Lecture One – Vectors and Vector-Values Functions

Solution *Section* 1.1 – Vectors

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2 + z^2 = 4, \quad y = 0$

Solution

The circle $x^2 + z^2 = 4$ in the ***xz***-plane

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2 + y^2 = 4, \quad z = -2$

Solution

The circle $x^2 + y^2 = 4$ in the plane $z = -2$

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2 + y^2 + z^2 = 1, \quad x = 0$

Solution

The circle $y^2 + z^2 = 1$ in the ***yz***-plane

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2 + (y-1)^2 + z^2 = 4, \quad y = 0$

Solution

$$x^2 + (0-1)^2 + z^2 = 4 \quad \Rightarrow x^2 + z^2 = 3$$

The circle $x^2 + z^2 = 3$ in the ***xz***-plane

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2+y^2+z^2=4, \quad y=x$

Solution

The circle formed by the intersection of the sphere $x^2+y^2+z^2=4$ and the plane $y=x$

Exercise

Find the distance between points $P_1(1,\ 1,\ 1), \quad P_2(3,\ 3,\ 0)$

Solution

$$\begin{aligned}\left|P_1P_2\right| &= \sqrt{(3-1)^2+(3-1)^2+(0-1)^2} \\ &= \sqrt{4+4+1} \\ &= \sqrt{9} \\ &\underline{=3|}\end{aligned}$$

Exercise

Find the distance between points $P_1(-1,\ 1,\ 5), \quad P_2(2,\ 5,\ 0)$

Solution

$$\begin{aligned}\left|P_1P_2\right| &= \sqrt{(2+1)^2+(5-1)^2+(0-5)^2} \\ &= \sqrt{9+16+25} \\ &= \sqrt{50} \\ &\underline{=5\sqrt{2}|}\end{aligned}$$

Exercise

Find the distance between points $P_1(1,\ 4,\ 5), \quad P_2(4,\ -2,\ 7)$

Solution

$$\begin{aligned}\left|P_1P_2\right| &= \sqrt{(4-1)^2+(-2-4)^2+(7-5)^2} \\ &= \sqrt{9+36+4} \\ &\underline{=7|}\end{aligned}$$

Exercise

Find the distance between points $P_1(3,\ 4,\ 5),\quad P_2(2,\ 3,\ 4)$

Solution

$$\begin{aligned}\left|P_1P_2\right| &= \sqrt{(2-3)^2+(3-4)^2+(4-5)^2}\\ &= \sqrt{1+1+1}\\ &= \boxed{\sqrt{3}}\end{aligned}$$

Exercise

Find the center and radii of the spheres $\quad x^2+y^2+z^2+4x-4z=0$

Solution

$$\left(x^2+4x\right)+y^2+\left(z^2-4z\right)=0$$

$$\left(x^2+4x+4\right)+y^2+\left(z^2-4z+4\right)=4+4$$

$$(x+2)^2+y^2+(z-2)^2=8$$

The center is at $(-2,\ 0,\ 2)$ and the radius is $\sqrt{8}=\boxed{2\sqrt{2}}$

Exercise

Find the center and radii of the spheres $\quad x^2+y^2+z^2-6y+8z=0$

Solution

$$x^2+\left(y^2-6y\right)+\left(z^2+8z\right)=0$$

$$x^2+\left(y^2-6y+\left(-\tfrac{6}{2}\right)^2\right)+\left(z^2+8z+\left(\tfrac{8}{2}\right)^2\right)=9+16$$

$$x^2+(y-3)^2+(z+4)^2=25$$

The center is at $(0,\ 3,\ -4)$ and the radius is $\boxed{5}$

Exercise

Find the center and radii of the spheres $2x^2 + 2y^2 + 2z^2 + x + y + z = 9$

Solution

$$x^2 + y^2 + z^2 + \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z = \frac{9}{2}$$

$$\left(x^2 + \frac{1}{2}x + \left(\frac{1}{2}\frac{1}{2}\right)^2\right) + \left(y^2 + \frac{1}{2}y + \left(\frac{1}{4}\right)^2\right) + \left(z^2 + \frac{1}{2}z + \left(\frac{1}{4}\right)^2\right) = \frac{9}{2} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}$$

$$\left(x + \frac{1}{4}\right)^2 + \left(y + \frac{1}{4}\right)^2 + \left(z + \frac{1}{4}\right)^2 = \frac{9}{2} + \frac{3}{16}$$

$$\left(x + \frac{1}{4}\right)^2 + \left(y + \frac{1}{4}\right)^2 + \left(z + \frac{1}{4}\right)^2 = \frac{75}{16}$$

The center is at $\left(-\frac{1}{4},\ -\frac{1}{4},\ -\frac{1}{4}\right)$ and the radius is $\boxed{\frac{5\sqrt{3}}{4}}$

Exercise

Find a formula for the distance from the point $P(x, y, z)$ to x-axis

Solution

The distance between (x, y, z) and $(x, 0, 0)$ is:

$$d = \sqrt{(x-x)^2 + (y-0)^2 + (z-0)^2}$$

$$= \sqrt{y^2 + z^2}$$

Exercise

Find a formula for the distance from the point $P(x, y, z)$ to xy-plane

Solution

The distance between (x, y, z) and $(x, 0, z)$ is:

$$d = \sqrt{(x-x)^2 + (y-0)^2 + (z-z)^2}$$

$$= y$$

Exercise

Let $\boldsymbol{u} = \langle 3,\ -2 \rangle$ *and* $\boldsymbol{v} = \langle -2,\ 5 \rangle$. Find the component form and the magnitude if the vector

a) $3\boldsymbol{u}$ *b)* $\boldsymbol{u} - \boldsymbol{v}$ *c)* $2\boldsymbol{u} - 3\boldsymbol{v}$ *d)* $-2\boldsymbol{u} + 5\boldsymbol{v}$ *e)* $-\frac{5}{13}\boldsymbol{u} + \frac{12}{13}\boldsymbol{v}$

Solution

a) $3\boldsymbol{u} = 3\langle 3,\ -2 \rangle = \underline{\langle 9,\ -6 \rangle}$

b) $\boldsymbol{u} - \boldsymbol{v} = \langle 3,\ -2 \rangle - \langle -2,\ 5 \rangle = \underline{\langle 5,\ -7 \rangle}$

c)
$$\begin{aligned} 2\boldsymbol{u} - 3\boldsymbol{v} &= 2\langle 3,\ -2 \rangle - 3\langle -2,\ 5 \rangle \\ &= \langle 6,\ -4 \rangle - \langle -6,\ 15 \rangle \\ &= \underline{\langle 12,\ -19 \rangle} \end{aligned}$$

d)
$$\begin{aligned} -2\boldsymbol{u} + 5\boldsymbol{v} &= -2\langle 3,\ -2 \rangle + 5\langle -2,\ 5 \rangle \\ &= \langle -6,\ 4 \rangle + \langle -10,\ 25 \rangle \\ &= \underline{\langle -14,\ 29 \rangle} \end{aligned}$$

e)
$$\begin{aligned} -\frac{5}{13}\boldsymbol{u} + \frac{12}{13}\boldsymbol{v} &= -\frac{5}{13}\langle 3,\ -2 \rangle + \frac{12}{13}\langle -2,\ 5 \rangle \\ &= \langle -6,\ 4 \rangle - \langle -10,\ 25 \rangle \\ &= \underline{\langle 4,\ -21 \rangle} \end{aligned}$$

Exercise

Find the component form of the vector: The sum of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ where $A = (1,-1)$, $B = (2,0)$, $C = (-1,3)$, *and* $D = (-2,2)$

Solution

$$\overrightarrow{AB} = \langle 2-1,\ 0-(-1) \rangle = \langle 1,\ 1 \rangle$$

$$\overrightarrow{CD} = \langle -2-(-1),\ 2-3 \rangle = \langle -1,\ -1 \rangle$$

$$\overrightarrow{AB} + \overrightarrow{CD} = \langle 1,\ 1 \rangle + \langle -1,\ -1 \rangle = \underline{\langle 0,\ 0 \rangle}$$

Exercise

Find the component form of the vector: The unit vector that makes an angle $\theta = \frac{2\pi}{3}$ with the positive x-axis

Solution

$$\left\langle \cos\frac{2\pi}{3},\ \sin\frac{2\pi}{3} \right\rangle = \underline{\left\langle -\frac{1}{2},\ \frac{\sqrt{3}}{2} \right\rangle}$$

Exercise

Find the component form of the vector: The unit vector obtained by rotating the vector $\langle 0,\ 1\rangle$ $120°$ counterclockwise about the origin

Solution

The angle of unit vector $\langle 0,\ 1\rangle$ is $90°$, this unit vector rotates $120°$ which makes an angle of $90° + 120° = 210°$ with the positive x-axis

$$\langle \cos 210°,\ \sin 210°\rangle = \boxed{\left\langle -\frac{\sqrt{3}}{2},\ -\frac{1}{2}\right\rangle}$$

Exercise

Find the component form of the vector: The unit vector obtained by rotating the vector $\langle 1,\ 0\rangle$ $135°$ counterclockwise about the origin

Solution

The angle of unit vector $\langle 1,\ 0\rangle$ is $0°$, this unit vector rotates $135°$ which makes an angle of $0° + 135° = 135°$ with the positive x-axis

$$\langle \cos 135°,\ \sin 135°\rangle = \boxed{\left\langle -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right\rangle}$$

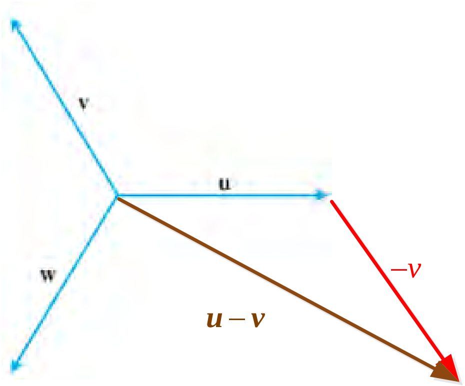
Exercise

Sketch the indicated vector

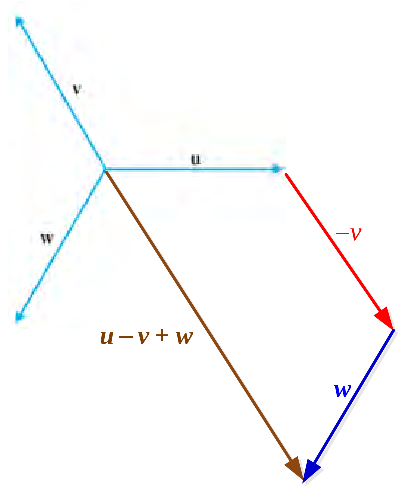
a) $\boldsymbol{u} - \boldsymbol{v}$ *b)* $2\boldsymbol{u} - \boldsymbol{v}$ *c)* $\boldsymbol{u} - \boldsymbol{v} + \boldsymbol{w}$

Solution

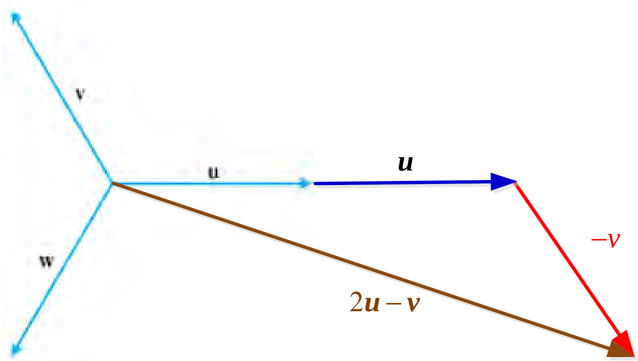
a)



b)



c)



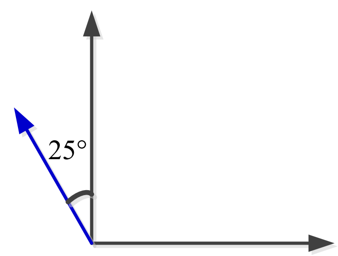
Exercise

An Airplane is flying in the direction 25° west of north at 800 *km/h*. Find the component form of the velocity of the airplane, assuming that the positive *x*-axis represents due east and the positive *y*-axis represents due north.

Solution

25° west of north is 25° + 90° = 115° north of east

$$800\langle \cos 115°,\ \sin 115°\rangle \approx \langle -338.095,\ 725.046\rangle$$



Exercise

A jet airliner, flying due east at 500 *mph* in still air, encounters a 70-*mph* tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What speed and direction should the jetliner have in order for the resultant vector to be 500 *mph* due east?

Solution

$\boldsymbol{u} = \langle x,\ y\rangle =$ the velocity of the airplane; $\qquad \boldsymbol{v} =$ the velocity of the tailwind

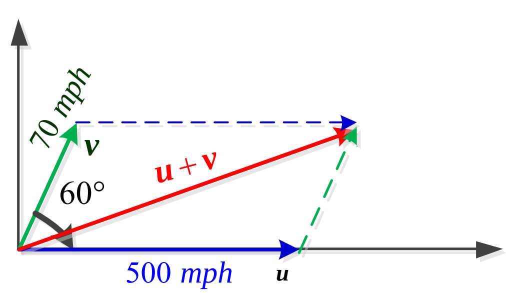
$$\boldsymbol{v} = \langle 70\cos 60°,\ 70\sin 60°\rangle = \langle 35,\ 35\sqrt{3}\rangle$$

$$\boldsymbol{u} + \boldsymbol{v} = \langle 500,\ 0\rangle$$

$$\langle x,\ y\rangle + \langle 35,\ 35\sqrt{3}\rangle = \langle 500,\ 0\rangle$$

$$\langle x,\ y\rangle = \langle 500,\ 0\rangle - \langle 35,\ 35\sqrt{3}\rangle = \langle 765,\ -35\sqrt{3}\rangle$$

$$\boxed{\boldsymbol{u} = \langle 765,\ -35\sqrt{3}\rangle}$$



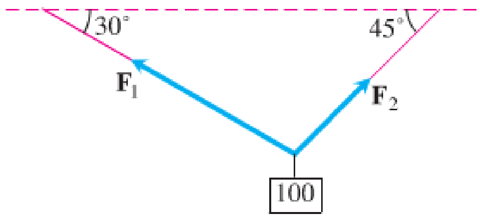
$$|\boldsymbol{u}| = \sqrt{465^2 + \left(-35\sqrt{3}\right)^2} \approx 468.9\ mph$$

$$\theta = \tan^{-1}\frac{-35\sqrt{3}}{465} \approx -7.4°$$

The direction is 7.4° south of east

Exercise

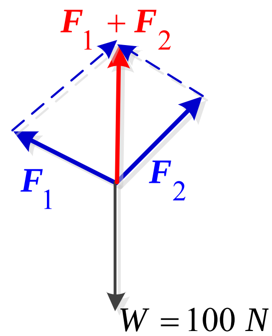
Consider a 100-N weight suspended by two wires. Find the magnitudes and components of the force vectors F_1 *and* F_2



Solution

$$F_1 = \left\langle -\left|F_1\right|\cos 30°,\ \left|F_1\right|\sin 30° \right\rangle = \left\langle -\frac{\sqrt{3}}{2}\left|F_1\right|,\ \frac{1}{2}\left|F_1\right| \right\rangle$$

$$F_2 = \left\langle \left|F_2\right|\cos 45°,\ \left|F_2\right|\sin 45° \right\rangle = \left\langle \frac{\sqrt{2}}{2}\left|F_2\right|,\ \frac{\sqrt{2}}{2}\left|F_2\right| \right\rangle$$



$$F_1 + F_2 = \langle 0,\ 100 \rangle$$

$$\left\langle -\frac{\sqrt{3}}{2}\left|F_1\right|,\ \frac{1}{2}\left|F_1\right| \right\rangle + \left\langle \frac{\sqrt{2}}{2}\left|F_2\right|,\ \frac{\sqrt{2}}{2}\left|F_2\right| \right\rangle = \langle 0,\ 100 \rangle$$

$$\left\langle -\frac{\sqrt{3}}{2}\left|F_1\right| + \frac{\sqrt{2}}{2}\left|F_2\right|,\ \frac{1}{2}\left|F_1\right| + \frac{\sqrt{2}}{2}\left|F_2\right| \right\rangle = \langle 0,\ 100 \rangle$$

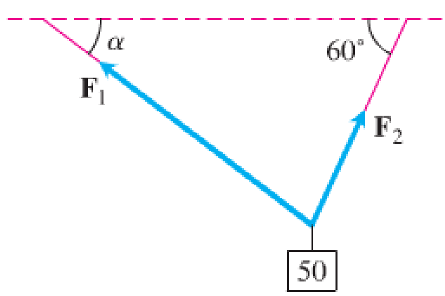
$$\begin{cases} -\frac{\sqrt{3}}{2}\left|F_1\right| + \frac{\sqrt{2}}{2}\left|F_2\right| = 0 \\ \frac{1}{2}\left|F_1\right| + \frac{\sqrt{2}}{2}\left|F_2\right| = 100 \end{cases} \Rightarrow \boxed{\left|F_1\right| \approx 73.205\ N} \quad \boxed{\left|F_2\right| \approx 89.658\ N}$$

$$F_1 = \left\langle -\frac{\sqrt{3}}{2}(73.205),\ \frac{1}{2}(73.205) \right\rangle \quad \to \boxed{F_1 \approx \langle -63.397,\ 36.603 \rangle}$$

$$F_2 = \left\langle \frac{\sqrt{2}}{2}(89.658),\ \frac{\sqrt{2}}{2}(89.658) \right\rangle \quad \to \boxed{F_2 \approx \langle 63.397,\ 63.397 \rangle}$$

Exercise

Consider a 50-N weight suspended by two wires, If the magnitude of vector $F_1 = 35\ N$, find the angle α and the magnitude of vector F_2



Solution

$$F_1 = \left\langle -\left|F_1\right|\cos\alpha,\ \left|F_1\right|\sin\alpha \right\rangle = \langle -35\cos\alpha,\ 35\sin\alpha \rangle$$

$$F_2 = \left\langle \left|F_2\right|\cos 60°,\ \left|F_2\right|\sin 60° \right\rangle = \left\langle \frac{1}{2}\left|F_2\right|,\ \frac{\sqrt{3}}{2}\left|F_2\right| \right\rangle$$

$$w = \langle 0,\ -50 \rangle \quad \Rightarrow \quad F_1 + F_2 = \langle 0,\ 50 \rangle$$

$$\langle -35\cos\alpha,\ 35\sin\alpha \rangle + \left\langle \frac{1}{2}\left|F_2\right|,\ \frac{\sqrt{3}}{2}\left|F_2\right| \right\rangle = \langle 0,\ 50 \rangle$$

$$\left\langle -35\cos\alpha + \frac{1}{2}\left|F_2\right|,\ 35\sin\alpha + \frac{\sqrt{3}}{2}\left|F_2\right|\right\rangle = \langle 0,\ 50\rangle$$

$$\rightarrow \begin{cases} -35\cos\alpha + \frac{1}{2}\left|F_2\right| = 0 \\ 35\sin\alpha + \frac{\sqrt{3}}{2}\left|F_2\right| = 50 \end{cases} \qquad \rightarrow \begin{cases} \left|F_2\right| = 70\cos\alpha \\ \end{cases}$$

$$35\sin\alpha + \frac{\sqrt{3}}{2}(70\cos\alpha) = 50$$

$$35\sqrt{3}\cos\alpha = 50 - 35\sin\alpha$$

$$\sqrt{3}\cos\alpha = \frac{10}{7} - \sin\alpha$$

$$\left(\sqrt{3}\cos\alpha\right)^{2} = \left(\frac{10}{7} - \sin\alpha\right)^{2}$$

$$3\cos^2\alpha = \frac{100}{49} - \frac{20}{7}\sin\alpha + \sin^2\alpha$$

$$3\left(1 - \sin^2\alpha\right) = \frac{100}{49} - \frac{20}{7}\sin\alpha + \sin^2\alpha$$

$$3 - 3\sin^2\alpha - \frac{100}{49} + \frac{20}{7}\sin\alpha - \sin^2\alpha = 0$$

$$-4\sin^2\alpha + \frac{20}{7}\sin\alpha + \frac{47}{49} = 0$$

$$-196\sin^2\alpha + 140\sin\alpha + 47 = 0 \quad \Rightarrow \quad \sin\alpha = \frac{5 \pm 6\sqrt{2}}{14}$$

Since $\alpha > 0 \Rightarrow \sin\alpha > 0$

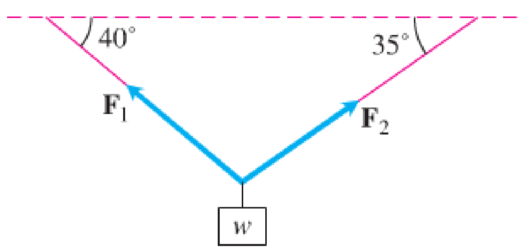
$$\rightarrow \sin\alpha = \frac{5 + 6\sqrt{2}}{14} \approx 0.963$$

$$\underline{\left|\alpha \approx \sin^{-1}(0.963) = 74.42^\circ\right.}$$

$$\begin{aligned} \left|F_2\right| &= 70\cos\alpha \\ &= 70\cos 74.42^\circ \\ &\approx 18.81\ N \end{aligned}$$

Exercise

Consider a ***w***-N weight suspended by two wires, If the magnitude of vector $F_2 = 100\ N$, find ***w*** and the magnitude of vector F_1



Solution

$$F_1 = \left\langle -\left|F_1\right|\cos 40^\circ,\ \left|F_1\right|\sin 40^\circ\right\rangle$$

$$F_2 = \langle |F_2|\cos 35°,\ |F_2|\sin 35° \rangle$$
$$= \langle 100(0.819),\ 100(0.5736) \rangle$$
$$= \langle 81.915,\ 57.358 \rangle$$

$$F_1 + F_2 = \langle 0,\ w \rangle$$

$$\langle -|F_1|\cos 40°,\ |F_1|\sin 40° \rangle + \langle 81.915,\ 57.358 \rangle = \langle 0,\ w \rangle$$

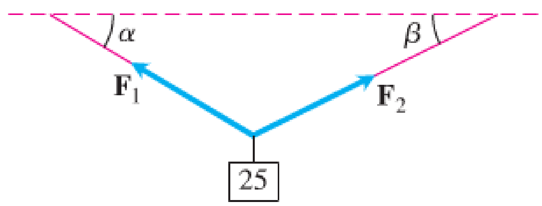
$$\langle -|F_1|\cos 40° + 81.915,\ |F_1|\sin 40° + 57.358 \rangle = \langle 0,\ w \rangle$$

$$-|F_1|\cos 40° + 81.915 = 0 \quad \Rightarrow \quad |F_1|\cos 40° = 81.915$$
$$|F_1| = \frac{81.915}{\cos 40°} \approx 106.933\ N$$

$$w = |F_1|\sin 40° + 57.358$$
$$= 106.933\sin 40° + 57.358$$
$$\approx 126.093\ \ N$$

Exercise

Consider a 25-N weight suspended by two wires, If the magnitude of vector F_1 *and* F_2 are both 75 N, then angles α and β are equal. Find α.



Solution

$$F_1 = \langle -|F_1|\cos\alpha,\ |F_1|\sin\alpha \rangle = \langle -75\cos\alpha,\ 75\sin\alpha \rangle$$
$$F_2 = \langle |F_2|\cos\beta,\ |F_2|\sin\beta \rangle = \langle 75\cos\beta,\ 75\sin\beta \rangle$$
$$w = \langle 0,\ -25 \rangle \quad \Rightarrow \quad F_1 + F_2 = \langle 0,\ 25 \rangle$$

$$\langle -75\cos\alpha,\ 75\sin\alpha \rangle + \langle 75\cos\beta,\ 75\sin\beta \rangle = \langle 0,\ 25 \rangle \qquad \textit{since} \quad \alpha = \beta$$
$$\langle -75\cos\alpha + 75\cos\alpha,\ 75\sin\alpha + 75\sin\alpha \rangle = \langle 0,\ 25 \rangle$$

$$-75\cos\alpha + 75\cos\beta = 0 \quad \Rightarrow \quad \cos\alpha = \cos\beta$$

$$150\sin\alpha = 25$$

$$\sin\alpha = \frac{25}{150}$$

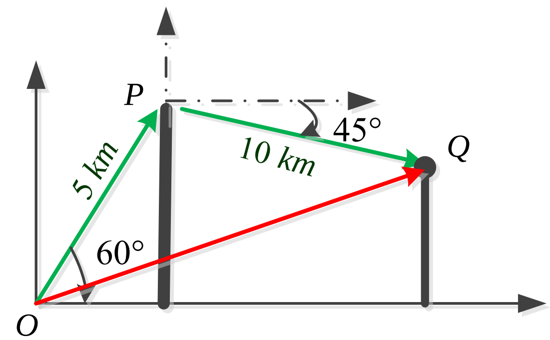
$$\alpha = \sin^{-1}\frac{25}{150} \approx 9.59°$$

Exercise

A bird flies from its nest 5 km in the direction 60° north east, where it stops to rest on a tree. It then flies 10 km in the direction due southeast and lands atop a telephone pole. Place an *xy*-coordinate system so that the origin is the bird's nest, the *x*-axis points east, and the *y*-axis points north.

a) At what point is the tree located?

b) At what point is the telephone pole?



Solution

a) $\overrightarrow{OP} = (5\cos 60°)\boldsymbol{i} + (5\sin 60°)\boldsymbol{j}$

$$= \frac{5}{2}\boldsymbol{i} + \frac{5\sqrt{3}}{2}\boldsymbol{j}$$

The tree is located at the point

$$\boxed{P = \left(\frac{5}{2},\ \frac{5\sqrt{3}}{2}\right)}$$

b) $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$

$$= \frac{5}{2}\boldsymbol{i} + \frac{5\sqrt{3}}{2}\boldsymbol{j} + (10\cos 315°)\boldsymbol{i} + (10\sin 315°)\boldsymbol{j}$$

$$= \frac{5}{2}\boldsymbol{i} + \frac{5\sqrt{3}}{2}\boldsymbol{j} + \left(10\frac{\sqrt{2}}{2}\right)\boldsymbol{i} + \left(10\left(-\frac{\sqrt{2}}{2}\right)\right)\boldsymbol{j}$$

$$= \left(\frac{5}{2} + 5\sqrt{2}\right)\boldsymbol{i} + \left(\frac{5\sqrt{3}}{2} - \frac{10\sqrt{2}}{2}\right)\boldsymbol{j}$$

$$= \left(\frac{5+10\sqrt{2}}{2}\right)\boldsymbol{i} + \left(\frac{5\sqrt{3}-10\sqrt{2}}{2}\right)\boldsymbol{j}$$

The pole is located at the point $\boxed{Q = \left(\frac{5+10\sqrt{2}}{2},\ \frac{5\sqrt{3}-10\sqrt{2}}{2}\right)}$

Exercise

Suppose that *A*, *B*, and *C* are the corner points of the thin triangular plate of constant density.

a) Find the vector from *C* to the midpoint *M* of side *AB*.

b) Find the vector from *C* to the point that lies two-thirds of the way from *C* to *M* on the median *CM*.

