

Solution

Section 3.1 – Inverse Functions

Exercise

Find the inverse relation of the given set: $A = \{(-2, 2), (1, -1), (0, 4), (1, 3)\}$

Solution

$$\underline{A^{-1} = \{(2, -2), (-1, 1), (4, 0), (3, 1)\}}$$

Exercise

Find the inverse relation of the given set: $B = \{(1, -1), (2, -2), (3, -3), (4, -4)\}$

Solution

$$\underline{B^{-1} = \{(-1, 1), (-2, 2), (-3, 3), (-4, 4)\}}$$

Exercise

Find the inverse relation of the given set: $C = \{(a, -a), (b, -b), (c, -c)\}$

Solution

$$\underline{C^{-1} = \{(-a, a), (-b, b), (-c, c)\}}$$

Exercise

Find the inverse relation of the given set: $D = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4)\}$

Solution

$$\underline{D^{-1} = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4)\}}$$

Exercise

Find the inverse relation of the given set: $E = \{(-a, a), (-b, b), (-c, c), (-d, d)\}$

Solution

$$\underline{E^{-1} = \{(a, -a), (b, -b), (c, -c), (d, -d)\}}$$

Exercise

Determine whether the function is one-to-one: $f(x) = 3x - 7$

Solution

$$f(a) = f(b)$$

$$3a - 7 = 3b - 7$$

$$3a = 3b - 7 + 7$$

$$3a = 3b$$

Divide both sides by 3

$$a = b$$

\therefore The function is one-to-one

Exercise

Determine whether the function is one-to-one: $f(x) = x^2 - 9$

Solution

$$1 \neq -1$$

$$1^2 - 9 \neq (-1)^2 - 9$$

$-8 = -8 \rightarrow$ Contradict the definition

$$f(a) = f(b)$$

$$a^2 - 9 = b^2 - 9$$

$$a^2 = b^2$$

$$a = \pm b$$

\therefore The function is ***not*** one-to-one

Exercise

Determine whether the function is one-to-one: $f(x) = \sqrt{x}$

Solution

$$f(a) = f(b)$$

$$\sqrt{a} = \sqrt{b}$$

$$(\sqrt{a})^2 = (\sqrt{b})^2$$

Square both sides

$$a = b$$

\therefore The function is one-to-one

Exercise

Determine whether the function is one-to-one: $f(x) = \sqrt[3]{x}$

Solution

$$f(a) = f(b)$$

$$\sqrt[3]{a} = \sqrt[3]{b}$$

$$\left(\sqrt[3]{a}\right)^3 = \left(\sqrt[3]{b}\right)^3 \quad \text{cube both sides}$$

$$a = b$$

\therefore The function is one-to-one

Exercise

Determine whether the function is one-to-one: $f(x) = |x|$

Solution

$$1 \neq -1$$

$$|1| \neq |-1|$$

$$1 \neq 1 \text{ (false)}$$

\therefore The function is **not** one-to-one

Exercise

Determine whether the function is one-to-one $f(x) = \frac{2}{x+3}$

Solution

$$f(a) = f(b)$$

$$\frac{2}{a+3} = \frac{2}{b+3}$$

$$2(b+3) = 2(a+3)$$

$$b+3 = a+3$$

$$a = b$$

$\therefore f$ is one-to-one

Exercise

Determine whether the function is one-to-one $f(x) = (x-2)^3$

Solution

$$f(a) = f(b)$$

$$(a-2)^3 = (b-2)^3$$

$$\left[(a-2)^3\right]^{1/3} = \left[(b-2)^3\right]^{1/3}$$

$$a-2 = b-2$$

Add 2 on both sides

$$a = b$$

\therefore Function is one-to-one

Exercise

Determine whether the function is one-to-one $y = x^2 + 2$

Solution

$$f(a) = f(b)$$

$$a^2 + 2 = b^2 + 2$$

Subtract 2

$$a^2 = b^2$$

$$a = \pm\sqrt{b^2}$$

\therefore Function is **not** a one-to-one

The inverse function doesn't exist.

Exercise

Determine whether the function is one-to-one $f(x) = \frac{x+1}{x-3}$

Solution

$$f(a) = f(b)$$

$$\frac{a+1}{a-3} = \frac{b+1}{b-3}$$

Cross multiplication

$$(a+1)(b-3) = (b+1)(a-3)$$

$$ab - 3a + b - 3 = ab - 3b + a - 3$$

$$-4a = -4b$$

Divide by -4

$$a = b$$

\therefore Function is one-to-one

Exercise

Given that $f(x) = 5x + 8$, use composition of functions to show that $f^{-1}(x) = \frac{x-8}{5}$

Solution

$$\begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}(f(x)) \\&= f^{-1}(5x + 8) \\&= \frac{(5x + 8) - 8}{5} \\&= \frac{5x + 8 - 8}{5} \\&= \frac{5x}{5} \\&= x\end{aligned}$$

$$\begin{aligned}(f \circ f^{-1})(x) &= f(f^{-1}(x)) \\&= f\left(\frac{x-8}{5}\right) \\&= 5\left(\frac{x-8}{5}\right) + 8 \\&= x - 8 + 8 \\&= x\end{aligned}$$

Exercise

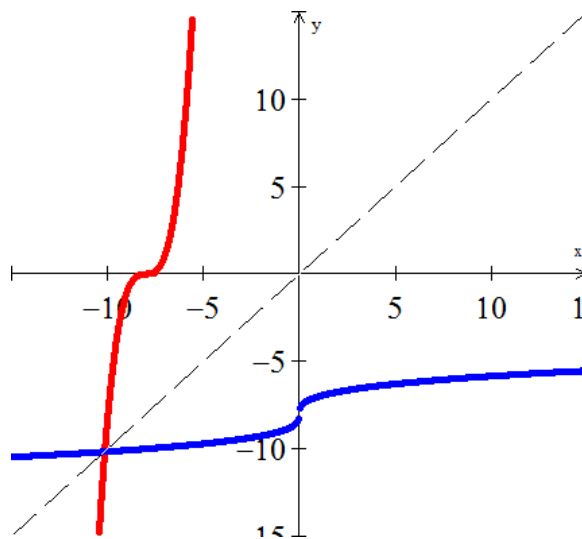
Given the function $f(x) = (x + 8)^3$

- a) Find $f^{-1}(x)$
- b) Graph f and f^{-1} in the same rectangular coordinate system
- c) Find the domain and the range of f and f^{-1}

Solution

a)	$y = (x + 8)^3$	Replace $f(x)$ with y
	$x = (y + 8)^3$	Interchange x and y
	$(x)^{1/3} = \left((y + 8)^3\right)^{1/3}$	
	$x^{1/3} = y + 8$	Subtract 8 from both sides.
	$f^{-1}(x) = x^{1/3} - 8$	

b)



- c) Domain of $f = \text{Range of } f^{-1}: (-\infty, \infty)$
 Range of $f = \text{Domain of } f^{-1}: (-\infty, \infty)$

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = 4x; \quad g(x) = \frac{x}{4}$

Solution

$$\begin{aligned} f(g(x)) &= f\left(\frac{x}{4}\right) \\ &= 4\left(\frac{x}{4}\right) \\ &= x \end{aligned}$$

$$\begin{aligned} g(f(x)) &= g(4x) \\ &= \frac{4x}{4} \\ &= x \end{aligned}$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = 2x; \quad g(x) = \frac{1}{2x}$

Solution

$$f(g(x)) = f\left(\frac{1}{2x}\right)$$

$$= 2\left(\frac{1}{2x}\right)$$

$$\underline{= \frac{1}{x}} \quad \neq x$$

$\therefore f(x)$ and $g(x)$ are **not** inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = 4x - 1; \quad g(x) = \frac{x+1}{4}$

Solution

$$f(g(x)) = f\left(\frac{x+1}{4}\right)$$

$$= 4\left(\frac{x+1}{4}\right) - 1$$

$$= x + 1 - 1$$

$$\underline{= x}$$

$$g(f(x)) = g(4x - 1)$$

$$= \frac{4x - 1 + 1}{4}$$

$$= \frac{4x}{4}$$

$$\underline{= x}$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = \frac{1}{2}x - \frac{3}{2}; \quad g(x) = 2x + 3$

Solution

$$f(g(x)) = f(2x + 3)$$

$$= \frac{1}{2}(2x + 3) - \frac{3}{2}$$

$$= x + \frac{3}{2} - \frac{3}{2}$$

$$\underline{= x}$$

$$g(f(x)) = g\left(\frac{1}{2}x - \frac{3}{2}\right)$$

$$= 2\left(\frac{1}{2}x - \frac{3}{2}\right) + 3$$

$$= x - 3 + 3$$

$$\underline{= x}$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = -\frac{1}{2}x - \frac{1}{2}; \quad g(x) = -2x + 1$

Solution

$$\begin{aligned} f(g(x)) &= f(-2x + 1) \\ &= -\frac{1}{2}(-2x + 1) - \frac{1}{2} \\ &= x - \frac{1}{2} - \frac{1}{2} \\ &= \underline{\frac{1}{x} - 1} \neq x \end{aligned}$$

$\therefore f(x)$ and $g(x)$ are **not** inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = 3x + 2; \quad g(x) = \frac{1}{3}(x - 2)$

Solution

$$\begin{aligned} f(g(x)) &= f\left(\frac{x-2}{3}\right) \\ &= 3\left(\frac{x-2}{3}\right) + 2 \\ &= x - 2 + 2 \\ &= \underline{x} \end{aligned}$$

$$\begin{aligned} g(f(x)) &= g(3x + 2) \\ &= \frac{1}{3}(3x + 2 - 2) \\ &= \frac{1}{3}(3x) \\ &= \underline{x} \end{aligned}$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = \frac{5}{x+3}; \quad g(x) = \frac{5}{x} - 3$

Solution

$$\begin{aligned} f(g(x)) &= f\left(\frac{5}{x} - 3\right) \\ &= \frac{5}{\frac{5}{x} - 3 + 3} \\ &= \frac{5}{\frac{5}{x}} \\ &= 5 \frac{x}{5} \\ &= x \end{aligned}$$

$$\begin{aligned} g(f(x)) &= g\left(\frac{5}{x+3}\right) \\ &= \frac{5}{\frac{5}{x+3}} - 3 \\ &= 5\left(\frac{x+3}{5}\right) - 3 \\ &= x + 3 - 3 \\ &= x \end{aligned}$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = \frac{2x}{x+1}; \quad g(x) = \frac{-x}{x-2}$

Solution

$$\begin{aligned} f(g(x)) &= f\left(\frac{-x}{x-2}\right) \\ &= 2\left(\frac{-x}{x-2}\right) \frac{1}{\frac{-x}{x-2} + 1} \\ &= \left(\frac{-2x}{x-2}\right) \frac{x-2}{-x+x-2} \\ &= \frac{-2x}{-2} \\ &= x \end{aligned}$$

$$g(f(x)) = g\left(\frac{2x}{x+1}\right)$$

$$\begin{aligned}
&= -\left(\frac{2x}{x+1}\right) \frac{1}{\frac{2x}{x+1} - 2} \\
&= -\left(\frac{2x}{x+1}\right) \frac{x+1}{2x - 2x - 2} \\
&= \frac{-2x}{-2} \\
&= x
\end{aligned}$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = \frac{3x}{x-1}; \quad g(x) = \frac{x}{x-3}$

Solution

$$\begin{aligned}
f(g(x)) &= f\left(\frac{x}{x-3}\right) \\
&= 3\left(\frac{x}{x-3}\right) \frac{1}{\frac{x}{x-3} - 1} \\
&= \left(\frac{3x}{x-3}\right) \frac{x-3}{x-x+3} \\
&= \frac{3x}{3} \\
&= x
\end{aligned}$$

$$\begin{aligned}
g(f(x)) &= g\left(\frac{3x}{x-1}\right) \\
&= \left(\frac{3x}{x-1}\right) \frac{1}{\frac{3x}{x-1} - 3} \\
&= \left(\frac{3x}{x-1}\right) \frac{x-1}{3x-3x+3} \\
&= \frac{3x}{3} \\
&= x
\end{aligned}$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = x^3 + 2; \quad g(x) = \sqrt[3]{x-2}$

Solution

$$f(g(x)) = f\left(\sqrt[3]{x-2}\right)$$

$$\begin{aligned}
&= \left(\sqrt[3]{x-2}\right)^3 + 2 \\
&= x - 2 + 2 \\
&= x
\end{aligned}$$

$$\begin{aligned}
g(f(x)) &= g(x^3 + 2) \\
&= \sqrt[3]{x^3 + 2 - 2} \\
&= \sqrt[3]{x^3} \\
&= x
\end{aligned}$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = (x+4)^3$; $g(x) = \sqrt[3]{x} - 4$

Solution

$$\begin{aligned}
f(g(x)) &= f(\sqrt[3]{x} - 4) \\
&= (\sqrt[3]{x} - 4 + 4)^3 \\
&= (\sqrt[3]{x})^3 \\
&= x
\end{aligned}$$

$$\begin{aligned}
g(f(x)) &= g((x+4)^3) \\
&= \sqrt[3]{(x+4)^3} - 4 \\
&= x + 4 - 4 \\
&= x
\end{aligned}$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = x^3 - 1$; $g(x) = \sqrt[3]{x+1}$

Solution

$$\begin{aligned}
f(g(x)) &= f(\sqrt[3]{x+1}) \\
&= (\sqrt[3]{x+1})^3 - 1
\end{aligned}$$

$$= x + 1 - 1$$

$$= x$$

$$g(f(x)) = g(x^3 - 1)$$

$$= \sqrt[3]{x^3 - 1 + 1}$$

$$= \sqrt[3]{x^3}$$

$$= x$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = 3x - 2$; $g(x) = \frac{x+2}{3}$

Solution

$$f(g(x)) = f\left(\frac{x+2}{3}\right)$$

$$= 3\left(\frac{x+2}{3}\right) - 2$$

$$= x + 2 - 2$$

$$= x$$

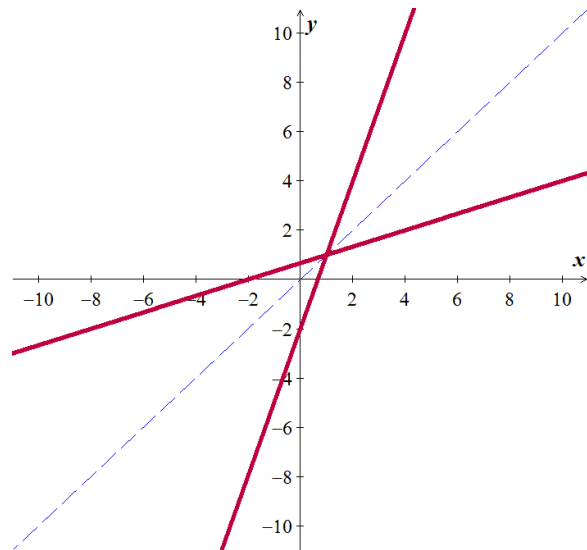
$$g(f(x)) = g(3x - 2)$$

$$= \frac{3x - 2 + 2}{3}$$

$$= \frac{3x}{3}$$

$$= x$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other



Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = x^2 + 5, x \leq 0$ $g(x) = -\sqrt{x-5}, x \geq 5$

Solution

$$f(g(x)) = f(-\sqrt{x-5})$$

$$= (-\sqrt{x-5})^2 + 5$$

$$= x - 5 + 5$$

$$= x$$

$$\begin{aligned}
 g(f(x)) &= g(x^2 + 5) \\
 &= -\sqrt{x^2 + 5} - 5 \\
 &= -\sqrt{x^2} \\
 &= -|x| \quad x \leq 0 \\
 &= -(-x) \\
 &= x
 \end{aligned}$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other

Exercise

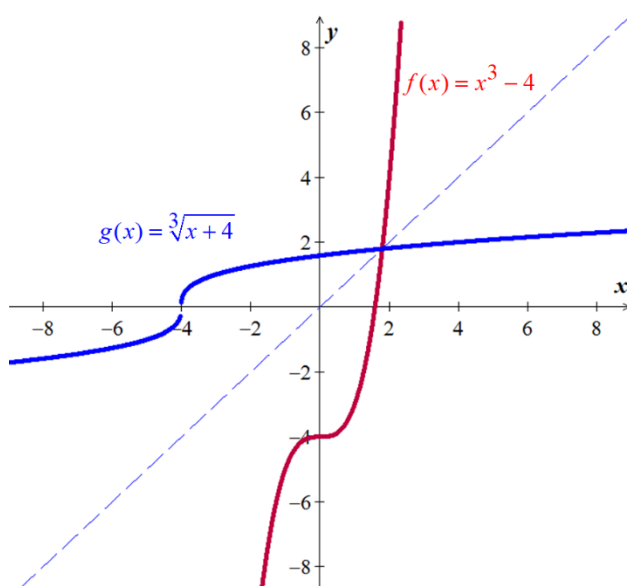
Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = x^3 - 4$; $g(x) = \sqrt[3]{x+4}$

Solution

$$\begin{aligned}
 f(g(x)) &= f(\sqrt[3]{x+4}) \\
 &= (\sqrt[3]{x+4})^3 - 4 \\
 &= x + 4 - 4 \\
 &= x
 \end{aligned}$$

$$\begin{aligned}
 g(f(x)) &= g(x^3 - 4) \\
 &= \sqrt[3]{x^3 - 4 + 4} \\
 &= \sqrt[3]{x^3} \\
 &= x
 \end{aligned}$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other



Exercise

Find the inverse of $f(x) = (x-2)^3$

Solution

$$\begin{aligned}
 y &= (x-2)^3 \\
 x &= (y-2)^3 \\
 x^{1/3} &= [(y-2)^3]^{1/3} \\
 x^{1/3} &= y-2
 \end{aligned}$$

$$\sqrt[3]{x} + 2 = y$$

$$\underline{f^{-1}(x) = \sqrt[3]{x} + 2}$$

Exercise

Find the inverse of $f(x) = \frac{x+1}{x-3}$

Solution

$$y = \frac{x+1}{x-3}$$

$$x = \frac{y+1}{y-3}$$

$$x(y-3) = y+1$$

$$xy - 3x = y+1$$

$$xy - y = 3x+1$$

$$y(x-1) = 3x+1$$

$$\underline{f^{-1}(x) = \frac{3x+1}{x-1}}$$

Exercise

Find the inverse of $f(x) = \frac{2x+1}{x-3}$

Solution

$$y = \frac{2x+1}{x-3}$$

$$x = \frac{2y+1}{y-3}$$

$$xy - 3x = 2y+1$$

$$y(x-2) = 3x+1$$

$$\underline{f^{-1}(x) = \frac{3x+1}{x-2}}$$

Exercise

Determine the domain and range of $f^{-1}: f(x) = -\frac{2}{x-1}$ (Hint: first find the domain and range of f)

Solution

$$x-1 \neq 0 \Rightarrow x \neq 1$$

$$\text{Range of } f^{-1} = \text{Domain of } f: \mathbb{R} - \{1\} \quad (-\infty, 1) \cup (1, \infty)$$

$$\text{Domain of } f^{-1} = \text{Range of } f: \mathbb{R} - \{0\} \quad (-\infty, 0) \cup (0, \infty)$$

Exercise

Determine the domain and range of $f^{-1}: f(x) = \frac{5}{x+3}$ (Hint: first find the domain and range of f)

Solution

$$\text{Domain of } f^{-1} = \text{Range of } f: \mathbb{R} - \{0\} \quad (-\infty, 0) \cup (0, \infty)$$

$$\text{Range of } f^{-1} = \text{Domain of } f: \mathbb{R} - \{-3\} \quad (-\infty, -3) \cup (-3, \infty)$$

Exercise

Determine the domain and range of $f^{-1}: f(x) = \frac{4x+5}{3x-8}$ (Hint: first find the domain and range of f)

Solution

$$\text{Domain of } f^{-1} = \text{Range of } f: \mathbb{R} - \left\{\frac{8}{3}\right\} \quad \left(-\infty, \frac{8}{3}\right) \cup \left(\frac{8}{3}, \infty\right)$$

$$\text{Range of } f^{-1} = \text{Domain of } f: \mathbb{R} - \left\{\frac{4}{3}\right\} \quad \left(-\infty, \frac{4}{3}\right) \cup \left(\frac{4}{3}, \infty\right)$$

Exercise

For the given function $f(x) = \frac{2x}{x-1}$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

$$a) \quad f(a) = f(b)$$

$$\frac{2a}{a-1} = \frac{2b}{b-1}$$

$$2ab - 2a = 2ab - 2b$$

$$-2a = -2b$$

$$\underline{a = b} \quad \checkmark$$

$\therefore f(x)$ is one-to-one function.

$$b) \quad y = \frac{2x}{x-1}$$

$$x = \frac{2y}{y-1}$$

$$xy - x = 2y$$

$$(x-2)y = x$$

$$y = \frac{x}{x-2} = f^{-1}(x)$$

$$c) \quad \text{Domain of } f^{-1}(x) = \text{Range of } f(x): \underline{\mathbb{R} - \{1\}}$$

$$\text{Range of } f^{-1}(x) = \text{Domain of } f(x): \underline{\mathbb{R} - \{2\}}$$

Exercise

For the given function $f(x) = \frac{x}{x-2}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

$$a) \quad f(a) = f(b)$$

$$\frac{a}{a-2} = \frac{b}{b-2}$$

$$ab - 2a = ab - 2b$$

$$-2a = -2b$$

$$\underline{a = b} \quad \checkmark$$

$\therefore f(x)$ is one-to-one function.

$$b) \quad y = \frac{x}{x-2}$$

$$x = \frac{y}{y-2}$$

$$xy - 2x = y$$

$$(x-1)y = 2x$$

$$\underline{f^{-1}(x) = \frac{2x}{x-1} \mid}$$

$$c) \text{ Domain of } f^{-1}(x) = \text{Range of } f(x) : \underline{\mathbb{R} - \{2\} \mid}$$

$$\text{Range of } f^{-1}(x) = \text{Domain of } f(x) : \underline{\mathbb{R} - \{1\} \mid}$$

Exercise

For the given function $f(x) = \frac{x+1}{x-1}$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

$$a) f(a) = f(b)$$

$$\frac{a+1}{a-1} = \frac{b+1}{b-1}$$

$$ab - a + b - 1 = ab - b + a - 1$$

$$-2a = -2b$$

$$\underline{a = b \mid} \quad \checkmark$$

$\therefore f(x)$ is one-to-one function.

$$b) y = \frac{x+1}{x-1}$$

$$x = \frac{y+1}{y-1}$$

$$xy - x = y + 1$$

$$(x-1)y = x+1$$

$$\underline{f^{-1}(x) = \frac{x+1}{x-1} \mid}$$

$$c) \text{ Domain of } f^{-1}(x) = \text{Range of } f(x) : \underline{\mathbb{R} - \{1\} \mid}$$

$$\text{Range of } f^{-1}(x) = \text{Domain of } f(x) : \underline{\mathbb{R} - \{1\} \mid}$$

Exercise $f(x) = \frac{2x+1}{x+3}$

For the given function

- a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\frac{2a+1}{a+3} = \frac{2b+1}{b+3}$$

$$2ab + 6a + b + 3 = 2ab + 6b + a + 3$$

$$5a = 5b$$

$$\underline{a = b} \quad \checkmark$$

$\therefore f(x)$ is one-to-one function.

b) $y = \frac{2x+1}{x+3}$

$$x = \frac{2y+1}{y+3}$$

$$xy + 3x = 2y + 1$$

$$(x-2)y = -3x+1$$

$$\underline{f^{-1}(x) = \frac{-3x+1}{x-2}}$$

c) Domain of $f^{-1}(x)$ = Range of $f(x)$: $\underline{\mathbb{R} - \{-3\}}$

Range of $f^{-1}(x)$ = Domain of $f(x)$: $\underline{\mathbb{R} - \{2\}}$

Exercise

For the given function $f(x) = \frac{3x-1}{x-2}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\frac{3a-1}{a-2} = \frac{3b-1}{b-2}$$

$$3ab - 6a - b + 2 = 3ab - 6b - a + 2$$

$$-5a = -5b$$

$$\underline{a = b} \quad \checkmark$$

$\therefore f(x)$ is one-to-one function.

$$b) \quad y = \frac{3x-1}{x-2}$$

$$x = \frac{3y-1}{y-2}$$

$$xy - 2x = 3y - 1$$

$$(x-3)y = 2x-1$$

$$\underline{f^{-1}(x) = \frac{2x-1}{x-3} \quad |}$$

$$c) \quad \text{Domain of } f^{-1}(x) = \text{Range of } f(x) : \quad \underline{\mathbb{R} - \{2\} \quad |}$$

$$\text{Range of } f^{-1}(x) = \text{Domain of } f(x) : \quad \underline{\mathbb{R} - \{3\} \quad |}$$

Exercise

For the given function $f(x) = \frac{3x-2}{x+4}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

$$a) \quad f(a) = f(b)$$

$$\frac{3a-2}{a+4} = \frac{3b-2}{b+4}$$

$$3ab + 12a - 2b - 8 = 3ab + 12b - 2a - 8$$

$$14a = 14b$$

$$\underline{a=b} \quad \checkmark$$

$\therefore f(x)$ is one-to-one function.

$$b) \quad y = \frac{3x-2}{x+4}$$

$$x = \frac{3y-2}{y+4}$$

$$xy + 4x = 3y - 2$$

$$(x-3)y = -4x-2$$

$$\underline{f^{-1}(x) = \frac{-4x-2}{x-3} \quad |}$$

$$c) \quad \text{Domain of } f^{-1}(x) = \text{Range of } f(x) : \quad \underline{\mathbb{R} - \{-4\} \quad |}$$

$$\text{Range of } f^{-1}(x) = \text{Domain of } f(x) : \quad \underline{\mathbb{R} - \{3\} \quad |}$$

Exercise

For the given function $f(x) = \frac{-3x-2}{x+4}$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\frac{-3a-2}{a+4} = \frac{-3b-2}{b+4}$$

$$-3ab - 12a - 2b - 8 = -3ab - 12b - 2a - 8$$

$$-10a = -10b$$

$$\underline{a = b} \quad \checkmark$$

$\therefore f(x)$ is one-to-one function.

b) $y = \frac{-3x-2}{x+4}$

$$x = \frac{-3y-2}{y+4}$$

$$xy + 4x = -3y - 2$$

$$(x+3)y = -4x - 2$$

$$\underline{f^{-1}(x) = \frac{-4x-2}{x+3}}$$

c) Domain of $f^{-1}(x)$ = Range of $f(x)$: $\underline{\mathbb{R} - \{-4\}}$

Range of $f^{-1}(x)$ = Domain of $f(x)$: $\underline{\mathbb{R} - \{-3\}}$

Exercise

For the given function $f(x) = \sqrt{x-1} \quad x \geq 1$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\sqrt{a-1} = \sqrt{b-1}$$

$$\left(\sqrt{a-1}\right)^2 = \left(\sqrt{b-1}\right)^2$$

$$a-1=b-1$$

$$\underline{a=b} \quad \checkmark$$

$\therefore f(x)$ is one-to-one function.

b) $y = \sqrt{x-1}$

$$x = \sqrt{y-1}$$

$$x^2 = y-1$$

$$y = x^2 + 1$$

$$\underline{f^{-1}(x) = x^2 + 1 \quad x \geq 0}$$

c) Domain of $f(x)$ = Range of $f^{-1}(x)$: $\underline{[1, \infty)}$

Range of $f(x)$ = Domain of $f^{-1}(x)$: $\underline{[0, \infty)}$

Exercise

For the given function $f(x) = \sqrt{2-x} \quad x \leq 2$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\sqrt{2-a} = \sqrt{2-b}$$

$$\left(\sqrt{2-a}\right)^2 = \left(\sqrt{2-b}\right)^2$$

$$2-a=2-b$$

$$\underline{a=b} \quad \checkmark$$

$\therefore f(x)$ is one-to-one function.

b) $y = \sqrt{2-x}$

$$x = \sqrt{2-y}$$

$$x^2 = 2-y$$

$$y = 2-x^2$$

$$\underline{f^{-1}(x) = 2-x^2 \quad x \geq 0}$$

c) Domain of $f(x) = \text{Range of } f^{-1}(x): \underline{(-\infty, 2]}$

Range of $f(x) = \text{Domain of } f^{-1}(x): \underline{[0, \infty)}$

Exercise

For the given function $f(x) = x^2 + 4x \quad x \geq -2$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

$$\begin{aligned} x_{\text{vertex}} &= -\frac{4}{2} \\ &= -2 \end{aligned}$$

$$\begin{aligned} f(-2) &= 4 - 8 \\ &= -4 \end{aligned}$$

$$\text{Vertex} = (-2, -4)$$

a) Since, $f(x)$ is a restricted function with $x \geq -2$.

