12. Potential due to a point source in a right, circular, metal cylinder — separation of variables.

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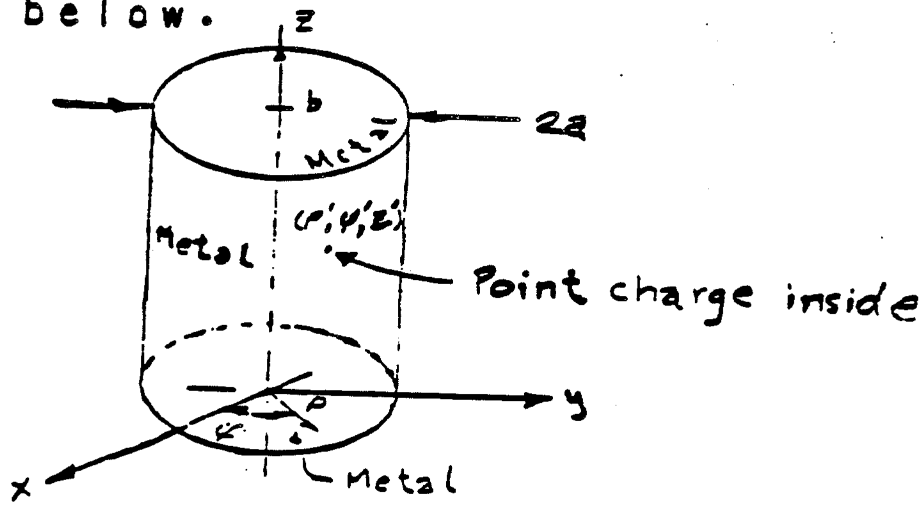
The cavity which we will analyze to provide another example of the method of separation of variables is shown below.



This problem is, in principle, no different from the rectangular cavity problem that we just solved.

In the previous problem, the elementary transcendental functions, sin, cos, sinh, cosh, etc. arose naturally from the ORDINARY differential equations obtained by SEPARATING the original PARTIAL differential equation.

In connection with the current problem, other, "higher" transcendental functions (Bessel functions) will come out the separated, ordinary differential equations.

In order to solve these differential equations, we will need to review (or learn!) the "method of Frobenius.

a) The differential equation in cylindrical coordinates and isolation of the source point.

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The differential equation in cylindrical coordinates is

$$\nabla^2 G = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho \frac{\partial G}{\partial \rho}\right] + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial \varphi^2} + \frac{\partial^2 G}{\partial z^2} = -\frac{\delta(\rho-\rho')\delta(\varphi-\varphi')\delta(z-}{\epsilon\cdot\rho}$$

(Go back and review the section on curvilinear coordinates!!!!)

There are several ways to "isolate" the source point.

We saw that in the case of a rectangular metal box, the source point could be isolated by dividing the box into regions separated by a CONSTANT COORDINATE SURFACE.

Of course, this is precisely the same way we will isolate the source in this problem.

In the rectangular cavity problem, we had three different ways to divide the cavity to isolate the source point:

By a constant

x surface,
y surface, or
z surface.

Similarly, for this problem there are also three ways to isolate the source point.

(1) Separation of the source-free regions by a constant z surface
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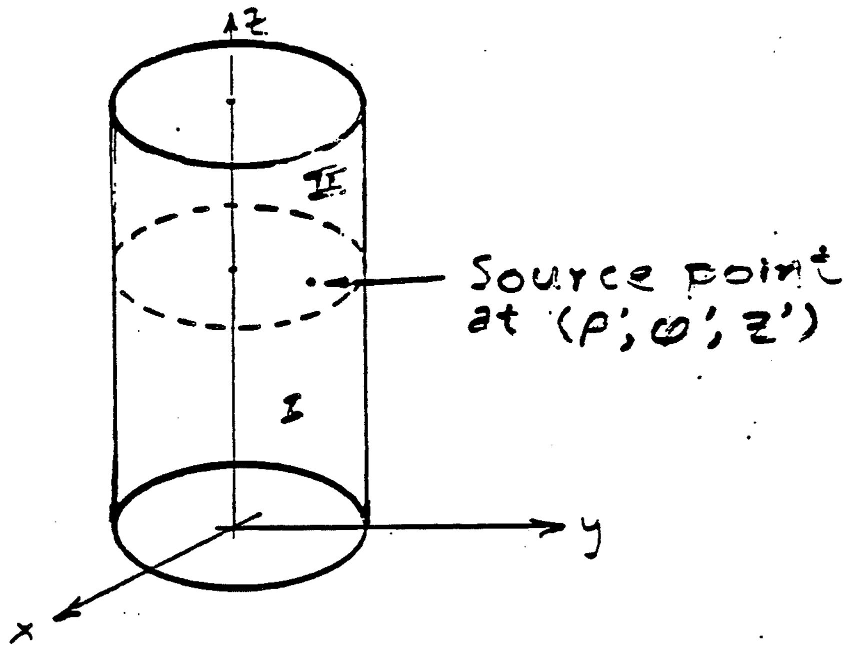
With this division of the cavity, let

Region I be $z < z'$
Region II be $z > z'$.



(2) Separation of the source-free regions by a constant radius surface

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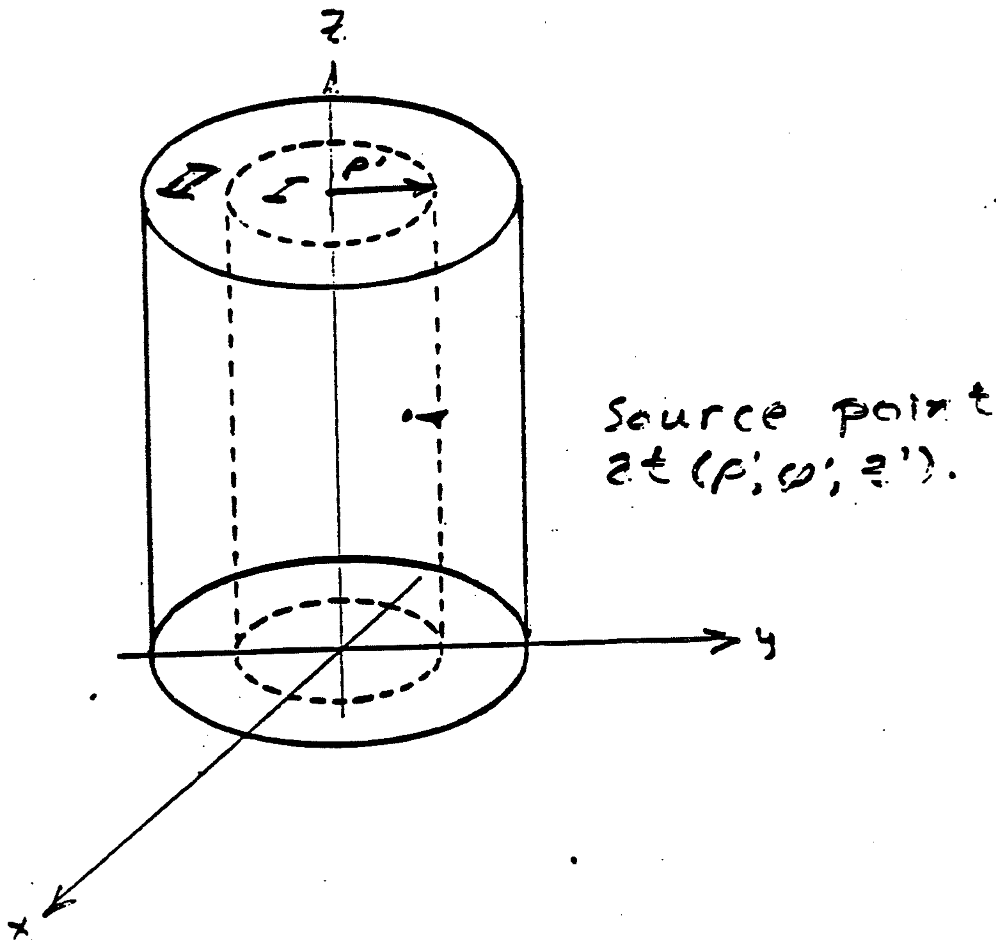
With this division of the cavity, let

Region I be $\rho < \rho'$

Region II be $\rho > \rho'$.



(3) Separation of the source-free regions by a constant angle surface

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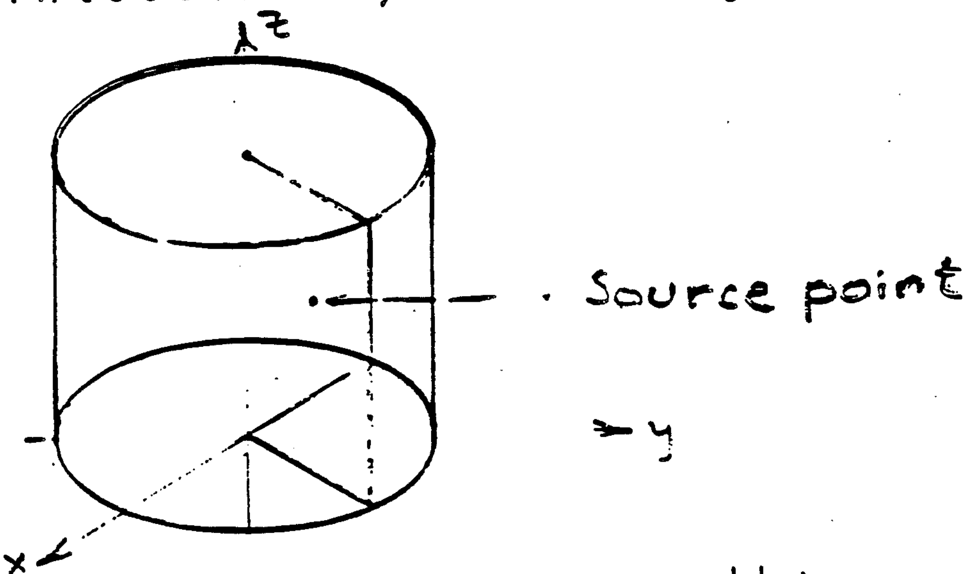
With this division of the cavity, the source point is contained in an "infintessimally thin" angular sector.



