Derivative: Rational Function to Power 'n' in the form $\dfrac{ax^n+b}{cx^n+d}$

$$\left(\frac{ax^n+b}{cx^n+d}\right)' = \boxed{\frac{n\left(ad-bc\right)x^{n-1}}{\left(cx^n+d\right)^2}} = \frac{n\begin{vmatrix} a & b \\ c & d \end{vmatrix}x^{n-1}}{\left(cx^n+d\right)^2}$$

Proof

$$u = ax^n + b \qquad v = cx^n + d$$

$$u' = nax^{n-1} \qquad v' = ncx^{n-1}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$\left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{nax^{n-1}\left(cx^n+d\right) - ncx^{n-1}\left(ax^n+b\right)}{\left(cx^n+d\right)^2}$$

$$= \frac{nacx^{2n-1} + nadx^{n-1} - nacx^{2n-1} - nbcx^{n-1}}{\left(cx^n+d\right)^2}$$

$$= \frac{nadx^{n-1} - nbcx^{n-1}}{\left(cx^n+d\right)^2}$$

$$= \boxed{\frac{n\left(ad-bc\right)x^{n-1}}{\left(cx^n+d\right)^2}}$$

Example

Find $\left(\dfrac{x+2}{3x-2}\right)'$

Solution

$$\left(\frac{x+2}{3x-2}\right)' = \frac{-2-6}{(3x-2)^2}$$

$$= \frac{-8}{(3x-2)^2}$$

$$\left(\frac{x+2}{3x-2}\right)' = \frac{3x-2-3(x+2)}{(3x-2)^2}$$

$$= \frac{3x-2-3x-6}{(3x-2)^2}$$

$$= \frac{-8}{(3x-2)^2}$$

Example

Find $\left(\dfrac{5x^2-3}{2x^2-4}\right)'$

Solution

$$\left(\frac{5x^2-3}{2x^2-4}\right)' = \frac{2(-20+6)x}{\left(2x^2-4\right)^2}$$

$$= \frac{-28x}{\left(2x^2-4\right)^2}$$

$$\left(\frac{5x^2-3}{2x^2-4}\right)' = \frac{10x\left(2x^2-4\right)-4x\left(5x^2-3\right)}{(3x-2)^2}$$

$$= \frac{20x^3-40x-20x^3+12x}{(3x-2)^2}$$

$$= \frac{-28x}{\left(2x^2-4\right)^2}$$

Derivative: Rational Function to Power '**n**' in the form $\left(\dfrac{ax^n+b}{cx^n+d}\right)^m$

$$\frac{d}{dx}\left(\frac{ax^n+b}{cx^n+d}\right)^m = mn(ad-bc)x^{n-1}\frac{\left(ax^n+b\right)^{m-1}}{\left(cx^n+d\right)^{m+1}}$$

Proof

$$u = ax^n + b \qquad v = cx^n + d$$

$$u' = nax^{n-1} \qquad v' = ncx^{n-1}$$

$$\frac{d}{dx}\left(\frac{ax^n+b}{cx^n+d}\right)^m = m\frac{nax^{n-1}\left(cx^n+d\right)-ncx^{n-1}\left(ax^n+b\right)}{\left(cx^n+d\right)^2}\left(\frac{ax^n+b}{cx^n+d}\right)^{m-1} \qquad \left(\frac{u}{v}\right)' = \frac{u'v-v'u}{v^2}$$

$$= \frac{m\left(nacx^{2n-1}+nadx^{n-1}-nacx^{2n-1}-nbcx^{n-1}\right)\left(ax^n+b\right)^{m-1}}{\left(cx^n+d\right)^2\left(cx^n+d\right)^{m-1}}$$

$$= \frac{m\left(nadx^{n-1}-nbcx^{n-1}\right)\left(ax^n+b\right)^{m-1}}{\left(cx^n+d\right)^{m+1}}$$

$$= \frac{mn(ad-bc)x^{n-1}\left(ax^n+b\right)^{m-1}}{\left(cx^n+d\right)^{m+1}}$$

