Lecture Three – Applications of Derivatives

Section 3.1 – Maxima and Minima

Definition

Let f be a function with Domain D. Then f has an ***absolute maximum*** value on D at a point c if

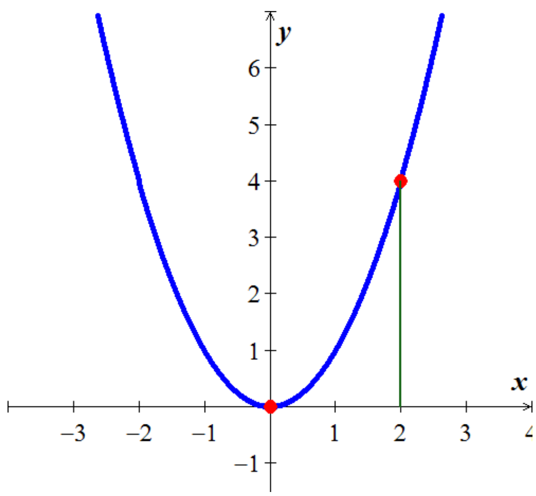
$$f(x) \le f(c) \quad \textit{for all } x \textit{ in } D$$

And an ***absolute maximum*** value on D at a point c if

$$f(x) \ge f(c) \quad \textit{for all } x \textit{ in } D$$

Maximum and minimum values are called ***extreme values*** of the function f .

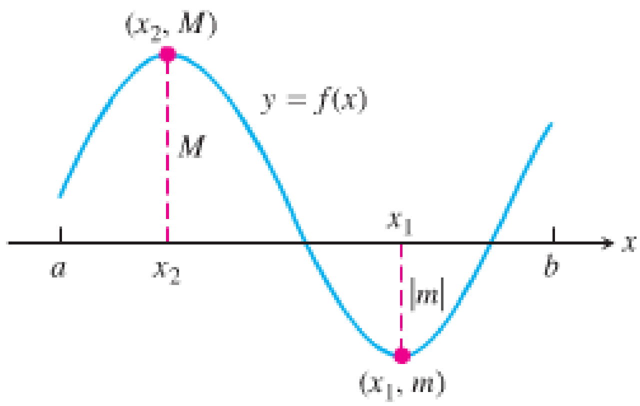
Absolute maxima or minima are also referred to as ***global*** maxima or minima.



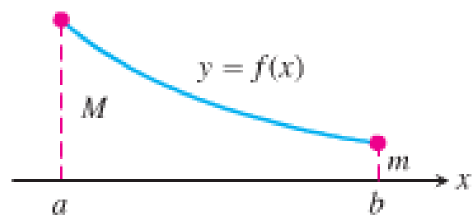
Function rule	***Domain D***	***Absolute extrema on D***
$y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at x = 0.
$y = x^2$	$[0, 2]$	Absolute minimum of 0 at x = 0. Absolute minimum of 4 at x = 2.
$y = x^2$	(0, 2]	No absolute minimum. Absolute minimum of 4 at x = 2.
$y = x^2$	(0, 2)	No absolute extrema.

Theorem – The Extreme Value Theorem

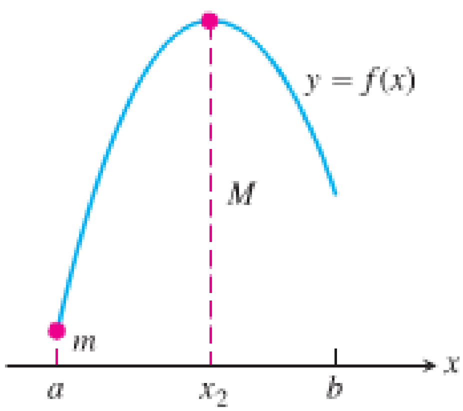
If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \le f(x) \le M$ for every other x in $[a, b]$.



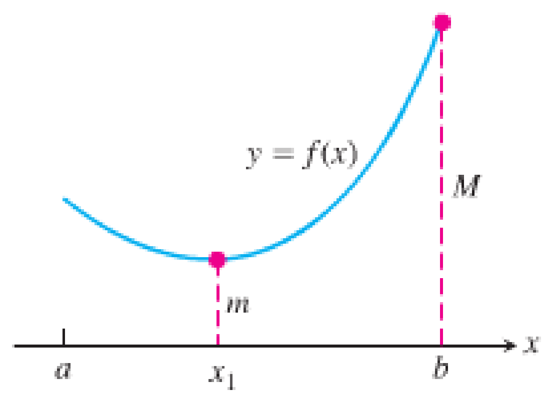
Maximum and minimum at interior points



Maximum and minimum at endpoints



Maximum at interior point, minimum at endpoint

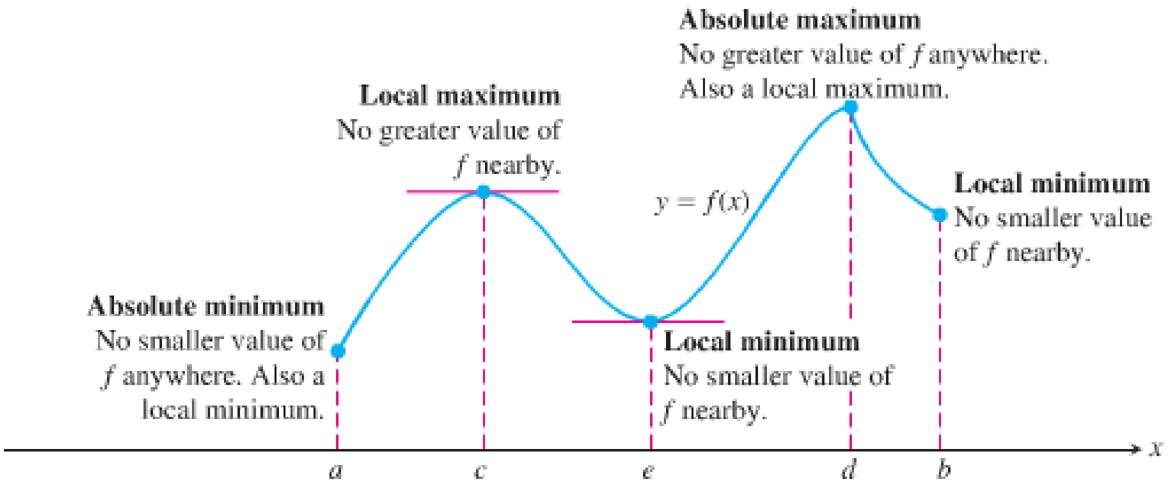


Minimum at interior point, maximum at endpoint

Definitions

A function f has a ***local maximum*** (***LMAX***) value at a point c within its domain D if $f(x) \le f(c)$ for all $x \in D$ lying is some open interval containing c.

A function f has a ***local minimum*** (***LMIN***) value at a point c within its domain D if $f(x) \ge f(c)$ for all $x \in D$ lying is some open interval containing c.



An absolute maximum is also a local maximum. Being the largest value overall, it is also the largest value in its immediate neighborhood.

Finding Extrema

Theorem – The First Derivative Theorem for Local Extreme Values

If f has a ***local minimum*** or ***local maximum*** value at a point c of its domain D, and f' is defined at c, then

$$f'(c) = 0$$

Proof

For $f'(c) = 0$ at a local extremum, we need to show that $f'(c)$ can't be positive or negative at c.

$$\left.\begin{aligned} f'(c) &= \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \left\{ \frac{f(x) \le f(c)}{x - c > 0} \right\} \le 0 \\ f'(c) &= \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \left\{ \frac{f(x) \le f(c)}{x - c < 0} \right\} \ge 0 \end{aligned}\right. \Rightarrow f'(c) = 0$$

Definition

An interior point of the domain of a function f where f' is zero or undefined is a critical point of f.

How to find the Absolute Extrema of a continuous Function f on a Finite Closed Interval

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

Example

Find the absolute maximum and minimum values of $f(x)=x^2$ on [–2, 1].

Solution

$$f'(x)=2x=0 \quad \Rightarrow \quad x=0$$

$$f(0)=0^2=0$$

Check: $f(-2)=(-2)^2=4$

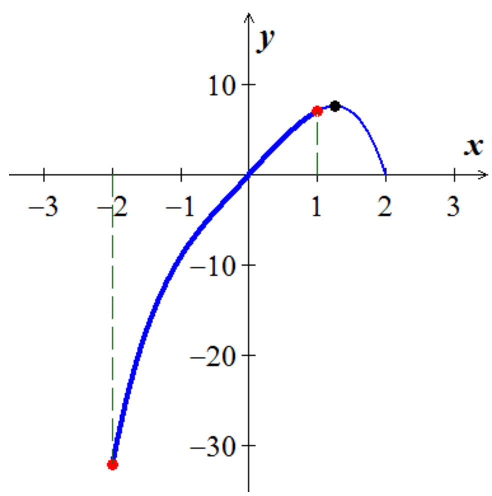
$$f(1)=(1)^2=1$$

The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x=0$.

Example

Find the absolute maximum and minimum values of $g(t)=8t-t^4$ on [–2, 1].

Solution



$$g'(t)=8-4t^3 \quad \Rightarrow \quad t^3=2 \rightarrow \boxed{t=\sqrt[3]{2}} > 1$$

Check: $g(-2)=8(-2)-(-2)^4=-32$

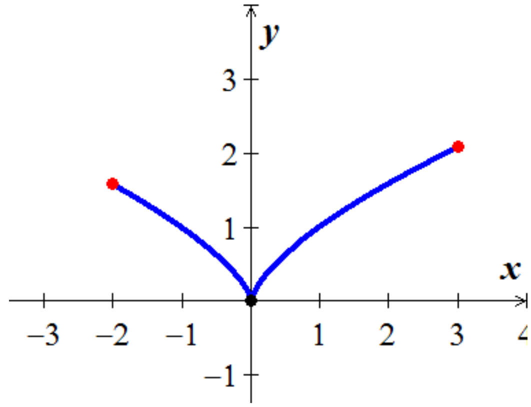
$$g(1)=8(1)-(1)^4=7$$

The function has an absolute maximum value of 7 at $x = 1$ and an absolute minimum value of –32 at $x=-2$.

Example

Find the absolute maximum and minimum values of $f(x)=x^{2/3}$ on [–2, 3].

Solution



$$f'(x)=\frac{2}{3}x^{-1/3}=\frac{2}{3\sqrt[3]{x}}=0 \quad \Rightarrow \quad \textit{Undefined}$$

Critical point: $\quad f(0)=0$

Endpoint values: $\quad f(-2)=(-2)^{2/3}=\sqrt[3]{2^2}=\sqrt[3]{4}$

$$f(3)=(3)^{2/3}=\sqrt[3]{3^2}=\sqrt[3]{9}$$

Absolute ***MIN***: $\quad (0, 0)$

Absolute ***MAX***: $\quad \left(3,\ \sqrt[3]{9}\right)$

Critical Points *(CP)* or *Critical Numbers*

The critical points for a function f are those numbers c in the domain of f for which $f'(c)=0$ or $f'(c)$ doesn't exist. A critical point is a point whose x-coordinate is the critical point c, and whose y-coordinate is $f(c)$

$f(x)=x^2$

$\Rightarrow f'(x)=2x=0$

$\rightarrow \boxed{x=0}$ is a Critical Number

$f(0)=0$

Critical Point: $(0,\ 0)$

If $f'(x)=0$ undefined

Exercises *Section* 3.1 – Maxima and Minima

Find the absolute maximum and minimum values of each function. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

1. $f(x)=\frac{2}{3}x-5 \qquad -2\le x\le 3$

2. $f(x)=x^2-1 \qquad -1\le x\le 2$

3. $f(x)=-\dfrac{1}{x^2} \qquad 0.5\le x\le 2$

4. $f(x)=\sqrt{4-x^2} \qquad -2\le x\le 1$

5. $f(\theta)=\sin\theta \qquad -\frac{\pi}{2}\le\theta\le\frac{5\pi}{6}$

6. $g(x)=\sec x \qquad -\frac{\pi}{3}\le x\le\frac{\pi}{6}$

Find the absolute maximum and minimum values of each function (if they exist).

7. $f(x)=x^{4/3}, \quad -1\le x\le 8$

8. $f(\theta)=\theta^{3/5}, \quad -32\le\theta\le 1$

9. $f(x)=2^x\sin x \quad [-2,\ 6]$

10. $f(x)=\sec x \quad \left[-\frac{\pi}{4},\ \frac{\pi}{4}\right]$

11. $f(x)=x^3e^{-x} \quad [-1,\ 5]$

12. $f(x)=x\ln\left(\frac{x}{5}\right) \quad [0.1,\ 5]$

13. $f(x)=x^{8/3}-16x^{2/3} \quad [-1,\ 8]$

14. $f(x)=x^2-8x+10 \quad [0,\ 7]$

15. $f(x)=2(3-x), \quad [-1,\ 2]$

16. $f(x)=x^3-3x^2, \quad [0,\ 4]$

17. $f(x)=\frac{1}{3}x^3-2x^2+3x-4, \quad [-2,\ 5]$

18. $f(x)=\dfrac{1}{x+2}, \quad [-4,\ 1]$

19. $f(x)=\left(x^2+4\right)^{2/3}, \quad [-2,\ 2]$

20. $f(x)=\sin 2x+3$ *on* $[-\pi,\ \pi]$

21. $f(x)=2x^3-3x^2-36x+12$ *on* $(-\infty,\ \infty)$

22. $f(x)=4x^{1/2}-x^{5/2}$ *on* $[0,\ 4]$

23. $f(x)=2x\ln x+10$ *on* $(0,\ 4)$

24. $f(x)=x\sin^{-1}x$ *on* $[-1,\ 1]$

Determine all critical points of each function

25. $y=x^2-6x+7$

26. $g(x)=(x-1)^2(x-3)^2$

27. $f(x)=\dfrac{x^2}{x-2}$

28. $g(x)=x^2-32\sqrt{x}$

Find the extreme values (absolute and local) of the function and where they occur

29. $f(x)=3x^2-4x+2$

30. $y=x^3-2x+4$

31. $y=\sqrt{x^2-1}$

32. $y=\dfrac{1}{\sqrt[3]{1-x^2}}$

33. $y=x^2\sqrt{3-x}$

34. $y=\dfrac{x+1}{x^2+2x+2}$

35. $y=x^{2/3}(x+2)$

36. $y=x\sqrt{4-x^2}$

37. $f(x)=\dfrac{e^x+e^{-x}}{2}$

38. $f(x)=\frac{1}{8}x^3-\frac{1}{2}x \quad [-1,\ 3]$

39. $f(x)=\dfrac{1}{x}-\ln x$

40. $f(x)=\sin x\cos x \quad [0,\ 2\pi]$

41. $f(x)=x-\tan^{-1}x$

42. Let $f(x) = (x-2)^{2/3}$

a) Does $f'(2)$ exist?

b) Show the only local extreme value of f occurs at $x = 2$.

c) Does the result in part (*b*) contradict the Extreme Value Theorem?

43. When a telephone wire is hung between two poles, the wire forms a U-shape curve called a Catenary. For instance, the function $y = 30\left(e^{x/60} + e^{-x/60}\right) \quad -30 \le x \le 30$ models the shape of the telephone wire strung between two poles that are 60 *feet.* apart (x & y are measured in *feet*.). Show that the lowest point on the wire is midway between two poles. How much does the wire sag between the two poles?

