Solution   *Section* 2.8 – Applications

Exercise

A 66-*kg* cyclist on a 7-*kg* bicycle starts coasting on level ground at 9 *m/sec*. The $k \approx 3.9\ kg\ /\ \sec$

a) About how far will the cyclist coast before reaching a complete stop?

b) How long will it take the cyclist's speed to drop to 1 *m/sec*?

Solution

Mass: $m = 66 + 7 = 73\ kg$

$$v = v_0 e^{-(k/m)t} = 9e^{-(3.9/73)t}$$

a) $s(t) = \int v(t)\,dt = \int 9e^{-(3.9/73)t}\,dt$

$$= 9\left(-\frac{73}{3.9}\right)e^{-(3.9/73)t} + C$$

$$= -\frac{219}{1.3}e^{-(3.9/73)t} + C$$

$$= -\frac{2190}{13}e^{-(3.9/73)t} + C$$

$$s(0) = -\frac{2190}{13}e^{-(3.9/73)(0)} + C$$

$$0 = -\frac{2190}{13} + C$$

$$\boxed{C = \frac{2190}{13}}$$

$$s(t) = -\frac{2190}{13}e^{-(3.9/73)t} + \frac{2190}{13} = \frac{2190}{13}\left(1 - e^{-(3.9/73)t}\right)$$

$$\lim_{t\to\infty} s(t) = \frac{2190}{13}\lim_{t\to\infty}\left(1 - e^{-(3.9/73)t}\right)$$

$$= \frac{2190}{13}(1 - 0)$$

$$\approx 168.5$$

The cyclist coast about 168.5 meters.

b) $1 = 9e^{-(3.9/76)t}$

$$\frac{1}{9} = e^{-(3.9/73)t} \quad \Rightarrow \quad -\frac{3.9}{73}t = \ln\frac{1}{9}$$

$$t = -\frac{73}{3.9}\ln\frac{1}{9} \approx 41.13\ \sec$$

It will take about 41.13 *seconds*.

Exercise

Suppose that an Iowa class battleship has mass 51,000 metric tons (51,000,000 *kg*) and $k \approx 59,000\ kg/\sec$. Assume that the ship loses power when it is moving at a speed of 9 *m/sec*.

a) About how far will the ship coast before it is dead in the water?

b) About how long will it take the ship's speed to drop to 1 *m/sec*?

Solution

$$v = v_0 e^{-(k/m)t} = 9e^{-(59{,}000/51{,}000{,}000)t} = 9e^{-(59/51{,}000)t}$$

a) $$s(t) = \int v(t)\,dt = \int 9e^{-(59/51{,}000)t}\,dt$$

$$= 9\left(-\frac{51000}{59}\right)e^{-(59/51{,}000)t} + C$$

$$= -\frac{459{,}000}{59}e^{-(59/51{,}000)t} + C$$

$$s(0) = -\frac{51{,}000}{59}e^{-(59/51{,}000)(0)} + C$$

$$0 = -\frac{51{,}000}{59} + C$$

$$\boxed{C = \frac{51{,}000}{59}}$$

$$s(t) = -\frac{459{,}000}{59}e^{-(59/51{,}000)t} + \frac{459{,}000}{59}$$

$$= \frac{459{,}000}{59}\left(1 - e^{-(59/51{,}000)t}\right)$$

$$\lim_{t\to\infty} s(t) = \frac{459{,}000}{59}\lim_{t\to\infty}\left(1 - e^{-(59/51{,}000)t}\right)$$

$$= \frac{51{,}000}{59}(1-0)$$

$$\approx 7780\ m$$

The ship will coast about 7780 meters or 7.78 km.

b) $$1 = 9e^{-(59/51{,}000)t}$$

$$e^{-(59/51{,}000)t} = \frac{1}{9}$$

$$-\frac{59}{51000}t = \ln\frac{1}{9}$$

$$t = -\frac{51000}{59}\ln\frac{1}{9} \approx 1899.3\ \sec$$

It will take about $\frac{1899.3}{60} \approx 61.65\ minutes$

Exercise

A 200-*gal* tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 *lb./gal* of concentrate enters the tank at the rate of 5 *gal/min*, and the well-stirred mixture is withdrawn at the rate of 3 *gal/min*.

a) At what time will the tank be full?

b) At the time the tank is full, how many pounds of concentrate will it contain?

Solution

a) $V(t) = 100 + \left(5\frac{gal}{\min} - 3\frac{gal}{\min}\right)(t \ \min) = 100 + 2t$

$200 = 100 + 2t$

$100 = 2t \quad \Rightarrow \quad \boxed{t = 50 \ \min}$

b) Let $y(t)$ be the amount of concentrate in the tank at time t.

$$\frac{dy}{dt} = \text{Rate in} - \text{Rate out}$$

$$\frac{dy}{dt} = \left(0.5\ \frac{lb}{gal}\right)\left(5\ \frac{gal}{\min}\right) - \left(\frac{y}{100+2t}\ \frac{lb}{gal}\right)\left(3\ \frac{gal}{\min}\right)$$

$$= \frac{5}{2} - \frac{3y}{100+2t}$$

$$\frac{dy}{dt} + \frac{3}{100+2t}y = \frac{5}{2} \quad \rightarrow \quad P(t) = \frac{3}{100+2t} \qquad Q(t) = \frac{5}{2}$$

$$e^{\int \frac{3dt}{100+2t}} = e^{\frac{3}{2}\int \frac{dt}{50+t}} = e^{\frac{3}{2}\ln(50+t)} = e^{\ln(50+t)^{3/2}} = (50+t)^{3/2}$$

$$\int \frac{5}{2}(50+t)^{3/2}dt = (t+50)^{5/2}$$

$$y(t) = \frac{1}{(t+50)^{3/2}}\left[(t+50)^{5/2} + C\right]$$

$$= t + 50 + \frac{C}{(t+50)^{3/2}}$$

$$y(0) = 0 + 50 + \frac{C}{(0+50)^{3/2}}$$

$$0 = 50 + \frac{C}{50^{3/2}} \qquad \rightarrow \frac{C}{50^{3/2}} = -50 \quad \Rightarrow \quad \boxed{C = -50^{5/2}}$$

$$y(t) = t + 50 - \frac{50^{5/2}}{(t+50)^{3/2}}$$

$$y(t=50) = 50 + 50 - \frac{50^{5/2}}{(50+50)^{3/2}} \approx 83.22 \ \ lb \ of \ \ concentrate$$

Exercise

A tank contains 100 *gal* of fresh water. A solution containing 1 *lb./gal* of soluble lawn fertilizer runs into the tank at the rate of 1 *gal/min*, and the mixture is pumped out of the tank at a rate of 3 *gal/min*. Find the maximum amount of fertilizer in the tank and the time required to reach the maximum.

Solution

Volume of the tank at time t is:

$$V(t) = 100\ gal + \left(1\frac{gal}{\min} - 3\frac{gal}{\min}\right)(t\ \min) = 100 - 2t$$

$$\frac{dy}{dt} = \text{Rate in} - \text{Rate out}$$

$$\frac{dy}{dt} = \left(1\ \frac{lb}{gal}\right)\left(1\ \frac{gal}{\min}\right) - \left(\frac{y}{100-2t}\ \frac{lb}{gal}\right)\left(3\ \frac{gal}{\min}\right)$$

$$\frac{dy}{dt} = 1 - \frac{3y}{100-2t}$$

$$\frac{dy}{dt} + \frac{3}{100-2t}y = 1 \quad \rightarrow \quad P(t) = \frac{3}{100-2t} \quad Q(t) = 1$$

$$e^{\int \frac{3dt}{100-2t}} = e^{\frac{3}{2}\int \frac{-dt}{100-2t}} = e^{-\frac{3}{2}\ln(100-2t)} = e^{\ln(100-2t)^{-3/2}} = (100-2t)^{-3/2}$$

$$\int 1(100-2t)^{-3/2}dt = -\frac{1}{2}\int (100-2t)^{-3/2}d(100-2t) = (100-2t)^{-1/2}$$

$$y(t) = \frac{1}{(100-2t)^{-3/2}}\left[(100-2t)^{-1/2} + C\right]$$

$$y(t) = 100 - 2t + C(100-2t)^{3/2}$$

$$y(0) = 100 - 2(0) + C(100-2(0))^{3/2}$$

$$0 = 100 + C(100)^{3/2}$$

$$\boxed{C = -100^{-1/2} = -\frac{1}{10}}$$

$$y(t) = 100 - 2t - 0.1(100-2t)^{3/2}$$

$$\frac{dy}{dx} = -2 - 0.1\frac{3}{2}(100-2t)^{1/2}(-2)$$

$$\frac{dy}{dx} = -2 + 0.3(100-2t)^{1/2} = 0$$

$$(100-2t)^{1/2} = \frac{2}{0.3} \quad \Rightarrow \quad 100 - 2t = \left(\frac{2}{0.3}\right)^2 = \frac{4}{0.09} = \frac{400}{9}$$

$$2t = 100 - \frac{400}{9} = \frac{500}{9}$$

$$\boxed{t = \frac{500}{18} \approx 12.78 \text{ min}}$$

The maximum amount is:

$$y(t = 12.78) = 100 - 2(12.78) - 0.1(100 - 2(12.78))^{3/2}$$

$$\boxed{y \approx 14.8\ lb}$$

Exercise

An Executive conference room of a corporation contains 4500 ft^3 of air initially free of carbon monoxide. Starting at time *t* = 0, cigarette smoke containing 4% carbon monoxide is blown into the room at the rate of 0.3 $ft^3/\min$. A ceiling fan keeps the air in the room well circulated and the air leaves the room at the same rate of 0.3 $ft^3/\min$. Find the time when the concentration of carbon monoxide in the room reaches 0.01%.

