

Solution

Section 2.6 – Forced Harmonic Motion

Exercise

A 1-kg mass is attached to a spring $k = 4 \text{ kg} / \text{s}^2$ and the system is allowed to come to rest. The spring-mass system is attached to a machine that supplies external driving force $f(t) = 4 \cos \omega t$ *Newtons*. The system is started from equilibrium; the mass is having neither initial displacement nor velocity. Ignore any damping forces.

- a) Find the position of the mass as a function of time
- b) Place your answer in the form $s(t) = A \sin \delta t \sin \bar{\omega} t$. Select an ω near the natural frequency of the system to demonstrate the "beating" of the system. Sketch a plot shows the "beats:" and include the envelope of the beating motion in your plot.

Solution

a) $mx'' + \omega_0^2 x = f(t)$

$$mx'' + kx = f(t) \quad x(0) = x'(0) = 0$$

$$x'' + 4x = 4 \cos \omega t$$

$$x(t) = \frac{A}{(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t)$$

$$x(t) = \frac{4}{4 - \omega^2} (\cos \omega t - \cos 2t)$$

b) $x(t) = \frac{4}{4 - \omega^2} (\cos \omega t - \cos 2t)$

$$\cos \alpha - \cos \beta = -2 \sin \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right)$$

$$= \frac{4}{4 - \omega^2} \left[-2 \sin \left(\frac{\omega + 2}{2} t \right) \sin \left(\frac{\omega - 2}{2} t \right) \right]$$

$$= \frac{4}{4 - \omega^2} \left[2 \sin \left(\frac{\omega + 2}{2} t \right) \sin \left(\frac{2 - \omega}{2} t \right) \right]$$

Mean frequency: $\bar{\omega} = \frac{\omega_0 + \omega}{2} = \frac{2 + \omega}{2}$

$$2\bar{\omega} = 2 + \omega$$

Half difference: $\delta = \frac{\omega_0 - \omega}{2} = \frac{2 - \omega}{2}$

$$2\delta = 2 - \omega$$

$$4 - \omega^2 = (2 + \omega)(2 - \omega) = 2\bar{\omega}2\delta$$

$$= 2\bar{\omega}2\delta$$

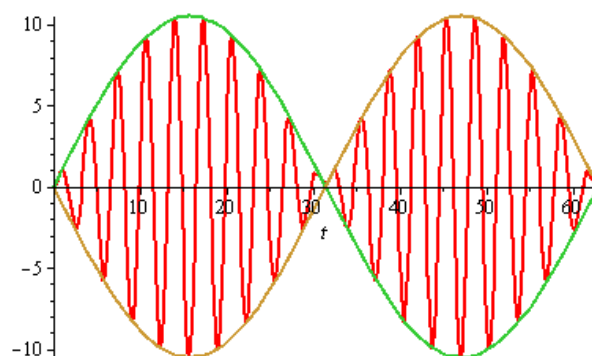
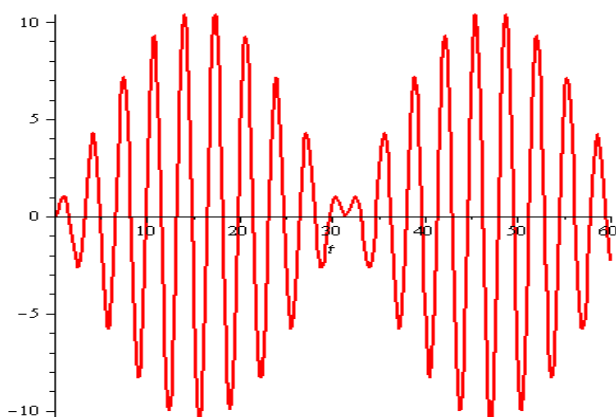
$$= 4\bar{\omega}\delta$$

$$\begin{aligned} x(t) &= \frac{8}{4\bar{\omega}\delta} \sin \varpi t \sin \delta t \\ &= \frac{2}{\bar{\omega}\delta} \sin \varpi t \sin \delta t \end{aligned}$$

If we choose $\omega = 1.8$ near to $\omega_0 = 2$

That implies to: $\bar{\omega} = \frac{2+1.8}{2} \approx 1.9$ and $\delta = \frac{2-1.8}{2} \approx 0.1$

$$\begin{aligned} x(t) &= \frac{2}{0.19} \sin 0.1t \sin 1.9t \\ &= \frac{200}{19} \sin \frac{t}{10} \sin \frac{19}{10}t \end{aligned}$$



Exercise

Find a particular solution to the differential equation using undetermined coefficients. Find and plot the solution of the initial value problem. Superimpose the plots of the transient response and the steady state solution.

$$x'' + 7x' + 10x = 3\cos 3t \quad x(0) = -1, \quad x'(0) = 0$$

Solution

The particular solution: $x(t) = a \cos 3t + b \sin 3t$

$$x' = -3a \sin 3t + 3b \cos 3t$$

$$x'' = -9a \cos 3t - 9b \sin 3t$$

$$-9a \cos 3t - 9b \sin 3t + 7(-3a \sin 3t + 3b \cos 3t) + 10(a \cos 3t + b \sin 3t) = 3 \cos 3t$$

$$-9a \cos 3t - 9b \sin 3t - 21a \sin 3t + 21b \cos 3t + 10a \cos 3t + 10b \sin 3t = 3 \cos 3t$$

$$a \cos 3t - 21a \sin 3t + 21b \cos 3t + b \sin 3t = 3 \cos 3t$$

$$(a + 21b) \cos 3t + (b - 21a) \sin 3t = 3 \cos 3t$$

$$\begin{aligned} a + 21b &= 3 \\ -21a + b &= 0 \end{aligned} \Rightarrow a = \frac{3}{442} \quad b = \frac{63}{442}$$

The particular solution (**steady-state solution**):

$$x_p(t) = \frac{3}{442} \cos 3t + \frac{63}{442} \sin 3t$$

The homogeneous eq.: $x'' + 7x' + 10x = 0$

The characteristic eq.: $\lambda^2 + 7\lambda + 10 = 0 \Rightarrow \lambda_1 = -5, \lambda_2 = -2$

$$x_h(t) = C_1 e^{-5t} + C_2 e^{-2t}$$

$$x(t) = \frac{3}{442} \cos 3t + \frac{63}{442} \sin 3t + C_1 e^{-5t} + C_2 e^{-2t} \quad x(0) = -1, \quad x'(0) = 0$$

$$x(0) = \frac{3}{442} \cos 3(0) + \frac{63}{442} \sin 3(0) + C_1 e^{-5(0)} + C_2 e^{-2(0)}$$

$$-1 = \frac{3}{442} + C_1 + C_2 \rightarrow C_1 + C_2 = -\frac{445}{442}$$

$$x'(t) = -\frac{9}{442} \sin 3t + \frac{189}{442} \cos 3t - 5C_1 e^{-5t} - 2C_2 e^{-2t}$$

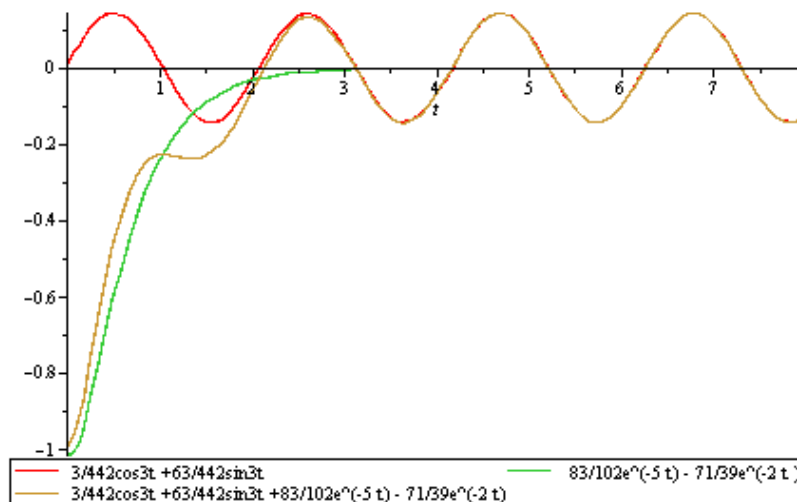
$$x'(0) = -\frac{9}{442} \sin 3(0) + \frac{189}{442} \cos 3(0) - 5C_1 e^{-5(0)} - 2C_2 e^{-2(0)}$$

$$5C_1 + 2C_2 = \frac{189}{442}$$

$$\begin{cases} C_1 + C_2 = -\frac{445}{442} \\ 5C_1 + 2C_2 = \frac{189}{442} \end{cases} \rightarrow C_1 = \frac{83}{102} \quad C_2 = -\frac{71}{39}$$

Transient response solution: $x_h(t) = \frac{83}{102} e^{-5t} - \frac{71}{39} e^{-2t}$

The general solution: $x(t) = \frac{3}{442} \cos 3t + \frac{63}{442} \sin 3t + \frac{83}{102} e^{-5t} - \frac{71}{39} e^{-2t}$



Complex Method

$$x'' + 7x' + 10x = 3\cos 3t$$

The particular solution: $z = Ae^{i3t}$

$$z' = (3i)Ae^{i3t}$$

$$z'' = (3i)^2 Ae^{i3t}$$

$$z'' + 7z' + 10z = 3e^{i3t}$$

$$(3i)^2 Ae^{i3t} + 7(3i)Ae^{i3t} + 10Ae^{i3t} = 3e^{i3t}$$

$$(-9 + 21i + 10)A = 3$$

$$(1 + 21i)A = 3$$

$$A = 3 \frac{1}{1 + 21i} \cdot \frac{1 - 21i}{1 - 21i}$$

$$= 3 \cdot \frac{1 - 21i}{1 + 441}$$

$$= \frac{3}{442} - i \frac{63}{442}$$

$$z = \left(\frac{3}{442} - i \frac{63}{442} \right) e^{i3t}$$

$$= \left(\frac{3}{442} - i \frac{63}{442} \right) (\cos 3t + i \sin 3t)$$

$$= \frac{3}{442} \cos 3t + \frac{63}{442} \sin 3t + i \left(\frac{3}{442} \sin 3t - \frac{63}{442} \cos 3t \right)$$

The particular solution (**steady-state solution**): $x_p(t) = \frac{3}{442} \cos 3t + \frac{63}{442} \sin 3t$

Exercise

Find a particular solution to the differential equation using undetermined coefficients. Find and plot the solution of the initial value problem. Superimpose the plots of the transient response and the steady state solution.

$$x'' + 4x' + 5x = 3\sin t \quad x(0) = 0, \quad x'(0) = -3$$

Solution

$$x'' + 4x' + 5x = 3\sin t$$

The particular solution: $z = Ae^{it}$

$$z' = (i)Ae^{it}$$

$$z'' = (i)^2 Ae^{it} = -Ae^{it}$$

$$\begin{aligned}
 z'' + 4z' + 5z &= 3e^{it} \\
 -Ae^{i3t} + 4iAe^{i3t} + 5Ae^{i3t} &= 3e^{it} \\
 (-1 + 4i + 5)A &= 3 \\
 (4 + 4i)A &= 3
 \end{aligned}$$

$$\begin{aligned}
 A &= \frac{3}{4} \frac{1}{1+i} \cdot \frac{1-i}{1-i} \\
 &= \frac{3}{4} \left(\frac{1}{2} - i \frac{1}{2} \right) \\
 &= \frac{3}{8} - i \frac{3}{8}
 \end{aligned}$$

$$\begin{aligned}
 z &= \left(\frac{3}{8} - i \frac{3}{8} \right) e^{it} \\
 &= \left(\frac{3}{8} - i \frac{3}{8} \right) (\cos t + i \sin t) \\
 &= \frac{3}{8} [\cos t + \sin t + i(\sin t - \cos t)]
 \end{aligned}$$

$$\boxed{x_p(t) = \text{Im}(z) = \frac{3}{8}(\sin t - \cos t)}$$

The homogeneous eq.: $x'' + 4x' + 5x = 0$

The characteristic eq.: $\lambda^2 + 4\lambda + 5 = 0 \Rightarrow \boxed{\lambda = -2 \pm i}$

$$x_h(t) = e^{-2t} (C_1 \cos t + C_2 \sin t)$$

$$x(t) = \frac{3}{8}(\sin t - \cos t) + e^{-2t} (C_1 \cos t + C_2 \sin t) \quad x(0) = 0, \quad x'(0) = -3$$

$$x(0) = \frac{3}{8}(\sin 0 - \cos 0) + e^{-2(0)} (C_1 \cos 0 + C_2 \sin 0)$$

$$0 = -\frac{3}{8} + C_1 \rightarrow \boxed{C_1 = \frac{3}{8}}$$

$$x'(t) = \frac{3}{8}(\cos t + \sin t) - 2e^{-2t} (C_1 \cos t + C_2 \sin t) + e^{-2t} (-C_1 \sin t + C_2 \cos t)$$

$$x'(0) = \frac{3}{8}(\cos 0 + \sin 0) - 2e^{-2(0)} (C_1 \cos 0 + C_2 \sin 0) + e^{-2(0)} (-C_1 \sin 0 + C_2 \cos 0)$$

$$-3 = \frac{3}{8} - 2C_1 + C_2$$

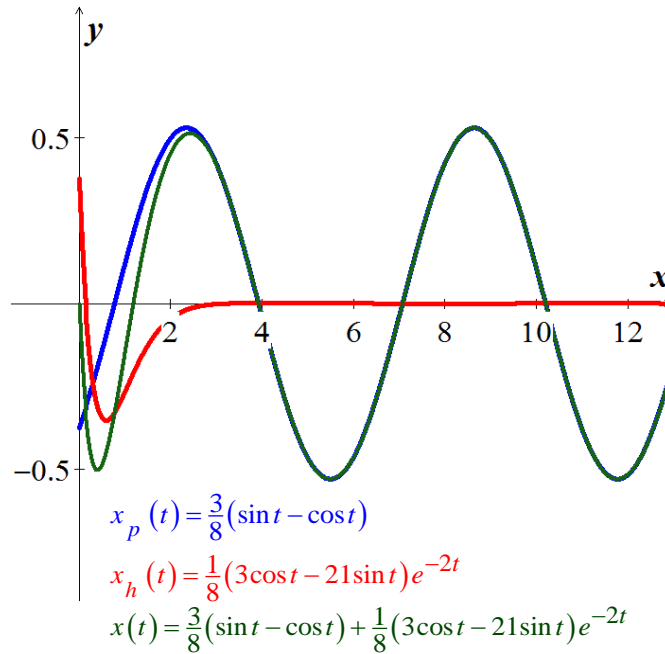
$$\boxed{C_2 = -3 - \frac{3}{8} + 2\left(\frac{3}{8}\right) = -\frac{21}{8}}$$

$$x(t) = \frac{3}{8}(\sin t - \cos t) + e^{-2t} \left(\frac{3}{8} \cos t - \frac{21}{8} \sin t \right)$$

$$\boxed{x(t) = \frac{3}{8}(\sin t - \cos t) + \frac{1}{8}e^{-2t} (3 \cos t - 21 \sin t)}$$

The steady-state solution is the particular solution: $x_p(t) = \frac{3}{8}(\sin t - \cos t)$

The transient response is: $x_h(t) = \frac{1}{8}(3\cos t - 21\sin t)e^{-2t}$



Exercise

Find a particular solution of $y'' - 2y' + 5y = 2\cos 3x - 4\sin 3x + e^{2x}$ given the set

$y_p = A\cos 3x + B\sin 3x + Ce^{2x}$ where A, B, C are to be determined

Solution

$$y_p = A\cos 3x + B\sin 3x + Ce^{2x}$$

$$y'_p = -3A\sin 3x + 3B\cos 3x + 2Ce^{2x}$$

$$y''_p = -9A\cos 3x - 9B\sin 3x + 4Ce^{2x}$$

$$\begin{aligned}
 y'' - 2y' + 5y &= -9A\cos 3x - 9B\sin 3x + 4Ce^{2x} + 6A\sin 3x - 6B\cos 3x - 4Ce^{2x} \\
 &\quad + 5A\cos 3x + 5B\sin 3x + 5Ce^{2x} \\
 &= (-4A - 6B)\cos 3x + (6A - 4B)\sin 3x + 5Ce^{2x} = 2\cos 3x - 4\sin 3x + e^{2x}
 \end{aligned}$$

$$\begin{cases} -4A - 6B = 2 \\ 6A - 4B = -4 \end{cases} \rightarrow A = -\frac{8}{13}, B = \frac{1}{13}$$

$$\begin{cases} 5C = 1 \end{cases} \rightarrow C = \frac{1}{5}$$

The particular solution: $y_p = -\frac{8}{13}\cos 3x + \frac{1}{13}\sin 3x + \frac{1}{5}e^{2x}$

Exercise

Find the general solution: $mx'' + kx = F_0 \cos \omega t$; $x(0) = x_0$, $x'(0) = 0$ ($\omega \neq \omega_0$)

Solution

$$m\lambda^2 + k = 0 \rightarrow (k > 0) \quad \lambda_{1,2} = \pm i\sqrt{\frac{k}{m}} = \pm \omega_0 i \quad \left(\omega_0 = \sqrt{\frac{k}{m}} \right)$$

$$\underline{x_h = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t}$$

$$x_P = A \cos \omega t + B \sin \omega t$$

$$x'_P = -A\omega \sin \omega t + B\omega \cos \omega t$$

$$x''_P = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t$$

$$m(-A\omega^2 \cos \omega t - B\omega^2 \sin \omega t) + k(A \cos \omega t + B \sin \omega t) = F_0 \cos \omega t$$

$$(kA - m\omega^2 A) \cos \omega t + (kB - m\omega^2 B) \sin \omega t = F_0 \cos \omega t$$

$$\begin{cases} (k - m\omega^2)A = F_0 \\ (k - m\omega^2)B = 0 \end{cases} \rightarrow \underline{A = \frac{F_0}{k - m\omega^2}, B = 0}$$

$$x_P = \frac{F_0}{k - m\omega^2} \cos \omega t$$

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_0}{k - m\omega^2} \cos \omega t$$

$$\underline{x(0) = x_0 \rightarrow C_1 = x_0 - \frac{F_0}{k - m\omega^2}}$$

$$x'(t) = -\omega_0 C_1 \sin \omega_0 t + \omega_0 C_2 \cos \omega_0 t - \frac{\omega_0 F_0}{k - m\omega^2} \sin \omega t$$

$$\underline{x'(0) = 0 \rightarrow C_2 = 0}$$

$$x(t) = x_0 - \frac{F_0}{k - m\omega^2} \cos \omega_0 t + \frac{F_0}{k - m\omega^2} \cos \omega t$$

$$\underline{= x_0 + \frac{F_0}{k - m\omega^2} (\cos \omega t - \cos \omega_0 t)}$$

Exercise

Find the general solution: $mx'' + kx = F_0 \cos \omega t$; $x(0) = 0$, $x'(0) = v_0$ ($\omega = \omega_0$)

Solution

$$m\lambda^2 + k = 0 \rightarrow (k > 0) \quad \lambda_{1,2} = \pm i\sqrt{\frac{k}{m}} = \pm \omega_0 i \quad \left(\omega_0 = \sqrt{\frac{k}{m}} \right)$$

$$\underline{x_h = C_1 \cos \omega t + C_2 \sin \omega t} \quad \omega_0 = \omega$$

$$x_p = A \cos \omega t + B \sin \omega t$$

$$x'_p = -A\omega \sin \omega t + B\omega \cos \omega t$$

$$x''_p = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t$$

$$m(-A\omega^2 \cos \omega t - B\omega^2 \sin \omega t) + k(A \cos \omega t + B \sin \omega t) = F_0 \cos \omega t$$

$$(kA - m\omega^2 A) \cos \omega t + (kB - m\omega^2 B) \sin \omega t = F_0 \cos \omega t$$

$$\begin{cases} (k - m\omega^2)A = F_0 \\ (k - m\omega^2)B = 0 \end{cases} \rightarrow \underline{A = \frac{F_0}{k - m\omega^2}, \quad B = 0}$$

$$x_p = \frac{F_0}{k - m\omega^2} \cos \omega t$$

$$x(t) = C_1 \cos \omega t + C_2 \sin \omega t + \frac{F_0}{k - m\omega^2} \cos \omega t$$

$$= C_2 \sin \omega t + \left(C_1 + \frac{F_0}{k - m\omega^2} \right) \cos \omega t$$

$$\underline{x(0) = 0 \rightarrow C_1 = -\frac{F_0}{k - m\omega^2}}$$

$$x'(t) = \omega C_2 \cos \omega t - \omega \left(C_1 + \frac{F_0}{k - m\omega^2} \right) \sin \omega t$$

$$\underline{x'(0) = v_0 \rightarrow C_2 = \frac{v_0}{\omega}}$$

$$x(t) = \frac{v_0}{\omega} \sin \omega t + \left(-\frac{F_0}{k - m\omega^2} + \frac{F_0}{k - m\omega^2} \right) \cos \omega t$$

$$\underline{= \frac{v_0}{\omega} \sin \omega t}$$

Exercise

Find the general solution: $x'' + \omega_0^2 x = F_0 \sin \omega t$; $x(0) = 0$, $x'(0) = 0$ ($\omega \neq \omega_0$)

Solution

$$\lambda^2 + \omega_0^2 = 0 \rightarrow \lambda_{1,2} = \pm \omega_0 i$$

$$x_h = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$$

$$x_p = A \cos \omega t + B \sin \omega t$$

$$x'_p = -A\omega \sin \omega t + B\omega \cos \omega t$$

$$x''_p = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t$$

$$(-A\omega^2 \cos \omega t - B\omega^2 \sin \omega t) + \omega_0^2 (A \cos \omega t + B \sin \omega t) = F_0 \sin \omega t$$

$$\begin{cases} (\omega_0^2 - \omega^2)A = F_0 \\ (\omega_0^2 - \omega^2)B = 0 \end{cases} \rightarrow \underline{A = \frac{F_0}{\omega_0^2 - \omega^2}, B = 0}$$

$$x_p = \frac{F_0}{\omega_0^2 - \omega^2} \cos \omega t$$

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_0}{\omega_0^2 - \omega^2} \sin \omega t$$

$$x(0) = 0 \rightarrow \underline{C_1 = 0}$$

$$x'(t) = -\omega_0 C_1 \sin \omega_0 t + \omega_0 C_2 \cos \omega_0 t + \frac{F_0 \omega}{\omega_0^2 - \omega^2} \cos \omega t$$

$$x'(0) = 0 \rightarrow \underline{C_2 = -\frac{F_0 \omega}{(\omega_0^2 - \omega^2)\omega_0}}$$

$$x(t) = -\frac{F_0 \omega}{(\omega_0^2 - \omega^2)\omega_0} \sin \omega_0 t + \frac{F_0}{\omega_0^2 - \omega^2} \sin \omega t$$

$$= \underline{\frac{F_0}{\omega_0(\omega_0^2 - \omega^2)} (\omega_0 \sin \omega t - \omega \sin \omega_0 t)}$$

Exercise

A forced mass–spring–dashpot system with equation $mx'' + cx' + kx = F_0 \cos \omega t$. Investigate the possibility of practical resonance of this system. In particular, find the amplitude $C(\omega)$ of steady state periodic forced oscillations with frequency ω . Sketch the graph $C(\omega)$ of and find the practical resonance frequency ω (if any).
 $m = 1, \quad c = 2, \quad k = 2, \quad F_0 = 2$

Solution

$$x'' + 2x' + 2x = 2 \cos \omega t \quad mx'' + cx' + kx = F(t)$$

$$x'' + 2x' + 2x = 0 \quad mx'' + cx' + kx = 0$$

$$\lambda^2 + 2\lambda + 2 = 0 \rightarrow \lambda_{1,2} = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$$x_h = e^{-t} (C_1 \cos t + C_2 \sin t)$$

$$x_p = A \cos \omega t + B \sin \omega t$$

$$x' = -A\omega \sin \omega t + B\omega \cos \omega t$$

$$x'' = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t$$

$$-A\omega^2 \cos \omega t - B\omega^2 \sin \omega t - 2A\omega \sin \omega t + 2B\omega \cos \omega t + 2A \cos \omega t + 2B \sin \omega t = 2 \cos \omega t$$

$$\rightarrow \begin{cases} (2 - \omega^2)A + 2\omega B = 2 \\ -2\omega A + (2 - \omega^2)B = 0 \end{cases}$$

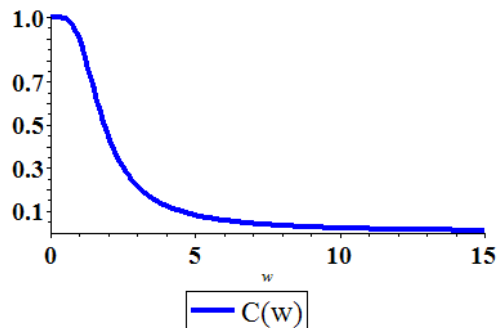
$$D = \begin{vmatrix} 2 - \omega^2 & 2\omega \\ -2\omega & 2 - \omega^2 \end{vmatrix} = (2 - \omega^2)^2 + 4\omega^2 = 4 + \omega^4$$

$$D_A = \begin{vmatrix} 2 & 2\omega \\ 0 & 2 - \omega^2 \end{vmatrix} = 2(2 - \omega^2) \quad D_B = \begin{vmatrix} 2 - \omega^2 & 2 \\ -2\omega & 0 \end{vmatrix} = 4\omega$$

$$A = \frac{4 - 2\omega^2}{4 + \omega^4} \quad B = \frac{4\omega}{4 + \omega^4}$$

$$C(\omega) = \frac{2}{\sqrt{4 + \omega^4}}$$

$$C(\omega) = \sqrt{A^2 + B^2} = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$



$C(\omega)$ starts with $C(0) = 1$ and steadily decreases as ω increases.

Hence, there is no practical resonance frequency.

Exercise

A forced mass–spring–dashpot system with equation $mx'' + cx' + kx = F_0 \cos \omega t$. Investigate the possibility of practical resonance of this system. In particular, find the amplitude $C(\omega)$ of steady state periodic forced oscillations with frequency ω . Sketch the graph $C(\omega)$ of and find the practical resonance frequency ω (if any). $m = 1, \quad c = 4, \quad k = 5, \quad F_0 = 10$

Solution

$$x'' + 4x' + 5x = 10 \cos \omega t$$

$$mx'' + cx' + kx = F(t)$$

$$A = \frac{10(5 - \omega^2)}{(5 - \omega^2)^2 + 16\omega^2} = \frac{10(5 - \omega^2)}{25 + 6\omega^2 + \omega^4}$$

$$A = \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + (c\omega)^2}$$

$$B = \frac{40\omega}{25 + 6\omega^2 + \omega^4}$$

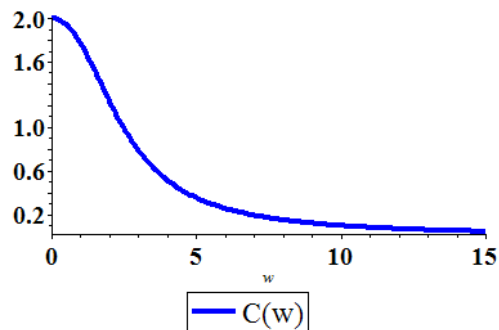
$$B = \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2}$$

$$C(\omega) = \frac{10}{\sqrt{25 + 6\omega^2 + \omega^4}}$$

$$C(\omega) = \sqrt{A^2 + B^2} = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

$C(\omega)$ starts with $C(0) = 2$ and steadily decreases as ω increases.

Hence there is no practical resonance frequency.



Exercise

A forced mass–spring–dashpot system with equation $mx'' + cx' + kx = F_0 \cos \omega t$. Investigate the possibility of practical resonance of this system. In particular, find the amplitude $C(\omega)$ of steady state periodic forced oscillations with frequency ω . Sketch the graph $C(\omega)$ of and find the practical resonance frequency ω (if any). $m = 1, \quad c = 6, \quad k = 45, \quad F_0 = 50$

Solution

$$x'' + 6x' + 45x = 50 \cos \omega t$$

$$mx'' + cx' + kx = F(t)$$

$$A = \frac{50(45 - \omega^2)}{(45 - \omega^2)^2 + 36\omega^2} = \frac{50(45 - \omega^2)}{2025 - 54\omega^2 + \omega^4}$$

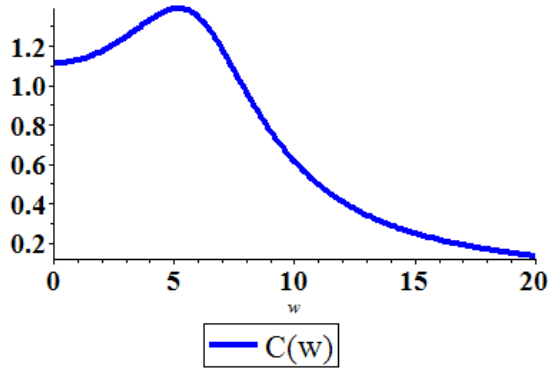
$$B = \frac{300\omega}{2025 - 54\omega^2 + \omega^4}$$

$$C(\omega) = \frac{50}{\sqrt{2025 - 54\omega^2 + \omega^4}}$$

$$A = \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + (c\omega)^2}$$

$$B = \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2}$$

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$



$$C'(\omega) = -\frac{25(-108\omega + 4\omega^3)}{(2025 - 54\omega^2 + \omega^4)^{3/2}} = 0$$

$$\omega(4\omega^2 - 108) = 0 \Rightarrow \omega = \sqrt{27} = 3\sqrt{3} \quad (C.N)$$

$C(\omega)$ starts with $C(0) = \frac{10}{9}$, hence the practical resonance frequency is $\omega = 3\sqrt{3}$.

Exercise

A forced mass–spring–dashpot system with equation $mx'' + cx' + kx = F_0 \cos \omega t$. Investigate the possibility of practical resonance of this system. In particular, find the amplitude $C(\omega)$ of steady state periodic forced oscillations with frequency ω . Sketch the graph $C(\omega)$ of and find the practical resonance frequency ω (if any). $m = 1$, $c = 10$, $k = 650$, $F_0 = 100$

Solution

$$x'' + 10x' + 650x = 100 \cos \omega t \quad mx'' + cx' + kx = F(t)$$

$$A = \frac{100(650 - \omega^2)}{(650 - \omega^2)^2 + 100\omega^2} = \frac{100(650 - \omega^2)}{422,500 - 1200\omega^2 + \omega^4}$$

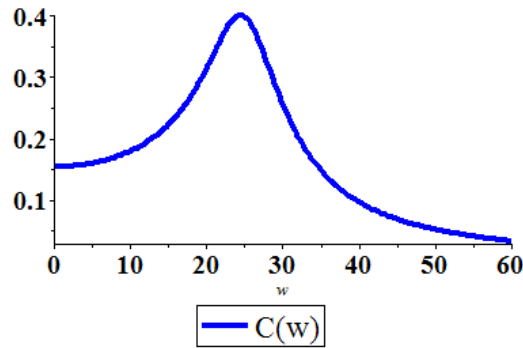
$$A = \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + (c\omega)^2}$$

$$B = \frac{1000\omega}{422,500 - 1200\omega^2 + \omega^4}$$

$$C(\omega) = \frac{100}{\sqrt{422,500 - 1200\omega^2 + \omega^4}}$$

$$B = \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2}$$

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$



$$C'(\omega) = -\frac{500(-2400\omega + 4\omega^3)}{(422,500 - 1200\omega^2 + \omega^4)^{3/2}} = 0$$

$$\omega(4\omega^2 - 2400) = 0 \Rightarrow \omega = \sqrt{600} = 10\sqrt{6} \text{ (C.N)}$$

$$C(\omega) \text{ starts with } C(0) = \frac{2}{13}, \text{ hence the practical resonance frequency is } \omega = 10\sqrt{6}.$$

Exercise

A mass weighing 100 lb. (mass $m = 3.125$ slugs in fps units) is attached to the end of a spring that is stretched 1 in. by a force of 100 lb. A force $F_0 \cos \omega t$ acts on the mass. At what frequency (in hertz) will resonance oscillation occur? Neglect damping.

