Solution   Section 1.1 – System of Equations

Exercise

Solve the system

$$\begin{aligned} 2x+y-z&=2 \quad (1)\\ x+3y+2z&=1 \quad (2)\\ x+y+z&=2 \quad (3)\end{aligned}$$

Solution

$$R_1-2R_2 \to R_2 \quad \begin{array}{r} 2x+y-z=2\\ \underline{-2x-6y-4z=-2}\\ -5y-5z=0 \end{array}$$

$$R_1-2R_3 \to R_3 \quad \begin{array}{r} 2x+y-z=2\\ \underline{-2x-2y-2z=-4}\\ -y-3z=-2 \end{array}$$

$$\begin{aligned} 2x+y-z&=2\\ -5y-5z&=0\\ -y-3z&=-2\end{aligned}$$

$$R_2-5R_3 \to R_3 \quad \begin{array}{r} -5y-5z=0\\ \underline{5y+15z=10}\\ 10z=10 \end{array}$$

$$\begin{array}{lll} 2x+y-z=2 & 2x=2-y+z & \Rightarrow 2x=2+1+1=4 \to x=2\\ -5y-5z=0 & -5y=5z & \Rightarrow y=-z=-1\\ 10z=10 & \to z=1 & \end{array}$$

Solution $(2, -1, 1)$

Exercise

Solve the system:
$$\begin{aligned} 3x_1 + x_2 - 2x_3 &= 2 \\ x_1 - 2x_2 + x_3 &= 3 \\ 2x_1 - x_2 - 3x_3 &= 3 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 3 & 1 & -2 & 2 \\ 2 & -1 & -3 & 3 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 1 & -2 & 2 \\ -3 & 6 & -3 & -9 \\ \hline 0 & 7 & -5 & -7 \end{array} \qquad \begin{array}{cccc} 2 & -1 & -3 & 3 \\ -2 & 4 & -2 & -6 \\ \hline 0 & 3 & -5 & -3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 7 & -5 & -7 \\ 0 & 3 & -5 & -3 \end{array}\right] \tfrac{1}{7}R_2 \qquad 0 \quad 1 \quad -\tfrac{5}{7} \quad -1$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & -\frac{5}{7} & -1 \\ 0 & 3 & -5 & -3 \end{array}\right] \begin{array}{l} R_1 + 2R_2 \\ \\ R_3 - 3R_2 \end{array}$$

$$\begin{array}{cccc} 0 & 3 & -5 & -3 \\ 0 & -3 & \frac{15}{7} & 3 \\ \hline 0 & 0 & -\frac{20}{7} & 0 \end{array} \qquad \begin{array}{cccc} 1 & -2 & 1 & 3 \\ 0 & 2 & -\frac{10}{7} & -2 \\ \hline 1 & 0 & -\frac{3}{7} & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{3}{7} & 1 \\ 0 & 1 & -\frac{5}{7} & -1 \\ 0 & 0 & -\frac{20}{7} & 0 \end{array}\right] \begin{array}{l} \\ \\ -\frac{7}{20}R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{3}{7} & 1 \\ 0 & 1 & -\frac{5}{7} & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 + \frac{3}{7}R_3 \\ R_2 + \frac{5}{7}R_3 \\ \\ \end{array}$$

$$\begin{array}{cccc} 0 & 1 & -\frac{5}{7} & -1 \\ 0 & 0 & \frac{5}{7} & 0 \\ \hline 0 & 1 & 0 & -1 \end{array} \qquad \begin{array}{cccc} 1 & 0 & -\frac{3}{7} & 1 \\ 0 & 0 & \frac{3}{7} & 0 \\ \hline 1 & 0 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Solution: $(1, -1, 0)$

Exercise

Solve the system:
$$\begin{aligned} 2x_1 - 2x_2 + x_3 &= 3 \\ 3x_1 + x_2 - x_3 &= 7 \\ x_1 - 3x_2 + 2x_3 &= 0 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 2 & -2 & 1 & 3 \\ 3 & 1 & -1 & 7 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -2 & 1 & 3 \\ -2 & 6 & -4 & 0 \\ \hline 0 & 4 & -3 & 3 \end{array} \qquad \begin{array}{cccc} 3 & 1 & -1 & 7 \\ 3 & 9 & -6 & 0 \\ \hline 0 & 10 & -7 & 7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 4 & -3 & 3 \\ 0 & 10 & -7 & 7 \end{array}\right] \tfrac{1}{4}R_2$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 1 & -\frac{3}{4} & \frac{3}{4} \\ 0 & 10 & -7 & 7 \end{array}\right] \begin{array}{l} R_1 + 3R_2 \\ \\ R_3 - 10R_2 \end{array}$$