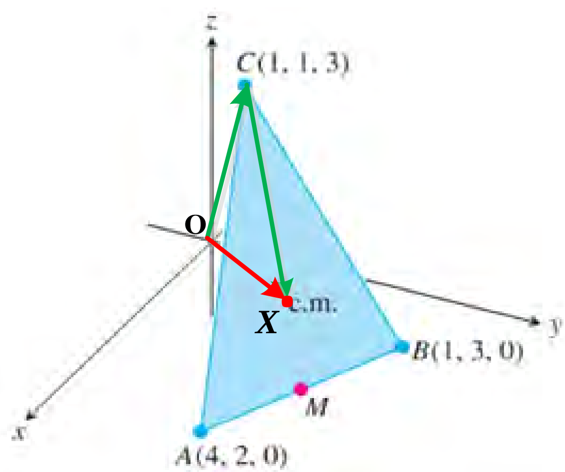
c) Find the coordinates of the point in which the medians of $\triangle ABC$ intersect (this point is the plate's center of mass).

Solution

a) The midpoint of AB is: $M=\left(\frac{4+1}{2},\frac{2+3}{2},0\right)=\left(\frac{5}{2},\ \frac{5}{2},\ 0\right)$

$$\overrightarrow{CM}=\left(\frac{5}{2}-1\right)\boldsymbol{i}+\left(\frac{5}{2}-1\right)\boldsymbol{j}+(0-3)\boldsymbol{k}$$

$$=\frac{3}{2}\boldsymbol{i}+\frac{3}{2}\boldsymbol{j}-3\boldsymbol{k}$$



b) The desired vector is $\overrightarrow{CX}=\frac{2}{3}\overrightarrow{CM}$

$$=\frac{2}{3}\left(\frac{3}{2}\boldsymbol{i}+\frac{3}{2}\boldsymbol{j}-3\boldsymbol{k}\right)$$

$$=\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}$$

c) The vector whose sum is the vector from the origin to C and the result of part (b) will terminate at the center of mass.

$$\overrightarrow{OX}=\overrightarrow{OC}+\overrightarrow{CX}$$

$$=\boldsymbol{i}+\boldsymbol{j}+3\boldsymbol{k}+\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}$$

$$=2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$$

Therefore; the center of mass point is $(2,\ 2,\ 1)$

Exercise

Show that a unit vector in the plane can be expressed as $\boldsymbol{u}=(\cos\theta)\boldsymbol{i}+(\sin\theta)\boldsymbol{j}$, obtained by rotating $\boldsymbol{i}$ through an angle θ in the counterclockwise direction. Explain why this form gives ***every*** *unit vector* in the plane.

Solution

Let $\boldsymbol{u}$ be any unit vector in the plane.

If $\boldsymbol{u}$ is positioned so that its initial point and terminal point is at (x, y), then $\boldsymbol{u}$ makes an angle θ with $\boldsymbol{i}$, measured in the *ccw* direction.

Since $|\boldsymbol{u}|=1 \quad \Rightarrow \quad x=\cos\theta \quad \textit{and} \quad y=\sin\theta$

That implies to: $\boldsymbol{u}=(\cos\theta)\boldsymbol{i}+(\sin\theta)\boldsymbol{j}$

Since $\boldsymbol{u}$ is any unit vector in the plane; this holds for every unit vector in the plane.

Solution   *Section* 1.2 – Dot Products

Exercise

Find for $\boldsymbol{v} = 2\boldsymbol{i} - 4\boldsymbol{j} + \sqrt{5}\boldsymbol{k}, \quad \boldsymbol{u} = -2\boldsymbol{i} + 4\boldsymbol{j} - \sqrt{5}\boldsymbol{k}$

a) $\boldsymbol{v} \bullet \boldsymbol{u}$, $|\boldsymbol{v}|$, $|\boldsymbol{u}|$

b) The cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$

c) The scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$

d) The vector $proj_{\boldsymbol{v}}\boldsymbol{u}$

Solution

a)
$$\begin{aligned}\boldsymbol{v} \bullet \boldsymbol{u} &= \left(2\boldsymbol{i} - 4\boldsymbol{j} + \sqrt{5}\boldsymbol{k}\right) \cdot \left(-2\boldsymbol{i} + 4\boldsymbol{j} - \sqrt{5}\boldsymbol{k}\right) \\ &= -4 - 16 - 5 \\ &= -25\end{aligned}$$

$$\begin{aligned}|\boldsymbol{v}| &= \sqrt{2^2 + (-4)^2 + \left(\sqrt{5}\right)^2} \\ &= \sqrt{4 + 16 + 5} \\ &= \sqrt{25} \\ &= 5\end{aligned}$$

$$\begin{aligned}|\boldsymbol{u}| &= \sqrt{(-2)^2 + 4^2 + \left(-\sqrt{5}\right)^2} \\ &= \sqrt{25} \\ &= 5\end{aligned}$$

b) $\cos\theta = \dfrac{\boldsymbol{u} \bullet \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|} = \dfrac{-25}{(5)(5)} = -1$

c) $|\boldsymbol{u}|\cos\theta = (5)(-1) = -5$

d)
$$\begin{aligned}proj_{\boldsymbol{v}}\boldsymbol{u} &= \left(\frac{\boldsymbol{u} \bullet \boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\boldsymbol{v} \\ &= \left(\frac{-25}{5^2}\right)\left(2\boldsymbol{i} - 4\boldsymbol{j} + \sqrt{5}\boldsymbol{k}\right) \\ &= -\left(2\boldsymbol{i} - 4\boldsymbol{j} + \sqrt{5}\boldsymbol{k}\right) \\ &= -2\boldsymbol{i} + 4\boldsymbol{j} - \sqrt{5}\boldsymbol{k}\end{aligned}$$

Exercise

Find for $\boldsymbol{v}=\frac{3}{5}\boldsymbol{i}+\frac{4}{5}\boldsymbol{k}, \quad \boldsymbol{u}=5\boldsymbol{i}+12\boldsymbol{j}$

a) $\boldsymbol{v}\bullet\boldsymbol{u},\ |\boldsymbol{v}|,\ |\boldsymbol{u}|$

b) The cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$

c) The scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$

d) The vector $proj_{\boldsymbol{v}}\boldsymbol{u}$

Solution

a) $\boldsymbol{v}\bullet\boldsymbol{u}=\left(\frac{3}{5}\boldsymbol{i}+\frac{4}{5}\boldsymbol{k}\right)\cdot(5\boldsymbol{i}+12\boldsymbol{j})$

$$=3$$

$$|\boldsymbol{v}|=\sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}$$

$$=\sqrt{\frac{9}{25}+\frac{16}{25}}$$

$$=\sqrt{\frac{25}{25}}$$

$$=1$$

$$|\boldsymbol{u}|=\sqrt{5^2+12^2}$$

$$=13$$

b) $\cos\theta=\frac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$

$$=\frac{3}{(1)(13)}$$

$$=\frac{3}{13}$$

c) $|\boldsymbol{u}|\cos\theta=(13)\left(\frac{3}{13}\right)=3$

d) $proj_{\boldsymbol{v}}\boldsymbol{u}=\left(\frac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\boldsymbol{v}$

$$=\left(\frac{3}{1^2}\right)\left(\frac{3}{5}\boldsymbol{i}+\frac{4}{5}\boldsymbol{k}\right)$$

$$=\frac{9}{5}\boldsymbol{i}+\frac{12}{5}\boldsymbol{k}$$

Exercise

Find for $\boldsymbol{v} = 2\boldsymbol{i} + 10\boldsymbol{j} - 11\boldsymbol{k}, \quad \boldsymbol{u} = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

a) $\boldsymbol{v} \bullet \boldsymbol{u},\ |\boldsymbol{v}|,\ |\boldsymbol{u}|$

b) The cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$

c) The scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$

d) The vector $proj_{\boldsymbol{v}}\boldsymbol{u}$

Solution

a) $\boldsymbol{v} \bullet \boldsymbol{u} = (2\boldsymbol{i} + 10\boldsymbol{j} - 11\boldsymbol{k}) \cdot (2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$

$$= 4 + 20 - 11$$

$$= 13$$

$$|\boldsymbol{v}| = \sqrt{2^2 + 10^2 + (-11)^2}$$

$$= \sqrt{4 + 100 + 121}$$

$$= \sqrt{225}$$

$$= 15$$

$$|\boldsymbol{u}| = \sqrt{2^2 + 2^2 + 1^2}$$

$$= 3$$

b) $\cos\theta = \dfrac{\boldsymbol{u} \bullet \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$

$$= \frac{13}{(3)(15)}$$

$$= \frac{13}{45}$$

c) $|\boldsymbol{u}|\cos\theta = (3)\left(\dfrac{13}{45}\right) = \dfrac{13}{15}$

d) $proj_{\boldsymbol{v}}\boldsymbol{u} = \left(\dfrac{\boldsymbol{u} \bullet \boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\boldsymbol{v}$

$$= \left(\frac{13}{15^2}\right)(2\boldsymbol{i} + 10\boldsymbol{j} - 11\boldsymbol{k})$$

$$= \frac{13}{225}(2\boldsymbol{i} + 10\boldsymbol{j} - 11\boldsymbol{k})$$

Exercise

Find for $\boldsymbol{v} = 5\boldsymbol{i} + \boldsymbol{j}, \quad \boldsymbol{u} = 2\boldsymbol{i} + \sqrt{17}\,\boldsymbol{j}$

a) $\boldsymbol{v} \bullet \boldsymbol{u},\ |\boldsymbol{v}|,\ |\boldsymbol{u}|$

b) The cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$

c) The scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$

d) The vector $proj_{\boldsymbol{v}}\boldsymbol{u}$

Solution

a) $\boldsymbol{v} \bullet \boldsymbol{u} = (5\boldsymbol{i} + \boldsymbol{j}) \cdot \left(2\boldsymbol{i} + \sqrt{17}\,\boldsymbol{j}\right) = 10 + \sqrt{17}$

$|\boldsymbol{v}| = \sqrt{25+1} = \sqrt{26}$

$|\boldsymbol{u}| = \sqrt{4+17} = \sqrt{21}$

b) $\cos\theta = \dfrac{\boldsymbol{u} \bullet \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$

$= \dfrac{10 + \sqrt{17}}{\sqrt{21}\sqrt{26}}$

$= \dfrac{10 + \sqrt{17}}{\sqrt{546}}$

c) $|\boldsymbol{u}|\cos\theta = \left(\sqrt{21}\right)\left(\dfrac{10 + \sqrt{17}}{\sqrt{546}}\right)$

$= \dfrac{10 + \sqrt{17}}{\sqrt{26}}$

d) $proj_{\boldsymbol{v}}\boldsymbol{u} = \left(\dfrac{\boldsymbol{u} \bullet \boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\boldsymbol{v}$

$= \left(\dfrac{10 + \sqrt{17}}{26}\right)(5\boldsymbol{i} + \boldsymbol{j})$

Exercise

Find for $\boldsymbol{v}=\left\langle \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}} \right\rangle$, $\boldsymbol{u}=\left\langle \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{3}} \right\rangle$

a) $\boldsymbol{v}\bullet\boldsymbol{u},\ |\boldsymbol{v}|,\ |\boldsymbol{u}|$

b) The cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$

c) The scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$

d) The vector $proj_{\boldsymbol{v}}\boldsymbol{u}$

Solution

a) $\boldsymbol{v}\bullet\boldsymbol{u}=\left\langle \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}} \right\rangle\cdot\left\langle \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{3}} \right\rangle=\frac{1}{2}-\frac{1}{3}\ \underline{=\frac{1}{6}|}$

$$|\boldsymbol{v}|=\sqrt{\frac{1}{2}+\frac{1}{3}}=\frac{\sqrt{5}}{\sqrt{6}}\frac{\sqrt{6}}{\sqrt{6}}\ \underline{=\frac{\sqrt{30}}{6}|}$$

$$|\boldsymbol{u}|=\sqrt{\frac{1}{2}+\frac{1}{3}}=\frac{\sqrt{5}}{\sqrt{6}}\frac{\sqrt{6}}{\sqrt{6}}\ \underline{=\frac{\sqrt{30}}{6}|}$$

b) $\cos\theta=\frac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$

$$=\frac{\frac{1}{6}}{\frac{\sqrt{30}}{6}\frac{\sqrt{30}}{6}}$$

$$=\frac{1}{6}\left(\frac{36}{30}\right)$$

$$\underline{=\frac{1}{5}|}$$

c) $|\boldsymbol{u}|\cos\theta=\left(\frac{\sqrt{30}}{6}\right)\left(\frac{1}{5}\right)=\frac{\sqrt{30}}{30}=\underline{\frac{1}{\sqrt{30}}|}$

d) $proj_{\boldsymbol{v}}\boldsymbol{u}=\left(\frac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\boldsymbol{v}$

$$=\frac{1}{6}\left(\frac{36}{30}\right)\left\langle \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}} \right\rangle$$

$$\underline{=\frac{1}{5}\left\langle \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}} \right\rangle|}$$

Exercise

Find the angles between the vectors $\boldsymbol{u} = 2\boldsymbol{i} + \boldsymbol{j}, \quad \boldsymbol{v} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

Solution

$$\theta = \cos^{-1} \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$$

$$= \cos^{-1}\left(\frac{2+2+0}{\sqrt{4+1}\sqrt{1+4+1}}\right)$$

$$= \cos^{-1}\left(\frac{4}{\sqrt{5}\sqrt{6}}\right) = \cos^{-1}\left(\frac{4}{\sqrt{30}}\right)$$

$$\approx 0.84 \ \ rad$$

Exercise

Find the angles between the vectors $\boldsymbol{u} = \sqrt{3}\boldsymbol{i} - 7\boldsymbol{j}, \quad \boldsymbol{v} = \sqrt{3}\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

Solution

$$\theta = \cos^{-1} \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$$

$$= \cos^{-1}\left(\frac{3-7+0}{\sqrt{3+49}\sqrt{3+1+1}}\right)$$

$$= \cos^{-1}\left(\frac{-4}{\sqrt{52}\sqrt{5}}\right) = \cos^{-1}\left(-\frac{4}{\sqrt{260}}\right)$$

$$\approx 1.82 \ \ rad$$

Exercise

Find the angles between the vectors $\boldsymbol{u} = \boldsymbol{i} + \sqrt{2}\boldsymbol{j} - \sqrt{2}\boldsymbol{k}, \quad \boldsymbol{v} = -\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

Solution

$$\theta = \cos^{-1} \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$$

$$= \cos^{-1}\left(\frac{-1+\sqrt{2}-\sqrt{2}}{\sqrt{1+2+2}\sqrt{1+1+1}}\right)$$

$$= \cos^{-1}\left(\frac{-1}{\sqrt{5}\sqrt{3}}\right) = \cos^{-1}\left(-\frac{1}{\sqrt{15}}\right)$$

$$\approx 1.83 \ \ rad$$

Exercise

The direction angles α, β, and γ of a vector $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ are defined as follows:

is the angle between $\boldsymbol{v}$ and the positive x-axis $(0 \le \alpha \le \pi)$

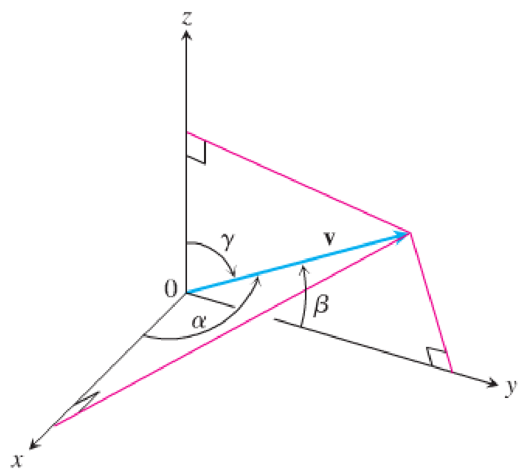
is the angle between $\boldsymbol{v}$ and the positive y-axis $(0 \le \beta \le \pi)$

is the angle between $\boldsymbol{v}$ and the positive z-axis $(0 \le \gamma \le \pi)$

a) Show that $\cos\alpha = \frac{a}{|\mathbf{v}|}$, $\cos\beta = \frac{b}{|\mathbf{v}|}$, $\cos\gamma = \frac{c}{|\mathbf{v}|}$, and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. These cosines are called the direction cosines of $\boldsymbol{v}$.

b) Show that if $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ is a unit vector, then a, b, and c are the direction cosines of $\boldsymbol{v}$.

Solution



a)

$$\cos\alpha = \frac{\boldsymbol{i}\cdot\boldsymbol{v}}{|\boldsymbol{i}||\boldsymbol{v}|} = \frac{\boldsymbol{i}\cdot(a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k})}{|\mathbf{v}|} = \frac{a}{|\mathbf{v}|}$$

$$\cos\beta = \frac{\boldsymbol{j}\cdot\boldsymbol{v}}{|\boldsymbol{j}||\boldsymbol{v}|} = \frac{\boldsymbol{j}\cdot(a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k})}{|\mathbf{v}|} = \frac{b}{|\mathbf{v}|}$$

$$\cos\gamma = \frac{\boldsymbol{k}\cdot\boldsymbol{v}}{|\boldsymbol{k}||\boldsymbol{v}|} = \frac{\boldsymbol{k}\cdot(a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k})}{|\mathbf{v}|} = \frac{c}{|\mathbf{v}|}$$

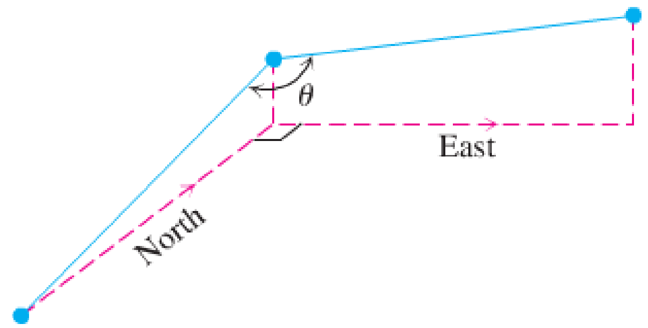
$$\begin{aligned}\cos^2\alpha + \cos^2\beta + \cos^2\gamma &= \left(\frac{a}{|\mathbf{v}|}\right)^2 + \left(\frac{b}{|\mathbf{v}|}\right)^2 + \left(\frac{c}{|\mathbf{v}|}\right)^2 \\ &= \frac{a^2}{|\mathbf{v}|^2} + \frac{b^2}{|\mathbf{v}|^2} + \frac{c^2}{|\mathbf{v}|^2} \\ &= \frac{a^2 + b^2 + c^2}{|\mathbf{v}|^2} \\ &= \frac{a^2 + b^2 + c^2}{a^2 + b^2 + c^2} \\ &\underline{= 1}\end{aligned}$$

b) If $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ is a unit vector $\Rightarrow |\mathbf{v}| = 1$

$\cos\alpha = \frac{a}{|\mathbf{v}|} = a$, $\cos\beta = \frac{b}{|\mathbf{v}|} = b$, $\cos\gamma = \frac{c}{|\mathbf{v}|} = c$ are the direction cosines of $\boldsymbol{v}$.

Exercise

A water main is to be constructed with 20% grade in the north direction and a 10% grade in the east direction. Determine the angle θ required in the water main for the turn from north to east.



Solution

20% grade in the north direction $\Rightarrow zk = 20\% xi = .2xi \quad \rightarrow If \;\; x = 10 \quad z = 2$

Let $\boldsymbol{u} = 10\boldsymbol{i} + 2\boldsymbol{k}$ be parallel to the pipe in the north direction.

$\boldsymbol{v} = 10\boldsymbol{j} + \boldsymbol{k}$ be parallel to the pipe in the east direction.

$$\theta = \cos^{-1}\frac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$$

$$= \cos^{-1}\frac{0+0+2}{\sqrt{100+4}\sqrt{100+1}}$$

$$= \cos^{-1}\frac{2}{\sqrt{104}\sqrt{101}}$$

$$\boxed{\approx 88.88^\circ}$$

Exercise

A gun with muzzle velocity of 1200 ft/sec is fired at an angle of 8° above the horizontal. Find the horizontal and vertical components of the velocity.

Solution

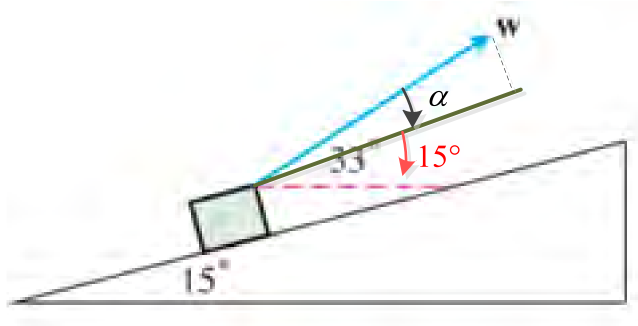
Horizontal component: $1200\cos 8^\circ \boxed{\approx 1188 \; ft/s}$

Vertical component: $1200\sin 8^\circ \boxed{\approx 167 \; ft/s}$

Exercise

Suppose that a box is being towed up an inclined plane. Find the force $\boldsymbol{w}$ needed to make the component of the force parallel to the indicated plane equal to 2.5 lb.

Solution



$$2.5 = |w|\cos\alpha$$

$$|\boldsymbol{w}| = \frac{2.5}{\cos(33° - 15°)} = \frac{2.5}{\cos 18°}$$

$$\boldsymbol{w} = \frac{2.5}{\cos 18°}\langle \cos 33°,\ \sin 33° \rangle$$

$$= \langle 2.205,\ 1.432 \rangle$$

Exercise

Find the work done by a force $\boldsymbol{F} = 5\boldsymbol{i}$ (magnitude 5 N) in moving an object along the line from the origin to the point (1, 1) (distance in meters)

Solution

$$P(1,\ 1) \ \Rightarrow\ \overrightarrow{OP} = \boldsymbol{i} + \boldsymbol{j}$$

$$W = F \cdot \overrightarrow{OP} = 5\boldsymbol{i} \cdot (\boldsymbol{i} + \boldsymbol{j}) \ \ = 5\ J$$

Exercise

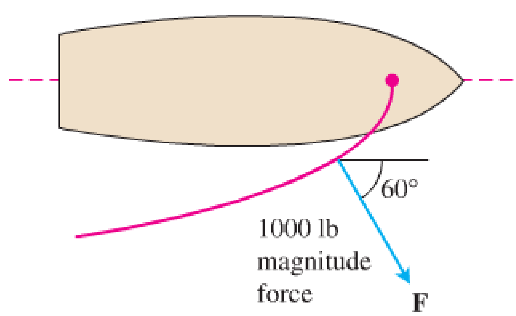
How much work does it take to slide a crate 20 m along a loading dock by pulling on it with a 200 N force at an angle of 30° from the horizontal?

Solution

$$W = |F|\left|\overrightarrow{PQ}\right|\cos\theta = (200)(20)\cos 30° \ \ = 3464.10\ J$$

Exercise

The wind passing over a boat's sail exerted a 1000-*lb* magnitude force F. How much work did the wind perform in moving the boat forward 1 *mi*? Answer in foot-pounds.



Solution

$$W = |F|\left|\overrightarrow{PQ}\right|\cos\theta$$

$$= (1000N)\left(1\ mi\,\frac{5280\,ft}{1\ mi}\right)\cos 60°$$

$$= 2,640,000\ \ ft \cdot lb$$

Solution Section 1.3 – Cross Products

Exercise

Find the length and direction of $\boldsymbol{u}\times\boldsymbol{v}$ and $\boldsymbol{v}\times\boldsymbol{u}$: $\quad \boldsymbol{u}=2\boldsymbol{i}-2\boldsymbol{j}-\boldsymbol{k}, \quad \boldsymbol{v}=\boldsymbol{i}-\boldsymbol{k}$

Solution

$$\boldsymbol{u}\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 2 & -2 & -1\\ 1 & 0 & -1\end{vmatrix}=2\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}$$

Length: $\quad |\boldsymbol{u}\times\boldsymbol{v}|=\sqrt{4+1+4}=\underline{3}$

Direction: $\underline{\frac{1}{3}(2\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k})}$

$$\boldsymbol{v}\times\boldsymbol{u}=-(\boldsymbol{u}\times\boldsymbol{v})=-2\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}$$

Length: $\quad |\boldsymbol{v}\times\boldsymbol{u}|=\sqrt{4+1+4}=\underline{3}$

Direction: $\frac{1}{3}(-2\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k})$

Exercise

Find the length and direction of $\boldsymbol{u}\times\boldsymbol{v}$ and $\boldsymbol{v}\times\boldsymbol{u}$: $\quad \boldsymbol{u}=\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}, \quad \boldsymbol{v}=0$

Solution

$$\boldsymbol{u}\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & 1 & -1\\ 0 & 0 & 0\end{vmatrix}=0$$

Length: $\quad \underline{0}$

Direction: No direction

$$\boldsymbol{v}\times\boldsymbol{u}=-(\boldsymbol{u}\times\boldsymbol{v})=0$$

Length: $\quad \underline{0}$

Direction: No direction

Exercise

Find the length and direction of $\boldsymbol{u}\times\boldsymbol{v}$ and $\boldsymbol{v}\times\boldsymbol{u}$: $\quad \boldsymbol{u}=\boldsymbol{i}\times\boldsymbol{j}, \quad \boldsymbol{v}=\boldsymbol{j}\times\boldsymbol{k}$

Solution

$$\boldsymbol{u}\times\boldsymbol{v}=(\boldsymbol{i}\times\boldsymbol{j})\times(\boldsymbol{j}\times\boldsymbol{k})=\boldsymbol{k}\times\boldsymbol{i}=\boldsymbol{j}$$

Length: $\underline{1|}$

Direction: $\underline{\boldsymbol{j}|}$

$$\boldsymbol{v}\times\boldsymbol{u}=-(\boldsymbol{u}\times\boldsymbol{v})=-\boldsymbol{j}$$

Length: $\underline{1|}$

Direction: $\underline{-\boldsymbol{j}|}$

Exercise

Find the length and direction of $\boldsymbol{u}\times\boldsymbol{v}$ and $\boldsymbol{v}\times\boldsymbol{u}$: $\quad \boldsymbol{u}=-8\boldsymbol{i}-2\boldsymbol{j}-4\boldsymbol{k}, \quad \boldsymbol{v}=2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$

Solution

$$\boldsymbol{u}\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ -8 & -2 & -4\\ 2 & 2 & 1\end{vmatrix}=6\boldsymbol{i}-12\boldsymbol{k}$$

Length: $|\boldsymbol{u}\times\boldsymbol{v}|=\sqrt{36+144}=\sqrt{180}=\underline{6\sqrt{5}|}$

Direction: $\frac{1}{6\sqrt{5}}(6\boldsymbol{i}-12\boldsymbol{k})=\underline{\frac{1}{\sqrt{5}}\boldsymbol{i}-\frac{2}{\sqrt{5}}\boldsymbol{k}|}$

$$\boldsymbol{v}\times\boldsymbol{u}=-(\boldsymbol{u}\times\boldsymbol{v})=-6\boldsymbol{i}+12\boldsymbol{k}$$

