Solution Section 3.2 – Angle and Orthogonality in Inner Product Spaces

Exercise

Which of the following form orthonormal sets?

a) $(1, 0), (0, 2)$ in $\mathbb{R}^2$

b) $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in $\mathbb{R}^2$

c) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in $\mathbb{R}^2$

d) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ in $\mathbb{R}^3$

e) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ in $\mathbb{R}^3$

f) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ in $\mathbb{R}^3$

Solution

a)
$$
\begin{aligned}
(1,\ 0)\cdot(0,\ 2) &= 1(0)+0(2)\\
&= 0
\end{aligned}
$$

They are ***orthonormal*** sets

b)
$$
\begin{aligned}
\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right) &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}+\left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\\
&= \frac{1}{2}-\frac{1}{2}\\
&= 0
\end{aligned}
$$

They are ***orthonormal*** sets

c)
$$
\begin{aligned}
\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right) &= \left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}+\left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\\
&= -\frac{1}{2}-\frac{1}{2}\\
&= -1 \neq 0
\end{aligned}
$$

They are ***not orthonormal*** sets

d) $\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)\cdot\left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{2}}\right) + 0 + \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{3}}\right)\frac{1}{\sqrt{2}}$$

$$= -\frac{1}{2}\frac{1}{\sqrt{3}} - \frac{1}{2}\frac{1}{\sqrt{3}}$$

$$= -\frac{1}{\sqrt{3}} \neq 0$$

They are ***not orthonormal*** sets

e) $\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right) \cdot \left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right) \cdot \left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right)$

$$= \frac{2}{3}\frac{2}{3}\frac{1}{3} + \left(-\frac{2}{3}\right)\frac{1}{3}\frac{2}{3} + \frac{1}{3}\left(-\frac{2}{3}\right)\frac{2}{3}$$

$$= \frac{4}{27} - \frac{4}{27} - \frac{4}{27}$$

$$= -\frac{4}{27} \neq 0$$

They are ***not orthonormal*** sets

f) $\left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right) \cdot \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right) = \frac{1}{\sqrt{6}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}}\left(-\frac{1}{\sqrt{2}}\right) + 0$

$$= 0$$

They are ***orthonormal*** sets

Exercise

Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.

a) $\vec{u} = (1,\ -3),\quad \vec{v} = (2,\ 4)$

b) $\vec{u} = (-1,\ 0),\quad \vec{v} = (3,\ 8)$

c) $\vec{u} = (-1,\ 5,\ 2),\quad \vec{v} = (2,\ 4,\ -9)$

d) $\vec{u} = (4,\ 1,\ 8),\quad \vec{v} = (1,\ 0,\ -3)$

e) $\vec{u} = (1,\ 0,\ 1,\ 0),\quad \vec{v} = (-3,\ -3,\ -3,\ -3)$

f) $\vec{u} = (2,\ 1,\ 7,\ -1),\quad \vec{v} = (4,\ 0,\ 0,\ 0)$

g) $\vec{u} = (1,\ 3,\ -5,\ 4),\quad \vec{v} = (2,\ -4,\ 4,\ 1)$

h) $\vec{u} = (1,\ 2,\ 3,\ 4),\quad \vec{v} = (-1,\ -2,\ -3,\ -4)$

Solution

a) $\vec{u} = (1,\ -3),\quad \vec{v} = (2,\ 4)$

$$\|\vec{u}\| = \sqrt{1^2 + (-3)^2}$$

$$= \sqrt{10}$$

$$\|\vec{v}\| = \sqrt{2^2 + 4^2}$$

$$= \sqrt{20}$$

$$\langle \vec{u},\ \vec{v}\rangle = 1(2) + (-3)(4)$$
$$= -10$$

$$\cos\theta = \frac{-10}{\sqrt{10}\ \sqrt{20}} \qquad \cos\theta = \frac{\langle \vec{u},\ \vec{v}\rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$
$$= -\frac{10}{\sqrt{200}}$$
$$= -\frac{1}{\sqrt{2}}$$

b) $\vec{u} = (-1,\ 0);\quad \vec{v} = (3,\ 8)$

$$\|\vec{u}\| = \sqrt{(-1)^2 + 0^2}$$
$$= 1$$

$$\|\vec{v}\| = \sqrt{3^2 + 8^2}$$
$$= \sqrt{73}$$

$$\langle \vec{u},\ \vec{v}\rangle = (-1)(3) + (0)(8)$$
$$= -3$$

$$\cos\theta = \frac{-3}{1\sqrt{73}} \qquad \cos\theta = \frac{\langle \vec{u},\ \vec{v}\rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$
$$= -\frac{3}{\sqrt{73}}$$

c) $\vec{u} = (-1,\ 5,\ 2);\quad \vec{v} = (2,\ 4,\ -9)$

$$\|\vec{u}\| = \sqrt{(-1)^2 + 5^2 + 2^2}$$
$$= \sqrt{30}$$

$$\|\vec{v}\| = \sqrt{2^2 + 4^2 + (-9)^2}$$
$$= \sqrt{101}$$

$$\langle \vec{u},\ \vec{v}\rangle = (-1)(2) + (5)(4) + (2)(-9)$$
$$= 0$$

$$\cos\theta = 0 \qquad \cos\theta = \frac{\langle \vec{u},\ \vec{v}\rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$

d) $\vec{u} = (4,\ 1,\ 8);\quad \vec{v} = (1,\ 0,\ -3)$

$$\|\vec{u}\| = \sqrt{4^2 + 1^2 + 8^2}$$

$$\underline{=9}$$

$$\|\vec{v}\| = \sqrt{1+0+9}$$

$$\underline{= \sqrt{10}}$$

$$\langle \vec{u}, \vec{v} \rangle = (4)(1)+(1)(0)+(8)(-3)$$

$$\underline{=-20}$$

$$\underline{\cos\theta = -\frac{20}{9\sqrt{10}}} \qquad \cos\theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$

e) $\vec{u} = (1,\ 0,\ 1,\ 0);\quad \vec{v} = (-3,\ -3,\ -3,\ -3)$

$$\|\vec{u}\| = \underline{\sqrt{2}}$$

$$\|\vec{v}\| = \sqrt{9+9+9+9}$$

$$\underline{=12}$$

$$\langle \vec{u}, \vec{v} \rangle = -3+0-3+0$$

$$= \underline{-6}$$

$$\cos\theta = \frac{-6}{12\sqrt{2}} \qquad \cos\theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$

