Lecture Two – Techniques of Integration

Section 2.1 – Integration by Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)\,dx$$

Example: $\quad \int x\cos x\,dx, \quad \int x^2 e^x\,dx, \quad \textit{and} \quad \int x\ln x\,dx$

Integration by Parts Formula

$$\int f(x)g'(x) = f(x)g(x) - \int f'(x)g(x)\,dx$$

Let u and v be differentiable functions of x. $\qquad \int u\,dv = uv - \int v\,du$

Guidelines for integration by Parts

1. Let dv be the most complicated portion of the integrand that fits a basic integration formula. Let u be the remaining factor.
2. Let u be the portion of the integrand whose derivative is a function simpler than u. Let dv be the remaining factor.

Example

Evaluate: $\quad \int x\cos x\,dx$

Solution

Let:

$$u = x \qquad\qquad dv = \cos x\,dx$$

$$du = dx \qquad v = \int dv = \int \cos x\,dx = \sin x$$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx \qquad\qquad \int u\,dv = uv - \int v\,du$$

$$= x\sin x + \cos x + C$$

Example

Evaluate: $\int \ln x \, dx$

Solution

Let:

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = \int dx = x$

$$\int \ln x dx = x \ln x - \int x \frac{1}{x} dx \qquad \int udv = uv - \int vdu$$

$$= x \ln x - \int dx$$

$$\underline{= x \ln x - x + C}$$

Tabular Integration

Example

Evaluate $\int x^2 e^x dx$

Solution

$f(x) = x^2 \quad and \quad g(x) = e^x$

$f(x)$ **& derivatives**		$\int g(x) = \int e^x$
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x

It is called ***tabular integration***

$$\int x^2 e^x dx \underline{= x^2 e^x - 2xe^x + 2e^x + C}$$

$u = x^2 \qquad dv = e^x dx$

$du = 2xdx \qquad v = \int e^x dx = e^x$

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx$$

Let: $u = x \qquad dv = e^x dx$

$du = dx \qquad v = \int e^x dx = e^x$

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx$$

$$= x^2 e^x - 2\left[xe^x - \int e^x dx \right]$$

$$= x^2 e^x - 2\left(xe^x - e^x \right) + C$$

$$\underline{= x^2 e^x - 2xe^x + 2e^x + C}$$

Example

Evaluate $\displaystyle\int x^3 \sin x\, dx$

Solution

$$\int x^3 \sin x dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6\sin x + C$$

		$\int \sin x$
+	x^3	$-\cos x$
–	$3x^2$	$-\sin x$
+	$6x$	$\cos x$
–	6	$\sin x$

Example

Evaluate $\displaystyle\int e^x \cos x\, dx$

Solution

$$\int e^x \cos x\, dx = e^x \sin x + e^x \cos x - \int e^x \cos x\, dx$$

$$2\int e^x \cos x\, dx = e^x (\sin x + \cos x)$$

$$\int e^x \cos x\, dx = \frac{1}{2} e^x (\sin x + \cos x) + C$$

		$\int \cos x\, dx$
+	e^x	$\sin x$
-	e^x	$-\cos x$
+	e^x	$-\int \cos x\, dx$

Let: $u = e^x \qquad dv = \cos x dx$

$du = e^x dx \qquad v = \int \cos x dx = \sin x$

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$$

Let: $u = e^x \qquad dv = \sin x dx$

$du = e^x dx \qquad v = \int \sin x dx = -\cos x$

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$$

$$= e^x \sin x - \left[-e^x \cos x - \int (-\cos x) e^x dx \right]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x dx$$

$$\int e^x \cos x dx + \int e^x \cos x dx = e^x \sin x + e^x \cos x - \int e^x \cos x dx + \int e^x \cos x dx$$

$$2\int e^x \cos x dx = e^x \sin x + e^x \cos x + C_1$$

$$\int e^x \cos x\, dx = \frac{1}{2} e^x \sin x + \frac{1}{2} e^x \cos x + C$$

Example

Obtain a formula that expresses the integral $\int \cos^n x dx$

Solution

Let:

$$u = \cos^{n-1} x \qquad dv = \cos x dx$$

$$du = (n-1)\cos^{n-2} x(-\sin x dx) = -(n-1)\cos^{n-2} x \sin x dx \qquad v = \int \cos x dx = \sin x$$

$$\int u dv = uv - \int v du$$

$$\int \cos^n x dx = \cos^{n-1} x \sin x - \int \sin x\left(-(n-1)\cos^{n-2} x \sin x dx\right)$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\left(1-\cos^2 x\right) dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \left(\cos^{n-2} x - \cos^n x\right) dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx - (n-1)\int \cos^n x dx$$

$$\int \cos^n x dx + (n-1)\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx$$

$$(1+n-1)\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx$$

$$n\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx$$

$$\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx$$

Example:

$$\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\int \cos x dx$$

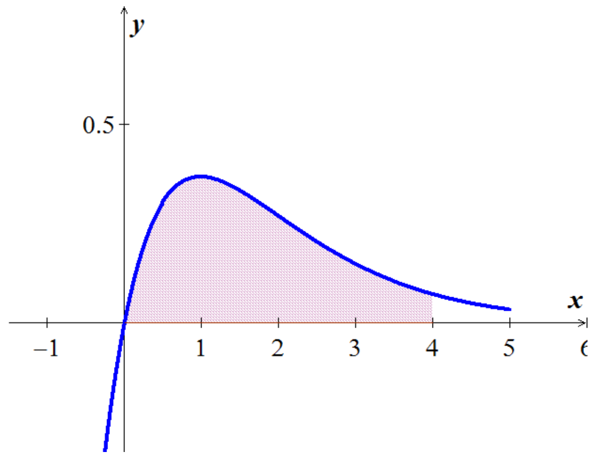
$$= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$$

Evaluating Definite Integrals by Parts

Example

Find the area of the region bounded by the curve $y = xe^{-x}$ and the *x*-axis from $x = 0$ to $x = 4$.

Solution



$$A = \int_0^4 xe^{-x}dx$$

		$\int e^{-x}$
+	x	$-e^{-x}$
−	1	e^{-x}

$$A = (-x-1)e^{-x}\Big|_0^4$$

$$= -5e^{-4} + 1$$

$$\boxed{\approx 0.91\ unit^2}$$

2

2nd *Method*

Let: $u = x \qquad dv = e^{-x}dx$

$du = dx \qquad v = \int e^{-x}dx = -e^{-x}$

$$\int u\,dv = uv - \int v\,du$$

$$\int_0^4 xe^{-x}dx = -xe^{-x}\Big]_0^4 - \int_0^4 \left(-e^{-x}\right)dx$$

$$= -\left[4e^{-4} - 0\right] + \int_0^4 e^{-x}dx$$

$$= -4e^{-4} + \left[-e^{-x}\right]_0^4$$

$$= -4e^{-4} - \left[e^{-4} - 1\right]$$

$$= -4e^{-4} - e^{-4} + 1$$

$$= 1 - 5e^{-4}$$

$$\boxed{\approx 0.91\ unit^2}$$

Formula

Evaluate $\int x^n e^{ax}\,dx$

		$\int e^{ax}$
+	x^n	$\frac{1}{a}e^{ax}$
−	nx^{n-1}	$\frac{1}{a^2}e^{ax}$
+	$n(n-1)x^{n-2}$	$\frac{1}{a^3}e^{ax}$
−	$n(n-1)(n-2)x^{n-3}$	$\frac{1}{a^4}e^{ax}$
	$\vdots\ \vdots$	$\vdots\ \vdots$

$$\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a^2}x^{n-1}e^{ax} + \frac{n(n-1)}{a^3}x^{n-2}e^{ax} - \frac{n(n-1)(n-2)}{a^4}x^{n-3}e^{ax} + \dots$$

$$= e^{ax}\sum_{k=0}^{n}(-1)^k \cdot \frac{n!}{(n-k)!}\cdot\frac{1}{a^{k+1}}\cdot x^{n-k}$$

Exercises *Section* 2.1 – Integration by Parts

Evaluate the integrals

1. $\int xe^{2x}dx$
2. $\int x\ln x dx$
3. $\int x^3e^xdx$
4. $\int \ln x^2dx$
5. $\int \frac{2x}{e^x}dx$
6. $\int \ln(3x)dx$
7. $\int \frac{1}{x\ln x}dx$
8. $\int \frac{x}{\sqrt{x-1}}dx$
9. $\int \frac{x^3e^{x^2}}{\left(x^2+1\right)^2}dx$
10. $\int x^2e^{-3x}dx$
11. $\int \theta\cos\pi\theta d\theta$
12. $\int x^2\sin x\, dx$
13. $\int x(\ln x)^2\, dx$
14. $\int \left(x^2-2x+1\right)e^{2x}dx$
15. $\int \tan^{-1}y\, dy$
16. $\int \sin^{-1}y\, dy$
17. $\int 4x\sec^2 2x\, dx$
18. $\int e^{2x}\cos 3x\, dx$
19. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$
20. $\int \frac{(\ln x)^3}{x}dx$
21. $\int x^5e^{x^3}dx$
22. $\int x^2\ln x^3\, dx$
23. $\int \ln\left(x+x^2\right)dx$
24. $\int e^{-x}\sin 4x\, dx$
25. $\int e^{-2\theta}\sin 6\theta\, d\theta$
26. $\int xe^{-4x}dx$
27. $\int x\ln(x+1)dx$
28. $\int \frac{(\ln x)^2}{x}dx$
29. $\int \frac{xe^{2x}}{(2x+1)^2}dx$
30. $\int \frac{5x}{e^{2x}}dx$
31. $\int \frac{e^{1/x}}{x^2}dx$
32. $\int x^5\ln 3x\, dx$
33. $\int x\sqrt{x-5}\, dx$
34. $\int \frac{x}{\sqrt{6x+1}}dx$
35. $\int x\cos x\, dx$
36. $\int x\csc x\cot x\, dx$
37. $\int x^3\sin x\, dx$
38. $\int x^2\cos x\, dx$
39. $\int e^{-3x}\sin 5x\, dx$
40. $\int e^{-3x}\sin 4x\, dx$
41. $\int e^{4x}\cos 2x\, dx$
42. $\int e^{3x}\cos 3x\, dx$
43. $\int x^2e^{4x}\, dx$
44. $\int x^3e^{-3x}\, dx$
45. $\int x^3\cos 2x\, dx$
46. $\int x^3\sin x\, dx$
47. $\int x^5\ln x\, dx$

48. $\int_0^{1/\sqrt{2}} 2x\sin^{-1}\left(x^2\right)dx$

49. $\int_1^e x^3 \ln x \, dx$

50. $\int_0^1 x\sqrt{1-x} \, dx$

51. $\int_0^{\pi/3} x\tan^2 x \, dx$

52. $\int_0^{\pi} x\sin x \, dx$

53. $\int_1^e \ln 2x \, dx$

54. $\int_0^{\pi/2} x\cos 2x \, dx$

55. $\int_0^{\ln 2} xe^x \, dx$

56. $\int_1^{e^2} x^2 \ln x \, dx$

57. $\int_0^3 xe^{x/2}dx$

58. $\int_0^2 x^2e^{-2x}dx$

59. $\int_0^{\pi/4} x\cos 2x \, dx$

60. $\int_0^{\pi} x\sin 2x \, dx$

61. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate aces, the cure $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$

62. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the cure $y = e^{-x}$, and the line $x = 1$, about

a) the line $y-axis$

b) the line $x = 1$

63. Find the volume of the solid that is generated by the region bounded by $f(x) = e^{-x}$, $x = \ln 2$, and the coordinate axes is revolved about the y-*axis*.

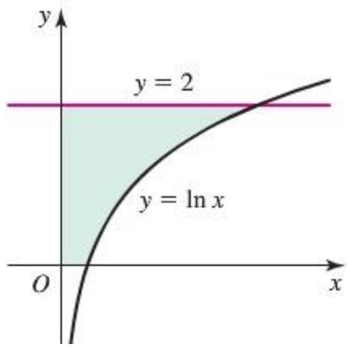
64. Find the volume of the solid that is generated by the region bounded by $f(x) = \sin x$, and the x-*axis* on $[0, \pi]$ is revolved about the y-*axis*.

65. Find the area of the region generated when the region bounded by $y = \sin x$ and $y = \sin^{-1} x$ on the interval $\left[0, \frac{1}{2}\right]$.

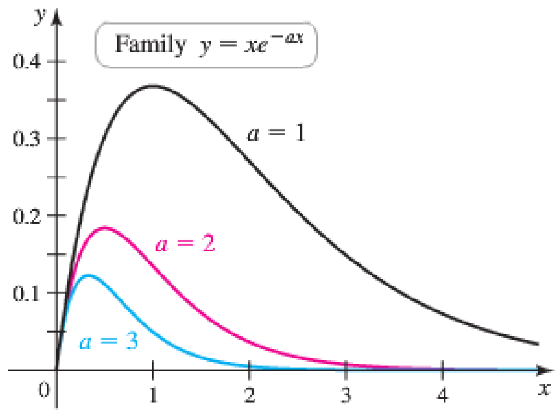
66. Find the area between the curves $y = \ln x^2$, $y = \ln x$, *and* $x = e^2$

67. Determine the area of the shaded region bounded by

$y = \ln x, \quad y = 2, \quad y = 0, \quad \text{and } x = 0$



68. The curves $y = xe^{-ax}$ are shown in the figure for $a = 1,\ 2,\ and\ 3$.



a) Find the area of the region bounded by $y = xe^{-x}$ and the *x*-axis on the interval [0, 4].

b) Find the area of the region bounded by $y = xe^{-ax}$ and the *x*-axis on the interval [0, 4] where $a > 0$

c) Find the area of the region bounded by $y = xe^{-ax}$ and the *x*-axis on the interval [0, *b*]. Because this area depends on *a* and *b*, we call it $A(a,\ b)$ where $a > 0$ and $b > 0$.

d) Use part (*c*) to show that $A(1,\ \ln b) = 4A\left(2,\ \frac{1}{2}\ln b\right)$

e) Does this pattern continue? Is it true that $A(1,\ \ln b) = a^2 A\left(a,\ \frac{1}{a}\ln b\right)$

69. Suppose a mass on a spring that is slowed by friction has the position function $s(t) = e^{-t}\sin t$

a) Graph the position function. At what times does the oscillator pass through the position $s = 0$?

b) Find the average value of the position on the interval $[0,\ \pi]$.

c) Generalize part (*b*) and find the average value of the position on the interval $[n\pi,\ (n+1)\pi]$, for $n = 0,\ 1,\ 2,\ \ldots$

70. Given the region bounded by the graphs of $y = x\sin x,\quad y = 0,\quad x = 0,\quad x = \pi$, find

a) The area of the region.

b) The volume of the solid generated by revolving the region about the *x-axis*

c) The volume of the solid generated by revolving the region about the *y-axis*

d) The centroid of the region

Section 2.2 – Trigonometric Integrals

Products of Powers of *Sines* and *Cosines*

We begin with integrals of the form

$$\int \sin^m x \cos^n x \, dx$$

Example

Evaluate $\int \sin^3 x \cos^2 x \, dx$

Solution

$$\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int \sin x \sin^2 x \cos^2 x \, dx \\
&= \int \left(1-\cos^2 x\right)\cos^2 x \, \left(-d\left(\cos x\right)\right) \qquad d\left(\cos x\right) = -\sin x dx \quad \Rightarrow \sin x dx = -d\left(\cos x\right) \\
&= -\int \left(\cos^2 x - \cos^4 x\right) \, d\left(\cos x\right) \qquad \textit{or} \quad \textit{Assume} \quad u = \cos x \\
&= -\left(\frac{1}{3}\cos^3 x - \frac{1}{5}\cos^5 x\right) + C \\
&= \boxed{\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C}
\end{aligned}$$

Example

Evaluate $\int \cos^5 x \, dx$

Solution

$$\begin{aligned}
\int \cos^5 x \, dx &= \int \cos^4 x \cos x \, dx \qquad \cos x dx = d\left(\sin x\right) \qquad \cos^2 x = 1 - \sin^2 x \\
&= \int \left(1-\sin^2 x\right)^2 \, d\left(\sin x\right) \\
&= \int \left(1 - 2\sin^2 x + \sin^4 x\right) \, d\sin x \\
&= \boxed{\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C}
\end{aligned}$$

Example

Evaluate $\displaystyle\int \sin^2 x \cos^4 x \, dx$

Solution

$$\int \sin^2 x \cos^4 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx \qquad \sin^2 x = \frac{1-\cos 2x}{2} \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \frac{1}{8}\int (1-\cos 2x)\left(1+2\cos 2x+\cos^2 2x\right) dx$$

$$= \frac{1}{8}\int \left(1+2\cos 2x+\cos^2 2x-\cos 2x-2\cos^2 2x-\cos^3 2x\right) dx$$

$$= \frac{1}{8}\int \left(1+\cos 2x-\cos^2 2x-\cos^3 2x\right) dx$$

$$= \frac{1}{8}\left[x+\frac{1}{2}\sin 2x-\int \left(\cos^3 2x+\cos^2 2x\right) dx\right]$$

$$\int \cos^3 2x dx = \int \left(1-\sin^2 2x\right)\cos 2x dx$$

$$= \frac{1}{2}\int \left(1-\sin^2 2x\right) d(\sin 2x)$$

$$= \frac{1}{2}\left(\sin 2x-\frac{1}{3}\sin^3 2x\right)$$

$$\int \cos^2 2x dx = \frac{1}{2}\int (1+\cos 4x)\, dx$$

$$= \frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)$$

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{8}\left[x+\frac{1}{2}\sin 2x-\frac{1}{2}\left(\sin 2x-\frac{1}{3}\sin^3 2x\right)-\frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)\right]+C$$

$$= \frac{1}{8}\left[x+\frac{1}{2}\sin 2x-\frac{1}{2}\sin 2x+\frac{1}{6}\sin^3 2x-\frac{1}{2}x-\frac{1}{8}\sin 4x\right]+C$$

$$= \frac{1}{8}\left(\frac{1}{2}x+\frac{1}{6}\sin^3 2x-\frac{1}{8}\sin 4x\right)+C$$

$$\boxed{= \frac{1}{16}\left(x+\frac{1}{3}\sin^3 2x-\frac{1}{4}\sin 4x\right)+C}$$

Example

Evaluate $\int_0^{\pi/4} \sqrt{1+\cos 4x}\ dx$

Solution

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \quad \Rightarrow \quad 1+\cos 2\theta = 2\cos^2\theta$$

$$\theta = 2x \quad \Rightarrow \quad 1+\cos 4x = 2\cos^2 2x$$

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\ dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\ dx$$

$$\sqrt{2\cos^2 2x} = \sqrt{2}\sqrt{\cos^2 2x} = \sqrt{2}\,|\cos 2x|$$

$$= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\ dx$$

$$\cos 2x \ge 0 \quad on \quad \left[0, \frac{\pi}{4}\right]$$

$$= \sqrt{2}\int_0^{\pi/4} \cos 2x\ dx$$

$$= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4}$$

$$= \frac{\sqrt{2}}{2}\left[\sin\frac{\pi}{2} - 0\right]$$

$$= \frac{\sqrt{2}}{2}$$

Example

Evaluate $\int \sin^3 x \cos^{-2} x\ dx$

Solution

$$\int \sin^3 x\cos^{-2} x\ dx = \int \sin^2 x\cos^{-2} x\ \sin x\ dx$$

$$= -\int \left(1-\cos^2 x\right)\cos^{-2} x\ \ d(\cos x)$$

$$= -\int \left(\cos^{-2} x - 1\right)\ d(\cos x)$$

$$= -\left(-\cos^{-1} x - \cos x\right) + C$$

$$= \cos x + \sec x + C$$

Products of Powers of *tan x* and *sec x*

Example

Evaluate $\displaystyle\int \tan^4 x\, dx$

Solution

$$\int \tan^4 x dx = \int \tan^2 x \cdot \tan^2 x\, dx \qquad \tan^2 x = \sec^2 x - 1$$

$$= \int \tan^2 x \cdot \left(\sec^2 x - 1\right) dx$$

$$= \int \tan^2 x \sec^2 x dx - \int \tan^2 x\, dx$$

$$= \int \tan^2 x \sec^2 x dx - \int \left(\sec^2 x - 1\right) dx$$

$$= \int \tan^2 x \sec^2 x dx - \int \sec^2 x\, dx + \int dx$$

$$= \int \tan^2 x\, d(\tan x) - \int \sec^2 x\, dx + \int dx \qquad d(\tan x) = \sec^2 x\, dx$$

$$= \frac{1}{3}\tan^3 x - \tan x + x + C$$

Example

Evaluate $\displaystyle\int \sec^3 x\, dx$

Solution

Let: $u = \sec x \qquad dv = \sec^2 x dx$

$du = \sec x \tan x dx \qquad v = \tan x$

$$\int \sec^3 x dx = \sec x \tan x - \int \tan x \left(\sec x \tan x dx\right)$$

$$= \sec x \tan x - \int \tan^2 x \sec x dx$$

$$= \sec x \tan x - \int \left(\sec^2 x - 1\right) \sec x\, dx$$

$$= \sec x \tan x - \int \left(\sec^3 x - \sec x\right) dx$$

$$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$\int \sec^3 x\, dx + \int \sec^3 x\, dx = \sec x \tan x - \int \sec^3 x\, dx + \int \sec^3 x\, dx + \int \sec x\, dx$$