Solution

Given: $m = 3.125$ slug

$$k = \frac{F}{x} = \frac{100 \text{ lb}}{1 \text{ in}} \cdot \frac{12 \text{ in}}{1 \text{ ft}} = 1200 \text{ lb / ft}$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{1200}{3.125}}$$

$$= \sqrt{384} \text{ rad / sec}$$

$$= \frac{\sqrt{384}}{2\pi} \text{ Hz} \approx 3.12 \text{ Hz.}$$

Exercise

A mass weighing 16 *pounds* stretches a spring $\frac{8}{3}$ *ft*. The mass is initially released from rest from a point 2 *ft* below the equilibrium position, and the subsequent motion takes place in a medium that offers a damping force that is numerically equal to $\frac{1}{2}$ the instantaneous velocity. Find the equation of motion if the mass is driven by an external force equal to $f(t) = 10\cos 3t$

Solution

$$m = \frac{16}{32} = \frac{1}{2} \text{ slug}$$

$$k\left(\frac{8}{3} \text{ ft}\right) = 16 \rightarrow k = 6 \quad kx = mg$$

$$\frac{1}{2}x'' + \frac{1}{2}x' + 6x = 10\cos 3t \quad mx'' + cx' + kx = F(t)$$

$$x'' + x' + 12x = 0; \quad x(0) = 2, \quad x'(0) = 0$$

$$\lambda^2 + \lambda + 12 = 0 \rightarrow \lambda_{1,2} = \frac{-1 \pm i\sqrt{47}}{2}$$

$$x_h(t) = e^{-t/2} \left(C_1 \cos \frac{\sqrt{47}}{2} t + C_2 \sin \frac{\sqrt{47}}{2} t \right)$$

$$x_p = A \cos 3t + B \sin 3t$$

$$x'_p = -3A \sin 3t + 3B \cos 3t$$

$$x''_p = -9A \cos 3t - 9B \sin 3t$$

$$x'' + x' + 12x = 20\cos 3t$$

$$-9A \cos 3t - 9B \sin 3t - 3A \sin 3t + 3B \cos 3t + 12A \cos 3t + 12B \sin 3t = 20\cos 3t$$

$$\begin{cases} \cos 3t & 3A + 3B = 20 \\ \sin 3t & -3A + 3B = 0 \end{cases} \rightarrow \underline{A = B = \frac{10}{3}}$$

$$x_p = \frac{10}{3}(\cos 3t + \sin 3t)$$

$$x(t) = e^{-t/2} \left(C_1 \cos \frac{\sqrt{47}}{2} t + C_2 \sin \frac{\sqrt{47}}{2} t \right) + \frac{10}{3}(\cos 3t + \sin 3t)$$

$$x(0) = 2 \rightarrow C_1 + \frac{10}{3} = 2 \Rightarrow \underline{C_1 = -\frac{4}{3}}$$

$$x'(t) = e^{-t/2} \left(-\frac{1}{2} C_1 \cos \frac{\sqrt{47}}{2} t + \frac{1}{2} C_2 \sin \frac{\sqrt{47}}{2} t - \frac{\sqrt{47}}{2} C_1 \sin \frac{\sqrt{47}}{2} t + \frac{\sqrt{47}}{2} C_2 \cos \frac{\sqrt{47}}{2} t \right) + \frac{10}{3}(-3\sin 3t + 3\cos 3t)$$

$$x'(0) = 0 \rightarrow -\frac{1}{2} C_1 + \frac{\sqrt{47}}{2} C_2 + 10 = 0$$

$$\Rightarrow C_2 = -\frac{64}{3\sqrt{47}} \Big|$$

$$x(t) = e^{-t/2} \left(-\frac{4}{3} \cos \frac{\sqrt{47}}{2} t - \frac{64}{3\sqrt{47}} \sin \frac{\sqrt{47}}{2} t \right) + \frac{10}{3} (\cos 3t + \sin 3t) \Big|$$

Exercise

A mass of 32 *pounds* is attached to a spring with a constant spring 5 *lb/ft*. Initially, the mass is released 1 *foot* below the equilibrium position with a downward velocity of 5 *ft/s*, and the subsequent motion takes is numerically equal to 2 times the instantaneous velocity.

a) Find the equation of motion if the mass is driven by an external force equal to

$$f(t) = 12 \cos 2t + 3 \sin 2t.$$

b) Graph the transient, steady-state, and the equation of motion solutions on the same coordinate axes.

Solution

$$m = \frac{32}{32} = 1 \text{ slug}$$

$$a) \quad x'' + 2x' + 5x = 12 \cos 2t + 3 \sin 2t; \quad x(0) = 1, \quad x'(0) = 5 \qquad mx'' + cx' + kx = f(t)$$

$$\lambda^2 + 2\lambda + 5 = 0 \rightarrow \lambda_{1,2} = -1 \pm 2i$$

$$x_h = e^{-t} (C_1 \cos 2t + C_2 \sin 2t) \Big|$$

$$x_p = A \cos 2t + B \sin 2t$$

$$x'_p = -2A \sin 2t + 2B \cos 2t$$

$$x''_p = -4A \cos 2t - 4B \sin 2t$$

$$-4A \cos 2t - 4B \sin 2t - 4A \sin 2t + 4B \cos 2t + 5A \cos 2t + 5B \sin 2t = 12 \cos 2t + 3 \sin 2t$$

$$\begin{cases} \cos 2t & A + 4B = 12 \\ \sin 2t & -4A + B = 3 \end{cases} \rightarrow \Delta = \begin{vmatrix} 1 & 4 \\ -4 & 1 \end{vmatrix} = 17 \quad \Delta_A = \begin{vmatrix} 12 & 4 \\ 3 & 1 \end{vmatrix} = 0 \quad \Delta_B = \begin{vmatrix} 1 & 12 \\ -4 & 3 \end{vmatrix} = 51$$

$$\rightarrow \underline{A = 0, \quad B = 3}$$

$$x_p = 3 \sin 2t \Big|$$

$$x(t) = e^{-t} (C_1 \cos 2t + C_2 \sin 2t) + 3 \sin 2t$$

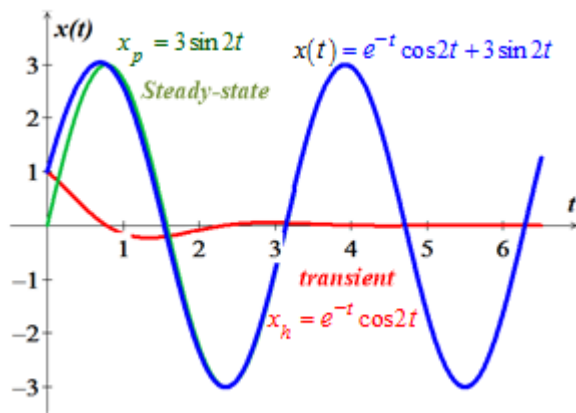
$$x(0) = 1 \rightarrow \underline{C_1 = 1}$$

$$x'(t) = e^{-t} (-C_1 \cos 2t - C_2 \sin 2t - 2C_1 \sin 2t + 2C_2 \cos 2t) + 6 \cos 2t$$

$$x'(0) = 5 \rightarrow -C_1 + 2C_2 + 6 = 5 \quad \underline{C_2 = 0}$$

$$x(t) = e^{-t} \cos 2t + 3 \sin 2t$$

b)



Exercise

A mass of 32 pounds is attached to a spring and stretched it 2 feet and then comes to rest in the equilibrium position. The surrounding medium offers a damping force that is numerically equal to 8 times the instantaneous velocity

- Find the equation of motion if the mass is driven by an external force equal to $f(t) = 8 \sin 4t$.
- Graph the transient, steady-state, and the equation of motion solutions on the same coordinate axes.

Solution

Given: $m = \frac{32}{32} = 1$ slug $c = 8$

$k(2 \text{ ft}) = 32 \rightarrow k = 16$ $kx = mg$

a) $x'' + 8x' + 16x = 8 \sin 4t$; $x(0) = 0$, $x'(0) = 0$

$mx'' + cx' + kx = f(t)$

$\lambda^2 + 8\lambda + 16 = 0 \rightarrow \lambda_{1,2} = -4$

$x_h = (C_1 + C_2 t)e^{-4t}$ (Transient solution)

$x_p = A \cos 4t + B \sin 4t$

$x'_p = -4A \sin 4t + 4B \cos 4t$

$x''_p = -16A \cos 4t - 16B \sin 4t$

$-16A \cos 4t - 16B \sin 4t - 32A \sin 4t + 32B \cos 4t + 16A \cos 4t + 16B \sin 4t = 8 \sin 4t$

$\begin{cases} \cos 4t & 32B = 0 \\ \sin 4t & -32A = 8 \end{cases} \rightarrow A = -\frac{1}{4}, B = 0$

$x_p = -\frac{1}{4} \cos 4t$ (Steady-state solution)

$x(t) = (C_1 + C_2 t)e^{-4t} - \frac{1}{4} \cos 4t$

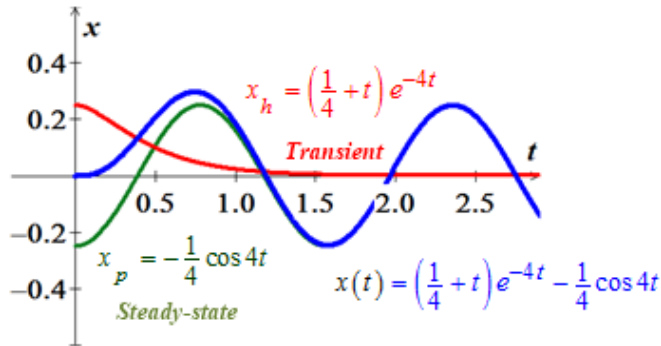
$$x(0) = 0 \rightarrow \underline{C_1 = \frac{1}{4}}$$

$$x'(t) = (C_2 - 4C_1 - 4C_2 t)e^{-4t} + \sin 4t$$

$$x'(0) = 0 \rightarrow C_2 - 4C_1 = 0 \quad \underline{C_2 = 1}$$

$$\underline{x(t) = \left(\frac{1}{4} + t\right)e^{-4t} - \frac{1}{4}\cos 4t}$$

b)



Exercise

A mass of 32 *pounds* is attached to a spring and stretched it 2 *feet* and then comes to rest in the equilibrium position. The surrounding medium offers a damping force that is numerically equal to 8 times the instantaneous velocity

a) Find the equation of motion with a starting external force equal to $f(t) = e^{-t} \sin 4t$ at $t = 0$

b) Graph the transient, steady-state, and the equation of motion solutions on the same coordinate axes.

Solution

Given: $m = \frac{32}{32} = 1$ slug $c = 8$

$$k(2 \text{ ft}) = 32 \rightarrow k = 16 \quad kx = mg$$

a) $x'' + 8x' + 16x = e^{-t} \sin 4t$; $x(0) = 0$, $x'(0) = 0$

$$mx'' + cx' + kx = f(t)$$

$$\lambda^2 + 8\lambda + 16 = 0 \rightarrow \lambda_{1,2} = -4$$

$$\underline{x_h = (C_1 + C_2 t)e^{-4t}} \quad (\text{Transient solution})$$

$$x_p = e^{-t}(A \cos 4t + B \sin 4t)$$

$$x'_p = e^{-t}(-A \cos 4t - B \sin 4t - 4A \sin 4t + 4B \cos 4t)$$

$$x''_p = e^{-t}(A \cos 4t + B \sin 4t + 4A \sin 4t - 4B \cos 4t + 4A \sin 4t - 4B \cos 4t - 16A \cos 4t - 16B \sin 4t)$$

$$= e^{-t}(-15A \cos 4t - 8B \cos 4t + 8A \sin 4t - 15B \sin 4t)$$

$$x'' + 8x' + 16x = e^{-t} \sin 4t$$

$$\begin{cases} \cos 4t & -15A - 8B - 8A + 32B + 16A = 0 \\ \sin 4t & 8A - 15B - 8B - 32A + 16B = 1 \end{cases} \rightarrow \begin{cases} -7A + 24B = 0 \\ -24B - 7B = 1 \end{cases}$$

$$\Delta = \begin{vmatrix} -7 & 24 \\ -24 & -7 \end{vmatrix} = 625 \quad \Delta_A = \begin{vmatrix} 0 & 24 \\ 1 & -7 \end{vmatrix} = -24 \quad \Delta_B = \begin{vmatrix} -7 & 0 \\ -24 & 1 \end{vmatrix} = -7$$

$$\underline{A = -\frac{24}{625}, B = -\frac{7}{625}}$$

$$\underline{x_p = e^{-t} \left(-\frac{24}{625} \cos 4t - \frac{7}{625} \sin 4t \right)} \quad (\text{Steady-state solution})$$

$$x(t) = (C_1 + C_2 t) e^{-4t} - \frac{1}{625} e^{-t} (24 \cos 4t + 7 \sin 4t)$$

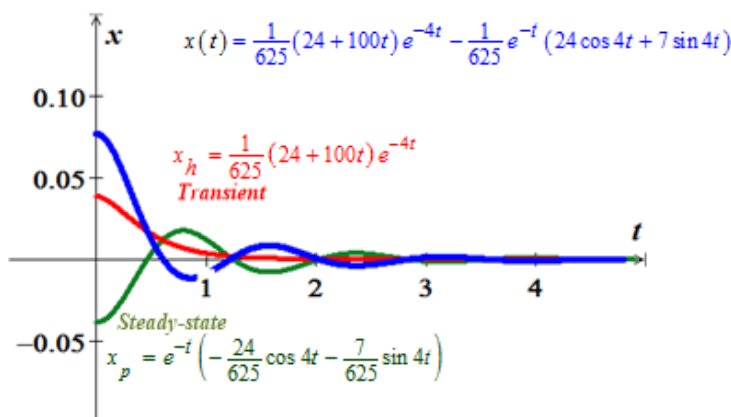
$$x(0) = 0 \rightarrow \underline{C_1 = \frac{24}{625}}$$

$$x(t) = (C_2 - 4C_1 - 4C_2 t) e^{-4t} - \frac{1}{625} e^{-t} (-24 \cos 4t - 7 \sin 4t - 96 \sin 4t + 28 \cos 4t)$$

$$x'(0) = 0 \rightarrow C_2 - 4C_1 - \frac{4}{625} = 0 \Rightarrow \underline{C_2 = \frac{100}{625}}$$

$$\underline{x(t) = \frac{1}{625} (24 + 100t) e^{-4t} - \frac{1}{625} e^{-t} (24 \cos 4t + 7 \sin 4t)}$$

b)



Exercise

A mass of 64 pounds is attached to a spring with a spring constant 32 lb/ft and then comes to rest in the equilibrium position. Neglect the damping.

- Find the equation of motion with a starting external force equal to $f(t) = 68e^{-2t} \cos 4t$ at $t = 0$
- Graph the transient, steady-state, and the equation of motion solutions on the same coordinate axes.

Solution

Given: $m = \frac{64}{32} = 2$ slugs, $k = 32$, $c = 0$

a) $2x'' + 32x = 68e^{-2t} \cos 4t$; $x(0) = 0$, $x'(0) = 0$

$$mx'' + cx' + kx = f(t)$$

$$2\lambda^2 + 32 = 0 \rightarrow \lambda_{1,2} = \pm 4i$$

$$\underline{x_h = C_1 \cos 4t + C_2 \sin 4t} \quad (\text{Transient solution})$$

$$x_p = e^{-2t} (A \cos 4t + B \sin 4t)$$

$$x'_p = e^{-2t} (-2A \cos 4t - 2B \sin 4t - 4A \sin 4t + 4B \cos 4t)$$

$$\begin{aligned} x''_p &= e^{-2t} (4A \cos 4t + 4B \sin 4t + 8A \sin 4t - 8B \cos 4t + 8A \sin 4t - 8B \cos 4t - 16A \cos 4t - 16B \sin 4t) \\ &= e^{-2t} (-12A \cos 4t - 16B \cos 4t + 16A \sin 4t - 12B \sin 4t) \end{aligned}$$

$$2x'' + 32x = 68e^{-2t} \cos 4t$$

$$\begin{cases} \cos 4t & -24A - 32B + 32A = 68 \\ \sin 4t & 32A - 24B + 32B = 0 \end{cases} \rightarrow \begin{cases} 8A - 32B = 68 \rightarrow 136A = 68 \\ 32A + 8B = 0 \quad B = -4A \end{cases}$$

$$\underline{A = \frac{1}{2}, B = -2}$$

$$\underline{x_p = e^{-2t} \left(\frac{1}{2} \cos 4t - 2 \sin 4t \right)} \quad (\text{Steady-state solution})$$

$$x(t) = C_1 \cos 4t + C_2 \sin 4t + \left(\frac{1}{2} \cos 4t - 2 \sin 4t \right) e^{-2t}$$

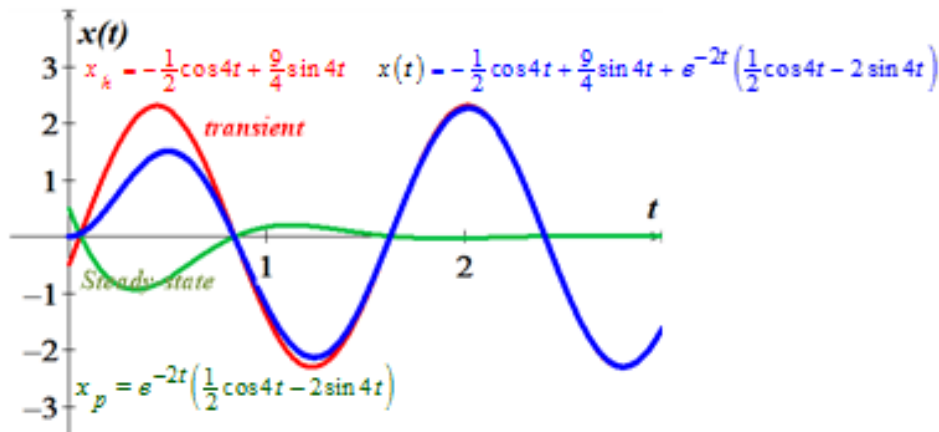
$$x(0) = 0 \rightarrow \underline{C_1 = -\frac{1}{2}}$$

$$x(t) = -4C_1 \sin 4t + 4C_2 \cos 4t + (-\cos 4t + 4 \sin 4t - 2 \sin 4t - 8 \cos 4t) e^{-2t}$$

$$x'(0) = 0 \rightarrow 4C_2 - 1 - 8 = 0 \Rightarrow \underline{C_2 = \frac{9}{4}}$$

$$\underline{x(t) = -\frac{1}{2} \cos 4t + \frac{9}{4} \sin 4t + e^{-2t} \left(\frac{1}{2} \cos 4t - 2 \sin 4t \right)}$$

b)



Exercise

A 3-*kg* object is attached to a spring and stretches the spring 392 *mm* by itself. There is no damping in the system and a forcing function of the form $F(t) = 10\cos \omega t$ is attached to the object and the system will experience resonance. If the object is initially displaced 20 *cm* downward from its equilibrium position and given a velocity of 10 *cm/sec* upward find the displacement $y(t)$ at any time t .

Solution

$$k(0.392) = 3(9.8) \rightarrow k = 75 \text{ kg/m} \quad kL = mg$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{75}{3}} = 5$$

$$3y'' + 75y = 10\cos 5t \quad my'' + \mu y' + ky = 0$$

$$y'' + 25y = \frac{10}{3}\cos 5t; \quad y(0) = 0.2 \text{ m}, \quad y'(0) = -0.10 \text{ m/sec}$$

$$\lambda^2 + 25 = 0 \rightarrow \lambda_{1,2} = \pm 5i$$

$$y_h = C_1 \cos 5t + C_2 \sin 5t$$

$$y_p = At \cos 5t + Bt \sin 5t$$

$$y'_p = A \cos 5t - 5At \sin 5t + B \sin 5t + 5Bt \cos 5t$$

$$y''_p = -5A \sin 5t - 5A \sin 5t - 25At \cos 5t + 5B \cos 5t + 5B \cos 5t - 25Bt \sin 5t$$

$$3y'' + 75y = 10\cos 5t$$

$$-5A \sin 5t - 5A \sin 5t - 25At \cos 5t + 5B \sin 5t + 5B \cos 5t - 25Bt \sin 5t$$

$$\begin{cases} \cos 5t & 30B = 10 \\ \sin 5t & -30A = 0 \\ t \cos 5t & -75A + 75A \\ t \sin 5t & -75B + 75B \end{cases} \rightarrow A = 0 \quad B = \frac{1}{3}$$

$$y_p = \frac{1}{3}t \sin 5t$$

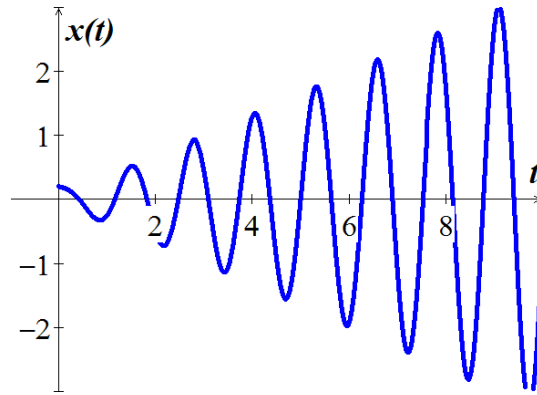
$$y(t) = C_1 \cos 5t + C_2 \sin 5t + \frac{1}{3}t \sin 5t$$

$$y(0) = 0.2 \rightarrow C_1 = 0.2 = \frac{1}{5}$$

$$y' = -5C_1 \sin 5t + 5C_2 \cos 5t + \frac{1}{3} \sin 5t + \frac{5}{3}t \cos 5t$$

$$y'(0) = -\frac{1}{10} \rightarrow 5C_2 = -\frac{1}{10} \quad C_2 = -\frac{1}{50}$$

$$y(t) = \frac{1}{5} \cos 5t - \frac{1}{50} \sin 5t + \frac{1}{3}t \sin 5t$$



Exercise

A 8-kg mass is attached to a spring hanging vertically, thereby causing the spring to stretch 1.96 m upon coming to rest at equilibrium. The damping constant is given by 3 N-sec/m .

- Find the equation of motion if the mass is driven by an external force equal to $f(t) = \cos 2t\text{ N}$ at $t = 0$.
- Determine the transient, steady-state solution of the motion

Solution

Given: $m = 8$ $k = \frac{mg}{x} = \frac{8(9.8)}{1.96} = 40$

a) $8y'' + 3y' + 40y = \cos 2t$; $y(0) = 0$, $y'(0) = 0$ $my'' + cy' + ky = F(t)$

$$8\lambda^2 + 3\lambda + 40 = 0 \rightarrow \lambda_{1,2} = -\frac{3}{2} \pm \frac{1}{2}i\sqrt{1271}$$

$$y_h = e^{-3t/2} \left(C_1 \cos \frac{\sqrt{1271}}{2} t + C_2 \sin \frac{\sqrt{1271}}{2} t \right)$$

$$y_p = A \cos 2t + B \sin 2t$$

$$y'_p = -2A \sin 2t + 2B \cos 2t$$

$$y''_p = -4A \cos 2t - 4B \sin 2t$$

$$8y'' + 3y' + 40y = \cos 2t$$

$$\begin{cases} \cos 2t & -32A + 6B + 40A = 1 \\ \sin 2t & -32B - 6A + 40B = 0 \end{cases} \rightarrow \begin{cases} 8A + 6B = 1 \\ -6A + 8B = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 8 & 6 \\ -6 & 8 \end{vmatrix} = 100 \quad \Delta_A = \begin{vmatrix} 1 & 6 \\ 0 & 8 \end{vmatrix} = 8 \quad \Delta_B = \begin{vmatrix} 8 & 1 \\ -6 & 0 \end{vmatrix} = 6$$

$$\rightarrow A = \frac{2}{25}, B = \frac{3}{50}$$

$$y_p = \frac{2}{25} \cos 2t + \frac{3}{50} \sin 2t$$

$$y(t) = e^{-3t/2} \left(C_1 \cos \frac{\sqrt{1271}}{2} t + C_2 \sin \frac{\sqrt{1271}}{2} t \right) + \frac{2}{25} \cos 2t + \frac{3}{50} \sin 2t$$

$$y(0) = 0 \rightarrow \underline{C_1 = -\frac{2}{25}}$$

$$y'(t) = e^{-3t/2} \left(-\frac{3}{2} C_1 \cos \frac{\sqrt{1271}}{2} t - \frac{3}{2} C_2 \sin \frac{\sqrt{1271}}{2} t - \frac{\sqrt{1271}}{2} C_1 \sin \frac{\sqrt{1271}}{2} t + \frac{\sqrt{1271}}{2} C_2 \cos \frac{\sqrt{1271}}{2} t \right) - \frac{4}{25} \cos 2t + \frac{6}{50} \sin 2t$$

$$y'(0) = 0 \rightarrow -\frac{3}{2} C_1 + \frac{\sqrt{1271}}{2} C_2 - \frac{4}{25} = 0 \quad \underline{C_2 = \frac{4}{25\sqrt{1271}}}$$

$$y(t) = e^{-3t/2} \left(-\frac{2}{25} \cos \frac{\sqrt{1271}}{2} t + \frac{4}{25\sqrt{1271}} \sin \frac{\sqrt{1271}}{2} t \right) + \frac{2}{25} \cos 2t + \frac{3}{50} \sin 2t$$

b) Transient solution: $y(t) = e^{-3t/2} \left(-\frac{2}{25} \cos \frac{\sqrt{1271}}{2} t + \frac{4}{25\sqrt{1271}} \sin \frac{\sqrt{1271}}{2} t \right)$

Steady-state solution: $y(t) = \frac{2}{25} \cos 2t + \frac{3}{50} \sin 2t$

Exercise

A 2-kg mass is attached to a spring hanging vertically, thereby causing the spring to stretch 0.2 m upon coming to rest at equilibrium. At $t = 0$, the mass is displaced 5 cm below the equilibrium position and released. The damping constant is given by 5 N-sec/m.

- Find the equation of motion if the mass is driven by an external force equal to $f(t) = 0.3 \cos t$ N.
- Determine the transient, steady-state solution of the motion.

Solution

Given: $m = 2$ $k = \frac{mg}{x} = \frac{2(9.8)}{0.2} = 98$ $c = 5$

a) $2y'' + 5y' + 98y = \frac{3}{10} \cos t$; $y(0) = .05$, $y'(0) = 0$ $my'' + cy' + ky = F(t)$

$$2\lambda^2 + 5\lambda + 98 = 0 \rightarrow \lambda_{1,2} = -\frac{5}{4} \pm \frac{1}{4}i\sqrt{759}$$

$$y_h = e^{-5t/4} \left(C_1 \cos \frac{\sqrt{759}}{4} t + C_2 \sin \frac{\sqrt{759}}{4} t \right)$$

$$y_p = A \cos t + B \sin t$$

$$y'_p = -A \sin t + B \cos t$$

$$y''_p = -A \cos t - B \sin t$$

$$2y'' + 5y' + 98y = \frac{3}{10} \cos t$$

$$\begin{cases} \cos 2t & -2A + 5B + 98A = 1 \\ \sin 2t & -2B - 5A + 98B = 0 \end{cases} \rightarrow \begin{cases} 96A + 5B = 1 \\ -5A + 96B = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 96 & 5 \\ -5 & 96 \end{vmatrix} = 9241 \quad \Delta_A = \begin{vmatrix} \frac{3}{10} & 5 \\ 0 & 96 \end{vmatrix} = \frac{144}{5} \quad \Delta_B = \begin{vmatrix} 96 & \frac{3}{10} \\ -5 & 0 \end{vmatrix} = \frac{3}{2}$$

$$\rightarrow A = \frac{144}{46,205}, B = \frac{3}{18,482}$$

$$y_p = \frac{144}{46,205} \cos t + \frac{3}{18,482} \sin t$$

$$y(t) = e^{-5t/4} \left(C_1 \cos \frac{\sqrt{759}}{4} t + C_2 \sin \frac{\sqrt{759}}{4} t \right) + \frac{144}{46,205} \cos t + \frac{3}{18,482} \sin t$$

$$y(0) = 0.05 = \frac{1}{20} \rightarrow C_1 + \frac{144}{46,205} = \frac{1}{20} \quad C_1 = \frac{1,733}{36,964}$$

$$y'(t) = e^{-5t/4} \left(-\frac{5}{4} C_1 \cos \frac{\sqrt{759}}{4} t - \frac{5}{4} \sin \frac{\sqrt{759}}{4} t - \frac{\sqrt{759}}{4} C_1 \sin \frac{\sqrt{759}}{4} t + \frac{\sqrt{759}}{4} C_2 \cos \frac{\sqrt{759}}{4} t \right) - \frac{144}{46,205} \sin t + \frac{3}{18,482} \cos t$$

$$y'(0) = 0 \rightarrow -\frac{5}{4} \frac{1,733}{36,964} + \frac{\sqrt{759}}{4} C_2 + \frac{3}{18,482} = 0 \quad C_2 = \frac{8,641}{36,964\sqrt{759}}$$

$$y(t) = e^{-5t/4} \left(\frac{1,733}{36,964} \cos \frac{\sqrt{759}}{4} t + \frac{8,641}{36,964\sqrt{759}} \sin \frac{\sqrt{759}}{4} t \right) + \frac{144}{46,205} \cos t + \frac{3}{18,482} \sin t$$

b) Transient solution: $y_h(t) = e^{-5t/4} \left(\frac{1,733}{36,964} \cos \frac{\sqrt{759}}{4} t + \frac{8,641}{36,964\sqrt{759}} \sin \frac{\sqrt{759}}{4} t \right)$

Steady-state solution: $y_p(t) = \frac{144}{46,205} \cos t + \frac{3}{18,482} \sin t$

Exercise

A 8-kg mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by 3 N-sec/m and the spring constant is 40 N/m. Find steady-state solution if the mass is driven by an external force equal to $f(t) = 2 \sin\left(2t + \frac{\pi}{4}\right)$ N.

Solution

Given: $m = 8$ $k = 40$ $c = 3$

$$8y'' + 3y' + 40y = 2 \sin\left(2t + \frac{\pi}{4}\right)$$

$$my'' + cy' + ky = F(t)$$

$$= 2 \left(\sin 2t \cos \frac{\pi}{4} + \cos 2t \sin \frac{\pi}{4} \right)$$

$$= \sqrt{2} \sin 2t + \sqrt{2} \cos 2t$$

$$8\lambda^2 + 3\lambda + 40 = 0 \rightarrow \lambda_{1,2} = -\frac{3}{16} \pm \frac{1}{16}i\sqrt{1271}$$

$$y_h = e^{-3t/16} \left(C_1 \cos \frac{\sqrt{1271}}{16} t + C_2 \sin \frac{\sqrt{1271}}{16} t \right)$$

$$y_p = A \cos 2t + B \sin 2t$$

$$y'_p = -2A \sin 2t + 2B \cos 2t$$

$$y''_p = -4A \cos 2t - 4B \sin 2t$$

$$8y'' + 3y' + 40y = \sqrt{2} \sin 2t + \sqrt{2} \cos 2t$$

$$\begin{cases} \cos 2t & -32A + 6B + 40A = \sqrt{2} \\ \sin 2t & -32B - 6A + 40B = \sqrt{2} \end{cases} \rightarrow \begin{cases} 8A + 6B = \sqrt{2} \\ -6A + 8B = \sqrt{2} \end{cases}$$

$$\Delta = \begin{vmatrix} 8 & 6 \\ -6 & 8 \end{vmatrix} = 100 \quad \Delta_A = \begin{vmatrix} \sqrt{2} & 6 \\ \sqrt{2} & 8 \end{vmatrix} = 2\sqrt{2} \quad \Delta_B = \begin{vmatrix} 8 & \sqrt{2} \\ -6 & \sqrt{2} \end{vmatrix} = 14\sqrt{2}$$

$$\rightarrow A = \frac{\sqrt{2}}{50}, B = \frac{7\sqrt{2}}{50}$$

$$\text{Steady-state solution: } y_p(t) = \frac{\sqrt{2}}{50} \cos 2t + \frac{7\sqrt{2}}{50} \sin 2t$$

Exercise

A 32-*lb* mass weight is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by 2 *lb-sec/ft* and the spring constant is 5 *lb/ft*. If the mass is driven by an external force equal to $f(t) = 3 \cos 4t$ *lb* at time $t = 0$.

- Find steady-state solution.
- Determine the amplitude and frequency

Solution

$$\text{Given: } m = \frac{32}{32} = 1 \quad c = 2 \quad k = 5$$

$$a) \quad y'' + 2y' + 5y = 3 \cos 4t$$

$$my'' + cy' + ky = F(t)$$

$$\lambda^2 + 2\lambda + 5 = 0 \rightarrow \lambda_{1,2} = -1 \pm 2i$$

$$y_h = e^{-t} (C_1 \cos 2t + C_2 \sin 2t)$$

$$y_p = A \cos 4t + B \sin 4t$$

$$y'_p = -4A \sin 4t + 4B \cos 4t$$

$$y''_p = -16A \cos 4t - 16B \sin 4t$$

$$y'' + 2y' + 5y = 3\cos 4t$$

$$\begin{cases} \cos 4t & -16A + 8B + 5A = 3 \\ \sin 4t & -16B - 8A + 5B = 0 \end{cases} \rightarrow \begin{cases} -11A + 8B = 3 \\ -8A - 11B = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} -11 & 8 \\ -8 & -11 \end{vmatrix} = 185 \quad \Delta_A = \begin{vmatrix} 3 & 8 \\ 0 & -11 \end{vmatrix} = -33 \quad \Delta_B = \begin{vmatrix} -11 & 3 \\ -8 & 0 \end{vmatrix} = 24$$

$$\rightarrow A = -\frac{33}{185}, B = \frac{24}{185}$$

$$y_p(t) = -\frac{33}{185}\cos 4t + \frac{24}{185}\sin 4t$$

b) Amplitude: $A = \sqrt{\left(-\frac{33}{185}\right)^2 + \left(-\frac{24}{185}\right)^2}$

$$= \frac{9\sqrt{185}}{185} \text{ ft}$$

Frequency: $f = \frac{1}{P} = \frac{\omega}{2\pi} = \frac{4}{2\pi} = \frac{2}{\pi}$

Exercise

A 8-kg mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by 3 N-sec/m and the spring constant is 40 N/m. If the mass is driven by an external force equal to $f(t) = 2\sin 2t \cos 2t$ N.

