

$$dx \quad \partial y \quad y'' + p(t)y' + q(t)y = g(t) \quad y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \quad \mathcal{L}^{-1}\{Y(s)\} \quad \mathcal{L}\{y(t)\}$$

Contents

PREFACE

Lecture R – Introduction to Differential Equation 1

Section R.1 – Derivative 1

- R.1-1 Constant Rule 1
- R.1-2 Power Rule 1
- R.1-3 Constant Times a Function 1
- R.1-4 The Product Rule 2
- R.1-5 Quotient Rule 3
- R.1-6 The Chain Rule 4
- R.1-7 The General Power Rule 4
- R.1-8 Derivatives of Trigonometric Functions 5
- R.1-9 Derivative of the Natural Exponential Function 6
- R.1-10 Derivative of \ln 7
- R.1-11 Derivative of $\log_a x$ 7
- R.1-12 Other Bases and Differentiation 7
- R.1-13 Formula 8

Exercises 9

Section R.2 – Integration 11

- R.2-1 Definition of Antiderivative 11
- R.2-2 Notation For Antiderivatives 11
- R.2-3 The General Power Rule 12
- R.2-4 Exponential Rule 13
- R.2-5 Log Rule 13
- R.2-6 Integration by Parts 14
- R.2-7 Particular Solutions 16
- R.2-8 Fundamental Theorem of Calculus 18

Exercises 19

Lecture 1 – First Order Equations 21

Section 1.1 – Differential Equations & Solutions 21

1.1-1 Ordinary Differential Equations 21

1.1-2 Definition 22

1.1-3 Solutions 22

Exercises 23

Section 1.2 – Solutions to Separable Equations 25

1.2-1 Separable Equation 25

1.2-2 Definition 25

1.2-3 Newton's Law of Cooling 27

1.2-4 Losing a solution 28

1.2-5 Implicitly Defined Solutions 29

Exercises 32

Section 1.3 – Models of Motions 38

1.3-1 Law of mechanics – Newton's 2nd Law 38

1.3-2 Universal Law of gravitation 38

1.3-3 Air Resistance 39

1.3-4 Finding the displacement 41

1.3-5 Torricelli's Law 43

Exercises 45

Section 1.4 – Linear Equations 59

1.4-1 Solution of the homogenous equation 59

1.4-2 Solving a linear first-order Equation (Properties) 60

1.4-3 Solution of the Nonhomogeneous Equation 62

Exercises 65

Section 1.5 – Mixing Problems 71

Example 1 71

Example 2 73

Exercises 75

Section 1.6 – Exact Differential Equations 85

1.6-1 Theorem 85

1.6-2 Integrating Factors 87

1.6-3 Definition 87

1.6-4 Bernoulli Equations 89

1.6-5 Homogeneous Equations 91

1.6-6 Equations with Linear Coefficients 93

Exercises 95**Section 1.7 – Modeling Population Growth** 100

- 1.7-1 Modeling Population Growth 100
- 1.7-2 Malthusian Method 100
- 1.7-3 Logistic Model of Growth 102
- 1.7-4 Pollution 105

Exercises 106**Section 1.8 – Basic Electrical Circuit** 111

- 1.8-1 Resistor 111
- 1.8-2 Inductor 112
- 1.8-3 Capacitance 113
- 1.8-4 RLC circuit 114
- 1.8-5 Summary 116
- 1.8-6 Communication Channel 116
- 1.8-7 Example 118

Exercises 120**Section 1.9 – Existence and Uniqueness of Solutions** 127

- 1.9-1 Existence of Solutions 127
- 1.9-2 Theorem: Existence of Solutions 128
- 1.9-3 Interval of Existence of a Solution 128
- 1.9-4 Theorem: Existence of a Unique Solution 129
- 1.9-5 Mathematics & Theorems 129

Exercises 131**Section 1.10 – Autonomous Equations and Stability** 132

- 1.10-1 Definition 132
- 1.10-2 The Direction Fields 132
- 1.10-3 Autonomous 1st order DE 136
- 1.10-4 Equilibrium Points & Solutions 136

