Lecture Four

Section 4.1 – Relations and Their Properties

Definition

Let A and B be sets. A binary relation from A to B is a subset of $A \times B$

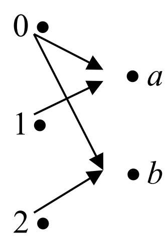
A binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B. We use the notation $a\ R\ b$ to denote that $(a,\ b) \in R$ and $a \not R\ b$ to denote that $(a,\ b) \notin R$. Moreover, when $(a,\ b)$ belongs to R, a is said to be related to b by R.

Example

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B.

This means, for instance, that $0Ra$ but the $1 \not R b$.

Relations can be represented graphically, as shown below, using arrows to represent ordered pairs.



Another way to represent this relation is to use a table.

R	***a***	***b***
0	x	x
1	x	
2		x

Relations on a Set

Definition

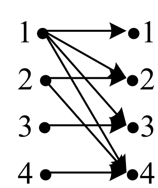
A ***relation*** on a set A is a relation from A to A. and it's a subset of $A \times A$

Example

Let A = {1, 2, 3, 4} which ordered pairs are the relation $R = \{(a,\ b) \mid a \text{ divides } b\}$?

Solution

R = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)}



Example

Consider these relations on the set of integers:

$R_1 = \{(a, b) \mid a \le b\}$

$R_2 = \{(a, b) \mid a > b\}$

$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$

$R_4 = \{(a, b) \mid a = b\}$

$R_5 = \{(a, b) \mid a = b + 1\}$

$R_6 = \{(a, b) \mid a + b \le 3\}$

Which of these relations contain each of the pairs (1, 1), (1, 2), (2, 1), (1, −1), and (2, 2)?

<u>Solution</u>

(1, 1) $\rightarrow R_1, R_3, R_4, \text{ and } R_6$

(1, 2) $\rightarrow R_1 \text{ and } R_6$

(2, 1) $\rightarrow R_2, R_5, \text{ and } R_6$

(1, −1) $\rightarrow R_2, R_3, \text{ and } R_6$

(2, 2) $\rightarrow R_1, R_3, \text{ and } R_4$

Example

How many relations are there on a set with *n* elements?

<u>Solution</u>

A relation on a set *A* is a subset of $A \times A$. Because $A \times A$ has n^2 elements when *A* has *n* elements, and a set with *m* elements has 2^m subsets, there are 2^{n^2} subset of $A \times A$.

Thus there are 2^{n^2} relations on a set with *n* elements.

Properties of Relations

Reflexive

Definition

A relation R on a set A is called ***reflexive*** if $(a, a) \in R$ for every element $a \in A$

Example

Consider the following relations on $\{1,2,3,4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$$

$$R_2 = \{(1, 1), (1, 2), (2, 1)\}$$

$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$$

$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$$

$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$$

$$R_6 = \{(3, 4)\}$$

Which of these relations are ***reflexive***?

Solution

The relations R_3 and R_5 are reflexive because they contain all pairs of the form (*a, a*), namely, (1, 1), (2, 2), (3, 3), and (4, 4).

$R_1, R_2, R_4,$ *and* R_6 are not reflexive because (3, 3) is not in any of these relations.

Example

Is the "divides" relation on the set of positive integers reflexive?

Solution

Because $a \mid a$ whenever a is a positive integer, the "divides" relation is reflexive.

(0 is doesn't divide 0)

Symmetric

Definition

A relation R on a set A is called ***symmetric*** if $(b,a) \in R$ whenever $(a,b) \in R$, for all $a,b \in A$.

$$\forall a \forall b \big((a,\ b) \in R \rightarrow (b,\ a) \in R\big)$$

A relation R on a set A such that for all $a,b \in A$, if $(a,b) \in R$ and $(b,a) \in R$, then $a = b$ is called ***antisymmetric***.

$$\forall a \forall b \big(\big((a,\ b) \in R \wedge (b,\ a) \in R\big) \rightarrow (a = b)\big)$$

Example

Is the "divides" relation on the set of positive integers symmetric? Is it antisymmetric?

Solution

It is antisymmetric because $1|2$ bur $2 \nmid 1$

Example

Consider the following relations on $\{1,2\ ,3\ ,4\}$:

$$R_1 = \{(1,\ 1),\ (1,\ 2),\ (2,\ 1),\ (2,\ 2),\ (3,\ 4),\ (4,\ 1),\ (4,\ 4)\}$$

$$R_2 = \{(1,\ 1),\ (1,\ 2),\ (2,\ 1)\}$$

$$R_3 = \{(1,\ 1),\ (1,\ 2),\ (1,\ 4),\ (2,\ 1),\ (2,\ 2),\ (3,\ 3),\ (4,\ 1),\ (4,\ 4)\}$$

$$R_4 = \{(2,\ 1),\ (3,\ 1),\ (3,\ 2),\ (4,\ 1),\ (4,\ 2),\ (4,\ 3)\}$$

$$R_5 = \{(1,\ 1),\ (1,\ 2),\ (1,\ 3),\ (1,\ 4),\ (2,\ 2),\ (2,\ 3),\ (2,\ 4),\ (3,\ 3),\ (3,\ 4),\ (4,\ 4)\}$$

$$R_6 = \{(3,\ 4)\}$$

Which of these relations are symmetric and which are antisymmetric?