Length: $|\boldsymbol{v}\times\boldsymbol{u}|==\underline{6\sqrt{5}|}$

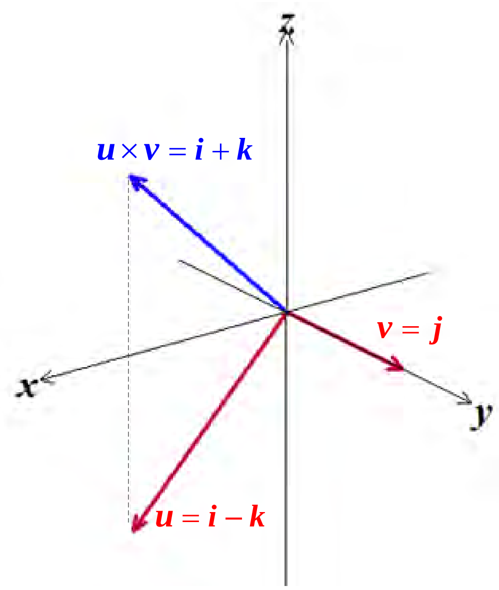
Direction: $\underline{-\frac{1}{\sqrt{5}}\boldsymbol{i}+\frac{2}{\sqrt{5}}\boldsymbol{k}|}$

Exercise

Sketch the coordinate axes and then include the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{u}\times\boldsymbol{v}$ as vectors starting origin for $\boldsymbol{u}=\boldsymbol{i}-\boldsymbol{k},\quad \boldsymbol{v}=\boldsymbol{j}$

Solution

$$\boldsymbol{u}\times\boldsymbol{v}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix}=\boldsymbol{i}+\boldsymbol{k}$$

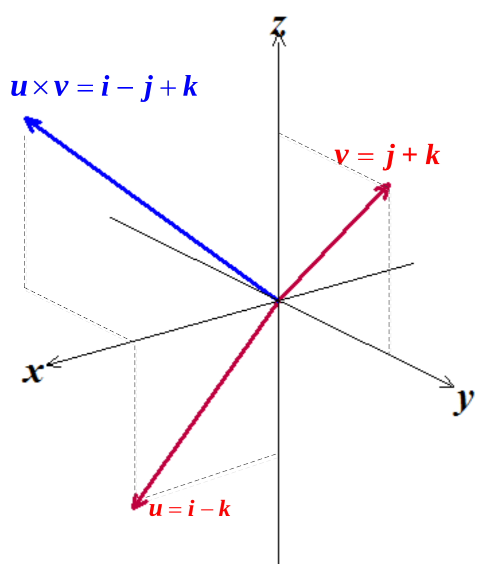


Exercise

Sketch the coordinate axes and then include the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{u}\times\boldsymbol{v}$ as vectors starting origin for $\boldsymbol{u}=\boldsymbol{i}-\boldsymbol{k},\quad \boldsymbol{v}=\boldsymbol{j}+\boldsymbol{k}$

Solution

$$\boldsymbol{u}\times\boldsymbol{v}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{vmatrix}=\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}$$



Exercise

Find the area of the triangle determined by the points P, Q, and R, and then find a unit vector perpendicular to plane PQR. $P(1,\ -1,\ 2),\quad Q(2,\ 0,\ -1),\quad and\quad R(0,\ 2,\ 1)$

Solution

$$\overrightarrow{PQ} = (2-1)\boldsymbol{i} + (0+1)\boldsymbol{j} + (-1-2)\boldsymbol{k} = \boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$$

$$\overrightarrow{PR} = (0-1)\boldsymbol{i} + (2+1)\boldsymbol{j} + (1-2)\boldsymbol{k} = -\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix} = 8\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k}$$

$$\begin{aligned} Area &= \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| \\ &= \frac{1}{2}\sqrt{8^2 + 4^2 + 4^2} \\ &= \frac{1}{2}\sqrt{96} \\ &= 2\sqrt{6} \end{aligned}$$

$$\begin{aligned} \boldsymbol{u} &= \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|} \\ &= \frac{1}{4\sqrt{6}}(8\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k}) \\ &= \frac{1}{\sqrt{6}}(2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) \end{aligned}$$

Exercise

Find the area of the triangle determined by the points P, Q, and R, and then find a unit vector perpendicular to plane PQR. $P(1,\ 1,\ 1),\quad Q(2,\ 1,\ 3),\quad and\quad R(3,\ -1,\ 1)$

Solution

$$\overrightarrow{PQ} = (2-1)\boldsymbol{i} + (1-1)\boldsymbol{j} + (3-1)\boldsymbol{k} = \boldsymbol{i} + 2\boldsymbol{k}$$

$$\overrightarrow{PR} = (3-1)\boldsymbol{i} + (-1-1)\boldsymbol{j} + (1-1)\boldsymbol{k} = 2\boldsymbol{i} - 2\boldsymbol{j}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & 2 \\ 2 & -2 & 0 \end{vmatrix} = 4\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$$

$$Area = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$$

$$= \frac{1}{2}\sqrt{16+16+4}$$

$$= \frac{1}{2}\sqrt{36}$$

$$\boxed{= 3}$$

$$\boldsymbol{u} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|}$$

$$= \frac{1}{6}(4\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k})$$

$$\boxed{= \frac{1}{3}(2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})}$$

Exercise

Find the area of the triangle determined by the points P, Q, and R, and then find a unit vector perpendicular to plane PQR. $P(-2,\ 2,\ 0),\quad Q(0,\ 1,\ -1),\quad and\ \ R(-1,\ 2,\ -2)$

Solution

$$\overrightarrow{PQ} = 2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$$

$$\overrightarrow{PR} = \boldsymbol{i} - 2\boldsymbol{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -1 & -1 \\ 1 & 0 & -2 \end{vmatrix} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$$

$$Area = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$$

$$= \frac{1}{2}\sqrt{4+9+1}$$

$$\boxed{= \frac{\sqrt{14}}{2}}$$

$$\boldsymbol{u} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|}$$

$$\boxed{= \frac{1}{\sqrt{14}}(2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k})}$$

Exercise

Verify that $(\boldsymbol{u}\times\boldsymbol{v})\bullet\boldsymbol{w}=(\boldsymbol{v}\times\boldsymbol{w})\bullet\boldsymbol{u}=(\boldsymbol{w}\times\boldsymbol{u})\bullet\boldsymbol{v}$ and find the volume of the parallelepiped determined by

$\boldsymbol{u}=2\boldsymbol{i}, \quad \boldsymbol{v}=2\boldsymbol{j}, \quad \textit{and} \quad \boldsymbol{w}=2\boldsymbol{k}$

Solution

Let $\boldsymbol{u}=\langle u_1, u_2, u_3\rangle, \quad \boldsymbol{v}=\langle v_1, v_2, v_3\rangle, \quad \boldsymbol{w}=\langle w_1, w_2, w_3\rangle$

$$(\boldsymbol{u}\times\boldsymbol{v})\bullet\boldsymbol{w}=\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \qquad (\boldsymbol{v}\times\boldsymbol{w})\bullet\boldsymbol{u}=\begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{vmatrix} \qquad (\boldsymbol{w}\times\boldsymbol{u})\bullet\boldsymbol{v}=\begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Which all have the same absolute value, by interchanging the rows the determinant does not change its absolute value.

$$\textit{Volume}=(\boldsymbol{u}\times\boldsymbol{v})\bullet\boldsymbol{w}=\begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}=8$$

Exercise

Find the volume of the parallelepiped determined by

$$\boldsymbol{u}=\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}, \quad \boldsymbol{v}=2\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}, \quad \textit{and} \quad \boldsymbol{w}=-\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$$

Solution

$$\textit{Volume}=(\boldsymbol{u}\times\boldsymbol{v})\bullet\boldsymbol{w}=abs\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 2 & -1 \end{vmatrix}=3$$

Exercise

Find the volume of the parallelepiped determined by

$$\boldsymbol{u}=\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}, \quad \boldsymbol{v}=-\boldsymbol{i}-\boldsymbol{k}, \quad \textit{and} \quad \boldsymbol{w}=2\boldsymbol{i}+4\boldsymbol{j}-2\boldsymbol{k}$$

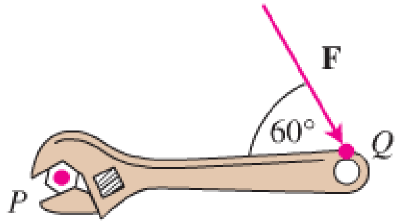
Solution

$$\textit{Volume}=(\boldsymbol{u}\times\boldsymbol{v})\bullet\boldsymbol{w}=abs\begin{vmatrix} 1 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 4 & -2 \end{vmatrix}=8$$

Exercise

Find the magnitude of the torque force exerted by $\boldsymbol{F}$ on the bolt at P if $\left|\overrightarrow{PQ}\right| = 8\ in.$ and $\left|\boldsymbol{F}\right| = 30\ lb.$

Solution



$$\left|\overrightarrow{PQ} \times \boldsymbol{F}\right| = \left|\overrightarrow{PQ}\right|\left|\boldsymbol{F}\right|\sin 60°$$

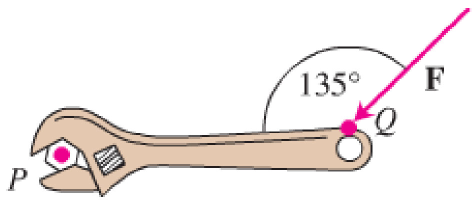
$$= \frac{8}{12}(30)\frac{\sqrt{3}}{2}$$

$$\boxed{= 10\sqrt{3}\ \ ft.lb}$$

Exercise

Find the magnitude of the torque force exerted by $\boldsymbol{F}$ on the bolt at P if $\left|\overrightarrow{PQ}\right| = 8\ in.$ and $\left|\boldsymbol{F}\right| = 30\ lb.$

Solution



$$\left|\overrightarrow{PQ} \times \boldsymbol{F}\right| = \left|\overrightarrow{PQ}\right|\left|\boldsymbol{F}\right|\sin 135°$$

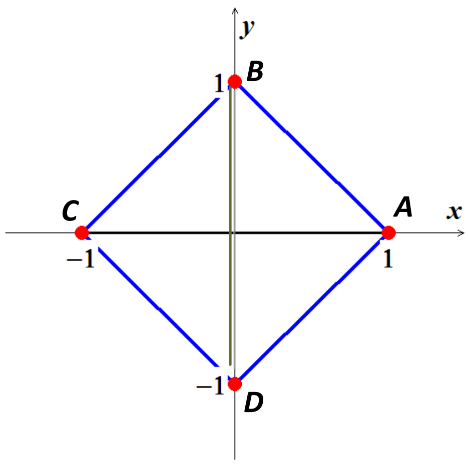
$$= \frac{8}{12}(30)\frac{\sqrt{2}}{2}$$

$$\boxed{= 10\sqrt{2}\ \ ft.lb}$$

Exercise

Find the area of the parallelogram whose vertices are: $A(1,\ 0),\quad B(0,\ 1),\quad C(-1,\ 0),\quad D(0,\ -1)$

Solution



$$\overrightarrow{AB} = -\boldsymbol{i} + \boldsymbol{j} \quad \overrightarrow{AD} = -\boldsymbol{i} - \boldsymbol{j}$$

$$Area(\triangle ABD) = Area(\triangle CBD)$$

$$Area = \left|\overrightarrow{AB} \times \overrightarrow{AD}\right|$$

$$= abs\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 1 & 0 \\ -1 & -1 & 0 \end{vmatrix}$$

$$= abs\left|2\boldsymbol{k}\right|$$

$$\boxed{= 2}$$

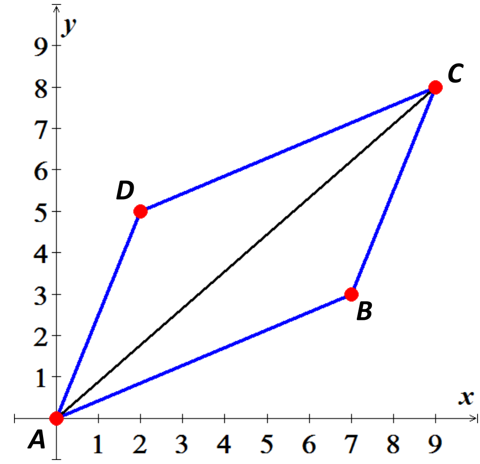
Exercise

Find the area of the parallelogram whose vertices are: $A(0,\ 0),\quad B(7,\ 3),\quad C(9,\ 8),\quad D(2,\ 5)$

Solution

$\overrightarrow{AB} = 7\boldsymbol{i} + 3\boldsymbol{j} \quad \overrightarrow{AC} = 9\boldsymbol{i} + 8\boldsymbol{j}$

$Area(\triangle ABC) = Area(\triangle ACD)$



$$Area = \left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

$$= abs\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 7 & 3 & 0 \\ 9 & 8 & 0 \end{vmatrix}$$

$$= abs|29\boldsymbol{k}|$$

$$\underline{= 29|}$$

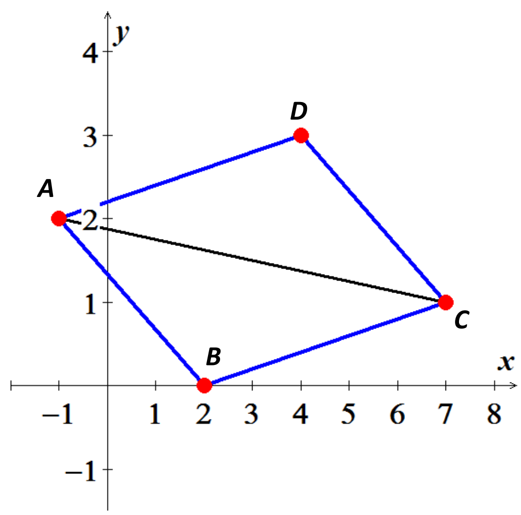
Exercise

Find the area of the parallelogram whose vertices are: $A(-1,\ 2),\quad B(2,\ 0),\quad C(7,\ 1),\quad D(4,\ 3)$

Solution

$\overrightarrow{AB} = 3\boldsymbol{i} - 2\boldsymbol{j} \quad \overrightarrow{AC} = 8\boldsymbol{i} - \boldsymbol{j}$

$Area(\triangle ABC) = Area(\triangle ACD)$



$$Area = \left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

$$= abs\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & -2 & 0 \\ 8 & -1 & 0 \end{vmatrix}$$

$$= abs|13\boldsymbol{k}|$$

$$\underline{= 13|}$$

Exercise

Find the area of the parallelogram whose vertices are:

$A(0,\ 0,\ 0),\quad B(3,\ 2,\ 4),\quad C(5,\ 1,\ 4),\quad D(2,\ -1,\ 0)$

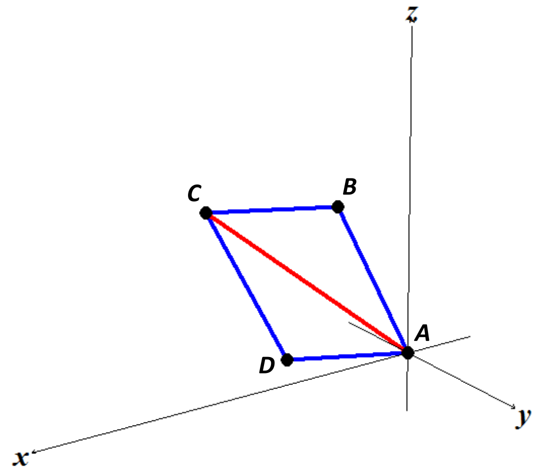
Solution

$\overrightarrow{AB} = 3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k} \quad \overrightarrow{DC} = 3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}$

$\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$

$\overrightarrow{AD} = 2\boldsymbol{i} - \boldsymbol{j} \quad \overrightarrow{BC} = 2\boldsymbol{i} - \boldsymbol{j}$

$\overrightarrow{AD}$ is parallel to $\overrightarrow{BC}$



$$Area = \left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= abs\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 2 & 4 \\ 2 & -1 & 0 \end{vmatrix}$$

$$= abs\left|4\boldsymbol{i} + 8\boldsymbol{j} - 7\boldsymbol{k}\right|$$

$$\boxed{= \sqrt{129}}$$

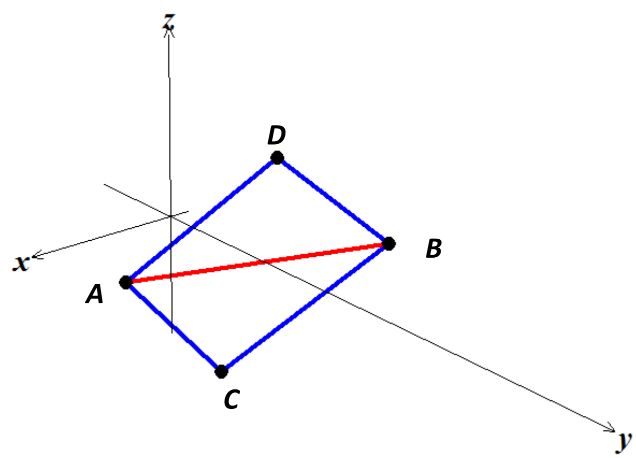
Exercise

Find the area of the parallelogram whose vertices are:

$A(1,\ 0,\ -1),\quad B(1,\ 7,\ 2),\quad C(2,\ 4,\ -1),\quad D(0,\ 3,\ 2)$

Solution

$\overrightarrow{AC} = \boldsymbol{i} + 4\boldsymbol{j} \quad \overrightarrow{CB} = -\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}$



$$Area = \left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= abs\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 4 & 0 \\ -1 & 3 & 3 \end{vmatrix}$$

$$= abs\left|12\boldsymbol{i} - 3\boldsymbol{j} + 7\boldsymbol{k}\right|$$

$$\sqrt{144 + 9 + 49}$$

$$\boxed{= \sqrt{202}}$$

Exercise

Find the area of the triangle whose vertices are: $A(0,\ 0),\quad B(-2,\ 3),\quad C(3,\ 1)$

Solution

$$\overrightarrow{AB} = -2\boldsymbol{i} + 3\boldsymbol{j} \quad \overrightarrow{AC} = 3\boldsymbol{i} + \boldsymbol{j}$$

$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= \left(\frac{1}{2}\right) abs \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -2 & 3 & 0 \\ 3 & 1 & 0 \end{vmatrix}$$

$$= \left(\frac{1}{2}\right) abs\left|-11\boldsymbol{k}\right|$$

$$= \frac{11}{2}$$

Exercise

Find the area of the triangle whose vertices are: $A(-1,\ -1),\quad B(3,\ 3),\quad C(2,\ 1)$

Solution

$$\overrightarrow{AB} = 4\boldsymbol{i} + 4\boldsymbol{j} \quad \overrightarrow{AC} = 3\boldsymbol{i} + 2\boldsymbol{j}$$

$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= \frac{1}{2}\left\|\begin{matrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 4 & 4 & 0 \\ 3 & 2 & 0 \end{matrix}\right\|$$

$$= \frac{1}{2}\left\|-4\boldsymbol{k}\right\|$$

$$= 2$$

Exercise

Find the area of the triangle whose vertices are: $A(1,\ 0,\ 0),\quad B(0,\ 0,\ 2),\quad C(0,\ 0,\ -1)$

Solution

$$\overrightarrow{AB} = -\boldsymbol{i} + 2\boldsymbol{k} \quad \overrightarrow{AC} = -\boldsymbol{i} - \boldsymbol{k}$$

$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= \frac{1}{2}\left\| \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 0 & 2 \\ -1 & 0 & -1 \end{vmatrix} \right\|$$

$$= \frac{1}{2}\|3\boldsymbol{k}\|$$

$$= \frac{3}{2}$$

Exercise

Find the area of the triangle whose vertices are: $A(0,\ 0,\ 0),\quad B(-1,\ 1,\ -1),\quad C(3,\ 0,\ 3)$

Solution

$$\overrightarrow{AB} = -\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} \qquad \overrightarrow{AC} = 3\boldsymbol{i} + 3\boldsymbol{k}$$

$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= \frac{1}{2}\left\| \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 1 & -1 \\ 3 & 0 & 3 \end{vmatrix} \right\|$$

$$= \frac{1}{2}\|3\boldsymbol{i} - 3\boldsymbol{k}\|$$

$$= \frac{1}{2}\sqrt{9+9}$$

$$= \frac{3\sqrt{2}}{2}$$

Exercise

Find the volume of the parallelepiped if four of its eight vertices are:
$A(0,\ 0,\ 0),\quad B(1,\ 2,\ 0),\quad C(0,\ -3,\ 2),\quad D(3,\ -4,\ 5)$

Solution

$$\overrightarrow{AB} = \boldsymbol{i} + 2\boldsymbol{j} \qquad \overrightarrow{AC} = -3\boldsymbol{j} + 2\boldsymbol{k} \qquad \overrightarrow{AD} = 3\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k}$$

$$\left(\overrightarrow{AB} \times \overrightarrow{AC}\right) \bullet \overrightarrow{AD} = \begin{vmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 3 & -4 & 5 \end{vmatrix} = 5$$

$$Volume = \left|\left(\overrightarrow{AB} \times \overrightarrow{AC}\right) \bullet \overrightarrow{AD}\right| = 5$$

Solution Section 1.4 – Lines and Curves in Space

Exercise

Find the parametric equation for the line through the point $P(3,-4,-1)$ parallel to the vector $\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$

Solution

$$\underline{x=3+t, \quad y=-4+t, \quad z=-1+t}$$

Exercise

Find the parametric equation for the line through the points $P(1,2,-1)$ and $Q(-1,0,1)$

Solution

The direction: $\overrightarrow{PQ}=-2\boldsymbol{i}-2\boldsymbol{j}+2\boldsymbol{k}$ and $P(1,2,-1)$

$$\underline{x=1-2t, \quad y=2-2t, \quad z=-1+2t}$$

Exercise

Find the parametric equation for the line through the points $P(-2,0,3)$ and $Q(3,5,-2)$

Solution

The direction: $\overrightarrow{PQ}=5\boldsymbol{i}+5\boldsymbol{j}-5\boldsymbol{k}$ and $P(-2,0,3)$ $\quad \underline{x=-2+5t, \quad y=5t, \quad z=3-5t}$

Exercise

Find the parametric equation for the line through the origin parallel to the vector $2\boldsymbol{j}+\boldsymbol{k}$

Solution

The direction: $2\boldsymbol{j}+\boldsymbol{k}$ and $P(0,0,0)$ $\quad \underline{x=0, \quad y=2t, \quad z=t}$

Exercise

Find the parametric equation for the line through the point $P(3,-2,1)$ parallel to the line $x=1+2t, \quad y=2-t, \quad z=3t$

Solution

The direction: $2\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}$ and $P(3,-2,1)$ $\quad \underline{x=3+2t, \quad y=-2-t, \quad z=1+3t}$

Exercise

Find the parametric equation for the line through $(2,4,5)$ perpendicular to the plane $3x+7y-5z=21$

Solution

The direction: $3\boldsymbol{i}+7\boldsymbol{j}-5\boldsymbol{k}$ and $(2,4,5)$

$$\underline{x=2+3t, \quad y=4+7t, \quad z=5-5t}$$

Exercise

Find the parametric equation for the line through $(2,3,0)$ perpendicular to the vectors $\boldsymbol{u}=\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}$ and $\boldsymbol{v}=3\boldsymbol{i}+4\boldsymbol{j}+5\boldsymbol{k}$

Solution

The direction: $\boldsymbol{u}\times\boldsymbol{v}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix}=-2\boldsymbol{i}+4\boldsymbol{j}-2\boldsymbol{k}$ and $(2,3,0)$

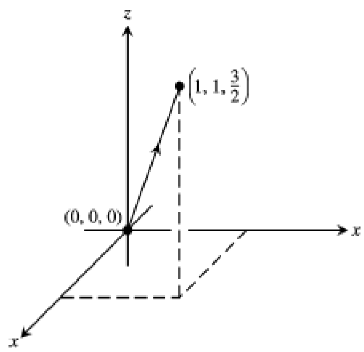
$$\underline{x=2-2t, \quad y=3+4t, \quad z=-2t}$$

Exercise

Find the parameterization for the line segment joining the points . Draw coordinate axes and sketch the segment, indicate the direction on increasing *t* for the parametrization.

Solution

The direction: $\overrightarrow{PQ}=\boldsymbol{i}+\boldsymbol{j}+\frac{3}{2}\boldsymbol{k}$ and



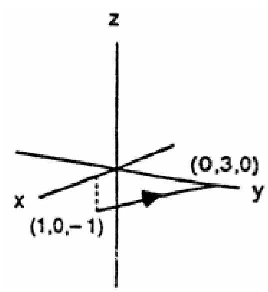
Exercise

Find the parameterization for the line segment joining the points $(1,0,-1)$, $(0,3,0)$. Draw coordinate axes and sketch the segment, indicate the direction on increasing *t* for the parametrization.

Solution

The direction: $\overrightarrow{PQ}=-\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}$ and $(1,0,-1)$

$$x=1-t, \quad y=3t, \quad z=-1+t, \quad 0\le t\le 1$$



Exercise

Find equation for the plane through normal to $\boldsymbol{n} = 3\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$

Solution

$$3(x-0) - 2(y-2) - (z+1) = 0$$

$$3x - 2y + 4 - z - 1 = 0$$

$$\boxed{3x - 2y - z = -3}$$

Exercise

Find equation for the plane through $(1,-1,3)$ parallel to the plane $3x + y + z = 7$

Solution

$$3(x-1) + (y+1) + (z-3) = 0$$

$$3x - 3 + y + 1 + z - 3 = 0 \qquad \boxed{3x + y + z = 5}$$

Exercise

Find equation for the plane through $(1,1,-1)$, $(2,0,2)$ and $(0,-2,1)$

Solution

$$\overrightarrow{PQ} = \boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k} \quad \overrightarrow{PS} = -\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PS} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix} = 7\boldsymbol{i} - 5\boldsymbol{j} - 4\boldsymbol{k} \text{ is normal to the plane.}$$

$$7(x-2) - 5(y+0) - 4(z-2) = 0$$

$$7x - 14 - 5y - 4z + 8 = 0$$

$$\boxed{7x - 5y - 4z = 6}$$

Exercise

Find equation for the plane through $P_0(2,4,5)$ perpendicular to the line $x = 5 + t, \quad y = 1 + 3t, \quad z = 4t$

Solution

$$x = 5 + t, \quad y = 1 + 3t, \quad z = 4t \qquad \Rightarrow \quad \boldsymbol{n} = \boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$$

$$1(x-2) + 3(y-4) + 4(z-5) = 0$$

$$x - 2 + 3y - 12 + 4z - 20 = 0$$

$$\boxed{x + 3y + 4z = 34}$$

Exercise

Find equation for the plane through $A(1,-2,1)$ perpendicular to the vector from the origin to A.

Solution

$\Rightarrow \quad \boldsymbol{n} = \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$

$1(x-1) - 2(y+2) + 1(z-1) = 0$

$x - 1 - 2y - 4 + z - 1 = 0$

$\boxed{x - 2y + z = 6}$

Exercise

Find the point of intersection of the lines $x = 2t+1, \quad y = 3t+2, \quad z = 4t+3$ and $x = s+2, \quad y = 2s+4, \quad z = -4s-1$, and find the plane determined by these lines.

