Lecture Three

SOLUTION

Lecture 3

Solution Section 3.1 – Inner Products

Exercise

Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (1,\ 1)$, $\vec{v} = (3,\ 2)$, $\vec{w} = (0,\ -1)$, and $k = 3$. Compute the following.

a) $\langle \vec{u}, \vec{v} \rangle$

b) $\langle k\vec{v}, \vec{w} \rangle$

c) $\langle \vec{u} + \vec{v}, \vec{w} \rangle$

d) $\|\vec{v}\|$

e) $d(\vec{u}, \vec{v})$

f) $\|\vec{u} - k\vec{v}\|$

Solution

a) $\langle \vec{u}, \vec{v} \rangle = 1(3) + 1(2)$

$= 5$

b) $\langle k\vec{v}, \vec{w} \rangle = \langle 3\vec{v}, \vec{w} \rangle$

$= 9 \cdot 0 + 6 \cdot (-1)$

$= -6$

c) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

$= 1 \cdot 0 + 1 \cdot (-1) + 3 \cdot 0 + 2 \cdot (-1)$

$= -3$

d) $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$

$= \sqrt{3^2 + 2^2}$

$= \sqrt{13}$

e) $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$

$= \|(-2,\ -1)\|$

$= \sqrt{(-2)^2 + (-1)^2}$

$= \sqrt{5}$

f) $\|\vec{u} - k\vec{v}\| = \|(1,\ 1) - 3(3,\ 2)\|$

$= \|(-8,\ -5)\|$

$= \sqrt{(-8)^2 + (-5)^2}$

$= \sqrt{89}$

Exercise

Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (3, -2)$, $\vec{v} = (4, 5)$, $\vec{w} = (-1, 6)$, and $k = -4$. Verify the following.

a) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

b) $\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

c) $\langle \vec{u}, \vec{v}+\vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

d) $\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$

e) $\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$

Solution

a) $\langle \vec{u}, \vec{v} \rangle = 3\cdot 4 + (-2)\cdot(5)$

$= 2$

$\langle \vec{v}, \vec{u} \rangle = 4\cdot 3 + (5)\cdot(-2)$

$= 2$

$\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ ✓

b) $\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle (7,3), (-1,6) \rangle$

$= 7(-1) + 3(6)$

$= 11$

$\langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle = (3)(-1) + (-2)(6) + (4)(-1) + (5)(6)$

$= 11$

$\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ ✓

c) $\langle \vec{u}, \vec{v}+\vec{w} \rangle = \langle (3, -2), (3, 11) \rangle$

$= 3(3) + (-2)(11)$

$= -13$

$\langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle = (3)(4) + (-2)(5) + (3)(-1) + (-2)(6)$

$= -13$

$\langle \vec{u}, \vec{v}+\vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$ ✓

d) $\langle k\vec{u}, \vec{v} \rangle = (-4\cdot 3)\cdot 4 + ((-4)(-2))\cdot(5)$

$= -8$

$k\langle \vec{u}, \vec{v} \rangle = (-4)(3\cdot 4 + (-2)\cdot(5))$

$= -8$

$\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$ ✓

e) $\langle \vec{0},\ \vec{v}\rangle = 0\cdot 4 + 0\cdot(5)$

$\underline{= 0\ |}$

$\langle \vec{v},\ \vec{0}\rangle = 4\cdot 0 + (5)\cdot(0)$

$\underline{= 0\ |}$

$\langle \vec{0},\ \vec{v}\rangle = \langle \vec{v},\ \vec{0}\rangle = 0 \quad \checkmark$

Exercise

Let $\langle \vec{u},\ \vec{v}\rangle$ be the Euclidean inner product on $\mathbb{R}^3$, and let $\vec{u} = (1,\ 0,\ -2)$, $\vec{v} = (5,\ 1,\ 2)$, $\vec{w} = (5,\ 2,\ -1)$, and $k = 3$. Compute the following.

a) $\langle \vec{u},\ \vec{v}\rangle$

b) $\langle k\vec{v},\ \vec{w}\rangle$

c) $\langle \vec{u}+\vec{v},\ \vec{w}\rangle$

d) $\|\vec{v}\|$

e) $d(\vec{u},\ \vec{v})$

f) $\|\vec{u} - k\vec{v}\|$

Solution

a) $\langle \vec{u},\ \vec{v}\rangle = (1,\ 0,\ -2)\bullet(5,\ 1,\ 2)$

$= 1(5) + 0(1) - 2(2)$

$\underline{= 1\ |}$

b) $\langle k\vec{v},\ \vec{w}\rangle = \langle 3\vec{v},\ \vec{w}\rangle$

$= (3(5,\ 1,\ 2))\bullet(5,\ 2,\ -1)$

$= (15,\ 3,\ 6)\bullet(5,\ 2,\ -1)$

$= 15(5) + 3(2) + 6(-1)$

$\underline{= 75\ |}$

c) $\langle \vec{u}+\vec{v},\ \vec{w}\rangle = \langle \vec{u},\ \vec{w}\rangle + \langle \vec{v},\ \vec{w}\rangle$

$= (1,\ 0,\ -2)\bullet(5,\ 2,\ -1) + (5,\ 1,\ 2)\bullet(5,\ 2,\ -1)$

$= (5+0+2) + (25+2-2)$

$\underline{= 32\ |}$

d) $\|\vec{v}\| = \sqrt{\langle \vec{v},\ \vec{v}\rangle}$

$= \sqrt{(5,\ 1,\ 2)\bullet(5,\ 1,\ 2)}$

$= \sqrt{25+1+4}$

$\underline{= \sqrt{30}\ |}$

e) $d(\vec{u},\ \vec{v}) = \|\vec{u} - \vec{v}\|$

$$= \|(1,\ 0,\ -2) - (5,\ 1,\ 2)\|$$

$$= \|(-4,\ -1,\ -4)\|$$

$$= \sqrt{16+1+16}$$

$$= \sqrt{33}$$

f) $\|\vec{u} - k\vec{v}\| = \|(1,\ 0,\ -2) - 3(5,\ 1,\ 2)\|$

$$= \|(-14,\ -3,\ -8)\|$$

$$= \sqrt{196+9+64}$$

$$= \sqrt{269}$$

Exercise

Let $\langle \vec{u},\ \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^3$, and let $\vec{u} = (2,\ 4,\ 3)$, $\vec{v} = (0,\ -3,\ -4)$, $\vec{w} = (6,\ 3,\ 1)$, and $k = 2$. Verify the following.

a) $\langle \vec{u},\ \vec{v} \rangle = \langle \vec{v},\ \vec{u} \rangle$

b) $\langle \vec{u}+\vec{v},\ \vec{w} \rangle = \langle \vec{u},\ \vec{w} \rangle + \langle \vec{v},\ \vec{w} \rangle$

c) $\langle \vec{u},\ \vec{v}+\vec{w} \rangle = \langle \vec{u},\ \vec{v} \rangle + \langle \vec{u},\ \vec{w} \rangle$

d) $\langle k\vec{u},\ \vec{v} \rangle = k\langle \vec{u},\ \vec{v} \rangle$

e) $\langle \vec{0},\ \vec{v} \rangle = \langle \vec{v},\ \vec{0} \rangle = 0$

Solution

a) $\langle \vec{u},\ \vec{v} \rangle = (2,\ 4,\ 3) \bullet (0,\ -3,\ -4)$

$$= -12 - 12$$

$$= -24$$

$\langle \vec{v},\ \vec{u} \rangle = (0,\ -3,\ -4) \bullet (2,\ 4,\ 3)$

$$= -12 - 12$$

$$= -24$$

$\langle \vec{u},\ \vec{v} \rangle = \langle \vec{v},\ \vec{u} \rangle$ ✓

b) $\langle \vec{u}+\vec{v},\ \vec{w} \rangle = \langle (2,\ 4,\ 3) + (0,\ -3,\ -4),\ (6,\ 3,\ 1) \rangle$

$$= \langle (2,\ 1,\ -1),\ (6,\ 3,\ 1) \rangle$$

$$= 12 + 3 - 1$$

$$= 14$$

$\langle \vec{u},\ \vec{w} \rangle + \langle \vec{v},\ \vec{w} \rangle = (2,\ 4,\ 3) \bullet (6,\ 3,\ 1) + (0,\ -3,\ -4) \bullet (6,\ 3,\ 1)$

$$= (12+12+3)+(0-9-4)$$

$$\underline{= 14}$$

$$\langle \vec{u}+\vec{v},\ \vec{w}\rangle = \langle \vec{u},\ \vec{w}\rangle + \langle \vec{v},\ \vec{w}\rangle \quad \checkmark$$

c) $\langle \vec{u},\ \vec{v}+\vec{w}\rangle = \langle (2,\ 4,\ 3),\ \ (0,\ -3,\ -4)+(6,\ 3,\ 1)\rangle$

$$= \langle (2,\ 4,\ 3),\ \ (6,\ 0,\ -3)\rangle$$

$$= 12+0-9$$

$$\underline{= 3}$$

$$\langle \vec{u},\ \vec{v}\rangle + \langle \vec{u},\ \vec{w}\rangle = (2,\ 4,\ 3)\cdot(0,\ -3,\ -4)+(2,\ 4,\ 3)\cdot(6,\ 3,\ 1)$$

$$= (0-12-12)+(12+12+3)$$

$$\underline{= 3}$$

$$\langle \vec{u},\ \vec{v}+\vec{w}\rangle = \langle \vec{u},\ \vec{v}\rangle + \langle \vec{u},\ \vec{w}\rangle \quad \checkmark$$

d) $\langle k\vec{u},\ \vec{v}\rangle = 2(2,\ 4,\ 3)\cdot(0,\ -3,\ -4)$

$$= (4,\ 8,\ 6)\cdot(0,\ -3,\ -4)$$

$$= 0-24-24$$

$$\underline{= -48}$$

$$k\langle \vec{u},\ \vec{v}\rangle = (2)\big((2,\ 4,\ 3)\cdot(0,\ -3,\ -4)\big)$$

$$= (2)(-12-12)$$

$$\underline{= -48}$$

$$\langle k\vec{u},\ \vec{v}\rangle = k\langle \vec{u},\ \vec{v}\rangle \quad \checkmark$$

e) $\langle \vec{0},\ \vec{v}\rangle = (0,\ 0,\ 0)\cdot(0,\ -3,\ -4)$

$$= 0+0+0$$

$$\underline{= 0}$$

$$\langle \vec{v},\ \vec{0}\rangle = (0,\ -3,\ -4)\cdot(0,\ 0,\ 0)$$

$$= 0+0+0$$

$$\underline{= 0}$$

$$\langle \vec{0},\ \vec{v}\rangle = \langle \vec{v},\ \vec{0}\rangle = 0 \quad \checkmark$$

Exercise

Let $\langle \vec{u},\ \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (1,\ 1)$, $\vec{v} = (3,\ 2)$, $\vec{w} = (0,\ -1)$ and $k = 3$. Compute the following for the weighted Euclidean inner product $\langle \vec{u},\ \vec{v} \rangle = 2u_1 v_1 + 3u_2 v_2$.

a) $\langle \vec{u},\ \vec{v} \rangle$ *c)* $\langle \vec{u} + \vec{v},\ \vec{w} \rangle$ *e)* $d(\vec{u},\ \vec{v})$

b) $\langle k\vec{v},\ \vec{w} \rangle$ *d)* $\|\vec{v}\|$ *f)* $\|\vec{u} - k\vec{v}\|$

Solution

a) $\langle \vec{u},\ \vec{v} \rangle = 2(1)(3) + 3(1)(2)$

$= 12$

b) $\langle k\vec{v},\ \vec{w} \rangle = 2(3\cdot 3)(0) + 3(3\cdot 2)(-1)$

$= -18$

c) $\langle \vec{u} + \vec{v},\ \vec{w} \rangle = \langle \vec{u},\ \vec{w} \rangle + \langle \vec{v},\ \vec{w} \rangle$

$= 1\cdot 0 + 1\cdot(-1) + 3\cdot 0 + 2\cdot(-1)$

$= -3$

d) $\|\vec{v}\| = \sqrt{\langle \vec{v},\ \vec{v} \rangle}$

$= \sqrt{2(3)(3) + 3(2)(2)}$

$= \sqrt{30}$

e) $d(\vec{u},\ \vec{v}) = \|\vec{u} - \vec{v}\|$

$= \|\langle(-2,\ -1)\rangle\|$

$= \sqrt{2(-2)(-2) + 3(-1)(-1)}$

$= \sqrt{11}$

f) $\|\vec{u} - k\vec{v}\| = \|(1,\ 1) - 3(3,\ 2)\|$

$= \|\langle(-8,\ -5)\rangle\|$

$= \sqrt{2(-8)^2 + 3(-5)^2}$

$= \sqrt{203}$

Exercise

Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (3, -2)$, $\vec{v} = (4, 5)$, $\vec{w} = (-1, 6)$, and $k = -4$. Verify the following for the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 4u_1v_1 + 5u_2v_2$.

a) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

b) $\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

c) $\langle \vec{u}, \vec{v}+\vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

d) $\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$

e) $\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$

Solution

a) $\langle \vec{u}, \vec{v} \rangle = 4\cdot 3\cdot 4 + 5\cdot(-2)\cdot(5)$

$= -2$

$\langle \vec{v}, \vec{u} \rangle = 4\cdot 4\cdot 3 + 5\cdot(5)\cdot(-2)$

$= -2$

$\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle \quad \checkmark$

b) $\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle (7, 3), (-1, 6) \rangle$

$= 4\cdot 7(-1) + 5\cdot 3(6)$

$= 62$

$\langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle = \big(4\cdot(3)(-1) + 5\cdot(-2)(6)\big) + \big(4\cdot(4)(-1) + 5\cdot(5)(6)\big)$

$= (-12 - 60) + (-16 + 150)$

$= 62$

$\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \quad \checkmark$

c) $\langle \vec{u}, \vec{v}+\vec{w} \rangle = \langle (3, -2), (4, 5)+(-1, 6) \rangle$

$= \langle (3, -2), (3, 11) \rangle$

$= 4\cdot(3)(3) + 5\cdot(-2)(11)$

$= -74$

$\langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle = \big(4\cdot(3)(4) + 5\cdot(-2)(5)\big) + \big(4\cdot(3)(-1) + 5\cdot(-2)(6)\big)$

$= (48 - 50) + (-12 - 60)$

$= -74$

$\langle \vec{u}, \vec{v}+\vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle \quad \checkmark$

d) $\langle k\vec{u}, \vec{v} \rangle = \langle -4(3, -2), (4, 5) \rangle$

$= \langle (-12, 8), (4, 5) \rangle$

$$= 4\cdot(-12)(4) + 5\cdot(8)(5)$$

$$\underline{= 8}$$

$$k\langle \vec{u},\ \vec{v}\rangle = (-4)\big(4\cdot 3\cdot 4 + 5\cdot(-2)\cdot(5)\big)$$

$$\underline{= 8}$$

$$\langle k\vec{u},\ \vec{v}\rangle = k\langle \vec{u},\ \vec{v}\rangle \ \checkmark$$

e) $\left\langle \vec{0},\ \vec{v}\right\rangle = 4\cdot(0)(4) + 5\cdot(0)(5)$

$$\underline{= 0}$$

$$\left\langle \vec{v},\ \vec{0}\right\rangle = 4\cdot(4)(0) + 5\cdot(5)(0)$$

$$\underline{= 0}$$

$$\left\langle \vec{0},\ \vec{v}\right\rangle = \left\langle \vec{v},\ \vec{0}\right\rangle = 0 \quad \checkmark$$

Exercise

Let $\langle \vec{u},\ \vec{v}\rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (-3,\ 2)$, $\vec{v} = (5,\ 4)$, $\vec{w} = (1,\ -6)$, and $k = 2$. Verify the following for the weighted Euclidean inner product $\langle \vec{u},\ \vec{v}\rangle = 4u_1v_1 + 5u_2v_2$

a) $\langle \vec{u},\ \vec{v}\rangle = \langle \vec{v},\ \vec{u}\rangle$

b) $\langle \vec{u}+\vec{v},\ \vec{w}\rangle = \langle \vec{u},\ \vec{w}\rangle + \langle \vec{v},\ \vec{w}\rangle$

c) $\langle \vec{u},\ \vec{v}+\vec{w}\rangle = \langle \vec{u},\ \vec{v}\rangle + \langle \vec{u},\ \vec{w}\rangle$

d) $\langle k\vec{u},\ \vec{v}\rangle = k\langle \vec{u},\ \vec{v}\rangle$

e) $\left\langle \vec{0},\ \vec{v}\right\rangle = \left\langle \vec{v},\ \vec{0}\right\rangle = 0$

Solution

a) $\langle \vec{u},\ \vec{v}\rangle = 4(-3)(5) + 5(2)(4)$

$$= -60 + 40$$

$$\underline{= -20}$$

$$\langle \vec{v},\ \vec{u}\rangle = 4(5)(-3) + 5(4)(2)$$

$$= -60 + 40$$

$$\underline{= -20}$$

$$\langle \vec{u},\ \vec{v}\rangle = \langle \vec{v},\ \vec{u}\rangle \quad \checkmark$$

b) $\langle \vec{u}+\vec{v},\ \vec{w}\rangle = \big\langle (-3,\ 2) + (5,\ 4),\ (1,\ -6)\big\rangle$

$$= \big\langle (2,\ 6),\ (1,\ -6)\big\rangle$$

$$= 4\cdot(2)(1) + 5\cdot(6)(-6)$$

$$= 8 - 180$$

$$= -172$$

$$\langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle = (4\cdot(-3)(1) + 5\cdot(2)(-6)) + (4\cdot(5)(1) + 5\cdot(4)(-6))$$

$$= (-12 - 60) + (20 - 120)$$

$$= -172$$

$$\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \quad \checkmark$$

c) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle (-3, 2), (5, 4) + (1, -6) \rangle$

$$= \langle (-3, 2), (6, -2) \rangle$$

$$= 4\cdot(-3)(6) + 5\cdot(2)(-2)$$

$$= -72 - 20$$

$$= -92$$

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle = (4\cdot(-3)(5) + 5\cdot(2)(4)) + (4\cdot(-3)(1) + 5\cdot(2)(-6))$$

$$= (-60 + 40) + (-12 - 60)$$

$$= -92$$

$$\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle \quad \checkmark$$

d) $\langle k\vec{u}, \vec{v} \rangle = \langle 2(-3, 2), (5, 4) \rangle$

$$= \langle (-6, 4), (5, 4) \rangle$$

$$= 4\cdot(-6)(5) + 5\cdot(4)(4)$$

$$= -120 + 80$$

$$= -40$$

$$k\langle \vec{u}, \vec{v} \rangle = 2\cdot(4\cdot(-3)(5) + 5\cdot(2)(4))$$

$$= 2\cdot(-60 + 40)$$

$$= -40$$

$$\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle \quad \checkmark$$

e) $\langle \vec{0}, \vec{v} \rangle = 4\cdot(0)(5) + 5\cdot(0)(4)$

$$= 0$$

$$\langle \vec{v}, \vec{0} \rangle = 4\cdot(5)(0) + 5\cdot(4)(0)$$

$$= 0$$

$$\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0 \quad \checkmark$$

Exercise

Let $\langle \vec{u},\ \vec{v}\rangle$ be the Euclidean inner product on $\mathbb{R}^3$, and let $\vec{u}=(2,\ 1,\ -2)$, $\vec{v}=(-1,\ 3,\ 2)$, $\vec{w}=(2,\ 1,\ 0)$ and $k=2$.

Compute the following for the weighted Euclidean inner product $\langle \vec{u},\ \vec{v}\rangle = 2u_1v_1 + 3u_2v_2 + u_3v_3$.

a) $\langle \vec{u},\ \vec{v}\rangle$

b) $\langle k\vec{v},\ \vec{w}\rangle$

c) $\langle \vec{u}+\vec{v},\ \vec{w}\rangle$

d) $\|\vec{v}\|$

e) $d(\vec{u},\ \vec{v})$

f) $\|\vec{u}-k\vec{v}\|$

Solution

a) $$\begin{aligned}\langle \vec{u},\ \vec{v}\rangle &= 2\cdot(2)(-1)+3(1)(3)+(-2)(2)\\ &= -4+9-4\\ &= 1\end{aligned}$$

b) $$\begin{aligned}\langle k\vec{v},\ \vec{w}\rangle &= \langle 2(-1,\ 3,\ 2),\ \ (2,\ 1,\ 0)\rangle\\ &= \langle(-2,\ 6,\ 4),\ \ (2,\ 1,\ 0)\rangle\\ &= 2\cdot(-2)(2)+3(6)(1)+(4)(0)\\ &= -8+18\\ &= 10\end{aligned}$$

c) $$\begin{aligned}\langle \vec{u}+\vec{v},\ \vec{w}\rangle &= \langle(2,\ 1,\ -2)+(-1,\ 3,\ 2),\ \ (2,\ 1,\ 0)\rangle\\ &= \langle(1,\ 4,\ 0),\ \ (2,\ 1,\ 0)\rangle\\ &= 2\cdot(1)(2)+3(4)(1)+(0)(0)\\ &= 14\end{aligned}$$

d) $$\begin{aligned}\|\vec{v}\| &= \sqrt{\langle \vec{v},\ \vec{v}\rangle}\\ &= \sqrt{2\cdot(-1)(-1)+3(3)(3)+(2)(2)}\\ &= \sqrt{2+27+4}\\ &= \sqrt{33}\end{aligned}$$

e) $$\begin{aligned}d(\vec{u},\ \vec{v}) &= \|(\vec{u}-\vec{v})\|\\ &= \|(2,\ 1,\ -2)-(-1,\ 3,\ 2)\|\\ &= \|\langle 3,\ -2,\ -4\rangle\|\\ &= \sqrt{2\cdot(3)(3)+3(-2)(-2)+(-4)(-4)}\\ &= \sqrt{18+12+16}\\ &= \sqrt{29}\end{aligned}$$

f) $\left\| \vec{u} - k\vec{v} \right\| = \left\| (2,\ 1,\ -2) - 2(-1,\ 3,\ 2) \right\|$

$$= \left\| (2,\ 1,\ -2) - (-2,\ 6,\ 4) \right\|$$

$$= \left\| \langle (4,\ -5,\ -6) \rangle \right\|$$

$$= \sqrt{2(16) + 3(25) + 36}$$

$$= \sqrt{32 + 75 + 36}$$

$$= \sqrt{143}$$

Exercise

Let $\vec{u} = (u_1,\ u_2)$ and $\vec{v} = (v_1,\ v_2)$. Show that the following are inner product on $\mathbb{R}^2$ by verifying that the inner product axioms hold. $\langle \vec{u},\ \vec{v} \rangle = 3u_1v_1 + 5u_2v_2$

Solution

Let $\vec{w} = (w_1,\ w_2)$

Axiom 1: $\langle \vec{u},\ \vec{v} \rangle = 3u_1v_1 + 5u_2v_2$

$$= 3v_1u_1 + 5v_2u_2$$

$$= \langle \vec{v},\ \vec{u} \rangle \quad \checkmark$$

Axiom 2: $\langle \vec{u} + \vec{v},\ \vec{w} \rangle = 3(u_1 + v_1)w_1 + 5(u_2 + v_2)w_2$

$$= 3(u_1w_1 + v_1w_1) + 5(u_2w_2 + v_2w_2)$$

$$= 3u_1w_1 + 3v_1w_1 + 5u_2w_2 + 5v_2w_2$$

$$= (3u_1w_1 + 5u_2w_2) + (3v_1w_1 + 5v_2w_2)$$

$$= \langle \vec{u},\ \vec{w} \rangle + \langle \vec{v},\ \vec{w} \rangle \quad \checkmark$$

Axiom 3: $\langle k\vec{u},\ \vec{v} \rangle = 3(ku_1)v_1 + 5(ku_2)v_2$

$$= k(3u_1v_1 + 5u_2v_2)$$

$$= k\langle \vec{u},\ \vec{v} \rangle \quad \checkmark$$

Axiom 4: $\langle \vec{v},\ \vec{v} \rangle = 3v_1v_1 + 5v_2v_2$

$$= 3v_1^2 + 5v_2^2 \ge 0$$

$$v_1 = v_2 = 0 \quad \textit{iff} \quad \vec{v} = \vec{0} \quad \checkmark$$

Exercise

Let $\vec{u}=(u_1,\ u_2,\ u_3)$ and $\vec{v}=(v_1,\ v_2,\ v_3)$. Show that the following are inner product on $\mathbb{R}^3$ by verifying that the inner product axioms hold. $\langle \vec{u},\ \vec{v}\rangle = 2u_1v_1+3u_2v_2+5u_3v_3$

Solution

Let $\vec{w}=(w_1,\ w_2,\ w_3)$

Axiom 1: $\langle \vec{u},\ \vec{v}\rangle = 2u_1v_1+3u_2v_2+5u_3v_3$

$$= 2v_1u_1+3v_2u_2+5v_3u_3$$

$$=\langle \vec{v},\ \vec{u}\rangle \quad \checkmark$$

Axiom 2: $\langle \vec{u}+\vec{v},\ \vec{w}\rangle = \left\langle (u_1,\ u_2,\ u_3)+(v_1,\ v_2,\ v_3),\ (w_1,\ w_2,\ w_3)\right\rangle$

$$= \left\langle (u_1+v_1,\ u_2+v_2,\ u_3+v_3),\ (w_1,\ w_2,\ w_3)\right\rangle$$

$$= 2(u_1+v_1)w_1+3(u_2+v_2)w_2+5(u_3+v_3)w_3$$

$$= 2u_1w_1+2v_1w_1+3u_2w_2+3v_2w_2+5u_3w_3+5v_3w_3$$

$$= (2u_1w_1+3u_2w_2+5u_3w_3)+(2v_1w_1+3v_2w_2+5v_3w_3)$$

$$= \langle \vec{u},\ \vec{w}\rangle + \langle \vec{v},\ \vec{w}\rangle \quad \checkmark$$

Axiom 3: $\langle k\vec{u},\ \vec{v}\rangle = \left\langle k(u_1,\ u_2,\ u_3),\ (v_1,\ v_2,\ v_3)\right\rangle$

$$= \left\langle (ku_1,\ ku_2,\ ku_3),\ (v_1,\ v_2,\ v_3)\right\rangle$$

$$= 2(ku_1)v_1+3(ku_2)v_2+5(ku_3)v_3$$

$$= k(2u_1v_1+3u_2v_2+5u_3v_3)$$

$$= k\langle \vec{u},\ \vec{v}\rangle \quad \checkmark$$

Axiom 4: $\langle \vec{v},\ \vec{v}\rangle = 2v_1v_1+3v_2v_2+5v_3v_3$

$$= 2v_1^2+3v_2^2+5v_3^2 \ge 0$$

$$v_1=v_2=v_3=0 \quad \mathit{iff} \quad \vec{v}=\vec{0} \quad \checkmark$$

Exercise

Show that the following identity holds for the vectors in any inner product space

$$\|\vec{u}+\vec{v}\|^2+\|\vec{u}-\vec{v}\|^2=2\|\vec{u}\|^2+2\|\vec{v}\|^2$$

Solution

$$\begin{aligned}\|\vec{u}+\vec{v}\|^2+\|\vec{u}-\vec{v}\|^2 &= \langle\vec{u}+\vec{v},\ \vec{u}+\vec{v}\rangle+\langle\vec{u}-\vec{v},\ \vec{u}-\vec{v}\rangle\\ &= \langle\vec{u},\ \vec{u}+\vec{v}\rangle+\langle\vec{v},\ \vec{u}+\vec{v}\rangle+\langle\vec{u},\ \vec{u}-\vec{v}\rangle-\langle\vec{v},\ \vec{u}-\vec{v}\rangle\\ &= \langle\vec{u},\ \vec{u}\rangle+\langle\vec{u},\ \vec{v}\rangle+\langle\vec{v},\ \vec{u}\rangle+\langle\vec{v},\ \vec{v}\rangle+\langle\vec{u},\ \vec{u}\rangle-\langle\vec{u},\ \vec{v}\rangle-\langle\vec{v},\ \vec{u}\rangle+\langle\vec{v},\ \vec{v}\rangle\\ &= 2\langle\vec{u},\ \vec{u}\rangle+2\langle\vec{v},\ \vec{v}\rangle\\ &= 2\|\vec{u}\|^2+2\|\vec{v}\|^2 \quad \checkmark\end{aligned}$$

Exercise

Show that the following identity holds for the vectors in any inner product space

$$\langle\vec{u},\ \vec{v}\rangle=\frac{1}{4}\left(\|\vec{u}+\vec{v}\|^2-\|\vec{u}-\vec{v}\|^2\right)$$

Solution

$$\begin{aligned}\|\vec{u}+\vec{v}\|^2 &= \langle\vec{u}+\vec{v},\ \vec{u}+\vec{v}\rangle\\ &= \|\vec{u}\|^2+2\langle\vec{u},\ \vec{v}\rangle+\|\vec{v}\|^2\end{aligned}$$

$$\begin{aligned}\|\vec{u}-\vec{v}\|^2 &= \langle\vec{u}-\vec{v},\ \vec{u}-\vec{v}\rangle\\ &= \|\vec{u}\|^2-2\langle\vec{u},\ \vec{v}\rangle+\|\vec{v}\|^2\end{aligned}$$

$$\begin{array}{rl} & \|\vec{u}+\vec{v}\|^2=\|\vec{u}\|^2+2\langle\vec{u},\ \vec{v}\rangle+\|\vec{v}\|^2\\ - & \|\vec{u}-\vec{v}\|^2=\|\vec{u}\|^2-2\langle\vec{u},\ \vec{v}\rangle+\|\vec{v}\|^2\\ \hline & \|\vec{u}+\vec{v}\|^2-\|\vec{u}-\vec{v}\|^2=4\langle\vec{u},\ \vec{v}\rangle\end{array}$$

$$\langle\vec{u},\ \vec{v}\rangle=\frac{1}{4}\left(\|\vec{u}+\vec{v}\|^2-\|\vec{u}-\vec{v}\|^2\right) \quad \checkmark$$

Exercise

Prove that $\|k\vec{v}\|=|k|\ \|\vec{v}\|$

Solution

$$\begin{aligned}\|k\vec{v}\|^2 &= \langle k\vec{v},\ \vec{v}\rangle \\ &= k^2\langle \vec{v},\ \vec{v}\rangle \\ &= k^2\|\vec{v}\|^2\end{aligned}$$

$$\|k\vec{v}\| = k\ \|\vec{v}\| \quad \checkmark$$

Solution *Section* 3.2 – Angle and Orthogonality in Inner Product Spaces

Exercise

Which of the following form orthonormal set?

$$\{(1,\ 0),\ (0,\ 2)\} \quad in\ \mathbb{R}^2$$

Solution

$$(1,\ 0)\cdot(0,\ 2) = 1(0)+0(2)$$
$$= 0$$

Therefore, they are ***orthonormal*** set.

Exercise

Which of the following form orthonormal set?

$$\{(2,\ -4),\ (2,\ 1)\} \quad in\ \mathbb{R}^2$$

Solution

$$(2,\ -4)\cdot(2,\ 1) = 4-4$$
$$= 0$$

Therefore, they are ***orthonormal*** set.

Exercise

Which of the following form orthonormal set?

$$\left\{\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)\right\} \quad in\ \mathbb{R}^2$$

Solution

$$\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}+\left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}$$
$$= \frac{1}{2}-\frac{1}{2}$$
$$= 0$$

Therefore, they are ***orthonormal*** set.

Exercise

Which of the following form orthonormal set?

$$\left\{\left(-\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)\right\} \quad in\ \mathbb{R}^2$$

Solution

$$\left(-\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right) = \left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}+\left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}$$

$$=-\frac{1}{2}-\frac{1}{2}$$

$$=-1 \neq 0$$

Therefore, they are ***not orthonormal*** set

Exercise

Which of the following form orthonormal set?

$$\left\{\left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right),\ \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)\right\} \quad in\ \mathbb{R}^3$$

Solution

$$\left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right)\cdot\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)=\frac{1}{\sqrt{6}}\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{6}}\left(-\frac{1}{\sqrt{2}}\right)+0$$

$$=0$$

Therefore, they are ***orthonormal*** set.

Exercise

Which of the following form orthonormal set?

$$\{(4,\ -1,\ 1),\ (-1,\ 0,\ 4),\ (-4,\ -17,\ -1)\} \quad in\ \mathbb{R}^3$$

Solution

$$(4,\ -1,\ 1)\cdot(-1,\ 0,\ 4)\cdot(-4,\ -17,\ -1)=16+0-4$$

$$=12 \neq 0$$

Therefore, they are ***not orthonormal*** set.

Exercise

Which of the following form orthonormal set?

$$\left\{\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right),\ \left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right),\ \left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right)\right\} \quad in\ \mathbb{R}^3$$

Solution

$$\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right)\cdot\left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right)\cdot\left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right) = \frac{2}{3}\frac{2}{3}\frac{1}{3}+\left(-\frac{2}{3}\right)\frac{1}{3}\frac{2}{3}+\frac{1}{3}\left(-\frac{2}{3}\right)\frac{2}{3}$$

$$=\frac{4}{27}-\frac{4}{27}-\frac{4}{27}$$

$$=-\frac{4}{27}\neq 0$$

Therefore, they are ***not orthonormal*** set.

Exercise

Which of the following form orthonormal set?

$$\left\{\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right),\ \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\right\} \quad in\ \mathbb{R}^3$$

Solution

$$\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)\cdot\left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$=\frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{2}}\right)+0+\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{3}}\right)\frac{1}{\sqrt{2}}$$

$$=-\frac{1}{2}\frac{1}{\sqrt{3}}-\frac{1}{2}\frac{1}{\sqrt{3}}$$

$$=-\frac{1}{\sqrt{3}}\neq 0$$

Therefore, they are ***not orthonormal*** set.

Exercise

Which of the following form orthonormal set?

$$\left\{\left(\frac{1}{\sqrt{2}},\ 0,\ 0,\ \frac{1}{\sqrt{2}}\right),\ \left(0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right),\ \left(-\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2}\right)\right\} \quad in\ \mathbb{R}^4$$

Solution

$$\left(\frac{1}{\sqrt{2}},\ 0,\ 0,\ \frac{1}{\sqrt{2}}\right)\cdot\left(0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)\cdot\left(-\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2}\right)=0+0+0+0$$

$$\boxed{= 0}$$

Therefore, they are ***orthonormal*** set.

Exercise

Which of the following form orthonormal set?

$$\left\{\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right),\ \left(0,\ 0,\ -\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right),\ \left(-\frac{1}{3},\ \frac{1}{3},\ -\frac{1}{3},\ \frac{1}{3}\right)\right\} \quad in\ \mathbb{R}^4$$

Solution

$$\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)\bullet\left(0,\ 0,\ -\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right)\bullet\left(-\frac{1}{3},\ \frac{1}{3},\ -\frac{1}{3},\ \frac{1}{3}\right) = 0+0+0+0$$

$$\boxed{= 0}$$

Therefore, they are ***orthonormal*** set.

Exercise

Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.

$$\vec{u} = (1,\ -3), \quad \vec{v} = (2,\ 4)$$

Solution

$$\|\vec{u}\| = \sqrt{1^2 + (-3)^2}$$

$$\boxed{= \sqrt{10}}$$

$$\|\vec{v}\| = \sqrt{2^2 + 4^2}$$

$$\boxed{= \sqrt{20}}$$

$$\langle \vec{u},\ \vec{v}\rangle = 1(2) + (-3)(4)$$

$$\boxed{= -10}$$

$$\cos\theta = \frac{-10}{\sqrt{10}\ \sqrt{20}} \qquad\qquad \cos\theta = \frac{\langle \vec{u},\ \vec{v}\rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$

$$= -\frac{10}{\sqrt{200}}$$

$$\boxed{= -\frac{1}{\sqrt{2}}}$$

Exercise

Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.

$$\vec{u}=(-1,\ 0);\quad \vec{v}=(3,\ 8)$$

Solution

$$\|\vec{u}\|=\sqrt{(-1)^2+0^2}$$
$$=1$$

$$\|\vec{v}\|=\sqrt{3^2+8^2}$$
$$=\sqrt{73}$$

$$\langle\vec{u},\ \vec{v}\rangle=(-1)(3)+(0)(8)$$
$$=-3$$

$$\cos\theta=\frac{-3}{1\sqrt{73}} \qquad \cos\theta=\frac{\langle\vec{u},\ \vec{v}\rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$
$$=-\frac{3}{\sqrt{73}}$$

Exercise

Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.

$$\vec{u}=(-1,\ 5,\ 2);\quad \vec{v}=(2,\ 4,\ -9)$$

Solution

$$\|\vec{u}\|=\sqrt{(-1)^2+5^2+2^2}$$
$$=\sqrt{30}$$

$$\|\vec{v}\|=\sqrt{2^2+4^2+(-9)^2}$$
$$=\sqrt{101}$$

$$\langle\vec{u},\ \vec{v}\rangle=(-1)(2)+(5)(4)+(2)(-9)$$
$$=0$$

$$\cos\theta=0 \qquad \cos\theta=\frac{\langle\vec{u},\ \vec{v}\rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$

Exercise

Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.

$$\vec{u} = (4,\ 1,\ 8); \quad \vec{v} = (1,\ 0,\ -3)$$

Solution

$$\|\vec{u}\| = \sqrt{4^2 + 1^2 + 8^2}$$
$$= 9$$

$$\|\vec{v}\| = \sqrt{1 + 0 + 9}$$
$$= \sqrt{10}$$

$$\langle \vec{u},\ \vec{v} \rangle = (4)(1) + (1)(0) + (8)(-3)$$
$$= -20$$

$$\cos\theta = -\frac{20}{9\sqrt{10}} \qquad \cos\theta = \frac{\langle \vec{u},\ \vec{v} \rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$

Exercise

Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.

$$\vec{u} = (1,\ 0,\ 1,\ 0); \quad \vec{v} = (-3,\ -3,\ -3,\ -3)$$

Solution

$$\|\vec{u}\| = \sqrt{2}$$

$$\|\vec{v}\| = \sqrt{9 + 9 + 9 + 9}$$
$$= 12$$

$$\langle \vec{u},\ \vec{v} \rangle = -3 + 0 - 3 + 0$$
$$= -6$$

$$\cos\theta = \frac{-6}{12\sqrt{2}} \qquad \cos\theta = \frac{\langle \vec{u},\ \vec{v} \rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$
$$= -\frac{1}{2\sqrt{2}}$$

Exercise

Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.

$$\vec{u} = (2,\ 1,\ 7,\ -1);\quad \vec{v} = (4,\ 0,\ 0,\ 0)$$

Solution

$$\|\vec{u}\| = \sqrt{2^2 + 1^2 + 7^2 + (-1)^2}$$

$$= \sqrt{55}$$

$$\|\vec{v}\| = \sqrt{4^2 + 0}$$

$$= 4$$

$$\langle \vec{u},\ \vec{v} \rangle = (2)(4) + (1)(0) + (7)(0) + (-1)(0)$$

$$= 8$$

$$\cos\theta = \frac{8}{4\sqrt{55}} \qquad\qquad \cos\theta = \frac{\langle \vec{u},\ \vec{v} \rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$

$$= \frac{2}{\sqrt{55}}$$

Exercise

Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.

$$\vec{u} = (1,\ 3,\ -5,\ 4),\quad \vec{v} = (2,\ -4,\ 4,\ 1)$$

Solution

$$\|\vec{u}\| = \sqrt{1 + 9 + 25 + 16}$$

$$= \sqrt{51}$$

$$\|\vec{v}\| = \sqrt{4 + 16 + 16 + 1}$$

$$= \sqrt{37}$$

$$\langle \vec{u},\ \vec{v} \rangle = 2 - 12 - 20 + 4$$

$$= -26$$

$$\cos\theta = \frac{-26}{\sqrt{51}\sqrt{37}} \qquad\qquad \cos\theta = \frac{\langle \vec{u},\ \vec{v} \rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$

Exercise

Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.

$$\vec{u}=(1,\ 2,\ 3,\ 4),\quad \vec{v}=(-1,\ -2,\ -3,\ -4)$$

Solution

$$\|\vec{u}\|=\sqrt{1+4+9+16}$$
$$=\sqrt{30}$$

$$\|\vec{v}\|=\sqrt{1+4+9+16}$$
$$=\sqrt{30}$$

$$\langle\vec{u},\ \vec{v}\rangle=-1-4-9-16$$
$$=-30$$

$$\cos\theta=\frac{-30}{\sqrt{30}\sqrt{30}}\qquad\qquad \cos\theta=\frac{\langle\vec{u},\ \vec{v}\rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$
$$=-1$$

Exercise

Find the cosine of the angle between ***A*** and ***B***.

$$A=\begin{pmatrix}2&6\\1&-3\end{pmatrix}\quad B=\begin{pmatrix}3&2\\1&0\end{pmatrix}$$

Solution

$$\|A\|=\sqrt{\langle A,\ A\rangle}$$
$$=\sqrt{2^2+6^2+1^2+(-3)^2}$$
$$=\sqrt{50}$$
$$=5\sqrt{2}$$

$$\|B\|=\sqrt{\langle B,\ B\rangle}$$
$$=\sqrt{9+4+1+0}$$
$$=\sqrt{14}$$

$$\langle A,\ B\rangle=2(3)+6(2)+1(1)+(-3)(0)$$
$$=19$$

$$\cos\theta=\frac{19}{5\sqrt{2}\sqrt{14}}\qquad\qquad \cos\theta=\frac{\langle A,\ B\rangle}{\|A\|\ \|B\|}$$

$$= \boxed{\frac{19}{10\sqrt{7}}}$$

Exercise

Find the cosine of the angle between ***A*** and ***B***.

$$A = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$$

Solution

$$\|A\| = \sqrt{\langle A,\ A\rangle}$$
$$= \sqrt{2^2 + 4^2 + (-1)^2 + 3^2}$$
$$= \boxed{\sqrt{30}}$$

$$\|B\| = \sqrt{\langle B,\ B\rangle}$$
$$= \sqrt{(-3)^2 + 1^2 + 4^2 + 2^2}$$
$$= \boxed{\sqrt{30}}$$

$$\langle A,\ B\rangle = 2(-3) + 4(1) + (-1)(4) + 3(2)$$
$$= \boxed{0}$$

$$\cos\theta = \frac{0}{30} \qquad\qquad \cos\theta = \frac{\langle A,\ B\rangle}{\|A\|\ \|B\|}$$
$$= \boxed{0}$$

Exercise

Find the cosine of the angle between ***A*** and ***B***.

$$A = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Solution

$$\|A\| = \sqrt{81 + 64 + 49 + 36 + 25 + 16} \qquad\qquad \|A\| = \sqrt{\langle A,\ A\rangle}$$
$$= \boxed{\sqrt{271}}$$

$$\|B\| = \sqrt{1 + 4 + 9 + 16 + 25 + 36} \qquad\qquad \|B\| = \sqrt{\langle B,\ B\rangle}$$
$$= \boxed{\sqrt{91}}$$

$$\langle A,\ B\rangle = 9+16+21+24+25+24$$
$$= 119$$

$$\cos\theta = \frac{119}{\sqrt{271}\sqrt{91}} \qquad\qquad \cos\theta = \frac{\langle A,\ B\rangle}{\|A\|\ \|B\|}$$

Exercise

Find the cosine of the angle between ***A*** and ***B***.

$$A = \begin{pmatrix} 1 & -2 & 7 \\ 6 & -3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 3 \\ 6 & 5 & -4 \end{pmatrix}$$

Solution

$$\|A\| = \sqrt{1^2+(-2)^2+7^2+6^2+(-3)^2+4^2} \qquad\qquad \|A\| = \sqrt{\langle A,\ A\rangle}$$
$$= \sqrt{115}$$

$$\|B\| = \sqrt{1^2+(-2)^2+3^2+6^2+5^2+(-4)^2} \qquad\qquad \|B\| = \sqrt{\langle B,\ B\rangle}$$
$$= \sqrt{91}$$

$$\langle A,\ B\rangle = 1+4+21+36-15-16$$
$$= 31$$

$$\cos\theta = \frac{31}{\sqrt{115}\sqrt{91}} \qquad\qquad \cos\theta = \frac{\langle A,\ B\rangle}{\|A\|\ \|B\|}$$

Exercise

Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

$$\vec{u} = (-1,\ 3,\ 2), \quad \vec{v} = (4,\ 2,\ -1)$$

Solution

$$\langle \vec{u},\ \vec{v}\rangle = (-1)(4)+3(2)+2(-1)$$
$$= 0$$

Therefore, the given vectors are orthogonal.

Exercise

Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

$$\vec{u} = (a,\ b), \quad \vec{v} = (-b,\ a)$$

Solution

$$\langle \vec{u},\ \vec{v} \rangle = a(-b) + b(a)$$
$$= 0$$

Therefore, the given vectors are orthogonal.

Exercise

Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

$$\vec{u} = (-2,\ -2,\ -2), \quad \vec{v} = (1,\ 1,\ 1)$$

Solution

$$\langle \vec{u},\ \vec{v} \rangle = (-2)(1) + (-2)(1) + (-2)(1)$$
$$= -6$$

Therefore, the given vectors are ***not*** orthogonal.

Exercise

Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

$$\vec{u} = (-4,\ 6,\ -10,\ 1), \quad \vec{v} = (2,\ 1,\ -2,\ 9)$$

Solution

$$\langle\langle \vec{u},\ \vec{v} \rangle\rangle = (-4)(2) + (6)(1) + (-10)(-2) + (1)(9)$$
$$= 27 \quad \neq 0$$

Therefore, the given vectors are ***not*** orthogonal.

Exercise

Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

$$\vec{u} = (-4,\ 6,\ -10,\ 1), \quad \vec{v} = (2,\ 1,\ -2,\ 9)$$

Solution

$$\|\vec{u}\| = \sqrt{(-4)^2 + 6^2 + (-10)^2 + 1^2}$$

$$= \sqrt{153}$$

$$\underline{= 3\sqrt{17}}$$

$$\|\vec{v}\| = \sqrt{2^2 + 1^2 + (-2)^2 + 9^2}$$

$$= \sqrt{90}$$

$$\underline{= 3\sqrt{10}}$$

$$\langle \vec{u},\ \vec{v} \rangle = (-4)(2) + 6(1) - 10(-2) + 1(9)$$

$$\underline{= 27}$$

$$\cos\theta = \frac{27}{3\sqrt{17}\left(3\sqrt{10}\right)} \qquad\qquad \cos\theta = \frac{\langle \vec{u},\ \vec{v} \rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$

$$\underline{= \frac{3}{\sqrt{170}}}$$

The vectors $\vec{u}$ *and* $\vec{v}$ are ***not*** orthogonal with respect to the Euclidean

Exercise

Do there exist scalars k and l such that the vectors $\vec{u} = (2,\ k,\ 6)$, $\vec{v} = (l,\ 5,\ 3)$, *and* $\vec{w} = (1,\ 2,\ 3)$ are mutually orthogonal with respect to the Euclidean inner product?

Solution

$$\langle \vec{u},\ \vec{w} \rangle = (2)(1) + (k)(2) + (6)(3)$$

$$= 20 + 2k = 0$$

$$\Rightarrow \underline{k = -10}$$

$$\langle \vec{v},\ \vec{w} \rangle = (l)(1) + (5)(2) + (3)(3)$$

$$= l + 19 = 0$$

$$\Rightarrow \underline{l = -19}$$

$$\langle \vec{u},\ \vec{v} \rangle = (2)(l) + (k)(5) + (6)(3)$$

$$= 2l + 5k + 18 = 0$$

$$2(-19) + 5(-10) + 18 = -70 \neq 0$$

Thus, there are no scalars such that the vectors are mutually orthogonal.

Exercise

Let $\mathbb{R}^3$ have the Euclidean inner product. For which values of k are $\vec{u}$ and $\vec{v}$ orthogonal?

a) $\vec{u}=(2,\ 1,\ 3),\quad \vec{v}=(1,\ 7,\ k)$

b) $\vec{u}=(k,\ k,\ 1),\quad \vec{v}=(k,\ 5,\ 6)$

Solution

a) $\langle\vec{u},\ \vec{v}\rangle=(2)(1)+(1)(7)+(3)(k)$

$$=9+3k=0$$

$\vec{u}$ and $\vec{v}$ are orthogonal for $\boxed{k=-3}$

b) $\langle\vec{u},\ \vec{v}\rangle=(k)(k)+(k)(5)+(1)(6)$

$$=k^2+5k+6=0$$

$\vec{u}$ and $\vec{v}$ are orthogonal for $\boxed{k=-2,\ -3}$

Exercise

Let V be an inner product space. Show that if $\vec{u}$ and $\vec{v}$ are orthogonal unit vectors in V, then $\|\vec{u}-\vec{v}\|=\sqrt{2}$

Solution

$$\begin{aligned}\|\vec{u}-\vec{v}\|^2 &= \langle\vec{u}-\vec{v},\ \vec{u}-\vec{v}\rangle\\ &= \langle\vec{u},\ \vec{u}-\vec{v}\rangle-\langle\vec{v},\ \vec{u}-\vec{v}\rangle\\ &= \langle\vec{u},\ \vec{u}\rangle-\langle\vec{u},\ \vec{v}\rangle-\langle\vec{u},\ \vec{v}\rangle+\langle\vec{v},\ \vec{v}\rangle\\ &= \|\vec{u}\|^2-0-0+\|\vec{v}\|^2 && \text{since } \vec{u} \text{ and } \vec{v} \text{ are orthogonal unit vectors}\\ &= 1+1\\ &= 2\end{aligned}$$

Thus $\boxed{\|\vec{u}-\vec{v}\|=\sqrt{2}}$

Exercise

Let $\boldsymbol{S}$ be a subspace of $\mathbb{R}^n$. Explain what $\left(\boldsymbol{S}^{\perp}\right)^{\perp}=\boldsymbol{S}$ means and why it is true.

Solution

$\left(S^{\perp}\right)^{\perp}$ is the orthogonal complement of , $S^{\perp}$, which is itself the orthogonal complement of S, so $\left(S^{\perp}\right)^{\perp}=S$ means that S is the orthogonal of its orthogonal complement.

We need to show that S is contained in $\left(S^{\perp}\right)^{\perp}$ and, conversely, that $\left(S^{\perp}\right)^{\perp}$ is contained in S to be true.

i. Suppose $\vec{v}\in S^{\perp}$ and $\vec{w}\in S^{\perp}$. Then $\langle \vec{v},\ \vec{w}\rangle=0$ by definition of $S^{\perp}$.

Thus, S is certainly contained is $\left(S^{\perp}\right)^{\perp}$ (which consists of all vectors in $\mathbb{R}^n$ which are orthogonal to $S^{\perp}$).

ii. Suppose $\vec{v}\in\left(S^{\perp}\right)^{\perp}$ (means $\vec{v}$ is orthogonal to all vectors in $S^{\perp}$); then we need to show that $\vec{v}\in S$.

Let assume $\left\{\vec{u}_1,\ \vec{u}_2,\ \ldots,\ \vec{u}_p\right\}$ be a basis for S and let $\left\{\vec{w}_1,\ \vec{w}_2,\ \ldots,\ \vec{w}_q\right\}$ be a basis for $S^{\perp}$.

If $\vec{v}\notin S$, then $\left\{\vec{u}_1,\ \vec{u}_2,\ \ldots,\ \vec{u}_p,\ \vec{v}\right\}$ is linearly independent set. Since each vector ifs that set is orthogonal to all of $S^{\perp}$, the set $\left\{\vec{u}_1,\ \vec{u}_2,\ \ldots,\ \vec{u}_p,\ \vec{v},\ \vec{w}_1,\ \vec{w}_2,\ \ldots,\ \vec{w}_q\right\}$ is linearly independent.

Since there are $p+q+1$ vectors in this set, this means that $p+q+1\leq n \quad \Leftrightarrow \quad p+q\leq n-1$.

On the other hand, If A is the matrix whose i^{th} row is $\vec{u}_i^T$, then the row space of A is S and the nullspace of A is $S^{\perp}$.

Since S is p-dimensional, the rank of A is p, meaning that the dimension of $\text{nul}(A)=S^{\perp}$ is $q=n-p$. Therefore,

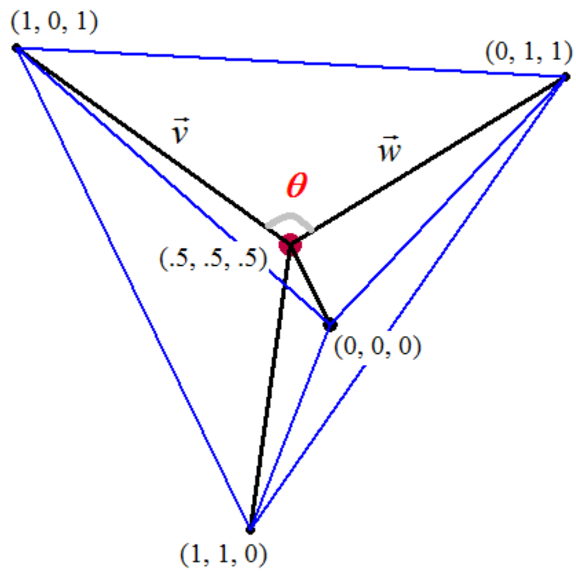
$$p+q=p+(n-p)=n$$

Which contradict the fact that $p+q\leq n-1$. From this, we see that, if $\vec{v}\in\left(S^{\perp}\right)^{\perp}$, it must be the case that $\vec{v}\in S$.

Exercise

The methane molecule CH_4 is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at $(0,\ 0,\ 0)$, $(1,\ 1,\ 0)$, $(1,\ 0,\ 1)$ and $(0,\ 1,\ 1)$ – (***note*** that all six edges have length $\sqrt{2}$, so the tetrahedron is regular). What is the cosine of the angle between the rays going from the center $\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$ to the vertices?

Solution



Let $\vec{v}$ be the vector of the segment (1, 0, 1) and $\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$

$$\vec{v}=\begin{pmatrix}1\\0\\1\end{pmatrix}-\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\end{pmatrix}$$

Let be the vector of the segment (0, 1, 1) and $\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$

$$\vec{v}=\begin{pmatrix}0\\1\\1\end{pmatrix}-\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

We have:

$$\cos\theta = \frac{\langle \vec{v},\, \vec{w} \rangle}{\|\vec{v}\|\, \|\vec{w}\|}$$

$$= \frac{\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) \cdot \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}{\sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}\, \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}}$$

$$= \frac{-\frac{1}{4}}{\frac{3}{4}}$$

$$= -\frac{1}{3}$$

$$\theta \approx 109.47°$$

Exercise

Determine if the given vectors are orthogonal.

$$\vec{x}_1 = (1,\ 0,\ 1,\ 0), \quad \vec{x}_2 = (0,\ 1,\ 0,\ 1), \quad \vec{x}_3 = (1,\ 0,\ -1,\ 0), \quad \vec{x}_4 = (1,\ 1,\ -1,\ -1)$$

Solution

$$\vec{x}_1 \bullet \vec{x}_2 = (1,\ 0,\ 1,\ 0) \bullet (0,\ 1,\ 0,\ 1)$$

$$= 0$$

$$\vec{x}_1 \bullet \vec{x}_3 = (1,\ 0,\ 1,\ 0) \bullet (1,\ 0,\ -1,\ 0)$$

$$= 1 - 1$$

$$= 0$$

$$\vec{x}_1 \bullet \vec{x}_4 = (1,\ 0,\ 1,\ 0) \bullet (1,\ 1,\ -1,\ -1)$$

$$= 1 - 1$$

$$= 0$$

$$\vec{x}_2 \bullet \vec{x}_3 = (0,\ 1,\ 0,\ 1) \bullet (1,\ 0,\ -1,\ 0)$$

$$= 0$$

$$\vec{x}_2 \bullet \vec{x}_4 = (0,\ 1,\ 0,\ 1) \bullet (1,\ 1,\ -1,\ -1)$$

$$= 1 - 1$$

$$\underline{= 0}$$

$$\vec{x}_3 \bullet \vec{x}_4 = (1,\ 0,\ -1,\ 0) \bullet (1,\ 1,\ -1,\ -1)$$
$$= 1 - 1$$
$$\underline{= 0}$$

The given vectors are ***orthogonal.***

Exercise

Which of the following sets of vectors are orthogonal with respect to the Euclidean inner

$$\left\{\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right),\ \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\right\}$$

Solution

$$\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) \bullet \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}}$$
$$\underline{= 0}$$

$$\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) \bullet \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) = -\frac{1}{2} + 0 + \frac{1}{2}$$
$$\underline{= 0}$$

$$\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right) \bullet \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}}$$
$$\underline{= -\frac{2}{\sqrt{6}} \neq 0}$$

Therefore, the given vectors are ***not** orthogonal*.

$$\begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{vmatrix} = \frac{1}{2\sqrt{3}} + \frac{1}{2\sqrt{3}}$$
$$\underline{= \frac{1}{\sqrt{3}} \neq 0}$$

Exercise

Which of the following sets of vectors are orthogonal with respect to the Euclidean inner

$$\left\{\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right),\ \left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right),\ \left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right)\right\}$$

Solution

$$\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right)\cdot\left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right)=\frac{4}{9}-\frac{2}{9}-\frac{2}{9}$$

$$=0$$

$$\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right)\cdot\left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right)=\frac{2}{9}-\frac{4}{9}+\frac{2}{9}$$

$$=0$$

$$\left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right)\cdot\left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right)=\frac{2}{9}+\frac{2}{9}-\frac{4}{9}$$

$$=0$$

Therefore, the given vectors are *orthogonal*.

$$\begin{vmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{vmatrix} = \frac{4}{27}+\frac{4}{27}+\frac{4}{27}-\frac{1}{27}+\frac{8}{27}+\frac{8}{27}$$

$$=0$$

Exercise

Which of the following sets of vectors are orthogonal with respect to the Euclidean inner

$$\left\{\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right),\ \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\right\}$$

Solution

$$\begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{vmatrix} = \frac{1}{2\sqrt{3}}+\frac{1}{2\sqrt{3}}$$

$$=\frac{1}{\sqrt{3}}\ \neq 0$$

Therefore, the given vectors are ***not*** *orthogonal*.

$$\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)\cdot\left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)=-\frac{1}{2\sqrt{3}}-\frac{1}{2\sqrt{3}}$$

$$=-\frac{1}{\sqrt{3}}\ \neq 0$$

Exercise

Which of the following sets of vectors are orthogonal with respect to the Euclidean inner

$$\left\{\left(\frac{1}{\sqrt{2}},\ 0,\ 0,\ \frac{1}{\sqrt{2}}\right),\ \left(0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right),\ \left(-\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2}\right)\right\}$$

Solution

$$\left(\frac{1}{\sqrt{2}},\ 0,\ 0,\ \frac{1}{\sqrt{2}}\right)\cdot\left(0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)\cdot\left(-\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2}\right)=0+0+0+0$$

$$=0$$

Therefore, the given vectors are *orthogonal*.

Exercise

Consider vectors $\vec{u}=(2,\ 3,\ 5)\quad \vec{v}=(1,\ -4,\ 3)\quad in\quad \mathbb{R}^3$

a) $\langle\vec{u},\ \vec{v}\rangle$ *b)* $\|\vec{u}\|$ *c)* $\|\vec{v}\|$ *d)* Cosine between $\vec{u}$ *and* $\vec{v}$

Solution

a) $\langle\vec{u},\ \vec{v}\rangle=(2,\ 3,\ 5)\cdot(1,\ -4,\ 3)$

$$=2-12+15$$

$$=5$$

b) $\|\vec{u}\|=\sqrt{4+9+25}$

$$=\sqrt{38}$$

c) $\vec{v}=\sqrt{1+16+9}$

$$=\sqrt{26}$$

d) $\cos\theta=\dfrac{5}{\sqrt{38}\sqrt{26}}$ $\qquad\cos\theta=\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$

Exercise

Consider vectors $\vec{u} = (1,\ 1,\ 1)$ $\vec{v} = (1,\ 2,\ -3)$ *in* $\mathbb{R}^3$

a) $\langle \vec{u},\ \vec{v} \rangle$ *b)* $\|\vec{u}\|$ *c)* $\|\vec{v}\|$ *d)* Cosine θ between $\vec{u}$ *and* $\vec{v}$

Solution

a) $\langle \vec{u},\ \vec{v} \rangle = (1,\ 1,\ 1)\cdot(1,\ 2,\ -3)$

$= 1 + 2 - 3$

$= 0$

b) $\|\vec{u}\| = \sqrt{1+1+1}$

$= \sqrt{3}$

c) $\|\vec{v}\| = \sqrt{1+4+9}$

$= \sqrt{14}$

d) $\cos\theta = 0$ $\cos\theta = \dfrac{\vec{u}\bullet\vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$

$\vec{u}$ *and* $\vec{v}$ are orthogonal vectors.