$x = -2$ is the line symmetry, therefore; $f(x)$ is one-to-one function.

b) $y = x^2 + 4x$

$$x = y^2 + 4y$$

$$y^2 + 4y - x = 0$$

$$y = \frac{-4 \pm \sqrt{16 + 4x}}{2}$$

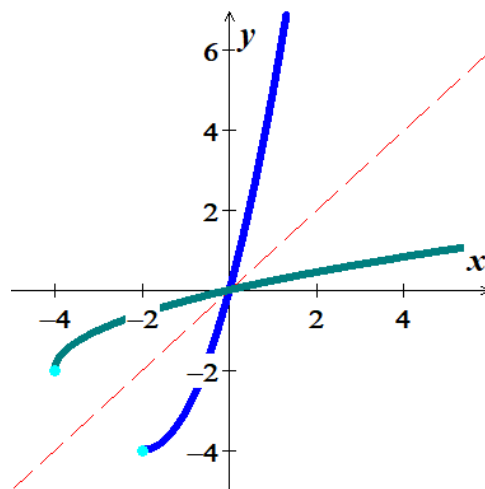
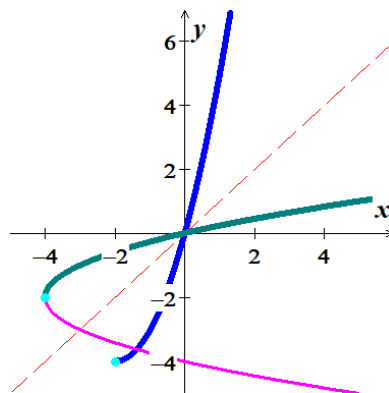
$$= \frac{-4 \pm 2\sqrt{4 + x}}{2}$$

$$= -2 + \sqrt{x + 4}$$

$$\underline{f^{-1}(x) = -2 + \sqrt{x + 4} \quad x \geq 0}$$

c) Domain of $f(x) = \text{Range of } f^{-1}(x): \underline{[-2, \infty)}$

Range of $f(x) = \text{Domain of } f^{-1}(x): \underline{[-4, \infty)}$



Exercise

For the given function $f(x) = 3x + 5$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$3a + 5 = 3b + 5$$

$$3a = 3b$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

b) $y = 3x + 5$

$$x = 3y + 5$$

$$x - 5 = 3y$$

$$\frac{x-5}{3} = y$$

$$\boxed{f^{-1}(x) = \frac{x-5}{3}}$$

Interchange x and y

Solve for y

c) Domain of $f^{-1} = \text{Range of } f: \mathbb{R}$

Range of $f^{-1} = \text{Domain of } f: \mathbb{R}$

Exercise

For the given function $f(x) = \frac{1}{3x-2}$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\frac{1}{3a-2} = \frac{1}{3b-2}$$

$$3b-2 = 3a-2$$

$$3b = 3a$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

$$b) \quad y = \frac{1}{3x-2}$$

$$x = \frac{1}{3y-2}$$

Interchange x and y

$$x(3y-2) = 1$$

Solve for y

$$3xy - 2x = 1$$

$$3xy = 1 + 2x$$

$$\underline{f^{-1}(x) = \frac{1+2x}{3x}}$$

$$c) \quad \text{Domain of } f^{-1} = \text{Range of } f: \underline{\mathbb{R} - \left\{\frac{2}{3}\right\}}$$

$$\text{Range of } f^{-1} = \text{Domain of } f: \underline{\mathbb{R} - \{0\}}$$

Exercise

For the given function $f(x) = \frac{3x+2}{2x-5}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

$$a) \quad f(a) = f(b)$$

$$\frac{3a+2}{2a-5} = \frac{3b+2}{2b-5}$$

$$6ab - 15a + 4b - 10 = 6ab - 15b + 4a - 10$$

$$19a = 19b$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

$$b) \quad y = \frac{3x+2}{2x-5}$$

$$x = \frac{3y+2}{2y-5}$$

Interchange x and y

$$2xy - 5x = 2y + 2$$

Solve for y

$$(2x-3)y = 5x+2$$

$$\underline{f^{-1}(x) = \frac{5x+2}{2x-3}}$$

$$c) \quad \text{Domain of } f^{-1} = \text{Range of } f: \underline{\mathbb{R} - \left\{\frac{5}{2}\right\}}$$

$$\text{Range of } f^{-1} = \text{Domain of } f: \underline{\mathbb{R} - \left\{\frac{3}{2}\right\}}$$

Exercise

For the given function $f(x) = \frac{4x}{x-2}$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\frac{4a}{a-2} = \frac{4b}{b-2}$$

$$4ab - 8a = 4ab - 8b$$

$$-8a = -8b$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

b) $y = \frac{4x}{x-2}$

$$x = \frac{4y}{y-2}$$

$$xy - 2x = 4y$$

$$(x-4)y = 2x$$

$$\underline{f^{-1}(x) = \frac{2x}{x-4}}$$

c) Domain of $f^{-1} = \text{Range of } f: \underline{\mathbb{R} - \{2\}}$

Range of $f^{-1} = \text{Domain of } f: \underline{\mathbb{R} - \{4\}}$

Exercise

For the given function $f(x) = 2 - 3x^2; \quad x \leq 0$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$2 - 3a^2 = 2 - 3b^2$$

$$-3a^2 = -3b^2$$

$$a^2 = b^2$$

$$a = b \quad \text{since } x \leq 0$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

b) $y = 2 - 3x^2$

$$x = 2 - 3y^2$$

$$3y^2 = 2 - x$$

$$y^2 = \frac{2-x}{3}$$

$$\underline{f^{-1}(x) = -\sqrt{\frac{2-x}{3}}} \quad \text{Since } x < 0$$

c) Domain of f^{-1} = Range of f : \mathbb{R}

Range of f^{-1} = Domain of f : \mathbb{R}

Exercise

For the given function $f(x) = 2x^3 - 5$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$2a^3 - 5 = 2b^3 - 5$$

$$a^3 = b^3$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

b) $y = 2x^3 - 5$

$$y + 5 = 2x^3$$

$$\frac{y+5}{2} = x^3$$

$$x = \sqrt[3]{\frac{y+5}{2}}$$

$$\underline{f^{-1}(x) = \sqrt[3]{\frac{x+5}{2}}}$$

c) Domain of f^{-1} = Range of f : \mathbb{R}

Range of f^{-1} = Domain of f : \mathbb{R}

Exercise

For the given function $f(x) = \sqrt{3-x}$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$(\sqrt{3-a})^2 = (\sqrt{3-b})^2$$

$$3-a = 3-b$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

b) $y = \sqrt{3-x} \quad y \geq 0$

$$y = \sqrt{3-x}$$

$$y^2 = 3-x$$

$$x = 3 - y^2 \quad x \geq 0$$

$$\underline{f^{-1}(x) = 3 - x^2}$$

c) Domain of f^{-1} = Range of f : $\underline{(-\infty, 3]}$

Range of f^{-1} = Domain of f : $\underline{[0, \infty)}$

Exercise

For the given function $f(x) = \sqrt[3]{x} + 1$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\sqrt[3]{a} + 1 = \sqrt[3]{b} + 1$$

$$(\sqrt[3]{a})^3 = (\sqrt[3]{b})^3$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

b) $y = \sqrt[3]{x} + 1$

$$y = \sqrt[3]{x} + 1$$

$$y - 1 = \sqrt[3]{x}$$

$$(y - 1)^3 = x$$

$$\underline{f^{-1}(x) = (x - 1)^3}$$

c) Domain of f^{-1} = Range of f : \mathbb{R}

Range of f^{-1} = Domain of f : \mathbb{R}

Exercise

For the given function $f(x) = (x^3 + 1)^5$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$(a^3 + 1)^5 = (b^3 + 1)^5$$

$$a^3 + 1 = b^3 + 1$$

$$a^3 = b^3$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

b) $y = (x^3 + 1)^5$

$$y = (x^3 + 1)^5$$

$$\sqrt[5]{y} = x^3 + 1$$

$$\sqrt[5]{y} - 1 = x^3$$

$$x = \sqrt[3]{\sqrt[5]{y} - 1}$$

$$\underline{f^{-1}(x) = \sqrt[3]{\sqrt[5]{x} - 1}}$$

c) Domain of f^{-1} = Range of f : \mathbb{R}

Range of f^{-1} = Domain of f : \mathbb{R}

Exercise

For the given function $f(x) = x^2 - 6x$; $x \geq 3$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$a^2 - 6a = b^2 - 6b$$

$$a^2 - b^2 = 6a - 6b$$

$$(a - b)(a + b) = 6(a - b)$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

b) $y = x^2 - 6x$

$$x^2 - 6x - y = 0$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-y)}}{2(1)}$$

$$= \frac{6 \pm 4\sqrt{9+y}}{2}$$

$$= 3 \pm \sqrt{9+y}$$

Since $x \geq 3 \Rightarrow$ we can select $x = 3 + \sqrt{y+9}$

$$\therefore f^{-1}(x) = 3 + \sqrt{x+9}$$

c) Domain of f^{-1} = Range of f : $\mathbb{R} : \geq 3$

Range of f^{-1} = Domain of f : ≥ -9

Exercise

For the given function $f(x) = (x - 2)^3$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$(a - 2)^3 = (b - 2)^3$$

$$a - 2 = b - 2$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

b) $y = (x-2)^3$

$$x = (y-2)^3$$

$$x^{1/3} = \left[(y-2)^3 \right]^{1/3}$$

$$x^{1/3} = y-2$$

$$\sqrt[3]{x} + 2 = y$$

$$\therefore \underline{f^{-1}(x) = \sqrt[3]{x} + 2}$$

c) Domain of f^{-1} = Range of f : \mathbb{R}

Range of f^{-1} = Domain of f : \mathbb{R}

Exercise

For the given function $f(x) = \frac{x+1}{x-3}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\frac{a+1}{a-3} = \frac{b+1}{b-3}$$

$$ab - 3a + b - 3 = ab - 3b + a - 3$$

$$-4a = -4b$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

b) $y = \frac{x+1}{x-3}$

$$x = \frac{y+1}{y-3}$$

$$x(y-3) = y+1$$

$$xy - 3x = y+1$$

$$xy - y = 3x+1$$

$$y(x-1) = 3x+1$$

$$y = \underline{\frac{3x+1}{x-1} = f^{-1}(x)}$$

c) Domain of $f^{-1} = \text{Range of } f: \underline{\mathbb{R} - \{3\}}$

Range of $f^{-1} = \text{Domain of } f: \underline{\mathbb{R} - \{1\}}$

Exercise

For the given function $f(x) = \frac{2x+1}{x-3}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\frac{2a+1}{a-3} = \frac{2b+1}{b-3}$$

$$2ab - 6a + b - 3 = 2ab - 6b + a - 3$$

$$-7a = -7b$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

b) $y = \frac{2x+1}{x-3}$

$$x = \frac{2y+1}{y-3}$$

$$xy - 3x = 2y + 1$$

$$y(x-2) = 3x+1$$

$$y = \underline{\frac{3x+1}{x-2} = f^{-1}(x)}$$

c) Domain of $f^{-1} = \text{Range of } f: \underline{\mathbb{R} - \{3\}}$

Range of $f^{-1} = \text{Domain of } f: \underline{\mathbb{R} - \{2\}}$

Exercise

The function $w(x) = 2x + 24$ can be used to convert a U.S. women's shoe size into an Italian women's shoe size. Determine the function $w^{-1}(x)$ that can use to convert an Italian women's shoe size to its equivalent U.S. shoe size.

Solution

$$x = 2w^{-1}(x) + 24$$

$$2w^{-1}(x) = x - 24$$

$$\underline{w^{-1}(x) = \frac{1}{2}x - 12}$$



Exercise

The function $m(x) = 1.3x - 4.7$ can be used to convert a U.S. men's shoe size into an U.K. women's shoe size. Determine the function $m^{-1}(x)$ that can used to convert an U.K. men's shoe size to its equivalent U.S. shoe size.

Solution

$$x = 1.3m^{-1}(x) - 4.7$$

$$1.3m^{-1}(x) = x + 4.7$$

$$\frac{13}{10}m^{-1}(x) = x + \frac{47}{10}$$

$$\underline{m^{-1}(x) = \frac{10}{13}x + \frac{47}{13}}$$

$$\underline{w^{-1}(x) = \frac{1}{2}x - 12}$$

Exercise

A catering service use the function $c(x) = \frac{300 + 12x}{x}$ to determine the amount, in *dollars*, it charges per person for a sit-down dinner, where x is the number of people in attendance.

a) Find $c(30)$ and explain what it represents

b) Find $c^{-1}(x)$

c) Use $c^{-1}(x)$ to determine how many people attended a dinner for which the cost per person was \$15.00

Solution

$$\begin{aligned}
 a) \quad c(30) &= \frac{300 + 12(30)}{30} \\
 &= \frac{30 + 36}{3} \\
 &= \frac{66}{3} \\
 &= \$22
 \end{aligned}$$

Catering service will charge \$22 per person to a sit-down dinner.

$$\begin{aligned}
 b) \quad cx &= 300 + 12x \\
 (c - 12)x &= 300 \\
 c^{-1}(x) &= \frac{300}{x - 12}
 \end{aligned}$$

$$\begin{aligned}
 c) \quad c^{-1}(15) &= \frac{300}{15 - 12} \\
 &= \frac{300}{3} \\
 &= 100
 \end{aligned}$$

Exercise

A landscaping service use the function $c(x) = \frac{600 + 140x}{x}$ to determine the amount, in *dollars*, it charges per tree to deliver, where x is the number of trees.

- Find $c(5)$ and explain what it represents
- Find $c^{-1}(x)$
- Use $c^{-1}(x)$ to determine how many trees were delivered for which the cost per tree was \$160.00

Solution

$$\begin{aligned}
 d) \quad c(5) &= \frac{600 + 140(5)}{5} \\
 &= \frac{1,300}{5} \\
 &= \$260
 \end{aligned}$$

Landscaping service will charge \$260 per tree to deliver.

$$\begin{aligned}
 e) \quad y &= \frac{600 + 140x}{x} \\
 x &= \frac{600 + 140y}{y} \\
 xy &= 600 + 140y \\
 (x - 140)y &= 600
 \end{aligned}$$

$$\underline{c^{-1}(x) = \frac{600}{x-140} \quad |}$$

$$\begin{aligned} f) \quad c^{-1}(160) &= \frac{600}{160-140} \\ &= \frac{600}{20} \\ &= 30 \quad | \end{aligned}$$

