Lecture One

Section 1.1 – Introduction to System of Linear Equations

Given the linear equations

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases}$$

The solution to this system is $(3,\ 1)$, which means that 2 lines meeting at a single point.

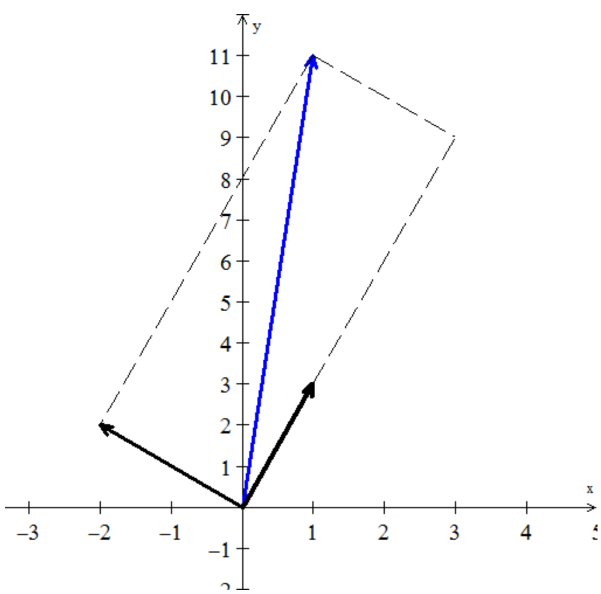
We can rewrite the system equation as linear combination:

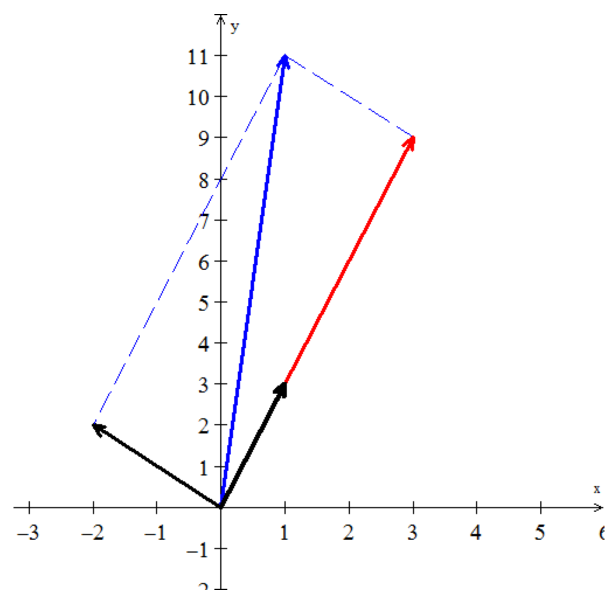
$$x\begin{bmatrix}1\\3\end{bmatrix} + y\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix}$$

$$x.v_1 + y.v_2 = v$$

$$\begin{bmatrix}1+x\\3+y\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix} \Rightarrow \begin{bmatrix}x=3\\y=9\end{bmatrix}$$

$$\begin{bmatrix}3\\9\end{bmatrix} = 3\begin{bmatrix}1\\3\end{bmatrix}$$





Therefore, the side vectors are $\begin{bmatrix}3\\9\end{bmatrix}$ *and* $\begin{bmatrix}-2\\2\end{bmatrix}$

The diagonal sum is $\begin{bmatrix}3-2\\9+2\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix}$

The linear combination is given by:

$$3\begin{bmatrix}1\\3\end{bmatrix}+1\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}$$

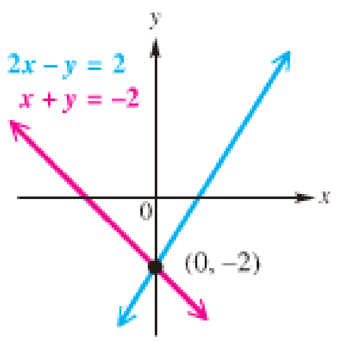
Thus, the solution is $x=3 \quad y=1$

<u>Note</u>

$\begin{bmatrix}1&-2\\3&2\end{bmatrix}$ is called the "***coefficient matrix***"

The matrix form of the system is written as $\qquad Ax=b$

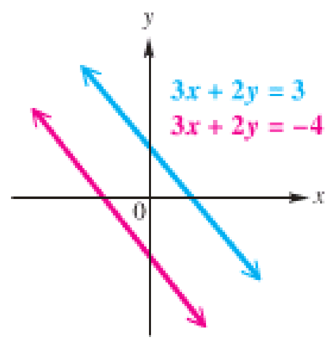
$$\begin{bmatrix}1&-2\\3&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}$$



One solution (lines intersect)

Consistent

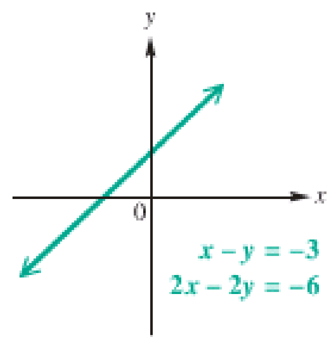
Independent



No Solution (lines //)

Inconsistent

Independent



Unique Infinite solution

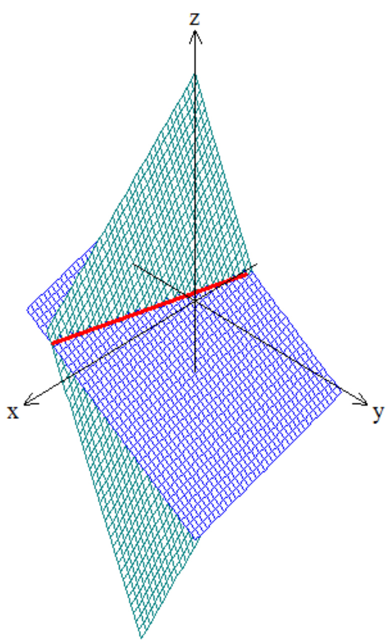
Consistent

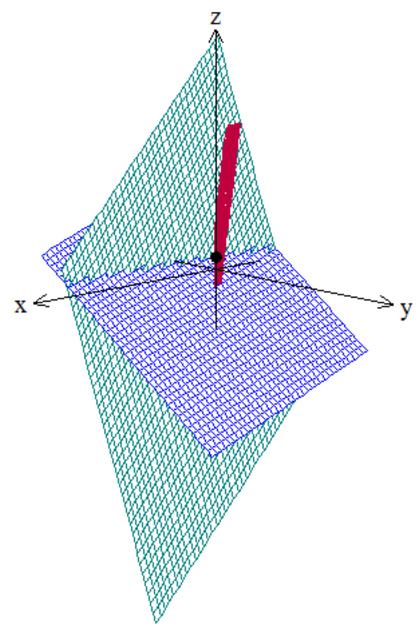
Dependent

Three Equations in 3 Unknowns

Given the system equations

$$x + 2y + 3z = 6$$
$$2x + 5y + 2z = 4$$
$$6x - 3y + z = 2$$



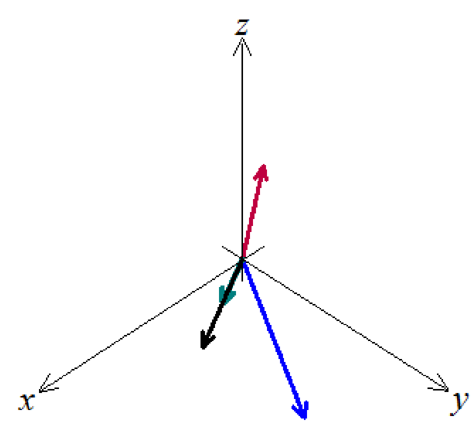


This system can be written as linear combination:

$$x\begin{bmatrix}1\\2\\6\end{bmatrix} + y\begin{bmatrix}2\\5\\-3\end{bmatrix} + z\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}6\\4\\2\end{bmatrix}$$

Let $b = \begin{bmatrix}6\\4\\2\end{bmatrix}$

We want to multiply the three column vectors by x, y, z to produce ***b***.

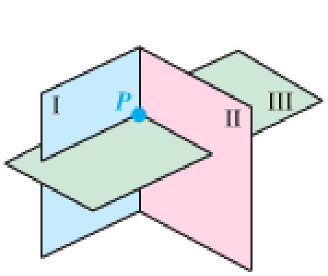


The combination of the three vectors that produces vector b is 2 times the third vector.

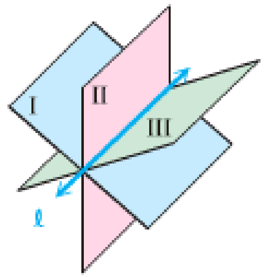
$$2(3,\ 2,\ 1) = (6,\ 4,\ 2) = b$$

Therefore, the coefficients that we need are $x = 0, \quad y = 0, \quad and\ z = 2$.

$$0\begin{bmatrix}1\\2\\6\end{bmatrix} + 0\begin{bmatrix}2\\5\\-3\end{bmatrix} + 2\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}6\\4\\2\end{bmatrix}$$

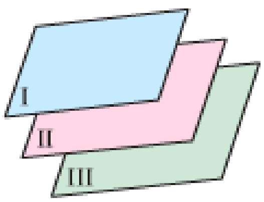


A single solution

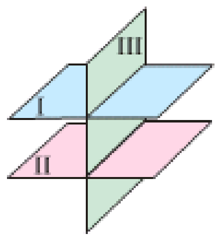


Points of a line in common

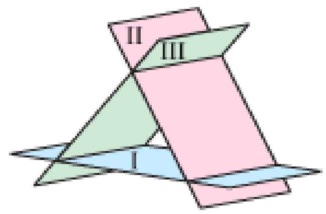
All points in common



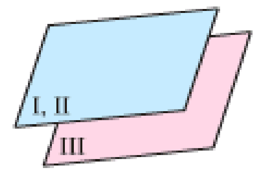
No points in common



No points in common



No points in common



No points in common

Exercises *Section* 1.1 – Introduction to System of Linear Equations

1. Find a solution for *x, y, z* to the system of equations

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3e + 2\sqrt{2} + \pi \\ 6e + 5\sqrt{2} + 4\pi \\ 9e + 8\sqrt{2} + 7\pi \end{pmatrix}$$

2. Draw the two pictures in two planes for the equations: $x - 2y = 0, \quad x + y = 6$

3. Normally 4 planes in 4-dimensional space meet at a ________. Normally 4 column vectors in 4-deimensional space can combine to produce b. what combinations of $(1,\ 0,\ 0,\ 0),\quad (1,\ 1,\ 0,\ 0),\quad (1,\ 1,\ 1,\ 0),\quad (1,\ 1,\ 1,\ 1)$ produces $b = (3,\ 3,\ 3,\ 2)$?
What 4 equations for $x,\ y,\ z,\ w$ are you solving?

4. What 2 by 2 matrix A rotates every vector through 45° ?
The vector (1, 0) goes to $\left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$. The vector (0, 1) goes to $\left(-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$.
Those determine the matrix. Draw these particular vectors is the *xy*-plane and find A.

5. What two vectors are obtained by rotating the plane vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by 30° (*cw*) ?
Write a matrix A such that for every vector $\vec{v}$ in the plane, $A\vec{v}$ is the vector obtained by rotating $\vec{v}$ clockwise by 30°.
Find a matrix B such that for every 3-dimensional vector $\vec{v}$, the vector $B\vec{v}$ is the reflection of $\vec{v}$ through the plane $x + y + z = 0$. $Hint: v = (1,\ 0,\ 0)$

6. In each part, find a system of linear equation corresponding to the given augmented matrix

a) $\begin{bmatrix} 0 & 3 & -1 & -1 & -1 \\ 5 & 2 & 0 & -3 & -6 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \\ 5 & -6 & 1 & 1 \\ -8 & 0 & 0 & 3 \end{bmatrix}$

7. Find the augmented matrix for the given system of linear equations.

a) $\begin{cases} -2x_1 = 6 \\ 3x_1 = 8 \\ 9x_1 = -3 \end{cases}$

c) $\begin{cases} 2x_1 \qquad + 2x_3 = 1 \\ 3x_1 - x_2 + 4x_3 = 7 \\ 6x_1 + x_2 - x_3 = 0 \end{cases}$

b) $\begin{cases} 3x_1 - 2x_2 = -1 \\ 4x_1 + 5x_2 = 3 \\ 7x_1 + 3x_2 = 2 \end{cases}$

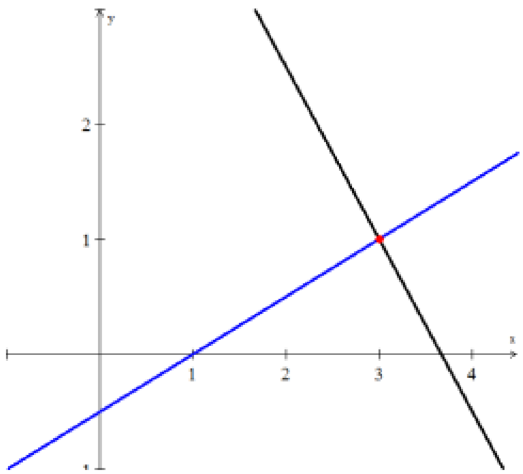
Section 1.2 – Gaussian Elimination

Elimination produces an ***upper triangular system***.

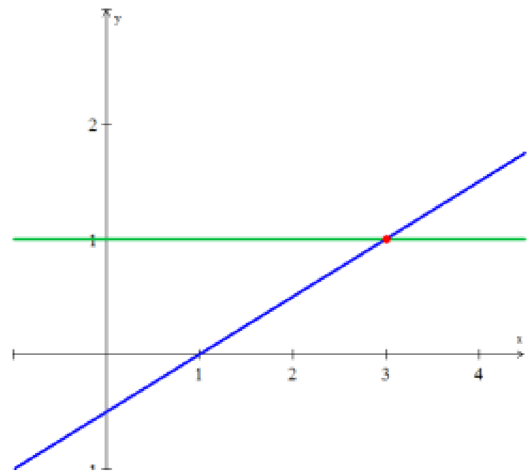
$$\begin{cases} x-2y=1 \\ 3x+2y=11 \end{cases} \Rightarrow \begin{cases} x-2y=1 & \textit{Mutliply by } 3 \\ \quad 8y=8 & \textit{and subtract} \end{cases}$$

The equation $8y=8$ *reveals* $y=1$

This process is called ***back substitution***.



Before elimination



After elimination

Definitions

Pivot: first nonzero in the row that does the elimination

Multiplier: (entry to eliminate) divide by pivot

$$\begin{array}{lll} 4x-8y=4 & \textit{Multiply equation } 1 \textit{ by } \frac{3}{4} & 4x-8y=4 \\ 3x+2y=11 & \textit{Subtract from equation } 2 & \quad\quad 8y=8 \end{array}$$

The first pivot is 4 (the coefficient of x) and the multiplier is $l=\frac{3}{4}$

The pivots are on the diagonal of the triangle after elimination.

Definition

The operations are the elementary reduction operations, or row operations, or Gaussian operations. They are swapping, multiplying by a scalar or rescaling, and pivoting.

Reduced Row Echelon Form

$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

Use the Gaussian elimination method to solve the system

$$\begin{aligned} 3x + \ \ y + 2z &= 31 \\ x + \ \ y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 3 & 1 & 2 & 31 \\ 1 & 3 & 2 & 25 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array} \qquad \begin{array}{cccc} 3 & 1 & 2 & 31 \\ -3 & -3 & -6 & -57 \\ \hline 0 & -2 & -4 & -26 \end{array} \qquad \begin{array}{cccc} 1 & 2 & 2 & 25 \\ -1 & -1 & -2 & -19 \\ \hline 0 & 2 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & -2 & -4 & -26 \\ 0 & 2 & 0 & 6 \end{array}\right] -\tfrac{1}{2}R_2 \qquad \begin{array}{cccc} 0 & 1 & 2 & 13 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 2 & 0 & 6 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_2 \end{array} \qquad \begin{array}{cccc} 0 & 2 & 0 & 6 \\ 0 & -2 & -4 & -26 \\ \hline 0 & 0 & -4 & -20 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & -4 & -20 \end{array}\right] -\tfrac{1}{4}R_3 \qquad \begin{array}{cccc} 0 & 0 & 1 & 5 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 1 & 5 \end{array}\right] \Rightarrow \begin{array}{rl} x + y + 2z = 19 & (3) \\ y + 2z = 13 & (2) \\ z = 5 & (1) \end{array}$$

$(2) \Rightarrow y = 13 - 2z = 13 - 2(5) = 3$

$(3) \Rightarrow \ x = 19 - y - 2z = 19 - 3 - 10 = 6$

$\Rightarrow$ **(6, 3, 5)**

Example

Use Gauss-Jordan elimination to solve the homogeneous linear system

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 \qquad &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\ 5x_3 + 10x_4 \qquad + 15x_6 &= 5 \\ 2x_1 + 6x_2 \qquad + 8x_4 + 4x_5 + 18x_6 &= 6 \end{aligned}$$

Solution

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \quad \textit{Adding } (-2) \textit{ times the 1st row to the 2nd} \\ \\ R_4 - 2R_1 \quad \textit{Adding } (-2) \textit{ times the 1st row to the 4th} \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] -R_2$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 5R_2 \\ R_4 - 4R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right] \begin{array}{l} \\ \\ \\ \frac{1}{6}R_4 \quad \textit{then interchanging row3 and row4} \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_3 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{cases} x_1 + 3x_2 \qquad + 4x_4 + 2x_5 \qquad = 0 \\ \qquad\qquad x_3 + 2x_4 \qquad\qquad\quad = 0 \\ \qquad\qquad\qquad\qquad\qquad + x_6 = \frac{1}{3} \end{cases}$$

The general solution of the system: $\underline{\left(-3x_2 - 4x_4 - 2x_5, \quad x_2, \quad -2x_4, \quad x_4, \quad x_5, \quad \frac{1}{3}\right)}$

Example

Use Gauss-Jordan elimination to solve the homogeneous linear system

$$\begin{aligned} 2x+\ 8y-\ z+\ w&=0\\ 4x+16y-3z-w&=-10\\ -2x+4y-z+3w&=-6\\ -6x+2y+5z+w&=3 \end{aligned}$$

Solution

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0\\ 4 & 16 & -3 & -1 & -10\\ -2 & 4 & -1 & 3 & -6\\ -6 & 2 & 5 & 1 & 3 \end{array}\right] \begin{array}{l} \\ R_2-2R_1\\ R_3+R_1\\ R_4+3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0\\ 0 & 12 & -2 & 4 & -6\\ 0 & 0 & -1 & -3 & -10\\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4-\frac{13}{6}R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0\\ 0 & 0 & -1 & -3 & -10\\ 0 & 12 & -2 & 4 & -6\\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \qquad \textit{Interchange } R_2 \textit{ and } R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0\\ 0 & 12 & -2 & 4 & -6\\ 0 & 0 & -1 & -3 & -10\\ 0 & 0 & \frac{19}{3} & -\frac{14}{3} & 16 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4+\frac{19}{3}R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0\\ 0 & 12 & -2 & 4 & -6\\ 0 & 0 & -1 & -3 & -10\\ 0 & 0 & 0 & -\frac{71}{3} & -\frac{142}{3} \end{array}\right] \begin{array}{lll} 2x+8y-z+w=0 \rightarrow & 2x=-8y+z-w=6 & \Rightarrow \boxed{x=3}\\ 12y-2z+4w=-6 \rightarrow & 12y=2z-4w-6=-6 & \Rightarrow \boxed{y=-\frac{1}{2}}\\ -z-3w=-10 & & \\ -\frac{71}{3}w=-\frac{142}{3} \rightarrow & z=10-3w=4 & \\ \rightarrow & \boxed{w=2} & \end{array}$$

Solution: $\left(3,\ -\frac{1}{2},\ 4,\ 2\right)$

Theorem: Free Variable Theorem for Homogeneous Systems

If a *homogeneous linear* system has ***n*** unknowns, and if the reduced row echelon form of its augmented matrix has ***r*** nonzero rows, then the system has $n - r$ free variables.

Theorem

A *homogeneous linear* system with more unknowns than equations has ***infinitely many*** *unknowns*.

Breakdown Elimination

Permanent failure with no solution

$$\begin{array}{lcl} x-2y=1 & \textit{Subtract 3 times} & x-2y=1 \\ 3x-6y=11 & \textit{eqn. 1 from eqn. 2} & 0y=8 \end{array}$$

The last equation $0y = 8$; therefore, there is *no* solution.
This system has no second pivot, since no zero allowed as a pivot.

Permanent failure with infinitely many solutions

$$\begin{array}{lcl} x-2y=1 & \textit{Subtract 3 times} & x-2y=1 \\ 3x-6y=3 & \textit{eqn. 1 from eqn. 2} & 0y=0 \end{array}$$

Every y satisfies $0y = 0$. There is only one equation $x - 2y = 1$.
There are ***unique infinitely*** many solutions.

Three Equations in Three Unknowns

To understand Gaussian elimination, you have to go beyond 2 by 2 systems.

Consider the system equations: $\begin{cases} 2x+4y-2z=2 \\ 4x+9y-3z=8 \\ -2x-3y+7z=10 \end{cases}$

$$\begin{cases} 2x+4y-2z=2 \\ 4x+9y-3z=8 \\ -2x-3y+7z=10 \end{cases} \quad \begin{array}{c} \textit{subtract 2 times eqn.1} \\ \textit{from eqn.2} \\ \ \end{array} \quad \begin{array}{r} 2x+4y-2z=2 \\ y+z=4 \\ -2x-3y+7z=10 \end{array}$$

$$\begin{cases} 2x+4y-2z=2 \\ y+z=4 \\ -2x-3y+7z=10 \end{cases} \quad \begin{array}{c} \textit{Add eqn.1} \\ \ \\ \textit{and eqn.3} \end{array} \quad \begin{array}{r} 2x+4y-2z=2 \\ y+z=4 \\ y+5z=12 \end{array}$$

$$\begin{cases} 2x+4y-2z=2 \\ \qquad y+z=4 \quad \textit{Subtract eqn.2} \\ \quad y+5z=12 \quad \textit{from eqn.3} \end{cases} \qquad \begin{aligned} 2x+4y-2z&=2 &\Rightarrow\ & x=1-2y+z=-1 \\ y+z&=4 &\Rightarrow\ & y=4-z=2 \\ 4z&=8 &\Rightarrow\ & \boxed{z=2} \end{aligned}$$

The solution is $(-1,\ 2,\ 2)$

Definition

A square matrix is nonsingular if it is the matrix of coefficient of a homogeneous system, with a unique solution. It is singular otherwise, that is, if it is the matrix of coefficients of a homogeneous system, with infinitely many solutions.

Exercises *Section* 1.2 – Gaussian Elimination

1. When elimination is applied to the matrix $A=\begin{bmatrix} 3 & 1 & 0 \\ 6 & 9 & 2 \\ 0 & 1 & 5 \end{bmatrix}$

a) What are the first and second pivots?

b) What is the multiplier l_{21} in the first step (l_{21} times row 1 is subtracted from row 2)?

c) What entry in the 2, 2 position (instead of 9) would force an exchange of rows 2 and 3?

d) What is the multiplier $l_{31}=0$, subtracting 0 times row 1 from row 3?

2. Use elimination to reach upper triangular matrices *U*. Solve by back substitution or explain why this impossible. What are the pivots (never zero)? Exchange equations when necessary. The only difference is the $-x$ in equation (3).

$$\begin{cases} x+y+z=7 \\ x+y-z=5 \\ x-y+z=3 \end{cases} \qquad \begin{cases} x+y+z=7 \\ x+y-z=5 \\ -x-y+z=3 \end{cases}$$

3. For which numbers *a* does the elimination break down (1) permanently (2) temporarily

$$ax+3y=-3$$
$$4x+6y=6$$

Solve for *x* and *y* after fixing the second breakdown by a row change.

4. Find the pivots and the solution for these four equations:

$$\begin{aligned} 2x+y \quad\quad &= 0 \\ x+2y+z \quad &= 0 \\ y+2z+t &= 0 \\ z+2t &= 5 \end{aligned}$$

5. Look for a matrix that has row sums 4 and 8, and column sums 2 and *s*.

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{matrix} a+b=4 & a+c=2 \\ c+d=8 & b+d=s \end{matrix}$$

The four equations are solvable only if $s =$ ____. Then find two different matrices that have the correct row and column sums.

6. Three planes can fail to have an intersection point, even if no planes are parallel. The system is singular if row 3 of *A* is a _______ of the first two rows. Find a third equation that can't be solved together with $x+y+z=0$ and $x-2y-z=1$

(7 – 14) Use the Gauss-Jordan method to solve the system

7. $\begin{cases} x - y + 5z = -6 \\ 3x + 3y - z = 10 \\ x + 3y + 2z = 5 \end{cases}$

8. $\begin{cases} 2x - y + 4z = -3 \\ x - 2y - 10z = -6 \\ 3x \quad\quad + 4z = 7 \end{cases}$

9. $\begin{cases} 4x + 3y - 5z = -29 \\ 3x - 7y - z = -19 \\ 2x + 5y + 2z = -10 \end{cases}$

10. $\begin{cases} x + 2y - 3z = -15 \\ 2x - 3y + 4z = 18 \\ -3x + y + z = 1 \end{cases}$

11. $\begin{cases} x + 2y + 3z = 10 \\ 4x + 5y + 6z = 11 \\ 7x + 8y + 9z = 12 \end{cases}$

12. $\begin{cases} 2x + y + 2z = 4 \\ 2x + 2y \quad\quad = 5 \\ 2x - y + 6z = 2 \end{cases}$

13. $\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$

14. $\begin{cases} x + 2y + z = 8 \\ -x + 3y - 2z = 1 \\ 3x + 4y - 7z = 10 \end{cases}$

(15 – 49) Use augmented elimination to solve linear system

15. $\begin{cases} 2x - 5y + 3z = 1 \\ x - 2y - 2z = 8 \end{cases}$

16. $\begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$

17. $\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$

18. $\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$

19. $\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$

20. $\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$

21. $\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$

22. $\begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$

23. $\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$

24. $\begin{cases} 3x - 4y + 4z = 7 \\ x - y - 2z = 2 \\ 2x - 3y + 6z = 5 \end{cases}$

25. $\begin{cases} x - 2y - z = 2 \\ 2x - y + z = 4 \\ -x + y + z = 4 \end{cases}$

26. $\begin{cases} x + y + z = 3 \\ -y + 2z = 1 \\ -x + z = 0 \end{cases}$

27. $\begin{cases} 3x + y + 3z = 14 \\ 7x + 5y + 8z = 37 \\ x + 3y + 2z = 9 \end{cases}$

28. $\begin{cases} 4x - 2y + z = 7 \\ x + y + z = -2 \\ 4x + 2y + z = 3 \end{cases}$

29. $\begin{cases} 2x - 2y + z = -4 \\ 6x + 4y - 3z = -24 \\ x - 2y + 2z = 1 \end{cases}$

30. $\begin{cases} 9x + 3y + z = 4 \\ 16x + 4y + z = 2 \\ 25x + 5y + z = 2 \end{cases}$

31. $\begin{cases} 2x - y + 2z = -8 \\ x + 2y - 3z = 9 \\ 3x - y - 4z = 3 \end{cases}$

32. $\begin{cases} x \quad\quad - 3z = -5 \\ 2x - y + 2z = 16 \\ 7x - 3y - 5z = 19 \end{cases}$

33. $\begin{cases} x + 2y - z = 5 \\ 2x - y + 3z = 0 \\ 2y + z = 1 \end{cases}$

34. $\begin{cases} x + y + z = 6 \\ 3x + 4y - 7z = 1 \\ 2x - y + 3z = 5 \end{cases}$

35. $\begin{cases} 3x + 2y + 3z = 3 \\ 4x - 5y + 7z = 1 \\ 2x + 3y - 2z = 6 \end{cases}$

36. $\begin{cases} x-3y+z=2 \\ 4x-12y+4z=8 \\ -2x+6y-2z=-4 \end{cases}$

37. $\begin{cases} 2x-2y+z=-1 \\ x+2y-z=2 \\ 6x+4y+3z=5 \end{cases}$

38. $\begin{cases} x_1-5x_2+2x_3-2x_4=4 \\ x_2-3x_3-x_4=0 \\ 3x_1+2x_3-x_4=6 \\ -4x_1+x_2+4x_3+2x_4=-3 \end{cases}$

39. $\begin{cases} x_1+x_2+x_3+x_4=5 \\ x_1+2x_2-x_3-2x_4=-1 \\ x_1-3x_2-3x_3-x_4=-1 \\ 2x_1-x_2+2x_3-x_4=-2 \end{cases}$

40. $\begin{cases} 2x+8y-z+w=0 \\ 4x+16y-3z-w=-10 \\ -2x+4y-z+3w=-6 \\ -6x+2y+5z+w=3 \end{cases}$

41. $\begin{cases} 2x_1+x_2+3x_3=0 \\ x_1+2x_2=0 \\ x_2+x_3=0 \end{cases}$

42. $\begin{cases} 2x+2y+4z=0 \\ -y-3z+w=0 \\ 3x+y+z+2w=0 \\ x+3y-2z-2w=0 \end{cases}$

43. $\begin{cases} 2x+z+w=5 \\ y-w=-1 \\ 3x-z-w=0 \\ 4x+y+2z+w=9 \end{cases}$

44. $\begin{cases} 4y+z=20 \\ 2x-2y+z=0 \\ x+z=5 \\ x+y-z=10 \end{cases}$

45. $\begin{cases} x-y+2z-w=-1 \\ 2x+y-2z-2w=-2 \\ -x+2y-4z+w=1 \\ 3x-3w=-3 \end{cases}$

46. $\begin{cases} 2u-3v+w-x+y=0 \\ 4u-6v+2w-3x-y=-5 \\ -2u+3v-2w+2x-y=3 \end{cases}$

47. $\begin{cases} 6x_3+2x_4-4x_5-8x_6=8 \\ 3x_3+x_4-2x_5-4x_6=4 \\ 2x_1-3x_2+x_3+4x_4-7x_5+x_6=2 \\ 6x_1-9x_2+11x_4-19x_5+3x_6=1 \end{cases}$

48. $\begin{cases} 3x_1+2x_2-x_3=-15 \\ 5x_1+3x_2+2x_3=0 \\ 3x_1+x_2+3x_3=11 \\ -6x_1-4x_2+2x_3=30 \end{cases}$

49. $\begin{cases} x_1+3x_2-2x_3+2x_5=0 \\ 2x_1+6x_2-5x_3-2x_4+4x_5-3x_6=-1 \\ 5x_3+10x_4+15x_6=5 \\ 2x_1+6x_2+8x_4+4x_5+18x_6=6 \end{cases}$

50. Add 3 times the second row to the first of $\begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$

51. For what value(s) of k, if any, does the system $\begin{cases} x+y-z=1 \\ 2x+3y+kz=3 \\ x+ky+3z=2 \end{cases}$ have

a) A unique solution?
b) Infinitely many solutions?
c) No solution?

52. Choose a coefficient b that makes the system singular.

$$\begin{cases} 3x+4y=16 \\ 4x+by=g \end{cases}$$

Then choose a right-hand side g that makes it solvable.
Find 2 solutions in that singular case.

53. This system us not linear, in some sense,

$$\begin{cases} 2\sin\alpha-\cos\beta+3\tan\theta=3 \\ 4\sin\alpha+2\cos\beta-2\tan\theta=10 \\ 6\sin\alpha-3\cos\beta+\tan\theta=9 \end{cases}$$

Does the system have a solution?

Section 1.3 – The Algebra of Matrices

Matrices

$$\begin{array}{c} \\ \\ \\ Row\ 1 \rightarrow R_1 \\ Row\ 2 \rightarrow R_2 \\ Row\ 3 \rightarrow R_3 \end{array} \begin{array}{c} \qquad Column \\ \begin{array}{ccc} C_1 & C_2 & C_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$

This is called Matrix (*Matrices*)

Each number in the array is an ***element*** or ***entry***

The matrix is said to be of order m *x* n

m: numbers of rows,

n: number of columns

When $m = n$, then matrix is said to be ***square***.

Given the system equations

$$\begin{aligned} 3x + \ \ y + 2z &= 31 \\ x + \ \ y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

Write into an ***augmented matrix*** form

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 31 \\ 1 & 1 & 2 & 19 \\ 1 & 3 & 2 & 25 \end{array}\right]$$

The Matrix: $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix}$ is called the ***coefficient matrix*** of the system.

The matrix A above has 3 rows and 3 columns, therefore the order of the matrix A is (3 *x* 3)

$$A = \left[a_{ij}\right] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Equality of Matrices

Definition of Equality of Matrices

Two matrices ***A*** and ***B*** are equal if and only if they have the same order (size) $m \times n$ and if each pair corresponding elements is equal

$a_{ij} = b_{ij}$ for $i = 1, 2, \ldots, m$ ***and*** $j = 1, 2, \ldots, n$

Example

Find the values of the variables for which each statement is true, if possible.

a) $\begin{bmatrix} 2 & 1 \\ p & q \end{bmatrix} = \begin{bmatrix} x & y \\ -1 & 0 \end{bmatrix}$

$x = 2,\ y = 1,\ \ p = -1,\ \ q = 0$

b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$

can't be true

c) $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

$\Rightarrow \begin{bmatrix} w = 9 & x = 17 \\ 8 = y & -12 = z \end{bmatrix}$

Addition and *Subtraction* of Matrices

Definition

If $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ are $m \times n$ matrices, their sum $A + B$, is the $m \times n$ matrix obtained by adding the corresponding entries; that is

$$\left[a_{ij}\right] + \left[b_{ij}\right] = \left[a_{ij} + b_{ij}\right]$$

Matrices can be added if their shapes are the same, meaning have the same ***order***.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 4 & 4 \\ 9 & 9 \end{bmatrix} = \begin{bmatrix} 1+2 & 2+2 \\ 3+4 & 4+4 \\ 0+9 & 0+9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 \\ 7 & 8 \\ 9 & 9 \end{bmatrix}$$

Scalar Multiplication Matrices

Definition

If k is a scalar and $A = \left[a_{ij}\right]$ is an $m \times n$ matrices, then scalar product $\boldsymbol{kA}$ is the $m \times n$ matrix obtained by multiplying each entry of A by k; that is

$$k\left[a_{ij}\right] = \left[ka_{ij}\right]$$

$$kA = k\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}$$

Example

$$2\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (2)1 & (2)2 \\ (2)3 & (2)4 \\ (2)0 & (2)0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 6 & 8 \\ 0 & 0 \end{bmatrix}$$

Definition

If $A_1, A_2, \ldots, A_n$ are matrices of the same size, and if $c_1, c_2, \ldots, c_n$ are scalars, then expression of the form

$$c_1A_1 + c_2A_2 + \ldots + c_nA_n$$

Is called a ***linear combination*** of $A_1, A_2, \ldots, A_n$ with *coefficients* $c_1, c_2, \ldots, c_n$.

Matrix Multiplication

Product of Two Matrices

Let ***A*** be an *m x n* matrix and let ***B*** be an *n x k* matrix. To find the element in the i^{th} row and j^{th} column of the product matrix ***AB***, multiply each element in the i^{th} row of ***A*** by the corresponding element in the j^{th} column of ***B***, and then add these products. The product matrix ***AB*** is an *m x k* matrix.

*Matrix **A*** *Matrix B*

m x n *n x k*

Inner *must be equal*

Outer: Order of *AB* is *m x k*

- ✓ To multiply ***AB*** or dot product, if ***A*** has ***n*** columns, ***B*** must have ***n*** rows.
- ✓ Squares matrices can be multiplied if and only if (***iff***) they have the same size.
- ✓ The entry in row *i* and column *j* of ***AB*** is $\boxed{(row\ i\ of\ A).(col\ j\ of\ B)}$

  The result: $\sum a_{ik}b_{kj}$

$$\begin{bmatrix} * & * & & & \\ a_{i1} & a_{i2} & \cdots & \cdots & a_{i5} \\ * & & & & \\ * & & & & \end{bmatrix} \begin{bmatrix} * & * & b_{1j} & * & * & * \\ & & b_{2j} & & & \\ & & \vdots & & & \\ & & \vdots & & & \\ & & b_{5j} & & & \end{bmatrix} = \begin{bmatrix} & & * & & & \\ * & * & (AB)_{ij} & * & * & * \\ & & * & & & \\ & & * & & & \end{bmatrix}$$

$4\ by\ 5$ $\quad 5\ by\ 6$ $\quad 4\ by\ 6$

$$\boldsymbol{AB} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$2x2 \qquad 2x2 \quad \to \quad 2x2$$

$$a_{11} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & \_ \\ \_ & \_ \end{bmatrix}$$

$$a_{12} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & af+bh \\ \_ & \_ \end{bmatrix}$$

$$a_{21} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ ce+dg & \_ \end{bmatrix}$$

$$a_{22} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ \_ & cf+dh \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Example

Find: $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1(5)+1(1) & 1(6)+1(0) \\ 2(5)-1(1) & 2(6)-1(0) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 6 \\ 9 & 12 \end{bmatrix}$$

Special Case

When A is a square matrix, then

$$A \quad \textit{times} \quad A = A^2 \quad \textit{times} \quad A \quad \underline{= A^3}$$

$$A^p = AA \cdots A \qquad (p \textit{ factors})$$

$$\left(A^p\right)\left(A^q\right) = A^{p+q}$$

$$\left(A^p\right)^q = A^{pq}$$

Block Multiplication

If the cuts between columns of ***A*** match the cuts between rows of ***B***, then the block multiplication of ***AB*** allowed.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \left[\begin{array}{c|c} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{21} + a_{12}b_{22} \\ \hline a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{21} + a_{22}b_{22} \end{array}\right]$$

Important special case

$$\begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 5 & 0 \end{bmatrix}$$

Matrix Form of the Equations

The coefficient matrix is $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}$

The equivalent matrix equation is in the form $AX = b$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Multiplication by ***rows*** $\quad AX = \begin{bmatrix} (row\ 1).X \\ (row\ 2).X \\ (row\ 3).X \end{bmatrix}$

Multiplication by ***columns*** $\quad AX = x\ (column\ 1) + y\ (column\ 2) + z\ (column\ 3)$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Identity Matrix

The identity matrix is given by the form: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \boxed{Ix = x}$

Properties of Matrix

Addition and Scalar Multiplication

$A + B = B + A$	*Commutative Property of Addition*
$A + (B + C) = (A + B) + C$	*Associative Property of Addition*
$(kl)A = k(lA)$	*Associative Property of Scalar Multiplication*
$k(A + B) = kA + kB$	*Distributive Property*
$k(A - B) = kA - kB$	*Distributive Property*
$(k + l)A = kA + lA$	*Distributive Property*
$(k - l)A = kA - lA$	*Distributive Property*
$A + 0 = 0 + A = A$	*Additive Identity Property*
$A + (-A) = (-A) + A = 0$	*Additive Inverse Property*
$k(AB) = kA(B) = A(kB)$	

Multiplication

$AB \neq BA$	*Commutative "**law**" is usually broken*
$A(BC) = (AB)C$	*Associative Property of Multiplication* (***Parentheses not needed***)
$A(B + C) = AB + AC$	*Distributive Property*
$(B + C)A = BA + CA$	*Distributive Property*
$A(B - C) = AB - AC$	*Distributive Property*
$(B - C)A = BA - CA$	*Distributive Property*

****

Consider the three vectors $\boldsymbol{u}, \quad \boldsymbol{v}, \quad and\ \boldsymbol{w}$:

$$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The linear combinations in three-dimensional space are $cu + dv + ew$

Combination $$c\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}$$

Combine the three vectors $u,\ v,\ and\ w$ into on matrix A.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Multiplies the matrix A by a vector x, where $c,\ d,\ e$ are the component of a vector x.

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}$$

We can rewrite the form, matrix A times the vector x, as the combination $cu + dv + ew$

$$Ax = \begin{bmatrix} u & v & w \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$$

Write the matrix in the form $Ax = b$

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b$$

Where the x is the input and b is the output.

Cyclic Difference

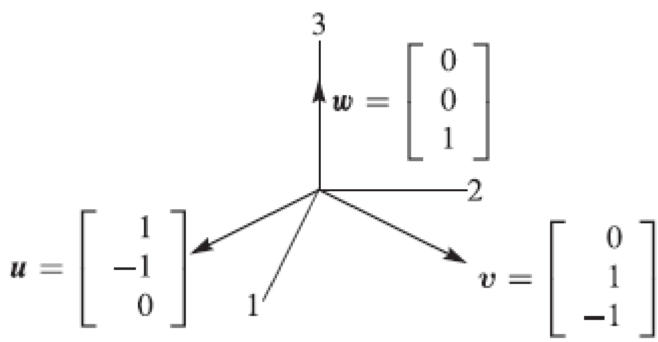
The linear combinations of three vectors $u,\ v,\ and\ w^*$ lead to a cyclic difference matrix C and is given by:

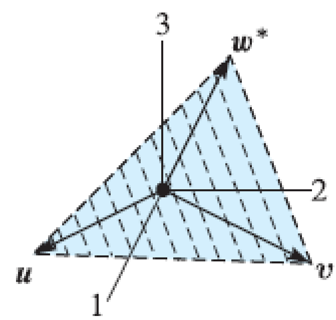
$$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad w^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = b$$

The matrix C is not triangular. It is not easy to find the solution to $Cx = b$, because either we are going to have ***infinitely many solution*** or ***no solution***.

Let looks at these problems geometrically.





Exercises *Section* 1.3 – The Algebra of Matrices

1. For the matrices: $A=\begin{bmatrix} p & 0 \\ q & r \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, when does $AB=BA$

(2 – 8) Find values for the variables so that the matrices are equal.

2. $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix}=\begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

3. $\begin{bmatrix} x & y+3 \\ 2z & 8 \end{bmatrix}=\begin{bmatrix} 12 & 5 \\ 6 & 8 \end{bmatrix}$

4. $\begin{bmatrix} 5 & x-4 & 9 \\ 2 & -3 & 8 \\ 6 & 0 & 5 \end{bmatrix}=\begin{bmatrix} y+3 & 2 & 9 \\ z+4 & -3 & 8 \\ 6 & 0 & w \end{bmatrix}$

5. $\begin{bmatrix} a+2 & 3b & 4c \\ d & 7f & 8 \end{bmatrix}+\begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix}=\begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$

6. $\begin{bmatrix} a+11 & 12z+1 & 5m \\ 11k & 3 & 1 \end{bmatrix}+\begin{bmatrix} 9a & 9z & 4m \\ 12k & 5 & 3 \end{bmatrix}=\begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$

7. $\begin{bmatrix} x+2 & 3y+1 & 5z \\ 8w & 2 & 3 \end{bmatrix}+\begin{bmatrix} 3x & 2y & 5z \\ 2w & 5 & -5 \end{bmatrix}=\begin{bmatrix} 10 & -14 & 80 \\ 10 & 7 & -2 \end{bmatrix}$

8. $\begin{bmatrix} 2x-3 & y-2 & 2z+1 \\ 5 & 2w & 7 \end{bmatrix}+\begin{bmatrix} 3x-3 & y+2 & z-1 \\ -5 & 5w+1 & 3 \end{bmatrix}=\begin{bmatrix} 20 & 8 & 9 \\ 0 & 8 & 10 \end{bmatrix}$

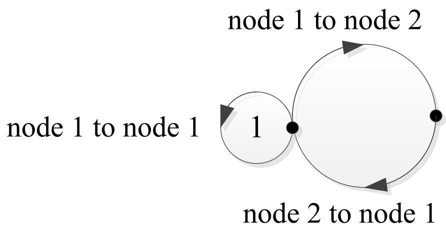
9. Find a combination $x_1w_1+x_2w_2+x_3w_3$ that gives the zero vector:

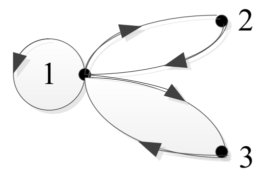
$$w_1=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}; \quad w_2=\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}; \quad w_3=\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Those vectors are independent or dependent?
The vectors lie in a ______.
The matrix W with those columns is not invertible.

10. The very last words say that the 5 by 5 centered difference matrix is not invertible, Write down the 5 equations $Cx=b$. Find a combination of left sides that gives zero. What combination of b_1, b_2, b_3, b_4, b_5 must be zero?

11. A direct graph starts with n nodes. There are n^2 possible edges, each edge leaves one of the n nodes and enters one of the n nodes (possibly itself). The n by n adjacency matrix has $a_{ij} = 1$ when edge leaves node i and enter node j; if no edge then $a_{ij} = 0$. Here are directed graphs and their adjacency matrices:





$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The i, j entry of A^2 is $a_{i1}a_{1j} + \ldots + a_{in}a_{nj}$.

Why does that sum count the two-step paths from i to any node to j?

The i, j entry of A^k counts k-steps paths:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{matrix} \textit{counts the paths} \\ \textit{with two edges} \end{matrix} \quad \begin{bmatrix} 1 \textit{ to } 2 \textit{ to } 1, 1 \textit{ to } 1 \textit{ to } 1 & 1 \textit{ to } 1 \textit{ to } 2 \\ 2 \textit{ to } 1 \textit{ to } 1 & 2 \textit{ to } 1 \textit{ to } 2 \end{bmatrix}$$

List all 3-step paths between each pair of nodes and compare with A^3. When A^k has ***no zeros***, that number k is the diameter of the graph – the number of edges needed to connect the most pair of nodes. What is the diameter of the second graph?

12. A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. All entries are 1. Which of these matrix operations are allowed, and what are the results?

a) AB *b*) BA *c*) ABD *d*) DBA

e) ABC *f*) $ABCD$ *g*) $A(B + C)$

13. What rows or columns or matrices do you multiply to find.

a) The third column of AB?
b) The second column of AB?
c) The first row of AB?
d) The second row of AB?
e) The entry in row 3, column 4 of AB?
f) The entry in row 2, column 3 of AB?

14. Add AB to AC and compare with $A(B+C)$:

$$A=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix} \quad and \quad B=\begin{bmatrix}0 & 2\\ 0 & 1\end{bmatrix} \quad and \quad C=\begin{bmatrix}3 & 1\\ 0 & 0\end{bmatrix}$$

15. True or False

a) If A^2 is defined then A is necessarily square.

b) If AB and BA are defined then A and B are squares.

c) If AB and BA are defined then AB and BA are squares.

d) If $AB=B$, then $A=I$

16. *a*) Find a nonzero matrix A such that $A^2=0$

b) Find a matrix that has $A^2\neq 0$ but $A^3=0$

17. Suppose you solve $Ax=b$ for three special right sides *b*:

$$Ax_1=\begin{bmatrix}1\\0\\0\end{bmatrix} \quad Ax_2=\begin{bmatrix}0\\1\\0\end{bmatrix} \quad Ax_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

If the three solutions x_1, x_2, x_3 are the columns of a matrix *X*, what is A times *X*?

18. Show that $(A+B)^2$ is different from $A^2+2AB+B^2$, when

$$A=\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix} \quad and \quad B=\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}$$

Write down the correct rule for $(A+B)(A+B)=A^2+$ __________ $+B^2$

(19 – 22) Find the product of the 2 matrices by rows or by columns:

19. $\begin{bmatrix}2 & 3\\ 5 & 1\end{bmatrix}\begin{bmatrix}4\\ 2\end{bmatrix}$

20. $\begin{bmatrix}3 & 6\\ 6 & 12\end{bmatrix}\begin{bmatrix}2\\ -1\end{bmatrix}$

21. $\begin{bmatrix}1 & 2 & 4\\ 2 & 0 & 1\end{bmatrix}\begin{bmatrix}3\\ 1\\ 1\end{bmatrix}$

22. $\begin{bmatrix}1 & 2 & 4\\ -2 & 3 & 1\\ -4 & 1 & 2\end{bmatrix}\begin{bmatrix}2\\ 2\\ 3\end{bmatrix}$

23. Given $A=\begin{bmatrix}4 & 1 & 3\\ 3 & -1 & -2\\ 0 & 0 & 4\end{bmatrix} \quad B=\begin{bmatrix}-3 & -2 & -3\\ -1 & 0 & 0\\ 8 & -2 & -4\end{bmatrix}$ Find $A+B,\ \ 2A,\ \ and\ \ -B$

(24 – 37) Find AB and BA, if possible

24. $A=\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ $B=\begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}$

25. $A=\begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$ $B=\begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix}$

26. $A=\begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$ $B=\begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix}$

27. $A=\begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}$ $B=\begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix}$

28. $A=\begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix}$ $B=\begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}$

29. $A=\begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}$ $B=\begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$

30. $A=\begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix}$ $B=\begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$

31. $A=\begin{pmatrix} 2 & 4 \\ 0 & -1 \\ -3 & 2 \end{pmatrix}$ $B=\begin{pmatrix} 3 & 0 & -2 \\ -2 & 6 & 2 \end{pmatrix}$

32. $A=\begin{bmatrix} 3 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix}$ $B=\begin{bmatrix} 3 & -4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

33. $A=\begin{bmatrix} 5 & 3 \\ -1 & 0 \end{bmatrix}$ $B=\begin{bmatrix} 4 & -2 \\ -2 & 0 \\ 9 & 1 \end{bmatrix}$

34. $A=\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$ $B=\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$

35. $A=\begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$ $B=\begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix}$

36. $A=\begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix}$ $B=\begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

37. $A=\begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix}$ $B=\begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix}$

38. Consider the matrices

$$A=\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \quad B=\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \quad C=\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \quad D=\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \quad E=\begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

Compute the following (where possible):

a) $D+E$ *b*) $D-E$ *c*) $5A$ *d*) $-7C$ *e*) $2B-C$ *f*) $-3(D+2E)$

39. Consider the matrices

$$A=\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \quad B=\begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ -4 & 1 & 3 \end{bmatrix} \quad C=\begin{bmatrix} -3 & -1 \\ 2 & 1 \\ 4 & 3 \end{bmatrix} \quad D=\begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$$

Compute the following (where possible):

a) $A+B$ *b*) $A+C$ *c*) AB *d*) BA *e*) CD *f*) DC

g) BD *h*) DB *i*) A^2 *j*) B^2 *k*) D^2

40. Let $B=\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$, show that $B^4=\begin{pmatrix} a^4 & 0 \\ a^3+a^2b+ab^2+b^3 & b^4 \end{pmatrix}$

41. Let $B=\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$, show that $B^n=\begin{pmatrix} a^n & 0 \\ \sum\limits_{k=0}^{n-1} a^{n-1-k}b^k & b^n \end{pmatrix}$

42. Let $A=\begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix}$. Prove that $A^n=\begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix}$ if $n \geq 1$

43. Let $A=\begin{bmatrix} 2a & -a^2 \\ 1 & 0 \end{bmatrix}$. Prove that $A^n=\begin{bmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{bmatrix}$ if $n \geq 1$

44. The following system of recurrence relations holds for all $n \geq 0$

$$\begin{cases} x_{n+1} = 7x_n + 4y_n \\ y_{n+1} = -9x_n - 5y_n \end{cases}$$

Solve the system for x_n and y_n in terms of x_0 and y_0

45. If $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, prove that $A^2-(a+d)A+(ad-bc)I_{2\times 2}=0$

46. If $A=\begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}$, use the fact $A^2=4A-3I$ and mathematical induction, to prove that

$$A^n=\frac{3^n-1}{2}A+\frac{3-3^n}{2}I \quad if \quad n \geq 1$$

47. A sequence of numbers $x_1, x_2, \ldots, x_n, \ldots$ satisfies the recurrence relation $x_{n+1}=ax_n+bx_{n-1}$ for $n \geq 1$, where a and b are constants. Prove that

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = A\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

Where $A=\begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$ and hence express $\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$ in terms of $\begin{bmatrix} x_1 \\ x_0 \end{bmatrix}$.

If $a=4$ and $b=-3$, use the previous question to find a formula for x_n in terms x_1 and x_0

Section 1.4 – Inverse Matrices - Finding A^{-1}

Definition

The matrix A is invertible if there exists a matrix A^{-1} such that:

$$A^{-1}A = AA^{-1} = I$$

where A^{-1} *read as* "A ***inverse***" and A has to be a ***square matrix***.

Not all matrices have inverses.

1. The inverse exists *iff* elimination produces n pivots (row exchanges allow).

2. The matrix A cannot have two different inverses.

3. If A is invertible, the one and only one solution to $Ax = B$ is $x = A^{-1}B$

$$AX = B$$

$$A^{-1}(AX) = A^{-1}B \qquad \textit{Multiply both side by } A^{-1}$$

$$\left(A^{-1}A\right)X = A^{-1}B \qquad \textit{Associate property}$$

$$IX = A^{-1}B \qquad \textit{Multiplicative inverse property}$$

$$X = A^{-1}B \qquad \textit{Identity property}$$

4. Suppose there is a ***nonzero*** vector x such that $Ax = 0$. Then A cannot have an inverse

5. A 2 by 2 matrix is invertible iff $ad - bc$ is not zero.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \textit{Only for 2 by 2 matrices}$$

If $ad - bc = 0$ is the determinant, then A^{-1} doesn't exist

The Inverse of a Product AB

Theorem

If an $n \times n$ matrix has an inverse, that inverse is unique.

Proof

Suppose that A has an inverse A^{-1} and B is a matrix such that $BA = I$

$$\begin{aligned} B &= BI \\ &= B\left(AA^{-1}\right) \\ &= (BA)A^{-1} \\ &= IA^{-1} \\ &= A^{-1} \end{aligned}$$

Therefore, the inverse is unique

Theorem

If A and B are invertible then so is AB. The inverse of a product AB is $(AB)^{-1} = B^{-1}A^{-1}$

Proof

$$\begin{aligned} (AB)\left(B^{-1}A^{-1}\right) &= A\left(BB^{-1}\right)A^{-1} \\ &= (AI)A^{-1} \\ &= AA^{-1} \\ &= \underline{I} \end{aligned}$$

Reverse Order

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

$$(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$$

Theorem

If A is invertible and n is a nonnegative integer, then:

a) A^{-1} is *invertible* and $\left(A^{-1}\right)^{-1} = A$

b) A^{n} is *invertible* and $\left(A^{n}\right)^{-1} = A^{-n} = \left(A^{-1}\right)^{n}$

c) kA is *invertible* for any nonzero scalar k, and $(kA)^{-1} = k^{-1}A^{-1}$

Proof

$$
\begin{aligned}
(kA)\left(k^{-1}A^{-1}\right) &= k^{-1}(kA)A^{-1} \\
&= \left(k^{-1}k\right)AA^{-1} \\
&= (1)I \\
&= I
\end{aligned}
$$

$$
\begin{aligned}
\left(k^{-1}A^{-1}\right)(kA) &= k^{-1}\left(kA^{-1}\right)A \\
&= \left(k^{-1}k\right)A^{-1}A \\
&= (1)I \\
&= I
\end{aligned}
$$

Finding A^{-1} using Gauss-Jordan Elimination

$$\left[A\middle|I\right] \to \left[I\middle|A^{-1}\right]$$

Find A^{-1} *if* $A=\begin{bmatrix}1&0&1\\2&-2&-1\\3&0&0\end{bmatrix}$

$$\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\2&-2&-1&0&1&0\\3&0&0&0&0&1\end{array}\right]\begin{array}{l}\\R_2-2R_1\\R_3-3R_1\end{array}$$

$$\begin{array}{cccccc}2&-2&-1&0&1&0\\-2&0&-2&-2&0&0\\\hline 0&-2&-3&-2&1&0\end{array}\qquad\begin{array}{cccccc}3&0&0&0&0&1\\-3&0&-3&-3&0&0\\\hline 0&0&-3&-3&0&1\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&-2&-3&-2&1&0\\0&0&-3&-3&0&1\end{array}\right]\ -\tfrac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&\frac{3}{2}&1&-\frac{1}{2}&0\\0&0&-3&-3&0&1\end{array}\right]\ -\tfrac{1}{3}R_3$$

$$\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&\frac{3}{2}&1&-\frac{1}{2}&0\\0&0&1&1&0&-\frac{1}{3}\end{array}\right]\begin{array}{l}R_1-R_3\\R_2-\frac{3}{2}R_3\\ \\ \end{array}$$

$$\begin{array}{cccccc}1&0&1&1&0&0\\0&0&-1&-1&0&\frac{1}{3}\\\hline 1&0&0&0&0&\frac{1}{3}\end{array}\qquad\begin{array}{cccccc}0&1&\frac{3}{2}&1&-\frac{1}{2}&0\\0&0&-\frac{3}{2}&-\frac{3}{2}&0&\frac{1}{2}\\\hline 0&1&0&-\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&0&0&0&0&\frac{1}{3}\\0&1&0&-\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\0&0&1&1&0&-\frac{1}{3}\end{array}\right]\qquad A^{-1}=\begin{bmatrix}0&0&\frac{1}{3}\\-\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\1&0&-\frac{1}{3}\end{bmatrix}$$

- ✓ Matrix ***A*** is ***symmetric*** across its main diagonal. So is A^{-1}
- ✓ Matrix ***A*** is ***tridiagonal*** (only three nonzero diagonals). But A^{-1} is a full matrix with no zeros. (another reason we don't compute A^{-1})

Singular *versus* Invertible

A^{-1} exists when A has a full set of n pivots. (Row exchanges allowed)

- With n pivots, elimination solves all the equations $Ax_i = b_i$. The columns x_i go into A^{-1}. Then $AA^{-1} = I$ is at least a ***right-inverse***.
- Elimination is really a sequence of multiplications.

Conclusion

- If A doesn't have n pivots, elimination will lead to a ***zero row***.
- Elimination steps are taken by an invertible M. So a row of MA is zero.
- If $AB = I$ then $MAB = M$. The zero row of MA, times B, gives a zero row of M.
- The invertible matrix M can't have a zero row! A must have n pivots if $AB = I$.

Elementary Matrices

Definition

Let ***e*** be an elementary row operation. Then the $n \times n$ ***elementary matrix*** E associated with ***e*** is the matrix obtained by applying ***e*** to the $n \times n$ identity matrix. Thus

$$E = eI$$

Example

a) $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \rightarrow$ *Multiply R_2 of I by -3*

b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow$ *Multiply the third row by -5*

c) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ *Interchange the first and second rows*

d) $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ *Add -3 times R_1 to R_2*

Theorem

Let e be an elementary operation and let E be the corresponding elementary matrix $E = e(I)$. Then for every $m \times n$ matrix A

$$e(A) = EA$$

That is, an elementary row operation can be performed on A by multiplying A on the left by the corresponding elementary matrix.

Example $m \times m$

Let $A = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$ $\quad M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

$$\begin{aligned} MA &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 & -4 & 6 & 2 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix} \end{aligned}$$

This result can be obtained from A by multiplying the first row by 2.

$$\begin{aligned} PA &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -2 & 3 & 1 \\ -6 & 2 & 1 & 5 \\ 1 & 0 & 2 & 4 \end{bmatrix} \end{aligned}$$

This result can be obtained from A by interchanging rows 2 and 3.

$$\begin{aligned} EA &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ 0 & -4 & 10 & 8 \end{bmatrix} \end{aligned}$$

This result can be obtained from A by adding 3 times row 1 to row 3.

Uniqueness of Echelon Form

Two matrices A and B are row-equivalent if and only if they have the same reduced echelon form.

Proof

If A and B have the same reduced echelon form E, then A is row-equivalent to E and E is row-equivalent to B. It follows that A is row-equivalent to B.
Now Suppose A and B are row-equivalent. Let E_1 be a reduced echelon form of A and E_2 be a reduced echelon form of B. Then E_1 and E_2 are row equivalent.
Suppose $E_1 = IF_1$ *and* $E_2 = IF_2$. Since E_1 and E_2 are row equivalent, $E_2 = CE_1$ for some matrix C. This means $I = CI$ and $F_2 = CF_1$. But then $C = I$ *and* $F_2 = F_1$.

Example

Show that the two matrices are row equivalent

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

Solution

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{matrix} R_1 + R_2 \\ R_2 - 2R_1 \end{matrix}$$

$$= \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

$$\underline{= B}$$

Definition

A relationship ~ (equivalent) between elements of a set is called an equivalence relation if

- ✓ $A \sim A$ is always true,
- ✓ $A \sim B$ always implies $B \sim A$,
- ✓ $A \sim B$ and $B \sim C$ always implies $A \sim C$.

Exercises *Section* 1.4 – Inverse Matrices - Finding A^{-1}

1. Apply Gauss-Jordan method to find the inverse of this triangular "Pascal matrix"

Triangular Pascal matrix $A=\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{bmatrix}$

2. If A is invertible and $AB=AC$, prove that $B=C$

3. If $A=\begin{bmatrix}1&1\\1&1\end{bmatrix}$, find two matrices $B\neq C$ such that $AB=AC$

4. If A has ***row*** 1 + ***row*** 2 = ***row*** 3, show that A is not invertible

a) Explain why $Ax=(1,\ 0,\ 0)$ can't have a solution.

b) Which right sides $\left(b_1,\ b_2,\ b_3\right)$ might allow a solution to $Ax=b$

c) What happens to ***row*** 3 in elimination?

5. True or false (with a counterexample if false and a reason if true):

a) A 4 by 4 matrix with a row of zeros is not invertible.

b) A matrix with 1's down the main diagonal is invertible.

c) If A is invertible then A^{-1} is invertible.

d) If A is invertible then A^2 is invertible.

6. Do there exist 2 by 2 matrices A and B with real entries such that $AB-BA=I$, where I is the identity matrix?

7. If B is the inverse of A^2, show that AB is the inverse of A.

8. Find and check the inverses (assuming they exist) of these block matrices.

$\begin{bmatrix}I&0\\C&I\end{bmatrix}\quad\begin{bmatrix}A&0\\C&D\end{bmatrix}\quad\begin{bmatrix}0&I\\I&D\end{bmatrix}$

9. For which three numbers c is this matrix not invertible, and why not?

$A=\begin{bmatrix}2&c&c\\c&c&c\\8&7&c\end{bmatrix}$

10. Find A^{-1} and B^{-1} (if they exist) by elimination.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

11. Find A^{-1} using the Gauss-Jordan method, which has a remarkable inverse.

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find the inverse, if exists, of

12. $\begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$

14. $\begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}$

15. $A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$

16. $A = \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$

17. $A = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$

18. $A = \begin{bmatrix} a & b \\ 3 & 3 \end{bmatrix}$

19. $A = \begin{bmatrix} -2 & a \\ 4 & a \end{bmatrix}$

20. $A = \begin{bmatrix} 4 & 4 \\ b & a \end{bmatrix}$

21. $A = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$

22. $A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$

23. $A = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$

24. $A = \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

25. $A = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$

26. $A = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$

27. $A = \begin{pmatrix} -6 & 9 \\ 2 & -3 \end{pmatrix}$

28. $A = \begin{pmatrix} -2 & 7 \\ 0 & 2 \end{pmatrix}$

29. $A = \begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$

30. $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

31. $A = \begin{pmatrix} 2 & 1 \\ a & a \end{pmatrix}$

32. $A = \begin{pmatrix} b & 3 \\ b & 2 \end{pmatrix}$

33. $A = \begin{pmatrix} 1 & a \\ 3 & a \end{pmatrix}$

34. $A = \begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix}$

35. $A = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$

36. $A = \begin{pmatrix} -3 & \frac{1}{2} \\ 6 & -1 \end{pmatrix}$

37. $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

38. $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{bmatrix}$

39. $A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

40. $A = \begin{bmatrix} -2 & 5 & 3 \\ 4 & -1 & 3 \\ 7 & -2 & 5 \end{bmatrix}$

41. $A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{pmatrix}$

42. $A=\begin{pmatrix}1 & -1 & 1\\ 0 & -2 & 1\\ -2 & -3 & 0\end{pmatrix}$

43. $A=\begin{pmatrix}1 & 0 & 2\\ -1 & 2 & 3\\ 1 & -1 & 0\end{pmatrix}$

44. $A=\begin{pmatrix}1 & 1 & 1\\ 3 & 2 & -1\\ 3 & 1 & 2\end{pmatrix}$

45. $A=\begin{pmatrix}3 & 3 & 1\\ 1 & 2 & 1\\ 2 & -1 & 1\end{pmatrix}$

46. $A=\begin{pmatrix}-3 & 1 & -1\\ 1 & -4 & -7\\ 1 & 2 & 5\end{pmatrix}$

47. $A=\begin{pmatrix}1 & 1 & -3\\ 2 & -4 & 1\\ -5 & 7 & 1\end{pmatrix}$

48. $A=\begin{pmatrix}1 & 2 & -1\\ 3 & 5 & 3\\ 2 & 4 & 3\end{pmatrix}$

49. $A=\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 0\end{pmatrix}$

50. $A=\begin{pmatrix}\frac{1}{5} & \frac{1}{5} & -\frac{2}{5}\\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10}\\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10}\end{pmatrix}$

51. $A=\begin{pmatrix}\sqrt{2} & 3\sqrt{2} & 0\\ -4\sqrt{2} & \sqrt{2} & 0\\ 0 & 0 & 1\end{pmatrix}$

52. $A=\begin{pmatrix}-8 & 17 & 2 & \frac{1}{3}\\ 4 & 0 & \frac{2}{5} & -9\\ 0 & 0 & 0 & 0\\ -1 & 13 & 4 & 2\end{pmatrix}$

53. $A=\begin{bmatrix}-2 & -3 & 4 & 1\\ 0 & 1 & 1 & 0\\ 0 & 4 & -6 & 1\\ -2 & -2 & 5 & 1\end{bmatrix}$

54. $A=\begin{bmatrix}1 & -14 & 7 & 38\\ -1 & 2 & 1 & -2\\ 1 & 2 & -1 & -6\\ 1 & -2 & 3 & 6\end{bmatrix}$

55. $A=\begin{bmatrix}10 & 20 & -30 & 15\\ 3 & -7 & 14 & -8\\ -7 & -2 & -1 & 2\\ 4 & 4 & -3 & 1\end{bmatrix}$

56. Show that A is not invertible for any values of the entries

$$A=\begin{bmatrix}0 & a & 0 & 0 & 0\\ b & 0 & c & 0 & 0\\ 0 & d & 0 & e & 0\\ 0 & 0 & f & 0 & g\\ 0 & 0 & 0 & h & 0\end{bmatrix}$$

57. Prove that if A is an invertible matrix and B is row equivalent to A, then B is also invertible.

58. Determine if the given matrix has an inverse, and find the inverse if it exists. Check your answer by multiplying $A \cdot A^{-1} = I$

a) $\begin{bmatrix} 2 & 3 \\ -3 & -5 \end{bmatrix}$ *b)* $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}$

59. Show that the inverse of $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is $\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$

60. If the product $C = AB$ is invertible (and *A* & *B* are square matrices), find a formula for A^{-1} that involves C^{-1} and *B*.

Hence, it is not possible to multiply a non-invertible matrix by another matric and obtain an invertible matrix as a result.

61. Prove that if *A* is an $m \times n$ matrix, there is an invertible matrix *C* such that CA is in reduced row-echelon form.

62. Prove that 2 $m \times n$ matrices *A* and *B* are row equivalent if and only if there exists a nonsingular matrix *P* such that $B = PA$

63. Let *A* and *B* be 2 $m \times n$ matrices. Suppose *A* is row equivalent to *B*. Prove that *A* is nonsingular if and only if *B* is nonsingular.

