Section 3.3 – Inverse Laplace Transform

Definition

If f is a continuous function of exponential order and $\mathcal{L}(f)(s)=F(s)$, then we call f the inverse Laplace transform of F,

$$f(t)=\mathcal{L}^{-1}(F(s))$$

$$F(s)=\mathcal{L}(f(t)) \quad \Leftrightarrow \quad f(t)=\mathcal{L}^{-1}(F(s))$$

$$f(t)\ \overset{\text{Laplace tansform - }\mathcal{L}}{\underset{\text{Inverse Laplace tansform - }\mathcal{L}^{-1}}{\rightleftarrows}}\ F(S)$$

Note: Inverse transforms are not unique. If f_1 and f_2 are identical except at a discrete set of points, then $\mathcal{L}(f_1(t))=\mathcal{L}(f_2(t))$. However, there is at most one continuous function f satisfying $\mathcal{L}\{f(t)\}=F(s)$

Laplace Transform Linear

Proposition

$$\begin{aligned}\mathcal{L}^{-1}[aF(s)+bG(s)] &= a.\mathcal{L}^{-1}(F(s))+b.\mathcal{L}^{-1}(G(s))\\ &= af(t)+bg(t)\end{aligned}$$

Example

Compute the inverse Laplace transform of $F(s)=\frac{1}{s-2}-\frac{16}{s^2+4}$

Solution

$$\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}=e^{2t}$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}=\sin 2t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-2}-8\frac{2}{s^2+4}\right\}=e^{2t}-8\sin 2t$$

Example

Compute the inverse Laplace transform of $F(s)=\dfrac{1}{s^2-2s-3}$; $s>3$

Solution

$$\frac{1}{s^2-2s-3}=\frac{A}{s-3}+\frac{B}{s+1}$$

$$1=As+A+Bs-3B$$

$$\begin{cases}A+B=0\\A-3B=1\end{cases}\quad\to A=\frac{1}{4}\quad B=-\frac{1}{4}$$

$$\frac{1}{s^2-2s-3}=\frac{1}{4}\left(\frac{1}{s-3}-\frac{1}{s+1}\right)$$

$$\mathcal{L}^{-1}\{F(s)\}=\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-3}-\frac{1}{s+1}\right\}$$

$$=\frac{1}{4}\left(e^{3t}-e^{-t}\right)$$

Example

Compute the inverse Laplace transform of $F(s)=\dfrac{1}{s^2+4s+13}$

Solution

$$s^2+4s+13=s^2+4s+4+9$$

$$=(s+2)^2+3^2$$

$$\mathcal{L}^{-1}\left\{\frac{1}{3}\frac{3}{(s+2)^2+3^2}\right\}\ =\frac{1}{3}e^{-2t}\sin 3t$$

Example

Find the inverse Laplace transform of $F(s)=\dfrac{2s^2+s+13}{(s-1)\left((s+1)^2+4\right)}$

Solution

$$\frac{2s^2+s+13}{(s-1)\left((s+1)^2+4\right)}=\frac{A}{(s-1)}+\frac{Bs+C}{(s+1)^2+4}$$

$$2s^2+s+13=As^2+2As+5A+Bs^2+(C-B)s-C$$

$$\begin{array}{l} s^2 \\ s \\ s^0 \end{array} \begin{cases} A+B=2 & \rightarrow B=2-A \\ 2A-B+C=1 & \\ 5A-C=13 & \rightarrow C=5A-13 \end{cases} \qquad 2A-2+A+5A-13=1 \Rightarrow A=2$$

$$\begin{cases} B=2-2=0 \\ C=5(2)-13=-3 \end{cases}$$

$$F(s)=\frac{2}{(s-1)}-\frac{3}{(s+1)^2+4}$$

$$f(t)=\mathcal{L}^{-1}\left\{\frac{2}{(s-1)}-\frac{3}{(s+1)^2+4}\right\}$$

$$=2\mathcal{L}^{-1}\left\{\frac{1}{(s-1)}\right\}-3\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\}$$