Solution

$$\begin{cases} x = 2t+1 = s+2 \\ y = 3t+2 = 2s+4 \\ z = 4t+3 = -4s-1 \end{cases} \Rightarrow \begin{cases} 2t - s = 1 \\ 3t - 2s = 2 \\ \end{cases} \rightarrow \boxed{t = 0} \quad \boxed{s = -1}$$

$z = 4t+3 = -4s-1 \quad \Rightarrow \quad 4(0) + 3 = -4(-1) - 1 \rightarrow 3 = 3$ √ (satisfied)

The lines intersect when $t = 0$ and $s = -1 \Rightarrow$ The point of intersection $x = 1, \quad y = 2, \quad z = 3$

Therefore; the point is $\boxed{P(1,\ 2,\ 3)}$

The normal vectors: $\boldsymbol{n}_1 = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k} \quad \boldsymbol{n}_2 = \boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$

$\boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 3 & 4 \\ 1 & 2 & -4 \end{vmatrix} = -20\boldsymbol{i} + 12\boldsymbol{j} + \boldsymbol{k}$ $\boldsymbol{n}_1$ *and* $\boldsymbol{n}_2$ are directions of the lines.

The plane containing the lines is represented by

$-20(x-1) + 12(y-2) + 1(z-3) = 0 \quad \Rightarrow \quad \boxed{-20x + 12y + z = 7}$

Exercise

Find the plane determined by the intersecting lines:

$L_1: \ x = -1+t, \quad y = 2+t, \quad z = 1-t; \quad -\infty < t < \infty$

$L_2: \ x = 1-4s, \quad y = 1+2s, \quad z = 2-2s; \quad -\infty < s < \infty$

Solution

The normal vectors: $\boldsymbol{n}_1 = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} \quad \boldsymbol{n}_2 = -4\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$

$$\boldsymbol{n} = \boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & -1 \\ -4 & 2 & -2 \end{vmatrix} = 6\boldsymbol{j} + 6\boldsymbol{k}$$

Let $t = 0$ $L_1 : x = -1, \quad y = 2, \quad z = 1; \quad \Rightarrow \boxed{P(-1,2,1)}$

Therefore; the desired plane is:

$$0(x+1) + 6(y-2) + 6(z-1) = 0$$

$$6y - 12 + 6z - 6 = 0$$

$$6y + 6z = 18 \quad \Rightarrow \quad \boxed{y + z = 3}$$

Exercise

Find a plane through $P_0(2,1,-1)$ and perpendicular to the line of intersection of the planes $2x + y - z = 3, \quad x + 2y + z = 2$

Solution

The normal vectors: $\boldsymbol{n}_1 = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} \quad \boldsymbol{n}_2 = \boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

$\boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3\boldsymbol{j} - 3\boldsymbol{j} + 3\boldsymbol{k}$ is the vector in the direction of the line of intersection of the planes.

$$\Rightarrow 3(x-2) - 3(y-1) + 3(z+1) = 0$$

$$3x - 3y + 3z = 0$$

$\boxed{x - y + z = 0}$ is the desired plane containing $P_0(2,1,-1)$

Exercise

Find the distance from the point to the plane $(0,0,12), \quad x = 4t, \quad y = -2t, \quad z = 2t$

Solution

At $t = 0 \Rightarrow \boxed{P(0,0,0)}$ and let $S(0,0,12)$

$\overrightarrow{PS} = 12\boldsymbol{k}$ and $\boldsymbol{v} = 4\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$

$$\overrightarrow{PS} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & 12 \\ 4 & -2 & 2 \end{vmatrix} = 24\boldsymbol{i} + 48\boldsymbol{j}$$

$$d = \frac{\left|\overrightarrow{PS} \times \boldsymbol{v}\right|}{|\boldsymbol{v}|}$$

$$= \frac{\sqrt{24^2 + 48^2}}{\sqrt{16+4+4}}$$

$$= \frac{24\sqrt{5}}{\sqrt{24}}$$

$$= \sqrt{5}\sqrt{24}$$

$$= 2\sqrt{30}$$

Exercise

Find the distance from the point to the plane $(2,1,-1)$, $x = 2t$, $y = 1+2t$, $z = 2t$

Solution

At $t = 0 \Rightarrow \boxed{P(0,1,0)}$ and let $S(2,1,-1)$

$\overrightarrow{PS} = 2\boldsymbol{i} - \boldsymbol{k}$ and $\boldsymbol{v} = 2\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$

$$\overrightarrow{PS} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 0 & -1 \\ 2 & 2 & 2 \end{vmatrix} = 2\boldsymbol{i} - 6\boldsymbol{j} + 4\boldsymbol{k}$$

$$d = \frac{\left|\overrightarrow{PS} \times \boldsymbol{v}\right|}{|\boldsymbol{v}|}$$

$$= \frac{\sqrt{4+36+16}}{\sqrt{4+4+4}}$$

$$= \frac{\sqrt{56}}{\sqrt{12}}$$

$$= \frac{2\sqrt{14}}{2\sqrt{3}}$$

$$= \sqrt{\frac{14}{3}} \quad unit$$

Exercise

Find the distance from the point to the plane $(3,-1,4)$, $x = 4-t$, $y = 3+2t$, $z = -5+3t$

Solution

At $t = 0 \Rightarrow \boxed{P(4,3,-5)}$ and let $S(3,-1,4)$

$\overrightarrow{PS} = -\boldsymbol{i} - 4\boldsymbol{j} + 9\boldsymbol{k}$ and $\boldsymbol{v} = -\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$

$$\overrightarrow{PS} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 2 & 3 \\ -1 & 2 & 3 \end{vmatrix} = -30\boldsymbol{i} - 6\boldsymbol{j} - 6\boldsymbol{k}$$

$$d = \frac{\left|\overrightarrow{PS} \times \boldsymbol{v}\right|}{|\boldsymbol{v}|}$$

$$= \frac{\sqrt{900 + 36 + 36}}{\sqrt{1 + 4 + 9}}$$

$$= \sqrt{\frac{972}{14}}$$

$$= \sqrt{\frac{486}{7}}$$

$$= \frac{9\sqrt{6}}{\sqrt{7}} \frac{\sqrt{7}}{\sqrt{7}}$$

$$= \frac{9\sqrt{42}}{7} \quad unit$$

Exercise

Find the distance from the point to the plane $(2,-3,4)$, $\quad x + 2y + 2z = 13$

Solution

$\Rightarrow \boxed{P(13,0,0)}$ and let $S(2,-3,4)$

$\overrightarrow{PS} = -11\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{n} = \boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$

$|\boldsymbol{n}| = \sqrt{1 + 4 + 4} = 3$

$$d = \left|\overrightarrow{PS} \cdot \frac{\boldsymbol{n}}{|\boldsymbol{n}|}\right|$$

$$= \left|(-11\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}) \cdot \left(\frac{1}{3}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} + \frac{2}{3}\boldsymbol{k}\right)\right|$$

$$= \left|-\frac{11}{3} - \frac{6}{3} + \frac{8}{3}\right|$$

$$= 3 \quad unit$$

Exercise

Find the distance from the point to the plane $(0,0,0)$, $3x+2y+6z=6$

Solution

$$3x+2y+6z=6,\quad 3x+2(0)+6(0)=6\to \boxed{x=2}$$

$$\Rightarrow \boxed{P(2,0,0)} \text{ and let } S(0,0,0)$$

$$\overrightarrow{PS}=-2\boldsymbol{i} \text{ and } \boldsymbol{n}=3\boldsymbol{i}+2\boldsymbol{j}+6\boldsymbol{k} \qquad \to |\boldsymbol{n}|=\sqrt{9+4+36}=7$$

$$\begin{aligned} d &= \left|\overrightarrow{PS}\cdot\frac{\boldsymbol{n}}{|\boldsymbol{n}|}\right| \\ &= \left|(-2\boldsymbol{i})\cdot\left(\frac{3}{7}\boldsymbol{i}+\frac{2}{7}\boldsymbol{j}+\frac{6}{7}\boldsymbol{k}\right)\right| \\ &= \frac{6}{7} \quad unit \end{aligned}$$

Exercise

Find the distance from the point to the plane $(0,1,1)$, $4y+3z=-12$

Solution

$$\Rightarrow \boxed{P(0,-3,0)} \text{ and let } S(0,1,1)$$

$$\overrightarrow{PS}=4\boldsymbol{j}+\boldsymbol{k} \text{ and } \boldsymbol{n}=4\boldsymbol{j}+3\boldsymbol{k} \qquad \to |\boldsymbol{n}|=\sqrt{16+9}=5$$

$$\begin{aligned} d &= \left|\overrightarrow{PS}\cdot\frac{\boldsymbol{n}}{|\boldsymbol{n}|}\right| \\ &= \left|(4\boldsymbol{j}+\boldsymbol{k})\cdot\left(\frac{4}{5}\boldsymbol{j}+\frac{3}{5}\boldsymbol{k}\right)\right| \\ &= \left|\frac{16}{5}+\frac{3}{5}\right| \\ &= \frac{19}{5} \quad unit \end{aligned}$$

Exercise

Find the distance from the plane $x+2y+6z=1$ to the plane $x+2y+6z=10$

Solution

$$x+2y+6z=1 \quad\Rightarrow\quad P(1,0,0)$$

$$x+2y+6z=10 \quad\Rightarrow\quad S(10,0,0)$$

$$\overrightarrow{PS}=9\boldsymbol{i} \text{ and } \boldsymbol{n}=\boldsymbol{i}+2\boldsymbol{j}+6\boldsymbol{k} \qquad \to |\boldsymbol{n}|=\sqrt{1+4+36}=\sqrt{41}$$

$$d = \left| \overrightarrow{PS} \bullet \frac{\boldsymbol{n}}{|\boldsymbol{n}|} \right|$$

$$= \left| (9\boldsymbol{i}) \bullet \frac{1}{\sqrt{41}} (\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}) \right|$$

$$= \frac{1}{\sqrt{41}} |9|$$

$$= \frac{9}{\sqrt{41}}$$

Exercise

Find the angle between the planes $x + y = 1, \quad 2x + y - 2z = 2$

Solution

The vectors: $\boldsymbol{n}_1 = \boldsymbol{i} + \boldsymbol{j}, \quad \boldsymbol{n}_2 = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$ are normal to the planes.

The angle between them is:

$$\theta = \cos^{-1}\left(\frac{\boldsymbol{n}_1 \bullet \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|} \right)$$

$$= \cos^{-1}\left(\frac{2+1}{\sqrt{1+1}\sqrt{4+1+4}} \right)$$

$$= \cos^{-1}\left(\frac{3}{3\sqrt{2}} \right) = \cos^{-1}\left(\frac{1}{\sqrt{2}} \right)$$

$$= \frac{\pi}{4}$$

Exercise

Find the angle between the planes $5x + y - z = 10, \quad x - 2y + 3z = -1$

Solution

The vectors: $\boldsymbol{n}_1 = 5\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}, \quad \boldsymbol{n}_2 = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ are normal to the planes.

The angle between them is:

$$\theta = \cos^{-1}\left(\frac{\boldsymbol{n}_1 \bullet \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|} \right)$$

$$= \cos^{-1}\left(\frac{5-2-3}{\sqrt{25+1+1}\sqrt{1+4+9}} \right) = \cos^{-1}(0)$$

$$= \frac{\pi}{2}$$

Exercise

Find the point in which the line meets the plane $x = 1 - t, \quad y = 3t, \quad z = 1 + t; \quad 2x - y + 3z = 6$

Solution

$$2(1-t) - 3t + 3(1+t) = 6$$

$$2 - 2t - 3t + 3 + 3t = 6$$

$$-2t = 1$$

$$\boxed{t = -\frac{1}{2}}$$

$$x = 1 - \left(-\frac{1}{2}\right) = \frac{3}{2}, \quad y = 3\left(-\frac{1}{2}\right) = -\frac{3}{2}, \quad z = 1 + \left(-\frac{1}{2}\right) = \frac{1}{2}$$

$$\Rightarrow \boxed{P\left(\frac{3}{2}, \ -\frac{3}{2}, \ \frac{1}{2}\right)}$$

Exercise

Find the point in which the line meets the plane $x = 2, \quad y = 3 + 2t, \quad z = -2 - 2t; \quad 6x + 3y - 4z = -12$

Solution

$$12 + 3(3 + 2t) - 4(-2 - 2t) = -12$$

$$12 + 9 + 6t + 8 + 8t = -12$$

$$14t = -41$$

$$\boxed{t = -\frac{41}{14}}$$

$$x = 2, \quad y = 3 + 2\left(-\frac{41}{14}\right) = -\frac{20}{7}, \quad z = -2 - 2\left(-\frac{41}{14}\right) = \frac{27}{7}$$

$$\Rightarrow \boxed{P\left(2, \ -\frac{20}{7}, \ \frac{27}{7}\right)}$$

Solution Section 1.5 – Calculus of Vector-Valued Functions

Exercise

$\boldsymbol{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find an equation in *x* and *y* whose is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of *t*.

$\boldsymbol{r}(t)=(t+1)\boldsymbol{i}+\left(t^2-1\right)\boldsymbol{j}, \quad t=1$

Solution

$$x=t+1, \quad y=t^2-1 \quad \Rightarrow \boxed{y=(x-1)^2-1=x^2-2x}$$

$$\boldsymbol{v}(t)=\boldsymbol{r}'=\boldsymbol{i}+2t\boldsymbol{j}$$

$$\boldsymbol{v}(t=1)=\boldsymbol{i}+2\boldsymbol{j}$$

$$\boldsymbol{a}=\boldsymbol{v}'=2\boldsymbol{j}$$

$$\boldsymbol{a}(t=1)=2\boldsymbol{j}$$

Exercise

$\boldsymbol{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find an equation in *x* and *y* whose is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of *t*.

$\boldsymbol{r}(t)=\frac{t}{t+1}\boldsymbol{i}+\frac{1}{t}\boldsymbol{j}, \quad t=-\frac{1}{2}$

Solution

$$x=\frac{t}{t+1}, \quad y=\frac{1}{t}\rightarrow t=\frac{1}{y}$$

$$x=\frac{\frac{1}{y}}{\frac{1}{y}+1}=\frac{1}{1+y} \quad \Rightarrow \quad 1+y=\frac{1}{x}\Rightarrow \boxed{y=\frac{1}{x}-1}$$

$$\left(\frac{t}{t+1}\right)'=\frac{(t+1)-t}{(t+1)^2}=\frac{1}{(t+1)^2} \qquad \left(\frac{1}{t}\right)'=-\frac{1}{t^2}$$

$$\boldsymbol{v}(t)=\frac{1}{(t+1)^2}\boldsymbol{i}-\frac{1}{t^2}\boldsymbol{j}$$

$$\boldsymbol{v}\left(t=-\frac{1}{2}\right)=\frac{1}{\left(-\frac{1}{2}+1\right)^2}\boldsymbol{i}-\frac{1}{\frac{1}{4}}\boldsymbol{j}=\boxed{4\boldsymbol{i}-4\boldsymbol{j}}$$

$$\left(\frac{1}{(t+1)^2}\right)' = \frac{-2(t+1)}{(t+1)^4} = \frac{-2}{(t+1)^3} \qquad \left(-\frac{1}{t^2}\right)' = \frac{2}{t^3}$$

$$\boldsymbol{a} = \boldsymbol{v}' = \frac{-2}{(t+1)^3}\boldsymbol{i} + \frac{2}{t^3}\boldsymbol{j}$$

$$\boldsymbol{a}\left(t = -\frac{1}{2}\right) = \frac{-2}{\left(-\frac{1}{2}+1\right)^3}\boldsymbol{i} + \frac{2}{\left(-\frac{1}{2}\right)^3}\boldsymbol{j}$$

$$= \frac{-2}{-\frac{1}{8}}\boldsymbol{i} + \frac{2}{-\frac{1}{8}}\boldsymbol{j}$$

$$= 16\boldsymbol{i} - 16\boldsymbol{j}$$

Exercise

$\boldsymbol{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find an equation in *x* and *y* whose is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of *t*.

$\boldsymbol{r}(t) = e^t\boldsymbol{i} + \frac{2}{9}e^{2t}\boldsymbol{j}, \quad t = \ln 3$

Solution

$$x = e^t, \quad y = \frac{2}{9}e^{2t} = \frac{2}{9}\left(e^t\right)^2 \quad \Rightarrow \boxed{y = \frac{2}{9}x^2}$$

$$\boldsymbol{v}(t) = e^t\boldsymbol{i} + \frac{4}{9}e^{2t}\boldsymbol{j}$$

$$\boldsymbol{v}(t = \ln 3) = e^{\ln 3}\boldsymbol{i} + \frac{4}{9}e^{2\ln 3}\boldsymbol{j}$$

$$= 3\boldsymbol{i} + \frac{4}{9}e^{\ln 3^2}\boldsymbol{j} \qquad e^{\ln 3^2} = e^{\ln 9} = 9$$

$$= 3\boldsymbol{i} + 4\boldsymbol{j}$$

$$\boldsymbol{a}(t) = e^t\boldsymbol{i} + \frac{8}{9}e^{2t}\boldsymbol{j}$$

$$\boldsymbol{a}(t = \ln 3) = e^{\ln 3}\boldsymbol{i} + \frac{8}{9}e^{2\ln 3}\boldsymbol{j}$$

$$= 3\boldsymbol{i} + \frac{8}{9}e^{\ln 9}\boldsymbol{j}$$

$$= 3\boldsymbol{i} + 8\boldsymbol{j}$$

Exercise

$\boldsymbol{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find an equation in *x* and *y* whose is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of *t*.

$\boldsymbol{r}(t) = (\cos 2t)\boldsymbol{i} + (3\sin 2t)\boldsymbol{j}, \quad t = 0$

Solution

$$x = \cos 2t, \quad y = 3\sin 2t \to \sin 2t = \frac{y}{3}$$

$$\cos^2 2t + \sin^2 2t = 1 \quad \Rightarrow \quad \boxed{x^2 + \frac{y^2}{9} = 1}$$

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}}{dt} = -(2\sin 2t)\boldsymbol{i} + (6\cos 2t)\boldsymbol{j}$$

$$\boldsymbol{v}(t=0) = -(2\sin 0)\boldsymbol{i} + (6\cos 0)\boldsymbol{j} = \underline{6\boldsymbol{j}}$$

$$\boldsymbol{a}(t) = \frac{d\boldsymbol{v}}{dt} = -(4\cos 2t)\boldsymbol{i} - (12\sin 2t)\boldsymbol{j}$$

$$\boldsymbol{a}(t=0) = -(4\cos 2t)\boldsymbol{i} - (12\sin 2t)\boldsymbol{j} = \underline{-4\boldsymbol{i}}$$

Exercise

Give the position vectors of particles moving along various curves in the *xy*-plane. Find the particle's velocity and acceleration vectors at the stated times and sketch them as vectors on the curve

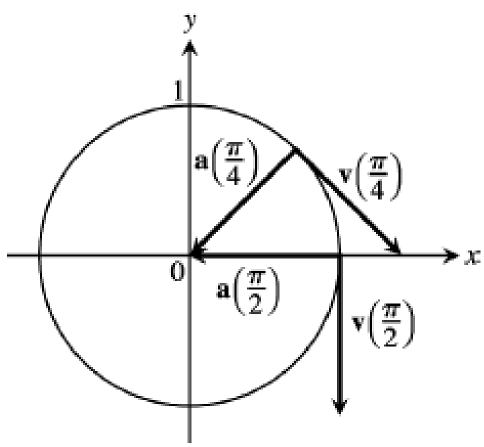
Motion on the circle $x^2 + y^2 = 1 \qquad \boldsymbol{r}(t) = (\sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j}, \quad t = \frac{\pi}{4} \text{ and } \frac{\pi}{2}$

Solution

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = (\cos t)\boldsymbol{i} - (\sin t)\boldsymbol{j}$$

$$\boldsymbol{v}\left(t = \tfrac{\pi}{4}\right) = \left(\cos\tfrac{\pi}{4}\right)\boldsymbol{i} - \left(\sin\tfrac{\pi}{4}\right)\boldsymbol{j} = \underline{\frac{\sqrt{2}}{2}\boldsymbol{i} - \frac{\sqrt{2}}{2}\boldsymbol{j}}$$

$$\boldsymbol{v}\left(t = \tfrac{\pi}{2}\right) = \left(\cos\tfrac{\pi}{2}\right)\boldsymbol{i} - \left(\sin\tfrac{\pi}{2}\right)\boldsymbol{j} = \underline{-\boldsymbol{j}}$$



$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = -(\sin t)\boldsymbol{i} - (\cos t)\boldsymbol{j}$$

$$\boldsymbol{a}\left(t = \tfrac{\pi}{4}\right) = -\left(\sin\tfrac{\pi}{4}\right)\boldsymbol{i} - \left(\cos\tfrac{\pi}{4}\right)\boldsymbol{j} = \underline{-\frac{\sqrt{2}}{2}\boldsymbol{i} - \frac{\sqrt{2}}{2}\boldsymbol{j}}$$

$$\boldsymbol{a}\left(t = \tfrac{\pi}{2}\right) = -\left(\sin\tfrac{\pi}{2}\right)\boldsymbol{i} - \left(\cos\tfrac{\pi}{2}\right)\boldsymbol{j} = \underline{-\boldsymbol{i}}$$

Exercise

Give the position vectors of particles moving along various curves in the xy-plane. Find the particle's velocity and acceleration vectors at the stated times and sketch them as vectors on the curve

Motion on the cycloid $x = t - \sin t, \quad y = 1 - \cos t$

$$\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}; \quad t = \pi \text{ and } \frac{3\pi}{2}$$

Solution

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (1 - \cos t)\mathbf{i} + (\sin t)\mathbf{j}$$

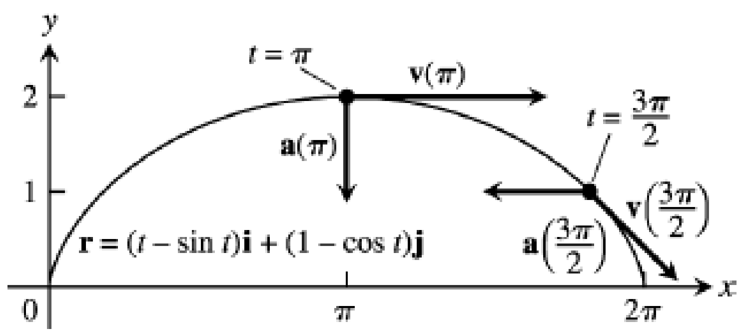
$$\mathbf{v}(t = \pi) = (1 - \cos\pi)\mathbf{i} + (\sin\pi)\mathbf{j} = \boxed{2\mathbf{i}}$$

$$\mathbf{v}\left(t = \frac{3\pi}{2}\right) = \left(1 - \cos\frac{3\pi}{2}\right)\mathbf{i} + \left(\sin\frac{3\pi}{2}\right)\mathbf{j} = \boxed{\mathbf{i} - \mathbf{j}}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}$$

$$\mathbf{a}(t = \pi) = (\sin\pi)\mathbf{i} + (\cos\pi)\mathbf{j} = \boxed{-\mathbf{j}}$$

$$\mathbf{a}\left(t = \frac{3\pi}{2}\right) = \left(\sin\frac{3\pi}{2}\right)\mathbf{i} + \left(\cos\frac{3\pi}{2}\right)\mathbf{j} = \boxed{-\mathbf{i}}$$



Exercise

$\mathbf{r}(t)$ is the position of a particle in the xy-plane at time t. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of t. Write the particle's velocity at that time as the product of its speed and direction.

$$\mathbf{r}(t) = (t + 1)\mathbf{i} + \left(t^2 - 1\right)\mathbf{j} + 2t\mathbf{k}, \quad t = 1$$

Solution

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 2\mathbf{j}$$

$$\boldsymbol{v}(t=1)=\boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k}$$

Speed: $|\boldsymbol{v}(1)|=\sqrt{1+4+4}=\underline{3}$

Direction: $\dfrac{\boldsymbol{v}(1)}{|\boldsymbol{v}(1)|}=\frac{1}{3}(\boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k})$

$$=\frac{1}{3}\boldsymbol{i}+\frac{2}{3}\boldsymbol{j}+\frac{2}{3}\boldsymbol{k}$$

$$\boldsymbol{v}(1)=3\left(\frac{1}{3}\boldsymbol{i}+\frac{2}{3}\boldsymbol{j}+\frac{2}{3}\boldsymbol{k}\right)$$

Exercise

$\boldsymbol{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of *t*. Write the particle's velocity at that time as the product of its speed and direction.

$$\boldsymbol{r}(t)=(t+1)\boldsymbol{i}+\frac{t^2}{\sqrt{2}}\boldsymbol{j}+\frac{t^3}{3}\boldsymbol{k}, \quad t=1$$

Solution

$$\boldsymbol{v}=\frac{d\boldsymbol{r}}{dt}=\boldsymbol{i}+\frac{2}{\sqrt{2}}t\boldsymbol{j}+t^2\boldsymbol{k}$$

$$\boldsymbol{a}=\frac{d\boldsymbol{v}}{dt}=\frac{2}{\sqrt{2}}\boldsymbol{j}+2t\boldsymbol{k}$$

$$\boldsymbol{v}(t=1)=\boldsymbol{i}+\frac{2}{\sqrt{2}}\boldsymbol{j}+\boldsymbol{k}$$

Speed: $|\boldsymbol{v}(1)|=\sqrt{1+2+1}=\underline{2}$

Direction: $\dfrac{\boldsymbol{v}(1)}{|\boldsymbol{v}(1)|}=\frac{1}{2}\left(\boldsymbol{i}+\frac{2}{\sqrt{2}}\boldsymbol{j}+\boldsymbol{k}\right)$

$$=\frac{1}{2}\boldsymbol{i}+\frac{1}{\sqrt{2}}\boldsymbol{j}+\frac{1}{2}\boldsymbol{k}$$

$$\boldsymbol{v}(1)=2\left(\frac{1}{2}\boldsymbol{i}+\frac{1}{\sqrt{2}}\boldsymbol{j}+\frac{1}{2}\boldsymbol{k}\right)$$

Exercise

$\boldsymbol{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of *t*. Write the particle's velocity at that time as the product of its speed and direction.

$$\boldsymbol{r}(t)=(2\cos t)\boldsymbol{i}+(3\sin t)\boldsymbol{j}+4t\boldsymbol{k}, \quad t=\frac{\pi}{2}$$

Solution

$$\boldsymbol{v}=\frac{d\boldsymbol{r}}{dt}=-(2\sin t)\boldsymbol{i}+(3\cos t)\boldsymbol{j}+4\boldsymbol{k}$$

$$\boldsymbol{a}=\frac{d\boldsymbol{v}}{dt}=-(2\cos t)\boldsymbol{i}+(3\sin t)\boldsymbol{j}$$

$$\boldsymbol{v}\left(t=\frac{\pi}{2}\right)=-\left(2\sin\frac{\pi}{2}\right)\boldsymbol{i}+\left(3\cos\frac{\pi}{2}\right)\boldsymbol{j}+4\boldsymbol{k}=-2\boldsymbol{i}+4\boldsymbol{k}$$

