

## ***Solution***

## **Section 2.2 – Techniques for Finding Derivatives**

### ***Exercise***

Find the first derivative of  $f(x) = -2$

### **Solution**

$$f'(x) = 0$$

### ***Exercise***

Find the first derivative of  $y = \pi$

### **Solution**

$$y' = 0$$

### ***Exercise***

Find the first derivative of  $y = \sqrt{5}$

### **Solution**

$$y' = 0$$

### ***Exercise***

Find the first derivative of  $f(x) = x^4$

### **Solution**

$$\begin{aligned} f'(x) &= 4x^{4-1} \\ &= 4x^3 \end{aligned}$$

### ***Exercise***

Find the first derivative of  $s(t) = \frac{1}{t}$

### **Solution**

$$\begin{aligned} s(t) &= t^{-1} \\ s'(t) &= (-1)t^{-1-1} \\ &= -t^{-2} \end{aligned}$$

***Exercise***

Find the first derivative of  $y = 4x^2$

**Solution**

$$\begin{aligned}y' &= 4(2)x^{2-1} \\ &= 8x\end{aligned}$$

***Exercise***

Find the first derivative of  $y = \frac{9}{4x^2}$

**Solution**

$$\begin{aligned}y &= \frac{9}{4x^2} \\ &= \frac{9}{4}x^{-2} \\ \rightarrow y' &= \frac{9}{4}(-2)x^{-3} \\ &= -\frac{9}{2x^3}\end{aligned}$$

***Exercise***

Find the first derivative of  $y = \frac{9}{(4x)^2}$

**Solution**

$$\begin{aligned}y &= \frac{9}{(4x)^2} \\ &= \frac{9}{4^2x^2} \\ &= \frac{9}{16}x^{-2} \\ \rightarrow y' &= \frac{9}{16}(-2)x^{-3} \\ &= -\frac{9}{8x^3}\end{aligned}$$

### ***Exercise***

Find the first derivative of  $y = \sqrt{5x}$

### **Solution**

$$y = \sqrt{5}x^{1/2}$$

$$\rightarrow y' = \sqrt{5} \left( \frac{1}{2} \right) x^{1/2-1}$$

$$= \frac{\sqrt{5}}{2x^{1/2}}$$

$$= \frac{\sqrt{5}}{2\sqrt{x}}$$

### ***Exercise***

Find the first derivative of  $y = \sqrt[3]{x}$

### **Solution**

$$y = x^{1/3}$$

$$\rightarrow y' = \frac{2}{3} x^{(1/3)-1}$$

$$= \frac{2}{3} x^{-2/3}$$

$$= \frac{2}{3\sqrt[3]{x^2}}$$

### ***Exercise***

Find the derivative of  $y = \frac{0.4}{\sqrt{x^3}}$

### **Solution**

$$y' = 0.4 \frac{d}{dx} \frac{1}{x^{3/2}}$$

$$= 0.4 \frac{d}{dx} x^{-3/2}$$

$$= 0.4 \left( -\frac{3}{2} \right) x^{-3/2-1}$$

$$= \underline{-0.6x^{-5/2}} \quad \text{or} \quad = -\frac{0.6}{\sqrt{x^5}}$$

**Exercise**

Find the derivative of  $y = -\frac{2}{\sqrt[3]{x}}$

**Solution**

$$y = -\frac{2}{x^{1/3}} = -2x^{-1/3}$$

$$y' = -2\left(-\frac{1}{3}\right)x^{-1/3-1}$$

$$= \frac{2}{3}x^{-4/3}$$

$$= \frac{2}{3x^{4/3}}$$

$$= \frac{2}{3x\sqrt[3]{x}} \Big|$$

**Exercise**

Find the derivative of  $y = \frac{1}{\sqrt[3]{x}}$

**Solution**

$$y = \frac{1}{x^{1/3}} = x^{-1/3}$$

$$y' = -\left(-\frac{1}{3}\right)x^{-4/3}$$

$$= \frac{1}{3x^{4/3}}$$

$$= \frac{1}{3x\sqrt[3]{x}} \Big|$$

$$x^{4/3} = \sqrt[3]{x^4} = \sqrt[3]{x^3 \cdot x} = x\sqrt[3]{x}$$

**Exercise**

Find the derivative of  $y = \frac{x^3 - 4x}{\sqrt{x}}$

**Solution**

$$y = \frac{x^3}{x^{1/2}} - 4\frac{x}{x^{1/2}} = x^{5/2} - 4x^{1/2}$$

$$y' = \frac{5}{2}x^{3/2} - 4\frac{1}{2}x^{-1/2}$$

$$= \frac{5}{2}x\sqrt{x} - 2\frac{2}{\sqrt{x}} \Big|$$

### ***Exercise***

Find the derivative of  $f(x) = 3x^2 + 2x$

#### **Solution**

$$f'(x) = \underline{6x + 2}$$

### ***Exercise***

Find the derivative of  $f(x) = 4 + 2x^3 - 3x^{-1}$

#### **Solution**

$$\begin{aligned} f(x) &= 0 + 6x^2 + 3x^{-2} \\ &= \underline{6x^2 + 3x^{-2}} \end{aligned}$$

### ***Exercise***

Find the derivative of  $f(x) = \frac{5}{3x^2} - \frac{2}{x^4} + \frac{x^3}{9}$

#### **Solution**

$$\begin{aligned} f(x) &= \frac{5}{3}x^{-2} - 2x^{-4} + \frac{x^3}{9} \\ f'(x) &= \frac{5}{3}(-2x^{-3}) - 2(-4x^{-5}) + \frac{3x^2}{9} \\ &= \underline{-\frac{10}{3x^3} + \frac{8}{x^5} + \frac{x^2}{3}} \end{aligned}$$