- Find steady-state solution.
- Determine the amplitude, phase angle, period and frequency

Solution

Given: $m = 8$ $k = 40$ $c = 3$

a) $8y'' + 3y' + 40y = 2\sin 2t \cos 2t$
 $= \sin 4t$

$$my'' + cy' + ky = F(t)$$

$$8\lambda^2 + 3\lambda + 40 = 0 \rightarrow \lambda_{1,2} = -\frac{3}{16} \pm \frac{1}{16}i\sqrt{1271}$$

$$y_h = e^{-3t/16} \left(C_1 \cos \frac{\sqrt{1271}}{16} t + C_2 \sin \frac{\sqrt{1271}}{16} t \right)$$

$$y_p = A \cos 4t + B \sin 4t$$

$$y'_p = -4A \sin 4t + 4B \cos 4t$$

$$y''_p = -16A \cos 4t - 16B \sin 4t$$

$$8y'' + 3y' + 40y = \sin 4t$$

$$\begin{cases} \cos 4t & -128A + 12B + 40A = 0 \\ \sin 4t & -128B - 12A + 40B = 1 \end{cases} \rightarrow \begin{cases} -88A + 12B = 0 \\ -12A - 88B = 1 \end{cases}$$

$$\Delta = \begin{vmatrix} -88 & 12 \\ -12 & -88 \end{vmatrix} = 7,888 \quad \Delta_A = \begin{vmatrix} 0 & 12 \\ 1 & -88 \end{vmatrix} = -12 \quad \Delta_B = \begin{vmatrix} -88 & 0 \\ -12 & 1 \end{vmatrix} = -88$$

$$\rightarrow A = -\frac{3}{1972}, B = -\frac{11}{986}$$

$$y_p(t) = -\frac{3}{1972} \cos 4t - \frac{11}{986} \sin 4t$$

c) Amplitude: $A = \sqrt{\left(-\frac{3}{1972}\right)^2 + \left(-\frac{11}{986}\right)^2} = 0.01 \text{ m}$

Phase angle: $\phi = 2\pi - \arctan\left(\frac{3}{1972} \cdot \frac{986}{11}\right) \approx 6.148 \text{ rad}$

Period: $P = \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2}$

Frequency: $f = \frac{1}{P} = \frac{2}{\pi}$

$$y_p(t) = 0.01 \sin(4t + 6.148)$$

Exercise

A 10-kg mass is attached to a spring hanging vertically stretches the spring 0.098 m from its equilibrium rest position, measured positive in the downward direction. At time $t = 0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t) = 20 \cos 10t \text{ N}$. (t in seconds)

- Determine the spring constant k .
- Find the equation of motion.
- Plot the equation of motion.
- Determine the maximum excursion from equilibrium made of the object on the t -interval $0 \leq t < \infty$

Solution

a) $k = \frac{10(9.8)}{0.098} = 1,000 \text{ N/m}$ $ky = mg$

b) $10y'' + 1000y = 20 \cos 10t$; $y(0) = 0$, $y'(0) = 0$ $my'' + cy' + ky = F(t)$

$$10\lambda^2 + 1000 = 0 \rightarrow \lambda_{1,2} = \pm 10i$$

$$y_h = C_1 \cos 10t + C_2 \sin 10t$$

$$y_p = At \cos 10t + Bt \sin 10t$$

$$y'_p = A \cos 10t + B \sin 10t - 10At \sin 10t + 10Bt \cos 10t$$

$$= (A + 10Bt)\cos 10t + (-10At + B)\sin 10t$$

$$y_p'' = 10B\cos 10t - 10A\sin 10t - (10A + 100Bt)\sin 10t + (-100At + 10B)\cos 10t$$

$$= (-100At + 20B)\cos 10t - (20A + 100Bt)\sin 10t$$

$$y'' + 100y = 2\cos 10t$$

$$\begin{aligned} \cos 10t &\rightarrow \begin{cases} t & -100A + 100A = 0 \\ t^0 & 20B = 2 \end{cases} \\ \sin 10t &\rightarrow \begin{cases} t & -100B + 100B = 0 \\ t^0 & -20A = 0 \end{cases} \end{aligned} \Rightarrow \underline{A = 0, B = \frac{1}{10}}$$

$$\underline{y_p = \frac{1}{10}t\sin 10t}$$

$$y(t) = C_1 \cos 10t + C_2 \sin 10t + \frac{1}{10}t\sin 10t$$

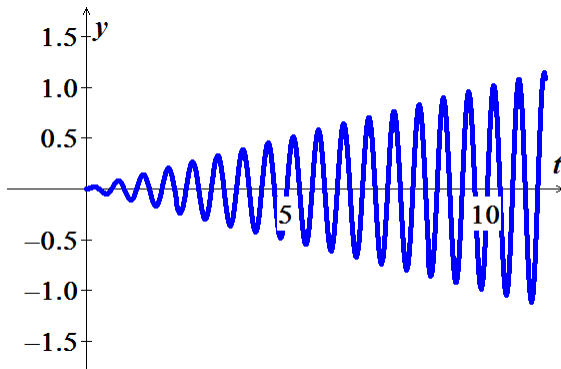
$$y(0) = 0 \rightarrow \underline{C_1 = 0}$$

$$y' = -10C_1 \sin 10t + 10C_2 \cos 10t + \frac{1}{10}\sin 10t - 10t\cos 10t$$

$$y'(0) = 0 \rightarrow 10C_2 = 0 \quad \underline{C_2 = 0}$$

$$\underline{y(t) = \frac{1}{10}t\sin 10t \text{ m}}$$

c)



d) There is no maximum excursion.

Exercise

A 10-kg mass is attached to a spring hanging vertically stretches the spring 0.098 m from its equilibrium rest position, measured positive in the downward direction. At time $t = 0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t) = 20\cos 8t \text{ N}$. (t in seconds)

- Determine the spring constant k .
- Find the equation of motion.
- Plot the equation of motion.

d) Determine the maximum excursion from equilibrium made of the object on the t -interval $0 \leq t < \infty$

Solution

a) $k = \frac{10(9.8)}{0.098} = \underline{1,000 \text{ N/m}}$ $ky = mg$

b) $10y'' + 1000y = 20\cos 8t$; $y(0) = 0$, $y'(0) = 0$ $my'' + cy' + ky = F(t)$

$$10\lambda^2 + 1000 = 0 \rightarrow \lambda_{1,2} = \pm 10i$$

$$y_h = C_1 \cos 10t + C_2 \sin 10t$$

$$y_p = A \cos 8t + B \sin 8t$$

$$y'_p = -8A \sin 8t + 8B \cos 8t$$

$$y_p = -64A \cos 8t - 64B \sin 8t$$

$$y'' + 100y = 2\cos 8t$$

$$\cos 8t \rightarrow -64A + 100A = 2$$

$$\sin 8t \rightarrow -64B + 100B = 0$$

$$\Rightarrow A = \frac{1}{18}, B = 0$$

$$y_p = \frac{1}{18} \cos 8t$$

$$y(t) = C_1 \cos 10t + C_2 \sin 10t + \frac{1}{18} \cos 8t$$

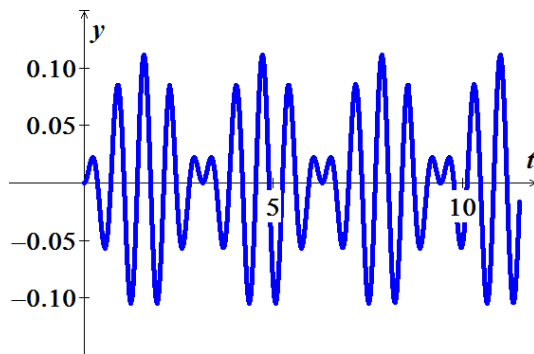
$$y(0) = 0 \rightarrow C_1 = -\frac{1}{18}$$

$$y' = -10C_1 \sin 10t + 10C_2 \cos 10t - \frac{1}{18} \sin 8t$$

$$y'(0) = 0 \rightarrow 10C_2 = 0 \quad C_2 = 0$$

$$y(t) = -\frac{1}{18} \cos 10t + \frac{1}{18} \cos 8t \quad m$$

c)



d) $y(t) = \frac{1}{18}(\cos 8t - \cos 10t)$

$$= -\frac{1}{9} \sin 9t \sin(-t)$$

$$\cos \alpha - \cos \beta = -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right)$$

$$= \frac{1}{9} \sin 9t \sin t$$

$$\left| y_{max} \right| = \frac{1}{9} \approx 0.1111 \text{ m}$$

1.568	0.11108
1.570	0.11111
1.572	0.11110

Exercise

A 10- kg mass is attached to a spring hanging vertically stretches the spring 0.098 m from its equilibrium rest position, measured positive in the downward direction. At time $t = 0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t) = 20e^{-t}$ N . (t in seconds)

- Determine the spring constant k .
- Find the equation of motion.
- Plot the equation of motion.
- Determine the maximum excursion from equilibrium made of the object on the t -interval $0 \leq t < \infty$

Solution

$$a) \quad k = \frac{10(9.8)}{0.098} = 1,000 \text{ N/m} \quad ky = mg$$

$$b) \quad 10y'' + 1000y = 20e^{-t}; \quad y(0) = 0, \quad y'(0) = 0 \quad my'' + cy' + ky = F(t)$$

$$10\lambda^2 + 1000 = 0 \rightarrow \lambda_{1,2} = \pm 10i$$

$$y_h = C_1 \cos 10t + C_2 \sin 10t$$

$$y_p = Ae^{-t}$$

$$y'_p = -Ae^{-t}$$

$$y_p = Ae^{-t}$$

$$y'' + 100y = 2e^{-t}$$

$$A + 100A = 2 \rightarrow A = \frac{2}{101}$$

$$y_p = \frac{2}{101} e^{-t}$$

$$y(t) = C_1 \cos 10t + C_2 \sin 10t + \frac{2}{101} e^{-t}$$

$$y(0) = 0 \rightarrow C_1 = -\frac{2}{101}$$

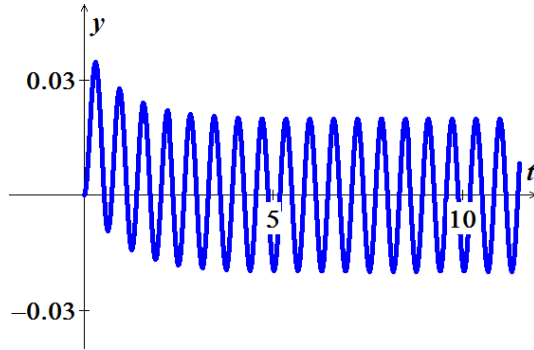
$$y' = -10C_1 \sin 10t + 10C_2 \cos 10t - \frac{2}{101} e^{-t}$$

$$y'(0) = 0 \rightarrow 10C_2 \cos 10t - \frac{2}{101} = 0 \Rightarrow C_2 = \frac{1}{505}$$

$$y(t) = -\frac{2}{101} \cos 10t + \frac{1}{505} \sin 10t + \frac{2}{101} e^{-t}$$

$$= \frac{1}{505} (\sin 10t - 10 \cos 10t + 10e^{-t}) \text{ m}$$

c)



d) $|y_{\max}| \approx 0.03456 \text{ m}$

0.29320	0.03455
0.29440	0.03456
0.29560	0.03456
0.29680	0.03456
0.29800	0.03456
0.29920	0.03456
0.30040	0.03455

Exercise

A 2-kg mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by $c = 8 \text{ kg/sec}$ and the spring constant is $k = 80 \text{ N/m}$. At time $t = 0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t) = 20 \cos 8t \text{ N}$. (t in seconds)

- Find the equation of motion.
- Plot the equation of motion.
- Determine the long-time behavior of the system, as $t \rightarrow \infty$

Solution

a) $2y'' + 8y' + 80y = 20 \cos 8t$; $y(0) = 0$, $y'(0) = 0$ $my'' + cy' + ky = F(t)$

$$2\lambda^2 + 8\lambda + 80 = \lambda^2 + 4\lambda + 40 = 0 \rightarrow \lambda_{1,2} = -2 \pm 6i$$

$$y_h = e^{-2t} (C_1 \cos 6t + C_2 \sin 6t)$$

$$y_p = A \cos 8t + B \sin 8t$$

$$y'_p = -8A \sin 8t + 8B \cos 8t$$

$$y_p = -64A \cos 8t - 64B \sin 8t$$

$$y'' + 4y' + 40y = 10 \cos 8t$$

$$\begin{aligned} \cos 8t &\rightarrow -64A + 32B + 40A = 10 \\ \sin 8t &\rightarrow -64B - 32A + 40B = 0 \end{aligned} \Rightarrow \begin{cases} -12A + 16B = 5 \\ -4A - 3B = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} -12 & 16 \\ -4 & -3 \end{vmatrix} = 100 \quad \Delta_A = \begin{vmatrix} 5 & 16 \\ 0 & -3 \end{vmatrix} = -15 \quad \Delta_B = \begin{vmatrix} -12 & 5 \\ -4 & 0 \end{vmatrix} = 20$$

$$\Rightarrow \underline{A = -\frac{3}{20}, B = \frac{1}{5}}$$

$$\underline{y_p = -\frac{3}{20} \cos 8t + \frac{1}{5} \sin 8t}$$

$$y(t) = e^{-2t} (C_1 \cos 6t + C_2 \sin 6t) - \frac{3}{20} \cos 8t + \frac{1}{5} \sin 8t$$

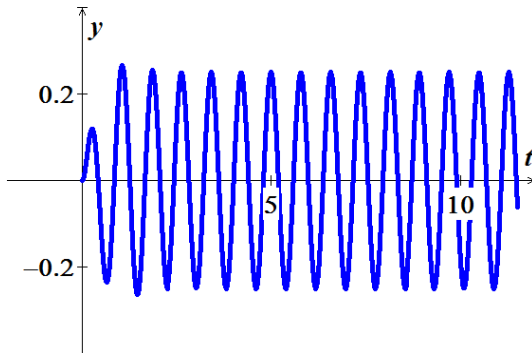
$$y(0) = 0 \rightarrow \underline{C_1 = \frac{3}{20}}$$

$$y' = e^{-2t} (-2C_1 \cos 6t - 2C_2 \sin 6t - 6C_1 \sin 6t + 6C_2 \cos 6t) + \frac{6}{5} \sin 8t + \frac{8}{5} \cos 8t$$

$$y'(0) = 0 \rightarrow -2\left(\frac{3}{20}\right) + 6C_2 + \frac{8}{5} = 0 \quad \underline{C_2 = -\frac{13}{60}}$$

$$\underline{y(t) = e^{-2t} \left(\frac{3}{20} \cos 6t - \frac{13}{60} \sin 6t \right) - \frac{3}{20} \cos 8t + \frac{1}{5} \sin 8t}$$

b)



$$\begin{aligned} c) \quad \lim_{t \rightarrow \infty} \left(e^{-2t} \left(\frac{3}{20} \cos 6t - \frac{13}{60} \sin 6t \right) - \frac{3}{20} \cos 8t + \frac{1}{5} \sin 8t \right) &= \lim_{t \rightarrow \infty} \left(-\frac{3}{20} \cos 8t + \frac{1}{5} \sin 8t \right) \\ &= \text{doesn't exist} \end{aligned}$$

The equation of motion is a steady-state solution.

Exercise

A 2-kg mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by $c = 8 \text{ kg/sec}$ and the spring constant is $k = 80 \text{ N/m}$. At time $t = 0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t) = 20 \sin 6t \text{ N}$. (t in seconds)

- Find the equation of motion.
- Plot the equation of motion.
- Determine the long-time behavior of the system, as $t \rightarrow \infty$

Solution

$$a) \quad 2y'' + 8y' + 80y = 20 \sin 6t; \quad y(0) = 0, \quad y'(0) = 0 \quad my'' + cy' + ky = F(t)$$

$$2\lambda^2 + 8\lambda + 80 = \lambda^2 + 4\lambda + 40 = 0 \rightarrow \lambda_{1,2} = -2 \pm 6i$$

$$y_h = e^{-2t} (C_1 \cos 6t + C_2 \sin 6t)$$

$$y_p = A \cos 6t + B \sin 6t$$

$$y'_p = -6A \sin 6t + 6B \cos 6t$$

$$y''_p = -36A \cos 6t - 36B \sin 6t$$

$$y'' + 4y' + 40y = 10 \sin 6t$$

$$\begin{aligned} \cos 6t &\rightarrow -36A + 24B + 40A = 0 \\ \sin 6t &\rightarrow -36B - 24A + 40B = 10 \end{aligned} \Rightarrow \begin{cases} A + 6B = 0 \\ -12A + 2B = 5 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 6 \\ -12 & 2 \end{vmatrix} = 74 \quad \Delta_A = \begin{vmatrix} 0 & 6 \\ 5 & 2 \end{vmatrix} = -30 \quad \Delta_B = \begin{vmatrix} 1 & 0 \\ -12 & 5 \end{vmatrix} = 5$$

$$\Rightarrow A = -\frac{15}{37}, B = \frac{5}{74}$$

$$y_p = -\frac{15}{37} \cos 6t + \frac{5}{74} \sin 6t$$

$$y(t) = e^{-2t} (C_1 \cos 6t + C_2 \sin 6t) - \frac{15}{37} \cos 6t + \frac{5}{74} \sin 6t$$

$$y(0) = 0 \rightarrow C_1 = \frac{15}{37}$$

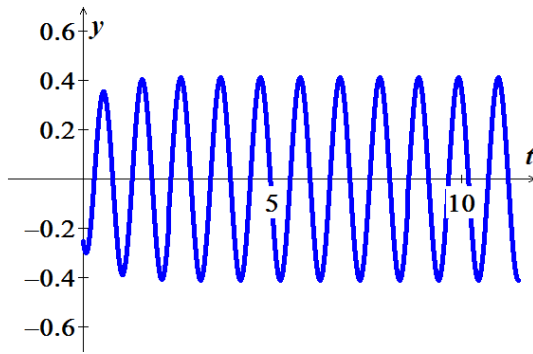
$$y' = e^{-2t} (-2C_1 \cos 6t - 2C_2 \sin 6t - 6C_1 \sin 6t + 6C_2 \cos 6t) + \frac{90}{37} \sin 6t + \frac{30}{74} \cos 6t$$

$$y'(0) = 0 \rightarrow -2\left(\frac{30}{74}\right) + 6C_2 + \frac{30}{74} = 0$$

$$\rightarrow C_2 = \frac{5}{74}$$

$$y(t) = e^{-2t} \left(\frac{15}{37} \cos 6t + \frac{5}{74} \sin 6t \right) - \frac{15}{37} \cos 6t + \frac{5}{74} \sin 6t$$

b)



$$c) \lim_{t \rightarrow \infty} \left(e^{-2t} \left(\frac{3}{20} \cos 6t - \frac{13}{60} \sin 6t \right) - \frac{15}{37} \cos 6t + \frac{5}{74} \sin 6t \right) = \lim_{t \rightarrow \infty} \left(-\frac{15}{37} \cos 6t + \frac{5}{74} \sin 6t \right) = \text{doesn't exist}$$

The equation of motion is a steady-state solution.

Exercise

A 2-kg mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by $c = 8 \text{ kg/sec}$ and the spring constant is $k = 80 \text{ N/m}$. At time $t = 0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t) = 20e^{-t} \text{ N}$. (t in seconds)

- Find the equation of motion.
- Plot the equation of motion.
- Determine the long-time behavior of the system, as $t \rightarrow \infty$

Solution

$$a) \quad 2y'' + 8y' + 80y = 20e^{-t} ; \quad y(0) = 0, \quad y'(0) = 0 \quad my'' + cy' + ky = F(t)$$

$$2\lambda^2 + 8\lambda + 80 = \lambda^2 + 4\lambda + 40 = 0 \rightarrow \lambda_{1,2} = -2 \pm 6i$$

$$y_h = e^{-2t} (C_1 \cos 6t + C_2 \sin 6t)$$

$$y_p = Ae^{-t}$$

$$y'_p = -Ae^{-t}$$

$$y''_p = Ae^{-t}$$

$$y'' + 4y' + 40y = 10e^{-t}$$

$$e^{-t} \quad A - 4A + 40A = 10 \Rightarrow A = \frac{10}{37}$$

$$y_p = \frac{10}{37} e^{-t}$$

$$y(t) = e^{-2t} (C_1 \cos 6t + C_2 \sin 6t) + \frac{10}{37} e^{-t}$$

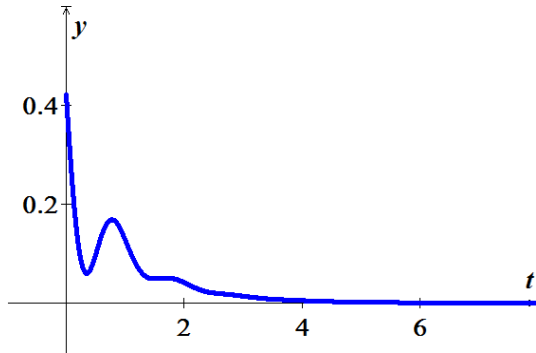
$$y(0) = 0 \rightarrow \underline{C_1 = -\frac{10}{37}}$$

$$y' = e^{-2t} \left(-2C_1 \cos 6t - 2C_2 \sin 6t - 6C_1 \sin 6t + 6C_2 \cos 6t \right) - \frac{10}{37} e^{-t}$$

$$y'(0) = 0 \rightarrow -2\left(-\frac{10}{37}\right) + 6C_2 - \frac{10}{37} = 0 \quad \underline{C_2 = \frac{5}{111}}$$

$$y(t) = e^{-2t} \left(-\frac{10}{37} \cos 6t - \frac{5}{111} \sin 6t \right) + \frac{10}{37} e^{-t}$$

b)



$$c) \lim_{t \rightarrow \infty} \left(e^{-2t} \left(-\frac{10}{37} \cos 6t - \frac{5}{111} \sin 6t \right) + \frac{10}{37} e^{-t} \right) = 0$$

Exercise

A 10-kg mass is attached to a spring having a spring constant of 140 N/m. The mass is started in motion initially from the equilibrium position with an initial velocity 1 m/sec in the upward direction and with an applied external force $F(t) = 5 \sin t$. If the force due to air resistance is $-90y' \text{ N}$.

- Find the equation motion of the mass.
- Plot the motion
- Determine the motion of the solution.

Solution

$$a) 10y'' + 90y' + 140y = 5 \sin t$$

$$y'' + 9y' + 14y = \frac{1}{2} \sin t ; \quad y(0) = 0, \quad y'(0) = -1$$

$$\lambda^2 + 9\lambda + 14 = 0 \rightarrow \underline{\lambda_{1,2} = -2, -7}$$

$$y_h = C_1 e^{-2t} + C_2 e^{-7t}$$

$$y_p = A \cos t + B \sin t$$

$$y'_p = -A \sin t + B \cos t$$

$$y_p = -A \cos t - B \sin t$$

$$y'' + 9y' + 14y = \frac{1}{2} \sin t$$

$$\begin{cases} \text{cost} & -A + 9B + 14A = 0 \\ \text{sint} & -B - 9A + 14B = \frac{1}{2} \end{cases} \rightarrow \begin{cases} 13A + 9B = 0 \\ -9A + 13B = \frac{1}{2} \end{cases}$$

$$\Delta = \begin{vmatrix} 13 & 9 \\ -9 & 13 \end{vmatrix} = 250 \quad \Delta_A = \begin{vmatrix} 0 & 9 \\ \frac{1}{2} & 13 \end{vmatrix} = -\frac{9}{2} \quad \Delta_B = \begin{vmatrix} 13 & 0 \\ -9 & \frac{1}{2} \end{vmatrix} = \frac{13}{2}$$

$$\underline{A = -\frac{9}{500}, \quad B = \frac{13}{500}}$$

$$\underline{y_p = -\frac{9}{500}\cos t + \frac{13}{500}\sin t}$$

$$y(t) = C_1 e^{-2t} + C_2 e^{-7t} + \frac{13}{500}\sin t - \frac{9}{500}\cos t$$

$$\textcolor{red}{y(0) = 0} \rightarrow C_1 + C_2 = \frac{9}{500}$$

$$y' = -2C_1 e^{-2t} - 7C_2 e^{-7t} + \frac{13}{500}\cos t + \frac{9}{500}\sin t$$

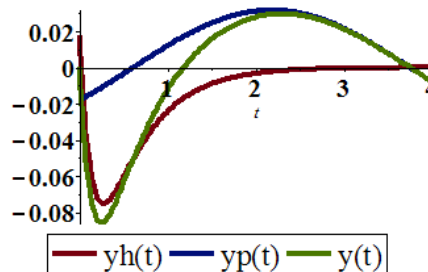
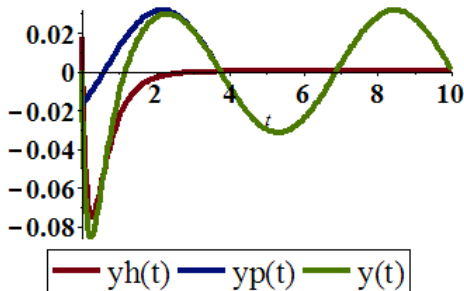
$$\textcolor{red}{y'(0) = -1} \rightarrow -2C_1 - 7C_2 + \frac{13}{500} = -1 \Rightarrow 2C_1 + 7C_2 = \frac{513}{500}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 2 & 7 \end{vmatrix} = 5 \quad \Delta_{C_1} = \begin{vmatrix} \frac{9}{500} & 1 \\ \frac{513}{500} & 7 \end{vmatrix} = -\frac{450}{500} \quad \Delta_{C_2} = \begin{vmatrix} 1 & \frac{9}{500} \\ 2 & \frac{513}{500} \end{vmatrix} = \frac{495}{500}$$

$$\underline{C_1 = -\frac{9}{50}, \quad C_2 = \frac{99}{500}}$$

$$\begin{aligned} y(t) &= -\frac{9}{50}e^{-2t} + \frac{99}{500}e^{-7t} + \frac{13}{500}\sin t - \frac{9}{500}\cos t \\ &= \frac{1}{500} \left(99e^{-7t} - 90e^{-2t} + 13\sin t - 9\cos t \right) \end{aligned}$$

b)



c) $\lim_{t \rightarrow \infty} y(t) = \lim_{t \rightarrow \infty} \left(-\frac{9}{50}e^{-2t} + \frac{99}{500}e^{-7t} + \frac{13}{500}\sin t - \frac{9}{500}\cos t \right) = \textcolor{blue}{doesn't exist}$

The homogeneous equation y_h are transient part of the solution which quickly die out.

They are steady-state part of the solution.

Exercise

A 128-*lb* weight is attached to a spring having a spring constant of 64 *lb/ft*. The weight is started in motion initially by displacing it 6 *in* above the equilibrium position with no initial velocity and with an applied external force $F(t) = 8\sin 4t$. Assume no air resistance.

- Find the equation motion of the mass.
- Plot the motion.
- Determine the motion of the solution.

Solution

$$a) \quad m = \frac{128}{32} = 4$$

$$4y'' + 64y = 8\sin 4t$$

$$y'' + 16y = 2\sin 4t; \quad y(0) = -\frac{6}{12} = -\frac{1}{2}, \quad y'(0) = 0$$

$$\lambda^2 + 16 = 0 \rightarrow \lambda_{1,2} = \pm 4i$$

$$y_h = C_1 \cos 4t + C_2 \sin 4t$$

$$y_p = At \cos 4t + Bt \sin 4t$$

$$y'_p = A \cos 4t - 4At \sin 4t + B \sin 4t + 4Bt \cos 4t$$

$$= (A + 4Bt) \cos 4t + (-4At + B) \sin 4t$$

$$y''_p = 4B \cos 4t - 4A \sin 4t - (4A + 16Bt) \sin 4t + (-16At + 4B) \cos 4t$$

$$= (-16At + 8B) \cos 4t - (8A + 16Bt) \sin 4t$$

$$y'' + 16y = 2\sin 4t$$

$$\begin{cases} \cos 4t & t & -16A + 16A = 0 \\ & t^0 & 8B = 0 \end{cases} \quad \underline{B = 0}$$

$$\begin{cases} \sin 4t & t & -16B + 16B = 0 \\ & t^0 & -8A = 2 \end{cases} \quad \underline{A = -\frac{1}{4}}$$

$$y_p = -\frac{1}{4}t \cos 4t$$

$$y(t) = C_1 \cos 4t + C_2 \sin 4t - \frac{1}{4}t \cos 4t$$

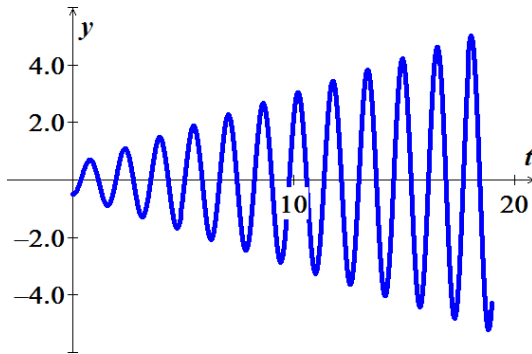
$$y(0) = -\frac{1}{2} \rightarrow \underline{C_1 = -\frac{1}{2}}$$

$$y' = -4C_1 \sin 4t + 4C_2 \cos 4t - \frac{1}{4} \cos 4t + t \sin 4t$$

$$y'(0) = 0 \rightarrow \underline{C_2 = \frac{1}{16}}$$

$$y(t) = -\frac{1}{2}\cos 4t + \frac{1}{16}\sin 4t - \frac{1}{4}t\cos 4t$$

b)



c) $\lim_{t \rightarrow \infty} y(t) = \lim_{t \rightarrow \infty} \left(-\frac{1}{2}\cos 4t + \frac{1}{16}\sin 4t - \frac{1}{4}t\cos 4t \right) = \infty$

This is a pure resonance,

Exercise

A 3-kg object is attached to spring and stretches the spring 39.2 cm by itself. There is no damping in the system and a forcing function is given by $F(t) = 10\cos \omega t$ is attached to the object and the system will experience resonance. If the object is initially displaced 20 cm downward from its equilibrium position and given a velocity of 10 cm/sec upward.

- Find the spring constant k .
- Find the natural frequency ω .
- Find the displacement at any time t .
- Sketch the displacement function.

Solution

a) $k = \frac{3(9.8)}{0.392} = 75 \text{ N/m}$ $k = \frac{mg}{y}$

b) $\omega = \sqrt{\frac{75}{3}} = 5$ $\omega = \sqrt{\frac{k}{m}}$

c) $3y'' + 75y = 10\cos 5t$; $y(0) = 0.2$, $y'(0) = -0.1$

$$3\lambda^2 + 75 = 0 \rightarrow \lambda_{1,2} = \pm 5i$$

$$y_h(t) = C_1 \cos 5t + C_2 \sin 5t$$

$$y_p = At \cos 5t + Bt \sin 5t$$

$$y'_p = A \cos 5t - 5At \sin 5t + B \sin 5t + 5Bt \cos 5t$$

$$= (A + 5Bt) \cos 5t + (-5At + B) \sin 5t$$

$$y_p'' = 5B \cos 5t - 5A \sin 5t - (5A + 25Bt) \sin 5t + (-25At + 5B) \cos 5t$$

$$= (-25At + 10B) \cos 5t - (10A + 25Bt) \sin 5t$$

$$3y'' + 75y = 10 \cos 5t$$

$$\begin{cases} \cos 5t & t & -75A + 75A = 0 \\ & t^0 & 30B = 10 \end{cases} \quad \underline{B = \frac{1}{3}}$$

$$\begin{cases} \sin 5t & t & -75B + 75B = 0 \\ & t^0 & -30A = 0 \end{cases} \quad \underline{A = 0}$$

$$\underline{y_p(t) = \frac{1}{3}t \sin 5t}$$

$$y(t) = C_1 \cos 5t + C_2 \sin 5t + \frac{1}{3}t \sin 5t$$

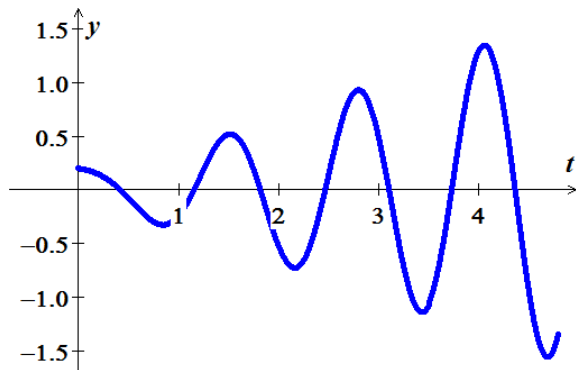
$$\underline{y(0) = \frac{1}{5} \rightarrow C_1 = \frac{1}{5}}$$

$$y' = -5C_1 \sin 5t + 5C_2 \cos 5t + \frac{1}{3} \sin 5t + \frac{5}{3}t \cos 5t$$

$$\underline{y'(0) = -\frac{1}{10} \rightarrow C_2 = -\frac{1}{50}}$$

$$\underline{y(t) = \frac{1}{5} \cos 5t - \frac{1}{50} \sin 5t + \frac{1}{3}t \sin 5t}$$

d)



Exercise

Find the transient motion and steady periodic oscillations of a damped mass-and-spring system with $m = 1$, $c = 2$, and $k = 26$ under the influence of an external force $F(t) = 82 \cos 4t$ with $x(0) = 6$ and $x'(0) = 0$. Also investigate the possibility of practical resonance for this system.