Speed: $\left|\boldsymbol{v}\left(\frac{\pi}{2}\right)\right|=\sqrt{4+16}=\underline{2\sqrt{5}|}$

Direction: $\dfrac{\boldsymbol{v}\left(\frac{\pi}{2}\right)}{\left|\boldsymbol{v}\left(\frac{\pi}{2}\right)\right|}=\dfrac{1}{2\sqrt{5}}(-2\boldsymbol{i}+4\boldsymbol{k})$

$$=-\frac{1}{\sqrt{5}}\boldsymbol{i}+\frac{2}{\sqrt{5}}\boldsymbol{k}$$

$$\underline{\boldsymbol{v}\left(\frac{\pi}{2}\right)=2\sqrt{5}\left(-\frac{1}{\sqrt{5}}\boldsymbol{i}+\frac{2}{\sqrt{5}}\boldsymbol{k}\right)|}$$

Exercise

$\boldsymbol{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of *t*. Write the particle's velocity at that time as the product of its speed and direction.

$$\boldsymbol{r}(t)=(2\ln(t+1))\boldsymbol{i}+t^2\boldsymbol{j}+\frac{t^2}{2}\boldsymbol{k}, \quad t=1$$

Solution

$$\boldsymbol{v}=\frac{d\boldsymbol{r}}{dt}=\frac{2}{t+1}\boldsymbol{i}+2t\boldsymbol{j}+t\boldsymbol{k}$$

$$\boldsymbol{a}=\frac{d\boldsymbol{v}}{dt}=\frac{-2}{(t+1)^2}\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$$

$$\boldsymbol{v}(t=1)=\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$$

Speed: $|\boldsymbol{v}(1)|=\sqrt{1+4+1}=\underline{\sqrt{6}|}$

Direction: $\dfrac{\mathbf{v}(1)}{|\mathbf{v}(1)|} = \dfrac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

$$= \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k}$$

$$\mathbf{v}(1) = \sqrt{6}\left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k}\right)$$

Exercise

$\mathbf{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of *t*. Write the particle's velocity at that time as the product of its speed and direction.

$$\mathbf{r}(t) = \left(e^{-t}\right)\mathbf{i} + (2\cos 3t)\mathbf{j} + (2\sin 3t)\mathbf{k}, \quad t = 0$$

Solution

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -e^{-t}\mathbf{i} - 6\sin 3t\mathbf{j} + 6\cos 3t\mathbf{k} \qquad \mathbf{v}(0) = -\mathbf{i} + 6\mathbf{k}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = e^{-t}\mathbf{i} - 18\cos 3t\mathbf{j} - 18\sin 3t\mathbf{k}$$

Speed: $|\mathbf{v}(0)| = 1 + 36 = \sqrt{37}$

Direction: $\dfrac{\mathbf{v}(0)}{|\mathbf{v}(0)|} = \dfrac{1}{\sqrt{37}}(-\mathbf{i} + 6\mathbf{k})$

$$= -\frac{1}{\sqrt{37}}\mathbf{i} + \frac{6}{\sqrt{37}}\mathbf{k}$$

$$\mathbf{v}(1) = \sqrt{37}\left(-\frac{1}{\sqrt{37}}\mathbf{i} + \frac{6}{\sqrt{37}}\mathbf{k}\right)$$

Solution Section 1.6 – Motion in Space

Exercise

Evaluate the integral: $\displaystyle\int_0^1 \left(t^3\boldsymbol{i}+7\boldsymbol{j}+(t+1)\boldsymbol{k}\right)dt$

Solution

$$\int_0^1 \left(t^3\boldsymbol{i}+7\boldsymbol{j}+(t+1)\boldsymbol{k}\right)dt=\left[\frac{1}{4}t^4\boldsymbol{i}+7t\boldsymbol{j}+\left(\frac{1}{2}t^2+t\right)\boldsymbol{k}\right]_0^1$$

$$=\left(\frac{1}{4}\boldsymbol{i}+7\boldsymbol{j}+\left(\frac{1}{2}+1\right)\boldsymbol{k}\right)-0$$

$$=\frac{1}{4}\boldsymbol{i}+7\boldsymbol{j}+\frac{3}{2}\boldsymbol{k}$$

Exercise

Evaluate the integral: $\displaystyle\int_1^2 \left((6-6t)\boldsymbol{i}+3\sqrt{t}\,\boldsymbol{j}+\frac{4}{t^2}\boldsymbol{k}\right)dt$

Solution

$$\int_1^2 \left((6-6t)\boldsymbol{i}+3\sqrt{t}\,\boldsymbol{j}+\frac{4}{t^2}\boldsymbol{k}\right)dt=\left[\left(6t-3t^2\right)\boldsymbol{i}+2t^{3/2}\boldsymbol{j}-\frac{4}{t}\boldsymbol{k}\right]_1^2$$

$$=\left[(12-12)\boldsymbol{i}+2(2)^{3/2}\boldsymbol{j}-\frac{4}{2}\boldsymbol{k}\right]-\left[(6-3)\boldsymbol{i}+2\boldsymbol{j}-4\boldsymbol{k}\right]$$

$$=4\sqrt{2}\boldsymbol{j}-2\boldsymbol{k}-3\boldsymbol{i}-2\boldsymbol{j}+4\boldsymbol{k}$$

$$=-3\boldsymbol{i}+\left(4\sqrt{2}-2\right)\boldsymbol{j}+2\boldsymbol{k}$$

Exercise

Evaluate the integral: $\displaystyle\int_{-\pi/4}^{\pi/4} \left((\sin t)\boldsymbol{i}+(1+\cos t)\boldsymbol{j}+\left(\sec^2 t\right)\boldsymbol{k}\right)dt$

Solution

$$\int_{-\pi/4}^{\pi/4} \left((\sin t)\boldsymbol{i}+(1+\cos t)\boldsymbol{j}+\left(\sec^2 t\right)\boldsymbol{k}\right)dt=\left[-(\cos t)\boldsymbol{i}+(t+\sin t)\boldsymbol{j}+(\tan t)\boldsymbol{k}\right]_{-\pi/4}^{\pi/4}$$

$$=\left[-\left(\cos\frac{\pi}{4}\right)\boldsymbol{i}+\left(\frac{\pi}{4}+\sin\frac{\pi}{4}\right)\boldsymbol{j}+\left(\tan\frac{\pi}{4}\right)\boldsymbol{k}\right]$$

$$-\left[-\left(\cos\left(-\frac{\pi}{4}\right)\right)\boldsymbol{i}+\left(-\frac{\pi}{4}+\sin\left(-\frac{\pi}{4}\right)\right)\boldsymbol{j}+\left(\tan\left(-\frac{\pi}{4}\right)\right)\boldsymbol{k}\right]$$

$$= -\frac{\sqrt{2}}{2}\boldsymbol{i} + \left(\frac{\pi}{4} + \frac{\sqrt{2}}{2}\right)\boldsymbol{j} + \boldsymbol{k} + \frac{\sqrt{2}}{2}\boldsymbol{i} - \left(-\frac{\pi}{4} - \frac{\sqrt{2}}{2}\right)\boldsymbol{j} + \boldsymbol{k}$$

$$= 2\left(\frac{\pi}{4} + \frac{\sqrt{2}}{2}\right)\boldsymbol{j} + 2\boldsymbol{k}$$

$$= 2\left(\frac{\pi + 2\sqrt{2}}{4}\right)\boldsymbol{j} + 2\boldsymbol{k}$$

$$= \left(\frac{\pi + 2\sqrt{2}}{2}\right)\boldsymbol{j} + 2\boldsymbol{k}$$

Exercise

Evaluate the integral: $\displaystyle\int_0^{\pi/3} \left((\sec t \tan t)\boldsymbol{i} + (\tan t)\boldsymbol{j} + (2\sin t \cos t)\boldsymbol{k}\right)dt$

Solution

$$\int_0^{\pi/3} \left((\sec t \tan t)\boldsymbol{i} + (\tan t)\boldsymbol{j} + (2\sin t \cos t)\boldsymbol{k}\right)dt = \int_0^{\pi/3} \left((\sec t \tan t)\boldsymbol{i} + (\tan t)\boldsymbol{j} + (\sin 2t)\boldsymbol{k}\right)dt$$

$$= \left[(\sec t)\boldsymbol{i} + (-\ln(\cos t))\boldsymbol{j} - \left(\tfrac{1}{2}\cos 2t\right)\boldsymbol{k}\right]_0^{\pi/3}$$

$$= \left[\left(\sec\tfrac{\pi}{3}\right)\boldsymbol{i} + \left(-\ln\left(\cos\tfrac{\pi}{3}\right)\right)\boldsymbol{j} - \left(\tfrac{1}{2}\cos\tfrac{2\pi}{3}\right)\boldsymbol{k}\right] - \left[(\sec 0)\boldsymbol{i} + (-\ln(\cos 0))\boldsymbol{j} - \left(\tfrac{1}{2}\cos 0\right)\boldsymbol{k}\right]$$

$$= \left[2\boldsymbol{i} + \left(-\ln\tfrac{1}{2}\right)\boldsymbol{j} - \left(\tfrac{1}{2}\left(-\tfrac{1}{2}\right)\right)\boldsymbol{k}\right] - \left[\boldsymbol{i} + (-\ln(1))\boldsymbol{j} - \tfrac{1}{2}\boldsymbol{k}\right]$$

$$= 2\boldsymbol{i} + \ln 2\,\boldsymbol{j} + \tfrac{1}{4}\boldsymbol{k} - \boldsymbol{i} + \tfrac{1}{2}\boldsymbol{k}$$

$$= \boldsymbol{i} + (\ln 2)\boldsymbol{j} + \tfrac{3}{4}\boldsymbol{k}$$

Exercise

Evaluate the integral: $\int_0^1 \left(\frac{2}{\sqrt{1-t^2}}\boldsymbol{i} + \frac{\sqrt{3}}{1+t^2}\boldsymbol{k} \right) dt$

Solution

$$\int_0^1 \left(\frac{2}{\sqrt{1-t^2}}\boldsymbol{i} + \frac{\sqrt{3}}{1+t^2}\boldsymbol{k} \right) dt = \left[\left(2\sin^{-1} t\right)\boldsymbol{i} + \left(\sqrt{3}\tan^{-1} t\right)\boldsymbol{k} \right]_0^1$$

$$= \left[\left(2\sin^{-1} 1\right)\boldsymbol{i} + \left(\sqrt{3}\tan^{-1} 1\right)\boldsymbol{k} \right] - \left[\left(2\sin^{-1} 0\right)\boldsymbol{i} + \left(\sqrt{3}\tan^{-1} 0\right)\boldsymbol{k} \right]$$

$$= \left[\left(2\tfrac{\pi}{2}\right)\boldsymbol{i} + \left(\sqrt{3}\tfrac{\pi}{4}\right)\boldsymbol{k} \right] - \left[(0)\boldsymbol{i} + (0)\boldsymbol{k} \right]$$

$$= \pi\boldsymbol{i} + \frac{\pi\sqrt{3}}{4}\boldsymbol{k}$$

Exercise

Evaluate the integral: $\int_1^{\ln 3} \left(te^t\boldsymbol{i} + e^t\boldsymbol{j} + (\ln t)\boldsymbol{k} \right) dt$

Solution

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = \int dx = x \qquad \int u\,dv = uv - \int v\,du$

	e^t
$(+)$ t	e^t
$(-)$ 1	e^t
0	

$$\int \ln x\,dx = x\ln x - \int x\frac{1}{x}dx = x\ln x - \int dx = x\ln x - x + C$$

$$\int_1^{\ln 3} \left(te^t\boldsymbol{i} + e^t\boldsymbol{j} + (\ln t)\boldsymbol{k} \right) dt = \left[\left(te^t - e^t\right)\boldsymbol{i} + e^t\boldsymbol{j} + (t\ln t - t)\boldsymbol{k} \right]_1^{\ln 3}$$

$$= \left[\left((\ln 3)e^{\ln 3} - e^{\ln 3}\right)\boldsymbol{i} + e^{\ln 3}\boldsymbol{j} + \left(\ln 3\ln(\ln 3) - \ln 3\right)\boldsymbol{k} \right]$$
$$- \left[(e-e)\boldsymbol{i} + e\boldsymbol{j} + \left(\ln(1) - 1\right)\boldsymbol{k} \right]$$

$$= (3\ln 3 - 3)\boldsymbol{i} + 3\boldsymbol{j} + \left(\ln 3\left(\ln(\ln 3) - 1\right)\right)\boldsymbol{k} - e\boldsymbol{j} + \boldsymbol{k}$$

$$= 3(\ln 3 - 1)\boldsymbol{i} + (3-e)\boldsymbol{j} + \left(\ln 3\left(\ln(\ln 3) - 1\right) + 1\right)\boldsymbol{k}$$

Exercise

Evaluate the integral: $\displaystyle\int_0^{\pi/2}\left(\cos t\,\boldsymbol{i}-\sin 2t\,\boldsymbol{j}+\sin^2 t\,\boldsymbol{k}\right)dt$

Solution

$$\begin{aligned}\int_0^{\pi/2}\left(\cos t\,\boldsymbol{i}-\sin 2t\,\boldsymbol{j}+\sin^2 t\,\boldsymbol{k}\right)dt &= \int_0^{\pi/2}\left(\cos t\,\boldsymbol{i}-\sin 2t\,\boldsymbol{j}+\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 2t\right)\boldsymbol{k}\right)dt\\
&=\left[\sin t\,\boldsymbol{i}+\tfrac{1}{2}\cos 2t\,\boldsymbol{j}+\left(\tfrac{1}{2}t-\tfrac{1}{4}\sin 2t\right)\boldsymbol{k}\right]_0^{\pi/2}\\
&=\left[\boldsymbol{i}+\tfrac{1}{2}(-1)\,\boldsymbol{j}+\tfrac{\pi}{4}\,\boldsymbol{k}\right]-\left[\tfrac{1}{2}\,\boldsymbol{j}\right]\\
&=\boldsymbol{i}-\tfrac{1}{2}\,\boldsymbol{j}+\tfrac{\pi}{4}\,\boldsymbol{k}-\tfrac{1}{2}\,\boldsymbol{j}\\
&=\boxed{\boldsymbol{i}-\boldsymbol{j}+\tfrac{\pi}{4}\,\boldsymbol{k}}\end{aligned}$$

Exercise

Solve the initial value problem for $\boldsymbol{r}$ as a vector function of t.

$$\begin{cases}\textit{Differential equation}: & \dfrac{d\boldsymbol{r}}{dt}=-t\boldsymbol{i}-t\boldsymbol{j}-t\boldsymbol{k}\\ \textit{Initial condition}: & \boldsymbol{r}(0)=\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}\end{cases}$$

Solution

$$\begin{aligned}\boldsymbol{r}&=\int\frac{d\boldsymbol{r}}{dt}dt=\int\left(-t\boldsymbol{i}-t\boldsymbol{j}-t\boldsymbol{k}\right)dt\\
&=-\tfrac{t^2}{2}\boldsymbol{i}-\tfrac{t^2}{2}\boldsymbol{j}-\tfrac{t^2}{2}\boldsymbol{k}+\boldsymbol{C}\end{aligned}$$

$$\boldsymbol{r}(0)=-0\boldsymbol{i}-0\boldsymbol{j}-0\boldsymbol{k}+\boldsymbol{C}$$

$$\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}=\boldsymbol{C}$$

$$\begin{aligned}\boldsymbol{r}&=-\tfrac{t^2}{2}\boldsymbol{i}-\tfrac{t^2}{2}\boldsymbol{j}-\tfrac{t^2}{2}\boldsymbol{k}+\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}\\
&=\boxed{\left(-\tfrac{t^2}{2}+1\right)\boldsymbol{i}+\left(2-\tfrac{t^2}{2}\right)\boldsymbol{j}+\left(3-\tfrac{t^2}{2}\right)\boldsymbol{k}}\end{aligned}$$

Exercise

Solve the initial value problem for $\boldsymbol{r}$ as a vector function of t.

$$\begin{cases} \textit{Differential equation}: & \dfrac{d\boldsymbol{r}}{dt} = (180t)\boldsymbol{i} + \left(180t - 16t^2\right)\boldsymbol{j} \\ \textit{Initial condition}: & \boldsymbol{r}(0) = 100\boldsymbol{j} \end{cases}$$

Solution

$$\boldsymbol{r} = \int \left[(180t)\boldsymbol{i} + \left(180t - 16t^2\right)\boldsymbol{j}\right] dt$$

$$= \left(90t^2\right)\boldsymbol{i} + \left(90t^2 - \frac{16}{3}t^3\right)\boldsymbol{j} + \boldsymbol{C}$$

$$\boldsymbol{r}(0) = 0\boldsymbol{i} + 0\boldsymbol{j} + \boldsymbol{C}$$

$$100\boldsymbol{j} = \boldsymbol{C}$$

$$\boldsymbol{r} = \left(90t^2\right)\boldsymbol{i} + \left(90t^2 - \frac{16}{3}t^3\right)\boldsymbol{j} + 100\boldsymbol{j}$$

$$= \left(90t^2\right)\boldsymbol{i} + \left(90t^2 - \frac{16}{3}t^3 + 100\right)\boldsymbol{j}$$

Exercise

Solve the initial value problem for $\boldsymbol{r}$ as a vector function of t.

$$\begin{cases} \textit{Differential equation}: & \dfrac{d\boldsymbol{r}}{dt} = \frac{3}{2}(t+1)^{1/2}\boldsymbol{i} + e^{-t}\boldsymbol{j} + \frac{1}{t+1}\boldsymbol{k} \\ \textit{Initial condition}: & \boldsymbol{r}(0) = \boldsymbol{k} \end{cases}$$

Solution

$$\boldsymbol{r} = \int \left(\frac{3}{2}(t+1)^{1/2}\boldsymbol{i} + e^{-t}\boldsymbol{j} + \frac{1}{t+1}\boldsymbol{k}\right) dt$$

$$= (t+1)^{3/2}\boldsymbol{i} - e^{-t}\boldsymbol{j} + \ln(t+1)\boldsymbol{k} + \boldsymbol{C}$$

$$\boldsymbol{r}(0) = \boldsymbol{i} - \boldsymbol{j} + \ln(1)\boldsymbol{k} + \boldsymbol{C}$$

$$\boldsymbol{k} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{C}$$

$$\boldsymbol{C} = -\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$$

$$\boldsymbol{r} = (t+1)^{3/2}\boldsymbol{i} - e^{-t}\boldsymbol{j} + \ln(t+1)\boldsymbol{k} - \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$$

$$= \left((t+1)^{3/2} - 1\right)\boldsymbol{i} + \left(1 - e^{-t}\right)\boldsymbol{j} + \left(\ln(t+1) + 1\right)\boldsymbol{k}$$

Exercise

Solve the initial value problem for $\boldsymbol{r}$ as a vector function of t.

$$\textit{Differential equation}: \quad \frac{d^2\boldsymbol{r}}{dt^2} = -32\boldsymbol{k}$$

$$\textit{Initial condition}: \quad \boldsymbol{r}(0) = 100\boldsymbol{k}$$

$$\left.\frac{d\boldsymbol{r}}{dt}\right|_{t=0} = 8\boldsymbol{i} + 8\boldsymbol{j}$$

Solution

$$\frac{d\boldsymbol{r}}{dt} = \int (-32\boldsymbol{k})\,dt$$

$$= -32t\,\boldsymbol{k} + \boldsymbol{C}_1$$

$$\left.\frac{d\boldsymbol{r}}{dt}\right|_{t=0} = 0\boldsymbol{k} + \boldsymbol{C}_1$$

$$\boxed{8\boldsymbol{i} + 8\boldsymbol{j} = \boldsymbol{C}_1}$$

$$\frac{d\boldsymbol{r}}{dt} = -32t\,\boldsymbol{k} + 8\boldsymbol{i} + 8\boldsymbol{j} = 8\boldsymbol{i} + 8\boldsymbol{j} - 32t\,\boldsymbol{k}$$

$$\boldsymbol{r} = \int (8\boldsymbol{i} + 8\boldsymbol{j} - 32t\,\boldsymbol{k})\,dt$$

$$= 8t\,\boldsymbol{i} + 8t\,\boldsymbol{j} - 16t^2\,\boldsymbol{k} + \boldsymbol{C}_2$$

$$\boldsymbol{r}(0) = 8(0)\,\boldsymbol{i} + 8(0)\,\boldsymbol{j} - 16(0)^2\,\boldsymbol{k} + \boldsymbol{C}_2$$

$$100\,\boldsymbol{k} = \boldsymbol{C}_2$$

$$\boldsymbol{r} = 8t\,\boldsymbol{i} + 8t\,\boldsymbol{j} + \left(100 - 16t^2\right)\boldsymbol{k}$$

Exercise

Solve the initial value problem for r as a vector function of t.

$$\textit{Differential equation}: \quad \frac{d^2\boldsymbol{r}}{dt^2} = -(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$$

$$\textit{Initial condition}: \quad \boldsymbol{r}(0) = 10\boldsymbol{i} + 10\boldsymbol{j} + 10\boldsymbol{k}$$

$$\left.\frac{d\boldsymbol{r}}{dt}\right|_{t=0} = \boldsymbol{0}$$

Solution

$$\frac{d\boldsymbol{r}}{dt} = -\int (\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})\,dt$$

$$= -(t\boldsymbol{i} + t\boldsymbol{j} + t\boldsymbol{k}) + \boldsymbol{C}_1$$

$$\left.\frac{d\boldsymbol{r}}{dt}\right|_{t=0} = -(0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}) + \boldsymbol{C}_1$$

$$\boxed{0 = \boldsymbol{C}_1}$$

$$\frac{d\boldsymbol{r}}{dt} = -(t\boldsymbol{i} + t\boldsymbol{j} + t\boldsymbol{k})$$

$$\boldsymbol{r} = -\int (t\boldsymbol{i} + t\boldsymbol{j} + t\boldsymbol{k})\,dt$$

$$= -\left(\frac{t^2}{2}\boldsymbol{i} + \frac{t^2}{2}\boldsymbol{j} + \frac{t^2}{2}\boldsymbol{k}\right) + \boldsymbol{C}_2$$

$$\boldsymbol{r}(0) = -(0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}) + \boldsymbol{C}_2$$

$$\boxed{10\boldsymbol{i} + 10\boldsymbol{j} + 10\boldsymbol{k} = \boldsymbol{C}_2}$$

$$\boldsymbol{r} = -\frac{t^2}{2}\boldsymbol{i} - \frac{t^2}{2}\boldsymbol{j} - \frac{t^2}{2}\boldsymbol{k} + 10\boldsymbol{i} + 10\boldsymbol{j} + 10\boldsymbol{k}$$

$$= \left(10 - \frac{t^2}{2}\right)\boldsymbol{i} + \left(10 - \frac{t^2}{2}\right)\boldsymbol{j} + \left(10 - \frac{t^2}{2}\right)\boldsymbol{k}$$

Exercise

At time $t = 0$, a particle is located at the point $(1,\ 2,\ 3)$. It travels in a straight line to the point $(4,\ 1,\ 4)$, has speed 2 at $(1,\ 2,\ 3)$ and constant acceleration $3\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$. Find an equation for the position vector $\boldsymbol{r}(t)$ of the particle at time t.

Solution

$$\boldsymbol{a} = 3\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k} = \frac{d\boldsymbol{v}}{dt}$$

$$\boldsymbol{v} = \int (3\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})\,dt$$

$$= 3t\boldsymbol{i} - t\boldsymbol{j} + t\boldsymbol{k} + \boldsymbol{C}_1$$

Since the particle travels in a straight line in the direction of the vector:

$$(4-1)\boldsymbol{i} + (1-2)\boldsymbol{j} + (4-3)\boldsymbol{k} = 3\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$$

At $t = 0$, the particle has a speed of 2.

$$\boldsymbol{v}(0) = \frac{2}{\sqrt{9+1+1}}(3\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) = \boldsymbol{C}_1$$

$$C_1 = \frac{6}{\sqrt{11}}\boldsymbol{i} - \frac{2}{\sqrt{11}}\boldsymbol{j} + \frac{2}{\sqrt{11}}\boldsymbol{k}$$

$$\begin{aligned}\boldsymbol{v} &= 3t\boldsymbol{i} - t\boldsymbol{j} + t\boldsymbol{k} + \frac{6}{\sqrt{11}}\boldsymbol{i} - \frac{2}{\sqrt{11}}\boldsymbol{j} + \frac{2}{\sqrt{11}}\boldsymbol{k} \\ &= \left(3t + \frac{6}{\sqrt{11}}\right)\boldsymbol{i} - \left(t + \frac{2}{\sqrt{11}}\right)\boldsymbol{j} + \left(t + \frac{2}{\sqrt{11}}\right)\boldsymbol{k}\end{aligned}$$

$$\begin{aligned}\boldsymbol{r} &= \int \left(\left(3t + \frac{6}{\sqrt{11}}\right)\boldsymbol{i} - \left(t + \frac{2}{\sqrt{11}}\right)\boldsymbol{j} + \left(t + \frac{2}{\sqrt{11}}\right)\boldsymbol{k}\right)dt \\ &= \left(\frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t\right)\boldsymbol{i} - \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)\boldsymbol{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)\boldsymbol{k} + \boldsymbol{C}_2\end{aligned}$$

At time $t = 0$, a particle is located at the point $(1,\ 2,\ 3)$ $\boldsymbol{r}(0) = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$

$$\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k} = 0\boldsymbol{i} - 0\boldsymbol{j} + 0\boldsymbol{k} + \boldsymbol{C}_2$$

$$\boldsymbol{C}_2 = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$$

$$\begin{aligned}\boldsymbol{r} &= \left(\frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t\right)\boldsymbol{i} - \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)\boldsymbol{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)\boldsymbol{k} + \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k} \\ &= \left(\frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t + 1\right)\boldsymbol{i} - \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t - 2\right)\boldsymbol{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t + 3\right)\boldsymbol{k}\end{aligned}$$

Exercise

A projectile is fired at a speed of 840 *m/sec* at an angle of 60°. How long will it take to get 21 *km* downrange?

Solution

$$x = \left(v_0 \cos\alpha\right)t$$

$$21\ km\frac{1000\ m}{1\ km} = \left(840\ (m/s)\ \cos 60°\right)t$$

$$\begin{aligned}t &= \frac{21000}{840\cos 60°} \\ &= 50\ \text{sec}\end{aligned}$$

Exercise

Find the muzzle speed of a gun whose maximum range is 24.5 *km*.

Solution

$$R = \frac{v_0^2}{g}\sin 2\alpha$$

Maximum R occurs when sine equals to 1 $\rightarrow \quad \sin 2\alpha = 1 \quad \Rightarrow \ 2\alpha = 90°$

$$24.5 = \frac{v_0^2}{9.8}\sin 90°$$

$$v_0^2 = (24.5)(9.8)$$

$$v_0 = \sqrt{(24.5)(9.8)}$$

$$= 490 \ \ m/s$$

Exercise

A spring gun at ground level fires a golf ball at an angle of 45°. The ball lands 10 *m* away.

a) What was the ball's initial speed?

b) For the same initial speed, find the two firing angles that make the range 6 *m*.