Exercise

Consider vectors $\vec{u} = (1,\ 2,\ 5)$ $\vec{v} = (2,\ -3,\ 5)$ $\vec{w} = (4,\ 2,\ -3)$ *in* $\mathbb{R}^3$

a) $\langle \vec{u},\ \vec{v} \rangle$ *d)* $\|\vec{u}\|$ *g)* Cosine α between $\vec{u}$ *and* $\vec{v}$

b) $\langle \vec{u},\ \vec{w} \rangle$ *e)* $\|\vec{v}\|$ *h)* Cosine β between $\vec{u}$ *and* $\vec{w}$

c) $\langle \vec{v},\ \vec{w} \rangle$ *f)* $\|\vec{w}\|$ *i)* Cosine θ between $\vec{v}$ *and* $\vec{w}$

j) $(\vec{u}+\vec{v})\bullet\vec{w}$

Solution

a) $\langle \vec{u},\ \vec{v} \rangle = (1,\ 2,\ 5)\bullet(2,\ -3,\ 5)$

$= 2 - 6 + 25$

$= 21$

b) $\langle \vec{u},\ \vec{w} \rangle = (1,\ 2,\ 5)\bullet(4,\ 2,\ -3)$

$= 4 + 4 - 15$

$= -7$

c) $\langle \vec{v},\ \vec{w} \rangle = (2,\ -3,\ 5)\bullet(4,\ 2,\ -3)$

$= 8 - 6 - 15$

$= -13$

d) $\|\vec{u}\| = \sqrt{1+4+25}$

$= \sqrt{30}$

e) $\|\vec{v}\| = \sqrt{4+9+25}$

$= \sqrt{38}$

f) $\|\vec{w}\| = \sqrt{16+4+9}$

$= \sqrt{29}$

g) $\cos\alpha = \dfrac{21}{\sqrt{30}\sqrt{38}}$ $\qquad \cos\theta = \dfrac{\vec{u}\bullet\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$

$= \dfrac{21}{2\sqrt{114}}$

h) $\cos\beta = \dfrac{-7}{\sqrt{30}\sqrt{29}}$ $\qquad \cos\theta = \dfrac{\vec{u}\bullet\vec{w}}{\|\vec{u}\|\,\|\vec{w}\|}$

$= \dfrac{-7}{\sqrt{870}}$

i) $\cos\theta = \dfrac{-13}{\sqrt{38}\sqrt{29}}$ $\qquad \cos\theta = \dfrac{\vec{v}\bullet\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}$

$= \dfrac{-13}{\sqrt{1,102}}$

j) $(\vec{u}+\vec{v})\bullet\vec{w} = \left[(1,\ 2,\ 5)+(2,\ -3,\ 5)\right]\bullet(4,\ 2,\ -3)$

$= (3,\ -1,\ 10)\bullet(4,\ 2,\ -3)$

$= 12-2-30$

$= -20$

Exercise

Consider vectors $\vec{u} = (-1,\ -1,\ -1)$ $\quad \vec{v} = (2,\ 2,\ 2)$ $\quad \vec{w} = (4,\ 2,\ -3)$ $\quad in \quad \mathbb{R}^3$

a) $\langle\vec{u},\ \vec{v}\rangle$

b) $\langle\vec{u},\ \vec{w}\rangle$

c) $\langle\vec{v},\ \vec{w}\rangle$

d) $\|\vec{u}\|$

e) $\|\vec{v}\|$

f) $\|\vec{w}\|$

g) Cosine α between $\vec{u}$ *and* $\vec{v}$

h) Cosine β between $\vec{u}$ *and* $\vec{w}$

i) Cosine θ between $\vec{v}$ *and* $\vec{w}$

j) $(\vec{u}+\vec{v})\bullet\vec{w}$

Solution

a) $\langle\vec{u},\ \vec{v}\rangle = (-1,\ -1,\ -1)\bullet(2,\ 2,\ 2)$

$= -2-2-2$

$= -6$

b) $\langle\vec{u},\ \vec{w}\rangle = (-1,\ -1,\ -1)\bullet(4,\ 2,\ -3)$

$= -4 - 2 + 3$

$= -3$

c) $\langle \vec{v},\ \vec{w} \rangle = (2,\ 2,\ 2) \bullet (4,\ 2,\ -3)$

$= 8 + 4 - 6$

$= 6$

d) $\|\vec{u}\| = \sqrt{1+1+1}$

$= \sqrt{3}$

e) $\|\vec{v}\| = \sqrt{4+4+4}$

$= 2\sqrt{3}$

f) $\|\vec{w}\| = \sqrt{16+4+9}$

$= \sqrt{29}$

g) $\cos\alpha = \dfrac{-6}{6}$ $\qquad\qquad$ $\cos\theta = \dfrac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$

$= -1$

h) $\cos\beta = \dfrac{-3}{\sqrt{3}\sqrt{29}}$ $\qquad\qquad$ $\cos\theta = \dfrac{\vec{u} \bullet \vec{w}}{\|\vec{u}\|\ \|\vec{w}\|}$

$= -\dfrac{3}{\sqrt{87}}$

i) $\cos\theta = \dfrac{6}{2\sqrt{3}\sqrt{29}}$ $\qquad\qquad$ $\cos\theta = \dfrac{\vec{v} \bullet \vec{w}}{\|\vec{v}\|\ \|\vec{w}\|}$

$= \dfrac{3}{\sqrt{87}}$

j) $(\vec{u} + \vec{v}) \bullet \vec{w} = \left[(-1,\ -1,\ -1) + (2,\ 2,\ 2)\right] \bullet (4,\ 2,\ -3)$

$= (1,\ 1,\ 1) \bullet (4,\ 2,\ -3)$

$= 4 + 2 - 3$

$= 3$

Exercise

Consider vectors $\vec{u}=(-2,\ 0,\ 1,\ 3)$ $\vec{v}=(1,\ 1,\ 1,\ 1)$ $\vec{w}=(3,\ -1,\ 5,\ 2)$ *in* $\mathbb{R}^4$

a) $\langle \vec{u}, \vec{v}\rangle$

b) $\langle \vec{u}, \vec{w}\rangle$

c) $\langle \vec{v}, \vec{w}\rangle$

d) $\|\vec{u}\|$

e) $\|\vec{v}\|$

f) $\|\vec{w}\|$

g) Cosine α between $\vec{u}$ *and* $\vec{v}$

h) Cosine β between $\vec{u}$ *and* $\vec{w}$

i) Cosine θ between $\vec{v}$ *and* $\vec{w}$

j) $(\vec{u}+\vec{v})\cdot\vec{w}$

Solution

a) $\langle \vec{u}, \vec{v}\rangle=(-2,\ 0,\ 1,\ 3)\cdot(1,\ 1,\ 1,\ 1)$

$$=-2+0+1+3$$

$$=2$$

b) $\langle \vec{u}, \vec{w}\rangle=(-2,\ 0,\ 1,\ 3)\cdot(3,\ -1,\ 5,\ 2)$

$$=-6+0+5+6$$

$$=5$$

c) $\langle \vec{v}, \vec{w}\rangle=(1,\ 1,\ 1,\ 1)\cdot(3,\ -1,\ 5,\ 2)$

$$=3-1+5+2$$

$$=9$$

d) $\|\vec{u}\|=\sqrt{4+1+9}$

$$=\sqrt{14}$$

e) $\|\vec{v}\|=\sqrt{1+1+1+1}$

$$=2$$

f) $\|\vec{w}\|=\sqrt{9+1+25+4}$

$$=\sqrt{39}$$

g) $\cos\alpha=\dfrac{2}{2\sqrt{14}}$ $\qquad\qquad$ $\cos\theta=\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$

$$=\frac{1}{\sqrt{14}}$$

h) $\cos\beta=\dfrac{5}{\sqrt{14}\sqrt{39}}$ $\qquad\qquad$ $\cos\theta=\dfrac{\vec{u}\cdot\vec{w}}{\|\vec{u}\|\ \|\vec{w}\|}$

$$=\frac{5}{\sqrt{546}}$$

i) $\cos\theta=\dfrac{9}{2\sqrt{39}}$ $\qquad\qquad$ $\cos\theta=\dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\ \|\vec{w}\|}$

j) $(\vec{u}+\vec{v})\cdot\vec{w}=\left[(-2,\ 0,\ 1,\ 3)+(1,\ 1,\ 1,\ 1)\right]\cdot(3,\ -1,\ 5,\ 2)$

$$=(-1,\ 1,\ 1,\ 3)\bullet(3,\ -1,\ 5,\ 2)$$
$$=-3-1+5+6$$
$$=7$$

Exercise

Consider polynomial $f(t)=3t-5;\quad g(t)=t^2\quad in\quad \mathbb{P}(t)$

a) $\langle f,g\rangle$ *b)* $\|f\|$ *c)* $\|g\|$ *d)* Cosine between f *and* g

Solution

a) $\langle f,g\rangle=\int_0^1 f(t)g(t)dt$

$$=\int_0^1(3t-5)t^2dt$$
$$=\int_0^1\left(3t^3-5t^2\right)dt$$
$$=\frac{3}{4}t^4-\frac{5}{3}t^3\Big|_0^1$$
$$=\frac{3}{4}-\frac{5}{3}$$
$$=-\frac{11}{12}$$

b) $\langle f,f\rangle=\int_0^1 f(t)f(t)dt$

$$=\int_0^1(3t-5)^2\,dt$$
$$=\frac{1}{3}\int_0^1(3t-5)^2\,d(3t-5)$$
$$=\frac{1}{9}(3t-5)^3\Big|_0^1$$
$$=\frac{1}{9}(8-125)$$
$$=13$$

$$\|f\|=\sqrt{|\langle f,f\rangle|}$$

$$=\sqrt{13}$$

c) $\langle g,\, g\rangle = \int_0^1 g(t)g(t)\,dt$

$$= \int_0^1 t^4\,dt$$

$$= \frac{1}{5}t^5 \Big|_0^1$$

$$= \frac{1}{5}$$

$$\|g\| = \sqrt{|\langle g,\, g\rangle|}$$

$$= \frac{1}{\sqrt{5}}$$

d) $\cos\theta = \dfrac{-\frac{11}{12}}{\sqrt{13}\,\frac{\sqrt{5}}{5}}$ $\qquad\qquad \cos\theta = \dfrac{f \bullet g}{\|f\|\,\|g\|}$

$$= \frac{-55}{12\sqrt{65}}$$

Exercise

Consider polynomial $f(t) = t + 2$; $\;g(t) = 3t - 2$; $\;h(t) = t^2 - 2t - 3 \;$ *in* $\;\mathbb{P}(t)$

a) $\langle f,\, g\rangle$	*d)* $\|f\|$	*g)* Cosine α between f *and* g
b) $\langle f,\, h\rangle$	*e)* $\|g\|$	*h)* Cosine β between f *and* h
c) $\langle g,\, h\rangle$	*f)* $\|h\|$	*i)* Cosine θ between g *and* h

Solution

a) $\langle f,\, g\rangle = \int_0^1 (t+2)(3t-2)\,dt$ $\qquad\qquad \langle f,\, g\rangle = \int_0^1 f(t)g(t)\,dt$

$$= \int_0^1 \left(3t^2 + 4t - 4\right)dt$$

$$= t^3 + 2t^2 - 4t \Big|_0^1$$

$$= 1 + 2 - 4$$

$$= -1$$

b) $\langle f, h\rangle = \int_0^1 (t+2)\left(t^2-2t-3\right)dt$ $\qquad \langle f, h\rangle = \int_0^1 f(t)h(t)dt$

$$= \int_0^1 \left(t^3-7t-6\right)dt$$

$$= \frac{1}{4}t^4 - \frac{7}{2}t^2 - 6t \Big|_0^1$$

$$= \frac{1}{4} - \frac{7}{2} - 6$$

$$\boxed{= -\frac{37}{4}}$$

c) $\langle g, h\rangle = \int_0^1 (3t-2)\left(t^2-2t-3\right)dt$ $\qquad \langle f, h\rangle = \int_0^1 g(t)h(t)dt$

$$= \int_0^1 \left(3t^3-8t^2-5t+6\right)dt$$

$$= \frac{3}{4}t^4 - \frac{8}{3}t^3 - \frac{5}{2}t^2 + 6t \Big|_0^1$$

$$= \frac{3}{4} - \frac{8}{3} - \frac{5}{2} + 6$$

$$\boxed{= \frac{9}{4}}$$

d) $\langle f, f\rangle = \int_0^1 (t+2)^2\, dt$ $\qquad \langle f, f\rangle = \int_0^1 f(t)f(t)dt$

$$= \frac{1}{3}(t+2)^3 \Big|_0^1$$

$$= \frac{1}{3}(27-8)$$

$$\boxed{= \frac{19}{3}}$$

$$\|f\| = \sqrt{\left|\langle f, f\rangle\right|}$$

$$\boxed{= \sqrt{\frac{19}{3}}}$$

e) $\langle g, g\rangle = \int_0^1 (3t-2)^2\, dt$ $\qquad \langle g, g\rangle = \int_0^1 g(t)g(t)dt$

$$= \frac{1}{3}\int_0^1 (3t-2)^2\, d(3t-2)$$

$$= \frac{1}{9}(3t-2)^3 \Big|_0^1$$

$$= \frac{1}{9}(1+8)$$

$$= 1$$

$$\|g\| = \sqrt{|\langle g,\ g\rangle|}$$

$$= 1$$

f) $\langle h,\ h\rangle = \int_0^1 \left(t^2-2t-3\right)^2 dt$ $\qquad$ $\langle h,\ h\rangle = \int_0^1 h(t)h(t)\,dt$

$$= \int_0^1 \left(t^4-4t^3-2t^2+12t+9\right)dt$$

$$= \left(\frac{1}{5}t^5 - t^4 - \frac{2}{3}t^3 + 6t^2 + 9t\right) \Big|_0^1$$

$$= \frac{1}{5} - 1 - \frac{2}{3} + 6 + 9$$

$$= \frac{203}{15}$$

$$\|h\| = \sqrt{|\langle h,\ h\rangle|}$$

$$= \sqrt{\frac{203}{15}}$$

g) $\cos\alpha = \dfrac{-1}{\sqrt{\frac{19}{3}}}$ $\qquad$ $\cos\alpha = \dfrac{f \bullet g}{\|f\|\ \|g\|}$

$$= -\sqrt{\frac{3}{19}}$$

h) $\cos\beta = -\dfrac{37}{4}\dfrac{1}{\sqrt{\frac{19}{3}}\sqrt{\frac{203}{15}}}$ $\qquad$ $\cos\beta = \dfrac{f \bullet g}{\|f\|\ \|g\|}$

$$= -\frac{111}{4}\sqrt{\frac{5}{3{,}857}}$$

i) $\cos\theta = \frac{9}{4}\frac{1}{\sqrt{\frac{203}{15}}}$ $\qquad\cos\theta = \frac{f \bullet g}{\|f\|\,\|g\|}$

$= \frac{9}{4}\sqrt{\frac{15}{203}}$

Exercise

Consider polynomial $f(x) = x^2 - 2x + 3$; $\quad g(x) = 2x + 3$; $\quad h(x) = x - 2 \quad in \quad \mathbb{P}(x)$

a) $\langle f, g\rangle$

b) $\langle f, h\rangle$

c) $\langle g, h\rangle$

d) $\|f\|$

e) $\|g\|$

f) $\|h\|$

g) Cosine α between f *and* g

h) Cosine β between f *and* h

i) Cosine θ between g *and* h

Solution

a) $\langle f, g\rangle = \int_0^1 \left(x^2 - 2x + 3\right)(2x + 3)\,dx$ $\qquad\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$

$= \int_0^1 \left(2x^3 - x^2 + 9\right)dx$

$= \frac{1}{2}x^4 - \frac{1}{3}x^3 + 9x \Big|_0^1$

$= \frac{1}{2} - \frac{1}{3} + 9$

$= \frac{55}{6}$

b) $\langle f, h\rangle = \int_0^1 \left(x^2 - 2x + 3\right)(x - 2)\,dx$ $\qquad\langle f, h\rangle = \int_0^1 f(x)h(x)\,dx$

$= \int_0^1 \left(x^3 - 4x^2 + 7x - 6\right)dx$

$= \frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{7}{2}x^2 - 6x \Big|_0^1$

$= \frac{1}{4} - \frac{4}{3} - \frac{7}{2} - 6$

$= \frac{3 - 16 - 42 + 72}{12}$

$= \frac{17}{12}$

c) $\langle g,\ h\rangle = \int_0^1 (2x+3)(x-2)\ dx$ $\qquad \langle f,\ h\rangle = \int_0^1 g(x)h(x)dx$

$$= \int_0^1 \left(2x^2 - x - 6\right)dx$$

$$= \frac{2}{3}x^3 - \frac{1}{2}x^2 - 6x \Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{2} - 6$$

$$\boxed{= -\frac{37}{6}}$$

d) $\langle f, f\rangle = \int_0^1 \left(x^2 - 2x + 3\right)^2 dx$ $\qquad \langle f, f\rangle = \int_0^1 f(x)f(x)dx$

$$= \int_0^1 \left(x^4 - 8x^3 + 6x^2 - 12x + 3\right)dx$$

$$= \frac{1}{5}x^5 - 2x^4 + 2x^3 - 6x^2 + 3x \Big|_0^1$$

$$= \frac{1}{5} - 2 + 2 - 6 + 3$$

$$\boxed{= -\frac{14}{5}}$$

$$\|f\| = \sqrt{\left|\langle f, f\rangle\right|}$$

$$\boxed{= \sqrt{\frac{14}{5}}}$$

e) $\langle g,\ g\rangle = \int_0^1 (2x+3)^2\ dx$ $\qquad \langle g,\ g\rangle = \int_0^1 g(x)g(x)dx$

$$= \frac{1}{2}\int_0^1 (2x+3)^2\ d(2x+3)$$

$$= \frac{1}{6}(2x+3)^3 \Big|_0^1$$

$$= \frac{1}{6}(125+27)$$

$$= \frac{152}{6}$$

$$=\boxed{\frac{76}{3}}$$

$$\|g\|=\sqrt{|\langle g,\ g\rangle|}$$

$$=\boxed{\sqrt{\frac{76}{3}}}$$

f) $\langle h,\ h\rangle=\int_0^1 (x-2)^2\,dx$ $\qquad\qquad$ $\langle h,\ h\rangle=\int_0^1 h(x)h(x)\,dx$

$$=\frac{1}{3}(x-2)^3\ \Big|_0^1$$

$$=\frac{1}{3}(-1+8)$$

$$=\boxed{\frac{7}{3}}$$

$$\|h\|=\sqrt{|\langle h,\ h\rangle|}$$

$$=\boxed{\sqrt{\frac{7}{3}}}$$

g) $\cos\alpha=\dfrac{\frac{55}{6}}{\sqrt{\frac{76}{3}}\sqrt{\frac{14}{5}}}$ $\qquad\qquad$ $\cos\alpha=\dfrac{f\bullet g}{\|f\|\ \|g\|}$

$$=\frac{55}{6}\sqrt{\frac{5}{14}\cdot\frac{3}{76}}$$

$$=\boxed{\frac{55}{12}\sqrt{\frac{15}{266}}}$$

h) $\cos\beta=\dfrac{17}{12}\sqrt{\dfrac{3}{7}\cdot\dfrac{5}{14}}$ $\qquad\qquad$ $\cos\beta=\dfrac{f\bullet h}{\|f\|\ \|h\|}$

$$=\boxed{\frac{17}{84}\sqrt{\frac{15}{2}}}$$

i) $\cos\theta=-\dfrac{37}{6}\sqrt{\dfrac{3}{76}\cdot\dfrac{3}{7}}$ $\qquad\qquad$ $\cos\theta=\dfrac{h\bullet g}{\|h\|\ \|g\|}$

$$=\boxed{-\frac{37}{4}\sqrt{\frac{1}{133}}}$$

Exercise

Suppose $\langle \vec{u},\ \vec{v}\rangle = 3+2\,i$ in a complex inner product space *V*. Find:

a) $\langle (2-4\,i)\vec{u},\ \vec{v}\rangle$ *b)* $\langle \vec{u},\ (4+3\,i)\vec{v}\rangle$ *c)* $\langle (3-6\,i)\vec{u},\ (5-2\,i)\vec{v}\rangle$ *d)* $\|\vec{u},\ \vec{v}\|$

Solution

a)
$$\begin{aligned}\langle (2-4\,i)\vec{u},\ \vec{v}\rangle &= (2-4\,i)\langle \vec{u},\ \vec{v}\rangle \\ &= (2-4\,i)(3+2\,i) \\ &= 6+4i-12i+8 \qquad i^2=-1 \\ &= \underline{14-8i} \end{aligned}$$

b)
$$\begin{aligned}\langle \vec{u},\ (4+3\,i)\vec{v}\rangle &= (4+3\,i)\langle \vec{u},\ \vec{v}\rangle \\ &= (4+3\,i)(3+2\,i) \\ &= 12+8i+9i-6 \qquad i^2=-1 \\ &= \underline{14-8i} \end{aligned}$$

c)
$$\begin{aligned}\langle (3-6\,i)\vec{u},\ (5-2\,i)\vec{v}\rangle &= (3-6\,i)(5-2\,i)\langle \vec{u},\ \vec{v}\rangle \\ &= (15-36i-12)(3+2\,i) \\ &= (3-36i)(3+2\,i) \\ &= 9-102i+72 \\ &= \underline{81-102i} \end{aligned}$$

d)
$$\begin{aligned}\|\vec{u},\ \vec{v}\| &= \sqrt{\langle \vec{u},\ \vec{v}\rangle} \\ &= \sqrt{9+4} \\ &= \underline{\sqrt{13}} \end{aligned}$$

Exercise

Suppose $\langle \vec{u},\ \vec{v}\rangle = 2-3\,i$ in a complex inner product space *V*. Find:

a) $\langle (2+2\,i)\vec{u},\ \vec{v}\rangle$ *b)* $\langle \vec{u},\ (3-4\,i)\vec{v}\rangle$ *c)* $\langle (1+3\,i)\vec{u},\ (5-2\,i)\vec{v}\rangle$ *d)* $\|\vec{u},\ \vec{v}\|$

Solution

a)
$$\begin{aligned}\langle (2+2\,i)\vec{u},\ \vec{v}\rangle &= (2+2i)\langle \vec{u},\ \vec{v}\rangle \\ &= (2+2i)(2-3\,i) \\ &= 4-6i+4i+6 \\ &= \underline{10-2i} \end{aligned}$$

b) $\langle \vec{u},\ (3-4\,i)\vec{v}\rangle = (3-4\,i)\langle \vec{u},\ \vec{v}\rangle$

$$=(3-4i)(2-3i)$$
$$=4-9i-8i-12$$
$$\underline{=-8-17i}$$

c) $\langle(1+3i)\vec{u},\ (5-2i)\vec{v}\rangle=(1+3i)(5-2i)\langle\vec{u},\ \vec{v}\rangle$
$$=(5-2i+15i+6)(2-3i)$$
$$=(11+13i)(2-3i)$$
$$=22-33i+26i+39$$
$$\underline{=61-7i}$$

d) $\|\vec{u},\ \vec{v}\|=\sqrt{\langle\vec{u},\ \vec{v}\rangle}$
$$=\sqrt{4+9}$$
$$\underline{=\sqrt{13}}$$

Exercise

Find the Fourier coefficient c and the projection $c\vec{v}$ of $\vec{u}=(3+4i,\ 2-3i)$ along $\vec{v}=(5+i,\ 2i)$ in $\mathbb{C}^2$

Solution

$$c=\frac{(3+4i)\left(\overline{5+i}\right)+(2-3i)\left(\overline{2i}\right)}{5^2+1^2+2^2} \qquad\qquad c=\frac{\langle\vec{u},\ \vec{v}\rangle}{\langle\vec{v},\ \vec{v}\rangle}$$
$$=\frac{(3+4i)(5-i)+(2-3i)(-2i)}{25+1+4}$$
$$=\frac{15+17i+4-4i-6}{30}$$
$$=\frac{13+13i}{30}$$
$$\underline{=\frac{13}{30}+\frac{13}{30}i}$$

$$proj(\vec{u},\ \vec{v})=c\vec{v}$$
$$=\left(\frac{13}{30}+\frac{13}{30}i\right)(5+i,\ 2i)$$
$$=\left(\frac{13}{6}+\frac{13}{6}i+\frac{13}{30}i-\frac{13}{30},\quad \frac{13}{15}i-\frac{13}{15}\right)$$
$$=\left(\frac{52}{30}+\frac{78}{30}i,\ -\frac{13}{15}+\frac{13}{15}i\right)$$
$$\underline{=\left(\frac{26}{15}+\frac{39}{30}i,\ -\frac{13}{15}+\frac{13}{15}i\right)}$$

Exercise

Suppose $\vec{v}=(1,\ 3,\ 5,\ 7)$. Find the projection of $\vec{v}$ onto W or find $\vec{w}\in W$ that minimizes $\|\vec{v}-\vec{w}\|$, where W is the subspace of $\mathbb{R}^4$ spanned by:

a) $\vec{u}_1=(1,\ 1,\ 1,\ 1)$ *and* $\vec{u}_2=(1,\ -3,\ 4,\ -2)$

b) $\vec{v}_1=(1,\ 1,\ 1,\ 1)$ *and* $\vec{v}_2=(1,\ 2,\ 3,\ 2)$

Solution

a)
$$\begin{aligned}\vec{u}_1\bullet\vec{u}_2 &=(1,\ 1,\ 1,\ 1)\bullet(1,\ -3,\ 4,\ -2)\\ &=1-3+4-2\\ &=0\end{aligned}$$

Therefore, $\vec{u}_1$ *and* $\vec{u}_2$ are orthogonal.

$$\begin{aligned}c_1 &=\frac{\langle\vec{v},\ \vec{u}_1\rangle}{\langle\vec{u}_1,\ \vec{u}_1\rangle}\\ &=\frac{(1,\ 3,\ 5,\ 7)\bullet(1,\ 1,\ 1,\ 1)}{\|(1,\ 1,\ 1,\ 1)\|^2}\\ &=\frac{1+3+5+7}{1+1+1+1}\\ &=\frac{16}{4}\\ &=4\end{aligned}$$

$$\begin{aligned}c_2 &=\frac{\langle\vec{v},\ \vec{u}_2\rangle}{\langle\vec{u}_2,\ \vec{u}_2\rangle}\\ &=\frac{(1,\ 3,\ 5,\ 7)\bullet(1,\ -3,\ 4,\ -2)}{\|(1,\ -3,\ 4,\ -2)\|^2}\\ &=\frac{1-9+20-14}{1+9+16+4}\\ &=\frac{-2}{30}\\ &=\frac{1}{15}\end{aligned}$$

$$\begin{aligned}w &= proj(\vec{v},\ W)\\ &=c_1\vec{u}_1+c_2\vec{u}_2\\ &=4(1,\ 1,\ 1,\ 1)+\frac{1}{15}(1,\ -3,\ 4,\ -2)\end{aligned}$$

$$=\left(\frac{59}{15},\ \frac{63}{5},\ \frac{56}{15},\ \frac{62}{15}\right)$$

b) $\vec{v}_1 \bullet \vec{v}_2 = (1,\ 1,\ 1,\ 1)\bullet(1,\ 2,\ 3,\ 2)$

$$= 1+2+3+2$$

$$= 8 \neq 0$$

Therefore, $\vec{v}_1$ *and* $\vec{v}_2$ are *not* orthogonal.

Applying Gram-Schmidt algorithm

$$\vec{w}_1 = \vec{v}_1 = (1,\ 1,\ 1,\ 1)$$

$$\vec{w}_2 = (1,\ 2,\ 3,\ 2) - \frac{(1,\ 2,\ 3,\ 2)\bullet(1,\ 1,\ 1,\ 1)}{4}(1,\ 1,\ 1,\ 1) \qquad \vec{w}_2 = \vec{v}_2 - \frac{\langle \vec{v},\ \vec{w}_1\rangle}{\|\vec{w}_1\|^2}\vec{w}_1$$

$$= (1,\ 2,\ 3,\ 2) - 2(1,\ 1,\ 1,\ 1)$$

$$= (-1,\ 0,\ 1,\ 0)$$

$$c_1 = \frac{(1,\ 3,\ 5,\ 7)\bullet(1,\ 1,\ 1,\ 1)}{\|(1,\ 1,\ 1,\ 1)\|^2} \qquad c_1 = \frac{\langle \vec{v},\ \vec{w}_1\rangle}{\langle \vec{w}_1,\ \vec{w}_1\rangle}$$

$$= \frac{1+3+5+7}{1+1+1+1}$$

$$= \frac{16}{4}$$

$$= 4$$

$$c_2 = \frac{(1,\ 3,\ 5,\ 7)\bullet(-1,\ 0,\ 1,\ 0)}{\|(-1,\ 0,\ 1,\ 0)\|^2} \qquad c_2 = \frac{\langle \vec{v},\ \vec{w}_2\rangle}{\langle \vec{w}_2,\ \vec{w}_2\rangle}$$

$$= \frac{-1+0+5+0}{2}$$

$$= -3$$

$$w = proj(\vec{v},\ W)$$

$$= c_1\vec{w}_1 + c_2\vec{w}_2$$

$$= 4(1,\ 1,\ 1,\ 1) - 3(-1,\ 0,\ 1,\ 0)$$

$$= (7,\ 4,\ 1,\ 4)$$

Exercise

Suppose $\left\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\right\}$ is an orthogonal set of vectors. Prove that (*Pythagoras*)

$$\left\|\vec{u}_1 + \vec{u}_2 + \dots + \vec{u}_n\right\|^2 = \left\|\vec{u}_1\right\|^2 + \left\|\vec{u}_2\right\|^2 + \dots + \left\|\vec{u}_n\right\|^2$$

Solution

$$\begin{aligned}\left\|\vec{u}_1 + \vec{u}_2 + \dots + \vec{u}_n\right\|^2 &= \left\langle \vec{u}_1 + \vec{u}_2 + \dots + \vec{u}_n,\ \vec{u}_1 + \vec{u}_2 + \dots + \vec{u}_n \right\rangle \\ &= \left\langle \vec{u}_1, \vec{u}_1 \right\rangle + \left\langle \vec{u}_2, \vec{u}_2 \right\rangle + \dots + \left\langle \vec{u}_n, \vec{u}_n \right\rangle \\ &= \left\|\vec{u}_1\right\|^2 + \left\|\vec{u}_2\right\|^2 + \dots + \left\|\vec{u}_n\right\|^2\end{aligned}$$

Exercise

Suppose A is an orthogonal matrix. Show that $\langle \vec{u}A, \vec{v}A \rangle = \langle \vec{u}, \vec{v} \rangle$ for any $\vec{u}, \vec{v} \in V$

Solution

A is an orthogonal matrix $\Rightarrow AA^T = I$

And $\langle \vec{u}, \vec{v} \rangle = \vec{u}^T \vec{v}$

$$\begin{aligned}\langle \vec{u}A, \vec{v}A \rangle &= (A\vec{u})^T (A\vec{v}) \\ &= \vec{u}^T \left(A^T A\right) \vec{v} \\ &= \vec{u}^T I\, \vec{v} \\ &= \vec{u}^T \vec{v} \\ &= \langle \vec{u}, \vec{v} \rangle \quad \checkmark\end{aligned}$$

Exercise

Suppose A is an orthogonal matrix. Show that $\|\vec{u}A\| = \|\vec{u}\|$ for every $\vec{u} \in V$

Solution

A is an orthogonal matrix

$\Rightarrow AA^T = I$ and $\langle \vec{u}, \vec{u} \rangle = \vec{u}^T \vec{u}$

$$\begin{aligned}\|\vec{u}A\|^2 &= \langle \vec{u}A, \vec{u}A \rangle \\ &= (A\vec{u})^T (A\vec{u})\end{aligned}$$

$$= \vec{u}^T \left(A^T A\right)\vec{u}$$

$$= \vec{u}^T I\ \vec{u}$$

$$= \vec{u}^T \vec{u}$$

$$= \langle \vec{u},\ \vec{u} \rangle \quad \checkmark$$

Exercise

Let V be an inner product space over $\mathbb{R}$ or $\mathbb{C}$. Show that

$$\|\vec{u}+\vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$$

If and only if

$$\|s\vec{u}+t\vec{v}\| = s\|\vec{u}\| + t\|\vec{v}\| \quad \textit{for all} \quad s,\ t \ge 0$$

Solution

Suppose that $\|\vec{u}+\vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$. For $s,\ t \ge 0$

$$\begin{aligned}
\|s\vec{u}+t\vec{v}\|^2 &= s^2\|\vec{u}\|^2 + t^2\|\vec{v}\|^2 + 2st\,\vec{u}\vec{v} \\
&\le s^2\|\vec{u}\|^2 + t^2\|\vec{v}\|^2 \\
&\le s\|\vec{u}\| + t\|\vec{v}\|
\end{aligned}$$

$$\|s\vec{u}+t\vec{v}\| \le s\|\vec{u}\| + t\|\vec{v}\|$$

$$\begin{aligned}
\|s\vec{u}+t\vec{v}\| &= \|s\vec{u}+t\vec{u}-t\vec{u}+t\vec{v}\| \\
&= \|t\vec{u}+t\vec{v}-t\vec{u}+s\vec{u}\| \\
&= \|t(\vec{u}+\vec{v})-(t-s)\vec{u}\| \\
&\ge \big|\, t\|\vec{u}+\vec{v}\| - (t-s)\|\vec{u}\| \,\big| \qquad\qquad \|\vec{u}+\vec{v}\| = \|\vec{u}\| + \|\vec{v}\| \\
&= t\|\vec{u}\| + \|\vec{v}\| - t\|\vec{u}\| + s\|\vec{u}\| \\
&= t\|\vec{v}\| + s\|\vec{u}\|
\end{aligned}$$

$$\begin{cases} \|s\vec{u}+t\vec{v}\| \le s\|\vec{u}\| + t\|\vec{v}\| \\ \qquad\qquad \textit{and} \\ \|s\vec{u}+t\vec{v}\| \ge s\|\vec{u}\| + t\|\vec{v}\| \end{cases} \Rightarrow \quad \|s\vec{u}+t\vec{v}\| = s\|\vec{u}\| + t\|\vec{v}\|$$

Exercise

Let V be an inner product vector space over $\mathbb{R}$.

a) If e_1, e_2, e_3 are three vectors in V with pairwise product negative, that is,

$$\langle e_1, e_2 \rangle < 0, \quad i, j = 1,\ 2,\ 3, \quad i \neq j$$

Show that e_1, e_2, e_3 are linearly independent.

b) Is it possible for three vectors on the xy–plane to have pairwise negative products?

c) Does part (a) remain valid when the word "negative: is replaced with positive?

d) Suppose $\vec{u},\ \vec{v},\ and\ \vec{w}$ are three–unit vectors in the xy–plane. What are the maximum and minimum values that

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle$$

Can attain? And when?

Solution

a) Suppose that e_1, e_2, e_3 are linearly dependent.

Then, assume that e_1, e_2, e_3 are unit vectors and that

$e_3 = c_1 e_1 + c_2 e_2$

Then

$$\langle e_1, e_3 \rangle = c_1 \langle e_1, e_1 \rangle + c_2 \langle e_1, e_2 \rangle \qquad \langle e_1, e_1 \rangle = 1$$

$$= c_1 + c_2 \langle e_1, e_2 \rangle < 0$$

$$c_1 < -c_2 \langle e_1, e_2 \rangle$$

$$\langle e_2, e_3 \rangle = c_1 \langle e_2, e_1 \rangle + c_2 \langle e_2, e_2 \rangle \qquad \langle e_2, e_2 \rangle = 1$$

$$= c_1 \langle e_2, e_1 \rangle + c_2 < 0$$

$$c_2 < -c_1 \langle e_2, e_1 \rangle$$

$$c_2 < -c_1 \langle e_2, e_1 \rangle$$

$$< -\left(-c_2 \langle e_1, e_2 \rangle\right) \langle e_2, e_1 \rangle$$

$$= c_2 \langle e_1, e_2 \rangle^2 \qquad \langle e_1, e_2 \rangle^2 > 1$$

$c_2 < c_2$ *Contradiction*

Therefore, e_1, e_2, e_3 are linearly independent.

b) To have all three vectors on the xy–plane which is in 2 dimensional.
Therefore, it is ***impossible*** for three to have pairwise negative products.

c) No

d) Given: $\vec{u},\ \vec{v},\ and\ \vec{w}$ are three–unit vectors in the *xy*–plane and

$$\left|\langle \vec{u},\ \vec{u}\rangle\right| = \left|\langle \vec{v},\ \vec{v}\rangle\right| = \left|\langle \vec{w},\ \vec{w}\rangle\right| = 1$$

$$\cos\alpha_1 = \frac{\langle \vec{u},\ \vec{v}\rangle}{\|\vec{u}\|\ \|\vec{v}\|} \quad\to\quad \cos\alpha_1 = \langle \vec{u},\ \vec{v}\rangle$$

$$\cos\alpha_2 = \frac{\langle \vec{v},\ \vec{w}\rangle}{\|\vec{v}\|\ \|\vec{w}\|} \quad\to\quad \cos\alpha_2 = \langle \vec{v},\ \vec{w}\rangle$$

$$\cos\alpha_3 = \frac{\langle \vec{u},\ \vec{w}\rangle}{\|\vec{u}\|\ \|\vec{w}\|} \quad\to\quad \cos\alpha_3 = \langle \vec{u},\ \vec{w}\rangle$$

$$\langle \vec{u},\ \vec{v}\rangle + \langle \vec{v},\ \vec{w}\rangle + \langle \vec{w},\ \vec{u}\rangle = \cos\alpha_1 + \cos\alpha_2 + \cos\alpha_3$$

Since $-1 \le \cos\theta \le 1$

$$\begin{aligned}\langle \vec{u},\ \vec{v}\rangle + \langle \vec{v},\ \vec{w}\rangle + \langle \vec{w},\ \vec{u}\rangle &= 1+1+1 \\ &= 3\end{aligned}$$

Since the 3 vectors are unit vectors in the *xy*–plane and which it will divide the plane into a three equal angles $\alpha = \frac{2\pi}{3}$

$$\begin{aligned}\langle \vec{u},\ \vec{v}\rangle + \langle \vec{v},\ \vec{w}\rangle + \langle \vec{w},\ \vec{u}\rangle &= \cos\frac{2\pi}{3} + \cos\frac{2\pi}{3} + \cos\frac{2\pi}{3} \\ &= 3\cos\frac{2\pi}{3} \\ &= 3\left(-\frac{1}{2}\right) \\ &= -\frac{3}{2}\end{aligned}$$

Therefore, the minimum $\langle \vec{u},\ \vec{v}\rangle + \langle \vec{v},\ \vec{w}\rangle + \langle \vec{w},\ \vec{u}\rangle = -\frac{3}{2}$

The maximum: $\langle \vec{u},\ \vec{v}\rangle + \langle \vec{v},\ \vec{w}\rangle + \langle \vec{w},\ \vec{u}\rangle = 3$

Solution — *Section* 3.3 – Gram-Schmidt Process

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\vec{u}_1 = (1,\ -3), \quad \vec{u}_2 = (2,\ 2)$$

Solution

$$\boxed{\vec{v}_1 = \vec{u}_1 = (1,\ -3)}$$

$$\vec{q}_1 = \frac{(1,\ -3)}{\sqrt{1+9}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$\boxed{= \left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2, \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$= (2,\ 2) - \frac{1}{10}\left[(2,\ 2)\bullet(1,\ -3)\right](1,\ -3)$$

$$= (2,\ 2) + \frac{2}{5}(1,\ -3)$$

$$= (2,\ 2) + \left(\frac{2}{5},\ -\frac{6}{5}\right)$$

$$\boxed{= \left(\frac{12}{5},\ \frac{4}{5}\right)}$$

$$\left\|\vec{v}_2\right\| = \sqrt{\left(\frac{12}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$$

$$= \sqrt{\frac{144}{25} + \frac{16}{25}}$$

$$= \sqrt{\frac{160}{25}}$$

$$\boxed{= \frac{4\sqrt{10}}{5}}$$

$$\vec{q}_2 = \frac{5}{4\sqrt{10}}\left(\frac{12}{5},\ \frac{4}{5}\right) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$\boxed{= \left(\frac{3}{\sqrt{10}},\ \frac{1}{\sqrt{10}}\right)}$$

The *orthogonal* basis: $\left\{(1,\ -3),\ \left(\frac{12}{5},\ \frac{4}{5}\right)\right\}$

The *orthonormal* basis: $\left\{\left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right),\ \left(\frac{3}{\sqrt{10}},\ \frac{1}{\sqrt{10}}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$\vec{u}_1 = (1,\ 0),\quad \vec{u}_2 = (3,\ -5)$

Solution

$\vec{v}_1 = \vec{u}_1 = (1,\ 0)$

$\vec{q}_1 = \dfrac{(1,\ 0)}{\sqrt{1+0}}$ $\qquad\qquad$ $\vec{q}_1 = \dfrac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$

$= (1,\ 0)$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2,\ \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$= (3,\ -5) - \left[(3,\ -5)\bullet(1,\ 0)\right](1,\ 0)$

$= (3,\ -5) - (3)(1,\ 0)$

$= (3,\ -5) - (3,\ 0)$

$= (0,\ -5)$

$\vec{q}_2 = \frac{1}{5}(0,\ -5)$ $\qquad\qquad$ $\vec{q}_2 = \dfrac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$

$= (0,\ -1)$

The *orthogonal* basis: $\left\{(1,\ 0),\ (0,\ -5)\right\}$

The *orthonormal* basis: $\left\{(1,\ 0),\ (0,\ -1)\right\}$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\vec{u}_1 = (1,\ 0,\ 0), \quad \vec{u}_2 = (3,\ 7,\ -2), \quad \vec{u}_3 = (0,\ 4,\ 1)$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (1,\ 0,\ 0)$$

$$\vec{q}_1 = \frac{(1,\ 0,\ 0)}{\sqrt{1+0+0}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= (1,\ 0,\ 0)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$= (3,\ 7,\ -2) - \left[(3,\ 7,\ -2) \cdot (1,\ 0,\ 0)\right](1,\ 0,\ 0)$$

$$= (3,7,-2) - 3(1,0,0)$$

$$= (0,\ 7,\ -2)$$

$$\vec{q}_2 = \frac{1}{\sqrt{53}}(0,\ 7,\ -2) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \left(0,\ \frac{7}{\sqrt{53}},\ -\frac{2}{\sqrt{53}}\right)$$

$$\vec{u}_3 \cdot \vec{v}_1 = (0,\ 4,\ 1) \cdot (1,\ 0,\ 0)$$

$$= 0$$

$$\vec{u}_3 \cdot \vec{v}_2 = (0,\ 4,\ 1) \cdot (0,\ 7,\ -2)$$

$$= 26$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2$$

$$= (0,\ 4,\ 1) - 0 - \left(\frac{26}{53}\right)(0,\ 7,\ -2)$$

$$= (0,\ 4,\ 1) - \left(0,\ \frac{182}{53},\ -\frac{52}{53}\right)$$

$$= \left(0,\ \frac{30}{53},\ \frac{105}{53}\right)$$

$$\vec{q}_3 = \frac{1}{\sqrt{\left(\frac{30}{53}\right)^2 + \left(\frac{105}{53}\right)^2}}\left(0,\ \frac{30}{53},\ \frac{105}{53}\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{53}{\sqrt{11925}}\left(0,\ \frac{30}{53},\ \frac{105}{53}\right)$$

$$= \frac{53}{15\sqrt{53}}\left(0,\ \frac{30}{53},\ \frac{105}{53}\right)$$

$$= \left(0,\ \frac{2}{\sqrt{53}},\ \frac{7}{\sqrt{53}}\right)$$

The *orthogonal* basis: $\left\{(1,\ 0,\ 0),\ (0,\ 7,\ -2),\ \left(0,\ \frac{30}{53},\ \frac{105}{53}\right)\right\}$

The *orthonormal* basis: $\left\{(1,\ 0,\ 0),\ \left(0,\ \frac{7}{\sqrt{53}},\ -\frac{2}{\sqrt{53}}\right),\ \left(0,\ \frac{2}{\sqrt{53}},\ \frac{7}{\sqrt{53}}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\vec{u}_1 = (1,\ 1,\ 0,\ -1),\quad \vec{u}_2 = (1,\ 3,\ 0,\ 1),\quad \vec{u}_3 = (4,\ 2,\ 2,\ 0)$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (1,\ 1,\ 0,\ -1)$$

$$\vec{q}_1 = \frac{1}{\sqrt{3}}(1,\ 1,\ 0,\ -1) \qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ 0,\ -\frac{1}{\sqrt{3}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2,\ \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (1,\ 3,\ 0,\ 1) - \frac{(1,\ 3,\ 0,\ 1)\bullet(1,\ 1,\ 0,\ -1)}{3}(1,\ 1,\ 0,\ -1)$$

$$= (1,\ 3,\ 0,\ 1) - (1,\ 1,\ 0,\ -1)$$

$$= (0,\ 2,\ 0,\ 2)$$

$$\vec{q}_2 = \frac{1}{2\sqrt{2}}(0,\ 2,\ 0,\ 2) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \boxed{\left(0,\ \frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(4,\ 2,\ 2,\ 0)\bullet(1,\ 1,\ 0,\ -1)}{3}(1,\ 1,\ 0,\ -1)$$

$$= \boxed{(2,\ 2,\ 0,\ -2)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{(4,\ 2,\ 2,\ 0)\bullet(0,\ 2,\ 0,\ 2)}{8}(0,\ 2,\ 0,\ 2)$$

$$= \frac{1}{2}(0,\ 2,\ 0,\ 2)$$

$$= \boxed{(0,\ 1,\ 0,\ 1)}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (4,\ 2,\ 2,\ 0) - (2,\ 2,\ 0,\ -2) - (0,\ 1,\ 0,\ 1)$$

$$= \boxed{(2,\ -1,\ 2,\ 1)}$$

$$\vec{q}_3 = \frac{1}{\sqrt{10}}(2,\ -1,\ 2,\ 1) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \boxed{\left(\frac{2}{\sqrt{10}},\ -\frac{1}{\sqrt{10}},\ \frac{2}{\sqrt{10}},\ \frac{1}{\sqrt{10}}\right)}$$

The *orthogonal* basis: $\{(1,\ 1,\ 0,\ -1),\ (0,\ 2,\ 0,\ 2),\ (2,\ -1,\ 2,\ 1)\}$

The *orthogonal* basis:

$$\left\{\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ 0,\ -\frac{1}{\sqrt{3}}\right),\ \left(0,\ \frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right),\ \left(\frac{2}{\sqrt{10}},\ -\frac{1}{\sqrt{10}},\ \frac{2}{\sqrt{10}},\ \frac{1}{\sqrt{10}}\right)\right\}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\vec{u}_1 = (1,\ 1,\ 1,\ 1), \quad \vec{u}_2 = (1,\ 2,\ 4,\ 5), \quad \vec{u}_3 = (1,\ -3,\ -4,\ -2)$$

Solution

$$\vec{v}_1 = \vec{u}_1 = \boxed{(1,\ 1,\ 1,\ 1)}$$

$$\vec{q}_1 = \frac{(1,\ 1,\ 1,\ 1)}{\sqrt{4}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2,\ \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$= (1,\ 2,\ 4,\ 5) - \left(\frac{1}{4}\right)\left[(1,\ 2,\ 4,\ 5) \cdot (1,\ 1,\ 1,\ 1)\right](1,\ 1,\ 1,\ 1)$$

$$= (1,\ 2,\ 4,\ 5) - \left(\frac{12}{4}\right)(1,\ 1,\ 1,\ 1)$$

$$= (1,\ 2,\ 4,\ 5) - (3,\ 3,\ 3,\ 3)$$

$$= (-2,\ -1,\ 1,\ 2)$$

$$\vec{q}_2 = \frac{1}{\sqrt{10}}(-2,\ -1,\ 1,\ 2) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \left(-\frac{2}{\sqrt{10}},\ -\frac{1}{\sqrt{10}},\ \frac{1}{\sqrt{10}},\ \frac{2}{\sqrt{10}}\right)$$

$$\vec{u}_3 \cdot \vec{v}_1 = (1,\ -3,\ -4,\ -2) \cdot (1,\ 1,\ 1,\ 1)$$

$$= -8$$

$$\vec{u}_3 \cdot \vec{v}_2 = (1,\ -3,\ -4,\ -2) \cdot (-2,\ -1,\ 1,\ 2)$$

$$= -7$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\left\langle \vec{u}_3,\ \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1 - \frac{\left\langle \vec{u}_3,\ \vec{v}_2 \right\rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2$$

$$= (1,\ -3,\ -4,\ -2) + \frac{8}{4}(1,\ 1,\ 1,\ 1) + \frac{7}{10}(-2,\ -1,\ 1,\ 2)$$

$$= (1,\ -3,\ -4,\ -2) + (2,\ 2,\ 2,\ 2) + \left(-\frac{7}{5},\ -\frac{7}{10},\ \frac{7}{10},\ \frac{7}{5}\right)$$

$$= \left(\frac{8}{5},\ -\frac{17}{10},\ -\frac{13}{10},\ \frac{7}{5}\right)$$

$$\vec{q}_3 = \frac{1}{\sqrt{\frac{64}{25} + \frac{289}{100} + \frac{169}{100} + \frac{49}{25}}}\left(\frac{8}{5},\ -\frac{17}{10},\ -\frac{13}{10},\ \frac{7}{5}\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$= \frac{10}{\sqrt{910}}\left(\frac{8}{5},\ -\frac{17}{10},\ -\frac{13}{10},\ \frac{7}{5}\right)$$

$$=\boxed{\left(\frac{16}{\sqrt{910}},\ -\frac{17}{\sqrt{910}},\ -\frac{13}{\sqrt{910}},\ \frac{14}{\sqrt{910}}\right)}$$

The *orthogonal* basis: $\left\{(1,\ 1,\ 1,\ 1),\ (-2,\ -1,\ 1,\ 2),\ \left(\frac{8}{5},\ -\frac{17}{10},\ -\frac{13}{10},\ \frac{7}{5}\right)\right\}$

The orthonormal basis:

$$\left\{\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right),\ \left(-\frac{2}{\sqrt{10}},\ -\frac{1}{\sqrt{10}},\ \frac{1}{\sqrt{10}},\ \frac{2}{\sqrt{10}}\right),\ \left(\frac{16}{\sqrt{910}},\ -\frac{17}{\sqrt{910}},\ -\frac{13}{\sqrt{910}},\ \frac{14}{\sqrt{910}}\right)\right\}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\vec{u}_1=(1,\ 1,\ 1,\ 1),\quad \vec{u}_2=(1,\ 1,\ 2,\ 4),\quad \vec{u}_3=(1,\ 2,\ -4,\ -3)$$

Solution

$$\boxed{\vec{v}_1=\vec{u}_1=(1,\ 1,\ 1,\ 1)}$$

$$\vec{q}_1=\frac{(1,\ 1,\ 1,\ 1)}{\sqrt{4}} \qquad\qquad \vec{q}_1=\frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$=\boxed{\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)}$$

$$\vec{v}_2=\vec{u}_2-\frac{\left\langle\vec{u}_2,\vec{v}_1\right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$=(1,\ 1,\ 2,\ 4)-\frac{1}{4}\left[(1,\ 1,\ 2,\ 4)\bullet(1,\ 1,\ 1,\ 1)\right](1,\ 1,\ 1,\ 1)$$

$$=(1,\ 1,\ 2,\ 4)-(2,\ 2,\ 2,\ 2)$$

$$=\boxed{(-1,\ -1,\ 0,\ 2)}$$

$$\vec{q}_2=\frac{1}{\sqrt{1+1+4}}(-1,\ -1,\ 0,\ 2) \qquad\qquad \vec{q}_2=\frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$=\boxed{\left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0,\ \frac{2}{\sqrt{6}}\right)}$$

$$\vec{u}_3\bullet\vec{v}_1=(1,\ 2,\ -4,\ -3)\bullet(1,\ 1,\ 1,\ 1)$$

$$=\boxed{-4}$$

$$\vec{u}_3\bullet\vec{v}_2=(1,\ 2,\ -4,\ -3)\bullet(-1,\ -1,\ 0,\ 2)$$

$$\underline{=-9}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\left\| \vec{v}_2 \right\|^2}\vec{v}_2$$

$$= (1,\ 2,\ -4,\ -3) + \frac{4}{4}(1,\ 1,\ 1,\ 1) + \frac{9}{6}(-1,\ -1,\ 0,\ 2)$$

$$= (1,\ 2,\ -4,\ -3) + (1,\ 1,\ 1,\ 1) + \left(-\frac{3}{2},\ -\frac{3}{2},\ 0,\ 3\right)$$

$$= \left(\frac{1}{2},\ \frac{3}{2},\ -3,\ 1\right)$$

$$\vec{q}_3 = \frac{1}{\sqrt{\frac{1}{4}+\frac{9}{4}+9+1}}\left(\frac{1}{2},\ \frac{3}{2},\ -3,\ 1\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\left\| \vec{v}_3 \right\|}$$

$$= \frac{2}{5\sqrt{2}}\left(\frac{1}{2},\ \frac{3}{2},\ -3,\ 1\right)$$

$$= \left(\frac{1}{5\sqrt{2}},\ \frac{3}{5\sqrt{2}},\ -\frac{6}{5\sqrt{2}},\ \frac{2}{5\sqrt{2}}\right)$$

The *orthogonal* basis: $\left\{(1,\ 1,\ 1,\ 1),\ (-1,\ -1,\ 0,\ 2),\ \left(\frac{1}{2},\ \frac{3}{2},\ -3,\ 1\right)\right\}$

The *orthonormal* basis:

$$\left\{\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right),\ \left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0,\ \frac{2}{\sqrt{6}}\right),\ \left(\frac{1}{5\sqrt{2}},\ \frac{3}{5\sqrt{2}},\ -\frac{6}{5\sqrt{2}},\ \frac{2}{5\sqrt{2}}\right)\right\}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\vec{u}_1 = (0,\ 2,\ 1,\ 0); \quad \vec{u}_2 = (1,\ -1,\ 0,\ 0); \quad \vec{u}_3 = (1,\ 2,\ 0,\ -1); \quad \vec{u}_4 = (1,\ 0,\ 0,\ 1)$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (0,\ 2,\ 1,\ 0)$$

$$\vec{q}_1 = \frac{(0,\ 2,\ 1,\ 0)}{\sqrt{4+1}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\| \vec{v}_1 \right\|}$$

$$= \left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (1, -1, 0, 0) - \frac{1}{5}\left[(1, -1, 0, 0) \cdot (0, 2, 1, 0)\right](0, 2, 1, 0)$$

$$= (1, -1, 0, 0) + \frac{2}{5}(0, 2, 1, 0)$$

$$= (1, -1, 0, 0) + \left(0, \frac{4}{5}, \frac{2}{5}, 0\right)$$

$$= \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)$$

$$\vec{q}_2 = \frac{1}{\sqrt{1+\frac{1}{25}+\frac{4}{25}}}\left(1, -\frac{1}{5}, \frac{2}{5}, 0\right) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{5}{\sqrt{30}}\left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)$$

$$= \left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0\right)$$

$$u_3 \cdot v_1 = (1, 2, 0, -1) \cdot (0, 2, 1, 0)$$

$$= 4$$

$$u_3 \cdot v_2 = (1, 2, 0, -1) \cdot \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)$$

$$= 1 - \frac{2}{5}$$

$$= \frac{3}{5}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (1, 2, 0, -1) - \left(\frac{4}{5}\right)(0, 2, 1, 0) - \frac{1}{1+\frac{1}{25}+\frac{4}{25}}\left(\frac{3}{5}\right)\left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)$$

$$= (1, 2, 0, -1) - \left(0, \frac{8}{5}, \frac{4}{5}, 0\right) - \left(\frac{3}{5}\right)\left(\frac{25}{30}\right)\left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)$$

$$= (1, 2, 0, -1) - \left(0, \frac{8}{5}, \frac{4}{5}, 0\right) - \left(\frac{1}{2}, -\frac{1}{10}, \frac{1}{5}, 0\right)$$

$$= \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right)$$

$$\vec{q}_3 = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+(-1)^2+(-1)^2}}\left(\frac{1}{2}, \frac{1}{2}, -1, -1\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$=\frac{1}{\sqrt{\frac{5}{2}}}\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)$$

$$=\frac{\sqrt{2}}{\sqrt{5}}\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)$$

$$=\frac{2}{\sqrt{10}}\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)$$

$$=\left(\frac{1}{\sqrt{10}},\ \frac{1}{\sqrt{10}},\ -\frac{2}{\sqrt{10}},\ -\frac{2}{\sqrt{10}}\right)$$

$$u_4 \bullet v_1 = (1,\ 0,\ 0,\ 1) \bullet (0,\ 2,\ 1,\ 0)$$

$$=0$$

$$u_4 \bullet v_2 = (1,\ 0,\ 0,\ 1) \bullet \left(1,\ -\frac{1}{5},\ \frac{2}{5},\ 0\right)$$

$$=1$$

$$u_4 \bullet v_3 = (1,\ 0,\ 0,\ 1) \bullet \left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)$$

$$=-\frac{1}{2}$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$

$$=(1,\ 2,\ 0,\ -1)-(0)-\left(\frac{25}{5}\right)\left(1,\ -\frac{1}{5},\ \frac{2}{5},\ 0\right)+\left(-\frac{1}{2}\right)\left(\frac{2}{5}\right)\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)$$

$$=(1,\ 2,\ 0,\ -1)-\left(\frac{5}{6},\ -\frac{1}{6},\ \frac{1}{3},\ 0\right)+\left(\frac{1}{10},\ \frac{1}{10},\ -\frac{1}{5},\ -\frac{1}{5}\right)$$

$$=\left(\frac{4}{15},\ \frac{4}{15},\ -\frac{8}{15},\ \frac{4}{5}\right)$$

$$\vec{q}_4 = \frac{1}{\sqrt{\left(\frac{4}{15}\right)^2+\left(\frac{4}{15}\right)^2+\left(-\frac{8}{15}\right)^2+\left(\frac{4}{5}\right)^2}}\left(\frac{4}{15},\ \frac{4}{15},\ -\frac{8}{15},\ \frac{4}{5}\right) \qquad \vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

$$=\frac{1}{\sqrt{\frac{240}{225}}}\left(\frac{4}{15},\ \frac{4}{15},\ -\frac{8}{15},\ \frac{4}{5}\right)$$

$$=\frac{1}{\frac{4}{\sqrt{15}}}\left(\frac{4}{15},\ \frac{4}{15},\ -\frac{8}{15},\ \frac{4}{5}\right)$$

$$=\frac{15}{4\sqrt{15}}\left(\frac{4}{15},\frac{4}{15},-\frac{8}{15},\frac{4}{5}\right)$$

$$=\left(\frac{1}{\sqrt{15}},\ \frac{1}{\sqrt{15}},\ -\frac{2}{\sqrt{15}},\ \frac{3}{\sqrt{15}}\right)$$

The *orthogonal* basis:

$$\left\{(0,\ 2,\ 1,\ 0),\ \left(1,\ -\frac{1}{5},\ \frac{2}{5},\ 0\right),\ \left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right),\ \left(\frac{4}{15},\ \frac{4}{15},\ -\frac{8}{15},\ \frac{4}{5}\right)\right\}$$

The *orthonormal* basis:

$$\left\{\begin{array}{l}\left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right),\ \left(\frac{5}{\sqrt{30}},-\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right),\\ \left(\frac{1}{\sqrt{10}},\ \frac{1}{\sqrt{10}},\ -\frac{2}{\sqrt{10}},\ -\frac{2}{\sqrt{10}}\right),\ \left(\frac{1}{\sqrt{15}},\ \frac{1}{\sqrt{15}},\ -\frac{2}{\sqrt{15}},\ \frac{3}{\sqrt{15}}\right)\end{array}\right\}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(3,\ 4),\ (1,\ 0)\}$$

Solution

Let $\quad \vec{u}_1 = (3,\ 4) \quad \vec{u}_2 = (1,\ 0)$

$$\vec{v}_1 = \vec{u}_1 = (3,\ 4)$$

$$\vec{q}_1 = \frac{(3,\ 4)}{\sqrt{9+16}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(\frac{3}{5},\ \frac{4}{5}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2,\ \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (1,\ 0) - \frac{1}{25}\left[(1,\ 0)\cdot(3,\ 4)\right]\ (3,\ 4)$$

$$= (1,\ 0) - \left(\frac{9}{25},\ \frac{12}{25}\right)$$

$$= \left(\frac{16}{25},\ -\frac{12}{25}\right)$$

$$\vec{q}_2 = \frac{25}{20}\left(\frac{16}{25},\ -\frac{12}{25}\right) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(\frac{4}{5},\ -\frac{3}{5}\right)$$

The *orthogonal* basis: $\quad \left\{(3,\ 4),\ \left(\frac{16}{25},\ -\frac{12}{25}\right)\right\}$

The *orthonormal* basis: $\left\{\left(\frac{3}{5}, \frac{4}{5}\right), \left(\frac{4}{5}, -\frac{3}{5}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(3, 0, -1), (8, 5, -6)\}$$

Solution

Let $\vec{u}_1 = (3, 0, -1)$ $\vec{u}_2 = (8, 5, -6)$

$$\boxed{\vec{v}_1 = \vec{u}_1 = (3, 0, -1)}$$

$$\vec{q}_1 = \frac{(3, 0, -1)}{\sqrt{9+1}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\boxed{= \left(\frac{3}{\sqrt{10}}, 0, -\frac{1}{\sqrt{10}}\right)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (8, 5, -6) - \frac{1}{10}\left[(8, 5, -6) \bullet (3, 0, -1)\right](3, 0, -1)$$

$$= (8, 5, -6) - (3)(3, 0, -1)$$

$$= (8, 5, -6) - (9, 0, -3)$$

$$\boxed{= (-1, 5, -3)}$$

$$\vec{q}_2 = \frac{1}{\sqrt{1+25+9}}(-1, 5, -3) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{\sqrt{35}}(-1, 5, -3)$$

$$\boxed{= \left(-\frac{1}{\sqrt{35}}, \frac{5}{\sqrt{35}}, -\frac{3}{\sqrt{35}}\right)}$$

The *orthogonal* basis: $\{(3, 0, -1), (-1, 5, -3)\}$

The *orthonormal* basis: $\left\{\left(\frac{3}{\sqrt{10}}, 0, -\frac{1}{\sqrt{10}}\right), \left(-\frac{1}{\sqrt{35}}, \frac{5}{\sqrt{35}}, -\frac{3}{\sqrt{35}}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(0,\ 4,\ 2),\ (5,\ 6,\ -7)\}$$

Solution

Let $\vec{u}_1 = (0,\ 4,\ 2) \quad \vec{u}_2 = (5,\ 6,\ -7)$

$$\underline{\vec{v}_1 = \vec{u}_1 = (0,\ 4,\ 2)}$$

$$\vec{q}_1 = \frac{(0,\ 4,\ 2)}{\sqrt{16+4}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (5,\ 6,\ -7) - \frac{1}{20}\left[(5,\ 6,\ -7)\cdot(0,\ 4,\ 2)\right](0,\ 4,\ 2)$$

$$= (5,\ 6,\ -7) - \left(\frac{10}{20}\right)(0,\ 4,\ 2)$$

$$= (5,\ 6,\ -7) - (0,\ 2,\ 1)$$

$$= (5,\ 4,\ -8)$$

$$\vec{q}_2 = \frac{1}{\sqrt{25+16+64}}(5,\ 4,\ -8) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{\sqrt{105}}(5,\ 4,\ -8)$$

$$= \left(\frac{5}{\sqrt{105}},\ \frac{4}{\sqrt{105}},\ -\frac{8}{\sqrt{105}}\right)$$

The *orthogonal* basis: $\{(0,\ 4,\ 2),\ (5,\ 4,\ -8)\}$

The *orthonormal* basis: $\left\{\left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right),\ \left(\frac{5}{\sqrt{105}},\ \frac{4}{\sqrt{105}},\ -\frac{8}{\sqrt{105}}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(1,\ 1,\ 1),\ (-1,\ 1,\ 0),\ (1,\ 2,\ 1)\}$$

Solution

Let $\vec{u}_1=(1,\ 1,\ 1)\quad \vec{u}_2=(-1,\ 1,\ 0)\quad \vec{u}_3=(1,\ 2,\ 1)$

$$\vec{v}_1=\vec{u}_1=(1,\ 1,\ 1)$$

$$\vec{q}_1=\frac{(1,\ 1,\ 1)}{\sqrt{1+1+1}} \qquad\qquad \vec{q}_1=\frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$=\frac{(1,\ 1,\ 1)}{\sqrt{3}}$$

$$=\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\vec{v}_2=\vec{u}_2-\frac{\langle\vec{u}_2,\ \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$=(-1,\ 1,\ 0)-\tfrac{1}{3}\left[(-1,\ 1,\ 0)\bullet(1,\ 1,\ 1)\right](1,\ 1,\ 1)$$

$$=(-1,\ 1,\ 0)-(0)(1,\ 1,\ 1)$$

$$=(-1,\ 1,\ 0)$$

$$\vec{q}_2=\frac{1}{\sqrt{2}}(-1,\ 1,\ 0) \qquad\qquad \vec{q}_2=\frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$=\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$\langle\vec{u}_3,\ \vec{v}_1\rangle=(1,\ 2,\ 1)\bullet(1,\ 1,\ 1)$$

$$=4$$

$$\langle\vec{u}_3,\ \vec{v}_2\rangle=(1,\ 2,\ 1)\bullet(-1,\ 1,\ 0)$$

$$=1$$

$$\vec{v}_3=\vec{u}_3-\frac{\langle\vec{u}_3,\ \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1-\frac{\langle\vec{u}_3,\ \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (1,\ 2,\ 1) - \frac{4}{3}(1,\ 1,\ 1) - \frac{1}{2}(-1,\ 1,\ 0)$$

$$= (1,\ 2,\ 1) - \left(\frac{4}{3},\ \frac{4}{3},\ \frac{4}{3}\right) - \left(-\frac{1}{2},\ \frac{1}{2},\ 0\right)$$

$$= \left(\frac{1}{6},\ \frac{1}{6},\ -\frac{1}{3}\right)$$

$$\vec{q}_3 = \frac{1}{\sqrt{\frac{1}{36}+\frac{1}{36}+\frac{1}{9}}}\left(\frac{1}{6},\ \frac{1}{6},\ -\frac{1}{3}\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$= \frac{6}{\sqrt{6}}\left(\frac{1}{6},\ \frac{1}{6},\ -\frac{1}{3}\right)$$

$$= \left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right)$$

The *orthogonal* basis: $\left\{(1,\ 1,\ 1),\ (-1,\ 1,\ 0),\ \left(\frac{1}{6},\ \frac{1}{6},\ -\frac{1}{3}\right)\right\}$

The *orthonormal* basis: $\left\{\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right),\ \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right),\ \left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(1,\ 1,\ 1),\ (0,\ 1,\ 1),\ (0,\ 0,\ 1)\}$$

Solution

Let $\vec{u}_1 = (1,\ 1,\ 1)$ $\vec{u}_2 = (0,\ 1,\ 1)$ $\vec{u}_3 = (0,\ 0,\ 1)$

$$\vec{v}_1 = \vec{u}_1 = (1,\ 1,\ 1)$$

$$\vec{q}_1 = \frac{(1,\ 1,\ 1)}{\sqrt{1+1+1}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \frac{(1,\ 1,\ 1)}{\sqrt{3}}$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (0,\ 1,\ 1) - \tfrac{1}{3}\left[(0,\ 1,\ 1) \cdot (1,\ 1,\ 1)\right](1,\ 1,\ 1)$$

$$= (0,\ 1,\ 1) - \left(\tfrac{2}{3}\right)(1,\ 1,\ 1)$$

$$= (0,\ 1,\ 1) - \left(\tfrac{2}{3},\ \tfrac{2}{3},\ \tfrac{2}{3}\right)$$

$$= \left(-\tfrac{2}{3},\ \tfrac{1}{3},\ \tfrac{1}{3}\right)$$

$$\|\vec{v}_2\| = \sqrt{\left(-\tfrac{2}{3}\right)^2 + \left(\tfrac{1}{3}\right)^2 + \left(\tfrac{1}{3}\right)^2}$$

$$= \sqrt{\tfrac{4}{9} + \tfrac{1}{9} + \tfrac{1}{9}}$$

$$= \sqrt{\tfrac{6}{9}}$$

$$= \frac{\sqrt{6}}{3}$$

$$\vec{q}_2 = \frac{3}{\sqrt{6}}\left(-\tfrac{2}{3},\ \tfrac{1}{3},\ \tfrac{1}{3}\right) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = (0,\ 0,\ 1) \cdot (1,\ 1,\ 1)$$

$$= 1$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = (0,\ 0,\ 1) \cdot \left(-\tfrac{2}{3},\ \tfrac{1}{3},\ \tfrac{1}{3}\right)$$

$$= \tfrac{1}{3}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (0,\ 0,\ 1) - \tfrac{1}{3}(1,\ 1,\ 1) - \tfrac{1}{3}\left(\tfrac{9}{6}\right)\left(-\tfrac{2}{3},\ \tfrac{1}{3},\ \tfrac{1}{3}\right)$$

$$= (0,\ 0,\ 1) - \left(\tfrac{1}{3},\ \tfrac{1}{3},\ \tfrac{1}{3}\right) - \left(-\tfrac{1}{3},\ \tfrac{1}{6},\ \tfrac{1}{6}\right)$$

$$= \left(0,\ -\tfrac{1}{2},\ \tfrac{1}{2}\right)$$

$$\vec{q}_3 = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}}\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \left(\sqrt{2}\right)\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)$$

$$= \left(0,\ -\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$$

The *orthogonal* basis: $\left\{(1,\ 1,\ 1),\ \left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right),\ \left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)\right\}$