Solution

Given: $m = 1$, $c = 2$, $k = 26$, and $F(t) = 82 \cos 4t$ $x(0) = 6$; $x'(0) = 0$

$$x'' + 2x' + 26x = 82 \cos 4t \qquad mx'' + cx' + kx = F(t)$$

$$\lambda^2 + 2\lambda + 26 = 0 \quad \lambda_{1,2} = \frac{-2 \pm \sqrt{-100}}{2} = -1 \pm 5i$$

$$\underline{x_h = e^{-t} (C_1 \cos 5t + C_2 \sin 5t)}$$

$$x_p = A \cos 4t + B \sin 4t$$

$$x'_p = -4A \sin 4t + 4B \cos 4t$$

$$x''_p = -16A \cos 4t - 16B \sin 4t$$

$$x'' + 2x' + 26x = 82 \cos 4t$$

$$-16A \cos 4t - 16B \sin 4t - 8A \sin 4t + 8B \cos 4t + 26A \cos 4t + 26B \sin 4t = 82 \cos 4t$$

$$(10A + 8B) \cos 4t + (-8A + 10B) \sin 4t = 82 \cos 4t$$

$$\begin{cases} 10A + 8B = 82 \\ -8A + 10B = 0 \end{cases} \rightarrow \begin{cases} 5A + 4B = 41 \\ -4A + 5B = 0 \end{cases}$$

$$D = \begin{vmatrix} 5 & 4 \\ -4 & 5 \end{vmatrix} = 41 \quad D_A = \begin{vmatrix} 41 & 4 \\ 0 & 5 \end{vmatrix} = 205 \quad D_B = \begin{vmatrix} 5 & 41 \\ -4 & 0 \end{vmatrix} = 164$$

$$A = \frac{205}{41} = 5; \quad B = \frac{164}{41} = 4$$

$$\underline{x_p = 5 \cos 4t + 4 \sin 4t} \quad A = \sqrt{25 + 16} = \sqrt{41} \quad \varphi = \tan^{-1}\left(\frac{4}{5}\right) \approx 0.6747$$

$$\underline{x_p = \sqrt{41} \cos(4t - 0.6747)}$$

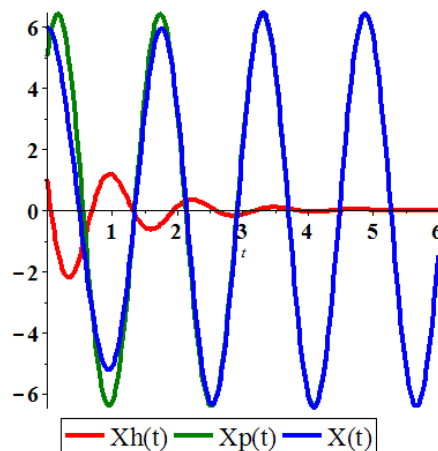
$$\underline{x(t) = e^{-t} (C_1 \cos 5t + C_2 \sin 5t) + 5 \cos 4t + 4 \sin 4t}$$

$$x(0) = C_1 + 5 = 6 \Rightarrow \underline{C_1 = 1}$$

$$x' = e^{-t} (5C_1 \sin 5t + 5C_2 \cos 5t - C_1 \cos 5t - C_2 \sin 5t) - 20 \sin 4t + 16 \cos 4t$$

$$x'(0) = 5C_2 - C_1 + 16 = 0 \rightarrow \underline{C_2 = -3}$$

$$\underline{x(t) = e^{-t} (\cos 5t - 3 \sin 5t) + 5 \cos 4t + 4 \sin 4t}$$



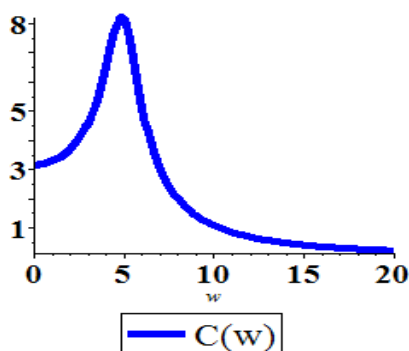
The forced amplitude at frequency ω is:

$$C(\omega) = \frac{82}{\sqrt{26^2 - 48\omega^2 + \omega^4}}$$

$$C'(\omega) = -\frac{41(4\omega^3 - 96\omega)}{(676 - 48\omega^2 + \omega^4)^{3/2}} = 0$$

$$164\omega(\omega^2 - 24) = 0 \Rightarrow \omega_{1,2,3} = 0, \pm 2\sqrt{6}$$

The mass-and-spring's undamped critical frequency of $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{26}$



Exercise

A mass m is attached to the end of a spring with a spring constant k . After the mass reaches equilibrium, its support begins to oscillate vertically about a horizontal line L according to a formula $h(t)$. The value of h represents the distance in feet measured from L .

- Determine the differential equation of motion if the entire system moves through a medium offering a damping force that is numerically equal to $\mu \frac{dx}{dt}$
- Solve the differential equation in part (a) if the spring is stretched 4 feet by a mass weighing 16 pounds and $\mu = 2$, $h(t) = 5 \cos t$, $x(0) = x'(0) = 0$

Solution

- The external force is $F(t) = kh(t)$

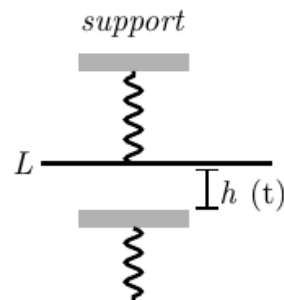
$$\underline{mx'' + \mu x' + kx = kh(t)} \quad mx'' + \mu x' + kx = F(t)$$

- Given:** $m = \frac{16}{32} = \frac{1}{2}$ slug, $k(4 \text{ ft}) = 16 \rightarrow k = 4$

$$\mu = 2, \quad h(t) = 5 \cos t, \quad x(0) = x'(0) = 0$$

$$\frac{1}{2}x'' + 2x' + 4x = 20 \cos t; \quad x(0) = x'(0) = 0$$

$$\lambda^2 + 4\lambda + 8 = 0 \rightarrow \lambda_{1,2} = -2 \pm 2i$$



$$\underline{x_h = e^{-2t} (C_1 \cos 2t + C_2 \sin 2t)}$$

$$x_p = A \cos t + B \sin t$$

$$x'_p = -A \sin t + B \cos t$$

$$x''_p = -A \cos t - B \sin t$$

$$x'' + 4x' + 8x = 40 \cos t$$

$$\begin{cases} \text{cos } t & -A + 4B + 8A = 40 \\ \text{sin } t & -B - 4A + 8B = 0 \end{cases} \rightarrow \begin{cases} 7A + 4B = 40 \\ -4A + 7B = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 7 & 4 \\ -4 & 7 \end{vmatrix} = 65 \quad \Delta_A = \begin{vmatrix} 40 & 4 \\ 0 & 7 \end{vmatrix} = 280 \quad \Delta_B = \begin{vmatrix} 7 & 40 \\ -4 & 0 \end{vmatrix} = 160$$

$$\underline{A = \frac{56}{13}, \quad B = \frac{32}{13}}$$

$$\underline{x_p = \frac{56}{13} \cos t + \frac{32}{13} \sin t}$$

$$x(t) = e^{-2t} (C_1 \cos 2t + C_2 \sin 2t) + \frac{56}{13} \cos t + \frac{32}{13} \sin t$$

$$x(0) = 0 \rightarrow \underline{C_1 = -\frac{56}{13}}$$

$$x'(t) = e^{-2t} (-2C_1 \cos 2t - 2C_2 \sin 2t - 2C_1 \sin 2t + 2C_2 \cos 2t) - \frac{56}{13} \sin t + \frac{32}{13} \cos t$$

$$x'(0) = 0 \rightarrow -2C_1 + 2C_2 + \frac{32}{13} = 0 \Rightarrow \underline{C_2 = -\frac{72}{13}}$$

$$\underline{x(t) = e^{-2t} \left(-\frac{56}{13} \cos 2t - \frac{72}{13} \sin 2t \right) + \frac{56}{13} \cos t + \frac{32}{13} \sin t}$$

Exercise

A mass m on the end of a pendulum (of length L) also attached to a horizontal spring (with constant k). Assume small oscillations of m so that the spring remains essentially horizontal and neglect damping. Find the natural circular frequency ω_0 of motion of the mass in terms of L , k , m , and the gravitational constant g .

Solution

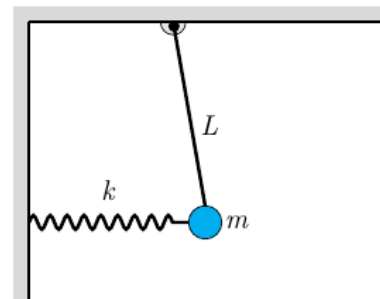
Let θ is the angular displacement.

The displacement of the mass is: $x = L\theta$

Its total energy ($KE + PE$) is

$$mv^2 + kx^2 + 2mgh = C$$

$$m(x')^2 + kx^2 + 2mgh = C$$



$$\frac{d}{dt} \left(mL^2 (\theta')^2 + kL^2 \theta^2 + 2mgL(1 - \cos \theta) \right) = \frac{d}{dt} C$$

$$2mL^2 (\theta') \theta'' + 2kL^2 \theta \theta' + 2mgL \theta' \sin \theta = 0 \quad (\theta' \neq 0)$$

$$mL^2 \theta'' + kL^2 \theta + mgL \sin \theta = 0 \quad \sin \theta \approx \theta$$

$$mL^2 \theta'' + (kL^2 + mgL) \theta = 0$$

$$\theta'' + \left(\frac{k}{m} + \frac{g}{L} \right) \theta = 0$$

$$\omega_0 = \sqrt{\frac{k}{m} + \frac{g}{L}}$$

Exercise

A mass m hangs on the end of a cord around a pullet of radius a and moment of inertia I . The rim of the pulley is attached to a spring (with constant k). Assume small oscillations so that the spring remains essentially and neglect friction. Find the natural circular frequency in terms of m , a , k , I , and g .

Solution

Let x be the displacement of the mass from its equilibrium position.

$v = x'$ be the velocity.

$\omega = \frac{v}{a}$ the angular velocity of the pulley.

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}kx^2 - mgx = C \quad \text{Conservation of energy}$$

$$\frac{d}{dt} \left(\frac{1}{2}mv^2 + \frac{1}{2} \frac{I}{a^2} v^2 + \frac{1}{2}kx^2 - mgx \right) = \frac{d}{dt} C$$

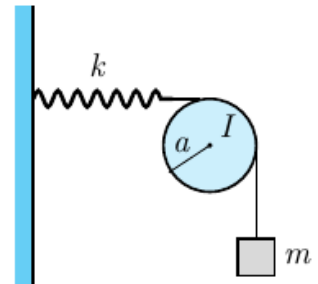
$$mvv' + \frac{I}{a^2} vv' + kxx' - mgx' = 0$$

$$\left(m + \frac{I}{a^2} \right) x'x'' + kxx' - mgx' = 0$$

$$\left(m + \frac{I}{a^2} \right) x'' + kx - mg = 0$$

$$\omega = \sqrt{\frac{k}{m + \frac{I}{a^2}}}$$

$$= a \sqrt{\frac{k}{ma^2 + I}}$$



Exercise

Consider a floating cylindrical buoy with radius r , height h , and uniform density $\rho \leq 0.5$ (recall that the density of water is 1 g/cm^3). The buoy is initially suspended at rest with its bottom at the top surface of the water and is released at time $t = 0$. Thereafter it is acted on by two forces: a downward gravitational force equal to its weight $mg = \pi r^2 h g$ and (by Archmedes' principle of buoyancy) an upward force equal to the weight $\pi r^2 x g$ of water displaced, where $x = x(t)$ is the depth of the bottom of the buoy beneath the surface at time t .

Conclude that the buoy undergoes simple harmonic motion around its equilibrium position $x_e = \rho h$ with

period $p = 2\pi \sqrt{\frac{\rho h}{g}}$.

- Compute p and the amplitude of the motion if $\rho = 0.5 \text{ g/cm}^3$, $h = 200 \text{ cm}$, and $g = 980 \text{ cm/s}^2$
- If the cylindrical buoy weighting 100 lb floats in water with its axis vertical. When depressed slightly and released, it oscillates up and down four times every 10 sec . assume that friction is negligible. Find the radius of the buoy.

Solution

a) $F = ma$

$$\rho \pi r^2 h x'' = \rho \pi r^2 h g - \pi r^2 x g \rightarrow \rho h x'' + g x - \rho h g = 0$$

$$x'' + \frac{g}{\rho h} x = g$$

$$\lambda^2 + \frac{g}{\rho h} = 0 \Rightarrow \lambda = \pm i \sqrt{\frac{g}{\rho h}}$$

$$x(t) = A \cos\left(\sqrt{\frac{g}{\rho h}} t\right) + B \sin\left(\sqrt{\frac{g}{\rho h}} t\right)$$

$$x_p = A \Rightarrow x_p'' = 0$$

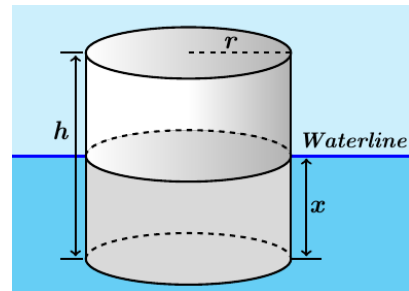
$$\frac{g}{\rho h} A = g \rightarrow A = \rho h$$

$$x(t) = A \cos\left(\sqrt{\frac{g}{\rho h}} t\right) + B \sin\left(\sqrt{\frac{g}{\rho h}} t\right) + \rho h \quad x(0) = x'(0) = 0$$

$$x(0) = A + \rho h = 0 \Rightarrow \underline{A = -\rho h}$$

$$x'(t) = -A \sqrt{\frac{g}{\rho h}} \sin\left(\sqrt{\frac{g}{\rho h}} t\right) + B \sqrt{\frac{g}{\rho h}} \cos\left(\sqrt{\frac{g}{\rho h}} t\right) \rightarrow x'(0) = B \sqrt{\frac{g}{\rho h}} = 0 \Rightarrow \underline{B = 0}$$

$$x(t) = -\rho h \cos\left(\sqrt{\frac{g}{\rho h}} t\right) + \rho h$$



$$\begin{aligned} &= \rho h (1 - \cos \omega_0 t) \quad \omega_0 = \sqrt{\frac{g}{\rho h}} = \sqrt{\frac{980}{0.5(200)}} \approx 3.13 \\ &= 100(1 - \cos(3.13t)) \end{aligned}$$

Amplitude: $A = 100 \text{ cm}$

Period: $P = \frac{2\pi}{3.13} \approx 2.01 \text{ sec}$

b) **Given:** $mg = 100$ $P = \frac{10}{4} = 2.5 \text{ sec}$

The weight of water: $\rho = 62.4 \text{ lb / ft}^3$

$$\frac{100}{32} x'' + 62.4\pi r^2 x = 100 \quad mx'' + \pi \rho r^2 x = mg$$

$$x'' + 19.968\pi r^2 x = 32 \Rightarrow \omega^2 = 19.968\pi r^2$$

$$\left(\frac{2\pi}{2.5}\right)^2 = 19.968\pi r^2$$

$$\begin{aligned} r &= \frac{1}{\sqrt{19.968\pi}} \left(\frac{2\pi}{2.5}\right) \\ &\approx 0.3173 \text{ ft} \quad \approx 3.8 \text{ in} \end{aligned}$$

Exercise

Assume that the earth is a solid sphere of uniform density, with mass M and radius $R = 3960 \text{ (mi)}$. For a particle of mass m within the earth at distance r from the center of the earth, the gravitational force

attracting m toward the center is $F_r = -\frac{GM_r m}{r^2}$, where M_r is the mass of the part of the earth within a sphere of radius r .

a) Show that $F_r = -\frac{GMmr}{R^3}$

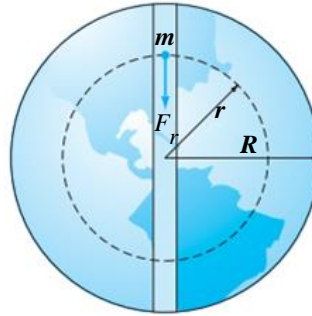
b) Now suppose that a small hole is drilled straight through the center of the earth, thus connecting two antipodal points on its surface. Let a particle of mass m be dropped at time $t = 0$ into this hole with initial speed zero, and let $r(t)$ be its distance from the center of the earth at time t . conclude from

Newton's second law and part (a) that $r''(t) = -k^2 r(t)$, where $k^2 = \frac{GM}{R^3} = \frac{g}{R}$.

- Take $g = 32.2 \text{ ft / s}^2$, and conclude from part (b) that the particle undergoes simple harmonic motion back and forth between the ends of the hole, with a period of about 84 min.
- Look up (or derive) the period of a satellite that just skims the surface of the earth; compare with the result in part (c). How do you explain the coincidence? Or is it a coincidence?
- With what speed (in miles per hours) does the particle pass through the center of the earth?
- Look up (or derive) the orbital velocity of a satellite that just skims the surface of the earth; compare with the result in part (e). How do you explain the coincidence? Or is it a coincidence?

Solution

$$\begin{aligned}
 a) \quad M_r &= M \left(\frac{r}{R} \right)^3 \\
 F_r &= -\frac{GM_r m}{r^2} \\
 &= -\frac{GM}{r^2} M \frac{r^3}{R^3} \\
 &= -\frac{GMmr}{R^3}
 \end{aligned}$$



$$\begin{aligned}
 b) \quad \text{Since } \frac{GM}{R^3} &= \frac{g}{R} \\
 mr'' &= F_r \\
 mr'' &= -\frac{GM}{R^3} mr \\
 r'' + \frac{g}{R} r &= 0
 \end{aligned}$$

$$c) \quad r'' + \frac{g}{R} r = 0$$

$$\lambda^2 + \frac{g}{R} = 0 \rightarrow \lambda = \pm i \sqrt{\frac{g}{R}}$$

$$r(t) = A \cos\left(\sqrt{\frac{g}{R}} t\right) + B \sin\left(\sqrt{\frac{g}{R}} t\right) \quad r(0) = R \quad r'(0) = 0$$

$$r(0) = A = R$$

$$r'(t) = -A \sqrt{\frac{g}{R}} \sin\left(\sqrt{\frac{g}{R}} t\right) + B \sqrt{\frac{g}{R}} \cos\left(\sqrt{\frac{g}{R}} t\right) \rightarrow r'(0) = B \sqrt{\frac{g}{R}} = 0 \Rightarrow B = 0$$

$$r(t) = R \cos \omega_0 t; \quad \omega_0 = \sqrt{\frac{g}{R}}$$

Given: $g = 32.2 \text{ ft/sec}^2$ $R = (3960)(5280) \text{ ft}$

The period of the particle's simple harmonic motion is:

$$\begin{aligned}
 P &= \frac{2\pi}{\omega_0} \\
 &= 2\pi \sqrt{\frac{R}{g}} \\
 &= 2\pi \sqrt{\frac{3960 \times 5280}{32.2}} \\
 &\approx 84.38 \text{ min}
 \end{aligned}$$

- d) The orbital velocity v of such a satellite must be such that the centrifugal force $\frac{mv^2}{R}$ on the satellite just offsets the weight mg of the satellite at the surface of the earth. Thus

$$\begin{aligned}
\frac{mv^2}{R} = mg &\Rightarrow v = \sqrt{gR} = \sqrt{32.2 \times 3960 \text{ mi} \times \frac{5280 \text{ ft}}{1 \text{ mi}}} \\
&\approx 2.5947 \times 10^4 \text{ ft / sec} \\
&\approx 2.5947 \times 10^4 \frac{\text{ft}}{\text{sec}} \frac{1 \text{ mi}}{5280 \text{ ft}} \frac{3600 \text{ sec}}{1 \text{ hr}} \\
&\approx 1.7691 \times 10^4 \text{ mi / hr}
\end{aligned}$$

Because the circumference of the earth is $2\pi R$, the period of the satellite's orbit is

$$\begin{aligned}
\frac{2\pi}{\sqrt{gR}} &= 2\pi \sqrt{\frac{R}{g}} \\
&= 2\pi \sqrt{\frac{3960 \times 5280}{32.2}} \\
&\approx 84.38 \text{ min}
\end{aligned}$$

Let assume that at time $t = 0$, the satellite is directly over the hole in the earth at the top, and its orbit proceeds in a clockwise direction.

The distance r of the particle, from part (c), from the center of the earth is

$$r(t) = R \cos \omega_0 t; \quad \omega_0 = \sqrt{\frac{g}{R}}$$

The key observation is that $\omega_0 t$ is the angle drawn clockwise from the vertical to the radius vector of the satellite at time t ; thus, the distance $r(t)$ is simply the vertical component of the satellite's position. It follows that $r(t)$ completes one cycle through the earth (and back) in the same length of time required for the satellite to complete one orbit around the earth.

- e) The particle passes through the center of the earth when $r(t) = R \cos \omega_0 t = 0$, that is when

$$\omega_0 t = \frac{\pi}{2} \rightarrow t = \frac{\pi}{2\omega_0}.$$

At this time the speed of the particle is

$$\begin{aligned}
r'(t) &= -R\omega_0 \sin \omega_0 t \\
|r'(t)| &= \left| -R\omega_0 \sin \left(\omega_0 \frac{\pi}{2\omega_0} \right) \right| \\
&= R\omega_0 \\
&= R\sqrt{\frac{g}{R}} = \sqrt{gR} \\
&\approx 1.7691 \times 10^4 \text{ mi / hr} \quad \text{part (d)}
\end{aligned}$$

- f) The orbital velocity is $v = \sqrt{gR}$.

The vertical component of the satellite's velocity vector $v(t)$ at any given time t is equal to the speed $|r'(t)|$ of the particle at that time.

At the moment when the particle passes through the center of earth, the satellite is travelling straight downward, and hence $v(t)$ is vertical.

Therefore, the orbital velocity v of the satellite, which is the magnitude of $v(t)$, is equal to the speed of the particle at this moment.

Exercise

Express the given circuit in the second-order differential equation

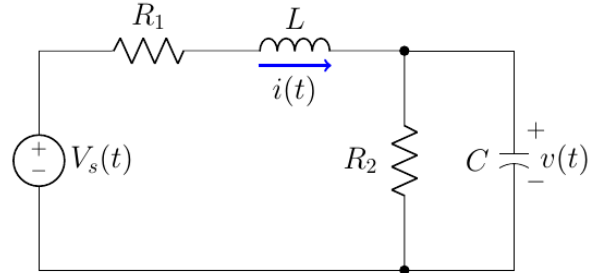
Solution

$$V_s = R_1 i + L \frac{di}{dt} + v(t)$$

$$i(t) = \frac{v(t)}{R_2} + C \frac{dv}{dt}$$

$$\begin{aligned} V_s &= R_1 \left(\frac{v}{R_2} + C \frac{dv}{dt} \right) + L \frac{d}{dt} \left(\frac{v}{R_2} + C \frac{dv}{dt} \right) + v \\ &= \frac{R_1}{R_2} v + R_1 C \frac{dv}{dt} + \frac{L}{R_2} \frac{dv}{dt} + LC \frac{d^2 v}{dt^2} + v \\ &= LC \frac{d^2 v}{dt^2} + \left(R_1 C + \frac{L}{R_2} \right) \frac{dv}{dt} + \left(\frac{R_1}{R_2} + 1 \right) v(t) \end{aligned}$$

$$\underline{= LC \frac{d^2 v}{dt^2} + \left(R_1 C + \frac{L}{R_2} \right) \frac{dv}{dt} + \left(\frac{R_1 + R_2}{R_2} \right) v(t)}$$



Exercise

Express the given circuit in the second-order differential equation

Solution

$$v(t) = R_2 i(t) + L \frac{di}{dt}$$

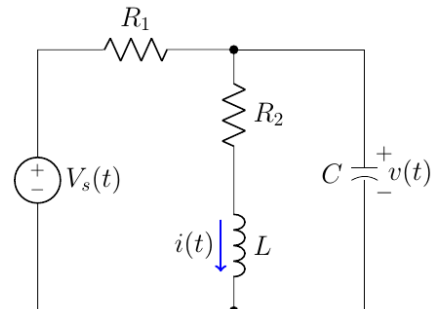
$$i_{R_1} = i(t) + i_C$$

$$= i(t) + C \frac{dv}{dt}$$

$$V_S = R_1 \left(i + C \frac{dv}{dt} \right) + v(t) \qquad V_S = V_{R_1} + V_C$$

$$= R_1 i + R_1 C \frac{d}{dt} \left(R_2 i + L \frac{di}{dt} \right) + R_2 i + L \frac{di}{dt}$$

$$= R_1 i + R_1 R_2 C \frac{di}{dt} + R_1 CL \frac{d^2 i}{dt^2} + R_2 i + L \frac{di}{dt}$$



$$= R_1 CL \frac{d^2}{dt^2} i(t) + (R_1 R_2 C + L) \frac{d}{dt} i(t) + (R_1 + R_2) i(t) \quad \Bigg|$$

Exercise

Find the steady-state solution $q_p(t)$ and the steady-state current in an LRC -series circuit when the source voltage is $E(t) = E_0 \sin \omega t$

Solution

$$Lq'' + Rq' + \frac{1}{C}q = E_0 \sin \omega t$$

$$L\lambda^2 + R\lambda + \frac{1}{C} = 0 \rightarrow \lambda_{1,2} = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L} = \frac{-R\sqrt{C} \pm \sqrt{R^2 C - 4L}}{2L\sqrt{C}}$$

$$q_p = A \cos \omega t + B \sin \omega t$$

$$q'_p = -\omega A \sin \omega t + \omega B \cos \omega t$$

$$q''_p = -\omega^2 A \cos \omega t - \omega^2 B \sin \omega t$$

$$-\omega^2 L A \cos \omega t - \omega^2 L B \sin \omega t - \omega R A \sin \omega t + \omega R B \cos \omega t + \frac{1}{C} A \cos \omega t + \frac{1}{C} B \sin \omega t = E_0 \sin \omega t$$

$$\begin{cases} \cos \omega t & \left(-\omega^2 L + \frac{1}{C}\right)A + \omega R B = 0 \\ \sin \omega t & -\omega R A + \left(\frac{1}{C} - \omega^2 L\right)B = E_0 \end{cases}$$

$$\Delta = \begin{vmatrix} \frac{1}{C} - \omega^2 L & \omega R \\ -\omega R & \frac{1}{C} - \omega^2 L \end{vmatrix} = \frac{1}{C^2} - \frac{2\omega^2 L}{C} + \omega^4 L^2 + \omega^2 R^2 = \frac{1}{C^2} (1 - 2\omega^2 LC + \omega^4 L^2 C^2 + \omega^2 R^2 C^2)$$

$$\Delta_A = \begin{vmatrix} 0 & \omega R \\ E_0 & \frac{1}{C} - \omega^2 L \end{vmatrix} = \omega R E_0 \quad \Delta_B = \begin{vmatrix} \frac{1}{C} - \omega^2 L & 0 \\ -\omega R & E_0 \end{vmatrix} = \left(\frac{1}{C} - \omega^2 L\right) E_0$$

$$A = \frac{\omega R C^2 E_0}{1 - 2\omega^2 LC + \omega^4 L^2 C^2 + \omega^2 R^2 C^2} \quad B = \frac{C(1 - \omega LC) E_0}{1 - 2\omega^2 LC + \omega^4 L^2 C^2 + \omega^2 R^2 C^2}$$

Therefore, the steady-state charge is:

$$q_p(t) = \frac{CE_0}{1 - 2\omega^2 LC + \omega^4 L^2 C^2 + \omega^2 R^2 C^2} (\omega R C \cos \omega t + (1 - \omega LC) \sin \omega t) \quad \Bigg|$$

The steady-state current is: $i_p(t) = q'_p(t)$

$$i_p(t) = \frac{CE_0}{1 - 2\omega^2 LC + \omega^4 L^2 C^2 + \omega^2 R^2 C^2} (-\omega^2 R C \sin \omega t + (\omega - \omega^2 LC) \cos \omega t) \quad \Bigg|$$

Exercise

Find the charge $q(t)$ on the capacitor in an LRC -series circuit when $L = \frac{5}{3} \text{ h}$, $R = 10 \text{ } \Omega$, $C = \frac{1}{30} \text{ f}$, $E(t) = 300 \text{ V}$, $q(0) = 0 \text{ C}$, and $i(0) = 0 \text{ A}$. Find the maximum charge on the capacitor

Solution

$$\frac{5}{3}q'' + 10q' + 30q = 300 \qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 6q' + 18q = 900; \quad q(0) = 0, \quad q'(0) = i(0) = 0$$

$$\lambda^2 + 6\lambda + 18 = 0 \rightarrow \lambda_{1,2} = -3 \pm 3i$$

$$q(t) = e^{-3t} (C_1 \cos 3t + C_2 \sin 3t)$$

$$q_p = A$$

$$q'_p = q''_p = 0$$

$$\frac{5}{3}q'' + 10q' + 30q = 300 \rightarrow 30A = 300 \Rightarrow A = 10$$

$$q(t) = e^{-3t} (C_1 \cos 3t + C_2 \sin 3t) + 10$$

$$q(0) = 0 \rightarrow C_1 = -10$$

$$q'(t) = e^{-3t} (-3C_1 \cos 3t - 3C_2 \sin 3t - 3C_1 \sin 3t + 3C_2 \cos 3t)$$

$$q'(0) = i(0) = 0 \rightarrow -3C_1 + 3C_2 = 0 \quad C_2 = -10$$

$$q(t) = 10 - 10e^{-3t} (\cos 3t + \sin 3t)$$

$$i(t) = q'(t) = -10e^{-3t} (-3\cos 3t - 3\sin 3t - 3\sin 3t + 3\cos 3t) \\ = 60e^{-3t} \sin 3t$$

$$\text{Maximum charge: } q'(t) = 60e^{-3t} \sin 3t = 0$$

$$3t = \pi \rightarrow t = \frac{\pi}{3} \text{ sec}$$

$$q\left(\frac{\pi}{3}\right) = 10 - 10e^{-\pi} (\cos \pi + \sin \pi) \\ = 10 + 10e^{-\pi} \approx 10.432 \text{ C}$$

Exercise

Find the charge $q(t)$ on the capacitor in an LRC -series circuit when $L = 1 \text{ h}$, $R = 100 \text{ } \Omega$, $C = 0.0004 \text{ f}$, $E(t) = 30 \text{ V}$, $q(0) = 0 \text{ C}$, and $i(0) = 2 \text{ A}$. Find the maximum charge on the capacitor

Solution

$$q'' + 100q' + \frac{1}{0.0004}q = 30 \qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 100q' + 2500q = 30 ; \quad q(0) = 0, \quad q'(0) = i(0) = 2$$

$$\lambda^2 + 100\lambda + 2500 = 0 \rightarrow \lambda_{1,2} = -50$$

$$q_h(t) = (C_1 + C_2 t)e^{-50t}$$

$$q_p = A$$

$$q'_p = q''_p = 0$$

$$q'' + 100q' + 2500q = 30$$

$$\rightarrow 2500A = 30 \Rightarrow A = 0.012$$

$$q(t) = (C_1 + C_2 t)e^{-50t} + 0.012$$

$$q(0) = 0 \rightarrow C_1 = -0.012$$

$$q'(t) = (C_2 - 50C_1 - 50C_2 t)e^{-50t}$$

$$q'(0) = i(0) = 2 \rightarrow C_2 - 50C_1 = 2 \quad C_2 = 1.4$$

$$q(t) = (-0.012 + 1.4t)e^{-50t} + 0.012$$

$$i(t) = q'(t) = (2 - 70t)e^{-50t}$$

$$\text{Maximum charge: } q'(t) = (2 - 70t)e^{-50t} = 0$$

$$2 - 70t \rightarrow t = \frac{1}{35} \text{ sec}$$

$$q\left(\frac{1}{35}\right) = \left(-0.012 + \frac{1.4}{35}\right)e^{-50/35} + 0.012 \\ \approx 0.01871 \text{ C}$$

Exercise

Find the charge $q(t)$ and $i(t)$ on the capacitor in an LRC -series circuit when $L = \frac{1}{2} \text{ h}$, $R = 10 \text{ } \Omega$, $C = 0.01 \text{ f}$, and $E(t) = 150 \text{ V}$, $q(0) = 1 \text{ C}$, and $i(0) = 0 \text{ A}$. What is the charge on the capacitor after a long time?

