Solution Section 1.5 – Transpose, Diagonal, Triangular, and Symmetric Matrices

Exercise

Solve $Lc = b$ to find $\boldsymbol{c}$. Then solve $Ux = c$ to find $\boldsymbol{x}$. What was A?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Solution

$$Lc = b$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

$$\rightarrow \begin{cases} \boxed{c_1 = 4} \\ c_1 + c_2 = 5 \Rightarrow \boxed{c_2 = 5 - 4 = 1} \\ c_1 + c_2 + c_3 = 6 \Rightarrow \boxed{c_3 = 6 - 4 - 1 = 1} \end{cases} \qquad c = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

$$Ux = c$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x + y + z = 4 \\ y + z = 1 \\ z = 1 \end{cases} \rightarrow \begin{cases} x = 3 \\ y = 0 \end{cases} \qquad x = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

$$Lc = b \Rightarrow LUx = b$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}}_{b}$$

Exercise

Find L and U for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on *a, b, c, d* to get $A = LU$ with four pivots

Solution

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

Exercise

Determine whether the given matrix is invertible

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Solution

The matrix is a diagonal matrix with nonzero entries on the diagonal, so it is invertible.

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Exercise

Find A^2, A^{-2}, *and* A^{-k} by inspection $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

Solution

$$A^2 = \begin{bmatrix} 1^2 & 0 \\ 0 & (-2)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$A^{-2} = \begin{bmatrix} 1^{-2} & 0 \\ 0 & (-2)^{-2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

$$A^{-k} = \begin{bmatrix} 1^{-k} & 0 \\ 0 & (-2)^{-k} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{(-2)^k} \end{bmatrix}$$

Exercise

Find A^2, A^{-2}, *and* A^{-k} by inspection $A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$

Solution

$$A^2 = \begin{bmatrix} \left(\frac{1}{2}\right)^2 & 0 & 0 \\ 0 & \left(\frac{1}{3}\right)^2 & 0 \\ 0 & 0 & \left(\frac{1}{4}\right)^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{16} \end{bmatrix}$$

$$A^{-2} = \begin{bmatrix} \left(\frac{1}{2}\right)^{-2} & 0 & 0 \\ 0 & \left(\frac{1}{3}\right)^{-2} & 0 \\ 0 & 0 & \left(\frac{1}{4}\right)^{-2} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$

$$A^{-k} = \begin{bmatrix} \left(\frac{1}{2}\right)^{-k} & 0 & 0 \\ 0 & \left(\frac{1}{3}\right)^{-k} & 0 \\ 0 & 0 & \left(\frac{1}{4}\right)^{-k} \end{bmatrix}$$

$$= \begin{bmatrix} 2^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 4^k \end{bmatrix}$$

Exercise

Find A^2, A^{-2}, *and* A^{-k} by inspection $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Solution

$$A^2 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$A^{-2} = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{16} & 0 & 0 \\ 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$A^{-k} = \begin{bmatrix} (-2)^{-k} & 0 & 0 & 0 \\ 0 & (-4)^{-k} & 0 & 0 \\ 0 & 0 & (-3)^{-k} & 0 \\ 0 & 0 & 0 & (2)^{-k} \end{bmatrix}$$

Exercise

Decide whether the given matrix is symmetric $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

Solution

Not *symmetric*, since $a_{12} \neq a_{21} \quad (1 \neq -1)$

Exercise

Decide whether the given matrix is symmetric $\begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{bmatrix}$

Solution

Symmetric

Exercise

Decide whether the given matrix is symmetric $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

Solution

Not *symmetric*, since $a_{13} = 1 \neq 3 = a_{31}$

Exercise

Find all values of the unknown constant(s) in order for A to be symmetric

$$A = \begin{bmatrix} 2 & a-2b+2c & 2a+b+c \\ 3 & 5 & a+c \\ 0 & -2 & 7 \end{bmatrix}$$

Solution

$$\begin{cases} a - 2b + 2c = 3 \\ 2a + b + c = 0 \\ a \qquad + c = -2 \end{cases} \quad \rightarrow a = 11, \quad b = 9, \quad c = -13$$

Exercise

Find a diagonal matrix A that satisfies the given condition $A^{-2} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution

$$\begin{aligned} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^{-2} &= \begin{pmatrix} a^{-2} & 0 & 0 \\ 0 & b^{-2} & 0 \\ 0 & 0 & c^{-2} \end{pmatrix} \\ &= \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$\rightarrow \begin{cases} a^{-2} = 9 \quad \Rightarrow \quad a = \pm 9^{-1/2} = \pm \frac{1}{3} \\ b^{-2} = 4 \quad \Rightarrow \quad b = \pm 2^{-1/2} = \pm \frac{1}{2} \\ c^{-2} = 1 \quad \Rightarrow \quad c = \pm 1^{-1/2} = \pm 1 \end{cases}$$

$$A = \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A = \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A = \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \ldots$$

$$A = \begin{pmatrix} \pm\frac{1}{3} & 0 & 0 \\ 0 & \pm\frac{1}{2} & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$$

