Ex $A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$

Eigen value

$$|A - \lambda I| = \begin{vmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{vmatrix}$$

$$= -20 - \lambda + \lambda^2 + 18$$

$$= \lambda^2 - \lambda - 2 = 0$$

eigenvalues, $\lambda_{1,2} = -1, 2$

For $\lambda_1 = -1 \Rightarrow (A - \lambda_1 I) V_1 = 0$

$$\begin{pmatrix} -3 & 6 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow -3x_1 + 6y_1 = 0$$

$$3x_1 = 6y_1 \Rightarrow x_1 = 2y_1$$

$$V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \Rightarrow (A - \lambda_2 I) V_2 = 0$

$$\begin{pmatrix} -6 & 6 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-3x_2 = -3y_2 \Rightarrow x_2 = y_2$$

$$\Rightarrow V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x(t) = V_1 e^{\lambda_1 t} + V_2 e^{\lambda_2 t}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

$x_1(t) = 2e^{-t} + e^{2t}$

$x_2(t) = e^{-t} + e^{2t}$

Ex

$\xleftarrow{1\ \text{gal/min}}$

$x_1(t)$ $\qquad$ $x_2(t)$

$\xrightarrow{1\ \text{gal/min}}$

$x_1(0) = 100$ $\qquad$ $x_2(0) = 0$

$V_1 = V_2 = 500$

$$R_{1in} = 1 \frac{\text{gal}}{\text{min}} \cdot \frac{x_2(t)}{500} = \frac{x_2}{500}$$

$$R_{1out} = 1 \cdot \frac{x_1}{500} = \frac{x_1}{500}$$

$$x_1'(t) = \frac{x_2}{500} - \frac{x_1}{500} \qquad x_1' = -\frac{1}{500} x_1 + \frac{1}{500} x_2$$

$$R_{2in} = \frac{x_1}{500} \qquad R_{2out} = \frac{x_2}{500}$$

$$x_2' = \frac{x_1}{500} - \frac{x_2}{500}$$

$$\begin{cases} x_1' = -\frac{1}{500} x_1 + \frac{1}{500} x_2 \\ x_2' = \frac{1}{500} x_1 - \frac{1}{500} x_2 \end{cases} \qquad A = \begin{pmatrix} -\frac{1}{500} & \frac{1}{500} \\ \frac{1}{500} & -\frac{1}{500} \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\frac{1}{500} - \lambda & \frac{1}{500} \\ \frac{1}{500} & -\frac{1}{500} - \lambda \end{vmatrix}$$

$$= \lambda^2 + \frac{1}{250} \lambda = 0$$

Eigenvalues: $\lambda_1 = -\frac{1}{250}$ $\quad$ $\lambda_2 = 0$

For $\lambda_1 = -\frac{1}{250} \Rightarrow (A - \lambda_1 I) V_1 = 0$

$$\begin{pmatrix} \frac{1}{500} & \frac{1}{500} \\ \frac{1}{500} & \frac{1}{500} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow x_1 = -y_1$$

$$\frac{1}{500} x_1 + \frac{1}{500} y_1 = 0 \qquad V_1' = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 0 \Rightarrow (A - \lambda_2 I) V_2 = 0$

$$\begin{pmatrix} -\frac{1}{500} & \frac{1}{500} \\ \frac{1}{500} & -\frac{1}{500} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow x_2 = y_2$$

$$V_2' = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$X(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t/250} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} -c_1 e^{-t/250} + c_2 \\ c_1 e^{-t/250} + c_2 \end{pmatrix}$$

$$\begin{pmatrix} 100 \\ 0 \end{pmatrix} = \begin{pmatrix} -c_1 + c_2 \\ c_1 + c_2 \end{pmatrix} \qquad t = 0$$

$$\begin{cases} -c_1 + c_2 = 100 \\ c_1 + c_2 = 0 \end{cases} \Rightarrow \begin{array}{l} c_2 = 50 \\ c_1 = -50 \end{array}$$

$$\boxed{X(t) = \begin{pmatrix} 50 e^{-t/250} + 50 \\ -50 e^{-t/250} + 50 \end{pmatrix}}$$

12 a) 1.

$$\begin{cases} x_1'(t) = 5x_1 + x_2 \\ x_2' = -2x_1 + 3x_2 \end{cases}$$

$$A = \begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix}$$

$$= \lambda^2 - 8\lambda + 17 = 0$$

$$\lambda = \frac{8}{2} \pm \frac{1}{2}\sqrt{-4}$$

$$\lambda_{1,2} = 4 \pm i$$

For $\lambda_1 = 4 + i$ $(A - \lambda_1 I) v_1 = 0$

$$\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(1-i)x_1 = -x_2 \qquad v_1 = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}$$

$$(i-1)x_1 = x_2 \qquad v_2 = \begin{pmatrix} 1 \\ -1-i \end{pmatrix}$$

$$x = \begin{pmatrix} 1 \\ -1+i \end{pmatrix} e^{(4+i)t} \qquad e^{(a+bi)t} = e^{a} e^{(bt)i}$$

$$= \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \underbrace{(\cos t + i \sin t)}_{e^{it}} e^{4t}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$= \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] (\cos t + i \sin t)\, e^{4t}$$

$$= \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t + i \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t \right) \right] e^{4t}$$

$$= \left[\begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ -\sin t + \cos t \end{pmatrix} \right] e^{4t}$$

$$X(t) = C_1 \begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} e^{4t} + C_2 \begin{pmatrix} \sin t \\ -\sin t + \cos t \end{pmatrix} e^{4t}$$

$$x_1(t) = (C_1 \cos t + C_2 \sin t)\, e^{4t}$$

$$x_2(t) = (-C_1 \cos t - C_1 \sin t)\, e^{4t} + C_2 (\cos t - \sin t)\, e^{4t}$$

