Solution   *Section* 1.9 – Hyperbolic Functions

Exercise

Rewrite the expression $\cosh 3x - \sinh 3x$ in terms of exponentials and simplify the results as much as you can.

Solution

$$\begin{aligned}\cosh 3x - \sinh 3x &= \frac{e^{3x}+e^{-3x}}{2} - \frac{e^{3x}-e^{-3x}}{2}\\ &= \frac{e^{3x}+e^{-3x}-e^{3x}+e^{-3x}}{2}\\ &= \frac{2e^{-3x}}{2}\\ &= e^{-3x}\end{aligned}$$

Exercise

Rewrite the expression $\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x)$ in terms of exponentials and simplify the results as much as you can.

Solution

$$\begin{aligned}\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x) &= \ln\left[(\cosh x + \sinh x)(\cosh x - \sinh x)\right]\\ &= \ln\left(\cosh^2 x - \sinh^2 x\right)\\ &= \ln(1)\\ &= 0\end{aligned}$$

Exercise

Prove the identities

a) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

b) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

Solution

a) $$\begin{aligned}\sinh x \cosh y + \cosh x \sinh y &= \frac{e^{x}-e^{-x}}{2}\frac{e^{y}+e^{-y}}{2} + \frac{e^{x}+e^{-x}}{2}\frac{e^{y}-e^{-y}}{2}\\ &= \frac{e^{x}e^{y}+e^{x}e^{-y}-e^{-x}e^{y}-e^{-x}e^{-y}+e^{x}e^{y}-e^{x}e^{-y}+e^{-x}e^{y}-e^{-x}e^{-y}}{4}\\ &= \frac{2e^{x}e^{y}-2e^{-x}e^{-y}}{4}\\ &= \frac{e^{x+y}-e^{-x-y}}{2}\end{aligned}$$

$$= \frac{e^{x+y} - e^{-(x+y)}}{2}$$

$$= \sinh(x+y)$$

b) $\cosh x \cosh y + \sinh x \sinh y = \frac{e^{y}+e^{-y}}{2}\frac{e^{x}+e^{-x}}{2} + \frac{e^{x}-e^{-x}}{2}\frac{e^{y}-e^{-y}}{2}$

$$= \frac{e^{x}e^{y} + e^{x}e^{-y} + e^{-x}e^{y} + e^{-x}e^{-y} + e^{x}e^{y} - e^{x}e^{-y} - e^{-x}e^{y} + e^{-x}e^{-y}}{4}$$

$$= \frac{2e^{x}e^{y} + 2e^{-x}e^{-y}}{4}$$

$$= \frac{e^{x+y} + e^{-x-y}}{2}$$

$$= \frac{e^{x+y} + e^{-(x+y)}}{2}$$

$$= \cosh(x+y)$$

Exercise

Find the derivative of $y = \frac{1}{2}\sinh(2x+1)$

Solution

$$y' = \frac{1}{2}\left[\cosh(2x+1)\right](2) \quad = \cosh(2x+1)$$

Exercise

Find the derivative of $y = 2\sqrt{t}\tanh\sqrt{t}$

Solution

$$y' = 2\left(\frac{1}{2}t^{-1/2}\tanh\sqrt{t} + t^{1/2}\left(\frac{1}{2}\right)\sec^{2}\sqrt{t}\right)$$

$$= \frac{\tanh\sqrt{t}}{\sqrt{t}} + \sqrt{t}\sec^{2}\sqrt{t}$$

Exercise

Find the derivative of $y = \ln(\cosh z)$

Solution

$$y' = \frac{\sinh z}{\cosh z} = \tanh z$$

Exercise

Find the derivative of $y = \operatorname{csch}\theta(1 - \ln \operatorname{csch}\theta)$

Solution

$$y' = (-\operatorname{csch}\theta \coth\theta)(1 - \ln \operatorname{csch}\theta) + \operatorname{csch}\theta\left(-\frac{-\operatorname{csch}\theta \coth\theta}{\operatorname{csch}\theta}\right)$$

$$= -\operatorname{csch}\theta \coth\theta + \operatorname{csch}\theta \coth\theta(\ln \operatorname{csch}\theta) + \operatorname{csch}\theta \coth\theta$$

$$= \operatorname{csch}\theta \coth\theta(\ln \operatorname{csch}\theta)$$

Exercise

Find the derivative of $y = \ln \sinh v - \frac{1}{2}\coth^2 v$

Solution

$$y' = \frac{\cosh v}{\sinh v} - \frac{1}{2}2\coth v\left(-\operatorname{csch}^2 v\right)$$

$$= \coth v + (\coth v)\left(\operatorname{csch}^2 v\right)$$

Exercise

Find the derivative of $y = \left(x^2 + 1\right)\operatorname{sech}(\ln x)$

Solution

$$y = \left(x^2 + 1\right)\left(\frac{2}{e^{\ln x} + e^{-\ln x}}\right)$$

$$= \left(x^2 + 1\right)\left(\frac{2}{x + x^{-1}}\right)$$

$$= \left(x^2 + 1\right)\left(\frac{2x}{x^2 + 1}\right)$$

$$= 2x$$

$$y' = 2$$

Exercise

Find the derivative of $y = \left(4x^2 - 1\right)\operatorname{csch}(\ln 2x)$

Solution

$$y = \left(4x^2 - 1\right)\left(\frac{2}{e^{\ln 2x} - e^{-\ln 2x}}\right)$$

$$= \left(4x^2 - 1\right)\left(\frac{2}{2x - (2x)^{-1}}\right)$$

$$= \left(4x^2 - 1\right)\left(\frac{4x}{4x^2 - 1}\right)$$

$$= 4x$$

$$\underline{y' = 4}$$

Exercise

Find the derivative of $y = \cosh^{-1} 2\sqrt{x+1}$

Solution

$$y = \cosh^{-1} 2\sqrt{x+1} = \cosh^{-1} 2(x+1)^{1/2}$$

$$y' = \frac{2\left(\frac{1}{2}\right)(x+1)^{-1/2}}{\sqrt{\left(2(x+1)^{1/2}\right)^2 - 1}}$$

$$= \frac{1}{(x+1)^{1/2}\sqrt{4(x+1)-1}}$$

$$= \frac{1}{\sqrt{x+1}\sqrt{4x+3}}$$

$$= \frac{1}{\sqrt{4x^2 + 7x + 3}}$$

Exercise

Find the derivative of $y = \left(\theta^2 + 2\theta\right)\tanh^{-1}(\theta + 1)$