Solution

Let $y(t)$ be the amount of carbon monoxide (CO) in the room at time *t*.

$$\frac{dy}{dt} = Rate\ in - \ Rate\ out$$

$$\frac{dy}{dt} = (0.04)(0.3) - \left(\frac{y}{4500}\right)(0.3)$$

$$\frac{dy}{dt} = \frac{12}{1000} - \frac{y}{15,000}$$

$$\frac{dy}{dt} + \frac{1}{15,000}y = \frac{12}{1000} \quad \to \quad P(t) = \frac{1}{15,000} \quad Q(t) = \frac{12}{1000}$$

$$e^{\int \frac{dt}{15000}} = e^{\frac{1}{15000}t}$$

$$\int \frac{12}{1000} e^{\frac{1}{15000}t} dt = \frac{12}{1000} 15000 e^{\frac{1}{15000}t} = 180 e^{\frac{1}{15000}t}$$

$$y(t) = \frac{1}{e^{\frac{1}{15000}t}}\left[180 e^{\frac{1}{15000}t} + C\right]$$

$$y(t) = 180 + Ce^{\frac{-1}{15000}t}$$

$$y(0) = 180 + Ce^{\frac{-1}{15000}0}$$

$$0 = 180 + C \quad \Rightarrow \quad \boxed{C = -180}$$

$$y(t) = 180 - 180 e^{\frac{-1}{15000}t}$$

When the concentration of CO is 0.01% in the room, the amount of CO satisfies

$$\frac{y}{4500} = \frac{.01}{100} \quad \Rightarrow \quad y = 0.45\ ft^3$$

When the room contains the amount $y = 0.45\ ft^3$

$$0.45 = 180 - 180e^{\frac{-1}{15000}t}$$

$$180e^{\frac{-1}{15000}t} = 179.55$$

$$e^{\frac{-1}{15000}t} = \frac{179.55}{180}$$

$$\frac{-1}{15000}t = \ln\left(\frac{179.55}{180}\right)$$

$$t = -15000\ln\left(\frac{179.55}{180}\right)$$

$$\boxed{t \approx 37.55 \text{ min}}$$

Exercise

Many chemical reactions are the result of the interaction of 2 molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If a is the amount of substance A and b is the substance B at time $t = 0$, and if x is the amount of product at time t, then the rate of formation of x may be given by the differential equation

$$\frac{dx}{dt} = k(a-x)(b-x) \quad or \quad \frac{1}{(a-x)(b-x)}\frac{dx}{dt} = k$$

Where k is a constant for the reaction. Integrate both sides of this equation to obtain a relation between x and t.

a) If $a = b$

b) If $a \ne b$

Assume in each case that $x = 0$ when $t = 0$

Solution

$$\frac{1}{(a-x)(b-x)}dx = kdt$$

a) $a = b \quad \Rightarrow \quad \frac{1}{(a-x)^2}dx = kdt$

$$\int \frac{1}{(a-x)^2}dx = \int kdt$$

$$\frac{1}{a-x} = kt + C$$

$$x(t=0) = 0 \quad \Rightarrow \quad \boxed{\frac{1}{a} = C}$$

$$\frac{1}{a-x}=kt+\frac{1}{a}=\frac{kat+1}{a}$$

$$a-x=\frac{a}{kat+1}$$

$$x=a-\frac{a}{kat+1}$$

$$\boxed{=\frac{a^2kt}{kat+1}}$$

b) $a\neq b \quad \Rightarrow \quad \dfrac{1}{(a-x)(b-x)}dx=kdt$

$$\int\frac{1}{(a-x)(b-x)}dx=\int kdt$$

$$\frac{1}{(a-x)(b-x)}=\frac{A}{a-x}+\frac{B}{b-x}$$

$$\begin{cases}-A-B=0\\ bA+aB=1\end{cases} \rightarrow \begin{cases}B=\dfrac{1}{a-b}\\ A=-\dfrac{1}{a-b}\end{cases}$$

$$\frac{-1}{a-b}\int\frac{1}{a-x}dx+\frac{1}{a-b}\int\frac{1}{b-x}dx=\int kdt$$

$$\frac{1}{a-b}\ln|a-x|-\frac{1}{a-b}\ln|b-x|=kt+C$$

$$\frac{1}{a-b}\ln\left|\frac{a-x}{b-x}\right|=kt+C$$

$$x(0)=0 \quad \Rightarrow \quad \boxed{\frac{1}{a-b}\ln\left(\frac{a}{b}\right)=C}$$

$$\frac{1}{a-b}\ln\left|\frac{a-x}{b-x}\right|=kt+\frac{1}{a-b}\ln\left(\frac{a}{b}\right)$$

$$\ln\left|\frac{a-x}{b-x}\right|=(a-b)kt+\ln\left(\frac{a}{b}\right)$$

$$\frac{a-x}{b-x}=e^{(a-b)kt+\ln\left(\frac{a}{b}\right)}$$

$$\frac{a-x}{b-x}=\frac{a}{b}e^{(a-b)kt}$$

$$a-x=b\frac{a}{b}e^{(a-b)kt}-x\frac{a}{b}e^{(a-b)kt}$$

$$x\left(\frac{a}{b}e^{(a-b)kt}-1\right)=ae^{(a-b)kt}-a$$

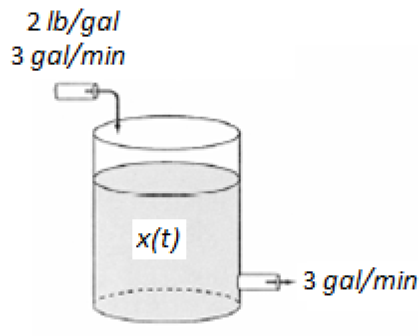
$$\boxed{x=\frac{abe^{(a-b)kt}-ab}{ae^{(a-b)kt}-b}}$$

Exercise

The tank initially holds 100 *gal* of pure water. At time $t = 0$, a solution containing 2 *lb* of salt per gallon begins to enter the tank at the rate of 3 gallons per minute. At the same time a drain is opened at the bottom of the tank so that the volume of solution in the tank remains constant.
How much salt is in the tank after 60 *min*?
What will be the eventual salt content in the tank?

Solution

$x(t)$: number of pounds of salt in the tank after t min.



