Section 1.8 – Basic Electrical Circuit

Resistor: (**O**hm's **L**aw)

A ***resistor*** is a component of a circuit that resists the flow of electrical current. It has two terminals across which electricity must pass, and it is designed to drop the voltage of the current as it flows from one terminal to the other. Resistors are primarily used to create and maintain known safe currents within electrical components.

A voltage $V(t)$ across the terminals of a resistor is proportional to the current $I(t)$ in it. The constant proportional ***R*** is called the resistance of the resistor in **V**olt/**A**mpere or **O**hms (Ω), and is given by the equation:

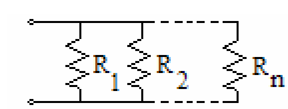
$$V_R = RI$$

For series resistors, the equivalent resistor is:

$R_1 \quad R_2 \qquad R_n$

Then: $R_{eq} = R_1 + R_2 + \cdots + R_n$

For resistors in parallels:

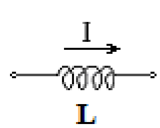


Then: $\dfrac{1}{R_{eq}} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \cdots + \dfrac{1}{R_n}$

Inductor: (**F**araday's **L**aw)

When a current in a circuit is changing, then the magnetic flux is linking the same circuit changes. This change in flux causes an *emf* ***v*** to be induced in the circuit.

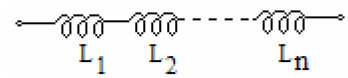
Inductance is symbolized by letter ***L***, is measured in ***henrys*** (***H***), and is represented graphically as a coiled wire – a reminder that inductance is a consequence of a conductor linking a magnetic field.



The voltage $V(t)$ is proportional to the time rate of change of the current, and is given by:

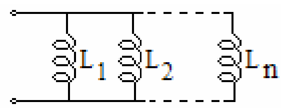
$$V_L = L\frac{dI}{dt} \qquad \text{and} \qquad I(t) = \frac{1}{L}\int V\,dt$$

For series inductors, the equivalent inductor is:



$$L_{eq} = L_1 + L_2 + \cdots + L_n$$

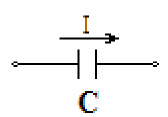
For inductors in parallels:



$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_n}$$

Capacitance**:** (**C**oulomb's **L**aw)

The circuit parameter of ***capacitance*** is represented by letter ***C***, is measured in *farads* (***F***), and is symbolized graphically by two short parallel conductive plates.



The farad is an extremely large quantity of capacitance, practical capacitor values usually lie in the picofarad (*pF*) to microfarad (μF) range.

The graphic symbol for a capacitor is a reminder that capacitance occurs whenever electrical conductors are separated by a dielectric, or insulating, material. This condition implies that electric charge is not transported through the capacitor. Although applying a voltage to the terminals of the capacitor cannot move a charge through the dielectric, it can displace a charge within the dielectric. As the voltage varies with time, the displacement of charge also varies with time, causing what is known as the displacement current.

The potential ***v*** between the terminals of a capacitor is proportional to the charge *q* on it.

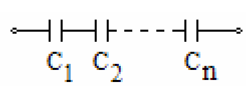
$$Q(t) = Cv(t)$$

$$I = \frac{dq}{dt} = C\frac{dv}{dt}$$

$$\Rightarrow \quad v(t) = \frac{1}{C}\int I\,dt$$

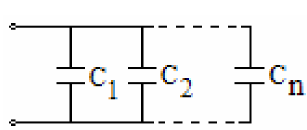
C is **C**oulombs/**V**olts or farads.

