Section 2.5 – Polynomial Functions

Polynomial Function

A *Polynomial function* $P(x)$ in x is a sum of the form is given by:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$$

Where the coefficients $a_n,\ a_{n-1},\ \ldots, a_2,\ a_1, a_0$ are real numbers and the exponents are whole numbers.

Degree

$$a_n x^n$$

Leading Term

Leading Coefficient

Non-polynomial Functions: $\frac{1}{x} + 2x;\quad \sqrt{x^2 - 3} + x;\quad \frac{x-5}{x^2+2}$

Degree of f	***Form of f(x)***	***Graph of f(x)***
0	$f(x) = a_0$	A horizontal line
1	$f(x) = a_1 x + a_0$	A line with slope a_1
2	$f(x) = a_2 x^2 + a_1 x + a_0$	A parabola with a vertical axis

All polynomial functions are ***continuous functions***.

End Behavior $\left(a_n x^n\right)$

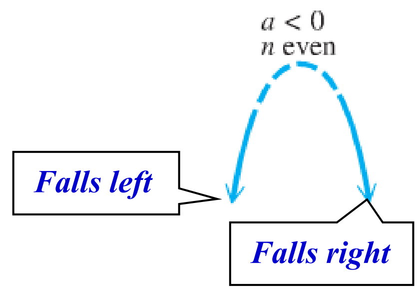
If n (degree) is ***even***:

If $a_n < 0$ (in front x^n is negative).

Then the function falls from the left and right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to -\infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to -\infty$$

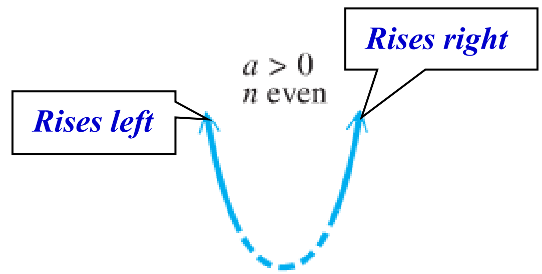


If $a_n > 0$ (in front x^n is positive).

Then the function rises from the left and right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to \infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to \infty$$



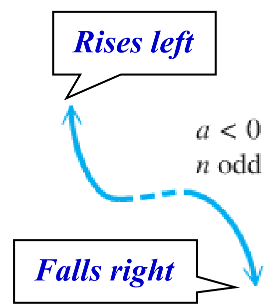
If n (degree) is ***odd***:

If $a_n < 0$ (negative).

Then the function rises from the left side and falls from the right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to \infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to -\infty$$

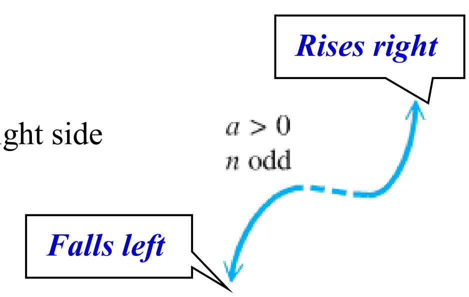


If $a_n > 0$ (positive).

Then the function falls from the left side and rises from the right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to -\infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to \infty$$



Example

Determine the end behavior of the graph of the polynomial function $f(x) = -4x^5 + 7x^2 - x + 9$

Solution

Leading term: $-4x^5$ with 5th degree (n is odd)

$x \to -\infty \quad \Rightarrow \quad f(x) = -(-)^5 = (-)(-) = + \to \infty \qquad f(x)$ rises left

$x \to \infty \quad \Rightarrow \quad f(x) \to -\infty \qquad f(x)$ falls right

The Intermediate Value *Theorem*

For any polynomial function $f(x)$ with real coefficients and $f(a) \neq f(b)$ for $a < b$, then f takes on every value between $f(a)$ and $f(b)$ in the interval $[a, b]$.

$\therefore$ $f(a)$ and $f(b)$ are the ***opposite signs***. Then the function has a real zero between *a* and *b*.

