# *Section* 1.3 – Models of Motions

In mathematics, the rate at which a quantity changes is the derivative of that quantity.
The 2nd way of computing the rate of change comes from the application itself and is different from on application to another.

### *Mechanics*

### Law of mechanics – Newton's 2nd Law (1665-1671)

*The force acting on a mass is equal to the rate of change of momentum with respect to time. Momentum is defined as the product of mass and velocity (m.v).*
The force is equal to the derivative of the momentum

$$F = \frac{d}{dt}mv = m\frac{dv}{dt} = ma$$

***Position***: $x(t) = -\frac{1}{2}gt^2 + v_0 t + x_0$

### *Universal Law of gravitation*

Any body with mass M attacks any other body with mass *m* directly toward the mass M, with a magnitude proportional to the product of the 2 masses and inversely proportional to the square of the distance separating them.

$$F = \frac{GMm}{r^2}$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$F = -mg$$

$$g = \frac{GM}{r^2}$$

Motion ball: $a = -g = \frac{d^2x}{dt^2}$

$g = 32\ ft/\sec^2 = 9.8\ m/s^2$

## *Air Resistance*

$$R(x,v) = -r(x,v).v$$

***R***: resistance force (*has sign opposite of the velocity*)
***r***: is a function that is always nonnegative

- *when a ball is falling from a high altitude, the density of the air has to be taken into account.*

$$F = -mg + R(v)$$

$$m\frac{dv}{dt} = -mg - rv$$

$$\frac{dv}{dt} = -g - \frac{r}{m}v$$

$$dv = \left(-g - \frac{r}{m}v\right)dt$$

$$\frac{dv}{g+\frac{r}{m}v} = -dt$$

$$\int \frac{dv}{g+\frac{r}{m}v} = -\int dt$$

$$\frac{m}{r}\ln\left(g + \frac{r}{m}v\right) = -t + C_1$$

$$\ln\left(g + \frac{r}{m}v\right) = -\frac{r}{m}t + C_2$$

$$g + \frac{r}{m}v = e^{-\frac{r}{m}t+C_2}$$

$$v(t) = Ce^{-rt/m} - \frac{mg}{r}$$

When $t \to \infty \;\; \Rightarrow \;\; v = -\frac{mg}{r}$ (*Terminal Velocity*)

$x(t) = -\frac{mC}{r}e^{-rt/m} - \frac{mg}{r}t + A$ ($A$: is a constant)

### *Example*

Suppose you drop a brick from the top of a building that is 250 *m* high. The brick has a mass of 2 *kg*, and the resistance force is given by $R = -4v$. How long will it take the brick to reach the ground? what will be its velocity at that time?

***Solution***

$$v(t) = Ce^{-rt/m} - \frac{mg}{r}$$

$$v(0) = 0 = C - \frac{mg}{r}$$

$$\Rightarrow C = \frac{mg}{r}$$

$$= \frac{2(9.8)}{4}$$

$$= 4.9$$

$$\frac{dx}{dt} = v(t)$$

$$= 4.9\left(e^{-2t} - 1\right)$$

$$\int dx = \int 4.9\left(e^{-2t} - 1\right)dt$$

$$x(t) = 4.9\left(-\frac{1}{2}e^{-2t} - t\right) + A$$

$$x(0) = 250 = 4.9\left(-\frac{1}{2}e^{-2(0)} - (0)\right) + A$$

$$250 = 4.9\left(-\frac{1}{2}\right) + A$$

$$250 = -2.45 + A$$

$$A = 252.45$$

$$x(t) = 4.9\left(-\frac{1}{2}e^{-2t} - t\right) + 252.45$$

$x(t) = 0 \quad \Rightarrow \underline{t = 51.52\ sec}$ (Using software to solve it)

$$v(t) = 4.9\left(e^{-2t} - 1\right)$$

$$\underline{v(t = 51.52) \approx -4.9\ m/s}$$

### *Finding the displacement*

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}.v$$

$$F = -mg + R(v) \qquad R = -k|v|.v$$

$$m\frac{dv}{dt} = -mg - k|v|v$$

$$\frac{dv}{dt} = -g - \frac{k}{m}v^2 = \boxed{\frac{dv}{dx}.v}$$

$$v\frac{dv}{dx} = -g - \frac{k}{m}v^2 \boxed{= -\frac{mg + kv^2}{m}}$$

### *Example*

A ball of mass $m = 0.2\ kg$ is protected from the surface of the earth, with velocity $v_0 = 50\ m/s$. Assume that the force of air resistance is given by $R = -k|v|.v$, where $k = 0.02$. What is the maximum height reached by the ball?

