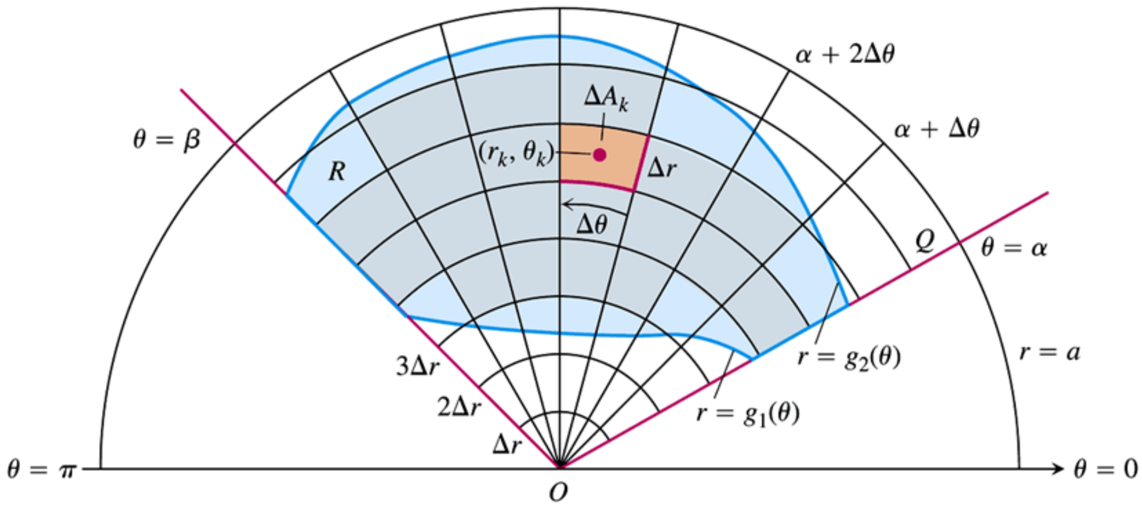
Section 3.3 – Double Integrals in Polar Coordinates

Integrals in Polar Coordinates



$$S_n = \sum_{k=1}^{n} f\left(r_k, \theta_k\right) \Delta A_k$$

If f is continuous throughout R, this sum will approach a limit as Δr and $\Delta\theta$ go to zero. The limit is called the double integral of f over R.

$$\lim_{n\to\infty} S_n = \iint_R f\left(r,\theta\right)\, dA$$

However, the area of a wedge-shaped sector of a circle having radius r and angle θ is

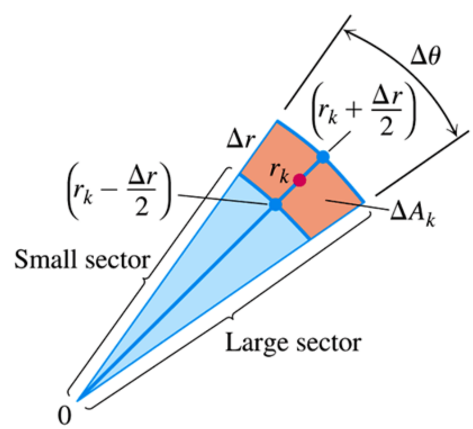
$$A = \frac{1}{2}\theta \cdot r^2$$

Inner radius: $\quad \frac{1}{2}\left(r_k - \frac{\Delta r}{2}\right)^2 \cdot \Delta\theta$

outer radius: $\quad \frac{1}{2}\left(r_k + \frac{\Delta r}{2}\right)^2 \cdot \Delta\theta$

$$\Delta A_k = \left(\begin{array}{c} \textit{area of} \\ \textit{large sector} \end{array}\right) - \left(\begin{array}{c} \textit{area of} \\ \textit{small sector} \end{array}\right)$$

