Solution Section 4.1 – System of linear Equations

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 3x+2y=-4 \\ 2x-y=-5 \end{cases}$$

Solution

$$\begin{cases} \quad 3x+2y=-4 \\ 2\times \quad 2x-y=-5 \end{cases}$$

$$\begin{array}{r} 3x+2y=-4 \\ \underline{4x-2y=-10} \\ 7x=-14 \end{array}$$

$$\underline{x=-2}|$$

$$\begin{aligned} y &= 2x+5 \\ &= -4+5 \\ &= 1 \end{aligned}$$

Solution: $(-2,\ 1)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 2x+5y=7 \\ 5x-2y=-3 \end{cases}$$

Solution

$$\begin{cases} -5\times \quad 2x+5y=7 \\ 2\times \quad 5x-2y=-3 \end{cases}$$

$$\begin{array}{r} -10x-25y=-35 \\ \underline{10x-4y=-6} \\ -29y=-41 \end{array}$$

$$y=\frac{41}{29}$$

$$x=\frac{1}{2}\left(7-5\left(\frac{41}{29}\right)\right)$$

$$x=\frac{1}{2}\left(-\frac{2}{29}\right)$$

$$= -\frac{1}{29}$$

$\therefore$ ***Solution***: $\left(-\frac{1}{29}, \frac{41}{29}\right)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$$

Solution

$$\begin{cases} \quad\quad 4x - 7y = -16 \\ -2\times \quad 2x + 5y = 9 \end{cases}$$

$$\begin{array}{r} 4x - 7y = -16 \\ -4x - 10y = -18 \\ \hline -17y = -34 \end{array}$$

$$y = 2$$

$$x = \frac{9 - 5y}{2}$$

$$= \frac{9 - 10}{2}$$

$$= -\frac{1}{2}$$

$\therefore$ ***Solution***: $\left(-\frac{1}{2}, 2\right)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$$

Solution

$$\begin{cases} 3x + 2y = 4 & (1) \\ 2x + y = 1 & (2) \end{cases}$$

$$(2) \rightarrow \quad y = 1 - 2x \quad (3)$$

$$(1) \rightarrow \quad 3x + 2 - 4x = 4$$

$\underline{x = -2}|$

$(3) \to \quad y = 1 + 4$

$\quad\quad = 5|$

$\therefore$ ***Solution***: $\quad \underline{(-2,\ 5)}|$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 3x + 4y = 2 \\ 2x + 5y = -1 \end{cases}$$

Solution

$$\begin{cases} -2\times \quad 3x + 4y = 2 \\ 3\times \quad 2x + 5y = -1 \end{cases}$$

$$\begin{array}{r} -6x - 8y = -4 \\ \underline{6x + 15y = -3} \\ 7y = -7 \end{array}$$

$\underline{y = -1}|$

$2x = -1 + 5$

$x = \frac{4}{2}$

$\quad = 2|$

$\therefore$ ***Solution***: $\quad \underline{(2,\ -1)}|$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method) $\begin{cases} 5x - 2y = 4 \\ -10x + 4y = 7 \end{cases}$

Solution

$$\begin{cases} 2\times \quad 5x - 2y = 4 \\ \quad\quad -10x + 4y = 7 \end{cases}$$

$$\begin{array}{r} 10x - 4y = 8 \\ \underline{-10x + 4y = 7} \\ 0 = 15 \end{array} \quad (\textit{impossible})$$

$\therefore$ ***No Solution***

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} x-4y=-8 \\ 5x-20y=-40 \end{cases}$$

Solution

$$\begin{cases} x-4y=-8 & (1) \\ 5x-20y=-40 & (2) \end{cases}$$

$(1) \rightarrow \quad x=4y-8$

$(2) \rightarrow \quad 5(4y-8)-20y=-40$

$20y-40-20y=-40$

$-40=-40 \quad (\textit{True})$

$\therefore$ ***Solution***: $\quad \underline{x-4y=-8}$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$$

Solution

$$\begin{cases} 2x+y=3 & (1) \\ x-y=3 & (2) \end{cases}$$

$(2) \rightarrow \quad x=3+y \quad (3)$

$(1) \rightarrow \quad 6+2y+y=3$

$3y=-3$

$\underline{y=-1}$

$(3) \rightarrow \quad \underline{x=2}$

$\therefore$ ***Solution***: $\quad \underline{(2,\ -1)}$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 2x + 10y = -14 \\ 7x - 2y = -16 \end{cases}$$

Solution

$$\begin{cases} \quad 2x + 10y = -14 \\ 5\times \quad 7x - 2y = -16 \end{cases}$$

$$\begin{array}{r} 2x + 10y = -14 \\ 35x - 10y = -80 \\ \hline 37x = -94 \end{array}$$

$$\boxed{x = -\frac{94}{37}}$$

$$2y = 7\left(-\frac{94}{37}\right) + 16$$

$$y = -\frac{329}{37} + 8$$

$$\boxed{= -\frac{33}{37}}$$

$\therefore$ ***Solution***: $\boxed{\left(-\frac{94}{37},\ -\frac{33}{37}\right)}$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$$

Solution

$$\begin{cases} 3\times \quad 4x - 3y = 24 \\ \quad -3x + 9y = -1 \end{cases}$$

$$\begin{array}{r} 12x - 9y = 72 \\ -3x + 9y = -1 \\ \hline -9x = -71 \end{array}$$

$$\boxed{x = \frac{71}{9}}$$

$$3y = 4\left(\frac{71}{9}\right) - 24$$

$$y = \frac{284}{27} - 8$$

$$= \frac{68}{27}$$

$\therefore$ ***Solution***: $\left(\frac{71}{9},\ \frac{68}{27}\right)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$$

Solution

$$\begin{array}{r} 4x + 2y = 12 \\ \underline{3x - 2y = 16} \\ 7x = 28 \end{array}$$

$x = 4$

$2y = 12 - 4(4)$

$y = -\frac{4}{2}$

$= -2$

$\therefore$ ***Solution***: $(4,\ -2)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$$

Solution

$$\begin{array}{r} x + 2y = -1 \\ \underline{4x - 2y = 6} \\ 5x = 5 \end{array}$$

$x = 1$

$2y = -x - 1$

$y = -\frac{2}{2}$

$= -1$

$\therefore$ ***Solution***: $(1,\ -1)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} x-2y=5 \\ -10x+2y=4 \end{cases}$$

Solution

$$\begin{array}{r} x-2y=5 \\ \underline{-10x+2y=4} \\ -9x=9 \end{array}$$

$\underline{x=-1}|$

$2y=x-5$

$y=-\frac{6}{2}$

$\quad \underline{=-3}|$

$\therefore$ ***Solution***: $\quad \underline{(-1,\ -3)}|$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 12x+15y=-27 \\ 30x-15y=-15 \end{cases}$$

Solution

$$\begin{array}{r} 12x+15y=-27 \\ \underline{30x-15y=-15} \\ 42x=-42 \end{array}$$

$\underline{x=-1}|$

$15y=-27-12(-1)$

$y=-\frac{15}{15}$

$\quad \underline{=-1}|$

$\therefore$ ***Solution***: $\quad \underline{(-1,\ -1)}|$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$$

Solution

$$\begin{array}{r} 4x - 4y = -12 \\ \underline{4x + 4y = -20} \\ 8x = -32 \end{array}$$

$$\underline{x = -4|}$$

$$4y = 4(-4) + 12$$

$$y = -\frac{4}{4}$$

$$\underline{= -1|}$$

$\therefore$ ***Solution***: $\quad \underline{(-4,\ -1)\ |}$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 1 & 4 & 7 \\ 3 & 5 & 0 \end{array}\right] \quad R_2 - 3R_1$$

Solution

$$\begin{array}{ccc} 3 & 5 & 0 \\ -3 & -12 & -21 \\ \hline 0 & -7 & -21 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 4 & 7 \\ 0 & -7 & -21 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -5 \end{array}\right] \quad R_2 - 2R_1$$

Solution

$$\begin{array}{ccc} 2 & 1 & -5 \\ -2 & 6 & -2 \\ \hline 0 & 7 & -7 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 7 & -7 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 5 & 2 & 19 \end{array}\right] \quad R_2 - 5R_1$$

Solution

$$\begin{array}{ccc} 5 & 2 & 19 \\ -5 & 15 & -15 \\ \hline 0 & 17 & -4 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 17 & -4 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 2 & -3 & 8 \\ -6 & 9 & 4 \end{array}\right] \quad R_2 + 3R_1$$

Solution

$$\begin{array}{ccc} -6 & 9 & 4 \\ 6 & -9 & 24 \\ \hline 0 & 0 & 28 \end{array}$$

$$\left[\begin{array}{cc|c} 2 & -3 & 8 \\ 0 & 0 & 28 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 2 & 3 & 11 \\ 1 & 2 & 8 \end{array}\right] \quad 2R_2 - R_1$$

