Lecture R – Introduction to Differential Equation

Solution *Section* R.1 – Derivative

Exercise

Find the derivative of $f(t) = -3t^2 + 2t - 4$

Solution

$$f'(t) = \boxed{-6t + 2}$$

Exercise

Find the derivative of $g(x) = 4\sqrt[3]{x} + 2$

Solution

$$g(x) = 4x^{1/3} + 2$$

$$g'(x) = \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3x^{2/3}}$$

$$= \boxed{\frac{4}{3\sqrt[3]{x^2}}}$$

Exercise

Find the derivative of $f(x) = x\left(x^2 + 1\right)$

Solution

$$f(x) = x^3 + x$$

$$f'(x) = \boxed{3x^2 + 1}$$

Exercise

Find the derivative of $f(x) = \frac{2x^2 - 3x + 1}{x}$

Solution

$$f(x) = \frac{2x^2}{x} - \frac{3x}{x} + \frac{1}{x}$$

$$= 2x - 3 + \frac{1}{x}$$

$$f'(x) = \boxed{2 - \frac{1}{x^2}} \qquad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

Exercise

Find the derivative of $f(x)=\frac{4x^3-3x^2+2x+5}{x^2}$

Solution

$$f(x)=4x-3+\frac{2}{x}+5x^{-2} \qquad \left(\frac{1}{x}\right)'=-\frac{1}{x^2}$$

$$f'(x)=4-\frac{2}{x^2}-10x^{-3}$$

$$\boxed{=4-\frac{2}{x^2}-\frac{10}{x^3}}$$

Exercise

Find the derivative of $f(x)=\frac{-6x^3+3x^2-2x+1}{x}$

Solution

$$f(x)=-6x^2+3x-2+\frac{1}{x} \qquad \left(\frac{1}{x}\right)'=-\frac{1}{x^2}$$

$$f'(x)=\boxed{-12x+3-\frac{1}{x^2}}$$

Exercise

Find the derivative of $f(x)=x\left(1-\frac{2}{x+1}\right)$

Solution

$$f(x)=x-\frac{2x}{x+1}$$

$$\left(\frac{2x}{x+1}\right)' \Rightarrow \quad \begin{array}{ll} f=2x & f'=2 \\ g=x+1 & g'=1 \end{array}$$

$$f'(x)=1-\frac{2(x+1)-2x}{(x+1)^2}$$

$$=1-\frac{2x+2-2x}{(x+1)^2}$$

$$\boxed{=1-\frac{2}{(x+1)^2}}$$

Exercise

Find the derivative of $g(s)=\dfrac{s^2-2s+5}{\sqrt{s}}$

Solution

$$g(s)=\frac{s^2}{s^{1/2}}-2\frac{s}{s^{1/2}}+\frac{5}{s^{1/2}}$$

$$=s^{3/2}-2s^{1/2}+5s^{-1/2}$$

$$g'(s)=\tfrac{3}{2}s^{1/2}-2\tfrac{1}{2}s^{-1/2}+5\left(-\tfrac{1}{2}\right)s^{-3/2}$$

$$=\tfrac{3}{2}s^{1/2}-s^{-1/2}-\tfrac{5}{2}s^{-3/2}$$

$$=\frac{3}{2}\sqrt{s}-\frac{1}{\sqrt{s}}-\frac{5}{2s^{3/2}}$$

$$=\frac{3}{2}\sqrt{s}-\frac{1}{\sqrt{s}}-\frac{5}{2s\sqrt{s}}$$

Exercise

Find the derivative of $f(x)=\dfrac{x+1}{\sqrt{x}}$

Solution

$$f(x)=\frac{x}{x^{1/2}}+\frac{1}{x^{1/2}}$$

$$=x^{1/2}+x^{-1/2}$$

$$f'(x)=\tfrac{1}{2}x^{-1/2}-\tfrac{1}{2}x^{-3/2}$$

$$=\frac{1}{2x^{1/2}}-\frac{1}{2x^{3/2}}$$

Exercise

Find the derivative to the following functions $y=3x\left(2x^2+5x\right)$

Solution

$$y=6x^3+15x^2$$

$$\Rightarrow y'=18x^2+30x$$

Exercise

Find the derivative to the following functions $y = 3\left(2x^2 + 5x\right)$

Solution

$$y = 6x^2 + 15x$$

$$\Rightarrow \boxed{y' = 12x + 15}$$

Exercise

Find the derivative to the following functions $y = \dfrac{x^2 + 4x}{5}$

Solution

$$y' = \frac{1}{5}(2x + 4)$$

Exercise

Find the derivative to the following functions $y = \dfrac{3x^4}{5}$

Solution

$$y' = \frac{12}{5}x^3$$

Exercise

Find the derivative to the following functions $y = \dfrac{x^2 - 4}{2x + 5}$

Solution

$$\begin{aligned} y' &= \frac{(2x+5)(2x) - (x^2 - 4)(2)}{(2x+5)^2} \\ &= \frac{4x^2 + 10x - 2x^2 + 8}{(2x+5)^2} \\ &= \frac{2x^2 + 10x + 8}{(2x+5)^2} \end{aligned}$$

