Evaluate using integration by parts

#1 a

$$\int x^2 \ln x \, dx$$

$$u = \ln x \qquad dv = x^2 dx$$
$$du = \frac{1}{x} dx \qquad v = \frac{1}{3}x^3$$

$$\int x^2 \ln x \, dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \frac{1}{x} dx$$
$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx$$
$$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$$

#1 b

$$\int (x+1)^2 e^x \, dx$$

$$\begin{array}{ccc} & & e^x \\ (x+1)^2 & \xrightarrow{+} & e^x \\ 2(x+1) & \xrightarrow{-} & e^x \\ 2 & \xrightarrow{+} & e^x \end{array}$$

$$= (x+1)^2 e^x - 2e^x(x+1) + 2e^x + C$$

$$= e^x\left[(x+1)^2 - 2x - 2 + 2\right] + C$$
$$= e^x\left[(x+1)^2 - 2x\right] + C$$

#1-c

$$\int e^{-2x} \sin 3x \, dx$$

$$u = \sin 3x \qquad dv = e^{-2x} dx$$
$$du = 3\cos 3x \, dx \qquad v = -\frac{1}{2}e^{-2x}$$

$$\int e^{-2x} \sin 3x \, dx = -\frac{1}{2}e^{-2x}\sin 3x + \frac{3}{2}\int e^{-2x}\cos 3x \, dx$$

$$u = \cos 3x \qquad dv = e^{-2x} dx$$
$$du = -3\sin 3x \, dx \qquad v = -\frac{1}{2}e^{-2x}$$

$$\int e^{-2x}\sin 3x \, dx = -\frac{1}{2}e^{-2x}\sin 3x + \frac{3}{2}\left[-\frac{1}{2}(\cos 3x)e^{-2x} - \frac{3}{2}\int e^{-2x}\sin 3x \, dx\right]$$

$$\int e^{-2x}\sin 3x\,dx = -\frac{1}{2}e^{-2x}\sin 3x - \frac{3}{4}e^{-2x}\cos 3x + \frac{9}{4}\int e^{-2x}\sin 3x\,dx$$

$$\int e^{-2x}\sin 3x\,dx + \frac{9}{4}\int e^{-2x}\sin 3x\,dx = -\frac{1}{2}e^{-2x}\sin 3x - \frac{3}{4}e^{-2x}\cos 3x$$

$$\left(\frac{4}{13}\right)\times \quad \frac{13}{4}\int e^{-2x}\sin 3x\,dx = -\frac{1}{2}e^{-2x}\sin 3x - \frac{3}{4}e^{-2x}\cos 3x$$

$$\int e^{-2x}\sin 3x\,dx = -\frac{2}{13}e^{-2x}\sin 3x - \frac{3}{13}e^{-2x}\cos 3x + C$$

#1-d

$$\int x^2\cos x\,dx$$

	$\int \cos x$
x^2	$\xrightarrow{+}$ $+\sin x$
$2x$	$\xrightarrow{-}$ $-\cos x$
2	$\xrightarrow{+}$ $-\sin x$

$$\int x^2\cos x\,dx = +x^2\sin x + 2x\cos x - 2\sin x + C$$

$$= (x^2-2)\sin x + 2x\cos x + C$$

#2-a

$$\int \sin^3 x \cos^4 x \, dx$$

$$= \int \cos^4 x \sin^2 x \sin x \, dx \qquad d\cos x = -\sin x \, dx$$

$$= -\int \cos^4 x (1 - \cos^2 x) \, d(\cos x)$$

$$= -\int (\cos^4 x - \cos^6 x) \, d(\cos x)$$

$$= -\left(\frac{1}{5}\cos^5 x - \frac{1}{7}\cos^7 x\right) + C$$

$$= -\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C$$

2-b

$$\int \cos^5 x \sin^5 x \, dx = \int \sin^5 x \cos^4 x \cos x \, dx$$

$$d\sin x = \cos x \, dx$$

$$= \int \sin^5 x (1 - \sin^2 x)^2 \, d(\sin x)$$

$$= \int \sin^5 x (1 - 2\sin^2 x + \sin^4 x) \, d\sin x$$

$$= \int (\sin^5 x - 2\sin^7 x + \sin^9 x) \, d(\sin x$$

$$= \frac{1}{6}\sin^6 x - \frac{1}{4}\sin^8 x + \frac{1}{10}\sin^{10} x + C$$

2-6 $\displaystyle\int_0^{\pi/2} \cos^4 x\,dx$ $\qquad \cos^2 x = \frac{1+\cos 2x}{2}$

$$\int_0^{\pi/2} \cos^4 x\,dx = \int_0^{\pi/2} \left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int_0^{\pi/2} (1 + 2\cos 2x + \cos^2 2x)\,dx \qquad \cos^2 2x = \frac{1+\cos 4x}{2}$$

$$= \frac{1}{4}\int_0^{\pi/2} \left(1 + 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\int_0^{\pi/2} \left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\left[\frac{3}{2}x + \sin 2x + \frac{1}{8}\sin 4x \right|_0^{\pi/2}$$

$$= \frac{1}{4}\left[\frac{3}{2}\frac{\pi}{2} + \sin\pi + \frac{1}{8}\sin 2\pi - \left(0 + \sin 0 + \frac{1}{8}\sin 0\right)\right]$$

$$= \frac{1}{4}\left[\frac{3\pi}{4}\right]$$

$$= \frac{3\pi}{16}$$

2-7

$$\int_0^{\pi/6} \sin^5\theta \, d\theta = \int_0^{\pi/6} \sin^4\theta \sin\theta \, d\theta$$