Leads to the formula: $\Delta A_k = r_k \Delta r \Delta\theta$

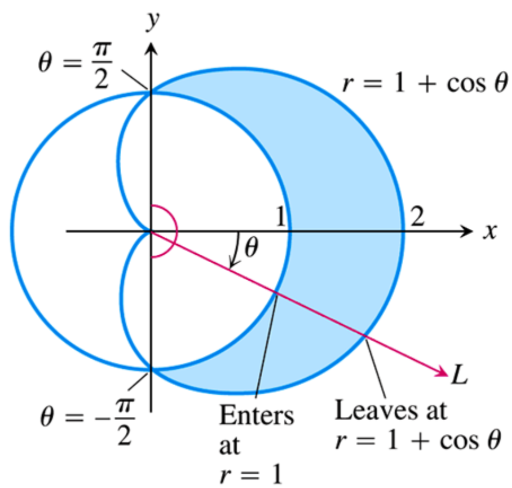


Example

Find the limits of integration for integrating $f(r,\theta)$ over the region *R* that lies inside the cardioid $r = 1+\cos\theta$ and outside the circle $r=1$.

Solution

The sketch of the region:



From the graph, we can find the *r - limits of integration*. A typical ray from the origin enters *R* where $r=1$ and leaves where $r=1+\cos\theta$

θ *- limits of integration*: The rays from the origin that intersects *R* run from $\theta=-\frac{\pi}{2}$ to $\theta=\frac{\pi}{2}$. The integral is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{1}^{1+\cos\theta} f(r,\theta)r\ dr\ d\theta$$

Area in Polar Coordinates

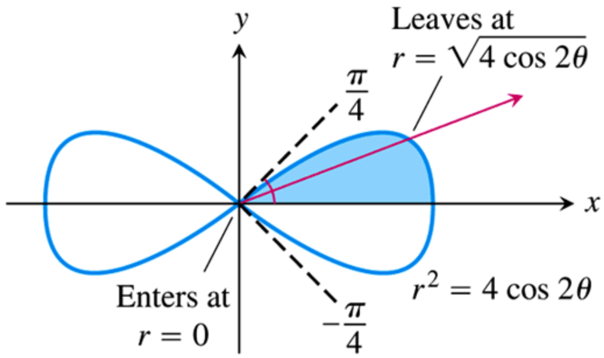
The area of a closed and bounded region *R* in the polar coordinate plane is

$$A=\iint_R r\ dr\ d\theta$$

Example

Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$

Solution



From the graph, we can determine the lemniscate limits of integration, and the total area is 4 times the first-quadrant portion, since it has a form of symmetry.

$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\ dr d\theta$$

$$= 4\int_0^{\pi/4} \frac{1}{2}r^2 \Big|_0^{\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} (2\cos 2\theta)\, d\theta$$

$$= 4\int_0^{\pi/4} \cos 2\theta\ d(2\theta)$$

$$= 4\sin 2\theta \Big|_0^{\pi/4}$$

$$= 4\sin\frac{\pi}{2}$$

$$\underline{= 4 \quad unit^2}$$

Changing Cartesian Integrals into Polar Integrals

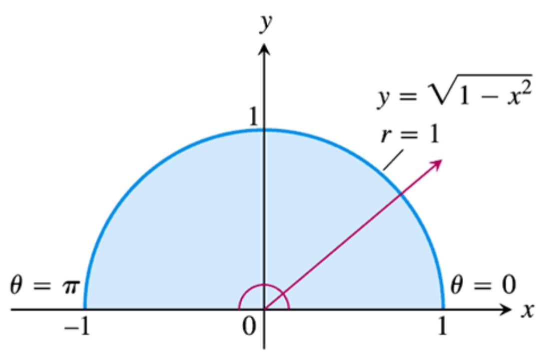
$$\iint_R f(x,y)\,dxdy = \iint_G f(r\cos\theta, r\sin\theta)\,r\;drd\theta$$

Example

Evaluate $\displaystyle\iint_R e^{x^2+y^2}\;dydx$

Where R is the semicircular region bounded by the x-axis and the curve $y=\sqrt{1-x^2}$