Solution **Section 3.2 - Exponential Functions**

Exercise

Evaluate to four decimal places using a calculator $2^{3.4}$

Solution

$$2^{3.4} = 10.5561$$

Exercise

Evaluate to four decimal places using a calculator $5^{\sqrt{3}}$

Solution

$$5^{\sqrt{3}} = 16.2425$$

Exercise

Evaluate to four decimal places using a calculator $6^{-1.2}$

Solution

$$6^{-1.2} = 0.1165$$

Exercise

Evaluate to four decimal places using a calculator: $e^{-0.75}$

Solution

$$e^{-0.75} = .4724$$

Exercise

Evaluate to four decimal places using a calculator: $e^{2.3}$

Solution

$$e^{2.3} = 9.9742$$

Exercise

Evaluate to four decimal places using a calculator: $e^{-0.95}$

Solution

$$\underline{e^{-0.95} = 0.3867}$$

Exercise

Evaluate to four decimal places using a calculator: $\pi^{\sqrt{\pi}}$

Solution

$$\underline{\pi^{\sqrt{\pi}} = 7.6063}$$

Exercise

Evaluate to four decimal places using a calculator: $e^{\sqrt{2}}$

Solution

$$\underline{e^{\sqrt{2}} = 4.1133}$$

Exercise

Sketch the graph: $f(x) = 2^x + 3$

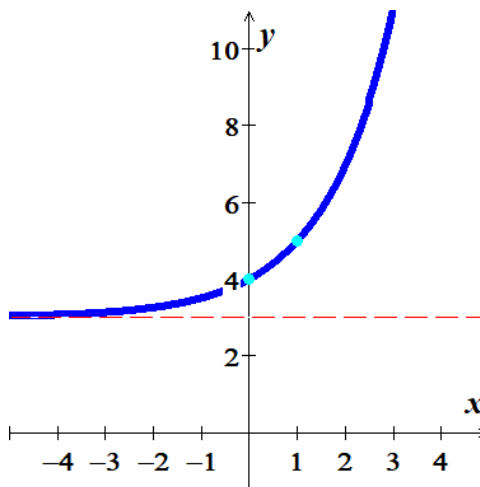
Solution

Asymptote: $y = 3$

Domain: $(-\infty, \infty)$

Range: $(3, \infty)$

x	$f(x)$
-1	3.5
0	4
1	5
2	7



Exercise

Sketch the graph: $f(x) = 2^{3-x}$

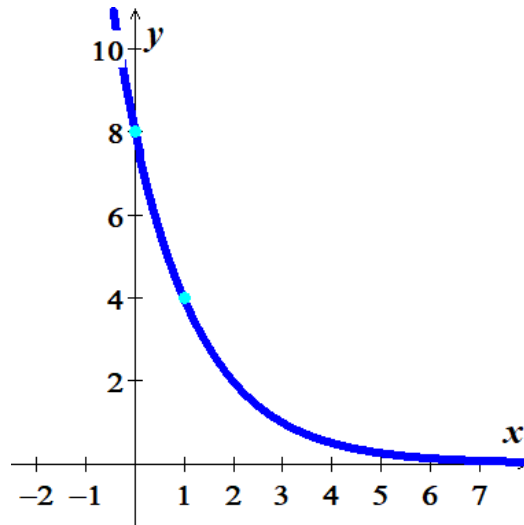
Solution

Asymptote: $y = 0$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

x	$f(x)$
1	4
2	2
0	8



Exercise

Sketch the graph: $f(x) = \left(\frac{2}{5}\right)^{-x}$

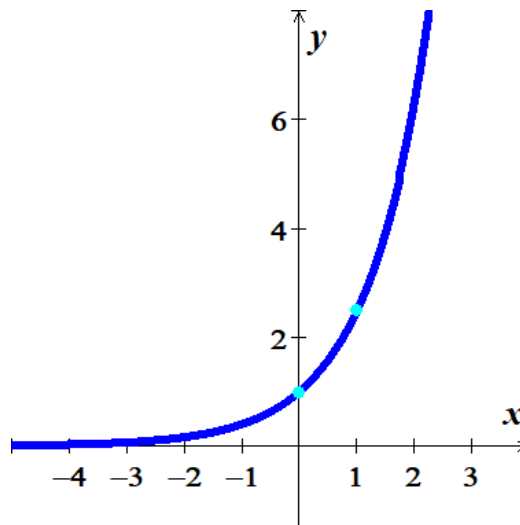
Solution

Asymptote: $y = 0$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

x	$f(x)$
-1	0.4
0	1
1	2.5



Exercise

Sketch the graph: $f(x) = -\left(\frac{1}{2}\right)^x + 4$

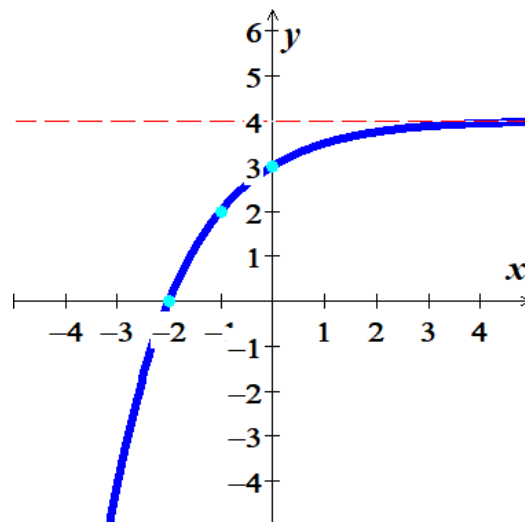
Solution

Asymptote: $y = 4$

Domain: $(-\infty, \infty)$

Range: $(-\infty, 4)$

x	$f(x)$
-2	0
-1	2
0	3



Exercise

Sketch the graph of $f(x) = 4^x$

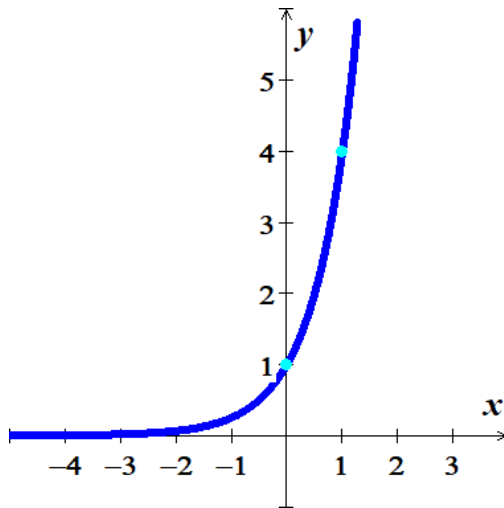
Solution

Asymptote: $y = 0$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

x	$f(x)$
0	1
1	4



Exercise

Sketch the graph of $f(x) = 2 - 4^x$

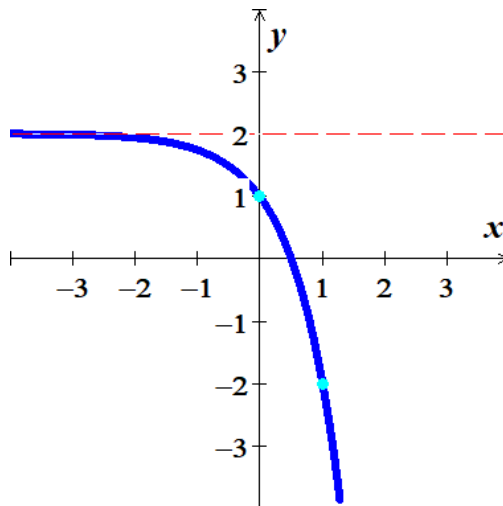
Solution

Asymptote: $y = 2$

Domain: $(-\infty, \infty)$

Range: $(-\infty, 2)$

x	$f(x)$
0	1
1	-2



Exercise

Sketch the graph of $f(x) = -3 + 4^{x-1}$

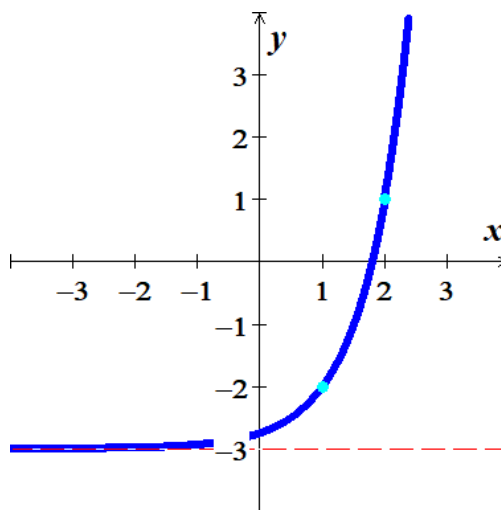
Solution

Asymptote: $y = -3$

Domain: $(-\infty, \infty)$

Range: $(-3, \infty)$

x	$f(x)$
1	-2
2	1



Exercise

Sketch the graph of $f(x) = 1 + \left(\frac{1}{4}\right)^{x+1}$

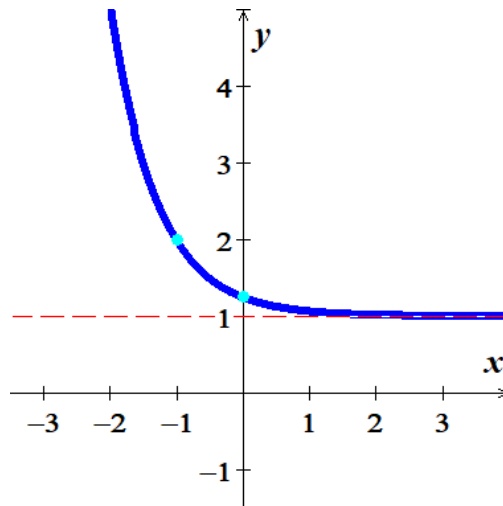
Solution

Asymptote: $y = 1$

Domain: $(-\infty, \infty)$

Range: $(1, \infty)$

x	$f(x)$
-1	2
0	$\frac{5}{4}$



Exercise

Sketch the graph of $f(x) = e^{x-2}$

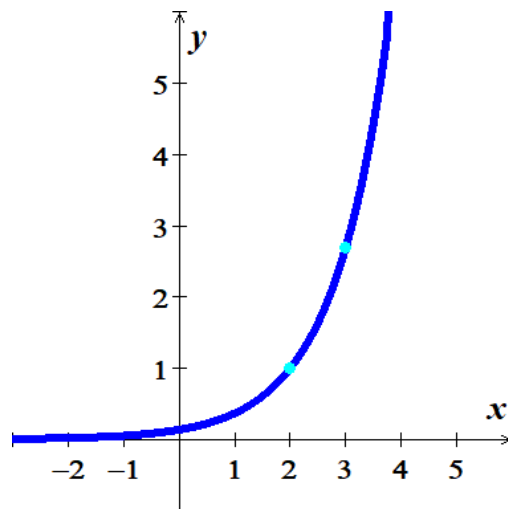
Solution

Asymptote: $y = 0$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

x	$f(x)$
2	1
3	2.7



Exercise

Sketch the graph of $f(x) = 3 - e^{x-2}$

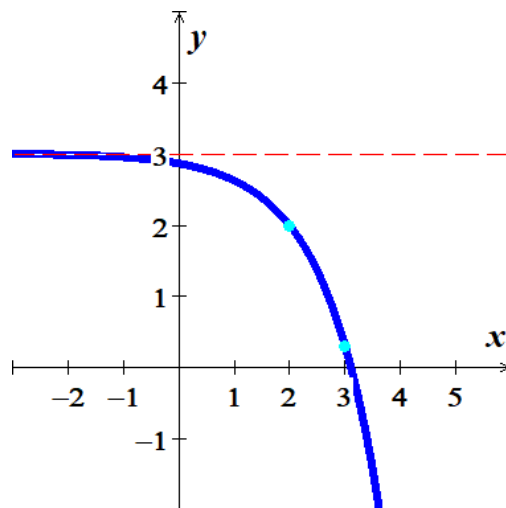
Solution

Asymptote: $y = 3$

Domain: $(-\infty, \infty)$

Range: $(-\infty, 3)$

x	$f(x)$
2	2
3	.3



Exercise

Sketch the graph of $f(x) = e^{x+4}$

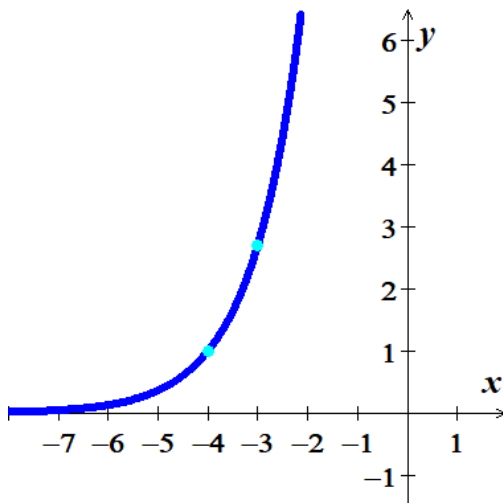
Solution

Asymptote: $y = 0$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

x	$f(x)$
-4	1
-3	2.7



Exercise

Sketch the graph of $f(x) = 2 + e^{x-1}$

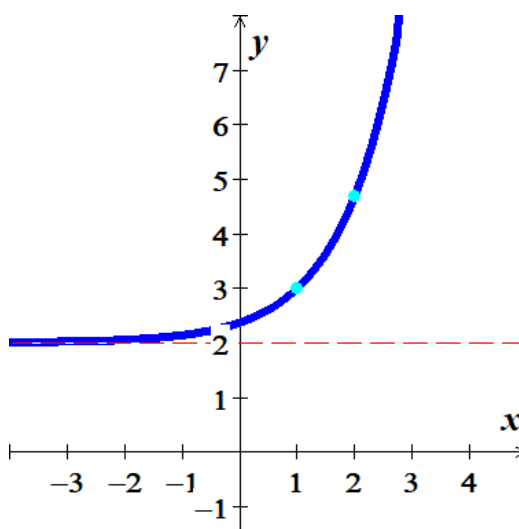
Solution

Asymptote: $y = 2$

Domain: $(-\infty, \infty)$

Range: $(2, \infty)$

x	$f(x)$
1	3
2	4.7



Exercise

The exponential function $f(x) = 1066e^{0.042x}$ models the gray wolf population of the Western Great Lakes, $f(x)$, in *billions*, x years after 1978. Project the gray population in the recovery area in 2012.

Solution

$$x = 2012 - 1978 = 34$$

$$\begin{aligned} f(x = 34) &= 1066e^{0.042(34)} \\ &= 4445.6 \\ &\approx 4446 \text{ billions} \end{aligned}$$

Exercise

The function $f(x) = 6.4e^{0.0123x}$ describes world population, $f(x)$, in billions, x years after 2004 subject to a growth rate of 1.23% annually. Use the function to predict world population in 2050.

Solution

$$x = 2050 - 2004 = 46$$

$$f(x = 46) = 6.4e^{0.0123(46)} \\ \approx 11.27 \text{ billion}$$

Exercise

A cup of coffee is heated to $160^\circ F$ and placed in a room that maintains a temperature of $70^\circ F$. The temperature T of the coffee, in degree Fahrenheit, after t minutes is given by

$$T(t) = 70 + 90e^{-0.0485t}$$

- a) Find the temperature of the coffee 20 minutes after it is placed in the room
- b) Determine when the temperature of the coffee will reach $90^\circ F$

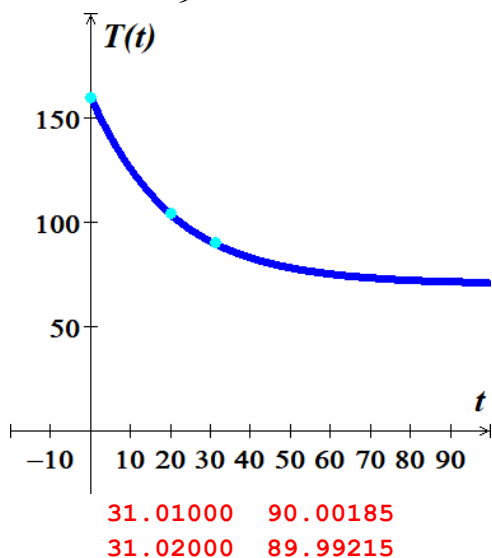
Solution

$$a) \quad T(20) = 70 + 90e^{-0.0485(20)} \\ \approx 104^\circ F$$

$$b) \quad T(t) = 70 + 90e^{-0.0485t} = 90$$

$$90e^{-0.0485t} = 20$$

$$e^{-0.0485t} = \frac{2}{9}$$



\therefore The temperature of the coffee will reach $90^\circ F$ in about 31.01 minutes.

Exercise

A cup of coffee is heated to $180^{\circ}F$ and placed in a room that maintains a temperature of $65^{\circ}F$. The temperature T of the coffee, in *degree Fahrenheit*, after t minutes is given by

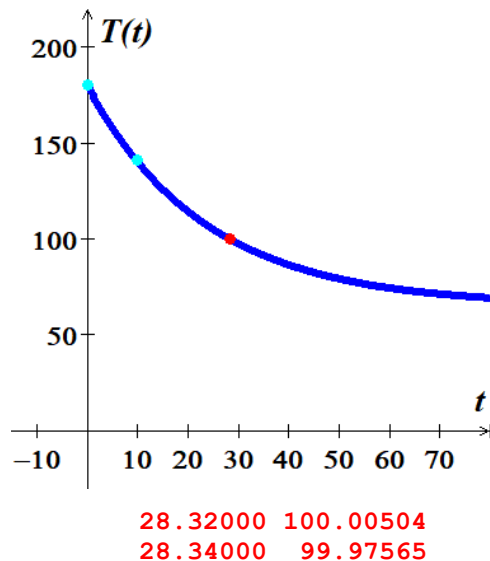
$$T(t) = 65 + 115e^{-0.042t}$$

- a) Find the temperature of the coffee 10 *minutes* after it is placed in the room
- b) Determine when the temperature of the coffee will reach $100^{\circ}F$

Solution

a) $T(10) = 65 + 115e^{-0.042(10)}$
 $\approx 141^{\circ}F$

b) $T(t) = 65 + 115e^{-0.042t} = 100$
 $115e^{-0.042t} = 35$
 $e^{-0.042t} = \frac{7}{23}$



\therefore The temperature of the coffee will reach $100^{\circ}F$ in about **31.01** *minutes*.

Exercise

The percent $I(x)$ of the original intensity of light striking the surface of a lake that is available x *feet* below the surface of the lake is given by the equation

$$I(x) = 100e^{-.95x}$$

- a) What percentage of the light is available 2 *feet* below the surface of the lake?
- b) At what depth is the intensity of the light one-half the intensity at the surface?

Solution

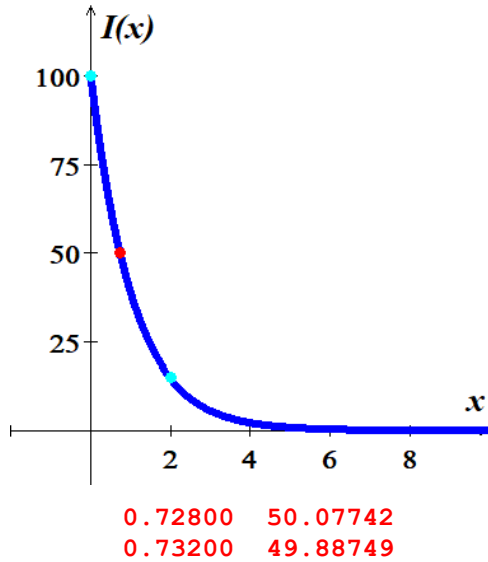
a) $I(2) = 100e^{-.95(2)}$

$$\approx 14.96$$

\therefore The percentage of the light is available 2 feet below the surface of the lake is **15%**

$$b) I(x) = 100e^{-.95x} = \frac{1}{2}(100)$$

$$e^{-.95x} = \frac{1}{2}$$

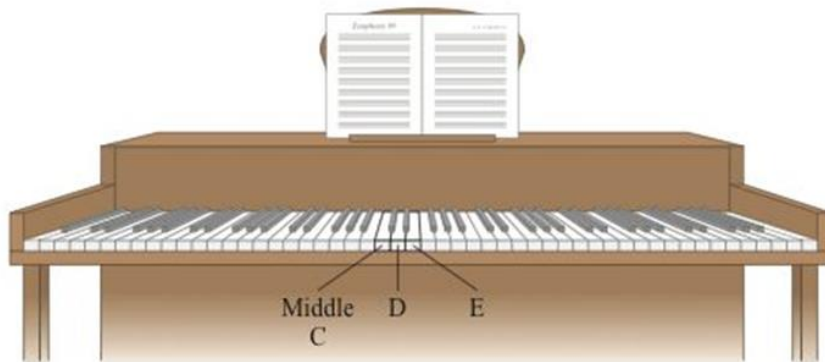


\therefore The depth is **0.73 feet** when the intensity of the light one-half the intensity at the surface

Exercise

Starting on the left side of a standard 88-key piano, the frequency, in vibrations per second, of the n th note is given by

$$f(n) = (2.75)^{\frac{n-1}{12}}$$



- Determine the frequency of middle C, key number 40 on an 88-key piano.
- Is the difference in frequency between middle C (key number 40) and D (key number 42) the same as the difference in frequency between D (key number 42) and E (key number 44)?

Solution

$$a) f(40) = (2.75)^{\frac{40-1}{12}}$$

$$\approx 26.16 \mid$$

the frequency of middle *C* is ≈ 26 vibrations per second.

$$\begin{aligned} b) \quad f(42) &= (2.75) 2^{(41/12)} \\ &\approx 29.37 \mid \end{aligned}$$

The difference between the frequency of middle *C* and *D* is:

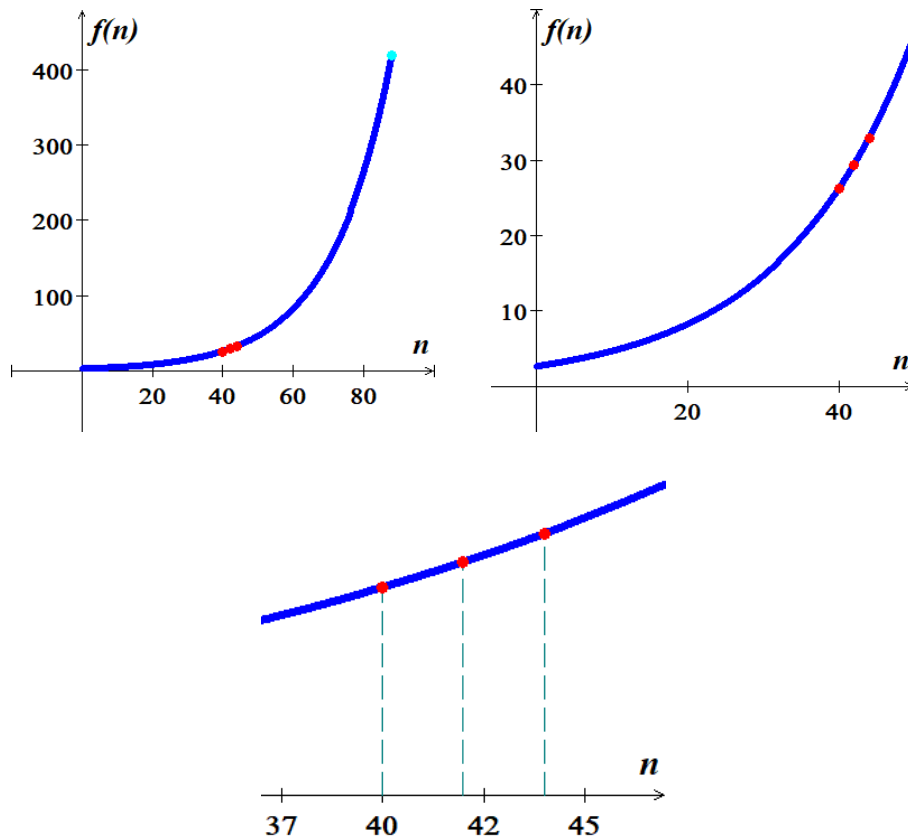
$$29.37 - 26.16 \approx 3.21$$

$$\begin{aligned} f(44) &= (2.75) 2^{(43/12)} \\ &\approx 32.96 \mid \end{aligned}$$

The difference between the frequency of middle *D* and *E* is:

$$32.96 - 29.37 \approx 3.59$$

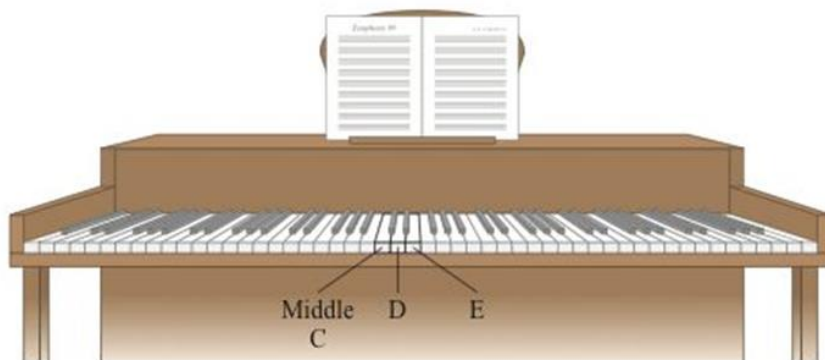
\therefore the differences are **not** the same since the function is *not* linear function



Exercise

Starting on the left side of a standard 88-key piano, the frequency, in *vibrations per second*, of the n th note is given by

$$f(n) = (27.5) 2^{\frac{n-1}{12}}$$



- c) Determine the frequency of middle C, key number 40 on an 88-key piano.
- d) Is the difference in frequency between middle C (key number 40) and D (key number 42) the same as the difference in frequency between D (key number 42) and E (key number 44)?

Solution

$$\begin{aligned} \text{c) } f(40) &= (27.5) 2^{\frac{40-1}{12}} \\ &\approx 261.63 \end{aligned}$$

the frequency of middle C is ≈ 262 vibrations per second.

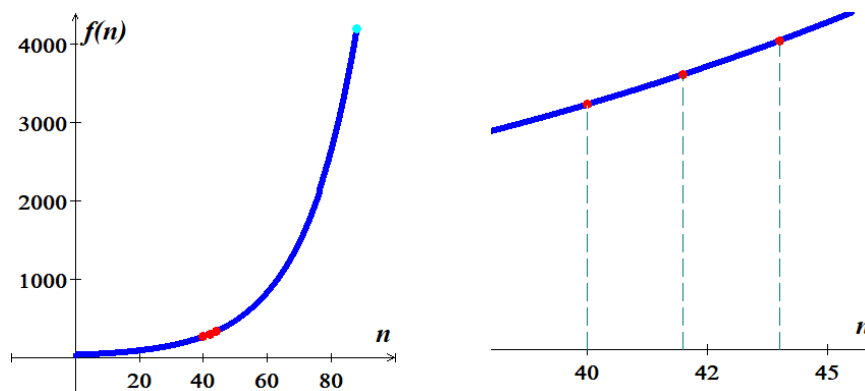
$$\begin{aligned} \text{d) } f(42) &= (27.5) 2^{(41/12)} \\ &\approx 293.66 \end{aligned}$$

The difference between the frequency of middle C and D is: $293.66 - 261.66 \approx 32$

$$\begin{aligned} f(44) &= (27.5) 2^{(43/12)} \\ &\approx 329.63 \end{aligned}$$

The difference between the frequency of middle D and E is: $329.63 - 293.66 \approx 36$

\therefore The differences are **not** the same since the function is *not* linear function.



Solution

Section 3.3 – Logarithmic Functions

Exercise

Write the equation in its equivalent logarithmic form $2^6 = 64$

Solution

$$\underline{6 = \log_2 64}$$

Exercise

Write the equation in its equivalent logarithmic form $5^4 = 625$

Solution

$$\underline{4 = \log_5 625}$$

Exercise

Write the equation in its equivalent logarithmic form $5^{-3} = \frac{1}{125}$

Solution

$$\underline{-3 = \log_5 \frac{1}{125}}$$

Exercise

Write the equation in its equivalent logarithmic form $\sqrt[3]{64} = 4$

Solution

$$64^{1/3} = 4$$

$$\underline{\log_{64} = \frac{1}{3}}$$

Exercise

Write the equation in its equivalent logarithmic form $b^3 = 343$

Solution

$$\underline{\log_b 343 = 3}$$

Exercise

Write the equation in its equivalent logarithmic form $8^y = 300$

Solution

$$\log_8 300 = y$$

Exercise

Write the equation in its equivalent logarithmic form: $\sqrt[n]{x} = y$

Solution

$$(x)^{1/n} = y$$

$$\log_x (y) = \frac{1}{n}$$

Exercise

Write the equation in its equivalent logarithmic form: $\left(\frac{2}{3}\right)^{-3} = \frac{27}{8}$

Solution

$$\log_{\frac{2}{3}} \left(\frac{27}{8}\right) = -3$$

Exercise

Write the equation in its equivalent logarithmic form: $\left(\frac{1}{2}\right)^{-5} = 32$

Solution

$$\log_{\frac{1}{2}} (32) = -5$$

Exercise

Write the equation in its equivalent logarithmic form: $e^{x-2} = 2y$

Solution

$$x - 2 = \ln |2y|$$

Exercise

Write the equation in its equivalent logarithmic form: $e = 3x$

Solution

$$\boxed{1 = \ln |3x|}$$

Exercise

Write the equation in its equivalent logarithmic form: $\sqrt[3]{e^{2x}} = y$

Solution

$$e^{2x/3} = y$$

$$\boxed{\frac{2x}{3} = \ln |y|}$$

Exercise

Write the equation in its equivalent exponential form $\log_5 125 = y$

Solution

$$\boxed{5^y = 125}$$

Exercise

Write the equation in its equivalent exponential form $\log_4 16 = x$

Solution

$$\boxed{16 = 4^x}$$

Exercise

Write the equation in its equivalent exponential form $\log_5 \frac{1}{5} = x$

Solution

$$\boxed{\frac{1}{5} = 5^x}$$

Exercise

Write the equation in its equivalent exponential form $\log_2 \frac{1}{8} = x$

Solution

$$\underline{\frac{1}{8} = 2^x}$$

Exercise

Write the equation in its equivalent exponential form $\log_6 \sqrt{6} = x$

Solution

$$\underline{\sqrt{6} = 6^x}$$

Exercise

Write the equation in its equivalent exponential form $\log_3 \frac{1}{\sqrt{3}} = x$

Solution

$$\underline{3^{-1/2} = 3^x}$$

Exercise

Write the equation in its equivalent exponential form: $6 = \log_2 64$

Solution

$$6 = \log_2 64 \Leftrightarrow \underline{2^6 = 64}$$

Exercise

Write the equation in its equivalent exponential form: $2 = \log_9 x$

Solution

$$2 = \log_9 x \Leftrightarrow \underline{x = 2^9}$$

Exercise

Write the equation in its equivalent exponential form: $\log_{\sqrt{3}} 81 = 8$

Solution

$$\log_{\sqrt{3}} 81 = 8 \Leftrightarrow \underline{81 = (\sqrt{3})^8}$$

Exercise

Write the equation in its equivalent exponential form: $\log_4 \frac{1}{64} = -3$

Solution

$$\log_4 \frac{1}{64} = -3 \Leftrightarrow \boxed{\frac{1}{64} = x^{-3}}$$

Exercise

Write the equation in its equivalent exponential form: $\log_4 26 = y$

Solution

$$\log_4 26 = y \Leftrightarrow \boxed{26 = 4^y}$$

Exercise

Write the equation in its equivalent exponential form: $\ln M = c$

Solution

$$\ln M = c \Leftrightarrow \boxed{M = e^c}$$

Exercise

Evaluate the expression without using a calculator: $\log_4 16$

Solution

$$\begin{aligned} \log_4 16 &= \log_4 4^2 \\ &= 2 \end{aligned} \qquad \log_b b^x = x$$

Exercise

Evaluate the expression without using a calculator: $\log_2 \frac{1}{8}$

Solution

$$\begin{aligned} \log_2 \frac{1}{8} &= \log_2 \frac{1}{2^3} \\ &= \log_2 2^{-3} \\ &= -3 \end{aligned} \qquad \log_b b^x = x$$

Exercise

Evaluate the expression without using a calculator: $\log_6 \sqrt{6}$

Solution

$$\begin{aligned}\log_6 \sqrt{6} &= \log_6 6^{1/2} \\ &= \frac{1}{2}\end{aligned}$$

Exercise

Evaluate the expression without using a calculator: $\log_3 \frac{1}{\sqrt{3}}$

Solution

$$\begin{aligned}\log_3 \frac{1}{\sqrt{3}} &= \log_3 3^{-1/2} \\ &= \log_3 3^{-1/2} \qquad \log_b b^x = x \\ &= -\frac{1}{2}\end{aligned}$$

Exercise

Evaluate the expression without using a calculator: $\log_3 \sqrt[7]{3}$

Solution

$$\begin{aligned}\log_3 3^{1/7} &= x \qquad \text{Converts to exponential} \\ 3^{1/7} &= 3^x \\ x &= \frac{1}{7} \\ \log_3 \sqrt[7]{3} &= \frac{1}{7}\end{aligned}$$

Exercise

Evaluate the expression without using a calculator: $\log_3 \sqrt{9}$

Solution

$$\begin{aligned}\log_3 \sqrt{9} &= \log_3 3 \qquad \log_b b^x = x \\ &= 1\end{aligned}$$

Exercise

Evaluate the expression without using a calculator: $\log_{\frac{1}{2}} \sqrt{\frac{1}{2}}$

Solution

$$\log_{\frac{1}{2}} \sqrt{\frac{1}{2}} = \log_{\frac{1}{2}} \left(\frac{1}{2}\right)^{\frac{1}{2}} \quad \log_b b^x = x$$

$$\underline{= \frac{1}{2}} \quad \Big|$$

Exercise

Simplify $\log_5 1$

Solution

$$\underline{\log_5 1 = 0} \quad \Big|$$

Exercise

Simplify $\log_7 7^2$

Solution

$$\underline{\log_7 7^2 = 2} \quad \Big|$$

Exercise

Simplify $3^{\log_3 8}$

Solution

$$\underline{3^{\log_3 8} = 8} \quad \Big|$$

Exercise

Simplify $10^{\log 3}$

Solution

$$\underline{10^{\log 3} = 3} \quad \Big|$$

Exercise

Simplify $e^{2+\ln 3}$

Solution

$$\begin{aligned} e^{2+\ln 3} &= e^2 e^{\ln 3} \\ &= 3e^2 \end{aligned}$$

Exercise

Simplify $\ln e^{-3}$

Solution

$$\ln e^{-3} = -3$$

Exercise

Simplify $\ln e^{x-5}$

Solution

$$\ln e^{x-5} = x-5$$

Exercise

Simplify $\log_b b^n$

Solution

$$\log_b b^n = n$$

Exercise

Simplify $\ln e^{x^2+3x}$

Solution

$$\ln e^{x^2+3x} = x^2 + 3x$$

Exercise

Find the domain of $f(x) = \log_5(x+4)$

Solution

Domain: $\underline{x > -4}$

Exercise

Find the domain of $f(x) = \log_5(x+6)$

Solution

Domain: $\underline{x > -6}$

Exercise

Find the domain of $f(x) = \log(2-x)$

Solution

Domain: $\underline{x < 2}$

Exercise

Find the domain of $f(x) = \log(7-x)$

Solution

Domain: $\underline{x < 7}$

Exercise

Find the domain of $f(x) = \ln(x-2)^2$

Solution

Domain: $\underline{\mathbb{R} - \{2\}}$
 $\underline{(-\infty, 2) \cup (2, \infty)}$