$\sin^2\theta = 1 - \cos^2\theta$
$d(\cos\theta) = -\sin\theta \, d\theta$

⑤

$$= -\int_0^{\pi/6} (1-\cos^2\theta)^2 \, d(\cos\theta)$$

$$= -\int_0^{\pi/6} (1 - 2\cos^2\theta + \cos^4\theta) \, d(\cos\theta)$$

$$= -\left[\cos\theta - \frac{2}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta\right]\Big/_0^{\pi/6}$$

$$= -\left(\cos\frac{\pi}{6} - \frac{2}{3}\cos^3\frac{\pi}{6} + \frac{1}{5}\cos^5\frac{\pi}{6}\right) + \left(\cos 0 - \frac{2}{3}\cos^3 0 + \frac{1}{5}\cos^5 0\right)$$

$$= -\frac{\sqrt{3}}{2} + \frac{2}{3}\left(\frac{\sqrt{3}}{2}\right)^3 - \frac{1}{5}\left(\frac{\sqrt{3}}{2}\right)^5 + \left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

$$= -\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4} - \frac{9\sqrt{3}}{160} + \frac{8}{15}$$

$$= \frac{256 - 147\sqrt{3}}{480}$$

2-e $\int \tan^4 u \, du = \int \tan^2 u \, (\sec^2 u - 1) \, du$ $\quad \} \tan^2 u = \sec^2 u - 1$

$= \int \tan^2 u \, \overbrace{\sec^2 u}^{du} - \int \tan^2 u \, du$ $\quad d(\tan u) = \sec^2 u \, du$

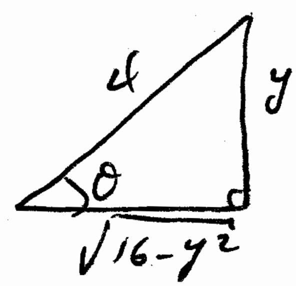
$= \int \tan^2 u \, d(\tan u) - \int (\sec^2 u - 1) \, du$

$= \frac{1}{3} \tan^3 u - \int \sec^2 u \, du + \int du$

$= \frac{1}{3} \tan^3 u - \tan u + u + C$

#3-a Evaluate using a trigonometric substitution

$$\int \frac{y \, dy}{\sqrt{16 - y^2}} \qquad a^2 = 16 \Rightarrow a = 4$$



$y = 4 \sin\theta \rightarrow dy = 4 \cos\theta \, d\theta$

$\sqrt{16 - y^2} = 4 \cos\theta$

$$\int \frac{y \, dy}{\sqrt{16 - y^2}} = \int \frac{4 \sin\theta \, (4 \cos\theta \, d\theta)}{4 \cos\theta}$$

$= 4 \int \sin\theta \, d\theta$

$= -4 \cos\theta + C$

$= -\sqrt{16 - y^2} + C$

or $d(16 - y^2) = -2y \, dy \rightarrow y \, dy = -\frac{1}{2} d(16 - y^2)$

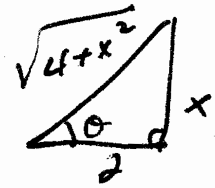
$$\int \frac{y \, dy}{\sqrt{16 - y^2}} = -\frac{1}{2} \int (16 - y^2)^{-1/2} d(16 - y^2)$$

$= -\frac{1}{2} \, 2 \, (16 - y^2)^{1/2} + C$

$= -\sqrt{16 - y^2} + C$

3-b

$$\int \frac{x\,dx}{\sqrt{4+x^2}} \qquad a^2=4 \Rightarrow a=2$$



$$x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta\,d\theta$$

$$\sqrt{4+x^2} = 2\sec\theta$$

$$\int \frac{x\,dx}{\sqrt{4+x^2}} = \int \frac{2\tan\theta\,(2\sec^2\theta\,d\theta)}{2\sec\theta}$$