Solution



$$\iint_R e^{x^2+y^2}\,dydx = \int_0^{\pi}\int_0^1 e^{r^2}\,r\;drd\theta \qquad\qquad d\left(r^2\right)=2rdr$$

$$=\frac{1}{2}\int_0^{\pi} d\theta \;\int_0^1 e^{r^2}\;d\left(r^2\right)$$

$$=\frac{1}{2}\;\theta\,\Big|_0^{\pi}\;\; e^{r^2}\,\Big|_0^1$$

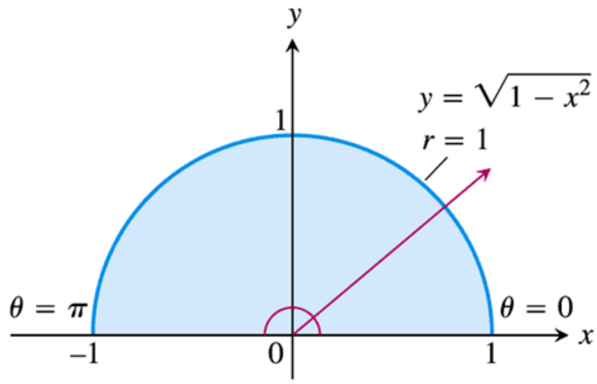
$$=\frac{1}{2}\int_0^{\pi}(e-1)\,d\theta$$

$$=\frac{\pi}{2}(e-1)$$

Example

Evaluate the integral $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\left(x^2+y^2\right)dydx$

Solution



Since: $0\le x\le 1 \quad \to \ interior\ of\ \ x^2+y^2=1 \quad and\ in\ QI$

Let: $r^2=x^2+y^2 \quad with \quad 0\le r\le 1$

$$\int_0^1\int_0^{\sqrt{1-x^2}}\left(x^2+y^2\right)dydx=\int_0^{\pi/2}\int_0^1\left(r^2\right)rdrd\theta$$

$$=\int_0^{\pi/2}d\theta\int_0^1 r^3\,dr$$

$$=\theta\Big|_0^{\pi/2}\ \frac{1}{4}r^4\Big|_0^1$$

$$=\frac{\pi}{8}$$

- Or we can use the integral table to solve it

$$\int_0^1\int_0^{\sqrt{1-x^2}}\left(x^2+y^2\right)dydx=\int_0^1\left[x^2\sqrt{1-x^2}+\frac{1}{3}\left(1-x^2\right)^3\right]dx$$

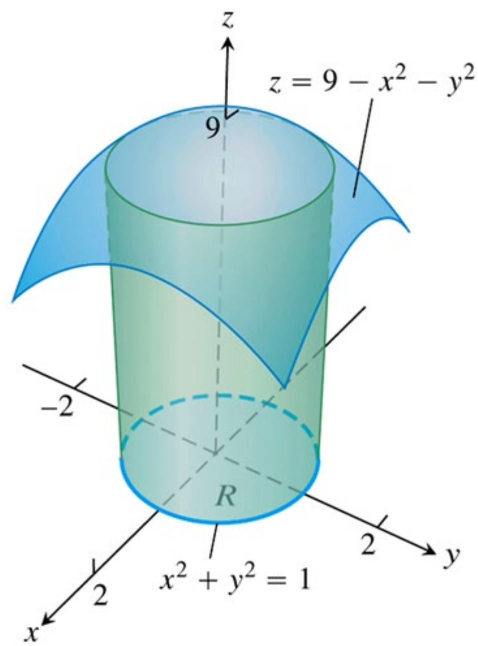
Example

Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy-plane.

Solution

The region of integration R is the unit circle:

$x^2 + y^2 = 1 \quad \to \quad r = 1, \quad 0 \le \theta \le 2\pi$



$$Volume = \int_0^{2\pi} \int_0^1 \left(9 - r^2\right) r\, dr d\theta$$

$$= \int_0^{2\pi} d\theta \quad \int_0^1 \left(9r - r^3\right) dr$$

$$= 2\pi \quad \left(\frac{9}{2} r^2 - \frac{1}{4} r^4 \right.\Big|_0^1$$

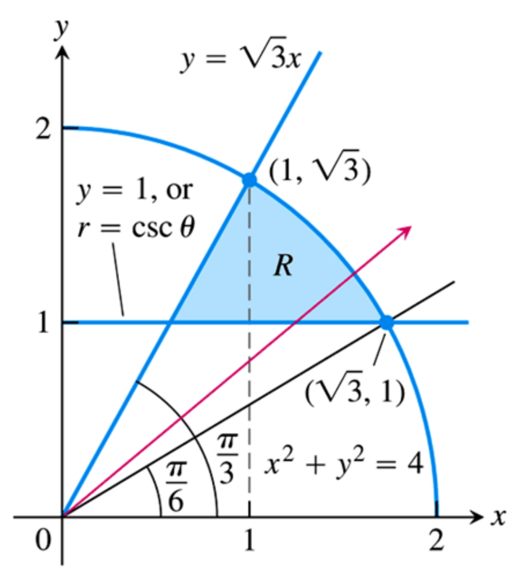
$$= 2\pi \left(\frac{9}{2} - \frac{1}{4}\right)$$

$$= \frac{17\pi}{2} \quad unit^3$$

Example

Using the polar integration, find the area of the region R in the xy-plane enclosed by the circle $x^2 + y^2 = 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$.

Solution



The $y = \sqrt{3}x$ has a slope of $\sqrt{3} = \tan\theta \;\Rightarrow\; \theta = \frac{\pi}{3}$

Line $y = 1$ intersects $x^2 + y^2 = 4$

when $x^2 + 1 = 4 \to x = \sqrt{3}$.

A line from origin to $\left(\sqrt{3},\, 1\right)$ has a slope of

$$\frac{1}{\sqrt{3}} = \tan\theta \to \theta = \frac{\pi}{6}$$

$$\therefore \quad \frac{\pi}{6} \le \theta \le \frac{\pi}{3}$$

The polar coordinate r varies from the horizontal line $y = 1$ to the circle $x^2 + y^2 = 4$.

Substituting $r\sin\theta$ for y:

$y = 1 \to r\sin\theta = 1$

$r = \dfrac{1}{\sin\theta} = \underline{\csc\theta\,|}$

The radius of the circle is 2.

$\therefore\ \ \underline{\csc\theta \le r \le 2\,|}$

$$Area = \int_{\pi/6}^{\pi/3}\int_{\csc\theta}^{2} r\,drd\theta$$

$$= \int_{\pi/6}^{\pi/3} \frac{1}{2}r^2 \Big|_{\csc\theta}^{2} d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{\pi/3}\left(4 - \csc^2\theta\right)d\theta$$

$$= \frac{1}{2}\left(4\theta + \cot\theta\right. \Big|_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2}\left[\frac{4\pi}{3} + \frac{1}{\sqrt{3}} - \left(\frac{4\pi}{6} + \sqrt{3}\right)\right]$$

$$= \frac{1}{2}\left(\frac{2\pi}{3} + \frac{\sqrt{3}}{3} - \sqrt{3}\right)$$

$$= \frac{1}{2}\left(\frac{2\pi - 2\sqrt{3}}{3}\right)$$

$$= \frac{\pi - \sqrt{3}}{3}\quad unit^2$$

Example

Evaluate the double integral: $\displaystyle\iint e^{-x^2-y^2}\,dA$

In the first quadrant and bounded by the circle $x^2 + y^2 = a^2$ and the coordinate axes.

Solution

$x^2 + y^2 = r^2$

$\underline{0 \le r \le a\,|}$

In QI: $\underline{0 \le \theta \le \frac{\pi}{2}\,|}$

$$\iint e^{-x^2-y^2}\,dA = \int_0^{\frac{\pi}{2}}\int_0^a e^{-r^2} r\,dr d\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta \int_0^a e^{-r^2}\, d\left(-r^2\right)$$

$$= -\frac{1}{2}\theta \Big|_0^{\frac{\pi}{2}} \; e^{-r^2}\Big|_0^a$$

$$= -\frac{1}{2}\left(\frac{\pi}{2}\right)\left(e^{-a^2}-1\right)$$

$$\boxed{= \frac{\pi}{4}\left(1-e^{-a^2}\right)}$$

Exercises *Section* 3.3 – Double Integrals in Polar Coordinates

(1 – 16) Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

1. $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dydx$

2. $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} \left(x^2+y^2\right)dxdy$

3. $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dydx$

4. $\displaystyle\int_{0}^{6}\int_{0}^{y} xdxdy$

5. $\displaystyle\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}dydx$

6. $\displaystyle\int_{0}^{\ln 2}\int_{0}^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}dxdy$

7. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right)dxdy$

8. $\displaystyle\int_{1}^{2}\int_{0}^{\sqrt{2x-x^2}} \frac{1}{\left(x^2+y^2\right)^2}dydx$

9. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2dydx}{\left(1+x^2+y^2\right)^2}$

10. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right)dxdy$

11. $\displaystyle\int_{0}^{\infty}\int_{0}^{\infty} \frac{1}{\left(1+x^2+y^2\right)^2}dxdy$

12. $\displaystyle\int_{0}^{3}\int_{0}^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\ dydx$

13. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(x^2+y^2\right)^{3/2}\ dydx$

14. $\displaystyle\int_{-4}^{4}\int_{0}^{\sqrt{16-y^2}} \left(16-x^2-y^2\right)\ dxdy$

15. $\displaystyle\int_{0}^{\frac{\pi}{4}}\int_{0}^{\sec\theta} r^3\ drd\theta$

16. $\displaystyle\int_{0}^{\frac{\pi}{2}}\int_{1}^{\infty} \frac{\cos\theta}{r^3}r\ drd\theta$

17. Find the area of the region cut from the first quadrant by the curve $r=2(2-\sin 2\theta)^{1/2}$

18. Find the area of the region lies inside the cardioid $r=1+\cos\theta$ and outside the circle $r=1$

19. Find the area enclosed by one leaf of the rose $r=12\cos 3\theta$

20. Find the area of the region common to the interiors of the cardioids $r=1+\cos\theta$ and $r=1-\cos\theta$

21. Find the area of the region bounded by all leaves of the rose $r=3\cos 2\theta$

22. Find the area of the region inside both the circles $r=2$ and $r=4\cos\theta$

23. Find the area of the region that lies inside both the cardioids $r = 2 - 2\cos\theta$ and $r = 2 + 2\cos\theta$

24. Find the area of the annular region $\{(r,\ \theta):\ 1 \le r \le 2,\quad 0 \le \theta \le 2\pi\}$

25. Find the area of the region bounded by the cardioid $r = 2(1 - \sin\theta)$

26. Find the area of the region bounded by all leaves of the rose $r = 2\cos 3\theta$

27. Find the area of the region inside both the cardioid $r = 1 - \cos\theta$ and the circle $r = 1$

28. Find the area of the region inside both the cardioid $r = 1 + \sin\theta$ and the cardioid $r = 1 + \cos\theta$

29. Find the area of the region bounded by the spiral $r = 2\theta$, for $0 \le \theta \le \pi$, and the x-axis.

30. Find the area of the region inside the limaçon $r = 1 + \frac{1}{2}\cos\theta$

31. Find the area of the region bounded by $r = 2\sin 2\theta$

32. Find the area of the region bounded by $r^2 = 2\sin 2\theta$

33. Find the area of the region outside the circle $r = 1$ and inside the rose $r = 2\sin 3\theta$ in QI

34. Find the area of the region outside the circle $r = \frac{1}{2}$ and inside the circle $r = 1 + \cos\theta$

35. Integrate $f(x,y) = \dfrac{\ln\left(x^2 + y^2\right)}{\sqrt{x^2 + y^2}}$ over the region $1 \le x^2 + y^2 \le e$

36. The region enclosed by the lemniscates $r^2 = 2\cos 2\theta$ is the base of a solid right cylinder whose top is bounded by the sphere $z = \sqrt{2 - r^2}$. Find the cylinder's volume.

37. Evaluate $\displaystyle\iint_R (x + y)\,dA$; R is the disk bounded by circle $r = 4\sin\theta$

38. Find the volume of the solid bounded above by the paraboloid $z = 2 - x^2 - y^2$ and below by the plane $z = 1$

39. Find the volume of the solid bounded above by the paraboloid $z = 8 - x^2 - 3y^2$ and below by the hyperbolic paraboloid $z = x^2 - y^2$

(40 – 51) Evaluate the integral over R using polar coordinates

40. $\displaystyle\iint_R \left(x^2 + y^2\right) dA;\quad R = \{(r,\ \theta):\ 0 \le r \le 4,\quad 0 \le \theta \le 2\pi\}$