Solution

a) $R = \frac{v_0^2}{g}\sin 2\alpha$

$$10 = \frac{v_0^2}{9.8}\sin(2\times 45°)$$

$$v_0^2 = \frac{98}{\sin 90°} = 98$$

$$v_0 = \sqrt{98} \approx 9.9 \ \ m/s$$

b) $6 = \frac{98}{9.8}\sin 2\alpha$

$$\sin 2\alpha = 6\left(\frac{9.8}{98}\right) = 0.6$$

$$2\alpha = \sin^{-1}(0.6)$$

$$2\alpha \approx 36.87° \quad or \quad 2\alpha \approx 143.12°$$

$$\boxed{\alpha \approx 18.4°} \quad or \quad \boxed{\alpha \approx 71.6°}$$

Exercise

An electron in a TV tube is beamed horizontally at a speed of $5\times10^{6}\ m/\sec$ toward the face of the tube 40 cm away. About how far will the electron drop before it hits?

Solution

$$v_0 = 5\times10^{6}\ m/\sec, \quad x = 40cm = 0.4\ m$$

$$x = \left(v_0 \cos\alpha\right)t$$

$$0.4 = \left(5\times10^{6}\cos 0^\circ\right)t \qquad \textbf{\textit{Horizontal}}\ \boldsymbol{\alpha = 0^\circ}$$

$$t = \frac{0.4}{5\times10^{6}} = .08\times10^{-6} = 8\times10^{-8}\ \sec$$

$$y = -\tfrac{1}{2}gt^2 + \left(v_0 \sin\alpha\right)t + y_0$$

$$= -\tfrac{1}{2}(9.8)\left(8\times10^{-8}\right)^2 + \left(5\times10^{6}\sin 0^\circ\right)\left(8\times10^{-8}\right) + 0$$

$$= -3.136\times10^{-14}\ m$$

Therefore, the electron drop $3.136\times10^{-12}\ cm$

Exercise

A golf ball is hit with an initial speed of 116 ft/sec at an angle of elevation of 45° from the tee to a green that is elevated 45 ft above the tee. Assuming that the pin, 369 ft downrange, does not get in the way, where will the ball land in relation to the pin?

Solution

$$v_0 = 116\ ft/\sec, \quad \alpha = 45^\circ$$

$$x = \left(v_0 \cos\alpha\right)t$$

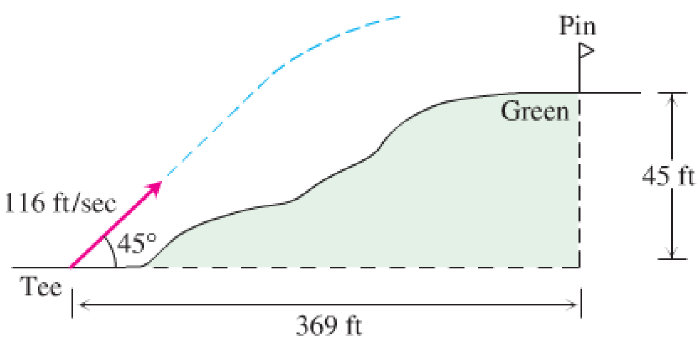
$$369 = \left(116\cos 45^\circ\right)t$$

$$t = \frac{369}{116\cos 45^\circ} \approx \underline{4.5\ \sec}|$$

$$y = -\tfrac{1}{2}gt^2 + \left(v_0 \sin\alpha\right)t + y_0$$

$$= -\tfrac{1}{2}(32)(4.5)^2 + \left(116\sin 45^\circ\right)t$$

$$\approx 45.11\ ft$$



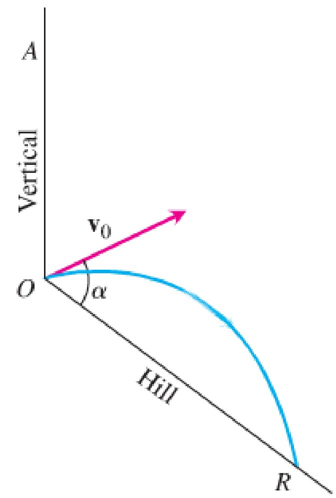
It will take the ball 4.5 sec to travel 369 ft. at the time the ball will be 45.11 ft in the air and will hit the green past the pin.

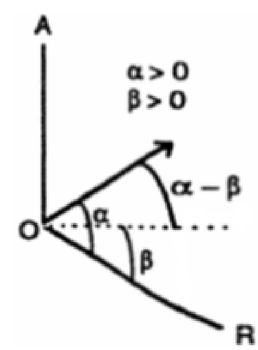
Exercise

An ideal projectile is launched straight down an inclined plane.

a) Show that the greates1 downhill range is achieved when the initial velocity vector bisects angle *AOR*

b) If the projectile were fired uphill instead of down, what launch angle would maximize its range?





Solution

a) $x=\left(v_0\cos(\alpha-\beta)\right)t, \quad y=\left(v_0\sin(\alpha-\beta)\right)t-\frac{1}{2}gt^2$

$$\tan\beta=\frac{y}{x}$$

$$=\frac{\left|\left(v_0\sin(\alpha-\beta)\right)t-\frac{1}{2}gt^2\right|}{\left(v_0\cos(\alpha-\beta)\right)t}$$

$$=\frac{\left|v_0\sin(\alpha-\beta)-\frac{1}{2}gt\right|}{v_0\cos(\alpha-\beta)}$$

$$\frac{1}{2}gt-v_0\sin(\alpha-\beta)=v_0\cos(\alpha-\beta)tan\beta$$

$$\frac{1}{2}gt=v_0\cos(\alpha-\beta)tan\beta+v_0\sin(\alpha-\beta)$$

$$t=\frac{2v_0\left(\cos(\alpha-\beta)tan\beta+\sin(\alpha-\beta)\right)}{g};$$

Which is time when the projectile hits the downhill slope.

$$x=v_0\cos(\alpha-\beta)\frac{2v_0\left(\cos(\alpha-\beta)tan\beta+\sin(\alpha-\beta)\right)}{g}$$

$$=\frac{2v_0^2}{g}\left(\cos^2(\alpha-\beta)tan\beta+\cos(\alpha-\beta)\sin(\alpha-\beta)\right)$$

$$=\frac{2v_0^2}{g}\left(\cos^2(\alpha-\beta)tan\beta+\frac{1}{2}\sin 2(\alpha-\beta)\right)$$

$$\frac{dx}{d\alpha}=\frac{2v_0^2}{g}\left(-2\cos(\alpha-\beta)\sin(\alpha-\beta)tan\beta+\cos 2(\alpha-\beta)\right)=0$$

$$-\sin 2(\alpha-\beta)tan\beta+\cos 2(\alpha-\beta)=0$$

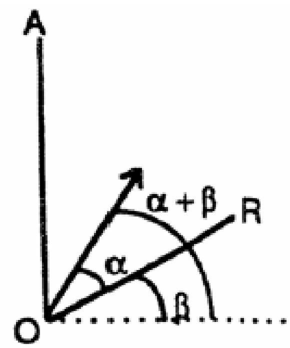
$$\sin 2(\alpha-\beta)tan\beta=\cos 2(\alpha-\beta)$$

$$tan\beta=\cot 2(\alpha-\beta) \quad \Rightarrow \quad 90°-\beta=2(\alpha-\beta)$$

$$\alpha-\beta=45°-\frac{1}{2}\beta$$

$$\boxed{\alpha=\frac{1}{2}(90°+\beta)} \qquad \frac{1}{2}\angle AOR$$

b) $x=\left(v_0\cos(\alpha+\beta)\right)t,\quad y=-\frac{1}{2}gt^2+\left(v_0\sin(\alpha+\beta)\right)t$



$$\tan\beta=\frac{y}{x}$$
$$=\frac{-\frac{1}{2}gt^2+\left(v_0\sin(\alpha+\beta)\right)t}{\left(v_0\cos(\alpha+\beta)\right)t}$$
$$=\frac{-\frac{1}{2}gt+v_0\sin(\alpha+\beta)}{v_0\cos(\alpha+\beta)}$$

$$-\frac{1}{2}gt+v_0\sin(\alpha+\beta)=v_0\cos(\alpha+\beta)tan\beta$$

$$\frac{1}{2}gt=v_0\sin(\alpha+\beta)-v_0\cos(\alpha+\beta)tan\beta$$

$t=\dfrac{2v_0}{g}\left(v_0\sin(\alpha+\beta)-\cos(\alpha+\beta)tan\beta\right)$; which is time when the projectile hits the uphill slope.

$$x=\frac{2v_0^2}{g}\left(\cos(\alpha+\beta)\sin(\alpha+\beta)-\cos^2(\alpha+\beta)tan\beta\right)$$
$$=\frac{2v_0^2}{g}\left(\frac{1}{2}\sin2(\alpha+\beta)-\cos^2(\alpha+\beta)tan\beta\right)$$

$$\frac{dx}{d\alpha}=\frac{2v_0^2}{g}\left(\cos2(\alpha+\beta)+2\cos(\alpha+\beta)\sin(\alpha+\beta)tan\beta\right)=0$$

$$\cos2(\alpha+\beta)+\sin2(\alpha+\beta)tan\beta=0$$

$$\sin2(\alpha+\beta)tan\beta=-\cos2(\alpha+\beta)$$

$$\tan\beta=-\cot2(\alpha+\beta)$$

$$\tan(-\beta)=\cot2(\alpha+\beta)\quad\Rightarrow\quad 90^\circ+\beta=2\alpha+2\beta$$

$\boxed{\alpha=\frac{1}{2}(90^\circ-\beta)}$ $\quad\frac{1}{2}\angle AOR$

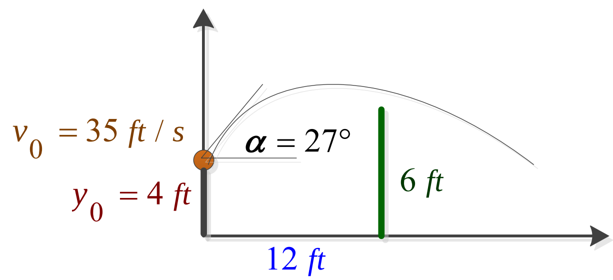
Exercise

A volleyball is hit when it is 4 *ft* above the ground and 12 *ft* from a 6-*ft*-high net. It leaves the point of impact with an initial velocity of 35 *ft/sec* at an angle of 27° and slips by the opposing team untouched.

a) Find a vector equation for the path of the volleyball.
b) How high does the volleyball go, and when does it reach maximum height?
c) Find its range and flight time.
d) When is the volleyball 7 *ft* above the ground? How far (ground distance) is the volleyball from where it will land?
e) Suppose that the net is raised to 8 *ft*. Does this change things? Explain.

Solution

Given: $y_0 = 4\ ft,\quad v_0 = 35\ ft/s,\quad \alpha = 27°$



a) $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$

$$x = (v_0 \cos\alpha)t = (35\cos 27°)t$$

$$y = -\frac{1}{2}gt^2 + (v_0 \sin\alpha)t + y_0$$

$$= -16t^2 + (35\sin 27°)t + 4$$

$$\mathbf{r}(t) = (35\cos 27°)t\ \mathbf{i} + \left(-16t^2 + (35\sin 27°)t + 4\right)\mathbf{j}$$

b) $$y_{max} = \frac{(v_0 \sin\alpha)^2}{2g} + y_0$$

$$= \frac{(35\sin 27°)^2}{2(32)} + 4$$

$$\approx 7.945\ ft$$

$$t = \frac{v_0 \sin\alpha}{g} = \frac{35\sin 27°}{32} \approx 0.497\ \sec$$

c) $y = -16t^2 + (35\sin 27°)t + 4 = 0$ *Solve for* ***t***

$$t = \frac{-35\sin 27° - \sqrt{(35\sin 27°)^2 - 4(-16)(4)}}{2(-16)} \approx 1.201\ \sec$$

Range: $x = (35\cos 27°)(1.201)\ \approx 37.453\ ft$

d) $y = -16t^2 + (35\sin 27°)t + 4 = 7$ *Solve for* ***t***

$$-16t^2 + (35\sin 27°)t - 3 = 0$$

$$t=\frac{-35\sin 27^\circ \pm \sqrt{(-35\sin 27^\circ)^2-4(-16)(-3)}}{2(-16)} \approx \boxed{\begin{cases} 0.7396 \text{ sec} \\ 0.2535 \text{ sec} \end{cases}}$$

$$x(t=0.2535)=(35\cos 27^\circ)(0.2535)\ \boxed{\approx 7.921\ ft}$$

$$x(t=0.74)=(35\cos 27^\circ)(0.74)\ \boxed{\approx 23.077\ ft}$$

e) Since $y_{\max} \approx 7.945\ ft$, the ball won't clear the 8 *ft* net, therefore, Yes, it changes things.

Solution Section 1.7 – Length of Curves

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + \sqrt{5}t\,\mathbf{k}; \quad 0 \le t \le \pi$$

Solution

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = -(2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + \sqrt{5}\,\mathbf{k}$$

$$|\mathbf{v}| = \sqrt{4\sin^2 t + 4\cos^2 t + 5} = \sqrt{4+5} = 3$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -\frac{2\sin t}{3}\mathbf{i} + \frac{2\cos t}{3}\mathbf{j} + \frac{\sqrt{5}}{3}\mathbf{k}$$

$$\text{Length: } s = \int_0^\pi |\mathbf{v}(t)|\,dt = \int_0^\pi 3\,dt = 3t\Big|_0^\pi = 3\pi$$

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\mathbf{r}(t) = t\mathbf{i} + \tfrac{2}{3}t^{3/2}\,\mathbf{k}; \quad 0 \le t \le 8$$

Solution

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \mathbf{i} + t^{1/2}\,\mathbf{k} \qquad \Rightarrow \quad |\mathbf{v}| = \sqrt{1+t}$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{1+t}}\mathbf{i} + \frac{t^{1/2}}{\sqrt{1+t}}\mathbf{k}$$

$$\text{Length: } s = \int_0^8 |\mathbf{v}(t)|\,dt$$

$$= \int_0^8 (1+t)^{1/2}\,dt \qquad = \int_0^8 (1+t)^{1/2}\,d(1+t)$$

$$= \left[\frac{2}{3}(1+t)^{3/2}\right]_0^8$$

$$= \frac{2}{3}\left[(9)^{3/2} - 1\right]$$

$$= \frac{2}{3}(27-1)$$

$$= \frac{52}{3}$$

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\boldsymbol{r}(t)=(2+t)\boldsymbol{i}-(t+1)\boldsymbol{j}+t\,\boldsymbol{k}; \quad 0\le t\le 3$$

Solution

$$\boldsymbol{v}(t)=\frac{d\boldsymbol{r}}{dt}=\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}$$

$$|\boldsymbol{v}|=\sqrt{1+1+1}=\sqrt{3}$$

$$\boldsymbol{T}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}=\frac{1}{\sqrt{3}}\boldsymbol{i}-\frac{1}{\sqrt{3}}\boldsymbol{j}+\frac{1}{\sqrt{3}}\boldsymbol{k}$$

$$\text{Length: } s=\int_0^3 |\boldsymbol{v}(t)|\,dt=\int_0^3 \sqrt{3}\,dt=\sqrt{3}t\Big|_0^3 \underline{=3\sqrt{3}|}$$

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\boldsymbol{r}(t)=\left(\cos^3 t\right)\boldsymbol{i}+\left(\sin^3 t\right)\boldsymbol{k}; \quad 0\le t\le \frac{\pi}{2}$$

Solution

$$\boldsymbol{v}(t)=\frac{d\boldsymbol{r}}{dt}=-\left(3\cos^2 t\sin t\right)\boldsymbol{i}+\left(3\sin^2 t\cos t\right)\boldsymbol{k}$$

$$\begin{aligned}|\boldsymbol{v}|&=\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t}\\&=3\sqrt{\cos^2 t\sin^2 t\left(\cos^2 t+\sin^2 t\right)}\\&=3\sqrt{\cos^2 t\sin^2 t}\\&=\underline{3|\cos t\sin t||}\end{aligned}$$

$$\boldsymbol{T}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}=-\left(\frac{3\cos^2 t\sin t}{3|\cos t\sin t|}\right)\boldsymbol{i}+\left(\frac{3\sin^2 t\cos t}{3|\cos t\sin t|}\right)\boldsymbol{k}=\underline{-(\cos t)\boldsymbol{i}+(\sin t)\boldsymbol{k}|}$$

$$\begin{aligned}\text{Length: } s&=\int_0^{\pi/2}|\boldsymbol{v}(t)|\,dt\\&=\int_0^{\pi/2}3\cos t\sin t\,dt \qquad\qquad \sin 2t=2\cos t\sin t\\&=\frac{3}{2}\int_0^{\pi/2}\sin 2t\,dt\end{aligned}$$

$$= \frac{3}{2}\left[-\frac{1}{2}\cos 2t\right]_0^{\pi/2}$$

$$= -\frac{3}{4}(\cos\pi - \cos 0)$$

$$= -\frac{3}{4}(-2)$$

$$= \frac{3}{2}$$

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\boldsymbol{r}(t) = (t\cos t)\boldsymbol{i} + (t\sin t)\boldsymbol{j} + \left(\frac{2\sqrt{2}}{3}t^{3/2}\right)\boldsymbol{k}; \quad 0 \le t \le \pi$$

Solution

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}}{dt} = (\cos t - t\sin t)\boldsymbol{i} + (\sin t + t\cos t)\boldsymbol{j} + \left(\sqrt{2}\ t^{1/2}\right)\boldsymbol{k}$$

$$|\boldsymbol{v}(t)| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 2t}$$

$$= \sqrt{\cos^2 t - 2t\cos t + \sin^2 t + \sin^2 t + 2t\cos t + \cos^2 t + 2t}$$

$$= \sqrt{2 + 2t}$$

$$\boldsymbol{T} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{1}{\sqrt{2}}\left(\frac{\cos t - t\sin t}{\sqrt{1+t}}\right)\boldsymbol{i} + \frac{1}{\sqrt{2}}\left(\frac{\cos t + t\sin t}{\sqrt{1+t}}\right)\boldsymbol{j} + \left(\sqrt{\frac{t}{1+t}}\ t^{1/2}\right)\boldsymbol{k}$$

$$\text{Length: } s = \int_0^{\pi} |\boldsymbol{v}(t)|\ dt$$

$$= \int_0^{\pi} \sqrt{2}\sqrt{1+t}\ dt$$

$$= \sqrt{2}\int_0^{\pi} \sqrt{1+t}\ d(1+t)$$

$$= \sqrt{2}\int_0^{\pi} \sqrt{1+t}\ d(1+t)$$

$$= \sqrt{2}\left(\sqrt{1+\pi} - 1\right)$$

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$\boldsymbol{r}(t) = (t\sin t + \cos t)\boldsymbol{i} + (t\cos t - \sin t)\boldsymbol{j}; \quad \sqrt{2} \le t \le 2$

Solution

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}}{dt} = (\sin t + t\cos t - \sin t)\boldsymbol{i} + (\cos t - t\sin t - \cos t)\boldsymbol{j}$$

$$= (t\cos t)\boldsymbol{i} - (t\sin t)\boldsymbol{j}$$

$$|\boldsymbol{v}| = \sqrt{t^2\cos^2 t + t^2\sin^2 t}$$

$$= \sqrt{t^2\left(\cos^2 t + \sin^2 t\right)}$$

$$= \sqrt{t^2}$$

$$= |t|$$

$$= t \quad \textit{because} \ \sqrt{2} \le t \le 2$$

$$\boldsymbol{T} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \left(\frac{t\cos t}{t}\right)\boldsymbol{i} - \left(\frac{t\sin t}{t}\right)\boldsymbol{j}$$

$$= (\cos t)\boldsymbol{i} - (\sin t)\boldsymbol{j}$$

$$\text{Length: } s = \int_{\sqrt{2}}^{2} |\boldsymbol{v}(t)| \, dt$$

$$= \int_{\sqrt{2}}^{2} t \, dt$$

$$= \frac{1}{2}t^2 \Big|_{\sqrt{2}}^{2}$$

$$= \frac{1}{2}(4-2)$$

$$= 1$$

Exercise

Find the point on the curve $\boldsymbol{r}(t)=(5\sin t)\boldsymbol{i}+(5\cos t)\boldsymbol{j}+12t\,\boldsymbol{k}$ at a distance 26π units along the curve from the point (0, 5, 0) in the direction of increasing arc length.

Solution

$$\boldsymbol{v}=(5\cos t)\boldsymbol{i}-(5\sin t)\boldsymbol{j}+12\,\boldsymbol{k}$$

$$\begin{aligned}|\boldsymbol{v}| &= \sqrt{25\cos^2 t+25\sin^2 t+144}\\ &= \sqrt{25\left(\cos^2 t+\sin^2 t\right)+144}\\ &= \sqrt{25+144}\\ &= \sqrt{169}\\ &= 13\end{aligned}$$

$$s=\int_0^{t_0}|\boldsymbol{v}(t)|\,dt=\int_0^{t_0}13\,dt=13t_0$$

$$s=26\pi=13t_0$$

$$\boxed{t_0=2\pi}$$

$$\boldsymbol{r}(t=2\pi)=(5\sin 2\pi)\boldsymbol{i}+(5\cos 2\pi)\boldsymbol{j}+12(2\pi)\,\boldsymbol{k}=0\boldsymbol{i}+5\boldsymbol{j}+24\pi\,\boldsymbol{k}$$

The point is: $(0,\ 5,\ 24\pi)$

Exercise

Find the arc length parameter along the curve from the point where $t=0$. Also, find the length of the indicated portion of the curve. $\boldsymbol{r}(t)=(4\cos t)\boldsymbol{i}+(4\sin t)\boldsymbol{j}+3t\,\boldsymbol{k}$; $0\le t\le\frac{\pi}{2}$

Solution

$$\boldsymbol{v}=-(4\sin t)\boldsymbol{i}+(4\cos t)\boldsymbol{j}+3\,\boldsymbol{k}$$

$$|\boldsymbol{v}|=\sqrt{16\sin^2 t+16\cos^2 t+9}=\sqrt{16+9}=5$$

$$s=\int_0^{t}|\boldsymbol{v}(\tau)|\,d\tau=\int_0^{t}5\,dt=5t$$

$$s\left(\frac{\pi}{2}\right)=\frac{5\pi}{2}$$

Exercise

Find the arc length parameter along the curve from the point where $t = 0$. Also, find the length of the indicated portion of the curve. $\boldsymbol{r}(t) = \left(e^t \cos t\right)\boldsymbol{i} + \left(e^t \sin t\right)\boldsymbol{j} + e^t\ \boldsymbol{k}; \quad -\ln 4 \le t \le 0$

Solution

$$\boldsymbol{v}(t) = \left(e^t \cos t - e^t \sin t\right)\boldsymbol{i} + \left(e^t \sin t + e^t \cos t\right)\boldsymbol{j} + e^t\ \boldsymbol{k}$$

$$\begin{aligned}
|\boldsymbol{v}| &= \sqrt{\left(e^t \cos t - e^t \sin t\right)^2 + \left(e^t \sin t + e^t \cos t\right)^2 + e^{2t}} \\
&= \sqrt{e^{2t}\cos^2 t - e^{2t}\cos t \sin t + e^{2t}\sin^2 t + e^{2t}\sin^2 t + e^{2t}\cos t \sin t + e^{2t}\cos^2 t + e^{2t}} \\
&= \sqrt{2e^{2t}\left(\cos^2 t + \sin^2 t\right) + e^{2t}} \\
&= \sqrt{3e^{2t}} \\
&= \sqrt{3}\ e^t
\end{aligned}$$

$$\begin{aligned}
s(t) &= \int_0^t |\boldsymbol{v}(\tau)|\ d\tau \\
&= \int_0^t \sqrt{3}\ e^\tau\ d\tau \\
&= \sqrt{3}\ e^\tau \Big|_0^t \\
&= \sqrt{3}\ \left(e^t - 1\right)
\end{aligned}$$

$$s(0) = \sqrt{3}\ \left(e^0 - 1\right) = 0$$

$$\begin{aligned}
s(-\ln 4) &= \sqrt{3}\ \left(e^{-\ln 4} - 1\right) \\
&= \sqrt{3}\left(e^{\ln\frac{1}{4}} - 1\right) \\
&= \sqrt{3}\left(\tfrac{1}{4} - 1\right) \\
&= \sqrt{3}\left(-\tfrac{3}{4}\right) \\
&= \frac{3\sqrt{3}}{4}
\end{aligned}$$

$$s(-\ln 4) - s(0)\ \boxed{= \frac{3\sqrt{3}}{4}}$$

Exercise

Find the arc length parameter along the curve from the point where $t = 0$. Also, find the length of the indicated portion of the curve. $\boldsymbol{r}(t) = (1+2t)\boldsymbol{i} + (1+3t)\boldsymbol{j} + (6-6t)\boldsymbol{k}; \quad -1 \le t \le 0$

Solution

$$\boldsymbol{v} = 2\boldsymbol{i} + 3\boldsymbol{j} - 6\boldsymbol{k}$$

$$|\boldsymbol{v}| = \sqrt{4+9+36} = \underline{7}$$

$$s(t) = \int_0^t |\boldsymbol{v}(\tau)|\, d\tau = \int_0^t 7\, d\tau = 7t$$

$$\text{Length: } s(0) - s(-1) = 0 - (-7) \; \underline{= 7}$$

Exercise

If a string wound around a fixed circle in unwound while held taut in the plane of the circle, its end P traces an involute of the circle. The circle in question is the circle $x^2 + y^2 = 1$ and the tracing point starts at (1, 0). The unwound portion of the string is tangent to the circle at Q, and t is the radian measure of the angle from the position x-axis to segment OQ. Derive the parametric equations

$x = \cos t + t \sin t, \quad y = \sin t - t \cos t, \quad t > 0$ of the point $P(x, y)$ for the involute.