Volume: $V(t) = 100 + (3-3)t = 100$

Concentration at time t: $c(t) = \dfrac{x(t)}{V(t)} = \dfrac{x(t)}{100}$ $lb\,/\,gal$

Rate in = Volume Rate x Concentration

$$= 3\frac{gal}{min} \times 2\frac{lb}{gal}$$

$$= 6 \;\; lb\,/\,\text{min}$$

Rate out = Volume Rate x Concentration

$$= 3\frac{gal}{\text{min}} \times \frac{x(t)}{100}\frac{lb}{gal}$$

$$= \frac{3x(t)}{100} \;\; lb\,/\,\text{min}$$

$\dfrac{dx}{dt} =$ rate of change

$= \text{rate in} - \text{rate out}$

$$= 6 - \frac{3x}{100}$$

$$\frac{dx}{dt} + \frac{3}{100}x = 6$$

$$u(t) = e^{\int\left(\frac{3}{100}\right)dt} = e^{0.03t}$$

$$\int 6e^{0.03t}dt = \frac{6}{0.03}e^{0.03t} = 200e^{0.03t}$$

$$x(t) = e^{-0.03t}\left(200e^{0.03t} + C\right)$$

$$\boxed{x(t) = 200 + Ce^{-0.03t}}$$

Since there was no salt present in the tank initially, the initial condition is $x(0) = 0$

$$x(t=0) = 200 + Ce^{-0.03(0)} = 0$$

$$200 + C = 0 \quad \to \boxed{C = -200}$$

$$\boxed{x(t) = 200 - 200e^{-0.03t}}$$

After 60 min: $\quad x(60) = 200 - 200e^{-0.03(60)}$ $\boxed{\approx 167\ lb}$

As $t \to \infty$ then $x(t) = \lim\limits_{t\to\infty}\left(200 - 200e^{-0.03t}\right)$

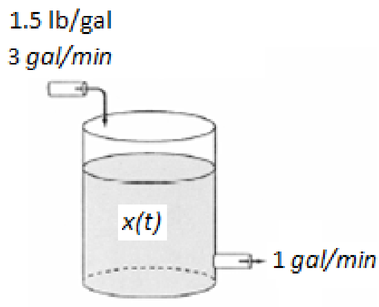
$$= 200 - 200\lim_{t\to\infty}\left(e^{-0.03t}\right) \qquad\qquad \lim_{t\to\infty}\left(e^{-0.03t}\right) = e^{-\infty} = 0$$

$$\boxed{= 200\ \ lb}$$

Exercise

The 600-*gal* tank is filled with 300 *gal* of pure water. A spigot is opened above the tank and a salt solution containing 1.5 *lb*. of salt per gallon of solution begins flowing into the tank at the rate of 3 *gal/min*. Simultaneously, a drain is opened at the bottom of the tank allowing the solution to leave tank at a rate of 1 *gal/min*. What will be the salt content in the tank at the precise moment that the volume of solution in the tank is equal to the tank's capacity (600 *gal*)?

Solution



$$V(t) = 300 + (3-1)t = 300 + 2t$$

$$c(t) = \frac{x(t)}{300+2t}$$

$$\text{Rate in } = 3\frac{gal}{min} \times 1.5\frac{lb}{gal} = 4.5\ \ lb/min$$

$$\text{Rate out } = 1 \times \frac{x}{300+2t} = \frac{x}{300+2t}\ \ lb/\text{min}$$

$$\frac{dx}{dt} = 4.5 - \frac{x}{300+2t}$$

$$\frac{dx}{dt} + \frac{1}{300+2t}x = 4.5$$

$$u(t) = e^{\int \frac{1}{300+2t}dt} \qquad\qquad d(300+2t) = 2dt$$

$$= e^{\frac{1}{2}\int \frac{1}{300+2t}d(300+2t)}$$

$$= e^{\frac{1}{2}\ln(300+2t)}$$

$$= e^{\ln(300+2t)^{1/2}}$$

$$= \sqrt{300+2t}$$

$$\int 4.5\sqrt{300+2t}\ dt = 4.5\frac{1}{2}\frac{2}{3}(300+2t)^{2/3}$$

$$x(t) = \frac{1}{\sqrt{300+2t}}\left(1.5(300+2t)^{3/2} + C\right)$$

$$= 1.5(300+2t) + \frac{C}{\sqrt{300+2t}}$$

$$= 450 + 3t + \frac{C}{\sqrt{300+2t}}$$

$$x(0) = 450 + 3(0) + \frac{C}{\sqrt{300+2(0)}} = 0$$

$$450 + \frac{C}{\sqrt{300}} = 0$$

$$C = -450\sqrt{300} = -4500\sqrt{3}$$

$$x(t) = 450 + 3t - \frac{4500\sqrt{3}}{\sqrt{300+2t}}$$

$$V = 300 + 2t = 600$$

$$t = 150\ min$$

$$x(t = 150) = 450 + 3(150) - \frac{4500\sqrt{3}}{\sqrt{300+2(150)}}$$

$$\approx 582\ lb$$

Exercise

The amount of drug in the blood of a patient (in *mg*) due to an intravenous line is governed by the initial value problem

$$y'(t) = -0.02y + 3, \quad y(0) = 0 \ \ for\ t \geq 0$$

Where t is measured in hours

a) Find and graph the solution of the initial value problem.
b) What is the steady-state level of the drug?
c) When does the drug level reach 90% of the steady-state value?

Solution

a) $y' + 0.02y = 3$

$$e^{\int 0.02dt} = e^{0.02t}$$

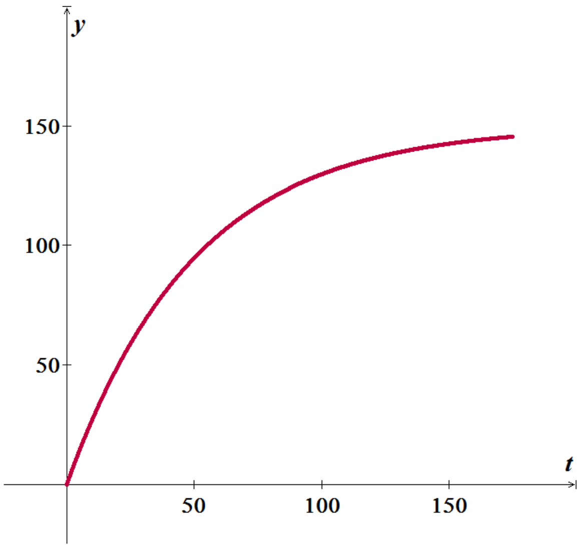
$$\int 3e^{0.02t}dt = 150e^{0.02t}$$

$$y = \frac{1}{e^{0.02t}}\left(150e^{0.02t} + C\right)$$

$$= 150 + Ce^{-0.02t}$$

$$y(0) = 0 \qquad 0 = 150 + C \to C = -150$$

$$y(t) = 150\left(1 - e^{-0.02t}\right)$$



b) The steady-state level is

$$\lim_{t\to\infty} 150\left(1-e^{-0.02t}\right) \underline{=150\ mg|}$$

c) $150\left(1-e^{-0.02t}\right) = 0.9(150)$

$$1-e^{-0.02t} = 0.9$$

$$e^{-0.02t} = 0.1$$

$$-0.02t = \ln 0.1$$

$$t = \frac{\ln 0.1}{-0.02} \underline{\approx 115\ hrs|}$$

Exercise

A fish hatchery has 500 *fish* at time $t = 0$, when harvesting begins at a rate of b *fish/yr*. where $b > 0$. The fish population is modeled by the initial value problem.

$$y'(t) = 0.1y - b, \quad y(0) = 500 \qquad for \ \ t \ge 0$$

Where t is measured in years.

a) Find the fish population for $t \ge 0$ in terms of the harvesting rate b.

b) Graph the solution in the case that $b = 40\ fish\,/\,yr$. Describe the solution.

c) Graph the solution in the case that $b = 60\ fish\,/\,yr$. Describe the solution.

Solution

a) $y' - 0.1y = -b$

$$e^{\int -0.1dt} = e^{-0.1t}$$

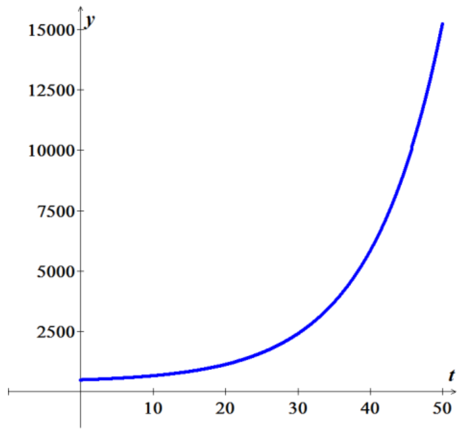
$$\int -be^{-0.1t}dt = 10be^{-0.1t}$$

$$y(t) = e^{0.1t}\left(10be^{-0.1t} + C\right)$$

$$\underline{= 10b + Ce^{0.1t}|}$$

$$y(0) = 500 \quad \to \quad 500 = 10b + C \ \Rightarrow\ \underline{C = 500 - 10b|}$$

$$y(t) \ \underline{= 10b + (500 - 10b)e^{0.1t}|}$$

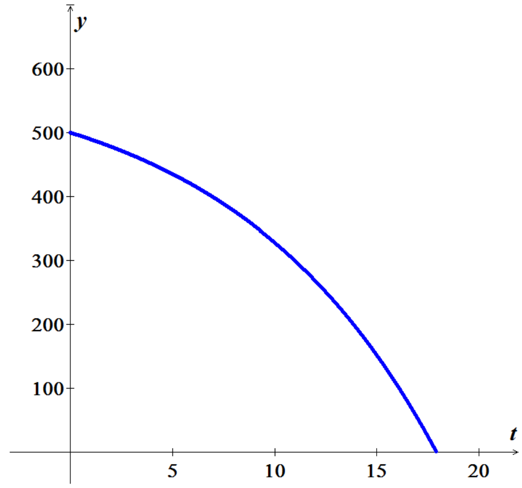


b) For $b = 40$

$$y(t) = 400 + 100e^{0.1t}$$

c) For $b = 60$

$$y(t) = 600 - 100e^{0.1t}$$



Exercise

A community of hares on an island has a population of 50 when observations begin at $t = 0$. The population for $t \ge 0$ is modeled by the initial value problem.

