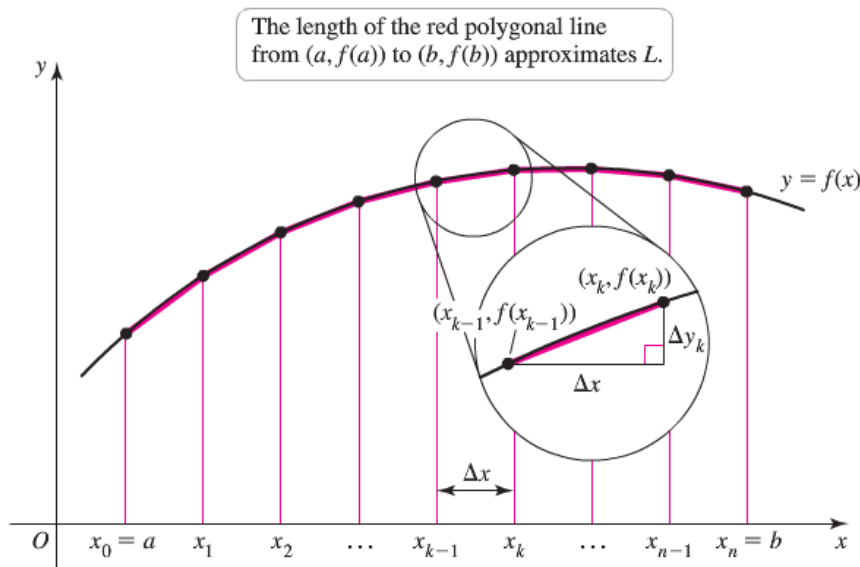


Section 1.5 – Length of Curves

Length of a curve $y = f(x)$

We assume that f has a continuous derivative at every point of $[a, b]$. Such function is called **smooth**, and its graph is a **smooth curve** because it doesn't have any breaks, corners, or cusps.



Definition

If f' is continuous on $[a, b]$, then the length (arc length) of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

Example

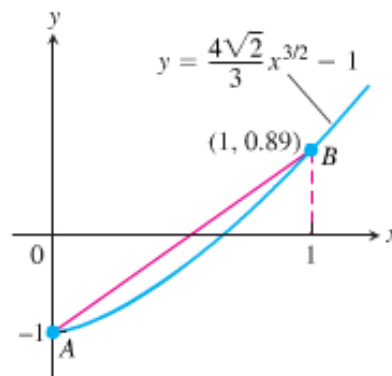
Find the length of the curve $y = \frac{4\sqrt{2}}{3}x^{3/2} - 1$, $0 \leq x \leq 1$

Solution

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2} x^{1/2} = 2\sqrt{2}x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2}x^{1/2}\right)^2 = 8x$$

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$



$$\begin{aligned}
&= \int_0^1 (1+8x)^{1/2} dx && \text{or } u = 1+8x \quad du = 8dx \rightarrow dx = \frac{du}{8} \\
&= \frac{1}{8} \int_0^1 (1+8x)^{1/2} d(1+8x) \\
&= \frac{1}{8} \left[\frac{2}{3} (1+8x)^{3/2} \right]_0^1 \\
&= \frac{1}{12} \left[(1+8(\textcolor{red}{1}))^{3/2} - (1+8(\textcolor{blue}{0}))^{3/2} \right] \\
&= \frac{1}{12} \left[(9)^{3/2} - (1)^{3/2} \right] \\
&= \frac{1}{12} [27 - 1] \\
&= \frac{1}{12} (26) \\
&= \underline{\frac{13}{6} \approx \textcolor{blue}{2.17} \text{ unit}}
\end{aligned}$$

Example

Find the length of the graph of $f(x) = \frac{x^3}{12} + \frac{1}{x}$, $1 \leq x \leq 4$

Solution

$$a = \frac{1}{12}, \quad m = 3, \quad b = 1, \quad n = -1$$

$$\textcolor{red}{1.} \quad m + n = 3 - 1 = 2 \quad \checkmark$$

$$\textcolor{red}{2.} \quad abmn = \frac{1}{12}(1)(3)(-1) = -\frac{1}{4} \quad \checkmark$$

$$\begin{aligned}
L &= \left(\frac{x^3}{12} - \frac{1}{x} \right)_1^4 \\
&= \left(\frac{\textcolor{red}{4}^3}{12} - \frac{1}{\textcolor{red}{4}} \right) - \left(\frac{\textcolor{blue}{1}}{12} - \frac{1}{\textcolor{blue}{1}} \right) \\
&= \frac{72}{12} \\
&= \underline{\textcolor{blue}{6} \text{ unit}}
\end{aligned}$$

