Section 3.7 – Linear Dependence and Independence

There are n columns in an m by n matrix, and each column has m components. But the true ***dimension*** of the column space is not necessarily m or n. The dimension is measured by counting ***independent columns***.

- **Independent vectors** (*not too many*)
- **Spanning a space** (*not too few*)

Linear Independence (LI)

The column of A are ***linearly independent*** when the only solution to $Ax = 0$ is $x = 0$. ***No other combination*** Ax ***of the columns gives the zero vector***.

Definitions

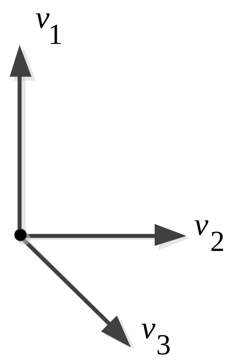
- A set of two or more vectors is ***linearly dependent*** if one vector in the set is a linear combination of the others. A set of one vector is ***linearly dependent*** if that one vector is the zero vector.
$$\vec{0} = 0v_1 + 0v_2 + \cdots + 0v_k$$
- The sequence of vectors v_1, v_2, v_3 is ***linearly independent*** if the only combination that gives the zero vector is $0v_1 + 0v_2 + \cdots + 0v_k$. Thus linear independence means that:
$$x_1v_1 + x_2v_2 + \cdots + x_kv_k = 0 \text{ only happens when all } x\text{'s are zero.}$$
- A (nonempty) set of vectors is ***linearly independent*** if it is not linearly dependent.
- If three vectors w_1, w_2, w_3 are in the same plane, they are dependent.

Theorem

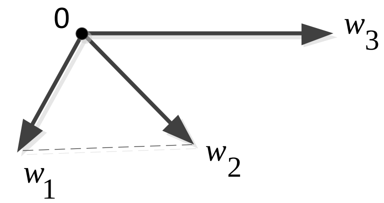
A set S with two or more vectors $S = \{v_1, \cdots, v_k\}$ is

a) Linearly dependent *iff* at least one of the vectors in S is expressible as a linear combination of the other vectors in S. There are numbers $c_1, \ldots, c_k$ at least one of which is nonzero, such that
$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$$

b) Linearly independent *iff* no vector in S is expressible as a linear combination of the other vectors in S.



Independent vectors v_1, v_2, v_3



Dependent vectors w_1, w_2, w_3

The combination $w_1 - w_2 + w_3$ is (0, 0, 0)

Example

a) The vectors (1, 0) and (0, 1) are ***independent***.

b) The vectors (1, 1) and (1, 0.0001) are ***independent***.

c) The vectors (1, 1) and (2, 2) are ***dependent***.

d) The vectors (1, 1) and (0, 0) are ***dependent***.

Theorem

a) A finite set that contains $\mathbf{0}$ is linearly dependent.

b) A set with exactly one vector is linearly independent if and only if that vector is not $\mathbf{0}$.

c) A set with exactly two vectors is linearly independent *iff* neither vector is a scalar multiple of the other.

Theorem

Let S be a set $\boldsymbol{k}$ vectors in $\boldsymbol{R}^n$, then if $k > n$, S is ***linearly dependent***.

Example

v_1, v_2, v_3 are 3 vectors in $\boldsymbol{R}^2 \Rightarrow$ Linearly dependent.

Example

Determine whether the vectors $v_1 = (1, -2, 3)$ $v_2 = (5, 6, -1)$ $v_3 = (3, 2, 1)$ are linearly dependent or linearly independent in $\boldsymbol{R}^3$

Solution

$$k_1 v_1 + k_2 v_2 + k_3 v_3 = \mathbf{0}$$

$$k_1 (1, -2, 3) + k_2 (5, 6, -1) + k_3 (3, 2, 1) = (0, 0, 0)$$

$$\rightarrow \begin{cases} k_1 + 5k_2 + 3k_3 = 0 \\ -2k_1 + 6k_2 + 2k_3 = 0 \\ 3k_1 - k_2 + k_3 = 0 \end{cases}$$

$$\begin{bmatrix} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} k_1 + \frac{1}{2}k_3 = 0 \\ k_2 + \frac{1}{2}k_3 = 0 \end{array}$$

Solve the system equations: $k_1 = -\frac{1}{2}t, \quad k_2 = -\frac{1}{2}t, \quad k_3 = t$

This shows that the system has nontrivial solutions and hence that the vectors are linearly dependent.