### ***Exercise***

Find the derivative of  $f(x) = \frac{3}{x^{3/5}} - \frac{6}{x^{1/2}}$

#### **Solution**

$$\begin{aligned} f(x) &= 3x^{-3/5} - 6x^{-1/2} \\ f'(x) &= 3\left(-\frac{3}{5}\right)x^{-8/5} - 6\left(-\frac{1}{2}\right)x^{-3/2} \\ &= \underline{-\frac{9}{5x^{8/5}} + \frac{3}{x^{3/2}}} \end{aligned}$$

### Exercise

Find the derivative of  $f(x) = \frac{5}{x^{1/5}} - \frac{8}{x^{3/2}}$

#### Solution

$$f(x) = 5x^{-1/5} - 8x^{-3/2}$$

$$f'(x) = 5\left(-\frac{1}{5}\right)x^{-6/5} - 8\left(-\frac{3}{2}\right)x^{-5/2}$$
$$= -\frac{1}{x^{6/5}} + \frac{12}{x^{5/2}} \Big|$$

### Exercise

Find the derivative of  $y = \frac{1.2}{\sqrt{x}} - 3.2x^{-2} + x$

#### Solution

$$y' = -\frac{1.2}{2x\sqrt{x}} + 6.4x^{-3} + 1$$
$$= -\frac{0.6}{x\sqrt{x}} + 6.4x^{-3} + 1 \Big|$$
$$\left(\frac{1}{\sqrt{x}}\right)' = -\frac{1}{2x\sqrt{x}}$$

### Exercise

Find the derivative of  $f(x) = x^2 - 3x - 4\sqrt{x}$

#### Solution

$$f'(x) = 2x - 3 - \frac{2}{\sqrt{x}} \Big|$$
$$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

### Exercise

Find the derivative of  $f(x) = 3\sqrt[3]{x^4} - 2x^3 + 4x$

#### Solution

$$f(x) = 3x^{4/3} - 2x^3 + 4x$$
$$f'(x) = 4x^{1/3} - 6x^2 + 4$$
$$= 4\sqrt[3]{x} - 6x^2 + 4 \Big|$$

### ***Exercise***

Find the derivative of  $f(x) = 0.05x^4 + 0.1x^3 - 1.5x^2 - 1.6x + 3$

#### **Solution**

$$f'(x) = \underline{0.2x^3 + 0.3x^2 - 3x - 1.6}$$

### ***Exercise***

Find the derivative of  $y = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$

#### **Solution**

$$\begin{aligned} y' &= 3(4)x^3 - 6(3)x^2 + \frac{2}{8}x + 0 \\ &= \underline{12x^3 - 18x^2 + \frac{1}{4}x} \end{aligned}$$

### ***Exercise***

Find the derivative of  $f(t) = -3t^2 + 2t - 4$

#### **Solution**

$$f'(t) = \underline{-6t + 2}$$

### ***Exercise***

Find the derivative of  $g(x) = 4\sqrt[3]{x} + 2$

#### **Solution**

$$\begin{aligned} g(x) &= 4x^{1/3} + 2 \\ g'(x) &= \underline{\frac{4}{3}x^{-2/3}} \end{aligned}$$

### ***Exercise***

Find the derivative of  $f(x) = x(x^2 + 1)$

#### **Solution**

$$\begin{aligned} f(x) &= x^3 + x \\ f(x) &= \underline{3x^2 + 1} \end{aligned}$$

### Exercise

Find the derivative of  $f(x) = \frac{2x^2 - 3x + 1}{x}$

#### Solution

$$\begin{aligned} f(x) &= \frac{2x^2}{x} - \frac{3x}{x} + \frac{1}{x} \\ &= 2x - 3 + \frac{1}{x} \end{aligned}$$

$$f'(x) = \underline{2 - \frac{1}{x^2}} \quad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

### Exercise

Find the derivative of  $f(x) = \frac{4x^3 - 3x^2 + 2x + 5}{x^2}$

#### Solution

$$\begin{aligned} f(x) &= 4 \frac{x^3}{x^2} - 3 \frac{x^2}{x^2} + 2 \frac{x}{x^2} + \frac{5}{x^2} \\ &= 4x - 3 + 2 \frac{1}{x} + 5x^{-2} \end{aligned}$$

$$\begin{aligned} f'(x) &= 4 - 2 \frac{1}{x^2} - 10x^{-3} \\ &= \underline{4 - \frac{2}{x^2} - \frac{10}{x^3}} \quad \left(\frac{1}{x}\right)' = -\frac{1}{x^2} \end{aligned}$$

### Exercise

Find the derivative of  $f(x) = \frac{-6x^3 + 3x^2 - 2x + 1}{x}$

#### Solution

$$\begin{aligned} f(x) &= -6 \frac{x^3}{x} + 3 \frac{x^2}{x} - 2 \frac{x}{x} + \frac{1}{x} \\ &= -6x^2 + 3x - 2 + \frac{1}{x} \end{aligned}$$

$$f'(x) = \underline{-12x + 3 - \frac{1}{x^2}} \quad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$



**Exercise**

Find the slope of the graph of  $f(x) = x^2 - 5x + 1$  at the point  $(2, -5)$

**Solution**

$$f'(x) = 2x - 5$$

$$\text{Slope} = f'(2) = 2(2) - 5 = \underline{-1}$$

**Exercise**

Find an equation of the tangent line to the graph of  $f(x) = -x^2 + 3x - 2$  at the point  $(2, 0)$

**Solution**

$$f'(x) = -2x + 3$$

$$\begin{aligned}\text{Slope} &= f'(2) \\ &= -2(2) + 3 = -1 \\ &= -1\end{aligned}$$

$$y - 0 = -1(x - 2)$$

$$\Rightarrow y = -x + 2$$

**Exercise**

Find the slope of the graph of  $f(x) = x^3$  when  $x = -1$ ,  $0$ , and  $1$ .