Solution

$$y' = (2\theta + 2)\tanh^{-1}(\theta + 1) + \left(\theta^2 + 2\theta\right)\left(\frac{1}{1 - (\theta + 1)^2}\right)$$

$$= (2\theta + 2)\tanh^{-1}(\theta + 1) + \frac{\theta^2 + 2\theta}{1 - \left(\theta^2 + 2\theta + 1\right)}$$

$$= (2\theta + 2)\tanh^{-1}(\theta + 1) + \frac{\theta^2 + 2\theta}{-\theta^2 - 2\theta}$$

$$= (2\theta + 2)\tanh^{-1}(\theta + 1) - 1$$

Exercise

Find the derivative of $y=(1-t)\coth^{-1}\sqrt{t}$

Solution

$$y'=-\coth^{-1}\sqrt{t}+(1-t)\frac{\frac{1}{2}t^{-1/2}}{1-\left(t^{1/2}\right)^2}$$

$$=-\coth^{-1}\sqrt{t}+(1-t)\frac{1}{2\sqrt{t}(1-t)}$$

$$\boxed{=-\coth^{-1}\sqrt{t}+\frac{1}{2\sqrt{t}}}$$

Exercise

Find the derivative of $y=\ln x+\sqrt{1-x^2}\ \operatorname{sech}^{-1}x$

Solution

$$y'=\frac{1}{x}+\frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x)\ \operatorname{sech}^{-1}x+\sqrt{1-x^2}\left(\frac{-1}{x\sqrt{1-x^2}}\right)$$

$$=\frac{1}{x}-\frac{x}{\left(1-x^2\right)^{1/2}}\ \operatorname{sech}^{-1}x-\frac{1}{x}$$

$$\boxed{=-\frac{x}{\sqrt{1-x^2}}\ \operatorname{sech}^{-1}x}$$

Exercise

Find the derivative of $y=\operatorname{csch}^{-1}\left(\frac{1}{2}\right)^{\theta}$

Solution

$$y'=-\frac{\left[\ln\left(\frac{1}{2}\right)\right]\left(\frac{1}{2}\right)^{\theta}}{\left(\frac{1}{2}\right)^{\theta}\sqrt{1+\left[\left(\frac{1}{2}\right)^{\theta}\right]^2}}$$

$$=-\frac{-\ln 2}{\sqrt{1+\left(\frac{1}{2}\right)^{2\theta}}}$$

$$\boxed{=\frac{\ln 2}{\sqrt{1+\left(\frac{1}{2}\right)^{2\theta}}}}$$

Exercise

Find the derivative of $y = \cosh^{-1}(\sec x)$

Solution

$$y' = \frac{(\sec x)(\tan x)}{\sqrt{\sec^2 x - 1}}$$

$$= \frac{(\sec x)(\tan x)}{\sqrt{\tan^2 x}}$$

$$= \frac{(\sec x)(\tan x)}{|\tan x|}$$

$$\underline{= \sec x|} \quad 0 < x < \frac{\pi}{2}$$

Exercise

Find the derivative of $y = -\sinh^3 4x$

Solution

$$\underline{y' = -12\left(\sinh^2 4x\right)(\cosh 4x)|}$$

Exercise

Find the derivative of $y = \sqrt{\coth 3x}$

Solution

$$y' \underline{= \frac{-3\operatorname{csch}^2 3x}{2\sqrt{\coth 3x}}|}$$

$$\left(\sqrt{u}\right)' = \frac{u'}{2\sqrt{u}}$$

Exercise

Find the derivative of $y = \frac{x}{\operatorname{csch} x}$

Solution

$$y' = \frac{\operatorname{csch} x + x \operatorname{csch} x \coth x}{\operatorname{csch}^2 x}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -u' \operatorname{csch} u \coth u$$

$$= \frac{1 + x\coth x}{\operatorname{csch} x}$$

$$= \frac{1}{\operatorname{csch} x} + x\frac{\coth x}{\operatorname{csch} x}$$

$$\underline{= \sinh x + x\cosh x|}$$

Exercise

Find the derivative of $y = \tanh^2 x$

Solution

$$y' = 2\tanh x \ \text{sech}^2 x$$

$$\frac{d}{dx}(\tanh u) = u' \text{sech}^2 u$$

Exercise

Find the derivative of $y = \ln\ \text{sech}\, 2x$

Solution

$$y' = \frac{-2\ \text{sech}\, 2x\ \tanh 2x}{\text{sech}\, 2x} = -2\tanh 2x$$

$$\frac{d}{dx}(\text{sech}\, u) = -u' \text{sech}\, u\ \tanh u$$

Exercise

Find the derivative of $y = x^2 \cosh^2 3x$

Solution

$$y' = 2x\cosh^2 3x + 6x^2 \cosh 3x\ \sinh 3x$$

$$= 2x\cosh 3x\left(\cosh 3x + 3x\ \sinh 3x\right)$$

$$\frac{d}{dx}(\sinh u) = u' \cosh u$$

Exercise

Find the derivative of $f(t) = 2\tanh^{-1}\sqrt{t}$

Solution

$$f'(t) = 2\frac{\frac{1}{2\sqrt{t}}}{1-\left(\sqrt{t}\right)^2} = \frac{1}{\sqrt{t}\,(1-t)}$$