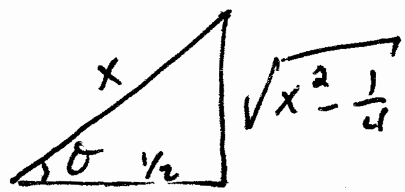
$$= 2\int \tan\theta\sec\theta\,d\theta$$

$$= 2\sec\theta + C.$$

$$= \cancel{2}\sqrt{4+x^2} + C$$

3-c

$$\int \frac{x\,dx}{\sqrt{4x^2-1}}$$



$$\sqrt{4x^2-1} = \sqrt{4\left(x^2-\tfrac{1}{4}\right)} = 2\sqrt{x^2-\tfrac{1}{4}} \Rightarrow a^2=\tfrac{1}{4}$$

$$a = 1/2$$

$$x = \tfrac{1}{2}\sec\theta \Rightarrow dx = \tfrac{1}{2}\sec\theta\tan\theta\,d\theta$$

$$\sqrt{x^2-\left(\tfrac{1}{2}\right)^2} = \tfrac{1}{2}\tan\theta.$$

$$\int \frac{x\,dx}{\sqrt{4x^2-1}} = \int \frac{\frac{1}{2}\sec\theta\ \frac{1}{2}\sec\theta\tan\theta\,d\theta}{2\ \frac{1}{2}\tan\theta}$$

$$= \cancel{2}\int \tfrac{1}{4}\sec^2\theta\,d\theta$$

$$= \cancel{6}\tfrac{1}{4}\tan\theta + C.$$

3-d

$$\int \frac{dy}{y^2\sqrt{9-y^2}} \qquad y = 3\sin\theta \rightarrow dy = 3\cos\theta\, d\theta$$

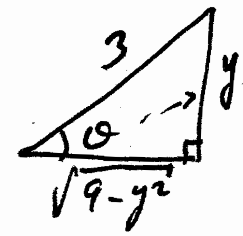
$$\sqrt{9-y^2} = 3\cos\theta$$

$$\int \frac{dy}{y^2\sqrt{9-y^2}} = \int \frac{3\cos\theta\, d\theta}{9\sin^2\theta\,(3\cos\theta)} = \frac{1}{9}\int \frac{d\theta}{\sin^2\theta}$$

$$= \frac{1}{9}\int \csc^2\theta\, d\theta$$

$$= -\frac{1}{9}\cot\theta + C$$

$$= -\frac{1}{9}\,\frac{\sqrt{9-y^2}}{y} + C$$



4-a

$$\int \frac{x\, dx}{x^2+4x+3}$$

$$\frac{x}{x^2+4x+3} = \frac{A}{x+3} + \frac{B}{x+1} = \frac{A(x+1)+B(x+3)}{(x+3)(x+1)}$$

$$x = Ax + A + Bx + 3B$$

$$= (A+B)x + A + 3B$$

$$\begin{cases} A + B = 1 \\ A + 3B = 0 \end{cases} \Rightarrow A = \frac{3}{2} \qquad B = -\frac{1}{2}$$

$$\int \frac{x\,dx}{x^2+4x+3} = \frac{3}{2}\int \frac{dx}{x+3} - \frac{1}{2}\int \frac{dx}{x+1}$$

$$= \frac{3}{2}\ln|x+3| - \frac{1}{2}\ln|x+1| + C$$

4-b

$$\int \frac{x+1}{x^2(x-1)}\,dx$$

$$\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} =$$

$$= \underline{Ax(x-1) + B(x-1) + Cx^2}$$

$$x+1 = Ax^2 - Ax + Bx - B + Cx^2$$
$$= (A+C)x^2 + (B-A)x - B$$

$$\begin{cases} A + C = 0 & \Rightarrow \boxed{C = -A = 2} \\ B - A = 1 & \rightarrow -A = 1 - B = 2 \Rightarrow \boxed{A = -2} \\ -B = 1 & \Rightarrow \boxed{B = -1} \end{cases}$$

$$\int \frac{x+1}{x^2(x-1)}\,dx = -2\int \frac{dx}{x} - \int \frac{dx}{x^2} + 2\int \frac{dx}{x-1}$$
$$= -2\ln|x| + \frac{1}{x} + 2\ln|x-1| + C$$

4-c

$$\int \frac{x+3}{2x^3 - 8x}\,dx \qquad 2x^3 - 8x = 2x(x^2 - 4)$$
$$= 2x(x-2)(x+2)$$

$$\frac{x+3}{2x^3 - 8x} = \frac{1}{2}\left(\frac{x+3}{x(x-2)(x+2)}\right) \Rightarrow$$

$$\frac{x+3}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2} = A(x^2-4) + B(x^2+2x) + C(x^2-2x)$$

$$x + 3 = (A+B+C)x^2 + (2B-2C)x - 4A$$

$$\begin{cases} A + B + C = 0 & C = \frac{1}{8} \qquad B = \frac{5}{8} \\ 2B - 2C = 1 & \\ -4A = 3 \Rightarrow A = -3/4 & \end{cases}$$

$$\int \frac{x+3}{2x^3 - 8x}\,dx = \frac{1}{2}\left[\left(-\frac{3}{4}\right)\int \frac{dx}{x} + \frac{5}{8}\int \frac{dx}{x-2} + \frac{1}{8}\int \frac{dx}{x+2}\right]$$
$$= \frac{1}{2}\left[-\frac{3}{4}\ln|x| + \frac{5}{8}\ln|x-2| + \frac{1}{8}\ln|x+2|\right] + C$$