64. Show that if *A* and *B* are two $n \times n$ invertible matrices then *A* is row equivalent to *B*.

65. Prove that a square matrix *A* is nonsingular if and only if *A* is a product of elementary matrices.

66. Show that if $A \sim B$ (that is, if they are row equivalent), then $EA = B$ for some matrix *E* which is a product of elementary matrices.

67. Show that if $EA = B$ for some matrix *E* which is a product of elementary matrices, then $AC \sim BC$ for every $n \times n$ matrix *C*.

68. Let $A\vec{x} = 0$ be a homogeneous system of *n* linear equations in *n* unknowns that has only the trivial solution. Show that of *k* is any positive integer, then the system $A^k \vec{x} = 0$ also has only trivial solution.

69. Let $A\vec{x} = 0$ be a homogeneous system of *n* linear equations in *n* unknowns, and let *Q* be an invertible $n \times n$ matrix. Show that $A\vec{x} = 0$ has just trivial solution if and only if $(QA)\vec{x} = 0$ has just trivial solution.

70. Let $A\vec{x}=b$ be any consistent system of linear equations, and let $\vec{x}_1$ be a fixed solution. Show that every solution to the system can be written in the form $\vec{x}=\vec{x}_1+\vec{x}_0$ where $\vec{x}_0$ is a solution to $A\vec{x}=0$. Show also that every matrix of this form is a solution.

71. If A and B are $n\times n$ matrices satisfying $A^2=B^2=(AB)^2=I_n$. Prove that $AB=BA$.

72. Let $A=\begin{pmatrix}0&1&0\\0&0&1\\5&0&0\end{pmatrix}$. Verify that $A^3=5I$, then find A^{-1} in term of A.

73. Consider $B(A,\ I)=(BA,\ B)$, thus if B is the inverse of A, then $(BA,\ B)$ becomes $\left(I,\ A^{-1}\right)$. On the other hand B is a product of elementary matrices since it is invertible. This indicates that the inverse of A can be obtained by applying elementary row operations to $(A,\ I)$ to get $\left(I,\ A^{-1}\right)$.

Find the inverses of

a) $A=\begin{pmatrix}1&0&0\\0&1&0\\a&b&1\end{pmatrix}$

b) $B=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\a&b&c&d\end{pmatrix}$

74. Let $A,\ B,\ C,\ X,\ Y,\ Z\in M_n(\mathbb{C})$, A and C are invertible. Find

a) $\begin{pmatrix}A&B\\0&C\end{pmatrix}^{-1}$

b) $\begin{pmatrix}I&X&Y\\0&I&Z\\0&0&I\end{pmatrix}^{-1}$

75. Suppose that A, B, and $A-B$ are invertible $n\times n$ matrices. Show that

$$(A-B)^{-1}=A^{-1}+A^{-1}\left(B^{-1}-A^{-1}\right)^{-1}A^{-1}$$

76. Suppose P is invertible and $A=PBP^{-1}$. Solve for B in terms of A.

77. Suppose $(A-B)C=0$, where A and B are $m\times n$ matrices and C is invertible. Show that $A=B$.

Section 1.5 – Transpose, Diagonal, Triangular, and Symmetric Matrices

Transpose

Definition

The transpose of a matrix A is defined as the matrix that is obtained by interchanging the corresponding rows and columns in A. Then the transpose of A, denoted by A^T or A'.

The columns of A^T are the rows of A.

When A is an m by n matrix, the transpose is n by m:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \quad \text{then} \quad A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

The matrix ***flips over*** the main diagonal. The entry in row i, column j of A^T comes from row j, column i of the original A.

$$\left(A^T\right)_{ij} = A_{ji}$$

Properties of Transpose

a) $\left(A^T\right)^T = A$

b) $(A+B)^T = A^T + B^T$

c) $(A-B)^T = A^T - B^T$

d) $(kA)^T = kA^T$

e) $(AB)^T = B^T A^T$

The transpose of a product of any number of matrices is the product of the transposes in the reverse order.

Theorem

If A is an invertible matrix, then A^T is also invertible and

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

Proof

$$A^T\left(A^{-1}\right)^T = \left(A^{-1}A\right)^T$$
$$= I^T$$
$$= I$$

$$\left(A^{-1}\right)^T A^T = \left(AA^{-1}\right)^T$$
$$= I^T$$
$$= I$$

****

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad and \quad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\left(A^T\right)^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$
$$= \begin{bmatrix} \frac{d}{ad-bc} & -\frac{c}{ad-bc} \\ -\frac{b}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

Trace

Definition

If A is a square matrix, then the trace of A, denoted by **tr(*A*)**, is defined to the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.

Example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$tr(A) = a_{11} + a_{22} + a_{33}$$

Diagonal

A square matrix in which all the entries off the main diagonal are zero is called a ***diagonal matrix***. A general $n \times n$ diagonal matrix can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

A diagonal matrix is invertible iff all of its diagonal entries are nonzero; the

$$D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}$$

Powers of diagonal matrices are:

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

Triangular Matrices

A square matrix in which all the entries above the main diagonal are zero is called ***lower diagonal triangular***.

A square matrix in which all the entries below the main diagonal are zero is called ***upper diagonal triangular***.

A matrix that is either upper triangular or lower triangular is called ***triangular***.

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 \\ a_{51} & a_{51} & a_{53} & a_{54} & a_{55} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{bmatrix}$$

lower diagonal triangular ***upper diagonal triangular***

Theorem

- ✓ The transpose of a lower triangular matrix is upper triangular, and the transpose of a upper triangular matrix is lower triangular.
- ✓ The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.
- ✓ A triangular matrix is invertible iff its diagonal entries are all nonzero.
- ✓ The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.

Example

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

$$A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \qquad AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix} \qquad BA = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 5 \end{bmatrix}$$

The goal is to describe Gaussian elimination in the most useful way by looking at them closely, which are factorizations of a matrix.

The factors are triangular matrices.

The factorization that comes from elimination is $A = LU$ **.**

Symmetric Matrices

Definition

A square matrix A is said to be ***symmetric*** if $A^T = A$. That means a square matrix must satisfies $a_{ij} = a_{ji}$

Example

$$A = \begin{pmatrix} 1 & -4 \\ -4 & 1 \end{pmatrix} = A^T$$

$$A = \begin{pmatrix} 6 & 5 & 1 \\ 5 & 0 & 7 \\ 1 & 7 & -1 \end{pmatrix} = A^T$$

- The ***inverse*** of a symmetric matrix is also ***symmetric***.

Example

Given $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, show that the inverse is symmetric too?

Solution

$$A^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

Theorem

If A and B are symmetric matrices with the same size, and if k is any scalar, then:

a) A^T is symmetric

b) $A + B$ and $A - B$ are symmetric.

c) kA is symmetric

- If A is an invertible symmetric matrix, then A^{-1} is *symmetric*.

Proof

Assume that A is symmetric and invertible then $A = A^T$

$$\left(A^{-1}\right)^T = \left(A^T\right)^{-1} = A^{-1}$$

Which proves that A^{-1} is *symmetric*

- Multiplying M by M^T gives a symmetric matrix.

Proof

The entry (i, j) of $M^T M$, it is the dot product of ***row*** i of M^T (column i of M) with column j of M.

The (i, j) entry is the same dot product, column j with column i. so $M^T M$ is symmetric.

The matrix $M.M^T$ is also symmetric and $M^T M$ is a different matrix from $M.M^T$.

- If A is an invertible symmetric matrix, then AA^T and $A^T A$ are also invertible.
- Matrix A is symmetric across its main diagonal. So is A^{-1}
- Matrix A is tridiagonal (only three nonzero diagonals). But A^{-1} is a full matrix with no zeros.
  (another reason we don't compute A^{-1})

Example

Given $M = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and $M^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find $M^T M$ and $M.M^T$

Solution

$$M^T M = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$MM^T = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \end{bmatrix}$$

***Symmetric in* LDU**

When elimination is applied to a symmetric matrix, $A^T = A$ is an advantage.

$$\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$= \underset{L}{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}$$

- If $A = A^T$ can be factored into LDU with no row exchanges, then $U = L^T$. The ***symmetric factorization of a symmetric matrix is*** $A = LDL^T$

Exercises *Section* 1.5 – Transpose, Diagonal, Triangular, and Symmetric Matrices

1. Solve $Lc = b$ to find $\boldsymbol{c}$. Then solve $Ux = c$ to find $\boldsymbol{x}$. What was A?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

2. Find L and U for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on $a,\ b,\ c,\ d$ to get $A = LU$ with four pivots

3. Determine whether the given matrix is invertible

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

4. Find $A^2,\ A^{-2},\ and\ A^{-k}$ by inspection

$$a)\ A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \qquad b)\ A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \qquad c)\ A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

5. Decide whether the given matrix is symmetric

$$a)\ \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \qquad b)\ \begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{bmatrix} \qquad c)\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

6. Find all values of the unknown constant(s) in order for A to be symmetric

$$A = \begin{bmatrix} 2 & a-2b+2c & 2a+b+c \\ 3 & 5 & a+c \\ 0 & -2 & 7 \end{bmatrix}$$

7. Find a diagonal matrix A that satisfies the given condition $A^{-2} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. Let A be an $n \times n$ symmetric matrix

a) Show that A^2 is symmetric

b) Show that $2A^2 - 3A + I$ is symmetric

9. Prove if $A^T A = A$, then A is symmetric and $A = A^2$

10. A square matrix A is called ***skew-symmetric*** if $A^T = -A$. Prove

a) If A is an invertible skew-symmetric matrix, then A^{-1} is skew-symmetric.

b) If A and B are skew-symmetric matrices, then so are A^T, $A+B$, $A-B$, *and* kA for any scalar k.

c) Every square matrix A can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

$$\left[\textit{Hint}: \text{ Note the identity } A = \tfrac{1}{2}\left(A + A^T\right) + \tfrac{1}{2}\left(A - A^T\right) \right]$$

11. Suppose R is rectangular (m by n) and A is symmetric (m by m)

a) Transpose $R^T AR$ to show its symmetric

b) Show why $R^T R$ has no negative numbers on its diagonal.

12. If L is a lower-triangular matrix, then $\left(L^{-1}\right)^T$ is _______Triangular

13. True or False

a) The block matrix $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ is automatically symmetric

b) If A and B are symmetric then their product is symmetric

c) If A is not symmetric then A^{-1} is not symmetric

d) When A, B, C are symmetric, the transpose of ABC is CBA.

e) The transpose of a diagonal matrix is a diagonal.

f) The transpose of an upper triangular matrix is an upper triangular matrix.

g) The sum of an upper triangular matrix and a lower triangular matrix is a diagonal matrix.

h) All entries of a symmetric matrix are determined by the entries occurring on and above the main diagonal.

i) All entries of an upper triangular matrix are determined by the entries occurring on and above the main diagonal.

j) The inverse of an invertible lower triangular matrix is an upper triangular matrix.
k) A diagonal matrix is invertible if and only if all of its diagonal entries are positive.
l) The sum of a diagonal matrix and a lower triangular matrix is a lower triangular matrix.
m) A matrix that is both symmetric and upper triangular must be a diagonal matrix.
n) If A and B are $n \times n$ matrices such that $A + B$ is symmetric, then A and B are symmetric.
o) If A and B are $n \times n$ matrices such that $A + B$ is upper triangular, then A and B are upper triangular.
p) If A^2 is a symmetric matrix, then A is a symmetric matrix.
q) If kA is a symmetric matrix for some $k \neq 0$, then A is a symmetric matrix.

14. Find 2 by 2 symmetric matrices $A = A^T$ with these properties

a) A is not invertible

b) A is invertible but cannot be factored into LU (row exchanges needed)

c) A can be factored into LDL^T but not into LL^T (because of negative D)

15. A group of matrices includes AB and A^{-1} if it includes A and B. "Products and inverses stay in the group." Which of these sets are groups?

Lower triangular matrices L with 1's on the diagonal, symmetric matrices S, positive matrices M, diagonal invertible matrices D, permutation matrices P, matrices with $Q^T = Q^{-1}$. ***Invent two more matrix groups***.

16. Write $A = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$ as the product EH of an elementary row operation matrix E and a symmetric matrix H.

17. When is the product of two symmetric matrices symmetric? Explain your answer.

18. Express $\left((AB)^{-1}\right)^T$ in terms of $\left(A^{-1}\right)^T$ and $\left(B^{-1}\right)^T$

19. Find the transpose of the given matrix: $\begin{bmatrix} 8 & -1 \\ 3 & 5 \\ -2 & 5 \\ 1 & 2 \\ -3 & -5 \end{bmatrix}$

20. Show that if A is symmetric and invertible, then A^{-1} is also symmetric.

21. Prove that $(AB)^T = B^T A^T$

22. For the given matrix, compute A^T, $\left(A^T\right)^{-1}$, A^{-1}, and $\left(A^{-1}\right)^T$, then compare $\left(A^T\right)^{-1}$ and $\left(A^{-1}\right)^T$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

23. Show that a 2×2 lower triangular matrix is invertible if and only if $a_{11}a_{22} \neq 0$ and in this case the inverse is also lower triangular.

24. Let A be any 2×2 diagonal matrix. Give a necessary and sufficient condition on the diagonal entries so that A has an inverse. Compute the inverse of any such matrix.

Section 1.6 – Determinants and Properties

The determinant is a number that contains information about matrix. It is used to find formulas for inverse matrices, pivots, and solutions $A^{-1}b$.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ has inverse } A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Determinant of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is written $\det(A)$ *or* $|A|$ and is define as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The determinant is zero when the matrix has no inverse.

Properties of the Determinants

By using these property rules, we can compute the determinant of any square matrix.

1. ***Determinant of the n by n identity matrix is* 1.**

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \quad and \quad \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} = 1$$

2. ***Determinant changes sign when 2 rows are exchanged.***

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = -(ad - bc) \quad \Rightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

3. ***Determinant is a linear function of each row separately.***

Multiply row 1 by any number t: $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Add row 1 of A to row 1 of A': $\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

- ***For 2 by 2 determinants, if you expand to a rectangle, the determinants equal areas.***
- ***For n-dimensional, the determinants equal volumes.***

4. ***If 2 rows of A are equal, then*** $\det A = 0$.

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$$

5. ***Subtracting a multiple of one row from another row leaves*** $\det A$ ***unchanged.***

$$\begin{vmatrix} a & b \\ c - ta & d - ta \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

6. ***A matrix with a row of zeros has*** $\det A = 0$.

$$\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0 \quad and \quad \begin{vmatrix} 0 & 0 \\ b & c \end{vmatrix} = 0$$

7. ***If A is triangular then*** $\det A = a_{11} a_{22} \ \dots \ a_{nn}$ ***= product of diagonal entries.***

$$\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad \quad and \quad \begin{vmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & a_{nn} \end{vmatrix} = a_{11} a_{22} \dots a_{nn}$$

8. ***If A is singular then*** $\det A = 0$.

9. ***If A is invertible then*** $\det A \neq 0$.

10. ***The determinant of AB*** $\det A$ ***is times*** $\det B$ ***:*** $|AB| = |A||B|$

11. ***The transpose*** A^T ***has the same determinant as A:*** $\det(A) = \det\left(A^T\right)$

- $\det(A + B) \neq \det(A) + \det(B)$

Big Formula for Determinants (Diagonal)

Determinant Using Diagonal Method

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} \qquad a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \quad (1)$$

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} \qquad -a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \quad (2)$$

Determinant: $\boldsymbol{D} = \mathbf{(1) + (2)}$

$$\textbf{det} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Example

Evaluate: $\begin{vmatrix} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{vmatrix} \begin{matrix} x & 0 \\ 2 & x \\ -3 & x \end{matrix} = x(x)(1) + 0\left(x^2\right)(2) + (-1)(2)(x) - (-1)(x)(-3) - x\left(x^2\right)(x) - 0(-3)(1)$$

$$= x^2 - 2x - 3x - x^4$$

$$\underline{= x^2 - 5x - x^4}$$

Determinant by *Cofactors*

$$\boldsymbol{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Minor

For a square matrix $A = \left[a_{ij} \right]$, the minor M_{ij}. Of an element a_{ij} is the ***determinant*** of the matrix formed by deleting the i^{th} row and the j^{th} column of A.

Example

Let $A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$ Find M_{32}

Solution

$$M_{32} = \begin{vmatrix} 3 & \cancel{1} & -4 \\ 2 & \cancel{5} & 6 \\ \cancel{1} & \cancel{4} & \cancel{8} \end{vmatrix}$$

$$= \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix}$$

$$= 26$$

Theorem

The determinant is the dot product of any row $\boldsymbol{i}$ of $\boldsymbol{A}$ with its cofactors:

Cofactor Formula: $\boxed{C_{ij} = (-1)^{i+j} M_{ij}}$

$$|\boldsymbol{A}| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example

Find the determinant of the matrix:

$$A = \begin{bmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{bmatrix}$$

Solution

$$|A| = \begin{vmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{vmatrix}$$

$$= -8\begin{vmatrix} -6 & 7 \\ -3 & 5 \end{vmatrix} - 0\begin{vmatrix} 4 & 7 \\ -1 & 5 \end{vmatrix} + 6\begin{vmatrix} 4 & -6 \\ -1 & -3 \end{vmatrix}$$

$$= -8(-30-(-21)) - 0 + 6(-12-6)$$

$$= -8(-9) + 6(-18)$$

$$\underline{= -36}$$

✓ By the property of determinants, If $\boldsymbol{A}$ is triangular then $\mathbf{det}\boldsymbol{A} = \boldsymbol{a}_{11}\boldsymbol{a}_{22} \ \dots \ \boldsymbol{a}_{nn}$ = product of diagonal entries.

Example

$$\begin{vmatrix} 2 & 7 & -3 & 8 & 3 \\ 0 & -3 & 7 & 5 & 1 \\ 0 & 0 & 6 & 7 & 6 \\ 0 & 0 & 0 & 9 & 8 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix} = (2)(-3)(6)(9)(4)$$

$$\underline{= -1296}$$

Theorem

Let $\boldsymbol{A}$ be any n by n matrix.

a) If $\boldsymbol{A'}$ is the matrix that results when a single row of $\boldsymbol{A}$ is multiplied by a constant $\boldsymbol{k}$, then $\det(A') = k\det(A)$.

b) If $\boldsymbol{A'}$ is the matrix that results when two rows of $\boldsymbol{A}$ are interchanged, then $\det(A') = -\det(A)$

c) If $\boldsymbol{A'}$ is the matrix that results when a multiple of one row of $\boldsymbol{A}$ is added to another row then $\det(A') = \det(A)$

Example

Evaluate $\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

Interchanged 1st and 2nd row

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

A common factor of 3 from the first row (no need)

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \begin{matrix} \\ \\ R_3 - 2R_1 \end{matrix}$$

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} \begin{matrix} \\ \\ R_3 - 10R_2 \end{matrix}$$

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$

$$= -3(1)(1)(-55)$$

$$\underline{= 165}$$

Exercises *Section* 1.6 – Determinants and Properties

1. Verify that $\det(AB)=\det(A)\det(B)$ when: $A=\begin{pmatrix}2&1&0\\3&4&0\\0&0&2\end{pmatrix}$ *and* $B=\begin{pmatrix}1&-1&3\\7&1&2\\5&0&1\end{pmatrix}$

2. For which value(s) of $\boldsymbol{k}$ does A fail to be invertible? $A=\begin{bmatrix}k-3&-2\\-2&k-2\end{bmatrix}$

3. Without directly evaluating, show that $\begin{vmatrix}b+c&c+a&b+a\\a&b&c\\1&1&1\end{vmatrix}=0$

4. If the entries in every row of A add to zero, solve $\boldsymbol{Ax}=0$ to prove $\det(A)=0$. If those entries add to one, show that $\det(A-I)=0$. Does this mean $\det(A)=I$?

5. Does $\det(AB)=\det(BA)$ in general?

a) True or false if $\boldsymbol{A}$ and $\boldsymbol{B}$ are square *n* x *n* matrices?

b) True or false if $\boldsymbol{A}$ is *m* x *n* and B is *n* x *m* with $m\neq n$?

6. True or false, with a reason if true or a counterexample if false:

a) The determinant of $I+A$ is $1+\det(A)$.

b) The determinant of ABC is $|A||B||C|$.

c) The determinant of 4A is $4|A|$

d) The determinant of $AB-BA$ is zero. (try an example)

e) If A is not invertible then AB is not invertible.

f) The determinant of $A-B$ equals to $\det(A)-\det(B)$.

7. Use row operations to show the 3 by 3 "Vandermonde determinant" is

$$\det\begin{bmatrix}1&a&a^2\\1&b&b^2\\1&c&c^2\end{bmatrix}=(b-a)(c-a)(c-b)$$

8. The inverse of a 2 by 2 matrix seems to have determinant = 1:

$$\det A^{-1} = \det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{ad-bc}{ad-bc} = 1$$

What is wrong with this calculation? What is the correct $\det A^{-1}$

9. A *Hessenberg* matrix is a triangular matrix with one extra diagonal. Use cofactors of row 1 to show that the 4 by 4 determinant satisfies Fibonacci's rule $|H_4| = |H_3| + |H_2|$. The same rule will continue for all sizes $|H_n| = |H_{n-1}| + |H_{n-2}|$. Which Fibonacci number is $|H_n|$?

$$H_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad H_3 = \begin{bmatrix} 2 & 1 & \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad H_4 = \begin{bmatrix} 2 & 1 & & \\ \mathbf{1} & 2 & \mathbf{1} & \\ \mathbf{1} & 1 & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & 1 & \mathbf{1} & \mathbf{2} \end{bmatrix}$$

(10 – 44) Evaluate

10. $\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix}$

11. $\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix}$

12. $\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix}$

13. $\begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix}$

14. $\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix}$

15. $\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix}$

16. $\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix}$

17. $\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix}$

18. $\begin{vmatrix} 5 & 3 \\ -2 & 3 \end{vmatrix}$

19. $\begin{vmatrix} -4 & -1 \\ 5 & 6 \end{vmatrix}$

20. $\begin{vmatrix} \sqrt{3} & -2 \\ -3 & \sqrt{3} \end{vmatrix}$

21. $\begin{vmatrix} \sqrt{7} & 6 \\ -3 & \sqrt{7} \end{vmatrix}$

22. $\begin{vmatrix} \sqrt{5} & 3 \\ -2 & 2 \end{vmatrix}$

23. $\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{vmatrix}$

24. $\begin{vmatrix} \frac{1}{5} & \frac{1}{6} \\ -6 & -5 \end{vmatrix}$

25. $\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{2} & \frac{3}{4} \end{vmatrix}$

26. $\begin{vmatrix} x & x^2 \\ 4 & x \end{vmatrix}$

27. $\begin{vmatrix} x & x^2 \\ x & 9 \end{vmatrix}$

28. $\begin{vmatrix} x^2 & x \\ -3 & 2 \end{vmatrix}$

29. $\begin{vmatrix} x+2 & 6 \\ x-2 & 4 \end{vmatrix}$

30. $\begin{vmatrix} x+1 & -6 \\ x+3 & -3 \end{vmatrix}$

31. $\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & -5 \\ 2 & 5 & -1 \end{vmatrix}$

32. $\begin{vmatrix} 4 & 0 & 0 \\ 3 & -1 & 4 \\ 2 & -3 & 6 \end{vmatrix}$

33. $\begin{vmatrix} 3 & 1 & 0 \\ -3 & -4 & 0 \\ -1 & 3 & 5 \end{vmatrix}$

34. $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & -4 & 5 \end{vmatrix}$

35. $\begin{vmatrix} x & 0 & -1 \\ 2 & 1 & x^2 \\ -3 & x & 1 \end{vmatrix}$

36. $\begin{vmatrix} x & 1 & -1 \\ x^2 & x & x \\ 0 & x & 1 \end{vmatrix}$

37. $\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix}$

38. $\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix}$

39. $\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix}$

40. $\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix}$

41. $\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix}$

42. $\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix}$

43. $\begin{vmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix}$

44. $\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix}$

45. Find all the values of λ for which $\det(A) = 0$

a) $A = \begin{bmatrix} \lambda-1 & -2 \\ 1 & \lambda-4 \end{bmatrix}$ *b)* $A = \begin{bmatrix} \lambda-6 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 4 & \lambda-4 \end{bmatrix}$

46. Prove that if a square matrix $\boldsymbol{A}$ has a column of zeros, then $\det(A) = 0$

47. With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A||D| \quad \textit{but} \quad \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D| - |C||B|$$

a) Why is the first statement true? Somehow B doesn't enter.

b) Show by example that equality fails (as shown) when C enters.

c) Show by example that the answer $\det(AD - CB)$ is also wrong.

48. Show that the value of the following determinant is independent of θ.

$$\begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta - \cos\theta & \sin\theta + \cos\theta & 1 \end{vmatrix}$$

49. Show that the matrices $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ *and* $B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$

commute if and only if $\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$

50. Show that $\det(A) = \frac{1}{2}\begin{vmatrix} tr(A) & 1 \\ tr\left(A^2\right) & tr(A) \end{vmatrix}$ for every 2×2 matrix A.

51. What is the maximum number of zeros that a 4×4 matrix can have without a zero determinant? Explain your reasoning.

52. Evaluate $\det(A)$, $\det(E)$, and $\det(AE)$. Then verify that $\det(A)\cdot\det(E) = \det(AE)$

$$A = \begin{bmatrix} 4 & 1 & 3 \\ 0 & -2 & 0 \\ 3 & 1 & 5 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & & \\ & 3 & \\ & & 1 \end{bmatrix}$$

53. Show that $\begin{bmatrix} \sin^2\alpha & \sin^2\beta & \sin^2\gamma \\ \cos^2\alpha & \cos^2\beta & \cos^2\gamma \\ 1 & 1 & 1 \end{bmatrix}$ is not invertible for any values of $\alpha,\ \beta,\ \gamma$

54. The determinant of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det(A) = ad - bc$.

Assuming no rows swaps are required, perform elimination on A and show explicitly that $ad - bc$ is the product of the pivots.

55. If A is a 7×7 matrix and let $\det(A) = 17$. What is $\det\left(3A^2\right)$?

56. Explain without computations why the following determinant is equal to zero

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & 0 & 0 & 0 \\ d_1 & d_2 & 0 & 0 & 0 \\ e_1 & e_2 & 0 & 0 & 0 \end{vmatrix}$$

57. Let A be $n \times n$ real matrix.

a) Show that if $A^t = -A$ and n is odd, then $|A| = 0$.

b) Show that if $A^2 + I = 0$, then n must be even.

c) Does part (b) remain true for complex matrices?

58. Let A and C be $m \times m$ and $n \times n$ matrices, respectively.

a) Show that $\begin{vmatrix} A & B \\ 0 & C \end{vmatrix} = \begin{vmatrix} A & 0 \\ B & C \end{vmatrix} = |A||C|$

b) Evaluate

i. $\begin{vmatrix} I_m & 0 \\ 0 & I_n \end{vmatrix}$

ii. $\begin{vmatrix} 0 & I_m \\ I_n & 0 \end{vmatrix}$

iii. $\begin{vmatrix} I_m & B \\ 0 & I_n \end{vmatrix}$

iv. $\begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix}_{n \times n}$

c) Find a formula for $\begin{vmatrix} 0 & A \\ C & B \end{vmatrix}_{n \times n}$

59. Let $f(x) = (p_1 - x)(p_2 - x)\ldots(p_n - x)$ and let

$$\Delta_n = \begin{vmatrix} p_1 & a & a & \ldots & a & a \\ b & p_2 & a & \ldots & a & a \\ b & b & p_3 & \ldots & a & a \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b & b & b & \cdots & p_{n-1} & a \\ b & b & b & \ldots & b & p_n \end{vmatrix}$$

a) Show that, if $a \neq b$,

$$\Delta_n = \frac{bf(a) - af(b)}{b - a}$$

b) Show that, if $a = b$,

$$\Delta_n = a\sum_{i=1}^{n} f_i(a) + p_n f_n(a)$$

Where $f_i(a)$ means $f(a)$ with factor $(p_i - a)$ missing.

c) Use part (b) to evaluate

$$\begin{vmatrix} a & b & b & \dots & b & b \\ b & a & b & \dots & b & b \\ b & b & a & \dots & a & a \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b & b & b & \cdots & a & a \\ b & b & b & \dots & b & a \end{vmatrix}_{n\times n}$$

60. Let $A, B, C, D \in M_n(\mathbb{C})$

a) Show that when A is invertible: $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A|\left|D - CA^{-1}B\right|$

b) Show that when $AC = CA$: $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|$

c) Can B and C on the right-hand side of the identity be switched?

d) Does part (b) remain true if the condition $AC = CA$ is dropped?

Section 1.7 – Cramer's Rule

Cramer's Rule

Theorem

If $AX = B$ is a system of a linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(B_1)}{\det(A)}$$

$$x_2 = \frac{\det(B_2)}{\det(A)}$$

$$\vdots \qquad \vdots$$

$$x_n = \frac{\det(B_n)}{\det(A)}$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & & & a_{nn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

$$\det(B_1) = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & & & \\ \vdots & & & \\ b_n & a_{n2} & & a_{nn} \end{vmatrix}$$

Example

Use Cramer's rule to solve

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ -2x_1 + x_2 \quad\ \ &= 0 \\ -4x_1 \quad\ \ + x_3 &= 0 \end{aligned}$$

Solution

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 7$$

$$|B_1| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$|B_2| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 0 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 2$$

$$|B_3| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 0 \end{vmatrix} = 4$$

$$x_1 = \frac{|B_1|}{|A|} = \frac{1}{7} \qquad\qquad x_1 = \frac{\det(A_1)}{\det(A)}$$

$$x_2 = \frac{|B_2|}{|A|} = \frac{2}{7}$$

$$x_3 = \frac{|B_3|}{|A|} = \frac{4}{7}$$

Solution: $\left(\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\right)$

Example

Use Cramer's Rule to solve.

$$\begin{aligned} x_1 + \quad\quad + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

Solution

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix} \quad \Rightarrow \quad \det(A) = 44$$

$$\det(A_1) = \begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{vmatrix} = -40$$

$$\det(A_2) = \begin{vmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{vmatrix} = 72$$

$$\det(A_3) = \begin{vmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{vmatrix} = 152$$

$$x_1 = \frac{-40}{44} \qquad\qquad x_1 = \frac{\det(A_1)}{\det(A)}$$

$$= -\frac{10}{11}$$

$$x_2 = \frac{72}{44} \qquad\qquad x_2 = \frac{\det(A_2)}{\det(A)}$$

$$= \frac{18}{11}$$

$$x_3 = \frac{152}{44} \qquad\qquad x_3 = \frac{\det(A_3)}{\det(A)}$$

$$= \frac{38}{11}$$

Solution: $\left(-\frac{10}{11}, \frac{18}{11}, \frac{38}{11}\right)$

A Formula for A^{-1}

Theorem: ***Inverse of a matrix using its Adjoint***

The i, j entry of A^{-1} is the cofactor C_{ji} $\left(not\ C_{ij}\right)$ divided by det(***A***):

$$\textbf{\textit{Formula for }} A^{-1}: \quad \left(A^{-1}\right)_{ij} = \frac{C_{ji}}{|A|} \quad and \quad A^{-1} = \frac{C^T}{|A|}$$

$$\boxed{A^{-1} = \frac{1}{\det(A)} adj(A)}$$

Example

Find the inverse matrix of $A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$ using its adjoint.

Solution

$$C_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1; \qquad C_{12} = -\begin{vmatrix} -2 & 0 \\ -4 & 1 \end{vmatrix} = 2; \qquad C_{13} = \begin{vmatrix} -2 & 1 \\ -4 & 0 \end{vmatrix} = 4$$

$$C_{21} = -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1; \qquad C_{22} = \begin{vmatrix} 1 & 1 \\ -4 & 1 \end{vmatrix} = 5; \qquad C_{23} = -\begin{vmatrix} 1 & 1 \\ -4 & 0 \end{vmatrix} = -4$$

$$C_{31} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1; \qquad C_{32} = -\begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = -2; \qquad C_{33} = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3$$

$$C = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 5 & -2 \\ 4 & -4 & 3 \end{bmatrix}$$

$$\det(A) = \frac{1}{7}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{7}\begin{bmatrix} 1 & -1 & -1 \\ 2 & 5 & -2 \\ 4 & -4 & 3 \end{bmatrix}$$

Theorem

If A is an $n \times n$ matrix, then the following statements are equivalent

a) A is invertible

b) $A\boldsymbol{x} = 0$ has only the trivial solution

c) The reduced row echelon form of A is I_n

d) A can be expressed as a product of elementary matrices

e) $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for every $n \times 1$ matrix $\boldsymbol{b}$

f) $\det(A) \neq 0$

Exercises Section 1.7 – Cramer's Rule

1. Use Cramer's Rule with ratios $\frac{\det B_j}{\det A}$ to solve $A\boldsymbol{x} = b$. Also find the inverse matrix $A^{-1} = \frac{C^T}{\det A}$.

Why is the solution $\boldsymbol{x}$ is the first part the same as column 3 of A^{-1}? Which cofactors are involved in computing that column $\boldsymbol{x}$?

$$Ax = b \quad is \quad \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

2. Verify that $\det(AB) = \det(BA)$ and determine whether the equality $\det(A+B) = \det(A) + \det(B)$ holds

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

3. Verify that $\det(kA) = k^n \det(A)$

a) $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}, \quad k = 2$

b) $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{bmatrix}, \quad k = -2$

c) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix}, \quad k = 3$

(4 – 58) Use Cramer's rule to solve the system

4. $\begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$

5. $\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$

6. $\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$

7. $\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$

8. $\begin{cases} 3x + 4y = 2 \\ 2x + 5y = -1 \end{cases}$

9. $\begin{cases} 5x - 2y = 4 \\ -10x + 4y = 7 \end{cases}$

10. $\begin{cases} x - 4y = -8 \\ 5x - 20y = -40 \end{cases}$

11. $\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$

12. $\begin{cases} 2x + 10y = -14 \\ 7x - 2y = -16 \end{cases}$

13. $\begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$

14. $\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$

15. $\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$

16. $\begin{cases} x-2y=5 \\ -10x+2y=4 \end{cases}$

17. $\begin{cases} 12x+15y=-27 \\ 30x-15y=-15 \end{cases}$

18. $\begin{cases} 4x-4y=-12 \\ 4x+4y=-20 \end{cases}$

19. $\begin{cases} x+y=7 \\ x-y=3 \end{cases}$

20. $\begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$

21. $\begin{cases} 12x+3y=15 \\ 2x-3y=13 \end{cases}$

22. $\begin{cases} x-2y=5 \\ 5x-y=-2 \end{cases}$

23. $\begin{cases} 3x+2y=2 \\ 2x+2y=3 \end{cases}$

24. $\begin{cases} 4x-5y=17 \\ 2x+3y=3 \end{cases}$

25. $\begin{cases} x-3y=4 \\ 3x-4y=12 \end{cases}$

26. $\begin{cases} 2x-9y=5 \\ 3x-3y=11 \end{cases}$

27. $\begin{cases} 3x-4y=4 \\ x+y=6 \end{cases}$

28. $\begin{cases} 3x=7y+1 \\ 2x=3y-1 \end{cases}$

29. $\begin{cases} 2x=3y+2 \\ 5x=51-4y \end{cases}$

30. $\begin{cases} y=-4x+2 \\ 2x=3y-1 \end{cases}$

31. $\begin{cases} 3x=2-3y \\ 2y=3-2x \end{cases}$

32. $\begin{cases} x+2y-3=0 \\ 12=8y+4x \end{cases}$

33. $\begin{cases} 7x-2y=3 \\ 3x+y=5 \end{cases}$

34. $\begin{cases} 3x+2y-z=4 \\ 3x-2y+z=5 \\ 4x-5y-z=-1 \end{cases}$

35. $\begin{cases} x+y+z=2 \\ 2x+y-z=5 \\ x-y+z=-2 \end{cases}$

36. $\begin{cases} 2x+y+z=9 \\ -x-y+z=1 \\ 3x-y+z=9 \end{cases}$

37. $\begin{cases} 3y-z=-1 \\ x+5y-z=-4 \\ -3x+6y+2z=11 \end{cases}$

38. $\begin{cases} x+3y+4z=14 \\ 2x-3y+2z=10 \\ 3x-y+z=9 \end{cases}$

39. $\begin{cases} x+4y-z=20 \\ 3x+2y+z=8 \\ 2x-3y+2z=-16 \end{cases}$

40. $\begin{cases} -2x+6y+7z=3 \\ -4x+5y+3z=7 \\ -6x+3y+5z=-4 \end{cases}$

41. $\begin{cases} 2x-y+z=1 \\ 3x-3y+4z=5 \\ 4x-2y+3z=4 \end{cases}$

42. $\begin{cases} 3x-4y+4z=7 \\ x-y-2z=2 \\ 2x-3y+6z=5 \end{cases}$

43. $\begin{cases} x-2y-z=2 \\ 2x-y+z=4 \\ -x+y+z=4 \end{cases}$

44. $\begin{cases} x+y+z=3 \\ -y+2z=1 \\ -x+z=0 \end{cases}$

45. $\begin{cases} 3x+y+3z=14 \\ 7x+5y+8z=37 \\ x+3y+2z=9 \end{cases}$

46. $\begin{cases} 4x-2y+z=7 \\ x+y+z=-2 \\ 4x+2y+z=3 \end{cases}$

47. $\begin{cases} 2y-z=7 \\ x+2y+z=17 \\ 2x-3y+2z=-1 \end{cases}$

48. $\begin{cases} 2x-2y+z=-4 \\ 6x+4y-3z=-24 \\ x-2y+2z=1 \end{cases}$

49. $\begin{cases} 9x+3y+z=4 \\ 16x+4y+z=2 \\ 25x+5y+z=2 \end{cases}$

50. $\begin{cases} 2x-y+2z=-8 \\ x+2y-3z=9 \\ 3x-y-4z=3 \end{cases}$

51. $\begin{cases} x \quad -3z=-5 \\ 2x-y+2z=16 \\ 7x-3y-5z=19 \end{cases}$

52. $\begin{cases} x+2y-z=5 \\ 2x-y+3z=0 \\ 2y+z=1 \end{cases}$

53. $\begin{cases} x+y+z=6 \\ 3x+4y-7z=1 \\ 2x-y+3z=5 \end{cases}$

54. $\begin{cases} 3x + 2y + 3z = 3 \\ 4x - 5y + 7z = 1 \\ 2x + 3y - 2z = 6 \end{cases}$

55. $\begin{cases} 4x + 5y \qquad = 2 \\ 11x + y + 2z = 3 \\ x + 5y + 2z = 1 \end{cases}$

56. $\begin{cases} x - 4y + z = 6 \\ 4x - y + 2z = -1 \\ 2x + 2y - 3z = -20 \end{cases}$

57. $\begin{cases} 2x - y + z = -1 \\ 3x + 4y - z = -1 \\ 4x - y + 2z = -1 \end{cases}$

58. $\begin{cases} -x_1 - 4x_2 + 2x_3 + x_4 = -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 = 14 \\ -x_1 + x_2 + 3x_3 + x_4 = 11 \\ -x_1 - 2x_2 + x_3 - 4x_4 = -4 \end{cases}$

59. Show that the matrix A is invertible for all values of θ, then find A^{-1} using $A^{-1} = \frac{1}{\det(A)} adj(A)$

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Section 1.8 – Applications

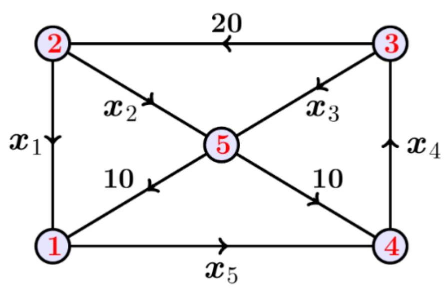
Network Analysis

Networks composed of branches and junctions are used as models in such fields as economics, traffic analysis, and electrical engineering.

In a network model, you assume that the total flow into a junction is equal to the total flow out of the junction

Example

Set up a system of linear equations to represent the network shown below. Then solve the system for $x_i, \quad i = 1,\ 2,\ 3,\ 4,\ 5.$



Solution

$1 \rightarrow \quad x_1 + 10 = x_5 \quad \Rightarrow \quad x_1 - x_5 = -10$

$2 \rightarrow \quad x_1 + x_2 = 20$

$3 \rightarrow \quad x_4 = x_3 + 20 \quad \Rightarrow \quad -x_3 + x_4 = 20$

$4 \rightarrow \quad x_4 = x_5 + 10 \quad \Rightarrow \quad x_4 - x_5 = 10$

$5 \rightarrow \quad x_2 + x_3 = 10 + 10 = 20$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 1 & 1 & 0 & 0 & 20 \end{array}\right) \quad \begin{array}{l} \\ R_2 - R_1 \\ \\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 1 & 1 & 0 & 0 & 20 \end{array}\right) \quad \begin{array}{l} \\ \\ \\ \\ R_5 - R_2 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 1 & 0 & -1 & -10 \end{array}\right) \quad R_5 + R_3$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 1 & -1 & 10 \end{array}\right) \quad R_5 - R_4$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{ll} \to x_1 - x_5 = -10 & \Rightarrow \boxed{x_1 = x_5 - 10} \\ \to x_2 + x_5 = 30 & \Rightarrow \boxed{x_2 = 30 - x_5} \\ \to -x_3 + x_4 = 20 & \Rightarrow \boxed{x_3 = x_5 - 10} \\ \to x_4 - x_5 = 10 & \Rightarrow \boxed{x_4 = x_5 + 10} \end{array}$$

Solution: $\boxed{\left(x_5 - 10,\ \ 30 - x_5,\ \ x_5 - 10,\ \ 10 + x_5,\ \ x_5\right)}$

2nd Method

$$\begin{vmatrix} 1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= -1 + 1$$

$$\boxed{= 0}$$

Infinite solution:

$1 \to \boxed{x_1 = x_5 - 10}$

$2 \to x_2 = 20 - x_1 \boxed{= 30 - x_5}$

$4 \to \boxed{x_4 = x_5 + 10}$

$3 \to x_3 = x_4 - 20 \boxed{= x_5 - 10}$

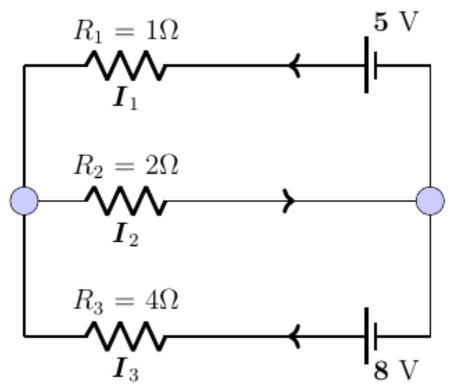
Electrical network

An electrical network is another type of network where analysis is commonly applied. An analysis of such a system uses two properties of electrical networks known as **Kirchhoff's Laws.**

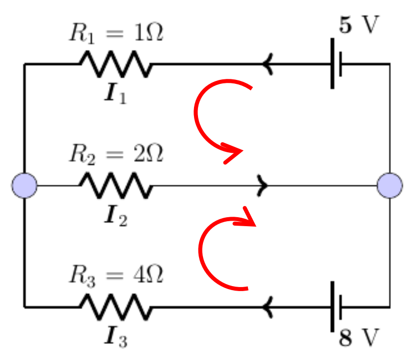
- All the current flowing into a junction must flow out of it.
- The sum of the products *IR* (*I* is current and *R* is resistance) around a closed path is equal to the total voltage in the path.

Example

Determine the currents I_1, I_2, and I_3 for the electrical network



Solution



$$I_2 = I_1 + I_3$$
$$I_1 + 2I_2 = 5$$
$$2I_2 + 4I_3 = 8$$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ I_1 + 2I_2 \qquad = 5 \\ \qquad I_2 + 2I_3 = 4 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 7 \qquad D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 5 & 2 & 0 \\ 4 & 1 & 2 \end{vmatrix} = 7 \qquad D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 5 & 0 \\ 0 & 4 & 2 \end{vmatrix} = 14 \qquad D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 2 & 5 \\ 0 & 1 & 4 \end{vmatrix} = 7$$

$\underline{I_1 = 1 \;\; A}$ $\underline{I_2 = 2 \;\; A}$ $\underline{I_3 = 1 \;\; A}$

Cryptography

A ***cryptogram*** is a message written according to a secret code (the Greek word ***kryptos*** means "*hidden*"). One method of using matrix multiplication to ***encode*** and ***decode*** messages.

Let assign a number to each letter in the alphabet (with 0 assigned to a blank space), as shown

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

Example

Consider the invertible matrix: $A = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix}$

The message: ***MEET ME MONDAY***

a) Write the uncoded row matrices 1×3 for the message.

b) Use the matrix A to encode the message.

c) Decode a message from part *b*) given the matrix A.

Solution

a)

$$\begin{matrix} M\ E\ E & T\ \_\ M & E\ \_\ M & O\ N\ D & A\ Y\ \_ \\ [13\ \ 5\ \ 5] & [20\ \ 0\ \ 13] & [5\ \ 0\ \ 13] & [15\ \ 14\ \ 4] & [1\ \ 25\ \ 0] \end{matrix}$$

b) Let encode the message ***MEET ME MONDAY***

$$\begin{bmatrix} 13 & 5 & 5 \end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 13 & -26 & 21 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 0 & 13 \end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 33 & -53 & -12 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 13 \end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 18 & -23 & -42 \end{bmatrix}$$

$$\begin{bmatrix}15 & 14 & 4\end{bmatrix}\begin{bmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{bmatrix}=\begin{bmatrix}5 & -20 & 56\end{bmatrix}$$

$$\begin{bmatrix}1 & 25 & 0\end{bmatrix}\begin{bmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{bmatrix}=\begin{bmatrix}-24 & 23 & 77\end{bmatrix}$$

The sequence of coded row matrices is

$$\begin{bmatrix}13 & -26 & -21\end{bmatrix}\ \begin{bmatrix}33 & -53 & -12\end{bmatrix}\ \begin{bmatrix}18 & -23 & -42\end{bmatrix}\ \begin{bmatrix}5 & -20 & 56\end{bmatrix}\ \begin{bmatrix}-24 & 23 & 77\end{bmatrix}$$

The cryptogram:

$$13\quad -26\quad -21\quad 33\quad -53\quad -12\quad 18\quad -23\quad -42\quad 5\quad -20\quad 56\quad -24\quad 23\quad 77$$

c) To decode a message given the matrix A.

$$|A|=\begin{vmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{vmatrix}=1$$

$$A^{-1}=\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}$$

With the cryptogram:

$$\begin{bmatrix}13 & -26 & -21\end{bmatrix}\ \begin{bmatrix}33 & -53 & -12\end{bmatrix}\ \begin{bmatrix}18 & -23 & -42\end{bmatrix}\ \begin{bmatrix}5 & -20 & 56\end{bmatrix}\ \begin{bmatrix}-24 & 23 & 77\end{bmatrix}$$

$$\begin{bmatrix}13 & -26 & 21\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}13 & 5 & 5\end{bmatrix}$$

$$\begin{bmatrix}33 & -53 & -1\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}20 & 0 & 13\end{bmatrix}$$

$$\begin{bmatrix}18 & -23 & -42\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}5 & 0 & 13\end{bmatrix}$$

$$\begin{bmatrix}5 & -20 & 56\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}15 & 14 & 4\end{bmatrix}$$

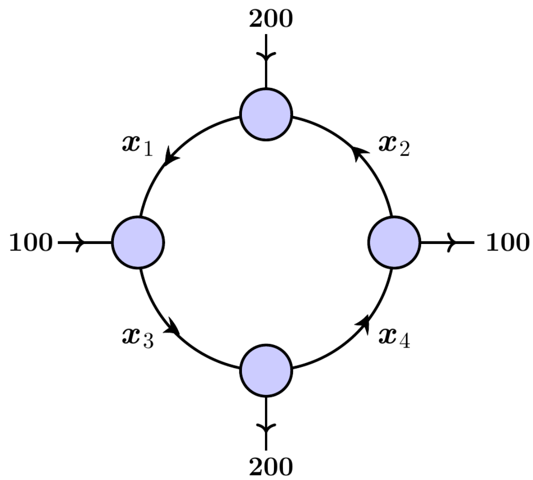
$$\begin{bmatrix} -24 & 23 & 77 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 25 & 0 \end{bmatrix}$$

The message is:

13	5	5	20	0	13	5	0	13	15	14	4	1	25	0
M	*E*	*E*	*T*	_	*M*	*E*	_	*M*	*O*	*N*	*D*	*A*	*Y*	_

Exercises *Section* 1.8 – Applications

1. The flow of traffic, in vehicles per hour, through a network of streets as is shown below



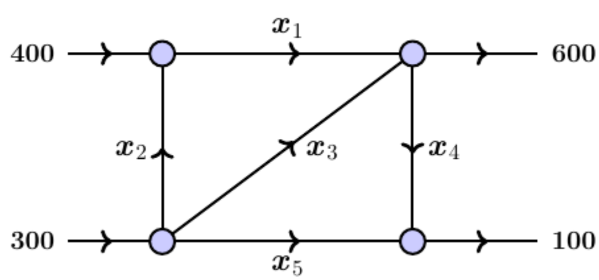
a) Solve this system for $x_i, \quad i = 1,\ 2,\ 3,\ 4.$

b) Find the traffic flow when $x_4 = 0$.

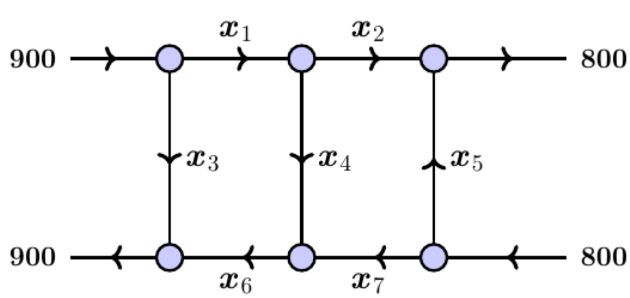
c) Find the traffic flow when $x_4 = 100$.

d) Find the traffic flow when $x_1 = 2x_2$.

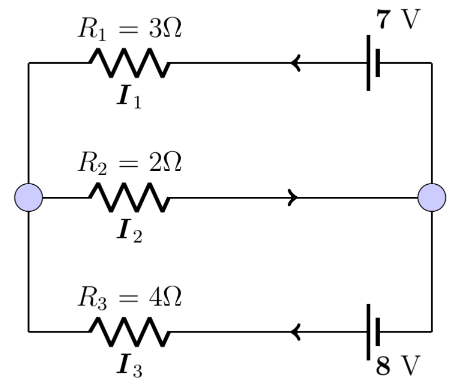
2. Through a network, Express x_n's in terms of the parameters ***s*** and ***t***.



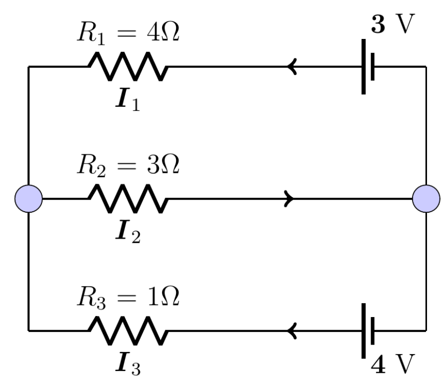
3. Water is flowing through a network of pipes. Express x_n's in terms of the parameters ***s*** and ***t***.



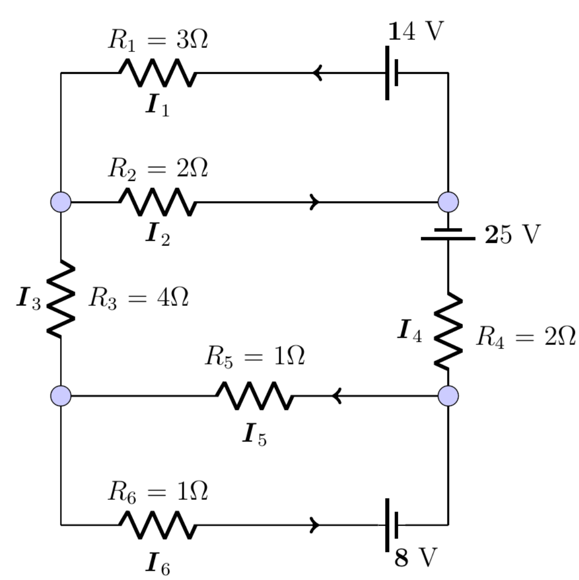
4. Determine the currents $I_1, I_2,$ *and* I_3 for the electrical network shown below



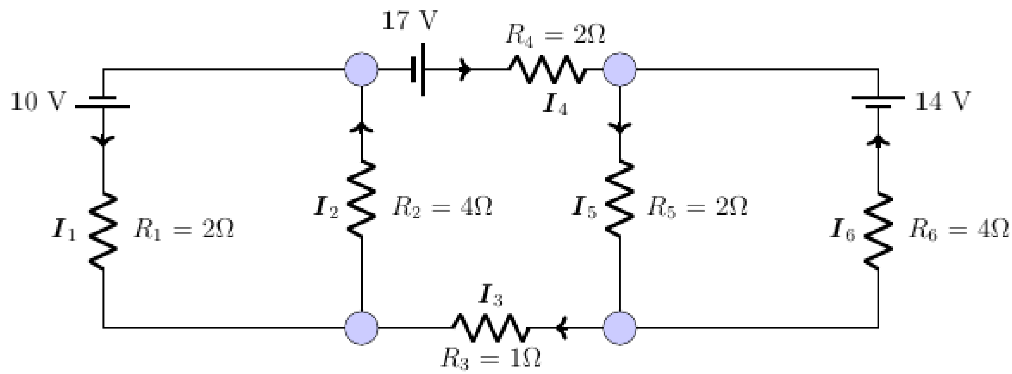
5. Determine the currents $I_1, I_2,$ *and* I_3 for the electrical network shown below



6. Determine the currents $I_1, I_2, I_3, I_4, I_5,$ *and* I_6 for the electrical network shown below



7. Determine the currents $I_1, I_2, I_3, I_4, I_5,$ *and* I_6 for the electrical network shown below



8. Consider the invertible matrix: $A = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -2 & -2 & 7 \end{pmatrix}$

The message: ***ICEBERG DEAD AHEAD***

a) Write the uncoded row matrices 1×3 for the message.

b) Use the matrix A to encode the message.

c) Decode a message from part *b*) given the matrix A.

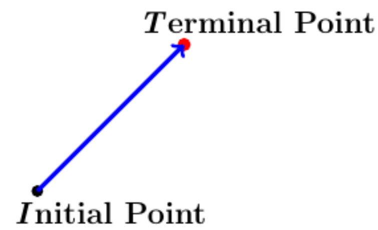
Lecture Two

Section 2.1 – Vectors in 2-Space, 3-Space, and *n*-Space

Vectors in two dimensions are also called **2–*space***

Vectors in three dimensions are also called **3–*space*** by arrow

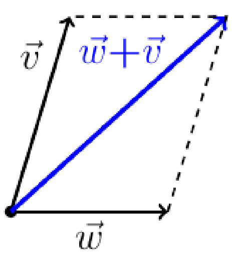
The direction of the arrowhead specifies the ***direction*** of the vector and the ***length*** of the arrow specifies the *magnitude*.



The tail of the arrow is called the ***initial point*** of the vector and the tip the ***terminal point***.

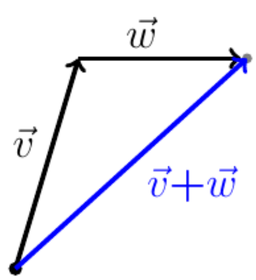
Parallelogram Rule for Vector Addition

If $\vec{v}$ and $\vec{w}$ are vectors in 2-space or 3-space that are positioned so their initial points coincides, then the vectors form adjacent sides of a parallelogram, and then the sum $\vec{v}+\vec{w}$ is the vector represented by the arrow from the common initial point of $\vec{v}$ and $\vec{w}$ to the opposite vertex of the parallelogram.

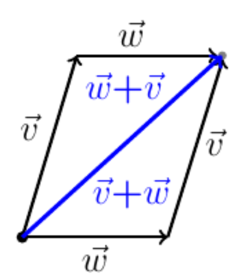


Triangle Rule for Vector Addition

If $\vec{v}$ and $\vec{w}$ are vectors in 2-space or 3-space that are positioned so the initial point of w is at the terminal point of $\vec{v}$, then the sum $\vec{v}+\vec{w}$ is represented by the arrow from the initial point of $\vec{v}$ to the terminal point of $\vec{w}$.



$$\boxed{\vec{v}+\vec{w}=\vec{w}+\vec{v}}$$



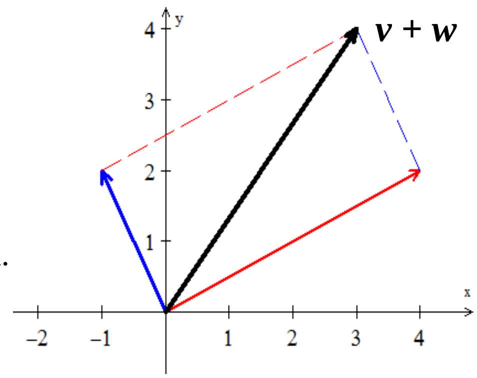
Example of Sum and Difference of vectors

Consider the vector $\vec{v}$ is given by the component $\begin{bmatrix}4\\2\end{bmatrix}$ and represented by an arrow. The arrow goes from 4 units to the right and 2 units up.

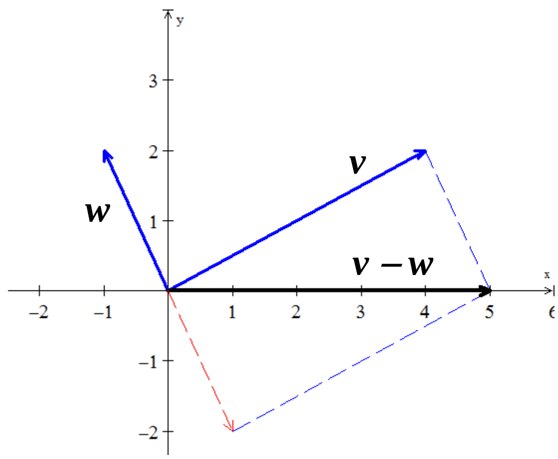
Consider anther vector $\vec{w}=\begin{bmatrix}-1\\2\end{bmatrix}$

Vector addition (head to tail) at the end of $\vec{v}$, place the start of $\vec{w}$.

The vector addition and w produces the diagonal of a parallelogram.



$$\vec{v}+\vec{w}=\begin{bmatrix}4\\2\end{bmatrix}+\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix}$$



$$\vec{v}-\vec{w}=\begin{bmatrix}4\\2\end{bmatrix}-\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}5\\0\end{bmatrix}$$

In 3-dimensional space, the arrow starts at the origin $(0,\ 0,\ 0)$, where the *xyz* axis meet.

$\vec{v}=\begin{bmatrix}1\\2\\2\end{bmatrix}$ is also written as $(1,\ 2,\ 2)$

Notes:

1. The picture of the combinations $c\vec{u}$ fills a line
2. The picture of the combinations $c\vec{u}+d\vec{v}$ fills a plane
3. The picture of the combinations $c\vec{u}+d\vec{v}+e\vec{w}$ fills a 3-dimensional space.

Linear Combination

Definition

The sum of $c\vec{v}$ and $d\vec{w}$ is a linear combination of vectors $\vec{v}$ and $\vec{w}$; c, d are constants.

4-Special Linear Combinations:

$$\begin{aligned} 1\vec{v}+1\vec{w} &= \textit{sum of vectors} \\ 1\vec{v}-1\vec{w} &= \textit{difference of vectors} \\ 0\vec{v}+0\vec{w} &= \textit{zero vectors} \\ c\vec{v}+0\vec{w} &= \textit{vector } c\vec{v} \textit{ in the direction of } \vec{v} \end{aligned}$$

Vectors in Coordinate Systems

It is sometimes necessary to consider vectors whose initial are not at the origin. If $\overrightarrow{P_1P_2}$ denotes the vector with initial point $P_1\left(x_1, y_1\right)$ and terminal point $P_2\left(x_2, y_2\right)$, then the components of this vector are given by the formula

$$\boxed{\overrightarrow{P_1P_2} = \left(x_2-x_1,\ y_2-y_1\right)}$$

If $P_1\left(x_1, y_1, z_1\right)$ *and* $P_2\left(x_2, y_2, z_2\right)$

$$\boxed{\overrightarrow{P_1P_2} = \left(x_2-x_1,\ y_2-y_1,\ z_2-z_1\right)}$$

Example

The components of the vector $\vec{v}=\overrightarrow{P_1P_2}$ with initial point $P_1(2,\ -1,\ 4)$ and terminal point $P_2(7,\ 5,\ -8)$, find $\vec{v}$?

Solution

$$\begin{aligned} \vec{v} &= \left(7-2,\ 5-(-1),\ -8-4\right) \\ &= (5,\ 6,\ -12) \end{aligned}$$

n – *Space*

The vector spaces are denoted by $\mathbb{R}^1,\ \mathbb{R}^2,\ \mathbb{R}^3,\ \mathbb{R}^4,\ \ldots$. Each space $\mathbb{R}^n$ consists of a whole collection of vectors.

Definition

The space $\mathbb{R}^n$ consists of all column vectors *v* with *n* components.

Example

$$\underset{\mathbb{R}^3}{\begin{bmatrix}1\\2\\3\end{bmatrix}} \qquad \underset{\mathbb{R}^5}{(1,\ 2,\ 3,\ 0,\ 1)} \qquad \underset{\mathbb{C}^2}{\begin{bmatrix}1+i\\1-i\end{bmatrix}}$$

The one-dimensional space $\mathbb{R}^1$ is a line (like the *x*-axis)

The two essential vector operations go on inside the vector space that we can add any vectors in $\mathbb{R}^n$, and we can multiply any vector by any scalar. The ***result*** stays in the space.

A real vector space is a set of "***vectors***" together with rules for vector addition and for multiplication by real numbers. The addition and the multiplication must produce vectors that are in the space.

Here are three other spaces other than $\mathbb{R}^n$:

M The vector space of ***all real 2 by 2 matrices***.

F The vector space of ***all real functions*** $f(x)$.

Z The vector space that consists only of a ***zero vector***.

The zero vector in $\mathbb{R}^3$ is the vector (0, 0, 0).

Operation on Vectors in $\mathbb{R}^n$

Definition

If n is a positive integer, then an ordered ***n-tuple*** is a sequence of real numbers $(v_1, v_2, \ldots, v_n)$. The set of all ordered n-tuples is called ***n-space*** and is denoted by $\mathbb{R}^n$

Definition

Vectors $\vec{v} = (v_1, v_2, \ldots, v_n)$ and $\vec{w} = (w_1, w_2, \ldots, w_n)$ in $\mathbb{R}^n$ are said to be ***equivalent*** (also called ***equal***) if

$$v_1 = w_1, \quad v_2 = w_2, \quad \ldots \quad v_n = w_n$$

We indicate this by $\vec{v} = \vec{w}$

Example

$$(a, b, c, d) = (1, -4, 2, 7)$$

Solution

$$Iff\ a = 1, \quad b = -4, \quad c = 2, \quad d = 7$$

Vector Space of Infinite Sequences of Real Numbers

If $\vec{v} = (v_1, v_2, \ldots, v_n)$ and $\vec{w} = (w_1, w_2, \ldots, w_n)$ are vectors in $\mathbb{R}^n$, and if k is any scalar, then we defined

$$\begin{aligned}\vec{v} + \vec{w} &= (v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n) \\ &= (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)\end{aligned}$$

$$k\vec{v} = (kv_1, kv_2, \ldots, kv_n)$$

$$-\vec{v} = (-v_1, -v_2, \ldots, -v_n)$$

$$\vec{w} - \vec{v} = \vec{w} + (-\vec{v}) = (w_1 - v_1, w_2 - v_2, \ldots, w_n - v_n)$$

The *Zero* Vector Space

Let V consist of a single object, which we denote by $\vec{0}$, and define

$$\vec{0} + \vec{0} = \vec{0} \quad and \quad k\vec{0} = \vec{0}$$

Theorem

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^n$, and if k and m are scalars, then

a) $\vec{u}+\vec{v}=\vec{v}+\vec{u}$

b) $\left(\vec{u}+\vec{v}\right)+\vec{w}=\vec{u}+\left(\vec{v}+\vec{w}\right)$

c) $\vec{u}+\vec{0}=\vec{0}+\vec{u}=\vec{u}$

d) $\vec{u}+\left(-\vec{u}\right)=\vec{0}$

e) $k\left(\vec{u}+\vec{v}\right)=k\vec{u}+k\vec{v}$

f) $\left(k+m\right)\vec{u}=k\vec{u}+m\vec{u}$

g) $k\left(m\vec{u}\right)=\left(km\right)\vec{u}$

h) $1\vec{u}=\vec{u}$

i) $0\vec{v}=0$

j) $k\vec{0}=\vec{0}$

k) $\left(-1\right)\vec{v}=-\vec{v}$

Proof: $\left(\vec{u}+\vec{v}\right)+\vec{w}=\vec{u}+\left(\vec{v}+\vec{w}\right)$

Let $\vec{u}=\left(u_1,\ u_2,\ \ldots,\ u_n\right)$

$\vec{v}=\left(v_1,\ v_2,\ \ldots,\ v_n\right)$

$\vec{w}=\left(w_1,\ w_2,\ \ldots,\ w_n\right)$

$$
\begin{aligned}
\left(\vec{u}+\vec{v}\right)+\vec{w} &= \left(\left(u_1,\ u_2,\ \ldots,\ u_n\right)+\left(v_1,\ v_2,\ \ldots,\ v_n\right)\right)+\left(w_1,\ w_2,\ \ldots,\ w_n\right)\\
&= \left(u_1+v_1,\ u_2+v_2,\ \ldots,\ u_n+v_n\right)+\left(w_1,\ w_2,\ \ldots,\ w_n\right)\\
&= \left(\left(u_1+v_1\right)+w_1,\ \left(u_2+v_2\right)+w_2,\ \ldots,\ \left(u_n+v_n\right)+w_n\right)\\
&= \left(u_1+\left(v_1+w_1\right),\ u_2+\left(v_2+w_2\right),\ \ldots,\ u_n+\left(v_n+w_n\right)\right)\\
&= \left(u_1,\ u_2,\ \ldots,\ u_n\right)+\left(\left(v_1,\ v_2,\ \ldots,\ v_n\right)+\left(w_1,\ w_2,\ \ldots,\ w_n\right)\right)\\
&= \vec{u}+\left(\vec{v}+\vec{w}\right)
\end{aligned}
$$

Exercises *Section* 2.1 – Vectors in 2-Space, 3-Space, and *n*-Space

1. Sketch the following vectors with initial points located at the origin

a) $P_1(4,8)$ $P_2(3,7)$ *b)* $P_1(-1,0,2)$ $P_2(0,-1,0)$ *c)* $P_1(3,-7,2)$ $P_2(-2,5,-4)$

2. Find the components of the vector $\overrightarrow{P_1P_2}$

a) $P_1(3,5)$ $P_2(2,8)$ *b)* $P_1(5,-2,1)$ $P_2(2,4,2)$ *c)* $P_1(0,\ 0,\ 0)$ $P_2(-1,\ 6,\ 1)$

3. Find the terminal point of the vector that is equivalent to $\vec{u} = (1, 2)$ and whose initial point is $A(1,1)$

4. Find the initial point of the vector that is equivalent to $\vec{u} = (1, 1, 3)$ and whose terminal point is $B(-1,-1,2)$

5. Find a nonzero vector $\vec{u}$ with initial point $P(-1, 3, -5)$ such that

a) $\vec{u}$ has the same direction as $\vec{v} = (6, 7, -3)$

b) $\vec{u}$ is oppositely directed as $\vec{v} = (6, 7, -3)$

6. Let $\vec{u} = (-3, 1, 2)$, $\vec{v} = (4, 0, -8)$, and $\vec{w} = (6, -1, -4)$. Find the components

a) $\vec{v} - \vec{w}$

b) $6\vec{u} + 2\vec{v}$

c) $5(\vec{v} - 4\vec{u})$

d) $-3(\vec{v} - 8\vec{w})$

e) $(2\vec{u} - 7\vec{w}) - (8\vec{v} + \vec{u})$

f) $-\vec{u} + (\vec{v} - 4\vec{w})$

7. Let $\vec{u} = (2, 1, 0, 1, -1)$ and $\vec{v} = (-2, 3, 1, 0, 2)$. Find scalars *a* and *b* so that $a\vec{u} + b\vec{v} = (-8,\ 8,\ 3,\ -1,\ 7)$

8. Find all scalars $c_1,\ c_2,\ and\ c_3$ such that $c_1(1, 2, 0) + c_2(2, 1, 1) + c_3(0, 3, 1) = (0, 0, 0)$

9. Find the distance between the given points $\begin{bmatrix}5 & 1 & 8 & -1 & 2 & 9\end{bmatrix}$, $\begin{bmatrix}4 & 1 & 4 & 3 & 2 & 8\end{bmatrix}$

10. Let *V* be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operation on $\vec{u} = (u_1,\ u_2)$ $\vec{v} = (v_1,\ v_2)$

$$\vec{u} + \vec{v} = (u_1 + v_1 + 1,\ u_2 + v_2 + 1) \qquad k\vec{u} = (ku_1,\ ku_2)$$

a) Compute $\vec{u} + \vec{v}$ and $k\vec{u}$ for $\vec{u} = (0, 4)$, $\vec{v} = (1, -3)$, and $k = 2$.

b) Show that $(0, 0) \neq \vec{0}$.

c) Show that $(-1, -1) = 0$.

d) Show that $\vec{u} + (-\vec{u}) = 0$ for $\vec{u} = (u_1,\ u_2)$

e) Find two vector space axioms that fail to hold.

