Solution Section 4.4 – Green's Theorem

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\boldsymbol{F} = (x-y)\boldsymbol{i} + (y-x)\boldsymbol{j}$ and curve C is the square bounded by $x=0,\ x=1,\ y=0,\ y=1$

Solution

$$M = x - y \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = -1$$

$$N = y - x \quad \Rightarrow \quad \frac{\partial N}{\partial x} = -1, \quad \frac{\partial N}{\partial y} = 1$$

$$\begin{aligned}
Flux &= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy \\
&= \iint_R (1+1)\, dxdy \\
&= 2\int_0^1 \int_0^1 dxdy \\
&= 2\int_0^1 dy \\
&= 2
\end{aligned}$$

$$\begin{aligned}
Circulation &= \iint_R \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dxdy \\
&= \int_0^1 \int_0^1 \left(-1-(-1)\right) dxdy \\
&= 0
\end{aligned}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\boldsymbol{F} = \left(x^2 + 4y\right)\boldsymbol{i} + \left(x + y^2\right)\boldsymbol{j}$ and curve C is the square bounded by $x=0,\ x=1,\ y=0,\ y=1$

Solution

$$M = x^2 + 4y \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 2x, \quad \frac{\partial M}{\partial y} = 4$$

$$N = x + y^2 \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 1, \quad \frac{\partial N}{\partial y} = 2y$$

$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_0^1 (2x + 2y)\, dxdy$$

$$= \int_0^1 \left[x^2 + 2yx \right]_0^1 dy$$

$$= \int_0^1 (1 + 2y)\, dy$$

$$= y + y^2 \Big|_0^1$$

$$\underline{= 2}$$

$$Circulation = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_0^1 (1 - 4)\, dxdy$$

$$= -3 \int_0^1 dy$$

$$\underline{= -3}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\boldsymbol{F} = (x + y)\boldsymbol{i} - \left(x^2 + y^2\right)\boldsymbol{j}$ and curve C is the triangle bounded by $y = 0,\ x = 1,\ y = x$

Solution

$$M = x + y \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = 1$$

$$N = -\left(x^2 + y^2\right) \quad \Rightarrow \quad \frac{\partial N}{\partial x} = -2x, \quad \frac{\partial N}{\partial y} = -2y$$

$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_0^x (1-2y)\,dydx$$

$$= \int_0^1 \left[y - y^2 \right]_0^x dx$$

$$= \int_0^1 \left(x - x^2 \right) dx$$

$$= \left[\frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$\boxed{= \frac{1}{6}}$$

$$Circulation = \iint\limits_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_0^x (-2x-1)\,dydx$$

$$= \int_0^1 \left[-2xy - y \right]_0^x dx$$

$$= \int_0^1 \left(-2x^2 - x \right) dx$$

$$= \left[-\frac{2}{3}x^3 - \frac{1}{2}x^2 \right]_0^1$$

$$= -\frac{2}{3} - \frac{1}{2}$$

$$\boxed{= -\frac{7}{6}}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\boldsymbol{F} = \left(xy + y^2 \right)\boldsymbol{i} + (x-y)\,\boldsymbol{j}$ and curve C

Solution

$$M = xy + y^2 \quad \Rightarrow \quad \frac{\partial M}{\partial x} = y, \quad \frac{\partial M}{\partial y} = x + 2y$$

$$N = x - y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 1, \quad \frac{\partial N}{\partial y} = -1$$

$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (y-1)\, dydx$$

$$= \int_0^1 \left[\frac{1}{2} y^2 - y \right]_{x^2}^{\sqrt{x}} dx$$

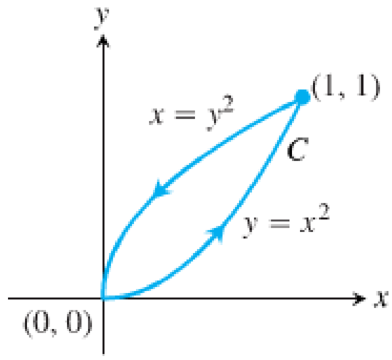
$$= \int_0^1 \left(\frac{1}{2} x - \sqrt{x} - \left(\frac{1}{2} x^4 - x^2 \right) \right) dx$$

$$= \int_0^1 \left(\frac{1}{2} x - x^{1/2} - \frac{1}{2} x^4 + x^2 \right) dx$$

$$= \left[\frac{1}{4} x^2 - \frac{2}{3} x^{3/2} - \frac{1}{10} x^5 + \frac{1}{3} x^3 \right]_0^1$$

$$= \frac{1}{4} - \frac{2}{3} - \frac{1}{10} + \frac{1}{3}$$

$$= -\frac{11}{60}$$



$$Circulation = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (1 - x - 2y)\, dydx$$

$$= \int_0^1 \left[y - xy - y^2 \right]_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 \left(\sqrt{x} - x\sqrt{x} - x - x^2 + x^3 + x^4 \right) dx$$

$$= \frac{2}{3} x^{3/2} - \frac{2}{5} x^{5/2} - \frac{1}{2} x^2 - \frac{1}{3} x^3 + \frac{1}{4} x^4 + \frac{1}{5} x^5 \Big|_0^1$$

$$= \frac{2}{3} - \frac{2}{5} - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

$$= -\frac{7}{60}$$

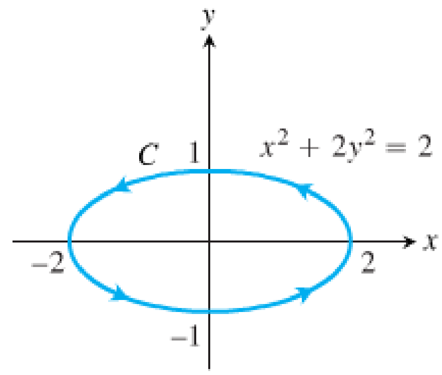
Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\boldsymbol{F} = (x+3y)\boldsymbol{i} + (2x-y)\boldsymbol{j}$ and curve C

Solution

$$M = x+3y \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = 3$$

$$N = 2x-y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 2, \quad \frac{\partial N}{\partial y} = -1$$



$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{(2-x^2)/2}}^{\sqrt{(2-x^2)/2}} (1-1)\, dydx$$

$$\underline{= 0}$$

$$Circulation = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{(2-x^2)/2}}^{\sqrt{(2-x^2)/2}} (2-3)\, dydx$$

$$= -\int_{-\sqrt{2}}^{\sqrt{2}} \left(\sqrt{\frac{2-x^2}{2}} + \sqrt{\frac{2-x^2}{2}}\right) dx$$

$$= -\frac{2}{\sqrt{2}} \int_{-\sqrt{2}}^{\sqrt{2}} \left(\sqrt{2-x^2}\right) dx \qquad \int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$= -\frac{2}{\sqrt{2}} \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{-\sqrt{2}}^{\sqrt{2}}$$

$$= -\frac{2}{\sqrt{2}} \left[0 + \sin^{-1}\frac{\sqrt{2}}{\sqrt{2}} - \left(0 + \sin^{-1}\frac{-\sqrt{2}}{\sqrt{2}}\right)\right]$$

$$= -\frac{2}{\sqrt{2}} \left[\frac{\pi}{2} + \frac{\pi}{2}\right]$$

$$= -\frac{2\pi}{\sqrt{2}}$$

$$\underline{= -\pi\sqrt{2}}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\mathbf{F} = \left(x + e^x \sin y\right)\mathbf{i} + \left(x + e^x \cos y\right)\mathbf{j}$ and curve C is the right-hand loop of the lemniscate $r^2 = \cos 2\theta$

Solution

$$M = x + e^x \sin y \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 1 + e^x \sin y, \quad \frac{\partial M}{\partial y} = e^x \cos y$$

$$N = x + e^x \cos y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 1 + e^x \cos y, \quad \frac{\partial N}{\partial y} = -e^x \sin y$$

$$Flux = \iint_R \left(1 + e^x \sin y - e^x \sin y\right) dxdy \qquad\qquad Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$= \iint_R dxdy$$

$$= \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} r \; drd\theta$$

$$= \int_{-\pi/4}^{\pi/4} \left[\frac{1}{2} r^2\right]_0^{\sqrt{\cos 2\theta}} d\theta$$

$$= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos 2\theta d\theta$$

$$= \frac{1}{4} \left[\sin 2\theta\right]_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{4} \left(1 - (-1)\right)$$

$$= \frac{1}{2}$$

$$Circulation = \iint_R \left(1 + e^x \cos y - e^x \cos y\right) dxdy \qquad\qquad Circulation = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= \iint_R dxdy$$

$$= \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} r \; drd\theta$$

$$= \frac{1}{2}$$

Exercise

Use Green's Theorem to find the counterclockwise circulation and outward flux for the field and curves

Square: $\boldsymbol{F} = (2xy + x)\boldsymbol{i} + (xy - y)\boldsymbol{j}$ C: *The square bounded by* $x = 0,\ x = 1,\ y = 0,\ y = 1$

Solution

$$M = 2xy + x \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 2y + 1, \quad \frac{\partial M}{\partial y} = 2x$$

$$N = xy - y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = y, \quad \frac{\partial N}{\partial y} = x - 1$$

$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$\begin{aligned}
Flux &= \iint_R (2y + 1 + x - 1)\, dxdy \\
&= \int_0^1 \int_0^1 (2y + x)\, dydx \\
&= \int_0^1 \left(y^2 + xy\right)\Big|_0^1\, dx \\
&= \int_0^1 (1 + x)\, dx \\
&= \left(x + \frac{1}{2}x^2\right)\Big|_0^1 \\
&= 1 + \frac{1}{2} \\
&= \frac{3}{2}
\end{aligned}$$

$$Circulation = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$\begin{aligned}
Cir &= \int_0^1 \int_0^1 (y - 2x)\, dydx \\
&= \int_0^1 \left(\frac{1}{2}y^2 - 2xy\right)\Big|_0^1\, dx \\
&= \int_0^1 \left(\frac{1}{2} - 2x\right) dx \\
&= \left(\frac{1}{2}x - x^2\right)\Big|_0^1 \\
&= \frac{1}{2} - 1 \\
&= -\frac{1}{2}
\end{aligned}$$

Exercise

Use Green's Theorem to find the counterclockwise circulation and outward flux for the field and curves

Triangle: $\boldsymbol{F} = \left(y - 6x^2\right)\boldsymbol{i} + \left(x + y^2\right)\boldsymbol{j}$