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{200}\right), \quad P(0) = 50$$

d) Find the solution of the initial value problem.

e) What is the steady-state population?

Solution

a) $$\int \frac{200}{P(200-P)}\,dP = \int 0.08\,dt$$

$$\int \left(\frac{1}{P} + \frac{1}{200-P}\right) dP = \int 0.08\,dt$$

$$\ln P + \ln|200 - P| = 0.08t + C$$

$$\ln\left|\frac{P}{200-P}\right| = 0.08t + C$$

$$P(0) = 50 \quad \to \ln\frac{50}{150} == C \;\Rightarrow\; \underline{C = -\ln 3}$$

$$\ln\left|\frac{P}{200-P}\right| = 0.08t - \ln 3$$

$$\frac{P}{200-P} = e^{0.08t - \ln 3}$$

$$\frac{P}{200-P} = e^{0.08t}e^{\ln 3^{-1}}$$

$$\frac{P}{200-P} = \frac{1}{3}e^{0.08t}$$

$$3P = 200e^{0.08t} - Pe^{0.08t}$$

$$P(t) = \frac{200e^{0.08t}}{3 + e^{0.08t}}$$

$$= \boxed{\frac{200}{3e^{-0.08t}+1}}$$

b) $\lim\limits_{t\to\infty} P(t) = \lim\limits_{t\to\infty} \dfrac{200}{3e^{-0.08t}+1} \underline{= 200}$

Exercise

When an infected person is introduced into a closed and otherwise healthy community, the number of people who become infected with the disease (in the absence of any intervention) may be modeled by the logistic equation

$$\frac{dP}{dt} = kP\left(1-\frac{P}{A}\right), \quad P(0) = P_0$$

Where k is a positive infection rate, A is the number of people in the community, and P_0 is the number of infected people at $t=0$. The model assumes no recovery or intervention.

a) Find the solution of the initial value problem in terms of k, A, and P_0.

b) Graph the solution in the case that $k = 0.025, \quad A = 300, \quad and \;\; P_0 = 1$.

c) For fixed values of k and A, describe the long-term behavior of the solutions for any P_0 with $0 < P_0 < A$

Solution

a) $\dfrac{dP}{dt} = kP\left(\dfrac{A-P}{A}\right)$

$$\int \frac{A}{P(A-P)}dP = \int kdt$$

$$\int \left(\frac{1}{P}+\frac{1}{A-P}\right)dP = \int kdt$$

$$\ln P - \ln|A-P| = kt + C_1$$

$$\ln\left|\frac{P}{A-P}\right| = kt + C_1$$

$$\frac{P}{A-P} = Ce^{kt}$$

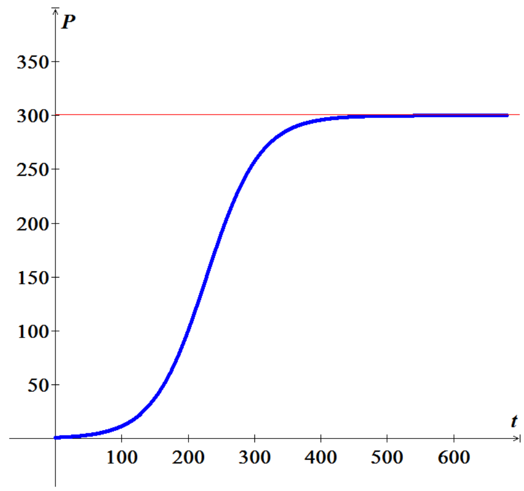
$$P(0) = P_0 \rightarrow \quad \frac{P_0}{A-P_0} = C$$

$$\frac{P}{A-P} = \frac{P_0}{A-P_0}e^{kt}$$

$$P = (A-P)\frac{P_0}{A-P_0}e^{kt}$$

$$\left(A - P_0 + P_0 e^{kt}\right)P = AP_0 e^{kt}$$

$$\boxed{P(t) = \frac{AP_0 e^{kt}}{A - P_0 + P_0 e^{kt}}} = \frac{AP_0}{P_0 + (A - P_0)e^{-kt}}$$



b) $k = 0.025, \quad A = 300, \quad and \quad P_0 = 1$

$$P(t) = \frac{300}{1 + 299e^{-0.025t}}$$

c) $\lim\limits_{t\to\infty} P(t) = \lim\limits_{t\to\infty} \dfrac{AP_0}{P_0 + (A - P_0)e^{-kt}}$

$$= \frac{AP_0}{P_0}$$

$\underline{= A}$ Which is the steady-state solution

Exercise

An object in free fall may be modeled by assuming that the only forces at work are the gravitational force and resistance (friction due to the medium in which the objects falls). By Newton's second law (mass × acceleration = the sum of the external forces), the velocity of the object satisfies the differential equation

$$\underset{mass}{m} \cdot \underset{acceleration}{v'(t)} = \underbrace{mg + f(v)}_{external\ force}$$

Where f is a function that models the resistance and the positive direction is downward. One common assumption (often used for motion in air) is that $f(v) = -kv^2$, where $k > 0$ is a drag coefficient.

a) Show that the equation can be written in the form $v'(t) = g - av^2$ where $a = \frac{k}{m}$

b) For what (positive) value of v is $v'(t) = 0$? (This equilibrium solution is called the ***terminal velocity***.)

c) Find the solution of this separable equation assuming $v(0) = 0 \ and \ 0 < v(t)^2 < \frac{g}{a}$ for $t \ge 0$

d) Graph the solution found in part (***c***) with $g = 9.8\ m/s^2, \ m = 1\ kg, \ and \ k = 0.1\ kg/m$, and verify the terminal velocity agrees with the value found in part (***b***).

Solution

a) Given: $f(v) = -kv^2$

$$mv'(t) = mg + f(v)$$

$$mv'(t) = mg - kv^2$$

$$v'(t) = g - \frac{k}{m}v^2$$

$$\boxed{v'(t) = g - av^2} \qquad \text{where } a = \frac{k}{m}$$

b) $v'(t) = g - av^2 = 0 \quad \Rightarrow \quad v^2 = \frac{g}{a} \quad \to \quad \boxed{v = \sqrt{\frac{g}{a}}}$

c) $\frac{dv}{dt} = g - av^2$

$$\int \frac{dv}{g - av^2} = \int dt$$

$$-\frac{1}{a}\int \frac{dv}{v^2 - \frac{g}{a}} = \int dt$$

$$-\frac{1}{2a}\sqrt{\frac{a}{g}}\int \frac{dv}{v - \sqrt{\frac{g}{a}}} + \frac{1}{2a}\sqrt{\frac{a}{g}}\int \frac{dv}{v + \sqrt{\frac{g}{a}}} = \int dt$$

$$\frac{1}{v^2 - \frac{g}{a}} = \frac{A}{v - \sqrt{\frac{g}{a}}} + \frac{B}{v + \sqrt{\frac{g}{a}}}$$

$$1 = A\sqrt{\frac{g}{a}} + Av + Bv - B\sqrt{\frac{g}{a}}$$

$$\begin{cases} A + B = 0 \ \to \ \boxed{A = -B} \\ A\sqrt{\frac{g}{a}} - B\sqrt{\frac{g}{a}} = 1 \end{cases}$$

$$\boxed{A = -B = \frac{1}{2}\sqrt{\frac{a}{g}}}$$

$$\frac{1}{2}\sqrt{\frac{1}{ag}}\left(-\ln\left|\sqrt{\frac{g}{a}} - v\right| + \ln\left|\sqrt{\frac{g}{a}} + v\right|\right) = t + C_1$$

$$\ln\frac{\sqrt{\frac{g}{a}} + v}{\sqrt{\frac{g}{a}} - v} = 2\sqrt{ag}\,t + C_2$$

$$\frac{\sqrt{\frac{g}{a}} + v}{\sqrt{\frac{g}{a}} - v} = e^{2\sqrt{ag}\,t + C_2}$$

$$\sqrt{\frac{g}{a}} + v = Ce^{2\sqrt{ag}\,t}\left(\sqrt{\frac{g}{a}} - v\right) \qquad v(0)=0 \ \Rightarrow \ \sqrt{\frac{g}{a}} = \sqrt{\frac{g}{a}}C \to \boxed{C=1}$$