The *orthonormal* basis: $\left\{\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right),\ \left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right),\ \left(0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(1,\ 1,\ 1),\ (0,\ 2,\ 1),\ (1,\ 0,\ 3)\}$$

Solution

Let $\vec{u}_1 = (1,\ 1,\ 1)$ $\vec{u}_2 = (0,\ 2,\ 1)$ $\vec{u}_3 = (1,\ 0,\ 3)$

$$\vec{v}_1 = \vec{u}_1 = (1,\ 1,\ 1)$$

$$\vec{q}_1 = \frac{(1,\ 1,\ 1)}{\sqrt{1+1+1}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ 1,\ 1)}{\sqrt{3}}$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (0,\ 2,\ 1) - \frac{1}{3}\left[(0,\ 2,\ 1) \cdot (1,\ 1,\ 1)\right](1,\ 1,\ 1)$$

$$= (0,\ 2,\ 1) - \frac{3}{3}(1,\ 1,\ 1)$$

$$= (0,\ 2,\ 1) - (1,\ 1,\ 1)$$

$$= (-1,\ 1,\ 0)$$

$$\vec{q}_2 = \frac{1}{\sqrt{2}}(-1,\ 1,\ 0) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = (1,\ 0,\ 3) \cdot (1,\ 1,\ 1)$$

$$= 4$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = (1,\ 0,\ 3) \cdot (-1,\ 1,\ 0)$$

$$= -1$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (1,\ 0,\ 3) - \frac{4}{3}(1,\ 1,\ 1) + \frac{1}{2}(-1,\ 1,\ 0)$$

$$= (1,\ 0,\ 3) - \left(\frac{4}{3},\ \frac{4}{3},\ \frac{4}{3}\right) + \left(-\frac{1}{2},\ \frac{1}{2},\ 0\right)$$

$$= \left(-\frac{5}{6},\ -\frac{5}{6},\ \frac{5}{3}\right)$$

$$\vec{q}_3 = \frac{1}{\sqrt{\frac{25}{36} + \frac{25}{36} + \frac{25}{9}}}\left(-\frac{5}{6},\ -\frac{5}{6},\ \frac{5}{3}\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\frac{5}{6}\sqrt{6}}\left(-\frac{5}{6},\ -\frac{5}{6},\ \frac{5}{3}\right)$$

$$= \left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)$$

The *orthogonal* basis: $\left\{(1,\ 1,\ 1),\ (-1,\ 1,\ 0),\ \left(-\frac{5}{6},\ -\frac{5}{6},\ \frac{5}{3}\right)\right\}$

The *orthonormal* basis: $\left\{\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right),\ \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right),\ \left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(2,\ 2,\ 2),\ (1,\ 0,\ -1),\ (0,\ 3,\ 1)\}$$

Solution

Let $\vec{u}_1 = (2,\ 2,\ 2)\quad \vec{u}_2 = (1,\ 0,\ -1)\quad \vec{u}_3 = (0,\ 3,\ 1)$

$$\vec{v}_1 = \vec{u}_1 = (2,\ 2,\ 2)$$

$$\vec{q}_1 = \frac{(2,\ 2,\ 2)}{\sqrt{4+4+4}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(2,\ 2,\ 2)}{2\sqrt{3}}$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (1,\ 0,\ -1) - \frac{1}{12}\left[(1,\ 0,\ -1)\cdot(2,\ 2,\ 2)\right](2,\ 2,\ 2)$$

$$= (1,\ 0,\ -1) - \frac{0}{12}(2,\ 2,\ 2)$$

$$= (1,\ 0,\ -1)$$

$$\vec{q}_2 = \frac{(1,\ 0,\ -1)}{\sqrt{2}} \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(\frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right)$$

$$\langle \vec{u}_3, \vec{v}_1\rangle = (0,\ 3,\ 1)\cdot(2,\ 2,\ 2)$$

$$= 8$$

$$\langle \vec{u}_3, \vec{v}_2\rangle = (0,\ 3,\ 1)\cdot(1,\ 0,\ -1)$$

$$= -1$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$=(0,\ 3,\ 1)-\frac{8}{12}(2,\ 2,\ 2)+\frac{1}{2}(1,\ 0,\ -1)$$

$$=(0,\ 3,\ 1)-\left(\frac{4}{3},\ \frac{4}{3},\ \frac{4}{3}\right)+\left(\frac{1}{2},\ 0,\ -\frac{1}{2}\right)$$

$$=\left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right)$$

$$\vec{q}_3=\frac{1}{\sqrt{\left(-\frac{5}{6}\right)^2+\left(\frac{5}{3}\right)^2+\left(-\frac{5}{6}\right)^2}}\left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right) \qquad \vec{q}_3=\frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$=\frac{1}{\sqrt{\left(-\frac{5}{6}\right)^2+\left(\frac{5}{3}\right)^2+\left(-\frac{5}{6}\right)^2}}\left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right)$$

$$=\frac{1}{\sqrt{\frac{25}{36}+\frac{25}{36}+\frac{25}{9}}}\left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right)$$

$$=\frac{1}{\frac{5}{6}\sqrt{6}}\left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right)$$

$$=\left(-\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}}\right)$$

The *orthogonal* basis: $\left\{(2,\ 2,\ 2),\ (1,\ 0,\ -1),\ \left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right)\right\}$

The *orthonormal* basis: $\left\{\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right),\ \left(\frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right),\ \left(-\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(1,\ -1,\ 0),\ (0,\ 1,\ 0),\ (2,\ 3,\ 1)\}$$

Solution

Let $\vec{u}_1=(1,\ -1,\ 0)\quad \vec{u}_2=(0,\ 1,\ 0)\quad \vec{u}_3=(2,\ 3,\ 1)$

$$\vec{v}_1=\vec{u}_1=(1,\ -1,\ 0)$$

$$\vec{q}_1=\frac{(1,\ -1,\ 0)}{\sqrt{2}} \qquad \vec{q}_1=\frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \frac{(1,\ -1,\ 0)}{\sqrt{2}}$$

$$= \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (0,\ 1,\ 0) - \frac{1}{2}\left[(0,\ 1,\ 0) \cdot (1,\ -1,\ 0)\right](1,\ -1,\ 0)$$

$$= (0,\ 1,\ 0) + \frac{1}{2}(1,\ -1,\ 0)$$

$$= (0,\ 1,\ 0) + \left(\frac{1}{2},\ -\frac{1}{2},\ 0\right)$$

$$= \left(\frac{1}{2},\ \frac{1}{2},\ 0\right)$$

$$\vec{q}_2 = \frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}}}\left(\frac{1}{2},\ \frac{1}{2},\ 0\right) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = (2,\ 3,\ 1) \cdot (1,\ -1,\ 0)$$

$$= -1$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = (2,\ 3,\ 1) \cdot \left(\frac{1}{2},\ \frac{1}{2},\ 0\right)$$

$$= \frac{5}{2}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (2,\ 3,\ 1) + \frac{1}{2}(1,\ -1,\ 0) - \frac{5}{2}(2)\left(\frac{1}{2},\ \frac{1}{2},\ 0\right)$$

$$= (2,\ 3,\ 1) + \left(\frac{1}{2},\ -\frac{1}{2},\ 0\right) - \left(\frac{5}{2},\ \frac{5}{2},\ 0\right)$$

$$= (0,\ 0,\ 1)$$

$$\vec{q}_3 = \frac{1}{1}(0,\ 0,\ 1) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= (0,\ 0,\ 1)$$

The *orthogonal* basis: $\left\{(1,\ -1,\ 0),\ \left(\frac{1}{2},\ \frac{1}{2},\ 0\right),\ (0,\ 0,\ 1)\right\}$

The *orthonormal* basis: $\left\{\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right),\ \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right),\ (0,\ 0,\ 1)\right\}$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(3,\ 0,\ 4),\ (-1,\ 0,\ 7),\ (2,\ 9,\ 11)\}$$

Solution

Let $\vec{u}_1 = (3,\ 0,\ 4)$ $\vec{u}_2 = (-1,\ 0,\ 7)$ $\vec{u}_3 = (2,\ 9,\ 11)$

$\vec{v}_1 = \vec{u}_1 = (3,\ 0,\ 4)$

$\vec{q}_1 = \dfrac{(3,\ 0,\ 4)}{\sqrt{9+16}}$ $\qquad \vec{q}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$

$= \left(\frac{3}{5},\ 0,\ \frac{4}{5}\right)$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2,\ \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (-1,\ 0,\ 7) - \frac{1}{25}\left[(-1,\ 0,\ 7)\cdot(3,\ 0,\ 4)\right](3,\ 0,\ 4)$$

$$= (-1,\ 0,\ 7) - \frac{25}{25}(3,\ 0,\ 4)$$

$$= (-1,\ 0,\ 7) - (3,\ 0,\ 4)$$

$$= (-4,\ 0,\ 3)$$

$\vec{q}_2 = \dfrac{1}{\sqrt{16+9}}(-4,\ 0,\ 3)$ $\qquad \vec{q}_2 = \dfrac{\vec{v}_2}{\|\vec{v}_2\|}$

$= \left(-\frac{4}{5},\ 0,\ \frac{3}{5}\right)$

$\langle \vec{u}_3,\ \vec{v}_1\rangle = (2,\ 9,\ 11)\cdot(3,\ 0,\ 4)$

$= 50$

$\langle \vec{u}_3,\ \vec{v}_2\rangle = (2,\ 9,\ 11)\cdot(-4,\ 0,\ 3)$

$$=25$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (2,\ 9,\ 11) - \frac{50}{25}(3,\ 0,\ 4) - \frac{25}{25}(-4,\ 0,\ 3)$$

$$= (2,\ 9,\ 11) - (6,\ 0,\ 8) - (-4,\ 0,\ 3)$$

$$= (0,\ 9,\ 0)$$

$$\vec{q}_3 = \frac{1}{9}(0,\ 9,\ 0) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= (0,\ 1,\ 0)$$

The *orthogonal* basis: $\{(3,\ 0,\ 4),\ (-4,\ 0,\ 3),\ (0,\ 9,\ 0)\}$

The *orthonormal* basis: $\left\{\left(\frac{3}{5},\ 0,\ \frac{4}{5}\right),\ \left(-\frac{4}{5},\ 0,\ \frac{3}{5}\right),\ (0,\ 1,\ 0)\right\}$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(2,\ 1,\ -1),\ (1,\ 2,\ 2),\ (2,\ -2,\ 1)\}$$

Solution

Let $\vec{u}_1 = (2,\ 1,\ -1)$ $\vec{u}_2 = (1,\ 2,\ 2)$ $\vec{u}_3 = (2,\ -2,\ 1)$

$$\vec{v}_1 = \vec{u}_1 = (2,\ 1,\ -1)$$

$$\vec{q}_1 = \frac{(2,\ 1,\ -1)}{\sqrt{4+1+1}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|}\vec{v}_1$$

$$= (1,\ 2,\ 2) - \frac{1}{4+1+1}\left[(1,\ 2,\ 2)\cdot(2,\ 1,\ -1)\right](2,\ 1,\ -1)$$

$$= (1,\ 2,\ 2) - \frac{1}{6}(2)(2,\ 1,\ -1)$$

$$= (1,\ 2,\ 2) - \left(\frac{2}{3},\ \frac{1}{3},\ -\frac{1}{3}\right)$$

$$= \left(\frac{1}{3},\ \frac{5}{3},\ \frac{7}{3}\right)$$

$$\vec{q}_2 = \frac{1}{\sqrt{\frac{1}{9} + \frac{25}{9} + \frac{49}{9}}}\left(\frac{1}{3},\ \frac{5}{3},\ \frac{7}{3}\right) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{3}{5\sqrt{3}}\left(\frac{1}{3},\ \frac{5}{3},\ \frac{7}{3}\right)$$

$$= \left(\frac{1}{5\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{7}{5\sqrt{3}}\right)$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = (2,\ -2,\ 1) \bullet (2,\ 1,\ -1)$$

$$= 4 - 2 - 1$$

$$= 1$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = (2,\ -2,\ 1) \bullet \left(\frac{1}{3},\ \frac{5}{3},\ \frac{7}{3}\right)$$

$$= \frac{2}{3} - \frac{10}{3} + \frac{7}{3}$$

$$= -\frac{1}{3}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (2,\ -2,\ 1) - \frac{1}{6}(2,\ 1,\ -1) + \left(\frac{1}{3}\right)\left(\frac{9}{75}\right)\left(\frac{1}{3},\ \frac{5}{3},\ \frac{7}{3}\right)$$

$$= (2,\ -2,\ 1) - \left(\frac{1}{3},\ \frac{1}{6},\ -\frac{1}{6}\right) + \left(\frac{1}{75},\ \frac{1}{15},\ \frac{7}{75}\right)$$

$$= \left(\frac{42}{25},\ -\frac{63}{30},\ \frac{63}{50}\right)$$

$$\vec{q}_2 = \frac{1}{\sqrt{\frac{1,764}{625} + \frac{3,969}{900} + \frac{3,969}{2,500}}}\left(\frac{42}{25},\ -\frac{63}{30},\ \frac{63}{50}\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\sqrt{\frac{63,504 + 99,225 + 35,721}{22,500}}}\left(\frac{42}{25},\ -\frac{63}{30},\ \frac{63}{50}\right)$$

$$= \frac{150}{\sqrt{198,450}}\left(\frac{42}{25},\ -\frac{63}{30},\ \frac{63}{50}\right)$$

$$= \frac{150}{315\sqrt{2}}\left(\frac{42}{25},\ -\frac{63}{30},\ \frac{63}{50}\right)$$

$$= \frac{10}{21\sqrt{2}}\left(\frac{42}{25},\ -\frac{63}{30},\ \frac{63}{50}\right)$$

$$\boxed{= \left(\frac{4}{5\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ \frac{3}{5\sqrt{2}}\right)}$$

The *orthogonal* basis: $\left\{(2,\ 1,\ -1),\ \left(\frac{1}{3},\ \frac{5}{3},\ \frac{7}{3}\right),\ \left(\frac{42}{25},\ -\frac{63}{30},\ \frac{63}{50}\right)\right\}$

The *orthonormal* basis:

$$\left\{\left(\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}}\right),\ \left(\frac{1}{5\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{7}{5\sqrt{3}}\right),\ \left(\frac{4}{5\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ \frac{3}{5\sqrt{2}}\right)\right\}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(1,\ 1,\ 0),\ (1,\ 2,\ 0),\ (0,\ 1,\ 2)\}$$

Solution

Let $\vec{u}_1 = (1,\ 1,\ 0)$ $\vec{u}_2 = (1,\ 2,\ 0)$ $\vec{u}_3 = (0,\ 1,\ 2)$

$$\boxed{\vec{v}_1 = \vec{u}_1 = (1,\ 1,\ 0)}$$

$$\vec{q}_1 = \frac{(1,\ 1,\ 0)}{\sqrt{2}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$\boxed{= \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2,\ \vec{v}_1\right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$= (1,\ 2,\ 0) - \frac{(1,\ 2,\ 0)\bullet(1,\ 1,\ 0)}{1+1+0}(1,\ 1,\ 0)$$

$$= (1,\ 2,\ 0) - \frac{3}{2}(1,\ 1,\ 0)$$

$$\boxed{= \left(-\frac{1}{2},\ \frac{1}{2},\ 0\right)}$$

$$\vec{q}_2 = \frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}}}\left(-\frac{1}{2},\ \frac{1}{2},\ 0\right) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \frac{2}{\sqrt{2}}\left(-\frac{1}{2},\ \frac{1}{2},\ 0\right)$$

$$= \boxed{\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(0,\ 1,\ 2)\bullet(1,\ 1,\ 0)}{2}(1,\ 1,\ 0)$$

$$= \boxed{\left(\frac{1}{2},\ \frac{1}{2},\ 0\right)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{1}{\frac{1}{4}+\frac{1}{4}}\left[(0,\ 1,\ 2)\bullet\left(-\frac{1}{2},\ \frac{1}{2},\ 0\right)\right]\left(-\frac{1}{2},\ \frac{1}{2},\ 0\right)$$

$$= 2\left(\frac{1}{2}\right)\left(-\frac{1}{2},\ \frac{1}{2},\ 1\right)$$

$$= \boxed{\left(-\frac{1}{2},\ \frac{1}{2},\ 1\right)}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (0,\ 1,\ 2) - \left(\frac{1}{2},\ \frac{1}{2},\ 0\right) - \left(-\frac{1}{2},\ \frac{1}{2},\ 1\right)$$

$$= \boxed{(0,\ 0,\ 1)}$$

$$\boxed{\vec{q}_3 = (0,\ 0,\ 1)} \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

The *orthogonal* basis: $\left\{(1,\ 1,\ 0),\ \left(-\frac{1}{2},\ \frac{1}{2},\ 0\right),\ (0,\ 0,\ 1)\right\}$

The *orthogonal* basis: $\left\{\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right),\ \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right),\ (0,\ 0,\ 1)\right\}$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(1,\ -2,\ 2),\ (2,\ 2,\ 1),\ (2,\ -1,\ -2)\}$$

Solution

Let $\vec{u}_1 = (1,\ -2,\ 2)\quad \vec{u}_2 = (2,\ 2,\ 1)\quad \vec{u}_3 = (2,\ -1,\ -2)$

$$\vec{v}_1 = \vec{u}_1 = (1,\ -2,\ 2)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (2,\ 2,\ 1) - \frac{(2,\ 2,\ 1)\cdot(1,\ -2,\ 2)}{9}(1,\ -2,\ 2)$$

$$= (2,\ 2,\ 1) - \frac{0}{9}(1,\ -2,\ 2)$$

$$= (2,\ 2,\ 1)$$

$$\frac{\langle \vec{u}_3, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(2,\ -1,\ -2)\cdot(1,\ -2,\ 2)}{9}(1,\ -2,\ 2)$$

$$= \frac{0}{9}(1,\ -2,\ 2)$$

$$= (0,\ 0,\ 0)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{1}{9}\left[(2,\ -1,\ -2)\cdot(2,\ 2,\ 1)\right](2,\ 2,\ 1)$$

$$= (0,\ 0,\ 0)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (2,\ -1,\ -2) - (0,\ 0,\ 0) - (0,\ 0,\ 0)$$

$$= (2,\ -1,\ -2)$$

$$\vec{q}_1 = \frac{1}{3}(1,\ -2,\ 2) \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(\frac{1}{3},\ -\frac{2}{3},\ \frac{2}{3}\right)$$

$\vec{q}_2 = \frac{1}{3}(2,\ 2,\ 1)$ $\qquad \vec{q}_2 = \dfrac{\vec{v}_2}{\|\vec{v}_2\|}$

$= \left(\frac{2}{3},\ \frac{2}{3},\ \frac{1}{3}\right)$

$\vec{q}_3 = \frac{1}{3}(2,\ -1,\ -2)$ $\qquad \vec{q}_3 = \dfrac{\vec{v}_3}{\|\vec{v}_3\|}$

$= \left(\frac{2}{3},\ -\frac{1}{3},\ -\frac{2}{3}\right)$

The *orthogonal* basis: $\quad \{(1,\ -2,\ 2),\ (2,\ 2,\ 1),\ (2,\ -1,\ -2)\}$

The *orthogonal* basis:

$$\left\{\left(\frac{1}{3},\ -\frac{2}{3},\ \frac{2}{3}\right),\ \left(\frac{2}{3},\ \frac{2}{3},\ \frac{1}{3}\right),\ \left(\frac{2}{3},\ -\frac{1}{3},\ -\frac{2}{3}\right)\right\}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(1,\ 0,\ 0),\ (1,\ 1,\ 1),\ (1,\ 1,\ -1)\}$$

Solution

Let $\quad \vec{u}_1 = (1,\ 0,\ 0) \quad \vec{u}_2 = (1,\ 1,\ 1) \quad \vec{u}_3 = (1,\ 1,\ -1)$

$\vec{v}_1 = \vec{u}_1 = (1,\ 0,\ 0)$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 \\
&= (1,\ 1,\ 1) - \frac{(1,\ 1,\ 1)\bullet(1,\ 0,\ 0)}{1}(1,\ 0,\ 0) \\
&= (1,\ 1,\ 1) - (1,\ 0,\ 0) \\
&= (0,\ 1,\ 1)
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 &= \frac{(1,\ 1,\ -1)\bullet(1,\ 0,\ 0)}{1}(1,\ 0,\ 0) \\
&= (1,\ 0,\ 0)
\end{aligned}$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{1}{1}\left[(1,\ 1,\ -1)\cdot(0,\ 1,\ 1)\right](0,\ 1,\ 1)$$

$$= 0(0,\ 1,\ 1)$$

$$= \boxed{(0,\ 0,\ 0)}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (1,\ 1,\ -1) - (1,\ 0,\ 0) - (0,\ 0,\ 0)$$

$$= \boxed{(0,\ 1,\ -1)}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{1}(1,\ 0,\ 0)$$

$$= \boxed{(1,\ 0,\ 0)}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{\sqrt{2}}(0,\ 1,\ 1)$$

$$= \boxed{\left(0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\sqrt{2}}(0,\ 1,\ -1)$$

$$= \boxed{\left(0,\ \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right)}$$

The *orthogonal* basis: $\{(1,\ 0,\ 0),\ (0,\ 1,\ 1),\ (0,\ 1,\ -1)\}$

The *orthogonal* basis:

$$\left\{(1,\ 0,\ 0),\ \left(0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right),\ \left(0,\ \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right)\right\}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(4,\ -3,\ 0),\ (1,\ 2,\ 0),\ (0,\ 0,\ 4)\}$$

Solution

Let $\vec{u}_1 = (4,\ -3,\ 0)\quad \vec{u}_2 = (1,\ 2,\ 0)\quad \vec{u}_3 = (0,\ 0,\ 4)$

$$\vec{v}_1 = \vec{u}_1 = (4,\ -3,\ 0)$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1\\ &= (1,\ 2,\ 0) - \frac{(1,\ 2,\ 0)\bullet(4,\ -3,\ 0)}{25}(4,\ -3,\ 0)\\ &= (1,\ 2,\ 0) + \frac{2}{25}(4,\ -3,\ 0)\\ &= \left(\frac{33}{25},\ \frac{44}{25},\ 0\right)\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 &= \frac{(0,\ 0,\ 4)\bullet(4,\ -3,\ 0)}{25}(4,\ -3,\ 0)\\ &= (0,\ 0,\ 0)\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2 &= \frac{225}{3,025}\left[(0,\ 0,\ 4)\bullet\left(\frac{33}{25},\ \frac{44}{25},\ 0\right)\right]\left(\frac{33}{25},\ \frac{44}{25},\ 0\right)\\ &= (0,\ 0,\ 0)\end{aligned}$$

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2\\ &= (0,\ 0,\ 4) - (0,\ 0,\ 0) - (0,\ 0,\ 0)\\ &= (0,\ 0,\ 4)\end{aligned}$$

$$\begin{aligned}\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|}\\ &= \frac{1}{\sqrt{16+9}}(4,\ -3,\ 0)\\ &= \left(\frac{4}{5},\ -\frac{3}{5},\ 0\right)\end{aligned}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{25}{\sqrt{3,025}}\left(\frac{33}{25},\ \frac{44}{25},\ 0\right)$$

$$= \frac{25}{55}\left(\frac{33}{25},\ \frac{44}{25},\ 0\right)$$

$$= \left(\frac{3}{5},\ \frac{4}{5},\ 0\right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{4}(0,\ 0,\ 4)$$

$$= (0,\ 0,\ 1)$$

The *orthogonal* basis: $\left\{(4,\ -3,\ 0),\ \left(\frac{33}{25},\ \frac{44}{25},\ 0\right),\ (0,\ 0,\ 4)\right\}$

The *orthogonal* basis: $\left\{\left(\frac{4}{5},\ -\frac{3}{5},\ 0\right),\ \left(\frac{3}{5},\ \frac{4}{5},\ 0\right),\ (0,\ 0,\ 1)\right\}$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(0,\ 1,\ 2),\ (2,\ 0,\ 0),\ (1,\ 1,\ 1)\}$$

Solution

Let $\vec{u}_1 = (0,\ 1,\ 2)$ $\vec{u}_2 = (2,\ 0,\ 0)$ $\vec{u}_3 = (1,\ 1,\ 1)$

$$\vec{v}_1 = \vec{u}_1 = (0,\ 1,\ 2)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2,\ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (2,\ 0,\ 0) - \frac{(2,\ 0,\ 0) \bullet (0,\ 1,\ 2)}{5}(0,\ 1,\ 2)$$

$$= (2,\ 0,\ 0) + \frac{0}{5}(0,\ 1,\ 2)$$

$$= (2,\ 0,\ 0)$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(1,\ 1,\ 1)\cdot(0,\ 1,\ 2)}{5}(0,\ 1,\ 2)$$
$$= \frac{3}{5}(0,\ 1,\ 2)$$
$$= \left(0,\ \frac{3}{5},\ \frac{6}{5}\right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{1}{4}\left[(1,\ 1,\ 1)\cdot(2,\ 0,\ 0)\right](2,\ 0,\ 0)$$
$$= \frac{1}{2}(2,\ 0,\ 0)$$
$$= (1,\ 0,\ 0)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$
$$= (1,\ 1,\ 1) - (1,\ 0,\ 0) - \left(0,\ \frac{3}{5},\ \frac{6}{5}\right)$$
$$= \left(0,\ \frac{2}{5},\ -\frac{1}{5}\right)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$
$$= \frac{1}{\sqrt{5}}(0,\ 1,\ 2)$$
$$= \left(0,\ \frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$
$$= \frac{1}{2}(2,\ 0,\ 0)$$
$$= (1,\ 0,\ 0)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$
$$= \frac{5}{\sqrt{5}}\left(0,\ \frac{2}{5},\ -\frac{1}{5}\right)$$
$$= \left(0,\ \frac{2}{\sqrt{5}},\ -\frac{1}{\sqrt{5}}\right)$$

The *orthogonal* basis: $\left\{(0,\ 1,\ 2),\ (2,\ 0,\ 0),\ \left(0,\ \frac{2}{5},\ -\frac{1}{5}\right)\right\}$

The *orthogonal* basis: $\left\{\left(0,\ \frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right),\ (1,\ 0,\ 0),\ \left(0,\ \frac{2}{\sqrt{5}},\ -\frac{1}{\sqrt{5}}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\left\{(0,\ 1,\ 1),\ (1,\ 1,\ 0),\ (1,\ 0,\ 1)\right\}$$

Solution

Let $\vec{u}_1 = (0,\ 1,\ 1) \quad \vec{u}_2 = (1,\ 1,\ 0) \quad \vec{u}_3 = (1,\ 0,\ 1)$

$$\vec{v}_1 = \vec{u}_1 = (0,\ 1,\ 1)$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 \\
&= (1,\ 1,\ 0) - \frac{(1,\ 1,\ 0)\cdot(0,\ 1,\ 1)}{2}(0,\ 1,\ 1) \\
&= (1,\ 1,\ 0) - \frac{1}{2}(0,\ 1,\ 1) \\
&= \left(1,\ \frac{1}{2},\ -\frac{1}{2}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 &= \frac{(1,\ 0,\ 1)\cdot(0,\ 1,\ 1)}{2}(0,\ 1,\ 1) \\
&= \left(0,\ \frac{1}{2},\ \frac{1}{2}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 &= \frac{4}{6}\left[(1,\ 0,\ 1)\cdot\left(1,\ \frac{1}{2},\ -\frac{1}{2}\right)\right]\left(1,\ \frac{1}{2},\ -\frac{1}{2}\right) \\
&= \frac{1}{3}\left(1,\ \frac{1}{2},\ -\frac{1}{2}\right) \\
&= \left(\frac{1}{3},\ \frac{1}{6},\ -\frac{1}{6}\right)
\end{aligned}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$=(1,\ 0,\ 1)-\left(0,\ \frac{1}{2},\ \frac{1}{2}\right)-\left(\frac{1}{3},\ \frac{1}{6},\ -\frac{1}{6}\right)$$

$$=\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3}\right)$$

$$\vec{q}_1=\frac{1}{\sqrt{2}}(0,\ 1,\ 1) \qquad \vec{q}_1=\frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$=\left(0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

$$\vec{q}_2=\frac{2}{\sqrt{6}}\left(1,\ \frac{1}{2},\ -\frac{1}{2}\right) \qquad \vec{q}_2=\frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$=\left(\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}}\right)$$

$$\vec{q}_3=\frac{3}{\sqrt{12}}\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3}\right) \qquad \vec{q}_3=\frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$=\left(\frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

The *orthogonal* basis: $\left\{(0,\ 1,\ 1),\ \left(1,\ \frac{1}{2},\ -\frac{1}{2}\right),\ \left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3}\right)\right\}$

The *orthogonal* basis: $\left\{\left(0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right),\ \left(\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}}\right),\ \left(\frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(2,\ -1,\ 3),\ \ (3,\ 4,\ 1),\ \ (2,\ -3,\ 4)\}$$

Solution

Let $\vec{u}_1=(2,\ -1,\ 3)$ $\vec{u}_2=(3,\ 4,\ 1)$ $\vec{u}_3=(2,\ -3,\ 4)$

$$\vec{v}_1=\vec{u}_1=(2,\ -1,\ 3)$$

$$\vec{q}_1=\frac{1}{\sqrt{4+1+9}}(2,\ -1,\ 3) \qquad \vec{q}_1=\frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$=\left(\frac{2}{\sqrt{14}},\ -\frac{1}{\sqrt{14}},\ \frac{3}{\sqrt{14}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (3,\ 4,\ 1) - \frac{(3,\ 4,\ 1)\bullet(2,\ -1,\ 3)}{14}(2,\ -1,\ 3)$$

$$= (3,\ 4,\ 1) - \frac{5}{14}(2,\ -1,\ 3)$$

$$= (3,\ 4,\ 1) - \left(\frac{5}{7},\ -\frac{5}{14},\ \frac{15}{14}\right)$$

$$= \left(\frac{16}{7},\ \frac{61}{14},\ -\frac{1}{14}\right)$$

$$\vec{q}_2 = \frac{14}{\sqrt{32^2 + 61^2 + 1}}\left(\frac{16}{7},\ \frac{61}{14},\ -\frac{1}{14}\right) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{14}{\sqrt{4,746}}\left(\frac{16}{7},\ \frac{61}{14},\ -\frac{1}{14}\right)$$

$$= \left(\frac{32}{\sqrt{4,746}},\ \frac{61}{\sqrt{4,746}},\ -\frac{1}{\sqrt{4,746}}\right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(2,\ -3,\ 4)\bullet(2,\ -1,\ 3)}{14}(2,\ -1,\ 3)$$

$$= \frac{19}{14}(2,\ -1,\ 3)$$

$$= \left(\frac{19}{7},\ -\frac{19}{14},\ \frac{57}{14}\right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{196}{4,746}\left((2,\ -3,\ 4)\bullet\left(\frac{16}{7},\ \frac{61}{14},\ -\frac{1}{14}\right)\right)\left(\frac{16}{7},\ \frac{61}{14},\ -\frac{1}{14}\right)$$

$$= \frac{14}{339}\left(\frac{32}{7} - \frac{183}{14} - \frac{2}{7}\right)\left(\frac{16}{7},\ \frac{61}{14},\ -\frac{1}{14}\right)$$

$$= \frac{14}{339}\left(-\frac{123}{14}\right)\left(\frac{16}{7},\ \frac{61}{14},\ -\frac{1}{14}\right)$$

$$= -\frac{41}{113}\left(\frac{16}{7},\ \frac{61}{14},\ -\frac{1}{14}\right)$$

$$= \left(-\frac{656}{791},\ -\frac{2,501}{1,582},\ \frac{41}{1,582}\right)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$=(2,\ -3,\ 4)-\left(\frac{19}{7},\ -\frac{19}{14},\ \frac{57}{14}\right)-\left(-\frac{656}{791},\ -\frac{2{,}501}{1{,}582},\ \frac{41}{1{,}582}\right)$$

$$=\left(-\frac{5}{7},\ -\frac{23}{14},\ -\frac{1}{14}\right)-\left(-\frac{656}{791},\ -\frac{2{,}501}{1{,}582},\ \frac{41}{1{,}582}\right)$$

$$=\left(\frac{13}{113},\ -\frac{7}{113},\ -\frac{11}{113}\right)$$

$$\vec{q}_3=\frac{113}{\sqrt{169+49+121}}\left(\frac{13}{113},\ -\frac{7}{113},\ -\frac{11}{113}\right) \qquad \vec{q}_3=\frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$=\frac{1}{\sqrt{339}}(13,\ -7,\ -11)$$

$$=\left(\frac{13}{\sqrt{339}},\ -\frac{7}{\sqrt{339}},\ -\frac{11}{\sqrt{339}}\right)$$

The *orthogonal* basis: $\left\{(2,\ -1,\ 3),\ \left(\frac{16}{7},\ \frac{61}{14},\ -\frac{1}{14}\right),\ \left(\frac{13}{113},\ -\frac{7}{113},\ -\frac{11}{113}\right)\right\}$

The *orthogonal* basis:

$$\left\{\left(\frac{2}{\sqrt{14}},\ -\frac{1}{\sqrt{14}},\ \frac{3}{\sqrt{14}}\right),\ \left(\frac{32}{\sqrt{4,746}},\ \frac{61}{\sqrt{4,746}},\ -\frac{1}{\sqrt{4,746}}\right),\ \left(\frac{13}{\sqrt{339}},\ -\frac{7}{\sqrt{339}},\ -\frac{11}{\sqrt{339}}\right)\right\}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(-3,\ 0,\ 4),\ (5,\ -1,\ 2),\ (1,\ 1,\ 3)\}$$

Solution

Let $\vec{u}_1=(-3,\ 0,\ 4)\quad \vec{u}_2=(5,\ -1,\ 2)\quad \vec{u}_3=(1,\ 1,\ 3)$

$$\vec{v}_1=\vec{u}_1=(-3,\ 0,\ 4)$$

$$\vec{q}_1=\frac{(-3,\ 0,\ 4)}{\sqrt{9+16}} \qquad \vec{q}_1=\frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$=\left(-\frac{3}{5},\ 0,\ \frac{4}{5}\right)$$

$$\vec{v}_2=\vec{u}_2-\frac{\left\langle\vec{u}_2,\ \vec{v}_1\right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$=(5,\ -1,\ 2)-\frac{(5,\ -1,\ 2)\bullet(-3,\ 0,\ 4)}{25}(-3,\ 0,\ 4)$$

$$= (5,\ -1,\ 2) + \frac{7}{25}(-3,\ 0,\ 4)$$

$$= (5,\ -1,\ 2) + \left(-\frac{21}{25},\ 0,\ \frac{28}{25}\right)$$

$$= \boxed{\left(\frac{104}{25},\ -1,\ \frac{78}{25}\right)}$$

$$\vec{q}_2 = \frac{25}{\sqrt{104^2 + 25^2 + 78^2}}\left(\frac{104}{25},\ -1,\ \frac{78}{25}\right) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{\sqrt{17{,}525}}(104,\ -25,\ 78)$$

$$= \frac{1}{5\sqrt{701}}(104,\ -25,\ 78)$$

$$= \boxed{\left(\frac{104}{5\sqrt{701}},\ -\frac{5}{\sqrt{701}},\ \frac{78}{5\sqrt{701}}\right)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(1,\ 1,\ 3)\bullet(-3,\ 0,\ 4)}{25}(-3,\ 0,\ 4)$$

$$= \frac{9}{25}(-3,\ 0,\ 4)$$

$$= \boxed{\left(-\frac{27}{25},\ 0,\ \frac{36}{25}\right)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{25^2}{17{,}525}\left((1,\ 1,\ 3)\bullet\left(\frac{104}{25},\ -1,\ \frac{78}{25}\right)\right)\left(\frac{104}{25},\ -1,\ \frac{78}{25}\right)$$

$$= \frac{25}{701}\left(\frac{104}{25} - 1 + \frac{234}{25}\right)\left(\frac{104}{25},\ -1,\ \frac{78}{25}\right)$$

$$= \frac{1}{701}\left(\frac{313}{25}\right)(104,\ -25,\ 78)$$

$$= \frac{313}{17{,}525}(104,\ -25,\ 78)$$

$$= \boxed{\left(\frac{32{,}552}{17{,}525},\ -\frac{313}{701},\ \frac{24{,}414}{17{,}525}\right)}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (1,\ 1,\ 3) - \left(-\frac{27}{25},\ 0,\ \frac{36}{25}\right) - \left(\frac{32{,}552}{17{,}525},\ -\frac{313}{701},\ \frac{24{,}414}{17{,}525}\right)$$

$$= \left(\frac{52}{25},\ 1,\ \frac{39}{25}\right) - \left(\frac{32{,}552}{17{,}525},\ -\frac{313}{701},\ \frac{24{,}414}{17{,}525}\right)$$

$$=\boxed{\left(\frac{156}{701},\ \frac{1,014}{701},\ \frac{117}{701}\right)}$$

$$\vec{q}_3=\frac{701}{\sqrt{156^2+1014^2+117^2}}\left(\frac{156}{701},\ \frac{1,014}{701},\ \frac{117}{701}\right) \qquad \vec{q}_3=\frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$=\frac{701}{\sqrt{1,066,221}}\left(\frac{156}{701},\ \frac{1,014}{701},\ \frac{117}{701}\right)$$

$$=\frac{701}{39\sqrt{701}}\left(\frac{156}{701},\ \frac{1,014}{701},\ \frac{117}{701}\right)$$

$$=\boxed{\left(\frac{4}{\sqrt{701}},\ \frac{26}{\sqrt{701}},\ \frac{3}{\sqrt{701}}\right)}$$

The *orthogonal* basis: $\left\{(-3,\ 0,\ 4),\ \left(\frac{104}{25},\ -1,\ \frac{78}{25}\right),\ \left(\frac{156}{701},\ \frac{1,014}{701},\ \frac{117}{701}\right)\right\}$

The *orthogonal* basis:

$$\left\{\left(-\frac{3}{5},\ 0,\ \frac{4}{5}\right),\ \left(\frac{104}{5\sqrt{701}},\ -\frac{5}{\sqrt{701}},\ \frac{78}{5\sqrt{701}}\right),\ \left(\frac{4}{\sqrt{701}},\ \frac{26}{\sqrt{701}},\ \frac{3}{\sqrt{701}}\right)\right\}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(-2,\ 0,\ 1),\ (3,\ 2,\ 5),\ (6,-1,\ 1),\ (7,\ 0,\ -2)\}$$

Solution

Let $\vec{u}_1=(-2,\ 0,\ 1)\quad \vec{u}_2=(3,\ 2,\ 5)\quad \vec{u}_3=(6,-1,\ 1)\quad \vec{u}_4=(7,\ 0,\ -2)$

$$\boxed{\vec{v}_1=\vec{u}_1=(-2,\ 0,\ 1)}$$

$$\vec{q}_1=\frac{(-2,\ 0,\ 1)}{\sqrt{5}} \qquad \vec{q}_1=\frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$=\boxed{\left(-\frac{2}{\sqrt{5}},\ 0,\ \frac{1}{\sqrt{5}}\right)}$$

$$\vec{v}_2=\vec{u}_2-\frac{\left\langle\vec{u}_2,\ \vec{v}_1\right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$=(3,\ 2,\ 5)-\frac{(3,\ 2,\ 5)\bullet(-2,\ 0,\ 1)}{5}(-2,\ 0,\ 1)$$

$$= (3,\ 2,\ 5) + \frac{1}{5}(-2,\ 0,\ 1)$$

$$= (3,\ 2,\ 5) + \left(-\frac{2}{5},\ 0,\ \frac{1}{5}\right)$$

$$= \boxed{\left(\frac{13}{5},\ 2,\ \frac{26}{5}\right)}$$

$$\vec{q}_2 = \frac{5}{\sqrt{13^2 + 10^2 + 26^2}}\left(\frac{13}{5},\ 2,\ \frac{26}{5}\right) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{\sqrt{945}}(13,\ 10,\ 26)$$

$$= \frac{1}{3\sqrt{105}}(13,\ 10,\ 26)$$

$$= \boxed{\left(\frac{13}{3\sqrt{105}},\ -\frac{10}{3\sqrt{105}},\ \frac{26}{3\sqrt{105}}\right)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(6, -1,\ 1)\bullet(-2,\ 0,\ 1)}{5}(-2,\ 0,\ 1)$$

$$= -\frac{11}{5}(-2,\ 0,\ 1)$$

$$= \boxed{\left(\frac{22}{5},\ 0,\ -\frac{11}{5}\right)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{25}{945}\left((6, -1,\ 1)\bullet\left(\frac{13}{5},\ 2,\ \frac{26}{5}\right)\right)\left(\frac{13}{5},\ 2,\ \frac{26}{5}\right)$$

$$= \frac{5}{189}\left(\frac{78}{5} - 2 + \frac{26}{5}\right)\left(\frac{13}{5},\ 2,\ \frac{26}{5}\right)$$

$$= \frac{94}{189}\left(\frac{13}{5},\ 2,\ \frac{26}{5}\right)$$

$$= \boxed{\left(\frac{1,222}{945},\ -\frac{188}{189},\ \frac{2,444}{945}\right)}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (6, -1,\ 1) - \left(\frac{22}{5},\ 0,\ -\frac{11}{5}\right) - \left(\frac{1,222}{945},\ -\frac{188}{189},\ \frac{2,444}{945}\right)$$

$$= \left(\frac{8}{5},\ -1,\ \frac{16}{5}\right) - \left(\frac{1,222}{945},\ -\frac{188}{189},\ \frac{2,444}{945}\right)$$

$$= \left(\frac{290}{945},\ -\frac{377}{189},\ \frac{580}{945}\right)$$

$$=\boxed{\left(\frac{58}{189},\ -\frac{377}{189},\ \frac{116}{189}\right)}$$

$$\vec{q}_3=\frac{189}{\sqrt{58^2+377^2+116^2}}\left(\frac{58}{189},\ -\frac{377}{189},\ \frac{116}{189}\right) \qquad \vec{q}_3=\frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$=\frac{1}{\sqrt{158{,}949}}(58,\ -377,\ 116)$$

$$=\frac{1}{87\sqrt{21}}(58,\ -377,\ 116)$$

$$=\boxed{\left(\frac{2}{3\sqrt{21}},\ -\frac{13}{3\sqrt{21}},\ \frac{4}{3\sqrt{21}}\right)}$$

$$\frac{\langle\vec{u}_4,\vec{v}_1\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1=\frac{(7,\ 0,\ -2)\cdot(-2,\ 0,\ 1)}{5}(-2,\ 0,\ 1)$$

$$=-\frac{16}{5}(-2,\ 0,\ 1)$$

$$=\boxed{\left(\frac{32}{5},\ 0,\ -\frac{16}{5}\right)}$$

$$\left\|\vec{v}_2\right\|^2=\frac{13^2+10^2+26^2}{25}$$

$$=\boxed{\frac{945}{25}}$$

$$\frac{\langle\vec{u}_4,\vec{v}_2\rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2=\frac{25}{945}\left[(7,\ 0,\ -2)\cdot\left(\frac{13}{5},\ 2,\ \frac{26}{5}\right)\right]\left(\frac{13}{5},\ 2,\ \frac{26}{5}\right)$$

$$=\frac{25}{945}\left(\frac{39}{5}\right)\left(\frac{13}{5},\ \frac{10}{5},\ \frac{26}{5}\right)$$

$$=\frac{13}{315}(13,\ 10,\ 26)$$

$$=\boxed{\left(\frac{169}{315},\ \frac{2}{63},\ \frac{338}{315}\right)}$$

$$\left\|\vec{v}_3\right\|^2=\frac{58^2+377^2+116^2}{189^2}$$

$$=\frac{158{,}949}{35{,}721}$$

$$=\boxed{\frac{841}{189}}$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3 = \frac{189}{841}\left[(7,\ 0,\ -2)\cdot\left(\frac{58}{189},\ -\frac{377}{189},\ \frac{116}{189}\right)\right]\left(\frac{58}{189},\ -\frac{377}{189},\ \frac{116}{189}\right)$$

$$= \frac{189}{841}\left(\frac{174}{189}\right)\left(\frac{58}{189},\ -\frac{377}{189},\ \frac{116}{189}\right)$$

$$= \frac{174}{841}\left(\frac{58}{189},\ -\frac{377}{189},\ \frac{116}{189}\right)$$

$$= \frac{58}{841}\left(\frac{58}{63},\ -\frac{377}{63},\ \frac{116}{63}\right)$$

$$= \boxed{\left(\frac{3,364}{52,983},\ -\frac{21,866}{52,983},\ \frac{6,728}{52,983}\right)}$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$

$$= (7,\ 0, -2) - \left(\frac{32}{5},\ 0,\ -\frac{16}{5}\right) - \left(\frac{169}{315},\ \frac{2}{63},\ \frac{338}{315}\right) - \left(\frac{3,364}{52,983},\ -\frac{21,866}{52,983},\ \frac{6,728}{52,983}\right)$$

$$= \left(\frac{3}{5},\ 0,\ \frac{6}{5}\right) - \left(\frac{3}{5},\ 0,\ \frac{6}{5}\right)$$

$$= \boxed{(0,\ 0,\ 0)}$$

$$\boxed{\vec{q}_4 = (0,\ 0,\ 0)} \qquad\qquad \vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

The *orthogonal* basis:

$$\left\{(-2,\ 0,\ 1),\ \left(\frac{13}{5},\ 2,\ \frac{26}{5}\right),\ \left(\frac{58}{189},\ -\frac{377}{189},\ \frac{116}{189}\right),\ (0,\ 0,\ 0)\right\}$$

The *orthogonal* basis:

$$\left\{\begin{aligned}&\left(-\frac{2}{\sqrt{5}},\ 0,\ \frac{1}{\sqrt{5}}\right),\ \left(\frac{13}{3\sqrt{105}},\ -\frac{10}{3\sqrt{105}},\ \frac{26}{3\sqrt{105}}\right),\\ &\left(\frac{2}{3\sqrt{21}},\ -\frac{13}{3\sqrt{21}},\ \frac{4}{3\sqrt{21}}\right),\ (0,\ 0,\ 0)\end{aligned}\right\}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(1,\ 2,\ -2),\ (1,\ 0,\ -4),\ (5,\ 2,\ 0),\ (1,\ 1,\ -1)\}$$

Solution

Let $\vec{u}_1 = (1,\ 2,\ -2)$ $\vec{u}_2 = (1,\ 0,\ -4)$ $\vec{u}_3 = (5,\ 2,\ 0)$ $\vec{u}_4 = (1,\ 1,\ -1)$

$$\boxed{\vec{v}_1 = \vec{u}_1 = (1,\ 2,\ -2)}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 \\ &= (1,\ 0,\ -4) - \frac{(1,\ 0,\ -4)\bullet(1,\ 2,\ -2)}{9}(1,\ 2,\ -2) \\ &= (1,\ 0,\ -4) - (1,\ 2,\ -2) \\ &= \boxed{(0,\ -2,\ -2)}\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 &= \frac{(5,\ 2,\ 0)\bullet(1,\ 2,\ -2)}{9}(1,\ 2,\ -2) \\ &= \boxed{(1,\ 2,\ -2)}\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 &= \tfrac{1}{8}\left[(5,\ 2,\ 0)\bullet(0,\ -2,\ -2)\right](0,\ -2,\ -2) \\ &= -\tfrac{1}{2}(0,\ -2,\ -2) \\ &= \boxed{(0,\ 1,\ 1)}\end{aligned}$$

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 \\ &= (5,\ 2,\ 0) - (1,\ 2,\ -2) - (0,\ 1,\ 1) \\ &= \boxed{(4,\ -1,\ 1)}\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 &= \frac{(1,\ 1,\ -1)\bullet(1,\ 2,\ -2)}{9}(1,\ 2,\ -2) \\ &= \tfrac{5}{9}(1,\ 2,\ -2) \\ &= \boxed{\left(\tfrac{5}{9},\ \tfrac{10}{9},\ -\tfrac{10}{9}\right)}\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 &= \tfrac{1}{8}\left[(1,\ 1,\ -1)\bullet(0,\ -2,\ -2)\right](0,\ -2,\ -2) \\ &= \boxed{(0,\ 0,\ 0)}\end{aligned}$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3 = \frac{1}{18}\left[(1,\ 1,\ -1)\cdot(4,\ -1,\ 1)\right](4,\ -1,\ 1)$$

$$= \frac{1}{9}(4,\ -1,\ 1)$$

$$= \left(\frac{4}{9},\ -\frac{1}{9},\ \frac{1}{9}\right)$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$

$$= (1,\ 1,\ -1) - \left(\frac{5}{9},\ \frac{10}{9},\ -\frac{10}{9}\right) - \left(\frac{4}{9},\ -\frac{1}{9},\ \frac{1}{9}\right)$$

$$= (0,\ 0,\ 0)$$

$$\vec{q}_1 = \frac{1}{3}(1,\ 2,\ -2) \qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(\frac{1}{3},\ \frac{2}{3},\ -\frac{2}{3}\right)$$

$$\vec{q}_2 = \frac{1}{2\sqrt{2}}(0,\ -2,\ -2) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(0,\ -\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right)$$

$$\vec{q}_3 = \frac{1}{3\sqrt{2}}(4,\ -1,\ 1) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \left(\frac{4}{3\sqrt{2}},\ -\frac{1}{3\sqrt{2}},\ \frac{1}{3\sqrt{2}}\right)$$

$$\vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

$$= (0,\ 0,\ 0)$$

The *orthogonal* basis: $\{(1,\ 2,\ -2),\ (0,\ -2,\ -2),\ (4,\ -1,\ 1),\ (0,\ 0,\ 0)\}$

The *orthogonal* basis:

$$\left\{\left(\frac{1}{3},\ \frac{2}{3},\ -\frac{2}{3}\right),\ \left(0,\ -\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right),\ \left(\frac{4}{3\sqrt{2}},\ -\frac{1}{3\sqrt{2}},\ \frac{1}{3\sqrt{2}}\right),\ (0,\ 0,\ 0)\right\}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(-3,\ 1,\ 2),\ (1,\ 1,\ 1),\ (2,\ 0,\ -1),\ (1,\ -3,\ 2)\}$$

Solution

Let $\vec{u}_1=(-3,\ 1,\ 2)\quad \vec{u}_2=(1,\ 1,\ 1)\quad \vec{u}_3=(2,\ 0,\ -1)\quad \vec{u}_4=(1,\ -3,\ 2)$

$$\boxed{\vec{v}_1=\vec{u}_1=(-3,\ 1,\ 2)}$$

$$\begin{aligned}\vec{v}_2&=\vec{u}_2-\frac{\langle\vec{u}_2,\ \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1\\&=(1,\ 1,\ 1)-\frac{(1,\ 1,\ 1)\cdot(-3,\ 1,\ 2)}{14}(-3,\ 1,\ 2)\\&=(1,\ 1,\ 1)-\frac{0}{14}(1,\ 2,\ -2)\\&=\boxed{(1,\ 1,\ 1)}\end{aligned}$$

$$\begin{aligned}\frac{\langle\vec{u}_3,\ \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1&=\frac{(2,\ 0,\ -1)\cdot(-3,\ 1,\ 2)}{14}(-3,\ 1,\ 2)\\&=-\frac{4}{7}(-3,\ 1,\ 2)\\&=\boxed{\left(\frac{12}{7},\ -\frac{4}{7},\ -\frac{8}{7}\right)}\end{aligned}$$

$$\begin{aligned}\frac{\langle\vec{u}_3,\ \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2&=\frac{1}{3}\left[(2,\ 0,\ -1)\cdot(1,\ 1,\ 1)\right](1,\ 1,\ 1)\\&=\frac{1}{3}(1,\ 1,\ 1)\\&=\boxed{\left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}\end{aligned}$$

$$\begin{aligned}\vec{v}_3&=\vec{u}_3-\frac{\langle\vec{u}_3,\ \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1-\frac{\langle\vec{u}_3,\ \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2\\&=(2,\ 0,\ -1)-\left(\frac{12}{7},\ -\frac{4}{7},\ -\frac{8}{7}\right)-\left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right)\\&=\boxed{\left(-\frac{1}{21},\ \frac{5}{21},\ -\frac{4}{21}\right)}\end{aligned}$$

$$\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(1,\ -3,\ 2)\bullet(-3,\ 1,\ 2)}{14}(-3,\ 1,\ 2)$$

$$= -\frac{1}{7}(-3,\ 1,\ 2)$$

$$= \left(\frac{3}{7},\ -\frac{1}{7},\ -\frac{2}{7}\right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{(1,\ -3,\ 2)\bullet(1,\ 1,\ 1)}{3}(1,\ 1,\ 1)$$

$$= (0,\ 0,\ 0)$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3 = \frac{441}{42}\left[(1,\ -3,\ 2)\bullet\left(-\frac{1}{21},\ \frac{5}{21},\ -\frac{4}{21}\right)\right]\left(-\frac{1}{21},\ \frac{5}{21},\ -\frac{4}{21}\right)$$

$$= \frac{441}{42}\left(-\frac{24}{21}\right)\left(-\frac{1}{21},\ \frac{5}{21},\ -\frac{4}{21}\right)$$

$$= \left(\frac{4}{7},\ -\frac{20}{7},\ \frac{16}{7}\right)$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$

$$= (1,\ -3,\ 2) - \left(\frac{3}{7},\ -\frac{1}{7},\ -\frac{2}{7}\right) - (0,\ 0,\ 0) - \left(\frac{4}{7},\ -\frac{20}{7},\ \frac{16}{7}\right)$$

$$= (0,\ 0,\ 0)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{14}}(-3,\ 1,\ 2)$$

$$= \left(-\frac{3}{\sqrt{14}},\ \frac{1}{\sqrt{14}},\ \frac{2}{\sqrt{14}}\right)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{\sqrt{3}}(1,\ 1,\ 1)$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$= \frac{21}{\sqrt{42}}\left(-\frac{1}{21},\ \frac{5}{21},\ -\frac{4}{21}\right)$$

$$= \left(-\frac{1}{\sqrt{42}},\ \frac{5}{\sqrt{42}},\ -\frac{4}{\sqrt{42}}\right)$$

$$\vec{q}_4 = \frac{\vec{v}_4}{\left\|\vec{v}_4\right\|}$$

$$= (0,\ 0,\ 0)$$

The *orthogonal* basis: $\left\{(-3,\ 1,\ 2),\ (1,\ 1,\ 1),\ \left(-\frac{1}{21},\ \frac{5}{21},\ -\frac{4}{21}\right),\ (0,\ 0,\ 0)\right\}$

The *orthogonal* basis:

$$\left\{\left(-\frac{3}{\sqrt{14}},\ \frac{1}{\sqrt{14}},\ \frac{2}{\sqrt{14}}\right),\ \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right),\ \left(-\frac{1}{\sqrt{42}},\ \frac{5}{\sqrt{42}},\ -\frac{4}{\sqrt{42}}\right),\ (0,\ 0,\ 0)\right\}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(2,\ 1,\ 1),\ (0,\ 3,\ -1),\ (3,\ -4,\ -2),\ (-1,\ -1,\ 3)\}$$

Solution

Let $\vec{u}_1 = (2,\ 1,\ 1)$ $\vec{u}_2 = (0,\ 3,\ -1)$ $\vec{u}_3 = (3,\ -4,\ -2)$ $\vec{u}_4 = (-1,\ -1,\ 3)$

$$\vec{v}_1 = \vec{u}_1 = (2,\ 1,\ 1)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2,\ \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$= (0,\ 3,\ -1) - \frac{(0,\ 3,\ -1) \cdot (2,\ 1,\ 1)}{6}(2,\ 1,\ 1)$$

$$= (0,\ 3,\ -1) - \frac{1}{3}(2,\ 1,\ 1)$$

$$= \left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(3,\ -4,\ -2)\bullet(2,\ 1,\ 1)}{6}(2,\ 1,\ 1)$$

$$= (0,\ 0,\ 0)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{9}{84}\left[(3,\ -4,\ -2)\bullet\left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right)\right]\left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right)$$

$$= \frac{3}{28}(-10)\left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right)$$

$$= \left(\frac{5}{7},\ -\frac{20}{7},\ \frac{10}{7}\right)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (3,\ -4,\ -2) - (0,\ 0,\ 0) - \left(\frac{5}{7},\ -\frac{20}{7},\ \frac{10}{7}\right)$$

$$= \left(\frac{16}{7},\ -\frac{8}{7},\ -\frac{24}{7}\right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(-1,\ -1,\ 3)\bullet(2,\ 1,\ 1)}{6}(2,\ 1,\ 1)$$

$$= (0,\ 0,\ 0)$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{9}{84}\left[(-1,\ -1,\ 3)\bullet\left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right)\right]\left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right)$$

$$= \frac{3}{28}\left(-\frac{18}{3}\right)\left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right)$$

$$= \left(\frac{3}{7},\ -\frac{12}{7},\ \frac{6}{7}\right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3 = \frac{49}{896}\left[(-1,\ -1,\ 3)\bullet\left(\frac{16}{7},\ -\frac{8}{7},\ -\frac{24}{7}\right)\right]\left(\frac{16}{7},\ -\frac{8}{7},\ -\frac{24}{7}\right)$$

$$= \frac{7}{128}\left(-\frac{80}{7}\right)\left(\frac{16}{7},\ -\frac{8}{7},\ -\frac{24}{7}\right)$$

$$= \left(-\frac{10}{7},\ \frac{5}{7},\ \frac{15}{7}\right)$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$

$$= (-1, -1, 3) - (0, 0, 0) - \left(\tfrac{3}{7}, -\tfrac{12}{7}, \tfrac{6}{7}\right) - \left(-\tfrac{10}{7}, \tfrac{5}{7}, \tfrac{15}{7}\right)$$

$$= (0, 0, 0)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{6}}(2, 1, 1)$$

$$= \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{3}{2\sqrt{21}}\left(-\tfrac{2}{3}, \tfrac{8}{3}, -\tfrac{4}{3}\right)$$

$$= \left(-\frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, -\frac{2}{\sqrt{21}}\right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{7}{8\sqrt{14}}\left(\tfrac{16}{7}, -\tfrac{8}{7}, -\tfrac{24}{7}\right)$$

$$= \left(\frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, -\frac{3}{\sqrt{14}}\right)$$

$$\vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

$$= (0, 0, 0)$$

The *orthogonal* basis: $\left\{(2, 1, 1), \left(-\tfrac{2}{3}, \tfrac{8}{3}, -\tfrac{4}{3}\right), \left(\tfrac{16}{7}, -\tfrac{8}{7}, -\tfrac{24}{7}\right), (0, 0, 0)\right\}$

The *orthogonal* basis:

$$\left\{\left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(-\frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, -\frac{2}{\sqrt{21}}\right), \left(\frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, -\frac{3}{\sqrt{14}}\right), (0, 0, 0)\right\}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(1,\ 1,\ 1,\ 1),\ (0,\ 1,\ 1,\ 1),\ (0,\ 0,\ 1,\ 1)\}$$

Solution

Let $\vec{u}_1 = (1,\ 1,\ 1,\ 1)\quad \vec{u}_2 = (0,\ 1,\ 1,\ 1)\quad \vec{u}_3 = (0,\ 0,\ 1,\ 1)$

$$\vec{v}_1 = \vec{u}_1 = (1,\ 1,\ 1,\ 1)$$

$$\vec{q}_1 = \frac{(1,\ 1,\ 1,\ 1)}{2} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2,\ \vec{v}_1\right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$= (0,\ 1,\ 1,\ 1) - \frac{1}{4}\left[(0,\ 1,\ 1,\ 1)\bullet(1,\ 1,\ 1,\ 1)\right](1,\ 1,\ 1,\ 1)$$

$$= (0,\ 1,\ 1,\ 1) - \frac{3}{4}(1,\ 1,\ 1,\ 1)$$

$$= \left(-\frac{3}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right)$$

$$\vec{q}_2 = \frac{4}{\sqrt{9+1+1+1}}\left(-\frac{3}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \frac{2}{\sqrt{3}}\left(-\frac{3}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right)$$

$$= \left(-\frac{3}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}}\right)$$

$$\left\langle \vec{u}_3,\ \vec{v}_1\right\rangle = (0,\ 0,\ 1,\ 1)\bullet(1,\ 1,\ 1,\ 1)$$

$$= 2$$

$$\left\langle \vec{u}_3,\ \vec{v}_2\right\rangle = (0,\ 0,\ 1,\ 1)\bullet\left(-\frac{3}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right)$$

$$= \frac{1}{2}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (0, \ 0, \ 1, \ 1) - \frac{2}{4}(1, \ 1, \ 1, \ 1) - \left(\frac{1}{2}\right)\left(\frac{16}{12}\right)\left(-\frac{3}{4}, \ \frac{1}{4}, \ \frac{1}{4}, \ \frac{1}{4}\right)$$

$$= (0, \ 0, \ 1, \ 1) - \left(\frac{1}{2}, \ \frac{1}{2}, \ \frac{1}{2}, \ \frac{1}{2}\right) - \left(-\frac{1}{2}, \ \frac{1}{6}, \ \frac{1}{6}, \ \frac{1}{6}\right)$$

$$= \left(0, \ -\frac{2}{3}, \ \frac{1}{3}, \ \frac{1}{3}\right)$$

$$\vec{q}_2 = \frac{3}{\sqrt{4+1+1}}\left(0, \ -\frac{2}{3}, \ \frac{1}{3}, \ \frac{1}{3}\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{3}{\sqrt{6}}\left(0, \ -\frac{2}{3}, \ \frac{1}{3}, \ \frac{1}{3}\right)$$

$$= \left(0, \ -\frac{2}{\sqrt{6}}, \ \frac{1}{\sqrt{6}}, \ \frac{1}{\sqrt{6}}\right)$$

The *orthogonal* basis: $\left\{(1, \ 1, \ 1, \ 1), \ \left(-\frac{3}{4}, \ \frac{1}{4}, \ \frac{1}{4}, \ \frac{1}{4}\right), \ \left(0, \ -\frac{2}{3}, \ \frac{1}{3}, \ \frac{1}{3}\right)\right\}$

The *orthonormal* basis:

$$\left\{\left(\frac{1}{2}, \ \frac{1}{2}, \ \frac{1}{2}, \ \frac{1}{2}\right), \ \left(-\frac{3}{2\sqrt{3}}, \ \frac{1}{2\sqrt{3}}, \ \frac{1}{2\sqrt{3}}, \ \frac{1}{2\sqrt{3}}\right), \ \left(0, \ -\frac{2}{\sqrt{6}}, \ \frac{1}{\sqrt{6}}, \ \frac{1}{\sqrt{6}}\right)\right\}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(1, \ 2, \ -1, \ 0), \ (2, \ 2, \ 0, \ 1), \ (1, \ 1, \ -1, \ 0)\}$$

Solution

Let $\vec{u}_1 = (1, \ 2, \ -1, \ 0)$ $\vec{u}_2 = (2, \ 2, \ 0, \ 1)$ $\vec{u}_3 = (1, \ 1, \ -1, \ 0)$

$$\vec{v}_1 = \vec{u}_1 = (1, \ 2, \ -1, \ 0)$$

$$\vec{q}_1 = \frac{(1, \ 2, \ -1, \ 0)}{\sqrt{1+4+1}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(\frac{1}{\sqrt{6}}, \ \frac{2}{\sqrt{6}}, \ -\frac{1}{\sqrt{6}}, \ 0\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (2,\ 2,\ 0,\ 1) - \tfrac{1}{6}\left[(2,\ 2,\ 0,\ 1) \bullet (1,\ 2,\ -1,\ 0)\right](1,\ 2,\ -1,\ 0)$$

$$= (2,\ 2,\ 0,\ 1) - (1,\ 2,\ -1,\ 0)$$

$$= (1,\ 0,\ 1,\ 1)$$

$$\vec{q}_2 = \frac{1}{\sqrt{3}}(1,\ 0,\ 1,\ 1) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(\frac{1}{\sqrt{3}},\ 0,\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = (1,\ 1,\ -1,\ 0) \bullet (1,\ 2,\ -1,\ 0)$$

$$= 4$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = (1,\ 1,\ -1,\ 0) \bullet (1,\ 0,\ 1,\ 1)$$

$$= 0$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (1,\ 1,\ -1,\ 0) - \tfrac{4}{6}(1,\ 2,\ -1,\ 0) - (0)(1,\ 0,\ 1,\ 1)$$

$$= (1,\ 1,\ -1,\ 0) - \left(\tfrac{2}{3},\ \tfrac{4}{3},\ -\tfrac{2}{3},\ 0\right)$$

$$= \left(\tfrac{1}{3},\ -\tfrac{1}{3},\ -\tfrac{1}{3},\ 0\right)$$

$$\vec{q}_2 = \frac{3}{\sqrt{3}}\left(\tfrac{1}{3},\ -\tfrac{1}{3},\ -\tfrac{1}{3},\ 0\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \left(\frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}},\ 0\right)$$