There is only one region in this case.

b) Separation of the differential equation

Poisson's differential equation which the potential,

$$G(\rho,\varphi,z|\rho',\varphi',z')$$

must satisfy is

$$\nabla^2 G(\rho,\varphi,z|\rho',\varphi',z') =$$

$$-\frac{1}{\epsilon}\,\frac{\delta(\rho-\rho')}{\rho}\,\delta(\varphi-\varphi')\,\delta(z-z')$$

In the source-free regions, the potential, G, satisfies Laplaces equation:

$$\nabla^2 G(\rho,\varphi,z|\rho',\varphi',z') = 0$$

Assuming a solution of the form

$$G = R(\rho)\,T(\varphi)\,Z(z),$$

then

$$\nabla^2 G = T\cdot Z\cdot \frac{1}{\rho}[\rho R'(\rho)]' + \frac{1}{\rho^2} R\cdot Z\cdot T''(\varphi) + R\cdot T\cdot Z''(z) = 0 \Longrightarrow$$

$$\frac{1}{R}\cdot\frac{1}{\rho}[\rho R']' + \frac{1}{\rho^2}\frac{T''}{T} + \frac{Z''}{Z} = 0$$

Thus, our partial differential equation is reduced to three ORDINARY differential equations:

$$\frac{Z''}{Z} = \gamma_z^2 \Longrightarrow \frac{1}{R}\cdot\frac{1}{\rho}[\rho R']' + \frac{1}{\rho^2}\frac{T''}{T} = -\gamma_z^2 \quad \text{or,}$$

$$\frac{1}{R}\rho[\rho R']' + \gamma_z^2\rho^2 + \frac{T''}{T} = 0$$

$$\Longrightarrow \frac{T''}{T} = -\alpha^2 \quad \text{and} \quad \rho[\rho R']' + [\gamma_z^2\rho^2 - \alpha^2]R = 0$$

c) Separation of the boundary conditions

The boundary conditions which will be applied to the solutions of these ordinary differential equations depend, of course, on how we isolate the source.

(1) Source-free regions separated
by constant z surface
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In this case,

$$G_{II}(\rho, \varphi, b \mid \rho', \varphi', z') = 0,$$

$$G_{I}(\rho, \varphi, 0 \mid \rho', \varphi', z') = 0,$$

$$G_{k}(a, \varphi, z \mid \rho', \varphi', z') = 0,$$

where k = "I" or "II."

These imply, repectively that

$$R_{II}(\rho)T_{II}(\varphi)Z_{II}(b) = 0,$$

$$R_{I}(\rho)T_{I}(\varphi)Z_{I}(0) = 0,$$

$$R_{k}(a)T_{k}(\varphi)Z_{k}(z) = 0,$$

or

$$Z_{II}(b) = 0, \quad Z_{I}(0) = 0,$$

$$R_{I}(a) = 0, \text{ and } R_{II}(a) = 0.$$

What about a conditon on

$$G_{k}(\rho,\varphi,z|\rho',\varphi',z') \ ?$$

The line $\rho = 0$ has some SINGULAR properties in cylindrical coordinates.

Usually, to define a CURVE, we must specify TWO independent conditions.

For example,

$\rho = 1, \varphi = 0,$

is a straight line parallel to the z axis passing through the x axis at x = 1.

However, the straigth line $\rho = 0$ (which coincides with the z axis) does not depend on what φ is!

Thus, $G(0,\varphi,z|\rho',\varphi',z')$ is the potential along this line which should be UNIQUE (i.e., independent of φ).

Under our assumption of a separated solution, what kind of requirements does this place on $R(\rho)$, $T(\varphi)$, and $Z(z)$, in region k?

We have that

$$R_k(0)\,T_k(\varphi)\,Z_k(z) \text{ is}$$

INDEPENDENT OF φ.

The ONLY way that this can happen is if

$T_k(\varphi)$ is constant — or —

$R_k(0) = 0.$

This condition, then, becomes the "boundary condition" for region k solutions, at $\rho = 0$.

(2) Source-free regions separated
by a constant radius surface
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In this case,

$$G_{II}(a,\varphi,z|\rho',\varphi',z') = 0,$$

$$G_k(\rho,\varphi,0|\rho',\varphi',z') = 0, \text{ and}$$

$$G_k(\rho,\varphi,b|\rho',\varphi',z') = 0, \text{ for}$$

$$k = \text{"I" or "II."}$$

Of course, the same conditions that arise from the requirement that the solution be unique apply to the solution in region I.

Namely,

$$T_I(\varphi) \text{ is constant, or}$$

$$R_I(0) = 0.$$

(3) Source-free regions separated
by a constant angle surface
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In this case, there is only one region and the boundary conditions are

$$G(a,\varphi,z|\rho',\varphi',z') = 0,$$
$$G(\rho,\varphi,0|\rho',\varphi',z') = 0, \text{ and}$$
$$G(\rho,\varphi,b|\rho',\varphi',z') = 0.$$

which implies that

$$R(a) = 0,$$
$$Z(0) = 0, \text{ and}$$
$$Z(b) = 0.$$

Again, the same conditions at $\rho = 0$ apply to the radial and angular solutions to ensure uniqueness of the solution.

d) Solutions of the separated, ordinary differential equations

(1) Differential equations in z

The differential equation for the z variable is a simple second order ordinary differential equation with CONSTANT coefficients which we have solved before in connection with the rectangular cavity problem.

In all cases, we will want this solution to vanish at z = 0 and/or z = b.

Thus, the most convenient forms of the solution which have these properties are

$\sin(k_z z)$, and $\sin[k_z(b-z)]$, or

$\sinh(\gamma_z z)$ and $\sinh[\gamma_z(b-z)]$

where $k_z = j\gamma_z$.

(2) Differential equation in angle
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The differential equation in the angular variable is identical to that in the radial variable.

The most convenient form for the solution for the cases where the cavity is divided into regions by a constant z or ρ surface is

$$\cos[\alpha(\varphi-\varphi')], \text{ or } \cosh[\beta(\varphi-\varphi')]$$

where $\beta = j\alpha$.

(3) Differential equation in radius
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The differential equation in ρ is a second order, linear differential equation, WITH NON-CONSTANT COEFFICIENTS.

You learned methods of solving this type of equation in your differential equations course. But we will review it here.

(A) Series solutions to differential equations and Fuch's theorem

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Have you ever asked yourself how sin(x) is defined?

Have ever wondered how your calculator or your favorite computer calculates sin(x)?

Surely these questions have caused you many sleepless nights!

The transcendental function, sin(x) can be defined geometrically.

Another way of defining it which is more practical from a computational standpoint is to say that it is that function which satisfies the differential equation,

$$y'' + y = 0$$

subject to $y(0) = 0$, $y'(0) = 1$. It is fine to TALK about such a function, but how do we actually EXPRESS that function in a USEFUL form?

Well, it will be shown below that a POWER SERIES expansion for $y(x) = \sin(x)$ exists, is convergent for all x to $\sin(x)$, and is given by

$$x - x^3 / 3! + x^5 / 5! - \ldots$$

If we were stuck out on a desert isle without or calculator and had a burning desire to compute the $\sin(.378)$, we COULD do it using such a series.

What we need to do to solve the problem at hand is to find a function R(x) that satisfies the differential equation,

$$x[xR(x)]' + [(kx)^2 - p^2] R(x) = 0.$$

Can we find a series solution to THIS equation as can be done for the differential equation that $\sin(x)$ satisfies?

This question was answered in a theorem due to Fuchs.

Suppose that

$$y'' + P(x)y' + Q(x)y = 0.$$

Then the point x = u is called an ORDINARY POINT of the differential equation if and only if both P(x) and Q(x) possess Taylor's series about x = u.

For example, in the equation,

$$x[xy']' + [(kx)^2 - p^2]y = 0$$

----->

$$x^2 y'' + xy' + [(kx)^2 - p^2]y = 0$$

----->

$$y'' + y'/x + [(kx)^2 - p^2]y/x^2 = 0,$$

the point x = 1 is an ordinary point since

$$1/x \text{ and } [(kx)^2 - p^2]/x^2$$

both have Taylor series about x=1.

If $x = u$ is NOT an ordinary point, then it is a SINGULAR POINT.

In the previous example, is $x = 0$ an ordinary or a singular point?

A singular point, $x = u$, is said to be a REGULAR SINGULAR POINT if and only if

$$(x - u)P(x) \text{ and } (x - u)^2 Q(x)$$

both have Taylor series expansions about $x = u$.

A singular point which is not regular is called an IRREGULAR SINGULAR POINT.

In the example above, is the point $x = 0$ a regular, or an irregular singular point?

We are now ready to state Fuchs' theorem:

For the differential equation,

$$y'' + P(x)y' + Q(x)y = 0,$$

If $x = u$ is an ordinary point, then a convergent series solution to the differential equation exists and is of the form

$$y = \sum_{n=0}^{\infty} a_n (x-u)^n$$

If $x = u$ is a regular singular point, then a convergent series solution to the differential equation exists and is of the form

$$y = \sum_{n=0}^{\infty} a_n (x-u)^{n+k}$$

Note that if $x = u$ is an irregular singular point, in general, a simple series solution like those above may not exist in general.