$$\begin{array}{cccc} 0 & 10 & -7 & 7 \\ 0 & -10 & 7.5 & -7.5 \\ \hline 0 & 0 & .5 & -.5 \end{array} \qquad \begin{array}{cccc} 1 & -3 & 2 & 0 \\ 0 & 3 & -2.25 & 2.25 \\ \hline 1 & 0 & -.25 & 2.25 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -.25 & 2.25 \\ 0 & 1 & -.75 & .75 \\ 0 & 0 & .5 & -.5 \end{array}\right] \tfrac{1}{.5}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -.25 & 2.25 \\ 0 & 1 & -.75 & .75 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} R_1 + .25R_3 \\ R_2 + .75R_3 \\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & -.25 & 2.25 \\ 0 & 0 & .25 & -.25 \\ \hline 1 & 0 & 0 & 2 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -.75 & .75 \\ 0 & 0 & .75 & -.75 \\ \hline 0 & 1 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Solution: $(2, 0, -1)$

Exercise

Katherine invests \$10,000 received from her grandmother in three ways. With one part, she buys U.S saving bonds at an interest rate of 2.5% per year. She uses the second part, which amounts to twice the first, to buy mutual funds that offer a return of 6% per year. She puts the rest of the money into a money market account paying 4.5% annual interest. The first year her investments bring a return of \$470. How much did she invest in each way?

Solution

x = amount invested in savings bonds (2.5 %)
y = amount invested in mutual bonds (6 %)
z = amount invested in money market (4.5%)

She invested \$10,000

$$x + y + z = 10,000$$

She invested twice as much in mutual as in savings

$$y = 2x$$

Total return \$470.

$$.025x + .06y + .045z = 470$$

$$\begin{aligned} x + y + z &= 10000 \\ y &= 2x \\ .025x + .06y + .045z &= 470 \end{aligned}$$

Solution: $\boxed{(2000,\ 4000,\ 4000)}$

\$2,000 invested in savings bonds
\$4,000 invested in mutual bonds
\$4,000 invested in money market

Exercise

A Company that rents small moving trucks wants to purchase 25 trucks with a combined capacity of 28,000 cubic ft. Three different types of trucks are available: a 10-foot truck with a capacity of 350 cubic feet, a 14-foot truck with a capacity of 700 cubic feet, and a 24-foot truck with a capacity of 1,400 cubic feet. How many truck of each type of truck should the company purchase?

Solution

$$x_1 = \#10\,ft \qquad x_2 = \#14\,ft \qquad x_3 = \#24\,ft$$

$$\begin{cases} x_1 + x_2 + x_3 = 25 \\ 350x_1 + 700x_2 + 1400x_3 = 28000 & \Rightarrow x_1 + 2x_2 + 4x_3 = 80 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25 \\ 1 & 2 & 4 & 80 \end{array}\right] \quad R_2 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25 \\ 0 & 1 & 3 & 55 \end{array}\right] \quad R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -30 \\ 0 & 1 & 3 & 55 \end{array}\right] \quad \begin{array}{ll} x_1 - 2x_3 = -30 & (1) \\ x_2 + 3x_3 = 55 & (2) \end{array}$$

$$\begin{cases} x_1 = 2x_3 - 30 \\ x_2 = -3x_3 + 55 \end{cases}$$

$$\begin{cases} x_1 = 2x_3 - 30 \ge 0 \\ x_2 = -3x_3 + 55 \ge 0 \end{cases} \to \begin{cases} 2x_3 \ge 30 \\ -3x_3 \ge -55 \end{cases}$$

$$\to \begin{cases} x_3 \ge 15 \\ x_3 \le \frac{55}{3} \approx 18.35 \end{cases}$$

$$\Rightarrow\ 15 \le x_3 \le 18$$

All possibilities:

x_3 : ***24 ft***	x_1 : ***10 ft***	x_2 : ***14 ft***
15	0	10
16	2	7
17	4	4
18	6	1

Exercise

A flight leaves New York at 8 PM and arrives in Paris at 9 AM (Paris time). This 13-hour difference includes the flight time plus the change in time zones. The return leaves Paris at 1 PM and arrives in New York at 3 PM (NY time). This 2-hour difference includes the flight time minus time zones, plus an extra hour due to the fact that flying westward is against the wind. Find the actual flight time eastward and the difference in time zones.

