

## ***Solution***      **Section 1.4 – Limits at Infinity**

### ***Exercise***

Find the limit as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$  of  $h(x) = \frac{-5 + \frac{7}{x}}{3 - \frac{1}{x^2}}$

### **Solution**

$$\lim_{x \rightarrow \infty} \frac{-5 + \frac{7}{x}}{3 - \frac{1}{x^2}} = \underline{-\frac{5}{3}}$$

$$\lim_{x \rightarrow -\infty} \frac{-5 + \frac{7}{x}}{3 - \frac{1}{x^2}} = \underline{-\frac{5}{3}}$$

### ***Exercise***

Find the limit as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$  of  $f(x) = \frac{2x+3}{5x+7}$

### **Solution**

$$\lim_{x \rightarrow \infty} \frac{2x+3}{5x+7} = \lim_{x \rightarrow \infty} \frac{2 + \frac{3}{x}}{5 + \frac{7}{x}} = \underline{\frac{2}{5}}$$

$$\lim_{x \rightarrow -\infty} \frac{2x+3}{5x+7} = \lim_{x \rightarrow -\infty} \frac{2 + \frac{3}{x}}{5 + \frac{7}{x}} = \underline{\frac{2}{5}}$$

### ***Exercise***

Find the limit as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$  of  $f(x) = \frac{2x^3+7}{x^3-x^2+x+7}$

### **Solution**

$$\lim_{x \rightarrow \infty} \frac{2x^3+7}{x^3-x^2+x+7} = \lim_{x \rightarrow \infty} \frac{2 + \frac{7}{x^3}}{1 - \frac{1}{x} + \frac{1}{x^2} + \frac{7}{x^3}}$$

$$= 2 \mid$$

$$\lim_{x \rightarrow -\infty} \frac{2x^3 + 7}{x^3 - x^2 + x + 7} = \lim_{x \rightarrow -\infty} \frac{2 + \frac{7}{x^3}}{1 - \frac{1}{x} + \frac{1}{x^2} + \frac{7}{x^3}}$$

$$= 2 \mid$$

### Exercise

Find the limit as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$  of  $f(x) = \frac{x+1}{x^2+3}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{x+1}{x^2+3} = \lim_{x \rightarrow \infty} \frac{\frac{x}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{3}{x^2}}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{1 + \frac{3}{x^2}}$$

$$= 0 \mid$$

$$\lim_{x \rightarrow -\infty} \frac{x+1}{x^2+3} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{1 + \frac{3}{x^2}}$$

$$= 0 \mid$$

### Exercise

Find the limit as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$  of  $f(x) = \frac{7x^3}{x^3 - 3x^2 + 6x}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{7x^3}{x^3 - 3x^2 + 6x} = \lim_{x \rightarrow \infty} \frac{7}{1 - \frac{3}{x} + \frac{6}{x^2}}$$

$$= 7 \mid$$

$$\lim_{x \rightarrow -\infty} \frac{7x^3}{x^3 - 3x^2 + 6x} = \lim_{x \rightarrow -\infty} \frac{7}{1 - \frac{3}{x} + \frac{6}{x^2}}$$

$$= 7 \mid$$

### Exercise

Find the limit as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$  of  $f(x) = \frac{9x^4 + x}{2x^4 + 5x^2 - x + 6}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{9x^4 + x}{2x^4 + 5x^2 - x + 6} &= \lim_{x \rightarrow \infty} \frac{\frac{9x^4}{x^4} + \frac{x}{x^4}}{\frac{2x^4}{x^4} + \frac{5x^2}{x^4} - \frac{x}{x^4} + \frac{6}{x^4}} \\ &= \lim_{x \rightarrow \infty} \frac{9 + \frac{1}{x^3}}{2 + \frac{5}{x^2} - \frac{1}{x^3} + \frac{6}{x^4}} \\ &= \frac{9}{2}\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{9x^4 + x}{2x^4 + 5x^2 - x + 6} &= \lim_{x \rightarrow -\infty} \frac{9 + \frac{1}{x^3}}{2 + \frac{5}{x^2} - \frac{1}{x^3} + \frac{6}{x^4}} \\ &= \frac{9}{2}\end{aligned}$$

### Exercise

Find the limit as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$  of  $f(x) = \frac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x} &= \lim_{x \rightarrow \infty} \frac{-2 - \frac{2}{x^2} + \frac{3}{x^3}}{3 + \frac{3}{x} - \frac{5}{x^2}} \\ &= -\frac{2}{3}\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x} &= \lim_{x \rightarrow -\infty} \frac{-2 - \frac{2}{x^2} + \frac{3}{x^3}}{3 + \frac{3}{x} - \frac{5}{x^2}} \\ &= -\frac{2}{3}\end{aligned}$$

### ***Exercise***

Find  $\lim_{x \rightarrow \infty} x^{12}$

#### **Solution**

$$\lim_{x \rightarrow \infty} x^{12} = \underline{\infty}$$

### ***Exercise***

Find  $\lim_{x \rightarrow -\infty} 3x^9$

#### **Solution**

$$\lim_{x \rightarrow -\infty} 3x^9 = \underline{-\infty}$$

### ***Exercise***

Find  $\lim_{x \rightarrow -\infty} x^{-8}$

#### **Solution**

$$\lim_{x \rightarrow -\infty} x^{-8} = \frac{1}{(-\infty)^8} = \underline{0}$$

### ***Exercise***

Find  $\lim_{x \rightarrow -\infty} x^{-9}$

#### **Solution**

$$\lim_{x \rightarrow -\infty} x^{-9} = \frac{1}{(-\infty)^9} = \underline{0}$$

### ***Exercise***

Find  $\lim_{x \rightarrow -\infty} 2x^{-6}$

#### **Solution**

$$\lim_{x \rightarrow -\infty} 2x^{-6} = \frac{2}{\infty} = \underline{0}$$

### ***Exercise***

Find  $\lim_{x \rightarrow \infty} (3x^{12} - 9x^7)$

### **Solution**

$$\lim_{x \rightarrow \infty} (3x^{12} - 9x^7) = \underline{\infty}$$

### ***Exercise***

Find  $\lim_{x \rightarrow -\infty} (3x^7 + x^2)$

### **Solution**

$$\lim_{x \rightarrow -\infty} (3x^7 + x^2) = \lim_{x \rightarrow -\infty} x^2(3x^5 + 1) = \underline{-\infty}$$

### ***Exercise***

Find  $\lim_{x \rightarrow -\infty} (-2x^{16} + 2)$

### **Solution**

$$\lim_{x \rightarrow -\infty} (-2x^{16} + 2) = \underline{-\infty}$$

### ***Exercise***

Find  $\lim_{x \rightarrow -\infty} (2x^{-6} + 4x^5)$

### **Solution**

$$\lim_{x \rightarrow -\infty} (2x^{-6} + 4x^5) = \lim_{x \rightarrow -\infty} x^{-6}(2 + 4x^{11}) = \underline{-\infty} \quad +\infty(-\infty)$$

### ***Exercise***

Find  $\lim_{x \rightarrow -\infty} \frac{\cos x}{3x}$

### **Solution**

$$-\frac{1}{3x} \leq \frac{\cos x}{3x} \leq \frac{1}{3x}, \quad -1 \leq \cos x \leq 1$$

$$\lim_{x \rightarrow -\infty} \frac{\cos x}{3x} = \underline{0} \quad \text{By the Sandwich Theorem}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{x + \sin x}{2x + 7 - 5 \sin x}$

#### Solution

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x + \sin x}{2x + 7 - 5 \sin x} &= \lim_{x \rightarrow \infty} \frac{1 + \frac{\sin x}{x}}{2 + \frac{7}{x} - \frac{5 \sin x}{x}} \\ &= \frac{1 + 0}{2 + 0 - 0} \\ &= \underline{\frac{1}{2}} \end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \sqrt{\frac{8x^2 - 3}{2x^2 + x}}$

#### Solution

$$\begin{aligned} \lim_{x \rightarrow \infty} \sqrt{\frac{8x^2 - 3}{2x^2 + x}} &= \lim_{x \rightarrow \infty} \sqrt{\frac{8 - \frac{3}{x^2}}{2 + \frac{1}{x}}} \\ &= \sqrt{\frac{8}{2}} \\ &= \underline{2} \end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} \left( \frac{x^2 + x - 1}{8x^2 - 3} \right)^{1/3}$