For capacitances in series, the equivalent capacitance is given by:



$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}$$

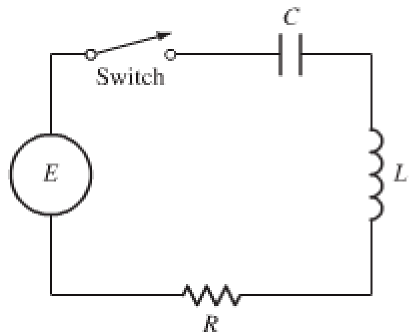
For capacitances in parallels:



$$C_{eq} = C_1 + C_2 + \cdots + C_n$$

RLC circuit

RLC circuit is a basic building block in electrical circuits and networks. A second order linear differential equations with constant coefficients is their use as a model of the flow of electric current in the simple series circuit



Circuit Element	*Voltage Drop*
Inductor	$L\frac{dI}{dt}$
Resistor	RI
Capacitor	$\frac{1}{C}Q$

The current *I*, measured in amperes (***A***), is a function of time *t*.

A ***resistor*** with a resistance of *R ohms* (**Ω**)

An ***inductor*** with an inductance of *L henries* (***H***)

A ***capacitor*** with a capacitance of *C farads* (***F***)

The impressed ***voltage*** *E* in volts (***V***) is a given function of time.

In series with a source of electromotive force (such as a battery or a generator) that supplies a voltage of $E(t)$ volts at time *t*. If the switch shown in the circuit is closed, this results in a current of $I(t)$ *amperes* in the circuit and a charge of $Q(t)$ *coulombs* on the capacitor at time *t*. The relation between the functions *Q* and the current *I* is

$$\frac{dQ}{dt} = I(t)$$

We use ***mks*** electric units, in which time is measured in seconds.

According to elementary principles of electricity, the voltage drops across the three circuit elements.

Kirchhoff's Current Law (***KCL***) (also known as Kirchhoff's ***First Law***)

> ***The algebraic sum of all the currents at any node in a circuit equals to zero.***
>
> ***Current is distributed when it reaches a junction: the amount of current entering a junction must equal the amount of current leaving that junction.***

Kirchhoff's Voltage Law (***KVL***) (also known as Kirchhoff's ***Second Law***)

> ***The algebraic sum of all the voltages around any closed path in a circuit equals to zero.***
> ***In a closed circuit the impressed voltage is equal to the sum of the voltage drops in the rest of the circuit.***

According to the elementary laws of electricity, we know that

The voltage drop across the resistor is IR.

The voltage drop across the capacitor is $\frac{Q}{C}$.

The voltage drop across the inductor is $L\frac{dI}{dt}$.

- When the switch is open, no current flows in the circuit; when the switch is closed, there is a current $I(t)$ and a charge $Q(t)$ on the capacitor.

The current and charge in the simple RLC circuit satisfy the basic electrical equation

$$L\frac{dI}{dt}+RI+\frac{1}{C}Q=E(t)$$

The units for voltage, resistance, current, charge, capacitance, inductance, and time are all related:

$$1\ volt\ =\ 1\ ohm\times 1\ ampere=\frac{1\ coulomb}{1\ farad}=\frac{1\ henry\times 1\ ampere}{1\ second}$$

$$\boxed{1\ V\ =\ 1\ \Omega\times 1\ A=\frac{1\ C}{1\ F}=\frac{1\ H\times 1\ A}{1\ sec}}$$

Since $\frac{dQ}{dt}=I(t)\ \Rightarrow\ \frac{d^2Q}{dt^2}=\frac{dI}{dt}$, we can get the second-order linear differential equation

$$LQ''+RQ'+\frac{1}{C}Q=E(t)$$

For the charge $Q(t)$, under the assumption that the voltage $E(t)$ is known.

It is the current, in most problems, rather than the charge Q that us of primary interest, so we differentiate both sides and substitute I for Q' to obtain

$$LI''+RI'+\frac{1}{C}I=E'(t)$$

With initial conditions are

$$Q(t_0)=Q_0,\quad Q'(t_0)=I(t_0)=I_0$$

$$Q''(t_0)=I'_0,\quad I'(t_0)=I'_0$$

$$LQ''+RQ'+\frac{1}{C}Q=E(t)$$

$$LI'_0+RI_0+\frac{1}{C}Q=E(t_0)$$

$$\boxed{I'_0=\frac{E(t_0)-RI_0-\frac{1}{C}Q}{L}}$$

Hence I'_0 is also determined by the initial charge and current, which are physically measurable quantities.