C: *The triangle made by the lines* $y = 0,\ y = x,\ and\ x = 1$

Solution

$$M = y - 6x^2 \quad \Rightarrow \quad \frac{\partial M}{\partial x} = -12x, \quad \frac{\partial M}{\partial y} = 1$$

$$N = x + y^2 \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 1, \quad \frac{\partial N}{\partial y} = 2y$$

$$Flux = \int_0^1 \int_y^1 (-12x + 2y)\,dxdy$$

$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dxdy$$

$$= \int_0^1 \left(-6x^2 + 2yx\right)\Big|_y^1 \,dy$$

$$= \int_0^1 \left(-6 + 2y + 6y^2 - 2y^2\right)\Big|_y^1 \,dy$$

$$= \int_0^1 \left(4y^2 + 2y - 6\right)dy$$

$$= \left(\frac{4}{3}y^3 + y^2 - 6y\right)\Big|_0^1$$

$$= \frac{4}{3} + 1 - 6$$

$$= -\frac{11}{3}$$

$$Cir = \iint_R (1 - 1)\,dydx$$

$$Circulation = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

$$= 0$$

Exercise

Find the circulation and the outward flux of the vector field $\boldsymbol{F} = \langle y - x,\ y\rangle$ for the curve

$\boldsymbol{r}(t) = \langle 2\cos t,\ 2\sin t\rangle, \quad 0 \le t \le 2\pi$

Solution

$$\vec{F} = \langle 2\sin t - 2\cos t,\ 2\sin t\rangle$$

$$\vec{r}' = \langle -2\sin t,\ 2\cos t\rangle$$

$$\vec{F}\cdot\vec{r}' = \langle 2\sin t - 2\cos t,\ 2\sin t\rangle \cdot \langle -2\sin t,\ 2\cos t\rangle$$

$$= -4\sin^2 t + 4\cos t\sin t + 4\sin t\cos t$$

$$= -4\sin^2 t + 8\cos t\sin t$$

$$Cir = \int_0^{2\pi} (2\cos 2t - 2 + 4\sin 2t)\,dt \qquad\qquad Circulation = \int_C \vec{F}\cdot\vec{T}\ ds$$

$$= (\sin 2t - 2t - 2\cos 2t)\Big|_0^{2\pi}$$

$$= -4\pi - 2 + 2$$

$$\underline{= -4\pi}$$

$$dy = d(2\sin t) = 2\cos t\ dt$$

$$dx = d(2\cos t) = -2\sin 2t\ dt$$

$$Flux = \int_0^{2\pi} \big((2\sin t - 2\cos t)(2\cos t) - (2\sin t)(-2\sin t)\big)\,dt \qquad\qquad Flux = \int_C (Mdy - Ndx)\ dt$$

$$= \int_0^{2\pi} \left(4\sin t\cos t - 4\cos^2 t + 4\sin^2 t\right)dt$$

$$= \int_0^{2\pi} (2\sin 2t - 4\cos 2t)\,dt$$

$$= (-\cos 2t - 2\sin 2t)\Big|_0^{2\pi}$$

$$= -1 + 1$$

$$\underline{= 0}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \langle x,\ y\rangle$; where R is the half-annulus $\{(r,\ \theta):\ \ 1 \le r \le 2,\quad 0 \le \theta \le \pi\}$

Solution

$$M = y \quad\to\quad M_y = 1$$

$$N = x \quad\to\quad N_x = 1$$

$$Cir = \iint_R (1-1)\,dA \qquad\qquad Cir = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

$$\underline{= 0}$$

$M = x \quad \to \quad M_x = 1$

$N = y \quad \to \quad N_y = 1$

$$Flux = \iint_R (1+1)\, dA \qquad\qquad Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= 2\int_0^{\pi} d\theta \int_1^2 r\, dr$$

$$= 2\pi \left(\frac{1}{2} r^2 \right) \Big|_1^2$$

$$\underline{= 3\pi}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \langle -y,\ x \rangle$; where R is the annulus $\{(r,\ \theta):\ \ 1 \le r \le 3,\ \ \ 0 \le \theta \le 2\pi\}$

Solution

$$Cir = \iint_R \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) \right) dA \qquad\qquad Cir = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \iint_R (1+1)\, dA$$

$$= 2\int_0^{2\pi} d\theta \int_1^3 r\, dr$$

$$= 4\pi \left(\frac{1}{2} r^2 \right) \Big|_1^3$$

$$\underline{= 16\pi}$$

$$Flux = \iint_R \left(\frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) \right) dA \qquad\qquad Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \iint_R (0-0)\, dA$$

$$\underline{= 0}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \langle 2x + y,\ x - 4y \rangle$; where R is the quarter-annulus $\left\{(r,\ \theta):\ \ 1 \le r \le 4,\quad 0 \le \theta \le \frac{\pi}{2}\right\}$

Solution

$$Cir = \iint_R \left(\frac{\partial}{\partial x}(x - 4y) - \frac{\partial}{\partial y}(2x + y) \right) dA \qquad Cir = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \iint_R (1 - 1)\, dA$$

$$\underline{= 0}$$

$$Flux = \iint_R \left(\frac{\partial}{\partial x}(2x + y) + \frac{\partial}{\partial y}(x - 4y) \right) dA \qquad Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \iint_R (2 - 4)\, dA$$

$$= -2 \int_0^{\frac{\pi}{2}} d\theta \int_1^4 r\ dr$$

$$= -\pi \left(\frac{1}{2} r^2 \right) \Big|_1^4$$

$$\underline{= -\frac{15}{2}\pi}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \langle x - y,\ 2y - x \rangle$; where R is the parallelogram $\{(x,\ y):\ \ 1 - x \le y \le 3 - x,\quad 0 \le x \le 1\}$

Solution

$$Cir = \iint_R \left(\frac{\partial}{\partial x}(2y - x) - \frac{\partial}{\partial y}(x - y) \right) dA \qquad Cir = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \iint_R (-1 + 1)\, dA$$

$$\underline{= 0}$$

$$Flux = \iint_R \left(\frac{\partial}{\partial x}(x - y) + \frac{\partial}{\partial y}(2y - x) \right) dA \qquad Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \iint_R (1+2)\, dA$$

$$= 3\int_0^1 \int_{1-x}^{3-x} dydx$$

$$= 3\int_0^1 y\Big|_{1-x}^{3-x} dx$$

$$= 3\int_0^1 2\ dx$$

$$\underline{= 6}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \left\langle \ln\left(x^2+y^2\right),\quad \tan^{-1}\frac{y}{x}\right\rangle$; where R is the annulus $\left\{(r,\ \theta):\quad 1 \le r \le 2,\quad 0 \le \theta \le 2\pi\right\}$

Solution

$$Cir = \iint_R \left(\frac{\partial}{\partial x}\left(\tan^{-1}\frac{y}{x}\right) - \frac{\partial}{\partial y}\left(\ln\left(x^2+y^2\right)\right)\right) dA$$

$$Cir = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= \iint_R \left(-\frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2} - \frac{2y}{x^2+y^2}\right) dA$$

$$\left(\tan^{-1} u\right)' = \frac{u'}{1+u^2}$$

$$= \iint_R \left(-\frac{y}{x^2+y^2} - \frac{2y}{x^2+y^2}\right) dA$$

$$= -3\iint_R \left(\frac{y}{x^2+y^2}\right) dA$$

$$= -3\int_0^{2\pi}\int_1^2 \frac{r\sin\theta}{r^2} r\ drd\theta$$

$$= -3\int_0^{2\pi} \sin\theta\ d\theta \int_1^2 dr$$

$$= 3(\cos\theta)\Big|_0^{2\pi} \quad (r)\Big|_1^2$$

$$= 3(1-1)(1)$$

$\underline{=0}$

$$Flux = \iint_R \left(\frac{\partial}{\partial x}\left(\ln\left(x^2+y^2\right)\right) + \frac{\partial}{\partial y}\left(\tan^{-1}\frac{y}{x}\right)\right) dA$$

$$Flux = \iint\limits_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \iint_R \left(\frac{2x}{x^2+y^2} + \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2} \right) dA$$

$$\left(\tan^{-1} u\right)' = \frac{u'}{1+u^2}$$

$$= \iint_R \left(\frac{2x}{x^2+y^2} + \frac{x}{x^2+y^2} \right) dA$$

$$= 3\iint_R \frac{x}{x^2+y^2} dA$$

$$= 3\int_0^{2\pi} \int_1^2 \frac{r\cos\theta}{r^2} r\ drd\theta$$

$$= 3\int_0^{2\pi} \cos\theta\ d\theta \int_1^2 dr$$

$$= 3(\sin\theta)\Big|_0^{2\pi} \quad (r)\Big|_1^2$$

$$= 3(0)(1)$$

$\underline{=0}$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \nabla\left(\sqrt{x^2+y^2}\right)$; where R is the half-annulus $\{(r,\ \theta):\ \ 1 \le r \le 3,\quad 0 \le \theta \le \pi\}$

Solution

$$\vec{F} = \nabla\left(\sqrt{x^2+y^2}\right)$$

$$= \left\langle \frac{x}{\sqrt{x^2+y^2}},\ \ \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

$$Cir = \iint_R \left(\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2}}\right) - \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right) \right) dA$$

$$Cir = \iint\limits_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \iint_R \left(-\frac{xy}{\left(x^2+y^2\right)^{3/2}} + \frac{xy}{\left(x^2+y^2\right)^{3/2}} \right) dA$$

$$\underline{= 0}$$

$$Flux = \iint_R \left(\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}} \right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}} \right) \right) dA \qquad Flux = \iint\limits_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \iint_R \left(\frac{x^2+y^2-x^2}{\left(x^2+y^2\right)^{3/2}} + \frac{x^2+y^2-y^2}{\left(x^2+y^2\right)^{3/2}} \right) dA$$

$$= \iint_R \frac{x^2+y^2}{\left(x^2+y^2\right)^{3/2}} dA$$

$$= \iint_R \frac{1}{\left(x^2+y^2\right)^{1/2}} dA$$

$$= \int_0^{\pi} \int_1^3 \frac{1}{r} r \; dr d\theta$$

$$= \int_0^{\pi} d\theta \; \int_1^3 \; dr$$

$$\underline{= 2\pi}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \langle y\cos x, \; -\sin x \rangle$; where R is the square $\left\{ (x, \; y): \;\; 0 \le x \le \frac{\pi}{2}, \;\; 0 \le y \le \frac{\pi}{2} \right\}$

Solution

$$Cir = \iint_R \left(\frac{\partial}{\partial x}(-\sin x) - \frac{\partial}{\partial y}(y\cos x) \right) dA \qquad Cir = \iint\limits_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \iint_R (-\cos x - \cos x) dA$$

$$= -2 \int_0^{\frac{\pi}{2}} dy \; \int_0^{\frac{\pi}{2}} \cos x \; dx$$

$$= -\pi \sin x \Big|_0^{\frac{\pi}{2}}$$

$$\underline{= -\pi}$$

$$Flux = \iint_R \left(\frac{\partial}{\partial x}(y \cos x) + \frac{\partial}{\partial y}(-\sin x) \right) dA \qquad Flux = \iint\limits_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \iint_R (-y \sin x + 0)\, dA$$

$$= -\int_0^{\frac{\pi}{2}} y\, dy \int_0^{\frac{\pi}{2}} \sin x\, dx$$

$$= \frac{1}{2} y^2 \Big|_0^{\frac{\pi}{2}} \quad \cos x \Big|_0^{\frac{\pi}{2}}$$

$$\underline{= -\frac{\pi^2}{8}}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \left\langle x + y^2,\ x^2 - y \right\rangle$; where $R = \left\{ (x,\ y):\ \ 3y^2 \le x \le 36 - y^2 \right\}$

