

Section 1.4 – Inverse Matrices - Finding A^{-1}

Definition

The matrix A is invertible if there exists a matrix A^{-1} such that:

$$A^{-1}A = AA^{-1} = I$$

where A^{-1} read as "A *inverse*" and A has to be a *square matrix*.

Not all matrices have inverses.

1. The inverse exists *iff* elimination produces n pivots (row exchanges allow).
2. The matrix A cannot have two different inverses.
3. If A is invertible, the one and only one solution to $Ax = B$ is $x = A^{-1}B$

$$AX = B$$

$$A^{-1}(AX) = A^{-1}B \quad \text{Multiply both side by } A^{-1}$$

$$(A^{-1}A)X = A^{-1}B \quad \text{Associate property}$$

$$IX = A^{-1}B \quad \text{Multiplicative inverse property}$$

$$X = A^{-1}B \quad \text{Identity property}$$

4. Suppose there is a ***nonzero*** vector x such that $Ax = 0$. Then A cannot have an inverse
5. A 2 by 2 matrix is invertible iff $ad - bc$ is not zero.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{Only for 2 by 2 matrices}$$

If $ad - bc = 0$ is the determinant, then A^{-1} doesn't exist

The Inverse of a Product AB

Theorem

If an $n \times n$ matrix has an inverse, that inverse is unique.

Proof

Suppose that A has an inverse A^{-1} and B is a matrix such that $BA = I$

$$B = BI = B(AA^{-1}) = (BA)A^{-1} = IA^{-1} = A^{-1}$$

Theorem

If A and B are invertible then so is AB . The inverse of a product AB is $(AB)^{-1} = B^{-1}A^{-1}$

Proof

$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= (AI)A^{-1} \\ &= AA^{-1} \\ &= I\end{aligned}$$

Reverse Order

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

Theorem

If A is invertible and n is a nonnegative integer, then:

- a) A^{-1} is invertible and $(A^{-1})^{-1} = A$
- b) A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$
- c) kA is invertible for any nonzero scalar k , and $(kA)^{-1} = k^{-1}A^{-1}$

Proof

$$\begin{aligned}(kA)(k^{-1}A^{-1}) &= k^{-1}(kA)A^{-1} = (k^{-1}k)AA^{-1} = (1)I = I \\ (k^{-1}A^{-1})(kA) &= k^{-1}(kA^{-1})A = (k^{-1}k)A^{-1}A = (1)I = I\end{aligned}$$

Finding A^{-1} using Gauss-Jordan Elimination

$$\left[A \mid I \right] \rightarrow \left[I \mid A^{-1} \right]$$

Find A^{-1} if $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \quad \begin{array}{cccccc} 2 & -2 & -1 & 0 & 1 & 0 \\ -2 & 0 & -2 & -2 & 0 & 0 \\ \hline 0 & -2 & -3 & -2 & 1 & 0 \end{array} \quad \begin{array}{cccccc} 3 & 0 & 0 & 0 & 0 & 1 \\ -3 & 0 & -3 & -3 & 0 & 0 \\ \hline 0 & 0 & -3 & -3 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array} \right] -\frac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array} \right] -\frac{1}{3}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array} \right] \begin{array}{l} R_1 - R_3 \\ R_2 - \frac{3}{2}R_3 \\ \end{array} \quad \begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & \frac{1}{3} \\ \hline 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{array} \quad \begin{array}{cccccc} 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & 0 & \frac{1}{2} \\ \hline 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array} \right] \quad A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

- ✓ Matrix A is *symmetric* across its main diagonal. So is A^{-1}
- ✓ Matrix A is *tridiagonal* (only three nonzero diagonals). But A^{-1} is a full matrix with no zeros. (another reason we don't compute A^{-1})

Singular *versus* Invertible

A^{-1} exists when A has a full set of n pivots. (Row exchanges allowed)

- With n pivots, elimination solves all the equations $Ax_i = b_i$. The columns x_i go into A^{-1} . Then $AA^{-1} = I$ is at least a **right-inverse**.
- Elimination is really a sequence of multiplications.

Conclusion

- If A doesn't have n pivots, elimination will lead to a **zero row**.
- Elimination steps are taken by an invertible M . So a row of MA is zero.
- If $AB = I$ then $MAB = M$. The zero row of MA , times B , gives a zero row of M .
- The invertible matrix M can't have a zero row! A must have n pivots if $AB = I$.