Exercises 138

Lecture 2 – Second & Higher Order Equations 141

Section 2.1 – Definitions of Second and Higher Order Equations 141

- 2.1-1 Newton's - Hooke's Law for Springs 141
- 2.1-2 Proposition 142
- 2.1-3 Definition 142
- 2.1-4 Definition 142
- 2.1-5 Wronskian 143
- 2.1-6 Theorem 143
- 2.1-7 System Equations 144
- 2.1-8 Second-Order Equations and Planar Systems 144

Exercises 146

Section 2.2 – Linear, Homogeneous Equations with Constant Coefficients 149

- 2.2-1 Introduction 149
- 2.2-2 Case 1: Distinct Real Root 150
- 2.2-3 Proposition 150
- 2.2-4 Case 2: Complex Roots 151
- 2.2-5 Proposition 151
- 2.2-6 Case 3: Repeated Roots 153
- 2.2-7 Proposition 154
- 2.2-8 Higher-Order Equations 155
- 2.2-9 Summary 156

Exercises 157

Section 2.3 – Harmonic Motion 161

- 2.3-1 Hooke's Law 161
- 2.3-2 Newton's Second Law 161
- 2.3-3 Damped, Free Vibrations 161
- 2.3-4 Linear Constant-Coefficient Models 164
- 2.3-5 Simple Harmonic Motion 164
- 2.3-6 Amplitude and Phase Angle 165
- 2.3-7 Damped Harmonic Motion 167
 - TABLE **A**: Relationships between the variables of the analog system components 172
 - TABLE **B**: Analogous between electrical and mechanical systems 172
- 2.3-8 Pendulum 173

Exercises 175

Section 2.4 – Inhomogeneous Equations; the Method of Undetermined Coefficients 184

- 2.4-1 Theorem 184
- 2.4-2 Theorem 184
- 2.4-3 Forcing Term 184
- 2.4-4 Trigonometric Forcing Term 185

2.4-5	Complex Method	186
2.4-6	Polynomial Forcing Term	187
2.4-7	Exceptional Cases	187
2.4-8	Summary	188
Exercises		190

Section 2.5 – Variation of Parameters 195

2.5-1	General Case	195
2.5-2	Higher-Order Equations	199
Exercises		200

Section 2.6 – Forced Harmonic Motion 202

2.6-1	Forced undamped harmonic motion	202
2.6-2	Case 1 $\omega \neq \omega_0$	202
2.6-3	Case 2 $\omega = \omega_0$	204
2.6-4	Forced Damped Harmonic Motion	205
2.6-5	Underdamped Case: $c < \omega_0$	207
2.6-6	Transient and Steady-State	209
Exercises		211

Section 2.7 – Euler's & Runge-Kutta Methods 221

2.7-1	Euler's method	221
2.7-2	Runge-Kutta Methods	223
2.7-3	The Second-Order Runge-Kutta Method	223
2.7-4	Fourth-Order Runge-Kutta Method	227
Exercises		229

Lecture 3 – Laplace and Linear Systems 231

Section 3.1 – Definition of the Laplace Transform 231

3.1-1	Definition	231
Exercises		235

Section 3.2 – Basic Properties of the Laplace Transform 236

3.2-1	The Laplace Transform of Derivatives	236
3.2-2	Proposition	236
3.2-3	Proposition	237
3.2-4	Proposition	237
3.2-5	Laplace Transform Linear	237