#### Solution

$$\lim_{x \rightarrow -\infty} \left( \frac{x^2 + x - 1}{8x^2 - 3} \right)^{1/3} = \lim_{x \rightarrow -\infty} \left( \frac{1 + \frac{1}{x} - \frac{1}{x^2}}{8 - \frac{3}{x^2}} \right)^{1/3}$$

$$= \left(\frac{1}{8}\right)^{1/3}$$

$$= \frac{1}{2}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7} = \lim_{x \rightarrow \infty} \frac{\frac{2\sqrt{x}}{x} + \frac{x^{-1}}{x}}{3 - \frac{7}{x}}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{2}{x^{1/2}} + \frac{1}{x^2}}{3 - \frac{7}{x}}$$

$$= 0$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{x^{-1} + x^{-4}}{x^{-2} + x^{-3}}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{x^{-1} + x^{-4}}{x^{-2} + x^{-3}} = \lim_{x \rightarrow \infty} \frac{\frac{x^{-1}}{x^{-2}} + \frac{x^{-4}}{x^{-2}}}{\frac{x^{-2}}{x^{-2}} + \frac{x^{-3}}{x^{-2}}}$$

$$= \lim_{x \rightarrow \infty} \frac{x + \frac{1}{x^2}}{1 + \frac{1}{x}}$$

$$= \infty$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6} + 9}$

### Solution

$$\begin{aligned}
\lim_{x \rightarrow -\infty} \frac{4-3x^3}{\sqrt{x^6+9}} &= \lim_{x \rightarrow -\infty} \frac{\frac{4-3x^3}{\sqrt{x^6}}}{\frac{\sqrt{x^6+9}}{\sqrt{x^6}}} \\
&= \lim_{x \rightarrow -\infty} \frac{\frac{4-3x^3}{x^3}}{\sqrt{\frac{x^6+9}{x^6}}} \\
&= \lim_{x \rightarrow -\infty} \frac{\frac{4}{x^3}-3}{\sqrt{1+\frac{9}{x^6}}} \\
&= \frac{-3}{\sqrt{1}} \\
&= \underline{-3}
\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} \left( \sqrt{x^2+3} + x \right)$

### Solution

$$\begin{aligned}
\lim_{x \rightarrow -\infty} \left( \sqrt{x^2+3} + x \right) &= \lim_{x \rightarrow -\infty} \left( \sqrt{x^2+3} + x \right) \frac{\sqrt{x^2+3}-x}{\sqrt{x^2+3}-x} \\
&= \lim_{x \rightarrow -\infty} \frac{x^2+3-x^2}{\sqrt{x^2+3}-x} \\
&= \lim_{x \rightarrow -\infty} \frac{3}{\sqrt{x^2+3}-x} \\
&= \lim_{x \rightarrow -\infty} \frac{\frac{3}{x}}{\sqrt{\frac{x^2}{x^2} + \frac{3}{x^2} - \frac{x}{x}}} \\
&= \lim_{x \rightarrow -\infty} \frac{\frac{3}{x}}{\sqrt{1 + \frac{3}{x^2} + 1}} \\
&= \frac{0}{\sqrt{1+1}} \\
&= \underline{0}
\end{aligned}$$



### Exercise

Find  $\lim_{x \rightarrow \infty} \left( \sqrt{x^2 + 3x} - \sqrt{x^2 - 2x} \right)$

### Solution

$$\begin{aligned} \lim_{x \rightarrow \infty} \left( \sqrt{x^2 + 3x} - \sqrt{x^2 - 2x} \right) &= \lim_{x \rightarrow \infty} \left( \sqrt{x^2 + 3x} - \sqrt{x^2 - 2x} \right) \frac{\sqrt{x^2 + 3x} + \sqrt{x^2 - 2x}}{\sqrt{x^2 + 3x} + \sqrt{x^2 - 2x}} \\ &= \lim_{x \rightarrow \infty} \frac{(x^2 + 3x) - (x^2 - 2x)}{\sqrt{x^2 + 3x} + \sqrt{x^2 - 2x}} \\ &= \lim_{x \rightarrow \infty} \frac{x^2 + 3x - x^2 + 2x}{\sqrt{x^2 + 3x} + \sqrt{x^2 - 2x}} \\ &= \lim_{x \rightarrow \infty} \frac{5x}{\sqrt{x^2 + 3x} + \sqrt{x^2 - 2x}} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{5x}{\sqrt{x^2}}}{\sqrt{\frac{x^2}{x^2} + \frac{3x}{x^2}} + \sqrt{\frac{x^2}{x^2} - \frac{2x}{x^2}}} \\ &= \lim_{x \rightarrow \infty} \frac{5}{\sqrt{1 + \frac{3}{x}} + \sqrt{1 - \frac{2}{x}}} \\ &= \frac{5}{\sqrt{1} + \sqrt{1}} \\ &= \frac{5}{2} \end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{2x - 3}{4x + 10}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{2x - 3}{4x + 10} = \frac{1}{2}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{x^4 - 1}{x^5 + 2}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{x^4 - 1}{x^5 + 2} = 0$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} (-3x^3 + 5)$

### Solution

$$\lim_{x \rightarrow -\infty} (-3x^3 + 5) = \infty$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \left( e^{-2x} + \frac{2}{x} \right)$

### Solution

$$\lim_{x \rightarrow \infty} \left( e^{-2x} + \frac{2}{x} \right) = e^{-\infty} + 0 = 0$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{1}{\ln x + 1}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{1}{\ln x + 1} = \frac{1}{\infty} = 0$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \left( 3 + \frac{10}{x^2} \right)$

### Solution

$$\lim_{x \rightarrow \infty} \left( 3 + \frac{10}{x^2} \right) = 3 + 0 = 3$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \left( 5 + \frac{1}{x} + \frac{10}{x^2} \right)$

### Solution

$$\lim_{x \rightarrow \infty} \left( 5 + \frac{1}{x} + \frac{10}{x^2} \right) = 5 + 0 + 0$$
$$\underline{= 5}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{4x^2 + 2x + 3}{x^2}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{4x^2 + 2x + 3}{x^2} = \lim_{x \rightarrow \infty} \frac{4x^2}{x^2}$$
$$\underline{= 4}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \left( 5 + \frac{100}{x} + \frac{\sin^4 x^3}{x^2} \right)$

### Solution

$$-1 \leq \sin \theta \leq 1$$

$$0 \leq \sin^4 \theta \leq 1$$

$$0 \leq \frac{\sin^4 \theta}{x^2} \leq \frac{1}{x^2} \rightarrow 0$$

$$\lim_{x \rightarrow \infty} \left( 5 + \frac{100}{x} + \frac{\sin^4 x^3}{x^2} \right) \underline{= 5}$$

### Exercise

Find  $\lim_{\theta \rightarrow \infty} \frac{\cos \theta}{\theta^2}$

### Solution

$$-1 \leq \cos \theta \leq 1$$

$$-\frac{1}{\theta^2} \leq \frac{\cos \theta}{\theta^2} \leq \frac{1}{\theta^2} \rightarrow 0$$

$$\lim_{\theta \rightarrow \infty} \frac{\cos \theta}{\theta^2} = \underline{0}$$

### Exercise

Find  $\lim_{\theta \rightarrow \infty} \frac{\cos \theta^5}{\sqrt{\theta}}$

### Solution

$$-1 \leq \cos \theta^5 \leq 1$$

$$-\frac{1}{\sqrt{\theta}} \leq \frac{\cos \theta^5}{\sqrt{\theta}} \leq \frac{1}{\sqrt{\theta}} \rightarrow 0$$

$$\lim_{\theta \rightarrow \infty} \frac{\cos \theta^5}{\sqrt{\theta}} = \underline{0}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{4x}{20x+1}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{4x}{20x+1} = \frac{4}{20} = \underline{\frac{1}{5}}$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} \frac{4x}{20x+1}$

### Solution

$$\lim_{x \rightarrow -\infty} \frac{4x}{20x+1} = \lim_{x \rightarrow -\infty} \frac{4x}{20x} = \underline{\frac{1}{5}}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{3x^2-7}{x^2+5x}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{3x^2 - 7}{x^2 + 5x} = 3$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} \frac{3x^2 - 7}{x^2 + 5x}$