11. Find $\vec{w}$ given that $10\vec{u}+3\vec{w}=4\vec{v}-2\vec{w}$, $\vec{u}=\begin{pmatrix}1\\-6\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}-20\\5\end{pmatrix}$

12. Find $\vec{w}$ given that $\vec{u}+3\vec{v}-2\vec{w}=5\vec{u}+\vec{v}-4\vec{w}$, $\vec{u}=\begin{pmatrix}1\\-1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}-2\\3\end{pmatrix}$

13. Find $\vec{w}$ given that $2\vec{u}+\vec{v}-3\vec{w}=5\vec{u}+7\vec{v}+3\vec{w}$, $\vec{u}=\begin{pmatrix}1\\-1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}-2\\3\end{pmatrix}$

(**14 – 17**) Draw $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$, and $\vec{u}+2\vec{v}$

14. $\vec{u}=\begin{pmatrix}1\\-1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}-2\\3\end{pmatrix}$

15. $\vec{u}=\begin{pmatrix}-1\\1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}2\\1\end{pmatrix}$

16. $\vec{u}=\begin{pmatrix}1\\0\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}1\\2\end{pmatrix}$

17. $\vec{u}=\begin{pmatrix}-1\\0\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}1\\-1\end{pmatrix}$

Section 2.2 – Norm, Dot product, and distance in $\mathbb{R}^n$

Norm of a Vector

The ***length*** (or ***norm***) of a vector $\vec{v}$ is the square root of $\vec{v} \bullet \vec{v}$

$$Length = \|\vec{v}\| = \sqrt{\vec{v} \bullet \vec{v}}$$

$$= \sqrt{x^2 + y^2} \qquad \textbf{2-dimension}$$

$$= \sqrt{x^2 + y^2 + z^2} \qquad \textbf{3-dimension}$$

Definition

If $\vec{v} = \left(v_1,\ v_2,\ \ldots,\ v_n\right)$ is a vector in $\mathbb{R}^n$, then the ***norm*** of $\vec{v}$ (also called the ***length*** of $\vec{v}$ or the ***magnitude*** of $\vec{v}$) is denoted by $\|\vec{v}\|$, and is defined by the formula

$$\boxed{\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \ldots + v_n^2}}$$

Example

Find the length of the vector $\vec{v} = (1,\ 2,\ 3)$

Solution

$$\|\vec{v}\| = \sqrt{1^2 + 2^2 + 3^2}$$

$$= \sqrt{14}$$

Theorem

If $\vec{v}$ is a vector in $\mathbb{R}^n$, and if $\boldsymbol{k}$ is any scalar, then:

a) $\|\vec{v}\| \ge 0$

b) $\|\vec{v}\| = 0 \quad iff \quad \vec{v} = \vec{0}$

c) $\|k\vec{v}\| = |k| \cdot \|\vec{v}\|$

Unit Vectors

Definition

A ***unit vector*** $\vec{u}$ is a vector whose length equals to one. Then $\vec{u}\cdot\vec{u}=1$

Divide any nonzero vector $\vec{v}$ by its length. Then $\vec{u}=\frac{\vec{v}}{\|\vec{v}\|}$ is a unit vector in the same direction as v.

Example

Find the unit vector $\vec{u}$ that has the same direction as $\vec{v}$ = (2, 2, –1)

Solution

$$\|\vec{v}\|=\sqrt{2^2+2^2+(-1)^2}$$

$$=3$$

$$\vec{u}=\frac{\vec{v}}{\|\vec{v}\|}$$

$$=\frac{1}{3}(2,\ 2,\ -1)$$

$$=\left(\frac{2}{3},\ \frac{2}{3},\ -\frac{1}{3}\right)$$

$$\|\vec{u}\|=\sqrt{\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^2+\left(\frac{-1}{3}\right)^2}$$

$$=\sqrt{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}}$$

$$=\sqrt{\frac{9}{9}}$$

$$=1 \quad \checkmark$$

Example of unit vectors

$$\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \hat{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

In $\mathbb{R}^3$

$$\hat{i} = (1,\ 0,\ 0) \quad \hat{j} = (0,\ 1,\ 0) \quad \textit{and} \quad \hat{k} = (0,\ 0,\ 1)$$

In general, these formulas can be defined as ***standard unit vector*** in $\mathbb{R}^n$

$$\hat{e}_1 = (1,\ 0,\ \cdots,\ 0), \quad \hat{e}_2 = (0,\ 1,\ \cdots,\ 0), \quad \ldots, \quad \hat{e}_n = (0,\ 0,\ \cdots,\ 1)$$

$$\begin{aligned} \vec{v} &= \left(v_1,\ v_2,\ \ldots,\ v_n\right) \\ &= v_1\hat{e}_1 + v_2\hat{e}_2 + \ldots + v_n\hat{e}_n \end{aligned}$$

Example

$$(7,\ 3,\ -4,\ 5) = 7\hat{e}_1 + 3\hat{e}_2 - 4\hat{e}_3 + 5\hat{e}_4$$

Distance in $\mathbb{R}^n$

In $\mathbb{R}^2$ $\quad d = \left\| \overrightarrow{P_1P_2} \right\| = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$

In $\mathbb{R}^3$ $\quad d\left(\vec{u},\ \vec{v}\right) = \left\| \overrightarrow{P_1P_2} \right\| = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}$

Definition

If $\vec{u} = \left(u_1,\ u_2,\ \dots,\ u_n\right)$ and $\vec{v} = \left(v_1,\ v_2,\ \dots,\ v_n\right)$ are points in $\mathbb{R}^n$, then we denote the distance between u and v by $d\left(\vec{u},\ \vec{v}\right)$ and define it to be

$$d\left(\vec{u},\ \vec{v}\right) = \left\|\vec{u} - \vec{v}\right\|$$

$$= \sqrt{\left(u_1 - v_1\right)^2 + \left(u_2 - v_2\right)^2 + \ldots + \left(u_n - v_n\right)^2}$$

Dot Product

If $\vec{u}$ and $\vec{v}$ are nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, and if θ is he angle between $\vec{u}$ and $\vec{v}$, then the ***dot product*** (also called the ***Euclidean inner product***) of $\vec{u}$ and $\vec{v}$ is denoted by $\vec{u} \bullet \vec{v}$ and is defined as

$$\vec{u} \bullet \vec{v} = \left\|\vec{u}\right\|\ \left\|\vec{v}\right\|\ \cos\theta$$

Cosine Formula

If $\vec{u}$ and $\vec{v}$ are nonzero vectors that implies

$$\Rightarrow\ \cos\theta = \frac{\vec{u} \bullet \vec{v}}{\left\|\vec{u}\right\| . \left\|\vec{v}\right\|}$$

Example

Find the dot product of the vectors $\vec{u} = \left(0,\ 0,\ 1\right)$ and $\vec{v} = \left(0,\ 2,\ 2\right)$ and have an angle of 45°.

<u>Solution</u>

$$\left\|\vec{u}\right\| = 1$$

$$\left\|\vec{v}\right\| = \sqrt{0 + 2^2 + 2^2}$$

$$= \sqrt{8}$$

$$= 2\sqrt{2}$$

$$\vec{u} \bullet \vec{v} = \|\vec{u}\| \ \|\vec{v}\| \cos\theta$$
$$= (1)\left(2\sqrt{2}\right)\cos 45^\circ$$
$$= \left(2\sqrt{2}\right)\frac{1}{\sqrt{2}}$$
$$\underline{= 2}$$

Component Form of the Dot Product

The ***dot product*** or ***inner product*** of $\vec{v} = \left(v_1,\ v_2\right)$ and $\vec{w} = \left(w_1,\ w_2\right)$ is the number

$$\vec{v} \bullet \vec{w} = v_1 w_1 + v_2 w_2$$

Example

Find the dot product of $\vec{v} = (4,\ 2)$ and $\vec{w} = (-1,\ 2)$

Solution

$$\vec{v} \bullet \vec{w} = 4.(-1) + 2(2)$$
$$\underline{= 0}$$

- ***For dot products, zero means that the 2 vectors are perpendicular*** (= 90°).

Example

Put a weight of 4 at the point $x = -1$ and weight of 2 at the point $x = 2$. The *x*-axis will balance on the center point $x = 0$.

Solution

The weight balance is $4(-1) + 2(2) = 0$ (*dot product*).

In 3-dimensionals the dot product:

$$\left(v_1,\ v_2,\ v_3\right) \bullet \left(w_1,\ w_2,\ w_3\right) = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Theorem

a) $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$

b) $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$

c) $\vec{u} \bullet (\vec{v} - \vec{w}) = \vec{u} \bullet \vec{v} - \vec{u} \bullet \vec{w}$

d) $(\vec{u} + \vec{v}) \bullet \vec{w} = \vec{u} \bullet \vec{w} + \vec{v} \bullet \vec{w}$

e) $(\vec{u} - \vec{v}) \bullet \vec{w} = \vec{u} \bullet \vec{w} - \vec{v} \bullet \vec{w}$

f) $k(\vec{u} \bullet \vec{v}) = (k\vec{u}) \bullet \vec{v}$

g) $k(\vec{u} \bullet \vec{v}) = \vec{u} \bullet (k\vec{v})$

h) $\vec{v} \bullet \vec{v} \ge 0$ *and* $\vec{v} \bullet \vec{v} = 0$ *iff* $\vec{v} = 0$

i) $\vec{0} \bullet \vec{v} = \vec{v} \bullet \vec{0} = 0$

Right Angles

The dot product is $\vec{v} \bullet \vec{w} = 0$ when $\vec{v}$ is perpendicular to $\vec{w}$

Proof

Perpendicular vectors: $\|\vec{v}\|^2 + \|\vec{w}\|^2 = \|\vec{v} - \vec{w}\|^2$

Let $\vec{v} = (v_1, v_2)$ & $\vec{w} = (w_1, w_2)$

$$\begin{aligned}\|\vec{v} - \vec{w}\|^2 &= (v_1 - w_1)^2 + (v_2 - w_2)^2 \\ &= v_1^2 - 2v_1w_1 + w_1^2 + v_2^2 - 2v_2w_2 + w_2^2 \\ &= v_1^2 + w_1^2 + v_2^2 + w_2^2 - 2(v_1w_1 + v_2w_2) \qquad v_1w_1 + v_2w_2 = 0 \quad \textit{dot product} \\ &= v_1^2 + w_1^2 + v_2^2 + w_2^2 \\ &= v_1^2 + v_2^2 + w_1^2 + w_2^2 \\ &= \|\vec{v}\|^2 + \|\vec{w}\|^2\end{aligned}$$

If $\vec{u}$ and U are unit vectors, then $\vec{u} \bullet \vec{U} = \cos\theta$

Certainly,

$$\left|\vec{u} \bullet \vec{U}\right| \le 1$$

$$-1 \le \cos\theta \le 1$$

$$-1 \le \textit{dot product} \le 1$$

Schwarz Inequality

If $\vec{v}$ and $\vec{w}$ are any vectors $\Rightarrow$ $\|\vec{v} \bullet \vec{w}\| \le \|\vec{v}\|.\|\vec{w}\|$

Proof

The dot product of $\vec{v} = (a,\ b)$ and $\vec{w} = (b,\ a)$ is $2ab$ and both lengths are $\sqrt{a^2 + b^2}$.

Then, the Schwarz inequality says that: $2ab \le a^2 + b^2$

$$a^2 + b^2 - 2ab = (a - b)^2 \ge 0$$

$$a^2 + b^2 - 2ab \ge 0$$

$$a^2 + b^2 \ge 2ab$$

This proves the Schwarz inequality:

$$2ab \le a^2 + b^2$$

$$\Rightarrow \|\vec{v} \bullet \vec{w}\| \le \|\vec{v}\|.\|\vec{w}\|$$

Theorem – Parallelogram Equation for Vectors

If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then

$$\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right)$$

Proof

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 &= (\vec{u}+\vec{v}) \bullet (\vec{u}+\vec{v}) + (\vec{u}-\vec{v}) \bullet (\vec{u}-\vec{v}) \\
&= \vec{u} \bullet \vec{u} + \vec{u} \bullet \vec{v} + \vec{v} \bullet \vec{u} + \vec{v} \bullet \vec{v} + \vec{u} \bullet \vec{u} - \vec{u} \bullet \vec{v} - \vec{v} \bullet \vec{u} + \vec{v} \bullet \vec{v} \\
&= 2(\vec{u} \bullet \vec{u}) + 2(\vec{v} \bullet \vec{v}) \\
&= 2\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right)
\end{aligned}$$

Theorem

If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$ with the Euclidean Inner product, then

$$\vec{u} \bullet \vec{v} = \frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2$$

Exercises Section 2.2 – Norm, Dot product, and distance in R^n

1. If $\|\vec{v}\| = 5$ and $\|\vec{w}\| = 3$, what are the smallest and largest possible values of $\|\vec{v} - \vec{w}\|$ and $\vec{v}.\vec{w}$?

2. If $\|\vec{v}\| = 7$ and $\|\vec{w}\| = 3$, what are the smallest and largest possible values of $\|\vec{v} + \vec{w}\|$ and $\vec{v} \bullet \vec{w}$?

3. Given that $\cos(\alpha) = \frac{\vec{v}_1}{\|\vec{v}\|}$ and $\sin(\alpha) = \frac{\vec{v}_2}{\|\vec{v}\|}$. Similarly, $\cos(\beta) =$ ______ and $\sin(\beta) =$ ______ . The angle θ is $\beta - \alpha$. Substitute into the trigonometry formula $\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$ for $\cos(\beta - \alpha)$ to find $\cos\theta = \frac{\vec{v} \bullet \vec{w}}{\|\vec{v}\|.\|\vec{w}\|}$

4. Can three vectors in the *xy* plane have $\vec{u} \bullet \vec{v} < 0$, $\vec{v} \bullet \vec{w} < 0$ and $\vec{u} \bullet \vec{w} < 0$?

5. Find the norm of $\vec{v}$, a unit vector that has the same direction as $\vec{v}$, and a unit vector that is oppositely directed.

a) $\vec{v} = (4, -3)$

b) $\vec{v} = (1, -1, 2)$

c) $\vec{v} = (-2, 3, 3, -1)$

6. Evaluate the given expression with $\vec{u} = (2, -2, 3)$, $\vec{v} = (1, -3, 4)$, and $\vec{w} = (3, 6, -4)$

a) $\|\vec{u} + \vec{v}\|$

b) $\|-2\vec{u} + 2\vec{v}\|$

c) $\|3\vec{u} - 5\vec{v} + \vec{w}\|$

d) $\|3\vec{v}\| - 3\|\vec{v}\|$

e) $\|\vec{u}\| + \|-2\vec{v}\| + \|-3\vec{w}\|$

7. Let $\boldsymbol{v} = (1, 1, 2, -3, 1)$. Find all scalars k such that $\|k\vec{v}\| = 5$

8. Find $\vec{u} \bullet \vec{v}$, $\vec{u} \bullet \vec{u}$, *and* $\vec{v} \bullet \vec{v}$

a) $\vec{u} = (3, 1, 4)$, $\vec{v} = (2, 2, -4)$

b) $\vec{u} = (1, 1, 4, 6)$, $\vec{v} = (2, -2, 3, -2)$

c) $\vec{u} = (2, -1, 1, 0, -2)$, $\vec{v} = (1, 2, 2, 2, 1)$

9. Find the Euclidean distance between $\vec{u}$ and $\vec{v}$, then find the angle between them

a) $\vec{u} = (3, 3, 3)$, $\vec{v} = (1, 0, 4)$

b) $\vec{u} = (1, 2, -3, 0)$, $\vec{v} = (5, 1, 2, -2)$

c) $\vec{u} = (0, 1, 1, 1, 2)$, $\vec{v} = (2, 1, 0, -1, 3)$

10. Find a unit vector that has the same direction as the given vector

a) $(-4, -3)$

b) $\left(-3,\ 2,\ \sqrt{3}\right)$

c) $(1, 2, 3, 4, 5)$

11. Find a unit vector that is oppositely to the given vector

a) $(-12, -5)$ *b)* $(3, -3, 3)$ *c)* $\left(-3,\ 1,\ \sqrt{6},\ 3\right)$

12. Verify that the Cauchy-Schwarz inequality holds

a) $\vec{u} = (-3, 1, 0)$, $\vec{v} = (2, -1, 3)$
b) $\vec{u} = (0, 2, 2, 1)$, $\vec{v} = (1, 1, 1, 1)$
c) $\vec{u} = (1, 3, 5, 2, 0, 1)$, $\vec{v} = (0, 2, 4, 1, 3, 5)$

13. Find $\vec{u} \bullet \vec{v}$ and then the angle θ between $\vec{u}$ and $\vec{v}$ $\vec{u} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$

14. Find the norm: $\|\vec{u}\| + \|\vec{v}\|$, $\|\vec{u} + \vec{v}\|$ for $\vec{u} = (3,\ -1,\ -2,\ 1,\ 4)$ $\vec{v} = (1,\ 1,\ 1,\ 1,\ 1)$

15. Find all numbers r such that: $\left\| r(1,\ 0,\ -3,\ -1,\ 4,\ 1) \right\| = 1$

16. Find the distance between $P_1(7, -5, 1)$ and $P_2(-7, -2, -1)$

17. Given $\vec{u} = (1, -5, 4)$, $\vec{v} = (3, 3, 3)$

a) Find $\vec{u} \bullet \vec{v}$
b) Find the cosine of the angle θ between $\vec{u}$ and $\vec{v}$.

18. Let $\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ *and* $\vec{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. Find $\left\| \dfrac{1}{\|2\vec{u} + \vec{v}\|}(2\vec{u} + \vec{v}) \right\|$

19. Let $\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ *and* $\vec{v} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$. Find $\left\| \dfrac{1}{\|\vec{u} - \vec{v}\|}(\vec{u} - \vec{v}) \right\|$

20. Let $\vec{u} = \begin{pmatrix} 18 \\ 6 \end{pmatrix}$ *and* $\vec{v} = \begin{pmatrix} -11 \\ 12 \end{pmatrix}$. Find $\left\| \dfrac{1}{\|5\vec{u} + 3\vec{v}\|}(5\vec{u} + 3\vec{v}) \right\|$

21. Let $\vec{u} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$. Calculate the following:

a) $\vec{u} + \vec{v}$ *b)* $2\vec{u} + 3\vec{v}$ *c)* $\vec{v} + (2\vec{u} - 3\vec{v})$ *d)* $\|\vec{u}\|$ *e)* $\|\vec{v}\|$ *f)* *unit vector of* $\vec{v}$

22. Let $\vec{u}=\begin{pmatrix}2\\-1\\0\\1\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}1\\4\\3\\1\end{pmatrix}$. Calculate the following:

a) $\vec{u}-\vec{v}$ *b)* $3\vec{u}-2\vec{v}$ *c)* $2(\vec{u}-\vec{v})+3\vec{u}$ *d)* $\|\vec{u}\|$ *e) unit vector of* $\vec{v}$

23. Let $\vec{u}=\begin{pmatrix}2\\1\\3\\0\\-1\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}1\\0\\2\\1\\-1\end{pmatrix}$. Calculate the following:

a) $\vec{v}-\vec{u}$ *b)* $\vec{u}+3\vec{v}$ *c)* $3(\vec{u}+\vec{v})-3\vec{u}$ *d)* $\|\vec{v}\|$ *e) unit vector of* $\vec{v}$

24. Let $\vec{u}=\begin{pmatrix}2\\-1\\1\end{pmatrix}$, $\vec{v}=\begin{pmatrix}1\\2\\-2\end{pmatrix}$ and $\vec{w}=\begin{pmatrix}3\\2\\1\end{pmatrix}$. Calculate the following:

a) $\vec{u}\bullet\vec{v}$ *b)* $\vec{u}\bullet(\vec{v}+\vec{w})$ *c)* $(\vec{u}+2\vec{v})\bullet\vec{w}$ *d)* $\|(\vec{w}\bullet\vec{v})\vec{u}\|$

25. Let $\vec{u}=\begin{pmatrix}1\\3\\2\\1\end{pmatrix}$, $\vec{v}=\begin{pmatrix}-2\\5\\2\\-6\end{pmatrix}$ and $\vec{w}=\begin{pmatrix}4\\-1\\0\\-2\end{pmatrix}$. Calculate the following:

a) $\vec{u}\bullet\vec{v}$ *b)* $\vec{u}\bullet(\vec{v}+\vec{w})$ *c)* $(\vec{u}+\vec{v})\bullet(\vec{u}-\vec{v})$ *d)* $\|(\vec{w}\bullet\vec{v})\vec{u}\|$

26. Let $\vec{u}=\begin{pmatrix}1\\0\\-2\\1\end{pmatrix}$, $\vec{v}=\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$ and $\vec{w}=\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}$. Calculate the following:

a) $\vec{u}\bullet(\vec{v}+\vec{w})$ *b)* $\|(\vec{w}\bullet\vec{v})\vec{u}\|$ *c)* $(\vec{u}\bullet\vec{w})\vec{v}+(\vec{v}\bullet\vec{w})\vec{u}$ *d)* $(\vec{u}+2\vec{v})\bullet(\vec{u}-\vec{v})$

27. Suppose $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^n$ such that $\vec{u}\bullet\vec{v}=2$, $\vec{u}\bullet\vec{w}=-3$, and $\vec{v}\bullet\vec{w}=5$. If possible, calculate the following values:

a) $\vec{u}\bullet(\vec{v}+\vec{w})$ *d)* $\vec{w}\bullet(2\vec{v}-4\vec{u})$ *g)* $\vec{w}\bullet((\vec{u}\bullet\vec{w})\vec{u})$

b) $(\vec{u}+\vec{v})\bullet\vec{w}$ *e)* $(\vec{u}+\vec{v})\bullet(\vec{v}+\vec{w})$ *h)* $\vec{u}\bullet((\vec{u}\bullet\vec{v})\vec{v}+(\vec{u}\bullet\vec{w})\vec{w})$

c) $\vec{u}\bullet(2\vec{v}-\vec{w})$ *f)* $\vec{w}\bullet(5\vec{v}+\pi\vec{u})$

28. You are in an airplane flying from Chicago to Boston for a job interview. The compass in the cockpit of the plane shows that your plane is pointed due East, and the airspeed indicator on the plane shows that the plane is traveling through the air at 400 *mph*. there is a crosswind that affects your plane however, and the crosswind is blowing due South at 40 *mph*.
Given the crosswind you wonder; relative to the ground, in what direction are you really flying and how fast are you really traveling?

29. A jet airliner, flying due east at 500 *mph* in still air, encounters a 70-*mph* tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What speed and direction should the jetliner have in order for the resultant vector to be 500 *mph* due east?

30. A jet airliner, flying due east at 500 *mph* in still air, encounters a 70-*mph* tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What are they?

31. A bird flies from its nest 5 *km* in the direction 60° north east, where it stops to rest on a tree. It then flies 10 *km* in the direction due southeast and lands atop a telephone pole. Place an *xy*-coordinate system so that the origin is the bird's nest, the *x*-axis points east, and the *y*-axis points north.

a) At what point is the tree located?

b) At what point is the telephone pole?

32. Prove $\vec{u} \bullet \vec{u} = \|\vec{u}\|^2 \ \geq 0$

33. Prove, for any vectors and $\vec{v}$ in $\mathbb{R}^2$ and any scalars c and d,

$(c\vec{u} + d\vec{v}) \bullet (c\vec{u} + d\vec{v}) = c^2\|\vec{u}\|^2 + 2cd\vec{u} \bullet \vec{v} + d^2\|\vec{v}\|^2$

34. Prove $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$

35. Prove Minkowski theorem: $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

Section 2.3 – Orthogonality

Definition

Two nonzero vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are said to be ***orthogonal*** (or ***perpendicular***) if their dot product is zero $\vec{u} \bullet \vec{v} = 0$.

We will also agree that he zero vector in $\mathbb{R}^n$ is orthogonal to every vector in $\mathbb{R}^n$. A nonempty set of vectors $\mathbb{R}^n$ is called an ***orthogonal set*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set of unit vectors is called an ***orthonormal set***.

Example

The floor of your room (extended to infinity) is a subspace ***V***. The line where two walls meet is a subspace ***W*** (one-dimensional). Those subspaces are orthogonal. Every vector up the meeting line is perpendicular to every vector on the floor. The origin (0, 0, 0) is in the corner.

Example

Show that $\vec{u}$ = (–2, 3, 1, 4) and $\vec{v}$ = (1, 2, 0, –1) are orthogonal in $\mathbb{R}^4$

Solution

$$\begin{aligned}\vec{u} \bullet \vec{v} &= (-2)(1) + (3)(2) + (1)(0) + (4)(-1) \\ &= -2 + 6 + 0 - 4 \\ &= 0\end{aligned}$$

These vectors are orthogonal in $\mathbb{R}^4$

Standard Unit Vectors

$$\hat{i} \bullet \hat{j} = \hat{i} \bullet \hat{k} = \hat{j} \bullet \hat{k} = \mathbf{0}$$

Proof

$$\begin{aligned}\hat{i} \bullet \hat{j} &= (1,\ 0,\ 0) \bullet (0,\ 1,\ 0) \\ &= 0\end{aligned}$$

Normal

To specify slope and inclination is to use a nonzero vector $\vec{n}$, called a ***normal***, that is orthogonal to the line or plane.

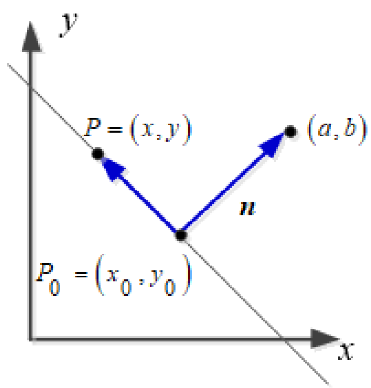
The line passes through a point $P_0\left(x_0, y_0\right)$ that has a normal $\vec{n} = (a, b)$

The plane through $P_0\left(x_0, y_0, z_0\right)$ that has a normal $\vec{n} = (a, b, c)$.

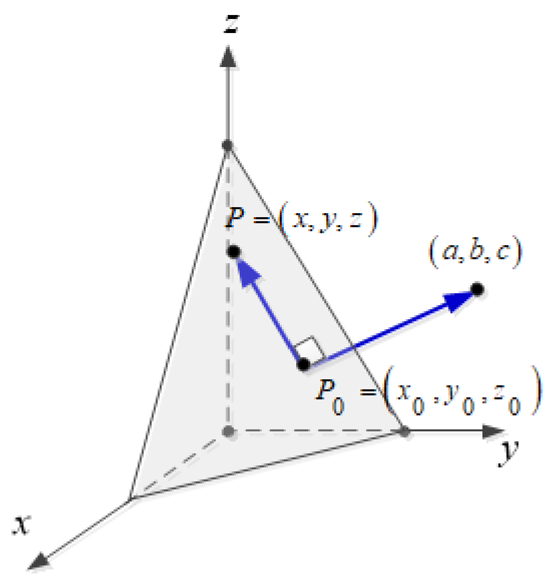
Both the line and the plane are represented by the vector equation

$$\vec{n} \bullet \overrightarrow{P_0 P} = 0$$

The line equation: $\quad a\left(x - x_0\right) + b\left(y - y_0\right) = 0$



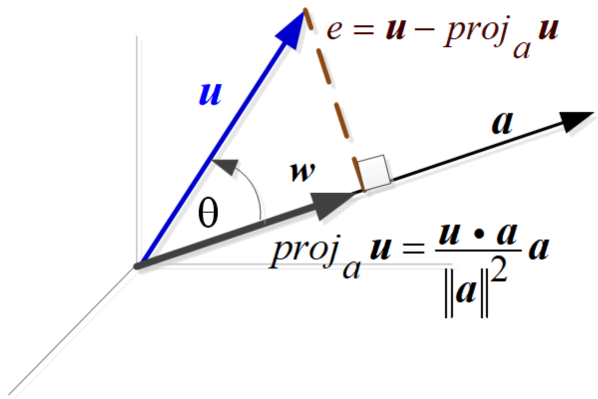
The plane equation: $\quad a\left(x - x_0\right) + b\left(y - y_0\right) + c\left(z - z_0\right) = 0$



Projections

Theorem **Projection onto a line**

If $\vec{u}$ and $\vec{a}$ are vectors in $\mathbb{R}^n$, and if $\vec{a} \neq 0$, then $\vec{u}$ can be expressed in exactly one way in the form $\vec{u} = \vec{w} + \vec{e}$, where $\vec{w}$ is a scalar multiple of $\vec{a}$ and $\vec{e}$ is orthogonal to $\vec{a}$.



The vector $\vec{w}$ is called the ***orthogonal projection*** of $\vec{u}$ on $\vec{a}$ or sometimes ***component*** of $\vec{u}$ along $\vec{a}$. The vector $\vec{e}$ is called the vector ***component*** of $\vec{u}$ ***orthogonal*** to $\vec{a}$ (error vector and should be perpendicular to $\vec{a}$)

$$proj_{\vec{a}}\vec{u} = \frac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a} = \vec{p} \qquad \text{(vector component of } \vec{u} \text{ along } \vec{a}\text{)}$$

$$\vec{u} - proj_{\vec{a}}\vec{u} = \vec{u} - \frac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a} \qquad \text{(vector component of } \vec{u} \text{ orthogonal to } \vec{a}\text{)}$$

The length is $\left\| proj_{\vec{a}}\vec{u} \right\| = \|\vec{u}\| |\cos\theta|$

$$\left\| proj_{\vec{a}}\vec{u} \right\| = \frac{|\vec{u} \bullet \vec{a}|}{\|\vec{a}\|}$$

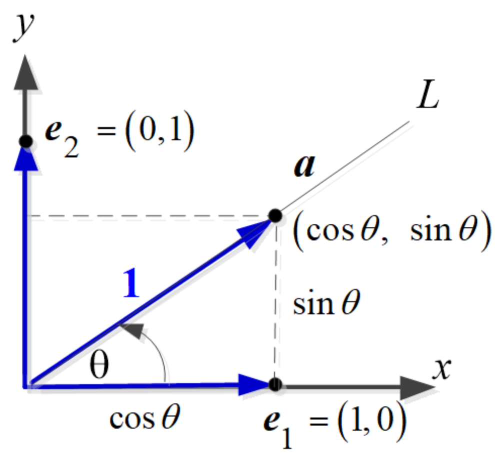
Special case: If $\vec{u} = \vec{a}$ then $\frac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|^2} = 1$. The projection of $\vec{a}$ onto $\vec{a}$ is itself.

Special case: If $\vec{u}$ is perpendicular to $\vec{a}$ then $\vec{u} \bullet \vec{a} = 0$. The projection is $\vec{p} = \vec{0}$.

Example

Find the orthogonal projections of the vectors $\hat{e}_1 = (1,\ 0)$ and $\hat{e}_2 = (0,\ 1)$ on the line *L* that makes an angle θ with the positive *x*-axis in $\mathbb{R}^2$

Solution



Let $\vec{a} = (\cos\theta,\ \sin\theta)$ be the unit vector along the line *L*.

$$\|\vec{a}\| = \sqrt{\cos^2\theta + \sin^2\theta}$$
$$= 1$$

$$\hat{e}_1 \bullet \vec{a} = (1,0)(\cos\theta, \sin\theta)$$
$$= (1)\cos\theta + (0)\sin\theta$$
$$= \cos\theta$$

$$proj_{\vec{a}}\hat{e}_1 = \frac{\hat{e}_1 \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a}$$
$$= \frac{\cos\theta}{1}(\cos\theta,\ \sin\theta)$$
$$= \left(\cos^2\theta,\ \cos\theta\sin\theta\right)$$

$$proj_{\vec{a}}\hat{e}_2 = \frac{\hat{e}_2 \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a}$$
$$= \frac{(0,\ 1)(\cos\theta,\ \sin\theta)}{1}(\cos\theta,\ \sin\theta)$$
$$= \sin\theta(\cos\theta,\ \sin\theta)$$
$$= \left(\sin\theta\cos\theta,\ \sin^2\theta\right)$$

Example

Let $\vec{u}$ = (2, –1, 3) and $\vec{a}$ = (4, –1, 2). Find the vector component of $\vec{u}$ along $\vec{a}$ and the vector component of $\vec{u}$ orthogonal to $\vec{a}$.

Solution

$$\begin{aligned}
proj_{\vec{a}}\vec{u} &= \frac{\vec{u}\bullet\vec{a}}{\|\vec{a}\|^2}\vec{a} \\
&= \frac{(2,\ -1,\ 3)\bullet(4,\ -1,\ 2)}{\left(\sqrt{4^2+(-1)^2+2^2}\right)^2}(4,\ -1,\ 2) \\
&= \frac{8+1+6}{21}(4,\ -1,\ 2) \\
&= \frac{15}{21}(4,\ -1,\ 2) \\
&= \frac{5}{7}(4,\ -1,\ 2) \\
&= \left(\frac{20}{7},\ -\frac{5}{7},\ \frac{10}{7}\right)
\end{aligned}$$

The vector component of $\vec{u}$ orthogonal to $\vec{a}$ is

$$\begin{aligned}
\vec{u} - proj_{\vec{a}}\vec{u} &= (2,\ -1,\ 3) - \left(\frac{20}{7},\ -\frac{5}{7},\ \frac{10}{7}\right) \\
&= \left(-\frac{6}{7},\ -\frac{2}{7},\ \frac{11}{7}\right)
\end{aligned}$$

Theorem of *Pythagoras* in $\mathbb{R}^n$

If $\vec{u}$ and $\vec{v}$ are orthogonal vectors in $\mathbb{R}^n$ with the Euclidean inner product, then

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$

Proof

Since $\vec{u}$ and $\vec{v}$ are orthogonal, then $\vec{u}\bullet\vec{v}=0$

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 &= (\vec{u}+\vec{v})\bullet(\vec{u}+\vec{v}) \\
&= \|\vec{u}\|^2 + 2(\vec{u}\bullet\vec{v}) + \|\vec{v}\|^2 \\
&= \|\vec{u}\|^2 + \|\vec{v}\|^2
\end{aligned}$$

Distance

Theorem

In $\mathbb{R}^2$ the distance D between the point $P_0 = (x_0,\ y_0)$ and the line $ax + by + c = 0$ is

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

In $\mathbb{R}^3$ the distance D between the point $P_0 = (x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Exercises *Section* 2.3 – Orthogonality

1. Determine whether $\vec{u}$ and $\vec{v}$ are orthogonal

a) $\vec{u}=(-6,\ -2),\quad \vec{v}=(5,\ -7)$

b) $\vec{u}=(6,\ 1,\ 4),\quad \vec{v}=(2,\ 0,\ -3)$

c) $\vec{u}=(1,\ -5,\ 4),\quad \vec{v}=(3,\ 3,\ 3)$

d) $\vec{u}=(-2,\ 2,\ 3),\quad \vec{v}=(1,\ 7,\ -4)$

2. Determine whether the vectors form an orthogonal set

a) $\vec{v}_1=(2,\ 3),\quad \vec{v}_2=(3,\ 2)$

b) $\vec{v}_1=(1,\ -2),\quad \vec{v}_2=(-2,\ 1)$

c) $\vec{u}=(-4,\ 6,\ -10,\ 1)\quad \vec{v}=(2,\ 1,\ -2,\ 9)$

d) $\vec{u}=(a,\ b)\quad \vec{v}=(-b,\ a)$

e) $\vec{v}_1=(-2,\ 1,\ 1),\quad \vec{v}_2=(1,\ 0,\ 2),\quad \vec{v}_3=(-2,\ -5,\ 1)$

f) $\vec{v}_1=(1,\ 0,\ 1),\quad \vec{v}_2=(1,\ 1,1),\quad \vec{v}_3=(-1,\ 0,\ 1)$

g) $\vec{v}_1=(2,\ -2,\ 1),\quad \vec{v}_2=(2,\ 1,\ -2),\quad \vec{v}_3=(1,\ 2,\ 2)$

3. Find a unit vector that is orthogonal to both $\vec{u}$ = (1, 0, 1) and $\vec{v}$ = (0, 1, 1)

4. *a*) Show that $\vec{v}$ = (*a*, *b*) and $\vec{w}$ = (–*b*, *a*) are orthogonal vectors.

b) Use the result to find two vectors that are orthogonal to $\vec{v}$ = (2, –3).

c) Find two unit vectors that are orthogonal to (–3, 4)

5. Find the vector component of $\vec{u}$ along $\vec{a}$ and the vector component of $\vec{u}$ orthogonal to $\vec{a}$.

a) $\vec{u}=(6,\ 2),\quad \vec{a}=(3,\ -9)$

b) $\vec{u}=(3,\ 1,\ -7),\quad \vec{a}=(1,\ 0,\ 5)$

c) $\vec{u}=(1,\ 0,\ 0),\quad \vec{a}=(4,\ 3,\ 8)$

d) $\vec{u}=(1,\ 1,\ 1),\quad \vec{a}=(0,\ 2,\ -1)$

e) $\vec{u}=(2,\ 1,\ 1,\ 2),\quad \vec{a}=(4,\ -4,\ 2,\ -2)$

f) $\vec{u}=(5,\ 0,\ -3,\ 7),\quad \vec{a}=(2,\ 1,\ -1,-1)$

6. Project the vector $\vec{v}$ onto the line through $\vec{a}$, check that $\vec{e}=\vec{u}-proj_{\vec{a}}\vec{u}$ is perpendicular to $\vec{a}$:

a) $\vec{v}=\begin{pmatrix}1\\2\\2\end{pmatrix}$ *and* $\vec{a}=\begin{pmatrix}1\\1\\1\end{pmatrix}$

b) $\vec{v}=\begin{pmatrix}1\\3\\1\end{pmatrix}$ *and* $\vec{a}=\begin{pmatrix}-1\\-3\\-1\end{pmatrix}$

c) $\vec{v}=\begin{pmatrix}1\\1\\1\end{pmatrix}$ *and* $\vec{a}=\begin{pmatrix}1\\2\\2\end{pmatrix}$

7. Find the projection matrix $proj_{\vec{a}}\vec{u}=\dfrac{\vec{u}\bullet\vec{a}}{\|\vec{a}\|^2}\vec{a}$ onto the line through $\vec{a}=\begin{bmatrix}1\\2\\2\end{bmatrix}$

(8 – 9) Draw the projection of $\vec{b}$ onto $\vec{a}$ and also compute it from $proj_{\vec{a}}\vec{u} = \frac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a}$

8. $\vec{b} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ *and* $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

9. $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and* $\vec{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

10. Show that if $\vec{v}$ is orthogonal to both $\vec{w}_1$ and $\vec{w}_2$, then $\boldsymbol{v}$ is orthogonal to $k_1\vec{w}_1 + k_2\vec{w}_2$ for all scalars k_1 and k_2.

11. *a*) Project the vector $\vec{v}$ = (3, 4, 4) onto the line through $\vec{a}$ = (2, 2, 1) and then onto the plane that also contains $\vec{a}^* = (1,\ 0,\ 0)$.

b) Check that the first error vector $\vec{v} - \vec{p}$ is perpendicular to $\vec{a}$, and the second error vector $\vec{v} - \vec{p}^*$ is also perpendicular to $\vec{a}^*$.

12. Compute the projection matrices $\vec{a}\vec{a}^T / \vec{a}^T\vec{a}$ onto the lines through $\vec{a}_1 = (-1,\ 2,\ 2)$ and $\vec{a}_2 = (2,\ 2,\ -1)$. Multiply those projection matrices and explain why their product P_1P_2 is what it is. Project $\vec{v} = (1,\ 0,\ 0)$ onto the lines $\vec{a}_1$, $\vec{a}_2$, and also onto $\vec{a}_3 = (2,\ -1,\ 2)$.Add up the three projections $p_1 + p_2 + p_3$.

13. If $P^2 = P$ show that $(I - P)^2 = I - P$. When P projects onto the column space of A, $I - P$ projects onto the ____.

14. What linear combination of $(1,\ 2,\ -1)$ and $(1,\ 0,\ 1)$ is closest to $\vec{v} = (2,\ 1,\ 1)$?

15. Show that $\vec{u} - \vec{v}$ is orthogonal to $\vec{u} + \vec{v}$ if and only if $\|\vec{u}\| = \|\vec{v}\|$

16. Given $\vec{u} = (3,\ -1,\ 2)$ $\vec{v} = (4,\ -1,\ 5)$ *and* $\vec{w} = (8,\ -7,\ -6)$

a) Find $3\vec{v} - 4(5\vec{u} - 6\vec{w})$

b) Find $\vec{u} \bullet \vec{v}$ and then the angle θ between $\vec{u}$ and $\vec{v}$.

17. Given: $\vec{u} = (3,\ 1,\ 3)$ $\vec{v} = (4,\ 1,\ -2)$

a) Compute the projection $\vec{w}$ of $\vec{u}$ on $\vec{v}$

b) Find $\vec{p} = \vec{u} - \vec{v}$ and show that $\vec{p}$ is perpendicular to $\vec{v}$.

18. *a*) Show that $\vec{v}$ = (*a*, *b*) and $\vec{w}$ = (–*b*, *a*) are orthogonal vectors

b) Use the result in part (*a*) to find two vectors that are orthogonal to $\vec{v}$ = (2, –3)

c) Find two unit vectors that are orthogonal to (–3, 4)

19. Show that A (3, 0, 2), B (4, 3, 0), and C (8, 1, −1) are vertices of a right triangle. At which vertex is the right angle?

20. Establish the identity: $\vec{u} \bullet \vec{v} = \frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2$

21. Find the Euclidean inner product $\vec{u} \bullet \vec{v}$: $\vec{u} = (-1,\ 1,\ 0,\ 4,\ -3)$ $\vec{v} = (-2,\ -2,\ 0,\ 2,\ -1)$

22. Find the Euclidean distance between $\vec{u}$ and $\vec{v}$: $\vec{u} = (3,\ -3,\ -2,\ 0,\ -3)$ $\vec{v} = (-4,\ 1,\ -1,\ 5,\ 0)$

(*Exercises* **22 – 26**) Find

a) $\vec{v} \bullet \vec{u},\ \ |\vec{v}|,\ \ |\vec{u}|$

b) The cosine of the angle between $\vec{v}$ and $\vec{u}$

c) The scalar component of $\vec{u}$ in the direction of $\vec{v}$

d) The vector $proj_{\vec{v}}\vec{u}$

23. $\vec{v} = 2\hat{i} - 4\hat{j} + \sqrt{5}\,\hat{k}, \quad \vec{u} = -2\hat{i} + 4\hat{j} - \sqrt{5}\,\hat{k}$

24. $\vec{v} = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{k}, \quad \vec{u} = 5\hat{i} + 12\hat{j}$

25. $\vec{v} = 2\hat{i} + 10\hat{j} - 11\hat{k}, \quad \vec{u} = 2\hat{i} + 2\hat{j} + \hat{k}$

26. $\vec{v} = 5\hat{i} + \hat{j}, \quad \vec{u} = 2\hat{i} + \sqrt{17}\,\hat{j}$

27. $\vec{v} = \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}}\right), \quad \vec{u} = \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{3}}\right)$

28. Suppose Ted weighs 180 *lb*. and he is sitting on an inclined plane that drops 3 *units* for every 4 horizontal units. The gravitational force vector is $\vec{F}_g = \begin{pmatrix} 0 \\ -180 \end{pmatrix}$.

a) Find the force pushing Ted down the slope.

b) Find the force acting to hold Ted against the slope

29. Prove that is two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$ are orthogonal to nonzero vector $\vec{w}$ in $\mathbb{R}^2$, then $\vec{u}$ and $\vec{v}$ are scalar multiples of each other.

Section 2.4 – Cross Product

The *Cross* Product

To find a vector in 3-space that is perpendicular to two vectors; the type of vector multiplication that facilities this construction is the cross product.

Definition

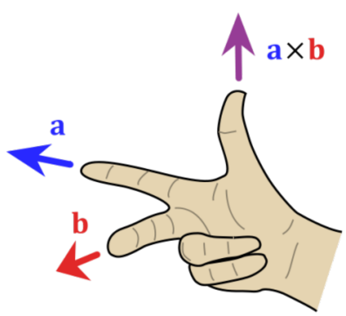
The cross product of $\vec{u} = \left(u_1,\ u_2,\ u_3\right)$ and $\vec{v} = \left(v_1,\ v_2,\ v_3\right)$ is the vector

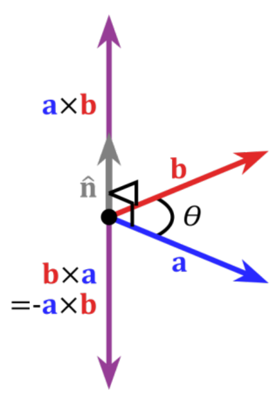
$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

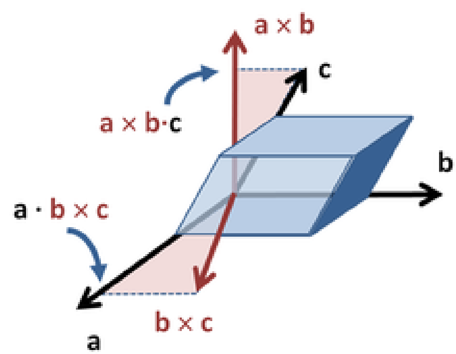
$$= \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix}\hat{i} - \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix}\hat{j} + \begin{vmatrix} u_1 & v_2 \\ u_2 & v_1 \end{vmatrix}\hat{k}$$

$$= \left(u_2v_3 - u_3v_2\right)\hat{i} - \left(u_1v_3 - u_3v_1\right)\hat{j} + \left(u_1v_2 - u_2v_1\right)\hat{k}$$

$$= \left(u_2v_3 - u_3v_2,\ \ u_3v_1 - u_1v_3,\ \ u_1v_2 - u_2v_1\right)$$







- In 1773, ***Joseph Louis Lagrange*** introduced the component form of both the dot and cross products in order to study the tetrahedron in three dimensions. In 1843 the Irish mathematical physicist Sir ***William Rowan Hamilton*** introduced the quaternion product, and with it the terms "*vector*" and "*scalar*". Given two quaternions [0, $\vec{u}$] and [0, $\vec{v}$], where $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^3$, their quaternion product can be summarized as $\left[-\vec{u}\cdot\vec{v},\ \ \vec{u}\times\vec{v}\right]$. ***James Clerk Maxwell*** used Hamilton's quaternion tools to develop his famous ***electromagnetism*** equations, and for this and other reasons quaternions for a time were an essential part of physics education.

Example

Find $\vec{u}\times\vec{v}$, where $\vec{u}=(1,\ 2,\ -2)$ and $\vec{v}=(3,\ 0,\ 1)$

Solution

$$\begin{bmatrix} 1 & 2 & -2 \\ 3 & 0 & 1 \end{bmatrix}$$

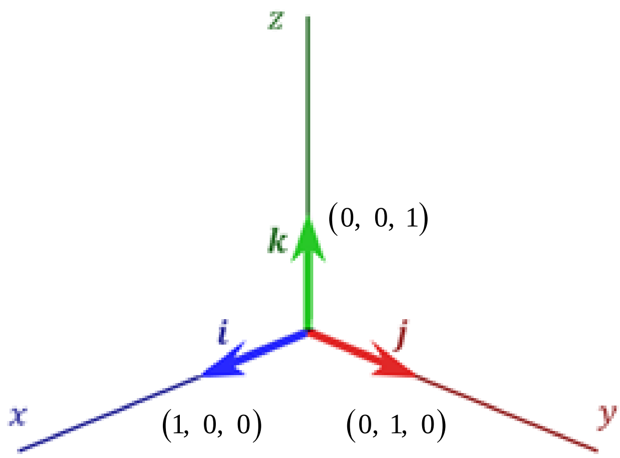
$$\vec{u}\times\vec{v}=\left(\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix},\ -\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix},\ \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}\right)$$

$$=(2,\ -7,\ -6)$$

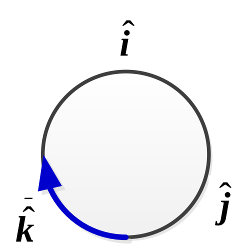
Example

Consider the vectors $\hat{\boldsymbol{i}}=(1,\ 0,\ 0)$ $\hat{\boldsymbol{j}}=(0,\ 1,\ 0)$ $\hat{\boldsymbol{k}}=(0,\ 0,\ 1)$

These vectors each have length of 1 and lie along the coordinate axes. They are called the ***standard unit vectors*** in 3-space.



For example: $(2,\ 3,\ -4)=2\hat{i}+3\hat{j}-4\hat{k}$



Note:

- ✓ $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=\vec{\mathbf{0}}$
- ✓ $\hat{j}\times\hat{i}=-\hat{k},\quad \hat{k}\times\hat{j}=-\hat{i},\quad \hat{i}\times\hat{k}=-\hat{j}$
- ✓ $\hat{i}\times\hat{j}=\hat{k},\quad \hat{j}\times\hat{k}=\hat{i},\quad \hat{k}\times\hat{i}=\hat{j}$

Properties

1. $\vec{u}\times\vec{v}$ reverses rows 2 and 3 in the determinant so it is equals $-(\vec{u}\times\vec{v})$

2. The cross product $\vec{u}\times\vec{v}$ is perpendicular to $\vec{u}$, then $\vec{u}\bullet(\vec{u}\times\vec{v})=0$

3. The cross product $\vec{u}\times\vec{v}$ is perpendicular to $\vec{v}$, then $(\vec{u}\times\vec{v})\bullet\vec{v}=0$

4. The cross product of any vector with itself (two equal rows) is $\vec{u}\times\vec{u}=0$.

5. Lagrange's identity: $\|\vec{u}\times\vec{v}\|^2=\|\vec{u}\|^2\ \|\vec{v}\|^2-(\vec{u}\bullet\vec{v})^2$

$$=\|\vec{u}\|\ \|\vec{v}\|\ |\sin\theta|$$

$$|\vec{u}\bullet\vec{v}|=\|\vec{u}\|\ \|\vec{v}\|\ |\cos\theta|$$

Theorem

a) $\vec{u}\times\vec{v}=-(\vec{v}\times\vec{u})$

b) $\vec{u}\times(\vec{v}+\vec{w})=(\vec{u}\times\vec{v})+(\vec{u}\times\vec{w})$

c) $(\vec{u}+\vec{v})\times\vec{w}=(\vec{u}\times\vec{w})+(\vec{v}\times\vec{w})$

d) $k(\vec{u}\times\vec{v})=(k\vec{u})\times\vec{v}=\vec{u}\times(k\vec{v})$

e) $\vec{u}\times\vec{0}=\vec{0}\times\vec{u}=\vec{0}$

f) $\vec{u}\times\vec{u}=0$

Definition

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in 3-space, then $\boxed{\vec{u}\bullet(\vec{v}\times\vec{w})}$ is called the ***scalar triple product*** of $\vec{u}$, $\vec{v}$, and $\vec{w}$.

Example

Calculate the scalar triple product $\vec{u}\bullet(\vec{u}\times\vec{v})$ of the vectors:

$$\vec{u}=-2\hat{i}+6\hat{k}\quad \vec{v}=\hat{i}-3\hat{j}+\hat{k}\quad \vec{w}=-5\hat{i}-\hat{j}+\hat{k}$$

Solution

$$\vec{u}\bullet(\vec{u}\times\vec{v})=\begin{vmatrix}-2 & 0 & 6\\ 1 & -3 & 1\\ -5 & -1 & 1\end{vmatrix}$$

$$\underline{=-92}$$

Area of a Parallelogram

Theorem

If $\vec{u}$ and $\vec{v}$ are vectors in 3-space, then $\|\vec{u}\times\vec{v}\|$ is equal to the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$.

Example

Find the area of the triangle determined by the points $P_1(2,\ 2,\ 0),\ P_2(-1,\ 0,\ 2),\ and\ P_3(0,\ 4,\ 3)$.

Solution

The area of the triangle is $\frac{1}{2}$ the area of the parallelogram determined by the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$

$$\begin{aligned}\overrightarrow{P_1P_2} &= (-1,\ 0,\ 2)-(2,\ 2,\ 0)\\ &= (-3,\ -2,\ 2)\end{aligned}$$

$$\begin{aligned}\overrightarrow{P_1P_3} &= (0,\ 4,\ 3)-(2,\ 2,\ 0)\\ &= (-2,\ 2,\ 3)\end{aligned}$$

$$\begin{aligned}\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3} &= \left(\begin{vmatrix}-2 & 2\\ 2 & 3\end{vmatrix},\ -\begin{vmatrix}-3 & 2\\ -2 & 3\end{vmatrix},\ \begin{vmatrix}-3 & -2\\ -2 & 2\end{vmatrix}\right)\\ &= (-10,\ 5,\ -10)\end{aligned}$$

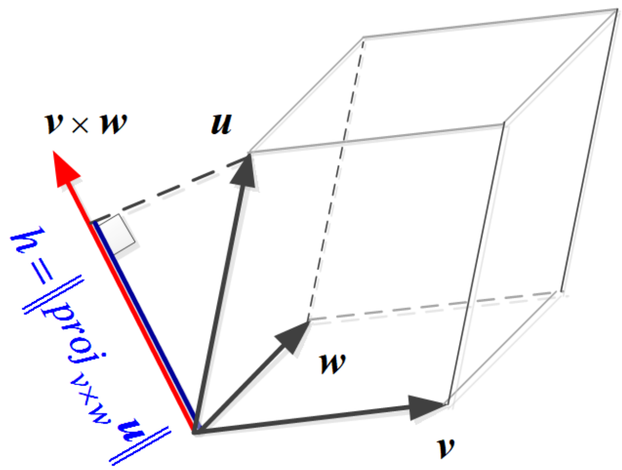
$$\begin{aligned}Area &= \frac{1}{2}\left\|\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}\right\|\\ &= \frac{1}{2}\sqrt{(-10)^2+5^2+(-10)^2}\\ &= \frac{15}{2}\ unit^2\end{aligned}$$

Volume

The Volume of the Parallelepiped is

$$V = (area\ of\ \ base).(height) = \|\vec{v}\times\vec{w}\|\frac{|\vec{u}\bullet(\vec{v}\times\vec{w})|}{\|\vec{v}\times\vec{w}\|} = |\vec{u}\ \bullet\ (\vec{u}\times\vec{v})|$$

$$V = \left|\ \det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}\right|$$



Theorem

If the vectors $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, and $\vec{w} = (w_1, w_2, w_3)$ have the initial point, then they lie in the same plane if and only if

$$\vec{u}\bullet(\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0$$

Example

Find the volume of the parallelepiped with sides $\vec{u} = (2,\ -6,\ 2)$, $\vec{v} = (0,\ 4,\ -2)$, and $\vec{w} = (2,\ 2,\ -4)$

Solution

$$V = \left|\ \det\begin{bmatrix} 2 & -6 & 2 \\ 0 & 4 & -2 \\ 2 & 2 & -4 \end{bmatrix}\right|$$

$$= 16$$

Exercises Section 2.4 – Cross Product

1. Prove when the cross product $\vec{u}\times\vec{v}$ is perpendicular to $\vec{u}$, then $\vec{u}\bullet(\vec{u}\times\vec{v})=0$

2. Find $\vec{u}\times\vec{v}$, where $\vec{u}=(1,\ 2,\ -2)$ *and* $\vec{v}=(3,\ 0,\ 1)$ and show that $\vec{u}\times\vec{v}$ is perpendicular to $\vec{u}$ and to $\vec{v}$.

3. Given $\vec{u}=(3,\ 2,\ -1)$, $\vec{v}=(0,\ 2,\ -3)$, *and* $\vec{w}=(2,\ 6,\ 7)$ Compute the vectors

a) $\vec{u}\times\vec{v}$

b) $\vec{v}\times\vec{w}$

c) $\vec{u}\times(\vec{v}\times\vec{w})$

d) $(\vec{u}\times\vec{v})\times\vec{w}$

e) $\vec{u}\times(\vec{v}-2\vec{w})$

4. Use the cross product to find a vector that is orthogonal to both

a) $\vec{u}=(-6,\ 4,\ 2)$, $\vec{v}=(3,\ 1,\ 5)$

b) $\vec{u}=(1,\ 1,\ -2)$, $\vec{v}=(2,\ -1,\ 2)$

c) $\vec{u}=(-2,\ 1,\ 5)$, $\vec{v}=(3,\ 0,\ -3)$

5. Find the area of the parallelogram determined by the given vectors

a) $\vec{u}=(1,\ -1,\ 2)$ *and* $\vec{v}=(0,\ 3,\ 1)$

b) $\vec{u}=(3,\ -1,\ 4)$ *and* $\vec{v}=(6,\ -2,\ 8)$

c) $\vec{u}=(2,\ 3,\ 0)$ *and* $\vec{v}=(-1,\ 2,\ -2)$

6. Find the area of the parallelogram with the given vertices

$P_1(3,\ 2)$, $P_2(5,\ 4)$, $P_3(9,\ 4)$, $P_4(7,\ 2)$

7. Find the area of the triangle with the given vertices:

a) $A(2,\ 0)$ $B(3,\ 4)$ $C(-1,\ 2)$

b) $A(1,\ 1)$ $B(2,\ 2)$ $C(3,\ -3)$

c) $P(2,\ 6,\ -1)$ $Q(1,\ 1,\ 1)$ $R=(4,\ 6,\ 2)$

8. *a*) Find the area of the parallelogram with edges $\vec{v}=(3,\ 2)$ and $\vec{w}=(1,\ 4)$

b) Find the area of the triangle with sides $\vec{v}$, $\vec{w}$, and $\vec{v}+\vec{w}$. Draw it.

c) Find the area of the triangle with sides $\vec{v}$, $\vec{w}$, and $\vec{v}-\vec{w}$. Draw it.

9. Find the volume of the parallelepiped with sides $\vec{u}$, $\vec{v}$, and $\vec{w}$.

a) $\vec{u}=(2,\ -6,\ 2)$, $\vec{v}=(0,\ 4,\ -2)$, $\vec{w}=(2,\ 2,\ -4)$

b) $\vec{u}=(3,\ 1,\ 2)$, $\vec{v}=(4,\ 5,\ 1)$, $\vec{w}=(1,\ 2,\ 4)$

10. Compute the scalar triple product $\vec{u} \bullet (\vec{v} \times \vec{w})$

a) $\vec{u} = (-2,\ 0,\ 6),\quad \vec{v} = (1,\ -3,\ 1),\quad \vec{w} = (-5,\ -1,\ 1)$

b) $\vec{u} = (-1,\ 2,\ 4),\quad \vec{v} = (3,\ 4,\ -2),\quad \vec{w} = (-1,\ 2,\ 5)$

c) $\vec{u} = (a,\ 0,\ 0),\quad \vec{v} = (0,\ b,\ 0),\quad \vec{w} = (0,\ 0,\ c)$

d) $\vec{u} = 3\hat{i} - 2\hat{j} - 5\hat{k},\quad \vec{v} = \hat{i} + 4\hat{j} - 4\hat{k},\quad \vec{w} = 3\hat{j} + 2\hat{k}$

e) $\vec{u} = (3,\ -1,\ 6)\quad \vec{v} = (2,\ 4,\ 3)\quad \vec{w} = (5,\ -1,\ 2)$

11. Use the cross product to find the sine of the angle between the vectors
$\vec{u} = (2,\ 3,\ -6),\quad \vec{v} = (2,\ 3,\ 6)$

12. Simplify $(\vec{u} + \vec{v}) \times (\vec{u} - \vec{v})$

13. Prove Lagrange's identity: $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2\ \|\vec{v}\|^2 - (\vec{u} \bullet \vec{v})^2$

14. Polar coordinates satisfy $x = r\cos\theta$ *and* $y = \sin\theta$. Polar area $J\ dr\ d\theta$ includes J:

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

The two columns are orthogonal. Their lengths are ______. Thus J = ______.

15. Prove that $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$ if and only if $\vec{u}$ and $\vec{v}$ are parallel vectors.

16. State the following statements as True or False

a) The cross product of two nonzero vectors $\vec{u}$ and $\vec{v}$ is a nonzero vector if and only if $\vec{u}$ and $\vec{v}$ are not parallel.

b) A normal vector to a plane can be obtained by taking the cross product of two nonzero and noncollinear vectors lying in the plane.

c) The scalar triple product of $\vec{u}$, $\vec{v}$, and $\vec{w}$ determines a vector whose length is equal to the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$.

d) If $\vec{u}$ and $\vec{v}$ are vectors in 3-space, then $\|\vec{u} \times \vec{v}\|$ is equal to the area of the parallelogram determine by $\vec{u}$ and $\vec{v}$.

e) For all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in R^3, the vectors $(\vec{u} \times \vec{v}) \times \vec{w}$ and $\vec{u} \times (\vec{v} \times \vec{w})$ are the same.

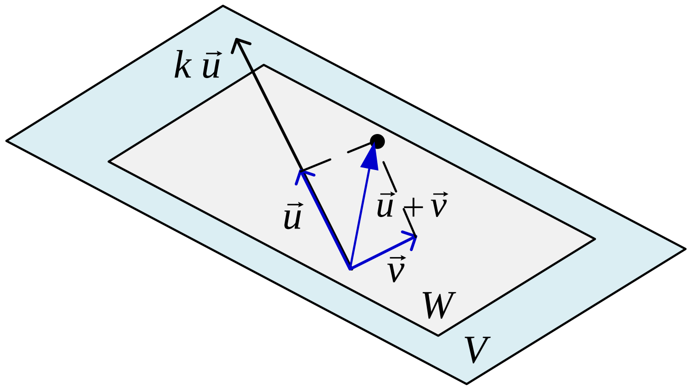
f) If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in R^3, where $\vec{u}$ is nonzero and $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$, then $\vec{v} = \vec{w}$

Section 2.5 – Subspaces, Span and Null Spaces

Subspaces

Definition

A subset W of a vector space V is called a ***subspace*** of V if W itself a vector space under the addition and scalar multiplication defined in V.



Theorem

If W is a set of one or more vectors in a vector space V, then W is a subspace of V iff the following conditions holds

1. If $\vec{u}$ and $\vec{v}$ are vectors in W, then $\vec{u}+\vec{v}$ is in W.
2. If k is any scalar and $\vec{v}$ is any vector in W, the $k\vec{v}$ is in the subspace in W.

➢ The most fundamental ideas in linear algebra are that the plane is a subspace of the full vector space $\mathbb{R}^n$.

➢ Every subspace contains the zero vector. The plane vector in $\mathbb{R}^3$ has to go through (0, 0, 0). From rule (**2**), if we choose $k = 0$ and the rule requires $0v$ to be in the subspace.

The ***axioms*** that are ***not*** inherited by W are

Axiom 1 – Closure of W under addition

Axiom 4 – Existence of a zero vector in W

Axiom 5 – Existence of a negative in W for every vector in W

Axiom 6 – Closure of W under scalar multiplication

Example

Keep only the vectors $(x,\ y)$ whose components are positive or zero (first quadrant ***"quarter-plane"***).

The vector $(2,\ 3)$ is included but $(-2,\ -3)$ is not. So, rule (**2**) is violated when we try $k=-1$. ***The quarter-plane is not a subspace***.

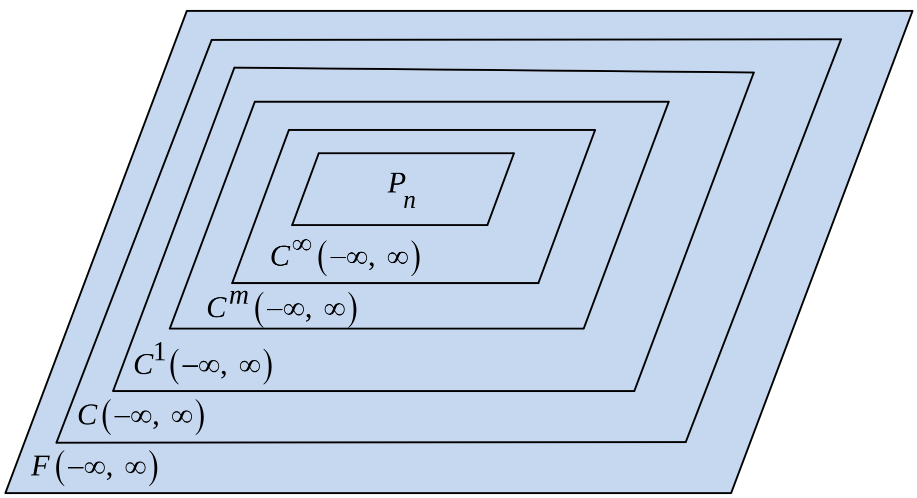
Example

Include also the vectors whose components are both negative. Now we have two quarter-planes. Rule (***ii***) satisfies when we multiply by any c. But rule (***i***) fails. The sum of $v=(2,\ 3)$ and $w=(-3,\ -2)$ is $(-1,\ 1)$ which is outside the quarter-plane. ***Two quarter-planes don't make a subspace***.