Example

Using the intermediate value theorem, determine, if possible, whether the function has a real zero between *a* and *b*.

a) $f(x) = x^3 + x^2 - 6x; \quad a = -4, \ b = -2$

b) $f(x) = x^3 + x^2 - 6x; \quad a = -1, \ b = 3$

Solution

a) $f(x) = x^3 + x^2 - 6x; \quad a = -4, \ b = -2$

$$f(-4) = (-4)^3 + (-4)^2 - 6(-4)$$
$$= -24$$
$$f(-2) = (-2)^3 + (-2)^2 - 6(-2)$$
$$= 8$$

$\therefore$ $f(x)$ has a zero between -4 and -2

b) $f(x) = x^3 + x^2 - 6x; \quad a = -1, \ b = 3$

$$f(-1) = (-1)^3 + (-1)^2 - 6(-1)$$
$$= 6$$
$$f(3) = (3)^3 + (3)^2 - 6(3) = 18$$
$$= 18$$

$\therefore$ $f(x)$ *zeros* ***can't be determined***

Example

Show that $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3$ has a zero between 1 and 2.

Solution

$$f(1) = 1 + 2 - 6 + 2 - 3$$
$$= -4$$
$$f(2) = (2)^5 + 2(2)^4 - 6(2)^3 + 2(2) - 3$$

$$=17$$

Since $f(1)$ *and* $f(2)$ have opposite signs.

Therefore, $f(c)=0$ for at least one real number c between 1 and 2.

Exercises *Section* 2.5 – Polynomial Functions

(1 – 12) Determine the end behavior of the graph of the polynomial function

1. $f(x)=5x^3+7x^2-x+9$

2. $f(x)=11x^3-6x^2+x+3$

3. $f(x)=-11x^3-6x^2+x+3$

4. $f(x)=2x^3+3x^2-23x-42$

5. $f(x)=5x^4+7x^2-x+9$

6. $f(x)=11x^4-6x^2+x+3$

7. $f(x)=-5x^4+7x^2-x+9$

8. $f(x)=-11x^4-6x^2+x+3$

9. $f(x)=5x^5-16x^2-20x+64$

10. $f(x)=-5x^5-16x^2-20x+64$

11. $f(x)=-3x^6-16x^3+64$

12. $f(x)=3x^6-16x^3+4$

(13 – 32) Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers.

13. $f(x)=x^3-x-1;$ *between* 1 *and* 2

14. $f(x)=x^3-4x^2+2;$ *between* 0 *and* 1

15. $f(x)=2x^4-4x^2+1;$ *between* -1 *and* 0

16. $f(x)=x^4+6x^3-18x^2;$ *between* 2 *and* 3

17. $f(x)=x^3+x^2-2x+1;$ *between* -3 *and* -2

18. $f(x)=x^5-x^3-1;$ *between* 1 *and* 2

19. $f(x)=3x^3-10x+9;$ *between* -3 *and* -2

20. $f(x)=3x^3-8x^2+x+2;$ *between* 2 *and* 3

21. $f(x)=3x^3-8x^2+x+2;$ *between* 1 *and* 2

22. $f(x)=x^5+2x^4-6x^3+2x-3;$ *between* 0 *and* 1

23. $P(x)=2x^3+3x^2-23x-42,\quad a=3,\quad b=4$

24. $P(x)=4x^3-x^2-6x+1,\quad a=0,\quad b=1$

25. $P(x)=3x^3+7x^2+3x+7,\quad a=-3,\quad b=-2$

26. $P(x)=2x^3-21x^2-2x+25,\quad a=1,\quad b=2$

27. $P(x)=4x^4+7x^3-11x^2+7x-15,\quad a=1,\quad b=\frac{3}{2}$

28. $P(x)=5x^3-16x^2-20x+64,\quad a=3,\quad b=\frac{7}{2}$

29. $P(x)=x^4-x^2-x-4,\quad a=1,\quad b=2$

30. $P(x) = x^3 - x - 8, \quad a = 2, \quad b = 3$

31. $P(x) = x^3 - x - 8, \quad a = 0, \quad b = 1$

32. $P(x) = x^3 - x - 8, \quad a = 2.1, \quad b = 2.2$