^2\frac{x}{2}=\frac{\tan x-\sin x}{2\tan x}$

Solution

$$\begin{aligned}\sin^2\frac{x}{2} &= \frac{1-\cos x}{2}\\ &= \frac{\tan x}{\tan x}\frac{1-\cos x}{2}\\ &= \frac{\tan x-\tan x\cos x}{2\tan x}\\ &= \frac{\tan x-\frac{\sin x}{\cos x}\cos x}{2\tan x}\\ &= \frac{\tan x-\sin x}{2\tan x}\end{aligned}$$

√

Exercises Section 3.3 – Double-angle Half-Angle Formulas

1. Let $\sin A = -\frac{3}{5}$ with A in Q*III* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin \frac{A}{2}$ *e)* $\cos \frac{A}{2}$ *f)* $\tan \frac{A}{2}$

2. Let $\sin A = \frac{3}{5}$ with A in Q*II* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin \frac{A}{2}$ *e)* $\cos \frac{A}{2}$ *f)* $\tan \frac{A}{2}$

3. Let $\cos A = \frac{3}{5}$ with A in Q*IV* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin \frac{A}{2}$ *e)* $\cos \frac{A}{2}$ *f)* $\tan \frac{A}{2}$

4. Let $\cos A = \frac{5}{13}$ with A in Q*I* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin \frac{A}{2}$ *e)* $\cos \frac{A}{2}$ *f)* $\tan \frac{A}{2}$

5. Let $\cos A = -\frac{12}{13}$ with A in Q*II* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin \frac{A}{2}$ *e)* $\cos \frac{A}{2}$ *f)* $\tan \frac{A}{2}$

6. Let $\sin A = -\frac{7}{25}$ with A in Q*III* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin \frac{A}{2}$ *e)* $\cos \frac{A}{2}$ *f)* $\tan \frac{A}{2}$

7. Let $\sin A = -\frac{24}{25}$ with A in Q*IV* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin \frac{A}{2}$ *e)* $\cos \frac{A}{2}$ *f)* $\tan \frac{A}{2}$

8. Let $\cos A = \frac{15}{17}$ with A in Q*I* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin \frac{A}{2}$ *e)* $\cos \frac{A}{2}$ *f)* $\tan \frac{A}{2}$

9. Let $\cos x = \frac{1}{\sqrt{10}}$ with x in Q*IV* and find $\cot 2x$

10. Verify: $(\cos x - \sin x)(\cos x + \sin x) = \cos 2x$

11. Verify: $\cot x \sin 2x = 1 + \cos 2x$

12. Simplify $\cos^2 7x - \sin^2 7x$

13. Write $\sin 3x$ in terms of $\sin x$

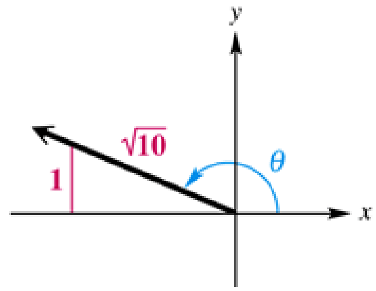
14. Find the values of the six trigonometric functions of θ if $\cos 2\theta = \frac{4}{5}$ *and* $90° < \theta < 180°$

15. Use half-angle formulas to find the exact value of $\sin 105°$

16. Find the exact of $\tan 22.5°$

17. Given: $\cos x = \frac{2}{3}$, $\frac{3\pi}{2} < x < 2\pi$, find $\cos \frac{x}{2}$, $\sin \frac{x}{2}$, and $\tan \frac{x}{2}$

18. Use a right triangle in Q*II* to find the value of $\cos\theta$ *and* $\tan\theta$



(**19 – 46**) Prove the following equation is an identity

19. $\cot\theta = \dfrac{\sin 2\theta}{1-\cos 2\theta}$

20. $\sin 3x = \sin x\left(3\cos^2 x - \sin^2 x\right)$

21. $\cos 3x = \cos^3 x - 3\cos x \sin^2 x$

22. $\cos^4 x - \sin^4 x = \cos 2x$

23. $\sin 2x = -2\sin x \sin\left(x - \frac{\pi}{2}\right)$

24. $\dfrac{\sin 4t}{4} = \cos^3 t \sin t - \sin^3 t \cos t$

25. $\dfrac{\cos 2x}{\sin^2 x} = \csc^2 x - 2$

26. $\dfrac{\cos 2x + \cos 2y}{\sin x + \cos y} = 2\cos y - 2\sin x$

27. $\dfrac{\cos 2x}{\cos^2 x} = \sec^2 x - 2\tan^2 x$

28. $\sin 4x = (4\sin x\cos x)\left(2\cos^2 x - 1\right)$

29. $\cos 2y = \dfrac{1-\tan^2 y}{1+\tan^2 y}$

30. $\cos 4x = \cos^4 x - 6\sin^2 x\cos^2 x + \sin^4 x$

31. $\tan^2 x(1+\cos 2x) = 1 - \cos 2x$

32. $\dfrac{\cos 2x}{\sin^2 x} = 2\cot^2 x - \csc^2 x$

33. $\tan x + \cot x = 2\csc 2x$

34. $\tan 2x = \dfrac{2}{\cot x - \tan x}$

35. $\dfrac{1-\tan x}{1+\tan x} = \dfrac{1-\sin 2x}{\cos 2x}$

36. $\sin 2\alpha \sin 2\beta = \sin^2(\alpha+\beta) - \sin^2(\alpha-\beta)$

37. $\cos^2(A-B) - \cos^2(A+B) = \sin 2A\ \sin 2B$

38. $2\csc x\cos^2\frac{x}{2} = \dfrac{\sin x}{1-\cos x}$

39. $\tan\frac{\alpha}{2} = \sin\alpha + \cos\alpha\cot\alpha - \cot\alpha$

40. $\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right) = \dfrac{\sin^2 x}{4}$

41. $\tan\frac{x}{2} + \cot\frac{x}{2} = 2\csc x$

42. $2\sin^2\left(\frac{x}{2}\right) = \dfrac{\sin^2 x}{1+\cos x}$

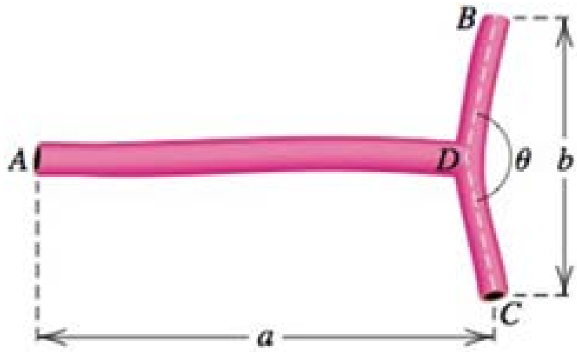
43. $\tan^2\left(\frac{x}{2}\right) = \dfrac{\sec x + \cos x - 2}{\sec x - \cos x}$

44. $\sec^2\left(\frac{x}{2}\right) = \dfrac{2\sec x + 2}{\sec x + 2 + \cos x}$

45. $\dfrac{1-\sin^2\left(\frac{x}{2}\right)}{1+\sin^2\left(\frac{x}{2}\right)} = \dfrac{1+\cos x}{3-\cos x}$

46. $\dfrac{1-\cos^2\left(\frac{x}{2}\right)}{1-\sin^2\left(\frac{x}{2}\right)} = \dfrac{1-\cos x}{1+\cos x}$

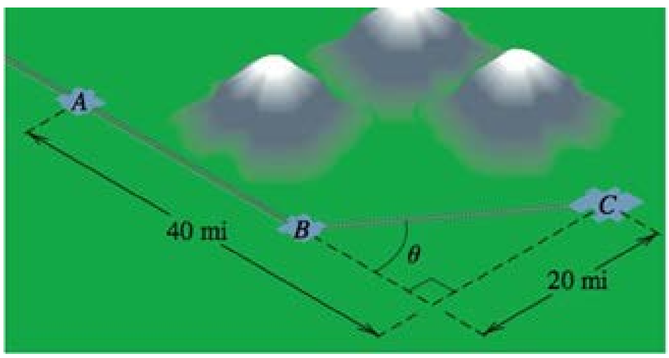
47. A common form of cardiovascular branching is bifurcation, in which an artery splits into two smaller blood vessels. The bifurcation angle θ is the angle formed by the two smaller arteries. The line through *A* and *D* bisects θ and is perpendicular to the line through *B* and *C*.



a) Show that the length l of the artery from *A* to *B* is given by $l = a + \frac{b}{2}\tan\frac{\theta}{4}$.

b) Estimate the length l from the three measurements $a = 10$ *mm*, $b = 6$ *mm*, and $\theta = 156°$.

48. A proposed rail road route through three towns located at points *A*, *B*, and *C*. At *B*, the track will turn toward *C* at an angle θ.



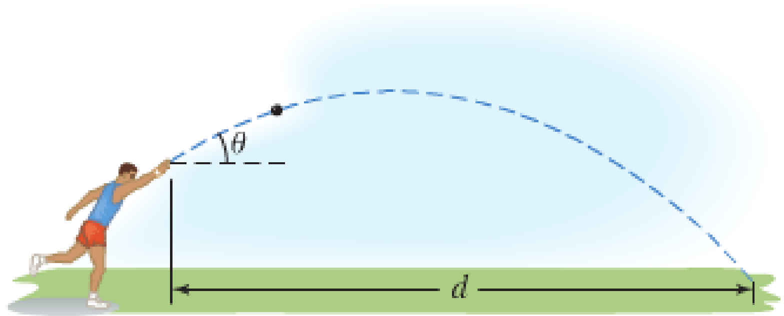
a) Show that the total distance *d* from *A* to *C* is given by $d = 20\tan\frac{1}{2}\theta + 40$

b) Because of mountains between *A* and *C*, the turning point *B* must be at least 20 *miles* from *A*.Is there a route that avoids the mountains and measures exactly 50 *miles*?