---

1 eigenvalue (mult)

$$\begin{cases} x_1(t) = C_1 V_1 e^{\lambda t} \\ x_2(t) = C_2 (V_1 t + V_2)\, e^{\lambda t} \end{cases}$$

Ex $A = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & -1 \\ 1 & -3-\lambda \end{vmatrix}$$

$$= \lambda^2 + 4\lambda + 4$$

$$= (\lambda + 2)^2 = 0$$

$$\lambda_{1,2} = -2$$

For $\lambda = -2 \Rightarrow (A - \lambda_2 I)^2 V_2 = 0$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$(A + 2I) V_2 = V_1$$

$$\angle V_1 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x(t) = C_1 V_1 e^{\lambda t} + C_2 (V_1 t + V_2) e^{\lambda t}$$

$$= C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + C_2 \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) e^{-2t}$$

$$= C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} t+1 \\ t \end{pmatrix} e^{-2t}$$

$$\begin{cases} x_1' = 4x_1 + 5x_2 \\ x_2' = -2x_1 + 6x_2 \end{cases}$$

$$A = \begin{pmatrix} 4 & 5 \\ -2 & 6 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 5 \\ -2 & 6-\lambda \end{vmatrix}$$

$$= \lambda^2 - 10\lambda + 34 = 0$$

$$\lambda_{1,2} = +5 \pm \frac{1}{2}\sqrt{100 - 136}$$

$$= 5 \pm 3i$$

For $\lambda_1 = 5 + 3i \Rightarrow (A - \lambda_1 I) V_1 = 0$

$$\begin{pmatrix} -1-3i & 5 \\ -2 & 1+3i \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-(1+3i)x_1 + 5y_1 = 0$$

$$(1+3i)x_1 = 5y_1 \Rightarrow V_1 = \begin{pmatrix} 5 \\ 1+3i \end{pmatrix}$$

$$Z(t) = \left[\begin{pmatrix} 5 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 3 \end{pmatrix}\right] e^{(5+3i)t}$$

$$= \left[\begin{pmatrix} 5 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 3 \end{pmatrix}\right] \overbrace{(\cos 3t + i \sin 3t)}^{e^{3it}} e^{5t}$$

$$= e^{5t}\left[\left(\begin{pmatrix}5\\1\end{pmatrix}\cos 3t - \begin{pmatrix}0\\3\end{pmatrix}\sin 3t\right) + i\left(\begin{pmatrix}0\\3\end{pmatrix}\cos 3t + \begin{pmatrix}5\\1\end{pmatrix}\sin 3t\right)\right]$$

$$x(t) = c_1\begin{pmatrix}5\cos 3t\\ \cos 3t - 3\sin 3t\end{pmatrix}e^{5t} + c_2\begin{pmatrix}5\sin 3t\\ 3\cos 3t + \sin 3t\end{pmatrix}e^{5t}$$

$$\begin{cases} x_1'(t) = -6x_1 + 5x_2 \\ x_2'(t) = -5x_1 + 4x_2 \end{cases}$$

$$A = \begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -6-\lambda & 5 \\ -5 & 4-\lambda \end{vmatrix}$$

$$= \lambda^2 + 2\lambda + 1 = 0$$

$$\underline{\lambda_{1,2} = -1} \text{ eigenvalues}$$

For $\lambda = -1 \Rightarrow (A - \lambda I)^2 V_2 = 0$

$$\begin{pmatrix} -5 & 5 \\ -5 & 5 \end{pmatrix}\begin{pmatrix} -5 & 5 \\ -5 & 5 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \underline{V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$V_1 = (A - \lambda I)V_2$$

$$V_1 = \begin{pmatrix} -5 & 5 \\ -5 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ -5 \end{pmatrix}$$

$$x(t) = C_1 \begin{pmatrix} -5 \\ -5 \end{pmatrix} e^{-t} + C_2\left(\begin{pmatrix} -5 \\ -5 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) e^{-t}$$

$$= \underbrace{-5C_1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -5t+1 \\ -5t \end{pmatrix} e^{-t}$$

$$= C_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -5t+1 \\ -5t \end{pmatrix} e^{-t}$$

$$C_4 \begin{pmatrix} t - \frac{1}{5} \\ t \end{pmatrix} e^{-t}$$

For $\lambda = -1 \Rightarrow (A - \lambda I) V_1 = 0$

$$\begin{pmatrix} -5 & 5 \\ -5 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow x_1 = y_1$$

$$V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A V_2 = V_1$$

$$\begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \underline{1^{st}}$$

$$-6x_2 + 5y_2 = 1 \qquad x_2 = 0 \Rightarrow y_2 = \frac{1}{5}$$

$$\cancel{-5x_2 + 4y_2} = 1 \rightarrow y_2 = 0 \Rightarrow x_2 = -4/5$$

$$V_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x_2 = -1 \qquad y_2 = -1$$

$$V_2 = \begin{pmatrix} 0 \\ 1/5 \end{pmatrix}$$

$$x(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1/5 \end{pmatrix} \right) e^{-t}$$

$$= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} t \\ t + \frac{1}{5} \end{pmatrix} e^{-t}$$