The *orthogonal* basis: $\left\{(1,\ 2,\ -1,\ 0),\ (1,\ 0,\ 1,\ 1),\ \left(\tfrac{1}{3},\ -\tfrac{1}{3},\ -\tfrac{1}{3},\ 0\right)\right\}$

The *orthonormal* basis:

$$\left\{\left(\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0\right),\ \left(\frac{1}{\sqrt{3}},\ 0,\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right),\ \left(\frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}},\ 0\right)\right\}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(1,\ 1,\ 1,\ 1),\ (1,\ 2,\ 1,\ 0),\ (1,\ 3,\ 0,\ 0)\}$$

Solution

Let $\vec{u}_1 = (1,\ 1,\ 1,\ 1)\quad \vec{u}_2 = (1,\ 2,\ 1,\ 0)\quad \vec{u}_3 = (1,\ 3,\ 0,\ 0)$

$$\vec{v}_1 = \vec{u}_1 = (1,\ 1,\ 1,\ 1)$$

$$\begin{aligned}\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|}\\ &= \frac{(1,\ 1,\ 1,\ 1)}{\sqrt{4}}\\ &= \left(\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2}\right)\end{aligned}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1\\ &= (1,\ 2,\ 1,\ 0) - \frac{(1,\ 2,\ 1,\ 0)\cdot(1,\ 1,\ 1,\ 1)}{\|(1,\ 1,\ 1,\ 1)\|^2}(1,\ 1,\ 1,\ 1)\\ &= (1,\ 2,\ 1,\ 0) - \frac{1+2+1}{4}(1,\ 1,\ 1,\ 1)\\ &= (1,\ 2,\ 1,\ 0) - \frac{4}{4}(1,\ 1,\ 1,\ 1)\\ &= (1,\ 2,\ 1,\ 0) - (1,\ 1,\ 1,\ 1)\\ &= (0,\ 1,\ 0,\ -1)\end{aligned}$$

$$\begin{aligned}\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|}\\ &= \frac{(0,\ 1,\ 0,\ -1)}{\sqrt{1+1}}\\ &= \left(0,\ \frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right)\end{aligned}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (1,\ 3,\ 0,\ 0) - \frac{(1,\ 3,\ 0,\ 0) \bullet (1,\ 1,\ 1,\ 1)}{\|(1,\ 1,\ 1,\ 1)\|^2}(1,\ 1,\ 1,\ 1) - \frac{(1,\ 3,\ 0,\ 0) \bullet (0,\ 1,\ 0,\ -1)}{\|(0,\ 1,\ 0,\ -1)\|^2}(0,\ 1,\ 0,\ -1)$$

$$= (1,\ 3,\ 0,\ 0) - \frac{4}{4}(1,\ 1,\ 1,\ 1) - \frac{3}{2}(0,\ 1,\ 0,\ -1)$$

$$= (1,\ 3,\ 0,\ 0) - (1,\ 1,\ 1,\ 1) - \left(0,\ \frac{3}{2},\ 0,\ -\frac{3}{2}\right)$$

$$= \left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)$$

$$\vec{q}_3 = \frac{1}{\sqrt{\frac{1}{4}+1+\frac{1}{4}}}\left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{2}{\sqrt{6}}\left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)$$

$$= \left(0,\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

The *orthogonal* basis: $\left\{(1,\ 1,\ 1,\ 1),\ (0,\ 1,\ 0,\ -1),\ \left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)\right\}$

The *orthonormal* basis: $\left\{\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right),\ \left(0,\ \frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right),\ \left(0,\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(0,\ 2,\ -1,\ 1),\ (0,\ 0,\ 1,\ 1),\ (-2,\ 1,\ 1,\ -1)\}$$

Solution

Let $\vec{u}_1 = (0,\ 2,\ -1,\ 1) \quad \vec{u}_2 = (0,\ 0,\ 1,\ 1) \quad \vec{u}_3 = (-2,\ 1,\ 1,\ -1)$

$$\vec{v}_1 = \vec{u}_1 = (0,\ 2,\ -1,\ 1)$$

$$\vec{q}_1 = \frac{(0,\ 2,\ -1,\ 1)}{\sqrt{6}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (0,\ 0,\ 1,\ 1) - \frac{1}{6}\big[(0,\ 0,\ 1,\ 1) \cdot (0,\ 2,\ -1,\ 1)\big](0,\ 2,\ -1,\ 1)$$

$$= (0,\ 0,\ 1,\ 1) - \frac{1}{6}(0)(0,\ 2,\ -1,\ 1)$$

$$= (0,\ 0,\ 1,\ 1)$$

$$\vec{q}_2 = \frac{(0,\ 0,\ 1,\ 1)}{\sqrt{2}} \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(0,\ 0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = (-2,\ 1,\ 1,\ -1) \cdot (0,\ 2,\ -1,\ 1)$$

$$= 0$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = (-2,\ 1,\ 1,\ -1) \cdot (0,\ 0,\ 1,\ 1)$$

$$= 0$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (-2,\ 1,\ 1,\ -1) - 0 - 0$$

$$= (-2,\ 1,\ 1,\ -1)$$

$$\vec{q}_3 = \frac{(-2,\ 1,\ 1,\ -1)}{\sqrt{(-2)^2 + 1^2 + 1^2 + (-1)^2}} \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{(-2,\ 1,\ 1,\ -1)}{\sqrt{7}}$$

$$= \left(-\frac{2}{\sqrt{7}},\ \frac{1}{\sqrt{7}},\ \frac{1}{\sqrt{7}},\ -\frac{1}{\sqrt{7}}\right)$$

The *orthogonal* basis: $\{(0,\ 2,\ -1,\ 1),\ (0,\ 0,\ 1,\ 1),\ (-2,\ 1,\ 1,\ -1)\}$

The *orthonormal* basis:

$$\left\{\left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right),\ \left(0,\ 0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right),\ \left(-\frac{2}{\sqrt{7}},\ \frac{1}{\sqrt{7}},\ \frac{1}{\sqrt{7}},\ -\frac{1}{\sqrt{7}}\right)\right\}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(-1,\ 3,\ 1,\ 1),\ (6,\ -8,\ -2,\ -4),\ (6,\ 3,\ 6,\ -3)\}$$

Solution

Let $\vec{u}_1 = (-1,\ 3,\ 1,\ 1)\quad \vec{u}_2 = (6,\ -8,\ -2,\ -4)\quad \vec{u}_3 = (6,\ 3,\ 6,\ -3)$

$$\vec{v}_1 = \vec{u}_1 = (-1,\ 3,\ 1,\ 1)$$

$$\vec{q}_1 = \frac{(-1,\ 3,\ 1,\ 1)}{\sqrt{1+9+1+1}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(-\frac{1}{2\sqrt{3}},\ \frac{3}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (6,\ -8,\ -2,\ -4) - \frac{1}{12}\left[(6,\ -8,\ -2,\ -4)\bullet(-1,\ 3,\ 1,\ 1)\right](-1,\ 3,\ 1,\ 1)$$

$$= (6,\ -8,\ -2,\ -4) - \frac{1}{12}(-36)(-1,\ 3,\ 1,\ 1)$$

$$= (6,\ -8,\ -2,\ -4) + (-3,\ 9,\ 3,\ 3)$$

$$= (3,\ 1,\ 1,\ -1)$$

$$\vec{q}_2 = \frac{1}{\sqrt{12}}(3,\ 1,\ 1,\ -1) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(\frac{3}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}},\ -\frac{1}{2\sqrt{3}}\right)$$

$$\langle \vec{u}_3, \vec{v}_1\rangle = (6,\ 3,\ 6,\ -3)\bullet(-1,\ 3,\ 1,\ 1)$$

$$= 6$$

$$\langle \vec{u}_3, \vec{v}_2\rangle = (6,\ 3,\ 6,\ -3)\bullet(3,\ 1,\ 1,\ -1)$$

$$= 30$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$=(6,\ 3,\ 6,\ -3)-\frac{6}{12}(-1,\ 3,\ 1,\ 1)-\frac{30}{12}(3,\ 1,\ 1,\ -1)$$

$$=(6,\ 3,\ 6,\ -3)-\left(-\frac{1}{2},\ \frac{3}{2},\ \frac{1}{2},\ \frac{1}{2}\right)-\left(\frac{15}{2},\ \frac{5}{2},\ \frac{5}{2},\ -\frac{5}{2}\right)$$

$$=(-1,\ -1,\ 3,\ -1)$$

$$\vec{q}_3=\frac{1}{2\sqrt{3}}(-1,\ -1,\ 3,\ -1) \qquad \vec{q}_3=\frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$=\left(-\frac{1}{2\sqrt{3}},\ -\frac{1}{2\sqrt{3}},\ \frac{3}{2\sqrt{3}},\ -\frac{1}{2\sqrt{3}}\right)$$

The *orthogonal* basis: $\{(-1,\ 3,\ 1,\ 1),\ (3,\ 1,\ 1,\ -1),\ (-1,\ -1,\ 3,\ -1)\}$

The *orthonormal* basis:

$$\left\{\begin{array}{l}\left(-\frac{1}{2\sqrt{3}},\ \frac{3}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}}\right),\ \left(\frac{3}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}},\ -\frac{1}{2\sqrt{3}}\right),\\ \left(-\frac{1}{2\sqrt{3}},\ -\frac{1}{2\sqrt{3}},\ \frac{3}{2\sqrt{3}},\ -\frac{1}{2\sqrt{3}}\right)\end{array}\right\}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(0,\ 0,\ 2,\ 2),\ (3,\ 3,\ 0,\ 0),\ (1,\ 1,\ 0,\ -1)\}$$

Solution

Let $\vec{u}_1=(0,\ 0,\ 2,\ 2)$ $\vec{u}_2=(3,\ 3,\ 0,\ 0)$ $\vec{u}_3=(1,\ 1,\ 0,\ -1)$

$$\vec{v}_1=\vec{u}_1=(0,\ 0,\ 2,\ 2)$$

$$\vec{q}_1=\frac{(0,\ 0,\ 2,\ 2)}{\sqrt{4+4}} \qquad \vec{q}_1=\frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$=\frac{(0,\ 0,\ 2,\ 2)}{2\sqrt{2}}$$

$$=\left(0,\ 0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (3,\ 3,\ 0,\ 0) - \tfrac{1}{8}\left[(3,\ 3,\ 0,\ 0) \cdot (0,\ 0,\ 2,\ 2)\right]\ (0,\ 0,\ 2,\ 2)$$

$$= (3,\ 3,\ 0,\ 0) - \tfrac{1}{8}(0)\ (0,\ 0,\ 2,\ 2)$$

$$= (3,\ 3,\ 0,\ 0)$$

$$\vec{q}_2 = \frac{1}{3\sqrt{2}}(3,\ 3,\ 0,\ 0) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = (1,\ 1,\ 0,\ -1) \cdot (0,\ 0,\ 2,\ 2)$$

$$= -2$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = (1,\ 1,\ 0,\ -1) \cdot (3,\ 3,\ 0,\ 0)$$

$$= 6$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (1,\ 1,\ 0,\ -1) + \tfrac{2}{8}(0,\ 0,\ 2,\ 2) - \tfrac{6}{18}(3,\ 3,\ 0,\ 0)$$

$$= (1,\ 1,\ 0, -1) + \left(0,\ 0,\ \tfrac{1}{2},\ \tfrac{1}{2}\right) - (1,\ 1,\ 0,\ 0)$$

$$= \left(0,\ 0,\ \tfrac{1}{2},\ -\tfrac{1}{2}\right)$$

$$\vec{q}_3 = \frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}}}\left(0,\ 0,\ \tfrac{1}{2},\ -\tfrac{1}{2}\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{2}{\sqrt{2}}\left(0,\ 0,\ \tfrac{1}{2},\ -\tfrac{1}{2}\right)$$

$$= \left(0,\ 0,\ \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right)$$

The *orthogonal* basis: $\left\{(0,\ 0,\ 2,\ 2),\ \ (3,\ 3,\ 0,\ 0),\ \ \left(0,\ 0,\ \tfrac{1}{2},\ -\tfrac{1}{2}\right)\right\}$

The *orthonormal* basis: $\left\{\left(0,\ 0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right),\ \left(0,\ 0,\ \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(1,\ 0,\ 0,\ 0),\ \ (1,\ 1,\ 0,\ 0),\ \ (1,\ 1,\ 1,\ 0),\ \ (1,\ 1,\ 1,\ 1)\}$$

<u>***Solution***</u>

Let $\vec{u}_1 = (1,\ 0,\ 0,\ 0)$ $\vec{u}_2 = (1,\ 1,\ 0,\ 0)$ $\vec{u}_3 = (1,\ 1,\ 1,\ 0)$ $\vec{u}_4 = (1,\ 1,\ 1,\ 1)$

$$\underline{\vec{v}_1 = \vec{u}_1 = (1,\ 0,\ 0,\ 0)}$$

$$\vec{q}_1 = \frac{(1,\ 0,\ 0,\ 0)}{1} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\underline{= (1,\ 0,\ 0,\ 0)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (1,\ 1,\ 0,\ 0) - [(1,\ 1,\ 0,\ 0) \bullet (1,\ 0,\ 0,\ 0)]\ (1,\ 0,\ 0,\ 0)$$

$$= (1,\ 1,\ 0,\ 0) - \ (1,\ 0,\ 0,\ 0)$$

$$\underline{= (0,\ 1,\ 0,\ 0)}$$

$$\vec{q}_2 = \frac{1}{1}(0,\ 1,\ 0,\ 0) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$\underline{= (0,\ 1,\ 0,\ 0)}$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = (1,\ 1,\ 1,\ 0) \bullet (1,\ 0,\ 0,\ 0)$$

$$\underline{= 1}$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = (1,\ 1,\ 1,\ 0) \bullet (0,\ 1,\ 0,\ 0)$$

$$\underline{= 1}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (1,\ 1,\ 1,\ 0) - \frac{1}{1}(1,\ 0,\ 0,\ 0) - \frac{1}{1}(0,\ 1,\ 0,\ 0)$$

$$= (1,\ 1,\ 1,\ 0) - (1,\ 0,\ 0,\ 0) - (0\ 1,\ 0,\ 0)$$

$$\underline{= (0,\ 0,\ 1,\ 0)}$$

$$\vec{q}_3 = \frac{1}{1}(0,\ 0,\ 1,\ 0) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \boxed{(0,\ 0,\ 1,\ 0)}$$

$$\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(1,\ 1,\ 1,\ 1)\bullet(1,\ 0,\ 0,\ 0)}{1}(1,\ 0,\ 0,\ 0)$$

$$= \boxed{(1,\ 0,\ 0,\ 0)}$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \left[(1,\ 1,\ 1,\ 1)\bullet(0,\ 1,\ 0,\ 0)\right](0,\ 1,\ 0,\ 0)$$

$$= \boxed{(0,\ 1,\ 0,\ 0)}$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3 = \left[(1,\ 1,\ 1,\ 1)\bullet(0,\ 0,\ 1,\ 0)\right](0,\ 0,\ 1,\ 0)$$

$$= \boxed{(0,\ 0,\ 1,\ 0)}$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$

$$= (1,\ 1,\ 1,\ 1) - (1,\ 0,\ 0,\ 0) - (0,\ 1,\ 0,\ 0) - (0,\ 0,\ 1,\ 0)$$

$$= \boxed{(0,\ 0,\ 0,\ 1)}$$

$$\vec{q}_4 = \boxed{(0,\ 0,\ 0,\ 1)} \qquad \vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

The *orthogonal* basis: $\{(1,\ 0,\ 0,\ 0),\ \ (0,\ 1,\ 0,\ 0),\ \ (0,\ 0,\ 1,\ 0),\ \ (0,\ 0,\ 0,\ 1)\}$

The *orthonormal* basis: $\{(1,\ 0,\ 0,\ 0),\ \ (0,\ 1,\ 0,\ 0),\ \ (0,\ 0,\ 1,\ 0),\ \ (0,\ 0,\ 0,\ 1)\}$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(3,\ 8,\ 7,\ -3),\ (1,\ 5,\ 3,\ -1),\ (2,\ -1,\ 2,\ 6),\ (1,\ 4,\ 0,\ 3)\}$$

Solution

Let $\vec{u}_1=(3,\ 8,\ 7,\ -3)\quad \vec{u}_2=(1,\ 5,\ 3,\ -1)\quad \vec{u}_3=(2,\ -1,\ 2,\ 6)\quad \vec{u}_4=(1,\ 4,\ 0,\ 3)$

$$\underline{\vec{v}_1=\vec{u}_1=(3,\ 8,\ 7,\ -3)}$$

$$\vec{q}_1=\frac{(3,\ 8,\ 7,\ -3)}{\sqrt{9+64+49+9}} \qquad \vec{q}_1=\frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$=\frac{(3,\ 8,\ 7,\ -3)}{\sqrt{131}}$$

$$\underline{=\left(\frac{3}{\sqrt{131}},\ \frac{8}{\sqrt{131}},\ \frac{7}{\sqrt{131}},\ -\frac{3}{\sqrt{131}}\right)}$$

$$\vec{v}_2=\vec{u}_2-\frac{\left\langle \vec{u}_2,\vec{v}_1\right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$=(1,\ 5,\ 3,\ -1)-\frac{1}{131}\left[(1,\ 5,\ 3,\ -1)\bullet(3,\ 8,\ 7,\ -3)\right]\ (3,\ 8,\ 7,\ -3)$$

$$=(1,\ 5,\ 3,\ -1)-\frac{67}{131}\ (3,\ 8,\ 7,\ -3)$$

$$=(1,\ 5,\ 3,\ -1)-\left(\frac{201}{131},\ \frac{536}{131},\ \frac{469}{131},\ -\frac{201}{131}\right)$$

$$\underline{=\left(-\frac{70}{131},\ \frac{119}{131},\ -\frac{76}{131},\ \frac{70}{131}\right)}$$

$$\vec{q}_2=\frac{131}{\sqrt{70^2+119^2+76^2+70^2}}\left(-\frac{70}{131},\ \frac{119}{131},\ -\frac{76}{131},\ \frac{70}{131}\right) \qquad \vec{q}_2=\frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$=\frac{131}{\sqrt{29{,}737}}\left(-\frac{70}{131},\ \frac{119}{131},\ -\frac{76}{131},\ \frac{70}{131}\right)$$

$$\underline{=\left(-\frac{70}{\sqrt{29{,}737}},\ \frac{119}{\sqrt{29{,}737}},\ -\frac{76}{\sqrt{29{,}737}},\ \frac{70}{\sqrt{29{,}737}}\right)}$$

$$\left\langle \vec{u}_3,\vec{v}_1\right\rangle=(2,\ -1,\ 2,\ 6)\bullet(3,\ 8,\ 7,\ -3)$$

$$\underline{=-6}$$

$$\left\langle \vec{u}_3,\vec{v}_2\right\rangle=(2,\ -1,\ 2,\ 6)\bullet\left(-\frac{70}{131},\ \frac{119}{131},\ -\frac{76}{131},\ \frac{70}{131}\right)$$

$$= \frac{-140-119-152+420}{131}$$

$$= \boxed{\frac{9}{131}}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (2,\ -1,\ 2,\ 6) + \frac{6}{131}(3,\ 8,\ 7,\ -3) - \frac{9}{131}\left(\frac{131^2}{29,737}\right)\left(-\frac{70}{131},\ \frac{119}{131},\ -\frac{76}{131},\ \frac{70}{131}\right)$$

$$= (2,\ -1,\ 2,\ 6) + \left(\frac{18}{131},\ \frac{48}{131},\ \frac{42}{131},\ -\frac{18}{131}\right) - \frac{9}{131}\left(\frac{131}{227}\right)\left(-\frac{70}{131},\ \frac{119}{131},\ -\frac{76}{131},\ \frac{70}{131}\right)$$

$$= \left(\frac{280}{131},\ -\frac{83}{131},\ \frac{304}{131},\ \frac{768}{131}\right) - \left(-\frac{630}{29,737},\ \frac{1,071}{29,737},\ -\frac{684}{29,737},\ \frac{630}{29,737}\right)$$

$$= \left(\frac{64,190}{29,737},\ -\frac{19,912}{29,737},\ \frac{69,692}{29,737},\ \frac{173,706}{29,737}\right)$$

$$= \boxed{\left(\frac{490}{227},\ -\frac{152}{227},\ \frac{532}{227},\ \frac{1,326}{227}\right)}$$

$$\vec{q}_3 = \frac{227}{\sqrt{490^2+152^2+532^2+1,326^2}}\left(\frac{490}{227},\ -\frac{152}{227},\ \frac{532}{227},\ \frac{1,326}{227}\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{227}{\sqrt{2,304,504}}\left(\frac{490}{227},\ -\frac{152}{227},\ \frac{532}{227},\ \frac{1,326}{227}\right)$$

$$= \frac{1}{6\sqrt{64,014}}(490,\ -152,\ 532,\ 1,326)$$

$$= \boxed{\left(\frac{245}{3\sqrt{64,014}},\ -\frac{76}{3\sqrt{64,014}},\ \frac{266}{3\sqrt{64,014}},\ \frac{221}{\sqrt{64,014}}\right)}$$

$$\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(1,\ 4,\ 0,\ 3)\bullet(3,\ 8,\ 7,\ -3)}{131}(3,\ 8,\ 7,\ -3)$$

$$= \frac{26}{131}(3,\ 8,\ 7,\ -3)$$

$$= \boxed{\left(\frac{78}{131},\ \frac{208}{131},\ \frac{182}{131},\ -\frac{78}{131}\right)}$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{131^2}{29,737}\left[(1,\ 4,\ 0,\ 3)\bullet\left(-\frac{70}{131},\ \frac{119}{131},\ -\frac{76}{131},\ \frac{70}{131}\right)\right]\left(-\frac{70}{131},\ \frac{119}{131},\ -\frac{76}{131},\ \frac{70}{131}\right)$$

$$= \frac{131}{29,737}\left(\frac{-70+476+210}{131}\right)(-70,\ 119,\ -76,\ 70)$$

$$= \frac{616}{29,737}\left(-70,\ 119,\ -76,\ 70\right)$$

$$= \boxed{\left(-\frac{43,120}{29,737},\ \frac{73,304}{29,737},\ -\frac{46,816}{29,737},\ \frac{43,120}{29,737}\right)}$$

$$\frac{\langle \vec{u}_4, \vec{v}_3\rangle}{\left\|\vec{v}_3\right\|^2}\vec{v}_3 = \frac{227^2}{2,304,504}\left[(1,\ 4,\ 0,\ 3)\bullet\left(\frac{490}{227},\ -\frac{152}{227},\ \frac{532}{227},\ \frac{1,326}{227}\right)\right]\left(\frac{490}{227},\ -\frac{152}{227},\ \frac{532}{227},\ \frac{1,326}{227}\right)$$

$$= \frac{227}{2,304,504}\left(\frac{490-608+3,978}{227}\right)\left(490,\ -152,\ 532,\ 1,326\right)$$

$$= \frac{3,860}{2,304,504}\left(490,\ -152,\ 532,\ 1,326\right)$$

$$= \frac{965}{576,126}\left(490,\ -152,\ 532,\ 1,326\right)$$

$$= \left(\frac{472,850}{576,126},\ -\frac{146,680}{576,126},\ \frac{513,380}{576,126},\ \frac{1,279,590}{576,126}\right)$$

$$= \boxed{\left(\frac{236,425}{288,063},\ -\frac{73,340}{288,063},\ \frac{256,690}{288,063},\ \frac{213,265}{96,021}\right)}$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2\rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3\rangle}{\left\|\vec{v}_3\right\|^2}\vec{v}_3$$

$$= (1,\ 4,\ 0,\ 3) - \left(\frac{78}{131},\ \frac{208}{131},\ \frac{182}{131},\ -\frac{78}{131}\right) - \left(-\frac{43,120}{29,737},\ \frac{73,304}{29,737},\ -\frac{46,816}{29,737},\ \frac{43,120}{29,737}\right)$$
$$- \left(\frac{236,425}{288,063},\ -\frac{73,340}{288,063},\ \frac{256,690}{288,063},\ \frac{213,265}{96,021}\right)$$

$$= \left(\frac{53}{131},\ \frac{316}{131},\ -\frac{182}{131},\ \frac{471}{131}\right) - \left(-\frac{43,120}{29,737},\ \frac{73,304}{29,737},\ -\frac{46,816}{29,737},\ \frac{43,120}{29,737}\right)$$
$$- \left(\frac{236,425}{288,063},\ -\frac{73,340}{288,063},\ \frac{256,690}{288,063},\ \frac{213,265}{96,021}\right)$$

$$= \left(\frac{55,151}{29,737},\ -\frac{1,572}{29,737},\ \frac{5,502}{29,737},\ \frac{63,797}{29,737}\right) - \left(\frac{236,425}{288,063},\ -\frac{73,340}{288,063},\ \frac{256,690}{288,063},\ \frac{213,265}{96,021}\right)$$

$$= \left(\frac{421}{227},\ -\frac{12}{227},\ \frac{42}{227},\ \frac{487}{227}\right) - \left(\frac{236,425}{288,063},\ -\frac{73,340}{288,063},\ \frac{256,690}{288,063},\ \frac{213,265}{96,021}\right)$$

$$= \left(\frac{297,824}{288,063},\ \frac{58,112}{288,063},\ -\frac{203,392}{288,063},\ -\frac{7,264}{96,021}\right)$$

$$= \boxed{\left(\frac{1,312}{1,269},\ \frac{256}{1,269},\ -\frac{896}{1,269},\ -\frac{32}{423}\right)}$$

$$\left\|\vec{v}_4\right\| = \frac{1}{1269}\sqrt{(1,312)^2 + (256)^2 + (896)^2 + (96)^2}$$

$$= \frac{\sqrt{2,598,912}}{1269}$$

$$= \boxed{\frac{96\sqrt{282}}{1269}}$$

$$\vec{q}_4 = \frac{1269}{96\sqrt{282}}\left(\frac{1,312}{1,269},\ \frac{256}{1,269},\ -\frac{896}{1,269},\ -\frac{32}{423}\right) \qquad \vec{q}_4 = \frac{\vec{v}_4}{\left\|\vec{v}_4\right\|}$$

$$= \boxed{\left(\frac{41}{3\sqrt{282}},\ \frac{8}{3\sqrt{282}},\ -\frac{28}{3\sqrt{282}},\ -\frac{1}{\sqrt{282}}\right)}$$

The *orthogonal* basis:

$$\left\{\begin{array}{l}(3,\ 8,\ 7,\ -3),\ \left(-\frac{70}{131},\ \frac{119}{131},\ -\frac{76}{131},\ \frac{70}{131}\right),\ \left(\frac{490}{227},\ -\frac{152}{227},\ \frac{532}{227},\ \frac{1,326}{227}\right), \\ \quad \left(\frac{1,312}{1,269},\ \frac{256}{1,269},\ -\frac{896}{1,269},\ -\frac{32}{423}\right)\end{array}\right\}$$

The *orthonormal* basis:

$$\left\{\begin{array}{l}\left(\frac{3}{\sqrt{131}},\ \frac{8}{\sqrt{131}},\ \frac{7}{\sqrt{131}},\ -\frac{3}{\sqrt{131}}\right),\ \left(-\frac{70}{\sqrt{29,737}},\ \frac{119}{\sqrt{29,737}},\ -\frac{76}{\sqrt{29,737}},\ \frac{70}{\sqrt{29,737}}\right), \\ \quad \left(\frac{245}{3\sqrt{64,014}},\ -\frac{76}{3\sqrt{64,014}},\ \frac{266}{3\sqrt{64,014}},\ \frac{221}{\sqrt{64,014}}\right), \\ \quad \left(\frac{41}{3\sqrt{282}},\ \frac{8}{3\sqrt{282}},\ -\frac{28}{3\sqrt{282}},\ -\frac{1}{\sqrt{282}}\right)\end{array}\right\}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(0,\ 3,\ -3,\ -6),\ (-2,\ 0,\ 0,\ -6),\ (0,\ -4,\ -2,\ -2),\ (0,\ -8,\ 4,\ -4)\}$$

Solution

Let

$$\vec{u}_1 = (0,\ 3,\ -3,\ -6) \quad \vec{u}_2 = (-2,\ 0,\ 0,\ -6) \quad \vec{u}_3 = (0,\ -4,\ -2,\ -2) \quad \vec{u}_4 = (0,\ -8,\ 4,\ -4)$$

$$\vec{v}_1 = \vec{u}_1 = (0,\ 3,\ -3,\ -6)$$

$$\vec{q}_1 = \frac{(0,\ 3,\ -3,\ -6)}{\sqrt{9+9+36}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(0,\ 3,\ -3,\ -6)}{3\sqrt{6}}$$

$$= \left(0,\ \frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (-2,\ 0,\ 0,\ -6) - \frac{1}{54}\left[(-2,\ 0,\ 0,\ -6)\bullet(0,\ 3,\ -3,\ -6)\right](0,\ 3,\ -3,\ -6)$$

$$= (-2,\ 0,\ 0,\ -6) - \frac{2}{3}(0,\ 3,\ -3,\ -6)$$

$$= (-2,\ 0,\ 0,\ -6) - (0,\ 2,\ -2,\ -4)$$

$$= (-2,\ -2,\ 2,\ -2)$$

$$\vec{q}_2 = \frac{1}{\sqrt{4+4+4+4}}(-2,\ -2,\ 2,\ -2) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{4}(-2,\ -2,\ 2,\ -2)$$

$$= \left(-\frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2}\right)$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = (0,\ -4,\ -2,\ -2)\bullet(0,\ 3,\ -3,\ -6)$$

$$= 6$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = (0,\ -4,\ -2,\ -2)\bullet(-2,\ -2,\ 2,\ -2)$$

$$= 8$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (0,\ -4,\ -2,\ -2) - \frac{6}{54}(0,\ 3,\ -3,\ -6) - \frac{8}{16}(-2,\ -2,\ 2,\ -2)$$

$$= (0,\ -4,\ -2,\ -2) - \left(0,\ \frac{1}{3},\ -\frac{1}{3},\ -\frac{2}{3}\right) - (-1,\ -1,\ 1,\ -1)$$

$$= \boxed{\left(1,\ -\frac{10}{3},\ -\frac{8}{3},\ -\frac{1}{3}\right)}$$

$$\vec{q}_3 = \frac{3}{\sqrt{9+100+64+1}}\left(1,\ -\frac{10}{3},\ -\frac{8}{3},\ -\frac{1}{3}\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{3}{\sqrt{174}}\left(1,\ -\frac{10}{3},\ -\frac{8}{3},\ -\frac{1}{3}\right)$$

$$= \boxed{\left(\frac{3}{\sqrt{174}},\ -\frac{10}{\sqrt{174}},\ -\frac{8}{\sqrt{174}},\ -\frac{1}{\sqrt{174}}\right)}$$

$$\frac{\langle \vec{u}_4, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(0,\ -8,\ 4,\ -4)\bullet(0,\ 3,\ -3,\ -6)}{54}(0,\ 3,\ -3,\ -6)$$

$$= -\frac{12}{54}(0,\ 3,\ -3,\ -6)$$

$$= \boxed{\left(0,\ -\frac{2}{3},\ \frac{2}{3},\ \frac{4}{3}\right)}$$

$$\frac{\langle \vec{u}_4, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{1}{16}\left[(0,\ -8,\ 4,\ -4)\bullet(-2,\ -2,\ 2,\ -2)\right](-2,\ -2,\ 2,\ -2)$$

$$= \frac{32}{16}(-2,\ -2,\ 2,\ -2)$$

$$= \boxed{(-4,\ -4,\ 4,\ -4)}$$

$$\frac{\langle \vec{u}_4, \vec{v}_3\rangle}{\|\vec{v}_3\|^2}\vec{v}_3 = \frac{9}{174}\left[(0,\ -8,\ 4,\ -4)\bullet\left(1,\ -\frac{10}{3},\ -\frac{8}{3},\ -\frac{1}{3}\right)\right]\left(1,\ -\frac{10}{3},\ -\frac{8}{3},\ -\frac{1}{3}\right)$$

$$= \frac{9}{174}\left(\frac{80-32+4}{3}\right)\left(1,\ -\frac{10}{3},\ -\frac{8}{3},\ -\frac{1}{3}\right)$$

$$= \frac{9}{174}\left(\frac{52}{3}\right)\left(1,\ -\frac{10}{3},\ -\frac{8}{3},\ -\frac{1}{3}\right)$$

$$= \frac{78}{87}\left(1,\ -\frac{10}{3},\ -\frac{8}{3},\ -\frac{1}{3}\right)$$

$$= \boxed{\left(\frac{26}{29},\ -\frac{260}{87},\ -\frac{208}{87},\ -\frac{26}{87}\right)}$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3\rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$

$$= (0,\ -8,\ 4,\ -4) - \left(0,\ -\frac{2}{3},\ \frac{2}{3},\ \frac{4}{3}\right) - (-4,\ -4,\ 4,\ -4) - \left(\frac{26}{29},\ -\frac{260}{87},\ -\frac{208}{87},\ -\frac{26}{87}\right)$$

$$=(4,\ -4,\ 0,\ 0)+\left(-\frac{26}{29},\ \frac{318}{87},\ \frac{150}{87},\ -\frac{30}{29}\right)$$

$$=\boxed{\left(\frac{90}{29},\ -\frac{10}{29},\ \frac{50}{29},\ -\frac{30}{29}\right)}$$

$$\left\|\vec{v}_4\right\|=\sqrt{\left(\frac{90}{29}\right)^2+\left(\frac{10}{29}\right)^2+\left(\frac{50}{29}\right)^2+\left(\frac{90}{87}\right)^2}$$

$$=\sqrt{\frac{8,100}{841}+\frac{100}{841}+\frac{2,500}{841}+\frac{8,100}{7,569}}$$

$$=10\sqrt{\frac{729+9+225+81}{7,569}}$$

$$=\frac{10}{87}\sqrt{1,044}$$

$$=\boxed{\frac{20}{29}\sqrt{29}}$$

$$\vec{q}_4=\frac{29}{20}\frac{1}{\sqrt{29}}\left(\frac{90}{29},\ -\frac{10}{29},\ \frac{50}{29},\ -\frac{90}{87}\right) \qquad \vec{q}_4=\frac{\vec{v}_4}{\left\|\vec{v}_4\right\|}$$

$$=\boxed{\left(\frac{9}{2\sqrt{29}},\ -\frac{1}{2\sqrt{29}},\ \frac{5}{2\sqrt{29}},\ -\frac{3}{2\sqrt{29}}\right)}$$

The *orthogonal* basis:

$$\left\{\begin{array}{l}(0,\ 3,\ -3,\ -6),\ (-2,\ -2,\ 2,\ -2),\ \left(1,\ -\frac{10}{3},\ -\frac{8}{3},\ -\frac{1}{3}\right), \\ \qquad \left(\frac{90}{29},\ -\frac{10}{29},\ \frac{50}{29},\ -\frac{30}{29}\right)\end{array}\right\}$$

The *orthonormal* basis:

$$\left\{\begin{array}{l}\left(0,\ \frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right),\ \left(-\frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2}\right), \\ \left(\frac{3}{\sqrt{174}},\ -\frac{10}{\sqrt{174}},\ -\frac{8}{\sqrt{174}},\ -\frac{1}{\sqrt{174}}\right),\ \left(\frac{9}{2\sqrt{29}},\ -\frac{1}{2\sqrt{29}},\ \frac{5}{2\sqrt{29}},\ -\frac{3}{2\sqrt{29}}\right)\end{array}\right\}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(3,\ 0,\ -3,\ 6),\ (0,\ 2,\ 3,\ 1),\ (0,\ -2,\ -2,\ 0),\ (-2,\ 1,\ 2,\ 1)\}$$

Solution

Let $\vec{u}_1=(3,\ 0,\ -3,\ 6)$ $\vec{u}_2=(0,\ 2,\ 3,\ 1)$ $\vec{u}_3=(0,\ -2,\ -2,\ 0)$ $\vec{u}_4=(-2,\ 1,\ 2,\ 1)$

$$\boxed{\vec{v}_1 = \vec{u}_1 = (3,\ 0,\ -3,\ 6)}$$

$$\vec{q}_1 = \frac{(3,\ 0,\ -3,\ 6)}{\sqrt{9+9+36}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(3,\ 0,\ -3,\ 6)}{3\sqrt{6}}$$

$$\boxed{= \left(\frac{1}{\sqrt{6}},\ 0,\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (0,\ 2,\ 3,\ 1) - \frac{1}{54}\left[(0,\ 2,\ 3,\ 1)\bullet(3,\ 0,\ -3,\ 6)\right]\ (3,\ 0,\ -3,\ 6)$$

$$= (0,\ 2,\ 3,\ 1) + \frac{1}{18}\ (3,\ 0,\ -3,\ 6)$$

$$= (0,\ 2,\ 3,\ 1) + \left(\frac{1}{6},\ 0,\ -\frac{1}{6},\ \frac{1}{3}\right)$$

$$\boxed{= \left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right)}$$

$$\vec{q}_2 = \frac{1}{\sqrt{\frac{1}{36}+4+\frac{289}{36}+\frac{16}{9}}}\left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{\sqrt{\frac{498}{36}}}\left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right)$$

$$= \frac{6}{\sqrt{498}}\left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right)$$

$$\boxed{= \left(\frac{1}{\sqrt{498}},\ \frac{12}{\sqrt{498}},\ \frac{17}{\sqrt{498}},\ \frac{8}{\sqrt{498}}\right)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{1}{54}\left[(0,\ -2,\ -2,\ 0)\bullet(3,\ 0,\ -3,\ 6)\right]\ (3,\ 0,\ -3,\ 6)$$

$$= \frac{6}{54}\ (3,\ 0,\ -3,\ 6)$$

$$\boxed{= \left(\frac{1}{3},\ 0,\ -\frac{1}{3},\ \frac{2}{3}\right)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{36}{498}\left[(0,\ -2,\ -2,\ 0)\bullet\left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right)\right]\ \left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right)$$

$$=\frac{6}{83}\left(-\frac{29}{3}\right)\left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right)$$

$$=-\frac{58}{83}\left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right)$$

$$=\left(-\frac{29}{249},\ -\frac{116}{83},\ -\frac{493}{249},\ -\frac{232}{249}\right)$$

$$\vec{v}_3=\vec{u}_3-\frac{\langle\vec{u}_3,\vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1-\frac{\langle\vec{u}_3,\vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$=(0,\ -2,\ -2,\ 0)-\left(\frac{1}{3},\ 0,\ -\frac{1}{3},\ \frac{2}{3}\right)-\left(-\frac{29}{249},\ -\frac{116}{83},\ -\frac{493}{249},\ -\frac{232}{249}\right)$$

$$=\left(-\frac{1}{3},\ -2,\ -\frac{5}{3},\ -\frac{2}{3}\right)-\left(-\frac{29}{249},\ -\frac{116}{83},\ -\frac{493}{249},\ -\frac{232}{249}\right)$$

$$=\left(-\frac{18}{83},\ -\frac{50}{83},\ \frac{26}{83},\ \frac{22}{83}\right)$$

$$\vec{q}_3=\frac{83}{\sqrt{324+2500+676+484}}\left(-\frac{18}{83},\ -\frac{50}{83},\ \frac{26}{83},\ \frac{22}{83}\right) \qquad \vec{q}_3=\frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$=\frac{1}{\sqrt{3,984}}(-18,\ -50,\ 26,\ 22)$$

$$=\frac{1}{4\sqrt{249}}(-18,\ -50,\ 26,\ 22)$$

$$=\left(-\frac{9}{2\sqrt{249}},\ -\frac{25}{2\sqrt{249}},\ \frac{13}{2\sqrt{249}},\ \frac{11}{2\sqrt{249}}\right)$$

$$\frac{\langle\vec{u}_4,\vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1=\frac{(-2,\ 1,\ 2,\ 1)\cdot(3,\ 0,\ -3,\ 6)}{54}(3,\ 0,\ -3,\ 6)$$

$$=-\frac{6}{54}(3,\ 0,\ -3,\ 6)$$

$$=-\frac{1}{9}(3,\ 0,\ -3,\ 6)$$

$$=\left(-\frac{1}{3},\ 0,\ \frac{1}{3},\ -\frac{2}{3}\right)$$

$$\frac{\langle\vec{u}_4,\vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2=\frac{36}{498}\left[(-2,\ 1,\ 2,\ 1)\cdot\left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right)\right]\left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right)$$

$$=\frac{6}{83}\left(-\frac{1}{3}+2+\frac{17}{3}+\frac{4}{3}\right)\left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right)$$

$$=\frac{6}{83}\left(\frac{26}{3}\right)\left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right)$$
$$=\frac{52}{83}\left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right)$$
$$=\left(\frac{26}{249},\ \frac{104}{83},\ \frac{442}{249},\ \frac{208}{249}\right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_3\rangle}{\|\vec{v}_3\|^2}\vec{v}_3=\frac{83^2}{3{,}984}\left[(-2,\ 1,\ 2,\ 1)\bullet\left(-\frac{18}{83},\ -\frac{50}{83},\ \frac{26}{83},\ \frac{22}{83}\right)\right]\left(-\frac{18}{83},\ -\frac{50}{83},\ \frac{26}{83},\ \frac{22}{83}\right)$$
$$=\frac{83}{3{,}984}\ \frac{36-50+52+22}{83}\ (-18,\ -50,\ 26,\ 22)$$
$$=\frac{60}{3{,}984}\ (-18,\ -50,\ 26,\ 22)$$
$$=\frac{15}{996}\ (-18,\ -50,\ 26,\ 22)$$
$$=\frac{5}{332}\ (-18,\ -50,\ 26,\ 22)$$
$$=\left(-\frac{45}{166},\ -\frac{125}{166},\ \frac{65}{166},\ \frac{55}{166}\right)$$

$$\vec{v}_4=\vec{u}_4-\frac{\langle \vec{u}_4, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1-\frac{\langle \vec{u}_4, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2-\frac{\langle \vec{u}_4, \vec{v}_3\rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$
$$=(-2,\ 1,\ 2,\ 1)-\left(-\frac{1}{3},\ 0,\ \frac{1}{3},\ -\frac{2}{3}\right)-\left(\frac{26}{249},\ \frac{104}{83},\ \frac{442}{249},\ \frac{208}{249}\right)-\left(-\frac{45}{166},\ -\frac{125}{166},\ \frac{65}{166},\ \frac{55}{166}\right)$$
$$=\left(-\frac{5}{3},\ 1,\ \frac{5}{3},\ \frac{5}{3}\right)-\left(\frac{26}{249},\ \frac{104}{83},\ \frac{442}{249},\ \frac{208}{249}\right)+\left(\frac{45}{166},\ \frac{125}{166},\ -\frac{65}{166},\ -\frac{55}{166}\right)$$
$$=\left(-\frac{441}{249},\ -\frac{21}{83},\ -\frac{27}{249},\ \frac{207}{249}\right)+\left(\frac{45}{166},\ \frac{125}{166},\ -\frac{65}{166},\ -\frac{55}{166}\right)$$
$$=\left(-\frac{147}{83},\ -\frac{21}{83},\ -\frac{9}{83},\ \frac{69}{83}\right)+\left(\frac{45}{166},\ \frac{125}{166},\ -\frac{65}{166},\ -\frac{55}{166}\right)$$
$$=\left(-\frac{249}{166},\ \frac{83}{166},\ -\frac{83}{166},\ \frac{83}{166}\right)$$
$$=\left(-\frac{3}{2},\ \frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2}\right)$$

$$\vec{q}_4=\frac{1}{\sqrt{\frac{9}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}}\left(-\frac{3}{2},\ \frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2}\right) \qquad \vec{q}_4=\frac{\vec{v}_4}{\|\vec{v}_4\|}$$
$$=\frac{2}{\sqrt{12}}\left(-\frac{3}{2},\ \frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2}\right)$$
$$=\frac{1}{\sqrt{3}}\left(-\frac{3}{2},\ \frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2}\right)$$

$$= \left(-\frac{3}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}},\ -\frac{1}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}}\right)$$

The *orthogonal* basis:

$$\left\{(3,\ 0,\ -3,\ 6),\ \left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right),\ \left(-\frac{18}{83},\ -\frac{50}{83},\ \frac{26}{83},\ \frac{22}{83}\right),\ \left(-\frac{3}{2},\ \frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2}\right)\right\}$$

The *orthonormal* basis:

$$\left\{\left(\frac{1}{\sqrt{6}},\ 0,\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right),\ \left(\frac{1}{\sqrt{498}},\ \frac{12}{\sqrt{498}},\ \frac{17}{\sqrt{498}},\ \frac{8}{\sqrt{498}}\right),\ \left(-\frac{9}{2\sqrt{249}},\ -\frac{25}{2\sqrt{249}},\ \frac{13}{2\sqrt{249}},\ \frac{11}{2\sqrt{249}}\right),\ \left(-\frac{3}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}},\ -\frac{1}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}}\right)\right\}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(3,\ 4,\ 0,\ 0),\ (-1,\ 1,\ 0,\ 0),\ (2,\ 1,\ 0,\ -1),\ (0,\ 1,\ 1,\ 0)\}$$

Solution

Let $\vec{u}_1 = (3,\ 4,\ 0,\ 0)\quad \vec{u}_2 = (-1,\ 1,\ 0,\ 0)\quad \vec{u}_3 = (2,\ 1,\ 0,\ -1)\quad \vec{u}_4 = (0,\ 1,\ 1,\ 0)$

$$\vec{v}_1 = \vec{u}_1 = (3,\ 4,\ 0,\ 0)$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 \\
&= (-1,\ 1,\ 0,\ 0) - \frac{(-1,\ 1,\ 0,\ 0)\cdot(3,\ 4,\ 0,\ 0)}{25}(3,\ 4,\ 0,\ 0) \\
&= (-1,\ 1,\ 0,\ 0) - \frac{1}{25}(3,\ 4,\ 0,\ 0) \\
&= \left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 &= \frac{(2,\ 1,\ 0,\ -1)\cdot(3,\ 4,\ 0,\ 0)}{25}(3,\ 4,\ 0,\ 0) \\
&= \frac{10}{25}(3,\ 4,\ 0,\ 0)
\end{aligned}$$

$$=\left(\frac{6}{5},\ \frac{8}{5},\ 0,\ 0\right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{625}{1,225}\left[(2,\ 1,\ 0,\ -1)\bullet\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)\right]\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)$$

$$=\frac{25}{49}\left(-\frac{35}{25}\right)\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)$$

$$=\left(\frac{4}{5},\ -\frac{3}{5},\ 0,\ 0\right)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$=(2,\ 1,\ 0,\ -1)-\left(\frac{6}{5},\ \frac{8}{5},\ 0,\ 0\right)-\left(\frac{4}{5},\ -\frac{3}{5},\ 0,\ 0\right)$$

$$=(0,\ 0,\ 0,\ -1)$$

$$\frac{\langle \vec{u}_4, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(0,\ 1,\ 1,\ 0)\bullet(3,\ 4,\ 0,\ 0)}{25}(3,\ 4,\ 0,\ 0)$$

$$=\frac{4}{25}(3,\ 4,\ 0,\ 0)$$

$$=\left(\frac{12}{25},\ \frac{16}{25},\ 0,\ 0\right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{25}{49}\left[(0,\ 1,\ 1,\ 0)\bullet\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)\right]\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)$$

$$=\frac{25}{49}\left(\frac{21}{25}\right)\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)$$

$$=\left(-\frac{12}{25},\ \frac{9}{25},\ 0,\ 0\right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_3\rangle}{\|\vec{v}_3\|^2}\vec{v}_3 = \left[(0,\ 1,\ 1,\ 0)\bullet(0,\ 0,\ 0,\ -1)\right](0,\ 0,\ 0,\ -1)$$

$$=(0,\ 0,\ 0,\ 0)$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3\rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$

$$=(0,\ 1,\ 1,\ 0)-\left(\frac{12}{25},\ \frac{16}{25},\ 0,\ 0\right)-\left(-\frac{12}{25},\ \frac{9}{25},\ 0,\ 0\right)-(0,\ 0,\ 0,\ 0)$$

$$=\underline{\left(0,\ 0,\ 1,\ 0\right)}$$

$$\vec{q}_1=\frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$=\frac{1}{5}\left(3,\ 4,\ 0,\ 0\right)$$

$$=\underline{\left(\frac{3}{5},\ \frac{4}{5},\ 0,\ 0\right)}$$

$$\vec{q}_2=\frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$=\frac{25}{35}\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)$$

$$=\underline{\left(-\frac{4}{5},\ \frac{3}{5},\ 0,\ 0\right)}$$

$$\vec{q}_3=\frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$=\underline{\left(0,\ 0,\ 0,\ -1\right)}$$

$$\vec{q}_4=\frac{\vec{v}_4}{\left\|\vec{v}_4\right\|}$$

$$=\underline{\left(0,\ 0,\ 1,\ 0\right)}$$

The *orthogonal* basis:

$$\left\{\left(3,\ 4,\ 0,\ 0\right),\ \left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right),\ \left(0,\ 0,\ 0,\ -1\right),\ \left(0,\ 0,\ 1,\ 0\right)\right\}$$

The *orthonormal* basis:

$$\left\{\left(\frac{3}{5},\ \frac{4}{5},\ 0,\ 0\right),\ \left(-\frac{4}{5},\ \frac{3}{5},\ 0,\ 0\right),\ \left(0,\ 0,\ 0,\ -1\right),\ \left(0,\ 0,\ 1,\ 0\right)\right\}$$

Exercise

Find the QR–decomposition of $\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}$

Solution

Since $\begin{vmatrix}1 & -1\\ 2 & 3\end{vmatrix}=5\neq 0$, The matrix is invertible

$\vec{u}_1\left(1,\ 2\right),\quad \vec{u}_2=\left(-1,\ 3\right)$

$$\boxed{\vec{v}_1 = \vec{u}_1 = (1,\ 2)}$$

$$\begin{aligned}
\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
&= \frac{(1,\ 2)}{\sqrt{1^2+2^2}} \\
&= \frac{(1,\ 2)}{\sqrt{5}} \\
&= \boxed{\left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right)}
\end{aligned}$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \left(\vec{u}_2 \cdot \vec{v}_1\right)\vec{v}_1 \\
&= (-1,\ 3) - \left[(-1,\ 3) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)\right]\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \\
&= (-1,\ 3) - \left(\frac{5}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \\
&= (-1,\ 3) - (1,\ 2) \\
&= \boxed{(-2,\ 1)}
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \frac{(-2,\ 1)}{\sqrt{(-2)^2+1^2}} \\
&= \boxed{\left(-\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right)}
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_1, \vec{q}_1 \rangle &= (1,\ 2) \cdot \left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right) \\
&= \frac{5}{\sqrt{5}} \\
&= \boxed{\sqrt{5}}
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_2, \vec{q}_1 \rangle &= (-1,\ 3) \cdot \left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right) \\
&= \frac{5}{\sqrt{5}} \\
&= \boxed{\sqrt{5}}
\end{aligned}$$

$$\left\langle \vec{u}_2, \vec{q}_2 \right\rangle = (-1,\ 3) \cdot \left(-\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right)$$

$$= \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{5}}$$

$$\underline{= \sqrt{5}}\,|$$

$$R = \begin{bmatrix} \left\langle \vec{u}_1, \vec{q}_1 \right\rangle & \left\langle \vec{u}_2, \vec{q}_1 \right\rangle \\ 0 & \left\langle \vec{u}_2, \vec{q}_2 \right\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}$$

The QR-decomposition of the matrix is

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}$$

Exercise

Find the QR–decomposition of $\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix}$

Solution

The column vectors of are: $\vec{u}_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$$\underline{\vec{v}_1 = \vec{u}_1 = (3,\ -4)}\,|$$

$$\vec{q}_1 = \frac{(3,\ -4)}{\sqrt{9+16}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$\underline{= \left(\frac{3}{5},\ -\frac{4}{5}\right)}\,|$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2, \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$= (5,\ 0) - \frac{(5,\ 0) \cdot (3,\ -4)}{25}(3,\ -4)$$

$$= (5,\ 0) - \frac{15}{25}(3,\ -4)$$

$$= (5,\ 0) - \frac{3}{5}(3,\ -4)$$

$$= (5,\ 0) - \left(\frac{9}{5},\ -\frac{12}{5}\right)$$

$$= \left(\frac{16}{5},\ \frac{12}{5}\right)$$

$$\vec{q}_2 = \frac{1}{\sqrt{\frac{256}{25} + \frac{144}{25}}}\left(\frac{16}{5},\ \frac{12}{5}\right) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \frac{1}{\sqrt{\frac{400}{25}}}\left(\frac{16}{5},\ \frac{12}{5}\right)$$

$$= \frac{1}{\sqrt{16}}\left(\frac{16}{5},\ \frac{12}{5}\right)$$

$$= \frac{1}{4}\left(\frac{16}{5},\ \frac{12}{5}\right)$$

$$= \left(\frac{4}{5},\ \frac{3}{5}\right)$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} 3\left(\frac{3}{5}\right) - 4\left(-\frac{4}{5}\right) & 5\left(\frac{3}{5}\right) + 0\left(-\frac{4}{5}\right) \\ 0 & 5\left(\frac{4}{5}\right) - 0\left(\frac{3}{5}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 \\ 0 & 4 \end{bmatrix}$$

$$\underset{A}{\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix}} = \underset{Q}{\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}} \cdot \underset{R}{\begin{bmatrix} 5 & 3 \\ 0 & 4 \end{bmatrix}}$$

Exercise

Find the QR–decomposition of $\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$

Solution

The column vectors of are: $\vec{u}_1 = (1,\ -2) \quad \vec{u}_2 = (3,\ 4)$

$$\vec{v}_1 = \vec{u}_1 = (1,\ -2)$$

$$\vec{q}_1 = \frac{(1,\ -2)}{\sqrt{1+4}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \boxed{\left(\frac{1}{\sqrt{5}},\ -\frac{2}{\sqrt{5}}\right)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (3,\ 4) - \frac{(3,\ 4)\bullet(1,\ -2)}{5}(1,\ -2)$$

$$= (3,\ 4) + \frac{5}{5}(1,\ -2)$$

$$= (3,\ 4) + (1,\ -2)$$

$$= \boxed{(4,\ 2)}$$

$$\vec{q}_2 = \frac{1}{\sqrt{16+4}}(4,\ 2) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{2\sqrt{5}}(4,\ 2)$$

$$= \boxed{\left(\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right)}$$

$$R = \begin{bmatrix} 1\left(\frac{1}{\sqrt{5}}\right) - 2\left(-\frac{2}{\sqrt{5}}\right) & 3\left(\frac{1}{\sqrt{5}}\right) + 4\left(-\frac{2}{\sqrt{5}}\right) \\ 0 & 3\left(\frac{2}{\sqrt{5}}\right) + 4\left(\frac{1}{\sqrt{5}}\right) \end{bmatrix} \qquad\qquad R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix}$$

$$= \begin{pmatrix} \frac{5}{\sqrt{5}} & -\frac{5}{\sqrt{5}} \\ 0 & \frac{10}{\sqrt{5}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{5} & -\sqrt{5} \\ 0 & 2\sqrt{5} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{5} & -\sqrt{5} \\ 0 & 2\sqrt{5} \end{pmatrix}$$

$$\boldsymbol{A} \quad = \quad \boldsymbol{Q} \qquad \boldsymbol{R}$$

Exercise

Find the QR–decomposition of $\begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}$

Solution

The column vectors of are: $\vec{u}_1 = (1,\ -1)$ $\vec{u}_2 = (-2,\ 4)$

$$\boxed{\vec{v}_1 = \vec{u}_1 = (1,\ -1)}$$

$$\vec{q}_1 = \frac{(1,\ -1)}{\sqrt{2}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \boxed{\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (-2,\ 4) - \frac{(-2,\ 4)\bullet(1,\ -1)}{2}(1,\ -1)$$

$$= (-2,\ 4) + \frac{6}{2}(1,\ -1)$$

$$= (-2,\ 4) + (3,\ -3)$$

$$= \boxed{(1,\ 1)}$$

$$\vec{q}_2 = \frac{1}{\sqrt{2}}(1,\ 1) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \boxed{\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)}$$

$$R = \begin{bmatrix} \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) & -2\left(\frac{1}{\sqrt{2}}\right) + 4\left(-\frac{1}{\sqrt{2}}\right) \\ 0 & -2\left(\frac{1}{\sqrt{2}}\right) + 4\left(\frac{1}{\sqrt{2}}\right) \end{bmatrix} \qquad R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix}$$

$$= \begin{pmatrix} \frac{2}{\sqrt{2}} & -\frac{6}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{2} & -3\sqrt{2} \\ 0 & \sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} & -3\sqrt{2} \\ 0 & \sqrt{2} \end{pmatrix}$$

$$\boldsymbol{A} \quad = \quad \boldsymbol{Q} \qquad \boldsymbol{R}$$

Exercise

Find the *QR*–decomposition of $\begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix}$

Solution

The column vectors of are: $\vec{u}_1 = (-1,\ 2) \quad \vec{u}_2 = (1,\ -4)$

$$\vec{v}_1 = \vec{u}_1 = (-1,\ 2)$$

$$\vec{q}_1 = \frac{(-1,\ 2)}{\sqrt{5}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(-\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (1,\ -4) - \frac{(1,\ -4) \bullet (-1,\ 2)}{5}(-1,\ 2)$$

$$= (1,\ -4) + \frac{9}{5}(-1,\ 2)$$

$$= (1,\ -4) + \left(-\frac{9}{5},\ \frac{18}{5}\right)$$

$$= \left(-\frac{4}{5},\ -\frac{2}{5}\right)$$

$$\vec{q}_2 = \frac{5}{\sqrt{16+4}}\left(-\frac{4}{5},\ -\frac{2}{5}\right) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{5}{2\sqrt{5}}\left(-\frac{4}{5},\ -\frac{2}{5}\right)$$

$$= \left(-\frac{2}{\sqrt{5}},\ -\frac{1}{\sqrt{5}}\right)$$

$$R = \begin{bmatrix} \frac{1}{\sqrt{5}} + 2\left(\frac{2}{\sqrt{5}}\right) & \left(-\frac{1}{\sqrt{5}}\right) - 4\left(\frac{2}{\sqrt{5}}\right) \\ 0 & \left(-\frac{2}{\sqrt{5}}\right) - 4\left(-\frac{1}{\sqrt{5}}\right) \end{bmatrix} \qquad R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix}$$

$$= \begin{pmatrix} \frac{5}{\sqrt{5}} & -\frac{9}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix}} \underset{=}{=} \underset{Q}{\begin{pmatrix} -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}} \cdot \underset{R}{\begin{pmatrix} \frac{5}{\sqrt{5}} & -\frac{9}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix}}$$

Exercise

Find the QR–decomposition of $\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}$

Solution

The column vectors of are: $\vec{u}_1 = (1, \ -2) \quad \vec{u}_2 = (-3, \ 4)$

$$\vec{v}_1 = \vec{u}_1 = (1, \ -2)$$

$$\vec{q}_1 = \frac{(1, \ -2)}{\sqrt{5}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(\frac{1}{\sqrt{5}}, \ -\frac{2}{\sqrt{5}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (-3, \ 4) - \frac{(-3, \ 4) \bullet (1, \ -2)}{5}(1, \ -2)$$

$$= (-3, \ 4) + \frac{11}{5}(1, \ -2)$$

$$= \left(-\frac{4}{5}, \ -\frac{2}{5}\right)$$

$$\vec{q}_2 = \frac{5}{\sqrt{16+4}}\left(-\frac{4}{5}, \ -\frac{2}{5}\right) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{5}{2\sqrt{5}}\left(-\frac{4}{5},\ -\frac{2}{5}\right)$$

$$= \left(-\frac{2}{\sqrt{5}},\ -\frac{1}{\sqrt{5}}\right)$$

$$R = \begin{bmatrix} \frac{1}{\sqrt{5}} - 2\left(-\frac{2}{\sqrt{5}}\right) & -3\left(\frac{1}{\sqrt{5}}\right) + 4\left(-\frac{2}{\sqrt{5}}\right) \\ 0 & -3\left(-\frac{2}{\sqrt{5}}\right) + 4\left(-\frac{1}{\sqrt{5}}\right) \end{bmatrix} \qquad R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix}$$

$$= \begin{pmatrix} \frac{5}{\sqrt{5}} & -\frac{11}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}} = \underset{Q}{\begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}} \cdot \underset{R}{\begin{pmatrix} \frac{5}{\sqrt{5}} & -\frac{11}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix}}$$

Exercise

Find the QR–decomposition of $\begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix}$

Solution

The column vectors of are: $\vec{u}_1 = (-2,\ 4)$ $\vec{u}_2 = (6,\ -8)$

$$\vec{v}_1 = \vec{u}_1 = (-2,\ 4)$$

$$\vec{q}_1 = \frac{(-2,\ 4)}{\sqrt{20}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(-\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (6,\ -8) - \frac{(6,\ -8)\bullet(-2,\ 4)}{20}(-2,\ 4)$$

$$= (6,\ -8) + \frac{44}{20}(-2,\ 4)$$

$$=(6,\ -8)+\frac{11}{5}(-2,\ 4)$$

$$=\left(\frac{8}{5},\ \frac{4}{5}\right)$$

$$\vec{q}_2=\frac{5}{\sqrt{64+16}}\left(\frac{8}{5},\ \frac{4}{5}\right) \qquad \vec{q}_2=\frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$=\frac{5}{4\sqrt{5}}\left(\frac{8}{5},\ \frac{4}{5}\right)$$

$$=\left(\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right)$$

$$R=\begin{bmatrix} -2\left(-\frac{1}{\sqrt{5}}\right)+4\left(\frac{2}{\sqrt{5}}\right) & 6\left(-\frac{1}{\sqrt{5}}\right)-8\left(\frac{2}{\sqrt{5}}\right) \\ 0 & 6\left(\frac{2}{\sqrt{5}}\right)-8\left(\frac{1}{\sqrt{5}}\right) \end{bmatrix} \qquad R=\begin{bmatrix} \langle \vec{u}_1, \vec{q}_1\rangle & \langle \vec{u}_2, \vec{q}_1\rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2\rangle \end{bmatrix}$$

$$=\begin{pmatrix} \frac{10}{\sqrt{5}} & -\frac{22}{\sqrt{5}} \\ 0 & \frac{4}{\sqrt{5}} \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix}}=\underset{Q}{\begin{pmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}}\cdot\underset{R}{\begin{pmatrix} \frac{10}{\sqrt{5}} & -\frac{22}{\sqrt{5}} \\ 0 & \frac{4}{\sqrt{5}} \end{pmatrix}}$$

A* = *Q* *R

Exercise

Find the QR–decomposition of $\begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}$

Solution

The column vectors of are: $\vec{u}_1=(-1,\ 1)\quad \vec{u}_2=(2,\ 0)$

$$\vec{v}_1=\vec{u}_1=(-1,\ 1)$$

$$\vec{q}_1=\frac{(-1,\ 1)}{\sqrt{2}} \qquad \vec{q}_1=\frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$=\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (2,\ 0) - \frac{(2,\ 0) \bullet (-1,\ 1)}{2}(-1,\ 1)$$

$$= (2,\ 0) + (-1,\ 1)$$

$$= (1,\ 1)$$

$$\vec{q}_2 = \frac{1}{\sqrt{2}}(1,\ 1) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$R = \begin{bmatrix} -\left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right) & 2\left(-\frac{1}{\sqrt{2}}\right) + 0 \\ 0 & 2\left(\frac{1}{\sqrt{2}}\right) + 0 \end{bmatrix} \qquad\qquad R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix}$$

$$= \begin{pmatrix} \frac{2}{\sqrt{2}} & -\frac{2}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{\sqrt{2}} & -\frac{2}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{2}} \end{pmatrix}$$

$$\boldsymbol{A} \quad = \quad \boldsymbol{Q} \qquad\qquad \boldsymbol{R}$$

Exercise

Find the QR–decomposition of $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$

Solution

Since the column vectors $\vec{u}_1 = (1,\ 0,\ 1)$, $\vec{u}_2 = (2,\ 1,\ 4)$ are linearly independent, so has a QR–decomposition.

$$\vec{v}_1 = \vec{u}_1 = (1,\ 0,\ 1)$$

$$\vec{q}_1 = \frac{(1,\ 0,\ 1)}{\sqrt{1^2+0+1^2}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ 0,\ 1)}{\sqrt{2}}$$

$$\underline{= \left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)}$$

$$\vec{v}_2 = \vec{u}_2 - \langle \vec{u}_2, \vec{v}_1 \rangle \vec{v}_1$$

$$= (2,\ 1,\ 4) - \left[(2,\ 1,\ 4)\bullet\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\right]\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$= (2,\ 1,\ 4) - \left(\frac{6}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$= (2,\ 1,\ 4) - (3,\ 0,\ 3)$$

$$\underline{= (-1,\ 1,\ 1)}$$

$$\vec{q}_2 = \frac{(-1,\ 1,\ 1)}{\sqrt{(-1)^2+1^2+1^2}} \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$\underline{= \left(-\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)}$$

$$\langle \vec{u}_1, \vec{q}_1 \rangle = (1,\ 0,\ 1)\bullet\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$= \frac{2}{\sqrt{2}}$$

$$\underline{= \sqrt{2}}$$

$$\langle \vec{u}_2, \vec{q}_1 \rangle = (2,\ 1,\ 4)\bullet\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$= \frac{2}{\sqrt{2}} + \frac{4}{\sqrt{2}}$$

$$\underline{= 3\sqrt{2}}$$

$$\langle \vec{u}_2, \vec{q}_2 \rangle = (2,\ 1,\ 4)\bullet\left(-\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= -\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{4}{\sqrt{3}}$$

$$= \frac{3}{\sqrt{3}}$$

$$\underline{= \sqrt{3}}$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$$

The QR–decomposition of the matrix is

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$$

$$\boldsymbol{A} \quad = \quad \boldsymbol{Q} \qquad\qquad \boldsymbol{R}$$

Exercise

Find the QR–decomposition of $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{pmatrix}$

Solution

Since $\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{vmatrix} = -4 \neq 0$,

The matrix is invertible, so it has a QR-decomposition.

