

Lecture One – Limits and Derivatives

Section 1.1 – Idea of Limits

Position Function

An object that is falling or vertically projected into the air has its height above the ground, $s(t)$, in feet, given by

$$s(t) = -16t^2 + v_0 t + s_0$$

v_0 is the original velocity (initial velocity) of the object, in *feet per second*

t is the time that the object is in motion, in *second*

s_0 is the original height (initial height) of the object, in *feet*

The average rate is given by: $\frac{\Delta s}{\Delta t}$

Example

A rock breaks loose from the top of a tall cliff. What is its average speed

- a) During the first 2 *sec* of fall?
- b) During the 1-*sec* interval between second 1 and *second* 2?

Solution

Since the rock falls free (*down*) without any initial velocity or height. $\Rightarrow y(t) = 16t^2$

$$\begin{aligned} \text{a) For the first 2 sec: Average speed} &= \frac{\Delta y}{\Delta t} \\ &= \frac{y(2) - y(0)}{2 - 0} \\ &= \frac{16(2)^2 - 16(0)^2}{2} \\ &= \frac{64}{2} \\ &= 32 \text{ ft / sec} \end{aligned}$$

$$\begin{aligned} \text{b) From 1 sec to 2 sec: Average speed} &= \frac{y(2) - y(1)}{2 - 1} \\ &= \frac{16(2)^2 - 16(1)^2}{1} \\ &= 48 \text{ ft / sec} \end{aligned}$$

Example

Find the speed of a falling rock $\left(y(t) = 16t^2\right)$ over a time interval $\left[t_0, t_0 + h\right]$. Then find the average speed at 1 sec and 2 sec.

Solution

$$\begin{aligned}\frac{\Delta y}{\Delta t} &= \frac{16(t_0 + h)^2 - 16(t_0)^2}{(t_0 + h) - t_0} \\&= \frac{16(t_0^2 + 2ht_0 + h^2) - 16t_0^2}{t_0 + h - t_0} \\&= \frac{16t_0^2 + 32ht_0 + 16h^2 - 16t_0^2}{h} \\&= 32\frac{ht_0}{h} + 16\frac{h^2}{h} \\&= \underline{32t_0 + 16h} \quad | \end{aligned}$$

If $t_0 = 1$

$$\begin{aligned}\frac{\Delta y}{\Delta t} &= 32(1) + 16h \\&= \underline{32 + 16h} \quad | \end{aligned}$$

The average speed has the limiting value 32 ft/sec as h approaches 0.

If $t_0 = 2$

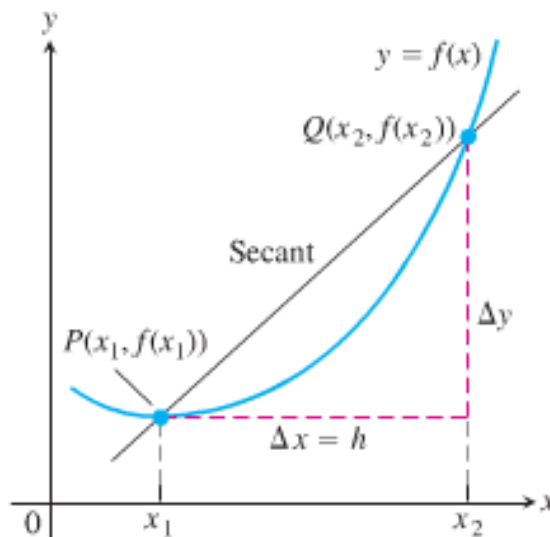
$$\begin{aligned}\frac{\Delta y}{\Delta t} &= 32(2) + 16h \\&= \underline{64 + 16h} \quad | \end{aligned}$$

The average speed has the limiting value 64 ft/sec as h approaches 0.

Average Rates of Changes and Secant Lines

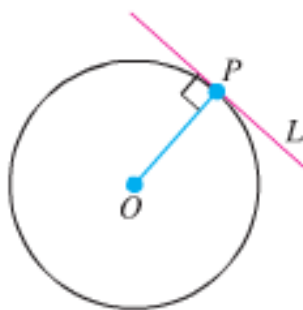
The average rate of change of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\ &= \frac{f(x_1 + h) - f(x_1)}{h}, \quad h \neq 0\end{aligned}$$



Defining the Slope of a Curve

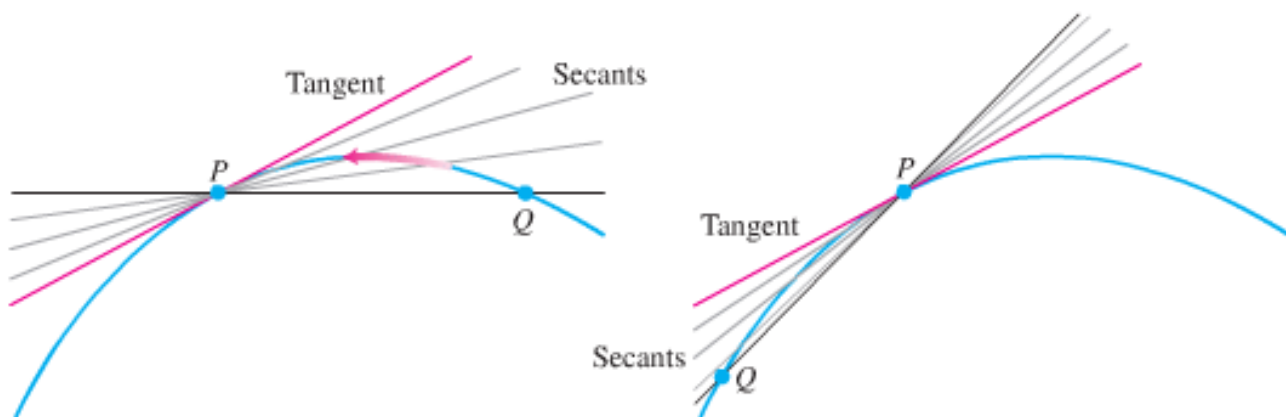
The slope of a line is the rate at which it rises or falls.



To define the tangency for general curves, we need an approach that makes the behavior of the secants through P and points Q as Q moves toward P along the curve:

1. Find the slope of the secant PQ .
2. Investigate the limiting value of the slope as Q approaches P along the curve.
3. If the limit exists, take it to be the slope of the curve at P and define the tangent to the curve at P to be the line through P with this slope.

$$m_{\text{tan}} = \lim_{t \rightarrow a} \frac{f(t) - f(a)}{t - a}$$

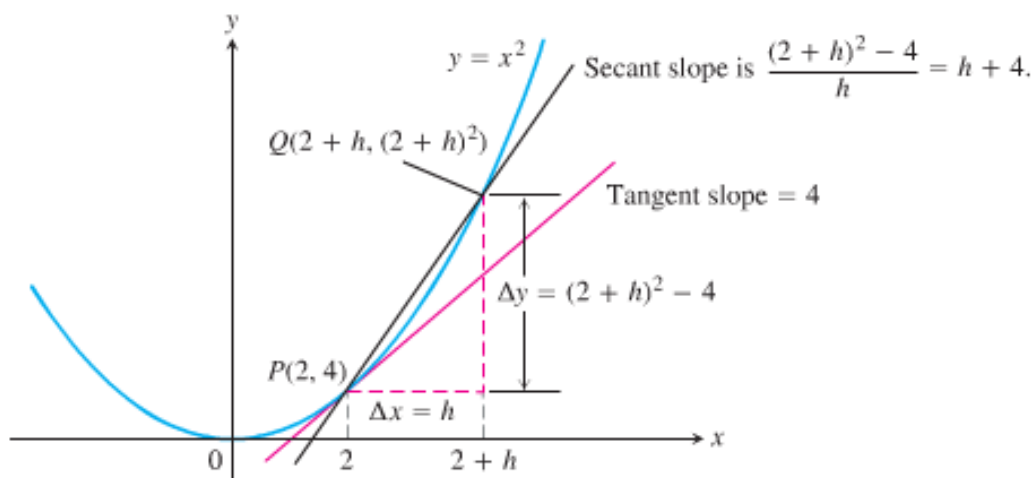


Example

Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point.

Solution

$$\begin{aligned} \text{Secant slope} &= \frac{\Delta y}{\Delta x} = \frac{f(x_1 + h) - f(x_1)}{h} \\ &= \frac{f(2 + h) - f(2)}{h} \\ &= \frac{(2 + h)^2 - 2^2}{h} \\ &= \frac{4 + 4h + h^2 - 4}{h} \\ &= \frac{4h + h^2}{h} \\ &= 4 + h \end{aligned}$$



As Q approaches P , h approaches 0. Then the secant slope $h + 4 \rightarrow 4 = \text{slope}$

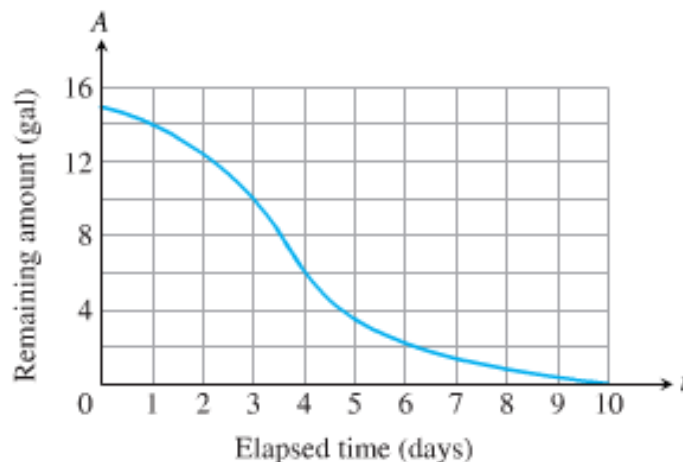
$$y = m(x - x_1) + y_1$$

$$y = 4(x - 2) + 4$$

$$\underline{y = 4x - 4}$$

Exercises Section 1.1 – Idea of Limits

- Find the average rate of change of the function $f(x) = x^3 + 1$ over the interval $[2, 3]$
- Find the average rate of change of the function $f(x) = x^2$ over the interval $[-1, 1]$
- Find the average rate of change of the function $f(t) = 2 + \cos t$ over the interval $[-\pi, \pi]$
- Find the slope of $y = x^2 - 3$ at the point $P(2, 1)$ and an equation of the tangent line at this P .
- Find the slope of $y = x^2 - 2x - 3$ at the point $P(2, -3)$ and an equation of the tangent line at this P .
- Find the slope of $y = x^3$ at the point $P(2, 8)$ and an equation of the tangent line at this P .
- Make a table of values for the function $f(x) = \frac{x+2}{x-2}$ at the points
 $x = 1.2, \quad x = \frac{11}{10}, \quad x = \frac{101}{100}, \quad x = \frac{1001}{1000}, \quad x = \frac{10001}{10000}, \quad \text{and } x = 1$
 - Find the average rate of change of $f(x)$ over the intervals $[1, x]$ for each $x \neq 1$ in the table
 - Extending the table if necessary, try to determine the rate of change of $f(x)$ at $x = 1$.
- The accompanying graph shows the total amount of gasoline A in the gas tank of an automobile after being driven for t days.



- Estimate the average rate of gasoline consumption over the time intervals $[0, 3]$, $[0, 5]$, and $[7, 10]$
- Estimate the instantaneous rate of gasoline consumption over the time $t = 1$, $t = 4$, and $t = 8$

Section 1.2 – Definitions / Techniques of Limits

Definition of the Limit of a Function

If $f(x)$ becomes arbitrary close to a single number L as x approaches x_0 from either side, then

$$\lim_{x \rightarrow x_0} f(x) = L$$

Which is read as “the limit of $f(x)$ as x approaches x_0 is L .”

Notation	Terminology
$x \rightarrow a^-$	x approaches a from the left (through values <i>less</i> than a)
$x \rightarrow a^+$	x approaches a from the right (through values <i>greater</i> than a)

Example

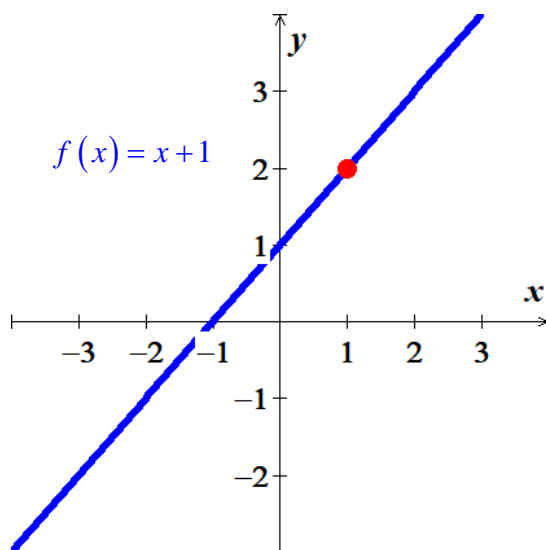
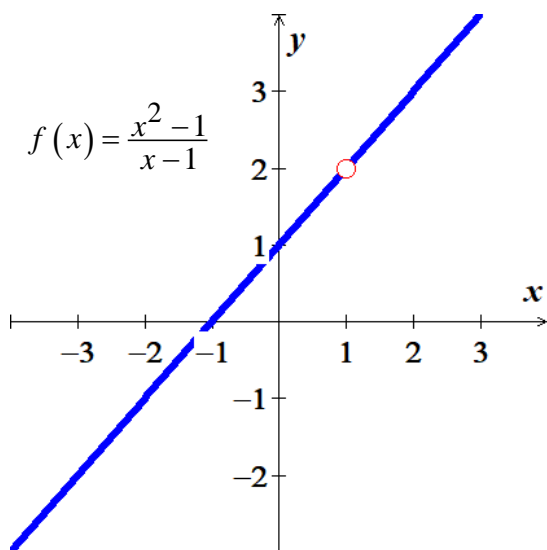
How does the function $f(x) = \frac{x^2 - 1}{x - 1}$ behave near $x = 1$?

Solution

$$\begin{aligned} f(x) &= \frac{(x-1)(x+1)}{x-1} \\ &= x+1 \quad \text{for } x \neq 1 \end{aligned}$$

For $x = 1$:

$$f(x=1) = 1+1 = 2$$

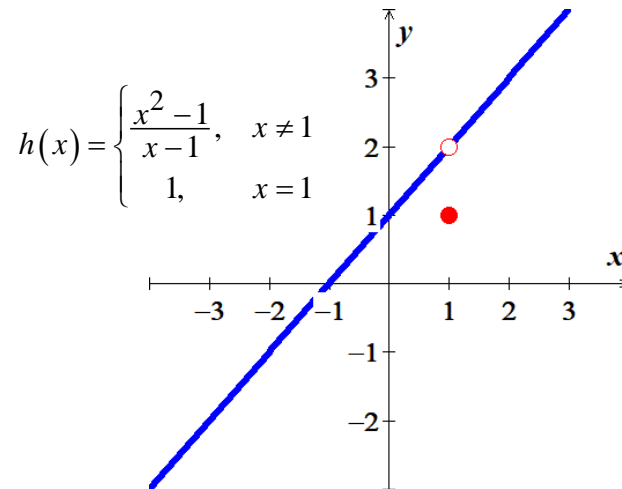
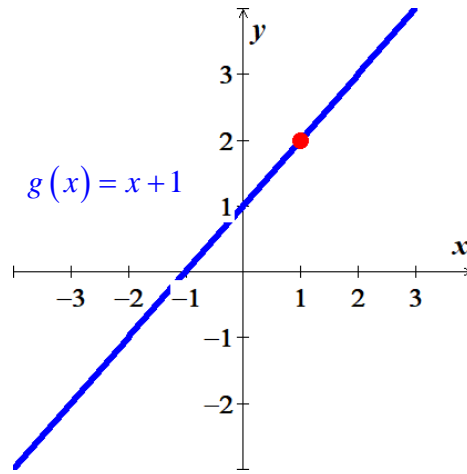
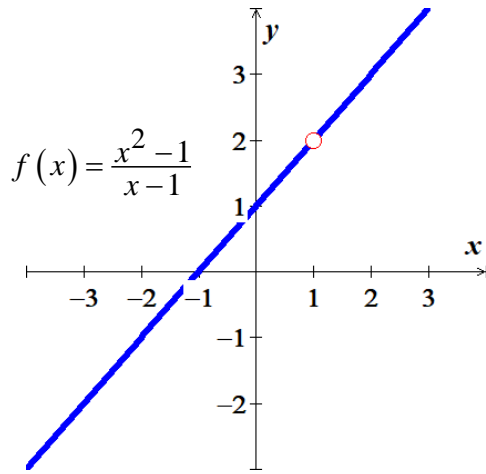


x	.9	.99	.999	1.001	1.01	1.1
-----	----	-----	------	-------	------	-----

$f(x)$	1.9	1.99	1.999	2.001	2.01	2.1
--------	-----	------	-------	-------	------	-----

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$$

$$= 2$$

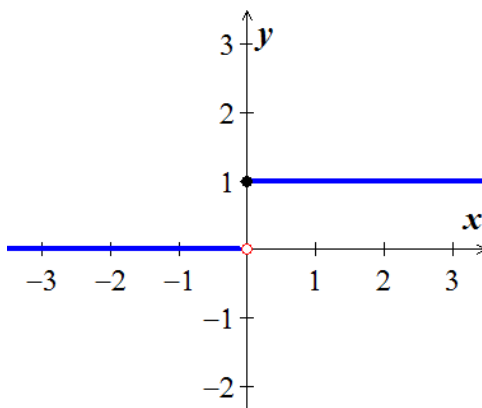


Example

Discuss the behavior of the following function as $x \rightarrow 0$.

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Solution



The unit step function $U(x)$ has no limit as $x \rightarrow 0$, it jumps, because the values jump at $x = 0$.

To the left of zero (*negative value* 0^-) $U(x) = 0$. For the positive values of x close to zero (0^+) $U(x) = 1$

One-Sided Limits

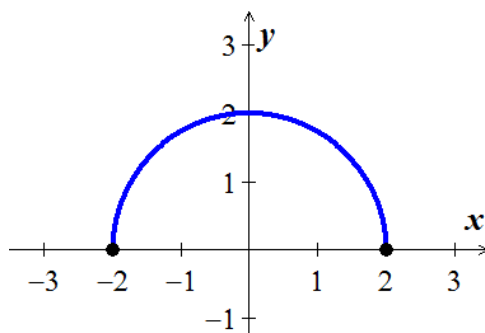
To have a limit L as x approaches c , a function f must be defined on **both sides** of c and its values $f(x)$ must approach L as x approaches c from either side. Because of this, ordinary limits are called **two-sided**. If f fails to have two-sided limit at c , it may still have one-sided limit.