Solution

The relations R_2 and R_3 are symmetric because in each case (*b, a*) belongs to the relation whenever (*a, b*) does. $(1,\ 2)$ *and* $(2,\ 1)$ in R_2 $(1,\ 2),\ (2,\ 1),\ (1,\ 4)$ *and* $(4,\ 1)$ in R_3.

The relations R_1, R_4, R_5 and R_6 are antisymmetric because for each relations there is no pair of elements *a* and *b* with $a \neq b$ such that both (*a, b*) and (*b, a*) belong to the relation.

Transitive

Definition

A relation ***R*** on a set ***A*** is called ***transitive*** if whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$, for all $a, b, c \in A$

$$\forall a \forall b \forall c \left(\left((a, b) \in R \land (b, c) \in R \right) \to (a, c) \in R \right)$$

Example

Consider the following relations on $\{1,2,3,4\}$:

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$

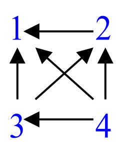
$R_6 = \{(3, 4)\}$

Which of these relations are transitive?

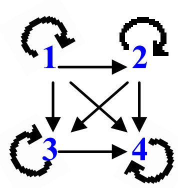
Solution

The relations R_4 and R_5 are transitive because in each of these relations case that is (a, b) and (b, c) belong to this relation then (a, c) also does.

For R_4



For R_5



The relation R_1 is not transitive because $(3, 4)$ *and* $(4, 1)$ belong to R_1 but not $(3, 1)$

The relation R_2 is not transitive because $(2, 1)$ *and* $(1, 2)$ belong to R_2 but not $(2, 2)$

The relation R_3 is not transitive because $(4, 1)$ *and* $(1, 2)$ belong to R_3 but not $(4, 2)$

Example

Consider these relations on the set of integers:

$R_1 = \{(a, b) \mid a \le b\}$ $\qquad$ $R_2 = \{(a, b) \mid a > b\}$

$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$ $\qquad$ $R_4 = \{(a, b) \mid a = b\}$

$R_5 = \{(a, b) \mid a = b + 1\}$ $\qquad$ $R_6 = \{(a, b) \mid a + b \le 3\}$

Which of these relations contain each of the pairs (1, 1), (1, 2), (2, 1), (1, −1), and (2, 2)?

Solution

The relations R_1, R_2, R_3 and R_4 are transitive.

R_1 is transitive because $a \le b$ and $b \le c$ imply that $a \le c$

R_2 is transitive because $a > b$ and $b > c$ imply that $a > c$

R_3 is transitive because $a = \pm b$ and $b = \pm c$ imply that $a = \pm c$

R_4 is transitive because $a = b$ and $b = c$ imply that $a = c$

The relations R_5 and R_6 are not transitive.

R_5 is not transitive because $a = b + 1$ and $b = c + 1$ imply that $a = (c+1) + 1 = c + 2 \ne c + 1$

R_6 is not transitive because $2 + 1 \le 3$ and $1 + 2 \le 3$ imply that $2 + 2 \not\le 3$

Example

Is the "divides" relation on the set of positive integers transitive?

Solution

Suppose *a* divides *b* and *b* divides *c*. Then there are positive integers *m* and *n* such that $b = ma$ and $c = nb$. Hence $c = n(ma) = (nm)a$, so *a* divides *c*.

Therefore this relation is transitive.

Combining Relations

Let *A* = {1, 2, 3} and *B* = {1, 2, 3, 4}.

The relations $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$

$R_1 \cup R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\}$

$R_1 \cap R_2 = \{(1, 1)\}$

$R_1 - R_2 = \{(2, 2), (3, 3)\}$

$R_2 - R_1 = \{(1, 2), (1, 3), (1, 4)\}$

Example

Let R_1 be the "less than" relation on the set of real numbers and let R_2 be the "greater than" relation on the set of real numbers, that is $R_1 = \{(x, y) | x < y\}$ and $R_2 = \{(x, y) | x > y\}$.

What are $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$, and $R_1 \oplus R_2$?