49. Throwing events in track and field include the shot put, the discus throw, the hammer throw, and the javelin throw. The distance that the athlete can achieve depends on the initial speed of the object thrown and the angle above the horizontal at which the object leaves the hand. This angle is represented by θ. The distance, *d*, in *feet*, that the athlete throws is modeled by the formula

$$d = \frac{v_0^2}{16}\sin\theta\cos\theta$$

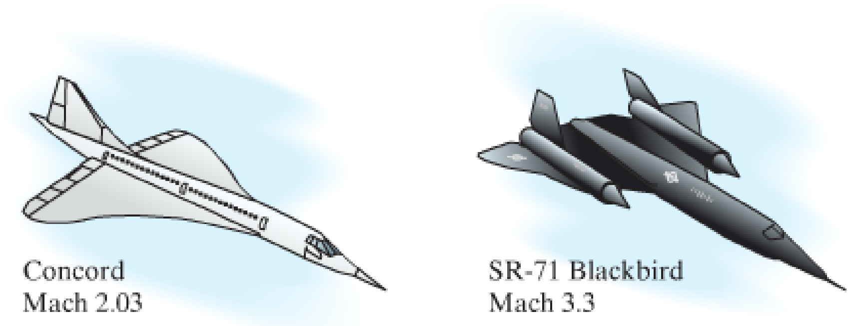
In which v_0 is the initial speed of the object thrown, in *feet* per *second*, and θ is the angle, in *degrees*, at which the object leaves the hand.



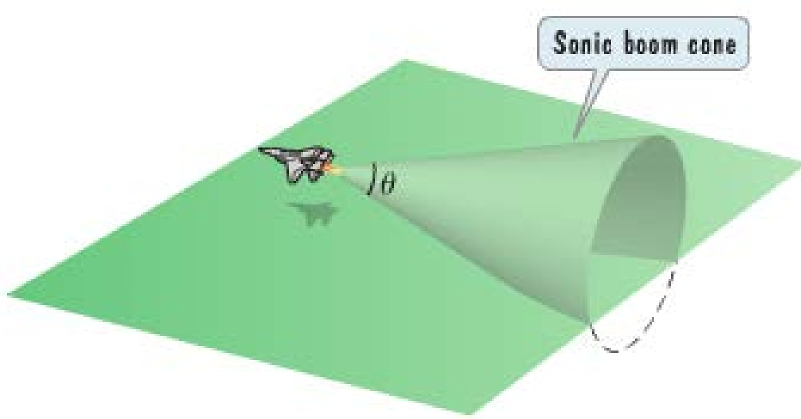
a) Use the identity to express the formula so that it contains the since function only.

b) Use the formula from part (*a*) to find the angle, θ, that produces the maximum distance, d, for a given initial speed, v_0.

50. The speed of a supersonic aircraft is usually represented by a Mach number. A Mach number is the speed of the aircraft, in *miles* per *hour*, divided by the speed of sound, approximately 740 *mph*. Thus, a plane flying at twice the speed of sound has a speed, M, of Mach 2.



If an aircraft has a speed greater than Mach 1, a sonic boom is heard, created by sound waves that form a cone with a vertex angle θ.



The relationship between the cone's vertex angle θ, and the Mach speed, M, of an aircraft that is flying faster than the speed of sound is given by

$$\sin\frac{\theta}{2} = \frac{1}{M}$$

a) If $\theta = \frac{\pi}{6}$, determine the Mach speed, M, of the aircraft. Express the speed as an exact value and as decimal to the nearest tenth.

b) If $\theta = \frac{\pi}{4}$, determine the Mach speed, M, of the aircraft. Express the speed as an exact value and as decimal to the nearest tenth.

Section 3.4 – Solving Trigonometry Equations

Example

Find the solutions of the equation $\sin\theta = \frac{1}{2}$ if

a) θ is in the interval $[0,\ 2\pi)$

b) θ is any real number

Solution

a) $\theta = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

b) Since the sine function has period 2π.

$\theta = \frac{\pi}{6} + 2\pi n \quad \textit{and} \quad \theta = \frac{5\pi}{6} + 2\pi n$

Example

Solve the equation $\sin x \tan x = \sin x$

Solution

$\sin x \tan x - \sin x = 0$

$\sin x(\tan x - 1) = 0$

$\sin x = 0$	$\tan x - 1 = 0$
	$\tan x = 1$
$\hat{x} = \sin^{-1}0 = 0$	$\hat{x} = \tan^{-1}1 = \frac{\pi}{4}$
$x = 0,\ \pm\pi,\ \pm 2\pi, \ldots$	$x = \pm\frac{\pi}{4},\ \pm\frac{5\pi}{4}, \ldots$
$x = \pi n$	$x = \frac{\pi}{4} + \pi n$

The solutions are: $x = \pi n \quad \textit{and} \quad x = \frac{\pi}{4} + \pi n$ for every integer n.

Example

Solve the equation $2\sin^2 t - \cos t - 1 = 0$, and express the solutions both in radians and degrees.

Solution

$$2\sin^2 t - \cos t - 1 = 0$$

$$2\left(1-\cos^2 t\right) - \cos t - 1 = 0 \qquad \sin^2 t + \cos^2 t = 1$$

$$2 - 2\cos^2 t - \cos t - 1 = 0$$

$$-2\cos^2 t - \cos t + 1 = 0 \qquad \textit{Multiply by } -1$$

$$2\cos^2 t + \cos t - 1 = 0 \qquad \textit{Factor or use quadratic formula}$$

$$(2\cos t - 1)(\cos t + 1) = 0$$

$$2\cos t - 1 = 0 \qquad \cos t + 1 = 0$$

$$2\cos t = 1 \qquad \cos t = -1$$

$$\cos t = \frac{1}{2}$$

$$t = \frac{\pi}{3} \quad or \quad t = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} \qquad t = \pi$$

$$t = \frac{\pi}{3} + 2\pi n, \quad t = \frac{5\pi}{3} + 2\pi n, \qquad t = \pi + 2\pi n$$

$$t = 60^\circ + 360^\circ n, \quad 300^\circ + 360^\circ n, \quad and \quad 180^\circ + 360^\circ n$$

Example

Solve the equation $4\sin^2 x \tan x - \tan x = 0$ in the interval $[0,\ 2\pi)$.

Solution

$$4\sin^2 x \tan x - \tan x = 0$$

$$\tan x\left(4\sin^2 x - 1\right) = 0 \qquad \textit{Factor out } \tan x$$

$$\tan x = 0 \qquad 4\sin^2 x - 1 = 0$$

$$\sin^2 x = \frac{1}{4}$$

$$\tan x = 0 \qquad \sin x = \frac{1}{2} \qquad \sin x = -\frac{1}{2}$$

$$\boxed{x = 0,\ \pi} \qquad \boxed{x = \frac{\pi}{6},\ \frac{5\pi}{6}} \qquad \boxed{x = \frac{7\pi}{6},\ \frac{11\pi}{6}}$$

Example

Find the solutions of $\csc^4 2u - 4 = 0$

Solution

$$\left(\csc^2 2u - 2\right)\left(\csc^2 2u + 2\right) = 0$$

$$\csc^2 2u - 2 = 0 \qquad \csc^2 2u + 2 = 0$$

$$\csc^2 2u = 2 \qquad \csc^2 2u = -2 \times$$

$$\csc 2u = \pm\sqrt{2}$$

$$\sin 2u = \pm\frac{1}{\sqrt{2}}$$

$$\sin 2u = \frac{\sqrt{2}}{2} \quad \Rightarrow 2u = \frac{\pi}{4} + 2\pi n \quad \rightarrow u = \frac{\pi}{8} + \pi n$$

$$\Rightarrow 2u = \frac{3\pi}{4} + 2\pi n \quad \rightarrow u = \frac{3\pi}{8} + \pi n$$

$$\sin 2u = -\frac{\sqrt{2}}{2} \quad \Rightarrow 2u = \frac{5\pi}{4} + 2\pi n \quad \rightarrow u = \frac{5\pi}{8} + \pi n$$

$$\Rightarrow 2u = \frac{7\pi}{4} + 2\pi n \quad \rightarrow u = \frac{7\pi}{8} + \pi n$$

Example

Approximate to the nearest degree, the solutions of the following equation in the interval $[0°,\ 360°)$:

$$5\sin\theta\tan\theta - 10\tan\theta + 3\sin\theta - 6 = 0$$

Solution

$$\tan\theta(5\sin\theta - 10) + (3\sin\theta - 6) = 0$$

$$5\tan\theta(\sin\theta - 2) + 3(\sin\theta - 2) = 0$$

$$(\sin\theta - 2)(5\tan\theta + 3) = 0$$

$$\sin\theta - 2 = 0 \qquad 5\tan\theta + 3 = 0$$

$$\sin\theta = 2 > 1 \qquad \tan\theta = -\frac{3}{5} \qquad \boldsymbol{\theta \in \text{QII, QIV}}$$

$$\hat{\theta} = \tan^{-1}\left(\frac{3}{5}\right) = 31°$$

$$\begin{cases} \theta = 180° - 31° & = 149° \\ \theta = 360° - 31° & = 329° \end{cases}$$

Exercises *Section* 3.4 – Trigonometric Equations

(**1 – 9**) Find all solutions of the equation

1. $\sin x = \frac{\sqrt{2}}{2}$

2. $\cos x = -\frac{\pi}{3}$

3. $2\cos\theta - \sqrt{3} = 0$

4. $\sqrt{3}\tan\frac{1}{3}x = 1$

5. $\cos\left(4x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

6. $(\cos\theta - 1)(\sin\theta + 1) = 0$

7. $\cot^2 x - 3 = 0$

8. $\cos x + 1 = 2\sin^2 x$

9. $\cos(\ln x) = 0$

(**10 – 24**) Find the solutions of the equation that are in the interval $[0,\ 2\pi)$

