Section 2.6 – Tangent Planes and Linear Approximation

Tangent Planes and Normal Lines

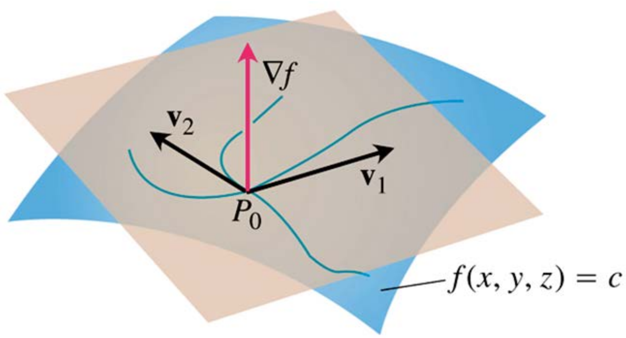
If $\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$ is a smooth curve on the level surface $f(x,y,z) = c$ of a differentiable function *f*, then $f(g(t),h(t),k(t)) = c$.

Differentiating both sides of this equation with respect to *t* leads to

$$\frac{d}{dt} f(g(t),h(t),k(t)) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt} = 0$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)}_{\nabla f} \bullet \underbrace{\left(\frac{dg}{dt}\hat{i} + \frac{dh}{dt}\hat{j} + \frac{dk}{dt}\hat{k}\right)}_{\frac{d\mathbf{r}}{dt}} = 0$$



Definition

The ***tangent plane*** at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x,y,z) = c$ of a differentiable function *f* is the plane through P_0 normal to $\nabla f\Big|_{P_0}$.

The ***normal line*** of the surface at P_0 is the line through P_0 parallel to $\nabla f\Big|_{P_0}$.

Normal Line to $f(x,y,z) = c$ at $P_0(x_0, y_0, z_0)$

$$x = x_0 + f_x(P_0)t, \qquad y = y_0 + f_y(P_0)t, \qquad z = z_0 + f_z(P_0)t$$

Tangent Plane to $f(x,y,z) = c$ at $P_0(x_0, y_0, z_0)$

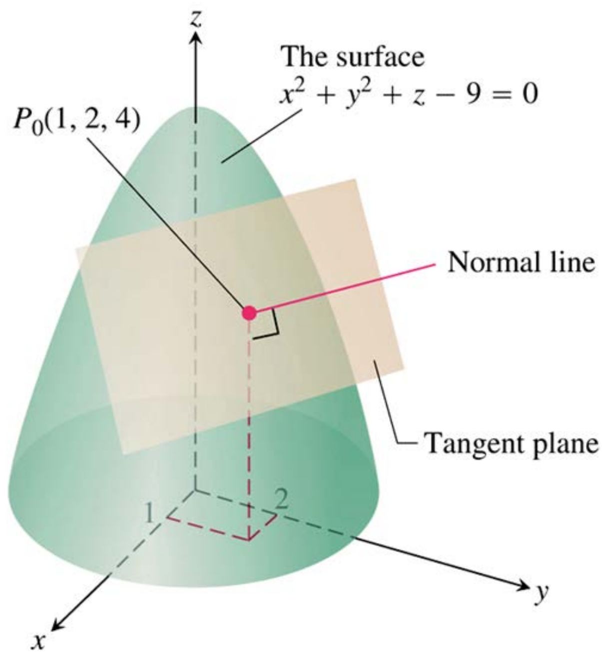
$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

Example

Find the tangent plane and normal line of the surface $f(x,y,z) = x^2 + y^2 + z - 9 = 0$ at the point $P_0(1,\ 2,\ 4)$

Solution

The tangent plane is the plane through P_0 perpendicular to the gradient of f at P_0.



The gradient is:

$$\nabla f\Big|_{P_0} = \left(2x\hat{i} + 2y\hat{j} + \hat{k}\right)\Big|_{(1,\ 2,\ 4)}$$

$$= 2\hat{i} + 4\hat{j} + \hat{k}$$

The tangent plane is the plane

$$2(x-1) + 4(y-2) + (z-4) = 0$$

$$2x + 4y + z = 14$$

The line normal to the surface at P_0 is

$$x = 1 + 2t, \quad y = 2 + 4t, \quad z = 4 + t$$

Plane Tangent to a Surface $z = f(x, y)$ **at** $\left(x_0, y_0, f\left(x_0, y_0\right)\right)$

The plane tangent to the surface $z = f(x, y)$ of a differentiable function *f* at the point

$P_0\left(x_0, y_0, z_0\right) = \left(x_0, y_0, f\left(x_0, y_0\right)\right)$ is

$$f_x\left(x_0, y_0\right)\left(x - x_0\right) + f_y\left(x_0, y_0\right)\left(y - y_0\right) - \left(z - z_0\right) = 0$$

Example

Find the plane tangent and the normal line to the surface $z = x\cos y - ye^x$ at (0, 0, 0)

Solution

$$f(x, y, z) = x\cos y - ye^x - z$$

$$\nabla f = \left(\cos y - ye^x\right)\hat{i} + \left(-x\sin y - e^x\right)\hat{j} - \hat{k}\Big|_{(0,\ 0)}$$

$$= \hat{i} - \hat{j} - \hat{k}$$

Therefore, the tangent plane is

$$1(x - 0) - (y - 0) - (z - 0) = 0$$

$$x - y - z = 0$$

The normal line:

$$\begin{cases} x = t \\ y = -t \\ z = -t \end{cases}$$

Example

The surfaces $f(x,y,z) = x^2 + y^2 - 2 = 0$ and $g(x,y,z) = x + z - 4 = 0$ meet in an ellipse E. Find the parametric equations for the line tangent to E at the point $P_0(1,\ 1,\ 3)$

Solution

The tangent line is orthogonal to both ∇f and ∇g at P_0 and therefore parallel to $\boldsymbol{v} = \nabla f \times \nabla g$.