$$\frac{d}{dx}\left(\tanh^{-1} u\right) = \frac{u'}{1-u^2}$$

Exercise

Find the derivative of $f(x) = \sinh^{-1} x^2$

Solution

$$f'(x) = \frac{2x}{\sqrt{x^4+1}}$$

$$\frac{d}{dx}\left(\sinh^{-1} u\right) = \frac{u'}{\sqrt{1+u^2}}$$

Exercise

Find the derivative of $f(x)=\operatorname{csch}^{-1}\left(\frac{2}{x}\right)$

Solution

$$f'(x)=\frac{-1}{\left|\frac{2}{x}\right|\sqrt{1+\frac{4}{x^2}}}\left(\frac{-2}{x^2}\right)=\boxed{\frac{1}{\sqrt{x^2+4}}}$$

$$\frac{d}{dx}\left(\operatorname{csch}^{-1}u\right)=-\frac{u'}{|u|\sqrt{1+u^2}}$$

Exercise

Find the derivative of $f(x)=x\sinh^{-1}x-\sqrt{x^2+1}$

Solution

$$f'(x)=\sinh^{-1}x+x\frac{1}{\sqrt{x^2+1}}-\frac{2x}{2\sqrt{x^2+1}}$$

$$\boxed{=\sinh^{-1}x}$$

$$\frac{d}{dx}\left(\sinh^{-1}u\right)=\frac{u'}{\sqrt{1+u^2}}$$

Exercise

Find the derivative of $f(x)=\sinh^{-1}(\tan x)$

Solution

$$f'(x)=\frac{\sec^2 x}{\sqrt{1+\tan^2 x}}$$

$$\frac{d}{dx}\left(\sinh^{-1}u\right)=\frac{u'}{\sqrt{1+u^2}}$$

$$=\frac{\sec^2 x}{\sqrt{\sec^2 x}}$$

$$\boxed{=|\sec x|}$$

Exercise

Verify the integration $\int x\operatorname{sech}^{-1}x\,dx=\frac{x^2}{2}\operatorname{sech}^{-1}x-\frac{1}{2}\sqrt{1-x^2}+C$

Solution

If $y=\frac{x^2}{2}\operatorname{sech}^{-1}x-\frac{1}{2}\sqrt{1-x^2}+C$

$$dy=\left[x\operatorname{sech}^{-1}x+\frac{x^2}{2}\left(\frac{-1}{x\sqrt{1-x^2}}\right)-\frac{1}{2}\left(\frac{1}{2}\right)\frac{-2x}{\sqrt{1-x^2}}\right]dx$$

$$dy=\left[x\operatorname{sech}^{-1}x-\frac{x}{2\sqrt{1-x^2}}+\frac{x}{2\sqrt{1-x^2}}\right]dx$$

$\boxed{dy=\left(x\operatorname{sech}^{-1}x\right)dx}$ ✓ Which verifies the formula

Exercise

Verify the integration $\int \tanh^{-1} x dx = x\tanh^{-1} x + \frac{1}{2}\ln\left(1-x^2\right) + C$

Solution

If $\quad y = x\tanh^{-1} x + \frac{1}{2}\ln\left(1-x^2\right) + C$

$$\frac{dy}{dx} = \tanh^{-1} x + x\left(\frac{1}{1-x^2}\right) + \frac{1}{2}\frac{-2x}{1-x^2}$$

$$= \tanh^{-1} x + \frac{x}{1-x^2} - \frac{x}{1-x^2}$$

$= \tanh^{-1} x$ $\checkmark$ which verifies the formula

Exercise

Evaluate the integral: $\int \sinh 2x dx$

Solution

$$\int \sinh 2x\ dx = \frac{1}{2}\int \sinh 2x\ d(2x)$$

$$= \frac{1}{2}\cosh 2x + C$$

Exercise

Evaluate the integral: $\int 4\cosh(3x - \ln 2) dx$

Solution

$$\int 4\cosh(3x-\ln 2) dx = \frac{4}{3}\int \cosh(3x-\ln 2)\ d(3x-\ln 2) \qquad d(3x-\ln 2) = 3dx$$

$$= \frac{4}{3}\sinh(3x-\ln 2) + C$$

Exercise

Evaluate the integral: $\int \tanh\frac{x}{7} dx$

Solution

$$\int \tanh\frac{x}{7}\ dx = \int \frac{\sinh\frac{x}{7}}{\cosh\frac{x}{7}}\ d(x) \qquad d\left(\cosh\frac{x}{7}\right) = \frac{1}{7}\left(\sinh\frac{x}{7}\right)dx$$

$$= 7\int \frac{1}{\cosh\frac{x}{7}}\ d\left(\cosh\frac{x}{7}\right)$$

$$= 7\ln\left|\cosh\frac{x}{7}\right| + C$$

$$= 7\ln\left[\frac{e^{x/7}+e^{-x/7}}{2}\right] + C$$

$$= 7\left[\ln\left(e^{x/7}+e^{-x/7}\right) - \ln 2\right] + C$$

$$= 7\ln\left(e^{x/7}+e^{-x/7}\right) - 7\ln 2 + C \qquad C_1 = -7\ln 2 + C$$

$$\boxed{= 7\ln\left(e^{x/7}+e^{-x/7}\right) + C_1}$$

Exercise

Evaluate the integral: $\displaystyle\int \coth\frac{\theta}{\sqrt{3}}\,d\theta$

Solution

$$\int \coth\frac{\theta}{\sqrt{3}}\,d\theta = \int \frac{\cosh\frac{\theta}{\sqrt{3}}}{\sinh\frac{\theta}{\sqrt{3}}}\,d\theta \qquad d\left(\sinh\frac{\theta}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}\left(\cosh\frac{\theta}{\sqrt{3}}\right)d\theta$$