$$\underline{= -\frac{1}{2\sqrt{2}}}$$

f) $\vec{u} = (2,\ 1,\ 7,\ -1);\quad \vec{v} = (4,\ 0,\ 0,\ 0)$

$$\|\vec{u}\| = \sqrt{2^2+1^2+7^2+(-1)^2}$$

$$\underline{= \sqrt{55}}$$

$$\|\vec{v}\| = \sqrt{4^2+0}$$

$$\underline{=4}$$

$$\langle \vec{u}, \vec{v} \rangle = (2)(4)+(1)(0)+(7)(0)+(-1)(0)$$

$$\underline{=8}$$

$$\cos\theta = \frac{8}{4\sqrt{55}} \qquad \cos\theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$

$$\underline{= \frac{2}{\sqrt{55}}}$$

g) $\vec{u} = (1,\ 3,\ -5,\ 4),\quad \vec{v} = (2,\ -4,\ 4,\ 1)$

$$\|\vec{u}\| = \sqrt{1+9+25+16}$$

$$= \sqrt{51}$$

$$\|\vec{v}\| = \sqrt{4+16+16+1}$$

$$= \sqrt{37}$$

$$\langle \vec{u}, \vec{v} \rangle = 2-12-20+4$$

$$= -26$$

$$\cos\theta = \frac{-26}{\sqrt{51}\sqrt{37}} \qquad \cos\theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \ \|\vec{v}\|}$$

h) $\vec{u} = (1, 2, 3, 4), \quad \vec{v} = (-1, -2, -3, -4)$

$$\|\vec{u}\| = \sqrt{1+4+9+16}$$

$$= \sqrt{30}$$

$$\|\vec{v}\| = \sqrt{1+4+9+16}$$

$$= \sqrt{30}$$

$$\langle \vec{u}, \vec{v} \rangle = -1-4-9-16$$

$$= -30$$

$$\cos\theta = \frac{-30}{\sqrt{30}\sqrt{30}} \qquad \cos\theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \ \|\vec{v}\|}$$

$$= -1$$

Exercise

Find the cosine of the angle between ***A*** and ***B***.

a) $A = \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$

b) $A = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$

c) $A = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

d) $A = \begin{pmatrix} 1 & -2 & 7 \\ 6 & -3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 3 \\ 6 & 5 & -4 \end{pmatrix}$

Solution

a) $A = \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$

$$\|A\| = \sqrt{\langle A, A \rangle}$$

$$= \sqrt{2^2 + 6^2 + 1^2 + (-3)^2}$$

$$= \sqrt{50}$$

$$= 5\sqrt{2}$$

$$\|B\| = \sqrt{\langle B,\ B\rangle}$$
$$= \sqrt{9+4+1+0}$$
$$= \sqrt{14}$$

$$\langle A,\ B\rangle = 2(3)+6(2)+1(1)+(-3)(0)$$
$$= 19$$

$$\cos\theta = \frac{19}{5\sqrt{2}\sqrt{14}} \qquad \cos\theta = \frac{\langle A,\ B\rangle}{\|A\|\ \|B\|}$$
$$= \frac{19}{10\sqrt{7}}$$

b) $A = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$

$$\|A\| = \sqrt{\langle A,\ A\rangle}$$
$$= \sqrt{2^2+4^2+(-1)^2+3^2}$$
$$= \sqrt{30}$$

$$\|B\| = \sqrt{\langle B,\ B\rangle}$$
$$= \sqrt{(-3)^2+1^2+4^2+2^2}$$
$$= \sqrt{30}$$

$$\langle A,\ B\rangle = 2(-3)+4(1)+(-1)(4)+3(2)$$
$$= 0$$

$$\cos\theta = \frac{0}{30} \qquad \cos\theta = \frac{\langle A,\ B\rangle}{\|A\|\ \|B\|}$$
$$= 0$$

c) $A = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

$$\|A\| = \sqrt{81+64+49+36+25+16} \qquad \|A\| = \sqrt{\langle A,\ A\rangle}$$
$$= \sqrt{271}$$

$$\|B\| = \sqrt{1+4+9+16+25+36} \qquad \|B\| = \sqrt{\langle B,\ B\rangle}$$
$$= \sqrt{91}$$

$\langle A, B\rangle = 9+16+21+24+25+24$

$= 119$

$\cos\theta = \dfrac{119}{\sqrt{271}\sqrt{91}}$ $\quad\quad \cos\theta = \dfrac{\langle A, B\rangle}{\|A\|\,\|B\|}$

d) $A = \begin{pmatrix} 1 & -2 & 7 \\ 6 & -3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 3 \\ 6 & 5 & -4 \end{pmatrix}$

$\|A\| = \sqrt{1^2 + (-2)^2 + 7^2 + 6^2 + (-3)^2 + 4^2}$ $\quad\quad \|A\| = \sqrt{\langle A, A\rangle}$

$= \sqrt{115}$

$\|B\| = \sqrt{1^2 + (-2)^2 + 3^2 + 6^2 + 5^2 + (-4)^2}$ $\quad\quad \|B\| = \sqrt{\langle B, B\rangle}$

$= \sqrt{91}$

$\langle A, B\rangle = 1+4+21+36-15-16$

$= 31$

$\cos\theta = \dfrac{31}{\sqrt{115}\sqrt{91}}$ $\quad\quad \cos\theta = \dfrac{\langle A, B\rangle}{\|A\|\,\|B\|}$

Exercise

Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

a) $\vec{u} = (-1, 3, 2), \quad \vec{v} = (4, 2, -1)$

b) $\vec{u} = (a, b), \quad \vec{v} = (-b, a)$

c) $\vec{u} = (-2, -2, -2), \quad \vec{v} = (1, 1, 1)$

d) $\vec{u} = (-4, 6, -10, 1), \quad \vec{v} = (2, 1, -2, 9)$

e) $\vec{u} = (-4, 6, -10, 1), \quad \vec{v} = (2, 1, -2, 9)$

Solution

a) $\langle \vec{u}, \vec{v}\rangle = (-1)(4) + 3(2) + 2(-1)$

$= 0$

Therefore, the given vectors are orthogonal.

b) $\langle \vec{u}, \vec{v}\rangle = a(-b) + b(a)$

$= 0$

Therefore, the given vectors are orthogonal.

c) $\langle \vec{u}, \vec{v}\rangle = (-2)(1) + (-2)(1) + (-2)(1)$

$= -6$

Therefore, the given vectors are ***not*** orthogonal.

d) $\langle\langle \vec{u},\ \vec{v}\rangle\rangle = (-4)(2)+(6)(1)+(-10)(-2)+(1)(9)$

$= 27 \quad \neq 0$

Therefore, the given vectors are ***not*** orthogonal.