***Solution***

$$v\frac{dv}{dx} = -\frac{mg + kv^2}{m}$$

$$\frac{vdv}{mg + kv^2} = -\frac{dx}{m}$$

$$\int_{v_0}^{0} \frac{vdv}{mg + kv^2} = -\int_0^{x_{\max}} \frac{dx}{m}$$

$$d\left(mg + kv^2\right) = 2kvdv \quad \Rightarrow \quad \frac{d\left(mg + kv^2\right)}{2k} = vdv$$

$$\frac{1}{2k}\int_{v_0}^{0} \frac{d\left(mg + kv^2\right)}{mg + kv^2} = -\int_0^{x_{\max}} \frac{dx}{m}$$

$$\frac{1}{2k}\ln\left|mg + kv^2\right|_{50}^{0} = -\frac{x}{m}\Big|_0^{x_{max}}$$

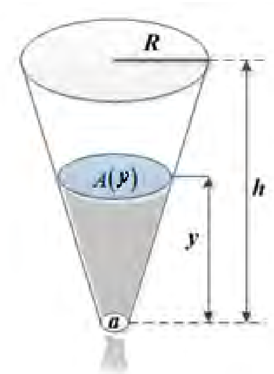
$$\frac{1}{2k}\left[\ln(mg) - \ln\left(mg + k\left(50^2\right)\right)\right] = -\frac{x_{max}}{m}$$

$$x_{max} = \frac{m}{2k}\left[\ln\left(mg + k\left(50^2\right)\right) - \ln(mg)\right]$$

$$= \frac{0.2}{2(.02)}\left[\ln\left(\frac{0.2(9.8) + (.02)(50)^2}{0.2(9.8)}\right)\right]$$

$$\boxed{= 16.4\ m}$$

## *Torricelli's Law*




Suppose that a water tank has a ***hole with area a*** at its bottom, from which water is leaking. Denote by $y(t)$ the depth of water in the tank at time *t*, and by $V(t)$ the volume of water in the tank then. It is plausible – and true, under ideal conditions – that the velocity of water exiting through the hole is

$$v = \sqrt{2gy}$$

Which is the velocity a drop of water would acquire in falling freely from the surface of the water to the hole. One can derive this formula beginning with the assumption that the sum of the kinetic and potential energy of the system remains constant. Under real conditions, taking into account the construction of a water jet from an orifice, $v = c\sqrt{2gy}$, where *c* is an empirical constant between 0 and 1 (usually about 0.6 for a small continuous stream of water). For simplicity we take $c = 1$ in the following discussion.

$$\frac{dV}{dt} = -av = -a\sqrt{2gy}$$

$$\frac{dV}{dt} = -k\sqrt{y} \qquad where \quad k = a\sqrt{2g}$$

This is a statement of *Torricelli's* law for a draining tank.
Let $A(y)$ denote the horizontal cross-sectional area of the tank at height *y*. Then, applied to a thin horizontal slice of water at height $\bar{y}$ with area $A(\bar{y})$ and thickness $d\bar{y}$, the integral method of cross sections gives

$$V(y) = \int_0^y A(\bar{y})\,d\bar{y}$$

The fundamental theorem of calculus therefore implies that $\frac{dV}{dy} = A(y)$ and hence that

$$\frac{dV}{dt} = \frac{dV}{dy} \cdot \frac{dy}{dt} = A(y)\frac{dy}{dt}$$

$$A(y)\frac{dy}{dt} = -a\sqrt{2gy}$$

$$= -k\sqrt{y} \qquad \text{(An alternative form of } \textit{Torricelli's} \text{ law)}$$

$$\frac{dh}{dt} = -c\frac{A_h}{A_w}\sqrt{2gh}$$

Where $A_w$ and $A_h$ are the cross-sectional areas of the water and the hole,

### *Example*

A tank is shaped like a vertical cylinder; it initially contains water to a depth of 9 *ft*, and a bottom plug is removed at time $t = 0$ (*hours*). After 1 *hr*. the depth of the water has dropped to 4 *ft*. how long does it take for all the water to drain from the tank?

***Solution***

$$\frac{dy}{dt} = k\sqrt{y}$$

$$\frac{dy}{y^{1/2}} = kdt$$

$$\int y^{-1/2} dy = \int kdt$$

$$2y^{1/2} = kt + C$$

With initial condition $y(0) = 9$

$$2\sqrt{9} = k(0) + C$$

$$\underline{C = 6}$$

$$2\sqrt{y} = kt + 6$$

$$y(1) = 4$$

$$2\sqrt{4} = k(1) + 6$$

$$k = 6 - 4 = -2$$

$$2\sqrt{y} = -2t + 6$$

$$\sqrt{y} = 3 - t$$

$$y(t) = (3-t)^2$$

It will take 3 *hours* for the tank to empty.

# *Exercises* *Section* 1.3 – Models of Motions

**1.** A body of mass $m$ falls from rest subject to gravity in a medium offering resistance proportional to the square of the velocity. Determine the velocity and position of the body at $t$ *seconds*.

**2.** A body of mass $m$, with initial velocity $v_0$, falls vertically. If the initial position is denoted $s_0$.
Determine the velocity and position of the body at $t$ seconds.
Assume the body acted upon by gravity alone and the air resistance proportional to the square of the velocity.

**3.** A body falls from a height of 300 *ft*. What distance has it traveled after 4 *sec*. if subject to $g$, the earth's acceleration?

**4.** A body falls from an initial velocity of $1,000\ ft/s$. What distance has it traveled after 3 *sec*. if subject to $g = 32\ ft/s^2$, the earth's acceleration?

**5.** A projectile is fired straight upwards with an initial velocity of $1,600\ ft/s$. What is its velocity at $40,000\ ft$. $\left(g = 32\ ft/s^2\right)$

**6.** A projectile is fired straight upwards with an initial velocity of $1,000\ ft/s$. What is its velocity at $8,000\ ft$. $\left(g = 32\ ft/s^2\right)$

**7.** An 8 *lb*. weight falls from rest toward earth. Assuming that the weight is acted upon by air resistance, numerically equal to $2v$, but measured in pounds, find the velocity and distance fallen after $t$ seconds.
(The variable $v$ represents the velocity measured in *ft/sec*.)