**Solution**

$$f'(x) = 3x^2$$

$$x = -1 \Rightarrow m = f'(x) = 3(-1)^2 = 3$$

$$x = 0 \Rightarrow m = f'(x) = 3(0)^2 = 0$$

$$x = 1 \Rightarrow m = f'(x) = 3$$

**Exercise**

The height  $h$  (in feet) of a free-falling object at time (in seconds) is given by  $h = -16t^2 + 180$ . Find the average velocity of the object over each interval.

a.  $[0, 1]$

b.  $[1, 2]$

**Solution**

a)  $h(0) = 180,$

$$h(1) = 164$$

$$\rightarrow \frac{\Delta h}{\Delta t} = \frac{164 - 180}{1 - 0} = -16 \text{ ft / sec}$$

$$b) h(2) = 116$$

$$\rightarrow \frac{\Delta h}{\Delta t} = \frac{164 - 116}{2 - 1} = -48 \text{ ft / sec}$$

### ***Exercise***

Give the position function of a diver who jumps from a board 12 feet high with initial velocity 16 feet per second. Then find the diver's velocity function.

### **Solution**

$$h_0 = 12 \text{ ft and } v_0 = 16 \text{ ft / sec}$$

$$\Rightarrow h = -16t^2 + 16t + 12$$

$$v(t) = h' = -32t + 16$$

### ***Exercise***

An analyst has found that a company's costs and revenues in dollars for one product are given by

$$C(x) = 2x \qquad R(x) = 6x - \frac{x^2}{1000}$$

Respectively, where  $x$  is the number of items produced.

- Find the marginal cost function
- Find the marginal revenue function
- Using the fact that profit is the difference between revenue and costs, find the marginal profit function.
- What value of  $x$  makes the marginal profit is 0.
- Find the profit when the marginal profit is 0.

### **Solution**

$$a) C'(x) = 2$$

$$b) R'(x) = 6 - \frac{2x}{1000} = 6 - \frac{x}{500}$$

$$\begin{aligned} c) P &= R - C \\ &= 6x - \frac{x^2}{1000} - 2x \\ &= 4x - \frac{x^2}{1000} \end{aligned}$$

$$P'(x) = 4 - \frac{x}{500}$$

$$d) \quad P'(x) = 4 - \frac{x}{500} = 0$$

$$\frac{x}{500} = 4$$

$$x = 2,000$$

$$e) \quad P(x) = 4x - \frac{x^2}{1000}$$

$$= 4(2000) - \frac{2000^2}{1000}$$

$$= \$4,000$$

### ***Exercise***

A business sells 2000 units per month at a price \$10 each. If monthly sales increases 200 units for each \$0.10 reduction in price.

### **Solution**

$$x = 2000 + 200 \left( \frac{10-p}{0.1} \right)$$

$$= 2000 + 2000(10 - p)$$

$$= 2000 + 20000 - 2000p$$

$$x = 22000 - 2000p$$

$$\Rightarrow x - 22000 = -2000p$$

$$\Rightarrow -x + 22000 = 2000p$$

$$p = \frac{22000}{2000} - \frac{x}{2000}$$

$$= 11 - \frac{x}{2000}$$

### Exercise

From 1998 through 2005, the revenue per share  $R$  (in dollars) for McDonald's Corporation can be modeled by

$$R = 0.0598t^2 - 0.379t + 8.44 \quad 8 \leq t \leq 15$$

Where  $t$  represents the year, with  $t = 8$  corresponding to 1998. At what rate was McDonald's revenue per share changing in 2003?

### Solution

$$R' = 0.1196t - 0.379$$

$$2003 \Rightarrow t = 13$$

$$\rightarrow R' = 0.1196(13) - 0.379 = \underline{1.1758}$$

### Exercise

The cost  $C$  (in dollars) of producing  $x$  units of a product is given by  $C = 3.6\sqrt{x} + 500$

- a) Find the additional cost when the production increases from 9 to 10 units.
- b) Find the marginal cost when  $x = 9$
- c) Compare the results of parts (a) and (b)

### Solution

$$a) \quad C(9) = 3.6\sqrt{9} + 500 = \underline{\$510.8}$$

$$C(10) = 3.6\sqrt{10} + 500 = \underline{\$511.38}$$

$$\text{Additional cost: } = 511.38 - 510.8 = \$0.58$$

$$b) \quad C' = \frac{1}{2} 3.6 \left( x^{-1/2} \right)$$

$$C'(9) = 1.8 \left( (9)^{-1/2} \right) = \underline{\$0.60}$$

- c) Similar

### Exercise

The revenue  $R$  (in dollars) of renting  $x$  apartments can be modeled by  $R = 2x(900 + 32x - x^2)$

- a) Find the additional revenue when the number of rentals is increased from 14 to 15
- b) Find the marginal revenue when  $x = 14$
- c) Compare the results of parts (a) and (b)