### Solution

$$\lim_{x \rightarrow -\infty} \frac{3x^2 - 7}{x^2 + 5x} = \lim_{x \rightarrow -\infty} \frac{3x^2}{x^2} = 3$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{6x^2 - 9x + 8}{3x^2 + 2}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{6x^2 - 9x + 8}{3x^2 + 2} = \lim_{x \rightarrow \infty} \frac{6x^2}{3x^2} = \frac{6}{3} = 2$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} \frac{6x^2 - 9x + 8}{3x^2 + 2}$

### Solution

$$\lim_{x \rightarrow -\infty} \frac{6x^2 - 9x + 8}{3x^2 + 2} = \lim_{x \rightarrow -\infty} \frac{6x^2}{3x^2} = \frac{6}{3} = 2$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{4x^2 - 7}{8x^2 + 5x + 2}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{4x^2 - 7}{8x^2 + 5x + 2} &= \lim_{x \rightarrow \infty} \frac{4x^2}{8x^2} \\ &= \frac{4}{8} \\ &= \frac{1}{2}\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} \frac{4x^2 - 7}{8x^2 + 5x + 2}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{4x^2 - 7}{8x^2 + 5x + 2} &= \lim_{x \rightarrow -\infty} \frac{4x^2}{8x^2} \\ &= \frac{4}{8} \\ &= \frac{1}{2}\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{\sqrt{16x^4 + 64x^2} + x^2}{2x^2 - 4}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\sqrt{16x^4 + 64x^2} + x^2}{2x^2 - 4} &= \lim_{x \rightarrow \infty} \frac{\sqrt{16x^4} + x^2}{2x^2} \\ &= \lim_{x \rightarrow \infty} \frac{4x^2 + x^2}{2x^2} \\ &= \lim_{x \rightarrow \infty} \frac{5x^2}{2x^2} \\ &= \frac{5}{2}\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} \frac{\sqrt{16x^4 + 64x^2 + x^2}}{2x^2 - 4}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{\sqrt{16x^4 + 64x^2 + x^2}}{2x^2 - 4} &= \lim_{x \rightarrow -\infty} \frac{\sqrt{16x^4 + x^2}}{2x^2} \\ &= \lim_{x \rightarrow -\infty} \frac{4x^2 + x^2}{2x^2} \\ &= \lim_{x \rightarrow -\infty} \frac{5x^2}{2x^2} \\ &= \underline{\underline{\frac{5}{2}}}\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{3x^4 + 3x^3 - 36x^2}{x^4 - 25x^2 + 144}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{3x^4 + 3x^3 - 36x^2}{x^4 - 25x^2 + 144} &= \lim_{x \rightarrow \infty} \frac{3x^4}{x^4} \\ &= \underline{\underline{3}}\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} \frac{3x^4 + 3x^3 - 36x^2}{x^4 - 25x^2 + 144}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{3x^4 + 3x^3 - 36x^2}{x^4 - 25x^2 + 144} &= \lim_{x \rightarrow -\infty} \frac{3x^4}{x^4} \\ &= \underline{\underline{3}}\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} 16x^2 \left( 4x^2 - \sqrt{16x^4 + 1} \right)$

#### Solution

$$\begin{aligned}
\lim_{x \rightarrow \infty} 16x^2 \left( 4x^2 - \sqrt{16x^4 + 1} \right) &= \infty - \infty \\
&= \lim_{x \rightarrow \infty} 16x^2 \left( 4x^2 - \sqrt{16x^4 + 1} \right) \cdot \frac{4x^2 + \sqrt{16x^4 + 1}}{4x^2 + \sqrt{16x^4 + 1}} \\
&= \lim_{x \rightarrow \infty} 16x^2 \frac{16x^4 - 16x^4 - 1}{4x^2 + \sqrt{16x^4 + 1}} \\
&= \lim_{x \rightarrow \infty} 16x^2 \frac{-1}{4x^2 + \sqrt{16x^4 + 1}} \\
&= \lim_{x \rightarrow \infty} \frac{-16x^2}{4x^2 + 4x^2} \\
&= \lim_{x \rightarrow \infty} \frac{-16x^2}{8x^2} \\
&= \underline{-2}
\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} 16x^2 \left( 4x^2 - \sqrt{16x^4 + 1} \right)$

### Solution

$$\begin{aligned}
\lim_{x \rightarrow -\infty} 16x^2 \left( 4x^2 - \sqrt{16x^4 + 1} \right) &= \infty - \infty \\
&= \lim_{x \rightarrow -\infty} 16x^2 \left( 4x^2 - \sqrt{16x^4 + 1} \right) \cdot \frac{4x^2 + \sqrt{16x^4 + 1}}{4x^2 + \sqrt{16x^4 + 1}} \\
&= \lim_{x \rightarrow -\infty} 16x^2 \frac{16x^4 - 16x^4 - 1}{4x^2 + \sqrt{16x^4 + 1}} \\
&= \lim_{x \rightarrow -\infty} 16x^2 \frac{-1}{4x^2 + \sqrt{16x^4 + 1}} \\
&= \lim_{x \rightarrow -\infty} \frac{-16x^2}{4x^2 + 4x^2} \\
&= \lim_{x \rightarrow -\infty} \frac{-16x^2}{8x^2} \\
&= \underline{-2}
\end{aligned}$$



### ***Exercise***

Find  $\lim_{x \rightarrow \infty} \frac{x-1}{x^{2/3}-1}$

### **Solution**

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{x-1}{x^{2/3}-1} &= \lim_{x \rightarrow \infty} \frac{x}{x^{2/3}} \\ &= \lim_{x \rightarrow \infty} x^{1/3} \\ &= \infty \quad | \end{aligned}$$

### ***Exercise***

Find  $\lim_{x \rightarrow -\infty} \frac{x-1}{x^{2/3}-1}$

### **Solution**

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{x-1}{x^{2/3}-1} &= \lim_{x \rightarrow -\infty} \frac{x}{x^{2/3}} \\ &= \lim_{x \rightarrow -\infty} x^{1/3} \\ &= -\infty \quad | \end{aligned}$$

### ***Exercise***

Find  $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2+2x+6}-3}{x-1}$

### **Solution**

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\sqrt{x^2+2x+6}-3}{x-1} &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2}}{x} \\ &= \lim_{x \rightarrow \infty} \frac{x}{x} \\ &= 1 \quad | \end{aligned}$$

### ***Exercise***

Find  $\lim_{x \rightarrow \infty} \frac{|1-x^2|}{x(x+1)}$

**Solution**

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{|1 - x^2|}{x(x+1)} &= \lim_{x \rightarrow \infty} \frac{x^2 - 1}{x^2 + 1} \\ &= \lim_{x \rightarrow \infty} \frac{x^2}{x^2} \\ &= 1\end{aligned}$$

**Exercise**

Find  $\lim_{x \rightarrow \infty} (\sqrt{|x|} - \sqrt{|x-1|})$

**Solution**

$$\begin{aligned}\lim_{x \rightarrow \infty} (\sqrt{|x|} - \sqrt{|x-1|}) &= \infty - \infty & x \rightarrow \infty \Rightarrow |x| = x \quad \& \quad |x-1| = x-1 \\ &= \lim_{x \rightarrow \infty} (\sqrt{x} - \sqrt{x-1}) \cdot \frac{\sqrt{x} + \sqrt{x-1}}{\sqrt{x} + \sqrt{x-1}} \\ &= \lim_{x \rightarrow \infty} \frac{x - x + 1}{\sqrt{x} + \sqrt{x-1}} \\ &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x} + \sqrt{x-1}} \\ &= \frac{1}{\infty} \\ &= 0\end{aligned}$$

**Exercise**

Find  $\lim_{x \rightarrow \infty} \frac{\tan^{-1} x}{x}$

**Solution**

$$\begin{aligned}-\frac{\pi}{2} &\leq \tan^{-1} x \leq \frac{\pi}{2} \\ -\frac{\pi}{2x} &\leq \frac{\tan^{-1} x}{x} \leq \frac{\pi}{2x} \rightarrow 0 \\ \lim_{x \rightarrow \infty} \frac{\tan^{-1} x}{x} &= 0\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{\cos x}{e^{3x}}$

### Solution

$$-1 \leq \cos x \leq 1$$

$$-\frac{1}{e^{3x}} \leq \frac{\cos x}{e^{3x}} \leq \frac{1}{e^{3x}} \rightarrow 0$$

$$\lim_{x \rightarrow \infty} \frac{\cos x}{e^{3x}} = 0$$

### Exercise

Find  $\lim_{x \rightarrow 0} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}}$