**8.** A stone is released from rest and dropped into a deep well. Eight seconds later, the sound of the stone splashing into the water at the bottom of the well returns to the ear of the person who released the stone. How long does it take the stone to drop to the bottom of the well? How deep is the well? Ignore air resistance. The speed of sound is 340 *m/s*.

**9.** A rocket is fired vertically and ascends with constant acceleration $a = 100\ m/s^2$ for 1.0 *min*. At that point, the rocket motor shuts off and the rocket continues upward under the influence of gravity. Find the maximum altitude acquired by the rocket and the total time elapsed from the take-off until the rocket returns to the earth. *Ignore air resistance*.

**10.** A ball having mass $m = 0.1\ kg$ falls from rest under the influence of gravity in a medium that provides a resistance that is proportional to its velocity. For a velocity of $0.2\ m/s$ the force due to the resistance of the medium is $-1$ *N*. Find the terminal velocity of the ball.
1 *N* is the force required to accelerate a 1 *kg* mass at a rate of $1\ m/s^2$: $1N = 1\ \text{kg} \cdot m/s^2$

**11.** A ball is projected vertically upward with initial velocity $v_0$ from ground level. Ignore air resistance.

*a)* What is the maximum height acquired by the ball?

*b)* How long does it take the ball to reach its maximum height? How long does it take the ball to return to the ground? Are these times identical?

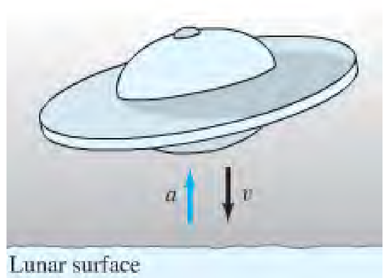
*c)* What is the speed of the ball when it impacts with the ground on its return?

**12.** An object having mass 70 *kg* falls from rest under the influence of gravity. The terminal velocity of the object is $-20\ m/s$. Assume that the air resistance is proportional to the velocity.

*a)* Find the velocity and distance traveled at the end of 2 seconds.

*b)* How long does it take the object to reach 80% of its terminal velocity?

**13.** A lunar lander is falling freely toward the surface of the moon at a speed of 450 *m/s*. Its retrorockets, when fired, provide a constant deceleration of 2.5 $m/s^2$ (the gravitational acceleration produced by the moon is assumed to be included in the given acceleration). At What height above the lunar surface should the retrorockets be activated to ensure a "soft touchdown? ($v = 0$ at impact)?



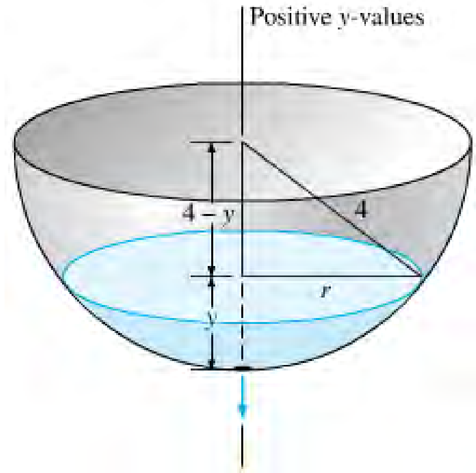

**14.** A body falling in a relatively dense fluid, oil for example, is acted on by three forces: a resistance force *R*, a buoyant force *B*, and its weight *w* due to gravity. The buoyant force is equal to the weight of the fluid displaced by the object. For a slowly moving spherical body of radius a, the resistive force is given by Stokes's law $R = 6\pi \frac{\mu a}{v}$, where *v* is the velocity of the body, and $\mu$ is the coefficient of viscosity of the surrounding fluid?


*a)* Find the limiting velocity of a solid sphere of radius *a* and density $\rho$ falling freely in a medium of density $\rho'$ and coefficient of viscosity μ.

*b)* In 1910 R. A. Millikan studied the motion of tiny droplets of oil falling in an electric field. A field of strength *E* exerts a force $E_e$ on a droplet with charge *e*. Assume that *E* has been adjusted so the droplet is held stationary $(v = 0)$ and that *w* and *B* are as given. Find an expression for *e*.

**15.** Suppose that the tank has a radius of 3 *ft*. and that its bottom hole is circular with radius 1 *in*. How long will it take the water (initially 9 *ft*. deep) to drain completely?