Elementary Matrices

Definition

Let e be an elementary row operation. Then the $n \times n$ **elementary matrix** E associated with e is the matrix obtained by applying e to the $n \times n$ identity matrix. Thus

$$E = eI$$

Example

$$a) \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \rightarrow \text{Multiply } R_2 \text{ of } I \text{ by } -3$$

$$b) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{Multiply the third row by } -5$$

$$c) \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{Interchange the first and second rows}$$

$$d) \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{Add } -3 \text{ times } R_1 \text{ to } R_2$$

Theorem

Let e be an elementary operation and let E be the corresponding elementary matrix $E = e(I)$. Then for every $m \times n$ matrix A

$$e(A) = EA$$

That is, an elementary row operation can be performed on A by multiplying A on the left by the corresponding elementary matrix.

Example $m \times m$

$$\text{Let } A = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix} \quad M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$MA = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 6 & 2 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$$

This result can be obtained from A by multiplying the first row by 2.

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 3 & 1 \\ -6 & 2 & 1 & 5 \\ 1 & 0 & 2 & 4 \end{bmatrix}$$

This result can be obtained from A by interchanging rows 2 and 3.

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ 0 & -4 & 10 & 8 \end{bmatrix}$$

This result can be obtained from A by adding 3 times row 1 to row 3.

Uniqueness of Echelon Form

Two matrices A and B are row-equivalent if and only if they have the same reduced echelon form.

Proof

If A and B have the same reduced echelon form E , then A is row-equivalent to E and E is row-equivalent to B . It follows that A is row-equivalent to B .

Now Suppose A and B are row-equivalent. Let E_1 be a reduced echelon form of A and E_2 be a reduced echelon form of B . Then E_1 and E_2 are row equivalent.

Suppose $E_1 = IF_1$ and $E_2 = IF_2$. Since E_1 and E_2 are row equivalent, $E_2 = CE_1$ for some matrix C . This means $I = CI$ and $F_2 = CF_1$. But then $C = I$ and $F_2 = F_1$.

Example

Show that the two matrices are row equivalent

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

Solution

$$\begin{aligned} A &= \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{array}{l} R_1 + R_2 \\ R_2 - 2R_1 \end{array} \\ &= \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix} \\ &= B \end{aligned}$$

Definition

A relationship \sim (equivalent) between elements of a set is called an equivalence relation if

- ✓ $A \sim A$ is always true,
- ✓ $A \sim B$ always implies $B \sim A$,
- ✓ $A \sim B$ and $B \sim C$ always implies $A \sim C$.

Exercises Section 1.4 – Inverse Matrices - Finding A^{-1}

1. Apply Gauss-Jordan method to find the inverse of this triangular “Pascal matrix”

$$\text{Triangular Pascal matrix} \quad A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$$

2. If A is invertible and $AB = AC$, prove that $B = C$

3. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, find two matrices $B \neq C$ such that $AB = AC$

4. If A has $\text{row } 1 + \text{row } 2 = \text{row } 3$, show that A is not invertible
- Explain why $Ax = (1, 0, 0)$ can't have a solution.
 - Which right sides (b_1, b_2, b_3) might allow a solution to $Ax = b$
 - What happens to $\text{row } 3$ in elimination?

5. True or false (with a counterexample if false and a reason if true):

- A 4 by 4 matrix with a row of zeros is not invertible.
- A matrix with 1's down the main diagonal is invertible.
- If A is invertible then A^{-1} is invertible.
- If A is invertible then A^2 is invertible.

6. Do there exist 2 by 2 matrices A and B with real entries such that $AB - BA = I$, where I is the identity matrix?

7. If B is the inverse of A^2 , show that AB is the inverse of A .

8. Find and check the inverses (assuming they exist) of these block matrices.

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}$$

9. For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

10. Find A^{-1} and B^{-1} (if they exist) by elimination.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

11. Find A^{-1} using the Gauss-Jordan method, which has a remarkable inverse.

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

12. Find the inverse.

a) $\begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$

c) $\begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

e) $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$

f) $\begin{bmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

g) $\begin{bmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{bmatrix}$

13. Show that A is not invertible for any values of the entries

$$A = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

14. Prove that if A is an invertible matrix and B is row equivalent to A , then B is also invertible.

15. Determine if the given matrix has an inverse, and find the inverse if it exists. Check your answer by multiplying $A \cdot A^{-1} = I$

a) $\begin{bmatrix} 2 & 3 \\ -3 & -5 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}$

16. Show that the inverse of $\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$ is $\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$