Solution

$$x = 36 - y^2 = 3y^2$$

$$4y^2 = 36 \ \rightarrow \ \underline{y = \pm 3}$$

$$Cir = \iint_R \left(\frac{\partial}{\partial x}\left(x^2 - y\right) - \frac{\partial}{\partial y}\left(x + y^2\right) \right) dA \qquad Cir = \iint\limits_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \iint_R (2x - 2y)\, dA$$

$$= 2\int_{-3}^{3} \int_{3y^2}^{36-y^2} (x - y)\, dxdy$$

$$= 2\int_{-3}^{3} \left(\frac{1}{2}x^2 - yx \right) \Big|_{3y^2}^{36-y^2} dy$$

$$= 2\int_{-3}^{3}\left(648 - 36y^2 + \frac{1}{2}y^4 - 36y + y^3 - \frac{9}{2}y^4 + 3y^3\right)dy$$

$$= 2\int_{-3}^{3}\left(648 - 36y - 36y^2 + 4y^3 - 4y^4\right)dy$$

$$= 8\left(162y - \frac{9}{2}y^2 - 3y^3 + \frac{1}{4}y^4 - \frac{1}{5}y^5\right)\Big|_{-3}^{3}$$

$$= 8\left(486 - \frac{81}{2} - 81 + \frac{81}{4} - \frac{243}{5} + 486 + \frac{81}{2} - 81 - \frac{81}{4} - \frac{243}{5}\right)$$

$$= 8\left(810 - \frac{486}{5}\right)$$

$$\underline{= \frac{28,512}{5}}$$

$$Flux = \iint_R \left(\frac{\partial}{\partial x}\left(x + y^2\right) + \frac{\partial}{\partial y}\left(x^2 - y\right)\right)dA \qquad Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dxdy$$

$$= \iint_R (1-1)\,dA$$

$$\underline{= 0}$$

Exercise

Find the outward flux for the field $\boldsymbol{F} = \left(3xy - \frac{x}{1+y^2}\right)\boldsymbol{i} + \left(e^x + \tan^{-1}y\right)\boldsymbol{j}$ across the cardioid $r = a(1+\cos\theta),\ \ a > 0$

Solution

$$M = 3xy - \frac{x}{1+y^2} \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 3y - \frac{1}{1+y^2}$$

$$N = e^x + \tan^{-1}y \quad \Rightarrow \quad \frac{\partial N}{\partial y} = \frac{1}{1+y^2}$$

$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dxdy$$

$$= \iint_R \left(3y - \frac{1}{1+y^2} + \frac{1}{1+y^2}\right)dxdy$$

$$= \iint_R 3y\ dxdy$$

$$= 3\int_0^{2\pi}\int_0^{a(1+\cos\theta)} (r\sin\theta)\ rdrd\theta$$

$$= 3\int_0^{2\pi} \frac{1}{3}\sin\theta\left[r^3\right]_0^{a(1+\cos\theta)} d\theta$$

$$= a^3\int_0^{2\pi} \sin\theta(1+\cos\theta)^3 d\theta \qquad d(1+\cos\theta) = -\sin\theta d\theta$$

$$= -a^3\int_0^{2\pi} (1+\cos\theta)^3 d(1+\cos\theta)$$

$$= -\frac{1}{4}a^3(1+\cos\theta)^4 \Big|_0^{2\pi}$$

$$= -\frac{1}{4}a^3\left(2^4 - 2^4\right)$$

$$= 0$$

Exercise

Find the work done by $\boldsymbol{F} = 2xy^3\boldsymbol{i} + 4x^2y^2\boldsymbol{j}$ in moving a particle once counterclockwise around the curve C: The boundary of the triangular region in the first quadrant enclosed by the x-axis, the line $x = 1$ and the curve $y = x^3$

Solution

$$M = 2xy^3 \quad\Rightarrow\quad \frac{\partial M}{\partial y} = 6xy^2$$

$$N = 4x^2y^2 \quad\Rightarrow\quad \frac{\partial N}{\partial x} = 8xy^2$$

$$Work = \iint\limits_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy \qquad Work = \oint_C Mdx + Ndy$$

$$= \int_0^1\int_0^{x^3} \left(8xy^2 - 6xy^2\right)dydx$$

$$= \int_0^1\int_0^{x^3} \left(2xy^2\right)dydx$$

$$= \int_0^1 \left[\frac{2}{3}xy^3\right]_0^{x^3} dx$$

$$= \frac{2}{3}\int_0^1 x^{10}dx$$

$$= \frac{2}{33}x^{11}\Big|_0^1$$

$$= \frac{2}{33}$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint_C \left(y^2dx + x^2dy\right)$ C: The triangle bounded by

$x = 0, \ x + y = 1, \ y = 0$

Solution

$$M = y^2 \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 2y$$

$$N = x^2 \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 2x$$

$$\oint_C \left(y^2dx + x^2dy\right) = \int_0^1 \int_0^{1-x} (2x - 2y)\,dydx$$

$$= \int_0^1 \left[2xy - y^2\right]_0^{1-x} dx$$

$$= \int_0^1 \left[2x(1-x) - (1-x)^2\right]dx$$

$$= \int_0^1 \left(2x - 2x^2 - 1 + 2x - x^2\right)dx$$

$$= \int_0^1 \left(-3x^2 + 4x - 1\right)dx$$

$$= \left[-x^3 + 2x^2 - x\right]_0^1$$

$$= -1 + 2 - 1$$

$$= 0$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint_C (3ydx + 2xdy)$ C: The boundary of $0 \le x \le \pi, \quad 0 \le y \le \sin x$

Solution

$$M = 3y \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 3$$

$$N = 2x \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 2$$

$$\begin{aligned}
\oint_C (3ydx + 2xdy) &= \int_0^{\pi} \int_0^{\sin x} (2-3)\, dydx \\
&= -\int_0^{\pi} [y]_0^{\sin x}\, dx \\
&= -\int_0^{\pi} \sin x\, dx \\
&= \cos x \Big|_0^{\pi} \\
&= -2
\end{aligned}$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint_C \left(3y - e^{\sin x}\right)dx + \left(7x + \sqrt{y^4+1}\right)dy$: where C is the circle $x^2 + y^2 = 9$

Solution

$$\begin{aligned}
\oint_C \left(3y - e^{\sin x}\right)dx + \left(7x + \sqrt{y^4+1}\right)dy &= \iint_R \left(\frac{\partial}{\partial x}\left(7x + \sqrt{y^4+1}\right) - \frac{\partial}{\partial y}\left(3y - e^{\sin x}\right)\right)dA \\
&= \iint_R (7-3)\, dA \\
&= 4\iint_R dA \\
&= 4\int_0^{2\pi} d\theta \int_0^3 r\, dr
\end{aligned}$$

$$= 8\pi\left(\frac{1}{2}r^2\right)\Bigg|_0^3$$

$$\underline{= 36\pi}$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint_C (3x-5y)\,dx + (x-6y)\,dy$: where C is the ellipse $\frac{x^2}{4}+y^2=1$

Solution

$$\oint_C (3x-5y)\,dx + (x-6y)\,dy = \iint_R \left(\frac{\partial}{\partial x}(x-6y) - \frac{\partial}{\partial y}(3x-5y)\right)dA$$

$$= \iint_R (1-(-5))\,dA$$

$$= 6\iint_R dA$$

$$\underline{= 6\times \textit{Area of ellipse}}$$

$$\frac{x^2}{4}+y^2=1$$

$$x = 2\cos t \quad\to\quad dx = -2\sin t\ dt$$

$$y = \sin t \quad\to\quad dy = \cos t\ dt$$

$$A = \frac{1}{2}\int_0^{2\pi}\left(2\cos t(\cos t) - \sin t(-2\sin t)\right)dt \qquad\qquad A = \frac{1}{2}\oint_C xdy - ydx$$

$$= \int_0^{2\pi}\left(\cos^2 t + \sin^2 t\right)dt$$

$$= \int_0^{2\pi} dt$$

$$\underline{= 2\pi}$$

$$\oint_C (3x-5y)\,dx + (x-6y)\,dy \ \underline{= 12\pi}$$

Exercise

Use either form of Green's Theorem to evaluate the line integral $\oint_C \left(x^3 + xy\right)dy + \left(2y^2 - 2x^2 y\right)dx$; C is the square with vertices $(\pm 1,\ \pm 1)$ with *counterclockwise* orientation

Solution

$$N = x^3 + xy \quad \rightarrow \quad N_x = 3x^2 + y$$

$$M = 2y^2 - 2x^2 y \quad \rightarrow \quad M_y = 4y - 2x^2$$

$$\begin{aligned}
\oint_C \left(x^3 + xy\right)dy + \left(2y^2 - 2x^2 y\right)dx &= \int_{-1}^{1}\int_{-1}^{1} \left(3x^2 + y - 4y + 2x^2\right)dydx \\
&= \int_{-1}^{1}\int_{-1}^{1} \left(5x^2 - 3y\right)dydx \\
&= \int_{-1}^{1} \left(5x^2 y - \frac{3}{2}y^2\right)\Big|_{-1}^{1}\ dx \\
&= \int_{-1}^{1} \left(5x^2 - \frac{3}{2} + 5x^2 + \frac{3}{2}\right)dx \\
&= \int_{-1}^{1} 10x^2\ dx \\
&= \frac{10}{3}x^3\Big|_{-1}^{1} \\
&= \boxed{\frac{20}{3}}
\end{aligned}$$

Exercise

Use either form of Green's Theorem to evaluate the line integral $\oint_C 3x^3 dy - 3y^3 dx$; C is the circle of radius 4 centered at the origin with *clockwise* orientation.