If the approach is from the *right*, the limit is a **right-hand limit**. $\lim_{x \rightarrow c^+} f(x) = L$

If the approach is from the *left*, the limit is a **left-hand limit**. $\lim_{x \rightarrow c^-} f(x) = M$

Example

The domain of $f(x) = \sqrt{4 - x^2}$ is $[-2, 2]$; its graph is the semicircle.



We have: $\lim_{x \rightarrow -2^+} \sqrt{4 - x^2} = 0$ and $\lim_{x \rightarrow 2^-} \sqrt{4 - x^2} = 0$

The function doesn't have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$.

It does not have ordinary two-sided limits at either -2 or 2 .

Theorem

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \rightarrow c} f(x) = L \Leftrightarrow \lim_{x \rightarrow c^-} f(x) = L \text{ and } \lim_{x \rightarrow c^+} f(x) = L$$

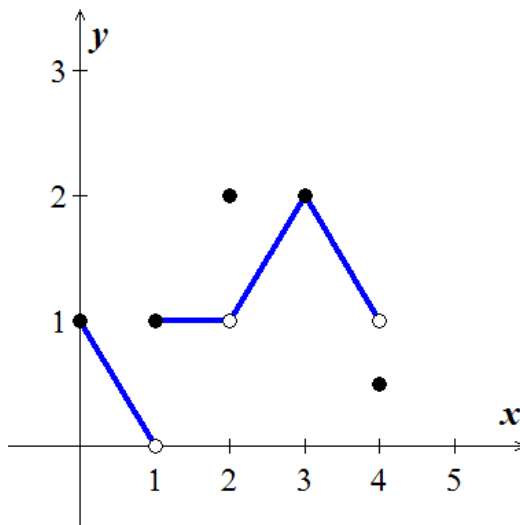
Properties of Limits

Constant function ($f(x) = k$): $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} k = k$

Identity function ($f(x) = x$): $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} x = x_0$

Example

Given the function graphed:



At $x = 0$: $\lim_{x \rightarrow 0^+} f(x) = 1$

$\lim_{x \rightarrow 0^-} f(x)$ and $\lim_{x \rightarrow 0} f(x)$ don't exist. The function is not defined to the left of $x = 0$

At $x = 1$: $\lim_{x \rightarrow 1^-} f(x) = 0$ $\lim_{x \rightarrow 1^+} f(x) = 1$

$\lim_{x \rightarrow 1} f(x)$ doesn't exist. The right-hand and left-hand limits are not equal.

At $x = 2$: $\lim_{x \rightarrow 2^-} f(x) = 1$ $\lim_{x \rightarrow 2^+} f(x) = 1$

$$\lim_{x \rightarrow 2} f(x) = 2 \text{ even though } f(2) = 2$$

At $x = 3$: $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3} f(x) = \underline{2}$

At $x = 4$: $\lim_{x \rightarrow 4^-} f(x) = 1$ even though $f(4) \neq 1$
 $\lim_{x \rightarrow 4^+} f(x)$ and $\lim_{x \rightarrow 4} f(x)$ do not exist.

The function is not defined to the right of $x = 4$

Definitions

We say that $f(x)$ has right-hand limit L at x_0 and $\lim_{x \rightarrow x_0^+} f(x) = L$

If for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \Rightarrow |f(x) - L| < \varepsilon$$

We say that $f(x)$ has left-hand limit L at x_0 and $\lim_{x \rightarrow x_0^-} f(x) = L$

If for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \Rightarrow |f(x) - L| < \varepsilon$$

Example

Prove that $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$

Solution

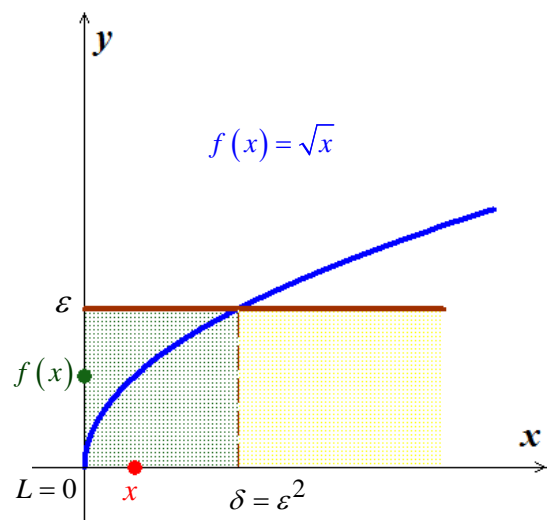
Let $\varepsilon > 0$ be given. $x_0 = 0$, $L = 0$, Find $\delta > 0 \ni \forall x$

$$0 < x < \delta \Rightarrow |\sqrt{x} - 0| < \varepsilon$$

or $0 < x < \delta \Rightarrow \sqrt{x} < \varepsilon$

$$(\sqrt{x})^2 < \varepsilon^2$$

$$\Rightarrow x < \varepsilon^2 \text{ if } 0 < x < \delta$$



If we choose $\delta = \varepsilon^2$, we have

$$0 < x < \delta = \varepsilon^2 \Rightarrow \sqrt{x} < \varepsilon$$

According to the definition, this shows that $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$

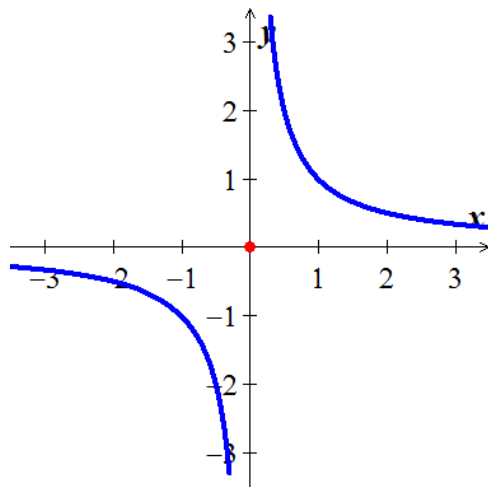
Example

Discuss the behavior of the following function as $x \rightarrow 0$.

$$a) \quad g(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \quad b) \quad f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \frac{1}{x}, & x > 0 \end{cases}$$

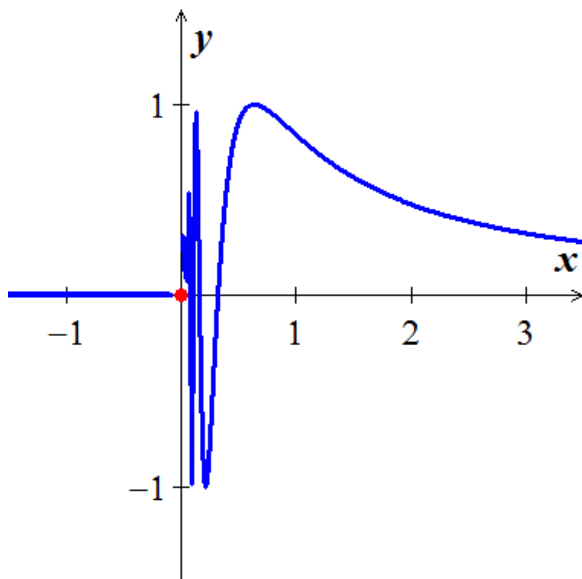
Solution

a)



$g(x)$ has *no limit* as $x \rightarrow 0$ because the values of $g(x)$ grow arbitrary large (negative and positive) value as $x \rightarrow 0$ and do not stay close.

b)



$f(x)$ has *no limit* as $x \rightarrow 0$ because the function's values oscillate between -1 and $+1$ in every open interval containing 0 . The values do not stay close to any one number as $x \rightarrow 0$.

Limit Laws

If $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} g(x) = M$

Constant Multiple Rule: $\lim_{x \rightarrow c} [bf(x)] = b \lim_{x \rightarrow c} f(x) = \underline{bL}$

Sum and Difference Rules: $\lim_{x \rightarrow c} [f(x) \pm g(x)] = \lim_{x \rightarrow c} f(x) \pm \lim_{x \rightarrow c} g(x) = \underline{L \pm M}$

Product Rule: $\lim_{x \rightarrow c} [f(x) \cdot g(x)] = \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x) = \underline{LM}$

Quotient Rule: $\lim_{x \rightarrow c} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} = \underline{\frac{L}{M}} \quad M \neq 0$

Power Rule: $\lim_{x \rightarrow c} (f(x))^n = \left[\lim_{x \rightarrow c} f(x) \right]^n = \underline{L^n}$

Root Rule: $\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow c} f(x)} = \underline{\sqrt[n]{L}} \quad n > 0, \quad L > 0, \quad n \text{ is even}$

Example

Find the following limits:

$$a) \lim_{x \rightarrow c} (x^3 + 4x^2 - 3) \quad b) \lim_{x \rightarrow c} \frac{x^4 + x^2 - 1}{x^2 + 5}$$

$$c) \lim_{x \rightarrow -2} \sqrt{4x^2 - 3}$$

Solution

$$a) \lim_{x \rightarrow c} (x^3 + 4x^2 - 3) = \lim_{x \rightarrow c} x^3 + \lim_{x \rightarrow c} 4x^2 - \lim_{x \rightarrow c} (3) \\ = \underline{c^3 + 4c^2 - 3}$$

Sum and Difference Rules

$$b) \lim_{x \rightarrow c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \rightarrow c} (x^4 + x^2 - 1)}{\lim_{x \rightarrow c} (x^2 + 5)} \\ = \frac{\lim_{x \rightarrow c} x^4 + \lim_{x \rightarrow c} x^2 - \lim_{x \rightarrow c} 1}{\lim_{x \rightarrow c} x^2 + \lim_{x \rightarrow c} 5} \\ = \underline{\frac{c^4 + c^2 - 1}{c^2 + 5}}$$

Quotient Rule

Sum and Difference Rules

$$c) \lim_{x \rightarrow -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x \rightarrow -2} (4x^2 - 3)} \\ = \sqrt{\lim_{x \rightarrow -2} 4x^2 - \lim_{x \rightarrow -2} 3} \\ = \sqrt{4(-2)^2 - 3} \\ = \sqrt{16 - 3} \\ = \underline{\sqrt{13}}$$

Root Rule

Difference Rule

Theorem – Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, then $\lim_{x \rightarrow c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c + a_0$

Theorem – Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then $\lim_{x \rightarrow c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$

Example

Find the limit: $\lim_{x \rightarrow -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5}$

Solution

$$\begin{aligned} \lim_{x \rightarrow -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} &= \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} \\ &= \frac{0}{6} \\ &= 0 \end{aligned}$$

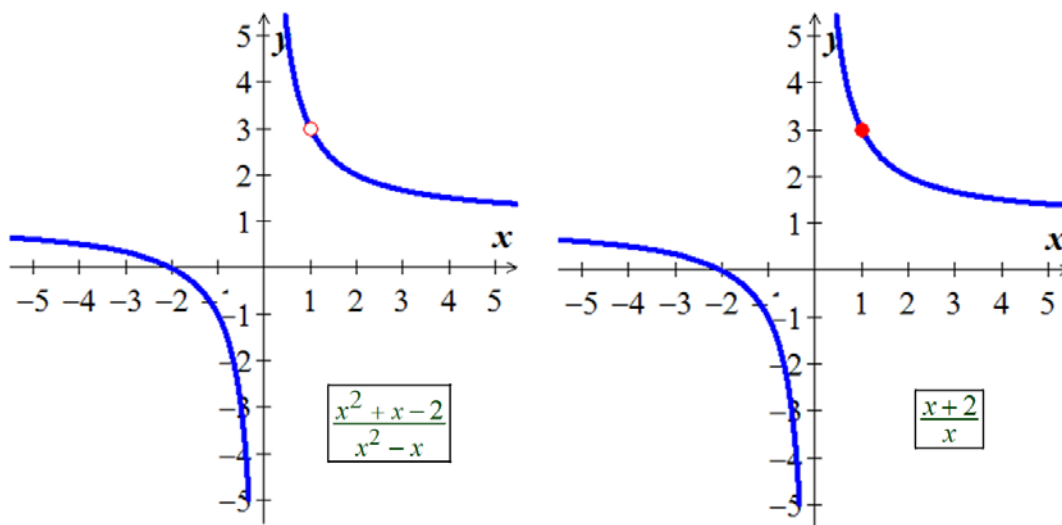
Eliminating Zero Denominators Algebraically

Example

Evaluate: $\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x}$

Solution

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x} &= \frac{1^2 + 1 - 2}{1^2 - 1} = \frac{0}{0} \\ \lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x} &= \lim_{x \rightarrow 1} \frac{(x-1)(x+2)}{x(x-1)} \\ &= \lim_{x \rightarrow 1} \frac{(x+2)}{x} \\ &= \frac{1+2}{1} \\ &= 3 \end{aligned}$$



Example

Evaluate: $\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}$

Solution

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} = \frac{\sqrt{0 + 100} - 10}{0} = \frac{0}{0}$$

$$\frac{\sqrt{x^2 + 100} - 10}{x^2} = \frac{\sqrt{x^2 + 100} - 10}{x^2} \cdot \frac{\sqrt{x^2 + 100} + 10}{\sqrt{x^2 + 100} + 10}$$

$$= \frac{x^2 + 100 - 100}{x^2(\sqrt{x^2 + 100} + 10)}$$

$$= \frac{x^2}{x^2(\sqrt{x^2 + 100} + 10)}$$

$$= \frac{1}{\sqrt{x^2 + 100} + 10}$$

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x^2 + 100} + 10}$$

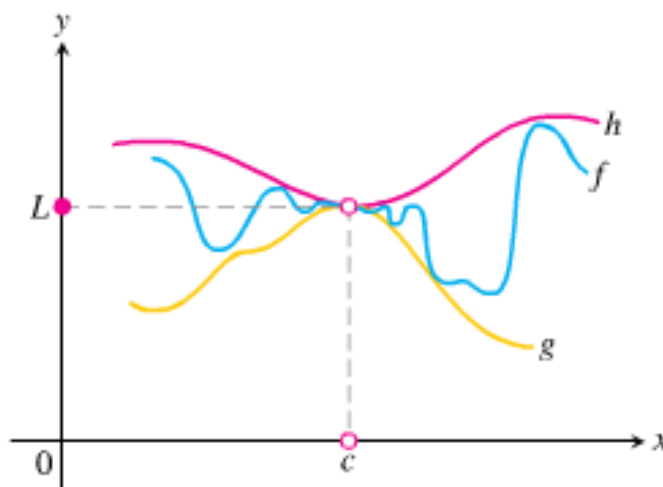
$$= \frac{1}{\sqrt{0 + 100} + 10}$$

$$= \frac{1}{10 + 10}$$

$$= \frac{1}{20}$$

$$(a - b)(a + b) = a^2 - b^2; \quad (\sqrt{a})^2 = a$$

The Sandwich (Squeeze) Theorem



Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c , except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L \quad \text{then} \quad \lim_{x \rightarrow c} f(x) = L$$

Example

Given that $1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2}$ for all $x \neq 0$, find the $\lim_{x \rightarrow 0} u(x)$, no matter how complicated u is.

Solution

$$\lim_{x \rightarrow 0} \left(1 - \frac{x^2}{4} \right) = 1 - \frac{0}{4}$$

$$\underline{= 1}$$

$$\lim_{x \rightarrow 0} \left(1 + \frac{x^2}{2} \right) = 1$$

The Sandwich theorem implies that $\lim_{x \rightarrow 0} u(x) = 1$

Theorem

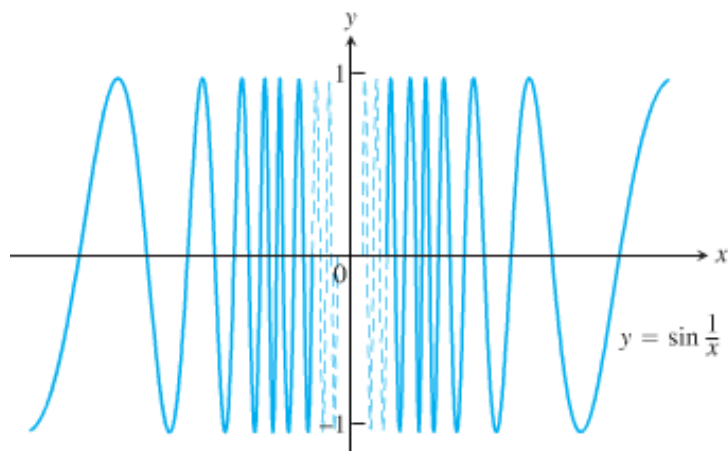
Suppose that $f(x) \leq g(x)$ for all x in some open interval containing c , except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c , then

$$\lim_{x \rightarrow c} f(x) \leq \lim_{x \rightarrow c} g(x)$$

Example

Show that $y = \sin\left(\frac{1}{x}\right)$ has no limit as x approaches zero from either side.

Solution

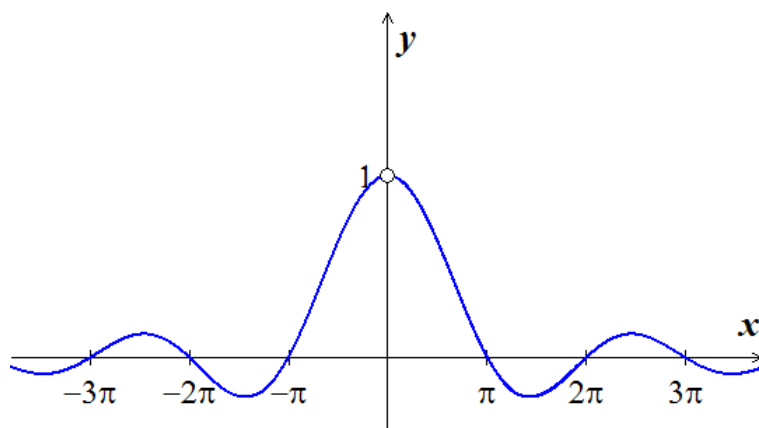


As x approaches zero, its reciprocal, $\frac{1}{x}$, grows without bound and the values of $\sin\left(\frac{1}{x}\right)$ cycle repeatedly from -1 to 1 .