10. $2\sin^2 x = 1 - \sin x$

11. $\tan^2 x \sin x = \sin x$

12. $1 - \sin x = \sqrt{3}\cos x$

13. $\sin x + \cos x \cot x = \csc x$

14. $2\sin^3 x + \sin^2 x - 2\sin x - 1 = 0$

15. $2\tan x \csc x + 2\csc x + \tan x + 1 = 0$

16. $5\cos t + \sqrt{12} = \cos t$

17. $2\sin^2 x - \cos x - 1 = 0$

18. $2\cos^2 t - 9\cos t = 5$

19. $\tan^2 x + \tan x - 2 = 0$

20. $\tan x + \sqrt{3} = \sec x$

21. $2\sin^2\theta + 2\sin\theta - 1 = 0$

22. $2\cos x - 1 = \sec x$

23. $4\cos^2 x + 4\sin x - 5 = 0$

24. $\sin\theta - \cos\theta = 1$

(**25 – 35**) Find the solutions of the equation that are in the interval *if* $0° \le \theta < 360°$

25. $2\cos\theta + \sqrt{3} = 0$

26. $\tan\theta - 2\cos\theta\tan\theta = 0$

27. $2\sin^2\theta - 2\sin\theta - 1 = 0$

28. $4\cos\theta - 3\sec\theta = 0$

29. $\sin\theta - \sqrt{3}\cos\theta = 1$

30. $7\sin^2\theta - 9\cos 2\theta = 0$

31. $\sin\theta\tan\theta = \sin\theta$

32. $2\sin\theta - 3 = 0$

33. $3\sin\theta - 2 = 7\sin\theta - 1$

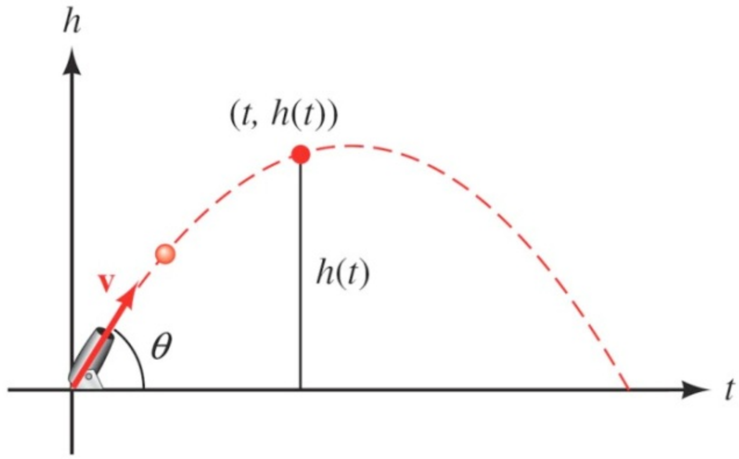
34. $\cos 2\theta + 3\sin\theta - 2 = 0$

35. $\sin 2\theta + \sqrt{2}\cos\theta = 0$

36. Solve $\cos\left(A - \frac{\pi}{9}\right) = -\frac{1}{2}$

37. Solve $\cos(A - 25°) = -\frac{1}{\sqrt{2}}$

38. If a projectile (such as a bullet) is fired into the air with an initial velocity $\boldsymbol{v}$ at an angle of elevation θ, then the height h of the projectile at time t is given by: $h(t) = -16t^2 + vt\sin\theta$



a) Give the equation for the height, if $\boldsymbol{v}$ is 600 *ft./sec* and $\theta = 45°$.

b) Use the equation in part (*a*) to find the height of the object after $\sqrt{3}$ seconds.

c) Find the angle of elevation of θ of a rifle barrel, if a bullet fired at 1,500 *ft./sec* takes 3 seconds to reach a height of 750 *feet*. Give your answer in the nearest of a degree.

Section 3.5 – Inverse Trigonometry Functions

Relationships Between f^{-1} *and* f

- $y = f^{-1}(x)$ if and only if $x = f(y)$, where ***x*** is in the domain of f^{-1} and ***y*** is in the domain of f
- Domain of f^{-1} = Range of f
- Range of f^{-1} = Domain of f
- $f\left(f^{-1}(x)\right) = x$ for every *x* in the domain of f^{-1}
- $f^{-1}(f(y)) = y$ for every *y* in the domain of f
- The point (*a*, *b*) is on the graph of f ***iff*** the point (*b*, *a*) is on the graph of f^{-1}.
- The graphs of f^{-1} and f are reflections of each other through the line *y* = *x*.

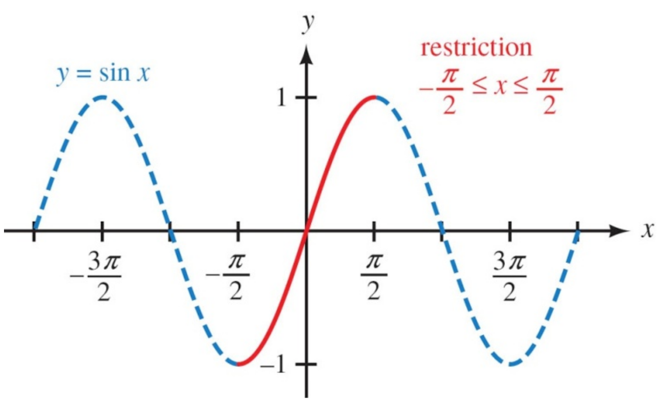
The Inverse *Sine* Function

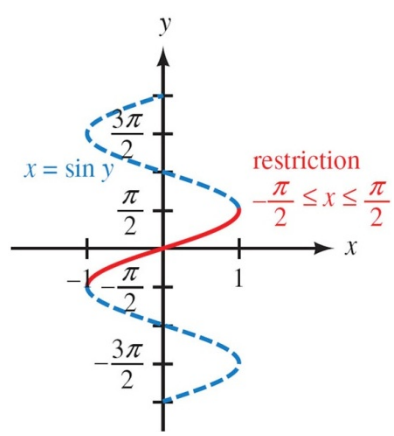
$$y = \sin^{-1} x \quad or \quad y = \arcsin x \quad \textbf{iff} \quad x = \sin y \quad for\ -\tfrac{\pi}{2} \le y \le \tfrac{\pi}{2} \quad and \quad -1 \le x \le 1$$

Properties of $\sin^{-1}$

$$\sin\left(\sin^{-1} x\right) = \sin(\arcsin x) = x \quad if \quad -1 \le x \le 1$$

$$\sin^{-1}(\sin y) = \arcsin(\sin y) = y \quad if \quad -\tfrac{\pi}{2} \le y \le \tfrac{\pi}{2}$$





Example

Find the exact value: $\sin\left(\sin^{-1}\frac{1}{2}\right)$, $\sin^{-1}\left(\sin\frac{\pi}{4}\right)$

Solution

$\sin\left(\sin^{-1}\frac{1}{2}\right) = \frac{1}{2}$ Since $-1 \le \frac{1}{2} \le 1$

$\sin^{-1}\left(\sin\frac{\pi}{4}\right) = \frac{\pi}{4}$ Since $-\frac{\pi}{2} \le \frac{\pi}{4} \le \frac{\pi}{2}$

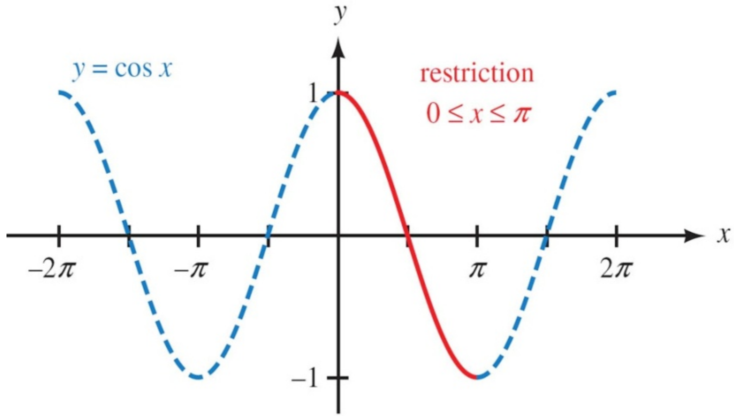
The Inverse *Cosine* Function

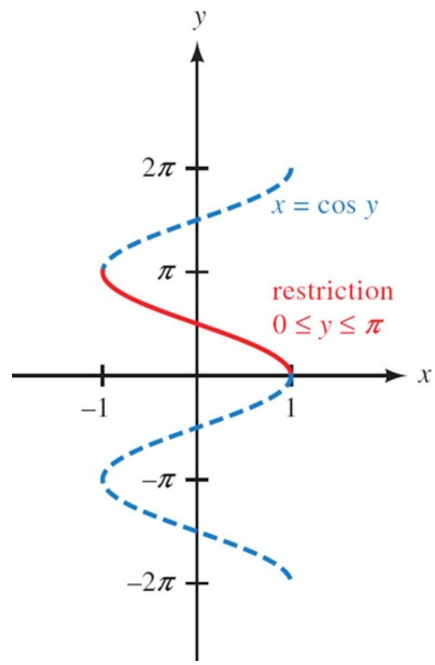
Definition

The inverse cosine function, denoted by $\cos^{-1}$, is defined by

$$y = \cos^{-1} x \quad \textit{iff} \quad x = \cos y \quad \textit{for } 0 \le y \le \pi \quad \textit{and} \quad -1 \le x \le 1$$

Notation	***Meaning***
$y = \cos^{-1} x$ *or* $y = \arccos x$	$x = \cos y$ *and* $0 \le y \le \pi$





Properties of $\cos^{-1}$

$$\cos\left(\cos^{-1} x\right) = \cos(\arccos x) = x \quad \textit{if} \quad -1 \le x \le 1$$

$$\cos^{-1}(\cos y) = \arccos(\cos y) = y \quad \textit{if} \quad 0 \le y \le \pi$$

Example

Find the exact value: $\cos\left(\cos^{-1}(-0.5)\right)$, $\cos^{-1}(\cos(3.14))$, $\cos^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right)$