e) $\|\vec{u}\| = \sqrt{(-4)^2+6^2+(-10)^2+1^2}$

$= \sqrt{153}$

$= 3\sqrt{17}$

$\|\vec{v}\| = \sqrt{2^2+1^2+(-2)^2+9^2}$

$= \sqrt{90}$

$= 3\sqrt{10}$

$\langle \vec{u},\ \vec{v}\rangle = (-4)(2)+6(1)-10(-2)+1(9)$

$= 27$

$\cos\theta = \dfrac{27}{3\sqrt{17}\left(3\sqrt{10}\right)}$ $\qquad\qquad \cos\theta = \dfrac{\langle \vec{u},\ \vec{v}\rangle}{\|\vec{u}\|\ \|\vec{v}\|}$

$= \dfrac{3}{\sqrt{170}}$

The vectors $\vec{u}$ *and* $\vec{v}$ are ***not*** orthogonal with respect to the Euclidean

Exercise

Do there exist scalars k and l such that the vectors $\vec{u} = (2,\ k,\ 6),\ \ \vec{v} = (l,\ 5,\ 3),\ \ and\ \ \vec{w} = (1,\ 2,\ 3)$ are mutually orthogonal with respect to the Euclidean inner product?

Solution

$\langle \vec{u},\ \vec{w}\rangle = (2)(1)+(k)(2)+(6)(3)$

$= 20+2k = 0$

$\Rightarrow k = -10$

$\langle \vec{v},\ \vec{w}\rangle = (l)(1)+(5)(2)+(3)(3)$

$= l+19 = 0$

$\Rightarrow l = -19$

$\langle \vec{u},\ \vec{v}\rangle = (2)(l)+(k)(5)+(6)(3)$

$= 2l+5k+18 = 0$

$$2(-19)+5(-10)+18=-70\neq 0$$

Thus, there are no scalars such that the vectors are mutually orthogonal.

Exercise

Let $\mathbb{R}^3$ have the Euclidean inner product. For which values of k are $\vec{u}$ and $\vec{v}$ orthogonal?

a) $\vec{u}=(2, 1, 3), \quad \vec{v}=(1, 7, k)$

b) $\vec{u}=(k, k, 1), \quad \vec{v}=(k, 5, 6)$

Solution

a) $\langle\vec{u}, \vec{v}\rangle=(2)(1)+(1)(7)+(3)(k)$

$$=9+3k=0$$

$\vec{u}$ and $\vec{v}$ are orthogonal for $\underline{k=-3}$

b) $\langle\vec{u}, \vec{v}\rangle=(k)(k)+(k)(5)+(1)(6)$

$$=k^2+5k+6=0$$

$\vec{u}$ and $\vec{v}$ are orthogonal for $\underline{k=-2, \ -3}$

Exercise

Let V be an inner product space. Show that if $\vec{u}$ and $\vec{v}$ are orthogonal unit vectors in V, then $\|\vec{u}-\vec{v}\|=\sqrt{2}$

Solution

$$\begin{aligned}
\|\vec{u}-\vec{v}\|^2 &= \langle\vec{u}-\vec{v}, \vec{u}-\vec{v}\rangle \\
&= \langle\vec{u}, \vec{u}-\vec{v}\rangle-\langle\vec{v}, \vec{u}-\vec{v}\rangle \\
&= \langle\vec{u}, \vec{u}\rangle-\langle\vec{u}, \vec{v}\rangle-\langle\vec{u}, \vec{v}\rangle+\langle\vec{v}, \vec{v}\rangle \\
&= \|\vec{u}\|^2-0-0+\|\vec{v}\|^2 && \text{since } \vec{u} \text{ and } \vec{v} \text{ are orthogonal unit vectors} \\
&= 1+1 \\
&= 2
\end{aligned}$$

Thus $\underline{\|\vec{u}-\vec{v}\|=\sqrt{2}}$

Exercise

Let $\boldsymbol{S}$ be a subspace of $\mathbb{R}^n$. Explain what $\left(\boldsymbol{S}^{\perp}\right)^{\perp} = \boldsymbol{S}$ means and why it is true.

Solution

$\left(\boldsymbol{S}^{\perp}\right)^{\perp}$ is the orthogonal complement of, $\boldsymbol{S}^{\perp}$, which is itself the orthogonal complement of $\boldsymbol{S}$, so $\left(\boldsymbol{S}^{\perp}\right)^{\perp} = \boldsymbol{S}$ means that $\boldsymbol{S}$ is the orthogonal of its orthogonal complement.

We need to show that $\boldsymbol{S}$ is contained in $\left(\boldsymbol{S}^{\perp}\right)^{\perp}$ and, conversely, that $\left(\boldsymbol{S}^{\perp}\right)^{\perp}$ is contained in $\boldsymbol{S}$ to be true.

i. Suppose $\vec{v} \in \boldsymbol{S}^{\perp}$ and $\vec{w} \in \boldsymbol{S}^{\perp}$. Then $\langle \vec{v}, \vec{w} \rangle = 0$ by definition of $\boldsymbol{S}^{\perp}$.

Thus, $\boldsymbol{S}$ is certainly contained is $\left(\boldsymbol{S}^{\perp}\right)^{\perp}$ (which consists of all vectors in $\mathbb{R}^n$ which are orthogonal to $\boldsymbol{S}^{\perp}$).

ii. Suppose $\vec{v} \in \left(\boldsymbol{S}^{\perp}\right)^{\perp}$ (means $\vec{v}$ is orthogonal to all vectors in $\boldsymbol{S}^{\perp}$); then we need to show that $\vec{v} \in \boldsymbol{S}$.

Let assume $\left\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\right\}$ be a basis for $\boldsymbol{S}$ and let $\left\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_q\right\}$ be a basis for $\boldsymbol{S}^{\perp}$. If $\vec{v} \notin \boldsymbol{S}$, then $\left\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p, \vec{v}\right\}$ is linearly independent set. Since each vector ifs that set is orthogonal to all of $\boldsymbol{S}^{\perp}$, the set $\left\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p, \vec{v}, \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_q\right\}$ is linearly independent.

Since there are $p+q+1$ vectors in this set, this means that $p+q+1 \le n \iff p+q \le n-1$.

On the other hand, If A is the matrix whose i^{th} row is $\vec{u}_i^T$, then the row space of A is $\boldsymbol{S}$ and the nullspace of A is $\boldsymbol{S}^{\perp}$.