$$= \sqrt{3}\int \frac{1}{\sinh\frac{\theta}{\sqrt{3}}}\,d\left(\sinh\frac{\theta}{\sqrt{3}}\right)$$

$$= \sqrt{3}\ln\left|\sinh\frac{\theta}{\sqrt{3}}\right| + C_1$$

$$= \sqrt{3}\ln\left|\frac{e^{\theta/\sqrt{3}}-e^{-\theta/\sqrt{3}}}{2}\right| + C_1$$

$$= \sqrt{3}\ln\left|e^{\theta/\sqrt{3}}-e^{-\theta/\sqrt{3}}\right| - \sqrt{3}\ln 2 + C_1 \qquad C = -\sqrt{3}\ln 2 + C_1$$

$$\boxed{= \sqrt{3}\ln\left|e^{\theta/\sqrt{3}}-e^{-\theta/\sqrt{3}}\right| + C}$$

Exercise

Evaluate the integral: $\displaystyle\int \operatorname{csch}^2(5-x)\,dx$

Solution

$$\int \operatorname{csch}^2(5-x)\,dx = -\int \operatorname{csch}^2(5-x)\,d(5-x) \qquad \int \operatorname{csch}^2 u\,du = -\coth u$$

$$\boxed{= \coth(5-x) + C}$$

Exercise

Evaluate the integral: $\int \frac{\operatorname{sech}\sqrt{t}\ \tanh\sqrt{t}}{\sqrt{t}}dt$

Solution

$$\int \frac{\operatorname{sech}\sqrt{t}\ \tanh\sqrt{t}}{\sqrt{t}}dt = 2\int \operatorname{sech}\sqrt{t}\ \tanh\sqrt{t}\ d\left(\sqrt{t}\right) \qquad d\left(\sqrt{t}\right) = \frac{1}{2\sqrt{t}}dt$$

$$\boxed{= -2\operatorname{sec}h\sqrt{t} + C} \qquad \int \operatorname{sech} u \tanh u\ du = -\operatorname{sech} u$$

Exercise

Evaluate the integral: $\int \frac{\operatorname{csch}(\ln t)\ \coth(\ln t)}{t}dt$

Solution

$$\int \frac{\operatorname{csch}(\ln t)\ \coth(\ln t)}{t}dt = \int \operatorname{csch}(\ln t)\ \coth(\ln t)\ d(\ln t) \qquad d(\ln t) = \frac{dt}{t}$$

$$\boxed{= -\operatorname{csc}h(\ln t) + C} \qquad \int \operatorname{csch} u \coth u\ du = -\operatorname{csch} u$$

Exercise

Evaluate the integral $\int \frac{\sinh x}{1+\cosh x}dx$

Solution

$$\int \frac{\sinh x}{1+\cosh x}dx = \int \frac{d(1+\cosh x)}{1+\cosh x} \qquad d(1+\cosh x) = \sinh x\ dx$$

$$\boxed{= \ln|1+\cosh x| + C}$$

Exercise

Evaluate the integral $\int \operatorname{sech}^2 x\ \tanh x\ dx$

Solution

$$\int \operatorname{sech}^2 x\ \tanh x\ dx = \int \tanh x\ d(\tanh x) \qquad d(\tanh x) = \operatorname{sech}^2 x\ dx$$

$$\boxed{= \frac{1}{2}\tanh^2 x + C}$$

Exercise

Evaluate the integral $\int \coth^2 x \operatorname{csch}^2 x \, dx$

Solution

$$\int \coth^2 x \operatorname{csch}^2 x \, dx = -\int \coth^2 x \, d(\coth x) \qquad d(\coth x) = -\operatorname{csch}^2 x \, dx$$

$$= -\frac{1}{3}\coth^3 x + C$$

Exercise

Evaluate the integral $\int \tanh^2 x \, dx$

Solution

$$\int \tanh^2 x \, dx = \int \left(1 - \operatorname{sech}^2 x\right) dx \qquad \tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$= x - \tanh x + C \qquad \int \operatorname{sech}^2 u \, du = \tanh u$$

Exercise

Evaluate the integral $\int \frac{\sinh(\ln x)}{x} dx$

Solution

$$\int \frac{\sinh(\ln x)}{x} dx = \int \sinh(\ln x) \, d(\ln x)$$

$$= \cosh(\ln x) + C$$

$$= \frac{e^{\ln x} + e^{-\ln x}}{2} + C$$

$$= \frac{1}{2}\left(x + \frac{1}{x}\right) + C$$

$$= \frac{x^2 + 1}{2x} + C$$

$$\sinh(\ln x) = \frac{e^{\ln x} - e^{-\ln x}}{2} = \frac{1}{2}\left(x - \frac{1}{x}\right)$$

$$\int \frac{\sinh(\ln x)}{x} dx = \frac{1}{2}\int \frac{1}{x}\left(x - \frac{1}{x}\right) d(x)$$

$$= \frac{1}{2}\int \left(1 - \frac{1}{x^2}\right) d(x)$$

$$= \frac{1}{2}\left(x + \frac{1}{x}\right) + C$$

$$= \frac{x^2 + 1}{2x} + C$$

Exercise

Evaluate the integral $\int \frac{dx}{8 - x^2} \quad x > 2\sqrt{2}$

Solution

$$\int \frac{dx}{8 - x^2} = \frac{1}{2\sqrt{2}} \tanh^{-1}\left(\frac{x}{2\sqrt{2}}\right) + C \qquad \int \frac{du}{a^2 - u^2} = \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right)$$