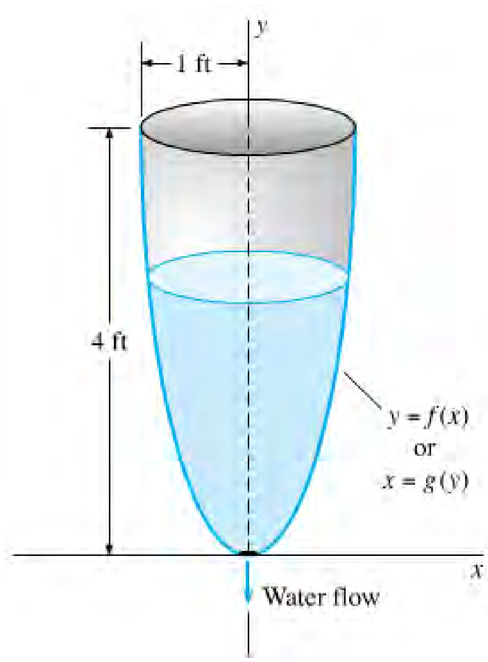
**16.** A hemispherical bowl has top radius of 4 *ft*. and at time $t = 0$ is full of water. At that moment a circular hole with diameter 1 *in*. is opened in the bottom of the tank. How long will it take for all the water to drain from the tank?




**17.** At time $t = 0$ the bottom plug (at the vertex) of a full conical water tank 16 *feet* high is removed. After 1 *hr* the water in the tank is 9 *feet* deep. When will the tank be empty?

**18.** Suppose that a cylindrical tank initially containing $V_0$ gallons of water drains (through a bottom hole) in $T$ minutes. Use Torricelli's law to show that the volume of water in the tank after $t \le T$ minutes is $V = V_0\left(1 - \frac{t}{T}\right)^2$

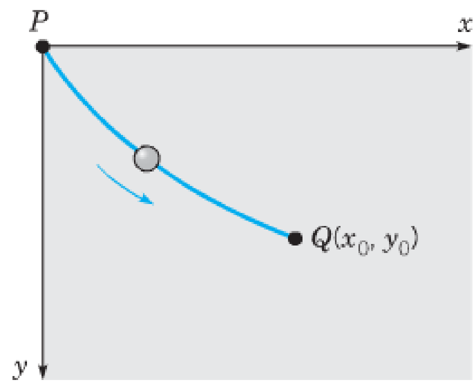
**19.** The clepsydra, or water clock – A 12-*hr* water clock is to be designed with the dimensions, shaped like the surface obtained by revolving the curve $y = f(x)$ around the *y*-axis. What should be this curve, and what should be the radius of the circular bottom hole, in order that the water level will fall at the constant rate of 4 inches per hour?



**20.** One of the famous problems in the history of mathematics is the brachistochrone problem: to find the curve along which a particle will slide without friction in the minimum time from one given point *P* to another point *Q*, the second point being lower than the first but not directly beneath it.
This problem was posed by Johann Bernoulli in 1696 as a challenge problem to the mathematicians of his day. Correct solutions were found by Johann Bernoulli and his brother Jakob Bernoulli and by Isaac Newton, Gottfried Leibniz, and the Marquis de L'Hospital. The brachistochrone problem is important in the development of mathematics as one of the forerunners of the calculus of variations.
In solving this problem, it is convenient to take the origin as the upper point *P* and to orient the axes as shown. The lower point Q has coordinates $\left(x_0, y_0\right)$. It is then possible to show that the curve of minimum time is given by a function $y = \phi(x)$ that satisfies the differential equation

$$\left(1 + y'^{\,2}\right) y = k^2 \qquad (eq.\ i)$$

Where $k^2$ is a certain positive constant to be determined later




*a)* Solve the equation $(eq.\ i)$ for $y'$. Why is it necessary to choose the positive square root?

*b)* Introduce the new variable t by the relation

$$y = k^2 \sin^2 t \qquad (eq.\ ii)$$

Show that the equation found in part (a) then takes the form

$$k^2 \sin^2 t\ dt = dx \qquad (eq.\ iii)$$

*c)* Letting $\theta = 2t$, show that the solution of $(eq.\ iii)$ for which $x = 0$ when $y = 0$ is given by

$$x = k^2 \frac{\theta - \sin\theta}{2}, \quad y = k^2 \frac{1 - \cos\theta}{2} \qquad (eq.\ iv)$$

Equations (*iv*) are parametric equations of the solution of (eq. *i*) that passes through (0, 0). The graph of Eqs. (*iv*) is called a cycloid.

*d)* If we make a proper choice of the constant *k*, then the cycloid also passes through the point $\left(x_0, y_0\right)$ and is the solution of the brachistochrone problem. Find *k* if $x_0 = 1$ and $y_0 = 2$

**21.** Many chemical reactions are the result of the interaction of 2 molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If *a* is the amount of substance *A* and *b* is the substance *B* at time *t* = 0, and if *x* is the amount of product at time *t*, then the rate of formation of *x* may be given by the differential equation

$$\frac{dx}{dt} = k(a-x)(b-x) \quad or \quad \frac{1}{(a-x)(b-x)}\frac{dx}{dt} = k$$

Where $k$ is a constant for the reaction. Integrate both sides of this equation to obtain a relation between $x$ and $t$.

***a)*** If $a = b$

***b)*** If $a \neq b$

Assume in each case that $x = 0$ when $t = 0$

**22.** An open cylindrical tank initially filled with water drains through a hole in the bottom of the tank according to Torricelli's Law. If $h(t)$ is the depth of water in the tank for $t \geq 0$, then Torricelli's Law implies $h'(t) = -2k\sqrt{h}$ , where $k$ is a constant that includes the acceleration due to gravity, the radius of the tank, and the radius of the drain. Assume that the initial depth of the water is $h(0) = H$


*a)* Find the solution of the initial value problem.