Exercise

Find the derivative to the following functions $y = \dfrac{(1+x)(2x-1)}{x-1}$

Solution

$$y' = \frac{(x-1)\frac{d}{dx}\left[(1+x)(2x-1)\right]-(1+x)(2x-1)\frac{d}{dx}[x-1]}{(x-1)^2}$$

$$= \frac{(x-1)\left[(1)(2x-1)+2(1+x)\right]-(1+x)(2x-1)(1)}{(x-1)^2}$$

$$= \frac{(x-1)(2x-1+2+2x)-(2x-1+2x^2-x)}{(x-1)^2}$$

$$= \frac{(x-1)(4x+1)-2x+1-2x^2+x}{(x-1)^2}$$

$$= \frac{4x^2+x-4x-1-2x+1-2x^2+x}{(x-1)^2}$$

$$= \frac{2x^2-4x}{(x-1)^2}$$

Or

$$y = \frac{(1+x)(2x-1)}{x-1}$$

$$= \frac{2x-1+2x^2-x}{x-1}$$

$$= \frac{2x^2+x-1}{x-1}$$

$$y' = \frac{(x-1)(4x+1)-(2x^2+x-1)(1)}{(x-1)^2}$$

$$= \frac{4x^2+x-4x-1-2x^2-x+1}{(x-1)^2}$$

$$= \frac{2x^2-4x}{(x-1)^2}$$

Exercise

Find the derivative to the following functions $y = \frac{4}{2x+1}$

Solution

$$y = 4(2x+1)^{-1}$$

$$y' = -4(2x+1)^{-2}(2)$$

$$= -8(2x+1)^{-2}$$

$$\boxed{= -\frac{8}{(2x+1)^2}}$$

Exercise

Find the derivative to the following functions $y = \dfrac{2}{(x-1)^3} = 2(x-1)^{-3}$

Solution

$$y = 2(x-1)^{-3}$$

$$y' = 2(-3)(x-1)^{-4}(1)$$

$$\boxed{= -\frac{6}{(x-1)^4}}$$

Exercise

Find the derivative to the following functions $y = \sqrt[3]{(x+4)^2}$

Solution

$$y = (x+4)^{2/3}$$

$$y' = \tfrac{2}{3}(x+4)^{-1/3}$$

$$= \frac{2}{3}\frac{1}{(x+4)^{1/3}}$$

$$\boxed{= \frac{2}{3\sqrt[3]{x+4}}}$$

Exercise

Find the derivative of $f(x) = \sqrt{2t^2 + 5t + 2}$

Solution

$$f(t) = \left(2t^2 + 5t + 2\right)^{1/2} \qquad U = 2t^2 + 5t + 2 \;\rightarrow\; U' = 4t + 5$$

$$f'(t) = \tfrac{1}{2}(4t+5)\left(2t^2 + 5t + 2\right)^{-1/2} \qquad \left(U^n\right)' = nU'U^{n-1}$$

$$\boxed{= \frac{1}{2}\frac{4t+5}{\sqrt{2t^2 + 5t + 2}}}$$

Exercise

Find the derivative of $f(x) = \dfrac{1}{\left(x^2 - 3x\right)^2}$

Solution

$$f(x) = \left(x^2 - 3x\right)^{-2}$$

$$f'(x) = -2(2x-3)\left(x^2 - 3x\right)^{-3}$$

$$\boxed{= -\frac{2(2x-3)}{\left(x^2 - 3x\right)^3}}$$

Exercise

Find the derivative of $y = t^2\sqrt{t-2}$

Solution

$$f = t^2 \qquad f' = 2t$$

$$g = (t-2)^{1/2} \qquad g' = \tfrac{1}{2}(t-2)^{-1/2}$$

$$y' = 2t\sqrt{t-2} + t^2 \tfrac{1}{2}(t-2)^{-1/2}$$

$$= \left[2t(t-2)^{1/2} + t^2 \tfrac{1}{2}(t-2)^{-1/2}\right]\frac{2(t-2)^{1/2}}{2(t-2)^{1/2}}$$

$$= \frac{4t(t-2) + t^2}{2(t-2)^{1/2}}$$

$$= \frac{4t^2 - 8t + t^2}{2\sqrt{t-2}}$$

$$\boxed{= \frac{5t^2 - 4t}{2\sqrt{t-2}}}$$

Exercise

Find the derivative of $y = \left(\dfrac{6-5x}{x^2-1}\right)^2$

Solution

$$f = 6 - 5x \qquad f' = -5$$

$$g = x^2 - 1 \qquad g' = 2x$$

$$y' = 2\frac{-5\left(x^2-1\right)-2x(6-5x)}{\left(x^2-1\right)^2}\left(\frac{6-5x}{x^2-1}\right) \qquad \left(U^n\right)' = nU'U^{n-1}$$

$$= 2\frac{-5x^2+5-12x+10x^2}{\left(x^2-1\right)^3}(6-5x)$$

$$= \boxed{\frac{2\left(5x^2-12x+5\right)(6-5x)}{\left(x^2-1\right)^3}}$$

Exercise

Find the derivative to the following functions $y = x^2\sqrt{x^2+1}$

Solution

$$y = x^2\left(x^2+1\right)^{1/2}$$

$$y' = x^2\frac{d}{dx}\left[(x^2+1)^{1/2}\right]+(x^2+1)^{1/2}\frac{d}{dx}\left[x^2\right]$$

