

Exercise

(a) Find the series' radius and interval of convergence. For what values of x does the series converge (b) absolutely, (c) conditionally?

$$\sum_{n=0}^{\infty} x^n$$

Solution

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{x^{n+1}}{x^n} \right| = |x| < 1 \Rightarrow -1 < x < 1$$

When $x = 1 \Rightarrow \sum_{n=0}^{\infty} 1$ and $x = -1 \Rightarrow \sum_{n=0}^{\infty} (-1)^n$ the series diverges.

- a) The radius is 1; the interval of converges $-1 < x < 1$
- b) The interval of absolute convergence is $-1 < x < 1$
- c) There are no values for which the series converges conditionally

Exercise

(a) Find the series' radius and interval of convergence. For what values of x does the series converge (b) absolutely, (c) conditionally?

$$\sum_{n=0}^{\infty} (x+5)^n$$

Solution

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{(x+5)^{n+1}}{(x+5)^n} \right| = |x+5| < 1 \Rightarrow -6 < x < -4$$

When $x = -6 \Rightarrow \sum_{n=0}^{\infty} (-1)^n$ and $x = -4 \Rightarrow \sum_{n=0}^{\infty} 1$ the series diverges.

- a) The radius is 1; the interval of converges $-6 < x < -4$
- b) The interval of absolute convergence is $-6 < x < -4$
- c) There are no values for which the series converges conditionally.

Exercise

(a) Find the series' radius and interval of convergence. For what values of x does the series converge (b) absolutely, (c) conditionally?

$$\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$$

Solution

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{(3x-2)^{n+1}}{n+1} \cdot \frac{n}{(3x-2)^n} \right| = \frac{n}{n+1} |3x-2| < 1$$

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} |3x-2| < 1 \Rightarrow |3x-2| < 1$$

$$-1 < 3x-2 < 1 \Rightarrow 1 < 3x < 3 \Rightarrow \boxed{\frac{1}{3} < x < 1}$$

When $x = \frac{1}{3} \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ which is the alternating harmonic series and is conditionally convergent.

$x = 1 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{n}$ the series diverges harmonic.

a) The radius is $\frac{1}{3}$; the interval of converges $\frac{1}{3} \leq x < 1$

b) The interval of absolute convergence is $\frac{1}{3} < x < 1$

c) The series converges conditionally at $x = \frac{1}{3}$

Exercise

(a) Find the series' radius and interval of convergence. For what values of x does the series converge (b) absolutely, (c) conditionally?

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$$

Solution

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{(x-2)^{n+1}}{10^{n+1}} \cdot \frac{10^n}{(x-2)^n} \right|$$
$$= \frac{|x-2|}{10} < 1$$

$$-1 < \frac{x-2}{10} < 1$$

$$-10 < x - 2 < 10$$

$$-8 < x < 12$$

When $x = -8 \Rightarrow \sum_{n=0}^{\infty} (-1)^n$ which is a divergent series

$x = 12 \Rightarrow \sum_{n=0}^{\infty} 1$ the series diverges

- a) The radius is 10; the interval of converges $-8 < x < 12$
- b) The interval of absolute convergence is $-8 < x < 12$
- c) There are no values for which the series converges conditionally

Exercise

(a) Find the series' radius and interval of convergence. For what values of x does the series converge (b) absolutely, (c) conditionally?

$$\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$$

Solution

$$\begin{aligned} \left| \frac{u_{n+1}}{u_n} \right| &= \left| \frac{3^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n x^n} \right| \\ &= \frac{3}{n+1} |x| < 1 \end{aligned}$$

$$3|x| \lim_{n \rightarrow \infty} \frac{1}{n+1} < 1 \Rightarrow \forall x$$

- a) The radius is ∞ ; the series converges for all x .
- b) The series convergence absolutely for all x .
- c) There are no values for which the series converges conditionally

Exercise

(a) Find the series' radius and interval of convergence. For what values of x does the series converge (b) absolutely, (c) conditionally?

$$\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2 + 3}}$$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{\sqrt{(n+1)^2 + 3}} \cdot \frac{\sqrt{n^2 + 3}}{x^n} \right| \\ &= |x| \lim_{n \rightarrow \infty} \frac{n^2 + 3}{n^2 + 2n + 4} < 1\end{aligned}$$

$$|x| < 1 \Rightarrow -1 < x < 1$$

When $x = -1 \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2 + 3}}$ which is a convergent conditionally series

$x = 1 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2 + 3}}$ the series diverges

- a) The radius is 1; the series converges for $-1 \leq x < 1$.
- b) The series convergence absolutely for $-1 < x < 1$.
- c) The series convergence conditionally for $x = -1$

Exercise

(a) Find the series' radius and interval of convergence. For what values of x does the series converge (b) absolutely, (c) conditionally?