Exercise

Find the domain of $f(x) = \ln(x-7)^2$

Solution

Domain: $\underline{\mathbb{R} - \{7\}}$

$$\underline{(-\infty, 7) \cup (7, \infty)}$$

Exercise

Find the domain of $f(x) = \log(x^2 - 4x - 12)$

Solution

$$x^2 - 4x - 12 > 0$$

$$x = \frac{4 \pm \sqrt{16 + 48}}{2}$$

$$= \begin{cases} \frac{4-8}{2} = -2 \\ \frac{4+8}{2} = 6 \end{cases}$$

Domain: $\underline{x < -2 \quad x > 6}$

$$\underline{(-\infty, -2) \cup (6, \infty)}$$

Exercise

Find the domain of $f(x) = \log\left(\frac{x-2}{x+5}\right)$

Solution

$$\begin{cases} x \neq 2 \\ x \neq -5 \end{cases}$$

	-5	0	2	
+		-		+

Domain: $\underline{x < -5 \quad x > 2}$

$$\underline{(-\infty, -5) \cup (2, \infty)}$$

Exercise

Find the domain of $f(x) = \log\left(\frac{3-x}{x-2}\right)$

Solution

$$\begin{cases} x \neq 3 \\ x \neq 2 \end{cases}$$

	0	2	3	
-		+		-

Domain: $\underline{2 < x < 3}$

$$\underline{(2, 3)}$$

Exercise

Find the domain of $f(x) = \ln(x^2 - 9)$

Solution

$$x^2 - 9 > 0$$

$$\text{Domain: } \underline{x < -3 \quad x > 3}$$

Exercise

Find the domain of $f(x) = \ln\left(\frac{x^2}{x-4}\right)$

Solution

$$\frac{x^2}{x-4} > 0$$

$$x^2 \rightarrow \mathbb{R}$$

$$x > 4$$

$$\text{Domain: } \underline{x > 4}$$

Exercise

Find the domain of $f(x) = \log_3(x^3 - x)$

Solution

$$x^3 - x > 0$$

$$\underline{x = 0, 0, 1}$$

$$\text{Domain: } \underline{x > 1}$$

0,0	1	2
—	—	+

Exercise

Find the domain of $f(x) = \log \sqrt{2x-5}$

Solution

$$2x - 5 > 0$$

$$\text{Domain: } \underline{x > \frac{5}{2}}$$

Exercise

Find the domain of $f(x) = 3 \ln(5x - 6)$

Solution

$$5x - 6 > 0$$

$$\text{Domain: } \underline{x > \frac{6}{5}} \mid$$

Exercise

Find the domain of $f(x) = \log\left(\frac{x}{x-2}\right)$

Solution

$$\frac{x}{x-2} > 0$$

$$\underline{x = 0, 2} \mid$$

$$\text{Domain: } \underline{x < 0 \quad x > 2} \mid$$

Exercise

Find the domain of $f(x) = \log(4 - x^2)$

Solution

$$4 - x^2 > 0$$

$$4 - x^2 = 0 \rightarrow x = \pm 2$$

$$\text{Domain: } \underline{-2 < x < 2} \mid$$

Exercise

Find the domain of $f(x) = \ln(x^2 + 4)$

Solution

$$x^2 + 4 \text{ always positive.}$$

$$\text{Domain: } \underline{\mathbb{R}} \mid$$

Exercise

Find the domain of $f(x) = \ln|4x - 8|$

Solution

$$4x - 8 = 0 \rightarrow x = 2$$

$$\text{Domain: } \underline{\mathbb{R} - \{2\}}$$

Exercise

Find the domain of $f(x) = \ln|5 - x|$

Solution

$$5 - x = 0 \rightarrow x = 5$$

$$\text{Domain: } \underline{\mathbb{R} - \{5\}}$$

Exercise

Find the domain of $f(x) = \ln(x - 4)^2$

Solution

$$x - 4 = 0 \rightarrow x = 4$$

$$\text{Domain: } \underline{\mathbb{R} - \{4\}}$$

Exercise

Find the domain of $f(x) = \ln(x^2 - 4)$

Solution

$$x^2 - 4 > 0$$

$$x^2 - 4 = 0 \rightarrow x = \pm 2$$

$$\text{Domain: } \underline{x < -2 \quad x > 2}$$

Exercise

Find the domain of $f(x) = \ln(x^2 - 4x + 3)$

Solution

$$x^2 - 4x + 3 = 0 \rightarrow \underline{x = 1, 3}$$

$$x^2 - 4x + 3 > 0$$

$$\text{Domain: } \underline{x < 1 \quad x > 3}$$

Exercise

Find the domain of $f(x) = \ln(2x^2 - 5x + 3)$

Solution

$$2x^2 - 5x + 3 = 0 \rightarrow \underline{x = 1, \frac{3}{2}}$$

$$2x^2 - 5x + 3 > 0$$

$$\text{Domain: } \underline{x < 1 \quad x > \frac{3}{2}}$$

Exercise

Find the domain of $f(x) = \log(x^2 + 4x + 3)$

Solution

$$x^2 + 4x + 3 = 0 \rightarrow \underline{x = -1, -3}$$

$$x^2 + 4x + 3 > 0$$

$$\text{Domain: } \underline{x < -3 \quad x > -1}$$

Exercise

Find the domain of $f(x) = \ln(x^4 - x^2)$

Solution

$$x^4 - x^2 = 0$$

$$x^2(x^2 - 1) = 0$$

$$\underline{x = 0, 0, \pm 1}$$

$$x^4 - x^2 > 0$$

$$\text{Domain: } \underline{x < -1 \quad x > 1}$$

-1	0,0	1	2
+	-	-	+

Exercise

Find the **asymptote**, **domain**, and **range** of the given function. Then, sketch the graph $f(x) = \log_4(x-2)$

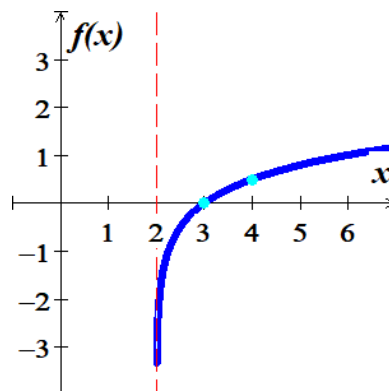
Solution

Asymptote: $x = 2$

Domain: $(2, \infty)$

Range: $(-\infty, \infty)$

x	$f(x)$
2	
3	0
4	.5



Exercise

Find the **asymptote**, **domain**, and **range** of the given function. Then, sketch the graph $f(x) = \log_4|x|$

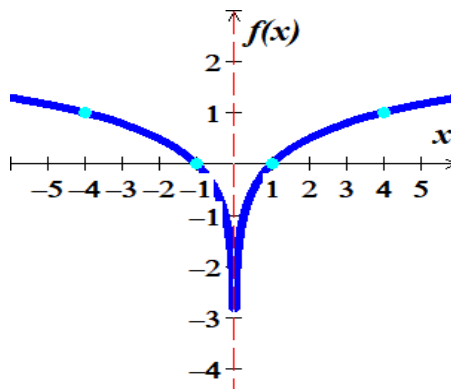
Solution

Asymptote: $x = 0$

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, \infty)$

x	$f(x)$
0	
± 1	0
± 4	1



Exercise

Find the **asymptote**, **domain**, and **range** of the given function. Then, sketch the graph $f(x) = \left(\log_4 x\right) - 2$

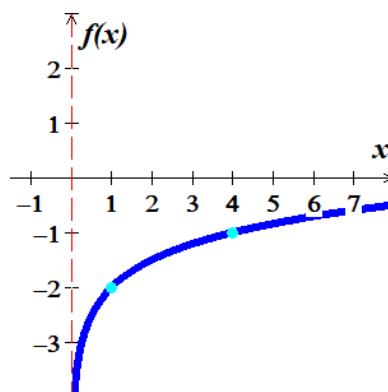
Solution

Asymptote: $x = 0$

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

x	$f(x)$
0	
1	0
4	-1



Exercise

Find the *asymptote*, *domain*, and *range* of the given function. Then, sketch the graph $f(x) = \log(3 - x)$

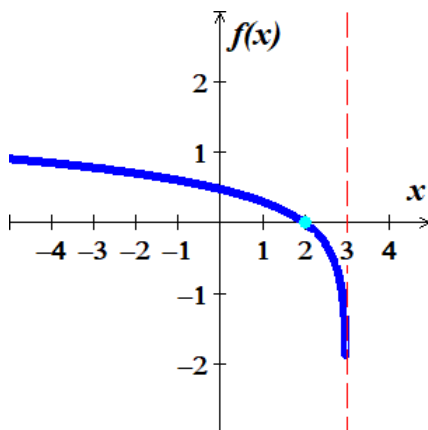
Solution

Asymptote: $x = 3$

Domain: $(-\infty, 3)$

Range: $(-\infty, \infty)$

x	$f(x)$
3	
2	0



Exercise

Find the *asymptote*, *domain*, and *range* of the given function. Then, sketch the graph $f(x) = 2 - \log(x + 2)$

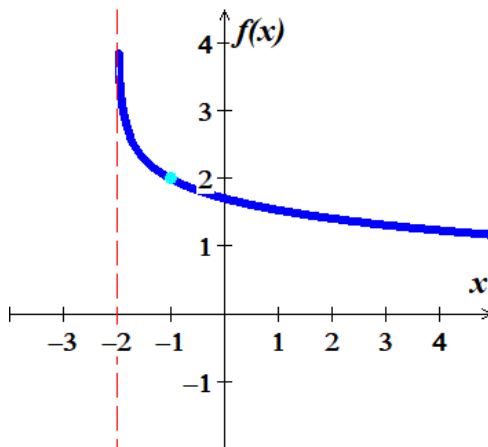
Solution

Asymptote: $x = -2$

Domain: $(-2, \infty)$

Range: $(-\infty, \infty)$

x	$f(x)$
-2	
-1	2



Exercise

Find the *asymptote*, *domain*, and *range* of the given function. Then, sketch the graph $f(x) = \ln(x - 2)$

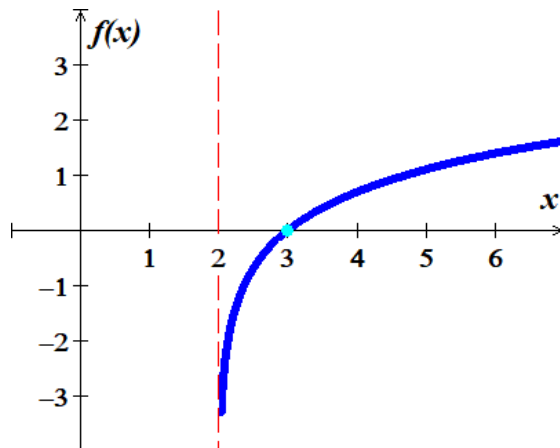
Solution

Asymptote: $x = 2$

Domain: $(2, \infty)$

Range: $(-\infty, \infty)$

x	$f(x)$
2	
3	0



Exercise

Find the **asymptote**, **domain**, and **range** of the given function. Then, sketch the graph $f(x) = \ln(3 - x)$

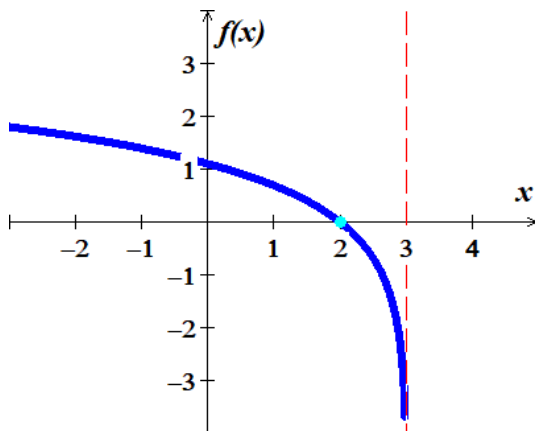
Solution

Asymptote: $x = 3$

Domain: $(-\infty, 3)$

Range: $(-\infty, \infty)$

x	$f(x)$
$-\infty$	$-\infty$
2	0



Exercise

Find the **asymptote**, **domain**, and **range** of the given function. Then, sketch the graph $f(x) = 2 + \ln(x + 1)$

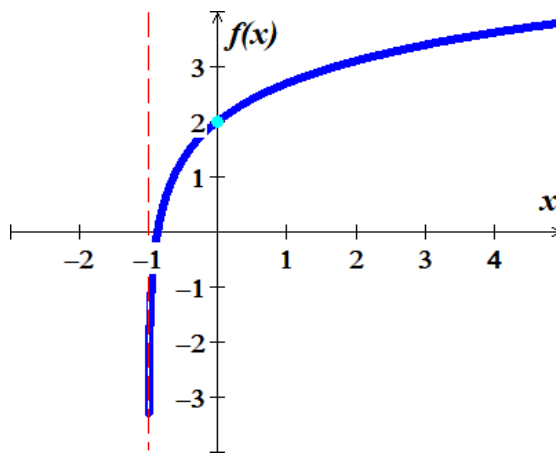
Solution

Asymptote: $x = -1$

Domain: $(-1, \infty)$

Range: $(-\infty, \infty)$

x	$f(x)$
$-\infty$	$-\infty$
0	2



Exercise

Find the **asymptote**, **domain**, and **range** of the given function. Then, sketch the graph $f(x) = 1 - \ln(x - 2)$

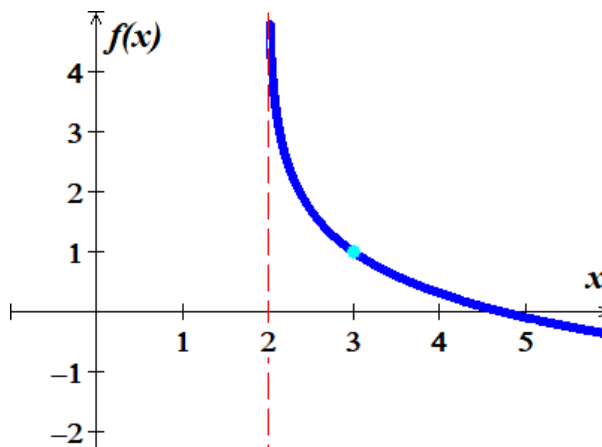
Solution

Asymptote: $x = 2$

Domain: $(2, \infty)$

Range: $(-\infty, \infty)$

x	$f(x)$
2	$-\infty$
3	1



Exercise

On a study by psychologists Bornstein and Bornstein, it was found that the average walking speed w , in feet per second, of a person living in a city of population P , in **thousands**, is given by the function

$$w(P) = 0.37 \ln P + 0.05$$

- a) The population is 124,848. Find the average walking speed of people living in Hartford.
- b) The population is 1,236,249. Find the average walking speed of people living in San Antonio.

Solution

$$124,848 = 124.848 \text{ thousand}$$

$$\begin{aligned} \text{a) } w(124.848) &= 0.37 \ln(124.848) + 0.05 \\ &\approx 1.8 \text{ ft/sec} \end{aligned}$$

$$\begin{aligned} \text{b) } w(1,236.249) &= 0.37 \ln(1,236.249) + 0.05 \\ &\approx 2.7 \text{ ft/sec} \end{aligned}$$

Exercise

The loudness of sounds is measured in a unit called a decibel. To measure with this unit, we first assign an intensity of I_0 to a very faint sound, called the threshold sound. If a particular sound has intensity I , then the decibel rating of this louder sound is

$$d = 10 \log \frac{I}{I_0}$$

Find the exact decibel rating of a sound with intensity $10,000I_0$

Solution

$$\begin{aligned} d &= 10 \log \frac{10000I_0}{I_0} \\ &= 10 \log 10000 \\ &= 40 \text{ db} \end{aligned}$$

Exercise

Students in an accounting class took a final exam and then took equivalent forms of the exam at monthly intervals thereafter. The average score $S(t)$, as a percent, after t months was found to be given by the function

$$S(t) = 78 - 15 \log(t+1); \quad t \geq 0$$

- a) What was the average score when the students initially took the test, $t = 0$?
- b) What was the average score after 4 months? 24 months?

Solution

$$a) \quad S(0) = 78 - 15 \log(1)$$

$$\approx 78\% \quad |$$

$$b) \quad \text{After 4 months}$$

$$S(4) = 78 - 15 \log(5)$$

$$\approx 67.5\% \quad |$$

$$\text{After 24 months}$$

$$S(24) = 78 - 15 \log(25)$$

$$\approx 57\% \quad |$$

Exercise

A model for advertising response is given by the function

$$N(a) = 1,000 + 200 \ln a, \quad a \geq 1$$

Where $N(a)$ is the number of units sold when a is the amount spent on advertising, in *thousands of dollars*.

$$a) \quad N(1)$$

$$b) \quad N(5)$$

Solution

$$a) \quad N(1) = 1,000 + 200 \ln(1)$$

$$= 1,000 \text{ units} \quad |$$

$$b) \quad N(5) = 1,000 + 200 \ln(5)$$

$$= 1,322 \text{ units} \quad |$$

Solution **Section 3.4 – Properties of Logarithms**

Exercise

Express the following in terms of sums and differences of logarithms: $\log_3(ab)$

Solution

$$\log_3(ab) = \log_3 a + \log_3 b$$

Exercise

Express the following in terms of sums and differences of logarithms: $\log_7(7x)$

Solution

$$\begin{aligned}\log_7(7x) &= \log_7 7 + \log_7 x \\ &= 1 + \log_7 x\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms: $\log \frac{x}{1000}$

Solution

$$\begin{aligned}\log \frac{x}{1000} &= \log x - \log 10^3 \\ &= \log x - 3\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_5 \left(\frac{125}{y} \right)$

Solution

$$\begin{aligned}\log_5 \left(\frac{125}{y} \right) &= \log_5 5^3 - \log_5 y \\ &= 3 - \log_5 y\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b x^7$

Solution

$$\log_b x^7 = 7 \log_b x$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln \sqrt[7]{x}$

Solution

$$\begin{aligned} \ln \sqrt[7]{x} &= \ln x^{1/7} \\ &= \frac{1}{7} \ln x \end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_a \frac{x^2 y}{z^4}$

Solution

$$\begin{aligned} \log_a \frac{x^2 y}{z^4} &= \log_a x^2 y - \log_a z^4 \\ &= \log_a x^2 + \log_a y - \log_a z^4 \\ &= 2 \log_a x + \log_a y - 4 \log_a z \end{aligned}$$

Quotient Rule

Product Rule

Power Rule

Exercise

Express the following in terms of sums and differences of logarithms $\log_b \frac{x^2 y}{b^3}$

Solution

$$\begin{aligned} \log_b \left(\frac{x^2 y}{b^3} \right) &= \log_b x^2 y - \log_b b^3 \\ &= \log_b x^2 + \log_b y - \log_b b^3 \\ &= 2 \log_b x + \log_b y - 3 \log_b b \\ &= 2 \log_b x + \log_b y - 3 \end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b \left(\frac{x^3 y}{z^2} \right)$

Solution

$$\begin{aligned}\log_b \left(\frac{x^3 y}{z^2} \right) &= \log_b (x^3 y) - \log_b z^2 \\ &= \log_b x^3 + \log_b y - \log_b z^2 \\ &= \underline{3 \log_b x + \log_b y - 2 \log_b z}\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b \left(\frac{\sqrt[3]{x} y^4}{z^5} \right)$

Solution

$$\begin{aligned}\log_b \left(\frac{\sqrt[3]{x} y^4}{z^5} \right) &= \log_b (\sqrt[3]{x} y^4) - \log_b (z^5) \\ &= \underline{\log_b (x^{1/3}) + \log_b (y^4) - \log_b (z^5)}\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log \left(\frac{100x^3 \sqrt[3]{5-x}}{3(x+7)^2} \right)$

Solution

$$\begin{aligned}\log \left(\frac{100x^3 \sqrt[3]{5-x}}{3(x+7)^2} \right) &= \log (100x^3 \sqrt[3]{5-x}) - \log (3(x+7)^2) \\ &= \log 10^2 + \log x^3 + \log (5-x)^{1/3} - \left[\log 3 + \log ((x+7)^2) \right] \\ &= 2 \log 10 + 3 \log x + \frac{1}{3} \log (5-x) - \log 3 - 2 \log (x+7) \\ &= \underline{2 + 3 \log x + \frac{1}{3} \log (5-x) - \log 3 - 2 \log (x+7)}\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_a \sqrt[4]{\frac{m^8 n^{12}}{a^3 b^5}}$

Solution

$$\begin{aligned}
\log_a \sqrt[4]{\frac{m^8 n^{12}}{a^3 b^5}} &= \log_a \left(\frac{m^8 n^{12}}{a^3 b^5} \right)^{1/4} && \text{Power Rule} \\
&= \frac{1}{4} \log_a \left(\frac{m^8 n^{12}}{a^3 b^5} \right) && \text{Quotient Rule} \\
&= \frac{1}{4} \left[\log_a m^8 n^{12} - \log_a a^3 b^5 \right] && \text{Product Rule} \\
&= \frac{1}{4} \left[\log_a m^8 + \log_a n^{12} - \left(\log_a a^3 + \log_a b^5 \right) \right] && \text{Power Rule} \\
&= \frac{1}{4} \left[8 \log_a m + 12 \log_a n - 3 - 5 \log_a b \right] \\
&= \underline{2 \log_a m + 3 \log_a n - \frac{3}{4} - \frac{5}{4} \log_a b}
\end{aligned}$$

Exercise

Use the properties of logarithms to rewrite: $\log_p \sqrt[3]{\frac{m^5 n^4}{t^2}}$

Solution

$$\begin{aligned}
\log_p \sqrt[3]{\frac{m^5 n^4}{t^2}} &= \log_p \left(\frac{m^5 n^4}{t^2} \right)^{1/3} && \text{Power Rule} \\
&= \frac{1}{3} \log_p \left(\frac{m^5 n^4}{t^2} \right) && \text{Quotient Rule} \\
&= \frac{1}{3} \left(\log_p m^5 n^4 - \log_p t^2 \right) && \text{Product Rule} \\
&= \frac{1}{3} \left(\log_p m^5 + \log_p n^4 - \log_p t^2 \right) && \text{Power Rule} \\
&= \frac{1}{3} \left(5 \log_p m + 4 \log_p n - 2 \log_p t \right) \\
&= \underline{\frac{5}{3} \log_p m + \frac{4}{3} \log_p n - \frac{2}{3} \log_p t}
\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b \sqrt[n]{\frac{x^3 y^5}{z^m}}$

Solution

$$\log_b \sqrt[n]{\frac{x^3 y^5}{z^m}} = \log_b \left(\frac{x^3 y^5}{z^m} \right)^{1/n}$$

$$\begin{aligned}
&= \frac{1}{n} \log_b \left(\frac{x^3 y^5}{z^m} \right) && \text{Power Rule} \\
&= \frac{1}{n} \left(\log_b x^3 y^5 - \log_b z^m \right) && \text{Quotient Rule} \\
&= \frac{1}{n} \left(\log_b x^3 + \log_b y^5 - \log_b z^m \right) && \text{Product Rule} \\
&= \frac{1}{n} \left(3 \log_b x + 5 \log_b y - m \log_b z \right) && \text{Power Rule} \\
&= \underline{\frac{3}{n} \log_b x + \frac{5}{n} \log_b y - \frac{m}{n} \log_b z}
\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_a \sqrt[3]{\frac{a^2 b}{c^5}}$

Solution

$$\begin{aligned}
\log_a \sqrt[3]{\frac{a^2 b}{c^5}} &= \log_a \left(\frac{a^2 b}{c^5} \right)^{1/3} && \text{Convert the radical to power} \\
&= \frac{1}{3} \log_a \left(\frac{a^2 b}{c^5} \right) && \text{Power Rule} \\
&= \frac{1}{3} \left[\log_a a^2 b - \log_a c^5 \right] && \text{Quotient Rule} \\
&= \frac{1}{3} \left[\log_a a^2 + \log_a b - \log_a c^5 \right] && \text{Product Rule} \\
&= \frac{1}{3} \left[2 \log_a a + \log_a b - 5 \log_a c \right] && \text{Power Rule} \\
&= \frac{2}{3} \log_a a + \frac{1}{3} \log_a b - \frac{5}{3} \log_a c \\
&= \underline{\frac{2}{3} + \frac{1}{3} \log_a b - \frac{5}{3} \log_a c}
\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b \left(x^4 \sqrt[3]{y} \right)$

Solution

$$\begin{aligned}
\log_b \left(x^4 \sqrt[3]{y} \right) &= \log_b \left(x^4 \right) + \log_b \left(\sqrt[3]{y} \right) \\
&= \log_b \left(x^4 \right) + \log_b \left(y^{1/3} \right)
\end{aligned}$$

$$= 4 \log_b x + \frac{1}{3} \log_b y$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_5 \left(\frac{\sqrt{x}}{25y^3} \right)$

Solution

$$\begin{aligned} \log_5 \left(\frac{\sqrt{x}}{25y^3} \right) &= \log_5 \left(x^{1/2} \right) - \log_5 \left(25y^3 \right) \\ &= \log_5 \left(x^{1/2} \right) - \left[\log_5 \left(5^2 \right) + \log_5 \left(y^3 \right) \right] \\ &= \log_5 \left(x^{1/2} \right) - \log_5 \left(5^2 \right) - \log_5 \left(y^3 \right) \\ &= \frac{1}{2} \log_5 (x) - 2 \log_5 (5) - 3 \log_5 (y) \\ &= \frac{1}{2} \log_5 (x) - 2 - 3 \log_5 (y) \end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_a \frac{x^3 w}{y^2 z^4}$

Solution

$$\begin{aligned} \log_a \frac{x^3 w}{y^2 z^4} &= \log_a x^3 w - \log_a y^2 z^4 && \text{Quotient rule} \\ &= \log_a x^3 + \log_a w - \left(\log_a y^2 + \log_a z^4 \right) && \text{Product rule} \\ &= \log_a x^3 + \log_a w - \log_a y^2 - \log_a z^4 && \text{Distribute minus} \\ &= 3 \log_a x + \log_a w - 2 \log_a y - 4 \log_a z && \text{Power rule} \end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_a \frac{\sqrt{y}}{x^4 \sqrt[3]{z}}$

Solution

$$\begin{aligned} \log_a \frac{\sqrt{y}}{x^4 \sqrt[3]{z}} &= \log_a y^{1/2} - \log_a x^4 z^{1/3} && \text{Quotient rule} \\ &= \log_a y^{1/2} - \left(\log_a x^4 + \log_a z^{1/3} \right) && \text{Product rule} \end{aligned}$$

$$\begin{aligned}
 &= \log_a y^{1/2} - \log_a x^4 - \log_a z^{1/3} \\
 &= \frac{1}{2} \log_a y - 4 \log_a x - \frac{1}{3} \log_a z
 \end{aligned}$$