Solution

$$\begin{array}{ccc} 2 & 4 & 16 \\ -2 & -3 & -11 \\ \hline 0 & 1 & 5 \end{array}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 11 \\ 0 & 1 & 5 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 3 & 5 & -13 \\ 2 & 3 & -9 \end{array}\right] \quad 3R_2 - 2R_1$$

Solution

$$\begin{array}{ccc} 6 & 9 & -27 \\ -6 & -10 & 26 \\ \hline 0 & -1 & -1 \end{array}$$

$$\left[\begin{array}{cc|c} 3 & 5 & -13 \\ 0 & -1 & -1 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 5 & 4 & 1 \end{array}\right] \quad R_3 - 5R_2$$

Solution

$$\begin{array}{cccc} 0 & 5 & 4 & 1 \\ 0 & -5 & 5 & -10 \\ \hline 0 & 0 & 9 & -9 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 9 & -9 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array}\right] \quad \begin{array}{c} R_2 - 3R_1 \\ R_3 - R_1 \end{array}$$

Solution

$$\begin{array}{cccc} 3 & 3 & -1 & 10 \\ -3 & 3 & -15 & 18 \\ \hline 0 & 6 & -16 & 28 \end{array} \qquad \begin{array}{cccc} 1 & 3 & 2 & 5 \\ -1 & 1 & -5 & 6 \\ \hline 0 & 4 & -3 & 11 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 0 & 4 & -3 & 11 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 2 & 4 & 4 & 22 \\ -1 & -2 & 3 & 15 \end{array}\right] \quad \begin{array}{c} 3R_2 - 2R_1 \\ 3R_3 + R_1 \end{array}$$

Solution

$$\begin{array}{cccc} 6 & 12 & 12 & 66 \\ -6 & -4 & -2 & -2 \\ \hline 0 & 8 & 10 & 64 \end{array} \qquad \begin{array}{cccc} -3 & -6 & 9 & 45 \\ 3 & 2 & 1 & 1 \\ \hline 0 & -4 & 10 & 46 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 0 & 8 & 10 & 64 \\ 0 & -4 & 10 & 46 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & -4 & 2 & -7 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

Solution

$$\begin{array}{rrrr} 2 & 1 & 1 & 3 \\ -2 & -2 & -2 & -4 \\ \hline 0 & -1 & -1 & -1 \end{array} \qquad \begin{array}{rrrr} 3 & -4 & 2 & -7 \\ -3 & -3 & -3 & -6 \\ \hline 0 & -7 & -1 & -13 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & -1 \\ 0 & -7 & -1 & -13 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & -2 \\ 2 & -3 & 5 & -1 & 0 \\ 1 & 0 & 3 & 1 & -4 \\ -4 & 3 & 2 & -1 & 3 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 + 4R_1 \end{array}$$

Solution

$$\begin{array}{rrrrr} 2 & -3 & 5 & -1 & 0 \\ -2 & 4 & -2 & -6 & 4 \\ \hline 0 & 1 & 3 & -7 & 4 \end{array} \qquad \begin{array}{rrrrr} 1 & 0 & 3 & 1 & -4 \\ -1 & 2 & -1 & -3 & 2 \\ \hline 0 & 2 & 2 & -2 & -2 \end{array} \qquad \begin{array}{rrrrr} -4 & 3 & 2 & -1 & 3 \\ 4 & -8 & 4 & 12 & -8 \\ \hline 0 & -5 & 6 & 11 & -5 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & -2 \\ 0 & 1 & 3 & -7 & 4 \\ 0 & 2 & 2 & -2 & -2 \\ 0 & -5 & 6 & 11 & -5 \end{array}\right]$$

Exercise

Use the Gauss-Jordan method to solve the system
$$\begin{aligned} x-y+5z&=-6\\ 3x+3y-z&=10\\ x+3y+2z&=5 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6\\ 3 & 3 & -1 & 10\\ 1 & 3 & 2 & 5 \end{array}\right] \begin{array}{l} \\ R_2-3R_1\\ R_3-R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 3 & -1 & 10\\ -3 & 3 & -15 & 18\\ \hline 0 & 6 & -16 & 28 \end{array} \qquad \begin{array}{cccc} 1 & 3 & 2 & 5\\ -1 & 1 & -5 & 6\\ \hline 0 & 4 & -3 & 11 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6\\ 0 & 6 & -16 & 28\\ 0 & 4 & -3 & 11 \end{array}\right] \ \tfrac{1}{6}R_2 \qquad 0\quad 1\quad -\tfrac{8}{3}\quad \tfrac{14}{3}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6\\ 0 & 1 & -\frac{8}{3} & \frac{14}{3}\\ 0 & 4 & -3 & 11 \end{array}\right] \begin{array}{l} R_1+R_2\\ \\ R_3-4R_2 \end{array}$$

$$\begin{array}{cccc} 0 & 4 & -3 & 11\\ 0 & -4 & \frac{32}{3} & -\frac{56}{3}\\ \hline 0 & 0 & \frac{23}{3} & -\frac{23}{3} \end{array} \qquad \begin{array}{cccc} 1 & -1 & 5 & -6\\ 0 & 1 & -\frac{8}{3} & \frac{14}{3}\\ \hline 1 & 0 & \frac{7}{3} & -\frac{4}{3} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{3} & -\frac{4}{3}\\ 0 & 1 & -\frac{8}{3} & \frac{14}{3}\\ 0 & 0 & \frac{23}{3} & -\frac{23}{3} \end{array}\right] \begin{array}{l} \\ \\ \frac{3}{23}R_3 \end{array} \qquad 0\quad 0\quad 1\quad -1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{3} & -\frac{4}{3}\\ 0 & 1 & -\frac{8}{3} & \frac{14}{3}\\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} R_1-\frac{7}{3}R_3\\ R_2+\frac{8}{3}R_3\\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & \frac{7}{3} & -\frac{4}{3}\\ 0 & 0 & -\frac{7}{3} & \frac{7}{3}\\ \hline 1 & 0 & 0 & 1 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -\frac{8}{3} & \frac{14}{3}\\ 0 & 0 & \frac{8}{3} & -\frac{8}{3}\\ \hline 0 & 1 & 0 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & -1 \end{array}\right]$$

S***olution***: $(1,\ 2,\ -1)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} 2x - y + 4z = -3 \\ x - 2y - 10z = -6 \\ 3x \qquad + 4z = 7 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & -1 & 4 & -3 \\ 1 & -2 & -10 & -6 \\ 3 & 0 & 4 & 7 \end{array}\right] \begin{array}{l} \frac{1}{2}R_1 \\ \\ \\ \end{array}$$

$$1 \quad -\tfrac{1}{2} \quad 2 \quad -\tfrac{3}{2}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 1 & -2 & -10 & -6 \\ 3 & 0 & 4 & 7 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} 1 & -2 & -10 & -6 \\ -1 & \frac{1}{2} & -2 & \frac{3}{2} \\ \hline 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \end{array} \qquad \begin{array}{cccc} 3 & 0 & 4 & 7 \\ -3 & \frac{3}{2} & -6 & \frac{9}{2} \\ \hline 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \\ 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}\right] \begin{array}{l} \\ -\frac{2}{3}R_2 \\ \\ \end{array}$$

$$0 \quad 1 \quad 8 \quad 3$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & 1 & 8 & 3 \\ 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}\right] \begin{array}{l} R_1 + \frac{1}{2}R_2 \\ \\ R_3 - \frac{3}{2}R_2 \end{array}$$

$$\begin{array}{cccc} 0 & \frac{3}{2} & -2 & \frac{23}{2} \\ 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \\ \hline 0 & 0 & -14 & 7 \end{array} \qquad \begin{array}{cccc} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & \frac{1}{2} & 4 & \frac{3}{2} \\ \hline 1 & 0 & 6 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & -14 & 7 \end{array}\right] \begin{array}{l} \\ \\ -\frac{1}{14}R_3 \end{array}$$

$$0 \quad 0 \quad 1 \quad -\tfrac{1}{2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right] \begin{array}{l} R_1 - 6R_3 \\ R_2 - 8R_3 \\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & 6 & 0 \\ 0 & 0 & -6 & 3 \\ \hline 1 & 0 & 0 & 3 \end{array} \qquad \begin{array}{cccc} 0 & 1 & 8 & 3 \\ 0 & 0 & -8 & 4 \\ \hline 0 & 1 & 0 & 7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right]$$