Discontinuities in $\frac{dy}{dx}$

Formula for the length of $x = g(y)$, $c \leq y \leq d$

If g' is continuous on $[c, d]$, the length of the curve $x = g(y)$ from the point $A = (g(c), c)$ to the point $B = (g(d), d)$ is the value of the integral

$$L = \int_c^d \sqrt{1 + [g'(y)]^2} dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

Example

Find the length of the curve $y = \left(\frac{x}{2}\right)^{2/3}$ from $x = 0$ to $x = 2$.

Solution

$$\begin{aligned} \frac{dy}{dx} &= \frac{2}{3} \left(\frac{x}{2}\right)^{-1/3} \left(\frac{1}{2}\right) \\ &= \frac{1}{3} \left(\frac{2}{x}\right)^{1/3} \quad \boxed{x \neq 0} \text{ (CP)} \end{aligned}$$

$$y = \left(\frac{x}{2}\right)^{2/3} \rightarrow y^{3/2} = \frac{x}{2} \quad \text{Raised both sides to the power } 3/2$$
$$x = 2y^{3/2}$$

$$\frac{dx}{dy} = 2 \left(\frac{3}{2}\right) y^{1/2} = 3y^{1/2} \rightarrow \begin{cases} x = 0 & \Rightarrow y = 0 \\ x = 2 & \Rightarrow y = \left(\frac{2}{2}\right)^{2/3} = 1 \end{cases}$$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_0^1 \sqrt{1 + (3y^{1/2})^2} dy$$

$$= \int_0^1 \sqrt{1 + 9y} dy$$

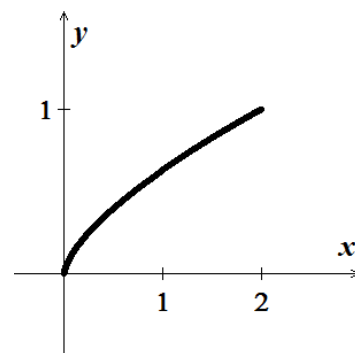
$$= \int_0^1 (1 + 9y)^{1/2} dy$$

$$= \frac{1}{9} \frac{2}{3} (1 + 9y)^{3/2} \Big|_0^1$$

$$= \frac{2}{27} \left[(1 + 9)^{3/2} - (1 + 0)^{3/2} \right]$$

$$= \frac{2}{27} (10^{3/2} - 1)$$

$$\approx 2.27 \text{ unit}$$



Example

Find the arc length function for the curve $f(x) = \ln(x + \sqrt{x^2 - 1})$ on the interval $[1, \sqrt{2}]$

Solution

$$y = \ln(x + \sqrt{x^2 - 1}) \rightarrow x + \sqrt{x^2 - 1} = e^y$$

$$\left(\sqrt{x^2 - 1}\right)^2 = (e^y - x)^2$$

$$x^2 - 1 = e^{2y} - 2xe^y + x^2$$

$$2xe^y = e^{2y} + 1 \Rightarrow x = \frac{e^{2y} + 1}{2e^y} \left(\frac{e^y}{e^y}\right) = \frac{e^y + e^{-y}}{2}$$

$$y = \ln(x + \sqrt{x^2 - 1}) \Leftrightarrow x = \frac{e^y + e^{-y}}{2} = g(y)$$

$$x = 1 \rightarrow y = 0$$

$$x = \sqrt{2} \rightarrow y = \ln(\sqrt{2} + 1)$$

$$L = \int_0^{\ln(\sqrt{2}+1)} \sqrt{1 + g'(y)^2} dy$$

$$= \int_0^{\ln(\sqrt{2}+1)} \sqrt{1 + \left(\frac{e^y - e^{-y}}{2}\right)^2} dy$$