The components of $\boldsymbol{v}$ and the coordinates of P_0 give us equations for the line.

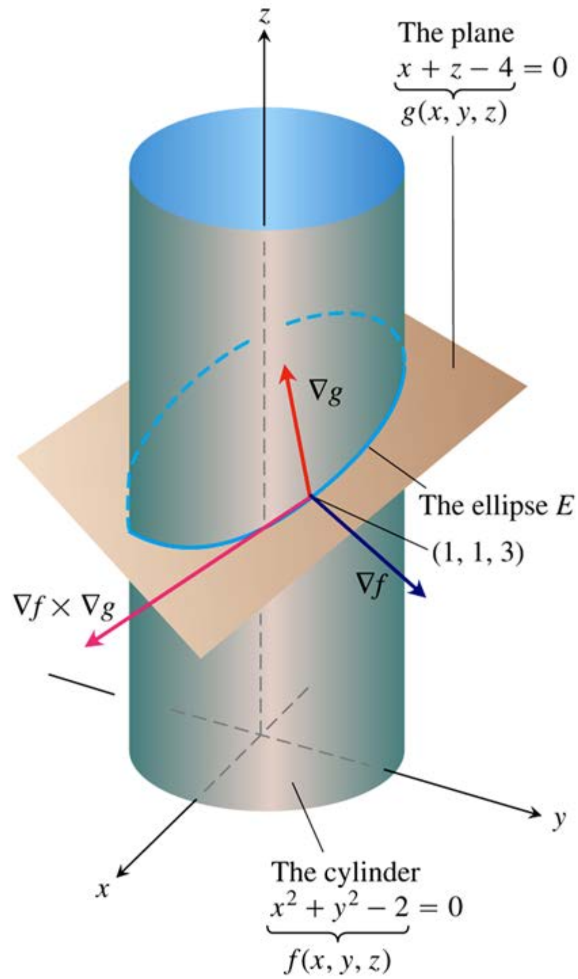
$$\nabla f\Big|_{(1,\ 1,\ 3)} = \left(2x\hat{i} + 2y\hat{j}\right)\Big|_{(1,\ 1,\ 3)}$$
$$= 2\hat{i} + 2\hat{j}$$

$$\nabla g\Big|_{(1,\ 1,\ 3)} = \left(\hat{i} + \hat{k}\right)\Big|_{(1,\ 1,\ 3)}$$
$$= \hat{i} + \hat{k}$$

$$\vec{v} = \left(2\hat{i} + 2\hat{j}\right) \times \left(\hat{i} + \hat{k}\right)$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$
$$= 2\hat{i} - 2\hat{j} - 2\hat{k}$$

The tangent line is

$$x = 1 + 2t, \quad y = 1 - 2t, \quad z = 3 - 2t$$



Estimating Change in a Specific Direction

How much the value of a function *f* changes if we move a small distance *ds* from a point P_0 to another point neary.

$$df = f'\left(P_0\right)ds \qquad (\textit{single variable})$$

$$df = \left(\nabla f\Big|_{P_0} \bullet \vec{u}\right)ds \qquad (\textit{two or more variables})$$

$\vec{u}$ is the direction of the motion away from P_0.

Estimating the Change in *f* in a Direction $\vec{u}$

To estimate the change in the value of a differentiable function *f* when we move a small distance *ds* from a point P_0 in a paricular direction $\vec{u}$ is given by

$$df = \underbrace{\left(\nabla f\Big|_{P_0} \bullet \vec{u}\right)}_{\text{Directional derivative}} \cdot \underbrace{ds}_{\text{Distance}}$$

Example

Estimate how much the value of $f(x,y,z) = y\sin x + 2yz$ will change if the point $P(x,y,z)$ moves 0.1 unit from $P_0(0,\ 1,\ 0)$ straight toward $P_1(2,\ 2,\ -2)$

Solution

$$\overrightarrow{P_0P_1} = 2\hat{i} + \hat{j} - 2\hat{k}$$

The direction of the vector is:

$$\begin{aligned}\vec{u} &= \frac{\overrightarrow{P_0P_1}}{\left|\overrightarrow{P_0P_1}\right|}\\ &= \frac{2\hat{i} + \hat{j} - 2\hat{k}}{\sqrt{4+1+4}}\\ &= \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\end{aligned}$$

$$\begin{aligned}\nabla f\Big|_{(0,1,0)} &= \left((y\cos x)\hat{i} + (\sin x + 2z)\hat{j} + 2y\hat{k}\right)\Big|_{(0,1,0)}\\ &= \hat{i} + 2\hat{k}\end{aligned}$$

$$\left.\nabla f\right|_{P_0} \cdot \vec{u} = \left(\hat{i} + 2\hat{k}\right) \cdot \left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\right)$$
$$= \frac{2}{3} - \frac{4}{3}$$
$$\underline{= -\frac{2}{3}}$$