Exercise

Evaluate the integral $\int \frac{dx}{\sqrt{x^2-16}}$

Solution

$$\int \frac{dx}{\sqrt{x^2-16}} = \coth^{-1}\left(\frac{x}{4}\right)+C$$

$$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right)$$

Exercise

Evaluate the integral $\int_0^1 \cosh^3 3x \ \sinh 3x \ dx$

Solution

$$\int_0^1 \cosh^3 3x \ \sinh 3x \ dx = \frac{1}{3}\int_0^1 \cosh^3 3x \ d(\cosh 3x)$$

$$d(\cosh 3x) = 3\sinh x \ dx$$

$$= \frac{1}{12}\cosh^4 3x \Big|_0^1$$

$$= \frac{1}{12}\left(\cosh^4 3 - \cosh^4 0\right)$$

$$= \frac{1}{12}\left(\cosh^4 3 - 1\right) \quad \approx 856.034$$

Exercise

Evaluate the integral $\int_0^4 \frac{\operatorname{sech}^2 \sqrt{x}}{\sqrt{x}} dx$

Solution

$$\int_0^4 \frac{\operatorname{sech}^2 \sqrt{x}}{\sqrt{x}} dx = 2\int_0^4 \operatorname{sech}^2 \sqrt{x} \ d\left(\sqrt{x}\right)$$

$$d\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}} dx$$

$$= 2\tanh\sqrt{x}\Big|_0^4$$

$$= 2\tanh 2 \quad \approx 1.93$$

Exercise

Evaluate the integral $\int_{\ln 2}^{\ln 3} \operatorname{csch} x \ dx$

Solution

$$\int_{\ln 2}^{\ln 3} \operatorname{csch} x\, dx = \ln\left|\tanh\frac{x}{2}\right|\Big|_{\ln 2}^{\ln 3}$$

$$= \ln\left|\tanh\frac{\ln 3}{2}\right| - \ln\left|\tanh\frac{\ln 2}{2}\right| \quad \approx 0.405$$

Exercise

Evaluate the integral: $\displaystyle\int_{\ln 2}^{\ln 4} \coth x\, dx$

Solution

$$\int_{\ln 2}^{\ln 4} \coth x\, dx = \int_{\ln 2}^{\ln 4} \frac{\cosh x}{\sinh x}\, dx$$

$$= \int_{\ln 2}^{\ln 4} \frac{1}{\sinh x}\, d(\sinh x) \qquad\qquad d(\sinh x) = \cosh x\, dx$$

$$= \ln|\sinh x|\Big|_{\ln 2}^{\ln 4}$$

$$= \ln|\sinh \ln 4| - \ln|\sinh \ln 2|$$

$$= \ln\left(\frac{e^{\ln 4} - e^{-\ln 4}}{2}\right) - \ln\left(\frac{e^{\ln 2} - e^{-\ln 2}}{2}\right)$$

$$= \ln\left(\frac{4 - \frac{1}{4}}{2}\right) - \ln\left(\frac{2 - \frac{1}{2}}{2}\right)$$

$$= \ln\left(\frac{15}{8}\right) - \ln\left(\frac{3}{4}\right)$$

$$= \ln\left(\frac{15}{8} \div \frac{3}{4}\right)$$

$$= \ln\left(\frac{5}{2}\right)$$

Exercise

Evaluate the integral: $\displaystyle\int_{0}^{\pi/2} 2\sinh(\sin\theta)\cos\theta\, d\theta$

Solution

$$\int_{0}^{\pi/2} 2\sinh(\sin\theta)\cos\theta\, d\theta = 2\int_{0}^{\pi/2} \sinh(\sin\theta)\, d(\sin\theta) \qquad\qquad d(\sin\theta) = \cos\theta\, d\theta$$

$$= 2\cosh(\sin\theta)\Big|_{0}^{\pi/2}$$

$$= 2\left(\cosh 1 - \cosh 0\right)$$

$$= 2\left(\frac{e+e^{-1}}{2} - 1\right)$$

$$\underline{= e + e^{-1} - 2}$$

Exercise

Evaluate the integral: $\displaystyle\int_1^2 \frac{8\cosh\sqrt{x}}{\sqrt{x}}\,dx$

Solution

$$\int_1^2 \frac{8\cosh\sqrt{x}}{\sqrt{x}}\,dx = 16\int_1^2 \cosh\sqrt{x}\; d\left(\sqrt{x}\right)$$

$$= 16\sinh\sqrt{x}\,\Big|_1^2$$

$$= 16\left(\sinh\sqrt{2} - \sinh 1\right)$$

$$= 16\left(\frac{e^{\sqrt{2}} - e^{-\sqrt{2}}}{2} - \frac{e-e^{-1}}{2}\right)$$

$$\underline{= 8\left(e^{\sqrt{2}} - e^{-\sqrt{2}} - e + e^{-1}\right)}$$

Exercise

Evaluate the integral: $\displaystyle\int_{-\ln 2}^0 \cosh^2\left(\frac{x}{2}\right)dx$

Solution

$$\int_{-\ln 2}^0 \cosh^2\left(\frac{x}{2}\right)dx = \frac{1}{2}\int_{-\ln 2}^0 \left(\cosh x + 1\right)dx$$

$$= \frac{1}{2}\left(\sinh x + x\right)\Big|_{-\ln 2}^0$$

$$= \frac{1}{2}\left(-\sinh\left(-\ln 2\right) + \ln 2\right)$$

$$= \frac{1}{2}\left(-\frac{e^{-\ln 2} - e^{\ln 2}}{2} + \ln 2\right)$$

$$= \frac{1}{2}\left(-\frac{\frac{1}{2} - 2}{2} + \ln 2\right)$$

$$=\frac{3}{8}+\frac{1}{2}\ln 2$$

$$\boxed{=\frac{3}{8}+\ln\sqrt{2}}$$

Exercise

Evaluate the integral: $\displaystyle\int_0^{\ln 2} 4e^{-\theta}\sinh\theta\, d\theta$