4-d

$$\int \frac{2x^3+x^2-21x+24}{x^2+2x-8}\,dx$$

$$\begin{array}{r|l}
 & 2x - 3 \\
\hline
x^2+2x-8 & 2x^3+x^2-21x+24 \\
 & -2x^3 \mp 4x^2 \pm 16x \\
\hline
 & -3x^2-5x+24 \\
 & +3x^2+6x \mp 24 \\
\hline
 & x
\end{array}$$

$$\int = \int \left[2x-3+\frac{x}{x^2+2x-8}\right]dx$$

$$\frac{x}{x^2+2x-8} = \frac{A}{x+4} + \frac{B}{x-2}$$

$$x = Ax - 2A + Bx + 4B$$

$$\begin{cases} A+B=1 \\ 4B-2A=0 \end{cases} \rightarrow \begin{array}{l} A = 2/3 \\ B = 1/3 \end{array}$$

$$\int = \int (2x-3)\,dx + \frac{2}{3}\int \frac{dx}{x+4} + \frac{1}{3}\int \frac{dx}{x-2}$$

$$= x^2 - 3x + \frac{2}{3}\ln|x+4| + \frac{1}{3}\ln|x-2| + C$$

#5-a Evaluate the improper integral

$$\int_1^{\infty} \frac{dx}{(x+1)^9} = \int_1^b (x+1)^{-9}\, d(x+1) \qquad d(x+1) = dx$$

$$= -\frac{1}{8}(x+1)^{-8}\Big|_1^b$$

$$= -\frac{1}{8}\left[(b+1)^{-8} - 2^{-8}\right]$$

$$= -\frac{1}{8}\left(\frac{1}{(b+1)^8} - \frac{1}{2^8}\right)$$

$$\int_1^{\infty} \frac{dx}{(x+1)^9} = -\frac{1}{8}\lim_{b\to\infty}\left(\frac{1}{(b+1)^8} - \frac{1}{256}\right)$$

$$= -\frac{1}{8}\left(-\frac{1}{256}\right)$$

$$= \frac{1}{2048}$$

5-b

$$\int_0^{\infty} x e^{-x}\,dx = -e^{-x}(x+1)\Big|_0^b = -e^{-b}(b+1) - (-e^{0}(1))$$

$$\int_0^{\infty} x e^{-x}\,dx = \lim_{b\to\infty}\left(-\frac{b+1}{e^{+b}} + 1\right)$$

$$= 0 + 1$$

$$= 1$$

5-c

$$\int_0^1 \ln x\,dx = \lim_{b\to 0^+}\left[x\ln x - x\right]_b^1$$
$$= \lim_{b\to 0^+}\left[\ln 1 - 1 - (b\ln b - b)\right]$$
$$= (0-1)-(0-0)]$$
$$= -1$$

5-d

$$\int_1^\infty \frac{3x-1}{4x^3-x^2}\,dx$$

$$\frac{3x-1}{4x^3-x^2} = \frac{3x-1}{x^2(4x-1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{4x-1}$$

$$3x-1 = Ax(4x-1)+B(4x-1)+Cx^2$$
$$= 4Ax^2 - Ax + 4Bx - B + Cx^2$$

$$\begin{cases} 4A + C = 0 & \Rightarrow C = -4 \\ -A + 4B = 3 & \Rightarrow A = 1 \\ -B = -1 \Rightarrow B = 1 & \end{cases}$$

$$\int_1^\infty \frac{3x-1}{4x^3-x^2}\,dx = \cancel{\int_1^\infty \frac{dx}{x} + \int_1^\infty dx}$$
$$= \int_1^\infty\left(\frac{1}{x}+\frac{1}{x^2}-\frac{4}{4x+1}\right)dx$$
$$= \lim_{b\to\infty}\left(\ln|x| - \frac{1}{x}\right) - \int_1^\infty \frac{d(4x-1)}{4x-1}$$
$$= \lim_{b\to\infty}\left(\ln|x| - \frac{1}{x} - \ln(4x-1)\right)\Big|_1^b$$
$$= \lim_{b\to\infty}\left(\ln\frac{x}{4x-1} - \frac{1}{x}\right)\Big|_1^b$$
$$= \lim_{b\to\infty}\left(\ln\frac{b}{4b-1} - \frac{1}{b} - \left(\ln\frac{1}{3} - 1\right)\right)$$
$$= \ln\frac{1}{4} - 0 - \ln\frac{1}{3} + 1 = \ln 1/4 - \ln 1/3 + 1$$