Example

Find $\qquad \dfrac{d}{dx}\left(\dfrac{x+2}{3x-2}\right)^4$

Solution

$$\frac{d}{dx}\left(\frac{x+2}{3x-2}\right)^4 = (1)(4)(-2-6)\frac{(x+2)^3}{(3x-2)^5}$$

$$= -\frac{32(x+2)^3}{(3x-2)^5}$$

$$\frac{d}{dx}\left(\frac{x+2}{3x-2}\right)^4 = 4\left(\frac{x+2}{3x-2}\right)^3 \frac{d}{dx}\left(\frac{x+2}{3x-2}\right)$$

$$= 4\frac{(x+2)^3}{(3x-2)^3}\frac{3x-2-3(x+2)}{(3x-2)^2}$$

$$= 4\frac{(x+2)^3}{(3x-2)^3}\frac{3x-2-3x-6}{(3x-2)^2}$$

$$= 4(-8)\frac{(x+2)^3}{(3x-2)^5}$$

$$= -\frac{32(x+2)^3}{(3x-2)^5}$$

Example

Find $\qquad \dfrac{d}{dx}\left(\dfrac{5x^2-3}{2x^2-4}\right)^5$

Solution

$$\frac{d}{dx}\left(\frac{5x^2-3}{2x^2-4}\right)^5 = \frac{-140x\left(5x^2-3\right)^4}{\left(2x^2-4\right)^6}$$

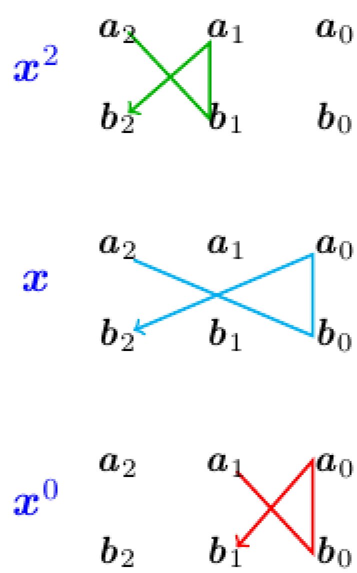
Derivative: in the form $\quad y=\dfrac{ax^2+bx+c}{dx^2+ex+f}$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)=\frac{(2ax+b)\left(dx^2+ex+f\right)-(2dx+e)\left(ax^2+bx+c\right)}{\left(dx^2+ex+f\right)^2}$$

$$=\frac{2adx^3+2aex^2+2afx+bdx^2+bex+bf-2adx^3-2bdx^2-2cdx-aex^2-bex-ce}{\left(dx^2+ex+f\right)^2}$$

$$=\frac{(ae-bd)x^2+2(af-cd)x+bf-ce}{\left(dx^2+ex+f\right)^2}$$

$$=\frac{\begin{vmatrix}a & b\\ d & e\end{vmatrix}x^2+2\begin{vmatrix}a & c\\ d & f\end{vmatrix}x+\begin{vmatrix}b & c\\ e & f\end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$



Example

$$f(x)=\frac{x^2-6x+8}{x^2-2x+1}$$

Solution

$$f'(x)=\frac{\begin{vmatrix}1 & -6\\ 1 & -2\end{vmatrix}x^2+2\begin{vmatrix}1 & 8\\ 1 & 1\end{vmatrix}x+\begin{vmatrix}-6 & 8\\ -2 & 1\end{vmatrix}}{\left(x^2-2x+1\right)^2}$$

$$=\frac{4x^2-14x+10}{\left(x^2-2x+1\right)^2}$$

$$f'(x)=\frac{(2x-6)\left(x^2-2x+1\right)-(2x-2)\left(x^2-6x+8\right)}{\left(x^2-2x+1\right)^2}$$

$$=\frac{2x^3-4x^2+2x-6x^2+12x-6-\left(2x^3-12x^2+16x-2x^2+12x-16\right)}{\left(x^2-2x+1\right)^2}$$

$$=\frac{2x^3-4x^2+2x-6x^2+12x-6-2x^3+12x^2-16x+2x^2-12x+16}{\left(x^2-2x+1\right)^2}$$

$$=\frac{4x^2-14x+10}{\left(x^2-2x+1\right)^2}$$

Example

$$f(x)=\frac{x+4}{x^2+x+1}$$

$$\begin{matrix}0 & 1 & 4\\ 1 & 1 & 1\end{matrix}$$

Solution

$$f'(x)=\frac{\begin{vmatrix}0 & 1\\ 1 & 1\end{vmatrix}x^2+2\begin{vmatrix}0 & 4\\ 1 & 1\end{vmatrix}x+\begin{vmatrix}1 & 4\\ 1 & 1\end{vmatrix}}{\left(x^2+x+1\right)^2}$$