$$= x^2\left[\frac{1}{2}(x^2+1)^{-1/2}(2x)\right]+(x^2+1)^{1/2}[2x]$$

$$= x^3(x^2+1)^{-1/2}+2x(x^2+1)^{1/2}$$

$$= \frac{(x^2+1)^{1/2}}{(x^2+1)^{1/2}}\left[x^3(x^2+1)^{-1/2}+2x(x^2+1)^{1/2}\right]$$

$$= \frac{x^3(x^2+1)^{-1/2}(x^2+1)^{1/2}+2x(x^2+1)^{1/2}(x^2+1)^{1/2}}{(x^2+1)^{1/2}}$$

$$= \frac{x^3+2x(x^2+1)}{(x^2+1)^{1/2}}$$

$$= \frac{x^3+2x^3+2x}{\sqrt{x^2+1}}$$

$$= \frac{3x^3+2x}{\sqrt{x^2+1}}$$

$$= \boxed{\frac{x(3x^2+2)}{\sqrt{x^2+1}}}$$

Exercise

Find the derivative to the following functions $y=\left(\frac{x+1}{x-5}\right)^2$

Solution

$$y'=2\left(\frac{x+1}{x-5}\right)\frac{d}{dx}\left[\frac{x+1}{x-5}\right]$$

$$=2\left(\frac{x+1}{x-5}\right)\left[\frac{(1)(x-5)-(1)(x+1)}{(x-5)^2}\right]$$

$$=2\left(\frac{x+1}{x-5}\right)\left(\frac{x-5-x-1}{(x-5)^2}\right)$$

$$=2\left(\frac{x+1}{x-5}\right)\left(\frac{-6}{(x-5)^2}\right)$$

$$=-\frac{12(x+1)}{(x-5)^3}$$

Exercise

Find the derivative to the following functions $y=x^2\sin x$

Solution

$$y'=2x\sin x+x^2\cos x$$

$u=x^2 \quad v=\sin x$

$u'=2x \quad v'=\cos x$

Exercise

Find the derivative to the following functions $y=\frac{\sin x}{x}$

Solution

$$y'=\frac{x\cos x-\sin x}{x^2}$$

$u=\sin x \quad v=x$

$u'=\cos x \quad v'=1$

Exercise

Find the derivative to the following functions $y=\frac{\cot x}{1+\cot x}$

Solution

$$y'=\frac{-\csc^2 x(1+\cot x)+\csc^2 x\cot x}{(1+\cot x)^2}$$

$u=\cot x \quad v=1+\cot x$

$u'=-\csc^2 x \quad v'=-\csc^2 x$

$$= \frac{-\csc^2 x - \csc^2 x \cot x + \csc^2 x \cot x}{(1+\cot x)^2}$$

$$= \boxed{\frac{-\csc^2 x}{(1+\cot x)^2}}$$

Exercise

Find the derivative to the following functions $y = x^2 \sin x + 2x \cos x - 2\sin x$

Solution

$$y' = 2x\sin x + x^2 \cos x + 2\cos x - 2x\sin x - 2\cos x$$

$$= \boxed{x^2 \cos x}$$

Exercise

Find the derivative to the following functions $y = x^3 \sin x \cos x$

Solution

$$y' = \left(x^3\right)' \sin x \cos x + x^3 (\sin x)' \cos x + x^3 \sin x (\cos x)'$$

$$= \boxed{3x^2 \sin x \cos x + x^3 \cos^2 x - x^3 \sin^2 x}$$

Exercise

Find the derivative to the following functions $y = \frac{4}{\cos x} + \frac{1}{\tan x}$

Solution

$$y' = \frac{-4\sin x}{\cos^2 x} - \frac{\sec^2 x}{\tan^2 x} \qquad \left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$$

$$= -4\frac{\sin x}{\cos x}\frac{1}{\cos x} - \frac{1}{\cos^2 x}\frac{\cos^2 x}{\sin^2 x}$$

$$\boxed{= -4\tan x \sec x - \csc^2 x}$$

Exercise

Find the derivative to the following functions $f(x) = x^2 e^x$

Solution

$$f'(x) = e^x \frac{d}{dx}\left(x^2\right) + x^2 \frac{d}{dx}\left(e^x\right)$$

$$= e^x(2x) + x^2 e^x$$

$$\boxed{= xe^x(2+x)}$$

Exercise

Find the derivative to the following functions $f(x) = \frac{e^x + e^{-x}}{2}$

Solution

$$f(x) = \frac{e^x + e^{-x}}{2}$$

$$= \frac{1}{2}(e^x + e^{-x})$$

$$f'(x) = \frac{1}{2}\left(\frac{d}{dx}\left[e^x\right] + \frac{d}{dx}\left[e^{-x}\right]\right)$$