44. You are sitting in a classroom next to the wall looking at the blackboard at the front of the room. The blackboard is 12 *feet* long and starts 3 *feet* from the wall you are sitting next to.

a) Show that your viewing angle is $\alpha = \cot^{-1}\frac{x}{15} - \cot^{-1}\frac{x}{3}$. If you are x *feet* from the front wall.

b) Find x so that α is as large as possible

Section 3.2 – Graphing Functions

Increasing and Decreasing Functions

Corollary

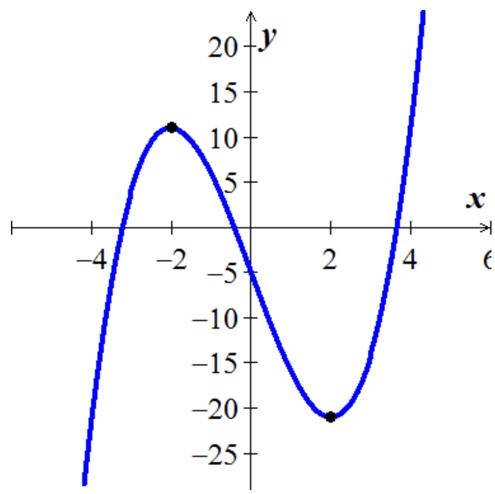
Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b)
2. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on (a, b)
3. If $f'(x) = 0$ for all x in (a, b), then f is constant on (a, b)

Example

Find the open intervals on which the function $f(x) = x^3 - 12x - 5$ is increasing or decreasing

Solution



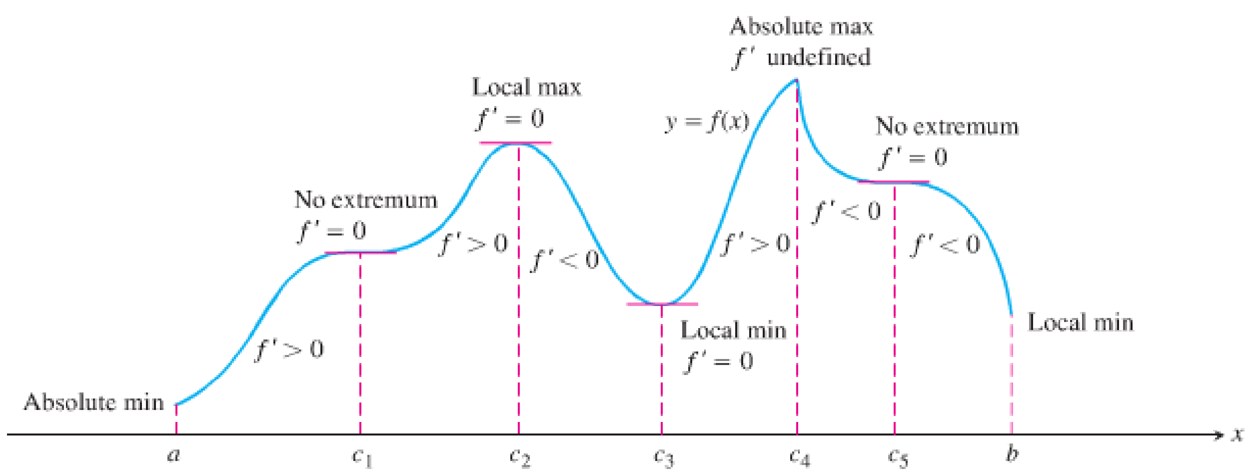
$$f'(x) = 3x^2 - 12$$

$$3\left(x^2 - 4\right) = 0 \Rightarrow \boxed{x = \pm 2} \quad (CN)$$

$-\infty$	-2	2	∞
$f'(-3) > 0$	$f'(0) < 0$	$f'(3) > 0$	
Increasing	***Decreasing***	***Increasing***	

Increasing: $(-\infty, -2)$ and $(2, \infty)$

Decreasing: $(-2, 2)$



First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f.

1. If f' changes from negative to positive at c, then f has a local minimum (***LMIN***).

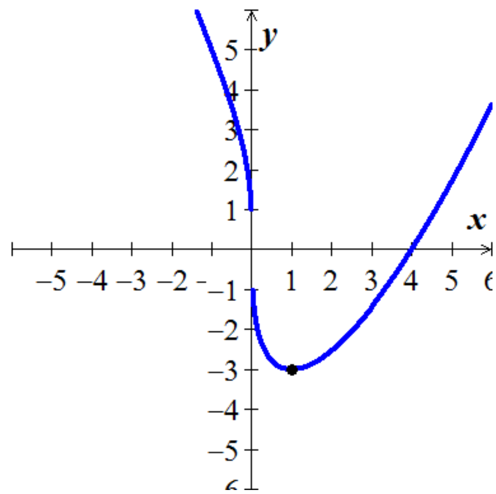
2. If f' changes from positive to negative at c, then f has a local maximum (***LMAX***).

3. If f' doesn't change sign at c, then f has no local extremum at c.

Example

Find the open intervals on which the function $f(x) = x^{1/3}(x-4)$ is increasing or decreasing

Solution



$$f(x) = x^{4/3} - 4x^{1/3}$$

$$f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3}\left(x^{1/3} - x^{-2/3}\right)\frac{x^{2/3}}{x^{2/3}}$$

$$= \frac{4}{3}\frac{x-1}{x^{2/3}} = 0$$

$$\Rightarrow \begin{cases} x = 1 \\ x \neq 0 \end{cases} \quad (\textbf{\textit{CN}})$$

-∞ 0	0 1	1 ∞
$f'(-1) < 0$	$f'(0.5) < 0$	$f'(2) > 0$
Decreasing	***Decreasing***	***Increasing***

Increasing: $(1, \infty)$

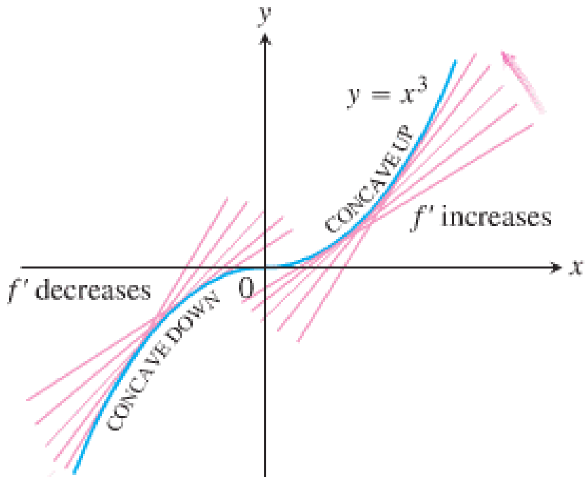
Decreasing: $(-\infty, 1)$

Concavity

Definition

Let f be differentiable on an open interval I. The graph of f is

1. **Concave *upward*** on I if f' is increasing on the interval.
2. **Concave *downward*** on I if f' is decreasing on the interval.



Test for Concavity

Let f be function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then f is ***concave up*** on I.
2. If $f''(x) < 0$ for all x in I, then f is ***concave down*** on I.

   i. Locate the x values @ which $f''(x) = 0$ or undefined

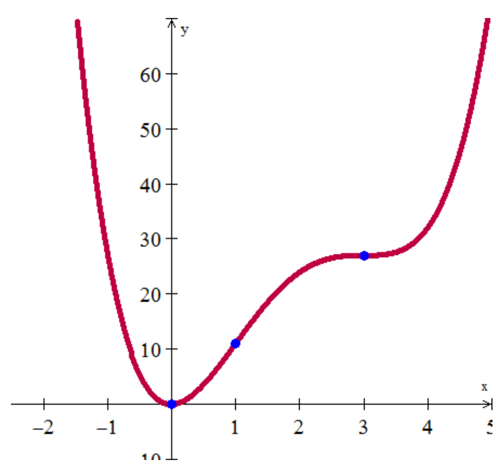
   ii. Use these test x-value to determine the test intervals

   iii. Test the sign of $f''(x)$ in each interval

Example

Determine the intervals on which the graph of the function is concave upward or concave downward.

$$f(x) = x^4 - 8x^3 + 18x^2$$



Solution

$$f'(x) = 4x^3 - 24x^2 + 36x$$

$$f''(x) = 12x^2 - 48x + 36 = 0 \quad \rightarrow x = 1 \quad x = 3$$

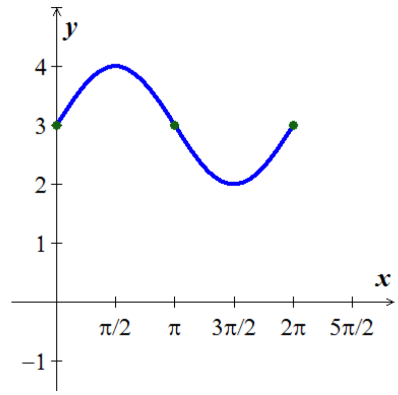
$-\infty$ to 1	1 to 3	3 to ∞
$f''(0) > 0$	$f''(2) < 0$	$f''(4) > 0$
upward	***downward***	***upward***

f is concave upward on $(-\infty, 1)$ *and* $(3, \infty)$

f is concave downward on $(1, 3)$

Example

Determine the concavity of $y = 3 + \sin x$ *on* $[0,\ 2\pi]$



Solution

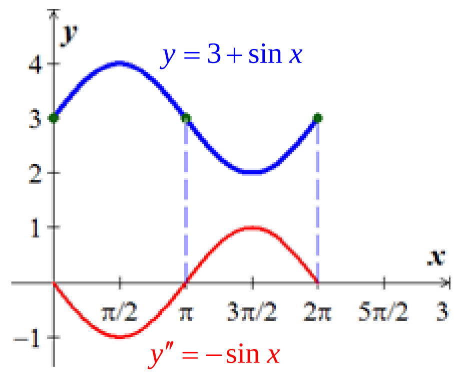
$y' = \cos x$

$y'' = -\sin x = 0 \quad \Rightarrow \quad \boxed{x = 0,\ \pi,\ 2\pi}$

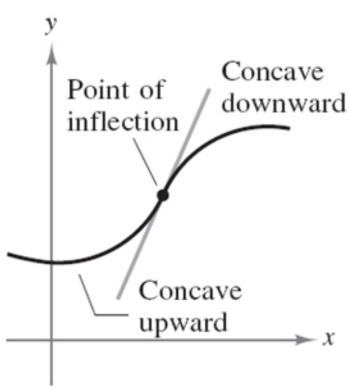
0	π	2π
$f''\left(\frac{\pi}{2}\right) < 0$ *downward*		$f''\left(\frac{3\pi}{2}\right) > 0$ *upward*

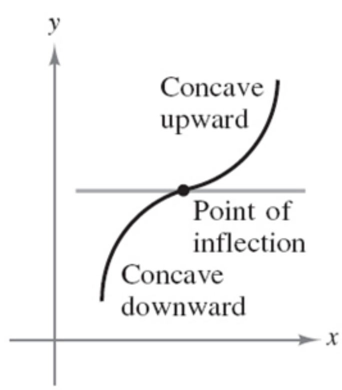
The graph y is ***concave down*** on $(0,\ \pi)$

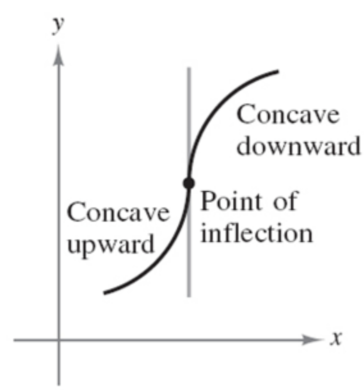
The graph y is ***concave up*** on $(\pi,\ 2\pi)$



Points of Inflection



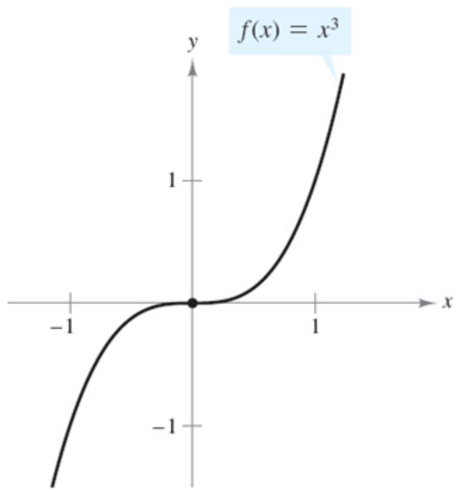




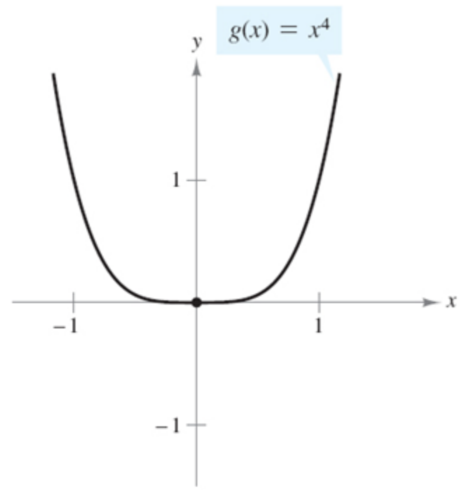
Definition

If the graph of a continuous function has a tangent line @ a point where its concavity changes from upward to downward (or down to upward) then the point is a ***point of inflection***.

At a point of inflection $(c, f(c))$, either $f''(c)=0$ *or* $f''(c)$ fails to exist.



$f''(0) = 0$, and $(0, 0)$ is a point of inflection.



$g''(0) = 0$, but $(0, 0)$ is not a point of inflection.

Example

A particle is moving along a horizontal coordinate line (positive to the right) with position function

$$s(t) = 2t^3 - 14t^2 + 22t - 5, \quad t \ge 0$$

Find the velocity and acceleration, and describe the motion of the particle.

Solution

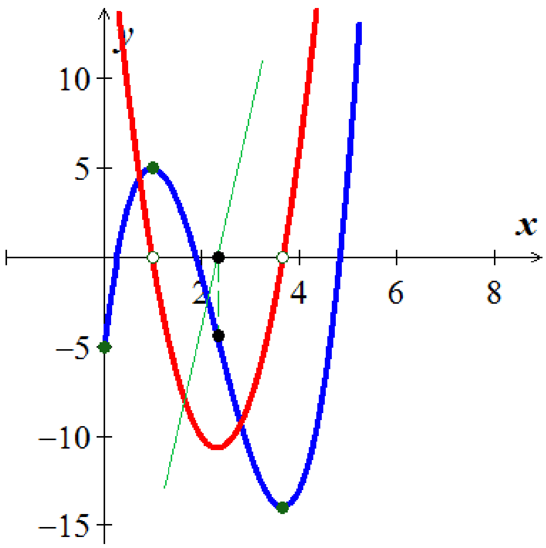
The velocity is: $v(t) = s'(t) = 6t^2 - 28t + 22 = 0 \quad \Rightarrow \boxed{t = 1, \ \frac{11}{3}}$

The acceleration is: $a(t) = v'(t) = 12t - 28 = 0 \quad \Rightarrow \boxed{t = \frac{7}{3}}$

0 — 1	1 — $\frac{7}{3}$ — $\frac{11}{3}$	$\frac{11}{3}$ —
$f'(.5) > 0$ ***Increasing*** ***right***	$f'(2) < 0$ ***Decreasing*** ***left***	$f'(4) > 0$ ***Increasing*** ***right***

0 — $\frac{7}{3}$	$\frac{7}{3}$ —
$f''(1) < 0$ ***Concave down***	$f''(4) > 0$ ***Concave up***

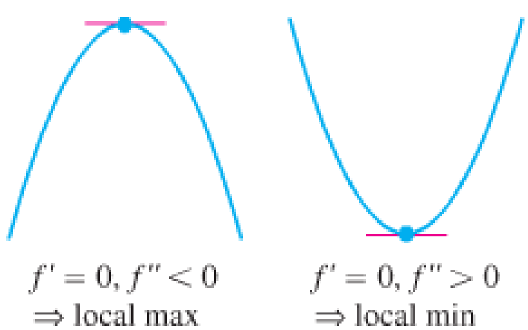
The particle starts moving to the right while slowing down, and then reverses by moving to the left at $t=1$ under the influence of the leftward acceleration over the time interval $\left[0,\ \frac{7}{3}\right)$. The acceleration then changes direction changes direction at $t=\frac{7}{3}$ but the particles continues moving leftward, while slowing down under the rightward acceleration. At $t=\frac{11}{3}$ the particle reverses direction again; moving to the right in the same direction as the acceleration.



Second Derivative Test for local Extrema

Let $f'(c)=0$ and let f'' exist ($\exists$)

1. If $f'(c)=0$ *and* $f''(c)>0 \Rightarrow f$ is a local Minimum at $x=c$
2. If $f'(c)=0$ *and* $f''(c)<0 \Rightarrow f$ is a local Maximum at $x=c$
3. If $f'(c)=0$ *and* $f''(c)=0 \Rightarrow$ Test fails $\to$ use f' to determine Max, Min.