Since $\boldsymbol{S}$ is p-dimensional, the rank of A is p, meaning that the dimension of $\operatorname{nul}(A) = \boldsymbol{S}^{\perp}$ is $q = n - p$. Therefore,

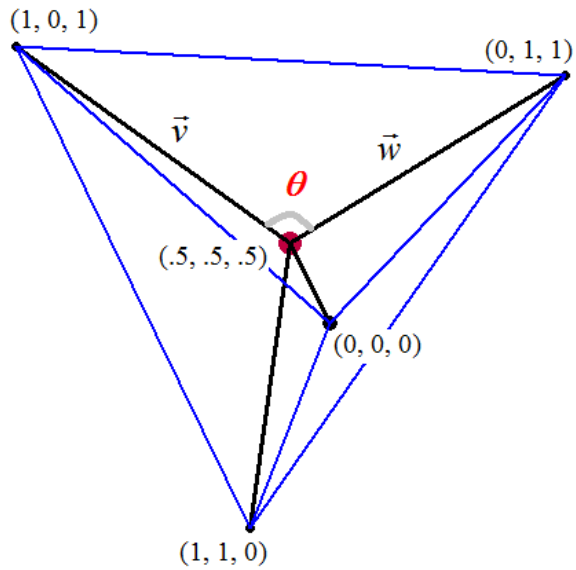
$$p + q = p + (n - p) = n$$

Which contradict the fact that $p + q \le n - 1$. From this, we see that, if $\vec{v} \in \left(\boldsymbol{S}^{\perp}\right)^{\perp}$, it must be the case that $\vec{v} \in \boldsymbol{S}$.

Exercise

The methane molecule CH_4 is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at $(0,\ 0,\ 0)$, $(1,\ 1,\ 0)$, $(1,\ 0,\ 1)$ and $(0,\ 1,\ 1)$ – (***note*** that all six edges have length $\sqrt{2}$, so the tetrahedron is regular). What is the cosine of the angle between the rays going from the center $\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$ to the vertices?

Solution



Let $\vec{v}$ be the vector of the segment (1, 0, 1) and $\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

Let be the vector of the segment (0, 1, 1) and $\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$

$$\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

We have:

$$\cos\theta = \frac{\langle \vec{v},\ \vec{w} \rangle}{\|\vec{v}\|\ \|\vec{w}\|}$$

$$= \frac{\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) \cdot \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}{\sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}\ \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}}$$

$$= \frac{-\frac{1}{4}}{\frac{3}{4}}$$

$$= -\frac{1}{3}$$

$$\theta \approx 109.47°$$

Exercise

Determine if the given vectors are orthogonal.

$$\vec{x}_1 = (1,\ 0,\ 1,\ 0), \quad \vec{x}_2 = (0,\ 1,\ 0,\ 1), \quad \vec{x}_3 = (1,\ 0,\ -1,\ 0), \quad \vec{x}_4 = (1,\ 1,\ -1,\ -1)$$

Solution

$$\vec{x}_1 \bullet \vec{x}_2 = (1,\ 0,\ 1,\ 0) \bullet (0,\ 1,\ 0,\ 1)$$

$$= 0$$

$$\vec{x}_1 \bullet \vec{x}_3 = (1,\ 0,\ 1,\ 0) \bullet (1,\ 0,\ -1,\ 0)$$

$$= 1 - 1$$

$$= 0$$

$$\vec{x}_1 \bullet \vec{x}_4 = (1,\ 0,\ 1,\ 0) \bullet (1,\ 1,\ -1,\ -1)$$

$$= 1 - 1$$

$$= 0$$

$$\vec{x}_2 \bullet \vec{x}_3 = (0,\ 1,\ 0,\ 1) \bullet (1,\ 0,\ -1,\ 0)$$

$$= 0$$

$$\vec{x}_2 \bullet \vec{x}_4 = (0,\ 1,\ 0,\ 1) \bullet (1,\ 1,\ -1,\ -1)$$

$$= 1 - 1$$

$= 0$

$$\vec{x}_3 \bullet \vec{x}_4 = (1,\ 0,\ -1,\ 0) \bullet (1,\ 1,\ -1,\ -1)$$

$= 1 - 1$

$= 0$

The given vectors are ***orthogonal.***

Exercise

Which of the following sets of vectors are orthogonal with respect to the Euclidean inner

a) $\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\ \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)\ \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$

b) $\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right)\ \left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right)\ \left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right)$

Solution

a) $\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) \bullet \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}}$

$= 0$

$\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) \bullet \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) = -\frac{1}{2} + 0 + \frac{1}{2}$

$= 0$

$\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right) \bullet \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}}$

$= -\frac{2}{\sqrt{6}} \neq 0$

Therefore, the given vectors are ***not*** orthogonal.

b) $\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right) \bullet \left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right) = \frac{4}{9} - \frac{2}{9} - \frac{2}{9}$

$= 0$

$\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right) \bullet \left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right) = \frac{2}{9} - \frac{4}{9} + \frac{2}{9}$

$= 0$

$\left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right) \bullet \left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right) = \frac{2}{9} + \frac{2}{9} - \frac{4}{9}$

$= 0$

Therefore, the given vectors are orthogonal.

Exercise

Consider vectors $\vec{u} = (2,\ 3,\ 5)$ $\vec{v} = (1,\ -4,\ 3)$ *in* $\mathbb{R}^3$

a) $\langle \vec{u},\ \vec{v} \rangle$ *b)* $\|\vec{u}\|$ *c)* $\|\vec{v}\|$ *d)* Cosine between $\vec{u}$ *and* $\vec{v}$

Solution

a) $\langle \vec{u},\ \vec{v} \rangle = (2,\ 3,\ 5) \bullet (1,\ -4,\ 3)$

$$= 2 - 12 + 15$$

$$= 5$$

b) $\|\vec{u}\| = \sqrt{4 + 9 + 25}$

$$= \sqrt{38}$$

c) $\vec{v} = \sqrt{1 + 16 + 9}$

$$= \sqrt{26}$$

d) $\cos\theta = \dfrac{5}{\sqrt{38}\sqrt{26}}$ $\qquad$ $\cos\theta = \dfrac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$

Exercise

Consider vectors $\vec{u} = (1,\ 1,\ 1)$ $\vec{v} = (1,\ 2,\ -3)$ *in* $\mathbb{R}^3$

a) $\langle \vec{u},\ \vec{v} \rangle$ *b)* $\|\vec{u}\|$ *c)* $\|\vec{v}\|$ *d)* Cosine θ between $\vec{u}$ *and* $\vec{v}$

Solution

a) $\langle \vec{u},\ \vec{v} \rangle = (1,\ 1,\ 1) \bullet (1,\ 2,\ -3)$

$$= 1 + 2 - 3$$

$$= 0$$

b) $\|\vec{u}\| = \sqrt{1 + 1 + 1}$

$$= \sqrt{3}$$

c) $\|\vec{v}\| = \sqrt{1 + 4 + 9}$

$$= \sqrt{14}$$

d) $\cos\theta = 0$ $\qquad$ $\cos\theta = \dfrac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$

$\vec{u}$ *and* $\vec{v}$ are orthogonal vectors.