$$= \frac{1}{2} \int_0^{\ln(\sqrt{2}+1)} \sqrt{4 + e^{2y} - 2 + e^{-2y}} dy$$

$$= \frac{1}{2} \int_0^{\ln(\sqrt{2}+1)} \sqrt{e^{2y} + 2 + e^{-2y}} dy$$

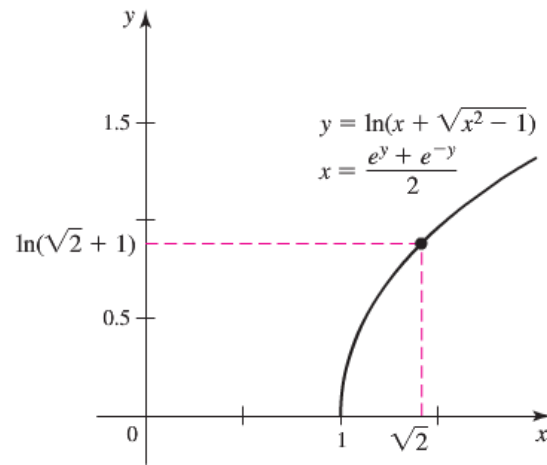
$$= \frac{1}{2} \int_0^{\ln(\sqrt{2}+1)} \sqrt{(e^y + e^{-y})^2} dy$$

$$= \frac{1}{2} \int_0^{\ln(\sqrt{2}+1)} (e^y + e^{-y}) dy$$

$$= \frac{1}{2} (e^y - e^{-y}) \Big|_0^{\ln(\sqrt{2}+1)}$$

$$= \frac{1}{2} \left(\sqrt{2} + 1 - \frac{1}{\sqrt{2} + 1} - 1 + 1 \right)$$

$$\begin{aligned} f'(x) &= \frac{1 + x(x^2 - 1)^{-1/2}}{x + \sqrt{x^2 - 1}} \\ &= \frac{\sqrt{x^2 - 1} + x}{x\sqrt{x^2 - 1} + x^2 - 1} \end{aligned}$$



OR

$$a = \frac{1}{2}, \quad m = 1, \quad b = \frac{1}{2}, \quad n = -1$$

1. $m = -n$ ✓

2. $abmn = -\frac{1}{4}$ ✓

$$L = \frac{1}{2} (e^y - e^{-y}) \Big|_0^{\ln(\sqrt{2}+1)}$$

$$= \frac{1}{2} \left(\sqrt{2} + 1 - \frac{1}{\sqrt{2} + 1} - 1 + 1 \right)$$

$$= \frac{1}{2} \left(\frac{3 + 2\sqrt{2} - 1}{\sqrt{2} + 1} \right)$$

$$= \frac{1}{2} \left(\frac{2 + 2\sqrt{2}}{\sqrt{2} + 1} \right)$$

$$= 1 \text{ unit}$$

$$\begin{aligned}
&= \frac{1}{2} \left(\frac{3 + 2\sqrt{2} - 1}{\sqrt{2} + 1} \right) \\
&= \frac{1}{2} \left(\frac{2 + 2\sqrt{2}}{\sqrt{2} + 1} \right) \\
&= \underline{1 \text{ unit}}
\end{aligned}$$

The differential Formula for Arc length

If $y = f(x)$ and if f' is continuous on $[a, b]$, then by the Fundamental Theorem of Calculus, we can define a new function

$$\begin{aligned}
s(x) &= \int_a^x \sqrt{1 + [f'(t)]^2} dt \\
\frac{ds}{dx} &= \sqrt{1 + [f'(t)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \Rightarrow ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
&= \sqrt{(dx)^2 + (dx)^2 \frac{(dy)^2}{(dx)^2}} \\
&\boxed{ds = \sqrt{dx^2 + dy^2}}
\end{aligned}$$

Example

Find the arc length function for the curve $f(x) = \frac{x^3}{12} + \frac{1}{x}$ taking $A = \left(1, \frac{13}{12}\right)$ as the starting point

Solution

$$\begin{aligned}
1 + [f'(x)]^2 &= \left(\frac{x^2}{4} + \frac{1}{x^2} \right)^2 \\
s(x) &= \int_1^x \sqrt{1 + [f'(t)]^2} dt = \int_1^x \left(\frac{t^2}{4} + \frac{1}{t^2} \right) dt \\
&= \left(\frac{t^3}{12} - \frac{1}{t} \right) \Big|_1^x \\
&= \left(\frac{x^3}{12} - \frac{1}{x} \right) - \left(\frac{1}{12} - 1 \right) \\
&= \underline{\frac{x^3}{12} - \frac{1}{x} + \frac{11}{12}} \\
s(4) &= \underline{\frac{4^3}{12} - \frac{1}{4} + \frac{11}{12}} = \underline{6 \text{ unit}}
\end{aligned}$$