Solution

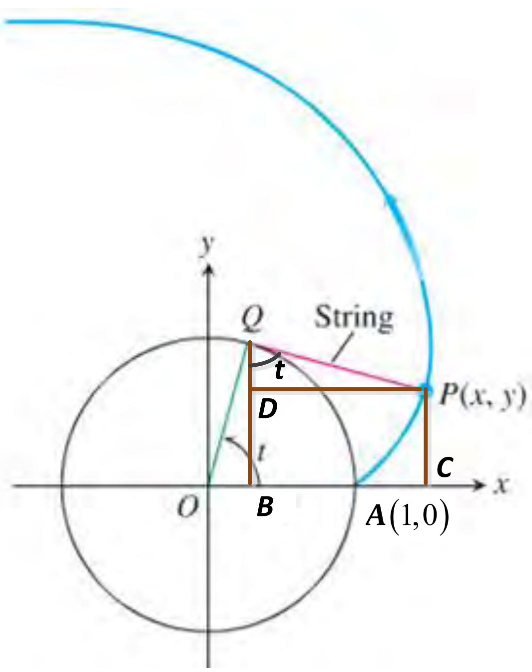
$$\angle PQB = \angle QOB = t$$

$$PQ = arc(AQ) = t$$

PQ = Length of the unwound string

$arc(AQ)$

$$\Delta PDQ: \begin{cases} \sin t = \dfrac{DP}{PQ} = \dfrac{DP}{t} \to \underline{DT = t \sin t} \\ \cos t = \dfrac{QD}{PQ} = \dfrac{QD}{t} \to \underline{QD = t \cos t} \end{cases}$$



$$\begin{aligned} x &= OB + BC \\ &= OB + DP \\ &\underline{= \cos t + t \sin t} \end{aligned}$$

$$\begin{aligned} y &= PC \\ &= QB - QD \\ &\underline{= \sin t - t \cos t} \end{aligned}$$

Solution Section 1.8 – Curvature and Normal Vectors

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the plane curves: $\boldsymbol{r}(t) = t\boldsymbol{i} + (\ln \cos t)\boldsymbol{j}, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$

Solution

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}}{dt} = \boldsymbol{i} - \frac{\sin t}{\cos t}\boldsymbol{j} = \boldsymbol{i} - \tan t\, \boldsymbol{j}$$

$$\begin{aligned}|\boldsymbol{v}| &= \sqrt{1 + \tan^2 t}\\ &= \sqrt{\sec^2 t}\\ &= \sec t\end{aligned}$$

$$\begin{aligned}\boldsymbol{T} &= \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{1}{\sec t}\boldsymbol{i} - \frac{\tan t}{\sec t}\boldsymbol{j}\\ &= \cos t\boldsymbol{i} - \cos t\frac{\sin t}{\cos t}\boldsymbol{j}\\ &= \cos t\, \boldsymbol{i} - \sin t\, \boldsymbol{j}\end{aligned}$$

$$\frac{d\boldsymbol{T}}{dt} = -(\sin t)\boldsymbol{i} - (\cos t)\boldsymbol{j}$$

$$\left|\frac{d\boldsymbol{T}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$\begin{aligned}\boldsymbol{N} &= \frac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|} = \frac{-(\sin t)\boldsymbol{i} - (\cos t)\boldsymbol{j}}{1}\\ &= -(\sin t)\boldsymbol{i} - (\cos t)\boldsymbol{j}\end{aligned}$$

$$\begin{aligned}\kappa &= \frac{1}{|v|}\left|\frac{d\boldsymbol{T}}{dt}\right|\\ &= \frac{1}{\sec t}(1)\\ &= \cos t\end{aligned}$$

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the plane curves: $\boldsymbol{r}(t)=(\ln\sec t)\boldsymbol{i}+t\boldsymbol{j}, \quad -\frac{\pi}{2}<t<\frac{\pi}{2}$

Solution

$$\boldsymbol{v}(t)=\frac{d\boldsymbol{r}}{dt}=\frac{\sec t\tan t}{\sec t}\boldsymbol{i}+\boldsymbol{j}=\tan t\ \boldsymbol{i}+\boldsymbol{j}$$

$$|\boldsymbol{v}|=\sqrt{\tan^2 t+1}=\sqrt{\sec^2 t}=\sec t$$

$$\boldsymbol{T}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}=\frac{\tan t}{\sec t}\boldsymbol{i}+\frac{1}{\sec t}\boldsymbol{j}=\underline{(\sin t)\boldsymbol{i}+(\cos t)\boldsymbol{j}}$$

$$\frac{d\boldsymbol{T}}{dt}=(\cos t)\boldsymbol{i}-(\sin t)\boldsymbol{j}$$

$$\left|\frac{d\boldsymbol{T}}{dt}\right|=\sqrt{\cos^2 t+\sin^2 t}=1$$

$$\boldsymbol{N}=\frac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|}=\underline{(\sin t)\boldsymbol{i}+(\cos t)\boldsymbol{j}}$$

$$\kappa=\frac{1}{|v|}\left|\frac{d\boldsymbol{T}}{dt}\right|=\frac{1}{\sec t}(1)\ \underline{=\cos t}$$

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the plane curves: $\boldsymbol{r}(t)=(2t+3)\boldsymbol{i}+\left(5-t^2\right)\boldsymbol{j}$

Solution

$$\boldsymbol{v}(t)=\frac{d\boldsymbol{r}}{dt}=2\boldsymbol{i}-2t\boldsymbol{j}$$

$$|\boldsymbol{v}|=\sqrt{4+4t^2}=\underline{2\sqrt{1+t^2}}$$

$$\boldsymbol{T}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}=\frac{2}{2\sqrt{1+t^2}}\boldsymbol{i}-\frac{2t}{2\sqrt{1+t^2}}\boldsymbol{j}$$

$$\underline{=\frac{1}{\sqrt{1+t^2}}\boldsymbol{i}-\frac{t}{\sqrt{1+t^2}}\boldsymbol{j}}$$

$$\frac{d\boldsymbol{T}}{dt}=\frac{-2t}{2\left(1+t^2\right)^{3/2}}\boldsymbol{i}-\frac{\left(1+t^2\right)^{1/2}-\frac{1}{2}\left(1+t^2\right)^{-1/2}(2t)t}{\left(1+t^2\right)}\boldsymbol{j}$$

$$=\frac{-t}{\left(1+t^2\right)^{3/2}}\boldsymbol{i}-\frac{1+t^2-t^2}{\left(1+t^2\right)^{3/2}}\boldsymbol{j}$$

$$= \frac{-t}{\left(1+t^2\right)^{3/2}}\boldsymbol{i} - \frac{1}{\left(1+t^2\right)^{3/2}}\boldsymbol{j}$$

$$\left|\frac{d\boldsymbol{T}}{dt}\right| = \sqrt{\frac{t^2}{\left(1+t^2\right)^3} + \frac{1}{\left(1+t^2\right)^3}}$$

$$= \sqrt{\frac{t^2+1}{\left(1+t^2\right)^3}}$$

$$= \sqrt{\frac{1}{\left(1+t^2\right)^2}}$$

$$= \frac{1}{\left(1+t^2\right)}$$

$$\boldsymbol{N} = \frac{d\boldsymbol{T}/dt}{\left|d\boldsymbol{T}/dt\right|}$$

$$= \left(1+t^2\right)\left(\frac{-t}{\left(1+t^2\right)^{3/2}}\boldsymbol{i} - \frac{1}{\left(1+t^2\right)^{3/2}}\boldsymbol{j}\right)$$

$$= \frac{-t}{\sqrt{1+t^2}}\boldsymbol{i} - \frac{1}{\sqrt{1+t^2}}\boldsymbol{j}$$

$$\kappa = \frac{1}{|v|}\left|\frac{d\boldsymbol{T}}{dt}\right|$$

$$= \frac{1}{2\sqrt{1+t^2}}\frac{1}{\left(1+t^2\right)}$$

$$= \frac{1}{2\left(1+t^2\right)^{3/2}}$$

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the plane curves: $\boldsymbol{r}(t)=(\cos t+t\sin t)\boldsymbol{i}+(\sin t-t\cos t)\boldsymbol{j}, \quad t>0$

Solution

$$\boldsymbol{v}=\frac{d\boldsymbol{r}}{dt}=(-\sin t+\sin t+t\cos t)\boldsymbol{i}+(\cos t-\cos t+t\sin t)\boldsymbol{j}=(t\cos t)\boldsymbol{i}+(t\sin t)\boldsymbol{j}$$

$$|\boldsymbol{v}|=\sqrt{t^2\cos^2 t+t^2\sin^2 t}=\sqrt{t^2\left(\cos^2 t+\sin^2 t\right)}=\sqrt{t^2}=|t|=t$$

$$\boldsymbol{T}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}=\left(\frac{t\cos t}{t}\right)\boldsymbol{i}+\left(\frac{t\sin t}{t}\right)\boldsymbol{j}=\underline{(\cos t)\boldsymbol{i}+(\sin t)\boldsymbol{j}}$$

$$\frac{d\boldsymbol{T}}{dt}=(-\sin t)\boldsymbol{i}+(\cos t)\boldsymbol{j}$$

$$\left|\frac{d\boldsymbol{T}}{dt}\right|=\sqrt{\sin^2 t+\cos^2 t}=1$$

$$\boldsymbol{N}=\frac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|}=\underline{(-\sin t)\boldsymbol{i}+(\cos t)\boldsymbol{j}}$$

$$\kappa=\frac{1}{|\boldsymbol{v}|}\left|\frac{d\boldsymbol{T}}{dt}\right|=\frac{1}{t}(1)\ \underline{=\frac{1}{t}}$$

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the space curves: $\boldsymbol{r}(t)=(3\sin t)\boldsymbol{i}+(3\cos t)\boldsymbol{j}+4t\boldsymbol{k}$

Solution

$$\boldsymbol{v}=\frac{d\boldsymbol{r}}{dt}=(3\cos t)\boldsymbol{i}-(3\sin t)\boldsymbol{j}+4\,\boldsymbol{k}$$

$$|\boldsymbol{v}|=\sqrt{9\cos^2 t+9\sin^2 t+16}=\sqrt{9+16}=\underline{5}$$

$$\boldsymbol{T}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}=\underline{\frac{3}{5}\cos t\ \boldsymbol{i}-\frac{3}{5}\sin t\ \boldsymbol{j}+\frac{4}{5}\ \boldsymbol{k}}$$

$$\frac{d\boldsymbol{T}}{dt}=-\frac{3}{5}\sin t\ \boldsymbol{i}-\frac{3}{5}\cos t\ \boldsymbol{j}$$

$$\left|\frac{d\boldsymbol{T}}{dt}\right|=\sqrt{\frac{9}{25}\sin^2 t+\frac{9}{25}\cos^2 t}=\sqrt{\frac{9}{25}}=\frac{3}{5}$$

$$\boldsymbol{N}=\frac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|}=\frac{5}{3}\left(-\frac{3}{5}\sin t\ \boldsymbol{i}-\frac{3}{5}\cos t\ \boldsymbol{j}\right)=\underline{(-\sin t)\boldsymbol{i}-(\cos t)\boldsymbol{j}}$$

$$\kappa=\frac{1}{|v|}\left|\frac{d\boldsymbol{T}}{dt}\right|=\frac{1}{5}\frac{3}{5}\ \underline{=\frac{3}{25}}$$

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the space curves: $\boldsymbol{r}(t)=\left(e^t\cos t\right)\boldsymbol{i}+\left(e^t\sin t\right)\boldsymbol{j}+2\boldsymbol{k}$

Solution

$$\boldsymbol{v}(t)=\left(e^t\cos t-e^t\sin t\right)\boldsymbol{i}+\left(e^t\sin t+e^t\cos t\right)\boldsymbol{j}$$

$$\begin{aligned}|\boldsymbol{v}|&=\sqrt{\left(e^t\cos t-e^t\sin t\right)^2+\left(e^t\sin t+e^t\cos t\right)^2}\\&=\sqrt{e^{2t}\cos^2 t-e^{2t}\cos t\sin t+e^{2t}\sin^2 t+e^{2t}\sin^2 t+e^{2t}\cos t\sin t+e^{2t}\cos^2 t}\\&=\sqrt{2e^{2t}\left(\cos^2 t+\sin^2 t\right)}\\&=e^t\sqrt{2}\end{aligned}$$

$$\boldsymbol{T}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}=\left(\frac{e^t\cos t-e^t\sin t}{\sqrt{2}e^t}\right)\boldsymbol{i}+\left(\frac{e^t\sin t+e^t\cos t}{\sqrt{2}e^t}\right)\boldsymbol{j}$$

$$=\left(\frac{\cos t-\sin t}{\sqrt{2}}\right)\boldsymbol{i}+\left(\frac{\sin t+\cos t}{\sqrt{2}}\right)\boldsymbol{j}$$

$$\frac{d\boldsymbol{T}}{dt}=\left(\frac{-\sin t-\cos t}{\sqrt{2}}\right)\boldsymbol{i}+\left(\frac{\cos t-\sin t}{\sqrt{2}}\right)\boldsymbol{j}$$

$$\begin{aligned}\left|\frac{d\boldsymbol{T}}{dt}\right|&=\sqrt{\frac{(-\sin t-\cos t)^2}{2}+\frac{(\cos t-\sin t)^2}{2}}\\&=\frac{1}{\sqrt{2}}\sqrt{\sin^2 t+2\sin t\cos t+\cos^2 t+\sin^2 t-2\sin t\cos t+\cos^2 t}\\&=\frac{1}{\sqrt{2}}\sqrt{2\sin^2 t+2\cos^2 t}\\&=\frac{1}{\sqrt{2}}\sqrt{2}\\&=1\end{aligned}$$

$$\boldsymbol{N}=\frac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|}=\left(\frac{-\sin t-\cos t}{\sqrt{2}}\right)\boldsymbol{i}+\left(\frac{\cos t-\sin t}{\sqrt{2}}\right)\boldsymbol{j}$$

$$\kappa=\frac{1}{|v|}\left|\frac{d\boldsymbol{T}}{dt}\right|=\frac{1}{\sqrt{2}e^t}(1)=\frac{1}{\sqrt{2}e^t}$$

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the space curves: $\boldsymbol{r}(t)=\frac{t^3}{3}\boldsymbol{i}+\frac{t^2}{2}\boldsymbol{j}, \quad t>0$

Solution

$$\boldsymbol{v}=\frac{d\boldsymbol{r}}{dt}=\left(t^2\right)\boldsymbol{i}+(t)\boldsymbol{j}$$

$$|\boldsymbol{v}|=\sqrt{t^4+t^2}=|t|\sqrt{t^2+1}=\boxed{t\sqrt{t^2+1}} \quad (t>0)$$

$$\boldsymbol{T}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}=\left(\frac{t^2}{t\sqrt{t^2+1}}\right)\boldsymbol{i}+\left(\frac{t}{t\sqrt{t^2+1}}\right)\boldsymbol{j}=\boxed{\left(\frac{t}{\sqrt{t^2+1}}\right)\boldsymbol{i}+\left(\frac{1}{\sqrt{t^2+1}}\right)\boldsymbol{j}}$$

$$\frac{d\boldsymbol{T}}{dt}=\frac{\left(1+t^2\right)^{1/2}-\frac{1}{2}\left(1+t^2\right)^{-1/2}(2t)t}{\left(1+t^2\right)}\boldsymbol{i}+\frac{-2t}{2\left(1+t^2\right)^{3/2}}\boldsymbol{j}$$

$$=\frac{1+t^2-t^2}{\left(1+t^2\right)^{3/2}}\boldsymbol{i}-\frac{t}{\left(1+t^2\right)^{3/2}}\boldsymbol{j}$$

$$=\frac{1}{\left(1+t^2\right)^{3/2}}\boldsymbol{i}-\frac{t}{\left(1+t^2\right)^{3/2}}\boldsymbol{j}$$

$$\left|\frac{d\boldsymbol{T}}{dt}\right|=\sqrt{\frac{1}{\left(1+t^2\right)^3}+\frac{t^2}{\left(1+t^2\right)^3}}$$

$$=\sqrt{\frac{t^2+1}{\left(1+t^2\right)^3}}$$

$$=\sqrt{\frac{1}{\left(1+t^2\right)^2}}$$

$$=\frac{1}{1+t^2}$$

$$\boldsymbol{N}=\frac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|}=\left(1+t^2\right)\left(\frac{1}{\left(1+t^2\right)^{3/2}}\boldsymbol{i}-\frac{t}{\left(1+t^2\right)^{3/2}}\boldsymbol{j}\right)=\boxed{\frac{1}{\sqrt{1+t^2}}\boldsymbol{i}-\frac{t}{\sqrt{1+t^2}}\boldsymbol{j}}$$

$$\kappa=\frac{1}{|v|}\left|\frac{d\boldsymbol{T}}{dt}\right|=\frac{1}{t\sqrt{t^2+1}}\frac{1}{1+t^2}=\boxed{\frac{t}{t\left(t^2+1\right)^{3/2}}}$$

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the space curves: $\boldsymbol{r}(t)=\left(\cos^3 t\right)\boldsymbol{i}+\left(\sin^3 t\right)\boldsymbol{j}, \quad 0<t<\frac{\pi}{2}$

Solution

$$\boldsymbol{v}(t)=\frac{d\boldsymbol{r}}{dt}=-\left(3\cos^2 t\sin t\right)\boldsymbol{i}+\left(3\sin^2 t\cos t\right)\boldsymbol{j}$$

$$|\boldsymbol{v}|=\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t}$$

$$=3\sqrt{\cos^2 t\sin^2 t\left(\cos^2 t+\sin^2 t\right)}$$

$$=3\sqrt{\cos^2 t\sin^2 t}$$

$$=3|\cos t\sin t|$$

$$=3\cos t\sin t$$

$$\boldsymbol{T}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}=-\left(\frac{3\cos^2 t\sin t}{3|\cos t\sin t|}\right)\boldsymbol{i}+\left(\frac{3\sin^2 t\cos t}{3|\cos t\sin t|}\right)\boldsymbol{j}=-(\cos t)\boldsymbol{i}+(\sin t)\boldsymbol{j}$$

$$\frac{d\boldsymbol{T}}{dt}=(\sin t)\boldsymbol{i}+(\cos t)\boldsymbol{j}$$

$$\left|\frac{d\boldsymbol{T}}{dt}\right|=\sqrt{\sin^2 t+\cos^2 t}=1$$

$$\boldsymbol{N}=\frac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|}=(\sin t)\boldsymbol{i}+(\cos t)\boldsymbol{j}$$

$$\kappa=\frac{1}{|v|}\left|\frac{d\boldsymbol{T}}{dt}\right|=\frac{1}{3\cos t\sin t}(1)\ =\frac{t}{3\cos t\sin t}$$

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the space curves: $\boldsymbol{r}(t)=(\cosh t)\boldsymbol{i}-(\sinh t)\boldsymbol{j}+t\boldsymbol{k}$

Solution

$$\boldsymbol{v}=\frac{d\boldsymbol{r}}{dt}=(\sinh t)\boldsymbol{i}-(\cosh t)\boldsymbol{j}+\boldsymbol{k}$$

$$|\boldsymbol{v}|=\sqrt{\sinh^2 t+\cosh^2 t+1} \qquad \cosh^2 t-\sinh^2 t=1 \quad\Rightarrow\quad \cosh^2 t=1+\sinh^2 t$$

$$=\sqrt{\cosh^2 t+\cosh^2 t}$$

$$=\sqrt{2}\cosh t$$

$$\boldsymbol{T} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \left(\frac{1}{\sqrt{2}\cosh t}\sinh t\right)\boldsymbol{i} - \left(\frac{1}{\sqrt{2}\cosh t}\cosh t\right)\boldsymbol{j} + \frac{1}{\sqrt{2}\cosh t}\boldsymbol{k}$$

$$\boxed{= \left(\frac{1}{\sqrt{2}}\tanh t\right)\boldsymbol{i} - \frac{1}{\sqrt{2}}\boldsymbol{j} + \left(\frac{1}{\sqrt{2}}\operatorname{sech} t\right)\boldsymbol{k}}$$

$$\frac{d\boldsymbol{T}}{dt} = \left(\frac{1}{\sqrt{2}}\operatorname{sech}^2 t\right)\boldsymbol{i} - \left(\frac{1}{\sqrt{2}}\operatorname{sech} t \tanh t\right)\boldsymbol{k}$$

$$\begin{aligned}\left|\frac{d\boldsymbol{T}}{dt}\right| &= \sqrt{\frac{1}{2}\operatorname{sech}^4 t + \frac{1}{2}\operatorname{sech}^2 t \tanh^2 t} \\ &= \frac{1}{\sqrt{2}}\operatorname{sech} t\sqrt{\operatorname{sech}^2 t + \tanh^2 t} \\ &= \frac{1}{\sqrt{2}}\operatorname{sech} t\sqrt{1} \\ &= \boxed{\frac{1}{\sqrt{2}}\operatorname{sech} t}\end{aligned}$$

$$\boldsymbol{N} = \frac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|} = \frac{\sqrt{2}}{\operatorname{sech} t}\left(\left(\frac{1}{\sqrt{2}}\operatorname{sech}^2 t\right)\boldsymbol{i} - \left(\frac{1}{\sqrt{2}}\operatorname{sech} t \tanh t\right)\boldsymbol{k}\right)$$

$$\boxed{= (\operatorname{sech} t)\boldsymbol{i} - (\tanh t)\boldsymbol{k}}$$

$$\kappa = \frac{1}{|v|}\left|\frac{d\boldsymbol{T}}{dt}\right| = \frac{1}{\sqrt{2}\cosh t}\left(\frac{1}{\sqrt{2}}\operatorname{sech} t\right)$$

$$\boxed{= \frac{1}{2}\operatorname{sech}^2 t}$$

Exercise

Find an equation for the circle of curvature of the curve $\boldsymbol{r}(t) = t\boldsymbol{i} + (\sin t)\boldsymbol{j}$, at the point $\left(\frac{\pi}{2},\ 1\right)$. (The curve parametrizes the graph $y = \sin x$ in the *xy*-plane.)

Solution

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \boldsymbol{i} + (\cos t)\boldsymbol{j}$$

$$|\boldsymbol{v}| = \sqrt{1 + \cos^2 t}$$

$$\left|\boldsymbol{v}\left(\frac{\pi}{2}\right)\right| = \sqrt{1 + \cos^2\left(\frac{\pi}{2}\right)} = \sqrt{1 + 0} = 1$$

$$\boldsymbol{T} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{1}{\sqrt{1 + \cos^2 t}}\boldsymbol{i} + \left(\frac{\cos t}{\sqrt{1 + \cos^2 t}}\right)\boldsymbol{j}$$

$$\frac{d\boldsymbol{T}}{dt} = -\frac{1}{2}\frac{2\cos t(-\sin t)}{\left(1+\cos^2 t\right)^{3/2}}\boldsymbol{i} + \frac{-\sin t\left(1+\cos^2 t\right)^{1/2} - \cos t\left(\left(\frac{1}{2}\right)2\cos t(-\sin t)\right)\left(1+\cos^2 t\right)^{-1/2}}{\left(1+\cos^2 t\right)}\boldsymbol{j}$$

$$= \frac{\cos t\sin t}{\left(1+\cos^2 t\right)^{3/2}}\boldsymbol{i} + \frac{-\sin t\left(1+\cos^2 t\right)+\sin t\cos^2 t}{\left(1+\cos^2 t\right)^{3/2}}\boldsymbol{j}$$

$$= \frac{\cos t\sin t}{\left(1+\cos^2 t\right)^{3/2}}\boldsymbol{i} + \frac{-\sin t\left(1+\cos^2 t-\cos^2 t\right)}{\left(1+\cos^2 t\right)^{3/2}}\boldsymbol{j}$$

$$= \frac{\sin t\cos t}{\left(1+\cos^2 t\right)^{3/2}}\boldsymbol{i} - \frac{\sin t}{\left(1+\cos^2 t\right)^{3/2}}\boldsymbol{j}$$

$$\left|\frac{d\boldsymbol{T}}{dt}\right| = \left|\frac{(\sin t\cos t)\boldsymbol{i}-(\sin t)\boldsymbol{j}}{\left(1+\cos^2 t\right)^{3/2}}\right|$$

$$= \frac{\sqrt{\sin^2 t\cos^2 t+\sin^2 t}}{\left(1+\cos^2 t\right)^{3/2}}$$

$$= \frac{|\sin t|\sqrt{\cos^2 t+1}}{\left(1+\cos^2 t\right)^{3/2}}$$

$$= \frac{|\sin t|}{1+\cos^2 t}$$

$$\left|\frac{d\boldsymbol{T}}{dt}\right|\Bigg|_{t=\frac{\pi}{2}} = \frac{\left|\sin\frac{\pi}{2}\right|}{1+\cos^2\frac{\pi}{2}} = 1$$

$$\kappa = \frac{1}{|v|}\left|\frac{d\boldsymbol{T}}{dt}\right| = \frac{1}{1}(1) = 1$$

The radius of curvature is: $\rho = \frac{1}{\kappa} = \frac{1}{1} = 1$

The center of the circle is $\left(\frac{\pi}{2},\ 0\right)$

The equation of the osculating circle is: $\left(x-\frac{\pi}{2}\right)^2 + y^2 = 1$

Exercise

Write $\boldsymbol{a}$ of the motion $\boldsymbol{a} = a_T \boldsymbol{T} + a_N \boldsymbol{N}$ without finding $\boldsymbol{T}$ and $\boldsymbol{N}$. $\boldsymbol{r}(t) = (a\cos t)\boldsymbol{i} + (a\sin t)\boldsymbol{j} + bt\boldsymbol{k}$

Solution

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = (-a\sin t)\boldsymbol{i} + (a\cos t)\boldsymbol{j} + (b)\boldsymbol{k}$$

$$|\boldsymbol{v}| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} = \sqrt{a^2 + b^2}$$

$$a_T = \frac{d}{dt}|\boldsymbol{v}| = 0$$

$$\boldsymbol{a} = \boldsymbol{v}' = (-a\cos t)\boldsymbol{i} - (a\sin t)\boldsymbol{j}$$

$$|\boldsymbol{a}| = \sqrt{a^2 \cos^2 t + a^2 \sin^2 t} = |a|$$

$$a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2} = \sqrt{a^2 + 0} = |a|$$

$$\boldsymbol{a} = a_T \boldsymbol{T} + a_N \boldsymbol{N} = (0)\boldsymbol{T} + |a|\boldsymbol{N} \ \underline{= |a|\boldsymbol{N}|}$$

Exercise

Write $\boldsymbol{a}$ of the motion $\boldsymbol{a} = a_T \boldsymbol{T} + a_N \boldsymbol{N}$ without finding $\boldsymbol{T}$ and $\boldsymbol{N}$. $\boldsymbol{r}(t) = (1+3t)\boldsymbol{i} + (t-2)\boldsymbol{j} - 3t\boldsymbol{k}$

Solution

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = 3\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$$

$$|\boldsymbol{v}| = \sqrt{9+1+9} = \sqrt{19}$$

$$a_T = \frac{d}{dt}|\boldsymbol{v}| = 0$$

$$\boldsymbol{a} = \boldsymbol{v}' = 0$$

$$a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2} = 0$$

$$\boldsymbol{a} = a_T \boldsymbol{T} + a_N \boldsymbol{N} = (0)\boldsymbol{T} + 0\boldsymbol{N} \ \underline{= \boldsymbol{0}|}$$

Exercise

Write $\boldsymbol{a}$ of the motion $\boldsymbol{a} = a_T \boldsymbol{T} + a_N \boldsymbol{N}$ at the given value of t without finding $\boldsymbol{T}$ and $\boldsymbol{N}$.

$$\boldsymbol{r}(t) = (t+1)\boldsymbol{i} + 2t\boldsymbol{j} + t^2\boldsymbol{k}, \quad t = 1$$

Solution

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \boldsymbol{i} + 2\boldsymbol{j} + 2t\boldsymbol{k}$$

$$|\boldsymbol{v}| = \sqrt{1+4+4t^2} = \sqrt{5+4t^2}$$

$$a_T = \frac{d}{dt}|\boldsymbol{v}| = \frac{1}{2}(8t)\left(5+4t^2\right)^{-1/2} = 4t\left(5+4t^2\right)^{-1/2}$$

$$a_T\Big|_{t=1} = 4(5+4)^{-1/2} = 4(9)^{-1/2} = \frac{4}{3}$$

$$\boldsymbol{a} = \boldsymbol{v}' = 2\boldsymbol{k}$$

$$|\boldsymbol{a}| = \sqrt{4} = 2$$

$$a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2} = \sqrt{4-\frac{16}{9}} = \sqrt{\frac{20}{9}} = \frac{2\sqrt{5}}{3}$$

$$\boldsymbol{a}(1) = a_T\boldsymbol{T} + a_N\boldsymbol{N} = \boxed{\frac{4}{3}\boldsymbol{T} + \frac{2\sqrt{5}}{3}\boldsymbol{N}}$$

Exercise

Write $\boldsymbol{a}$ of the motion $\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$ at the given value of t without finding $\boldsymbol{T}$ and $\boldsymbol{N}$.