41. $\iint_R 2xy\,dA;\quad R=\left\{(r,\ \theta):\ \ 1\le r\le 3,\quad 0\le\theta\le\frac{\pi}{2}\right\}$

42. $\iint_R 2xy\ dA;\quad R=\left\{(x,\ y):\ \ x^2+y^2\le 9,\quad y\ge 0\right\}$

43. $\iint_R \frac{dA}{1+x^2+y^2};\quad R=\left\{(r,\ \theta):\ \ 1\le r\le 2,\quad 0\le\theta\le\pi\right\}$

44. $\iint_R \frac{dA}{\sqrt{16-x^2-y^2}};\quad R=\left\{(x,\ y):\ \ x^2+y^2\le 4,\quad y\ge 0\right\}$

45. $\iint_R \frac{dA}{\sqrt{16-x^2-y^2}};\quad R=\left\{(x,\ y):\ \ x^2+y^2\le 4,\quad x,\ y\ge 0\right\}$

46. $\iint_R e^{-x^2-y^2}\,dA;\quad R=\left\{(x,\ y):\ \ x^2+y^2\le 9\right\}$

47. $\iint_R \sqrt{x^2+y^2}\ dA;\quad R=\left\{(x,\ y):\ \ y\le x\le 1,\quad 0\le y\le 1\right\}$

48. $\iint_R \sqrt{x^2+y^2}\ dA;\quad R=\left\{(x,\ y):\ \ 1\le x^2+y^2\le 2\right\}$

49. $\iint_R \frac{dA}{\left(x^2+y^2\right)^{5/2}};\quad R=\left\{(r,\ \theta):\ \ 1\le r\le\infty,\quad 0\le\theta\le 2\pi\right\}$

50. $\iint_R e^{-x^2-y^2}\,dA;\quad R=\left\{(r,\ \theta):\ \ 0\le r\le\infty,\quad 0\le\theta\le\frac{\pi}{2}\right\}$

51. $\iint_R \frac{dA}{\left(1+x^2+y^2\right)^2};\quad R\in QI$

52. Which bowl holds more water if it is filled to a depth of four units?

a) The paraboloid $z=x^2+y^2$, for $0\le z\le 4$

b) The cone $z=\sqrt{x^2+y^2}$, for $0\le z\le 4$

c) The hyperboloid $z=\sqrt{1+x^2+y^2}$, for $1\le z\le 5$

d) To what weight (above the bottom of the bowl) must the cone and paraboloid bowls be filled to hold the same volume of water as the hyperboloid bowl filled to a depth of 4 units $(1 \le z \le 5)$

53. Consider the surface $z = x^2 - y^2$

a) Find the region in the *xy*-plane in polar coordinates for which $z \ge 0$.

b) Let $R = \left\{(r,\ \theta):\quad 0 \le r \le a,\quad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}\right\}$, which is a sector of a circle of radius a. Find the volume of the region below the hyperbolic paraboloid and above the region *R*.

54. A cake is shaped like a hemisphere of radius 4 with its base on the *xy*-plane. A wedge of the cake is removed by making two slices from the center of the cake outward, perpendicular to the *xy*-plane and separated by an angle of φ.

a) Use a double integral to find the volume of the slice for $\varphi = \frac{\pi}{4}$.

b) Suppose the cake is sliced by a plane perpendicular to the *xy*-plane at $x = a > 0$. Let *D* be the smaller of the two pieces produced. For what value of *a* is the volume of *D* equal to the volume in part (*a*)?

55. Suppose the density of a thin plate represented by the region *R* is $\rho(r,\ \theta)$ (in units of mass per area). The mass of the plate is $\iint\limits_R \rho(r,\ \theta)\,dA$. Find the mass of the thin half annulus $R = \{(r,\ \theta):\quad 1 \le r \le 4,\quad 0 \le \theta \le \pi\}$ with a density $\rho(r,\ \theta) = 4 + r\sin\theta$

56. An important integral in statistics associated with the normal distribution is $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$. It is evaluated in the following steps.

a) Assume that $I = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dxdy$

Where we have chosen the variables of integration to be *x* and *y* and then written the product as an iterated integral. Evaluate this integral in polar coordinates and show that $I = \sqrt{\pi}$. Why is the solution $I = -\sqrt{\pi}$ rejected?

b) Evaluate $\int_0^{\infty} e^{-x^2}\,dx$, $\int_0^{\infty} xe^{-x^2}\,dx$, and $\int_0^{\infty} x^2e^{-x^2}\,dx$.