Example

Sketch a graph of the function $f(x) = x^4 - 4x^3 + 10$ using the following steps

a) Identify where the extrema of f occur

b) Find the intervals on which f is increasing and decreasing

c) Find where the graph of f is concave up and down

d) Sketch the general shape of the graph for f

e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

Solution

$$f'(x) = 4x^3 - 12x^2$$

$$= 4x^2(x-3) = 0 \quad \Rightarrow \quad \boxed{x = 0,\ 0} \quad \boxed{x=3} \quad (\textit{CN})$$

$-\infty$ 0	3	∞
$f'(-1) < 0$	$f'(1) < 0$	$f'(4) > 0$
decreasing	***decreasing***	***increasing***

a) $x = 3 \quad \Rightarrow \quad y = 3^4 - 4(3)^3 + 10 = -17$

A local minimum at (3, –17)

b) f is ***decreasing***: $(-\infty,\ 0] \cup [0,\ 3)$

f is ***increasing***: $(3,\ \infty)$

$-\infty$ 0	2	∞
$f''(-1) > 0$	$f''(1) < 0$	$f''(3) > 0$
Concave up	***Concave down***	***Concave up***

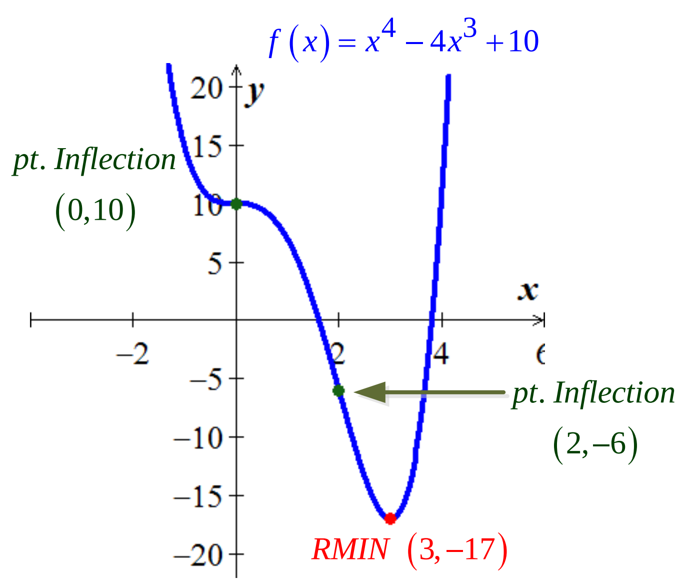
c) $f''(x) = 12x^2 - 24x = 12x(x-2) = 0$

$$\Rightarrow \begin{cases} \boxed{x=0} & \to f(0) = 10 \\ \boxed{x=2} & \to f(2) = -6 \end{cases}$$

f is ***concave up***: $(-\infty,\ 0) \cup (2,\ \infty)$

f is ***concave down***: $(0,\ 2)$

d) $f(x=0) = 0^4 - 4(0)^3 + 10 = 10$



Example

Sketch the graph of $f(x)=\dfrac{(x+1)^2}{1+x^2}$

Solution

Domain of f is $(-\infty,\ \infty)$ ***Horizontal Asymptotes*** $y=1$

$$f'(x)=\frac{2(x+1)\left(1+x^2\right)-2x(x+1)^2}{\left(1+x^2\right)^2}$$

$u=(x+1)^2 \quad v=1+x^2$

$u'=2(x+1) \quad v'=2x$

$$=\frac{2(x+1)\left[\left(1+x^2\right)-x(x+1)\right]}{\left(1+x^2\right)^2}$$

$$=\frac{2(x+1)\left[1+x^2-x^2-x\right]}{\left(1+x^2\right)^2}$$

$$=\frac{2(x+1)(1-x)}{\left(1+x^2\right)^2}$$

$$=2\frac{1-x^2}{\left(1+x^2\right)^2}$$

$\to (x+1)(1-x)=0 \quad\Rightarrow\quad \boxed{x=\pm 1}$ (*CN*)

$$f'(x)=2\left(1-x^2\right)\left(1+x^2\right)^{-2}$$

$$f''(x)=2\left(1+x^2\right)^{-3}\left[-2x\left(1+x^2\right)-2(2x)\left(1-x^2\right)\right]$$

$$\left(U^mV^n\right)'=U^{m-1}V^{n-1}\left(mU'V+nUV'\right)$$

$$=2\frac{-2x-2x^3-4x+4x^3}{\left(1+x^2\right)^3}$$

$$=2\frac{2x^3-6x}{\left(1+x^2\right)^3}$$

$$=\frac{4x\left(x^2-3\right)}{\left(1+x^2\right)^3}=0$$

$\to \boxed{x=0}$ $\boxed{x=\pm\sqrt{3}}$ ***Point of inflections***

$-\infty$	$-\sqrt{3}$	-1	0	1	$\sqrt{3}$	∞
$-$ *Decreasing*		$+$ *Increasing*			$-$ *Decreasing*	
$-$ *Concave down*	$+$ *Concave up*		$-$ *Concave down*		$+$ *Concave up*	

RMAX: $(1,\ 2)$

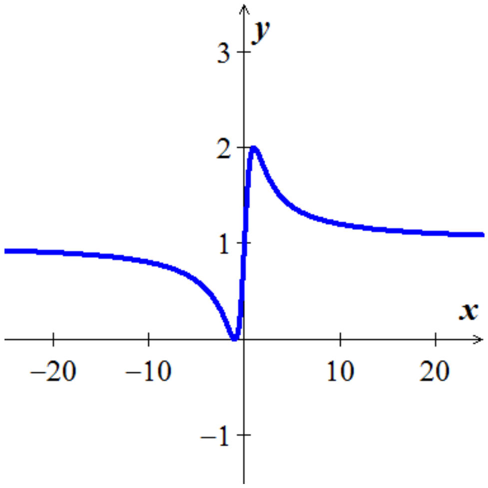
RMIN: $(-1,\ 0)$

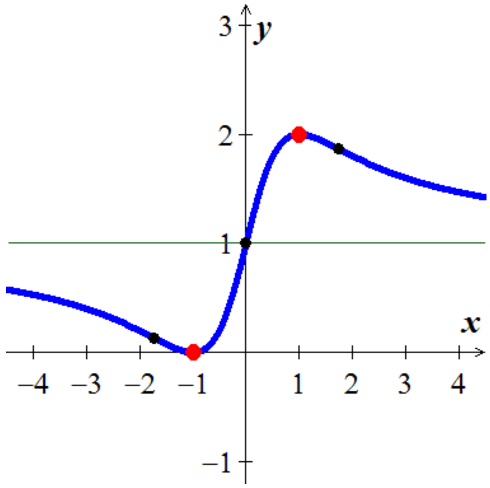
Decreasing: $(-\infty,\ -1)\cup(1,\ \infty)$

Increasing: $(-1,\ 1)$

Concave down: $\boxed{\left(-\infty,\ -\sqrt{3}\right)\cup\left(0,\ \sqrt{3}\right)}$

Concave up: $\boxed{\left(-\sqrt{3},\ 0\right)\cup\left(\sqrt{3},\ \infty\right)}$





Exercises *Section* 3.2 – Graphing Functions

Find the critical numbers and the open intervals on which the function is increasing or decreasing.

1. $f(x)=x^3+3x^2-9x+4$
2. $f(x)=(x-1)^{2/3}$
3. $f(x)=x\sqrt{x+1}$
4. $f(x)=\dfrac{x}{x^2+4}$
5. $f(x)=\dfrac{x}{x^2+1}$
6. $f(x)=x\sqrt{x+1}$
7. $f(x)=x^3-12x$
8. $f(x)=x^{2/3}$

Find the open intervals on which the function is increasing and decreasing. Then, identify the function's local and absolute extreme values, if any, saying where they occur.

9. $g(t)=-t^2-3t+3$
10. $h(x)=2x^3-18x$
11. $f(\theta)=3\theta^2-4\theta^3$
12. $g(x)=x^4-4x^3+4x^2$
13. $f(x)=x-6\sqrt{x-1}$
14. $f(x)=\dfrac{x^3}{3x^2+1}$
15. $f(x)=x^{1/3}(x+8)$

Find all relative Extrema as well as where the function is increasing and decreasing

16. $f(x)=2x^3-6x+1$
17. $f(x)=6x^{2/3}-4x$
18. $f(x)=x^4-4x^3$
19. $f(x)=3x^{2/3}-2x$
20. $y=\sqrt{4-x^2}$
21. $f(x)=x\sqrt{x+1}$
22. $f(x)=\dfrac{x}{x^2+1}$
23. $f(x)=x^4-8x^2+9$

Find the local extrema of each function on the given interval, and say where they occur

24. $f(x)=\sin 2x \quad 0\le x\le \pi$
25. $f(x)=\sqrt{3}\cos x+\sin x \quad 0\le x\le 2\pi$
26. $f(x)=\frac{x}{2}-2\sin\frac{x}{2} \quad 0\le x\le 2\pi$
27. $f(x)=\sec^2 x-2\tan x \quad -\frac{\pi}{2}\le x\le \frac{\pi}{2}$

Determine the intervals on which the graph of the function is concave upward or concave downward.

28. $f(x)=\dfrac{x^2-1}{2x+1}$
29. $f(x)=-4x^3-8x^2+32$
30. $f(x)=\dfrac{12}{x^2+4}$

31. Find the points of inflection. $f(x)=x^3-9x^2+24x-18$

32. Does $f(x)=2x^5-10x^4+20x^3+x+1$ have any inflection points? If so, identify them.

33. Find the second derivative of $f(x)=-2\sqrt{x}$ and discuss the concavity of the graph

34. Find the extrema using the second derivative test $f(x) = \frac{4}{x^2 + 1}$

35. Discuss the concavity of the graph of *f* and find its points of inflection. $f(x) = x^4 - 2x^3 + 1$

36. Find all relative extrema of $f(x) = x^4 - 4x^3 + 1$

Sketch the graph

37. $y = x^3 - 3x + 3$

38. $y = -x^4 + 6x^2 - 4$

39. $y = x\left(\frac{x}{2} - 5\right)^4$

40. $y = x + \sin x \quad 0 \le x \le 2\pi$

41. $y = \cos x + \sqrt{3}\sin x \quad 0 \le x \le 2\pi$

42. $y = \frac{x}{\sqrt{x^2 + 1}}$

43. $y = x^2 + \frac{2}{x}$

44. $y = \frac{x^2 - 3}{x - 2}$

45. $y = \frac{5}{x^4 + 5}$

46. $y = \frac{x^2 - 49}{x^2 + 5x - 14}$

47. $y = \frac{x^4 + 1}{x^2}$

48. $y = \frac{x^2 - 4}{x^2 - 1}$

49. $y = -\frac{x^2 - x + 1}{x - 1}$

50. $y = \frac{x^3 - 3x^2 + 3x - 1}{x^2 + x - 2}$

51. $y = \frac{4x}{x^2 + 4}$

52. $f(x) = \frac{x^2 + 4}{2x}$

53. $f(x) = \frac{1}{2}x^4 - 3x^2 + 4x + 1$

54. $f(x) = \frac{3x}{x^2 + 3}$

55. $f(x) = 4\cos(\pi(x - 1)) \quad on\ [0,\ 2]$

56. $f(x) = \frac{x^2 + x}{4 - x^2}$

57. $f(x) = \sqrt[3]{x} - \sqrt{x} + 2$

58. $f(x) = \frac{\cos \pi x}{1 + x^2} \quad on\ [-2,\ 2]$

59. $f(x) = x^{2/3} + (x + 2)^{1/3}$

60. $f(x) = x(x - 1)e^{-x}$

61. The revenue R generated from sales of a certain product is related to the amount x spent on advertising by

$$R(x) = \frac{1}{15{,}000}\left(600x^2 - x^3\right), \qquad 0 \le x \le 600$$

Where x and R are in thousands of dollars.
Is there a point of diminishing returns for this function?

62. Find the point of diminishing returns (x, y) for the function

$$R(x) = -x^3 + 45x^2 + 400x + 8000, \quad 0 \le x \le 20$$

where $R(x)$ represents revenue in thousands of dollars and x represents the amount spent on advertising in tens of thousands of dollars.

63. A county realty group estimates that the number of housing starts per year over the next three years will be

$$H(r) = \frac{300}{1 + 0.03r^2}$$

Where r is the mortgage rate (in percent).

a) Where is $H(r)$ increasing?

b) Where is $H(r)$ decreasing?

64. Suppose the total cost $C(x)$ to manufacture a quantity x of insecticide (in hundreds of liters) is given by $C(x) = x^3 - 27x^2 + 240x + 750$. Where is $C(x)$ decreasing?

65. A manufacturer sells telephones with cost function $C(x) = 6.14x - 0.0002x^2, \quad 0 \le x \le 950$ and revenue function $R(x) = 9.2x - 0.002x^2, \quad 0 \le x \le 950$. Determine the interval(s) on which the profit function is increasing.

66. The cost of a computer system increases with increased processor speeds. The cost C of a system as a function of processor speed is estimated as $C(x) = 14x^2 - 4x + 1200$, where x is the processor speed in *MHz*. Determine the intervals where the cost function $C(x)$ is decreasing.

67. The percent of concentration of a drug in the bloodstream t hours after the drug is administered is given by $K(t) = \frac{t}{t^2 + 36}$. On what time interval is the concentration of the drug increasing?

68. Coughing forces the trachea to contract, this in turn affects the velocity of the air through the trachea. The velocity of the air during coughing can be modeled by: $v = k\left(R - r\right)r^2, \quad 0 \le r < R$ where k is a constant, R is the normal radius of the trachea (also a constant) and r is the radius of the trachea during coughing. What radius r will produce the maximum air velocity?

69. $P(x) = -x^3 + 15x^2 - 48x + 450, \quad x \ge 3$ is an approximation to the total profit (in thousands of dollars) from the sale of x hundred thousand tires. Find the number of hundred thousands of tires that must be sold to maximize profit.

70. $P(x) = -x^3 + 3x^2 + 360x + 5000; \quad 6 \le x \le 20$ is an approximation to the number of salmon swimming upstream to spawn, where x represents the water temperature in degrees Celsius. Find the temperature that produces the maximum number of salmon.

Section 3.3 – Applied Optimization

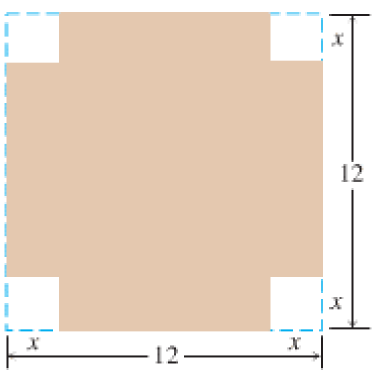
Solving Applied Optimization Problems

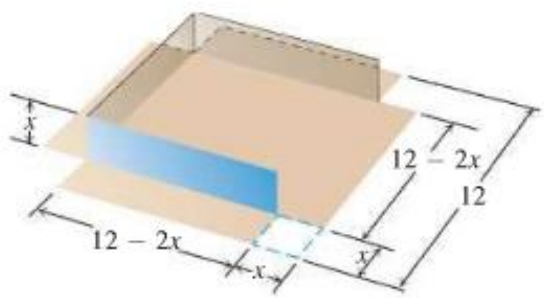
1. Read the problem
2. Draw a picture
3. Introduce variables
4. Write an equation for the unknown quantity
5. Test the critical points and endpoints in the domain of the unknown

Example

An open-top box is to be made by cutting small congruent squares from the corners of a 12-*in*. by 12-*in*. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

Solution

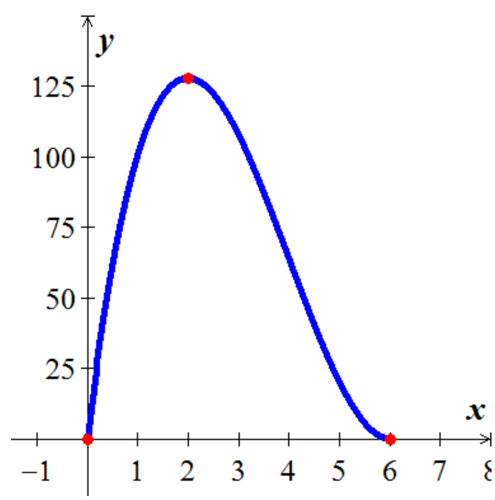




$$\begin{aligned} V(x) &= hlw \\ &= x(12-2x)^2 \\ &= x\left(144-48x+4x^2\right) \\ &= 4x^3-48x^2+144x \qquad 0 \le x \le 6 \end{aligned}$$

$$\frac{dV}{dx} = 12x^2-96x+144 = 0$$

$$\boxed{x = 2,\ 6} \quad (\boldsymbol{CP})$$



$$V(2) = 4(2)^3 - 48(2)^2 + 144(2) = \underline{128}$$

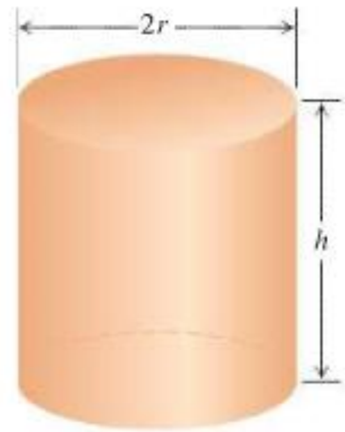
$$V(6) = 4(6)^3 - 48(6)^2 + 144(6) = \underline{0}$$

The maximum volume is 128 in^3. The cutout squares is 2 in.

Example

You have been asked to design a one-liter can shaped like a right circular cylinder. What dimensions will use the least material?