### Solution

$$a) \quad R(14) = 2(14)(900 + 32(14) - (14)^2) = \$32,256.00$$

$$R(15) = 2(15)(900 + 32(15) - (15)^2) = \$34,650.00$$

$$\text{Additional revenue: } 34,650.00 - 32,256 = \$2394.00$$

$$b) \quad R = 1800x + 64x^2 - 2x^3$$

$$R' = 1800 + 128x - 6x^2$$

$$R'(14) = 1800 + 128(14) - 6(14)^2 = \$2416.00$$

$$c) \quad 2416 - 2394 = \$22$$

### Exercise

The profit  $P$  (in dollars) of selling  $x$  units of calculus textbooks is given by

$$P = -0.05x^2 + 20x - 1000$$

- a) Find the additional profit when the sales increase from 150 to 151 units.
- b) Find the marginal profit when  $x = 150$
- c) Compare the results of parts (a) and (b)

### Solution

$$a) \quad P(150) = -0.05(150)^2 + 20(150) - 1000 = \$875.00$$

$$P(151) = -0.05(151)^2 + 20(151) - 1000 = \$879.95$$

$$\text{Additional profit: } 879.95 - 875.00 = \$4.95$$

$$b) \quad P' = -0.1x + 20$$

$$P'(150) = -0.1(150) + 20 = \$5.00$$

$$c) \quad \text{Nearly the same } \$0.05$$

### Exercise

The profit derived from selling  $x$  units, is given by  $P = 0.0002x^3 + 10x$ , find the marginal profit for a production level of 100 units. Compare this with the actual gain in profit by increasing production from 100 to 101 units.

### Solution

$$\frac{dP}{dx} = P' = 0.0006x^2 + 10$$

$$\Rightarrow P'(100) = 0.0006(100)^2 + 10 = \underline{\$16}$$

$$P(100) = 0.0002(100)^3 + 10(100) = \underline{\$1200.00}$$

$$P(101) = 0.0002(101)^3 + 10(101) = \underline{\$1216.06}$$

$$\text{Actual Gain} = 1216.06 - 1200 = \underline{\$16.06}$$

### Exercise

The Cost of producing  $x$  hamburgers is  $C = 5000 + 0.56x$ ,  $0 \leq x \leq 50,000$  and the revenue function is given by

$$R = \frac{1}{20000} (60000x - x^2)$$

Compare the marginal profit when 10,000 units are produced with the actual increase in profit from 10,000 units to 10,001 units

### Solution

$$\begin{aligned} \Rightarrow P &= R - C = \frac{1}{20000} (60000x - x^2) - (5000 + 0.56x) \\ &= 2.44x - \frac{x^2}{20000} - 5000 \end{aligned}$$

$$\frac{dP}{dx} = P' = 2.44 - \frac{x}{10000}$$

$$\text{For } x = 10000 \Rightarrow P'(10000) = 2.44 - \frac{10000}{10000} = \underline{\$1.44 / unit}$$

$$P(10000) = 2.44(10000) - \frac{(10000)^2}{20000} - 5000 = \underline{\$14400}$$

$$P(10001) = 2.44(10001) - \frac{(10001)^2}{20000} - 5000 = \underline{\$14401.44}$$

$$P(10001) - P(10000) = 14,401.44 - 14,400 = 1.44$$

$$\Rightarrow \underline{\$1.44 / unit}$$

### Exercise

An object moves along the y-axis (marked in feet) so that its position at time  $x$  (in seconds) is

$$f(x) = x^3 - 6x^2 + 9x$$

- a) Find the instantaneous velocity function  $v$ .
- b) Find the velocity at  $x = 2$  and  $x = 5$  seconds
- c) Find the time(s) when the velocity is 0.

### Solution

$$a) \quad v = f'(x) = \underline{3x^2 - 12x + 9}$$

$$b) \quad v(2) = 3(\underline{2})^2 - 12(\underline{2}) + 9 = \underline{-3 \text{ ft / sec}}$$

$$v(5) = 3(\underline{5})^2 - 12(\underline{5}) + 9 = \underline{24 \text{ ft / sec}}$$

$$c) \quad v = 3x^2 - 12x + 9 = \underline{0} \quad \text{Solve for } x$$
$$\underline{x = 1, 3}$$

So,  $v = 0$  at  $x = \underline{1} \text{ sec}$  and  $x = \underline{3} \text{ sec}$

### Exercise

A company's total sales (in millions of dollars)  $t$  months from now are given by

$$S(t) = 0.03t^3 + 0.5t^2 + 2t + 3$$

- a) Find  $S'(t)$ .
- b) Find  $S(5)$  and  $S'(5)$  (to two decimal places). Write a brief verbal interpretation of these results.
- c) Find  $S(10)$  and  $S'(10)$  (to two decimal places). Write a brief verbal interpretation of these results.

### Solution

$$a) \quad S'(t) = 0.09t^2 + t + 2$$

$$b) \quad S(\underline{5}) = 0.03(\underline{5})^3 + 0.5(\underline{5})^2 + 2(\underline{5}) + 3 = \underline{18}$$

$$S'(\underline{5}) = 0.09(\underline{5})^2 + (\underline{5}) + 2 = \underline{9.25}$$

After 5 months, sales are \$18 million and are increasing at the rate of \$9.25 million per month.

$$c) \quad S(\underline{10}) = 0.03(\underline{10})^3 + 0.5(\underline{10})^2 + 2(\underline{10}) + 3 = \underline{103}$$

$$S'(\underline{10}) = 0.09(\underline{10})^2 + \underline{10} + 2 = \underline{21}$$

After 10 months, sales are \$103 million and are increasing at the rate of \$21 million per month.

