

Section 1.7 – Cramer’s Rule

Cramer’s Rule

Theorem

If $AX = B$ is a system of a linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(B_1)}{\det(A)}$$

$$x_2 = \frac{\det(B_2)}{\det(A)}$$

$$\vdots \quad \quad \quad \vdots$$

$$x_n = \frac{\det(B_n)}{\det(A)}$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & & & a_{nn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

$$\det(B_1) = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & & & \\ \vdots & & & \\ b_n & a_{n2} & & a_{nn} \end{vmatrix}$$

Example

Use Cramer's rule to solve

$$x_1 + x_2 + x_3 = 1$$

$$-2x_1 + x_2 = 0$$

$$-4x_1 + x_3 = 0$$

Solution

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 7$$

$$|B_1| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$|B_2| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 0 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 2$$

$$|B_3| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 0 \end{vmatrix} = 4$$

$$x_1 = \frac{|B_1|}{|A|} = \frac{1}{7} \qquad x_1 = \frac{\det(A_1)}{\det(A)}$$

$$x_2 = \frac{|B_2|}{|A|} = \frac{2}{7}$$

$$x_3 = \frac{|B_3|}{|A|} = \frac{4}{7}$$

$$\textbf{Solution: } \left(\frac{1}{7}, \frac{2}{7}, \frac{4}{7} \right)$$

Example

Use Cramer's Rule to solve.

$$\begin{aligned}x_1 + \quad + 2x_3 &= 6 \\-3x_1 + 4x_2 + 6x_3 &= 30 \\-x_1 - 2x_2 + 3x_3 &= 8\end{aligned}$$

Solution

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix} \Rightarrow \det(A) = 44$$

$$\det(A_1) = \begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{vmatrix} = -40$$

$$\det(A_2) = \begin{vmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{vmatrix} = 72$$

$$\det(A_3) = \begin{vmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{vmatrix} = 152$$

$$x_1 = \frac{-40}{44} \qquad x_1 = \frac{\det(A_1)}{\det(A)}$$
$$\underline{= -\frac{10}{11}}$$

$$x_2 = \frac{72}{44} \qquad x_2 = \frac{\det(A_2)}{\det(A)}$$
$$\underline{= \frac{18}{11}}$$

$$x_3 = \frac{152}{44} \qquad x_3 = \frac{\det(A_3)}{\det(A)}$$
$$\underline{= \frac{38}{11}}$$

$$\text{Solution: } \underline{\left(-\frac{10}{11}, \frac{18}{11}, \frac{38}{11} \right)}$$

A Formula for A^{-1}

Theorem: Inverse of a matrix using its Adjoint

The i, j entry of A^{-1} is the cofactor C_{ji} (not C_{ij}) divided by $\det(A)$:

$$\text{Formula for } A^{-1}: \quad \left(A^{-1}\right)_{ij} = \frac{C_{ji}}{|A|} \quad \text{and} \quad A^{-1} = \frac{C^T}{|A|}$$

$$\boxed{A^{-1} = \frac{1}{\det(A)} \text{adj}(A)}$$

Example

Find the inverse matrix of $A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$ using its adjoint.

Solution

$$\begin{aligned} C_{11} &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1; & C_{12} &= -\begin{vmatrix} -2 & 0 \\ -4 & 1 \end{vmatrix} = 2; & C_{13} &= \begin{vmatrix} -2 & 1 \\ -4 & 0 \end{vmatrix} = 4 \\ C_{21} &= -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1; & C_{22} &= \begin{vmatrix} 1 & 1 \\ -4 & 1 \end{vmatrix} = 5; & C_{23} &= -\begin{vmatrix} 1 & 1 \\ -4 & 0 \end{vmatrix} = -4 \\ C_{31} &= \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1; & C_{32} &= -\begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = -2; & C_{33} &= \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3 \end{aligned}$$

$$C = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 5 & -2 \\ 4 & -4 & 3 \end{bmatrix}$$

$$\det(A) = \frac{1}{7}$$

$$\Rightarrow A^{-1} = \frac{1}{7} \begin{bmatrix} 1 & -1 & -1 \\ 2 & 5 & -2 \\ 4 & -4 & 3 \end{bmatrix}$$

Theorem

If A is an $n \times n$ matrix, then the following statements are equivalent

- a)*** A is invertible
- b)*** $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
- c)*** The reduced row echelon form of A is I_n
- d)*** A can be expressed as a product of elementary matrices
- e)*** $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix \mathbf{b}
- f)*** $\det(A) \neq 0$

Exercises Section 1.7 – Cramer's Rule

1. Use Cramer's Rule with ratios $\frac{\det B_j}{\det A}$ to solve $A\mathbf{x} = b$. Also find the inverse matrix $A^{-1} = \frac{C^T}{\det A}$.