$\vec{u}_1 = (1,\ 1,\ 0), \quad \vec{u}_2 = (2,\ 1,\ 3), \quad \vec{u}_3 = (1,\ 1,\ 1)$

$\vec{v}_1 = \vec{u}_1 = (1,\ 1,\ 0)$

$$\vec{q}_1 = \frac{(1,\ 1,\ 0)}{\sqrt{1+1+0}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ 1,\ 0)}{\sqrt{2}}$$

$$= \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0 \right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (2,\ 1,\ 3) - \tfrac{1}{2}\left[(2,\ 1,\ 3)\bullet(1,\ 1,\ 0)\right](1,\ 1,\ 0)$$

$$= (2,\ 1,\ 3) - \tfrac{3}{2}(1,\ 1,\ 0)$$

$$= (2,\ 1,\ 3) - \left(\tfrac{3}{2},\ \tfrac{3}{2},\ 0\right)$$

$$= \left(\tfrac{1}{2},\ -\tfrac{1}{2},\ 3\right)$$

$$\vec{q}_2 = \frac{\left(\frac{1}{2},\ -\frac{1}{2},\ 3\right)}{\sqrt{\frac{1}{4}+\frac{1}{4}+9}} \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{2}{\sqrt{38}}\left(\tfrac{1}{2},\ -\tfrac{1}{2},\ 3\right)$$

$$= \left(\frac{1}{\sqrt{38}},\ -\frac{1}{\sqrt{38}},\ \frac{6}{\sqrt{38}}\right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(1,\ 1,\ 1)\bullet(1,\ 1,\ 0)}{2}(1,\ 1,\ 0)$$

$$= (1,\ 1,\ 0)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{4}{38}\left((1,\ 1,\ 1)\bullet\left(\tfrac{1}{2},\ -\tfrac{1}{2},\ 3\right)\right)\left(\tfrac{1}{2},\ -\tfrac{1}{2},\ 3\right)$$

$$= \frac{6}{19}\left(\tfrac{1}{2},\ -\tfrac{1}{2},\ 3\right)$$

$$= \left(\tfrac{3}{19},\ -\tfrac{3}{19},\ \tfrac{18}{19}\right)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (1,\ 1,\ 1) - (1,\ 1,\ 0) - \left(\tfrac{3}{19},\ -\tfrac{3}{19},\ \tfrac{18}{19}\right)$$

$$= (0,\ 0,\ 1) - \left(\tfrac{3}{19},\ -\tfrac{3}{19},\ \tfrac{18}{19}\right)$$

$$= \left(-\tfrac{3}{19},\ \tfrac{3}{19},\ \tfrac{1}{19}\right)$$

$$\vec{q}_3 = \frac{1}{\sqrt{\left(-\frac{3}{19}\right)^2 + \left(\frac{3}{19}\right)^2 + \left(\frac{1}{19}\right)^2}}\left(-\frac{3}{19}, \frac{3}{19}, \frac{1}{19}\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{19}{\sqrt{19}}\left(-\frac{3}{19}, \frac{3}{19}, \frac{1}{19}\right)$$

$$= \left(-\frac{3}{\sqrt{19}}, \frac{3}{\sqrt{19}}, \frac{1}{\sqrt{19}}\right)$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix}$$

$$\langle \vec{u}_1, \vec{q}_1 \rangle = (1, 1, 0) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$= \frac{2}{\sqrt{2}}$$

$$= \sqrt{2}$$

$$\langle \vec{u}_2, \vec{q}_1 \rangle = (2, 1, 3) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$= \frac{3}{\sqrt{2}}$$

$$\langle \vec{u}_2, \vec{q}_2 \rangle = (2, 1, 3) \cdot \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}}\right)$$

$$= \frac{2-1+18}{\sqrt{38}}$$

$$= \frac{19}{\sqrt{38}}$$

$$= \frac{19}{\sqrt{2}\sqrt{19}}$$

$$= \frac{\sqrt{19}}{\sqrt{2}}$$

$$\langle \vec{u}_3, \vec{q}_1 \rangle = (1, 1, 1) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$= \frac{2}{\sqrt{2}}$$

$$= \sqrt{2}$$

$$\left\langle \vec{u}_3, \vec{q}_2 \right\rangle = (1,\ 1,\ 1) \cdot \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}} \right)$$

$$= \frac{1-1+6}{\sqrt{38}}$$

$$= \frac{6}{\sqrt{2}\sqrt{19}} \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{3\sqrt{2}}{\sqrt{19}}$$

$$\left\langle \vec{u}_3, \vec{q}_3 \right\rangle = (1,\ 1,\ 1) \cdot \left(-\frac{3}{\sqrt{19}}, \frac{3}{\sqrt{19}}, \frac{1}{\sqrt{19}} \right)$$

$$= \frac{-3+3+1}{\sqrt{19}}$$

$$= \frac{1}{\sqrt{19}}$$

$$R = \begin{bmatrix} \left\langle \vec{u}_1, \vec{q}_1 \right\rangle & \left\langle \vec{u}_2, \vec{q}_1 \right\rangle & \left\langle \vec{u}_3, \vec{q}_1 \right\rangle \\ 0 & \left\langle \vec{u}_2, \vec{q}_2 \right\rangle & \left\langle \vec{u}_3, \vec{q}_2 \right\rangle \\ 0 & 0 & \left\langle \vec{u}_3, \vec{q}_3 \right\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix}$$

The QR–decomposition of the matrix is

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix}}_{R}$$

Exercise

Find the QR–decomposition of $\begin{pmatrix} 7 & 1 & 2 \\ 2 & -1 & 3 \\ -3 & 0 & 4 \end{pmatrix}$

Solution

Since $\begin{vmatrix} 7 & 1 & 2 \\ 2 & -1 & 3 \\ -3 & 0 & 4 \end{vmatrix} = -51 \neq 0$,

The matrix is invertible, so it has a QR-decomposition.

$\vec{u}_1 = (7,\ 2,\ -3), \quad \vec{u}_2 = (1,\ -1,\ 0), \quad \vec{u}_3 = (2,\ 3,\ 4)$

$$\boxed{\vec{v}_1 = \vec{u}_1 = (7,\ 2,\ -3)}$$

$$\vec{q}_1 = \frac{(7,\ 2,\ -3)}{\sqrt{49+4+9}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(7,\ 2,\ -3)}{\sqrt{62}}$$

$$\boxed{= \left(\frac{7}{\sqrt{62}},\ \frac{2}{\sqrt{62}},\ -\frac{3}{\sqrt{62}}\right)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (1,\ -1,\ 0) - \frac{1}{62}\left[(1,\ -1,\ 0) \bullet (7,\ 2,\ -3)\right](7,\ 2,\ -3)$$

$$= (1,\ -1,\ 0) - \frac{5}{62}(7,\ 2,\ -3)$$

$$= (1,\ -1,\ 0) - \left(\frac{35}{62},\ \frac{5}{31},\ -\frac{15}{62}\right)$$

$$\boxed{= \left(\frac{27}{62},\ -\frac{36}{31},\ \frac{15}{62}\right)}$$

$$\vec{q}_2 = \frac{62}{\sqrt{27^2+72^2+15^2}}\left(\frac{27}{62},\ -\frac{36}{31},\ \frac{15}{62}\right) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{62}{\sqrt{6,138}}\left(\frac{27}{62},\ -\frac{36}{31},\ \frac{15}{62}\right)$$

$$= \frac{62}{3\sqrt{682}}\left(\frac{27}{62},\ -\frac{36}{31},\ \frac{15}{62}\right)$$

$$\boxed{= \left(\frac{9}{\sqrt{682}},\ -\frac{24}{\sqrt{682}},\ \frac{5}{\sqrt{682}}\right)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(2,\ 3,\ 4)\cdot(7,\ 2,\ -3)}{62}\ (7,\ 2,\ -3)$$

$$= \frac{8}{62}\ (7,\ 2,\ -3)$$

$$= \frac{4}{31}\ (7,\ 2,\ -3)$$

$$= \left(\frac{28}{31},\ \frac{8}{31},\ -\frac{12}{31}\right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{3{,}844}{6{,}138}\left((2,\ 3,\ 4)\cdot\left(\frac{27}{62},\ -\frac{36}{31},\ \frac{15}{62}\right)\right)\ \left(\frac{27}{62},\ -\frac{36}{31},\ \frac{15}{62}\right)$$

$$= \frac{62}{6{,}138}\left(\frac{54}{62}-\frac{108}{31}+\frac{60}{62}\right)\ (27,\ -72,\ 15)$$

$$= -\frac{102}{6{,}138}\ (27,\ -72,\ 15)$$

$$= -\frac{17}{1{,}023}\ (27,\ -72,\ 15)$$

$$= \left(-\frac{153}{341},\ \frac{408}{341},\ -\frac{85}{341}\right)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (2,\ 3,\ 4) - \left(\frac{28}{31},\ \frac{8}{31},\ -\frac{12}{31}\right) - \left(-\frac{153}{341},\ \frac{408}{341},\ -\frac{85}{341}\right)$$

$$= \left(\frac{34}{31},\ \frac{85}{31},\ \frac{136}{31}\right) - \left(-\frac{153}{341},\ \frac{408}{341},\ -\frac{85}{341}\right)$$

$$= \left(\frac{17}{11},\ \frac{17}{11},\ \frac{51}{11}\right)$$

$$\vec{q}_3 = \frac{11}{\sqrt{17^2+17^2+51^2}}\left(\frac{17}{11},\ \frac{17}{11},\ \frac{51}{11}\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\sqrt{3{,}179}}\ (17,\ 17,\ 51)$$

$$= \frac{1}{17\sqrt{11}}\ (17,\ 17,\ 51)$$

$$= \left(\frac{1}{\sqrt{11}},\ \frac{1}{\sqrt{11}},\ \frac{3}{\sqrt{11}}\right)$$

$$Q = \begin{bmatrix} \frac{7}{\sqrt{62}} & \frac{3}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ \frac{2}{\sqrt{62}} & -\frac{24}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ -\frac{3}{\sqrt{62}} & \frac{5}{\sqrt{19}} & \frac{3}{\sqrt{11}} \end{bmatrix}$$

$$\begin{aligned} \langle \vec{u}_1, \vec{q}_1 \rangle &= (7,\ 2,\ -3) \cdot \left(\frac{7}{\sqrt{62}},\ \frac{2}{\sqrt{62}},\ -\frac{3}{\sqrt{62}} \right) \\ &= \frac{49+4+9}{\sqrt{62}} \\ &= \boxed{\frac{62}{\sqrt{62}}} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{q}_1 \rangle &= (1,\ -1,\ 0) \cdot \left(\frac{7}{\sqrt{62}},\ \frac{2}{\sqrt{62}},\ -\frac{3}{\sqrt{62}} \right) \\ &= \boxed{\frac{5}{\sqrt{62}}} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{q}_2 \rangle &= (1,\ -1,\ 0) \cdot \left(\frac{9}{\sqrt{682}},\ -\frac{24}{\sqrt{682}},\ \frac{5}{\sqrt{682}} \right) \\ &= \boxed{\frac{33}{\sqrt{682}}} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_3, \vec{q}_1 \rangle &= (2,\ 3,\ 4) \cdot \left(\frac{7}{\sqrt{62}},\ \frac{2}{\sqrt{62}},\ -\frac{3}{\sqrt{62}} \right) \\ &= \frac{14+6-12}{\sqrt{62}} \\ &= \boxed{\frac{8}{\sqrt{62}}} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_3, \vec{q}_2 \rangle &= (2,\ 3,\ 4) \cdot \left(\frac{9}{\sqrt{682}},\ -\frac{24}{\sqrt{682}},\ \frac{5}{\sqrt{682}} \right) \\ &= \frac{18-72+20}{\sqrt{682}} \\ &= \boxed{-\frac{34}{\sqrt{682}}} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_3, \vec{q}_3 \rangle &= (2,\ 3,\ 4) \cdot \left(\frac{1}{\sqrt{11}},\ \frac{1}{\sqrt{11}},\ \frac{3}{\sqrt{11}} \right) \\ &= \frac{2+3+12}{\sqrt{11}} \\ &= \boxed{\frac{17}{\sqrt{11}}} \end{aligned}$$

$$R=\begin{bmatrix}\frac{62}{\sqrt{62}} & \frac{5}{\sqrt{62}} & \frac{8}{\sqrt{62}}\\ 0 & \frac{33}{\sqrt{682}} & -\frac{34}{\sqrt{682}}\\ 0 & 0 & \frac{17}{\sqrt{11}}\end{bmatrix} \qquad R=\begin{bmatrix}\langle\vec{u}_1,\vec{q}_1\rangle & \langle\vec{u}_2,\vec{q}_1\rangle & \langle\vec{u}_3,\vec{q}_1\rangle\\ 0 & \langle\vec{u}_2,\vec{q}_2\rangle & \langle\vec{u}_3,\vec{q}_2\rangle\\ 0 & 0 & \langle\vec{u}_3,\vec{q}_3\rangle\end{bmatrix}$$

The QR–decomposition of the matrix is

$$\underbrace{\begin{pmatrix}7 & 1 & 2\\ 2 & -1 & 3\\ -3 & 0 & 4\end{pmatrix}}_{A} = \underbrace{\begin{pmatrix}\frac{7}{\sqrt{62}} & \frac{9}{\sqrt{682}} & \frac{1}{\sqrt{11}}\\ \frac{2}{\sqrt{62}} & -\frac{24}{\sqrt{682}} & \frac{1}{\sqrt{11}}\\ -\frac{3}{\sqrt{62}} & \frac{5}{\sqrt{682}} & \frac{3}{\sqrt{11}}\end{pmatrix}}_{Q}\underbrace{\begin{pmatrix}\frac{62}{\sqrt{62}} & \frac{5}{\sqrt{62}} & \frac{8}{\sqrt{62}}\\ 0 & \frac{33}{\sqrt{682}} & -\frac{34}{\sqrt{682}}\\ 0 & 0 & \frac{17}{\sqrt{11}}\end{pmatrix}}_{R}$$

Exercise

Find the QR–decomposition of $\begin{pmatrix}1 & 0 & 1\\ -1 & 1 & 1\\ 1 & 0 & 1\\ -1 & 1 & 1\end{pmatrix}$

Solution

$$\begin{bmatrix}1 & 0 & 1\\ -1 & 1 & 1\\ 1 & 0 & 1\\ -1 & 1 & 1\end{bmatrix}\begin{array}{l}\\ R_2+R_1\\ R_3-R_1\\ R_4+R_1\end{array}$$

$$\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 0\\ 0 & 1 & 2\end{bmatrix}\begin{array}{l}\\ \\ \\ R_4-R_2\end{array}$$

$$\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

The matrix is linearly dependent, so ***doesn't*** have a QR–decomposition.

Exercise

Find the QR–decomposition of $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Solution

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{matrix} R_2 - R_1 \\ \\ R_4 - R_3 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ \\ R_3 - R_1 \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ \\ R_3 - R_2 \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The matrix has a QR-decomposition.

$\vec{u}_1 = (1,\ 1,\ 1,\ 1), \quad \vec{u}_2 = (0,\ 1,\ 1,\ 1), \quad \vec{u}_3 = (0,\ 0,\ 1,\ 1)$

$\vec{v}_1 = \vec{u}_1 = (1,\ 1,\ 1,\ 1)$

$$\vec{q}_1 = \frac{(1,\ 1,\ 1,\ 1)}{\sqrt{4}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (0,\ 1,\ 1,\ 1) - \frac{1}{4}\left[(0,\ 1,\ 1,\ 1) \bullet (1,\ 1,\ 1,\ 1)\right](1,\ 1,\ 1,\ 1)$$

$$= (0,\ 1,\ 1,\ 1) - \left(\frac{3}{4},\ \frac{3}{4},\ \frac{3}{4},\ \frac{3}{4}\right)$$

$$=\boxed{\left(-\frac{3}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right)}$$

$$\vec{q}_2=\frac{4}{\sqrt{9+1+1+1}}\left(-\frac{3}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right) \qquad \vec{q}_2=\frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$=\frac{2}{\sqrt{3}}\left(-\frac{3}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right)$$

$$=\boxed{\left(-\frac{3}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}}\right)}$$

$$\frac{\left\langle \vec{u}_3,\ \vec{v}_1\right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1=\frac{(0,\ 0,\ 1,\ 1)\cdot(1,\ 1,\ 1,\ 1)}{4}\ (1,\ 1,\ 1,\ 1)$$

$$=\frac{1}{2}\ (1,\ 1,\ 1,\ 1)$$

$$=\boxed{\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)}$$

$$\frac{\left\langle \vec{u}_3,\ \vec{v}_2\right\rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2=\frac{16}{12}\ \left((0,\ 0,\ 1,\ 1)\cdot\left(-\frac{3}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right)\right)\ \left(-\frac{3}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right)$$

$$=\frac{4}{3}\ \left(\frac{1}{2}\right)\ \left(-\frac{3}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right)$$

$$=\boxed{\left(-\frac{1}{2},\ \frac{1}{6},\ \frac{1}{6},\ \frac{1}{6}\right)}$$

$$\vec{v}_3=\vec{u}_3-\frac{\left\langle \vec{u}_3,\ \vec{v}_1\right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1-\frac{\left\langle \vec{u}_3,\ \vec{v}_2\right\rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2$$

$$=(0,\ 0,\ 1,\ 1)-\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)-\left(-\frac{1}{2},\ \frac{1}{6},\ \frac{1}{6},\ \frac{1}{6}\right)$$

$$=\boxed{\left(0,\ -\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}$$

$$\vec{q}_3=\frac{3}{\sqrt{6}}\left(0,\ -\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right) \qquad \vec{q}_3=\frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$=\boxed{\left(0,\ -\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)}$$

$$Q=\begin{pmatrix} \frac{1}{2} & -\frac{3}{2\sqrt{3}} & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$\left\langle \vec{u}_1, \vec{q}_1 \right\rangle = (1,\ 1,\ 1,\ 1) \cdot \left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$
$$= 2$$

$$\left\langle \vec{u}_2, \vec{q}_1 \right\rangle = (0,\ 1,\ 1,\ 1) \cdot \left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$
$$= \frac{3}{2}$$

$$\left\langle \vec{u}_2, \vec{q}_2 \right\rangle = (0,\ 1,\ 1,\ 1) \cdot \left(-\frac{3}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}}\right)$$
$$= \frac{3}{2\sqrt{3}}$$

$$\left\langle \vec{u}_3, \vec{q}_1 \right\rangle = (0,\ 0,\ 1,\ 1) \cdot \left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$
$$= 1$$

$$\left\langle \vec{u}_3, \vec{q}_2 \right\rangle = (0,\ 0,\ 1,\ 1) \cdot \left(-\frac{3}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}}\right)$$
$$= \frac{1}{\sqrt{3}}$$

$$\left\langle \vec{u}_3, \vec{q}_3 \right\rangle = (0,\ 0,\ 1,\ 1) \cdot \left(0,\ -\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$
$$= \frac{2}{\sqrt{6}}$$

$$R=\begin{pmatrix} 2 & \frac{3}{2} & 1 \\ 0 & \frac{3}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{2}{\sqrt{6}} \end{pmatrix} \qquad R=\begin{bmatrix} \left\langle \vec{u}_1, \vec{q}_1 \right\rangle & \left\langle \vec{u}_2, \vec{q}_1 \right\rangle & \left\langle \vec{u}_3, \vec{q}_1 \right\rangle \\ 0 & \left\langle \vec{u}_2, \vec{q}_2 \right\rangle & \left\langle \vec{u}_3, \vec{q}_2 \right\rangle \\ 0 & 0 & \left\langle \vec{u}_3, \vec{q}_3 \right\rangle \end{bmatrix}$$

The QR–decomposition of the matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{3}{2\sqrt{3}} & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 2 & \frac{3}{2} & 1 \\ 0 & \frac{3}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$$

$$\boldsymbol{A} \quad = \quad \boldsymbol{Q} \quad \boldsymbol{R}$$

Exercise

Find the QR–decomposition of $\begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{pmatrix}$

Solution

The matrix has a QR-decomposition.

$\vec{u}_1 = (3,\ 1,\ -1,\ 3),\quad \vec{u}_2 = (-5,\ 1,\ 5,\ -7),\quad \vec{u}_3 = (1,\ 1,\ -2,\ 8)$

$\vec{v}_1 = \vec{u}_1 = (3,\ 1,\ -1,\ 3)$

$$\vec{q}_1 = \frac{(3,\ 1,\ -1,\ 3)}{\sqrt{20}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(\frac{3}{2\sqrt{5}},\ \frac{1}{2\sqrt{5}},\ -\frac{1}{2\sqrt{5}},\ \frac{3}{2\sqrt{5}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (-5,\ 1,\ 5,\ -7) - \frac{1}{20}\left[(-5,\ 1,\ 5,\ -7) \bullet (3,\ 1,\ -1,\ 3)\right](3,\ 1,\ -1,\ 3)$$

$$= (-5,\ 1,\ 5,\ -7) + 2\,(3,\ 1,\ -1,\ 3)$$

$$= (1,\ 3,\ 3,\ -1)$$

$$\vec{q}_2 = \frac{1}{\sqrt{20}}(1,\ 3,\ 3,\ -1) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(\frac{1}{2\sqrt{5}},\ \frac{3}{2\sqrt{5}},\ \frac{3}{2\sqrt{5}},\ -\frac{1}{2\sqrt{5}}\right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(1,\ 1,\ -2,\ 8)\bullet(3,\ 1,\ -1,\ 3)}{20}\ (3,\ 1,\ -1,\ 3)$$

$$= \frac{3}{2}\ (3,\ 1,\ -1,\ 3)$$

$$= \left(\frac{9}{2},\ \frac{3}{2},\ -\frac{3}{2},\ \frac{9}{2}\right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{(1,\ 1,\ -2,\ 8)\bullet(1,\ 3,\ 3,\ -1)}{20}\ (1,\ 3,\ 3,\ -1)$$

$$= -\frac{1}{2}\ (1,\ 3,\ 3,\ -1)$$

$$= \left(-\frac{1}{2},\ -\frac{3}{2},\ -\frac{3}{2},\ \frac{1}{2}\right)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (1,\ 1,\ -2,\ 8) - \left(\frac{9}{2},\ \frac{3}{2},\ -\frac{3}{2},\ \frac{9}{2}\right) - \left(-\frac{1}{2},\ -\frac{3}{2},\ -\frac{3}{2},\ \frac{1}{2}\right)$$

$$= (-3,\ 1,\ 1,\ 3)$$

$$\vec{q}_3 = \frac{1}{2\sqrt{5}}\ (-3,\ 1,\ 1,\ 3) \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \left(-\frac{3}{2\sqrt{5}},\ \frac{1}{2\sqrt{5}},\ \frac{1}{2\sqrt{5}},\ \frac{3}{2\sqrt{5}}\right)$$

$$Q = \begin{pmatrix} \frac{3}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} & -\frac{3}{2\sqrt{5}} \\ \frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} \\ \frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} \\ \frac{3}{2\sqrt{5}} & -\frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} \end{pmatrix}$$

$$\langle \vec{u}_1, \vec{q}_1 \rangle = (3,\ 1,\ -1,\ 3)\bullet\left(\frac{3}{2\sqrt{5}},\ \frac{1}{2\sqrt{5}},\ -\frac{1}{2\sqrt{5}},\ \frac{3}{2\sqrt{5}}\right)$$

$$= \frac{10}{\sqrt{5}}$$

$$\langle \vec{u}_2, \vec{q}_1 \rangle = (-5,\ 1,\ 5,\ -7)\bullet\left(\frac{3}{2\sqrt{5}},\ \frac{1}{2\sqrt{5}},\ -\frac{1}{2\sqrt{5}},\ \frac{3}{2\sqrt{5}}\right)$$

$$= -\frac{20}{\sqrt{5}}$$

$$\left\langle \vec{u}_2, \vec{q}_2 \right\rangle = (-5,\ 1,\ 5,\ -7) \cdot \left(\frac{1}{2\sqrt{5}},\ \frac{3}{2\sqrt{5}},\ \frac{3}{2\sqrt{5}},\ -\frac{1}{2\sqrt{5}} \right)$$

$$= \frac{10}{\sqrt{5}}$$

$$\left\langle \vec{u}_3, \vec{q}_1 \right\rangle = (1,\ 1,\ -2,\ 8) \cdot \left(\frac{3}{2\sqrt{5}},\ \frac{1}{2\sqrt{5}},\ -\frac{1}{2\sqrt{5}},\ \frac{3}{2\sqrt{5}} \right)$$

$$= \frac{15}{\sqrt{5}}$$

$$\left\langle \vec{u}_3, \vec{q}_2 \right\rangle = (1,\ 1,\ -2,\ 8) \cdot \left(\frac{1}{2\sqrt{5}},\ \frac{3}{2\sqrt{5}},\ \frac{3}{2\sqrt{5}},\ -\frac{1}{2\sqrt{5}} \right)$$

$$= -\frac{5}{\sqrt{5}}$$

$$\left\langle \vec{u}_3, \vec{q}_3 \right\rangle = (1,\ 1,\ -2,\ 8) \cdot \left(-\frac{3}{2\sqrt{5}},\ \frac{1}{2\sqrt{5}},\ \frac{1}{2\sqrt{5}},\ \frac{3}{2\sqrt{5}} \right)$$

$$= \frac{10}{\sqrt{5}}$$

$$R = \begin{pmatrix} \frac{10}{\sqrt{5}} & -\frac{20}{\sqrt{5}} & \frac{15}{\sqrt{5}} \\ 0 & \frac{10}{\sqrt{5}} & -\frac{5}{\sqrt{5}} \\ 0 & 0 & \frac{10}{\sqrt{5}} \end{pmatrix} \qquad R = \begin{bmatrix} \left\langle \vec{u}_1, \vec{q}_1 \right\rangle & \left\langle \vec{u}_2, \vec{q}_1 \right\rangle & \left\langle \vec{u}_3, \vec{q}_1 \right\rangle \\ 0 & \left\langle \vec{u}_2, \vec{q}_2 \right\rangle & \left\langle \vec{u}_3, \vec{q}_2 \right\rangle \\ 0 & 0 & \left\langle \vec{u}_3, \vec{q}_3 \right\rangle \end{bmatrix}$$

The QR–decomposition of the matrix is

$$\underbrace{\begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} \frac{3}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} & -\frac{3}{2\sqrt{5}} \\ \frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} \\ \frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} \\ \frac{3}{2\sqrt{5}} & -\frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} \frac{10}{\sqrt{5}} & -\frac{20}{\sqrt{5}} & \frac{15}{\sqrt{5}} \\ 0 & \frac{10}{\sqrt{5}} & -\frac{5}{\sqrt{5}} \\ 0 & 0 & \frac{10}{\sqrt{5}} \end{pmatrix}}_{R}$$

Exercise

Find the QR–decomposition of $\begin{pmatrix} -1 & 2 & 2 \\ 2 & 5 & -1 \\ 2 & 8 & 0 \\ -3 & 3 & 5 \end{pmatrix}$

Solution

$$\begin{pmatrix} 1 & -2 & -2 \\ 2 & 5 & -1 \\ 2 & 8 & 0 \\ -3 & 3 & 5 \end{pmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \\ R_4 + 3R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & -2 & -2 \\ 0 & 9 & 3 \\ 0 & 12 & 4 \\ 0 & -3 & -1 \end{pmatrix} \begin{matrix} \\ \frac{1}{3}R_2 \\ \frac{1}{4}R_3 \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & -2 & -2 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \\ 0 & -3 & -1 \end{pmatrix} \begin{matrix} \\ \\ R_3 - R_2 \\ R_4 + R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & -2 & -2 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The matrix is linearly dependent, so ***doesn't*** have a QR–decomposition.

Exercise

Verify that the Cauchy-Schwarz inequality holds for the given vectors using the Euclidean inner product

$$\vec{u} = (0,\ -2,\ 2,\ 1), \quad \vec{v} = (-1,\ -1,\ 1,\ 1)$$

Solution

$$\langle \vec{u},\ \vec{v} \rangle = 0 - 2(-1) + 2(1) + 1(1)$$

$$= 5$$

$$\|\langle \vec{u},\ \vec{v} \rangle\| = \sqrt{5}$$

$$\|\vec{u}\|\ \|\vec{v}\| = \sqrt{0+4+4+1}\ \sqrt{1+1+1+1}$$

$$= \sqrt{9}\sqrt{4}$$

$$= 6$$

$$\sqrt{5} < 6 \quad \Rightarrow \quad \left\| \langle \vec{u}, \vec{v} \rangle \right\| \le \|\vec{u}\| \, \|\vec{v}\|$$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $\quad f_1(x) = x + 2, \quad f_2(x) = x^2 - 3x + 4$

Solution

Let $\vec{u}_1 = f_1 = x + 2, \quad \vec{u}_2 = f_2 = x^2 - 3x + 4$

$$\vec{v}_1 = \vec{u}_1 = x + 2$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^{1} (x+2)^2 \, dx$$

$$= \frac{1}{3}(x+2)^3 \Big|_{-1}^{1}$$

$$= \frac{1}{3}(27 - 1)$$

$$= \frac{26}{3}$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_{-1}^{1} \left(x^2 - 3x + 4\right)(x+2) \, dx$$

$$= \int_{-1}^{1} \left(x^3 - x^2 - 2x + 8\right) dx$$

$$= \left(\frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 + 8x\right)\Big|_{-1}^{1}$$

$$= \frac{1}{4} - \frac{1}{3} - 1 + 8 - \frac{1}{4} - \frac{1}{3} + 1 + 8$$

$$= \frac{46}{3}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$\vec{v}_2 = x^2 - 3x + 4 - \frac{46}{3}\left(\frac{3}{26}\right)(x+2)$$

$$= x^2 - 3x + 4 - \frac{23}{13}x - \frac{46}{13}$$

$$\boxed{= x^2 - \frac{62}{13}x + \frac{6}{13}}$$

The orthogonal basis is $\left\{ x+2,\quad x^2 - \frac{62}{13}x + \frac{6}{13} \right\}$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^{1} \left(x^2 - \frac{62}{13}x + \frac{6}{13} \right)^2 \, dx$$

$$= \frac{1}{169} \int_{-1}^{1} \left(13x^2 - 62x + 6 \right)^2 \, dx$$

$$= \frac{1}{169} \int_{-1}^{1} \left(169x^4 + 3{,}844x^2 + 36 - 1{,}612x^3 + 156x^2 - 744x \right) dx$$

$$= \frac{1}{169} \left(\frac{169}{5}x^5 + \frac{4{,}000}{3}x^3 + 36x - 403x^4 - 372x^2 \right) \Big|_{-1}^{1}$$

$$= \frac{1}{169} \left(\frac{169}{5} + \frac{4{,}000}{3} + 36 - 403 - 372 + \frac{169}{5} + \frac{4{,}000}{3} + 36 + 403 + 372 \right)$$

$$= \frac{1}{169} \left(\frac{338}{5} + \frac{8{,}000}{3} + 72 \right)$$

$$\boxed{= \frac{3{,}238}{195}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\boxed{= \frac{\sqrt{3}}{\sqrt{26}}(x+2)}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$\boxed{= \sqrt{\frac{195}{3238}} \left(x^2 - \frac{62}{13}x + \frac{6}{13} \right)}$$

The *orthonormal* basis is $\left\{ \frac{\sqrt{3}}{\sqrt{26}}(x+2),\quad \sqrt{\frac{195}{3238}} \left(x^2 - \frac{62}{13}x + \frac{6}{13} \right) \right\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $\quad f_1(x) = 1 + 3x^2, \quad f_2(x) = x - x^2$

Solution

Let $\vec{u}_1 = 1 + 3x^2, \quad \vec{u}_2 = x - x^2$

$$\vec{v}_1 = \vec{u}_1 = 1 + 3x^2$$

$$\begin{aligned}
\langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^{1} \left(1 + 3x^2\right)^2 dx \\
&= 2\int_0^1 \left(1 + 6x^2 + 9x^4\right) dx \\
&= 2\left(x + 2x^3 + \frac{9}{5}x^5\right) \Big|_0^1 \\
&= 2\left(1 + 2 + \frac{9}{5}\right) \\
&= \frac{48}{5}
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^{1} \left(1 + 3x^2\right)\left(x - x^2\right) dx \\
&= \int_{-1}^{1} \left(x - x^2 + 3x^3 - 3x^4\right) dx \\
&= \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{3}{4}x^4 - \frac{3}{5}x^5 \Big|_{-1}^{1} \\
&= \frac{1}{2} - \frac{1}{3} + \frac{3}{4} - \frac{3}{5} - \left(\frac{1}{2} + \frac{1}{3} + \frac{3}{4} + \frac{3}{5}\right) \\
&= -\frac{28}{15}
\end{aligned}$$

$$\vec{v}_2 = x - x^2 - \frac{5}{48}\left(-\frac{28}{15}\right)\left(1 + 3x^2\right) \qquad \vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= x - x^2 + \frac{7}{36} + \frac{7}{12}x^2$$

$$= -\frac{5}{12}x^2 + x + \frac{7}{36}$$

The *orthogonal* basis is $\left\{ 1 + 3x^2, \quad -\frac{5}{12}x^2 + x + \frac{7}{36} \right\}$

$$\left\langle \vec{v}_2, \vec{v}_2 \right\rangle = \int_{-1}^{1} \left(-\frac{5}{12}x^2 + x + \frac{7}{36} \right)^2 dx$$

$$= \frac{1}{1,296} \int_{-1}^{1} \left(-15x^2 + 36x + 7 \right)^2 dx$$

$$= \frac{1}{1,296} \int_{-1}^{1} \left(225x^4 - 1,080x^3 - 210x^2 + 504x + 49 \right) dx$$

$$= \frac{1}{1,296} \left(45x^5 - 270x^4 - 70x^3 + 252x^2 + 49x \right|_{-1}^{1}$$

$$= \frac{1}{648} \left(-270x^4 + 252x^2 \right|_{0}^{1}$$

$$= \frac{1}{648} \left(-270 + 252 \right)$$

$$= \frac{12}{648}$$

$$= \frac{1}{54}$$

$$\vec{q}_1 = \frac{\sqrt{5}}{\sqrt{48}} \left(1 + 3x^2 \right) \qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\| \vec{v}_1 \right\|}$$

$$= \frac{\sqrt{15}}{12} \left(1 + 3x^2 \right)$$

$$\vec{q}_2 = 54 \left(-\frac{4}{12}x^2 + x + \frac{7}{36} \right) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\| \vec{v}_2 \right\|}$$

$$= -18x^2 + 54x + \frac{63}{2}$$

The *orthonormal* basis is $\left\{ \frac{\sqrt{15}}{12} \left(1 + 3x^2 \right), \quad -18x^2 + 54x + \frac{63}{2} \right\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $\quad f_1(x) = 5x - 3, \quad f_2(x) = x^3 - x^2$

Solution

Let $\vec{u}_1 = 5x - 3, \quad \vec{u}_2 = x^3 - x^2$

$$\vec{v}_1 = \vec{u}_1 = 5x - 3$$

$$\begin{aligned}\langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^{1} (5x-3)^2\, dx \\ &= \int_{-1}^{1} \left(25x^2 - 30x + 9\right) dx \\ &= \left(\frac{25}{3}x^3 - 15x^2 + 9x \right|_{-1}^{1} \\ &= \frac{25}{3} - 15 + 9 + \frac{25}{3} + 15 + 9 \\ &= \frac{104}{3}\end{aligned}$$

$$\begin{aligned}\langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^{1} \left(x^3 - x^2\right)(5x-3)\; dx \\ &= \int_{-1}^{1} \left(5x^4 - 8x^3 - 3x^2\right) + dx \\ &= x^5 - 2x^4 + x^3 \Big|_{-1}^{1} \\ &= 4\end{aligned}$$

$$\vec{v}_2 = x^3 - x^2 - \frac{3}{104}(4)\left(1 + 3x^2\right) \qquad\qquad \vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$= x^3 - x^2 - \frac{3}{26}\left(1 + 3x^2\right)$$

$$= x^3 - \frac{35}{26}x^2 - \frac{3}{26}$$

The *orthogonal* basis is $\left\{ 1 + 3x^2, \quad x^3 - \frac{35}{26}x^2 - \frac{3}{26} \right\}$

$$\left\langle \vec{v}_2, \vec{v}_2 \right\rangle = \int_{-1}^{1} \left(x^3 - \frac{35}{26}x^2 - \frac{3}{26}\right)^2 dx$$

$$= \frac{1}{676} \int_{-1}^{1} \left(26x^3 - 35x^2 - 3\right)^2 dx$$

$$= \frac{1}{676} \int_{-1}^{1} \left(676x^6 - 1,820x^5 + 1,225x^4 - 156x^3 + 210x^2 + 9\right) dx$$

$$= \frac{1}{676} \left(\frac{676}{7}x^7 - \frac{910}{3}x^6 + 245x^5 - 39x^4 + 70x^3 + 9x \right|_{-1}^{1}$$

$$= \frac{2}{676} \left(\frac{676}{7} + 245 + 70 + 9\right)$$

$$= \frac{1}{338} \left(\frac{2,944}{7}\right)$$

$$= \frac{1,472}{1,183}$$

$$\vec{q}_1 = \frac{3}{104}\left(1 + 3x^2\right) \qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$\vec{q}_2 = \frac{1,472}{1,183}\left(x^3 - \frac{35}{26}x^2 - \frac{3}{26}\right) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

The *orthonormal* basis is $\left\{ \frac{3}{104}\left(1 + 3x^2\right), \ \frac{1,472}{1,183}\left(x^3 - \frac{35}{26}x^2 - \frac{3}{26}\right) \right\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $f_1(x) = 1, \quad f_2(x) = 2x - 1$

Solution

Let $\vec{u}_1 = 1, \quad \vec{u}_2 = 2x - 1$

$$\vec{v}_1 = \vec{u}_1 = 1$$

$$\left\langle \vec{v}_1, \vec{v}_1 \right\rangle = \int_{-1}^{1} (1)^2 dx$$

$$= x \Big|_{-1}^{1}$$

$$\underline{= 2}$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_{-1}^{1} (2x-1)(1)\, dx$$

$$= x^2 - x \Big|_{-1}^{1}$$

$$\underline{= -2}$$

$$\vec{v}_2 = 2x - 1 - \tfrac{1}{2}(-2)(1) \qquad\qquad \vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$\underline{= 2x}$$

The *orthogonal* basis is $\{1, \quad 2x\}$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^{1} (2x)^2\, dx$$

$$= \int_{-1}^{1} 4x^2\, dx$$

$$= \frac{4}{3}x^3 \Big|_{-1}^{1}$$

$$\underline{= \frac{8}{3}}$$

$$\underline{\vec{q}_1 = \frac{1}{\sqrt{2}}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\underline{\vec{q}_2 = \frac{3}{\sqrt{2}}x} \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

The *orthonormal* basis is $\left\{ \frac{1}{2}, \quad \frac{3}{4}x \right\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $\quad f_1(x) = e^x, \quad f_2(x) = x$

Solution

Let $\quad \vec{u}_1 = e^x, \quad \vec{u}_2 = x$

$$\vec{v}_1 = \vec{u}_1 = e^x$$

$$\left\langle \vec{v}_1, \vec{v}_1 \right\rangle = \int_{-1}^{1} e^{2x} dx$$

$$= \frac{1}{2} e^x \Big|_{-1}^{1}$$

$$= \frac{1}{2}\left(e - e^{-1}\right)$$

$$\left\langle \vec{u}_2, \vec{v}_1 \right\rangle = \int_{-1}^{1} x\, e^x\, dx$$

$$= e^x (x-1) \Big|_{-1}^{1}$$

$$= (0)e + 2e$$

$$= 2e$$

$$\vec{v}_2 = x - \frac{2}{e - e^{-1}}(2e)\left(e^x\right) \qquad\qquad \vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1$$

$$= x - \frac{4e^2}{e^2 - 1} e^x$$

The orthogonal basis is $\left\{ e^x, \; x - \dfrac{4e^2}{e^2-1} e^x \right\}$

$$\left\langle \vec{v}_2, \vec{v}_2 \right\rangle = \int_{-1}^{1} \left(x - \frac{4e^2}{e^2-1} e^x \right)^2 dx$$

$$= \int_{-1}^{1} \left(x^2 - \frac{8e^2}{e^2-1} x e^x + \left(\frac{4e^2}{e^2-1} \right)^2 e^{2x} \right) dx$$

$$=\frac{1}{3}x^3-\frac{8e^2}{e^2-1}(x-1)e^x+\frac{8e^4}{\left(e^2-1\right)^2}e^{2x}\Big|_{-1}^{1}$$

$$=\frac{1}{3}+\frac{8e^4}{\left(e^2-1\right)^2}e^2+\frac{1}{3}+\frac{16e^2}{e^2-1}e^{-1}+\frac{8e^4}{\left(e^2-1\right)^2}e^{-2}$$

$$=\frac{2}{3}+\frac{8e^6}{\left(e^2-1\right)^2}+\frac{16e}{e^2-1}+\frac{8e^2}{\left(e^2-1\right)^2}$$

$$=\frac{1}{\left(e^2-1\right)^2}\left(\frac{2}{3}e^4-\frac{4}{3}e^2+\frac{4}{3}+8e^6+16e^3-16e+8e^2\right)$$

$$=\frac{1}{\left(e^2-1\right)^2}\left(8e^6+\frac{2}{3}e^4+16e^3+\frac{20}{3}e^2-16e+\frac{4}{3}\right)$$

$$\vec{q}_1=\sqrt{\frac{2}{e-e^{-1}}}\,e^x \qquad \vec{q}_1=\frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$\vec{q}_2=\frac{e^2-1}{8e^6+\frac{2}{3}e^4+16e^3+\frac{20}{3}e^2-16e+\frac{4}{3}}\left(x-\frac{4e^2}{e^2-1}e^x\right) \qquad \vec{q}_2=\frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

The *orthonormal* basis is

$$\left\{\sqrt{\frac{2}{e-e^{-1}}}\,e^x,\quad \frac{e^2-1}{8e^6+\frac{2}{3}e^4+16e^3+\frac{20}{3}e^2-16e+\frac{4}{3}}\left(x-\frac{4e^2}{e^2-1}e^x\right)\right\}$$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1,\ 1]$ spanned by the functions, using the inner product $\quad f_1(x)=x,\quad f_2(x)=x^3,\quad f_3(x)=x^5$

Solution

Let $\vec{u}_1=f_1=x,\quad \vec{u}_2=f_2=x^3,\quad \vec{u}_3=f_3=x^5$

$$\vec{v}_1=\vec{u}_1=x$$

$$\left\langle \vec{v}_1,\ \vec{v}_1\right\rangle=\int_{-1}^{1}x^2dx$$

$$= \frac{1}{3}x^3 \Big|_{-1}^{1}$$

$$= \frac{2}{3}$$

$$\left\langle \vec{u}_2, \vec{v}_1 \right\rangle = \int_{-1}^{1} x^4 \, dx$$

$$= \frac{1}{5}x^5 \Big|_{-1}^{1}$$

$$= \frac{2}{5}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1$$

$$= x^3 - \frac{2}{5}\left(\frac{3}{2}\right)(x)$$

$$= x^3 - \frac{3}{5}x$$

$$\left\langle \vec{v}_2, \vec{v}_2 \right\rangle = \int_{-1}^{1} \left(x^3 - \frac{3}{5}x\right)^2 dx$$

$$= \int_{-1}^{1} \left(x^6 - \frac{6}{5}x^4 + \frac{9}{25}x^2\right) dx$$

$$= \left(\frac{1}{7}x^7 - \frac{6}{25}x^5 + \frac{3}{25}x^3\right) \Big|_{-1}^{1}$$

$$= 2\left(\frac{1}{7} - \frac{6}{25} + \frac{3}{25}\right)$$

$$= \frac{8}{175}$$

$$\left\langle \vec{u}_3, \vec{v}_1 \right\rangle = \int_{-1}^{1} x^6 \, dx$$

$$= \frac{1}{7}x^7 \Big|_{-1}^{1}$$

$$= \frac{2}{7}$$

$$\left\langle \vec{u}_3, \vec{v}_2 \right\rangle = \int_{-1}^{1} x^5 \left(x^3 - \frac{3}{5}x\right) dx$$

$$= \int_{-1}^{1} \left(x^8 - \frac{3}{5}x^6\right) dx$$

$$= \left(\frac{1}{9}x^9 - \frac{3}{35}x^7\right)\Big|_{-1}^{1}$$

$$= 2\left(\frac{1}{9} - \frac{3}{35}\right)$$

$$= \frac{16}{315}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= x^5 - \frac{16}{315}\left(\frac{175}{8}\right)\left(x^3 - \frac{3}{5}x\right) - \frac{2}{7}\left(\frac{3}{2}\right)x$$

$$= x^5 - \frac{70}{63}\left(x^3 - \frac{3}{5}x\right) - \frac{3}{7}x$$

$$= x^5 - \frac{70}{63}x^3 + \frac{14}{21}x - \frac{3}{7}x$$

$$= x^5 - \frac{70}{63}x^3 + \frac{5}{21}x$$

The orthogonal basis is $\left\{ x,\ \ x^3 - \frac{3}{5}x\ ,\ \ x^5 - \frac{70}{63}x^3 + \frac{5}{21}x \right\}$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_{-1}^{1} \left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x\right)^2 dx$$

$$= \int_{-1}^{1} \frac{1}{3,969}\left(63x^5 - 70x^3 + 15x\right)^2 dx$$

$$= \frac{1}{3,969}\int_{-1}^{1} \left(3,969x^{10} - 8,820x^8 + 1,890x^6 - 2,100x^4 + 4,900x^6 + 225x^2\right) dx$$

$$= \frac{1}{3,969}\left(\frac{3,969}{11}x^{11} - 980x^9 + 970x^7 - 420x^5 + 75x^3\right)\Big|_{-1}^{1}$$

$$= \frac{2}{3,969}\left(\frac{3,969}{11} - 980 + 970 - 420 + 75\right)$$

$$= \frac{2}{3,969}\left(\frac{3,969}{11} - 355\right)$$

$$= \frac{2}{3,969}\left(\frac{64}{11}\right)$$

$$= \frac{128}{43,659}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$
$$= \frac{x}{\sqrt{2/3}}$$
$$\boxed{= \frac{\sqrt{3}}{\sqrt{2}}x}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$
$$= \sqrt{\frac{175}{8}}\left(x^3 - \frac{3}{5}x\right)$$
$$\boxed{= \frac{5\sqrt{7}}{2\sqrt{2}}\left(x^3 - \frac{3}{5}x\right)}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$
$$= \sqrt{\frac{43,659}{128}}\left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x\right)$$
$$\boxed{= \frac{63\sqrt{11}}{8\sqrt{2}}\left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x\right)}$$

The *orthonormal* basis is $\left\{ \frac{\sqrt{3}}{\sqrt{2}}x,\ \ \frac{5}{2}\sqrt{\frac{7}{2}}\left(x^3 - \frac{3}{5}x\right),\ \ \frac{63}{8}\sqrt{\frac{11}{2}}\left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x\right) \right\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $f_1(x) = 1,\ \ f_2(x) = x,\ \ f_3(x) = \frac{1}{2}\left(3x^2 - 1\right)$

Solution

Let $\vec{u}_1 = f_1 = 1,\ \ \vec{u}_2 = f_2 = x,\ \ \vec{u}_3 = f_3 = \frac{3}{2}x^2 - \frac{1}{2}$

$$\boxed{\vec{v}_1 = \vec{u}_1 = 1}$$

$$\left\langle \vec{v}_1, \vec{v}_1 \right\rangle = \int_{-1}^{1} 1\, dx$$
$$= x \Big|_{-1}^{1}$$

$$\underline{= 2}$$

$$\left\langle \vec{u}_2, \vec{v}_1 \right\rangle = \int_{-1}^{1} x \, dx$$

$$= \frac{1}{2} x^2 \Big|_{-1}^{1}$$

$$\underline{= 0}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1$$

$$= x - \frac{0}{2}(1)$$

$$\underline{= x}$$

$$\left\langle \vec{v}_2, \vec{v}_2 \right\rangle = \int_{-1}^{1} x^2 \, dx$$

$$= \frac{1}{3} x^3 \Big|_{-1}^{1}$$

$$\underline{= \frac{2}{3}}$$

$$\left\langle \vec{u}_3, \vec{v}_1 \right\rangle = \frac{1}{2} \int_{-1}^{1} \left(3x^2 - 1\right) dx$$

$$= \frac{1}{2}\left(x^3 - x\right) \Big|_{-1}^{1}$$

$$\underline{= 0}$$

$$\left\langle \vec{u}_3, \vec{v}_2 \right\rangle = \frac{1}{2} \int_{-1}^{1} x\left(3x^2 - 1\right) dx$$

$$= \frac{1}{2} \int_{-1}^{1} \left(3x^3 - x\right) dx$$

$$= \frac{1}{2}\left(\frac{3}{4} x^4 - \frac{1}{2} x^2\right) \Big|_{-1}^{1}$$

$$\underline{= 0}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= \frac{3}{2}x^2 - \frac{1}{2} - \frac{0}{1}(1) - \frac{0}{\frac{2}{3}}(x)$$

$$\boxed{= \frac{1}{2}\left(3x^2 - 1\right)}$$

The *orthogonal* basis is $\left\{1,\ x\ ,\ \frac{1}{2}\left(3x^2 - 1\right)\right\}$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \frac{1}{4}\int_{-1}^{1}\left(3x^2 - 1\right)^2 dx$$

$$= \frac{1}{4}\int_{-1}^{1}\left(9x^4 - 6x^2 + 1\right) dx$$

$$= \frac{1}{4}\left(\frac{9}{5}x^5 - 2x^3 + x \right|_{-1}^{1}$$

$$= \frac{1}{2}\left(\frac{9}{5} - 2 + 1\right)$$

$$\boxed{= \frac{2}{5}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\boxed{= \frac{1}{\sqrt{2}}}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$\boxed{= \sqrt{\frac{3}{2}}\,x}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\sqrt{\frac{2}{5}}}\frac{1}{2}\left(3x^2 - 1\right)$$

$$\boxed{= \frac{1}{2}\sqrt{\frac{5}{2}}\left(3x^2 - 1\right)}$$

The *orthonormal* basis is $\left\{\frac{1}{\sqrt{2}},\ \sqrt{\frac{3}{2}}x\ ,\ \frac{1}{2}\sqrt{\frac{5}{2}}\left(3x^2 - 1\right)\right\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1,\ 1]$ spanned by the functions, using the inner product $\quad f_1(x)=5,\quad f_2(x)=x^2-6x,\quad f_3(x)=(3-x)^2$

Solution

Let $\quad \vec{u}_1=5,\quad \vec{u}_2=x^2-6x,\quad \vec{u}_3=x^2-6x+9$

$$\boxed{\vec{v}_1=\vec{u}_1=5}$$

$$\left\langle \vec{v}_1,\ \vec{v}_1\right\rangle=\int_{-1}^{1}5\ dx$$

$$=5x\Big|_{-1}^{1}$$

$$\boxed{=10}$$

$$\left\langle \vec{u}_2,\ \vec{v}_1\right\rangle=5\int_{-1}^{1}\left(x^2-6x\right)dx$$

$$=5\left(\tfrac{1}{3}x^3-3x^2\ \Big|_{-1}^{1}\right.$$

$$\boxed{=\frac{10}{3}}$$

$$\vec{v}_2=x^2-6x+9-\tfrac{1}{100}\left(\tfrac{10}{3}\right)(5) \qquad\qquad \vec{v}_2=\vec{u}_2-\frac{\left\langle \vec{u}_2,\ \vec{v}_1\right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$=x^2-6x+9-\tfrac{1}{6}$$

$$\boxed{=x^2-6x+\frac{53}{6}}$$

$$\left\langle \vec{v}_2,\ \vec{v}_2\right\rangle=\frac{1}{36}\int_{-1}^{1}\left(6x^2-36x+53\right)^2\ dx$$

$$=\frac{1}{36}\int_{-1}^{1}\left(36x^4-432x^3+1{,}932x^2-3{,}816x+2{,}809\right)dx$$

$$=\frac{1}{36}\left(\frac{36}{5}x^5-108^4+644x^3-1{,}908x^2+2{,}809x\ \Big|_{-1}^{1}\right.$$

$$
\begin{aligned}
&= \frac{1}{18}\left(\frac{36}{5}x^5 + 644x^3 + 2,809x \right|_{0}^{1} \\
&= \frac{1}{18}\left(\frac{36}{5} + 644 + 2,809\right) \\
&= \frac{17,301}{90} \\
&= \boxed{\frac{5,767}{30}}
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \vec{u}_3, \vec{v}_1 \right\rangle &= 5\int_{-1}^{1} \left(x^2 - 6x + 9\right)\, dx \\
&= 5\left(\frac{1}{3}x^3 - 3x^2 + 9x \right|_{-1}^{1} \\
&= 10\left(\frac{1}{3}x^3 + 9x \right|_{0}^{1} \\
&= 10\left(\frac{1}{3} + 9\right) \\
&= \boxed{\frac{280}{3}}
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \vec{u}_3, \vec{v}_2 \right\rangle &= \int_{-1}^{1} \left(x^2 - 6x + 9\right)\left(x^2 - 6x + \frac{53}{6}\right)\, dx \\
&= \int_{-1}^{1} \left(x^4 - 12x^3 + \frac{323}{6}x^2 - 107x + \frac{159}{2}\right)\, dx \\
&= \left(\frac{1}{5}x^5 - 3x^4 + \frac{323}{18}x^3 - \frac{107}{2}x^2 + \frac{159}{2}x \right|_{-1}^{1} \\
&= 2\left(\frac{1}{5}x^5 + \frac{323}{18}x^3 + \frac{159}{2}x \right|_{0}^{1} \\
&= 2\left(\frac{1}{5} + \frac{323}{18} + \frac{159}{2}\right) \\
&= 2\left(\frac{18 + 1,615 + 7,155}{90}\right) \\
&= \boxed{\frac{8,788}{45}}
\end{aligned}
$$

$$
\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\left\langle \vec{u}_3, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2}\vec{v}_1 - \frac{\left\langle \vec{u}_3, \vec{v}_2 \right\rangle}{\left\| \vec{v}_2 \right\|^2}\vec{v}_2 \\
&= x^2 - 6x + 9 - \left(\frac{280}{3}\right)\left(\frac{1}{10}\right)(5) - \left(\frac{8,788}{45}\right)\left(\frac{30}{5,767}\right)\left(x^2 - 6x + \frac{53}{6}\right)
\end{aligned}
$$

$$= x^2 - 6x + 9 - \frac{140}{3} - \frac{17{,}576}{17{,}301}\left(x^2 - 6x + \frac{53}{6}\right)$$

$$= x^2 - 6x - \frac{113}{3} - \frac{17{,}576}{17{,}301}x^2 - \frac{35{,}152}{5{,}767}x + \frac{465{,}764}{51{,}903}$$

$$= -\frac{275}{17{,}301}x^2 - \frac{69{,}754}{5{,}767}x - \frac{1{,}489{,}249}{51{,}903}$$

The *orthogonal* basis is $\left\{ 5,\ x^2 - 6x + \frac{53}{6},\ -\frac{275}{17{,}301}x^2 - \frac{69{,}754}{5{,}767}x - \frac{1{,}489{,}249}{51{,}903} \right\}$

$$\left\langle \vec{v}_3, \vec{v}_3 \right\rangle = \int_{-1}^{1} \left(-\frac{275}{17{,}301}x^2 - \frac{69{,}754}{5{,}767}x - \frac{1{,}489{,}249}{51{,}903}\right)^2 dx$$

$$= \frac{4{,}700{,}107{,}122{,}016}{2{,}693{,}921{,}409}$$

$$\vec{q}_1 = \frac{5}{\sqrt{10}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$\vec{q}_2 = \sqrt{\frac{60}{5{,}767}}\left(x^2 - 6x + \frac{53}{6}\right) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$\vec{q}_3 = \frac{2{,}693{,}921{,}409}{4{,}700{,}107{,}122{,}016}\left(-\frac{275}{17{,}301}x^2 - \frac{69{,}754}{5{,}767}x - \frac{1{,}489{,}249}{51{,}903}\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$= \frac{1}{\sqrt{\frac{2}{5}}}\frac{1}{2}\left(3x^2 - 1\right)$$

$$= \frac{1}{2}\sqrt{\frac{5}{2}}\left(3x^2 - 1\right)$$

The *orthonormal* basis is

$$\left\{ \frac{5}{\sqrt{10}},\ \sqrt{\frac{60}{5{,}767}}\left(x^2 - 6x + \frac{53}{6}\right),\ \frac{2{,}693{,}921{,}409}{4{,}700{,}107{,}122{,}016}\left(-\frac{275}{17{,}301}x^2 - \frac{69{,}754}{5{,}767}x - \frac{1{,}489{,}249}{51{,}903}\right) \right\}$$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1,\ 1]$ spanned by the functions, using the inner product $f_1(x) = 1,\quad f_2(x) = \sin \pi x,\quad f_3(x) = \cos \pi x$

Solution

Let $\vec{u}_1 = 1,\quad \vec{u}_2 = \sin \pi x,\quad \vec{u}_3 = \cos \pi x$

$$\underline{\vec{v}_1 = \vec{u}_1 = 1}$$

$$\left\langle \vec{v}_1, \vec{v}_1 \right\rangle = \int_{-1}^{1} 1\, dx$$

$$= x \Big|_{-1}^{1}$$

$$\underline{= 2}$$

$$\left\langle \vec{u}_2, \vec{v}_1 \right\rangle = \int_{-1}^{1} \sin \pi x\, dx$$

$$= -\frac{1}{\pi} \cos \pi x \Big|_{-1}^{1}$$

$$\underline{= 0}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1$$

$$\underline{= \sin \pi x}$$

$$\left\langle \vec{v}_2, \vec{v}_2 \right\rangle = \int_{-1}^{1} \sin^2 \pi x\, dx$$

$$= \frac{1}{2} \int_{-1}^{1} \left(1 - \cos 2\pi x\right) dx$$

$$= \frac{1}{2} \left(x - \frac{1}{2\pi} \sin 2\pi x \right|_{-1}^{1}$$

$$\underline{= 1}$$

$$\left\langle \vec{u}_3, \vec{v}_1 \right\rangle = \int_{-1}^{1} \cos \pi x\, dx$$

$$= \frac{1}{\pi} \sin \pi x \Big|_{-1}^{1}$$

$$\underline{= 0}$$

$$\left\langle \vec{u}_3, \vec{v}_2 \right\rangle = \int_{-1}^{1} \cos \pi x \sin \pi x\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} \sin 2\pi x \, dx$$

$$= -\frac{1}{4\pi}\cos 2\pi x \Big|_{-1}^{1}$$

$$\underline{= 0 \,|}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= \cos \pi x - \underline{0} - \underline{0}$$

$$\underline{= \cos \pi x \,|}$$

The *orthogonal* basis is $\left\{1, \ \sin \pi x - \frac{1}{\pi}, \ \cos \pi x \right\}$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_{-1}^{1} \cos^2 \pi x \, dx$$

$$= \frac{1}{2}\int_{-1}^{1} (1 + \cos 2\pi x) \, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{2\pi}\sin 2\pi x \right|_{-1}^{1}$$

$$\underline{= 1 \,|}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\underline{= \frac{1}{\sqrt{2}} \,|}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$\underline{= \sin \pi x \,|}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$\underline{= \cos \pi x \,|}$$

The *orthonormal* basis is $\left\{\frac{1}{\sqrt{2}}, \ \sin \pi x, \ \cos \pi x \right\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1,\ 1]$ spanned by the functions, using the inner product $f_1(x)=1,\quad f_2(x)=\sin x,\quad f_3(x)=\sin 2x$

Solution

Let $\vec{u}_1=1,\quad \vec{u}_2=\sin x,\quad \vec{u}_3=\sin 2x$

$$\underline{\vec{v}_1=\vec{u}_1=1}$$

$$\begin{aligned}\left\langle \vec{v}_1,\ \vec{v}_1\right\rangle &= \int_{-1}^{1} 1\,dx\\ &= x\Big|_{-1}^{1}\\ &= 2\end{aligned}$$

$$\begin{aligned}\left\langle \vec{u}_2,\ \vec{v}_1\right\rangle &= \int_{-1}^{1} \sin x\ dx\\ &= -\cos x\Big|_{-1}^{1}\\ &= -\left(\cos 1-\cos(-1)\right)\\ &= -\left(\cos 1-\cos 1\right)\\ &= 0\end{aligned}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2-\frac{\left\langle \vec{u}_2,\ \vec{v}_1\right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1\\ &= \sin x-(0)\\ &= \sin x\end{aligned}$$

$$\begin{aligned}\left\langle \vec{v}_2,\ \vec{v}_2\right\rangle &= \int_{-1}^{1} \sin^2 x\ dx\\ &= \frac{1}{2}\int_{-1}^{1} (1-\cos 2x)\ dx\\ &= \frac{1}{2}\left(x-\frac{1}{2}\sin 2x\right)\Big|_{-1}^{1}\\ &= \frac{1}{2}\left(1-\frac{1}{2}\sin 2+1-\frac{1}{2}\sin 2\right)\end{aligned}$$

$$= 1 - \frac{1}{2}\sin 2$$

$$\boxed{= 1 - \sin(1)\cos(1)}$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_{-1}^{1} \sin 2x \, dx$$

$$= -\frac{1}{2}\cos 2x \Big|_{-1}^{1}$$

$$= -\frac{1}{2}(\cos 2 - \cos 2)$$

$$\boxed{= 0}$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^{1} \sin 2x \, \sin x \, dx$$

$$= 2\int_{-1}^{1} \sin x \, \cos x \, \sin x \, dx$$

$$= 2\int_{-1}^{1} \sin^2 x \, d(\sin x)$$

$$= \frac{2}{3}\sin^3 x \Big|_{-1}^{1}$$

$$= \frac{2}{3}\left(\sin^3 1 + \sin^3 1\right)$$

$$\boxed{= \frac{4}{3}\sin^3 1}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= \sin 2x - (0) - \frac{4}{3}\sin^3(1)\frac{1}{1-\sin(1)\cos(1)}\sin x$$

$$\boxed{= \sin 2x - \frac{4}{3}\,\frac{\sin^3(1)}{1-\sin(1)\cos(1)}\,\sin x}$$

The *orthogonal* basis is $\left\{1, \quad \sin x \,, \quad \sin 2x - \frac{4}{3}\,\frac{\sin^3(1)}{1-\sin(1)\cos(1)}\,\sin x \right\}$

$$\left\langle \vec{v}_3, \vec{v}_3 \right\rangle = \int_{-1}^{1} \left(\sin 2x - \frac{4\sin^3(1)}{3-3\sin(1)\cos(1)} \sin x \right)^2 dx$$

$$= \int_{-1}^{1} \left(\sin^2 2x - \frac{8\sin^3(1)}{3-3\sin(1)\cos(1)} \sin 2x \sin x + \frac{16\sin^6(1)}{\left(3-3\sin(1)\cos(1)\right)^2} \sin^2 x \right) dx$$

$$= \int_{-1}^{1} \left(\frac{1}{2} - \frac{1}{2}\cos 4x \right) dx \; - \frac{16\sin^3(1)}{3-3\sin(1)\cos(1)} \int_{-1}^{1} \sin^2 x \, d(\sin x)$$

$$+ \frac{8\sin^6(1)}{\left(3-3\sin(1)\cos(1)\right)^2} \int_{-1}^{1} \left(1-\cos 2x\right) dx$$

$$= \frac{1}{2}x - \frac{1}{8}\sin 4x - \frac{16\sin^3(1)}{9-9\sin(1)\cos(1)} \sin^3 x + \frac{8\sin^6(1)}{\left(3-3\sin(1)\cos(1)\right)^2}\left(x - \frac{1}{2}\sin 2x\right) \Bigg|_{-1}^{1}$$

$$= \frac{1}{2} - \frac{1}{8}\sin 4 - \frac{16\sin^6 1}{9-9\sin 1\cos 1} + \frac{8\sin^6 1}{\left(3-3\sin 1\cos 1\right)^2}\left(1 - \frac{1}{2}\sin 2\right)$$

$$-\left(-\frac{1}{2} + \frac{1}{8}\sin 4 + \frac{16\sin^6 1}{9-9\sin 1\cos 1} + \frac{8\sin^6 1}{\left(3-3\sin 1\cos 1\right)^2}\left(-1 + \frac{1}{2}\sin 2\right) \right)$$

$$= \frac{1}{2} - \frac{1}{8}\sin 4 - \frac{16\sin^6 1}{9-9\sin 1\cos 1} + \frac{8\sin^6 1}{\left(3-3\sin 1\cos 1\right)^2} - \frac{4\sin^6 1\sin 2}{\left(3-3\sin 1\cos 1\right)^2}$$

$$+ \frac{1}{2} - \frac{1}{8}\sin 4 - \frac{16\sin^6 1}{9-9\sin 1\cos 1} + \frac{8\sin^6 1}{\left(3-3\sin 1\cos 1\right)^2} - \frac{4\sin^6 1\sin 2}{\left(3-3\sin 1\cos 1\right)^2}$$

$$\boxed{= 1 - \frac{1}{4}\sin 4 - \frac{32\sin^6 1}{9-9\sin 1\cos 1} + \frac{16\sin^6 1}{\left(3-3\sin 1\cos 1\right)^2} - \frac{8\sin^6 1\sin 2}{\left(3-3\sin 1\cos 1\right)^2}}$$

$$\boxed{\vec{q}_1 = \frac{1}{\sqrt{2}}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$\boxed{\vec{q}_2 = \frac{\sin x}{\sqrt{1-\sin(1)\cos(1)}}} \qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$\vec{q}_3 = \frac{\sin 2x - \frac{4}{3}\,\frac{\sin^3(1)}{1-\sin(1)\cos(1)}\,\sin x}{\sqrt{1 - \frac{1}{4}\sin 4 - \frac{32\sin^6 1}{9-9\sin 1\cos 1} + \frac{16\sin^6 1}{\left(3-3\sin 1\cos 1\right)^2} - \frac{8\sin^6 1\sin 2}{\left(3-3\sin 1\cos 1\right)^2}}} \qquad \vec{q}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