Solution

$$\frac{1}{2}q'' + 10q' + \frac{1}{0.01}q = 150 \qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 20q' + 200q = 300; \quad q(0) = 1, \quad q'(0) = i(0) = 0$$

$$\lambda^2 + 20\lambda + 200 = 0 \rightarrow \lambda_{1,2} = -10 \pm 10i$$

$$q_h(t) = e^{-10t} (C_1 \cos 10t + C_2 \sin 10t)$$

$$q_p = A$$

$$q'_p = q''_p = 0$$

$$q'' + 20q' + 200q = 300 \rightarrow 200A = 300 \Rightarrow A = \frac{3}{2}$$

$$q(t) = e^{-10t} (C_1 \cos 10t + C_2 \sin 10t) + \frac{3}{2}$$

$$q(0) = 1 \rightarrow C_1 + \frac{3}{2} = 1 \Rightarrow C_1 = -\frac{1}{2}$$

$$q(t) = e^{-10t} (-10C_1 \cos 10t - 10C_2 \sin 10t - 10C_1 \sin 10t + 10C_2 \cos 10t)$$

$$q'(0) = i(0) = 0 \rightarrow -10C_1 + 10C_2 = 0 \quad C_2 = -\frac{1}{2}$$

$$q(t) = -\frac{1}{2}e^{-10t} (\cos 10t + \sin 10t) + \frac{3}{2}$$

$$i(t) = q'(t) = -\frac{1}{2}e^{-10t} (-10\cos 10t - 10\sin 10t - 10\sin 10t + 10\cos 10t)$$

$$= 10e^{-10t} \sin 10t$$

$$\lim_{t \rightarrow \infty} q(t) = \lim_{t \rightarrow \infty} \left(-\frac{1}{2}e^{-10t} (\cos 10t + \sin 10t) + \frac{3}{2} \right)$$

$$= \frac{3}{2}$$

Exercise

Find the charge $q(t)$ and $i(t)$ on the capacitor in an LRC -series circuit when $L = 1 \text{ h}$, $R = 50 \text{ } \Omega$, $C = 0.0002 \text{ f}$, $E(t) = 50 \text{ V}$, $q(0) = 0 \text{ C}$, and $i(0) = 0 \text{ A}$.

Solution

$$q'' + 50q' + \frac{1}{0.0002}q = 50 \qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 50q' + 5000q = 50 ; \quad q(0) = 0, \quad q'(0) = i(0) = 0$$

$$\lambda^2 + 50\lambda + 5000 = 0 \rightarrow \lambda_{1,2} = -25 \pm 25\sqrt{7}$$

$$q(t) = e^{-25t} (C_1 \cos 25\sqrt{7}t + C_2 \sin 25\sqrt{7}t)$$

$$q_p = A$$

$$q'_p = q''_p = 0$$

$$q'' + 50q' + 5000q = 50 \rightarrow 5000A = 50 \Rightarrow A = 0.01$$

$$q(t) = e^{-25t} (C_1 \cos 25\sqrt{7}t + C_2 \sin 25\sqrt{7}t) + 0.01$$

$$q(0) = 0 \rightarrow C_1 = -0.01 = -\frac{1}{100}$$

$$q'(t) = e^{-25t} (-25C_1 \cos 25\sqrt{7}t - 25C_2 \sin 25\sqrt{7}t - 25\sqrt{7}C_1 \sin 25\sqrt{7}t + 25\sqrt{7}C_2 \cos 25\sqrt{7}t)$$

$$q'(0) = i(0) = 0 \rightarrow -25C_1 + 25\sqrt{7}C_2 = 0 \quad C_2 = -\frac{1}{100\sqrt{7}}$$

$$q(t) = -\frac{1}{100\sqrt{7}} e^{-25t} (\sqrt{7} \cos 25\sqrt{7}t + \sin 25\sqrt{7}t) + \frac{1}{100}$$

$$\begin{aligned} i(t) = q'(t) &= -\frac{1}{100\sqrt{7}} e^{-25t} (-25\sqrt{7} \cos 25\sqrt{7}t - 25 \sin 25\sqrt{7}t - 4,375 \sin 25\sqrt{7}t + 4,375 \cos 25\sqrt{7}t) \\ &= -\frac{1}{100\sqrt{7}} e^{-25t} (4,350\sqrt{7} \cos 25\sqrt{7}t - 4,400 \sin 25\sqrt{7}t) \end{aligned}$$

Exercise

Find the steady-state charge and the steady-state current in an LRC -series circuit when $L = 1 \text{ h}$, $R = 2 \text{ } \Omega$, $C = 0.25 \text{ f}$, and $E(t) = 50 \cos t \text{ V}$.

Solution

$$q'' + 2q' + \frac{1}{0.25}q = 50 \cos t \qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 2q' + 4q = 0$$

$$\lambda^2 + 2\lambda + 4 = 0 \rightarrow \lambda_{1,2} = -1 \pm i\sqrt{3}$$

$$q_h = e^{-t} (C_1 \cos \sqrt{3}t + C_2 \sin \sqrt{3}t)$$

$$q_p = A \cos t + B \sin t$$

$$q'_p = -A \sin t + B \cos t$$

$$q_p'' = -A \cos t - B \sin t$$

$$q'' + 2q' + 4q = 50 \cos t$$

$$\begin{cases} \cos t & -A + 2B + 4A = 50 \\ \sin t & -B - 2A + 4B = 0 \end{cases} \rightarrow \begin{cases} 3A + 2B = 50 \\ -2A + 3B = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 3 & 2 \\ -2 & 3 \end{vmatrix} = 13 \quad \Delta_A = \begin{vmatrix} 50 & 2 \\ 0 & 3 \end{vmatrix} = 150 \quad \Delta_B = \begin{vmatrix} 3 & 50 \\ -2 & 0 \end{vmatrix} = 100$$

$$\rightarrow \underline{A = \frac{150}{13} \quad B = \frac{100}{13}}$$

$$\text{The steady-state charge is: } \underline{q_p(t) = \frac{150}{13} \cos t + \frac{100}{13} \sin t}$$

$$\text{The steady-state current is: } \underline{i_p(t) = -\frac{150}{13} \sin t + \frac{100}{13} \cos t}$$

Exercise

Find the steady-state charge and the steady-state current in an LRC -series circuit when $L = \frac{1}{2} \text{ h}$, $R = 20 \text{ } \Omega$, $C = 0.001 \text{ f}$, and $E(t) = 100 \sin 60t \text{ V}$.

Solution

$$\frac{1}{2} q'' + 20q' + \frac{1}{0.001} q = 100 \sin 60t \quad Lq'' + Rq' + \frac{1}{C} q = E(t)$$

$$q'' + 40q' + 2,000q = 200 \sin 60t$$

$$\lambda^2 + 40\lambda + 2000 = 0 \rightarrow \underline{\lambda_{1,2} = -20 \pm 40i}$$

$$\underline{q_h = e^{-20t} (C_1 \cos 40t + C_2 \sin 40t)}$$

$$q_p = A \cos 60t + B \sin 60t$$

$$q_p' = -60A \sin 60t + 60B \cos 60t$$

$$q_p'' = -3600A \cos 60t - 3600B \sin 60t$$

$$q'' + 40q' + 2,000q = 200 \sin 60t$$

$$\begin{cases} \cos 60t & -3600A + 2400B + 2000A = 0 \\ \sin 60t & -3600B - 2400A + 2000B = 200 \end{cases} \rightarrow \begin{cases} -2A + 3B = 0 \\ -12A - 8B = 1 \end{cases}$$

$$\Delta = \begin{vmatrix} -2 & 3 \\ -12 & -8 \end{vmatrix} = 52 \quad \Delta_A = \begin{vmatrix} 0 & 3 \\ 1 & -8 \end{vmatrix} = -3 \quad \Delta_B = \begin{vmatrix} -2 & 0 \\ -12 & 1 \end{vmatrix} = -2$$

$$\rightarrow \underline{A = -\frac{1}{26} \quad B = -\frac{3}{52}}$$

The steady-state charge is: $\boxed{q_p(t) = -\frac{1}{26}\cos 60t - \frac{3}{52}\sin 60t}$

The steady-state current is: $\boxed{i_p(t) = \frac{1}{26}\sin 60t - \frac{3}{52}\cos 60t}$

Exercise

Find the steady-state charge and the steady-state current in an LRC -series circuit when $L = \frac{1}{2} \text{ h}$, $R = 20 \text{ } \Omega$, $C = 0.001 \text{ f}$, and $E(t) = 100\sin 60t + 200\cos 40t \text{ V}$.

Solution

$$\frac{1}{2}q'' + 20q' + \frac{1}{0.001}q = 100\sin 60t + 200\cos 40t \quad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 40q' + 2,000q = 200\sin 60t + 400\cos 40t$$

$$\lambda^2 + 40\lambda + 2000 = 0 \rightarrow \lambda_{1,2} = -20 \pm 40i$$

$$\boxed{q_h(t) = e^{-20t}(C_1 \cos 40t + C_2 \sin 40t)}$$

$$q_p = A\cos 60t + B\sin 60t + C\cos 40t + D\sin 40t$$

$$q'_p = -60A\sin 60t + 60B\cos 60t - 40C\sin 40t + 40D\cos 40t$$

$$q''_p = -3600A\cos 60t - 3600B\sin 60t - 1600C\cos 40t - 1600D\sin 40t$$

$$q'' + 40q' + 2,000q = 200\sin 60t + 400\cos 40t$$

$$\begin{cases} \cos 60t & -3600A + 2400B + 2000A = 0 \\ \sin 60t & -3600B - 2400A + 2000B = 200 \end{cases} \rightarrow \begin{cases} -2A + 3B = 0 \\ -12A - 8B = 1 \end{cases}$$

$$\Delta = \begin{vmatrix} -2 & 3 \\ -12 & -8 \end{vmatrix} = 52 \quad \Delta_A = \begin{vmatrix} 0 & 3 \\ 1 & -8 \end{vmatrix} = -3 \quad \Delta_B = \begin{vmatrix} -2 & 0 \\ -12 & 1 \end{vmatrix} = -2$$

$$\rightarrow \boxed{A = -\frac{1}{26} \quad B = -\frac{3}{52}}$$

$$\begin{cases} \cos 40t & -1600C + 1600D + 2000C = 400 \\ \sin 40t & -1600D - 1600C + 2000D = 0 \end{cases} \rightarrow \begin{cases} C + 4D = 1 \\ -4C + D = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 4 \\ -4 & 1 \end{vmatrix} = 17 \quad \Delta_C = \begin{vmatrix} 1 & 4 \\ 0 & 1 \end{vmatrix} = 1 \quad \Delta_D = \begin{vmatrix} 1 & 1 \\ -4 & 0 \end{vmatrix} = 4$$

$$\rightarrow \boxed{C = \frac{1}{17} \quad D = \frac{4}{17}}$$

The steady-state charge is: $\boxed{q_p(t) = -\frac{1}{26}\cos 60t - \frac{3}{52}\sin 60t + \frac{1}{17}\cos 40t + \frac{4}{17}\sin 40t}$

The steady-state current is: $\boxed{i_p(t) = \frac{1}{26}\sin 60t - \frac{3}{52}\cos 60t - \frac{1}{17}\sin 40t + \frac{4}{17}\cos 40t}$

Exercise

Find the charge $q(t)$ and $i(t)$ on the capacitor in an LC -series circuit when

$$E(t) = E_0 \sin \omega t \text{ V, } q(0) = q_0 \text{ C, and } i(0) = i_0 \text{ A}$$

Solution

$$Lq'' + \frac{1}{C}q = E_0 \sin \omega t ; \quad q(0) = q_0 \quad i(0) = i_0 \quad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$\lambda^2 + \frac{1}{LC} = 0 \rightarrow \lambda_{1,2} = \pm \frac{1}{\sqrt{LC}}i$$

$$q_h(t) = C_1 \cos \frac{1}{\sqrt{LC}}t + C_2 \sin \frac{1}{\sqrt{LC}}t$$

$$q_p = A \cos \omega t + B \sin \omega t$$

$$q'_p = -\omega A \sin \omega t + \omega B \cos \omega t$$

$$q''_p = -\omega^2 A \cos \omega t - \omega^2 B \sin \omega t$$

$$Lq'' + \frac{1}{C}q = E_0 \sin \omega t$$

$$\begin{cases} \cos \omega t & -L\omega^2 A + \frac{1}{C}A = 0 \\ \sin \omega t & \left(-L\omega^2 + \frac{1}{C}\right)B = E_0 \end{cases} \rightarrow \begin{cases} A = 0 \\ B = \frac{E_0 C}{1 - LC\omega^2} \end{cases}$$

$$q_p(t) = \frac{E_0 C}{1 - LC\omega^2} \sin \omega t$$

$$q(t) = C_1 \cos \frac{1}{\sqrt{LC}}t + C_2 \sin \frac{1}{\sqrt{LC}}t + \frac{E_0 C}{1 - LC\omega^2} \sin \omega t$$

$$q(0) = q_0 \rightarrow C_1 = 0$$

$$q'(t) = -\frac{1}{\sqrt{LC}}C_1 \sin \frac{1}{\sqrt{LC}}t + \frac{1}{\sqrt{LC}}C_2 \cos \frac{1}{\sqrt{LC}}t + \frac{E_0 C \omega}{1 - LC\omega^2} \cos \omega t$$

$$q'(0) = i_0 \rightarrow \frac{1}{\sqrt{LC}}C_2 + \frac{E_0 C \omega}{1 - LC\omega^2} = i_0$$

$$C_2 = \sqrt{LC} i_0 - \frac{E_0 C \omega \sqrt{LC}}{1 - LC\omega^2}$$

$$q(t) = \sqrt{LC} \left(i_0 - \frac{E_0 C \omega}{1 - LC\omega^2} \right) \sin \frac{1}{\sqrt{LC}}t + \frac{E_0 C}{1 - LC\omega^2} \sin \omega t$$

$$i(t) = \left(i_0 - \frac{E_0 C \omega}{1 - LC\omega^2} \right) \cos \frac{1}{\sqrt{LC}}t + \frac{E_0 C \omega}{1 - LC\omega^2} \cos \omega t$$

Exercise

Find the charge $q(t)$ and $i(t)$ on the capacitor in an LC -series circuit when

$$E(t) = E_0 \cos \omega t \text{ V, } q(0) = q_0 \text{ C, and } i(0) = i_0 \text{ A}$$

Solution

$$Lq'' + \frac{1}{C}q = E_0 \sin \omega t ; \quad q(0) = q_0 \quad i(0) = i_0 \quad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$\lambda^2 + \frac{1}{LC} = 0 \rightarrow \lambda_{1,2} = \pm \frac{1}{\sqrt{LC}}i$$

$$q_h(t) = C_1 \cos \frac{1}{\sqrt{LC}}t + C_2 \sin \frac{1}{\sqrt{LC}}t$$

$$q_p = A \cos \omega t + B \sin \omega t$$

$$q'_p = -\omega A \sin \omega t + \omega B \cos \omega t$$

$$q''_p = -\omega^2 A \cos \omega t - \omega^2 B \sin \omega t$$

$$Lq'' + \frac{1}{C}q = E_0 \cos \omega t$$

$$\begin{cases} \cos \omega t & \left(-L\omega^2 + \frac{1}{C}\right)A = E_0 \\ \sin \omega t & \left(-L\omega^2 + \frac{1}{C}\right)B = 0 \end{cases} \rightarrow \begin{cases} A = \frac{E_0 C}{1 - LC\omega^2} \\ B = 0 \end{cases}$$

$$q_p(t) = \frac{E_0 C}{1 - LC\omega^2} \cos \omega t$$

$$q(t) = C_1 \cos \frac{1}{\sqrt{LC}}t + C_2 \sin \frac{1}{\sqrt{LC}}t + \frac{E_0 C}{1 - LC\omega^2} \cos \omega t$$

$$q(0) = q_0 \rightarrow C_1 = q_0 - \frac{E_0 C}{1 - LC\omega^2}$$

$$q'(t) = -\frac{1}{\sqrt{LC}}C_1 \sin \frac{1}{\sqrt{LC}}t + \frac{1}{\sqrt{LC}}C_2 \cos \frac{1}{\sqrt{LC}}t - \frac{E_0 C \omega}{1 - LC\omega^2} \sin \omega t$$

$$q'(0) = i_0 \rightarrow \frac{1}{\sqrt{LC}}C_2 = i_0 \quad C_2 = i_0 \sqrt{LC}$$

$$q(t) = \left(q_0 - \frac{E_0 C}{1 - LC\omega^2} \right) \cos \frac{1}{\sqrt{LC}}t + i_0 \sqrt{LC} \sin \frac{1}{\sqrt{LC}}t + \frac{E_0 C}{1 - LC\omega^2} \cos \omega t$$

$$i(t) = -\frac{1}{\sqrt{LC}} \left(q_0 - \frac{E_0 C}{1 - LC\omega^2} \right) \sin \frac{1}{\sqrt{LC}}t + i_0 \cos \frac{1}{\sqrt{LC}}t - \frac{E_0 C \omega}{1 - LC\omega^2} \sin \omega t$$

Exercise

Find the charge $q(t)$ and $i(t)$ on the capacitor in an LC -series circuit when

$$L = 0.1 \text{ h}, \quad C = 0.1 \text{ f}, \quad E(t) = 100 \sin \omega t \text{ V}, \quad q(0) = 0 \text{ C}, \quad \text{and} \quad i(0) = 0 \text{ A}$$

Solution

$$0.1q'' + \frac{1}{0.1}q = 100 \sin \omega t$$

$$Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 100q = 1,000 \sin \omega t; \quad q(0) = 0 \quad i(0) = 0$$

$$\lambda^2 + 100 = 0 \rightarrow \lambda_{1,2} = \pm 10i$$

$$q_h(t) = C_1 \cos 10t + C_2 \sin 10t$$

$$q_p = A \cos \omega t + B \sin \omega t$$

$$q'_p = -\omega A \sin \omega t + \omega B \cos \omega t$$

$$q''_p = -\omega^2 A \cos \omega t - \omega^2 B \sin \omega t$$

$$q'' + 100q = 1,000 \sin \omega t$$

$$\begin{cases} \cos \omega t & (-\omega^2 + 100)A = 0 \\ \sin \omega t & (-\omega^2 + 100)B = 1,000 \end{cases} \rightarrow \begin{cases} A = 0 \\ B = \frac{1000}{100 - \omega^2} \end{cases}$$

$$q_p(t) = \frac{1000}{100 - \omega^2} \sin \omega t$$

$$q(t) = C_1 \cos 10t + C_2 \sin 10t + \frac{1000}{100 - \omega^2} \sin \omega t$$

$$q(0) = 0 \rightarrow C_1 = 0$$

$$q'(t) = -10C_1 \sin 10t + 10C_2 \cos 10t + \frac{1000\omega}{100 - \omega^2} \cos \omega t$$

$$q'(0) = 0 \rightarrow 10C_2 + \frac{1000\omega}{100 - \omega^2} = 0 \Rightarrow C_2 = -\frac{100\omega}{100 - \omega^2}$$

$$q(t) = -\frac{100}{100 - \omega^2} \sin 10t + \frac{1000}{100 - \omega^2} \sin \omega t$$

$$= \frac{100}{100 - \omega^2} (10 \sin \omega t - \sin 10t)$$

$$i(t) = \frac{1000}{100 - \omega^2} (\omega \cos \omega t - \cos 10t)$$

Exercise

Find the charge $q(t)$ and $i(t)$ on the capacitor in an LC -series circuit when

$$L = 1 \text{ H}, \quad C = 4 \text{ } \mu\text{F}, \quad E(t) = 3 \sin 3t \text{ V}, \quad q(0) = 0 \text{ C}, \quad \text{and} \quad i(0) = 0 \text{ A}$$

Solution

$$q'' + \frac{1}{4}q = 3 \sin 3t; \quad q(0) = 0 \quad i(0) = 0 \qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$\lambda^2 + \frac{1}{4} = 0 \rightarrow \lambda_{1,2} = \pm \frac{1}{2}i$$

$$q_h(t) = C_1 \cos \frac{1}{2}t + C_2 \sin \frac{1}{2}t$$

$$q_p = A \cos 3t + B \sin 3t$$

$$q'_p = -3A \sin 3t + 3B \cos 3t$$

$$q''_p = -9A \cos 3t - 9B \sin 3t$$

$$4q'' + q = 12 \sin 3t$$

$$\begin{cases} \cos 3t & -36A + A = 0 \\ \sin 3t & -36B + B = 12 \end{cases} \rightarrow \begin{cases} A = 0 \\ B = -\frac{12}{35} \end{cases}$$

$$q_p = -\frac{12}{35} \sin 3t$$

$$q(t) = C_1 \cos \frac{1}{2}t + C_2 \sin \frac{1}{2}t - \frac{12}{35} \sin 3t$$

$$q(0) = 0 \rightarrow C_1 = 0$$

$$q'(t) = -\frac{1}{2}C_1 \sin \frac{1}{2}t + \frac{1}{2}C_2 \cos \frac{1}{2}t - \frac{36}{35} \cos 3t$$

$$q'(0) = 0 \rightarrow \frac{1}{2}C_2 - \frac{36}{35} = 0 \Rightarrow C_2 = \frac{72}{35}$$

$$q(t) = \frac{72}{35} \sin \frac{1}{2}t - \frac{12}{35} \sin 3t$$

$$i(t) = q'(t) = \frac{36}{35} \cos \frac{1}{2}t - \frac{4}{35} \cos 3t$$

Exercise

Find the charge $q(t)$ and $i(t)$ on the capacitor in an LC -series circuit when

$$L = 1 \text{ H}, \quad C = 4 \text{ } \mu\text{F}, \quad E(t) = 10te^{-t} \text{ V}, \quad q(0) = 0 \text{ C}, \quad \text{and} \quad i(0) = 0 \text{ A}$$

Solution

$$q'' + \frac{1}{4}q = 10te^{-t}; \quad q(0) = 0 \quad i(0) = 0 \qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$\lambda^2 + \frac{1}{4} = 0 \rightarrow \lambda_{1,2} = \pm \frac{1}{2}j$$

$$q_h(t) = C_1 \cos \frac{t}{2} + C_2 \sin \frac{t}{2}$$

$$q_p = (At + B)e^{-t}$$

$$q'_p = (A - B - At)e^{-t}$$

$$q''_p = (-2A + B + At)e^{-t}$$

$$4q'' + q = 40te^{-t}$$

$$\begin{cases} e^{-t} & t & 4A + A = 40 \\ & t^0 & -8A + 4B + B = 0 \end{cases} \rightarrow \begin{cases} A = 8 \\ B = \frac{64}{5} \end{cases}$$

$$q_p(t) = \left(8t + \frac{64}{5}\right)e^{-t}$$

$$q(t) = C_1 \cos \frac{t}{2} + C_2 \sin \frac{t}{2} + \left(8t + \frac{64}{5}\right)e^{-t}$$

$$q(0) = 0 \rightarrow C_1 = -\frac{64}{5}$$

$$q'(t) = -\frac{1}{2}C_1 \sin \frac{1}{2}t + \frac{1}{2}C_2 \cos \frac{1}{2}t - \left(8t + \frac{24}{5}\right)e^{-t}$$

$$q'(0) = 0 \rightarrow \frac{1}{2}C_2 - \frac{24}{5} = 0 \Rightarrow C_2 = \frac{48}{5}$$

$$q(t) = -\frac{64}{5} \cos \frac{t}{2} + \frac{48}{5} \sin \frac{t}{2} + \left(8t + \frac{64}{5}\right)e^{-t}$$

$$i(t) = q'(t) = \frac{32}{5} \sin \frac{t}{2} + \frac{24}{5} \cos \frac{t}{2} - \left(8t + \frac{24}{5}\right)e^{-t}$$

Exercise

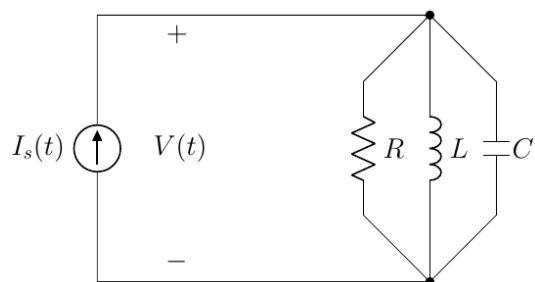
Consider the parallel RLC network. Assume that at time $t = 0$, the voltage $V(t)$ and its time rate of change are both zero. Determine the voltage $V(t)$ for

$$R = 1 \text{ k}\Omega, \quad L = 1 \text{ H}, \quad C = \frac{1}{2} \mu\text{F}, \quad I_s(t) = 1 - e^{-t} \text{ mA}$$

Solution

$$I_S(t) = \frac{1}{R}V + C \frac{dV}{dt} + \frac{1}{L} \int V(s) ds$$

$$\frac{d}{dt} \left(I_S(t) = \frac{1}{R}V + C \frac{dV}{dt} + \frac{1}{L} \int V(s) ds \right)$$



$$\frac{d}{dt} I_S(t) = \frac{1}{R} \frac{dV}{dt} + C \frac{d^2 V}{dt^2} + \frac{1}{L} V(s)$$

$$\frac{d^2 V}{dt^2} + \frac{1}{RC} \frac{dV}{dt} + \frac{1}{LC} V = \frac{1}{C} \frac{dI}{dt}$$

$$V'' + 2V' + 2V = 2 \frac{d}{dt} (1 - e^{-t})$$

$$V'' + 2V' + 2V = 2e^{-t} ; \quad V(0) = 0, \quad V'(0) = 0$$

$$\lambda^2 + 2\lambda + 2 = 0 \rightarrow \lambda_{1,2} = -1 \pm i$$

$$V_h(t) = e^{-t} (C_1 \cos t + C_2 \sin t)$$

$$V_p = Ae^{-t}$$

$$V'_p = -Ae^{-t}$$

$$V''_p = Ae^{-t}$$

$$V'' + 2V' + 2V = 2e^{-t}$$

$$e^{-t} \quad A - 2A + 2A = 2 \rightarrow A = 2$$

$$V_p(t) = 2e^{-t}$$

$$V(t) = e^{-t} (C_1 \cos t + C_2 \sin t) + 2e^{-t}$$

$$V(0) = 0 \rightarrow C_1 + 2 = 0 \Rightarrow C_1 = -2$$

$$V'(t) = e^{-t} (-C_1 \sin t + C_2 \cos t - C_1 \cos t - C_2 \sin t) - 2e^{-t}$$

$$V'(0) = 0 \rightarrow C_2 + 2 - 2 = 0 \Rightarrow C_2 = 0$$

$$V(t) = -2e^{-t} \cos t + 2e^{-t}$$

Exercise

Consider the parallel RLC network. Assume that at time $t = 0$, the voltage $V(t)$ and its time rate of change are both zero. Determine the voltage $V(t)$ for

$$R = 1 \text{ k}\Omega, \quad L = 1 \text{ H}, \quad C = \frac{1}{2} \text{ }\mu\text{F}, \quad I_s(t) = 5 \sin t \text{ mA}$$

Solution

$$I_S(t) = \frac{1}{R} V + C \frac{dV}{dt} + \frac{1}{L} \int V(s) ds$$

$$\frac{d}{dt} \left(I_s(t) = \frac{1}{R} V + C \frac{dV}{dt} + \frac{1}{L} \int V(s) ds \right)$$

$$\frac{d}{dt} I_s(t) = \frac{1}{R} \frac{dV}{dt} + C \frac{d^2 V}{dt^2} + \frac{1}{L} V(s)$$

$$\frac{d^2 V}{dt^2} + \frac{1}{RC} \frac{dV}{dt} + \frac{1}{LC} V = \frac{1}{C} \frac{dI}{dt}$$

$$V'' + 2V' + 2V = 2 \frac{d}{dt}(5 \sin t)$$

$$V'' + 2V' + 2V = 10 \cos t ; \quad V(0) = 0, \quad V'(0) = 0$$

$$\lambda^2 + 2\lambda + 2 = 0 \rightarrow \lambda_{1,2} = -1 \pm i$$

$$\underline{V_h(t) = e^{-t} (C_1 \cos t + C_2 \sin t)}$$

$$V_p = A \cos t + B \sin t$$

$$V'_p = -A \sin t + B \cos t$$

$$V''_p = -A \cos t - B \sin t$$

$$V'' + 2V' + 2V = 10 \cos t$$

$$\begin{cases} \cos t & -A + 2B + 2A = 10 \\ \sin t & -B - 2A + 2B = 0 \end{cases} \rightarrow \begin{cases} A + 2B = 10 \\ -2A + B = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 5 \quad \Delta_A = \begin{vmatrix} 10 & 2 \\ 0 & 1 \end{vmatrix} = 10 \quad \Delta_B = \begin{vmatrix} 1 & 10 \\ -2 & 0 \end{vmatrix} = 20$$

$$\underline{A = 2, \quad B = 4}$$

$$\underline{V_p(t) = 2 \cos t + 4 \sin t}$$

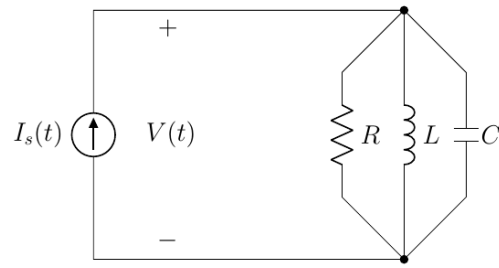
$$V(t) = e^{-t} (C_1 \cos t + C_2 \sin t) + 2 \cos t + 4 \sin t$$

$$V(0) = 0 \rightarrow C_1 + 2 = 0 \Rightarrow \underline{C_1 = -2}$$

$$V'(t) = e^{-t} (-C_1 \sin t + C_2 \cos t - C_1 \cos t - C_2 \sin t) - 2 \sin t + 4 \cos t$$

$$V'(0) = 0 \rightarrow C_2 + 2 + 4 = 0 \Rightarrow \underline{C_2 = -6}$$

$$\underline{V(t) = e^{-t} (-2 \cos t - 6 \sin t) + 2 \cos t + 4 \sin t}$$



Exercise

Consider the parallel RLC network. Assume that at time $t = 0$, the voltage $V(t)$ and its time rate of change are both zero. Determine the voltage $V(t)$ for

$$R = 1 \text{ k}\Omega, \quad L = 1 \text{ H}, \quad C = \frac{1}{2} \text{ }\mu\text{F}, \quad I_s(t) = 5 \cos t \text{ mA}$$

Solution

$$I_s(t) = \frac{1}{R}V + C \frac{dV}{dt} + \frac{1}{L} \int V(s) ds$$

$$\frac{d}{dt} \left(I_s(t) = \frac{1}{R}V + C \frac{dV}{dt} + \frac{1}{L} \int V(s) ds \right)$$

$$\frac{d}{dt} I_s(t) = \frac{1}{R} \frac{dV}{dt} + C \frac{d^2V}{dt^2} + \frac{1}{L} V(s)$$

$$\frac{d^2V}{dt^2} + \frac{1}{RC} \frac{dV}{dt} + \frac{1}{LC} V = \frac{1}{C} \frac{dI}{dt}$$

$$V'' + 2V' + 2V = 2 \frac{d}{dt}(5 \cos t)$$

$$V'' + 2V' + 2V = -10 \sin t; \quad V(0) = 0, \quad V'(0) = 0$$

$$\lambda^2 + 2\lambda + 2 = 0 \rightarrow \lambda_{1,2} = -1 \pm i$$

$$V_h(t) = e^{-t} (C_1 \cos t + C_2 \sin t)$$

$$V_p = A \cos t + B \sin t$$

$$V'_p = -A \sin t + B \cos t$$

$$V''_p = -A \cos t - B \sin t$$

$$V'' + 2V' + 2V = -10 \sin t$$

$$\begin{cases} \cos t & -A + 2B + 2A = 0 \\ \sin t & -B - 2A + 2B = -10 \end{cases} \rightarrow \begin{cases} A + 2B = 0 \\ -2A + B = -10 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 5 \quad \Delta_A = \begin{vmatrix} 0 & 2 \\ -10 & 1 \end{vmatrix} = 20 \quad \Delta_B = \begin{vmatrix} 1 & 0 \\ -2 & -10 \end{vmatrix} = -10$$

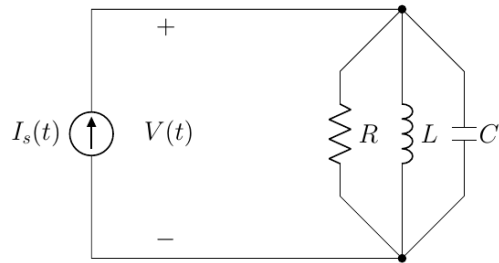
$$A = 4, \quad B = -2$$

$$V_p(t) = 4 \cos t - 2 \sin t$$

$$V(t) = e^{-t} (C_1 \cos t + C_2 \sin t) + 4 \cos t - 2 \sin t$$

$$V(0) = 0 \rightarrow C_1 + 4 = 0 \Rightarrow C_1 = -4$$

$$V'(t) = e^{-t} (-C_1 \sin t + C_2 \cos t - C_1 \cos t - C_2 \sin t) - 4 \sin t - 2 \cos t$$



$$V'(0)=0 \rightarrow -C_2 + 4 - 2 = 0 \Rightarrow \underline{C_2 = 2}$$

$$\underline{V(t) = e^{-t}(-4\cos t - 2\sin t) + 4\cos t - 2\sin t}$$

Exercise

Consider the parallel RLC network. Assume that at time $t = 0$, the voltage $V(t)$ and its time rate of change are both zero. Determine the voltage $V(t)$ for

$$R = 2 \text{ k}\Omega, \quad L = 1 \text{ H}, \quad C = \frac{1}{4} \mu\text{F}, \quad I_s(t) = e^{-t} \text{ mA}$$

Solution

$$I_s(t) = \frac{1}{R}V + C\frac{dV}{dt} + \frac{1}{L}\int V(s)ds$$

$$\frac{d}{dt}\left(I_s(t) = \frac{1}{R}V + C\frac{dV}{dt} + \frac{1}{L}\int V(s)ds\right)$$

$$\frac{d}{dt}I_s(t) = \frac{1}{R}\frac{dV}{dt} + C\frac{d^2V}{dt^2} + \frac{1}{L}V(s)$$

$$\frac{d^2V}{dt^2} + \frac{1}{RC}\frac{dV}{dt} + \frac{1}{LC}V = \frac{1}{C}\frac{dI}{dt}$$

$$V'' + 2V' + 4V = 4\frac{d}{dt}(e^{-t})$$

$$V'' + 2V' + 4V = -4e^{-t}; \quad V(0)=0, \quad V'(0)=0$$

$$\lambda^2 + 2\lambda + 4 = 0 \rightarrow \underline{\lambda_{1,2} = -1 \pm i\sqrt{3}}$$

$$\underline{V_h(t) = e^{-t}(C_1 \cos \sqrt{3}t + C_2 \sin \sqrt{3}t)}$$

$$V_p = Ae^{-t}$$

$$V'_p = -Ae^{-t}$$

$$V''_p = Ae^{-t}$$

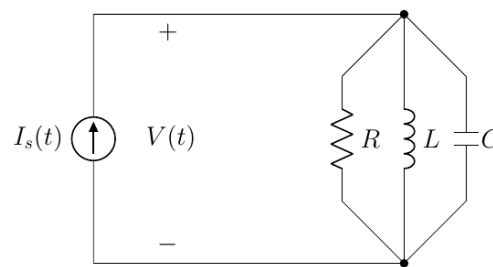
$$V'' + 2V' + 4V = -4e^{-t}$$

$$e^{-t} \quad A - 2A + 4A = -4 \rightarrow \underline{A = -\frac{4}{3}}$$

$$\underline{V_p(t) = -\frac{4}{3}e^{-t}}$$

$$V(t) = e^{-t}(C_1 \cos \sqrt{3}t + C_2 \sin \sqrt{3}t) - \frac{4}{3}e^{-t}$$

$$V(0)=0 \rightarrow C_1 - \frac{4}{3} = 0 \Rightarrow \underline{C_1 = \frac{4}{3}}$$



$$V'(t) = e^{-t} \left(-\sqrt{3}C_1 \sin \sqrt{3}t + \sqrt{3}C_2 \cos \sqrt{3}t - C_1 \cos \sqrt{3}t - C_2 \sin \sqrt{3}t \right) + \frac{4}{3}e^{-t}$$

$$V'(0) = 0 \rightarrow \sqrt{3}C_2 - \frac{4}{3} + \frac{4}{3} = 0 \Rightarrow \underline{C_2 = 0}$$

$$\underline{V(t) = \frac{4}{3}e^{-t} \cos \sqrt{3}t - \frac{4}{3}e^{-t}}$$

Exercise

An RCL circuit connected in series has $R = 180 \, \Omega$, $C = \frac{1}{280} \, F$, $L = 20 \, H$, and applied voltage

$E(t) = 10 \sin t \, V$. Assuming no initial charge on the capacitor, but an initial current of $1 \, A$ at $t = 0$ when the voltage is first applied.