We can apply this theorem to our problem to say that the function R(x), whatever it is, can be put in the form of

$$R(x) = \sum_{n=0}^{\infty} a_n x^{n+k}$$

since x = 0 is a regular singular point. The important detail that remains is how to actually find out what the expansion coefficients in this series are.

(B) The method of Frobenius
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The coefficients of the series expansions of differential equations in a certain class can be found by the method of Frobenius.

As a first example of this method, let us find the solutions to

$$y'' + y = 0$$

We already know that the solutions two this differential equation are sin(x) and cos(x).

Since the point x = 0 (as well as any other point) is an ordinary point of the differential equation, we know from Fuchs' theorem that a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

exists.

Substituting this back into the differential equation,

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$y'' + y = \underbrace{\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}}_{\text{replace } n \text{ by } n+2} + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=-2}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} + a_n\right] x^n = 0$$

Taylor series are unique $\Rightarrow$

$$[(n+1)(n+2) a_{n+2} + a_n] = 0 \quad \text{for } n = 0, 1, \ldots$$

We can now use this expression, called a RECURRENCE FORMULA, or DIFFERENCE EQUATION, to generate two independent tables —

-1- relating a_{2n} to a_0

and

-2- relating a_{2n+1} to a_1.

$$a_2 = -1/(2 \cdot 1)\ a_0$$

$$a_4 = -1/(4 \cdot 3)\ a_2 = (-1)^2/4!\ a_0$$

$$\vdots$$

$$a_{2n} = (-1)^n / (2n)!\ a_0$$

$$a_3 = -1/(3\cdot 2)\ a_1$$

$$a_5 = -1/(5\cdot 4)\ a_3 = (-1)^2/5!\ a_1$$

$$\vdots$$

$$a_{2n+1} = (-1)^n / (2n+1)!\ a_1$$

Therefore, the solution to differential equation is

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}}_{\cos(x)}$$

$$+ a_1 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}_{\sin(x)}$$

The coefficients,

a_0 and a_1

represent two ARBITRARY coefficients and the solution given above is the GENERAL SOLUTION to the differential equation.

The two infinite series are IDENTIFIED as sin(x) and cos(x).

These series could be taken to represent the DEFINITION of these two functions.

This series coverges absolutely for all x (real or complex).

This can be seen by an application of the ratio test:

$$\frac{|a_{2n+2}x^{2n+2}|}{|a_{2n}x^{2n}|} = \frac{x^2 \cdot 1/(2n+2)!}{1/2n!}$$

$$= x^2/(2n+2)(2n+1) \longrightarrow 0, \; n \to \infty.$$

Similarly $|a_{2n+3}x^{2n+3}|/|a_{2n+1}x^{2n+1}| \longrightarrow 0$ as $n \to \infty$.

Although this series converges everywhere, it is not the most practical way to compute sin(x) and cos(x).

We will see other ways that are somewhat more practical in the computation of transcendental functions.

Now that we have covered a familiar function and differential equation, let's try to solve Bessel's differential equation:

$$x[xy']' + (x^2 - p^2)y = 0$$

$$\longleftrightarrow$$

$$x^2 y'' + xy' + (x^2 - p^2)y = 0.$$

Since the point x = 0 is a regular singular point, according to Fuchs' theorem, we must assume a solution of the form:

$$y = x^k \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+k}.$$

Substituting,

$$\sum_{n=0}^{\infty} a_n \{(n+k)(n+k-1)x^{n+k} + (n+k)x^{n+k} + x^{n+2+k} - p^2 x^{n+k}\}$$

$$= \sum_{n=0}^{\infty} \{[(n+k)^2 - p^2]x^{n+k}\} a_n + \sum_{n=0}^{\infty} x^{n+2+k} a_n$$

$$= \sum_{n=0}^{\infty} \{[(n+k)^2 - p^2]x^{n+k}\} a_n + \sum_{n=2}^{\infty} x^{n+k} a_{n-2}$$

$$= x^k \{[k^2-p^2]a_0 + [(k+n)^2 - p^2]a_1 x + \sum_{n=2}^{\infty}[[(k+n)^2-p^2]a_n + a_{n-2}]x^n$$

$$= 0$$

Equating coefficients to zero,

$$[k^2 - p^2]a_0 = 0$$

$$[(k+1)^2 - p^2]a_1 = 0$$

$$[(k+n)^2 - p^2]a_n + a_{n-2} = 0 \quad \text{for } n = 2, 3 \ldots$$

$$\longrightarrow a_n = -\frac{1}{[(k+n)^2 - p^2]} a_{n-2} \qquad n = 2, 3, \ldots.$$

If

$$a_0 \neq 0$$

then

$$\underbrace{k^2 - p^2 = 0}_{\text{The "indicial equation"}} \longrightarrow k = \pm p$$

and

$$a_1 = 0.$$

Or, if

$$a_1 \neq 0$$

then

$$(k+1)^2 - p^2 = 0 \longrightarrow k = -1 \pm p \text{ and } a_0 = 0.$$

Let's first take the case

$$a_0 \neq 0, \quad a_1 = 0, \quad k = \pm p$$

Then the recurrence formula is

$$a_n = -\frac{1}{[(n \pm p)^2 - p^2]} a_{n-2}$$

$$= -\frac{1}{[(n \pm p) + p][(n \pm p) - p]} a_{n-2}$$

Specifically, let's take $k=+p$. Then

$$a_n = -\frac{1}{n(n+2p)}a_{n-2}$$

$$a_2 = -\frac{1}{2\cdot(2+2p)}a_0$$

$$a_4 = -\frac{1}{4(4+2p)}a_2 = (-1)^2\frac{a_0}{4\cdot 2\cdot(4+2p)(2+2p)}$$

$$= (-1)^2\frac{1}{2^2\cdot[2\cdot 1]\cdot 2^2[(2+p)(1+p)]}a_0$$

$$\vdots$$

$$a_{2n} = (-1)^n\frac{a_0}{2^{2n}n![(n+p)(n-1+p)\cdots(1+p)]}$$

Products of the form

$$[(n+p)(n-1+p)\ldots(1+p)]$$

can be conveniently written in terms of the GAMMA function:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt\ .$$

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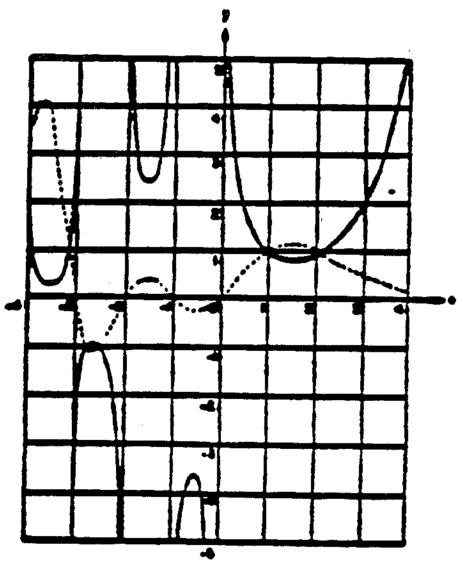


FIGURE 6.1. *Gamma function.*

———, $y=\Gamma(x)$. - - - -, $y=1/\Gamma(x)$

The gamma function has the property that

$$\Gamma(x+1) = x\Gamma(x)$$

which allows us to define $\Gamma(x)$ for $x<0$.

We also note that $\Gamma(1) = 1$.

From these equations, we can see that if x is the integer, m, then

$$\Gamma(m+1) = m!$$

The graph of $\Gamma(x)$ is shown below.

Now, how do we use the gamma function to more succinctly express the product

$$[(n+p)(n-1+p)\ldots(1+p)]\ ?$$

Using the recurrence relation of the gamma function, we see that

$$\begin{aligned}\Gamma(n+p+1) &= (n+p)\Gamma(n+p)\\ &= (n+p)\cdot(n+p-1)\cdot\Gamma(n+p-1)\\ &\vdots\\ &= (n+p)(n-1+p)\cdots(1+p)\Gamma(1+p).\end{aligned}$$

Therefore,

$$[(n+p)(n-1+p)\cdots(1+p)] = \frac{\Gamma(n+p+1)}{\Gamma(1+p)}$$

Therefore, we can write

$$a_{2n} = (-1)^n \frac{\Gamma(p+1)}{2^{2n}\, n!\, \Gamma(n+p+1)} a_0$$

We can now expand one solution to Bessel's differential equation:

$$y = x^p \sum_{n=0}^{\infty} a_n x^n = x^p \sum_{n=0}^{\infty} a_{2n} x^{2n}$$

$$= x^p \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(p+1)\, a_0}{2^{2n}\, n!\, \Gamma(n+p+1)} x^{2n}$$

$$= \left[2^p \Gamma(p+1)\, a_0\right] \cdot \underbrace{\left(\frac{x}{2}\right)^p \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n!\, \Gamma(n+p+1)}}_{J_p(x)}$$

The function $J_p(x)$ is called the BESSEL FUNCTION of the FIRST KIND of ORDER, p.

Remember, this solution was generated by taking $k=+p$.

We have another solution that corresponds to $k=-p$.

