Section 1.4 – Exact Differential Equations

A class of equations known as exact equations for which there is also a well-defined method of solution

Theorem

Let the function M, N, M_y *and* N_x, where M_y *and* N_x are partial derivatives, be continuous in the rectangular region $R: \alpha < x < \beta, \ \gamma < y < \delta$ then

$$M(x, y) + N(x, y)y' = 0$$

Is an exact differential equation in R, if and only if

$$M_y(x, y) = N_x(x, y)$$

At each point in R. That is, there exists a function ψ satisfying

$$\psi_y(x, y) = M(x, y) \quad \text{and} \quad \psi_x(x, y) = N(x, y)$$

Iff $M_y(x, y) = N_x(x, y)$

Example

Solve the differential equation: $2x + y^2 + 2xyy' = 0$

Solution

$$\frac{\partial\psi}{\partial x} = M = 2x + y^2 \quad \Rightarrow \quad M_y = 2y$$

$$\frac{\partial\psi}{\partial y} = N = 2xy \quad \Rightarrow \quad N_x = 2y$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial\psi}{\partial x} = 2x + y^2 \quad \Rightarrow \quad \psi = \int \left(2x + y^2\right)dx = x^2 + xy^2 + h(y)$$

$$\psi_y = 2xy + h'(y) = 2xy \quad \Rightarrow \quad h'(y) = 0$$

$$\Rightarrow \ h(y) = C$$

$$\psi(x, y) = x^2 + xy^2 + C$$

$$\boxed{x^2 + xy^2 = C}$$

Example

Solve the differential equation: $y\cos x + 2xe^{y} + \left(\sin x + x^{2}e^{y} - 1\right)y' = 0$

Solution

$$M = y\cos x + 2xe^{y} = \frac{\partial\psi}{\partial x} \quad \Rightarrow \quad M_{y} = \cos x + 2xe^{y}$$

$$\frac{\partial\psi}{\partial y} = N = \sin x + x^{2}e^{y} - 1 \quad \Rightarrow \quad N_{x} = \cos x + 2xe^{y}$$

$$\Rightarrow M_{y} = N_{x}$$

$$\psi = \int\left(y\cos x + 2xe^{y}\right)dx = y\sin x + x^{2}e^{y} + h(y)$$

$$\psi_{y} = \sin x + x^{2}e^{y} + h'(y) = \sin x + x^{2}e^{y} - 1 \quad \Rightarrow \quad h'(y) = -1$$

$$\Rightarrow h(y) = -y$$

$$\psi(x, y) = y\sin x + x^{2}e^{y} - y = C$$

$$\boxed{y\sin x + x^{2}e^{y} - y = C}$$

Example

Solve the differential equation: $3xy + y^{2} + \left(x^{2} + xy\right)y' = 0$

Solution

$$M = 3xy + y^{2} = \frac{\partial\psi}{\partial x} \quad \Rightarrow \quad M_{y} = 3x + 2y$$

$$N = x^{2} + xy = \frac{\partial\psi}{\partial y} \quad \Rightarrow \quad N_{x} = 2x + y$$

$$\Rightarrow M_{y} \neq N_{x}$$

Can be solved by this procedure.

Integrating Factors

It is sometimes possible to convert a differential equation that is not exact equation by multiplying the equation by a suitable integrating factor.

Definition

An integrating factor for the differential equation $\omega = Mdx + Ndy = 0$ is a function $\mu(x,\ y)$ such that the form $\mu\omega = \mu(x,\ y)M(x,\ y)dx + \mu(x,\ y)N(x,\ y)dy$ is exact.

$$(\mu M)_y = (\mu N)_x$$

$$M\mu_y - N\mu_x + \left(M_y - N_x\right)\mu = 0$$

Assuming that μ is a function of x only, we have

$$(\mu M)_y = \mu M_y \quad \& \quad (\mu N)_x = \mu N_x + N\frac{d\mu}{dx}$$

$$\Rightarrow \quad \mu M_y = \mu N_x + N\frac{d\mu}{dx}$$

$$\boxed{\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu}$$

Example

Find an integrating factor for the equation $\left(3xy + y^2\right) + \left(x^2 + xy\right)y' = 0$

And then solve the equation.