*b)* Find the solution in the case that $k = 0.1$ and $H = 0.5\ m$.

*c)* In general, how long does it take the tank to drain in terms of $k$ and $H$?

**23.** An object in free fall may be modeled by assuming that the only forces at work are the gravitational force and resistance (friction due to the medium in which the objects falls). By Newton's second law (mass × acceleration = the sum of the external forces), the velocity of the object satisfies the differential equation

$$\underset{mass}{m} \cdot \underset{acceleration}{v'(t)} = \underbrace{mg + f(v)}_{external\ force}$$

Where $f$ is a function that models the resistance and the positive direction is downward. One common assumption (often used for motion in air) is that $f(v) = -kv^2$, where $k > 0$ is a drag coefficient.

*a)* Show that the equation can be written in the form $v'(t) = g - av^2$ where $a = \frac{k}{m}$

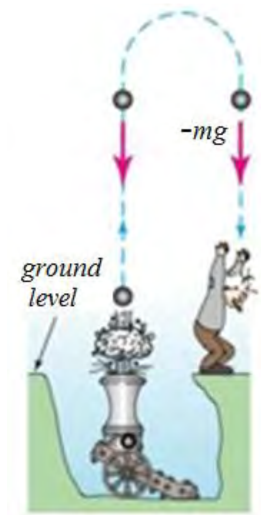
*b)* For what (positive) value of $v$ is $v'(t) = 0$? (This equilibrium solution is called the ***terminal velocity***.)

*c)* Find the solution of this separable equation assuming $v(0) = 0$ *and* $0 < v(t)^2 < \frac{g}{a}$ for $t \geq 0$

*d)* Graph the solution found in part (***c***) with $g = 9.8\ m/s^2$, $m = 1\ kg$, *and* $k = 0.1\ kg/m$, and verify the terminal velocity agrees with the value found in part (***b***).

**24.** Suppose a small cannonball weighing 16 *pounds* is shot vertically upward, with an initial velocity $v_0 = 300$ *ft/s*

The answer to the question "How high does the cannonball go?" depends on whether we take air resistance into account.




*a)* Suppose air resistance is ignored. If the positive direction is upward, then a model for the state of the cannonball is given by $\frac{d^2s}{dt^2} = -g$. Since $\frac{ds}{dt} = v(t)$ the last differential equation is the same as $\frac{dv}{dt} = -g$, where we take $g = 32$ *ft/s*$^2$. Find the velocity $v(t)$ of the cannonball at time *t*.

*b)* Use the result in part (*a*) to determine the height $s(t)$ of the cannonball measured from ground level. Find the maximum height attained by the cannonball.

**25.** Two chemicals *A* and *B* are combined to form a chemical *C*. The resulting reaction between the two chemicals is such that for each *gram* of *A*, 4 *grams* of *B* is used. It is observed that 30 *grams* of the compound *C* is formed in 10 *minutes*.

*a)* Determine the amount of *C* at time *t* if the rate of the reaction is proportional to the amounts of *A* and *B* remaining and if initially there are 50 *grams* of *A* and 32 *grams* of *B*.

*b)* How much of the compound *C* is present at 15 *minutes*.

*c)* Interpret the solution as $t \to \infty$

**26.** Two chemicals *A* and *B* are combined to form a chemical *C*. The rate, or velocity, of the reaction is proportional to the product of the instantaneous amounts of *A* and *B* not converted to chemical *C*. Initially, there are 40 *grams* of *A* and 50 *grams* of *B*, and each gram of *B*, 2 *grams* of *A* is used. It is observed that 10 *grams* of *C* is formed in 5 *minutes*.

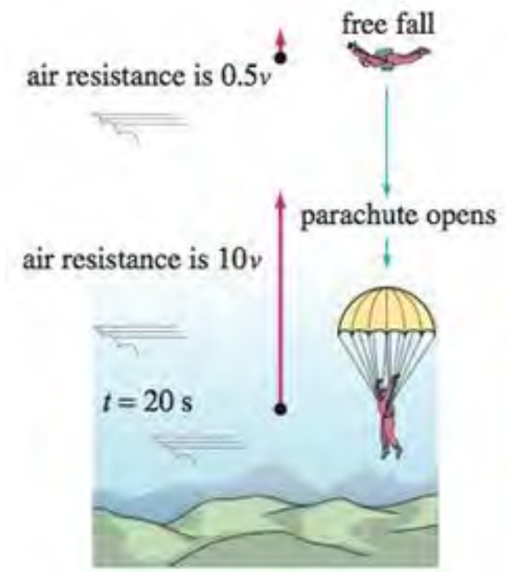
*a)* How much is formed in 20 *minutes*?

*b)* What is the limiting amount of *C* after a long time?

*c)* How much of chemicals *A* and *B* remains after a long time?

*d)* If 100 *grams* of chemical *A* is present initially, at what time is chemical *C* half-formed?

**27.** A skydiver weighs 125 *pounds*, and her parachute and equipment combined weigh another 35 *pounds*. After exiting from a plane at an altitude of 15,000 *feet*, she waits 15 *seconds* and opens her parachute. Assume that the constant of proportionality has the value $k = 0.5$ during free fall and $k = 10$ after the parachute is opened.




Assume that her initial velocity on leaving the plane is *zero*.