There is no single number L that the function's values stay increasingly close to as x approaches zero.

The function has neither a right-hand limit nor a left-hand limit at $x = 0$.

Limit Involving $\frac{\sin \theta}{\theta}$



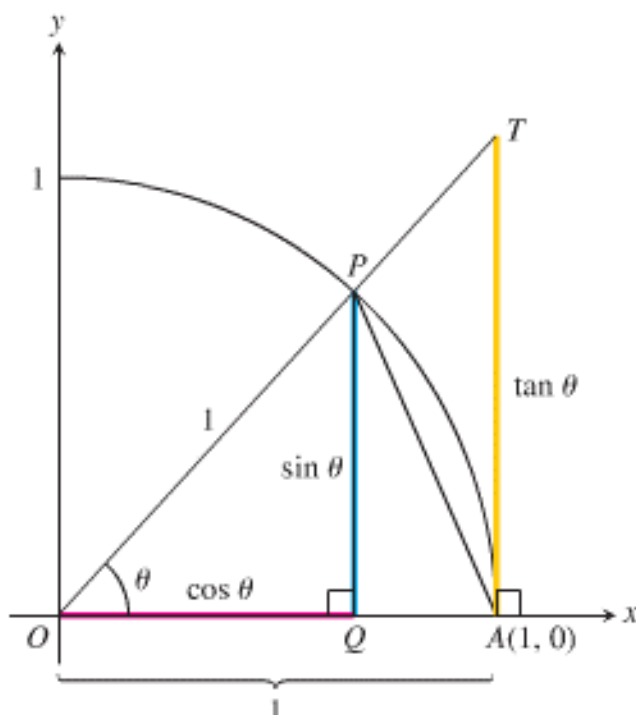
A central fact about $\frac{\sin \theta}{\theta}$ is that in radian measure its limit as $\theta \rightarrow 0$ is **1**.

Theorem

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \quad (\theta \text{ in rad.})$$

Proof

We need to show that the right-hand limit is 1, $\theta < \frac{\pi}{2}$



Notice that:

$$\text{Area } \triangle OAP < \text{Area Sector } OAP < \text{Area } \triangle OAT$$

$$\text{Area } \triangle OAP = \frac{1}{2} \text{base} \times \text{height} = \frac{1}{2}(1)(\sin \theta)$$

$$\text{Area Sector } \triangle OAP = \frac{1}{2}r^2 \times \theta = \frac{1}{2}(1)^2(\theta) = \frac{\theta}{2}$$

$$\text{Area } \triangle OAP = \frac{1}{2} \text{base} \times \text{height} = \frac{1}{2}(1)(\tan \theta) = \frac{1}{2} \tan \theta$$

$$\Rightarrow \frac{1}{2} \sin \theta < \frac{1}{2} \theta < \frac{1}{2} \tan \theta$$

$$\frac{2}{\sin \theta} \frac{1}{2} \sin \theta < \frac{1}{2} \theta \frac{2}{\sin \theta} < \frac{1}{2} \frac{\sin \theta}{\cos \theta} \frac{2}{\sin \theta}$$

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta} \quad \text{Taking reciprocals reverses the inequalities}$$

$$1 > \frac{\sin \theta}{\theta} > \cos \theta$$

Since $\lim_{\theta \rightarrow 0^+} \cos \theta = 1$, then

$$\lim_{\theta \rightarrow 0^-} \frac{\sin \theta}{\theta} = 1 = \lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta}$$

$$\text{So } \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

Example

Show that $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$

Solution

Using the half-angle formula: $\cos x = 1 - 2 \sin^2 \left(\frac{x}{2} \right)$

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = \lim_{x \rightarrow 0} \frac{1 - 2 \sin^2 \left(\frac{x}{2} \right) - 1}{x}$$

$$= \lim_{x \rightarrow 0} \frac{-2 \sin^2 \left(\frac{x}{2} \right)}{x}$$

$$\text{Let } \theta = \frac{x}{2}$$

$$= - \lim_{\theta \rightarrow 0} \frac{2 \sin^2(\theta)}{2\theta}$$

$$= - \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \sin \theta$$

$$= -(1)(0)$$

$$= 0$$

Example

Show that $\lim_{x \rightarrow 0} \frac{\sin 2x}{5x} = \frac{2}{5}$

Solution

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{5x} = \lim_{x \rightarrow 0} \frac{\left(\frac{2}{5}\right) \sin 2x}{\left(\frac{2}{5}\right) 5x}$$

Since we need 2x in the denominator

$$= \frac{2}{5} \lim_{x \rightarrow 0} \frac{\sin 2x}{2x}$$

$$= \frac{2}{5}(1)$$

$$\underline{= \frac{2}{5}} \quad \Bigg|$$

Example

Show that $\lim_{x \rightarrow 0} \frac{\tan x \sec 2x}{3x} = \frac{1}{3}$

Solution

$$\lim_{x \rightarrow 0} \frac{\tan x \sec 2x}{3x} = \frac{1}{3} \lim_{x \rightarrow 0} \frac{1}{x} \cdot \frac{\sin x}{\cos x} \cdot \frac{1}{\cos 2x}$$

$$= \frac{1}{3} \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} \cdot \frac{1}{\cos 2x} \quad \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \quad \lim_{x \rightarrow 0} \frac{1}{\cos x} = 1, \quad \lim_{x \rightarrow 0} \frac{1}{\cos 2x} = 1$$

$$= \frac{1}{3}(1)(1)(1)$$

$$\underline{= \frac{1}{3}} \quad \Bigg|$$

Exercises Section 1.2 – Definitions / Techniques of Limits

(1 – 121) Find the limit:

1. $\lim_{x \rightarrow 3} (-1)$
2. $\lim_{x \rightarrow -1} 3$
3. $\lim_{x \rightarrow 1000} 18\pi^2$
4. $\lim_{x \rightarrow 1} \sqrt{5x+6}$
5. $\lim_{x \rightarrow 9} \sqrt{x}$
6. $\lim_{x \rightarrow -3} (x^2 + 3x)$
7. $\lim_{x \rightarrow -4} |x-4|$
8. $\lim_{x \rightarrow 4} (x+2)$
9. $\lim_{x \rightarrow 4} (x-4)$
10. $\lim_{x \rightarrow 2} (5x-6)^{3/2}$
11. $\lim_{x \rightarrow 9} \frac{x-9}{\sqrt{x}-3}$
12. $\lim_{x \rightarrow 1} (2x+4)$
13. $\lim_{x \rightarrow 1} \frac{x^2-4}{x-2}$
14. $\lim_{x \rightarrow 2} \frac{x^2+4}{x-2}$
15. $\lim_{x \rightarrow 0} \frac{|x|}{x}$
16. $\lim_{x \rightarrow 3} \frac{x^2-x-1}{\sqrt{x+1}}$
17. $\lim_{x \rightarrow 2} \frac{x^2+x-6}{x-2}$
18. $\lim_{x \rightarrow 0} (3x-2)$
19. $\lim_{x \rightarrow 1} (2x^2 - x + 4)$
20. $\lim_{x \rightarrow -2} (x^3 - 2x^2 + 4x + 8)$
21. $\lim_{x \rightarrow 2} \frac{x^2-4}{x-2}$
22. $\lim_{x \rightarrow 2} \frac{x^3-8}{x-2}$
23. $\lim_{x \rightarrow 3} \frac{x^2+x-12}{x-3}$
24. $\lim_{x \rightarrow 0} \frac{\sqrt{x+4}-2}{x}$
25. $\lim_{x \rightarrow -2} \frac{5}{x+2}$
26. $\lim_{x \rightarrow 0} \frac{3}{\sqrt{3x+1}+1}$
27. $\lim_{x \rightarrow 3} \frac{\sqrt{x+1}-1}{x}$
28. $\lim_{x \rightarrow 1} \frac{x^2-1}{x-1}$
29. $\lim_{x \rightarrow -2} \frac{|x+2|}{x+2}$
30. $\lim_{x \rightarrow 0} (2z-8)^{1/3}$
31. $\lim_{x \rightarrow 2} \frac{x^2-7x+10}{x-2}$
32. $\lim_{x \rightarrow 0} \frac{5x^3+8x^2}{3x^4-16x^2}$
33. $\lim_{x \rightarrow 1} \frac{1-x}{x-1}$
34. $\lim_{u \rightarrow 1} \frac{u^4-1}{u^3-1}$
35. $\lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x+3}-2}$
36. $\lim_{x \rightarrow -1} \frac{\sqrt{x^2+8}-3}{x+1}$
37. $\lim_{x \rightarrow -3} \frac{2-\sqrt{x^2-5}}{x+3}$
38. $\lim_{x \rightarrow 0} (2\sin x - 1)$
39. $\lim_{x \rightarrow 0} \sin^2 x$
40. $\lim_{x \rightarrow 0} \sec x$
41. $\lim_{x \rightarrow 0} \frac{1+x+\sin x}{3\cos x}$
42. $\lim_{x \rightarrow -\pi} \sqrt{x+4} \cos(x+\pi)$
43. $\lim_{x \rightarrow -0.5^-} \sqrt{\frac{x+2}{x+1}}$
44. $\lim_{x \rightarrow 1^+} \sqrt{\frac{x-1}{x+2}}$
45. $\lim_{x \rightarrow -2^+} \left(\frac{x}{x+1} \right) \left(\frac{2x+5}{x^2+x} \right)$
46. $\lim_{x \rightarrow 0^+} \frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}$
47. $\lim_{x \rightarrow -2^+} (x+3) \frac{|x+2|}{x+2}$
48. $\lim_{x \rightarrow 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|}$

49. $\lim_{x \rightarrow 0^-} \frac{x}{\sin 3x}$
50. $\lim_{\theta \rightarrow 0} \frac{\sin \sqrt{2} \cdot \theta}{\sqrt{2} \cdot \theta}$
51. $\lim_{x \rightarrow 0} \frac{\sin 3x}{4x}$
52. $\lim_{x \rightarrow 0} \frac{\tan 2x}{x}$
53. $\lim_{x \rightarrow 0} 6x^2 (\cot x) (\csc 2x)$
54. $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\sin 2\theta}$
55. $\lim_{h \rightarrow 0} \frac{\sin(\sin h)}{\sin h}$
56. $\lim_{\theta \rightarrow 0} \frac{\theta \cot 4\theta}{\sin^2 \theta \cot^2 2\theta}$
57. $\lim_{\theta \rightarrow \pi/4} \frac{\sin^2 \theta - \cos^2 \theta}{\sin \theta - \cos \theta}$
58. $\lim_{x \rightarrow \pi/2} \frac{\frac{1}{\sqrt{\sin x}} - 1}{x + \frac{\pi}{2}}$
59. $\lim_{x \rightarrow 1} \frac{x^3 - 7x^2 + 12x}{4 - x}$
60. $\lim_{x \rightarrow 4} \frac{x^3 - 7x^2 + 12x}{4 - x}$
61. $\lim_{x \rightarrow 1} \frac{1 - x^2}{x^2 - 8x + 7}$
62. $\lim_{x \rightarrow 3} \frac{\sqrt{3x+16} - 5}{x - 3}$
63. $\lim_{x \rightarrow 3} \frac{1}{x-3} \left(\frac{1}{\sqrt{x+1}} - \frac{1}{2} \right)$
64. $\lim_{x \rightarrow 1/3} \frac{x - \frac{1}{3}}{(3x-1)^2}$
65. $\lim_{x \rightarrow 3} \frac{x^4 - 81}{x - 3}$
66. $\lim_{x \rightarrow 1} \frac{x^5 - 1}{x - 1}$
67. $\lim_{x \rightarrow 81} \frac{\sqrt[4]{x} - 3}{x - 81}$
68. $\lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{x - 1}$
69. $\lim_{x \rightarrow 2} \frac{x^5 - 32}{x - 2}$
70. $\lim_{x \rightarrow 1} \frac{x^6 - 1}{x - 1}$
71. $\lim_{x \rightarrow -1} \frac{x^7 + 1}{x + 1}$
72. $\lim_{x \rightarrow a} \frac{x^5 - a^5}{x - a}$
73. $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} \quad n \in \mathbb{Z}^+$
74. $\lim_{h \rightarrow 0} \frac{100}{(10h-1)^{11} + 2}$
75. $\lim_{h \rightarrow 0} \frac{(5+h)^2 - 25}{h}$
76. $\lim_{x \rightarrow 3} \frac{\frac{1}{x^2 + 2x} - \frac{1}{15}}{x - 3}$
77. $\lim_{x \rightarrow 1} \frac{\sqrt{10x-9} - 1}{x - 1}$
78. $\lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{2}{x^2 - 2x} \right)$
79. $\lim_{x \rightarrow c} \frac{x^2 - 2cx + c^2}{x - c}$
80. $\lim_{x \rightarrow -c} \frac{x^2 + 5cx + 4c^2}{x^2 + cx}$
81. $\lim_{x \rightarrow 16} \frac{\sqrt[4]{x} - 2}{x - 16}$
82. $\lim_{x \rightarrow 1} \frac{x - 1}{\sqrt{x} - 1}$
83. $\lim_{x \rightarrow 1} \frac{x - 1}{\sqrt{4x+5} - 3}$
84. $\lim_{x \rightarrow 4} \frac{3(x-4)\sqrt{x+5}}{3 - \sqrt{x+5}}$
85. $\lim_{x \rightarrow 0} \frac{x}{\sqrt{ax+1} - 1} \quad (a \neq 0)$
86. $\lim_{x \rightarrow \pi} \frac{\cos^2 x + 3\cos x + 2}{\cos x + 1}$
87. $\lim_{x \rightarrow \frac{3\pi}{2}} \frac{\sin^2 x + 6\sin x + 5}{\sin^2 x - 1}$
88. $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x - 1}{\sqrt{\sin x} - 1}$
89. $\lim_{x \rightarrow 0} \frac{\frac{1}{2 + \sin x} - \frac{1}{2}}{\sin x}$
90. $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{e^x - 1}$
91. $\lim_{x \rightarrow \frac{\pi}{4}} \csc x$
92. $\lim_{x \rightarrow 4} \frac{x - 5}{(x^2 - 10x + 24)^2}$
93. $\lim_{x \rightarrow 0} \frac{\cos x - 1}{\sin^2 x}$
94. $\lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{\sin x}$
95. $\lim_{x \rightarrow 0} \frac{x^3 - 5x^2}{x^2}$
96. $\lim_{x \rightarrow 5} \frac{4x^2 - 100}{x - 5}$

$$97. \lim_{x \rightarrow 3} \frac{\sqrt{9-6x+x^2}}{x-3}$$

$$98. \lim_{x \rightarrow 3} \frac{\sqrt{9+6x+x^2}}{x-3}$$

$$99. \lim_{x \rightarrow 3} \frac{\sqrt{x^2-9}}{x-3}$$

$$100. \lim_{x \rightarrow \frac{4\pi}{3}} \sin x$$

$$101. \lim_{x \rightarrow \frac{2\pi}{3}} \cos x$$

$$102. \lim_{x \rightarrow \frac{7\pi}{4}} \sin x$$

$$103. \lim_{x \rightarrow 1} \frac{\sin \sqrt{1-x}}{\sqrt{1-x^2}}$$

$$104. \lim_{x \rightarrow 2} \frac{\sin \sqrt{2-x}}{\sqrt{4-x^2}}$$

$$105. \lim_{x \rightarrow 0} \frac{\sin(\sqrt{5} x)}{\sin(\sqrt{3} x)}$$

$$106. \lim_{x \rightarrow 0} \frac{\sin(\sqrt{15} x)}{\sin(\sqrt{3} x)}$$

$$107. \lim_{x \rightarrow 0^+} \frac{x - \sqrt{x}}{\sqrt{\sin x}}$$

$$108. \lim_{x \rightarrow 1} \frac{x - \sqrt{x}}{\sqrt{\sin x}}$$

$$109. \lim_{x \rightarrow \pi} \frac{x - \sqrt{x}}{\sqrt{\sin x}}$$

$$110. \lim_{x \rightarrow 0} e^{x^3}$$

$$111. \lim_{x \rightarrow 1} e^{x^2}$$

$$112. \lim_{x \rightarrow 1} e^{x^3-1}$$

$$113. \lim_{x \rightarrow -1} e^{x^3-1}$$

$$114. \lim_{x \rightarrow 2} (e^{x^2} - \ln x)$$

$$115. \lim_{x \rightarrow 1} (e^{x^2} - \ln x)$$

$$116. \lim_{x \rightarrow e} \ln x$$

$$117. \lim_{x \rightarrow e} \ln x^2$$

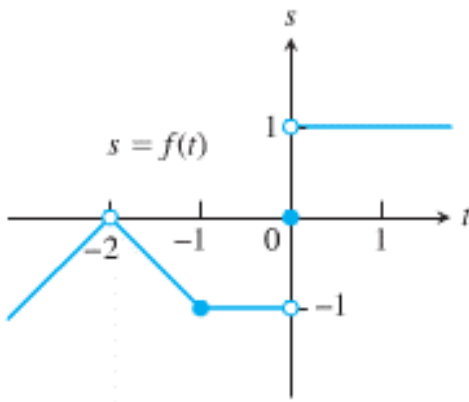
$$118. \lim_{x \rightarrow 0^+} \ln x$$

$$119. \lim_{x \rightarrow 1} \frac{1}{\ln x}$$

$$120. \lim_{x \rightarrow e} \ln e^{2x}$$

$$121. \lim_{x \rightarrow 1} \ln e^{x^2}$$

122. For the function $f(t)$ graphed, find the following limits or explain why they do not exist.



$$a) \lim_{t \rightarrow -2} f(t)$$

$$b) \lim_{t \rightarrow -1} f(t)$$

$$c) \lim_{t \rightarrow 0} f(t)$$

$$d) \lim_{t \rightarrow -0.5} f(t)$$

123. Suppose $\lim_{x \rightarrow c} f(x) = 5$ and $\lim_{x \rightarrow c} g(x) = -2$. Find

$$a) \lim_{x \rightarrow c} f(x)g(x)$$

$$c) \lim_{x \rightarrow c} (f(x) + 3g(x))$$

$$b) \lim_{x \rightarrow c} 2f(x)g(x)$$

$$d) \lim_{x \rightarrow c} \frac{f(x)}{f(x) - g(x)}$$

124. Explain why the limits do not exist for $\lim_{x \rightarrow 0} \frac{x}{|x|}$

(125 – 126) Evaluate the limit using the form $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ for

125. $f(x) = x^2, \quad x = 1$

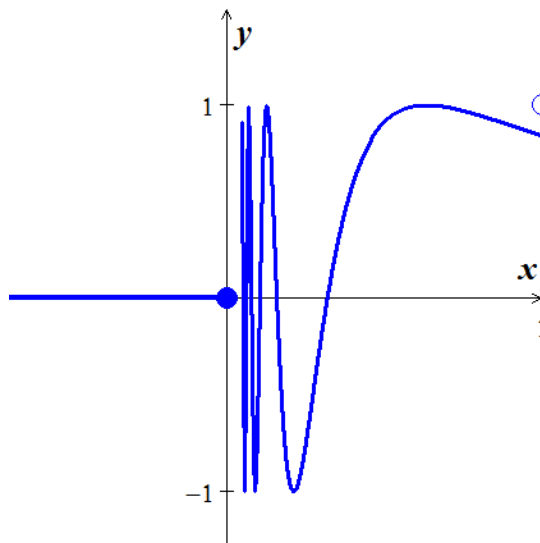
126. $f(x) = \sqrt{3x+1}, \quad x = 0$

127. If $\lim_{x \rightarrow 4} \frac{f(x) - 5}{x - 2} = 1$, find $\lim_{x \rightarrow 4} f(x)$

128. If $\lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 1$, find $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 0} \frac{f(x)}{x}$

129. If $x^4 \leq f(x) \leq x^2 \quad -1 \leq x \leq 1$ and $x^2 \leq f(x) \leq x^4 \quad x < -1$ and $x > 1$. At what points c do you automatically know $\lim_{x \rightarrow c} f(x)$? What can you say about the value of the limits at these points?