Exercise

Let A be an $n \times n$ symmetric matrix

a) Show that A^2 is symmetric

b) Show that $2A^2 - 3A + I$ is symmetric

Solution

a) The property of the transpose states that $(AB)^T = B^T A^T$

$$\begin{aligned}\left(A^2\right)^T &= (AA)^T \\ &= A^T A^T \\ &= \left(A^T\right)^2 \qquad \textit{A is symmetric} \\ &= A^2\end{aligned}$$

$\therefore$ A^2 is symmetric

b)
$$\begin{aligned}\left(2A^2 - 3A + I\right)^T &= 2\left(A^2\right)^T - 3(A)^T + (I)^T \\ &= 2\left(A^T\right)^2 - 3A^T + (I)^T \qquad \textit{A and I are symmetric} \\ &= 2A^2 - 3A + I\end{aligned}$$

$\therefore$ $2A^2 - 3A + I$ is ***Symmetric***

Exercise

Prove if $A^T A = A$, then A is symmetric and $A = A^2$

Solution

If $A^T A = A$, then

$$\begin{aligned}A^T &= \left(A^T A\right)^T \\ &= A^T \left(A^T\right)^T \\ &= A^T A \\ &= A\end{aligned}$$

So A is symmetric.

Since $A = A^T$

$$AA = A^T A \qquad A^T A = A$$

$$A^2 = A$$

Exercise

A square matrix A is called ***skew-symmetric*** if $A^T = -A$. Prove

a) If A is an invertible skew-symmetric matrix, then A^{-1} is skew-symmetric.

b) If A and B are skew-symmetric matrices, then so are A^T, $A+B$, $A-B$, *and* kA for any scalar k.

c) Every square matrix A can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

$\left[\textit{Hint}: \text{ Note the identity } \ A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right) \right]$

Solution

a)

$$\left(A^{-1}\right)^T = \left(A^T\right)^{-1}$$
$$= (-A)^{-1} \qquad \textit{skew-symmetric}$$
$$= -A^{-1}$$

$\therefore\ A^{-1}$ is also skew-symmetric

b) Let A and B are skew-symmetric matrices

$$\left(A^T\right)^T = (-A)^T$$
$$= -A^T$$

$$(A+B)^T = A^T + B^T$$
$$= -A - B$$
$$= -(A+B)$$

$$(A-B)^T = A^T - B^T$$
$$= -A + B$$
$$= -(A-B)$$

$$(kA)^T = k(A)^T$$
$$= k(-A)$$
$$= -kA$$

c) We need to prove from the hint that $\frac{1}{2}\left(A + A^T\right)$ is symmetric and $\frac{1}{2}\left(A - A^T\right)$ is skew-symmetric

$$\frac{1}{2}\left(A + A^T\right)^T = \frac{1}{2}\left(A^T + \left(A^T\right)^T\right)$$
$$= \frac{1}{2}\left(A + A^T\right)$$

Thus $\frac{1}{2}\left(A+A^T\right)$ ***is symmetric***

$$\frac{1}{2}\left(A-A^T\right)^T=\frac{1}{2}\left(A^T-\left(A^T\right)^T\right)$$
$$=\frac{1}{2}\left(A^T-A\right)$$
$$=-\frac{1}{2}\left(A-A^T\right)$$

Thus $\frac{1}{2}\left(A-A^T\right)$ ***is skew-symmetric***

Exercise

Suppose R is rectangular (m by n) and A is symmetric (m by m)

a) Transpose $R^T AR$ to show its symmetric

b) Show why $R^T R$ has no negative numbers on its diagonal.

Solution

a) $\left(R^T AR\right)^T=\left(\left(R^T A\right)R\right)^T$

$$=R^T\left(R^T A\right)^T$$
$$=R^T A^T\left(R^T\right)^T$$
$$=R^T AR$$

b) $\left(R^T R\right)_{jj}=\left(column\ j\ of\ R\right).\left(column\ j\ of\ R\right)$

= ***Product of the diagonal entry by itself.***

= length squared of column j.

Exercise

If L is a lower-triangular matrix, then $\left(L^{-1}\right)^T$ is _______Triangular

Solution

$\left(L^{-1}\right)^T$ is ***upper*** triangular.

L^{-1} is a lower-triangular because L is.