Solution

Let: x: flight time

y: difference in time zones

13-*hour includes the flight time plus the change in time zones*

$$x+y=13$$

2-*hour difference includes the flight time minus time zones, plus an extra hour*

$$(x+1)-y=2$$

$$x-y=1$$

$$\begin{cases} x+y=13 & (1) \\ x-y=1 & (2) \end{cases} \quad \Rightarrow x=7 \quad and \quad y=6$$

The flight eastward is 7 hours.

The difference in time zones is 6 hours.

Exercise

$$\begin{bmatrix} 2 & 4 & 5 & -7 \\ 6 & -3 & 12 & 0 \end{bmatrix} + \begin{bmatrix} 8 & 0 & -10 & 1 \\ -2 & 8 & -9 & 11 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 2 & 4 & 5 & -7 \\ 6 & -3 & 12 & 0 \end{bmatrix} + \begin{bmatrix} 8 & 0 & -10 & 1 \\ -2 & 8 & -9 & 11 \end{bmatrix} = \begin{bmatrix} 10 & 4 & -5 & -6 \\ 4 & 5 & 3 & 11 \end{bmatrix}$$

Exercise

$$\begin{bmatrix} a+2 & 3b & 4c \\ d & 7f & 8 \end{bmatrix} + \begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$$

Solution

$$\begin{bmatrix} a-5 & 5b & 4c+6 \\ -2d & 7f-6 & 6 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$$

$$a-5=15 \to a=20$$
$$5b=25 \to b=5$$
$$4c+6=6 \to 4c=0 \to c=0$$
$$-2d=-8 \to d=4$$
$$7f-6=1 \to 7f=7 \to f=1$$

Exercise

$$\begin{bmatrix} x & y+3 \\ 2z & 8 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 6 & 8 \end{bmatrix}$$

Solution

$$\begin{cases} x=12 \\ y+3=5 \to y=2 \\ 2z=6 \to z=3 \end{cases}$$

Exercise

Evaluate $\begin{bmatrix} -4 & 3 \\ 7 & -6 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & -4 \end{bmatrix}$

Solution

$$\begin{bmatrix} -4 & 3 \\ 7 & -6 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} -4+6 & 3+(-3) \\ 7+2 & -6+(-4) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 9 & -10 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 5 & 4 \\ -3 & 7 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 8 \\ 6 & 0 \\ -5 & 3 \end{bmatrix}$

Solution

$$\begin{bmatrix} 5 & 4 \\ -3 & 7 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 8 \\ 6 & 0 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 5-(-4) & 4-8 \\ -3-6 & 7-0 \\ 0-(-5) & 1-3 \end{bmatrix} = \begin{bmatrix} 9 & -4 \\ -9 & 7 \\ 5 & -2 \end{bmatrix}$$

Exercise

Evaluate $$-4\begin{bmatrix}-2 & 4\\ 0 & 3\end{bmatrix}+5\begin{bmatrix}-6 & 2\\ 4 & 0\end{bmatrix}$$

Solution

$$-4\begin{bmatrix}-2 & 4\\ 0 & 3\end{bmatrix}+5\begin{bmatrix}-6 & 2\\ 4 & 0\end{bmatrix}=\begin{bmatrix}8 & -16\\ 0 & -12\end{bmatrix}+\begin{bmatrix}-30 & 10\\ 20 & 0\end{bmatrix}$$

$$=\begin{bmatrix}-22 & -6\\ 20 & -12\end{bmatrix}$$

Exercise

Evaluate $$\begin{bmatrix}2 & 2 & -1\\ 3 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 2\\ -1 & 4\\ 0 & 2\end{bmatrix}$$

Solution

$$\begin{bmatrix}2 & 2 & -1\\ 3 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 2\\ -1 & 4\\ 0 & 2\end{bmatrix}=\begin{bmatrix}-2 & 10\\ 0 & 8\end{bmatrix}$$

Exercise

Evaluate $$\begin{bmatrix}2\\ -9\\ 12\end{bmatrix}\begin{bmatrix}1 & 0 & -1\end{bmatrix}$$