The *orthonormal* basis is

$$\left\{ \frac{1}{\sqrt{2}},\quad \frac{\sin x}{\sqrt{1-\sin(1)\cos(1)}},\quad \frac{\sin 2x - \frac{4}{3}\frac{\sin^3(1)}{1-\sin(1)\cos(1)}\sin x}{\sqrt{1-\frac{1}{4}\sin 4 - \frac{32\sin^6 1}{9-9\sin 1\cos 1} + \frac{16\sin^6 1}{(3-3\sin 1\cos 1)^2} - \frac{8\sin^6 1\sin 2}{(3-3\sin 1\cos 1)^2}}} \right\}$$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $f_1(x)=6,\quad f_2(x)=3\sin^2 x,\quad f_3(x)=2\cos^2 x$

Solution

Let $\vec{u}_1 = 6,\quad \vec{u}_2 = 3\sin^2 x,\quad \vec{u}_3 = 2\cos^2 x$

$$\vec{v}_1 = \vec{u}_1 = 6$$

$$\left\langle \vec{v}_1, \vec{v}_1 \right\rangle = \int_{-1}^{1} 6\,dx$$

$$= 12$$

$$\left\langle \vec{u}_2, \vec{v}_1 \right\rangle = 18\int_{-1}^{1} \sin^2 x\, dx$$

$$= 9\int_{-1}^{1} (1-\cos 2x)\, dx$$

$$= 9\left(x - \frac{1}{2}\sin 2x\right. \Big|_{-1}^{1}$$

$$= 9\left(1-\frac{1}{2}\sin 2 + 1 - \frac{1}{2}\sin 2\right)$$

$$= 18 - 9\sin 2$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2}\vec{v}_1$$

$$= 3\sin^2 x - \frac{1}{12}(18-9\sin 2)(6)$$

$$\underline{= 3\sin^2 x + \frac{9}{2}\sin 2 - 9}\Big|$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^{1} \left(3\sin^2 x + \frac{9}{2}\sin 2 - 9\right)^2 dx$$

$$= \int_{-1}^{1} \left(9\sin^4 x + 6\left(\frac{9}{2}\sin 2 - 9\right)\sin^2 x + \left(\frac{9}{2}\sin 2 - 9\right)^2\right) dx$$

$$= \int_{-1}^{1} \left(9\sin^4 x + 6\left(\frac{9}{2}\sin 2 - 9\right)\sin^2 x + \left(\frac{9}{2}\sin 2 - 9\right)^2\right) dx$$

$$9\int_{-1}^{1} \sin^4 x\, dx = \frac{9}{4}\int_{-1}^{1} (1 - \cos 2x)^2\, dx$$

$$= \frac{9}{4}\int_{-1}^{1} \left(1 - 2\cos 2x + \cos^2 2x\right) dx$$

$$= \frac{9}{4}\int_{-1}^{1} \left(1 - 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{9}{4}\int_{-1}^{1} \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{9}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\ \Big|_{-1}^{1}\right.$$

$$= \frac{9}{4}\left(3 - 2\sin 2 + \frac{1}{4}\sin 4\right)$$

$$6\left(\frac{9}{2}\sin 2 - 9\right)\int_{-1}^{1} \sin^2 x\, dx = \left(\frac{27}{2}\sin 2 - 27\right)\int_{-1}^{1} (1 - \cos 2x)\, dx$$

$$= \frac{27}{2}(\sin 2 - 2)\ \left(x - \frac{1}{2}\sin 2x\ \Big|_{-1}^{1}\right.$$

$$= 27(\sin 2 - 2)\left(1 - \frac{1}{2}\sin 2\right)$$

$$\int_{-1}^{1} \left(\frac{9}{2}\sin 2 - 9\right)^2 dx = 2\left(\frac{9}{2}\sin 2 - 9\right)^2$$

$$= \frac{27}{4} - \frac{9}{2}\sin 2 + \frac{9}{16}\sin 4 + 27(\sin 2 - 2)\left(1 - \frac{1}{2}\sin 2\right) + 2\left(\frac{9}{2}\sin 2 - 9\right)^2$$

$$= \frac{27}{4} - \frac{9}{2}\sin 2 + \frac{9}{16}\sin 4 + 54\sin 2 - \frac{27}{2}\sin^2 2 - 54 + \frac{81}{2}\sin^2 2 - 162\sin 2 + 162$$

$$\underline{= \frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2 \,\Big|}$$

$$\left\langle \vec{u}_3,\ \vec{v}_1 \right\rangle = 12 \int_{-1}^{1} \cos^2 x \, dx$$

$$= 6 \int_{-1}^{1} (1 + \cos 2x) \, dx$$

$$= 12\left(x + \frac{1}{2}\sin 2x \right|_0^1$$

$$\underline{= 12 + 6\sin 2 \,\big|}$$

$$\left\langle \vec{u}_3,\ \vec{v}_2 \right\rangle = \int_{-1}^{1} \left(3\sin^2 x + \frac{9}{2}\sin 2 - 9\right)\left(2\cos^2 x\right) dx$$

$$= 3 \int_{-1}^{1} \left(2\sin^2 x\cos^2 x + (3\sin 2 - 6)\cos^2 x\right) dx$$

$$\int_{-1}^{1} 2\sin^2 x\cos^2 x \, dx = \frac{1}{2}\int_{-1}^{1} (1 - \cos 2x)(1 + \cos 2x) \, dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(1 - \cos^2 2x\right) dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{2}\left(x - \frac{1}{4}\sin 4x \right|_0^1$$

$$= \frac{1}{2}(1 - \sin 4)$$

$$(3\sin 2 - 6)\int_{-1}^{1} \cos^2 x \, dx = \frac{1}{2}(3\sin 2 - 6)\int_{-1}^{1} (1 + \cos 2x) \, dx$$

$$= (3\sin 2 - 6)\left(x + \frac{1}{2}\sin 2x \right|_0^1$$

$$= (3\sin 2 - 6)\left(1 + \frac{1}{2}\sin 2\right)$$

$$= \frac{3}{2}\sin^2 2 - 6$$

$$= \frac{1}{2} - \frac{1}{2}\sin 4 + \frac{3}{2}\sin^2 2 - 6$$

$$\boxed{= \frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2}}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\left\langle \vec{u}_3, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2}\vec{v}_1 - \frac{\left\langle \vec{u}_3, \vec{v}_2 \right\rangle}{\left\| \vec{v}_2 \right\|^2}\vec{v}_2$$

$$= 2\cos^2 x - \left(\frac{1}{12}\right)(12 + 6\sin 2)(6) - \frac{\left(3\sin^2 x + \frac{9}{2}\sin 2 - 9\right)\left(\frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2}\right)}{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2}$$

$$\boxed{= 2\cos^2 x - 6 - 3\sin 2 - \frac{\left(\frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2}\right)}{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2}\left(3\sin^2 x + \frac{9}{2}\sin 2 - 9\right)}$$

The *orthogonal* basis is

$$\left\{ \begin{array}{l} 6, \quad 3\sin^2 x + \frac{9}{2}\sin 2 - 9, \\ 2\cos^2 x - 6 - 3\sin 2 - \frac{\left(\frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2}\right)}{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2}\left(3\sin^2 x + \frac{9}{2}\sin 2 - 9\right) \end{array} \right\}$$

$$\vec{v}_3 = 2\cos^2 x - \frac{3\left(\frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2}\right)}{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2}\sin^2 x$$
$$- \left(\frac{\left(\frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2}\right)\left(\frac{9}{2}\sin 2 - 9\right)}{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2} + 6 + 3\sin 2 \right)$$

$$2\int_{-1}^{1} \cos^2 x \, dx = \int_{-1}^{1} (1 + \cos 2x) \, dx$$

$$= x + \frac{1}{2}\sin 2x \Big|_{-1}^{1}$$

$$\boxed{= 2 + \sin 2}$$

$$\int_{-1}^{1} \sin^2 x \, dx = \frac{1}{2}\int_{-1}^{1} (1 - \cos 2x) \, dx$$

$$= \left(x - \frac{1}{2}\sin 2x \right|_{0}^{1}$$

$$= 1 - \frac{1}{2}\sin 2$$

$$\int_{-1}^{1} dx = 2$$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = 2 + \sin 2 - \frac{3\left(\frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2}\right)\left(1 - \frac{1}{2}\sin 2\right)}{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2} - 2\left(\frac{\left(\frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2}\right)\left(\frac{9}{2}\sin 2 - 9\right)}{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2} + 6 + 3\sin 2\right)$$

$$\vec{q}_1 = \frac{6}{\sqrt{12}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{3}{\sqrt{3}}$$

$$\vec{q}_2 = \frac{3\sin^2 x + \frac{9}{2}\sin 2 - 9}{\sqrt{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2}} \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{3\sin^2 x + \frac{9}{2}\sin 2 - 9}{\frac{1}{4}\sqrt{1{,}836 - 1{,}800\sin 2 + 9\sin 4 + 432\sin^2 2}}$$

$$= \frac{12\sin^2 x + 18\sin 2 - 36}{\sqrt{1{,}836 - 1{,}800\sin 2 + 9\sin 4 + 432\sin^2 2}}$$

$$\vec{q}_3 = \frac{2\cos^2 x - 6 - 3\sin 2 - \frac{\left(\frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2}\right)}{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2}\left(3\sin^2 x + \frac{9}{2}\sin 2 - 9\right)}{\sqrt{\langle \vec{v}_3, \vec{v}_3 \rangle}} \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

The *orthonormal* basis is

$$\left\{ \frac{3}{\sqrt{3}}, \frac{12\sin^2 x + 18\sin 2 - 36}{\sqrt{1{,}836 - 1{,}800\sin 2 + 9\sin 4 + 432\sin^2 2}}, \frac{2\cos^2 x - 6 - 3\sin 2 - \frac{\left(\frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2}\right)}{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2}\left(3\sin^2 x + \frac{9}{2}\sin 2 - 9\right)}{\sqrt{\langle \vec{v}_3, \vec{v}_3 \rangle}} \right\}$$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1,\ 1]$ spanned by the functions, using the inner product $f_1(x) = \cos 2x, \quad f_2(x) = \sin^2 x, \quad f_3(x) = \cos^2 x$

Solution

Let $\vec{u}_1 = \cos 2x, \quad \vec{u}_2 = \sin^2 x, \quad \vec{u}_3 = \cos^2 x$

$$\vec{v}_1 = \vec{u}_1 = \cos 2x$$

$$\langle \vec{v}_1,\ \vec{v}_1 \rangle = \int_{-1}^{1} \cos 2x\, dx$$

$$= \frac{1}{2}\sin 2x \Big|_{-1}^{1}$$

$$= \sin 2$$

$$\langle \vec{u}_2,\ \vec{v}_1 \rangle = \int_{-1}^{1} \sin^2 x \cos 2x\ dx$$

$$= \frac{1}{2}\int_{-1}^{1} (1 - \cos 2x)\ \cos 2x\ dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(\cos 2x - \cos^2 2x\right)\ dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(\cos 2x - \frac{1}{2} - \frac{1}{2}\cos 4x\right)\ dx$$

$$= \left(\frac{1}{2}\sin 2x - \frac{1}{2}x - \frac{1}{8}\sin 8x \right|_{0}^{1}$$

$$= \frac{1}{2}\sin 2 - \frac{1}{2} - \frac{1}{8}\sin 8$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2,\ \vec{v}_1 \rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$= \sin^2 x - \left(\frac{1}{\sin 2}\right)\left(\frac{1}{2}\sin 2 - \frac{1}{2} - \frac{1}{8}\sin 8\right)(\cos 2x)$$

$$= \sin^2 x - \left(\frac{4\sin 2 - 4 - \sin 8}{8\ \sin 2}\right)\ \cos 2x$$

$$= \frac{1}{2} - \left(\frac{1}{2} + \frac{4\sin 2 - \sin 8 - 4}{8\sin 2}\right)\cos 2x$$

$$\boxed{= \frac{1}{2} - \left(\frac{8\sin 2 - \sin 8 - 4}{8\sin 2}\right)\cos 2x}$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^{1} \left(\sin^2 x - \left(\frac{4\sin 2 - 4 - \sin 8}{8\sin 2}\right)\cos 2x\right)^2 dx$$

$$= \int_{-1}^{1} \left(\sin^4 x - \left(\frac{4\sin 2 - 4 - \sin 8}{4\sin 2}\right)\cos 2x\,\sin^2 x + \frac{1}{64}\left(\frac{4\sin 2 - 4 - \sin 8}{\sin 2}\right)^2 \cos^2 2x\right) dx$$

$$\int_{-1}^{1} \sin^4 x\, dx = \frac{1}{4}\int_{-1}^{1} (1 - \cos 2x)^2\, dx$$

$$= \frac{1}{4}\int_{-1}^{1} \left(1 - 2\cos 2x + \cos^2 2x\right) dx$$

$$= \frac{1}{4}\int_{-1}^{1} \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right.\Bigg|_{-1}^{1}$$

$$\boxed{= \frac{1}{4}\left(3 - 2\sin 2 + \frac{1}{4}\sin 4\right)}$$

$$\int_{-1}^{1} \cos 2x\,\sin^2 x\, dx = \frac{1}{2}\int_{-1}^{1} \cos 2x\,(1 - \cos 2x)\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(\cos 2x - \cos^2 2x\right) dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(\cos 2x - \frac{1}{2} - \frac{1}{2}\cos 4x\right) dx$$

$$= \left(\frac{1}{2}\sin 2x - \frac{1}{2}x - \frac{1}{8}\sin 8x\right.\Bigg|_{0}^{1}$$

$$\boxed{= \frac{1}{2}\sin 2 - \frac{1}{2} - \frac{1}{8}\sin 8}$$

$$\int_{-1}^{1} \cos^2 2x\, dx = \frac{1}{2}\int_{-1}^{1} (1 + \cos 4x)\, dx$$

$$= \left(x + \frac{1}{4}\sin 4x\right|_0^1$$

$$\boxed{= 1 + \frac{1}{4}\sin 4}$$

$$\left\langle \vec{v}_2, \vec{v}_2 \right\rangle = \frac{1}{4}\left(3 - 2\sin 2 + \frac{1}{4}\sin 4\right) - \left(\frac{4\sin 2 - 4 - \sin 8}{4\sin 2}\right)\left(\frac{1}{2}\sin 2 - \frac{1}{2} - \frac{1}{8}\sin 8\right)$$

$$+ \left(\frac{4\sin 2 - 4 - \sin 8}{8\sin 2}\right)^2\left(1 + \frac{1}{4}\sin 4\right)$$

$$= \frac{3}{4} - \frac{1}{2}\sin 2 + \frac{1}{16}\sin 4 - \frac{(4\sin 2 - 4 - \sin 8)^2}{32\sin 2} + \frac{(4 + \sin 4)(4\sin 2 - 4 - \sin 8)^2}{256\sin^2 2}$$

$$= \frac{3}{4} - \frac{1}{2}\sin 2 + \frac{1}{16}\sin 4 + \frac{(4 + \sin 4)(4\sin 2 - 4 - \sin 8)^2 - 8\sin 2(4\sin 2 - 4 - \sin 8)^2}{256\sin^2 2}$$

$$\boxed{= \frac{3}{4} - \frac{1}{2}\sin 2 + \frac{1}{16}\sin 4 + \frac{(4 - 7\sin 2)(4\sin 2 - 4 - \sin 8)^2}{256\sin^2 2}}$$

$$\left\langle \vec{u}_3, \vec{v}_1 \right\rangle = \int_{-1}^{1} \cos^2 x \cos 2x \, dx$$

$$= \frac{1}{2}\int_{-1}^{1} (1 + \cos 2x) \cos 2x \, dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(\cos 2x + \cos^2 2x\right) dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$$

$$= \left(\frac{1}{2}\sin 2x + \frac{1}{2}x + \frac{1}{8}\sin 8x\right|_0^1$$

$$\boxed{= \frac{1}{2}\sin 2 + \frac{1}{2} + \frac{1}{8}\sin 8}$$

$$\left\langle \vec{u}_3, \vec{v}_2 \right\rangle = \int_{-1}^{1} \left(3\sin^2 x + \frac{9}{2}\sin 2 - 9\right)\left(2\cos^2 x\right) dx$$

$$= 3\int_{-1}^{1} \left(2\sin^2 x\cos^2 x + (3\sin 2 - 6)\cos^2 x\right) dx$$

$$6\int_{-1}^{1}\sin^2 x\cos^2 x\ dx = \frac{3}{2}\int_{-1}^{1}(1-\cos 2x)(1+\cos 2x)\ dx$$

$$= \frac{3}{2}\int_{-1}^{1}\left(1-\cos^2 2x\right)\ dx$$

$$= \frac{3}{2}\int_{-1}^{1}\left(\frac{1}{2}-\frac{1}{2}\cos 4x\right)\ dx$$

$$= \frac{3}{2}\left(x-\frac{1}{4}\sin 4x\ \Big|_0^1\right.$$

$$\boxed{= \frac{3}{2}(1-\sin 4)}$$

$$(9\sin 2-18)\int_{-1}^{1}\cos^2 x\ dx = \frac{1}{2}(9\sin 2-18)\int_{-1}^{1}(1+\cos 2x)\ dx$$

$$= (9\sin 2-18)\left(x+\frac{1}{2}\sin 2x\ \Big|_0^1\right.$$

$$= (9\sin 2-18)\left(1+\frac{1}{2}\sin 2\right)$$

$$\boxed{= \frac{9}{2}\sin^2 2-18}$$

$$\left\langle \vec{u}_3,\ \vec{v}_2\right\rangle = \frac{3}{2}-\frac{3}{2}\sin 4+\frac{9}{2}\sin^2 2-18$$

$$\boxed{= \frac{9}{2}\sin^2 2-\frac{3}{2}\sin 4-\frac{33}{2}}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\left\langle \vec{u}_3,\ \vec{v}_1\right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1 - \frac{\left\langle \vec{u}_3,\ \vec{v}_2\right\rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2$$

$$= \cos^2 x - \frac{4\sin 2+4+\sin 8}{8\sin 2}\cos 2x$$

$$- \frac{72\sin^2 2-24\sin 4-264}{6-8\sin 2+\sin 4+\dfrac{(4-7\sin 2)(4\sin 2-4-\sin 8)^2}{16\sin^2 2}}\left(\sin^2 x-\left(\frac{4\sin 2-4-\sin 8}{8\sin 2}\right)\cos 2x\right)$$

$$= \frac{1}{2} + \frac{1}{2}\cos 2x - \frac{4\sin 2 + 4 + \sin 8}{8\sin 2}\cos 2x$$

$$- \frac{72\sin^2 2 - 24\sin 4 - 264}{6 - 8\sin 2 + \sin 4 + \frac{(4 - 7\sin 2)(4\sin 2 - 4 - \sin 8)^2}{16\sin^2 2}}\left(\frac{1}{2} - \frac{1}{2}\cos 2x - \left(\frac{4\sin 2 - 4 - \sin 8}{8\sin 2}\right)\cos 2x\right)$$

$$= \frac{1}{2} - \frac{4 + \sin 8}{8\sin 2}\cos 2x$$

$$- \frac{128\sin^2 2\left(9\sin^2 2 - 3\sin 4 - 33\right)}{16\left(6 - 8\sin 2 + \sin 4\right)\sin^2 2 + (4 - 7\sin 2)(4\sin 2 - 4 - \sin 8)^2}\left(\frac{1}{2} - \left(\frac{8\sin 2 - 4 - \sin 8}{8\sin 2}\right)\cos 2x\right)$$

The *orthogonal* basis is

$$\left\{ \cos 2x,\ \ \frac{1}{2} - \left(\frac{8\sin 2 - \sin 8 - 4}{8\sin 2}\right)\cos 2x\ , \right\}$$

$$\left\langle \vec{v}_3, \vec{v}_3 \right\rangle =$$

$$\vec{q}_1 = \frac{\cos 2x}{\sin 2} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$\vec{q}_2 = \frac{1}{\sqrt{\frac{3}{4} - \frac{1}{2}\sin 2 + \frac{1}{16}\sin 4 + \frac{(4 - 7\sin 2)(4\sin 2 - 4 - \sin 8)^2}{256\sin^2 2}}}\left(\frac{1}{2} - \left(\frac{8\sin 2 - \sin 8 - 4}{8\sin 2}\right)\cos 2x\right)$$

$$\vec{q}_3 = \frac{\frac{1}{2} - \frac{4+\sin 8}{8\sin 2}\cos 2x - \frac{128\sin^2 2\left(9\sin^2 2 - 3\sin 4 - 33\right)}{16\left(6 - 8\sin 2 + \sin 4\right)\sin^2 2 + (4 - 7\sin 2)(4\sin 2 - 4 - \sin 8)^2}\left(\frac{1}{2} - \left(\frac{8\sin 2 - 4 - \sin 8}{8\sin 2}\right)\cos 2x\right)}{\sqrt{\left\langle \vec{v}_3, \vec{v}_3 \right\rangle}}$$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1,\ 1]$ spanned by the functions, using the inner product $f_1(x) = \sin \pi x$, $\ f_2(x) = \sin 2\pi x$, $\ f_3(x) = \sin 3\pi x$

Solution

Let $\vec{u}_1 = \sin \pi x, \quad \vec{u}_2 = \sin 2\pi x, \quad \vec{u}_3 = \sin 3\pi x$

$$\boxed{\vec{v}_1 = \vec{u}_1 = \sin \pi x}$$

$$\begin{aligned}\left\langle \vec{v}_1, \vec{v}_1 \right\rangle &= \int_{-1}^{1} \sin^2 \pi x \, dx \\ &= \frac{1}{2}\int_{-1}^{1} (1-\cos 2\pi x)\, dx \\ &= \frac{1}{2}\left(x - \frac{1}{2\pi}\sin 2\pi x\right) \Big|_{-1}^{1} \\ &= 1\end{aligned}$$

$$\begin{aligned}\left\langle \vec{u}_2, \vec{v}_1 \right\rangle &= \int_{-1}^{1} \sin \pi x \sin 2\pi x \, dx \qquad \sin a \sin b = \frac{1}{2}\left[\cos(a+b) - \cos(a-b)\right] \\ &= \frac{1}{2}\int_{-1}^{1} \left(\cos 3\pi x - \cos(-\pi x)\right) dx \\ &= \frac{1}{2}\int_{-1}^{1} \left(\cos 3\pi x - \cos \pi x\right) dx \\ &= \frac{1}{2}\left(\frac{1}{3\pi}\sin 3\pi x - \frac{1}{\pi}\sin \pi x\right) \Big|_{-1}^{1} \\ &= 0\end{aligned}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\left\langle \vec{u}_2, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2}\vec{v}_1 \\ &= \sin 2\pi x\end{aligned}$$

$$\begin{aligned}\left\langle \vec{v}_2, \vec{v}_2 \right\rangle &= \int_{-1}^{1} \sin^2 2\pi x \, dx \\ &= \frac{1}{2}\int_{-1}^{1} (1-\cos 4\pi x)\, dx \\ &= \frac{1}{2}\left(x - \frac{1}{4\pi}\sin 4\pi x\right) \Big|_{-1}^{1} \\ &= 1\end{aligned}$$

$$\left\langle \vec{u}_3, \vec{v}_1 \right\rangle = \int_{-1}^{1} \sin \pi x \sin 3\pi x \, dx$$

$$\sin a \sin b = \frac{1}{2}\left[\cos(a+b) - \cos(a-b)\right]$$

$$= \frac{1}{2}\int_{-1}^{1} \left(\cos 4\pi x - \cos(-2\pi x)\right) dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(\cos 4\pi x - \cos 2\pi x\right) dx$$

$$= \frac{1}{2}\left(\frac{1}{4\pi}\sin 4\pi x - \frac{1}{2\pi}\sin 2\pi x\right) \Big|_{-1}^{1}$$

$$\underline{= 0}$$

$$\left\langle \vec{u}_3, \vec{v}_2 \right\rangle = \int_{-1}^{1} \sin 3\pi x \sin 2\pi x \, dx$$

$$\sin a \sin b = \frac{1}{2}\left[\cos(a+b) - \cos(a-b)\right]$$

$$= \frac{1}{2}\int_{-1}^{1} \left(\cos 5\pi x - \cos \pi x\right) dx$$

$$= \frac{1}{2}\left(\frac{1}{5\pi}\sin 5\pi x - \frac{1}{\pi}\sin \pi x\right) \Big|_{-1}^{1}$$

$$\underline{= 0}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\left\langle \vec{u}_3, \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1 - \frac{\left\langle \vec{u}_3, \vec{v}_2 \right\rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2$$

$$\underline{= \sin 3\pi x}$$

The *orthogonal* basis is $\{\sin \pi x, \ \sin 2\pi x, \ \sin 3\pi x\}$

$$\left\langle \vec{v}_3, \vec{v}_3 \right\rangle = \int_{-1}^{1} \sin^2 3\pi x \, dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(1 - \cos 6\pi x\right) dx$$

$$= \frac{1}{2}\left(x - \frac{1}{6\pi}\sin 6\pi x\right) \Big|_{-1}^{1}$$

$$\underline{= 1}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$\underline{= \sin \pi x}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$
$$\boxed{= \sin 2\pi x}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$
$$\boxed{= \sin 3\pi x}$$

The *orthonormal* basis is $\{\sin \pi x, \ \sin 2\pi x, \ \sin 3\pi x\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $f_1(x) = \cos \pi x, \quad f_2(x) = \cos 2\pi x, \quad f_3(x) = \cos 3\pi x$

Solution

Let $\vec{u}_1 = f_1 = \cos \pi x, \quad \vec{u}_2 = f_2 = \cos 2\pi x, \quad \vec{u}_3 = f_3 = \cos 3\pi x$

$$\boxed{\vec{v}_1 = \vec{u}_1 = \cos \pi x}$$

$$\left\langle \vec{v}_1, \vec{v}_1 \right\rangle = \int_{-1}^{1} \cos^2 \pi x \, dx$$
$$= \frac{1}{2}\int_{-1}^{1} (1 + \cos 2\pi x) \, dx$$
$$= \frac{1}{2}\left(x + \frac{1}{2\pi}\sin 2\pi x\right) \Big|_{-1}^{1}$$
$$\boxed{= 1}$$

$$\left\langle \vec{u}_2, \vec{v}_1 \right\rangle = \int_{-1}^{1} \cos 2\pi x \cos \pi x \, dx \qquad \cos a \cos b = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right]$$
$$= \frac{1}{2}\int_{-1}^{1} (\cos 3\pi x + \cos \pi x) \, dx$$
$$= \frac{1}{2}\left(\frac{1}{3\pi}\sin 3\pi x + \frac{1}{\pi}\sin \pi x\right) \Big|_{-1}^{1}$$
$$\boxed{= 0}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= \cos 2\pi x$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^{1} \cos^2 2\pi x \, dx$$

$$= \frac{1}{2}\int_{-1}^{1} (1 + \cos 4\pi x) \, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{4\pi}\sin 4\pi x\right) \Big|_{-1}^{1}$$

$$= 1$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_{-1}^{1} \cos 3\pi x \cos \pi x \, dx$$

$\cos a \cos b = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right]$

$$= \frac{1}{2}\int_{-1}^{1} (\cos 4\pi x + \cos 2\pi x) \, dx$$

$$= \frac{1}{2}\left(\frac{1}{4\pi}\sin 4\pi x + \frac{1}{2\pi}\sin 2\pi x\right) \Big|_{-1}^{1}$$

$$= 0$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^{1} \cos 3\pi x \cos 2\pi x \, dx$$

$\cos a \cos b = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right]$

$$= \frac{1}{2}\int_{-1}^{1} (\cos 5\pi x + \cos \pi x) \, dx$$

$$= \frac{1}{2}\left(\frac{1}{5\pi}\sin 5\pi x + \frac{1}{\pi}\sin \pi x\right) \Big|_{-1}^{1}$$

$$= 0$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= \cos 3\pi x$$

The *orthogonal* basis is $\{\cos \pi x, \ \cos 2\pi x, \ \cos 3\pi x\}$

$$\begin{aligned}\langle \vec{v}_3, \vec{v}_3 \rangle &= \int_{-1}^{1} \cos^2 3\pi x \, dx \\ &= \frac{1}{2}\int_{-1}^{1} (1+\cos 6\pi x) \, dx \\ &= \frac{1}{2}\left(x + \frac{1}{6\pi}\sin 6\pi x\right) \Big|_{-1}^{1} \\ &= 1 \end{aligned}$$

$$\begin{aligned}\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \cos \pi x \end{aligned}$$

$$\begin{aligned}\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \cos 2\pi x \end{aligned}$$

$$\begin{aligned}\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= \cos 3\pi x \end{aligned}$$

The *orthonormal* basis is $\{\cos \pi x, \ \cos 2\pi x, \ \cos 3\pi x\}$

Exercise

For $\mathbb{P}_3[x]$, define the inner product over $\mathbb{R}$ as

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, dx$$

a) If $f(x)=1$ is a unit vector in $\mathbb{P}_3[x]$?

b) Find an orthonormal basis for the subspace spanned by x *and* x^2.

c) Complete the basis in part (*b*) to an orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product.

d) Is

$$[f, g] = \int_{0}^{1} f(x)g(x) \, dx$$

Also, an inner product for $\mathbb{P}_3[x]$

e) Find a pair of vectors $\vec{v}$ and $\vec{w}$ such that

$$\langle \vec{v},\ \vec{w}\rangle = 0 \quad but \quad [\vec{v},\ \vec{w}] \neq 0$$

f) Is the basis found in part (*c*) are orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product in part (*d*)?

Solution

a) $f(x) = 1$

$$\langle f, f\rangle = \int_{-1}^{1} f(x)f(x)\, dx$$

$$= \int_{-1}^{1} dx$$

$$= x \Big|_{-1}^{1}$$

$$= 1 + 1$$

$$\underline{= 2 \neq 1 \,|}$$

Therefore, when $f(x) = 1$ is ***not*** a unit vector in $\mathbb{P}_3[x]$

b) Let $\vec{u}_1 = f = x,\quad \vec{u}_2 = g = x^2$

$$\underline{\vec{v}_1 = \vec{u}_1 = x\,|}$$

$$\langle \vec{v}_1,\ \vec{v}_1\rangle = \int_{-1}^{1} x^2\, dx$$

$$= \frac{1}{3}x^3 \Big|_{-1}^{1}$$

$$= \frac{1}{3}(1+1)$$

$$\underline{= \frac{2}{3}\,|}$$

$$\langle \vec{u}_2,\ \vec{v}_1\rangle = \int_{-1}^{1} x^2(x)\, dx$$

$$= \int_{-1}^{1} x^3\, dx$$

$$= \frac{1}{4}x^4 \Big|_{-1}^{1}$$

$$= \frac{1}{4}(1-1)$$

$$\underline{= 0}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= x^2 - \frac{0}{*}(\ )$$

$$\underline{= x^2}$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^{1} x^4\, dx$$

$$= \frac{1}{5}x^5 \Big|_{-1}^{1}$$

$$= \frac{1}{5}(1+1)$$

$$\underline{= \frac{2}{5}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{x}{\sqrt{\frac{2}{3}}}$$

$$\underline{= \sqrt{\frac{3}{2}}\, x}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{x^2}{\sqrt{\frac{2}{5}}}$$

$$\underline{= \sqrt{\frac{5}{2}}\, x^2}$$

The orthonormal basis is $\left\{ \sqrt{\frac{3}{2}}\, x,\ \sqrt{\frac{5}{2}}\, x^2 \right\}$

c) Since $\vec{u}_1 = x, \quad \vec{u}_2 = x^2$ in $\mathbb{P}_3[x]$

Then, let $\vec{u}_3 = 1$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_{-1}^{1} (1)(x)\, dx$$

$$= \int_{-1}^{1} x\, dx$$

$$= \frac{1}{2}x^2 \Big|_{-1}^{1}$$

$$= \frac{1}{2}(1-1)$$

$$\underline{= 0}$$

$$\left\langle \vec{u}_3, \vec{v}_2 \right\rangle = \int_{-1}^{1} (1)\left(x^2\right) dx$$

$$= \int_{-1}^{1} x^2\, dx$$

$$= \frac{1}{3}x^3 \Big|_{-1}^{1}$$

$$= \frac{1}{3}(1+1)$$

$$\underline{= \frac{2}{3}}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\left\langle \vec{u}_3, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2}\vec{v}_1 - \frac{\left\langle \vec{u}_3, \vec{v}_2 \right\rangle}{\left\| \vec{v}_2 \right\|^2}\vec{v}_2$$

$$= 1 - \frac{0}{\frac{2}{3}}(x) - \frac{\frac{2}{3}}{\frac{2}{5}}\left(x^2\right)$$

$$\underline{= 1 - \frac{5}{3}x^2}$$

$$\left\langle \vec{v}_3, \vec{v}_3 \right\rangle = \int_{-1}^{1} \left(1 - \frac{5}{3}x^2\right)^2 dx$$

$$= \int_{-1}^{1} \left(1 - \frac{10}{3}x^2 + \frac{25}{9}x^4\right) dx$$

$$= \left(x - \frac{10}{9}x^3 + \frac{5}{9}x^5\right) \Big|_{-1}^{1}$$

$$= 2\left(1 - \frac{10}{9} + \frac{5}{9}\right)$$

$$= 2\left(\frac{9-5}{9}\right)$$

$$= \frac{8}{9}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \left(\sqrt{\frac{9}{8}}\right)\left(1 - \frac{5}{3}x^2\right)$$

$$= \frac{3}{2\sqrt{2}}\left(1 - \frac{5}{3}x^2\right)$$

$$= \frac{3}{2\sqrt{2}} - \frac{5}{2\sqrt{2}}\, x^2$$

The orthonormal basis is $\left\{ \sqrt{\frac{3}{2}}\, x,\ \sqrt{\frac{5}{2}}\, x^2,\ \frac{3}{2\sqrt{2}} - \frac{5}{2\sqrt{2}}\, x^2 \right\}$

d) $[f, g] = \displaystyle\int_0^1 f(x)g(x)\, dx$

Let $\vec{u}_1 = 1, \quad \vec{u}_2 = x, \quad \vec{u}_3 = x^2$

$$\vec{v}_1 = \vec{u}_1 = 1$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_0^1 1\, dx$$

$$= x \Big|_0^1$$

$$= 1$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_0^1 x(1)\, dx$$

$$= \int_0^1 x\, dx$$

$$= \frac{1}{2}x^2 \Big|_0^1$$

$$= \frac{1}{2}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= x - \frac{\frac{1}{2}}{1}(1)$$

$$\boxed{= x - \frac{1}{2}}$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_0^1 \left(x - \frac{1}{2}\right)^2 dx$$

$$= \int_0^1 \left(x - \frac{1}{2}\right)^2 d\left(x - \frac{1}{2}\right)$$

$$= \frac{1}{3}\left(x - \frac{1}{2}\right)^3 \Big|_0^1$$

$$= \frac{1}{3}\left(\left(\frac{1}{2}\right)^3 - \left(-\frac{1}{2}\right)^3\right)$$

$$= \frac{1}{3}\left(\frac{1}{8} + \frac{1}{8}\right)$$

$$\boxed{= \frac{1}{12}}$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_0^1 \left(x^2\right)(1)\, dx$$

$$= \int_0^1 x^2\, dx$$

$$= \frac{1}{3}x^3 \Big|_0^1$$

$$\boxed{= \frac{1}{3}}$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_0^1 \left(x^2\right)\left(x - \frac{1}{2}\right) dx$$

$$= \int_0^1 \left(x^3 - \frac{1}{2}x^2\right) dx$$

$$= \left(\frac{1}{4}x^4 - \frac{1}{6}x^3\right) \Big|_0^1$$

$$= \frac{1}{4} - \frac{1}{6}$$

$$\boxed{= \frac{1}{12}}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= x^2 - \frac{\frac{1}{3}}{1}(1) - \frac{\frac{1}{12}}{\frac{1}{12}}\left(x - \frac{1}{2}\right)$$

$$= x^2 - \frac{1}{3} - x + \frac{1}{2}$$

$$\boxed{= x^2 - x + \frac{1}{6}}$$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 dx$$

$$= \int_0^1 \left(x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36}\right) dx$$

$$= \left(\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{4}{9}x^3 - \frac{1}{6}x^2 + \frac{1}{36}x\right)\Big|_0^1$$

$$= \frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36}$$

$$= \frac{2-5}{10} + \frac{16-6+1}{36}$$

$$= -\frac{3}{10} + \frac{11}{36}$$

$$= \frac{-108+110}{360}$$

$$\boxed{= \frac{1}{180}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\boxed{= 1}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{x - \frac{1}{2}}{\sqrt{\frac{1}{12}}}$$

$$\boxed{= 2\sqrt{3}\left(x - \frac{1}{2}\right)}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$=\left(\sqrt{180}\right)\left(x^2-x+\frac{1}{6}\right)$$

$$=\left(6\sqrt{5}\right)\left(x^2-x+\frac{1}{6}\right)$$

$$\boxed{=6\sqrt{5}\,x^2-6\sqrt{5}\,x+\sqrt{5}}$$

The orthonormal basis is $\left\{1,\ 2\sqrt{3}\left(x-\frac{1}{2}\right),\ \sqrt{5}\left(6x^2-6x+1\right)\right\}$

Therefore, $[f,\ g]=\int_0^1 f(x)g(x)\,dx$ is an inner product for $\mathbb{P}_3[x]$

e) Let assume: $\vec{v}=1$ *and* $\vec{w}=x$

$$\langle\vec{v},\ \vec{w}\rangle=\int_{-1}^{1}1(x)\,dx$$

$$=\int_{-1}^{1}x\,dx$$

$$=\frac{1}{2}x^2\Big|_{-1}^{1}$$

$$=\frac{1}{2}(1-1)$$

$$\boxed{=0}\quad\checkmark$$

$$[\vec{v},\ \vec{w}]=\int_0^1 1(x)\,dx$$

$$=\frac{1}{2}x^2\Big|_0^1$$

$$\boxed{=\frac{1}{2}\neq 0}\quad\checkmark$$

f) The orthonormal basis in part (c) $\left\{\sqrt{\frac{3}{2}}\,x,\ \sqrt{\frac{5}{2}}\,x^2,\ \frac{3}{2\sqrt{2}}-\frac{5}{2\sqrt{2}}\,x^2\right\}$ are ***not*** the same

as

the orthonormal basis in part (d) $\left\{1,\ 2\sqrt{3}\left(x-\frac{1}{2}\right),\ \sqrt{5}\left(6x^2-6x+1\right)\right\}$

Solution Section 3.4 – Orthogonal Matrices

Exercise

Show that the matrix is orthogonal $A=\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$

<u>Solution</u>

$$AA^T=\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underline{=I} \quad \surd$$

$$A^TA=\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underline{=I} \quad \surd$$

$$AA^T=A^TA=I$$

$\therefore$ ***A*** is orthogonal

Exercise

Show that the matrix is orthogonal $A=\begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix}$

<u>Solution</u>

$$AA^T=\begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underline{=I} \quad \surd$$

$$A^T A = \begin{pmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underline{= I} \quad \surd$$

$$AA^T = A^T A = I$$

$\therefore$ $\boldsymbol{A}$ is orthogonal.

Exercise

Show that the matrix is orthogonal $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$

Solution

$$AA^T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{= I} \quad \surd$$

$$A^T A = \begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{= I} \quad \surd$$

$$AA^T = A^T A = I$$

$\therefore$ $\boldsymbol{A}$ is orthogonal.

Exercise

Show that the matrix is orthogonal $A=\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{7}{\sqrt{53}} & \frac{2}{\sqrt{53}}\\ 0 & -\frac{2}{\sqrt{53}} & \frac{7}{\sqrt{53}}\end{pmatrix}$

Solution

$$AA^T=\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{7}{\sqrt{53}} & \frac{2}{\sqrt{53}}\\ 0 & -\frac{2}{\sqrt{53}} & \frac{7}{\sqrt{53}}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{7}{\sqrt{53}} & -\frac{2}{\sqrt{53}}\\ 0 & \frac{2}{\sqrt{53}} & \frac{7}{\sqrt{53}}\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\underline{=I} \quad \checkmark$$

$$A^TA=\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{7}{\sqrt{53}} & -\frac{2}{\sqrt{53}}\\ 0 & \frac{2}{\sqrt{53}} & \frac{7}{\sqrt{53}}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{7}{\sqrt{53}} & \frac{2}{\sqrt{53}}\\ 0 & -\frac{2}{\sqrt{53}} & \frac{7}{\sqrt{53}}\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\underline{=I} \quad \checkmark$$

$$AA^T=A^TA=I$$

$\therefore$ $\boldsymbol{A}$ is orthogonal.

Exercise

Show that the matrix is orthogonal $A=\begin{bmatrix}\frac{4}{5} & 0 & -\frac{3}{5}\\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25}\\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25}\end{bmatrix}$

Solution

$$AA^T = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix} \begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

$$A^T A = \begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix} \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

$$AA^T = A^T A = I$$

$\therefore$ ***A*** is orthogonal.

Exercise

Show that the matrix is orthogonal $A = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$

Solution

$$AA^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\underline{= I}$ $\surd$

$$A^T A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\underline{= I}$ $\surd$

$AA^T = A^T A = I$

$\therefore$ ***A*** is orthogonal.

Exercise

Show that the matrix is orthogonal $A = \begin{pmatrix} 0 & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{15}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{15}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & -\frac{2}{\sqrt{10}} & -\frac{2}{\sqrt{15}} \\ 0 & 0 & -\frac{2}{\sqrt{10}} & \frac{3}{\sqrt{15}} \end{pmatrix}$

Solution

$$AA^T = \begin{pmatrix} 0 & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{15}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{15}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & -\frac{2}{\sqrt{10}} & -\frac{2}{\sqrt{15}} \\ 0 & 0 & -\frac{2}{\sqrt{10}} & \frac{3}{\sqrt{15}} \end{pmatrix} \begin{pmatrix} 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{5}{\sqrt{30}} & -\frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & 0 \\ \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & -\frac{2}{\sqrt{10}} & -\frac{2}{\sqrt{10}} \\ \frac{1}{\sqrt{15}} & \frac{1}{\sqrt{15}} & -\frac{2}{\sqrt{15}} & \frac{3}{\sqrt{15}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\underline{= I}$ $\surd$

$$A^TA=\begin{pmatrix} 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{5}{\sqrt{30}} & -\frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & 0 \\ \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & -\frac{2}{\sqrt{10}} & -\frac{2}{\sqrt{10}} \\ \frac{1}{\sqrt{15}} & \frac{1}{\sqrt{15}} & -\frac{2}{\sqrt{15}} & \frac{3}{\sqrt{15}} \end{pmatrix}\begin{pmatrix} 0 & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{15}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{15}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & -\frac{2}{\sqrt{10}} & -\frac{2}{\sqrt{15}} \\ 0 & 0 & -\frac{2}{\sqrt{10}} & \frac{3}{\sqrt{15}} \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underline{=I\,|} \quad \surd$$

$$AA^T=A^TA=I$$

$\therefore$ ***A*** is orthogonal.

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{=I\,|} \quad \surd$$

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is orthogonal with inverse $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I \quad \checkmark$$

$$\therefore \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ is orthogonal with inverse } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Solution

$$\det\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2 \neq \pm 1$$

$$\therefore \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \text{ is } \textit{not} \text{ orthogonal}$$

$$\text{Inverse } \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix}$$

Solution

$$\begin{vmatrix} 4 & 1 \\ 3 & -1 \end{vmatrix} \underline{= -7 \neq \pm 1}$$

$\therefore \begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix}$ is *not* orthogonal.

Inverse $\begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{3}{7} & -\frac{4}{7} \end{pmatrix}$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$$

Solution

$$\begin{vmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{vmatrix} = -\frac{9}{25} - \frac{16}{25}$$

$$\underline{= -1} \quad \checkmark$$

$\therefore \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$ is orthogonal.

$$\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \frac{2}{\sqrt{6}} & 0 \\ 0 & \frac{3}{\sqrt{6}} \end{pmatrix}$$

Solution

$$\begin{vmatrix} \frac{2}{\sqrt{6}} & 0 \\ 0 & \frac{3}{\sqrt{6}} \end{vmatrix} \underline{=1}$$

$$\therefore \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 \\ 0 & \frac{3}{\sqrt{6}} \end{pmatrix}$$

Is orthogonal.

$$\begin{pmatrix} \frac{2}{\sqrt{6}} & 0 \\ 0 & \frac{3}{\sqrt{6}} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 \\ 0 & \frac{3}{\sqrt{6}} \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Solution

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{= I} \quad \surd$$

$$\therefore \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ is orthogonal with inverse } \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

(It is a standard matrix for a rotation of 45°)

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Solution

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\therefore \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \text{ is orthogonal with inverse } \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Solution

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}^T = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\therefore \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \text{ is orthogonal with an inverse } \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 7 \\ 1 & 3 & -5 \\ -1 & 4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & & \\ & & \\ & & \end{pmatrix} \neq I$$

$$\therefore \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix} \text{ is } \textit{not} \text{ orthogonal.}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Solution

$$\begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}^T = \begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2} & & \\ & & \\ & & \end{pmatrix} \neq I$$

Or $\|r_1\| = \sqrt{0 + 1^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{3}{2}} \underline{\neq 1}$

$\therefore \boldsymbol{A}$ is ***not*** orthogonal.

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Solution

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\therefore \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \text{is orthogonal with inverse} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution

$$\begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1\left(\frac{1}{2}+\frac{1}{2}\right)$$

$\underline{=1}\rfloor \quad \surd$

∴ The matrix is orthogonal.

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{T} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix}$$

Solution

$$\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix}\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix}^{T} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix}\begin{pmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\underline{=I}\rfloor \quad \surd$

∴ The matrix is orthogonal.

$$\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{5\sqrt{3}} & \frac{4}{5\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{7}{5\sqrt{3}} & \frac{3}{5\sqrt{2}} \end{pmatrix}$$

Solution

$$\begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{5\sqrt{3}} & \frac{4}{5\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{7}{5\sqrt{3}} & \frac{3}{5\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{5\sqrt{3}} & \frac{4}{5\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{7}{5\sqrt{3}} & \frac{3}{5\sqrt{2}} \end{pmatrix}^{T} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{5\sqrt{3}} & \frac{4}{5\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{7}{5\sqrt{3}} & \frac{3}{5\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{5\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{7}{5\sqrt{3}} \\ \frac{4}{5\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{3}{5\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{= I} \quad \checkmark$$

∴ The matrix is orthogonal.

$$\begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{5\sqrt{3}} & \frac{4}{5\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{7}{5\sqrt{3}} & \frac{3}{5\sqrt{2}} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{5\sqrt{3}} & \frac{4}{5\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{7}{5\sqrt{3}} & \frac{3}{5\sqrt{2}} \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$$

Solution

$$\begin{vmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{vmatrix} = \frac{1}{3} + \frac{1}{6} + \frac{1}{6} + \frac{1}{3}$$

$$= 1 \quad \checkmark$$

∴ The matrix is orthogonal.

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{pmatrix}$$

Solution

$$\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{pmatrix}^T = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}$$

$$\underline{=I} \quad \checkmark$$

∴ The matrix is orthogonal.

$$\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{pmatrix}$$

$$\begin{vmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{vmatrix} = -\left(-\frac{1}{5}-\frac{4}{5}\right)$$

$$\underline{=1} \quad \checkmark$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{42}} \\ \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{42}} \\ \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{4}{\sqrt{42}} \end{pmatrix}$$

Solution

$$\begin{pmatrix} -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{42}} \\ \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{42}} \\ \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{4}{\sqrt{42}} \end{pmatrix}\begin{pmatrix} -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{42}} \\ \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{42}} \\ \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{4}{\sqrt{42}} \end{pmatrix}^T = \begin{pmatrix} -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{42}} \\ \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{42}} \\ \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{4}{\sqrt{42}} \end{pmatrix}\begin{pmatrix} -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{42}} & \frac{5}{\sqrt{42}} & -\frac{4}{\sqrt{42}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{= I} \quad \surd$$

∴ The matrix is orthogonal.

$$\begin{pmatrix} -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{42}} \\ \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{42}} \\ \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{4}{\sqrt{42}} \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{42}} \\ \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{42}} \\ \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{4}{\sqrt{42}} \end{pmatrix}$$

$$\begin{vmatrix} -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{42}} \\ \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{42}} \\ \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{4}{\sqrt{42}} \end{vmatrix} = \frac{12+10-1+2+15+4}{\sqrt{14}\,\sqrt{3}\,\sqrt{42}}$$

$$= \frac{42}{42}$$

$$\underline{= 1} \quad \surd$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

Solution

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{= I} \quad \surd$$

∴ The matrix is orthogonal &

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \frac{7}{\sqrt{62}} & \frac{9}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ \frac{2}{\sqrt{62}} & -\frac{24}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ -\frac{3}{\sqrt{62}} & \frac{5}{\sqrt{682}} & \frac{3}{\sqrt{11}} \end{pmatrix}$$

Solution

$$\begin{pmatrix} \frac{7}{\sqrt{62}} & \frac{9}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ \frac{2}{\sqrt{62}} & -\frac{24}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ -\frac{3}{\sqrt{62}} & \frac{5}{\sqrt{682}} & \frac{3}{\sqrt{11}} \end{pmatrix}\begin{pmatrix} \frac{7}{\sqrt{62}} & \frac{9}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ \frac{2}{\sqrt{62}} & -\frac{24}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ -\frac{3}{\sqrt{62}} & \frac{5}{\sqrt{682}} & \frac{3}{\sqrt{11}} \end{pmatrix}^T = \begin{pmatrix} \frac{7}{\sqrt{62}} & \frac{9}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ \frac{2}{\sqrt{62}} & -\frac{24}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ -\frac{3}{\sqrt{62}} & \frac{5}{\sqrt{682}} & \frac{3}{\sqrt{11}} \end{pmatrix}\begin{pmatrix} \frac{7}{\sqrt{62}} & \frac{2}{\sqrt{62}} & -\frac{3}{\sqrt{62}} \\ \frac{9}{\sqrt{682}} & -\frac{24}{\sqrt{682}} & \frac{5}{\sqrt{682}} \\ \frac{1}{\sqrt{11}} & \frac{1}{\sqrt{11}} & \frac{3}{\sqrt{11}} \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{= I} \quad \surd$$

∴ The matrix is orthogonal &

$$\begin{pmatrix} \frac{7}{\sqrt{62}} & \frac{9}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ \frac{2}{\sqrt{62}} & -\frac{24}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ -\frac{3}{\sqrt{62}} & \frac{5}{\sqrt{682}} & \frac{3}{\sqrt{11}} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{7}{\sqrt{62}} & \frac{9}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ \frac{2}{\sqrt{62}} & -\frac{24}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ -\frac{3}{\sqrt{62}} & \frac{5}{\sqrt{682}} & \frac{3}{\sqrt{11}} \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

Solution

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{= I \,|} \quad \surd$$

∴ The matrix is orthogonal &

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{bmatrix}$$

Solution

$$\begin{aligned} \left\| r_2 \right\| &= \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{2}\right)^2} \\ &= \sqrt{\frac{1}{3} + \frac{1}{4}} \\ &= \sqrt{\frac{7}{12}} \neq 1 \end{aligned}$$

Or

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{5}{6} & & \\ & & & \\ & & & \end{pmatrix} \neq I$$

∴ The matrix is ***not*** orthogonal

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} 0 & -\frac{1}{2} & \frac{3}{\sqrt{174}} & \frac{9}{2\sqrt{29}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{2} & -\frac{10}{\sqrt{174}} & -\frac{1}{2\sqrt{29}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{2} & -\frac{8}{\sqrt{174}} & \frac{5}{2\sqrt{29}} \\ -\frac{2}{\sqrt{6}} & -\frac{1}{2} & -\frac{1}{\sqrt{174}} & -\frac{3}{2\sqrt{29}} \end{pmatrix}$$

Solution

$$\begin{pmatrix} 0 & -\frac{1}{2} & \frac{3}{\sqrt{174}} & \frac{9}{2\sqrt{29}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{2} & -\frac{10}{\sqrt{174}} & -\frac{1}{2\sqrt{29}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{2} & -\frac{8}{\sqrt{174}} & \frac{5}{2\sqrt{29}} \\ -\frac{2}{\sqrt{6}} & -\frac{1}{2} & -\frac{1}{\sqrt{174}} & -\frac{3}{2\sqrt{29}} \end{pmatrix}\begin{pmatrix} 0 & -\frac{1}{2} & \frac{3}{\sqrt{174}} & \frac{9}{2\sqrt{29}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{2} & -\frac{10}{\sqrt{174}} & -\frac{1}{2\sqrt{29}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{2} & -\frac{8}{\sqrt{174}} & \frac{5}{2\sqrt{29}} \\ -\frac{2}{\sqrt{6}} & -\frac{1}{2} & -\frac{1}{\sqrt{174}} & -\frac{3}{2\sqrt{29}} \end{pmatrix}^T$$

$$= \begin{pmatrix} 0 & -\frac{1}{2} & \frac{3}{\sqrt{174}} & \frac{9}{2\sqrt{29}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{2} & -\frac{10}{\sqrt{174}} & -\frac{1}{2\sqrt{29}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{2} & -\frac{8}{\sqrt{174}} & \frac{5}{2\sqrt{29}} \\ -\frac{2}{\sqrt{6}} & -\frac{1}{2} & -\frac{1}{\sqrt{174}} & -\frac{3}{2\sqrt{29}} \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{\sqrt{174}} & -\frac{10}{\sqrt{174}} & -\frac{8}{\sqrt{174}} & -\frac{1}{\sqrt{174}} \\ \frac{9}{2\sqrt{29}} & -\frac{1}{2\sqrt{29}} & \frac{5}{2\sqrt{29}} & -\frac{3}{2\sqrt{29}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{= I} \quad \checkmark$$

$\therefore$ The matrix is orthogonal &

$$\begin{pmatrix} 0 & -\frac{1}{2} & \frac{3}{\sqrt{174}} & \frac{9}{2\sqrt{29}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{2} & -\frac{10}{\sqrt{174}} & -\frac{1}{2\sqrt{29}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{2} & -\frac{8}{\sqrt{174}} & \frac{5}{2\sqrt{29}} \\ -\frac{2}{\sqrt{6}} & -\frac{1}{2} & -\frac{1}{\sqrt{174}} & -\frac{3}{2\sqrt{29}} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -\frac{1}{2} & \frac{3}{\sqrt{174}} & \frac{9}{2\sqrt{29}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{2} & -\frac{10}{\sqrt{174}} & -\frac{1}{2\sqrt{29}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{2} & -\frac{8}{\sqrt{174}} & \frac{5}{2\sqrt{29}} \\ -\frac{2}{\sqrt{6}} & -\frac{1}{2} & -\frac{1}{\sqrt{174}} & -\frac{3}{2\sqrt{29}} \end{pmatrix}$$

Exercise

Determine if the given matrix is orthogonal. If it is, find its inverse

$$\begin{bmatrix} \frac{1}{9} & \frac{4}{5} & \frac{3}{7} \\ \frac{4}{9} & \frac{3}{5} & -\frac{2}{7} \\ \frac{8}{9} & -\frac{2}{5} & \frac{3}{7} \end{bmatrix}$$

Solution

$$\vec{q}_1 = \begin{bmatrix} \frac{1}{9} & \frac{4}{9} & \frac{8}{9} \end{bmatrix}^T \qquad \vec{q}_2 = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} & -\frac{2}{5} \end{bmatrix}^T \qquad \vec{q}_3 = \begin{bmatrix} \frac{3}{7} & -\frac{2}{7} & \frac{3}{7} \end{bmatrix}^T$$

$$\begin{aligned} \vec{q}_1 \bullet \vec{q}_2 &= \frac{4}{45} + \frac{12}{45} - \frac{16}{45} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \vec{q}_1 \bullet \vec{q}_3 &= \frac{3}{63} - \frac{8}{63} + \frac{24}{63} \\ &= \frac{19}{63} \neq 0 \end{aligned}$$

$$\begin{aligned} \vec{q}_2 \bullet \vec{q}_3 &= \frac{12}{35} - \frac{6}{35} + \frac{6}{35} \\ &= \frac{12}{35} \neq 0 \end{aligned}$$

The given matrix is ***not*** orthogonal

Exercise

Find a last column so that the resulting matrix is orthogonal

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & \cdots \end{bmatrix}$$

Solution

$$\vec{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}^T$$

$$\begin{aligned} \left\| \vec{q}_1 \right\| &= \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} \\ &= 1 \end{aligned}$$

$$\vec{q}_2 = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}^T$$

$$\left\| \vec{q}_2 \right\| = \sqrt{\frac{1}{6} + \frac{1}{6} + \frac{4}{6}}$$

$$= 1$$

Let $\vec{q}_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$

$$\vec{q}_1 \bullet \vec{q}_3 = \frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y - \frac{1}{\sqrt{3}}z = 0 \quad \to \quad x + y - z = 0$$

$$\vec{q}_2 \bullet \vec{q}_3 = \frac{1}{\sqrt{6}}x + \frac{1}{\sqrt{6}}y - \frac{2}{\sqrt{6}}z = 0 \quad \to \quad x + y - 2z = 0$$

$$\begin{cases} x + y - z = 0 \\ x + y - 2z = 0 \end{cases} \to \quad z = 0 \quad and \quad x + y = 0 \Rightarrow x = -y$$

$$x^2 + y^2 = 1$$

$$2x^2 = 1$$

$$x = \pm \frac{1}{\sqrt{2}}$$

$$\vec{q}_3 = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}^T$$

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \end{pmatrix}$$

Exercise

Find a last column so that the resulting matrix is orthogonal

$$\begin{pmatrix} \frac{3}{5} & 0 & \cdots \\ \frac{4}{5} & 0 & \cdots \\ 0 & 1 & \cdots \end{pmatrix}$$

Solution

$$\vec{q}_1 = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 0 \end{bmatrix}^T$$

$$\left\|\vec{q}_1\right\| = \sqrt{\frac{9}{25}+\frac{16}{25}}$$

$$\boxed{=1}$$

$$\vec{q}_2 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$$

$$\boxed{\left\|\vec{q}_2\right\| = 1}$$

Let $\vec{q}_3 = \begin{bmatrix}x & y & z\end{bmatrix}^T$

$$\vec{q}_1 \bullet \vec{q}_3 = \frac{3}{5}x + \frac{4}{5}y - 0z = 0$$

$$\frac{3}{5}x + \frac{4}{5}y = 0$$

$$\boxed{3x + 4y = 0}$$

$$\vec{q}_2 \bullet \vec{q}_3 = 0\cdot x + 0\cdot y + 1\cdot z = 0$$

$$\boxed{z = 0}$$

From $3x + 4y = 0$

$$\boxed{x = -\frac{4}{3}y}$$

$$x^2 + y^2 + z^2 = 1$$

$$\frac{16}{9}y^2 + y^2 = 1$$

$$\frac{25}{9}y^2 = 1$$

$$y = \pm\frac{3}{5}$$

$$y = \frac{3}{5} \quad \Rightarrow \quad x = -\frac{4}{5}$$

Then $\vec{q}_3 = \begin{bmatrix}-\frac{4}{5} & \frac{3}{5} & 0\end{bmatrix}^T$

$$\begin{pmatrix}\frac{3}{5} & 0 & -\frac{4}{5}\\ \frac{4}{5} & 0 & \frac{3}{5}\\ 0 & 1 & 0\end{pmatrix}$$

$$\boxed{\vec{q}_1 \bullet \vec{q}_3 = \frac{3}{5}\left(-\frac{4}{5}\right) + \frac{4}{5}\left(\frac{3}{5}\right) = 0} \quad \checkmark$$

Exercise

Find a last column so that the resulting matrix is orthogonal

$$\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & \cdots \\ 0 & 0 & \cdots \\ \frac{4}{5} & \frac{3}{5} & \cdots \end{pmatrix}$$

Solution

$$\vec{q}_1 = \begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}^T$$

$$\left\| \vec{q}_1 \right\| = \sqrt{\frac{9}{25} + \frac{16}{25}}$$

$$= 1$$

$$\vec{q}_2 = \begin{bmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}^T$$

$$\left\| \vec{q}_2 \right\| = \sqrt{\frac{9}{25} + \frac{16}{25}}$$

$$= 1$$

Let $\vec{q}_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$

$$\vec{q}_1 \bullet \vec{q}_3 = \frac{3}{5}x - 0y + \frac{4}{5}z = 0$$

$$\frac{3}{5}x + \frac{4}{5}z = 0$$

$$3x + 4z = 0$$

$$\vec{q}_2 \bullet \vec{q}_3 = \left(-\frac{4}{5}\right) \cdot x + 0 \cdot y + \left(\frac{3}{5}\right) \cdot z = 0$$

$$-\frac{4}{5}x + \frac{3}{5}z = 0$$

$$-4x + 3z = 0$$

$$\begin{cases} 3x + 4z = 0 \\ -4x + 3z = 0 \end{cases}$$

$$x = \frac{\begin{vmatrix} 0 & 4 \\ 0 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ -4 & 3 \end{vmatrix}}$$

$$= \frac{0}{25}$$

$$= 0$$

$$\Rightarrow \quad z = 0$$

$$x^2 + y^2 + z^2 = 1$$

$$y^2 = 1$$

$$y = \pm 1$$

Then $\vec{q}_3 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$

$$\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix}$$

$$\vec{q}_1 \bullet \vec{q}_3 = \tfrac{3}{5}(0) + 0 + \tfrac{4}{5}(0) = 0 \quad \checkmark$$

Exercise

Find a last column so that the resulting matrix is orthogonal

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots \\ 0 & 0 & \cdots \end{pmatrix}$$