Solution

$(x, y) \in R_1 \cup R_2$ if and only if $(x, y) \in R_1$ *or* $(x, y) \in R_2$. That implies $(x, y) \in R_1 \cup R_2$ *iff* $x < y$ or $x > y$. Since $x < y$ or $x > y$ means that, that follows that $R_1 \cup R_2 = \{(x, y) | x \neq y\}$.

$R_1 \cap R_2 = \varnothing$, since it is impossible for a pair (x, y) to belong to both R_1 and R_2 because $x < y$ and $x > y$.

$R_1 - R_2 = R_1$, since $R_1 \cap R_2 = \varnothing$

$R_2 - R_1 = R_2$, since $R_1 \cap R_2 = \varnothing$

$R_1 \oplus R_2 = R_1 \cup R_2 - R_1 \cap R_2 = \{(x, y) | x \neq y\}$

Definition

Let *R* be a relation from a set *A* to a set *B* and *S* a relation from *B* to a set *C*. The composite of *R* and *A* is the relation consisting of ordered pairs (*a, c*), where $a \in A, c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of *R* and *S* by $S \circ R$.

Example

What is the composite of the relation *R* and *S*, where
R is the relation from {1, 2, 3} to {1, 2, 3, 4} with *R* = {(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)}.
S is the relation from {1, 2, 3, 4} to {0, 1, 2} with *S* = {(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)}.

Solution

R	S		$S \circ R$
$(1,1)$	$(1,0)$	$\rightarrow$	$(1,0)$
$(1,4)$	$(4,1)$	$\rightarrow$	$(1,1)$
$(2,3)$	$(3,1)$	$\rightarrow$	$(2,1)$
$(2,3)$	$(3,2)$	$\rightarrow$	$(2,2)$
$(3,1)$	$(1,0)$	$\rightarrow$	$(3,0)$
$(3,4)$	$(4,1)$	$\rightarrow$	$(3,1)$

$S \circ R = \{(1,0), (1,1), (2,1), (2,2), (3,0), (3,1)\}$

Definition

Let R be a relation on the set A. Then powers R^n, $n = 1, 2, 3, \ldots$ are defined recursively by

$$R^1 = R \quad and \quad R^{n+1} = R^n \circ R$$

Example

Let $R = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$. Find the powers R^n, $n = 2, 3, 4, \ldots$

Solution

$$R^2 = R \circ R = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$$

$$R^3 = R^2 \circ R = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$$

$$R^4 = R^3 \circ R = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$$

From that , it follows that $R^n = R^3$ for $n = 5, 6, 7, \ldots$.

Theorem

The relation on a set A is transitive ***iff*** $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

Proof

Suppose that $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$. In particular, $R^2 \subseteq R$. If $(a, b) \in R$ and $(b, c) \in R$, then by definition of composite, $(a, c) \in R^2$. Because $R^2 \subseteq R$, this means that $(a, c) \in R$. Hence, R is transitive.

Using mathematical induction to prove the only if part of the theorem

Assume that $R^n \subseteq R$ where n is a positive integer. This is the inductive hypothesis.

To complete the inductive step we must show that this implies that R^{n+1} is also a subset of R.

Assume that $(a, b) \in R^{n+1}$, then because $R^{n+1} = R^n \circ R$, there is an element x with $x \in A$ such that $(a, x) \in R$ and $(x, b) \in R^n$. The inductive hypothesis, namely, that $R^n \subseteq R$, implies that $(x, b) \in R$

Furthermore, because R is transitive, and $(a, x) \in R$ and $(x, b) \in R$, it follows that $(a, b) \in R$.

This shows that $R^n \subseteq R$.

Exercises *Section* 4.1 – Relations and Their Properties

1. List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$ where $(a, b) \in R$ if and only if

a) $a = b$ ***b***) $a + b = 4$ ***c***) $a > b$

d) $a \mid b$ ***e***) $gcd(a, b) = 1$ ***f***) $lcm(a, b) = 2$

2. ***a***) List all the ordered pairs in the relation $R = \{(a,\ b) \mid a\ divides\ \ b\}$ on the set $\{1, 2, 3, 4, 5, 6\}$

b) Display this relation graphically.

c) Display this relation in tabular form.

3. For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, symmetric, antisymmetric and transitive

a) $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$
b) $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
c) $\{(2, 4), (4, 2)\}$
d) $\{(1, 2), (2, 3), (3, 4)\}$
e) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
f) $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$

4. Determine whether the relation R on the set of all people is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if

a) a is taller than b.
b) a and b were born on the same day
c) a has the same first name as b.
d) a and b have a common grandparent.