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{\sqrt{n} + 3}$$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+2}}{\sqrt{n+1} + 3} \cdot \frac{\sqrt{n} + 3}{x^{n+1}} \right| \\ &= |x| \lim_{n \rightarrow \infty} \frac{\sqrt{n} + 3}{\sqrt{n+1} + 3} < 1\end{aligned}$$

$$|x| < 1 \Rightarrow -1 < x < 1$$

When $x = -1 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{\sqrt{n} + 3}$ which is a divergent series

$x = 1 \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n} + 3}$ the series converges conditionally

- a) The radius is 1; the series converges for $-1 < x \leq 1$.

b) The series convergence absolutely for $-1 < x < 1$.

c) The series convergence conditionally for $x = 1$

Exercise

(a) Find the series' radius and interval of convergence. For what values of x does the series converge (b) absolutely, (c) conditionally?

$$\sum_{n=1}^{\infty} \sqrt[n]{n} (2x+5)^n$$

Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{n+1 \sqrt[n+1]{n+1} (2x+5)^{n+1}}{\sqrt[n]{n} (2x+5)^n} \right| \\ &= |2x+5| \lim_{n \rightarrow \infty} \frac{n+1 \sqrt[n+1]{n+1}}{\sqrt[n]{n}} < 1 \end{aligned}$$

$$= |2x+5| \frac{\lim_{m \rightarrow \infty} m \sqrt[m]{m}}{\lim_{n \rightarrow \infty} \sqrt[n]{n}} < 1 \rightarrow |2x+5| < 1$$

$$\begin{aligned} |2x+5| < 1 &\Rightarrow -1 < 2x+5 < 1 \\ &\Rightarrow -3 < x < -2 \end{aligned}$$

When $x = -3 \Rightarrow \sum_{n=0}^{\infty} (-1)^n \sqrt[n]{n}$ which is a divergent series

$x = -0 \Rightarrow \sum_{n=0}^{\infty} \sqrt[n]{n}$ which is a divergent series

a) The radius is $\frac{1}{2}$; the series converges for $-3 < x < -2$.

b) The series convergence absolutely for $-3 < x < -2$.

c) There are no values for which the series convergence conditionally

Exercise

(a) Find the series' radius and interval of convergence. For what values of x does the series converge (b) absolutely, (c) conditionally?

$$\sum_{n=1}^{\infty} \left(2 + (-1)^n\right) \cdot (x+1)^{n-1}$$

Solution

$$\sum_{n=1}^{\infty} \left(2 + (-1)^n\right) \cdot (x+1)^{n-1} = \sum_{n=1}^{\infty} 2(x+1)^{n-1} + \sum_{n=1}^{\infty} (-1)^n (x+1)^{n-1}$$

For the series $\sum_{n=1}^{\infty} 2(x+1)^{n-1}$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2(x+1)^n}{2(x+1)^{n-1}} \right|$$
$$= |x+1| < 1$$

$$-1 < x+1 < 1 \rightarrow -2 < x < 0$$

For the series $\sum_{n=1}^{\infty} (-1)^n (x+1)^{n-1}$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x+1)^n}{(x+1)^{n-1}} \right|$$
$$= |x+1| < 1$$

$$-1 < x+1 < 1 \rightarrow -2 < x < 0$$

When $x = -2 \Rightarrow \sum_{n=0}^{\infty} \left(2 + (-1)^n\right) \cdot (-1)^{n-1}$ which is a divergent series

$x = 0 \Rightarrow \sum_{n=0}^{\infty} \left(2 + (-1)^n\right)$ which is a divergent series

a) The radius is $\frac{1}{2}$; the series converges for $-2 < x < 0$.

b) The series convergence absolutely for $-2 < x < 0$.

c) There are no values for which the series convergence conditionally

Exercise

Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} n!x^n$$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| \\ &= |x| \lim_{n \rightarrow \infty} (n+1) \\ &= \underline{\infty}\end{aligned}$$

$$\rightarrow R = \frac{1}{\infty} = 0$$

By the Ratio Test, the series diverges for $|x| > 0$ and converges only at its center, 0.

Therefore, the radius of convergence is $R = 0$.

Exercise

Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} 3(x-2)^n$$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{3(x-2)^{n+1}}{3(x-2)^n} \right| \\ &= \underline{|x-2|}\end{aligned}$$

By the Ratio Test, the series converges for $|x-2| < 1$ and diverges for $|x-2| > 1$.

Therefore, the radius of convergence is $R = 1$.

Exercise

Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| \\ &= x^2 \lim_{n \rightarrow \infty} \frac{1}{(2n+2)(2n+3)} \\ &= \underline{0}\end{aligned}$$

$$\rightarrow R = \frac{1}{0} = \infty$$

By the Ratio Test, the series converges for all x . Therefore, the radius of convergence is $R = \infty$.

Exercise

Find the radius of convergence of the power series $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$

Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{n+2} \cdot \frac{n+1}{x^n} \right| \\ &= |x| \lim_{n \rightarrow \infty} \frac{n+1}{n+2} \\ &= \underline{|x|} \end{aligned}$$

$$\rightarrow R = 1$$

By the Ratio Test, the series converges for $|x| < 1$ and diverges for $|x| > 1$.

Therefore, the radius of convergence is $R = 1$.

Exercise

Find the radius of convergence of the power series $\sum_{n=0}^{\infty} (3x)^n$

Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(3x)^{n+1}}{(3x)^n} \right| \\ &= \underline{3|x|} \end{aligned}$$

$$\rightarrow 3|x| < 1 \Rightarrow R = \frac{1}{3}$$

By the Ratio Test, the series converges for $|x| < \frac{1}{3}$ and diverges for $|x| > \frac{1}{3}$.

Therefore, the radius of convergence is $R = \frac{1}{3}$.

Exercise

Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(4x)^n}{n^2}$$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(4x)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(4x)^n} \right| \\ &= |4x| \lim_{n \rightarrow \infty} \left| \frac{n^2}{(n+1)^2} \right| \\ &= \underline{4|x|}\end{aligned}$$

$$\rightarrow 4|x| < 1 \Rightarrow R = \frac{1}{4}$$

By the Ratio Test, the series converges for $|x| < \frac{1}{4}$ and diverges for $|x| > \frac{1}{4}$.

Therefore, the radius of convergence is $R = \frac{1}{4}$.