Solution



Volume of can: $V = \pi r^2 h = 1\ liter = 1000\ cm^3$

Surface area of can: $A = 2\pi r^2 + 2\pi rh$

$$\pi r^2 h = 1000 \quad \Rightarrow \quad h = \frac{1000}{\pi r^2}$$

$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{2000}{r}$$

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2} = 0$$

Solve for r

$$4\pi r = \frac{2000}{r^2}$$

$$r^3 = \frac{500}{\pi}$$

$$\underline{\left\lfloor r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42 \right\rfloor}$$

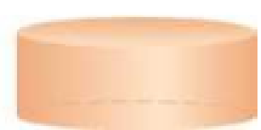
0	5.42	
	$A'(1) < 0$ ***decreasing***	$A'(6) > 0$ ***increasing***

$$\underline{\left\lfloor h = \frac{1000}{\pi r^2} = \frac{1000}{\pi (5.42)^2} \approx 10.84 \right\rfloor} \quad (h = 2r)$$

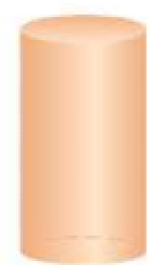
The graph is concave up on its domain.

The on 2-liter can that uses the least material has height equal to twice its radius.

$$\boxed{r \approx 5.42\ cm \quad h \approx 10.84\ cm}$$



Short and wide



Tall and thin

Example

A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area of the rectangle can have, and what are its dimensions?

Solution

The equation of the circle is given by the equation: $x^2 + y^2 = 4$

Therefore, the semicircle is: $y = \sqrt{4 - x^2}$

The dimensions of semicircle: $\quad length: 2x \quad height: \sqrt{4 - x^2}$

Area: $A(x) = lh = 2x\sqrt{4 - x^2}$

$$A'(x) = 2\sqrt{4 - x^2} - \frac{2x^2}{\sqrt{4 - x^2}}$$

$u = 2x \quad v = \sqrt{4 - x^2}$

$u' = 2 \quad v' = \frac{-x}{\sqrt{4-x^2}}$

$$= \frac{8 - 2x^2 - 2x^2}{\sqrt{4 - x^2}}$$

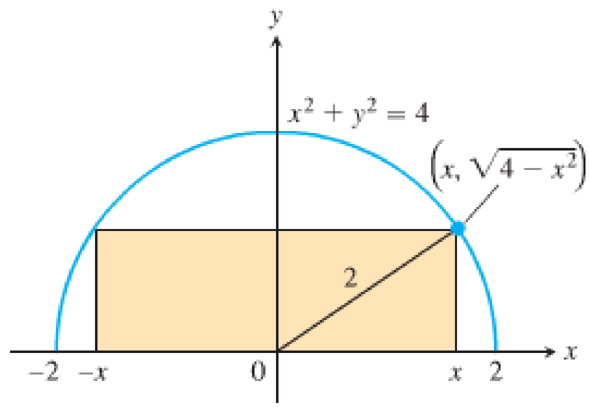
$$= \frac{8 - 4x^2}{\sqrt{4 - x^2}} = 0$$

Solve for x

$$8 - 4x^2 = 0 \quad \Rightarrow \quad x^2 = 2 \rightarrow \boxed{x = \pm\sqrt{2}}$$

$$A\left(\sqrt{2}\right) = 2\left(\sqrt{2}\right)\sqrt{4 - \left(\sqrt{2}\right)^2} = 4$$

$$A(2) = 2(2)\sqrt{4 - (2)^2} = 0$$



The area has a maximum value of 4 when the height is $x = \sqrt{2}$ ***unit*** and length $2x = 2\sqrt{2}$ ***unit***

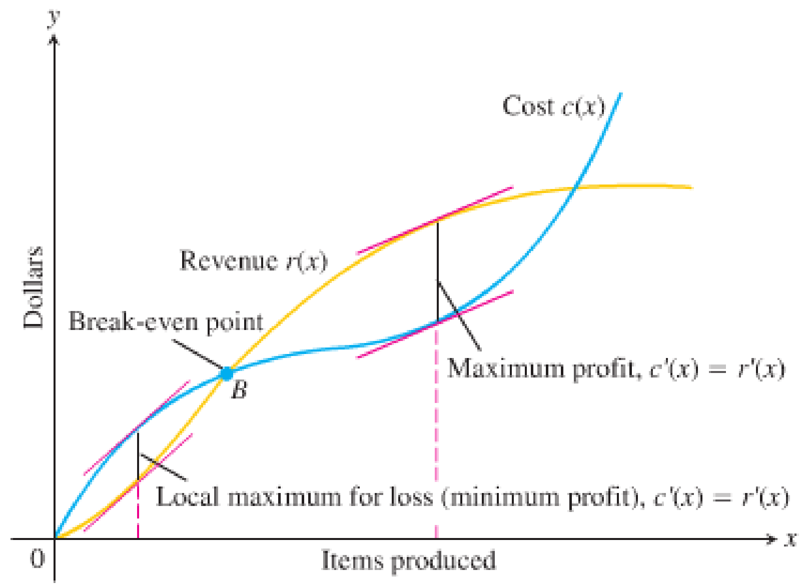
Example from *Economics*

$r(x) =$ Revenue from selling x items

$c(x) =$ Cost of producing the x items

$p(x) = r(x) - c(x)$ Profit from producing and selling x items

At a production level yielding maximum profit, marginal revenue equals to marginal cost.



Marginal Analysis

Profit = P Revenue = R Cost = C $P = R - C$

The derivatives of these quantities are called ***Marginal***

$$\frac{dP}{dx} = \text{Marginal Profit}$$

$$\frac{dR}{dx} = \text{Marginal Revenue}$$

$$\frac{dC}{dx} = \text{Marginal Cost}$$

Example

Suppose that $r(x)=9x$ and $c(x)=x^3-6x^2+15x$, where x represents millions of MP3 players produced. Is there a production level that maximizes profit? If so, what is it?

Solution

$r'(x)=9 \qquad c'(x)=3x^2-12x+15$

To find the intersection between the 2 derivatives, set $r'(x)=c'(x)$

$3x^2-12x+15=9$

$3x^2-12x+6=0 \quad \to \quad \begin{cases} x=2-\sqrt{2}\approx .586 \\ x=2+\sqrt{2}\approx 3.414 \end{cases}$

The possible productions are $x\approx 0.586$ *or* $x\approx 3.414$ million.

$p(x)=r(x)-c(x)=9x-x^3+6x^2-15x$

$p(x)=-x^3+6x^2-6x$

$p'(x)=-3x^2+12x-6$

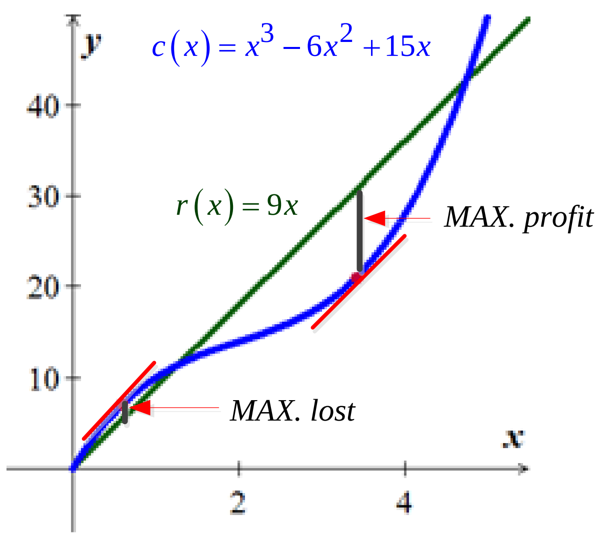
$p''(x)=-6x+12=0 \quad \to \quad \boxed{x=2}$

Concave down: $(0,\ 2)$

Concave up: $(2,\ \infty)$

The maximum profit is about $x\approx 3.414$ *million*

The maximum lost is about $x\approx .586$ *million*



Exercises *Section* 3.3 – Applied Optimization

1. Find two nonnegative numbers *x* and *y* for which $2x + y = 30$, such that xy^2 is maximized.

2. A rectangular page will contain 54 in^2 of print. The margins at the top and bottom of the page are 1.5 *inches* wide. The margins on each side are 1 *inch* wide. What should the dimensions of the page be to minimize the amount of paper used?

3. A rectangular page in a textbook (with width *x* and length *y*) has an area of $98\ in^2$, top and bottom margins set at 1 *in*., left and right margins set at $\frac{1}{2}$ *in.* The printable area of the page is the rectangle that lies within the margins. What are the dimensions of the page that maximize the printable area?

4. What point on the graph of $f(x) = \frac{5}{2} - x^2$ is closest to the origin? (*Hint*: You can minimize the square of the distance.)

5. A line segment of length 10 joins the points $(0,\ p)$ and $(q,\ 0)$ to form a triangle in the first quadrant. Find the values of *p* and *q* that maximize the area of the triangle.

6. A metal cistern in the shape of a right circular cylinder with volume $V = 50\ m^3$ needs to be painted each year to reduce corrosion. The paint is applied only to surfaces exposed to the elements (the outside cylinder wall and the circular top). Find the dimensions *r* and *h* of the cylinder that minimize the area of the painted surfaces.

7. The product of two numbers is 72. Minimize the sum of the second number and twice the first number

8. Verify the function $V = 27x - \frac{1}{4}x^3$ has an absolute maximum when *x* = 6. What is the maximum volume?

9. A net enclosure for golf practice is open at one end. The volume of the enclosure is $83\frac{1}{3}$ cubic meters. Find the dimensions that require the least amount of netting.

10. Find two numbers *x* and *y* such that their sum is 480 and x^2y is maximized.

11. If the price charged for a candy bar is $p(x)$ cents, then *x* thousand candy bars will be sold in a certain city, where $p(x) = 82 - \frac{x}{20}$. How many candy bars must be sold to maximize revenue?

12. $S(x) = -x^3 + 6x^2 + 288x + 4000;\ 4 \le x \le 20$ is an approximation to the number of salmon swimming upstream to spawn, where x represents the water temperature in degrees Celsius. Find the temperature that produces the maximum number of salmon.

13. A company wishes to manufacture a box with a volume of 52 cubic feet that is open on top and is twice as long as it is wide. Find the width of the box that can be produced using the minimum amount of material.

14. A rectangular field is to be enclosed on four sides with a fence. Fencing costs $3 per foot for two opposite sides, and $4 per foot for the other two sides. Find the dimensions of the field of area 730 square feet that would be the cheapest to enclose.

15. A manufacturer wants to design an open box that has a square base and a surface area of 108 in^2. What dimensions will produce a box with a maximum volume?

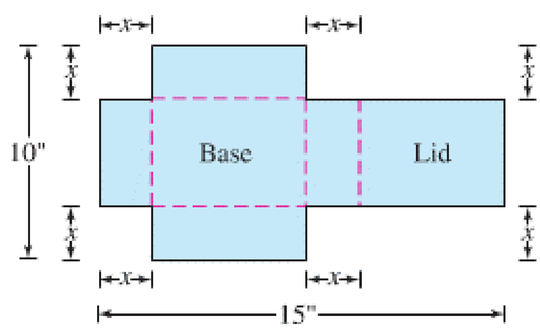
16. A company wants to manufacture cylinder aluminum can with a volume $1000cm^3$. What should the radius and height of the can be to minimize the amount of aluminum used?

17. What is the smallest perimeter possible for a rectangle whose area is 16 in^2, and what are its dimensions?

18. You are planning to make an open rectangular box from an 8-in. by 15-in. piece of cardboard by cutting congruent squares from the corners and folding up the sides. What are the dimensions of the box of largest volume you can make this way, and what is its volume?

19. A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?

20. A piece of cardboard measures 10-*in*. by 15-*in*. Two equal squares are removed from the corners of 10-*in*. side. Two equal rectangles are removed from the other corners so that the tabs can be folded to form a rectangular box with lid.

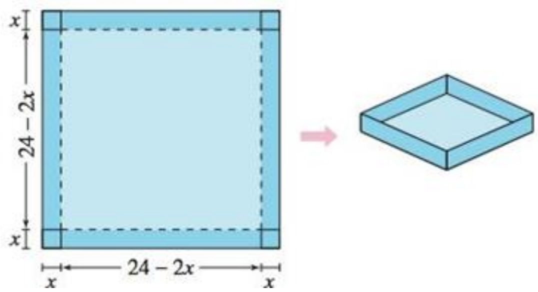


a) Write a formula $V(x)$ for the volume of the box

b) Find the domain of V for the problem situation and graph V over this domain

c) Use the graphical or analytically method to find the maximum volume and the value of x that gives it.

21. An open box of maximum volume is to be made from a square piece of material, 24 *inches* on a side, by cutting equal squares from the corners and turning up the sides.

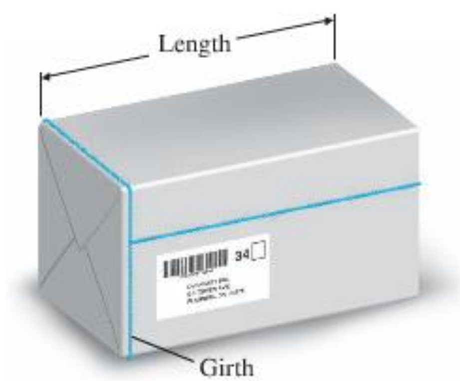


a) Write the volume V as a function of x.
b) Find the critical number of the function and find the maximum value.
c) Graph the function and verify the maximum volume from the graph.

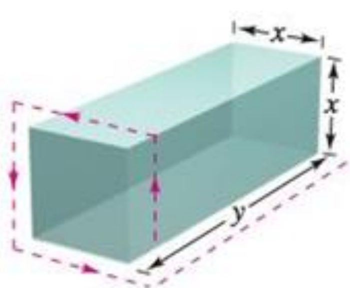
22. A manufacturer wants to design an open box having square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?

23. A rectangular solid (with a square base) has a surface area of 337.5 cm^2. Find the dimensions that will result in a solid with maximum volume.

24. A parcel delivery service will deliver a package only if the length plus girth (distance around) does not exceed 108 *inches*.
a) Find the dimensions of a rectangular box with square ends that satisfies the delivery service's restriction and has maximum volume. What is the maximum volume?
b) Find the dimensions (radius and height) of a cylinder container that meets the delivery service's requirement and has maximum volume. What is the maximum volume?

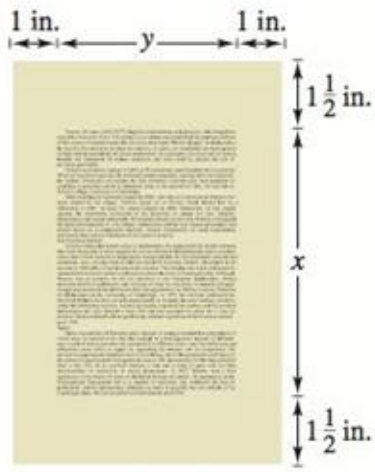


25. A rectangular package to be sent by a postal service can have a maximum combined length and girth (perimeter of a cross section) of 108 *inches*. Find the dimensions of the package of maximum volume that can be sent.



26. A page is to contain 30 *square inches* of print. The margins at the top and bottom of the page are 2 *inches* wide. The margins on the sides are 1 *inch* wide. What dimensions will minimize the amount of paper used?

27. A rectangular page is to contain 24 *squares inches* of print. The margins at the top and bottom of the page are $1\frac{1}{2}$ *inches*, and the margins on the left and right are to be 1 *inch*. What should the dimensions of the page be so that the least amount of paper used?

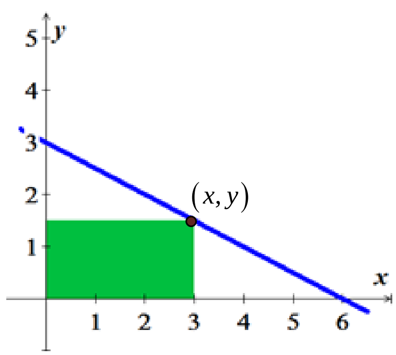


28. A rectangle has its base on the x-axis and its upper vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?

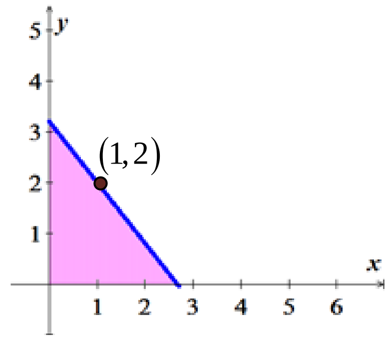
29. Find the points of $y = 4 - x^2$ that are closet to (0, 3)

30. Which points on the graph of $y = 4 - x^2$ are closest to the point $(0,\ 2)$?

31. A rectangle is bounded by the x- and y-axes and the graph of $y = \frac{1}{2}(6 - x)$. What length and width should the rectangle have so that its area is a maximum?