Distribute minus

Power rule

Exercise

Express the following in terms of sums and differences of logarithms $\ln \sqrt[4]{\frac{x^7}{y^5 z}}$

Solution

$$\ln \sqrt[4]{\frac{x^7}{y^5 z}} = \ln \left(\frac{x^7}{y^5 z} \right)^{1/4}$$

$$= \frac{1}{4} \ln \left(\frac{x^7}{y^5 z} \right)$$

Power rule

$$= \frac{1}{4} (\ln x^7 - \ln y^5 z)$$

Quotient rule

$$= \frac{1}{4} (\ln x^7 - (\ln y^5 + \ln z))$$

Product rule

$$= \frac{1}{4} (\ln x^7 - \ln y^5 - \ln z)$$

$$= \frac{1}{4} (7 \ln x - 5 \ln y - \ln z)$$

Power rule

$$= \frac{7}{4} \ln x - \frac{5}{4} \ln y - \ln z$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln x \sqrt[3]{\frac{y^4}{z^5}}$

Solution

$$\ln x \sqrt[3]{\frac{y^4}{z^5}} = \ln x + \ln \left(\frac{y^4}{z^5} \right)^{1/3}$$

Product rule

$$= \ln x + \ln \left(\frac{y^{4/3}}{z^{5/3}} \right)$$

$$= \ln x + \ln y^{4/3} - \ln z^{5/3}$$

Quotient rule

$$= \ln x + \frac{4}{3} \ln y - \frac{5}{3} \ln z$$

Power rule

Exercise

Express the following in terms of sums and differences of logarithms

$$\log_b \sqrt[5]{\frac{m^4 n^5}{x^2 ab^{10}}}$$

Solution

$$\begin{aligned}\log_b \sqrt[5]{\frac{m^4 n^5}{x^2 ab^{10}}} &= \log_b \left(\frac{m^4 n^5}{x^2 ab^{10}} \right)^{1/5} \\&= \frac{1}{5} \log_b \left(\frac{m^4 n^5}{x^2 ab^{10}} \right) \\&= \frac{1}{5} \left(\log_b (m^4 n^5) - \log_b (x^2 ab^{10}) \right) \\&= \frac{1}{5} \left(\left(\log_b (m^4) + \log_b (n^5) \right) - \left(\log_b (x^2) + \log_b (a) + \log_b (b^{10}) \right) \right) \\&= \frac{1}{5} \left(4 \log_b m + 5 \log_b n - 2 \log_b x - \log_b a - 10 \right) \\&= \underline{\underline{\frac{4}{5} \log_b m + \log_b n - \frac{2}{5} \log_b x - \frac{1}{5} \log_b a - 2}}}\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms

$$\log_b \frac{a^5 b^{10}}{c^2 \sqrt[4]{d^3}}$$

Solution

$$\begin{aligned}\log_b \frac{a^5 b^{10}}{c^2 \sqrt[4]{d^3}} &= \log_b (a^5 b^{10}) - \log_b (c^2 d^{3/4}) \\&= \log_b (a^5) + \log_b (b^{10}) - \left(\log_b (c^2) + \log_b (d^{3/4}) \right) \\&= \underline{\underline{5 \log_b a + 10 - 2 \log_b c - \frac{3}{4} \log_b d}}\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms

$$\ln \left(x^2 \sqrt{x^2 + 1} \right)$$

Solution

$$\begin{aligned}\ln \left(x^2 \sqrt{x^2 + 1} \right) &= \ln x^2 + \ln \left(x^2 + 1 \right)^{1/2} \\&= \underline{\underline{2 \ln x + \frac{1}{2} \ln (x^2 + 1)}}\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms

$$\ln \frac{x^2}{x^2 + 1}$$

Solution

$$\begin{aligned}\ln \frac{x^2}{x^2 + 1} &= \ln x^2 - \ln(x^2 + 1) \\ &= \underline{2 \ln x - \ln(x^2 + 1)}\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms

$$\ln \left(\frac{x^2 (x+1)^3}{(x+3)^{1/2}} \right)$$

Solution

$$\begin{aligned}\ln \left(\frac{x^2 (x+1)^3}{(x+3)^{1/2}} \right) &= \ln(x^2 (x+1)^3) - \ln(x+3)^{1/2} \\ &= \ln x^2 + \ln(x+1)^3 - \frac{1}{2} \ln(x+3) \\ &= \underline{2 \ln x + 3 \ln(x+1) - \frac{1}{2} \ln(x+3)}\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms

$$\ln \sqrt{\frac{(x+1)^5}{(x+2)^{20}}}$$

Solution

$$\begin{aligned}\ln \sqrt{\frac{(x+1)^5}{(x+2)^{20}}} &= \ln \left(\frac{(x+1)^5}{(x+2)^{20}} \right)^{1/2} \\ &= \frac{1}{2} \ln \left(\frac{(x+1)^5}{(x+2)^{20}} \right) \\ &= \frac{1}{2} (\ln(x+1)^5 - \ln(x+2)^{20}) \\ &= \frac{1}{2} (5 \ln(x+1) - 20 \ln(x+2)) \\ &= \underline{\frac{5}{2} \ln(x+1) - 10 \ln(x+2)}\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms

$$\ln \frac{(x^2 + 1)^5}{\sqrt{1-x}}$$

Solution

$$\begin{aligned} \ln \frac{(x^2 + 1)^5}{\sqrt{1-x}} &= \ln (x^2 + 1)^5 - \ln (1-x)^{1/2} \\ &= \underline{5 \ln (x^2 + 1) - \frac{1}{2} \ln (1-x)} \end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms

$$\ln \left(\sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}} \right)$$

Solution

$$\begin{aligned} \ln \left(\sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}} \right) &= \ln \left(\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)} \right)^{1/3} \\ &= \frac{1}{3} \ln \left(\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)} \right) \\ &= \frac{1}{3} \left(\ln (x(x+1)(x-2)) - \ln ((x^2+1)(2x+3)) \right) \\ &= \frac{1}{3} \left(\ln x + \ln (x+1) + \ln (x-2) - (\ln (x^2+1) + \ln (2x+3)) \right) \\ &= \frac{1}{3} \left(\ln x + \ln (x+1) + \ln (x-2) - \ln (x^2+1) - \ln (2x+3) \right) \\ &= \underline{\frac{1}{3} \ln x + \frac{1}{3} \ln (x+1) + \frac{1}{3} \ln (x-2) - \frac{1}{3} \ln (x^2+1) - \frac{1}{3} \ln (2x+3)} \end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms

$$\ln \left(\sqrt{\frac{1}{x(x+1)}} \right)$$

Solution

$$\ln \left(\sqrt{\frac{1}{x(x+1)}} \right) = \ln \left(\frac{1}{x(x+1)} \right)^{1/2}$$

$$\begin{aligned}
&= \frac{1}{2} (\ln 1 - \ln(x(x+1))) \\
&= -\frac{1}{2} (\ln x + \ln(x+1)) \\
&= \underline{-\frac{1}{2} \ln x - \frac{1}{2} \ln(x+1)}
\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln \left(\sqrt{(x^2 + 1)(x-1)^2} \right)$

Solution

$$\begin{aligned}
\ln \left(\sqrt{(x^2 + 1)(x-1)^2} \right) &= \ln \left((x^2 + 1)(x-1)^2 \right)^{1/2} \\
&= \frac{1}{2} \ln \left((x^2 + 1)(x-1)^2 \right) \\
&= \frac{1}{2} \left(\ln(x^2 + 1) + \ln(x-1)^2 \right) \\
&= \frac{1}{2} \left(\ln(x^2 + 1) + 2 \ln(x-1) \right) \\
&= \underline{\frac{1}{2} \ln(x^2 + 1) + \ln(x-1)}
\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary: $\log(x+5) + 2 \log x$

Solution

$$\begin{aligned}
\log(x+5) + 2 \log x &= \log(x+5) + \log x^2 \\
&= \underline{\log(x^2(x+5))}
\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary: $3 \log_b x - \frac{1}{3} \log_b y + 4 \log_b z$

Solution

$$\begin{aligned}
3 \log_b x - \frac{1}{3} \log_b y + 4 \log_b z &= \log_b x^3 + \log_b z^4 - \log_b y^{1/3} \\
&= \log_b (x^3 z^4) - \log_b \sqrt[3]{y} \\
&= \underline{\log_b \left(\frac{x^3 z^4}{\sqrt[3]{y}} \right)}
\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary: $\frac{1}{2}\log_b(x+5) - 5\log_b y$

Solution

$$\begin{aligned}\frac{1}{2}\log_b(x+5) - 5\log_b y &= \log_b(x+5)^{1/2} - \log_b y^5 \\ &= \log_b \left(\frac{\sqrt{x+5}}{y^5} \right)\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary: $\ln(x^2 - y^2) - \ln(x - y)$

Solution

$$\begin{aligned}\ln(x^2 - y^2) - \ln(x - y) &= \ln \frac{x^2 - y^2}{x - y} \\ &= \ln \frac{(x - y)(x + y)}{x - y} \\ &= \ln(x + y)\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary: $\ln(xz) - \ln(x\sqrt{y}) + 2\ln \frac{y}{z}$

Solution

$$\begin{aligned}\ln(xz) - \ln(x\sqrt{y}) + 2\ln \frac{y}{z} &= \ln(xz) + \ln\left(\frac{y}{z}\right)^2 - \ln(x\sqrt{y}) \\ &= \ln\left(\frac{xzy^2}{z^2}\right) - \ln(x\sqrt{y}) \\ &= \ln\left(\frac{xy^2}{z} \cdot \frac{1}{x\sqrt{y}}\right) \\ &= \ln\left(\frac{y^{3/2}}{z}\right)\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary: $\log(x^2 y) - \log z$

Solution

$$\log(x^2 y) - \log z = \log\left(\frac{x^2 y}{z}\right)$$

Exercise

Write the expression as a single logarithm and simplify if necessary: $\log(z^2 \sqrt{y}) - \log z^{1/2}$

Solution

$$\begin{aligned}\log(z^2 \sqrt{y}) - \log z^{1/2} &= \log\left(\frac{z^2 \sqrt{y}}{z^{1/2}}\right) \\ &= \log(z^{3/2} \sqrt{y}) \\ &= \log(\sqrt{z^3 y})\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$2\log_a x + \frac{1}{3}\log_a (x-2) - 5\log_a (2x+3)$$

Solution

$$\begin{aligned}2\log_a x + \frac{1}{3}\log_a (x-2) - 5\log_a (2x+3) &= \log_a x^2 + \log_a (x-2)^{1/3} - \log_a (2x+3)^5 \\ &= \log_a x^2 (x-2)^{1/3} - \log_a (2x+3)^5 \\ &= \log_a \frac{x^2 (x-2)^{1/3}}{(2x+3)^5}\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$5\log_a x - \frac{1}{2}\log_a (3x-4) - 3\log_a (5x+1)$$

Solution

$$\begin{aligned}
5\log_a x - \frac{1}{2}\log_a (3x-4) - 3\log_a (5x+1) &= \log_a x^5 - \log_a (3x-4)^{1/2} - \log_a (5x+1)^3 \\
&= \log_a x^5 - \left[\log_a (3x-4)^{1/2} + \log_a (5x+1)^3 \right] \\
&= \log_a x^5 - \left[\log_a (3x-4)^{1/2} (5x+1)^3 \right] \\
&= \log_a \frac{x^5}{(3x-4)^{1/2} (5x+1)^3}
\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\log(x^3 y^2) - 2\log(x\sqrt[3]{y}) - 3\log\left(\frac{x}{y}\right)$$

Solution

$$\begin{aligned}
\log(x^3 y^2) - 2\log(x\sqrt[3]{y}) - 3\log\left(\frac{x}{y}\right) &= \log(x^3 y^2) - \log(xy^{1/3})^2 - \log(xy^{-1})^3 \\
&= \log(x^3 y^2) - \left[\log(x^2 y^{2/3}) + \log(x^3 y^{-3}) \right] \\
&= \log(x^3 y^2) - \log(x^2 y^{2/3} x^3 y^{-3}) \\
&= \log(x^3 y^2) - \log(x^5 y^{-7/3}) \\
&= \log\left(\frac{x^3 y^2}{x^5 y^{-7/3}}\right) \\
&= \log\left(\frac{y^2 y^{7/3}}{x^2}\right) \\
&= \log\left(\frac{y^{13/3}}{x^2}\right) \\
&= \log\left(\frac{\sqrt[3]{y^{13}}}{x^2}\right) \\
&= \log\left(\frac{y^4 \sqrt[3]{y}}{x^2}\right)
\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\ln y^3 + \frac{1}{3} \ln(x^3 y^6) - 5 \ln y$$

Solution

$$\begin{aligned} \ln y^3 + \frac{1}{3} \ln(x^3 y^6) - 5 \ln y &= \ln y^3 + \ln(x^3 y^6)^{1/3} - \ln y^5 \\ &= \ln y^3 + \ln(x^{3/3} y^{6/3}) - \ln y^5 \\ &= \ln y^3 + \ln(xy^2) - \ln y^5 \\ &= \ln(y^3 xy^2) - \ln y^5 \\ &= \ln\left(\frac{y^5 x}{y^5}\right) \\ &= \ln x \end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$2 \ln x - 4 \ln\left(\frac{1}{y}\right) - 3 \ln(xy)$$

Solution

$$\begin{aligned} 2 \ln x - 4 \ln\left(\frac{1}{y}\right) - 3 \ln(xy) &= \ln x^2 - \ln\left(\frac{1}{y}\right)^4 - \ln(xy)^3 \\ &= \ln x^2 - \left[\ln(y^{-4}) + \ln(x^3 y^3)\right] \\ &= \ln x^2 - \ln(y^{-4} x^3 y^3) \\ &= \ln x^2 - \ln(y^{-1} x^3) \\ &= \ln \frac{x^2}{y^{-1} x^3} \\ &= \ln \frac{y}{x} \end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$4 \ln x + 7 \ln y - 3 \ln z$$

Solution

$$\begin{aligned} 4 \ln x + 7 \ln y - 3 \ln z &= \ln x^4 + \ln y^7 - \ln z^3 \\ &= \ln(x^4 y^7) - \ln z^3 \\ &= \ln\left(\frac{x^4 y^7}{z^3}\right) \end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\frac{1}{3} \left[5 \ln(x+6) - \ln x - \ln(x^2 - 25) \right]$$

Solution

$$\begin{aligned} \frac{1}{3} \left[5 \ln(x+6) - \ln x - \ln(x^2 - 25) \right] &= \frac{1}{3} \left[5 \ln(x+6) - (\ln x + \ln(x^2 - 25)) \right] \\ &= \frac{1}{3} \left[\ln(x+6)^5 - \ln x(x^2 - 25) \right] \\ &= \frac{1}{3} \left[\ln \frac{(x+6)^5}{x(x^2 - 25)} \right] \\ &= \ln \left(\frac{(x+6)^5}{x(x^2 - 25)} \right)^{1/3} \end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\frac{2}{3} \left[\ln(x^2 - 4) - \ln(x+2) \right] + \ln(x+y)$$

Solution

$$\begin{aligned} \frac{2}{3} \left[\ln(x^2 - 4) - \ln(x+2) \right] + \ln(x+y) &= \frac{2}{3} \left[\ln \frac{x^2 - 4}{x+2} \right] + \ln(x+y) \\ &= \frac{2}{3} \left[\ln \frac{(x+2)(x-2)}{x+2} \right] + \ln(x+y) \\ &= \frac{2}{3} \ln(x-2) + \ln(x+y) \end{aligned}$$

$$\begin{aligned}
&= \ln(x-2)^{2/3} + \ln(x+y) \\
&= \ln(x-2)^{2/3}(x+y) \\
&= \ln(x+y) \sqrt[3]{(x-2)^2}
\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\frac{1}{2}\log_b m + \frac{3}{2}\log_b 2n - \log_b m^2 n$$

Solution

$$\begin{aligned}
\frac{1}{2}\log_b m + \frac{3}{2}\log_b 2n - \log_b m^2 n &= \log_b m^{1/2} + \log_b (2n)^{3/2} - \log_b m^2 n \\
&= \log_b \left(m^{1/2} (2n)^{3/2} \right) - \log_b m^2 n \\
&= \log_b \frac{m^{1/2} 2^{3/2} n^{3/2}}{m^2 n} \\
&= \log_b \frac{2^{3/2} n^{1/2}}{m^{3/2}} \\
&= \log_b \left(\frac{2^3 n}{m^3} \right)^{1/2} \\
&= \log_b \sqrt{\frac{8n}{m^3}}
\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\frac{1}{2}\log_y p^3 q^4 - \frac{2}{3}\log_y p^4 q^3$$

Solution

$$\begin{aligned}
\frac{1}{2}\log_y p^3 q^4 - \frac{2}{3}\log_y p^4 q^3 &= \log_y \left(p^3 q^4 \right)^{1/2} - \log_y \left(p^4 q^3 \right)^{2/3} \\
&= \log_y \frac{\left(p^3 q^4 \right)^{1/2}}{\left(p^4 q^3 \right)^{2/3}} \\
&= \log_y \frac{\left(p^3 \right)^{1/2} \left(q^4 \right)^{1/2}}{\left(p^4 \right)^{2/3} \left(q^3 \right)^{2/3}}
\end{aligned}$$

$$\begin{aligned}
&= \log_y \frac{p^{3/2} q^2}{p^{8/3} q^2} \\
&= \log_y \frac{p^{3/2}}{p^{8/3}} \\
&= \log_y \frac{1}{p^{8/3-3/2}} \\
&= \log_y \frac{1}{p^{7/6}}
\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\frac{1}{2} \log_a x + 4 \log_a y - 3 \log_a x$$

Solution

$$\begin{aligned}
\frac{1}{2} \log_a x + 4 \log_a y - 3 \log_a x &= 4 \log_a y - \frac{5}{2} \log_a x \\
&= \log_a y^4 - \log_a x^{5/2} \\
&= \log_a \frac{y^4}{\sqrt{x^5}}
\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\frac{2}{3} \left[\ln(x^2 - 9) - \ln(x + 3) \right] + \ln(x + y)$$

Solution

$$\begin{aligned}
\frac{2}{3} \left[\ln(x^2 - 9) - \ln(x + 3) \right] + \ln(x + y) &= \frac{2}{3} \ln \frac{x^2 - 9}{x + 3} + \ln(x + y) \\
&= \frac{2}{3} \ln \frac{(x + 3)(x - 3)}{x + 3} + \ln(x + y) \\
&= \frac{2}{3} \ln(x - 3) + \ln(x + y) \\
&= \ln(x - 3)^{2/3} + \ln(x + y) \\
&= \ln \left((x - 3)^{2/3} (x + y) \right) \\
&= \ln \left((x + y) \sqrt[3]{(x - 3)^2} \right)
\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\frac{1}{4}\log_b x - 2\log_b 5 - 10\log_b y$$

Solution

$$\begin{aligned}\frac{1}{4}\log_b x - 2\log_b 5 - 10\log_b y &= \log_b x^{1/4} - \log_b 5^2 - \log_b y^{10} \\ &= \log_b x^{1/4} - \left[\log_b 5^2 + \log_b y^{10} \right] \\ &= \log_b x^{1/4} - \log_b (5^2 y^{10}) \\ &= \log_b \frac{\sqrt[4]{x}}{25y^{10}}\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$2\ln(x+4) - \ln x - \ln(x^2 - 3)$$

Solution

$$\begin{aligned}2\ln(x+4) - \ln x - \ln(x^2 - 3) &= \ln(x+4)^2 - (\ln x + \ln(x^2 - 3)) \\ &= \ln(x+4)^2 - \ln(x(x^2 - 3)) \\ &= \ln \frac{(x+4)^2}{x(x^2 - 3)}\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\ln x + \ln(y+3) + \ln(y+2) - \ln(y^2 + 5y + 6)$$

Solution

$$\begin{aligned}\ln x + \ln(y+3) + \ln(y+2) - \ln(y^2 + 5y + 6) &= \ln(x(y+3)(y+2)) - \ln((y+3)(y+2)) \\ &= \ln\left(\frac{x(y+3)(y+2)}{(y+3)(y+2)}\right) \\ &= \ln x\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\ln x + \ln(x+4) + \ln(x+1) - \ln(x^2 + 5x + 4)$$

Solution

$$\begin{aligned}\ln x + \ln(x+4) + \ln(x+1) - \ln(x^2 + 5x + 4) &= \ln(x(x+4)(x+1)) - \ln((x+4)(x+1)) \\ &= \ln\left(\frac{x(x+4)(x+1)}{(x+4)(x+1)}\right) \\ &= \ln x\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\ln(x^2 - 25) - 2\ln(x+5) + \ln(x-5)$$

Solution

$$\begin{aligned}\ln(x^2 - 25) - 2\ln(x+5) + \ln(x-5) &= \ln(x^2 - 25) + \ln(x-5) - \ln(x+5)^2 \\ &= \ln\frac{(x-5)(x+5)(x-5)}{(x+5)^2} \\ &= \ln\left(\frac{(x-5)^2}{x+5}\right)\end{aligned}$$

Exercise

Assume that $\log_{10} 2 = .3010$. Find each logarithm $\log_{10} 4$, $\log_{10} 5$

Solution

$$\begin{aligned}a) \quad \log_{10} 4 &= \log_{10} 2^2 \\ &= 2\log_{10} 2 \\ &= 2(.301) \\ &= .6020\end{aligned}$$

$$\begin{aligned}b) \quad \log_{10} 5 &= \log_{10} \frac{10}{2} \\ &= \log_{10} 10 - \log_{10} 2 \\ &= 1 - .3010\end{aligned}$$

$$= .6990 \mid$$

Exercise

Given that: $\log_a 2 \approx 0.301$, $\log_a 7 \approx 0.845$, and $\log_a 11 \approx 1.041$ find each of the following:

$$a) \log_a \frac{2}{11}$$

$$c) \log_a 98$$

$$e) \log_a 9$$

$$b) \log_a 14$$

$$d) \log_a \frac{1}{7}$$

$$f) \log_a \frac{77}{8}$$

Solution

$$\begin{aligned} a) \log_a \frac{2}{11} &= \log_a 2 - \log_a 11 \\ &= 0.301 - 1.041 \\ &\approx 1.342 \mid \end{aligned}$$

$$\begin{aligned} b) \log_a 14 &= \log_a 2(7) \\ &= \log_a 2 + \log_a 7 \\ &= 0.301 + 0.845 \\ &\approx 1.146 \mid \end{aligned}$$

$$\begin{aligned} c) \log_a 98 &= \log_a 2(7^2) \\ &= \log_a 2 + \log_a 7^2 \\ &= \log_a 2 + 2\log_a 7 \\ &= 0.301 + 2(0.845) \\ &\approx 1.991 \mid \end{aligned}$$

$$\begin{aligned} d) \log_a \frac{1}{7} &= \log_a 1 - \log_a 7 \\ &\approx 0 - 0.845 \\ &\approx -0.845 \mid \end{aligned}$$

$$e) \log_a 9 \text{ Can't be found from the given information}$$

$$\begin{aligned} f) \log_a \frac{77}{8} &= \log_a 77 - \log_a 8 \\ &= \log_a (7 \times 11) - \log_a 2^3 \\ &= \log_a 7 + \log_a 11 - 3\log_a 2 \\ &\approx 0.845 + 1.041 - 3(0.301) \\ &\approx 1.886 - 0.903 \\ &\approx 0.983 \mid \end{aligned}$$

Solution

Section 3.5 – Exponential and logarithmic Equations

Exercise

Solve the equation: $2^x = 128$

Solution

$$2^x = 2^7$$

$$\underline{x = 7}$$

Exercise

Solve the equation: $3^x = 243$

Solution

$$3^x = 3^5$$

$$\underline{x = 5}$$

Exercise

Solve the equation: $5^x = 70$

Solution

$$\underline{x = \log_5 70}$$

Exercise

Solve the equation: $6^x = 50$

Solution

$$\underline{x = \log_6 50}$$

Exercise

Solve the equation: $5^x = 134$

Solution

$$\underline{x = \log_5 134}$$

Exercise

Solve the equation: $7^x = 12$

Solution

$$\underline{x = \log_7 12}$$

Exercise

Solve the equation: $9^x = \frac{1}{\sqrt[3]{3}}$

Solution

$$\left(3^2\right)^x = \frac{1}{3^{1/3}}$$

$$3^{2x} = 3^{-1/3}$$

$$2x = -\frac{1}{3}$$

$$\underline{x = -\frac{1}{6}}$$

Exercise

Solve the equation: $49^x = \frac{1}{343}$

Solution

$$\left(7^2\right)^x = \frac{1}{7^3}$$

$$7^{2x} = 7^{-3}$$

$$2x = -3$$

$$\underline{x = -\frac{3}{2}}$$

Exercise

Solve the equation: $2^{5x+3} = \frac{1}{16}$

Solution

$$2^{5x+3} = 2^{-4}$$

$$5x + 3 = -4$$

$$5x = -7$$

$$\underline{x = -\frac{7}{5}}$$

Exercise

Solve the equation: $\left(\frac{2}{5}\right)^x = \frac{8}{125}$

Solution

$$\left(\frac{2}{5}\right)^x = \left(\frac{2}{5}\right)^3$$

$$\underline{x = 3}$$

Exercise

Solve the equation: $2^{3x-7} = 32$

Solution

$$2^{3x-7} = 32$$

$$= 2^5$$

$$3x - 7 = 5$$

add 7 on both sides

$$3x = 12$$

Divide by 3

$$\underline{x = 4}$$

Exercise

Solve the equation: $4^{2x-1} = 64$

Solution

$$4^{2x-1} = 4^3$$

$$2x - 1 = 3$$

$$2x = 4$$

$$\underline{x = 2}$$

Exercise

Solve the equation: $3^{1-x} = \frac{1}{27}$

Solution

$$3^{1-x} = \frac{1}{3^3}$$

$$3^{1-x} = 3^{-3}$$

$$1 - x = -3$$

$$\underline{x = 4}$$

Exercise

Solve the equation: $2^{-x^2} = 5$

Solution

$$\ln 2^{-x^2} = \ln 5$$

$$-x^2 \ln 2 = \ln 5$$

$$x^2 = -\frac{\ln 5}{\ln 2} \Rightarrow \text{No Solution}$$

Exercise

Solve the equation: $2^{-x} = 8$

Solution

$$2^{-x} = 2^3$$

$$-x = 3$$

$$\underline{x = -3}$$

Exercise

Solve the equation: $\left(\frac{1}{3}\right)^x = 81$

Solution

$$\left(\frac{1}{3}\right)^x = 81$$

$$\left(3^{-1}\right)^x = 3^4$$

$$3^{-x} = 3^4$$

$$-x = 4$$

$$\underline{x = -4}$$

Exercise

Solve the equation: $3^{-x} = 120$

Solution

$$-x = \log_3 120$$

Convert to Log

$$x = -\log_3 120$$

$$\underline{= \log_3 \frac{1}{120}}$$

Exercise

Solve the equation: $27 = 3^{5x} 9^{x^2}$

Solution

$$\begin{aligned} 3^3 &= 3^{5x} (3^2)^{x^2} \\ &= 3^{5x} 3^{2x^2} \\ &= 3^{5x+2x^2} \end{aligned}$$

$$2x^2 + 5x = 3$$

$$2x^2 + 5x - 3 = 0$$

$$x = \frac{-5 \pm \sqrt{25 + 24}}{6}$$

$$x = \left\{ \begin{array}{l} \frac{-5-7}{6} = -2 \\ \frac{-5+7}{6} = \frac{1}{3} \end{array} \right.$$