### Exercise

A company's total sales (in millions of dollars)  $t$  months from now are given by

$$S(t) = 0.015t^4 + 0.4t^3 + 3.4t^2 + 10t - 3$$

- a) Find  $S'(t)$ .
- b) Find  $S(4)$  and  $S'(4)$  (to two decimal places). Write a brief verbal interpretation of these results.
- c) Find  $S(8)$  and  $S'(8)$  (to two decimal places). Write a brief verbal interpretation of these results.

### Solution

a)  $S'(t) = 0.06t^3 + 1.2t^2 + 6.8t + 10$

b)  $S(4) = 0.015(4)^4 + 0.4(4)^3 + 3.4(4)^2 + 10(4) - 3 = 120.84$

$$S'(4) = 0.06(4)^3 + 1.2(4)^2 + 6.8(4) + 10 = 60.24$$

After 4 months, sales are \$120.84 million and are increasing at the rate of \$60.24 million per month.

c)  $S(8) = 0.015(8)^4 + 0.4(8)^3 + 3.4(8)^2 + 10(8) - 3 = 560.84$

$$S'(8) = 0.06(8)^3 + 1.2(8)^2 + 6.8(8) + 10 = 171.92$$

After 8 months, sales are \$560.84 million and are increasing at the rate of \$171.92 million per month.

### Exercise

A marine manufacturer will sell  $N(x)$  power boats after spending  $\$x$  thousand on advertising, as given by

$$N(x) = 1,000 - \frac{3,780}{x} \quad 5 \leq x \leq 30$$

- a) Find  $N'(x)$ .
- b) Find  $N(20)$  and  $N'(20)$  (to two decimal places). Write a brief verbal interpretation of these results.

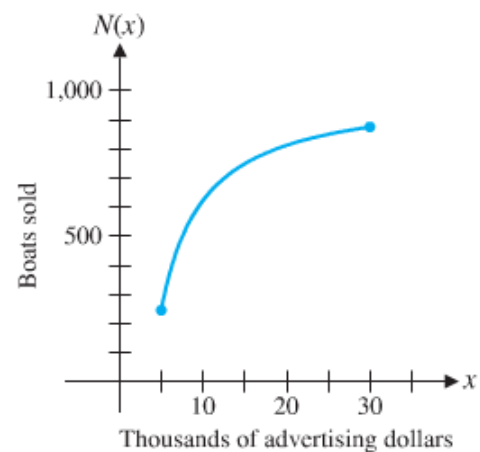
### Solution

a)  $N'(x) = \frac{3,780}{x^2}$

b)  $N(20) = 1,000 - \frac{3,780}{20} = 811$

$$N'(20) = \frac{3,780}{20^2} = 9.45$$

After \$811,000 on advertising, a marine manufacturer will sell 811 power boats.





### Exercise

A company manufactures and sells  $x$  transistor radios per week. If the weekly cost and revenue equations are

$$C(x) = 5,000 + 2x \quad R(x) = 10x - \frac{x^2}{1,000} \quad 0 \leq x \leq 8,000$$

Then find the approximate changes in revenue and profit if production is increased from 2,000 to 2,010 per week.

### Solution

$$|dx = 2,010 - 2,000 = 10|$$

$$\begin{aligned} dR &= \left(10 - \frac{x}{500}\right) dx \\ &= \left(10 - \frac{2,000}{500}\right) (10) \\ &= \$60 \quad \text{per week} \end{aligned}$$

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 10x - \frac{x^2}{1,000} - (5,000 + 2x) \\ &= 10x - \frac{x^2}{1,000} - 5,000 - 2x \\ &= 8x - \frac{x^2}{1,000} - 5,000 \end{aligned}$$

$$\begin{aligned} dP &= \left(8 - \frac{x}{500}\right) dx \\ &= \left(8 - \frac{2,000}{500}\right) (10) \\ &= \$40 \quad \text{per week} \end{aligned}$$

### Exercise

A company manufactures fuel tanks for cars. The total weekly cost (in dollars) of producing  $x$  tanks given by

$$C(x) = 10,000 + 90x - 0.05x^2$$

- Find the marginal cost function.
- Find the marginal cost at a production level of 500 tanks per week.
- Interpret the result of part b.
- Find the exact cost of producing the 501<sup>st</sup> item.

### Solution

$$a) \quad C'(x) = 90 - 0.1x$$

$$b) \quad C'(500) = 90 - 0.1(500) = \underline{\$40}$$

c) At a production level of 500 tanks per week, the total production costs are increasing at the rate of \$40 per tank.

$$d) \quad C(501) = 10,000 + 90(501) - 0.05(501)^2 \\ = \underline{\$42,539.95}$$

$$C(500) = 10,000 + 90(500) - 0.05(500)^2 \\ = \underline{\$42,500.00}$$

$$C(501) - C(500) = 42,539.95 - 42,500.00 \\ = \underline{\$39.95} \quad \text{Exact cost of producing the 501}^{\text{st}} \text{ tank.}$$

### Exercise

A company's market research department recommends the manufacture and marketing of a new headphone set for MP3 players. After suitable test marketing, the research department presents the following *price-demand* equation:

$$x = 10,000 - 1,000p \rightarrow p = 10 - 0.001x$$

Where  $x$  is the number of headphones that retailers are likely to buy at \$ $p$  per set.

The financial department provides the cost function

$$C(x) = 7,000 + 2x$$

Where \$7,000 is the estimate of fixed costs (tooling and overhead) and \$2 is the estimate of variable costs per headphone set (materials, labor, marketing, transportation, storage, etc.).