The change *df* in *f* that results from moving $ds = 0.1$ unit away from P_0 in the direction of $\vec{u}$ is

$$df = \left(\left.\nabla f\right|_{P_0} \cdot \vec{u}\right)(ds)$$
$$= \left(-\frac{2}{3}\right)(0.1)$$
$$\underline{\approx -0.067 \ \ unit}$$

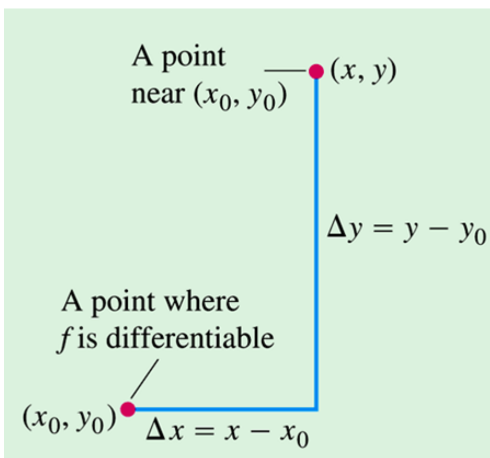
Definition

The ***linearization*** of a function $f(x,y)$ at a point (x_0, y_0) where *f* is differentiable is the function

$$L(x,y) = f(x_0,y_0) + \left.f_x\right|_{(x_0,y_0)}(x - x_0) + \left.f_y\right|_{(x_0,y_0)}(y - y_0)$$

The approximation $\qquad f(x,y) \approx L(x,y)$

is the ***standard linear*** approximation of *f* at (x_0, y_0)



Example

Find the linearization of $f(x,y)=x^2-xy+\frac{1}{2}y^2+3$ at the point (3, 2)

Solution

$$f(3,\ 2)=3^2-(3)(2)+\frac{1}{2}2^2+3$$

$$\underline{=8\,|}$$

$$f_x(3,\ 2)=\frac{\partial}{\partial x}\left(x^2-xy+\frac{1}{2}y^2+3\right)\Big|_{(3,2)}$$

$$=2x-y\Big|_{(3,2)}$$

$$=2(3)-2$$

$$\underline{=4\,|}$$

$$f_y(3,2)=-x+y\Big|_{(3,2)}$$

$$=-3+2$$

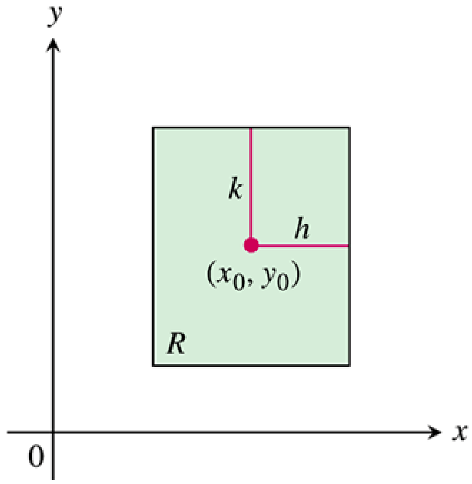
$$\underline{=-1\,|}$$

$$L(x,y)=8+4(x-3)-1(y-2)$$

$$\underline{=4x-y-2\,|}$$

The Error in the Standard Linear Approximation

If *f* has continuous first and second partial derivatives throughout an open set containing a rectangle *R* centered at $\left(x_0, y_0\right)$ and if *M* is any upper bound for the values of $\left|f_{xx}\right|$, $\left|f_{yy}\right|$, and $\left|f_{xy}\right|$ on *R*, then the error $E(x,y)$ incurred in replacing $F(x,y)$ on *R* by its linearization



$$L(x,y) = f\left(x_0,y_0\right) + f_x\left(x_0,y_0\right)\left(x-x_0\right) + f_y\left(x_0,y_0\right)\left(y-y_0\right)$$

Satisfies the inequality:

$$\left|E(x,y)\right| \le \tfrac{1}{2}M\left(\left|x-x_0\right| + \left|y-y_0\right|\right)^2$$

$$R: \quad \left|x-x_0\right| \le h, \quad \left|y-y_0\right| \le k$$

Example

Find an upper bound for the error in the approximation $f(x,y) \approx L(x,y)$ of $f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$ over the rectangle

$$R: \quad |x-3| \le 0.1, \quad |y-2| \le 0.1$$

Express the upper bound as a percentage of $f(3,2)$, the value of *f* at the center of the rectangle.

Solution

$$f_{xx} = \frac{\partial}{dx}(2x-y) = 2 \quad \to \quad \left|f_{xx}\right| = 2$$

$$f_{yy} = \frac{\partial}{dy}(-x+y) = 1 \quad \to \quad \left|f_{yy}\right| = 1$$

$$f_{xy} = \frac{\partial}{dy}(2x-y) = -1 \quad \to \quad \left|f_{xy}\right| = |-1| = 1$$

The largest of these is 2, so let $M = 2$.

$$|E(x,y)| \le \tfrac{1}{2}M\left(|x-x_0|+|y-y_0|\right)^2$$

$$= \tfrac{1}{2}(2)\left(|x-3|+|y-2|\right)^2$$

$$= \left(|x-3|+|y-2|\right)^2$$

Since $|x-3| \le 0.1, \quad |y-2| \le 0.1$

$$|E(x,y)| \le (0.1+0.1)^2$$

$$= 0.04$$

As a percentage of $f(3,2)=8$, the error is no greater than

$$\frac{0.04}{8}\times 100 = 0.5\%$$

Differentials

Definition

If we move from $\left(x_0, y_0\right)$ to a point $\left(x_0 + dx, y_0 + dy\right)$ nearby, the resulting change

$$df = f_x\left(x_0, y_0\right)dx + f_y\left(x_0, y_0\right)dy$$

In the linearization of *f* is called the ***total differential of f***.