*a)* What is her velocity and how far has she traveled 20 *seconds* after leaving the plane?
*b)* How does her velocity at 20 *seconds* compare with her terminal velocity?
*c)* How long does it take her to reach the ground?

**28.** A tank in the form of a right-circular cylinder standing on end is leaking water through a circular hole in its bottom. When friction and contraction of water at the hole are ignored, the height $h$ of water in the tank is described by

$$\frac{dh}{dt} = -\frac{A_h}{A_w}\sqrt{2gh}$$

Where $A_w$ and $A_h$ are the cross-sectional areas of the water and the hole, respectively.

*a)* Find $h(t)$ if the initial height of the water is $H$.

*b)* Sketch the graph $h(t)$ and give the interval $I$ of definition in terms of the symbols $A_w$, $A_h$, and $H.\left(g = 32\ ft/s^2\right)$

*c)* Suppose the tank is 10 *feet* high and has radius 2 *feet* and the circular hole has radius $\frac{1}{2}$ *inch*. If the tank is initially full, how long will it take to empty?

**29.** A tank in the form of a right-circular cylinder cone standing on end, vertex down, is leaking water through a circular hole in its bottom.

*a)* Suppose the tank is 20 *feet* high and has radius 8 *inches*. Show that the differential equation governing the height $h$ of water leaking from a tank is

$$\frac{dh}{dt} = -\frac{5}{6h^{3/2}}$$

In this model, friction and contraction of the water at the hole were taken into account with $c = 0.6$ and $g = 32\ ft/s^2$. If the tank is initially full, how long will it take the tank to empty?

*b)* Suppose the tank has a vertex angle of 60° and the circular hole has radius 2 *inches*. Determine the differential equation governing the height $h$ of water. Use $c = 0.6$ and $g = 32\ ft/s^2$.

*c)* If the height of the water is initially 9 *feet*, how long will it take the tank to empty?

**30.** Suppose that the conical tank is inverted and that water leaks out a circular hole of radius 2 *inches* in the center of its circular base. Is the time it takes to empty a full tank the same as for the tank with vertex down?


Take the friction/.contraction coefficient to be $c = 0.6$ and $g = 32\ ft/s^2$

**31.** A differential equation for the velocity $v$ of a falling mass m subjected to air resistance proportional to the square of the instantaneous velocity is

$$m\frac{dv}{dt} = mg - kv^2$$

Where $k > 0$ is a constant of proportionality. The positive direction is downward.

*a)* Solve the equation subject to the initial condition $v(0) = v_0$.

*b)* Use the solution in part (*a*) to determine the limiting, or terminal, velocity of the mass.

*c)* If the distance $s$, measured from the point where the mass was released above the ground, is related to velocity $v$ by $\frac{ds}{dt} = v(t)$, find an explicit expression for $s(t)$ if $s(0) = 0$

**32.** An object is dropped from altitude $y_0$

*a)* Determine the impact velocity if the drag force is proportional to the square of velocity, with drag coefficient $\kappa$.

*b)* If the terminal velocity is known to $-120$ *mph* and the impact velocity was $-90$ *mph*, what was the initial altitude $y_0$?

**33.** An object is dropped from altitude $y_0$

*a)* Assume that the drag force is proportional to the velocity, with drag coefficient $\kappa$. Obtain an implicit solution relating velocity and altitude.

*b)* If the terminal velocity is known to $-120$ *mph* and the impact velocity was $-90$ *mph*, what was the initial altitude $y_0$?

**34.** An object of mass 3 *kg* is released from rest 500 *m* above the ground and allowed to fall under the influence of gravity. Assume the gravitational force constant, with $g = 9.81\ m/s^2$, and the force due to air resistance is proportional to the velocity of the object with proportionality constant $\kappa = 3$ *N-sec/m*. Determine when the object will hit the ground.

**35.** A parachutist whose mass is 75 *kg* drops from helicopter hovering 4000 *m* above the ground and falls toward the earth under the influence of gravity. Assume the gravitational force is constant. Assume also that the force due to air resistance is proportional to the velocity of the parachutist, with the proportionality constant $\kappa_1 = 15$ *N-sec/m* when the chute is closed and with constant $\kappa_2 = 105$ *N-sec/m* when the chute is open. If the chute does not open until 1 *min* after the parachutist leaves the helicopter, after how many *seconds* will he reach the ground?

**36.** A parachutist whose mass is 75 *kg* drops from helicopter hovering 2000 *m* above the ground and falls toward the earth under the influence of gravity. Assume the gravitational force is constant. Assume also that the force due to air resistance is proportional to the velocity of the parachutist, with the proportionality constant $\kappa_1 = 30$ *N-sec/m* when the chute is closed and with constant $\kappa_2 = 90$ *N-sec/m* when the chute is open. If the chute does not open until the velocity of the parachutist reaches 20 *m/sec*, after how many seconds will he reach the ground?

**37.** An object of mass 5 *kg* is released from rest 1000 *m* above the ground and allowed to fall under the influence of gravity. Assume the gravitational force constant, with $g = 9.8\ m/s^2$, and the force due to air resistance is proportional to the velocity of the object with proportionality constant $\kappa = 50$ *N-sec/m*. Determine when the object will hit the ground.