Solution

$$N = 3x^3 \quad \rightarrow \quad N_x = 9x^2$$

$$M = -3y^3 \quad \rightarrow \quad M_y = -9y^2$$

$$\oint_C 3x^3 dy - 3y^3 dx = \iint_R \left(9x^2 + 9y^2\right)dA$$

$$= 9\int_0^{2\pi}\int_0^4 r^2\ r\ dr d\theta$$

$$= 9\int_0^{2\pi} d\theta \int_0^4 r^3\ dr$$

$$= 9(2\pi)\left(\frac{1}{4}r^4\right)\Big|_0^4$$

$$= 18\pi(64)$$

$$\underline{= 1152\pi}$$

Since the orientation is cw: -1152π

Exercise

Evaluate $\int_C (x-y)dx + (x+y)dy$ counterclockwise around the triangle with vertices (0, 0), $(1,\ 0)$ and $(0,\ 1)$

Solution

Along $(0,0) \to (1,0)$: $\boldsymbol{r}(t) = t\boldsymbol{i}, \quad 0 \le t \le 1$

$$\boldsymbol{F} = (x-y)\boldsymbol{i} + (x+y)\boldsymbol{j} = t\boldsymbol{i} + t\boldsymbol{j}$$

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{i}$$

$$F \bullet \frac{d\boldsymbol{r}}{dt} = (t\boldsymbol{i} + t\boldsymbol{j}) \cdot (\boldsymbol{i}) = t$$

Along $(1,0) \to (0,1)$: $\boldsymbol{r}(t) = (1-t)\boldsymbol{i} + t\boldsymbol{j}, \quad 0 \le t \le 1$

$$\boldsymbol{F} = (x-y)\boldsymbol{i} + (x+y)\boldsymbol{j} = (1-2t)\boldsymbol{i} + \boldsymbol{j}$$

$$\frac{d\boldsymbol{r}}{dt} = -\boldsymbol{i} + \boldsymbol{j}$$

$$F \bullet \frac{d\boldsymbol{r}}{dt} = ((1-2t)\boldsymbol{i} + \boldsymbol{j}) \cdot (-\boldsymbol{i} + \boldsymbol{j}) = -1 + 2t + 1 = 2t$$

Along $(0,1) \to (0,0)$: $\boldsymbol{r}(t) = (1-t)\boldsymbol{j}, \quad 0 \le t \le 1$

$$\boldsymbol{F} = (x-y)\boldsymbol{i} + (x+y)\boldsymbol{j} = (t-1)\boldsymbol{i} + (1-t)\boldsymbol{j}$$

$$\frac{d\boldsymbol{r}}{dt} = -\boldsymbol{j}$$

$$F \bullet \frac{d\boldsymbol{r}}{dt} = ((t-1)\boldsymbol{i} + (1-t)\boldsymbol{j}) \cdot (-\boldsymbol{j}) = t-1$$

$$\int_C (x-y)dx + (x+y)dy = \int_0^1 tdt + \int_0^1 2tdt + \int_0^1 (t-1)dt$$

$$= \int_0^1 (t + 2t + t - 1)\,dt$$

$$= \int_0^1 (4t - 1)\,dt$$

$$= \left[2t^2 - t \right]_0^1$$

$$= 2 - 1$$

$$\underline{= 1}$$

Exercise

Use Green's theorem to evaluate the line integral $\oint xy^2dx + x^2ydy$; C is the triangle with vertices $(0,\ 0)$, $(2,\ 0)$, $(0,\ 2)$ with counterclockwise orientation.

Solution

$$\oint xy^2dx + x^2ydy = \iint_R \left(\frac{\partial}{\partial x}\left(x^2y\right) - \frac{\partial}{\partial x}\left(x^2y\right) \right) dxdy$$

$$= \iint_R (2xy - 2xy)\,dxdy$$

$$\underline{= 0}$$

Exercise

Use Green's theorem to evaluate the line integral $\oint \left(-3y + x^{3/2}\right)dx + \left(x - y^{2/3}\right)dy$; C is the boundary of the half disk $\left\{(x, y):\ x^2 + y^2 \le 2,\ y \ge 0\right\}$ with counterclockwise orientation.

Solution

$$\oint \left(-3y + x^{3/2}\right)dx + \left(x - y^{2/3}\right)dy = \iint_C \left(\frac{\partial}{\partial x}\left(x - y^{2/3}\right) - \frac{\partial}{\partial y}\left(-3y + x^{3/2}\right) \right) dA$$

$$= \iint_C (1 + 3)\,dA$$

$$= \iint_C 4\ dA \qquad \textit{Semicircle}\ \ A = \pi$$

$$\underline{= 4\pi}$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint_{(0,\ 1)} \left(2x+e^{y^2}\right)dy-\left(4y^2+e^{x^2}\right)dx$: C is the boundary of the square with vertices $(0,\ 0)$, $(1,\ 0)$, $(1,\ 1)$ with counterclockwise orientation.

Solution

$$\oint_{(0,\ 1)} \left(2x+e^{y^2}\right)dy-\left(4y^2+e^{x^2}\right)dx = \iint_C \left(\frac{\partial}{\partial x}\left(2x+e^{y^2}\right)+\frac{\partial}{\partial y}\left(4y^2+e^{x^2}\right)\right)dA$$

$$= \iint_C (2+8y)\,dA$$

$$= \int_0^1 \int_0^1 (2+8y)\,dxdy$$

$$= \int_0^1 (2+8y)\,dy$$

$$= \left(2y+4y^2\right)\Big|_0^1$$

$$\underline{=6}$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint_C (2x-3y)\,dy-(3x+4y)\,dx$: C is the unit circle

Solution

$$\oint_C (2x-3y)\,dy-(3x+4y)\,dx = \iint_C \left(\frac{\partial}{\partial x}(2x-3y)+\frac{\partial}{\partial y}(3x+4y)\right)dA$$

$$= \iint_C (2+4)\,dA$$

$$= 6\times(\text{Area of the unit circle})$$

$$\underline{=6\pi}$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint fdy - gdx;$ where $\langle f,\ g\rangle = \langle 0,\ xy\rangle$ and C is the triangle with vertices $(0,\ 0),\ (2,\ 0),\ (0,\ 4)$ with counterclockwise orientation.

Solution

$$(2,\ 0)-(0,\ 4):\ \to\ y=\frac{4}{-2}x+4 \underline{=4-2x|}$$

$$\begin{aligned}\oint fdy - gdx &= \iint_R \left(\frac{\partial}{\partial x}(f)+\frac{\partial}{\partial y}(g)\right)dA \\
&= \iint_R \left(\frac{\partial}{\partial x}(0)+\frac{\partial}{\partial y}(xy)\right)dA \\
&= \int_0^2\int_0^{4-2x} x\ dydx \\
&= \int_0^2 xy\Big|_0^{4-2x}\ dx \\
&= \int_0^2 \left(4x-2x^2\right)dx \\
&= \left(2x^2-\frac{2}{3}x^3\right)\Big|_0^2 \\
&= 8-\frac{16}{3} \\
&= \frac{8}{3}\end{aligned}$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint fdy - gdx;$ where $\langle f,\ g\rangle = \left\langle x^2,\ 2y^2\right\rangle$ and C is the upper half of the unit circle and the line segment $-1\le x\le 1$ with clockwise orientation.

Solution

$x^2+y^2=1 \quad\to\quad y=\sqrt{1-x^2}$ *upper half of the unit circle*

$$\oint fdy - gdx = -\iint_R \left(\frac{\partial}{\partial x}(f)+\frac{\partial}{\partial y}(g)\right)dA$$ ***clockwise orientation***

$$= -\iint_R \left(\frac{\partial}{\partial x}\left(x^2\right)+\frac{\partial}{\partial y}\left(2y^2\right)\right)dA$$

$$= -\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (2x+4y)\,dydx$$

$$= -\int_{-1}^{1} \left(2xy+2y^2\right)\Big|_{0}^{\sqrt{1-x^2}}\,dx$$

$$= -\int_{-1}^{1} \left(2x\sqrt{1-x^2}+2\left(1-x^2\right)\right)dx$$

$$= \int_{-1}^{1} \left(1-x^2\right)^{1/2} d\left(1-x^2\right) - 2\int_{-1}^{1} \left(1-x^2\right)dx$$

$$= \left(\frac{2}{3}\left(1-x^2\right)^{3/2} - 2x + \frac{2}{3}x^3\right)\Big|_{-1}^{1}$$

$$= -2+\frac{2}{3}-2+\frac{2}{3}$$

$$= -\frac{8}{3}$$

Exercise

Apply Green's Theorem to evaluate the integral, the circulation line integral of $\vec{F} = \left\langle x^2+y^2,\ 4x+y^3\right\rangle$, where C is the boundary of $\left\{(x,\ y):\ 0 \le y \le \sin x, \quad 0 \le x \le \pi\right\}$

Solution

Using Circulation form

$$\iint_C \left(\frac{\partial}{\partial x}\left(4x+y^3\right) - \frac{\partial}{\partial y}\left(x^2+y^2\right)\right)dA = \iint_C (4-2y)\,dA$$

$$= \int_0^{\pi}\int_0^{\sin x} (4-2y)\,dydx$$

$$= \int_0^{\pi} \left(4y-y^2\right)\Big|_0^{\sin x}\,dx$$

$$= \int_0^{\pi} \left(4\sin x - \sin^2 x\right)dx$$

$$= \int_0^{\pi} \left(4\sin x - \frac{1}{2} + \frac{1}{2}\cos 2x\right)dx$$

$$= \left(-4\cos x - \frac{1}{2}x + \frac{1}{4}\sin 2x\right)\Big|_0^{\pi}$$

$$= 4 - \frac{\pi}{2} + 4$$

$$\boxed{= 8 - \frac{\pi}{2}}$$

Exercise

Apply Green's Theorem to evaluate the integral, the circulation line integral of $\vec{F} = \left\langle 2xy^2 + x,\ 4x^3 + y\right\rangle$, where C is the boundary of $\left\{(x,\ y):\ 0 \le y \le \sin x, \quad 0 \le x \le \pi\right\}$

Solution

Using Circulation form

$$\iint_C \left(\frac{\partial}{\partial x}\left(4x^3 + y\right) - \frac{\partial}{\partial y}\left(2xy^2 + x\right)\right) dA = \iint_C \left(12x^2 - 4xy\right) dA$$

$$= \int_0^{\pi}\int_0^{\sin x} \left(12x^2 - 4xy\right) dydx$$

$$= \int_0^{\pi} \left(12x^2 y - 2xy^2\right)\Big|_0^{\sin x}\ dx$$

$$= \int_0^{\pi} \left(12x^2 \sin x - 2x\sin^2 x\right) dx$$

$$= \int_0^{\pi} \left(12x^2 \sin x - 2x\left(\frac{1-\cos 2x}{2}\right)\right) dx$$

$$= \int_0^{\pi} \left(12x^2 \sin x - x + x\cos 2x\right) dx$$

		$\int \sin x$			$\int \cos 2x$
+	$12x^2$	$-\cos x$	+	x	$\frac{1}{2}\sin 2x$
–	$24x$	$-\sin x$	–	1	$-\frac{1}{4}\cos 2x$
+	24	$\cos x$			

$$= \left(-12x^2\cos x + 24x\sin x + 24\cos x - \frac{1}{2}x^2 + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right)\Big|_0^{\pi}$$

$$= 12\pi^2 - 24 - \frac{\pi^2}{2} + \frac{1}{4} + 12\pi^2 - 24 - \frac{1}{4}$$

$$= \boxed{\frac{23\pi^2}{2} - 48}$$

Exercise

Apply Green's Theorem to evaluate the integral, the flus line integral of $\vec{F} = \left\langle e^{x-y},\ e^{y-x} \right\rangle$, where C is the boundary of $\{(x,\ y):\ 0 \le y \le x, \quad 0 \le x \le 1\}$