- Find the domain of the function defined by the price demand function.
- Find and interpret the marginal cost function  $C'(x)$ .
- Find the revenue function as a function of  $x$  and find its domain.
- Find the marginal revenue at  $x = 2,000$ ,  $5,000$ , and  $7,000$ . Interpret these results.
- Graph the cost function and the revenue function in the same coordinate system, Find the intersection points of these two graphs and interpret the results.
- Find the profit function and its domain and sketch the graph of the function.
- Find the marginal profit at  $x = 1,000$ ,  $4,000$ , and  $6,000$ . Interpret these results.

### Solution

- a) Since price  $p$  and demand  $x$  must be non-negative, we have  $x \geq 0$

$$p = 10 - 0.001x \geq 0$$

$$10 \geq 0.001x$$

$$10,000 \geq x$$

The permissible values of  $x$  are  $0 \leq x \leq 10,000$

- b) The marginal cost is  $C'(x) = 2$ . Since this is a constant, it costs an additional \$2 to produce one more headphone set at any production level.

- c) The revenue is the amount of money  $R$  received by the company for manufacturing and selling  $x$  headphone sets at  $\$p$  per set and is given by

$$\begin{aligned} R(x) &= (\text{number of headphone sets sold}) (\text{price per headphone set}) \\ &= xp \\ &= x(10 - 0.001x) \\ &= 10x - 0.001x^2 \quad 0 \leq x \leq 10,000 \end{aligned}$$

- d) The marginal revenue is:

$$\begin{aligned} R'(x) &= 10 - 0.002x \\ R'(2,000) &= 10 - 0.002(2,000) = 6 \\ R'(5,000) &= 10 - 0.002(5,000) = 0 \\ R'(7,000) &= 10 - 0.002(7,000) = -4 \end{aligned}$$

At production levels of 2,000, 5,000, and 7,000, the respective approximate changes in revenue per unit change in production are \$6, \$0, and -\$4.

At the \$2,000 output level, revenue increases as production increases.

At the \$5,000 output level, revenue does not change with a *small* change in production.

At the \$7,000 output level, revenue decreases as production increases.

- e) The intersection points are called the **break-even points**, because revenue equals cost at these production levels.

$$\begin{aligned} C(x) &= R(x) \\ 7,000 + 2x &= 10x - 0.001x^2 \\ 0.001x^2 - 8x + 7,000 &= 0 \\ \text{Solve for } x: & x = 1,000, \quad 7,000 \\ R(1,000) &= 10(1,000) - 0.001(1,000)^2 = 9,000 \\ C(1,000) &= 7,000 + 2(1,000) = 9,000 \\ R(7,000) &= 10(7,000) - 0.001(7,000)^2 = 21,000 \quad C(7,000) = 7,000 + 2(7,000) = 21,000 \end{aligned}$$

The *break-even* points are:

$$(1,000, 9,000) \text{ and } (7,000, 21,000)$$

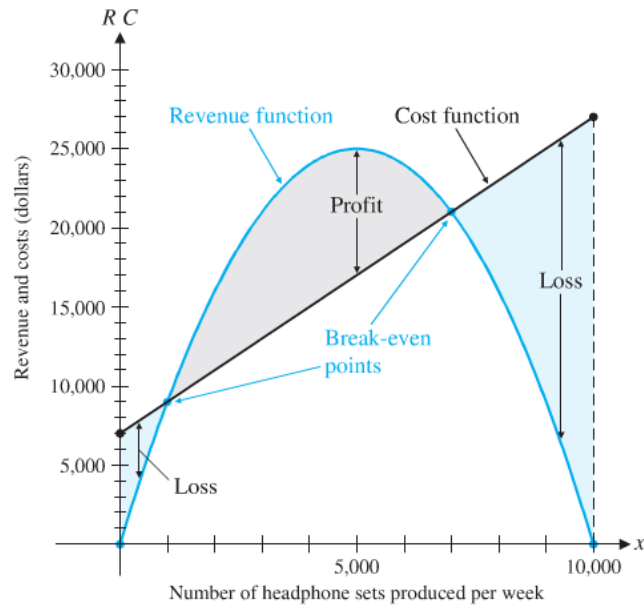
- f) The profit function is:

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 10x - 0.001x^2 - (7,000 + 2x) \\ &= -0.001x^2 + 8x - 7,000 \end{aligned}$$

The domain of the cost function is  $x \geq 0$ ,

The domain of the revenue function is  $0 \leq x \leq 10,000$

The domain of the profit function is  $0 \leq x \leq 10,000$



g) The marginal profit is

$$P'(x) = -0.002x + 8$$

$$P'(1,000) = -0.002(1,000) + 8 = 6$$

$$P'(4,000) = -0.002(4,000) + 8 = 0$$

$$P'(6,000) = -0.002(6,000) + 8 = -4$$

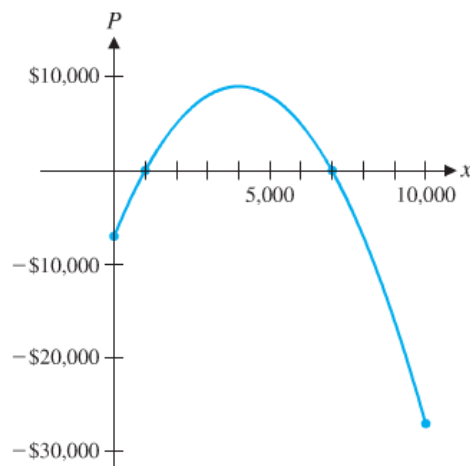
This means that at productions of 1,000, 4,000, and 6,000, the respective approximate changes in revenue per unit change in production are \$6, \$0, and -\$4.