Solution

$\cos\left(\cos^{-1}(-0.5)\right) = -0.5$ Since $-1 \le -0.5 \le 1$

$\cos^{-1}(\cos(3.14)) = 3.14$ Since $0 \le 3.14 \le \pi$

$\cos^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

Example

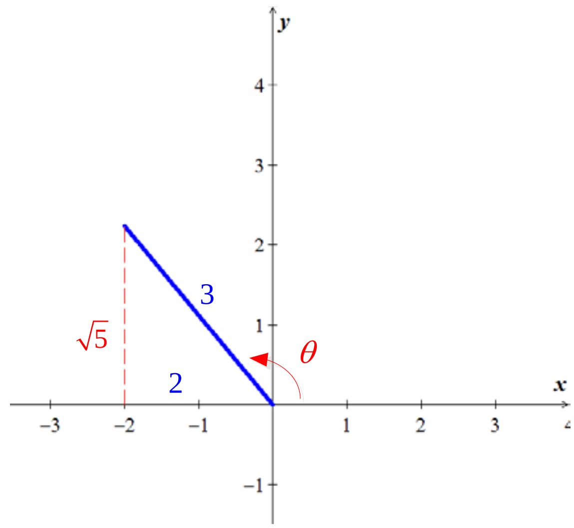
Find the exact value of $\sin\left[\arccos\left(-\frac{2}{3}\right)\right]$

Solution

$$\theta = \arccos\left(-\frac{2}{3}\right) \Rightarrow \cos\theta = -\frac{2}{3} \qquad 0 \le \theta \le \pi$$

$$y = \sqrt{3^2 - 2^2} = \sqrt{5}$$

$$\sin\left[\arccos\left(-\frac{2}{3}\right)\right] = \sin\theta = \frac{\sqrt{5}}{3}$$



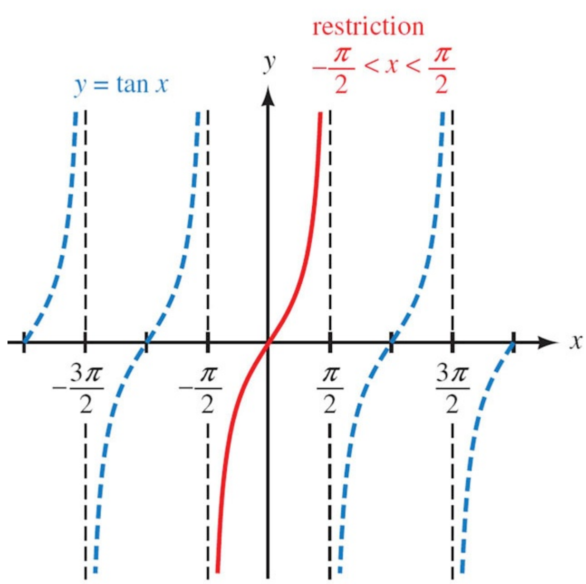
The Inverse *Tangent* Function

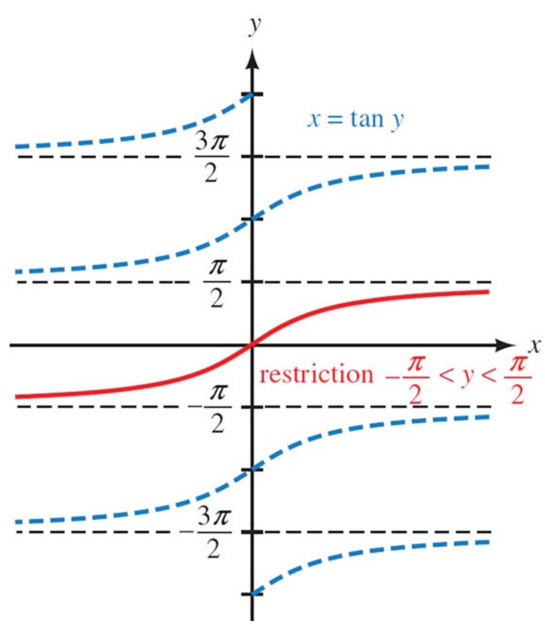
Definition

The inverse cosine function, denoted by $\tan^{-1}$, is defined by

$y = \tan^{-1} x$ *iff* $x = \tan y$ for any real number x and for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$y = \tan^{-1} x$ *or* $y = \arctan x$





Properties of $\tan^{-1}$

$$\tan\left(\tan^{-1} x\right) = \tan(\arctan x) = x \quad \textit{for} \quad \textit{every } x$$

$$\tan^{-1}(\tan y) = \arctan(\tan y) = y \quad \textit{if} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

Example

Find the exact value: $\tan\left(\tan^{-1}(1000)\right)$, $\tan^{-1}\left(\tan\frac{\pi}{4}\right)$, $\arctan(\tan\pi)$

Solution

$\tan\left(\tan^{-1}1000\right) = 1000$

$\tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$ Since $-\frac{\pi}{2} \le \frac{\pi}{4} \le \frac{\pi}{2}$

$\arctan(\tan\pi) = \arctan(0) = 0$ $\therefore \pi > \frac{\pi}{2}$

Example

Evaluate in radians without using a calculator or tables.

a. $\sin^{-1}\frac{1}{2}$

$$-\frac{\pi}{2} \le angle \le \frac{\pi}{2} \Rightarrow \sin\frac{\pi}{6} = \frac{1}{2}$$

$$\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$

b. $\arccos\left(-\frac{\sqrt{3}}{2}\right)$

$$0 < angle < \pi \Rightarrow \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$$

$$\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$$

c. $\tan^{-1}(-1)$

$$-\frac{\pi}{2} < angle < \frac{\pi}{2} \Rightarrow \tan\left(-\frac{\pi}{4}\right) = -1$$

$$\tan^{-1}(-1) = -\frac{\pi}{4}$$

Example

Use a calculator to evaluate each expression to the nearest tenth of a degree

a. $\arcsin(0.5075)$

$$\arcsin(0.5075) = 30.5°$$

b. $\arcsin(-0.5075)$

$$\arcsin(-0.5075) = -30.5°$$

c. $\cos^{-1}(0.6428)$

$$\cos^{-1}(0.6428) = 50.0°$$

d. $\cos^{-1}(-0.6428)$

$$\cos^{-1}(-0.6428) = 130.0°$$

e. $\arctan(4.474)$

$$\arctan(4.474) = 77.4°$$

f. $\arctan(-4.474)$

$$\arctan(-4.474) = -77.4°$$

Example

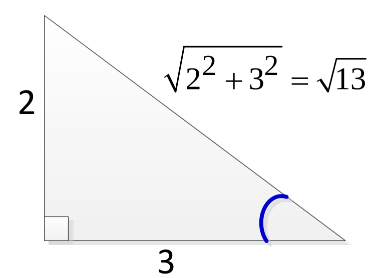
Find the exact value: $\sec\left(\arctan\frac{2}{3}\right)$

Solution

$$\alpha = \arctan\frac{2}{3} \to \tan\alpha = \frac{2}{3}$$

$$\sec\left(\arctan\frac{2}{3}\right) = \sec\alpha$$

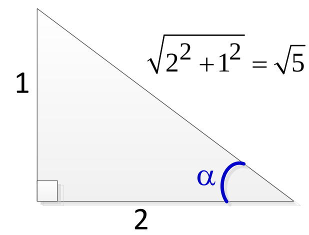
$$= \frac{\sqrt{13}}{3}$$



Example

Find the exact value: $\sin\left(\arctan\frac{1}{2} - \arccos\frac{4}{5}\right)$

Solution

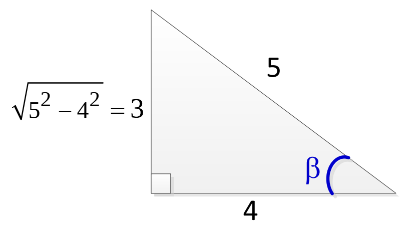


$$\alpha = \arctan\frac{1}{2} \qquad \beta = \arccos\frac{4}{5}$$

$$\tan\alpha = \frac{1}{2} \qquad \cos\beta = \frac{4}{5}$$

$$\sin\alpha = \frac{1}{\sqrt{5}} \qquad \sin\beta = \frac{3}{5}$$

$$\cos\alpha = \frac{2}{\sqrt{5}}$$

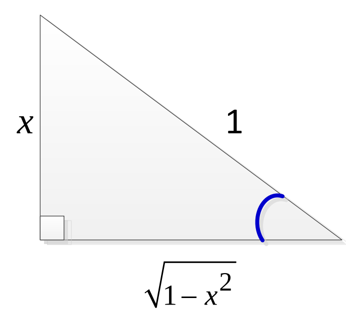


$$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\ &= \frac{1}{\sqrt{5}}\frac{4}{5} - \frac{2}{\sqrt{5}}\frac{3}{5} \\ &= \frac{4}{5\sqrt{5}} - \frac{6}{5\sqrt{5}} \\ &= -\frac{2}{5\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}} \\ &= -\frac{2\sqrt{5}}{25}\end{aligned}$$

Example

If $-1 \le x \le 1$, rewrite $\cos\left(\sin^{-1}x\right)$ as an algebraic expression in x.