Solution: $\left(3,\ 7,\ -\frac{1}{2}\right)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} 4x+3y-5z=-29 \\ 3x-7y-z=-19 \\ 2x+5y+2z=-10 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 4 & 3 & -5 & -29 \\ 3 & -7 & -1 & -19 \\ 2 & 5 & 2 & -10 \end{array}\right] \quad \tfrac{1}{4}R_1$$

$$1 \quad \tfrac{3}{4} \quad -\tfrac{5}{4} \quad -\tfrac{29}{4}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 3 & -7 & -1 & -19 \\ 2 & 5 & 2 & -10 \end{array}\right] \quad \begin{array}{l} \\ R_2-3R_1 \\ R_3-2R_1 \end{array}$$

$$\begin{array}{cccc} 3 & -7 & -1 & -19 \\ -3 & -\frac{9}{4} & \frac{15}{4} & \frac{87}{4} \\ \hline 0 & -\frac{37}{4} & \frac{11}{4} & \frac{11}{4} \end{array} \qquad \begin{array}{cccc} 2 & 5 & 2 & -10 \\ -2 & -\frac{3}{2} & \frac{5}{2} & \frac{29}{2} \\ \hline 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & -\frac{37}{4} & \frac{11}{4} & \frac{11}{4} \\ 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}\right] \quad -\tfrac{4}{37}R_2$$

$$0 \quad 1 \quad -\tfrac{11}{37} \quad -\tfrac{11}{37}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}\right] \quad \begin{array}{l} R_1-\frac{3}{4}R_2 \\ \\ R_3-\frac{7}{2}R_2 \end{array}$$

$$\begin{array}{cccc} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & -\frac{3}{4} & \frac{33}{148} & \frac{33}{148} \\ \hline 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \end{array} \qquad \begin{array}{cccc} 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \\ 0 & -\frac{7}{2} & \frac{77}{72} & \frac{77}{72} \\ \hline 0 & 0 & \frac{401}{72} & \frac{401}{72} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & \frac{401}{72} & \frac{401}{72} \end{array}\right] \quad \tfrac{72}{401}R_3$$

$$0 \quad 0 \quad 1 \quad 1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_1+\frac{38}{37}R_3 \\ R_2+\frac{11}{37}R_3 \\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 0 & \frac{38}{37} & \frac{38}{37} \\ \hline 1 & 0 & 0 & -6 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & \frac{11}{37} & \frac{11}{37} \\ \hline 0 & 1 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Solution: $(-6,\ 0,\ 1)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} x+2y-3z=-15 \\ 2x-3y+4z=18 \\ -3x+y+z=1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 2 & -3 & 4 & 18 \\ -3 & 1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 2 & -3 & 4 & 18 \\ -3 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3+3R_1 \end{array}$$

$$\begin{array}{cccc} -2 & -4 & 6 & 30 \\ 2 & -3 & 4 & 18 \\ \hline 0 & -7 & 10 & 48 \end{array} \qquad \begin{array}{cccc} 3 & 6 & -9 & -45 \\ -3 & 1 & 1 & 1 \\ \hline 0 & 7 & -8 & -44 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 0 & -7 & 10 & 48 \\ 0 & 7 & -8 & -44 \end{array}\right] -\frac{1}{7}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 7 & -8 & -44 \end{array}\right] \begin{array}{l} R_1-2R_2 \\ \\ R_3-7R_2 \end{array}$$

$$\begin{array}{cccc} 1 & 2 & -3 & -15 \\ 0 & -2 & \frac{20}{7} & \frac{96}{7} \\ \hline 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \end{array} \qquad \begin{array}{cccc} 0 & -7 & 10 & 48 \\ 0 & 7 & -8 & -44 \\ \hline 0 & 0 & 2 & 4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & 2 & 4 \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{2}R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1+\frac{1}{7}R_3 \\ R_2+\frac{10}{7}R_3 \\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 0 & \frac{1}{7} & \frac{2}{7} \\ \hline 1 & 0 & 0 & -1 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & \frac{10}{7} & \frac{20}{7} \\ \hline 0 & 1 & 0 & -4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solution: $(-1,\ -4,\ 2)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} x+2y+3z=10 \\ 4x+5y+6z=11 \\ 7x+8y+9z=12 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 4 & 5 & 6 & 11 \\ 7 & 8 & 9 & 12 \end{array}\right] \begin{array}{l} \\ R_2 - 4R_1 \\ R_3 - 7R_1 \end{array}$$

$$\begin{array}{rrrr} -4 & -8 & -12 & -40 \\ 4 & 5 & 6 & 11 \\ \hline 0 & -3 & -6 & -29 \end{array} \qquad \begin{array}{rrrr} -7 & -14 & -21 & -70 \\ 7 & 8 & 9 & 12 \\ \hline 0 & -6 & -12 & -58 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & -3 & -6 & -29 \\ 0 & -6 & -12 & -58 \end{array}\right] \tfrac{1}{3}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & \frac{29}{3} \\ 0 & -6 & -12 & -58 \end{array}\right] R_3 + 6R_2$$

$$\begin{array}{rrrr} 0 & -6 & -12 & -58 \\ 0 & 6 & 12 & 58 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & \frac{29}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

let z be the variable

From Row 1 $\Rightarrow y+2z=\frac{29}{3}$

$$\underline{y=\frac{29}{3}-2z}$$

From Row 1 $\Rightarrow x+2y+3z=10$

$$x=10-2y-3z$$

$$x=10-2\left(\frac{29}{3}-2z\right)-3z$$

$$x=10-\frac{58}{3}+4z-3z$$

$$\underline{x=z-\frac{28}{3}}$$

Solution: $\underline{\left(z-\frac{28}{3},\ \frac{29}{3}-2z,\ z\right)}$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} 2x+y+2z=4 \\ 2x+2y \quad\quad = 5 \\ 2x-y+6z=2 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 4 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array}\right] \begin{array}{l} \frac{1}{2}R_1 \\ \\ \\ \end{array} \qquad \begin{array}{cccc} 1 & \frac{1}{2} & 1 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-2R_1 \end{array} \qquad \begin{array}{cccc} 2 & 2 & 0 & 5 \\ -2 & -1 & -2 & -4 \\ \hline 0 & 1 & -2 & 1 \end{array} \qquad \begin{array}{cccc} 2 & -1 & 6 & 2 \\ -2 & -1 & -2 & -4 \\ \hline 0 & -2 & 4 & -2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 4 & -2 \end{array}\right] \begin{array}{l} R_1-\frac{1}{2}R_2 \\ \\ R_3+2R_2 \end{array} \qquad \begin{array}{cccc} 1 & \frac{1}{2} & 1 & 2 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ \hline 1 & 0 & 2 & \frac{3}{2} \end{array} \qquad \begin{array}{cccc} 0 & -2 & 4 & -2 \\ 0 & 2 & -4 & 2 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & \frac{3}{2} \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From Row 3: $0 = 0$ is a true statement. Let z be the variable.

From Row 2: $y-2z=1$

$\underline{y=1+2z}$

From Row 1: $x+2z=\frac{3}{2}$

$\underline{x=-2z+\frac{3}{2}}$

∴ ***Solution***: $\underline{\left(-2z+\frac{3}{2},\ 2z+1,\ z\right)}$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array} \qquad \begin{array}{cccc} -1 & -2 & 3 & 1 \\ 1 & 1 & 2 & 8 \\ \hline 0 & -1 & 5 & 9 \end{array} \qquad \begin{array}{cccc} 3 & -7 & 4 & 10 \\ -3 & -3 & -6 & -24 \\ \hline 0 & -10 & -2 & -14 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right] -R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 10R_2 \end{array} \qquad \begin{array}{cccc} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ \hline 1 & 0 & 7 & 17 \end{array} \qquad \begin{array}{cccc} 0 & -10 & -2 & -14 \\ 0 & 10 & -50 & -90 \\ \hline 0 & 0 & -52 & -104 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right] -\frac{1}{52}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 - 7R_3 \\ R_2 + 5R_3 \\ \\ \end{array} \qquad \begin{array}{cccc} 1 & 0 & 7 & 17 \\ 0 & 0 & -7 & -14 \\ \hline 1 & 0 & 0 & 3 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -5 & -9 \\ 0 & 0 & 5 & 10 \\ \hline 0 & 1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

∴ ***Solution***: $(3,\ 1,\ 2)$

Exercise

Use augmented elimination to solve linear system

$$\begin{array}{l} 2x-5y+3z=1 \\ x-2y-2z=8 \end{array}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 2 & -5 & 3 & 1 \end{array}\right] \quad R_2-2R_1$$

$$\begin{array}{cccc} 2 & -5 & 3 & 1 \\ -2 & 4 & 4 & -16 \\ \hline 0 & -1 & 7 & -15 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 0 & -1 & 7 & -15 \end{array}\right] \quad -R_2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 0 & 1 & -7 & 15 \end{array}\right] \quad R_1+2R_2$$

$$\begin{array}{cccc} 1 & -2 & -2 & 8 \\ 0 & 2 & -14 & 30 \\ \hline 1 & 0 & -16 & 38 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -16 & 38 \\ 0 & 1 & -7 & 15 \end{array}\right] \begin{array}{l} \rightarrow \quad x-16z=38 \\ \rightarrow \quad y-7z=15 \end{array}$$

$$\begin{cases} x=16z+38 \\ y=7z+15 \end{cases}$$

$\therefore$ ***Solution***: $\underline{(16z+38,\ 7z+15,\ z)}$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases} x+y+z=2 \\ 2x+y-z=5 \\ x-y+z=-2 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 5 \\ 1 & -1 & 1 & -2 \end{array}\right] \quad \begin{array}{l} R_2-2R_1 \\ R_3-R_1 \end{array}$$

$$\begin{array}{cccc} 2 & 1 & -1 & 5 \\ -2 & -2 & -2 & -4 \\ \hline 0 & -1 & -3 & 1 \end{array} \qquad \begin{array}{cccc} 1 & -1 & 1 & -2 \\ -1 & -1 & -1 & -2 \\ \hline 0 & -2 & 0 & -4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & 1 \\ 0 & -2 & 0 & -4 \end{array}\right] \quad \begin{array}{l} (2) \\ (1) \\ -2y=-4 \end{array}$$