130. Let $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \frac{1}{x}, & x > 0 \end{cases}$

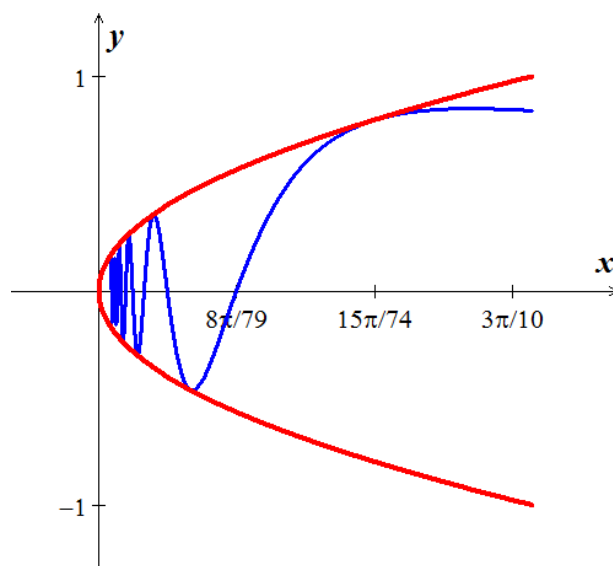


a) Does $\lim_{x \rightarrow 0^+} f(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim_{x \rightarrow 0^-} f(x)$ exist? If so, what is it? If not, why not?

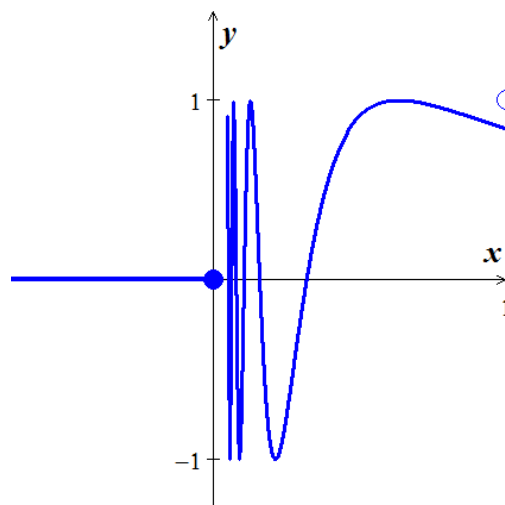
c) Does $\lim_{x \rightarrow 0} f(x)$ exist? If so, what is it? If not, why not?

131. Let $g(x) = \sqrt{x} \sin \frac{1}{x}$



- a) Does $\lim_{x \rightarrow 0^+} g(x)$ exist? If so, what is it? If not, why not?
- b) Does $\lim_{x \rightarrow 0^-} g(x)$ exist? If so, what is it? If not, why not?
- c) Does $\lim_{x \rightarrow 0} g(x)$ exist? If so, what is it? If not, why not?

132. Let $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \frac{1}{x}, & x > 0 \end{cases}$



- d) Does $\lim_{x \rightarrow 0^+} f(x)$ exist? If so, what is it? If not, why not?
- e) Does $\lim_{x \rightarrow 0^-} f(x)$ exist? If so, what is it? If not, why not?

f) Does $\lim_{x \rightarrow 0} f(x)$ exist? If so, what is it? If not, why not?

133. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

a) $\lim_{x \rightarrow -1^+} f(x) = 1$

g) $\lim_{x \rightarrow 0} f(x) = 1$

b) $\lim_{x \rightarrow 0^-} f(x) = 0$

h) $\lim_{x \rightarrow 1} f(x) = 1$

c) $\lim_{x \rightarrow 0^-} f(x) = 1$

i) $\lim_{x \rightarrow 1} f(x) = 0$

d) $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x)$

j) $\lim_{x \rightarrow 2^-} f(x) = 2$

e) $\lim_{x \rightarrow 0} f(x)$ exists

k) $\lim_{x \rightarrow -1^-} f(x) = 0$ does not exist

f) $\lim_{x \rightarrow 0} f(x) = 0$

l) $\lim_{x \rightarrow 2^+} f(x) = 0$

Section 1.3 – Infinite Limits

Definitions

We say that $f(x)$ has the **limit L as x approaches infinity** and write $\lim_{x \rightarrow \infty} f(x) = L$

$$\text{If, } \forall \varepsilon > 0 \exists N \ni \forall x, \quad x > M \Rightarrow |f(x) - L| < \varepsilon$$

We say that $f(x)$ has the **limit L as x approaches *minus* infinity** and write $\lim_{x \rightarrow -\infty} f(x) = L$

$$\text{If, } \forall \varepsilon > 0 \exists N \ni \forall x, \quad x < M \Rightarrow |f(x) - L| < \varepsilon$$

Basic Facts: $\lim_{x \rightarrow \pm\infty} k = k$ and $\lim_{x \rightarrow \pm\infty} \frac{1}{x} = 0$

Example

Find $\lim_{x \rightarrow \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2}$

Solution

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} &= \lim_{x \rightarrow \infty} \frac{5 + \frac{8}{x} - \frac{3}{x^2}}{3 + \frac{2}{x^2}} \\ &= \frac{5 + 0 - 0}{3 + 0} \\ &= \frac{5}{3} \end{aligned}$$

Divide by x^2

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x} = 0$$

Example

Find $\lim_{x \rightarrow \infty} \frac{11x + 2}{2x^3 - 1}$

Solution

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{11x + 2}{2x^3 - 1} &= \lim_{x \rightarrow \infty} \frac{\frac{11}{x^2} + \frac{2}{x^3}}{2 - \frac{1}{x^3}} \\ &= \frac{0 + 0}{2 - 0} \\ &= 0 \end{aligned}$$

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x} = 0$$

Vertical Asymptote (VA) - Think Domain

The line $x = a$ is a **vertical asymptote** for the graph of a function f if

$$\lim_{x \rightarrow a^+} f(x) \rightarrow \pm\infty \quad \text{or} \quad \lim_{x \rightarrow a^-} f(x) \rightarrow \pm\infty$$

As x approaches a from either the left or the right

$$\lim_{x \rightarrow 0^+} \frac{1}{x} \rightarrow \infty \quad \text{or} \quad \lim_{x \rightarrow 0^-} \frac{1}{x} \rightarrow -\infty$$

Example

Find $\lim_{x \rightarrow 3^+} \frac{2-5x}{x-3}$ and $\lim_{x \rightarrow 3^-} \frac{2-5x}{x-3}$

Solution

$$\begin{aligned} \lim_{x \rightarrow 3^+} \frac{2-5x}{x-3} &= \frac{2-5(3)}{3^+-3} \rightarrow -13 \\ &\rightarrow \text{positive and approaches 0} \\ &= -\infty \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 3^-} \frac{2-5x}{x-3} &= \frac{2-5(3)}{3^--3} \rightarrow -13 \\ &\rightarrow \text{negative and approaches 0} \\ &= \infty \end{aligned}$$

Example

Find $\lim_{x \rightarrow -4^+} \frac{-x^3 + 5x^2 - 6x}{-x^3 - 4x^2}$

Solution

$$\begin{aligned} \lim_{x \rightarrow -4^+} \frac{-x^3 + 5x^2 - 6x}{-x^3 - 4x^2} &= \frac{168}{0} \\ \frac{-x^3 + 5x^2 - 6x}{-x^3 - 4x^2} &= \frac{(x-2)(x-3)}{x(x+4)} \rightarrow \text{positive} \\ &\rightarrow \text{negative and approaches 0} \end{aligned}$$

$$\lim_{x \rightarrow -4^+} \frac{-x^3 + 5x^2 - 6x}{-x^3 - 4x^2} = -\infty$$

Example

Let $f(x) = \frac{x^2 - 4x + 3}{x^2 - 1}$, determine the following limits and find the vertical asymptotes of f .

a) $\lim_{x \rightarrow 1} f(x)$

b) $\lim_{x \rightarrow -1^-} f(x)$

c) $\lim_{x \rightarrow -1^+} f(x)$

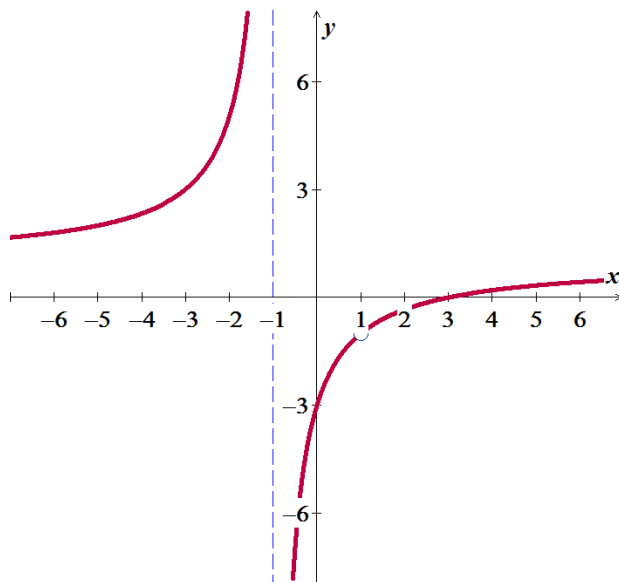
Solution

$$\begin{aligned} a) \quad \lim_{x \rightarrow 1} \frac{x^2 - 4x + 3}{x^2 - 1} &= \frac{0}{0} = \lim_{x \rightarrow 1} \frac{(x-1)(x-3)}{(x-1)(x+1)} \\ &= \lim_{x \rightarrow 1} \frac{x-3}{x+1} \\ &= \underline{-1} \end{aligned}$$

The vertical asymptote: $\underline{x = -1}$, while the hole is $\underline{(1, -1)}$

$$\begin{aligned} b) \quad \lim_{x \rightarrow -1^-} f(x) &= \lim_{x \rightarrow -1^-} \frac{x-3}{x+1} \quad \rightarrow \text{negative} \\ &\quad \rightarrow \text{negative and approaches 0} \\ &= \underline{\infty} \end{aligned}$$

$$\begin{aligned} c) \quad \lim_{x \rightarrow -1^+} f(x) &= \lim_{x \rightarrow -1^+} \frac{x-3}{x+1} \quad \rightarrow \text{negative} \\ &\quad \rightarrow \text{positive and approaches 0} \\ &= \underline{-\infty} \end{aligned}$$



Example

Find $\lim_{\theta \rightarrow 0^+} \cot \theta$ and $\lim_{\theta \rightarrow 0^-} \cot \theta$

Solution

$$\cot \theta = \frac{\cos \theta}{\sin \theta} \Rightarrow \cot 0 = \frac{1}{0},$$

As $\theta \rightarrow 0^+$ $\cos \theta > 0$; $\sin \theta > 0$

$$\lim_{\theta \rightarrow 0^+} \cot \theta = \underline{\underline{\infty}}$$

As $\theta \rightarrow 0^-$ $\cos \theta > 0$; $\sin \theta < 0$

$$\lim_{\theta \rightarrow 0^-} \cot \theta = \underline{\underline{-\infty}}$$

Exercises Section 1.3 – Infinite Limits

(1 – 50) Find

1. $\lim_{x \rightarrow 5} \frac{x-7}{x(x-5)^2}$

2. $\lim_{x \rightarrow -5^+} \frac{x-5}{x+5}$

3. $\lim_{x \rightarrow 3^-} \frac{x-4}{x^2-3x}$

4. $\lim_{x \rightarrow 0^+} \frac{1}{3x}$

5. $\lim_{x \rightarrow -5^-} \frac{3x}{2x+10}$

6. $\lim_{x \rightarrow 0} \frac{1}{x^{2/3}}$

7. $\lim_{x \rightarrow 0^-} \frac{1}{3x^{1/3}}$

8. $\lim_{x \rightarrow \left(-\frac{\pi}{2}\right)^+} \sec x$

9. $\lim_{\theta \rightarrow 0^-} (1 + \csc \theta)$

10. $\lim_{\theta \rightarrow 0^+} \csc \theta$

11. $\lim_{x \rightarrow 0^+} (-10 \cot x)$

12. $\lim_{\theta \rightarrow \frac{\pi}{2}^+} \frac{1}{3} \tan \theta$

13. $\lim_{x \rightarrow 2^+} \frac{1}{x-2}$

14. $\lim_{x \rightarrow 2^-} \frac{1}{x-2}$

15. $\lim_{x \rightarrow 2} \frac{1}{x-2}$

16. $\lim_{x \rightarrow 3^+} \frac{2}{(x-3)^3}$

17. $\lim_{x \rightarrow 3^-} \frac{2}{(x-3)^3}$

18. $\lim_{x \rightarrow 3} \frac{2}{(x-3)^3}$

19. $\lim_{x \rightarrow 4^+} \frac{x-5}{(x-4)^2}$

20. $\lim_{x \rightarrow 4^-} \frac{x-5}{(x-4)^2}$

21. $\lim_{x \rightarrow 4} \frac{x-5}{(x-4)^2}$

22. $\lim_{x \rightarrow 1^+} \frac{x-2}{(x-1)^3}$

23. $\lim_{x \rightarrow 1^-} \frac{x-2}{(x-1)^3}$

24. $\lim_{x \rightarrow 1} \frac{x-2}{(x-1)^3}$

25. $\lim_{x \rightarrow 3^+} \frac{(x-1)(x-2)}{x-3}$

26. $\lim_{x \rightarrow 3^-} \frac{(x-1)(x-2)}{x-3}$

27. $\lim_{x \rightarrow 3} \frac{(x-1)(x-2)}{x-3}$

28. $\lim_{x \rightarrow 2^+} \frac{x-4}{x(x+2)}$

29. $\lim_{x \rightarrow 2^-} \frac{x-4}{x(x+2)}$

30. $\lim_{x \rightarrow 2} \frac{x-4}{x(x+2)}$

31. $\lim_{x \rightarrow 2} \frac{x-4}{x(x+2)}$

32. $\lim_{x \rightarrow 2^-} \frac{x^2-4x+3}{(x-2)^2}$

33. $\lim_{x \rightarrow 2} \frac{x^2-4x+3}{(x-2)^2}$

34. $\lim_{x \rightarrow -2^+} \frac{x^3-5x^2+6x}{x^4-4x^2}$

35. $\lim_{x \rightarrow -2^-} \frac{x^3-5x^2+6x}{x^4-4x^2}$

36. $\lim_{x \rightarrow -2} \frac{x^3-5x^2+6x}{x^4-4x^2}$

37. $\lim_{u \rightarrow 0^+} \frac{u-1}{\sin u}$

38. $\lim_{x \rightarrow 0^-} \frac{2}{\tan x}$

39. $\lim_{x \rightarrow 1^+} \frac{x^2-5x+6}{x-1}$

40. $\lim_{x \rightarrow 4} \frac{x-5}{(x^2-10x+24)^2}$

41. $\lim_{x \rightarrow 2\pi^-} \csc x$

42. $\lim_{x \rightarrow 0^+} e^{\sqrt{x}}$

43. $\lim_{x \rightarrow \frac{\pi}{2}^-} \frac{1+\sin x}{\cos x}$

44. $\lim_{x \rightarrow \frac{\pi}{2}^+} \frac{1+\sin x}{\cos x}$

45. $\lim_{x \rightarrow 0^-} \frac{e^x}{1-e^x}$

$$46. \quad \lim_{x \rightarrow 0^+} \frac{e^x}{1 - e^x}$$

$$48. \quad \lim_{x \rightarrow 0^+} \frac{x}{\ln x}$$

$$50. \quad \lim_{x \rightarrow 0^+} \frac{2e^x + 5e^{3x}}{e^{2x} - e^{3x}}$$

$$47. \quad \lim_{x \rightarrow 1^-} \frac{x}{\ln x}$$

$$49. \quad \lim_{x \rightarrow 0^-} \frac{2e^x + 5e^{3x}}{e^{2x} - e^{3x}}$$

$$51. \quad \text{Let } f(x) = \frac{x^2 - 7x + 12}{x - a}$$

a) For what values of a , if any, does $\lim_{x \rightarrow a^+} f(x)$ equal a finite number?

b) For what values of a , if any, does $\lim_{x \rightarrow a^+} f(x) = \infty$?

c) For what values of a , if any, does $\lim_{x \rightarrow a^+} f(x) = -\infty$?

$$52. \quad \text{Analyze } \lim_{x \rightarrow 1^+} \sqrt{\frac{x-1}{x-3}} \text{ and } \lim_{x \rightarrow 1^-} \sqrt{\frac{x-1}{x-3}}$$

Section 1.4 – Limits at Infinity

Notation	Terminology
$f(x) \rightarrow \infty$	$f(x)$ increases without bound (can be made as large positive as desired)
$f(x) \rightarrow -\infty$	$f(x)$ decreases without bound (can be made as large negative as desired)

Horizontal Asymptote (HA)

The line $y = b$ is a **horizontal asymptote** for the graph of a function f if

$$\lim_{x \rightarrow \infty} f(x) = b \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = b$$

Let $f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0} = \frac{a_n x^n}{b_m x^m}$ be a rational function. (**Proof !**)

1. If the degree of numerator is less than of denominator ($n < m$) $\Rightarrow y = 0$

$$y = \frac{2x+1}{4x^2+5} \Rightarrow \underline{y = 0}$$

2. If the degree of numerator is equal of denominator ($n = m$) $\Rightarrow y = \frac{a_n}{b_m}$

$$y = \frac{2x^2+1}{4x^2+5} \Rightarrow y = \frac{2}{4} = \underline{\frac{1}{2}}$$

3. If the degree of numerator is greater than of denominator ($n > m$) \Rightarrow No horizontal asymptote

$$y = \frac{2x^3+1}{4x^2+5} \Rightarrow \text{No HA}$$

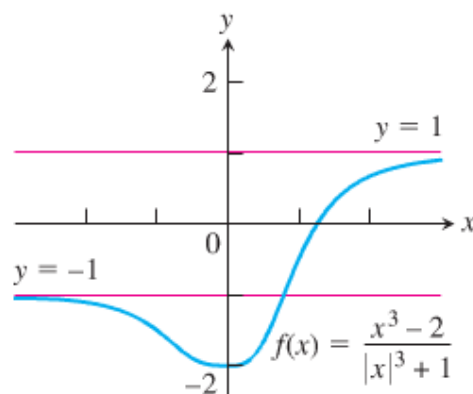
Example

Find the horizontal asymptotes of the graph of $f(x) = \frac{x^3 - 2}{|x|^3 + 1}$

Solution

For $x \geq 0$

$$\lim_{x \rightarrow \infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \rightarrow \infty} \frac{x^3}{x^3}$$



$$\underline{=1}$$

For $x \leq 0$

$$\lim_{x \rightarrow \infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \rightarrow -\infty} \frac{x^3}{(-x)^3}$$

$$\underline{=-1}$$

The **HA** are $y = -1$ and $y = 1$.