Exercise

Consider vectors $\vec{u} = (1,\ 2,\ 5)$ $\vec{v} = (2,\ -3,\ 5)$ $\vec{w} = (4,\ 2,\ -3)$ *in* $\mathbb{R}^3$

a) $\langle \vec{u},\ \vec{v} \rangle$

b) $\langle \vec{u},\ \vec{w} \rangle$

c) $\langle \vec{v},\ \vec{w} \rangle$

d) $\|\vec{u}\|$

e) $\|\vec{v}\|$

f) $\|\vec{w}\|$

g) Cosine α between $\vec{u}$ *and* $\vec{v}$

h) Cosine β between $\vec{u}$ *and* $\vec{w}$

i) Cosine θ between $\vec{v}$ *and* $\vec{w}$

j) $(\vec{u}+\vec{v}) \bullet \vec{w}$

Solution

a) $\langle \vec{u},\ \vec{v} \rangle = (1,\ 2,\ 5) \bullet (2,\ -3,\ 5)$

$$= 2 - 6 + 25$$
$$= 21$$

b) $\langle \vec{u},\ \vec{w} \rangle = (1,\ 2,\ 5) \bullet (4,\ 2,\ -3)$

$$= 4 + 4 - 15$$
$$= -7$$

c) $\langle \vec{v},\ \vec{w} \rangle = (2,\ -3,\ 5) \bullet (4,\ 2,\ -3)$

$$= 8 - 6 - 15$$
$$= -13$$

d) $\|\vec{u}\| = \sqrt{1+4+25}$

$$= \sqrt{30}$$

e) $\|\vec{v}\| = \sqrt{4+9+25}$

$$= \sqrt{38}$$

f) $\|\vec{w}\| = \sqrt{16+4+9}$

$$= \sqrt{29}$$

g) $\cos\alpha = \dfrac{21}{\sqrt{30}\sqrt{38}}$ $\qquad \cos\theta = \dfrac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$

h) $\cos\beta = \dfrac{-7}{\sqrt{30}\sqrt{29}}$ $\qquad \cos\theta = \dfrac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$

i) $\cos\theta = \dfrac{-13}{\sqrt{38}\sqrt{29}}$ $\qquad \cos\theta = \dfrac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$

j) $(\vec{u}+\vec{v}) \bullet \vec{w} = \left[(1,\ 2,\ 5) + (2,\ -3,\ 5)\right] \bullet (4,\ 2,\ -3)$

$$= (3,\ -1,\ 10) \bullet (4,\ 2,\ -3)$$
$$= 12 - 2 - 30$$
$$= -20$$

Exercise

Consider polynomial $f(t)=3t-5;\quad g(t)=t^2\quad in\quad \mathbb{P}(t)$

a) $\langle f,\, g\rangle$ *b)* $\|f\|$ *c)* $\|g\|$ *d)* Cosine between f *and* g

Solution

a) $$\begin{aligned}\langle f,\, g\rangle &= \int_0^1 f(t)g(t)\,dt \\ &= \int_0^1 (3t-5)t^2\,dt \\ &= \int_0^1 \left(3t^3-5t^2\right)dt \\ &= \frac{3}{4}t^4-\frac{5}{3}t^3\Big|_0^1 \\ &= \frac{3}{4}-\frac{5}{3} \\ &= -\frac{11}{12}\end{aligned}$$

b) $$\begin{aligned}\langle f,\, f\rangle &= \int_0^1 f(t)f(t)\,dt \\ &= \int_0^1 (3t-5)^2\,dt \\ &= \frac{1}{3}\int_0^1 (3t-5)^2\,d(3t-5) \\ &= \frac{1}{9}(3t-5)^3\Big|_0^1 \\ &= \frac{1}{9}(8-125) \\ &= 13\end{aligned}$$

$$\begin{aligned}\|f\| &= \sqrt{|\langle f,\, f\rangle|} \\ &= \sqrt{13}\end{aligned}$$

c) $$\langle g,\, g\rangle = \int_0^1 g(t)g(t)\,dt$$

$$= \int_0^1 t^4 dt$$

$$= \frac{1}{5}t^5 \Big|_0^1$$

$$= \frac{1}{5}$$

$$\|g\| = \sqrt{|\langle g, g\rangle|}$$

$$= \frac{1}{\sqrt{5}}$$

d) $\cos\theta = \dfrac{-\frac{11}{12}}{\sqrt{13}\,\frac{\sqrt{5}}{5}}$ $\qquad\qquad$ $\cos\theta = \dfrac{f \bullet g}{\|f\|\,\|g\|}$

$$= \frac{-55}{12\sqrt{65}}$$

Exercise

Consider polynomial $f(t) = t + 2$; $g(t) = 3t - 2$; $h(t) = t^2 - 2t - 3$ *in* $\mathbb{P}(t)$

a) $\langle f, g\rangle$ $\qquad$ *d)* $\|f\|$ $\qquad$ *g)* Cosine α between f *and* g

b) $\langle f, h\rangle$ $\qquad$ *e)* $\|g\|$ $\qquad$ *h)* Cosine β between f *and* h

c) $\langle g, h\rangle$ $\qquad$ *f)* $\|h\|$ $\qquad$ *i)* Cosine θ between g *and* h

Solution

a) $\langle f, g\rangle = \int_0^1 (t+2)(3t-2)\,dt$ $\qquad\qquad$ $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$

$$= \int_0^1 \left(3t^2 + 4t - 4\right) dt$$

$$= t^3 + 2t^2 - 4t \Big|_0^1$$

$$= 1 + 2 - 4$$

$$= -1$$

b) $\langle f, h\rangle = \int_0^1 (t+2)\left(t^2 - 2t - 3\right) dt$ $\qquad\qquad$ $\langle f, h\rangle = \int_0^1 f(t)h(t)\,dt$

$$= \int_0^1 \left(t^3 - 7t - 6\right) dt$$

$$= \frac{1}{4}t^4 - \frac{7}{2}t^2 - 6t \Big|_0^1$$

$$= \frac{1}{4} - \frac{7}{2} - 6$$

$$\underline{= -\frac{37}{4}}$$

c) $\langle g,\, h\rangle = \int_0^1 (3t-2)\left(t^2 - 2t - 3\right) dt$ $\qquad \langle f,\, h\rangle = \int_0^1 g(t)h(t)\,dt$

$$= \int_0^1 \left(3t^3 - 8t^2 - 5t + 6\right) dt$$

$$= \frac{3}{4}t^4 - \frac{8}{3}t^3 - \frac{5}{2}t^2 + 6t \Big|_0^1$$

$$= \frac{3}{4} - \frac{8}{3} - \frac{5}{2} + 6$$

$$\underline{= \frac{9}{4}}$$

d) $\langle f, f\rangle = \int_0^1 (t+2)^2\, dt$ $\qquad \langle f, f\rangle = \int_0^1 f(t)f(t)\,dt$