This solution can be found simply by replacing p by −p:

$$J_{-p}(x) = \left(\frac{x}{2}\right)^{-p} \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n!\,\Gamma(n-p+1)} .$$

We also found that we could choose

$$k = -1 \pm p, \quad a_0 = 0, \quad a_1 \neq 0$$

In this case, the recurrence formula is

$$a_n = \frac{-1}{[(n-1\pm p)^2 - p^2]} a_{n-2}$$

$$= \frac{-1}{[n-1\pm p - p][n-1\pm p+p]} a_{n-2} .$$

For $k=-1+p$,

$$a_3 = -\frac{1}{2\cdot(2+2p)} a_1$$

$$a_5 = -\frac{1}{4\cdot(4+2p)} a_3 = (-1)^2 \frac{1}{4\cdot 2\cdot(4+2p)(2+2p)} a$$

$$\vdots$$

$$a_{2n+1} = (-1)^n \frac{\Gamma(p+1)\, a_1}{2^{2n}\, n!\, \Gamma(n+p+1)}$$

Thus, THIS solution is

$$x^{-1+p}\sum_{n=0}^{\infty}\frac{(-1)^n\,\Gamma(p+1)\,a_1}{2^{2n}\,n!\,\Gamma(n+p+1)}\,x^{2n+1}$$

$$=\left[2^p\Gamma(p+1)\,a_1\right]\left(\frac{x}{2}\right)^p\sum_{n=0}^{\infty}\frac{(-1)^n\,(x/2)^{2n}}{n!\,\Gamma(n+p+1)}$$

Therefore, choosing $k=-1+p$ yields the SAME reuslt as the choice $k=p$!

Similarly, $k=-1-p$ yields the SAME result as the choice $k=-p$.

Thus, it appears that there are at MOST two linearly independent solutions to the differential equation (as expected).

The two solutions are

$$J_{-p}(x) \quad \text{and} \quad J_p(x).$$

But are $J_p(x)$ and $J_{-p}(x)$ really linearly independent?

If we look at the behavior of the solutions for x nearly zero, we can see right away that if p is NOT AN INTEGER, the two solutions are independent.

For example, for p=0.5,

$$J_{.5}(x) = \sqrt{\frac{x}{2}} \cdot \left[\frac{1}{\Gamma(1.5)} - \frac{1}{\Gamma(2.5)} \left(\frac{x}{2}\right)^2 + \cdots \right]$$

$$J_{-.5}(x) = \sqrt{\frac{2}{x}} \left[\frac{1}{\Gamma(.5)} - \frac{1}{\Gamma(1.5)} \left(\frac{x}{2}\right)^2 + \cdots \right]$$

However, if p is an integer, for example, p=3, then

$$J_{-3}(x) = \left(\frac{x}{2}\right)^{-3} \left\{ \frac{1}{\Gamma(-2)} - \frac{1}{1!\,\Gamma(-1)} \left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(0)} \left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\Gamma(1)} \left(\frac{x}{2}\right. + \cdots \right\}$$

But, from the graph of $\Gamma(x)$, we see that $\Gamma(x)$ goes to infinity at x = negative integer, or, $1/\Gamma(x)$ goes to zero at a negative integer.

Thus,

$$J_{-3}(x) = \left(\frac{x}{2}\right)^{-3} \left\{ -\frac{1}{\Gamma(1)} \cdot \frac{1}{3!} \left(\frac{x}{2}\right)^6 + \frac{1}{\Gamma(2)} \cdot \frac{1}{4!} \left(\frac{x}{2}\right)^8 - \cdots \right\}$$

$$= -\left(\frac{x}{2}\right)^3 \left\{ \frac{1}{0!} \cdot \frac{1}{\Gamma(4)} - \frac{1}{1!} \frac{1}{\Gamma(5)} \left(\frac{x}{2}\right)^2 + \cdots \right\}$$

which is just $(-1)^3 J_3(x)$.

In general, we can show that

$$J_{-n}(x) = (-1)^n J_n(x).$$

It would appear from this, that only one solution to Bessel's differential equation can be found when p is an integer.

This situation is analogous to that which arises in the differential equation,

$$y'' + 2y' + y = 0$$

If we assume a solution of the form

$$y = e^{st}$$

into the equation, we obtain

$$s^2 + 2s + 1 = (s+1)^2 = 0 \Rightarrow s = -1 \pm 0.$$

which has a DOUBLE root at $s=-1$.

For this differential equation, however, we know that there IS another independent solution of the form

$$y = te^{-t}$$

It is similary also true that Bessel's differential equation also has TWO linearly independent solutions EVEN when p is an integer.

The way this other solution can be obtained is the following.

First define the BESSEL FUNCTION of the SECOND KIND of ORDER p as

$$Y_p(x) = \frac{J_p(x)\cos(p\pi) - J_{-p}(x)}{\sin(p\pi)}.$$

Because of the linearity of Bessel's equation, this also satisfies the differential equation.

If we set p=m, an integer, then the expression for $Y_m(x)$ becomes

$$Y_m(x) = \frac{J_m(x)\cos(m\pi) - J_{-m}(x)}{\sin(m\pi)}$$

$$= \frac{J_m(x)(-1)^m - (-1)^m J_m(x)}{\sin(m\pi)} = \frac{0}{0}.$$

Clearly, we must evaluate this expression by taking its LIMIT as $p \longrightarrow m$.

$$Y_m(x) = \lim_{p\to m}\left\{\frac{J_p(x)\cos(p\pi) - J_{-p}(x)}{\sin(p\pi)}\right.$$

Since we have an indeterminant form, we can apply L'Hospital's rule to obtain

$$Y_m(x) = \left. \frac{\frac{\partial}{\partial p}\left[J_p(x)\cos(p\pi) - J_{-p}(x)\right]}{\frac{\partial}{\partial p}\left[\sin(p\pi)\right]}\right|_{p=}$$

$$= \frac{\left.\frac{\partial J_p}{\partial p}\right|_{p=m}(-1)^m - \left.\frac{\partial J_{-p}}{\partial p}\right|_{p=m}}{\pi(-1)^m}.$$

We need only differentiate (termwise) the series for $J_{\pm p}(x)$ to obtain the expression for Y.

The result of this is

$$Y_m(x) = \frac{2}{\pi}J_m(x)\left[\ln\left(\frac{x}{2}\right)+\gamma\right] - \frac{1}{\pi}\sum_{n=0}^{m-1}\frac{(x/2)^{2n-m}(m-n-1)!}{n!} - \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n(x/2)^{2n+m}\left[\varphi(m+n)+\varphi(n)\right]}{n!\,(m+n)!}$$

where

$$\gamma \cong 0.577$$

is Euler's constant and

$$\varphi(n) = \sum_{k=1}^{n}\frac{1}{k} \text{ with } \varphi(0)=0.$$

Thus, the general solution to Bessel's differential equation is

$$y(x) = A J_p(x) + B Y_p(x)$$

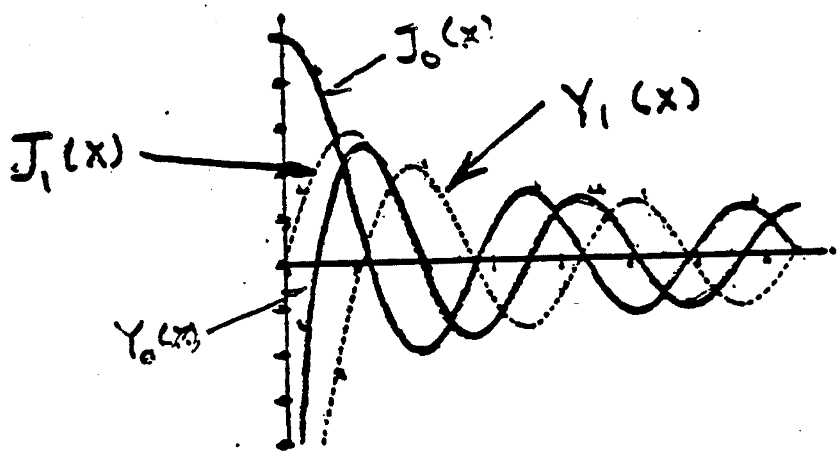
This is analogous to the general solution to

$$y'' + y = 0$$

which is

$$A\cos(x) + B\sin(x).$$

The graphs of $J_p(x)$ and $Y_p(x)$ for various values of p are shown below.


$J_0(x)$
$Y_1(x)$
$J_1(x)$
$Y_0(x)$

So we don't loose sight of our mission, let's see what we have got so far:

-1- We defined the problem, and wrote the differential equation for the potential due to a point source.

-2- We isolated the source by separating the "can" into source-free regions bounded by a constant z plane, or a constant radius surface, or a constant angle plane.

-3- We separated the partial differential equation into three ordinary differential equations, one in the z variable, one in the angle variable, and one in the radius variable.

-4- We found what boundary conditions the Z, radial, and angular separted functions must satisfy for the three possible methods of isolating the source point.

-5- Finally, we solved the differential equations for the z, angular, and the radial variables.

The next logical step is to apply the boundary conditions to each of the solutions we have obtained. Application of the boundary conditions will determine what the separation constants must be.

However, before we go on to this step, we will discuss the orthogonality of the Bessel functions since this question will also arise later as it did for the metal box problem.

This brings us to the topic of

(C) Sturm-Liouville systems

In the metal box problem, we had to find the solution to the differential equation

$$Y''(y) + k_y^2 Y(y) = 0$$

subject to

$$Y(0) = Y(b) = 0.$$

We found that a solution to this problem could only be found for DISCRETE values of k_y:

$$Y(y) = \sin[(n\pi/b)y].$$

This is an example of a STURM-LIOUVILLE PROBLEM.

The general Sturm-Liouville problem is to find solutions to

$$\frac{d}{dx}[r(x)y'] + [q(x) + p\,\omega(x)]y = 0$$

subject to

$r(x) > 0$ and $\omega(x) > 0$,

$$Ay'(a) + By(a) = 0$$

$$Ay'(b) + By(b) = 0$$

There are at most, two linearly independent solutions to this second order differential equation, say

$$y(x) = Cu(x;p) + Dv(x;p)$$

Then, applying the boundary conditions,

$$A[u'(a;p)\cdot C + v'(a;p)\cdot D] + B[u(a;p)C + v(a;p)D] = 0$$

$$A[u'(b;p)\cdot C + v'(b;p)\cdot D] + B[u(b;p)C + v(b;p)D] = 0$$

$$\Rightarrow \begin{bmatrix} [Au'(a;p) + Bu(a;p)] & [Av'(a;p) + Bv'(a;p)] \\ [Au'(b;p) + Bu(b;p)] & [Av'(b;p) + Bv(b;p)] \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = 0$$

If we wish to have a non-trivial solution, then the determinant of coefficients,

$$\det \begin{bmatrix} [\cdot] & [\cdot] \\ [\cdot] & [\cdot] \end{bmatrix}$$

must vanish.