Exercise

Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{5^n}$$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{5^{n+1}} \cdot \frac{5^n}{x^n} \right| \\ &= \underline{\frac{|x|}{5}}\end{aligned}$$

$$\rightarrow \frac{|x|}{5} < 1 \Rightarrow R = 5$$

By the Ratio Test, the series converges for $|x| < 5$ and diverges for $|x| > 5$.

Therefore, the radius of convergence is $R = 5$.

Exercise

Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

Solution

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right|$$

$$= x^2 \lim_{n \rightarrow \infty} \frac{1}{(2n+1)(2n+1)}$$

$$= 0$$

$$R = \frac{1}{0} = \infty$$

By the Ratio Test, the series converges for all x . Therefore, the radius of convergence is $R = \infty$.

Exercise

Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(2n)! x^{2n}}{n!}$

Solution

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+2)! x^{2n+2}}{(n+1)!} \cdot \frac{n!}{(2n)! x^{2n}} \right|$$

$$= x^2 \lim_{n \rightarrow \infty} \left(\frac{(2n+1)(2n+2)}{n+1} \right)$$

$$= \infty$$

$$\rightarrow R = \frac{1}{\infty} = 0$$

By the Ratio Test, the series diverges for $|x| > 0$ and converges only at its center, 0.

Therefore, the radius of convergence is $R = 0$.

Exercise

Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$

Solution

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right|$$

$$= |x| \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)$$

$$= |x|$$

$$\rightarrow R = 1$$

So, by the Ratio Test, the radius of convergence is $R = 1$.

The series centered at 0, it converges in the interval $(-1, 1)$

When $x = 1$ $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$ *diverges*

When $x = -1$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \dots$ *converges*

Therefore, the interval of convergence $[-1, 1)$

Exercise

Find the interval of convergence of the power series $\sum_{n=0}^{\infty} (-1)^n \frac{(x+1)^n}{2^n}$

Solution

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x+1)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(x+1)^n} \right|$$

$$= \frac{1}{2} |x+1|$$

$$|x+1| < 2 \rightarrow R = 2$$

$$\begin{cases} x+1 = -2 & x = -3 \\ x+1 = 2 & x = 1 \end{cases}$$

So, by the Ratio Test, the radius of convergence is $R = 2$.

The series centered at -1 , it converges in the interval $(-3, 1)$

When $x = -3$ $\sum_{n=0}^{\infty} \frac{(-1)^n (-2)^n}{2^n} = \sum_{n=0}^{\infty} 1$ *diverges*

When $x = 1$ $\sum_{n=0}^{\infty} \frac{(-1)^n (2)^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n$ *diverges*

Therefore, the interval of convergence $(-3, 1)$

Exercise

Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

Solution

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n} \right|$$

$$= |x| \lim_{n \rightarrow \infty} \left| \frac{n^2}{(n+1)^2} \right|$$

$$= \underline{|x|}$$

$$\rightarrow R = 1$$

So, by the Ratio Test, the radius of convergence is $R = 1$.

The series centered at 0, it converges in the interval $(-1, 1)$

When $x = -1$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -1 + \frac{1}{2^2} - \frac{1}{3^2} + \dots$ *converges by alternating series*

$$u_{n+1} = \frac{1}{(n+1)^2} < \frac{1}{n^2} = u_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$$

When $x = 1$ $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$ *converges by p-series*

Therefore, the interval of convergence $[-1, 1]$

Exercise

Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$

Solution

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{4^{n+1}} \cdot \frac{4^n}{x^n} \right|$$

$$= \underline{\frac{|x|}{4}}$$

$$\rightarrow R = 4$$

So, by the Ratio Test, the radius of convergence is $R = 4$.

The series centered at 0, it converges in the interval $(-4, 4)$

When $x = -4$ $\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + \dots$ *diverges by alternating series*

When $x = 4$ $\sum_{n=1}^{\infty} 1$ *diverges*

Therefore, the interval of convergence $(-4, 4)$

Exercise

Find the interval of convergence of the power series $\sum_{n=0}^{\infty} (2x)^n$

Solution

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2x)^{n+1}}{(2x)^n} \right|$$
$$= \underline{2|x|}$$

$$2|x| = 1 \rightarrow R = \frac{1}{2}$$

So, by the Ratio Test, the radius of convergence is $R = \frac{1}{2}$.

The series converges in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$

When $x = -\frac{1}{2}$ $\sum_{n=0}^{\infty} (-1)^n = -1 + 1 - 1 + \dots$ *diverges by alternating series*

When $x = \frac{1}{2}$ $\sum_{n=0}^{\infty} 1$ *diverges*

Therefore, the interval of convergence $\left(-\frac{1}{2}, \frac{1}{2}\right)$

Exercise

Find the interval of convergence of the power series $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$

Solution

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right|$$
$$= |x| \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right|$$
$$= \underline{|x|}$$

$$|x| = 1 \rightarrow R = 1$$

So, by the Ratio Test, the radius of convergence is $R = 1$.