3.2-6 Laplace Transform of the Product of an Exponential with a Function 238

3.2-7 Proposition: Derivative of a Laplace Transform 239

Exercises 240

Section 3.3 – Inverse Laplace Transform 242

3.3-1 Definition 242

3.3-2 Laplace Transform Linear 242

3.3-3 Proposition 242

Exercises 246

Section 3.4 – Using Laplace Transform to Solve Differential Equations 249

3.4-1 Homogeneous Equations 249

3.4-2 Inhomogeneous Equations 250

3.4-3 Higher-Order Equations 251

3.4-4 Electrical Circuit 253

3.4-5 Transfer Function 253

3.4-6 Convolution Integral 253

3.4-7 Circuit Element Models 254

3.4-8 Resistor 254

3.4-9 Inductor 254

3.4-10 Capacitance 255

3.4-11 Springs-Masses 257

Exercises 259

Section 3.5 – Definitions of Second and Higher Order Equations 269

3.5-1 Example of Predator-Prey Systems (*Ecology*) 269

3.5-2 Summary of Predator-Prey 270

3.5-3 Definition 271

3.5-4 Matrix Notation for Linear Systems 272

3.5-5 Example of a Spring-Mass (*mechanical*) 272

3.5-6 Example of a parallel LRC *circuit* 275

3.5-7 Properties of Homogeneous Systems 277

3.5-8 Theorem 277

3.5-9 Theorem 277

3.5-10 Linearly Independence and Dependence 277

3.5-11 Proposition 277

3.5-12 Definition 277

Exercises 280

Section 3.6 – Planar Systems –Distinct, Complex, & Repeated Eigenvalues–Eigenvectors 287

- 3.6-1 Definition 287
- 3.6-2 Eigenvalues 287
- 3.6-3 Eigenvectors 289
- 3.6-4 Planar Systems 291
- 3.6-5 Summary 291
- 3.6-6 Proposition 291
- 3.6-7 Distinct Real Eigenvalues 292
- 3.6-8 Complex Eigenvalues 295
- 3.6-9 Theorem 296
- 3.6-10 One Real Eigenvalue of *Multiplicity* 2 299

Exercises 301

Section 3.7 – Phase Plane Portraits & Applications 308

- 3.7-1 Equilibrium Points 308
- 3.7-2 Stability of the equilibrium point condition 309
 - Case 1: $\lambda_{1,2} > 0$ (\mathbb{R}) 309
 - Case 2: $\lambda_{1,2} < 0$ (\mathbb{R}) 311
 - Case 3: $\lambda_1 < 0$ $\lambda_2 > 0$ (\mathbb{R}) 313
 - Case 4: $\lambda_{1,2} \in \mathbb{C}$ 315
- 3.7-3 Stability properties of linear systems (in 2-dimensions) 319
- 3.7-4 Stability properties of linear systems (in 3dimensions) 319
- 3.7-5 Theorem: second-Order Homogeneous Linear Saystems 323

Exercises 327

Lecture 4 – Series 343**Section 4.1 – Introduction and Review of Power Series 343**

- 4.1-1 Definition 343
- 4.1-2 Radius of Convergence of a Power Series - Corollary to Theorem 344
- 4.1-3 Interval of convergence - Theorem 344
- 4.1-4 The ratio Test - Theorem 345
- 4.1-5 Definition 345
- 4.1-6 How to Test a Power Series for Convergence 345
- 4.1-7 Algebraic Operations on Series 348
- 4.1-8 Differentiating Power Series - Theorem 348
- 4.1-9 Identity Theorem 349
- 4.1-10 Taylor and Maclaurin Series - Definition 349
- 4.1-11 Integrating Power Series - Theorem 352

Exercises 353

Section 4.2 – Series Solutions near Ordinary Points 355

4.2-1 Example of a First-Order Equation 355

4.2-2 Example 2 356

Exercises 358**Section 4.3 – Definitions of Second and Higher Order Equations 361**4.3-1 Legendre's Equation of order n 3614.3-2 Legendre Polynomials $P_n(x)$ 363**Exercises** 365**Section 4.4 – Solution about Singular Points 367**

4.4-1 Solution about Singular Points 367

4.4-2 Definition (Regular and Irregular Singular Points) 367

4.4-3 Frobenius Theorem 369

4.4-4 The model of Frobenius 369

4.4-5 Theorem – Frobenius Series Solutions 370

4.4-6 Theorem – The Extended Theorem and Procedure of *Frobenius* 376**Exercises** 377**Section 4.5 – Bessel's Equation and Bessel Functions 379**

4.5-1 Bessel's Equation 379

4.5-2 Gamma Function 381

4.5-3 Bessel Equation of Order *Zero* 3824.5-4 Bessel Equation of Order *One-Half* 3834.5-5 Bessel Equation of Order *One* 385

4.5-6 Applications of Bessel Functions 386

4.5-7 Theorem: Solutions in Bessel Functions 388

Exercises 390**Appendix 395**

A Derivatives 397

B Differential Equations 399

C Electrical (Basic) 401

D Factorial 423

E Integrals 425

F Laplace Transform 437

G Proofs 441

H Series 445

Answers	449
References	515
Index	517