Solution



$$\alpha = \sin^{-1}x \quad \rightarrow \quad \sin\alpha = x = \frac{x}{1}$$

$$\begin{aligned}\cos\left(\sin^{-1}x\right) &= \cos\alpha \\ &= \frac{\sqrt{1-x^2}}{1} \\ &= \sqrt{1-x^2}\end{aligned}$$

Exercises *Section* 3.5 – Inverse Trigonometric Functions

(1 – 18) Find the exact value of the expression whenever it is defined

1. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

2. $\arccos\left(\frac{\sqrt{2}}{2}\right)$

3. $\arctan\left(-\frac{\sqrt{3}}{3}\right)$

4. $\sin\left[\arcsin\left(-\frac{3}{10}\right)\right]$

5. $\tan\left[\arctan(14)\right]$

6. $\sin\left[\sin^{-1}\left(\frac{2}{3}\right)\right]$

7. $\cos^{-1}\left[\cos\left(\frac{5\pi}{6}\right)\right]$

8. $\tan^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right]$

9. $\arcsin\left[\sin\left(-\frac{\pi}{2}\right)\right]$

10. $\arccos\left[\cos(0)\right]$

11. $\arctan\left[\tan\left(-\frac{\pi}{4}\right)\right]$

12. $\sin\left[\arcsin\left(\frac{1}{2}\right)+\arccos 0\right]$

13. $\cos\left[\arctan\left(-\frac{3}{4}\right)-\arcsin\frac{4}{5}\right]$

14. $\tan\left[\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)\right]$

15. $\sin\left[2\arccos\left(-\frac{3}{5}\right)\right]$

16. $\cos\left[2\sin^{-1}\left(\frac{15}{17}\right)\right]$

17. $\tan\left[2\tan^{-1}\left(\frac{3}{4}\right)\right]$

18. $\cos\left[\frac{1}{2}\tan^{-1}\left(\frac{8}{15}\right)\right]$

(19 – 28) Evaluate without using a calculator

19. $\cos\left(\cos^{-1}\frac{3}{5}\right)$

20. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

21. $\tan\left(\cos^{-1}\frac{3}{5}\right)$

22. $\sin\left(\cos^{-1}\frac{1}{\sqrt{5}}\right)$

23. $\cos\left(\sin^{-1}\frac{1}{2}\right)$

24. $\sin\left(\sin^{-1}\frac{3}{5}\right)$

25. $\cos\left(\tan^{-1}\frac{3}{4}\right)$

26. $\tan\left(\sin^{-1}\frac{3}{5}\right)$

27. $\sec\left(\cos^{-1}\frac{1}{\sqrt{5}}\right)$

28. $\cot\left(\tan^{-1}\frac{1}{2}\right)$

(29 – 41) Write an equivalent expression that involves x only for

29. $\cos\left(\cos^{-1}x\right)$

30. $\tan\left(\cos^{-1}x\right)$

31. $\csc\left(\sin^{-1}\frac{1}{x}\right)$

32. $\sin\left(\tan^{-1}x\right); \quad x>0$

33. $\sec\left(\sin^{-1}\frac{x}{\sqrt{x^2+4}}\right) \quad x>0$

34. $\cot\left(\sin^{-1}\frac{\sqrt{x^2-9}}{x}\right) \quad x>0$

35. $\sin\left(2\sin^{-1}x\right) \quad x>0$

36. $\cos\left(2\tan^{-1}x\right), \quad x>0$

37. $\cos\left(\frac{1}{2}\arccos x\right), \quad x>0$

38. $\tan\left(\frac{1}{2}\cos^{-1}\frac{1}{x}\right), \quad x>0$

39. $\sec\left(\tan^{-1}\frac{2}{\sqrt{x^2-4}}\right) \quad x>0$

40. $\sec\left(\sin^{-1}\frac{\sqrt{x^2-25}}{x}\right) \quad x>0$

41. $\sin\left(\cos^{-1}\frac{x}{\sqrt{x^2+4}}\right) \quad x>0$

(**42 – 44**) Sketch the graph of the equation:

42. $y = \sin^{-1} 2x$ **43.** $y = \sin^{-1}(x-2) + \frac{\pi}{2}$ **44.** $y = \cos^{-1} \frac{1}{2} x$

45. Evaluate $\sin\left(\tan^{-1} \frac{3}{4}\right)$ without using a calculator

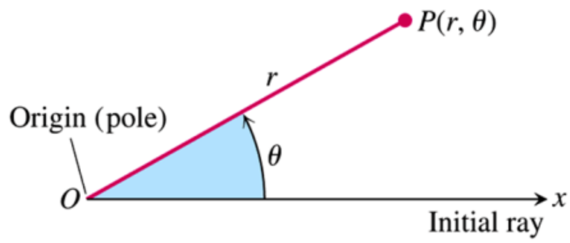
46. Evaluate $\sin(\cos^{-1} x)$ as an equivalent expression in x only

Section 3.6 – Polar Coordinates

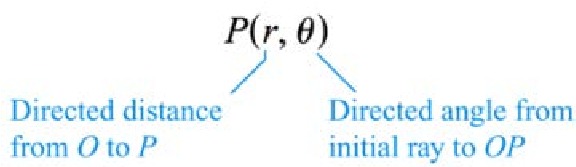
To reach the point whose address is (2, 1), we start from origin and travel 2 units right and then 1 unit up. Another way to get to that point, we can travel $\sqrt{5}$ units on the terminal side of an angle in standard position and this type is called *Polar Coordinates*.

Definition of Polar Coordinates

To define polar coordinates, let an ***origin*** *O* (called the ***pole***) and an ***initial ray*** from *O*. Then each point *P* can be located by assigning to it a ***polar coordinate pair*** $(r,\ \theta)$ in which r gives the directed from *O* to *P* and θ gives the directed angle from the initial ray to yay *OP*.



Polar Coordinates



Definition – Relationships between Rectangular and Polar Coordinates

The rectangular coordinates (*x*, *y*) and polar coordinates $(r,\ \theta)$ of a point P are related as follows:

1. $x = r\cos\theta, \quad y = r\sin\theta$

2. $r^2 = x^2 + y^2 \quad \tan\theta = \dfrac{y}{x} \quad if\ \ x \ne 0$

Example

If $(r,\ \theta)=\left(4,\ \frac{7\pi}{6}\right)$ are polar coordinates of a point P, find the rectangular coordinates of P.

Solution

$$x = r\cos\theta = 4\cos\frac{7\pi}{6} = 4\left(-\frac{\sqrt{3}}{2}\right) = \underline{-2\sqrt{3}}$$

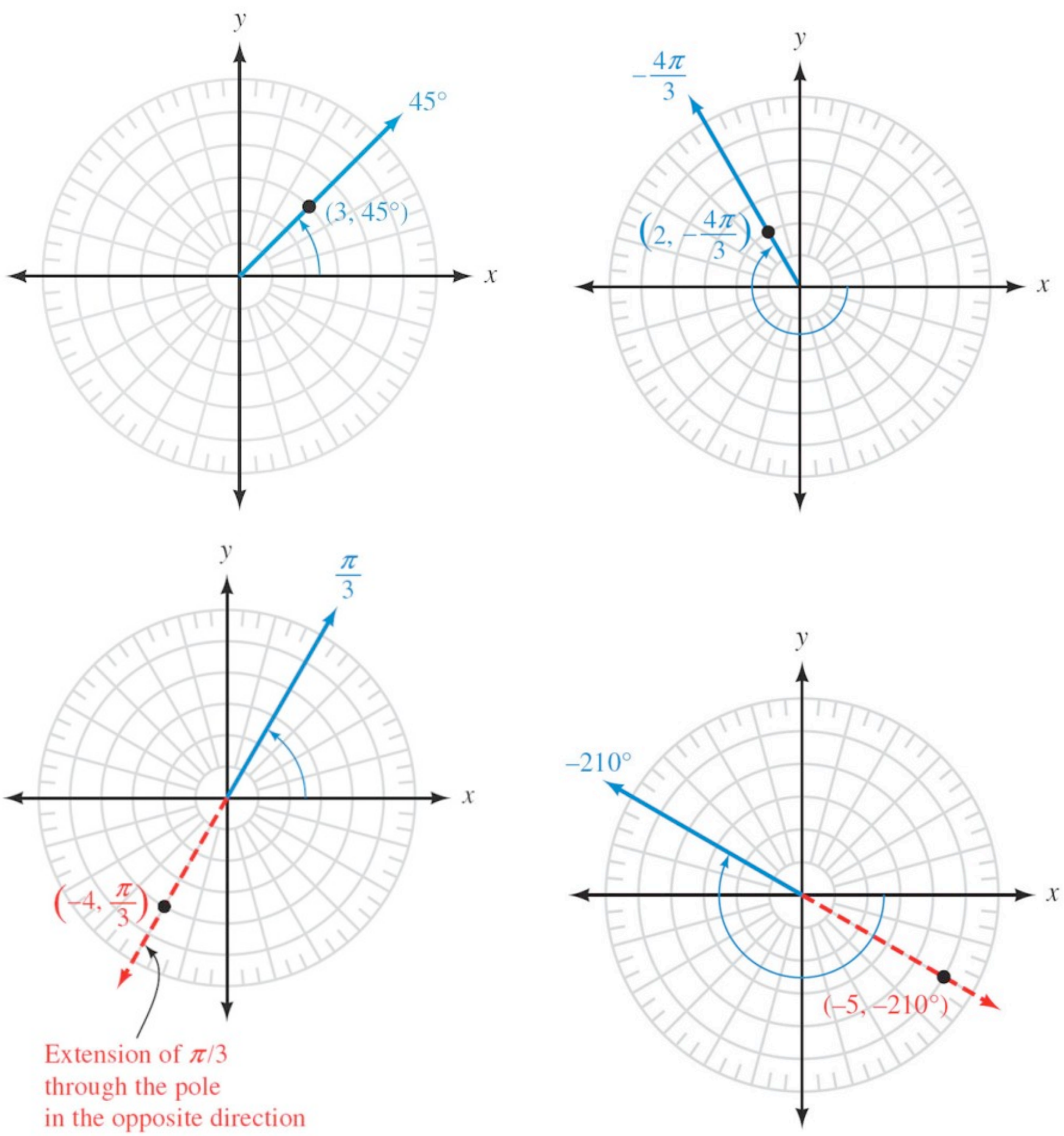
$$y = r\sin\theta = 4\sin\frac{7\pi}{6} = 4\left(-\frac{1}{2}\right) = \underline{-2}$$

The rectangular coordinates of P are $\boxed{(x,\ y)=\left(-2\sqrt{3},\ -2\right)}$

Example

Graph the points $(3, 45°)$, $\left(2,\ -\frac{4\pi}{3}\right)$, $\left(-4,\ \frac{\pi}{3}\right)$, and $(-5,\ -210°)$ on a polar coordinate system

Solution



Example

If $(x,\ y)=\left(-1,\ \sqrt{3}\right)$ are rectangular coordinates of a point P, find three different pairs the polar coordinates of P.