The series converges in the interval $(-1, 1)$

When $x = -1$ $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges by p-series*

When $x = 1$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ *converges* by Alternating Series

Therefore, the interval of convergence $(-1, 1]$

Exercise

Find the interval of convergence of the power series $\sum_{n=0}^{\infty} (-1)^n (n+1)x^n$

Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+2)x^{n+1}}{(n+1)x^n} \right| \\ &= |x| \lim_{n \rightarrow \infty} \left| \frac{n+2}{n+1} \right| \\ &= |x| \end{aligned}$$

$$|x| = 1 \rightarrow R = 1$$

The series converges in the interval $(-1, 1)$

When $x = -1$ $\sum_{n=0}^{\infty} (n+1)$ *diverges*

When $x = 1$ $\sum_{n=0}^{\infty} (-1)^n (n+1)$ *diverges*

Therefore, the interval of convergence $(-1, 1)$

Exercise

Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$

Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{5n+5}}{(n+1)!} \cdot \frac{n!}{x^{5n}} \right| \\ &= |x^5| \lim_{n \rightarrow \infty} \left| \frac{1}{n+1} \right| \\ &= 0 \rightarrow R = \infty \end{aligned}$$

The series converges for all x . Therefore, the interval of convergence $(-\infty, \infty)$

Exercise

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(3x)^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{(3x)^n} \right| \\ &= |3x| \lim_{n \rightarrow \infty} \left| \frac{1}{(2n+1)(2n+2)} \right| \\ &= \underline{0}\end{aligned}$$

$$\rightarrow R = \infty$$

The series converges for all x . Therefore, the interval of convergence $(-\infty, \infty)$

Exercise

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} (2n)! \left(\frac{x}{3}\right)^n$$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| (2n+2)! \left(\frac{x}{3}\right)^{n+1} \cdot (2n)! \left(\frac{x}{3}\right)^{-n} \right| \\ &= \left| \frac{x}{3} \right| \lim_{n \rightarrow \infty} |(2n+1)(2n+2)| \\ &= \underline{\infty}\end{aligned}$$

The series converges only for $x = 0$

Exercise

Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{(n+1)(n+2)}$$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+2)(n+3)} \cdot \frac{(n+1)(n+2)}{x^n} \right| \\ &= |x| \lim_{n \rightarrow \infty} \left| \frac{n+1}{n+3} \right| \\ &= \underline{|x|}\end{aligned}$$

$$|x| = 1 \rightarrow R = 1$$

The series converges in the interval $(-1, 1)$

When $x = -1$ $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$ *converges* by *Alternating Series*

$$u_{n+1} = \frac{1}{(n+3)(n+2)} < \frac{1}{(n+1)(n+2)} = u_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{(n+1)(n+2)} = 0$$

When $x = 1$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)(n+2)}$ *converges* by *Limit Comparison Test* to $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Therefore, the interval of convergence $[-1, 1]$

Exercise

Find the interval of convergence of the power series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{6^n}$

Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{6^{n+1}} \cdot \frac{6^n}{x^n} \right| \\ &= \frac{1}{6} |x| \end{aligned}$$

$$\frac{1}{6} |x| = 1 \rightarrow R = 6$$

The series converges in the interval $(-6, 6)$

When $x = -6$ $\sum_{n=1}^{\infty} (-1)^n$ *diverges*

When $x = 6$ $\sum_{n=1}^{\infty} (-1)^{n+1}$ *diverges*

Therefore, the interval of convergence $(-6, 6)$

Exercise

Find the interval of convergence of the power series $\sum_{n=0}^{\infty} (-1)^n \frac{n!(x-5)^n}{3^n}$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)!(x-5)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{n!(x-5)^n} \right| \\ &= \left| \frac{x-5}{3} \right| \lim_{n \rightarrow \infty} (n+1) \\ &= \underline{\infty}\end{aligned}$$

The series converges only for $x = 5$

Exercise

Find the interval of convergence of the power series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-4)^n}{n9^n}$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x-4)^{n+1}}{(n+1)9^{n+1}} \cdot \frac{n9^n}{(x-4)^n} \right| \\ &= \frac{1}{9} |x-4| \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| \\ &= \underline{\frac{1}{9} |x-4|}\end{aligned}$$

$$\frac{1}{9} |x-4| = 1 \rightarrow R = 9$$

$$|x-4| = 9 \Rightarrow \begin{cases} x-4 = -9 & x = -5 \\ x-4 = 9 & x = 13 \end{cases}$$

The series converges in the interval $(-5, 13)$ and center $x = 4$

$$\text{When } x = -5 \quad \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-9)^n}{n9^n} = \sum_{n=1}^{\infty} \frac{-1}{n} \quad \text{diverges}$$

$$\text{When } x = 13 \quad \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(9)^n}{n9^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \quad \text{converges by Alternating Series}$$

$$u_{n+1} = \frac{1}{n+1} < \frac{1}{n} = u_n \quad \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Therefore, the interval of convergence $(-5, 13]$

Exercise

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(x-3)^{n+1}}{(n+1)4^{n+1}}$$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x-3)^{n+2}}{(n+2)4^{n+2}} \cdot \frac{(n+1)4^{n+1}}{(x-3)^{n+1}} \right| \\ &= \frac{1}{4} |x-3| \lim_{n \rightarrow \infty} \left| \frac{n+1}{n+2} \right| \\ &= \frac{1}{4} |x-3|\end{aligned}$$

$$\frac{1}{4} |x-3| = 1 \rightarrow R = 4$$

$$|x-3| = 4 \Rightarrow \begin{cases} x-3 = -4 & x = -1 \\ x-3 = 4 & x = 7 \end{cases}$$

The series converges in the interval $(-1, 7)$ and center $x = 3$

$$\text{When } x = -1 \quad \sum_{n=0}^{\infty} \frac{(-4)^{n+1}}{(n+1)4^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} \quad \text{converges by Alternating Series}$$

$$u_{n+1} = \frac{1}{n+2} < \frac{1}{n+1} = u_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$$

$$\text{When } x = 4 \quad \sum_{n=0}^{\infty} \frac{4^{n+1}}{(n+1)4^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{n+1} \quad \text{diverges by Integral Test}$$

$$\int_0^{\infty} \frac{dx}{x+1} = \ln(x+1) \Big|_0^{\infty} = \underline{\underline{\infty}}$$