32. A right triangle is formed in the first quadrant by the x- and y-axes and a line through the point $(1,\ 2)$.

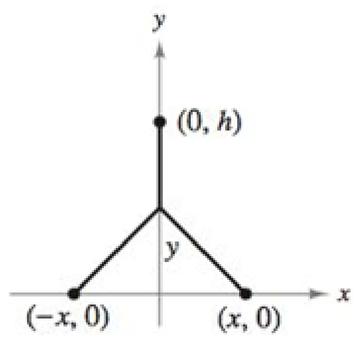


a) Write the length L of the hypotenuse as a function of x.

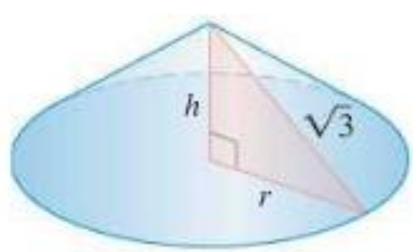
b) Graph the function and approximate x graphically such that the length of the hypotenuse is a minimum.

c) Find the vertices of the triangle such that its area is a minimum.

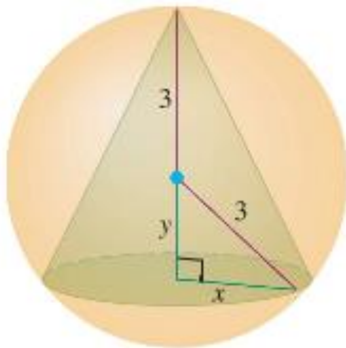
33. Two factories are located at the coordinates $(-x,\ 0)$ and $(x,\ 0)$, and their power is at $(0,\ h)$. Find y such that the total length of power line from the power supply to the factories is a minimum.



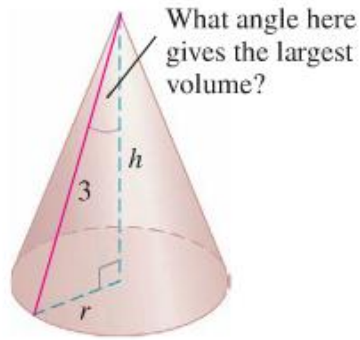
34. A right triangle whose hypotenuse is $\sqrt{3}\ m$ long is revolved about one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume that can be made this way.



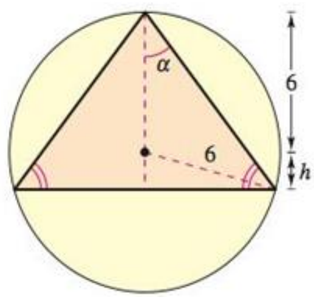
35. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.



36. The slant height of the cone is 3 *m*. How large should the indicated angle be to maximize the cone's volume?



37. Find the area of the largest isosceles triangle that can be inscribed in a circle of radius 6.

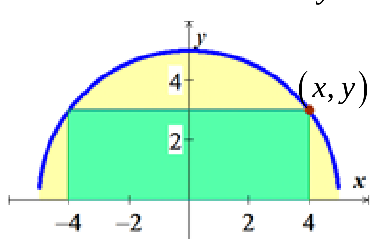


a) Solve the area as a function of h.

b) Solve the area as a function of α.

c) Identify the type of triangle of maximum area.

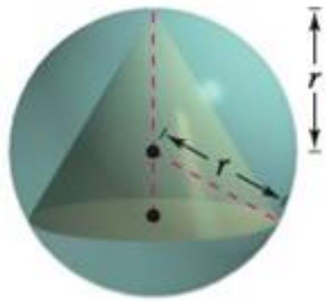
38. A rectangle is bounded by the *x-axis* and the semicircle $y=\sqrt{25-x^2}$



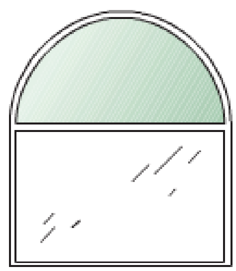
What length and width should the rectangle have so that its area is a maximum?

39. What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of 1000 cm^3?

40. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius r.



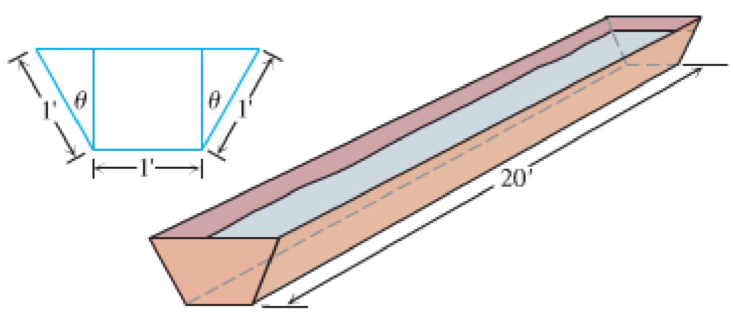
41. A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame.



42. The cost per hour for fuel to run a train is $\frac{v^2}{4}$ dollars, where v is the speed of the train in miles per hour. (Note that the cost goes up as the square of the speed.) Other costs, including labor are $300 per hour. How fast should the train travel on a 360-mile trip to minimize the total cost for the trip?

43. Jane is 2 mi offshore in a boat and wishes to reach a coastal village 6 mi down a straight shoreline from the point nearest the boat. She can row 2 mph and can walk 5 mph. Where should she land her boat to reach the village in the least amount of time?

44. The trough in the figure is to be made to the dimensions shown. Only the angle θ can be varied. What value of θ will maximize the trough's volume?



45. The height above the ground of an object moving vertically is given by $s = -16t^2 + 96t + 112$ with s in feet and t in seconds. Find

a) The object's velocity when $t = 0$

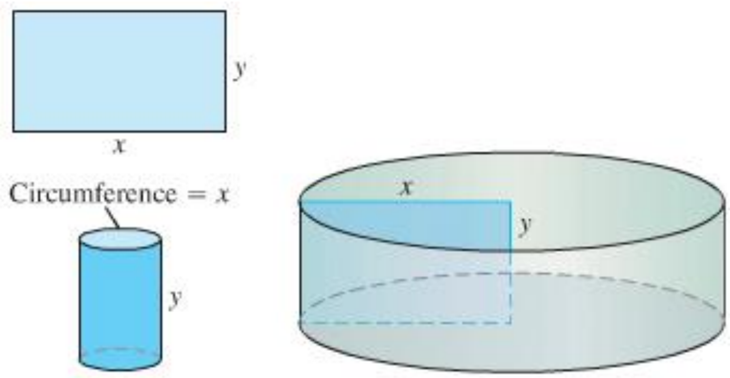
b) Its maximum height and when it occurs

c) Its velocity when $s = 0$

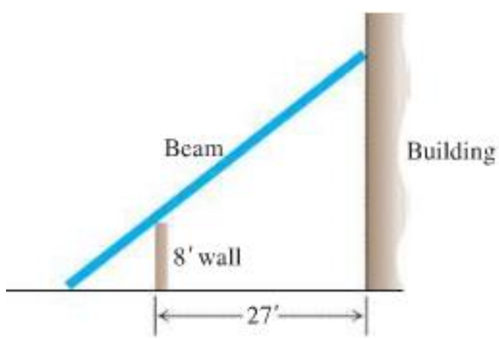
46. Compare the answers to the following two construction problems.

a) A rectangular sheet of perimeter 36 *cm* and dimensions x *cm* and y *cm* is to be rolled into a cylinder as shown in part (a) of the figure. What values of x and y give the largest volume?

b) The same sheet is to be revolved about one of the sides of length y to sweep out the cylinder as shown in part (b) of the figure. What values of x and y give the largest volume?



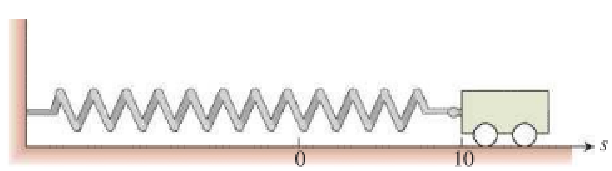
47. The 8-*feet* wall stands 27 *feet* from the building. Find the length of the shortest straight beam that will reach to the side of the building from the ground outside the wall.



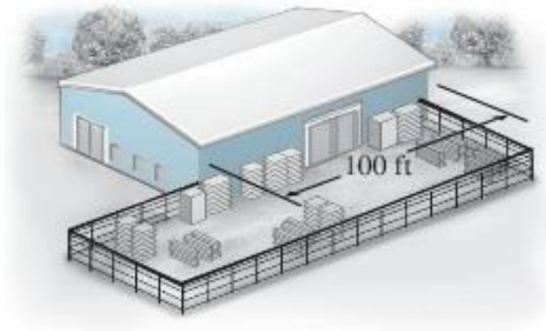
48. A small frictionless cart, attached to the wall by a spring, is pulled 10 cm from its rest position and released at time $t = 0$ to roll back and forth for 4 *sec*. Its position at time t is $s = 10\cos \pi t$

a) What is the cart's maximum speed? When is the cart moving that fast? Where is it then? What is the magnitude of the acceleration then?

b) Where is the cart when the magnitude of the acceleration is greatest? What is the cart's speed then?

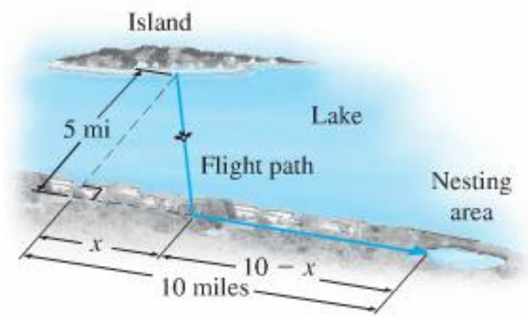


49. The owner of a retail lumber store wants to construct a fence an outdoor storage are adjacent to the store, using all of the store as part of one side of the area. Find the dimensions that will enclose the largest area if



a) 240 *feet* fencing material are used.
b) 400 *feet* fencing material are used.

50. Some birds tend to avoid flights over large bodies of water during daylight hours. Suppose that an adult bird with this tendency is taken from its nesting area on the edge of a large lake to an island 5 miles offshore and is then release.



a) If it takes only 1.4 times as much energy to fly over water as land, how far up the shore (x, in miles) should the bird head to minimize the total energy expended in returning to the nesting area?
b) If it takes only 1.1 times as much energy to fly over water as land, how far up the shore should the bird head to minimize the total energy expended in returning to the nesting area?

51. A pharmacy has a uniform annual demand for 200 bottles of a certain antibiotic. It costs \$10 to store one bottle for one year and \$40 to place an order. How many times during the year should the pharmacy order the antibiotic in order to minimize the total storage and reorder costs?

52. A 300-room hotel in Las Vegas is filled to capacity every night at \$80 a room. For each \$1 increase in rent, 3 fewer rooms are rented. If each rented room costs \$10 to service per day, how much should be the management charge for each room to maximize gross profit? What is the maximum gross profit?

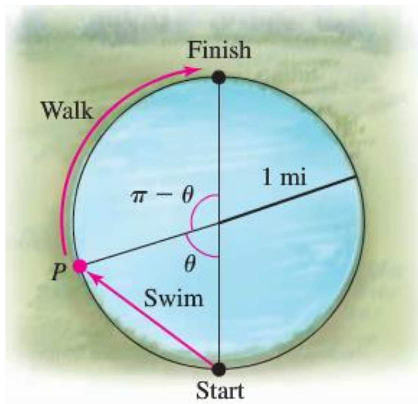
53. A multimedia company anticipates that there will be a demand for 20,000 copies of a certain DVD during the next year. It costs the company \$0.50 to store a DVD for one year. Each time it must make additional DVDs, it costs \$200 to set up the equipment. How many DVDs should the company make during each production run to minimize its total storage and setup costs?

54. A university student center sells 1,600 cups of coffee per day at a price of \$2.40.

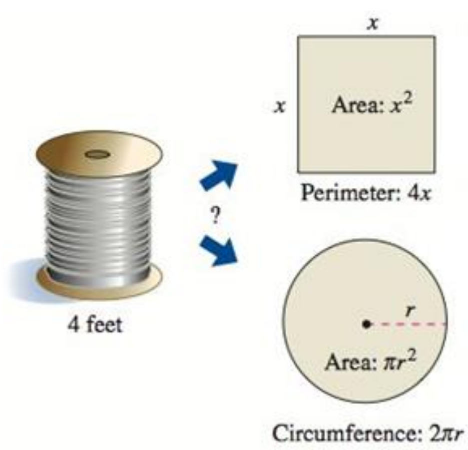
a) A market survey shows that for every \$0.05 reduction in price, 50 more cups of coffee will be sold. How much should be the student center charge for a cup of coffee in order to maximize revenue?

b) A different market survey shows that for every \$0.10 reduction in the original \$2.40 price, 60 more cups of coffee will be sold. Now how much should the student center charge for a cup of coffee in order to maximize revenue?

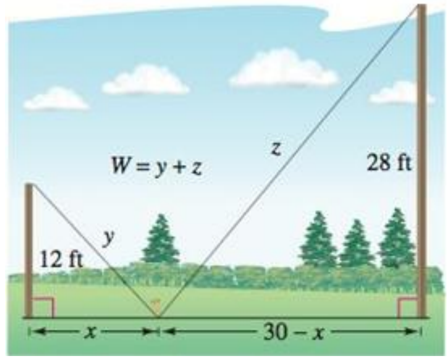
55. Suppose you are standing on the shore of a circular pond with radius 1 *mi* and you want to get to a point on the shore directly opposite your position (on the other end of a diameter). You plan to swim at 2 *mi/hr* from your current position to another point *P* on the shore and then walk at 3 *mi/hr* along the shore to the terminal point. How should you choose *P* to minimize the total time for the trip?



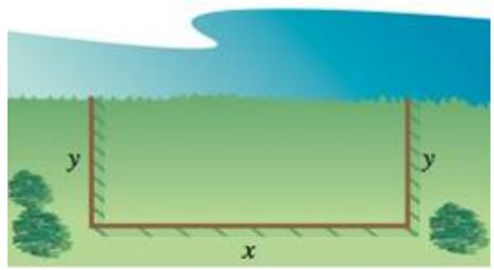
56. Four feet of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area?



57. Two posts, one 12 *feet* high and the other 28 *feet* high, stand 30 *feet* apart. They are to be stayed by two wires, attached to a single stake, running from ground level on the top of each post. Where should the stake be placed to use the least amount of wire?



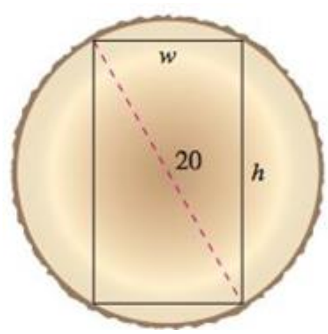
58. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 245,000 m^2 in order to provide enough grass for the herd. No fencing is needed along the river. What dimensions will require the least amount of fencing?



59. A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window. Find the dimensions of a Norman window of maximum area when the total perimeter is 16 *feet*.

60. A wooden beam has a rectangular cross section of height h and width w, the strength S of the beam is directly proportional to the width and the square of the height. What are the dimensions of the strongest beam that can be cut from a round log of diameter 20 *inches*?

$$S = kwh^2 \quad (k:\ proportional\ constant)$$



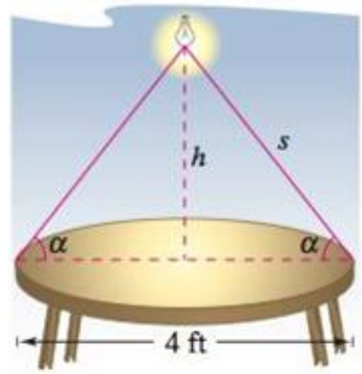
61. A light source is located over the center of a circular table of diameter 4 *feet*. Find the height h of the light source such that the illumination I at the perimeter of the table is maximum when

$$I = \frac{k \sin \alpha}{s^2}$$

where s is the slant height

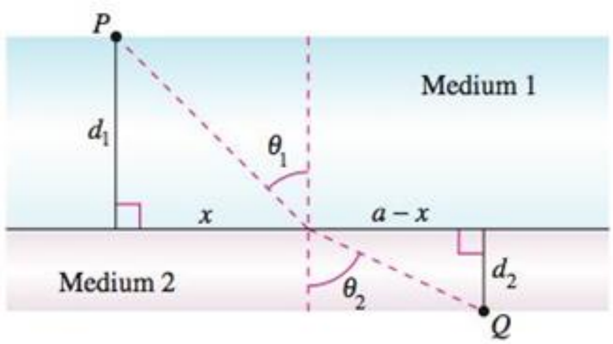
α is the angle at which the light strikes the table

k is a constant.