In general, this will vanish ONLY for discrete values of p, say for p in the set (p_m).

These discrete values of p are called EIGENVALUES.

There associated functions,

$$y_n(x) = C_n u(x;p_n) + D_n v(x;p_n)$$

are called EIGENFUNCTIONS.

One of the coefficients of $y_n(x)$ can be taken arbitrarily.

However, the other must be taken so that boundary conditions are satisfied:

$$[Au'(a;p_n) + Bu(a;p_n)]\,C_n + [Av'(a;p_n) + Bv(a;p_n)]\,D_n = 0 \longrightarrow$$

$$C_n = -\frac{Av'(a;p_n) + Bv(a;p_n)}{Au'(a;p_n) + Bu(a;p_n)} \cdot D_n .$$

Thus, the solution to the Sturm-Liouville problem can be taken to be

$$y_n(x) = D_n \cdot \left\{ v(x;p_n) - \frac{Av'(a;p_n) + Bv(a;p_n)}{Au'(a;p_n) + Bu(a;p_n)}\, u(x;p_n) \right.$$

The set of solutions, $\{y_n(x)\}$, have a very interesting property: THEY ARE ORTHOGONAL over the interval [a,b] with respect to a WEIGHT FUNCTION, w(x)!

This result is easy to prove. Take

$$[r(x)y_n']' + [g(x) + P_n w(x)]y_n = 0$$

$$[r(x)y_m']' + [g(x) + P_m w(x)]y_m = 0$$

Then

$$y_m[ry_n']' + g\cdot y_n y_m + P_n w\cdot y_n y_m = 0$$

$$\underline{y_n[ry_m']' + g\, y_m y_n + P_m w\cdot y_m y_n = 0}$$

$$\{y_m[ry_n']' - y_n[ry_m']'\} + w\cdot y_n y_m (P_n - P_m) = 0$$

Notice that the bracketed term of this equation is just

$$\{y_m[ry_n']' - y_n[ry_m']'\} = \{r\cdot[y_n' y_m - y_n y_m']\}'$$

Therefore,

$$(P_n - P_m)\, w\cdot y_n y_m = -\frac{d}{dx}\{r[y_n' y_m - y_n y_m']\}$$

Integrating this result from a to b,

$$(P_n - P_m)\int_a^b w\cdot y_n y_m\, dx = -\int_a^b \frac{d}{dx}\{\ \}\, dx$$

$$= \{\ \}_a^b = r(a)[y_n'(a)y_m(a) - y_n(a)y_m'(a)]$$

$$- r(b)[y_n'(b)y_m(b) - y_n(b)y_m'(b)]$$

But $[y_n'(a)y_m(a) - y_n(a)y_m'(a)] =$

$$\det\begin{bmatrix} y_n'(a) & y_n(a) \\ y_m'(a) & y_m(a) \end{bmatrix} = 0 \quad \text{since}$$

$$A y_n'(a) + B y_n(a) = 0$$
$$A y_m'(a) + B y_m(a) = 0$$

A similar result holds for $y = b$.

Therefore, $\langle y_n, y_m \rangle = 0$ for $m \neq n$.

The one arbitrary coefficient can be adjusted so that the norms of the y's are unity.

It can also be shown that the set $\{y\}$, like the set of trigonometric functions (which are also solutions to a Sturm-Liouville problem) is COMPLETE set.

e) Application of the boundary conditions and continuity conditions - determination of separation constants and expansion coefficients

Now that we have the solutions to the three ordinary differential equations,

$$Z''(z) - \gamma_z^2 Z(z) = 0$$

$$T''(\varphi) + \alpha^2 T(\varphi) = 0$$

$$\rho[\rho R'(\rho)]' + [\gamma_z^2 \rho^2 - \alpha^2] R(\rho) = 0$$

We are ready to APPLY the boundary conditions.

Application of the boundary conditions will determine the (as yet undetermined) separation constants, γ_z and α.

(1) Source-free regions separated
by constant z surface
<<<<<<<<<<<<<<<<<>>>>>>>>>>>>>>>>>>

The general solution to $T(\varphi)$'s differential equation is

$$T(\varphi) = A\cos(\alpha\varphi) + B\sin(\alpha\varphi),$$

or, what is more convenient in view of the obvious symmetry of the problem,

$$T(\varphi) = A\cos[\alpha(\varphi - \varphi')] + B\sin[\alpha(\varphi - \varphi')].$$

We had NO boundary conditions for $T(\varphi)$.

What properties should our solution have?

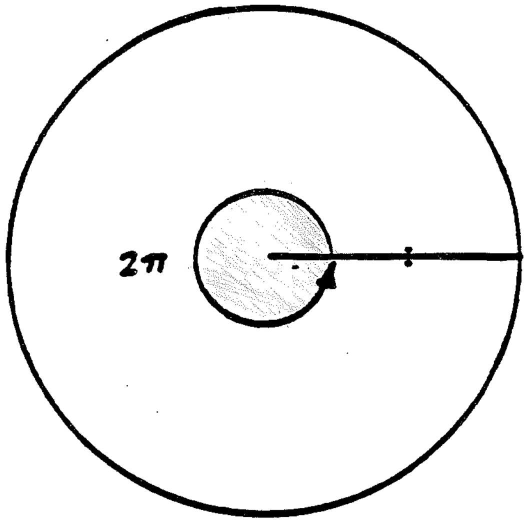
First of all, the symmetry of the problem demands EVEN SYMMETRY about

$$\varphi = \varphi'$$

Therefore, we can eliminate the sin term leaving

$$T(\varphi) = A\cos[\alpha(\varphi - \varphi')]$$

Furthermore, if $\varphi' = 0$, then how should $T(0)$ and $T(2\pi)$ be related?



The potential is continuous!

Thus,

$$T(2\pi) = T(0) \rightarrow \cos(\alpha \cdot 2\pi) = \cos(0) = 1$$
$$\longrightarrow \alpha \cdot 2\pi = n \cdot 2\pi \rightarrow \alpha = n, \quad n = 0, 1, 2, \ldots$$

Therefore,

$$T_n(\varphi) = \cos[n(\varphi - \varphi')]$$

satisfies the differential equation and the PERIODICITY condition.

We note that these functions form an orthogonal set:

$$\int_0^{2\pi} T_n(\varphi) T_k(\varphi)\, d\varphi = \int_0^{2\pi} d\varphi \cos[n(\varphi-\varphi')]\cos[k(\varphi-\varphi')]$$
$$= \frac{2\pi}{\epsilon_n} \delta_{nk} \quad \text{where } \epsilon_n = \begin{cases} 1 & n = 0 \\ 2 & n \neq 0 \end{cases}$$

Next, we turn to the differential equation in ρ.

We found that the solution to Bessel's differential equation,

$$x[xy'(x)]' + [x^2 - p^2]\,y(x) = 0$$

was

$$y(x) = A\,J_p(x) + B\,Y_p(x).$$

But we need to solve

$$\rho[\rho R'(\rho)]' + [\gamma_z^2\rho^2 - \underbrace{\alpha^2}_{n^2}]\,R(\rho) = 0$$

Note that if $x = \gamma_z\rho$,

$$x\frac{d}{dx} = x\frac{d\rho}{dx}\cdot\frac{d}{d\rho} = \gamma_z\rho\frac{1}{\gamma_z}\frac{d}{d\rho} = \rho\frac{d}{d\rho}$$

so that

$$R(\rho) = y(\gamma_z\rho) = A\,J_n(\gamma_z\rho) + B\,Y_n(\gamma_z\rho).$$

Now remember that a "boundary condition" on $R(\rho)$ was that $R(0) = 0$ if $T(\varphi)$ was not a constant.

We note that $Y_n(0)$ is infinite for every n and that $J_n(0)=0$ for $n=1,2,3,\ldots$.

Thus, we must choose B=0 for n=1,2,3,... .

(Note that for n=0, $T_0(\varphi) = \cos(0) = 1$ is a constant. In this case, however, the physics demands that B=0 so that the potential be finite along $\rho = 0$.

Furthermore, R(a) = 0 in both regions I and II and therefore, we must find a γ_z such that

$$J_n(\gamma_z a) = J_n(\zeta_{mn}) = 0$$

where $\boxed{\zeta_{mn} = \gamma_z a}$ are the roots or zeros of the Bessel function, J_n.

We can determine these roots from the graph of the Bessel function. It is clear from that graph that there are an infinite number of such roots.

(Compare this to the situation we had in the metal box problem where we had to solve sin(ka)=0 and found an infinite number of roots, k = $m\pi/a$).

A more practical way to determine these Bessel function zeros is to look them up in a table, such as the one below.

Thus, the radial solutions are

$$R_{mn}(\rho) = J_n\left(\frac{\zeta_{mn}}{a}\cdot\rho\right).$$

Some of these solutions are graphed below.