$\boldsymbol{r}(t) = (t\cos t)\boldsymbol{i} + (t\sin t)\boldsymbol{j} + t^2\boldsymbol{k}, \quad t=0$

Solution

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = (\cos t - t\sin t)\boldsymbol{i} + (\sin t + t\cos t)\boldsymbol{j} + 2t\boldsymbol{k}$$

$$|\boldsymbol{v}| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 4t^2}$$

$$= \sqrt{\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + 4t^2}$$

$$= \sqrt{\cos^2 t + \sin^2 t + t^2\left(\sin^2 t + \cos^2 t\right) + 4t^2}$$

$$= \sqrt{1+5t^2}$$

$$a_T = \frac{d}{dt}|\boldsymbol{v}| = \frac{1}{2}(10t)\left(1+5t^2\right)^{-1/2} = 5t\left(1+5t^2\right)^{-1/2}$$

$$a_T\Big|_{t=0} = 0$$

$$\boldsymbol{a} = \boldsymbol{v}' = (-\sin t - \sin t - t\cos t)\boldsymbol{i} + (\cos t + \cos t - t\sin t)\boldsymbol{j} + 2\boldsymbol{k}$$

$$= (-2\sin t - t\cos t)\boldsymbol{i} + (2\cos t - t\sin t)\boldsymbol{j} + 2\boldsymbol{k}$$

$$\boldsymbol{a}\Big|_{t=0} = 2\boldsymbol{j} + 2\boldsymbol{k}$$

$$|\boldsymbol{a}|\Big|_{t=0} = \sqrt{4+4} = 2\sqrt{2}$$

$$a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2} = \sqrt{8-0} = 2\sqrt{2}$$

$$\boldsymbol{a}(1) = a_T\boldsymbol{T} + a_N\boldsymbol{N} = (0)\boldsymbol{T} + 2\sqrt{2}\boldsymbol{N} = \underline{2\sqrt{2}\boldsymbol{N}}$$

Exercise

Write $\boldsymbol{a}$ of the motion $\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$ at the given value of t without finding $\boldsymbol{T}$ and $\boldsymbol{N}$.

$$\boldsymbol{r}(t) = \left(e^t \cos t\right)\boldsymbol{i} + \left(e^t \sin t\right)\boldsymbol{j} + \sqrt{2}e^t\boldsymbol{k}, \quad t = 0$$

Solution

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \left(e^t \cos t - e^t \sin t\right)\boldsymbol{i} + \left(e^t \sin t + e^t \cos t\right)\boldsymbol{j} + \sqrt{2}e^t\boldsymbol{k}$$

$$\begin{aligned}|\boldsymbol{v}| &= \sqrt{\left(e^t \cos t - e^t \sin t\right)^2 + \left(e^t \sin t + e^t \cos t\right)^2 + 2e^{2t}} \\ &= \sqrt{e^{2t}\left(\cos^2 t - 2\cos t \sin t + \sin^2 t\right) + e^{2t}\left(\cos^2 t + 2\cos t \sin t + \sin^2 t\right) + 2e^{2t}} \\ &= e^t\sqrt{1 - 2\cos t \sin t + 1 + 2\cos t \sin t + 2} \\ &= e^t\sqrt{4} \\ &= 2e^t\end{aligned}$$

$$a_T = \frac{d}{dt}|\boldsymbol{v}| = 2e^t \quad \Rightarrow \quad a_T\Big|_{t=0} = 2$$

$$\begin{aligned}\boldsymbol{a} = \boldsymbol{v}' &= \left(e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t\right)\boldsymbol{i} + \left(e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t\right)\boldsymbol{j} + \sqrt{2}e^t\boldsymbol{k} \\ &= \left(-2e^t \sin t\right)\boldsymbol{i} + \left(2e^t \cos t\right)\boldsymbol{j} + \sqrt{2}e^t\boldsymbol{k}\end{aligned}$$

$$\boldsymbol{a}\Big|_{t=0} = 2\boldsymbol{j} + \sqrt{2}\boldsymbol{k}$$

$$|\boldsymbol{a}|\Big|_{t=0} = \sqrt{4+2} = \sqrt{6}$$

$$a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2} = \sqrt{6-4} = \sqrt{2}$$

$$\boldsymbol{a}(1) = a_T\boldsymbol{T} + a_N\boldsymbol{N} \ \underline{= 2\boldsymbol{T} + \sqrt{2}\boldsymbol{N}}$$

Exercise

Find ***r, T, N***, and ***B*** at the given value of *t*. Then find equations for the osculating, normal, and rectifying planes at that value of *t*. $\boldsymbol{r}(t)=(\cos t)\boldsymbol{i}+(\sin t)\boldsymbol{j}-\boldsymbol{k},\quad t=\frac{\pi}{4}$

Solution

$$\boldsymbol{r}\left(t=\frac{\pi}{4}\right)=\left(\cos\frac{\pi}{4}\right)\boldsymbol{i}+\left(\sin\frac{\pi}{4}\right)\boldsymbol{j}-\boldsymbol{k}=\frac{\sqrt{2}}{2}\boldsymbol{i}+\frac{\sqrt{2}}{2}\boldsymbol{j}-\boldsymbol{k}$$

$$\boldsymbol{v}=\frac{d\boldsymbol{r}}{dt}=(-\sin t)\boldsymbol{i}+(\cos t)\boldsymbol{j}$$

$$|\boldsymbol{v}|=\sqrt{\sin^2 t+\cos^2 t}=1$$

$$\boldsymbol{T}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}=(-\sin t)\boldsymbol{i}+(\cos t)\boldsymbol{j}\quad\Rightarrow\quad \boldsymbol{T}\left(t=\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}\boldsymbol{i}+\frac{\sqrt{2}}{2}\boldsymbol{j}$$

$$\frac{d\boldsymbol{T}}{dt}=(-\cos t)\boldsymbol{i}+(-\sin t)\boldsymbol{j}$$

$$\left|\frac{d\boldsymbol{T}}{dt}\right|=\sqrt{\cos^2 t+\sin^2 t}=1$$

$$\boldsymbol{N}=\frac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|}=(-\cos t)\boldsymbol{i}+(-\sin t)\boldsymbol{j}\quad\Rightarrow\quad \boldsymbol{N}\left(t=\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}\boldsymbol{i}-\frac{\sqrt{2}}{2}\boldsymbol{j}$$

$$\boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\-\sin t&\cos t&0\\-\cos t&-\sin t&0\end{vmatrix}=\boldsymbol{k}\quad\Rightarrow\quad \boldsymbol{B}\left(t=\frac{\pi}{4}\right)=\boldsymbol{k}$$

The normal to the osculating plane $\boldsymbol{r}=\frac{\sqrt{2}}{2}\boldsymbol{i}+\frac{\sqrt{2}}{2}\boldsymbol{j}-\boldsymbol{k}\ \Rightarrow\ P\left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2},\ -1\right)$ lies on the osculating plane (using ***B***): $0\left(x-\frac{\sqrt{2}}{2}\right)+0\left(y-\frac{\sqrt{2}}{2}\right)+(z-(-1))=0\ \Rightarrow\ \boxed{z=-1}$ is the osculating plane.

T is normal to the normal plane

$$-\frac{\sqrt{2}}{2}\left(x-\frac{\sqrt{2}}{2}\right)+\frac{\sqrt{2}}{2}y\left(x-\frac{\sqrt{2}}{2}\right)+0(z-(-1))=0\quad\Rightarrow\quad -\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}}{2}y=0$$

$-x+y=0$ is the normal plane

N is normal to the rectifying plane:

$$-\frac{\sqrt{2}}{2}\left(x-\frac{\sqrt{2}}{2}\right)-\frac{\sqrt{2}}{2}\left(y-\frac{\sqrt{2}}{2}\right)+0(z-(-1))=0\quad\Rightarrow\quad -\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}}{2}y+\frac{1}{2}+\frac{1}{2}=0$$

$x+y=\sqrt{2}$ is the rectifying plane.

Exercise

Find **r, T, N**, and **B** at the given value of t. Then find equations for the osculating, normal, and rectifying planes at that value of t. $\boldsymbol{r}(t)=(\cos t)\boldsymbol{i}+(\sin t)\boldsymbol{j}+t\boldsymbol{k}, \quad t=0$

Solution

$$\boldsymbol{r}(t=0)=(\cos 0)\boldsymbol{i}+(\sin 0)\boldsymbol{j}+0\boldsymbol{k}\ \underline{=\boldsymbol{i}}$$

$$\boldsymbol{v}=\frac{d\boldsymbol{r}}{dt}=(-\sin t)\boldsymbol{i}+(\cos t)\boldsymbol{j}+\boldsymbol{k}$$

$$|\boldsymbol{v}|=\sqrt{\sin^2 t+\cos^2 t+1}=\sqrt{2}$$

$$\boldsymbol{T}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}=-\left(\frac{\sin t}{\sqrt{2}}\right)\boldsymbol{i}+\left(\frac{\cos t}{\sqrt{2}}\right)\boldsymbol{j}+\frac{1}{\sqrt{2}}\boldsymbol{k} \quad\Rightarrow\quad \boldsymbol{T}(t=0)=\underline{\frac{1}{\sqrt{2}}\boldsymbol{j}+\frac{1}{\sqrt{2}}\boldsymbol{k}}$$

$$\frac{d\boldsymbol{T}}{dt}=\left(-\frac{1}{\sqrt{2}}\cos t\right)\boldsymbol{i}-\left(\frac{1}{\sqrt{2}}\sin t\right)\boldsymbol{j}$$

$$\left|\frac{d\boldsymbol{T}}{dt}\right|=\sqrt{\frac{1}{2}\cos^2 t+\frac{1}{2}\sin^2 t}=\frac{1}{\sqrt{2}}$$

$$\boldsymbol{N}=\frac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|}=\sqrt{2}\left(\left(-\frac{1}{\sqrt{2}}\cos t\right)\boldsymbol{i}-\left(\frac{1}{\sqrt{2}}\sin t\right)\boldsymbol{j}\right)=(-\cos t)\boldsymbol{i}-(\sin t)\boldsymbol{j}$$

$$\boldsymbol{N}(t=0)=\underline{-\boldsymbol{i}}$$

$$\boldsymbol{B}(t=0)=\boldsymbol{T}\times\boldsymbol{N}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -1 & 0 & 0\end{vmatrix}=\underline{-\frac{1}{\sqrt{2}}\boldsymbol{j}+\frac{1}{\sqrt{2}}\boldsymbol{k}}$$

The normal to the osculating plane $\boldsymbol{r}\ =\boldsymbol{i} \quad\Rightarrow\quad P(1,\ 0,\ 0)$ lies on the osculating plane (using **B**):

$$0(x-1)-\frac{1}{\sqrt{2}}(y-0)+\frac{1}{\sqrt{2}}(z-0)=0 \quad\Rightarrow\quad -\frac{1}{\sqrt{2}}y+\frac{1}{\sqrt{2}}z=0$$

$\boxed{y-z=0}$ is the osculating plane.

T is normal to the normal plane $\quad 0(x-1)+\frac{1}{\sqrt{2}}(y-0)+\frac{1}{\sqrt{2}}(z-0)=0 \quad\Rightarrow\quad \frac{1}{\sqrt{2}}y+\frac{1}{\sqrt{2}}z=0$

$\boxed{y+z=0}$ is the normal plane

N is normal to the rectifying plane: $\quad -(x-1)+0(y-0)+0(z-0)=0 \quad\Rightarrow\quad -x+1=0$

$\boxed{x=1}$ is the rectifying plane.

Exercise

Find $\boldsymbol{B}$ and τ for: $\boldsymbol{r}(t)=(3\sin t)\boldsymbol{i}+(3\cos t)\boldsymbol{j}+4t\boldsymbol{k}$

Solution

$$\boldsymbol{v}=\frac{d\boldsymbol{r}}{dt}=(3\cos t)\boldsymbol{i}-(3\sin t)\boldsymbol{j}+4\,\boldsymbol{k}$$

$$|\boldsymbol{v}|=\sqrt{9\cos^2 t+9\sin^2 t+16}=\sqrt{9+16}=\underline{5|}$$

$$\boldsymbol{T}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}=\underline{\frac{3}{5}\cos t\,\boldsymbol{i}-\frac{3}{5}\sin t\,\boldsymbol{j}+\frac{4}{5}\,\boldsymbol{k}|}$$

$$\frac{d\boldsymbol{T}}{dt}=-\frac{3}{5}\sin t\,\boldsymbol{i}-\frac{3}{5}\cos t\,\boldsymbol{j}$$

$$\left|\frac{d\boldsymbol{T}}{dt}\right|=\sqrt{\frac{9}{25}\sin^2 t+\frac{9}{25}\cos^2 t}=\sqrt{\frac{9}{25}}=\frac{3}{5}$$

$$\boldsymbol{N}=\frac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|}=\frac{5}{3}\left(-\frac{3}{5}\sin t\,\boldsymbol{i}-\frac{3}{5}\cos t\,\boldsymbol{j}\right)=\underline{(-\sin t)\boldsymbol{i}-(\cos t)\boldsymbol{j}|}$$

$$\boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{3}{5}\cos t & -\frac{3}{5}\sin t & \frac{4}{5}\\ -\sin t & -\cos t & 0\end{vmatrix}=\underline{\left(\frac{4}{5}\cos t\right)\boldsymbol{i}-\left(\frac{4}{5}\sin t\right)\boldsymbol{j}-\frac{3}{5}\boldsymbol{k}|}$$

$$\boldsymbol{a}=\frac{d\boldsymbol{v}}{dt}=(-3\sin t)\boldsymbol{i}-(3\cos t)\boldsymbol{j}$$

$$\boldsymbol{v}\times\boldsymbol{a}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 3\cos t & -3\sin t & 4\\ -3\sin t & -3\cos t & 0\end{vmatrix}=(12\cos t)\boldsymbol{i}-(12\sin t)\boldsymbol{j}-9\boldsymbol{k}$$

$$|\boldsymbol{v}\times\boldsymbol{a}|^2=144\cos^2 t+144\sin^2 t+81=144+81=225$$

$$\tau=\frac{\begin{vmatrix}\dot{x} & \dot{y} & \dot{z}\\ \ddot{x} & \ddot{y} & \ddot{z}\\ \dddot{x} & \dddot{y} & \dddot{z}\end{vmatrix}}{|\boldsymbol{v}\times\boldsymbol{a}|^2}=\frac{\begin{vmatrix}3\cos t & -3\sin t & 4\\ -3\sin t & -3\cos t & 0\\ -3\cos t & 3\sin t & 0\end{vmatrix}}{225}$$

$$=\frac{4\left(-9\sin^2 t-9\cos^2 t\right)}{225}$$

$$=-\frac{36}{225}$$

$$\underline{=-\frac{4}{25}|}$$

Exercise

Find $\boldsymbol{B}$ and τ for: $\boldsymbol{r}(t)=(\cos t+t\sin t)\boldsymbol{i}+(\sin t-t\cos t)\boldsymbol{j}+3\boldsymbol{k}$

Solution

$$\boldsymbol{v}=\frac{d\boldsymbol{r}}{dt}=(-\sin t+\sin t+t\cos t)\boldsymbol{i}+(\cos t-\cos t+t\sin t)\boldsymbol{j}=(t\cos t)\boldsymbol{i}+(t\sin t)\boldsymbol{j}$$

$$|\boldsymbol{v}|=\sqrt{t^2\cos^2 t+t^2\sin^2 t}=\sqrt{t^2\left(\cos^2 t+\sin^2 t\right)}=\sqrt{t^2}=|t|=t$$

$$\boldsymbol{T}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}=\left(\frac{t\cos t}{t}\right)\boldsymbol{i}+\left(\frac{t\sin t}{t}\right)\boldsymbol{j}=\underline{(\cos t)\boldsymbol{i}+(\sin t)\boldsymbol{j}}$$

$$\frac{d\boldsymbol{T}}{dt}=(-\sin t)\boldsymbol{i}+(\cos t)\boldsymbol{j}$$

$$\left|\frac{d\boldsymbol{T}}{dt}\right|=\sqrt{\sin^2 t+\cos^2 t}=1$$

$$\boldsymbol{N}=\frac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|}=\underline{(-\sin t)\boldsymbol{i}+(\cos t)\boldsymbol{j}}$$

$$\boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \cos t & \sin t & 0\\ -\sin t & -\cos t & 0\end{vmatrix}=\left(\cos^2 t+\sin^2 t\right)\boldsymbol{k}=\underline{\boldsymbol{k}}$$

$$\boldsymbol{a}=\frac{d\boldsymbol{v}}{dt}=(\cos t-t\sin t)\boldsymbol{i}+(\sin t+t\cos t)\boldsymbol{j}$$

$$\boldsymbol{v}\times\boldsymbol{a}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ t\cos t & t\sin t & 0\\ \cos t-t\sin t & \sin t+t\cos t & 0\end{vmatrix}$$

$$=\left(t\cos t\sin t+t^2\cos^2 t-t\sin t\cos t+t^2\sin^2 t\right)\boldsymbol{k}=t^2\boldsymbol{k}$$

$$=t^2\boldsymbol{k}$$

$$|\boldsymbol{v}\times\boldsymbol{a}|^2=\left(t^2\right)^2=t^4$$

$$\tau=\frac{\begin{vmatrix}\dot{x} & \dot{y} & \dot{z}\\ \ddot{x} & \ddot{y} & \ddot{z}\\ \dddot{x} & \dddot{y} & \dddot{z}\end{vmatrix}}{|\boldsymbol{v}\times\boldsymbol{a}|^2}=\frac{\begin{vmatrix}t\cos t & t\sin t & 0\\ \cos t-t\sin t & \sin t+t\cos t & 0\\ -2\sin t-t\cos t & 2\cos t-t\sin t & 0\end{vmatrix}}{t^4}$$

$$=\frac{0}{t^4}$$

$$\underline{=0}$$

Exercise

Find $\boldsymbol{B}$ and τ for: $\boldsymbol{r}(t)=(6\sin 2t)\boldsymbol{i}+(6\cos 2t)\boldsymbol{j}+5t\boldsymbol{k}$

Solution

$$\boldsymbol{v}=\frac{d\boldsymbol{r}}{dt}=(12\cos 2t)\boldsymbol{i}-(12\sin 2t)\boldsymbol{j}+5\,\boldsymbol{k}$$

$$|\boldsymbol{v}|=\sqrt{144\cos^2 t+144\sin^2 t+25}=\sqrt{144+25}=\underline{13}$$

$$\boldsymbol{T}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}=\underline{\frac{12}{13}\cos 2t\;\boldsymbol{i}-\frac{12}{13}\sin 2t\;\boldsymbol{j}+\frac{5}{13}\;\boldsymbol{k}}$$

$$\frac{d\boldsymbol{T}}{dt}=-\frac{24}{13}\sin 2t\;\boldsymbol{i}-\frac{24}{13}\cos 2t\;\boldsymbol{j}$$

$$\left|\frac{d\boldsymbol{T}}{dt}\right|=\sqrt{\frac{576}{169}\sin^2 t+\frac{576}{169}\cos^2 t}=\sqrt{\frac{576}{169}}=\frac{24}{13}$$

$$\boldsymbol{N}=\frac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|}=\frac{13}{24}\left(-\frac{24}{13}\sin 2t\;\boldsymbol{i}-\frac{24}{13}\cos 2t\;\boldsymbol{j}\right)=\underline{(-\sin 2t)\boldsymbol{i}-(\cos 2t)\boldsymbol{j}}$$

$$\boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{12}{13}\cos 2t & -\frac{12}{13}\sin 2t & \frac{5}{13}\\ -\sin 2t & -\cos 2t & 0\end{vmatrix}=\underline{\left(\frac{5}{13}\cos 2t\right)\boldsymbol{i}-\left(\frac{5}{13}\sin 2t\right)\boldsymbol{j}-\frac{12}{13}\boldsymbol{k}}$$

$$\boldsymbol{a}=\frac{d\boldsymbol{v}}{dt}=(-24\sin 2t)\boldsymbol{i}-(24\cos 2t)\boldsymbol{j}$$

$$\boldsymbol{v}\times\boldsymbol{a}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 12\cos 2t & -12\sin 2t & 5\\ -24\sin 2t & -24\cos 2t & 0\end{vmatrix}=(120\cos 2t)\boldsymbol{i}-(120\sin 2t)\boldsymbol{j}-288\boldsymbol{k}$$

$$|\boldsymbol{v}\times\boldsymbol{a}|^2=14400\cos^2 2t+14400\sin^2 2t+288^2=14400+82944=97344$$

$$\tau=\frac{\begin{vmatrix}\dot{x} & \dot{y} & \dot{z}\\ \ddot{x} & \ddot{y} & \ddot{z}\\ \dddot{x} & \dddot{y} & \dddot{z}\end{vmatrix}}{|\boldsymbol{v}\times\boldsymbol{a}|^2}=\frac{\begin{vmatrix}12\cos 2t & -12\sin 2t & 5\\ -24\sin 2t & -24\cos 2t & 0\\ -48\cos 2t & 48\sin 2t & 0\end{vmatrix}}{97344}$$

$$=\frac{5\left(-1152\sin^2 2t-1152\cos^2 2t\right)}{97344}$$

$$=-\frac{5760}{97344}$$

$$\underline{=-\frac{10}{169}}$$

Exercise

The speedometer on your car reads a steady 35 mph. Could you be accelerating? Explain.

Solution

Yes.

If the car is moving along a curved path, then $\kappa \neq 0$ and $a_N = \kappa |\mathbf{v}|^2 \neq 0$

$\Rightarrow \boldsymbol{a} = a_T \boldsymbol{T} + a_N \boldsymbol{N} \neq 0$

Exercise

Can anything be said about the acceleration of a particle that is moving at a constant speed? Give reasons for your answer.

Solution

$|v|$ is constant $\Rightarrow$ $\boldsymbol{a}_T = \frac{d\boldsymbol{v}}{dt} = 0$

$\boldsymbol{a} = a_T \boldsymbol{T} + a_N \boldsymbol{N} = a_N \boldsymbol{N}$ is orthogonal to $\boldsymbol{T}$.

$\therefore$ The acceleration is normal to the path.

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{B}$, τ and κ as functions of t for the plane curves: $\boldsymbol{r}(t) = (\sin t)\boldsymbol{i} + \left(\sqrt{2}\cos t\right)\boldsymbol{j} + (\sin t)\boldsymbol{k}$,

then write $\boldsymbol{a}$ of the motion $\boldsymbol{a} = a_T \boldsymbol{T} + a_N \boldsymbol{N}$

Solution

$$\mathbf{v} = \frac{d\boldsymbol{r}}{dt} = (\cos t)\boldsymbol{i} - \left(\sqrt{2}\sin t\right)\boldsymbol{j} + (\cos t)\,\boldsymbol{k}$$

$$|\mathbf{v}| = \sqrt{\cos^2 t + 2\sin^2 t + \cos^2 t} = \sqrt{2\cos^2 t + 2\sin^2 t} = \underline{\sqrt{2}|}$$

$$\boldsymbol{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\cos t}{\sqrt{2}}\right)\boldsymbol{i} - (\sin t)\,\boldsymbol{j} + \left(\frac{\cos t}{\sqrt{2}}\right)\boldsymbol{k}$$

$$\frac{d\boldsymbol{T}}{dt} = -\frac{\sin t}{\sqrt{2}}\boldsymbol{i} - (\cos t)\,\boldsymbol{j} - \frac{\sin t}{\sqrt{2}}\,\boldsymbol{k}$$

$$\left|\frac{d\boldsymbol{T}}{dt}\right| = \sqrt{\frac{1}{2}\sin^2 t + \cos^2 t + \frac{1}{2}\sin^2 t} = \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$\boldsymbol{N} = \frac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|} = -\frac{\sin t}{\sqrt{2}}\boldsymbol{i} - (\cos t)\,\boldsymbol{j} - \frac{\sin t}{\sqrt{2}}\,\boldsymbol{k}$$

$$\boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\cos t}{\sqrt{2}} & -\sin t & \frac{\cos t}{\sqrt{2}} \\ -\frac{\sin t}{\sqrt{2}} & -\cos t & -\frac{\sin t}{\sqrt{2}} \end{vmatrix}$$

$$=\left(\frac{\sin^2 t}{\sqrt{2}}+\frac{\cos^2 t}{\sqrt{2}}\right)\boldsymbol{i}-\left(-\frac{\sin t\cos t}{\sqrt{2}}+\frac{\sin t\cos t}{\sqrt{2}}\right)\boldsymbol{j}+\left(-\frac{\cos^2 t}{\sqrt{2}}-\frac{\sin^2 t}{\sqrt{2}}\right)\boldsymbol{k}$$

$$\boxed{=\frac{1}{\sqrt{2}}\boldsymbol{i}-\frac{1}{\sqrt{2}}\boldsymbol{k}}$$

$$\kappa=\frac{1}{|\boldsymbol{v}|}\left|\frac{d\boldsymbol{T}}{dt}\right| \boxed{=\frac{1}{\sqrt{2}}}$$

$$\boldsymbol{a}=\boxed{(-\sin t)\boldsymbol{i}-\left(\sqrt{2}\cos t\right)\boldsymbol{j}-(\sin t)\,\boldsymbol{k}}$$

$$\boldsymbol{v}\times\boldsymbol{a}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos t & -\sqrt{2}\sin t & \cos t \\ -\sin t & -\sqrt{2}\cos t & -\sin t \end{vmatrix}$$

$$=\left(\sqrt{2}\sin^2 t+\sqrt{2}\cos^2 t\right)\boldsymbol{i}+\left(-\sqrt{2}\cos^2 t-\sqrt{2}\sin^2 t\right)\boldsymbol{k}$$

$$=\left(\sqrt{2}\right)\boldsymbol{i}-\left(\sqrt{2}\right)\boldsymbol{k}$$

$$|\boldsymbol{v}\times\boldsymbol{a}|=\sqrt{2+2}=2$$

$$\tau=\frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\boldsymbol{v}\times\boldsymbol{a}|^2}=\frac{\begin{vmatrix} \cos t & -\sqrt{2}\sin t & \cos t \\ -\sin t & -\sqrt{2}\cos t & -\sin t \\ -\cos t & \sqrt{2}\sin t & -\cos t \end{vmatrix}}{4}$$

$$=\frac{\sqrt{2}\cos^3 t-\sqrt{2}\sin^2 t\cos t-\sqrt{2}\sin^2 t\cos t-\sqrt{2}\cos^3 t+\sqrt{2}\sin^2 t\cos t+\sqrt{2}\sin^2 t\cos t}{4}$$

$$\boxed{=0}$$