Solution

$$\vec{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}^T$$

$$\left\|\vec{q}_1\right\| = \sqrt{\frac{1}{2} + \frac{1}{2}}$$

$$= 1$$

$$\vec{q}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}^T$$

$$\left\|\vec{q}_2\right\| = \sqrt{\frac{1}{2} + \frac{1}{2}}$$

$$= 1$$

Let $\vec{q}_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$

$$\vec{q}_1 \bullet \vec{q}_3 = \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y + 0 \cdot z = 0$$

$$\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y = 0$$

$$\underline{x = y}$$

$$\vec{q}_2 \bullet \vec{q}_3 = \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y + 0 \cdot z = 0$$

$$\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y = 0$$

$$\underline{x = -y}$$

$$x = \pm y \quad \Rightarrow \quad \underline{x = y = 0}$$

$$x^2 + y^2 + z^2 = 1$$

$$z^2 = 1$$

$$\underline{z = \pm 1}$$

Then $\vec{q}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise

Find a last column so that the resulting matrix is orthogonal

$$\begin{pmatrix} \frac{2}{7} & \frac{3}{7} & \cdots \\ \frac{6}{7} & \frac{2}{7} & \cdots \\ -\frac{3}{7} & \frac{6}{7} & \cdots \end{pmatrix}$$

Solution

$$\vec{q}_1 = \begin{bmatrix} \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}^T$$

$$\left\| \vec{q}_1 \right\| = \sqrt{\frac{4}{49} + \frac{36}{49} + \frac{9}{49}}$$

$$\underline{= 1}$$

$$\vec{q}_2 = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \end{bmatrix}^T$$

$$\left\|\vec{q}_2\right\| = \sqrt{\frac{9}{49}+\frac{4}{49}+\frac{36}{49}}$$

$$\underline{=1}$$

Let $\vec{q}_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$

$$\vec{q}_1 \bullet \vec{q}_3 = \frac{2}{7}x + \frac{6}{7}y - \frac{3}{7}\cdot z = 0$$

$$\underline{2x + 6y - 3z = 0}$$

$$\vec{q}_2 \bullet \vec{q}_3 = \frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}\cdot z = 0$$

$$\underline{3x + 2y + 6z = 0}$$

$$\begin{array}{r} 2 \\ + \end{array}\begin{cases} 2x + 6y - 3z = 0 \\ 3x + 2y + 6z = 0 \end{cases}$$

$$7x + 14y = 0$$

$$\underline{x = -2y}$$

$$z = \frac{2}{3}x + 2y$$

$$= -\frac{4}{3}y + 2y$$

$$\underline{= \frac{2}{3}y}$$

$$x^2 + y^2 + z^2 = 1$$

$$4y^2 + y^2 + \frac{4}{9}y^2 = 1$$

$$\left(\frac{36+9+4}{9}\right)y^2 = 1$$

$$y^2 = \frac{9}{49}$$

$$y = \pm\frac{3}{7}$$

If $\underline{y = \frac{3}{7} \quad \Rightarrow \quad x = -\frac{6}{7}, \quad z = \frac{2}{7}}$

Or

$$\underline{y = -\frac{3}{7} \quad \Rightarrow \quad x = \frac{6}{7}, \quad z = -\frac{2}{7}}$$

Then $\vec{q}_3 = \begin{bmatrix} -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \end{bmatrix}^T$

$$\begin{pmatrix} \frac{2}{7} & \frac{3}{7} & -\frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & \frac{3}{7} \\ -\frac{3}{7} & \frac{6}{7} & \frac{2}{7} \end{pmatrix} \quad or \quad \begin{pmatrix} \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \\ -\frac{3}{7} & \frac{6}{7} & -\frac{2}{7} \end{pmatrix}$$

Exercise

Find a last column so that the resulting matrix is orthogonal:

$$\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \cdots \\ \frac{2}{3} & \frac{1}{3} & \cdots \\ -\frac{2}{3} & \frac{2}{3} & \cdots \end{pmatrix}$$

Solution

$$\vec{q}_1 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}^T$$

$$\left\|\vec{q}_1\right\| = \sqrt{\frac{1}{9}+\frac{4}{9}+\frac{4}{9}}$$

$$= 1$$

$$\vec{q}_2 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}^T$$

$$\left\|\vec{q}_2\right\| = \sqrt{\frac{4}{9}+\frac{1}{9}+\frac{4}{9}}$$

$$= 1$$

Let $\vec{q}_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$

$$\vec{q}_1 \bullet \vec{q}_3 = \frac{1}{3}x + \frac{2}{3}y - \frac{2}{3}\cdot z = 0$$

$$x + 2y - 2z = 0$$

$$\vec{q}_2 \bullet \vec{q}_3 = \frac{2}{3}x + \frac{1}{3}y + \frac{2}{3}\cdot z = 0$$

$$2x + y + 2z = 0$$

$$+\begin{cases} x + 2y - 2z = 0 \\ 2x + y + 2z = 0 \end{cases}$$

$$3x + 3y = 0$$

$$x = -y$$

$$\begin{aligned} z &= \frac{1}{2}x + y \\ &= -\frac{1}{2}y + y \\ &= \frac{1}{2}y \end{aligned}$$

$$x^2 + y^2 + z^2 = 1$$

$$y^2 + y^2 + \frac{1}{4}y^2 = 1$$

$$\left(\frac{9}{4}\right)y^2 = 1$$

$$y = \pm\frac{2}{3}$$

If $y = \frac{2}{3} \Rightarrow x = -\frac{2}{3}, \quad z = \frac{1}{3}$

Or

$y = -\frac{2}{3} \Rightarrow x = \frac{2}{3}, \quad z = -\frac{1}{3}$

$$\vec{q}_3 = \begin{bmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}^T \quad or \quad \vec{q}_3 = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}^T$$

$$\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \quad or \quad \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

Exercise

Find a last column so that the resulting matrix is orthogonal:

$$\begin{pmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & \cdots \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} & \cdots \\ -\frac{1}{3} & 0 & \cdots \end{pmatrix}$$

Solution

$$\vec{q}_1 = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}^T$$

$$\begin{aligned} \left\|\vec{q}_1\right\| &= \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} \\ &= 1 \end{aligned}$$

$$\vec{q}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}^T$$

$$\left\| \vec{q}_2 \right\| = \sqrt{\frac{1}{2} + \frac{1}{2}}$$

$$= 1$$

Let $\vec{q}_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$

$$\vec{q}_1 \bullet \vec{q}_3 = \frac{2}{3}x + \frac{2}{3}y - \frac{1}{3} \cdot z = 0$$

$$2x + 2y - z = 0$$

$$\vec{q}_2 \bullet \vec{q}_3 = \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y + 0 \cdot z = 0$$

$$y = x$$

$$2x + 2y - z = 0$$

$$2x + 2x - z = 0$$

$$z = 4x$$

$$x^2 + y^2 + z^2 = 1$$

$$x^2 + x^2 + 16x^2 = 1$$

$$x^2 = \frac{1}{18}$$

$$x = \pm \frac{1}{3\sqrt{2}}$$

If $x = \frac{1}{3\sqrt{2}} \quad \Rightarrow \quad y = \frac{1}{3\sqrt{2}}, \quad z = \frac{4}{3\sqrt{2}}$

Or

$$x = -\frac{1}{3\sqrt{2}} \quad \Rightarrow \quad y = -\frac{1}{3\sqrt{2}}, \quad z = -\frac{4}{3\sqrt{2}}$$

$$\vec{q}_3 = \begin{bmatrix} \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} \end{bmatrix}^T \quad \textit{or} \quad \vec{q}_3 = \begin{bmatrix} -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{4}{3\sqrt{2}} \end{bmatrix}^T$$

For $\vec{q}_3 = \begin{bmatrix} \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} \end{bmatrix}^T$

$$\vec{q}_1 \bullet \vec{q}_3 = \frac{2}{3} \cdot \frac{1}{3\sqrt{2}} + \frac{2}{3} \cdot \frac{1}{3\sqrt{2}} - \frac{1}{3} \cdot \frac{4}{3\sqrt{2}} = 0 \quad \surd$$

$$\vec{q}_2 \bullet \vec{q}_3 = \frac{1}{\sqrt{2}} \cdot \frac{1}{3\sqrt{2}} - \frac{1}{\sqrt{2}} \cdot \frac{1}{3\sqrt{2}} + 0 \cdot \frac{4}{3\sqrt{2}} = 0 \quad \surd$$

For $\vec{q}_3 = \begin{bmatrix} -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{4}{3\sqrt{2}} \end{bmatrix}^T$

$$\vec{q}_1 \bullet \vec{q}_3 = -\frac{2}{3}\cdot\frac{1}{3\sqrt{2}} - \frac{2}{3}\cdot\frac{1}{3\sqrt{2}} + \frac{1}{3}\cdot\frac{4}{3\sqrt{2}} = 0 \quad \surd$$

$$\vec{q}_2 \bullet \vec{q}_3 = \frac{1}{\sqrt{2}}\cdot\frac{1}{3\sqrt{2}} - \frac{1}{\sqrt{2}}\cdot\frac{1}{3\sqrt{2}} + 0\cdot\frac{4}{3\sqrt{2}} = 0 \quad \surd$$

$$\begin{pmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ -\frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \end{pmatrix} \quad or \quad \begin{pmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ -\frac{1}{3} & 0 & -\frac{4}{3\sqrt{2}} \end{pmatrix}$$

Exercise

Find a last column so that the resulting matrix is orthogonal:

$$\begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{3}{\sqrt{70}} & \cdots \\ -\frac{1}{\sqrt{5}} & \frac{6}{\sqrt{70}} & \cdots \\ 0 & -\frac{5}{\sqrt{70}} & \cdots \end{pmatrix}$$

Solution

$$\vec{q}_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \end{bmatrix}^T$$

$$\left\| \vec{q}_1 \right\| = \sqrt{\frac{4}{5}+\frac{1}{5}}$$

$$= 1 \rfloor$$

$$\vec{q}_2 = \begin{bmatrix} \frac{3}{\sqrt{70}} & \frac{6}{\sqrt{70}} & -\frac{5}{\sqrt{70}} \end{bmatrix}^T$$

$$\left\| \vec{q}_2 \right\| = \sqrt{\frac{9}{70}+\frac{36}{70}+\frac{25}{70}}$$

$$= 1 \rfloor$$

Let $\vec{q}_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$

$$\vec{q}_1 \bullet \vec{q}_3 = \frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}y + 0\cdot z = 0$$

$$\underline{y = 2x}$$

$$\vec{q}_2 \bullet \vec{q}_3 = \frac{3}{\sqrt{70}}x + \frac{6}{\sqrt{70}}y - \frac{5}{\sqrt{70}} \cdot z = 0$$

$$\underline{3x + 6y - 5z = 0}$$

$$3x + 12x - 5z = 0$$

$$5z = 15x$$

$$\underline{z = 3x}$$

$$x^2 + y^2 + z^2 = 1$$

$$x^2 + 4x^2 + 9x^2 = 1$$

$$x^2 = \frac{1}{14}$$

$$x = \pm\frac{1}{\sqrt{14}}$$

If $x = \frac{1}{\sqrt{14}} \Rightarrow y = \frac{2}{\sqrt{14}}, \quad z = \frac{3}{\sqrt{14}}$

Or

$$x = -\frac{1}{\sqrt{14}} \Rightarrow y = -\frac{2}{\sqrt{14}}, \quad z = -\frac{3}{\sqrt{14}}$$

$$\vec{q}_3 = \begin{bmatrix} \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \end{bmatrix}^T \quad or \quad \vec{q}_3 = \begin{bmatrix} -\frac{1}{\sqrt{14}} & -\frac{2}{\sqrt{14}} & -\frac{3}{\sqrt{14}} \end{bmatrix}^T$$

$$\begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{3}{\sqrt{70}} & \frac{1}{\sqrt{14}} \\ -\frac{1}{\sqrt{5}} & \frac{6}{\sqrt{70}} & \frac{2}{\sqrt{14}} \\ 0 & -\frac{5}{\sqrt{70}} & \frac{3}{\sqrt{14}} \end{pmatrix}$$

Exercise

Prove that if A is orthogonal, then A^T is orthogonal.

Solution

Since A is orthogonal then $A^T = A^{-1}$ and $A = \left(A^T\right)^T$

Then $\left(A^T\right)^T A^T = AA^T = I \Rightarrow A^T$ is orthogonal.

Another word, since A is orthogonal, then both column and row vectors of A form an orthonormal set.

A^T is just A with its row and column vectors are swapped.
The column vectors of A^T (which are the row vectors of A) and row vectors of A^T (which are the column vectors of A) form orthonormal sets, therefore A^T is orthogonal

Exercise

Prove that if A is orthogonal, then A^{-1} is orthogonal

Solution

Since A is orthogonal then $A^T = A^{-1}$ and $A = \left(A^{-1}\right)^{-1}$

$$\left(A^{-1}\right)^{-1} = \left(A^T\right)^{-1} \qquad A^T = A^{-1}$$

$$= \left(A^{-1}\right)^T$$

$\therefore$ A^{-1} is orthogonal.

Exercise

Prove that if A and B are orthogonal, then AB is orthogonal.

Solution

Since A is orthogonal then $A^T = A^{-1}$
and B is orthogonal then $B^T = B^{-1}$

$$(AB)^T = B^T A^T$$

$$= B^{-1}A^{-1}$$

$$= (AB)^{-1}$$

$\therefore$ AB is orthogonal.

Exercise

Let Q be an $n \times n$ orthogonal matrix, and let A be an $n \times n$ matrix.
Show that $\det\left(QAQ^T\right) = \det(A)$

Solution

$$\det\left(QAQ^T\right) = \det(Q)\det(A)\det\left(Q^T\right)$$

$$= \det(A)\det\left(QQ^T\right)$$ Since Q is an orthogonal matrix $\det\left(QQ^T\right) = \det(I)$

$$= \det(A)\det(I)$$

$$= \det(A) \quad \checkmark$$

Exercise

Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$

a) Is matrix A an orthogonal matrix?
b) Let B be the matrix obtained by normalizing each row of A, find B.
c) Is B an orthogonal matrix?
d) Are the columns of B orthogonal?

Solution

a) $AA^T = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 7 \\ 1 & 3 & -5 \\ -1 & 4 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 3 & & \\ & & \\ & & \end{pmatrix} \neq I$$

$\therefore \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$ is ***not*** orthogonal.

b) $\|(1,\ 1,\ -1)\| = \sqrt{1+1+1}$

$$= \sqrt{3}$$

$\|(1,\ 3,\ 4)\| = \sqrt{1+9+16}$

$$= \sqrt{26}$$

$\|(7,\ -5,\ 2)\| = \sqrt{49+25+4}$

$$= \sqrt{78}$$

$$B = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{26}} & \frac{3}{\sqrt{26}} & \frac{4}{\sqrt{26}} \\ \frac{7}{\sqrt{78}} & -\frac{5}{\sqrt{78}} & \frac{2}{\sqrt{78}} \end{pmatrix}$$

c) *Yes*, since the rows are orthogonal with unit vectors.

$$BB^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{26}} & \frac{3}{\sqrt{26}} & \frac{4}{\sqrt{26}} \\ \frac{7}{\sqrt{78}} & -\frac{5}{\sqrt{78}} & \frac{2}{\sqrt{78}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{26}} & \frac{7}{\sqrt{78}} \\ \frac{1}{\sqrt{3}} & \frac{3}{\sqrt{26}} & -\frac{5}{\sqrt{78}} \\ -\frac{1}{\sqrt{3}} & \frac{4}{\sqrt{26}} & \frac{2}{\sqrt{78}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

d) *Yes*, since the rows of B form an orthonormal set of vectors.
Then, the column of B must form an orthonormal set.

$$\left\| \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{26}}, \frac{7}{\sqrt{78}} \right) \right\| = \sqrt{\frac{1}{3} + \frac{1}{26} + \frac{49}{78}}$$

$$= \sqrt{\frac{26+3+49}{78}}$$

$$= \sqrt{\frac{78}{78}}$$

$$\underline{=1}$$

$$\left\| \left(\frac{1}{\sqrt{3}}, \frac{3}{\sqrt{26}}, -\frac{5}{\sqrt{78}} \right) \right\| = \sqrt{\frac{1}{3} + \frac{9}{26} + \frac{25}{78}}$$

$$= \sqrt{\frac{26+27+25}{78}}$$

$$= \sqrt{\frac{78}{78}}$$

$$\underline{=1}$$

$$\left\| \left(-\frac{1}{\sqrt{3}}, \frac{4}{\sqrt{26}}, \frac{2}{\sqrt{78}} \right) \right\| = \sqrt{\frac{1}{3} + \frac{16}{26} + \frac{4}{78}}$$

$$= \sqrt{\frac{78}{78}}$$

$$\underline{=1}$$

Solution Section 3.5 – Least Squares Analysis

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(0,\ 2),\ (1,\ 2),\ (2,\ 0)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points.

Then

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

where $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} m \\ b \end{bmatrix}$ $\quad \vec{y} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

The normal equation formula: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$m = \frac{\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 3 & 3 \end{vmatrix}}$$

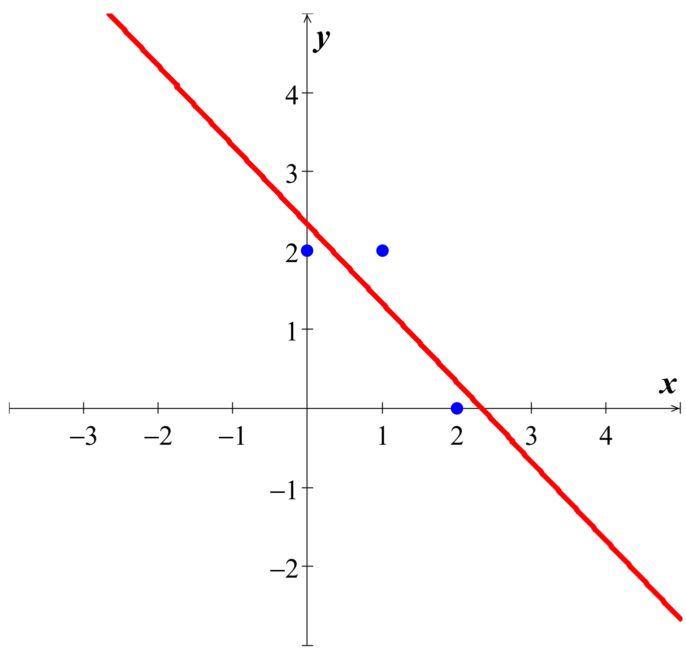
$$= \frac{-6}{6}$$

$$\underline{= -1}$$

$$b = \frac{\begin{vmatrix} 5 & 2 \\ 3 & 4 \end{vmatrix}}{6}$$

$$\underline{= \frac{7}{3}}$$

Thus, $\underline{y = -x + \frac{7}{3}}$



$$A\vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ \frac{7}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{7}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{7}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$$

Error: $\quad \|\vec{y} - A\vec{x}\| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{1}{9}}$

$$= \frac{\sqrt{6}}{3}$$

$$\approx 0.8164966$$

****

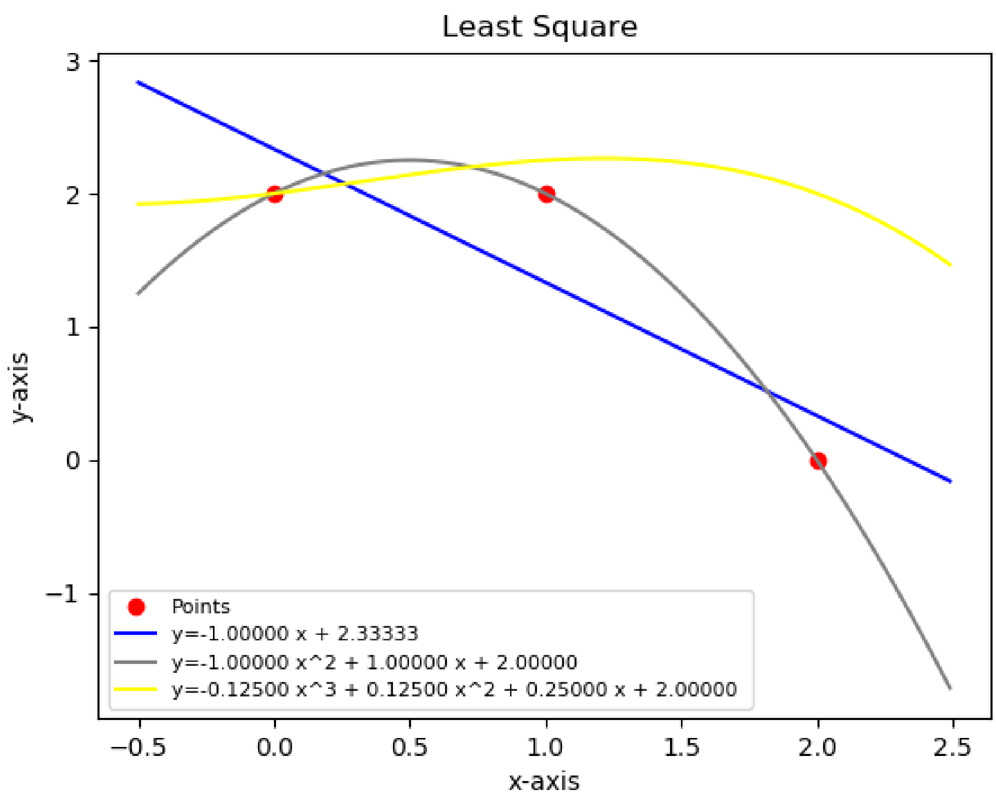
The ***second*** *order* equation: $\quad y = -x^2 + x + 2$

Error = 0.00000

****

The ***third order*** equation: $\quad y = -.1250x^3 - 0.1250x^2 + 0.25x + 2$

Error = 2.01556



Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(0,\ 0),\quad (1,\ 1),\quad (2,\ 4)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points.

Then

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$$

where $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} m \\ b \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$

The normal equation formula: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$

$$m = \frac{\begin{vmatrix} 9 & 3 \\ 5 & 3 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 3 & 3 \end{vmatrix}}$$

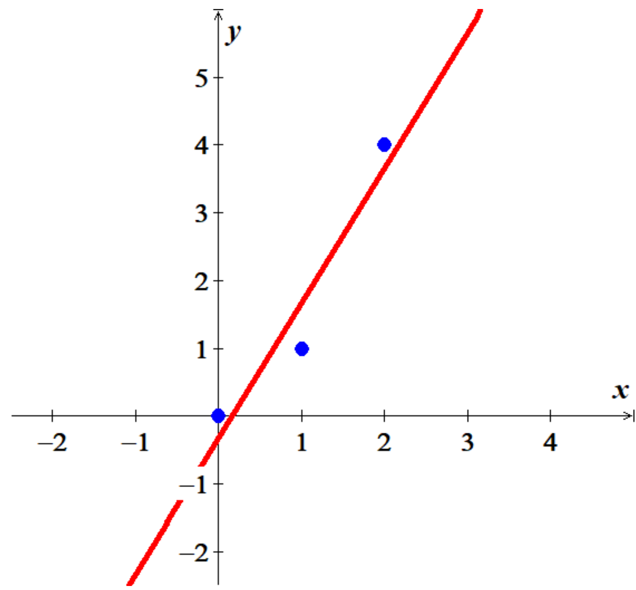
$$= \frac{12}{6}$$

$$= 2$$

$$b = \frac{\begin{vmatrix} 5 & 9 \\ 3 & 5 \end{vmatrix}}{6}$$

$$= -\frac{1}{3}$$

Thus, $y = 2x - \frac{1}{3}$



$$A\vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -\frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} \\ \frac{5}{3} \\ \frac{11}{3} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} -\frac{1}{3} \\ \frac{5}{3} \\ \frac{11}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

Error: $\|\vec{y} - A\vec{x}\| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{1}{9}}$

$$= \frac{\sqrt{6}}{3}$$

$$\approx 0.8164966$$

****

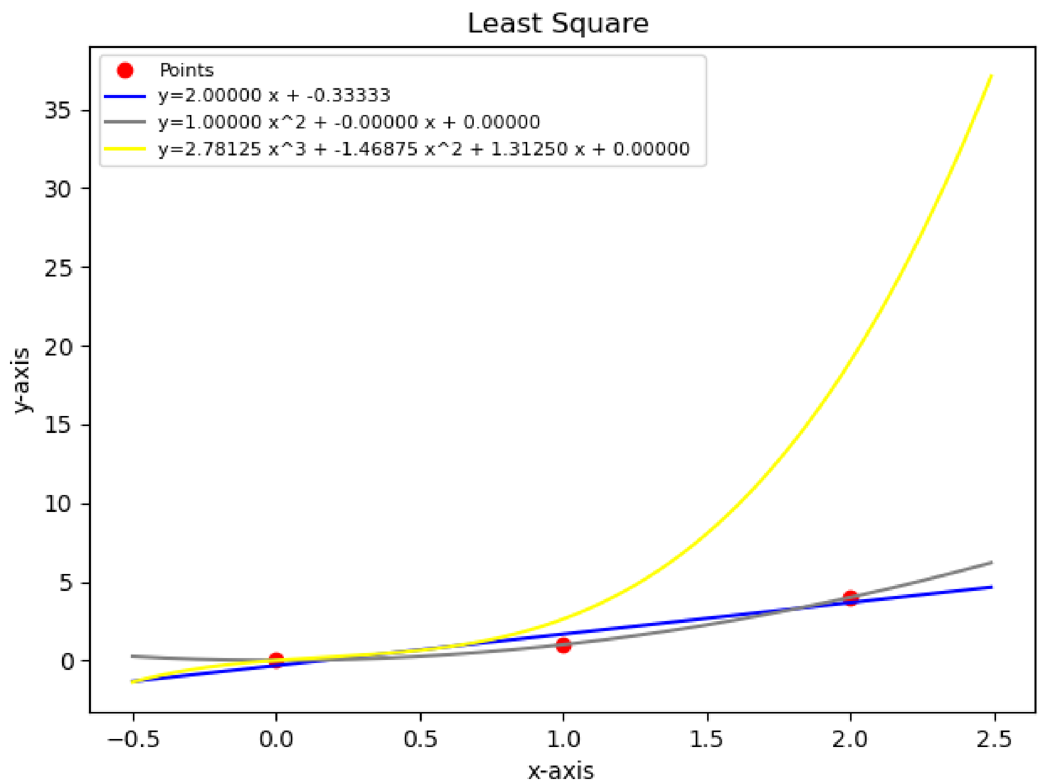
The ***second*** *order* equation: $y = -x^2$

Error = 0.00000

****

The ***third order*** equation: $y = 2.78x^3 - 1.469x^2 + 1.3125x$

Error = 15.08776



Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(0,\ 0),\quad (2,\ 1),\quad (4,\ 1)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then,

$$\begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

where $A=\begin{pmatrix}0&1\\2&1\\4&1\end{pmatrix}$ $\quad\vec{x}=\begin{pmatrix}m\\b\end{pmatrix}$ $\quad\vec{y}=\begin{pmatrix}0\\1\\1\end{pmatrix}$

The normal equation formula: $A^T A\vec{x}=A^T\vec{y}$

$$\begin{pmatrix}0&2&4\\1&1&1\end{pmatrix}\begin{pmatrix}0&1\\2&1\\4&1\end{pmatrix}\begin{pmatrix}m\\b\end{pmatrix}=\begin{pmatrix}0&2&4\\1&1&1\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}20&6\\6&3\end{pmatrix}\begin{pmatrix}m\\b\end{pmatrix}=\begin{pmatrix}6\\2\end{pmatrix}$$

$$m=\frac{\begin{vmatrix}6&6\\2&3\end{vmatrix}}{\begin{vmatrix}20&6\\6&3\end{vmatrix}}$$

$$=\frac{6}{24}$$

$$\boxed{=\frac{1}{4}}$$

$$b=\frac{4}{24}$$

$$\boxed{=\frac{1}{6}}$$

Thus, $\boxed{y=\frac{1}{4}x+\frac{1}{6}}$

$$A\vec{x}=\begin{pmatrix}0&1\\2&1\\4&1\end{pmatrix}\begin{pmatrix}\frac{1}{4}\\\frac{1}{6}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{6}\\\frac{2}{3}\\\frac{7}{6}\end{pmatrix}$$

$$\vec{y}-A\vec{x}=\begin{pmatrix}0\\1\\1\end{pmatrix}-\begin{pmatrix}\frac{1}{6}\\\frac{2}{3}\\\frac{7}{6}\end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{6} \\ \frac{1}{3} \\ -\frac{1}{6} \end{pmatrix}$$

Error: $\quad \|\vec{y} - A\vec{x}\| = \sqrt{\frac{1}{36} + \frac{1}{9} + \frac{1}{36}}$

$$= \frac{\sqrt{6}}{6}$$

$$\approx 0.40825$$

****

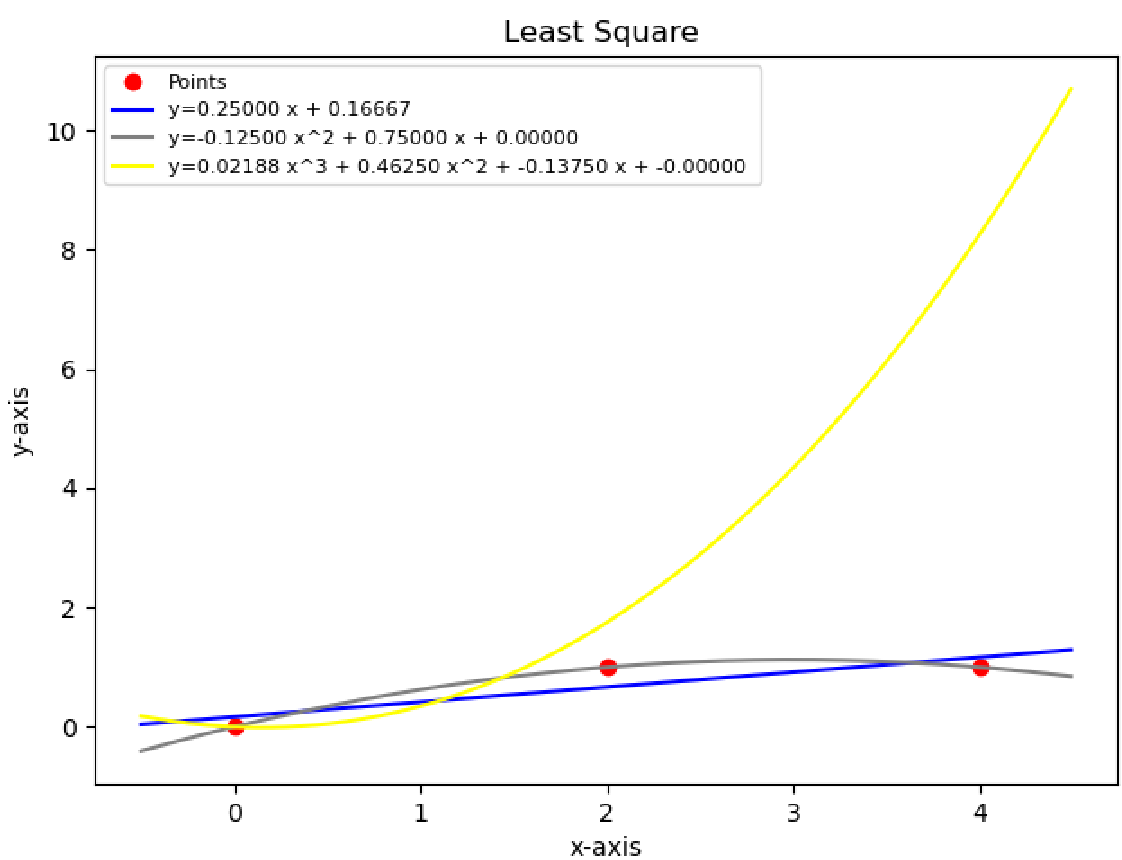
The ***second*** *order* equation: $\quad y = -0.125x^2 + .75x$

Error = 0.00000

****

The ***third order*** equation: $y = 0.01288x^3 + .4625x^2 - .1375x$

Error = 7.28869



Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-1,\ -1),\ \ (1,\ 0),\ \ (2,\ 4)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then,

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$$

where $A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$ $\vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$ $\vec{y} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$

The normal equation formula: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \end{pmatrix}$$

$$m = \frac{21}{14}$$

$$= \frac{3}{2}$$

$$b = \frac{0}{14}$$

$$= 0$$

Thus, $y = \frac{3}{2}x$

$$A\vec{x} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ 3 \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 1 \end{pmatrix}$$

Error: $\|\vec{y} - A\vec{x}\| = \sqrt{\frac{1}{4} + \frac{9}{4} + 1}$

$$= \frac{\sqrt{14}}{2}$$

$$\approx 1.87083$$

****

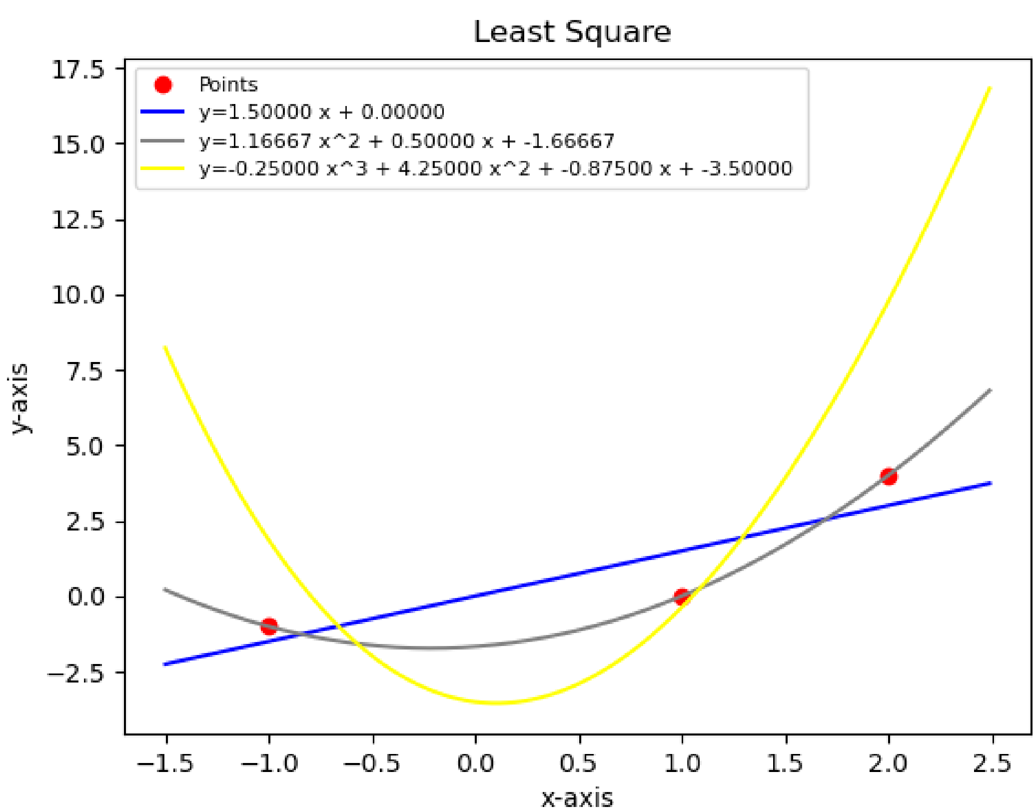
The ***second*** *order* equation: $y = 1.1667x^2 + .5x - 1.667$

Error = 0.00000

****

The ***third order*** equation: $y = -.25x^3 + 4.25x^2 - .875x - 3.5$

Error = 6.43962



Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(0,\ 1),\ (1,\ 1),\ (2,\ 2),\ (3,\ 2)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then,

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$$

where $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$ $\quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$ $\quad \vec{y} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$

The normal equation formula: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 6 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 11 \\ 6 \end{pmatrix}$$

$$m = \frac{8}{20}$$

$$= \frac{2}{5}$$

$$b = \frac{18}{20}$$

$$= \frac{9}{10}$$

Thus, $y = \frac{2}{5}x + \frac{9}{10}$

$$A\vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} \frac{2}{5} \\ \frac{9}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{9}{10} \\ \frac{13}{10} \\ \frac{17}{10} \\ \frac{21}{10} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} \frac{9}{10} \\ \frac{13}{10} \\ \frac{17}{10} \\ \frac{21}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{10} \\ -\frac{3}{10} \\ \frac{3}{10} \\ -\frac{1}{10} \end{pmatrix}$$

E*rror*: $\|\vec{y} - A\vec{x}\| = \sqrt{\frac{1}{100} + \frac{9}{100} + \frac{9}{100} + \frac{1}{100}}$

$$= \frac{2\sqrt{5}}{10}$$

$$\approx 0.44721$$

****

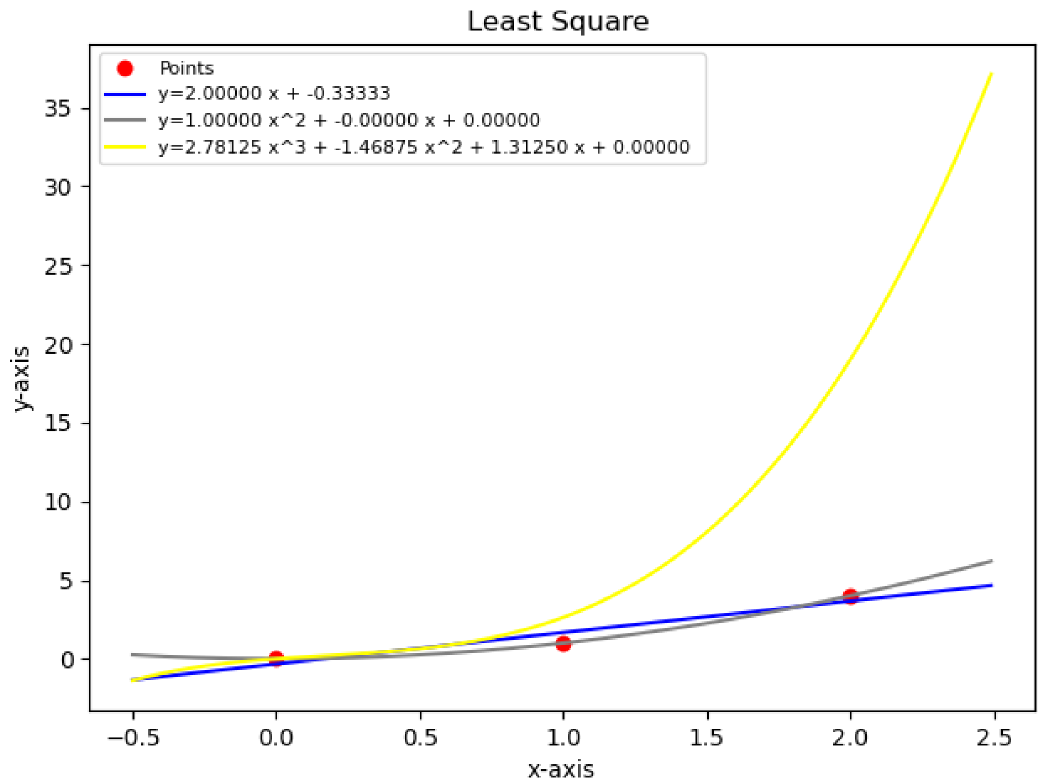
The ***second*** *order* equation: $y = 0x^2 + .4x + .9$

Error = 0. 44721

****

The ***third order*** equation: $y = -.333x^3 + 1.5x^2 - 1.1667x + 1$

Error = 0.00000



Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(2,\ 1),\ (5,\ 2),\ (7,\ 3),\ (8,\ 3)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then,

$$\begin{pmatrix} 2 & 1 \\ 5 & 1 \\ 7 & 1 \\ 8 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}$$

where $A = \begin{pmatrix} 2 & 1 \\ 5 & 1 \\ 7 & 1 \\ 8 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}$

The normal equation formula: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 2 & 5 & 7 & 8 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 5 & 1 \\ 7 & 1 \\ 8 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 2 & 5 & 7 & 8 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 142 & 22 \\ 22 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix}=\begin{pmatrix} 57 \\ 9 \end{pmatrix}$$

$$m=\frac{30}{84}$$
$$\boxed{=\frac{5}{14}}$$

$$b=\frac{24}{84}$$
$$\boxed{=\frac{4}{14}}$$

Thus, $\boxed{y=\frac{5}{14}x+\frac{4}{14}}$

$$A\vec{x}=\begin{pmatrix} 2 & 1 \\ 5 & 1 \\ 7 & 1 \\ 8 & 1 \end{pmatrix}\begin{pmatrix} \frac{5}{14} \\ \frac{4}{14} \end{pmatrix}$$
$$=\begin{pmatrix} 1 \\ \frac{29}{14} \\ \frac{39}{14} \\ \frac{22}{7} \end{pmatrix}$$

$$\vec{y}-A\vec{x}=\begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}-\begin{pmatrix} 1 \\ \frac{29}{14} \\ \frac{39}{14} \\ \frac{22}{7} \end{pmatrix}$$
$$=\begin{pmatrix} 0 \\ -\frac{1}{14} \\ \frac{3}{14} \\ -\frac{1}{7} \end{pmatrix}$$

Error: $\|\vec{y}-A\vec{x}\|=\sqrt{\frac{1}{196}+\frac{9}{196}+\frac{1}{49}}$

$$\boxed{=\frac{\sqrt{14}}{14}}$$
$$\boxed{\approx 0.26726}$$

****

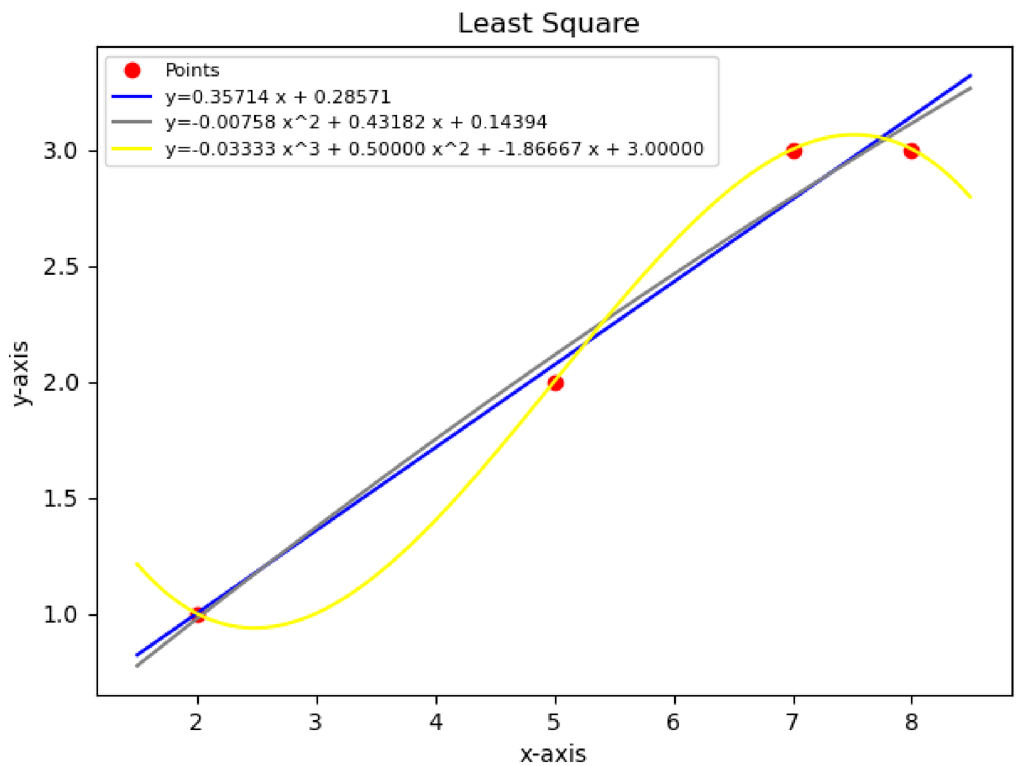
The ***second*** *order* equation: $y = -.00758x^2 + .43182x + .14394$

Error = 0. 26112

****

The ***third order*** equation: $y = -.0333x^3 + .5x^2 - 1.86667x + 3$

Error = 0.00000



Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-2,\ -2),\ (-1,\ 0),\ (0,\ -2),\ (1,\ 0)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then,

$$\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

where $A=\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ $\vec{x}=\begin{pmatrix} m \\ b \end{pmatrix}$ $\vec{y}=\begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix}$

The normal equation formula: $A^T A\vec{x}=A^T\vec{y}$

$$\begin{pmatrix} -2 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix}=\begin{pmatrix} -2 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix}=\begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

$$m=\frac{8}{20}$$

$$=\frac{2}{5}$$

$$b=-\frac{16}{20}$$

$$=-\frac{4}{5}$$

Thus, $y=\frac{2}{5}x-\frac{4}{5}$

$$A\vec{x}=\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{2}{5} \\ -\frac{4}{5} \end{pmatrix}$$

$$=\begin{pmatrix} -\frac{8}{5} \\ -\frac{6}{5} \\ -\frac{4}{5} \\ -\frac{2}{5} \end{pmatrix}$$

$$\vec{y}-A\vec{x}=\begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix}-\begin{pmatrix} -\frac{8}{5} \\ -\frac{6}{5} \\ -\frac{4}{5} \\ -\frac{2}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{5} \\ \frac{6}{5} \\ -\frac{6}{5} \\ \frac{2}{5} \end{pmatrix}$$

Error: $\quad \|\vec{y} - A\vec{x}\| = \sqrt{\frac{4}{25} + \frac{36}{25} + \frac{36}{25} + \frac{4}{25}}$

$$= \frac{4\sqrt{5}}{5}$$

$$\approx 1.78885$$

****

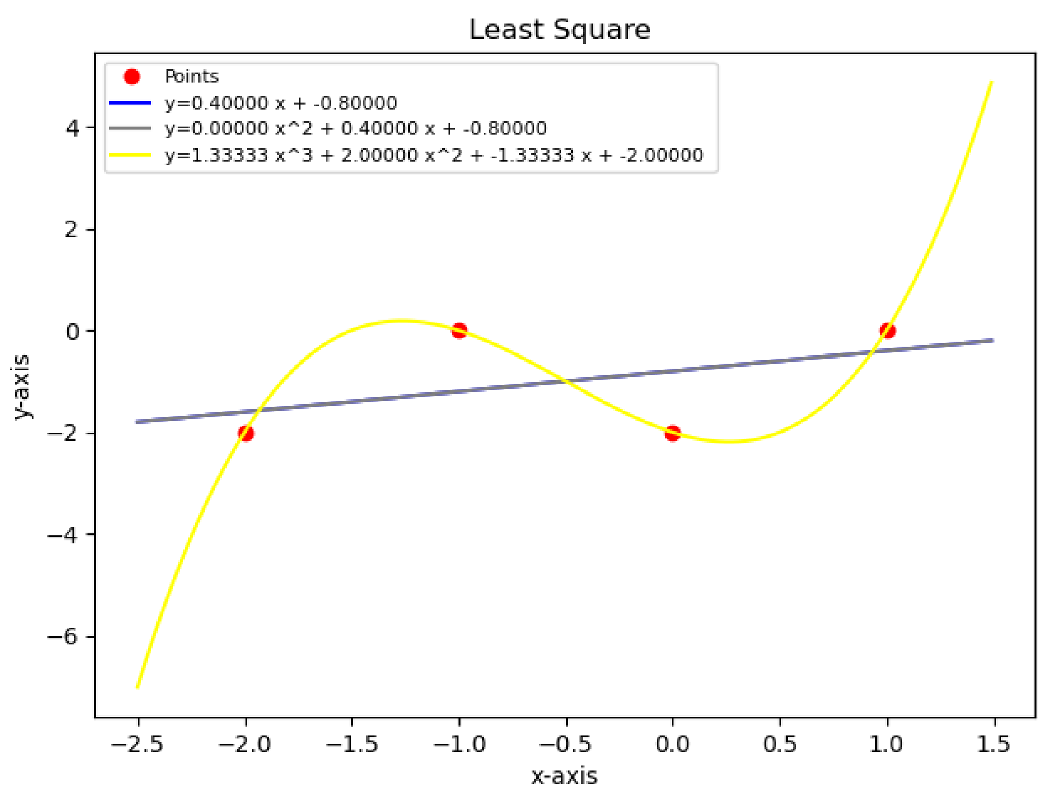
The ***second*** *order* equation: $\quad y = 0x^2 + .4x + .8$

Error = 1.78885

****

The ***third order*** equation: $\quad y = 1.333x^3 + 2x^2 - 1.333x - 2$

Error = 0.00000



Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-2,\ 0),\quad (-1,\ 1),\quad (0,\ 1),\quad (1,\ 2)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then,

$$\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

where $A = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$ $\quad \vec{y} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$

The normal equation formula: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} -2 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -2 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$m = \frac{12}{20}$$

$$= \frac{3}{5}$$

$$b = \frac{26}{20}$$

$$= \frac{13}{10}$$

Thus, $y = \frac{3}{5}x + \frac{13}{10}$ $\quad$ y = 0.60000 x + 1.3000

$$A\vec{x} = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{3}{5} \\ \frac{13}{10} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{10} \\ \frac{7}{10} \\ \frac{13}{10} \\ \frac{19}{10} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} -\frac{1}{10} \\ \frac{7}{10} \\ \frac{13}{10} \\ \frac{19}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{10} \\ \frac{3}{10} \\ -\frac{3}{10} \\ \frac{1}{10} \end{pmatrix}$$

Error: $\|\vec{y} - A\vec{x}\| = \sqrt{\frac{1}{100} + \frac{9}{100} + \frac{9}{100} + \frac{1}{100}}$

$$= \frac{2\sqrt{5}}{10}$$

$$\approx 0.44721$$

****

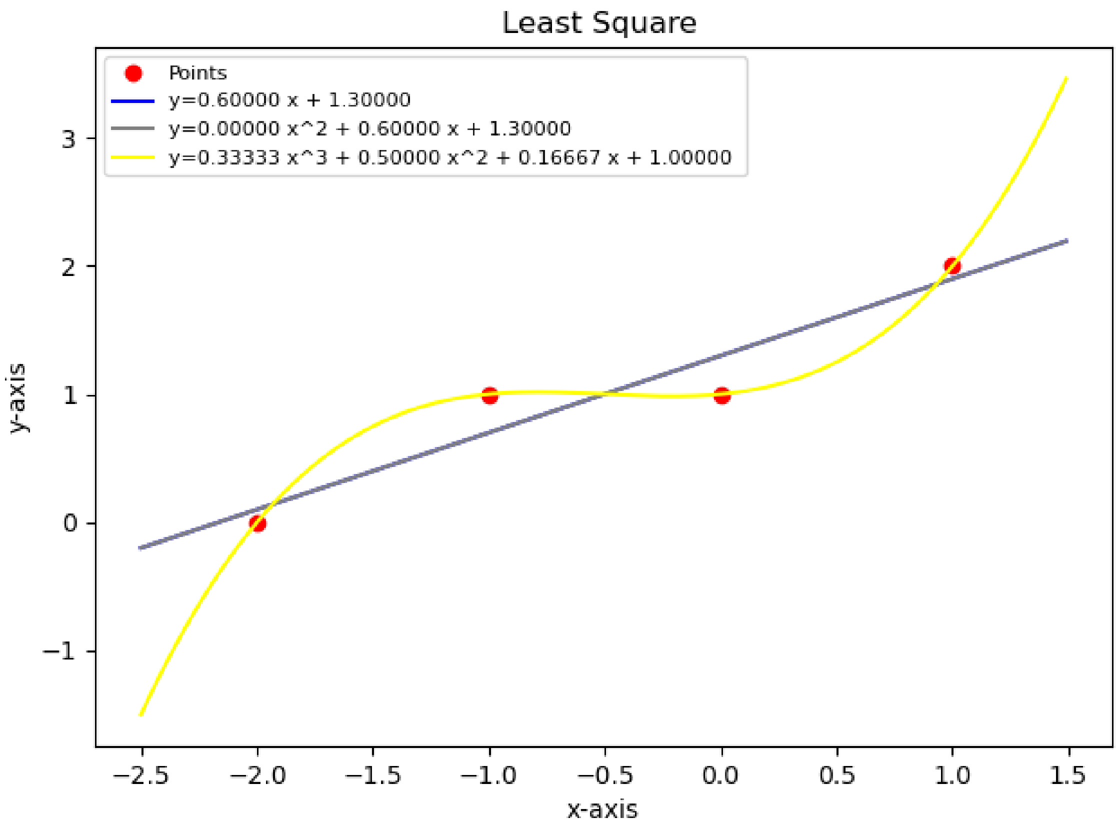
The ***second*** *order* equation: $y = 0x^2 + .6x + 1.3$

Error = 0.44721

****

The ***third order*** equation: $y = 0.333x^3 + .5x^2 + 1.6667x + 1$

Error = 0.00000



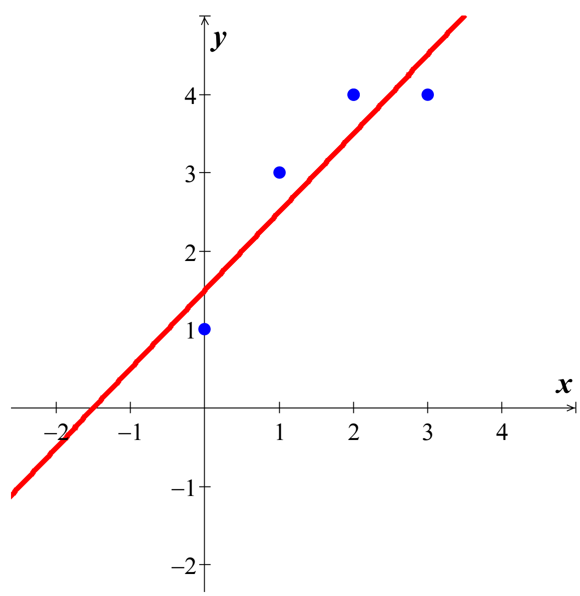
Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(0,\ 1),\ (1,\ 3),\ (2,\ 4),\ (3,\ 4)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then



$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

where $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$ $\quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$ $\quad \vec{y} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$

The normal equation: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 6 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 23 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{20}\begin{pmatrix} 4 & -6 \\ -6 & 14 \end{pmatrix}\begin{pmatrix} 23 \\ 12 \end{pmatrix} \qquad X = A^{-1}B$$

$$= \frac{1}{20}\begin{pmatrix} 20 \\ 30 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$$

We have: $m = 1 \quad and \quad b = \frac{3}{2}$.

Thus, $\boxed{y = x + \frac{3}{2}}$

$$A\vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2} \\ \frac{5}{2} \\ \frac{7}{2} \\ \frac{9}{2} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} \frac{3}{2} \\ \frac{5}{2} \\ \frac{7}{2} \\ \frac{9}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

Error:

$$\|\vec{y} - A\vec{x}\| = \sqrt{4\left(\frac{1}{4}\right)}$$

$$\underline{=1}$$

****

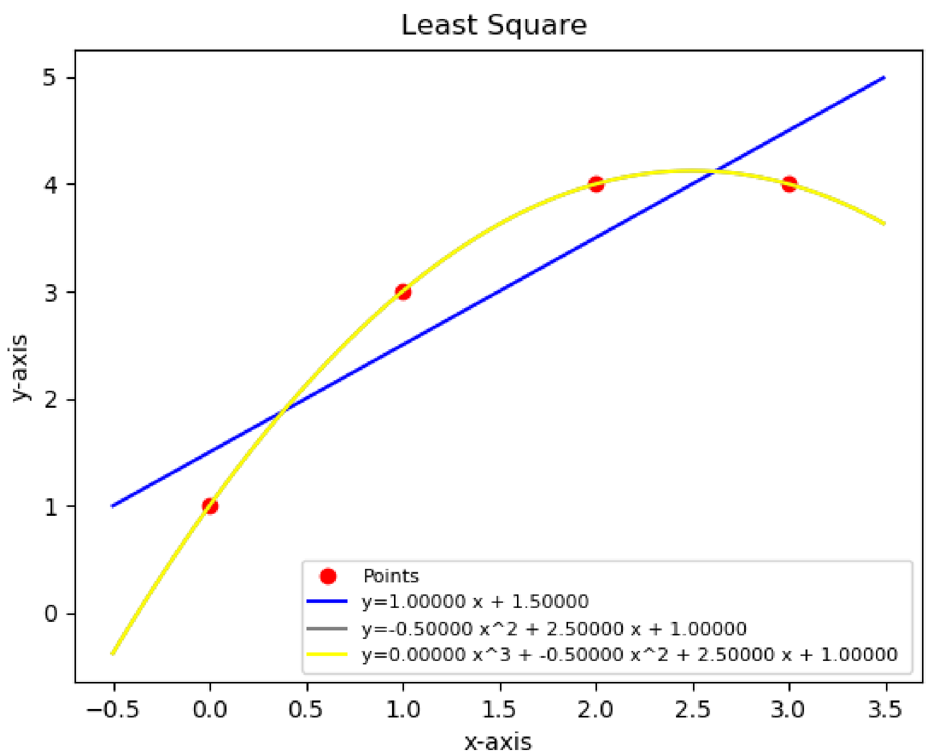
The ***second*** *order* equation: $y = -0.50x^2 + 2.5x + 1.0$

Error = 0.00000

****

The ***third order*** equation: $y = 0.0x^3 - 0.5x^2 + 2.5x + 1$

Error = 0.00000



Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(2,\ 3),\ (3,\ 2),\ (5,\ 1),\ (6,\ 0)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

where $A = \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$

The normal equation: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 2 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 74 & 16 \\ 16 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 17 \\ 6 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 74 & 16 \\ 16 & 4 \end{vmatrix} = 40 \quad \Delta_m = \begin{vmatrix} 17 & 16 \\ 6 & 4 \end{vmatrix} = -28 \quad \Delta_b = \begin{vmatrix} 74 & 17 \\ 16 & 6 \end{vmatrix} = 172$$

$$\boxed{m = -\frac{28}{40} = -\frac{7}{10}}$$

$$\boxed{b = \frac{172}{40} = \frac{43}{10}}$$

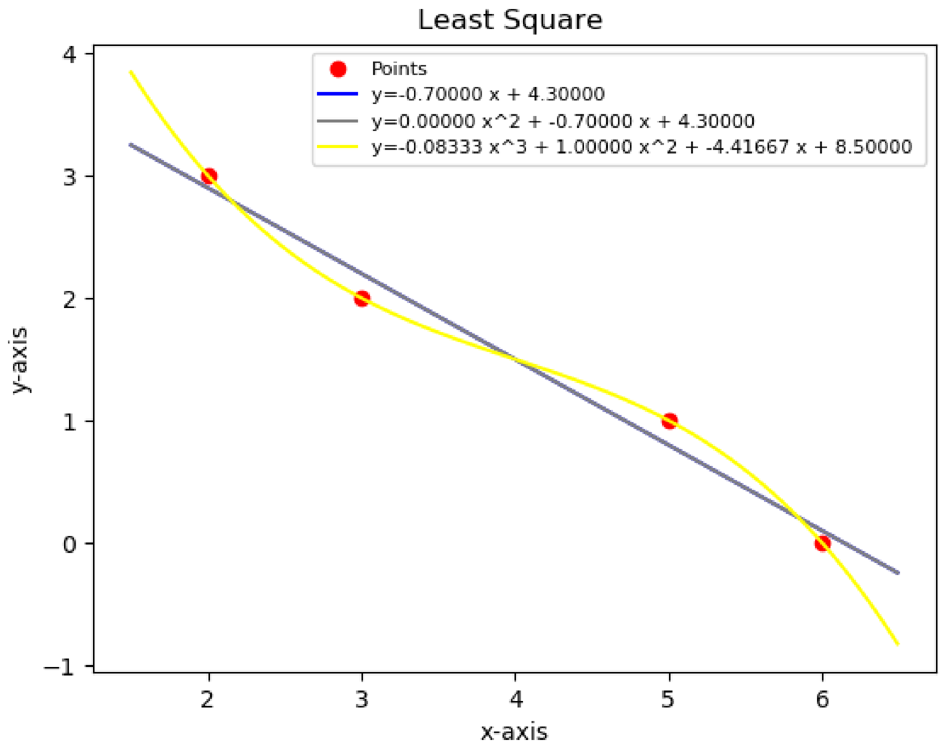
Thus, $\boxed{y = -\frac{7}{10}x + \frac{43}{10}}$

$$A\vec{x} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} -\frac{7}{10} \\ \frac{43}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{29}{10} \\ \frac{22}{10} \\ \frac{8}{10} \\ \frac{1}{10} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{29}{10} \\ \frac{22}{10} \\ \frac{8}{10} \\ \frac{1}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{10} \\ -\frac{1}{5} \\ \frac{1}{5} \\ -\frac{1}{10} \end{pmatrix}$$



Error:

$$\|\vec{y} - A\vec{x}\| = \sqrt{2\left(\frac{1}{100} + \frac{1}{25}\right)}$$

$$= \frac{\sqrt{10}}{10}$$

$$= 0.31623$$

****

The ***second order*** equation: $y = 0.0x^2 - 0.7x + 4.3$

Error = 0.31623

****

The ***third order*** equation: $y = -0.08333x^3 + x^2 - 4.41667x + 8.5$

Error = 0.00000

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-1,\ 0),\ (0,\ 1),\ (1,\ 2),\ (2,\ 4)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$$

where $A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$

The normal equation: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 10 \\ 7 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 6 & 2 \\ 2 & 4 \end{vmatrix} = 20 \quad \Delta_m = \begin{vmatrix} 10 & 2 \\ 7 & 4 \end{vmatrix} = 26 \quad \Delta_b = \begin{vmatrix} 6 & 10 \\ 2 & 7 \end{vmatrix} = 22$$

$$\boxed{m = \frac{26}{20} = \frac{13}{10}}$$

$$\boxed{b = \frac{22}{20} = \frac{11}{10}}$$

Thus, $\boxed{y = \frac{13}{10}x + \frac{11}{10}}$

$$A\vec{x} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \frac{13}{10} \\ \frac{11}{10} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{5} \\ \frac{11}{10} \\ \frac{12}{5} \\ \frac{37}{10} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} -\frac{1}{5} \\ \frac{11}{10} \\ \frac{12}{5} \\ \frac{37}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} \\ -\frac{1}{10} \\ -\frac{2}{5} \\ \frac{3}{10} \end{pmatrix}$$

Error:

$$\|\vec{y} - A\vec{x}\| = \sqrt{\frac{1}{25} + \frac{1}{100} + \frac{4}{25} + \frac{9}{100}}$$

$$= \sqrt{\frac{4+1+16+9}{100}}$$

$$= \frac{\sqrt{30}}{10}$$

$$= 0.54772$$

****

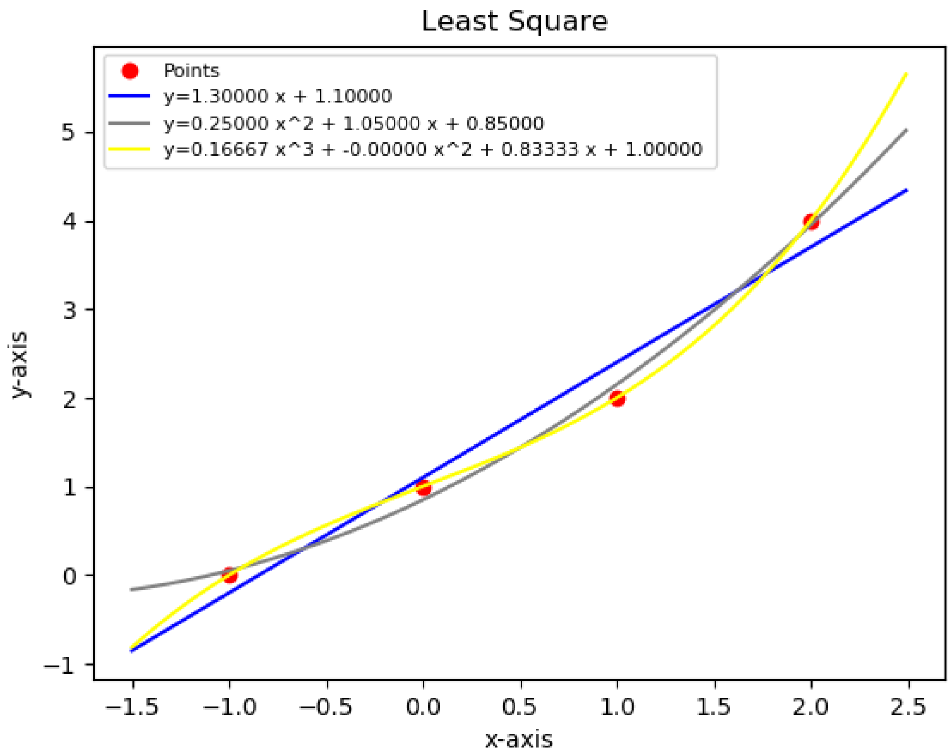
The ***second order*** equation: $y = 0.25x^2 + 1.05x + 0.85$