- Find the subsequent charge on the capacitor.
- Plot the *transient*, *steady-state*, and the charge on the capacitor.
- Find the current on the capacitor.

Solution

$$a) \quad 20q'' + 180q' + 280q = 10 \sin t$$

$$Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 9q' + 14q = \frac{1}{2} \sin t; \quad q(0) = 0 \quad i(0) = 1$$

$$\lambda^2 + 9\lambda + 14 = 0 \rightarrow \lambda_{1,2} = \frac{-9 \pm 5}{2} = -7, -2$$

$$\underline{q_h(t) = C_1 e^{-7t} + C_2 e^{-2t}}$$

$$q_p = A \cos t + B \sin t$$

$$q'_p = -A \sin t + B \cos t$$

$$q''_p = -A \cos t - B \sin t$$

$$q'' + 9q' + 14q = \frac{1}{2} \sin t$$

$$\begin{cases} \cos t & -A + 9B + 14A = 0 \\ \sin t & -B - 9A + 14B = \frac{1}{2} \end{cases} \rightarrow \begin{cases} 13A + 9B = 0 \\ -9A + 13B = \frac{1}{2} \end{cases}$$

$$\Delta = \begin{vmatrix} 13 & 9 \\ -9 & 13 \end{vmatrix} = 250 \quad \Delta_A = \begin{vmatrix} 0 & 9 \\ \frac{1}{2} & 13 \end{vmatrix} = -\frac{9}{2} \quad \Delta_B = \begin{vmatrix} 13 & 0 \\ -9 & \frac{1}{2} \end{vmatrix} = \frac{13}{2}$$

$$\underline{A = -\frac{9}{500}, \quad B = \frac{13}{500}}$$

$$\underline{q_p(t) = -\frac{9}{500} \cos t + \frac{13}{500} \sin t}$$

$$q(t) = C_1 e^{-7t} + C_2 e^{-2t} - \frac{9}{500} \cos t + \frac{13}{500} \sin t$$

$$q(0) = 0 \rightarrow C_1 + C_2 = \frac{9}{500}$$

$$q' = -7C_1 e^{-7t} - 2C_2 e^{-2t} + \frac{9}{500} \sin t + \frac{13}{500} \cos t$$

$$q'(0) = 1 \rightarrow -7C_1 - 2C_2 = 1 - \frac{13}{500} = \frac{487}{500}$$

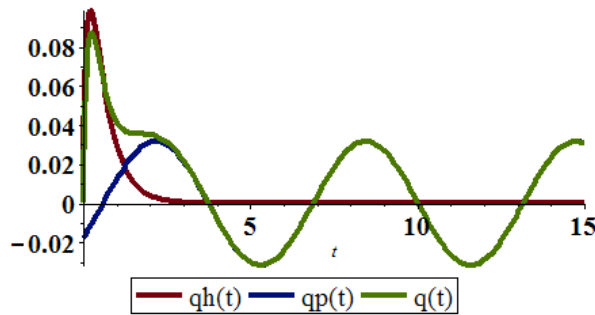
$$\Delta = \begin{vmatrix} 1 & 1 \\ -7 & -2 \end{vmatrix} = 5 \quad \Delta_{C_1} = \begin{vmatrix} \frac{9}{500} & 1 \\ \frac{487}{500} & -2 \end{vmatrix} = -\frac{505}{500} \quad \Delta_{C_2} = \begin{vmatrix} 1 & \frac{9}{500} \\ -7 & \frac{487}{500} \end{vmatrix} = \frac{550}{500}$$

$$C_1 = -\frac{101}{500} \quad C_2 = \frac{110}{500}$$

$$q(t) = -\frac{101}{500} e^{-7t} + \frac{11}{50} e^{-2t} - \frac{9}{500} \cos t + \frac{13}{500} \sin t$$

The solution is the sum of *transient* and *steady-state* terms

b)



$$c) \quad i(t) = \frac{707}{500} e^{-7t} - \frac{22}{50} e^{-2t} + \frac{9}{500} \sin t + \frac{13}{500} \cos t$$

Exercise

An RCL circuit connected in series has $R = 10 \, \Omega$, $C = 10^{-2} \, F$, $L = \frac{1}{2} \, H$, and applied voltage

$E(t) = 12 \, V$. Assuming no initial charge and no initial current at $t = 0$ when the voltage is first applied.

- Find the subsequent charge on the capacitor.
- Plot the *transient*, *steady-state*, and the charge on the capacitor.
- Find the current on the capacitor.

Solution

$$a) \quad \frac{1}{2} q'' + 10q' + 100q = 12$$

$$Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 20q' + 200q = 24 ; \quad q(0) = 0 \quad i(0) = 0$$

$$\lambda^2 + 20\lambda + 200 = 0 \rightarrow \lambda_{1,2} = -10 \pm 10i$$

$$q_h(t) = e^{-10t} (C_1 \cos 10t + C_2 \sin 10t)$$

$$q_p = A$$

$$q'_p = q''_p = 0$$

$$q'' + 20q' + 200q = 24$$

$$200A = 24 \rightarrow A = \frac{3}{25}$$

$$q_p = \frac{3}{25}$$

$$q(t) = e^{-10t} (C_1 \cos 10t + C_2 \sin 10t) + \frac{3}{25}$$

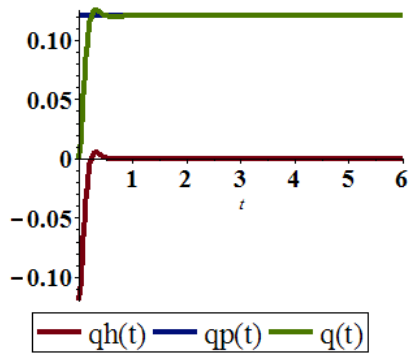
$$q(0) = 0 \rightarrow C_1 = -\frac{3}{25}$$

$$q' = e^{-10t} (-10C_1 \cos 10t - 10C_2 \sin 10t - 10C_1 \sin 10t + 10C_2 \cos 10t)$$

$$q'(0) = 0 \rightarrow -10C_1 + 10C_2 = 0 \Rightarrow C_1 = -\frac{3}{25}$$

$$q(t) = -\frac{3}{25} e^{-10t} (\cos 10t + \sin 10t) + \frac{3}{25}$$

b)



$$\begin{aligned} c) \quad I(t) &= q' = -\frac{3}{25} e^{-10t} (-10 \cos 10t - 10 \sin 10t - 10 \sin 10t + 10 \cos 10t) \\ &= \frac{12}{5} e^{-10t} \sin 10t \end{aligned}$$

∴ This is a completely transient

Exercise

An RCL circuit connected in series has $R = 5 \Omega$, $C = 4 \times 10^{-4} F$, $L = 0.05 H$, and applied voltage $E(t) = 200 \cos 100t V$. Assuming no initial charge and no initial current at $t = 0$ when the voltage is first applied.

- Find the subsequent charge on the capacitor.
- Plot the *transient*, *steady-state*, and the charge on the capacitor.
- Find the current flowing through this circuit.

Solution

$$a) .05q'' + 5q' + \frac{1}{4 \times 10^{-4}} q = 200 \cos 100t \quad Lq'' + Rq' + \frac{1}{C} q = E(t)$$

$$q'' + 100q' + 5 \times 10^4 q = 4,000 \cos 100t ; \quad q(0) = 0 \quad i(0) = 0$$

$$\lambda^2 + 100\lambda + 5 \times 10^4 = 0 \rightarrow \lambda_{1,2} = -50 \pm 50i\sqrt{19}$$

$$\underline{q_h(t) = e^{-50t} (C_1 \cos 50\sqrt{19}t + C_2 \sin 50\sqrt{19}t)}$$

$$q_p = A \cos 100t + B \sin 100t$$

$$q'_p = -100A \sin 100t + 100B \cos 100t$$

$$q''_p = -10^4 A \cos 100t - 10^4 B \sin 100t$$

$$q'' + 100q' + 5 \times 10^4 q = 4,000 \cos 100t$$

$$\begin{cases} \cos 100t & -10^4 A + 10^4 B + 5 \times 10^4 A = 4 \times 10^3 \\ \sin 100t & -10^4 B - 10^4 A + 5 \times 10^4 B = 0 \end{cases} \rightarrow \begin{cases} 4A + B = \frac{2}{5} \\ -A + 4B = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 4 & 1 \\ -1 & 4 \end{vmatrix} = 17 \quad \Delta_A = \begin{vmatrix} \frac{2}{5} & 1 \\ 0 & 4 \end{vmatrix} = \frac{8}{5} \quad \Delta_B = \begin{vmatrix} 4 & \frac{2}{5} \\ -1 & 0 \end{vmatrix} = \frac{2}{5}$$

$$\underline{A = \frac{8}{85} \quad B = \frac{2}{85}}$$

$$\underline{q_p(t) = \frac{8}{85} \cos 100t + \frac{2}{85} \sin 100t}$$

$$q(t) = e^{-50t} (C_1 \cos 50\sqrt{19}t + C_2 \sin 50\sqrt{19}t) + \frac{8}{85} \cos 100t + \frac{2}{85} \sin 100t$$

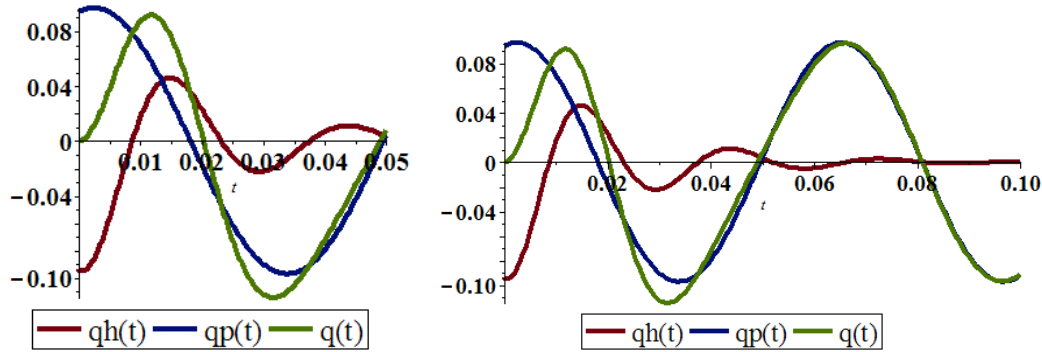
$$q(0) = 0 \rightarrow \underline{C_1 = -\frac{8}{85}}$$

$$q'(t) = e^{-50t} (-50C_1 \cos 50\sqrt{19}t - 50 \sin 50\sqrt{19}t - 50\sqrt{19}C_1 \sin 50\sqrt{19}t + 50\sqrt{19}C_2 \cos 50\sqrt{19}t) \\ + \frac{160}{17} \sin 100t + \frac{40}{17} \cos 100t$$

$$q'(0) = 0 \rightarrow -50C_1 + 50\sqrt{19}C_2 + \frac{40}{17} = 0 \Rightarrow \underline{C_2 = -\frac{12}{85\sqrt{19}}}$$

$$q(t) = e^{-50t} \left(-\frac{8}{85} \cos 50\sqrt{19}t - \frac{12}{85\sqrt{19}} \sin 50\sqrt{19}t \right) + \frac{8}{85} \cos 100t + \frac{2}{85} \sin 100t$$

b)



c)

$$I(t) = q' = -\frac{200}{85} e^{-50t} \left(-2 \cos 50\sqrt{19}t - \frac{3}{\sqrt{19}} \sin 50\sqrt{19}t - 2\sqrt{19} \sin 50\sqrt{19}t + 3 \cos 50\sqrt{19}t \right) + \frac{160}{17} \sin 100t + \frac{40}{17} \cos 100t$$

$$I(t) = -\frac{200}{85} e^{-50t} \left(\cos 50\sqrt{19}t - \frac{41}{\sqrt{19}} \sin 50\sqrt{19}t \right) + \frac{160}{17} \sin 100t + \frac{40}{17} \cos 100t$$

Exercise

An RCL circuit connected in series has $R = 40 \, \Omega$, $C = 16 \times 10^{-4} \, F$, $L = 1 \, H$, and applied voltage $E(t) = 100 \cos 10t \, V$. Assuming no initial charge and no initial current at $t = 0$ when the voltage is first applied.

- Find the charge in the circuit at time t .
- Find the current flowing through this circuit.
- Find the limit of the charge as $t \rightarrow \infty$

Solution

$$a) \quad q'' + 40q' + \frac{10^4}{16}q = 100 \cos 10t \quad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 40q' + 625q = 100 \cos 10t ; \quad q(0) = 0, \quad i(0) = 0$$

$$\lambda^2 + 40\lambda + 625 = 0 \rightarrow \lambda_{1,2} = -20 \pm 15i$$

$$q_h = e^{-20t} (C_1 \cos 15t + C_2 \sin 15t)$$

$$q_p = A \cos 10t + B \sin 10t$$

$$q'_p = -10A \sin 10t + 10B \cos 10t$$

$$q_p'' = -100A \cos 10t - 100B \sin 10t$$

$$q'' + 40q' + 625q = 100 \cos 10t$$

$$\begin{cases} \cos 10t & -100A + 400B + 625A = 100 \\ \sin 10t & -100B - 400A + 625B = 0 \end{cases}$$

$$\rightarrow \begin{cases} 525A + 400B = 100 \\ -400A + 525B = 0 \end{cases} \rightarrow \begin{aligned} & \left(\frac{11025}{16} + 400 \right) B = 100 \\ & A = \frac{21}{16} B \end{aligned}$$

$$\underline{B = \frac{64}{697}, \quad A = \frac{84}{697}}$$

$$\underline{q_p = \frac{84}{697} \cos 10t + \frac{64}{697} \sin 10t}$$

$$q(t) = e^{-20t} (C_1 \cos 15t + C_2 \sin 15t) + \frac{84}{697} \cos 10t + \frac{64}{697} \sin 10t$$

$$q(0) = 0 \rightarrow \underline{C_1 = -\frac{84}{697}}$$

$$q' = e^{-20t} (-20C_1 \cos 15t - 20C_2 \sin 15t - 15C_1 \sin 15t + 15C_2 \cos 15t) - \frac{840}{697} \sin 10t + \frac{640}{697} \cos 10t$$

$$q'(0) = 0 \rightarrow 20 \frac{84}{697} + 15C_2 + \frac{640}{697} = 0$$

$$\Rightarrow \underline{C_2 = -\frac{464}{2091}}$$

$$\underline{q(t) = e^{-20t} \left(-\frac{84}{697} \cos 15t - \frac{464}{2091} \sin 15t \right) + \frac{84}{697} \cos 10t + \frac{64}{697} \sin 10t}$$

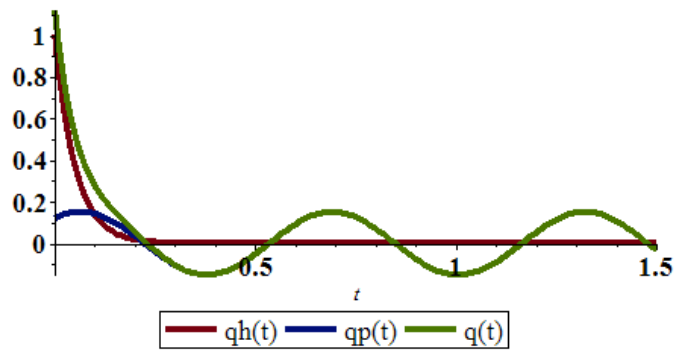
$$\begin{aligned} b) \quad I(t) = q' &= -\frac{1}{2091} e^{-20t} (-5040 \cos 15t - 9280 \sin 15t - 3780 \sin 15t + 6960 \cos 15t) \\ &\quad - \frac{840}{697} \sin 10t + \frac{640}{697} \cos 10t \end{aligned}$$

$$\underline{= -\frac{1}{2091} e^{-20t} (1,920 \cos 15t - 13,060 \sin 15t) - \frac{840}{697} \sin 10t + \frac{640}{697} \cos 10t}$$

$$c) \quad \lim_{t \rightarrow \infty} q(t) = \lim_{t \rightarrow \infty} \left(e^{-20t} \left(-\frac{84}{697} \cos 15t - \frac{464}{2091} \sin 15t \right) + \frac{84}{697} \cos 10t + \frac{64}{697} \sin 10t \right)$$

$$\lim_{t \rightarrow \infty} e^{-20t} \left(-\frac{84}{697} \cos 15t - \frac{464}{2091} \sin 15t \right) = \underline{0} \quad q_p : \text{is steady state solution}$$

$$\lim_{t \rightarrow \infty} \left(\frac{84}{697} \cos 10t + \frac{64}{697} \sin 10t \right) = \cancel{\neq}$$



Exercise

A series circuit consists of a resistor with $R = 20 \, \Omega$, an inductor with $L = 1 \, H$, a capacitor with $C = 0.002 \, F$, and a 12-V battery. If the initial charge and current are both 0, find the charge and current at time t .

Solution

$$q'' + 20q' + \frac{1}{.002}q = 12 \qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 20q' + 500q = 12 ; \quad q(0) = 0, \quad i(0) = 0$$

$$\lambda^2 + 20\lambda + 500 = 0 \rightarrow \lambda_{1,2} = -10 \pm 20i$$

$$q_h = e^{-10t} (C_1 \cos 20t + C_2 \sin 20t)$$

$$q_p = A$$

$$q'_p = q''_p = 0$$

$$q'' + 20q' + 500q = 12 \rightarrow 500A = 12 \Rightarrow A = \frac{3}{125}$$

$$q_p = \frac{3}{125}$$

$$q(t) = e^{-10t} (C_1 \cos 20t + C_2 \sin 20t) + \frac{3}{125}$$

$$q(0) = 0 \rightarrow C_1 = -\frac{3}{125}$$

$$q' = e^{-10t} (-10C_1 \cos 20t - 10C_2 \sin 20t - 20C_1 \sin 20t + 20C_2 \cos 20t)$$

$$q'(0) = 0 \rightarrow 10\frac{3}{125} + 20C_2 = 0$$

$$\Rightarrow C_2 = -\frac{3}{250}$$

$$q(t) = e^{-10t} \left(-\frac{3}{125} \cos 20t - \frac{3}{250} \sin 20t \right) + \frac{3}{125}$$

$$I(t) = q' = e^{-10t} \left(\frac{6}{25} \cos 20t + \frac{3}{25} \sin 20t + \frac{12}{25} \sin 20t - \frac{6}{25} \cos 20t \right)$$

$$= \frac{3}{5} e^{-10t} \sin 20t$$

Exercise

A series circuit consists of a resistor with $R = 20 \, \Omega$, an inductor with $L = 1 \, H$, a capacitor with $C = 0.002 \, F$, and $E(t) = 12 \sin 10t$. If the initial charge and current are both 0, find the charge and current at time t .

Solution

$$q'' + 20q' + \frac{1}{.002} q = 12 \sin 10t \quad Lq'' + Rq' + \frac{1}{C} q = E(t)$$

$$q'' + 20q' + 500q = 12 \sin 10t ; \quad q(0) = 0, \quad i(0) = 0$$

$$\lambda^2 + 20\lambda + 500 = 0 \rightarrow \lambda_{1,2} = -10 \pm 20i$$

$$q_h = e^{-10t} (C_1 \cos 20t + C_2 \sin 20t)$$

$$q_p = A \cos 10t + B \sin 10t$$

$$q'_p = -10A \sin 10t + 10B \cos 10t$$

$$q''_p = -100A \cos 10t - 100B \sin 10t$$

$$q'' + 20q' + 500q = 12 \sin 10t$$

$$\begin{cases} \cos 10t & -100A + 200B + 500A = 0 \\ \sin 10t & -100B - 200A + 500B = 12 \end{cases} \rightarrow \begin{cases} 2A + B = 0 \\ -50A + 100B = 3 \end{cases}$$

$$\Delta = \begin{vmatrix} 2 & 1 \\ -50 & 100 \end{vmatrix} = 250 \quad \Delta_A = \begin{vmatrix} 0 & 1 \\ 3 & 100 \end{vmatrix} = -3 \quad \Delta_B = \begin{vmatrix} 2 & 0 \\ -50 & 3 \end{vmatrix} = 6$$

$$A = -\frac{3}{250}, \quad B = \frac{3}{125}$$

$$q_p = -\frac{3}{250} \cos 10t + \frac{3}{125} \sin 10t$$

$$q(t) = e^{-10t} (C_1 \cos 20t + C_2 \sin 20t) - \frac{3}{250} \cos 10t + \frac{3}{125} \sin 10t$$

$$q(0) = 0 \rightarrow C_1 = \frac{3}{250}$$

$$q' = e^{-10t} (-10C_1 \cos 20t - 10C_2 \sin 20t - 20C_1 \sin 20t + 20C_2 \cos 20t) + \frac{3}{25} \sin 10t + \frac{6}{25} \cos 10t$$

$$q'(0) = 0 \rightarrow -10 \frac{3}{250} + 20C_2 + \frac{6}{25} = 0 \quad C_2 = -\frac{3}{500}$$

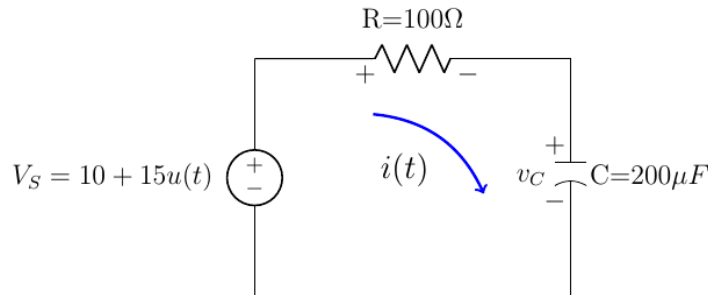
$$q(t) = e^{-10t} \left(\frac{3}{250} \cos 20t - \frac{3}{500} \sin 20t \right) - \frac{3}{250} \cos 10t + \frac{3}{125} \sin 10t \Big|$$

$$I(t) = e^{-10t} \left(-\frac{3}{25} \cos 20t + \frac{3}{50} \sin 20t - \frac{6}{25} \sin 20t - \frac{3}{25} \cos 20t \right) + \frac{3}{25} \sin 10t + \frac{6}{25} \cos 10t$$

$$= e^{-10t} \left(-\frac{6}{25} \cos 20t - \frac{9}{50} \sin 20t \right) + \frac{3}{25} \sin 10t + \frac{6}{25} \cos 10t \Big|$$

Exercise

Consider the given circuit. Assuming that the voltage source changes from 10 to 25 V at time $t = 0$, $v_s = 10 + 15u(t)$ V, where $u(t)$ is a unit step function.



Find the expressions that describe the voltage drop across the resistor across the capacitor and the current in the loop for $t > 0$

Solution

Given: $R = 100 \Omega$, $C = 200 \mu F$, $v_s(0^-) = 10$, $v_s = 25$ $i(0^-) = 0$

$$i_s(0) = \frac{v_s - v_s(0^-)}{R} = \frac{25 - 10}{100} = 0.15$$

Applying Kirkoff Voltage law: $v_R + v_C - v_s = 0$

$$v_R + v_C = v_s$$

$$Ri + \frac{1}{C}q = v_s$$

$$R \frac{di}{dt} + \frac{1}{C} \frac{dq}{dt} = \frac{dv_s}{dt}$$

$$i = \frac{dq}{dt}$$

$$R \frac{di}{dt} + \frac{1}{C} i = \frac{dv_s}{dt}$$

$$\frac{di}{dt} + \frac{1}{RC} i = \frac{1}{R} \frac{dv_s}{dt}$$

$$\frac{1}{RC} = \frac{1}{(100)(200 \times 10^{-6})} = \frac{1}{.02} = 50$$

$$\frac{di}{dt} + 50i = 0$$

$$e^{\int 50 dt} = \underline{e^{50t}}$$

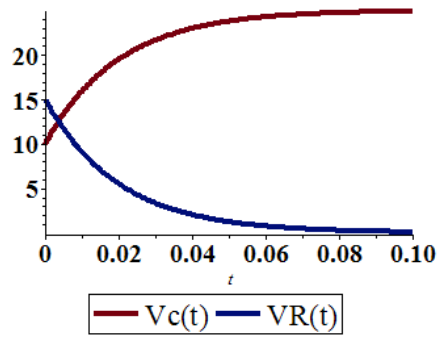
$$i(t) = Ke^{-50t}$$

$$i(0) = 0.15 \rightarrow \underline{0.15 = K}$$

$$i(t) = \underline{0.15e^{-50t}}$$

$$v_R = Ri(t) = \underline{15e^{-50t}}$$

$$v_C = v_s - v_R \\ = \underline{25 - 15e^{-50t}}$$



Exercise

Find the steady-state solution $q_p(t)$ and the steady-state current in and LRC -series circuit when the source voltage is $E(t) = E_0 \sin \omega t$

Solution

$$Lq'' + Rq' + \frac{1}{C}q = E_0 \sin \omega t$$

$$L\lambda^2 + R\lambda + \frac{1}{C} = 0 \rightarrow \lambda_{1,2} = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L} = \frac{-R\sqrt{C} \pm \sqrt{R^2C - 4L}}{2L\sqrt{C}}$$

$$q_p = A \cos \omega t + B \sin \omega t$$

$$q'_p = -\omega A \sin \omega t + \omega B \cos \omega t$$

$$q''_p = -\omega^2 A \cos \omega t - \omega^2 B \sin \omega t$$

$$-\omega^2 LA \cos \omega t - \omega^2 LB \sin \omega t - \omega RA \sin \omega t + \omega RB \cos \omega t + \frac{1}{C} A \cos \omega t + \frac{1}{C} B \sin \omega t = E_0 \sin \omega t$$

$$\begin{cases} \cos \omega t & \left(-\omega^2 L + \frac{1}{C}\right)A + \omega RB = 0 \\ \sin \omega t & -\omega RA + \left(\frac{1}{C} - \omega^2 L\right)B = E_0 \end{cases}$$

$$\Delta = \begin{vmatrix} \frac{1}{C} - \omega^2 L & \omega R \\ -\omega R & \frac{1}{C} - \omega^2 L \end{vmatrix}$$

$$= \frac{1}{C^2} - \frac{2\omega^2 L}{C} + \omega^4 L^2 + \omega^2 R^2$$

$$= \frac{1}{C^2} \left(1 - 2\omega^2 LC + \omega^4 L^2 C^2 + \omega^2 R^2 C^2\right)$$

$$\Delta_A = \begin{vmatrix} 0 & \omega R \\ E_0 & \frac{1}{C} - \omega^2 L \end{vmatrix} = \omega R E_0 \quad \Delta_B = \begin{vmatrix} \frac{1}{C} - \omega^2 L & 0 \\ -\omega R & E_0 \end{vmatrix} = \left(\frac{1}{C} - \omega^2 L \right) E_0$$

$$A = \frac{\omega R C^2 E_0}{1 - 2\omega^2 LC + \omega^4 L^2 C^2 + \omega^2 R^2 C^2}$$

$$B = \frac{C(1 - \omega LC) E_0}{1 - 2\omega^2 LC + \omega^4 L^2 C^2 + \omega^2 R^2 C^2}$$

Therefore, the steady-state charge is:

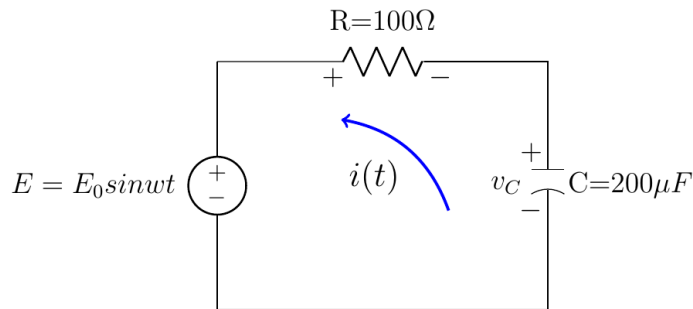
$$q_p(t) = \frac{CE_2}{1 - 2\omega^2 LC + \omega^4 L^2 C^2 + \omega^2 R^2 C^2} (\omega RC \cos \omega t + (1 - \omega LC) \sin \omega t)$$

The steady-state current is: $i_p(t) = q'_p(t)$

$$i_p(t) = \frac{CE_2}{1 - 2\omega^2 LC + \omega^4 L^2 C^2 + \omega^2 R^2 C^2} (-\omega^2 RC \sin \omega t + (\omega - \omega^2 LC) \cos \omega t)$$

Exercise

Consider the given RC -circuit with impressed emf is $E = E_0 \sin \omega t$ V. If no initial current is flowing at $t = 0$, find the current $i(t)$ for all $t > 0$.



