Section 3.5 – Determinants and Cramer's Rule

The determinant is a number that contains information about matrix. It is used to find formulas for inverse matrices, pivots, and solutions $A^{-1}b$.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ has inverse } A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Determinant of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is written $\det(A)$ *or* $|A|$ and is define as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The determinant is zero when the matrix has no inverse.

Properties of the Determinants

There are 3 basic properties (rules 1, 2, 3), by using those rules we can compute the determinant of any square matrix.

1. ***Determinant of the n by n identity matrix is* 1.**

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \quad \text{and} \quad \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} = 1$$

2. ***Determinant changes sign when 2 rows are exchanged.***

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = -(ad - bc) \quad \Rightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

3. ***Determinant is a linear function of each row separately*.**

Multiply row 1 by any number t: $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Add row 1 of A to row 1 of A ′: $\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

- ***For 2 by 2 determinants, if you expand to a rectangle, the determinants equal areas.***
- ***For n-dimensional, the determinants equal volumes.***

4. ***If 2 rows of A are equal, then*** $\det A = 0$.

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$$

5. ***Subtracting a multiple of one row from another row leaves*** $\det A$ ***unchanged.***

$$\begin{vmatrix} a & b \\ c - ta & d - ta \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

6. ***A matrix with a row of zeros has*** $\det A = 0$.

$$\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0 \quad and \quad \begin{vmatrix} 0 & 0 \\ b & c \end{vmatrix} = 0$$

7. ***If A is triangular then*** $\det A = a_{11} a_{22} \ \cdots \ a_{nn}$ ***= product of diagonal entries.***

$$\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad \quad and \quad \begin{vmatrix} a_{11} & & 0 \\ & a_{22} & \\ & & \\ 0 & & a_{nn} \end{vmatrix} = a_{11} a_{22} \cdots a_{nn}$$

8. ***If A is singular then*** $\det A = 0$. ***If A is invertible then*** $\det A \neq 0$.

9. ***The determinant of AB*** $\det A$ ***is times*** $\det B$ ***:*** $|AB| = |A||B|$

10. ***The transpose*** A^T ***has the same determinant as A:*** $\det(A) = \det\left(A^T\right)$

- $\det(A + B) \neq \det(A) + \det(B)$

Big Formula for Determinants (Diagonal)

Determinant Using Diagonal Method

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} \qquad a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \quad (1)$$

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} \qquad -a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \quad (2)$$

Determinant: ***D* = (1) + (2)**

det = $a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$

Example

Evaluate: $\begin{vmatrix} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{vmatrix} \begin{matrix} x & 0 \\ 2 & x \\ -3 & x \end{matrix} = x(x)(1) + 0\left(x^2\right)(2) + (-1)(2)(x) - (-1)(x)(-3) - x\left(x^2\right)(x) - 0(-3)(1)$$

$$= x^2 + 3x - 3x - x^4$$

$$= x^2 - x^4$$

Determinant by *Cofactors*

$$\boldsymbol{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Minor

For a square matrix $A = \left[a_{ij}\right]$, the minor M_{ij}. Of an element a_{ij} is the ***determinant*** of the matrix formed by deleting the i^{th} row and the j^{th} column of A.

Example

Let $A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$ Find M_{32}

Solution

$$M_{32} = \begin{vmatrix} 3 & \cancel{1} & -4 \\ 2 & \cancel{5} & 6 \\ \cancel{1} & \cancel{4} & \cancel{8} \end{vmatrix}$$

$$= \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix}$$

$$\underline{= 26}$$

Theorem

The determinant is the dot product of any row $\boldsymbol{i}$ of $\boldsymbol{A}$ with its cofactors:

Cofactor Formula: $\boxed{C_{ij} = (-1)^{i+j} M_{ij}}$

$$|\boldsymbol{A}| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example

Find the determinant of the matrix:

$$A = \begin{bmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{bmatrix}$$

Solution

$$|A| = \begin{vmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{vmatrix}$$

$$= -8\begin{vmatrix} -6 & 7 \\ -3 & 5 \end{vmatrix} - 0\begin{vmatrix} 4 & 7 \\ -1 & 5 \end{vmatrix} + 6\begin{vmatrix} 4 & -6 \\ -1 & -3 \end{vmatrix}$$

$= -8(-30 - (-21)) - 0 + 6(-12 - 6)$

$= -8(-9) + 6(-18)$

$= -36$

✓ By the property of determinants, If $\boldsymbol{A}$ is triangular then $\mathbf{det}\boldsymbol{A} = \boldsymbol{a}_{11}\boldsymbol{a}_{22} \ \dots \ \boldsymbol{a}_{nn}$ = product of diagonal entries.