From our consideration of the Sturm-Liouville problem, we know that $\{J_n(\zeta_{mn}\rho/a)\}$ forms an orthogonal set:

$$\int_0^a \rho J_n(\zeta_{mn}\rho/a)\, J_n(\zeta_{kn}\rho/a)\, d\rho = \delta_{km} \cdot \delta_{ml}\, \|J_n(\zeta_{mn}\rho/a)\|^2$$

It can be shown that

$$\|J_n(\zeta_{mn}\rho/a)\|^2 = \frac{a^2}{2}\left[J_n'(\zeta_{mn})\right]^2.$$

$$\int_0 \rho J_m\, J_l\, d\rho = 0 \quad m \neq l$$

Table 9.5

ZEROS AND ASSOCIATED VALUES OF BESSEL FUNCTIONS AND THEIR DERIVATIVES

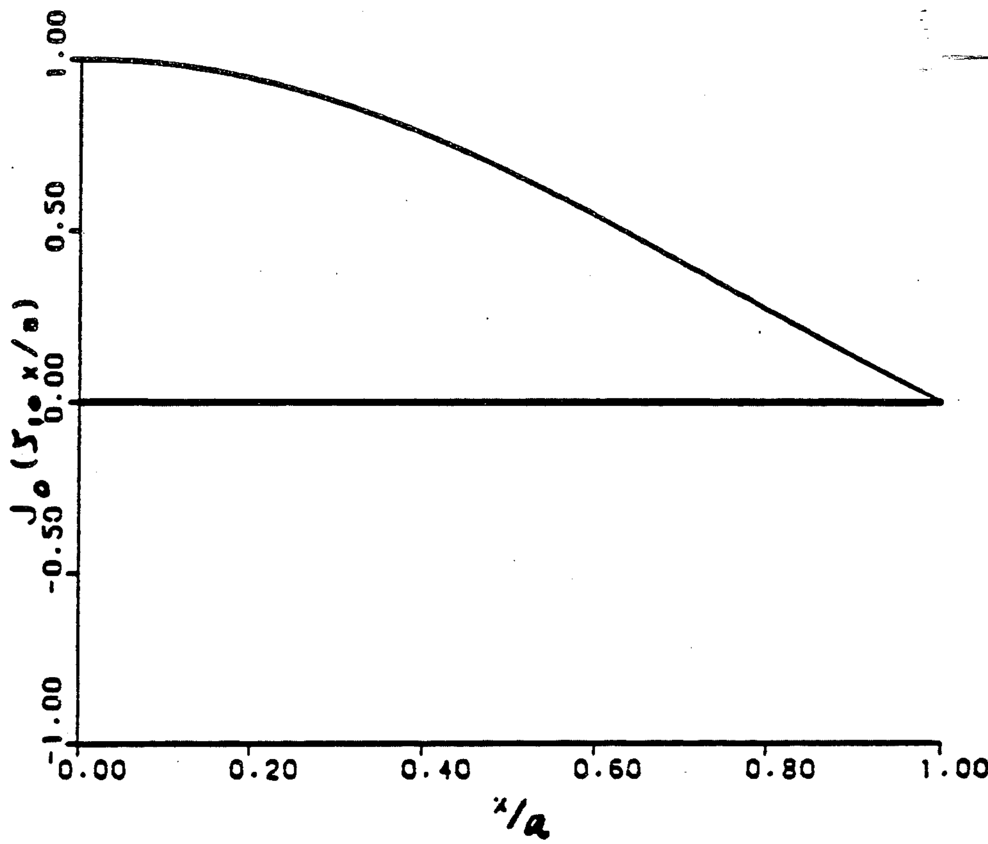
0 1 2

s	$j_{0,s}$	$J_0'(j_{0,s})$	$j_{1,s}$	$J_1'(j_{1,s})$	$j_{2,s}$	$J_2'(j_{2,s})$
1	2.40482 55577	−0.51914 74973	3.83171	−0.40276	5.13562	−0.33967
2	5.52007 81103	+0.34026 48065	7.01559	+0.30012	8.41724	+0.27138
3	8.65372 79129	−0.27145 22999	10.17347	−0.24970	11.61984	−0.23244
4	11.79153 44391	+0.23245 98314	13.32369	+0.21836	14.79595	+0.20654
5	14.93091 77086	−0.20654 64331	16.47063	−0.19647	17.95982	−0.18773
6	18.07106 39679	+0.18772 88030	19.61586	+0.18006	21.11700	+0.17326
7	21.21163 66299	−0.17326 58942	22.76008	−0.16718	24.27011	−0.16170
8	24.35247 15308	+0.16170 15507	25.90367	+0.15672	27.42057	+0.15218
9	27.49347 91320	−0.15218 12138	29.04683	−0.14801	30.56920	−0.14417
10	30.63460 64684	+0.14416 59777	32.18968	+0.14061	33.71652	+0.13730
11	33.77582 02134	−0.13729 69434	35.33231	−0.13421	36.86286	−0.13132
12	36.91709 83537	+0.13132 46267	38.47477	+0.12862	40.00845	+0.12607
13	40.05842 57646	−0.12606 94971	41.61709	−0.12367	43.15345	−0.12140
14	43.19979 17132	+0.12139 86248	44.75932	+0.11925	46.29800	+0.11721
15	46.34118 83717	−0.11721 11989	47.90146	−0.11527	49.44216	−0.11343
16	49.48260 98974	+0.11342 91926	51.04354	+0.11167	52.58602	+0.10999
17	52.62405 18411	−0.10999 11430	54.18555	−0.10839	55.72963	−0.10685
18	55.76551 07550	+0.10684 78883	57.32753	+0.10537	58.87302	+0.10396
19	58.90698 39261	−0.10395 95729	60.46946	−0.10260	62.01622	−0.10129
20	62.04846 91902	+0.10129 34989	63.61136	+0.10004	65.15927	+0.09882