5-e

$$\int_{-\infty}^{\infty} \frac{4\,dx}{x^2+16} = 2\int_0^{\infty} \frac{4\,dx}{x^2+4^2}$$

$$= 2 \lim_{b\to\infty} \tan^{-1}\frac{x}{4}\Big|_0^{b}$$

$$= 2 \lim_{b\to\infty} \left(\tan^{-1}\frac{b}{4} - \tan^{-1} 0 \right)$$

$$= 2\left(\frac{\pi}{2} - 0\right)$$

$$\boxed{= \pi}$$

#6-a

$$\cancel{\int \theta c} \quad \int \theta \cos(2\theta+1)\, d\theta$$

$$\begin{array}{ccc} & & \int \\ & & \cos(2\theta+1) \\ \theta & \xrightarrow{+} & \frac{1}{2}\sin(2\theta+1) \\ 1 & \xrightarrow{-} & -\frac{1}{4}\cos(2\theta+1) \end{array}$$

$$\int \theta \cos(2\theta+1)\, d\theta = \frac{1}{2}\theta \sin(2\theta+1) + \frac{1}{4}\cos(2\theta+1) + C$$

6-b $\displaystyle\int \frac{x+1}{x^2(x^2+4)}\,dx$

$$\frac{x+1}{x^2(x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4}$$

$$x+1 = Ax(x^2+4) + B(x^2+4) + (Cx+D)x^2$$
$$= Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Dx^2$$

$$\begin{cases} A+C=0 & \to C = -1/4 \\ B+D=0 & \to D = -B = -1/4 \\ 4A = 1 & \Rightarrow A = 1/4 \\ 4B=1 & \Rightarrow B = 1/4 \end{cases}$$

$$\int \frac{x+1}{x^2(x^2+4)}\,dx = \frac{1}{4}\int \frac{dx}{x} + \frac{1}{4}\int \frac{dx}{x^2} - \frac{1}{4}\int \frac{x+1}{x^2+4}\,dx$$
$$= \frac{1}{4}\ln|x| - \frac{1}{4x} - \frac{1}{4}\int \frac{x\,dx}{x^2+4} - \frac{1}{4}\int \frac{dx}{x^2+4}$$
$$= \frac{1}{4}\ln|x| - \frac{1}{4x} - \frac{1}{4}\int \frac{\frac{1}{2}d(x^2+4)}{x^2+4} - \frac{1}{4}\tan^{-1}\frac{x}{2}$$
$$= \frac{1}{4}\ln|x| - \frac{1}{4x} - \frac{1}{8}\ln(x^2+4) - \frac{1}{4}\tan^{-1}\frac{x}{2} + C$$

6-c

$$\int \frac{1+x^2}{(x+1)^3}\,dx$$

let $u = 1+x \Rightarrow du = dx$
$\hookrightarrow x = u-1$

$$\int \frac{1+x^2}{(x+1)^3}\,dx = \int \frac{1+(u-1)^2}{u^3}\,du$$
$$= \int \frac{1+u^2-2u+1}{u^3}\,du$$
$$= \int \frac{u^2-2u+2}{u^3}\,du$$

$$\frac{u^2-2u+2}{u^3} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u^3} = \frac{Au^2+Bu+C}{u^3}$$
$$\Rightarrow A=1,\ B=-2,\ C=2.$$

or $\frac{u^2}{u}$

$$\int \frac{1+x^2}{(x+1)^3}\,dx = \int\left(\frac{u^2}{u^3} - \frac{2u}{u^3} + \frac{2}{u^3}\right)du$$
$$= \int\left(\frac{1}{u} - 2u^{-2} + u^{-3}\right)du$$
$$= \ln|u| + \frac{2}{u} - \frac{1}{\cancel{2}u^2} + C$$
$$= \ln|x+1| + \frac{2}{x+1} - \frac{1}{(x+1)^2} + C$$

6-d $\int \sqrt{x}\sqrt{1+\sqrt{x}}\,dx$

$$u = \sqrt{x} \Rightarrow u^2 = x$$
$$2u\,du = dx$$

$$\int \sqrt{x}\sqrt{1+\sqrt{x}}\,dx = \int u\sqrt{1+u}\,(2u\,du)$$
$$= \int 2u^2(1+u)^{1/2}\,du$$

$$\begin{array}{ccc} & & (1+u)^{1/2} \\ 2u^2 & \xrightarrow{+} & \frac{2}{3}(1+u)^{3/2} \\ 4u & \xrightarrow{-} & \frac{4}{15}(1+u)^{5/2} \\ 4 & \xrightarrow{+} & \frac{8}{105}(1+u)^{7/2} \\ 0 & & \end{array}$$

$$\int \sqrt{x}\sqrt{1+\sqrt{x}}\,dx = \frac{4}{3}u^2(1+u)^{3/2} - \frac{16}{15}u(1+u)^{5/2} + \frac{32}{105}(1+u)^{7/2} + C$$
$$= \frac{4}{3}x(1+\sqrt{x})^{3/2} - \frac{16}{15}\sqrt{x}(1+\sqrt{x})^{5/2} + \frac{32}{105}(1+\sqrt{x})^{7/2} + C$$