- The most important conclusion is that the flow of current in the circuit is precisely the same form as the one that describes the motion of a spring-mass system.

Summary

In *RLC* circuit:

$$\boxed{L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{C}Q=E(t)}$$

$$I_S(t)=\frac{1}{R}V+C\frac{dV}{dt}+\frac{1}{L}\int V(s)\,ds$$

In terms of current: $I(t)=\frac{dQ(t)}{dt}$

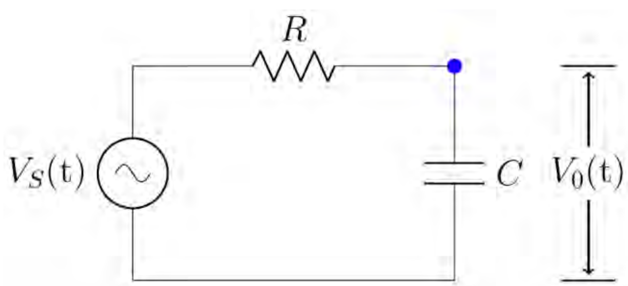
$$\boxed{L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{C}I=\frac{d}{dt}E(t)}$$

Without capacitor $\boxed{L\frac{dI}{dt}+RI=E(t)}$

Where $Q(t)$ is the change on the capacitor and $E(t)$ is the applied voltage.

Communication Channel

The *RC*-circuit consists of a voltage source, a resistor and a capacitor.



The source voltage $V_S(t)$ is the message sent and the output voltage $V_0(t)$ is the message received.

Kirchoff's Voltage Law: the source voltage V_S is the sum of the voltage drops across the other two circuit elements.

Voltage drop across the resistor $=V_S-V_0$

The current through the resistor, which by Ohm's Law,

$$Current\ into\ node=\frac{V_S-V_0}{R}$$

$$Current\ leaving\ node=CV_0'$$

Therfore, the model *RC* circuit *ODE*:

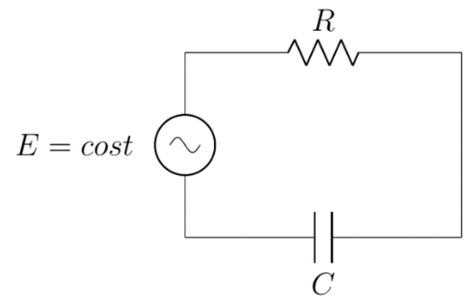
$$CV_0'=\frac{V_S-V_0}{R}$$

$$V_0'+\frac{1}{RC}V_0=\frac{1}{RC}V_S$$

Example

Suppose the electrical circuit has a resistor of $R = 2\Omega$ and a capacitor of $C = \frac{1}{5}F$. Assume the voltage source is $E = \cos t\ (V)$.

If the initial current is 0 A, find the resulting current.



Solution

$$2Q' + 5Q = \cos t \rightarrow Q' + \frac{5}{2}Q = \frac{1}{2}\cos t$$

$$e^{\int \frac{5}{2}dt} = e^{\frac{5}{2}t}$$

$$\int e^{5t/2}\left(\frac{1}{2}\cos t\right)dt = \frac{1}{2}e^{5t/2}\frac{1}{\left(\frac{5}{2}\right)^2 + 1^2}\left(\frac{5}{2}\cos t - \sin t\right) + K \qquad \int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2}$$

$$= \frac{1}{4}e^{5t/2}\frac{1}{\frac{29}{4}}(5\cos t + 2\sin t) + K$$

$$= \frac{1}{29}e^{5t/2}(5\cos t + 2\sin t) + K$$

$$Q(t) = \frac{1}{e^{5t/2}}\left[\frac{1}{29}e^{5t/2}(5\cos t + 2\sin t) + K\right]$$