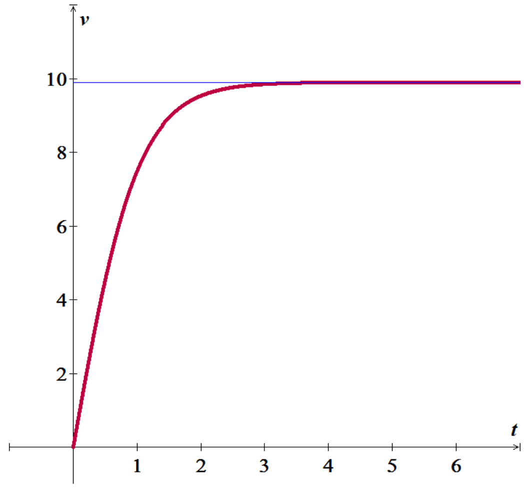
$$v\left(1 + e^{2\sqrt{ag}\,t}\right) = \sqrt{\frac{g}{a}}e^{2\sqrt{ag}\,t} - \sqrt{\frac{g}{a}}$$

$$\boxed{v(t) = \frac{e^{2\sqrt{ag}\,t} - 1}{1 + e^{2\sqrt{ag}\,t}}\sqrt{\frac{g}{a}}}$$

d) $g = 9.8\ m/s^2, \quad m = 1\ kg, \quad and \quad k = 0.1\ kg/m$

$$\to \ a = \frac{k}{m} \ \boxed{= 0.1}$$

$$v(t) = \sqrt{98}\,\frac{e^{2\sqrt{.98}t} - 1}{1 + e^{2\sqrt{.98}t}}$$



Exercise

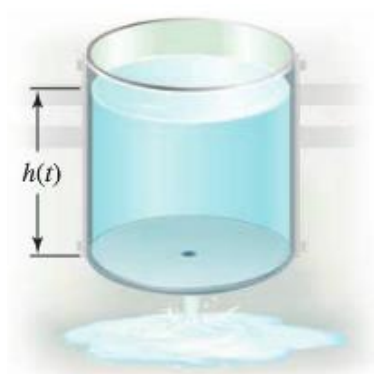
An open cylindrical tank initially filled with water drains through a hole in the bottom of the tank according to Torricelli's Law. If $h(t)$ is the depth of water in the tank for $t \geq 0$, then Torricelli's Law implies $h'(t) = -2k\sqrt{h}$, where k is a constant that includes the acceleration due to gravity, the radius of the tank, and the radius of the drain. Assume that the initial depth of the water is $h(0) = H$.

a) Find the solution of the initial value problem.

b) Find the solution in the case that $k = 0.1$ and $H = 0.5\ m$.

c) In general, how long does it take the tank to drain in terms of k and H?

Solution



a) $\dfrac{dh}{dt} = -2k\sqrt{h}$

$$\int \frac{dh}{\sqrt{h}} = -2\int k\,dt$$

$$2\sqrt{h} = 2kt + C_1$$

$$h(t) = (kt + C)^2$$

$$h(0)=H \quad \rightarrow H=C^2 \Rightarrow C=\sqrt{H}$$

$$\boxed{h(t) = \left(kt + \sqrt{H}\right)^2}$$

b) Given: $k = 0.1 \quad H = 0.5\ m$

$$\boxed{h(t) = \left(0.1t + \sqrt{0.5}\right)^2 = (0.1t + 0.707)^2}$$

c) The tank is drained when $h(t) = 0$

$$\left(kt + \sqrt{H}\right)^2 = 0$$

$$kt + \sqrt{H} = 0 \quad \rightarrow \quad \boxed{t = -\frac{\sqrt{H}}{k}}$$

Exercise

The reaction of chemical compounds can often be modeled by differential equations. Let $y(t)$ be the concentration of a substance in reaction for $t \ge 0$ (typical units of y are *moles/L*). The change in the concentration of a substance, under appropriate conditions, is $\frac{dy}{dt} = -ky^n$, where $k > 0$ is a rate constant and the positive integer n is the order of the reaction.

a) Show that for a first-order reaction $(n = 1)$, the concentration obeys an exponential decay law.

b) Solve the initial value problem for a second-order reaction $(n = 2)$ assuming $y(0) = y_0$

c) Graph and compare the concentration for a first-order and second-order reaction with $k = 0.1$ and $y_0 = 1$

Solution

a) $\displaystyle\int \frac{dy}{y} = -\int k dt$

$\ln|y| = -kt + C_1$

$\boxed{y(t) = Ce^{-kt}}$

b) $n = 2 \quad \to \quad \dfrac{dy}{dt} = -ky^2$

$-\displaystyle\int \frac{dy}{y^2} = \int k dt$

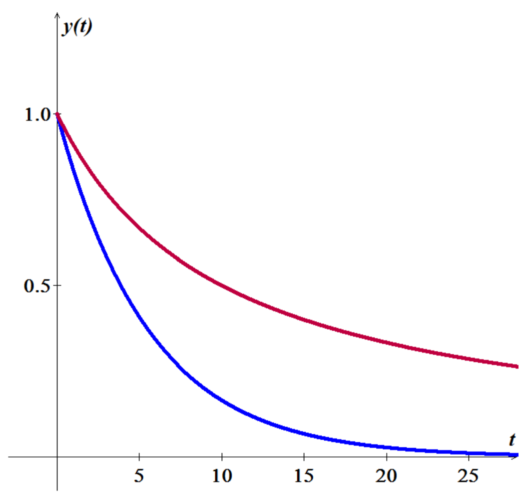
$\dfrac{1}{y} = kt + C \qquad y(0) = y_0 \quad \to \quad \boxed{\dfrac{1}{y_0} = C}$

$\dfrac{1}{y} = kt + \dfrac{1}{y_0}$

$\boxed{y(t) = \dfrac{y_0}{1 + ky_0 t}}$

c) $y(t) = \dfrac{1}{1 + 0.1t}$

$y_0 = 1 \quad \to \quad C = 1 \quad \Rightarrow \quad \boxed{y(t) = e^{-0.1t}}$



Exercise

The growth of cancer turmors may be modeled by the Gomperts growth equation. Let $M(t)$ be the mass of the tumor for $t \ge 0$. The relevant intial value problem is

$$\frac{dM}{dt} = -aM \ln\frac{M}{K}, \quad M(0) = M_0$$

Where a and K are positive constants and $0 < M_0 < K$

a) Graph the growth rate function $R(M) = -aM\ln\frac{M}{K}$ assuming $a=1$ and $K=4$. For what values of M is the growth rate positive? For what values of M is maximum?

b) Solve the initial cvalue problem and graph the solution for $a=1$, $K=4$, and $M_0=1$. Describe the groath pattern of the tumor. Is the growth unbounded? If not, what is the limiting size of the tumor?

c) In the general equation, what is the meaning of K?