7.a $x(x-1)\,dy - y\,dx = 0$

$$x(x-1)\,dy = y\,dx$$

$$\frac{dy}{y} = \frac{dx}{x(x-1)} = \left(\frac{A}{x} + \frac{B}{x-1}\right)dx$$

$$Ax - A + Bx = 1 \quad \begin{cases} A+B=0 \\ -A = 1 \Rightarrow A = -1 \\ B = 1 \end{cases}$$

$$\frac{dy}{y} = \left(-\frac{1}{x} + \frac{1}{x-1}\right)dx$$

$$\int \frac{dy}{y} = -\int \frac{1}{x}\,dx + \int \frac{dx}{x-1}$$

$$\ln y = -\ln|x| + \ln|x-1| + c \qquad c = \ln C_1$$

$$\ln y = \ln|x-1| + \ln C_1 - \ln|x|$$

$$\ln y = \ln \frac{C_1|x-1|}{|x|}$$

$$y = C_1 \frac{(x-1)}{x}$$

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b) $xy' + 2y = 1 - x^{-1}$

$$y' + \frac{2}{x}y = \frac{1}{x} - \frac{1}{x^2}$$

$$e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$$

$$\int x^2\left(\frac{1}{x} - \frac{1}{x^2}\right)dx = \int (x-1)\,dx = \frac{1}{2}x^2 - x$$

$$y = \frac{1}{x^2}\left(\frac{1}{2}x^2 - x + c\right) =$$

$$= \frac{1}{2} - \frac{1}{x} + \frac{c}{x^2}$$

7-c $\quad xy' - y = 2x\ln x$

$$y' - \frac{1}{x}y = 2\ln x$$

$$e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

$$\int x^{-1}(2\ln x)\,dx = 2\int \frac{\ln x}{x}\,dx \quad ; \; d(\ln x) = \frac{1}{x}dx$$

$$= 2\int \ln x \, d(\ln x)$$

$$= (\ln x)^2$$

$$y(x) = \frac{1}{x^{-1}}\left((\ln x)^2 + c\right)$$

$$= x(\ln x)^2 + Cx$$

7.d $\quad (1+e^x)\,dy + (ye^x + e^{-x})\,dx = 0$

$$(1+e^x)\frac{dy}{dx} + ye^x + e^{-x} = 0$$

$$(1+e^x)y' + e^x y = -e^{-x}$$

$$y' + \frac{e^x}{1+e^x}y = -\frac{e^{-x}}{1+e^x}$$

$$e^{\int \frac{e^x}{1+e^x}dx} = e^{\int \frac{d(e^x+1)}{e^x+1}} = e^{\ln e^x+1} = e^x+1$$

$$\int (e^x+1)\left(\frac{-e^{-x}}{1+e^x}\right)dx = -\int e^{-x}dx = e^{-x}$$

$$y(x) = \frac{1}{1+e^x}\left(e^{-x} + c\right) = \frac{e^{-x}+C}{1+e^x}$$

7-e $(x+3y^2)\,dy + y\,dx = 0$

$$x\,dy + 3y^2dy + y\,dx = 0$$

$$x\,dy + y\,dx = -3y^2dy$$

$$d(xy) = -3y^2dy$$

$$\int d(xy) = -3\int y^2dy$$

$$xy = -3\frac{y^3}{3} + c$$

$$xy = -y^3 + c$$

8-a $\frac{dy}{dx} + 3x^2y = x^2 , \quad y(0) = -1$

$$e^{\int 3x^2dx} = e^{x^3}$$

$\cancel{x^2 \quad e^{x^3}}$

$$\int x^2e^{x^3}dx = \frac{1}{3}\int e^{x^3}d(x^3) \qquad dx^3 = 3x^2dx$$

$$= \frac{1}{3}e^{x^3}$$

$$y(x) = \frac{1}{e^{x^3}}\left(\frac{1}{3}e^{x^3} + c\right) = \frac{1}{3} + \frac{c}{e^{x^3}}$$

$$y(0) = \frac{1}{3} + \frac{c}{e^0} \Rightarrow$$

$$-1 = \frac{1}{3} + c \Rightarrow c = -1 - \frac{1}{3} = -\frac{4}{3}$$

$$\Rightarrow y(x) = \frac{1}{3} - \frac{4}{3e^{x^3}}$$

8-b $x\,dy + (y - \cos x)\,dx = 0 \quad , y(\frac{\pi}{2}) = 0$

$\cancel{xy' =} y$

$$x \frac{dy}{dx} + y - \cos x = 0$$

$$x y' + y = \cos x$$

$$y' + \frac{1}{x} y = \frac{\cos x}{x}$$

$$e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

$$\int x \cdot \frac{\cos x}{x} dx = \int \cos x \, dx = + \sin x \cdot$$

$$y(x) = \frac{1}{x} (\sin x + C) = \frac{\sin x + C}{x}$$

$$y(\frac{\pi}{2}) = \frac{\sin \frac{\pi}{2} + C}{\pi/2}$$

$\frac{1}{\pi}$

$$0 = \sin \frac{\pi}{2} + c \rightarrow c = -1$$

$$\boxed{y(x) = \frac{\sin x - 1}{x}}$$

8-c $\quad y'(t) = \dfrac{t+1}{2ty}$, $y(1) = 4$

$$yy' = \frac{t+1}{2t} \Rightarrow \int y\,dy = \int \left(\frac{t}{2t} + \frac{1}{2t}\right) dt$$

$$\frac{1}{2}y^2 = \int \frac{1}{2}dt + \frac{1}{2}\int \frac{dt}{t}$$

$$= \frac{1}{2}t + \frac{1}{2}\ln|t| + C_1$$

$$y^2 = t + \ln|t| + C$$
$$y^2(1) = 1 + \ln|1| + C$$
$$4^2 = 1 + C \Rightarrow \boxed{C = 15}$$
$$y^2 = t + \ln|t| + 15$$
$$y(t) = \sqrt{t + \ln|t| + 15}$$