Solution

Using flux form

$$\iint_C \left(\frac{\partial}{\partial x}\left(e^{x-y}\right) + \frac{\partial}{\partial y}\left(e^{y-x}\right) \right) dA = \iint_C \left(e^{x-y} + e^{y-x}\right) dA$$

$$= \int_0^1 \int_0^x \left(e^{x-y} + e^{y-x}\right) dydx$$

$$= \int_0^1 \left(-e^{x-y} + e^{y-x}\right)\Big|_0^x \ dx$$

$$= \int_0^1 \left(-1 + 1 + e^x - e^{-x}\right) dx$$

$$= \int_0^1 \left(e^x - e^{-x}\right) dx$$

$$= \left(e^x + e^{-x}\right)\Big|_0^1$$

$$= \boxed{e + e^{-1} - 2}$$

Exercise

Evaluate $\displaystyle\int_C y^2 dx + x^2 dy \quad C$ is the circle $x^2 + y^2 = 4$

Solution

$$M = y^2 \quad \to \quad M_y = 2y$$

$$N = x^2 \quad \to \quad N_x = 2x$$

$$\int_C y^2 dx + x^2 dy = \int_C (2x - 2y)\, dxdy$$

$$= 2\int_0^{2\pi}\int_0^2 (r\cos\theta - r\sin\theta)\ r dr d\theta$$

$$= 2\int_0^{2\pi} (\cos\theta - \sin\theta) d\theta \int_0^2 r^2 dr$$

$$= 2(\sin\theta + \cos\theta)\Big|_0^{2\pi} \left(\frac{1}{3}r^3\right)\Big|_0^2$$

$$= 2(1-1)\left(\frac{8}{3}\right)$$

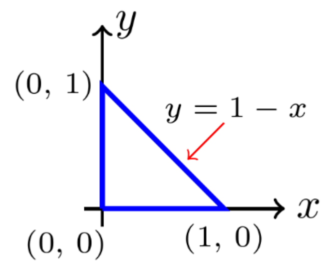
$$\underline{= 0}$$

Exercise

Use the flux form to Green's Theorem to evaluate $\iint_R \left(2xy + 4y^3\right) dA$, where R is the triangle with vertices $(0,\ 0)$, $(1,\ 0)$, and $(0,\ 1)$.

Solution

$(1,\ 0) - (0,\ 1):\quad y = -x + 1$



$$\iint_R \left(2xy + 4y^3\right) dA = \int_0^1 \int_0^{1-x} \left(2xy + 4y^3\right) dy dx$$

$$= \int_0^1 \left(xy^2 + y^4\right)\Big|_0^{1-x} dx$$

$$= \int_0^1 \left(x - 2x^2 + x^3 + 1 - 4x + 6x^2 - 4x^3 + x^4\right) dx$$

$$= \int_0^1 \left(1 - 3x + 4x^2 - 3x^3 + x^4\right) dx$$

$$= \left(x - \frac{3}{2}x^2 + \frac{4}{3}x^3 - \frac{3}{4}x^4 + \frac{1}{5}x^5\right)\Big|_0^1$$

$$= 1 - \frac{3}{2} + \frac{4}{3} - \frac{3}{4} + \frac{1}{5}$$

$$= \frac{-30 + 80 - 45 + 12}{60}$$

$$\underline{= \frac{17}{60}}$$

Exercise

Show that $\oint_C \ln x \sin y\, dy - \frac{\cos y}{x} dx = 0$ for any closed curve C to which Green's Theorem applies.

Solution

$$M = -\frac{\cos y}{x} \quad \to \quad M_y = \frac{\sin y}{x}$$

$$N = \ln x \sin y \quad \to \quad N_x = \frac{\ln y}{x}$$

$$\oint_C \ln x \sin y\, dy - \frac{\cos y}{x} dx = \iint_R \left(\frac{\sin y}{x} - \frac{\sin y}{x} \right) dxdy$$

$$= 0$$

Exercise

Prove that the radial field $\vec{F} = \frac{\vec{r}}{|\vec{r}|^p}$ where $\vec{r} = \langle x, y \rangle$ and p is a real number, is conservative on $\mathbb{R}^2$ with the origin removed. For what value of p is $\vec{F}$ conservative on $\mathbb{R}^2$ (including the origin)?

Solution

$$\vec{F} = \frac{\vec{r}}{|\vec{r}|^p}$$

$$= \frac{\langle x, y \rangle}{\left(\sqrt{x^2 + y^2} \right)^p}$$

$$\varphi_x = \frac{x}{\left(x^2 + y^2 \right)^{p/2}}; \quad \varphi_y = \frac{y}{\left(x^2 + y^2 \right)^{p/2}}$$

$$\varphi = \int \frac{x}{\left(x^2 + y^2 \right)^{p/2}} dx$$

$$= \frac{1}{2} \int \left(x^2 + y^2 \right)^{-p/2} d\left(x^2 + y^2 \right)$$

$$= \frac{1}{2} \frac{1}{\frac{2-p}{2}} \left(x^2 + y^2 \right)^{1-p/2} + C$$

$$= \frac{1}{2-p} \left(x^2 + y^2 \right)^{1-p/2} + C(x, y) \qquad \textit{for} \quad p \neq 2$$

For $p \neq 2$

$$\varphi = \frac{1}{2-p}\left(x^2+y^2\right)^{1-p/2} + C(x,y)$$

$$\varphi_y = \frac{1}{2-p}\frac{2-p}{2}(2y)\left(x^2+y^2\right)^{1-\frac{p}{2}-1} + C_y$$

$$= y\left(x^2+y^2\right)^{-\frac{p}{2}} + C_y = \frac{y}{\left(x^2+y^2\right)^{p/2}}$$

$$\Rightarrow \quad \underline{C_y = 0|}$$

$$\therefore \quad \varphi = \frac{1}{(2-p)\left(x^2+y^2\right)^{\frac{p-2}{2}}}$$

$$= \frac{-1}{(p-2)\left(r^2\right)^{\frac{p-2}{2}}}$$

$$= \frac{-1}{(p-2)\ |r|^{p-2}}$$

For $p = 2$

$$\vec{F} = \frac{\langle x,\ y\rangle}{\left(\sqrt{x^2+y^2}\right)^2}$$

$$= \frac{\langle x,\ y\rangle}{x^2+y^2}$$

$$\varphi_x = \frac{x}{x^2+y^2}; \quad \varphi_y = \frac{y}{x^2+y^2}$$

$$\varphi = \int \frac{x}{x^2+y^2}\,dx$$

$$= \frac{1}{2}\int \frac{1}{x^2+y^2}\,d\left(x^2+y^2\right)$$

$$= \frac{1}{2}\ln\left(x^2+y^2\right) + C(x,y)$$

$$\varphi_y = \frac{y}{x^2+y^2} + C_y = \frac{y}{x^2+y^2}$$

$$\Rightarrow \quad \underline{C_y = 0|}$$

$$\varphi = \frac{1}{2}\ln\left(|r|^2\right)$$

Thus $\vec{F}$ is conservative on all $\mathbb{R}^2$ for $p < 0$

Exercise

Find the area of the elliptical region cut from the plane $x+y+z=1$ by the cylinder $x^2+y^2=1$

Solution

$$f(x,y,z)=x+y+z-1$$

$$\nabla f=\langle 1,\ 1,\ 1\rangle$$

$$|\nabla f|=\sqrt{3}$$

$$Area=\sqrt{3}\int_0^{2\pi}\int_0^1 r\ drd\theta \qquad\qquad Area=\iint_R |\nabla f|\,dA$$

$$=\sqrt{3}\int_0^{2\pi} d\theta \quad \frac{1}{2}r^2\Big|_0^1$$

$$=\sqrt{3}(2\pi)\frac{1}{2}$$

$$=\pi\sqrt{3}\ \ unit^2$$

Exercise

Find the area of the cap cut from the paraboloid $x^2+y^2+z^2=1$ by the plane $z=\frac{\sqrt{2}}{2}$

Solution

$$f(x,y,z)=x^2+y^2+z^2-1$$

$$\nabla f=\langle 2x,\ 2y,\ 2z\rangle$$

$$|\nabla f|=\sqrt{4x^2+4y^2+4z^2}$$

$$=2\sqrt{x^2+y^2+z^2}$$

$$=2$$

$$|\nabla f\bullet\vec{p}|=\langle 2x,\ 2y,\ 2z\rangle\bullet\hat{k}$$

$$=2|z|$$

$$=2z$$

$$Area=\iint_R \frac{2}{2z}dA \qquad\qquad Area=\iint_R \frac{|\nabla f|}{|\nabla f\bullet\vec{p}|}dA$$

$$=\iint_R \frac{1}{\sqrt{1-x^2-y^2}}dxdy$$

$$=\int_0^{2\pi} d\theta\ \int_0^{\frac{1}{\sqrt{2}}}\frac{1}{\sqrt{1-r^2}}rdr$$

$$= -\pi \int_0^{\frac{1}{\sqrt{2}}} \left(1-r^2\right)^{-1/2} d\left(1-r^2\right)$$

$$= -2\pi \left(1-r^2\right)^{1/2} \Big|_0^{\frac{1}{\sqrt{2}}}$$

$$= -2\pi \left(\frac{1}{\sqrt{2}} - 1\right)$$

$$= 2\pi \left(1 - \frac{\sqrt{2}}{2}\right)$$

$$= \pi\left(2-\sqrt{2}\right) \ \ unit^2$$

Exercise

Evaluate both integrals in Green's theorem of the vector field. Is the vector field conservative?

$\vec{F} = \langle x,\ y\rangle; \quad R = \left\{(x,\ y): \quad x^2 + y^2 \le 2\right\}$

Solution

$$M = x \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 0$$

$$N = y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 0$$

$$Curl = 0 - 0 \qquad\qquad Curl = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$= 0$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R (0-0)\, dA$$

$$= 0$$

$$\vec{r}(t) = \left\langle \sqrt{2}\cos t,\ \sqrt{2}\sin t \right\rangle$$

$$\vec{r}\,' = \left\langle -\sqrt{2}\sin t,\ \sqrt{2}\cos t \right\rangle$$

$$\vec{F} = \left\langle \sqrt{2}\cos t,\ \sqrt{2}\sin t \right\rangle$$

$$\vec{F} \bullet \vec{r}\,' = \left\langle \sqrt{2}\cos t,\ \sqrt{2}\sin t \right\rangle \bullet \left\langle -\sqrt{2}\sin t,\ \sqrt{2}\cos t \right\rangle$$

$$= -2\cos t \sin t + 2\sin t \cos t$$

$$= 0$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} 0\, dt$$

$$= 0$$

$\therefore$ The vector field is conservative since its curl is zero.

Exercise

Evaluate both integrals in Green's theorem of the vector field. Is the vector field conservative?