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$

$$= \sec x \tan x + \ln\left|\sec x + \tan x\right| + C_1$$

$$\int \sec^3 x\, dx = \boxed{\frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right| + C}$$

Products of Sines and Cosines

Recall the identities

$$\sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$$

$$\sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right]$$

$$\cos mx \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]$$

Example

Evaluate $\int \sin 3x \cos 5x dx$

Solution

$$\int \sin 3x \cos 5x dx = \frac{1}{2}\int \left[\sin(-2x) + \sin 8x\right] dx$$

$$= \frac{1}{2}\int \left[-\sin(2x) + \sin 8x\right] dx$$

$$= \frac{1}{2}\left(\frac{1}{2}\cos 2x - \frac{1}{8}\cos 8x\right) + C$$

$$\boxed{= \frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + C}$$

Guidelines for Cosine & Sine

Case* 1** If ***m is ***odd***, we write m as $2k+1$ and use the identity $\sin^2 x = 1-\cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = \left(\sin^2 x\right)^k \sin x = \left(1-\cos^2 x\right)^k \sin x$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x dx = -d\left(\cos x\right)$

Case* 2** If ***m is ***even* and *n*** is ***odd***, in $\int \sin^m x \cos^n x dx$ we write n as $2k+1$ and use the identity $\cos^2 x = 1-\sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = \left(\cos^2 x\right)^k \cos x = \left(1-\sin^2 x\right)^k \cos x$$

Then we combine the single $\cos x$ with dx in the integral and set $\cos x dx = d\left(\sin x\right)$

Case* 3** If both ***m* and *n are ***even***, in $\int \sin^m x \cos^n x dx$, we substitute

To reduce the integrand to one in lower powers of $\cos 2x$ $\quad \int \cos ax dx = \frac{1}{a}\sin ax + C$

Guidelines for Tangent & Secant

Case* 1** When the power of the tangent is ***odd and positive.

$$\begin{aligned}\int \sec^m x\ \tan^{2k+1} x\ dx &= \int \sec^{m-1} x\left(\tan^2 x\right)^k \sec x \tan x\ dx \\ &= \int \sec^{m-1} x\left(\sec^2 x - 1\right)^k d\left(\sec x\right)\end{aligned}$$

Case* 2** When the power of the secant is ***even and positive.

$$\int \sec^{2k} x\ \tan^n x\ dx = \int \left(\sec^2 x\right)^{k-1} \tan^n x\ \sec^2 x\ dx = \int \left(1+\tan^2 x\right)^{k-1} \tan^n x\ d\left(\tan x\right)$$

***Case* 3** When there are no secant factors

$$\int \tan^n x\ dx = \int \tan^{n-2} x\ \left(\tan^2 x\ \right)dx = \int \tan^{n-2} x\ \left(\sec^2 x - 1\ \right)dx$$

***Case* 4** When there are only secant, use integration by parts.

***Case* 5** Otherwise, convert to cosines and sines.

Wallis's Formulas

1. If n is odd $(n \geq 3)$, then $$\int_0^{\pi/2} \cos^n x\,dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right)$$

2. If n is even $(n \geq 2)$, then $$\int_0^{\pi/2} \cos^n x\,dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

Formulas

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

$$\int \sin^n x\,dx = \frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$

$$\int \tan x\,dx = -\ln|\cos x| + C = \ln|\sec x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

Exercises *Section* **2.2 – Trigonometric Integrals**

Evaluate the integrals

1. $\int \sin^4 2x\cos 2x\, dx$

2. $\int \sin^5 \frac{x}{2}\, dx$

3. $\int \cos^3 2x\sin^5 2x\, dx$

4. $\int 8\cos^4 2\pi x\, dx$

5. $\int 16\sin^2 x\cos^2 x\, dx$

6. $\int \sec x\tan^2 x\, dx$

7. $\int \sec^2 x\tan^2 x\, dx$

8. $\int e^x \sec^3 e^x\, dx$

9. $\int \sec^4 x\tan^2 x\, dx$

10. $\int \sin 2x\cos 3x\, dx$

11. $\int \sin^2 \theta\cos 3\theta\, d\theta$

12. $\int \cos^3 \theta\sin 2\theta\, d\theta$

13. $\int \sin\theta\sin 2\theta\sin 3\theta\, d\theta$

14. $\int \frac{\sin^3 x}{\cos^4 x}\, dx$

15. $\int x\cos^3 x\, dx$

16. $\int \sin^4 x\cos^2 x\, dx$

17. $\int \tan^3 x\ \sec^4 x\, dx$

18. $\int \sin 3x\cos 7x\, dx$

19. $\int \sin^3 x\cos^4 x\, dx$

20. $\int \cos^4 x\, dx$

21. $\int \frac{\tan^3 x}{\sqrt{\sec x}}\, dx$

22. $\int \sec^4 3x\tan^3 3x\, dx$

23. $\int \frac{\sec x}{\tan^2 x}\, dx$

24. $\int \sin 5x\cos 4x\, dx$

25. $\int \sin x\cos^5 x\, dx$

26. $\int \sin^4 x\cos^3 x\, dx$

27. $\int \sin^7 2x\ \cos 2x\, dx$

28. $\int \sin^3 2x\sqrt{\cos 2x}\, dx$

29. $\int \frac{\cos^5 \theta}{\sqrt{\sin\theta}}\, d\theta$

30. $\int \sin^4 6\theta\, d\theta$

31. $\int \cos^2 3x\, dx$

32. $\int x^2\sin^2 x\, dx$

33. $\int \sin^3 3x\, dx$

34. $\int \sin^3 x\cos^2 x\, dx$

35. $\int \cos^3 \frac{x}{3}\, dx$

36. $\int \sec^4 2x\, dx$

37. $\int \sec^4 2x\, dx$

38. $\int \sec^3 \pi x\, dx$

39. $\int \tan^6 3x\, dx$

40. $\int \tan^3 \frac{\pi x}{2}\ \sec^2 \frac{\pi x}{2}\, dx$

41. $\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\, dx$

42. $\int_0^{\pi/4} \tan^4 x dx$

43. $\int_0^{\pi} 8\sin^4 y\cos^2 y\, dy$

44. $\int_0^{\pi/6} 3\cos^5 3x\, dx$

45. $\int_0^{\pi/2} \sin^2 2\theta\cos^3 2\theta\, d\theta$

46. $\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\, dx$

47. $\displaystyle\int_0^{\pi} \sqrt{1-\cos^2\theta}\; d\theta$

48. $\displaystyle\int_0^{\pi/6} \sqrt{1+\sin x}\; dx$

49. $\displaystyle\int_{-\pi}^{\pi} \left(1-\cos^2 x\right)^{3/2} dx$

50. $\displaystyle\int_{\pi/4}^{\pi/2} \csc^4\theta\; d\theta$

51. $\displaystyle\int_{-\pi}^{\pi} \sin 3x \sin 3x\; dx$

52. $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x \cos 7x\; dx$

53. $\displaystyle\int_0^{\pi/2} \cos^{10}\theta\; d\theta$

54. $\displaystyle\int_0^{\pi/2} \cos^7 x\; dx$

55. $\displaystyle\int_0^{\pi/2} \cos^9 x\; dx$

56. $\displaystyle\int_0^{\pi/2} \sin^5 x\; dx$

57. $\displaystyle\int_0^{\pi/2} \sin^6 x\; dx$

58. $\displaystyle\int_0^{\pi/2} \sin^8 x\; dx$

59. Find the area of the region bounded by the graphs of $y = \tan x$ and $y = \sec x$ on the interval $\left[0,\ \frac{\pi}{4}\right]$

Find the area of the region bounded by the graphs of the equations

60. $y = \sin x, \quad y = \sin^3 x, \quad x = 0, \quad x = \frac{\pi}{2}$

61. $y = \sin^2 \pi x, \quad y = 0, \quad x = 0, \quad x = 1$

62. $y = \cos^2 x, \quad y = \sin^2 x, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$

63. $y = \cos^2 x, \quad y = \sin x \cos x, \quad x = -\frac{\pi}{2}, \quad x = \frac{\pi}{4}$

Find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the *x-axis*

64. $y = \tan x, \quad y = 0, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$

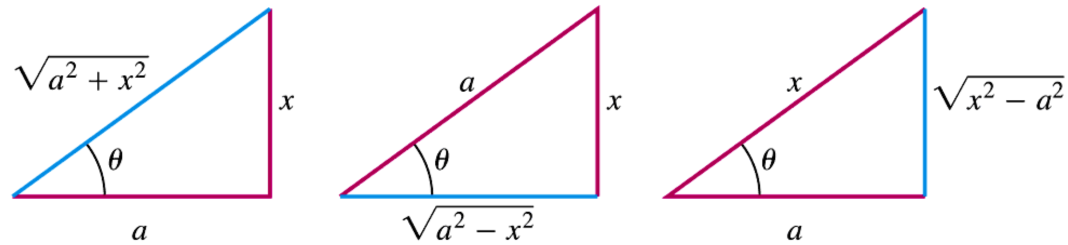
65. $y = \cos \frac{x}{2}, \quad y = \sin \frac{x}{2}, \quad x = 0, \quad x = \frac{\pi}{2}$

Find the ***volume*** of the solid generated by revolving the region bounded by the graphs of the equations about the *x-axis*, then find the ***centroid*** of the region

66. $y = \sin x, \quad y = 0, \quad x = 0, \quad x = \pi$

67. $y = \cos x, \quad y = \sin 0, \quad x = 0, \quad x = \frac{\pi}{2}$

Section 2.3 – Trigonometric Substitutions



$x = a\tan\theta$ $\quad$ $\sqrt{a^2+x^2} = a|\sec\theta|$

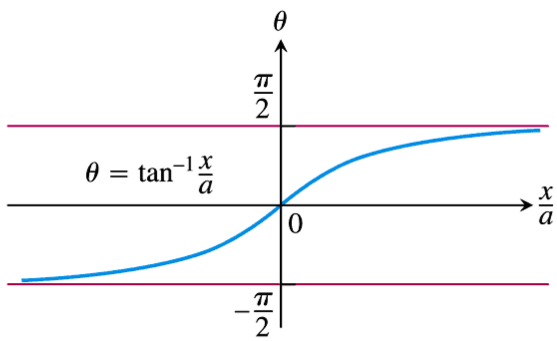
$x = a\sin\theta$ $\quad$ $\sqrt{a^2-x^2} = a|\cos\theta|$

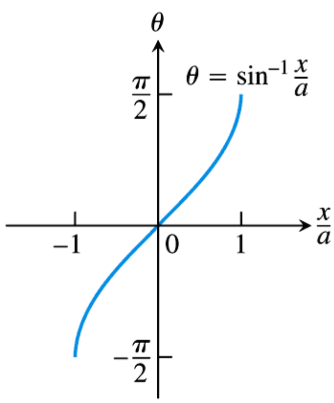
$x = a\sec\theta$ $\quad$ $\sqrt{x^2-a^2} = a|\tan\theta|$

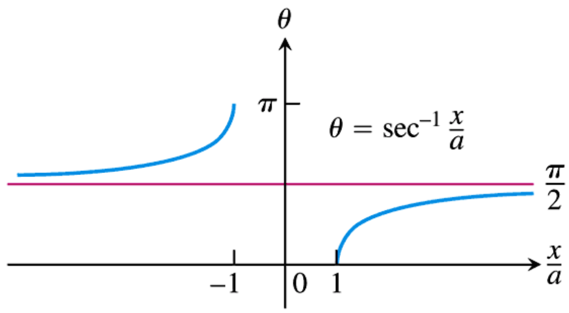
$$x = a\tan\theta \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{x}{a}\right) \quad \textit{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$x = a\sin\theta \quad \Rightarrow \quad \theta = \sin^{-1}\left(\frac{x}{a}\right) \quad \textit{with} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$x = a\sec\theta \quad \Rightarrow \quad \theta = \sec^{-1}\left(\frac{x}{a}\right) \quad \textit{with} \quad \begin{cases} 0 \le \theta < \frac{\pi}{2} & \textit{if } \frac{x}{a} \ge 1 \\ \frac{\pi}{2} < \theta \le \pi & \textit{if } \frac{x}{a} \le -1 \end{cases}$$







Procedure for a Trigonometric Substitution

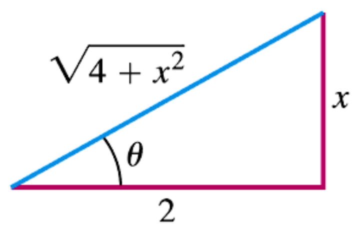
1. Write down the substitution for *x*, calculate the differential *dx*, and specify the selected values of θ for the substitution.
2. Substitute the trigonometric expression and the calculated differential into the integrand, and then simplify the results algebraically.
3. Integrate the trigonometric integral, keeping in mind the restrictions on the angle θ for reversibility.
4. Draw an appropriate reference triangle to reserve the substitution in the integration result and convert it back to the original variable *x*.

Example

Evaluate $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$

Solution

Let: $x = 2\tan\theta \quad \to dx = 2\sec^2\theta\, d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$



$$\begin{aligned} 4+x^2 &= 4+4\tan^2\theta \\ &= 4\left(1+\tan^2\theta\right) \\ &= 4\sec^2\theta \end{aligned}$$

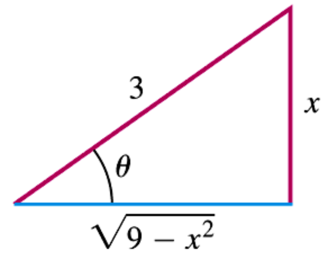
$$\begin{aligned} \int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} \\ &= \int \frac{2\sec^2\theta\, d\theta}{2|\sec\theta|} \\ &= \int \sec\theta\, d\theta \\ &= \ln|\sec\theta + \tan\theta| + C \\ &= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C \end{aligned}$$

Example

Evaluate $\displaystyle\int \frac{x^2dx}{\sqrt{9-x^2}}$

Solution

$x = 3\sin\theta \quad \to dx = 3\cos\theta d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$



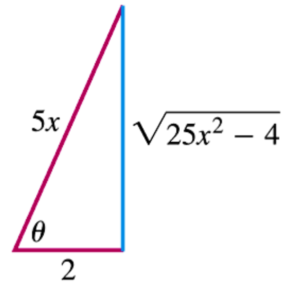
$$\begin{aligned} 9-x^2 &= 9-9\sin^2\theta \\ &= 9\left(1-\sin^2\theta\right) \\ &= 9\cos^2\theta \end{aligned}$$

$$\begin{aligned} \int \frac{x^2dx}{\sqrt{9-x^2}} &= \int \frac{9\sin^2\theta\cdot 3\cos\theta}{|3\cos\theta|}d\theta \\ &= 9\int \sin^2\theta d\theta \\ &= 9\int \frac{1-\cos 2\theta}{2}d\theta \\ &= \frac{9}{2}\left(\theta-\frac{1}{2}\sin 2\theta\right)+C \qquad\qquad \sin 2\theta = 2\sin\theta\cos\theta \\ &= \frac{9}{2}\left(\theta-\frac{1}{2}2\sin\theta\cos\theta\right)+C \qquad\qquad \sin\theta = \frac{x}{3}, \quad \cos^2\theta = \frac{9-x^2}{9} \Rightarrow \cos\theta = \frac{\sqrt{9-x^2}}{3} \\ &= \frac{9}{2}\left(\sin^{-1}\left(\frac{x}{3}\right)-\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right)+C \\ &= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)-\frac{x}{2}\sqrt{9-x^2}+C \end{aligned}$$

Example

Evaluate $\displaystyle\int \frac{dx}{\sqrt{25x^2-4}}, \quad x > \frac{2}{5}$

Solution



$$\sqrt{25x^2-4} = \sqrt{25\left(x^2-\frac{4}{25}\right)}$$

$$= 5\sqrt{x^2-\left(\frac{2}{5}\right)^2}$$

$$x = \frac{2}{5}\sec\theta \quad \to dx = \frac{2}{5}\sec\theta\tan\theta d\theta, \quad 0 < \theta < \frac{\pi}{2}$$

$$x^2-\left(\frac{2}{5}\right)^2 = \frac{4}{25}\sec^2\theta - \frac{4}{25}$$

$$= \frac{4}{25}\left(\sec^2\theta - 1\right)$$

$$= \frac{4}{25}\tan^2\theta$$

$$\int \frac{dx}{\sqrt{25x^2-4}} = \int \frac{dx}{5\sqrt{x^2-\left(\frac{2}{5}\right)^2}}$$

$$= \frac{1}{5}\int \frac{\frac{2}{5}\sec\theta\tan\theta d\theta}{\frac{2}{5}\tan\theta}$$

$$= \frac{1}{5}\int \sec\theta d\theta$$

$$= \frac{1}{5}\ln\left|\sec\theta + \tan\theta\right| + C$$

$$\boxed{= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2-4}}{2}\right| + C}$$

Exercises *Section* 2.3 – Trigonometric Substitutions

Evaluate the integrals

1. $\int \frac{3dx}{\sqrt{1+9x^2}}$
2. $\int \frac{5dx}{\sqrt{25x^2-9}}, \quad x > \frac{3}{5}$
3. $\int \frac{\sqrt{y^2-49}}{y} dy, \quad y > 7$
4. $\int \frac{2dx}{x^3\sqrt{x^2-1}}, \quad x > 1$
5. $\int \frac{x^2}{4+x^2} dx$
6. $\int \frac{dx}{x^2\sqrt{x^2+1}}$
7. $\int \frac{\left(1-x^2\right)^{1/2}}{x^4} dx$
8. $\int \frac{x^3 dx}{x^2-1}$
9. $\int \frac{\sqrt{1-(\ln x)^2}}{x \ln x} dx$
10. $\int \sqrt{x}\,\sqrt{1-x}\, dx$
11. $\int \frac{\sqrt{x-2}}{\sqrt{x-1}} dx$
12. $\int \frac{2dx}{\sqrt{1-4x^2}}$
13. $\int \frac{dx}{\sqrt{4x^2-49}}$
14. $\int \frac{dx}{\sqrt{x^2+4}}$
15. $\int \frac{dx}{\left(16-x^2\right)^{3/2}}$
16. $\int \frac{dx}{\left(1+x^2\right)^2}$
17. $\int \frac{dx}{\sqrt{x^2+4}}$
18. $\int \frac{dx}{x^2\sqrt{9-x^2}}$
19. $\int \frac{dx}{\sqrt{4x^2+1}}$
20. $\int \frac{dx}{\left(x^2+1\right)^{3/2}}$
21. $\int \frac{4}{x^2\sqrt{16-x^2}} dx$
22. $\int \frac{x^3}{\sqrt{9-x^2}} dx$
23. $\int \frac{dx}{\sqrt{x^2-25}}$
24. $\int \frac{\sqrt{x^2-25}}{x} dx$
25. $\int \frac{x^3}{\sqrt{x^2-25}} dx$
26. $\int x^3\sqrt{x^2-25}\, dx$
27. $\int x\sqrt{x^2+1}\, dx$
28. $\int \frac{9x^3}{\sqrt{x^2+1}} dx$
29. $\int \sqrt{16x^2+9}\, dx$
30. $\int \sqrt{25-4x^2}\, dx$
31. $\int \sqrt{5x^2-1}\, dx$
32. $\int \frac{\sqrt{25x^2+4}}{x^4} dx$
33. $\int \frac{1}{x\sqrt{4x^2+9}} dx$
34. $\int \frac{1}{\left(x^2+5\right)^{3/2}} dx$
35. $\int e^x\sqrt{1-e^{2x}}\, dx$
36. $\int \frac{\sqrt{1-x}}{\sqrt{x}} dx$
37. $\int \frac{1}{x^4+4x^2+4} dx$
38. $\int \frac{x^3+x+1}{x^4+2x^2+1} dx$
39. $\int \operatorname{arcsec} 2x\, dx \quad x > \frac{1}{2}$
40. $\int x \arcsin x\, dx$
41. $\int_0^{\sqrt{3}/2} \frac{x^2}{\left(1-x^2\right)^{3/2}} dx$
42. $\int_0^{\sqrt{3}/2} \frac{1}{\left(1-x^2\right)^{5/2}} dx$

43. $\displaystyle\int_0^3 \frac{x^3}{\sqrt{x^2+9}}\,dx$

44. $\displaystyle\int_0^{3/5} \sqrt{9-25x^2}\,dx$

45. $\displaystyle\int_4^6 \frac{x^2}{\sqrt{x^2-9}}\,dx$

46. $\displaystyle\int_{\sqrt{3}}^2 \frac{\sqrt{x^2-3}}{x}\,dx$

47. $\displaystyle\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2}\,dx$

48. $\displaystyle\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}}$

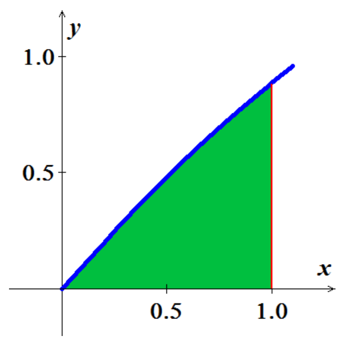
49. $\displaystyle\int_1^e \frac{e^x dx}{\left(1+e^{2x}\right)^{3/2}}$

50. $\displaystyle\int_{1/2}^{1/4} \frac{dy}{y\sqrt{1+(\ln y)^2}}$

51. $\displaystyle\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2-1}}$

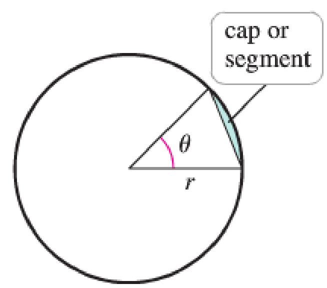
52. $\displaystyle\int_0^2 \sqrt{1+4x^2}\,dx$

53. Consider the region bounded by the graph $y=\sqrt{x\tan^{-1}x}$ and $y=0$ for $0\le x\le 1$. Find the volume of the solid formed by revolving this region about the x-axis.