Solution

$$M_y = \frac{\partial}{\partial y}\left(3xy + y^2\right) = 3x + 2y \qquad N_x = \frac{\partial}{\partial x}\left(x^2 + xy\right) = 2x + y$$

$$\Rightarrow M_y \neq N_x$$

$$\frac{M_y - N_x}{N} = \frac{3x + 2y - 2x - y}{x^2 + xy} = \frac{x + y}{x(x + y)} = \frac{1}{x}$$

$$\frac{d\mu}{dx} = \frac{\mu}{x} \quad \Rightarrow \quad \int \frac{d\mu}{\mu} = \int \frac{dx}{x}$$

$$\ln\mu = \ln x$$

$$\mu = x$$

$$x\left(3xy+y^2\right)+x\left(x^2+xy\right)y'=0$$

$$M_y=\frac{\partial}{\partial y}\left(3x^2y+xy^2\right)=3x^2+2xy \qquad N_x=\frac{\partial}{\partial x}\left(x^3+x^2y\right)=3x^2+2xy$$

$$\Rightarrow\ M_y=N_x$$

$$\psi=\int\left(3x^2y+xy^2\right)dx=x^3y+\frac{1}{2}x^2y^2+h(y)$$

$$\psi_y=x^3+x^2y+h'(y)=x^3+x^2y \quad \Rightarrow \quad h'(y)=0$$

$$\Rightarrow\ h(y)=C$$

$$\psi(x,\ y)=x^3y+\frac{1}{2}x^2y^2=C$$

$$\boxed{x^3y+\frac{1}{2}x^2y^2=C}$$

Bernoulli Equations

An equation of the form $y' + P(x)y = Q(x)y^n, \quad n \neq 0,\ 1$ is called a ***Bernoulli equation***.

If $n = 0 \quad \Rightarrow \quad y' + Py = Q$ First–order linear differential equation

If $n = 1 \quad \Rightarrow \quad y' + Py = Qy \ \rightarrow \ y' + (P - Q)y = 0$ Separable equation.

For $n \neq 0,\ 1$, the Bernoulli equation can be written as $y^{-n}\dfrac{dy}{dx} + Py^{1-n} = Q \qquad (1)$

Let $u = y^{1-n} \quad \Rightarrow \quad \dfrac{du}{dx} = (1-n)y^{-n}\dfrac{dy}{dx}$

$$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{du}{dx}$$

$$(1) \quad \Rightarrow \quad \frac{1}{1-n}\frac{du}{dx} + Pu = Q$$

$\boxed{u' + (1-n)Pu = (1-n)Q}$ Which is 1st–order linear differential equation.

Example

Find the general solution $y' - 4y = 2e^x\sqrt{y}$

Solution

$$\sqrt{y} = y^{1/2} \quad \Rightarrow \quad n = \frac{1}{2}$$

Let $u = y^{1-\frac{1}{2}} = y^{1/2} \quad \Rightarrow \quad y = u^2$

$$\frac{du}{dx} = \frac{1}{2}y^{-1/2}\frac{dy}{dx} \quad \Rightarrow \quad 2y^{1/2}\frac{du}{dx} = \frac{dy}{dx}$$

$$\frac{dy}{dx} - 4y = 2e^x u$$

$2u\dfrac{du}{dx} - 4u^2 = 2ue^x$ ***Divide by*** $2u$

$$u' - 2u = e^x$$

$$e^{\int -2dx} = e^{-2x}$$

$$\int e^x e^{-2x}dx = \int e^{-x}dx = -e^{-x}$$

$$u = \frac{1}{e^{-2x}}\left(-e^{-x} + C\right)$$

$$y^{1/2} = -e^x + Ce^{2x}$$

$$\boxed{y = \left(Ce^{2x} - e^x\right)^2}$$

Example

Find the general solution $xy' + y = 3x^3y^2$

Solution

$$y' + \frac{1}{x}y = 3x^2y^2$$

Let $u = y^{1-2} = y^{-1} \quad \Rightarrow \quad y = \frac{1}{u}$

$$\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx} \quad \Rightarrow \quad y' = -y^2u' = -\frac{1}{u^2}u'$$