Solution

$$\begin{aligned} r &= \pm\sqrt{x^2+y^2} \\ &= \pm\sqrt{(-1)^2+\left(\sqrt{3}\right)^2} \\ &= \pm\sqrt{1+3} \\ &= \pm\sqrt{4} \\ &= \pm 2 \end{aligned}$$

$$\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

$$\hat{\theta} = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$$

$$\theta_1 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\theta_2 = \frac{2\pi}{3} + 2\pi = \frac{3\pi}{3}$$

$$\theta_3 = -\frac{\pi}{3}$$

The polar coordinates of P are: $\left(2,\ \frac{2\pi}{3}\right)$, $\left(-2,\ \frac{5\pi}{3}\right)$, $\left(2,\ -\frac{4\pi}{3}\right)$, and $\left(-2,\ -\frac{\pi}{3}\right)$

Example

Find a polar equation of an arbitrary line.

Solution

An equation of a line can be written in the form: $ax+by=c$.

$$ax+by=c$$

$$ar\cos\theta + br\sin\theta = c$$

$$r\left(a\cos\theta + b\sin\theta\right) = c$$

$$r = \frac{c}{a\cos\theta + b\sin\theta}$$

Example

Find a polar equation of the hyperbola $x^2 - y^2 = 16$.

Solution

$$(r\cos\theta)^2 - (r\sin\theta)^2 = 16$$

$$r^2\cos^2\theta - r^2\sin^2\theta = 16$$

$$r^2\left(\cos^2\theta - \sin^2\theta\right) = 16$$

$$r^2(\cos 2\theta) = 16$$

$$\boxed{r^2 = \frac{16}{\cos 2\theta}} \qquad \cos 2\theta \neq 0$$

$$or \quad r^2 = 16\sec 2\theta$$

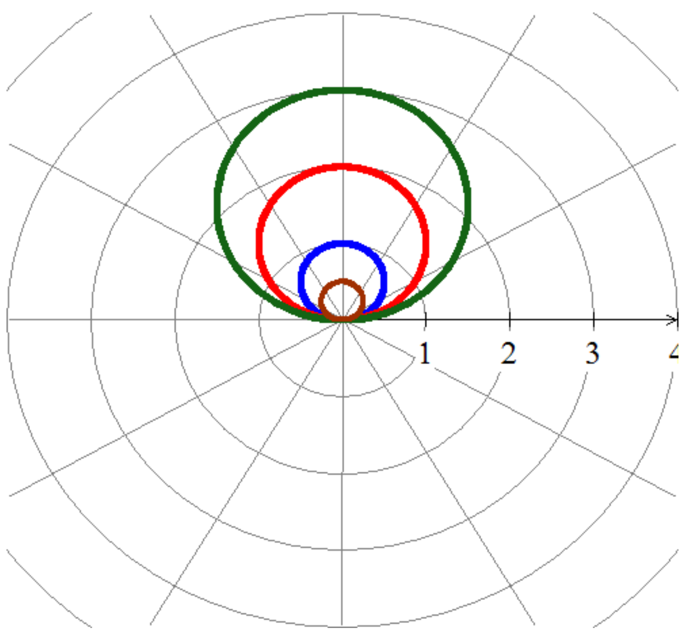
Example

Find an equation in x and y that has the same graph as the polar equation $r = a\sin\theta, \quad a \neq 0$. Sketch the graph.

Solution

$$r^2 = ar\sin\theta$$

$$x^2 + y^2 = ay$$

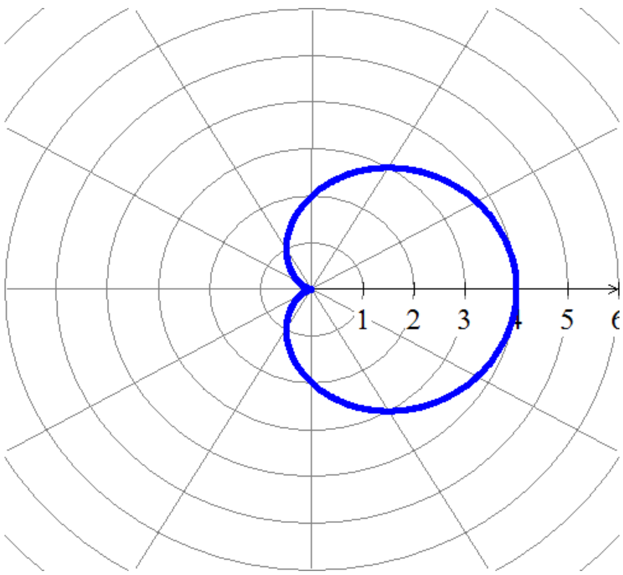


Example

Sketch the graph of the polar equation $r = 2 + 2\cos\theta$.

Solution

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π
r	4	$2+\sqrt{2}$	2	$2-\sqrt{2}$	0	2	4



Exercises *Section* 3.6 – Polar Coordinates

Convert to rectangular coordinates

1. $(4,\ 30°)$

2. $\left(-\sqrt{2},\ \frac{3\pi}{4}\right)$

3. $(3,\ 270°)$

4. $(2,\ 60°)$

5. $\left(\sqrt{2},\ -225°\right)$

6. $\left(4\sqrt{3},\ -\frac{\pi}{6}\right)$

7. Change the polar coordinates to rectangular coordinates $\left(-2,\ \frac{7\pi}{6}\right)$

8. Change the polar coordinates to rectangular coordinates $\left(6,\ \arctan\frac{3}{4}\right)$

9. Change the polar coordinates to rectangular coordinates $\left(10,\ \arccos\left(-\frac{1}{3}\right)\right)$

(**10 – 16**) Convert to polar coordinates

10. $(3,\ 3)$

11. $(-2,\ 0)$

12. $(-1,\ \sqrt{3})$

13. $(-3,\ -3)$ $\quad r \ge 0 \quad 0° \le \theta < 360°$

14. $\left(2,\ -2\sqrt{3}\right)$ $\quad r \ge 0 \quad 0° \le \theta < 360°$

15. $(-2,\ 0)$ $\quad r \ge 0 \quad 0 \le \theta < 2\pi$

16. $\left(-1,\ -\sqrt{3}\right)$ $\quad r \ge 0 \quad 0 \le \theta < 2\pi$

17. Change the rectangular coordinates to polar coordinates $\left(7,\ -7\sqrt{3}\right)$ $\quad r > 0 \quad 0 \le \theta < 2\pi$

18. Change the rectangular coordinates to polar coordinates $\left(-2\sqrt{2},\ -2\sqrt{2}\right)$ $\quad r > 0 \quad 0 \le \theta < 2\pi$

19. The point $(0,\ -3)$ in rectangular coordinates is equivalent to $(3,\ 270°)$ in polar coordinates.

20. The point $(1,\ -1)$ in rectangular coordinates is equivalent to $\left(-\sqrt{2},\ \frac{3\pi}{4}\right)$ in polar coordinates.

21. A point lies at (4, 4) on a rectangular coordinate system. Give its address in polar coordinates $(r,\ \theta)$

(**22 – 34**) Write the equation in rectangular coordinates

22. $r^2 = 4$

23. $r = 6\cos\theta$

24. $r^2 = 4\cos 2\theta$

25. $r(\cos\theta - \sin\theta) = 2$

26. $r^2 = 4\sin 2\theta$

27. $r\sin\theta = -2$

28. $\theta = \frac{\pi}{4}$

29. $r^2\left(4\sin^2\theta - 9\cos^2\theta\right) = 36$

30. $r^2\left(\cos^2\theta + 4\sin^2\theta\right) = 16$

31. $r(\sin\theta - 2\cos\theta) = 6$

32. $r = 8\sin\theta - 2\cos\theta$

33. $r = \tan\theta$

34. $r\left(\sin\theta + r\cos^2\theta\right) = 1$

(**35 – 38**) Find a polar equation that has the same graph as the equation in x and y

35. $y^2 = 6x$

36. $xy = 8$

37. $(x+2)^2 + (y-3)^2 = 13$

38. $y^2 - x^2 = 4$

(**39 – 42**) Write the equation in polar coordinates

39. $x + y = 5$

40. $x^2 + y^2 = 9$

41. $x^2 + y^2 = 4x$

42. $y = -x$

43. $x + y = 4$

(**44 – 54**) Sketch the graph of the polar equation

44. $r = 5$

45. $\theta = \frac{\pi}{4}$

46. $r = 4\cos\theta + 2\sin\theta$

47. $r = 2 + 4\sin\theta$

48. $r = 2 - \cos\theta$

49. $r = 4\csc\theta$

50. $r^2 = 4\cos 2\theta$

51. $r = 2^{\theta} \quad \theta \ge 0$

52. $r = e^{2\theta} \quad \theta \ge 0$

53. $r\theta = 1 \quad \theta > 0$

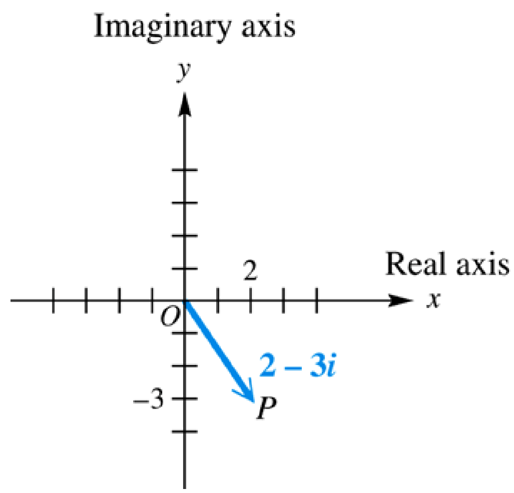
54. $r = 2 + 2\sec\theta$

Section 3.7 – Trigonometric Form

$\sqrt{-1} = i$

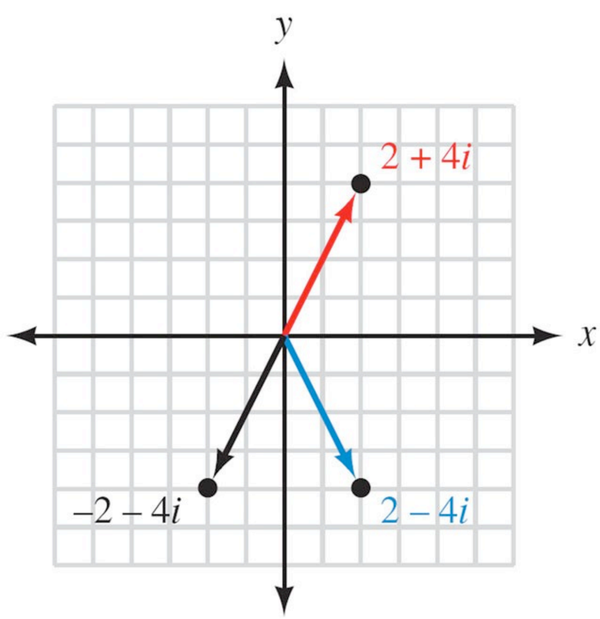
The graph of the complex number $x = yi$ is a vector (arrow) that extends from the origin out to the point (x, y)