Example

Suppose that a cylindrical can is designed to have a radius of 1 *in*. and a height of 5 *in*., but that the radius and height are off the amounts $dr = +0.03$ and $dh = -0.1$. Estimate the resulting absolute change in the volume of the can.

Solution

To estimate the absolute change in $V = \pi r^2 h$,

$$\Delta V \approx dV = V_r\left(r_0,\ h_0\right)dr + V_h\left(r_0,\ h_0\right)dh$$

$$\begin{aligned} dV &= \left(2\pi r_0 h_0\right)(0.03) + \left(\pi r_0^2\right)(-0.1) \\ &= 2\pi(1)(5)(0.03) + \pi(1)^2(-0.1) \\ &= 0.2\pi \\ &\approx 0.63\ in^3 \end{aligned}$$

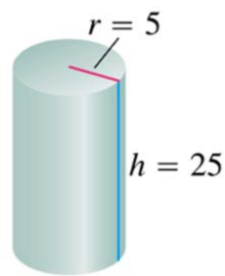
Example

Your company manufactures right circular cylindrical molasses storage tanks that are 25 *ft* with a radius of 5 *ft*. How sensitive are the tanks' volumes to small variations in height and radius?

Solution

$$V = \pi r^2 h$$

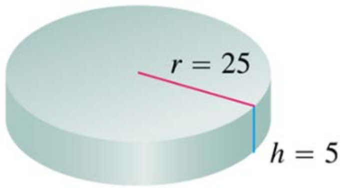
$$\begin{aligned} dV &= V_r\left(r_0,\ h_0\right)dr + V_h\left(r_0,\ h_0\right)dh \\ &= V_r(5,\ 25)dr + V_h(5,\ 25)dh \\ &= (2\pi rh)_{(5,\ 25)}dr + \left(\pi r^2\right)_{(5,\ 25)}dh \\ &= 250\pi dr + 25\pi dh \end{aligned}$$



A 1-unit change in *r* will change *V* about 250π units.

A 1-unit change in *h* will change *V* about 25π units.

The tanks' volume is 10 times more sensitive to a small change in r than it is to a small change of equal size in h.



$$dV = (2\pi rh)_{(25,\ 5)}\, dr + \left(\pi r^2\right)_{(25,\ 5)}\, dh$$

$$= 250\pi dr + 625\pi dh$$

Now the volume is more sensitive to changes in h than to changes in r.

The general rule is that functions are most sensitive to small changes in the variables that generated the largest partial derivatives.

Example

The volume $V = \pi r^2 h$ of a right circular cylinder is to be calculated from measured values of r and h. Suppose that r is measured with an error of no more than 2% and h with an error of no more than 0.5%. Estimate the resulting possible percentage error in the calculation of V.

Solution

$$\left|\frac{dr}{r}\times 100\right| \le 2 \qquad \left|\frac{dh}{h}\times 100\right| \le 0.5$$

$$\frac{dV}{V} = \frac{2\pi rh dr + \pi r^2 dh}{\pi r^2 h}$$

$$= \frac{2dr}{r} + \frac{dh}{h}$$

$$\left|\frac{dV}{V}\right| = \left|\frac{2dr}{r} + \frac{dh}{h}\right|$$

$$\le \left|2\frac{dr}{r}\right| + \left|\frac{dh}{h}\right|$$

$$\le 2(0.02) + 0.005$$

$$= 0.045$$

The error in the volume is at the most 4.5%

Functions of More Than Two Variables

- The linearization of $f(x,y,z)$ at a point $P_0(x_0,y_0,z_0)$ is

$$L(x,y,z)=f(P_0)+f_x(P_0)(x-x_0)+f_y(P_0)(y-y_0)+f_z(P_0)(z-z_0)$$

- Suppose that *R* is a closed rectangular solid centered at P_0 and lying in an open region on which the second partial derivatives of *f* are continuous. Suppose also that $|f_{xx}|$, $|f_{yy}|$, $|f_{zz}|$, $|f_{xy}|$, $|f_{xz}|$, and $|f_{yz}|$ are all less than or equal to *M* throughout *R*. Then the error $E(x,y,z)=f(x,y,z)-L(x,y,z)$ in the approximation of *f* by *L* is bounded throughout *R* by the inequality

$$|E(x,y)|\le \tfrac{1}{2}M\left(|x-x_0|+|y-y_0|+|z-z_0|\right)^2$$

- If the second partial derivatives of *f* are continuous and if *x*, *y*, and *z* change from $x_0, y_0, \text{ and } z_0$ by small amounts *dx*, *dy*, and *dz*, the total differential

$$df = f_x(P_0)dx+f_y(P_0)dy+f_z(P_0)dz$$

Example

Find the linearization $L(x,y,z)$ of $f(x,y,z)=x^2-xy+3\sin z$ at the point (2, 1, 0).
Find the upper bound for the error incurred in replacing *f* by *L* on the rectangle

$$R: \quad |x-2|\le 0.01, \quad |y-1|\le 0.02, \quad |z|\le 0.01$$

Solution

$f(2,1,0)=2^2-(2)(1)+3\sin 0=2$

$f_x(2,1,0)=2x-y=3$	$f_{xx}=2$	$f_{xy}=-1$
$f_y(2,1,0)=-x=-2$	$f_{yy}=0$	$f_{xz}=0$
$f_z(2,1,0)=3\cos z=3$	$f_{zz}=-3\sin z$	$f_{yz}=0$