Example

$$\begin{vmatrix} 2 & 7 & -3 & 8 & 3 \\ 0 & -3 & 7 & 5 & 1 \\ 0 & 0 & 6 & 7 & 6 \\ 0 & 0 & 0 & 9 & 8 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix} = (2)(-3)(6)(9)(4) \ \underline{= -1296}$$

Theorem

Let $\boldsymbol{A}$ be any n by n matrix.

a) If $\boldsymbol{A'}$ is the matrix that results when a single row of $\boldsymbol{A}$ is multiplied by a constant $\boldsymbol{k}$, then $\det(A') = k\det(A)$.

b) If $\boldsymbol{A'}$ is the matrix that results when two rows of $\boldsymbol{A}$ are interchanged, then $\det(A') = -\det(A)$

c) If $\boldsymbol{A'}$ is the matrix that results when a multiple of one row of $\boldsymbol{A}$ is added to another row then $\det(A') = \det(A)$

Cramer's Rule

Theorem

If $AX = B$ is a system of a linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(B_1)}{\det(A)} \quad x_2 = \frac{\det(B_2)}{\det(A)}, \quad \cdots, \quad x_n = \frac{\det(B_n)}{\det(A)}$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & & & a_{nn} \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

$$\det(B_1) = \begin{bmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & & & \\ \vdots & & & \\ b_n & a_{n2} & & a_{nn} \end{bmatrix}$$

Example

Use Cramer's rule to solve

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ -2x_1 + x_2 \quad\quad &= 0 \\ -4x_1 \quad\quad + x_3 &= 0 \end{aligned}$$

Solution

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 7$$

$$|B_1| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \qquad |B_2| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 0 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 2 \qquad |B_3| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 0 \end{vmatrix} = 4$$

$$x_1 = \frac{|B_1|}{|A|} = \frac{1}{7} \qquad x_2 = \frac{|B_2|}{|A|} = \frac{2}{7} \qquad x_3 = \frac{|B_3|}{|A|} = \frac{4}{7}$$

Solution: $\left(\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\right)$

Example

Use Cramer's Rule to solve.

$$\begin{aligned} x_1 + \quad\quad + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

Solution

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix} \quad \Rightarrow \quad \det(A) = 44$$

$$A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix} \quad \Rightarrow \quad \det\left(A_1\right) = -40$$

$$A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix} \quad \Rightarrow \quad \det\left(A_2\right) = 72$$

$$A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix} \quad \Rightarrow \quad \det\left(A_3\right) = 152$$

$$x_1 = \frac{\det\left(A_1\right)}{\det(A)} = \frac{-40}{44} = -\frac{10}{11}$$

$$x_2 = \frac{\det\left(A_2\right)}{\det(A)} = \frac{72}{44} = \frac{18}{11}$$

$$x_3 = \frac{\det\left(A_3\right)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$$

A Formula for A^{-1}

***Theorem**: Inverse of a matrix using its Adjoint*

The i, j entry of A^{-1} is the cofactor C_{ji} $\left(not\ C_{ij}\right)$ divided by det(***A***):

Formula for A^{-1}: $\left(A^{-1}\right)_{ij} = \dfrac{C_{ji}}{|A|}$ *and* $A^{-1} = \dfrac{C^T}{|A|}$

$$\boxed{A^{-1} = \frac{1}{\det(A)} adj(A)}$$

Example

Find the inverse matrix of $A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$ using its adjoint.

Solution

$$C_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1; \quad C_{12} = -\begin{vmatrix} -2 & 0 \\ -4 & 1 \end{vmatrix} = 2; \quad C_{13} = \begin{vmatrix} -2 & 1 \\ -4 & 0 \end{vmatrix} = 4$$

$$C_{21} = -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1; \quad C_{22} = \begin{vmatrix} 1 & 1 \\ -4 & 1 \end{vmatrix} = 5; \quad C_{23} = -\begin{vmatrix} 1 & 1 \\ -4 & 0 \end{vmatrix} = -4$$

$$C_{31} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1; \quad C_{32} = -\begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = -2; \quad C_{33} = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3$$

$$C = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 5 & -2 \\ 4 & -4 & 3 \end{bmatrix} \text{ and } \det(A) = \tfrac{1}{7} \qquad \Rightarrow \quad A^{-1} = \tfrac{1}{7}\begin{bmatrix} 1 & -1 & -1 \\ 2 & 5 & -2 \\ 4 & -4 & 3 \end{bmatrix}$$