At the \$1,000 output level, profit will increase if production is increased.

At the \$4,000 output level, profit does not change for *small* changes in production.

At the \$6,000 output level, profit will decrease as production is increased.

Therefore, the best production level to produce a maximum profit is 4,000.



### Exercise

A small machine shop manufactures drill bits used in the petroleum industry. The manager estimates that the total daily cost (in dollars) of producing  $x$  bits is

$$C(x) = 1,000 + 25x - 0.1x^2$$

- a) Find  $\bar{C}(x)$  and  $\bar{C}'(x)$
- b) Find  $\bar{C}(10)$  and  $\bar{C}'(10)$ . Interpret these quantities.
- c) Use the results in part (b) to estimate the average cost per bit at a production level of 11 bits per day.

### Solution

$$\begin{aligned} a) \quad \bar{C}(x) &= \frac{C(x)}{x} = \frac{1,000 + 25x - 0.1x^2}{x} \\ &= \frac{1,000}{x} + 25 - 0.1x \end{aligned}$$

$$\bar{C}'(x) = -\frac{1,000}{x^2} - 0.1 \qquad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$b) \quad \bar{C}(10) = \frac{1,000}{10} + 25 - 0.1(10) = \underline{\$124}$$

$$\bar{C}'(10) = -\frac{1,000}{10^2} - 0.1 = \underline{-\$10.10}$$

At a production level of 10 bits per day, the average cost of producing a bit is \$124. This cost is decreasing at the rate of \$10.10 per bit.

- c) If the production is increased by 1 bit, then the average cost per bit will decrease by approximately \$10.10. So, the average cost per bit at a production level of 11 bits per day is approximately

$$\$124 - \$10.10 = \underline{\$113.90}$$

### Exercise

The total profit (in dollars) from the sale of  $x$  calendars is

$$P(x) = 22x - 0.2x^2 - 400 \qquad 0 \leq x \leq 100$$

- a) Find the exact profit from the sale of the 41<sup>st</sup> calendar.
- b) Use the marginal profit to approximate the profit from the sale of the 41<sup>st</sup> calendar.

### Solution

$$\begin{aligned} a) \quad P(41) - P(40) &= 22(41) - 0.2(41)^2 - 400 - \left(22(40) - 0.2(40)^2 - 400\right) \\ &= \underline{\$5.80} \end{aligned}$$

$$b) \quad P'(x) = 22 - 0.4x$$

$$P'(40) = 22 - 0.4(40) = \underline{\$6}$$

### Exercise

The total profit (in dollars) from the sale of  $x$  cameras is

$$P(x) = 12x - 0.02x^2 - 1,000 \quad 0 \leq x \leq 600$$

Evaluate the marginal profit at the given values of  $x$ , and interpret the results.

a)  $x = 200$ .

b)  $x = 350$ .

### Solution

$$P'(x) = 12 - 0.04x$$

a)  $P'(200) = 12 - 0.04(200) = \$4$

At a production level of 200 cameras, the profit is increasing at the rate of \$4.00 per camera.

b)  $P'(350) = 12 - 0.04(350) = -\$2$

At a production level of 350 cameras, the profit is decreasing at the rate of \$2.00 per camera.

### Exercise

The total profit (in dollars) from the sale of  $x$  gas grills is

$$P(x) = 20x - 0.02x^2 - 320 \quad 0 \leq x \leq 1,000$$

a) Find the average profit per grill if 40 grills are produced.

b) Find the marginal average profit at a production level of 40 grills and interpret the results.

c) Use the results from parts (a) and (b) to estimate the average profit per grill if 41 grills are produced.

### Solution

$$\begin{aligned} \text{Average profit: } \bar{P}(x) &= \frac{P(x)}{x} = \frac{20x - 0.02x^2 - 320}{x} \\ &= 20 - 0.02x - \frac{320}{x} \end{aligned}$$

a)  $P(40) = 20 - 0.02(40) - \frac{320}{40} = \$11.20$

b)  $P'(x) = -0.02 + \frac{320}{x^2}$   $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$

$P'(40) = -0.02 + \frac{320}{40^2} = \$0.18$

At a production level of 40 grills, the average profit is increasing at the rate of \$0.18 per grill.

c) The average profit per grill if 41 grills are produced is  $\$11.20 + \$0.18 = \$11.38$

### Exercise

The price  $p$  (in dollars) and the demand  $x$  for a particular steam iron are related by the equation

$$x = 1,000 - 20p$$

- a) Express the price  $p$  in terms of the demand  $x$ , and find the domain of this function.
- b) Find the revenue  $R(x)$  from the sale of  $x$  steam irons. What is the domain of  $R$ ?
- c) Find the marginal revenue at a production level of 400 steam irons and interpret the results.
- d) Find the marginal revenue at a production level of 650 steam irons and interpret the results.

### Solution

a)  $20p = 1,000 - x$

$$p = 50 - 0.05x \quad 0 \leq x \leq 1,000$$

b)  $R(x) = xp = x(50 - 0.05x)$

$$= 50x - 0.05x^2 \quad 0 \leq x \leq 1,000$$

c)  $R'(x) = 50 - 0.1x$

$$R'(400) = 50 - 0.1(400) = 10$$

At a production level of 400 steam irons, the revenue is increasing at the rate of \$10 per steam iron.

d)  $R'(650) = 50 - 0.1(650) = -15$

At a production level of 650 steam irons, the revenue is decreasing at the rate of \$15 per steam iron.