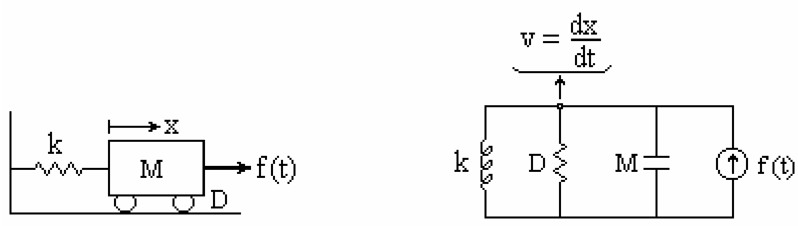
$$= \frac{1}{29}(5\cos t + 2\sin t) + Ke^{-5t/2} \qquad I(0) = 0$$

$$I(0) = 0 \quad 0 = \frac{1}{29}(5) + K \quad \Rightarrow \quad K = -\frac{5}{29}$$

$$Q(t) = \frac{5}{29}\cos t + \frac{2}{29}\sin t - \frac{5}{29}e^{-5t/2}$$

Example

The electrical analog of a carriage on wheels, coupled to the wall through a spring.



a) Mechanical system. ***b)*** Electrical analog.

A mechanical system with a one coordinates movement.

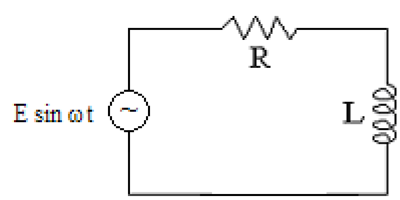
In the case of the electrical network, the equation was obtained by applying Kirchhoff's current law at the node ***v***, and is seen to be identical to the equation that would have been obtained by applying D'Alembert's principle to the mechanical system.

The differential equation for both systems is:

$$M\frac{d^2x}{dt^2}+D\frac{dx}{dt}+kx=f(t)$$

In particular, if one uses the force current analogy (or force-torque for a rational system). The topology of the electrical analog is very similar to that of the mechanical system.

Example: Alternating Circuit



Alternating Circuit

Translating the circuit into differential equation:

$$L\frac{di}{dt}+Ri=E\sin\omega t$$

$$\Rightarrow\ \frac{di}{dt}+\frac{R}{L}i=\frac{E}{L}\sin\omega t$$

$$e^{\int\frac{R}{L}dt}=e^{\frac{R}{L}t}$$

$$\int e^{\frac{R}{L}t}\,\frac{E}{L}\sin\omega t\ dt=\frac{E}{L}\int e^{\frac{R}{L}t}\ \sin\omega t\ dt \qquad\qquad \int e^{ax}\sin bx dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)$$

$$=\frac{E}{L}e^{\frac{R}{L}t}\ \frac{1}{\frac{R^2}{L^2}+\omega^2}\left(\frac{R}{L}\sin\omega t-\omega\cos\omega t\right)$$

$$= \frac{EL}{R^2 + L^2\omega^2} e^{\frac{R}{L}t} \left(\frac{R}{L} \sin \omega t - \omega \cos \omega t \right)$$

$$i(t) = \frac{1}{e^{\frac{R}{L}t}} \left[\frac{EL}{R^2 + L^2\omega^2} e^{\frac{R}{L}t} \left(\frac{R}{L} \sin \omega t - \omega \cos \omega t \right) + C \right]$$

$$\boxed{= \frac{EL}{R^2 + L^2\omega^2} \left(\frac{R}{L} \sin \omega t - \omega \cos \omega t \right) + Ce^{-\frac{R}{L}t}}$$

At $t = 0$; $i = 0$

$$0 = \frac{EL}{R^2 + L^2\omega^2} \left(\frac{R}{L} \sin \omega 0 - \omega \cos \omega 0 \right) + Ce^{-\frac{R}{L}(0)}$$