Example

The **Subspace** $C(-\infty,\ \infty)$

There is a theorem in calculus which states that a sum of continuous functions is continuous and than a constant times a continuous fruction is continuous. In vector word, the set of continuous functions on $(-\infty,\ \infty)$ is a subspace of $F(-\infty,\ \infty)$. We dente this subspace by $C(-\infty,\ \infty)$



Theorem

If $W_1,\ W_2,\ \ldots,\ W_n$ are subspaces of a vector space *V*, then intersection of these subspaces is also a subspace of *V*.

- ***A subspace containing*** $\vec{v}$ ***and*** $\vec{w}$ ***must contain all linear combination*** $c\vec{v}+d\vec{w}$.

Example

Inside the vector space M of all 2 by 2 matrices, given two subspaces:

U all upper triangular matrices $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

D all diagonal matrices $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$

Solution

If we add 2 matrices in **U**: $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 0 & 2d \end{bmatrix}$ is in **U**.

If we add 2 matrices in **D**: $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 0 & 2d \end{bmatrix}$ is in **D**.

In this case **D** is also a subspace of **U**!. The zero matrix is in these subspaces, when a, b, and d all equal zero.

Span

Definition

The subspace of a vector space V that is formed from all possible linear combinations of the vectors in a nonempty set S is called the ***span of S***, and we say that the vectors in S *span* that subspace. If $S = \{w_1, w_2, \ldots, w_r\}$, then we denoted the span of S by

$$span\{w_1, w_2, \ldots, w_r\} \quad or \quad span(S)$$

Theorem

Let $\vec{v}_1, \ldots, \vec{v}_n$ be vectors in vector space V and S be their span. Then,

a) S is a subspace of V.

Proof: $\forall\ \vec{u}, \vec{v} \in S$, $\vec{u} = a_1\vec{v}_1 + \ldots + a_n\vec{v}_n$ and $\vec{v} = b_1\vec{v}_1 + \ldots + b_n\vec{v}_n$

$$\vec{u} + \vec{v} = (a_1 + b_1)\vec{v}_1 + \ldots + (a_n + b_n)\vec{v}_n \quad \in \boldsymbol{S}$$

$$k\vec{u} = ka_1\vec{v}_1 + \ldots + ka_n\vec{v}_n \quad \in \boldsymbol{S}$$

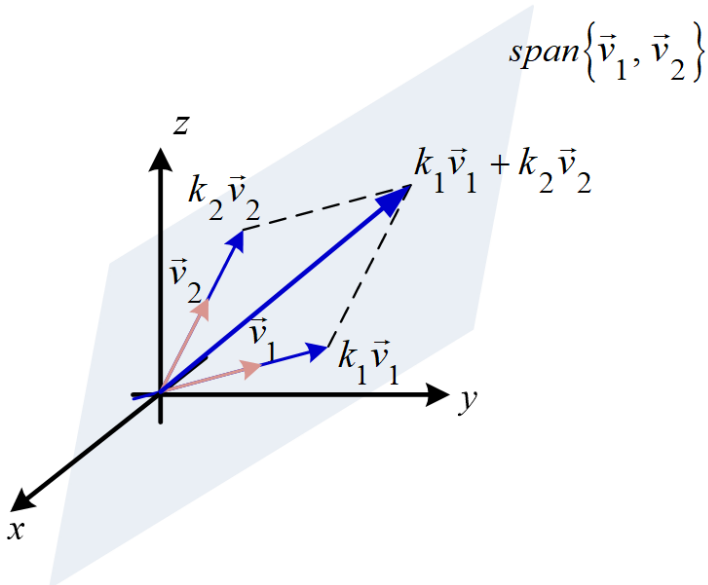
b) S is the smallest subspace of V that contains $\vec{v}_1, \ldots, \vec{v}_k$. i.e. any other subspace $\vec{w}$ containing $\vec{v}_1, \ldots, \vec{v}_n$ also contains S.

Proof: *let* $\vec{u} \in S$, $\vec{u} = a_1\vec{v}_1 + \ldots + a_n\vec{v}_n$

But $a_1\vec{v}_1, \ldots, a_n\vec{v}_n \in \vec{w}$ $\therefore$ $\vec{w}$ *closed under scalar multiplication.*

$a_1\vec{v}_1, \ldots, a_n\vec{v}_n \in \vec{w}$ $\therefore$ $\vec{w}$ *closed under addition.*

$\therefore$ $\underline{\vec{u} \in \vec{w}}\,|$



Example

a) $\vec{v}_1 = \begin{pmatrix}1\\0\end{pmatrix}$ *and* $\vec{v}_2 = \begin{pmatrix}0\\1\end{pmatrix}$ span the full two-dimensional space $\mathbb{R}^2$.

b) $\vec{v}_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}0\\1\end{pmatrix}$, *and* $\vec{v}_3 = \begin{pmatrix}4\\7\end{pmatrix}$ span the full space $\mathbb{R}^2$.

c) $\vec{w}_1 = \begin{pmatrix}1\\1\end{pmatrix}$ *and* $\vec{w}_2 = \begin{pmatrix}-1\\-1\end{pmatrix}$ only span a line in $\mathbb{R}^2$.

Definition

The ***row space*** of a matrix is the subspace of $\mathbb{R}^n$ spanned by the rows.

Example

Determine whether $\vec{v}_1 = (1,\ 1,\ 2)$, $\vec{v}_2 = (1,\ 0,\ 1)$, *and* $\vec{v}_3 = (2,\ 1,\ 3)$ span the vector space $\mathbb{R}^3$

Solution

Let $b = (b_1,\ b_2,\ b_3)$ be the arbitrary vector in $\mathbb{R}^3$ can be expressed as a linear combination

$$\vec{b} = k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3$$

$$(b_1,\ b_2,\ b_3) = k_1(1,\ 1,\ 2) + k_2(1,\ 0,\ 1) + k_3(2,\ 1,\ 3)$$

$$(b_1,\ b_2,\ b_3) = (k_1 + k_2 + 2k_3,\ \ k_1 + k_3,\ \ 2k_1 + k_2 + 3k_3)$$

$$\rightarrow \begin{cases} k_1 + k_2 + 2k_3 = b_1 \\ k_1 \qquad\ + k_3 = b_2 \\ 2k_1 + k_2 + 3k_3 = b_3 \end{cases}$$

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix}$$

$$\underline{= 0\ |}$$

Since the determinant is zero, the $\vec{v}_1,\ \vec{v}_2,$ *and* $\vec{v}_3$ ***do not*** span space $\mathbb{R}^3$

Solution Spaces of *Homogeneous* (*Null Space*) Systems

Theorem

The solution set of a homogeneous linear system $A\vec{x} = \vec{0}$ in n unknowns is a subspace of $\mathbb{R}^n$

Proof

Let W be the solution set for the system. The set W is not empty because it contains at least the trivial solution $\vec{x} = \vec{0}$.

To show that W is a subspace of $\mathbb{R}^n$, we must show that it is closed under addition and scalar multiplication.

Let $\vec{x}_1$ and $\vec{x}_2$ be vectors in W and these vectors are solution of $A\vec{x} = \vec{0}$.

$$A\vec{x}_1 = \vec{0} \quad \textit{and} \quad A\vec{x}_2 = \vec{0}$$

Therefore,

$$A\left(\vec{x}_1 + \vec{x}_2\right) = A\vec{x}_1 + A\vec{x}_2$$
$$= \vec{0} + \vec{0}$$
$$\underline{= \vec{0}}$$

So, *W* is closed under addition.

$$A\left(k\vec{x}_1\right) = kA\vec{x}_1 = k0 = 0$$

So, *W* is closed under scalar multiplication.

Null Spaces

Definition

The nullspace of A consists of all solutins to $A\vec{x}=\vec{0}$. These solution vectors $\vec{x}$ are in $\mathbb{R}^n$. The Nullspace containing all solutions is denoted by $N(A)$ *or* $NS(A)$.

$\left\{\vec{x}\in\mathbb{R}^n \mid A\vec{x}=\vec{0}\right\}$ is the nullspace of A, $NS(A)$

(Can also be called ***Kernel*** of A: $Ker(A)$)

Theorem

Suppose $NS(A)$ is a subspace of $\mathbb{R}^n$ for $A_{m\times n}$

- ✓ Let $\vec{x}$ and $\vec{y}$ are in the nullspace $(\vec{x},\ \vec{y}\in NS(A))$ then

$$\begin{aligned} A(\vec{x}+\vec{y}) &= A\vec{x}+A\vec{y} \\ &= \vec{0}+\vec{0} \\ &= \vec{0} \end{aligned}$$

- ✓ Let $\vec{x}\in NS(A)$ then $c\vec{x}\in NS(A)$

$$\begin{aligned} \therefore\ A(c\vec{x}) &= cA\vec{x} \\ &= c\vec{0} \\ &= \vec{0} \end{aligned}$$

Since we can add and multiply without leaving the Nullspace, it is a subspace.

Example

The equation $x+2y+3z=0$ comes from the 1 by 3 matrix $A=\begin{bmatrix}1 & 2 & 3\end{bmatrix}$. This equation produces a plane through the origin. The plane is a subspace of $\mathbb{R}^3$. *It is the Nullspace* of A.

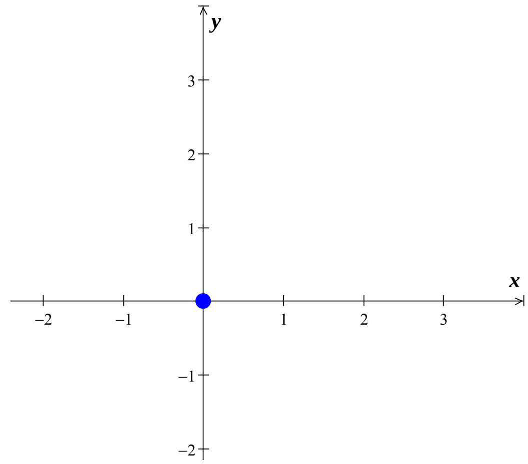
Solution

The solution to $x+2y+3z=6$ also form a plane, but not a subspace.

Example

Find the null space of

$$a)\quad A=\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix} \qquad b)\ B=\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}$$



Solution

a) $\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\to \begin{array}{l} x_1+x_2=0\\ 3x_2=0\end{array}$

$\Rightarrow x_1=x_2=0$

So $NS(A)=\{\vec{0}\}$

b) $\begin{bmatrix}1 & 2 & 0\\ 3 & 6 & 0\end{bmatrix}\ R_2-3R_1$

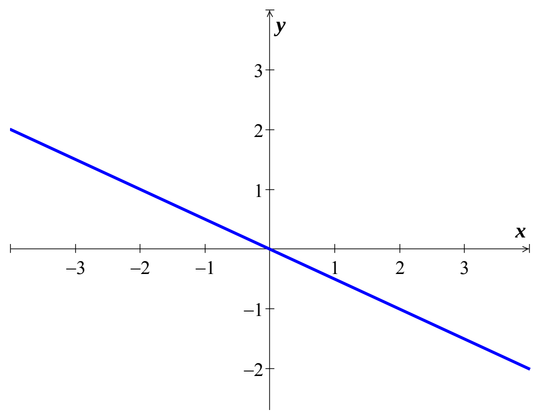
$\begin{bmatrix}1 & 2 & 0\\ 0 & 0 & 0\end{bmatrix}$

$\Rightarrow x_1=-2x_2$

If we let $x_2=s$, then

$\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ is in $NS(B)$ if and only if

$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=s\begin{bmatrix}-2\\ 1\end{bmatrix}$



Example

Describe the nullspace of $A=\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}$

Solution

Apply the elimination to the linear equations $Ax=0$:

$$\begin{bmatrix}x_1+2x_2=0\\ 3x_1+6x_2=0\end{bmatrix}\xrightarrow[R_2-3R_1]{}\begin{bmatrix}x_1+2x_2=0\\ 0=0\end{bmatrix}$$

There is only one equation $\left(x_1+2x_2=0\right)$, this line is the Nullspace $N(A)$.

Example

Consider the linear system $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solution

$z = t, \quad y = s, \quad x = 2s - 3t$

$\Rightarrow x - 2y + 3z = 0$

This is the equation of a plane through the origin that has $\vec{n}$ = (1, –2, 3) as a normal.

Example

Consider the linear system $\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solution

$x = 0, \quad y = 0, \quad z = 0$

The solution space is $\{\vec{0}\}$

Exercises *Section* 2.5 – Subspaces, Span and Null Spaces

1. Suppose S and T are two subspaces of a vector space $\mathbf{V}$.

a) The sum $S+T$ contains all sums $\vec{s}+\vec{t}$ of a vector $\vec{s}$ in S and a vector $\vec{t}$ in T. Show that $S+T$ satisfies the requirements (addition and scalar multiplication) for a vector space.

b) If S and T are lines in $\mathbb{R}^m$, what is the difference between $S+T$ and $S\cup T$? That union contains all vectors from S and T or both. Explain this statement: The span of $S\cup T$ is $S+T$.

2. Determine which of the following are subspaces of $\mathbb{R}^3$?

a) All vectors of the form $(a, 0, 0)$

b) All vectors of the form $(a, 1, 1)$

c) All vectors of the form (a, b, c), where $b = a + c$

d) All vectors of the form (a, b, c), where $b = a + c + 1$

e) All vectors of the form $(a, b, 0)$

3. Determine which of the following are subspaces of $\mathbb{R}^\infty$?

a) All sequences $\vec{v}$ in $\mathbb{R}^\infty$ of the form $\vec{v} = (v, 0, v, 0, \ldots)$

b) All sequences $\vec{v}$ in $\mathbb{R}^\infty$ of the form $\vec{v} = (v, 1, v, 1, \ldots)$

c) All sequences $\vec{v}$ in $\mathbb{R}^\infty$ of the form $\vec{v} = (v,\ 2v,\ 4v,\ 8v,\ 16v,\ \ldots)$

4. Which of the following are linear combinations of $\vec{u} = (0, -2, 2)$ and $\vec{v} = (1, 3, -1)$?

a) $(2, 2, 2)$ *b)* $(3, 1, 5)$ *c)* $(0, 4, 5)$ *d)* $(0, 0, 0)$

5. Which of the following are linear combinations of $\vec{u} = (2, 1, 4)$, $\vec{v} = (1, -1, 3)$ and $\vec{w} = (3, 2, 5)$?

a) $(-9, -7, -15)$ *b)* $(6, 11, 6)$ *c)* $(0, 0, 0)$

6. Determine whether the given vectors span $\mathbb{R}^3$

a) $\vec{v}_1 = (2,\ 2,\ 2),\quad \vec{v}_2 = (0,\ 0,\ 3),\quad \vec{v}_3 = (0,\ 1,\ 1)$

b) $\vec{v}_1 = (2,\ -1,\ 3),\quad \vec{v}_2 = (4,\ 1,\ 2),\quad \vec{v}_3 = (8,\ -1,\ 8)$

c) $\vec{v}_1 = (3,\ 1,\ 4),\quad \vec{v}_2 = (2,\ -3,\ 5),\quad \vec{v}_3 = (5,\ -2,\ 9),\quad \vec{v}_4 = (1,\ 4,\ -1)$

7. Which of the following are linear combinations of $A = \begin{pmatrix} 4 & 0 \\ -2 & -2 \end{pmatrix},\quad B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix},\quad C = \begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}$

a) $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ *b)* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *c)* $\begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix}$

8. Suppose that $\vec{v}_1 = (2,\ 1,\ 0,\ 3)$, $\vec{v}_2 = (3,\ -1,\ 5,\ 2)$, $\vec{v}_3 = (-1,\ 0,\ 2,\ 1)$. Which of the following vectors are in span $\{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\}$

a) (2, 3, −7, 3) *b)* (0, 0, 0, 0) *c)* (1, 1, 1, 1) *d)* (−4, 6, −13, 4)

9. Let $\boldsymbol{f} = \cos^2 x$ and $\boldsymbol{g} = \sin^2 x$. Which of the following lie in the space spanned by $\boldsymbol{f}$ and $\boldsymbol{g}$

a) $\cos 2x$ *b)* $3 + x^2$ *c)* $\sin x$ *d)* 0

10. Let $S = \left\{ (x,\ y) \middle|\ x^2 + y^2 = 0;\ \ x,\ y \in \mathbb{R} \right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^2$?

11. Let $S = \left\{ (x,\ y) \middle|\ x^2 + y^2 = 0;\ \ x,\ y \in \mathbb{C} \right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{C}^2$?

12. Let $S = \left\{ (x,\ y) \middle|\ x^2 - y^2 = 0;\ \ x,\ y \in \mathbb{R} \right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^2$?

13. Let $S = \left\{ (x,\ y) \middle|\ x - y = 0;\ \ x,\ y \in \mathbb{R} \right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^2$?

14. Let $S = \left\{ (x,\ y) \middle|\ x - y = 1;\ \ x,\ y \in \mathbb{R} \right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^2$?

15. $V = \mathbb{R}^3, \quad S = \{(0,\ s,\ t) \mid s,\ t \text{ are real numbers}\}$ where V is a vector space and S is subset of V

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V?

16. $V = \mathbb{R}^3, \quad S = \{(x,\ y,\ z) \mid x,\ y\ ,\ z\ \geq 0\}$ where V is a vector space and S is subset of V

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V?

17. $V = \mathbb{R}^3, \quad S = \{(x,\ y,\ z) \mid z = x + y + 1\}$ where V is a vector space and S is subset of V

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V?

18. Let $S = \left\{\left(a_1,\ a_2,\ a_3\right) \in \mathbb{R}^3 :\ a_1 = 3a_2 \text{ and } a_3 = -a_2\right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^3$?

19. Let $S = \left\{\left(a_1,\ a_2,\ a_3\right) \in \mathbb{R}^3 :\ a_1 = a_3 + 2\right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^3$?

20. Let $S = \left\{\left(a_1,\ a_2,\ a_3\right) \in \mathbb{R}^3 :\ 2a_1 - 7a_2 + a_3 = 0\right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^3$?

21. Let $S = \left\{\left(a_1,\ a_2,\ a_3\right) \in \mathbb{R}^3 :\ a_1 - 4a_2 - a_3 = 0\right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^3$?

22. Let $S = \left\{ \left(a_1,\ a_2,\ a_3 \right) \in \mathbb{R}^3 :\ a_1 + 2a_2 - 3a_3 = 0 \right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

23. Let $S = \left\{ \left(a_1,\ a_2,\ a_3 \right) \in \mathbb{R}^3 :\ a_1 + 2a_2 - 3a_3 = 1 \right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

24. Let $S = \left\{ \left(a_1,\ a_2,\ a_3 \right) \in \mathbb{R}^3 :\ 5a_1^2 - 3a_2^2 + 6a_3^2 = 0 \right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

25. Let $S = \left\{ \left(a_1,\ a_2,\ a_3 \right) \in \mathbb{R}^3 :\ a_3 = a_1 + a_2 \right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

26. Let $S = \left\{ \left(a_1,\ a_2,\ a_3 \right) \in \mathbb{R}^3 :\ a_1 + a_2 + a_3 = 0 \right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

27. $S = \left\{ \left(x_1,\ x_2,\ 1 \right) :\ x_1 \textit{ and } x_2 \textit{ are real numbers} \right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

28. $S = \left\{ \left(x_1,\ x_2,\ x_3 \right) \in \mathbb{R}^3 :\ x_2 = x_1 + 2x_3 \right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

29. $S=\left\{\begin{pmatrix} a & 1 \\ c & d \end{pmatrix} \in M_{2\times 2} \mid a,\ b,\ c\in\mathbb{R}\right\}$ and $V=M_{2,2}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V ?

30. $S=\left\{\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in M_{2\times 2} \mid a,\ b,\ c\in\mathbb{R}\right\}$ and $V=M_{2,2}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V ?

31. Let $S=\left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in M_{2\times 2} \mid a,\ d\in\mathbb{R}\ \ \&\ \ ad\geq 0\right\}$ and $V=M_{2,2}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V ?

32. $V=M_{33}$, $S=\{A \mid A\ \textit{is invertible}\}$ where V is a vector space and S is subset of V

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V?

33. Let $S=\left\{p(t)=a+2at+3at^3 \mid a\in\mathbb{R}\ \ \&\ \ p(t)\in\mathbb{P}_2\right\}$ and $V=\mathbb{P}_2$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V ?

34. Let $S=\{p(t) \mid p(t)\in\mathbb{P}[t]\ \textit{has degree}\ 3\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{P}[t]$?

35. Let $S=\{p(t) \mid p(0)=0,\ p(t)\in\mathbb{P}[t]\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{P}[t]$?

36. Given: $A=\begin{bmatrix}1 & 3 & 2\\ 2 & 6 & 4\end{bmatrix}$

a) Find $NS(A)$

b) For which n is $NS(A)$ a subspace of $\mathbb{R}^n$

c) Sketch $NS(A)$ in $\mathbb{R}^2$ or $\mathbb{R}^3$

37. Determine which of the following are subspaces of M_{22}

a) All 2×2 matrices with integer entries

b) All matrices $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ where $a+b+c+d=0$

38. Let $V=\left\{\begin{bmatrix}a & b\\ c & d\end{bmatrix}:\ ad-bc=1\right\}$. Is V a vector space?

39. Let $V=\{(x,0,y):\ x\ \&\ y\ \textit{are arbitrary}\ \mathbb{R}\}$. Define addition and scalar multiplication as follows:

$$\begin{cases}(x_1,\ 0,\ y_1)+(x_2,\ 0,\ y_2)=(x_1+x_2,\ y_1+y_2)\\ \quad c(x,\ 0,\ y)=(cx,\ cy)\end{cases}$$

Is V a vector space?

40. Construct a matrix whose column space contains $(1,\ 1,\ 0)$ and $(0,\ 1,\ 1)$ and whose nullspace contains $(1,\ 0,\ 1)$ and $(0,\ 0,\ 1)$

41. How is the nullspace $N(C)$ related to the spaces $N(A)$ and $N(B)$, is $C=\begin{bmatrix}A\\ B\end{bmatrix}$?

42. True or False (check addition or give a counterexample)

a) If V is a vector space and W is a subset of V that is a vector space, then W is s subspace of V.
b) The empty set is a subspace of every vector space.
c) If V is a vector space other than the zero vector space, then V contains a subspace W such that $W \neq V$.
d) The intersection of any two subsets of V is a subspace of V.
e) Let W be the *xy*-plane in $\mathbb{R}^3$; that is, $W=\{(a_1,\ a_2,\ 0):\ a_1,\ a_2 \in \mathbb{R}\}$. Then $W=\mathbb{R}^2$

43. Let $A\vec{x}=\vec{0}$ be a homogeneous system of n linear equations in n unknowns that has only the trivial solution. Show that of k is any positive integer, then the system $A^k\vec{x}=\vec{0}$ also has only trivial solution.

44. Let $A\vec{x}=\vec{0}$ be a homogeneous system of n linear equations in n unknowns and let Q be an invertible $n\times n$ matrix. Show that of $A\vec{x}=\vec{0}$ has just trivial solution if and only if $(QA)\vec{x}=\vec{0}$ has just trivial solution.

45. Let $A\vec{x}=\vec{b}$ be a consistent system of linear equations and let $\vec{x}_1$ be a fixed solution. Show that every solution to the system can be written in the form $\vec{x}=\vec{x}_1+\vec{x}_0$ where $\vec{x}_0$ is a solution to $A\vec{x}=\vec{0}$. Show also that every matrix of this form is a solution.

Section 2.6 – Linear Independence

There are n columns in an m by n matrix, and each column has m components. But the true ***dimension*** of the column space is not necessarily m or n. The dimension is measured by counting ***independent columns***.

- **Independent vectors** (*not too many*)
- **Spanning a space** (*not too few*)

Linear Independence (LI)

The columns of A are ***linearly independent*** when the only solution to $A\vec{x}=0$ is $\vec{x}=\vec{0}$. ***No other combination*** Ax ***of the columns gives the zero vector***.

Definitions

- A set of two or more vectors is ***linearly dependent*** if one vector in the set is a linear combination of the others. A set of one vector is ***linearly dependent*** if that one vector is the zero vector.

$$\vec{0}=0\vec{v}_1+0\vec{v}_2+\cdots+0\vec{v}_k$$

- The sequence of vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is ***linearly independent*** if the only combination that gives the zero vector is $0\vec{v}_1+0\vec{v}_2+\cdots+0\vec{v}_k$. Thus, linear independence means that:

$$x_1\vec{v}_1+x_2\vec{v}_2+\cdots+x_k\vec{v}_k=0 \text{ only happens when all } x\text{'s are zero.}$$

- A (nonempty) set of vectors is ***linearly independent*** if it is not linearly dependent.
- If three vectors $\vec{w}_1, \vec{w}_2, \vec{w}_3$ are in the same plane, they are dependent.
- The empty set is linearly independent, for linearly dependent sets must be nonempty.
- A set consisting of a single nonzero vector is linearly independent. For if $\{\vec{v}\}$ is linearly dependent, then $a\vec{v}=\vec{0}$ for some nonzero scalar a. Thus,

$$\vec{v}=a^{-1}(a\vec{v})=a^{-1}\vec{0}=\vec{0}$$

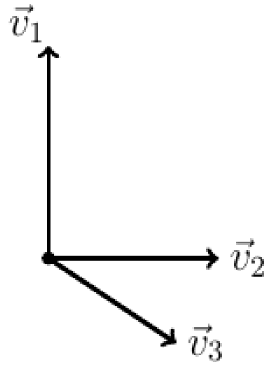
Theorem

A set S with two or more vectors $S = \{\vec{v}_1, \cdots, \vec{v}_k\}$ is

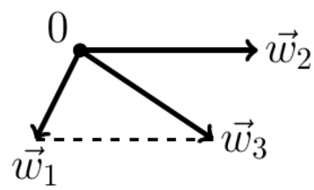
a) Linearly dependent *iff* at least one of the vectors in S is expressible as a linear combination of the other vectors in S. There are numbers $c_1, \ldots, c_k$ at least one of which is nonzero, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = 0$$

b) Linearly independent *iff* no vector in S is expressible as a linear combination of the other vectors in S.



Independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$



Dependent vectors $\vec{w}_1, \vec{w}_2, \vec{w}_3$

The combination $\vec{w}_1 - \vec{w}_2 + \vec{w}_3$ is (0, 0, 0)

Example

a) The vectors (1, 0) and (0, 1) are ***independent***.

b) The vectors (1, 1) and (1, 0.0001) are ***independent***.

c) The vectors (1, 1) and (2, 2) are ***dependent***.

d) The vectors (1, 1) and (0, 0) are ***dependent***.

Theorem

a) A finite set that contains $\vec{0}$ is linearly dependent.

b) A set with exactly one vector is linearly independent if and only if that vector is not $\vec{0}$.

c) A set with exactly two vectors is linearly independent *iff* neither vector is a scalar multiple of the other.

Theorem

Let S be a set ***k*** vectors in $\mathbb{R}^n$, then if $k > n$, S is ***linearly dependent***.

Example

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ are 3 vectors in $\mathbb{R}^2$ $\Rightarrow$ *Linearly dependent*.

Example

Determine whether the vectors $\vec{v}_1 = (1,\ -2,\ 3)$ $\vec{v}_2 = (5,\ 6,\ -1)$ $\vec{v}_3 = (3,\ 2,\ 1)$ are linearly dependent or linearly independent in $\mathbb{R}^3$

Solution

$$k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \mathbf{0}$$

$$k_1(1,\ -2,\ 3) + k_2(5,\ 6,\ -1) + k_3(3,\ 2,\ 1) = (0,\ 0,\ 0)$$

$$\rightarrow \begin{cases} k_1 + 5k_2 + 3k_3 = 0 \\ -2k_1 + 6k_2 + 2k_3 = 0 \\ 3k_1 - k_2 + k_3 = 0 \end{cases}$$

$$\begin{bmatrix} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{bmatrix} \quad \begin{matrix} R_2 + 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 5 & 3 & 0 \\ 0 & 16 & 8 & 0 \\ 0 & -16 & -8 & 0 \end{bmatrix} \quad R_3 + R_2$$

$$\begin{bmatrix} 1 & 5 & 3 & 0 \\ 0 & 16 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \frac{1}{16}R_2$$

$$\begin{bmatrix} 1 & 5 & 3 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 - 5R_2$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} k_1 + \frac{1}{2}k_3 = 0 \\ k_2 + \frac{1}{2}k_3 = 0 \end{matrix}$$

Solve the system equations: $k_1 = -\frac{1}{2}t, \quad k_2 = -\frac{1}{2}t, \quad k_3 = t$

This shows that the system has nontrivial solutions and hence that the vectors are linearly dependent.

2nd method to determine the linearly is to compute the determinant of the coefficient matrix

$$A = \begin{pmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{pmatrix}$$

$|A| = 0$ Which has nontrivial solutions and the vectors are *linearly dependent*.

Example

Determine whether the vectors are linearly dependent or linearly independent in $\mathbb{R}^4$

$$\vec{v}_1 = (1,\ 2,\ 2,\ -1) \quad \vec{v}_2 = (4,\ 9,\ 9,\ -4) \quad \vec{v}_3 = (5,\ 8,\ 9,\ -5)$$

Solution

$$k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{0}$$

$$k_1(1,\ 2,\ 2,\ -1) + k_2(4,\ 9,\ 9,\ -4) + k_3(5,\ 8,\ 9,\ -5) = (0,\ 0,\ 0,\ 0)$$

Which yields the homogeneous linear system

$$\rightarrow \begin{cases} k_1 + 4k_2 + 5k_3 = 0 \\ 2k_1 + 9k_2 + 8k_3 = 0 \\ 2k_1 + 9k_2 + 9k_3 = 0 \\ -k_1 - 4k_2 - 5k_3 = 0 \end{cases}$$

$$\begin{pmatrix} 1 & 4 & 5 & 0 \\ 2 & 9 & 8 & 0 \\ 2 & 9 & 9 & 0 \\ -1 & -4 & -5 & 0 \end{pmatrix} \quad \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \\ R_4 + R_1 \end{matrix}$$

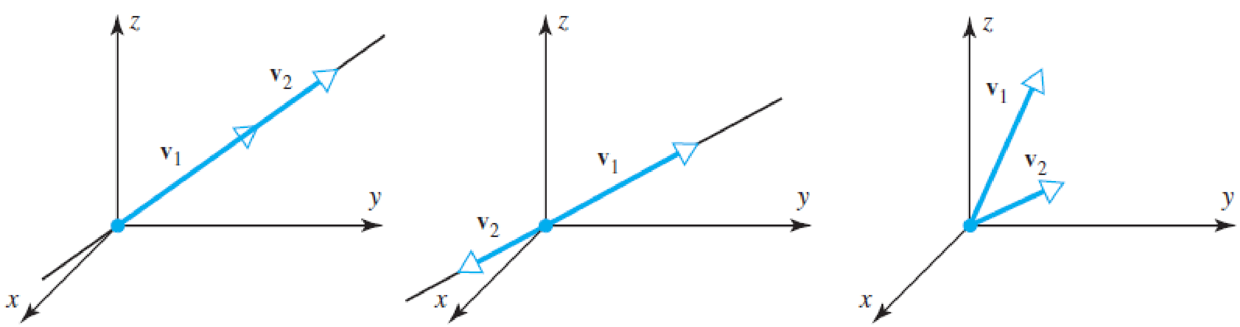
$$\begin{pmatrix} 1 & 4 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} R_1 - 4R_2 \\ \\ R_3 - R_2 \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 13 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} R_1 - 13R_3 \\ R_2 + 2R_3 \\ \\ \\ \end{array}$$

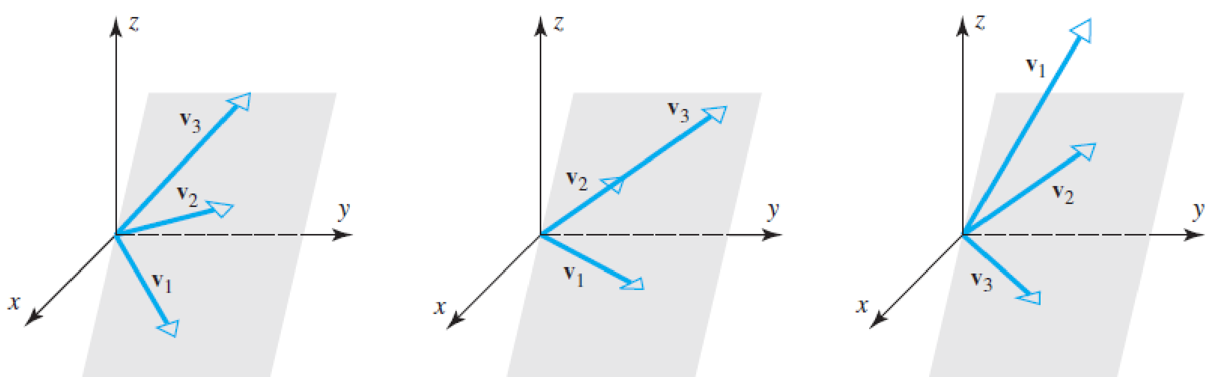
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \rightarrow\ k_1 = 0 \\ \rightarrow\ k_2 = 0 \\ \rightarrow\ k_3 = 0 \\ \\ \end{array}$$

Solve the system equations: $k_1 = 0, \quad k_2 = 0, \quad k_3 = 0$ has a trivial solution.

The vectors $\vec{v}_1, \vec{v}_2, \textit{and } \vec{v}_3$ are *linearly independent*.



(a) Linearly dependent (b) Linearly dependent (c) Linearly independent



(a) Linearly dependent (b) Linearly dependent (c) Linearly independent

Linear independence of Functions

Definition

If $\mathbf{f}_1 = f_1(x),\ \mathbf{f}_2 = f_2(x),\ \ldots,\ \mathbf{f}_n = f_n(x)$ are functions that are $n - 1$ times differentiable on the interval $(-\infty, \infty)$, the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the ***Wronskian*** of $f_1, f_2, \ldots, f_n$

$$\begin{cases} if\ \ W = 0 & \Rightarrow \quad Linearly\ Dependent \\ if\ \ W \neq 0 & \Rightarrow \quad Linearly\ Independent \end{cases}$$

Example

Use the Wronskian to show that $\mathbf{f}_1 = x,\quad \mathbf{f}_2 = \sin x$ are linearly independence

Solution

The Wronskian is

$$W(x) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix}$$

$$= x\cos x - \sin x \neq 0$$

This function is not identically zero. Thus, the functions are linearly independent.

Example

Use the Wronskian to show that $\mathbf{f}_1 = 1, \quad \mathbf{f}_2 = e^x, \quad \mathbf{f}_3 = e^{2x}$ are linearly independence

Solution

The Wronskian is

$$W(x) = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix}$$

$$= e^x 4e^{2x} - 2e^{2x} e^x$$

$$= 2e^{3x} \neq 0$$

Thus, the functions are linearly independent.

Theorem

Let S be a linearly independent subset of a vector space V, and let $\vec{v}$ be a vector in V that is not in S. Then $S \cup \{\vec{v}\}$ is linearly dependent if and only if $\vec{v} \in span(S)$

Proof

If $S \cup \{\vec{v}\}$ is linearly dependent, then there are vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ in $S \cup \{\vec{v}\}$ such that $a_1\vec{u}_1 + a_2\vec{u}_2 + \ldots + a_n\vec{u}_n = \vec{0}$ for some nonzero scalars $a_1, a_2, \ldots, a_n$.

Because S is linearly independent, one of the $\vec{u}_i$'s say $\vec{u}_1$, equal $\vec{v}$. Thus $a_1\vec{v} + a_2\vec{u}_2 + \ldots + a_n\vec{u}_n = \vec{0}$, and so

$$\begin{aligned} a_1\vec{v} &= -\left(a_2\vec{u}_2 + \ldots + a_n\vec{u}_n\right) \\ \vec{v} &= -a_1^{-1}\left(a_2\vec{u}_2 + \ldots + a_n\vec{u}_n\right) \\ &= -\left(a_1^{-1}a_2\right)\vec{u}_2 - \ldots - \left(a_1^{-1}a_n\right)\vec{u}_n \end{aligned}$$

Since $\vec{v}$ is linear combination of $\vec{u}_2, \ldots, \vec{u}_n$, which are in S, we have $\vec{v} \in span(S)$.

Conversely, let $\vec{v} \in span(S)$.

Then there exist vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ in S and scalars $b_1, b_2, \ldots, b_m$ such that $\vec{v} = b_1\vec{v}_1 + b_2\vec{v}_2 + \ldots + b_m\vec{v}_m$. Hence,

$$b_1\vec{v}_1 + b_2\vec{v}_2 + \ldots + b_m\vec{v}_m + (-1)\vec{v} = \vec{0}$$

Since $\vec{v} \neq \vec{v}_i$ for $i = 1, 2, \ldots, m$, the coefficient of $\vec{v}$ in this linear combination is nonzero, and so the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m, \vec{v}\}$ is linearly dependent.

Therefore $S \cup \{\vec{v}\}$ is *linearly dependent*.

Exercises *Section* 2.6 – Linear Independence

1. State the following statements as ***true*** or ***false***

a) If S is a linearly dependent set, then each vector in S is a linear combination of other vectors in S.

b) Any set containing the zero vector is linearly dependent.

c) The empty set is linearly dependent.

d) Subsets of linearly dependent sets are linearly dependent.

e) Subsets of linearly independent sets are linearly independent.

f) If $a_1x_1 + a_2x_2 + \ldots + a_nx_n = \vec{0}$ and $x_1, x_2, \ldots, x_n$ are linearly independent, the null the scalars a_i are zero

2. Given three independent vectors $\vec{w}_1, \vec{w}_2, \vec{w}_3$. Take combinations of those vectors to produce $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Write the combinations in a matrix form as $V = WM$.

$$\begin{aligned}
\vec{v}_1 &= \vec{w}_1 + \vec{w}_2 \\
\vec{v}_2 &= \vec{w}_1 + 2\vec{w}_2 + \vec{w}_3 \\
\vec{v}_1 &= \qquad \vec{w}_2 + c\vec{w}_3
\end{aligned}$$

which is $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & c \end{bmatrix}$

What is the test on a matrix **V** to see if its columns are linearly independent?

If $c \neq 1$ show that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent.

If $c = 1$ show that $\vec{v}\,'s$ are linearly *dependent*.

3. Find the largest possible number of independent vectors among

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad \vec{v}_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad \vec{v}_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad \vec{v}_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

4. Show that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent but $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are dependent:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \vec{v}_4 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

Solve either $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = 0$ *or* $A\vec{x} = 0$. The v's go in the columns of ***A***.

5. Decide the dependence or independence of

a) The vectors (1, 3, 2), (2, 1, 3), and (3, 2, 1).

b) The vectors $(1,\ -3,\ 2)$, $(2,\ 1,\ -3)$, and $(-3,\ 2,\ 1)$.

6. Find two independent vectors on the plane $x+2y-3z-t=0$ in $\mathbb{R}^4$. Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

7. Determine whether the vectors are linearly dependent or linearly independent in $\mathbb{R}^3$

a) (4, –1, 2), (–4, 10, 2)

b) (8, –1, 3), (4, 0, 1)

c) (–3, 0, 4), (5, –1, 2), (1, 1, 3)

d) (–2, 0, 1), (3, 2, 5), (6, –1, 1), (7, 0, –2)

8. Determine whether the vectors are linearly dependent or linearly independent in $\mathbb{R}^4$

a) $\{(3,\ 8,\ 7,\ -3),\ (1,\ 5,\ 3,\ -1),\ (2,\ -1,\ 2,\ 6),\ (1,\ 4,\ 0,\ 3)\}$

b) $\{(0,\ 0,\ 2,\ 2),\ (3,\ 3,\ 0,\ 0),\ (1,\ 1,\ 0,-1)\}$

c) $\{(0,\ 3,\ -3,\ -6),\ (-2,\ 0,\ 0,\ -6),\ (0,\ -4,\ -2,\ -2),\ (0,\ -8,\ 4,\ -4)\}$

d) $\{(3,\ 0,\ -3,\ 6),\ (0,\ 2,\ 3,\ 1),\ (0,\ -2,\ -2,\ 0),\ (-2,\ 1,\ 2,\ 1)\}$

e) $\{(1,\ 3,\ -4,\ 2),\ (2,\ 2,\ -4,\ 0),\ (2,\ 3,\ 2,\ -4),\ (-1,\ 0,\ 1,\ 0)\}$

f) $\{(1,\ 3,\ -4,\ 2),\ (2,\ 2,\ -4,\ 0),\ (1,\ -3,\ 2,\ -4),\ (-1,\ 0,\ 1,\ 0)\}$

g) $\{(1,\ 0,\ 0,\ -1),\ (0,\ 1,\ 0,\ -1),\ (0,\ 1,\ 0,\ -1),\ (0,\ 0,\ 0,\ 1)\}$

9. *a*) Show that the three vectors $\vec{v}_1=(1,\ 2,\ 3,\ 4)$ $\vec{v}_2=(0,\ 1,\ 0,\ -1)$ $\vec{v}_3=(1,\ 3,\ 3,\ 3)$ form a linearly dependent set in $\mathbb{R}^4$.

b) Express each vector in part (*a*) as a linear combination of the other two.

10. For which real values of λ do the following vectors form a linearly dependent set in $\mathbb{R}^3$

$$\vec{v}_1=\left(\lambda,\ -\tfrac{1}{2},\ -\tfrac{1}{2}\right)\quad \vec{v}_2=\left(-\tfrac{1}{2},\ \lambda,\ -\tfrac{1}{2}\right)\quad \vec{v}_3=\left(-\tfrac{1}{2},\ -\tfrac{1}{2},\ \lambda\right)$$

11. Show that if $S=\{\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_n\}$ is a linearly independent set of vectors, then so is every nonempty subset of S.

12. Show that if $S=\{\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_r\}$ is a linearly dependent set of vectors in a vector space *V*, and if $\vec{v}_{r+1},\ \ldots,\ \vec{v}_n$ are vectors in *V* that are not in *S*, then $\{\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_r,\ \vec{v}_{r+1},\ \ldots,\ \vec{v}_n\}$ is also linearly dependent.

13. Show that $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent and $\vec{v}_3$ does not lie in span $\{\vec{v}_1, \vec{v}_2\}$, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent.

14. By using the appropriate identities, where required, determine $F(-\infty, \infty)$ are linearly dependent.

a) $6,\ 3\sin^2 x,\ 2\cos^2 x$

b) $x,\ \cos x$

c) $1,\ \sin x,\ \sin 2x$

d) $(3-x)^2,\ x^2-6x,\ 5$

e) $\cos 2x,\ \sin^2 x,\ \cos^2 x$

15. $f_1(x)=\sin x,\quad f_2(x)=\cos x$ are linearly independent in $F(-\infty, \infty)$ because neither function is a scalar multiple of the other. Confirm the linear independence using Wrońskian's test.

16. Show $f_1(x)=e^x,\quad f_2(x)=xe^x\quad f_3(x)=x^2e^x$ are linearly independent in $F(-\infty, \infty)$.

17. Use the Wronskian to show that $f_1(x)=\sin x,\quad f_2(x)=\cos x,\quad f_3(x)=x\cos x$ span a three-dimensional subspace of $F(-\infty, \infty)$

18. Show by inspection that the vectors are linearly dependent.

$$\vec{v}_1(4,\ -1,\ 3),\ \ \vec{v}_2(2,\ 3,\ -1),\ \ \vec{v}_3(-1,\ 2,\ -1),\ \ \vec{v}_4(5,\ 2,\ 3),\ \ in\ \mathbb{R}^3$$

(**19 – 37**) Determine if the given vectors are linearly dependent or independent, (any method)

19. $(2,\ -1,\ 3),\ \ (3,\ 4,\ 1),\ \ (2,\ -3,\ 4),\ \ in\ \mathbb{R}^3$

20. $(1,\ 0,\ 0,\ 0),\ \ (1,\ 1,\ 0,\ 0),\ \ (1,\ 1,\ 1,\ 0),\ \ (1,\ 1,\ 1,\ 1),\ \ in\ \mathbb{R}^4$

21. $A_1\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix},\ \ A_2\begin{bmatrix}-2 & 4\\ 1 & 0\end{bmatrix},\ \ A_3\begin{bmatrix}3 & -1\\ 2 & 0\end{bmatrix},\ \ in\ M_{22}$

22. $\left\{\begin{pmatrix}1 & -3 & 2\\ -4 & 0 & 5\end{pmatrix}, \begin{pmatrix}-3 & 7 & 4\\ 6 & -2 & -7\end{pmatrix}, \begin{pmatrix}-2 & 3 & 11\\ -1 & -3 & 2\end{pmatrix}\right\}$ in $M_{2\times 3}(\mathbb{R})$

23. $\left\{\begin{pmatrix}1 & -3\\ -2 & 4\end{pmatrix}, \begin{pmatrix}-2 & 6\\ 4 & -8\end{pmatrix}\right\}$ in $M_{2\times 2}(\mathbb{R})$

24. $\left\{\begin{pmatrix}1 & -2\\ -1 & 4\end{pmatrix}, \begin{pmatrix}-1 & 1\\ 2 & -4\end{pmatrix}\right\}$ in $M_{2\times 2}(\mathbb{R})$

25. $\left\{\begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}, \begin{pmatrix}-1 & 2\\ 1 & 0\end{pmatrix}, \begin{pmatrix}2 & 1\\ -4 & 4\end{pmatrix}\right\}$ $in\ \ M_{2\times 2}(\mathbb{R})$

26. $\left\{\begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}, \begin{pmatrix}-1 & 2\\ 1 & 0\end{pmatrix}, \begin{pmatrix}2 & 1\\ 2 & -2\end{pmatrix}\right\} \quad in \quad M_{2\times 2}(\mathbb{R})$

27. $\left\{e^x, \ \ln x\right\} \quad in \quad \mathbb{R}$

28. $\left\{x, \ \frac{1}{x}\right\} \quad in \quad \mathbb{R}$

29. $\left\{1+x, \ 1-x\right\} \quad in \quad \mathbb{P}_2(\mathbb{R})$

30. $\left\{9x^2-x+3, \ 3x^2-6x+5, \ -5x^2+x+1\right\} \quad in \quad \mathbb{P}_3(\mathbb{R})$

31. $\left\{-x^2, \ 1+4x^2\right\} \quad in \quad \mathbb{P}_3(\mathbb{R})$

32. $\left\{7x^2+x+2, \ 2x^2-x+3, \ -3x^2 4\right\} \quad in \quad \mathbb{P}_3(\mathbb{R})$

33. $\left\{3x^2+3x+8, \ 2x^2+x, \ 2x^2+2x+2, \ 5x^2-2x+8\right\} \quad in \quad \mathbb{P}_3(\mathbb{R})$

34. $\left\{x^3+2x^2, \ -x^2+3x+1, \ x^3-x^2+2x-1\right\} \quad in \quad \mathbb{P}_3(\mathbb{R})$

35. $\left\{x^3-x, \ 2x^2+4, \ -2x^3+3x^2+2x+6\right\} \quad in \quad \mathbb{P}_3(\mathbb{R})$

36. $\left\{\begin{matrix} x^4-x^3+5x^2-8x+6, \quad -x^4+x^3-5x^2+5x-3, \quad x^4+3x^2-3x+5, \\ 2x^4+3x^3+4x^2-x+1, \ x^3-x+2 \end{matrix}\right\} \quad in \quad \mathbb{P}_4(\mathbb{R})$

37. $\left\{\begin{matrix} x^4-x^3+5x^2-8x+6, \quad -x^4+x^3-5x^2+5x-3, \\ x^4+3x^2-3x+5, \quad 2x^4+x^3+4x^2+8x \end{matrix}\right\} \quad in \quad \mathbb{P}_4(\mathbb{R})$

38. Suppose that the vectors $\vec{u}_1, \vec{u}_2, \textit{and } \vec{u}_3$ are linearly dependent. Are the vectors $\vec{v}_1 = \vec{u}_1 + \vec{u}_2$, $\vec{v}_2 = \vec{u}_1 + \vec{u}_3$, and $\vec{v}_3 = \vec{u}_2 + \vec{u}_3$ also linearly dependent?

(***Hint***: Assume that $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = 0$, and see what the $a_i\,'s$ can be.)

39. Show that the set $F = \left\{1+t, \ t^2, \ t-2\right\}$ is a linearly independent subset of $\mathbb{P}_2$

40. Suppose that *A* is linearly dependent set of vectors and *B* is any set containing *A*. Show that *B* must be linearly dependent.

41. Show that $\{\sin t, \ \sin 2t, \ \cos t\}$ is a linearly independent, subset of $C[0, 1]$. Does it span $C[0, 1]$

42. Show that the set $\{\sin(t+a),\ \sin(t+b),\ \sin(t+c)\}$ is linearly dependent on $C[0,\ 1]$

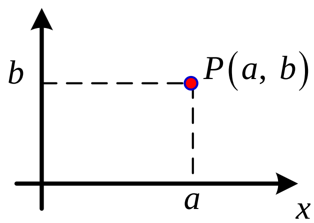
43. Show that if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent and $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta$ are linearly dependent, then β can be uniquely expressed as a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$.

44. Show that if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly dependent with $(\alpha_1 \neq 0)$ if and only if there exists an integer k $(1 < k \leq n)$, such that α_k is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$

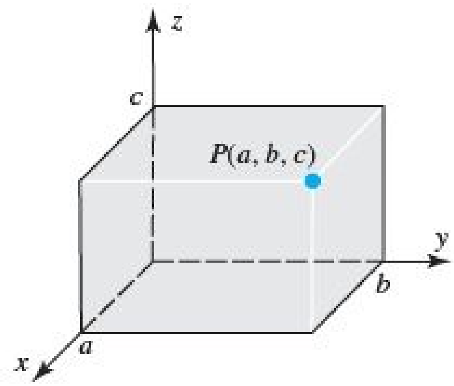
Section 2.7 – Coordinates, Basis and Dimension

Coordinate Systems in Linear Algebra

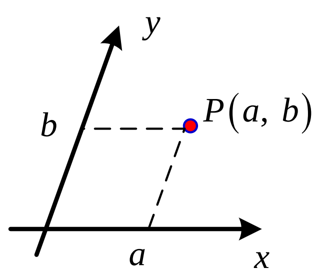
In ***analytic geometry***, we use rectangular coordinate systems to create a point either in 2-space or 3-space



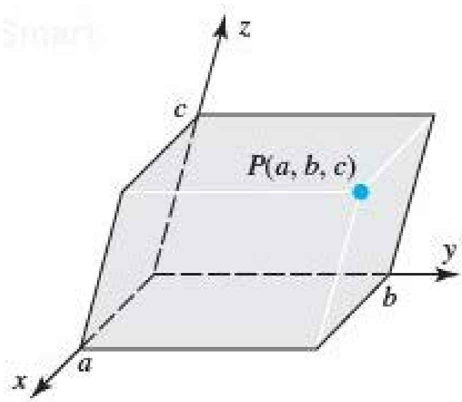
Coordinates of P in a rectangular coordinate system in 2-space



Coordinates of P in a rectangular coordinate system in 3-space

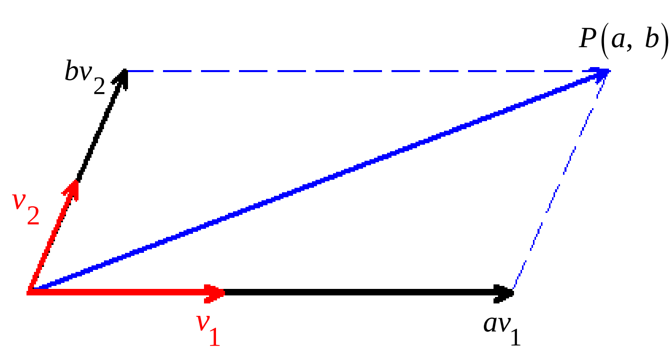


Coordinates of P in a nonrectangular coordinate system in 2-space



Coordinates of P in a nonrectangular coordinate system in 3-space

In ***linear algebra*** coordinate systems are commonly specified using vectors rather than coordinate axes.



Basis

Definition

If V is any vector space and $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a finite set of vectors in V, then S is called a ***basis*** for V if the following two conditions hold:

1. S is linearly independent.
2. S spans V.

Example

The columns of $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ produce the "standard basis" for $\mathbb{R}^2$.

Solution

The basis vectors: $\hat{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $\hat{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are independent. They span $\mathbb{R}^2$.

Example

The columns of any invertible n by n matrix give a basis for $\mathbb{R}^n$.

$$\underbrace{A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}}_{Basis} \quad \underbrace{A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{Basis} \quad \underbrace{A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}}_{Not\ basis}$$

Example

The standard unit vectors $\hat{e}_1 = (1, 0, 0, \cdots, 0)$, $\hat{e}_2 = (0, 1, 0, \cdots, 0)$, $\ldots$, $\hat{e}_n = (0, 0, \cdots, 0, 1)$ form a basis in $\mathbb{R}^n$.

Solution

1. $k_1\hat{e}_1 + k_2\hat{e}_2 + \ldots + k_n\hat{e}_n = 0 \rightarrow (k_1, k_2, \ldots, k_n) = (0, 0, \ldots, 0)$ it follows that $k_1 = k_2 = \ldots = k_n = 0$. That implies they are linearly independent.

2. Every vector $\vec{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ can be expressed as $\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + \ldots + v_n\hat{e}_n$ which is linear combination of $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n$. Thus, the standard vector span $\mathbb{R}^n$

Thus, they form a basis for $\mathbb{R}^n$ that we call the ***standard basis*** for $\mathbb{R}^n$.

Example

Show that the vectors $\vec{v}_1=(1,\ 2,\ 1),\quad \vec{v}_2=(2,\ 9,\ 0),\quad and\quad \vec{v}_3=(3,\ 3,\ 4)$ form a basis in $\mathbb{R}^3$

Solution

1. We need to show that the vectors are linearly independent.

   $k_1\vec{v}_1+k_2\vec{v}_2+k_3\vec{v}_3=\vec{0}$

   $k_1(1,\ 2,\ 1)+k_2(2,\ 9,\ 0)+k_3(3,\ 3,\ 4)=(0,\ 0,\ 0)$

   Which yields the homogeneous linear system

   $$\rightarrow\begin{cases} k_1+2k_2+3k_3=0 \\ 2k_1+9k_2+3k_3=0 \\ k_1\qquad\ \ +4k_3=0 \end{cases}$$

   $$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix}=\underline{-1\neq 0}$$

   $k_1=0,\quad k_2=0,\quad k_3=0$ has a trivial solution.

   The vectors $\vec{v}_1,\ \vec{v}_2,\ and\ \vec{v}_3$ are linearly independent.

2. Every vector can be expressed as $k_1\vec{v}_1+k_2\vec{v}_2+k_3\vec{v}_3=\vec{b}$ which is linear combination. Thus, the standard vector span $\mathbb{R}^3$

That proves that the vectors $\vec{v}_1,\ \vec{v}_2,\ and\ \vec{v}_3$ form a basis in $\mathbb{R}^3$

- The vectors $\vec{v}_1,\ \cdots,\ \vec{v}_n$ are a basis for $\mathbb{R}^n$ exactly when they are the ***columns*** of an ***n* by *n* invertible matrix**. Thus $\mathbb{R}^n$ has infinitely many different bases.

- ***The pivots columns of A are a basis for its column space***. The pivot rows of A are a basis for its row space. So are the pivot rows of its echelon form R.

Example

Find bases for the column and row spaces of a rank two matrices: $R=\begin{bmatrix}1&2&0&3\\0&0&1&4\\0&0&0&0\end{bmatrix}$

Solution

Columns 1 and 3 are the pivot columns. They are a basis for the column space. It is a subspace of $\mathbb{R}^3$.

Column 2 and 4 are a basis for the same column space.

Coordinates Relative to a Basis

Theorem – Uniqueness of Basis Representation

If $S=\left\{\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_n\right\}$ is a basis for a vector space *V*, then every vector $\vec{v}$ in *V* can be expressed in the form $\vec{v}=c_1\vec{v}_1+\ c_2\vec{v}_2+\ \ldots+\ c_n\vec{v}_n$ in exactly one way.

Proof

Suppose that some vector $\vec{v}$ can be written as

$$\vec{v}=c_1\vec{v}_1+\ c_2\vec{v}_2+\ \ldots+\ c_n\vec{v}_n$$

Also $\qquad \vec{v}=k_1\vec{v}_1+\ k_2\vec{v}_2+\ \ldots+\ k_n\vec{v}_n$

Subtracting the second from the first equation

$$0=\left(c_1-k_1\right)\vec{v}_1+\ \left(c_2-k_2\right)\vec{v}_2+\ \ldots+\ \left(c_n-k_n\right)\vec{v}_n$$

Since the right side of this equation is a linear combination of vectors in *S*, the linear independence of *S* implies that

$$c_1-k_1=0,\quad c_2-k_2=0,\ \ldots,\quad c_n-k_n=0$$

That implies $\qquad c_1=k_1,\quad c_2=k_2,\ \ldots,\quad c_n=k_n$

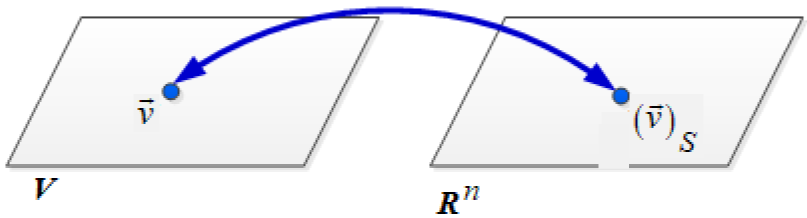
Thus, the two expressions for $\vec{v}$ are the same.

Definition

If $S=\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a basis for a vector space V, and $\vec{v}=c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$ is the expression for a vector $\vec{v}$ in terms of the basis **S**, then the scalars $c_1, c_2, \dots, c_n$ are called coordinates of $\vec{v}$ relative to the basis S. The vector $(c_1, c_2, \dots, c_n)$ in $\mathbb{R}^n$ constructed from these coordinates is called ***coordinate vector of*** $\vec{v}$ ***relative to S***; it is denoted by

$$(\vec{v})_S=(c_1, c_2, \dots, c_n)$$

A *one-to-one correspondence*



Example

a) Given the vectors $\vec{v}_1=(1,\ 2,\ 1)$, $\vec{v}_2=(2,\ 9,\ 0)$, *and* $\vec{v}_3=(3,\ 3,\ 4)$ form a basis for $\mathbb{R}^3$. Find the coordinate vector of $\vec{v}=(5,\ -1,\ 9)$ relative to the basis $S=\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

b) Find the coordinate vector of $\vec{v}$ in $\mathbb{R}^3$ whose coordinate relative to S is $(\vec{v})_S=(-1,3,2)$.

Solution

a) To find $(\vec{v})_S$ we must first express $\vec{v}$ as a linear combination of the vectors in S;

$$\vec{v}=c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$$

$$(5,\ -1,\ 9)=c_1(1,\ 2,\ 1)+ c_2(2,\ 9,\ 0)+ c_3(3,\ 3,\ 4)$$

Which gives: $$\begin{cases} c_1+2c_2+3c_3=5 \\ 2c_1+9c_2+3c_3=-1 \\ c_1 \qquad\quad +4c_3=9 \end{cases}$$

Solving this system, we obtain $c_1=1,\quad c_2=-1,\quad c_3=2$.

Therefore $(\vec{v})_S=(1,\ -1,\ 2)$

b) $\vec{v}=(-1)\vec{v}_1+3\vec{v}_2+ 2\vec{v}_3$

$$=(-1)(1,\ 2,\ 1)+3(2,\ 9,\ 0)+ 2(3,\ 3,\ 4)$$

$$\underline{=(11,\ 31,\ 7)}$$

Dimension

If $\vec{v}_1, \ldots, \vec{v}_m$ and $\vec{w}_1, \ldots, \vec{w}_n$ are both bases for the same vector space, then $m = n$.

Note

V may have many different bases, but they all must have the same number of elements.

Proof

Let **S** and **W** be bases of **V** can be written as a linear combination of vectors in **S**.

$$\vec{w}_1 = a_{11}\vec{s}_1 + \ldots + a_{1m}s_m$$
$$\vec{w}_2 = a_{21}\vec{s}_1 + \ldots + a_{2m}\vec{s}_m$$
$$\vdots$$
$$\vec{w}_n = a_{n1}\vec{s}_1 + \ldots + a_{nm}\vec{s}_m$$

But since **V** is a basis, $c_1\vec{w}_1 + \ \ldots + c_n\vec{w}_n = 0 \iff c_i = 0$ (to be linearly independent, otherwise to be linearly dependent with at least 1 of t $c_i \neq 0$)

$$c_1\left(a_{11}s_1 + \ldots + a_{1m}s_m\right) + c_2\left(a_{21}s_1 + \ldots + a_{2m}s_m\right) + \ldots + c_n\left(a_{n1}s_1 + \ldots + a_{nm}s_m\right) = 0$$

$$\left(c_1a_{11} + c_2a_{21} + \ldots + c_na_{n1}\right)s_1 + \ldots + \left(c_1a_{1m} + c_2a_{2m} + \ldots + c_na_{nm}\right)s_m = 0$$

$$c_1a_{11} + c_2a_{21} + \ldots + c_na_{n1} = 0$$
$$\iff$$
$$c_1a_{1m} + c_2a_{2m} + \ldots + c_na_{nm} = 0$$

$\therefore\ S_i$'s linear independent.

Now all bases of **V** have some number of elements, we can define the dimension (is # of vectors in a basis)

Definition

The dimension of a finite-dimensional vector space V is denoted by $\dim(V)$ and is defined to be the number of vectors in a basis for V. in addition, the zero-vector space is defined to have dimension zero.

1. Dim(**V**) = # elements in basis. If **V** is finite.
2. If $V = \{\vec{0}\}$, then Dim(**V**) = 0, even though there is no basis.
3. Dim(**V**) may be infinite.

- $\dim\left(\mathbb{R}^n\right) = n$ **The standard basis has *n* vectors**.
- $\dim\left(P_n\right) = n+1$ **The standard basis has *n* + 1 vectors**.
- $\dim\left(M_{mn}\right) = mn$ **The standard basis has *mn* vectors**.

Bases for Matrix Spaces *and Function Spaces*

Independence, basis, and dimension are not all restricted to column vectors.

- The dimension of the whole n by n space is n^2
- The dimension of the subspace of upper triangular matrices is $\frac{1}{2}n^2 + \frac{1}{2}n$
- The dimension of the subspace of diagonal matrices is n
- The dimension of the subspace of symmetric matrices is $\frac{1}{2}n^2 + \frac{1}{2}n$

Function Spaces

The equations:

$y'' = 0$ is solved by any linear function $y = cx + d$

$y'' = -y$ is solved by any combination $y = c\sin x + d\cos x$

$y'' = y$ is solved by any combination $y = ce^x + de^{-x}$

Example

Find a basis for and the dimension of the solution space of the homogeneous system

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \qquad + 2x_5 \qquad &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 - 10x_4 \qquad + 15x_6 &= 0 \\ 2x_1 + 6x_2 \qquad + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

Solution

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & -10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{pmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ \\ R_4 - 2R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & 0 \\ 0 & 0 & 5 & -10 & 0 & 15 & 0 \\ 0 & 0 & 4 & 8 & 0 & 18 & 0 \end{pmatrix} \begin{matrix} \\ -R_2 \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 5 & -10 & 0 & 15 & 0 \\ 0 & 0 & 4 & 8 & 0 & 18 & 0 \end{pmatrix} \begin{matrix} R_1 + 2R_2 \\ \\ R_3 - 5R_2 \\ R_4 - 4R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 6 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & -20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \end{pmatrix} \begin{matrix} \\ \\ -\frac{1}{20}R_3 \\ \frac{1}{6}R_4 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 6 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{matrix} R_1 - 4R_3 \\ R_2 - 2R_3 \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 0 & 2 & 6 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{matrix} R_1 - 6R_4 \\ R_2 - 3R_4 \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad x_1 = -3x_2 - 2x_5$$

The solution $(x_1, x_2, x_3, x_4, x_5, x_6) = (-3x_2 - 2x_5, x_2, 0, 0, x_5, 0)$

$$= x_2(-3, 1, 0, 0, 0, 0) + x_5(-2, 0, 0, 0, 1, 0)$$

The solution space has dimension 2.

Plus/Minus Theorem

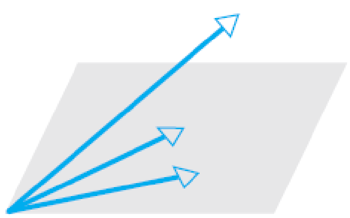
Theorem

Let S be a nonempty set of vector space V.

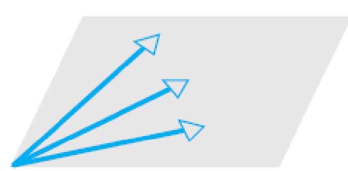
a) If S is a linearly independent set, and if $\vec{v}$ is a vector in V that is outside of span(S), the set $S \cup \{\vec{v}\}$ that results by inserting $\vec{v}$ into S is still linearly independent.

b) If $\vec{v}$ is a vector in S that is expressible as a linear combination of other vectors in S, and if $S - \{\vec{v}\}$ denotes the set obtained by removing $\vec{v}$ from S, then S and $S - \{\vec{v}\}$ span the same space; that is,

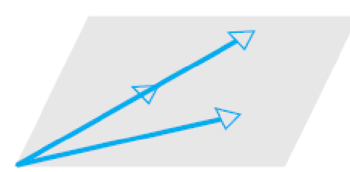
$$span(S) = span(S - \{\mathbf{v}\})$$



The vector outside the plane can be adjoined to the other two without affecting their linear independence



Any of the vectors can be removed, and the remaining two still span the plane



Either of the collinear vectors can be removed, and the remaining two will still span the plane

Theorem

If W is a subspace of a finite-dimensional vector space V, then

a) W is finite-dimensional

b) $\dim(W) \le \dim(V)$

c) $W = V$ if and only if $\dim(W) = \dim(V)$

Exercises   *Section* 2.7 – Coordinates, Basis and Dimension

1. Suppose $\vec{v}_1, \ldots, \vec{v}_n$ is a basis for $\mathbb{R}^n$ and the n by n matrix A is invertible. Show that $A\vec{v}_1, \ldots, A\vec{v}_n$ is also a basis for $\mathbb{R}^n$.

2. Consider the matrix $A = \begin{pmatrix} 1 & 0 & a \\ 2 & -1 & b \\ 1 & 1 & c \\ -2 & 1 & d \end{pmatrix}$

a) Which vectors $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ will make the columns of ***A*** linearly dependent?

b) Which vectors $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ will make the columns of ***A*** a basis for $\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : y + w = 0 \right\}$?

c) For $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \\ 2 \end{pmatrix}$, compute a basis for the four subspaces.

3. Find a basis for $x - 2y + 3z = 0$ in $\mathbb{R}^3$.

Find a basis for the intersection of that plane with *xy* plane. Then find a basis for all vectors perpendicular to the plane.

4. **U** comes from ***A*** by subtracting row 1 from row 3:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \quad \textit{and} \quad U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Find the bases for the two column spaces. Find the bases for the two row spaces. Find bases for the two nullspaces.