Solution

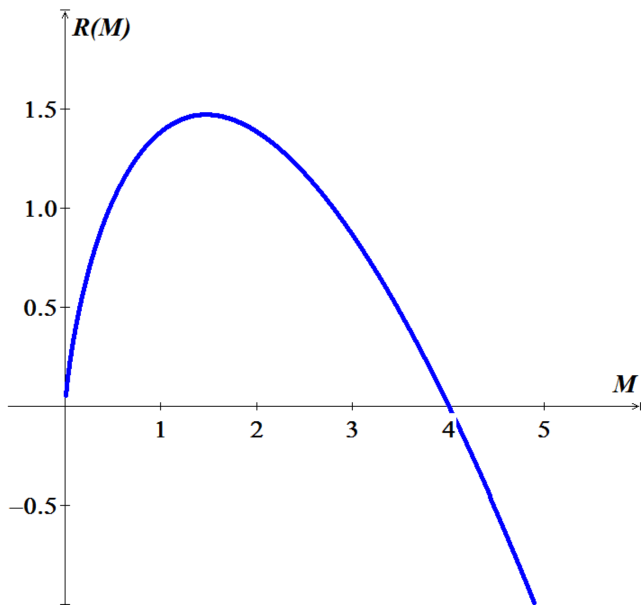
a) $R'(M) = -a\left(\ln\frac{M}{K} + M\frac{1}{K}\frac{K}{M}\right)$

$$= -a\left(\ln\frac{M}{K}+1\right) = 0$$

$$\Rightarrow \ln\frac{M}{K} = -1 \quad\to\quad \boxed{M = Ke^{-1} = \frac{K}{e}}$$

For $a=1$ and $K=4$

$$\to \boxed{R(M) = -M\ln\frac{M}{4}}$$



b) $\displaystyle\int\frac{dM}{M(\ln M - \ln K)} = -\int a\,dt$

$$d(\ln M - \ln K) = \frac{1}{M}dM$$

$$\int\frac{d(\ln M - \ln K)}{\ln M - \ln K} = -\int a\,dt$$

$$\ln\left|\ln M - \ln K\right| = -at + C_1$$

$$\ln\frac{M}{K} = Ce^{-at}$$

$$\boxed{M(t) = Ke^{Ce^{-at}}}$$

For $a=1$, $K=4$, and $M_0=1$

$$M(0) = 4e^{C} = 1 \quad\Rightarrow\quad C = \ln\frac{1}{4} = -\ln 4$$

$$\boxed{M(t) = 4e^{-(\ln 4)e^{-t}}}$$

$$\lim_{t\to\infty} M(t) = \lim_{t\to\infty} 4e^{-(\ln 4)e^{-t}} \;\underline{= 4}$$

So the limiting size of the tumor is 4.

c) $\displaystyle\lim_{t\to\infty} M(t) = \lim_{t\to\infty} Ke^{Ce^{-at}} = K \qquad \text{since } a>0$

Exercise

An endowment is an investment account in which the balance ideally remains constant and withdrawals are made on the interest earned by the account. Such an account may be modeled by the initial value problem $B'(t) = aB - m$ for $t \ge 0$, with $B(0) = B_0$. The constant a reflects the annual interest rate, m is the annual rate of withdrawal, and B_0 is the initial balance in the account.

a) Solve the initial value problem with $a = 0.05$, $m = \$1000/yr.$ and $B_0 = \$15,000$. Does the balance in the account increase or decrease?

b) If $a = 0.05$ and $B_0 = \$50,000$, what is the annual withdrawal rate m that ensures a constant balance in the account? What is the constant balance?

Solution

a) $B'(t) - aB = -m$

$$e^{\int -a\,dt} = e^{-at}$$

$$\int -me^{-at}\,dt = \frac{m}{a}e^{-at}$$

$$B(t) = \frac{1}{e^{-at}}\left(\frac{m}{a}e^{-at} + C\right)$$

$$= \frac{m}{a} + Ce^{at}$$

Given: $a = 0.05,\ m = \$1000/yr.\ B_0 = \$15,000$

$$B(0) = \frac{1000}{.05} + C = 15,000 \quad \Rightarrow \quad C = 15,000 - 20,000 = -5,000$$

$$B(t) = 20,000 - 5,000\, e^{0.05t}$$

The balance decreases since the exponential increases with time and subtract from 20,000.

b) ***Given***: $a = 0.05 \ \ B_0 = \$50,000$

$$B = \frac{m}{a} = 50,000 \quad \Rightarrow \quad m = 0.05 \times 50,000 = 2,500$$

Exercise

The halibut fishery has been modeled by the differential equation $\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right)$

Where $y(t)$ is the biomass (the total mass of the members of the population) in kilograms at time t (measured in years), the carrying capacity is estimated to be $M = 8 \times 10^7\ kg$ and $k = 0.71\ per\ year$.

a) If $y(0) = 2 \times 10^7\ kg$, find the biomass a year later.

b) How long will it take for the biomass to reach $4 \times 10^7\ kg$.

Solution

a) $\dfrac{M}{ky(M-y)}dy = dt \quad \rightarrow \quad \dfrac{M}{k}\dfrac{1}{y(M-y)}dy = dt$

$$\frac{1}{y(M-y)} = \frac{A}{y} + \frac{B}{M-y}$$

$$AM - Ay + By = 1 \quad \rightarrow \quad \begin{cases} AM = 1 \Rightarrow A = \frac{1}{M} \\ -A + B = 0 \Rightarrow B = A = \frac{1}{M} \end{cases}$$

$$\frac{M}{k}\frac{1}{M}\int\left(\frac{1}{y} + \frac{1}{M-y}\right)dy = \int dt$$

$$\frac{1}{k}\left(\ln y - \ln(M-y)\right) = t + C_1$$

$$\ln\frac{y}{M-y} = kt + C_2$$

$$\frac{y}{M-y} = e^{kt+C_2}$$

$$y = Me^{kt}e^{C_2} - ye^{kt}e^{C_2} \qquad C = e^{C_2}$$

$$y\left(1 + Ce^{kt}\right) = MCe^{kt}$$

$$y = \frac{MCe^{kt}}{1 + Ce^{kt}}$$

$$= \frac{M}{1 + Ce^{-kt}}$$

$$= \frac{8\times 10^7}{1 + Ce^{-0.71t}}$$

$$y(0) = \frac{8\times 10^7}{1 + C} = 2\times 10^7 \quad \Rightarrow \quad C = \frac{8\times 10^7}{2\times 10^7} - 1 = 3$$

$$y(t) = \frac{8\times 10^7}{1 + 3e^{-0.71t}}$$

$$y(1) = \frac{8\times 10^7}{1 + 3e^{-0.71}} \approx 3.23\times 10^7 \ kg$$

b) $y(t) = \dfrac{8\times 10^7}{1 + 3e^{-0.71t}} = 4\times 10^7$

$$1 + 3e^{-0.71t} = \frac{8\times 10^7}{4\times 10^7} = 2$$

$$3e^{-0.71t} = 1$$

$$e^{-0.71t} = \frac{1}{3}$$

$$-.071t = \ln\frac{1}{3}$$

$$t = \frac{\ln 3}{0.71} \approx 1.55 \ \ years$$

Exercise

Suppose a population $P(t)$ satisfies $\quad \dfrac{dP}{dt} = 0.4P - 0.001P^2, \quad P(0) = 50$

Where t is measured in years.

a) What is the carrying capacity?

b) What is $P'(0)$?

c) When will the population reach 50% of the carrying capacity?

Solution

a) $\dfrac{1}{0.4P(1-0.0025P)} dP = dt$

$$\frac{1}{P(1-0.0025P)} = \frac{A}{P} + \frac{B}{1-0.0025P}$$

$$A - .0025PA + PB = 1 \quad \rightarrow \begin{cases} \underline{A=1} \\ -.0025A+B=0 \quad \underline{B=.0025} \end{cases}$$

$$\int \left(\frac{1}{P} + \frac{.0025}{1-.0025P}\right) dP = 0.4\int dt$$

$$\ln P - \ln(1-.0025P) = 0.4t + C_1$$

$$\ln \frac{P}{1-.0025P} = 0.4t + C_1$$

$$\frac{P}{1-.0025P} = e^{0.4t+C_1} = Ce^{0.4t} \qquad C = e^{C_1}$$

$$Ce^{-0.4t}P = 1 - .0025P$$

$$Ce^{-0.4t}P + .0025P = 1$$

$$\left(Ce^{-0.4t} + .0025\right)P = 1$$

$$P(t) = \frac{1}{Ce^{-0.4t} + .0025}$$

$$P(0) = \frac{1}{C+.0025} = 50 \quad \underline{C} = \frac{1}{50} - .0025 \ \underline{= .0175}$$

$$P(t) = \frac{1}{.0175e^{-0.4t} + .0025}$$

$$\boxed{P(t) = \frac{400}{7e^{-0.4t} + 1}}$$

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{400}{1+7e^{-0.4t}} \ \underline{= 400}$$

The carrying capacity is 400.

b) $P'(0) = \dfrac{dP}{dt}\Big|_{t=0} = 0.4(50) - 0.001(50)^2 \ \underline{= 17.5}$

c) $P(t) = \dfrac{400}{7e^{-0.4t} + 1} = 200$

$$7e^{-0.4t} + 1 = 2$$

$$e^{-0.4t} = \frac{1}{7}$$

$$-0.4t = \ln\left(\frac{1}{7}\right)$$

$$t = \frac{\ln\left(\frac{1}{7}\right)}{-0.4} \underline{\approx 4.86 \ \ years}$$

Exercise

Let $P(t)$ be the performance level of someone learning a skill as a function of the training time t. The graph of P is called a ***learning curve***. We proposed the differential equation

$$\frac{dP}{dt} = k\left(M - P(t)\right)$$

As a reasonable model for learning, where k is a positive constant. Solve it as a linear differential equation and use your solution to graph the learning curve.