8-d $\quad \dfrac{dy}{dt} = \sqrt{y}\sin t \qquad y(0) = 4$

$$\frac{dy}{\sqrt{y}} = \sin t\,dt \Rightarrow \int y^{-1/2}dy = \int \sin t\,dt$$

$$2y^{1/2} = -\cos t + c$$
$$y^{1/2} = -\frac{1}{2}\cos t + C$$

$$\sqrt{(y(0)} = -\frac{1}{2}\cos 0 + C$$

$$\sqrt{4} = -\frac{1}{2} + C \rightarrow \boxed{C = 2 + \frac{1}{2} = 5/2}$$

$$y^{1/2} = -\frac{1}{2}\cos t + \frac{5}{2} = \frac{1}{2}(5 - \cos t)$$

$$y(t) = \frac{1}{4}(5 - \cos t)^2$$

#9 Find the length of the graph of the fctn. 22

$y = \ln(1-x^2) \qquad 0 \le x \le 1/2$

$$\frac{dy}{dx} = \frac{-2x}{1-x^2} \Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{4x^2}{(1-x^2)^2} = \frac{1-2x^2+x^4+4x^2}{(1-x^2)^2}$$

$$= \left(\frac{1+x^2}{1-x^2}\right)^2$$

$$L = \int_0^{1/2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{1/2} \sqrt{\left(\frac{1+x^2}{1-x^2}\right)^2}\, dx$$

$$= \int_0^{1/2} \frac{1+x^2}{1-x^2}\, dx \qquad \begin{array}{r|l} & -1 \\ \hline -x^2+1 & +x^2+1 \\ & +x^2 \mp 1 \\ \hline & 2 \end{array}$$

$$= \int_0^{1/2} \left(-1 + \frac{2}{1-x^2}\right) dx \qquad \frac{2}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x}$$

$$= \int_0^{1/2} \left(-1 + \frac{1}{1-x} + \frac{1}{1+x}\right) dx \qquad \begin{cases} A - B = 0 \\ A + B = 2 \end{cases} \Rightarrow \begin{array}{l} A = 1 \\ B = 1 \end{array}$$

$$= -x - \ln|1-x| + \ln|1+x| \cancel{+C} \Big]_0^{1/2}$$

$$= -x + \ln\left|\frac{1+x}{1-x}\right| \Big]_0^{1/2}$$

$$= -\frac{1}{2} + \ln\left|\frac{3/2}{1/2}\right| - (-0 + \ln 1)$$

$$= -\frac{1}{2} + \ln 3$$

$$\boxed{= \ln 3 - 1/2}$$

#10 The region -- $y = \frac{5}{x\sqrt{5-x}}$, $x=1$, $x=4$

$$V = \pi \int_a^b y^2 dx$$

$$= \pi \int_1^4 \frac{25}{x^2(5-x)} dx$$

$$\frac{25}{x^2(5-x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{5-x}$$

$$25 = Ax(5-x) + B(5-x) + Cx^2$$

$$= 5Ax - Ax^2 + 5B - Bx + Cx^2$$

$$\begin{cases} C - A = 0 \\ 5A - B = 0 \rightarrow A = 1 = C \\ 5B = 25 \rightarrow B = 5 \end{cases}$$

$$\cancel{V = \pi \int_1^4 \left(\frac{dx}{x}\right.} +$$

$$V = \pi \int_1^4 \left(\frac{1}{x} + \frac{5}{x^2} + \frac{1}{5-x}\right) dx$$

$$= \pi \left[\ln|x| - \frac{5}{x} - \ln|5-x| \right]_1^4$$

$$= \pi \left[\ln\left|\frac{x}{5-x}\right| - \frac{5}{x} \right]_1^4$$

$$= \pi \left[\ln 4 - \frac{5}{4} - \left(\ln\frac{1}{4} - 5\right) \right]$$

$$= \pi \left(\ln 4 - \frac{5}{4} + \ln 4 + 5 \right)$$

$$= \pi \left(2\ln 4 + \frac{15}{4} \right)$$

#11 $y = x \ln x \qquad 0 < x \leq 2$

$$V = \pi\int_0^2 y^2\,dx = \pi\int_0^2 (x\ln x)^2\,dx$$

$$u = (\ln x)^2 \qquad dv = x^2\,dx$$
$$du = 2\ln x\,\frac{dx}{x} \qquad v = \frac{1}{3}x^3$$