### Solution

$$\lim_{x \rightarrow 0} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}} = \frac{2+10}{1+1} = 6$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}} = \lim_{x \rightarrow \infty} \frac{2e^x}{e^x} = 2$$

$$\lim_{x \rightarrow \infty} e^{-x} = 0$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}}$

### Solution

$$\lim_{x \rightarrow -\infty} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}} = \lim_{x \rightarrow -\infty} \frac{10e^{-x}}{e^{-x}} = 10$$

$$\lim_{x \rightarrow -\infty} e^x = 0$$

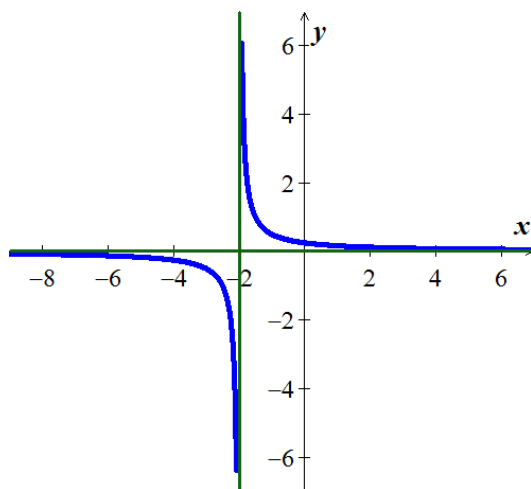
### Exercise

Graph the rational function  $y = \frac{1}{2x+4}$ . Include the equations of the asymptotes.

### Solution

$$VA: 2x + 4 = 0 \Rightarrow \underline{x = -2}$$

$$HA: \underline{y = 0}$$



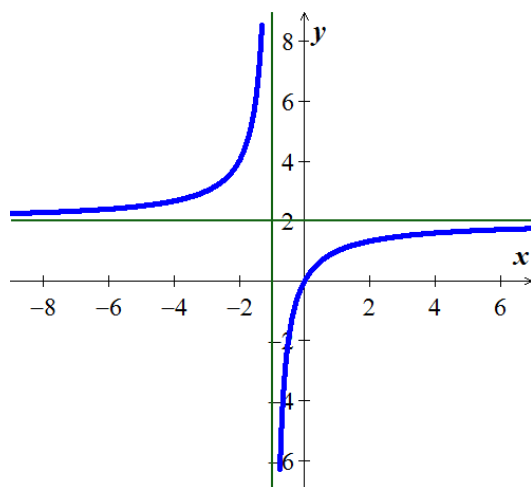
### Exercise

Graph the rational function  $y = \frac{2x}{x+1}$ . Include the equations of the asymptotes.

### Solution

$$VA: \underline{x = -1}$$

$$HA: \underline{y = 2}$$



### Exercise

Graph the rational function  $y = \frac{x^2}{x-1}$ . Include the equations of the asymptotes.

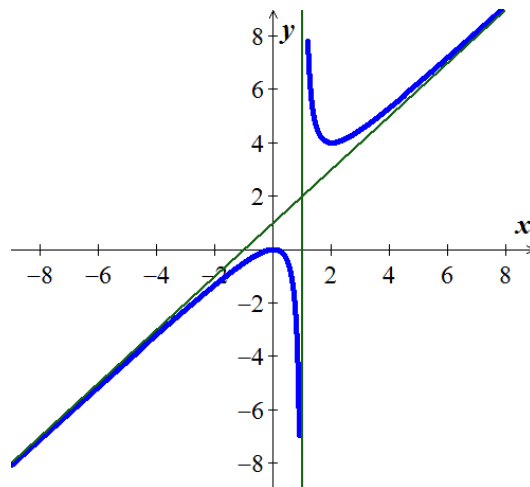
### Solution

$$\begin{array}{r} x+1 \\ x-1 \overline{) x^2} \\ \underline{x^2 - x} \phantom{+1} \\ x \phantom{+1} \\ \underline{x-1} \\ 1 \end{array}$$

$$\begin{aligned} y &= \frac{x^2}{x-1} \\ &= x+1 + \frac{1}{x-1} \end{aligned}$$

**VA:**  $x=1$

**Oblique Asymptote:**  $y = x+1$



### Exercise

Graph the rational function  $y = \frac{x^3+1}{x^2}$ . Include the equations of the asymptotes.

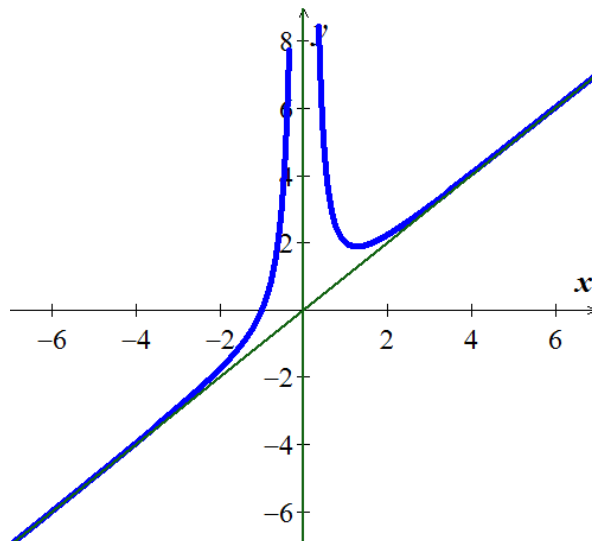
### Solution

$$\begin{array}{r} x \\ x^2 \overline{) x^3 + 1} \\ \underline{x^3} \phantom{+1} \\ 1 \end{array}$$

$$y = \frac{x^3+1}{x^2} = x + \frac{1}{x^2}$$

**VA:**  $x = 0$

**Oblique Asymptote:**  $y = x$



### Exercise

Let  $f(x) = \frac{x^2 - 5x + 6}{x^2 - 2x}$

a) Analyze  $\lim_{x \rightarrow 0^-} f(x)$ ,  $\lim_{x \rightarrow 0^+} f(x)$ ,  $\lim_{x \rightarrow 2^-} f(x)$ , and  $\lim_{x \rightarrow 2^+} f(x)$

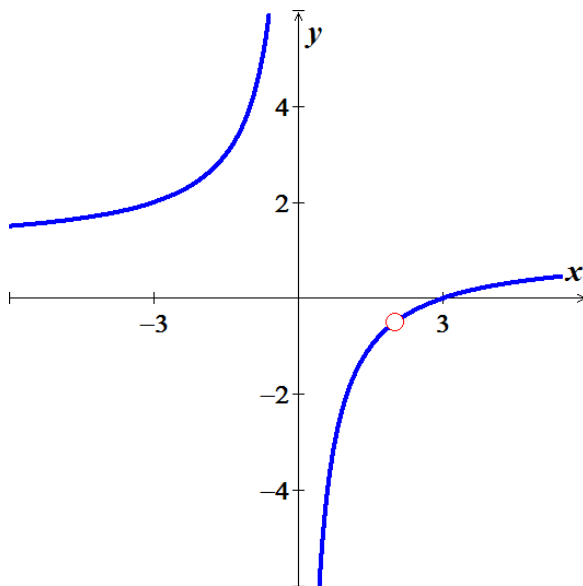
b) Does the graph of  $f$  have any vertical asymptotes? Explain?