Theorem

If A is an $n \times n$ matrix, then the following statements are equivalent

a) A is invertible

b) $A\boldsymbol{x} = 0$ has only the trivial solution

c) The reduced row echelon form of A is I_n

d) A can be expressed as a product of elementary matrices

e) $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for every $n \times 1$ matrix $\boldsymbol{b}$

f) $\det(A) \neq 0$

Exercises Section 3.5 – Determinants and Cramer's Rule

1. Verify that $\det(AB)=\det(A)\det(B)$ when: $A=\begin{pmatrix}2 & 1 & 0\\ 3 & 4 & 0\\ 0 & 0 & 2\end{pmatrix}$ *and* $B=\begin{pmatrix}1 & -1 & 3\\ 7 & 1 & 2\\ 5 & 0 & 1\end{pmatrix}$

2. For which value(s) of $\boldsymbol{k}$ does A fail to be invertible? $A=\begin{bmatrix}k-3 & -2\\ -2 & k-2\end{bmatrix}$

3. Without directly evaluating, show that $\begin{vmatrix}b+c & c+a & b+a\\ a & b & c\\ 1 & 1 & 1\end{vmatrix}=0$

4. If the entries in every row of A add to zero, solve $\boldsymbol{Ax}=0$ to prove $\det A = 0$. If those entries add to one, show that $\det(A-I)=0$. Does this mean $\det A = I$?

5. Does $\det(AB)=\det(BA)$ in general?

a) True or false if $\boldsymbol{A}$ and $\boldsymbol{B}$ are square n x n matrices?

b) True or false if $\boldsymbol{A}$ is m x n and B is n x m with $m\neq n$?

6. True or false, with a reason if true or a counterexample if false:

a) The determinant of $I+A$ is $1+\det \boldsymbol{A}$.

b) The determinant of ABC is $|A||B||C|$.

c) The determinant of $4A$ is $4|A|$

d) The determinant of $AB-BA$ is zero. (try an example)

e) If A is not invertible then AB is not invertible.

f) The determinant of $A-B$ equals to $\det A - \det B$.

7. Use row operations to show the 3 by 3 "Vandermonde determinant" is

$$\det\begin{bmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{bmatrix}=(b-a)(c-a)(c-b)$$

8. The inverse of a 2 by 2 matrix seems to have determinant = 1:

$$\det A^{-1}=\det\frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}=\frac{ad-bc}{ad-bc}=1$$

What is wrong with this calculation? What is the correct $\det A^{-1}$

9. A *Hessenberg* matrix is a triangular matrix with one extra diagonal. Use cofactors of row 1 to show that the 4 by 4 determinant satisfies Fibonacci's rule $|H_4| = |H_3| + |H_2|$. The same rule will continue for all sizes $|H_n| = |H_{n-1}| + |H_{n-2}|$. Which Fibonacci number is $|H_n|$?

$$H_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad H_3 = \begin{bmatrix} 2 & 1 & \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad H_4 = \begin{bmatrix} 2 & 1 & & \\ \mathbf{1} & 2 & \mathbf{1} & \\ \mathbf{1} & 1 & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & 1 & \mathbf{1} & \mathbf{2} \end{bmatrix}$$

10. Evaluate the determinant:

a) $\begin{vmatrix} -1 & 7 \\ -8 & -3 \end{vmatrix}$

b) $\begin{vmatrix} a-3 & 5 \\ -3 & a-2 \end{vmatrix}$

c) $\begin{vmatrix} k-1 & 2 \\ 4 & k-3 \end{vmatrix}$

d) $\begin{vmatrix} c & -4 & 3 \\ 2 & 1 & c^2 \\ 4 & c-1 & 2 \end{vmatrix}$

e) $\begin{vmatrix} 1 & 0 & 3 \\ 4 & 0 & -1 \\ 2 & 8 & 6 \end{vmatrix}$

f) $\begin{vmatrix} x & -3 & 9 \\ 2 & 4 & x+1 \\ 1 & x^2 & 3 \end{vmatrix}$

g) $\begin{vmatrix} 1 & 4 & -3 & 1 \\ 2 & 0 & 6 & 3 \\ 4 & -1 & 2 & 5 \\ 1 & 0 & -2 & 4 \end{vmatrix}$

h) $\begin{vmatrix} -3 & 1 & 2 \\ 6 & 2 & 1 \\ -9 & 1 & 2 \end{vmatrix}$

i) $\begin{vmatrix} 1 & 3 & 2 & -1 \\ 4 & 3 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 5 & 1 & 3 & -2 \end{vmatrix}$

j) $\begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 3 & 1 & -2 & 0 \\ 1 & 0 & 3 & -3 \end{vmatrix}$