**38.** An object of mass 500 *kg* is released from rest 1000 *m* above the ground and allowed to fall under the influence of gravity. Assume the gravitational force constant, with $g = 9.8\ m/s^2$, and the force due to air resistance is proportional to the velocity of the object with proportionality constant $\kappa = 50$ *N-sec/m*. Determine when the object will hit the ground.

**39.** A 400-*lb* object is released from rest 500 *ft* above the ground and allowed to fall under the influence of gravity. Assuming that the force in pounds due to air resistance is $-10v$, where $v$ is the velocity of the object in *ft/s*, determine the equation of motion of the object. When will the object hit the ground?

**40.** An object of mass 8 *kg* is given an upward initial velocity of 20 *m/sec* and then allowed to fall under the influence of gravity. Assume that the force in Newton due to air resistance is $-16v$, where $v$ is the velocity of the object in *m/sec*.

*a)* Determine the equation of motion of the object.
*b)* If the object is initially 100 *m* above the ground, determine when the object will hit the ground.

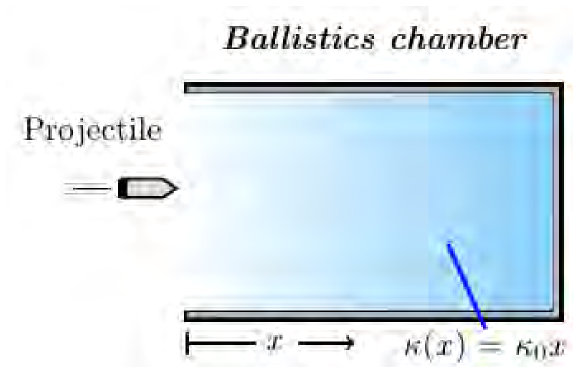
**41.** An object of mass 5 *kg* is given an downward initial velocity of 50 *m/sec* and then allowed to fall under the influence of gravity. Assume that the force in Newton due to air resistance is $-10v$, where $v$ is the velocity of the object in *m/sec*.

*a)* Determine the equation of motion of the object.
*b)* If the object is initially 100 *m* above the ground, determine when the object will hit the ground.

**42.** A shell of mass 2 *kg* is shot upward with an initial velocity of 200 *m/sec*. The magnitude of the force on the shell due to air resistance is $\frac{|v|}{20}$.

*a)* When will the shell reach its maximum height above the ground?
*b)* What is the maximum height?

**43.** We need to design a ballistics chamber to decelerate test projectiles fired into it. Assume the resistive force encountered by the projectile is proportional to the square of its velocity and neglect gravity.




The chamber is to be constructed so that the coefficient $\kappa$ associated with this resistive force is not constant but is, in fact, a linearly increasing function of distance into the chamber:

Let $\kappa(x) = \kappa_0 x$, where $\kappa_0$ is a constant; the resistive force then has the form $\kappa(x)v^2 = \kappa_0 xv^2$.

If we use time $t$ as the independent variable, Newton's law of motion leads us to the differential equation

$$m\frac{dv}{dt} + \kappa_0 xv^2 = 0 \quad with \quad v = \frac{dx}{dt}$$

*a)* Adopt distance $x$ into the chamber as the new independent variable and rewrite the given differential equation as a first order equation in terms of the new independent variable.
*b)* Determine the value $\kappa_0$ needed if the chamber is to reduce projectile velocity to 1% of its incoming value within *d units* of distance.

**44.** When the velocity $v$ of an object is very large, the magnitude of the force due to air resistance is proportional to $v^2$ with the force acting in opposition to the motion of the object. A shell of mass 3 *kg* is hot upward from the ground with an initial velocity of 500 *m/sec*. If the magnitude of the force due to air resistance is $(0.1)v^2$.

*a)* When will the shell reach its maximum height above the ground?
*b)* What is the maximum height?

**45.** A sailboat has been running (on a straight course) under a light wind at 1 *m/sec*. Suddenly the wind picks up, blowing hard enough to apply a constant force of 600 *N* to the sailboat. The only other force acting on the boat is water resistance that is proportional to the velocity of the boat. If the proportionality constant for water resistance is $\kappa = 100$ *N-sec/m* and the mass of the sailboat is 50 *kg*.

*a)* Find the equation of motion of the sailboat.

*b)* What is the limiting velocity of the sailboat under this wind?

*c)* When the velocity of the sailboat reaches 5 *m/sec*, the boat begins to rise out of the water and plane. When this happens, the proportionality constant for the water resistance drop to $\kappa = 60$ *N-sec/m*. Find the equation of motion of the sailboat.

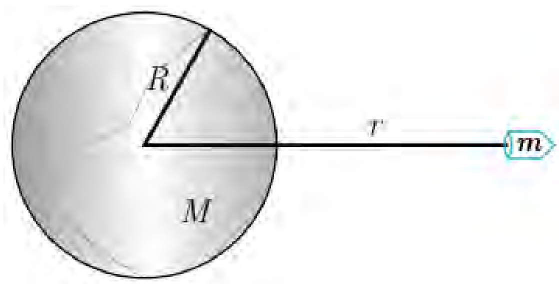
*d)* What is the limiting velocity of the sailboat under this wind as it is planning?