2nd method to determine the linearly is to compute the determinant of the coefficient matrix

$$A = \begin{pmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{pmatrix}$$

$|A| = 0$ Which has nontrivial solutions and the vectors are linearly dependent.

Example

Determine whether the vectors are linearly dependent or linearly independent in $\boldsymbol{R}^4$

$$v_1 = (1,2,2,-1) \quad v_2 = (4,9,9,-4) \quad v_3 = (5,8,9,-5)$$

Solution

$$k_1 v_1 + k_2 v_2 + k_3 v_3 = \mathbf{0}$$

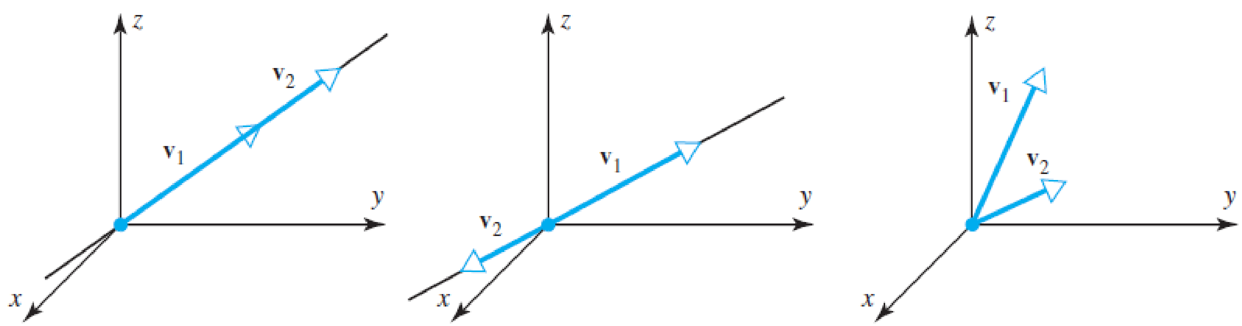
$$k_1 (1,2,2,-1) + k_2 (4,9,9,-4) + k_3 (5,8,9,-5) = (0,0,0)$$

Which yields the homogeneous linear system

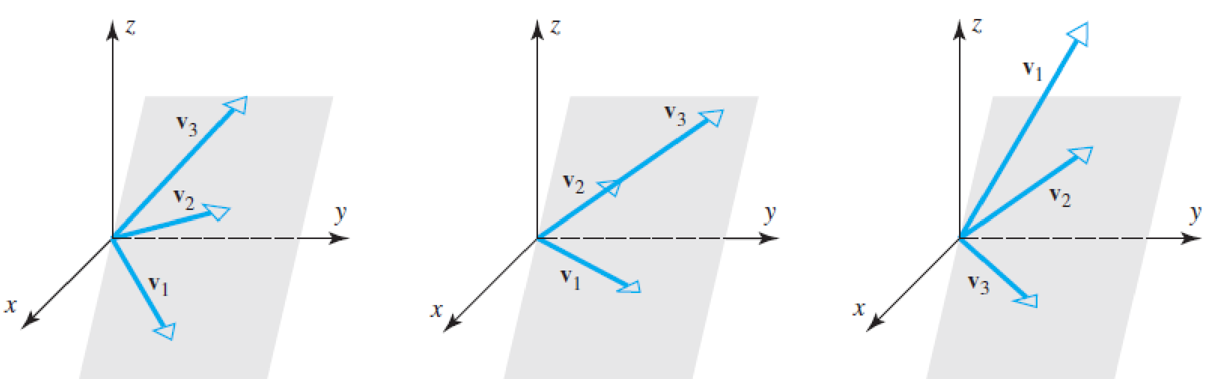
$$\rightarrow \begin{cases} k_1 + 4k_2 + 5k_3 = 0 \\ 2k_1 + 9k_2 + 8k_3 = 0 \\ 2k_1 + 9\,k_2 + 9k_3 = 0 \\ -k_1 - 4\,k_2 - 5k_3 = 0 \end{cases}$$

Solve the system equations: $k_1 = 0, \quad k_2 = 0, \quad k_3 = 0$ has a trivial solution.