54. Use two approach to show that the area of a cap (or segment) of a circle of radius r subtended by an angle θ is given by

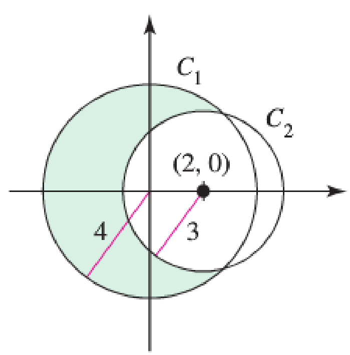
$$A_{seg}=\frac{1}{2}r^2(\theta-\sin\theta)$$



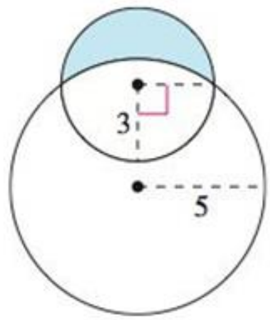
a) Find the area using geometry (no calculus).

b) Find the area using calculus

55. A lune is a crescent-shaped region bounded by the arcs of two circles. Let C_1 be a circle of radius 4 centered at the origin. Let C_2 be a circle of radius 3 centered at the point $(2,\ 0)$. Find the area of the lune that lies inside C_1 and outside C_2.



56. The crescent-shaped region bounded by two circles forms a lune. Find the area of the lune given that the radius of the smaller circle is 3 and the radius of the larger circle is 5.

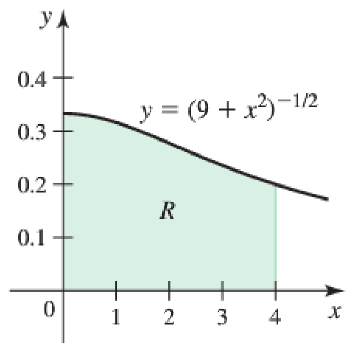


57. Consider the function $f(x) = \left(9 + x^2\right)^{-1/2}$ and the region ***R*** on the interval [0, 4].

a) Find the area of *R*.

b) Find the volume of the solid generated when *R* is revolved about the *x-axis* .

c) Find the volume of the solid generated when *R* is revolved about the *y-axis* .



58. A total of ***Q*** is distributed uniformly on a line segment of length 2*L* along the *y-axis* . The *x*-component of the electric field at a point $(a,\ 0)$ is given by

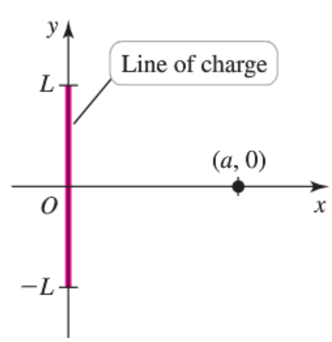
$$E_x = \frac{kQa}{2L}\int_{-L}^{L} \frac{dy}{\left(a^2 + y^2\right)^{3/2}}$$

Where k is a physical constant and $a > 0$

a) Confirm that $E_x(a) = \dfrac{kQ}{a\sqrt{a^2 + L^2}}$

b) Letting $\rho = \dfrac{Q}{2L}$ be the charge density on the line segment,

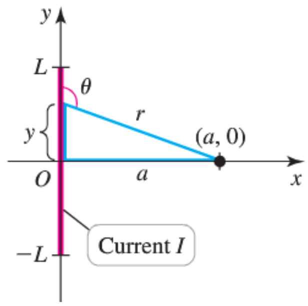
show that if $L \to \infty,$ *then* $E_x = \dfrac{2k\rho}{a}$



59. A long, straight wire of length 2*L* on the *y-axis* carries a current *I*. according to the Biot-Savart Law, the magnitude of the field due to the current at a point $(a,\ 0)$ is given by

$$B(a) = \frac{\mu_0 I}{4\pi}\int_{-L}^{L} \frac{\sin\theta}{r^2}\,dy$$

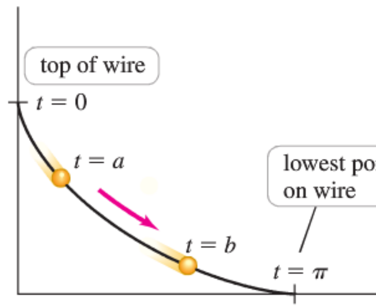
Where μ_0 is a physical constant, $a > 0$, and θ, r, and y are related to the figure



a) Show that the magnitude of the magnetic field at $(a,\ 0)$ is $B(a) = \dfrac{\mu_0 IL}{2\pi a\sqrt{a^2 + L^2}}$

b) What is the magnitude of the magnetic field at $(a,\ 0)$ due to an infinitely long wire $(L \to \infty)$?

60. The cycloid is the curve traced by a point on the rim of a rolling wheel. Imagine a wire shaped like an inverted cycloid.



A bead sliding down this wire without friction has some remarkable properties. Among all wire shapes, the cycloid is the shape that produces the fastest descent time. It can be shown that the descent time between any two points $0 \le a < b \le \pi$ on the curve is

$$descent\ time = \int_a^b \sqrt{\frac{1 - \cos t}{g(\cos a - \cos t)}}\ dt$$

Where g is the acceleration due to gravity, $t = 0$ corresponds to the top of the wire, and $t = \pi$ corresponds to the lowest point on the wire.

a) Find the descent time on the interval $[a,\ b]$.

b) Show that when $b = \pi$, the descent time is the same for all values of a; that is, the descent time to the bottom of the wire is the same for all starting points.

61. Find the area of the region bounded by the curve $f(x) = \left(16 + x^2\right)^{-3/2}$ and the x-*axis* on the interval $[0,\ 3]$

62. Find the length of the curve $y = ax^2$ from $x = 0$ to $x = 10$, where $a > 0$ is a real number.

63. Find the arc length of the graph of $f(x) = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$

64. A projectile is launched from the ground with an initial speed V at an angle θ from the horizontal. Assume that the *x-axis* is the horizontal ground and y is the height above the ground. Neglecting air resistance and letting g be the acceleration due to gravity, it can be shown that the trajectory of the projectile is given by

$$y = -\frac{1}{2}kx^2 + y_{max} \quad \text{where } k = \frac{g}{(V\cos\theta)^2}$$

$$\text{and} \qquad y_{max} = \frac{(V\sin\theta)^2}{2g}$$

a) Note that the high point of the trajectory occurs at $(0,\ y_{max})$. If the projectile is on the ground at $(-a,\ 0)$ and $(a,\ 0)$, what is a?

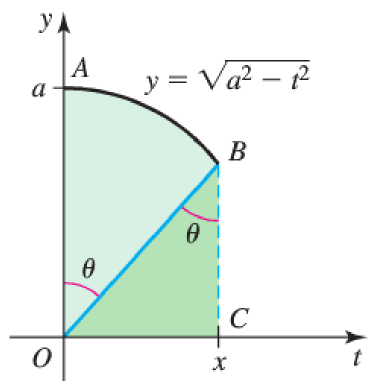
b) Show that the length of the trajectory (arc length) is $2\int_0^a \sqrt{1+k^2x^2}\ dx$

c) Evaluate the arc length integral and express your result in the terms of V, g, and θ.

d) For fixed value of V and g, show that the launch angle θ that maximizes the length of the trajectory satisfies $(\sin\theta)\ln(\sec\theta + \tan\theta) = 1$

65. Let $F(x) = \int_0^x \sqrt{a^2 - t^2}\,dt$. The figure shows that

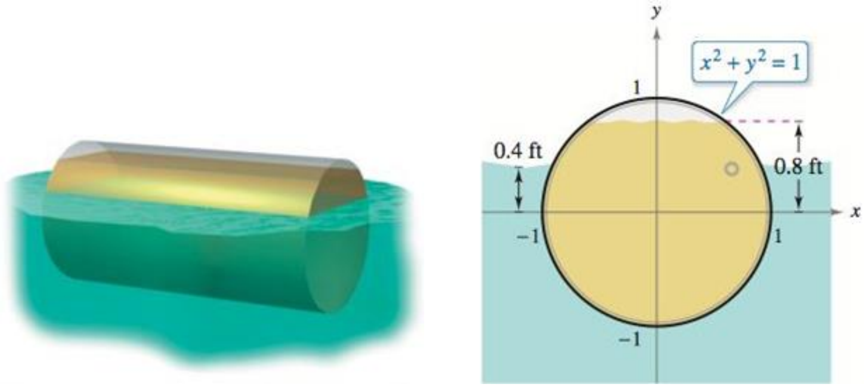
$F(x) = \textit{area of sector OAB} + \textit{area of triangle OBC}$



a) Use the figure to prove that $F(x) = \dfrac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2} + \dfrac{x\sqrt{a^2 - x^2}}{2}$

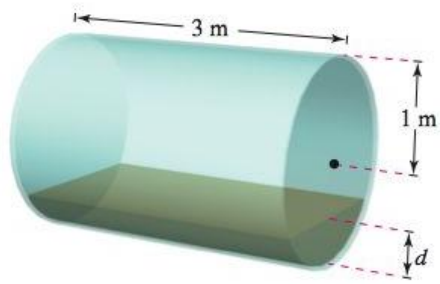
b) Conclude that $\int \sqrt{a^2 - x^2}\,dx = \dfrac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2} + \dfrac{x\sqrt{a^2 - x^2}}{2} + C$

66. A sealed barrel of oil (weighing *48 pounds per cubic foot) is floating in seawater (weighing 64 pounds per cubic foot). The barrel is not completely full of oil. With the barrel lying on its side, the top 0.2 *foot* of the barrel is empty.



Compare the fluid forces against one end of the barrel from the inside and from the outside.

67. The axis of a storage tank in the form of a right circular cylinder is horizontal. The radius and length of the tank are 1 *meter* and 3 *meters*, respectively.



a) Determine the volume of fluid in the tank as a function of its depth *d*.

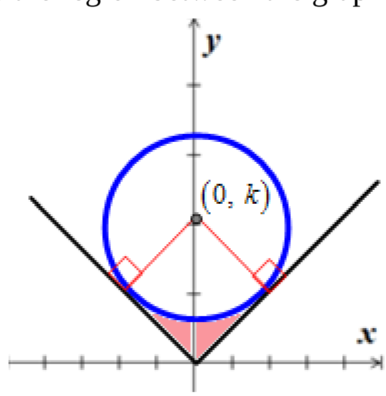
b) Graph the function in part (*a*).

c) Design a dip stick for the tank with markings of $\frac{1}{4}$, $\frac{1}{2}$, *and* $\frac{3}{4}$

d) Fluid is entering the tank at a rate of $\frac{1}{4}$ m^3/min . Determine the rate of change of the depth of the fluid as a function of its depth *d*.

e) Graph the function in part (*d*).\When will the rate of change of the depth be minimum?

68. The surface of a machine part is the region between the graphs of $y = |x|$ *and* $x^2 + (y - k)^2 = 25$



a) Find k when the circle is tangent to the graph of $y = |x|$

b) Find the area of the surface of the machine part.

c) Find the area of the surface of the machine part as a function of the radius *r* of the circle.

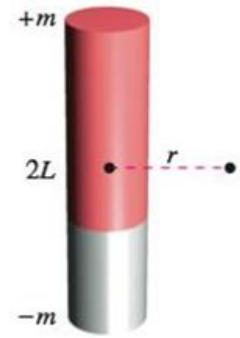
69. The field strength H of a magnet of length $2L$ on a particle r units from the center of the magnet is

$$H = \frac{2mL}{\left(r^2 + L^2\right)^{3/2}}$$

Where $\pm m$ are the poles of the magnet.

Find the average field strength as the particle moves from 0 to R units from the center by evaluating the integral

$$\frac{1}{R}\int_0^R \frac{2mL}{\left(r^2+L^2\right)^{3/2}}\,dr$$



Section 2.4 – Partial Fractions

This section shows how to express a rational; function as a sum of simpler functions, called ***partial fractions***.

Example

Evaluate $\int \frac{5x-3}{x^2-2x-3}dx$

Solution

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}$$

$$5x-3 = (A+B)x - 3A + B \quad \to \begin{cases} A+B=5 \\ -3A+B=-3 \end{cases} \to \boxed{A=2, \quad B=3}$$

$$\int \frac{5x-3}{x^2-2x-3}dx = \int \left(\frac{2}{x+1} + \frac{3}{x-3}\right)dx$$

$$= 2\ln|x+1| + 3\ln|x-3| + C$$

Example

Use partial fractions to evaluate $\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}dx$

Solution

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

$$x^2+4x+1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)$$

$$= Ax^2 + 4Ax + 3A + Bx^2 + 2Bx - 3B + Cx^2 - C$$

$$= (A+B+C)x^2 + (4A+2B)x + (3A-3B-C)$$

$$\to \begin{cases} A+B+C=1 \\ 4A+2B=4 \\ 3A-3B-C=1 \end{cases} \xrightarrow{rref} \boxed{A=\frac{3}{4}, \quad B=\frac{1}{2}, \quad C=-\frac{1}{4}}$$

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}dx = \int \left[\frac{3}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{4}\frac{1}{x+3}\right]dx$$

$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + K$$

Method of Partial Fractions $\left(f(x)/g(x) \right.$ ***Proper*** $\left.\right)$

1. Let $(x-r)$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $(x-r)$ that divides $g(x)$. Then,

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}$$

2. Let $x^2 + px + q$ be an irreducible quadratic function of $g(x)$ has no real roots. Suppose that $\left(x^2 + px + q\right)^n$ is the highest power. Then

$$\frac{B_1 x + C_1}{\left(x^2 + px + q\right)} + \frac{B_2 x + C_2}{\left(x^2 + px + q\right)^2} + \cdots + \frac{B_n x + C_n}{\left(x^2 + px + q\right)^n}$$

3. Set the original fraction $\dfrac{f(x)}{g(x)}$ equal to the sum of these partial fractions.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Example

Use partial fractions to evaluate $\displaystyle\int \frac{6x+7}{(x+2)^2}\,dx$

Solution

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$\begin{aligned} 6x+7 &= A(x+2) + B \\ &= Ax + 2A + B \end{aligned}$$

$$\to \begin{cases} \boxed{A=6} \\ 2A + B = 7 & \to \underline{B = 7 - 12 = -5} \end{cases}$$

$$\begin{aligned} \int \frac{6x+7}{(x+2)^2}\,dx &= \int \left(\frac{6}{x+2} - \frac{5}{(x+2)^2} \right) dx \qquad\qquad d(x+2) = dx \\ &= \int \frac{6}{x+2}\,dx - 5 \int (x+2)^{-2}\, d(x+2) \\ &= 6 \ln|x+2| + 5(x+2)^{-1} + C \end{aligned}$$

Example

Use partial fractions to evaluate $\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}dx$

Solution

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3} = 2x + \frac{5x-3}{x^2-2x-3}$$

$$\begin{array}{r} 2x \\ x^2-2x-3\overline{\smash{)}2x^3-4x^2-x-3} \\ \underline{2x^3-4x^2-6x} \\ 5x-3 \end{array}$$

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}$$

$$5x-3 = (A+B)x - 3A + B \quad \to \begin{cases} A+B=5 \\ -3A+B=-3 \end{cases} \to \boxed{A=2, \quad B=3}$$

$$\begin{aligned} \int \frac{2x^3-4x^2-x-3}{x^2-2x-3}dx &= \int 2x\,dx + \int \frac{5x-3}{x^2-2x-3}dx \\ &= \int 2x\,dx + \int \frac{2}{x+1}dx + \int \frac{3}{x-3}dx \\ &= x^2 + 2\ln|x+1| + 3\ln|x-3| + C \end{aligned}$$

Example

Use partial fractions to evaluate $\int \frac{-2x+4}{\left(x^2+1\right)(x-1)^2}dx$

Solution

$$\frac{-2x+4}{\left(x^2+1\right)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$\begin{aligned} -2x+4 &= (Ax+B)(x-1)^2 + C(x-1)\left(x^2+1\right) + D\left(x^2+1\right) \\ &= (Ax+B)\left(x^2-2x+1\right) + C\left(x^3-x^2+x-1\right) + Dx^2 + D \\ &= (A+C)x^3 + (-2A+B-C+D)x^2 + (A-2B+C)x + B-C+D \end{aligned}$$

$$\to \begin{cases} A+C=0 \\ -2A+B-C+D=0 \\ A-2B+C=-2 \\ B-C+D=4 \end{cases} \Rightarrow \boxed{A=2, \quad B=1, \quad C=-2, \quad D=1}$$

$$\int \frac{-2x+4}{\left(x^2+1\right)(x-1)^2}dx = \int \left(\frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right)dx$$

$$= \int \left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2} \right) dx$$

$$\boxed{= \ln\left(x^2+1\right) + \tan^{-1} x - 2\ln|x-1| - \frac{1}{x-1} + K}$$

Example

Use partial fractions to evaluate $\displaystyle\int \frac{dx}{x\left(x^2+1\right)^2}$

Solution

$$\frac{1}{x\left(x^2+1\right)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{\left(x^2+1\right)^2}$$

$$1 = A\left(x^2+1\right)^2 + (Bx+C)x\left(x^2+1\right) + x(Dx+E)$$

$$1 = (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

$$\begin{cases} A+B=0 \\ C=0 \\ 2A+B+D=0 \\ C+E=0 \\ A=1 \end{cases} \Rightarrow \boxed{A=1,\ B=-1,\ C=0,\ D=-1,\ E=0}$$

$$\frac{1}{x\left(x^2+1\right)^2} = \frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{\left(x^2+1\right)^2}$$

$$\int \frac{dx}{x\left(x^2+1\right)^2} = \int \frac{dx}{x} - \int \frac{xdx}{x^2+1} - \int \frac{xdx}{\left(x^2+1\right)^2} \qquad u = x^2+1 \Rightarrow du = 2xdx \to \tfrac{1}{2}du = xdx$$

$$= \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2}$$

$$= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2}\frac{1}{u} + K$$

$$= \ln|x| - \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{2}\frac{1}{x^2+1} + K$$

$$= \ln|x| - \ln\sqrt{x^2+1} + \frac{1}{2}\frac{1}{x^2+1} + K$$

$$\boxed{= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2\left(x^2+1\right)} + K}$$

Exercises *Section* 2.4 – Partial Fractions

Express the integrand as a sum of partial fractions and evaluate the integrals

1. $\int \frac{dx}{x^2+2x}$

2. $\int \frac{2x+1}{x^2-7x+12}dx$

3. $\int \frac{x+3}{2x^3-8x}dx$

4. $\int \frac{x^2}{(x-1)\left(x^2+2x+1\right)}dx$

5. $\int \frac{8x^2+8x+2}{\left(4x^2+1\right)^2}dx$

6. $\int \frac{x^2+x}{x^4-3x^2-4}dx$

7. $\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{\left(\theta^2+1\right)^3}d\theta$