$\underline{y=2}$

$(1) \rightarrow \quad -y-3z=1$

$3z=-1-2$

$\underline{z=-1}$

$$(2) \rightarrow \quad x + y + z = 2$$

$$x = 2 - 2 + 1$$

$$\underline{= 1}$$

$\therefore$ ***Solution***: $\underline{(1,\ 2,\ -1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ -1 & -1 & 1 & 1 \\ 3 & -1 & 1 & 9 \end{array}\right] \begin{array}{l} \\ 2R_2 + R_1 \\ 2R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} -2 & -2 & 2 & 2 \\ 2 & 1 & 1 & 9 \\ \hline 0 & -1 & 3 & 11 \end{array} \qquad \begin{array}{cccc} 6 & -2 & 2 & 18 \\ -6 & -3 & -3 & -27 \\ \hline 0 & -5 & -1 & -9 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ 0 & -1 & 3 & 11 \\ 0 & -5 & -1 & -9 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 5R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -5 & -1 & -9 \\ 0 & 5 & -15 & -55 \\ \hline 0 & 0 & -16 & -64 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ 0 & -1 & 3 & 11 \\ 0 & 0 & -16 & -64 \end{array}\right] \begin{array}{c} (2) \\ (1) \\ -16z = -64 \end{array}$$

$$\underline{z = 4}$$

$$(1) \rightarrow \quad -y + 3z = 11$$

$$y = 12 - 11$$

$$\underline{= 1}$$

$$(2) \rightarrow \quad 2x + y + z = 9$$

$$2x = 9 - 1 - 4$$

$$\underline{x = 2}$$

$\therefore$ ***Solution***: $\underline{(2,\ 1,\ 4)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ -3 & 6 & 2 & 11 \end{array}\right] \quad R_3 + 3R_1$$

$$\begin{array}{cccc} -3 & 6 & 2 & 11 \\ 3 & 15 & -3 & -12 \\ \hline 0 & 21 & -1 & -1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ 0 & 21 & -1 & -1 \end{array}\right] \quad R_3 - 7R_2$$

$$\begin{array}{cccc} 0 & 21 & -1 & -1 \\ 0 & -21 & 7 & 7 \\ \hline 0 & 0 & 6 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ 0 & 0 & 6 & 6 \end{array}\right] \quad \begin{array}{ll} \to x + 5y - z = -4 & (2) \\ \to 3y - z = -1 & (1) \\ \to 6z = 6 & \end{array}$$

$\underline{z = 1}$

$(1) \to \quad 3y = -1 + 1$

$\qquad \underline{y = 0}$

$(2) \to \quad x = -4 + 1$

$\qquad \underline{x = -3}$

$\therefore$ ***Solution***: $\underline{(-3,\ 0,\ 1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 2 & -3 & 2 & 10 \\ 3 & -1 & 1 & 9 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -3 & 2 & 10 \\ -2 & -6 & -8 & -28 \\ \hline 0 & -9 & -6 & -18 \end{array} \qquad \begin{array}{cccc} 3 & -1 & 1 & 9 \\ -3 & -9 & -12 & -42 \\ \hline 0 & -10 & -11 & -33 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & -9 & -6 & -18 \\ 0 & -10 & -11 & -33 \end{array}\right] \quad 9R_3 - 10R_2$$

$$\begin{array}{cccc} 0 & -90 & -99 & -297 \\ 0 & 90 & 60 & 180 \\ \hline 0 & 0 & -39 & -117 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & -9 & -6 & -18 \\ 0 & 0 & -39 & -117 \end{array}\right] \quad \begin{array}{ll} x + 3y + 4z = 14 & (3) \\ -9y - 6z = -18 & (2) \\ -39z = -117 & (1) \end{array}$$

$(1) \to \quad z = \dfrac{117}{39}$

$\qquad \underline{= 3}$

$(2) \to \quad 9y = 18 - 6(3)$

$9y = 0$

$\underline{y = 0}|$

$(3) \to \quad x = 14 - 12$

$\underline{x = 2}|$

$\therefore$ ***Solution***: $\underline{(2,\ 0,\ 3)}\ |$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 3 & 2 & 1 & 8 \\ 2 & -3 & 2 & -16 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 2 & 1 & 8 \\ -3 & -12 & 3 & -60 \\ \hline 0 & -10 & 4 & -52 \end{array} \qquad \begin{array}{cccc} 2 & -3 & 2 & -16 \\ -2 & -8 & 2 & -40 \\ \hline 0 & -11 & 4 & -56 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 0 & -10 & 4 & -52 \\ 0 & -11 & 4 & -56 \end{array}\right] \begin{array}{l} \\ \\ 10R_3 - 11R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -110 & 40 & -560 \\ 0 & 110 & -44 & 572 \\ \hline 0 & 0 & -4 & 12 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 0 & -10 & 4 & -52 \\ 0 & 0 & -4 & 12 \end{array}\right] \quad \begin{array}{rl} x + 4y - z = 20 & (3) \\ -10y + 4z = -52 & (2) \\ -4z = 12 & (1) \end{array}$$

$(1) \to \quad \underline{z = -3}|$

$(2) \to \quad -10y = -52 + 12$

$-10y = -40$

$\underline{y = 4}|$

$(3) \to \quad x = 20 - 16 - 3$

$\underline{x = 1}|$

$\therefore$ ***Solution***: $\underline{(1,\ 4,\ -3)}\ |$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 17 \\ 0 & 2 & -1 & 7 \\ 2 & -3 & 2 & -1 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_1 \end{array} \qquad \begin{array}{cccc} 2 & -3 & 2 & -1 \\ -2 & -4 & -2 & -34 \\ \hline 0 & -7 & 0 & -35 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 17 \\ 0 & 2 & -1 & 7 \\ 0 & -7 & 0 & -35 \end{array}\right] \begin{array}{rl} x + 2y + z = 17 & (3) \\ 2y - z = 7 & (2) \\ -7y = -35 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{y = 5|}$

$(2) \rightarrow \quad z = 10 - 7$

$\qquad\qquad \underline{= 3|}$

$(3) \rightarrow \quad x = 17 - 10 - 3$

$\qquad\qquad \underline{= 4|}$

$\therefore$ ***Solution***: $\underline{(4,\ 5,\ 3)\ |}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ -4 & 5 & 3 & 7 \\ -6 & 3 & 5 & -4 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \qquad \begin{array}{cccc} -4 & 5 & 3 & 7 \\ 4 & -12 & -14 & -6 \\ \hline 0 & -7 & -11 & 1 \end{array} \qquad \begin{array}{cccc} -6 & 3 & 5 & -4 \\ 6 & -18 & -21 & -9 \\ \hline 0 & -15 & -16 & -13 \end{array}$$

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ 0 & -7 & -11 & 1 \\ 0 & -15 & -16 & -13 \end{array}\right] \begin{array}{l} \\ \\ 7R_3 - 15R_1 \end{array} \qquad \begin{array}{cccc} 0 & -105 & -112 & -91 \\ 0 & 105 & 165 & -15 \\ \hline 0 & 0 & 53 & -106 \end{array}$$

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ 0 & -7 & -11 & 1 \\ 0 & 0 & 53 & -106 \end{array}\right] \begin{array}{rl} -2x + 6y + 7z = 3 & (3) \\ -7y - 11z = 1 & (2) \\ 53z = -106 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z = -2}|$

$(2) \rightarrow \quad -7y = 1 - 22$

$-7y = -21$

$\underline{y = 3}|$

$(3) \rightarrow \quad -2x = 3 - 18 + 14$

$-2x = -1$

$\underline{x = \frac{1}{2}}|$

$\therefore$ ***Solution***: $\underline{\left(\frac{1}{2},\ 3,\ -2\right)}|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 3 & -3 & 4 & 5 \\ 4 & -2 & 3 & 4 \end{array}\right] \quad \begin{array}{c} \\ 2R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\begin{array}{cccc} 6 & -6 & 8 & 10 \\ -6 & 3 & -3 & -3 \\ \hline 0 & -3 & 5 & 7 \end{array} \qquad \begin{array}{cccc} 4 & -2 & 3 & 4 \\ -4 & 2 & -2 & -2 \\ \hline 0 & 0 & 1 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 0 & -3 & 5 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \begin{array}{ll} 2x - y + z = 1 & (2) \\ -3y + 5z = 7 & (1) \\ \underline{z = 2}| & \end{array}$$