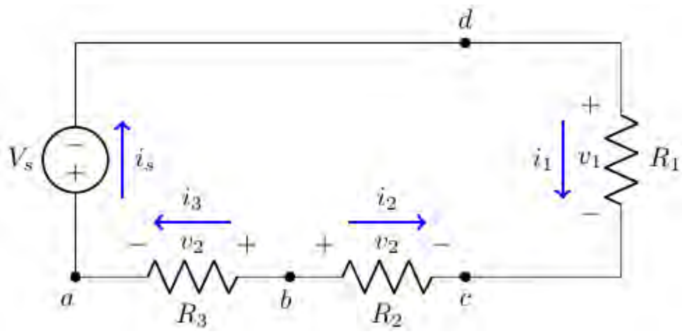
$$0 = \frac{EL}{R^2 + L^2\omega^2} (-\omega) + C$$

$$C = \frac{EL\omega}{R^2 + L^2\omega^2}$$

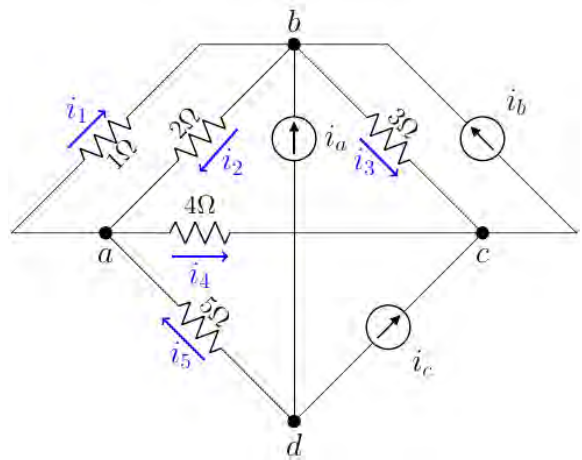
$$\boxed{i(t) = \frac{EL}{R^2 + L^2\omega^2} \left(\frac{R}{L} \sin \omega t - \omega \cos \omega t \right) + \frac{EL\omega}{R^2 + L^2\omega^2} e^{-\frac{R}{L}t}}$$

Exercises *Section* 1.8 - Basic Electrical Circuit

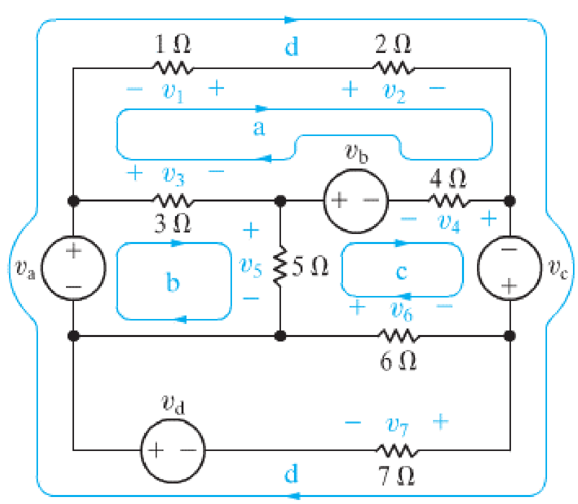
1. Sum the currents at each node in the circuit



2. Sum the currents at each node in the circuit



3. Sum the voltges around rach designated path in the circuit



A resistor $R = 20\ \Omega$ and a capacitor of $C = 0.1\ F$ are joined in series with an electronic force (*emf*) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge on the capacitor at time t for the given:

4. $E(t) = 100\sin 2t$

5. $E(t) = 100e^{-0.1t}$

6. $E(t) = 100\left(1 - e^{-0.1t}\right)$

7. $E(t) = 100\cos 3t$

An inductor $(L = 1\ H)$ and a resistor $(R = 0.1\ \Omega)$ are joined in series with an electronic force (*emf*) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing current in the current at time t for the given:

8. $E(t) = 10 - 2t$ **9.** $E(t) = 4\cos 3t$ **10.** $E(t) = 4\sin 2\pi t$

11. An RL circuit with a $1-\Omega$ resistor and a 0.1-H inductor is driven by a voltage $E(t) = \sin 100t\ V$. If the initial inductor current is zero, determine the subsequence resistor and inductor current and the voltages.

12. An RL circuit with a $1-\Omega$ resistor and a 0.01-H inductor is driven by a voltage $E(t) = \sin 100t\ V$. If the initial inductor current is zero, determine the subsequence resistor and inductor current and the voltages.