Therefore, the interval of convergence $[-1, 7)$

Exercise

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-1)^{n+1}}{n+1}$$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x-1)^{n+2}}{n+2} \cdot \frac{n+1}{(x-1)^{n+1}} \right| \\ &= |x-1| \lim_{n \rightarrow \infty} \left| \frac{n+1}{n+2} \right|\end{aligned}$$

$$= |x-1|$$

$$|x-1| = 1 \rightarrow R = 1$$

$$|x-1| = 1 \Rightarrow \begin{cases} x-1 = -1 & x = 0 \\ x-1 = 1 & x = 2 \end{cases}$$

The series converges in the interval $(0, 2)$ and center $x = 1$

When $x = 0$ $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$ *diverges by Integral Test*

$$\int_0^{\infty} \frac{dx}{x+1} = \ln(x+1) \Big|_0^{\infty} = \infty$$

When $x = 1$ $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}$ *converges by Alternative Test*

$$u_{n+1} = \frac{1}{n+2} < \frac{1}{n+1} = u_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$$

Therefore, the interval of convergence $(0, 2]$

Exercise

Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-2)^n}{n2^n}$$

Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{(x-2)^n} \right| \\ &= \frac{1}{2} |x-2| \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| \\ &= \frac{1}{2} |x-2| \end{aligned}$$

$$|x-2| = 2 \rightarrow R = 2$$

$$|x-2| = 2 \Rightarrow \begin{cases} x-2 = -2 & x = 0 \\ x-2 = 2 & x = 4 \end{cases}$$

The series converges in the interval $(0, 4)$ and center $x = 2$

When $x = 0$ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-2)^n}{n2^n} = \sum_{n=0}^{\infty} \frac{-1}{n}$ *diverges by Integral Test*

$$\int_0^{\infty} \frac{-dx}{x} = -\ln(x) \Big|_0^{\infty} = \infty$$

When $x = 4$ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ *converges by Alternative Test*

$$u_{n+1} = \frac{1}{n+1} < \frac{1}{n} = u_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Therefore, the interval of convergence $(0, 4]$

Exercise

Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-3)^{n-1}}{3^{n-1}}$$

Solution

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-3)^n}{3^n} \cdot \frac{3^{n-1}}{(x-3)^{n-1}} \right|$$

$$= \frac{1}{3} |x-3|$$

$$\frac{1}{3} |x-3| = 1 \rightarrow R = 3$$

$$|x-3| = 3 \Rightarrow \begin{cases} x-3 = -3 & x = 0 \\ x-3 = 3 & x = 6 \end{cases}$$

The series converges in the interval $(0, 6)$

When $x = 0$ $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty} (-1)$ *diverges*

When $x = 6$ $\sum_{n=1}^{\infty} \frac{3^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty} 1$ *diverges*

Therefore, the interval of convergence $(0, 6)$

Exercise

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{2n+3}}{2n+3} \cdot \frac{2n+1}{x^{2n+1}} \right| \\ &= x^2 \lim_{n \rightarrow \infty} \left| \frac{2n+1}{2n+3} \right| \\ &= \underline{x^2}\end{aligned}$$

$$\rightarrow R = 1$$

The series converges in the interval $(-1, 1)$

$$\text{When } x = -1 \quad \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} \quad \text{converges by Alternating Series}$$

$$\text{When } x = 1 \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \quad \text{converges by Alternating Series}$$

Therefore, the interval of convergence $[-1, 1]$

Exercise

Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{n}{n+1} (-2x)^{n-1}$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{n+1}{n+2} (-2x)^n \cdot \frac{n+1}{n(-2x)^{n-1}} \right| \\ &= |-2x| \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{n(n+2)} \right| \\ &= \underline{2|x|}\end{aligned}$$

$$2|x| = 1 \rightarrow R = \frac{1}{2}$$

The series converges in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$

$$\text{When } x = -\frac{1}{2} \quad \sum_{n=1}^{\infty} \frac{n}{n+1} \quad \text{diverges by } n\text{th Term Test}$$

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$$

$$\text{When } x = \frac{1}{2} \quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n+1} \quad \text{diverges by Alternating Series}$$

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$$

Therefore, the interval of convergence $\left(-\frac{1}{2}, \frac{1}{2}\right)$

Exercise

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{2n+2}}{(n+1)!} \cdot \frac{n!}{x^{2n}} \right| \\ &= x^2 \lim_{n \rightarrow \infty} \left| \frac{1}{n+1} \right| \\ &= 0\end{aligned}$$

$$\rightarrow R = \infty$$

Therefore, the interval of convergence $(-\infty, \infty)$

Exercise

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!}$$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{3n+4}}{(3n+4)!} \cdot \frac{(3n+1)!}{x^{3n+1}} \right| \\ &= |x^3| \lim_{n \rightarrow \infty} \left| \frac{1}{(3n+2)(3n+3)(3n+4)} \right| \\ &= 0\end{aligned}$$

$$\rightarrow R = \infty$$

Therefore, the interval of convergence $(-\infty, \infty)$

Exercise

Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n! x^n}{(2n)!}$$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)!x^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{n!x^n} \right| \\ &= |x| \lim_{n \rightarrow \infty} \left| \frac{n+1}{(2n+1)(2n+2)} \right| \\ &= \underline{0}\end{aligned}$$

$$\rightarrow R = \infty$$

Therefore, the interval of convergence $(-\infty, \infty)$

Exercise

Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{2 \cdot 3 \cdot 4 \cdots (n+1)x^n}{n!}$

Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{2 \cdot 3 \cdot 4 \cdots (n+1)(n+2)x^{n+1}}{(n+1)!} \cdot \frac{n!}{2 \cdot 3 \cdot 4 \cdots (n+1)x^n} \right| \\ &= |x| \lim_{n \rightarrow \infty} \left| \frac{n+2}{n+1} \right| \\ &= \underline{|x|}\end{aligned}$$

$$|x| = 1 \rightarrow R = 1$$

The series converges in the interval $(-1, 1)$

$$\text{When } x = -1 \quad \sum_{n=1}^{\infty} (-1)^n \frac{2 \cdot 3 \cdot 4 \cdots (n+1)}{n!} = \sum_{n=1}^{\infty} (-1)^n (n+1) \quad \text{diverges}$$

$$\text{When } x = 1 \quad \sum_{n=1}^{\infty} \frac{2 \cdot 3 \cdot 4 \cdots (n+1)}{n!} = \sum_{n=1}^{\infty} (n+1) \quad \text{diverges}$$

Therefore, the interval of convergence $(-1, 1)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^{2n}}{\sqrt{n+1}}$

Solution

$$R = \lim_{n \rightarrow \infty} \left| \frac{1}{\sqrt{n+2}} \cdot \sqrt{n+1} \right| = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n}} = 1$$

The radius of convergence is 1.

The centre of convergence is 0.

The interval of convergence is $(-1, 1)$.

The series does not converge at $x = -1$ or $x = 1$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=0}^{\infty} 3n(x+1)^n$

Solution

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \left| \frac{3n}{3(n+1)} \right| & R &= \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| \\ &= \lim_{n \rightarrow \infty} \frac{3n}{3n} \\ &= 1 \end{aligned}$$

The radius of convergence is 1, and the centre of convergence is -1 . ($x+1=0$)

$$a - R < x < a + R \Rightarrow -1 - 1 < x < -1 + 1$$

Therefore, the given series converges absolutely on $(-2, 0)$

At $x = -2$, the series is $\sum_{n=0}^{\infty} 3n(-1)^n$ which diverges.

At $x = 0$, the series is $\sum_{n=0}^{\infty} 3n(1)^n = \sum_{n=0}^{\infty} 3n$ which diverges.

Hence, the interval of convergence is $(-2, 0)$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} x^n$

Solution

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^4 2^{2n+2}}{n^4 2^{2n}} \right| & R &= \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| \\ &= 4 \lim_{n \rightarrow \infty} \left| \left(\frac{n+1}{n} \right)^4 \right| \\ &= 4 \end{aligned}$$

The radius of convergence is 4, and the centre of convergence is 0.

$a - R < x < a + R \Rightarrow -4 < x < 4$, the given series converges absolutely on $(-4, 4)$

$$\begin{aligned} \text{At } x = -4, \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} (-4)^n &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} (-1)^n 2^{2n} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^4} \text{ which converges (p-series).} \end{aligned}$$

$$\begin{aligned} \text{At } x = 4, \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} (4)^n &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} 2^{2n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \text{ which also converges.} \end{aligned}$$

Hence, the interval of convergence is $[-4, 4]$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{e^n}{n^3} (4-x)^n$

Solution

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \left| \frac{e^n}{n^3} \cdot \frac{(n+1)^3}{e^{n+1}} \right| & R &= \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| \\ &= \frac{1}{e} \lim_{n \rightarrow \infty} \left| \left(\frac{n+1}{n} \right)^3 \right| \\ &= \frac{1}{e} \end{aligned}$$

The radius of convergence is $\frac{1}{e}$.

The centre of convergence is 4. ($4 - x = 0 \Rightarrow x = 4$)

$a - R < x < a + R \Rightarrow 4 - \frac{1}{e} < x < 4 + \frac{1}{e}$, which the given series converges absolutely

$$\text{At } x = 4 - \frac{1}{e}, \text{ the series is } \sum_{n=1}^{\infty} \frac{e^n}{n^3} \left(\frac{1}{e} \right)^n = \sum_{n=1}^{\infty} \frac{1}{n^3} \text{ which converges (p-series).}$$

$$\text{At } x = 4 + \frac{1}{e}, \text{ the series is } \sum_{n=1}^{\infty} \frac{e^n}{n^3} \left(-\frac{1}{e} \right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \text{ which also converges (p-series).}$$

Hence, the interval of convergence is $\left[4 - \frac{1}{e}, 4 + \frac{1}{e} \right]$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{1+5^n}{n!} x^n$

Solution

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \left| \frac{1+5^n}{n!} \cdot \frac{(n+1)!}{1+5^{n+1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{1+5^n}{1+5^{n+1}} (n+1) \right| \\ &= \infty \end{aligned}$$

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

The radius of convergence is ∞ .

The centre of convergence is 0.

The interval of convergence is the real line $(-\infty, \infty)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n}$

Solution

$$\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n} = \sum_{n=1}^{\infty} \frac{4^n \left(x - \frac{1}{4}\right)^n}{n^n}$$

$$R = \lim_{n \rightarrow \infty} \left| \frac{4^n}{n^n} \cdot \frac{(n+1)^{n+1}}{4^{n+1}} \right|$$

$$= \frac{1}{4} \lim_{n \rightarrow \infty} \left| \left(\frac{n+1}{n}\right)^n (n+1) \right|$$

$$= \infty$$

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

The radius of convergence is ∞ .

The centre of convergence is $x = \frac{1}{4}$.