$$=\frac{-x^2-8x-3}{\left(x^2+x+1\right)^2}$$

$$f'(x)=\frac{x^2+x+1-(x+4)(2x+1)}{\left(x^2+x+1\right)^2}$$

$$=\frac{x^2+x+1-2x^2-9x-4}{\left(x^2+x+1\right)^2}$$

$$=\frac{-x^2-8x-3}{\left(x^2+x+1\right)^2}$$

Derivative: in the form $\quad f(x)=\dfrac{a_3x^3+a_2x^2+a_1x+a_0}{b_3x^3+b_2x^2+b_1x+b_0}$

$$u=a_3x^3+a_2x^2+a_1x+a_0 \quad\rightarrow\quad u'=3a_3x^2+2a_2x+a_1$$
$$v=b_3x^3+b_2x^2+b_1x+b_0 \quad\rightarrow\quad v'=3b_3x^2+2b_2x+b_1$$

$$u'v-v'u=\left(3a_3x^2+2a_2x+a_1\right)\left(b_3x^3+b_2x^2+b_1x+b_0\right)$$
$$-\left(3b_3x^2+2b_2x+b_1\right)\left(a_3x^3+a_2x^2+a_1x+a_0\right)$$

x^5	x^4	x^3	x^2	x^1	x^0
$3a_3b_3$	$3a_3b_2$	$3a_3b_1$	$3a_3b_0$		
$-3a_3b_3$	$2a_2b_3$	$2a_2b_2$	$2a_2b_1$	$2a_2b_0$	
	$-3a_2b_3$	a_1b_3	a_1b_2	a_1b_1	a_1b_0
	$-2a_3b_2$	$-3a_1b_3$	$-3a_0b_3$		
		$-2a_2b_2$	$-2a_1b_2$	$-2a_0b_2$	
		$-a_3b_1$	$-a_2b_1$	$-a_1b_1$	$-a_0b_1$

$$f'(x)=\frac{\left(a_3b_2-a_2b_3\right)x^4+2\left(a_3b_1-a_1b_3\right)x^3+\left(\left(a_2b_1-a_1b_2\right)+3\left(a_3b_0-a_0b_3\right)\right)x^2+2\left(a_2b_0-a_0b_2\right)x+a_1b_0-a_0b_1}{\left(b_3x^3+b_2x^2+b_1x+b_0\right)^2}$$

$$=\frac{\begin{vmatrix}a_3 & a_2\\ b_3 & b_2\end{vmatrix}x^4+2\begin{vmatrix}a_3 & a_1\\ b_3 & b_1\end{vmatrix}x^3+\left(\begin{vmatrix}a_2 & a_1\\ b_2 & b_1\end{vmatrix}+3\begin{vmatrix}a_3 & a_0\\ b_3 & b_0\end{vmatrix}\right)x^2+2\begin{vmatrix}a_2 & a_0\\ b_2 & b_0\end{vmatrix}x+\begin{vmatrix}a_1 & a_0\\ b_1 & b_0\end{vmatrix}}{\left(b_3x^3+b_2x^2+b_1x+b_0\right)^2}$$

Example

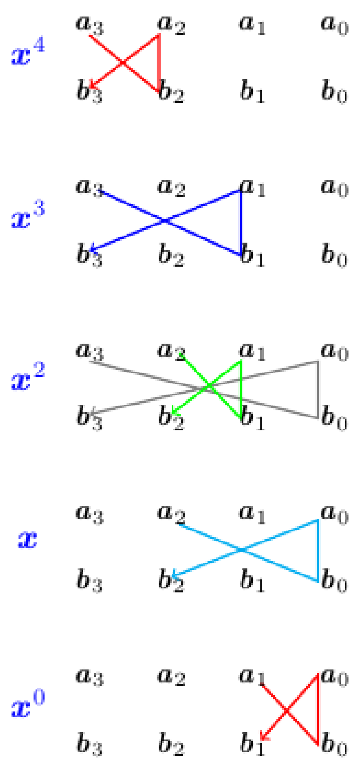
$$f(x)=\frac{x^3+2x^2-6x+2}{2x^3+x^2-2x+1}$$