Exercise

Solve the equation: $4^{x+3} = 3^{-x}$

Solution

$$\ln 4^{x+3} = \ln 3^{-x}$$

$$(x+3) \ln 4 = -x \ln 3$$

$$x \ln 4 + 3 \ln 4 = -x \ln 3$$

$$x \ln 4 + x \ln 3 = -3 \ln 4$$

$$x(\ln 4 + \ln 3) = -3 \ln 4$$

$$x = \frac{-3 \ln 4}{(\ln 4 + \ln 3)}$$

Exercise

Solve the equation: $2^{x+4} = 8^{x-6}$

Solution

$$2^{x+4} = (2^3)^{x-6}$$

$$2^{x+4} = 2^{3x-18}$$

$$x+4 = 3x-18$$

$$2x = 22$$

$$\underline{x = 11}$$

Exercise

Solve the equation: $8^{x+2} = 4^{x-3}$

Solution

$$(2^3)^{x+2} = (2^2)^{x-3}$$

$$2^{3(x+2)} = 2^{2(x-3)}$$

$$3(x+2) = 2(x-3)$$

$$3x + 6 = 2x - 6$$

$$3x - 2x = -6 - 6$$

$$\underline{x = -12}$$

Exercise

Solve the equation: $7^x = 12$

Solution

$$\underline{x = \log_7 12}$$

Convert to Log

Exercise

Solve the equation: $5^{x+4} = 4^{x+5}$

Solution

$$\ln 5^{x+4} = \ln 4^{x+5}$$

$$(x+4) \ln 5 = (x+5) \ln 4$$

$$x \ln 5 + 4 \ln 5 = x \ln 4 + 5 \ln 4$$

$$(\ln 5 - \ln 4) x = 5 \ln 4 - 4 \ln 5$$

$$\underline{x = \frac{5 \ln 4 - 4 \ln 5}{\ln 5 - \ln 4}}$$

Exercise

Solve the equation: $5^{x+2} = 4^{1-x}$

Solution

$$\ln 5^{x+2} = \ln 4^{1-x}$$

$$(x+2)\ln 5 = (1-x)\ln 4$$

$$x\ln 5 + 2\ln 5 = \ln 4 - x\ln 4$$

$$(\ln 5 + \ln 4)x = \ln 4 - 2\ln 5$$

$$x = \frac{\ln 4 - 2\ln 5}{\ln 5 + \ln 4}$$

Exercise

Solve the equation: $3^{2x-1} = 0.4^{x+2}$

Solution

$$\ln 3^{2x-1} = \ln(0.4^{x+2})$$

$$(2x-1)\ln 3 = (x+2)\ln \frac{4}{10}$$

$$2x\ln 3 - \ln 3 = x\ln \frac{2}{5} + 2\ln \frac{2}{5}$$

$$(2\ln 3 - \ln \frac{2}{5})x = \ln 3 + 2\ln \frac{2}{5}$$

$$x = \frac{\ln 3 + 2\ln 0.4}{2\ln 3 - \ln 0.4}$$

Exercise

Solve the equation: $4^{3x-5} = 16$

Solution

$$4^{3x-5} = 4^2$$

$$3x-5 = 2$$

$$3x = 7$$

$$x = \frac{7}{3}$$

Exercise

Solve the equation: $4^{x+3} = 3^{-x}$

Solution

$$\ln 4^{x+3} = \ln 3^{-x}$$

$$(x+3)\ln 4 = -x\ln 3$$

$$x\ln 4 + 3\ln 4 = -x\ln 3$$

$$(\ln 4 + \ln 3)x = -3 \ln 4$$

$$x = -\frac{3 \ln 4}{\ln 4 + \ln 3}$$

Exercise

Solve the equation: $7^{2x+1} = 3^{x+2}$

Solution

$$\ln 7^{2x+1} = \ln 3^{x+2}$$

$$(2x+1) \ln 7 = (x+2) \ln 3$$

$$2x \ln 7 + \ln 7 = x \ln 3 + 2 \ln 3$$

$$2x \ln 7 - x \ln 3 = 2 \ln 3 - \ln 7$$

$$x(2 \ln 7 - \ln 3) = 2 \ln 3 - \ln 7$$

$$x = \frac{2 \ln 3 - \ln 7}{2 \ln 7 - \ln 3}$$

Exercise

Solve the equation: $3^{x-1} = 7^{2x+5}$

Solution

$$\ln 3^{x-1} = \ln 7^{2x+5}$$

$$(x-1) \ln 3 = (2x+5) \ln 7$$

$$x \ln 3 - \ln 3 = 2x \ln 7 + 5 \ln 7$$

$$x \ln 3 - 2x \ln 7 = \ln 3 + 5 \ln 7$$

$$x(\ln 3 - 2 \ln 7) = \ln 3 + 5 \ln 7$$

$$x = \frac{\ln 3 + 5 \ln 7}{\ln 3 - 2 \ln 7}$$

Exercise

Solve the equation: $4^{x-2} = 2^{3x+3}$

Solution

$$(2^2)^{x-2} = 2^{3x+3}$$

$$2^{2x-4} = 2^{3x+3}$$

$$2x-4 = 3x+3$$

$$2x - 3x = 4 + 3$$

$$-x = 7$$

$$\underline{x = -7}$$

Exercise

Solve the equation: $3^{5x-8} = 9^{x+2}$

Solution

$$3^{5x-8} = (3^2)^{x+2}$$

$$3^{5x-8} = 3^{2x+4}$$

$$5x - 8 = 2x + 4$$

$$5x - 2x = 8 + 4$$

$$3x = 12$$

$$\underline{x = 4}$$

Exercise

Solve the equation: $3^{x+4} = 2^{1-3x}$

Solution

$$\ln 3^{x+4} = \ln 2^{1-3x}$$

$$(x+4) \ln 3 = (1-3x) \ln 2$$

$$x \ln 3 + 4 \ln 3 = \ln 2 - 3x \ln 2$$

$$x \ln 3 + 3x \ln 2 = \ln 2 - 4 \ln 3$$

$$x(\ln 3 + 3 \ln 2) = \ln 2 - 4 \ln 3$$

$$\underline{x = \frac{\ln 2 - 4 \ln 3}{\ln 3 + 3 \ln 2}}$$

'ln' both sides

Power Rule

Distribute

Exercise

Solve the equation: $3^{2-3x} = 4^{2x+1}$

Solution

$$\ln 3^{2-3x} = \ln 4^{2x+1}$$

$$(2-3x) \ln 3 = (2x+1) \ln 4$$

$$2 \ln 3 - 3x \ln 3 = 2x \ln 4 + \ln 4$$

'ln' both sides

Power Rule

$$-3x \ln 3 - 2x \ln 4 = \ln 4 - 2 \ln 3$$

$$-x(3 \ln 3 + 2 \ln 4) = \ln 4 - 2 \ln 3$$

$$x = -\frac{\ln 4 - 2 \ln 3}{3 \ln 3 + 2 \ln 4}$$

$$= -\frac{\ln 4 - \ln 3^2}{\ln 3^3 + \ln 4^2}$$

$$= \frac{\ln 9 - \ln 4}{\ln 27 + \ln 16}$$

$$= \frac{\ln \frac{9}{4}}{\ln 432}$$

$$= \log_{432} \frac{9}{4}$$

Exercise

Solve the equation: $4^{x+3} = 3^{-x}$

Solution

$$\ln 4^{x+3} = \ln 3^{-x}$$

$$(x+3) \ln 4 = -x \ln 3$$

$$x \ln 4 + 3 \ln 4 = -x \ln 3$$

$$x \ln 4 + x \ln 3 = -3 \ln 4$$

$$x(\ln 4 + \ln 3) = -3 \ln 4$$

$$x = \frac{-3 \ln 4}{(\ln 4 + \ln 3)}$$

Exercise

Solve the equation: $7^{x+6} = 7^{3x-4}$

Solution

$$x+6 = 3x-4$$

$$4+6 = 3x-x$$

$$10 = 2x$$

$$x = 5$$

Exercise

Solve the equation: $2^{-100x} = (0.5)^{x-4}$

Solution

$$2^{-100x} = \left(\frac{1}{2}\right)^{x-4}$$

$$2^{-100x} = \left(2^{-1}\right)^{x-4}$$

$$2^{-100x} = 2^{-x+4}$$

$$-100x = -x + 4$$

$$-100x + x = 4$$

$$-99x = 4$$

$$\underline{x = -\frac{4}{99}}$$

Exercise

Solve the equation: $4^x \left(\frac{1}{2}\right)^{3-2x} = 8 \cdot (2^x)^2$

Solution

$$\left(2^2\right)^x \left(2^{-1}\right)^{3-2x} = 2^3 \cdot 2^{2x}$$

$$2^{2x} 2^{2x-3} = 2^{3+2x}$$

$$2^{2x+2x-3} = 2^{3+2x}$$

$$2^{4x-3} = 2^{3+2x}$$

$$4x - 3 = 3 + 2x$$

$$4x - 2x = 3 + 3$$

$$2x = 6$$

$$\underline{x = 3}$$

Exercise

$5^x + 125(5^{-x}) = 30$

Solution

$$5^x 5^x + 125(5^{-x}) 5^x = 30(5^x)$$

$$5^{2x} + 125 = 30(5^x)$$

$$5^{2x} - 30(5^x) + 125 = 0 \quad \text{Solve for } 5^x$$

$$5^x = 5 \quad 5^x = 25 = 5^2$$

$$x = 1 \quad x = 2$$

$$\underline{x = 1, 2}$$

Exercise

$$4^x - 3(4^{-x}) = 8$$

Solution

$$4^x 4^x - 3(4^{-x}) 4^x = 8(4^x)$$

$$4^{2x} - 3 = 8(4^x)$$

$$4^{2x} - 8(4^x) - 3 = 0 \quad \text{Solve for } 4^x$$

$$4^x = 4 + \sqrt{19} \quad 4^x = 4 - \sqrt{19} < 0$$

$$x \ln 4 = \ln(4 + \sqrt{19})$$

$$\underline{x = \frac{\ln(4 + \sqrt{19})}{\ln 4}}$$

Exercise

$$\text{Solve the equation: } 5^{3x-6} = 125$$

Solution

$$5^{3x-6} = 5^3$$

$$3x - 6 = 3$$

$$3x = 9$$

$$\underline{x = 3}$$

Exercise

$$\text{Solve the equation: } e^x = 15$$

Solution

$$\underline{x = \ln 5}$$

Convert to Log

Exercise

Solve the equation: $e^{x+1} = 20$

Solution

$$x + 1 = \ln 20$$

Convert to Log

$$\underline{x = -1 + \ln 20}$$

Exercise

Solve the equation: $9e^x = 107$

Solution

$$e^x = \frac{107}{9}$$

$$\ln e^x = \ln\left(\frac{107}{9}\right)$$

$$x \ln e = \ln\left(\frac{107}{9}\right)$$

$$\underline{x = \ln\left(\frac{107}{9}\right)}$$

Exercise

Solve the equation: $e^{x \ln 3} = 27$

Solution

$$x \ln 3 = \ln 27$$

Convert to Log

$$x \ln 3 = \ln 3^3$$

$$x = \frac{3 \ln 3}{\ln 3}$$

$$\underline{= 3}$$

Exercise

Solve the equation: $e^{x^2} = e^{7x-12}$

Solution

$$e^{x^2} = e^{7x-12}$$

$$x^2 = 7x - 12$$

$$x^2 - 7x + 12 = 0$$

$$\underline{x = 3, 4}$$

Exercise

Solve the equation: $f(x) = xe^x + e^x$

Solution

$$xe^x + e^x = 0$$

$$e^x(x+1) = 0$$

$$e^x \neq 0 \quad x+1 = 0$$

$$\underline{x = -1} \quad (\text{Only solution})$$

Exercise

Solve the equation $f(x) = x^3(4e^{4x}) + 3x^2e^{4x}$

Solution

$$x^3(4e^{4x}) + 3x^2e^{4x} = 0$$

$$x^2e^{4x}(4x+3) = 0$$

$$x^2 = 0 \quad 4x+3 = 0$$

$$x = 0, 0 \quad x = -\frac{3}{4}$$

$$\text{The solutions are: } \underline{x = 0, 0, -\frac{3}{4}}$$

Exercise

Solve the equation: $e^{2x} - 2e^x - 3 = 0$

Solution

$$(e^x)^2 - 2e^x - 3 = 0$$

$$\begin{cases} e^x = -1 \quad \times \rightarrow \text{Impossible} \\ e^x = 3 \rightarrow \underline{x = \ln 3} \end{cases}$$

Exercise

Solve the equation: $e^{0.08t} = 2500$

Solution

$$\ln(e^{0.08t}) = \ln 2500$$

$$0.08t = \ln(50)^2$$

$$t = \frac{200 \ln 50}{8}$$

$$= 25 \ln 50$$

Exercise

Solve the equation: $e^{x^2} = 200$

Solution

$$\ln e^{x^2} = \ln 200 \quad \text{Natural Log both sides}$$

$$x^2 = \ln 200 \quad \ln e = 1$$

$$x = \pm \sqrt{\ln 200}$$

Exercise

Solve the equation: $e^{2x+1} \cdot e^{-4x} = 3e$

Solution

$$e^{2x+1-4x} = 3e$$

$$e^{-2x+1} = 3e$$

$$e^{-2x}e = 3e \quad \text{Divide by } e$$

$$e^{-2x} = 3$$

$$\ln e^{-2x} = \ln 3$$

$$-2x = \ln 3$$

$$x = -\frac{1}{2} \ln 3$$

Exercise

Solve the equation: $e^{2x} - 8e^x + 7 = 0$

Solution

$$(e^x)^2 - 8e^x + 7 = 0 \quad a + b + c = 0 \rightarrow x = 1, \frac{c}{a}$$

$$\begin{cases} e^x = 1 & \rightarrow \underline{x = 0} \\ e^x = 7 & \rightarrow \underline{x = \ln 7} \end{cases}$$

Exercise

Solve the equation without using the calculator: $e^{2x} + 2e^x - 15 = 0$

Solution

$$(e^x)^2 + 2e^x - 15 = 0 \quad \text{Solve for } e^x$$

$$e^x = 3$$

$$e^x \not= -5 < 0$$

$$\underline{x = \ln 3}$$

Exercise

Solve the equation: $e^x + e^{-x} - 6 = 0$

Solution

$$e^x e^x + e^x e^{-x} - e^x 6 = e^x 0$$

$$e^{2x} + 1 - 6e^x = 0$$

$$(e^x)^2 - 6e^x + 1 = 0$$

$$e^x = \frac{6 \pm \sqrt{36 - 4}}{2}$$

$$= \frac{6 \pm 4\sqrt{2}}{2}$$

$$e^x = 3 \pm 2\sqrt{2}$$

$$\underline{x = \ln(3 \pm 2\sqrt{2})}$$

Exercise

Solve the equation: $e^{1-3x} \cdot e^{5x} = 2e$

Solution

$$e^{1-3x+5x} = 2e$$

$$e^{1+2x} = 2e$$

$$e^1 e^{2x} = 2e$$

Divide by e

$$e^{2x} = 2$$

Natural Log both sides

$$\ln e^{2x} = \ln 2$$

$$2x = \ln 2$$

$$\underline{x = \frac{1}{2} \ln 2}$$

Exercise

Solve the equation: $6 \ln(2x) = 30$

Solution

$$\ln(2x) = \frac{30}{6}$$

$$\ln(2x) = 5$$

$$2x = e^5$$

$$\underline{x = \frac{1}{2} e^5}$$

Exercise

Solve the equation: $\log_5(x - 7) = 2$

Solution

$$x - 7 = 5^2$$

$$x = 25 + 7$$

$$\underline{x = 32}$$

Exercise

Solve the equation: $\log_4(5 + x) = 3$

Solution

$$5 + x = 4^3$$

$$x = 64 - 5$$

$$\underline{= 59}$$

Check: $\log_4(5 + 59)$

Exercise

Solve the equation: $\log(4x - 18) = 1$

Solution

$$4x - 18 = 10$$

$$4x = 28$$

$$\underline{x = 7}$$

Exercise

Solve the equation: $\log(x^2 + 19) = 2$

Solution

$$x^2 + 19 = 10^2$$

$$x^2 = 81$$

$$\underline{x = \pm 9} \quad (\pm 9)^2 + 19 > 0$$

Exercise

Solve the equation: $\ln(x^2 - 12) = \ln x$

Solution

$$\ln(x^2 - 12) = \ln x$$

$$x^2 - 12 = x$$

$$x^2 - x - 12 = 0$$

$$\underline{x = -3, 4}$$

Check: $x = -3 \quad \ln(9 - 12) = \ln(-3) \quad \times$

$x = 4 \quad \ln(16 - 12) = \ln(4)$

\therefore **Solution:** $\underline{x = 4}$

Exercise

Solve the equation: $\log(2x^2 + 3x) = \log(10x + 30)$

Solution

$$\log(2x^2 + 3x) = \log(10x + 30)$$

$$2x^2 + 3x = 10x + 30$$

$$2x^2 - 7x - 30 = 0$$

$$x = \frac{7 \pm \sqrt{49 + 240}}{4}$$

$$= \begin{cases} \frac{7-17}{4} = -\frac{5}{2} \\ \frac{7+17}{4} = 6 \end{cases}$$

Check: $x = -\frac{5}{2} \quad \log\left(\frac{25}{2} - \frac{15}{2}\right) = \log(-25 + 30)$

$x = 4 \quad \log(32 + 12) = \log(40 + 30)$

∴ Solution: $x = -\frac{5}{2}, 4$

Exercise

Solve the equation: $\log_5(2x + 3) = \log_5 11 + \log_5 3$

Solution

$$\log_5(2x + 3) = \log_5(11 \times 3)$$

$$2x + 3 = 33$$

$$2x = 30$$

$$x = 15$$

Check: $\log_5(30 + 3)$

Exercise

Solve the equation: $\log_3 x - \log_9(x + 42) = 0$

Solution

$$\frac{\log x}{\log 3} - \frac{\log(x + 42)}{\log 9} = 0$$

$$\frac{\log x}{\log 3} - \frac{\log(x + 42)}{\log 3^2} = 0$$

$$\frac{\log x}{\log 3} - \frac{1}{2} \frac{\log(x + 42)}{\log 3} = 0$$

$$\log x - \frac{1}{2} \log(x + 42) = 0$$

$$2 \log x = \log(x + 42)$$

$$\log x^2 = \log(x + 42)$$

$$x^2 = x + 42$$

$$x^2 - x - 42 = 0$$

$$\underline{x = -6, 7}$$

$$\text{Check: } x = -6 \quad \log_3 (-6) - \log_9 (-6 + 42) \quad \times$$

$$x = 7 \quad \log_3 7 - \log_9 (7 + 42) = 0$$

$$\therefore \text{Solution: } \underline{x = 7}$$

Exercise

Solve the equation: $\log_5 x + \log_5 (4x - 1) = 1$

Solution

$$\log_5 x(4x - 1) = 1$$

$$4x^2 - x = 5$$

$$4x^2 - x - 5 = 0 \quad a - b + c = 0 \rightarrow x = -1, -\frac{c}{a}$$

$$\underline{x = -\frac{5}{2}, 4}$$

$$\text{Check: } x = -\frac{5}{2} \quad \log_5 \left(-\frac{5}{2}\right) + \log_5 (10 - 1) \quad \times$$

$$x = 4 \quad \log_5 (4) + \log_5 (15)$$

$$\therefore \text{Solution: } \underline{x = 4}$$

Exercise

Solve the equation: $\log x - \log (x + 3) = 1$

Solution

$$\log \frac{x}{x+3} = 1$$

$$\frac{x}{x+3} = 10$$

$$x = 10x + 30$$

$$9x = -30$$

$$\underline{x = -\frac{10}{3}}$$

$$\text{Check: } x = -\frac{10}{3} \quad \log \left(-\frac{10}{3}\right) - \log (x + 3) \quad \times$$

$$\therefore \text{No Solution}$$

Exercise

Solve the equation: $\log x + \log(x-9) = 1$

Solution

$$\log x(x-9) = 1$$

$$x^2 - 9x = 10$$

$$x^2 - 9x - 10 = 0 \quad a - b + c = 0 \rightarrow x = -1, -\frac{c}{a}$$

$$\underline{x = -1, 10}$$

$$\text{Check: } x = -1 \quad \log(-1) + \log(x-9) \quad \times$$

$$x = 10 \quad \log(10) + \log(10-9)$$

$$\therefore \text{Solution: } \underline{x = 10}$$

Exercise

Solve the equation: $\log_2(x+1) + \log_2(x-1) = 3$

Solution

$$\log_2(x+1)(x-1) = 3$$

$$x^2 - 1 = 2^3$$

$$x^2 = 9$$

$$\underline{x = \pm 3}$$

$$\text{Check: } x = -3 \quad \log_2(-2) + \log_2(x-1) \quad \times$$

$$x = 3 \quad \log_2(4) + \log_2(2)$$

$$\therefore \text{Solution: } \underline{x = 3}$$

Exercise

Solve the equation: $\log_8(x+1) - \log_8 x = 2$

Solution

$$\log_8 \frac{x+1}{x} = 2$$

$$\frac{x+1}{x} = 8^2$$

$$x+1 = 64x$$

$$63x = 1$$

$$\underline{x = \frac{1}{63}}$$

$$\text{Check: } x = \frac{1}{63} \quad \log_8 \left(\frac{1}{63} + 1 \right) - \log_8 \frac{1}{63}$$

$$\therefore \text{Solution: } \underline{x = \frac{1}{63}}$$

Exercise

Solve the equation: $\ln(x+8) + \ln(x-1) = 2 \ln x$

Solution

$$\ln(x+8)(x-1) = \ln x^2$$

$$x^2 + 7x - 8 = x^2$$

$$7x - 8 = 0$$

$$\underline{x = \frac{8}{7}}$$

$$\text{Check: } x = \frac{8}{7} \quad \ln\left(\frac{8}{7} + 8\right) + \ln\left(\frac{8}{7} - 1\right) = 2 \ln \frac{8}{7}$$

$$\therefore \text{Solution: } \underline{x = \frac{8}{7}}$$

Exercise

Solve the equation: $\ln(4x+6) - \ln(x+5) = \ln x$

Solution

$$\ln \frac{4x+6}{x+5} = \ln x$$

$$\frac{4x+6}{x+5} = x$$

$$4x+6 = x^2 + 5x$$

$$x^2 + x - 6 = 0$$

$$\underline{x = -3, 2}$$

$$\text{Check: } x = -3 \quad \ln(-6) - \ln(x+5) = \ln x \quad \times$$

$$x = 2 \quad \ln(14) - \ln(7) = \ln 2$$

$$\therefore \text{Solution: } \underline{x = 2}$$

Exercise

Solve the equation: $\ln(5 + 4x) - \ln(x + 3) = \ln 3$

Solution

$$\ln \frac{5+4x}{x+3} = \ln 3$$

$$\frac{5+4x}{x+3} = 3$$

$$5 + 4x = 3x + 9$$

$$\underline{x = 4}$$

Check: $x = 4 \quad \ln(21) - \ln(7) = \ln 3$

\therefore **Solution:** $\underline{x = 4}$

Exercise

Solve the equation: $\ln \sqrt[4]{x} = \sqrt{\ln x}$

Solution

Domain: $\underline{x \geq 1}$

$$\ln x^{1/4} = \sqrt{\ln x}$$

$$\frac{1}{4} \ln x = \sqrt{\ln x}$$

$$\left(\frac{1}{4} \ln x\right)^2 = (\sqrt{\ln x})^2$$

$$\frac{1}{16} \ln^2 x = \ln x$$

$$\ln^2 x = 16 \ln x$$

$$\ln^2 x - 16 \ln x = 0$$

$$(\ln x)(\ln x - 16) = 0$$

$$\begin{cases} \ln x = 0 \rightarrow \underline{x = 1} \\ \ln x = 16 \rightarrow \underline{x = e^{16}} \end{cases}$$