$$-\frac{1}{u^2}u' + \frac{1}{x}\frac{1}{u} = 3x^2\frac{1}{u^2}$$ ***Multiply both sides by*** $-u^2$

$$u' - \frac{1}{x}u = -3x^2$$

$$e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

$$\int -3x^2x^{-1}dx = -3\int xdx = -\frac{3}{2}x^2$$

$$u = x\left(-\frac{3}{2}x^2 + C_1\right)$$

$$\frac{1}{y} = \frac{-3x^3 + 2C_1x}{2}$$

$$\boxed{y = \frac{2}{Cx - 3x^3}}$$

Homogeneous Equations $\frac{dy}{dx} = f(x, y)$

The form of a homogeneous equation suggests that it may be simplified by using a variable denoted by $'v'$, to represent the ratio of y to x. This

$$y = xv \quad \Rightarrow \quad \frac{dy}{dx} = F(v)$$

Let assume that v is a function of x, then

$$\frac{dy}{dx} = x\frac{dv}{dx} + v \quad \text{or} \quad y' = xv' + v$$

The most significant fact about this equation is that the variables x & v can always be ***separated***, regardless of the form of the function F.

$$\frac{dx}{x} = \frac{dv}{F(v) - v}$$

Solving this equation and then replacing v by $\frac{y}{x}$ gives the solution of the original equation.

Example

Solve the differential equation $\qquad \frac{dy}{dx} = \frac{y^2 + 2xy}{x^2}$

Solution

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + 2\frac{y}{x} = v^2 + 2v$$

$$x\frac{dv}{dx} + v = v^2 + 2v \qquad \Rightarrow x\frac{dv}{dx} = v^2 + v$$

$$xdv = v(v+1)dx$$

$$\int \frac{dx}{x} = \int \frac{dv}{v(v+1)}$$

$$\int \frac{dx}{x} = \int \left(\frac{1}{v} - \frac{1}{v+1}\right)dv$$

$$\ln x + \ln C = \ln v - \ln(v+1)$$

$$\ln(Cx) = \ln\frac{v}{v+1}$$

$$Cx = \frac{v}{v+1} = \frac{\frac{y}{x}}{\frac{y}{x}+1} = \frac{y}{y+x} \quad \Rightarrow Cxy + Cx^2 = y$$