$(1) \rightarrow \quad -3y = 7 - 10$

$-3y = -3$

$\underline{y = 1}|$

$(3) \rightarrow \quad 2x = 1 + 1 - 2$

$\underline{x = 0}|$

$\therefore$ ***Solution***: $\underline{(0,\ 1,\ 2)}|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x-4y+4z=7 \\ x-y-2z=2 \\ 2x-3y+6z=5 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 3 & -4 & 4 & 7 \\ 2 & -3 & 6 & 5 \end{array}\right] \begin{array}{l} \\ R_2-3R_1 \\ R_3-2R_1 \end{array}$$

$$\begin{array}{cccc} 3 & -4 & 4 & 7 \\ -3 & 3 & 6 & -6 \\ \hline 0 & -1 & 10 & 1 \end{array} \qquad \begin{array}{cccc} 2 & -3 & 6 & 5 \\ -2 & 2 & 4 & -4 \\ \hline 0 & -1 & 10 & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & -1 & 10 & 1 \\ 0 & -1 & 10 & 1 \end{array}\right] \begin{array}{l} \to x-y-2z=2 \quad (2) \\ \to -y+10z=1 \quad (1) \\ R_3=R_2 \end{array}$$

$(1)\to \quad y=10z-1$

$(2)\to \quad x=2+10z-1+2z$

$\qquad =12z+1$

$\therefore$ ***Solution***: $\boxed{(12z+1,\ 10z-1,\ z)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x-2y-z=2 \\ 2x-y+z=4 \\ -x+y+z=4 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 2 & -1 & 1 & 4 \\ -1 & 1 & 1 & 4 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3+R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -1 & 1 & 4 \\ -2 & 4 & 2 & -4 \\ \hline 0 & 3 & 3 & 0 \end{array} \qquad \begin{array}{cccc} -1 & 1 & 1 & 4 \\ 1 & -2 & -1 & 2 \\ \hline 0 & -1 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 3 & 3 & 0 \\ 0 & -1 & 0 & 6 \end{array}\right] \begin{array}{ll} x-2y-z=2 & (3) \\ 3y+3z=0 & (2) \\ -y=6 & (1) \end{array}$$

$(1)\to \quad y=-6$

$(2)\to \quad z=-y$

$\qquad =6$

$(3)\to \quad x=2-12+6$

$\qquad =-4$

$\therefore$ ***Solution***: $\boxed{(-4,\ -6,\ 6)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x+y+z=3 \\ -y+2z=1 \\ -x+z=0 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ -1 & 0 & 1 & 0 \end{array}\right] \quad R_3 + R_1 \qquad \begin{array}{cccc} -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 3 \\ \hline 0 & 1 & 2 & 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 1 & 2 & 3 \end{array}\right] \quad R_3 + R_2 \qquad \begin{array}{cccc} 0 & 1 & 2 & 3 \\ 0 & -1 & 2 & 1 \\ \hline 0 & 0 & 4 & 4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 4 & 4 \end{array}\right] \quad \begin{array}{rl} x+y+z=3 & (3) \\ -y+2z=1 & (2) \\ 4z=4 & (1) \end{array}$$

$(1) \to \quad \underline{z=1}$

$(2) \to \quad -y = 1-2$

$\qquad \underline{y=1}$

$(3) \to \quad x = 3-1-1$

$\qquad \underline{=1}$

$\therefore$ ***Solution***: $\underline{(1,\ 1,\ 1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x+y+3z=14 \\ 7x+5y+8z=37 \\ x+3y+2z=9 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 3 & 1 & 3 & 14 \\ 7 & 5 & 8 & 37 \end{array}\right] \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 - 7R_1 \end{array} \qquad \begin{array}{cccc} 3 & 1 & 3 & 14 \\ -3 & -9 & -6 & -27 \\ \hline 0 & -8 & -3 & -13 \end{array} \qquad \begin{array}{cccc} 7 & 5 & 8 & 37 \\ -7 & -21 & -14 & -63 \\ \hline 0 & -16 & -6 & -26 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 0 & -8 & -3 & -13 \\ 0 & -16 & -6 & -26 \end{array}\right] \quad R_3 - 2R_2 \qquad \begin{array}{cccc} 0 & -16 & -6 & -26 \\ 0 & 16 & 6 & 26 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 0 & -8 & -3 & -13 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{ll} x+3y+2z=9 & (2) \\ -8y-3z=-13 & (1) \end{array}$$

$(1) \to \quad -8y = 3z - 13$

$$\underline{y = -\frac{3}{8}z + \frac{13}{8}}$$

$(3) \to \quad x = 9 - 3\left(\frac{13}{8} - \frac{3}{8}z\right) - 2z$

$$= 9 - \frac{39}{8} + \frac{9}{8}z - 2z$$

$$\underline{= \frac{33}{8} - \frac{7}{8}z}$$

$\therefore$ ***Solution***: $\underline{\left(\frac{33}{8} - \frac{7}{8}z,\ \ \frac{13}{8} - \frac{3}{8}z,\ \ z\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 4x - 2y + z = 7 \\ x + y + z = -2 \\ 4x + 2y + z = 3 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 4 & -2 & 1 & 7 \\ 4 & 2 & 1 & 3 \end{array}\right] \quad \begin{array}{l} \\ R_2 - 4R_1 \\ R_3 - 4R_1 \end{array} \qquad \begin{array}{cccc} 4 & -2 & 1 & 7 \\ -4 & -4 & -4 & 8 \\ \hline 0 & -6 & -3 & 15 \end{array} \qquad \begin{array}{cccc} 4 & 2 & 1 & 3 \\ -4 & -4 & -4 & 8 \\ \hline 0 & -2 & -3 & 11 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -6 & -3 & 15 \\ 0 & -2 & -3 & 11 \end{array}\right] \quad \begin{array}{l} \\ \\ -3R_3 + R_2 \end{array} \qquad \begin{array}{cccc} 0 & 6 & 9 & -33 \\ 0 & -6 & -3 & 15 \\ \hline 0 & 0 & 6 & -18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -6 & -3 & 15 \\ 0 & 0 & 6 & -18 \end{array}\right] \quad \begin{array}{rl} x+y+z=-2 & (3) \\ -6y-3z=15 & (2) \\ 6z=-18 & (1) \end{array}$$

$(1) \to \quad \underline{z = -3}$

$(2) \to \quad -6y = 15 - 9$

$$\underline{y = -1}$$

$(3) \to \quad x = -2 + 1 + 3$

$$\underline{= 2}$$

$\therefore$ ***Solution***: $\underline{(2,\ -1,\ -3)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x-2y+z=-4 \\ 6x+4y-3z=-24 \\ x-2y+2z=1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 2 & -2 & 1 & -4 \\ 6 & 4 & -3 & -24 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-6R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -2 & 1 & -4 \\ -2 & 4 & -4 & -2 \\ \hline 0 & 2 & -3 & -6 \end{array} \qquad \begin{array}{cccc} 6 & 4 & -3 & -24 \\ -6 & 12 & -12 & -6 \\ \hline 0 & 16 & -15 & -30 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 2 & -3 & -6 \\ 0 & 16 & -15 & -30 \end{array}\right] \begin{array}{l} \\ \\ R_3-8R_2 \end{array}$$

$$\begin{array}{cccc} 0 & 16 & -15 & -30 \\ 0 & -16 & 24 & 48 \\ \hline 0 & 0 & 9 & 18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 2 & -3 & -6 \\ 0 & 0 & 9 & 18 \end{array}\right] \begin{array}{rl} x-2y+2z=1 & (3) \\ 2y-3z=-6 & (2) \\ 9z=18 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z=2}$

$(2) \rightarrow \quad 2y=-6+6$

$\underline{y=0}$

$(3) \rightarrow \quad x=1-4$

$\underline{=-3}$

$\therefore$ ***Solution***: $\underline{(-3,\ 0,\ 2)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 9x+3y+z=4 \\ 16x+4y+z=2 \\ 25x+5y+z=2 \end{cases}$

Solution

$$\begin{cases} z+9x+3y=4 \\ z+16x+4y=2 \\ z+25x+5y=2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 1 & 16 & 4 & 2 \\ 1 & 25 & 5 & 2 \end{array}\right] \begin{array}{l} \\ R_2-R_1 \\ R_3-R_1 \end{array}$$

$$\begin{array}{cccc} 1 & 16 & 4 & 2 \\ -1 & -9 & -3 & -4 \\ \hline 0 & 7 & 1 & -2 \end{array} \qquad \begin{array}{cccc} 1 & 25 & 5 & 2 \\ -1 & -9 & -3 & -4 \\ \hline 0 & 16 & 2 & -2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 0 & 7 & 1 & -2 \\ 0 & 16 & 2 & -2 \end{array}\right] \quad 7R_3 - 16R_3$$

$$\begin{array}{cccc} 0 & 112 & 14 & -14 \\ 0 & -112 & -16 & 32 \\ \hline 0 & 0 & -2 & 18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 0 & 7 & 1 & -2 \\ 0 & 0 & -2 & 18 \end{array}\right] \quad \begin{aligned} z + 9x + 3y &= 4 \quad (3) \\ 7x + y &= -2 \quad (2) \\ -2y &= 18 \quad (1) \end{aligned}$$