$$= \frac{1}{3}(t+2)^3 \Big|_0^1$$

$$= \frac{1}{3}(27-8)$$

$$\underline{= \frac{19}{3}}$$

$$\|f\| = \sqrt{|\langle f, f\rangle|}$$

$$\underline{= \sqrt{\frac{19}{3}}}$$

e) $\langle g,\, g\rangle = \int_0^1 (3t-2)^2\, dt$ $\qquad \langle g,\, g\rangle = \int_0^1 g(t)g(t)\,dt$

$$= \frac{1}{3}\int_0^1 (3t-2)^2\, d(3t-2)$$

$$= \frac{1}{9}(3t-2)^3 \Big|_0^1$$

$$=\frac{1}{9}(1+8)$$

$$\underline{=1}$$

$$\|g\|=\sqrt{|\langle g,\ g\rangle|}$$

$$\underline{=1}$$

f) $\langle g,\ g\rangle=\int_0^1\left(t^2-2t-3\right)^2 dt$ $\qquad\qquad \langle h,\ h\rangle=\int_0^1 h(t)h(t)dt$

$$=\int_0^1\left(t^4-4t^3-2t^2+12t+9\right)dt$$

$$=\left(\frac{1}{5}t^5-t^4-\frac{2}{3}t^3+6t^2+9t\right)\Big|_0^1$$

$$=\frac{1}{5}-1-\frac{2}{3}+6+9$$

$$\underline{=\frac{203}{15}}$$

$$\|h\|=\sqrt{|\langle h,\ h\rangle|}$$

$$\underline{=\sqrt{\frac{203}{15}}}$$

g) $\cos\alpha=\frac{-1}{\sqrt{\frac{19}{3}}}$ $\qquad\qquad \cos\alpha=\frac{f\bullet g}{\|f\|\ \|g\|}$

$$\underline{=-\sqrt{\frac{3}{19}}}$$

h) $\cos\beta=-\frac{37}{4}\frac{1}{\sqrt{\frac{19}{3}}\sqrt{\frac{203}{15}}}$ $\qquad\qquad \cos\beta=\frac{f\bullet g}{\|f\|\ \|g\|}$

$$\underline{=-\frac{111}{4}\sqrt{\frac{5}{3{,}857}}}$$

i) $\cos\theta=\frac{9}{4}\frac{1}{\sqrt{\frac{203}{15}}}$ $\qquad\qquad \cos\theta=\frac{f\bullet g}{\|f\|\ \|g\|}$

$$\underline{=\frac{9}{4}\sqrt{\frac{15}{203}}}$$

Exercise

Suppose $\langle \vec{u},\ \vec{v}\rangle = 3+2\,i$ in a complex inner product space V. Find:

a) $\langle (2-4\,i)\vec{u},\ \vec{v}\rangle$ *b)* $\langle \vec{u},\ (4+3\,i)\vec{v}\rangle$ *c)* $\langle (3-6\,i)\vec{u},\ (5-2\,i)\vec{v}\rangle$ *d)* $\|\vec{u},\ \vec{v}\|$

Solution

a)
$$\begin{aligned}\langle (2-4\,i)\vec{u},\ \vec{v}\rangle &= (2-4\,i)\langle \vec{u},\ \vec{v}\rangle\\ &= (2-4\,i)(3+2\,i)\\ &= 6+4i-12i+8 \qquad i^2=-1\\ &= 14-8i\end{aligned}$$

b)
$$\begin{aligned}\langle \vec{u},\ (4+3\,i)\vec{v}\rangle &= (4+3\,i)\langle \vec{u},\ \vec{v}\rangle\\ &= (4+3\,i)(3+2\,i)\\ &= 12+8i+9i-6 \qquad i^2=-1\\ &= 14-8i\end{aligned}$$

c)
$$\begin{aligned}\langle (3-6\,i)\vec{u},\ (5-2\,i)\vec{v}\rangle &= (3-6\,i)(5-2\,i)\langle \vec{u},\ \vec{v}\rangle\\ &= (15-36i-12)(3+2\,i)\\ &= (3-36i)(3+2\,i)\\ &= 9-102i+72\\ &= 81-102i\end{aligned}$$

d)
$$\begin{aligned}\|\vec{u},\ \vec{v}\| &= \sqrt{\langle \vec{u},\ \vec{v}\rangle}\\ &= \sqrt{9+4}\\ &= \sqrt{13}\end{aligned}$$

Exercise

Find the Fourier coefficient c and the projection $c\vec{v}$ of $\vec{u} = (3+4\,i,\ 2-3\,i)$ along $\vec{v} = (5+i,\ 2i)$ in $\mathbb{C}^2$

Solution

$$c = \frac{\langle \vec{u},\ \vec{v}\rangle}{\langle \vec{v},\ \vec{v}\rangle}$$

$$\begin{aligned}c &= \frac{(3+4\,i)\left(\overline{5+i}\right)+(2-3\,i)\left(\overline{2i}\right)}{5^2+1^2+2^2}\\ &= \frac{(3+4\,i)(5-i)+(2-3\,i)(-2i)}{25+1+4}\\ &= \frac{15+17i+4-4i-6}{30}\\ &= \frac{13+13i}{30}\end{aligned}$$

$$= \boxed{\frac{13}{30} + \frac{13}{30}i}$$

$$proj(\vec{u}, \vec{v}) = c\vec{v}$$

$$= \left(\frac{13}{30} + \frac{13}{30}i\right)(5+i,\ 2i)$$

$$= \left(\frac{13}{6} + \frac{13}{6}i + \frac{13}{30}i - \frac{13}{30},\quad \frac{13}{15}i - \frac{13}{15}\right)$$

$$= \left(\frac{52}{30} + \frac{78}{30}i,\ -\frac{13}{15} + \frac{13}{15}i\right)$$

$$= \boxed{\left(\frac{26}{15} + \frac{39}{30}i,\ -\frac{13}{15} + \frac{13}{15}i\right)}$$

Exercise

Suppose $\vec{v} = (1,\ 3,\ 5,\ 7)$. Find the projection of $\vec{v}$ onto W or find $\vec{w} \in W$ that minimizes $\|\vec{v} - \vec{w}\|$, where W is the subspace of $\mathbb{R}^4$ spanned by:

a) $\vec{u}_1 = (1,\ 1,\ 1,\ 1)$ *and* $\vec{u}_2 = (1,\ -3,\ 4,\ -2)$

b) $\vec{v}_1 = (1,\ 1,\ 1,\ 1)$ *and* $\vec{v}_2 = (1,\ 2,\ 3,\ 2)$

Solution

a) $\vec{u}_1 \cdot \vec{u}_2 = (1,\ 1,\ 1,\ 1) \cdot (1,\ -3,\ 4,\ -2)$

$$= 1 - 3 + 4 - 2$$

$$= \boxed{0}$$

Therefore, $\vec{u}_1$ *and* $\vec{u}_2$ are orthogonal.