Example

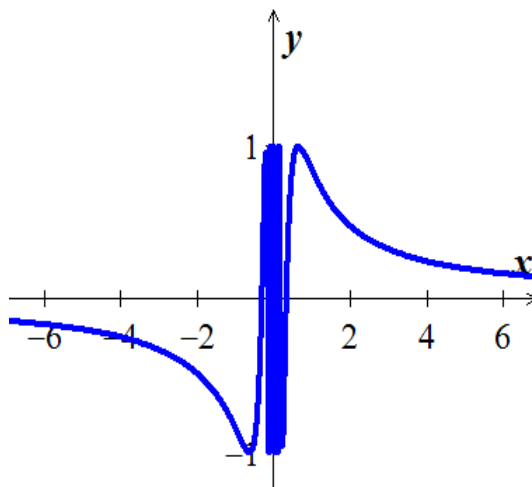
Find $\lim_{x \rightarrow \infty} \sin\left(\frac{1}{x}\right)$

Solution

Let $t = \frac{1}{x} \Rightarrow t \rightarrow 0$ as $x \rightarrow \infty$

$$\lim_{x \rightarrow \infty} \sin\left(\frac{1}{x}\right) = \lim_{t \rightarrow 0} \sin t$$

$$\underline{=0}$$



Example

Find $\lim_{x \rightarrow \pm\infty} x \sin\left(\frac{1}{x}\right)$

Solution

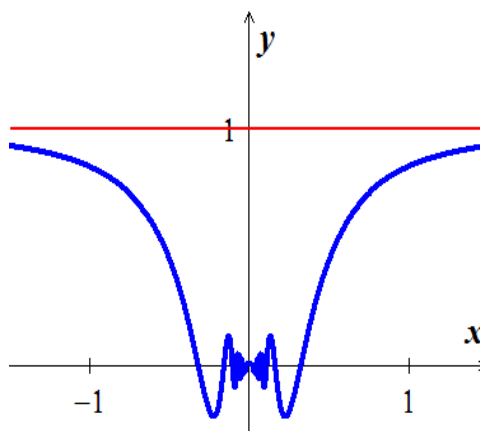
Let $t = \frac{1}{x} \Rightarrow x = \frac{1}{t}$

$$\lim_{x \rightarrow \infty} x \sin\left(\frac{1}{x}\right) = \lim_{t \rightarrow 0^+} \frac{\sin t}{t}$$

$$\underline{=1}$$

$$\lim_{x \rightarrow -\infty} x \sin\left(\frac{1}{x}\right) = \lim_{t \rightarrow 0^-} \frac{\sin t}{t}$$

$$\underline{=1}$$



Example

Find the horizontal asymptote of $y = 2 + \frac{\sin x}{x}$

Solution

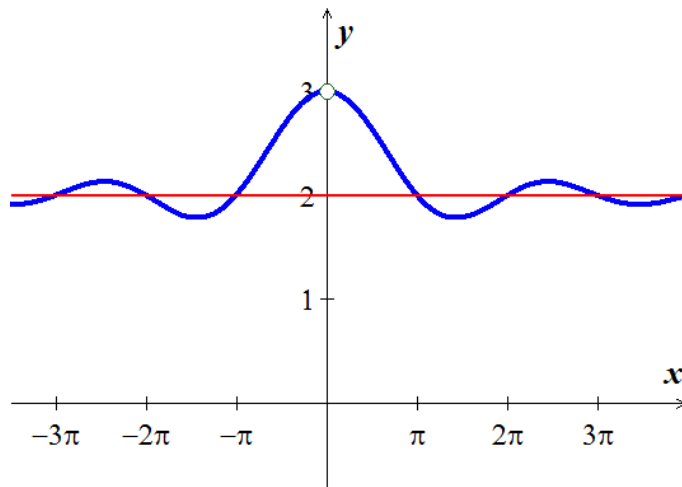
$$\text{Since } 0 \leq \left| \frac{\sin x}{x} \right| \leq \left| \frac{1}{x} \right|$$

$$\lim_{x \rightarrow \pm\infty} \left| \frac{1}{x} \right| = 0$$

$$\lim_{x \rightarrow \pm\infty} \frac{\sin x}{x} = 0$$

$$\lim_{x \rightarrow \pm\infty} \left(2 + \frac{\sin x}{x} \right) = 2 + 0$$
$$\underline{= 2}$$

HA: $y = 2$



Example

Find $\lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + 16} \right)$

Solution

$$\lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + 16} \right) = \lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + 16} \right) \cdot \frac{x + \sqrt{x^2 + 16}}{x + \sqrt{x^2 + 16}}$$

$$(a - b)(a + b) = a^2 - b^2$$

$$= \lim_{x \rightarrow \infty} \frac{x^2 - (x^2 + 16)}{x + \sqrt{x^2 + 16}}$$

$$= \lim_{x \rightarrow \infty} \frac{x^2 - x^2 - 16}{x + \sqrt{x^2 + 16}}$$

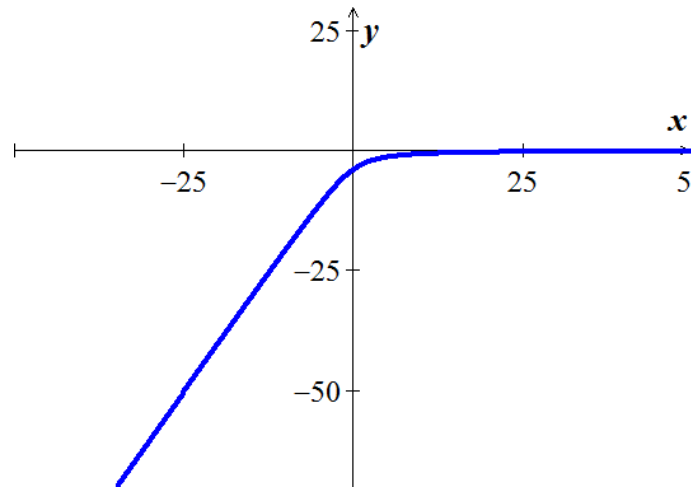
$$= \lim_{x \rightarrow \infty} \frac{-16}{x + \sqrt{x^2 + 16}}$$

$$= \lim_{x \rightarrow \infty} \frac{-\frac{16}{x}}{\frac{x}{x} + \sqrt{\frac{x^2}{x^2} + \frac{16}{x^2}}}$$

$$= \lim_{x \rightarrow \infty} \frac{-\frac{16}{x}}{1 + \sqrt{1 + \frac{16}{x^2}}}$$

$$= \frac{0}{1 + \sqrt{1 + 0}}$$

$$= 0$$



Slant or Oblique Asymptotes

When the degree of the numerator is one greater than the degree of the denominator, the graph has a *slant* or *oblique* asymptote and it is a line $y = ax + b$, $a \neq 0$. To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

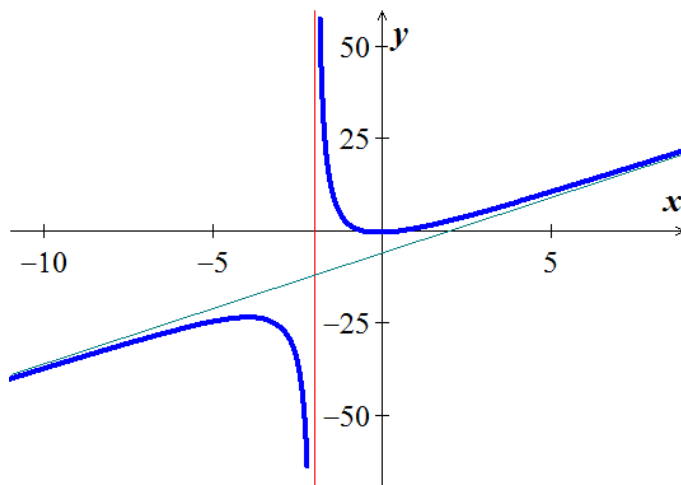
Example

$$y = \frac{3x^2 - 1}{x + 2}$$

$$\begin{array}{r} 3x - 6 \\ x + 2 \overline{) 3x^2 + 0x - 1} \\ \underline{3x^2 + 6x} \\ -6x - 1 \\ \underline{-6x - 12} \\ R = 11 \end{array}$$

$$y = \frac{3x^2 - 1}{x + 2}$$

$$= (3x - 6) + \frac{11}{x + 2}$$



The *oblique asymptote* is the line $y = 3x - 6$

Example

Find the horizontal and vertical asymptotes of the curve $y = \frac{x+3}{x+2}$

Solution

$$\text{HA: } y \rightarrow \frac{x}{x} = 1 \Rightarrow \underline{y = 1}$$

$$\text{VA: } x + 2 = 0 \Rightarrow \underline{x = -2}$$

Example

Find the horizontal and vertical asymptotes of the curve $f(x) = -\frac{8}{x^2 - 4}$

Solution

$$\text{HA: } y \rightarrow \lim_{x \rightarrow \infty} -\frac{8}{x^2} = 0 \Rightarrow \underline{y = 0}$$

VA: $x^2 - 4 = 0 \Rightarrow \underline{x = \pm 2}$

$$\lim_{x \rightarrow 2^+} f(x) = -\infty \quad \text{and} \quad \lim_{x \rightarrow 2^-} f(x) = \infty$$

Infinite Limits

The limit has a value of infinity or minus infinity, such a function $f(x) = \frac{1}{x}$. It is convenient to describe the behavior of f by saying that $f(x)$ approaches ∞ as $x \rightarrow 0^+$.

Definition

We say $\lim_{x \rightarrow 0^+} f(x) = \infty$

That $\lim_{x \rightarrow 0^+} \frac{1}{x}$ doesn't exist because $\frac{1}{x}$ becomes arbitrary large and positive as $x \rightarrow 0^+$.

We say $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$

That $\lim_{x \rightarrow 0^-} \frac{1}{x}$ doesn't exist because $\frac{1}{x}$ becomes arbitrary large and negative as $x \rightarrow 0^-$.

Example

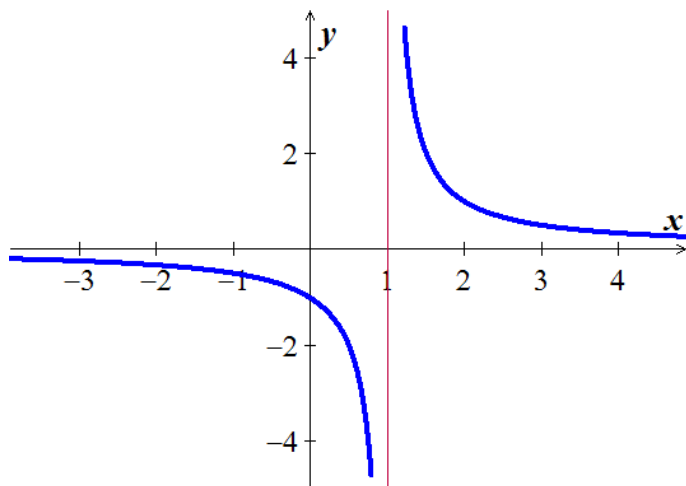
Find $\lim_{x \rightarrow 1^+} \frac{1}{x-1}$ and $\lim_{x \rightarrow 1^-} \frac{1}{x-1}$

Solution

As $x \rightarrow 1^+ \Rightarrow x-1 \rightarrow 0^+$

$$\lim_{x \rightarrow 1^+} \frac{1}{x-1} = \infty$$

$$\lim_{x \rightarrow 1^-} \frac{1}{x-1} = -\infty$$



Examples

$$\begin{aligned}\text{➤} \quad \lim_{x \rightarrow 2} \frac{(x-2)^2}{x^2 - 4} &= \lim_{x \rightarrow 2} \frac{(x-2)^2}{(x-2)(x+2)} \\ &= \lim_{x \rightarrow 2} \frac{(x-2)}{(x+2)} \\ &= \frac{0}{4} \\ &= 0 \quad | \end{aligned}$$

$$\begin{aligned}\text{➤} \quad \lim_{x \rightarrow 2} \frac{x-2}{x^2 - 4} &= \lim_{x \rightarrow 2} \frac{x-2}{(x-2)(x+2)} \\ &= \lim_{x \rightarrow 2} \frac{1}{x+2} \\ &= \frac{1}{4} \quad | \end{aligned}$$

$$\begin{aligned}\text{➤} \quad \lim_{x \rightarrow 2^+} \frac{x-3}{x^2 - 4} &= \lim_{x \rightarrow 2^+} \frac{x-3}{(x-2)(x+2)} \\ &= -\infty \quad | \end{aligned}$$

$$\begin{aligned}\text{➤} \quad \lim_{x \rightarrow 2^-} \frac{x-3}{x^2 - 4} &= \lim_{x \rightarrow 2^-} \frac{x-3}{(x-2)(x+2)} \\ &= \infty \quad | \end{aligned}$$

$$\begin{aligned}\text{➤} \quad \lim_{x \rightarrow 2} \frac{x-3}{x^2 - 4} &= \lim_{x \rightarrow 2} \frac{x-3}{(x-2)(x+2)} \\ &= \text{doesn't exist} \quad | \end{aligned}$$

Exercises Section 1.4 – Limits at Infinity

(1 – 8) Find the limit as $x \rightarrow \infty$ and as $x \rightarrow -\infty$ of

1. $h(x) = \frac{-5 + \frac{7}{x}}{3 - \frac{1}{x^2}}$

4. $f(x) = \frac{x+1}{x^2+3}$

6. $f(x) = \frac{9x^4 + x}{2x^4 + 5x^2 - x + 6}$

2. $f(x) = \frac{2x+3}{5x+7}$

5. $f(x) = \frac{7x^3}{x^3 - 3x^2 + 6x}$

7. $f(x) = \frac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x}$

3. $f(x) = \frac{2x^3 + 7}{x^3 - x^2 + x + 7}$

(8 – 60) Evaluate the limits

8. $\lim_{x \rightarrow \infty} x^{12}$

20. $\lim_{x \rightarrow -\infty} \left(\frac{x^2 + x - 1}{8x^2 - 3} \right)^{1/3}$

9. $\lim_{x \rightarrow -\infty} 3x^9$

21. $\lim_{x \rightarrow \infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7}$

10. $\lim_{x \rightarrow -\infty} x^{-8}$

22. $\lim_{x \rightarrow \infty} \frac{x^{-1} + x^{-4}}{x^{-2} + x^{-3}}$

11. $\lim_{x \rightarrow -\infty} x^{-9}$

23. $\lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}}$

12. $\lim_{x \rightarrow -\infty} 2x^{-6}$

24. $\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 3x} - \sqrt{x^2 - 2x} \right)$