$\vec{F} = \langle y,\ x \rangle$; R is the square with vertices $(0,\ 0)$, $(1,\ 0)$, $(1,\ 1)$, $(0,\ 1)$

Solution

$$M = y \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 1$$

$$N = x \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 1$$

$$Curl = 1 - 1 \qquad\qquad Curl = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$= 0$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \iint_R (1-1)\, dA$$

$$= 0$$

$(0,\ 0) - (1,\ 0)$

$$\vec{r}_1(t) = \langle t,\ 0 \rangle$$

$$\vec{r}_1\,' = \langle 1,\ 0 \rangle$$

$$\vec{F}_1 = \langle 0,\ t \rangle$$

$$\vec{F}_1 \cdot \vec{r}_1\,' = \langle 0,\ t \rangle \cdot \langle 1,\ 0 \rangle$$

$$= 0$$

$(1,\ 0) - (1,\ 1)$

$$\vec{r}_2(t) = \langle 1,\ t \rangle$$

$$\vec{r}_2\,' = \langle 0,\ 1 \rangle$$

$$\vec{F}_2 = \langle t,\ 1 \rangle$$

$$\vec{F}_2 \cdot \vec{r}_2\,' = \langle t,\ 1 \rangle \cdot \langle 0,\ 1 \rangle$$

$$= 1$$

$(1,\ 1) - (0,\ 1)$

$$\vec{r}_3(t) = \langle 1-t,\ 1\rangle$$

$$\vec{r}_3{}' = \langle -1,\ 0\rangle$$

$$\vec{F}_3 = \langle 1,\ 1-t\rangle$$

$$\vec{F}_3 \bullet \vec{r}_3{}' = \langle 1,\ 1-t\rangle \bullet \langle -1,\ 0\rangle$$

$$\underline{= -1\,|}$$

$$\int_C \vec{F} \bullet d\vec{r} = \int_0^1 \vec{F}_1 \bullet \vec{r}_1{}'\, dt + \int_0^1 \vec{F}_2 \bullet \vec{r}_2{}'\, dt + \int_0^1 \vec{F}_3 \bullet \vec{r}_3{}'\, dt$$

$$= 0 + 1 - 1$$

$$\underline{= 0\,|}$$

$\therefore$ The vector field is conservative since its curl is zero.

Exercise

Evaluate both integrals in Green's theorem of the vector field. Is the vector field conservative?

$\vec{F} = \langle 2y,\ -2x\rangle$; R is the region bounded by $y = \sin x$ and $y = 0$ for $0 \le x \le \pi$

Solution

$$M = 2y \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 2$$

$$N = -2x \quad \Rightarrow \quad \frac{\partial N}{\partial x} = -2$$

$$Curl = -2 - 2 \qquad\qquad Curl = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$\underline{= -4\,|}$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R (-4)\, dA$$

$$= -4\int_0^{\pi} \sin x\, dx$$

$$= 4\cos x \Big|_0^{\pi}$$

$$= 4(-1-1)$$

$$\underline{= -8\,|}$$

$$y = 0$$

$$\vec{r}_1(t) = \langle t,\ 0\rangle$$

$\vec{r}_1{}' = \langle 1,\ 0 \rangle$

$\vec{F} = \langle 0,\ -2t \rangle$

$\vec{F}_1 \cdot \vec{r}_1{}' = \langle 0,\ -2t \rangle \cdot \langle 1,\ 0 \rangle$

$\qquad = 0$

$y = \sin x$

$\vec{r}_2(t) = \langle t,\ \sin t \rangle$

$\vec{r}_2{}' = \langle 1,\ \cos t \rangle$

$\vec{F}_2 = \langle 2\sin t,\ -2t \rangle$

$\vec{F}_2 \cdot \vec{r}_2{}' = \langle 2\sin t,\ -2t \rangle \cdot \langle 1,\ \cos t \rangle$

$\qquad = 2\sin t - 2t\cos t$

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^{\pi} \vec{F}_1 \cdot \vec{r}_1{}'\, dt + \int_{\pi}^{0} \vec{F}_2 \cdot \vec{r}_2{}'\, dt \\
&= 0 + \int_{\pi}^{0} (2\sin t - 2t\cos t)\, dt \\
&= -2\cos t - 2t\sin t - 2\cos t \Big|_{\pi}^{0} \\
&= -4\cos t - 2t\sin t \Big|_{\pi}^{0} \\
&= -4 - 4 \\
&= -8
\end{aligned}$$

$\therefore$ The vector field is ***not*** conservative since its curl is nonzero.

Exercise

Evaluate both integrals in Green's theorem of the vector field. Is the vector field conservative?

$\vec{F} = \langle -3y,\ 3x \rangle$; R is the triangle with vertices $(0,\ 0)$, $(1,\ 0)$, $(0,\ 2)$

Solution

$M = -3y \quad \Rightarrow \quad \dfrac{\partial M}{\partial y} = -3$

$N = 3x \quad \Rightarrow \quad \dfrac{\partial N}{\partial x} = 3$

$Curl = 3 + 3 \qquad\qquad Curl = \dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$

$\qquad = 6$

$$y = \frac{2-0}{0-1}(x-1)$$

$$\underline{= -2x + 2}$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \iint_R 6\ dA$$

$$= 6\int_0^1 (2-2x)\,dx$$

$$= 6\left(2x - x^2\right)\Big|_0^1$$

$$\underline{= 6}$$

$(0,\ 0) - (1,\ 0)$

$$\vec{r}_1(t) = \langle t,\ 0\rangle$$

$$\vec{r}_1{}' = \langle 1,\ 0\rangle$$

$$\vec{F}_1 = \langle 0,\ 3t\rangle$$

$$\vec{F}_1 \bullet \vec{r}_1{}' = \langle 0,\ 3t\rangle \bullet \langle 1,\ 0\rangle$$

$$\underline{= 0}$$

$(1,\ 0) - (0,\ 2)$

$$\vec{r}_2(t) = \langle 1-t,\ 2t\rangle$$

$$\vec{r}_2{}' = \langle -1,\ 2\rangle$$

$$\vec{F}_2 = \langle -6t,\ 3-3t\rangle$$

$$\vec{F}_2 \bullet \vec{r}_2{}' = \langle -6t,\ 3-3t\rangle \bullet \langle -1,\ 2\rangle$$

$$= 6t + 6 - 6t$$

$$\underline{= 6}$$

$(0,\ 2) - (0,\ 0)$

$$\vec{r}_3(t) = \langle 0,\ 2-2t\rangle$$

$$\vec{r}_3{}' = \langle 0,\ -2\rangle$$

$$\vec{F}_3 = \langle 6t-6,\ 0\rangle$$

$$\vec{F}_3 \bullet \vec{r}_3{}' = \langle 6t-6,\ 0\rangle \bullet \langle 0,\ -2\rangle$$

$$\underline{= 0}$$

$$\int_C \vec{F} \bullet d\vec{r} = \int_0^1 \vec{F}_1 \bullet \vec{r}_1{}'\ dt + \int_0^1 \vec{F}_2 \bullet \vec{r}_2{}'\ dt + \int_0^1 \vec{F}_3 \bullet \vec{r}_3{}'\ dt$$

$$= 0 + \int_0^1 6\, dt + 0$$

$$\underline{= 6}\rfloor$$

$\therefore$ The vector field is conservative since its curl is zero.

Exercise

Evaluate both integrals in Green's theorem of the vector field. Is the vector field conservative?

$\vec{F} = \left\langle 2xy,\ x^2 - y^2 \right\rangle$; R is the region bounded by $y = x(2-x)$ and $y = 0$

Solution

$$M = 2xy \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 2x$$

$$N = x^2 - y^2 \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 2x$$

$$Curl = 2x - 2x \qquad\qquad Curl = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$\underline{= 0}\rfloor$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R (2x - 2x)\, dA$$

$$\underline{= 0}\rfloor$$

$y = 0$

$$\vec{r}_1(t) = \langle t,\ 0\rangle$$

$$\vec{r}_1{}' = \langle 1,\ 0\rangle$$

$$\vec{F}_1 = \left\langle 0,\ t^2 \right\rangle$$

$$\vec{F}_1 \bullet \vec{r}_1{}' = \left\langle 0,\ t^2 \right\rangle \bullet \langle 1,\ 0\rangle$$

$$\underline{= 0}\rfloor$$

$y = 2x - x^2$

$$\vec{r}_2(t) = \left\langle t,\ 2t - t^2 \right\rangle$$

$$\vec{r}_2{}' = \langle 1,\ 2 - 2t\rangle$$

$$\vec{F}_2 = \left\langle 4t^2 - 2t^3,\ \ -3t^2 + 4t^3 - t^4 \right\rangle$$

$$\vec{F}_2 \bullet \vec{r}_2{}' = \left\langle 4t^2 - 2t^3,\ \ -3t^2 + 4t^3 - t^4 \right\rangle \bullet \langle 1,\ 2 - 2t\rangle$$

$$= 4t^2 - 2t^3 + \left(-3t^2 + 4t^3 - t^4\right)(2 - 2t)$$

$$= 4t^2 - 2t^3 - 6t^2 + 8t^3 - 2t^4 + 6t^3 - 8t^4 + 2t^5$$

$$= 2t^5 - 10t^4 + 12t^3 - 2t^2$$

$$y = 2t - t^2 = 0 \quad \to \quad \underline{t = 0,\ 2}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^2 \vec{F}_1 \cdot \vec{r}_1{}' \, dt + \int_2^0 \vec{F}_2 \cdot \vec{r}_2{}' \, dt$$

$$= 0 + \int_2^0 \left(2t^5 - 10t^4 + 12t^3 - 2t^2\right) dt$$

$$= \frac{1}{3}t^6 - 2t^5 + 3t^4 - \frac{2}{3}t^3 \Big|_2^0$$

$$= -\frac{64}{3} + 64 - 48 + \frac{16}{3}$$

$$= -\frac{48}{3} + 16$$

$$\underline{= 0}$$

$\therefore$ The vector field is conservative since its curl is zero.

Exercise

Evaluate both integrals in Green's theorem of the vector field. Is the vector field conservative?

$$\vec{F} = \left\langle 0,\ x^2 + y^2 \right\rangle; \quad R = \left\{(x,\ y):\ \ x^2 + y^2 \le 1\right\}$$

Solution

$$M = 0 \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 0$$

$$N = x^2 + y^2 \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 2x$$

$$Curl = 2x - 0 \qquad\qquad Curl = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$\underline{= 2x}$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R 2x \, dA$$

$$= \int_0^{2\pi} \int_0^1 2r\cos\theta \ r \ drd\theta$$

$$= \int_0^{2\pi} \cos\theta \, d\theta \int_0^1 2r^2 \, dr$$

$$= \sin\theta \Big|_0^{2\pi} \quad \frac{2}{3} r^3 \Big|_0^1$$

$$\underline{= 0}$$

$$\vec{r}(t) = \langle \cos t, \ \sin t \rangle$$

$$\vec{r}\,' = \langle -\sin t, \ \cos t \rangle$$

$$\vec{F} = \left\langle 0, \ \cos^2 t + \sin^2 t \right\rangle$$

$$= \langle 0, \ 1 \rangle$$

$$\vec{F} \bullet \vec{r}\,' = \langle 0, \ 1 \rangle \bullet \langle -\sin t, \ \cos t \rangle$$

$$\underline{= \cos t}$$

$$\int_C \vec{F} \bullet d\vec{r} = \int_0^{2\pi} \cos t \ dt$$

$$= \sin t \Big|_0^{2\pi}$$

$$\underline{= 0}$$

∴ The vector field is ***not*** conservative since its curl is nonzero.