\therefore **Solution:** $\underline{x = 1, e^{16}}$

Exercise

Solve the equation: $\sqrt{\ln x} = \ln \sqrt{x}$

Solution

Domain: $\underline{x \geq 1}$

$$\sqrt{\ln x} = \ln x^{1/2}$$

$$\sqrt{\ln x} = \frac{1}{2} \ln x$$

$$\left(\sqrt{\ln x}\right)^2 = \left(\frac{1}{2} \ln x\right)^2$$

$$\ln x = \frac{1}{4} \ln^2 x$$

$$4 \ln x = \ln^2 x$$

$$\ln^2 x - 4 \ln x = 0$$

$$\ln x(\ln x - 4) = 0$$

$$\begin{cases} \ln x = 0 \rightarrow \underline{x=1} \\ \ln x = 4 \rightarrow \underline{x=e^4} \end{cases}$$

$$\therefore \text{Solution: } \underline{x=1, e^4}$$

Exercise

Solve the equation: $\log x^2 = (\log x)^2$

Solution

$$\text{Domain: } \underline{x \geq 1}$$

$$2 \log x = (\log x)^2$$

$$(\log x)^2 - 2 \log x = 0$$

$$\log x(\log x - 2) = 0$$

$$\begin{cases} \log x = 0 \rightarrow \underline{x=1} \\ \log x = 2 \rightarrow \underline{x=100} \end{cases}$$

$$\therefore \text{Solution: } \underline{x=1, 100}$$

Exercise

Solve the equation: $\log x^3 = (\log x)^2$

Solution

$$\text{Domain: } \underline{x \geq 1}$$

$$3 \log x = (\log x)^2$$

$$(\log x)^2 - 3 \log x = 0$$

$$\log x(\log x - 3) = 0$$

$$\begin{cases} \log x = 0 \rightarrow \underline{x = 1} \\ \log x = 3 \rightarrow \underline{x = 10^3} \end{cases} \quad \text{Convert to exponential}$$

$$\therefore \text{Solution: } \underline{x = 1, 10^3}$$

Exercise

Solve the equation: $\log(\log x) = 1$

Solution

$$\log x = 10 \quad \text{Convert to exponential}$$

$$\therefore \text{Solution: } \underline{x = 10^{10}}$$

Exercise

Solve the equation: $\log(\log x) = 2$

Solution

$$\log x = 10^2 \quad \text{Convert to exponential}$$

$$\therefore \text{Solution: } \underline{x = 10^{100}}$$

Exercise

Solve the equation: $\ln(\ln x) = 2$

Solution

$$\ln x = e^2 \quad \text{Convert to exponential}$$

$$\therefore \text{Solution: } \underline{x = e^{e^2}}$$

Exercise

Solve the equation: $\ln(e^{x^2}) = 64$

Solution

$$e^{x^2} = e^{64} \quad \text{Convert to exponential}$$

$$x^2 = 64$$

$$\therefore \text{Solution: } \underline{x = \pm 8}$$

Exercise

Solve the equation: $e^{\ln(x-1)} = 4$

Solution

$$x - 1 = 4$$

$$\therefore \text{Solution: } \underline{x = 5}$$

Exercise

Solve the equation: $10^{\log(2x+5)} = 9$

Solution

$$2x + 5 = 9$$

$$2x = 4$$

$$\therefore \text{Solution: } \underline{x = 2}$$

Exercise

Solve the equation: $\log \sqrt{x^3 - 9} = 2$

Solution

$$\sqrt{x^3 - 9} = 10^2$$

$$x^3 - 9 = 10^4$$

$$x^3 = 10,009$$

$$\therefore \text{Solution: } \underline{x = \sqrt[3]{10,009}}$$

Exercise

Solve the equation: $\log \sqrt{x^3 - 17} = \frac{1}{2}$

Solution

$$\log(x^3 - 17)^{1/2} = \frac{1}{2}$$

$$\frac{1}{2} \log(x^3 - 17) = \frac{1}{2}$$

$$\log(x^3 - 17) = 1$$

$$x^3 - 17 = 10$$

$$x^3 = 27$$

$$\underline{x = 3}$$

$$\text{Check: } x = 3 \quad \log \sqrt{27 - 17}$$

$$\therefore \text{Solution: } \underline{x = 3}$$

Exercise

$$\text{Solve the equation: } \log_4 x = \log_4 (8 - x)$$

Solution

$$x = 8 - x$$

$$x + x = 8$$

$$2x = 8$$

$$\underline{x = 4}$$

$$\text{Check: } x = 4 \quad \log_4 4 = \log_4 (8 - 4)$$

$$\therefore \text{Solution: } \underline{x = 4}$$

Exercise

$$\text{Solve the equation: } \log_7 (x - 5) = \log_7 (6x)$$

Solution

$$x - 5 = 6x$$

$$x - 6x = 5$$

$$-5x = 5$$

$$\underline{x = -1}$$

$$\text{Check: } x = -1 \quad \log_7 (-6) = \log_7 (6x) \quad \times$$

$$\therefore \text{No Solution}$$

Exercise

$$\text{Solve the equation: } \ln x^2 = \ln (12 - x)$$

Solution

$$\ln x^2 = \ln (12 - x)$$

$$x^2 = 12 - x$$

$$x^2 + x - 12 = 0$$

$$\underline{x = -4, 3}$$

Check: $x = -4 \quad \ln(16) = \ln(16)$

$$x = 3 \quad \ln(9) = \ln(12 - 3)$$

\therefore **Solution:** $\underline{x = -4, 3}$

Exercise

Solve the equation $\log_2 (x + 7) + \log_2 x = 3$

Solution

$$\log_2 x(x + 7) = 3$$

$$x(x + 7) = 2^3$$

Convert to Exponential Form

$$x^2 + 7x = 8$$

$$x^2 + 7x - 8 = 0$$

$$\underline{x = 1, -8}$$

Check: $x = -8 \quad \log_2 (x + 7) + \log_2 (-8) \quad \times$

$$x = 1 \quad \log_2 (1 + 7) + \log_2 1$$

\therefore **Solution:** $\underline{x = 1}$

Exercise

Solve the equation $\ln x = 1 - \ln(x + 2)$

Solution

$$\ln x + \ln(x + 2) = 1$$

$$\ln x(x + 2) = 1$$

$$x^2 + 2x = e$$

$$x^2 + 2x - e = 0$$

$$x = \frac{-2 \pm \sqrt{4 + 4e}}{2}$$

$$= \frac{-2 \pm 2\sqrt{1 + e}}{2}$$

$$= \begin{cases} -1 - \sqrt{1 + e} < 0 \\ -1 + \sqrt{1 + e} > 0 \end{cases}$$

\therefore **Solution:** $\underline{x = -1 + \sqrt{1 + e}}$

Exercise

Solve the equation $\ln x = 1 + \ln(x + 1)$

Solution

$$\ln x - \ln(x + 1) = 1$$

$$\ln \frac{x}{x+1} = 1$$

$$\frac{x}{x+1} = e^1$$

$$x = (x+1)e$$

$$x = ex + e$$

$$x - ex = e$$

$$x(1 - e) = e$$

$$x = \frac{e}{1-e} < 0$$

\therefore ***No solution***

Exercise

Solve the equation $\log_6(2x - 3) = \log_6 12 - \log_6 3$

Solution

$$\log_6(2x - 3) = \log_6 \frac{12}{3}$$

$$\log_6(2x - 3) = \log_6 4$$

$$2x - 3 = 4$$

$$2x = 7$$

$$\underline{x = \frac{7}{2}}$$

Check: $x = \frac{7}{2} \quad \log_6(7 - 3) = \log_6 12 - \log_6 3$

\therefore ***Solution:*** $\underline{x = \frac{7}{2}}$

Exercise

Solve the equation: $\log(3x + 2) + \log(x - 1) = 1$

Solution

Domain: $x > 1$

$$\log(3x + 2)(x - 1) = 1$$

Convert to exponential form

$$3x^2 - x - 2 = 10$$

$$3x^2 - x - 12 = 0$$

Solve for x

$$x = \frac{1 \pm \sqrt{1+144}}{6}$$

$$= \begin{cases} \frac{1 - \sqrt{145}}{6} < 0 \\ \frac{1 + \sqrt{145}}{6} > 1 \end{cases}$$

$$\therefore \text{Solution: } x = \frac{1 + \sqrt{145}}{6}$$

Exercise

Solve the equation: $\log_5 (x+2) + \log_5 (x-2) = 1$

Solution

$$\log_5 (x+2)(x-2) = 1$$

$$(x+2)(x-2) = 5^1$$

$$x^2 - 4 = 5$$

$$x^2 = 5 + 4$$

$$x^2 = 9$$

$$x = \pm 3$$

$$\text{Check: } x = -3 \quad \log_5 (-1) + \log_5 (x-2) \quad \times$$

$$x = 3 \quad \log_5 (3+2) + \log_5 (3-2)$$

$$\therefore \text{Solution: } x = 3$$

Exercise

Solve the equation: $\log_2 x + \log_2 (x-4) = 2$

Solution

Domain: $x > 4$

$$\log_2 x(x-4) = 2$$

$$x^2 - 4x = 2^2$$

$$x^2 - 4x - 4 = 0$$

$$x = \frac{4 \pm \sqrt{32}}{2}$$

$$= \begin{cases} 2 - 2\sqrt{2} < 4 \text{ X} \\ 2 + 2\sqrt{2} > 4 \end{cases}$$

\therefore **Solution:** $x = 2 + 2\sqrt{2}$ |

Exercise

Solve the equation: $\log_3 x + \log_3 (x + 6) = 3$

Solution

Domain: $x > 0$

$$\log_3 x(x + 6) = 3$$

$$x^2 + 6x = 3^3$$

$$x^2 + 6x - 27 = 0$$

$$x = \frac{-6 \pm \sqrt{36 + 108}}{2}$$

$$= \begin{cases} \frac{-6 - 12}{2} = -9 < 0 \text{ X} \\ \frac{-6 + 12}{2} = 3 > 0 \end{cases}$$

\therefore **Solution:** $x = 3$ |

Exercise

Solve the equation: $\log_3 (x + 3) + \log_3 (x + 5) = 1$

Solution

Domain: $x > -3$

$$\log_3 (x + 3)(x + 5) = 1$$

$$x^2 + 8x + 15 = 3$$

$$x^2 + 8x + 12 = 0$$

$$x = \frac{-8 \pm \sqrt{64 - 48}}{2}$$

$$= \begin{cases} \frac{-8 - 4}{2} = -6 < -3 \text{ X} \\ \frac{-8 + 4}{2} = -2 > -3 \end{cases}$$

\therefore **Solution:** $x = -2$ |

Exercise

Solve the equation: $\ln x = \frac{1}{2} \ln \left(2x + \frac{5}{2} \right) + \frac{1}{2} \ln 2$

Solution

Domain: $x > 0$

$$2 \ln x = \ln \left(2x + \frac{5}{2} \right) + \ln 2$$

$$\ln x^2 = \ln 2 \left(2x + \frac{5}{2} \right)$$

$$x^2 = 4x + 5$$

$$x^2 - 4x - 5 = 0$$

$$a - b + c = 0 \rightarrow x = -1, -\frac{c}{a}$$

$$\underline{x = -1, 5}$$

\therefore **Solution:** $\underline{x = 5}$

Exercise

Solve the equation $\ln(-4 - x) + \ln 3 = \ln(2 - x)$

Solution

Domain: $x < 5$

$$\ln 3(-4 - x) = \ln(2 - x)$$

$$-12 - 3x = 2 - x$$

$$-12 - 2 = 3x - x$$

$$-14 = 2x$$

\therefore **Solution:** $\underline{x = -7}$

Exercise

Solve the equation: $\log_4 x + \log_4 (x - 2) = \log_4 (15)$

Solution

Domain: $x > 2$

$$\log_4 x(x - 2) = \log_4 (15)$$

$$x^2 - 2x = 15$$

$$x^2 - 2x - 15 = 0$$

$$x = \frac{2 \pm \sqrt{4 + 60}}{2}$$

$$= \begin{cases} \frac{2-8}{2} = -4 < 2 \times \\ \frac{2+8}{2} = 5 > 2 \end{cases}$$

\therefore **Solution:** $x = 5$

Exercise

Solve the equation: $\ln(x-5) - \ln(x+4) = \ln(x-1) - \ln(x+2)$

Solution

Domain: $x > 5$

$$\ln \frac{x-5}{x+4} = \ln \frac{x-1}{x+2}$$

$$\frac{x-5}{x+4} = \frac{x-1}{x+2}$$

$$(x-5)(x+2) = (x-1)(x+4)$$

$$x^2 + 2x - 5x - 10 = x^2 + 4x - x - 4$$

$$x^2 - 3x - 10 = x^2 + 3x - 4$$

$$x^2 - 3x - 10 - x^2 - 3x + 4 = 0$$

$$-6x - 6 = 0$$

$$x = -1$$

\therefore **No solution**

Exercise

Solve the equation: $\log(x^2 + 4) - \log(x+2) = 2 + \log(x-2)$

Solution

Domain: $x > -2$

$$\log(x^2 + 4) - \log(x+2) - \log(x-2) = 2$$

$$\log(x^2 + 4) - [\log(x+2) + \log(x-2)] = 2$$

$$\log(x^2 + 4) - \log(x+2)(x-2) = 2$$

$$\log\left(\frac{x^2 + 4}{x^2 - 4}\right) = 2$$

$$\frac{x^2 + 4}{x^2 - 4} = 10^2$$

$$x^2 + 4 = 100x^2 - 400$$

$$400 + 4 = 100x^2 - x^2$$

$$99x^2 = 404$$

$$x^2 = \frac{404}{99}$$

$$\therefore \text{Solution: } \underline{x = \frac{2\sqrt{101}}{3\sqrt{11}}} \text{ is the only solution}$$

Exercise

Solve the equation $\log_3 (x-2) = \log_3 27 - \log_3 (x-4) - 5^{\log_5 1}$

Solution

Domain: $x > 4$

$$\log_3 (x-2) + \log_3 (x-4) = \log_3 3^3 - 1$$

$$\log_3 (x-2)(x-4) = 3 - 1$$

$$\log_3 (x^2 - 6x + 8) = 2$$

$$x^2 - 6x + 8 = 3^2$$

$$x^2 - 6x + 8 = 9$$

$$x^2 - 6x - 1 = 0$$

$$\rightarrow \underline{x = 3 \pm \sqrt{10}}$$

$$\text{Check: } x = 3 + \sqrt{10} > 4$$

$$x = 3 - \sqrt{10} < 4 \quad \times$$

$$\therefore \text{Solution: } \underline{x = 3 + \sqrt{10}}$$

Exercise

Solve the equation $\log_2 (x+3) = \log_2 (x-3) + \log_3 9 + 4^{\log_4 3}$

Solution

Domain: $x > 3$

$$\log_2 (x+3) - \log_2 (x-3) = 2 + 3$$

$$\log_2 \frac{x+3}{x-3} = 5$$

$$\frac{x+3}{x-3} = 2^5$$

$$x+3 = 32(x-3)$$

$$x + 3 = 32x - 96$$

$$96 + 3 = 32x - x$$

$$31x = 99$$

$$x = \frac{99}{31} > 3$$

$$\therefore \textbf{Solution: } x = \frac{99}{31}$$

Exercise

Solve for t using logarithms with base a : $2a^{t/3} = 5$

Solution

$$a^{t/3} = \frac{5}{2}$$

$$\log a^{t/3} = \log \frac{5}{2}$$

$$\frac{t}{3} \log a = \log \frac{5}{2}$$

$$\frac{t}{3} = \frac{\log \frac{5}{2}}{\log a}$$

$$\frac{t}{3} = \log_a \frac{5}{2}$$

$$t = 3 \log_a \frac{5}{2}$$

Exercise

Solve for t using logarithms with base a : $K = H - Ca^t$

Solution

$$Ca^t = H - K$$

$$a^t = \frac{H - K}{C}$$

$$\log a^t = \log \frac{H - K}{C}$$

$$t \log a = \log \frac{H - K}{C}$$

$$t = \frac{\log \frac{H - K}{C}}{\log a}$$

$$= \log_a \frac{H - K}{C}$$

Solution

Section 3.6 – Exponential Growth and Decay

Exercise

Suppose that \$10,000 is invested at interest rate of 5.4% per year, compounded continuously.

- a) Find the exponential growth function
- b) What will the balance be after, 1 yr 10 yrs?
- c) After how long will the investment be double?

Solution

a) $P(t) = 10000e^{0.054t}$

b) $P(t=1) = 10000e^{0.054(1)}$
 $\approx \$10,555$

$$P(t=10) = 10000e^{0.054(10)}$$
$$\approx \$17,160$$

c) $T = \frac{\ln 2}{k}$
 $= \frac{\ln 2}{0.054}$
 $\approx 12.8 \text{ yrs}$

Exercise

In 1990, the population of Africa was 643 million and by 2000 it had grown to 813 million

- a) Use the exponential growth function $A(t) = A_0 e^{kt}$, in which t is the number of years after 1990, to find the exponential growth function that models data
- b) By which year will Africa's population reach 2000 million, or two billion?

Solution

a) $A(t) = A_0 e^{kt}$ From 1990 to 2000, is 10 years

$$813 = 643e^{k(10)}$$

$$\frac{813}{643} = e^{10k}$$

$$\ln \frac{813}{643} = \ln e^{10k}$$

$$\ln \frac{813}{643} = 10k$$

$$\frac{1}{10} \ln \frac{813}{643} = k$$

$$k \approx 0.023$$

$$\underline{A(t) = 643e^{0.023t}}$$

$$b) \quad 2000 = 643e^{0.023t}$$

$$\frac{2000}{643} = e^{0.023t}$$

$$\ln \frac{2000}{643} = \ln e^{0.023t}$$

$$\ln \frac{2000}{643} = 0.023t$$

$$\frac{\ln \frac{2000}{643}}{0.023} = t$$

$$\underline{t \approx 49}$$

Africa's population reach 2000 *million* in Year : 2039

Exercise

The radioactive element carbon-14 has a half-life of 5750 yr. The percentage of carbon-14 present in the remains of organic matter can be used to determine the age of that organic matter. Archaeologists discovered that the linen wrapping from one of the Dead Sea Scrolls had lost 22.3% of its carbon-14 at the time it was found. How old was the linen wrapping?

Solution

When $t = 5750$ (half-life) $\rightarrow P(t)$ will be half $P_0 \rightarrow P(t) = \frac{1}{2} P_0$

$$P(t) = P_0 e^{-kt}$$

$$\frac{1}{2} P_0 = P_0 e^{-k(5750)}$$

$$\frac{1}{2} = e^{-k(5750)}$$

$$\ln \frac{1}{2} = \ln e^{-k(5750)}$$

$$\ln \frac{1}{2} = -k(5750)$$

$$k = -\frac{\ln \frac{1}{2}}{5750}$$

$$\underline{\approx 0.00012}$$

$$P(t) = P_0 e^{-(0.00012)t}$$

Lost 22.3% $\Rightarrow 100 - 22.3 = 77.7\%$ left from it is original.

$$0.777 P_0 = P_0 e^{-0.00012t}$$

$$0.777 = e^{-0.00012t}$$

$$\ln 0.777 = \ln e^{-0.00012t}$$

$$\ln 0.777 = -0.00012t$$

$$-0.00012t = \ln 0.777$$

$$t = \frac{\ln 0.777}{-0.00012}$$

$$\approx 2103$$

Exercise

Suppose that \$2000 is invested at interest rate k , compounded continuously, and grows to \$2983.65 in 5 yrs.

- What is the interest rate?
- Find the exponential growth function
- What will the balance be after 10 yrs.?
- After how long will the \$2000 have doubled?

Solution

$$a) \quad P(t) = P_0 e^{kt}$$

$$P(t=5) = P_0 e^{k5} = 2983.65$$

$$2000e^{k5} = 2983.65$$

$$e^{5k} = \frac{2983.65}{2000}$$

$$\ln e^{5k} = \ln \left(\frac{2983.65}{2000} \right)$$

$$5k \ln e = \ln \left(\frac{2983.65}{2000} \right)$$

$$k = \frac{1}{5} \ln \left(\frac{2983.65}{2000} \right)$$

$$\approx 0.08$$

$$\text{or } k = 8\%$$

$$b) \quad P(t) = 2000e^{0.08t}$$

$$c) \quad P(t=10) = 2000e^{0.08(10)}$$

$$\approx \$4451.08$$

$$d) \quad T = \frac{\ln 2}{0.08} \qquad T = \frac{\ln 2}{k}$$

$$\underline{\approx 8.7 \text{ yrs}} \quad |$$

Exercise

In 2005, the population of China was about 1.306 *billion*, and the exponential growth rate was 0.6% per year.

- Find the exponential growth function
- Estimate the population in 2008
- After how long will the population be double what it was in 2005?

Solution

- a) In 2005 $\Rightarrow t = 0$

$$k = \frac{0.6}{100}$$

$$\underline{= 0.006} \quad |$$

$$P_0 = 1.306$$

$$\underline{P(t) = 1.306e^{0.006t}} \quad |$$

- b) $P(t = 3) = 1.306e^{0.006(3)}$
- $$\underline{\approx 1.33} \quad |$$

- c) $2(1.306) = 1.306e^{0.006t}$

$$2 = e^{0.006t}$$

$$e^{0.006t} = 2$$

$$\ln e^{0.006t} = \ln 2$$

$$0.006t = \ln 2$$

$$t = \frac{\ln 2}{0.006}$$

$$\underline{\approx 116 \text{ yrs}} \quad |$$

Exercise

How long will it take for the money in an account that is compounded continuously at 3% interest to double?

Solution

$$T = \frac{\ln 2}{0.03} \qquad T = \frac{\ln 2}{k}$$

$$\underline{\approx 23 \text{ yrs}} \quad |$$

Exercise

If 600 g of radioactive substance are present initially and 3 yrs later only 300 g remain, how much of the substance will be present after 6 yrs?

Solution

$$y(t) = 600e^{kt}$$

$$y(t) = y_0 e^{kt}$$

When $t = 3 \rightarrow y = 300$

$$300 = 600e^{k(3)}$$

$$\frac{300}{600} = e^{3k}$$

$$\ln \frac{300}{600} = \ln e^{3k}$$

$$\ln e = 1$$

$$3k = \ln \frac{300}{600}$$

$$k = \frac{1}{3} \ln \frac{300}{600}$$

$$\approx -.231$$

$$y(t) = 600e^{-.231t}$$

$$y(6) = 600e^{-.231(6)}$$

$$\approx 150 \text{ g}$$

Exercise

The population of an endangered species of bird was 4200 in 1990. Thirteen years later, in 2003, the bird population declined to 3000. The population of the birds is decreasing exponentially according to the function $A(t) = 4200e^{kt}$ where A is the bird population t years after 1990. Use this information to find the value of k .

Solution

$$k = \frac{\ln\left(\frac{3000}{4200}\right)}{13}$$

$$kT = \ln \frac{A}{A_0}$$

$$\approx -0.26$$

Exercise

A city had a population of 21,400 in 2000 and a population of 23,200 in 2005.

- a) Find the exponential growth function for the city.
- b) Use the growth function to predict the population of the city in 2018.

Solution

a) **Given:** $P(0) = 21,400$ $P(5) = 23,200$

$$k = \frac{1}{5} \ln \frac{23,200}{21,400} \qquad kT = \ln \frac{P}{P_0}$$

$$= \frac{1}{5} \ln \frac{116}{107} \quad \Bigg|$$
$$\approx 0.01615 \quad \Bigg|$$

$$P(t) = 21,400e^{\frac{1}{5} \ln \left(\frac{116}{107} \right) t} \quad \Bigg|$$

$$P(t) = 21,400e^{0.01615t} \quad \Bigg|$$

b) $P(18) = 21,400e^{0.01615(18)}$

$$\approx 28,620 \quad \Bigg|$$

Exercise

A city had a population of 53,700 in 2002 and a population of 58,100 in 2006.

- a) Find the exponential growth function for the city.
- b) Use the growth function to predict the population of the city in 2013.

Solution

a) **Given:** $P(0) = 53,700$ $P(4) = 58,100$

$$k = \frac{1}{4} \ln \frac{58,100}{53,700} \qquad kT = \ln \frac{P}{P_0}$$

$$= \frac{1}{4} \ln \frac{581}{537} \quad \Bigg|$$
$$\approx 0.019688 \quad \Bigg|$$

$$P(t) = 53,700e^{\frac{1}{4} \ln \left(\frac{581}{537} \right) t} \quad \Bigg|$$

$$P(t) = 53,700e^{0.019688t} \quad \Bigg|$$

b) $t = 2013 - 2002 = 11$

$$P(11) = 53,700e^{0.019688(11)}$$
$$\approx 66,685 \quad \Bigg|$$

Exercise

The population of Charlotte, North Carolina, is growing exponentially. The population of Charlotte was 395,934 in 1990 and 610,949 in 2005. Find the exponential growth function that models the population of Charlotte and use it to predict the population of Charlotte in 2017.

Solution

$$\text{Given: } P(0) = 395,934 \quad P(15) = 610,949$$

$$k = \frac{1}{15} \ln \frac{610,949}{395,934} \qquad kT = \ln \frac{P}{P_0}$$
$$\approx 0.02892 \quad |$$

$$P(t) = 395,934e^{0.02892t} \quad |$$

$$P(27) = 395,934e^{0.02892(27)}$$
$$\approx 864,392 \quad |$$

Exercise

The population of Las Vegas, Nevada, is growing exponentially. The population of Las Vegas was 258,295 in 1990 and 545,147 in 2005. Find the exponential growth function that models the population of Las Vegas and use it to predict the population of Las Vegas in 2017.

Solution

$$\text{Given: } P(0) = 258,295 \quad P(15) = 545,147$$

$$k = \frac{1}{15} \ln \frac{545,147}{258,295} \qquad kT = \ln \frac{P}{P_0}$$
$$\approx 0.049797 \quad |$$

$$P(t) = 258,295e^{0.049797t} \quad |$$

$$P(27) = 258,295e^{0.049797(27)}$$
$$\approx 990,908 \quad |$$

Exercise

Find the decay function for the amount of Polonium $\left(^{210}\text{Po}\right)$ that remains in a sample after t days.

Solution

$$k = \frac{\ln \frac{1}{2}}{138} \qquad k = \frac{\ln \frac{1}{2}}{T}$$
$$= -0.005023 \quad |$$

$$\underline{A(t) = A_0 e^{-0.005023t}}$$

Exercise

Estimate the percentage of polonium (^{210}Po) that remains in a sample after 2 years.

Solution

$$\begin{aligned} k &= \frac{\ln \frac{1}{2}}{138} \\ &= -0.005023 \end{aligned}$$

$$k = \frac{\ln \frac{1}{2}}{T}$$

$$A(t) = A_0 e^{-0.005023t}$$

$$\begin{aligned} \ln \frac{A}{A_0} &= -\frac{0.005023}{2} \\ &= -0.0025115 \end{aligned}$$

$$\ln \frac{A}{A_0} = kT$$

$$\begin{aligned} \frac{A}{A_0} &= e^{-0.0025115} \\ &\approx 0.9975 \end{aligned}$$

\therefore The percentage of polonium that remains in a sample after 2 years is about 99.75%.

Exercise

Estimate the age of a bone if it now contains 65% of its original amount of carbon-14.