Exercises Section 1.5 – Length of Curves

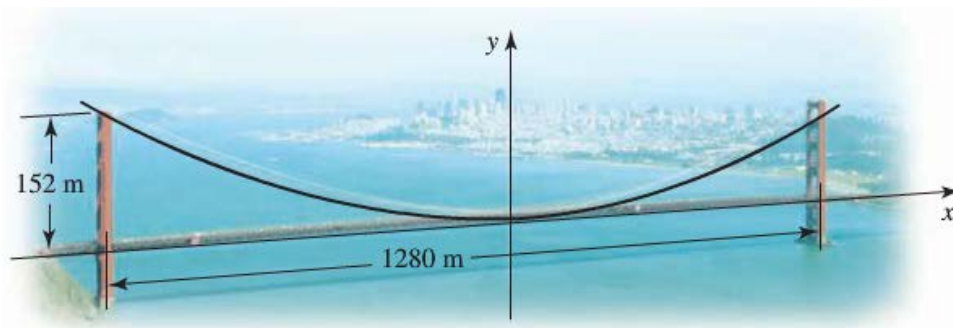
Find the length of the curve of

1. $y = \frac{1}{3}(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$
2. $y = (x)^{3/2}$ from $x = 0$ to $x = 4$
3. $x = \frac{y^{3/2}}{3} - y^{1/2}$ from $y = 1$ to $y = 9$
4. $x = \frac{y^3}{6} + \frac{1}{2y}$ from $y = 2$ to $y = 3$
5. $f(x) = x^3 + \frac{1}{12x}$ for $\frac{1}{2} \leq x \leq 2$
6. $f(x) = \frac{1}{5}x^5 + \frac{1}{12x^3}$ $1 \leq x \leq 2$
7. $y = \frac{1}{3}x^{1/2} - x^{3/2}$, $0 \leq x \leq \frac{1}{3}$
8. $y = \frac{1}{3}x^3 + \frac{1}{4x}$, $1 \leq x \leq 2$
9. $y = 2e^x + \frac{1}{8}e^{-x}$ $0 \leq x \leq \ln 2$
10. $y = e^{2x} + \frac{1}{16}e^{-2x}$ $0 \leq x \leq \ln 3$
11. $y = \ln(\cos x)$ $0 \leq x \leq \frac{\pi}{4}$
12. $f(y) = 2e^{\sqrt{2}y} + \frac{1}{16}e^{-\sqrt{2}y}$ $0 \leq y \leq \frac{\ln 2}{\sqrt{2}}$
13. $y = \frac{x^3}{3} + x^2 + x + 1 + \frac{1}{4x+4}$ $0 \leq x \leq 2$
14. $y = \frac{x^3}{3} + x^2 + x + 1 + \frac{1}{4x+4}$ $0 \leq x \leq 4$
15. $y = \ln(e^x - 1) - \ln(e^x + 1)$ $\ln 2 \leq x \leq \ln 3$
16. $f(x) = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2}$ $1 \leq x \leq 4$
17. $f(x) = x^3 + \frac{1}{12x}$ $1 \leq x \leq 4$
18. $f(x) = \frac{1}{8}x^4 + \frac{1}{4x^2}$ $1 \leq x \leq 10$
19. $f(x) = \frac{1}{4}x^4 + \frac{1}{8x^2}$ $3 \leq x \leq 8$
20. $f(x) = \frac{1}{10}x^5 + \frac{1}{6x^3}$ $1 \leq x \leq 7$
21. $f(x) = \frac{3}{10}x^{1/3} - \frac{3}{2}x^{5/3}$ $0 \leq x \leq 12$
22. $f(x) = x^{1/2} - \frac{1}{3}x^{3/2}$ $2 \leq x \leq 9$
23. $y = x^{1/2} - \frac{1}{3}x^{3/2}$ $1 \leq x \leq 4$
24. $x = y^{2/3}$, $1 \leq y \leq 8$
25. $y = 2x + 4$ $-2 \leq x \leq 2$
26. $y = \frac{x^3}{6} + \frac{1}{2x}$ $x \in [1, 2]$
27. $f(x) = x^{1/2} - \frac{1}{3}x^{3/2}$ $1 \leq x \leq 3$
28. $y = \frac{3}{4}x^{4/3} - \frac{3}{8}x^{2/3} + 5$, $1 \leq x \leq 8$
29. $y = \ln x - \frac{1}{8}x^2$; $1 \leq x \leq 2$
30. $y = \frac{1}{2}x^2 - \frac{1}{4}\ln x$; $1 \leq x \leq 3$