62. When light waves traveling in a transparent medium strike the surface of a second transparent medium, they change direction. This change of direction is called refraction and is defined by ***Snell's Law of Refraction***,

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$$

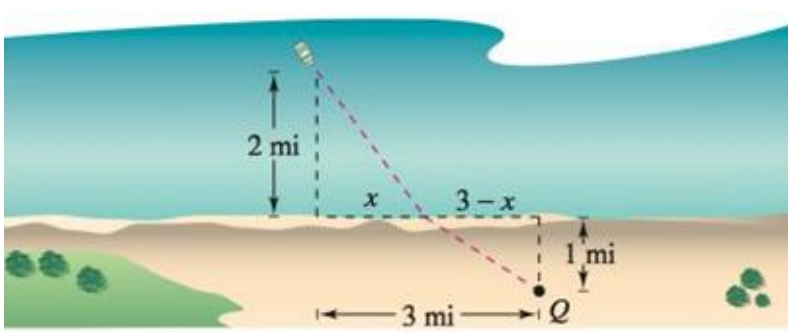


Where θ_1 and θ_2 are the magnitudes of the angles.

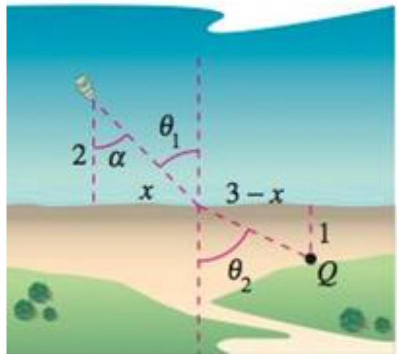
v_1 and v_2 are the velocities of light in the two media.

Show that the light waves traveling from P to Q follow the path of the minimum time.

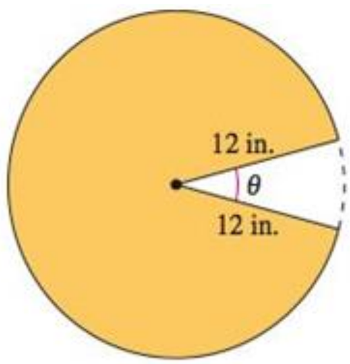
63. You are in a boat 2 *miles* from the nearest point on the coast. You are to go to a point Q that is 3 *miles* down the coast and 1 *mile* inland. You can row at 3 *mph* and walk 4 *mph*. Toward what point on the coast should you row in order to reach Q in the least time?



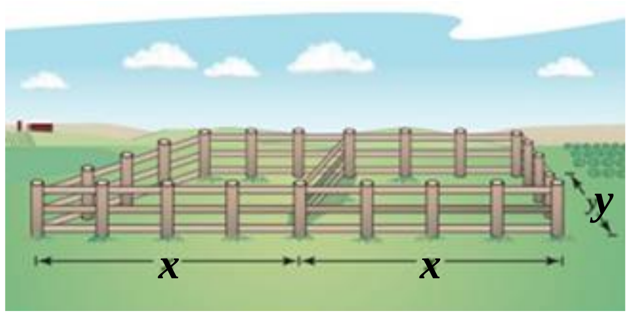
64. You are in a boat 2 *miles* from the nearest point on the coast. You are to go to a point Q that is 3 *miles* down the coast and 1 *mile* inland. You can row at 2 mph and walk 4 mph. Toward what point on the coast should you row in order to reach Q in the least time?



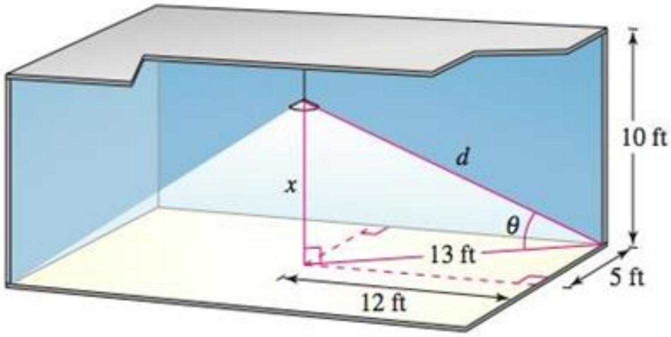
65. A sector with central angle θ is cut from a circle of radius 12 *inches*, and the edges of the sector are brought together to form a cone. Find the magnitude of θ such that the volume of the cone is a maximum.



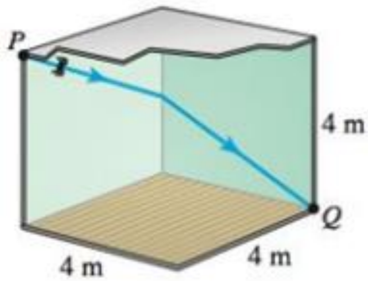
66. A rancher has 400 *feet* of fencing with which to enclose two adjacent rectangular corrals. What dimensions should be used so that the enclosed area will be a maximum?



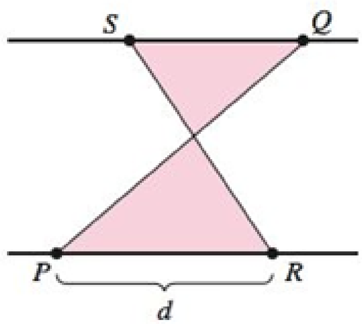
67. The amount of illumination of a surface is proportional to the intensity of the light source, inversely proportional to the square of the distance from the light source, and proportional to $\sin\theta$, where θ is the angle at which the light strikes the surface. A rectangular room measures 10 *feet* by 24 *feet*, with a 10-*foot* ceiling. Determine the height at which the light should be placed to allow the corners of the floor to receive as much light as possible.



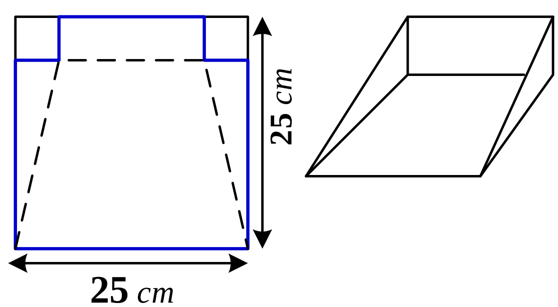
68. Consider a room in the shape of a cube, 4 *meters* on each side. A bug at point P wants to walk to point Q at the opposite corner. Determine the shortest path.



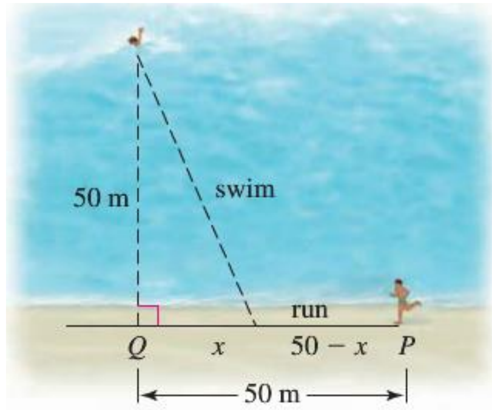
69. The line joining P and Q crosses the two parallel lines. The point R is d units from P. How far from Q should the point S be positioned so that the sum of the areas of the two shaded triangles is a minimum? So that the sum is a maximum?



70. Equal squares are cut out of two adjacent corners of a square of sheet metal having sides of length 25 *cm*. the three resulting flaps are bent up, to form the sides of a dustpan. Find the maximum volume of a dustpan made in this way.



71. You must get from a point P on the straight shore of a lake to a stranded swimmer who is 50 *m* from a point Q on the shore that is 50 *m* from you. If you can swim at a speed of 2 *m/s* and run at a speed of 4 *m/s*, at what point along the shore, x meters from Q, should you stop running and start swimming if you want to reach the swimmer in the minimum time?



a) Find the function T that gives the travel time as a function of x, where $0 \le x \le 50$

b) Find the critical point of T on (0, 50)

c) Evaluate T at the critical point and the endpoints ($x = 0$ and $x = 50$) to verify that the critical point corresponds to an absolute minimum. What is the minimum travel time?

d) Graph the function T to check your work.

72. Consider the function $f(x) = ax^2 + bx + c$ with $a \ne 0$. Explain geometrically why f has exactly one absolute extreme value on $(-\infty, \infty)$. Find the critical points to determine the value of x at which f has an extreme value.

73. An 8-*foot*-tall fence runs parallel to the side of a house 3 *feet* away. What is the length of the shortest ladder that clears the fence and reaches the house? Assume that the vertical wall of the house and the horizontal ground have infinite extent.

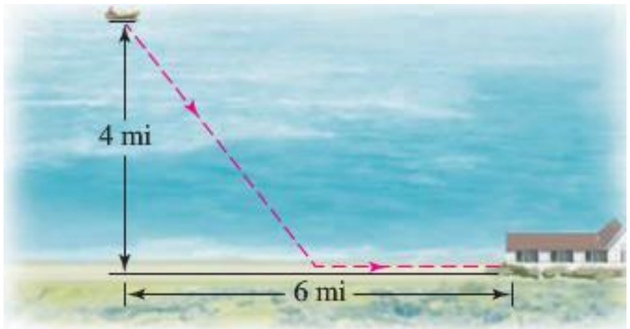
74. A man wishes to get from an initial point on the shore of a circular pond with radius 1 *mi* to a point on the shore directly opposite (on the other end of the diameter). He plans to swim from the initial point to another point on the shore and then walk along the shore to the terminal point.

a) If he swims at 2 *mi/hr.* and walks at 4 *mi/hr.*, what are the minimum and maximum times for the trip?

b) If he swims at 2 *mi/hr* and walks at 1.5 *mi/hr*, what are the minimum and maximum times for the trip?

c) If he swims at 2 *mi/hr*, what is the minimum walking speed for which it is quickest to walk the entire distance?

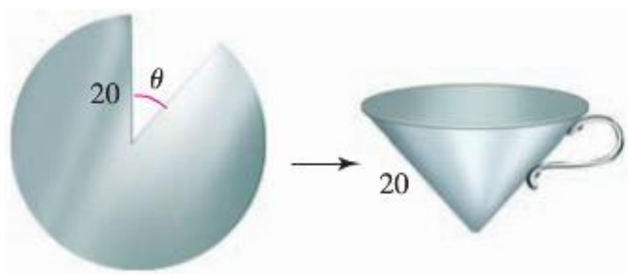
75. A boat on the ocean is 4 *mi* from the nearest point on a straight shoreline; that point is 6 *mi* from a restaurant on the shore. A woman plans to row the boat straight to a point on the shore and then walk along the shore to the restaurant.



a) If she walks at 3 *mi/hr* and rows at 2 *mi/hr*, at which point on the shore should she land to minimize the total travel time?

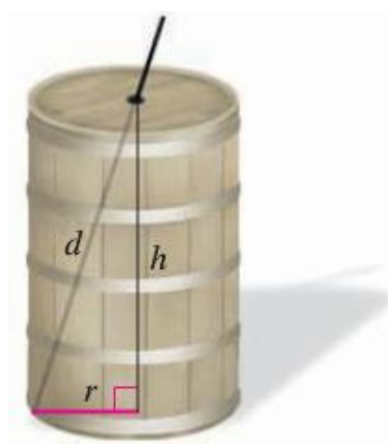
b) If she walks at 3 *mi/hr* what is the minimum speed at which she must row so that the quickest way to the restaurant is to row directly (with no walking)?

76. A cone is constructed by cutting a sector of angle θ from a circular sheet of metal with radius 20 *cm*. the cut sheet is then folded up and wheeled. What angle θ maximizes the volume of the cone?



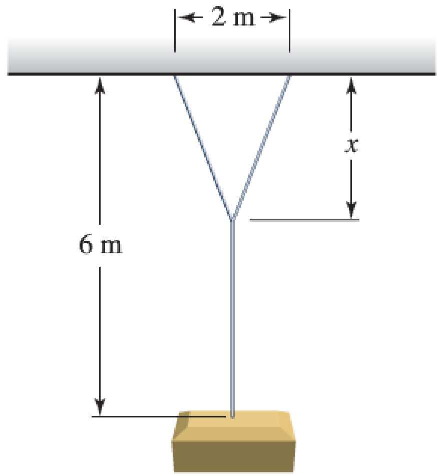
Find the radius and height of the cone with maximum volume that can be formed in this way.

77. Several mathematical stories originated with the second wedding of the mathematician and astronomer Johannes Kepler. Here is one: while shopping for wine for his wedding, Kepler noticed that the price of a barrel of wine (here assumed to be a cylinder) was determined solely by the length d of a dipstick that was inserted diagonally through a hole in the top of the barrel to the edge of the base of the barrel.



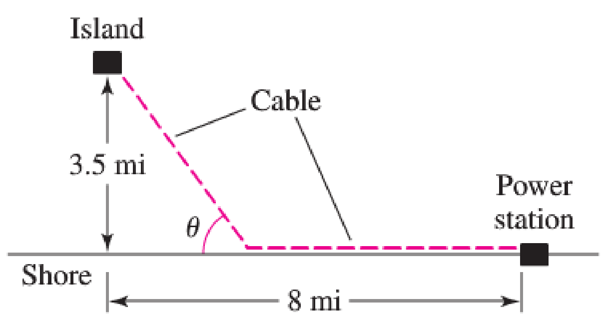
Kepler realized that this measurement does not determine the volume of the barrel and that for a fixed value of d, the volume varies the radius r and height h of the barrel. For a fixed value of d, what is the ratio r/h that maximizes the volume of the barrel?

78. A load must be suspended 6 *m* below a high ceiling using cables attached to two supports that are 2 *m* apart.



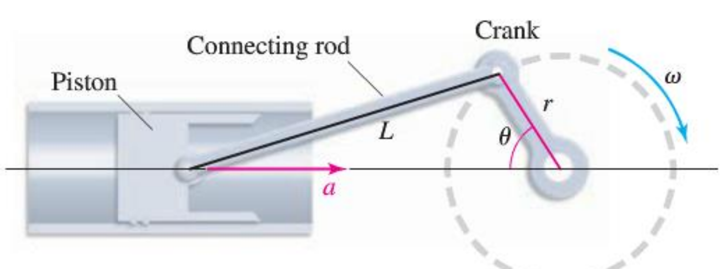
How far below the ceiling (x) should the cables be joined to minimize the total length of cable used?

79. An island is 3.5 *mi* from the nearest point on a straight shoreline; that point is 8 *mi* from a power station. A utility company plans to lay electrical cable underwater from the island to the shore and then underground along the shore to the power station. Assume that is costs $2,400/*mi* to lay underwater cable and $1,200/*mi* to lay underground cable. At what point should the underwater cable meet the shore in order to minimize the cost of the project?



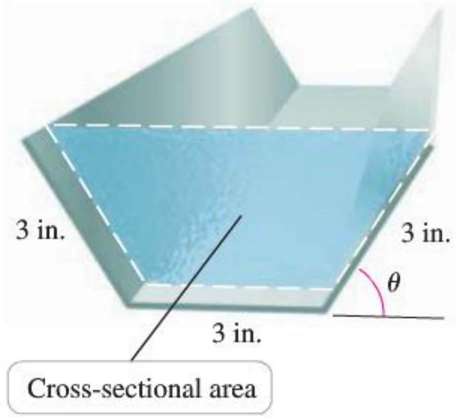
80. A crank of radius r rotates with an angular frequency ω. It is connected to a piston by a connecting rod of length L. the acceleration of the piston varies with the position of the crank according to the function

$$a(\theta) = \omega^2 r\left(\cos\theta + \frac{r\cos 2\theta}{L}\right)$$



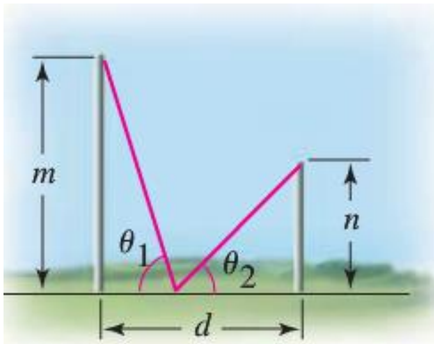
For fixed ω and r, find the values of θ, with $0 \le \theta \le 2\pi$, for which the acceleration of the piston is a maximum and minimum.

81. A rain gutter is made from sheets of metal 9 *in* wide. The gutters have a 3-*in* base and two 3-*in* sides, folded up at an angle θ.

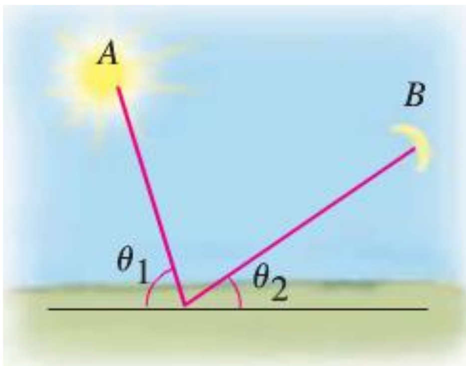


What angle θ maximizes the cross-sectional area of the gutter?