1	2	−6	2
2	1	−2	1

Solution

$$f'(x)=\frac{(1-4)x^4+2(10)x^3+\big((-4+6)+3(1-4)\big)x^2+2(2-2)x+(-6+4)}{\left(2x^3+x^2-2x+1\right)^2}$$

$$=\frac{-3x^4+20x^3-7x^2-2}{\left(2x^3+x^2-2x+1\right)^2}$$



Derivative: in the form $\qquad f(x)=\dfrac{a_4x^4+a_3x^3+a_2x^2+a_1x+a_0}{b_4x^4+b_3x^3+b_2x^2+b_1x+b_0}$

$$u'v-v'u=\left(4a_4x^3+3a_3x^2+2a_2x+a_1\right)\left(b_4x^4+b_3x^3+b_2x^2+b_1x+b_0\right)$$
$$-\left(4b_4x^3+3b_3x^2+2b_2x+b_1\right)\left(a_4x^4+a_3x^3+a_2x^2+a_1x+a_0\right)$$

$x^7 \quad 4a_4b_4-4a_4b_4$

$x^6 \quad 4a_4b_3+3a_3b_4-4a_3b_4-3a_4b_3$

$x^5 \quad 4a_4b_2+3a_3b_3+2a_2b_4-4a_2b_4-3a_3b_3-2a_4b_2$

$x^4 \quad 4a_4b_1+3a_3b_2+2a_2b_3+a_1b_4-4a_1b_4-3a_2b_3-2a_3b_2-a_4b_1$

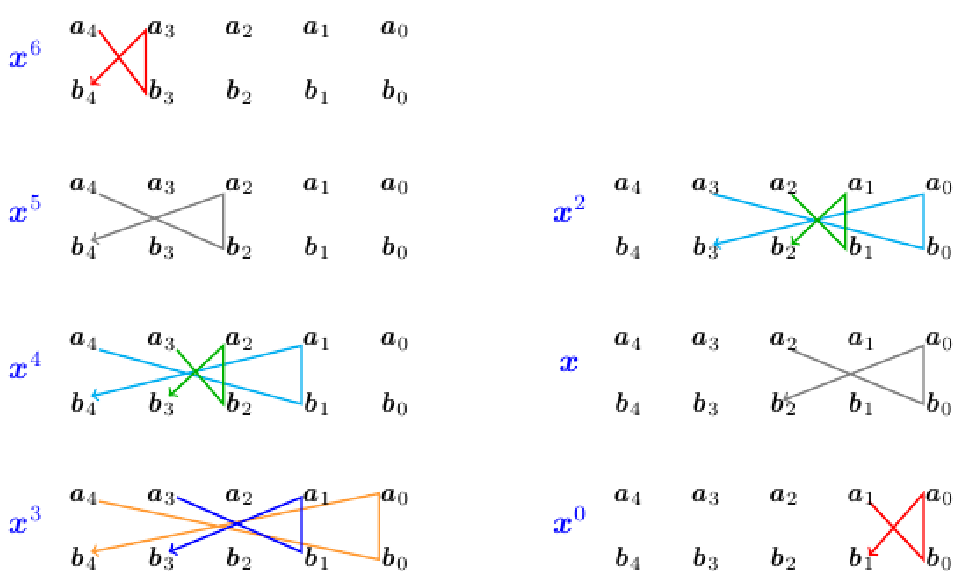
$x^3 \quad 4a_4b_0+3a_3b_1+2a_2b_2+a_1b_3-4a_0b_4-3a_1b_3-2a_2b_2-a_3b_1$

$x^2 \quad 3a_3b_0+2a_2b_1+a_1b_2-3a_0b_3-2a_1b_2-a_2b_1$

$x^1 \quad 2a_2b_0+a_1b_1-2a_0b_2-a_1b_1$

$x^0 \quad a_1b_0-a_0b_1$

$$f'(x)=\frac{\begin{aligned}&\left(a_4b_3-a_3b_4\right)x^6+2\left(a_4b_2-a_2b_4\right)x^5+\left(3\left(a_4b_1-a_1b_4\right)+\left(a_3b_2-a_2b_3\right)\right)x^4\\&+\left(4\left(a_4b_0-a_0b_4\right)+2\left(a_3b_1-a_1b_3\right)\right)x^3\\&+\left(\left(a_2b_1-a_1b_2\right)+3\left(a_3b_0-a_0b_3\right)\right)x^2+2\left(a_2b_0-a_0b_2\right)x+a_1b_0-a_0b_1\end{aligned}}{\left(b_4x^4+b_3x^3+b_2x^2+b_1x+b_0\right)^2}$$