$(1) \rightarrow \quad \underline{y = -9}$

$(2) \rightarrow \quad 7x = -2 + 9$

$\qquad \underline{= 1}$

$(3) \rightarrow \quad z = 4 - 9 + 27$

$\qquad \underline{= 22}$

$\therefore$ ***Solution***: $\underline{(1, \; -9, \; 22)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x - y + 2z = -8 \\ x + 2y - 3z = 9 \\ 3x - y - 4z = 3 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 2 & -8 \\ 3 & -1 & -4 & 3 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -1 & 2 & -8 \\ -2 & -4 & 6 & -18 \\ \hline 0 & -5 & 8 & -26 \end{array} \qquad \begin{array}{cccc} 3 & -1 & -4 & 3 \\ -3 & -6 & 9 & -27 \\ \hline 0 & -7 & 5 & -24 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 0 & -7 & 5 & -24 \end{array}\right] \quad 5R_3 - 7R_2$$

$$\begin{array}{cccc} 0 & -35 & 25 & -120 \\ 0 & 35 & -56 & 182 \\ \hline 0 & 0 & -31 & 62 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 0 & 0 & -31 & 62 \end{array}\right] \quad \begin{aligned} x + 2y - 3z &= 9 \quad (3) \\ -5y + 8z &= -26 \quad (2) \\ -31z &= 62 \quad (1) \end{aligned}$$

$(1) \rightarrow \quad \underline{z = -2}$

$(2) \rightarrow \quad -5y = -26 + 16$

$\qquad -5y = 10$

$\qquad \underline{y = 2}$

$(3) \rightarrow \quad x = 9 - 4 - 6$

$\underline{=-1}$

$\therefore$ ***Solution***: $\underline{(-1,\ 2,\ -2)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x \qquad -3z=-5 \\ 2x-y+2z=16 \\ 7x-3y-5z=19 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 2 & -1 & 2 & 16 \\ 7 & -3 & -5 & 19 \end{array}\right] \quad \begin{array}{l} \\ R_2-2R_1 \\ R_3-7R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -1 & 2 & 16 \\ -2 & 0 & 6 & 10 \\ \hline 0 & -1 & 8 & 26 \end{array} \qquad \begin{array}{cccc} 7 & -3 & -5 & 19 \\ -7 & 0 & 21 & 35 \\ \hline 0 & -3 & 16 & 54 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 0 & -1 & 8 & 26 \\ 0 & -3 & 16 & 54 \end{array}\right] \quad \begin{array}{l} \\ \\ R_3-3R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -3 & 16 & 54 \\ 0 & 3 & -24 & -78 \\ \hline 0 & 0 & -8 & -24 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 0 & -1 & 8 & 26 \\ 0 & 0 & -8 & -24 \end{array}\right] \quad \begin{array}{rl} x-3z=-5 & (3) \\ -y+8z=26 & (2) \\ -8z=-24 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z=3}$

$(2) \rightarrow \quad -y=26-24$

$\qquad \underline{y=-2}$

$(3) \rightarrow \quad x=-5+9$

$\qquad \underline{=4}$

$\therefore$ ***Solution***: $\underline{(4,\ -2,\ 3)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x+2y-z=5 \\ 2x-y+3z=0 \\ \qquad 2y+z=1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 2 & -1 & 3 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right] \quad \begin{array}{l} \\ R_2-2R_1 \\ \\ \end{array}$$

$$\begin{array}{cccc} 2 & -1 & 3 & 0 \\ -2 & -4 & 2 & -10 \\ \hline 0 & -5 & 5 & -10 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 5 & -10 \\ 0 & 2 & 1 & 1 \end{array}\right] \begin{array}{l} \\ \\ 5R_3+2R_2 \end{array}$$

$$\begin{array}{cccc} 0 & 10 & 5 & 5 \\ 0 & -10 & 10 & -20 \\ \hline 0 & 0 & 15 & -15 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 5 & -10 \\ 0 & 0 & 15 & -15 \end{array}\right] \begin{array}{rl} x+2y-z=5 & (3) \\ -5y+5z=-10 & (2) \\ 15z=-15 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z=-1}$

$(2) \rightarrow \quad -5y=-10+5$

$\underline{y=1}$

$(3) \rightarrow \quad x=5-2-1$

$\underline{=2}$

∴ ***Solution***: $\underline{(2,\ 1,\ -1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x+y+z=6 \\ 3x+4y-7z=1 \\ 2x-y+3z=5 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & 4 & -7 & 1 \\ 2 & -1 & 3 & 5 \end{array}\right] \begin{array}{l} \\ R_2-3R_1 \\ R_3-2R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 4 & -7 & 1 \\ -3 & -3 & -3 & -18 \\ \hline 0 & 1 & -10 & -17 \end{array} \qquad \begin{array}{cccc} 2 & -1 & 3 & 5 \\ -2 & -2 & -2 & -12 \\ \hline 0 & -3 & 1 & -7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -10 & -17 \\ 0 & -3 & 1 & -7 \end{array}\right] \begin{array}{l} \\ \\ R_3+3R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -3 & 1 & -7 \\ 0 & 3 & -30 & -51 \\ \hline 0 & 0 & -29 & -58 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -10 & -17 \\ 0 & 0 & -29 & -58 \end{array}\right] \begin{array}{rl} x+y+z=6 & (3) \\ y-10z=-17 & (2) \\ -29z=-58 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z=2}$

$(2) \rightarrow \quad y=-17+20$

$\underline{=3}$

$(3) \rightarrow \quad x=6-3-2$

$\underline{=1}$

∴ ***Solution***: $\underline{(1,\ 3,\ 2)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x+2y+3z=3 \\ 4x-5y+7z=1 \\ 2x+3y-2z=6 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 4 & -5 & 7 & 1 \\ 2 & 3 & -2 & 6 \end{array}\right] \begin{array}{l} \\ 3R_2-4R_1 \\ 3R_3-2R_1 \end{array}$$

$$\begin{array}{cccc} 12 & -15 & 21 & 3 \\ -12 & -8 & -12 & -12 \\ \hline 0 & -23 & 9 & -9 \end{array} \qquad \begin{array}{cccc} 6 & 9 & -6 & 18 \\ -6 & -4 & -6 & -6 \\ \hline 0 & 5 & -12 & 12 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 0 & -23 & 9 & -9 \\ 0 & 5 & -12 & 12 \end{array}\right] \begin{array}{l} \\ \\ 23R_3+5R_2 \end{array}$$

$$\begin{array}{cccc} 0 & 115 & -276 & 276 \\ 0 & -115 & 45 & -45 \\ \hline 0 & 0 & -231 & 231 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 0 & -23 & 9 & -9 \\ 0 & 0 & -231 & 231 \end{array}\right] \quad \begin{array}{rl} 3x+2y+3z=3 & (3) \\ -23y+9z=-9 & (2) \\ -231z=231 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z=-1}$

$(2) \rightarrow \quad -23y=-9+9$

$\qquad \underline{y=0}$

$(3) \rightarrow \quad 3x=3+3$

$\qquad \underline{x=2}$

$\therefore$ ***Solution***: $\underline{(2,\ 0,\ -1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x-3y+z=2 \\ 4x-12y+4z=8 \\ -2x+6y-2z=-4 \end{cases}$

Solution

$$\begin{cases} \quad x-3y+z=2 \\ \frac{1}{4}\times \quad 4x-12y+4z=8 \\ -\frac{1}{2}\times \quad -2x+6y-2z=-4 \end{cases}$$

$$\begin{cases} x-3y+z=2 \\ x-3y+z=2 \\ x-3y+z=2 \end{cases}$$

Since all three equations are the same.