Solution

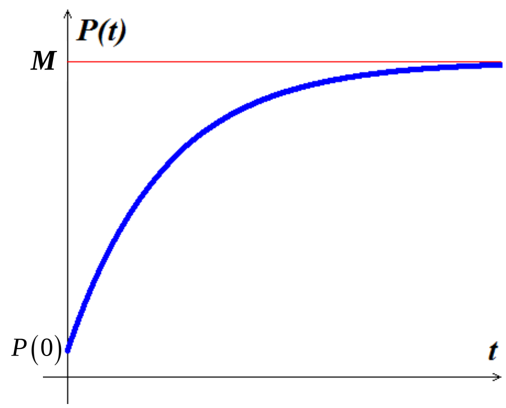
$$\frac{dP}{dt} + kP = kM$$

$$e^{\int kdt} = e^{kt}$$

$$\int kMe^{kt}dt = Me^{kt}$$

$$P(t) = \frac{1}{e^{kt}}\left(Me^{kt} + C\right)$$

$$\underline{= M + Ce^{-kt}} \quad k > o$$



Exercise

A circuit containing an electromotive force, a capacitor with a capacitance of C farads (F), and a resistor with a resistance of R ohms (Ω). The voltage drop across the capacitor is $\frac{Q}{C}$, where Q is the charge (in coulombs), so in this case ***Kirchhoff's Law*** gives

$$RI + \frac{Q}{C} = E(t)$$

But $I = \frac{dQ}{dt}$, so we have $$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

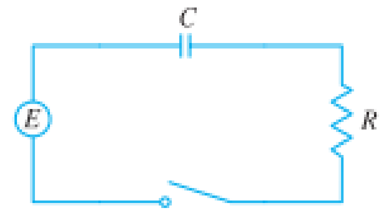
Find the charge and the current at time t

a) Suppose the resistance is $5\ \Omega$, the capacitance is 0.05 F, a battery gives voltage of 60 V and initial charge is $Q(0) = 0\ C$

b) Suppose the resistance is $2\ \Omega$, the capacitance is 0.01 F, $E(t) = 10\sin 60t$ and initial charge is $Q(0) = 0\ C$

Solution

a) $5\dfrac{dQ}{dt}+\dfrac{1}{.05}Q=60 \quad\rightarrow\quad \dfrac{dQ}{dt}+4Q=12$



$$e^{\int 4dt}=e^{4t}$$

$$\int 12e^{4t}dt=3e^{4t}$$

$$Q(t)=\frac{1}{e^{4t}}\left(3e^{4t}+C\right)\ \underline{=3+Ce^{-4t}}$$

$$Q(0)=3+C=0 \ \Rightarrow\ \underline{C=-3}$$

$$\boxed{Q(t)=3\left(1-e^{-4t}\right)}$$

$$I=\frac{dQ}{dt}\ \underline{=12e^{-4t}}$$

b) $2\dfrac{dQ}{dt}+\dfrac{1}{.01}Q=10\sin 60t \quad\rightarrow\quad \dfrac{dQ}{dt}+50Q=5\sin 60t$

$$e^{\int 50dt}=e^{50t}$$

$$5\int e^{50t}(\sin 60t)\,dt=$$

$$\int e^{50t}(\sin 60t)\,dt=\left(-\frac{1}{60}\cos 60t+\frac{1}{72}\sin 60t\right)e^{50t}-\frac{25}{36}\int e^{50t}(\sin 60t)\,dt$$

$$\frac{61}{36}\int e^{50t}(\sin 60t)\,dt=\left(-\frac{1}{60}\cos 60t+\frac{1}{72}\sin 60t\right)e^{50t}$$

$$\int e^{50t}(\sin 60t)\,dt=\frac{36}{21{,}960}(-6\cos 60t+5\sin 60t)e^{50t}$$

$$5\int e^{50t}(\sin 60t)\,dt=\frac{1}{122}(-6\cos 60t+5\sin 60t)e^{50t}$$

		$\int \sin 60t$
+	e^{50t}	$-\frac{1}{60}\cos 60t$
−	$50e^{50t}$	$-\frac{1}{3600}\sin 60t$
+	$2500e^{50t}$	$-\frac{1}{3600}\int \sin 60t$

$$Q(t)=\frac{1}{e^{50t}}\left(\frac{1}{122}(-6\cos 60t+5\sin 60t)e^{50t}+C\right)$$

$$=\frac{1}{122}(-6\cos 60t+5\sin 60t)+Ce^{-50t}$$

$$Q(0)=-\frac{6}{122}+C=0 \ \Rightarrow\ \underline{C=\frac{3}{61}}$$

$$\boxed{Q(t)=\frac{1}{122}\left(-5\cos 60t+6\sin 60t+6e^{-50t}\right)}$$

$$I=\frac{dQ}{dt}=\frac{1}{122}\left(300\sin 60t+360\cos 60t-300e^{-50t}\right)$$

$$\boxed{=\frac{30}{61}\left(5\sin 60t+6\cos 60t-5e^{-50t}\right)}$$

Exercise

A 30–volt electromotive force is applied to an *LR*-series circuit in which the inductance is 0.1 *henry* and the resistance is 50 *ohms*.

a) Find the current $i(t)$ if $i(0)=0$

b) Determine the current as $t\to\infty$

c) Solve the equation when $E(t)=E_0\sin\omega t$ and $i(0)=i_0$

Solution

a) $0.1\frac{di}{dt}+50i=30$ $\qquad L\frac{di}{dt}+Ri=E(t)$

$$\frac{di}{dt}+500i=300$$

$$e^{\int 500dt}=e^{500t}$$

$$\int 300e^{500t}dt=\frac{3}{5}e^{500t}$$

$$i(t)=e^{-500t}\left(\frac{3}{5}e^{500t}+C\right) \qquad i(0)=0$$

$$0=\frac{3}{5}+C \ \rightarrow\ C=-\frac{3}{5}$$

$$i(t)=\frac{3}{5}-\frac{3}{5}e^{-500t}$$

b) $\lim\limits_{t\to\infty} i(t)=\lim\limits_{t\to\infty}\left(\frac{3}{5}-\frac{3}{5}e^{-500t}\right)=\frac{3}{5}$

c) $\frac{di}{dt}+500i=10E_0\sin\omega t$

		$\int \sin\omega t$
+	e^{500t}	$-\frac{1}{\omega}\cos\omega t$
−	$500e^{500t}$	$-\frac{1}{\omega^2}\sin\omega t$
+	$25\times10^4e^{500t}$	$-\int\frac{1}{\omega^2}\sin\omega t$

$$\int 10E_0(\sin\omega t)e^{500t}dt=$$

$$\int(\sin\omega t)e^{500t}dt=\left(-\frac{1}{\omega}\cos\omega t+\frac{500}{\omega^2}\sin\omega t\right)e^{500t}-\frac{25\times10^4}{\omega^2}\int(\sin\omega t)e^{500t}dt$$

$$\left(\frac{\omega^2+25\times10^4}{\omega^2}\right)\int(\sin\omega t)e^{500t}dt=\frac{1}{\omega^2}(-\omega\cos\omega t+500\sin\omega t)e^{500t}$$

$$\int 10E_0(\sin\omega t)e^{500t}dt=\frac{10E_0}{\omega^2+25\times10^4}(-\omega\cos\omega t+500\sin\omega t)e^{500t}$$

$$i(t)=e^{-500t}\left(\frac{10E_0}{\omega^2+25\times10^4}(-\omega\cos\omega t+500\sin\omega t)e^{500t}+C\right) \qquad i(0)=i_0$$

$$0=-\frac{10\omega tE_0}{\omega^2+25\times10^4}+C \ \rightarrow\ C=\frac{10\omega tE_0}{\omega^2+25\times10^4}$$

$$i(t)=\frac{10E_0}{\omega^2+25\times10^4}\left(-\omega\cos\omega t+500\sin\omega t-\omega e^{-500t}\right)$$

Exercise

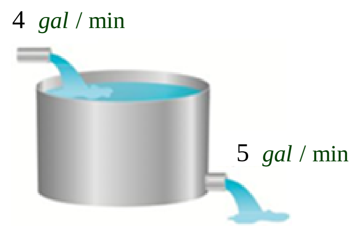
A tank contains 50 *gallons* of a solution composed of 90% water and 10% alcohol. A second solution containing 50% water and 50% alcohol is added to the tank at the rate of $4 \ gal/\min$. As the second solution is being added, the tank is being drained at a rate of $5 \ gal/\min$. The solution in the tank is stirred constantly. How much alcohol is in the tank after 10 *minutes*?