$$\boxed{= \frac{1}{2}(e^x - e^{-x})}$$

Exercise

Find the derivative to the following functions $f(x) = \frac{e^x}{x^2}$

Solution

$$f'(x) = \frac{x^2 e^x - e^x(2x)}{x^4}$$

$$= \frac{x^2 e^x - 2xe^x}{x^4}$$

$$= \frac{xe^x(x-2)}{x^4}$$

$$\boxed{= \frac{e^x(x-2)}{x^3}}$$

Exercise

Find the derivative to the following functions $f(x) = x^2 e^x - e^x$

Solution

$$f'(x) = e^x \frac{d}{dx}[x^2] + x^2 \frac{d}{dx}[e^x] - \frac{d}{dx}[e^x]$$

$$= e^x(2x) + x^2 e^x - e^x$$

$$\boxed{= e^x(x^2 + 2x - 1)}$$

Exercise

Find the derivative to the following functions $f(x) = (1+2x)e^{4x}$

Solution

$$f'(x) = (2)e^{4x} + (1+2x)(4e^{4x})$$

$$= 2e^{4x} + (1+2x)(4e^{4x})$$

$$= 2e^{4x}(1+2(1+2x))$$

$$= 2e^{4x}(1+2+4x)$$

$$\underline{= 2e^{4x}(3+4x)}$$

Exercise

Find the derivative to the following functions $y = x^2e^{5x}$

Solution

$$y' = x^2\left(5e^{5x}\right) + 2x\left(e^{5x}\right)$$

$$\underline{= xe^{5x}(5x+2)}$$

Exercise

Find the derivative to the following functions $y = e^{x^2+1}\sqrt{5x+2}$

Solution

$$y = (2x)e^{x^2+1}\sqrt{5x+2} + e^{x^2+1}\frac{5}{2\sqrt{5x+2}}$$

$$= 2xe^{x^2+1}\sqrt{5x+2}\,\frac{2\sqrt{5x+2}}{2\sqrt{5x+2}} + \frac{5e^{x^2+1}}{2\sqrt{5x+2}}$$

$$= \frac{4xe^{x^2+1}(5x+2)}{2\sqrt{5x+2}} + \frac{5e^{x^2+1}}{2\sqrt{5x+2}}$$

$$= \frac{20x^2e^{x^2+1}+8xe^{x^2+1}+5e^{x^2+1}}{2\sqrt{5x+2}}$$

$$\underline{= \frac{e^{x^2+1}\left(20x^2+8x+5\right)}{2\sqrt{5x+2}}}$$

Exercise

Find the derivative to the following functions $f(x) = \ln \sqrt[3]{x+1}$

Solution

$$f(x) = \ln(x+1)^{1/3}$$

$$= \frac{1}{3}\ln(x+1)$$

$$u = x+1 \Rightarrow \frac{du}{dx} = 1$$

$$f'(x) = \frac{1}{3}\frac{1}{x+1}$$

$$= \frac{1}{3(x+1)}$$

Exercise

Find the derivative to the following functions $f(x) = \ln\left[x^2\sqrt{x^2+1}\right]$

Solution

$$f(x) = \ln(x^2) + \ln\sqrt{x^2+1} \qquad \textit{Product Property}$$

$$f(x) = \ln(x^2) + \ln(x^2+1)^{1/2}$$

$$f(x) = 2\ln x + \frac{1}{2}\ln(x^2+1) \qquad \textit{Power Property}$$

$$f'(x) = 2\frac{1}{x} + \frac{1}{2}\frac{2x}{x^2+1} \qquad \textit{Differentiate}$$

$$= \frac{2}{x} + \frac{x}{x^2+1}$$

Exercise

Find the derivative to the following functions $y = \ln\frac{x^2}{x^2+1}$

Solution

$$y = \ln x^2 - \ln x^2 + 1$$

$$y' = \frac{2x}{x^2} - \frac{2x}{x^2+1}$$

$$= \frac{2}{x} - \frac{2x}{x^2+1}$$

Exercise

Find the derivative to the following functions $y = \ln\frac{1+e^x}{1-e^x}$

Solution

$$y = \ln\left(1+e^x\right) - \ln\left(1-e^x\right)$$

$$y' = \frac{e^x}{1+e^x} - \frac{-e^x}{1-e^x}$$

$$= \frac{e^x}{1+e^x} + \frac{e^x}{1-e^x}$$

$$= \frac{e^x - e^{2x} + e^x + e^{2x}}{\left(1+e^x\right)\left(1-e^x\right)}$$

$$= \frac{2e^x}{\left(1+e^x\right)\left(1-e^x\right)}$$

Exercise

Find the derivative to the following functions $y = x\ 3^{x+1}$

Solution

$$y' = 3^{x+1} + x\ 3^{x+1}\ln 3$$

$$= 3^{x+1}\left(1 + x\ \ln 3\right)$$

Exercise

Find the derivative to the following functions $f(t) = \frac{\log_8\left(t^{3/2}+1\right)}{t}$

Solution

$$f' = \frac{\frac{1}{\ln 8}\frac{\frac{3}{2}t^{1/2}}{t^{3/2}+1}\cdot t - \log_8\left(t^{3/2}+1\right)}{t}$$

$$= \frac{\frac{1}{\ln 8}\frac{\frac{3}{2}t^{3/2}}{t^{3/2}+1} - \log_8\left(t^{3/2}+1\right)}{t^2}\cdot\frac{2\ln 8\left(t^{3/2}+1\right)}{2\ln 8\left(t^{3/2}+1\right)}$$

$$= \frac{3t^{3/2} - 2\left(t^{3/2}+1\right)(\ln 8)\log_8\left(t^{3/2}+1\right)}{t^2\left(t^{3/2}+1\right)\ln 8}$$