$$c_1 = \frac{\langle \vec{v},\ \vec{u}_1 \rangle}{\langle \vec{u}_1,\ \vec{u}_1 \rangle}$$

$$= \frac{(1,\ 3,\ 5,\ 7) \cdot (1,\ 1,\ 1,\ 1)}{\|(1,\ 1,\ 1,\ 1)\|^2}$$

$$= \frac{1+3+5+7}{1+1+1+1}$$

$$= \frac{16}{4}$$

$$= \boxed{4}$$

$$c_2 = \frac{\langle \vec{v},\ \vec{u}_2 \rangle}{\langle \vec{u}_2,\ \vec{u}_2 \rangle}$$

$$= \frac{(1,\ 3,\ 5,\ 7) \cdot (1,\ -3,\ 4,\ -2)}{\|(1,\ -3,\ 4,\ -2)\|^2}$$

$$= \frac{1-9+20-14}{1+9+16+4}$$

$$= \frac{-2}{30}$$

$$= \frac{1}{15}$$

$$w = proj(\vec{v},\ W)$$

$$= c_1\vec{u}_1 + c_2\vec{u}_2$$

$$= 4(1,\ 1,\ 1,\ 1) + \frac{1}{15}(1,\ -3,\ 4,\ -2)$$

$$= \left(\frac{59}{15},\ \frac{63}{5},\ \frac{56}{15},\ \frac{62}{15}\right)$$

b) $\vec{v}_1 \bullet \vec{v}_2 = (1,\ 1,\ 1,\ 1) \bullet (1,\ 2,\ 3,\ 2)$

$$= 1+2+3+2$$

$$= 8 \ne 0$$

Therefore, $\vec{v}_1$ *and* $\vec{v}_2$ are *not* orthogonal.

Applying Gram-Schmidt algorithm

$$\vec{w}_1 = \vec{v}_1 = (1,\ 1,\ 1,\ 1)$$

$$\vec{w}_2 = (1,\ 2,\ 3,\ 2) - \frac{(1,\ 2,\ 3,\ 2) \cdot (1,\ 1,\ 1,\ 1)}{4}(1,\ 1,\ 1,\ 1)$$

$$\vec{w}_2 = \vec{v}_2 - \frac{\langle \vec{v},\ \vec{w}_1 \rangle}{\|\vec{w}_1\|^2}\vec{w}_1$$

$$= (1,\ 2,\ 3,\ 2) - 2(1,\ 1,\ 1,\ 1)$$

$$= (-1,\ 0,\ 1,\ 0)$$

$$c_1 = \frac{(1,\ 3,\ 5,\ 7) \cdot (1,\ 1,\ 1,\ 1)}{\|(1,\ 1,\ 1,\ 1)\|^2}$$

$$c_1 = \frac{\langle \vec{v},\ \vec{w}_1 \rangle}{\langle \vec{w}_1,\ \vec{w}_1 \rangle}$$

$$= \frac{1+3+5+7}{1+1+1+1}$$

$$= \frac{16}{4}$$

$$= 4$$

$$c_2 = \frac{(1,\ 3,\ 5,\ 7) \cdot (-1,\ 0,\ 1,\ 0)}{\|(-1,\ 0,\ 1,\ 0)\|^2}$$

$$c_2 = \frac{\langle \vec{v},\ \vec{w}_2 \rangle}{\langle \vec{w}_2,\ \vec{w}_2 \rangle}$$

$$= \frac{-1+0+5+0}{2}$$

$$\underline{= -3}$$

$$w = proj(\vec{v},\ W)$$

$$= c_1\vec{w}_1 + c_2\vec{w}_2$$

$$= 4(1,\ 1,\ 1,\ 1) - 3(-1,\ 0,\ 1,\ 0)$$

$$\underline{= (7,\ 4,\ 1,\ 4)}$$

Exercise

Suppose $\{\vec{u}_1,\ \vec{u}_2,\ \ldots,\ \vec{u}_n\}$ is an orthogonal set of vectors. Prove that (*Pythagoras*)

$$\left\|\vec{u}_1 + \vec{u}_2 + \ \ldots + \vec{u}_n\right\|^2 = \left\|\vec{u}_1\right\|^2 + \left\|\vec{u}_2\right\|^2 + \ \ldots + \left\|\vec{u}_n\right\|^2$$

Solution

$$\left\|\vec{u}_1 + \vec{u}_2 + \ \ldots + \vec{u}_n\right\|^2 = \left\langle \vec{u}_1 + \vec{u}_2 + \ \ldots + \vec{u}_n,\ \vec{u}_1 + \vec{u}_2 + \ \ldots + \vec{u}_n \right\rangle$$

$$= \left\langle \vec{u}_1,\ \vec{u}_1 \right\rangle + \left\langle \vec{u}_2,\ \vec{u}_2 \right\rangle + \ldots + \left\langle \vec{u}_n,\ \vec{u}_n \right\rangle$$

$$= \left\|\vec{u}_1\right\|^2 + \left\|\vec{u}_2\right\|^2 + \ \ldots + \left\|\vec{u}_n\right\|^2$$

Exercise

Suppose A is an orthogonal matrix. Show that $\langle \vec{u}A,\ \vec{v}A \rangle = \langle \vec{u},\ \vec{v} \rangle$ for any $\vec{u},\ \vec{v}\ \in V$

Solution

A is an orthogonal matrix $\Rightarrow AA^T = I$

And $\langle \vec{u},\ \vec{v} \rangle = \vec{u}^T\vec{v}$

$$\langle \vec{u}A,\ \vec{v}A \rangle = (A\vec{u})^T (A\vec{v})$$

$$= \vec{u}^T \left(A^T A\right)\vec{v}$$

$$= \vec{u}^T I\ \vec{v}$$

$$= \vec{u}^T\vec{v}$$

$$= \langle \vec{u},\ \vec{v} \rangle \quad \checkmark$$

Exercise

Suppose A is an orthogonal matrix. Show that $\|\vec{u}A\| = \|\vec{u}\|$ for every $\vec{u} \in V$