$\therefore$ ***Solution***: is the plane $\underline{x-3y+z=2}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x-2y+z=-1 \\ x+2y-z=2 \\ 6x+4y+3z=5 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & -2 & 1 & -1 \\ 6 & 4 & 3 & 5 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-6R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -2 & 1 & -1 \\ -2 & -4 & 2 & -4 \\ \hline 0 & -6 & 3 & -5 \end{array} \qquad \begin{array}{cccc} 6 & 4 & 3 & 5 \\ -6 & -12 & 6 & -12 \\ \hline 0 & -8 & 9 & -7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -6 & 3 & -5 \\ 0 & -8 & 9 & -7 \end{array}\right] \begin{array}{l} \\ \\ 3R_3-4R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -24 & 27 & -21 \\ 0 & 24 & -12 & 20 \\ \hline 0 & 0 & 15 & -1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -6 & 3 & -5 \\ 0 & 0 & 15 & -1 \end{array}\right] \begin{array}{rl} x+2y-z=2 & (3) \\ -6y+3z=-5 & (2) \\ 15z=-1 & (1) \end{array}$$

$(1) \to \quad \underline{z=-\frac{1}{15}}$

$(2) \to \quad -6y=-5+\frac{1}{5}$

$-6y=-\frac{24}{5}$

$\underline{y=\frac{4}{5}}$

$(3) \to \quad x=2-\frac{8}{5}-\frac{1}{15}$

$\underline{=\frac{1}{3}}$

$\therefore$ ***Solution***: $\underline{\left(\frac{1}{3},\ \frac{4}{5},\ -\frac{1}{15}\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 \\ x_2 - 3x_3 - x_4 = 0 \\ 3x_1 \quad\ + 2x_3 - x_4 = 6 \\ -4x_1 + x_2 + 4x_3 + 2x_4 = -3 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 3 & 0 & 2 & -1 & 6 \\ -4 & 1 & 4 & 2 & -3 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 3R_1 \\ R_4 + 4R_1 \end{array}$$

$$\begin{array}{rrrrr} 3 & 0 & 2 & -1 & 6 \\ -3 & 15 & -6 & 6 & -12 \\ \hline 0 & 15 & -4 & 5 & -6 \end{array} \qquad \begin{array}{rrrrr} -4 & 1 & 4 & 2 & -3 \\ 4 & -20 & 8 & -8 & 16 \\ \hline 0 & -19 & 12 & -6 & 13 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 15 & -4 & 5 & -6 \\ 0 & -19 & 12 & -6 & 13 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 15R_2 \\ R_4 + 19R_2 \end{array}$$

$$\begin{array}{rrrrr} 0 & 15 & -4 & 5 & -6 \\ 0 & -15 & 45 & 15 & 0 \\ \hline 0 & 0 & 41 & 20 & -6 \end{array} \qquad \begin{array}{rrrrr} 0 & -19 & 12 & -6 & 13 \\ 0 & 19 & -57 & -19 & 0 \\ \hline 0 & 0 & -45 & -25 & 13 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 41 & 20 & -6 \\ 0 & 0 & -45 & -25 & 13 \end{array}\right] \begin{array}{l} \\ \\ \\ 41R_4 + 45R_2 \end{array}$$

$$\begin{array}{rrrrr} 0 & 0 & -1845 & -1025 & 533 \\ 0 & 0 & 1845 & 900 & -270 \\ \hline 0 & 0 & 0 & -125 & 263 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 41 & 20 & -6 \\ 0 & 0 & 0 & -125 & 263 \end{array}\right] \quad \begin{array}{rl} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 & (4) \\ x_2 - 3x_3 - x_4 = 0 & (3) \\ 41x_3 + 20x_4 = -6 & (2) \\ -125x_4 = 263 & (1) \end{array}$$

$(1) \rightarrow \quad \boxed{x_4 = -\frac{263}{125}}$

$(2) \rightarrow \quad 41x_3 = -6 + \frac{1,052}{25}$

$= \frac{902}{25}$

$\boxed{x_3 = \frac{22}{25}}$

$(3) \rightarrow \quad x_2 = \frac{66}{25} - \frac{263}{125}$

$\boxed{= \frac{67}{125}}$

$(4) \rightarrow \quad x_1 = 4 + \frac{67}{25} - \frac{44}{25} - \frac{526}{125}$

$$= 4 + \frac{23}{25} - \frac{526}{125}$$

$$= \frac{500 + 115 - 526}{125}$$

$$= \frac{89}{125}$$

$\therefore$ ***Solution***: $\left(\frac{89}{125}, \frac{67}{125}, \frac{22}{25}, -\frac{263}{125} \right)$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 5 \\ x_1 + 2x_2 - x_3 - 2x_4 = -1 \\ x_1 - 3x_2 - 3x_3 - x_4 = -1 \\ 2x_1 - x_2 + 2x_3 - x_4 = -2 \end{cases}$$

Solution

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 1 & 2 & -1 & -2 & -1 \\ 1 & -3 & -3 & -1 & -1 \\ 2 & -1 & 2 & -1 & -2 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - 2R_1 \end{array}$$

$$\begin{array}{ccccc} 1 & 2 & -1 & -2 & -1 \\ -1 & -1 & -1 & -1 & -5 \\ \hline 0 & 1 & -2 & -3 & -6 \end{array} \qquad \begin{array}{ccccc} 1 & -3 & -3 & -1 & -1 \\ -1 & -1 & -1 & -1 & -5 \\ \hline 0 & -4 & -4 & -2 & -6 \end{array} \qquad \begin{array}{ccccc} 2 & -1 & 2 & -1 & -2 \\ -2 & -2 & -2 & -2 & -10 \\ \hline 0 & -3 & 0 & -3 & -12 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & -4 & -4 & -2 & -6 \\ 0 & -3 & 0 & -3 & -12 \end{array}\right] \begin{array}{l} \\ \\ R_3 + 4R_2 \\ R_4 + 3R_2 \end{array}$$

$$\begin{array}{ccccc} 0 & -4 & -4 & -2 & -6 \\ 0 & 4 & -8 & -12 & -24 \\ \hline 0 & 0 & -12 & -14 & -30 \end{array}$$

$$\begin{array}{ccccc} 0 & -3 & 0 & -3 & -12 \\ 0 & 3 & -6 & -9 & -18 \\ \hline 0 & 0 & -6 & -12 & -30 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & 0 & -12 & -14 & -30 \\ 0 & 0 & -6 & -12 & -30 \end{array}\right] \begin{array}{l} \\ \\ \\ -2R_4 + R_3 \end{array}$$

$$\begin{array}{ccccc} 0 & 0 & 12 & 24 & 60 \\ 0 & 0 & -12 & -14 & -30 \\ \hline 0 & 0 & 0 & 10 & 30 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & 0 & -12 & -14 & -30 \\ 0 & 0 & 0 & 10 & 30 \end{array}\right] \quad \begin{array}{ll} x_1 + x_2 + x_3 + x_4 = 5 & (4) \\ x_2 - 2x_3 - 3x_4 = -6 & (3) \\ -12x_3 - 14x_4 = -30 & (2) \\ 10x_4 = 30 & (1) \end{array}$$

$(1) \to \quad \underline{x_4 = 3}$

$(2) \to \quad -12x_3 = -30 + 42$

$= 12$

$\underline{x_3 = -1}$

$(3) \to \quad x_2 = -6 - 2 + 9$

$\underline{= 1}$

$(4) \to \quad x_1 = 5 - 1 + 1 - 3$

$\underline{= 2}$

$\therefore$ ***Solution***: $\underline{(2,\ 1,\ -1,\ 3)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x + 8y - z + w = 0 \\ 4x + 16y - 3z - w = -10 \\ -2x + 4y - z + 3w = -6 \\ -6x + 2y + 5z + w = 3 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 4 & 16 & -3 & -1 & -10 \\ -2 & 4 & -1 & 3 & -6 \\ -6 & 2 & 5 & 1 & 3 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 + 3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - \frac{13}{6}R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \quad \textit{Interchange } R_2 \textit{ and } R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & \frac{19}{3} & -\frac{14}{3} & 16 \end{array}\right] \quad R_4 + \frac{19}{3} R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & 0 & -\frac{71}{3} & -\frac{142}{3} \end{array}\right] \quad \begin{array}{ll} 2x + 8y - z + w = 0 & (3) \\ 12y - 2z + 4w = -6 & (2) \\ -z - 3w = -10 & (1) \\ -\frac{71}{3} w = -\frac{142}{3} \quad \to \quad \underline{w = 2} & \end{array}$$

$(1) \to \quad z = 10 - 3w \underline{= 4}$

$(2) \to \quad 12y = 2z - 4w - 6$

$\underline{y = -\frac{1}{2}}$

$(3) \to \quad 2x = -8y + z - w$

$2x = 4 + 4 - 2$

$2x = 6$

$\underline{x = 3}$

$\therefore$ ***Solution***: $\underline{\left(3,\ -\frac{1}{2},\ 4,\ 2\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 + 2x_2 = 0 \\ x_2 + x_3 = 0 \end{cases}$

Solution

$$\begin{cases} x_1 = -2x_2 \\ x_3 = -x_2 \end{cases}$$

$2x_1 + x_2 + 3x_3 = 0$

$-4x_2 + x_2 - 3x_2 = 0 \quad \to \quad \underline{x_2 = 0}$

$\therefore$ ***Solution***: $\underline{(0,\ 0,\ 0)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x+2y+4z \quad\; =0 \\ \quad -y-3z+w=0 \\ 3x+y+z+2w=0 \\ x+3y-2z-2w=0 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & -1 & -3 & 1 & 0 \\ 3 & 1 & 1 & 2 & 0 \\ 1 & 3 & -2 & -2 & 0 \end{array}\right] \begin{array}{c} \\ -R_2 \\ 2R_3-3R_1 \\ 2R_4-R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & -4 & -10 & 4 & 0 \\ 0 & 4 & -8 & -4 & 0 \end{array}\right] \begin{array}{c} \\ \\ R_3+4R_2 \\ R_4-4R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & -20 & 0 & 0 \end{array}\right] \begin{array}{ll} 2x+2y-4z=0 & (1) \\ y+3z-w=0 & (2) \\ \rightarrow \underline{z=0} & \\ & \end{array}$$