Solution

$$\int_0^{\ln 2} 4e^{-\theta}\sinh\theta d\theta=\int_0^{\ln 2} 4e^{-\theta}\,\frac{e^{\theta}-e^{-\theta}}{2}d\theta$$

$$=2\int_0^{\ln 2}\left(1-e^{-2\theta}\right)d\theta$$

$$=2\left[\theta+\frac{1}{2}e^{-2\theta}\right]_0^{\ln 2}$$

$$=2\left[\ln 2+\frac{1}{2}e^{-2\ln 2}-\left(0+\frac{1}{2}\right)\right]$$

$$=2\left[\ln 2+\frac{1}{2}e^{\ln 2^{-2}}-\frac{1}{2}\right]$$

$$=2\left(\ln 2+\frac{1}{2}2^{-2}-\frac{1}{2}\right)$$

$$=2\left(\ln 2+\frac{1}{8}-\frac{1}{2}\right)$$

$$=2\left(\ln 2-\frac{3}{8}\right)$$

$$=2\ln 2-\frac{3}{4}$$

$$\boxed{=\ln 4-\frac{3}{4}}$$

Exercise

Evaluate the integral: $\displaystyle\int_1^{e^2}\frac{dx}{x\sqrt{\ln^2 x+1}}$

Solution

$$\int_1^{e^2}\frac{dx}{x\sqrt{\ln^2 x+1}}=\int_1^{e^2}\frac{d(\ln x)}{\sqrt{\ln^2 x+1}}$$

$$\int\frac{du}{\sqrt{a^2+u^2}}=\sinh^{-1}\left(\frac{u}{a}\right)$$

$$= \sinh^{-1}(\ln x)\Big|_1^{e^2}$$

$$= \sinh^{-1} 2 - 0$$

$$\underline{= \sinh^{-1} 2}$$

Exercise

Evaluate the integral: $\displaystyle\int_{1/8}^{1} \frac{dx}{x\sqrt{1+x^{2/3}}}$

Solution

$$\int_{1/8}^{1} \frac{dx}{x\sqrt{1+x^{2/3}}} = 3\int_{1/8}^{1} \frac{u^2 du}{u^3\sqrt{1+u^2}}$$

$u = x^{1/3} \quad \rightarrow \quad du = \frac{1}{3}x^{-2/3}dx$

$u^3 = x \quad \& \quad dx = 3x^{2/3}du = 3u^2 du$

$$= 3\int_{1/8}^{1} \frac{du}{u\sqrt{1+u^2}}$$

$\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right|$

$$= -3\operatorname{csch}^{-1}\left|x^{1/3}\right|\Big|_{1/8}^{1}$$

$$= -3\left(\operatorname{csch}^{-1}1 - \operatorname{csch}^{-1}\frac{1}{2}\right)$$

$$= 3\left(\sinh^{-1}2 - \sinh^{-1}1\right)$$

$x = \ln\left(y + \sqrt{y^2+1}\right)$

$$\underline{= 3\left(\ln\left(2+\sqrt{5}\right) - \ln\left(1+\sqrt{2}\right)\right)}$$

Exercise

Derive the formula $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$ for all real *x*. Explain in your derivation why the plus sign is used with the square root instead of the minus sign

Solution

$$y = \sinh^{-1} x \quad \Rightarrow \quad x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - e^{-y}$$

$$2xe^y = e^y e^y - e^{-y}e^y$$

$$e^{2y} - 2xe^y - 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

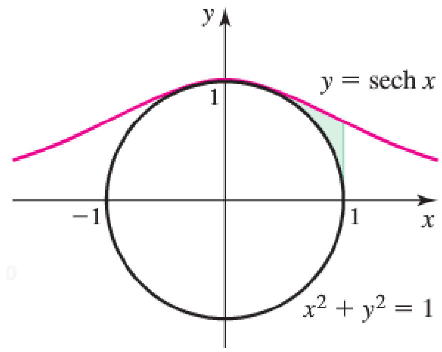
$$y=\ln\left(x\pm\sqrt{x^2+1}\right)$$

Since $x-\sqrt{x^2+1}<0$ (***impossible***) $\Rightarrow \quad y=\ln\left(x-\sqrt{x^2+1}\right)$

$$\therefore\ y=\ln\left(x+\sqrt{x^2+1}\right)$$

Exercise

Find the area of the region bounded by $y=\operatorname{sech}x$, $x=1$, and the unit circle.



Solution

The area of a quarter circle $=\frac{1}{4}\left(\pi r^2\right)=\frac{\pi}{4}$

$$Area=\int_0^1 \operatorname{sech}x\,dx-\frac{\pi}{4}$$

$$=\tan^{-1}|\sinh x|\Big|_0^1-\frac{\pi}{4}$$

$$=\tan^{-1}(\sinh 1)-\frac{\pi}{4}$$

$$\approx 0.08$$

$$Area=\int_0^1\left(\operatorname{sech}x-\sqrt{1-x^2}\right)dx$$

$$=\left(\tan^{-1}|\sinh x|-\frac{x}{2}\sqrt{1-x^2}-\frac{1}{2}\sin^{-1}x\right)\Big|_0^1$$

$$=\tan^{-1}(\sinh 1)-\frac{1}{2}\sin^{-1}1$$

$$=\tan^{-1}(\sinh 1)-\frac{\pi}{4}$$

$$\approx 0.08$$

Exercise

A region in the first quadrant is bounded above the curve $y=\cosh x$, below by the curve $y=\sinh x$, and on the left and right by the y-axis and the line $x=2$, respectively. Find the volume of the solid generated by revolving the region about the x-axis.