13. $\lim_{x \rightarrow \infty} (3x^{12} - 9x^7)$

25. $\lim_{x \rightarrow -\infty} \left(\sqrt{x^2 + 3} + x \right)$

14. $\lim_{x \rightarrow -\infty} (3x^7 + x^2)$

15. $\lim_{x \rightarrow -\infty} (-2x^{16} + 2)$

26. $\lim_{x \rightarrow \infty} \frac{2x - 3}{4x + 10}$

16. $\lim_{x \rightarrow -\infty} (2x^{-6} + 4x^5)$

27. $\lim_{x \rightarrow \infty} \frac{x^4 - 1}{x^5 + 2}$

17. $\lim_{x \rightarrow -\infty} \frac{\cos x}{3x}$

28. $\lim_{x \rightarrow -\infty} (-3x^3 + 5)$

18. $\lim_{x \rightarrow \infty} \frac{x + \sin x}{2x + 7 - 5 \sin x}$

29. $\lim_{x \rightarrow \infty} \left(e^{-2x} + \frac{2}{x} \right)$

19. $\lim_{x \rightarrow \infty} \sqrt{\frac{8x^2 - 3}{2x^2 + x}}$

30. $\lim_{x \rightarrow \infty} \frac{1}{\ln x + 1}$

31. $\lim_{x \rightarrow \infty} \left(3 + \frac{10}{x^2} \right)$
32. $\lim_{x \rightarrow \infty} \left(5 + \frac{1}{x} + \frac{10}{x^2} \right)$
33. $\lim_{x \rightarrow \infty} \frac{4x^2 + 2x + 3}{x^2}$
34. $\lim_{x \rightarrow \infty} \left(5 + \frac{100}{x} + \frac{\sin^4 x^3}{x^2} \right)$
35. $\lim_{\theta \rightarrow \infty} \frac{\cos \theta}{\theta^2}$
36. $\lim_{\theta \rightarrow \infty} \frac{\cos \theta^5}{\sqrt{\theta}}$
37. $\lim_{x \rightarrow \infty} \frac{4x}{20x + 1}$
38. $\lim_{x \rightarrow -\infty} \frac{4x}{20x + 1}$
39. $\lim_{x \rightarrow \infty} \frac{3x^2 - 7}{x^2 + 5x}$
40. $\lim_{x \rightarrow -\infty} \frac{3x^2 - 7}{x^2 + 5x}$
41. $\lim_{x \rightarrow \infty} \frac{6x^2 - 9x + 8}{3x^2 + 2}$
42. $\lim_{x \rightarrow -\infty} \frac{6x^2 - 9x + 8}{3x^2 + 2}$
43. $\lim_{x \rightarrow \infty} \frac{4x^2 - 7}{8x^2 + 5x + 2}$
44. $\lim_{x \rightarrow -\infty} \frac{4x^2 - 7}{8x^2 + 5x + 2}$
45. $\lim_{x \rightarrow \infty} \frac{\sqrt{16x^4 + 64x^2 + x^2}}{2x^2 - 4}$
46. $\lim_{x \rightarrow -\infty} \frac{\sqrt{16x^4 + 64x^2 + x^2}}{2x^2 - 4}$
47. $\lim_{x \rightarrow \infty} \frac{3x^4 + 3x^3 - 36x^2}{x^4 - 25x^2 + 144}$
48. $\lim_{x \rightarrow -\infty} \frac{3x^4 + 3x^3 - 36x^2}{x^4 - 25x^2 + 144}$
49. $\lim_{x \rightarrow \infty} 16x^2 \left(4x^2 - \sqrt{16x^4 + 1} \right)$
50. $\lim_{x \rightarrow -\infty} 16x^2 \left(4x^2 - \sqrt{16x^4 + 1} \right)$
51. $\lim_{x \rightarrow \infty} \frac{x - 1}{x^{2/3} - 1}$
52. $\lim_{x \rightarrow -\infty} \frac{x - 1}{x^{2/3} - 1}$
53. $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 2x + 6} - 3}{x - 1}$
54. $\lim_{x \rightarrow \infty} \frac{|1 - x^2|}{x(x + 1)}$
55. $\lim_{x \rightarrow \infty} \left(\sqrt{|x|} - \sqrt{|x - 1|} \right)$
56. $\lim_{x \rightarrow \infty} \frac{\tan^{-1} x}{x}$
57. $\lim_{x \rightarrow \infty} \frac{\cos x}{e^{3x}}$
58. $\lim_{x \rightarrow 0} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}}$
59. $\lim_{x \rightarrow \infty} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}}$
60. $\lim_{x \rightarrow -\infty} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}}$

(61 – 64) Graph the rational function and include the equations of the asymptotes

61. $y = \frac{1}{2x+4}$

62. $y = \frac{2x}{x+1}$

63. $y = \frac{x^2}{x-1}$

64. $y = \frac{x^3+1}{x^2}$

65. Let $f(x) = \frac{x^2-5x+6}{x^2-2x}$

a) Analyze $\lim_{x \rightarrow 0^-} f(x)$, $\lim_{x \rightarrow 0^+} f(x)$, $\lim_{x \rightarrow 2^-} f(x)$, and $\lim_{x \rightarrow 2^+} f(x)$

b) Does the graph of f have any vertical asymptotes? Explain?

(66 – 85) Find the vertical, horizontal, hole, and oblique asymptotes (if any) of

66. $y = \frac{3x}{1-x}$

73. $y = \frac{x^3+3x^2-2}{x^2-4}$

80. $f(x) = \frac{1}{\tan^{-1} x}$

67. $y = \frac{x^2}{x^2+9}$

74. $y = \frac{x-3}{x^2-9}$

81. $f(x) = \frac{2x^2+6}{2x^2+3x-2}$

68. $y = \frac{x-2}{x^2-4x+3}$

75. $y = \frac{6}{\sqrt{x^2-4x}}$

82. $f(x) = \frac{3x^2+2x-1}{4x+1}$

69. $y = \frac{5x-1}{1-3x}$

76. $f(x) = \frac{4x^3+1}{1-x^3}$

83. $f(x) = \frac{9x^2+4}{(2x-1)^2}$

70. $y = \frac{3}{x-5}$

77. $f(x) = \frac{x+1}{\sqrt{9x^2+x}}$

84. $f(x) = \frac{1+x-2x^2-x^3}{x^2+1}$

71. $y = \frac{x^3-1}{x^2+1}$

78. $f(x) = 1 - e^{-2x}$

85. $f(x) = \frac{x(x+2)^3}{3x^2-4x}$

72. $y = \frac{3x^2-27}{(x+3)(2x+1)}$

79. $f(x) = \frac{1}{\ln x^2}$

(85 – 142) Find the limits

86. $\lim_{x \rightarrow 0} \frac{x^2-4x+4}{x^3+5x^2-14x}$

89. $\lim_{x \rightarrow 0} \frac{(x+h)^2-x^2}{h}$

92. $\lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x}$

87. $\lim_{x \rightarrow 2} \frac{x^2-4x+4}{x^3+5x^2-14x}$

90. $\lim_{h \rightarrow 0} \frac{(x+h)^2-x^2}{h}$

93. $\lim_{x \rightarrow 1} \frac{x^{1/3}-1}{\sqrt{x}-1}$

88. $\lim_{x \rightarrow a} \frac{x^2-a^2}{x^4-a^4}$

91. $\lim_{x \rightarrow 1} \frac{1-\sqrt{x}}{1-x}$

94. $\lim_{x \rightarrow 64} \frac{x^{2/3}-16}{\sqrt{x}-8}$

95. $\lim_{x \rightarrow 0} \frac{\tan(2x)}{\tan(\pi x)}$
96. $\lim_{x \rightarrow \pi^-} \csc x$
97. $\lim_{x \rightarrow \pi} \sin\left(\frac{x}{2} + \sin x\right)$
98. $\lim_{x \rightarrow \pi} \cos^2(x - \tan x)$
99. $\lim_{x \rightarrow 0} \frac{8x}{3 \sin x - x}$
100. $\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\sin x}$
101. $\lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}}$
102. $\lim_{x \rightarrow -\infty} \frac{x^2 - 4x + 8}{3x^3}$
103. $\lim_{x \rightarrow -\infty} \frac{2x^2 + 3}{5x^2 + 7}$
104. $\lim_{x \rightarrow \infty} \frac{x^4 + x^3}{12x^3 + 128}$
105. $\lim_{x \rightarrow -\infty} \frac{2 + \sqrt{x}}{2 - \sqrt{x}}$
106. $\lim_{x \rightarrow \infty} \frac{2 + \sqrt{x}}{2 - \sqrt{x}}$
107. $\lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$
108. $\lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{1}{x^4}}{\frac{1}{x^2} - \frac{1}{x^3}}$
109. $\lim_{x \rightarrow \infty} \frac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}}$
110. $\lim_{x \rightarrow 2^+} \ln(x - 2)$
111. $\lim_{x \rightarrow 1} x^2 \ln(2 - \sqrt{x})$
112. $\lim_{\theta \rightarrow 0^+} \sqrt{\theta} e^{\cos \frac{\pi}{\theta}}$
113. $\lim_{x \rightarrow \infty} \frac{2x - 3}{5x + 6}$
114. $\lim_{x \rightarrow \infty} \frac{2x^2 - 3}{5x^2 + 6}$
115. $\lim_{x \rightarrow \infty} \frac{2x - 3}{5x^3 + 6}$
116. $\lim_{x \rightarrow \infty} \frac{1}{5x^2 - 3x + 6}$
117. $\lim_{\theta \rightarrow 0} \frac{\theta \cot 4\theta}{\sin^2 \theta \cot^2 2\theta}$
118. $\lim_{x \rightarrow 0^+} \frac{\sqrt{x^2 + 4x + 5} - \sqrt{5}}{x}$
119. $\lim_{x \rightarrow 2} \frac{x^4 - 16}{x - 2}$
120. $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2}$
121. $\lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4}$
122. $\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 1}}{x + 1}$
123. $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{x + 1}$
124. $\lim_{x \rightarrow \infty} \frac{x - 3}{\sqrt{4x^2 + 25}}$
125. $\lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}}$
126. $\lim_{x \rightarrow \infty} \frac{x^4 - x}{15x^3 + 4}$
127. $\lim_{x \rightarrow \infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x}$
128. $\lim_{x \rightarrow \infty} \frac{x^{2/3} - x^{-1}}{x^{2/3} + \cos^2 x}$
129. $\lim_{x \rightarrow \infty} \frac{\sin 2x}{x}$
130. $\lim_{x \rightarrow 0} \frac{\sin 5x}{3x}$
131. $\lim_{x \rightarrow -\infty} \frac{\cos x}{2x}$
132. $\lim_{x \rightarrow -\infty} \left(\frac{x^2 + x - 1}{8x^2 - 3} \right)^{1/3}$
133. $\lim_{x \rightarrow -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1}$
134. $\lim_{x \rightarrow -\infty} \left(\frac{1 - x^3}{x^2 + 7x} \right)^5$
135. $\lim_{x \rightarrow \infty} \sqrt{\frac{x^2 - 5x}{x^3 + x - 2}}$
136. $\lim_{x \rightarrow \infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7}$
137. $\lim_{x \rightarrow -5^-} \frac{3x}{2x + 10}$
138. $\lim_{x \rightarrow -8^+} \frac{3x}{x + 8}$
139. $\lim_{x \rightarrow 0} \frac{-1}{x^2(x + 1)}$
140. $\lim_{x \rightarrow 7} \frac{4}{(x - 7)^2}$
141. $\lim_{x \rightarrow 0} \frac{1}{x^{2/3}}$
142. $\lim_{x \rightarrow -\infty} \left(x + \sqrt{x^2 - 4x + 2} \right)$

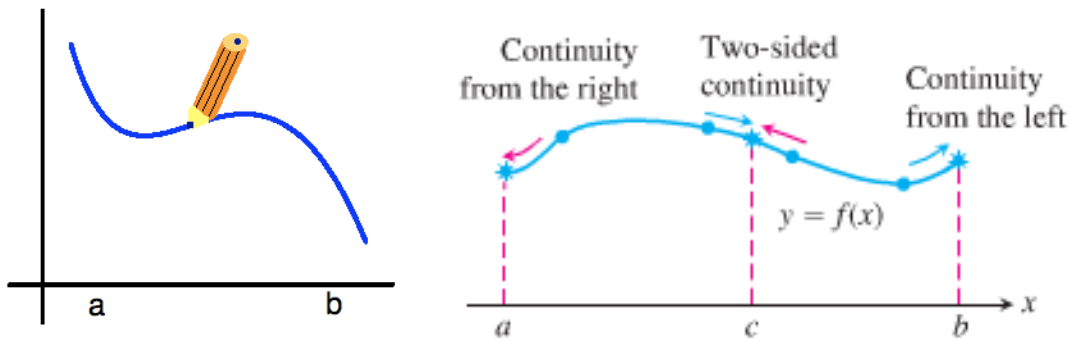
Section 1.5 – Continuity

Definition of Continuity

Let c be a number in the interval (a, b) , and let f be a function whose domain contains the interval (a, b) . The function f is continuous at the point c if the following conditions are true.

1. $f(c)$ is defined
2. $\lim_{x \rightarrow c} f(x)$ exists
3. $\lim_{x \rightarrow c} f(x) = f(c)$

If f is continuous at every point in the interval (a, b) , then it is continuous on an open interval (a, b)



Definition

Interior point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x \rightarrow c} f(x) = f(c)$$

Endpoint: A function $y = f(x)$ is **continuous at a left point a** or is **continuous at a right point b** of its domain if

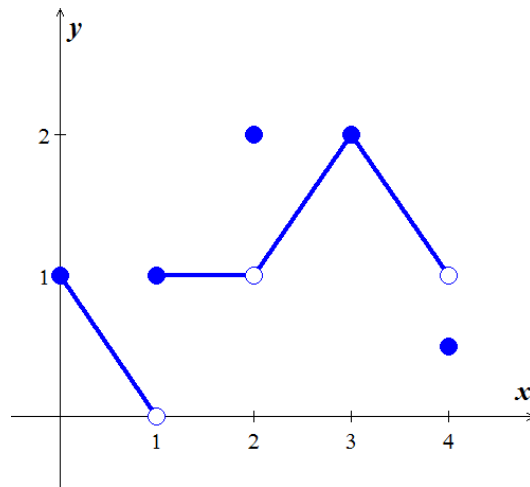
$$\lim_{x \rightarrow a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \rightarrow b^-} f(x) = f(b), \quad \text{respectively}$$



If a function f is not continuous at a point c , we say that f is **discontinuous** at c . (is a **point of discontinuity**)

Example

Find the points at which the function f is continuous and the points at which f is not continuous



Solution

The function f is continuous at every point in its domain $[0, 4]$ except at $x = 1$, $x = 2$, and $x = 4$. At these points, there are breaks in the graph.

$$x = 0 \quad \lim_{x \rightarrow 0^+} f(x) = f(0) = 1 \quad f \text{ is continuous @ } x = 0$$

$$x = 1 \quad \lim_{x \rightarrow 1} f(x) \text{ doesn't exist} \quad f \text{ is discontinuous @ } x = 1$$

$$x = 2 \quad \lim_{x \rightarrow 2} f(x) = 1, \text{ but } 1 \neq f(2) \quad f \text{ is discontinuous @ } x = 2$$

$$x = 3 \quad \lim_{x \rightarrow 3} f(x) = f(3) = 2 \quad f \text{ is continuous @ } x = 3$$

$$x = 4 \quad \lim_{x \rightarrow 4^-} f(x) = 1, \text{ but } 1 \neq f(4) \quad f \text{ is discontinuous @ } x = 4$$

$$c < 0, c > 4 \quad \text{These points are not in the domain of } f. \quad f \text{ is discontinuous}$$

$$0 < c < 4, c \neq 1, 2 \quad \lim_{x \rightarrow c} f(x) = f(c)$$

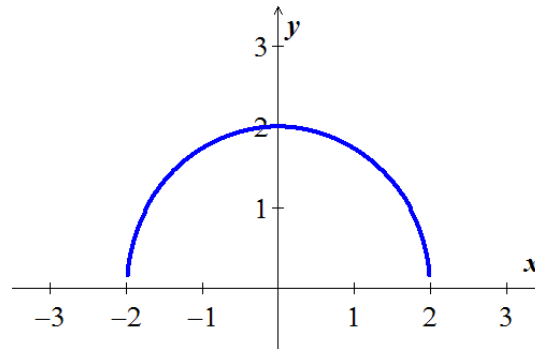
Example

At what points the function $f(x) = \sqrt{4 - x^2}$ is continuous?

Solution

The function is continuous at every point of its domain $[-2, 2]$.

Including $x = -2$, where f is right-continuous, and $x = 2$, where f is left-continuous.



Continuous Functions

A function is **continuous on an interval** iff it is continuous at every point of the interval. A **continuous function** is one that is continuous at every point of its domain. A continuous function need not be continuous on every interval.

Example

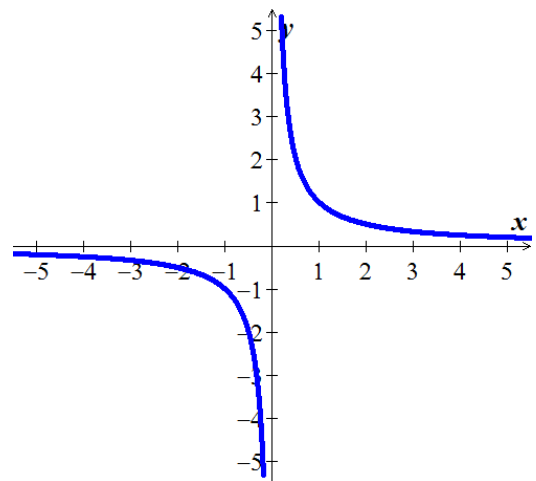
Determine at which points do the function $f(x) = \frac{1}{x}$ is continuous and discontinuous

Solution

The function $f(x)$ is a continuous function because it is continuous at every point of its domain.

It has a point of discontinuity at $x = 0$, however, because it is not defined.

It is discontinuous on any interval containing $x = 0$



Theorem – Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

Sums and Differences $f \pm g$

Constant multiples $k \cdot g$, for any number k .

Products $f \cdot g$

Quotients $\frac{f}{g}$

Powers f^n n a positive integer

Roots $\sqrt[n]{f}$, provided it is defined on an open interval containing c , where n is a positive integer

Proof

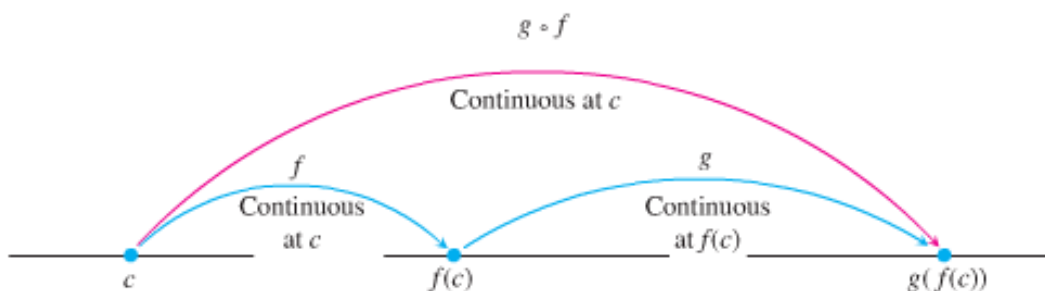
$$\begin{aligned}\lim_{x \rightarrow c} (f + g)(x) &= \lim_{x \rightarrow c} (f(x) + g(x)) \\ &= \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x) \\ &= f(c) + g(c) \\ &= (f + g)(c)\end{aligned}$$

This shows that $f + g$ is continuous

Composites

All composites of continuous functions are continuous.