8. $\int \frac{x^4}{x^2-1}dx$

9. $\int \frac{16x^3}{4x^2-4x+1}dx$

10. $\int \frac{e^{4x}+2e^{2x}-e^x}{e^{2x}+1}dx$

11. $\int \frac{\sin\theta\, d\theta}{\cos^2\theta+\cos\theta-2}$

12. $\int \frac{(x-2)^2\tan^{-1}(2x)-12x^3-3x}{\left(4x^2+1\right)(x-2)^2}\, dx$

13. $\int \frac{\sqrt{x+1}}{x}dx$

14. $\int \frac{x^3-2x^2+3x-4}{x^2+1}dx$

15. $\int \frac{4x^2+2x+4}{x+1}dx$

16. $\int \frac{3x^2+7x-2}{x^3-x^2-2x}dx$

17. $\int \frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)}dx$

18. $\int \frac{5x^2-3x+2}{x^3-2x^2}dx$

19. $\int \frac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}dx$

20. $\int \frac{1}{x^2-5x+6}\, dx$

21. $\int \frac{1}{x^2-5x+5}\, dx$

22. $\int \frac{5x^2+20x+6}{x^3+2x^2+x}\, dx$

23. $\int \frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}\, dx$

24. $\int \frac{8x^3+13x}{\left(x^2+2\right)^2}\, dx$

25. $\int \frac{\sin x}{\cos x+\cos^2 x}\, dx$

26. $\int \frac{5\cos x}{\sin^2 x+3\sin x-4}\, dx$

27. $\int \frac{e^x}{\left(e^x-1\right)\left(e^x+4\right)}\, dx$

28. $\int \frac{e^x}{\left(e^{2x}+1\right)\left(e^x-1\right)}\, dx$

29. $\displaystyle\int \frac{\sqrt{x}}{x-4}\,dx$

30. $\displaystyle\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$

31. $\displaystyle\int \frac{dx}{1+\sin x}$

32. $\displaystyle\int \frac{dx}{2+\cos x}$

33. $\displaystyle\int \frac{dx}{1-\cos x}$

34. $\displaystyle\int \frac{dx}{1+\sin x+\cos x}$

35. $\displaystyle\int \frac{1}{x^2-9}\,dx$

36. $\displaystyle\int \frac{5}{x^2+3x-4}\,dx$

37. $\displaystyle\int \frac{2}{9x^2-1}\,dx$

38. $\displaystyle\int \frac{3-x}{3x^2-2x-1}\,dx$

39. $\displaystyle\int \frac{x^2+12x+12}{x^3-4x}\,dx$

40. $\displaystyle\int \frac{x^3-x+3}{x^2+x-2}\,dx$

41. $\displaystyle\int \frac{5x-2}{(x-2)^2}\,dx$

42. $\displaystyle\int \frac{2x^3-4x^2-15x+4}{x^2-2x-8}\,dx$

43. $\displaystyle\int \frac{x+2}{x^2+5x}\,dx$

44. $\displaystyle\int \frac{\sec^2 x}{\tan^2 x+5\tan x+6}\,dx$

45. $\displaystyle\int \frac{\sec^2 x}{\tan x(\tan x+1)}\,dx$

46. $\displaystyle\int_0^2 \frac{3}{4x^2+5x+1}\,dx$

47. $\displaystyle\int_1^5 \frac{x-1}{x^2(x+1)}\,dx$

48. $\displaystyle\int_0^1 \frac{x^2-x}{x^2+x+1}\,dx$

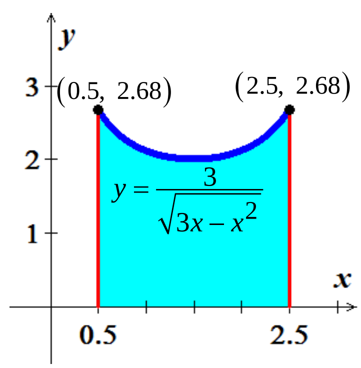
49. $\displaystyle\int_4^8 \frac{y\,dy}{y^2-2y-3}$

50. $\displaystyle\int_1^{\sqrt{3}} \frac{3x^2+x+4}{x^3+x}\,dx$

51. $\displaystyle\int_0^{\pi/2} \frac{dx}{\sin x+\cos x}$

52. $\displaystyle\int_0^{\pi/3} \frac{\sin\theta}{1-\sin\theta}\,d\theta$

53. Find the volume of the solid generated by the revolving the shaded region about *x*-axis



Find the area of the region bounded by the graphs of

54. $y=\dfrac{12}{x^2+5x+6},\quad y=0,\quad x=0,\quad and\quad x=1$

55. $y=\dfrac{7}{16-x^2}\quad and\quad y=1$

56. Consider the region bounded by the graphs $y=\dfrac{2x}{x^2+1},\quad y=0,\quad x=0,\quad and\quad x=3.$

a) Find the volume of the solid generated by revolving the region about the *x-axis*

b) Find the centroid of the region.

57. Consider the region bounded by the graph $y^2 = \dfrac{(2-x)^2}{(1+x)^2} \quad 0 \le x \le 1$.

Find the volume of the solid generated by revolving this region about the *x-axis*.

58. A single infected individual enters a community of *n* susceptible individuals. Let *x* be the number of newly infected individuals at time *t*. The common epidemic model assumes that the disease spreads at a rate proportional to the product of the total number infected and the number not yet infected. So,

$\dfrac{dx}{dt} = k(x+1)(n-x)$ and you obtain

$$\int \frac{1}{(x+1)(n-x)}\, dx = \int k\, dt$$

Solve for *x* as a function of *t*.

59. Evaluate $\displaystyle\int_0^1 \frac{x}{1+x^4}\, dx$ in ***two*** different ways.

Section 2.5 – Numerical Integration

Absolute and Relative Error

Definition

Suppose c is a computed numerical solution to a problem having an exact solution x.
There are two common measures of the error in c as an approximation to x:

$$\textbf{\textit{absolute error}} = |c-x| \qquad \& \qquad \textbf{\textit{relative error}} = \frac{|c-x|}{|x|} \quad (\text{if } x \neq 0)$$

Example

The ancient Greeks used $\frac{22}{7}$ to approximate the value of π. Determine the absolute and relative error in this approximation to π.

Solution

$$\text{absolute error} = \left|\frac{22}{7} - \pi\right| \approx 0.00126$$

$$\text{relative error} = \frac{\left|\frac{22}{7} - \pi\right|}{\pi} \approx .000402 \qquad \approx .04\%$$

Midpoint Rule

Definition

Suppose f is defined and integrable on $[a, b]$. The ***midpoint Rule Approximation*** to $\int_a^b f(x)dx$ using n equally spaced subintervals on $[a, b]$ is

$$M(n) = f(m_1)\Delta x + f(m_2)\Delta x + \cdots + f(m_n)\Delta x$$

$$= \sum_{k=1}^{n} f\left(\frac{x_{k-1} + x_k}{2}\right)\Delta x$$

Where $\Delta x = \dfrac{b-a}{n}$,

$x_0 = a, \; x_k = a + k\Delta x$

$m_k = \dfrac{x_{k-1} + x_k}{2}$ is the midpoint of $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$.

Example

Approximate $\int_2^4 x^2 dx$ using the Midpoint Rule with $n = 4$ subinterval

Solution

With $a = 2,\ b = 4 \ \rightarrow\ \Delta x = \frac{4-2}{4} = 0.5$

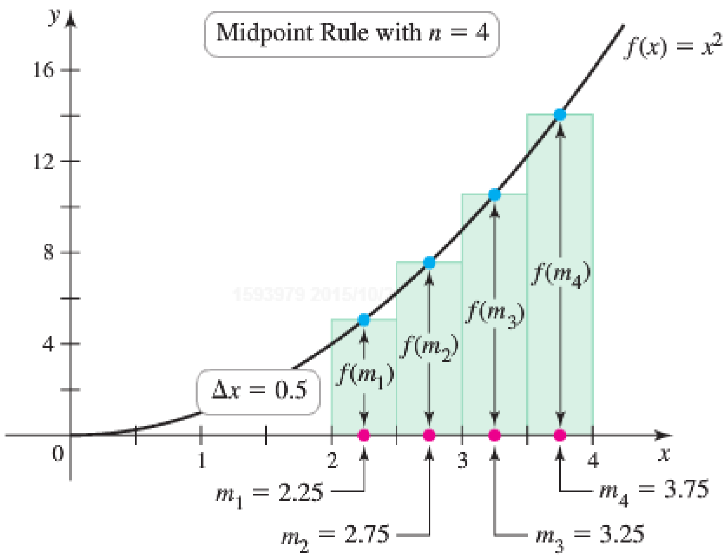
The grid points are: $x_0 = 2,\ x_1 = 2 + 0.5 = 2.5,\ x_2 = 3,\ x_3 = 3.5,\ x_4 = 4$

$m_1 = \frac{2.5+2}{2} = 2.25,\ m_2 = 2.75,\ m_3 = 3.25,\ m_4 = 3.75$

$$M(4) = f(m_1)\Delta x + f(m_2)\Delta x + f(m_3)\Delta x + f(m_4)\Delta x$$

$$= \left(2.25^2 + 2.75^2 + 3.25^2 + 3.75^2\right)(0.5)$$

$$= 18.625$$



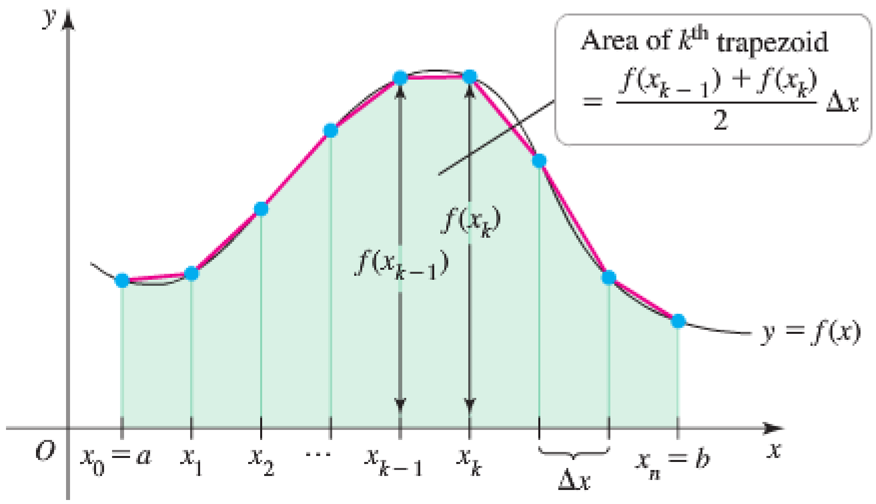
$$Exact = \int_2^4 x^2 dx = \frac{1}{3}x^3 \Big|_2^4 = \frac{56}{3}$$

$$absolute\ error = \left|18.625 - \frac{56}{3}\right| \approx 0.0417$$

$$relative\ error = \frac{\left|18.625 - \frac{56}{3}\right|}{\frac{56}{3}} \approx .00223 = .223\% \qquad \approx .04\%$$

Trapezoid Approximations

The ***Trapezoid Rule*** for the value of a definite integral is based on approximating the region between a curve and the x-axis with trapezoids instead of rectangles.



The length of each subinterval is $\Delta x = \dfrac{b-a}{n}$ is called the ***step size*** or ***mesh size***.

The area of a trapezoid: $\Delta x \cdot \left(\dfrac{y_{i-1} + y_i}{2} \right)$

The area is the approximation by adding the areas of all trapezoids:

$$\begin{aligned} T &= \tfrac{1}{2}\left(y_0 + y_1\right)\Delta x + \tfrac{1}{2}\left(y_1 + y_2\right)\Delta x + \cdots + \tfrac{1}{2}\left(y_{n-2} + y_{n-1}\right)\Delta x + \tfrac{1}{2}\left(y_{n-1} + y_n\right)\Delta x \\ &= \tfrac{1}{2}\Delta x\left(y_0 + y_1 + y_1 + y_2 + \cdots + y_{n-2} + y_{n-1} + y_{n-1} + y_n\right) \\ &= \tfrac{1}{2}\Delta x\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-2} + 2y_{n-1} + y_n\right) \end{aligned}$$

The Trapezoid Rule

If f is continuous on $[a, b]$ and if a regular partition of $[a, b]$ is determined by the numbers $a = x_0,\ x_1,\ \ldots,\ x_n = b$, then

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}\left[f\left(x_0\right) + 2f\left(x_1\right) + 2f\left(x_2\right) + \ldots + 2f\left(x_{n-1}\right) + f\left(x_n\right)\right]$$

$$T(n) = \left(\frac{1}{2} f\left(x_0\right) + \sum_{k=1}^{n-1} f\left(x_k\right) + \frac{1}{2} f\left(x_n\right) \right) \Delta x$$

$$x_0 = a,\ \ x_1 = a + \Delta x,\ \ x_2 = a + 2\Delta x,\ \ \ldots,\ \ x_{n-1} = a + (n-1)\Delta x,\ \ x_n = b$$

Where $\Delta x = \dfrac{b-a}{n}$ and $x_0 = a,\ x_k = a + k\Delta x$

Error Estimate for the Trapezoidal Rule

If M is a positive real number such that $\left|f''(x)\right| \le M$ for all x in $[a, b]$, then the error involved in using the Trapezoidal Rule is not greater than $\dfrac{M(b-a)^3}{12n^2}$

Example

Use the Trapezoid Rule with $n = 4$ to estimate $\displaystyle\int_1^2 x^2 dx$. Compare the estimate with the exact value.

Solution

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$$

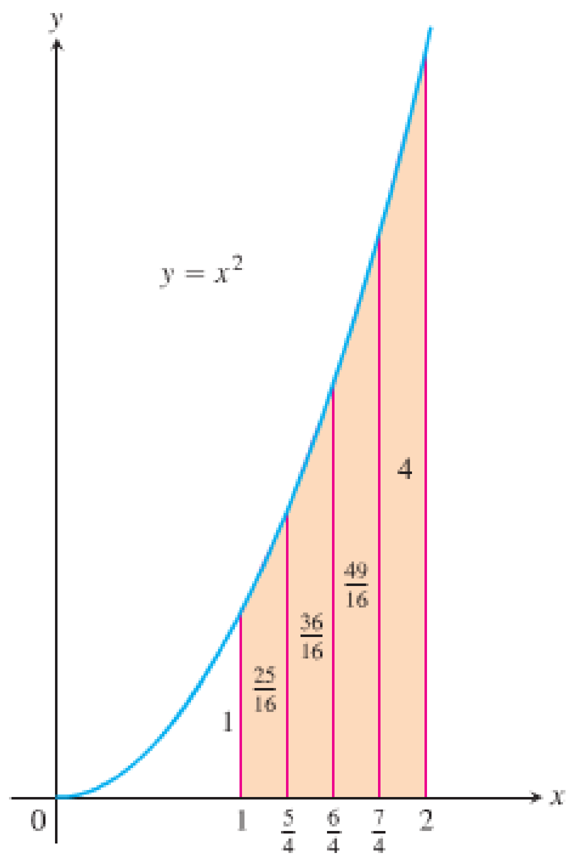
$$x_0 = 1$$

$$x_1 = 1 + \frac{1}{4} = \frac{5}{4}$$

$$x_2 = 1 + 2\left(\frac{1}{4}\right) = \frac{6}{4}$$

$$x_3 = 1 + 3\left(\frac{1}{4}\right) = \frac{7}{4}$$

$$x_4 = 2$$



$$T = \frac{1}{2}\Delta x\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right)$$

$$= \frac{1}{2}\cdot\frac{1}{4}\left(1^2 + 2\left(\frac{5}{4}\right)^2 + 2\left(\frac{6}{4}\right)^2 + 2\left(\frac{7}{4}\right)^2 + 2^2\right)$$

$$= \frac{1}{8}\left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right)$$

$$= \frac{75}{32}$$

$$\approx 2.34375$$

$$\int_1^2 x^2 dx = \frac{1}{3}x^3 \Big|_1^2$$

$$= \frac{1}{3}\left(2^3 - 1^3\right)$$

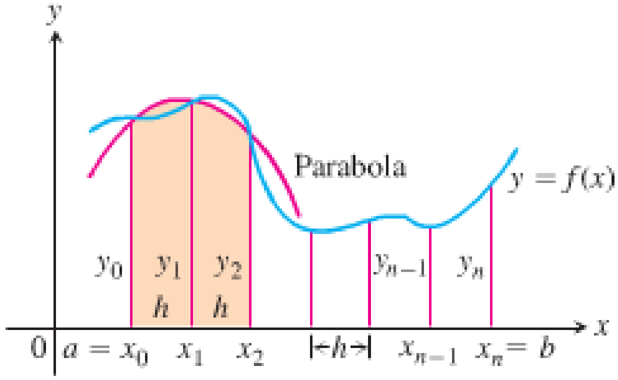
$$= \frac{7}{3} \approx 2.3333$$

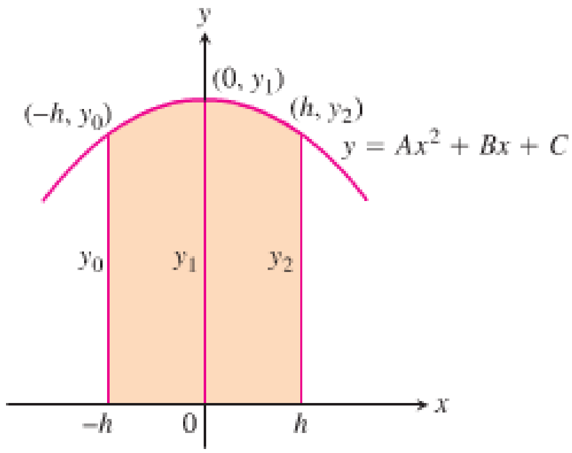
The difference: $2.34375 - 2.3333 \approx 0.01042$

The percentage error: $\dfrac{2.34375 - 2.33333}{2.33333} \approx 0.004466$ $\quad .446\%$

Simpson's Rule: Approximations Using Parabolas

We partition the interval [a, b] into n subintervals of equal length $h = \Delta x = \dfrac{b-a}{n}$ $\quad n$: *even number*





The parabola has an equation of the form: $y = Ax^2 + Bx + C$

So the area under it from $x = -h$ to $x = h$ is

$$\begin{aligned}
A_p &= \int_{-h}^{h} \left(Ax^2 + Bx + C\right) dx \\
&= \frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx \Big]_{-h}^{h} \\
&= \frac{A}{3}h^3 + \frac{B}{2}h^2 + Ch - \left(\frac{A}{3}(-h)^3 + \frac{B}{2}(-h)^2 + C(-h)\right) \\
&= \frac{A}{3}h^3 + \frac{B}{2}h^2 + Ch + \frac{A}{3}h^3 - \frac{B}{2}h^2 + Ch \\
&= \frac{2}{3}Ah^3 + 2Ch \\
&= \frac{h}{3}\left(2Ah^2 + 6C\right)
\end{aligned}$$

Since the curve passes through the three points $(-h, y_0)$, $(0, y_1)$, *and* (h, y_2)

$$y_0 = Ah^2 - Bh + C \quad y_1 = C \quad y_2 = Ah^2 + Bh + C$$

$$C = y_1, \qquad Ah^2 - Bh = y_0 - y_1$$

$$\frac{\quad Ah^2 + Bh = y_2 - y_1 \quad}{2Ah^2 = y_0 - 2y_1 + y_2}$$

$$\begin{aligned}
A_p &= \frac{h}{3}\left(2Ah^2 + 6C\right) \\
&= \frac{h}{3}\left(y_0 - 2y_1 + y_2 + 6y_1\right) \\
&= \frac{h}{3}\left(y_0 + 4y_1 + y_2\right)
\end{aligned}$$

Computing the areas under all the parabolas and adding the results gives the approximation

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left(y_0+4y_1+y_2\right)+\frac{h}{3}\left(y_2+4y_3+y_4\right)+\cdots+\frac{h}{3}\left(y_{n-2}+4y_{n-1}+y_n\right)$$

$$=\frac{h}{3}\left(y_0+4y_1+2y_2+4y_3+2y_4+\cdots+2y_{n-2}+4y_{n-1}+y_n\right)$$

Simpson's Rule

To approximate $\int_a^b f(x)\,dx$, use $S=\frac{\Delta x}{3}\left(y_0+4y_1+2y_2+4y_3+2y_4+\cdots+2y_{n-2}+4y_{n-1}+y_n\right)$

$$x_0=a,\ \ x_1=a+\Delta x,\ \ x_2=a+2\Delta x,\ \ \ldots,\ \ x_{n-1}=a+(n-1)\Delta x,\ \ x_n=b$$

Where $\Delta x=\frac{b-a}{n}$

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}\left[f\left(x_0\right)+4f\left(x_1\right)+2f\left(x_2\right)+4f\left(x_3\right)+\ldots+2f\left(x_{n-2}\right)+4f\left(x_{n-1}\right)+f\left(x_n\right)\right]$$

Error Estimate for the Trapezoidal Rule

If M is a positive real number such that $\left|f^{(4)}(x)\right|\le M$ for all x in $[a, b]$, then the error involved in using the Simpson's Rule is not greater than $\dfrac{M(b-a)^5}{180n^4}$

Example

Use Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4\,dx$

Solution

$$\Delta x=\frac{b-a}{n}=\frac{2-0}{4}=\boxed{\frac{1}{2}}$$

$$x_0=0,\quad x_1=\frac{1}{2},\quad x_2=0+2\frac{1}{2}=1,\quad x_3=0+3\frac{1}{2}=\frac{3}{2},\quad x_4=2$$

$$S=\frac{\Delta x}{3}\left(y_0+4y_1+2y_2+4y_3+y_4\right)$$

$$=\frac{1}{3}\frac{1}{2}\left(5(0)^4+4(5)\left(\tfrac{1}{2}\right)^4+2(5)(1)^4+4(5)\left(\tfrac{3}{2}\right)^4+5(2)^4\right)$$

$$= \frac{1}{6}\left(0 + \frac{5}{4} + 10 + \frac{405}{4} + 80\right)$$

$$= \frac{1}{6}\left(\frac{385}{2}\right)$$

$$= \frac{385}{12}$$

≈ 32.08333 *The exact value is* 32.

Example

The table lists rates of change $s'(t)$ in global sea level $s(t)$ in various years from 1995 $(t = 0)$ to 2011 $(t = 16)$, with rates of change reported in *mm/yr*.