$(2) \quad \rightarrow \quad \underline{y=w}$

$(1) \quad \rightarrow 2x=-2y \quad \underline{x=-w}$

$\therefore$ ***Solution***: $\underline{(-w,\ w,\ 0,\ w)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x \quad +z+w=5 \\ \quad y \quad -w=-1 \\ 3x \quad -z-w=0 \\ 4x+y+2z+w=9 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 3 & 0 & -1 & -1 & 0 \\ 4 & 1 & 2 & 1 & 9 \end{array}\right] \begin{array}{c} \\ \\ 2R_3-3R_1 \\ 2R_4-4R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -5 & -5 & -15 \\ 0 & 2 & 0 & -2 & -2 \end{array}\right] \quad R_4 - 2R_2$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -5 & -5 & -15 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{rr} 2x + z + w = 5 & (1) \\ y - w = -1 & (2) \\ -5z - 5w = -15 & (3) \end{array}$$

$(2) \rightarrow \qquad \underline{y = 1 + w}|$

$(3) \rightarrow \qquad \underline{z = 3 - w}|$

$(1) \rightarrow \quad 2x = 5 - (3 - w) - w \quad \Rightarrow \quad \underline{x = 1}|$

$\therefore$ ***Solution***: $(1,\ 1+w,\ 3-w,\ w)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 4y + z = 20 \\ 2x - 2y + z = 0 \\ x \quad + z = 5 \\ x + y - z = 10 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 10 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 1 & 5 \\ 0 & 4 & 1 & 20 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 10 \\ 0 & -4 & 3 & -20 \\ 0 & -1 & 2 & -5 \\ 0 & 4 & 1 & 20 \end{array}\right] \quad \begin{array}{l} 4R_3 - R_2 \\ R_4 + R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 10 \\ 0 & -4 & 3 & -20 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \quad \begin{array}{l} x + y = 10 \\ \rightarrow -4y = -20 \\ \rightarrow z = 0 \end{array}$$

$\therefore$ ***Solution***: $(5,\ 5,\ 0)$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x \qquad\qquad - 3w = -3 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ -1 & 3 & -2 & 1 \\ 3 & 4 & -7 & 10 \end{array}\right] \quad \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 5 & -1 & 9 \\ 0 & -2 & -10 & -14 \end{array}\right] \quad \begin{array}{l} \\ \\ 5R_3 + 2R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 5 & -1 & 9 \\ 0 & 0 & -52 & -52 \end{array}\right] \quad \begin{array}{rl} x + 2y + z = 8 & (3) \\ 5y - z = 9 & (2) \\ -52z = -52 & (1) \end{array}$$

$(1) \quad \Rightarrow \quad z = 1$

$(2) \quad \Rightarrow \quad 5y = 9 + 1 = 10 \to y = 2$

$(3) \quad \Rightarrow \quad x = 8 - 4 - 1 = 3$

$\therefore$ ***Solution***: $(3,\ 2,\ 1)$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} 2u - 3v + w - x + y = 0 \\ 4u - 6v + 2w - 3x - y = -5 \\ -2u + 3v - 2w + 2x - y = 3 \end{cases}$$

Solution

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 4 & -6 & 2 & -3 & -1 & -5 \\ -2 & 3 & -2 & 2 & -1 & 3 \end{array}\right] \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -3 & -5 \\ 0 & 0 & -1 & 1 & 0 & 3 \end{array}\right] \qquad \begin{array}{rl} 2u-3v+w-x+y=0 & (3) \\ -x-3y=-5 & (2) \\ -w+x=3 & (1) \end{array}$$

$(2) \quad \Rightarrow \quad x = 5-3y$

$(1) \quad \Rightarrow \quad w = x-3 = 2-3y$

$(3) \quad \Rightarrow \quad 2u = 3v-2+3y+5-3y-y = 3v-y+3$

$$u = \frac{3}{2}v - \frac{1}{2}y + \frac{3}{2}$$

$\therefore$ ***Solution***: $\left(\frac{3}{2}v-\frac{1}{2}y+\frac{3}{2}, v, 2-3y, 5-3y, y\right)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 6x_3+2x_4-4x_5-8x_6=8 \\ 3x_3+x_4-2x_5-4x_6=4 \\ 2x_1-3x_2+x_3+4x_4-7x_5+x_6=2 \\ 6x_1-9x_2+11x_4-19x_5+3x_6=1 \end{cases}$

Solution

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 6 & -9 & 0 & 11 & -19 & 3 & 1 \end{array}\right] \quad R_4-3R_1$$

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & -3 & -1 & 2 & 0 & -5 \end{array}\right] \quad \begin{array}{l} R_3-2R_2 \\ R_4+R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & -1 \end{array}\right] \quad \begin{array}{c} 2x_1-3x_2+x_3+4x_4-7x_5=2-x_6 \\ 3x_3+x_4-2x_5=4+4x_6 \\ \\ \to \ x_6=\frac{1}{4} \end{array}$$

$$\begin{cases} 3x_3 = 5-x_4+2x_5 \\ 2x_1 = \frac{7}{4}+3x_2-\frac{1}{3}\left(5-x_4+2x_5\right)-4x_4+7x_5 \end{cases}$$

$$\begin{cases} \boxed{x_3 = \frac{5}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5} \\ 2x_1 = \frac{1}{12} + 3x_2 - \frac{11}{3}x_4 + \frac{19}{3}x_5 \end{cases}$$

∴ ***Solution***: $\boxed{\left(\frac{1}{24} + \frac{3}{2}x_2 - \frac{11}{6}x_4 + \frac{19}{6}x_5,\ x_2,\ \frac{5}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5,\ x_4,\ x_5,\ \frac{1}{4}\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x_1 + 2x_2 - x_3 = -15 \\ 5x_1 + 3x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 3x_3 = 11 \\ -6x_1 - 4x_2 + 2x_3 = 30 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ -6 & -4 & 2 & 30 \end{array}\right] \begin{array}{l} \\ 3R_2 - 5R_1 \\ R_3 - R_1 \\ R_4 + 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & -1 & 4 & 26 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & 0 & -7 & -49 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{rr} 3x_1 + 2x_2 - x_3 = -15 & (3) \\ -x_2 + 11x_3 = 75 & (2) \\ -7x_3 = -49 & (1) \end{array}$$

$(1) \rightarrow x_3 = 7$

$(2) \rightarrow x_2 = 77 - 75 = 2$

$(1) \rightarrow 3x_1 = -15 - 4 + 7 = 12$

$\boxed{x_1 = -4}$

∴ ***Solution***: $\boxed{(-4, 2, 7)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x_1 + 3x_2 - 2x_3 \qquad + 2x_5 \qquad = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1 \\ \qquad 5x_3 + 10x_4 \qquad + 15x_6 = 5 \\ 2x_1 + 6x_2 \qquad + 8x_4 + 4x_5 + 18x_6 = 6 \end{cases}$

Solution

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ \\ R_4 - 2R_1 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \begin{array}{l} \\ -R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 5R_2 \\ R_4 - 4R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right] \begin{array}{l} \\ \\ \\ \frac{1}{6}R_4 \quad \textit{then interchanging row3 and row4} \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_3 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \rightarrow \begin{cases} x_1 + 3x_2 \qquad + 4x_4 + 2x_5 \qquad = 0 \\ \qquad x_3 + 2x_4 \qquad = 0 \\ \qquad\qquad + x_6 = \frac{1}{3} \end{cases}$$

The general solution of the system: $x_6 = \frac{1}{3}, \quad x_3 = -2x_4, \quad x_1 = -3x_2 - 4x_4 - 2x_5$

$\therefore$ ***Solution***: $\left(-3x_2 - 4x_4 - 2x_5,\ x_2,\ -2x_4,\ x_4,\ x_5,\ \frac{1}{3}\right)$

Exercise

At SnackMix, caramel corn worth \$2.50 per *pound* is mixed with honey roasted missed nuts worth \$7.50 per *pound* in order to get 20 *lbs*. of a mixture worth \$4.50 per *pound*. How much of each snack is used?

Solution

$$x + y = 20 \qquad (1)$$
$$2.50x + 7.50y = 90 \qquad (2)$$

(1) $y = 20 - x$

(2) $2.5x + 7.5(20 - x) = 90$

$2.5x + 150 - 7.5x = 90$

$-5x = 90 - 150$

$-5x = -60$

$x = \frac{-60}{-5} = 12$

$y = 20 - x$
$= 20 - 12$
$= 8$

The mixture consists of 12 *lbs*. of caramel and 8 *lbs*. of nuts