Exercise

Find the area of the region using line integral of the region enclosed by the ellipse $x^2 + 4y^2 = 16$

Solution

$$x^2 + 4y^2 = 16$$

$$\frac{x^2}{16} + \frac{y^2}{4} = 1 \quad \rightarrow \quad \begin{cases} x = 4\cos t \\ y = 2\sin t \end{cases} \quad 0 \le t \le 2\pi$$

$$A = \frac{1}{2} \oint_C \left(4\cos t \frac{d}{dt}(2\sin t) - 2\sin t \frac{d}{dt}(4\cos t) \right) dt$$

$$= 4 \int_0^{2\pi} \left(\cos^2 t + \sin^2 t \right) dt$$

$$= 4 \int_0^{2\pi} dt$$

$$\underline{= 8\pi \ \ unit^2}$$

Exercise

Find the area of the region using line integral of the region bounded by the hypocycloid $\vec{r}(t) = \left\langle \cos^3 t,\ \sin^3 t \right\rangle$ for $0 \le t \le 2\pi$.

Solution

$$A = \frac{1}{2}\int_0^{2\pi}\left(\cos^3 t\frac{d}{dt}\left(\sin^3 t\right) - \sin^3 t\frac{d}{dt}\left(\cos^3 t\right)\right)dt \qquad A = \frac{1}{2}\oint_C xdy - ydx$$

$$= \frac{1}{2}\int_0^{2\pi}\left(\cos^3 t\left(3\sin^2 t\cos t\right) - \sin^3 t\left(-3\cos^2 t\sin t\right)\right)dt$$

$$= \frac{3}{2}\int_0^{2\pi}\left(\cos^4 t\sin^2 t + \sin^4 t\cos^2 t\right)dt$$

$$= \frac{3}{2}\int_0^{2\pi}\sin^2 t\cos^2 t\left(\cos^2 t + \sin^2 t\right)dt$$

$$= \frac{3}{2}\int_0^{2\pi}\sin^2 t\cos^2 t\ dt$$

$$= \frac{3}{2}\int_0^{2\pi}\left(\frac{1-\cos 2t}{2}\right)\left(\frac{1+\cos 2t}{2}\right)\ dt$$

$$= \frac{3}{8}\int_0^{2\pi}\left(1-\cos^2 2t\right)\ dt$$

$$= \frac{3}{8}\int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 4t\right)\ dt$$

$$= \frac{3}{8}\left(\frac{t}{2} - \frac{1}{8}\sin 4t\right)\Big|_0^{2\pi}$$

$$= \frac{3\pi}{8}\ \ unit^2$$

Exercise

Find the area of the region using line integral of the region enclosed by a disk of radius 5

Solution

$$x = 5\cos t \quad \to \quad dx = -5\sin t\ dt$$

$$y = 5\sin t \quad \to \quad dy = 5\cos t\ dt$$

$$A=\frac{1}{2}\int_0^{2\pi}\left(5\cos t\,(5\cos t)-5\sin t\,(-5\sin t)\right)dt \qquad\qquad A=\frac{1}{2}\oint_C xdy-ydx$$

$$=\frac{25}{2}\int_0^{2\pi}\left(\cos^2 t+\sin^2 t\right)dt$$

$$=\frac{25}{2}\int_0^{2\pi}dt$$

$$\underline{=25\pi}$$

Exercise

Find the area of the region using line integral of the region bounded by an ellipse with semi-major and semi-minor axes of length 12 and 8, respectively.

Solution

$$\frac{x^2}{6^2}+\frac{y^2}{4^2}=1$$

$x=6\cos t \quad \to \quad dx=-6\sin t\ dt$

$y=4\sin t \quad \to \quad dy=4\cos t\ dt$

$$A=\frac{1}{2}\int_0^{2\pi}\left(6\cos t\,(4\cos t)-4\sin t\,(-6\sin t)\right)dt \qquad\qquad A=\frac{1}{2}\oint_C xdy-ydx$$

$$=12\int_0^{2\pi}\left(\cos^2 t+\sin^2 t\right)dt$$

$$=12\int_0^{2\pi}dt$$

$$\underline{=24\pi}$$

Exercise

Find the area of the region using line integral of the region bounded by an ellipse $9x^2+25y^2=225$.

Solution

$$\frac{x^2}{25}+\frac{y^2}{9}=1$$

$x=5\cos t \quad \to \quad dx=-5\sin t\ dt$

$y=3\sin t \quad \to \quad dy=3\cos t\ dt$

$$A=\frac{1}{2}\int_0^{2\pi}\left(5\cos t\left(3\cos t\right)-3\sin t\left(-5\sin t\right)\right)dt \qquad A=\frac{1}{2}\oint_C xdy-ydx$$

$$=\frac{15}{2}\int_0^{2\pi}\left(\cos^2 t+\sin^2 t\right)dt$$

$$=\frac{15}{2}\int_0^{2\pi}dt$$

$$\underline{=15\pi}$$

Exercise

Find the area of the region using line integral of the region $\left\{(x,\ y):\quad x^2+y^2\le 16\right\}$

Solution

$$x=4\cos t \quad\rightarrow\quad dx=-4\sin t\ dt$$

$$y=4\sin t \quad\rightarrow\quad dy=4\cos t\ dt$$

$$A=\frac{1}{2}\int_0^{2\pi}\left(4\cos t\left(4\cos t\right)-4\sin t\left(-4\sin t\right)\right)dt \qquad A=\frac{1}{2}\oint_C xdy-ydx$$

$$=8\int_0^{2\pi}\left(\cos^2 t+\sin^2 t\right)dt$$

$$=8\int_0^{2\pi}dt$$

$$\underline{=16\pi}$$

Exercise

Find the area of the region using line integral of the region bounded by the parabolas $\vec{r}(t)=\left\langle t,\ 2t^2\right\rangle$ and $\vec{r}(t)=\left\langle t,\ 12-t^2\right\rangle$ for $-2\le t\le 2$

Solution

$$A=\frac{1}{2}\oint_C xdy-ydx$$

$$A=\frac{1}{2}\int_{-2}^{2}\left(t\frac{d}{dt}\left(2t^2\right)-2t^2\frac{d}{dt}(t)\right)dt-\frac{1}{2}\int_{-2}^{2}\left(t\frac{d}{dt}\left(12-t^2\right)-\left(12-t^2\right)\frac{d}{dt}(t)\right)dt$$

$$= \frac{1}{2}\int_{-2}^{2}\left(t(4t)-2t^2\right)dt - \frac{1}{2}\int_{-2}^{2}\left(t(-2t)-12+t^2\right)dt$$

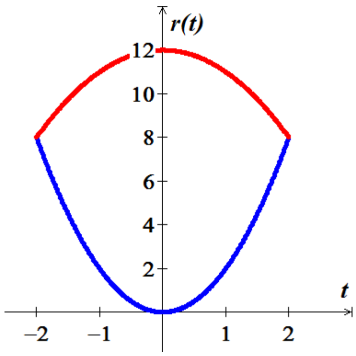
$$= \frac{1}{2}\int_{-2}^{2}\left(4t^2-2t^2+2t^2+12-t^2\right)dt$$

$$= \frac{1}{2}\int_{-2}^{2}\left(3t^2+12\right)dt$$

$$= \frac{1}{2}\left(t^3+12t\right)\Big|_{-2}^{2}$$

$$= \frac{1}{2}(8+24+8+24)$$

$$= 32$$



Exercise

Find the area of the region using line integral of the region bounded by the curve

$\vec{r}(t) = \left\langle t\left(1-t^2\right),\ 1-t^2\right\rangle$ for $-1 \le t \le 1$

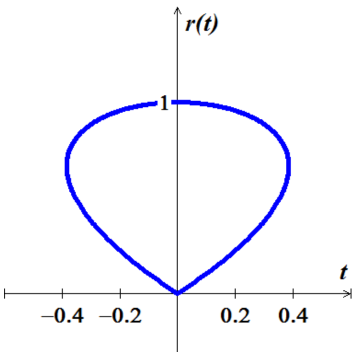
Solution

$\vec{r}(-1) = \langle 0,\ 0\rangle$

$\vec{r}\left(-\frac{1}{2}\right) = \left\langle \frac{1}{8},\ \frac{3}{4}\right\rangle$

$\vec{r}(0) = \langle 0,\ 1\rangle$

The curve travels in counterclockwise, therefore;



$$A = \frac{1}{2}\oint_C xdy - ydx$$

$$A = \frac{1}{2}\int_{1}^{-1}\left(\left(t-t^3\right)(-2t)-\left(1-t^2\right)\left(1-3t^2\right)\right)dt$$

$$= \frac{1}{2}\int_{1}^{-1}\left(-2t^2+2t^4-1+3t^2+t^2-3t^4\right)dt$$

$$= \frac{1}{2}\int_{1}^{-1}\left(2t^2-t^4-1\right)dt$$

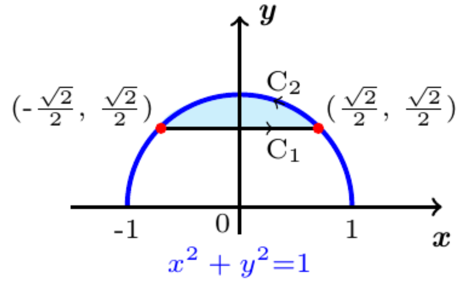
$$= \frac{1}{2}\left(\frac{2}{3}t^3-\frac{1}{5}t^5-t\right)\Big|_{1}^{-1}$$

$$= -\frac{2}{3}+\frac{1}{5}+1$$

$$= \frac{8}{3}$$

Exercise

Find the area of the region using line integral of the shaded region



Solution

For the path C_1 :

$$\begin{cases} t=0 & \rightarrow\ x=-\frac{\sqrt{2}}{2} \\ t=1 & \rightarrow\ x=\frac{\sqrt{2}}{2} \end{cases}$$

$$x=\frac{\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}}{1-0}t-\frac{\sqrt{2}}{2}$$

$$=\sqrt{2}t-\frac{\sqrt{2}}{2}$$

$$y=\frac{\sqrt{2}}{2}$$

$$C_1:\ \vec{r}_1(t)=\left\langle \sqrt{2}t-\frac{\sqrt{2}}{2},\ \ \frac{\sqrt{2}}{2}\right\rangle \quad 0\le t\le 1$$

$$\vec{r}_1{}'(t)=\left\langle \sqrt{2},\ \ 0\right\rangle$$

For the path C_2 :

$$C_2:\ \vec{r}_2(t)=\langle \cos t,\ \ \sin t\rangle \quad -\frac{\pi}{4}\le t\le \frac{\pi}{4}$$

$$\vec{r}_2{}'(t)=\langle -\sin t,\ \ \cos t\rangle$$

$$A=\frac{1}{2}\int_0^1\left(\left(\sqrt{2}t-\frac{\sqrt{2}}{2}\right)(0)-\left(\frac{\sqrt{2}}{2}\right)\left(\sqrt{2}\right)\right)dt+\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(\cos^2 t+\sin^2 t\right)dt \qquad A=\frac{1}{2}\oint_C xdy-ydx$$

$$=-\frac{1}{2}\int_0^1 dt+\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} dt$$

$$=-\frac{1}{2}+\frac{1}{2}t\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$\boxed{=\frac{\pi}{4}-\frac{1}{2}}$$