Solution

A is an orthogonal matrix

$\Rightarrow AA^T = I$ and $\langle \vec{u}, \vec{u} \rangle = \vec{u}^T \vec{u}$

$$\begin{aligned}
\|\vec{u}A\|^2 &= \langle \vec{u}A, \vec{u}A \rangle \\
&= (A\vec{u})^T (A\vec{u}) \\
&= \vec{u}^T \left(A^T A\right)\vec{u} \\
&= \vec{u}^T I\, \vec{u} \\
&= \vec{u}^T \vec{u} \\
&= \langle \vec{u}, \vec{u} \rangle \quad \checkmark
\end{aligned}$$

Exercise

Let V be an inner product space over $\mathbb{R}$ or $\mathbb{C}$. Show that

$$\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$$

If and only if

$$\|s\vec{u} + t\vec{v}\| = s\|\vec{u}\| + t\|\vec{v}\| \quad \textit{for all} \quad s,\ t \ge 0$$

Solution

Suppose that $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$. For $s,\ t \ge 0$

$$\begin{aligned}
\|s\vec{u} + t\vec{v}\|^2 &= s^2\|\vec{u}\|^2 + t^2\|\vec{v}\|^2 + 2st\,\vec{u}\vec{v} \\
&\le s^2\|\vec{u}\|^2 + t^2\|\vec{v}\|^2 \\
&\le s\|\vec{u}\| + t\|\vec{v}\|
\end{aligned}$$

$$\|s\vec{u} + t\vec{v}\| \le s\|\vec{u}\| + t\|\vec{v}\|$$

$$\begin{aligned}
\|s\vec{u} + t\vec{v}\| &= \|s\vec{u} + t\vec{u} - t\vec{u} + t\vec{v}\| \\
&= \|t\vec{u} + t\vec{v} - t\vec{u} + s\vec{u}\| \\
&= \|t(\vec{u} + \vec{v}) - (t - s)\vec{u}\| \\
&\ge \left|\, t\|\vec{u} + \vec{v}\| - (t - s)\|\vec{u}\| \,\right| \qquad\qquad \|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\| \\
&= \ t\|\vec{u}\| + \|\vec{v}\| - t\|\vec{u}\| + s\|\vec{u}\|
\end{aligned}$$

$$= t\|\vec{v}\| + s\|\vec{u}\|$$

$$\begin{cases} \|s\vec{u} + t\vec{v}\| \le s\|\vec{u}\| + t\|\vec{v}\| \\ \quad\quad and \\ \|s\vec{u} + t\vec{v}\| \ge s\|\vec{u}\| + t\|\vec{v}\| \end{cases} \Rightarrow \quad \|s\vec{u} + t\vec{v}\| = s\|\vec{u}\| + t\|\vec{v}\|$$

Exercise

Let V be an inner product vector space over $\mathbb{R}$.

a) If e_1, e_2, e_3 are three vectors in V with pairwise product negative, that is,

$$\langle e_1, e_2 \rangle < 0, \quad i, j = 1, \ 2, \ 3, \quad i \ne j$$

Show that e_1, e_2, e_3 are linearly independent.

b) Is it possible for three vectors on the xy–plane to have pairwise negative products?

c) Does part (a) remain valid when the word "negative: is replaced with positive?

d) Suppose $\vec{u}, \ \vec{v}, \ and \ \vec{w}$ are three–unit vectors in the xy–plane. What are the maximum and minimum values that

$$\langle \vec{u}, \ \vec{v} \rangle + \langle \vec{v}, \ \vec{w} \rangle + \langle \vec{w}, \ \vec{u} \rangle$$

Can attain? And when?

Solution

a) Suppose that e_1, e_2, e_3 are linearly dependent.

Then, assume that e_1, e_2, e_3 are unit vectors and that

$$e_3 = c_1 e_1 + c_2 e_2$$

Then

$$\langle e_1, e_3 \rangle = c_1 \langle e_1, e_1 \rangle + c_2 \langle e_1, e_2 \rangle \qquad \langle e_1, e_1 \rangle = 1$$

$$= c_1 + c_2 \langle e_1, e_2 \rangle < 0$$

$$c_1 < -c_2 \langle e_1, e_2 \rangle$$

$$\langle e_2, e_3 \rangle = c_1 \langle e_2, e_1 \rangle + c_2 \langle e_2, e_2 \rangle \qquad \langle e_2, e_2 \rangle = 1$$

$$= c_1 \langle e_2, e_1 \rangle + c_2 < 0$$

$$c_2 < -c_1 \langle e_2, e_1 \rangle$$

$$c_2 < -c_1 \langle e_2, e_1 \rangle$$

$$< -\left(-c_2 \langle e_1, e_2 \rangle\right) \langle e_2, e_1 \rangle$$

$$= c_2 \langle e_1, e_2 \rangle^2 \qquad\qquad \langle e_1, e_2 \rangle^2 > 1$$

$c_2 < c_2$ *Contradiction*

Therefore, e_1, e_2, e_3 are linearly independent.

b) To have all three vectors on the *xy*–plane which is in 2 dimensional.
Therefore, it is ***impossible*** for three to have pairwise negative products.

c) No

d) Given: $\vec{u}, \vec{v}, \text{and } \vec{w}$ are three–unit vectors in the *xy*–plane and

$$\left|\langle \vec{u}, \vec{u} \rangle\right| = \left|\langle \vec{v}, \vec{v} \rangle\right| = \left|\langle \vec{w}, \vec{w} \rangle\right| = 1$$

$$\cos\alpha_1 = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \, \|\vec{v}\|} \rightarrow \cos\alpha_1 = \langle \vec{u}, \vec{v} \rangle$$

$$\cos\alpha_2 = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \, \|\vec{w}\|} \rightarrow \cos\alpha_2 = \langle \vec{v}, \vec{w} \rangle$$

$$\cos\alpha_3 = \frac{\langle \vec{u}, \vec{w} \rangle}{\|\vec{u}\| \, \|\vec{w}\|} \rightarrow \cos\alpha_3 = \langle \vec{u}, \vec{w} \rangle$$

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle = \cos\alpha_1 + \cos\alpha_2 + \cos\alpha_3$$

Since $-1 \le \cos\theta \le 1$

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle = 1 + 1 + 1$$
$$= 3$$

Since the 3 vectors are unit vectors in the *xy*–plane and which it will divide the plane into a three equal angles $\alpha = \frac{2\pi}{3}$

$$\begin{aligned}\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle &= \cos\frac{2\pi}{3} + \cos\frac{2\pi}{3} + \cos\frac{2\pi}{3} \\ &= 3\cos\frac{2\pi}{3} \\ &= 3\left(-\frac{1}{2}\right) \\ &= -\frac{3}{2}\end{aligned}$$

Therefore, the minimum $\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle = -\frac{3}{2}$

The maximum: $\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle = 3$