5. Write a 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly independent. (Assume a combination gives $c_1P_1 + \ldots + c_5P_5 = 0$, and check entries to prove c_i is zero.) The five permutation matrices are a basis for the subspace of 3 by 3 matrices with row and column sums all equal.

6. Choose three independent columns of $A = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 12 & 15 & 2 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix}$. Then choose a different three independent columns. Explain whether either of these choices forms a basis for $C(A)$.

7. Which of the following sets of vectors are bases for $\mathbb{R}^2$?

a) $\{(2,\ 1),\ (3,\ 0)\}$ *b)* $\{(0,\ 0),\ (1,\ 3)\}$

8. Which of the following sets of vectors are bases for $\mathbb{R}^3$?

a) $\{(1,\ 0,\ 0),\ (2,\ 2,\ 0),\ (3,\ 3,\ 3)\}$ *c)* $\{(2,\ -3,\ 1),\ (4,\ 1,\ 1),\ (0,\ -7,\ 1)\}$

b) $\{(3,\ 1,\ -4),\ (2,\ 5,\ 6),\ (1,\ 4,\ 8)\}$ *d)* $\{(1,\ 6,\ 4),\ (2,\ 4, -1),\ (-1,\ 2,\ 5)\}$

9. Let *V* be the space spanned by $\vec{v}_1 = \cos^2 x,\quad \vec{v}_2 = \sin^2 x,\quad \vec{v}_3 = \cos 2x$

a) Show that $S = \{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\}$ is not a basis for *V*.

b) Find a basis for *V*.

10. Find the coordinate vector of $\vec{w}$ relative to the basis $S = \{\vec{u}_1,\ \vec{u}_2\}$ for $\mathbb{R}^2$

a) $\vec{u}_1 = (1,\ 0),\quad \vec{u}_2 = (0,\ 1),\quad \vec{w} = (3,\ -7)$

b) $\vec{u}_1 = (2,\ -4),\quad \vec{u}_2 = (3,\ 8),\quad \vec{w} = (1,\ 1)$

c) $\vec{u}_1 = (1,\ 1),\quad \vec{u}_2 = (0,\ 2),\quad \vec{w} = (a,\ b)$

d) $\vec{u}_1 = (1,\ -1),\quad \vec{u}_2 = (1,\ 1),\quad \vec{w} = (0,\ 1)$

e) $\vec{u}_1 = (1,\ -1),\quad \vec{u}_2 = (1,\ 1),\quad \vec{w} = (1,\ 1)$

11. Find the coordinate vector of $\vec{v}$ relative to the basis $S = \{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\}$

a) $\vec{v} = (2,\ -1,\ 3),\quad \vec{v}_1 = (1,\ 0,\ 0),\quad \vec{v}_2 = (2,\ 2,\ 0),\quad \vec{v}_3 = (3,\ 3,\ 3)$

b) $\vec{v} = (5,\ -12,\ 3),\quad \vec{v}_1 = (1,\ 2,\ 3),\quad \vec{v}_2 = (-4,\ 5,\ 6),\quad \vec{v}_3 = (7,\ -8,\ 9)$

12. Show that $\{A_1, A_2, A_3, A_4\}$ is a basis for M_{22}, and express A as a linear combination of the basis vectors

a) $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $A_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $A = \begin{pmatrix} 6 & 2 \\ 5 & 3 \end{pmatrix}$

b) $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ $A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

c) $A = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$, $A_1 = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

13. Consider the eight vectors

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

a) List all of the one-element. Linearly dependent sets formed from these.
b) What are the two-element, linearly dependent sets?
c) Find a three-element set spanning a subspace of dimension three, and dimension of two? One? Zero?
d) Which four-element sets are linearly dependent? Explain why.

(**14 – 18**) Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space

14. $\begin{cases} x_1 + x_2 - x_3 = 0 \\ -2x_1 - x_2 + 2x_3 = 0 \\ -x_1 \qquad + x_3 = 0 \end{cases}$

15. $\begin{cases} 3x_1 + x_2 + x_3 + x_4 = 0 \\ 5x_1 - x_2 + x_3 - x_4 = 0 \end{cases}$

16. $\begin{cases} x_1 - 3x_2 + x_3 = 0 \\ 2x_1 - 6x_2 + 2x_3 = 0 \\ 3x_1 - 9x_2 + 3x_3 = 0 \end{cases}$

17. $\begin{cases} x + y + z = 0 \\ 3x + 2y - 2z = 0 \\ 4x + 3y - z = 0 \\ 6x + 5y + z = 0 \end{cases}$

18. $\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 \qquad + 5x_3 = 0 \\ \qquad x_2 + x_3 = 0 \end{cases}$

19. If $AS = SA$ for the shift matrix S. Show that $\boldsymbol{A}$ must have this special form:

$$If \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$then \ A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$$

"The subspace of matrices that commute with the shift S has dimension ______."

20. Find bases for the following subspaces of $\mathbb{R}^3$

a) All vectors of the form $(a, b, c, 0)$

b) All vectors of the form (a, b, c, d), where $d = a + b$ and $c = a - b$.

c) All vectors of the form (a, b, c, d), where $a = b = c = d$.

21. Find a basis for the null space of A.

a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

22. Find a basis for the subspace of $\mathbb{R}^4$ spanned by the given vectors

a) $(1, 1, -4, -3), \ (2, 0, 2, -2), \ (2, -1, 3, 2)$

b) $(-1, 1, -2, 0), \ (3, 3, 6, 0), \ (9, 0, 0, 3)$

23. Determine whether the given vectors form a basis for the given vector space

a) $\vec{v}_1(3, -2, 1), \ \vec{v}_2(2, 3, 1), \ \vec{v}_3(2, 1, -3), \ in \ \mathbb{R}^3$

b) $\vec{v}_1 = (1, 1, 0, 0), \ \vec{v}_2 = (0, 1, 1, 0), \ \vec{v}_3 = (0, 0, 1, 1), \ \vec{v}_4 = (1, 0, 0, 1), \quad for \ \mathbb{R}^4$

c) $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ M_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \ M_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \ M_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad M_{22}$

24. Find a basis for, and the dimension of, the null space of the given matrix $\begin{bmatrix} 2 & 1 & -1 & 1 \\ 4 & -2 & -2 & 1 \end{bmatrix}$

25. Let $\mathbb{R}$ be the set of all real numbers and let $\mathbb{R}^+$ be the set of all positive real numbers. Show that $\mathbb{R}^+$ is a vector space over $\mathbb{R}$ under the addition

$$\alpha \oplus \beta = \alpha\beta \quad \alpha,\ \beta \in \mathbb{R}^+$$

And the scalar multiplication

$$a \odot \alpha = \alpha^a \quad \alpha \in \mathbb{R}^+,\ a \in \mathbb{R}$$

Find the dimension of the vector space. Is $\mathbb{R}^+$ also a vector space if the scalar multiplication is instead defined as

$$a \otimes \alpha = a^\alpha \quad \alpha \in \mathbb{R}^+,\ a \in \mathbb{R}?$$

Section 2.8 – Row and Column Spaces

Definition

For an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

The vectors

$$\begin{aligned} \vec{v}_1 &= \left(a_{11}, \ a_{12}, \ \ldots, \ a_{1n}\right) \\ \vec{v}_2 &= \left(a_{21}, \ a_{22}, \ \ldots, \ a_{2n}\right) \\ \vdots & \qquad\qquad \vdots \\ \vec{v}_{\text{m}} &= \left(a_{m1}, \ a_{m2}, \ \ldots, \ a_{mn}\right) \end{aligned}$$

In $\mathbb{R}^n$ that are formed from the rows of *A* are called the ***row vectors*** of *A*,

The vectors

$$\vec{v}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \cdots \quad \vec{v}_3 = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

In $\mathbb{R}^m$ that are formed from the columns of *A* are called the ***column vectors*** of *A*.

Definition

If *A* is $m \times n$ matrix, then the subspace of $\mathbb{R}^n$ spanned by the row vectors of *A* is called the ***row space*** of *A* and is denoted by $RS(A)$ *or* $R(A)$, and the subspace $\mathbb{R}^m$ spanned by the row vectors of *A* is called the ***column space*** of *A* and is denoted by $CS(A)$ *or* $C(A)$. The solution space of the homogeneous system of equations $Ax = 0$, which is a subspace of $\mathbb{R}^n$, is called the null space of *A*.

The *Column Space* of *A*

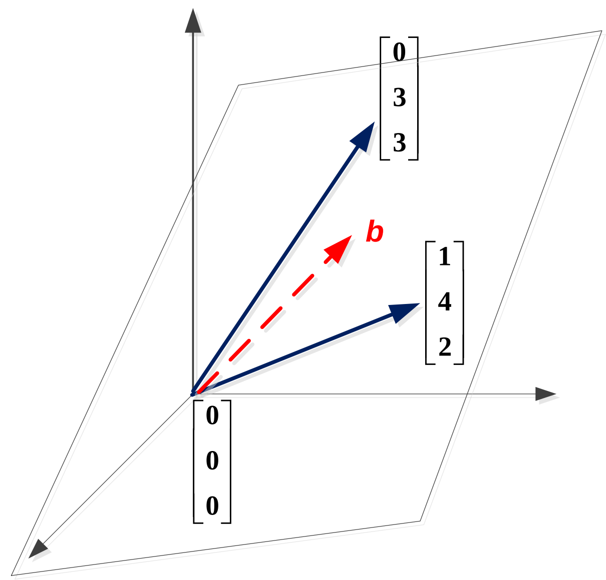
The most important subspaces are tied directly to a matrix A, to solve $A\vec{x} = b$.

Definition

The column space consists of all linear combinations of the columns. The combination are all possible vectors Ax. They fill the column space $C(A)$.

$$A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$$

$$b = x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$



To solve $A\vec{x} = \vec{b}$ is to express b as a combination of the columns.

The column space $CS(A)$ is a plane that containing the two columns. $A\vec{x} = b$ is solvable when b in on that plane.

Theorem

The system $A\vec{x} = b$ is solvable if and only if b is in the column space of A.

Example

Let $A\vec{x} = b$ be the linear system

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Show that ***b*** is in the column space of A by expressing it as a linear combination of the column vectors of A.

<u>Solution</u>

$$\left[\begin{array}{rrr|r} -1 & 3 & 2 & 1 \\ 1 & 2 & -3 & -9 \\ 2 & 1 & -2 & -3 \end{array}\right] \quad \begin{array}{l} \\ R_2 + R_1 \\ R_3 + 2R_1 \end{array}$$

$$\left[\begin{array}{rrr|r} -1 & 3 & 2 & 1 \\ 0 & 5 & -1 & -8 \\ 0 & 7 & 2 & -1 \end{array}\right] \quad \begin{array}{l} 5R_1 - 3R_2 \\ \\ 5R_3 - 7R_2 \end{array}$$

$$\left[\begin{array}{rrr|r} -5 & 0 & 13 & 29 \\ 0 & 5 & -1 & -8 \\ 0 & 0 & 17 & 51 \end{array}\right] \quad \begin{array}{l} 17R_1 - 13R_3 \\ 17R_2 + R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{rrr|r} -85 & 0 & 0 & -170 \\ 0 & 85 & 0 & -85 \\ 0 & 0 & 17 & 51 \end{array}\right] \quad \begin{array}{l} -\frac{1}{85}R_1 \\ \frac{1}{85}R_2 \\ \frac{1}{17}R_3 \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

That implies to $x_1 = 2, \quad x_2 = -1, \quad x_3 = 3$

It follows that

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$`

Example

Describe the column spaces (they are subspaces of $\mathbb{R}^2$) for

$$I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 4 \end{bmatrix}$$

Solution

The column space of I is the whole space $\mathbb{R}^2$. Every vector is a combination of the columns of I. In the space language $CS(I)$ *is* $\mathbb{R}^2$.

The column space of A is only a line, the second column (2, 4) is a multiple of the first column (1, 2) and (2, 4) and all other vectors $(c,\ 2c)$ along that line. The equation $A\vec{x} = \vec{b}$ is only solvable when $\vec{b}$ is on the line.

The column space $C(B)$ is all of $\mathbb{R}^2$. Every b is attainable. The vector $\vec{b} = (3,\ 4)$ is summation of column 1 and 2.

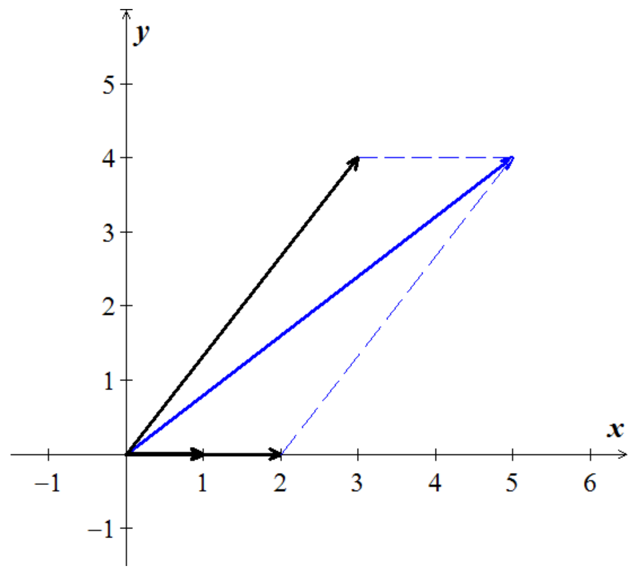
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ \qquad 4x_3 = 4 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 + 2x_2 = 2 \\ \quad x_3 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 1 \end{cases} \quad or \quad \Rightarrow \begin{cases} x_1 = 2 \\ x_2 = 0 \end{cases}$$

$$x = (0,\ 1,\ 1) \quad also \quad x = (2,\ 0,\ 1)$$



This matrix has the same column as I and any $\vec{b}$ is allowed. $\vec{x}$ has an extra component (more solutions).

Pivot Columns

The pivot columns of R have 1's in the pivots and 0's everywhere else.

$$\text{Pivot columns: } A = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix}$$

$$\text{Yields to: } R = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- ***The pivot columns are not combinations of earlier columns. The free columns are combinations of columns which are the special solutions!***

Complete **Solution** *to* $AX = B$

To solve $A\vec{x} = \vec{b}$, we need to put into an ***augmented*** form where $\vec{b}$ is not zero.

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \qquad B = \vec{b} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix}$$

$$X = \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The augmented matrix is just $\begin{bmatrix} A & B \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} R & \boldsymbol{d} \end{bmatrix}$$

Special Solutions

Each special solution to $A\vec{x}=0$ and $R\vec{x}=0$ has one free variable equal to1.

$$R\vec{x}=\begin{array}{c}\begin{bmatrix}1 & 3 & 0 & 2 & -1\\ 0 & 0 & 1 & 4 & -3\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}\\ \begin{array}{ccccc} & F & & F & F\end{array}\end{array}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

The ***free variables*** are $x_2,\ x_4,\ x_5$

$$\rightarrow\begin{cases}x_1+3x_2+2x_4-x_5=0\\ x_3+4x_4-3x_5=0\end{cases}$$

1. Set $x_2=1,\ x_4=x_5=0 \Rightarrow \begin{cases}x_1=-3\\ x_3=0\end{cases}$ **(*Column 2*)**

The special solution: $s_1=(-3,\ 1,\ 0,\ 0,\ 0)$

2. Set $x_4=1,\ x_2=x_5=0 \Rightarrow \begin{cases}x_1=-2\\ x_3=-4\end{cases}$ **(*Column 4*)**

The special solution: $s_2=(-2,\ 0,\ -4,\ 1,\ 0)$

3. Set $x_5=1,\ x_2=x_4=0 \Rightarrow \begin{cases}x_1=1\\ x_3=3\end{cases}$ **(*Column 5*)**

The special solution: $s_3=(1,\ 0,\ 3,\ 0,\ 1)$

The nullspace matrix N contains the 3 special solutions in its columns.

$$N=\begin{bmatrix}-3 & -2 & 1\\ 1 & 0 & 0\\ 0 & -4 & 3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{array}{c}\textit{not free}\\ \textit{free}\\ \textit{not free}\\ \textit{free}\\ \textit{free}\end{array}$$

The linear combinations of these three columns give all vectors in the nullspace.

One *Particular* Solution

$$A=\begin{bmatrix}1&3&0&2\\0&0&1&4\\1&3&1&6\end{bmatrix}\qquad \vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\quad \vec{b}=\begin{bmatrix}1\\6\\7\end{bmatrix}$$

$$\left[\begin{array}{cccc|c}1&3&0&2&1\\0&0&1&4&6\\0&0&0&0&0\end{array}\right]=\begin{bmatrix}R & \boldsymbol{d}\end{bmatrix}$$

The ***free variables*** for R to be $x_2=x_4$.

Then the equations give the ***pivot variables*** $x_1=1 \quad x_3=6$

The ***particular solution*** is: $\underline{(1,\ 0,\ 6,\ 0)}$

The two special (nullspace) solutions to $Rx=0$:

$$\left[\begin{array}{cccc|c}1&3&0&2&0\\0&0&1&4&0\\0&0&0&0&0\end{array}\right]\Rightarrow \begin{array}{r}x_1+3x_2+x_4=0\\x_3+4x_4=0\end{array}$$

$$\Rightarrow\begin{cases}x_1=-3x_2-x_4\\x_3=-4x_4\end{cases}$$

$x_2=1,\ x_4=0$

$$\Rightarrow\ x_1=-3,\ x_3=0\ \rightarrow\ \underline{(-3,\ 1,\ 0,\ 0)}$$

$x_2=0,\ x_4=1$

$$\Rightarrow\ x_1=-2,\ x_3=-4\ \rightarrow\ \underline{(-2,\ 0,\ -4,\ 1)}$$

The ***complete solution***:

$$x=x_p+x_n$$

$$=\begin{pmatrix}1\\0\\6\\0\end{pmatrix}+x_2\begin{pmatrix}-3\\1\\0\\0\end{pmatrix}+x_4\begin{pmatrix}-2\\0\\-4\\1\end{pmatrix}$$

Example

Find the condition on (b_1, b_2, b_3) for $A\vec{x}=\vec{b}$ to be solvable, if

$$A=\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} \qquad \vec{b}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Solution

The augmented form:

$$\begin{bmatrix} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{bmatrix} \quad \begin{matrix} \\ R_2-R_1 \\ R_3+2R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & b_1 \\ 0 & 1 & b_2-b_1 \\ 0 & -1 & b_3+2b_1 \end{bmatrix} \quad \begin{matrix} R_1-R_2 \\ \\ R_3+R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 2b_1-b_2 \\ 0 & 1 & b_2-b_1 \\ 0 & 0 & b_3+b_1+b_2 \end{bmatrix} \quad \rightarrow \quad \boxed{b_1+b_2+b_3=0}$$

The last equation is 0 = 0 provided $b_1+b_2+b_3=0$.

There are ***no*** *free variables* and ***no*** *special solutions*.

The nullspace solution: $x_n=0$

The complete solution:

$$x=x_p+x_n$$

$$=\begin{bmatrix} 2b_1-b_2 \\ b_2-b_1 \end{bmatrix}+\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $b_1+b_2+b_3\neq 0$, there is no solution to $A\vec{x}=\vec{b}$ and $\vec{x}_p$ doesn't exist.

Example

a) Find a subset of the vectors

$$\vec{v}_1 = (1,-2,0,3) \quad \vec{v}_2 = (2,-5,-3,6), \quad \vec{v}_3 = (0,1,3,0), \quad \vec{v}_4 = (2,-1,4,-7), \quad \vec{v}_5 = (5,-8,1,2)$$

That forms a basis for the space spanned by these vectors

b) Express each vector not in the basis as a linear combination of the basis vectors

Solution

a) Construct the vectors as its column vectors

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix} \end{array} \quad \begin{array}{l} \\ R_2 + 2R_1 \\ \\ R_4 - 3R_1 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & -3 & 3 & 4 & 1 \\ 0 & 0 & 0 & -13 & -13 \end{bmatrix} \quad \begin{array}{l} R_1 + 2R_2 \\ \\ R_3 - 3R_2 \\ \\ \end{array}$$

$$\begin{bmatrix} 5 & 0 & 10 & 0 & 5 \\ 0 & -5 & 5 & 0 & -5 \\ 0 & 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \frac{1}{5}R_1 \\ -\frac{1}{5}R_2 \\ -\frac{1}{5}R_3 \\ \\ \end{array}$$

$$\begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 & \vec{w}_4 & \vec{w}_5 \end{array} \end{array}$$

The leading 1's occurs in columns 1, 2, and 4.

$\left\{\vec{w}_1, \vec{w}_2, \vec{w}_4\right\}$ is a basis for the column space, and consequently $\left\{\vec{v}_1, \vec{v}_2, \vec{v}_4\right\}$

b) $\vec{w}_1 = (1,\ 0,\ 0,\ 0) \quad \vec{w}_2 = (0,\ 1,\ 0,\ 0), \quad \vec{w}_3 = (2,\ -1,\ 0,\ 0)$

$\vec{w}_4 = (0,\ 0,\ 1,\ 0), \quad \vec{w}_5 = (1,\ 1,\ 1,\ 0)$

$$\vec{w}_3 = 2\vec{w}_1 - \vec{w}_2$$

$$\vec{w}_3 = \vec{w}_1 + \vec{w}_2 + \vec{w}_4$$

We call these ***dependency equations***

The corresponding relationships are:

$$\vec{v}_3 = 2\vec{v}_1 - \vec{v}_2$$

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 + \vec{v}_4$$

Solving $Ax = 0$ by *elimination*

Matrix A is rectangular and we still use the elimination.

1. Forward elimnation from A to a triangular U.
2. Back substitution in $Ax = 0$ to find x.

Consider the matrix $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix} \quad \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 4 \end{bmatrix} \quad \begin{matrix} \\ \\ R_3 - 4R_2 \end{matrix}$$

Triangular U: $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

P: The ***pivot*** variables are x_1 *and* x_3, since columns 1 and 3 contains pivots.

F: The ***free*** variables are x_2 *and* x_4, since columns 2 and 4 have no pivots.

Special solutions to:

$$\begin{cases} x_1 + x_2 + 2x_3 + 3x_4 = 0 \\ \qquad 4x_3 + 4x_4 = 0 \end{cases} \to \begin{cases} x_1 = -x_2 - x_4 \\ x_3 = -x_4 \end{cases}$$

Complete solution: $$x = x_2 \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{Special} + x_4 \underbrace{\begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{Special} = \underbrace{\begin{pmatrix} -x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix}}_{Complete}$$

The special solution are in the nullspace $NS(A)$, and their combinations fill out the whole Nullspace.

Exercises Section 2.8 – Row and Column Spaces

1. List the row vectors and column vectors of the matrix

$$\begin{bmatrix} 2 & -1 & 0 & 1 \\ 3 & 5 & 7 & -1 \\ 1 & 4 & 2 & 7 \end{bmatrix}$$

(2 – 4) Express the product $A\vec{x}$ as a linear combination of the column vectors of A.

2. $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

3. $\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$

4. $\begin{bmatrix} -3 & 6 & 2 \\ 5 & -4 & 0 \\ 2 & 3 & -1 \\ 1 & 8 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$

(5 – 8) Determine whether $\vec{b}$ is in the column space of A, and if so, express $\vec{b}$ as a linear combination of the column vectors of A.

5. $A = \begin{bmatrix} 1 & 3 \\ 4 & -6 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$

7. $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ 3 \\ 5 \\ 7 \end{bmatrix}$

9. Suppose that $x_1 = -1, \quad x_2 = 2, \quad x_3 = 4, \quad x_4 = -3$ is a solution of a nonhomogeneous linear system $A\vec{x} = \vec{b}$ and that the solution set of the homogeneous system $A\vec{x} = \vec{0}$ is given by the formulas $\quad x_1 = -3r + 4s, \quad x_2 = r - s, \quad x_3 = r, \quad x_4 = s$

a) Find a vector form of the general solution of $A\vec{x} = \vec{0}$

b) Find a vector form of the general solution of $A\vec{x} = \vec{b}$

(10 – 13) Find the vector form of the general solution of the given linear system $A\vec{x} = \vec{b}$; then use that result to find the vector form of the general solution of $A\vec{x} = \vec{0}$.

10. $\begin{cases} x_1 - 3x_2 = 1 \\ 2x_1 - 6x_2 = 2 \end{cases}$

11. $\begin{cases} x_1 + x_2 + 2x_3 = 5 \\ x_1 \qquad + x_3 = -2 \\ 2x_1 + x_2 + 3x_3 = 3 \end{cases}$

12. $\begin{cases} x_1 + 2x_2 - 3x_3 + x_4 = 4 \\ -2x_1 + x_2 + 2x_3 + x_4 = -1 \\ -x_1 + 3x_2 - x_3 + 2x_4 = 3 \\ 4x_1 - 7x_2 \qquad - 5x_4 = -5 \end{cases}$

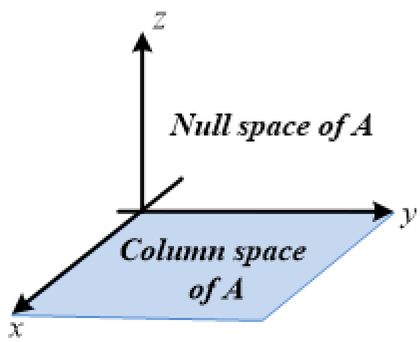
13. $\begin{cases} x_1 - 2x_2 + x_3 + 2x_4 = -1 \\ 2x_1 - 4x_2 + 2x_3 + 4x_4 = -2 \\ -x_1 + 2x_2 - x_3 - 2x_4 = 1 \\ 3x_1 - 6x_2 + 3x_3 + 6x_4 = -3 \end{cases}$

14. Given the vectors $\vec{v}_1 = (1,\ 2,\ 0)$ and $\vec{v}_2 = (2,\ 3,\ 0)$

a) Are they linearly independent?
b) Are they a basis for any space?
c) What space **V** do they span?
d) What is the dimension of that space?
e) What matrices $\boldsymbol{A}$ have **V** as their column space?
f) Which matrices have **V** as their nullspace?
g) Describe all vectors $\vec{v}_3$ that complete a basis $\vec{v}_1, \vec{v}_2, \vec{v}_3$ for $\mathbb{R}^3$.

15. *a*) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Show that relative to an *xyz*-coordinate system in 3-space the null space of *A* consists of all points on the *z*-axis and that the column space consists of all points in the *xy*-plane.



b) Find a 3 x 3 matrix whose null space is the *x*-axis and whose column space is the *yz*-plane.

16. If we add an extra column $\vec{b}$ to a matrix A, then the column space gets larger unless _____. Give an example where the column space gets larger and an example where it doesn't. Why is $A\vec{x} = \vec{b}$ solvable exactly when the column space doesn't get larger – it is the same for A and $\begin{bmatrix} A & \vec{b} \end{bmatrix}$?

17. For which right sides (find a condition on b_1, b_2, b_3) are these solvable. (Use the column space $C(A)$ and the equation $A\vec{x} = \vec{b}$)

a) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

18. Show that the matrices A and $\begin{bmatrix} A & AB \end{bmatrix}$ (with extra columns) have the same column space. But find a square matrix with $C\left(A^2\right)$ smaller than $C(A)$. Important point: An n by n matrix has $C(A) = \mathbb{R}^n$ exactly when A is an ______ matrix.

19. The column of AB are combinations of the columns of A. This means: The column space of AB is contained in (possibly equal to) to the column space of A. Give an example where the column spaces A and AB are not equal.

20. Find a square matrix A where $C\left(A^2\right)$ (the column space of A^2 is smaller than $C(A)$.

21. Suppose $A\vec{x} = \vec{b}$ and $C\vec{x} = \vec{b}$ have the same (complete) solutions for every $\vec{b}$. Is true that $A = C$?

22. Apply Gauss-Jordan elimination to $U\vec{x} = 0$ and $U\vec{x} = c$. Reach $R\vec{x} = 0$ and $R\vec{x} = d$:

$$\begin{bmatrix} U & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \qquad \begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

Solve $R\vec{x} = 0$ to find x_n (its free variable is $x_2 = 1$).

Solve $R\vec{x} = d$ to find x_p (its free variable is $x_2 = 0$).

The subspace requirements: $x+y$ ***and*** cx ***(and then all linear combinations*** $cx+dy$***) must stay in the subspace.***

23. Which of the following subsets of $\mathbb{R}^3$ are actually subspaces?

a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$

b) The plane of vectors with $b_1 = 1$.

c) The vectors with $b_1 b_2 b_3 = 0$.

d) All linear combinations of $v = (1, 4, 0)$ and $w = (2, 2, 2)$.

e) All vectors that satisfies $b_1 + b_2 + b_3 = 0$

f) All vectors with $b_1 \le b_2 \le b_3$.

24. We are given three different vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3$. Construct a matrix so that the equations $A\vec{x} = \vec{b}_1$ and $A\vec{x} = \vec{b}_2$ are solvable, but $A\vec{x} = \vec{b}_3$ is not solvable.

a) How can you decide if this possible?

b) How could you construct A?

25. For which vectors (b_1, b_2, b_3) do these systems have a solution?

a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

26. Find a basis for the null space of A. $\quad A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

27. Is it true that is $m = n$ then the row space of A equals the column space.

28. If the row space equals the column space the $A^T = A$

29. If $A^T = -A$, then the row space of A equals the column space.

30. Does the matrices A and $-A$ share the same 4 subspaces?

31. Is A and B share the same 4 subspaces then A is multiple of B.

32. Suppose $A\vec{x} = \vec{b}$ & $C\vec{x} = \vec{b}$ have the same (complete) solutions for every $\vec{b}$. Is it true that $A = C$

33. A and A^T have the same left nullspace?

Section 2.9 – Rank and the Fundamental Matrix Spaces

The **R**educed **R**ow **E**chelon **F**orm (***rref***) is a matrix (R) with each pivot column has only one nonzero entry (the pivots which is always 1).

$$R = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{R = rref(A)}$$

Rank of a Matrix

The rank of a matrix A (***m*** by ***n***) is the number of ***nonzero rows*** in the row-reduced echelon form of A (it is the number of pivot). The common dimension of the row space and column space of a matrix A is called the ***rank*** of A and is denoted by

$$\boxed{rank(A) = r}$$

Note:

The rank of a matrix is well defined due to the uniqueness of the row-reduced echelon form. No matter what sequence of elementary row operations is performed to put the given matrix in row-reduced echelon form; there will always be the same number of nonzero rows.

Theorem

The row space and column space of a matrix A have the same dimension

The objective is to connect ***rank*** and ***dimension***.

- The ***rank*** of a matrix is the number of pivots.
- The ***dimension*** of a subspace is the number of vectors in a basis.

✓ ***A has full row rank if every row has a pivot:*** $r = m$. ***No zero in R.***

✓ ***A has full column rank if every column has a pivot:*** $r = n$. ***No free variables.***

Example

Find the rank of $A=\begin{bmatrix}1&-1&2&1\\0&1&1&-2\\1&-3&0&5\end{bmatrix}$

Solution

$$\begin{bmatrix}1&-1&2&1\\0&1&1&-2\\1&-3&0&5\end{bmatrix}\quad \begin{matrix}\\ \\ R_3-R_1\end{matrix}$$

$$\begin{bmatrix}1&-1&2&1\\0&1&1&-2\\0&-2&-2&4\end{bmatrix}\quad \begin{matrix}R_1+R_2\\ \\ R_3+2R_2\end{matrix}$$

$$R=\begin{bmatrix}1&0&3&-1\\0&1&1&-2\\0&0&0&0\end{bmatrix}$$

The matrix R has 2 nonzero rows, therefore the $rank(A)=2$

Example

The columns of A are dependent. $A\vec{x}=\vec{0}$ has a nonzero solution.

$$Ax=\begin{bmatrix}1&0&3\\2&1&5\\1&0&3\end{bmatrix}\begin{bmatrix}-3\\1\\1\end{bmatrix}$$

$$-3\begin{pmatrix}1\\2\\1\end{pmatrix}+1\begin{pmatrix}0\\1\\0\end{pmatrix}+1\begin{pmatrix}3\\5\\3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

The rank of A is only $r = 2$.

Independent columns would give full column rank $r = n = 3$.

- The columns of A are independent exactly when the rank is $r = n$. There are $\boldsymbol{n}$ pivots and no free variables. Only $\vec{x}=\vec{0}$ is the nullspace.

Example

When all rows are multiplying of one pivot row, the rank is $r = 1$:

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix}, \quad \begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \quad [6]$$

Solution

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix} \quad R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix} \quad 3R_2 - 5R_1$$

$$\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \quad \tfrac{1}{3}R_2$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} \quad \begin{matrix} \tfrac{1}{5}R_1 \\ 5R_2 - 2R_1 \end{matrix} \quad \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The row-reduced echelon form $R = rref(A)$:

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad [1]$$

These matrices have only one pivot.

Dimension *Theorem* for Matrices

If A is a matrix with n columns, then $\boxed{rank(A)+nullity(A)=n}$

Theorem

If A is an $m\times n$ matrix, then

- $rank(A)$ = the number of leading variables in the general solution of $A\vec{x}=\vec{0}$
- $nullity(A)$ = the number of parameters in the general solution of $A\vec{x}=\vec{0}$

Theorem

If A is any matrix, then $rank(A)=rank\left(A^T\right)$

- $Ax=0$ ***has*** $n-r$ ***free variables and special solutions: n columns minus r pivot columns. The null matrix N has*** $n-r$ ***columns (the special solutions).***
- ***The particular solution solves:*** $A\vec{x}_p=\vec{b}$
- ***Full column rank*** $R=\begin{bmatrix} n \text{ by } n \text{ identity matrix} \\ m-n \text{ rows of zeros}\end{bmatrix}=\begin{bmatrix} I \\ 0\end{bmatrix}$

The reduced row echelon form looks like:

$$R=\begin{bmatrix} I & F \\ 0 & 0\end{bmatrix}\begin{array}{l} \boldsymbol{r} \text{ pivot rows} \\ \boldsymbol{m-r} \text{ zero rows}\end{array}$$

The pivot variables in the $n-r$ special columns come by changing F to $-F$:

Nullspace matrix: $$N=\begin{pmatrix} -F \\ I\end{pmatrix}\begin{array}{l} \boldsymbol{r} \text{ pivot variables} \\ \boldsymbol{n-r} \text{ free variables}\end{array}$$

- Every matrix A with ***full column rank*** ($r = n$) has all these properties:
  1. All columns of A are pivot columns
  2. There are no free variables or special solutions.
  3. The nullspace $NS(A)$ contains only the zero vector $\vec{x}=\vec{0}$
  4. If $A\vec{x}=\vec{b}$ has a solution (might not) then it has only one solution.

Example

Suppose A is a square invertible matrix, $m=n=r$. What are $\vec{x}_p$ and $\vec{x}_n$ ***?***

Solution

The particular solution is the one and only solution $A^{-1}\vec{b}$.

There are no special solutions or free variables. $R=I$ has no zero rows.

The only vector in the null space is $\vec{x}_n = \vec{0}$.

The complete solution is

$$\begin{aligned}\vec{x} &= \vec{x}_p + \vec{x}_n \\ &= A^{-1}\vec{b} + \vec{0} \\ &= A^{-1}\vec{b}\end{aligned}$$

Example

Compute $N(A)$ for $A:\ \mathbb{R}^2 \to \mathbb{R}^3$ given by $A=(x+y,\ x,\ 2x-y)$

Solution

To find $N(A)$, we must solve the equation $A(x,\ y)=(0,\ 0,\ 0)$

$$\begin{pmatrix} x+y \\ x \\ 2x-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} x+y=0 \Rightarrow \boxed{y=0} \\ \qquad\quad \boxed{x=0} \end{array}$$

Thus $NS(A)=\{0\}$, the set that consists solely of the zero vector.

- If $A\vec{x}=\vec{0}$ has more unknowns than equations (more columns than rows) then it has nonzero solutions. There must be free columns, without pivots.

Definition

If W is a subspace of $\mathbb{R}^n$ that are orthogonal to every vector in W is called orthogonal complement of W and is denoted nu the symbol $W^{\perp}$. $N(A)^{\perp}$ is exactly the row space $C\left(A^T\right)$

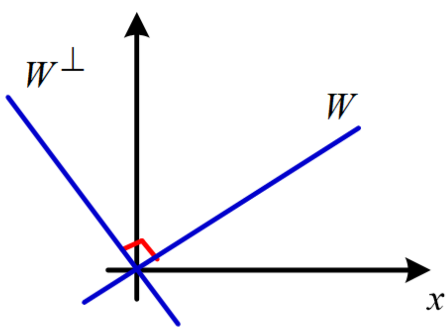
Fundamental Theorem of Linear Algebra

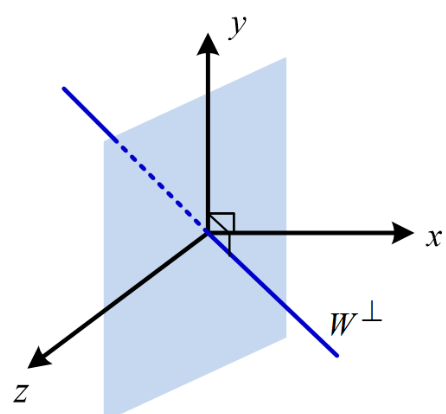
The nullspace is the orthogonal complement of the row space $\left(in\ \mathbb{R}^n\right)$.

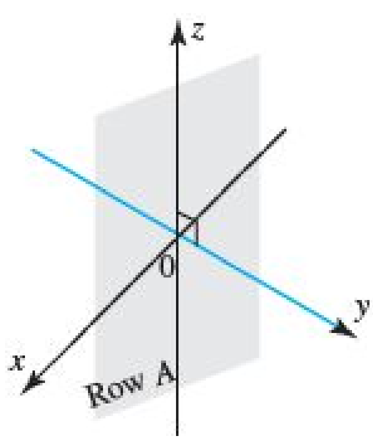
The left nullspace is the orthogonal complement of the column space $\left(in\ \mathbb{R}^m\right)$.

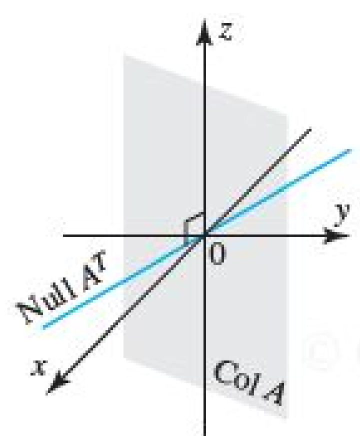
If W is a subspace of $\mathbb{R}^n$

- $W^{\perp}$ is a subspace of $\mathbb{R}^n$`
- The only vector common to W and $W^{\perp}$ is 0.
- The orthogonal complement of $W^{\perp}$ is W.









Left Nullspace

A matrix A^T has m columns and has r ranks, so the number of free columns of A^T must be $m-r$.

$$\dim N\left(A^T\right) = m - r$$

The left nullspace is the collection of vectors $\vec{y}$ for which $A^T\vec{y} = \vec{0}$. Equivalently, $\vec{y}^T A = \vec{0}$, where $\vec{y}$ and $\vec{0}$ are row vectors. We can call "***left nullspace***" because $\vec{y}^T$ is on the left of matrix A in that equation.

To find a basis for the left nullspace we reduce an augmented type of A.

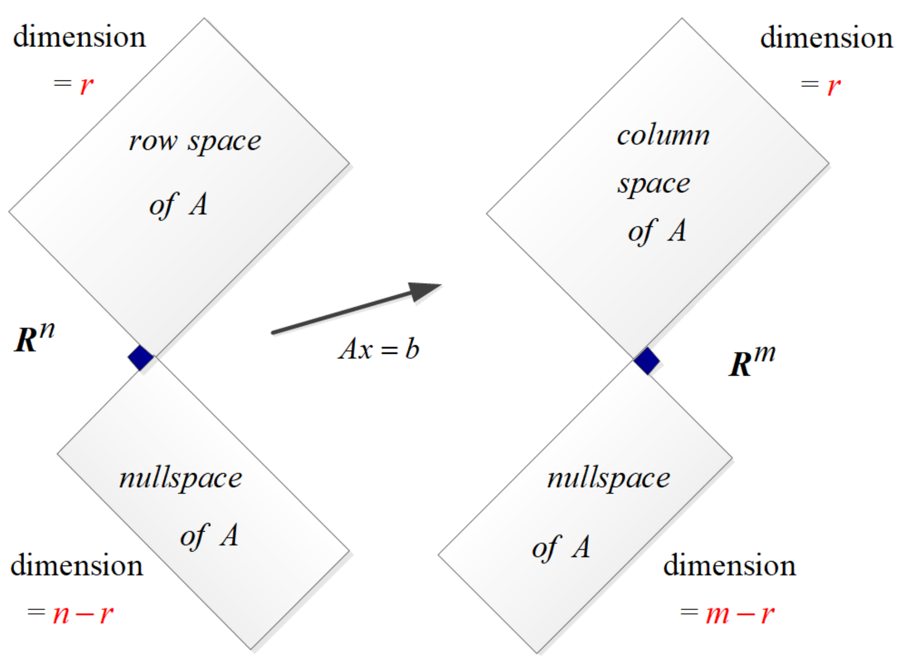
$$\left[A_{m\times n} \mid I_{m\times n}\right] \rightarrow \left[R_{m\times n} \mid E_{m\times n}\right]$$

Where matrix E can be found from $EA = R$.

If matrix A is a square matrix, then $E = A^{-1}$.

The Four Fundamental Subspaces

1. The ***row space*** is $C\left(A^T\right)$, a subspace of $\mathbb{R}^n$.
2. The ***column space*** is $C(A)$, a subspace of $\mathbb{R}^m$.
3. The ***null space*** is $N(A)$, a subspace of $\mathbb{R}^n$.
4. The ***left null space*** is $N\left(A^T\right)$, a subspace of $\mathbb{R}^m$.



Two pairs of orthogonal subspaces.

***For an m x n matrix of rank r*:**

Fundamental Space	***Subspace of***	***Dimension***
Nullspace	$\mathbb{R}^n$	$n-r$
Column Space	$\mathbb{R}^m$	r
Row space	$\mathbb{R}^n$	r
Left nullspace	$\mathbb{R}^m$	$m-r$

Example

Find a basis for each of the four subspaces associated with matrix A:

$$A=\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix} \quad R_2-2R_1$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \quad x_1=-2x_2-4x_3 \leftarrow \textit{Row space}$$

1. Basis for ***row space***: $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

2. Basis of the **column spaces**: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Rank($\boldsymbol{A}$) = 1

Dimension of A = 1

The pivots variables are: x_1

The free variables are: x_2, x_3

Set $x_2=1 \quad x_3=0$

The special solution: $s_1=(-2,\ 1,\ 0)$

Set $x_2=0 \quad x_3=1$

The special solution: $s_2=(-4,\ 0,\ 1)$

3. Basis of the **Null space**: $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$

$$A^T = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 8 \end{pmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad y_1 = -2y_2$$

Set $y_2 = 1 \quad \Rightarrow \quad s^*_1 = (-2,\ 1)$

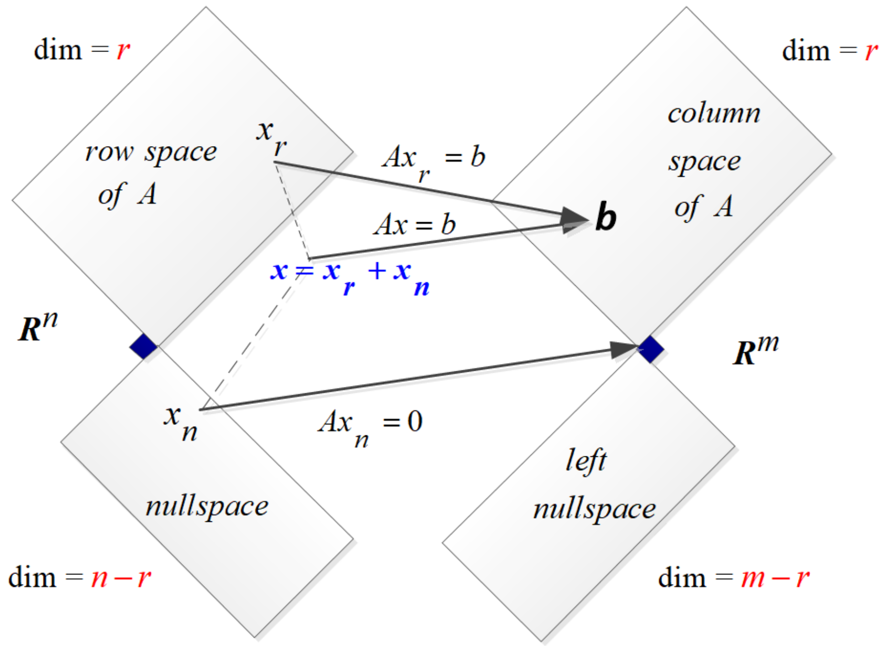
4. Basis of the **Left Nullspace**: $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Combining Bases from Subspaces

- Any n linearly independent vectors in $\mathbb{R}^n$ must span $\mathbb{R}^n$. They are basis. Any n vectors that span $\mathbb{R}^n$ must be independent. They are a basis.

- If the n columns of $\boldsymbol{A}$ are independent, they span $\mathbb{R}^n$, So $A\vec{x}=\vec{b}$ is solvable,

- If the n columns span $\mathbb{R}^n$, they are independent. So $A\vec{x}=\vec{b}$ has only one solution.



- When the orthogonal complement of a subspace S is defined to be the subspace whose vectors pairs to zero with the vectors in S. The larger the S is, the more restriction $S^{\perp}$ has, and hence the smaller $S^{\perp}$ is.

Theorem – *Equivalent Statements*

If A is an $n \times n$ matrix, then the following statements are equivalent.

a) A is invertible

b) $A\vec{x} = \vec{0}$ has only the trivial solution

c) The reduced row echelon form of A is I_n

d) A is expressible as a product of elementary matrices

e) $A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ matrix $\vec{b}$

f) $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ matrix $\vec{b}$

g) $\det(A) \neq 0$

h) The column vectors of A are linearly independent

i) The row vectors of A are linearly independent

j) The column vectors of A span $\mathbb{R}^n$

k) The row vectors of A span $\mathbb{R}^n$

l) The column vectors of A form a basis for $\mathbb{R}^n$

m) The row vectors of A form a basis for $\mathbb{R}^n$

n) A has a rank n.

o) A has nullity 0.

p) The orthogonal complement of the null space of A is $\mathbb{R}^n$

q) The orthogonal complement of the row space of A is $\{0\}$

Exercises   *Section* 2.9 – Rank and the Fundamental Matrix Spaces

1. Verify that $rank(A) = rank\left(A^T\right)$

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix}$$

2. Find the rank and nullity of the matrix; then verify that the values obtained satisfy $rank(A) + N(A) = n$

a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

3. If A is an $m \times n$ matrix, what is the largest possible value for its rank and the smallest possible value of the nullity of A.

4. Discuss how the rank of A varies with t.

a) $A = \begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix}$

b) $A = \begin{bmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{bmatrix}$

5. Are there values of r and s for which

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

Has rank 1? Has rank 2? If so, find those values.

6. Find the row reduced form R and the rank r of A (those depend on c).
Which are the pivot columns of A? Which variables are free? What are the special solutions and the nullspace matrix N (always depending on c)?

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & c \end{bmatrix} \quad and \quad A = \begin{bmatrix} c & c \\ c & c \end{bmatrix}$$

7. Find the row reduced form R and the rank r of A (those depend on c).
Which are the pivot columns of A? Which variables are free? What are the special solutions and the nullspace matrix N (always depending on c)?

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \quad and \quad A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}$$

8. If A has a rank r, then it has an r by r sub-matrix S that is invertible. Remove $m-r$ rows and $n-r$ columns to find an invertible sub-matrix S inside each A (you could keep the pivot rows and pivot columns of A).

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

9. Suppose that column 3 of 4 x 6 matrix is all zero. Then x_3 must be a ______ variable. Give one special solution for this matrix.

10. Fill in the missing numbers to make A rank 1, rank 2, rank 3. (your solution should be 3 matrices)

$$A = \begin{pmatrix} & -3 & \\ 1 & 3 & -1 \\ & 9 & -3 \end{pmatrix}$$

11. Fill out these matrices so that they have rank 1:

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & & \\ 4 & & \end{pmatrix} \quad B = \begin{pmatrix} 2 & & \\ 1 & & \\ 2 & 6 & -3 \end{pmatrix} \quad M = \begin{pmatrix} a & b \\ c & \end{pmatrix}$$

12. Suppose A and B are n by n matrices, and $AB = I$. Prove from $rank(AB) \le rank(A)$ that the $rank(A) = n$. So A is invertible and B must be its two-sided inverse. Therefore $BA = I$ (which is not so obvious!).

13. Every m by n matrix of rank r reduces to (m by r) times (r by n):

$$A = \text{(pivot columns of } A\text{) (first } r \text{ rows of } R\text{)} = (COL)(ROW)^T$$

Write the 3 by 4 matrix $A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{pmatrix}$ as the product of the 3 by 2 from the pivot columns and the 2 by 4 matrix from R.

14. Suppose R is m by n matrix of rank r, with pivot columns first: $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

a) What are the shapes of those 4 blocks?
b) Find the right-inverse B with $RB = I$ if $r = m$.
c) Find the right-inverse C with $CR = I$ if $r = n$.
d) What is the reduced row echelon form of R^T (with shapes)?
e) What is the reduced row echelon form of $R^T R$ (with shapes)?
Prove that $R^T R$ has the same nullspace as R. Then show that $A^T A$ always has the same nullspace as A (a value fact).
f) Suppose you allow elementary column operations on ***A*** as well as elementary row operations (which get to R). What is the "row-and-column reduced form" for an m by n matrix of rank ***r***?

15. True or False (check addition or give a counterexample)

a) The symmetric matrices in M $\left(with\ A^T = A\right)$ from a subspace.
b) The skew-symmetric matrices in M $\left(with\ A^T = -A\right)$ from a subspace.
c) The un-symmetric matrices in M $\left(with\ A^T \neq A\right)$ from a subspace.
d) Invertible matrices
e) Singular matrices

16. Let $A = \begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 2 & 4 & -3 & 7 & 0 \\ 3 & 6 & -5 & 10 & -2 \\ 5 & 10 & -9 & 16 & 0 \end{pmatrix}$

a) Reduce A to row-reduced echelon form.
b) What is the rank of A?
c) What are the pivots?
d) What are the free variables?
e) Find the special solutions. What is the nullspace $N(A)$?
f) Exhibit an r x r submatrix of A which is invertible, where $r = rank(A)$. (An r x r submatrix of A is obtained by keeping r rows and r columns of A)

17. Let $A=\begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 2 & 1 & 0 & 0 & -15 \\ 6 & -1 & -8 & -1 & -47 \\ 0 & 2 & 4 & 3 & 16 \end{pmatrix}$

a) Reduce A to row-reduced echelon form.
b) What is the rank of A?
c) What the pivots?
d) What are the free variables?
e) Find the special solutions. What is the nullspace $N(A)$?
f) Give the complete solution to $Ax=b,\ where\ \ b=A\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}$

18. Let $A=\begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

a) Reduce A to row-reduced echelon form.
b) What is the rank of A?
c) What the pivots?
d) What are the free variables?
e) Find the special solutions.
f) What is the nullspace $N(A)$?

19. Let $A=\begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix}$

a) Reduce A to row-reduced echelon form.
b) What is the rank of A?
c) What the pivots?
d) What are the free variables?
e) Find the special solutions.
f) What is the nullspace $N(A)$?

20. The 3 by 3 matrix A has rank 2.

$$A\vec{x} = \vec{b} \quad \text{is} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 + 5x_4 &= b_1 \\ 2x_1 + 4x_2 + 8x_3 + 12x_4 &= b_2 \\ 3x_1 + 6x_2 + 7x_3 + 13x_4 &= b_3 \end{aligned}$$

a) Reduce $\begin{bmatrix} A & \vec{b} \end{bmatrix}$ to $\begin{bmatrix} U & \vec{c} \end{bmatrix}$, so that $A\vec{x} = \vec{b}$ becomes triangular system $U\vec{x} = \vec{c}$.

b) Find the condition on (b_1, b_2, b_3) for $A\vec{x} = \vec{b}$ to have a solution

c) Describe the column space of A. Which plane in $\mathbb{R}^3$?

d) Describe the nullspace of A. Which special solutions in $\mathbb{R}^4$?

e) Find a particular solution to $A\vec{x} = (0, 6, -6)$ and then complete solution.

21. Find the special solutions and describe the complete solution to $A\vec{x} = \vec{0}$ for

A_1 = 3 by 4 zero matrix $\qquad A_2 = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \qquad A_3 = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$

Which are the pivot columns? Which are the free variables? What is the R (Reduced Row Echelon matrix) in each case?

22. Create a 3 by 4 matrix whose special solutions to $A\vec{x} = \vec{0}$ are $\vec{s}_1$ *and* $\vec{s}_2$:

$$\vec{s}_1 = \begin{pmatrix} -3 \\ 2 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{s}_2 = \begin{pmatrix} -2 \\ 0 \\ -6 \\ 1 \end{pmatrix}$$

You could create the matrix A in row reduced form R. Then describe all possible matrices A with the required Nullspace $N(A)$ = all combinations of $\vec{s}_1$ *and* $\vec{s}_2$.

23. The plane $x - 3y - z = 12$ is parallel to the plane $x - 3y - z = 0$. One particular point on this plane is $(12, 0, 0)$. All points on the plane have the form (fill the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}$$

24. Construct a matrix whose column space contains (1, 1, 5) and (0, 3, 1) and whose Nullspace contains $(1, 1, 2)$.

25. Construct a matrix whose column space contains (1, 1, 0) and (0, 1, 1) and whose Nullspace contains (1, 0, 1) and (0, 0, 1).

26. Construct a matrix whose column space contains (1, 1, 1) and whose Nullspace contains $(1, 1, 1, 1)$.

27. How is the Nullspace N(C) related to the spaces N(A) and N(B), if $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

28. Why does no 3 by 3 matrix have a nullspace that equals its column space?

29. If AB = 0 then the column space B is contained in the _______ of A. Give an example of A and B.

30. True or false (with reason if true or example to show it is false)

a) A square matrix has no free variables.

b) An invertible matrix has no free variables.

c) An m by n matrix has no more than $\boldsymbol{n}$ pivot variables.

d) An m by n matrix has no more than $\boldsymbol{m}$ pivot variables.

31. Suppose an m by n matrix has r pivots. The number of special solutions is ______.

The Nullspace contains only x = 0 when r = _______.

The column space is all of $\mathbb{R}^m$ when r = ______.

32. Find the complete solution in the form $\vec{x}_p + \vec{x}_n$ to these full rank system:

$a)\ \ x + y + z = 4$ $\qquad\qquad$ $b)\ \begin{cases} x + y + z = 4 \\ x - y + z = 4 \end{cases}$

33. Find the complete solution in the form $\vec{x}_p + \vec{x}_n$ to the system:

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

34. If $\boldsymbol{A}$ is 3 x 7 matrix, its largest possible rank is ________. In this case, there is a pivot in every ________ of $\boldsymbol{U}$ and $\boldsymbol{R}$, the solution to $A\vec{x} = \vec{b}$ _________ (always exists or is unique), and the column space of $\boldsymbol{A}$ is _________. Construct an example of such a matrix $\boldsymbol{A}$.

35. If $\boldsymbol{A}$ is 6 x 3 matrix, its largest possible rank is ________. In this case, there is a pivot in every ________ of $\boldsymbol{U}$ and R, the solution to $A\vec{x} = \vec{b}$ _________ (always exists or is unique), and the nullspace of $\boldsymbol{A}$ is _________. Construct an example of such a matrix $\boldsymbol{A}$.

36. Find the rank of $A, A^T A$ and AA^T for $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix}$

37. Explain why these are all false:

a) The complete solution is any linear combination of $\vec{x}_p$ *and* $\vec{x}_n$.

b) A system $A\vec{x}=\vec{b}$ has at most one particular solution.

c) The solution $\vec{x}_p$ with all free variables zero is the shortest solution (minimum length $\|\vec{x}\|$). Find a 2 by 2 counterexample.

d) If A is invertible there is no solution $\vec{x}_n$ in the null space.

38. Write down all known relation between r and m and n if $A\vec{x}=\vec{b}$ has

a) No solution for some $\vec{b}$.

b) Infinitely many solutions for every $\vec{b}$.

c) Exactly one solution for some $\vec{b}$, no solution for other $\vec{b}$.

d) Exactly one solution for every $\vec{b}$.

39. Find a basis for its row space, find a basis for its column space, and determine its rank

$$a)\ \begin{bmatrix} 0 & 2 & -3 & 4 & 1 & 2 & 1 & 7 \\ 0 & 0 & 3 & -2 & 0 & 4 & -5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 6 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad b)\ \begin{bmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{bmatrix}$$

40. Find a basis for the row space, find a basis for the null space, find $dim\,RS$, find $dim\,NS$, and verify $dim\,RS + dim\,NS = n$

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ 3 & 1 & -3 & -1 \\ 5 & -3 & 5 & 1 \end{bmatrix}$$

41. Determine if $\vec{b}$ lies in the column space of the given matrix. If it does, express $\vec{b}$ as linear combination of the column.

$$\begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, \quad \vec{b}=\begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

42. Find the transition matrix from B to C and find $[\vec{x}]_c$

a) $B=\{(3,\ 1),(-1,\ -2)\},\quad C=\{(1,\ -3),(5,\ 0)\},\quad [x]_B=[-1 \quad -2]^T$

b) $B=\{(1,\ 1,\ 1),\ (-2,\ -1,\ 0),\ (2,\ 1,\ 2)\},\quad C=\{(-6,\ -2,\ 1),\ (-1,\ 1,\ 5),\ (-1,\ -1,\ 1)\},$

$[\vec{x}]_B=[-3 \quad 2 \quad 4]^T$

43. Does A and A^T have the same number of pivots.

(**44 – 49**) For the given matrix A, which is given in row reduction echelon form

a) What is the rank of A?

b) What is the dimension of A?

c) What are the pivots?

d) What are the free variables?

e) Find the special (homogeneous) solutions.

f) What is the nullspace $N(A)$?

g) Find the particular solution $Ax = b$

h) Give the complete solution.

44. $A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad where \;\; b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

45. $A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad where \;\; b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

46. $A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad where \;\; b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

47. $A = \begin{pmatrix} 1 & 0 & 0 & \frac{13}{11} \\ 0 & 1 & 0 & -\frac{17}{11} \\ 0 & 0 & 1 & \frac{6}{11} \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad where \;\; b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

48. $A = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad where \;\; b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

49. $A = \begin{pmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad where \ \ b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

(**50 – 55**) Find a basis for each of the four subspaces associated with each given matrix

50. $A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{pmatrix}$

51. $B = \begin{pmatrix} 1 & 3 & 0 & 5 \\ 2 & 6 & 1 & 16 \\ 5 & 15 & 0 & 25 \end{pmatrix}$

52. $C = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$

53. $D = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix}$

54. $M = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 3 & 1 & -3 & -1 \\ 5 & -3 & 5 & 1 \end{pmatrix}$

55. $N = \begin{pmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{pmatrix}$

Lecture Three

Section 3.1 – Inner Products

Definition

An ***inner product*** on a real vector space *V* is a function that associates a real number $\langle \vec{u},\ \vec{v} \rangle$ with each pair of vectors in *V* in such a way that the following axioms are satisfies for all vectors $\vec{u},\ \vec{v}$, and $\vec{w}$ in *V* and all scalars *k*.

1. $\langle \vec{u},\ \vec{v} \rangle = \langle \vec{v},\ \vec{u} \rangle$ ***Symmetry axiom***
2. $\langle \vec{u}+\vec{v},\ \vec{w} \rangle = \langle \vec{u},\ \vec{w} \rangle + \langle \vec{v},\ \vec{w} \rangle$ ***Additivity axiom***
3. $\langle k\vec{u},\ \vec{v} \rangle = k\langle \vec{u},\ \vec{v} \rangle$ ***Homogeneity axiom***
4. $\langle \vec{v},\ \vec{v} \rangle \ge 0$ and $\langle \vec{v},\ \vec{v} \rangle = 0$ iff $\vec{v} = 0$ ***Positivity axiom***

A real vector space with an inner product is called a ***real inner product space***.

$$\boxed{\langle \vec{u},\ \vec{u} \rangle = \vec{u} \bullet \vec{v} = u_1 v_1 + u_2 v_2 + \cdots u_n v_n}$$

This is called the ***Euclidean inner product*** (or the ***standard inner product***)

Definition

If *V* is a real inner product space, then the norm (or length) of a vector $\vec{v}$ in *V* is denoted by $\|\vec{v}\|$ and is defined by

$$\|\vec{v}\| = \sqrt{\langle \vec{v},\ \vec{v} \rangle}$$

And the ***distance*** between two vectors is denoted by $d(\vec{u},\ \vec{v})$ and is defined by

$$d(\vec{u},\ \vec{v}) = \|\vec{u}-\vec{v}\| = \sqrt{\langle \vec{u}-\vec{v},\ \ \vec{u}-\vec{v} \rangle}$$

A vector of norm 1 is called a ***unit vector***.

Theorem

If ***u*** and ***v*** are vectors in a real inner product space *V*, and if *k* is a scalar, then:

a) $\|\vec{v}\| \ge 0$ with equality *iff* $\vec{v} = 0$

b) $\|k\vec{v}\| = |k|\|\vec{v}\|$

c) $d(\vec{u},\ \vec{v}) = d(\vec{v},\ \vec{u})$

d) $d(\vec{u},\ \vec{v}) \ge 0$ with equality *iff* $\vec{u} = \vec{v}$

Although the Euclidean inner product is the most important inner product on $\mathbb{R}^n$, there are various applications in which is desirable to modify it by weighing each term differently. More precisely, if $w_1, w_2, \ldots, w_n$ are positive real numbers, which we will call weighs, and if $\vec{u} = (u_1, u_2, \ldots, u_n)$ and are vectors in $\mathbb{R}^n$, then it can be shown that the formula

$$\langle \vec{u}, \vec{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \ldots + w_n u_n v_n$$

Defines an inner product on $\mathbb{R}^n$ that we call the ***weighted Euclidean inner product*** with weights $\boldsymbol{w_1, w_2, \ldots, w_n}$

Example

Let $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ be vectors in $\mathbb{R}^2$, verify that the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 3u_1 v_1 + 2u_2 v_2$ satisfies the four inner product axioms.

Solution

Axiom 1: $\langle \vec{u}, \vec{v} \rangle = 3u_1 v_1 + 2u_2 v_2$

$= 3v_1 u_1 + 2v_2 u_2$

$= \langle \vec{v}, \vec{u} \rangle$

Axiom 2: $\langle \vec{u} + \vec{v}, \vec{w} \rangle = 3(u_1 + v_1) w_1 + 2(u_2 + v_2) w_2$

$= 3(u_1 w_1 + v_1 w_1) + 2(u_2 w_2 + v_2 w_2)$

$= 3u_1 w_1 + 3v_1 w_1 + 2u_2 w_2 + 2v_2 w_2$

$= (3u_1 w_1 + 2u_2 w_2) + (3v_1 w_1 + 2v_2 w_2)$

$= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

Axiom 3: $\langle k\vec{u}, \vec{v} \rangle = 3(ku_1) v_1 + 2(ku_2) v_2$

$= k(3u_1 v_1 + 2u_2 v_2)$

$= k \langle \vec{u}, \vec{v} \rangle$

Axiom 4: $\langle \vec{v}, \vec{v} \rangle = 3v_1 v_1 + 2v_2 v_2$

$= 3v_1^2 + 2v_2^2 \geq 0$

$v_1 = v_2 = 0 \quad iff \quad \vec{v} = \vec{0}$

Exercises *Section* 3.1 – Inner Products

1. Let $\langle \vec{u},\ \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u}=(1,\ 1)$, $\vec{v}=(3,\ 2)$, $\vec{w}=(0,\ -1)$, and $k=3$. Compute the following.

a) $\langle \vec{u},\ \vec{v} \rangle$ *c)* $\langle \vec{u}+\vec{v},\ \vec{w} \rangle$ *e)* $d(\vec{u},\ \vec{v})$

b) $\langle k\vec{v},\ \vec{w} \rangle$ *d)* $\|\vec{v}\|$ *f)* $\|\vec{u}-k\vec{v}\|$

2. Let $\langle \vec{u},\ \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u}=(1,\ 1)$, $\vec{v}=(3,\ 2)$, $\vec{w}=(0,\ -1)$ and $k=3$. Compute the following for the weighted Euclidean inner product $\langle \vec{u},\ \vec{v} \rangle = 2u_1v_1 + 3u_2v_2$.

a) $\langle \vec{u},\ \vec{v} \rangle$ *c)* $\langle \vec{u}+\vec{v},\ \vec{w} \rangle$ *e)* $d(\vec{u},\ \vec{v})$

b) $\langle k\vec{v},\ \vec{w} \rangle$ *d)* $\|\vec{v}\|$ *f)* $\|\vec{u}-k\vec{v}\|$

3. Let $\langle \vec{u},\ \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u}=(3,\ -2)$, $\vec{v}=(4,\ 5)$, $\vec{w}=(-1,\ 6)$, and $k=-4$. Verify the following.

a) $\langle \vec{u},\ \vec{v} \rangle = \langle \vec{v},\ \vec{u} \rangle$

b) $\langle \vec{u}+\vec{v},\ \vec{w} \rangle = \langle \vec{u},\ \vec{w} \rangle + \langle \vec{v},\ \vec{w} \rangle$

c) $\langle \vec{u},\ \vec{v}+\vec{w} \rangle = \langle \vec{u},\ \vec{v} \rangle + \langle \vec{u},\ \vec{w} \rangle$

d) $\langle k\vec{u},\ \vec{v} \rangle = k\langle \vec{u},\ \vec{v} \rangle$

e) $\langle \vec{0},\ \vec{v} \rangle = \langle \vec{v},\ \vec{0} \rangle = 0$

4. Let $\langle \vec{u},\ \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u}=(3,\ -2)$, $\vec{v}=(4,\ 5)$, $\vec{w}=(-1,\ 6)$, and $k=-4$. Verify the following for the weighted Euclidean inner product $\langle \vec{u},\ \vec{v} \rangle = 4u_1v_1 + 5u_2v_2$

a) $\langle \vec{u},\ \vec{v} \rangle = \langle \vec{v},\ \vec{u} \rangle$

b) $\langle \vec{u}+\vec{v},\ \vec{w} \rangle = \langle \vec{u},\ \vec{w} \rangle + \langle \vec{v},\ \vec{w} \rangle$

c) $\langle \vec{u},\ \vec{v}+\vec{w} \rangle = \langle \vec{u},\ \vec{v} \rangle + \langle \vec{u},\ \vec{w} \rangle$

d) $\langle k\vec{u},\ \vec{v} \rangle = k\langle \vec{u},\ \vec{v} \rangle$

e) $\langle \vec{0},\ \vec{v} \rangle = \langle \vec{v},\ \vec{0} \rangle = 0$

5. Let $\vec{u}=(u_1,\ u_2)$ and $\vec{v}=(v_1,\ v_2)$. Show that the following are inner product on $\mathbb{R}^2$ by verifying that the inner product axioms hold. $\langle \vec{u},\ \vec{v} \rangle = 3u_1v_1 + 5u_2v_2$

6. Show that the following identity holds for the vectors in any inner product space

$$\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$$

7. Show that the following identity holds for the vectors in any inner product space

$$\langle \vec{u},\, \vec{v} \rangle = \frac{1}{4}\left(\|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2 \right)$$

8. Prove that $\|k\vec{v}\| = |k|\ \|\vec{v}\|$

Section 3.2 – Angle and Orthogonality in Inner Product Spaces

Cosine Formula

If $\vec{u}$ and $\vec{v}$ are nonzero vectors that implies

$$\cos\theta = \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\| \; \|\vec{v}\|}$$

$$\theta = \cos^{-1}\left(\frac{\langle \vec{u}, \vec{v}\rangle}{\|\vec{u}\| \; \|\vec{v}\|}\right)$$

$$-1 \le \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\| \; \|\vec{v}\|} \le 1$$

Example

Let $\mathbb{R}^4$ have the Euclidean inner product. Find the cosine angle θ between the vectors $\vec{u} = (4, 3, 1, -2)$ and $\vec{v} = (-2, 1, 2, 3)$.