Solution

$$V=\pi\int_0^2\left(\cosh^2 x-\sinh^2 x\right)dx$$

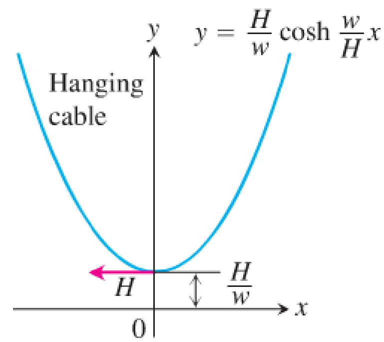
$$=\pi\int_0^2 dx$$

$$=\pi x\Big|_0^2$$

$$=2\pi$$

Exercise

Imagine a cable, like a telephone line or TV cable, strung from one support to another and hanging freely. The cable's weight per unit length is a constant *w* and the horizontal tension at its lowest point is a vector of length *H*. If we choose a coordinate system for the plane of the cable in which the *x*-axis is horizontal, the force of gravity is straight down, the positive *y*-axis points straight up, and the lowest point of the cable lies at the point $y=\frac{H}{w}$ on the *y*-axis, then it can be shown that the cable lies along the graph of the hyperbolic cosine



$$y=\frac{H}{w}\cosh\left(\frac{w}{H}x\right)$$

Such a curve is sometimes called a ***chain curve*** or a ***catenary***, the latter deriving from the Latin *catena*, meaning "*chain*".

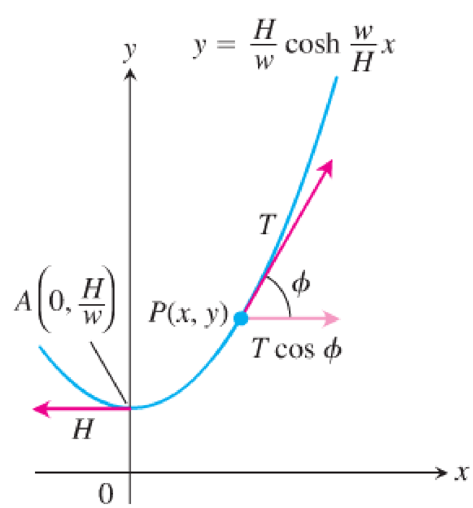
a) Let $P(x,y)$ denote an arbitrary point on the cable. The next accompanying displays the tension *H* at the lowest point *A*. Show that the cable's slope at *P* is

$$\tan\phi=\frac{dy}{dx}=\sinh\left(\frac{w}{H}x\right)$$

b) Using the result in part (*a*) and the fact that the horizontal tension at *P* must equal *H* (the cable is not moving), show that $T=wy$. Hence, the magnitude of the tension at $P(x,y)$ is exactly equal to the weight of *y* units of cable.

c) The length of arc *AP* is $s=\frac{1}{a}\sinh ax$, where $a=\frac{w}{H}$. Show that the coordinates of *P* may be expressed in terms of *s* as $x=\frac{1}{a}\sinh^{-1}as,\quad y=\sqrt{s^2+\frac{1}{a^2}}$

Solution



a) $y=\frac{H}{w}\cosh\left(\frac{w}{H}x\right)$

$$\tan\phi=\frac{dy}{dx}=\frac{H}{w}\left[\frac{w}{H}\sinh\left(\frac{w}{H}x\right)\right]$$

$$=\sinh\left(\frac{w}{H}x\right)$$

b) The tension at *P* is given by $T\cos\phi=H$.

$$T=\frac{H}{\cos\phi}$$

$$=H\sec\phi$$

$$=H\sqrt{1+\tan^2\phi}$$

$$=H\sqrt{1+\sinh^2\left(\frac{w}{H}x\right)}$$

$\cosh^2 x-\sinh^2 x=1 \ \rightarrow\ \cosh x=\sqrt{1+\sinh^2 x}$

$$= H \cosh\left(\frac{w}{H}x\right) \qquad\qquad yw = H\cosh\left(\frac{w}{H}x\right)$$

$$\boxed{= wy}$$

c) $s = \frac{1}{a}\sinh ax \quad \to \sinh ax = as$

$$ax = \sinh^{-1} as \ \to \ x = \frac{1}{a}\sinh^{-1} as$$

$$y = \frac{H}{w}\cosh\left(\frac{w}{H}x\right)$$

$$= \frac{1}{a}\cosh(ax) \qquad\qquad a = \frac{w}{H}$$

$$= \frac{1}{a}\sqrt{\cosh^2(ax)}$$

$$= \frac{1}{a}\sqrt{1+\sinh^2(ax)}$$

$$= \frac{1}{a}\sqrt{1+(as)^2}$$

$$\boxed{= \sqrt{\frac{1}{a^2}+s^2}}$$

Exercise

The portion of the curve $y = \frac{17}{15} - \cosh x$ that lies above the x-axis forms a catenary arch. Find the average height of the arch above the x-axis.

Solution

By symmetry;

$$I = 2\int_0^{\cosh^{-1}(17/15)} \left(\frac{17}{15} - \cosh x\right)dx$$

$$= 2\left(\frac{17}{15}x - \sinh x\right)\Big|_0^{\cosh^{-1}(17/15)}$$

$$= 2\left[\frac{17}{15}\cosh^{-1}\left(\frac{17}{15}\right) - \sinh\left(\cosh^{-1}\left(\frac{17}{15}\right)\right)\right]$$

$$= \frac{34}{15}\cosh^{-1}\left(\frac{17}{15}\right) - \frac{16}{15}$$

$$\text{Average height} = \frac{I}{2\cosh^{-1}\left(\frac{17}{15}\right)}$$

$$\boxed{= \frac{17}{15} - \frac{8}{15\cosh^{-1}\left(\frac{17}{15}\right)}} \quad \boxed{\approx 0.09}$$

Exercise

A power line is attached at the same height to two utility poles that are separated by a distance of 100 *ft*; the power line follows the curve $f(x) = a\cosh\left(\frac{x}{a}\right)$. Use the following steps to find the value of a that produces a sag of 10 *ft*. midway between the poles. Use the coordinate system that places the poles at $x = \pm 50$

a) Show that a satisfies the equation $\cosh\left(\frac{50}{a}\right) - 1 = \frac{10}{a}$

b) Let $t = \frac{10}{a}$, confirm that the equation in part (*a*) reduces to $\cosh 5t - 1 = t$, and solve for t using a graphing utility. (2 decimal places)

c) Use the answer in part (*b*) to find a and then compute the length of the power line.