$|-3\sin z|\le 3\sin 0.01\approx 0.03$

Let $M = 2$.

$|E|\le \frac{1}{2}2(0.01+0.02+0.01)^2$

$\quad = 0.0016$

Exercises *Section* 2.6 – Tangent Planes and Linear Approximation

(1 – 5) Find the tangent plane and normal line of the surface

1. $x^2+y^2+z^2=3$ at the point $P_0(1,\ 1,\ 1)$

2. $x^2+2xy-y^2+z^2=7$ at the point $P_0(1,\ -1,\ 3)$

3. $\cos\pi x-x^2y+e^{xz}+yz=4$ at the point $P_0(0,\ 1,\ 2)$

4. $x^2-xy-y^2-z=0$ at the point $P_0(1,\ 1,\ -1)$

5. $x^2+y^2-2xy-x+3y-z=-4$ at the point $P_0(2,\ -3,\ 18)$

(6 – 23) Find an equation for the plane that is tangent to the surface

6. $z=\ln\left(x^2+y^2\right)$ at the point $(1,\ 0,\ 0)$

7. $z=e^{-x^2-y^2}$ at the point $(0,\ 0,\ 1)$

8. $z=\sqrt{y-x}$ at the point $(1,\ 2,\ 1)$

9. $z=2x^2+y^2;\quad (1,\ 1,\ 3)$ *and* $(0,\ 2,\ 4)$

10. $x^2+\frac{1}{4}y^2-\frac{1}{9}z^2=1;\quad (0,\ 2,\ 0)$ *and* $\left(1,\ 1,\ \frac{3}{2}\right)$

11. $xy\sin z-1=0;\quad \left(1,\ 2,\ \frac{\pi}{6}\right)$ *and* $\left(-2,\ -1,\ \frac{5\pi}{6}\right)$

12. $yze^{xz}-8=0;\quad (0,\ 2,\ 4)$ *and* $(0,\ -8,\ -1)$

13. $z=x^2e^{x-y};\quad (2,\ 2,\ 4)$ *and* $(-1,\ -1,\ 1)$

14. $z=\ln(1+xy);\quad (1,\ 2,\ \ln 3)$ *and* $(-2,\ -1,\ \ln 3)$

15. $z=f(x,y)=\dfrac{1}{x^2+y^2}$ at the point $\left(1,\ 1,\ \frac{1}{2}\right)$

16. $x^2+y+z=3;\quad (1,\ 1,\ 1)$ *and* $(2,\ 0,\ -1)$

17. $x^2+y^3+z^4=2;\quad (1,\ 0,\ 1)$ *and* $(-1,\ 0,\ 1)$

18. $xy+xz+yz=12;\quad (2,\ 2,\ 2)$ *and* $(2,\ 0,\ 6)$

19. $x^2+y^2-z^2=0;\quad (3,\ 4,\ 5)$ *and* $(-4,\ -3,\ 5)$

20. $xy\sin z=1;\quad \left(1,\ 2,\ \frac{\pi}{6}\right)$ *and* $\left(-2,\ -1,\ \frac{5\pi}{6}\right)$

21. $yze^{xz} = 8;\ \ (0,\ 2,\ 4)$ *and* $(0,\ -8,\ -1)$

22. $z^2 - \frac{x^2}{16} - \frac{y^2}{9} = 1;\ \ (4,\ 3,\ -\sqrt{3})$ *and* $(-8,\ 9,\ \sqrt{14})$

23. $2x + y^2 - z^2 = 0;\ \ (0,\ 1,\ 1)$ *and* $(4,\ 1,\ -3)$

(**24 – 27**) Find parametric equation for the line tangent to the curve of intersection of the surfaces

24. $x + y^2 + 2z = 4,\ \ x = 1$ at the point $(1,\ 1,\ 1)$

25. $xyz = 1,\ \ x^2 + 2y^2 + 3z^2 = 6$ at the point $(1,\ 1,\ 1)$

26. $x^3 + 3x^2y^2 + y^3 + 4xy - z^2 = 0,\ \ x^2 + y^2 + z^2 = 11$ at the point $(1,\ 1,\ 3)$

27. $x^2 + y^2 = 4,\ \ x^2 + y^2 - z = 0$ at the point $(\sqrt{2},\ \sqrt{2},\ 4)$

28. Find an equation for the plane tangent to the level surface $f(x,y,z) = x^2 - y - 5z$ at the point $P_0(2,\ -1,\ 1)$. Also, find parametric equations for the line is normal to the surface at P_0.