### Solution

$$\begin{aligned} f(x) &= \frac{x^2 - 5x + 6}{x^2 - 2x} \\ &= \frac{(x-2)(x-3)}{x(x-2)} \\ &= \frac{x-3}{x} \end{aligned}$$

$$\begin{aligned} a) \quad \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} \frac{x-3}{x} \\ &= \frac{-3}{0^-} \\ &= \infty \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} \frac{x-3}{x} \\ &= \frac{-3}{0^+} \\ &= -\infty \end{aligned}$$



$$\underline{\underline{= -\infty}}$$

$$\begin{aligned}\lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} \frac{x-3}{x} \\ &= \frac{2-3}{2} \\ &= -\frac{1}{2}\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} \frac{x-3}{x} \\ &= \frac{2-3}{2} \\ &= -\frac{1}{2}\end{aligned}$$

$$b) \quad VA: x = 0 \quad \text{Hole: } x = 2 \rightarrow f(2) = -\frac{1}{2}$$

$$HA: y = 1 \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of  $y = \frac{3x}{1-x}$

#### Solution

$$VA: x = 1, \quad \text{Hole: } n/a, \quad HA: y = -3, \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of  $y = \frac{x^2}{x^2 + 9}$

#### Solution

$$VA: n/a; \quad \text{Hole: } n/a; \quad HA: y = 1; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of  $y = \frac{x-2}{x^2 - 4x + 3}$

#### Solution

$$VA: x = 1, 3; \quad \text{Hole: } n/a; \quad HA: y = 0; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of  $y = \frac{5x-1}{1-3x}$

#### Solution

$$VA: x = \frac{1}{3}; \quad \text{Hole: } n/a; \quad HA: y = -\frac{5}{3}; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of  $y = \frac{3}{x-5}$

#### Solution

$$VA: x = 5, \quad \text{Hole: } n/a, \quad HA: y = 0, \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of  $y = \frac{x^3-1}{x^2+1}$

#### Solution

$$\begin{array}{r} x \\ x^2+1 \overline{) x^3-1} \\ \underline{x^3+x} \phantom{-1} \\ -x-1 \end{array}$$

$$\begin{aligned} y &= \frac{x^3-1}{x^2+1} \\ &= x + \frac{-x-1}{x^2+1} \\ &= x - \frac{x+1}{x^2+1} \end{aligned}$$

$$VA: n/a, \quad \text{Hole: } n/a, \quad HA: n/a, \quad OA: y = x$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of  $y = \frac{3x^2-27}{(x+3)(2x+1)}$

#### Solution

$$VA: x = -3, \quad -\frac{1}{2}; \quad \text{Hole: } n/a; \quad HA: y = \frac{3}{2}; \quad OA: n/a$$



### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of  $y = \frac{x^3 + 3x^2 - 2}{x^2 - 4}$

#### Solution

$$\begin{array}{r} x^2 - 4 \overline{) x^3 + 3x^2 - 2} \\ \underline{x^3 \phantom{+ 3x^2} - 4x} \phantom{- 2} \\ 3x^2 + 4x - 2 \end{array}$$

$$\begin{aligned} y &= \frac{x^3 + 3x^2 - 2}{x^2 - 4} \\ &= x + 3 + \frac{4x + 10}{x^2 - 4} \end{aligned}$$

$$VA: x = \pm 2, \quad \text{Hole}: n/a, \quad HA: n/a, \quad OA: y = x + 3$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of  $y = \frac{x - 3}{x^2 - 9}$

#### Solution

$$VA: x = -3; \quad \text{Hole}: x = 3; \quad HA: y = 0; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of  $y = \frac{6}{\sqrt{x^2 - 4x}}$

#### Solution

$$VA: x = 0, 4; \quad \text{Hole}: n/a; \quad HA: y = 0; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of  $f(x) = \frac{4x^3 + 1}{1 - x^3}$

#### Solution

$$VA: x = 1; \quad \text{Hole}: n/a; \quad HA: y = -4; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of

$$f(x) = \frac{x+1}{\sqrt{9x^2+x}}$$

### Solution

$$VA: x = 0, -\frac{1}{9}; \quad \text{Hole: } n/a; \quad HA: y = \frac{1}{3}; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of

$$f(x) = 1 - e^{-2x}$$

### Solution

$$VA: n/a; \quad \text{Hole: } n/a; \quad HA: y = 1; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of

$$f(x) = \frac{1}{\ln x^2}$$

### Solution

$$VA: x = 0; \quad \text{Hole: } n/a; \quad HA: y = 0; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of

$$f(x) = \frac{1}{\tan^{-1} x}$$

### Solution

$$VA: x = 0; \quad \text{Hole: } n/a; \quad HA: y = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of

$$f(x) = \frac{2x^2+6}{2x^2+3x-2}$$

### Solution

$$VA: x = -2, \frac{1}{2}; \quad \text{Hole: } n/a; \quad HA: y = 1; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of

$$f(x) = \frac{3x^2 + 2x - 1}{4x + 1}$$

### Solution

$$\begin{array}{r}
 \frac{3}{4}x + \frac{5}{16} \\
 4x + 1 \overline{) 3x^2 + 2x - 1} \\
 \underline{3x^2 + \frac{3}{4}x} \phantom{-1} \\
 \frac{5}{4}x - 1
 \end{array}$$

$$VA: x = -\frac{1}{4}; \quad \text{Hole: } n/a; \quad HA: n/a; \quad OA: y = \frac{3}{4}x + \frac{5}{16}$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of

$$f(x) = \frac{9x^2 + 4}{(2x - 1)^2}$$

### Solution

$$VA: x = \frac{1}{2}; \quad \text{Hole: } n/a; \quad HA: y = \frac{9}{4}; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of

$$f(x) = \frac{1 + x - 2x^2 - x^3}{x^2 + 1}$$

### Solution

$$\begin{array}{r}
 -x - 2 \\
 x^2 + 1 \overline{) -x^3 - 2x^2 + x + 1} \\
 \underline{-x^3 \phantom{- 2x^2} - x} \phantom{+ 1} \\
 -2x^2 + 2x
 \end{array}$$

$$VA: n/a; \quad \text{Hole: } n/a; \quad HA: n/a; \quad OA: y = -x - 2$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of

$$f(x) = \frac{x(x+2)^3}{3x^2 - 4x}$$

### Solution

$$\begin{aligned}
 f(x) &= \frac{x(x^3 + 6x^2 + 12x + 8)}{x(3x - 4)} \\
 &= \frac{x^3 + 6x^2 + 12x + 8}{3x - 4} \\
 &= \frac{\frac{1}{3}x^2 + \frac{22}{9}x + \frac{196}{27}}{3x - 4} \left( \begin{array}{l} x^3 + 6x^2 + 12x + 8 \\ x^3 - \frac{4}{3}x^2 \\ \hline \frac{22}{3}x^2 + 12x \\ \frac{22}{3}x^2 - \frac{88}{9}x \\ \hline \frac{196}{9}x \end{array} \right)
 \end{aligned}$$

$$VA: x = \frac{4}{3}; \quad Hole: (0, -2); \quad HA: n/a; \quad OA: y = \frac{1}{3}x^2 + \frac{22}{9}x + \frac{196}{27}$$

### Exercise

Find the limit  $\lim_{x \rightarrow 0} \frac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x}$

#### Solution

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x} &= \frac{4}{0} \\
 &= \infty
 \end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow 2} \frac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x}$

#### Solution

$$\begin{aligned}
 \lim_{x \rightarrow 2} \frac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x} &= \frac{4 - 8 + 4}{8 + 20 - 28} = \frac{0}{0} \\
 &= \lim_{x \rightarrow 2} \frac{(x-2)(x-2)}{x(x-2)(x+7)} \\
 &= \lim_{x \rightarrow 2} \frac{x-2}{x(x+7)} \\
 &= \frac{0}{18} \\
 &= 0
 \end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow a} \frac{x^2 - a^2}{x^4 - a^4}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow a} \frac{x^2 - a^2}{x^4 - a^4} &= \frac{a^2 - a^2}{a^4 - a^4} = \frac{0}{0} \\&= \lim_{x \rightarrow a} \frac{x^2 - a^2}{(x^2 - a^2)(x^2 + a^2)} \\&= \lim_{x \rightarrow a} \frac{1}{x^2 + a^2} \\&= \frac{1}{a^2 + a^2} \\&= \frac{1}{2a^2} \quad \Big| \end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{(x+h)^2 - x^2}{h} &= \frac{h^2}{h} \\&= h \quad \Big| \end{aligned}$$

### Exercise

Find the limit  $\lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$

### Solution

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} &= \frac{x^2 - x^2}{0} = \frac{0}{0} \\&= \lim_{h \rightarrow 0} \frac{x^2 + 2hx + h^2 - x^2}{h} \\&= \lim_{h \rightarrow 0} \frac{2hx + h^2}{h} \end{aligned}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} (2x + h) \\
 &= \underline{2x}
 \end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x}$

### Solution

$$\begin{aligned}
 \lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x} &= \frac{1 - 1}{1 - 1} = \frac{0}{0} \\
 &= \lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{(1 - \sqrt{x})(1 + \sqrt{x})} \\
 &= \lim_{x \rightarrow 1} \frac{1}{1 + \sqrt{x}} \\
 &= \underline{\frac{1}{2}}
 \end{aligned}$$