11. Find all the values of λ for which det($\boldsymbol{A}$) = 0

$$a)\ A = \begin{bmatrix} \lambda-1 & -2 \\ 1 & \lambda-4 \end{bmatrix} \qquad b)\ A = \begin{bmatrix} \lambda-6 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 4 & \lambda-4 \end{bmatrix}$$

12. Prove that if a square matrix $\boldsymbol{A}$ has a column of zeros, then det($\boldsymbol{A}$) = 0

13. With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A||D| \quad \textit{but} \quad \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D| - |C||B|$$

a) Why is the first statement true? Somehow B doesn't enter.

b) Show by example that equality fails (as shown) when C enters.

c) Show by example that the answer $\det(AD - CB)$ is also wrong.

14. Show that the value of the following determinant is independent of θ.

$$\begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta - \cos\theta & \sin\theta + \cos\theta & 1 \end{vmatrix}$$

15. Show that the matrices $A=\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ *and* $B=\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$ commute if and only if $\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix}=0$

16. Show that $\det(A)=\frac{1}{2}\begin{vmatrix} tr(A) & 1 \\ tr\left(A^2\right) & tr(A) \end{vmatrix}$ for every 2×2 matrix A.

17. What is the maximum number of zeros that a 4×4 matrix can have without a zero determinant? Explain your reasoning.

18. Evaluate *det* ***A***, *det* ***E***, and *det* (***AE***). Then verify that (*det* ***A***)(*det* ***E***) = *det*(***AE***)

$$A=\begin{bmatrix} 4 & 1 & 3 \\ 0 & -2 & 0 \\ 3 & 1 & 5 \end{bmatrix}, \qquad E=\begin{bmatrix} 1 & & \\ & 3 & \\ & & 1 \end{bmatrix}$$

19. Show that $\begin{bmatrix} \sin^2\alpha & \sin^2\beta & \sin^2\gamma \\ \cos^2\alpha & \cos^2\beta & \cos^2\gamma \\ 1 & 1 & 1 \end{bmatrix}$ is not invertible for any values of α, β, γ

20. Use Cramer's Rule with ratios $\frac{\det B_j}{\det A}$ to solve $A\boldsymbol{x} = b$. Also find the inverse matrix $A^{-1}=\frac{C^T}{\det A}$.

Why is the solution $\boldsymbol{x}$ is the first part the same as column 3 of A^{-1}? Which cofactors are involved in computing that column $\boldsymbol{x}$?

$$Ax=b \quad is \quad \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

21. Verify that $\det(AB)=\det(BA)$ and determine whether the equality $\det(A+B)=\det(A)+\det(B)$ holds

$$A=\begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad and \quad B=\begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

22. Verify that $\det(kA)=k^n\det(A)$

$a)\quad A=\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}, \quad k=2$

$b)\quad A=\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{bmatrix}, \quad k=-2$

$c)\quad A=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix}, \quad k=3$

23. Solve by using Cramer's rule

a) $\begin{cases} 7x - 2y = 3 \\ 3x + y = 5 \end{cases}$

b) $\begin{cases} 4x + 5y \quad = 2 \\ 11x + y + 2z = 3 \\ x + 5y + 2z = 1 \end{cases}$

c) $\begin{cases} x - 4y + z = 6 \\ 4x - y + 2z = -1 \\ 2x + 2y - 3z = -20 \end{cases}$

d) $\begin{cases} -x_1 - 4x_2 + 2x_3 + x_4 = -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 = 14 \\ -x_1 + x_2 + 3x_3 + x_4 = 11 \\ -x_1 - 2x_2 + x_3 - 4x_4 = -4 \end{cases}$

e) $\begin{cases} 2x - y + z = -1 \\ 3x + 4y - z = -1 \\ 4x - y + 2z = -1 \end{cases}$

24. Show that the matrix A is invertible for all values of θ, then find A^{-1} using $A^{-1} = \frac{1}{\det(A)} adj(A)$

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$