$$Cx^2 = y - Cxy$$

$$\boxed{y = \frac{Cx^2}{1 - Cx}}$$

Example

Find the general solution $\qquad y' = \dfrac{x^2 e^{y/x} + y^2}{xy}$

Solution

Let $y = xv \quad \Rightarrow \quad y' = v + xv'$

$$v + xv' = \frac{x^2 e^{xv/x} + (xv)^2}{x(xv)}$$

$$xv' = \frac{x^2 e^v + x^2 v^2}{x^2 v} - v$$

$$x\frac{dv}{dx} = \frac{e^v + v^2}{v} - v$$

$$x\frac{dv}{dx} = \frac{e^v}{v}$$

$$\int \frac{v}{e^v} dv = \int \frac{dx}{x}$$

$$-ve^{-v} - e^{-v} = \ln x + C$$

$$-e^{-v}(v+1) = \ln x + C$$

$$-e^{-y/x}\left(\frac{y}{x} + 1\right) = \ln x + C$$

$$\boxed{y + x = -xe^{y/x}(\ln x + C)}$$

Exercises *Section* 1.4 – Exact Differential Equations

Solve the differential equation

1. $(2x+y)\,dx+(x-6y)\,dy=0$

2. $(2x+3)\,dx+(2y-2)\,dy=0$

3. $(1-y\sin x)+(\cos x)\,y'=0$

4. $\dfrac{dy}{dx}=-\dfrac{ax+by}{bx+cy}$

5. $\dfrac{dy}{dx}=\dfrac{3x^2+y}{3y^2-x}$

6. $2xydx+\left(x^2-1\right)dy=0$

7. $y'=\dfrac{x^2+y^2}{2xy}$

8. $2xyy'=x^2+2y^2$

9. $xy'=y+2\sqrt{xy}$

10. $xy^2y'=x^3+y^3$

11. $x^2y'=xy+x^2e^{y/x}$

12. $x^2y'=xy+y^2$

13. $xyy'=x^2+3y^2$

14. $\left(x^2-y^2\right)y'=2xy$

15. $xyy'=y^2+x\sqrt{4x^2+y^2}$

16. $xy'=y+\sqrt{x^2+y^2}$

17. $y^2y'+2xy^3=6x$

18. $x^2y'+2xy=5y^4$

19. $2xy'+y^3e^{-2x}=2xy$

20. $y^2\left(xy'+y\right)\left(1+x^4\right)^{1/2}=x$

21. $3y^2y'+y^3=e^{-x}$

22. $3xy^2y'=3x^4+y^3$

23. $xe^yy'=2\left(e^y+x^3e^{2x}\right)$

24. $(2x\sin y\cos y)\,y'=4x^2+\sin^2 y$

25. $\left(x+e^y\right)y'=xe^{-y}-1$

26. $\left(x^2+y^2\right)dx+\left(x^2-xy\right)dy=0$

27. $x\dfrac{dy}{dx}+y=x^2y^2$

28. $\left(3x^2-2xy+2\right)+\left(6y^2-x^2+3\right)y'=0$

29. $\left(e^x\sin y-2y\sin x\right)dx+\left(e^x\cos y+2\cos x\right)dy=0$

30. $\left(\dfrac{y}{x}+6x\right)dx+(\ln x-2)\,dy=0, \qquad x>0$

31. $\left(e^{2y}-y\cos x\right)dx+\left(2xe^{2y}x\cos xy+2y\right)dy=0$

32. $\dfrac{xdx}{\left(x^2+y^2\right)^{3/2}}+\dfrac{ydy}{\left(x^2+y^2\right)^{3/2}}=0$

33. $(2x-1)\,dx+(3y+7)\,dy=0$

34. $(5x+4y)\,dx+\left(4x-8y^3\right)dy=0$

35. $(\sin y-y\sin x)\,dx+(\cos x+x\cos y-y)\,dy=0$

36. $\left(2xy^2-3\right)dx+\left(2x^2y+4\right)dy=0$

37. $\left(1+\ln x+\dfrac{y}{x}\right)dx-(1-\ln x)\,dy=0$

38. $\left(x-y^3+y^2\sin x\right)dx-\left(3xy^2+2y\cos x\right)dy=0$

39. $\left(x^3+y^3\right)dx+3xy^2dy=0$

40. $\left(3x^2y+e^y\right)dx+\left(x^3+xe^y-2y\right)dy=0$

41. $xdy+\left(y-2xe^x-6x^2\right)dx=0$

42. $\left(1-\dfrac{3}{y}+x\right)dy+\left(y-\dfrac{3}{x}+1\right)dx=0$

43. $\left(x^2y^3-\dfrac{1}{1+9x^2}\right)\dfrac{dx}{dy}+x^3y^2=0$

44. $(5y-2x)\,y'-2y=0$

45. $(x-y)\,dx-xdy=0$

46. $(x+y)\,dx + x\,dy = 0$

47. $\dfrac{dy}{dx} = -\dfrac{2xy^2+1}{2x^2y}$

48. $\left(1+e^x y + xe^x y\right)dx + \left(xe^x + 2\right)dy = 0$

49. $\left(2xy^3+1\right)dx + \left(3x^2y^2 - \frac{1}{y}\right)dy = 0$

50. $(2x+y)\,dx + (x-2y)\,dy = 0$

51. $e^x(y-x)\,dx + \left(1+e^x\right)dy = 0$

52. $\left(ye^{xy} - \dfrac{1}{y}\right)dx + \left(xe^{xy} + \dfrac{x}{y^2}\right)dy = 0$

53. $(\tan x - \sin x \sin y)\,dx + (\cos x \cos y)\,dy = 0$

54. $\left(2x^3 - xy^2 - 2y + 3\right)dx - \left(x^2y + 2x\right)dy = 0$

55. $(x + \sin y)\,dx + (x\cos y - 2y)\,dy = 0$

56. $\left(x + \dfrac{1}{\sqrt{y^2-x^2}}\right)dx + \left(1 - \dfrac{1}{y\sqrt{y^2-x^2}}\right)dy = 0$