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$$\begin{aligned}
 \lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x} &= \lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x} \cdot \frac{1 + \sqrt{x}}{1 + \sqrt{x}} \\
 &= \lim_{x \rightarrow 1} \frac{1 - x}{(1 - x)(1 + \sqrt{x})} \\
 &= \lim_{x \rightarrow 1} \frac{1}{1 + \sqrt{x}} \\
 &= \underline{\frac{1}{2}}
 \end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x}$

### Solution

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} &= \frac{\frac{1}{2} - \frac{1}{2}}{0} = \frac{0}{0} \\
 &= \lim_{x \rightarrow 0} \frac{1}{x} \left( \frac{2 - 2 - x}{2(2+x)} \right) \\
 &= \frac{1}{2} \lim_{x \rightarrow 0} \frac{1}{x} \left( \frac{-x}{2+x} \right)
 \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2} \lim_{x \rightarrow 0} \frac{1}{2+x} \\
&= -\frac{1}{2} \left( \frac{1}{2} \right) \\
&= -\frac{1}{4}
\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow 1} \frac{x^{1/3} - 1}{\sqrt{x} - 1}$

### Solution

$$\begin{aligned}
\lim_{x \rightarrow 1} \frac{x^{1/3} - 1}{\sqrt{x} - 1} &= \frac{1-1}{1-1} = \frac{0}{0} \\
&= \lim_{x \rightarrow 1} \frac{x^{1/3} - 1}{\sqrt{x} - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} \\
&= \lim_{x \rightarrow 1} \frac{(x^{1/3} - 1)(\sqrt{x} + 1)}{x - 1} \\
&= \lim_{x \rightarrow 1} \frac{(x^{1/3} - 1)(\sqrt{x} + 1)}{(x^{1/3})^3 - 1^3} \\
&= \lim_{x \rightarrow 1} \frac{(x^{1/3} - 1)(\sqrt{x} + 1)}{(x^{1/3} - 1)(x^{2/3} + x^{1/3} + 1)} \\
&= \lim_{x \rightarrow 1} \frac{\sqrt{x} + 1}{x^{2/3} + x^{1/3} + 1} \\
&= \frac{2}{3}
\end{aligned}$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

### Exercise

Find the limit  $\lim_{x \rightarrow 64} \frac{x^{2/3} - 16}{\sqrt{x} - 8}$

### Solution

$$\lim_{x \rightarrow 64} \frac{x^{2/3} - 16}{\sqrt{x} - 8} = \frac{(4^3)^{2/3} - 16}{8 - 8}$$

$$\begin{aligned}
&= \frac{16-16}{0} = \frac{0}{0} \\
&= \lim_{x \rightarrow 64} \frac{(x^{1/3})^2 - 16}{\sqrt{x} - 8} \cdot \frac{\sqrt{x} + 8}{\sqrt{x} + 8} \\
&= \lim_{x \rightarrow 64} \frac{(x^{1/3} - 4)(x^{1/3} + 4)(\sqrt{x} + 8)}{x - 64} \\
&= \lim_{x \rightarrow 64} \frac{(x^{1/3} - 4)(x^{1/3} + 4)(\sqrt{x} + 8)}{(x^{1/3})^3 - 4^3} \\
&= \lim_{x \rightarrow 64} \frac{(x^{1/3} - 4)(x^{1/3} + 4)(\sqrt{x} + 8)}{(x^{1/3} - 4)(x^{2/3} + 4x^{1/3} + 16)} \\
&= \lim_{x \rightarrow 64} \frac{(x^{1/3} + 4)(\sqrt{x} + 8)}{x^{2/3} + 4x^{1/3} + 16} \\
&= \frac{(4+4)(8+8)}{16+16+16} \\
&= \frac{8}{3}
\end{aligned}$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

### Exercise

Find the limit  $\lim_{x \rightarrow 0} \frac{\tan(2x)}{\tan(\pi x)}$

### Solution

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{\tan(2x)}{\tan(\pi x)} &= \frac{0}{0} \\
&= \lim_{x \rightarrow 0} \frac{\sin 2x}{\cos 2x} \cdot \frac{\cos(\pi x)}{\sin(\pi x)} \\
&= \lim_{x \rightarrow 0} \frac{\cos(\pi x)}{\cos 2x} \cdot \frac{\sin 2x}{2x} \cdot \frac{2x}{\pi x} \cdot \frac{\pi x}{\sin(\pi x)} \\
&= \frac{2}{\pi} \cdot \frac{\cos 0}{\cos 0} \cdot \lim_{2x \rightarrow 0} \frac{\sin 2x}{2x} \cdot \lim_{\pi x \rightarrow 0} \frac{1}{\frac{\sin \pi x}{\pi x}} \\
&= \frac{2}{\pi}
\end{aligned}$$



### ***Exercise***

Find the limit  $\lim_{x \rightarrow \pi^-} \csc x$

#### **Solution**

$$\begin{aligned}\lim_{x \rightarrow \pi^-} \csc x &= \frac{1}{\sin \pi^-} \\ &= \frac{1}{0^-} \\ &= -\infty\end{aligned}$$

### ***Exercise***

Find the limit  $\lim_{x \rightarrow \pi} \sin\left(\frac{x}{2} + \sin x\right)$

#### **Solution**

$$\begin{aligned}\lim_{x \rightarrow \pi} \sin\left(\frac{x}{2} + \sin x\right) &= \sin\left(\frac{\pi}{2} + \sin \pi\right) \\ &= \sin \frac{\pi}{2} \\ &= 1\end{aligned}$$

### ***Exercise***

Find the limit  $\lim_{x \rightarrow \pi} \cos^2(x - \tan x)$

#### **Solution**

$$\begin{aligned}\lim_{x \rightarrow \pi} \cos^2(x - \tan x) &= \cos^2(\pi - \tan \pi) \\ &= \cos^2(\pi) \\ &= (-1)^2 \\ &= 1\end{aligned}$$

### ***Exercise***

Find the limit  $\lim_{x \rightarrow 0} \frac{8x}{3 \sin x - x}$

#### **Solution**

$$\lim_{x \rightarrow 0} \frac{8x}{3\sin x - x} = \frac{0}{0}$$

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{8 \cancel{x}}{3 \frac{\sin \cancel{x}}{\cancel{x}} - \cancel{x}} \\ &= \frac{8}{3 \lim_{x \rightarrow 0} \frac{\sin x}{x} - 1} \\ &= \frac{8}{3 - 1} \\ &= 4 \end{aligned}$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

### Exercise

Find the limit  $\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\sin x}$

### Solution

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\sin x} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \frac{1 - 2\sin^2 x - 1}{\sin x} \\ &= \lim_{x \rightarrow 0} \frac{-2\sin^2 x}{\sin x} \\ &= -2 \lim_{x \rightarrow 0} \sin x \\ &= 0 \end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}}$

### Solution

$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}} &= \lim_{x \rightarrow -\infty} \frac{3x^3}{\sqrt{x^6}} \\ &= \lim_{x \rightarrow -\infty} \frac{3x^3}{x^3} \\ &= 3 \end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow -\infty} \frac{x^2 - 4x + 8}{3x^3}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{x^2 - 4x + 8}{3x^3} &= \lim_{x \rightarrow -\infty} \frac{x^2}{3x^3} \\ &= \lim_{x \rightarrow -\infty} \frac{1}{3x} \\ &= 0\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow -\infty} \frac{2x^2 + 3}{5x^2 + 7}$

#### Solution

$$\lim_{x \rightarrow -\infty} \frac{2x^2 + 3}{5x^2 + 7} = \frac{2}{5}$$

### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{x^4 + x^3}{12x^3 + 128}$

#### Solution

$$\lim_{x \rightarrow \infty} \frac{x^4 + x^3}{12x^3 + 128} = \infty$$

### Exercise

Find the limit  $\lim_{x \rightarrow -\infty} \frac{2 + \sqrt{x}}{2 - \sqrt{x}}$

#### Solution

Since  $x \rightarrow -\infty$  and inside the square root can't be negative

$$\lim_{x \rightarrow -\infty} \frac{2 + \sqrt{x}}{2 - \sqrt{x}} = \text{not defined}$$

### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{2 + \sqrt{x}}{2 - \sqrt{x}}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{2 + \sqrt{x}}{2 - \sqrt{x}} &= \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{-\sqrt{x}} \\ &= -1\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}} &= \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x}}{\sqrt[3]{x}} \\ &= 1\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{1}{x^4}}{\frac{1}{x^2} - \frac{1}{x^3}}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{1}{x^4}}{\frac{1}{x^2} - \frac{1}{x^3}} &= \frac{0}{0} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{x^3 + 1}{x^4}}{\frac{x - 1}{x^3}} \\ &= \lim_{x \rightarrow \infty} \frac{x^3 + 1}{x - 1} \cdot \frac{x^3}{x^4} \\ &= \lim_{x \rightarrow \infty} \frac{x^3 + 1}{x(x - 1)} \\ &= \lim_{x \rightarrow \infty} \frac{x^3}{x^2} \\ &= \infty\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}} &= \lim_{x \rightarrow \infty} \frac{2x^{5/3}}{x^{8/5}} \\ &= \lim_{x \rightarrow \infty} 2x^{\left(\frac{5}{3} - \frac{8}{5}\right)} \\ &= \lim_{x \rightarrow \infty} 2x^{\frac{1}{15}} \\ &= \infty\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow 2^+} \ln(x - 2)$

### Solution

$$\begin{aligned}\lim_{x \rightarrow 2^+} \ln(x - 2) &= \ln(0^+) \\ &= -\infty\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow 1} x^2 \ln(2 - \sqrt{x})$

### Solution

$$\begin{aligned}\lim_{x \rightarrow 1} x^2 \ln(2 - \sqrt{x}) &= \ln(2 - 1) \\ &= \ln 1 \\ &= 0\end{aligned}$$

### Exercise

Find the limit  $\lim_{\theta \rightarrow 0^+} \sqrt{\theta} e^{\cos \frac{\pi}{\theta}}$

### Solution

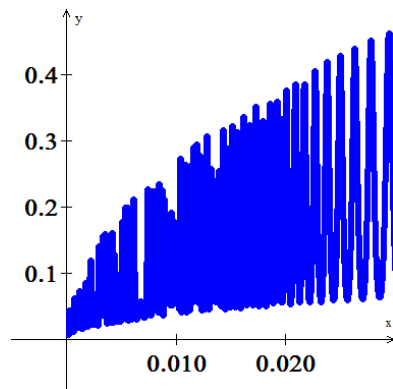
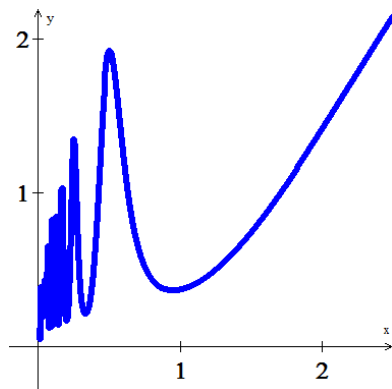
$$\lim_{\theta \rightarrow 0^+} \sqrt{\theta} e^{\cos \frac{\pi}{\theta}} = 0 \cdot e^{\cos \infty}$$

$$-1 \leq \cos \frac{\pi}{\theta} \leq 1$$

$$e^{-1} \leq e^{\cos \frac{\pi}{\theta}} \leq e$$

$$0 \cdot \frac{1}{e} \leq 0 \cdot e^{\cos \frac{\pi}{\theta}} \leq 0 \cdot e$$

$$\lim_{\theta \rightarrow 0^+} \sqrt{\theta} e^{\cos \frac{\pi}{\theta}} = 0$$



### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{2x-3}{5x+6}$

#### Solution

$$\lim_{x \rightarrow \infty} \frac{2x-3}{5x+6} = \lim_{x \rightarrow \infty} \frac{2x}{5x} = \frac{2}{5}$$

### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{2x^2-3}{5x^2+6}$

#### Solution

$$\lim_{x \rightarrow \infty} \frac{2x^2-3}{5x^2+6} = \lim_{x \rightarrow \infty} \frac{2x^2}{5x^2} = \frac{2}{5}$$

### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{2x-3}{5x^3+6}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{2x-3}{5x^3+6} &= \lim_{x \rightarrow \infty} \frac{2x}{5x^3} \\ &= 0\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{1}{5x^2-3x+6}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{1}{5x^2-3x+6} &= \lim_{x \rightarrow \infty} \frac{1}{5x^2} \\ &= 0\end{aligned}$$

### Exercise

Find the limit  $\lim_{\theta \rightarrow 0} \frac{\theta \cot 4\theta}{\sin^2 \theta \cot^2 2\theta}$

### Solution

$$\begin{aligned}\lim_{\theta \rightarrow 0} \frac{\theta \cot 4\theta}{\sin^2 \theta \cot^2 2\theta} &= \frac{0}{0} \\ &= \lim_{\theta \rightarrow 0} \frac{1}{\frac{\sin \theta}{\theta}} \cdot \frac{1}{\sin \theta} \cdot \frac{\cos 4\theta}{\sin 4\theta} \cdot \frac{\sin^2 2\theta}{\cos^2 2\theta} \\ &= \lim_{\theta \rightarrow 0} \frac{1}{\frac{\sin \theta}{\theta}} \lim_{\theta \rightarrow 0} \frac{\cos 4\theta}{\cos^2 2\theta} \lim_{\theta \rightarrow 0} \frac{1}{\sin \theta} \cdot \frac{\sin 2\theta \sin 2\theta}{2 \sin 2\theta \cos 2\theta} \\ &= (1)(1) \lim_{\theta \rightarrow 0} \frac{1}{\sin \theta} \cdot \frac{2 \sin \theta \cos \theta}{2 \cos 2\theta} \\ &= \lim_{\theta \rightarrow 0} \frac{\cos \theta}{\cos 2\theta} \\ &= 1\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow 0^+} \frac{\sqrt{x^2 + 4x + 5} - \sqrt{5}}{x}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{\sqrt{x^2 + 4x + 5} - \sqrt{5}}{x} &= \frac{\sqrt{5} - \sqrt{5}}{0} = \frac{0}{0} \\&= \lim_{x \rightarrow 0^+} \frac{\sqrt{x^2 + 4x + 5} - \sqrt{5}}{x} \cdot \frac{\sqrt{x^2 + 4x + 5} + \sqrt{5}}{\sqrt{x^2 + 4x + 5} + \sqrt{5}} \\&= \lim_{x \rightarrow 0^+} \frac{x^2 + 4x + 5 - 5}{x(\sqrt{x^2 + 4x + 5} + \sqrt{5})} \\&= \lim_{x \rightarrow 0^+} \frac{x(x + 4)}{x(\sqrt{x^2 + 4x + 5} + \sqrt{5})} \\&= \lim_{x \rightarrow 0^+} \frac{x + 4}{\sqrt{x^2 + 4x + 5} + \sqrt{5}} \\&= \frac{4}{\sqrt{5} + \sqrt{5}} \\&= \frac{4}{2\sqrt{5}} \\&= \frac{2}{\sqrt{5}}\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow 2} \frac{x^4 - 16}{x - 2}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x^4 - 16}{x - 2} &= \frac{16 - 16}{2 - 2} = \frac{0}{0} \\&= \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)(x^2 + 4)}{x - 2} \\&= \lim_{x \rightarrow 2} (x + 2)(x^2 + 4) \\&= (4)(8) \\&= 32\end{aligned}$$
$$a^4 - b^4 = (a - b)(a + b)(a^2 + b^2)$$



### Exercise

Find the limit  $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2} &= \frac{0}{0} \\&= \lim_{x \rightarrow 2} \frac{(x - 2)(x^2 + 2x + 4)}{x - 2} \\&= \lim_{x \rightarrow 2} (x^2 + 2x + 4) \\&= 4 + 4 + 4 \\&= 12\end{aligned}$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

### Exercise

Find the limit  $\lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4} &= \lim_{x \rightarrow -\infty} \frac{-5x}{2x} \\&= -\frac{5}{2}\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 1}}{x + 1}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 1}}{x + 1} &= \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2}}{x} \\&= \lim_{x \rightarrow -\infty} \frac{|x|}{x} \\&= -1\end{aligned}$$

### ***Exercise***

Find the limit  $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{x + 1}$

### **Solution**

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{x + 1} &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2}}{x} \\ &= \lim_{x \rightarrow \infty} \frac{|x|}{x} \\ &= 1\end{aligned}$$

### ***Exercise***

Find the limit  $\lim_{x \rightarrow \infty} \frac{x - 3}{\sqrt{4x^2 + 25}}$

### **Solution**

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{x - 3}{\sqrt{4x^2 + 25}} &= \lim_{x \rightarrow \infty} \frac{x}{2|x|} \\ &= \frac{1}{2}\end{aligned}$$