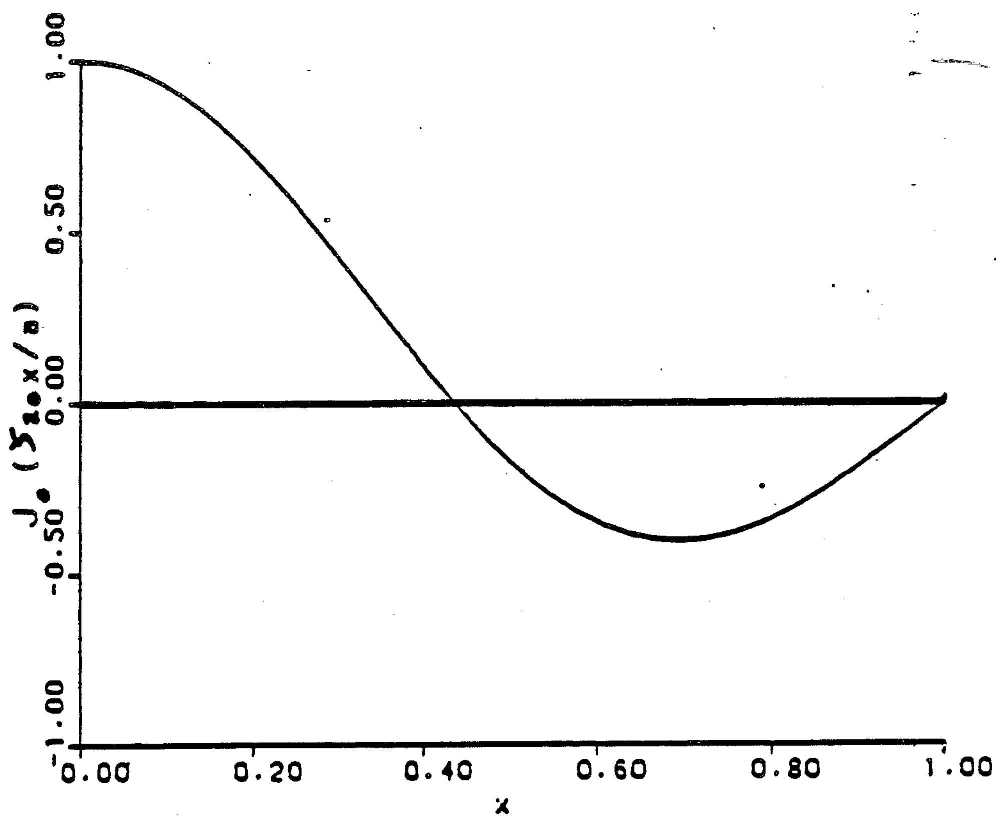
3 4 5

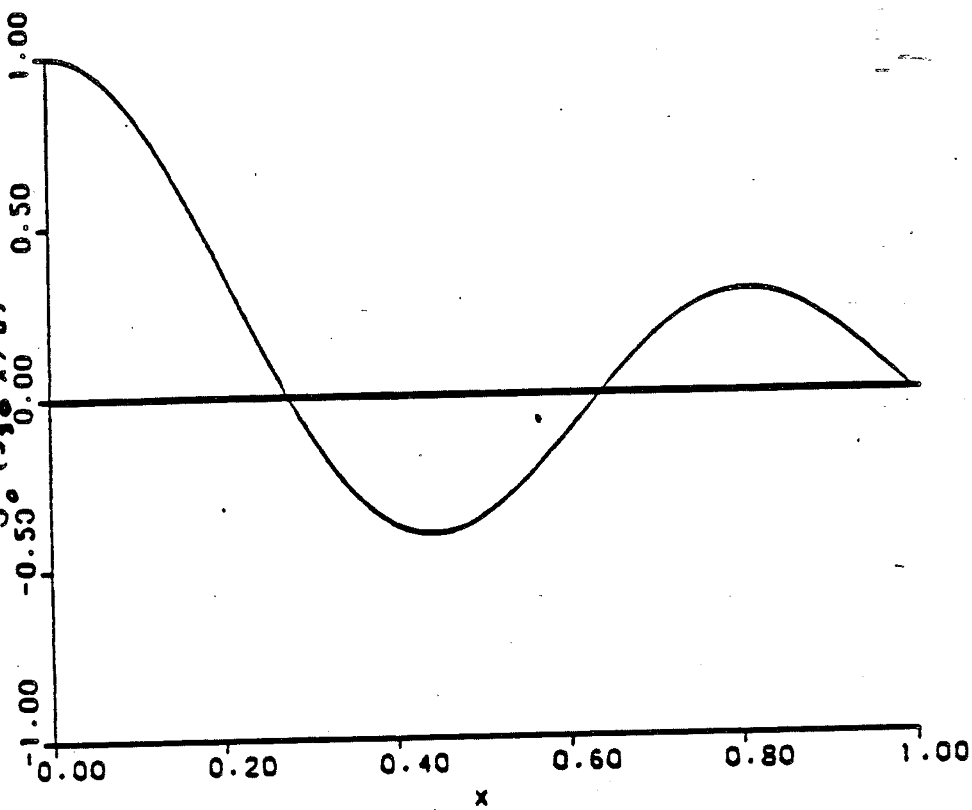
s	$j_{3,s}$	$J_3'(j_{3,s})$	$j_{4,s}$	$J_4'(j_{4,s})$	$j_{5,s}$	$J_5'(j_{5,s})$
1	6.38016	−0.29827	7.58834	−0.26836	8.77148	−0.24543
2	9.76102	+0.24942	11.06471	+0.23188	12.33860	+0.21743
3	13.01520	−0.21828	14.37254	−0.20636	15.70017	−0.19615
4	16.22347	+0.19644	17.61597	+0.18766	18.98013	+0.17993
5	19.40942	−0.18005	20.82693	−0.17323	22.21780	−0.16712
6	22.58273	+0.16718	24.01902	+0.16168	25.43034	+0.15669
7	25.74817	−0.15672	27.19909	−0.15217	28.62662	−0.14799
8	28.90835	+0.14801	30.37101	+0.14416	31.81172	+0.14059
9	32.06485	−0.14060	33.53714	−0.13729	34.98878	−0.13420
10	35.21867	+0.13421	36.69900	+0.13132	38.15987	+0.12861
11	38.37047	−0.12862	39.85763	−0.12607	41.32638	−0.12366
12	41.52072	+0.12367	43.01374	+0.12140	44.48932	+0.11925
13	44.66974	−0.11925	46.16785	−0.11721	47.64940	−0.11527
14	47.81779	+0.11527	49.32036	+0.11343	50.80717	+0.11167
15	50.96503	−0.11167	52.47155	−0.10999	53.96303	−0.10838
16	54.11162	+0.10839	55.62165	+0.10685	57.11730	+0.10537
17	57.25765	−0.10537	58.77084	−0.10396	60.27025	−0.10260
18	60.40322	+0.10260	61.91925	+0.10129	63.42705	+0.10003
19	63.54840	−0.10004	65.06700	−0.09882	66.57289	−0.09765
20	66.69324	+0.09765	68.21417	+0.09652	69.72289	+0.09543

s	$j_{6,s}$	$J_6'(j_{6,s})$	$j_{7,s}$	$J_7'(j_{7,s})$	$j_{8,s}$	$J_8'(j_{8,s})$
1	9.93611	−0.22713	11.08637	−0.21209	12.22509	−0.19944
2	13.58929	+0.20175	14.82127	+0.19479	16.03777	+0.18569
3	17.00382	−0.18726	18.28758	−0.17942	19.55454	−0.17744
4	20.32079	+0.17305	21.64154	+0.16688	22.94517	+0.16130
5	23.58608	−0.16159	24.93493	−0.15657	26.26681	−0.15196
6	26.82015	+0.15212	28.19119	+0.14792	29.54566	+0.14404
7	30.03372	−0.14413	31.42279	−0.14055	32.79580	−0.13722
8	33.23304	+0.13727	34.63709	+0.13418	36.02562	+0.13127
9	36.42202	−0.13131	37.83872	−0.12859	39.24045	−0.12603
10	39.60324	+0.12606	41.03077	+0.12365	42.44389	+0.12137
11	42.77848	−0.12139	44.21541	−0.11924	45.63844	−0.11719
12	45.94902	+0.11721	47.39417	+0.11526	48.82593	+0.11342
13	49.11577	−0.11343	50.56818	−0.11166	52.00769	−0.10998
14	52.27945	+0.10999	53.73833	+0.10838	55.18475	+0.10684
15	55.44059	−0.10685	56.90525	−0.10537	58.35789	−0.10395
16	58.59961	+0.10396	60.06948	+0.10260	61.52774	+0.10129
17	61.75682	−0.10129	63.23142	−0.10003	64.69478	−0.09882
18	64.91251	+0.09882	66.39141	+0.09765	67.85943	+0.09652
19	68.06689	−0.09652	69.54971	−0.09543	71.02200	−0.09438
20	71.22013	+0.09438	72.70655	+0.09336	74.18277	+0.09237



1.00
0.50
0.00
-0.50
-1.00
$J_o(\zeta_{10}\, x/a)$
0.00
0.20
0.40
0.60
0.80
1.00
x/a


1.00
0.50
0.00
-0.50
-1.00
J0 (J20 x/a)
0.00
0.20
0.40
0.60
0.80
1.00
x


1.00
0.50
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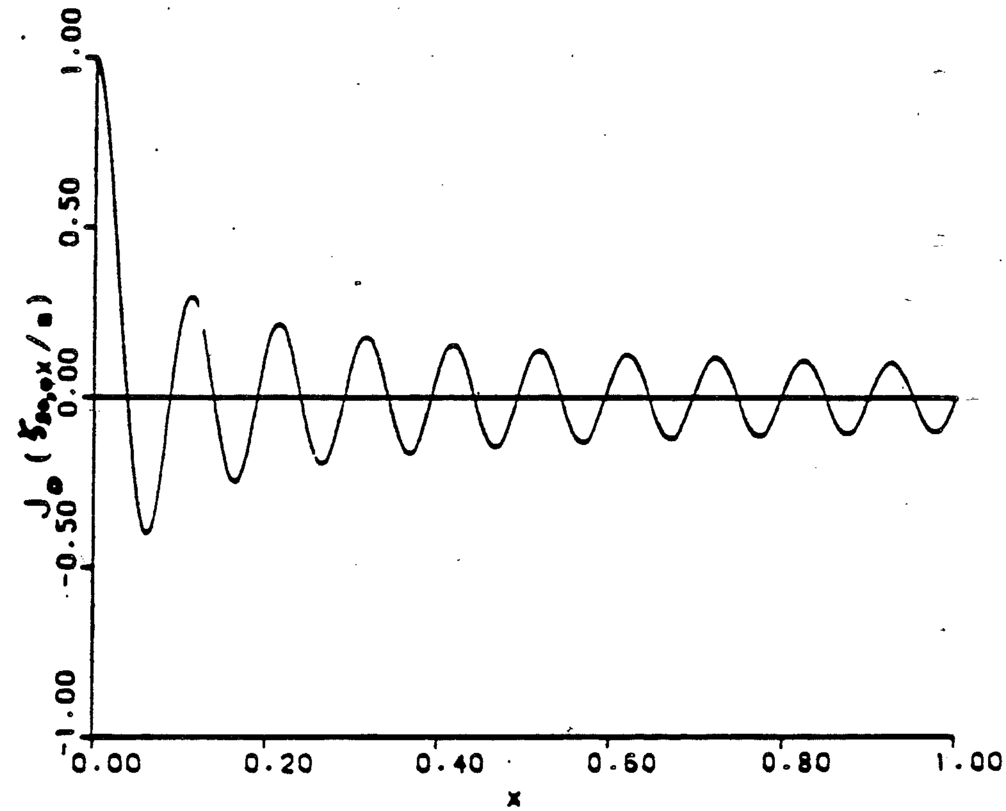
-0.50
-1.00
$J_0\ (j_{30}\ x/a)$
0.00
0.20
0.40
0.60
0.80
1.00
x


1.00
0.50
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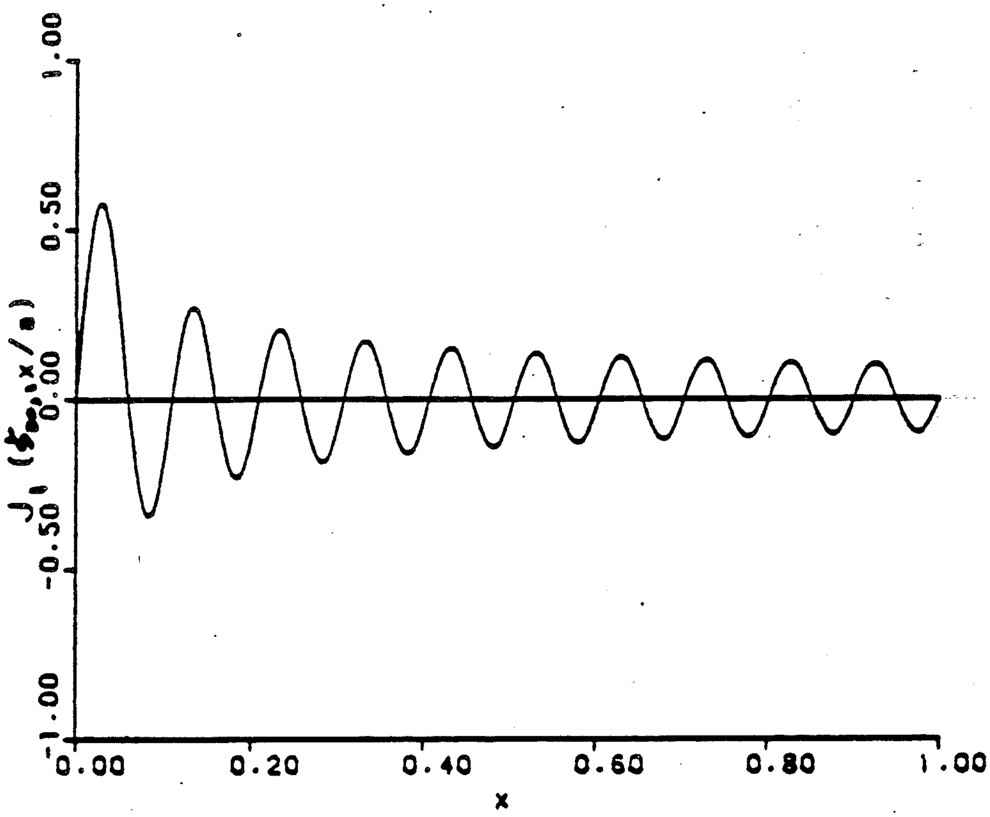
-0.50
-1.00
J0 (ζm,0 x / a)
0.00
0.20
0.40
0.60
0.80
1.00
x