Exercise

Prove the identity $\oint_C dx = \oint_C dy = 0$, where C is a simple closed smooth oriented curve

Solution

$$\oint_C dx = \oint_C dy$$

$$\oint_C dx - \oint_C dy = \oint_C (1dx - 1dy)$$

This is an outward flux of the constant vector field $\vec{F} = \langle 1,\ 1\rangle$

$$\oint_C dx - \oint_C dy = \underbrace{\iint_R \left(\frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(1)\right) dA}_{=0}$$

$$\oint_C dx = \oint_C dy = 0 \quad \checkmark$$

Exercise

Prove the identity $\oint_C f(x)\,dx + g(y)\,dy = 0$, where f and g have continuous derivatives on the region enclosed by C (is a simple closed smooth oriented curve)

Solution

By Green's Theorem:

$$\oint_C f(x)dx + g(y)dy = \underbrace{\iint \left(\frac{\partial}{\partial x}(g(y)) - \frac{\partial}{\partial y}(f(x))\right) dA}_{=0} \quad \checkmark$$

Exercise

Show that the value of $\oint_C xy^2 dx + \left(x^2 y + 2x\right) dy$ depends only on the area of the region enclosed by C.

Solution

$$\oint_C xy^2 dx + \left(x^2 y + 2x\right) dy = \iint_R \left(\frac{\partial}{\partial x}\left(x^2 y + 2x\right) - \frac{\partial}{\partial y}\left(xy^2\right)\right) dA$$

$$= \iint_R (2xy + 2 - 2xy)\, dA$$

$$= 2\iint_R dA$$

$$\boxed{= 2 \times Area\ of\ A}$$

$\therefore \oint_C xy^2dx + \left(x^2y + 2x\right)dy$ depends only on the area of the region

Exercise

In terms of the parameters a and b, how is the value of $\oint_C aydx + bxdy$ related to the area of the region enclosed by C, assuming counterclockwise orientation of C?

Solution

$$\oint_C aydx + bxdy = \iint_R \left(\frac{\partial}{\partial x}(bx) - \frac{\partial}{\partial y}(ay)\right)dA$$

$$= \iint_R (b-a)\,dA$$

$$\boxed{= (b-a) \times Area\ of\ A}$$

Exercise

Show that if the circulation form of Green's Theorem is applied to the vector field $\left\langle 0,\ \frac{f(x)}{c}\right\rangle$ and $R = \{(x,\ y):\ \ a \le x \le b,\ \ 0 \le y \le c\}$, then the result is the Fundamental Theorem of Calculus,

$$\int_a^b \frac{df}{dx}dx = f(b) - f(a)$$

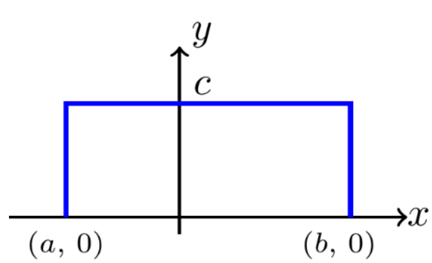
Solution

If $f(x)$ is continuous, then the circulation form of Green's Theorem is given

$$\oint_C \frac{f(x)}{c}dy = \frac{1}{c}\iint_R \frac{df}{dx}dA$$

$$\frac{1}{c}\iint_R \frac{df}{dx}dA = \frac{1}{c}\int_a^b \int_0^c \frac{df}{dx}dydx$$

$$= \frac{1}{c}\int_a^b \frac{df}{dx}\, y\Big|_0^c\, dx$$



$$= \int_a^b \frac{df}{dx}\,dx$$

$(a,\ 0)\ -\ (b,\ 0)$:

$$\vec{r}_1(t) = \langle (b-a)t + a,\ 0 \rangle$$

$$\vec{r}_1{}' = \langle b-a,\ 0 \rangle$$

$$\vec{F}_1 = \left\langle 0,\ \frac{f\left((b-a)t+a\right)}{c} \right\rangle$$

$$\vec{F}_1 \bullet \vec{r}_1{}' = \left\langle 0,\ \frac{f\left((b-a)t+a\right)}{c} \right\rangle \bullet \langle b-a,\ 0 \rangle$$

$$\underline{= 0}\,\rfloor$$

$(b,\ 0)\ -\ (b,\ c)$:

$$\vec{r}_2(t) = \langle b,\ ct \rangle$$

$$\vec{r}_2{}' = \langle 0,\ c \rangle$$

$$\vec{F}_2 = \left\langle 0,\ \frac{f(b)}{c} \right\rangle$$

$$\vec{F}_2 \bullet \vec{r}_2{}' = \left\langle 0,\ \frac{f(b)}{c} \right\rangle \bullet \langle 0,\ c \rangle$$

$$\underline{= f(b)}\,\rfloor$$

$(b,\ c)\ -\ (a,\ c)$:

$$\vec{r}_3(t) = \langle (a-b)t + b,\ c \rangle$$

$$\vec{r}_3{}' = \langle a-b,\ 0 \rangle$$

$$\vec{F}_3 = \left\langle 0,\ \frac{f\left((a-b)t+b\right)}{c} \right\rangle$$

$$\vec{F}_3 \bullet \vec{r}_3{}' = \left\langle 0,\ \frac{f\left((a-b)t+b\right)}{c} \right\rangle \bullet \langle a-b,\ 0 \rangle$$

$$\underline{= 0}\,\rfloor$$

$(a,\ c)\ -\ (a,\ 0)$:

$$\vec{r}_4(t) = \langle a,\ -ct + c \rangle$$

$$\vec{r}_4{}' = \langle 0,\ -c \rangle$$

$$\vec{F}_3 = \left\langle 0,\ \frac{f(a)}{c} \right\rangle$$

$$\vec{F}_3 \cdot \vec{r}_3\,' = \left\langle 0,\ \frac{f(a)}{c} \right\rangle \cdot \langle 0,\ -c \rangle$$

$$\underline{= -f(a)\,\big|}$$

$$\oint_C \frac{f(x)}{c}\,dy = \int_0^1 \big(0 + f(b) + 0 - f(a)\big)\,dt$$

$$= \int_0^1 \big(f(b) - f(a)\big)\,dt$$

$$= \big(f(b) - f(a)\big)t\Big|_0^1$$

$$\underline{= f(b) - f(a)\,\big|}$$

$$\underline{\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)\,\Big|}$$

Exercise

Show that if the flux form of Green's Theorem is applied to the vector field $\left\langle \frac{f(x)}{c},\ 0 \right\rangle$ and $R = \{(x,\ y):\ \ a \le x \le b,\ \ 0 \le y \le c\}$, then the result is the Fundamental Theorem of Calculus,

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

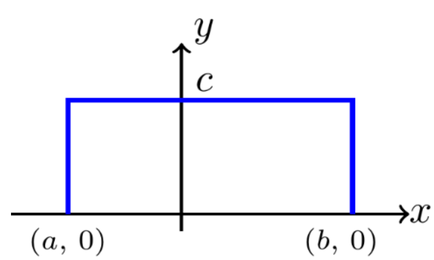
Solution

If $f(x)$ is continuous, then the circulation form of Green's Theorem is given

$$\oint_C \frac{f(x)}{c}\,dy = \frac{1}{c}\iint_R \frac{df}{dx}\,dA$$

$$\frac{1}{c}\iint_R \frac{df}{dx}\,dA = \frac{1}{c}\int_a^b \int_0^c \frac{df}{dx}\,dydx$$

$$= \frac{1}{c}\int_a^b \frac{df}{dx}\,y\Big|_0^c\,dx$$



$$= \int_a^b \frac{df}{dx}\,dx$$

$(a,\ 0)\ -\ (b,\ 0)$:

$$\vec{r}_1(t) = \langle (b-a)t + a,\ 0 \rangle$$

$$\vec{r}_1{}' = \langle b-a,\ 0 \rangle$$

$$\vec{F}_1 = \left\langle \frac{f\left((b-a)t+a\right)}{c},\ 0 \right\rangle$$

$$\frac{f\left((b-a)t+a\right)}{c}(0) + 0(b-a) = 0 \quad (1)$$

$(b,\ 0)\ -\ (b,\ c)$:

$$\vec{r}_2(t) = \langle b,\ ct \rangle$$

$$\vec{r}_2{}' = \langle 0,\ c \rangle$$

$$\vec{F}_2 = \left\langle \frac{f(b)}{c},\ 0 \right\rangle$$

$$\frac{f(b)}{c}(c) + 0 = f(b) \quad (2)$$

$(b,\ c)\ -\ (a,\ c)$:

$$\vec{r}_3(t) = \langle (a-b)t + b,\ c \rangle$$

$$\vec{r}_3{}' = \langle a-b,\ 0 \rangle$$

$$\vec{F}_3 = \left\langle \frac{f\left((a-b)t+b\right)}{c},\ 0 \right\rangle$$

$$\frac{f\left((a-b)t+b\right)}{c}(0) + 0(a-b) = 0 \quad (3)$$

$(a,\ c)\ -\ (a,\ 0)$:

$$\vec{r}_4(t) = \langle a,\ -ct + c \rangle$$

$$\vec{r}_4{}' = \langle 0,\ -c \rangle$$

$$\vec{F}_3 = \left\langle \frac{f(a)}{c},\ 0 \right\rangle$$

$$\frac{f(a)}{c}(c) + 0 = f(a) \quad (2)$$

$$\oint_C \frac{f(x)}{c}\,dy = \int_0^1 \left(0 + f(b) + 0 - f(a)\right) dt$$

$$= \int_0^1 \left(f(b) - f(a) \right) dt$$

$$= \left(f(b) - f(a) \right) t \Big|_0^1$$

$$= f(b) - f(a)$$

$$\boxed{\int_a^b \frac{df}{dx} dx = f(b) - f(a)}$$