Derivative: in the form $f(x)=\dfrac{a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0}{ba_5x^5+b_4x^4+b_3x^3+b_2x^2+b_1x+b_0}$

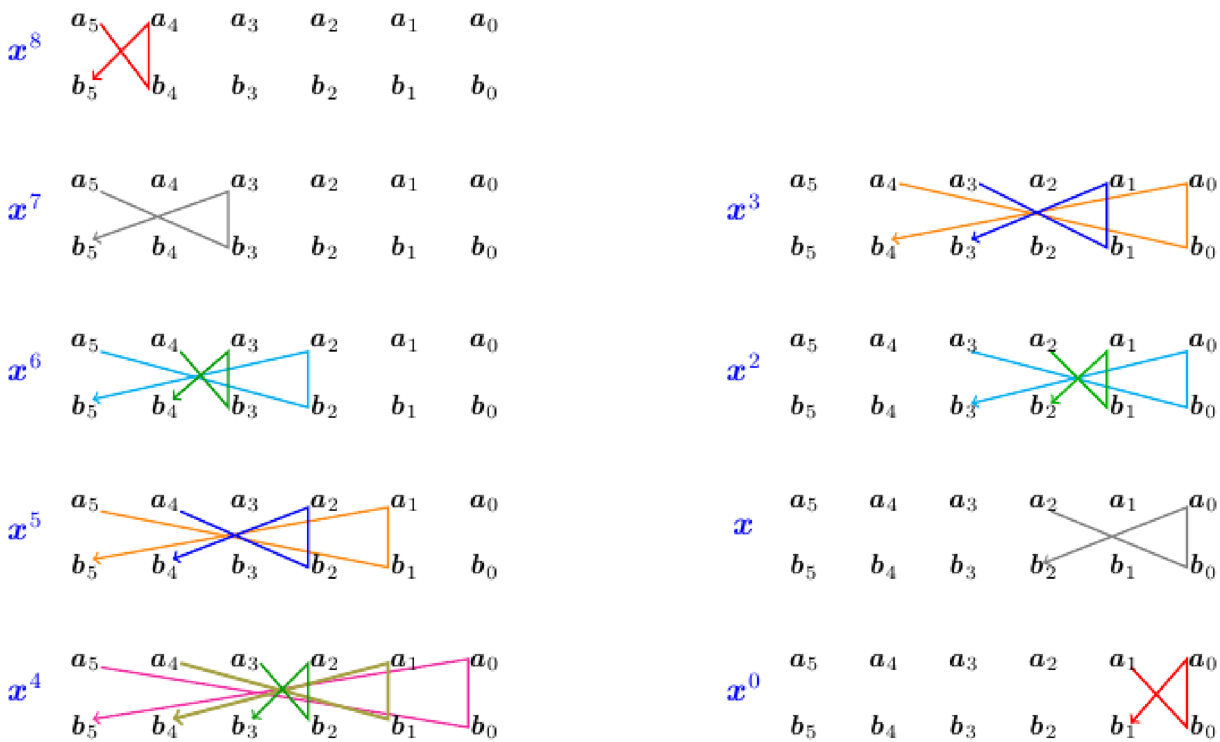
$$u=a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0 \quad\rightarrow\quad u'=5a_5x^4+4a_4x^3+3a_3x^2+2a_2x+a_1$$

$$v=b_5x^5+b_4x^4+b_3x^3+b_2x^2+b_1x+b_0 \quad\rightarrow\quad v'=5b_5x^4+4b_4x^3+3b_3x^2+2b_2x+b_1$$

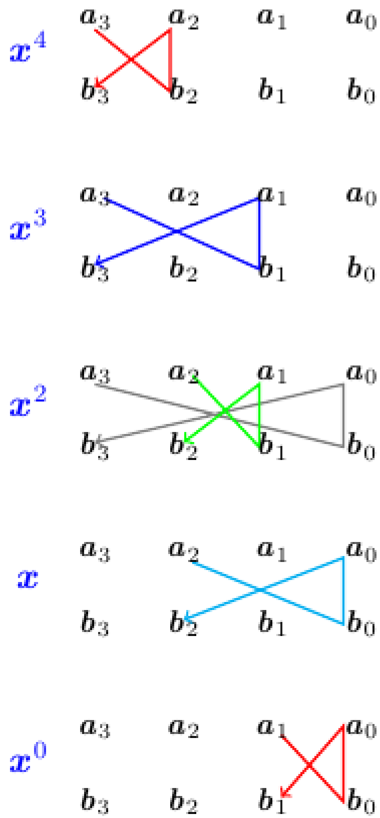
$$\begin{aligned}u'v-v'u &= \left(5a_5x^4+4a_4x^3+3a_3x^2+2a_2x+a_1\right)\left(b_5x^5+b_4x^4+b_3x^3+b_2x^2+b_1x+b_0\right)\\ &\quad-\left(5b_5x^4+4b_4x^3+3b_3x^2+2b_2x+b_1\right)\left(a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0\right)\end{aligned}$$