- Horizontal axis: ***real axis***
- Vertical axis: ***imaginary axis***



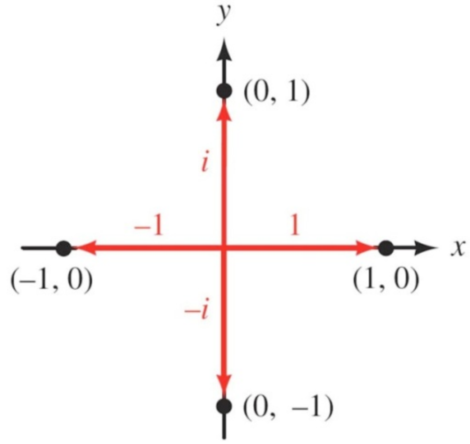
Example

Graph each complex number: $2 + 4i$, $-2 - 4i$, and $2 - 4i$



Example

Graph each complex number: $1,\ i,\ -1,\ and\ -i$



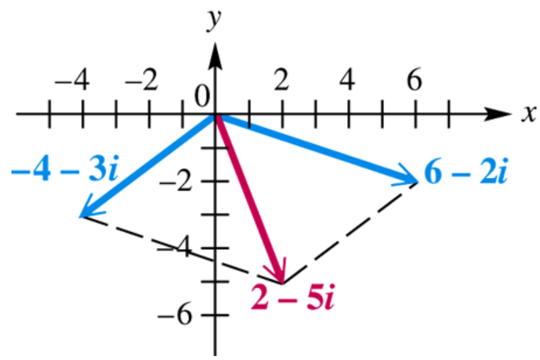
Example

Find the sum of $6 - 2i$ and $-4 - 3i$. Graph both complex numbers and their resultant.

Solution

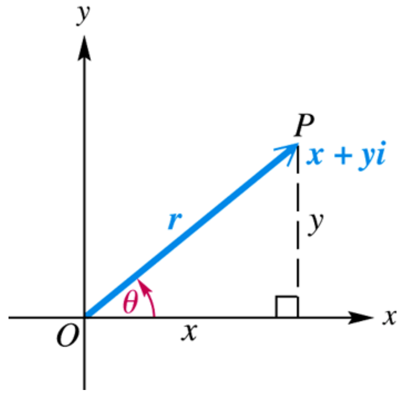
$$(6 - 2i) + (-4 - 3i) = 6 - 4 - 2i - 3i$$

$$= 2 - 5i$$



Definition

The *absolute value* or ***modulus*** of the complex number $z = x + yi$ is the distance from the origin to the point (x, y). If this distance is denoted by r, then



$$r = |z| = |x + yi|$$
$$= \sqrt{x^2 + y^2}$$

The ***argument*** of the complex number $z = x + yi$ denoted $\arg(z)$ is the smallest possible angle θ from the positive real axis to the graph of z.

$$\cos\theta = \frac{x}{r} \qquad \Rightarrow x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \qquad \Rightarrow y = r\sin\theta$$

$$\begin{aligned} z &= x + yi \\ &= r\cos\theta + (r\sin\theta)\ i \\ &= r(\cos\theta + i\ \sin\theta) \end{aligned}$$ $\rightarrow$ is called the *trigonometric* from of z.

Definition

If $z = x + y\ i$ is a complex number in standard form then the ***trigonometric form*** for z is given by

$$z = r(\cos\theta + i\ \sin\theta) \ = \ r\ cis\theta$$

Where $\boldsymbol{r}$ is the modulus or absolute value of z and

$\boldsymbol{\theta}$ is the argument of z.

We can convert back and forth between standard form and trigonometric form by using the relationships that follow

For $z = x + y\ i = r(\cos\theta + i\ \sin\theta) = r\ cis\theta$

$$r = \sqrt{x^2 + y^2}$$

$$\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}, \quad and \quad \tan\theta = \frac{y}{x}$$

Example

Write $z = -1 + i$ in trigonometric form

Solution

The modulus r:

$$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$$\cos\theta = \frac{x}{r} = \frac{-1}{\sqrt{2}}$$

$$\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}}$$

$$\rightarrow \theta = 135°$$

$$z = x + y\,i$$

$$= \sqrt{2}(\cos 135° + i\ \sin 135°)$$

$$= \sqrt{2}\ cis 135°$$

In radians: $z = \sqrt{2}\ cis\left(\frac{3\pi}{4}\right)$

Example

Write $z = 2\ cis 60°$ in rectangular form.

Solution

$$z = 2\ cis 60°$$

$$= 2(\cos 60° + i\ \sin 60°)$$

$$= 2\left(\frac{1}{2} + i\ \frac{\sqrt{3}}{2}\right)$$

$$= 1 + i\ \sqrt{3}$$

Example

Express $2(\cos 300° + i \sin 300°)$ in rectangular form.

Solution

$$2(\cos 300° + i \sin 300°) = 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$= 1 - i\sqrt{3}$$

Example

Find the modulus of each of the complex numbers $5i$, 7, and $3 + 4i$

Solution

For $z = 5i$

$= 0 + 5i$

$r = |z|$

$= \sqrt{0^2 + 5^2}$

$\underline{= 5}$

For $z = 7$

$= 7 + 0i$

$r = |z|$

$= \sqrt{7^2 + 0^2}$

$\underline{= 7}$

For $3 + 4i$

$\Rightarrow r = \sqrt{3^2 + 4^2}$

$\underline{= 5}$

Product Theorem

If $r_1\left(\cos\theta_1 + i\sin\theta_1\right)$ and $r_2\left(\cos\theta_2 + i\sin\theta_2\right)$ are any two complex numbers, then

$$\left[r_1\left(\cos\theta_1 + i\sin\theta_1\right)\right]\left[r_2\left(\cos\theta_2 + i\sin\theta_2\right)\right] = r_1 r_2\left[\cos\left(\theta_1+\theta_2\right) + i\sin\left(\theta_1+\theta_2\right)\right]$$

$$\left(r_1 cis\theta_1\right)\left(r_2 cis\theta_2\right) = r_1 r_2 cis\left(\theta_1+\theta_2\right)$$

$$\boxed{(a+bi)(a-bi) = a^2+b^2}$$

$$\boxed{\left(\sqrt{a}+\sqrt{b}i\right)\left(\sqrt{a}-\sqrt{b}i\right) = a+b}$$

Example

Find the product of $3(\cos 45° + i\sin 45°)$ and $2(\cos 135° + i\sin 135°)$. Write the result in rectangular form.

Solution

$$\left[3(\cos 45° + i\sin 45°)\right]\left[2(\cos 135° + i\sin 135°)\right]$$

$$= (3)(2)\left[\cos(45°+135°) + i\sin(45°+135°)\right]$$

$$= 6(\cos 180° + i\sin 180°)$$

$$= 6(-1 + i.0)$$

$$= -6$$

Quotient Theorem

If $r_1\left(\cos\theta_1 + i\sin\theta_1\right)$ and $r_2\left(\cos\theta_2 + i\sin\theta_2\right)$ are any two complex numbers, then

$$\frac{r_1\left(\cos\theta_1 + i\sin\theta_1\right)}{r_2\left(\cos\theta_2 + i\sin\theta_2\right)} = \frac{r_1}{r_2}\left[\cos\left(\theta_1 - \theta_2\right) + i\sin\left(\theta_1 - \theta_2\right)\right]$$

$$\frac{r_1 cis\theta_1}{r_2 cis\theta_2} = \frac{r_1}{r_2} cis\left(\theta_1 - \theta_2\right)$$

Example

Find the quotient $\frac{10cis\left(-60^\circ\right)}{5cis\left(150^\circ\right)}$. Write the result in rectangular form.

Solution

$$\begin{aligned}\frac{10cis\left(-60^\circ\right)}{5cis\left(150^\circ\right)} &= \frac{10}{5} cis\left(-60^\circ - 150^\circ\right)\\ &= 2cis\left(-210^\circ\right)\\ &= 2\left[\cos\left(-210^\circ\right) + i\sin\left(-210^\circ\right)\right]\\ &= 2\left[-\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right]\\ &= -\sqrt{3} + i\end{aligned}$$

De Moivre's *Theorem*

If $r(\cos\theta + i\sin\theta)$ is a complex number, then

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta)$$

$$\boxed{(r\,cis\theta)^n = r^n(cis\,n\theta)}$$

Example

Find $\left(1+i\sqrt{3}\right)^8$ and express the result in rectangular form.