The interval of convergence is the real line $(-\infty, \infty)$

Exercise

For what value of x does the series $1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \dots + \left(-\frac{1}{2}\right)^n (x-3)^n + \dots$ converges? What is its sum? What series do you get if you differentiate the given series term by term? For what value of x does the new series converge? What is its sum?

Solution

$$1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \dots + \left(-\frac{1}{2}\right)^n (x-3)^n + \dots = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-3)^n$$

$$\lim_{n \rightarrow \infty} \left| \frac{(x-3)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(x-3)^n} \right| = \left| \frac{x-3}{2} \right| < 1$$

$$\Rightarrow -1 < \frac{x-3}{2} < 1$$

$$-2 < x-3 < 2 \Rightarrow \boxed{1 < x < 5}$$

When $x=1 \Rightarrow \sum_{n=1}^{\infty} (1)^n$ which is a divergent series

$x=5 \Rightarrow \sum_{n=1}^{\infty} (-1)^n$ the series diverges

The series is a geometric series, the sum is $\frac{1}{1 + \frac{x-3}{2}} = \frac{2}{x-1}$

$$\text{If } f(x) = 1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \dots + \left(-\frac{1}{2}\right)^n (x-3)^n + \dots = \frac{2}{x-1}$$

$$\text{Then } f'(x) = -\frac{1}{2} + \frac{1}{2}(x-3) + \dots + \left(-\frac{1}{2}\right)^n n(x-3)^{n-1} + \dots$$

$f'(x)$ is convergent when $1 < x < 5$ and divergent when $x=1$ or 5

$$\text{The sum for } f'(x) \text{ is } \frac{-2}{(x-1)^2}$$

Exercise

The series $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots$ converges to $\sin x$ for all x .

- Find the first six terms of a series for $\cos x$. For what values of x should the series converge?
- By replacing x by $2x$ in the series for $\sin x$, find a series that converges to $\sin 2x$ for all x .
- Using the result in part (a) and series multiplication, calculate the first six term of a series for $2 \sin x \cos x$. Compare your answer with the answer in part (b).

Solution

$$a) (\sin x)' = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n}}{(2n)!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right| = x^2 \lim_{n \rightarrow \infty} \left| \frac{1}{(2n+1)(2n+2)} \right| = 0 < 1 \quad (\forall x)$$

The series converges for all values of x .

$$b) \sin 2x = 2x - \frac{2^3 x^3}{3!} + \frac{2^5 x^5}{5!} - \frac{2^7 x^7}{7!} + \frac{2^9 x^9}{9!} - \frac{2^{11} x^{11}}{11!} + \dots$$

$$= 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \frac{512x^9}{9!} - \frac{2048x^{11}}{11!} + \dots$$

$$c) 2\sin x \cos x = 2 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots \right) \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots \right)$$

$$= 2x \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots \right) - 2 \frac{x^3}{3!} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots \right)$$

$$+ 2 \frac{x^5}{5!} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots \right) - 2 \frac{x^7}{7!} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots \right)$$

$$+ 2 \frac{x^9}{9!} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots \right) - 2 \frac{x^{11}}{11!} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots \right)$$

$$= 2x - \frac{2x^3}{2!} + \frac{2x^5}{4!} - \frac{2x^7}{6!} + \frac{2x^9}{8!} - \frac{2x^{11}}{10!} - \frac{2x^3}{3!} + \frac{2x^5}{2!3!} - \frac{2x^7}{4!3!} + \frac{2x^9}{6!3!} - \frac{2x^{11}}{8!3!}$$

$$+ \frac{2x^5}{5!} - \frac{2x^7}{5!2!} + \frac{2x^9}{5!4!} - \frac{2x^{11}}{5!6!} - \frac{2x^7}{7!} + \frac{2x^9}{7!2!} - \frac{2x^{11}}{7!4!} + \frac{2x^9}{9!} - \frac{2x^{11}}{9!2!} - \frac{2x^{11}}{11!} + \dots$$

$$= 2x - \frac{2^3 x^3}{3!} + \frac{2^5 x^5}{5!} - \frac{2^7 x^7}{7!} + \frac{2^9 x^9}{9!} - \frac{2^{11} x^{11}}{11!} + \dots$$

Exercise

Find the sum of the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ by the first finding the sum of the power series

$$\sum_{n=1}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + 16x^4 + \dots$$

Solution

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots + x^n + \dots = \frac{1}{1-x}$$

$$\sum_{n=1}^{\infty} nx^n = x + x^2 + 3x^3 + 4x^4 + \dots$$

$$\begin{aligned} &= x(1 + x + 3x^2 + 4x^3 + \dots) \\ &= x \frac{d}{dx} (1 + x + x^2 + \dots + x^n + \dots) \\ &= x \left(\frac{1}{1-x} \right)' \\ &= \frac{x}{(1-x)^2} \end{aligned}$$

$$\begin{aligned} \frac{d}{dx} \sum_{n=1}^{\infty} nx^n &= \left(\frac{x}{(1-x)^2} \right)' \\ &= \frac{(1-x)^2 + 2x(1-x)}{(1-x)^4} \\ &= \frac{1+x}{(1-x)^3} \end{aligned}$$

$$\frac{d}{dx} \sum_{n=1}^{\infty} nx^n = \sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{1+x}{(1-x)^3}$$

Multiply by x both sides

$$x \sum_{n=1}^{\infty} n^2 x^{n-1} = x \frac{1+x}{(1-x)^3}$$

$$\sum_{n=1}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + 16x^4 + \dots = \frac{x(1+x)}{(1-x)^3}$$

Let $x = \frac{1}{2}$

$$\sum_{n=1}^{\infty} n^2 \left(\frac{1}{2} \right)^n = \frac{\frac{1}{2} \cdot \frac{3}{2}}{\left(\frac{1}{2} \right)^3}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \underline{6}$$

Exercise

Find a series representation of $f(x) = \frac{1}{2+x}$ in powers of $x-1$. What is the interval of convergence of this series?