Years	1995	1998	2000	2002	2003	2007	2009	2011
t	0	3	5	7	8	12	14	16
$s'(t)$ *(mm/yr)*	0.51	5.19	4.39	2.21	5.24	0.63	4.19	2.38

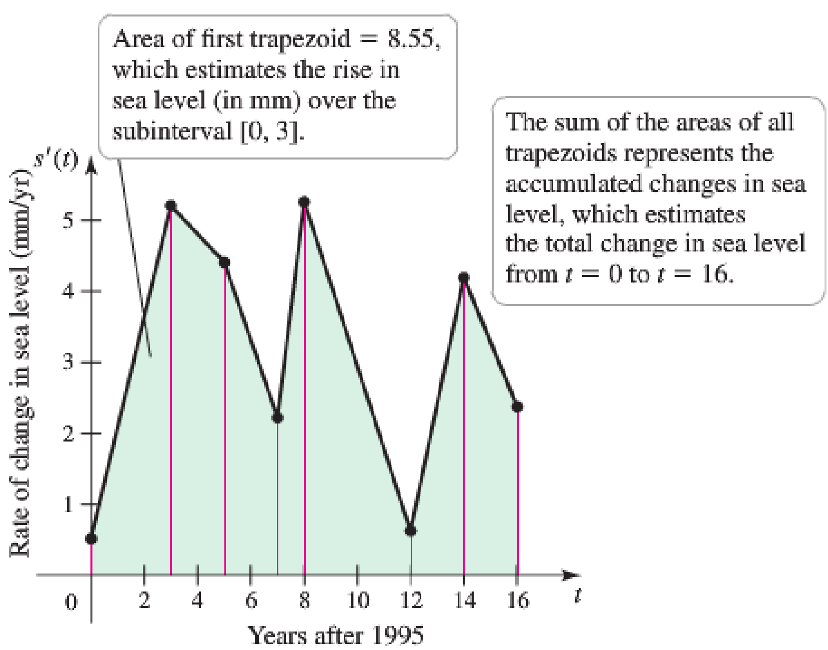
a) Assuming $s'(t)$ is continuous on $[0,\ 16]$, explain how a definite integral can be used to find the net change in sea level from 1995 to 2011; then write the definite integral.

b) Use the data in the table and generalize the trapezoid Rule to estimate the value of the integral from part (*a*).

Solution

a) The net charge in any quantity Q over the interval $[a,\ b]$ is $Q(b) - Q(a)$

Net change in $s(t)$ $= S(b) - S(a) = \int_0^{16} s'(t)\,dt$



b) From the figure the values accompanied by 7 trapezoids whose area approximates $\int_0^{16} s'(t)\,dt$

Area of the ***first*** trapezoid: $T_1 = \frac{1}{2}\left(s'(0)+s'(3)\right)\cdot 3 = \frac{1}{2}(0.51+5.19)\cdot 3 \underline{= 8.55}$

$$T_2 = \frac{1}{2}\left(s'(3)+s'(5)\right)\cdot 2 = \frac{1}{2}(5.19+4.39)\cdot 2 \underline{= 9.58}$$

$$T_3 = \frac{1}{2}\left(s'(5)+s'(7)\right)\cdot 2 = \frac{1}{2}(4.39+2.21)\cdot 2 \underline{= 6.6}$$

$$T_4 = \frac{1}{2}\left(s'(7)+s'(8)\right)\cdot 1 = \frac{1}{2}(2.21+5.24) \underline{= 3.725}$$

$$T_5 = \frac{1}{2}\left(s'(8)+s'(12)\right)\cdot 4 = \frac{1}{2}(5.24+0.63)\cdot 4 \underline{= 11.74}$$

$$T_6 = \frac{1}{2}\left(s'(12)+s'(14)\right)\cdot 2 = \frac{1}{2}(0.63+4.19)\cdot 2 \underline{= 4.82}$$

$$T_7 = \frac{1}{2}\left(s'(14)+s'(16)\right)\cdot 2 = \frac{1}{2}(4.19+2.38)\cdot 2 \underline{= 6.57}$$

$$T(7) = T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 \quad \underline{\approx 51.585\ \ mm}$$

Exercises Section 2.5 – Numerical Integration

Find the *Midpoint* Rule approximations to

1. $\int_0^1 \sin \pi x\, dx \quad n = 6 \ \textit{subintervals}$

2. $\int_0^1 e^{-x} dx \quad n = 8 \ \textit{subintervals}$

Estimate the minimum number of subintervals to approximate the integrals with an error of magnitude of 10^{-4} by (***a***) the *Trapezoid* Rule and (***b***) *Simpson's* Rule.

3. $\int_1^3 (2x-1)\,dx$

4. $\int_{-1}^1 \left(x^2+1\right)dx$

5. $\int_2^4 \frac{1}{(s-1)^2}\,ds$

Find the *Trapezoid* & *Simpson's* Rule approximations and error to

6. $\int_0^1 \sin \pi x\, dx \quad n = 6 \ \textit{subintervals}$

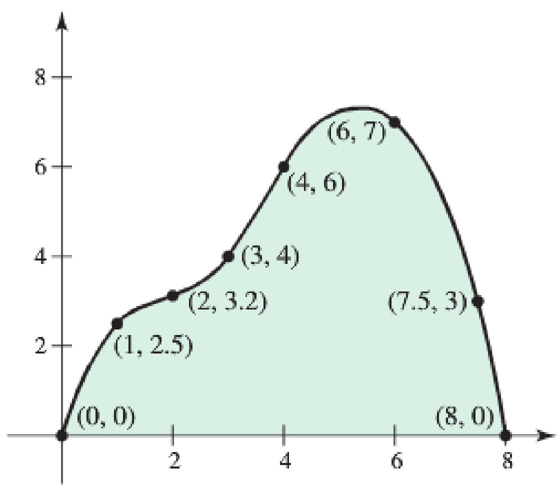
7. $\int_0^1 e^{-x} dx \quad n = 8 \ \textit{subintervals}$

8. $\int_1^5 \left(3x^2 - 2x\right)dx \quad n = 8 \ \textit{subintervals}$

9. $\int_0^{\pi/4} 3\sin 2x\, dx \quad n = 8 \ \textit{subintervals}$

10. $\int_0^8 e^{-2x}\, dx \quad n = 8 \ \textit{subintervals}$

11. A piece of wood paneling must be cut in the shape shown below. The coordinates of several point on its curved surface are also shown (with units of inches).



a) Estimate the surface area of the paneling using the Trapezoid Rule

b) Estimate the surface area of the paneling using a left Riemann sum.

c) Could two identical pieces be cut from a 9-in by 9-in piece of wood?

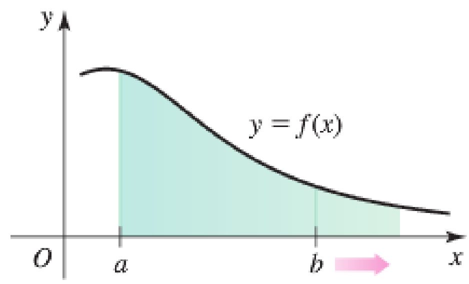
Section 2.6 – Improper Integrals

Definition

Integrals with infinite limits of integration are ***improper integrals***.

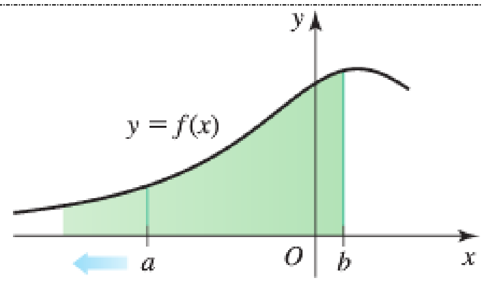
1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$



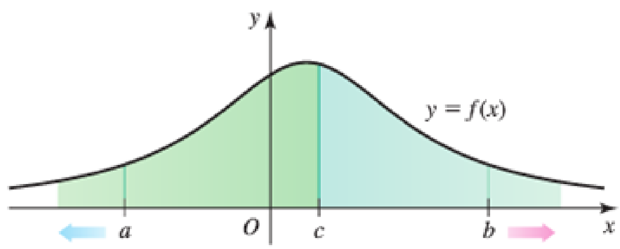
2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$



3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$



In each case, if the limit is finite we say that the improper integral ***converges*** and that the limit is the ***value*** of the improper integral. If the limit fails to exist, the improper integral ***diverges***.

Example

Is the area under the curve $y = \dfrac{\ln x}{x^2}$ from $x = 1$ to $x = \infty$ finite? If so, what is its value?

Solution

$$\int_1^b \frac{\ln x}{x^2}\,dx = -\frac{1}{x}\ln x\Big|_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx$$

$u = \ln x \qquad dv = \dfrac{dx}{x^2}$

$du = \dfrac{1}{x}dx \qquad v = -\dfrac{1}{x}$

$$= -\left(\frac{1}{b}\ln b - \ln 1\right) + \int_1^b \frac{1}{x^2}\,dx$$

$$= -\frac{1}{b}\ln b + \left[-\frac{1}{x}\right]_1^b$$

$$= -\frac{1}{b}\ln b - \left(\frac{1}{b} - 1\right)$$

$$= -\frac{1}{b}\ln b - \frac{1}{b} + 1$$

$$\int_1^{\infty} \frac{\ln x}{x^2}dx = \lim_{b\to\infty}\int_1^{b} \frac{\ln x}{x^2}dx$$

$$= \lim_{b\to\infty}\left(-\frac{1}{b}\ln b - \frac{1}{b} + 1\right)$$

$$= -\lim_{b\to\infty}\left(\frac{1/b}{1}\right) - 0 + 1 \qquad \textit{L'Hôpital Rule}$$

$$= -0 + 1$$

$$\underline{= 1}$$

Example

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Solution

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty}\int_{a}^{0} \frac{dx}{1+x^2}$$

$$= \lim_{a\to-\infty} \tan^{-1}x\Big|_a^0$$

$$= \lim_{a\to-\infty}\left(\tan^{-1}0 - \tan^{-1}a\right)$$

$$= 0 - \left(-\frac{\pi}{2}\right)$$

$$\underline{= \frac{\pi}{2}}$$

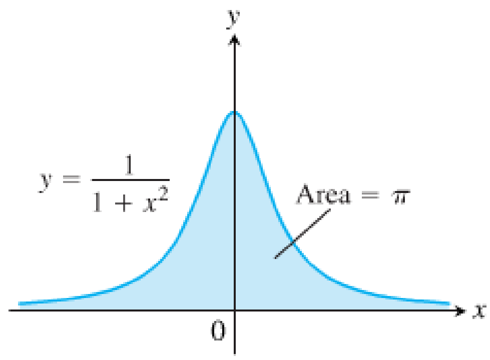
$$\int_{0}^{\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_{0}^{b} \frac{dx}{1+x^2}$$

$$= \lim_{b\to\infty} \tan^{-1}x\Big|_0^b$$

$$= \lim_{b\to\infty}\left(\tan^{-1}b - \tan^{-1}0\right)$$

$$= \frac{\pi}{2} - 0$$

$$\underline{= \frac{\pi}{2}}$$



$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$\underline{= \pi}$$

Example

For what value of p does the integral $\int_1^{\infty} \frac{dx}{x^p}$ converge? When the integral does converge, what is its value?

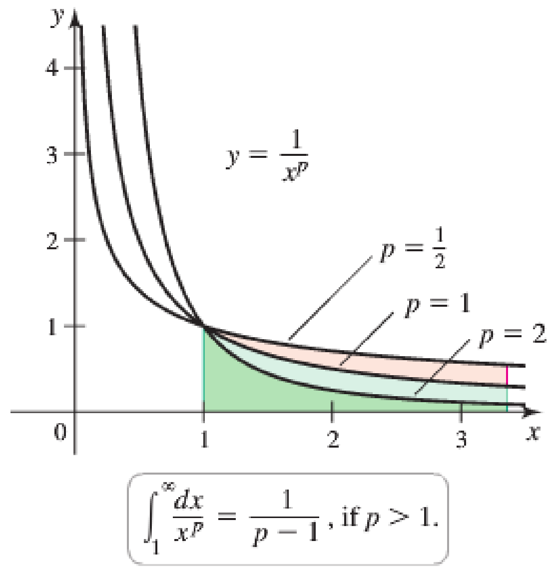
Solution

If $p \neq 1$ $\quad \int_1^b \frac{dx}{x^p} = \left.\frac{x^{-p+1}}{-p+1}\right|_1^b = \frac{1}{1-p}\left(b^{1-p} - 1\right)$

$$\int_1^{\infty} \frac{dx}{x^p} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^p}$$

$$= \lim_{b\to\infty} \left[\frac{1}{1-p}\left(b^{1-p} - 1\right)\right]$$

$$= \begin{cases} \frac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{cases}$$



If $p = 1$ $\quad \int_1^{\infty} \frac{dx}{x^p} = \int_1^{\infty} \frac{dx}{x}$

$$= \lim_{b\to\infty} \int_1^b \frac{dx}{x}$$

$$= \lim_{b\to\infty} \left[\ln x\right]_1^b$$

$$= \lim_{b\to\infty} \left(\ln b - \ln 1\right)$$

$$\underline{= \infty}$$

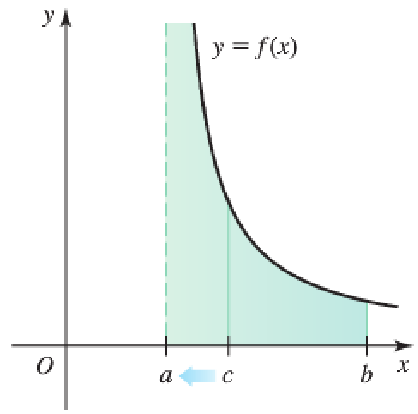
Integrands with Vertical Asymptotes

Definition

Integrals of functions that become infinite at a point within the interval of integration are ***improper integrals***.
If the limit is finite we say that the improper integral ***converges*** and that the limit is the ***value*** of the improper integral. If the limit does not exist, the integral ***diverges***.

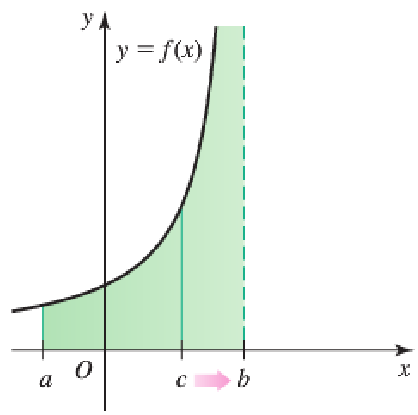
1. If $f(x)$ is continuous on $(a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx$$



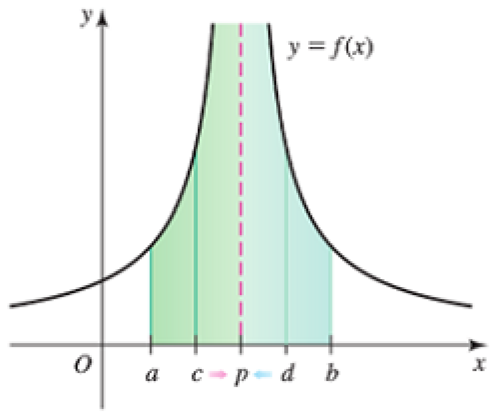
2. If $f(x)$ is continuous on $[a, b)$, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx$$



3. If $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{c\to p^-}\int_a^c f(x)\,dx + \lim_{d\to p^+}\int_d^b f(x)\,dx$$



Example

Investigate the convergence of $\displaystyle\int_0^1 \frac{1}{1-x}\,dx$

Solution

$$\int_0^1 \frac{1}{1-x}\,dx = \lim_{b\to 1^-}\int_0^b \frac{1}{1-x}\,dx$$

$$= \lim_{b\to 1^-}\Big[-\ln|1-x|\Big]_0^b$$

$$= \lim_{b\to 1^-}\Big[-\ln|1-b|+0\Big]$$

$$\boxed{=\infty}$$

The limit is infinite, so the integral diverges.

Example

Evaluate $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}}$

Solution

The integrand has a vertical asymptote at $x = 1$ and is continuous on [0, 1) and (1, 3].

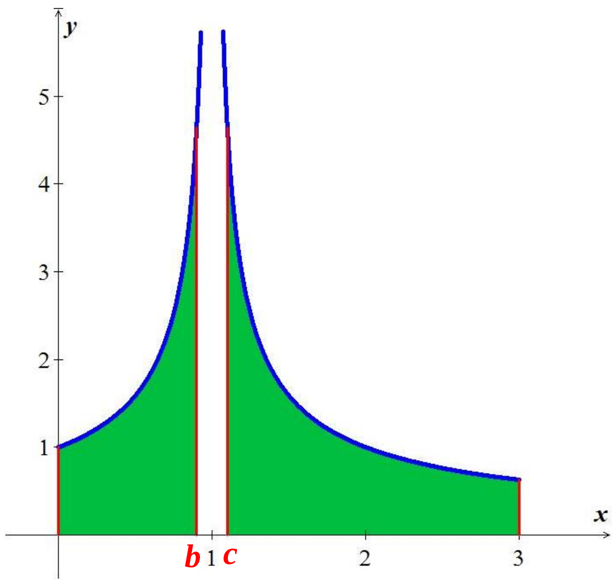
$$\int \frac{dx}{(x-1)^{2/3}} = \int (x-1)^{-2/3}\,d(x-1) = 3(x-1)^{1/3}$$

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$

$$= \Big[3(x-1)^{1/3}\Big]_0^{1^-} + \Big[3(x-1)^{1/3}\Big]_{1^+}^3$$

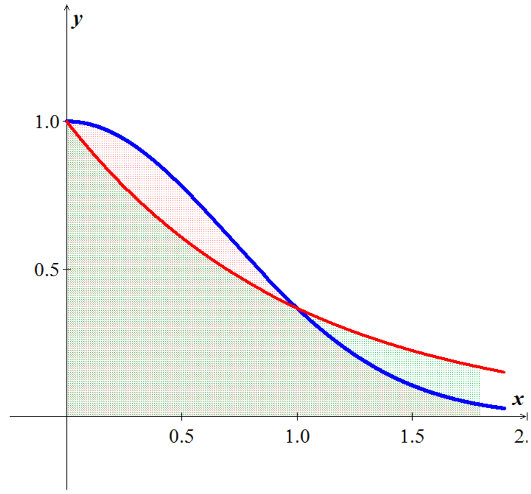
$$= 3(0+1) + 3\left(\sqrt[3]{2}-0\right)$$

$$\boxed{= 3+3\sqrt[3]{2}}$$



Example

Does the integral $\int_1^\infty e^{-x^2}\,dx$ converge?



Solution

$$\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b e^{-x^2}\,dx$$

$$\int_1^b e^{-x^2}\,dx \le \int_1^b e^{-x}dx = -e^{-b} + e^{-1} < e^{-1} \approx 0.36788$$

The integral converges

Theorem – Direct Comparison Test

Let f and g be continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

1. $\int_a^\infty f(x)\,dx$ *converges if* $\int_a^\infty g(x)\,dx$ *converges*

2. $\int_a^\infty g(x)\,dx$ *diverges if* $\int_a^\infty f(x)\,dx$ *diverges*

Theorem – Limit Comparison Test

If the positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = L, \quad 0 < L < \infty$$

Then

$$\int_a^\infty f(x)\,dx \quad \textit{and} \quad \int_a^\infty g(x)\,dx$$

Both converge or both diverge

Example

Show that $\int_1^\infty \frac{dx}{1+x^2}$ converges by comparison with $\int_1^\infty \frac{dx}{x^2}$. Find and compare the two integral values.

Solution

The functions $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{1+x^2}$ are positive and continuous on $[1, \infty)$. Also,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{1}{1+x^2}}$$

$$= \lim_{x\to\infty} \frac{1+x^2}{x^2}$$

$$\underline{=1|}$$

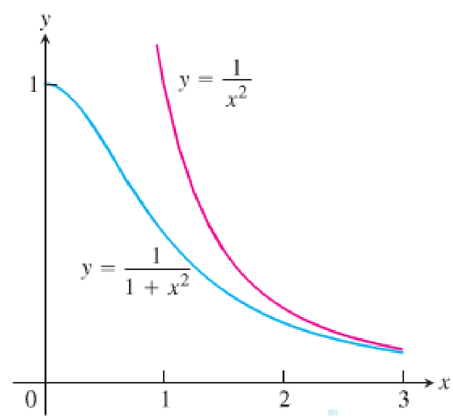
Therefore, $\int_1^\infty \frac{dx}{1+x^2}$ converges because $\int_1^\infty \frac{dx}{x^2}$ converges.

$$\int_1^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{1+x^2}$$

$$= \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} 1\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$\underline{= \frac{\pi}{4}|}$$



Example

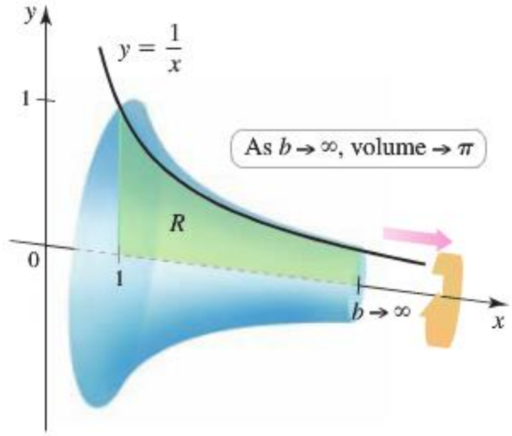
Let R be the region bounded by the graph of $y = x^{-1}$ and the x-axis, for $x \ge 1$.

a) What is the volume of the solid generated when R is revolved about the x-axis?

b) What is the surface area of the solid generated when R is revolved about the x-axis?

c) What is the volume of the solid generated when R is revolved about the y-axis?