## Exercise

The price-demand equation and the cost function for the production of TVs are given respectively, by

$$x = 9,000 - 30p \quad \text{and} \quad C(x) = 150,000 + 30x$$

Where  $x$  is the number of TVs that can be sold at a price of  $\$p$  per TV and  $C(x)$  is the total cost (in dollars) of producing  $x$  TVs.

- Express the price  $p$  as a function of the demand  $x$ , and find the domain of this function.
- Find the marginal cost.
- Find the revenue function and state its domain.
- Find the marginal revenue.
- Find  $R'(3,000)$  and  $R'(6,000)$  and interpret these quantities.
- Graph the cost function and the revenue function on the same coordinate system for  $0 \leq x \leq 9,000$ . Find the break-even points and indicate regions of loss and profit.
- Find the profit function in terms of  $x$ .
- Find the marginal profit.
- Find  $P'(1,500)$  and  $P'(4,500)$  and interpret these quantities

## Solution

$$a) \quad 30p = 9,000 - x \rightarrow p = 300 - \frac{1}{30}x \quad 0 \leq x \leq 9,000$$

$$b) \quad C'(x) = 30$$

$$c) \quad R(x) = xp = x\left(300 - \frac{1}{30}x\right) \\ = 300x - \frac{1}{30}x^2 \quad 0 \leq x \leq 9,000$$

$$d) \quad R'(x) = 300 - \frac{1}{15}x$$

$$e) \quad R'(3,000) = 300 - \frac{1}{15}(3,000) = 100$$

At a production level of 3,000 sets, the revenue is *increasing* at the rate of \$100 per set.

$$R'(6,000) = 300 - \frac{1}{15}(6,000) = -100$$

At a production level of 6,000 sets, the revenue is *decreasing* at the rate of \$100 per set.

- f) The break-even points are:

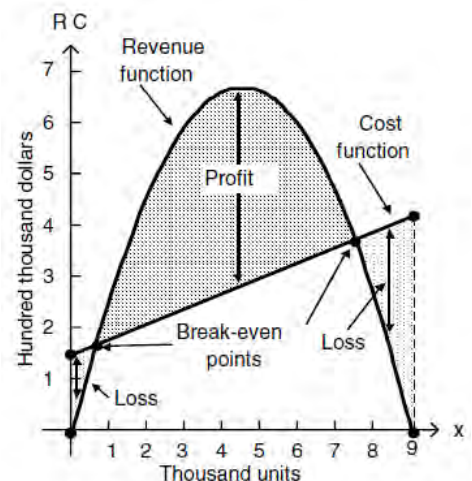
$$C(x) = R(x)$$

$$150,000 + 30x = 300x - \frac{1}{30}x^2$$

$$\frac{1}{30}x^2 - 270x + 150,000 = 0$$

$$x = 600 \quad \text{or} \quad x = 7,500$$

$$C(600) = 150,000 + 30(600) = 168,000$$





$$C(7,500) = 150,000 + 30(7,500) = 375,000$$

Thus, the break-even points are (600, 168,000) and (7,500, 375,000)

$$\begin{aligned} g) \quad P(x) &= R(x) - C(x) \\ &= 300x - \frac{1}{30}x^2 - (150,000 + 30x) \\ &= -\frac{1}{30}x^2 + 270x - 150,000 \end{aligned}$$

$$h) \quad P'(x) = -\frac{1}{15}x + 270$$

$$i) \quad P'(1,500) = -\frac{1}{15}(1,500) + 270 = 170$$

At a production level of 1,500 sets, the profit is *increasing* at the rate of \$170 per set.

$$P'(4,500) = -\frac{1}{15}(4,500) + 270 = -30$$

At a production level of 4,500 sets, the revenue is *decreasing* at the rate of \$30 per set.

### Exercise

The total cost and the total revenue (in dollars) for the production and sale of  $x$  hair dryers are given, respectively, by

$$C(x) = 5x + 2,340 \quad \text{and} \quad R(x) = 40x - 0.1x^2 \quad 0 \leq x \leq 400$$

- Find the value of  $x$  where the graph of  $R(x)$  has a horizontal tangent line.
- Find the profit function  $P(x)$ .
- Find the value of  $x$  where the graph of  $P(x)$  has a horizontal tangent line.
- Graph  $C(x)$ ,  $R(x)$ , and  $P(x)$  on the same coordinate system for  $0 \leq x \leq 400$ . Find the break-even points. Find the  $x$  intercept of the graph of  $P(x)$ .

### Solution

a)  $R'(x) = 40 - 0.2x$

The graph has a horizontal tangent line at the value(s) of  $x$  where  $R'(x) = 0$

$$40 - 0.2x = 0 \rightarrow \boxed{x = 200}$$

b)  $P(x) = R(x) - C(x)$

$$= 40x - 0.1x^2 - (5x + 2,340)$$

$$= -0.1x^2 + 35x - 2,340$$

c)  $P'(x) = -0.2x + 35$

$$P'(x) = -0.2x + 35 = 0$$

$$\boxed{x = 175}$$

d) The break-even points are:

$$R(x) = C(x)$$

$$40x - 0.1x^2 = 5x + 2,340$$

$$-0.1x^2 + 35x - 2,340 = 0$$

$$x = 90 \quad \text{or} \quad x = 260$$

$$C(90) = 5(90) + 2,340 = 2,790$$

$$C(260) = 5(260) + 2,340 = 3,640$$

Thus, the break-even points are  $(90, 2,790)$  and  $(260, 3,640)$

the  $x$  intercept of the graph of  $P(x)$  are  $P(x) = -0.1x^2 + 35x - 2,340 = 0$

Thus  $x = 90$  and  $x = 260$  are  $x$  intercepts of  $P(x)$