29. By about how much will $f(x,y,z) = \ln\sqrt{x^2 + y^2 + z^2}$ change if the point $P(x,y,z)$ moves from $P_0(3,\ 4,\ 12)$ a distance of $ds = 0.1$ unit in the direction of $3\boldsymbol{i} + 6\boldsymbol{j} - 2\boldsymbol{k}$?

30. By about how much will $f(x,y,z) = e^x \cos yz$ change if the point $P(x,y,z)$ moves from the origin at distance of $ds = 0.1$ *unit* in the direction of $2\hat{i} + 2\hat{j} - 2\hat{k}$?

(**31 – 38**) Find the linearization $L(x,y)$ of

31. $f(x,y) = x^2 + y^2 + 1$ at the point (0, 0) and (1, 1)

32. $f(x,y) = (x + y + 2)^2$ at the point (0, 0) and (1, 2)

33. $f(x,y) = x^3y^4$ at the point (1, 1) and (0, 0)

34. $f(x,y) = e^{2y-x}$ at the point (0, 0) and (1, 2)

35. $f(x,y,z) = x^2 + y^2 + z^2$ at the point (1, 1, 1)

36. $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$ at the point (1, 2, 2)

37. $f(x,y,z) = \frac{\sin xy}{z}$ at the point $\left(\frac{\pi}{2},\ 1,\ 1\right)$

38. $f(x,y,z) = e^x + \cos(y + z)$ at the point $\left(0,\ \frac{\pi}{4},\ \frac{\pi}{4}\right)$

(**39 – 46**) Find the linear approximation to the function f at the point $(a,\ b)$ and estimate the given function value

39. $f(x,\ y) = 4\cos(2x - y)$; $(a,\ b) = \left(\frac{\pi}{4},\ \frac{\pi}{4}\right)$; estimate $f(0.8,\ 0.8)$

40. $f(x,\ y) = (x + y)e^{xy}$; $(a,\ b) = (2,\ 0)$; estimate $f(1.95,\ 0.05)$

41. $f(x,\ y) = xy + x - y$; $(a,\ b) = (2,\ 3)$; estimate $f(2.1,\ 2.99)$

42. $f(x,\ y) = 12 - 4x^2 - 8y^2$; $(a,\ b) = (-1,\ 4)$; estimate $f(-1.05,\ 3.95)$

43. $f(x,\ y) = -x^2 + 2y^2$; $(a,\ b) = (3,\ -1)$; estimate $f(3.1,\ -1.04)$

44. $f(x,\ y) = \sqrt{x^2 + y^2}$; $(a,\ b) = (3,\ -4)$; estimate $f(3.06,\ -3.92)$

45. $f(x,\ y) = \ln(1 + x + y)$; $(a,\ b) = (0,\ 0)$; estimate $f(0.1,\ -0.2)$

46. $f(x,\ y) = \dfrac{x + y}{x - y}$; $(a,\ b) = (3,\ 2)$; estimate $f(2.95,\ 2.05)$

47. Estimate the change in the function $f(x,\ y) = -2y^2 + 3x^2 + xy$ when $(x,\ y)$ changes from $(1,\ -2)$ to $(1.05,\ -1.9)$.

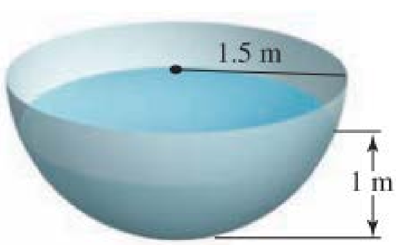
48. What is the largest value that the directional derivative of $f(x, y, z) = xyz$ can have at the point $(1,\ 1,\ 1)$?

49. You plan to calculate the volume inside a stretch of pipeline that is about 36 *in*. in diameter and 1 *mile* long. With which measurement should you be more careful, the length or the diameter? Why?

50. The volume of a cylinder with radius r and height h is $V = \pi r^2 h$. Find the approximate percentage change in the volume when the radius decreases by 3% and the height increases by 2%.

51. The volume of an ellipsoid with axes of length $2a$, $2b$, and $2c$ is $V = \pi abc$. Find the percentage change in the volume when a increases by 2%, b increases by 1.5%, and c decreases by 2.5%.

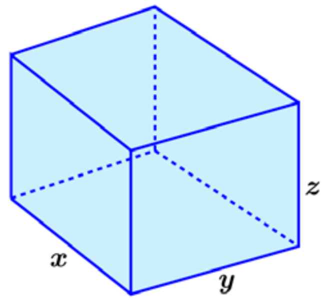
52. A hemispherical tank with a radius of 1.50 *m* is filled with water to a depth of 1.00 *m*. Water level drops by 0.05 *m* (from 1.00 *m* to 0.95 *m*)



a) Approximate the change in the volume of water in the tank. The volume of a spherical cap is $V = \frac{1}{3}\pi h^2 (3r - h)$, where r is the radius of the sphere and h is the thickness of the cap (in this case, the depth of the water).

b) Approximate the change in the surface area of the water in the tank.

53. Consider a closed rectangular box with a square base. If x is measured with error at most 2% and y is measured with error at most 3% use a differential to estimate the corresponding percentage error in computing the box's



a) Surface area

b) Volume

54. Consider a closed container in the shape of a cylinder of radius 10 *cm* and height 15 *cm* with a hemisphere on each end.