82. Two poles of heights m and n are separated by a horizontal distance d. A rope is stretched from the top of one pole to the ground and then to the top of the other pole. Show that the configuration that requires the least amount of rope occurs when $\theta_1 = \theta_2$



83. Fermat's principle states that when light travels between two points in the same medium (at a constant speed), it travels on the path that minimizes the travel time. Show that when light from a source *A* reflects off of a surface and is received at point *B*, the angle of incidence equals the angle of reflection, or $\theta_1 = \theta_2$



Section 3.4 – L'Hôpital's Rule

John Bernoulli discovered a rule using derivatives to calculate limits of fractions whose numerator and denominators both approach zero or $\pm\infty$. The rule is known today as ***L'Hôpital's Rule***, after Guillaume de L'Hôpital.

Indeterminate form 0/0

Theorem – L'Hôpital's Rule

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Assuming the limit on the right side of this equation exists.

Example

- $\displaystyle \lim_{x\to 0}\frac{3x-\sin x}{x} = \frac{0}{0} = \lim_{x\to 0}\frac{3-\cos x}{1} = \frac{3-\cos x}{1}\bigg|_{x=0} = \boxed{2}$

- $\displaystyle \lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \frac{0}{0} = \lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2\sqrt{1+x}}\bigg|_{x=0} = \boxed{\frac{1}{2}}$

- $\displaystyle \lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2} = \frac{0}{0} = \lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}-\frac{1}{2}}{2x} = \frac{0}{0}$

  $\displaystyle = \lim_{x\to 0}\frac{-\frac{1}{4}(1+x)^{-3/2}}{2} = \frac{-\frac{1}{4}(1+x)^{-3/2}}{2}\bigg|_{x=0} = \boxed{-\frac{1}{8}}$

- $\displaystyle \lim_{x\to 0}\frac{x-\sin x}{x^3} = \frac{0}{0} = \lim_{x\to 0}\frac{1-\cos x}{3x^2} = \frac{0}{0}$

  $\displaystyle = \lim_{x\to 0}\frac{\sin x}{6x} = \frac{0}{0}$

  $\displaystyle = \lim_{x\to 0}\frac{\cos x}{6}$

  $\displaystyle \boxed{= \frac{1}{6}}$

Example

Use l'Hôpital Rule to find $\lim\limits_{x \to 0} \dfrac{1-\cos x}{x+x^2}$

Solution

$$\lim_{x \to 0} \frac{1-\cos x}{x+x^2} = \frac{0}{0} = \lim_{x \to 0} \frac{\sin x}{1+2x}$$

$$= \left. \frac{\sin x}{1+2x} \right|_{x=0}$$

$$= \frac{0}{1}$$

$$= \underline{0}$$

Example

Use l'Hôpital Rule to find $\lim\limits_{x \to 0} \dfrac{\sin x}{x^2}$

Solution

$$\lim_{x \to 0} \frac{\sin x}{x^2} = \frac{0}{0} = \lim_{x \to 0} \frac{\cos x}{2x}$$

$$= \frac{1}{0}$$

$$\underline{= \infty}$$

Example

Use l'Hôpital Rule to find $\lim\limits_{x \to 0^-} \dfrac{\sin x}{x^2}$

Solution

$$\lim_{x \to 0^-} \frac{\sin x}{x^2} = \frac{0}{0} = \lim_{x \to 0^-} \frac{\cos x}{2x} = \underline{-\infty}$$

Indeterminate form ∞ / ∞, $\infty - 0$, $\infty - \infty$

L'Hôpital Rule applies to the indeterminate form ∞/∞, 0/0. If

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Example

Find the limits of these ∞ / ∞ forms:

a) $\displaystyle \lim_{x\to\pi/2} \frac{\sec x}{1+\tan x}$

b) $\displaystyle \lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}}$

c) $\displaystyle \lim_{x\to\infty} \frac{e^x}{x^2}$

Solution

a)

$$\lim_{x\to\pi/2} \frac{\sec x}{1+\tan x} = \frac{\infty}{\infty} = \lim_{x\to\pi/2} \frac{\sec x \tan x}{\sec^2 x}$$

$$= \lim_{x\to\pi/2} \frac{\tan x}{\sec x}$$

$$= \lim_{x\to\pi/2} \frac{\sin x}{\cos x} \cos x$$

$$= \lim_{x\to\pi/2} \sin x$$

$$\underline{=1|}$$

b)

$$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{1/x}{1/\sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x}}{x}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x}}$$

$$\underline{=0|}$$

c)

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{e^x}{2x}$$

$$= \lim_{x\to\infty} \frac{e^x}{2}$$

$$\underline{=\infty|}$$

Example

Find the limits of these $\infty \cdot 0$ forms:

a) $\lim\limits_{x\to\infty}\left(x\sin\frac{1}{x}\right)$

b) $\lim\limits_{x\to 0^+}\sqrt{x}\ln x$

Solution

a) $\lim\limits_{x\to\infty}\left(x\sin\frac{1}{x}\right)=\infty\cdot 0=\lim\limits_{h\to 0^+}\left(\frac{1}{h}\sin h\right)$ *Let* $h=\frac{1}{x}$

$$=\lim_{h\to 0^+}\left(\frac{\sin h}{h}\right)$$

$$\underline{=1|}$$

b) $\lim\limits_{x\to 0^+}\sqrt{x}\ln x=\lim\limits_{x\to 0^+}\frac{\ln x}{1/\sqrt{x}}$

$$=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{2x^{3/2}}}$$

$$=\lim_{x\to 0^+}\left(-2\sqrt{x}\right)$$

$$\underline{=0|}$$

Example

Find the limits of these $\infty-\infty$ form: $\lim\limits_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$

Solution

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\infty-\infty=\lim_{x\to 0}\left(\frac{x-\sin x}{x\sin x}\right)$$

$$=\lim_{x\to 0}\left(\frac{1-\cos x}{\sin x+x\cos x}\right)=\frac{0}{0}$$

$$=\lim_{x\to 0}\left(\frac{\sin x}{\cos x+\cos x-x\sin x}\right)$$

$$=\lim_{x\to 0}\left(\frac{\sin x}{2\cos x-x\sin x}\right)$$

$$=\frac{0}{2}$$

$$\underline{=0|}$$

Indeterminate Powers

If $\lim\limits_{x\to a} \ln f(x) = L$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^{L}$$

Example

Apply l'Hôpital Rule to show that $\lim\limits_{x\to 0^+} (1+x)^{1/x} = e$

Solution

$$\lim_{x\to 0^+} (1+x)^{1/x} = 1^{\infty}$$

$$\ln f(x) = \ln(1+x)^{1/x} = \frac{1}{x}\ln(1+x)$$

$$\lim_{x\to 0^+} \ln f(x) = \lim_{x\to 0^+} \frac{\ln(1+x)}{x} = \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1}$$

$$= \frac{1}{1}$$

$$= 1$$

$$\lim_{x\to 0^+} (1+x)^{1/x} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^{1} = e$$

Example

Find $\lim\limits_{x\to\infty} x^{1/x}$

Solution

$$\lim_{x\to\infty} x^{1/x} = \infty^{0}$$

$$\ln f(x) = \ln x^{1/x} = \frac{\ln x}{x}$$

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \frac{0}{1} = 0$$

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^{0} = 1$$

Exercises Section 3.4 – L'Hôpital's Rule

Apply l'Hôpital Rule to evaluate

1. $\lim\limits_{x\to -2}\dfrac{x+2}{x^2-4}$

2. $\lim\limits_{x\to 1}\dfrac{x^3-1}{4x^3-x-3}$

3. $\lim\limits_{x\to -5}\dfrac{x^2-25}{x+5}$

4. $\lim\limits_{t\to 0}\dfrac{\sin 5t}{2t}$

5. $\lim\limits_{\theta\to -\pi/3}\dfrac{3\theta+\pi}{\sin\left(\theta+\frac{\pi}{3}\right)}$

6. $\lim\limits_{x\to 0}\dfrac{x^2}{\ln(\sec x)}$

7. $\lim\limits_{\theta\to 0}\dfrac{3^{\sin\theta}-1}{\theta}$

8. $\lim\limits_{x\to 0}\dfrac{3^x-1}{2^x-1}$

9. $\lim\limits_{x\to 0^+}(\ln x-\ln\sin x)$

10. $\lim\limits_{x\to 0}\dfrac{\left(e^x-1\right)^2}{x\sin x}$

11. $\lim\limits_{x\to \pi/2^-}\dfrac{1+\tan x}{\sec x}$

12. $\lim\limits_{x\to \infty}\dfrac{4x^3-6x^2+1}{2x^3-10x+3}$

13. $\lim\limits_{x\to 0}\dfrac{3\sin 4x}{5x}$

14. $\lim\limits_{x\to 2\pi}\dfrac{x\sin x+x^2-4\pi^2}{x-2\pi}$

15. $\lim\limits_{x\to 0}\dfrac{\tan 4x}{\tan 7x}$

16. $\lim\limits_{x\to 0}\dfrac{\sin^2 3x}{x^2}$

17. $\lim\limits_{x\to -1}\dfrac{x^3-x^2-5x-3}{x^4+2x^3-x^2-4x-2}$

18. $\lim\limits_{x\to 1}\dfrac{x^n-1}{x-1}\quad (n>0)$

19. $\lim\limits_{x\to 1^-}(1-x)\tan\left(\frac{\pi x}{2}\right)$

20. $\lim\limits_{x\to \infty}\dfrac{3}{x}\csc\dfrac{5}{x}$

21. $\lim\limits_{x\to \pi/4}\dfrac{\tan x-\cot x}{x-\frac{\pi}{4}}$

22. $\lim\limits_{x\to 0}\dfrac{1-\cos 3x}{8x^2}$

23. $\lim\limits_{x\to 3}\dfrac{x-1-\sqrt{x^2-5}}{x-3}$

24. $\lim\limits_{x\to 2}\dfrac{x^2+x-6}{\sqrt{8-x^2}-x}$

25. $\lim\limits_{h\to 0}\dfrac{\sin(x+h)-\sin x}{h}$ *x is a real number*

26. $\lim\limits_{x\to 2}\dfrac{\sqrt[3]{3x+2}-2}{x-2}$

27. $\lim\limits_{x\to \infty}\dfrac{3x^4-x^2}{6x^4+12}$

28. $\lim\limits_{x\to \infty}\dfrac{4x^3-2x^2+6}{\pi x^3+4}$

29. $\lim\limits_{x\to \infty}\dfrac{8-4x^2}{3x^3+x-1}$

30. $\lim\limits_{x\to \pi/2}\dfrac{2\tan x}{\sec^2 x}$

31. $\lim\limits_{x\to 0}\dfrac{e^x-x-1}{5x^2}$

32. $\lim\limits_{x\to 0}\dfrac{e^x-\sin x-1}{x^4+8x^3+12x^2}$

33. $\lim\limits_{x\to \infty}\dfrac{e^{1/x}-1}{1/x}$

34. $\lim\limits_{x\to \infty}\dfrac{e^{3x}}{3e^{3x}+5}$

35. $\lim_{x\to\infty} \dfrac{\ln(3x+5)}{\ln(7x+3)+1}$

36. $\lim_{x\to\infty} \dfrac{\ln\left(3x+e^{x}\right)}{\ln\left(7x+3e^{2x}\right)}$

37. $\lim_{x\to\infty} \dfrac{x^2-\ln\left(\frac{2}{x}\right)}{3x^2+2x}$

38. $\lim_{x\to 1^+} x^{1/(x-1)}$

39. $\lim_{x\to e^+} (\ln x)^{1/(x-e)}$

40. $\lim_{x\to\infty} (1+2x)^{1/(2\ln x)}$

41. $\lim_{x\to\infty} \left(\dfrac{x^2+1}{x+2}\right)^{1/x}$

42. $\lim_{t\to 2} \dfrac{t^3-t^2-2t}{t^2-4}$

43. $\lim_{x\to 0} \dfrac{1-\cos 6x}{2x}$

44. $\lim_{x\to\infty} \dfrac{5x^2+2x-5}{\sqrt{x^4-1}}$

45. $\lim_{\theta\to 0} \dfrac{3\sin^2 2\theta}{\theta^2}$

46. $\lim_{x\to\infty} \left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)$

47. $\lim_{\theta\to 0} 2\theta\cot 3\theta$

48. $\lim_{x\to 0} \dfrac{e^{-2x}-1+2x}{x^2}$

49. $\lim_{x\to 1} \dfrac{x^4-x^3-3x^2+5x-2}{x^3+x^2-5x+3}$

50. $\lim_{y\to 0^+} \dfrac{\ln^{10} y}{\sqrt{y}}$

51. $\lim_{\theta\to 0} \dfrac{3\sin 8\theta}{8\sin 3\theta}$

52. $\lim_{x\to\infty} \dfrac{\ln x^{100}}{\sqrt{x}}$

53. $\lim_{x\to 0} \csc x \sin^{-1} x$

54. $\lim_{x\to\infty} \dfrac{\ln^3 x}{\sqrt{x}}$

55. $\lim_{x\to\infty} \ln\left(\dfrac{x+1}{x-1}\right)$

56. $\lim_{x\to 0^+} (1+x)^{\cot x}$

57. $\lim_{x\to \frac{\pi}{2}^+} (\sin x)^{\tan x}$

58. $\lim_{x\to\infty} \left(\sqrt{x}+1\right)^{1/x}$

59. $\lim_{x\to 0^+} |\ln x|^{x}$

60. $\lim_{x\to\infty} x^{1/x}$

61. $\lim_{x\to\infty} \left(1-\dfrac{3}{x}\right)^{x}$

62. $\lim_{x\to\infty} \left(\dfrac{2}{\pi}\tan^{-1} x\right)^{x}$

63. $\lim_{x\to 1} (x-1)^{\sin \pi x}$

64. $\lim_{x\to\infty} \dfrac{2x^5-x+1}{5x^6+x}$

65. $\lim_{x\to\infty} \dfrac{4x^4-\sqrt{x}}{2x^4+x^{-1}}$

66. $\lim_{x\to 0} \dfrac{1-\cos x^n}{x^{2n}}$

67. $\lim_{x\to 0} \dfrac{1-\cos^n x}{x^2}$

68. $\lim_{x\to 0} \dfrac{1-\cos x^n}{x^2}$

69. $\lim_{x\to 0} \dfrac{3x}{\tan 4x}$

70. $\lim_{x\to 0} \dfrac{\sin ax}{\sin bx}$

71. $\lim_{x\to 2} \dfrac{\ln(2x-3)}{x^2-4}$

72. $\lim\limits_{x\to 0} \dfrac{1-\cos ax}{1-\cos bx}$

73. $\lim\limits_{x\to 0} \dfrac{\sin^{-1} x}{\tan^{-1} x}$

74. $\lim\limits_{x\to 1} \dfrac{x^{1/3}-1}{x^{2/3}-1}$

75. $\lim\limits_{x\to 0} x\cot x$

76. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{\ln\left(1+x^2\right)}$

77. $\lim\limits_{x\to \pi} \dfrac{\sin^2 x}{x-\pi}$

78. $\lim\limits_{x\to 0} \dfrac{10^x - e^x}{x}$

79. $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\cos 3x}{\pi - 2x}$

80. $\lim\limits_{x\to 1} \dfrac{\ln(ex)-1}{\sin \pi x}$

81. $\lim\limits_{x\to \infty} x\sin \dfrac{1}{x}$

82. $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x^3}$

83. $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x-\tan x}$

84. $\lim\limits_{x\to 0} \dfrac{2-x^2-2\cos x}{x^4}$

85. $\lim\limits_{x\to 0^+} \dfrac{\sin^2 x}{\tan x - x}$

86. $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\ln \sin x}{\cos x}$

87. $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\sin x}{x}$

88. $\lim\limits_{x\to 1^-} \dfrac{\arccos x}{x-1}$

89. $\lim\limits_{x\to \infty} x\left(2\tan^{-1} x - \pi\right)$

90. $\lim\limits_{x\to \frac{\pi}{2}^+} x(\sec x - \tan x)$

91. $\lim\limits_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{xe^{ax}}\right)$