Solution

$$\|\vec{u}\| = \sqrt{4^2 + 3^2 + 1^2 + (-2)^2}$$

$$= \sqrt{30}$$

$$\|\vec{v}\| = \sqrt{4 + 1 + 4 + 9}$$

$$= \sqrt{18}$$

$$= 3\sqrt{2}$$

$$\langle \vec{u}, \vec{v}\rangle = 4(-2) + 3(1) + 1(2) - 2(3)$$

$$= -9$$

$$\cos\theta = -\frac{9}{3\sqrt{30}\sqrt{2}} \qquad \cos\theta = \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\| \; \|\vec{v}\|}$$

$$= -\frac{3}{\sqrt{60}}$$

$$= -\frac{3}{2\sqrt{15}}$$

Theorem – Cauchy-Schwarz Inequality

If $\vec{v}$ and $\vec{w}$ are vectors in a real inner product space V, then

$$\left\|\langle \vec{u},\ \vec{v}\rangle\right\| \le \|\vec{u}\|\ \|\vec{v}\|$$

Proof

If either $\vec{u}$ or $\vec{v}$ is equal to zero, then both sides equal to zero

Inequality holds.

Suppose that $\vec{u},\ \vec{v} \neq 0$ and if $\vec{w}$ any vector

$\|\vec{w}\| = \vec{w}\ \vec{w} \ge 0$

Let $\vec{w} = \vec{u} - t\vec{v}$, then:

$$\begin{aligned}
0 &\le \vec{w}\vec{w} \\
&= (\vec{u} - t\vec{v})(\vec{u} - t\vec{v}) \\
&= \vec{u} \bullet \vec{u} - t(\vec{u} \bullet \vec{v}) - t(\vec{v} \bullet \vec{u}) + t^2(\vec{v} \bullet \vec{v}) \\
&= \vec{u} \bullet \vec{u} - 2t\ (\vec{u} \bullet \vec{v}) + t^2(\vec{v} \bullet \vec{v}) \qquad \text{Let } t = \frac{\vec{u} \bullet \vec{v}}{\vec{v} \bullet \vec{v}} \\
&= \vec{u} \bullet \vec{u} - 2\left(\frac{\vec{u} \bullet \vec{v}}{\vec{v} \bullet \vec{v}}\right)(\vec{u} \bullet \vec{v}) + \left(\frac{\vec{u} \bullet \vec{v}}{\vec{v} \bullet \vec{v}}\right)^2(\vec{v} \bullet \vec{v}) \\
&= \vec{u} \bullet \vec{u} - 2\frac{(\vec{u} \bullet \vec{v})^2}{\vec{v} \bullet \vec{v}} + \frac{(\vec{u} \bullet \vec{v})^2}{\vec{v} \bullet \vec{v}} \\
&= \vec{u} \bullet \vec{u} - \frac{(\vec{u} \bullet \vec{v})^2}{\vec{v} \bullet \vec{v}} \\
&= \frac{(\vec{u} \bullet \vec{u})(\vec{v} \bullet \vec{v}) - (\vec{u} \bullet \vec{v})^2}{\vec{v} \bullet \vec{v}} \qquad \text{Since } \vec{v} \bullet \vec{v} > 0 \\
&\le (\vec{u} \bullet \vec{u})(\vec{v} \bullet \vec{v}) - (\vec{u} \bullet \vec{v})^2
\end{aligned}$$

$(\vec{u} \bullet \vec{v})^2 \le (\vec{u} \bullet \vec{u})(\vec{v} \bullet \vec{v})$

$\left\|\langle \vec{u},\ \vec{v}\rangle\right\| \le \|\vec{u}\|\ \|\vec{v}\|$

The following two alternative forms of the Cauchy-Schwarz inequality are useful to know:

$$\langle \vec{u},\ \vec{v}\rangle^2 \le \langle \vec{u},\ \vec{u}\rangle\langle \vec{v},\ \vec{v}\rangle$$

$$\langle \vec{u},\ \vec{v}\rangle^2 \le \|\vec{u}\|^2\ \|\vec{v}\|^2$$

Theorem

If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors in a real inner product space V, and if k is any scalar, then

a) $\|\vec{u}+\vec{v}\| \le \|\vec{u}\|+\|\vec{v}\|$ **(*Triangle inequality for vectors*)**

b) $d(\vec{u},\ \vec{v}) \le d(\vec{u},\ \vec{w})+d(\vec{w},\ \vec{v})$ **(*Triangle inequality for distances*)**

Proof **(*a*)**

$$\begin{aligned}\|\vec{u}+\vec{v}\|^2 &= \langle \vec{u}+\vec{v},\ \vec{u}+\vec{v}\rangle \\ &= \langle \vec{u},\ \vec{u}\rangle + 2\langle \vec{u},\ \vec{v}\rangle + \langle \vec{v},\ \vec{v}\rangle \\ &\le \langle \vec{u},\ \vec{u}\rangle + 2\left|\langle \vec{u},\ \vec{v}\rangle\right| + \langle \vec{v},\ \vec{v}\rangle \\ &\le \langle \vec{u},\ \vec{u}\rangle + 2\|\vec{u}\|\ \|\vec{v}\| + \langle \vec{v},\ \vec{v}\rangle \\ &= \|\vec{u}\|^2 + 2\|\vec{u}\|\ \|\vec{v}\| + \|\vec{v}\|^2 \\ &= \left(\|\vec{u}\|+\|\vec{v}\|\right)^2\end{aligned}$$

$$\|\vec{u}+\vec{v}\|^2 \le \left(\|\vec{u}\|+\|\vec{v}\|\right)^2$$

$$\|\vec{u}+\vec{v}\| \le \|\vec{u}\|+\|\vec{v}\|$$

Definition

Two vectors $\vec{u}$ and $\vec{v}$ in an inner product space are called orthogonal if $\langle \vec{u},\ \vec{v}\rangle = 0$

Example

The vectors $\vec{u}=(1,\ 1)$ and $\vec{v}=(1,\ -1)$ are orthogonal with respect to the Euclidean inner product on $\mathbb{R}^2$, since

$$\begin{aligned}\vec{u}\bullet\vec{v} &= 1(1)+1(-1) \\ &\underline{=0\ |}\end{aligned}$$

They are not orthogonal with the respect to the weighted Euclidean inner product $\langle \vec{u},\ \vec{v}\rangle = 3u_1v_1 + 2u_2v_2$, since

$$\begin{aligned}\langle \vec{u},\ \vec{v}\rangle &= 3(1)(1)+ 2(1)(-1) \\ &\underline{=1\neq 0\ |}\end{aligned}$$

Example

$U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ *and* $V = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ are orthogonal, since

$$U \bullet V = 1(0) + 0(2) + 1(0) + 1(0)$$
$$= 0$$

Definition

If W is a subspace of an inner product space V, then the set of all vectors are orthogonal to every vector in W is called the ***orthogonal complement*** of W and is denoted by the symbol $W^{\perp}$

Theorem

If W is a subspace of an inner product space V, then:

a) $W^{\perp}$ is a subspace of V.

b) $W \cap W^{\perp} = \{0\}$

Proof

a) Let set $W^{\perp}$ contains at least the zero vector, since $\langle \vec{0}, \vec{w} \rangle = 0$ for every vector $\vec{w}$ in W. We need to show that $W^{\perp}$ is closed under addition and scalar multiplication.

Suppose that $\vec{u}$ and $\vec{v}$ are vectors in $W^{\perp}$, so every vector $\vec{w}$ in W we have $\langle \vec{u}, \vec{w} \rangle = 0$ and $\langle \vec{v}, \vec{w} \rangle = 0$

$$\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$$
$$= 0 + 0$$
$$= 0 \qquad \textit{Closed under addition}$$

$$\langle k\vec{u}, \vec{w} \rangle = k \langle \vec{u}, \vec{w} \rangle$$
$$= k(0)$$
$$= 0 \qquad \textit{Closed under scalar multiplication}$$

Which proves that $\vec{u} + \vec{w}$ and $k\vec{u}$ are in $W^{\perp}$

b) If $\vec{v}$ is any vector in both W and $W^{\perp}$, then $\vec{v}$ is orthogonal to itself; that is, $\langle \vec{v}, \vec{v} \rangle = 0$. It follows from the positivity axiom for inner products that $\vec{v} = \vec{0}$

Theorem

If W is a subspace of a finite-dimensional inner product space V, then the orthogonal complement of $W^{\perp}$ is W; that is

$$\left(W^{\perp}\right)^{\perp} = W$$

Example

Let W be the subspace of $\mathbb{R}^6$ spanned by the vectors

$$\vec{w}_1 = (1,\ 3,\ -2,\ 0,\ 2,\ 0), \qquad \vec{w}_2 = (2,\ 6,\ -5,\ -2,\ 4,\ -3)$$
$$\vec{w}_3 = (0,\ 0,\ 5,\ 10,\ 0,\ 15), \qquad \vec{w}_4 = (2,\ 6,\ 0,\ 8,\ 4,\ 18)$$

Find a basis for the orthogonal complement of W.

Solution

The Space W is the same as the row space of the matrix

$$A = \begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{pmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ \\ R_4 - 2R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 0 & 0 & 4 & 8 & 0 & 18 \end{pmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ R_3 + 5R_2 \\ R_4 + 4R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 6 \\ 0 & 0 & -1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix} \begin{matrix} \\ -R_2 \\ \\ \frac{1}{6}R_3 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 6 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} R_1 - 6R_3 \\ R_2 - 3R_3 \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The solution

$$\left(x_1, x_2, x_3, x_4, x_5, x_6\right) = \left(-3x_2 - 4x_4 - 2x_5, x_2, -2x_4, x_4, x_5, 0\right)$$

$$= x_2\left(-3, 1, 0, 0, 0, 0\right) + x_4\left(-4, 0, -2, 1, 0, 0\right) + x_5\left(-2, 0, 0, 0, 1, 0\right)$$

$$\vec{v}_1 = \left(-3, 1, 0, 0, 0, 0\right), \quad \vec{v}_2 = \left(-4, 0, -2, 1, 0, 0\right), \quad \vec{v}_3 = \left(-2, 0, 0, 0, 1, 0\right)$$

Definition

A collection of vectors in $\mathbb{R}^n$ (or inner space) is called orthogonal if any 2 are perpendicular.

$$\vec{v}_i . \vec{v}_j = \vec{v}^T \vec{v} = \begin{cases} 0 & \text{for } i \neq j & (\text{orthogonal vectors}) \\ 1 & \text{for } i = j & (\text{unit vectors}) \end{cases}$$

Theorem

If $\vec{v}_1, \ldots, \vec{v}_m$ are nonzero orthogonal vectors, then they are linearly independent.

Definition

A vector $\vec{v}$ is called normal if $\|\vec{v}\| = 1$

A collection of vectors $\vec{v}_1, \ldots, \vec{v}_m$ is called orthonormal if they are orthogonal and each $\|\vec{v}_i\| = 1$.

An orthonormal basis is a basis made up of orthonormal vectors.

Example

Q rotates every vector in the plane through the angle θ.

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$Q^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad \cos^2\theta + \sin^2\theta = 1$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\boxed{= Q^T}$$

The dot product $(\cos\theta\sin\theta - \sin\theta\cos\theta = 0)$, the columns are orthogonal.

They are unit vectors because $\cos^2\theta + \sin^2\theta = 1$. Those columns give an orthonormal basis for the plane $\mathbb{R}^2$.

We have: $QQ^T = I = Q^T Q$ (This type is called ***rotation***)

Exercises *Section* 3.2 – Angle and Orthogonality in Inner Product Spaces

1. Which of the following form orthonormal sets?

a) $(1, 0), (0, 2)$ in $\mathbb{R}^2$

b) $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in $\mathbb{R}^2$

c) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in $\mathbb{R}^2$

d) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ in $\mathbb{R}^3$

e) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ in $\mathbb{R}^3$

f) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ in $\mathbb{R}^3$

2. Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.

a) $\vec{u} = (1, -3), \quad \vec{v} = (2, 4)$

b) $\vec{u} = (-1, 0), \quad \vec{v} = (3, 8)$

c) $\vec{u} = (-1, 5, 2), \quad \vec{v} = (2, 4, -9)$

d) $\vec{u} = (4, 1, 8), \quad \vec{v} = (1, 0, -3)$

e) $\vec{u} = (1, 0, 1, 0), \quad \vec{v} = (-3, -3, -3, -3)$

f) $\vec{u} = (2, 1, 7, -1), \quad \vec{v} = (4, 0, 0, 0)$

g) $\vec{u} = (1, 3, -5, 4), \quad \vec{v} = (2, -4, 4, 1)$

h) $\vec{u} = (1, 2, 3, 4), \quad \vec{v} = (-1, -2, -3, -4)$

3. Find the cosine of the angle between ***A*** and ***B***.

a) $A = \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$

b) $A = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$

c) $A = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

d) $A = \begin{pmatrix} 1 & -2 & 7 \\ 6 & -3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 3 \\ 6 & 5 & -4 \end{pmatrix}$

4. Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

a) $\vec{u} = (-1, 3, 2), \quad \vec{v} = (4, 2, -1)$

b) $\vec{u} = (a, b), \quad \vec{v} = (-b, a)$

c) $\vec{u} = (-2, -2, -2), \quad \vec{v} = (1, 1, 1)$

d) $\vec{u} = (-4, 6, -10, 1), \quad \vec{v} = (2, 1, -2, 9)$

e) $\vec{u} = (-4, 6, -10, 1), \quad \vec{v} = (2, 1, -2, 9)$

5. Do there exist scalars *k* and *l* such that the vectors $\vec{u} = (2, k, 6), \quad \vec{v} = (l, 5, 3), \quad$ *and* $\vec{w} = (1, 2, 3)$ are mutually orthogonal with respect to the Euclidean inner product?

6. Let $\mathbb{R}^3$ have the Euclidean inner product. For which values of k are $\vec{u}$ and $\vec{v}$ orthogonal?

a) $\vec{u} = (2, 1, 3), \quad \vec{v} = (1, 7, k)$ *b)* $\vec{u} = (k, k, 1), \quad \vec{v} = (k, 5, 6)$

7. Let V be an inner product space. Show that if $\vec{u}$ and $\vec{v}$ are orthogonal unit vectors in V, then $\|\vec{u} - \vec{v}\| = \sqrt{2}$

8. Let $\boldsymbol{S}$ be a subspace of $\mathbb{R}^n$. Explain what $\left(\boldsymbol{S}^{\perp}\right)^{\perp} = \boldsymbol{S}$ means and why it is true.

9. The methane molecule CH_4 is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$ – (***note*** that all six edges have length $\sqrt{2}$, so the tetrahedron is regular). What is the cosine of the angle between the rays going from the center $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ to the vertices?

10. Determine if the given vectors are orthogonal.

$\vec{x}_1 = (1, 0, 1, 0), \quad \vec{x}_2 = (0, 1, 0, 1), \quad \vec{x}_3 = (1, 0, -1, 0), \quad \vec{x}_4 = (1, 1, -1, -1)$

11. Which of the following sets of vectors are orthogonal with respect to the Euclidean inner

a) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

b) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

12. Consider vectors $\vec{u} = (2, 3, 5) \quad \vec{v} = (1, -4, 3) \quad in \quad \mathbb{R}^3$

a) $\langle \vec{u}, \vec{v} \rangle$ *b)* $\|\vec{u}\|$ *c)* $\|\vec{v}\|$ *d)* Cosine between $\vec{u}$ *and* $\vec{v}$

13. Consider vectors $\vec{u} = (1, 1, 1) \quad \vec{v} = (1, 2, -3) \quad in \quad \mathbb{R}^3$

a) $\langle \vec{u}, \vec{v} \rangle$ *b)* $\|\vec{u}\|$ *c)* $\|\vec{v}\|$ *d)* Cosine θ between $\vec{u}$ *and* $\vec{v}$

14. Consider vectors $\vec{u} = (1, 2, 5) \quad \vec{v} = (2, -3, 5) \quad \vec{w} = (4, 2, -3) \quad in \quad \mathbb{R}^3$

a) $\langle \vec{u}, \vec{v} \rangle$ *d)* $\|\vec{u}\|$ *g)* Cosine α between $\vec{u}$ *and* $\vec{v}$

b) $\langle \vec{u}, \vec{w} \rangle$ *e)* $\|\vec{v}\|$ *h)* Cosine β between $\vec{u}$ *and* $\vec{w}$

c) $\langle \vec{v}, \vec{w} \rangle$ *f)* $\|\vec{w}\|$ *i)* Cosine θ between $\vec{v}$ *and* $\vec{w}$

j) $(\vec{u} + \vec{v}) \bullet \vec{w}$

15. Consider polynomial $f(t) = 3t - 5;\quad g(t) = t^2 \quad in \quad \mathbb{P}(t)$

a) $\langle f, g\rangle$ *b)* $\|f\|$ *c)* $\|g\|$ *d)* Cosine between f *and* g

16. Consider polynomial $f(t) = t + 2\ ;\quad g(t) = 3t - 2\ ;\quad h(t) = t^2 - 2t - 3 \quad in \quad \mathbb{P}(t)$

a) $\langle f, g\rangle$ *d)* $\|f\|$ *g)* Cosine α between f *and* g

b) $\langle f, h\rangle$ *e)* $\|g\|$ *h)* Cosine β between f *and* h

c) $\langle g, h\rangle$ *f)* $\|h\|$ *i)* Cosine θ between g *and* h

17. Suppose $\langle \vec{u}, \vec{v}\rangle = 3 + 2\,i$ in a complex inner product space *V*. Find:

a) $\langle (2 - 4\,i)\vec{u}, \vec{v}\rangle$ *b)* $\langle \vec{u}, (4 + 3\,i)\vec{v}\rangle$ *c)* $\langle (3 - 6\,i)\vec{u}, (5 - 2\,i)\vec{v}\rangle$ *d)* $\|\vec{u}, \vec{v}\|$

18. Find the Fourier coefficient *c* and the projection $c\vec{v}$ of $\vec{u} = (3 + 4\,i,\ 2 - 3\,i)$ along $\vec{v} = (5 + i,\ 2i)$ in $\mathbb{C}^2$

19. Suppose $\vec{v} = (1,\ 3,\ 5,\ 7)$. Find the projection of $\vec{v}$ onto *W* or find $\vec{w} \in W$ that minimizes $\|\vec{v} - \vec{w}\|$, where *W* is the subspace of $\mathbb{R}^4$ spanned by:

a) $\vec{u}_1 = (1,\ 1,\ 1,\ 1)$ *and* $\vec{u}_2 = (1,\ -3,\ 4,\ -2)$

b) $\vec{v}_1 = (1,\ 1,\ 1,\ 1)$ *and* $\vec{v}_2 = (1,\ 2,\ 3,\ 2)$

20. Suppose $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ is an orthogonal set of vectors. Prove that (*Pythagoras*)

$$\|\vec{u}_1 + \vec{u}_2 + \ \ldots + \vec{u}_n\|^2 = \|\vec{u}_1\|^2 + \|\vec{u}_2\|^2 + \ \ldots + \|\vec{u}_n\|^2$$

21. Suppose *A* is an orthogonal matrix. Show that $\langle \vec{u}A, \vec{v}A\rangle = \langle \vec{u}, \vec{v}\rangle$ for any $\vec{u}, \vec{v} \in V$

22. Suppose *A* is an orthogonal matrix. Show that $\|\vec{u}A\| = \|\vec{u}\|$ for every $\vec{u} \in V$

23. Let *V* be an inner product space over $\mathbb{R}$ or $\mathbb{C}$. Show that

$$\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$$

If and only if

$$\|s\vec{u} + t\vec{v}\| = s\|\vec{u}\| + t\|\vec{v}\| \quad for\ all \quad s,\ t \ge 0$$

24. Let *V* be an inner product vector space over $\mathbb{R}$.

a) If e_1, e_2, e_3 are three vectors in *V* with pairwise product negative, that is,

$$\langle e_1, e_2\rangle < 0, \quad i, j = 1,\ 2,\ 3, \quad i \ne j$$

Show that e_1, e_2, e_3 are linearly independent.

b) Is it possible for three vectors on the *xy*–plane to have pairwise negative products?

c) Does part (a) remain valid when the word "negative: is replaced with positive?

d) Suppose $\vec{u},\ \vec{v},\ and\ \vec{w}$ are three unit vectors in the *xy*–plane. What are the maximum and minimum values that

$$\langle \vec{u},\ \vec{v}\rangle + \langle \vec{v},\ \vec{w}\rangle + \langle \vec{w},\ \vec{u}\rangle$$

Can attain? And when?

Section 3.3 – Gram-Schmidt Process

Definition

A set of two or more vectors in a real inner product space is said to be ***orthogonal*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be ***orthonormal***.

Theorem

1. If $S=\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is an orthogonal basis for an inner product space V, and if $\vec{u}$ is any vector in V, then

$$\vec{u}=\frac{\langle\vec{u}, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1+\frac{\langle\vec{u}, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2+\cdots+\frac{\langle\vec{u}, \vec{v}_n\rangle}{\|\vec{v}_n\|^2}\vec{v}_n$$

2. If $S=\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is an orthonormal basis for an inner product space V, and if $\vec{u}$ is any vector in V, then

$$\vec{u}=\langle\vec{u}, \vec{v}_1\rangle\vec{v}_1+\langle\vec{u}, \vec{v}_2\rangle\vec{v}_2+\cdots+\langle\vec{u}, \vec{v}_n\rangle\vec{v}_n$$

Proof

1. Since $S=\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for V, every vector $\vec{u}$ in V can be expressed in the form

$$\vec{u}=c_1\vec{v}_1+c_2\vec{v}_2+\cdots+c_n\vec{v}_n$$

Let show that $c_i=\dfrac{\langle\vec{u}, \vec{v}_i\rangle}{\|\vec{v}_i\|^2}$ *for* $i=1, 2, \ldots, n$

$$\begin{aligned}\langle\vec{u}, \vec{v}_i\rangle &= \langle c_1\vec{v}_1+c_2\vec{v}_2+\cdots+c_n\vec{v}_n,\ \vec{v}_i\rangle\\ &= c_1\langle\vec{v}_1, \vec{v}_i\rangle+c_2\langle\vec{v}_2, \vec{v}_i\rangle+\cdots+c_n\langle\vec{v}_n, \vec{v}_i\rangle\end{aligned}$$

Since S is an orthogonal set, all of the inner products in the last equality are zero except the i^{th}, so we have

$$\begin{aligned}\langle\vec{u}, \vec{v}_i\rangle &= c_i\langle\vec{v}_i, \vec{v}_i\rangle\\ &= c_i\|\vec{v}_i\|^2\end{aligned}$$

The Gram-Schmidt Process

To convert a basis $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r\}$ into an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$, perform the following computations:

***Step* 1**: $\vec{v}_1 = \vec{u}_1$

***Step* 2**: $\vec{v}_2 = \vec{u}_2 - \dfrac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$

***Step* 3**: $\vec{v}_3 = \vec{u}_3 - \dfrac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \dfrac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$

***Step* 4**: $\vec{v}_4 = \vec{u}_4 - \dfrac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \dfrac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \dfrac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$

To convert the orthogonal basis into an orthonormal basis $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$, normalize the orthogonal basis vectors. $\boxed{\vec{q}_i = \dfrac{\vec{v}_i}{\|\vec{v}_i\|}}$

Example

Assume that the vector space $\mathbb{R}$ has the Euclidean inner product. Apply the Gram-Schmidt process to transform the basis vectors

$$\vec{u}_1 = (1,\ 1,\ 1) \quad \vec{u}_2 = (0,\ 1,\ 1) \quad \vec{u}_3 = (0,\ 0,\ 1)$$

Into the orthogonal basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, and then normalize the ***orthogonal*** basis vectors to obtain an orthonormal basis $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$

Solution

$$\vec{v}_1 = \vec{u}_1$$
$$= (1,\ 1,\ 1)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (0,\ 1,\ 1) - \frac{0+1+1}{1^2+1^2+1^2}(1,\ 1,\ 1)$$

$$= (0,\ 1,\ 1) - \frac{2}{3}(1,\ 1,\ 1)$$

$$\underline{= \left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}\Big|$$

$$\vec{v}_3 = \vec{u}_3 - proj_{\vec{v}_2}\vec{u}_3$$

$$= \vec{u}_3 - \frac{\left\langle \vec{u}_3, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2}\vec{v}_1 - \frac{\left\langle \vec{u}_3, \vec{v}_2 \right\rangle}{\left\| \vec{v}_2 \right\|^2}\vec{v}_2$$

$$= (0,\ 0,\ 1) - \frac{0+0+1}{1^2+1^2+1^2}(1,\ 1,\ 1) - \frac{0+0+\frac{1}{3}}{\left(\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$= (0,\ 0,\ 1) - \frac{1}{3}(1,\ 1,\ 1) - \frac{\frac{1}{3}}{\frac{2}{3}}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$= (0,\ 0,\ 1) - \frac{1}{3}(1,\ 1,\ 1) - \frac{1}{2}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$\underline{= \left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}\Big|$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\left\| \vec{v}_1 \right\|}$$

$$= \frac{(1,\ 1,\ 1)}{\sqrt{3}}$$

$$\underline{= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)}\Big|$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\left\| \vec{v}_2 \right\|}$$

$$= \frac{\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}{\sqrt{\left(\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2}}$$

$$= \frac{\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}{\frac{\sqrt{6}}{3}}$$

$$\underline{= \left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)}\Big|$$

$$\begin{aligned}
\vec{q}_3 &= \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|} \\
&= \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\sqrt{0^2+\left(-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}} \\
&= \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\frac{\sqrt{2}}{2}} \\
&= \left(0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)
\end{aligned}$$

Gram-Schmidt Process (*Orthonormal*)

Suppose $\vec{v}_1,\ \ldots,\ \vec{v}_n$ linearly independent in $\mathbb{R}^n$, construct n ***orthonormal*** $\vec{u}_1,\ \ldots,\ \vec{u}_n$ that span the same space: span $\left\{\vec{u}_1,\ \ldots,\ \vec{u}_k\right\}$ = span $\left\{\vec{v}_1,\ \ldots,\ \vec{v}_k\right\}$

***Step* 1**: Since $\vec{v}_i$ are linearly independent ($\neq 0$), so $\left\|\vec{v}_1\right\| \neq 0$ (to create a normal vector)

Let $\vec{u}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|} = \vec{q}_1$, then $\left\|\vec{u}_1\right\| = 1$ since $\vec{u}_1$ is orthonormal and span $\left\{\vec{u}_1\right\} = span\left\{\vec{v}_1\right\}$

$$\vec{w}_1 = \vec{v}_1 \Rightarrow \vec{v}_1 = \left\|\vec{w}_1\right\|\vec{u}_1$$

***Step* 2**: $\vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2 \bullet \vec{q}_1\right)\vec{q}_1$

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \bullet \vec{u}_1}{\left\|\vec{v}_1\right\|}\vec{v}_1 \qquad \left(\vec{w}_2 \perp \vec{u}_1\right)$$

$$\vec{v}_2 = \left\|\vec{w}_2\right\|\vec{u}_2 + \left(\vec{v}_2 \bullet \vec{u}_1\right)\vec{u}_1 \qquad \vec{w}_2 = \left\|\vec{w}_2\right\|\vec{u}_2$$

$$\boxed{\vec{q}_2 = \frac{\vec{w}_2}{\left\|\vec{w}_2\right\|}}$$

***Step* 3**: $\vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \bullet \vec{q}_1\right)\vec{q}_1 - \left(\vec{v}_3 \bullet \vec{q}_2\right)\vec{q}_2$

$$\boxed{\vec{q}_3 = \frac{\vec{w}_3}{\left\|\vec{w}_3\right\|}}$$

	$\vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$
$\vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2 \bullet \vec{q}_1\right)\vec{q}_1$	$\vec{q}_2 = \frac{\vec{w}_2}{\left\|\vec{w}_2\right\|}$
$\vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \bullet \vec{q}_1\right)\vec{q}_1 - \left(\vec{v} \bullet \vec{q}_2\right)\vec{q}_2$	$\vec{q}_3 = \frac{\vec{w}_3}{\left\|\vec{w}_3\right\|}$
$\vec{w}_n = \vec{v}_n - \left(\vec{v}_n \bullet \vec{q}_1\right)\vec{q}_1 - \left(\vec{v}_n \bullet \vec{q}_2\right)\vec{q}_2 - \ \ldots\ - \left(\vec{v}_n \bullet \vec{q}_{n-1}\right)\vec{q}_{n-1}$	$\vec{q}_n = \frac{\vec{w}_n}{\left\|\vec{w}_n\right\|}$

Example

Use the Gram-Schmidt process to find an ***orthonormal*** basis for the subspaces of

$$\vec{v}_1 = (1,\ 1,\ 0,\ 0) \quad \vec{v}_2 = (0,\ 1,\ 1,\ 0) \quad \vec{v}_3 = (1,\ 0,\ 1,\ 1)$$

Solution

Step 1: $\vec{q}_1 = \dfrac{(1,\ 1,\ 0,\ 0)}{\sqrt{1^2+1^2+0+0}}$ $\qquad \vec{q}_1 = \dfrac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$

$$= \frac{(1,\ 1,\ 0,\ 0)}{\sqrt{2}}$$

$$= \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)$$

Step 2: $\vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2 \cdot \vec{q}_1\right)\vec{q}_1$

$$= (0,\ 1,\ 1,\ 0) - \left[(0,\ 1,\ 1,\ 0) \cdot \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)\right]\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)$$

$$= (0,\ 1,\ 1,\ 0) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)$$

$$= (0,\ 1,\ 1,\ 0) - \left(\frac{1}{2},\ \frac{1}{2},\ 0,\ 0\right)$$

$$= \left(-\frac{1}{2},\ \frac{1}{2},\ 1,\ 0\right)$$

$$\left\|\vec{w}_2\right\| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 1}$$

$$= \frac{\sqrt{3}}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{6}}{2}$$

$$\vec{q}_2 = \frac{\vec{w}_2}{\left\|\vec{w}_2\right\|}$$

$$= \frac{\left(-\frac{1}{2},\ \frac{1}{2},\ 1,\ 0\right)}{\frac{\sqrt{6}}{2}}$$

$$= \frac{2}{\sqrt{6}}\left(-\frac{1}{2},\ \frac{1}{2},\ 1,\ 0\right)$$

$$= \left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ 0\right)$$

***Step* 3**: $\vec{v}_3 \bullet \vec{q}_1 = (1,\ 0,\ 1,\ 1) \bullet \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)$

$$= \frac{1}{\sqrt{2}}$$

$$\vec{v}_3 \bullet \vec{q}_2 = (1,\ 0,\ 1,\ 1) \bullet \left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ 0\right)$$

$$= -\frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}}$$

$$\vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \bullet \vec{q}_1\right)\vec{q}_1 - \left(\vec{v} \bullet \vec{q}_2\right)\vec{q}_2$$

$$= (1,\ 0,\ 1,\ 1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right) - \frac{1}{\sqrt{6}}\left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ 0\right)$$

$$= (1,\ 0,\ 1,\ 1) - \left(\frac{1}{2},\ \frac{1}{2},\ 0,\ 0\right) - \left(-\frac{1}{6},\ \frac{1}{6},\ \frac{2}{6},\ 0\right)$$

$$= \left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$\vec{q}_3 = \frac{\vec{w}_3}{\left\|\vec{w}_3\right\|}$$

$$= \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + 1^2}}\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$= \frac{1}{\sqrt{\frac{21}{9}}}\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$= \frac{3}{\sqrt{21}}\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$= \left(\frac{2}{\sqrt{21}},\ -\frac{2}{\sqrt{21}},\ \frac{2}{\sqrt{21}},\ \frac{3}{\sqrt{21}}\right)$$

The ***orthonormal*** basis:

$$\left\{ \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right),\ \left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ 0\right),\ \left(\frac{2}{\sqrt{21}},\ -\frac{2}{\sqrt{21}},\ \frac{2}{\sqrt{21}},\ \frac{3}{\sqrt{21}}\right) \right\}$$

QR–Decomposition

Problem

If A is an $m \times n$ matrix with linearly independent column vectors, and if Q is the matrix that results by applying the Gram-Schmidt process to the column vectors of A, what relationship, if any, exists between A and Q?

To solve this problem, suppose that the column vectors of A are $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ and the orthonormal column vectors of Q are $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n$.

$$\begin{array}{l}
\vec{u}_1 = \langle \vec{u}_1, \vec{q}_1 \rangle \vec{q}_1 + \langle \vec{u}_1, \vec{q}_2 \rangle \vec{q}_2 + \cdots + \langle \vec{u}_1, \vec{q}_n \rangle \vec{q}_n \\
\vec{u}_2 = \langle \vec{u}_2, \vec{q}_1 \rangle \vec{q}_1 + \langle \vec{u}_2, \vec{q}_2 \rangle \vec{q}_2 + \cdots + \langle \vec{u}_2, \vec{q}_n \rangle \vec{q}_n \\
\vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\quad \vdots \\
\vec{u}_n = \langle \vec{u}_n, \vec{q}_1 \rangle \vec{q}_1 + \langle \vec{u}_n, \vec{q}_2 \rangle \vec{q}_2 + \cdots + \langle \vec{u}_n, \vec{q}_n \rangle \vec{q}_n
\end{array}$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle & \cdots & \langle \vec{u}_n, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle & \cdots & \langle \vec{u}_n, \vec{q}_2 \rangle \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \langle \vec{u}_n, \vec{q}_n \rangle \end{bmatrix}$$

The equation $A = QR$ is a factorization of A into the product of a matrix Q with orthonormal column vectors and an invertible upper triangular matrix R. We call it the ***QR-decomposition of A***.

Theorem

If A is an $m \times n$ matrix with linearly independent column vectors, then A can be factored as

$$A = QR$$

Where Q is an $m \times n$ matrix with orthonormal column vectors, and R is an $n \times n$ invertible upper triangular matrix.

Example

Find the QR-decomposition of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Solution

The column vectors of are

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

From the previous example

$$\vec{q}_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \qquad \vec{q}_2 = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right) \qquad \vec{q}_3 = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle & \langle \vec{u}_3, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle & \langle \vec{u}_3, \vec{q}_2 \rangle \\ 0 & 0 & \langle \vec{u}_3, \vec{q}_3 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} (1)\frac{1}{\sqrt{3}} + (1)\frac{1}{\sqrt{3}} + (1)\frac{1}{\sqrt{3}} & 0 + (1)\frac{1}{\sqrt{3}} + (1)\frac{1}{\sqrt{3}} & 0 + 0 + (1)\frac{1}{\sqrt{3}} \\ 0 & 0\left(\frac{-2}{\sqrt{6}}\right) + (1)\frac{1}{\sqrt{6}} + (1)\frac{1}{\sqrt{6}} & 0\left(\frac{-2}{\sqrt{6}}\right) + 0\frac{1}{\sqrt{6}} + (1)\frac{1}{\sqrt{6}} \\ 0 & 0 & 0 + (0)\frac{-1}{\sqrt{2}} + (1)\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{Q} \cdot \underbrace{\begin{pmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}}_{R}$$

Calculus: Applying the Gram-Schmidt Process

We can apply the Gram-Schmidt orthogonalization procedure to generate some polynomials that are orthonormal on the interval $x \in [-1,\ 1]$ with inner product

$$\langle p,\ q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

Example

Apply the Gram-Schmidt orthonormalization process to the basis $B = \{1,\ x,\ x^2\}$ in $\mathbb{P}_2$ using the inner product

Solution

$B = \{1,\ x,\ x^2\}$

Let $\vec{u}_1 = 1,\quad \vec{u}_2 = x,\quad \vec{u}_3 = x^2$

$\vec{v}_1 = \vec{u}_1 = 1$

$$\langle \vec{v}_1,\ \vec{v}_1\rangle = \int_{-1}^{1} dx$$

$$= x\Big|_{-1}^{1}$$

$$= 2$$

$$\langle \vec{u}_2,\ \vec{v}_1\rangle = \int_{-1}^{1} x\ dx$$

$$= \frac{1}{2}x^2\Big|_{-1}^{1}$$

$$= 0$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2,\ \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= x - \frac{0}{2}(1)$$

$$= x$$

$$\langle \vec{v}_2,\ \vec{v}_2\rangle = \int_{-1}^{1} x^2\ dx$$

$$= \frac{1}{3}x^3 \Big|_{-1}^{1}$$

$$\boxed{= \frac{2}{3}}$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_{-1}^{1} x^2 \, dx$$

$$= \frac{1}{3}x^3 \Big|_{-1}^{1}$$

$$\boxed{= \frac{2}{3}}$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^{1} x^3 \, dx$$

$$= \frac{1}{4}x^4 \Big|_{-1}^{1}$$

$$\boxed{= 0}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= x^2 - \frac{3}{2}(x)(0) - \frac{1}{2}\frac{2}{3}$$

$$\boxed{= x^2 - \frac{1}{3}}$$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx$$

$$= \int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) dx$$

$$= \left(\frac{1}{5}x^5 - \frac{2}{9}x^3 + \frac{1}{9}x\right) \Big|_{-1}^{1}$$

$$= \frac{1}{5} - \frac{2}{9} + \frac{1}{9} + \frac{1}{5} - \frac{2}{9} + \frac{1}{9}$$

$$\boxed{= \frac{8}{45}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\boxed{= \frac{1}{\sqrt{2}}}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{x}{\sqrt{2/3}}$$

$$\boxed{= \frac{\sqrt{3}}{\sqrt{2}}x}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$$

$$\boxed{= \frac{3\sqrt{5}}{2\sqrt{2}}\left(x^2 - \frac{1}{3}\right)}$$

The ***orthonormal*** basis is $\left\{ \frac{1}{\sqrt{2}},\ \sqrt{\frac{3}{2}}x,\ \frac{1}{2}\sqrt{\frac{5}{2}}\left(3x^2 - 1\right) \right\}$

Exercises *Section* 3.3 – Gram-Schmidt Process

(1 – 14) Use the Gram-Schmidt process to find an ***orthonormal*** basis for the subspaces of $\mathbb{R}^m$.

1. $\vec{u}_1 = (1,\ -3),\quad \vec{u}_2 = (2,\ 2)$

2. $\vec{u}_1 = (1,\ 0),\quad \vec{u}_2 = (3,\ -5)$

3. $\{(1,\ 1,\ 1),\ (-1,\ 1,\ 0),\ (1,\ 2,\ 1)\}$

4. $\{(1,\ 1,\ 1),\ (0,\ 1,\ 1),\ (0,\ 0,\ 1)\}$

5. $\{(1,\ 1,\ 1),\ (0,\ 2,\ 1),\ (1,\ 0,\ 3)\}$

6. $\{(2,\ 2,\ 2),\ (1,\ 0,\ -1),\ (0,\ 3,\ 1)\}$

7. $\{(1,\ -1,\ 0),\ (0,\ 1,\ 0),\ (2,\ 3,\ 1)\}$

8. $\{(3,\ 0,\ 4),\ (-1,\ 0,\ 7),\ (2,\ 9,\ 11)\}$

9. $\{(1,\ 1,\ 1,\ 1),\ (1,\ 2,\ 1,\ 0),\ (1,\ 3,\ 0,\ 0)\}$

10. $\{(0,\ 2,\ -1,\ 1),\ (0,\ 0,\ 1,\ 1),\ (-2,\ 1,\ 1,\ -1)\}$

11. $\vec{u}_1 = (1,\ 0,\ 0),\quad \vec{u}_2 = (3,\ 7,\ -2),\quad \vec{u}_3 = (0,\ 4,\ 1)$

12. $\vec{u}_1 = (1, 1, 1, 1),\quad \vec{u}_2 = (1,\ 2,\ 4,\ 5),\quad \vec{u}_3 = (1,\ -3,\ -4,\ -2)$

13. $\vec{u}_1 = (1, 1, 1, 1),\quad \vec{u}_2 = (1, 1, 2, 4),\quad \vec{u}_3 = (1,\ 2,\ -4,\ -3)$

14. $\vec{u}_1 = (0,\ 2,\ 1,\ 0),\quad \vec{u}_2 = (1,\ -1,\ 0,\ 0),\quad \vec{u}_3 = (1,\ 2,\ 0,\ -1),\quad \vec{u}_4 = (1,\ 0,\ 0,\ 1)$

(15 – 26) Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

15. $\{(1,\ 1,\ 0),\ (1,\ 2,\ 0),\ (0,\ 1,\ 2)\}$

16. $\{(1,\ -2,\ 2),\ (2,\ 2,\ 1),\ (2,\ -1,\ -2)\}$

17. $\{(1,\ 0,\ 0),\ (1,\ 1,\ 1),\ (1,\ 1,\ -1)\}$

18. $\{(4,\ -3,\ 0),\ (1,\ 2,\ 0),\ (0,\ 0,\ 4)\}$

19. $\{(0,\ 1,\ 2),\ (2,\ 0,\ 0),\ (1,\ 1,\ 1)\}$

20. $\{(0,\ 1,\ 1),\ (1,\ 1,\ 0),\ (1,\ 0,\ 1)\}$

21. $\{(1,\ 2,\ -2),\ (1,\ 0,\ -4),\ (5,\ 2,\ 0),\ (1,\ 1,\ -1)\}$

22. $\{(-3,\ 1,\ 2),\ (1,\ 1,\ 1),\ (2,\ 0,\ -1),\ (1,\ -3,\ 2)\}$

23. $\{(2,\ 1,\ 1),\ (0,\ 3,\ -1),\ (3,\ -4,\ -2),\ (-1,\ -1,\ 3)\}$

24. $\vec{u}_1 = (1, 1, 0, -1),\quad \vec{u}_2 = (1, 3, 0, 1),\quad \vec{u}_3 = (4, 2, 2, 0)$

25. $\vec{u}_1 = (1, 1, 1, 1),\quad \vec{u}_2 = (1, 1, 2, 4),\quad \vec{u}_3 = (1,\ 2,\ -4,\ -3)$

26. $\{(3, 4, 0, 0), (-1, 1, 0, 0), (2, 1, 0, -1), (0, 1, 1, 0)\}$

27. Find the ***QR***-decomposition of

a) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

28. Verify that the Cauchy-Schwarz inequality holds for the given vectors using the Euclidean inner product

$\vec{u} = (0, -2, 2, 1), \quad \vec{v} = (-1, -1, 1, 1)$

(**29 – 33**) Apply the Gram-Schmidt ***orthonormalization*** process in $\mathbb{C}^0[-1, 1]$ spanned by the functions, using the inner product

29. $f_1(x) = x + 2, \quad f_2(x) = x^2 - 3x + 4$

30. $f_1(x) = x, \quad f_2(x) = x^3, \quad f_3(x) = x^5$

31. $f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x) = \frac{1}{2}\left(3x^2 - 1\right)$

32. $f_1(x) = 1, \quad f_2(x) = \sin \pi x, \quad f_3(x) = \cos \pi x$

33. $f_1(x) = \sin \pi x, \quad f_2(x) = \sin 2\pi x, \quad f_3(x) = \sin 3\pi x$

34. For $\mathbb{P}_3[x]$, define the inner product over $\mathbb{R}$ as

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx$$

a) If $f(x) = 1$ is a unit vector in $\mathbb{P}_3[x]$?

b) Find an orthonormal basis for the subspace spanned by x *and* x^2.

c) Complete the basis in part (*b*) to an orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product.

d) Is

$$[f, g] = \int_{0}^{1} f(x) g(x)\, dx$$

Also, an inner product for $\mathbb{P}_3[x]$

e) Find a pair of vectors $\vec{v}$ and $\vec{w}$ such that

$$\langle \vec{v},\ \vec{w} \rangle = 0 \quad but \quad [\vec{v},\ \vec{w}] \neq 0$$

f) Is the basis found in part (*c*) are orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product in part (*d*)?

Section 3.4 – Orthogonal Matrices

Definition

A square matrix A is said to be orthogonal if its transpose is the same as its inverse, that is, if

$$A^{-1} = A^T$$

or, equivalently, if

$$AA^T = A^T A = I$$

Example

The matrix $A = \begin{pmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{pmatrix}$

Solution

$$A^T A = \begin{pmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{pmatrix} \begin{pmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example

The matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Solution

$$A^T A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem

The following are equivalent for $n \times n$ matrix A.

a) A is orthogonal.

b) The row vectors of A form an orthonormal set in R^n with the Euclidean inner product.

c) The column vectors of A form an orthonormal set in R^n with the Euclidean inner product.

Theorem

a) The inverse of an orthogonal matrix is orthogonal

b) A product of orthogonal matrices is orthogonal

c) If A is orthogonal, then $\det(A) = 1$ or $\det(A) = -1$

Theorem

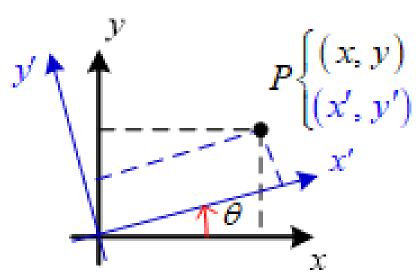
If A is an $n \times n$ matrix, then the following are equivalent

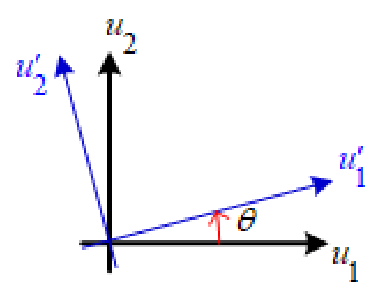
a) A is orthogonal.

b) $\|A\vec{x}\| = \|\vec{x}\|$ for all $\boldsymbol{x}$ in R^n.

c) $A\vec{x} \bullet A\vec{y} = \vec{x} \bullet \vec{y}$ for all $\vec{x}$ and $\vec{y}$ in R^n.

Let $\vec{u}_1$ and $\vec{u}_2$ be the unit vectors along the x- and y-axes and unit vectors $\vec{u}'_1$ and $\vec{u}'_2$ along the x'- and y'-axes.



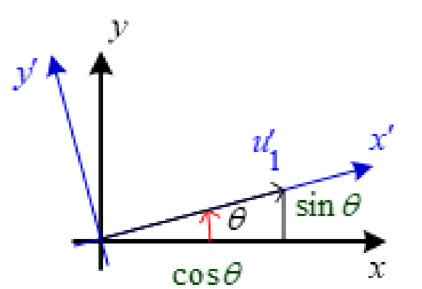


The new coordinates (x', y') and the old coordinates (x, y) of a point P will be related by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$$

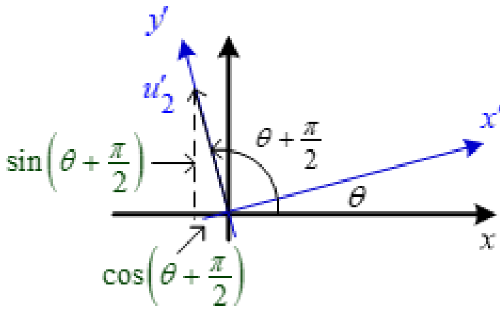
$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$P^{-1} = P^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$



$$\begin{bmatrix} \vec{x}' \\ \vec{y}' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix}$$

$$\rightarrow \begin{cases} x' = x\cos\theta + y\sin\theta \\ y' = -x\sin\theta + y\cos\theta \end{cases}$$



These are sometimes called the ***rotation equations***.

Example

Use the form $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ to find the new coordinates of the point $Q(2,\ 1)$ if the coordinate axes of a rectangular coordinate system are rotated through an angle of $\theta = \frac{\pi}{4}$.

Solution

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

The new coordinates of Q are $(x',\ y') = \left(\frac{3}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right)$

Exercises *Section* 3.4 – Orthogonal Matrices

(1 – 2) Show that the matrix is orthogonal

1. $A=\begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$

2. $A=\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

(3 – 12) Determine if the matrix is orthogonal. For those that is orthogonal find the inverse.

3. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

5. $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

6. $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$

7. $\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$

8. $\begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

9. $\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

10. $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$

11. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$

12. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{bmatrix}$

13. Find a last column so that the resulting matrix is orthogonal

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & \cdots \end{bmatrix}$$

14. Determine if the given matrix is orthogonal. If it is, find its inverse

$$\begin{bmatrix} \frac{1}{9} & \frac{4}{5} & \frac{3}{7} \\ \frac{4}{9} & \frac{3}{5} & -\frac{2}{7} \\ \frac{8}{9} & -\frac{2}{5} & \frac{3}{7} \end{bmatrix}$$

15. Prove that if A is orthogonal, then A^T is orthogonal.

16. Prove that if A is orthogonal, then A^{-1} is orthogonal.

17. Prove that if A and B are orthogonal, then AB is orthogonal.

18. Let Q be an $n \times n$ orthogonal matrix, and let A be an $n \times n$ matrix. Show that $\det\left(QAQ^T\right) = \det(A)$

19. Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$

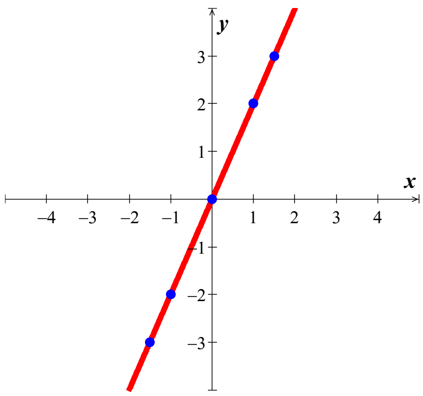
a) Is matrix A an orthogonal matrix?
b) Let B be the matrix obtained by normalizing each row of A, find B.
c) Is B an orthogonal matrix?
d) Are the columns of B orthogonal?

Section 3.5 – Least Squares Analysis

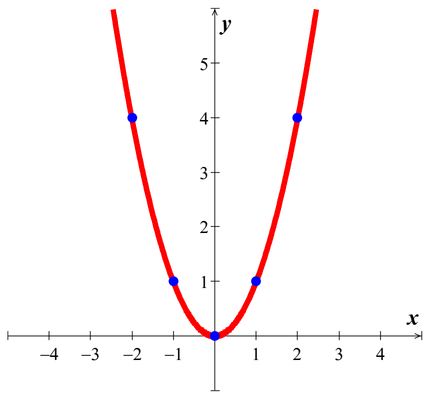
The use to ***best*** fit data, we will use results about orthogonal projections in inner product spaces to obtain a technique for fitting a line or other polynomial.

Fitting a Curve to Data

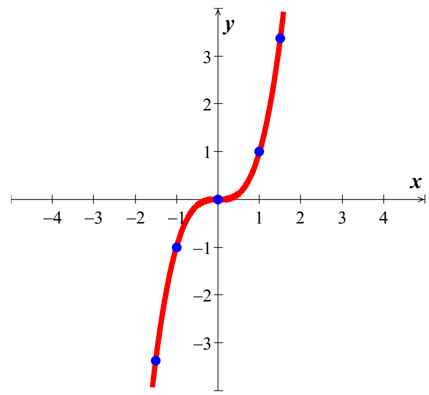
The common problem is to obtain a mathematical relationship between 2 variables x and y by ***fitting*** a curve to points in the xy-plane.
Some possibility of fitting the data



$y = ax + b$



$y = ax^2 + bx + c$



$y = ax^3 + bx^2 + cx + d$

Least Squares Fit of a Straight Line

Recall that a system of equations $A\vec{x} = \vec{y}$ is called inconsistent if it does not have a solution. Suppose we want to fit a straight line $y = mx + b$ to the determined points $\left(x_1, y_1\right), \ \ldots, \ \left(x_n, y_n\right)$

If the data points were collinear, the line would pass through all n points and the unknown coefficients m and b would satisfy the equations

$$\begin{aligned} y_1 &= mx_1 + b \\ y_2 &= mx_2 + b \\ \vdots \quad & \quad \vdots \quad\quad \vdots \\ y_n &= mx_n + b \end{aligned} \quad \Rightarrow \quad \underbrace{\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ b \end{bmatrix}}_{\vec{x}} \underset{=}{=} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\vec{y}}$$

The problem is to find m and b that minimize the errors is some sense.

Least Square Problem

Given a linear system $A\vec{x} = \vec{y}$ of m equations in n unknowns, find a vector $\vec{x}$ that minimizes $\|\vec{y} - A\vec{x}\|$ with respect to the Euclidean inner product on $\mathbb{R}^m$. We call such as $\vec{x}$ a least squares solution of the system, we call $\vec{y} - A\vec{x}$ the least squares error vectors, and we call $\|\vec{y} - A\vec{x}\|$ the least squares error.

$$A\mathbf{x} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{pmatrix}$$

The term "***least square solution***" results from the fact the minimizing $\|\vec{y} - A\vec{x}\| = e_1^2 + e_2^2 + \ldots + e_m^2$

Example

Find the sums of squares of the errors of (2, 4), (4, 8), (6, 6)

Solution

$$4 = 2m + b \quad \Rightarrow \quad 4 - 2m - b = e_1$$
$$8 = 4m + b \quad \Rightarrow \quad 8 - 4m - b = e_2$$
$$6 = 6m + b \quad \Rightarrow \quad 6 - 6m - b = e_3$$

$$e_1^2 + e_2^2 + \ldots + e_m^2 = (4 - 2m - b)^2 + (8 - 4m - b)^2 + (6 - 6m - b)^2$$

The least squares problem for this example to find the values m and b for which $e_1^2 + e_2^2 + \ldots + e_m^2$ is a minimum.

Theorem

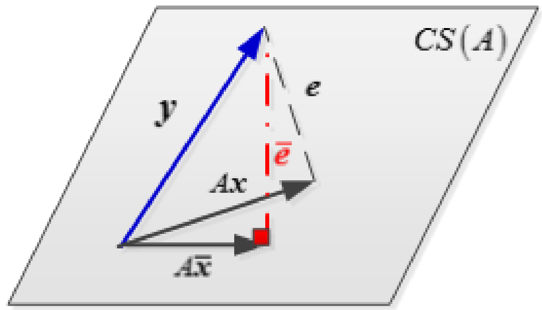
If A is an $m \times n$ matrix, the equation $A\vec{x} = \vec{y}$ has a solution if and only if $\vec{y}$ is in the column space of A.

$$\vec{y} - A\vec{x} = \vec{e}$$

$A\vec{x}$ is a vector that is in the column space of A. For this A the column space is a plane is $\mathbb{R}^m$

$\vec{y}$ is a vector, not in the column space of A (otherwise $A\vec{x} = \vec{y}$ has an exact solution)

$\vec{e}$ is the error vector, the difference between $\vec{y}$ and $A\vec{x}$



The length $\|\vec{e}\|$ is a *minimum* exactly when $\vec{e} \perp CS(A)$

Best Approximation *Theorem*

If $CS(A)$ is a finite dimensional subspace of an inner product space, and if $\vec{y}$ is a vector in $\boldsymbol{V}$, then $proj_{CS(A)}\vec{y}$ is the best approximation to $\boldsymbol{y}$ from $CS(A)$ is the sense that

$$\left\| \vec{y} - proj_{CS(A)}\vec{y} \right\| < \| \vec{y} - CS(A) \|$$

For every vector $\vec{w}$ in $CS(A)$ that is different from $proj_{CS(A)}\vec{y}$

Theorem

For every linear system $A\vec{x} = \vec{y}$, the associated normal system

$$A^T A\vec{x} = A^T \vec{y}$$

Is consistent, and all solutions are least squares solutions of $A\vec{x} = \vec{y}$

If the columns of A are linearly independent, then $A^T A$ is invertible so has a unique solution $\overline{x}$. This solution is often expressed theoretically as

$$\left(A^T A\right)^{-1} A^T A\overline{x} = \left(A^T A\right)^{-1} A^T \vec{y}$$

$$\overline{x} = \left(A^T A\right)^{-1} A^T \vec{y}$$

Proof

Let the vector $\overline{\boldsymbol{x}}$ is a least squares solution to $A\vec{x} = \vec{y} \quad \Leftrightarrow \quad (\vec{y} - A\overline{x}) \perp CS(A)$

$$(\vec{y} - A\overline{x}) \cdot \vec{z} = 0 \qquad \vec{z} \text{ in } CS(A) \quad \& \quad \vec{z} = A\vec{w}$$

$$(\vec{y} - A\overline{x}) \cdot A\vec{w} = 0 \qquad \vec{w} \text{ in } \mathbb{R}^n$$

$$A^T(\vec{y} - A\overline{x}) \cdot \vec{w} = 0$$

$$A^T(\vec{y} - A\overline{\boldsymbol{x}}) = 0$$

$$A^T\vec{y} - A^T A\overline{\boldsymbol{x}} = 0$$

$$A^T\vec{y} = A^T A\overline{x}$$

Theorem

If A is an $m \times n$ matrix, then the following are equivalent

a) A has linearly independent column vectors.

b) $A^T A$ is invertible.

Example

Find the equation of the line that best fits the given points in the least-squares sense.

(40, 482), (45, 467), (50, 452), (55, 432), (60, 421)

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 40 & 1 \\ 45 & 1 \\ 50 & 1 \\ 55 & 1 \\ 60 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 482 \\ 467 \\ 452 \\ 432 \\ 421 \end{pmatrix}$$

Where $A = \begin{pmatrix} 40 & 1 \\ 45 & 1 \\ 50 & 1 \\ 55 & 1 \\ 60 & 1 \end{pmatrix} \quad \boldsymbol{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \boldsymbol{y} = \begin{pmatrix} 482 \\ 467 \\ 452 \\ 432 \\ 421 \end{pmatrix}$

Using the normal equation formula: $A^T Ax = A^T y$

$$\begin{pmatrix} 40 & 45 & 50 & 55 & 60 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 40 & 1 \\ 45 & 1 \\ 50 & 1 \\ 55 & 1 \\ 60 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 40 & 45 & 50 & 55 & 60 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 482 \\ 467 \\ 452 \\ 432 \\ 421 \end{pmatrix}$$

$$\begin{pmatrix} 12,750 & 250 \\ 250 & 5 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 111,970 \\ 2,255 \end{pmatrix}$$

$X = A^{-1}B$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{1250}\begin{pmatrix} 5 & -250 \\ -250 & 12,750 \end{pmatrix}\begin{pmatrix} 111,970 \\ 2,255 \end{pmatrix}$$

$$= \begin{pmatrix} -3.12 \\ 607 \end{pmatrix}$$

Or

$$m = \frac{\begin{vmatrix} 111,970 & 250 \\ 2,255 & 5 \end{vmatrix}}{\begin{vmatrix} 12,750 & 250 \\ 250 & 5 \end{vmatrix}}$$

$$= \frac{-3,900}{1,250}$$

$$= -\frac{78}{25}$$

$$b = \frac{\begin{vmatrix} 12,750 & 111,970 \\ 250 & 2,255 \end{vmatrix}}{1,250}$$

$$= \frac{758,750}{1,250}$$

$$= 607$$

Thus, $y = -\frac{78}{25}x + 607$ *or* $y = -3.12x + 607$

Example

Given the system equation: $\begin{cases} x_1 - x_2 = 4 \\ 3x_1 + 2x_2 = 1 \\ -2x_1 + 4x_2 = 3 \end{cases}$

a) Find the least-squares solution of the linear system $A\vec{x} = \vec{y}$

b) Find the orthogonal projection of $\vec{y}$ on the column space of A

c) Find the ***error vector*** and the ***error***

Solution

a) $A = \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$

$$A^T A\vec{x} = A^T \vec{y}$$

$$\begin{pmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 14 & -3 \\ -3 & 21 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{285}\begin{pmatrix} 21 & 3 \\ 3 & 14 \end{pmatrix}\begin{pmatrix} 1 \\ 10 \end{pmatrix} \qquad X = A^{-1}B$$

$$= \begin{pmatrix} \frac{51}{285} \\ \frac{143}{285} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{17}{95} \\ \frac{143}{285} \end{pmatrix}$$

Thus $\underline{y = \frac{17}{95}x + \frac{143}{285}}$ *or* $\underline{y = 0.1789x + 0.5018}$

b) The orthogonal projection of $\vec{y}$ on the column space of A

$$A\vec{x} = \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} \frac{17}{95} \\ \frac{143}{285} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{pmatrix}$$

c) $\vec{y} - A\vec{x} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{pmatrix}$

$$= \begin{pmatrix} \frac{1232}{285} \\ -\frac{154}{285} \\ \frac{4}{3} \end{pmatrix}$$

The ***error***: $\|\vec{y} - A\vec{x}\| = \sqrt{\left(\frac{1232}{285}\right)^2 + \left(-\frac{154}{285}\right)^2 + \left(\frac{4}{3}\right)^2}$

$$\approx 4.556$$

Exercises *Section* 3.5 – Least Squares Analysis

(1 – 7) Find the equation of the line that best fits the given points in the least-squares sense and find the error.

1. $\{(0,\ 2),\ (1,\ 2),\ (2,\ 0)\}$

2. $\{(1,\ 5),\ (2,\ 4),\ (3,\ 1),\ (4,\ 1),\ (5,\ -1)\}$

3. $\{(0,\ 1),\ (1,\ 3),\ (2,\ 4),\ (3,\ 4)\}$

4. $\{(-2,\ 0),\ (-1,\ 0),\ (0,\ 1),\ (1,\ 3),\ (2,\ 5)\}$

5. $\{(2,\ 3),\ (3,\ 2),\ (5,\ 1),\ (6,\ 0)\}$

6. $\{(-1,\ 0),\ (0,\ 1),\ (1,\ 2),\ (2,\ 4)\}$

7. $\{(1,\ 0),\ (2,\ 1),\ (4,\ 2),\ (5,\ 3)\}$

(8 – 10) Find the orthogonal projection of the vector $\vec{u}$ on the subspace of $\mathbb{R}^4$ spanned by the vectors

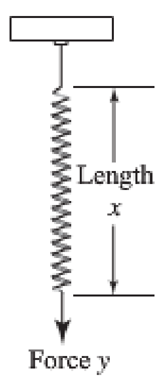
8. $\vec{u}=(-3,\ -3,\ 8,\ 9);\quad \vec{v}_1=(3,\ 1,\ 0,\ 1),\quad \vec{v}_2=(1,\ 2,\ 1,\ 1),\quad \vec{v}_3=(-1,\ 0,\ 2,\ -1)$

9. $\vec{u}=(6,\ 3,\ 9,\ 6);\quad \vec{v}_1=(2,\ 1,\ 1,\ 1),\quad \vec{v}_2=(1,\ 0,\ 1,\ 1),\quad \vec{v}_3=(-2,\ -1,\ 0,\ -1)$

10. $\vec{u}=(-2,\ 0,\ 2,\ 4);\quad v_1=(1,\ 1,\ 3,\ 0),\quad \vec{v}_2=(-2,\ -1,\ -2,\ 1),\quad \vec{v}_3=(-3,\ -1,\ 1,\ 3)$

11. Find the standard matrix for the orthogonal projection P of $\mathbb{R}^2$ on the line passes through the origin and makes an angle θ with the positive x-axis.

12. Hooke's law in physics states that the length x of a uniform spring is a linear function of the force y applied to it. If we express the relationship as $y = mx + b$, then the coefficient m is called the spring constant.



Suppose a particular unstretched spring has a measured length of 6.1 *inches*.(i.e., $x = 6.1$ when $y = 0$). Forces of 2 pounds, 4 pounds, and 6 pounds are then applied to the spring, and the corresponding lengths are found to be 7.6 inches, 8.7 inches, and 10.4 inches. Find the spring constant.

13. Prove: If A has a linearly independent column vectors, and if $\boldsymbol{b}$ is orthogonal to the column space of A, then the least squares solution of $A\vec{x}=\vec{b}$ is $\vec{x}=\vec{0}$.

14. Let A be an $m\times n$ matrix with linearly independent row vectors. Find a standard matrix for the orthogonal projection of $\mathbb{R}^n$ onto the row space of A.

15. Let W be the line with parametric equations $x=2t, \quad y=-t, \quad z=4t$

a) Find a basis for W.
b) Find the standard matrix for the orthogonal projection on W.
c) Use the matrix in part (b) to find the orthogonal projection of a point $P_0\left(x_0, y_0, z_0\right)$ on W.
d) Find the distance between the point $P_0\left(2, 1, -3\right)$ and the line W.

16. In $\mathbb{R}^3$, consider the line l given by the equations $x=t, \quad y=t, \quad z=t$
And the line m given by the equations $x=s, \quad y=2s-1, \quad z=1$
Let P be the point on l, and let Q be a point on m.
Find the values of t and s that minimize the distance between the lines by minimizing the squared distance $\|P-Q\|^2$

17. Determine whether the statement is true or false,

a) If A is an $m\times n$ matrix, then A^TA is a square matrix.
b) If A^TA is invertible, then A is invertible.
c) If A is invertible, then A^TA is invertible.
d) If $A\vec{x}=\vec{b}$ is a consistent linear system, then $A^TA\vec{x}=A^T\vec{b}$ is also consistent.
e) If $A\vec{x}=\vec{b}$ is an inconsistent linear system, then $A^TA\vec{x}=A^T\vec{b}$ is also inconsistent.
f) Every linear system has a least squares solution.
g) Every linear system has a unique least squares solution.
h) If A is an $m\times n$ matrix with linearly independent columns and $\vec{b}$ is in R^m, then $A\vec{x}=\vec{b}$ has a unique least squares solution.

18. A certain experiment produces the data $\{(1, 1.8), (2, 2.7), (3, 3.4), (4, 3.8), (5, 3.9)\}$.
Find the function that it will fit these data in the form of $y=\beta_1x+\beta_2x^2$

19. According to Kepler's first law, a comet should have an ellipse, parabolic, or hyperbolic orbit (with gravitational attractions from the planets ignored). In suitable polar coordinates, the position (r, υ) of a comet satisfies an equation of the form

$$r = \beta + e(r \cdot \cos\upsilon)$$

Where β is a constant and e is the eccentricity of the orbit, with $0 \le e < 1$ for an ellipse, $e = 1$ for a parabolic, and $e > 1$ for a hyperbola.

Suppose observations of a newly discovered comet provide the data below.

υ	.88	1.10	1.42	1.77	2.14
r	3.00	2.30	1.65	1.25	1.01

Determine the type of orbit, and predict where the orbit will be when $\upsilon = 4.6$ $(radians)$?

20. To measure the takeoff performance of an airplane, the horizontal position of the plane was measured every second, from $t = 0$ to $t = 12$

The position (in *feet*) were:

0, 8.8, 29.9, 62.0, 104.7, 159.1, 222.0, 294.5, 380.4, 471.1, 571.7, 686.8, *and* 809.2

a) Find the least square cubic curve $y = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$ for these data.

b) Estimate the velocity of the plane when $t = 4.5$ *sec*, using the result from part (*a*).