The transpose carries the lower-triangular matrices to the upper-triangular (and vice versa).

Exercise

True or False

a) The block matrix $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ is automatically symmetric

b) If A and B are symmetric then their product is symmetric

c) If A is not symmetric then A^{-1} is not symmetric

d) When A, B, C are symmetric, the transpose of ABC is CBA.

e) The transpose of a diagonal matrix is a diagonal.

f) The transpose of an upper triangular matrix is an upper triangular matrix.

g) The sum of an upper triangular matrix and a lower triangular matrix is a diagonal matrix.

h) All entries of a symmetric matrix are determined by the entries occurring on and above the main diagonal.

i) All entries of an upper triangular matrix are determined by the entries occurring on and above the main diagonal.

j) The inverse of an invertible lower triangular matrix is an upper triangular matrix.

k) A diagonal matrix is invertible if and only if all of its diagonal entries are positive.

l) The sum of a diagonal matrix and a lower triangular matrix is a lower triangular matrix.

m) A matrix that is both symmetric and upper triangular must be a diagonal matrix.

n) If A and B are $n \times n$ matrices such that $A+B$ is symmetric, then A and B are symmetric.

o) If A and B are $n \times n$ matrices such that $A+B$ is upper triangular, then A and B are upper triangular.

p) If A^2 is a symmetric matrix, then A is a symmetric matrix.

q) If kA is a symmetric matrix for some $k \neq 0$, then A is a symmetric matrix.

Solution

a) False: $\left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right)$

b) False $\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}_{B} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

c) True by definition.

d) True $(ABC)^T = C^T (AB)^T = C^T B^T A^T = CBA$ Since $A^T = A,\ B^T = B,\ C^T = C$

e) True Since a diagonal matrix must be square and have zeros off the main diagonal, its transpose is also diagonal.

f) False The transpose of an upper triangular matrix is lower triangular.

g) False $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix}$

h) True The entries above the main diagonal determine the entries below the main diagonal in a symmetric matrix.

i) True in an upper triangular matrix, the series below the main diagonal are all zeros.

j) False The inverse of an invertible lower triangular matrix is lower triangular.

k) False The diagonal entries may be negative, as long as they are nonzero.

l) True Adding a diagonal matrix to a lower triangular matrix will not create nonzero entries above the main diagonal.

m) True Since the entries below the main diagonal must be zero, so also must be the entries above the main diagonal.

n) False $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ which is symmetric

o) False $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 0 & 4 \end{bmatrix}$ which is upper triangular.

p) False $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

q) True $(kA)^T = kA$ then

$(kA)^T - kA = 0$

$kA^T - kA = 0$

$k\left(A^T - A\right) = 0$ since $k \neq 0$ then $A^T = A$

Therefore, A is a symmetric matrix

Exercise

Find 2 by 2 symmetric matrices $A = A^T$ with these properties

a) A is not invertible

b) A is invertible but cannot be factored into LU (row exchanges needed)

c) A can be factored into LDL^T but not into LL^T (because of negative D)

Solution

a) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

b) $A=\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$ only need a *zero* in the diagonal.

c) $A=LDL^T$

$$A=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$$

$$A=\begin{pmatrix}a & 0\\ a & d\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}a & a\\ a & a+d\end{pmatrix}=\underbrace{\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}}_{LL^T}\to\begin{cases}a=1\\ d=1\end{cases}$$

$$A=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\to D=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

Exercise

A group of matrices includes AB and A^{-1} if it includes A and B. "Products and inverses stay in the group." Which of these sets are groups?
Lower triangular matrices L with 1's on the diagonal, symmetric matrices S, positive matrices M, diagonal invertible matrices D, permutation matrices P, matrices with $Q^T=Q^{-1}$. ***Invent two more matrix groups***.

Solution

The lower triangular matrices L with 1's on the diagonal form a group.
Clearly the product of two is a third. The Gauss-Jordan method shows that the inverse of one is another.
The symmetric matrices don't form a group. An example of the 2 symmetric matrices A and B whose product is not symmetric

$$A=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\quad B=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}\quad AB=\begin{bmatrix}2 & 4 & 5\\ 1 & 2 & 3\\ 3 & 5 & 6\end{bmatrix}$$

The positive matrices do not form a group.

$M=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\quad M^{-1}=\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$, the inverse is not symmetric.

The diagonal invertible matrices form a group.
The permutation matrices form a group.
The matrices with $Q^T=Q^{-1}$ form a group. If A and B are two matrices, then so are AB and A^{-1}, as

$$(AB)^T = B^T A^T = B^{-1}A^{-1} = (AB)^{-1}$$
$$\left(A^{-1}\right)^T = \left(A^T\right)^{-1} = A^{-1}$$

There are many more matrix groups. For example, given two, the block matrices $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ form a third as A ranges over the first group and B ranges over the second.