If $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$, then $g \circ f$ is continuous at $x = c$



Example

Show that $y = \sqrt{x^2 - 2x - 5}$ is continuous everywhere on its domain

Solution

$$\text{Let } \begin{cases} f(x) = x^2 - 2x - 5, & \text{Domain: } \mathbb{R} \\ g(x) = \sqrt{x} & \text{Domain: } [0, \infty) \end{cases}$$

\therefore The function y is continuous on $[0, \infty)$

Example

Show that $y = \left| \frac{x \sin x}{x^2 + 2} \right|$ is continuous everywhere on its domain

Solution

$$\text{Let } \begin{cases} x \sin x & \text{Domain: } \mathbb{R} \\ x^2 + 2 & \text{Domain: } \mathbb{R} \end{cases}$$

\therefore The function is the composite of a quotient continuous functions with the continuous absolute value function.

Theorem

If g is continuous at the point b and $\lim_{x \rightarrow c} f(x) = b$, then

$$\lim_{x \rightarrow c} g(f(x)) = g(b) = g\left(\lim_{x \rightarrow c} f(x)\right)$$

Proof

Let $\varepsilon > 0$ be given. Since g is continuous at b , there exists a number $\delta_1 > 0$ such that

$$|g(y) - g(b)| < \varepsilon \quad \text{whenever} \quad 0 < |y - b| < \delta_1$$

$$\lim_{x \rightarrow c} f(x) = b, \exists \delta > 0 \ni |f(x) - b| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta$$

If we let $y = f(x)$, we then have that $|y - b| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta$

Which implies from the first statement that $|g(y) - g(b)| = |g(f(x)) - g(b)| < \varepsilon$ whenever

$0 < |x - c| < \delta$. From the definition of the limit, this proves that $\lim_{x \rightarrow c} g(f(x)) = g(b)$

Example

Find the $\lim_{x \rightarrow \frac{\pi}{2}} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right)$

Solution

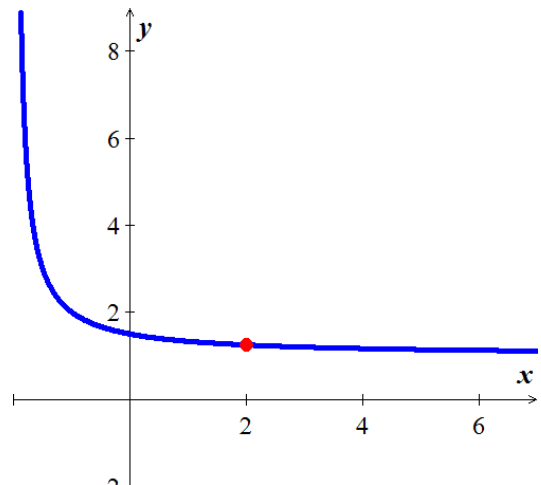
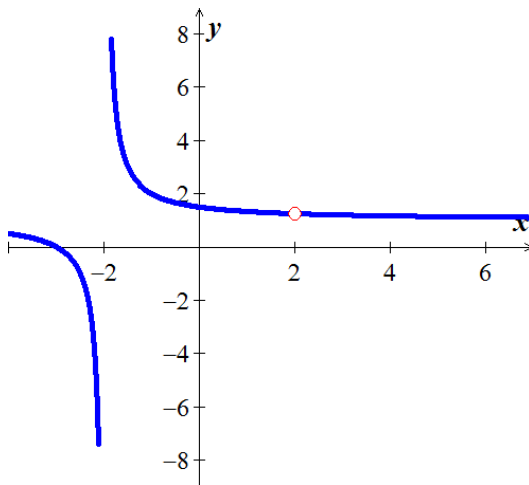
$$\begin{aligned}\lim_{x \rightarrow \frac{\pi}{2}} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right) &= \cos\left(\lim_{x \rightarrow \frac{\pi}{2}} 2x + \lim_{x \rightarrow \frac{\pi}{2}} \sin\left(\frac{3\pi}{2} + x\right)\right) \\ &= \cos(\pi + \sin 2\pi) \\ &= \cos(\pi + 0) \\ &= \cos(\pi) \\ &= -1\end{aligned}$$

Example

Show that $f(x) = \frac{x^2 + x - 6}{x^2 - 4}$, $x \neq 2$ has a continuous extension to $x = 2$, and find that extension.

Solution

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4} = \frac{(x-2)(x+3)}{(x-2)(x+2)} = \frac{x+3}{x+2}$$



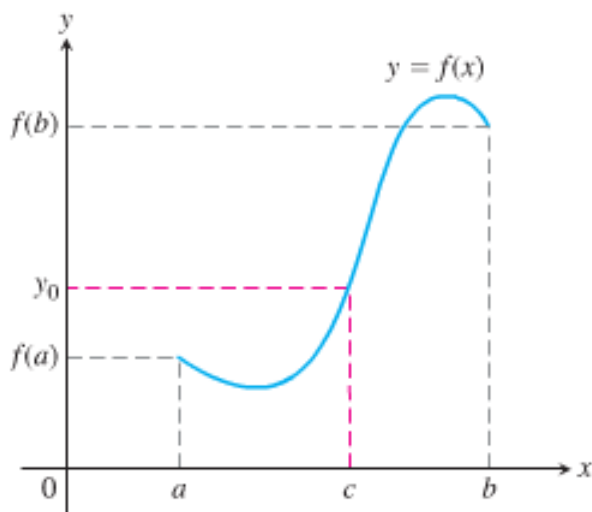
After simplification the function is continuous at $x = 2$

$$\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \rightarrow 2} \frac{x+3}{x+2} = \frac{5}{4}$$

The new function is the function f with its point of discontinuity at $x = 2$ removed.

Theorem – the Intermediate Value Theorem for Continuous Functions

If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



A Consequence for Root Finding

We call a solution of the equation $f(x) = 0$ a **root** of the equation or zero of the function f . The Intermediate Value Theorem said that if f is continuous, then any interval on which f changes sign contains a zero of the function.

Example

Show that there is a root of the equation $x^3 - x - 1$ between 1 and 2.

Solution

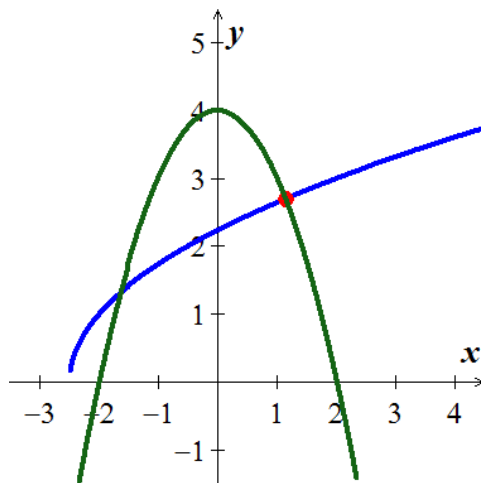
$$f(1) = 1^3 - 1 - 1 = -1 < 0$$

$$f(2) = 2^3 - 2 - 1 = 5 > 0$$

Since f is continuous, the Intermediate Value Theorem says there is a zero of f between 1 and 2.

Example

Use the Intermediate Value Theorem to prove that the equation $\sqrt{2x+5} = 4 - x^2$ has a solution.



Solution

The function $g(x) = \sqrt{2x+5}$ is continuous on the interval $\left[-\frac{5}{2}, \infty\right)$ since it is the composite of the square root function with nonnegative linear function $y = 2x + 5$. Then the function

$f(x) = \sqrt{2x+5} + x^2$ is the sum of the function $g(x)$ and $y = x^2$. It follows that $f(x)$ is continuous on the interval $\left[-\frac{5}{2}, \infty\right)$.

By trial and error:

$$f(0) = \sqrt{2(0)+5} + 0^2 = \sqrt{5} > 0$$

$$f(2) = \sqrt{2(2)+5} + 2^2 = \sqrt{9} + 4 = 7 > 0$$

f is continuous on the interval $[0, 2] \subset \left[-\frac{5}{2}, \infty\right)$.

Since the value $y_0 = 4$ is between $\sqrt{5}$ and 7, by the Intermediate Value Theorem there is a number $c \in [0, 2] \ni f(c) = 4$. That is, the number c solves the original equation.

Exercises Section 1.5 – Continuity

1. Given the graphed function $f(x)$

a) Does $f(-1)$ exist?

b) Does $\lim_{x \rightarrow -1^+} f(x)$ exist?

c) Does $\lim_{x \rightarrow -1^+} f(x) = f(-1)$?

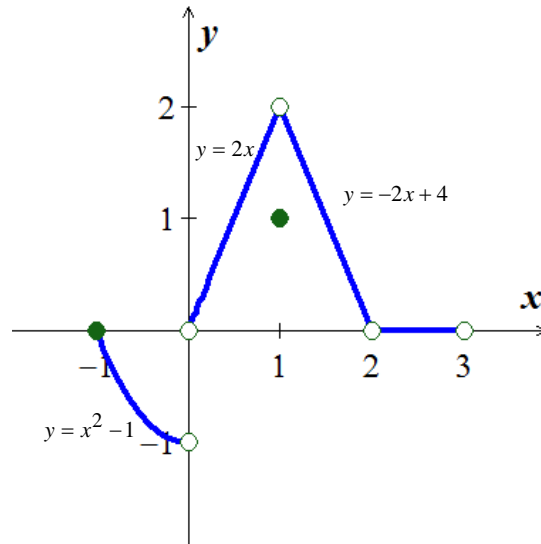
d) Is f continuous at $x = -1$?

e) Does $f(1)$ exist?

f) Does $\lim_{x \rightarrow 1} f(x)$ exist?

g) Does $\lim_{x \rightarrow 1} f(x) = f(1)$?

h) Is f continuous at $x = 1$?



(2 – 11) At what point(s) is the given function continuous?

2. $y = \frac{1}{x-2} - 3x$

6. $y = \tan \frac{\pi x}{2}$

9. $y = \sqrt{2x+3}$

3. $y = \frac{x+3}{x^2-3x-10}$

7. $y = \frac{x \tan x}{x^2+1}$

10. $y = \sqrt[4]{3x-1}$

4. $y = |x-1| + \sin x$

8. $y = \frac{\sqrt{x^4+1}}{1+\sin^2 x}$

11. $y = (2-x)^{1/5}$

5. $y = \frac{x+2}{\cos x}$

12. Find $\lim_{x \rightarrow \pi} \sin(x - \sin x)$, then is the function continuous at the point being approached?

13. Find $\lim_{x \rightarrow 0} \tan\left(\frac{\pi}{4} \cos\left(\sin x^{1/3}\right)\right)$, then is the function continuous at the point being approached?

14. Find $\lim_{t \rightarrow 0} \cos\left(\frac{\pi}{\sqrt{19-3\sec 2t}}\right)$, then is the function continuous at the point being approached?

15. Explain why the equation $\cos x = x$ has at least one solution.

(16 – 19) Show that the equation has three solutions in the given interval

16. $x^3 - 15x + 1 = 0$; $[-4, 4]$

18. $70x^3 - 87x^2 + 32x - 3 = 0$; $(0, 1)$

17. $x^3 + 10x^2 - 100x + 50 = 0$; $(-20, 10)$

19. $x^3 - 3x - 1 = 0$; $[-2, 2]$

20. Show that the equation has six solutions in the given interval $x^6 - 8x^4 + 10x^2 - 1 = 0$; $[-3, 3]$
21. If functions $f(x)$ and $g(x)$ are continuous for $0 \leq x \leq 1$, could $\frac{f(x)}{g(x)}$ possibly be discontinuous at a point of $[0, 1]$? Give reason for your answer.
22. Suppose that a function f is continuous on the closed interval $[0, 1]$ and that $0 \leq f(x) \leq 1$ for every x in $[0, 1]$. Show that there must exist a number c in $[0, 1]$ such that $f(c) = c$ (c is called a **fixed point** of f).
23. Use the Intermediate Value Theorem to show that the equation $x^5 + 7x + 5 = 0$ has a solution in the interval $(-1, 0)$.

24. The amount of an antibiotic (in mg) in the blood t hours after an intravenous line is opened is given by

$$m(t) = 100(e^{-0.1t} - e^{-0.3t})$$

- a) Use the Intermediate Value Theorem to show that the amount of drug is 30 mg at some time in the interval $[0, 5]$ and again at some time in the interval $[5, 15]$
- b) Estimate the times at which $m = 30\text{ mg}$
- c) Is the amount of drug in the blood ever 50 mg ?

(25 – 27) Determine whether the following functions are continuous at a .

25. $f(x) = \frac{1}{x-5}$; $a = 5$

27. $g(x) = \begin{cases} \frac{x^2-16}{x-4} & \text{if } x \neq 4; \\ 8 & \text{if } x = 4 \end{cases}$; $a = 4$

26. $h(x) = \sqrt{x^2 - 9}$; $a = 3$

(28 – 31) Find the intervals on which the following functions are continuous. Specify right- or left-continuity at the endpoints

28. $f(x) = \sqrt{x^2 - 5}$ 29. $f(x) = e^{\sqrt{x-2}}$ 30. $f(x) = \frac{2x}{x^3 - 25x}$ 31. $f(x) = \cos e^x$

32. Let $g(x) = \begin{cases} 5x-2 & \text{if } x < 1 \\ a & \text{if } x = 1 \\ ax^2 + bx & \text{if } x > 1 \end{cases}$

Determine values of the constants a and b for which $g(x)$ is continuous at $x = 1$

Section 1.6 – Precise Definition of a Limit

Example

Consider the function $y = 2x - 1$ near $x_0 = 4$. Intuitively it appears that y is close to 7 when x is close to 4, so $\lim_{x \rightarrow 4} (2x - 1) = 7$. However, how close to $x_0 = 4$ does x have to be so that $y = 2x - 1$ differs from 7 by, say less than 2 units?

Solution

We need to find the values of x for $|y - 7| < 2$.

$$|y - 7| = |2x - 1 - 7| = |2x - 8|$$

$$|2x - 8| < 2$$

$$-2 < 2x - 8 < 2$$

$$-2 + 8 < 2x - 8 + 8 < 2 + 8$$

$$6 < 2x < 10$$

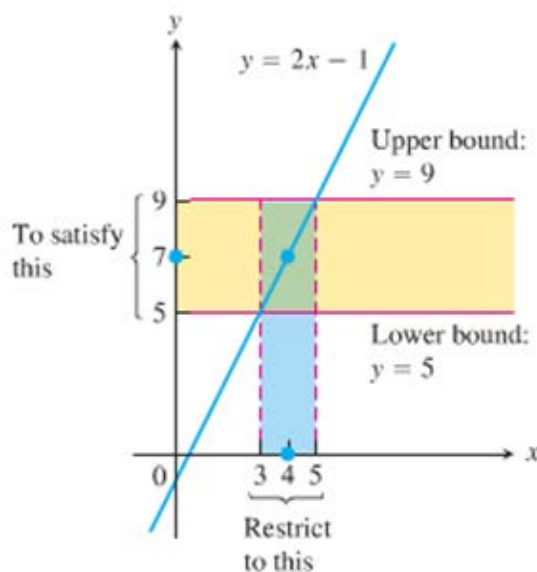
$$\frac{6}{2} < \frac{2x}{2} < \frac{10}{2}$$

$$3 < x < 5$$

$$3 - 4 < x - 4 < 5 - 4$$

$$-1 < x - 4 < 1$$

Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0 = 7$



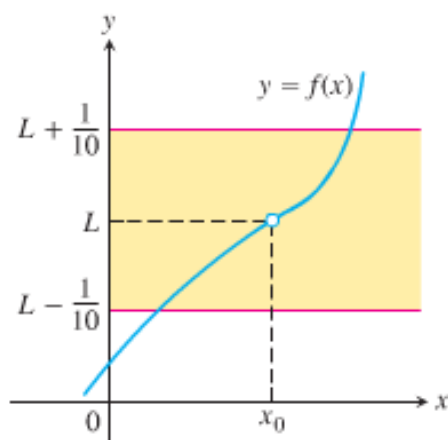
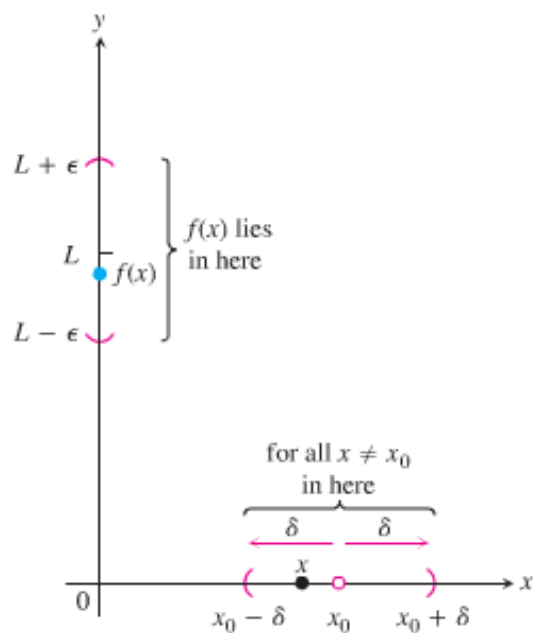
Definition

Let $f(x)$ be defined on an open interval about x_0 , except possibly at x_0 itself. We say that **the limit of $f(x)$ as x approaches x_0 is the number L** , and write

$$\lim_{x \rightarrow x_0} f(x) = L$$

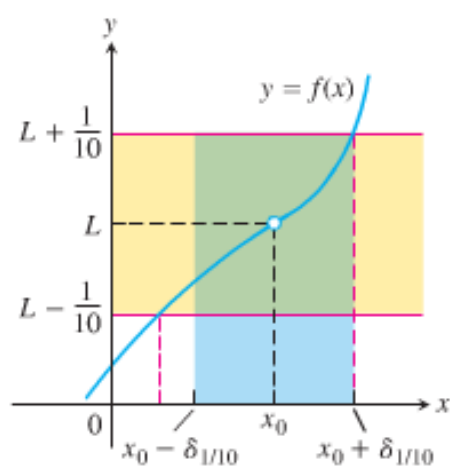
If, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x ,