Solution *Section* R.2 – Integration

Exercise

Find each indefinite integral. $\displaystyle\int \frac{x+2}{\sqrt{x}}\,dx$

Solution

$$\int \frac{x+2}{\sqrt{x}}dx = \int\left[\frac{x}{x^{1/2}}+\frac{2}{x^{1/2}}\right]dx$$

$$= \int \frac{x}{x^{1/2}}dx + \int \frac{2}{x^{1/2}}dx$$

$$= \int x^{1/2}dx + 2\int x^{-1/2}dx$$

$$= \frac{x^{3/2}}{3/2} + 2\frac{x^{1/2}}{1/2} + C$$

$$= \frac{2}{3}x^{3/2} + 4x^{1/2} + C$$

Exercise

Find each indefinite integral $\displaystyle\int 4y^{-3}dy$

Solution

$$\int 4y^{-3}dy = 4\frac{y^{-2}}{-2} + C$$

$$= -\frac{2}{y^2} + C$$

Exercise

Find each indefinite integral $\displaystyle\int (x^3-4x+2)dx$

Solution

$$\int \left(x^3-4x+2\right)dx = \frac{1}{4}x^4 - 2x^2 + 2x + C$$

Exercise

Find each indefinite integral $\int\left(\sqrt[4]{x^3}+1\right)dx$

Solution

$$\int\left(x^{3/4}+1\right)dx=\boxed{\frac{4}{7}x^{7/4}+x+C}$$

Exercise

Find each indefinite integral $\int\sqrt{x}(x+1)dx$

Solution

$$\int x^{1/2}(x+1)dx=\int\left(x^{3/2}+x^{1/2}\right)dx$$

$$\boxed{=\frac{2}{5}x^{5/2}+\frac{2}{3}x^{3/2}+C}=x+5x^{-1}+C$$

Exercise

Find each indefinite integral $\int(1+3t)t^2dt$

Solution

$$\int\left(t^2+3t^3\right)dt=\boxed{\frac{1}{3}t^3+\frac{3}{4}t^4+C}$$

Exercise

Find each indefinite integral $\int\frac{x^2-5}{x^2}dx$

Solution

$$\int\frac{x^2-5}{x^2}dx=\int\left(1-\frac{5}{x^2}\right)dx$$

$$=\int\left(1-5x^{-2}\right)dx$$

$$\boxed{=x+\frac{5}{x}+C}$$

Exercise

Find each indefinite integral $\int (-40x + 250)\,dx$

Solution

$$\int (-40x + 250)\,dx = \boxed{-20x^2 + 250x + C}$$

Exercise

Find each indefinite integral $\int \left(7 - 3x - 3x^2\right)(2x+1)\ dx$

Solution

$$\begin{aligned}\int \left(7 - 3x - 3x^2\right)(2x+1)\ dx &= \int \left(14x + 7 - 6x^2 - 3x - 6x^3 - 3x^2\right)dx \\ &= \int \left(-6x^3 - 9x^2 + 11x + 7\right)dx \\ &= \boxed{-\frac{3}{2}x^4 - 3x^3 + \frac{11}{2}x^2 + 7x + C}\end{aligned}$$

Exercise

Evaluate the integral $\int xe^{2x}dx$

Solution

Let: $u = x \Rightarrow du = dx$

$$dv = e^{2x}dx \Rightarrow v = \int dv = \int e^{2x}dx = \frac{1}{2}e^{2x}$$

$$\int udv \ = \ uv - \int vdu$$

$$\begin{aligned}\int xe^{2x}dx &= \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx \\ &= \frac{1}{2}xe^{2x} - \frac{1}{2}\frac{1}{2}e^{2x} + C \\ &= \boxed{\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C}\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int x\ln x\,dx$

Solution

Let: $u = \ln x \Rightarrow du = \frac{1}{x}dx$

$$dv = xdx \Rightarrow v = \int dv = \int xdx = \frac{1}{2}x^2$$

$$\begin{aligned}\int x\ln x\,dx &= \frac{1}{2}x^2\ln x - \frac{1}{2}\int x^2\frac{dx}{x}\\ &= \frac{1}{2}x^2\ln x - \frac{1}{2}\int xdx\\ &= \boxed{\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C}\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int x^2\sin x\,dx$

		$\int \sin x$
x^2	$(+)$	$-\cos x$
$2x$	$(-)$	$-\sin x$
2	$(+)$	$\cos x$

Solution

$$\int x^2\sin x\,dx = \boxed{-x^2\cos x - 2x\sin x + 2\cos x + C}$$

Exercise

Evaluate the integral $\displaystyle\int\left(x^2-2x+1\right)e^{2x}dx$

		$\int e^{2x}$
$+$	x^2-2x+1	$\frac{1}{2}e^{2x}$
$-$	$2x-2$	$\frac{1}{4}e^{2x}$
$+$	2	$\frac{1}{8}e^{2x}$

Solution

$$\begin{aligned}\int\left(x^2-2x+1\right)e^{2x}dx &= \frac{1}{2}\left(x^2-2x+1\right)e^{2x} - \frac{1}{4}(2x-2)e^{2x} + \frac{1}{8}(2)e^{2x} + C\\ &= \left(\frac{1}{2}x^2 - x + \frac{1}{2} - \frac{1}{2}x + \frac{1}{2} + \frac{1}{4}\right)e^{2x} + C\\ &= \boxed{\left(\frac{1}{2}x^2 - \frac{3}{2}x + \frac{5}{5}\right)e^{2x} + C}\end{aligned}$$