13. An RL circuit with a $5-\Omega$ resistor and a 0.05-H inductor is driven by a voltage $E(t) = 5\cos 120t\ V$. If the initial inductor current is 1 A, determine the subsequence resistor and inductor current and the voltages.

14. An RC circuit with a $1-\Omega$ resistor and a 10^{-6}-F capacitor is driven by a voltage $E(t) = \sin 100t\ V$. If the initial capacitor current is zero, determine the subsequence resistor and capacitor current and the voltages.

15. Solve the general initial value problem modeling the RC circuit

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E, \quad Q(0) = 0$$

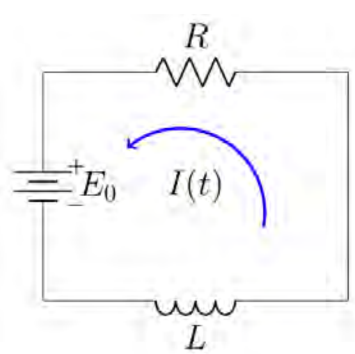
Where E is a constant source of *emf*

16. Solve the general initial value problem modeling the LR circuit

$$L\frac{dI}{dt} + RI = E, \quad I(0) = I_0$$

Where E is a constant source of *emf*

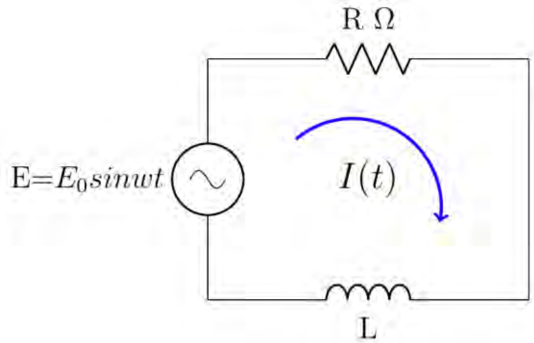
17. For the given $RL-$circuit



Where E_0 is a constant source of *emf* at time $t = 0$.

Find the current $I(t)$ flowing in the circuit.

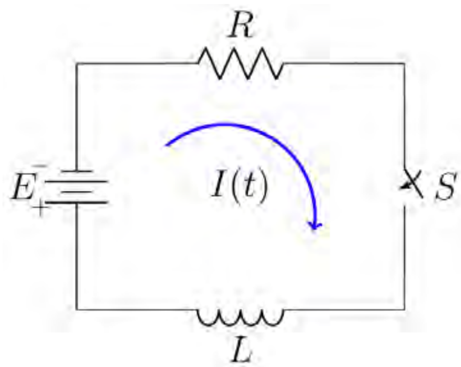
18. For the given $RL-$circuit



Where $E = E_0 \sin \omega t$ is the impressed voltage.

Find the current $I(t)$ flowing in the circuit.

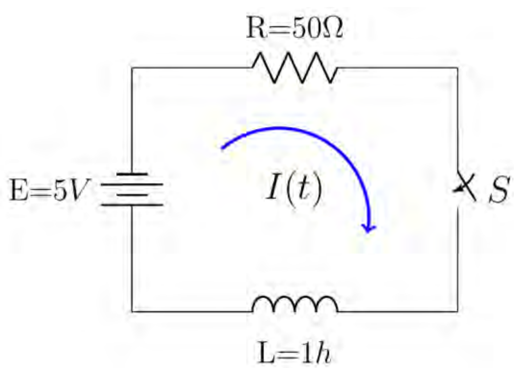
19. For the given $RL-$circuit



Which has a constant impressed voltage E, a resistor of resistance R, and a coil of impedance L.

Find the current $I(t)$ flowing in the circuit.

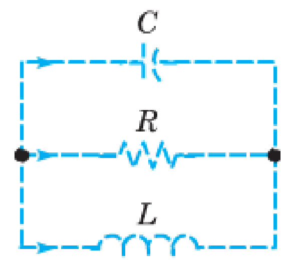
20. For the given $RL-$circuit



Which has a constant impressed voltage E, a resistor of resistance R, and a coil of impedance L.