31. Find the length of the curve $y = \int_{-2}^x \sqrt{2t^4 - 1} dt$ $-2 \leq x \leq -1$

32. Find the length of the curve $x = \int_0^y \sqrt{\sec^4 t - 1} dt$ $-\frac{\pi}{4} \leq y \leq \frac{\pi}{4}$

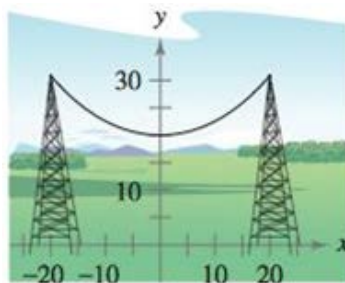
33. Find the length of the curve $y = 3 - 2x$ $0 \leq x \leq 2$. Check your answer by finding the length of the segment as the hypotenuse of a right triangle.
34. The profile of the cables on a suspension bridge may be modeled by a parabola. The central span of the Golden Gate Bridge is 1280 m long and 152 m high. The parabola $y = 0.00037x^2$ gives a good fit to the shape of the cables, where $|x| \leq 640$, and x and y are measured in meters. Approximate the length of the cables that stretch between the tops of the two towers.



35. Find a curve through the origin in the xy -plane whose length from $x = 0$ to $x = 1$ is

$$L = \int_0^1 \sqrt{1 + \frac{1}{4}e^x} dx$$

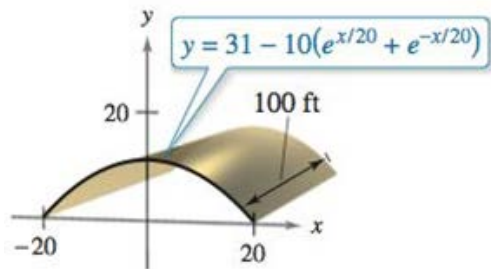
36. Confirm that the circumference of a circle of radius a is $2\pi a$
37. Electrical wires suspended between two towers form a catenary modeled by the equation



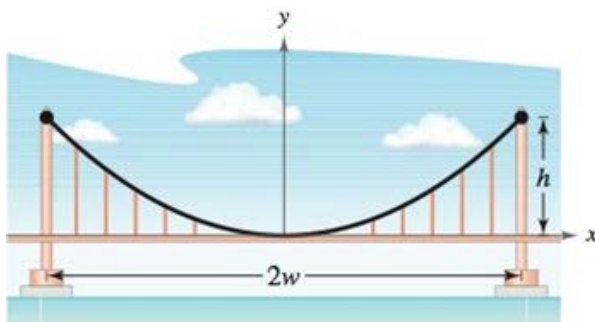
$$y = 20 \cosh \frac{x}{20}, \quad -20 \leq x \leq 20$$

Where x and y are measured in meters. The towers are 40 meters apart. Find the length of the suspended cable.

38. A barn is 100 feet long and 40 feet wide. A cross section of the roof is the inverted catenary $y = 31 - 10(e^{x/20} + e^{-x/20})$. Find the number of square feet of roofing on the barn.



39. A cable for a suspension bridge has the shape of a parabola with equation $y = kx^2$. Let h represent the height of the cable from its lowest point to its highest point and let $2w$ represent the total span of the bridge.



Show that the length C of the cable is given by $C = 2 \int_0^w \sqrt{1 + \frac{4h^2}{w^4} x^2} dx$

40. Find the total length of the graph of the astroid $x^{2/3} + y^{2/3} = 4$
41. Find the arc length from $(0, 3)$ clockwise to $(2, \sqrt{5})$ along the circle $x^2 + y^2 = 9$
42. Find the arc length from $(-3, 4)$ clockwise to $(4, 3)$ along the circle $x^2 + y^2 = 25$. Show that the result is one-fourth the circumference of the circle.
43. $y = \ln x$ between $x = 1$ and $x = b > 1$ that

$$\int \frac{\sqrt{x^2 + a^2}}{x} dx = \sqrt{x^2 + a^2} - a \ln \left(\frac{a + \sqrt{x^2 + a^2}}{x} \right) + C$$

Use any means to approximate the value of b for which the curve has length 2.