**46.** According to Newton's law of gravitation, the attractive force between two objects varies inversely as the square of the distances between them. That is, $F_g = \dfrac{GM_1 M_2}{r^2}$

Where $M_1$ and $M_2$ are the masses of the objects, $r$ is the distance between them (center to center), $F_g$ is the attractive force, and $G$ is the constant of proportionality.

Consider ta projectile of constant mass $m$ being fired vertically from Earth.

Let $t$ represent time and $v$ the velocity of the projectile.




*a)* Show that the motion of the projectile, under Earth's gravitational force, is governed by the equation

$$\frac{dv}{dt} = -\frac{gR^2}{r^2},$$

Where $r$ is the distance between the projectile and the center of Earth, $R$ is the radius of Earth, $M$ is the mass of Earth, and $g = \dfrac{GM}{R^2}$.

*b)* Use the fact the $\dfrac{dr}{dt} = v$ to obtain $v\dfrac{dv}{dr} = -\dfrac{gR^2}{r^2}$

*c)* If the projectile leaves Earth's surface with velocity $v_0$, show that $v^2 = \dfrac{2gR^2}{r} + v_0^2 - 2gR$

*d)* Use the result of part (*c*) to how that the velocity of the projectile remains positive if and only if $v_0^2 - 2gR > 0$. The velocity $v_e = \sqrt{2gR}$ is called the escape velocity?

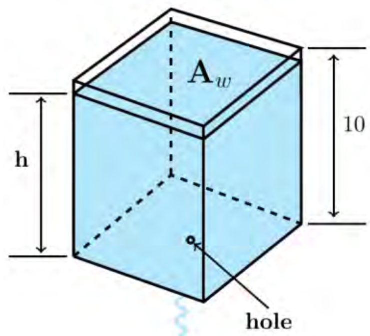
*e)* If $g = 9.81$ *m/sec*$^2$ and $R = 6370$ *km* for Earth, what is Earth's escape velocity?

*f)* If the acceleration due to gravity for the Moon is $g_m = \frac{g}{6}$ and the radius of the Moon is $R_m = 1738\ km$, what is the escape velocity of the Moon?

**47.** A 180-*lb* skydiver drops from a hot-air balloon. After 10 *sec* of free fall, a parachute is opened. The parachute immediately introduces a drag force proportional to velocity. After an additional 4 *sec*, the parachutist reaches the ground. Assume that air resistance is negligible during free fall and that the parachute is designed so that a 200-*lb* person will reach a terminal velocity of −10 *mph*.

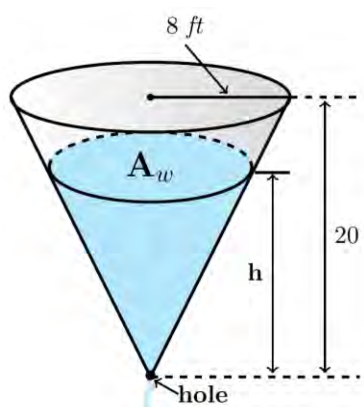
*a)* What is the speed of the skydiver immediately before the parachute is opened?
*b)* What is the parachutist's impact velocity?
*c)* At what altitude was the parachute opened?
*d)* What is the balloon's altitude?

**48.** Suppose water is leaking from a tank through a circular hole of area $A_h$ at its bottom. When water leaks through a hole, friction and contraction of the stream near the hole reduce the volume of water leaving the tank per second to $cA_h\sqrt{2gh}$, where $c$ $(0 < c < 1)$ is an empirical constant.




Determine a differential equation for the height $h$ of water at time $t$ for the cubical tank. The radius of the hole is 2 *in*., $g = 32\ ft/s^2$, and the friction/contraction factor is $c = 0.6$

**49.** The right-circular tank loses water out of a circular hole at its bottom.




The radius of the hole is 2 *in*., and $g = 32\ ft/s^2$, and the friction/contraction factor is $c = 0.6$.

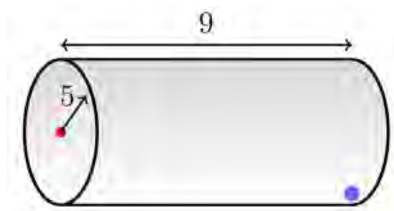
*a)* Determine a differential equation for the height $h$ of water at time $t$ for the cubical tank.
*b)* Find the height in function of time.

**50.** In meteorology, the term virga refers to falling raindrops or ice particles that evaporate before they reach the ground. Assume that a typical raindrop is spherical. Starting at some time, which we can designate as $t=0$, the raindrop of radius $r_0$ falls from rest from a cloud and begins to evaporate.

*a)* If it is assumed that a raindrop evaporates in such a manner that its shape remains spherical, then it also makes sense to assume that the rate at which the raindrop evaporates – that is, the rate at which it loses mass – is proportional to its surface area, Show that this latter assumption implies that the rate at which the radius *r* of the raindrop decreases is a constant. Find $r(t)$.

*b)* If the positive direction is downward, construct a mathematical model for the velocity *v* of the falling raindrop at time $t>0$. Ignore air resistance.

**51.** A horizontal cylindrical tank of length $9\ ft$, and radius $5\ ft$, is filled with oil. At $t=0$ a plug at the lowest point of the tank is removed and a flow results.




Find *y* the depth of the oil in the tank at any time *t* while the tank is draining. The constriction coefficient is $k=\frac{1}{15}$