12.6c



Finally, we must find $Z_I(z)$ and $Z_{II}(z)$ for regions I and II, respectively:

$$Z''(z) - \gamma_z^2 Z(z) = 0 \longrightarrow Z(z) = A\cosh(\gamma_z z) + B\sinh(\gamma_z z)$$

subject to

$$Z_I(0) = 0, \quad Z_{II}(b) = 0$$

Therefore, we take

$$Z_I(z) = \sinh(\gamma_z z) = \sinh\left(\frac{\xi_{mn}}{a} z\right).$$

Similarly,

$$Z_{II}(z) = \sinh[\gamma_z (b-z)] = \sinh\left[\frac{\xi_{mn}}{a}(b-z)\right].$$

Now we are ready to put all the solutions back together to find that

$$G_I(\rho,\varphi,z|\rho',\varphi',z') = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{A_{mn}^{I}\, \overbrace{J_n\left(\frac{\xi_{mn}}{a}\rho\right)\cos[n(\varphi-\varphi')]}^{\psi_{mn}(\rho,\varphi)} \sinh\left(\frac{\xi_{mn}}{a}z\right)\right\}$$

satisfies Laplace's equation and the boundary conditions in region I.

Similarly, the most general form of the solution in region II is

$$G_{II}(\rho,\varphi,z|\rho',\varphi',z') = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{A_{mn}^{II}\, J_n\left(\frac{\xi_{mn}}{a}\rho\right)\cos[n(\varphi-\varphi')]\sinh\left[\frac{\xi_{mn}}{a}(b-z)\right]\right\}$$

We note that since, $\{\cos[n(\varphi-\varphi')]\}$ and $\{J_n(\xi_{mn}\rho/a)\}$ are orthogon l sets, then,

$$\{\Psi_{mn}(\rho,\varphi)\} = \{\cos[n(\varphi-\varphi')]\, J_n(\xi_{mn}\rho/a)\}$$

is an orthogonal set als :

$$\int_0^{2\pi}\int_0^{a} \rho\, \Psi_{mn}(\rho,\varphi)\, \Psi_{k\ell}(\rho,\varphi)\, d\rho\, d\varphi = \delta_{km}\,\delta_{\ell n}\cdot \frac{a^2\pi}{\epsilon_n}\cdot [J_n'(\xi_{mn})]^2 .$$

Using this orthogonality property and the continuity conditions, we are now ready to obtain the final solution to the problem (for this particular way of isolating the source).

Applying the condition of continuity of the potential at z=z',

$$G_{I}(\rho,\varphi,z'|\rho',\varphi',z') = G_{II}(\rho,\varphi,z'|\rho',\varphi',z') \rightarrow$$

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{mn}^{I}\, \Psi_{mn}(\rho,\varphi)\sinh\left(\frac{\xi_{mn}}{a}z'\right)$$

$$=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{mn}^{II}\, \Psi_{mn}(\rho,\varphi)\sinh\left[\frac{\xi_{mn}}{a}(b-z')\right] \rightarrow$$

$$A_{mn}^{I}\sinh\left(\frac{\xi_{mn}}{a}z'\right) = A_{mn}^{II}\sinh\left[\frac{\xi_{mn}}{a}(b-z')\right]$$

Applying the condition,

$$\frac{\partial G_{II}}{\partial z}(\rho,\varphi,z'|\rho';\varphi';z') - \frac{\partial G_{I}}{\partial z}(\rho,\varphi,z'|\rho';\varphi';z') = -\frac{1}{\epsilon}\,\delta(\rho-\rho')\,\frac{\delta(\varphi-\varphi')}{\rho},$$

we have

B_{mn}

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{A^{II}_{mn}\,\frac{\zeta_{mn}}{a}\cosh\left[\frac{\zeta_{mn}}{a}(b-z')\right] + A^{I}_{mn}\,\frac{\zeta_{mn}}{a}\cosh\left(\frac{\zeta_{mn}}{a}z'\right)\right\}\Psi_{mn}(\rho,\varphi)$$

$$= -\frac{1}{\epsilon}\,\delta(\rho-\rho')\,\delta(\varphi-\varphi')/\rho$$

Using the orthogonality of $(\Psi_{mn}(\rho,\varphi))$, we find that

$$B_{mn} = -\frac{\epsilon_n}{a^2\pi}\,\frac{1}{[J_n'(\zeta_{mn})]^2}\cdot$$

$$\frac{1}{\epsilon}\int_0^{2\pi}\int_0^{a}\delta(\rho-\rho')\,\frac{\delta(\varphi-\varphi')}{\rho}\cdot\Psi_{mn}(\rho,\varphi)\,\rho\,d\rho\,d\varphi$$

$$= \frac{-\epsilon_n}{\epsilon\, a^2\pi}\,\frac{1}{[J_n'(\zeta_{mn})]^2}\cdot\Psi_{mn}(\rho';\varphi')$$

-or-

$$A^{II}_{mn}\cosh\left[\frac{\zeta_{mn}}{a}(b-z')\right] + A^{I}_{mn}\cosh\left(\frac{\zeta_{mn}}{a}z'\right)$$

$$= +\frac{1}{\epsilon}\cdot\frac{\epsilon_n}{\pi a}\,\frac{1}{\zeta_{mn}}\,\frac{1}{[J_n'(\zeta_{mn})]^2}\,\Psi_{mn}(\rho';\varphi')$$

Thus, just as in the metal box problem, we have two equations in two unknowns (for each m and n):

$$A_{mn}^{II} \sinh\left[\frac{\zeta_{mn}}{a}(b-z')\right] - A_{mn}^{I} \sinh\left(\frac{\zeta_{mn}}{a} z'\right) = 0$$

$$A_{mn}^{II} \cosh\left[\frac{\zeta_{mn}}{a}(b-z')\right] + A_{mn}^{I} \cosh\left(\frac{\zeta_{mn}}{a} z'\right) =$$
$$+\frac{1}{\epsilon}\frac{\epsilon_n}{\pi a}\frac{1}{\zeta_{mn}}\frac{1}{[J_n'(\zeta_{mn})]^2}\Psi_{mn}(\rho';\varphi')$$

The coefficients, therefore, are

$$A_{mn}^{I} = \frac{\begin{vmatrix} \sinh\left[\frac{\zeta_{mn}}{a}(b-z')\right] & 0 \\ \cosh\left[\frac{\zeta_{mn}}{a}(b-z')\right] & +\frac{1}{\epsilon}\frac{\epsilon_n}{\pi a}\frac{\Psi_{mn}(\rho';\varphi')}{\zeta_{mn}[J_n'(\zeta_{mn})]^2} \end{vmatrix}}{\begin{vmatrix} \sinh\left[\frac{\zeta_{mn}}{a}(b-z')\right] & -\sinh\left(\frac{\zeta_{mn}}{a} z'\right) \\ \cosh\left[\frac{\zeta_{mn}}{a}(b-z')\right] & \cosh\left(\frac{\zeta_{mn}}{a} z'\right) \end{vmatrix}}$$

$$= +\frac{1}{\epsilon}\frac{\epsilon_n}{\pi a}\frac{1}{[J_n'(\zeta_{mn})]^2}\frac{\sinh\left[\frac{\zeta_{mn}}{a}(b-z')\right]\Psi_{mn}(\rho';\varphi')}{\zeta_{mn}\sinh\left(\zeta_{mn}\frac{b}{a}\right)}$$

Similarly,

$$A_{mn}^{II} = +\frac{1}{\epsilon}\frac{\epsilon_n}{\pi a}\frac{1}{[J_n'(\zeta_{mn})]^2}\frac{\sinh\left(\frac{\zeta_{mn}}{a} z'\right)\Psi_{mn}(\rho';\varphi')}{\zeta_{mn}\sinh\left(\zeta_{mn}\frac{b}{a}\right)}$$

Therefore, the final solution is

$$G(\rho,\varphi,z\,|\,\rho',\varphi',z') =$$

$$+\frac{1}{\pi a \epsilon}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{\epsilon_n}{[J_n'(\zeta_{mn})]^2}\cdot\frac{\Psi_{mn}(\rho,\varphi)\Psi_{mn}(\rho',\varphi')}{\zeta_{mn}\sinh(\zeta_{mn}\frac{b}{a})}\cdot\right.$$

$$\begin{cases}\sinh\left[\frac{\zeta_{mn}}{a}(b-z')\right]\sinh\left(\frac{\zeta_{mn}}{a}z\right) & z<z' \\ \sinh\left(\frac{\zeta_{mn}}{a}z'\right)\sinh\left[\frac{\zeta_{mn}}{a}(b-z)\right] & z>z'\end{cases}$$

where $J_n(\zeta_{mn})=0$,

$$\Psi_{mn}(\rho,\varphi) = J_n\left(\frac{\zeta_{mn}}{a}\rho\right)\cos[n(\varphi-\varphi')].$$