Lecture Four

Section 4.1 – Matrix Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

Definition

If V and W are vector spaces, and if f is a function with domain V and codomain W, then we say that f is a transformation from V to W or that f maps V to W, which we denote by writing

$$f : V \to W$$

In the special case where $V = W$, the transformation is also called an operator on V.

Matrix Transformation

$$\begin{aligned}
w_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
w_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
\vdots \quad & \quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\
w_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n
\end{aligned}$$

Which we can write in matrix formation

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\vec{w} = A\vec{x}$$

Although we could view this as a linear system, we will view it instead as a transformation that maps the column vector $\vec{x}$ in $\mathbb{R}^n$ into the column vector $\vec{w}$ in $\mathbb{R}^m$ by multiplying $\vec{x}$ on the left by A. We call this a ***matrix transformation*** or ***function*** or ***mapping T*** from $\mathbb{R}^n$ to $\mathbb{R}^m$ (or ***matrix operator*** if $m = n$) and we denote it by

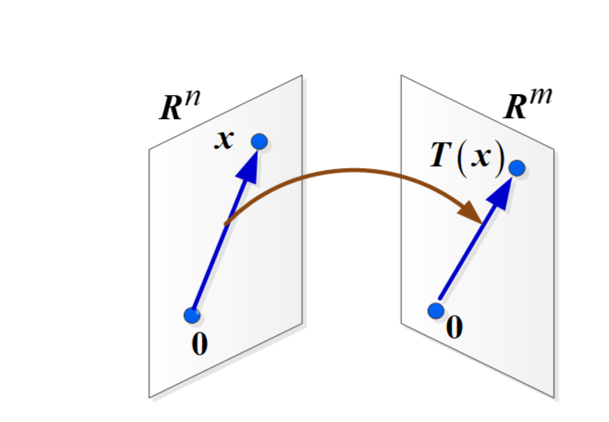
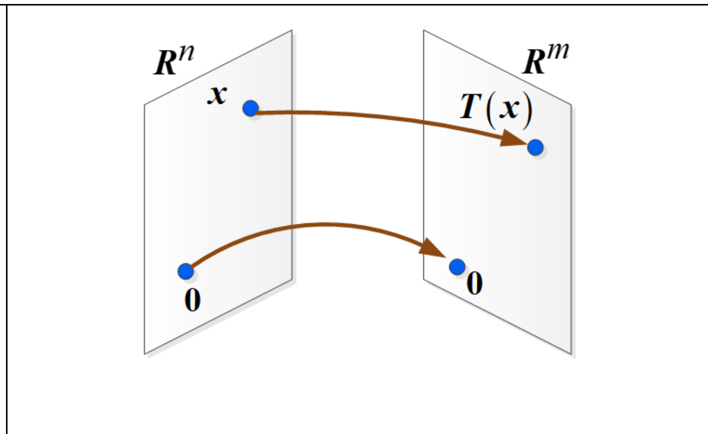
$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

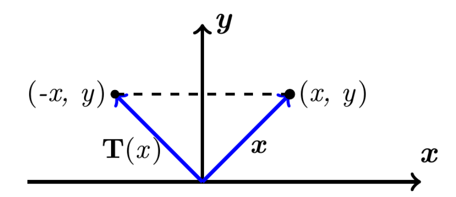
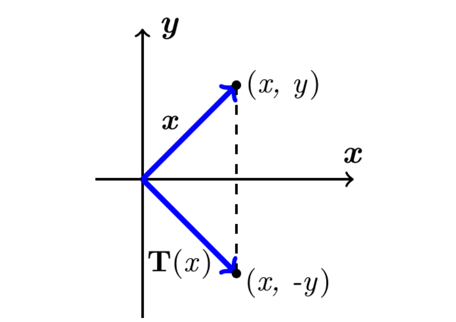
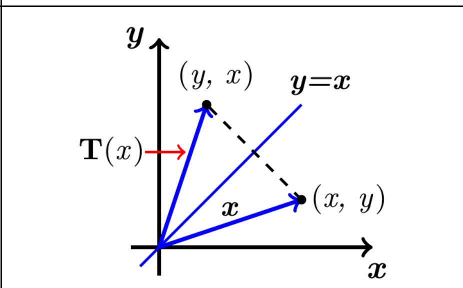
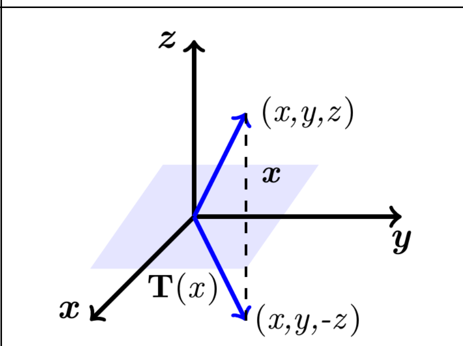
$\mathbb{R}^n$ is called the domain of T

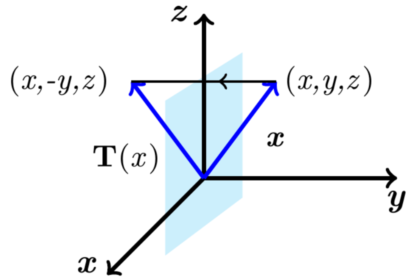
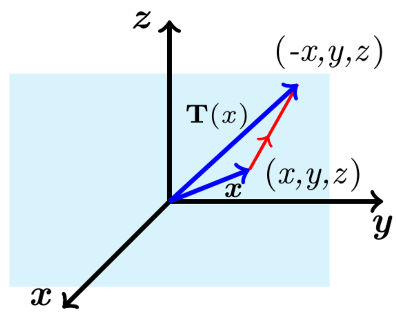
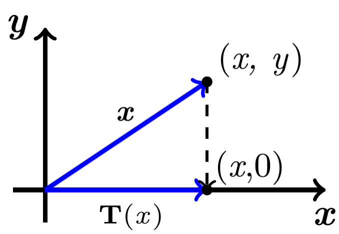
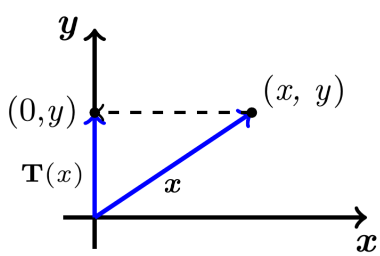
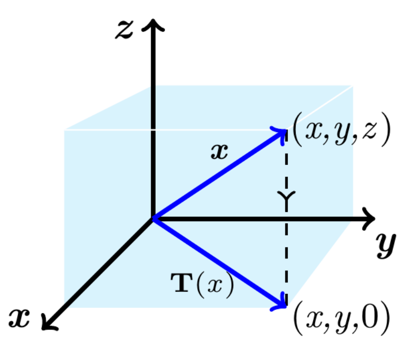
$\mathbb{R}^m$ is called the codomain of T

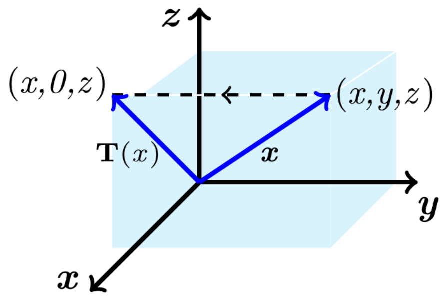
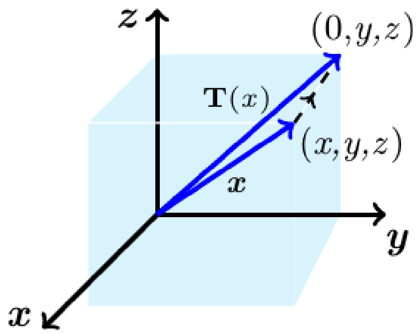
For $\vec{x}$ in $\mathbb{R}^n$, the vector $T(\vec{x})$ in $\mathbb{R}^m$ is called the image of $\vec{x}$ (under the action of T)

The set of all images $T(\vec{x})$ is called the range of T.

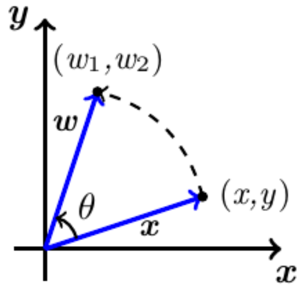
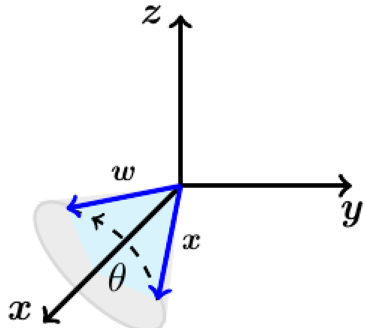
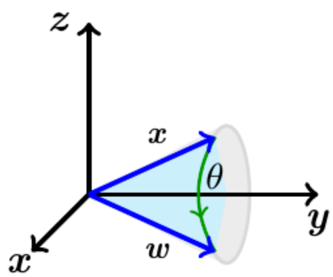
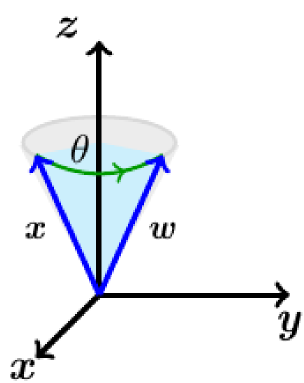
	
T *maps vectors to vectors*	***T*** *maps points to points*

Reflection about the y-axis $T(x,y)=(-x,y)$		$T(e_1)=T(1,0)=(-1,0)$ $T(e_2)=T(0,1)=(0,1)$	$\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$
Reflection about the x-axis $T(x,y)=(x,-y)$		$T(e_1)=T(1,0)=(1,0)$ $T(e_2)=T(0,1)=(0,-1)$	$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$
Reflection about the line $y=x$ $T(x,y)=(x,-y)$		$T(e_1)=T(1,0)=(0,1)$ $T(e_2)=T(0,1)=(1,0)$	$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$
Reflection about the xy-plane $T(x,y,z)=(x,y,-z)$		$T(e_1)=T(1,0,0)=(1,0,0)$ $T(e_2)=T(0,1,0)=(0,1,0)$ $T(e_3)=T(0,0,1)=(0,0,-1)$	$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$

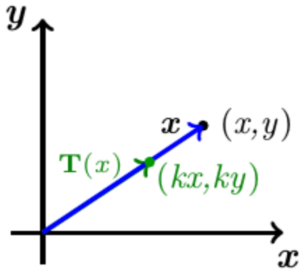
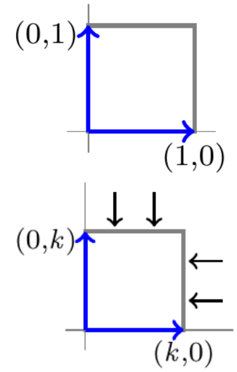
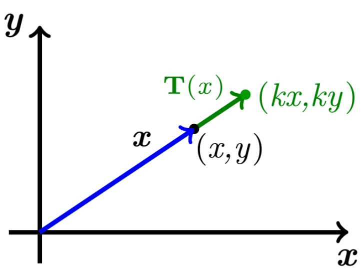
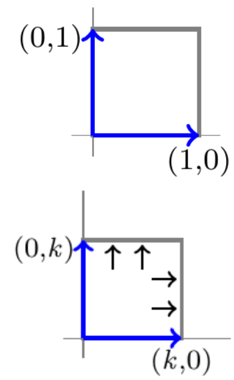
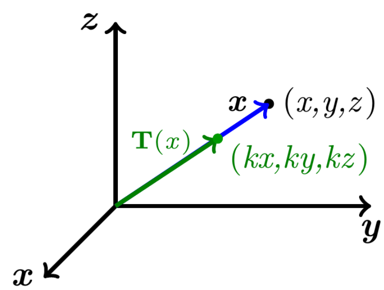
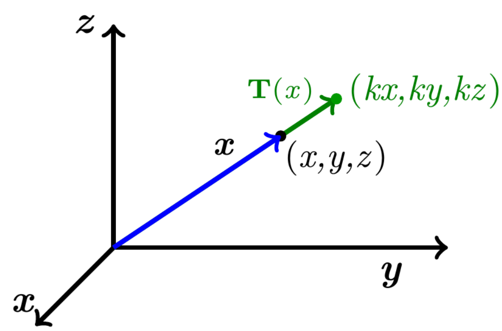
<table>
<tr>
<td>Reflection about the xy-plane
$T(x,y,z)=(x,-y,z)$</td>
<td></td>
<td>$T(e_1)=T(1,0,0)=(1,0,0)$
$T(e_2)=T(0,1,0)=(0,-1,0)$
$T(e_3)=T(0,0,1)=(0,0,1)$</td>
<td>$\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$</td>
</tr>
<tr>
<td>Reflection about the yz-plane
$T(x,y,z)=(-x,y,z)$</td>
<td></td>
<td>$T(e_1)=T(1,0,0)=(-1,0,0)$
$T(e_2)=T(0,1,0)=(0,1,0)$
$T(e_3)=T(0,0,1)=(0,0,1)$</td>
<td>$\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}$</td>
</tr>
<tr>
<td>Orthogonal projection on the x-axis
$T(x,y)=(x,0)$</td>
<td></td>
<td>$T(e_1)=T(1,0)=(1,0)$
$T(e_2)=T(0,1)=(0,0)$</td>
<td>$\begin{bmatrix}1&0\\0&0\end{bmatrix}$</td>
</tr>
<tr>
<td>Orthogonal projection on the y-axis
$T(x,y)=(0,y)$</td>
<td></td>
<td>$T(e_1)=T(1,0)=(0,0)$
$T(e_2)=T(0,1)=(0,1)$</td>
<td>$\begin{bmatrix}0&0\\0&1\end{bmatrix}$</td>
</tr>
<tr>
<td>Orthogonal projection on the xy-Plane
$T(x,y,z)=(x,y,0)$</td>
<td></td>
<td>$T(1,0,0)=(1,0,0)$
$T(0,1,0)=(0,1,0)$
$T(0,0,1)=(0,0,0)$</td>
<td>$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$</td>
</tr>
</table>

Orthogonal projection on the *xz*-Plane $T(x,y,z)=(x,0,z)$		$T(1,0,0)=(1,0,0)$ $T(0,1,0)=(0,0,0)$ $T(0,0,1)=(0,0,1)$	$\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}$
Orthogonal projection on the *yz*-Plane $T(x,y,z)=(0,y,z)$		$T(1,0,0)=(0,0,0)$ $T(0,1,0)=(0,1,0)$ $T(0,0,1)=(0,0,1)$	$\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}$

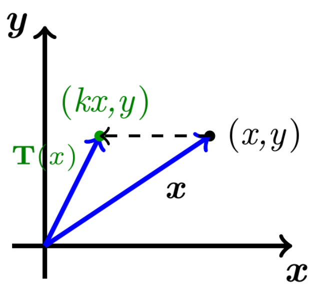
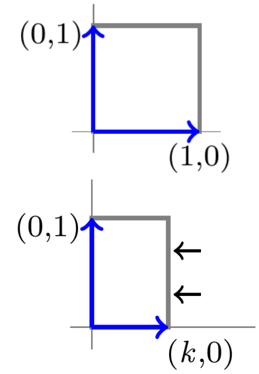
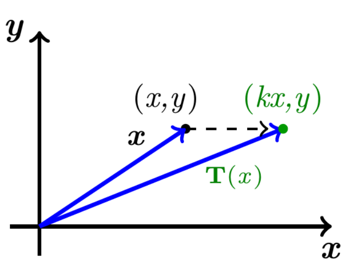
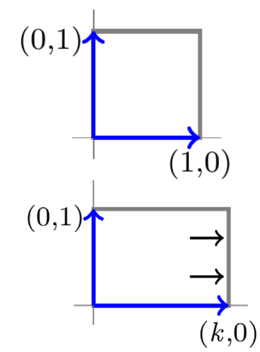
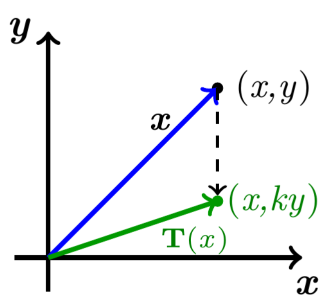
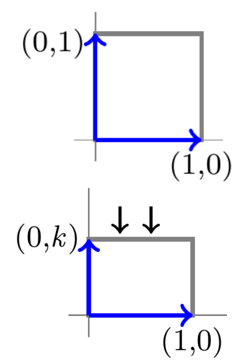
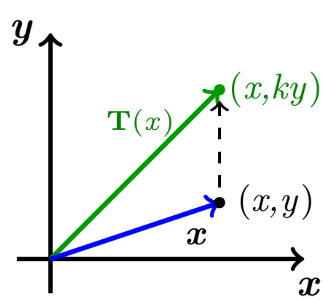
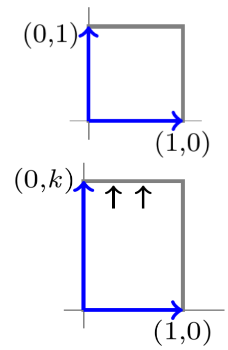
Rotation Operators

Rotation through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
Counterclockwise rotation about the positive x-axis through an angle θ		$w_1 = x$ $w_2 = y\cos\theta - z\sin\theta$ $w_3 = y\sin\theta + z\cos\theta$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$
Counterclockwise rotation about the positive y-axis through an angle θ		$w_1 = x\cos\theta + z\sin\theta$ $w_2 = y$ $w_3 = -x\sin\theta + z\cos\theta$	$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$
Counterclockwise rotation about the positive z-axis through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$ $w_3 = z$	$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

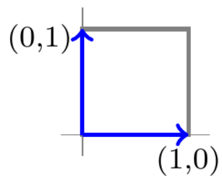
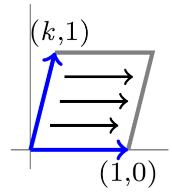
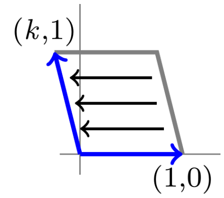
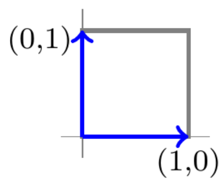
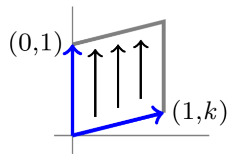
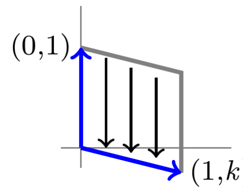
Contractions and Dilations

Contraction with factor k on $\mathbb{R}^2$ $(0 \le k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$
Dilation with factor k on $\mathbb{R}^2$ $(k > 1)$			
Contraction with factor k on $\mathbb{R}^2$ $(0 \le k \le 1)$			$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$
Dilation with factor k on $\mathbb{R}^2$ $(k \ge 1)$			

Expansion or Compression

Compression of $\mathbb{R}^2$ in the x–direction with factor k $(0 \le k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Expansion of $\mathbb{R}^2$ in the x–direction with factor k $(k > 1)$			
Compression of $\mathbb{R}^2$ in the y–direction with factor k $(0 \le k < 1)$			$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
Expansion of $\mathbb{R}^2$ in the y–direction with factor k $(k > 1)$			

Shear

Shear of $\mathbb{R}^2$ in the x–direction with factor k $T(x,\ y)=(x+ky,\ y)$		 $(k>0)$	 $(k<0)$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Shear of $\mathbb{R}^2$ in the y–direction with factor k $T(x,\ y)=(x,\ y+kx)$		 $(k>0)$	 $(k<0)$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Orthogonal Projections on Lines through the Origin

$$T\left(e_1\right)=\begin{bmatrix}\cos^2\theta \\ \sin\theta\cos\theta\end{bmatrix} \quad T\left(e_2\right)=\begin{bmatrix}\sin\theta\cos\theta \\ \sin^2\theta\end{bmatrix}$$

$$P_0=\begin{pmatrix}\cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta\end{pmatrix}$$

Example

Find the orthogonal projection of the vector $\vec{x}=(1,5)$ on the line through the origin that makes an angle of $\frac{\pi}{6}$ $(=30°)$ with the x-axis

Solution

$$\begin{aligned}P_0&=\begin{pmatrix}\cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta\end{pmatrix}\\
&=\begin{pmatrix}\cos^2\left(\frac{\pi}{6}\right) & \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right) & \sin^2\left(\frac{\pi}{6}\right)\end{pmatrix}\\
&=\begin{pmatrix}\left(\frac{\sqrt{3}}{2}\right)^2 & \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\ \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) & \left(\frac{1}{2}\right)^2\end{pmatrix}\\
&=\begin{pmatrix}\frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4}\end{pmatrix}\end{aligned}$$

$$\begin{aligned}P_0\vec{x}&=\begin{pmatrix}\frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4}\end{pmatrix}\begin{pmatrix}1\\5\end{pmatrix}\\
&=\begin{pmatrix}\frac{3+5\sqrt{3}}{4} \\ \frac{\sqrt{3}+5}{4}\end{pmatrix}\\
&\approx\begin{pmatrix}2.91\\1.68\end{pmatrix}\end{aligned}$$

Example

Define a linear transformation $T:\ \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$T(\vec{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

Find the images under T of $\vec{u} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\quad \vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\quad \vec{u} + \vec{v} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$

Solution

$$T(\vec{u}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

$$T(\vec{v}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

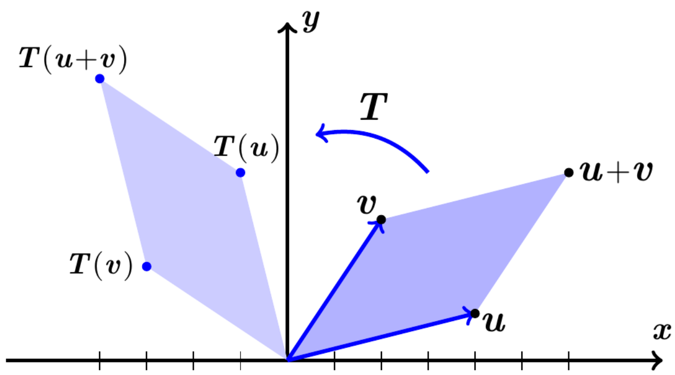
$$= \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$T(\vec{u} + \vec{v}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ 6 \end{pmatrix}$$

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$$

$$\begin{pmatrix} -4 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$



Four Fundamental Subspaces

1. The ***row space*** is $C\left(A^T\right)$, a subspace of $\mathbb{R}^n$.
2. The ***column space*** is $C(A)$, a subspace of $\mathbb{R}^m$.
3. The ***nullspace*** is $N(A)$, a subspace of $\mathbb{R}^n$.
4. The ***left nullspace*** is $N\left(A^T\right)$, a subspace of $\mathbb{R}^m$.

*The Four Subspaces for **R***

Consider the matrix 3 by 5:

$$\begin{bmatrix} 1 & 3 & 5 & 0 & 9 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{ll} m=3 & \textit{pivot rows } 1 \textit{ and } 2 \\ n=5 & \\ r=2 & \textit{pivot columns } 1 \textit{ and } 4 \end{array}$$

1. Rows 1 and 2 are a basis. The row space contains combination of all 3 rows.

   The ***row space*** of $\mathbb{R}$ has dimension 2 (= ***rank***).
   The dimension of the row space is r. The nonzero rows of ***R*** form a basis.

2. The ***column space*** of ***R*** has dimension $r = 2$.

   The pivot columns 1 and 4 form a basis. They are independent because they start with the r by r identity matrix.
   There are 3 special solutions:

$$\begin{array}{rl} C_2 = 3C_1 & \textit{The special solution is } (-3,\ 1,\ 0,\ 0,\ 0) \\ C_3 = 5C_1 & \textit{The special solution is } (-5,\ 0,\ 1,\ 0,\ 0) \\ C_5 = 9C_1 + 8C_2 & \textit{The special solution is } (-9,\ 0,\ 0,\ -8,\ 1) \end{array}$$

   The dimension of the column space is r. The pivot columns form a basis.

3. The ***nullspace*** has dimension $n - r = 5 - 2 = 3$ (free variables). Here x_2, x_3, x_5 are free (no pivots in those columns). They yield the three special solutions to $R\vec{x} = 0$. Set a free variable to 1, and solve for x_1 and x_4.

$$s_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad s_3 = \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad s_5 = \begin{bmatrix} -9 \\ 0 \\ 0 \\ -8 \\ 1 \end{bmatrix}$$

$Rx = 0$ has the complete solution: $x = x_2 s_2 + x_3 s_3 + x_5 s_5$

The nullspace has dimension $n - r$. The special solutions form a basis.

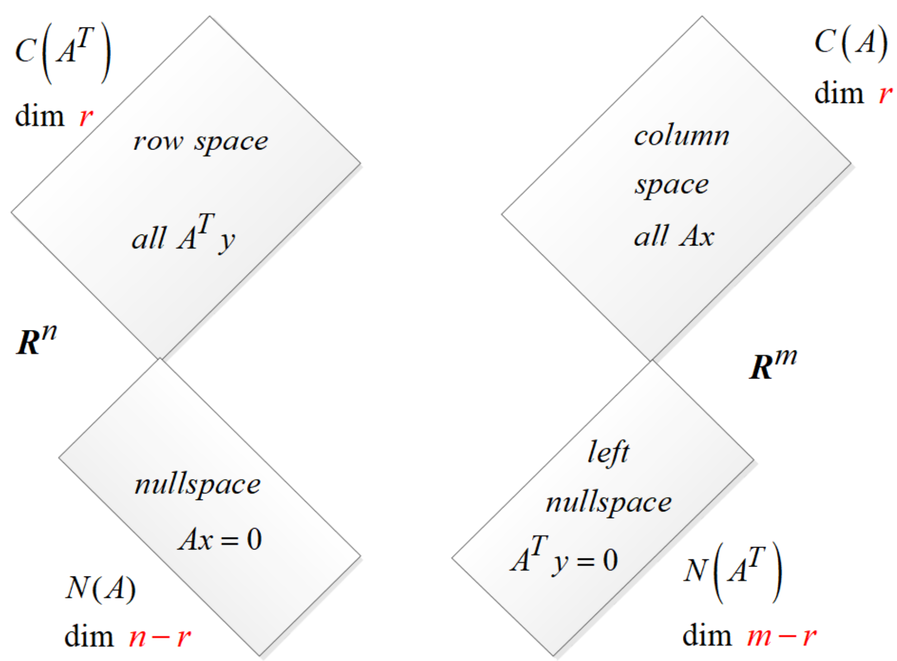
4. The ***nullspace*** of R^T has dimension $m - r = 3 - 2 = 1$

The equation $R^T y = 0$: $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 0 & 0 \\ 0 & 1 & 0 \\ 9 & 8 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} y_1 = 0 \\ y_2 = 0 \\ y_3 \ \textit{anything} \end{cases}$

The nullspace of R^T contains all vectors $y = (0,\ 0,\ y_3)$ and it is the line of the basis vector $(0,\ 0,\ 1)$.

The left nullspace has dimension m – r. The solutions are $y = (0, \ldots, y_{r+1}, \ldots, y_m)$

- In $\mathbb{R}^n$ the row space and nullspace have dimensions r and $n - r$ (adding to n)
- In $\mathbb{R}^m$ the column space and left nullspace have dimensions r and $m - r$ (total m)



The Four Subspaces for A

The subspace dimensions for A are the same as for R.

These matrices are connected by an invertible matrix E. $EA = R$ *and* $A = E^{-1}R$

1. ***A** has the same row space as **R***. Same dimension r and same basis
   Every row of ***A*** is a combination of the rows of ***R***. Also every row of ***R*** is a combination of the rows of ***A***.

2. *The column space of **A** has dimension r*. The number of independent columns equals the number of independent rows.

3. ***A** has the same nullspace as **R***. Dimension $n - r$ and same basis.

   (*dimension of column space*) + (*dimension of nullspace*) = *dimension of* $\boldsymbol{R}^n$

4. *The left nullspace **A*** (the nullspace of A^T) *has dimension* $m - r$.

Fundamental Theorem of Linear Algebra, (Part 1)

The column space and row space both have dimension r.

The nullspaces have dimensions $n - r$ and $m - r$.

Example

Consider $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

A has $m = 1$, $n = 3$, and rank: $r = 1$.

The row space is a line in $\mathbb{R}^3$.

The nullspace is the plane $Ax = x_1 + 2x_2 + 3x_3 = 0$. This plane has dimension 2 (which is 3 – 1).

The columns of this is 1 by 3 matrix are in $\mathbb{R}^1$. The column space is all of $\mathbb{R}^1$.

The left nullspace contains only the zero vector.

The only solution to $A^T y = 0$ *is* $y = 0$, the only combination of the row that gives the zero row.

Thus, $N\left(A^T\right)$ *is* $\mathbb{Z}$, the zero space with dimension 0 ($m - r$). In $\mathbb{R}^m$ the dimensions (1 + 0) = 1.

Example

Consider $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

A has $m = 2$, $n = 3$, and rank: $r = 1$.

The row space is a line in $\mathbb{R}^3$.

The nullspace is the plane $x_1 + 2x_2 + 3x_3 = 0$. This plane has dimension 3 (1 + 2).

The columns are multiples of the first column (1, 1).

The left nullspace contains more than one zero vector. The solution to $A^T \vec{y} = 0$ has the solution $y = (1, \ -1)$.

The column space and nullspace are perpendicular lines in $\mathbb{R}^2$. Their dimensions are 1 and 1 = 2.

Column space = line through $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Left nullspace = line through $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Exercises Section 4.1 – Matrix Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

1. Find the standard matrix for the transformation defined by the equations

a) $\begin{cases} w_1 = 2x_1 - 3x_2 + 4x_4 \\ w_2 = 3x_1 + 5x_2 - x_4 \end{cases}$

b) $\begin{cases} w_1 = 7x_1 + 2x_2 - 8x_3 \\ w_2 = \quad\quad -x_2 + 5x_3 \\ w_3 = 4x_1 + 7x_2 - x_3 \end{cases}$

c) $\begin{cases} w_1 = x_1 \\ w_2 = x_1 + x_2 \\ w_3 = x_1 + x_2 + x_3 \\ w_4 = x_1 + x_2 + x_3 + x_4 \end{cases}$

(2 – 8) Find the standard matrix for the operator T defined by the formula

2. $T\left(x_1, x_2\right) = \left(2x_1 - x_2,\ x_1 + x_2\right)$

3. $T\left(x_1, x_2, x_3\right) = \left(x_1 + 2x_2 + x_3,\ x_1 + 5x_2,\ x_3\right)$

4. $T\left(x_1, x_2, x_3\right) = \left(4x_1,\ 7x_2,\ -8x_3\right)$

5. $T\left(x_1, x_2\right) = \left(x_2,\ -x_1,\ x_1 + 3x_2,\ x_1 - x_2\right)$

6. $T\left(x_1, x_2, x_3\right) = (0,\ 0,\ 0,\ 0,\ 0)$

7. $T\left(x_1, x_2, x_3, x_4\right) = \left(x_4,\ x_1,\ x_3,\ x_2,\ x_1 - x_3\right)$

8. $T\left(x_1, x_2, x_3, x_4\right) = \left(7x_1 + 2x_2 - x_3 + x_4,\ x_2 + x_3,\ -x_1\right)$

(9 – 8) Plot $\vec{u} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and their images under the given transformation T

9. $T(\vec{x}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

10. $T(\vec{x}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

11. $T\left(\vec{x}\right)=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

12. $T\left(\vec{x}\right)=\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

13. $T\left(\vec{x}\right)=\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Section 4.2 – General Linear Transformations

Definition

A transformation T assigns an output $T(\vec{v})$ to each input vector $\vec{v}$. The transformation is ***linear*** if it meets these requirements for all $\vec{v}$ and $\vec{w}$:

$$\begin{cases} T(c\vec{v}) = cT(\vec{v}) \\ T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w}) \end{cases}$$

We can combine both into one: $\boxed{T(c\vec{v}+d\vec{w}) = cT(\vec{v}) + dT(\vec{w})}$

Theorem

If $T: V \to W$ is a linear transformation, then:

1. $T(0) = 0$
2. $T(\vec{u}-\vec{v}) = T(\vec{u}) - T(\vec{v})$ for all $\vec{u}$ and $\vec{v}$ in V.

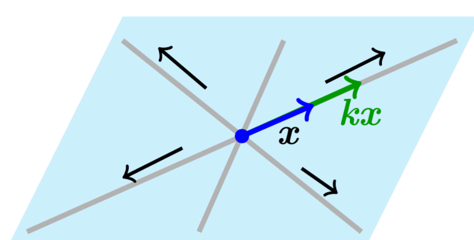
Example

If V is a vector space and k is any scalar, then the mapping $T: V \to W$ given by $T(\vec{x}) = k\vec{x}$ is a linear operator on V, for if c is any scalar and if $\vec{u}$ and $\vec{v}$ are any vectors in V, then

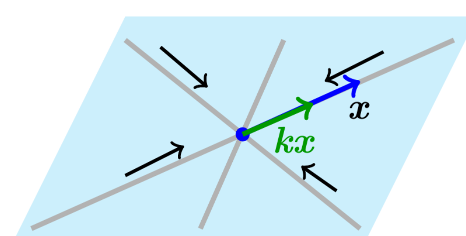
$$\begin{aligned} T(c\vec{u}) &= k(c\vec{u}) \\ &= c(k\vec{u}) \\ &= cT(\vec{u}) \end{aligned}$$

$$\begin{aligned} T(\vec{u}+\vec{v}) &= k(\vec{u}+\vec{v}) \\ &= k\vec{u}+k\vec{v} \\ &= T(\vec{u})+T(\vec{v}) \end{aligned}$$

If $0 < k < 1$, then T is called ***contraction*** of V with factor k, and if $k > 1$, then T is called ***dilation*** of V with factor k



Dilation of V



Contraction of V

Example

Determine if the given function T is a linear transformation. Also give the domain and range of T; if T is linear, find the A such $T = f_A$. $\quad T(x, y, z) = (z - x, z - y)$

Solution

Let $\vec{u} = (x_1, y_1, z_1)$ *and* $\vec{v} = (x_2, y_2, z_2)$

$$\begin{aligned} T(\vec{u} + \vec{v}) &= T(x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2) \\ &= (z_1 + z_2 - (x_1 + x_2),\ z_1 + z_2 - (y_1 + y_2)) \\ &= (z_1 + z_2 - x_1 - x_2,\ z_1 + z_2 - y_1 - y_2) \\ &= (z_1 - x_1,\ z_1 - y_1) + (z_2 - x_2,\ z_2 - y_2) \\ &= T(x_1, y_1, z_1) + T(x_2, y_2, z_2) \\ &= T(\vec{u}) + T(\vec{v}) \end{aligned}$$

$$\begin{aligned} T(r\vec{u}) &= T(rx_1, ry_1, rz_1) \\ &= (rz_1 - rx_1,\ rz_1 - ry_1) \\ &= r(z_1 - x_1,\ z_1 - y_1) \\ &= rT(\vec{u}) \end{aligned}$$

Since $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(r\vec{u}) = rT(\vec{v})$

Then function T is a linear transformation.

Domain: $T:\ \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{aligned} T(x, y, z) &= (z - x, z - y) \\ &= \begin{pmatrix} -x + z \\ -y + z \end{pmatrix} \end{aligned}$$

$$A = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \end{array}$$

Example – the Zero Transformations

Let V and W be any vector spaces. The mapping $T : V \to W$ such that $T(v) = 0$ for every $\vec{v}$ in V is a linear transformation called the zero transformation. To see that T is linear, observe that:

$$T(\vec{u}+\vec{v}) = 0, \quad T(\vec{u}) = 0, \quad T(\vec{v}) = 0, \quad \textit{and} \quad T(k\vec{u}) = 0$$

Therefore; $\quad T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \quad \textit{and} \quad T(k\vec{u}) = kT(\vec{u})$

Example

Choose a fixed vector $\vec{a} = (1,\ 3,\ 4)$, and let $T(v)$ be the dot product $\vec{a} \bullet \vec{v}$:

Solution

Let $\vec{v} = (v_1,\ v_2,\ v_3)$

$$\begin{aligned} T(\vec{v}) &= \vec{a} \bullet \vec{v} \\ &= (1,\ 3,\ 4) \bullet (v_1,\ v_2,\ v_3) \\ &= v_1 + 3v_2 + 4v_3 \end{aligned}$$

This is linear. The inputs v come from three–dimensional space, so $V = \mathbb{R}^3$. The output just numbers, so the output space is $W = \mathbb{R}^1$. We are multiplying by the row matrix A = [1, 3, 4].
Then $T(\vec{v}) = A\vec{v}$

Example

Show that the length $T(\vec{v}) = \|\vec{v}\|$ is not linear.

Solution

$$\|\vec{v}+\vec{w}\| \overset{?}{=} \|\vec{v}\| + \|\vec{w}\|$$

There are not equal because the sides of a triangle satisfy an inequality $\|\vec{v}+\vec{w}\| \le \|\vec{v}\| + \|\vec{w}\|$

$$\|c\vec{v}\| \overset{?}{=} c\|\vec{v}\|$$

Not - because the length $\|-\vec{v}\| \ne -\|\vec{v}\|$

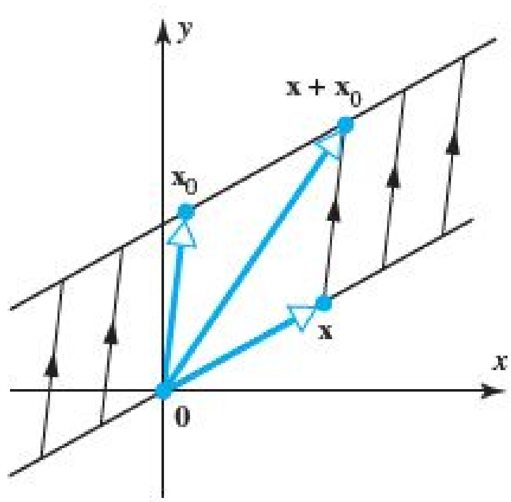
Example

If $\vec{x}_0$ is a fixed nonzero vector in $\mathbb{R}^2$, then the transformation

$$T(\vec{x}) = \vec{x} + \vec{x}_0$$

It has a geometric effect of translating each point $\vec{x}$ in a direction parallel to $\vec{x}_0$ through a distance of $\|\vec{x}_0\|$.

This cannot be a linear transformation since $T(0) = \vec{x}_0$



Theorem

Let $T: V \to W$ be the linear transformation, where *V* is finite dimensional. If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for *V*, then the image of any vector $\vec{v}$ in *V* can be expressed as

$$T(\vec{v}) = c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \cdots + c_n T(\vec{v}_n)$$

Where $c_1, c_2, \cdots, c_n$ are the coefficients required to express $\vec{v}$ as a linear combination of the vectors in *S*.

Example

Consider the basis $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for ∞^3, where

$$\vec{v}_1 = (1,\ 1,\ 1) \quad \vec{v}_2 = (1,\ 1,\ 0) \quad \vec{v}_3 = (1,\ 0,\ 0)$$

Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation for which

$$T(\vec{v}_1) = (1,\ 0), \quad T(\vec{v}_2) = (2,\ -1), \quad T(\vec{v}_3) = (4,\ 3)$$

Find a formula for $T(\vec{x}_1, \vec{x}_2, \vec{x}_3)$, and then use that formula to compute $T(2,\ -3,\ 5)$

Solution

$$(\vec{x}_1, \vec{x}_2, \vec{x}_3) = c_1(1,\ 1,\ 1) + c_2(1,\ 1,\ 0) + c_3(1,\ 0,\ 0)$$

$$\begin{cases} c_1 + c_2 + c_3 = x_1 \\ c_1 + c_2 \qquad\ \ = x_2 \\ c_1 \qquad\qquad\ \ = x_3 \end{cases}$$

$$\begin{cases} c_3 = x_1 - x_2 \\ c_2 = x_2 - x_3 \\ \ \ c_1 = x_3 \end{cases}$$

$$\begin{aligned}(\vec{x}_1, \vec{x}_2, \vec{x}_3) &= x_3(1,\ 1,\ 1) + (x_2 - x_3)(1,\ 1,\ 0) + (x_1 - x_2)(1,\ 0,\ 0) \\ &= x_3\vec{v}_1 + (x_2 - x_3)\vec{v}_2 + (x_1 - x_2)\vec{v}_3\end{aligned}$$

$$\begin{aligned}T(\vec{x}_1, \vec{x}_2, \vec{x}_3) &= x_3 T(\vec{v}_1) + (x_2 - x_3)T(\vec{v}_2) + (x_1 - x_2)T(\vec{v}_3) \\ &= x_3(1,\ 0) + (x_2 - x_3)(2,\ -1) + (x_1 - x_2)(4,\ 3) \\ &= (4x_1 - 2x_2 - x_3,\ \ 3x_1 - 4x_2 + x_3)\end{aligned}$$

$$\begin{aligned}T(2,-3,5) &= (4(2) - 2(-3) - 5,\ \ 3(2) - 4(-3) + 5) \\ &= (9,\ 23)\end{aligned}$$

Example

T is the transformation that rotates every vector by 30°, the domain is the xy-plane (where the input vector $\vec{v}$ is). The range is also the xy-plane (where the rotated $T(\vec{v})$ is). Is the rotation linear?

Solution

Yes it is. We can rotate two vectors and add the results. The sum of rotation $T(\vec{v})+T(\vec{w})$ is the same as the rotation $T(\vec{v}+\vec{w})$ of the sum.

The whole plane is turning together, in this linear transformation.

Definition

If $T: V \rightarrow W$ is a linear transformation, then the set of vectors in V that T maps into $\vec{0}$ is called ***kernel*** of T and is denoted by ***ker***(T). The set of all vectors in W that are images under T of at least one vector in V is called the ***range*** of T and is denoted by ***R***(T).

Note:

Transformations have a language of their own. Where there is no matrix, we can't talk about a column space. But the idea can be rescued and used. The column space consisted of all ouputs $A\vec{v}$.

The nullspace consisted of all inputs for which $A\vec{v}-\vec{0}$. Translate those into "range" and "kernel"

Range of T = set of all outputs $T(\vec{v})$: corresponds to column space

Kernel of T = set of all outputs for which $T(\vec{v})=0$: corresponds to nullspace

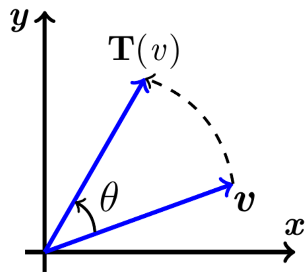
Example

Project every 3-dimensional vector down onto the xy plane.
The range is that plane, which contains every $T(\vec{v})$.

The kernel is the z axis (which projects down to zero). This projection is linear.

Example – *Kernel and Range of a Rotation*

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator that rotates each vector in the *xy*–plane through the angle *θ*. Since every vector in the *xy*–plane can be obtained by rotating some vector through the angle *θ*, it follows that $R(T) = \mathbb{R}^2$.



Moreover, the only vector that rotates into $\vec{0}$ is $\vec{0}$, so $\ker(T) = \{0\}$

Theorem

If $T: V \to W$ is a linear transformation, then:

1. The *kernel* of *T* is a subspace of *V*
2. The *range* of *T* is a subspace of *W*

Theorem

If $T: V \to W$ is a linear transformation from an *n*-dimensional vector space *V* to a vector space *W*, then

$$rank(T) + nullity(T) = n$$

Example

Project every 3-dimensional vector down onto horizontal plane $z = 1$.
The vector $\vec{v} = (x, y, z)$ is transformed to $T(\vec{v}) = (x, y, 1)$. This transformation is not linear, it doesn't even transform $\vec{v} = \vec{0}$ into $T(\vec{v}) = \vec{0}$.

Multiply every 3-dimensional vector by a 3 by 3 matrix *A*. This is definitely a linear transformation

$$T(\vec{v} + \vec{w}) = A(\vec{v} + \vec{w}) \quad \text{which does equal} \quad A\vec{v} + A\vec{w} = T(\vec{v}) + T(\vec{w})$$

Example

Suppose A is an invertible matrix. The kernel of T is the zero vector; the range $\boldsymbol{W}$ equals the domain $\boldsymbol{V}$.
Another linear transformation is multiplication by A^{-1}.
This is the inverse transformation T^{-1}, which brings every vector $T(\vec{v})$ back to $\vec{v}$:

$$T^{-1}\left(T(\vec{v})\right) = \vec{v} \quad \text{matches the matrix multiplication} \quad A^{-1}\left(A\vec{v}\right) = \vec{v}$$

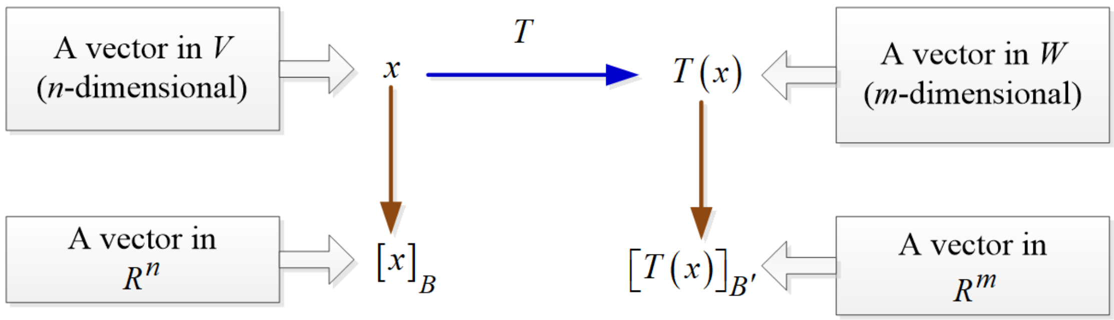
Are all linear transformation produced by matrices?

Each m by n matrix does produce a linear transformation from $V = \mathbb{R}^n$ to $W = \mathbb{R}^m$. When a linear T is described as a "rotation" or "projection" or "..." is there always a matrix hiding behind T?

The answer is yes. This is an approach to linear algebra that doesn't start with matrices. The next section shows that we still end up with matrices.

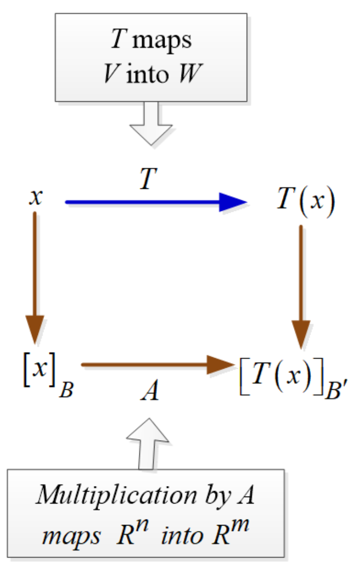
Matrices for General Linear Transformations

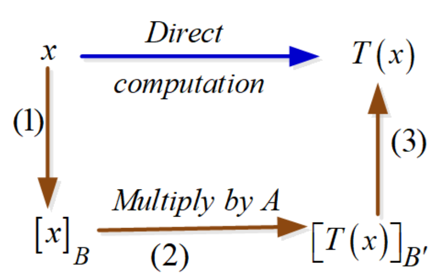
Suppose that V is an n-dimensional vector space, W is an m-dimensional vector space, and that $T: V \to W$ is a linear transformation. Suppose further that B is a basis for V, that B' is a basis for W, and that for each $\boldsymbol{x}$ in V, the coordinate matrices for $\boldsymbol{x}$ and $T(\vec{x})$ are $[\vec{x}]_B$ and $\left[T(\vec{x})\right]_{B'}$, respectively



By using matrix multiplication, we can execute the linear transformation and the following indirect procedure:

1. Compute the coordinate vector $[\vec{x}]_B$
2. Multiply $[\vec{x}]_B$ on the left by A to produce $\left[T(\vec{x})\right]_{B'}$
3. Reconstruct $T(\vec{x})$ from its coordinate vector $\left[T(\vec{x})\right]_{B'}$





$$A[\vec{x}]_B = \left[T(\vec{x})\right]_{B'}$$

Example

Let $T: P_1 \to P_2$ be the linear transformation defined by $T\left(p(x)\right) = xp(x)$

Find the matrix for T with respect to the standard bases

$$B = \left\{\vec{u}_1,\ \vec{u}_2\right\} \quad and \quad B' = \left\{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\right\}$$

Where $\vec{u}_1 = 1,\quad \vec{u}_2 = x;\quad \vec{v}_1 = 1,\quad \vec{v}_2 = x,\quad \vec{v}_3 = x^2$

Solution

$$\begin{aligned} T\left(\vec{u}_1\right) &= T(1) \\ &= x \end{aligned}$$

$$\begin{aligned} T\left(\vec{u}_2\right) &= T(x) \\ &= x(x) \\ &= x^2 \end{aligned}$$

$$\left[T\left(\vec{u}_1\right)\right]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\left[T\left(\vec{u}_2\right)\right]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The matrix for T with respect to B and B' is

$$\begin{aligned} [T]_{B,B'} &= \left[\left[T\left(\vec{u}_1\right)\right]_{B'} \middle| \left[T\left(\vec{u}_2\right)\right]_{B'}\right] \\ &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Example

Let $T : P_1 \to P_2$ be the linear transformation defined by $T\left(p(x)\right) = xp(x)$ describe in the following figure to perform the computation

$$T(a+bx) = x(a+bx) = ax + bx^2$$

$$\begin{tikzcd}
x \arrow[rr, "Direct"', "computation"] \arrow[d, "(1)"'] & & T(x) \\
{[x]_B} \arrow[rr, "Multiply\ by\ {[T]_{B,B'}}", "(2)"'] & & {[T(x)]_{B'}} \arrow[u, "(3)"']
\end{tikzcd}$$

$B = \{1,\ x\}$ *and* $B' = \left\{1,\ x,\ x^2\right\}$

Solution

Step 1: The coordinates matrix for $\vec{x} = ax + b$ relative to the basis $B = \{1,\ x\}$ is

$$[\boldsymbol{x}]_B = \begin{bmatrix} a \\ b \end{bmatrix}$$

Step 2: Multiply $[\vec{x}]_B$ by the matrix $[T]_{B,B'}$ found in previous example, we obtain

$$\begin{aligned}
[T]_{B,B'}[\vec{x}]_B &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ a \\ b \end{bmatrix} \\
&= \left[T(\vec{x})\right]_{B'}
\end{aligned}$$

Step 3: Reconstructing $T(\vec{x}) = T(ax+b)$ from $\left[T(\vec{x})\right]_{B'}$ we obtain

$$\begin{aligned}
T(ax+b) &= 0 + ax + bx^2 \\
&= ax + bx^2
\end{aligned}$$

Exercises Section 4.2 – General Linear Transformations

1. The matrix $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ gives a shearing transformation $T(x,y)=(x,\ 3x+y)$.

What happens to (1, 0) and (2, 0) on the x-axis.

What happens to the points on the vertical lines $x = 0$ and $x = a$?.

2. A nonlinear transformation T is invertible if every $\vec{b}$ in the output space comes from exactly one x in the input space. $T(\vec{x})=\vec{b}$ always has exactly one solution. Which of these transformation (on real numbers $\vec{x}$ is invertible and what is T^{-1}? None are linear, not even T_3. When you solve $T(\vec{x})=\vec{b}$, you are inverting T:

$$T_1(\vec{x})=x^2 \quad T_2(\vec{x})=x^3 \quad T_3(\vec{x})=x+9 \quad T_4(\vec{x})=e^x \quad T_5(\vec{x})=\frac{1}{x} \quad \textit{for nonzero } x\text{'s}$$

3. If S and T are linear transformations, is $S(T(\vec{v}))$ linear or quadratic?

a) If $S(\vec{v})=\vec{v}$ and $T(\vec{v})=\vec{v}$, then $S(T(\vec{v}))=\vec{v}$ or $\vec{v}^2$?

b) $S(\vec{w}_1+\vec{w}_2)=S(\vec{w}_1)+S(\vec{w}_2)$ and $T(\vec{v}_1+\vec{v}_2)=T(\vec{v}_1)+T(\vec{v}_2)$ combine into

$$S(T(\vec{v}_1+\vec{v}_2))=S(\_\_\_\_\_)=\_\_\_\_+\_\_\_\_\_\_$$

4. Find the range and kernel (like the column space and nullspace) of T:

a) $T(v_1,\ v_2)=(v_2,\ v_1)$

b) $T(v_1,\ v_2,\ v_3)=(v_1,\ v_2)$

c) $T(v_1,\ v_2)=(0,\ 0)$

d) $T(v_1,\ v_2)=(v_1,\ v_1)$

5. M is any 2 by 2 matrix and $A=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. The transformation T is defined by $T(M)=AM$. What rules of matrix multiplication show that T is linear?

6. Which of these transformations satisfy $T(\vec{v}+\vec{w})=T(\vec{v})+T(\vec{w})$ and which satisfy $T(c\vec{v})=cT(\vec{v})$?

a) $T(\vec{v})=\dfrac{\vec{v}}{\|\vec{v}\|}$

b) $T(\vec{v})=v_1+v_2+v_3$

c) $T(\vec{v})=(v_1,\ 2v_2,\ 3v_3)$

d) $T(\vec{v})=$ largest component of $\vec{v}$.

7. Consider the basis $S=\left\{\vec{v}_1, \vec{v}_2, \vec{v}_3\right\}$ for R^3, where $\vec{v}_1=(1,\ 1,\ 1)$ $\vec{v}_2=(1,\ 1,\ 0)$ $\vec{v}_3=(1,\ 0,\ 0)$ and let $T:\ \mathbb{R}^3\to\mathbb{R}^3$ be the linear transformation for which

$$T\left(\vec{v}_1\right)=(2,\ -1,\ 4),\quad T\left(\vec{v}_2\right)=(3,\ 0,\ 1),\quad T\left(\vec{v}_3\right)=(-1,\ 5,\ 1)$$

Find a formula for $T\left(\vec{x}_1, \vec{x}_2, \vec{x}_3\right)$, and then use that formula to compute $T(2,\ 4,\ -1)$

8. Consider the basis $S=\left\{\vec{v}_1, \vec{v}_2, \vec{v}_3\right\}$ for R^3, where $\vec{v}_1=(1,\ 2,\ 1)$ $\vec{v}_2=(2,\ 9,\ 0)$ $\vec{v}_3=(3,\ 3,\ 4)$ and let $T:\ \mathbb{R}^3\to\mathbb{R}^3$ be the linear transformation for which

$$T\left(\vec{v}_1\right)=(1,\ 0),\quad T\left(\vec{v}_2\right)=(-1,\ 1),\quad T\left(\vec{v}_3\right)=(0,\ 1)$$

Find a formula for $T\left(\vec{x}_1, \vec{x}_2, \vec{x}_3\right)$, and then use that formula to compute $T(7,\ 13,\ 7)$

9. let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be vectors in a vector space V, and let $T: V\to R^3$ be the linear transformation for which $T\left(\vec{v}_1\right)=(1,\ -1,\ 2)$, $T\left(\vec{v}_2\right)=(0,\ 3,\ 2)$, $T\left(\vec{v}_3\right)=(-3,\ 1,\ 2)$.

Find $T\left(2\vec{v}_1-3\vec{v}_2+4\vec{v}_3\right)$

10. Let $T:\ \mathbb{R}^2\to\mathbb{R}^2$ be the linear operation given by the formula $T(x,\ y)=(2x-y,\ -8x+4y)$

Which of the following vectors are in $R(T)$

a) $(1,\ -4)$ ***b***) $(5,\ 0)$ ***c***) $(-3,\ 12)$

11. Let $T:\ \mathbb{R}^2\to\mathbb{R}^2$ be the linear operation given by the formula $T(x,\ y)=(2x-y,\ -8x+4y)$

Which of the following vectors are in $\ker(T)$

a) $(5,\ 10)$ ***b***) $(3,\ 2)$ ***c***) $(1,\ 1)$

12. Let $T:\ \mathbb{R}^4\to\mathbb{R}^3$ be the linear operation given by the formula

$$T\left(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\right)=\left(4x_1+x_2-2x_3-3x_4,\ 2x_1+x_2+x_3-4x_4,\ 6x_1-9x_3+9x_4\right)$$

Which of the following vectors are in $R(T)$

a) $(0,\ 0,\ 6)$ ***b***) $(1,\ 3,\ 0)$ ***c***) $(2,\ 4,\ 1)$

13. Let $T:\ \mathbb{R}^4\to\mathbb{R}^3$ be the linear operation given by the formula

$$T\left(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\right)=\left(4x_1+x_2-2x_3-3x_4,\ 2x_1+x_2+x_3-4x_4,\ 6x_1-9x_3+9x_4\right)$$

Which of the following vectors are in $\ker(T)$

a) $(3,\ -8,\ 2,\ 0)$ ***b***) $(0,\ 0,\ 0,\ 1)$ ***c***) $(0,\ -4,\ 1,\ 0)$

14. Determine if the given function T is a linear transformation

$$T: M_{22} \to M_{22} \quad by \quad T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2ab & 3cd \\ 0 & 0 \end{bmatrix}$$

15. Determine if the given function T is a linear transformation

$$T: M_{22} \to M_{22} \quad by \quad T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+d & 0 \\ 0 & b+c \end{bmatrix}$$

16. Determine if the given function T is a linear transformation where A is fixed 2×3 matrix

$$T: M_{22} \to M_{23} \quad by \quad T(B) = BA$$

(17 – 25) Determine if the given function T is a linear transformation. Also give the domain and range of T; if T is linear, find the A such $T = f_A$.

17. $T(x, y) = (x^2, y)$

18. $T(x, y, z) = (2x + y,\ x - y + z)$

19. $T(x, y, z) = (z - x,\ z - y)$

20. $T(x_1, x_2, x_3) = (2x_1 - x_2 + x_3,\ \ x_2 - 4x_3)$

21. $T(x_1, x_2) = (2x_1 - x_2,\ -3x_1 + x_2,\ 2x_1 - 3x_2)$

22. $T(x_1, x_2) = (x_1 + 4x_2,\ 0,\ x_1 - 3x_2,\ x_1)$

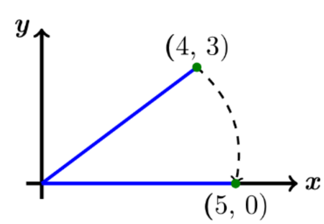
23. $T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3,\ \ x_2 - 6x_3)$

24. $T(x_1, x_2, x_3, x_4) = (x_1 + 2x_2,\ 0,\ 2x_2 + x_4,\ x_2 - x_4)$

25. $T(x_1, x_2, x_3, x_4) = 3x_1 + 4x_3 - 2x_4$

26. A Givens rotation is a linear transformation from $\mathbb{R}^n$ *to* $\mathbb{R}^n$ used in computer to create a zero entry in a vector (usually a column of a matrix). The standard matrix of a Givens rotation in $\mathbb{R}^2$ has the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}; \qquad a^2 + b^2 = 1$$



A Givens rotation in $\mathbb{R}^2$

Find a and b that $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ is rotated into $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$.

Section 4.3 – LU-Decompositions

The goal is to describe Gaussian elimination in the most useful way by looking at them closely, which are factorizations of a matrix.

The factors are triangular matrices.

The factorization that comes from elimination is $A = LU$ **.**

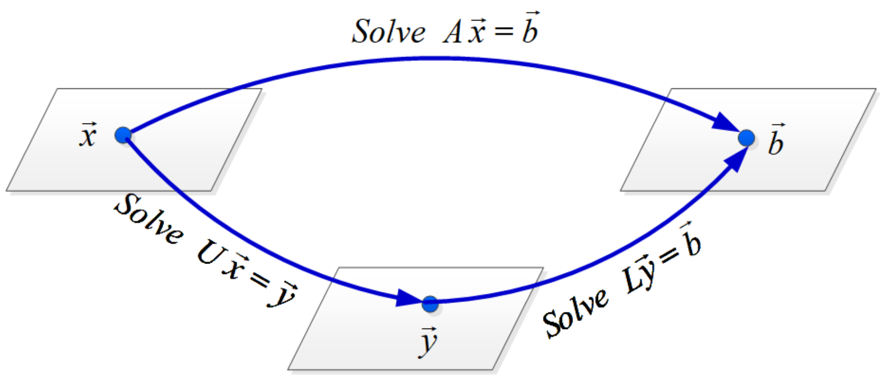
The Method of *LU*–Decomposition

Step 1: Rewrite the system $A\vec{x} = \vec{b}$ as $LU\vec{x} = \vec{b}$

Step 2: Define a new $n \times 1$ matrix $\vec{y}$ by $U\vec{x} = \vec{y}$

Step 3: Use $U\vec{x} = \vec{y}$ to rewrite $LU\vec{x} = \vec{b}$ as $L\vec{y} = \vec{b}$ and solve this system for $\vec{y}$.

Step 4: Substitute $\vec{y}$ in $U\vec{x} = \vec{y}$ and solve for $\vec{x}$.



Example

Given 2 by 2 matrix $A = \begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix}$

Find *L* and *U* and verify $A = LU$

Solution

To make ***row*** 2 ***column*** 1 is ***zero*** then we need to subtract 3 times *row* 2 from *row* 2

$$\begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix} \quad R_2 - 3R_1$$

$\ell_{21} = -3$

That step is $E_{21} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$ in the forward direction such that:

$$E_{21}A = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix} = U$$

The return step from U to A is $L = E_{21}^{-1}$

$$L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

Back from U to A:

$$E_{21}^{-1}U = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix}$$

$$\underline{= A}\,\big|$$

Therefore; $A = LU$

Example

What matrix L and U puts A into triangular form $A = LU$ where

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

<u>Solution</u>

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad R_2 - \tfrac{1}{2}R_1 : \ell_{21}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 1 & 2 \end{pmatrix} R_3 - \tfrac{2}{3}R_2 : \ell_{32}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix} = U$$

$$\ell_{21} = -\frac{1}{2} \qquad \ell_{32} = -\frac{2}{3}$$

The lower triangular L has all 1's on its diagonal. The multipliers ℓ_{ij} are below the diagonal of L with OPPOSITE sign

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}} = \underset{L}{\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}}$$

❖ $\left(E_{32}E_{31}E_{21}\right)A = U$ *becomes* $A = \left(E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}\right)U$ *which is* $A = LU$

The inverses go in opposite order.

❖ $(A = LU)$ This is ***elimination without row exchanges***. The *upper triangular* $\boldsymbol{U}$ has the pivots on its diagonal. The *lower triangular* $\boldsymbol{L}$ has all 1's on its diagonal.

The multipliers ℓ_{ij} are below the diagonal of L.

One Square System = *Two* Triangular Systems

Factor: into L and U, by forward elimination on A.

Solve: forward on $\vec{b}$ using L, then back substitution using U.

Solve $L\vec{c} = \vec{b}$ and then solve $U\vec{x} = \vec{c}$

Example

Forward elimination on $Ax = b$ ends at $Ux = c$

$$\begin{matrix} x + 2y = 5 \\ 4x + 9y = 21 \end{matrix} \quad \textit{becomes} \quad \begin{matrix} x + 2y = 5 \\ y = 1 \end{matrix}$$

Solution

The multiplier was 4. $\left(R_2 - 4R_1\right)$

The lower triangular system: $L\vec{c} = \vec{b}$

$$\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} [c] = \begin{bmatrix} 5 \\ 21 \end{bmatrix}$$

$$\vec{c} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

The upper triangular system: $U\vec{x} = \vec{c}$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} [x] = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

To solve 1000 equations on a PC

- Elimination on A requires about $\frac{1}{3}n^3$ multiplications and $\frac{1}{3}n^3$ subtractions.
- Each right–side needs n^2 multiplications and n^2 subtractions.

Exercises *Section* 4.3 – LU-Decompositions

1. What matrix E puts A into triangular form $EA = U$? Multiply by $E^{-1} = L$ to factor A into LU:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}$$

2. Solve $L\vec{c} = \vec{b}$ to find $\vec{c}$. Then solve $U\vec{x} = \vec{c}$ to find $\vec{x}$. What was A?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

3. Find L and U for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots

4. For which c is $A = LU$ impossible – with three pivots?

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

(5 – 14) Find an LU–decomposition of the coefficient matrix, and then use to solve the system

5. $\begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

6. $\begin{bmatrix} -5 & -10 \\ 6 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 19 \end{bmatrix}$

7. $\begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}$

8. $\begin{bmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$

9. $\begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$

10. $\begin{bmatrix} 2 & -6 & 4 \\ -4 & 8 & 0 \\ 0 & -4 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$

11. $\begin{bmatrix} 2 & -4 & 2 \\ -4 & 5 & 2 \\ 6 & -9 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}$

12. $\begin{bmatrix} 1 & -1 & 2 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$

13. $\begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 3 \\ 7 \end{bmatrix}$

14. $\begin{bmatrix} 1 & -2 & -2 & -3 \\ 3 & -9 & 0 & -9 \\ -1 & 2 & 4 & 7 \\ -3 & -6 & 26 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \\ 3 \end{bmatrix}$

(15 – 24) Find an LU factorization matrix

15. $\begin{pmatrix} 2 & 5 \\ -3 & -4 \end{pmatrix}$

16. $\begin{pmatrix} 6 & 4 \\ 12 & 5 \end{pmatrix}$

17. $\begin{pmatrix} 3 & 1 & 2 \\ -9 & 0 & -4 \\ 9 & 9 & 14 \end{pmatrix}$

18. $\begin{pmatrix} -5 & 0 & 4 \\ 10 & 2 & -5 \\ 10 & 10 & 16 \end{pmatrix}$

19. $\begin{pmatrix} 3 & 7 & 2 \\ 6 & 19 & 4 \\ 9 & 9 & 14 \end{pmatrix}$

20. $\begin{pmatrix} 2 & 3 & 2 \\ 4 & 13 & 9 \\ -6 & 5 & 4 \end{pmatrix}$

21. $\begin{pmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ -2 & -4 & 7 & 5 \end{pmatrix}$

22. $\begin{pmatrix} 1 & 3 & 1 & 5 \\ 5 & 20 & 6 & 31 \\ -2 & -1 & -1 & -4 \\ -1 & 7 & 1 & 7 \end{pmatrix}$

23. $\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{pmatrix}$

24. $\begin{pmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{pmatrix}$

25. Let A be a lower triangular $n \times n$ matrix with nonzero entries on the diagonal. Show that A is invertible and A^{-1} is lower triangular.

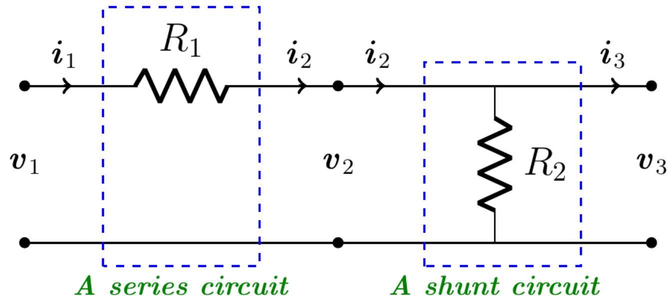
26. Let $A = LU$ be an LU factorization. Explain why A can be row reduced to U using only replacement operations.