Another example is the set of all products cP where c is a nonzero scalar and P is a permutation matrix of given size.

Exercise

Write $A = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$ as the product EH of an elementary row operation matrix E and a symmetric matrix H.

Solution

$$A = EH$$
$$E^{-1}A = E^{-1}EH$$
$$E^{-1}A = H$$

An elementary row operation matrix has the form $E = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$

The inverse is: $E^{-1} = \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}$

$$H = \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 \\ -x+4 & -2x+9 \end{pmatrix}$$

Since matrix H is symmetric, therefore:

$$-x + 4 = 2$$
$$\underline{x = 2}$$

$$\begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}}_{Elementary} \underbrace{\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}}_{Symmetric}$$

Exercise

When is the product of two symmetric matrices symmetric? Explain your answer.

Solution

AB is symmetric *iff* $AB = (AB)^T$

$$\begin{aligned} AB &= (AB)^T \\ &= B^T A^T \qquad \text{A and B are symmetric} \\ &= BA \end{aligned}$$

AB is symmetric iff A and B commute

Exercise

Express $\left((AB)^{-1}\right)^T$ in terms of $\left(A^{-1}\right)^T$ and $\left(B^{-1}\right)^T$

Solution

$$\begin{aligned} \left((AB)^{-1}\right)^T &= \left(B^{-1}A^{-1}\right)^T \\ &= \left(A^{-1}\right)^T \left(B^{-1}\right)^T \end{aligned}$$

Exercise

Find the transpose of the given matrix: $\begin{bmatrix} 8 & -1 \\ 3 & 5 \\ -2 & 5 \\ 1 & 2 \\ -3 & -5 \end{bmatrix}$

Solution

$$A^T = \begin{bmatrix} 8 & 3 & -2 & 1 & -3 \\ -1 & 5 & 5 & 2 & -5 \end{bmatrix}$$

Exercise

Show that if A is symmetric and invertible, then A^{-1} is also symmetric.

Solution

A is symmetric and invertible, then $A = A^T \quad AA^{-1} = I$

$$\left(A^{-1}\right)^T = \left(A^T\right)^{-1}$$
$$= A^{-1}$$

$\Rightarrow$ A^{-1} is symmetric.

Exercise

Prove that $(AB)^T = B^T A^T$

Solution

Let $A = \left[a_{ik}\right]$ and $B = \left[b_{kj}\right]$

Then the *ij*-entry of *AB* is:

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{im}b_{mj}$$

The reverse order, *ji*-entry of $(AB)^T$

Column *j* of *B* becomes row *j* of B^T, and row *i* of *A* becomes column *i* of A^T.

Thus, the *ij*-entry of $B^T A^T$ is:

$$\left(b_{1j},\ b_{2j}, \ldots,\ b_{mj}\right)\left(a_{i1},\ a_{i2},\ \ldots,\ a_{im}\right)^T = b_{1j}a_{i1} + b_{2j}a_{i2} + \ldots + b_{mj}a_{im}$$

Thus $(AB)^T = B^T A^T$

Exercise

For the given matrix, compute A^T, $\left(A^T\right)^{-1}$, A^{-1}, and $\left(A^{-1}\right)^T$, then compare $\left(A^T\right)^{-1}$ and $\left(A^{-1}\right)^T$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

Solution

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_1 - 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + R_3 \\ R_1 - 2R_3 \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \qquad \left(A^T\right)^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 2 & 1 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\left(A^{-1}\right)^T = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

Exercise

Show that a 2×2 lower triangular matrix is invertible if and only if $a_{11}a_{22} \neq 0$ and in this case the inverse is also lower triangular.

Solution

Let A to be the lower triangular matrix

$$A = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}$$

$\det(A) = a_{11}a_{22} \neq 0$ is invertible iff $a_{11}a_{22} \neq 0$ and then

$$A^{-1} = \frac{1}{a_{11}a_{22}} \begin{pmatrix} a_{22} & 0 \\ -a_{21} & a_{11} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{a_{11}} & 0 \\ -\frac{a_{21}}{a_{11}a_{22}} & \frac{1}{a_{22}} \end{pmatrix}$$

Exercise

Let A be any 2×2 diagonal matrix. Give a necessary and sufficient condition on the diagonal entries so that A has an inverse. Compute the inverse of any such matrix.

Solution

Let $A = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$

$$A^{-1} = \begin{pmatrix} \frac{1}{a_{11}} & 0 \\ 0 & \frac{1}{a_{22}} \end{pmatrix}$$

So, A^{-1} exists when both entries on the main diagonal are nonzero.