### ***Exercise***

Find the limit  $\lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}}$

### **Solution**

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}} &= \lim_{x \rightarrow -\infty} \frac{3x^3}{x^3} \\ &= 3\end{aligned}$$

### ***Exercise***

Find the limit  $\lim_{x \rightarrow \infty} \frac{x^4 - x}{15x^3 + 4}$

### **Solution**

$$\lim_{x \rightarrow \infty} \frac{x^4 - x}{15x^3 + 4} = \infty$$

### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x}$

### Solution

$$-1 \leq \sin x \leq 1$$

$$\lim_{x \rightarrow \infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x} = \lim_{x \rightarrow \infty} \frac{x}{x} = 1$$

### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{x^{2/3} - x^{-1}}{x^{2/3} + \cos^2 x}$

### Solution

$$-1 \leq \cos x \leq 1$$

$$0 \leq \cos^2 x \leq 1$$

$$\lim_{x \rightarrow \infty} \frac{x^{2/3} - \frac{1}{x}}{x^{2/3} + \cos^2 x} = \lim_{x \rightarrow \infty} \frac{x^{2/3}}{x^{2/3}} = 1$$

### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{\sin 2x}{x}$

### Solution

$$-1 \leq \sin 2x \leq 1$$

$$-\lim_{x \rightarrow \infty} \frac{1}{x} \leq \lim_{x \rightarrow \infty} \frac{\sin 2x}{x} \leq \lim_{x \rightarrow \infty} \frac{1}{x}$$

$$0 \leq \lim_{x \rightarrow \infty} \frac{\sin 2x}{x} \leq 0$$

$$\lim_{x \rightarrow \infty} \frac{\sin 2x}{x} = 0$$

### Exercise

Find the limit  $\lim_{x \rightarrow 0} \frac{\sin 5x}{3x}$

### Solution

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin 5x}{3x} &= \lim_{5x \rightarrow 0} \frac{\overset{5}{\cancel{5}} \cdot \sin \underset{5x}{\cancel{5}x}}{3 \cdot \underset{5x}{\cancel{5}}} \\ &= \frac{5}{3} \end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow -\infty} \frac{\cos x}{2x}$

### Solution

$$-1 \leq \cos x \leq 1$$

$$- \lim_{x \rightarrow \infty} \frac{1}{2x} \leq \lim_{x \rightarrow \infty} \frac{\cos x}{2x} \leq \lim_{x \rightarrow \infty} \frac{1}{2x}$$

$$0 \leq \lim_{x \rightarrow \infty} \frac{\cos x}{2x} \leq 0$$

$$\lim_{x \rightarrow \infty} \frac{\cos x}{2x} = 0$$

### Exercise

Find the limit  $\lim_{x \rightarrow -\infty} \left( \frac{x^2 + x - 1}{8x^2 - 3} \right)^{1/3}$

### Solution

$$\begin{aligned} \lim_{x \rightarrow -\infty} \left( \frac{x^2 + x - 1}{8x^2 - 3} \right)^{1/3} &= \lim_{x \rightarrow -\infty} \left( \frac{x^2}{8x^2} \right)^{1/3} \\ &= \left( \frac{1}{8} \right)^{1/3} \\ &= \frac{1}{2} \end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1} &= \frac{3 - 3}{-1 + 1} = \frac{0}{0} \\&= \lim_{x \rightarrow -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1} \cdot \frac{\sqrt{x^2 + 8} + 3}{\sqrt{x^2 + 8} + 3} \\&= \lim_{x \rightarrow -1} \frac{x^2 + 8 - 9}{(x + 1)(\sqrt{x^2 + 8} + 3)} \\&= \lim_{x \rightarrow -1} \frac{x^2 - 1}{(x + 1)(\sqrt{x^2 + 8} + 3)} \\&= \lim_{x \rightarrow -1} \frac{(x - 1)(x + 1)}{(x + 1)(\sqrt{x^2 + 8} + 3)} \\&= \lim_{x \rightarrow -1} \frac{x - 1}{\sqrt{x^2 + 8} + 3} \\&= \frac{0}{6} \\&= 0\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow -\infty} \left( \frac{1 - x^3}{x^2 + 7x} \right)^5$

### Solution

$$\begin{aligned}\lim_{x \rightarrow -\infty} \left( \frac{1 - x^3}{x^2 + 7x} \right)^5 &= \lim_{x \rightarrow -\infty} \left( \frac{-x^3}{x^2} \right)^5 \\&= \lim_{x \rightarrow -\infty} (-x^5) \\&= \infty\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \sqrt{\frac{x^2 - 5x}{x^3 + x - 2}}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \sqrt{\frac{x^2 - 5x}{x^3 + x - 2}} &= \lim_{x \rightarrow \infty} \sqrt{\frac{x^2}{x^3}} \\ &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} \\ &= 0\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7}$

### Solution

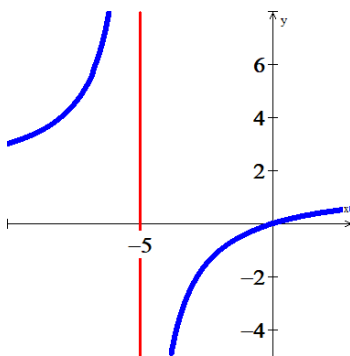
$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7} &= \lim_{x \rightarrow \infty} \frac{2\sqrt{x}}{3x} \\ &= \lim_{x \rightarrow \infty} \frac{2}{3\sqrt{x}} \\ &= 0\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow -5^-} \frac{3x}{2x + 10}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow -5^-} \frac{3x}{2x + 10} &= \frac{-15}{0^-} \\ &= \infty\end{aligned}$$

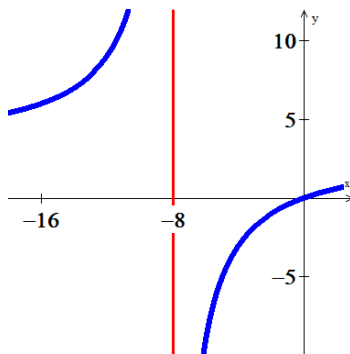


### Exercise

Find the limit  $\lim_{x \rightarrow -8^+} \frac{3x}{x + 8}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow -8^+} \frac{3x}{x + 8} &= \frac{-24}{0^+} \\ &= -\infty\end{aligned}$$



### Exercise

Find the limit  $\lim_{x \rightarrow 0} \frac{-1}{x^2(x+1)}$

### Solution

$$\lim_{x \rightarrow 0} \frac{-1}{x^2(x+1)} = -\frac{1}{0} \\ = -\infty$$

### Exercise

Find the limit  $\lim_{x \rightarrow 7} \frac{4}{(x-7)^2}$

### Solution

$$\lim_{x \rightarrow 7} \frac{4}{(x-7)^2} = \frac{4}{0} \\ = \infty$$

### Exercise

Find the limit  $\lim_{x \rightarrow 0} \frac{1}{x^{2/3}}$

### Solution

$$\lim_{x \rightarrow 0} \frac{1}{x^{2/3}} = \infty$$

### Exercise

Find the limit  $\lim_{x \rightarrow -\infty} \left( x + \sqrt{x^2 - 4x + 2} \right)$

### Solution

$$\begin{aligned} \lim_{x \rightarrow -\infty} \left( x + \sqrt{x^2 - 4x + 2} \right) &= -\infty + \infty \\ &= \lim_{x \rightarrow -\infty} \left( x + \sqrt{x^2 - 4x + 2} \right) \cdot \frac{x - \sqrt{x^2 - 4x + 2}}{x - \sqrt{x^2 - 4x + 2}} \\ &= \lim_{x \rightarrow -\infty} \frac{x^2 - x^2 + 4x - 2}{x - \sqrt{x^2 - 4x + 2}}. \end{aligned}$$

$$= \lim_{x \rightarrow -\infty} \frac{4x-2}{x-\sqrt{x^2-4x+2}}$$

$$= \lim_{x \rightarrow -\infty} \frac{4x-2}{x-|x|} \qquad x \rightarrow -\infty \quad (x < 0) \rightarrow |x| = -x$$

$$= \lim_{x \rightarrow -\infty} \frac{4x}{x+x}$$

$$= \lim_{x \rightarrow -\infty} \frac{4x}{2x}$$

$$= 2$$