The vectors v_1, v_2, *and* v_3 are linearly independent.



(a) Linearly dependent (b) Linearly dependent (c) Linearly independent



(a) Linearly dependent (b) Linearly dependent (c) Linearly independent

Linear independence of Functions

Definition

If $\mathbf{f}_1 = f_1(x),\ \mathbf{f}_2 = f_2(x),\ \ldots,\ \mathbf{f}_n = f_n(x)$ are functions that are *n* - 1 times differentiable on the interval (–∞, ∞), the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & & f_n(x) \\ f_1{}'(x) & f_2{}'(x) & & f_n{}'(x) \\ & & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the ***Wronskian*** of $f_1, f_2, \ldots, f_n$

Example

Use the Wronskian to show that $\mathbf{f}_1 = x,\quad \mathbf{f}_2 = \sin x$ are linearly independence

Solution

The Wronskian is

$$W(x) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x \neq 0$$

This function is not identically zero. Thus the functions are linearly independent.

Example

Use the Wronskian to show that $\mathbf{f}_1 = 1,\quad \mathbf{f}_2 = e^x,\quad \mathbf{f}_3 = e^{2x}$ are linearly independence

Solution

The Wronskian is

$$W(x) = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix} = e^x 4e^{2x} - 2e^{2x}e^x = 2e^{3x} \neq 0$$

Thus the functions are linearly independent.

Exercises Section 3.7 – Linear Dependence and Independence

1. Given three independent vectors w_1, w_2, w_3. Take combinations of those vectors to produce v_1, v_2, v_3. Write the combinations in a matrix form as $V = WM$.

$$\begin{aligned} v_1 &= w_1 + w_2 \\ v_2 &= w_1 + 2w_2 + w_3 \\ v_1 &= \quad\; w_2 + cw_3 \end{aligned} \quad \text{which is } \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & c \end{bmatrix}$$

What is the test on a matrix **V** to see if its columns are linearly independent?

If $c \neq 1$ show that v_1, v_2, v_3 are linearly independent.

If $c = 1$ show that $v's$ are linearly *dependent*.

2. Find the largest possible number of independent vectors among

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad v_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad v_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad v_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

3. Show that v_1, v_2, v_3 are independent but v_1, v_2, v_3, v_4 are dependent:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad v_4 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

Solve either $c_1v_1 + c_2v_2 + c_3v_3 = 0$ *or* $Ax = 0$. The v's go in the columns of $\boldsymbol{A}$.

4. Decide the dependence or independence of

a) The vectors (1, 3, 2) and (2, 1, 3) and (3, 2, 1).

b) The vectors $(1, \ -3, \ 2)$ and $(2, \ 1, \ -3)$ and $(-3, \ 2, \ 1)$.

5. Find two independent vectors on the plane $x + 2y - 3z - t = 0$ in $\boldsymbol{R}^4$. Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

6. Determine whether the vectors are linearly dependent or linearly independent in $\boldsymbol{R}^3$

a) (4, –1, 2), (–4, 10, 2)

b) (8, –1, 3), (4, 0, 1)

c) (–3, 0, 4), (5, –1, 2), (1, 1, 3)

d) (–2, 0, 1), (3, 2, 5), (6, –1, 1), (7, 0, –2)

7. Determine whether the vectors are linearly dependent or linearly independent in $\boldsymbol{R}^4$

a) (3, 8, 7, –3), (1, 5, 3, –1), (2, –1, 2, 6), (1, 4, 0, 3)
b) (0, 0, 2, 2), (3, 3, 0, 0), (1, 1, 0, –1)
c) (0, 3, –3, –6), (–2, 0, 0, –6), (0, –4, –2, –2), (0, –8, 4, –4)
d) (3, 0, –3, 6), (0, 2, 3, 1), (0, –2, –2, 0), (–2, 1, 2, 1)

8. *a*) Show that the three vectors $v_1 = (1,2,3,4)$ $v_2 = (0,1,0,-1)$ $v_3 = (1,3,3,3)$ form a linearly dependent set in $\boldsymbol{R}^4$.

b) Express each vector in part (*a*) as a linear combination of the other two.