x^9	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x^1	x^0
$5a_5b_5$	$5a_5b_4$	$5a_5b_3$	$5a_5b_2$	$5a_5b_1$	$5a_5b_0$				
$-5a_5b_5$	$4a_4b_5$	$4a_4b_4$	$4a_4b_3$	$4a_4b_2$	$4a_4b_1$	$4a_4b_0$			
	$-5a_4b_5$	$3a_3b_5$	$3a_3b_4$	$3a_3b_3$	$3a_3b_2$	$3a_3b_1$	$3a_3b_0$		
	$-4a_5b_4$	$-5a_3b_5$	$2a_2b_5$	$2a_2b_4$	$2a_2b_3$	$2a_2b_2$	$2a_2b_1$	$2a_2b_0$	
		$-4a_4b_4$	$-5a_2b_5$	a_1b_5	a_1b_4	a_1b_3	a_1b_2	a_1b_1	a_1b_0
		$-3a_5b_3$	$-4a_3b_4$	$-5a_1b_5$	$-5a_0b_5$	$-4a_0b_4$	$-3a_0b_3$	$-2a_0b_2$	$-a_0b_1$
			$-3a_4b_3$	$-4a_2b_4$	$-4a_1b_4$	$-3a_1b_3$	$-2a_1b_2$	$-a_1b_1$	
			$-2a_5b_2$	$-3a_3b_3$	$-3a_2b_3$	$-2a_2b_2$	$-a_2b_1$		
				$-2a_4b_2$	$-2a_3b_2$	$-a_3b_1$			
				$-a_5b_1$	$-a_4b_1$				

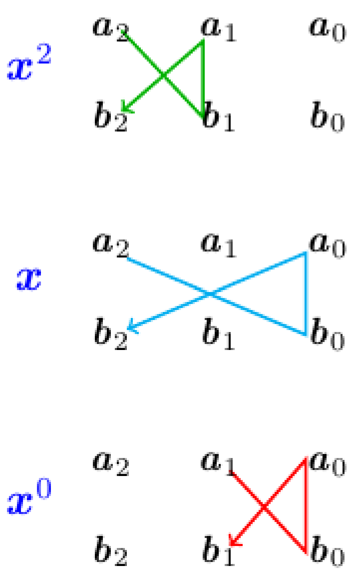
$$f'(x)=\frac{\begin{aligned}&\left(a_5b_4-a_4b_5\right)x^8 \quad +2\left(a_5b_3-a_3b_5\right)x^7\\ &+\left(3\left(a_5b_2-a_2b_5\right)+\left(a_4b_3-a_3b_4\right)\right)x^6\\ &+\left(4\left(a_5b_1-a_1b_5\right)+2\left(a_4b_2-a_2b_4\right)\right)x^5\\ &+\left(5\left(a_5b_0-a_0b_5\right)+3\left(a_4b_1-a_1b_4\right)+\left(a_3b_2-a_2b_3\right)\right)x^4\\ &+\left(4\left(a_4b_0-a_0b_4\right)+2\left(a_3b_1-a_1b_3\right)\right)x^3\\ &+\left(3\left(a_3b_0-a_0b_3\right)+\left(a_2b_1-a_1b_2\right)\right)x^2\\ &+2\left(a_2b_0-a_0b_2\right)x \quad +\left(a_1b_0-a_0b_1\right)\end{aligned}}{\left(b_5x^5+b_4x^4+b_3x^3+b_2x^2+b_1x+b_0\right)^2}$$