Solution

Let y be the amount (in *lb.*) of additive in the tank at time t and $y(0)=100$

$$V(t)=50+\left(4\frac{gal}{\min}-5\frac{gal}{\min}\right)(t \ \min)$$

$$\boxed{=50-t}$$



$$Rate\ out=\frac{y}{50-t}(5)=\frac{5y}{50-t} \ \frac{lb}{\min}$$

$$Rate\ in=\left(\frac{1}{2} \ \frac{lb}{gal}\right)\left(4 \ \frac{gal}{\min}\right)=\boxed{2 \ \frac{lb}{\min}}$$

$$\frac{dy}{dt}=2-\frac{5}{50-t}y$$

$$\frac{dy}{dt}+\frac{5}{50-t}y=2$$

$$e^{\int pdt}=e^{\int \frac{5}{50-t}dt}=e^{\int \frac{-5}{50-t}d(50-t)}=e^{-5\ln|50-t|}=(50-t)^{-5}$$

$$\int 2(50-t)^{-5}dt=-2\int (50-t)^{-5}d(50-t)=\frac{1}{2}(50-t)^{-4}$$

$$y(t)=\frac{1}{(50-t)^{-5}}\left(\frac{1}{2}(50-t)^{-4}+C\right)$$

$$\boxed{=\frac{1}{2}(50-t)+C(50-t)^5}$$

$$y(0)=\frac{1}{2}(50)+C(50)^5=5 \quad \rightarrow \quad \boxed{C=-\frac{20}{50^5}}$$

$$\boxed{y(t)=\frac{1}{2}(50-t)-\frac{20}{50^5}(50-t)^5}$$

$$y(t=20)=\frac{1}{2}(30)-\frac{20}{50^5}(30)^5$$

$$=15-\frac{20}{5^5}3^5$$

$$\approx \boxed{13.45 \ gal}$$

Exercise

A 200-*gallon* tank is half full of distilled water. At time $t = 0$, a concentrate solution containing $0.5\ lb/gal$ enters the tank at the rate of $5\ gal/\text{min}$, and well-stirred mixture is withdrawn at the rate of $3\ gal/\text{min}$.

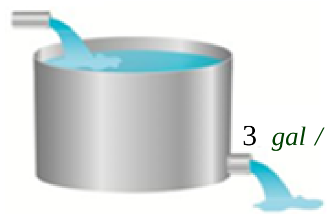
a) At what time will the tank be full?

b) At the time the tank is full, how many pounds of concentrate will it contain?

Solution

a) $V(t) = 100 + (5-3)t = 200$

$2t = 100 \quad \Rightarrow \quad \underline{t = 50\ \ min}$



b) $Rate\ out = \dfrac{y}{100+2t}(3) = \dfrac{3y}{100+2t}\ \dfrac{lb}{\text{min}}$

$Rate\ in = \left(0.5\ \dfrac{lb}{gal}\right)\left(5\ \dfrac{gal}{\text{min}}\right) = \underline{2.5\ \dfrac{lb}{\text{min}}}$

$$\frac{dy}{dt} = 2.5 - \frac{3y}{100+2t}$$

$$\frac{dy}{dt} + \frac{3}{100+2t}y = 2.5$$

$$e^{\int \frac{3}{100+2t}dt} = e^{\frac{3}{2}\int \frac{1}{50+t}d(50+t)} = e^{\frac{3}{2}\ln|50+t|} = (50+t)^{3/2}$$

$$2.5\int (50+t)^{3/2}\,dt = \frac{5}{2}\int (50+t)^{3/2}\,d(50+t) = (50+t)^{5/2}$$

$$y(t) = \frac{1}{(50+t)^{3/2}}\left((50+t)^{5/2} + C\right)$$

$$\underline{= 50 + t + C(50+t)^{-3/2}}$$

$$y(0) = 50 + C(50)^{-3/2} = 0$$

$$\underline{C = -(50)^{5/2}}$$

$$y(t)\ \underline{= 50 + t - (50)^{5/2}(50+t)^{-3/2}}$$

$$y(50) = 50 + 50 - (50)^{5/2}(100)^{-3/2}$$

$$\underline{\approx 82.32\ \ lb}$$

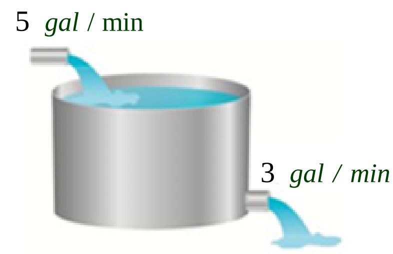
Exercise

A 200-*gallon* tank is half full of distilled water. At time $t=0$, a concentrate solution containing 1 *lb/gal* enters the tank at the rate of $5\ gal\ /\ \text{min}$, and well-stirred mixture is withdrawn at the rate of $3\ gal\ /\ \text{min}$.

a) At what time will the tank be full?

b) At the time the tank is full, how many pounds of concentrate will it contain?

Solution



c) $V(t)=100+(5-3)t=200$

$$2t=100 \quad \Rightarrow \quad \boxed{t=50\ \ min}$$

d) $Rate\ out=\dfrac{y}{100+2t}(3)=\dfrac{3y}{100+2t}\ \dfrac{lb}{\text{min}}$

$$Rate\ in=\left(1\ \frac{lb}{gal}\right)\left(5\ \frac{gal}{\text{min}}\right)=\boxed{5\ \frac{lb}{\text{min}}}$$

$$\frac{dy}{dt}=5-\frac{3y}{100+2t}$$

$$\frac{dy}{dt}+\frac{3}{100+2t}y=5$$

$$e^{\int \frac{3}{100+2t}dt}=e^{\frac{3}{2}\int \frac{1}{50+t}d(50+t)}=e^{\frac{3}{2}\ln|50+t|}=(50+t)^{3/2}$$

$$5\int (50+t)^{3/2}dt=5\int (50+t)^{3/2}d(50+t)=2(50+t)^{5/2}$$

$$y(t)=\frac{1}{(50+t)^{3/2}}\left(2(50+t)^{5/2}+C\right)$$

$$\boxed{=100+2t+C(50+t)^{-3/2}}$$

$$y(0)=100+C(50)^{-3/2}=0$$

$$\to C=-(100)(25\times 2)^{3/2}\ \boxed{=25000\sqrt{2}}$$

$$y(t)\ \boxed{=100+2t-25{,}000\sqrt{2}(50+t)^{-3/2}}$$

$$y(50)=100+100-25{,}000\sqrt{2}(100)^{-3/2}$$

$$=200-25\sqrt{2}$$

$$\boxed{\approx 164.64\ \ lb}$$

Exercise

A 200-*gallon* tank is full of a concentrate solution containing $25\ lb$. Starting at time $t=0$, distilled water is admitted to the tank at the rate of $10\ gal/\min$, and well-stirred mixture is withdrawn at the same rate.

a) Find the amount of concentrate in the solution as a function of *t*.

b) Find the time at which the amount of concentrate in the tank reaches 15 *pounds*.

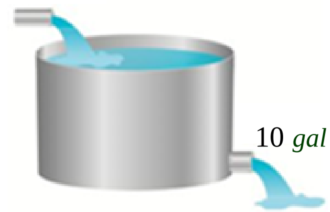
c) Find the quantity of the concentrate in the solution as $t \to \infty$.

Solution

a) $V(t) = 200 + (10-10)t \ \underline{= 200}$

$$Rate\ out = \frac{10y}{200} = \frac{y}{20}\ \frac{lb}{\min}$$

$Rate\ in = \underline{0}$



$$\frac{dy}{dt} = -\frac{y}{20}$$

$$\int \frac{dy}{y} = -\frac{1}{20}\int dt$$

$$\ln y = -\frac{1}{20}t + C_1 \ \to\ \underline{y(t) = Ce^{-t/20}}$$

$$y(0) = \underline{C = 25}$$

$$\underline{y(t) = 25e^{-t/20}}$$

b) $y(t) = 25e^{-t/20} = 15$

$$e^{-t/20} = \frac{3}{5}$$

$$-\frac{t}{20} = \ln\left(\frac{3}{5}\right)$$

$$t = -20\ln\left(\frac{3}{5}\right)\ \underline{\approx 10.2\ min}$$

c) $\lim\limits_{t\to\infty} y(t) = \lim\limits_{t\to\infty} 25e^{-t/20}\ \underline{=0}$