Solution



a) $V = \pi \int_1^\infty \frac{1}{x^2} dx \qquad V = \pi \int_a^b \left(f(x)\right)^2 dx$

$$= -\pi \frac{1}{x}\Big|_1^\infty$$

$$= -\pi(0-1)$$

$$\boxed{= \pi \ \ unit^3}$$

b) $S = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\left(-\frac{1}{x^2}\right)^2} dx \qquad S = 2\pi \int_a^b f(x)\sqrt{1+f'(x)^2}\, dx$

$$= 2\pi \int_1^\infty \frac{1}{x^3}\sqrt{x^4+1}\, dx$$

$$> 2\pi \int_1^\infty \frac{x^2}{x^3} dx \qquad \sqrt{x^4+1} > \sqrt{x^4} = x^2$$

$$= 2\pi \int_1^\infty \frac{1}{x} dx$$

$$= 2\pi(\ln x)\Big|_1^\infty$$

$$\boxed{= \infty \ \ unit^2}$$

c) $V = 2\pi \int_1^\infty x\frac{1}{x} dx \qquad V = 2\pi \int_a^b x \cdot f(x)\, dx \quad (Shell\ method)$

$$= 2\pi x\Big|_1^\infty$$

$$\boxed{= \infty \ \ unit^3}$$

Exercises *Section* **2.6 – Improper Integrals**

Evaluate the integrals

1. $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$
2. $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}}$
3. $\displaystyle\int_{-\infty}^2 \frac{2dx}{x^2+4}$
4. $\displaystyle\int_{-\infty}^\infty \frac{xdx}{\left(x^2+4\right)^{3/2}}$
5. $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x^2-1}}$
6. $\displaystyle\int_{-\infty}^\infty 2xe^{-x^2}dx$
7. $\displaystyle\int_0^1 (-\ln x)dx$
8. $\displaystyle\int_{-1}^4 \frac{dx}{\sqrt{|x|}}$
9. $\displaystyle\int_0^\infty e^{-3x}\,dx$
10. $\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^2}$
11. $\displaystyle\int_1^{10} \frac{dx}{(x-2)^{1/3}}$
12. $\displaystyle\int_1^\infty \frac{dx}{x^2}$
13. $\displaystyle\int_0^\infty \frac{dx}{(x+1)^3}$
14. $\displaystyle\int_{-\infty}^0 e^x\,dx$
15. $\displaystyle\int_1^\infty 2^{-x}\,dx$
16. $\displaystyle\int_{-\infty}^0 \frac{dx}{\sqrt[3]{2-x}}$
17. $\displaystyle\int_{4/\pi}^\infty \frac{1}{x^2}\sec^2\left(\frac{1}{x}\right)dx$
18. $\displaystyle\int_{e^2}^\infty \frac{dx}{x\ln^p x} \quad p>1$
19. $\displaystyle\int_0^\infty \frac{p}{\sqrt[5]{p^2+1}}\,dp$
20. $\displaystyle\int_{-1}^1 \ln y^2\,dy$
21. $\displaystyle\int_{-2}^6 \frac{dx}{\sqrt{|x-2|}}$
22. $\displaystyle\int_0^\infty xe^{-x}dx$
23. $\displaystyle\int_0^1 x\ln x\,dx$
24. $\displaystyle\int_1^\infty \frac{\ln x}{x^2}dx$
25. $\displaystyle\int_1^\infty (1-x)e^x\,dx$
26. $\displaystyle\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\,dx$
27. $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}}$
28. $\displaystyle\int_1^\infty \frac{3}{\sqrt[3]{x}}\,dx$
29. $\displaystyle\int_1^\infty \frac{4}{\sqrt[4]{x}}\,dx$
30. $\displaystyle\int_0^2 \frac{dx}{x^3}$
31. $\displaystyle\int_1^\infty \frac{dx}{x^3}$
32. $\displaystyle\int_1^\infty \frac{6}{x^4}\,dx$
33. $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$
34. $\displaystyle\int_{-\infty}^0 xe^{-4x}dx$
35. $\displaystyle\int_0^\infty xe^{-x/3}dx$
36. $\displaystyle\int_0^\infty x^2e^{-x}dx$
37. $\displaystyle\int_0^\infty e^{-x}\cos x\,dx$
38. $\displaystyle\int_4^\infty \frac{1}{x(\ln x)^3}dx$
39. $\displaystyle\int_1^\infty \frac{\ln x}{x}dx$
40. $\displaystyle\int_{-\infty}^\infty \frac{4}{16+x^2}dx$

41. $\displaystyle\int_0^{\infty} \frac{x^3}{\left(x^2+1\right)^2}\,dx$

42. $\displaystyle\int_0^{\infty} \frac{1}{e^x+e^{-x}}\,dx$

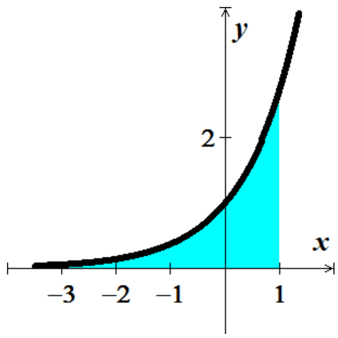
43. $\displaystyle\int_0^{\infty} \frac{e^x}{1+e^x}\,dx$

44. $\displaystyle\int_0^{\infty} \cos \pi x\,dx$

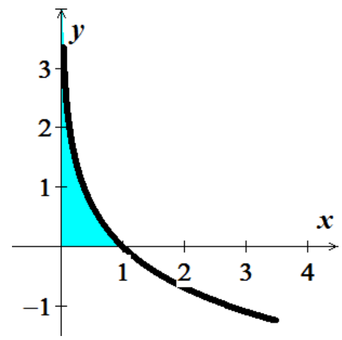
45. $\displaystyle\int_0^{\infty} \sin \frac{x}{2}\,dx$

Find the area of the unbounded shaded region

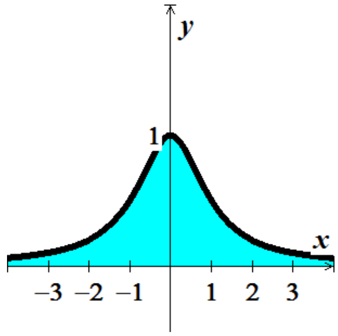
46. $y=e^x, \quad -\infty < x \le 1$



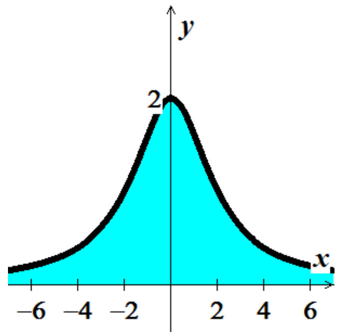
47. $y=-\ln x$



48. $y=\dfrac{1}{x^2+1}$



49. $y=\dfrac{8}{x^2+4}$



50. Find the area of the region R between the graph of $f(x)=\dfrac{1}{\sqrt{9-x^2}}$ and the x-$axis$ on the interval $(-3,\ 3)$ (if it exists)

51. Find the volume of the region bounded by $f(x)=\left(x^2+1\right)^{-1/2}$ and the x-$axis$ on the interval $[2,\ \infty)$ is revolved about the x-$axis$.

52. Find the volume of the region bounded by $f(x)=\sqrt{\dfrac{x+1}{x^3}}$ and the x-$axis$ on the interval $[1,\ \infty)$ is revolved about the x-$axis$.

53. Find the volume of the region bounded by $f(x)=(x+1)^{-3}$ and the x-$axis$ on the interval $[0,\ \infty)$ is revolved about the y-$axis$.

54. Find the volume of the region bounded by $f(x)=\frac{1}{\sqrt{x}\ln x}$ and the *x-axis* on the interval $[2,\ \infty)$ is revolved about the *x-axis*.

55. Find the volume of the region bounded by $f(x)=\frac{\sqrt{x}}{\sqrt[3]{x^2+1}}$ and the *x-axis* on the interval $[0,\ \infty)$ is revolved about the *x-axis*.

56. Find the volume of the region bounded by $f(x)=\left(x^2-1\right)^{-1/4}$ and the *x-axis* on the interval $(1,\ 2]$ is revolved about the *y-axis*.

57. Find the volume of the region bounded by $f(x)=\tan x$ and the *x-axis* on the interval $\left[0,\ \frac{\pi}{2}\right)$ is revolved about the *x-axis*.

58. Find the volume of the region bounded by $f(x)=-\ln x$ and the *x-axis* on the interval $(0,\ 1]$ is revolved about the *x-axis*.

59. Find the volume of the solid generated by revolving the region bounded by the graphs of $y=xe^{-x}$, $y=0$, and $x=0$ about the *x-axis*.

Consider the region satisfying the inequalities

a) Find the area of the region
b) Find the volume of the solid generated by revolving the region about the *x-axis*.
c) Find the volume of the solid generated by revolving the region about the *y-axis*.

60. $y\le e^{-x},\quad y\ge 0,\quad x\ge 0$

61. $y\le \frac{1}{x^2},\quad y\ge 0,\quad x\ge 1$

62. Find the perimeter of the hypocycloid of four cusps $x^{2/3}+y^{2/3}=4$

63. Find the arc length of the graph $y=\sqrt{16-x^2}$ over the interval $[0,\ 4]$

64. The region bounded by $(x-2)^2+y^2=1$ is revolved about the *y-axis* to form a torus. Find the surface area of the torus.

65. Find the surface area formed by revolving the graph $y=2e^{-x}$ on the interval $[0,\ \infty)$ about the *x-axis*

66. The magnetic potential *P* at a point on the axis of a circular coil is given by

$$P=\frac{2\pi NIr}{k}\int_c^{\infty}\frac{1}{\left(r^2+x^2\right)^{3/2}}\,dx$$

Where *N, I, r, k,* and *c* are constants. Find *P*.

67. A "semi-infinite" uniform rod occupies the nonnegative x-*axis*. The rod has a linear density δ, which means that a segment of length dx has a mass of δdx. A particle of mass M is located at the point $(-a,\ 0)$. The gravitational force F that the rod exerts on the mass is given by

$$F = \int_0^\infty \frac{GM\delta}{(a+x)^2}\, dx$$

Where G is the gravitational constant. Find F.

68. Let R be the region bounded by the graph of $f(x) = x^{-p}$ and the x-*axis*

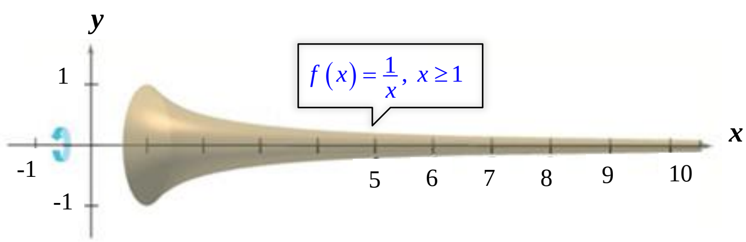
a) Let S be the solid generated when R is revolved about the x-*axis*. For what values of p is the volume of S finite for $0 < x \le 1$?

b) Let S be the solid generated when R is revolved about the y-*axis*. For what values of p is the volume of S finite for $0 < x \le 1$?

c) Let S be the solid generated when R is revolved about the x-*axis*. For what values of p is the volume of S finite for $x \ge 1$?

d) Let S be the solid generated when R is revolved about the y-*axis*. For what values of p is the volume of S finite for $x \ge 1$?

69. The solid formed by revolving (about the x-*axis*) the unbounded region lying between the graph of $f(x) = \frac{1}{x}$ and the x-*axis* $(x \ge 1)$ is called ***Gabriel's Horn***.



Show that this solid has a finite volume and an infinite surface area

70. Water is drained from a 3000-*gal* tank at a rate that starts at 100 *gal/hr*. and decreases continuously by 5% /*hr*. If the drain left open indefinitely, how much water drains from the tank? Can a full tank be emptied at this rate?

First Order Differential Equations

Section 2.7 – First-Order Linear Equations

General First-Order Differential Equations and Solutions

A ***first-order differential equation*** is an equation

$$\frac{dy}{dx} = f(x, y)$$

In which $f(x, y)$ is a function of two variables defined on a region in the *xy*-plane.

Example

Show that every member of the family of functions $y = \frac{C}{x} + 2$ is a solution of the first-order differential equation $\frac{dy}{dx} = \frac{1}{x}(2 - y)$ on the interval (0, ∞), where C is any constant.

Solution

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{C}{x} + 2\right)$$

$$= -\frac{C}{x^2}$$

$$\frac{dy}{dx} = \frac{1}{x}(2 - y)$$

$$-\frac{C}{x^2} = \frac{1}{x}\left(2 - \left(\frac{C}{x} + 2\right)\right)$$

$$-\frac{C}{x^2} = \frac{1}{x}\left(2 - \frac{C}{x} - 2\right)$$

$$-\frac{C}{x^2} = \frac{1}{x}\left(-\frac{C}{x}\right)$$

$$-\frac{C}{x^2} = -\frac{C}{x^2} \quad \checkmark$$

Therefore, for every value of C, the function $y = \frac{C}{x} + 2$ is a solution of the first-order differential equation $\frac{dy}{dx} = \frac{1}{x}(2 - y)$.

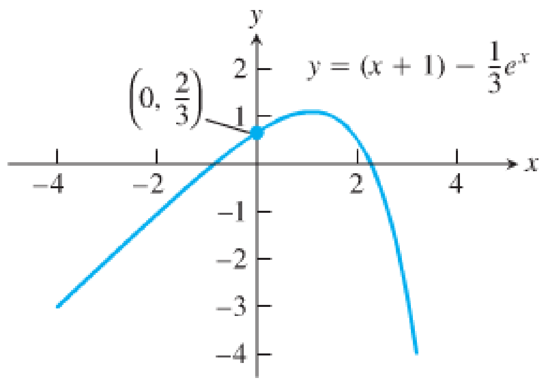
Example

Show that the function $y=(x+1)-\frac{1}{3}e^{x}$ is a solution of the first-order initial value problem

$\frac{dy}{dx}=y-x \quad y(0)=\frac{2}{3}$.

Solution

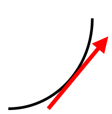
$$\frac{dy}{dx}=\frac{d}{dx}\left(x+1-\frac{1}{3}e^{x}\right)$$
$$=1-\frac{1}{3}e^{x}$$
$$y-x=1-\frac{1}{3}e^{x}$$
$$y=x+1-\frac{1}{3}e^{x}$$
$$y(0)=(0+1)-\frac{1}{3}e^{0}$$
$$=1-\frac{1}{3}$$
$$\boxed{=\frac{2}{3}}$$

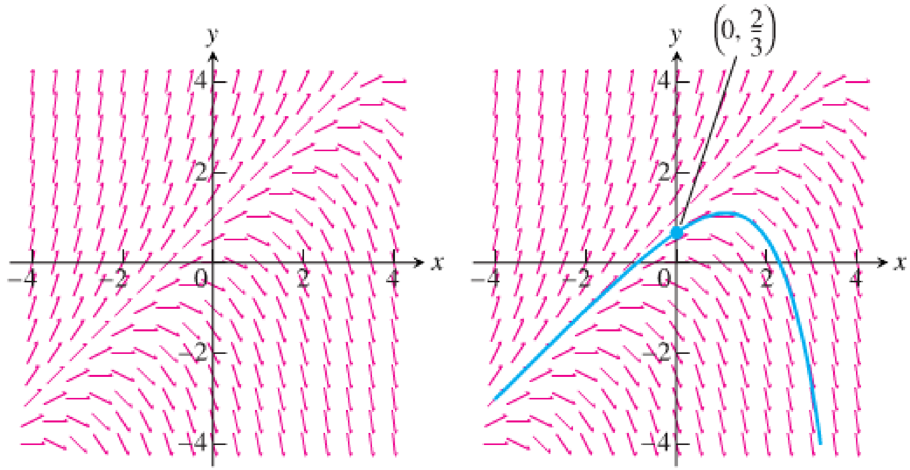


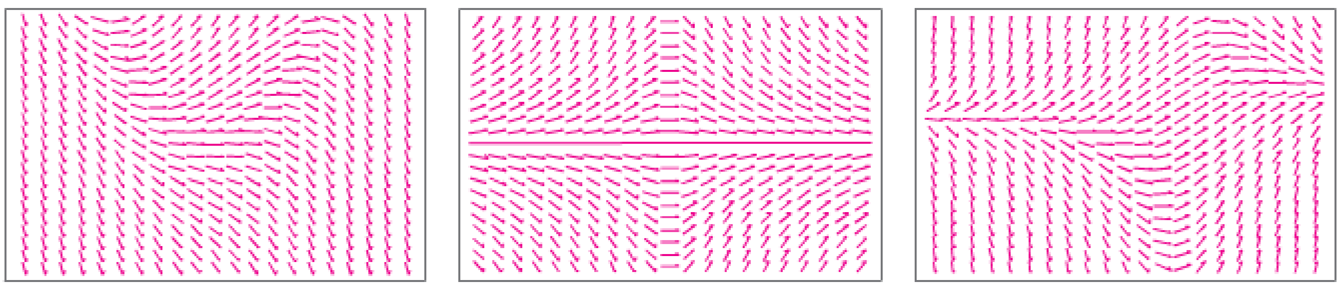
Slope Fields: Viewing Solution Curves

Each time we specify an initial condition $y(x_0)=y_0$ for the solution of a differential equation $y'=f(x,y)$, the solution curve is required to pass through the point (x_0,y_0) and to have a slope $f(x_0,y_0)$ there.

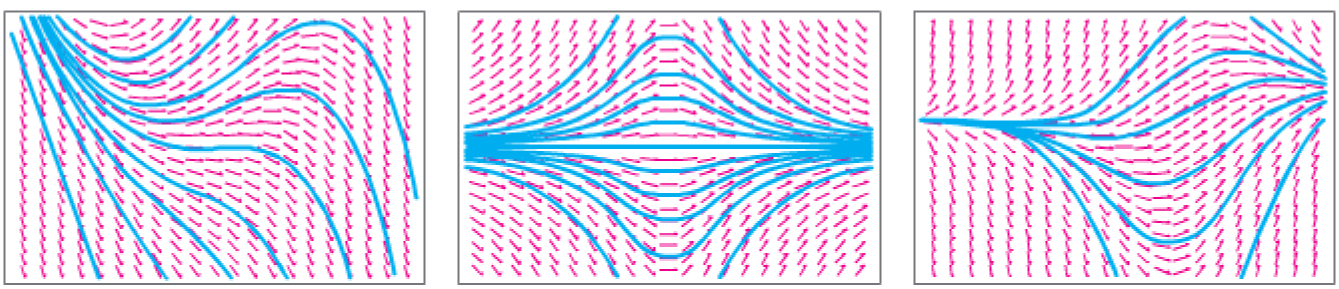
What we draw a lineal element at each point (x,y) with slope $f(x,y)$ then the collection of these lineal elements is called a ***direction field*** or a ***slope field*** of the differential equation $\frac{dy}{dx}=f(x,y)$.





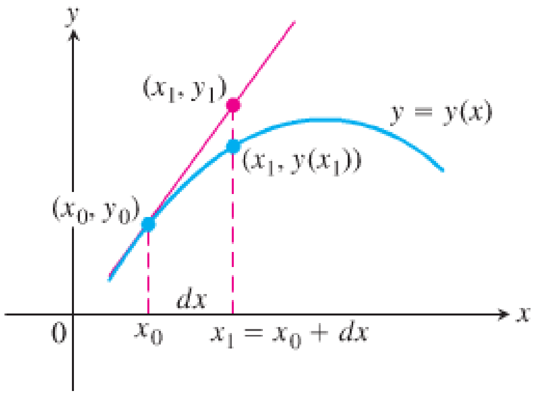


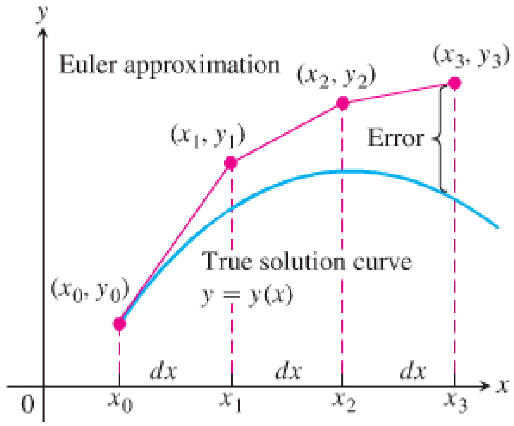
(a) $y' = y - x^2$ (b) $y' = -\dfrac{2xy}{1 + x^2}$ (c) $y' = (1 - x)y + \dfrac{x}{2}$



Euler's Method

Euler's method named after *Leonhard Euler* is an example of a ***fixed-step*** solver.
Euler's method is a first-order numerical procedure for solving ordinary differential equations (ODEs) with a given initial value. It is the most basic kind of explicit method for numerical integration of ordinary differential equations.





$$y' = f(x, y) \quad y(x_0) = y_0$$

The setting size: $h = \dfrac{b-a}{k} > 0$; $k = 1, 2, 3, \ldots$

Then, $x_0 = a$

$x_1 = x_0 + h = a + h$

$x_k = x_{k-1} + h = a + kh$

Last point $x_k = a + kh = b$

By the definition of the derivative:

$$y'(x_k) \approx \frac{y(x_{k+1}) - y(x_k)}{h}$$

$$y'(x_k) \approx \frac{y_{k+1} - y_k}{h} = f(x_k, y_k) : slope$$

The tangent line at the point $(x_0, y(x_0))$ is:

$$y_{k+1} = y_k + h.f(x_k, y_k)$$

$$y_{k+1} = y_k + \Delta x_{step}.f(x_k, y_k) \qquad \boxed{y_{k+1} = y_k + f(x_k, y_k)dx}$$

This method is known as *Euler's Method* with step size h.