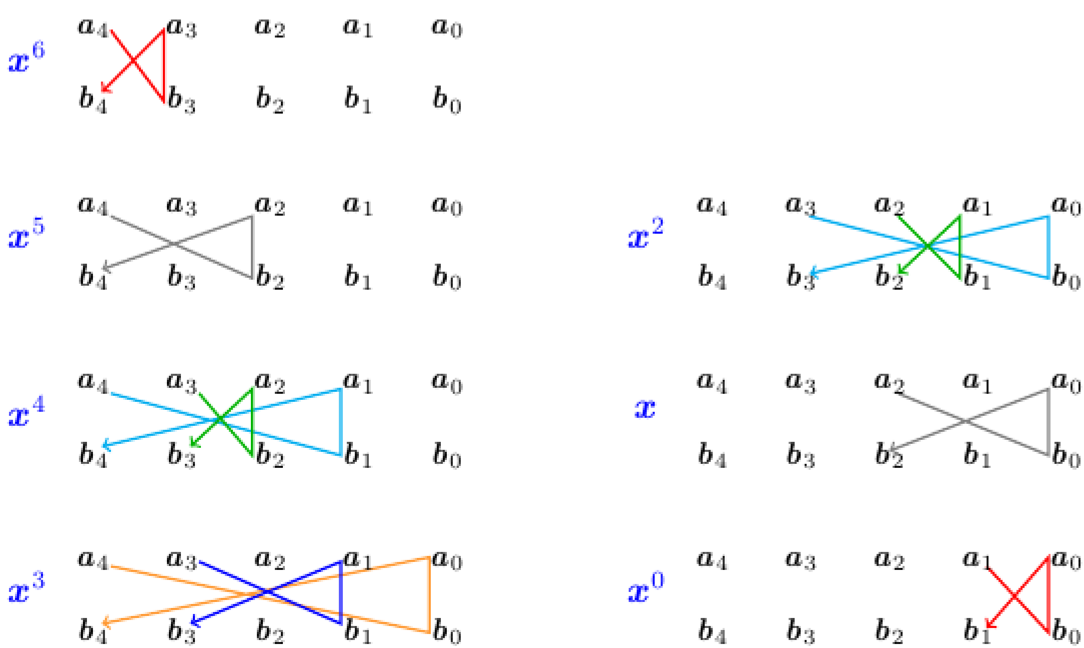
x^8 a_5 a_4 a_3 a_2 a_1 a_0 b_5 b_4 b_3 b_2 b_1 b_0
x^7 a_5 a_4 a_3 a_2 a_1 a_0 b_5 b_4 b_3 b_2 b_1 b_0
x^6 a_5 a_4 a_3 a_2 a_1 a_0 b_5 b_4 b_3 b_2 b_1 b_0
x^5 a_5 a_4 a_3 a_2 a_1 a_0 b_5 b_4 b_3 b_2 b_1 b_0
x^4 a_5 a_4 a_3 a_2 a_1 a_0 b_5 b_4 b_3 b_2 b_1 b_0
x^3 a_5 a_4 a_3 a_2 a_1 a_0 b_5 b_4 b_3 b_2 b_1 b_0
x^2 a_5 a_4 a_3 a_2 a_1 a_0 b_5 b_4 b_3 b_2 b_1 b_0
x a_5 a_4 a_3 a_2 a_1 a_0 b_5 b_4 b_3 b_2 b_1 b_0
x^0 a_5 a_4 a_3 a_2 a_1 a_0 b_5 b_4 b_3 b_2 b_1 b_0