57. $\left(2x + y^2 - \cos(x+y)\right)dx + \left(2xy - \cos(x+y) - e^y\right)dy = 0$

58. $\left(\dfrac{2}{\sqrt{1-x^2}} + y\cos(xy)\right)dx + \left(x\cos(xy) - y^{-1/3}\right)dy = 0$

59. $\left(2x + y\cos(xy)\right)dx + \left(x\cos(xy) - 2y\right)dy = 0$

60. $\left(e^x \sin y - 3x^2\right)dx + \left(e^x \cos y + \frac{1}{3}y^{-2/3}\right)dy = 0$

61. $\left(2y\sin x\cos x - y + 2y^2e^{xy^2}\right)dx = \left(x - \sin^2 x - 4xye^{xy^2}\right)dy$

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

62. $x^2y^3 + x\left(1+y^2\right)y' = 0,\qquad \mu(x,\,y) = \dfrac{1}{xy^3}$

63. $y^2 - xy + \left(x^2\right)y' = 0,\qquad \mu(x,\,y) = \dfrac{1}{xy^2}$

64. $x^2y^3 - y + x\left(1 + x^2y^2\right)y' = 0,\qquad \mu(x,\,y) = \dfrac{1}{xy}$

65. $\left(\dfrac{\sin y}{y} - 2e^{-x}\sin x\right)dx + \left(\dfrac{\cos y + 2e^{-x}\cos x}{y}\right)dy = 0,\qquad \mu(x,\,y) = ye^x$

66. $(x+2)\sin y\,dx + x\cos y\,dy = 0,\qquad \mu(x,\,y) = xe^x$

67. $\left(x^2 + y^2 - x\right)dx - y\,dy = 0,\qquad \mu(x,\,y) = \dfrac{1}{x^2+y^2}$

68. $(2y - 6x)\,dx + \left(3x - 4x^2y^{-1}\right)dy = 0,\qquad \mu(x,\,y) = xy^2$

Find the general solution of each homogenous equation

69. $\left(x^2+y^2\right)dx-2xydy=0$

70. $(x+y)\,dx+(y-x)\,dy=0$

71. $\dfrac{dy}{dx}=\dfrac{y\left(x^2+y^2\right)}{xy^2-2x^3}$

Find an integrating factor and solve the given equation

72. $\left(3x^2y+2xy+y^3\right)dx+\left(x^2+y^2\right)dy=0$

73. $dx+\left(\dfrac{x}{y}-\sin y\right)dy=0$

74. $e^x dx+\left(e^x\cot y+2y\csc y\right)dy=0$

75. $\left(3x+\dfrac{6}{y}\right)dx+\left(\dfrac{x^2}{y}+3\dfrac{y}{x}\right)dy=0$

76. $\left(x+3x^3\sin y\right)dx+\left(x^4\cos y\right)dy=0$

77. $\left(2x^2+y\right)dx+\left(x^2y-x\right)dy=0$

78. $\left(3x^2+y\right)dx+\left(x^2y-x\right)dy=0$

79. $\left(2y^2+2y+4x^2\right)dx+(2xy+x)\,dy=0$

80. $\left(x^4-x+y\right)dx-xdy=0$

81. $(2xy)\,dx+\left(y^2-3x^2\right)dy=0$

Solve the given initial-value problem

82. $\dfrac{dy}{dx}=\dfrac{xy^2-\cos x\sin x}{y\left(1-x^2\right)},\quad y(0)=2$

83. $(x+y)^2\,dx+\left(2xy+x^2-1\right)dy,\quad y(1)=1$

84. $\left(e^x+y\right)dx+\left(2+x+ye^y\right)dy,\quad y(0)=1$

85. $(2x-y)\,dx+(2y-x)\,dy,\quad y(1)=3$

86. $\left(9x^2+y-1\right)dx-(4y-x)\,dy,\quad y(1)=0$

87. $\left(x+y^3\right)y'+y+x^3=0,\quad y(0)=-2$

88. $y'=\left(3x^2+1\right)\left(y^2+1\right),\quad y(0)=1$

89. $\left(y^3+\cos t\right)y'=2+y\sin t,\quad y(0)=-1$

90. $\left(y^3-t^3\right)y'=3t^2y+1,\quad y(-2)=-1$

91. $\dfrac{dy}{dx}=(-2x+y)^2-7,\quad y(0)=0$

92. $(2y-x)\,y'-y+2x=0,\quad y(1)=0$

93. $\left(e^{2y}+t^2y\right)y'+ty^2+\cos t=0,\quad y\left(\frac{\pi}{2}\right)=0$