Error = 0.22361

****

The ***third order*** equation: $y = 0.16667x^3 + 0.82222x + 1$

Error = 0.00000



Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(1,\ 0),\ (2,\ 1),\ (4,\ 2),\ (5,\ 3)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

where $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix}$ $\quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$ $\quad \vec{y} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$

The normal equation: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 1 & 2 & 4 & 5 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 46 & 12 \\ 12 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 25 \\ 6 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 46 & 12 \\ 12 & 4 \end{vmatrix} = 40 \quad \Delta_m = \begin{vmatrix} 25 & 12 \\ 6 & 4 \end{vmatrix} = 28 \quad \Delta_b = \begin{vmatrix} 46 & 25 \\ 12 & 6 \end{vmatrix} = -24$$

$$m = \frac{28}{40} = \frac{7}{10}$$

$$b = -\frac{24}{40} = -\frac{3}{5}$$

Thus, $y = \frac{7}{10}x - \frac{3}{5}$

$$A\vec{x} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} \frac{7}{10} \\ -\frac{3}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{10} \\ \frac{4}{5} \\ \frac{11}{5} \\ \frac{29}{10} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} \frac{1}{10} \\ \frac{4}{5} \\ \frac{11}{5} \\ \frac{29}{10} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{10} \\ \frac{1}{5} \\ -\frac{1}{5} \\ \frac{1}{10} \end{pmatrix}$$

Error:

$$\|\vec{y} - A\vec{x}\| = \sqrt{2\left(\frac{1}{100} + \frac{1}{25}\right)}$$

$$= \frac{\sqrt{10}}{10}$$

$$= 0.31623$$

****

The ***second order*** equation:

$y = 0.0x^2 + 0.7x - .6$

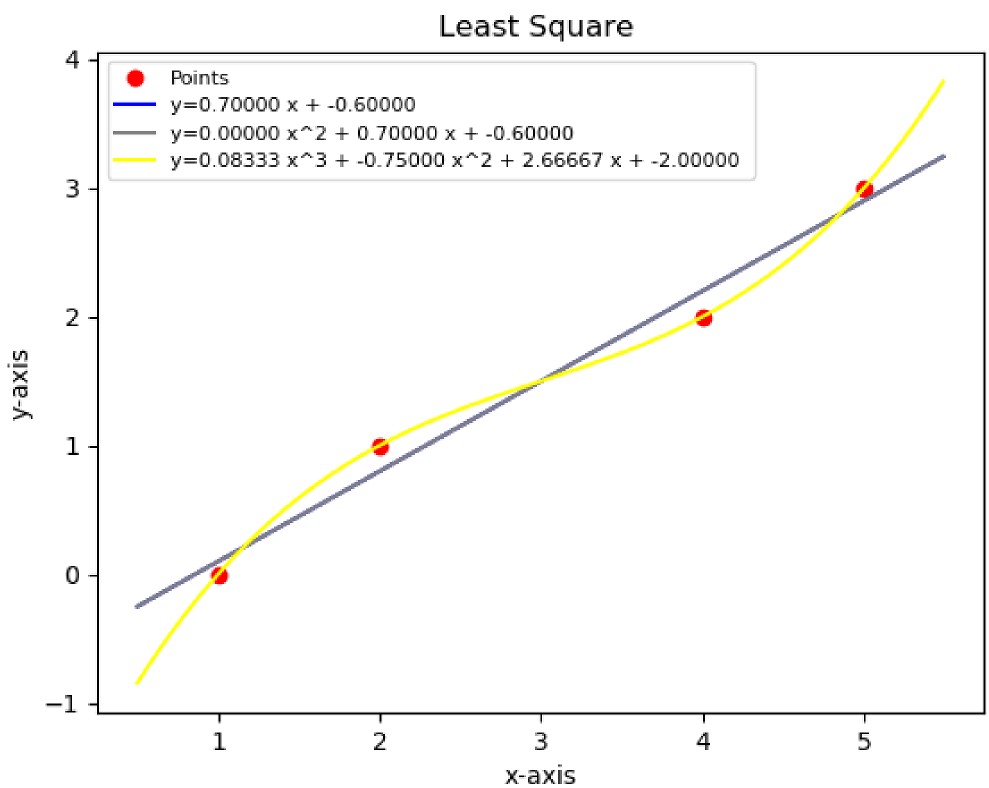
Error = 0.31623

****

The ***third order*** equation:

$y = 0.08333x^3 - 0.75x^2 + 2.66667x - 2$

Error = 0.00000



Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(1,\ 5),\ (2,\ 4),\ (3,\ 1),\ (4,\ 1),\ (5,-1)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{bmatrix}1&1\\2&1\\3&1\\4&1\\5&1\end{bmatrix}\begin{bmatrix}m\\b\end{bmatrix}=\begin{bmatrix}5\\4\\1\\1\\-1\end{bmatrix} \qquad \text{where} \quad A=\begin{bmatrix}1&1\\2&1\\3&1\\4&1\\5&1\end{bmatrix} \quad \vec{x}=\begin{bmatrix}m\\b\end{bmatrix} \quad \vec{y}=\begin{bmatrix}5\\4\\1\\1\\-1\end{bmatrix}$$

The normal equation: $A^T A\vec{x}=A^T\vec{y}$

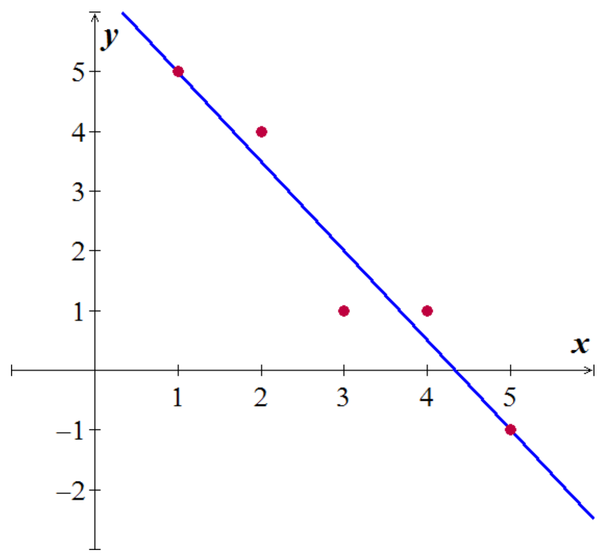
$$\begin{bmatrix}1&2&3&4&5\\1&1&1&1&1\end{bmatrix}\begin{bmatrix}1&1\\2&1\\3&1\\4&1\\5&1\end{bmatrix}\begin{bmatrix}m\\b\end{bmatrix}=\begin{bmatrix}1&2&3&4&5\\1&1&1&1&1\end{bmatrix}\begin{bmatrix}5\\4\\1\\1\\-1\end{bmatrix}$$

$$\begin{bmatrix}55&15\\15&5\end{bmatrix}\begin{bmatrix}m\\b\end{bmatrix}=\begin{bmatrix}15\\10\end{bmatrix}$$

$$m=\frac{\begin{vmatrix}15&15\\10&5\end{vmatrix}}{\begin{vmatrix}55&15\\15&5\end{vmatrix}}$$

$$=\frac{-75}{50}$$

$$\boxed{=-\frac{3}{2}}$$



$$b=\frac{\begin{vmatrix}55&15\\15&10\end{vmatrix}}{50}$$

$$=\frac{325}{50}$$

$$\boxed{=\frac{13}{2}}$$

Thus, $\boxed{y=-\frac{3}{2}x+\frac{13}{2}}$ *or* $\boxed{y=-1.5x+6.5}$

$$A\vec{x}=\begin{pmatrix}1&1\\2&1\\3&1\\4&1\\5&1\end{pmatrix}\begin{pmatrix}-\frac{3}{2}\\ \frac{13}{2}\end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ \frac{7}{2} \\ 2 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ \frac{7}{2} \\ 2 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \frac{1}{2} \\ -1 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

Error:

$$\left\| \vec{y} - A\vec{x} \right\| = \sqrt{\frac{1}{4} + 1 + \frac{1}{4}}$$

$$= \frac{\sqrt{6}}{2}$$

$$\approx 1.224745$$

****

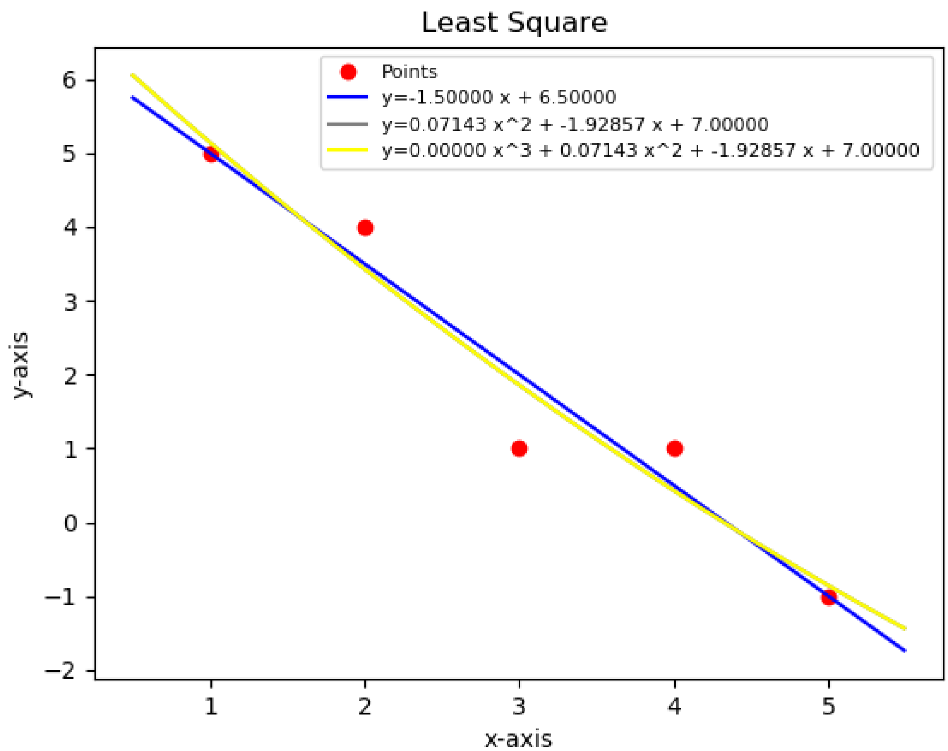
The ***second order*** equation: $y = 0.07143x^2 - 1.92857x + 7$

Error = 1.19523

****

The ***third order*** equation: $y = 0.0x^3 + 0.07143x^2 - 1.92857x + 7$

Error = 1.19523



Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-2,\ 0),\ (-1,\ 0),\ (0,\ 1),\ (1,\ 3),\ (2,\ 5)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

where $A = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$ $\quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$ $\quad \vec{y} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$

The normal equation: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

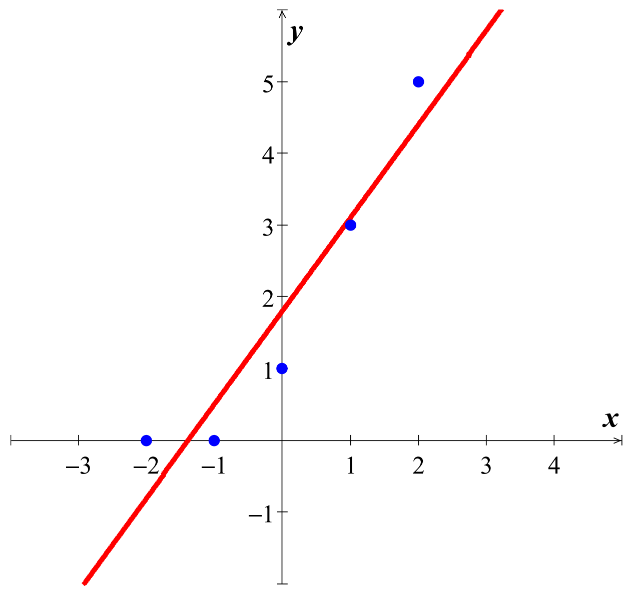
$$\begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 13 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{50} \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} 13 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{13}{10} \\ \frac{9}{5} \end{pmatrix}$$

We have: $m = 1.3 \quad and \quad b = 1.8$

Thus, $\boxed{y = \frac{13}{10}x + \frac{9}{5}}$



$$A\vec{x} = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{13}{10} \\ \frac{9}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{5} \\ \frac{1}{2} \\ \frac{9}{5} \\ \frac{31}{10} \\ \frac{22}{5} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix} - \begin{pmatrix} -\frac{4}{5} \\ \frac{1}{2} \\ \frac{9}{5} \\ \frac{31}{10} \\ \frac{22}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{5} \\ -\frac{1}{2} \\ -\frac{4}{5} \\ -\frac{1}{10} \\ \frac{3}{5} \end{pmatrix}$$

Error: $\|\vec{y} - A\vec{x}\| = \sqrt{\frac{16}{25} + \frac{1}{4} + \frac{16}{25} + \frac{1}{100} + \frac{9}{25}}$

$$= \sqrt{\frac{41}{25} + \frac{26}{100}}$$

$$= \frac{\sqrt{190}}{10}$$

$$\approx 1.37840$$

****

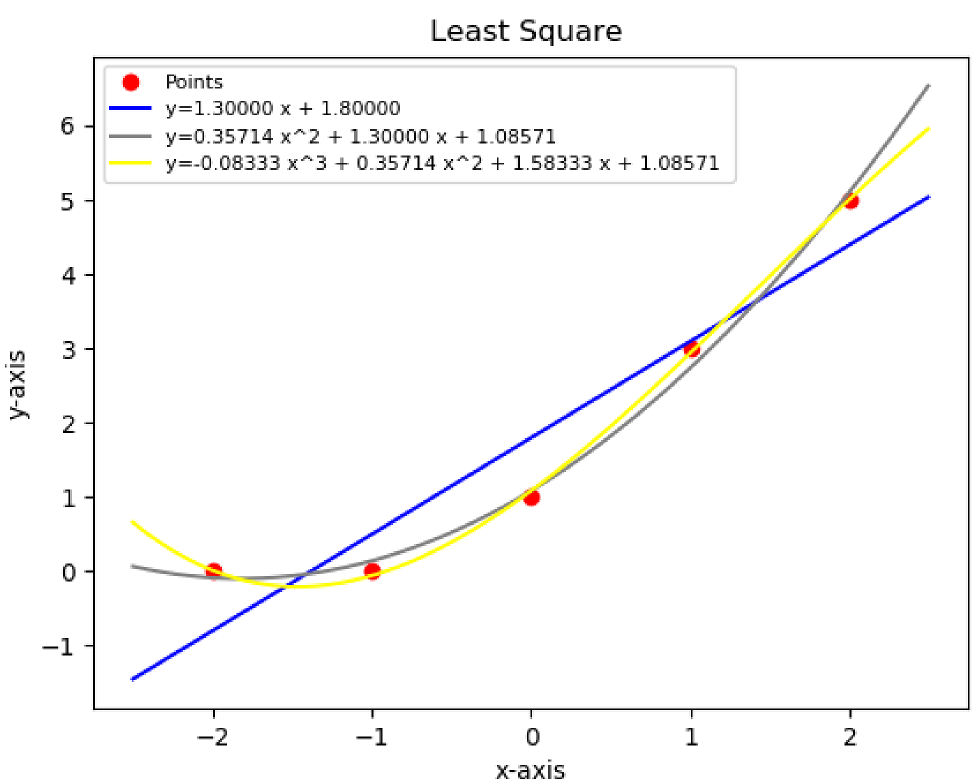
The ***second order*** equation: $y = 0.35714x^2 + 1.30x + 1.08571$

Error = 0.33806

****

The ***third order*** equation: $y = -0.08333x^3 + 0.35714x^2 + 1.58333x + 1.08571$

Error = 0.11952



Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-5,\ 10),\ \ (-1,\ 8),\ \ (3,\ 6),\ \ (7,\ 4),\ \ (5,\ 5)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} -5 & 1 \\ -1 & 1 \\ 3 & 1 \\ 7 & 1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 5 \end{pmatrix} \qquad \text{where } A = \begin{pmatrix} -5 & 1 \\ -1 & 1 \\ 3 & 1 \\ 7 & 1 \\ 5 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 5 \end{pmatrix}$$

The normal equation: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} -5 & -1 & 3 & 7 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -5 & 1 \\ -1 & 1 \\ 3 & 1 \\ 7 & 1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -5 & -1 & 3 & 7 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 109 & 9 \\ 9 & 5 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 13 \\ 33 \end{pmatrix}$$

$$m = \frac{232}{464}$$

$$\underline{= -\frac{1}{2}}$$

$$b = \frac{3,480}{464}$$

$$\underline{= \frac{15}{2}}$$

Thus, $\underline{y = -\frac{1}{2}x + \frac{15}{2}}$

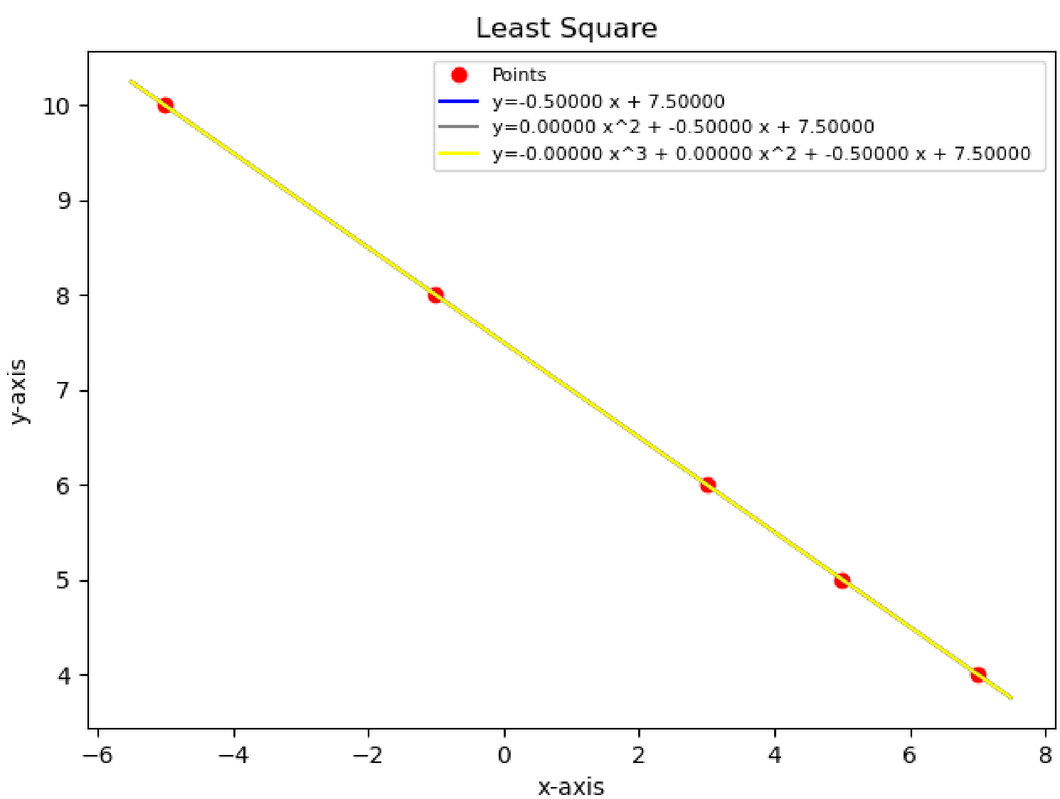
$$A\vec{x} = \begin{pmatrix} -5 & 1 \\ -1 & 1 \\ 3 & 1 \\ 7 & 1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} \\ \frac{15}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 5 \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

E*rror*: $\|\vec{y} - A\vec{x}\| = 0$



Exercise

Find the orthogonal projection of the vector $\vec{u}$ on the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{u}=(-3,\ -3,\ 8,\ 9);\quad \vec{v}_1=(3,\ 1,\ 0,\ 1),\quad \vec{v}_2=(1,\ 2,\ 1,\ 1),\quad \vec{v}_3=(-1,\ 0,\ 2,\ -1)$$

Solution

Let $A=\begin{pmatrix}3&1&-1\\1&2&0\\0&1&2\\1&1&-1\end{pmatrix}$

$$A^TA=\begin{pmatrix}3&1&0&1\\1&2&1&1\\-1&0&2&-1\end{pmatrix}\begin{pmatrix}3&1&-1\\1&2&0\\0&1&2\\1&1&-1\end{pmatrix}$$

$$=\begin{pmatrix}11&6&-4\\6&7&0\\-4&0&6\end{pmatrix}$$

$$A^T\vec{u}=\begin{pmatrix}3&1&0&1\\1&2&1&1\\-1&0&2&-1\end{pmatrix}\begin{pmatrix}-3\\-3\\8\\9\end{pmatrix}$$

$$=\begin{pmatrix}-3\\8\\10\end{pmatrix}$$

The normal solution is $A^TA\vec{x}=A^T\vec{u}$

$$\begin{pmatrix}11&6&-4\\6&7&0\\-4&0&6\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}-3\\8\\10\end{pmatrix}$$

$$\Delta=\begin{vmatrix}11&6&-4\\6&7&0\\-4&0&6\end{vmatrix}=134\qquad \Delta_1=\begin{vmatrix}-3&6&-4\\8&7&0\\10&0&6\end{vmatrix}=-134\qquad \Delta_2=\begin{vmatrix}11&-3&-4\\6&8&0\\-4&10&6\end{vmatrix}=268$$

$$x_1=\frac{-134}{134}=-1\qquad x_2=\frac{268}{134}=2\qquad x_3=\frac{134}{134}=1$$

$$\vec{x}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

So $proj_W \vec{u} = A\vec{x}$

$$=\begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$$=\begin{pmatrix} -2 \\ 3 \\ 4 \\ 0 \end{pmatrix}$$

$proj_W \vec{u} = (-2,\ 3,\ 4,\ 0)$

Exercise

Find the orthogonal projection of the vector $\vec{u}$ on the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{u}=(6,\ 3,\ 9,\ 6);\quad \vec{v}_1=(2,\ 1,\ 1,\ 1),\quad \vec{v}_2=(1,\ 0,\ 1,\ 1),\quad \vec{v}_3=(-2,\ -1,\ 0,\ -1)$$

Solution

Let $A=\begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$

$$A^TA=\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$=\begin{pmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{pmatrix}$$

$$A^T\vec{u}=\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \\ 9 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 30 \\ 21 \\ -21 \end{pmatrix}$$

The normal solution is $A^T A\vec{x} = A^T \vec{u}$

$$\begin{pmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 30 \\ 21 \\ -21 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{vmatrix} = 3 \qquad \Delta_1 = \begin{vmatrix} 30 & 4 & -6 \\ 21 & 3 & -3 \\ -21 & -3 & 6 \end{vmatrix} = 18 \qquad \Delta_2 = \begin{vmatrix} 7 & 30 & -6 \\ 4 & 21 & -3 \\ -6 & -21 & 6 \end{vmatrix} = 9$$

$x_1 = \frac{18}{3} = 6$ | $x_2 = \frac{9}{3} = 3$ | $x_3 = 4$ |

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$$

So $proj_W \vec{u} = A\vec{x}$

$$= \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 2 \\ 9 \\ 5 \end{pmatrix}$$

$proj_W \vec{v} = (7,\ 2,\ 9,\ 5)$

Exercise

Find the orthogonal projection of the vector $\vec{u}$ on the subspace of $\mathbb{R}^4$ spanned by the vectors

$\vec{u} = (-2,\ 0,\ 2,\ 4);\quad v_1 = (1,\ 1,\ 3,\ 0),\quad \vec{v}_2 = (-2,\ -1,\ -2,\ 1),\quad \vec{v}_3 = (-3,\ -1,\ 1,\ 3)$

Solution

Let $A=\begin{pmatrix}1&-2&-3\\1&-1&-1\\3&-2&1\\0&1&3\end{pmatrix}$

$$A^T A=\begin{pmatrix}1&1&3&0\\-2&-1&-2&1\\-3&-1&1&3\end{pmatrix}\begin{pmatrix}1&-2&-3\\1&-1&-1\\3&-2&1\\0&1&3\end{pmatrix}$$

$$=\begin{pmatrix}11&-9&-1\\-9&10&8\\-1&8&20\end{pmatrix}$$

$$A^T\vec{u}=\begin{pmatrix}1&1&3&0\\-2&-1&-2&1\\-3&-1&1&3\end{pmatrix}\begin{pmatrix}-2\\0\\2\\4\end{pmatrix}$$

$$=\begin{pmatrix}4\\4\\20\end{pmatrix}$$

The normal solution is $A^T A\vec{x}=A^T\vec{u}$

$$\begin{pmatrix}11&-9&-1\\-9&10&8\\-1&8&20\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}4\\4\\20\end{pmatrix}$$

$$\Delta=\begin{vmatrix}11&-9&-1\\-9&10&8\\-1&8&20\end{vmatrix}=10 \qquad \Delta_1=\begin{vmatrix}4&-9&-1\\4&10&8\\20&8&20\end{vmatrix}=-8 \qquad \Delta_2=\begin{vmatrix}11&4&-1\\-9&4&8\\-1&20&20\end{vmatrix}=-16$$

$$\underline{x_1=\frac{-8}{10}=-\frac{4}{5}} \quad \underline{x_2=\frac{-16}{10}=-\frac{8}{5}} \quad \underline{x_3=\frac{8}{5}}$$

$$\vec{x}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}-\frac{4}{5}\\-\frac{8}{5}\\\frac{8}{5}\end{pmatrix}$$

$$proj_W\vec{u}=A\vec{x}$$

$$= \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} -\frac{4}{5} \\ -\frac{8}{5} \\ \frac{8}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{12}{5} \\ -\frac{4}{5} \\ \frac{12}{5} \\ \frac{16}{5} \end{pmatrix}$$

$$\underline{proj_W \vec{u} = \left(-\frac{12}{5},\ -\frac{4}{5},\ \frac{12}{5},\ \frac{16}{5}\right)}$$

Exercise

For the given matrix A and $\vec{y}$

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 2 \\ 0 \\ 11 \end{pmatrix}$$

a) Find the least-squares solution of the linear system $A\vec{x} = \vec{y}$

b) Find the orthogonal projection of $\vec{y}$ on the column space of A

c) Find the ***error vector*** and the ***error***

Solution

a) Let $\vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$

$$A^T A\vec{x} = A^T \vec{y}$$

$$\begin{pmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 11 \end{pmatrix}$$

$$\begin{pmatrix} 17 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 19 \\ 11 \end{pmatrix}$$

$$m = \frac{84}{84}$$

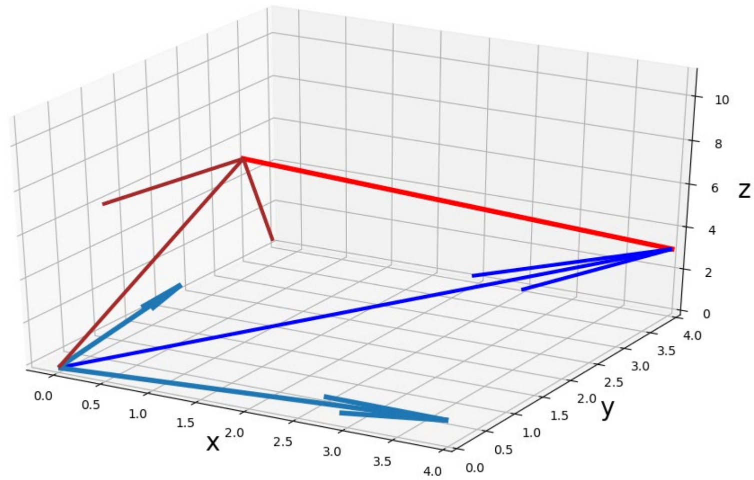
$$\underline{= 1}$$

$$b = \frac{168}{84}$$
$$= 2$$

Thus $y = x + 2$

b) The orthogonal projection of $\vec{y}$ on the column space of A

$$A\vec{x} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix}$$



c) $\vec{y} - A\vec{x} = \begin{pmatrix} 2 \\ 0 \\ 11 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix}$
$$= \begin{pmatrix} -2 \\ -4 \\ 8 \end{pmatrix}$$

The ***error***: $\quad \|\vec{y} - A\vec{x}\| = \sqrt{4 + 16 + 64}$
$$= 2\sqrt{21}$$
$$\approx 9.16515$$

Exercise

For the given matrix A and $\vec{y}$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -3 \\ -1 & 3 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

a) Find the least-squares solution of the linear system $A\vec{x} = \vec{y}$

b) Find the orthogonal projection of $\vec{y}$ on the column space of A

c) Find the ***error vector*** and the ***error***

Solution

a) Let $\vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$

$$A^T A\vec{x} = A^T \vec{y}$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & -3 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -7 \\ -7 & 22 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \end{pmatrix}$$

$$m = \frac{165}{83}$$

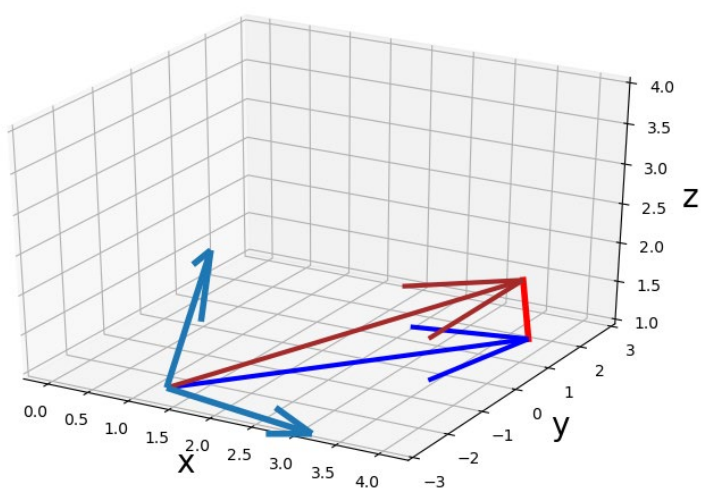
$$b = \frac{94}{83}$$

Thus $y = \frac{165}{83}x + \frac{94}{83}$

b) The orthogonal projection of $\vec{y}$ on the column space of A

$$A\vec{x} = \begin{pmatrix} 1 & 2 \\ 2 & -3 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} \frac{165}{83} \\ \frac{94}{83} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{353}{83} \\ \frac{48}{83} \\ \frac{117}{83} \end{pmatrix}$$



c) $\vec{y} - A\vec{x} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} \frac{353}{83} \\ \frac{48}{83} \\ \frac{117}{83} \end{pmatrix}$

$$= \begin{pmatrix} -\frac{21}{83} \\ -\frac{35}{83} \\ \frac{49}{83} \end{pmatrix}$$

The ***error***: $\|\vec{y} - A\vec{x}\| = \frac{1}{83}\sqrt{441 + 1,225 + 2,401}$

$$= \frac{\sqrt{4,067}}{83}$$

$$= \frac{7\sqrt{83}}{83}$$

$$\approx 0.76835$$

Exercise

For the given matrix A and $\vec{y}$

$$A = \begin{pmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} -5 \\ 8 \\ 1 \end{pmatrix}$$

a) Find the least-squares solution of the linear system $A\vec{x} = \vec{y}$

b) Find the orthogonal projection of $\vec{y}$ on the column space of A

c) Find the ***error vector*** and the ***error***

Solution

a) Let $\vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$

$$A^T A\vec{x} = A^T\vec{y}$$

$$\begin{pmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{pmatrix}\begin{pmatrix} -5 \\ 8 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 12 & 8 \\ 8 & 10 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -24 \\ -2 \end{pmatrix}$$

$$m = -\frac{224}{56}$$

$$= -4$$

$$b = \frac{168}{56}$$

$$= 3$$

Thus $y = -4x + 3$

b) The orthogonal projection of $\vec{y}$ on the column space of A

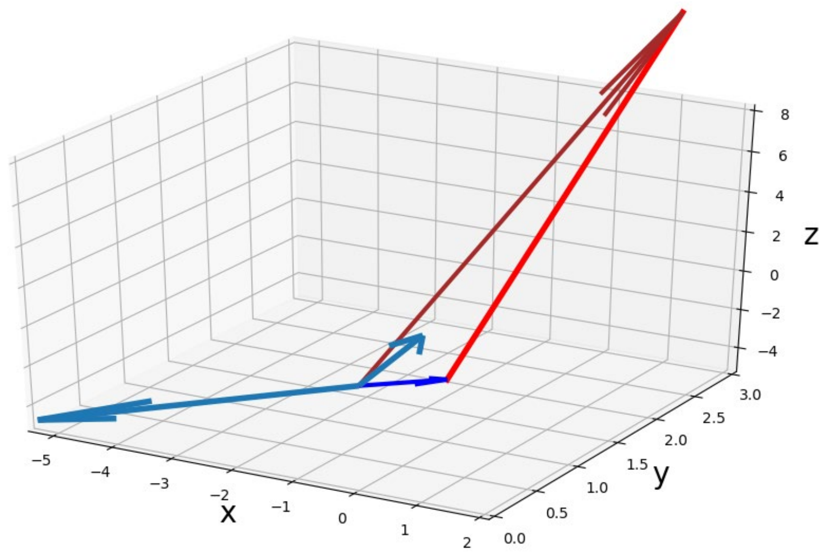
$$A\vec{x} = \begin{pmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 8 \\ 1 \end{pmatrix}$$

c) $\vec{y} - A\vec{x} = \begin{pmatrix} -5 \\ 8 \\ 1 \end{pmatrix} - \begin{pmatrix} -5 \\ 8 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The ***error***: $\boxed{\left\| \vec{y} - A\vec{x} \right\| = 0}$



Exercise

For the given matrix A and $\vec{y}$

$$A = \begin{pmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 6 \end{pmatrix}$$

a) Find the least-squares solution of the linear system $A\vec{x} = \vec{y}$

b) Find the orthogonal projection of $\vec{y}$ on the column space of A

c) Find the ***error vector*** and the ***error***

Solution

a) Let $\vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$

$$A^T A\vec{x} = A^T \vec{y}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -6 & -2 & 1 & 7 \end{pmatrix} \begin{pmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -6 & -2 & 1 & 7 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 0 \\ 0 & 90 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 8 \\ 45 \end{pmatrix}$$

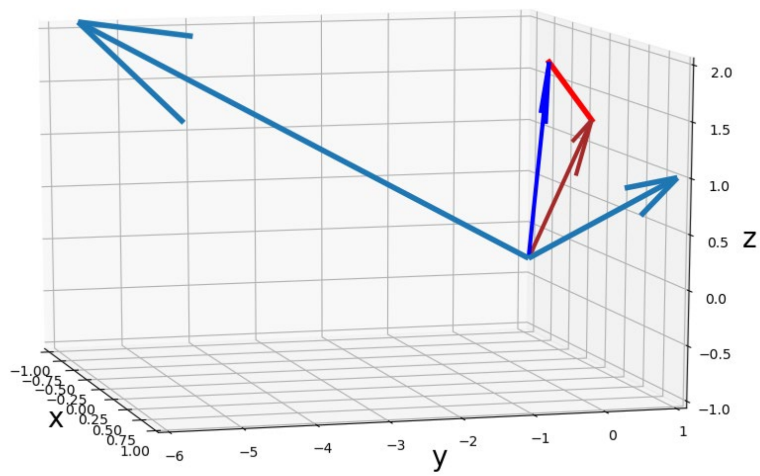
$$m = \frac{720}{360}$$
$$= 2$$

$$b = \frac{180}{360}$$
$$= \frac{1}{2}$$

Thus $y = 2x + \frac{1}{2}$

b) The orthogonal projection of $\vec{y}$ on the column space of A

$$A\vec{x} = \begin{pmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{pmatrix}\begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix}$$
$$= \begin{pmatrix} -1 \\ 1 \\ 3 \\ \frac{11}{2} \end{pmatrix}$$



c) $\vec{y} - A\vec{x} = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 6 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 3 \\ \frac{11}{2} \end{pmatrix}$

$$= \begin{pmatrix} 0 \\ 1 \\ -2 \\ \frac{1}{2} \end{pmatrix}$$

The ***error***: $\quad \|\vec{y} - A\vec{x}\| = \sqrt{1 + 4 + \frac{1}{4}}$

$$= \frac{\sqrt{21}}{2}$$
$$\approx 2.2913$$

Exercise

For the given matrix A and $\vec{y}$

$$A=\begin{pmatrix}1 & -2\\ -1 & 2\\ 0 & 3\\ 2 & 5\end{pmatrix}\qquad \vec{y}=\begin{pmatrix}3\\ 1\\ -4\\ 2\end{pmatrix}$$

a) Find the least-squares solution of the linear system $A\vec{x}=\vec{y}$

b) Find the orthogonal projection of $\vec{y}$ on the column space of A

c) Find the ***error vector*** and the ***error***

Solution

a) Let $\vec{x}=\begin{pmatrix}m\\ b\end{pmatrix}$

$$A^T A\vec{x}=A^T\vec{y}$$

$$\begin{pmatrix}1 & -1 & 0 & 2\\ -2 & 2 & 3 & 5\end{pmatrix}\begin{pmatrix}1 & -2\\ -1 & 2\\ 0 & 3\\ 2 & 5\end{pmatrix}\begin{pmatrix}m\\ b\end{pmatrix}=\begin{pmatrix}1 & -1 & 0 & 2\\ -2 & 2 & 3 & 5\end{pmatrix}\begin{pmatrix}3\\ 1\\ -4\\ 2\end{pmatrix}$$

$$\begin{pmatrix}6 & 6\\ 6 & 42\end{pmatrix}\begin{pmatrix}m\\ b\end{pmatrix}=\begin{pmatrix}6\\ -6\end{pmatrix}$$

$$m=\frac{288}{216}$$
$$=\frac{4}{3}$$

$$b=-\frac{72}{216}$$
$$=-\frac{1}{3}$$

Thus $y=\frac{4}{3}x-\frac{1}{3}$

b) The orthogonal projection of $\vec{y}$ on the column space of A

$$A\vec{x}=\begin{pmatrix}1 & -2\\ -1 & 2\\ 0 & 3\\ 2 & 5\end{pmatrix}\begin{pmatrix}\frac{4}{3}\\ -\frac{1}{3}\end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -2 \\ -1 \\ 1 \end{pmatrix}$$

c) $\vec{y} - A\vec{x} = \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ -1 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 \\ 3 \\ -3 \\ 1 \end{pmatrix}$$

The ***error***: $\quad \|\vec{y} - A\vec{x}\| = \sqrt{1+9+9+1}$

$$= 2\sqrt{5} \qquad \approx 4.47214$$

Exercise

For the given matrix A and $\vec{y}$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \qquad \vec{y} = \begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix}$$

a) Find the least-squares solution of the linear system $A\vec{x} = \vec{y}$

b) Find the orthogonal projection of $\vec{y}$ on the column space of A

c) Find the ***error vector*** and the ***error.***

Solution

a) Let $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$A^T A\vec{x} = A^T \vec{y}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 14 \\ 4 \\ 10 \end{pmatrix}$$

$$\begin{cases} 4a + 2b + 2c = 14 \\ 2a + 2b = 4 \\ 2a + 2c = 10 \end{cases}$$

$$\begin{cases} 2a + b + c = 7 \\ a + b \ = 2 \\ a + c \ = 5 \end{cases}$$

$$\Delta = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 0 \qquad \Delta_c = \begin{vmatrix} 2 & 1 & 7 \\ 1 & 1 & 2 \\ 1 & 0 & 5 \end{vmatrix} = 0$$

$$\begin{cases} b = 2 - a \\ c = 5 - a \end{cases}$$

Assume $\underline{a = 1 \quad \rightarrow \quad b = 1 \quad c = 4}$

Thus $\underline{x + y + 4z = 0}$

b) The orthogonal projection of $\vec{y}$ on the column space of A

$$A\vec{x} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 2 \\ 5 \\ 5 \end{pmatrix}$$

c) $$\vec{y} - A\vec{x} = \begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 5 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 1 \\ 3 \\ -3 \end{pmatrix}$$

The ***error***: $\quad \|\vec{y} - A\vec{x}\| = \sqrt{1 + 1 + 9 + 9}$

$$= 2\sqrt{5}$$

$$\approx 4.47214$$

Exercise

For the given matrix A and $\vec{y}$

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 3 \\ 5 \\ 7 \\ -3 \end{pmatrix}$$

a) Find the least-squares solution of the linear system $A\vec{x} = \vec{y}$

b) Find the orthogonal projection of $\vec{y}$ on the column space of A

c) Find the ***error vector*** and the ***error.***

Solution

a) Let $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$A^T A\vec{x} = A^T \vec{y}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 5 & 0 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 5 & 0 & 2 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \\ 7 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 8 & 10 \\ 8 & 20 & 26 \\ 10 & 26 & 38 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \\ 20 \end{pmatrix}$$

$$\begin{cases} 2a + 4b + 5c = 6 \\ 4a + 10b + 13c = 6 \\ 5a + 13b + 19c = 10 \end{cases}$$

$$\Delta = \begin{vmatrix} 2 & 4 & 5 \\ 4 & 10 & 13 \\ 5 & 13 & 19 \end{vmatrix} = 8 \qquad \Delta_a = \begin{vmatrix} 6 & 4 & 5 \\ 6 & 10 & 13 \\ 10 & 13 & 19 \end{vmatrix} = 80$$

$$\Delta_b = \begin{vmatrix} 2 & 6 & 5 \\ 4 & 6 & 13 \\ 5 & 10 & 19 \end{vmatrix} = -48 \qquad \Delta_c = \begin{vmatrix} 2 & 4 & 6 \\ 4 & 10 & 6 \\ 5 & 13 & 10 \end{vmatrix} = 16$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 10 \\ -6 \\ 2 \end{pmatrix}$$

Thus $\underline{10x - 6y + 2z = 0}$

b) The orthogonal projection of $\vec{y}$ on the column space of A

$$A\vec{x} = \begin{pmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix} \begin{pmatrix} 10 \\ -6 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 4 \\ 8 \\ -2 \end{pmatrix}$$

c) $$\vec{y} - A\vec{x} = \begin{pmatrix} 3 \\ 5 \\ 7 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 8 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

The ***error***: $\underline{\|\vec{y} - A\vec{x}\| = 2}$

Exercise

For the given matrix A and $\vec{y}$

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{pmatrix}$$

a) Find the least-squares solution of the linear system $A\vec{x} = \vec{y}$

b) Find the orthogonal projection of $\vec{y}$ on the column space of A

c) Find the ***error vector*** and the ***error.***

Solution

d) Let $\vec{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

$$A^T A\vec{x} = A^T \vec{y}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 4 & 2 & 2 \\ 4 & 4 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 2 \\ 6 \end{pmatrix}$$

$$\begin{cases} 3a + 2b + c + d = 2 \\ 2a + 2b + d = 1 \\ a + c = 1 \\ a + b + d = 3 \end{cases}$$

$$\Delta = \begin{vmatrix} 3 & 2 & 1 & 1 \\ 2 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix} = 0 \qquad \Delta_d = \begin{vmatrix} 3 & 2 & 1 & 2 \\ 2 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 3 \end{vmatrix} = 0$$

$(3) \quad \rightarrow \quad \underline{c = 1 - a}$

$$\begin{cases} 2a + 2b + d = 1 \\ \ \ a + b + d = 3 \end{cases}$$
$$a + b = -2$$

$\underline{b = -2 - a}$

$\underline{d = 5}$

Assume $\quad a = 0 \quad \rightarrow \quad b = -2 \quad c = 1$

$-2y + z + 5 = 0$

Therefore, $\quad \underline{-2y + z = -5}$

e) The orthogonal projection of $\vec{y}$ on the column space of A

$$A\vec{x} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \\ 1 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ -2 \\ 1 \\ 1 \\ 3 \\ 3 \end{pmatrix}$$

f) $$\vec{y} - A\vec{x} = \begin{pmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \\ 1 \\ 1 \\ 3 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 2 \\ -2 \end{pmatrix}$$

The ***error***: $\|\vec{y}-A\vec{x}\| = \sqrt{1+1+1+1+4+4}$

$$= \sqrt{12}$$

$$\underline{= 2\sqrt{3}}$$

$$\underline{\approx 3.4641}$$

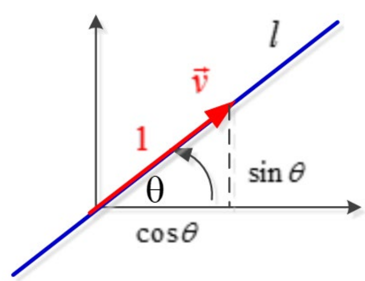
Exercise

Find the standard matrix for the orthogonal projection P of $\mathbb{R}^2$ on the line passes through the origin and makes an angle θ with the positive x-axis.

Solution

Since the line l in 2-dimensional, than we can take $\vec{v} = (\cos\theta,\ \sin\theta)$ as a basis for this subspace

$$A = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$



$$[P] = A^T A$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

Exercise

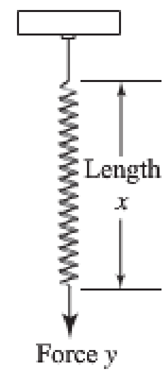
Hooke's law in physics states that the length x of a uniform spring is a linear function of the force y applied to it. If we express the relationship as $y = mx + b$, then the coefficient m is called the spring constant.

Suppose a particular unstretched spring has a measured length of 6.1 *inches*.(i.e., $x = 6.1$ when $y = 0$). Forces of 2 pounds, 4 pounds, and 6 pounds are then applied to the spring, and the corresponding lengths are found to be 7.6 inches, 8.7 inches, and 10.4 inches. Find the spring constant.

Solution

$$M = \begin{pmatrix} 6.1 & 1 \\ 7.6 & 1 \\ 8.7 & 1 \\ 10.4 & 1 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \end{pmatrix}$$

The normal equation: $A^T A\vec{x} = A^T\vec{y}$



$$\begin{pmatrix} 6.1 & 7.6 & 8.7 & 10.4 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 6.1 & 1 \\ 7.6 & 1 \\ 8.7 & 1 \\ 10.4 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 6.1 & 7.6 & 8.7 & 10.4 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 278.82 & 32.8 \\ 32.8 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 112.4 \\ 12 \end{pmatrix}$$

$$\begin{aligned}\begin{pmatrix} m \\ b \end{pmatrix} &= \frac{1}{39.44}\begin{pmatrix} 4 & -32.8 \\ -32.8 & 278.2 \end{pmatrix}\begin{pmatrix} 112.4 \\ 12 \end{pmatrix} \\ &= \frac{1}{39.44}\begin{pmatrix} 56 \\ -348.32 \end{pmatrix} \\ &= \begin{pmatrix} 1.4 \\ -8.8 \end{pmatrix}\end{aligned}$$

Thus, the estimated value of the spring constant is ≈ 1.4 *pounds*

Exercise

Prove:

If A has a linearly independent column vectors, and if $\vec{b}$ is orthogonal to the column space of A, then the least squares solution of $A\vec{x} = \vec{b}$ is $\vec{x} = \vec{0}$.

Solution

If A has linearly independent column vectors, then $A^T A$ is invertible and the least squares solution of $A\vec{x} = \vec{b}$ is the solution of $A^T A\vec{x} = A^T\vec{b}$, but since $\vec{b}$ is orthogonal to the column space of A.

$A^T\vec{b} = 0$, so $\vec{x}$ is a solution of $A^T A\vec{x} = 0$.

Thus $\vec{x} = \vec{0}$ since $A^T A$ is invertible.

Exercise

Let A be an $m \times n$ matrix with linearly independent row vectors. Find a standard matrix for the orthogonal projection of $\mathbb{R}^n$ onto the row space of A.

Solution

A^T will have linearly independent column vectors, and the column space A^T is the row space of A. Thus, the standard matrix for the orthogonal projection of $\mathbb{R}^n$ onto the row space of A is

$$[P] = A^T\left[\left(A^T\right)^T A^T\right]^{-1}\left(A^T\right)^T$$

$$= A^T\left(AA^T\right)^{-1}A$$

Exercise

Let W be the line with parametric equations $x = 2t, \quad y = -t, \quad z = 4t$

a) Find a basis for W.

b) Find the standard matrix for the orthogonal projection on W.

c) Use the matrix in part (b) to find the orthogonal projection of a point $P_0\left(x_0, y_0, z_0\right)$ on W.

d) Find the distance between the point $P_0\left(2, 1, -3\right)$ and the line W.

Solution

a) $W = span\{(2, -1, 4)\}$

So that the vector $(2, -1, 4)$ forms a basis for W (linear independence)

b) Let $A = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$

$$[P] = A\left(A^T A\right)^{-1} A^T$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}\left(\begin{bmatrix} 2 & -1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}\right)^{-1}\begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}[21]^{-1}\begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \frac{1}{21}\begin{bmatrix} 4 & -2 & 8 \\ -2 & 1 & -4 \\ 8 & -4 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix}$$

c) $\begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} \frac{4}{21}x_0 - \frac{2}{21}y_0 + \frac{8}{21}z_0 \\ -\frac{2}{21}x_0 + y_0 - \frac{4}{21}z_0 \\ \frac{8}{21}x_0 - \frac{4}{21}y_0 + \frac{16}{21}z_0 \end{bmatrix}$

d) $\begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -\frac{6}{7} \\ \frac{3}{7} \\ -\frac{12}{7} \end{bmatrix}$

The distance between P_0 and W equals to the distance between P_0 and its projection on W.

The distance between $(2,\ 1,\ -3)$ and $\left(-\frac{6}{7},\ \frac{3}{7},\ -\frac{12}{7}\right)$ is

$$d = \sqrt{\left(2+\frac{6}{7}\right)^2 + \left(1-\frac{3}{7}\right)^2 + \left(-3+\frac{12}{7}\right)^2}$$

$$= \sqrt{\frac{400}{49} + \frac{16}{49} + \frac{81}{49}}$$

$$= \frac{\sqrt{497}}{7}$$

Exercise

In R^3, consider the line l given by the equations $\quad x = t, \quad y = t, \quad z = t$

And the line m given by the equations $\quad x = s, \quad y = 2s - 1, \quad z = 1$

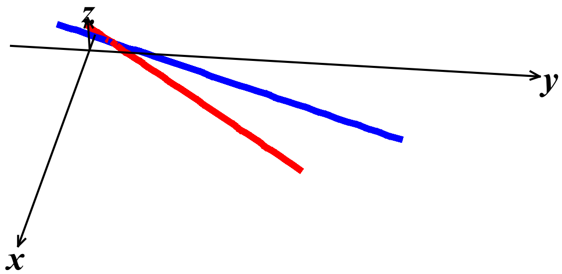
Let P be the point on l, and let Q be a point on m. Find the values of t and s that minimize the distance between the lines by minimizing the squared distance $\|P - Q\|^2$

Solution

When $t = 1 \quad \Rightarrow \quad Let\ P = (1,\ 1,\ 1)$ is on line l

When $s = 1 \quad \Rightarrow \quad Let\ Q = (1,\ 1,\ 1)$ is on line m

$$\|P - Q\| = \sqrt{(1-1)^2 + (1-1)^2 + (1-1)^2} = 0 \ge 0$$



Thus, these are the values $P=(1,\ 1,\ 1)$ and $Q=(1,\ 1,\ 1)$ are the values for $s=t=1$ that minimize the distance between the lines.

Exercise

Determine whether the statement is true or false,

a) If A is an $m\times n$ matrix, then $A^T A$ is a square matrix.

b) If $A^T A$ is invertible, then A is invertible.

c) If A is invertible, then $A^T A$ is invertible.

d) If $A\vec{x}=\vec{b}$ is a consistent linear system, then $A^T A\vec{x}=A^T\vec{b}$ is also consistent.

e) If $A\vec{x}=\vec{b}$ is an inconsistent linear system, then $A^T A\vec{x}=A^T\vec{b}$ is also inconsistent.

f) Every linear system has a least squares solution.

g) Every linear system has a unique least squares solution.

h) If A is an $m\times n$ matrix with linearly independent columns and $\vec{b}$ is in R^m, then $A\vec{x}=\vec{b}$ has a unique least squares solution.

Solution

a) True; $A^T A$ is an $n\times n$ matrix.

b) False; only square matrix has inverses, but $A^T A$ can be invertible when A is not square matrix.

c) True; if A is invertible, so is A^T, so the product $A^T A$ is also invertible.

d) True

e) False; the system $A^T A\vec{x}=A^T\vec{b}$ may be consistent.

f) True

g) False; the least squares solution may involve a parameter.

h) True; if A has linearly independent column vectors; then $A^T A$ is invertible, so $A^T A\vec{x}=A^T\vec{b}$ has a unique solution.

Exercise

A certain experiment produces the data $\{(1,\ 1.8),\ (2,\ 2.7),\ (3,\ 3.4),\ (4,\ 3.8),\ (5,\ 3.9)\}$.

Find the function that it will fit these data in the form of $y = \beta_1 x + \beta_2 x^2$

Solution

Given: the equation $y = \beta_1 x + \beta_2 x^2$ that best fits the given points. Then

$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{pmatrix}$$

where $A = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{pmatrix}$

The normal equation formula: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \end{pmatrix}\begin{pmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{pmatrix}$$

$$\begin{pmatrix} 55 & 225 \\ 225 & 979 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 52.1 \\ 201.5 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 55 & 225 \\ 225 & 979 \end{vmatrix} = 3{,}220 \qquad \Delta_{\beta_1} = \begin{vmatrix} 52.1 & 225 \\ 201.5 & 979 \end{vmatrix} = 5{,}668.4 \qquad \Delta_{\beta_2} = \begin{vmatrix} 55 & 52.1 \\ 225 & 201.5 \end{vmatrix} = -640$$

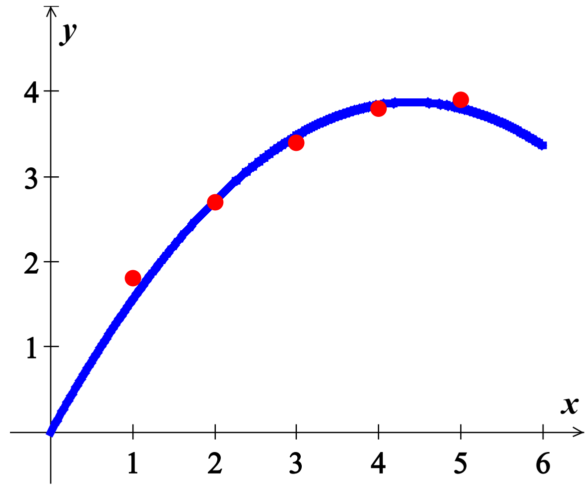
$$\beta_1 = \frac{5{,}668.4}{3{,}220}$$

$$\approx 1.76$$

$$\beta_2 = -\frac{640}{3{,}220}$$

$$\approx -0.199$$

$$y = 1.76x - .2x^2$$



Exercise

According to Kepler's first law, a comet should have an ellipse, parabolic, or hyperbolic orbit (with gravitational attractions from the planets ignored). In suitable polar coordinates, the position $(r,\ \upsilon)$ of a comet satisfies an equation of the form

$$r = \beta + e(r \cdot \cos\upsilon)$$

Where β is a constant and e is the eccentricity of the orbit, with $0 \le e < 1$ for an ellipse, $e = 1$ for a parabolic, and $e > 1$ for a hyperbola.

Suppose observations of a newly discovered comet provide the data below.

υ	.88	1.10	1.42	1.77	2.14
r	3.00	2.30	1.65	1.25	1.01

Determine the type of orbit, and predict where the orbit will be when $\upsilon = 4.6$ $(radians)$?

Solution

Given: the equation in the form $r = \beta + e(r \cdot \cos\upsilon)$

$3 = \beta + e(3 \cdot \cos(.88)) = \beta + 1.911e$

$2.3 = \beta + e(2.3\cos(1.1)) = \beta + 1.043e$

$1.65 = \beta + e(1.65\cos(1.42)) = \beta + .248e$

$1.25 = \beta + e(1.25\cos(1.77)) = \beta - .247e$

$1.01 = \beta + e(1.01\cos(2.14)) = \beta - .544e$

$$\begin{pmatrix} 1 & 1.911 \\ 1 & 1.043 \\ 1 & .248 \\ 1 & -.247 \\ 1 & -.544 \end{pmatrix}\begin{pmatrix} \beta \\ e \end{pmatrix} = \begin{pmatrix} 3 \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{pmatrix}$$

where $A = \begin{pmatrix} 1 & 1.911 \\ 1 & 1.043 \\ 1 & .248 \\ 1 & -.247 \\ 1 & -.544 \end{pmatrix} \quad \vec{\upsilon} = \begin{pmatrix} \beta \\ e \end{pmatrix} \quad \vec{r} = \begin{pmatrix} 3 \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{pmatrix}$

The normal equation formula: $A^T A\,\vec{\upsilon} = A^T \vec{r}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1.911 & 1.043 & .248 & -.247 & -.544 \end{pmatrix}\begin{pmatrix} 1 & 1.911 \\ 1 & 1.043 \\ 1 & .248 \\ 1 & -.247 \\ 1 & -.544 \end{pmatrix}\begin{pmatrix} \beta \\ e \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1.911 & 1.043 & .248 & -.247 & -.544 \end{pmatrix}\begin{pmatrix} 3 \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 2.411 \\ 2.411 & 5.158 \end{pmatrix}\begin{pmatrix} \beta \\ e \end{pmatrix} = \begin{pmatrix} 9.21 \\ 7.683 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 5 & 2.411 \\ 2.411 & 5.158 \end{vmatrix} = 19.98 \qquad \Delta_\beta = \begin{vmatrix} 9.21 & 2.411 \\ 7.683 & 5.158 \end{vmatrix} = 28.98 \qquad \Delta_e = \begin{vmatrix} 5 & 9.21 \\ 2.411 & 7.683 \end{vmatrix} = 16.21$$

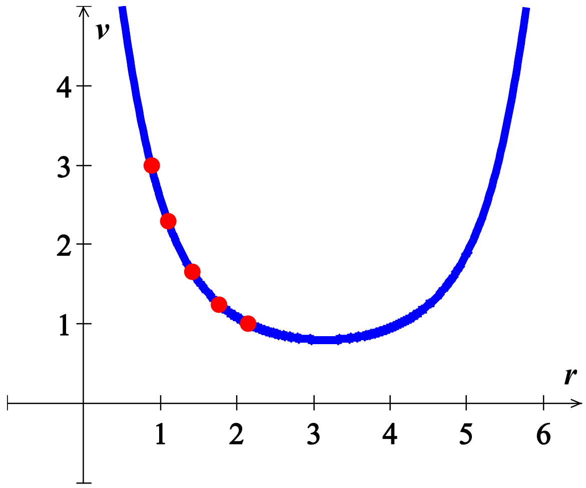
$$\beta = \frac{28.98}{19.98}$$
$$\approx 1.45$$

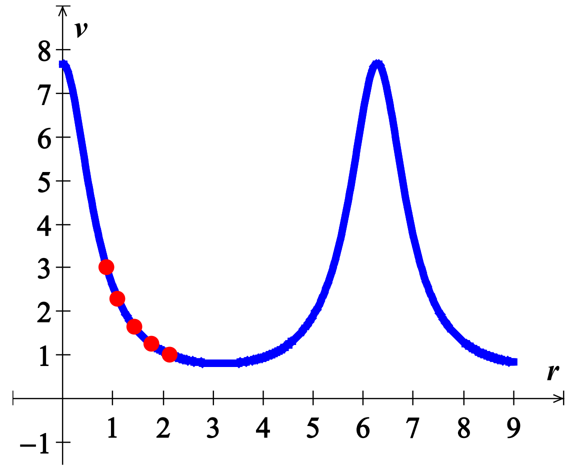
$$e = \frac{16.21}{19.98}$$
$$\approx 0.811 \quad < 1$$

Therefore, the orbit is an ***ellipse*** type since $e \approx 0.811 \quad < 1$

Since $r = \beta + e(r \cdot \cos\upsilon)$

Then, $r(\upsilon) = \dfrac{1.45}{1 - 0.811 \cdot \cos\upsilon}$





$$r(4.6) = \frac{1.45}{1 - 0.811 \cdot \cos 4.6}$$

$$\approx 1.329$$

Exercise

To measure the takeoff performance of an airplane, the horizontal position of the plane was measured every second, from $t = 0$ to $t = 12$

The position (in *feet*) were:

$$0,\ 8.8,\ 29.9,\ 62.0,\ 104.7,\ 159.1,\ 222.0,\ 294.5,\ 380.4,\ 471.1,\ 571.7,\ 686.8,\ \textit{and}\ 809.2$$

a) Find the least square cubic curve $y = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$ for these data.

b) Estimate the velocity of the plane when $t = 4.5$ *sec*, using the result from part (*a*).

Solution

Given: the equation is in form $y = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \\ 1 & 6 & 36 & 216 \\ 1 & 7 & 49 & 343 \\ 1 & 8 & 64 & 512 \\ 1 & 9 & 81 & 729 \\ 1 & 10 & 100 & 1000 \\ 1 & 11 & 121 & 1331 \\ 1 & 12 & 144 & 1728 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 8.8 \\ 29.9 \\ 62 \\ 104.7 \\ 159.1 \\ 222.0 \\ 294.5 \\ 380.4 \\ 471.1 \\ 571.7 \\ 686.8 \\ 809.2 \end{pmatrix}$$

$$A \qquad \vec{t} \quad = \quad \vec{y}$$

The normal equation formula: $A^T A\,\vec{t} = A^T \vec{y}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 & 81 & 100 & 121 & 144 \\ 0 & 1 & 8 & 27 & 64 & 125 & 216 & 343 & 512 & 729 & 1000 & 1331 & 1728 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \\ 1 & 6 & 36 & 216 \\ 1 & 7 & 49 & 343 \\ 1 & 8 & 64 & 512 \\ 1 & 9 & 81 & 729 \\ 1 & 10 & 100 & 1000 \\ 1 & 11 & 121 & 1331 \\ 1 & 12 & 144 & 1728 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 & 81 & 100 & 121 & 144 \\ 0 & 1 & 8 & 27 & 64 & 125 & 216 & 343 & 512 & 729 & 1000 & 1331 & 1728 \end{pmatrix} \begin{pmatrix} 0 \\ 8.8 \\ 29.9 \\ 62 \\ 104.7 \\ 159.1 \\ 222.0 \\ 294.5 \\ 380.4 \\ 471.1 \\ 571.7 \\ 686.8 \\ 809.2 \end{pmatrix}$$

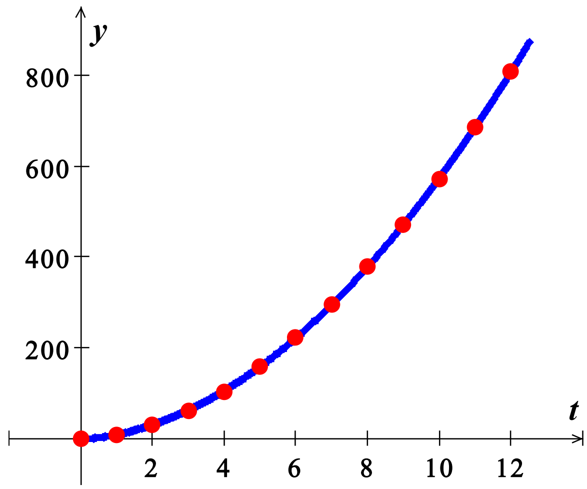
$$\begin{pmatrix} 13 & 78 & 650 & 6,084 \\ 78 & 650 & 6,084 & 60,710 \\ 650 & 6,084 & 60,710 & 630,708 \\ 6,084 & 60,710 & 630,708 & 6,735,950 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 3,800.2 \\ 35,127.7 \\ 348,063.9 \\ 3,599,800.9 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 13 & 78 & 650 & 6,084 \\ 78 & 650 & 6,084 & 60,710 \\ 650 & 6,084 & 60,710 & 630,708 \\ 6,084 & 60,710 & 630,708 & 6,735,950 \end{vmatrix} = 97,538,785,344$$

$$\Delta_0 = \begin{vmatrix} 3800.2 & 78 & 650 & 6,084 \\ 35,127.7 & 650 & 6,084 & 60,710 \\ 348,063.9 & 6,084 & 60,710 & 630,708 \\ 3,599,800.9 & 60,710 & 630,708 & 6,735,950 \end{vmatrix} = -83,470,691,303.8916$$

Or I use my program to find the values

```
rref = (Matrix([
[1, 0, 0, 0, -0.855769230765803],
[0, 1, 0, 0, 4.70248501498163],
[0, 0, 1, 0, 5.55536963037029],
[0, 0, 0, 1, -0.0273601398601744]]))
```



$\beta_0 \approx -0.855769$

$\beta_1 \approx 4.702485$

$\beta_2 \approx 5.55537$

$\beta_3 \approx -0.02736$

$y(t) = -0.855769 + 4.702485\,t + 5.55537\,t^2 - 0.02736\,t^3$

Error = 3.9734