27. Suppose an $m \times n$ matrix A admits a factorization $A = CD$ where C is $m \times 4$ and D is $4 \times n$.

a) Show that A is the sum of four outer products.

b) Let $m = 400$ *and* $n = 100$. Explain why a computer programmer might prefer to store the data from A in the form of two matrices C and D.

28. A ladder network, where two circuits are connected in series, so that the output of one circuit becomes the input of the next circuit.



The transformation $\begin{pmatrix} v_1 \\ i_1 \end{pmatrix} \longrightarrow \begin{pmatrix} v_2 \\ i_2 \end{pmatrix}$ is linear with a transfer matrix A of the ladder network.

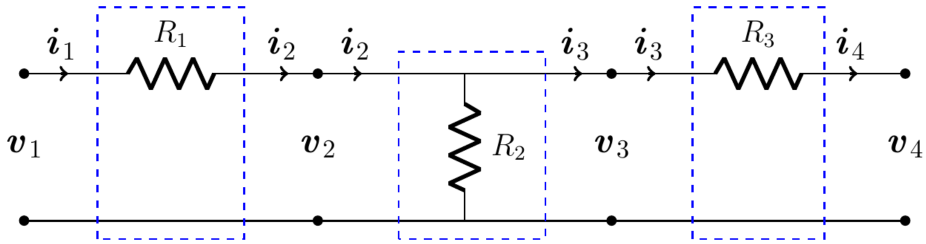
Let the transfer matrix A_1 of the series circuit is given by $\begin{pmatrix} v_2 \\ i_2 \end{pmatrix} = A_1 \begin{pmatrix} v_1 \\ i_1 \end{pmatrix}$

Let the transfer matrix A_2 of the shunt circuit is given by $\begin{pmatrix} v_3 \\ i_3 \end{pmatrix} = A_2 \begin{pmatrix} v_2 \\ i_2 \end{pmatrix}$

a) Compute the transfer matrix of the ladder network

b) Design a ladder network whose transfer matrix is $\begin{pmatrix} 1 & -8 \\ -\frac{1}{2} & 5 \end{pmatrix}$

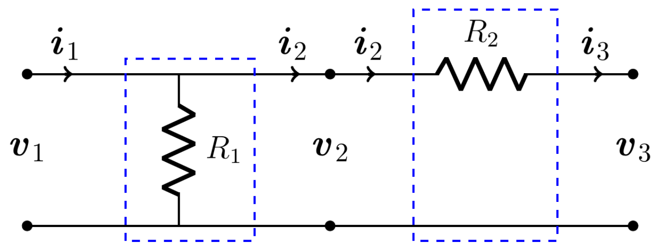
29. A ladder network, where three circuits are connected in series, so that the output of one circuit becomes the input of the next circuit.



a) Compute the transfer matrix of the ladder network

b) Design a ladder network whose transfer matrix is $\begin{pmatrix} 3 & -12 \\ -\frac{1}{3} & \frac{5}{3} \end{pmatrix}$

30. A ladder network, where two circuits are connected in series, so that the output of one circuit becomes the input of the next circuit.



a) Compute the transfer matrix of the ladder network

b) Find the values of the resistors when the input voltage is 12 volts and current is 6 amps if the output voltage is 9 volts and current is 4 amps

Section 4.4 – Eigenvalues and Eigenvectors

In many problems in science and mathematics, linear equations $A\vec{x} = \vec{b}$ come from steady state problems. Eigenvalues have their greatest importance in dynamic problems. The solution of $A\vec{x} = \lambda\vec{x}$ *or* $\frac{d\vec{x}}{dt} = A\vec{x}$ (is changing with time) has nonzero solutions. (***All matrices are square***)

Definition

Suppose A is an n x n matrix and

$$\boldsymbol{\lambda}\vec{x} = \boldsymbol{A}\vec{x}$$

The values of λ are called eigenvalues of the matrix $\boldsymbol{A}$ and the nonzero vectors $\vec{x}$ in $\mathbb{R}^n$ are called the eigenvectors corresponding to that eigenvalue (λ).

λ is the eigenvalue associated with or corresponding to the eigenvector $\vec{x}$.

- One of the meanings of the word "***eigen***" in German is "***proper***"; eigenvalues are also called ***proper values, characteristic values,*** or ***latent roots***.

Example

The vector $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 3$ since

$$\begin{aligned} A\vec{x} &= \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 6 \end{bmatrix} \\ &= 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= 3\vec{x} \end{aligned}$$

Eigenvalues and eigenvectors have a useful geometric interpretation in $\mathbb{R}^2$ and $\mathbb{R}^3$.

The equation for the *eigenvalues*

Let's rewrite the equation $\lambda\vec{x} = A\vec{x}$.

$A\vec{x} - \lambda\vec{x} = 0$ $\qquad\qquad$ λ: are the eigenvalues and not a vector

$A\vec{x} - \lambda I\vec{x} = 0$

$(A - \lambda I)\vec{x} = 0$

The matrix $A - \lambda I$ times the eigenvectors $\vec{x}$ is the zero vector.

The eigenvectors make up the nullspace of $A - \lambda I$.

Definition

The number λ is an eigenvalue of ***A*** if and only if $A - \lambda I$ is singular:

$$\boxed{\det(A - \lambda I) = 0}$$

This equation $\det(A - \lambda I) = 0$ is called ***characteristic equation*** of ***A***; the scalars satisfying this equation are the eigenvalues of ***A***. when expanding the determinant $\det(A - \lambda I)$ is a polynomial in λ of degree *n*, called the ***characteristic polynomial*** of ***A***.

Example

Find the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$

Solution

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix}$$

$$= (3-\lambda)(-\lambda) + 2$$

$$= \lambda^2 - 3\lambda + 2$$

The characteristic equation of ***A*** is:

$$\lambda^2 - 3\lambda + 2 = 0$$

The eigenvalues of ***A*** are $\underline{\lambda_1 = 1,\ \lambda_2 = 2}$

Theorem

If A is an n x n triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are the entries on the main diagonal of A.

Example

Find the eigenvalues of the lower triangular matrix

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{3}{2} & 0 \\ 5 & -8 & -\frac{1}{4} \end{pmatrix}$$

Solution

The eigenvalues are: $\lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{3}{2}, \quad and \quad \lambda_3 = -\frac{1}{4}$

Theorem

If A is an n x n matrix, the following are equivalent.

a) λ is an eigenvalue of A.

b) The system of equations $(A - \lambda I)\vec{x} = \vec{0}$ has nontrivial solutions.

c) There is a nonzero vector $\vec{x}$ in $\mathbb{R}^n$ such that $A\vec{x} = \lambda\vec{x}$.

d) λ is a real solution of the characteristic equation $\det(A - \lambda I) = 0$

Eigenvectors

To find the eigenvector $\vec{x}$, for each eigenvalue λ solve $(A-\lambda I)\vec{x}=0$ *or* $A\vec{x}=\lambda\vec{x}$

From the eigenvalues, the eigenvectors, in the form $\boldsymbol{V}_i=\begin{pmatrix} x_i \\ y_i \end{pmatrix}$, of the system can be determined by letting:

$$(A-\lambda_1 I)V_1=0 \qquad \text{and}\ (A-\lambda_2 I)V_2=0$$

Example

Find the eigenvalues and the eigenvectors of the matrix $A=\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

Solution

$$\det(A-\lambda I)=\det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix}$$

$$=(1-\lambda)(4-\lambda)-4$$

$$=\lambda^2-5\lambda+4-4$$

$$=\lambda^2-5\lambda$$

$$=\lambda(\lambda-5)=\mathbf{0}$$

The eigenvalues of $\boldsymbol{A}$ are: $\lambda_1=0 \quad \lambda_2=5$

For $\lambda_1=0$, we have:

$$(A-\lambda_1 I)V_1=0$$

$$\begin{pmatrix} 1-0 & 2 \\ 2 & 4-0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x+2y=0 \\ \end{cases}$$

$\underline{x=-2y \,|}$

If $y=-1 \Rightarrow x=2$

Therefore, the eigenvector $\boxed{V_1=\begin{pmatrix} 2 \\ -1 \end{pmatrix}}$

Or $\begin{pmatrix} -2y \\ y \end{pmatrix}=y\begin{pmatrix} -2 \\ 1 \end{pmatrix} \Rightarrow V_1=\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ *or* $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

For $\lambda_2 = 5 \quad \Rightarrow \quad (A - \lambda_2 I)V_2 = 0$:

$$\begin{pmatrix} 1-5 & 2 \\ 2 & 4-5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \; 2x - y = 0$$

$2x = y$

Therefore, the eigenvector $V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Power of a Matrix

Theorem

If k is a positive integer, λ is an eigenvalue of a matrix A, and $\vec{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\vec{x}$ is a corresponding eigenvector.

Example

Find the eigenvalues of A^7 for $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

Solution

$$\det(A - \lambda I) = \begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix}$$

$$= \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

The eigenvalues of A: $\lambda_1 = 1 \quad and \quad \lambda_2 = 2$

The eigenvalues of A^7 are:

$\lambda_1 = 1^7 = 1 \quad and \quad \lambda_2 = 2^7 = 128$

Theorem

A square matrix A is invertible *iff* $\lambda = 0$ is not an eigenvalue of A.

Summary

To solve the eigenvalue problem for an n by n matrix:

1. Compute the determinant of $A-\lambda I$. With λ subtracted along the diagonal, this determinant starts with λ^n or $-\lambda^n$. It is a polynomial in λ of degree n.
2. Find the roots of this polynomial, by solving $\det(A-\lambda I)=0$. The n roots are the n eigenvalues of A. They make $A-\lambda I$ singular.
3. For each eigenvalue λ, solve $(A-\lambda I)\vec{x}=\vec{0}$ ***to find an eigenvector x***.

Imaginary Eigenvalues

Example

Find the eigenvalues and the eigenvectors of the matrix $A=\begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}$

Solution

$$\det(A-\lambda I)=\begin{vmatrix} -2-\lambda & -1 \\ 5 & 2-\lambda \end{vmatrix}$$

$$=(-2-\lambda)(2-\lambda)+5$$

$$=\lambda^2+1=0$$

$$\Rightarrow \lambda^2=-1$$

The eigenvalues are: $\boxed{\lambda_{1,2}=\pm i}$

For $\lambda_1=i$: $(A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} -2-i & -1 \\ 5 & 2-i \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad (-2-i)x_1-y_1=0$$

$$\Rightarrow \boxed{(2+i)x_1=-y_1}$$

Therefore, the eigenvector $V_1=\begin{pmatrix} -1 \\ 2+i \end{pmatrix}$

$\lambda_1=-i:(A-\lambda_2 I)V_2=0$

$$\begin{pmatrix} -2+i & -1 \\ 5 & 2+i \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad (-2+i)x_2-y_2=0$$

$$\Rightarrow \boxed{(-2+i)x_2=y_2}$$

Therefore, the eigenvector $\boxed{V_2=\begin{pmatrix} -1 \\ 2-i \end{pmatrix}}$

Example

Find the eigenvalues and the eigenvectors of the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix}$$

$$= \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda^2 = -1$$

The eigenvalues are: $\boxed{\lambda_{1,2} = \pm i}$

The matrix $\boldsymbol{A}$ is a 90° rotation which has no real eigenvalues or eigenvectors.
No vector $A\vec{x}$ stays in the same direction as $\vec{x}$ (except the zero vector which is useless).
If we add the eigenvalues together the result is zero which is the trace of $\boldsymbol{A}$.

$\lambda_1 = i:\quad \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -ix + y = 0 \\ \end{cases}$$

$$\Rightarrow \boxed{x = -iy}$$

Therefore, the eigenvector $\boxed{V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}}$

$\lambda_2 = -i:\quad \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow ix + y = 0$$

$$\Rightarrow \boxed{y = -ix}$$

Therefore, the eigenvector $\boxed{V_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}}$

Exercises *Section* 4.4 – Eigenvalues and Eigenvectors

1. Find the eigenvalues and eigenvectors of $A,\ A^2,\ A^{-1},\ and\ \ A+4I$:

$$A=\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad and \quad A^2=\begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

Check the trace $\lambda_1+\lambda_2$ and the determinant $\lambda_1\lambda_2$ for A and also A^2.

2. Show directly that the given vectors are eigenvectors of the given matrix. What are the corresponding eigenvalues

$$\vec{v}_1=\begin{bmatrix} 1 \\ -3 \end{bmatrix},\quad \vec{v}_2=\begin{bmatrix} 2 \\ 1 \end{bmatrix},\quad A=\begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$$

3. For which real numbers c does this matrix A have

$$A=\begin{pmatrix} 2 & -c \\ -1 & 2 \end{pmatrix}$$

a) Two real eigenvalues and eigenvectors.

b) A repeated eigenvalue with only one eigenvector

c) Two complex eigenvalues and eigenvectors.

4. Find the eigenvalues of ***A***, ***B***, ***AB***, and ***BA***:

$$A=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad B=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad AB=\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad BA=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

a) The eigenvalues of ***AB*** (are equal to) (are not equal to) eigenvalues of ***A*** times eigenvalues of ***B***.

b) The eigenvalues of ***AB*** (are equal to) (are not equal to) eigenvalues of ***BA***.

5. When $a+b=c+d$ show that (1, 1) is an eigenvector and find both eigenvalues of

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

6. The eigenvalues of A equal to the eigenvalues of A^T. This is because $\det(A-\lambda I)$ equals $\det\left(A^T-\lambda I\right)$. That is true because _____. Show by an example that the eigenvectors of A and A^T are not the same.

7. Let $A=\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}$. Compute the eigenvalues and eigenvectors of A.

8. Let $A=\begin{bmatrix}2 & -2 & 3\\ 1 & 1 & 1\\ 1 & 3 & -1\end{bmatrix}$

a) What is the characteristic polynomial for A (i.e. compute $\det(A-\lambda I)$?

b) Verify that 1 is an eigenvalue of A. What is a corresponding eigenvector?

c) What are the other eigenvalues of A?

(9 – 58) For the following matrices:

i. Find the characteristic equation.

ii. Find the eigenvalues.

iii. Find the eigenvectors.

9. $\begin{pmatrix}3 & 0\\ 8 & -1\end{pmatrix}$

10. $\begin{pmatrix}10 & -9\\ 4 & -2\end{pmatrix}$

11. $\begin{pmatrix}0 & 3\\ 4 & 0\end{pmatrix}$

12. $\begin{pmatrix}-2 & -7\\ 1 & 2\end{pmatrix}$

13. $\begin{pmatrix}12 & 14\\ -7 & -9\end{pmatrix}$

14. $\begin{pmatrix}-4 & 1\\ -2 & 1\end{pmatrix}$

15. $\begin{pmatrix}5 & 3\\ -6 & -4\end{pmatrix}$

16. $\begin{pmatrix}-2 & 3\\ 0 & -5\end{pmatrix}$

17. $\begin{pmatrix}2 & 0\\ -4 & -2\end{pmatrix}$

18. $\begin{pmatrix}1 & 2\\ 4 & 3\end{pmatrix}$

19. $\begin{pmatrix}2 & 2\\ 1 & 3\end{pmatrix}$

20. $\begin{pmatrix}-4 & 2\\ -\frac{5}{2} & 2\end{pmatrix}$

21. $\begin{pmatrix}-\frac{5}{2} & 2\\ \frac{3}{4} & -2\end{pmatrix}$

22. $\begin{pmatrix}3 & -1\\ 9 & -3\end{pmatrix}$

23. $\begin{pmatrix}-6 & 5\\ -5 & 4\end{pmatrix}$

24. $\begin{pmatrix}6 & -1\\ 5 & 2\end{pmatrix}$

25. $\begin{pmatrix}1 & 1\\ -2 & -1\end{pmatrix}$

26. $\begin{pmatrix}5 & 1\\ -2 & 3\end{pmatrix}$

27. $\begin{pmatrix}4 & 5\\ -2 & 6\end{pmatrix}$

28. $\begin{pmatrix}5 & -4\\ 2 & -1\end{pmatrix}$

29. $\begin{pmatrix}6 & -6\\ 4 & -4\end{pmatrix}$

30. $\begin{pmatrix}5 & -3\\ 2 & 0\end{pmatrix}$

31. $\begin{pmatrix}5 & -4\\ 3 & -2\end{pmatrix}$

32. $\begin{pmatrix}6 & -10\\ 2 & -3\end{pmatrix}$

33. $\begin{pmatrix}11 & -15\\ 6 & -8\end{pmatrix}$

34. $\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}$

35. $\begin{pmatrix}9 & 2\\ 2 & 6\end{pmatrix}$

36. $\begin{pmatrix}13 & 4\\ 4 & 7\end{pmatrix}$

37. $\begin{pmatrix}5 & -1\\ 3 & -1\end{pmatrix}$

38. $\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$

39. $\begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$

40. $\begin{pmatrix} -1 & 0 & 0 \\ 2 & -5 & -6 \\ -2 & 3 & 4 \end{pmatrix}$

41. $\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$

42. $\begin{pmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{pmatrix}$

43. $\begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

44. $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

45. $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix}$

46. $\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$

47. $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

48. $\begin{pmatrix} -\frac{5}{2} & 1 & 1 \\ 1 & -\frac{5}{2} & 1 \\ 1 & 1 & -\frac{5}{2} \end{pmatrix}$

49. . $\begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

50. $\begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$

51. $\begin{pmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{pmatrix}$

52. $\begin{pmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{pmatrix}$

53. $\begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ -8 & -4 & -3 \end{pmatrix}$

54. $\begin{pmatrix} -1 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & -12 & 2 \end{pmatrix}$

55. $\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$

56. $\begin{pmatrix} -6 & 4 & 4 \\ -4 & 2 & 4 \\ -10 & 8 & 4 \end{pmatrix}$

57. $\begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

58. $\begin{pmatrix} 10 & -9 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

59. Find the eigenvalues of A^9 for $A = \begin{pmatrix} 1 & 3 & 7 & 11 \\ 0 & \frac{1}{2} & 3 & 8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

60. Given: $A = \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}$. Compute A^{11}

61. Find the eigenvalues of the matrices

$$A = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0.61 & 0.52 \\ 0.39 & 0.48 \end{pmatrix}, \quad A^\infty = \begin{pmatrix} 0.57143 & 0.57143 \\ 0.42857 & 0.42857 \end{pmatrix}, \quad and \;\; B = \begin{pmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{pmatrix}$$

62. Given the matrix $\begin{bmatrix} -1 & -3 \\ -3 & 7 \end{bmatrix}$

a) Find the characteristic polynomial.
b) Find the eigenvalues
c) Find the bases for its eigenspaces
d) Graph the eigenspaces
e) Verify directly that $A\vec{v} = \lambda\vec{v}$, for all associated eigenvectors and eigenvalues.

63. Given the matrix $\begin{bmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{bmatrix}$

a) Find the characteristic polynomial.
b) Find the eigenvalues
c) Find the bases for its eigenspaces
d) Graph the eigenspaces
e) Verify directly that $A\vec{v} = \lambda\vec{v}$, for all associated eigenvectors and eigenvalues.

64. Explain why a 2×2 matrix can have at most two distinct eigenvalues. Explain why an $n \times n$ matrix can have at most *n* distinct eigenvalues.

65. Construct an example of a 2×2 matrix with only one distinct eigenvalue.

66. Let λ be an eigenvalue of an invertible matrix *A*. Show that λ^{-1} is an eigenvalue of A^{-1}.

67. Show that if A^2 is the zero matrix, then the only eigenvalue of *A* is 0.

68. Show that λ is an eigenvalue of *A* if and only if λ is an eigenvalue of A^T.

69. For $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, find one eigenvalue, without calculation. Justify your answer.

70. For $A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$, find one eigenvalue, and two linearly independent eigenvectors, without calculation. Justify your answer.

71. Consider an $n \times n$ matrix *A* with the property that the row sums all equal the same number ***s***. Show that ***s*** is an eigenvalue of *A*.

72. Consider an $n \times n$ matrix *A* with the property that the column sums all equal the same number ***s***. Show that ***s*** is an eigenvalue of *A*.

73. Let A be the matrix of the linear transformation T on $\mathbb{R}^2$

T: reflects points across some line through the origin.

Without writing A, find an eigenvalue of A and describe the eigenspace.

74. Let A be the matrix of the linear transformation T on $\mathbb{R}^2$

T: reflects points about some line through the origin.

Without writing A, find an eigenvalue of A and describe the eigenspace.

75. Show that if $\vec{v}$ is an eigenvector of the matrix product AB and $B\vec{v} \neq \vec{0}$, then $B\vec{v}$ is an eigenvector of BA

76. Explain and demonstrate that the eigenspace of a matrix A corresponding to some eigenvalue λ is a subspace.

77. If λ is an eigenvalue of the matrix A, prove that λ^2 is an eigenvalue of A^2.

Section 4.5 – Diagonalization

When $\vec{x}$ is an eigenvector, multiplication by $\boldsymbol{A}$ is just multiplication by a single number: $A\vec{x} = \lambda\vec{x}$.
The matrix $\boldsymbol{A}$ turns into a diagonal matrix $\boldsymbol{\Lambda}$ when we use the eigenvectors property.

Diagonalization

Suppose the n by n matrix $\boldsymbol{A}$ has n linearly independent eigenvectors $\vec{x}_1, \ldots, \vec{x}_n$. Put them into the column of an ***eigenvector matrix P***. Then $P^{-1}AP$ is the eigenvalue matrix $\boldsymbol{\Lambda}$:

$$P^{-1}AP = \Lambda = D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Example

The projection matrix $A = \begin{pmatrix} .5 & .5 \\ .5 & .5 \end{pmatrix}$ has $\lambda_{1,2} = 0$ *and* 1

Solution

For $\lambda_1 = 0 \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \frac{1}{2}x + \frac{1}{2}y = 0$$

$x = -y$

Therefore, $V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

For $\lambda_2 = 1 \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad -\frac{1}{2}x + \frac{1}{2}y = 0$$

$x = y$

Therefore, $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The eigenvectors are: $(-1,\ 1)$ & $(1,\ 1)$ that are the value of P.

$$P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$P^{-1} = -\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underset{P^{-1}}{\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}} \quad \underset{A}{\begin{pmatrix} .5 & .5 \\ .5 & .5 \end{pmatrix}} \quad \underset{P}{\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}} \quad \underset{=}{=} \quad \underset{D}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}$$

Definition

A square matrix A is called ***diagonalizable*** if there is an invertible matrix P such that $P^{-1}AP$ is diagonal; the matrix P is said to ***diagonalize*** A.

Theorem

Independent x from different λ - Eigenvectors $\vec{x}_1, \ldots, \vec{x}_n$ that correspond to distinct (all different) eigenvalues are linearly independent. An n by n matrix that has n different eigenvalues (no repeated λ's) must be diagonalizable.

Proof

Suppose $c_1\vec{x}_1 + c_2\vec{x}_2 = 0$ (1)

$$\begin{pmatrix} c_1\vec{x}_1 & c_2\vec{x}_2 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0$$

$$c_1\lambda_1\vec{x}_1 + c_2\lambda_2\vec{x}_2 = 0 \quad (2)$$

Multiply (1) by λ_2, that implies to

$$c_1\lambda_2\vec{x}_1 + c_2\lambda_2\vec{x}_2 = 0 \quad (3)$$

(2) – (3)

$$c_1\lambda_1\vec{x}_1 + c_2\lambda_2\vec{x}_2 - \left(c_1\lambda_2\vec{x}_1 + c_2\lambda_2\vec{x}_2\right) = 0$$

$$c_1\lambda_1\vec{x}_1 + c_2\lambda_2\vec{x}_2 - c_1\lambda_2\vec{x}_1 - c_2\lambda_2\vec{x}_2 = 0$$

$$c_1\lambda_1\vec{x}_1 - c_1\lambda_2\vec{x}_1 = 0$$

$$c_1\left(\lambda_1 - \lambda_2\right)\vec{x}_1 = 0$$

Since $\vec{x}_i \ne 0$ and λ's are different $\lambda_1 - \lambda_2 \ne 0$, we forced $\underline{c_1 = 0}|$

Similarly; Multiply (1) by λ_1, that implies to $c_1\lambda_1\vec{x}_1 + c_2\lambda_1\vec{x}_2 = 0$ (4)

(2) – (4)

$$c_1\lambda_1\vec{x}_1 + c_2\lambda_2\vec{x}_2 - c_1\lambda_1\vec{x}_1 - c_2\lambda_1\vec{x}_2 = 0$$

$$c_2\left(\lambda_2 - \lambda_1\right)\vec{x}_2 = 0 \quad \Rightarrow \underline{c_2 = 0}|$$

Therefore, $\vec{x}_1$ *and* $\vec{x}_2$ must be independent.

Theorem

If $\vec{v}_1, \ldots, \vec{v}_n$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, then $\left\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\right\}$ is linearly independent set.

Theorem

If an n x n matrix $\boldsymbol{A}$ has n distinct eigenvalues, then the following are equivalent:

a) $\boldsymbol{A}$ is diagonalizable

b) $\boldsymbol{A}$ has n linearly independent eigenvectors.

Example

Given the Markov matrix $A = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} .8-\lambda & .3 \\ .2 & .7-\lambda \end{vmatrix}$$

$$= (.8-\lambda)(.7-\lambda) - .06$$

$$= \lambda^2 - 1.5\lambda + .56 - .06$$

$$= \lambda^2 - 1.5\lambda + .5 = 0$$

The eigenvalues are: $\lambda_1 = 1, \quad \lambda_2 = .5$

For $\lambda_1 = 1$, we have: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -.2 & .3 \\ .2 & -.3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad -.2x + .3y = 0$$

$$\Rightarrow 2x = 3y$$

Therefore, the eigenvector $V_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

For $\lambda_2 = .5$, we have: $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} .3 & .3 \\ .2 & .2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad .3x + .3y = 0$$

$$\Rightarrow x = -y$$

Therefore, the eigenvector $V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

The eigenvector matrix is given by:

$$P = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}$$

$$P^{-1} = -\frac{1}{5}\begin{pmatrix} -1 & -1 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{3}{5} \end{pmatrix}$$

$$\overset{P}{\begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}} \overset{D}{\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}} \overset{P^{-1}}{\begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{3}{5} \end{pmatrix}} = \begin{pmatrix} 3 & \frac{1}{2} \\ 2 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{3}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{8}{10} & \frac{3}{10} \\ \frac{2}{10} & \frac{7}{10} \end{pmatrix}$$

$$= \underset{A}{\begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}}$$

Eigenvalues of AB and A + B

An eigenvalue of ***A*** times an eigenvalue of ***B*** usually does not give an eigenvalue of ***AB***.

$$\lambda_A \lambda_B \neq \lambda_{AB}$$

Commuting matrices share eigenvectors: Suppose ***A*** and ***B*** can be diagonalized. They share the eigenvector matrix ***P*** if and only if $AB = BA$.

Matrix Powers A^k

$$A^2 = PDP^{-1}PDP^{-1}$$
$$= PD^2P^{-1}$$

$$A^k = \left(PDP^{-1}\right)\cdots\left(PDP^{-1}\right)$$
$$= PD^kP^{-1}$$

The eigenvector matrix for A^k is still *S*, and the eigenvalue matrix is A^k. The eigenvectors don't change, and the eigenvalues are taken to the k^{th} power. When *A* is diagonalized, $A^k\vec{u}_0$ is easy.

Here are steps (taken from Fibonacci):

1. Find the eigenvalues of *A* and look for *n* independent eigenvectors.
2. Write $\vec{u}_0$ as a combination $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ of the eigenvectors.
3. Multiply each eigenvector $\vec{v}_i$ *by* $\left(\lambda_i\right)^k$. Then

$$\vec{u}_k = A_k\vec{u}_0$$
$$= c_1\left(\lambda_1\right)^k\vec{v}_1 + \cdots + c_n\left(\lambda_n\right)^k\vec{v}_n$$

Example

Compute A^k *where* $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix}$$
$$= (1-\lambda)(2-\lambda) = 0$$

The eigenvalues are: $\lambda_{1,2} = 1,\ 2$

For $\lambda_1 = 1 \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad y = 0$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow \quad (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \underline{x = y}$$

$$\Rightarrow \quad \boxed{V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

The eigenvector matrix is given by:

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \Rightarrow \quad D^k = \begin{pmatrix} 1^k & 0 \\ 0 & 2^k \end{pmatrix}$$

$$\begin{aligned} A^k &= PD^kP^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^k \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2^k \\ 0 & 2^k \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{pmatrix} \end{aligned}$$

Similar Matrices

Definition

If A and B are square matrices, then we say that ***B is similar to A*** if there exists an invertible matrix P such that $B = P^{-1}AP \quad or \quad A = PBP^{-1}$

- Similar matrices B and $M^{-1}AM$ have the same eigenvalues. If $\vec{x}$ is an eigenvector of A then $M^{-1}\vec{x}$ is an eigenvector of $B = M^{-1}AM$.

Proof

Since $B = M^{-1}AM \Rightarrow A = MBM^{-1}$

Suppose $A\vec{x} = \lambda\vec{x}$:

$$MBM^{-1}\vec{x} = \lambda\vec{x}$$

$$BM^{-1}\vec{x} = \lambda M^{-1}\vec{x}$$

The eigenvalue of B is the same λ. The eigenvector is now $M^{-1}x$

Example

The projection $A = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ is similar to $D = P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Choose $M = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$; the similar matrix $M^{-1}AM \ \ is \ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Also choose $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$; the similar matrix $M^{-1}AM \ \ is \ \begin{bmatrix} .5 & -.5 \\ -.5 & .5 \end{bmatrix}$

These matrices $M^{-1}AM$ all have the same eigenvalues 1 and 0.

Every 2 by 2 matrix with those eigenvalues is similar to A.

The eigenvectors change with M.

Example

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is similar to every matrix $B = \begin{bmatrix} cd & d^2 \\ -c^2 & -cd \end{bmatrix}$ except $B = 0$.

These matrices B all have zero determinant (like A). T
hey all have rank one (like A). Their trace is $cd - cd = 0$.

Their eigenvalues are 0 and 0 (like A).

Choose $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with* $ad - cd = 1$ and $B = M^{-1}AM$

Connections between similar matrices A and B:

Not Changed	***Changed***
Eigenvalues	Eigenvectors
Trace and determinant	Nullspace
Rank	Column space
Number of independent	Row space
eigenvectors	Left nullspace
Jordan form	Singular values

Example

Jordan matrix J has triple eigenvalues 5, 5, 5. Its only eigenvectors are multiples of (1, 0, 0). Algebraic multiplicity 3, geometric multiplicity 1:

$$\text{If } J = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \quad \textit{then} \quad J - 5I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ has rank 2.}$$

Every similar matrix $B = M^{-1}JM$ has the same triple eigenvalues 5, 5, 5. Also $B - 5I$ must have the same rank 2. Its nullspace has dimension 3 − 2 = 1. So each similar matrix B also has only one independent eigenvector.

The transpose matrix J^T has the same eigenvalues 5, 5, 5, and $J^T - 5I$ has the same rank 2. ***Jordan's theory says that J^T is similar to J***. The matrix that produces the similarity happens to be the reserve identity M:

$$J^T = M^{-1}JM \quad \textit{is} \quad \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

There is one line of eigenvectors $(x_1,\ 0,\ 0)$ for J and another line $(0,\ 0,\ x_3)$ for J^T.

Fibonacci Numbers

Every new Fibonacci number is the sum of the two previous *F*'s.

The ***sequence*** 0, 1, 1, 2, 3, 5, 8, 13, …. comes from $F_{k+2} = F_{k+1} + F_k$

Problem

Find the Fibonacci number F_{100}

We can apply the rule one step at a time, or just use Linear algebra.
Let consider the matrix equation: $u_{k+1} = Au_k$. Fibonacci rule gave us a two-step rule for scalars.

Let $\vec{u}_k = \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix}$, the rule $\begin{matrix} F_{k+2} = F_{k+1} + F_k \\ F_{k+1} = F_{k+1} \end{matrix}$ becomes $\vec{u}_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \vec{u}_k$.

Every step multiplies by $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, after 100 steps we reach $\vec{u}_{100} = \begin{pmatrix} F_{101} \\ F_{100} \end{pmatrix}$

$$\vec{u}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \ldots \quad \vec{u}_{100} = \begin{pmatrix} F_{101} \\ F_{100} \end{pmatrix}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} \\ &= -\lambda(1-\lambda) - 1 \\ &= \lambda^2 - \lambda - 1 \end{aligned}$$

The characteristic equation is $\lambda^2 - \lambda - 1 = 0$ and the solutions are

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618 \quad \textit{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -.618$$

For $\lambda_1 \quad \Rightarrow \quad (A - \lambda_1 I)\vec{v}_1 = 0$

$$\begin{pmatrix} 1-\lambda_1 & 1 \\ 1 & -\lambda_1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x_1 - \lambda_1 y_1 = 0$$

$$\boxed{x_1 = \lambda_1 y_1}$$

$$\Rightarrow \boxed{\vec{v}_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}}$$

For $\lambda_2 \quad \Rightarrow \quad (A - \lambda_2 I)\vec{v}_2 = 0$

$$\begin{pmatrix} 1-\lambda_2 & 1 \\ 1 & -\lambda_2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x_1 - \lambda_2 y_1 = 0$$

$$\boxed{x_1 = \lambda_2 y_1}$$

$$\Rightarrow \boxed{\vec{v}_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}}$$

The eigenvector matrix is given by:

$$S = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$$

The combination of these eigenvectors that give $\vec{u}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\lambda_1 - \lambda_2}\left(\begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} - \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}\right)$$

$$\boxed{= \frac{\vec{v}_1 - \vec{v}_2}{\lambda_1 - \lambda_2}}$$

$$\vec{u}_{100} = \frac{(\lambda_1)^{100}\,\vec{v}_1 - (\lambda_2)^{100}\,\vec{v}_2}{\lambda_1 - \lambda_2}$$

$$F_{100} = \frac{1}{\lambda_1 - \lambda_2}\left[(\lambda_1)^{100} - (\lambda_2)^{100}\right]$$

$$= \frac{1}{\frac{1+\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{100} - \left(\frac{1+\sqrt{5}}{2}\right)^{100}\right]$$

$$\boxed{\approx 2.54 \times 10^{20}}$$

The *Jordan* Form

For every A, we want to choose M so that $M^{-1}AM$ is as nearly diagonal as possible. When A has a full set of n eigenvectors, they go into the columns of M. Then $M = P$. The matrix $P^{-1}AP$ is diagonal.

If A has ***s*** independent eigenvectors, it is similar to a matrix J that has ***s*** Jordan blocks on its diagonal. There is a matrix M such that

$$M^{-1}AM = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix} = J$$

Each block in J has one eigenvalue λ_i, one eigenvector, and 1's above the diagonal:

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

A is similar to B if they share the same Jordan form J – not otherwise.

Exercises Section 4.5 – Diagonalization

1. The Lucas numbers are like Fibonacci numbers except they start with $L_1 = 1$ *and* $L_2 = 3$. Following the rule $L_{k+2} = L_{k+1} + L_k$. The next Lucas numbers are 4, 7, 11, 18. Show that the Lucas number $L_{100} = \lambda_1^{100} + \lambda_2^{100}$.

2. Find all eigenvector matrices *S* that diagonalize *A* (rank 1) to give $S^{-1}AS = \Lambda$:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

What is A^n? Which matrices *B* commute with *A* (so that *AB* = *BA*)

(3 – 6) Determine whether the matrix is diagonalizable

3. $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$

4. $\begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$

5. $\begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$

6. $\begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

(7 – 26) Determine if the matrices are diagonalizable. If so, find a matrix ***P*** that diagonalizes *A* and determine $P^{-1}AP$.

7. $A = \begin{pmatrix} -14 & 12 \\ -20 & 17 \end{pmatrix}$

8. $A = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}$

9. $A = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}$

10. $A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$

11. $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$

12. $A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 3 & 2 \end{pmatrix}$

13. $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$

14. $A = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{pmatrix}$

15. $A = \begin{pmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

16. $A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$

17. $A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & -2 \\ 1 & 3 & 1 \end{pmatrix}$

18. $A = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{pmatrix}$

19. $A = \begin{pmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{pmatrix}$

20. $A = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

21. $A = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

22. $A = \begin{pmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{pmatrix}$

23. $A = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}$

24. $A = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

25. $A = \begin{pmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

26. $A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}$

27. The 4 by 4 triangular Pascal matrix P_L and its inverse (alternating diagonals) are

$$P_L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \quad and \quad P_L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

Check that P_L and P_L^{-1} have the same eigenvalues. Find a diagonal matrix D with alternating signs that gives $P_L^{-1} = D^{-1} P_L D$, so P_L is similar to P_L^{-1}. Show that $P_L D$ with columns of alternating signs is its own inverse.

Since P_L and P_L^{-1} are similar they have the same Jordan form J. Find J by checking the number of independent eigenvectors of P_L with $\lambda = 1$.

28. These Jordan matrices have eigenvalues 0, 0, 0, 0. They have two eigenvectors (one from each block). But the block sizes don't match and they are not similar:

$$J = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad and \quad K = \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

For any matrix M compare JM with MK. If they are equal show that M is not invertible. Then $M^{-1}JM = K$ is Impossible; J is not similar to K.

29. If $\boldsymbol{x}$ is in the nullspace of A show that $M^{-1}\boldsymbol{x}$ is in the nullspace of $M^{-1}AM$.
The nullspaces of A and $M^{-1}AM$ have the same (vectors) (basis) (dimension)

30. Prove that A^T is always similar to A (λ's are the same):

a) For one Jordan block J_i, find M_i so that $M_i^{-1}J_iM_i = J_i^T$.

b) For any J with blocks J_i, build M_0 from blocks so that $M_0^{-1}JM_0 = J^T$.

c) For any $A = MJM^{-1}$: Show that A^T is similar to J^T and so to J and so to A.

31. Why are these statements all true?

a) If A is similar to B then A^2 is similar to B^2.

b) A^2 and B^2 can be similar when A and B are not similar.

c) $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ is similar to $\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}$

d) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ is not similar to $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

e) If we exchange rows 1 and 2 of A, and then exchange columns 1 and 2 the eigenvalues stay the same. In this case M=?

32. If an n x n matrix A has all eigenvalues $\lambda = 0$ prove that A^n is the zero matrix.

33. If A is similar to A^{-1}, must all the eigenvalues equal to 1 or −1?.

(34 – 42) Determine whether the *two matrices* are similar matrices

34. $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}$

35. $A = \begin{pmatrix} 4 & -1 \\ 2 & 4 \end{pmatrix}$ $B = \begin{pmatrix} 4 & 1 \\ 2 & 4 \end{pmatrix}$

36. $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 2 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

37. $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}1&2&3\\0&4&5\\0&0&6\end{pmatrix}$

38. $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}4&0&0\\7&1&0\\8&9&6\end{pmatrix}$

39. $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}4&7&0\\0&1&0\\8&9&6\end{pmatrix}$

40. $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}1&2&0\\3&4&5\\0&7&6\end{pmatrix}$

41. $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}1&2&0\\3&4&0\\0&5&6\end{pmatrix}$

42. $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}1&0&0\\2&5&1\\3&1&5\end{pmatrix}$

43. Prove that two similar matrices have the same determinant. Explain geometrically why this is reasonable.

44. Prove that two similar matrices have the same characteristic polynomial and thus the same eigenvalues. Explain geometrically why this is reasonable.

45. Suppose that A is a matrix. Suppose that the linear transformation associated to A has two linearly independent eigenvectors. Prove that A is similar to a diagonal matrix.

46. Prove that if A is a 2×2 matrix that has two distinct eigenvalues, then A is similar to a diagonal matrix.

47. Suppose that the characteristic polynomial of a matrix has a double root. Is it true that the matrix has two linearly independent eigenvectors? Consider the example $\begin{pmatrix}0&0\\1&0\end{pmatrix}$. Is it true that matrices with equal characteristic polynomial are necessarily similar?

48. Show that the given matrix is not diagonalizable. $\begin{bmatrix}1&1\\-1&-1\end{bmatrix}$

49. Determine if the given matrix is diagonalizable. If, so, find matrices S and $\Lambda(D)$ such that the given matrix equals $S\Lambda S^{-1}$

a) $\begin{bmatrix}3&3\\4&2\end{bmatrix}$

b) $\begin{bmatrix}1&0&1\\0&2&0\\-1&0&-1\end{bmatrix}$

50. A is a 5×5 matrix with *two* eigenvalues. One eigenspace is *three*–dimensional, and the other eigenspace is *two*–dimensional. Is A diagonalizable? Why?

51. A is a 3×3 matrix with *two* eigenvalues. Each eigenspace is *one*–dimensional. Is A diagonalizable? Why?

52. A is a 4×4 matrix with *three* eigenvalues. One eigenspace is *one*–dimensional, and one of the other eigenspace is *two*–dimensional. Is it possible that A is *not* diagonalizable? Justify your answer?

53. A is a 7×7 matrix with *three* eigenvalues. One eigenspace is *two*–dimensional, and one of the other eigenspace is *three*–dimensional. Is it possible that A is *not* diagonalizable? Justify your answer?

54. Show that if A is diagonalizable and invertible, then so is A^{-1}.

55. Show that if A has n linearly independent eigenvectors, then so does A^T.

56. A factorization $A = PDP^{-1}$ is not unique. Demonstrate this for the matrix $A = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix}$ with

$D_1 = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$, find a matrix P_1 such that $A = P_1 D_1 P_1^{-1}$.

57. Construct a nonzero 2×2 matrix that is invertible but not diagonalizable.

58. Construct a nonzero 2×2 matrix that is diagonalizable but not invertible.

59. What are the matrices that are similar to themselves only?

60. For any scalars a, b, and c, show that

$$A = \begin{pmatrix} b & c & a \\ c & a & b \\ a & b & c \end{pmatrix}, \quad B = \begin{pmatrix} c & a & b \\ a & b & c \\ b & c & a \end{pmatrix}, \quad C = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$$

are similar.

Moreover, if $BC = CB$, then A has two zero eigenvalues.

(61 – 64) For positive integer $k \geq 2$, compute

61. $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}^k$

62. $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^k$

63. $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^k$

64. $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^k$

65. Let $A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$. Show that A^k is similar to A fro every positive integer k. It is true more generally for any matrix with all eigenvalues equal to 1.

66. Can a matrix be similar to two different diagonal matrices?

67. Prove that if A is diagonalizable, then A^T is diagonalizable.

68. Prove that if the eigenvalues of a diagonalizable matrix A are all ± 1, then the matrix is equal to its inverse.

69. Prove that if A is diagonalizable with n real eigenvalues $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$, then $|A|=\lambda_1\cdot\lambda_2\cdot\ \ldots\cdot\lambda_n$

70. If x is a real number, then we can define e^x by the series

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ \cdots$$

In similar way, If X is a square matrix, then we can define e^X by the series

$$e^X = I+X+\frac{1}{2!}X^2+\frac{1}{3!}X^3+\frac{1}{4!}X^4+\ \cdots$$

Evaluate e^X, where X is the indicated square matrix.

a) $X=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

b) $X=\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$

c) $X=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

d) $X=\begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix}$

Section 4.6 – Orthogonal Diagonalization

Definition

A square matrix $\boldsymbol{A}$ is called orthogonally diagonalizable if there is an orthogonal matrix P such that $P^{-1}AP\left(=P^{T}AP\right)$ is diagonal; the matrix P is said to orthogonally diagonalize A.

$$P^{T}AP=D$$

We say that A is orthogonally diagonalizable and that P orthogonally diagonalizes A.

Theorem

If $\boldsymbol{A}$ is an n x n matrix, then the following are equivalent.

a) $\boldsymbol{A}$ is orthogonally diagonalizable

b) $\boldsymbol{A}$ has an orthonormal set of n eigenvectors.

c) $\boldsymbol{A}$ is symmetric.

Theorem

If $\boldsymbol{A}$ is symmetric matrix, then:

a) The eigenvalues of $\boldsymbol{A}$ are all real numbers.

b) Eigenvectors from different eigenspaces are orthogonal.

Example

Find an orthogonal matrix P that diagonalizes

$$A=\begin{pmatrix}4&2&2\\2&4&2\\2&2&4\end{pmatrix}$$

Solution

$$\begin{aligned}\det\left(A-\lambda I\right)&=\begin{pmatrix}4-\lambda&2&2\\2&4-\lambda&2\\2&2&4-\lambda\end{pmatrix}\\&=\left(4-\lambda\right)^{3}+8+8-4\left(4-\lambda\right)-4\left(4-\lambda\right)-4\left(4-\lambda\right)\\&=64-48\lambda+12\lambda^{2}-\lambda^{3}+16-12\left(4-\lambda\right)\\&=64-48\lambda+12\lambda^{2}-\lambda^{3}+16-48+12\lambda\\&=-\lambda^{3}+12\lambda^{2}-36\lambda+32=0\end{aligned}$$

The eigenvalues are: $\boxed{\lambda_{1,2}=2 \quad and \quad \lambda_3=8}$

For $\lambda_{1,2}=2$, we have: $(A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad 2x_1+2y_1+2z_1=0$$

$\Rightarrow \quad \boxed{x_1+y_1+z_1=0}$

If $z_1=0 \quad \Rightarrow \quad \boxed{x_1=-y_1}$

Therefore, the eigenvector $\boxed{V_1=\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}$

If $y_1=0 \quad \Rightarrow \quad \boxed{x_1=-z_1}$

Therefore, the eigenvector $\boxed{V_2=\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$

For $\lambda_3=8$, we have: $(A-\lambda_3 I)V_3=0$

$$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} -4x_3+2y_3+2z_3=0 \\ 2x_3-4y_3+2z_3=0 \\ 2x_3+2y_3-4z_3=0 \end{cases}$$

$$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \quad \begin{matrix} \\ 2R_2+R_1 \\ 2R_3+R_1 \end{matrix}$$

$$\begin{pmatrix} -4 & 2 & 2 \\ 0 & -6 & 6 \\ 0 & 6 & -6 \end{pmatrix} \quad \begin{matrix} 3R_1+R_2 \\ \\ R_3+R_2 \end{matrix}$$

$$\begin{pmatrix} -12 & 0 & 12 \\ 0 & -6 & 6 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} -\frac{1}{12}R_1 \\ -\frac{1}{6}R_2 \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \to x_3 - z_3 = 0 \\ \to y_3 - z_3 = 0 \\ \\ \end{array}$$

$$\Rightarrow \underline{x_3 = y_3 = z_3}$$

Therefore the eigenvector $V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(-1,\ 1,\ 0)}{\sqrt{(-1)^2 + 1^2 + 0}}$$

$$= \frac{(-1,\ 1,\ 0)}{\sqrt{2}}$$

$$= \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$\vec{w}_2 = v_2 - \left(\vec{v}_2 \bullet \vec{u}_1\right)\vec{u}_1$$

$$= (-1,\ 0,\ 1) - \left[(-1,\ 0,\ 1)\bullet\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)\right]\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= (-1,\ 0,\ 1) - \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= \left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{\left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}}$$

$$= \frac{\left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)}{\sqrt{\frac{6}{4}}}$$

$$= \frac{2}{\sqrt{6}}\left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)$$

$$= \left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)$$

$$\begin{aligned}\vec{w}_3 &= \vec{v}_3 - \left(\vec{v}_3 \bullet \vec{u}_1\right)\vec{u}_1 - \left(\vec{v}_3 \bullet \vec{u}_2\right)\vec{u}_2 \\ &= (1,\ 1,\ 1) - (0)\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) - (0)\left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right) \\ &= (1,\ 1,\ 1)\end{aligned}$$

$$\begin{aligned}\vec{u}_3 &= \frac{\vec{w}_3}{\left\|\vec{w}_3\right\|} \\ &= \frac{1}{\sqrt{1^2+1^2+1^2}}(1,\ 1,\ 1) \\ &= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\end{aligned}$$

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \qquad (\textit{Orthogonal})$$

$$\begin{aligned}P^TAP &= \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} & 0 \\ -\frac{2}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{4}{\sqrt{6}} \\ \frac{8}{\sqrt{3}} & \frac{8}{\sqrt{3}} & \frac{8}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}\end{aligned}$$

Spectral Decomposition

The spectral decomposition of A is:

$$A = \lambda_1 \vec{u}_1 \vec{u}_1^T + \lambda_2 \vec{u}_2 \vec{u}_2^T + \cdots + \lambda_n \vec{u}_n \vec{u}_n^T$$

Example

The matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}$$

$$= (1-\lambda)(-2-\lambda) - 4$$

$$\boxed{= \lambda^2 + \lambda - 6 = 0}$$

The eigenvalues are: $\boxed{\lambda_1 = -3 \quad and \quad \lambda_2 = 2}$

For $\lambda_1 = -3$: $\left(A - \lambda_1 I\right)\vec{v}_1 = 0$

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad 2x_1 = -y_1$$

Therefore, the eigenvector $\boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$

For $\lambda_2 = 2$: $\left(A - \lambda_2 I\right)\vec{v}_2 = 0$

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x_1 = 2y_1$$

Therefore, the eigenvector $\boxed{\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$

The corresponding eigenvectors are: $\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ *and* $\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \frac{(1,\ -2)}{\sqrt{1^2 + (-2)^2}}$$

$$\boxed{= \left(\frac{1}{\sqrt{5}},\ -\frac{2}{\sqrt{5}}\right)}$$

$$\vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2 \bullet \vec{u}_1\right)\vec{u}_1$$

$$= (2,\ 1) - \left[(2,\ 1) \bullet \left(\frac{1}{\sqrt{5}},\ -\frac{2}{\sqrt{5}}\right)\right]\left(\frac{1}{\sqrt{5}},\ -\frac{2}{\sqrt{5}}\right)$$

$$= (2,\ 1) - (0)\left(\frac{1}{\sqrt{5}},\ -\frac{2}{\sqrt{5}}\right)$$

$$= (2,\ 1)$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\left\|\vec{w}_2\right\|}$$

$$= \frac{(2,\ 1)}{\sqrt{5}}$$

$$= \left(\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right)$$

$$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} = \lambda_1 \vec{u}_1 \vec{u}_1^T + \lambda_2 \vec{u}_2 \vec{u}_2^T$$

$$= -3\begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix} + 2\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$= -3\begin{pmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{pmatrix} + 2\begin{pmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

The spectral decomposition about the image of the vector $\vec{x} = (1,\ 1)$

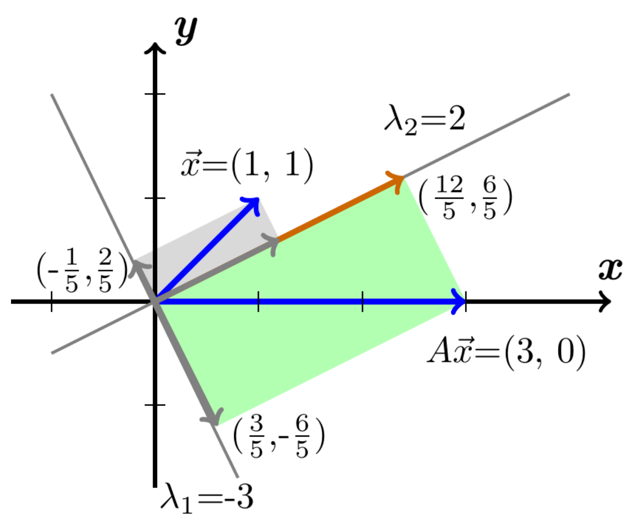
$$A\vec{x} = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= -3\begin{pmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= -3\begin{pmatrix} -\frac{1}{5} \\ \frac{2}{5} \end{pmatrix} + 2\begin{pmatrix} \frac{6}{5} \\ \frac{3}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{5} \\ -\frac{6}{5} \end{pmatrix} + \begin{pmatrix} \frac{12}{5} \\ \frac{6}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$



Example

Consider a 2 by 2 symmetric matrix $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

Solution

The eigenvalues are:

$$\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix}$$

$$= (a-\lambda)(c-\lambda) - b^2$$

$$= \lambda^2 - (a+c)\lambda + ac - b^2 = 0$$

$$\lambda = \frac{(a+c) \pm \sqrt{(a+c)^2 - 4(ac-b^2)}}{2} \qquad \therefore (a+c)^2 - 4(ac-b^2) > 0$$

The eigenvectors are:

For $\lambda_1 \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} a-\lambda_1 & b \\ b & c-\lambda_1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} (a-\lambda_1)x_1 + by_1 = 0 & (1) \\ bx_1 + (c-\lambda_1)y_1 = 0 \end{cases}$$

$$(1) \Rightarrow by_1 = (\lambda_1 - a)x_1$$

$$V_1 = \begin{pmatrix} b \\ \lambda_1 - a \end{pmatrix}$$

For $\lambda_2 \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} a-\lambda_2 & b \\ b & c-\lambda_2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} (a-\lambda_2)x_2 + by_2 = 0 \\ bx_2 + (c-\lambda_2)y_2 = 0 & (2) \end{cases}$$

$$(2) \Rightarrow bx_2 = (\lambda_2 - c)y_2$$

$$V_2 = \begin{pmatrix} \lambda_2 - c \\ b \end{pmatrix}$$

$$\lambda_1 + \lambda_2 = \frac{(a+c) - \sqrt{(a+c)^2 - 4\left(ac - b^2\right)} + (a+c) + \sqrt{(a+c)^2 - 4\left(ac - b^2\right)}}{2}$$

$$= \frac{2(a+c)}{2}$$

$$\underline{= a + c \,\rfloor}$$

$$V_1 \bullet V_2 = b\left(\lambda_1 - a\right) + b\left(\lambda_2 - c\right)$$

$$= b\left(\lambda_1 + \lambda_2 - a - c\right)$$

$$= b(a + c - a - c)$$

$$\underline{= 0 \,\rfloor}$$

Therefore, these eigenvectors are perpendicular.

Theorem

Orthogonal Eigenvectors: Eigenvectors of a real symmetric matrix (when they correspond to different λ's) are always perpendicular.

Proof

Suppose $A\vec{x} = \lambda_1 \vec{x}$, $A\vec{y} = \lambda_2 \vec{y}$ and $A = A^T$.

The dot products of the first equation with y and the second with x:

$$\begin{aligned}\left(\lambda_1 \vec{x}\right)^T \vec{y} &= \left(A\vec{x}\right)^T \vec{y} \\ &= \vec{x}^T A^T \vec{y} \\ &= \vec{x}^T A\vec{y} \\ &= \vec{x}^T \lambda_2 \vec{y}\end{aligned}$$

$$\Rightarrow \boxed{\vec{x}^T \lambda_1 \vec{y} = \vec{x}^T \lambda_2 \vec{y}}$$

Since $\lambda_1 \neq \lambda_2$, this proves that $\vec{x}^T \vec{y} = 0$.

The eigenvector $\vec{x}$ $\left(for\ \lambda_1 \right)$ is perpendicular to the eigenvector $\vec{y}$ $\left(for\ \lambda_2 \right)$

Example

Find the λ's and v's for this symmetric matrix with trace zero: $A = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$

Solution

$$\begin{aligned}\det\left(A - \lambda I\right) &= \begin{vmatrix} -3-\lambda & 4 \\ 4 & 3-\lambda \end{vmatrix} \\ &= \left(-3-\lambda\right)\left(3-\lambda\right) - 16 \\ &= -9 + \lambda^2 - 16 \\ &= \boxed{\lambda^2 - 25 = 0}\end{aligned}$$

The eigenvalues are: $\boxed{\lambda_1 = -5 \quad \lambda_2 = 5}$

The eigenvectors are:

For $\lambda_1 = -5 \Rightarrow \left(A - \lambda_1 I\right)\vec{v}_1 = 0$

$$\begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow 2x_1 + 4y_1 = 0$$

$$\Rightarrow \boxed{x_1 = -2y_1}$$

$\Rightarrow \quad \boxed{\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$

For $\lambda_2 = 5 \quad \Rightarrow \quad \left(A - \lambda_2 I\right)\vec{v}_2 = 0$

$$\begin{pmatrix} -8 & 4 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad 4x_2 - 2y_2 = 0$$

$\Rightarrow \quad \boxed{2x_2 = y_2}$

$\Rightarrow \quad \boxed{\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$

$$\begin{aligned} \vec{v}_1 \bullet \vec{v}_2 &= (-2)(1) + (1)(2) \\ &= -2 + 2 \\ &\boxed{= 0} \end{aligned}$$

Thus, the eigenvectors are perpendicular.

The unit vector of the eigenvectors by dividing by their length $\sqrt{2^2 + 1^2} = \sqrt{5}$

The eigenvectors $\vec{v}_1$ *and* $\vec{v}_2$ are the columns of Q.

$$Q = \frac{\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}}{\sqrt{5}}$$

$$Q^{-1} = Q^T = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\begin{aligned} A &= QDQ^T \\ &= \frac{1}{\sqrt{5}}\left(\frac{1}{\sqrt{5}}\right)\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -5 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 10 & 5 \\ -5 & 10 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} \quad \checkmark \end{aligned}$$

- Every symmetric matrix A has a complete set of orthogonal eigenvectors:

  $A = PDP^{-1} \quad \Rightarrow \quad A = QDQ^T$

Complex Eigenvalues of Real Matrices

For real matrices, complex λ's and x's come in "conjugate pairs"

$$\text{if} \quad Ax = \lambda x \quad \text{then} \quad A\bar{x} = \bar{\lambda}\bar{x}$$

Example

Given $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Solution

The eigenvalues of A:

$$\det(A - \lambda I) = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix}$$

$$= (\cos\theta - \lambda)(\cos\theta - \lambda) + \sin^2\theta$$

$$= \cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta$$

$$= \lambda^2 - 2\lambda\cos\theta + 1 = 0$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \frac{2\cos\theta \pm \sqrt{4\left(\cos^2\theta - 1\right)}}{2} \qquad \cos^2\theta + \sin^2\theta = 1 \to \cos^2\theta - 1 = -\sin^2\theta$$

$$= \frac{2\cos\theta \pm 2\sqrt{-\sin^2\theta}}{2}$$

$$= \cos\theta \pm i\sin\theta$$

The *eigenvalues* are conjugate to each other.

For $\lambda_1 = \cos\theta + i\sin\theta$: $\quad (A - \lambda_1 I)\vec{v}_1 = 0$

$$\begin{pmatrix} \cos\theta - (\cos\theta + i\sin\theta) & -\sin\theta \\ \sin\theta & \cos\theta - (\cos\theta + i\sin\theta) \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad i\sin\theta x_1 = \sin\theta y_1$$

$$\Rightarrow \quad x_1 = iy_1$$

The eigenvectors:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\rightarrow \quad \boxed{\vec{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}}$$

$$A\vec{v}_1 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$= (\cos\theta + i\sin\theta) \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$A\vec{v}_2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= (\cos\theta - i\sin\theta) \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$|\lambda| = \sqrt{\cos^2\theta + \sin^2\theta}$$

$$\boxed{= 1}$$

This fact holds for the eigenvalues of every orthogonal matrix.

Theorem – Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.

a) A is invertible

b) $A\vec{x} = \vec{0}$ has only the trivial solution

c) The reduced row echelon form of A is I_n

d) A is expressible as a product of elementary matrices

e) $A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ matrix $\vec{b}$

f) $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ matrix $\vec{b}$

g) $\det(A) \neq 0$

h) The column vectors of A are linearly independent

i) The row vectors of A are linearly independent

j) The column vectors of A span $\mathbb{R}^n$

k) The row vectors of A span $\mathbb{R}^n$

l) The column vectors of A form a basis for $\mathbb{R}^n$

m) The row vectors of A form a basis for $\mathbb{R}^n$

n) A has a rank n.

o) A has nullity 0.

p) The orthogonal complement of the null space of A is $\mathbb{R}^n$

q) The orthogonal complement of the row space of A is $\{0\}$

r) The range of T_A is $\mathbb{R}^n$

s) T_A is one-to-one.

t) $\lambda = 0$ is not an eigenvalue of A.

u) $A^T A$ is invertible,

Exercises *Section* 4.6 – Orthogonal Diagonalization

(**1 – 10**) Determine whether the matrix *is* orthogonal

1. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

2. $\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

3. $\begin{pmatrix} \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$

4. $\begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$

5. $\begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}$

6. $\begin{pmatrix} -4 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 4 \end{pmatrix}$

7. $\begin{pmatrix} 4 & -1 & -4 \\ -1 & 0 & -17 \\ 1 & 4 & -1 \end{pmatrix}$

8. $\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \end{pmatrix}$

9. $\begin{pmatrix} \frac{\sqrt{2}}{3} & 0 & \frac{\sqrt{5}}{2} \\ 0 & \frac{2\sqrt{5}}{5} & 0 \\ -\frac{\sqrt{2}}{6} & -\frac{\sqrt{5}}{5} & \frac{1}{2} \end{pmatrix}$

10. $\begin{pmatrix} \frac{1}{8} & 0 & 0 & \frac{3\sqrt{7}}{8} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{3\sqrt{7}}{8} & 0 & 0 & \frac{1}{8} \end{pmatrix}$

(**11 – 24**)Find a matrix *P* that orthogonally diagonalizes *A*, and determine $P^{-1}AP$

11. $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

12. $A = \begin{pmatrix} 6 & 2\sqrt{3} \\ 2\sqrt{3} & 7 \end{pmatrix}$

13. $A = \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}$

14. $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

15. $A = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$

16. $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

17. $A = \begin{pmatrix} 0 & 10 & 10 \\ 10 & 5 & 0 \\ 10 & 0 & -5 \end{pmatrix}$

18. $A=\begin{pmatrix}0&3&0\\3&0&4\\0&4&0\end{pmatrix}$

19. $A=\begin{pmatrix}2&2&-2\\2&-1&4\\-2&4&-1\end{pmatrix}$

20. $A=\begin{pmatrix}-2&0&-36\\0&-3&0\\-36&0&-23\end{pmatrix}$

21. $A=\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix}$

22. $A=\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}$

23. $A=\begin{pmatrix}-7&24&0&0\\24&7&0&0\\0&0&-7&24\\0&0&24&7\end{pmatrix}$

24. $A=\begin{pmatrix}3&1&0&0\\1&3&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$

25. Find the eigenvalues of A and B and check the Orthogonality of their first two eigenvectors. Graph these eigenvectors to see the discrete sines and cosines:

$$A=\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix} \qquad B=\begin{bmatrix}1&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&1\end{bmatrix}$$

The −1, 2, −1 pattern in both matrices is a "second derivative. Then $A\vec{x}=\lambda\vec{x}$ and $B\vec{x}=\lambda\vec{x}$ are like $\frac{d^2\vec{x}}{dt^2}=\lambda\vec{x}$ $\frac{d^2x}{dt^2}=\lambda x$. This has eigenvectors $x=\sin kt$ and $x=\cos kt$ that are the bases for Fourier series. The matrices lead to "discrete sines" and "discrete cosines" that are the bases for the discrete Fourier Transform. This DFT is absolutely central to all areas of digital signal processing.

26. Suppose $A\vec{x}=\lambda\vec{x}$ and $A\vec{y}=0\vec{y}$ and $\lambda\neq 0$. Then y is in the nullspace and $\vec{x}$ is in the column space. They are perpendicular because _______. (why are these subspaces orthogonal?) If the second eigenvalue is a nonzero number β, apply this argument to $A-\beta I$. The eigenvalue moves to zero and the eigenvectors stay the same – so they are perpendicular.

27. Which of these classes of matrices do A and B belong to: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix} \qquad B=\frac{1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$$

Which of these factorizations are possible for A and B: LU, QR, ADP^{-1}, QDQ^T?

28. *True* or *false*. Give a reason or a counterexample.

a) A matrix with real eigenvalues and eigenvectors is symmetric.
b) A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.
c) The inverse of a symmetric matrix is symmetric
d) The eigenvector matrix S of a symmetric matrix is symmetric.
e) A complex symmetric matrix has real eigenvalues.
f) If A is symmetric, then e^{iA} is symmetric.
g) If A is Hermitian, then e^{iA} is Hermitian.
h) An $n \times n$ matrix that is orthogonally diagonalizable must be symmetric.
i) If $A^T = A$ and if vectors $\vec{u}$ and $\vec{v}$ satisfy $A\vec{u} = 3\vec{u}$ and $A\vec{v} = 4\vec{v}$, then $\vec{u} \bullet \vec{v} = 0$
j) An $n \times n$ symmetric matrix has n distinct real eigenvalues.
k) For nonzero $\vec{v}$ in $\mathbb{R}^n$, the matrix $\vec{v}\vec{v}^T$ is called a projection matrix.
l) Every symmetric matrix is orthogonally diagonalizable
m) If $B = PDP^T$, where $P^T = P^{-1}$ and D is a diagonal matrix, then B is a symmetric matrix.
n) An orthogonal matrix is orthogonally diagonalizable.
o) The dimension of an eigenspace of a symmetric matrix equals the multiplicity of the corresponding eigenvalue.

29. Find a symmetric matrix $\begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$ that has a negative eigenvalue.

a) How do you know it must have a negative pivot?
b) How do you know it can't have two negative eigenvalues?

30. Prove that A is any $m \times n$ matrix, then $A^T A$ has an orthonormal set of n eigenvectors

31. Construct a 3 by 3 matrix A with no zero entries whose columns are mutually perpendicular. Compute $A^T A$. Why is it a diagonal matrix?

32. Assuming that $b \neq 0$, find a matrix that orthogonally diagonalizes $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$

33. Suppose A is a symmetric $n \times n$ matrix and B is any $n \times m$ matrix. Show that $B^T AB$, $B^T B$, and BB^T are symmetric matrices.

34. Show that if A is an $n \times n$ symmetric matrix, then $(A\vec{x}) \bullet \vec{y} = \vec{x} \bullet (A\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$.

35. Suppose A is invertible and orthogonally diagonalizable. Explain why A^{-1} is also orthogonally diagonalizable.

36. Suppose A and B are both orthogonally diagonalizable and $AB = BA$. Explain why AB is also orthogonally diagonalizable.

37. Let $A = PDP^{-1}$, where P is orthogonal and D is diagonal, let λ be an eigenvalue of A of multiplicity k. Then λ appears k times on the diagonal of D. Explain why the dimension of the eigenspace for λ is k.

38. Suppose $A = PUP^{-1}$, where P is orthogonal and U is an upper triangular. Show that if A is symmetric, then U is symmetric and hence is actually a diagonal matrix.

39. Let $\vec{u}$ be a unit vector in $\mathbb{R}^n$, and let $B = \vec{u}\vec{u}^T$.

a) Given $\vec{x} \in \mathbb{R}^n$, compute $B\vec{x}$ and show that $B\vec{x}$ is the orthogonal projection of $\vec{x}$ onto $\vec{u}$.

b) Show that B is a symmetric matrix and $B^2 = B$.

c) Show that $\vec{u}$ is an eigenvector of B. What is the corresponding eigenvalue?

40. Let B be an $n \times n$ symmetric matrix such that $B^2 = B$. Any such matrix is called a ***projection matrix*** (or an *orthogonal projection matrix*). Given any $\vec{y} \in \mathbb{R}^n$, let $\hat{y} = B\vec{y}$ and $\vec{z} = \vec{y} - \hat{y}$.

a) Show that $\vec{z}$ is orthogonal to $\hat{y}$.

b) Let W be the column space of B. Show that $\vec{y}$ is the sum of a vector in W and a vector in $W^{\perp}$. Why does this prove that $B\vec{y}$ is the orthogonal projection of $\vec{y}$ onto the column space of B?