94. $y'=-\dfrac{y\cos(ty)+1}{t\cos(ty)+2ye^{y^2}},\quad y(\pi)=0$

95. $\left(2ty+\dfrac{1}{y}\right)y'+y^2=1,\quad y(1)=1$

96. $\left(ye^x+1\right)dx+\left(e^x-1\right)dy=0\quad y(1)=1$

97. $2xy^2+4=2\left(3-x^2y\right)y'\quad y(-1)=8$

98. $y'+\dfrac{4}{x}y=x^3y^2\quad y(2)=-1$

99. $y'=5y+e^{-2x}y^{-2}\quad y(0)=2$

100. $6y'-2y=xy^4\quad y(0)=-2$

101. $y'+\dfrac{y}{x}-\sqrt{y}=0\quad y(1)=0$

102. $xyy'+4x^2+y^2=0\quad y(2)=-7$

103. $xy'=y(\ln x-\ln y)\quad y(1)=4$

104. $y'-(4x-y+1)^2=0\quad y(0)=2$

105. $\left(e^{t+y}+2y\right)y'+\left(e^{t+y}+3t^2\right)=0,\quad y(0)=0$

106. $(4y+2x-5)\,dx+(6y+4x-1)\,dy,\quad y(-1)=2$

107. $\left(ye^{xy}-\frac{1}{y}\right)dx+\left(xe^{xy}+\frac{x}{y^2}\right)dy=0\quad y(1)=1$

108. $(2y\ln t-t\sin y)\,y'+\frac{1}{t}y^2+\cos y=0,\quad y(2)=0$

109. $(\tan y-2)\,dx+\left(x\sec^2 y+\frac{1}{y}\right)dy=0\quad y(0)=1$

110. $2xy-9x^2+\left(2y+x^2+1\right)\frac{dy}{dx}=0\quad y(0)=-3$

111. $\frac{2t}{t^2+1}y-2t+\left(2-\ln\left(t^2+1\right)\right)\frac{dy}{dt}=0\quad y(5)=0$

112. $3y^3e^{3xy}-1+\left(2ye^{3xy}+3xy^2e^{3xy}\right)y'=0\quad y(0)=1$

113. $2xydx+\left(1+x^2\right)dy=0\ ;\quad y(2)=-5$

114. $\frac{dy}{dx}=-\frac{2x\cos y+3x^2y}{x^3-x^2\sin y-y}\ ;\quad y(0)=2$

Find an integrating factor of the form $x^n y^m$ and solve the equation

115. $\left(2y^2-6xy\right)dx+\left(3xy-4x^2\right)dy=0$

116. $(12+5xy)\,dx+\left(6xy^{-1}+3x^2\right)dy=0$

117. $\left(3y+4xy^2\right)dx+\left(2x+3x^2y\right)dy=0$

Find the general solution by using Bernoulli

118. $\frac{dy}{dx}-5y=-\frac{5}{2}xy^3$

119. $\frac{dy}{dx}+\frac{y}{x}=x^2y^2$

120. $\frac{dy}{dx}-y=e^{2x}y^3$

121. $\frac{dy}{dx}+\frac{y}{x-2}=5(x-2)\,y^{1/2}$

122. $\frac{dy}{dx}+y=e^xy^{-2}$

123. $\frac{dy}{dx}+y^3x+y=0$

Find the general solution by using homogeneous equations.

124. $\left(xy+y^2\right)dx-x^2dy=0$

125. $\left(x^2+y^2\right)dx+2xydy=0$

126. $\left(y^2-xy\right)dx+x^2dy=0$

127. $\frac{dy}{d\theta}=\frac{\theta\sec\left(\frac{y}{\theta}\right)+y}{\theta}$

128. $\frac{dy}{dx}=\frac{y(\ln y-\ln x+1)}{x}$