Solution

$$\begin{bmatrix}2\\ -9\\ 12\end{bmatrix}\begin{bmatrix}1 & 0 & -1\end{bmatrix}=\begin{bmatrix}2(1) & 2(0) & 2(-1)\\ -9(1) & -9(0) & -9(-1)\\ 12(1) & 12(0) & 12(-1)\end{bmatrix}$$

$$=\begin{bmatrix}2 & 0 & -2\\ -9 & 0 & 9\\ 12 & 0 & -12\end{bmatrix}$$

Exercise

Find: $$\begin{bmatrix}-1 & 0 & 3 & -2\\ 1 & 2 & 2 & 0\end{bmatrix}\begin{bmatrix}-1 & 1\\ 2 & 3\\ 1 & 0\end{bmatrix}$$

Solution

Not Defined

Exercise

Find: $\begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 3 & -2 \\ 1 & 2 & 2 & 0 \end{bmatrix}$

Solution

$$\begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 3 & -2 \\ 1 & 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -1 & 2 \\ 1 & 6 & 12 & -4 \\ -1 & 0 & 3 & -2 \end{bmatrix}$$

Exercise

$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \quad F = \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix}$

Solution

$$3F + 2A = 3\begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix} + 2\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 3(3) & 3(3) \\ 3(-1) & 3(-1) \end{bmatrix} + \begin{bmatrix} 2(1) & 2(2) \\ 2(4) & 2(3) \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 9 \\ -3 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 8 & 6 \end{bmatrix} \quad = \begin{bmatrix} 9+2 & 9+4 \\ -3+8 & -3+6 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 13 \\ 5 & 3 \end{bmatrix}$$

Exercise

$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -3 & 6 \\ -3 & 1 & -4 \end{bmatrix}$

Find: A – B *and* 3A+2B

Solution

$$A - B = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \end{bmatrix} - \begin{bmatrix} 2 & -3 & 6 \\ -3 & 1 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & -5 \\ 2 & 1 & 9 \end{bmatrix}$$

$$3A + 2B = 3\begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \end{bmatrix} + 2\begin{bmatrix} 2 & -3 & 6 \\ -3 & 1 & -4 \end{bmatrix}$$

$$=\begin{bmatrix} 9 & 3 & 3 \\ -3 & 6 & 15 \end{bmatrix}+\begin{bmatrix} 4 & -6 & 12 \\ -6 & 2 & -8 \end{bmatrix}$$

$$=\begin{bmatrix} 13 & -3 & 15 \\ -9 & 8 & 7 \end{bmatrix}$$

Exercise

$$AB=\begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix}.\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$$

Solution

$$AB=\begin{bmatrix} 1(1)-1(1)+4(1) & 1(1)-1(2)+4(-1) & 1(0)-1(4)+4(3) \\ 4(1)-1(1)+3(1) & 4(1)-1(2)+3(-1) & 4(0)-1(4)+3(3) \\ 2(1)+0(1)-2(1) & 2(1)+0(2)-2(-1) & 2(0)+0(4)-2(3) \end{bmatrix}$$

$$=\begin{bmatrix} 4 & -5 & 8 \\ 6 & -1 & 5 \\ 0 & 4 & -6 \end{bmatrix}$$

Exercise

$$BA=\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix}.\begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix}$$

Solution

$$BA=\begin{bmatrix} 1(1)+1(4)+0(2) & 1(-1)+1(-1)+0(0) & 1(4)+1(3)+0(-2) \\ 1(1)+2(4)+4(2) & 1(-1)+2(-1)+4(0) & 1(4)+2(3)+4(-2) \\ 1(1)-1(4)+3(2) & 1(-1)-1(-1)+3(0) & 1(4)-1(3)+3(-2) \end{bmatrix}$$

$$=\begin{bmatrix} 5 & -2 & 7 \\ 17 & -3 & 2 \\ 3 & 0 & -5 \end{bmatrix}$$

Exercise

Sal's Shoes and Fred's Footwear both have outlets in California and Arizona. Sal's sells shoes for $80, sandals for $40, and boots for $120. Fred's prices are $60, $30, and $150 for shoes, sandals and boots, respectively. Half of all sales in California stores are shoes, 1/4 are sandals, and 1/4 are boots. In Arizona the fractions are 1/5 shoes, 1/5 are sandals, and 3/5 are boots.

a) Write a 2 *x* 3 matrix called P representing prices for the two stores and three types of footwear.

b) Write a 2 *x* 3 matrix called F representing fraction of each type of footwear sold in each state.

c) Only one of the two products PF and FP is meaningful. Determine which one it is, calculate the product, and describe what the entries represent.