Solution

a) Let $a = 10\ ft\ \ (sag)$.

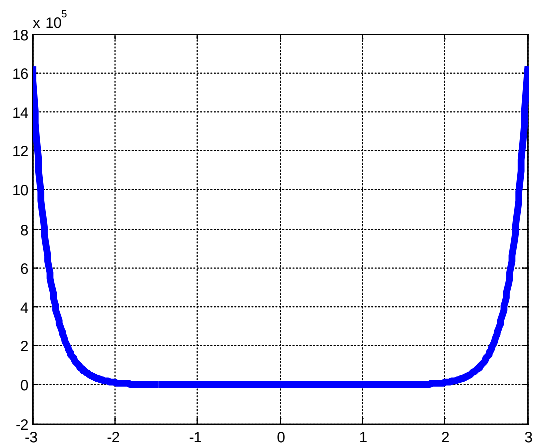
$$f(0) + sag = f(50)$$

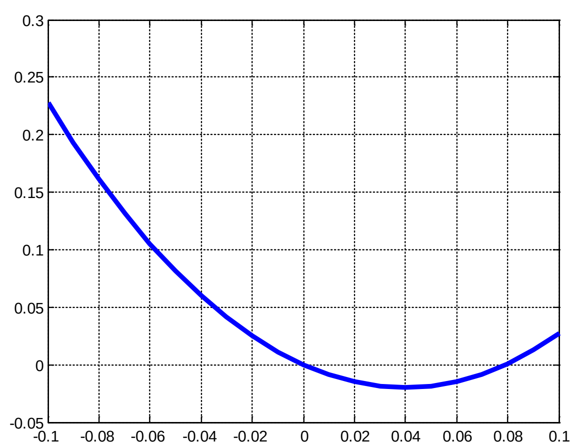
$$a + sag = a\cosh\left(\frac{50}{a}\right)$$

$$1 + \frac{sag}{a} = \cosh\left(\frac{50}{a}\right)$$

$$\boxed{\cosh\left(\frac{50}{a}\right) - 1 = \frac{10}{a}}$$

b) If $t = \frac{10}{a} \ \rightarrow\ \cosh(5t) - 1 = t$





$$\boxed{t \approx 0.08}$$

c) If $\frac{10}{a} = 0.08 \ \Rightarrow\ \boxed{a = \frac{10}{0.08} = 125}$

The length of the power line is:

$$L = 2\int_0^{50} \sqrt{1 - \sinh^2\left(\frac{x}{125}\right)}\, dx$$

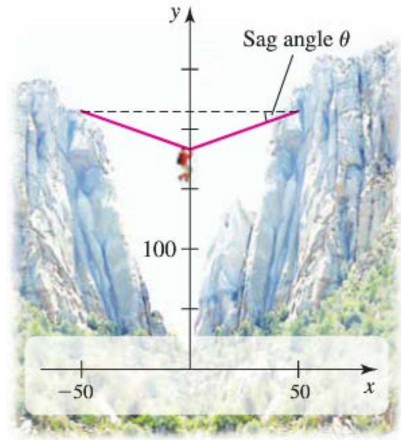
$$= 2\int_0^{50} \cosh\left(\frac{x}{125}\right) dx$$

$$= 250\sinh\left(\frac{x}{125}\right)\Big|_0^{50}$$

$$= 250\sinh\left(\frac{2}{5}\right) \approx 102.7\ ft$$

Exercise

Imagine a climber clipping onto the rope and pulling hinself to the rope's midpoint. Because the rope is supporting the weight of the climber, it no longer takes the shape of the catenary $y = 200\cosh\left(\frac{x}{200}\right)$. Instead, the rope (nearly) forms two sides of an isosceles triangle. Compute the sag angle illustrated in the figure, assuming that the rope does not stretch when weighted. Assume the length of the rope is 101 *feet*.



Solution

$$\theta = \cos^{-1}\left(\frac{50}{50.5}\right) \approx 0.14\ rad$$

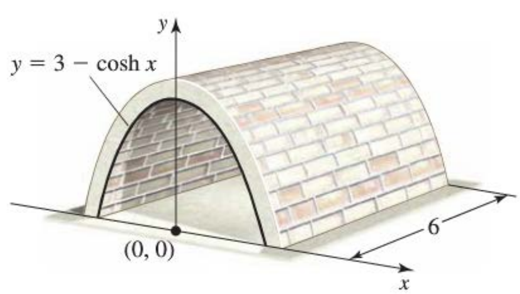
Exercise

Find the volume interior to the inverted catenary kiln (an oven used to fire pottery).

Solution

$$y = 3 - \cosh x = 0 \quad \Rightarrow \quad \cosh x = 3 \to x = \cosh^{-1}(\pm 3)$$

Therfore; the area is between $\cosh^{-1}(-3)$ and $\cosh^{-1}(3)$.



$$A = 2\int_0^{\cosh^{-1}(3)} (3 - \cosh x)\,dx$$

$$= 2(3x - \sinh x)\Big|_0^{\cosh^{-1}(3)}$$

$$= 2\left(3\cosh^{-1}(3) - \sinh x\left(\cosh^{-1}(3)\right)\right)$$

$$\approx 4.92$$

$$Volume \approx 6(4.92) \approx 29.5$$