Why is the solution \mathbf{x} is the first part the same as column 3 of A^{-1} ? Which cofactors are involved in computing that column \mathbf{x} ?

$$Ax = b \quad \text{is} \quad \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

2. Verify that $\det(AB) = \det(BA)$ and determine whether the equality $\det(A+B) = \det(A) + \det(B)$ holds

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

3. Verify that $\det(kA) = k^n \det(A)$

$$a) \quad A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}, \quad k = 2$$

$$c) \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix}, \quad k = 3$$

$$b) \quad A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{bmatrix}, \quad k = -2$$

(4 – 58) Use Cramer's rule to solve the system

$$4. \quad \begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$$

$$8. \quad \begin{cases} 3x + 4y = 2 \\ 2x + 5y = -1 \end{cases}$$

$$12. \quad \begin{cases} 2x + 10y = -14 \\ 7x - 2y = -16 \end{cases}$$

$$5. \quad \begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$$

$$9. \quad \begin{cases} 5x - 2y = 4 \\ -10x + 4y = 7 \end{cases}$$

$$13. \quad \begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$$

$$6. \quad \begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$$

$$10. \quad \begin{cases} x - 4y = -8 \\ 5x - 20y = -40 \end{cases}$$

$$14. \quad \begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$$

$$7. \quad \begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$$

$$11. \quad \begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$$

$$15. \quad \begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$$

$$16. \begin{cases} x - 2y = 5 \\ -10x + 2y = 4 \end{cases}$$

$$17. \begin{cases} 12x + 15y = -27 \\ 30x - 15y = -15 \end{cases}$$

$$18. \begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$$

$$19. \begin{cases} x + y = 7 \\ x - y = 3 \end{cases}$$

$$20. \begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$$

$$21. \begin{cases} 12x + 3y = 15 \\ 2x - 3y = 13 \end{cases}$$

$$22. \begin{cases} x - 2y = 5 \\ 5x - y = -2 \end{cases}$$

$$23. \begin{cases} 3x + 2y = 2 \\ 2x + 2y = 3 \end{cases}$$

$$24. \begin{cases} 4x - 5y = 17 \\ 2x + 3y = 3 \end{cases}$$

$$25. \begin{cases} x - 3y = 4 \\ 3x - 4y = 12 \end{cases}$$

$$26. \begin{cases} 2x - 9y = 5 \\ 3x - 3y = 11 \end{cases}$$

$$27. \begin{cases} 3x - 4y = 4 \\ x + y = 6 \end{cases}$$

$$28. \begin{cases} 3x = 7y + 1 \\ 2x = 3y - 1 \end{cases}$$

$$29. \begin{cases} 2x = 3y + 2 \\ 5x = 51 - 4y \end{cases}$$

$$30. \begin{cases} y = -4x + 2 \\ 2x = 3y - 1 \end{cases}$$

$$31. \begin{cases} 3x = 2 - 3y \\ 2y = 3 - 2x \end{cases}$$

$$32. \begin{cases} x + 2y - 3 = 0 \\ 12 = 8y + 4x \end{cases}$$

$$33. \begin{cases} 7x - 2y = 3 \\ 3x + y = 5 \end{cases}$$

$$34. \begin{cases} 3x + 2y - z = 4 \\ 3x - 2y + z = 5 \\ 4x - 5y - z = -1 \end{cases}$$

$$35. \begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$$

$$36. \begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$$

$$37. \begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$$

$$38. \begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$$

$$39. \begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$$

$$40. \begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$$

$$41. \begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$$

$$42. \begin{cases} 3x - 4y + 4z = 7 \\ x - y - 2z = 2 \\ 2x - 3y + 6z = 5 \end{cases}$$

$$43. \begin{cases} x - 2y - z = 2 \\ 2x - y + z = 4 \\ -x + y + z = 4 \end{cases}$$

$$44. \begin{cases} x + y + z = 3 \\ -y + 2z = 1 \\ -x + z = 0 \end{cases}$$

$$45. \begin{cases} 3x + y + 3z = 14 \\ 7x + 5y + 8z = 37 \\ x + 3y + 2z = 9 \end{cases}$$

$$46. \begin{cases} 4x - 2y + z = 7 \\ x + y + z = -2 \\ 4x + 2y + z = 3 \end{cases}$$

$$47. \begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$$

$$48. \begin{cases} 2x - 2y + z = -4 \\ 6x + 4y - 3z = -24 \\ x - 2y + 2z = 1 \end{cases}$$

$$49. \begin{cases} 9x + 3y + z = 4 \\ 16x + 4y + z = 2 \\ 25x + 5y + z = 2 \end{cases}$$

$$50. \begin{cases} 2x - y + 2z = -8 \\ x + 2y - 3z = 9 \\ 3x - y - 4z = 3 \end{cases}$$

$$51. \begin{cases} x - 3z = -5 \\ 2x - y + 2z = 16 \\ 7x - 3y - 5z = 19 \end{cases}$$

$$52. \begin{cases} x + 2y - z = 5 \\ 2x - y + 3z = 0 \\ 2y + z = 1 \end{cases}$$

$$53. \begin{cases} x + y + z = 6 \\ 3x + 4y - 7z = 1 \\ 2x - y + 3z = 5 \end{cases}$$

$$54. \begin{cases} 3x + 2y + 3z = 3 \\ 4x - 5y + 7z = 1 \\ 2x + 3y - 2z = 6 \end{cases}$$

$$55. \begin{cases} 4x + 5y = 2 \\ 11x + y + 2z = 3 \\ x + 5y + 2z = 1 \end{cases}$$

$$56. \begin{cases} x - 4y + z = 6 \\ 4x - y + 2z = -1 \\ 2x + 2y - 3z = -20 \end{cases}$$

$$57. \begin{cases} 2x - y + z = -1 \\ 3x + 4y - z = -1 \\ 4x - y + 2z = -1 \end{cases}$$

$$58. \begin{cases} -x_1 - 4x_2 + 2x_3 + x_4 = -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 = 14 \\ -x_1 + x_2 + 3x_3 + x_4 = 11 \\ -x_1 - 2x_2 + x_3 - 4x_4 = -4 \end{cases}$$

59. Show that the matrix A is invertible for all values of θ , then find A^{-1} using $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$

$$A = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$