Solution

Given: $R = 100 \, \Omega$, $C = 200 \, \mu F$, $E = E_0 \sin \omega t$ V, $i(0^+) = 0$

$$Ri + \frac{1}{C}q = E$$

$$\frac{d}{dt} \left(Ri + \frac{1}{C}q = E_0 \sin \omega t \right)$$

$$R \frac{di}{dt} + \frac{1}{C} \frac{dq}{dt} = \omega E_0 \cos \omega t \quad i = \frac{dq}{dt}$$

$$100 \frac{di}{dt} + \frac{1}{2 \times 10^{-3}} i = \omega E_0 \cos \omega t$$

$$\frac{di}{dt} + 5i = \frac{\omega E_0}{100} \cos \omega t$$

$$e^{\int 5dt} = e^{5t}$$

$$\int e^{5t} \left(\frac{\omega E_0}{100} \cos \omega t \right) dt = \frac{\omega E_0}{100} \int e^{5t} \cos \omega t dt$$

$$\int e^{5t} \cos \omega t dt = e^{5t} \left(\frac{1}{\omega} \sin \omega t + \frac{5}{\omega^2} \cos \omega t \right) - \frac{25}{\omega^2} \int e^{5t} \cos \omega t dt$$

$$\frac{\omega^2 + 25}{\omega^2} \int e^{5t} \cos \omega t dt = \frac{1}{\omega^2} e^{5t} (\omega \sin \omega t + 5 \cos \omega t)$$

$$\int e^{5t} \cos \omega t dt = \frac{1}{\omega^2 + 25} e^{5t} (\omega \sin \omega t + 5 \cos \omega t)$$

$$\int e^{5t} \left(\frac{\omega E_0}{100} \cos \omega t \right) dt = \frac{1}{100} \frac{\omega E_0}{\omega^2 + 25} e^{5t} (\omega \sin \omega t + 5 \cos \omega t)$$

$$i(t) = \frac{1}{e^{5t}} \left(\frac{1}{100} \frac{\omega E_0}{\omega^2 + 25} e^{5t} (\omega \sin \omega t + 5 \cos \omega t) + K \right)$$

$$= \frac{\omega E_0}{100(\omega^2 + 25)} (\omega \sin \omega t + 5 \cos \omega t) + \frac{K}{e^{5t}}$$

$$i(0) = 0 \rightarrow \frac{\omega E_0}{20(\omega^2 + 25)} + K = 0 \quad K = - \frac{\omega E_0}{20(\omega^2 + 25)}$$

$$i(t) = \frac{\omega E_0}{100(\omega^2 + 25)} (\omega \sin \omega t + 5 \cos \omega t) - \frac{\omega E_0}{20(\omega^2 + 25)} e^{-5t}$$

$$= \frac{\omega E_0}{20(\omega^2 + 25)} \left(\frac{\omega}{5} \sin \omega t + \cos \omega t - e^{-5t} \right)$$

		$\int \cos \omega t$
+	e^{5t}	$\frac{1}{\omega} \sin \omega t$
-	$5e^{5t}$	$-\frac{1}{\omega^2} \cos \omega t$
+	$25e^{5t}$	

Solution **Section 2.7 – Euler's & Runge-Kutta Methods**

Exercise

Calculate the first five iterations of Euler's method with step $h = 0.1$ of

$$y' = ty \quad y(0) = 1$$

Solution

t	y
0.1	1.00000000
0.2	1.01000000
0.3	1.03020000
0.4	1.06110600
0.5	1.10355024

$$\frac{dy}{dt} = ty$$

$$\int \frac{dy}{y} = \int t dt$$

$$\ln y = \frac{1}{2}t^2 + C$$

$$y(t) = e^{t^2/2+C}$$

$$y(0) = e^C = 1 \quad \rightarrow \quad \boxed{C = 0}$$

$$\boxed{y(t) = e^{t^2/2}}$$

Exercise

Calculate the first five iterations of Euler's method with step $h = 0.1$ of

$$z' = x - 2z \quad z(0) = 1$$

Solution

x	z
0.0	1.00000000
0.1	0.80000000
0.2	0.65000000
0.3	0.54000000
0.4	0.46200000
0.5	0.40960000

Exercise

Calculate the first five iterations of Euler's method with step $h = 0.1$ of: $z' = 5 - z$ $z(0) = 0$

Solution

$$x_0 = 0; \quad z_0 = 0$$

The *first* step:

$$z_1 = z_0 + h(5 - z_0) = 0 + 0.1(5 - 0) = 0.5$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

The *second* step:

$$z_2 = z_1 + h(5 - z_1) = 0.5 + 0.1(5 - 0.5) = 0.95$$

$$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$$

Euler Method

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	0.00000000	0.00000000	0.00000000
0.10	0.50000000	0.47581291	-0.02418709
0.20	0.95000000	0.90634623	-0.04365377
0.30	1.35500000	1.29590890	-0.05909110
0.40	1.71950000	1.64839977	-0.07110023
0.50	2.04755000	1.96734670	-0.08020330

$$y(t) = 5 - 5e^{-t}$$

Exercise

Given: $y' + 2xy = x$ $y(0) = 8$

- Use a computer and Euler's method to calculate three separate approximate solutions on the interval $[0, 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$.
- Use the appropriate analytic to compute the exact solution
- Plot the exact solution and approximate solutions as discrete points.

Solution

x	y
0.0	8.00000000
0.2	8.00000000
0.4	7.40000000
0.6	6.29600000
0.8	4.90496000
1.0	3.49537280

x	y
0.0	8.00000000
0.1	8.00000000
0.2	7.85000000
0.3	7.55600000
0.4	7.13264000
0.5	6.60202880
0.6	5.99182592
0.7	5.33280681
0.8	4.65621386
0.9	3.99121964
1.0	3.36280010

x	y	x	y
0.0	8.00000000		
0.05	8.00000000	0.55	6.16870319
0.10	7.96250000	0.60	5.85692451
0.15	7.88787500	0.65	5.53550904
0.20	7.77705688	0.70	5.20820096
0.25	7.63151574	0.75	4.87862689
0.30	7.45322784	0.80	4.55022987
0.35	7.24463101	0.85	4.22621148
0.40	7.00856892	0.90	3.90948351
0.45	6.74822617	0.95	3.60262999
0.50	6.46705599	1.00	3.30788014

$$y(t) = \frac{15}{2}e^{-t^2} + \frac{1}{2}$$

t	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>

0.00	8.00000000	8.00000000	0.00000000
0.20	8.00000000	7.70592079	-0.29407921
0.40	7.40000000	6.89107842	-0.50892158
0.60	6.29600000	5.73257245	-0.56342755
0.80	4.90496000	4.45469318	-0.45026682
1.00	3.49537280	3.25909581	-0.23627699

t	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>

0.00	8.00000000	8.00000000	0.00000000
0.10	8.00000000	7.92537375	-0.07462625
0.20	7.85000000	7.70592079	-0.14407921
0.30	7.55600000	7.35448389	-0.20151611
0.40	7.13264000	6.89107842	-0.24156158

0.50		6.60202880		6.34100587		-0.26102293
0.60		5.99182592		5.73257245		-0.25925347
0.70		5.33280681		5.09469796		-0.23810885
0.80		4.65621386		4.45469318		-0.20152068
0.90		3.99121964		3.83643550		-0.15478414
1.00		3.36280010		3.25909581		-0.10370430

<i>t</i>		<i>Approx.</i>		<i>Exact</i>		<i>Difference</i>

0.00		8.00000000		8.00000000		0.00000000
0.05		8.00000000		7.98127342		-0.01872658
0.10		7.96250000		7.92537375		-0.03712625
0.15		7.88787500		7.83313428		-0.05474072
0.20		7.77705688		7.70592079		-0.07113608
0.25		7.63151574		7.54559797		-0.08591777
0.30		7.45322784		7.35448389		-0.09874395
0.35		7.24463101		7.13529429		-0.10933672
0.40		7.00856892		6.89107842		-0.11749051
0.45		6.74822617		6.62514862		-0.12307755
0.50		6.46705599		6.34100587		-0.12605012
0.55		6.16870319		6.04226366		-0.12643953
0.60		5.85692451		5.73257245		-0.12435207
0.65		5.53550904		5.41554691		-0.11996214
0.70		5.20820096		5.09469796		-0.11350300
0.75		4.87862689		4.77337119		-0.10525570
0.80		4.55022987		4.45469318		-0.09553669
0.85		4.22621148		4.14152671		-0.08468477
0.90		3.90948351		3.83643550		-0.07304801
0.95		3.60262999		3.54165879		-0.06097120
1.00		3.30788014		3.25909581		-0.04878433

Exercise

Given: $y' + 2y = 2 - e^{-4t}$ $y(0) = 1$

- Solve the differential equation
- Use Euler's method and Runge-Kutta methods to calculate three separate approximate solutions on the interval $[0, 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$. Plot the exact solution and approximate solutions as discrete points.

Solution

a) $\int 2dt = e^{2t}$

$$\int (2 - e^{-4t}) e^{2t} dt = \int (2e^{2t} - e^{-2t}) dt = e^{2t} + \frac{1}{2} e^{-2t}$$

$$y(t) = \frac{1}{e^{2t}} \left(e^{2t} + \frac{1}{2} e^{-2t} + C \right)$$

$$= 1 + \frac{1}{2} e^{-4t} + C e^{-2t}$$

$$y(0) = 1 \rightarrow 1 = 1 + \frac{1}{2} + C \Rightarrow \underline{C = -\frac{1}{2}}$$

$$\underline{y(t) = 1 + \frac{1}{2} e^{-4t} - \frac{1}{2} e^{-2t}}$$

b) $h = 0.2$

Euler Method

t	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>

0.00	1.00000000	1.00000000	0.00000000
0.20	0.80000000	0.88950446	0.08950446
0.40	0.79013421	0.87628378	0.08614957
0.60	0.83370122	0.89476187	0.06106065
0.80	0.88207714	0.91943284	0.03735570
1.00	0.92109384	0.94149018	0.02039633

Runge-Kutta 2nd Order

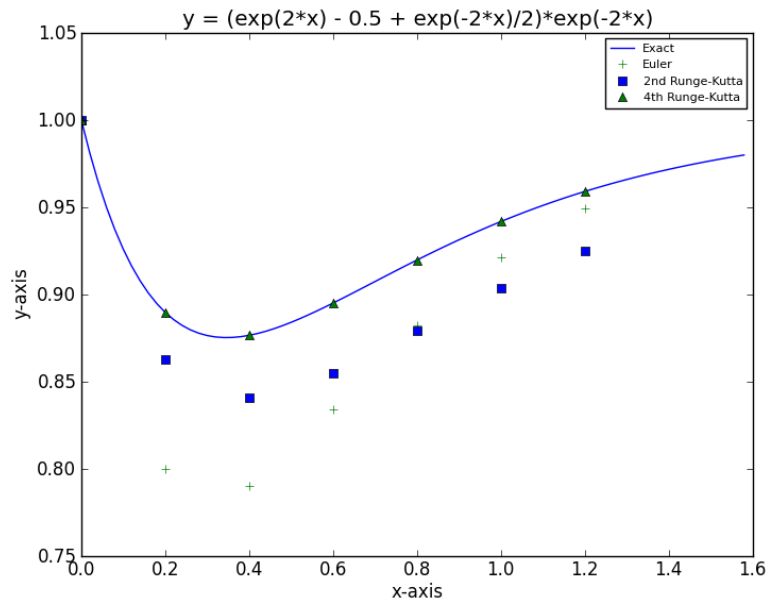
t	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>

0.00	1.00000000	1.00000000	0.00000000
0.20	0.89506710	0.88950446	-0.00556264
0.40	0.88149624	0.87628378	-0.00521246
0.60	0.89823186	0.89476187	-0.00346999
0.80	0.92127837	0.91943284	-0.00184552
1.00	0.94219199	0.94149018	-0.00070181

Runge-Kutta 4th Order

t	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>

0.00	1.00000000	1.00000000	0.00000000
0.20	0.88960071	0.88950446	-0.00009625
0.40	0.87638272	0.87628378	-0.00009894
0.60	0.89483774	0.89476187	-0.00007587
0.80	0.91948402	0.91943284	-0.00005118
1.00	0.94152197	0.94149018	-0.00003179



$h = 0.1$

Euler Method

t	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	1.00000000	1.00000000	0.00000000
0.10	0.90000000	0.92579465	0.02579465
0.20	0.85296800	0.88950446	0.03653646
0.30	0.83744150	0.87619129	0.03874979
0.40	0.83983378	0.87628378	0.03645000
0.50	0.85167737	0.88372792	0.03205055
0.60	0.86780837	0.89476187	0.02695350
0.70	0.88517490	0.90710655	0.02193165
0.80	0.90205891	0.91943284	0.01737393
0.90	0.91757091	0.93101242	0.01344151
1.00	0.93132436	0.94149018	0.01016582

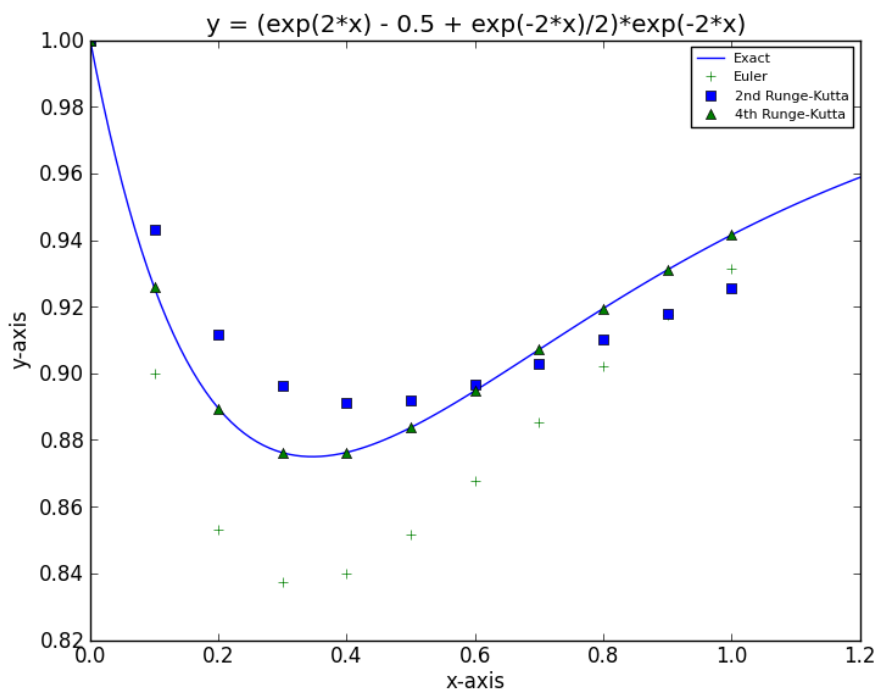
Runge-Kutta 2nd Order

t	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	1.00000000	1.00000000	0.00000000
0.10	0.92648400	0.92579465	-0.00068935
0.20	0.89043763	0.88950446	-0.00093317
0.30	0.87712599	0.87619129	-0.00093470
0.40	0.87710071	0.87628378	-0.00081694
0.50	0.88437996	0.88372792	-0.00065204
0.60	0.89524226	0.89476187	-0.00048039
0.70	0.90742943	0.90710655	-0.00032288

0.80	0.91962162	0.91943284	-0.00018878
0.90	0.93109305	0.93101242	-0.00008064
1.00	0.94148757	0.94149018	0.00000260

Runge-Kutta 4th Order

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
<hr/>			
0.00	1.00000000	1.00000000	0.00000000
0.10	0.92579791	0.92579465	-0.00000326
0.20	0.88950913	0.88950446	-0.00000467
0.30	0.87619629	0.87619129	-0.00000500
0.40	0.87628854	0.87628378	-0.00000476
0.50	0.88373216	0.88372792	-0.00000424
0.60	0.89476548	0.89476187	-0.00000361
0.70	0.90710953	0.90710655	-0.00000298
0.80	0.91943524	0.91943284	-0.00000240
0.90	0.93101431	0.93101242	-0.00000189
1.00	0.94149164	0.94149018	-0.00000146



$h = 0.05$

Euler Method

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
<hr/>			
0.00	1.00000000	1.00000000	0.00000000
0.05	0.95000000	0.95694667	0.00694667
0.10	0.91406346	0.92579465	0.01173118

0.15		0.88914111		0.90399671		0.01485559
0.20		0.87278642		0.88950446		0.01671804
0.25		0.86304133		0.88067439		0.01763306
0.30		0.85834323		0.87619129		0.01784806
0.35		0.85744919		0.87500583		0.01755664
0.40		0.85937442		0.87628378		0.01690935
0.45		0.86334216		0.87936461		0.01602246
0.50		0.86874300		0.88372792		0.01498492
0.55		0.87510193		0.88896604		0.01386410
0.60		0.88205158		0.89476187		0.01271029
0.65		0.88931053		0.90087089		0.01156037
0.70		0.89666579		0.90710655		0.01044076
0.75		0.90395871		0.91332845		0.00936974
0.80		0.91107349		0.91943284		0.00835936
0.85		0.91792803		0.92534487		0.00741684
0.90		0.92446656		0.93101242		0.00654586
0.95		0.93065372		0.93640108		0.00574736
1.00		0.93646981		0.94149018		0.00502037

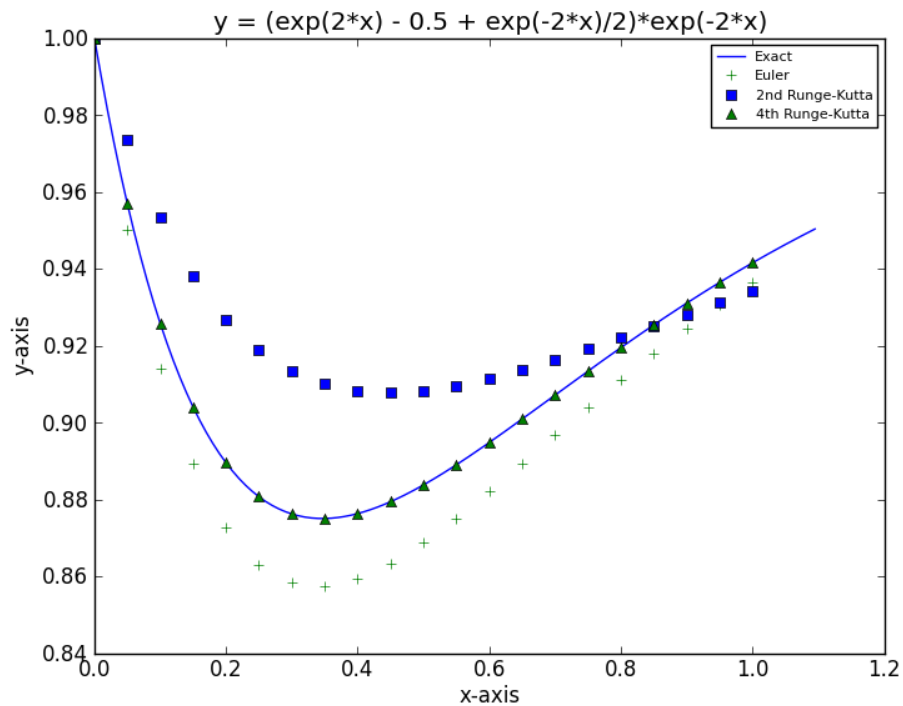
Runge-Kutta 2nd Order

t	$Approx.$	$Exact$	$Difference$

0.00	1.00000000	1.00000000	0.00000000
0.05	0.95703173	0.95694667	-0.00008506
0.10	0.92593427	0.92579465	-0.00013963
0.15	0.90416803	0.90399671	-0.00017132
0.20	0.88969058	0.88950446	-0.00018612
0.25	0.88086308	0.88067439	-0.00018869
0.30	0.87637395	0.87619129	-0.00018266
0.35	0.87517663	0.87500583	-0.00017080
0.40	0.87643901	0.87628378	-0.00015523
0.45	0.87950216	0.87936461	-0.00013754
0.50	0.88384684	0.88372792	-0.00011892
0.55	0.88906627	0.88896604	-0.00010023
0.60	0.89484396	0.89476187	-0.00008208
0.65	0.90093579	0.90087089	-0.00006489
0.70	0.90715548	0.90710655	-0.00004893
0.75	0.91336281	0.91332845	-0.00003435
0.80	0.91945408	0.91943284	-0.00002123
0.85	0.92535446	0.92534487	-0.00000958
0.90	0.93101179	0.93101242	0.00000063
0.95	0.93639162	0.93640108	0.00000946
1.00	0.94147318	0.94149018	0.00001700

Runge-Kutta 4th Order

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	1.00000000	1.00000000	0.00000000
0.05	0.95694677	0.95694667	-0.00000011
0.10	0.92579483	0.92579465	-0.00000018
0.15	0.90399694	0.90399671	-0.00000023
0.20	0.88950472	0.88950446	-0.00000026
0.25	0.88067466	0.88067439	-0.00000027
0.30	0.87619156	0.87619129	-0.00000027
0.35	0.87500610	0.87500583	-0.00000027
0.40	0.87628404	0.87628378	-0.00000026
0.45	0.87936486	0.87936461	-0.00000025
0.50	0.88372815	0.88372792	-0.00000023
0.55	0.88896625	0.88896604	-0.00000021
0.60	0.89476207	0.89476187	-0.00000020
0.65	0.90087107	0.90087089	-0.00000018
0.70	0.90710671	0.90710655	-0.00000016
0.75	0.91332860	0.91332845	-0.00000015
0.80	0.91943297	0.91943284	-0.00000013
0.85	0.92534499	0.92534487	-0.00000011
0.90	0.93101252	0.93101242	-0.00000010
0.95	0.93640117	0.93640108	-0.00000009
1.00	0.94149026	0.94149018	-0.00000008



Exercise

Given: $z' - 2z = xe^{2x}$ $z(0) = 1$

- Use a computer and Euler's method to calculate three separate approximate solutions on the interval $[0, 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a third with step size $h = 0.05$.
- Use the appropriate analytic to compute the exact solution
- Plot the exact solution and approximate solutions as discrete points.

Solution

a)

Euler Method

t	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	1.00000000	1.00000000	0.00000000
0.20	1.40000000	1.52166119	0.12166119
0.40	2.01967299	2.40358420	0.38391121
0.60	3.00558546	3.91773797	0.91215251
0.80	4.60623367	6.53800280	1.93176913
1.00	7.24121233	11.08358415	3.84237182

Euler Method

t	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	1.00000000	1.00000000	0.00000000
0.10	1.20000000	1.22750977	0.02750977
0.20	1.45221403	1.52166119	0.06944716
0.30	1.77249333	1.90411415	0.13162082
0.40	2.18165556	2.40358420	0.22192865
0.50	2.70700830	3.05806706	0.35105875
0.60	3.38432406	3.91773797	0.53341391
0.70	4.26039588	5.04872396	0.78832807
0.80	5.39633906	6.53800280	1.14166374
0.90	6.87184946	8.49975469	1.62790522
1.00	8.79068763	11.08358415	2.29289652

Euler Method

t	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	1.00000000	1.00000000	0.00000000
0.05	1.10000000	1.10655238	0.00655238
0.10	1.21276293	1.22750977	0.01474684
0.15	1.34014623	1.36504472	0.02489849
0.20	1.48428480	1.52166119	0.03737639
0.25	1.64763153	1.70024381	0.05261229
0.30	1.83300369	1.90411415	0.07111045
0.35	2.04363584	2.13709506	0.09345922
0.40	2.28324010	2.40358420	0.12034410
0.45	2.55607493	2.70863793	0.15256300
0.50	2.86702349	3.05806706	0.19104356

0.55		3.22168289		3.45854614		0.23686325
0.60		3.62646574		3.91773797		0.29127223
0.65		4.08871582		4.44443559		0.35571976
0.70		4.61683955		5.04872396		0.43188441
0.75		5.22045550		5.74216412		0.52170862
0.80		5.91056439		6.53800280		0.62743841
0.85		6.69974213		7.45141089		0.75166876
0.90		7.60235911		8.49975469		0.89739558
0.95		8.63482915		9.70290431		1.06807516
1.00		9.81589205		11.08358415		1.26769209

b) $z' - 2z = xe^{2x}$

$P(x) = -2, \quad Q(x) = xe^{2x}$

$$e^{\int -2dx} = e^{-2x}$$

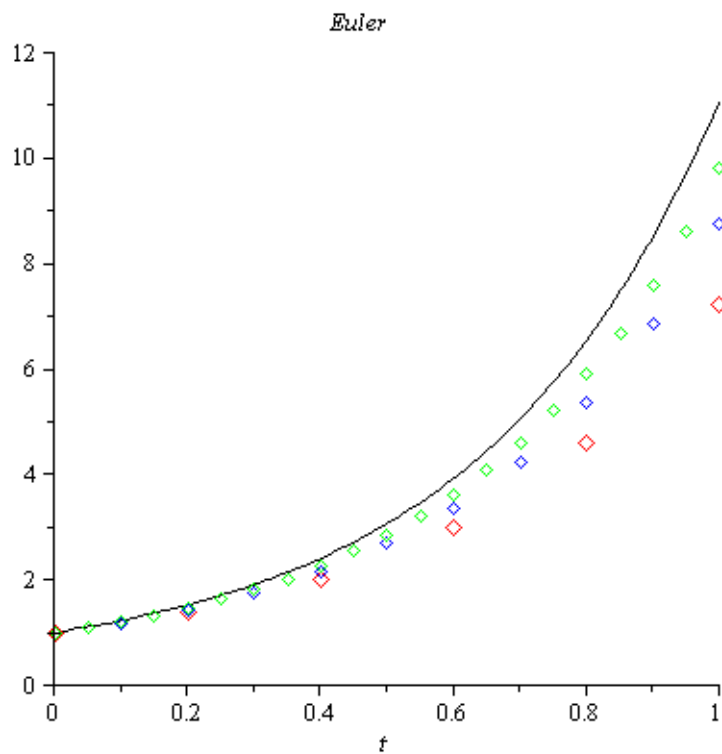
$$\int xe^{2x} e^{-2x} dx = \int x dx = \frac{1}{2} x^2$$

$$z(x) = \frac{1}{e^{-2x}} \left(\frac{1}{2} x^2 + C \right)$$

$$= \frac{1}{2} x^2 e^{2x} + C e^{2x}$$

$z(0) = \underline{C=1}$

$z(x) = \frac{1}{2} x^2 e^{2x} + e^{2x}$



Exercise

Consider the initial value problem $y' = 12y(4 - y)$ $y(0) = 1$

Use Euler's method with step size $h = 0.04$ to sketch solution on the interval $[0, 2]$

Solution

$$y(t) = \frac{4}{e^{\log 3 - 48t} + 1}$$

t	Approx.	Exact	Difference

0.00	1.00000000	1.00000000	0.00000000
0.04	2.44000000	2.77812333	0.33812333
0.08	4.26707200	3.75770045	-0.50937155
0.12	3.72005658	3.96254078	0.24248419
0.16	4.21993115	3.99446397	-0.22546718
0.20	3.77444588	3.99918742	0.22474154
0.24	4.18308995	3.99988085	-0.18320910
0.28	3.81546672	3.99998253	0.18451582
0.32	4.15342541	3.99999744	-0.15342797
0.36	3.84754974	3.99999962	0.15244989
0.40	4.12909852	3.99999994	-0.12909858
0.44	3.87322947	3.99999999	0.12677052
0.48	4.10891492	4.00000000	-0.10891492
0.52	3.89410430	4.00000000	0.10589570
0.56	4.09204138	4.00000000	-0.09204138
0.60	3.91125556	4.00000000	0.08874444
0.64	4.07786461	4.00000000	-0.07786461
0.68	3.92545437	4.00000000	0.07454563
0.72	4.06591460	4.00000000	-0.06591460
0.76	3.93727310	4.00000000	0.06272690
0.80	4.05582011	4.00000000	-0.05582011
0.84	3.94714987	4.00000000	0.05285013
0.88	4.04728141	4.00000000	-0.04728141
0.92	3.95542805	4.00000000	0.04457195
0.96	4.04005260	4.00000000	-0.04005260
1.00	3.96238159	4.00000000	0.03761841
1.04	4.03392967	4.00000000	-0.03392967
1.08	3.96823212	4.00000000	0.03176788
1.12	4.02874204	4.00000000	-0.02874204
1.16	3.97316079	4.00000000	0.02683921
1.20	4.02434630	4.00000000	-0.02434630
1.24	3.97731688	4.00000000	0.02268312
1.28	4.02062150	4.00000000	-0.02062150
1.32	3.98082411	4.00000000	0.01917589
1.36	4.01746532	4.00000000	-0.01746532
1.40	3.98378549	4.00000000	0.01621451
1.44	4.01479115	4.00000000	-0.01479115
1.48	3.98628712	4.00000000	0.01371288
1.52	4.01252558	4.00000000	-0.01252558
1.56	3.98840115	4.00000000	0.01159885
1.60	4.01060636	4.00000000	-0.01060636
1.64	3.99018815	4.00000000	0.00981185
1.68	4.00898069	4.00000000	-0.00898069
1.72	3.99169905	4.00000000	0.00830095
1.76	4.00760380	4.00000000	-0.00760380
1.80	3.99297675	4.00000000	0.00702325

1.84		4.00643771		4.00000000		-0.00643771
1.88		3.99405741		4.00000000		0.00594259
1.92		4.00545023		4.00000000		-0.00545023
1.96		3.99497153		4.00000000		0.00502847
2.00		4.00461405		4.00000000		-0.00461405

Exercise

You've seen that the error in Euler's method varies directly as the first power of the step size (i.e. $E_h \approx \lambda h$). This makes Euler's method an order to halve the error? How does this affect the number of required iterations?

Solution

Because $E_h \approx \lambda h$ halving the step size should halve the error.

$$E \approx \lambda \left(\frac{1}{2} h \right) \approx \frac{1}{2} \lambda h \approx \frac{1}{2} E_h$$

The number of iterations is given by: $N = \frac{b-a}{h}$, therefore halving the step size should double the number of iterations.

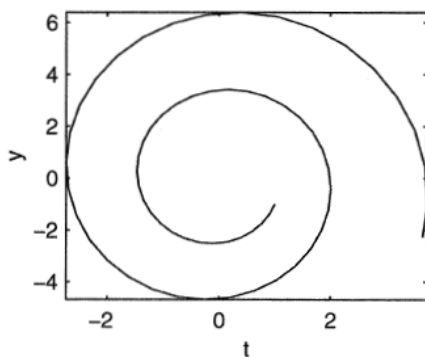
$$N = \frac{b-a}{\frac{1}{2}h} = 2 \frac{b-a}{h} \approx 2N_h$$

Exercise

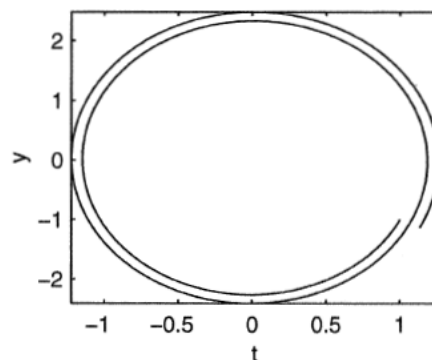
Use Euler's method to provide an approximate solution over the given time interval using the given step sizes. Provide a plot of y versus t for each step size

$$y'' + 4y = 0, \quad y(0) = 4, \quad y'(0) = 0, \quad [0, 2\pi]; \quad h = 0.1, 0.01, 0.001$$

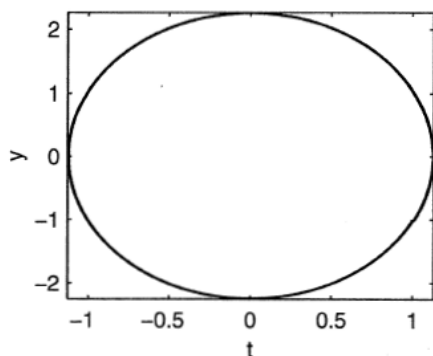
Solution



$h = 0.1$



$h = 0.01$



$h = 0.001$

Exercise

$$z' + z = \cos x \quad z(0) = 1$$

- Use a computer and Runge-Kutta method to calculate three separate approximate solutions on the interval $[0, 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$.
- Use the appropriate analytic to compute the exact solution
- Plot the exact solution and approximate solutions as discrete points.