Example

Find the first three approximations y_1, y_2, y_3 using Euler's method for the initial value problem

$$y' = 1 + y, \quad y(0) = 1$$

Starting at $x_0 = 0$ with $dx = 0.1$.

Solution

$$\begin{aligned} y_1 &= y_0 + f(x_0, y_0)dx \\ &= y_0 + (1 + y_0)dx \\ &= 1 + (1+1)(0.1) \\ &= 1.2 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + f(x_1, y_1)dx \\ &= y_1 + (1 + y_1)dx \\ &= 1.2 + (1+1.2)(0.1) \\ &= 1.42 \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + f(x_2, y_2)dx \\ &= y_2 + (1 + y_2)dx \\ &= 1.42 + (1+1.42)(0.1) \\ &= 1.662 \end{aligned}$$

Example

Use Euler's method to solve

$$y' = 1 + y, \quad y(0) = 1$$

On the interval $0 \le x \le 1$, starting at $x_0 = 0$ and taking

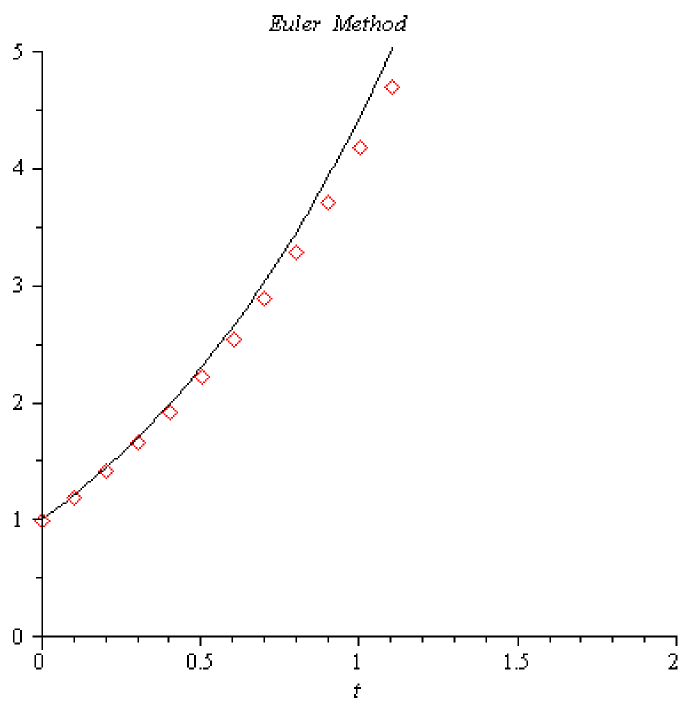
a) $dx = 0.1$.
b) $dx = 0.05$.

Compare the approximations with the values of the exact solution $y = 2e^x - 1$

Solution

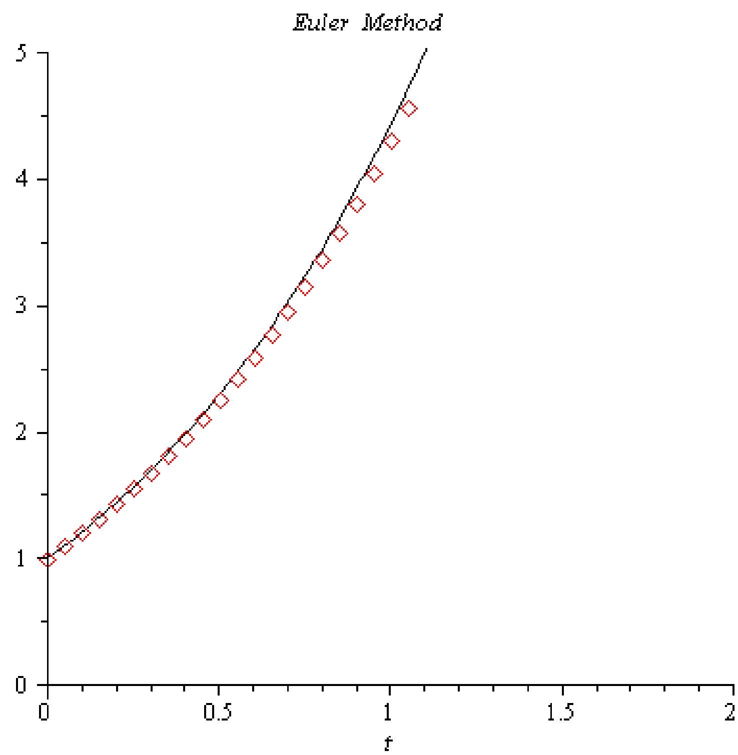
a) Euler Method ***dx* = 0.1**

t	***Approx.***	***Exact***	***Difference***
0.00	1.00000000	1.00000000	0.00000000
0.10	1.20000000	1.21034184	0.01034184
0.20	1.42000000	1.44280552	0.02280552
0.30	1.66200000	1.69971762	0.03771762
0.40	1.92820000	1.98364940	0.05544940
0.50	2.22102000	2.29744254	0.07642254
0.60	2.54312200	2.64423760	0.10111560
0.70	2.89743420	3.02750541	0.13007121
0.80	3.28717762	3.45108186	0.16390424
0.90	3.71589538	3.91920622	0.20331084
1.00	4.18748492	4.43656366	0.24907874



b) Euler Method ***dx* = 0.05**

t	***Approx.***	***Exact***	***Difference***
0.00	1.00000000	1.00000000	0.00000000
0.05	1.10000000	1.10254219	0.00254219
0.10	1.20500000	1.21034184	0.00534184
0.15	1.31525000	1.32366849	0.00841849
0.20	1.43101250	1.44280552	0.01179302
0.25	1.55256313	1.56805083	0.01548771
0.30	1.68019128	1.69971762	0.01952633
0.35	1.81420085	1.83813510	0.02393425
0.40	1.95491089	1.98364940	0.02873851
0.45	2.10265643	2.13662437	0.03396794
0.50	2.25778925	2.29744254	0.03965329
0.55	2.42067872	2.46650604	0.04582732
0.60	2.59171265	2.64423760	0.05252495
0.65	2.77129828	2.83108166	0.05978337
0.70	2.95986320	3.02750541	0.06764222
0.75	3.15785636	3.23400003	0.07614367
0.80	3.36574918	3.45108186	0.08533268
0.85	3.58403664	3.67929370	0.09525707
0.90	3.81323847	3.91920622	0.10596775
0.95	4.05390039	4.17141932	0.11751893
1.00	4.30659541	4.43656366	0.12996825



A ***first-order linear*** differential equation is one that can be written in the ***standard form***

$$\frac{dy}{dx} + P(x)\cdot y = Q(x)$$

Where P and Q are continuous functions of x

Solving Linear Equations

We solve the equation $\frac{dy}{dx} + P(x)\cdot y = Q(x)$

Separable Equation

Solution of the homogenous equation

$$\frac{dy}{dx} + P(x)y = 0$$

$$\frac{dy}{dx} = -P(x)y$$

$$\int \frac{dy}{y} = -\int P(x)dx$$

Integrate both sides

$$\ln|y| = -\int P(x)dx + C$$

Convert to exponential form

$$y(x) = e^{\int P(x)dx + C} = e^{\int P(x)dx} e^{C}$$

$$\boxed{y(x) = A.e^{\int P(x)dx}}$$

Example

Solve the differential equation $y' = ty^2$

Solution

$$\frac{dy}{dt} = ty^2$$

$$\frac{dy}{y^2} = tdt$$

$$\int y^{-2}dy = \int tdt$$

$$-y^{-1} = \tfrac{1}{2}t^2 + C$$

$$-\frac{1}{y} = \frac{t^2+2C}{2} \qquad \textit{Cross multiplication}$$

$$y(t) = -\frac{2}{t^2 + 2C}$$

General Method

1. Separate the variables
2. Integrate both sides
3. Solve for the solution $y(t)$, if possible

Example

Find the general solution of the differential equation. $y' = \dfrac{2xy + 2x}{x^2 - 1}$

Solution

$$\frac{dy}{dx} = \frac{2x(y+1)}{x^2 - 1}$$

$$\frac{dy}{y+1} = \frac{2x}{x^2 - 1}dx \qquad d\left(x^2 - 1\right) = 2xdx$$

$$\int \frac{d(y+1)}{y+1} = \int \frac{d\left(x^2 - 1\right)}{x^2 - 1}$$

$$\ln|y+1| = \ln\left|x^2 - 1\right| + C \quad \Rightarrow \quad y + 1 = e^{\ln\left|x^2-1\right| + C}$$

$$y = e^C e^{\ln\left|x^2-1\right|} - 1$$

$$y(x) = Ae^{\ln\left|x^2-1\right|} - 1$$

Solution of the Nonhomogeneous Equation $y' + p(x)y = f(x)$

Let assume: $y = y_h + y_p$ *where* $\begin{cases} y_h & \textit{Homogeneous Solution} \\ y_p & \textit{Paticular Solution} \end{cases}$

The homogeneous equation is given by $y'_h + p(x)y_h = 0$

$$y'_h = -p(x)y_h \quad \Rightarrow y_h = e^{-\int p\,dx}$$

$$y_p = u(x)y_h = u.e^{-\int p\,dx}$$

$$y'_p + p(x)y_p = f(x)$$

$$\left(uy_h\right)' + puy_h = f$$

$$u'y_h + uy'_h + puy_h = f$$

$$u'y_h + u\left(y'_h + py_h\right) = f \qquad \textit{Since } y'_h + py_h = 0 \qquad \textit{homogeneous}$$

$$u'y_h = f$$

$$\frac{du}{dx} = \frac{f}{y_h}$$

$$du = \frac{f}{e^{-\int p\,dx}}dx$$

$$= f.e^{\int p\,dx}dx$$

$$u = \int f.e^{\int p\,dx}dx$$

$$\boxed{y_p} = u.e^{-\int p\,dx} = \left(\int f.e^{\int p\,dx}dx\right)e^{-\int p\,dx} = \boxed{e^{-\int p\,dx}\int f.e^{\int p\,dx}dx}$$

$$y = y_h + y_p$$

$$= Ce^{-\int p\,dx} + e^{-\int p\,dx}\int f.e^{\int p\,dx}dx$$

$$= e^{-\int p\,dx}\left(C + \int f.e^{\int p\,dx}dx\right)$$

$$y' + p(x)y = f(x) \quad \Rightarrow \boxed{y = \frac{1}{e^{\int p\,dx}}\left(\int f.e^{\int p\,dx}dx + C\right)}$$

Example

Solve the equation $\quad x\dfrac{dy}{dx}=x^2+3y, \quad x>0$

Solution

$$y'-\frac{3}{x}y=x$$

$$e^{\int p\,dx}=e^{-3\int\frac{dx}{x}}=e^{-3\ln|x|}=e^{\ln x^{-3}}\underline{=x^{-3}}$$

$$\int x.x^{-3}dx=\int x^{-2}dx=\frac{1}{x}$$

$$y(x)=\frac{1}{x^{-3}}\left(\frac{1}{x}+C\right)$$

$$=x^3\left(\frac{1}{x}+C\right)$$

$$\underline{=x^2+Cx^3} \qquad x>0$$

Example

Solve the equation $\quad 3xy'-y=\ln x+1, \quad x>0$, satisfying $y(1)=-2$

Solution

$$y'-\frac{1}{3x}y=\frac{\ln x+1}{3x}$$

$$e^{\int p\,dx}=e^{\int\left(-\frac{1}{3x}\right)dx}=e^{-\frac{1}{3}\ln x}=e^{\ln x^{-1/3}}\underline{=x^{-1/3}}$$

$$\int\left(x^{-1/3}\right)\frac{\ln x+1}{3x}dx=\frac{1}{3}\int(\ln x+1)x^{-4/3}\,dx$$

$u=\ln x+1 \qquad dv=\int x^{-4/3}dx$

$du=\frac{1}{x}dx \qquad v=-3x^{-1/3}$

$$=\frac{1}{3}\left(-3x^{-1/3}(\ln x+1)+3\int\left(x^{-4/3}\right)dx\right)$$

$$=\frac{1}{3}\left(-3x^{-1/3}(\ln x+1)-9x^{-1/3}\right)$$

$$=-x^{-1/3}(\ln x+1)-3x^{-1/3}$$

$$y(x)=x^{1/3}\left(-x^{-1/3}(\ln x+1)-3x^{-1/3}+C\right)$$

$$\underline{=-\ln x-4+Cx^{1/3}}$$

$$y(1)=-\ln(1)-4+C(1)^{1/3}$$

$$-2=-0-4+C \qquad \boxed{2=C}$$

$$\underline{y=2x^{1/3}-\ln x-4}$$

Exercises *Section* 2.7 – First-Order Linear Equations

Write an equivalent first-order differential equation and initial condition for y.

1. $y = \int_1^x \frac{1}{t}\,dt$

2. $y = 2 - \int_0^x \left(1 + y(t)\right)\sin t\,dt$

Use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round the results to four decimals

1. $y' = 1 - \frac{y}{x}, \quad y(2) = -1, \quad dx = 0.5$

2. $y' = x(1-y), \quad y(1) = 0, \quad dx = 0.2$

3. $y' = y^2(1+2x), \quad y(-1) = 1, \quad dx = 0.5$

4. $y' = ye^x, \quad y(0) = 2, \quad dx = 0.5$

5. Use the Euler method with $dx = 0.2$ to estimate $y(2)$ if $y' = \frac{y}{x}$ and $y(1) = 2$. What is the exact value of $y(2)$?

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants.

6. $y = Ce^{-5t}; \quad y'(t) + 5y = 0$

7. $y = Ct^{-3}; \quad ty'(t) + 3y = 0$

8. $y = C_1 \sin 4t + C_2 \cos 4t; \quad y''(t) + 16y = 0$

9. $y = C_1 e^{-x} + C_2 e^x; \quad y''(x) - y = 0$

10. $y' + 4y = \cos t, \quad y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t + Ce^{-4t}, \qquad y(0) = -1$

11. $ty' + (t+1)y = 2te^{-t}, \quad y(t) = e^{-t}\left(t + \frac{C}{t}\right), \quad y(1) = \frac{1}{e}$

12. $y' = y(2+y), \quad y(t) = \frac{2}{-1+Ce^{-2t}}, \quad y(0) = -3$

Verify that the given function y is a solution of the initial value problem that follows it.

13. $y = 16e^{2t} - 10; \quad y' - 2y = 20, \quad y(0) = 6$

14. $y = 8t^6 - 3; \quad ty' - 6y = 18, \quad y(1) = 5$

15. $y = -3\cos 3t; \quad y'' + 9y = 0, \quad y(0) = -3, \ y'(0) = 0$

16. $y = \frac{1}{4}\left(e^{2x} - e^{-2x}\right); \quad y'' - 4y = 0, \quad y(0) = 0, \ y'(0) = 1$

Solve the differential equations

17. $y' = xy$

18. $xy' = 2y$

19. $y' = e^{x-y}$

20. $y' = \left(1+y^2\right)e^x$

21. $y' = xy + y$

22. $y' = ye^x - 2e^x + y - 2$

23. $y' = \dfrac{x}{y+2}$

24. $y' = \dfrac{xy}{x-1}$

25. $x\dfrac{dy}{dx} + y = e^x, \quad x > 0$

26. $y' + (\tan x)y = \cos^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

27. $(1+x)y' + y = \sqrt{x}$

28. $e^{2x}y' + 2e^{2x}y = 2x$

29. $x\dfrac{dy}{dx} + 2y = 1 - \dfrac{1}{x}, \quad x > 0$

30. $(t+1)\dfrac{ds}{dt} + 2s = 3(t+1) + \dfrac{1}{(t+1)^2}, \quad t > -1$

31. $\tan\theta \dfrac{dr}{d\theta} + r = \sin^2\theta, \quad 0 < \theta < \frac{\pi}{2}$

32. $y' = \cos x - y\sec x$

33. $\left(1+x^3\right)y' = 3x^2y + x^2 + x^5$

34. $\dfrac{dy}{dt} - 2y = 4 - t$

35. $y' + y = \dfrac{1}{1+e^t}$

36. $y' = 3y - 4$

37. $y' = -2y - 4$

38. $y' = -y + 2$

39. $y' = 2y + 6$

Solve the initial value problem

40. $t\dfrac{dy}{dt} + 2y = t^3, \quad t > 0, \quad y(2) = 1$

41. $\theta\dfrac{dy}{d\theta} + y = \sin\theta, \quad \theta > 0, \quad y\left(\frac{\pi}{2}\right) = 1$

42. $\dfrac{dy}{dx} + xy = x, \quad y(0) = -6$

43. $y' = \dfrac{y}{x}, \quad y(1) = -2$

44. $y' = \dfrac{\sin x}{y}, \quad y\left(\frac{\pi}{2}\right) = 1$

45. $y' = y + 2xe^{2x}; \quad y(0) = 3$

46. $\left(x^2+1\right)y' + 3xy = 6x; \quad y(0) = -1$

47. $y' = \left(4t^3+1\right)y, \quad y(0) = 4$

48. $y' = \dfrac{e^t}{2y}, \quad y(\ln 2) = 1$

49. $(\sec x)y' = y^3, \quad y(0) = 3$

50. $\dfrac{dy}{dx} = e^{x-y}, \quad y(0) = \ln 3$

51. $y' = 2e^{3y-t}, \quad y(0) = 0$

52. $y' = 3y - 6, \quad y(0) = 9$

53. $y' = -y + 2, \quad y(0) = -2$

54. $y' = -2y - 4, \quad y(0) = 0$

Section 2.8 – Applications

Motion with Resistance Proportional to Velocity

An object with a mass ***m*** is moving along a coordinate line with position function ***s*** and velocity ***v*** at time ***t***.

From ***Newton's second law*** of motion, the resistance force opposing the motion is

$$\begin{aligned} Force &= mass \times acceleration \\ &= m\frac{dv}{dt} \end{aligned}$$

If the resisting force is proportional to velocity, we have

$$m\frac{dv}{dt} = -kv \quad or \quad \frac{dv}{dt} = -\frac{k}{m}v \quad (k > 0)$$

This is a separable differential equation, the solution with initial condition $v = v_0 \quad at \quad t = 0$ is

$$v = v_0 e^{-(k/m)t}$$

In the English system, where the weight is measured in pounds, mass is measured in slugs. Thus,

$$Pounds = slugs \times 32$$

Example

For a 192-*lb* ice skater, the ***k*** is about $\frac{1}{3}$ slug/sec and $m = \frac{192}{32} = 6 \ \ slugs$. How long will it take the skater to coast from 11 *ft/sec* (7.5 *mph*) to 1 *ft/sec*? How far will the skater coast before coming to a complete stop?

Solution

$$v = v_0 e^{-(k/m)t} \qquad \frac{k}{m} = \frac{\frac{1}{3}}{6} = \frac{1}{18}$$

$$1 = 11e^{-(1/18)t}$$

$$e^{-(1/18)t} = \frac{1}{11}$$

$$-\frac{1}{18}t = \ln\left(\frac{1}{11}\right) = -\ln 11$$

$$t = 18\ln 11 \approx \underline{43 \ \sec}$$

$$\text{Distance } = \frac{v_0 m}{k} = \frac{11(6)}{1/3} = \underline{198 \ \ ft}$$

Mixtures Problems

The physical representation of the rate of change:

$$\frac{dy}{dt} = \text{rate of change} = \text{rate in - rate out}$$

This is referred to as a ***balance law***.