Solution

a) $$P = \begin{array}{cc} \begin{bmatrix} 80 & 40 & 120 \\ 60 & 30 & 150 \end{bmatrix} & \begin{array}{c} Sal's \\ Fred's \end{array} \end{array}$$

b) $$F = \begin{array}{c} \begin{array}{cc} CA & AR \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{3}{5} \end{bmatrix} \end{array}$$

c) $$PF = \begin{bmatrix} 80 & 40 & 120 \\ 60 & 30 & 150 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 80\frac{1}{2}+40\frac{1}{4}+120\frac{1}{4} & 80\frac{1}{5}+40\frac{1}{5}+120\frac{3}{5} \\ 60\frac{1}{2}+30\frac{1}{4}+150\frac{1}{4} & 60\frac{1}{5}+30\frac{1}{5}+150\frac{3}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 80\frac{1}{2}+40\frac{1}{4}+120\frac{1}{4} & 80\frac{1}{5}+40\frac{1}{5}+120\frac{3}{5} \\ 60\frac{1}{2}+30\frac{1}{4}+150\frac{1}{4} & 60\frac{1}{5}+30\frac{1}{5}+150\frac{3}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 80 & 96 \\ 75 & 108 \end{bmatrix}$$

Exercise

Use the inverse of the coefficient matrix to solve the linear system $\begin{cases} 2x+5y=15 \\ x+4y=9 \end{cases}$

Solution

$$A^{-1} = \begin{bmatrix} \frac{4}{3} & -\frac{5}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \qquad X = \begin{bmatrix} \frac{4}{3} & -\frac{5}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 15 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

The solution of the system is $\boxed{(5,\ 1)}$

Exercise

$$A=\begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2+R_1 \\ R_3-R_1 \end{array}$$

$$\begin{array}{cccccc} -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 \\ \hline 0 & 2 & 5 & 1 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 1 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & -1 & 0 & 0 \\ \hline 0 & -1 & -2 & -1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right] \ \tfrac{1}{2}R_2$$

$$\begin{array}{cccccc} 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3+R_2 \end{array}$$

$$\begin{array}{cccccc} 0 & -1 & -2 & -1 & 0 & 1 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ \hline 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}\right] \ 2R_3$$

$$\begin{array}{cccccc} 0 & 0 & 1 & -1 & 1 & 2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right] \begin{array}{l} R_1-2R_3 \\ R_2-\frac{5}{2}R_3 \\ \end{array}$$

$$\begin{array}{cccccc} 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{5}{2} & \frac{5}{2} & -\frac{5}{2} & -5 \\ \hline 0 & 1 & 0 & 3 & -2 & -5 \end{array} \qquad \begin{array}{cccccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 2 & -2 & -4 \\ \hline 1 & 0 & 0 & 3 & -2 & -4 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 1 & 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right]$$

$$\Rightarrow A^{-1}=\begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

Exercise

Find the inverse of: $A=\begin{bmatrix}1 & 2 & -1\\ 3 & 5 & 3\\ 2 & 4 & 3\end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ 3 & 5 & 3 & 0 & 1 & 0\\ 2 & 4 & 3 & 0 & 0 & 1\end{array}\right] \begin{array}{l} R_2-3R_1 \\ R_3-2R_1\end{array}$$

$$\begin{array}{cccccc}3 & 5 & 3 & 0 & 1 & 0\\ -3 & -6 & 3 & -3 & 0 & 0\\ \hline 0 & -1 & 6 & -3 & 1 & 0\end{array} \qquad \begin{array}{cccccc}2 & 4 & 3 & 0 & 0 & 1\\ -2 & -4 & 2 & -2 & 0 & 0\\ \hline 0 & 0 & 5 & -2 & 0 & 1\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ 0 & -1 & 6 & -3 & 1 & 0\\ 0 & 0 & 5 & -2 & 0 & 1\end{array}\right] -R_2$$

$$\left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ 0 & 1 & -6 & 3 & -1 & 0\\ 0 & 0 & 5 & -2 & 0 & 1\end{array}\right] R_1-2R_2$$

$$\begin{array}{cccccc}1 & 2 & -1 & 1 & 0 & 0\\ 0 & -2 & 12 & -6 & 2 & 0\\ \hline 1 & 0 & 11 & -5 & 2 & 0\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 11 & -5 & 2 & 0\\ 0 & 1 & -6 & 3 & -1 & 0\\ 0 & 0 & 5 & -2 & 0 & 1\end{array}\right] \tfrac{1}{5}R_3$$