5. Determine whether the relation $\boldsymbol{R}$ on the set of all ***real numbers*** is reflexive, symmetric, antisymmetric, and/or transitive, where $(x,\ y) \in \boldsymbol{R}$ if and only if

a) $x + y = 0$ ***b***) $x = \pm y$ ***c***) $x - y\ \ is\ a\ rational\ number$ ***d***) $x = 2y$

e) $xy \ge 0$ ***f***) $xy = 0$ ***g***) $x = 1$ ***h***) $x = 1\ or\ y = 1$

6. Determine whether the relation R on the set of all ***integers numbers*** is reflexive, symmetric, antisymmetric, and/or transitive, where $(x,\ y) \in R$ if and only if

a) $x \ne y$ ***b***) $xy \ge 1$ ***c***) $x = y + 1\ \ or\ \ x = y - 1$ ***d***) $x \equiv y(\boldsymbol{mod}\ 7)$

e) $x\ is\ a\ multiple\ of\ \ y$ ***f***) $x = y^2$ ***g***) $x \ge y^2$

7. Show that the relation $R = \varnothing$ on nonempty set S is symmetric and transitive, but not reflexive.

8. Show that the relation $R = \varnothing$ on nonempty set $S = \varnothing$ is reflexive, symmetric and transitive.

9. Give an example of a relation on a set that is

a) both symmetric and antisymmetric
b) neither symmetric nor antisymmetric

10. A relation R is called ***asymmetric*** if $(a,b) \in R$ implies that $(b,a) \notin R$. Explore the notion of an asymmetric relation to the following

a) {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}
b) {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}
c) {(2, 4), (4, 2)}
d) {(1, 2), (2, 3), (3, 4)}
e) {(1, 1), (2, 2), (3, 3), (4, 4)}
f) {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}
g) a is taller than b.
h) a and b were born on the same day
i) a has the same first name as b.
j) a and b have a common grandparent.

11. Let R be the relation $R = \{(a,\ b) \mid a < b\}$ on the set of integers. Find

a) R^{-1} ***b)*** $\overline{R}$

12. Let R be the relation $R = \{(a,\ b) \mid a \text{ divides } b\}$ on the set of positive integers. Find

a) R^{-1} ***b)*** $\overline{R}$

13. Let R be the relation on the set of all states in the U.S. consisting of pairs (a, b) where state a borders state b. Find

a) R^{-1} ***b)*** $\overline{R}$

14. Let $R_1 = \{(1,\ 2),\ (2,\ 3),\ (3,\ 4)\}$ and
$R_2 = \{(1,\ 1),\ (1,\ 2),\ (2,\ 1),\ (2,\ 2),\ (2,\ 3),\ (3,\ 1),\ (3,\ 2),\ (3,\ 3),\ (3,\ 4)\}$ be relation from {1, 2, 3} to {1, 2, 3, 4}. Find

a) $R_1 \cup R_2$ ***b***) $R_1 \cap R_2$ ***c***) $R_1 - R_2$ ***d***) $R_2 - R_1$

15. Let the relation R = {(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)} and the relation S = {(2, 1), (3, 1), (3, 2), (4, 2)}. Find $S \circ R$

16. $R_1 = \{(a,b) \in \boldsymbol{R}^2 \mid a > b\}$ $R_3 = \{(a,b) \in \boldsymbol{R}^2 \mid a < b\}$ $R_5 = \{(a,b) \in \boldsymbol{R}^2 \mid a = b\}$
$R_2 = \{(a,b) \in \boldsymbol{R}^2 \mid a \ge b\}$ $R_4 = \{(a,b) \in \boldsymbol{R}^2 \mid a \le b\}$ $R_6 = \{(a,b) \in \boldsymbol{R}^2 \mid a \ne b\}$

Find the following:

a) $R_1 \cup R_3$ ***b***) $R_1 \cup R_5$ ***c***) $R_2 \cap R_4$ ***d***) $R_3 \cap R_5$ ***e***) $R_1 - R_2$

f) $R_2 - R_1$ ***g***) $R_1 \oplus R_3$ ***h***) $R_2 \oplus R_4$ ***i***) $R_1 \circ R_1$ ***j***) $R_1 \circ R_2$

k) $R_1 \circ R_3$ ***l***) $R_1 \circ R_4$ ***m***) $R_1 \circ R_5$ ***n***) $R_1 \circ R_6$ ***o***) $R_2 \circ R_3$

17. Let R_1 *and* R_2 be the "*divides*" and "*is a multiple of*" relations on the set of all positive integers, respectively. That is $R_1 = \{(a,b) \mid a \text{ divides } b\}$ and $R_2 = \{(a,b) \mid a \text{ is a multiple of } b\}$

Find the following:

a) $R_1 \cup R_2$ ***b***) $R_1 \cap R_2$ ***c***) $R_1 - R_2$ ***d***) $R_2 - R_1$ ***e***) $R_1 \oplus R_2$