Solution

$$\text{Given: } A = .65A_0$$

$$\begin{aligned} k &= \frac{\ln \frac{1}{2}}{5730} \\ &= -\frac{\ln 2}{5730} \end{aligned}$$

$$k = \frac{\ln \frac{1}{2}}{T}$$

$$tk = \ln \frac{0.65A_0}{A_0}$$

$$kT = \ln \frac{A}{A_0}$$

$$\begin{aligned} t &= -\frac{5730 \ln(0.65)}{\ln 2} \\ &\approx 3561 \end{aligned}$$

\therefore The age of a bone is approximately 3561 years old.

Exercise

Geologists have determined that Crater Lake in Oregon was formed by a volcanic eruption. Chemical analysis of a wood chip assumed to be from a tree that died during the eruption has shown that it contains approximately 45% of its original carbon-14. Estimate how long ago the volcanic eruption occurred.

Solution

Given: $A = .45A_0$

$$k = \frac{\ln \frac{1}{2}}{5730}$$
$$= -\frac{\ln 2}{5730}$$

$$k = \frac{\ln \frac{1}{2}}{T}$$

$$tk = \ln \frac{0.45A_0}{A_0}$$

$$kT = \ln \frac{A}{A_0}$$

$$t = -\frac{5730 \ln(0.45)}{\ln 2}$$
$$\approx 6,600 \text{ |}$$

\therefore The age of a bone is approximately 6,600 years old.

Exercise

Lead shielding is used to contain radiation. The percentage of a certain radiation that can penetrate x millimeters of lead shielding is given by $I(x) = 100e^{-1.5x}$

- a) What percentage of radiation will penetrate a lead shield that is 1 millimeter thick?
- b) How many millimeters of lead shielding penetrates the shielding?

Solution

a) $I(1) = 100e^{-1.5}$

$$\approx 22.313 \text{ |}$$

\therefore The percentage of radiation will penetrate a lead shield is approximately 22.313%

Exercise

After a race, a runner's pulse rate R , in beats per minute, decreases according to the function

$$R(t) = 145e^{-0.092t}, \quad 0 \leq t \leq 15$$

Where t is measured in minutes.

- a) Find the runner's pulse rate at the end of the race and 1 minute after the end of the race.
- b) How long after the end of the race will the runner's pulse rate be 80 beats per minute?

Solution

$$\begin{aligned} a) \quad R(15) &= 145e^{-0.092(15)} \\ &\approx 36.48 \end{aligned}$$

$$\begin{aligned} R(16) &= 145e^{-0.092(16)} \\ &\approx 33.27 \end{aligned}$$

Exercise

A can of soda at $79^\circ F$ is placed in a refrigerator that maintains a constant temperature of $36^\circ F$. The temperature T of the soda t minutes after it is placed in the refrigerator is given by

$$T(t) = 36 + 43e^{-0.058t}$$

- a) Find the temperature of the soda 10 minutes after it is placed in the refrigerator.
- b) When will the temperature of the soda be $45^\circ F$

Solution

$$\begin{aligned} a) \quad T(10) &= 36 + 43e^{-0.058(10)} \\ &\approx 60^\circ F \end{aligned}$$

$$\begin{aligned} b) \quad 36 + 43e^{-0.058t} &= 45 \\ 43e^{-0.058t} &= 9 \\ e^{-0.058t} &= \frac{9}{43} \\ -0.058t &= \ln \frac{9}{43} \\ t &= \frac{-1}{0.058} \ln \frac{9}{43} \\ &\approx 27 \text{ min} \end{aligned}$$

Exercise

During surgery, a patient's circulatory system requires at least 50 milligrams of an anesthetic. The amount of anesthetic present t hours after 80 milligrams of anesthetic is administered is given by

$$T(t) = 80(0.727)^t$$

- a) How much of the anesthetic is present in the patient's circulatory system 30 minutes after the anesthetic is administered?
- b) How long can the operation last if the patient does not receive additional anesthetic?

Solution

$$a) \quad T\left(30 = \frac{1}{2} \text{ hr}\right) = 80(0.727)^{1/2}$$

$$\underline{\approx 68 \text{ mg}} \quad |$$

$$b) \quad T(t) = 80(0.727)^t = 50$$

$$(0.727)^t = \frac{5}{8}$$

$$t = \log_{.727} \left(\frac{5}{8}\right)$$

$$\underline{\approx 1.47 \text{ hrs}} \quad |$$

$$\underline{= 1 \text{ hr } 28' 12''} \quad |$$

Exercise

The following function models the average typing speed S , in *words per minute*, for a student who has been typing for t *months*.

$$S(t) = 5 + 29 \ln(t + 1), \quad 0 \leq t \leq 9$$

Use S to determine how long it takes the student to achieve an average speed of 65 *words per minute*.

Solution

$$S(t) = 5 + 29 \ln(t + 1) = 65$$

$$29 \ln(t + 1) = 60$$

$$\ln(t + 1) = \frac{60}{29}$$

$$t + 1 = e^{\frac{60}{29}}$$

$$\underline{t = e^{\frac{60}{29}} - 1} \quad |$$

$$\underline{t \approx 7 \text{ months}} \quad |$$

Exercise

The exponential function

$$S(x) = 8320(0.73)^x, \quad 10 \leq x \leq 20$$

models the speed of the dragster during the 10-*second* period immediately following the time when the dragster crosses the finish line. This is the deceleration period.

How long after the start of the race did the dragster attain a speed of 275 *miles per hour*?

Solution

$$S(x) = 8320(0.73)^x = 275$$

$$(0.73)^x = \frac{275}{8320}$$

$$x = \log_{0.73} \left(\frac{275}{8320} \right) \text{ minutes}$$

$$\approx 11 \text{ minutes}$$

Exercise

If \$8,000 is invested at an annual interest rate of 5% and compounded annually, find the balance after

- a) 4 years. b) 8 years.

Solution

Given: $P = 8,000$ $r = 0.05$ $n = 1$

a) $t = 4$

$$\begin{aligned} A &= 8,000 \left(1 + \frac{.05}{1} \right)^4 & A &= P \left(1 + \frac{r}{n} \right)^{tn} \\ &= 8,000(1.05)^4 \\ &= \$9,724.05 \end{aligned}$$

b) $t = 8$

$$\begin{aligned} A &= 8,000 \left(1 + \frac{.05}{1} \right)^8 & A &= P \left(1 + \frac{r}{n} \right)^{tn} \\ &= 8,000(1.05)^8 \\ &\approx \$11,819.64 \end{aligned}$$

Exercise

If \$20,000 is invested at an annual interest rate of 4.5% and compounded annually, find the balance after

- a) 3 years. b) 5 years.

Solution

Given: $P = 20,000$ $r = 0.045$ $n = 1$

a) $t = 3$

$$\begin{aligned} A &= 20,000 \left(1 + \frac{.045}{1} \right)^3 & A &= P \left(1 + \frac{r}{n} \right)^{tn} \\ &= 20,000(1.045)^3 \\ &= \$22,823.32 \end{aligned}$$

b) $t = 5$

$$\begin{aligned} A &= 20,000 \left(1 + \frac{.045}{1}\right)^5 \\ &= 20,000(1.045)^5 \\ &\approx \underline{\$24,923.64} \end{aligned}$$

$$A = P \left(1 + \frac{r}{n}\right)^{tn}$$

Exercise

If \$10,000 is invested at an annual interest rate of 3% for 5 years, find the balance if the interest rate is compounded

- | | | | |
|-------------------|--------------|----------------|-----------------|
| a) Annually. | c) Quarterly | e) Daily (365) | g) Continuously |
| b) Semi-annually. | d) Monthly | f) Hourly | |

Solution

Given: $P = 10,000$ $r = 0.03$ $t = 5$

a) Annually: $n = 1$

$$\begin{aligned} A &= 10,000 \left(1 + \frac{.03}{1}\right)^5 \\ &= 10,000(1.03)^5 \\ &\approx \underline{\$11,592.74} \end{aligned}$$

$$A = P \left(1 + \frac{r}{n}\right)^{tn}$$

b) Semi-annually: $n = 2$

$$\begin{aligned} A &= 10,000 \left(1 + \frac{.03}{2}\right)^{10} \\ &= 10,000(1.015)^{10} \\ &\approx \underline{\$11,605.41} \end{aligned}$$

$$A = P \left(1 + \frac{r}{n}\right)^{tn}$$

c) Quarterly: $n = 4$

$$\begin{aligned} A &= 10,000 \left(1 + \frac{.03}{4}\right)^{20} \\ &\approx \underline{\$11,611.84} \end{aligned}$$

$$A = P \left(1 + \frac{r}{n}\right)^{tn}$$

d) Monthly: $n = 12$

$$\begin{aligned} A &= 10,000 \left(1 + \frac{.03}{12}\right)^{60} \\ &\approx \underline{\$11,616.17} \end{aligned}$$

$$A = P \left(1 + \frac{r}{n}\right)^{tn}$$

e) Daily: $n = 365$

$$\begin{aligned} A &= 10,000 \left(1 + \frac{.03}{365}\right)^{365(5)} \\ &\approx \underline{\$11,618.27} \end{aligned}$$

$$A = P \left(1 + \frac{r}{n}\right)^{tn}$$

f) Hourly: $n = 365 \times 24 = 8,760$

$$A = 10,000 \left(1 + \frac{.03}{8,760} \right)^{43,800} \quad A = P \left(1 + \frac{r}{n} \right)^{tn}$$
$$\approx \$11,618.34$$

g) Continuously

$$A = 10,000 e^{(.03)(5)} \quad A = Pe^{rt}$$
$$\approx \$11,618.34$$

Exercise

If \$20,000 is invested at an annual interest rate of 2% for 10 years, find the balance if the interest rate is compounded

- | | | | |
|-------------------|--------------|----------------|-----------------|
| a) Annually. | c) Quarterly | e) Daily (365) | g) Continuously |
| b) Semi-annually. | d) Monthly | f) Hourly | |

Solution

Given: $P = 20,000$ $r = 0.02$ $t = 10$

a) Annually: $n = 1$

$$A = 20,000 \left(1 + \frac{.02}{1} \right)^{10} \quad A = P \left(1 + \frac{r}{n} \right)^{tn}$$
$$= 20,000(1.02)^{10}$$
$$\approx \$24,379.89$$

b) Semi-annually: $n = 2$

$$A = 20,000 \left(1 + \frac{.02}{2} \right)^{20} \quad A = P \left(1 + \frac{r}{n} \right)^{tn}$$
$$\approx \$24,403.80$$

c) Quarterly: $n = 4$

$$A = 20,000 \left(1 + \frac{.02}{4} \right)^{40} \quad A = P \left(1 + \frac{r}{n} \right)^{tn}$$
$$\approx \$24,416.88$$

d) Monthly: $n = 12$

$$A = 20,000 \left(1 + \frac{.02}{12} \right)^{120} \quad A = P \left(1 + \frac{r}{n} \right)^{tn}$$
$$\approx \$24,423.99$$

e) Daily: $n = 365$

$$A = 20,000 \left(1 + \frac{.02}{365} \right)^{3650} \quad A = P \left(1 + \frac{r}{n} \right)^{tn}$$

$$\approx \$24,427.92 \mid$$

f) Hourly: $n = 365 \times 24 = 8,760$

$$A = 20,000 \left(1 + \frac{.02}{8,760} \right)^{87,600} \quad A = P \left(1 + \frac{r}{n} \right)^{tn}$$

$$\approx \$24,428.05 \mid$$

g) Continuously

$$A = 20,000e^{(.02)(10)} \quad A = Pe^{rt}$$

$$\approx \$24,428.05 \mid$$

Exercise

Find the accumulated value of an investment of \$10,000 for 5 years at an interest rate of 5.5% if the money is

- Compounded *semiannually*
- Compounded *quarterly*
- Compounded *monthly*
- Compounded *Continuously*

Solution

Given: $P = 10000$

$t = 5$

$r = 0.055$

a) Semiannually: $n = 2$

$$A = 10000 \left(1 + \frac{0.055}{2} \right)^{2(5)} \\ = \$13,116.51 \mid$$

b) Quarterly: $n = 4$

$$A = 10000 \left(1 + \frac{0.055}{4} \right)^{4(5)} \\ = \$13,140.67 \mid$$

c) Monthly: $n = 12$

$$A = 10000 \left(1 + \frac{0.055}{12} \right)^{12(5)} \\ = \$13,157.04 \mid$$

d) $A = 10000e^{(0.055)(5)}$

$$= \$13,165.31 \mid$$

Exercise

Suppose \$1,000 is deposited in an account paying 4% interest per year compounded *quarterly*.

- a) Find the amount in the account after 10 years with no withdrawals.
- b) How much interest is earned over the 10 years period?

Solution

Given: $P = 1000$ $r = .04$ $n = 4$

a) $t = 10$

$$A = 1000 \left(1 + \frac{.04}{4} \right)^{10(4)} \qquad A = P \left(1 + \frac{r}{n} \right)^{tn}$$
$$\underline{= \$1,488.86}$$

b) The interest earned: $\$1488.86 - \$1000 = \underline{\$488.86}$

Exercise

Becky must pay a lump sum of \$6000 in 5 yrs.

- a) What amount deposited today at 3.1% compounded annually will grow to \$6000 in 5 yrs.?
- b) If only \$5000 is available to deposit now, what annual interest rate is necessary for the money to increase to \$6000 in 5 yrs.?

Solution

a) $A = P \left(1 + \frac{r}{n} \right)^{tn}$

$$6000 = P \left(1 + \frac{.031}{1} \right)^{5(1)}$$

$$6000 = P(1.031)^5$$

$$\frac{6000}{(1.031)^5} = P \qquad 6000 / 1.031^5$$

$$\underline{P \approx \$5,150.60}$$

b) $A = P \left(1 + \frac{r}{n} \right)^{tn}$

$$6000 = 5000 \left(1 + \frac{r}{1} \right)^{5(1)}$$

$$\frac{6000}{5000} = (1 + r)^5$$

$$\frac{6}{5} = (1 + r)^5$$

$$\left(\frac{6}{5} \right)^{1/5} = 1 + r$$

$$r = \left(\frac{6}{5}\right)^{1/5} - 1 \quad (6/5)^{(1/5)} - 1$$

$$\approx .0371 \mid$$

The interest rate of 3.71% will produce enough to increase the \$5,000 to \$6,000 by the end of 5 yrs.

Exercise

An investment of 1,000 increased to \$13,464 in 20 *years*. If the interest was compounded continuously, find the interest rate.

Solution

$$A = Pe^{rt}$$

$$13464 = 1000e^{20r}$$

$$13.464 = e^{20r}$$

$$\ln(13.464) = \ln e^{20r}$$

$$20r = \ln 13.464$$

$$r = \frac{\ln 13.464}{20}$$

$$\approx 0.13 \mid$$

The interest rate is 13%.

Exercise

Find the present value of \$4,000 if the annual interest rate is 3.5% compounded *quarterly* for 6 *years*.

Solution

Given: $A = 4000.00$, $r = 0.035$, $t = 6$, $n = 4$

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$4000 = P\left(1 + \frac{0.035}{4}\right)^{4(6)}$$

$$P = \frac{4000}{\left(1 + \frac{0.035}{4}\right)^{4(6)}}$$

$$= \$3,245.30 \mid$$

Exercise

How much money will there be in an account at the end of 8 years if \$18,000 is deposited at 3% interest compounded *semi-annually*?

Solution

$$A = 18000 \left(1 + \frac{0.03}{2}\right)^{2(8)} \qquad A = P \left(1 + \frac{r}{n}\right)^{nt}$$
$$\underline{= \$22,841.74}$$

Exercise

The function defined by $P(x) = 908e^{-0.0001348x}$ approximates the atmospheric pressure (in *millibars*) at an altitude of x meters. Use P to predict the pressure:

- a) At 0 meters
- b) At 12,000 meters

Solution

- a) At 0 meters

$$P(x = 0) = 908e^{-0.0001348(0)}$$
$$\underline{= 908 \text{ millibars}}$$

- b) At 12,000 meters

$$P(x = 12,000) = 908e^{-0.0001348(12,000)}$$
$$\underline{\approx 180 \text{ millibars}}$$

Exercise

How long, to the nearest tenth of a year, will it take \$1000 to grow to \$3600 at 8% annual interest compounded quarterly?

Solution

Given: $A = \$3600$
 $P = \$1000$
 $r = 8\% = 0.08$
 $n = 4$

$$A = P \left(1 + \frac{r}{n}\right)^{nt}$$
$$3600 = 1000 \left(1 + \frac{0.08}{4}\right)^{4t}$$
$$3.6 = (1.02)^{4t}$$

$$\ln 3.6 = \ln(1.02)^{4t}$$

$$\ln 3.6 = 4t \ln(1.02)$$

$$\frac{\ln 3.6}{4 \ln 1.02} = t$$

$$t \approx 16.2 \text{ yrs}$$

Exercise

The annual revenue R , in *dollars*, of a new company can be closely modeled by the logistic function

$$R(t) = \frac{625,000}{1 + \frac{3}{10}e^{-.045t}}$$

Where the natural number t is the time, in *years*, since the company was founded.

- According to the model, what will the company's annual revenue for its first year and its second year?
- According to the model, what will the company's annual revenue approach in the long-term future?

Solution

- First year:

$$R(1) = \frac{625,000}{1 + \frac{3}{10}e^{-.045}}$$

$$\approx \$485,701.25$$

Second year:

$$R(1) = \frac{625,000}{1 + \frac{3}{10}e^{-.045(2)}}$$

$$\approx \$490,511.79$$

$$b) \lim_{t \rightarrow \infty} R(t) = \lim_{t \rightarrow \infty} \frac{625,000}{1 + \frac{3}{10}e^{-.045t}}$$

$$= \frac{625,000}{1 + \frac{3}{10} \lim_{t \rightarrow \infty} e^{-.045t}}$$

$$= \frac{625,000}{1 + \frac{3}{10}(0)}$$

$$= \$625,000$$

$$\lim_{t \rightarrow \infty} e^{-\infty} = 0$$

Exercise

The number of cars A sold annually by an automobile dealership can be closely modeled by the logistic function

$$A(t) = \frac{1,650}{1 + \frac{12}{5}e^{-.055t}}$$

- a) According to the model, what number of cars will the dealership sell during its first year and its second year?
- b) According to the model, what will the dealership's car sales approach in the long-term future?

Solution

- a) First year:

$$A(1) = \frac{1,650}{1 + \frac{12}{5}e^{-.055}}$$
$$\approx 504 \text{ cars}$$

Second year:

$$A(2) = \frac{1,650}{1 + \frac{12}{5}e^{-.055(2)}}$$
$$\approx 524 \text{ cars}$$

b) $\lim_{t \rightarrow \infty} A(t) = \lim_{t \rightarrow \infty} \frac{1,650}{1 + \frac{12}{5}e^{-.055t}}$

$$= \frac{1,650}{1 + \frac{12}{5} \lim_{t \rightarrow \infty} e^{-.055t}}$$
$$= \frac{1,650}{1 + \frac{12}{5}(0)}$$
$$= 1,650 \text{ cars}$$

$$\lim_{t \rightarrow \infty} e^{-\infty} = 0$$

Exercise

The population of wolves in a preserve satisfies a logistic model in which $P_0 = 312$ in 2008, $c = 1,600$, and $P(6) = 416$.

- a) Determine the logistic model for this population, where t is the number of years after 2008.
- b) Use the logistic model from part (a) to predict the size of the groundhog population in 2014.

Solution

Given: $P_0 = 312$ $c = 1,600$ $P(6) = 416$

$$a) \quad P(t) = \frac{1,600}{1 + ae^{-bt}}$$

$$P(0) = \frac{1,600}{1 + a} = 312$$

$$1 + a = \frac{1,600}{312}$$

$$a = \frac{200}{39} - 1$$

$$= \frac{161}{39} \quad |$$

$$P(t) = \frac{1,600}{1 + \frac{161}{39}e^{-bt}}$$

$$P(6) = \frac{1,600}{1 + \frac{161}{39}e^{-6b}} = 416$$

$$1 + \frac{161}{39}e^{-6b} = \frac{1,600}{416}$$

$$\frac{161}{39}e^{-6b} = \frac{50}{13} - 1$$

$$\frac{161}{39}e^{-6b} = \frac{37}{13}$$

$$e^{-6b} = \frac{37}{13} \frac{39}{161}$$

$$e^{-6b} = \frac{111}{161}$$

$$-6b = \ln \frac{111}{161}$$

$$b = -\frac{1}{6} \ln \left(\frac{111}{161} \right)$$

$$\approx .062 \quad |$$

$$P(t) = \frac{1,600}{1 + \frac{161}{39}e^{-.062t}} \quad |$$

$$b) \quad t = 2014 - 2008 = 6 \quad |$$

$$P(6) = \frac{1,600}{1 + \frac{161}{39}e^{-.062(6)}}$$

$$\approx 416.04 \quad |$$

$$P(t) = \frac{c}{1 + ae^{-bt}}$$

Exercise

The population of walrus on an island satisfies a logistic model in which $P_0 = 800$ in 2006, $c = 5,500$, and $P(1) = 900$.

- Determine the logistic model for this population, where t is the number of years after 2006.
- Use the logistic model from part (a) to predict the year in which the walrus population will first exceed 2000.

Solution

Given: $P_0 = 800$ $c = 5,500$ $P(1) = 900$

$$a) \quad P(t) = \frac{5,500}{1 + ae^{-bt}} \qquad P(t) = \frac{c}{1 + ae^{-bt}}$$

$$P(0) = \frac{5,500}{1 + a} = 800$$

$$1 + a = \frac{5,500}{800}$$

$$a = \frac{55}{8} - 1$$

$$= \frac{47}{8}$$

$$P(t) = \frac{5,500}{1 + \frac{47}{8}e^{-bt}}$$

$$P(1) = \frac{5,500}{1 + \frac{47}{8}e^{-b}} = 900$$

$$1 + \frac{47}{8}e^{-b} = \frac{5,500}{900}$$

$$\frac{47}{8}e^{-b} = \frac{55}{9} - 1$$

$$e^{-b} = \frac{46}{9} \cdot \frac{8}{47}$$

$$e^{-b} = \frac{368}{423}$$

$$-b = \ln\left(\frac{368}{423}\right)$$

$$b = -\ln\left(\frac{368}{423}\right)$$

$$\approx 0.139$$

$$P(t) = \frac{5,500}{1 + \frac{47}{8}e^{-0.139t}}$$

$$b) \quad P(t) = \frac{5,500}{1 + \frac{47}{8} e^{-0.139t}} = 2,000$$

$$1 + \frac{47}{8} e^{-0.139t} = \frac{5,500}{2,000}$$

$$\frac{47}{8} e^{-0.139t} = \frac{11}{4} - 1$$

$$e^{-0.139t} = \frac{7}{4} \cdot \frac{8}{47}$$

$$-0.139t = \ln\left(\frac{14}{47}\right)$$

$$t = -\frac{1}{0.139} \ln\left(\frac{14}{47}\right)$$

$$\approx 8.8 \text{ years}$$

\therefore The walrus population will first exceed 2000 in year 2015

Exercise

Newton's Law of Cooling states that if an object at temperature T_0 is placed into an environment at constant temperature A , then the temperature of the object, $T(t)$ (in degrees Fahrenheit), after t minutes is given by $T(t) = A + (T_0 - A)e^{-kt}$, where k is a constant that depends on the object.

- Determine the constant k for a canned soda drink that takes 5 minutes to cool from $75^\circ F$ to $65^\circ F$ after being placed in a refrigerator that maintains a constant temperature of $34^\circ F$
- What will be the temperature of the soda after 30 minutes?
- When will the temperature of the soda drink be $36^\circ F$?

Solution

$$a) \quad T(5) = 34 + (75 - 34)e^{-5k} = 65$$

$$41e^{-5k} = 31$$

$$e^{-5k} = \frac{31}{41}$$

$$-5k = \ln\left(\frac{31}{41}\right)$$

$$k = -\frac{1}{5} \ln\left(\frac{31}{41}\right)$$

$$\approx 0.0559$$

$$b) \quad T(t) = 34 + 41e^{-0.0559t}$$

$$T(30) = 34 + 41e^{-0.0559(30)}$$

$$\approx 42^\circ F$$

$$c) \quad T(t) = 34 + 41e^{-0.0559t} = 36$$

$$41e^{-0.0559t} = 2$$

$$e^{-0.0559t} = \frac{2}{41}$$

$$-0.0559t = \ln\left(\frac{2}{41}\right)$$

$$t = -\frac{1}{0.0559} \ln\left(\frac{2}{41}\right)$$

$$\approx 54 \text{ min} \mid$$

Exercise

According to a software company, the users of its typing tutorial can expect to type $N(t)$ words per minute after t hours of practice with the product, according to the function $N(t) = 100(1.04 - 0.99^t)$

- How many words per minute can a student expect to type after 2 hours of practice?
- How many words per minute can a student expect to type after 40 hours of practice?
- How many hours of practice will be required before a student can expect to type 60 words per minute?

Solution

$$a) \quad N(2) = 100(1.04 - 0.99^2)$$

$$\approx 6 \mid \text{ words per minute}$$

$$b) \quad N(40) = 100(1.04 - 0.99^{40})$$

$$\approx 70 \mid \text{ words per minute}$$

$$c) \quad N(t) = 100(1.04 - 0.99^t) = 60$$

$$1.04 - 0.99^t = \frac{60}{100}$$

$$-0.99^t = 0.6 - 1.04$$

$$0.99^t = 0.44$$

$$t = \log_{.99}(.44)$$

$$\approx 82 \text{ hours} \mid$$

Exercise

A lawyer has determined that the number of people $P(t)$ in a city of 1.2 *million* people who have been exposed to a news item after t *days* is given by the function

$$P(t) = 1,200,000(1 - e^{-0.03t})$$

- a) How many days after a major crime has been reported has 40% of the population heard of the crime?
- b) A defense lawyer knows it will be difficult to pick an unbiased jury after 80% of the population has heard of the crime. After how many days will 80% of the population have heard of the crime?

Solution

a) $P(t) = 1,200,000(1 - e^{-0.03t}) = .4(1,200,000)$

$$1 - e^{-0.03t} = 0.4$$

$$e^{-0.03t} = 0.6$$

$$-0.03t = \ln(0.6)$$

$$t = -\frac{\ln(0.6)}{0.03}$$

$$\approx 17 \text{ days}$$