$$\underline{=2e^t-\frac{3}{2}e^{-t}\sin 2t}$$

Exercises *Section* 3.3 - Inverse Laplace Transform

Find the inverse Laplace transform of

1. $Y(s)=\dfrac{1}{3s+2}$
2. $Y(s)=\dfrac{2}{3-5s}$
3. $Y(s)=\dfrac{1}{s^2+4}$
4. $Y(s)=\dfrac{3}{s^2}$
5. $Y(s)=\dfrac{3s+2}{s^2+25}$
6. $Y(s)=\dfrac{2-5s}{s^2+9}$
7. $Y(s)=\dfrac{5}{(s+2)^3}$
8. $Y(s)=\dfrac{1}{(s-1)^6}$
9. $Y(s)=\dfrac{4(s-1)}{(s-1)^2+4}$
10. $Y(s)=\dfrac{2s-3}{(s-1)^2+5}$
11. $Y(s)=\dfrac{2s-1}{(s+1)(s-2)}$
12. $Y(s)=\dfrac{2s-2}{(s-4)(s+2)}$
13. $Y(s)=\dfrac{7s^2+3s+16}{(s+1)\left(s^2+4\right)}$
14. $Y(s)=\dfrac{1}{(s+2)^2\left(s^2+9\right)}$
15. $Y(s)=\dfrac{s}{(s+2)^2\left(s^2+9\right)}$
16. $Y(s)=\dfrac{1}{(s+1)^2\left(s^2-4\right)}$
17. $Y(s)=\dfrac{7s^2+20s+53}{(s-1)\left(s^2+2s+5\right)}$
18. $F(s)=\dfrac{1}{s^3}$
19. $F(s)=\dfrac{1}{s^4}$
20. $F(s)=\dfrac{1}{s^2}-\dfrac{48}{s^5}$
21. $F(s)=\dfrac{1}{s^2}-\dfrac{1}{s}+\dfrac{1}{s-2}$
22. $F(s)=\dfrac{4}{s}+\dfrac{4}{s^5}+\dfrac{1}{s-8}$
23. $F(s)=\dfrac{1}{4s+1}$
24. $F(s)=\dfrac{1}{5s-2}$
25. $F(s)=\dfrac{s+1}{s^2+2}$
26. $F(s)=\dfrac{2s-6}{s^2+9}$
27. $F(s)=\dfrac{10s}{s^2+16}$
28. $F(s)=\left(\dfrac{2}{s}-\dfrac{1}{s^3}\right)^2$
29. $F(s)=\dfrac{(s+1)^3}{s^4}$
30. $F(s)=\dfrac{(s+2)^2}{s^3}$
31. $F(s)=\dfrac{1}{s^4-9}$
32. $F(s)=\dfrac{1}{s^3+5s}$
33. $F(s)=\dfrac{5}{s^2+36}$
34. $F(s)=\dfrac{10s}{s^2+16}$
35. $F(s)=\dfrac{4s}{4s^2+1}$
36. $F(s)=\dfrac{1}{4s^2+1}$

37. $F(s)=\dfrac{1}{s^2+3s}$

38. $F(s)=\dfrac{s+1}{s^2-4s}$

39. $F(s)=\dfrac{1}{s^3+5s}$

40. $F(s)=\dfrac{3}{s^2+9}$

41. $F(s)=\dfrac{2}{s^2+4}$

42. $F(s)=\dfrac{3}{(2s+5)^3}$

43. $F(s)=\dfrac{6}{(s-1)^4}$

44. $F(s)=\dfrac{5}{(s+2)^4}$

45. $F(s)=\dfrac{s-1}{s^2-2s+5}$

46. $F(s)=\dfrac{3s+2}{s^2+2s+10}$

47. $F(s)=\dfrac{s}{s^2+2s-3}$

48. $F(s)=\dfrac{1}{s^2+2s-20}$

49. $F(s)=\dfrac{s+1}{s^2+2s+10}$

50. $F(s)=\dfrac{1}{s^2+4s+8}$

51. $F(s)=\dfrac{2s+16}{s^2+4s+13}$

52. $F(s)=\dfrac{2s+16}{s^2+4s+13}$

53. $F(s)=\dfrac{s-1}{2s^2+s+6}$

54. $F(s)=\dfrac{s^2+1}{s^3-2s^2-8s}$

55. $F(s)=\dfrac{6s+3}{s^4+5s^2+4}$

56. $F(s)=\dfrac{s-3}{\left(s-\sqrt{3}\right)\left(s+\sqrt{3}\right)}$

57. $F(s)=\dfrac{1}{\left(s^2+1\right)\left(s^2+4\right)}$

58. $F(s)=\dfrac{2s-4}{\left(s^2+s\right)\left(s^2+1\right)}$

59. $F(s)=\dfrac{s}{(s+2)\left(s^2+4\right)}$

60. $F(s)=\dfrac{s^2+1}{s(s-1)(s+1)(s-2)}$

61. $F(s)=\dfrac{s}{(s-2)(s-3)(s-6)}$

62. $F(s)=\dfrac{7s-1}{(s+1)(s+2)(s-3)}$

63. $F(s)=\dfrac{s^2+9s+2}{(s-1)^2(s+3)}$

64. $F(s)=\dfrac{2s^2+10s}{\left(s^2-2s+5\right)(s+1)}$

65. $F(s)=\dfrac{s^2-26s-47}{(s-1)(s+2)(s+5)}$

66. $F(s)=\dfrac{-s-7}{(s-1)(s+2)}$

67. $F(s)=\dfrac{-8s^2-5s+9}{\left(s^2-3s+2\right)(s+1)}$

68. $F(s)=\dfrac{-2s^2+8s-14}{(s+1)\left(s^2-2s+5\right)}$

69. $F(s)=\dfrac{-5s-36}{(s+2)\left(s^2+9\right)}$

70. $F(s)=\dfrac{3s^2+5s+3}{s^4+s^3}$

71. $F(s)=\dfrac{7s^3-2s^2-3s+6}{s^3(s-2)}$

72. $F(s)=\dfrac{7s^2-41s+84}{(s-1)\left(s^2-4s+13\right)}$

73. $F(s)=\dfrac{6s-5}{s^2+7}$

74. $F(s) = \dfrac{1-3s}{s^2+8s+21}$

75. $F(s) = \dfrac{3s-2}{2s^2-6s-2}$

76. $F(s) = \dfrac{s+7}{s^2-3s-10}$

77. $F(s) = \dfrac{86s-78}{(s+3)(s-4)(5s-1)}$

78. $F(s) = \dfrac{2-5s}{(s-6)\left(s^2+11\right)}$

79. $F(s) = \dfrac{25}{s^3\left(s^2+4s+5\right)}$

80. $F(s) = \dfrac{5e^{-6s}-3e^{-11s}}{(s+2)\left(s^2+9\right)}$