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$$



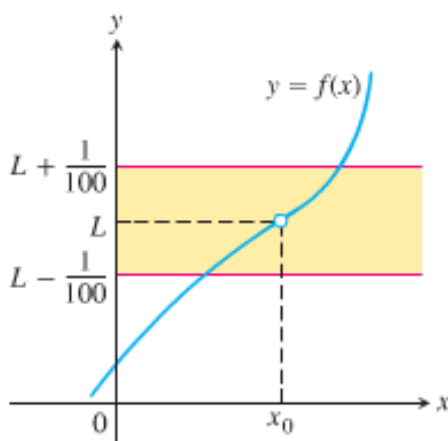
The challenge:

$$\text{Make } |f(x) - L| < \epsilon = \frac{1}{10}$$



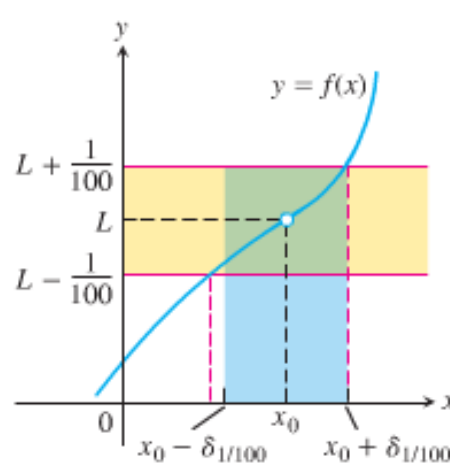
Response:

$$|x - x_0| < \delta_{1/10} \text{ (a number)}$$



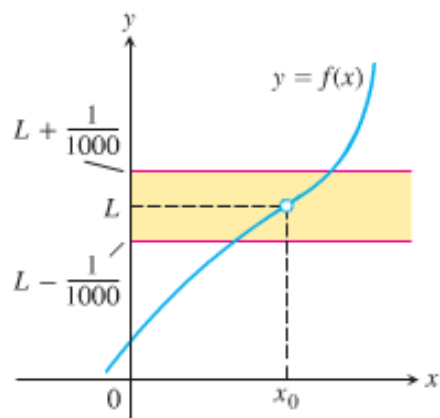
New challenge:

$$\text{Make } |f(x) - L| < \epsilon = \frac{1}{100}$$



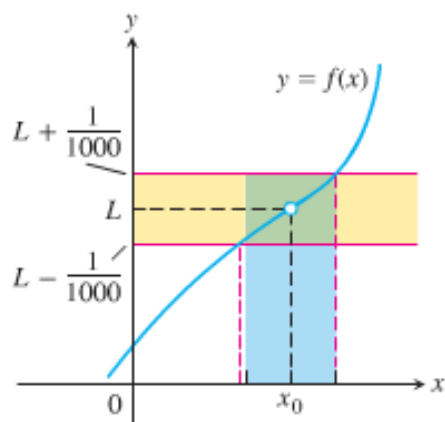
Response:

$$|x - x_0| < \delta_{1/100}$$



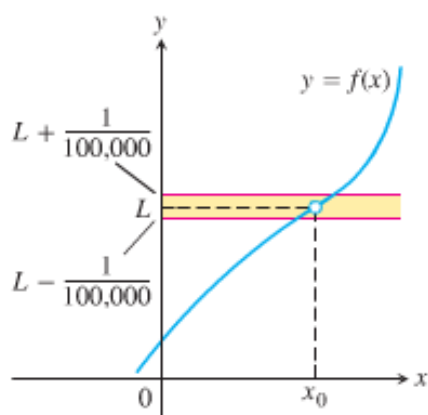
New challenge:

$$\epsilon = \frac{1}{1000}$$



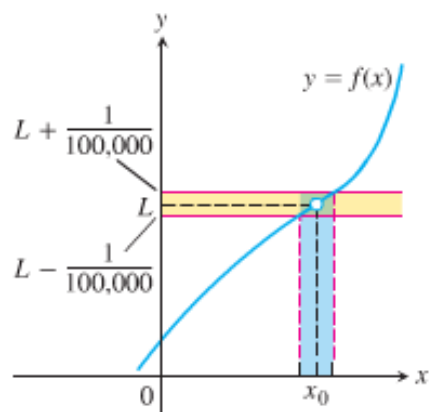
Response:

$$|x - x_0| < \delta_{1/1000}$$



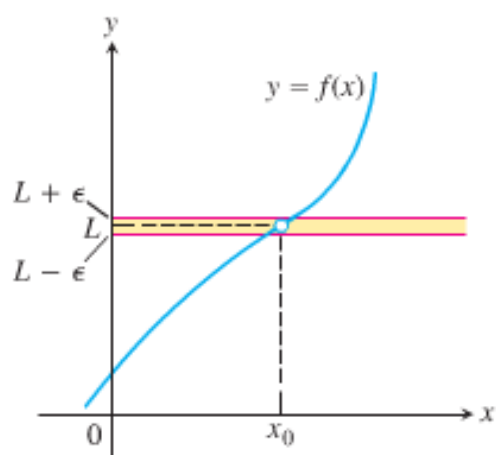
New challenge:

$$\epsilon = \frac{1}{100,000}$$



Response:

$$|x - x_0| < \delta_{1/100,000}$$



New challenge:

$$\epsilon = \dots$$

Example

Show that $\lim_{x \rightarrow 1} (5x - 3) = 2$

Solution

Let $x_0 = 1$, $f(x) = 5x - 3$, and $L = 2$.

For any given $\varepsilon > 0$, there exists a $\delta > 0$ so that $x \neq 1$ and x is within distance δ of $x_0 = 1$, that is

$$0 < |x - 1| < \delta \Rightarrow |f(x) - 2| < \varepsilon$$

$$|(5x - 3) - 2| < \varepsilon$$

$$|5x - 5| < \varepsilon$$

$$5|x - 1| < \varepsilon$$

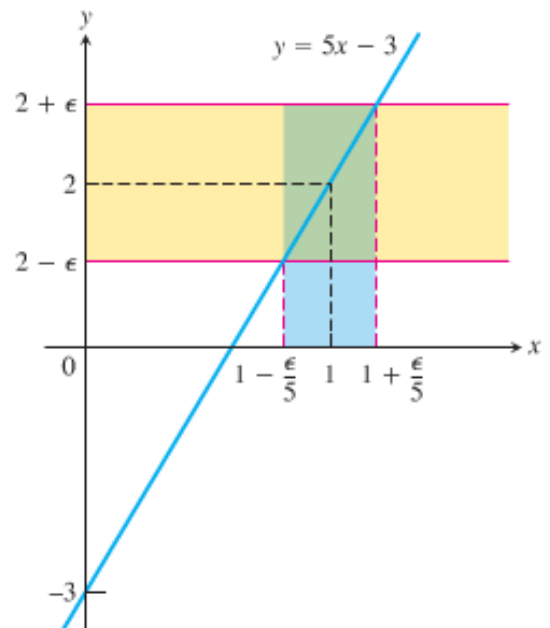
$$|x - 1| < \frac{\varepsilon}{5}$$

Thus, we can take: $\delta = \frac{\varepsilon}{5}$

If $0 < |x - 1| < \delta = \frac{\varepsilon}{5}$

$$\begin{aligned} |(5x - 3) - 2| &= |5x - 5| \\ &= 5|x - 1| \\ &= 5 \frac{\varepsilon}{5} \\ &= \varepsilon \end{aligned}$$

Which proves that $\lim_{x \rightarrow 1} (5x - 3) = 2$



Example

Prove the results presented graphically $\lim_{x \rightarrow x_0} x = x_0$

Solution

Let $\varepsilon > 0$ be given, we must find $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \Rightarrow |x - x_0| < \varepsilon$$

This implication will hold if $\delta = \varepsilon$ or any smaller number.

Example

For the limit $\lim_{x \rightarrow 5} \sqrt{x-1} = 2$, find a $\delta > 0$ that works for $\varepsilon = 1$. That is, find a $\delta > 0$ such that for all x :

$$0 < |x - 5| < \delta \quad \Rightarrow \quad \left| \sqrt{x-1} - 2 \right| < 1$$

Solution

$$|\sqrt{x-1}-2|<1$$

$$-1 < \sqrt{x-1} - 2 < 1$$

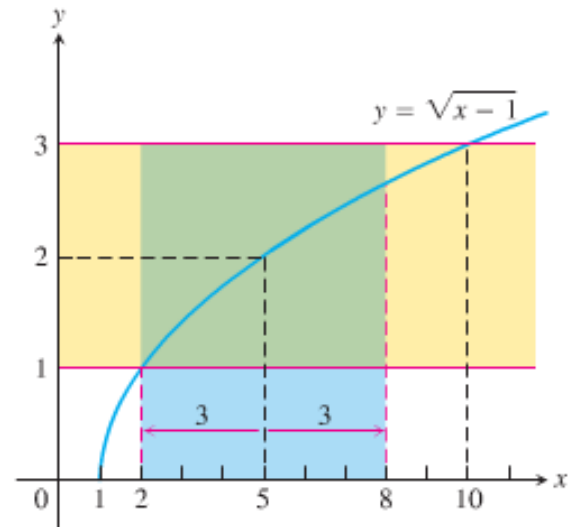
$$-1 + 2 < \sqrt{x-1} - 2 + 2 < 1 + 2$$

$$1 < \sqrt{x-1} < 3 \quad \text{Square all sides}$$

$$1 < x - 1 < 9$$

$$1+1 < x-1+1 < 9+1$$

$$2 < x < 10$$



The inequality holds for all x in the open interval $(2, 10)$.

So it holds for all $x \neq 5$ in the interval as well.

Finding δ value.

$$5 - \delta < x < 5 + \delta$$

Centered at $x_0 = 5$ inside the interval $(2, 10)$



$$\begin{cases} 5 - \delta = 2 \\ 5 + \delta < 10 \end{cases} \rightarrow \delta = 3 \text{ (to be centered)}$$

$$0 < |x - 5| < 3 \Rightarrow \left| \sqrt{x-1} - 2 \right| < 1$$

How to Find Algebraically a δ for a Given f, L, x_0 , and $\varepsilon > 0$

The process of finding a $\delta > 0$ such that for all x :

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Can be accomplished in two steps

1. Solve the inequality $|f(x) - L| < \varepsilon$ to find an open interval (a, b) containing x_0 on which the inequality holds for all $x \neq x_0$.
2. Find a value of $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b) . The inequality $|f(x) - L| < \varepsilon$ will hold for all $x \neq x_0$ in this δ -interval.

Example

Prove that $\lim_{x \rightarrow 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

Solution

We need to show that given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x :

$$0 < |x - 2| < \delta \Rightarrow |f(x) - 4| < \varepsilon$$

1. Solve the inequality $|f(x) - 4| < \varepsilon$ to find an open interval containing $x_0 = 2$ on which the inequality holds for all $x \neq x_0$.

For $x \neq x_0 = 2$, $f(x) = x^2$, and the inequality to solve is $|x^2 - 4| < \varepsilon$:

$$|x^2 - 4| < \varepsilon$$

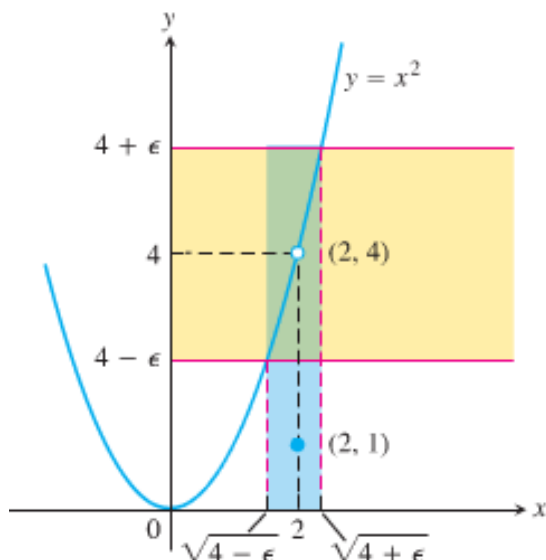
$$-\varepsilon < x^2 - 4 < \varepsilon \quad \text{Add 4 to all sides}$$

$$4 - \varepsilon < x^2 < 4 + \varepsilon \quad \text{Square root}$$

$$\sqrt{4 - \varepsilon} < |x| < \sqrt{4 + \varepsilon} \quad \text{Assume } \varepsilon < 4$$

$$\sqrt{4 - \varepsilon} < x < \sqrt{4 + \varepsilon}$$

The inequality $|f(x) - 4| < \varepsilon$ holds for all $x \neq 2$ in the open interval $(\sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon})$



2. Find a value of $\delta > 0$ that places the open interval $(2 - \delta, 2 + \delta)$ inside the interval $(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon})$.

Take δ to be the distance from $x_0 = 2$ to the nearer endpoint of $(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon})$.

$$\Rightarrow \delta = \min(2 - \sqrt{4 - \epsilon}, \sqrt{4 + \epsilon} - 2).$$

$$0 < |x - 2| < \delta$$

$$-(2 - \sqrt{4 - \epsilon}) < x - 2 < \sqrt{4 + \epsilon} - 2$$

$$-2 + \sqrt{4 - \epsilon} < x - 2 < \sqrt{4 + \epsilon} - 2$$

$$\sqrt{4 - \epsilon} < x < \sqrt{4 + \epsilon}$$

$$\therefore 0 < |x - 2| < \delta \Rightarrow |f(x) - 4| < \epsilon$$

Example

Given that $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} g(x) = M$, prove that $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$

Solution

We need to show that given $\epsilon > 0$ there exists a $\delta > 0$ such that for all x :

$$0 < |x - c| < \delta \Rightarrow |f(x) + g(x) - (L + M)| < \epsilon$$

$$|f(x) + g(x) - (L + M)| = |f(x) + g(x) - L - M|$$

$$\begin{aligned}
&= \left| (f(x) - L) + (g(x) - M) \right| && \textbf{Triangle Inequality } |a + b| \leq |a| + |b| \\
&\leq \left| (f(x) - L) \right| + \left| (g(x) - M) \right|
\end{aligned}$$

Since $\lim_{x \rightarrow c} f(x) = L$, there exists a number $\delta_1 > 0$ such that for all x :

$$0 < |x - c| < \delta_1 \Rightarrow |f(x) - L| < \frac{\varepsilon}{2}$$

Similarly, since $\lim_{x \rightarrow c} g(x) = M$, there exists a number $\delta_2 > 0$ such that for all x :

$$0 < |x - c| < \delta_2 \Rightarrow |g(x) - M| < \frac{\varepsilon}{2}$$

Let $\delta = \min\{\delta_1, \delta_2\}$, the smaller of δ_1 and δ_2 . If $0 < |x - c| < \delta$ then $0 < |x - c| < \delta_1$, so

$|f(x) - L| < \frac{\varepsilon}{2}$ and $|x - c| < \delta_2$, so $|g(x) - M| < \frac{\varepsilon}{2}$. Therefore

$$|f(x) + g(x) - (L + M)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This show that $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$

Exercises Section 1.6 – Precise Definition of Limits

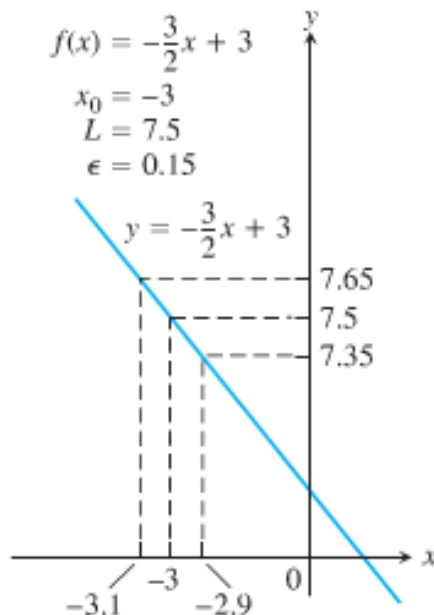
Sketch the interval (a, b) on the x -axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for

all x , $0 < |x - x_0| < \delta \Rightarrow a < x < b$ for

1. $a = 1, \quad b = 7, \quad x_0 = 5$

2. $a = -\frac{7}{2}, \quad b = -\frac{1}{2}, \quad x_0 = -\frac{3}{2}$

3. Use the graph to find a $\delta > 0$ such that for all x $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$



(4 – 11) Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

4. $f(x) = x + 1, \quad L = 5, \quad x_0 = 4, \quad \varepsilon = 0.01$

5. $f(x) = 2x - 1, \quad L = 3, \quad x_0 = 2, \quad \varepsilon = 0.1$

6. $f(x) = x + 2, \quad L = 3, \quad x_0 = 1, \quad \varepsilon = 0.001$

7. $f(x) = 3x + 2, \quad L = 2, \quad x_0 = 0, \quad \varepsilon = 0.1$

8. $f(x) = \sqrt{x+1}, \quad L = 1, \quad x_0 = 0, \quad \varepsilon = 0.1$

9. $f(x) = \sqrt{x-7}, \quad L = 4, \quad x_0 = 23, \quad \varepsilon = 1$

10. $f(x) = x^2, \quad L = 3, \quad x_0 = \sqrt{3}, \quad \varepsilon = 0.1$

11. $f(x) = \frac{120}{x}, \quad L = 5, \quad x_0 = 24, \quad \varepsilon = 1$

(12 – 17) Give a formal proof that

12. $\lim_{x \rightarrow 4} (9 - x) = 5$

13. $\lim_{x \rightarrow 1} \frac{1}{x} = 1$

14. $\lim_{x \rightarrow 5} \frac{x^2 - 25}{x - 5} = 10$

15. $\lim_{x \rightarrow 0} f(x) = 0$ if $f(x) = \begin{cases} 2x, & x < 0 \\ \frac{x}{2}, & x \geq 0 \end{cases}$

16. $\lim_{x \rightarrow 1} (5x - 2) = 3$

17. $\lim_{x \rightarrow 2} \frac{1}{(x - 2)^4} = \infty$

18. Prove that $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0$