57. For what values of p does the integral $\displaystyle\iint\limits_R \frac{k}{\left(x^2+y^2\right)^p}\,dA$ exist in the following cases?

a) $R=\{(r,\ \theta):\ 1\le r\le\infty,\quad 0\le\theta\le 2\pi\}$

b) $R=\{(r,\ \theta):\ 0\le r\le 1,\quad 0\le\theta\le 2\pi\}$

58. Consider the integral $\displaystyle\iint\limits_R \frac{k}{\left(1+x^2+y^2\right)^2}\,dA$ where $R=\{(x,\ y):\ 0\le x\le 1,\quad 0\le y\le a\}$

a) Evaluate I for $a=1$.

b) Evaluate I for arbitrary $a>0$.

c) Let $a\to\infty$ in part (b) to find I over the infinite strip $R=\{(x,\ y):\ 0\le x\le 1,\quad 0\le y\le\infty\}$

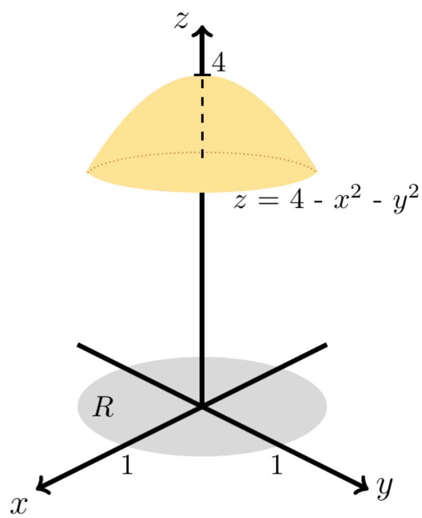
59. In polar coordinates an equation of an ellipse with eccentricity $0<e<1$ and semimajor axis a is

$$r=\frac{a\left(1-e^2\right)}{1+e\cos\theta}$$

a) Write the integral that gives the area of the ellipse.

b) Show that the area of an ellipse is πab, where $b^2=a^2\left(1-e^2\right)$

(60 – 63) Find the volume of the solid below the paraboloid $z=4-x^2-y^2$ and above



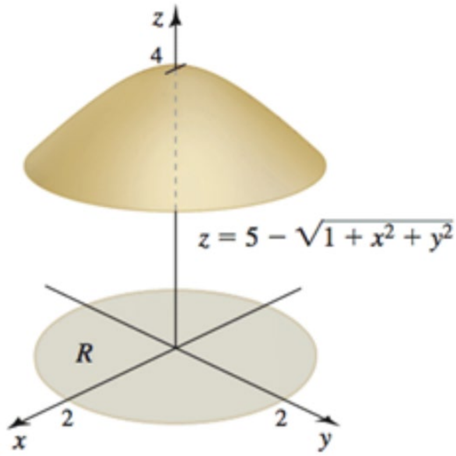
60. $R=\{(r,\ \theta):\ 0\le r\le 1,\quad 0\le\theta\le 2\pi\}$

61. $R=\{(r,\ \theta):\ 0\le r\le 2,\quad 0\le\theta\le 2\pi\}$

62. $R=\{(r,\ \theta):\ 1\le r\le 2,\quad 0\le\theta\le 2\pi\}$

63. $R=\left\{(r,\ \theta):\ 1\le r\le 2,\quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\right\}$

(64 – 67) Find the volume of the solid below the hyperboloid $z = 5 - \sqrt{1 + x^2 + y^2}$ and above