Exercise

Evaluate the integral $\int e^{2x}\cos 3x\,dx$

Solution

		$\int \cos 3x$
+	e^{2x}	$\frac{1}{3}\sin 3x$
-	$\frac{1}{2}e^{2x}$	$-\frac{1}{9}\cos 3x$
+	$\frac{1}{4}e^{2x}$	

$$\int e^{2x}\cos 3x\,dx = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x - \frac{9}{4}\int e^{2x}\cos 3x\,dx$$

$$\int e^{2x}\cos 3x\,dx + \frac{9}{4}\int e^{2x}\cos 3x\,dx = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x + C_1$$

$$\frac{13}{4}\int e^{2x}\cos 3x\,dx = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x + C_1$$

$$\frac{4}{13}\frac{13}{4}\int e^{2x}\cos 3x\,dx = \frac{4}{13}\frac{1}{2}e^{2x}\cos 3x + \frac{4}{13}\frac{3}{4}e^{2x}\sin 3x + \frac{4}{13}C_1$$

$$\int e^{2x}\cos 3x\,dx = \frac{e^{2x}}{13}(2\cos 3x + 3\sin 3x) + C$$

Exercise

Find the general solution of the differential equation $y' = 2t + 3$

Solution

$$\int dy = \int (2t+3)\,dt$$

$$y(t) = t^2 + 3t + C$$

Exercise

Find the general solution of the differential equation $y' = 3t^2 + 2t + 3$

Solution

$$\int dy = \int (3t^2 + 2t + 3)\,dt$$

$$y(t) = t^3 + t^2 + 3t + C$$

Exercise

Find the general solution of the differential equation $y' = \sin 2t + 2\cos 3t$

Solution

$$\int dy = \int (\sin 2t + 2\cos 3t)\, dt$$

$$\boxed{y(t) = -\frac{1}{2}\cos 2t + \frac{2}{3}\sin 3t + C}$$

Exercise

Find the general solution of the differential equation: $y' = x^3(3x^4+1)^2$

Solution

$$\int x^3\left(3x^4+1\right)^2 dx$$

$$u = 3x^4 + 1 \Rightarrow du = 12x^3 dx$$

$$\int x^3(3x^4+1)^2 dx = \int \frac{1}{12}u^2 du$$

$$= \frac{1}{12}\frac{(3x^4+1)^3}{3} + C$$

$$= \frac{1}{36}(3x^4+1)^3 + C$$

$$\boxed{y(x) = \frac{1}{36}\left(3x^4+1\right)^3 + C}$$

Exercise

Find the general solution of the differential equation: $y' = 5x\sqrt{x^2-1}$

Solution

$$\int 5x\sqrt{x^2-1}\, dx$$

$$u = x^2 - 1 \quad \Rightarrow du = 2xdx$$

$$\int 5x\left(x^2-1\right)^{1/2} dx$$

$$= 5\int u^{1/2}\ \frac{1}{2}du$$

Substitute for x and dx

$$= \frac{5}{2}\int u^{1/2} du$$

$$= \frac{5}{2}\frac{u^{3/2}}{3/2} + C$$

$$= \frac{5}{3}u^{3/2} + C$$

$$\boxed{= \frac{5}{3}(x^2 - 1)^{3/2} + C}$$

Exercise

Find the general solution of the differential equation: $y' = x\sqrt{x^2 + 4}$

Solution

$$u = x^2 + 4 \quad \Rightarrow du = 2xdx$$

$$xdx = \frac{1}{2}du$$

$$\int \sqrt{x^2 + 4}\ xdx = \int u^{1/2}\ \frac{1}{2}du$$

$$= \frac{1}{2}\frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{3}u^{3/2} + C$$

$$= \frac{1}{3}(x^2 + 4)^{3/2} + C$$

$$\boxed{y(x) = \frac{1}{3}\left(x^2 + 4\right)^{3/2} + C}$$

Exercise

Find the general solution of the differential equation: $y' = (2x + 1)e^{x^2 + x}$

Solution

$$\int dy = \int (2x + 1)e^{x^2 + x}\ dx \qquad u = x^2 + x \quad \Rightarrow\ du = (2x + 1)dx$$

$$\int dy = \int e^u\ du$$

$$y = e^u + C$$

$$\boxed{y(x) = e^{x^2 + x} + C}$$

Exercise

Find the general solution of the differential equation: $y' = \frac{1}{6x-5}$

Solution

$$\int dy = \int \frac{1}{6x-5}\,dx$$

$$\int dy = \frac{1}{6}\int \frac{1}{6x-5}\,d(6x-5)$$

$$\boxed{y(x) = \frac{1}{6}\ln|6x-5| + C}$$

Exercise

Find the general solution of the differential equation: $y' = \frac{x^2+2x+3}{x^3+3x^2+9x+1}$