Find the current $I(t)$ flowing in the circuit.

21. Consider the circuit shown and let I_1, I_2, and I_3 be the currents through the capacitor, resistor, and inductor, respectively. Let V_1, V_2, and V_3 be the corresponding voltage drops. The arrows denote the arbitrary chosen directions in which currents and voltage drops will be taken to be positive.



a) Applying Kirchhoff's second law to the upper loop in the circuit, show that $V_1 - V_2 = 0$ *and* $V_2 - V_3 = 0$

b) Applying Kirchhoff's first law to either node in the circuit, show that

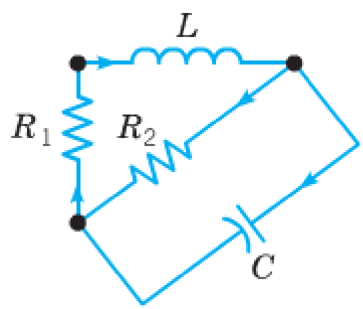
$$I_1 + I_2 + I_3 = 0$$

c) Use the current-voltage relation through each element in the circuit to obtain the equations

$$CV_1' = I_1, \quad V_2 = RI_2, \quad LI_3' = V_3$$

d) Eliminate V_2, V_3, I_1 and I_2 to obtain $CV_1' = -I_3 - \frac{V_1}{R}, \quad LI_3' = V_1$

22. Consider the circuit. Use the method outlined to show that the current I through the inductor and the voltage V across the capacitor satisfy the system of differential equations.

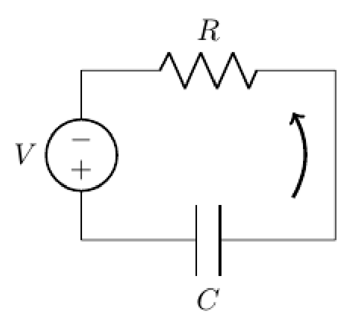


$$L\frac{dI}{dt} = -R_1 I - V, \quad C\frac{dV}{dt} = I - \frac{V}{R_2}$$

23. Consider an electric circuit containing a capacitor, resistor, and battery. The charge $Q(t)$ on the capacitor satisfies the equation

$$R\frac{dQ}{dt} + \frac{Q}{C} = V$$

Where R is the resistance, C is the capacitance, and V is the constant voltage supplied by the battery.

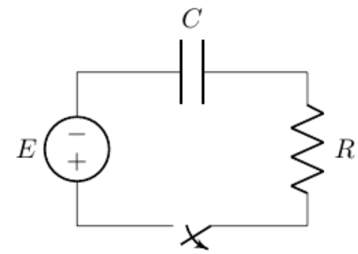


a) If $Q(0) = 0$, find $Q(t)$ at time *t*.

b) Find the limiting value Q_L that $Q(t)$ approaches after a long time.

c) Suppose that $Q(t_1) = Q(t)$ and that at time $t = t_1$ the battery is removed and the circuit is closed again. Find $Q(t)$ for $t > t_1$.

24. A circuit containing an electromotive force, a capacitor with a capacitance of C farads (F), and a resistor with a resistance of R ohms (Ω). The voltage drop across the capacitor is $\frac{Q}{C}$, where Q is the charge (in coulombs), so in this case ***Kirchhoff's Law*** gives



$$RI + \frac{Q}{C} = E(t)$$

But $I = \frac{dQ}{dt}$, so we have $\quad R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$

Find the charge and the current at time t

a) Suppose the resistance is $5\ \Omega$, the capacitance is 0.05 F, a battery gives voltage of 60 V and initial charge is $Q(0) = 0\ C$

b) Suppose the resistance is $2\ \Omega$, the capacitance is 0.01 F, $E(t) = 10\sin 60t$ and initial charge is $Q(0) = 0\ C$