92. $\lim\limits_{x\to 0^+} x^{\sqrt{x}}$

93. $\lim\limits_{x\to \pi} \dfrac{\cos x + 1}{(x-\pi)^2}$

94. $\lim\limits_{x\to 0} \dfrac{\sin x - x}{7x^3}$

95. $\lim\limits_{x\to \infty} \dfrac{\tan^{-1} x - \frac{\pi}{2}}{\frac{1}{x}}$

96. $\lim\limits_{x\to 3} \dfrac{x-1-\sqrt{x^2-5}}{x-3}$

97. $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{\sqrt{8-x^2}-x}$

98. $\lim\limits_{x\to 2} \dfrac{x^2-4x+4}{\sin^2 \pi x}$

99. $\lim\limits_{x\to 2} \dfrac{(3x+2)^{1/3}-2}{x-2}$

100. $\lim\limits_{x\to \infty} \dfrac{3x^4-x^2}{6x^4+12}$

101. $\lim\limits_{x\to \infty} \dfrac{4x^3-2x^2+6}{\pi x^3+4}$

102. $\lim\limits_{x\to \frac{\pi}{2}^-} \frac{1}{3}(2x-\pi)\tan x$

103. $\lim\limits_{x\to \infty} x\ln\left(1+\dfrac{1}{x}\right)$

104. $\lim\limits_{x\to \frac{\pi}{2}^-} \left(\dfrac{\pi}{2} - x\right)\sec x$

105. $\lim\limits_{x\to \infty} \dfrac{e^{1/x}-1}{\sin \frac{1}{x}}$

106. $\lim\limits_{x\to 0^+} \sin x \sqrt{\dfrac{1-x}{x}}$

107. $\lim\limits_{x\to 0} \left(\cot x - \dfrac{1}{x}\right)$

108. $\lim\limits_{x\to \infty} \left(x - \sqrt{x^2+1}\right)$

109. $\lim_{\theta \to \frac{\pi}{2}^-} (\tan\theta - \sec\theta)$

110. $\lim_{x\to 0^+} \ln x^{2x}$

111. $\lim_{x\to 0} \ln(1+4x)^{3/x}$

112. $\lim_{\theta \to \frac{\pi}{2}^-} \ln(\tan\theta)^{\cos\theta}$

113. $\lim_{x\to 0^+} (1+x)^{\cot x}$

114. $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{\ln x}$

115. $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^{x}$

116. $\lim_{x\to 0} \left(e^{5x}+x\right)^{1/x}$

117. $\lim_{x\to 0} \left(e^{ax}+x\right)^{1/x}$

118. $\lim_{x\to 0} \left(2^{ax}+x\right)^{1/x}$

119. $\lim_{x\to 0^+} (\tan x)^{x}$

120. The functions $f(x)=\left(x^x\right)^x$ *and* $g(x)=x^{\left(x^x\right)}$ are different functions. For example, $f(3)=19{,}683$ and $g(3)\approx 7.6\times 10^{12}$. Determine whether $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^+} g(x)$ are intermediate forms and evaluate the limits.

121. Consider the function $g(x)=\left(1+\frac{1}{x}\right)^{x+a}$. show that if $0\le a<\frac{1}{2}$, then $g(x)\to e$ from below as $x\to\infty$; if $\frac{1}{2}\le a<1$, then $g(x)\to e$ from above as $x\to\infty$

122. Let $f(x)=(a+x)^x$, where $a>0$

a) What is the domain of f (in terms of a)?

b) Describe the end behavior of f (near the left boundary of its domain and as $x\to\infty$).

c) Compute f'.

d) Show that f has a single local minimum at the point z that satisfies $(z+a)\ln(z+a)+z=0$

e) Describe how $f(z)$ varies as a increases.

Section 3.5 – Linear Approximation / Mean Value Theorem

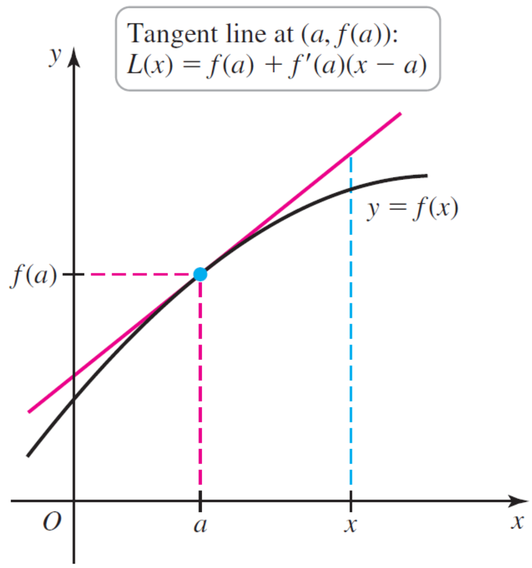
A line tangent to a graph of a function f at a point $(x, f(x))$ is used to approximate the value of f at points near x.

Linear Approximation

Definition

Suppose f is differentiable on an interval I containing the point a. The linear approximation to f at a is the linear function

$$L(x) = f(a) + f'(a)(x-a), \quad \textit{for } x \textit{ in } I$$



Example

Find the linear approximation to $f(x) = \sqrt{x}$ at $x = 1$ and use it to approximate $\sqrt{1.1}$

Solution

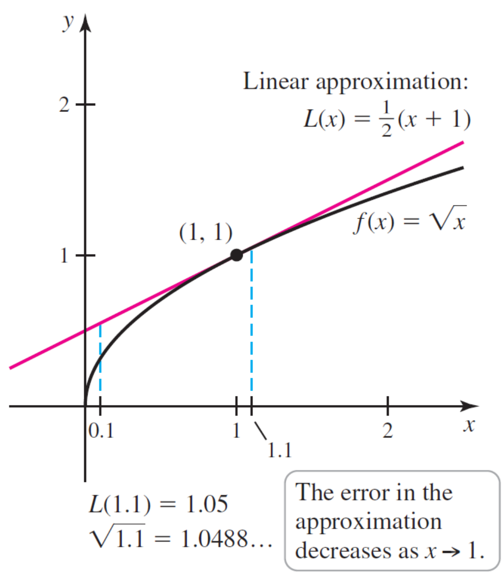
$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= f(1) + f'(1)(x-1) \\ &= 1 + \frac{1}{2}(x-1) \end{aligned}$$

$$\begin{aligned} \sqrt{1.1} \approx L(1.1) &= 1 + \frac{1}{2}(1.1-1) \\ &= 1.05 \end{aligned}$$

The exact value $\sqrt{1.1} \approx 1.0488$

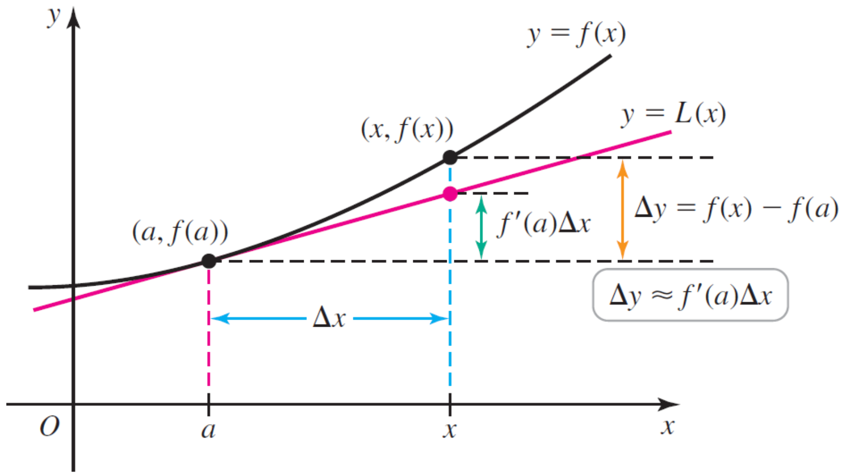
$$Error = \frac{1.05 - 1.0488}{1.0488} = 0.001144 \quad (0.11\%)$$



Relationship Between Δx and Δy

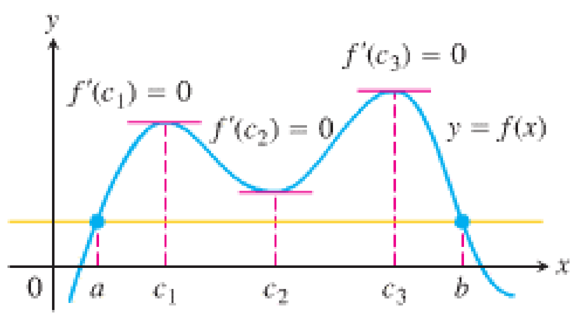
Suppose f is differentiable on an interval I containing the point a. The change in the value of f between two points a and $a+\Delta x$ is approximately

$$\Delta y \approx f'(a)\Delta x \qquad \textit{where } a+\Delta x \textit{ is in } I.$$



Rolle's Theorem

Suppose that $y=f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a)=f(b)$, then there is at least on number c in (a, b) at which $f'(c)=0$



Proof

Being continuous, f assumes absolute maximum and minimum values on $[a, b]$. These can occur only

1. At interior points where f' is zero,
2. At interior points where f' does not exist,
3. At the endpoints of the function's domain, in this case a and b.

By hypothesis, f has a derivative at every interior point. That rules out possibility (2), leaving us with interior points where $f'=0$ and with the two endpoints a and b.

If either maximum or the minimum occurs at a point c between a and b. then $f' = 0$.
If both the absolute maximum and the absolute minimum occur at the endpoints, then because $f(a) = f(b)$ it must be the case that f is a constant function with $f(x) = f(a) = f(b)$ for every $x \in [a, b]$. Therefore $f'(x) = 0$ and the point c can be taken anywhere in the interior (a, b).

Example

Show that the equation $x^3 + 3x + 1 = 0$ has exactly one real solution.

Solution

$f(x) = x^3 + 3x + 1$

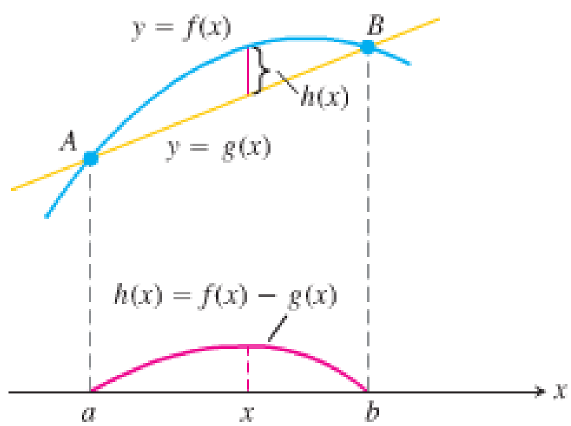
$f(-1) = -3$ and $f(0) = 1$, the Intermediate Value Theorem the equation has one real solution in the open interval $(-1, 0)$.

$f'(x) = 3x^2 + 3 > 0$ (***always positive***). Rolle's Theorem would guarantees the existence of a point $x = c$ in between them where f' was zero. Therefore, f has no more than one zero.

The Mean Value *Theorem*

Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$



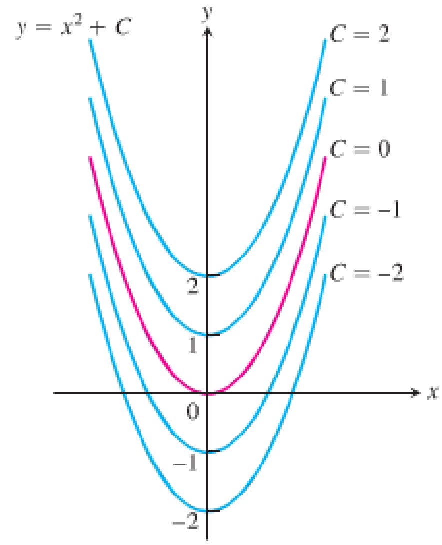
Example

The function $f(x) = x^2$ is continuous for $0 \le x \le 2$ and differentiable for $0 < x < 2$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem says that at some point c in the interval, the derivative $f'(x) = 2x$ must have the value $\frac{4-0}{2-0} = 2$. In this case we can identify c by solving the equation $2c = 2$ to get $c = 1$. However, it is not always easy to find c algebraically, even though we know it always exists.

If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

Corollary

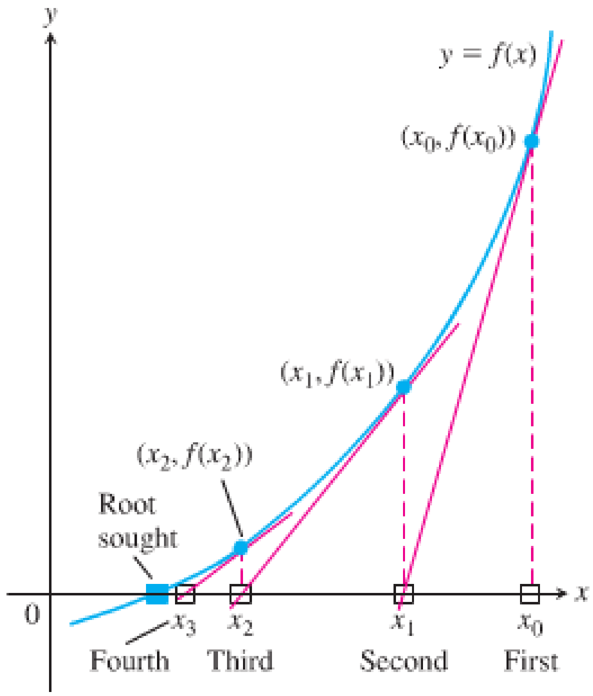
If $f'(x) = g'(x)$ at each point x of an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant function on (a, b).



Section 3.6 – Newton's Method

Procedure for *Newton's Method*

The goal of Newton's method, also called the ***Newton-Raphson*** method, for estimating a solution of an equation $f(x)=0$ is to produce a sequence of approximations that approach the solution.



We begin with the first number x_0 of the sequence. Then the function is approximated by its tangent line, and one computes the *x*-intercept of this tangent line. At each step the method approximates a zero of *f* with a zero of one of its linearization.

> Initial estimates, x_0, the method than uses the tangent curve $y=f(x)$ @ $\left(x_0, f\left(x_0\right)\right)$ to approximate the curve, calling the point x_1 where the tangent meets the *x*-axis. The number x_1 usually a better approximation to the solution that is x_0. The point x_2 where the tangent to the curve at $\left(x_1, f\left(x_1\right)\right)$ crosses the *x*-axis is the next approximation in the sequence. We continue on using each approximation to generate the next, until we are close enough to the root to stop.

The point-slope equation for the tangent to the curve at $\left(x_n, f\left(x_n\right)\right)$ is

$$y=f\left(x_n\right)+f'\left(x_n\right)\cdot\left(x-x_n\right)$$

We can find where it crosses the *x*-axis by setting $y=0$.

$$0 = f(x_n) + f'(x_n)\cdot(x - x_n) \quad \Rightarrow \quad x - x_n = -\frac{f(x_n)}{f'(x_n)}$$

$$\boxed{x = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{if } f'(x_n) \neq 0$$

Example

Find the positive root of the equation $f(x) = x^2 - 2 = 0$

Solution

$$f'(x) = 2x$$

$$\begin{aligned} x_{n+1} &= x_n - \frac{x_n^2 - 2}{2x_n} \\ &= x_n - \frac{x_n}{2} + \frac{1}{x_n} \\ &= \frac{x_n}{2} + \frac{1}{x_n} \end{aligned}$$

Example

Find the x-coordinate of the point where the curve $y = x^3 - x$ crosses the horizontal line $y = 1$.

Solution

-- $x^3 - x = 1$

$x^3 - x - 1 = 0$

$f(x) = x^3 - x - 1$

$$\begin{cases} f(1) = -1 \\ f(2) = 5 \end{cases}$$

$f'(x) = 3x^2 - 1$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	1	–1	2	1.5
1	1.5	0.875	5.75	1.347826087
2	1.347826087	0.100682173	4.449905482	1.325200399
3	1.325200399	0.002058362	4.268468292	1.324718174
4	1.324718174	0.000000924	4.264634722	1.324717957
5	1.324717957	–1.8672 E–13	4.264632999	1.324717957

The result: $\boxed{x = 1.324717957}$

Exercises *Section* 3.6 – Newton's Method

1. Use Newton's method to estimate the on real solution of $x^3+3x+1=0$. Start with $x_0=0$ and then find x_2

2. Use Newton's method to estimate the on real solution of $x^4+x-3=0$. Start with $x_0=-1$ for the left-hand zero and with $x_0=1$ for the zero on the right. Then, in each case, find x_2

3. Use Newton's method to estimate the on real solution of $2x-x^2+1=0$. Start with $x_0=0$ for the left-hand zero and with $x_0=2$ for the zero on the right. Then, in each case, find x_2

4. Use Newton's method to estimate the on real solution of $x^4-2=0$. Start with $x_0=1$ and then find x_2

Use the Newton's method to approximate the roots to ten digits of

5. $f(x)=3x^3-4x^2+1$

6. $f(x)=e^{-2x}+2e^x-6$

7. $f(x)=2x^5-6x^3-4x+2$