Solution

$$\int dy = \int \frac{x^2+2x+3}{x^3+3x^2+9x+1}\,dx \qquad u = x^3+3x^2+9x+1 \quad du = 3\left(x^2+2x+3\right)dx$$

$$\int dy = \frac{1}{3}\int \frac{du}{u}$$

$$y(x) = \frac{1}{3}\ln|u| + C$$

$$\boxed{y(x) = \frac{1}{3}\ln\left|x^3+3x^2+9x+1\right| + C}$$

Exercise

Find the general solution of the differential equation: $y' = \frac{1}{x(\ln x)^2}$

Solution

$$\int dy = \int \frac{1}{x(\ln x)^2}\,dx \qquad u = \ln x \quad du = \frac{dx}{x}$$

$$\int dy = \int \frac{1}{u^2}\,du$$

$$y = -\frac{1}{u} + C$$

$$\boxed{y(x) = -\frac{1}{\ln x} + C}$$

Exercise

Evaluate the integrals $\displaystyle\int_{-2}^{2}\left(x^3-2x+3\right)dx$

Solution

$$\int_{-2}^{2}\left(x^3-2x+3\right)dx=\left[\frac{x^4}{4}-x^2+3x\right]_{-2}^{2}$$

$$=\left(\frac{(2)^4}{4}-(2)^2+3(2)\right)-\left(\frac{(-2)^4}{4}-(-2)^2+3(-2)\right)$$

$$\underline{=12}$$

Exercise

Evaluate the integrals $\displaystyle\int_{0}^{1}\left(x^2+\sqrt{x}\right)dx$

Solution

$$\int_{0}^{1}\left(x^2+\sqrt{x}\right)dx=\left[\frac{x^3}{3}+\frac{2}{3}x^{3/2}\right]_{0}^{1}$$

$$=\left(\frac{(1)^3}{3}+\frac{2}{3}(1)^{3/2}\right)-0$$

$$\underline{=1}$$

Exercise

Evaluate the integrals $\displaystyle\int_{0}^{\pi/3}4\sec u\tan u\ du$

Solution

$$\int_{0}^{\pi/3}4\sec u\tan u\ du=4\sec u\Big|_{0}^{\pi/3}$$

$$=4\left(\sec\frac{\pi}{3}-\sec 0\right)$$

$$=4(2-1)$$

$$\underline{=4}$$

Exercise

Evaluate the integrals $\displaystyle\int_{\pi/4}^{3\pi/4} \csc\theta\cot\theta\, d\theta$

Solution

$$\begin{aligned}
\int_{\pi/4}^{3\pi/4} \csc\theta\cot\theta\, d\theta &= -\csc\theta \Big|_{\pi/4}^{3\pi/4} \\
&= -\left(\csc\frac{3\pi}{4} - \csc\frac{\pi}{4}\right) \\
&= -\left(\sqrt{2} - \sqrt{2}\right) \\
&= 0
\end{aligned}$$

Exercise

Evaluate the integrals $\displaystyle\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt$

Solution

$$\begin{aligned}
\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt &= \int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \pi t^{-2}\right) dt \\
&= \left[4\tan t - \pi t^{-1}\right]_{-\pi/3}^{-\pi/4} \\
&= \left(4\tan\left(-\frac{\pi}{4}\right) - \pi\left(-\frac{4}{\pi}\right)\right) - \left(4\tan\left(-\frac{\pi}{3}\right) - \pi\left(-\frac{3}{\pi}\right)\right) \\
&= \left(4(-1) + 4\right) - \left(4\left(-\sqrt{3}\right) + 3\right) \\
&= -\left(-4\sqrt{3} + 3\right) \\
&= 4\sqrt{3} - 3
\end{aligned}$$

Exercise

Evaluate the integrals $\displaystyle\int_{-3}^{-1} \frac{y^5 - 2y}{y^3}\, dy$

Solution

$$\begin{aligned}
\int_{-3}^{-1} \frac{y^5 - 2y}{y^3}\, dy &= \int_{-3}^{-1} \left(\frac{y^5}{y^3} - \frac{2y}{y^3}\right) dy \\
&= \int_{-3}^{-1} \left(y^2 - 2y^{-2}\right) dy
\end{aligned}$$

$$=\left[\frac{1}{3}y^3+2y^{-1}\right]_{-3}^{-1}$$

$$=\left(\frac{1}{3}(-1)^3+\frac{2}{-1}\right)-\left(\frac{1}{3}(-3)^3+\frac{2}{-3}\right)$$

$$\boxed{=\frac{22}{3}}$$

Exercise

Evaluate the integrals $\displaystyle\int_1^8 \frac{\left(x^{1/3}+1\right)\left(2-x^{2/3}\right)}{x^{1/3}}dx$