25. A heart pacemaker consists of a switch, a battery voltage E_0, a capacitor with constant capacitance C, and the heart as a resistor with constant resistance R. When the switch is closed, the capacitor charges; when the switch is open, the capacitor discharges, sending an electrical stimulus to the heart. During the time the heart is being stimulated, the voltage E across the heart satisfies the linear differential equation

$$\frac{dE}{dt} = -\frac{1}{RC}E$$

Solve the *DE*, subject to $E(4) = E_0$

26. A 30–volt electromotive force is applied to an *LR*-series circuit in which the inductance is 0.1 *henry* and the resistance is 50 *ohms*.

a) Find the current $i(t)$ if $i(0) = 0$

b) Determine the current as $t \to \infty$

c) Solve the equation when $E(t) = E_0 \sin \omega t$ and $i(0) = i_0$

27. A 100–volt electromotive force is applied to an *RC*-series circuit in which the resistance is 200 *ohms* and the capacitance is 10^{-4} *farad*.

a) Find the charge $q(t)$ if $q(0) = 0$

b) Find the current as $i(t)$

28. A 200–volt electromotive force is applied to an *RC*-series circuit in which the resistance is 1000 *ohms* and the capacitance is 5×10^{-6} *farad*.

a) Find the charge $q(t)$ if $i(0) = 0.4$

b) Determine the charge as $t \to \infty$

29. An electromotive force

$$E(t) = \begin{cases} 120, & 0 \le t \le 20 \\ 0, & t > 20 \end{cases}$$

Is applied to an *LR*-series circuit in which the inductance is 20 *henries* and resistance is 2 *ohms*. Find the current $i(t)$ if $i(0) = 0$

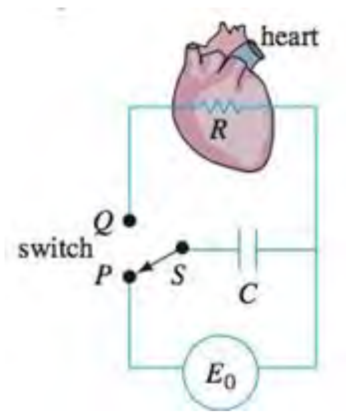
30. Suppose an *RC*-series circuit has a variable resistor. If the resistance at time *t* is given by $R = k_1 + k_2 t$, where k_1 and k_2 are known positive constants, then

$$(k_1 + k_2 t)\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

If $E(t) = E_0$ and $q(0) = q_0$, where E_0 and q_0 are constants, show that

$$q(t) = E_0 C + (q_0 - E_0 C)\left(\frac{k_1}{k_1 + k_2 t}\right)^{1/Ck_2}$$

31. A heart pacemaker, consists of a switch, a battery, a capacitor, and the heart as a resistor.



When the switch *S* is at *P*, the capacitor charges; when *S* is at *Q*, the capacitor discharges, sending an electrical stimulus to the heart. The electrical stimulus is being applied to the heart, the voltage *E* across the heart satisfies the linear DE.

$$\frac{dE}{dt} = -\frac{1}{RC}E$$

a) Let assume that over the time interval of length t_1, $0 < t < t_1$, the switch *S* is at position *P* and the capacitor is being charges. When the switch is moved to position *Q* at time t_1 the capacitor discharges, sending an impulse to the heart over the time interval of length t_2 : $t_1 \le t < t_1 + t_2$. Thus over the initial charging/discharging interval $0 < t < t_1 + t_2$ the voltage to the heart is actually modeled by the piecewise-defined differential equation

$$\frac{dE}{dt} = \begin{cases} 0, & 0 < t < t_1 \\ -\frac{1}{RC}E, & t_1 \le t < t_1 + t_2 \end{cases}$$

By moving *S* between *P* and *Q*, the charging and discharging over time intervals of lengths t_1 and t_2 is repeated indefinitely. Suppose $t_1 = 4\ s$, $t_2 = 2\ s$. $E_0 = 12\ V$, and $E(0) = 0$, $E(4) = 12$, $E(6) = 0$, $E(10) = 12$, $E(12) = 0$, and so on.

Solve for $E(t)$ for $0 \le t \le 24$

b) Suppose for the sake of illustration that $R = C = 1$. Graph the solution in part (*a*) for $0 \le t \le 24$