9. For which real values of λ do the following vectors form a linearly dependent set in $\boldsymbol{R}^3$

$$v_1 = \left(\lambda, -\tfrac{1}{2}, -\tfrac{1}{2}\right) \quad v_2 = \left(-\tfrac{1}{2}, \lambda, -\tfrac{1}{2}\right) \quad v_3 = \left(-\tfrac{1}{2}, -\tfrac{1}{2}, \lambda\right)$$

10. Show that if $S = \{v_1,\ v_2,\ \ldots,\ v_n\}$ is a linearly independent set of vectors, then so is every nonempty subset of S.

11. Show that if $S = \{v_1,\ v_2,\ \ldots,\ v_r\}$ is a linearly dependent set of vectors in a vector space *V*, and if $v_{r+1},\ \ldots,\ v_n$ are vectors in *V* that are not in *S*, then $\{v_1,\ v_2,\ \ldots,\ v_r,\ v_{r+1},\ \ldots,\ v_n\}$ is also linearly dependent.

12. Show that $\{v_1,\ v_2\}$ is linearly independent and v_3 does not lie in span $\{v_1,\ v_2\}$, then $\{v_1,\ v_2,\ v_3\}$ is a linearly independent.

13. By using the appropriate identities, where required, determine $F(-\infty,\ \infty)$ are linearly dependent.

a) $6,\ 3\sin^2 x,\ 2\cos^2 x$
b) $x,\ \cos x$
c) $1,\ \sin x,\ \sin 2x$
d) $(3-x)^2,\ x^2-6x,\ 5$
e) $\cos 2x,\ \sin^2 x,\ \cos^2 x$

14. $f_1(x) = \sin x,\quad f_2(x) = \cos x$ are linearly independent in $F(-\infty,\ \infty)$ because neither function is a scalar multiple of the other. Confirm the linear independence using Wroński's test.

15. Use the Wronskian to show that $f_1(x) = \sin x,\quad f_2(x) = \cos x,\quad f_3(x) = x\cos x$ span a three-dimensional subspace of $F(-\infty,\ \infty)$

16. Show by inspection that the vectors are linearly dependent.

$\boldsymbol{v}_1(4,\ -1,\ 3),\ \ \boldsymbol{v}_2(2,\ 3,\ -1),\ \ \boldsymbol{v}_3(-1,\ 2,\ -1),\ \ \boldsymbol{v}_4(5,\ 2,\ 3),\ \ in\ \mathbb{R}^3$

17. Determine if the given vectors are linearly dependent or independent, (any method)

a) $(2,\ -1,\ 3),\quad (3,\ 4,\ 1),\quad (2,\ -3,\ 4),\quad in\ \mathbb{R}^3.$

b) $(1,\ 0,\ 0,\ 0),\quad (1,\ 1,\ 0,\ 0),\quad (1,\ 1,\ 1,\ 0),\quad (1,\ 1,\ 1,\ 1),\quad in\ \mathbb{R}^4.$

c) $A_1\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix},\quad A_2\begin{bmatrix} -2 & 4 \\ 1 & 0 \end{bmatrix},\quad A_3\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix},\quad in\ M_{22}$

18. Suppose that the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2,\ and\ \boldsymbol{u}_3$ are linearly dependent. Are the vectors $\boldsymbol{v}_1 = \boldsymbol{u}_1 + \boldsymbol{u}_2$, $\boldsymbol{v}_2 = \boldsymbol{u}_1 + \boldsymbol{u}_3$, and $\boldsymbol{v}_3 = \boldsymbol{u}_2 + \boldsymbol{u}_3$ also linearly dependent?

(***Hint***: Assume that $a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3 = 0$, and see what the $a_i{}'s$ can be.)