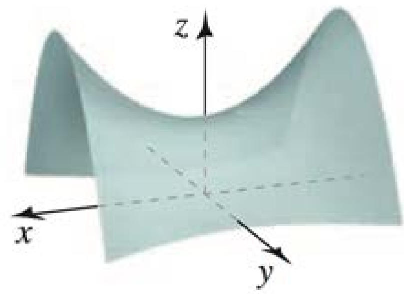
The container is coated with a layer of ice $\frac{1}{2}$ *cm* thick. Use a differential to estimate the total volume of ice. (*Hint*: assume r is radius with $dr = \frac{1}{2}$ and h is height with $dh = 0$)

55. A standard 12-*fl*-*oz* can of soda is essentially a cylinder of radius $r = 1$ *in* and height $h = 5$ *in*.

a) At these dimensions, how sensitive is the can's volume to a small change in radius versus a small change in height?

b) Could you design a soda can that appears to hold more soda but in fact holds the same 12-*fl*-*oz*? What might its dimensions be? (There is more than one correct answer.)

56. Consider the function $f(x, y) = 2x^2 - 4y^2 + 10$, whose graph is shown



a) Fill in the table showing the value of the directional derivative at points (a, b) in the direction given by the unit vectors $\boldsymbol{u}$, $\boldsymbol{v}$, *and* $\boldsymbol{w}$

	$(a, b) = (0, 0)$	$(a, b) = (2, 0)$	$(a, b) = (1, 1)$
$\boldsymbol{u} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$			
$\boldsymbol{v} = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$			
$\boldsymbol{w} = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$			

b) Interpret each of the directional derivatives computed in part(*a*) at the point $(2, 0)$

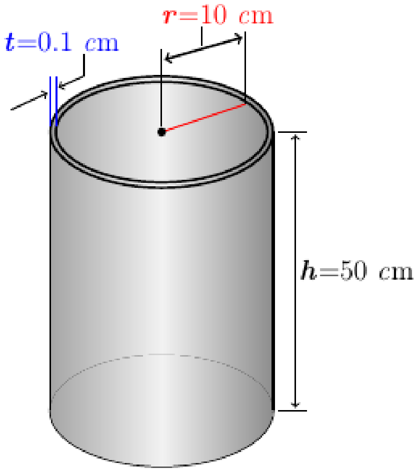
57. Two spheres have the same center and radii *r* and *R*, where $0 < r < R$. The volume of the region between the sphere is $V(r, R) = \frac{4\pi}{3}\left(R^3 - r^3\right)$.

a) First use your intuition. If *r* is held fixed, how does *V* change as *R* increases? What is the sign of V_R ? If *R* is held fixed, how does *V* change as *r* increases (up to the value of *R*)? What is the sign of V_r ?

b) Compute V_r and V_R . Are the results consistent with part (*a*)?

c) Consider spheres with $R = 3$ and $r = 1$. Does the volume change more if *R* is increased by $\Delta R = 0.1$ (with *r* fixed) or if *r* is decreased by $\Delta r = 0.1$ (with *R* fixed)?

58. A company manufactures cylindrical aluminum tubes to rigid specifications. The tubes are designed to have an outside radius of $r = 10\ cm$, a height of $h = 50\ cm$, and a thickness of $t = 0.1\ cm$. The manufacturing process produces tubes with a maximum error of $\pm 0.05\ cm$ in the radius and height and a maximum error of $\pm 0.0005\ cm$ in the thickness. The volume of the material used to construct a cylindrical tube is $V(r,h,t) = \pi ht(2r - t)$. Estimate maximum error in the volume of the tube.



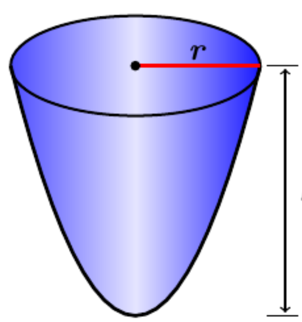
59. The volume of a right circular cone with radius *r* and height *h* is $V = \frac{1}{3}\pi hr^2$

a) Approximate the change in the volume of the cone when the radius changes from $r = 6.5$ to $r = 6.6$ and the height changes from $h = 4.20$ to $h = 4.15$

b) Approximate the change in the volume of the cone when the radius changes from $r = 5.4$ to $r = 5.37$ and the height changes from $h = 12.0$ to $h = 11.96$

60. The area of an ellipse with axes of length 2*a* and 2*b* is $A = \pi ab$. Approximate the percent change in the area when *a* increases by 2% and *b* increases by 1.5%.

61. The Volume of a segment of a circular paraboloid with radius *r* and height *h* is $V = \frac{1}{2}\pi hr^2$.



$V = \frac{\pi}{2}\ r^2 h$

Approximate the percent change in the volume when the radius decreases by 1.5% and the height increases by 2.2%

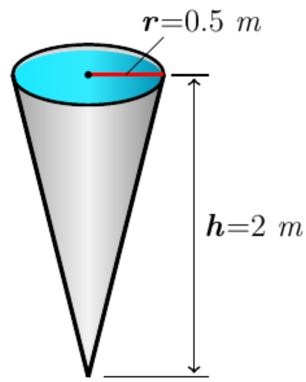
62. Batting averages in baseball are defined by $A = \frac{x}{y}$, where $x \ge 0$ is the total number of hits and $y > 0$ is the total number of at-bats. Treat x and y as positive real numbers and note that $0 \le A \le 1$.

a) Estimate the change in the batting average if the number of hits increases from 60 to 62 and the number of at-bats increases from 175 to 180.

b) If a batter currently has a batting average of $A = 0.35$, does the average decrease if the batter fails to get a hit more than it increases if the batter gets a hit?