$$\begin{array}{cccccc}0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5}\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 11 & -5 & 2 & 0\\ 0 & 1 & -6 & 3 & -1 & 0\\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5}\end{array}\right] \begin{array}{l} R_1-11R_3 \\ R_2+6R_3\end{array}$$

$$\begin{array}{cccccc}0 & 1 & -6 & 3 & -1 & 0\\ 0 & 0 & 6 & -\frac{12}{5} & 0 & \frac{6}{5}\\ \hline 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5}\end{array} \qquad \begin{array}{cccccc}1 & 0 & 11 & -5 & 2 & 0\\ 0 & 0 & -11 & \frac{22}{5} & 0 & -\frac{11}{5}\\ \hline 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5}\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5}\\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5}\\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5}\end{array}\right]$$

$$A^{-1}=\begin{bmatrix}-\frac{3}{5} & 2 & -\frac{11}{5}\\ \frac{3}{5} & -1 & \frac{6}{5}\\ -\frac{2}{5} & 0 & \frac{1}{5}\end{bmatrix}$$

Exercise

Find the inverse of: $M = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad \tfrac{1}{3}R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} \\ R_2 + R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{cccccc} -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \hline 0 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 1 & 0 & 1 & 0 & 0 & 1 \\ -1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ \hline 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 1 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 1 \end{array}\right] \quad \tfrac{3}{2}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 + \frac{1}{3}R_2 \\ \\ R_3 - \frac{1}{3}R_2 \end{array}$$

$$\begin{array}{cccccc} 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 1 \\ 0 & -\frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{2} & 0 \\ \hline 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{array} \qquad \begin{array}{cccccc} 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{2} & 0 \\ \hline 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right] \quad 2R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right] \quad \begin{array}{l} R_1 - \frac{1}{2}R_3 \\ R_2 - \frac{1}{2}R_3 \\ \\ \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -1 \\ \hline 1 & 0 & 0 & 1 & 1 & -1 \end{array} \qquad \begin{array}{cccccc} 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -1 \\ \hline 0 & 1 & 0 & 1 & 2 & -1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right] \qquad M^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Exercise

Use the inverse of the coefficient matrix to solve the linear system: $\begin{cases} 3x_1 - x_2 + x_3 = 3 \\ -x_1 + x_2 = -3 \\ x_1 + x_3 = 2 \end{cases}$

Solution

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \\ 4 \end{bmatrix}$$

Solution: $(-2, -5, 4)$

Exercise

An investment advisor currently has two types of investment available for clients: a conservative investment A that pays 8% per year and an investment B paying 24% per year. Clients may divide their investments between the two to achieve any total return desired between 8% and 24%. However, the higher the desired return, the higher the risk. How should each client listed in the table invest to achieve the indicated return?

Solution

	Client			
	1	**2**	**3**	**k**
Total investment	\$20,000	\$50,000	\$10,000	k_1
Annual return desired	\$2,400	\$7,500	\$1,300	k_2
	12%	15%	13%	

Total Investment: $x_1 + x_2 = k_1$

Total annual return desired: $.08x_1 + .24x_2 = k_2$

$$\begin{array}{ccc} A & X & = B \end{array}$$

$$\begin{bmatrix} 1 & 1 \\ .08 & .24 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ .08 & .24 \end{bmatrix} \to A^{-1} = \begin{bmatrix} 1.5 & -6.25 \\ -.5 & 6.25 \end{bmatrix}$$

$$X \quad = \quad A^{-1} \quad B$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.5 & -6.25 \\ -.5 & 6.25 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

Client # 1

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.5 & -6.25 \\ -.5 & 6.25 \end{bmatrix} \begin{bmatrix} 20,000 \\ 2,400 \end{bmatrix} = \begin{bmatrix} \$15,000 \\ \$5,000 \end{bmatrix}$$

Client # 2

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.5 & -6.25 \\ -.5 & 6.25 \end{bmatrix} \begin{bmatrix} 50,000 \\ 7,500 \end{bmatrix} = \begin{bmatrix} \$28,125 \\ \$21,875 \end{bmatrix}$$

Client # 3

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.5 & -6.25 \\ -.5 & 6.25 \end{bmatrix} \begin{bmatrix} 10,000 \\ 1,300 \end{bmatrix} = \begin{bmatrix} \$6,875 \\ \$3,125 \end{bmatrix}$$