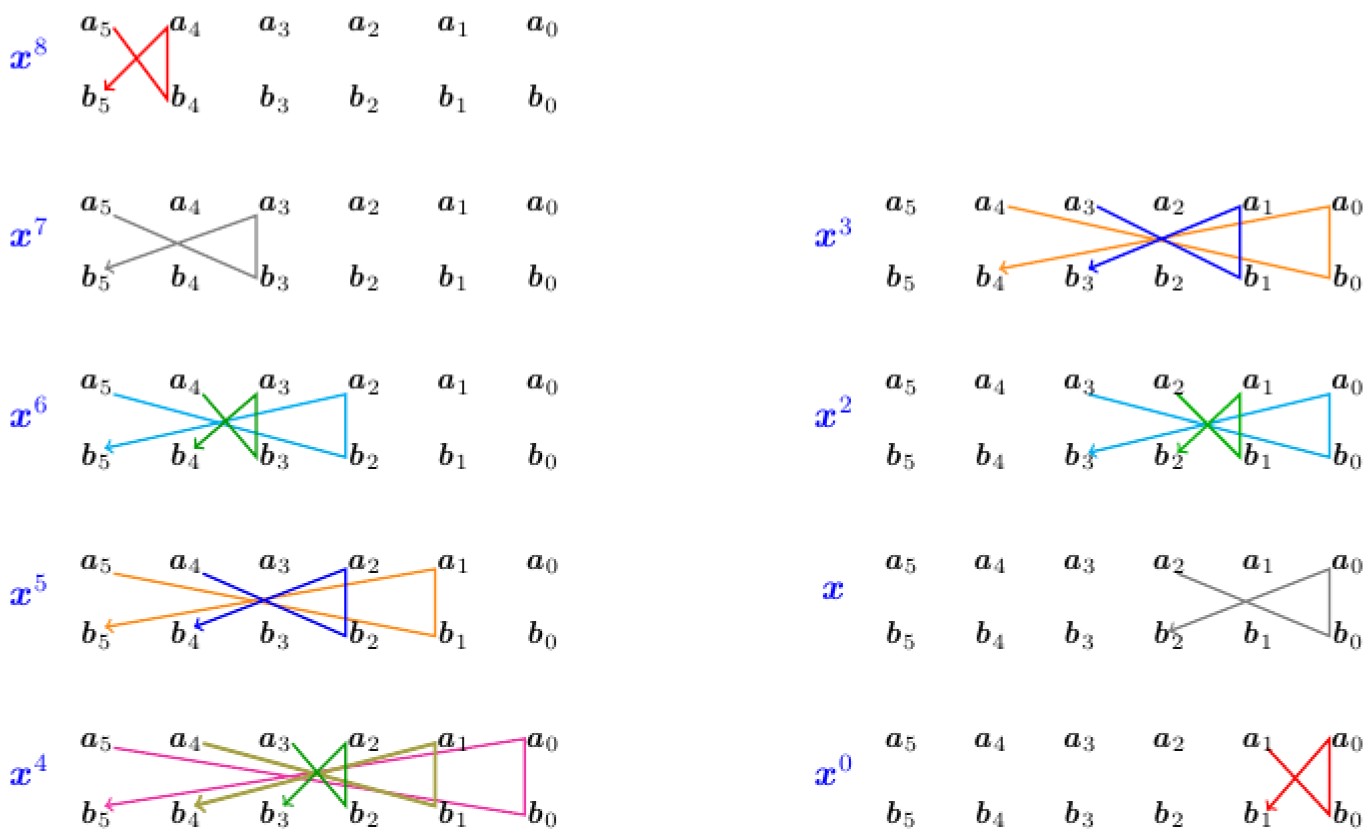
x^4 a_3 a_2 a_1 a_0 / b_3 b_2 b_1 b_0
x^3 a_3 a_2 a_1 a_0 / b_3 b_2 b_1 b_0
x^2 a_3 a_2 a_1 a_0 / b_3 b_2 b_1 b_0
x a_3 a_2 a_1 a_0 / b_3 b_2 b_1 b_0
x^0 a_3 a_2 a_1 a_0 / b_3 b_2 b_1 b_0



x^2 a_2 a_1 a_0 / b_2 b_1 b_0
x a_2 a_1 a_0 / b_2 b_1 b_0
x^0 a_2 a_1 a_0 / b_2 b_1 b_0



x^6 a_4 a_3 a_2 a_1 a_0 / b_4 b_3 b_2 b_1 b_0
x^5 a_4 a_3 a_2 a_1 a_0 / b_4 b_3 b_2 b_1 b_0
x^4 a_4 a_3 a_2 a_1 a_0 / b_4 b_3 b_2 b_1 b_0
x^3 a_4 a_3 a_2 a_1 a_0 / b_4 b_3 b_2 b_1 b_0
x^2 a_4 a_3 a_2 a_1 a_0 / b_4 b_3 b_2 b_1 b_0
x a_4 a_3 a_2 a_1 a_0 / b_4 b_3 b_2 b_1 b_0
x^0 a_4 a_3 a_2 a_1 a_0 / b_4 b_3 b_2 b_1 b_0



x^8
x^7
x^6
x^5
x^4
x^3
x^2
x
x^0
a_5 a_4 a_3 a_2 a_1 a_0
b_5 b_4 b_3 b_2 b_1 b_0