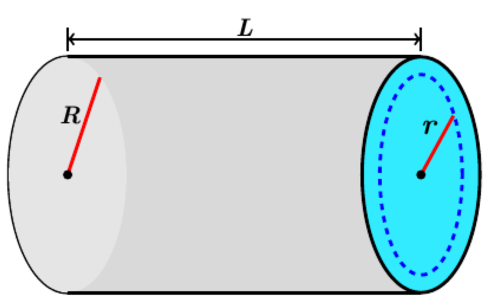
c) Does the answer in part (*b*) depend on the current batting average? Explain.

63. A conical tank with radius 0.50 *m* and height 2.0 *m* is filled with water.



Water released from the tank, and the water level drops by 0.05 *m* (from 2.0 *m* to 1.95 *m*). Approximate the change in volume of water in the tank.
(***Hint***: When the water level drops, both the radius and height of the cone of water change).

64. Poiseuille's law is a fundamental law of fluid dynamics that describes the flow velocity of a viscous incompressible fluid in a cylinder (it is used to model blood flow through veins and arteries). It says that in a cylinder of radius *R* and length *L*, the velocity of the fluid $r \le R$ units from the centerline of the cylinder is $V = \frac{P}{4L\upsilon}\left(R^2 - r^2\right)$, where *P* is the difference in the pressure between the ends of the cylinder and υ is the viscosity of the fluid. Assuming that *P* and υ are constant, the velocity *V* along the centerline of the cylinder $(r = 0)$ is $V = \frac{kR^2}{L}$, where *k* is a constant that we will take to be $k = 1$.



a) Estimate the change in the centerline velocity $(r = 0)$ if the radius of the flow cylinder increases from $R = 3$ *cm* to $R = 3.05$ *cm* and the length increases from $L = 50$ *cm* to $L = 50.5$ *cm*.

b) Estimate the percent change in the centerline velocity if the radius of the flow cylinder R decreases by 1% and the length increases by 2%.

65. Suppose that in a large group of people a fraction $0 \le r \le 1$ of the people have flu. The probability that in n random encounters, you will meet at least one person with flu is $P = f(n, r) = 1-(1-r)^n$. although n is a positive integer, regard it as a positive real number.

a) Compute f_r and f_n.

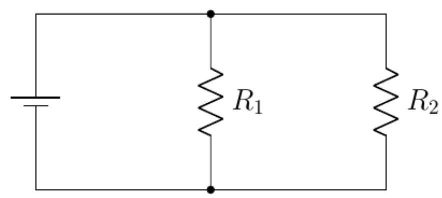
b) How sensitive is the probability P to the flu rate r? Suppose you meet $n = 20$ people. Approximately how much does the probability P increase if the flu rate increases from $r = 0.1$ to $r = 0.11$ (with n fixed)?

c) Approximately how much does the probability P increase the flu rate increases from $r = 0.9$ to $r = 0.91$

d) Interpret the results of parts (*b*) and (*c*).

66. When two electrical resistors with resistance $R_1 > 0$ and $R_2 > 0$ are wired in parallel in a circuit, the combined resistance R is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$



a) Estimate the change in R if R_1 increases from 2 Ω to 2.05 Ω and R_2 decreases from 3 Ω to 2.95 Ω.

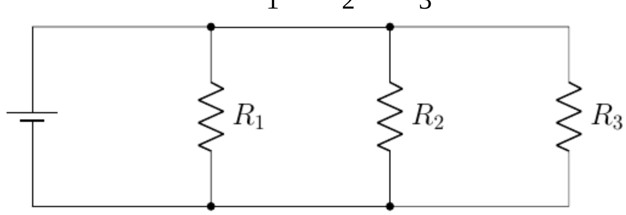
b) Is it true that if $R_1 = R_2$ and R_1 increases by the same small amount as R_2 decreases, then R is approximately unchanged? Explain.

c) Is it true that if R_1 and R_2 increase, then R increases? Explain.

d) Suppose $R_1 > R_2$ and R_1 increases by the same small amount as R_2 decreases. Does R increase or decrease?

67. When three electrical resistors with resistance $R_1 > 0$, $R_2 > 0$ and $R_3 > 0$ are wired in parallel in a circuit, the combined resistance R is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$



Estimate the change in R if R_1 increases from 2 Ω to 2.05 Ω, R_2 decreases from 3 Ω to 2.95 Ω, and R_3 increases from 1.5 Ω to 1.55 Ω

68. Consider the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and the plane P given by $Ax+By+Cz+1=0$. Let $h=\frac{1}{\sqrt{A^2+B^2+C^2}}$ and $m=\sqrt{a^2A^2+b^2B^2+c^2C^2}$

a) Find the equation of the plane tangent to the ellipsoid at the point $(p,\ q,\ r)$.

b) Find the two points on the ellipsoid at which the tangent plane parallel to P and find equations of the tangent planes.

c) Show that the distance between the origin and the plane P is h.

d) Show that the distance between the origin and the tangent planes is hm.

e) Find a condition that guarantees the plane P does not intersect the ellipsoid.