Solution

$$z(x) = \frac{1}{2}e^{-x} + \frac{1}{2}\cos x + \frac{1}{2}\sin x$$

Runge-Kutta 2nd Order

t	Approx.	Exact	Difference

0.00	1.00000000	1.00000000	0.00000000
0.20	0.99800666	0.99873333	0.00072667
0.40	0.98887689	0.99039969	0.00152281
0.60	0.96709749	0.96939486	0.00229738
0.80	0.92871746	0.93169588	0.00297842
1.00	0.87131508	0.87482637	0.00351128

Runge-Kutta 4th Order

t	Approx.	Exact	Difference

0.00	1.00000000	1.00000000	0.00000000
0.20	0.99873272	0.99873333	0.00000061
0.40	0.99039822	0.99039969	0.00000147
0.60	0.96939245	0.96939486	0.00000241
0.80	0.93169258	0.93169588	0.00000330
1.00	0.87482232	0.87482637	0.00000405

Runge-Kutta 2nd Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.10	0.99975021	0.99983750	0.00008729
0.20	0.99855245	0.99873333	0.00018088
0.30	0.99555979	0.99583746	0.00027767
0.40	0.99002480	0.99039969	0.00037489
0.50	0.98129932	0.98176938	0.00047006
0.60	0.96883388	0.96939486	0.00056098
0.70	0.95217687	0.95282259	0.00064572
0.80	0.93097330	0.93169588	0.00072258
0.90	0.90496314	0.90575327	0.00079013
1.00	0.87397921	0.87482637	0.00084716

Runge-Kutta 4th Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.10	0.99983748	0.99983750	0.00000002
0.20	0.99873329	0.99873333	0.00000004
0.30	0.99583739	0.99583746	0.00000007
0.40	0.99039960	0.99039969	0.00000009
0.50	0.98176926	0.98176938	0.00000012
0.60	0.96939471	0.96939486	0.00000015
0.70	0.95282241	0.95282259	0.00000018
0.80	0.93169568	0.93169588	0.00000020
0.90	0.90575304	0.90575327	0.00000023
1.00	0.87482612	0.87482637	0.00000025

Runge-Kutta 2nd Order

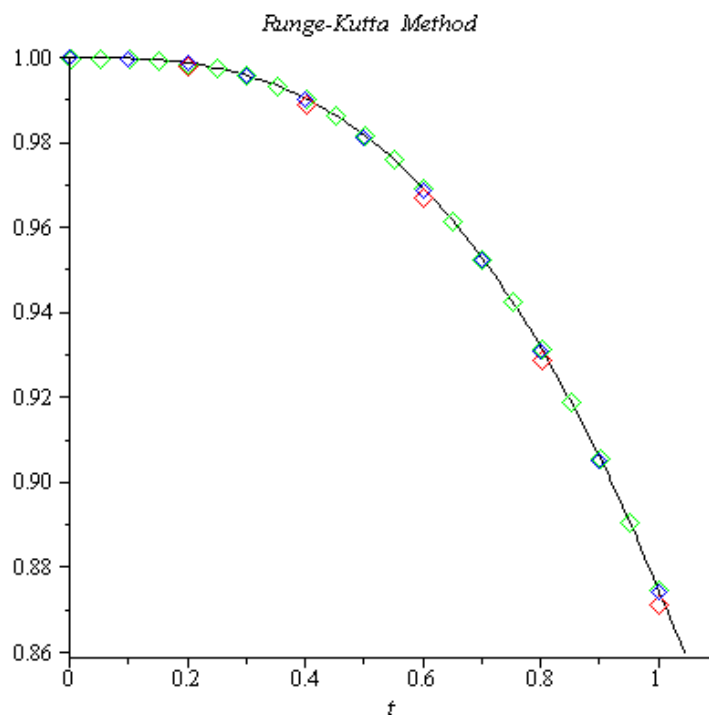
t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.05	0.99996876	0.99997943	0.00001067
0.10	0.99981570	0.99983750	0.00002180
0.15	0.99942531	0.99945859	0.00003328
0.20	0.99868831	0.99873333	0.00004502
0.25	0.99750164	0.99755858	0.00005694
0.30	0.99576852	0.99583746	0.00006894
0.35	0.99339836	0.99347931	0.00008094
0.40	0.99030682	0.99039969	0.00009287
0.45	0.98641574	0.98652039	0.00010465
0.50	0.98165315	0.98176938	0.00011623
0.55	0.97595326	0.97608078	0.00012752
0.60	0.96925639	0.96939486	0.00013847
0.65	0.96150896	0.96165799	0.00014903
0.70	0.95266344	0.95282259	0.00015915
0.75	0.94267832	0.94284709	0.00016877

0.80	0.93151803	0.93169588	0.00017785
0.85	0.91915289	0.91933924	0.00018635
0.90	0.90555903	0.90575327	0.00019423
0.95	0.89071835	0.89091981	0.00020146
1.00	0.87461836	0.87482637	0.00020801

Runge-Kutta 4th Order

t	y	y(t)	Difference

0.00	1.00000000	1.00000000	0.00000000
0.05	0.99997943	0.99997943	0.00000000
0.10	0.99983750	0.99983750	0.00000000
0.15	0.99945859	0.99945859	0.00000000
0.20	0.99873333	0.99873333	0.00000000
0.25	0.99755858	0.99755858	0.00000000
0.30	0.99583745	0.99583746	0.00000000
0.35	0.99347930	0.99347931	0.00000001
0.40	0.99039969	0.99039969	0.00000001
0.45	0.98652039	0.98652039	0.00000001
0.50	0.98176937	0.98176938	0.00000001
0.55	0.97608077	0.97608078	0.00000001
0.60	0.96939485	0.96939486	0.00000001
0.65	0.96165798	0.96165799	0.00000001
0.70	0.95282258	0.95282259	0.00000001
0.75	0.94284708	0.94284709	0.00000001
0.80	0.93169587	0.93169588	0.00000001
0.85	0.91933923	0.91933924	0.00000001
0.90	0.90575325	0.90575327	0.00000001
0.95	0.89091979	0.89091981	0.00000001
1.00	0.87482635	0.87482637	0.00000002



Exercise

Given $x' = \frac{t}{x}$ $x(0) = 1$

- Use a computer and Runge-Kutta method to calculate three separate approximate solutions on the interval $[0, 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$.
- Use the appropriate analytic to compute the exact solution
- Plot the exact solution and approximate solutions as discrete points.

Solution

a)

Runge-Kutta 2th Order

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>

0.00	1.00000000	1.00000000	0.00000000
0.20	1.01961161	1.01980390	0.00019229
0.40	1.07636229	1.07703296	0.00067067
0.60	1.16495094	1.16619038	0.00123944
0.80	1.27887002	1.28062485	0.00175483
1.00	1.41205020	1.41421356	0.00216336

Runge-Kutta 4th Order

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>

0.00	1.00000000	1.00000000	0.00000000
0.20	1.01980437	1.01980390	-0.00000046
0.40	1.07703431	1.07703296	-0.00000135
0.60	1.16619234	1.16619038	-0.00000196
0.80	1.28062701	1.28062485	-0.00000216
1.00	1.41421570	1.41421356	-0.00000214

Runge-Kutta 2th Order

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>

0.00	1.00000000	1.00000000	0.00000000
0.10	1.00497519	1.00498756	0.00001238
0.20	1.01975618	1.01980390	0.00004772
0.30	1.04392938	1.04403065	0.00010127
0.40	1.07686631	1.07703296	0.00016665
0.50	1.11779652	1.11803399	0.00023747
0.60	1.16588199	1.16619038	0.00030839
0.70	1.22027989	1.22065556	0.00037567
0.80	1.28018776	1.28062485	0.00043708
0.90	1.34487075	1.34536240	0.00049165
1.00	1.41367433	1.41421356	0.00053923

Runge-Kutta 4th Order

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>

0.00	1.00000000	1.00000000	0.00000000
0.10	1.00498757	1.00498756	-0.00000001
0.20	1.01980393	1.01980390	-0.00000003
0.30	1.04403071	1.04403065	-0.00000006
0.40	1.07703304	1.07703296	-0.00000008
0.50	1.11803409	1.11803399	-0.00000010
0.60	1.16619050	1.16619038	-0.00000012
0.70	1.22065569	1.22065556	-0.00000013
0.80	1.28062498	1.28062485	-0.00000013
0.90	1.34536254	1.34536240	-0.00000013
1.00	1.41421369	1.41421356	-0.00000013

Runge-Kutta 2th Order

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>

0.00	1.00000000	1.00000000	0.00000000
0.05	1.00124844	1.00124922	0.00000078
0.10	1.00498447	1.00498756	0.00000309
0.15	1.01118058	1.01118742	0.00000684
0.20	1.01979199	1.01980390	0.00001191
0.25	1.03075829	1.03077641	0.00001812
0.30	1.04400537	1.04403065	0.00002528
0.35	1.05944783	1.05948101	0.00003317
0.40	1.07699136	1.07703296	0.00004160
0.45	1.09653524	1.09658561	0.00005037
0.50	1.11797470	1.11803399	0.00005929
0.55	1.14120301	1.14127122	0.00006821
0.60	1.16611337	1.16619038	0.00007701
0.65	1.19260047	1.19268604	0.00008557
0.70	1.22056174	1.22065556	0.00009382
0.75	1.24989830	1.25000000	0.00010170
0.80	1.28051568	1.28062485	0.00010917
0.85	1.31232426	1.31244047	0.00011621
0.90	1.34523959	1.34536240	0.00012281
0.95	1.37918245	1.37931142	0.00012898
1.00	1.41407885	1.41421356	0.00013471

Runge-Kutta 4th Order

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>

0.00	1.00000000	1.00000000	0.00000000
0.05	1.00124922	1.00124922	-0.00000000

0.10		1.00498756		1.00498756		-0.00000000
0.15		1.01118742		1.01118742		-0.00000000
0.20		1.01980390		1.01980390		-0.00000000
0.25		1.03077641		1.03077641		-0.00000000
0.30		1.04403065		1.04403065		-0.00000000
0.35		1.05948101		1.05948101		-0.00000000
0.40		1.07703297		1.07703296		-0.00000001
0.45		1.09658562		1.09658561		-0.00000001
0.50		1.11803400		1.11803399		-0.00000001
0.55		1.14127123		1.14127122		-0.00000001
0.60		1.16619039		1.16619038		-0.00000001
0.65		1.19268605		1.19268604		-0.00000001
0.70		1.22065557		1.22065556		-0.00000001
0.75		1.25000001		1.25000000		-0.00000001
0.80		1.28062486		1.28062485		-0.00000001
0.85		1.31244048		1.31244047		-0.00000001
0.90		1.34536241		1.34536240		-0.00000001
0.95		1.37931143		1.37931142		-0.00000001
1.00		1.41421357		1.41421356		-0.00000001

b) The equation is separable:

$$x dx = t dt$$

$$\int x dx = \int t dt$$

$$\frac{1}{2}x^2 = \frac{1}{2}t^2 + C$$

$$x^2 = t^2 + 2C$$

$$x = \sqrt{t^2 + 2C}$$

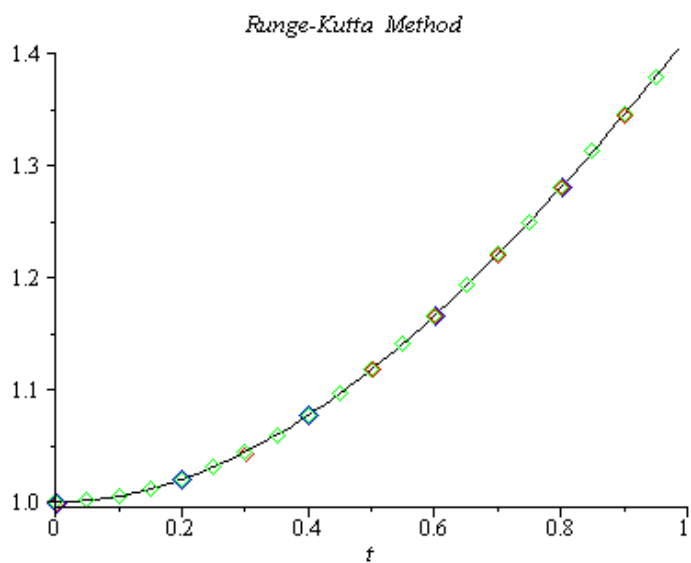
$$x(0) = \sqrt{2C}$$

$$1 = \sqrt{2C}$$

$$1 = 2C$$

$$C = \frac{1}{2}$$

$$x(t) = \sqrt{t^2 + 1}$$



Exercise

Consider the initial value problem $y' = \frac{t}{y^2}$ $y(0) = 1$

Use Runge-Kutta method with step size $h = 0.04$ to sketch solution on the interval $[0, 2]$

Solution

$$y(t) = 3^{1/3} \left(\frac{1}{2}t^2 + \frac{1}{3} \right)^{1/3}$$

Runge-Kutta 4th Order

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	1.00000000	1.00000000	0.00000000
0.04	1.00079936	1.00079936	-0.00000000
0.08	1.00318981	1.00318981	-0.00000000
0.12	1.00714877	1.00714877	-0.00000000
0.16	1.01263957	1.01263957	-0.00000000
0.20	1.01961283	1.01961282	-0.00000000
0.24	1.02800822	1.02800822	-0.00000000
0.28	1.03775651	1.03775651	-0.00000000
0.32	1.04878166	1.04878166	-0.00000001
0.36	1.06100297	1.06100297	-0.00000001
0.40	1.07433708	1.07433707	-0.00000001
0.44	1.08869975	1.08869974	-0.00000001
0.48	1.10400743	1.10400742	-0.00000001
0.52	1.12017855	1.12017854	-0.00000001
0.56	1.13713450	1.13713449	-0.00000001
0.60	1.15480036	1.15480035	-0.00000001
0.64	1.17310545	1.17310544	-0.00000001
0.68	1.19198361	1.19198360	-0.00000001
0.72	1.21137336	1.21137335	-0.00000001
0.76	1.23121787	1.23121787	-0.00000001
0.80	1.25146496	1.25146495	-0.00000001
0.84	1.27206683	1.27206682	-0.00000001
0.88	1.29297992	1.29297991	-0.00000001
0.92	1.31416464	1.31416463	-0.00000001
0.96	1.33558509	1.33558508	-0.00000001
1.00	1.35720882	1.35720881	-0.00000001
1.04	1.37900650	1.37900650	-0.00000001
1.08	1.40095174	1.40095173	-0.00000001
1.12	1.42302075	1.42302075	-0.00000001
1.16	1.44519217	1.44519216	-0.00000001
1.20	1.46744679	1.46744678	-0.00000001
1.24	1.48976740	1.48976739	-0.00000001
1.28	1.51213855	1.51213854	-0.00000001
1.32	1.53454641	1.53454640	-0.00000001

1.36		1.55697860		1.55697859		-0.00000001
1.40		1.57942403		1.57942403		-0.00000001
1.44		1.60187281		1.60187281		-0.00000001
1.48		1.62431609		1.62431608		-0.00000001
1.52		1.64674596		1.64674596		-0.00000001
1.56		1.66915540		1.66915539		-0.00000001
1.60		1.69153812		1.69153811		-0.00000001
1.64		1.71388854		1.71388853		-0.00000001
1.68		1.73620170		1.73620169		-0.00000001
1.72		1.75847320		1.75847319		-0.00000001
1.76		1.78069914		1.78069913		-0.00000001
1.80		1.80287607		1.80287606		-0.00000000
1.84		1.82500094		1.82500094		-0.00000000
1.88		1.84707109		1.84707109		-0.00000000
1.92		1.86908417		1.86908417		-0.00000000
1.96		1.89103813		1.89103813		-0.00000000
2.00		1.91293119		1.91293118		-0.00000000

Exercise

Consider the initial value problem $y' - y = -\frac{1}{2}e^{t/2} \sin 5t + 5e^{t/2} \cos 5t$ $y(0) = 0$

Use Runge-Kutta method with step size $h = 0.05$ to sketch solution on the interval $[0, 5]$

Solution

$$e^{\int -dt} = e^{-t}$$

$$\int \left(-\frac{1}{2}e^{t/2} \sin 5t + 5e^{t/2} \cos 5t \right) e^{-t} dt = -\frac{1}{2} \int \left(e^{-t/2} \sin 5t \right) dt + 5 \int \left(e^{-t/2} \cos 5t \right) dt$$

		$\int \sin 5t$
+	$e^{-t/2}$	$-\frac{1}{5} \cos 5t$
-	$-\frac{1}{2}e^{-t/2}$	$-\frac{1}{25} \sin 5t$
+	$\frac{1}{4}e^{-t/2}$	

		$\int \cos 5t$
+	$e^{-t/2}$	$\frac{1}{5} \sin 5t$
-	$-\frac{1}{2}e^{-t/2}$	$-\frac{1}{25} \cos 5t$
+	$\frac{1}{4}e^{-t/2}$	

$$\int \left(e^{-t/2} \sin 5t \right) dt = \left(-\frac{1}{5} \cos 5t - \frac{1}{50} \sin 5t \right) e^{-t/2} - \frac{1}{100} \int \left(e^{-t/2} \sin 5t \right) dt$$

$$\frac{101}{100} \int \left(e^{-t/2} \sin 5t \right) dt = -\frac{1}{50} (10 \cos 5t + \sin 5t) e^{-t/2}$$

$$\int \left(e^{-t/2} \sin 5t \right) dt = -\frac{2}{101} (10 \cos 5t + \sin 5t) e^{-t/2}$$

$$\int \left(e^{-t/2} \cos 5t \right) dt = e^{-t/2} \left(\frac{1}{5} \sin 5t - \frac{1}{50} \cos 5t \right) - \frac{1}{100} \int \left(e^{-t/2} \cos 5t \right) dt$$

$$\frac{101}{100} \int \left(e^{-t/2} \cos 5t \right) dt = \frac{1}{50} e^{-t/2} (10 \sin 5t - \cos 5t)$$

$$\int \left(e^{-t/2} \cos 5t \right) dt = \frac{2}{101} e^{-t/2} (10 \sin 5t - \cos 5t)$$

$$\begin{aligned} \int \left(-\frac{1}{2} e^{t/2} \sin 5t + 5 e^{t/2} \cos 5t \right) e^{-t} dt &= \left(\frac{10}{101} \cos 5t + \frac{1}{101} \sin 5t + \frac{100}{101} \sin 5t - \frac{10}{101} \cos 5t \right) e^{-t/2} \\ &= e^{-t/2} \sin 5t \end{aligned}$$

$$y(t) = e^t \left(e^{-t/2} \sin 5t + C \right)$$

$$= e^{t/2} \sin 5t + C e^t$$

$$y(0) = 0 \rightarrow \underline{C = 0}$$

$$y(t) = e^{t/2} \sin 5t$$

Euler Method

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	0.00000000	0.00000000	0.00000000
0.05	0.25000000	0.25366702	0.00366702
0.10	0.50451846	0.50400621	-0.00051225
0.15	0.74778853	0.73472762	-0.01306091
0.20	0.96397872	0.92996926	-0.03400946
0.25	1.13821003	1.07534045	-0.06286957
0.30	1.25756378	1.15892383	-0.09863994
0.35	1.31201510	1.17216954	-0.13984556
0.40	1.29522788	1.11061839	-0.18460949
0.45	1.20515309	0.97439874	-0.23075435
0.50	1.04438175	0.76845344	-0.27592831
0.55	0.82021731	0.50246840	-0.31774891
0.60	0.54444836	0.19049209	-0.35395627
0.65	0.23282095	-0.14974538	-0.38256633
0.70	-0.09577085	-0.49778510	-0.40201424
0.75	-0.42033859	-0.83161681	-0.41127822
0.80	-0.71904181	-1.12901665	-0.40997485
0.85	-0.97054890	-1.36896715	-0.39841824
0.90	-1.15543496	-1.53307240	-0.37763744
0.95	-1.25752830	-1.60687699	-0.34934869
1.00	-1.26511659	-1.58099885	-0.31588225
1.05	-1.17192748	-1.45199341	-0.28006593
1.10	-0.97780916	-1.22287990	-0.24507073
1.15	-0.68905162	-0.90327826	-0.21422665
1.20	-0.31830941	-0.50912823	-0.19081882

1.25	0.11588941	-0.06198694	-0.17787635
1.30	0.59003789	0.41207112	-0.17796677
1.35	1.07691137	0.88390140	-0.19300997
1.40	1.54713288	1.32300854	-0.22412434
1.45	1.97095749	1.69944069	-0.27151680
1.50	2.32017202	1.98574597	-0.33442605
1.55	2.56999372	2.15886828	-0.41112544
1.60	2.70084550	2.20185727	-0.49898823
1.65	2.69988727	2.10527280	-0.59461447
1.70	2.56219211	1.86817786	-0.69401425
1.75	2.29147346	1.49863486	-0.79283860
1.80	1.90029341	1.01364791	-0.88664550
1.85	1.40971217	0.43852637	-0.97118580
1.90	0.84837261	-0.19431898	-1.04269159
1.95	0.25104980	-0.84709881	-1.09814861
2.00	-0.34326837	-1.47880270	-1.13553433
2.05	-0.89366994	-2.04767466	-1.15400472
2.10	-1.35984868	-2.51386357	-1.15401490
2.15	-1.70472418	-2.84208687	-1.13736269
2.20	-1.89698837	-3.00413660	-1.10714824
2.25	-1.91341049	-2.98105850	-1.06764801
2.30	-1.74073987	-2.76484685	-1.02410698
2.35	-1.37706328	-2.35952135	-0.98245807
2.40	-0.83250141	-1.78148483	-0.94898341
2.45	-0.12916591	-1.05910196	-0.92993604
2.50	0.69965839	-0.23148617	-0.93114456
2.55	1.61109302	0.65346778	-0.95762524
2.60	2.55494455	1.54171751	-1.01322704
2.65	3.47657170	2.37623761	-1.10033409
2.70	4.32018626	3.10053857	-1.21964769
2.75	5.03239764	3.66233161	-1.37006604
2.80	5.56578542	4.01711092	-1.54867451
2.85	5.88227111	4.13141845	-1.75085267
2.90	5.95606196	3.98556468	-1.97049728
2.95	5.77595524	3.57560239	-2.20035285
3.00	5.34682192	2.91438791	-2.43243401
3.05	4.69013206	2.03161484	-2.65851722
3.10	3.84343922	0.97276538	-2.87067384
3.15	2.85880347	-0.20300879	-3.06181226
3.20	1.80020034	-1.42599446	-3.22619481
3.25	0.74003061	-2.61986419	-3.35989480
3.30	-0.24509055	-3.70625192	-3.46116137
3.35	-1.07903060	-4.60969444	-3.53066383
3.40	-1.69104556	-5.26263929	-3.57159374
3.45	-2.02058926	-5.61020009	-3.58961083
3.50	-2.02171162	-5.61434002	-3.59262841
3.55	-1.66674125	-5.25718495	-3.59044370
3.60	-0.94897881	-4.54320809	-3.59422929
3.65	0.11582328	-3.50008777	-3.61591105

3.70	1.48935225	-2.17811492	-3.66746718
3.75	3.11207493	-0.64811430	-3.76018923
3.80	4.90601419	1.00206320	-3.90395099
3.85	6.77885700	2.67232428	-4.10653272
3.90	8.62919773	4.25615056	-4.37304717
3.95	10.35263755	5.64712789	-4.70550966
4.00	11.84839105	6.74580367	-5.10258738
4.05	13.02600082	7.46644758	-5.55955324
4.10	13.81173327	7.74327526	-6.06845801
4.15	14.15422757	7.53570683	-6.61852074
4.20	14.02899591	6.83227210	-7.19672381
4.25	13.44142635	5.65284162	-7.78858473
4.30	12.42801840	4.04895406	-8.37906435
4.35	11.05568119	2.10212134	-8.95355985
4.40	9.41904025	-0.07988319	-9.49892344
4.45	7.63582435	-2.36861780	-10.00444215
4.50	5.84053115	-4.62218308	-10.46271423
4.55	4.17669135	-6.69366883	-10.87036018
4.60	2.78815793	-8.44035671	-11.22851463
4.65	1.80993130	-9.73312476	-11.54305605
4.70	1.35908774	-10.46545870	-11.82454643
4.75	1.52640052	-10.56146949	-12.08787001
4.80	2.36922910	-9.98235001	-12.35157911
4.85	3.90619931	-8.73077509	-12.63697440
4.90	6.11410924	-6.85285609	-12.96696533
4.95	8.92737431	-4.43739883	-13.36477314
5.00	12.24017767	-1.61237440	-13.85255207

Runge-Kutta 2nd Order

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>

0.00	0.00000000	0.00000000	0.00000000
0.05	0.25225923	0.25366702	0.00140779
0.10	0.50126928	0.50400621	0.00273693
0.15	0.73083289	0.73472762	0.00389473
0.20	0.92517466	0.92996926	0.00479460
0.25	1.06997848	1.07534045	0.00536197
0.30	1.15338415	1.15892383	0.00553968
0.35	1.16687699	1.17216954	0.00529254
0.40	1.10600766	1.11061839	0.00461072
0.45	0.97088706	0.97439874	0.00351168
0.50	0.76641304	0.76845344	0.00204040
0.55	0.50220045	0.50246840	0.00026796
0.60	0.19220390	0.19049209	-0.00171182
0.65	-0.14595823	-0.14974538	-0.00378715
0.70	-0.49195043	-0.49778510	-0.00583466
0.75	-0.82389011	-0.83161681	-0.00772670
0.80	-1.11967729	-1.12901665	-0.00933937
0.85	-1.35840643	-1.36896715	-0.01056072

0.90		-1.52177383		-1.53307240		-0.01129857
0.95		-1.59538951		-1.60687699		-0.01148748
1.00		-1.56990449		-1.58099885		-0.01109436
1.05		-1.44187115		-1.45199341		-0.01012226
1.10		-1.21426784		-1.22287990		-0.00861206
1.15		-0.89663646		-0.90327826		-0.00664180
1.20		-0.50480467		-0.50912823		-0.00432356
1.25		-0.06018891		-0.06198694		-0.00179803
1.30		0.41129806		0.41207112		0.00077306
1.35		0.88068514		0.88390140		0.00321626
1.40		1.31765160		1.32300854		0.00535694
1.45		1.69241026		1.69944069		0.00703043
1.50		1.97765283		1.98574597		0.00809314
1.55		2.15043546		2.15886828		0.00843282
1.60		2.19387987		2.20185727		0.00797740
1.65		2.09857119		2.10527280		0.00670161
1.70		1.86354684		1.86817786		0.00463102
1.75		1.49679192		1.49863486		0.00184294
1.80		1.01518384		1.01364791		-0.00153593
1.85		0.44386158		0.43852637		-0.00533521
1.90		-0.18496945		-0.19431898		-0.00934953
1.95		-0.83374872		-0.84709881		-0.01335009
2.00		-1.46170451		-1.47880270		-0.01709819
2.05		-2.02731452		-2.04767466		-0.02036014
2.10		-2.49094092		-2.51386357		-0.02292265
2.15		-2.81747925		-2.84208687		-0.02460762
2.20		-2.97885110		-3.00413660		-0.02528550
2.25		-2.95617232		-2.98105850		-0.02488618
2.30		-2.74144026		-2.76484685		-0.02340659
2.35		-2.33860692		-2.35952135		-0.02091443
2.40		-1.76393735		-1.78148483		-0.01754748
2.45		-1.04559354		-1.05910196		-0.01350842
2.50		-0.22243099		-0.23148617		-0.00905518
2.55		0.65795501		0.65346778		-0.00448723
2.60		1.54184603		1.54171751		-0.00012852
2.65		2.37254558		2.37623761		0.00369203
2.70		3.09387721		3.10053857		0.00666136
2.75		3.65382846		3.66233161		0.00850315
2.80		4.00811352		4.01711092		0.00899739
2.85		4.12342111		4.13141845		0.00799734
2.90		3.98012211		3.98556468		0.00544257
2.95		3.57423517		3.57560239		0.00136722
3.00		2.91848542		2.91438791		-0.00409752
3.05		2.04234183		2.03161484		-0.01072699
3.10		0.99097829		0.97276538		-0.01821290
3.15		-0.17682977		-0.20300879		-0.02617901
3.20		-1.39179241		-1.42599446		-0.03420205
3.25		-2.57802748		-2.61986419		-0.04183671
3.30		-3.65760857		-3.70625192		-0.04864335

3.35	-4.55547762	-4.60969444	-0.05421682
3.40	-5.20442486	-5.26263929	-0.05821444
3.45	-5.54981858	-5.61020009	-0.06038151
3.50	-5.55376757	-5.61434002	-0.06057245
3.55	-5.19841904	-5.25718495	-0.05876591
3.60	-4.48813526	-4.54320809	-0.05507283
3.65	-3.45035150	-3.50008777	-0.04973627
3.70	-2.13499213	-2.17811492	-0.04312279
3.75	-0.61240881	-0.64811430	-0.03570549
3.80	1.03010243	1.00206320	-0.02803923
3.85	2.69305372	2.67232428	-0.02072944
3.90	4.27054658	4.25615056	-0.01439602
3.95	5.65676237	5.64712789	-0.00963448
4.00	6.75278040	6.74580367	-0.00697673
4.05	7.47330147	7.46644758	-0.00685388
4.10	7.75283900	7.74327526	-0.00956374
4.15	7.55095199	7.53570683	-0.01524515
4.20	6.85613350	6.83227210	-0.02386140
4.25	5.68803555	5.65284162	-0.03519394
4.30	4.09780169	4.04895406	-0.04884764
4.35	2.16638901	2.10212134	-0.06426767
4.40	0.00088437	-0.07988319	-0.08076755
4.45	-2.27105067	-2.36861780	-0.09756712
4.50	-4.50834452	-4.62218308	-0.11383856
4.55	-6.56491101	-6.69366883	-0.12875783
4.60	-8.29879815	-8.44035671	-0.14155855
4.65	-9.58153975	-9.73312476	-0.15158501
4.70	-10.30711803	-10.46545870	-0.15834067
4.75	-10.39994039	-10.56146949	-0.16152910
4.80	-9.82126597	-9.98235001	-0.16108404
4.85	-8.57358877	-8.73077509	-0.15718632
4.90	-6.70259041	-6.85285609	-0.15026568
4.95	-4.29641205	-4.43739883	-0.14098678
5.00	-1.48215502	-1.61237440	-0.13021938

Runge-Kutta 4th Order

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>

0.00	0.00000000	0.00000000	0.00000000
0.05	0.25366730	0.25366702	-0.00000028
0.10	0.50400675	0.50400621	-0.00000054
0.15	0.73472836	0.73472762	-0.00000075
0.20	0.92997015	0.92996926	-0.00000089
0.25	1.07534142	1.07534045	-0.00000096
0.30	1.15892478	1.15892383	-0.00000095
0.35	1.17217037	1.17216954	-0.00000084
0.40	1.11061902	1.11061839	-0.00000064
0.45	0.97439909	0.97439874	-0.00000035
0.50	0.76845344	0.76845344	0.00000001

0.55		0.50246799		0.50246840		0.00000042
0.60		0.19049122		0.19049209		0.00000086
0.65		-0.14974669		-0.14974538		0.00000131
0.70		-0.49778684		-0.49778510		0.00000175
0.75		-0.83161894		-0.83161681		0.00000213
0.80		-1.12901910		-1.12901665		0.00000245
0.85		-1.36896982		-1.36896715		0.00000267
0.90		-1.53307519		-1.53307240		0.00000279
0.95		-1.60687978		-1.60687699		0.00000279
1.00		-1.58100151		-1.58099885		0.00000266
1.05		-1.45199583		-1.45199341		0.00000242
1.10		-1.22288197		-1.22287990		0.00000207
1.15		-0.90327990		-0.90327826		0.00000164
1.20		-0.50912938		-0.50912823		0.00000115
1.25		-0.06198757		-0.06198694		0.00000063
1.30		0.41207099		0.41207112		0.00000013
1.35		0.88390173		0.88390140		-0.00000033
1.40		1.32300925		1.32300854		-0.00000070
1.45		1.69944165		1.69944069		-0.00000096
1.50		1.98574704		1.98574597		-0.00000108
1.55		2.15886931		2.15886828		-0.00000103
1.60		2.20185808		2.20185727		-0.00000081
1.65		2.10527321		2.10527280		-0.00000041
1.70		1.86817772		1.86817786		0.00000014
1.75		1.49863402		1.49863486		0.00000084
1.80		1.01364626		1.01364791		0.00000165
1.85		0.43852385		0.43852637		0.00000252
1.90		-0.19432240		-0.19431898		0.00000342
1.95		-0.84710311		-0.84709881		0.00000430
2.00		-1.47880780		-1.47880270		0.00000510
2.05		-2.04768043		-2.04767466		0.00000577
2.10		-2.51386985		-2.51386357		0.00000628
2.15		-2.84209346		-2.84208687		0.00000659
2.20		-3.00414329		-3.00413660		0.00000669
2.25		-2.98106506		-2.98105850		0.00000656
2.30		-2.76485306		-2.76484685		0.00000621
2.35		-2.35952701		-2.35952135		0.00000566
2.40		-1.78148978		-1.78148483		0.00000496
2.45		-1.05910610		-1.05910196		0.00000414
2.50		-0.23148944		-0.23148617		0.00000327
2.55		0.65346536		0.65346778		0.00000242
2.60		1.54171585		1.54171751		0.00000165
2.65		2.37623657		2.37623761		0.00000104
2.70		3.10053793		3.10053857		0.00000064
2.75		3.66233110		3.66233161		0.00000051
2.80		4.01711023		4.01711092		0.00000068
2.85		4.13141725		4.13141845		0.00000119
2.90		3.98556265		3.98556468		0.00000203
2.95		3.57559921		3.57560239		0.00000319

3.00		2.91438329		2.91438791		0.00000462
3.05		2.03160857		2.03161484		0.00000627
3.10		0.97275731		0.97276538		0.00000808
3.15		-0.20301874		-0.20300879		0.00000995
3.20		-1.42600625		-1.42599446		0.00001179
3.25		-2.61987770		-2.61986419		0.00001351
3.30		-3.70626694		-3.70625192		0.00001502
3.35		-4.60971069		-4.60969444		0.00001625
3.40		-5.26265642		-5.26263929		0.00001712
3.45		-5.61021769		-5.61020009		0.00001760
3.50		-5.61435770		-5.61434002		0.00001767
3.55		-5.25720229		-5.25718495		0.00001735
3.60		-4.54322475		-4.54320809		0.00001666
3.65		-3.50010344		-3.50008777		0.00001567
3.70		-2.17812940		-2.17811492		0.00001448
3.75		-0.64812749		-0.64811430		0.00001319
3.80		1.00205127		1.00206320		0.00001193
3.85		2.67231345		2.67232428		0.00001083
3.90		4.25614054		4.25615056		0.00001002
3.95		5.64711827		5.64712789		0.00000962
4.00		6.74579394		6.74580367		0.00000973
4.05		7.46643715		7.46644758		0.00001043
4.10		7.74326349		7.74327526		0.00001177
4.15		7.53569307		7.53570683		0.00001376
4.20		6.83225573		6.83227210		0.00001637
4.25		5.65282207		5.65284162		0.00001955
4.30		4.04893088		4.04895406		0.00002318
4.35		2.10209420		2.10212134		0.00002714
4.40		-0.07991446		-0.07988319		0.00003128
4.45		-2.36865321		-2.36861780		0.00003542
4.50		-4.62222247		-4.62218308		0.00003939
4.55		-6.69371186		-6.69366883		0.00004303
4.60		-8.44040289		-8.44035671		0.00004618
4.65		-9.73317349		-9.73312476		0.00004873
4.70		-10.46550929		-10.46545870		0.00005060
4.75		-10.56152123		-10.56146949		0.00005174
4.80		-9.98240218		-9.98235001		0.00005217
4.85		-8.73082706		-8.73077509		0.00005196
4.90		-6.85290732		-6.85285609		0.00005123
4.95		-4.43744898		-4.43739883		0.00005015
5.00		-1.61242331		-1.61237440		0.00004891

