$$\frac{\pm\sqrt{b^2-4ac}}{2a}$$

if $p^2-4q>0 \Rightarrow 2\ \mathbb{R}$ roots

$p^2-4q<0 \Rightarrow 2\ \mathbb{C}$

$p^2-4q=0 \rightarrow$ 1 repeated root.

Case 1 Distinct Real Roots

$$y_1 = c_1 e^{\lambda_1 t},\ y_2 = c_2 e^{\lambda_2 t}$$

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

Ex $y''-3y'+2y=0$ $\quad \begin{cases} y(0)=2 \\ y'(0)=1 \end{cases}$

$$\lambda^2-3\lambda+2=0 \Rightarrow \lambda_{1,2}=1,2$$

$$y(t) = C_1 e^t + C_2 e^{2t}$$

$$y(0) = 2 = C_1 + C_2$$

$$y'(t) = C_1 e^t + 2C_2 e^{2t}$$

$$y'(0) = C_1 + 2C_2 = 1$$

$$\begin{cases} C_1 + C_2 = 2 \\ C_1 + 2C_2 = 1 \end{cases} \Rightarrow C_2 = -1,\ C_1 = 3$$

$$y(t) = 3e^t - e^{2t}$$

2. Complex Roots $\lambda = a \pm ib$

$$y(t) = e^{at}(C_1 \cos bt + C_2 \sin bt)$$

Ex $y'' + 2y' + 2y = 0 \quad y(0) = 2 \quad y'(0) = 3$

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda_{1,2} = \overset{a}{-1} \pm \overset{b}{1} i$$

$$y(t) = e^{-t}(C_1 \cos t + C_2 \sin t)$$

$$y(0) = C_1 = 2$$

$$y'(t) = e^{-t}[-C_1 \cos t - C_2 \sin t - C_1 \sin t + C_2 \cos t]$$

$$y'(0) = -C_1 + C_2 = 3 \Rightarrow C_2 = 5$$

$$\boxed{y(t) = e^{-t}(2\cos t + 5 \sin t)}$$

Ex $y'' - 4y' + 13y = 0$

$$\lambda_{1,2} = -2 \pm 3i$$

$$\boxed{y(t) = e^{-2t}(C_1 \cos 3t + C_2 \sin 3t)}$$

3. Repeated Roots.

$$\lambda = -\frac{b}{2a}$$

$$y(t) = (C_1 + C_2 t)e^{\lambda t}$$

Ex

$$y'' - 2y' + y = 0$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$\lambda_{1,2} = 1$$

$$y(t) = (C_1 + C_2 t)e^t$$

$$y(0) = 2 \quad y'(0) = -1$$

$$y(0) = C_1 = 2$$

$$y'(t) = (C_2 + C_1 + C_2 t)e^t$$

$$y'(0) = C_2 + C_1 = -1 \Rightarrow C_2 = -3$$

$$y(t) = (2 - 3t)e^t$$

Ex

$$y'' - 10y' + 25y = 0$$

$$\lambda^2 - 10\lambda + 25 = 0$$

$$\lambda_{1,2} = 5$$

$$y(t) = (C_1 + C_2 t)e^{5t}$$

Ex $y''' + 3y'' - 4y = 0$

$$\lambda^3 + 3\lambda^2 - 4 = 0 \qquad \underline{\lambda_1 = 1}$$

$$\begin{array}{r|rrrr} 1 & 1 & 3 & 0 & -4 \\ & & 1 & 4 & 4 \\ \hline & 1 & 4 & 4 & \boxed{0} \end{array}$$

$$\lambda_1 = 1, \quad \lambda_{2,3} = -2 \longrightarrow \lambda^2 + 4\lambda + 4 = 0$$

$$y(x) = C_1 e^t + (C_2 + C_3 t) e^{-2t}$$

Ex $\lambda^4 (\lambda+1)(\lambda+2)^2(\lambda^2+4) = 0$

Roots: $\lambda = 0, 0, 0, 0, -1, -2, -2, \pm 2i$

$$y(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 e^{-x}$$
$$+ (C_6 + C_7 x) e^{-2x} + C_8 \cos 2x + C_9 \sin 2x$$

Ex $m = 4$, $k = 169$ $y(0) = .1 = \frac{1}{10}$

$y'(0) = 130\ cm/s = \frac{13}{10}$

soln

$$m y'' + \mu y' + k y = f(t)$$

$$4y'' + 169 y = 0$$

$$y'' + \frac{169}{4} y = 0$$

natural freq. : $\omega_0 = \sqrt{\frac{169}{4}} = \frac{13}{2}$

Period: $T = \frac{2\pi}{\omega_0} = \frac{4\pi}{13}$

$$\lambda^2 + \frac{169}{4} = 0 \Rightarrow \lambda_{1,2} = \pm \frac{13}{2} i$$

$$y(t) = C_1 \cos \frac{13}{2} t + C_2 \sin \frac{13}{2} t$$

$$\frac{1}{10} = C_1$$

$$y'(t) = -\frac{13}{2} C_1 \sin \frac{13}{2} t + \frac{13}{2} C_2 \cos \frac{13}{2} t$$

$$y'(0) = \frac{13}{2} C_2 = \frac{13}{10} \Rightarrow C_2 = \frac{1}{5}$$

$$y(t) = \frac{1}{10} \cos \frac{13}{2} t + \frac{1}{5} \sin \frac{13}{2} t$$

Amplitude: $A = \sqrt{\left(\frac{1}{10}\right)^2 + \left(\frac{1}{5}\right)^2}$

$$= \sqrt{\frac{1}{100} + \frac{1}{25}}$$

$$= \frac{\sqrt{5}}{10}\ m$$

$$\phi = \tan^{-1} \frac{1/5}{1/10}$$

$$= \tan^{-1} 2$$

$$y(t) = \frac{\sqrt{5}}{10} \cos\left(\frac{13}{2} t - \tan^{-1} 2\right)$$