Solution

$$\int_1^8 \frac{\left(x^{1/3}+1\right)\left(2-x^{2/3}\right)}{x^{1/3}}dx=\int_1^8 \frac{2x^{1/3}-x+2-x^{2/3}}{x^{1/3}}dx$$

$$=\int_1^8 \left(2-x^{2/3}+2x^{-1/3}-x^{1/3}\right)dx$$

$$=\left[2x-\frac{3}{5}x^{5/3}+3x^{2/3}-\frac{3}{4}x^{4/3}\right]_1^8$$

$$=\left(2(8)-\frac{3}{5}(8)^{5/3}+3(8)^{2/3}-\frac{3}{4}(8)^{4/3}\right)-\left(2(1)-\frac{3}{5}(1)^{5/3}+3(1)^{2/3}-\frac{3}{4}(1)^{4/3}\right)$$

$$=\left(-\frac{16}{5}\right)-\left(\frac{73}{20}\right)$$

$$\boxed{=-\frac{137}{20}}$$

Exercise

Evaluate: $\displaystyle\int_0^1 (2t+3)^3dt$

Solution

$$\int_0^1 (2t+3)^3dt=\int_0^1 u^3\frac{1}{2}du \qquad u=2t+3\Rightarrow du=2dt\to\frac{du}{2}=dt$$

$$=\frac{1}{2}\int_0^1 u^3du$$

$$=\frac{1}{2}\frac{u^4}{4}\Big|_0^1$$

$$=\frac{1}{8}(2t+3)^4\Big|_0^1$$

$$= \frac{1}{8}\left[(2(1)+3)^4 - (2(0)+3)^4 \right]$$

$$= \frac{1}{8}\left[5^4 - 3^4 \right]$$

$$\underline{= 68|}$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^{1} r\sqrt{1-r^2}\, dr$

Solution

Let $u = 1 - r^2 \quad \Rightarrow \quad du = -2rdr \quad \rightarrow \quad -\frac{1}{2}du = rdr$

$$\int_{-1}^{1} r\sqrt{1-r^2}\, dr = \int_{-1}^{1} u^{1/2}\left(-\tfrac{1}{2}du\right)$$

$$= -\frac{1}{2}\frac{2}{3}u^{3/2}\Big|_{-1}^{1}$$

$$= -\frac{1}{3}\left[\left(1-r^2\right)^{3/2}\right]_{-1}^{1}$$

$$= -\frac{1}{3}\left[\left(1-(1)^2\right)^{3/2} - \left(1-(-1)^2\right)^{3/2}\right]$$

$$= -\frac{1}{3}[0-0]$$

$$\underline{= 0|}$$

Exercise

Find the general solution of $F'(x) = 4x + 2$, and find the particular solution that satisfies the initial condition $F(1) = 8$.

Solution

$$F(x) = \int (4x+2)dx$$

$$= 2x^2 + 2x + C$$

$$F(x) = 2(1)^2 + 2(1) + C = 8$$

$$2 + 2 + C = 8$$

$$C = 4$$

$$\Rightarrow \boxed{F(x) = 2x^2 + 2x + 4}$$

Exercise

Find the general solution of the differential equation: $y' = t\cos 3t$

Solution

$$u = t \to du = dt$$

$$dv = \cos 3t \to v = \frac{1}{3}\sin 3t$$

$$y = \frac{1}{3}t\sin 3t - \frac{1}{3}\int \sin 3t dt$$

$$= \frac{1}{3}t\sin 3t - \frac{1}{3}\frac{1}{3}\cos 3t + C$$

$$\boxed{y(t) = \frac{1}{3}t\sin 3t - \frac{1}{9}\cos 3t + C}$$

Exercise

A ball is thrown into the air from an initial height of 6 *m* with an initial velocity of 120 *m/s*. What will be the maximum height of the ball and at what time will this event occur?

Solution

$$\frac{dv}{dt} = -g \Rightarrow dv = -gdt$$

$$v(t) = -gt + C_1$$

$$v(t=0) = -g(0) + C_1 = 120$$

$$C_1 = 120$$

$$v(t) = -9.8t + 120$$

$$\frac{dx}{dt} = v \Rightarrow dx = vdt$$

$$x(t) = \int (-9.8t + 120)dt$$

$$= -4.9t^2 + 120t + C_2$$

$$x(0) = -4.9(0)^2 + 120(0) + C_2 = 6$$

$$C_2 = 6$$

$$x(t) = -4.9t^2 + 120t + 6$$

$$v(t) = -9.8t + 120 = 0 \quad \to t = \frac{120}{9.8} = 12.24\ sec$$

$$x(t = 12.24) = -4.9(12.24)^2 + 120(12.24) + 6$$

$$\boxed{x(t) = 740.69\ m}$$

Exercise

Derive the position function if a ball is thrown upward with initial velocity of 32 *ft* per *second* from an initial height of 48 *ft*. When does the ball hit the ground? With what velocity does the ball hit the ground?

Solution

$$s(t) = -16t^2 + 32t + 48$$

$$s(0) = 48$$

$$s''(t) = -32$$

$$s'(t) = \int -32\,dt \quad s'(0) = 32$$

$$= -32t + C_1$$

$$s'(0) = -32(0) + C_1 = 32$$

$$\Rightarrow C_1 = 32$$

$$s'(t) = -32t + 32$$

$$s(t) = \int (-32t + 32)\,dt$$

$$= -32\frac{t^2}{2} + 32t + C_2$$

$$s(0) = -32\frac{0^2}{2} + 32(0) + C_2 = 48 \qquad \Rightarrow C_2 = 48$$

$$s(t) = -16t^2 + 32t + 48$$

$$s(t) = -16t^2 + 32t + 48 = 0$$

$$-t^2 + 2t + 3 = 0 \ \Rightarrow t = -1,\ t = 3$$

The ball hits the ground in **3** seconds

The velocity: $v(t) = s'(t) = -32t + 32$

$$v(t = 3) = -32(3) + 32 = \underline{-64\ ft/\sec^2}$$