Rate = Volume Rate (*gal/min*) *x* Concentration (*lb/gal*)

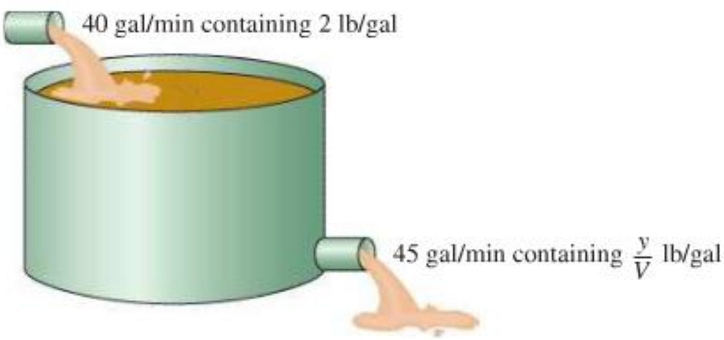
If $y(t)$ is the amount of chemical in the container at time *t* and $V(t)$ is the total volume of liquid in the container at time *t*, then the departure rate of the chemical at time *t* is

$$\text{Departure Rate} = \frac{y(t)}{V(t)} \bullet (\text{outflow rate})$$

$$= \begin{pmatrix} \text{concentration in} \\ \text{container at } \ t \end{pmatrix} \bullet (\text{outflow rate})$$

Example

In an oil refinery, a storage tank contains 2000 *gal* of gasoline that initially has 100 *lb*. of an additive dissolved in it. In preparation for winter weather, gasoline containing 2 *lb*. of additive per gallon is pumped into the tank at a rate of 40 *gal/min*. The well-mixed solution is pumped out at a rate of 45 *gal/min*. How much of the additive is in the tank 20 *min* after the pumping process begins?



Solution

Let *y* be the amount (in *lb*.) of additive in the tank at time *t* and $y(0) = 100$

$$V(t) = 2000 + \left(40\frac{gal}{\text{min}} - 45\frac{gal}{\text{min}}\right)(t \ \text{min})$$

$$= 2000 - 5t$$

$$\text{Rate out} = \frac{y(t)}{V(t)} \bullet \text{outflow rate}$$

$$= \left(\frac{y}{2000-5t}\right)(45)$$

$$= \frac{45y}{2000-5t} \quad \frac{lb}{\text{min}}$$

$$\text{Rate in} = \left(2\ \frac{lb}{gal}\right)\left(40\ \frac{gal}{\min}\right) = \underline{80\ \frac{lb}{\min}}$$

$$\frac{dy}{dt} = 80 - \frac{45y}{2000-5t}$$

$$\frac{dy}{dt} + \frac{45}{2000-5t}y = 80$$

$$v(t) = e^{\int p\,dt} = e^{\int \frac{45}{2000-5t}dt}$$

$$d(2000-5t) = -5dt \Rightarrow dt = -\frac{1}{5}d(2000-5t)$$

$$= e^{\int \frac{45}{2000-5t}\left(-\frac{1}{5}d(2000-5t)\right)}$$

$$= e^{-\int \frac{9}{2000-5t}d(2000-5t)}$$

$$= e^{-9\ln(2000-5t)}$$

$$= (2000-5t)^{-9}$$

$$y = \frac{1}{(2000-5t)^{-9}}\left(\int 80.(2000-5t)^{-9}\,dt + C\right)$$

$$= (2000-5t)^{9}\left(80\int (2000-5t)^{-9}\left(-\frac{1}{5}d(2000-5t)\right) + C\right)$$

$$= (2000-5t)^{9}\left(-16\frac{(2000-5t)^{-8}}{-8} + C\right)$$

$$= (2000-5t)^{9}\left(2(2000-5t)^{-8} + C\right)$$

$$= 2(2000-5t) + C(2000-5t)^{9}$$

$$y(0) = 2(2000-5(0)) + C(2000-5(0))^{9}$$

$$100 = 4000 + C(2000)^{9}$$

$$C(2000)^{9} = -3900$$

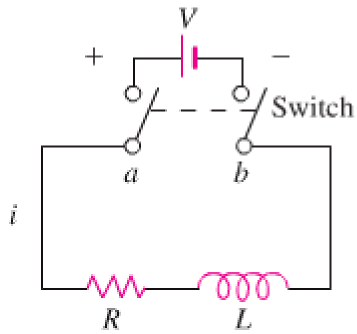
$$C = -\frac{3900}{(2000)^{9}}$$

$$y = 2(2000-5t) - \frac{3900}{(2000)^{9}}(2000-5t)^{9}$$

$$y(20) = 2(2000-5(20)) - \frac{3900}{(2000)^{9}}(2000-5(20))^{9}$$

$$\approx \underline{1342\ lb.}$$

RL Circuits



The diagram represents an electrical circuit whose total resistance is a constant ***R*** ohms and whose self-inductance is coil, is ***L*** henries.

Ohm's Law: $V = RI$

$$L\frac{di}{dt} + Ri = V$$

> ***i***: current in amperes
> ***t***: time in seconds

Example

The switch in the *RL* circuit is closed at time $t = 0$. How will the current flow as a function of time?

Solution

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}$$

$$e^{\int p\,dt} = e^{\int\left(\frac{R}{L}\right)dt} = e^{\frac{R}{L}t}$$

$$i = \frac{1}{e^{\int p\,dt}}\left(\int f.e^{\int p\,dt}dt + C\right)$$

$$= \frac{1}{e^{\frac{R}{L}t}}\left(\int \frac{V}{L}.e^{\frac{R}{L}t}dt + C\right)$$

$$= e^{-\frac{R}{L}t}\left(\frac{V}{L}\frac{L}{R}e^{\frac{R}{L}t} + C\right)$$

$$= e^{-\frac{R}{L}t}\left(\frac{V}{R}e^{\frac{R}{L}t} + C\right)$$

$t = 0 \Rightarrow i = 0 \qquad i = e^{-\frac{R}{L}(0)}\left(\frac{V}{R}e^{\frac{R}{L}(0)} + C\right)$

$$0 = \frac{V}{R} + C$$

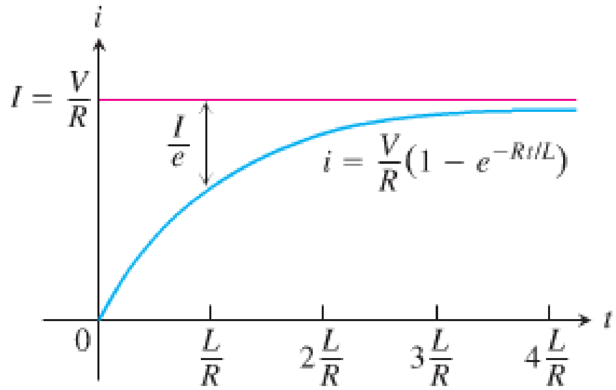
$$\boxed{C = -\frac{V}{R}}$$

$$i(t) = e^{-\frac{R}{L}t}\left(\frac{V}{R}e^{\frac{R}{L}t} - \frac{V}{R}\right)$$

$$\boxed{= \frac{V}{R} - \frac{V}{R}e^{-\frac{R}{L}t}}$$

$$\lim_{t\to\infty} i = \lim_{t\to\infty}\left(\frac{V}{R} - \frac{V}{R}e^{-\frac{R}{L}t}\right) = \frac{V}{R} - \frac{V}{R}\cdot 0 = \boxed{\frac{V}{R}}$$

The current approaches the ***steady-state value*** $\frac{V}{R}$.



Exercises *Section* **2.8 – Applications**

1. A 66-*kg* cyclist on a 7-*kg* bicycle starts coasting on level ground at 9 *m/sec*. The $k \approx 3.9\ kg/\sec$

a) About how far will the cyclist coast before reaching a complete stop?

b) How long will it take the cyclist's speed to drop to 1 *m/sec*?

2. Suppose that an Iowa class battleship has mass 51,000 metric tons (51,000,000 *kg*) and $k \approx 59{,}000\ kg/\sec$. Assume that the ship loses power when it is moving at a speed of 9 *m/sec*.

a) About how far will the ship coast before it is dead in the water?

b) About how long will it take the ship's speed to drop to 1 *m/sec*?

3. A 200-*gal* tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 *lb./gal* of concentrate enters the tank at the rate of 5 *gal/min*, and the well-stirred mixture is withdrawn at the rate of 3 *gal/min*.

a) At what time will the tank be full?

b) At the time the tank is full, how many pounds of concentrate will it contain?

4. A tank contains 100 *gal* of fresh water. A solution containing 1 *lb./gal* of soluble lawn fertilizer runs into the tank at the rate of 1 *gal/min*, and the mixture is pumped out of the tank at a rate of 3 *gal/min*. Find the maximum amount of fertilizer in the tank and the time required to reach the maximum.

5. An Executive conference room of a corporation contains 4500 ft^3 of air initially free of carbon monoxide. Starting at time $t = 0$, cigarette smoke containing 4% carbon monoxide is blown into the room at the rate of 0.3 $ft^3/\min$. A ceiling fan keeps the air in the room well circulated and the air leaves the room at the same rate of 0.3 $ft^3/\min$. Find the time when the concentration of carbon monoxide in the room reaches 0.01%.

6. Many chemical reactions are the result of the interaction of 2 molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If *a* is the amount of substance *A* and *b* is the substance *B* at time $t = 0$, and if *x* is the amount of product at time *t*, then the rate of formation of *x* may be given by the differential equation

$$\frac{dx}{dt} = k(a-x)(b-x) \quad \text{or} \quad \frac{1}{(a-x)(b-x)}\frac{dx}{dt} = k$$

Where *k* is a constant for the reaction. Integrate both sides of this equation to obtain a relation between *x* and *t*.

a) If $a = b$

b) If $a \neq b$

Assume in each case that $x = 0$ when $t = 0$

7. The tank initially holds 100 *gal* of pure water. At time $t = 0$, a solution containing 2 *lb* of salt per gallon begins to enter the tank at the rate of 3 gallons per minute. At the same time a drain is opened at the bottom of the tank so that the volume of solution in the tank remains constant.
How much salt is in the tank after 60 *min*?
What will be the eventual salt content in the tank?

8. The 600-*gal* tank is filled with 300 *gal* of pure water. A spigot is opened above the tank and a salt solution containing 1.5 *lb*. of salt per gallon of solution begins flowing into the tank at the rate of 3 *gal/min*. Simultaneously, a drain is opened at the bottom of the tank allowing the solution to leave tank at a rate of 1 *gal/min*. What will be the salt content in the tank at the precise moment that the volume of solution in the tank is equal to the tank's capacity (600 *gal*)?

9. The amount of drug in the blood of a patient (in *mg*) due to an intravenous line is governed by the initial value problem

$$y'(t) = -0.02y + 3, \quad y(0) = 0 \ \ for\ t \ge 0$$

Where *t* is measured in hours

a) Find and graph the solution of the initial value problem.

b) What is the steady-state level of the drug?

c) When does the drug level reach 90% of the steady-state value?

10. A fish hatchery has 500 *fish* at time $t = 0$, when harvesting begins at a rate of *b fish/yr*. where $b > 0$. The fish population is modeled by the initial value problem.

$$y'(t) = 0.1y - b, \quad y(0) = 500 \qquad for \ \ t \ge 0$$

Where *t* is measured in years.

a) Find the fish population for $t \ge 0$ in terms of the harvesting rate *b*.

b) Graph the solution in the case that $b = 40\ fish/yr$. Describe the solution.

c) Graph the solution in the case that $b = 60\ fish/yr$. Describe the solution.

11. A community of hares on an island has a population of 50 when observations begin at $t = 0$. The population for $t \ge 0$ is modeled by the initial value problem.

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{200}\right), \quad P(0) = 50$$

a) Find the solution of the initial value problem.

b) What is the steady-state population?

12. When an infected person is introduced into a closed and otherwise healthy community, the number of people who become infected with the disease (in the absence of any intervention) may be modeled by the logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{A}\right), \quad P(0) = P_0$$

Where *k* is a positive infection rate, *A* is the number of people in the community, and P_0 is the number of infected people at $t = 0$. The model assumes no recovery or intervention.

a) Find the solution of the initial value problem in terms of k, A, and P_0.

b) Graph the solution in the case that $k = 0.025, \quad A = 300, \quad and \ P_0 = 1$.

c) For fixed values of k and A, describe the long-term behavior of the solutions for any P_0 with $0 < P_0 < A$

13. An object in free fall may be modeled by assuming that the only forces at work are the gravitational force and resistance (friction due to the medium in which the objects falls). By Newton's second law (mass × acceleration = the sum of the external forces), the velocity of the object satisfies the differential equation

$$\underset{mass}{m} \cdot \underset{acceleration}{v'(t)} = \underbrace{mg + f(v)}_{external\ force}$$

Where f is a function that models the resistance and the positive direction is downward. One common assumption (often used for motion in air) is that $f(v) = -kv^2$, where $k > 0$ is a drag coefficient.

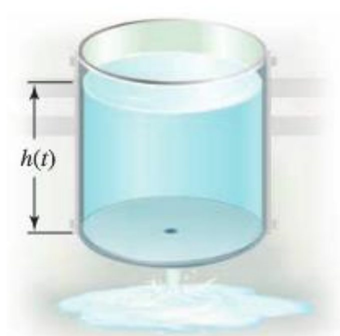
a) Show that the equation can be written in the form $v'(t) = g - av^2$ where $a = \frac{k}{m}$

b) For what (positive) value of v is $v'(t) = 0$? (This equilibrium solution is called the ***terminal velocity***.)

c) Find the solution of this separable equation assuming $v(0) = 0 \ and \ 0 < v(t)^2 < \frac{g}{a}$ for $t \ge 0$

d) Graph the solution found in part (***c***) with $g = 9.8\ m/s^2, \ m = 1\ kg, \ and \ k = 0.1\ kg/m$, and verify the terminal velocity agrees with the value found in part (***b***).

14. An open cylindrical tank initially filled with water drains through a hole in the bottom of the tank according to Torricelli's Law. If $h(t)$ is the depth of water in the tank for $t \ge 0$, then Torricelli's Law implies $h'(t) = -2k\sqrt{h}$, where k is a constant that includes the acceleration due to gravity, the radius of the tank, and the radius of the drain. Assume that the initial depth of the water is $h(0) = H$



a) Find the solution of the initial value problem.

b) Find the solution in the case that $k = 0.1$ and $H = 0.5\ m$.

c) In general, how long does it take the tank to drain in terms of k and H?

15. The reaction of chemical compounds can often be modeled by differential equations. Let $y(t)$ be the concentration of a substance in reaction for $t \geq 0$ (typical units of y are *moles/L*). The change in the concentration of a substance, under appropriate conditions, is $\frac{dy}{dt} = -ky^n$, where $k > 0$ is a rate constant and the positive integer n is the order of the reaction.

a) Show that for a first-order reaction $(n=1)$, the concentration obeys an exponential decay law.

b) Solve the initial value problem for a second-order reaction $(n=2)$ assuming $y(0) = y_0$

c) Graph and compare the concentration for a first-order and second-order reaction with $k = 0.1$ and $y_0 = 1$

16. The growth of cancer turmors may be modeled by the Gomperts growth equation. Let $M(t)$ be the mass of the tumor for $t \geq 0$. The relevant intial value problem is

$$\frac{dM}{dt} = -aM \ln\frac{M}{K}, \quad M(0) = M_0$$

Where a and K are positive constants and $0 < M_0 < K$

a) Graph the growth rate function $R(M) = -aM \ln\frac{M}{K}$ assuming $a = 1$ and $K = 4$. For what values of M is the growth rate positive? For what values of M is maximum?

b) Solve the initial value problem and graph the solution for $a = 1$, $K = 4$, and $M_0 = 1$. Describe the growth pattern of the tumor. Is the growth unbounded? If not, what is the limiting size of the tumor?

c) In the general equation, what is the meaning of K?

17. An endowment is an investment account in which the balance ideally remains constant and withdrawals are made on the interest earned by the account. Such an account may be modeled by the initial value problem $B'(t) = aB - m$ for $t \geq 0$, with $B(0) = B_0$. The constant a reflects the annual interest rate, m is the annual rate of withdrawal, and B_0 is the initial balance in the account.

a) Solve the initial value problem with $a = 0.05$, $m = \$1000/yr.$ and $B_0 = \$15{,}000$. Does the balance in the account increase or decrease?

b) If $a = 0.05$ and $B_0 = \$50{,}000$, what is the annual withdrawal rate m that ensures a constant balance in the account? What is the constant balance?

18. The halibut fishery has been modeled by the differential equation $\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right)$

Where $y(t)$ is the biomass (the total mass of the members of the population) in kilograms at time t (measured in years), the carrying capacity is estimated to be $M = 8 \times 10^7\ kg$ and $k = 0.71\ per\ year$.

a) If $y(0) = 2 \times 10^7\ kg$, find the biomass a year later.

b) How long will it take for the biomass to reach $4 \times 10^7\ kg$.

19. Suppose a population $P(t)$ satisfies $\frac{dP}{dt} = 0.4P - 0.001P^2, \quad P(0) = 50$ where *t* is measured in *years*.

a) What is the carrying capacity?

b) What is $P'(0)$?

c) When will the population reach 50% of the carrying capacity?

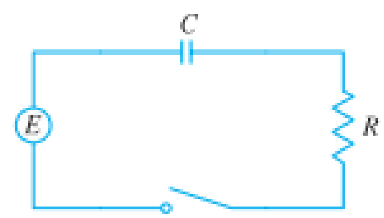
20. Let $P(t)$ be the performance level of someone learning a skill as a function of the training time *t*. The graph of *P* is called a ***learning curve***. We proposed the differential equation

$$\frac{dP}{dt} = k\left(M - P(t)\right)$$

As a reasonable model for learning, where *k* is a positive constant. Solve it as a linear differential equation and use your solution to graph the learning curve.

21. A circuit containing an electromotive force, a capacitor with a capacitance of *C* farads (*F*), and a resistor with a resistance of *R* ohms (Ω). The voltage drop across the capacitor is $\frac{Q}{C}$, where *Q* is the charge (in coulombs), so in this case ***Kirchhoff's Law*** gives

$$RI + \frac{Q}{C} = E(t)$$



But $I = \frac{dQ}{dt}$, so we have $\quad R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$

Find the charge and the current at time *t*

a) Suppose the resistance is $5\ \Omega$, the capacitance is 0.05 *F*, a battery gives voltage of 60 *V* and initial charge is $Q(0) = 0\ C$

b) Suppose the resistance is $2\ \Omega$, the capacitance is 0.01 *F*, $E(t) = 10\sin 60t$ and initial charge is $Q(0) = 0\ C$

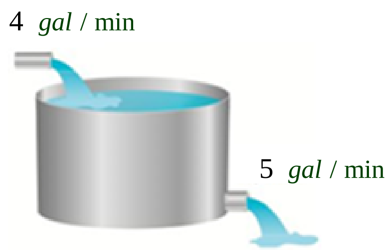
22. A 30–volt electromotive force is applied to an *LR*-series circuit in which the inductance is 0.1 *henry* and the resistance is 50 *ohms*.

a) Find the current $i(t)$ if $i(0) = 0$

b) Determine the current as $t \to \infty$

c) Solve the equation when $E(t) = E_0 \sin \omega t$ and $i(0) = i_0$

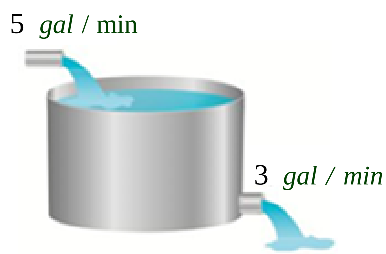
23. A tank contains 50 *gallons* of a solution composed of 90% water and 10% alcohol. A second solution containing 50% water and 50% alcohol is added to the tank at the rate of $4\ gal/\min$. As the second solution is being added, the tank is being drained at a rate of $5\ gal/\min$. The solution in the tank is stirred constantly. How much alcohol is in the tank after 10 *minutes*?



24. A 200-*gallon* tank is half full of distilled water. At time $t = 0$, a concentrate solution containing 0.5 *lb/gal* enters the tank at the rate of 5 *gal* / min, and well-stirred mixture is withdrawn at the rate of 3 *gal* / min.

a) At what time will the tank be full?

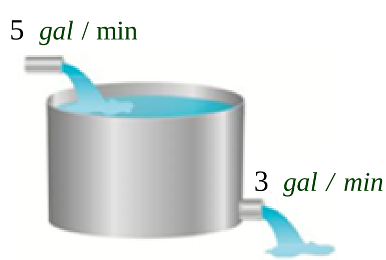
b) At the time the tank is full, how many pounds of concentrate will it contain?



25. A 200-*gallon* tank is half full of distilled water. At time $t = 0$, a concentrate solution containing 1 *lb/gal* enters the tank at the rate of 5 *gal* / min, and well-stirred mixture is withdrawn at the rate of 3 *gal* / min.

a) At what time will the tank be full?

b) At the time the tank is full, how many pounds of concentrate will it contain?



26. A 200-*gallon* tank is full of a concentrate solution containing 25 *lb*. Starting at time $t = 0$, distilled water is admitted to the tank at the rate of 10 *gal* / min, and well-stirred mixture is withdrawn at the same rate.

a) Find the amount of concentrate in the solution as a function of *t*.

b) Find the time at which the amount of concentrate in the tank reaches 15 pounds.

c) Find the quantity of the concentrate in the solution as $t \to \infty$.