Solution

$$1+i\sqrt{3} \Rightarrow \begin{cases} x=1 \\ y=\sqrt{3} \end{cases}$$

$$r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$$

$$\tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3}$$

θ is in QI, that implies: $\theta = 60°$

$$1+i\sqrt{3} = 2cis60°$$

Apply De Moivre's theorem:

$$\begin{aligned}
\left(1+i\sqrt{3}\right)^8 &= (2cis60°)^8 \\
&= 2^8\left[cis(8.60°)\right] \\
&= 256\left[cis(480°)\right] \qquad 480° - 360° = 120° \\
&= 256\left[cis(120°)\right] \\
&= 256\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\
&= -128 + 128i\sqrt{3}
\end{aligned}$$

n^{th} Root Theorem

For a positive integer n, the complex number $a + bi$ is an $\boldsymbol{n^{th}}$ **root** of the complex number $x + iy$ if

$$(a+bi)^n = x+yi$$

If n is any positive integer, r is a positive real number, and θ is in degrees, then the nonzero complex number $r(\cos\theta + i\sin\theta)$ has exactly n distinct nth roots, given by

$$\sqrt[n]{r}\,(\cos\alpha + i\sin\alpha) \quad \textbf{or} \quad \sqrt[n]{r}\ cis\alpha$$

Where $\alpha = \dfrac{\theta + 360°k}{n},\ \ k = 0, 1, 2, \cdots, n-1$ $\quad \boxed{\alpha = \dfrac{\theta}{n} + \dfrac{360°k}{n}}$

$$\alpha = \frac{\theta + 2\pi k}{n},\ \ k = 0, 1, 2, \cdots, n-1 \qquad \boxed{\alpha = \frac{\theta}{n} + \frac{2\pi k}{n}}$$

Example

Find the two square root of $\boldsymbol{4i}$. Write the roots in rectangular form.

Solution

$$4i \ \rightarrow \ \begin{cases} x = 0 \\ y = 4 \end{cases}$$

$$r = \sqrt{0^2 + 4^2}$$
$$= 4$$

$$\tan\theta = \frac{4}{0} = \infty$$

$$\theta = \frac{\pi}{2}$$

$$4i = 4cis\frac{\pi}{2}$$

The absolute value: $\sqrt{4} = 2$

Argument: $\alpha = \dfrac{\frac{\pi}{2} + 2\pi k}{2}$

$$= \frac{\frac{\pi}{2}}{2} + \frac{2\pi k}{2}$$

$$= \frac{\pi}{4} + \pi k$$

Since there are ***two*** square root, then $k = 0$ and 1.

If $k = 0$

$$\Rightarrow \quad \alpha = \frac{\pi}{4} + \pi(0) \; \underline{= \frac{\pi}{4}}\Big|$$

$If \;\; k = 1$

$$\Rightarrow \quad \alpha = \frac{\pi}{4} + \pi(1) \; \underline{= \frac{5\pi}{4}}\Big|$$

The square roots are: $2cis\frac{\pi}{4} \;\; and \;\; 2cis\frac{5\pi}{4}$

$$2cis\frac{\pi}{4} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

$$\underline{= \sqrt{2} + i\sqrt{2}}\;\Big|$$

$$2cis\frac{5\pi}{4} = 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$

$$= 2\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$\underline{= -\sqrt{2} - i\sqrt{2}}\;\Big|$$

Example

Find all fourth roots of $-8+8i\sqrt{3}$. Write the roots in rectangular form.

Solution

$$-8+8i\sqrt{3} \quad \Rightarrow \quad \begin{cases} x=-8 \\ y=8\sqrt{3} \end{cases}$$

$$r=\sqrt{(-8)^2+\left(8\sqrt{3}\right)^2}$$
$$\underline{=16|}$$

$$\tan\theta = \frac{8\sqrt{3}}{-8}$$
$$=-\sqrt{3}$$

$$\underline{\theta = 120°|}$$

$$-8+8i\sqrt{3} = 16cis120°$$

The fourth roots have absolute value: $\sqrt[4]{16}=2$

$$\alpha = \frac{120°}{4}+\frac{360°k}{4}$$
$$\underline{=30°+90°k|}$$

Since there are ***four*** roots, then $k = 0$, 1, 2, and 3.

$If\ \ k=0 \Rightarrow \alpha = 30°+90°(0)=30°$

$If\ \ k=1 \Rightarrow \alpha = 30°+90°(1)=120°$

$If\ \ k=2 \Rightarrow \alpha = 30°+90°(2)=210°$

$If\ \ k=3 \Rightarrow \alpha = 30°+90°(3)=300°$

The fourth roots are: $2cis30°,\ \ 2cis120°,\ \ 2cis210°,\ and\ \ 2cis300°$

$$2cis30° = 2\left(\cos 30° + i\sin 30°\right)$$
$$=2\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)$$
$$\underline{=\sqrt{3}+i\ |}$$

$$2cis120° = 2\left(\cos 120° + i\sin 120°\right)$$
$$=2\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$
$$\underline{=-1+i\sqrt{3}\ |}$$

$$2cis210^\circ = 2(\cos 210^\circ + i\sin 210^\circ)$$

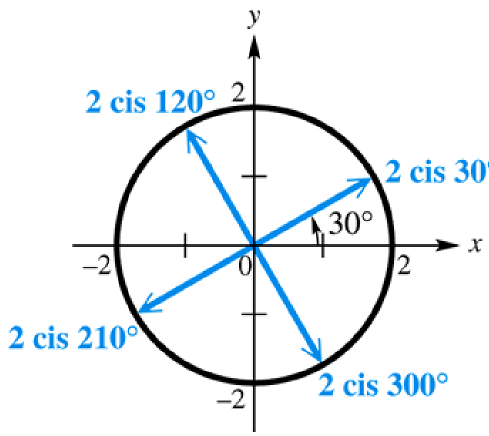
$$= 2\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$$

$$\underline{= -\sqrt{3} - i}$$

$$2cis300^\circ = 2(\cos 300^\circ + i\sin 300^\circ)$$

$$= 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$\underline{= 1 - i\sqrt{3}}$$



Example

Find all complex number solutions of $x^5 - 1 = 0$. Graph them as vectors in the complex plane.

Solution

$x^5 - 1 = 0 \Rightarrow x^5 = 1$

There is one real solution, 1, while there are five complex solutions.

$1 = 1 + 0i$

$$r = \sqrt{1^2 + 0^2}$$
$$= 1$$

$$\tan\theta = \frac{0}{1} = 0$$

$\theta = 0°$

$1 = 1cis0°$

The fifth roots have absolute value: $\sqrt[5]{1} = 1$

$$\alpha = \frac{0°}{5} + \frac{360°k}{5}$$
$$= 0° + 72°k$$
$$= 72°k$$

Since there are ***fifth*** roots, then k = 0, 1, 2, 3, and 4.

$If\ \ k = 0 \Rightarrow \alpha = 72°(0) = 0°$

$If\ \ k = 1 \Rightarrow \alpha = 72°(0) = 72°$

$If\ \ k = 2 \Rightarrow \alpha = 72°(2) = 144°$

$If\ \ k = 3 \Rightarrow \alpha = 72°(3) = 216°$

$If\ \ k = 4 \Rightarrow \alpha = 72°(4) = 288°$

Solution: $cis0°,\ \ cis72°,\ \ cis144°,\ cis216°,\ and\ \ cis288°$

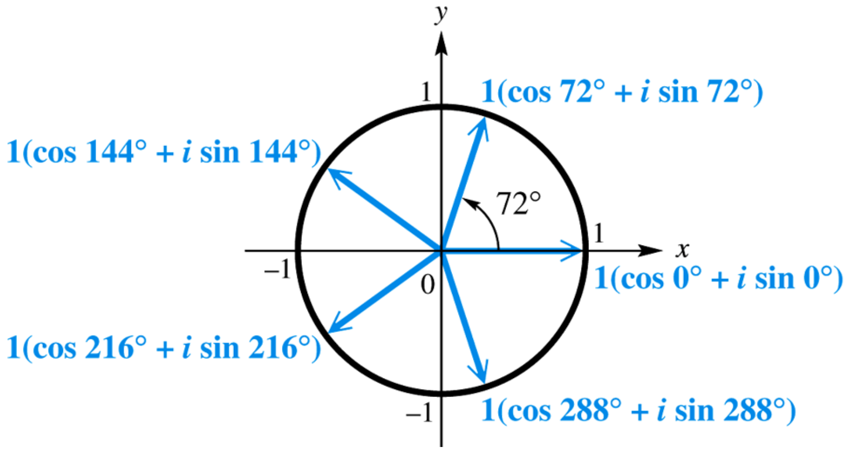
$cis0° = 1$

$cis72° = \cos 72° + i\sin 72°$

$cis144° = \cos 144° + i\sin 144°$

$cis216° = \cos 216° + i\sin 216°$

$cis288° = \cos 288° + i\sin 288°$



The graphs of the roots lie on a unit circle. The roots are equally spaced about the circle, 72° apart.

Exercises *Section* 3.7 – Trigonometric Form

(**1 – 8**) Write complex form in trigonometric form

1. $-\sqrt{3}+i$
2. $3-4i$
3. $-21-20i$
4. $11+2i$
5. $\sqrt{3}-i$
6. $1-\sqrt{3}i$
7. $9\sqrt{3}+9i$
8. $-2+3i$

(**9 – 13**) Write in standard form

9. $4\left(\cos 30^\circ + i\sin 30^\circ\right)$
10. $\sqrt{2}\ cis\frac{7\pi}{4}$
11. $3cis210^\circ$
12. $4\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$
13. $4cis\frac{\pi}{2}$

14. Find the quotient $\dfrac{20cis\left(75^\circ\right)}{4cis\left(40^\circ\right)}$. Write the result in rectangular form.

15. Divide $z_1 = 1+i\sqrt{3}$ *by* $z_2 = \sqrt{3}+i$. Write the result in rectangular form.

(**16 – 25**) Find and express the result in rectangular form

16. $(1+i)^8$
17. $(1+i)^{10}$
18. $(1-i)^5$
19. $\left(1-\sqrt{5}i\right)^8$
20. $(3cis80^\circ)^3$
21. $\left(\sqrt{3}cis10^\circ\right)^6$
22. $\left(\sqrt{2}-i\right)^6$
23. $(4cis40^\circ)^6$
24. $(2cis30^\circ)^5$
25. $\left(\frac{1}{2}cis72^\circ\right)^5$

26. Find fifth complex roots of $z=1+i\sqrt{3}$ and express the result in rectangular form.

(**27 – 30**) Find the fourth roots of

27. $z=16cis60^\circ$
28. $\sqrt{3}-i$
29. $4-4\sqrt{3}i$
30. $-16i$

(**31 – 33**) Find the cube roots of

31. 27
32. $8-8i$
33. -8

34. Find all complex number solutions of $x^3+1=0$.