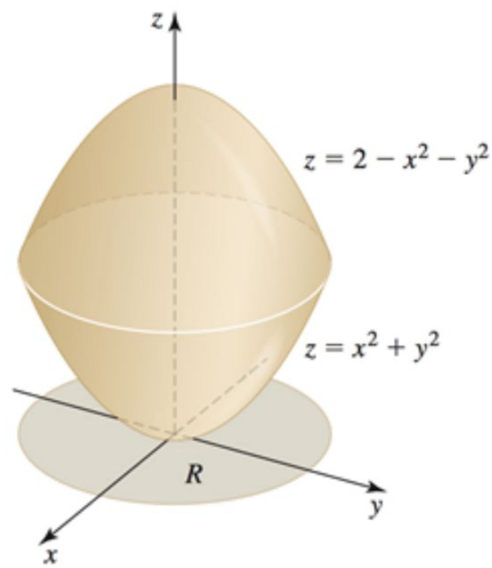
64. $R = \{(r,\ \theta):\ \ 0 \le r \le 2,\ \ \ 0 \le \theta \le 2\pi\}$

65. $R = \{(r,\ \theta):\ \ 0 \le r \le 1,\ \ \ 0 \le \theta \le \pi\}$

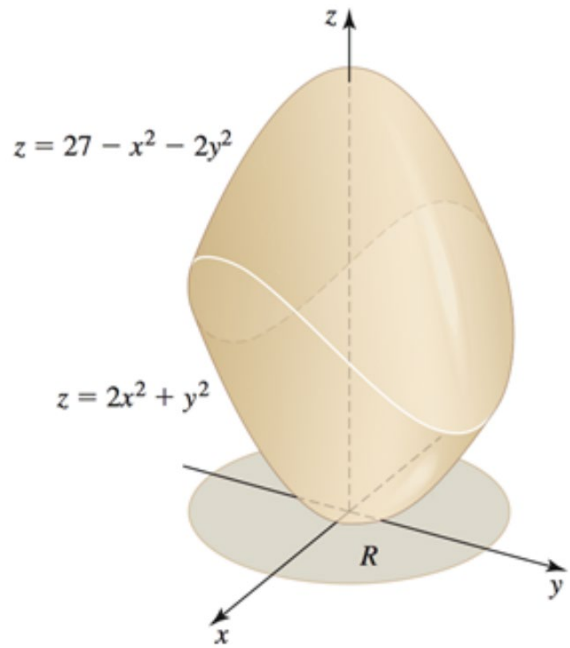
66. $R = \left\{(r,\ \theta):\ \ \sqrt{3} \le r \le 2\sqrt{2},\ \ \ 0 \le \theta \le 2\pi\right\}$

67. $R = \left\{(r,\ \theta):\ \ \sqrt{3} \le r \le \sqrt{15},\ \ \ -\frac{\pi}{2} \le \theta \le \pi\right\}$

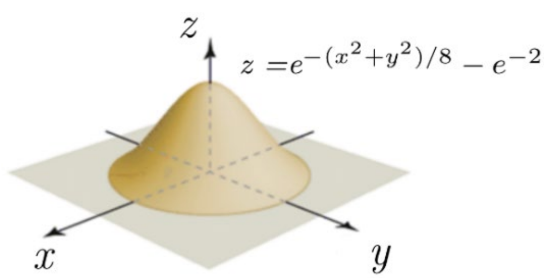
68. Find the volume of the solid between the paraboloids $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$



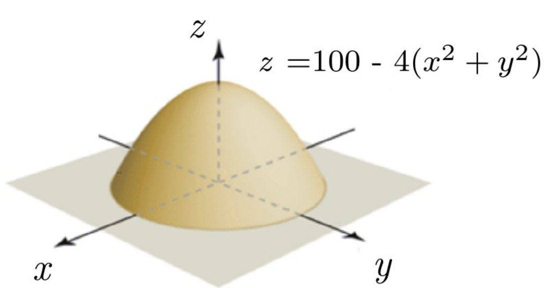
69. Find the volume of the solid between the paraboloids $z = 2x^2 + y^2$ and $z = 27 - x^2 - 2y^2$



70. Find the volume of island $z = e^{-\left(x^2+y^2\right)/8} - e^{-2}$



71. Find the volume of island $z = 100 - 4\left(x^2 + y^2\right)$



72. Find the volume of island $z = 25 - \sqrt{x^2 + y^2}$