$$V = \pi\int_b^2 x^2(\ln x)^2\,dx$$
$$= \pi\left[\lim_{b\to 0^+} \frac{1}{3}x^3(\ln x)^2\Big|_b^2 - \int_b^2 \frac{2}{3}(\ln x)\,x^3\,\frac{dx}{x}\right.$$
$$= \pi\left(\frac{1}{3}2^3(\ln 2)^2 - 0\right) - \pi\,\frac{2}{3}\int_0^2 x^2\ln x\,dx$$
$$= \frac{8\pi}{3}(\ln 2)^2 - \frac{2\pi}{3}\lim_{b\to 0^+}\left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_b^2$$
$$= \frac{8\pi}{3}(\ln 2)^2 - \frac{2\pi}{3}\left(\frac{2^3}{3}\ln 2 - \frac{8}{9} - 0\right)$$
$$= \frac{2\pi}{3}\left(4(\ln 2)^2 - \frac{8}{3}\ln 2 + \frac{8}{9}\right)$$

1.
#12

V: Vol of saltwater
y : amount of salt dissolved

$$V = 500 + 20t$$

$$\frac{dy}{dt} = \text{Rate in} - \text{Rate out}$$

$$= 60\ \frac{gal}{min} \cdot \left(0.1 \cdot \frac{lb}{gal}\right) - 40\ \frac{gal}{min}\ \frac{y}{V}\ \frac{lb}{gal}$$

$$= 6\ \frac{lb}{min} - 40\ \frac{y}{500+20t}$$

$$\frac{dy}{dt} + \frac{40}{500+20t}\, y = 6$$

$$d(500+20t) = 20\,dt$$

$$e^{\int \frac{40}{500+20t}\,dt} = e^{\frac{40}{20}\int \frac{d(500+20t)}{500+20t}}$$

$$= e^{2\ln(500+20t)}$$

$$= e^{\ln(500+20t)^2}$$

$$= (500+20t)^2$$

$$\int 6(500+20t)^2\,dt = \frac{6}{20}\int (500+20t)^2\, d(500+20t)$$

$$= \frac{3}{10}\,\frac{1}{3}(500+20t)^3$$

$$= \frac{1}{10}(500+20t)^3$$

$$y(t) = \frac{1}{(500+20t)^2}\left[\frac{1}{10}(500+20t)^3 + C\right]$$

$$= \frac{1}{10}(500+20t) + \frac{C}{(500+20t)^2}$$

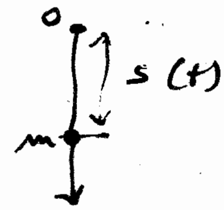
$$y(0) = \frac{1}{10}(500+20(0)) + \frac{C}{500^2}$$

$$100 = 50 + \frac{C}{250000} \Rightarrow \frac{C}{250000} = 50$$

$$C = 12{,}500{,}000$$

$$y(t) = \frac{1}{10}(500+20t) + \frac{12{,}500{,}000}{(500+20t)^2}$$

\# 13 Force resistance due to the air is kv



downward Force $F = ma = m\dfrac{dv}{dt}$

$F = mg - kv$

$$m\frac{dv}{dt} = mg - kv \Rightarrow \frac{dv}{dt} + \frac{k}{m}v = g.$$

$$\frac{dv}{dt} = g - \frac{k}{m}v \Rightarrow \frac{dv}{g - \frac{k}{m}v} = dt.$$

$$\int \frac{dv}{g - \frac{k}{m}v} = \int dt \qquad d\left(g - \frac{k}{m}v\right) = -\frac{k}{m}dv$$

$$-\frac{m}{k}\ln\left|g - \frac{k}{m}v\right| = t + C.$$

$$\ln\left|g - \frac{k}{m}v\right| = -\frac{k}{m}(t + C)$$

$$g - \frac{k}{m}v = e^{-\frac{k}{m}t - \frac{k}{m}C} = e^{-k/m\,t}\,e^{-k/m\,C}$$

$t = 0 \Rightarrow v = 0 \Rightarrow \quad g = e^{-k/m\,C}$

$$g - \frac{k}{m}v = g e^{-k/m\,t}$$

$$\frac{k}{m}v = g - g e^{-\frac{k}{m}t} = g\left(1 - e^{-k/m\,t}\right)$$

$$v = \frac{mg}{k}\left(1 - e^{-k/m\,t}\right) = s'(t)$$

$$s(t) = \frac{mg}{k}\int\left(1 - e^{-k/m\,t}\right)dt$$

$$= \frac{mg}{k}\left(t + \frac{m}{k}e^{-k/m\,t}\right) + E$$

$t = 0 \Rightarrow s = 0 \Rightarrow 0 = \dfrac{mg}{k}\dfrac{m}{k} + E \Rightarrow E = -\dfrac{m^2 g}{k^2}$

$$s(t) = \frac{mg}{k}t + \frac{m^2}{k^2}g e^{-k/m\,t} - \frac{m^2 g}{k^2}$$