Solution

Let $t = x - 1 \Rightarrow x = t + 1$, we have

$$\begin{aligned}\frac{1}{2+x} &= \frac{1}{3+t} \\ &= \frac{1}{3} \frac{1}{1+\frac{t}{3}} \\ &= \frac{1}{3} \left(1 - \frac{t}{3} + \frac{t^2}{3^2} - \frac{t^3}{3^3} + \dots \right) \quad \left(-1 < \frac{t}{3} < 1 \right) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{3^{n+1}} \quad (-3 < t < 3) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{3^{n+1}} \quad (-2 < x < 4)\end{aligned}$$

$$R = \lim_{n \rightarrow \infty} \frac{3^{n+2}}{3^{n+1}} = \underline{3}$$

The radius of convergence of this series is 3.

The distance from the centre of convergence $x-1=0 \Rightarrow \underline{x=1}$, to the point -2 where the denominator is 0.

Exercise

Determine the Cauchy product of the series $1 + x + x^2 + x^3 + \dots$ and $-x + x^2 - x^3 + \dots$. On what interval and to what function does the product series converge?

Solution

$$\begin{aligned}1 + x + x^2 + x^3 + \dots &= \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \\ -x + x^2 - x^3 + \dots &= \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n\end{aligned}$$

Let $a_n = 1$ and $b_n = (-1)^n$, then the series holds for $-1 < x < 1$

$$\text{We have } c_n = \sum_{j=0}^n (-1)^{n-j} = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

Then the Cauchy product is

$$\begin{aligned}
 1 + x^2 + x^4 + \dots &= \sum_{n=0}^{\infty} x^{2n} \\
 &= \frac{1}{1-x} \cdot \frac{1}{1+x} \\
 &= \frac{1}{1-x^2} \quad \text{for } -1 < x < 1
 \end{aligned}$$

Exercise

Determine the power series expansion of $\frac{1}{(1-x)^2}$ by formally dividing $1 - 2x + x^2$ into 1.

Use the power series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad -1 < x < 1$

Solution

$$\begin{array}{r}
 1 - 2x + x^2 \overline{) 1} \\
 \underline{1} \\
 2x - x^2 \\
 \underline{2x - 4x^2 + 2x^3} \\
 3x^2 - 2x^3 \\
 \underline{3x^2 - 6x^3 + 3x^4} \\
 4x^3 + 3x^4 \\
 \underline{4x^3 - 8x^4 + 4x^5} \\
 11x^4 - \dots
 \end{array}$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n \quad \text{for } -1 < x < 1$$

Exercise

Determine the interval of convergence and the sum of the series

$$1 - 4x + 16x^2 - 64x^3 + \dots = \sum_{n=0}^{\infty} (-1)^n (4x)^n$$

Solution

$$\begin{aligned}
 1 - 4x + 16x^2 - 64x^3 + \dots &= 1 + (-4x) + (-4x)^2 + (-4x)^3 + \dots \\
 &= \frac{1}{1 - (-4x)} \\
 &= \frac{1}{1 + 4x}
 \end{aligned}$$

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \dots$$

Therefore, the interval of convergence is $-\frac{1}{4} < x < \frac{1}{4}$

Exercise

Determine the interval of convergence and the sum of the series

$$3 + 4x + 5x^2 + 6x^3 + \cdots = \sum_{n=0}^{\infty} (n+3)x^n$$

Solution

$$\sum_{n=0}^{\infty} (n+3)x^n = \sum_{n=0}^{\infty} nx^n + \sum_{n=0}^{\infty} 3x^n$$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

$$\left(\sum_{n=0}^{\infty} x^n \right)' = 1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2} \quad \text{Differentiate}$$

$$x(1 + 2x + 3x^2 + 4x^3 + \cdots) = \frac{x}{(1-x)^2} \quad \text{Multiply by } x$$

$$\sum_{n=0}^{\infty} nx^n = x + 2x^2 + 3x^3 + \cdots = \frac{x}{(1-x)^2}$$

Then,

$$\begin{aligned} \sum_{n=0}^{\infty} (n+3)x^n &= \sum_{n=0}^{\infty} nx^n + \sum_{n=0}^{\infty} 3x^n \\ &= \frac{x}{(1-x)^2} + 3 \frac{1}{1-x} \\ &= \frac{3-2x}{(1-x)^2} \quad (-1 < x < 1) \end{aligned}$$

Exercise

Determine the interval of convergence and the sum of the series

$$\frac{1}{3} + \frac{x}{4} + \frac{x^2}{5} + \frac{x^3}{6} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n+3}$$

Solution

$$\begin{aligned} \frac{1}{3} + \frac{x}{4} + \frac{x^2}{5} + \frac{x^3}{6} + \cdots &= \frac{1}{x^3} \left(\frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \cdots \right) \\ &= \frac{1}{x^3} \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \cdots - x - \frac{x^2}{2} \right) \quad \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{x^3} \left(-\ln(1-x) - x - \frac{x^2}{2} \right) \\
&= \underbrace{-\frac{1}{x^3} \ln(1-x) - \frac{1}{x^2} - \frac{1}{2x}}_{\hspace{1.5cm}} \quad (-1 \leq x < 1, \ x \neq 0)
\end{aligned}$$