

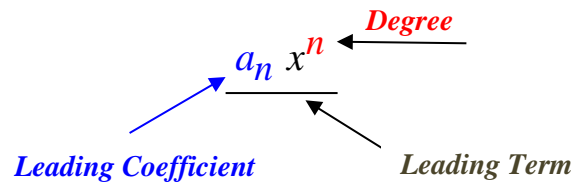
Section 1.3 – Polynomial Functions & Graphs

Polynomial Function

A Polynomial function $P(x)$ in x is a sum of the form is given by:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x + a_0$$

Where the coefficients $a_n, a_{n-1}, \dots, a_2, a_1, a_0$ are real numbers and the exponents are whole numbers.



Degree of f	Form of $f(x)$	Graph of $f(x)$
0	$f(x) = a_0$	A horizontal line
1	$f(x) = a_1 x + a_0$	A line with slope a_1
2	$f(x) = a_2 x^2 + a_1 x + a_0$	A parabola with a vertical axis

All polynomial functions are *continuous functions*.

End Behavior ($a_n x^n$)

If n (degree) is **even**:

$$\text{If } a_n < 0 \rightarrow \begin{cases} x \rightarrow -\infty \Rightarrow f(x) \rightarrow -\infty \\ x \rightarrow \infty \Rightarrow f(x) \rightarrow -\infty \end{cases}$$

$$\text{If } a_n > 0 \rightarrow \begin{cases} x \rightarrow -\infty \Rightarrow f(x) \rightarrow \infty \\ x \rightarrow \infty \Rightarrow f(x) \rightarrow \infty \end{cases}$$

If n (degree) is **odd**:

$$\text{If } a_n < 0 \rightarrow \begin{cases} x \rightarrow -\infty \Rightarrow f(x) \rightarrow \infty \\ x \rightarrow \infty \Rightarrow f(x) \rightarrow -\infty \end{cases}$$

$$\text{If } a_n > 0 \rightarrow \begin{cases} x \rightarrow -\infty \Rightarrow f(x) \rightarrow -\infty \\ x \rightarrow \infty \Rightarrow f(x) \rightarrow \infty \end{cases}$$

The intermediate value *Theorem*

For any polynomial function $f(x)$ with real coefficients and $f(a) \neq f(b)$ for $a < b$, then f takes on every value between $f(a)$ and $f(b)$ in the interval $[a, b]$.

$\therefore f(a)$ and $f(b)$ are the opposite signs. Then the function has a real zero between a and b .

Example

Using the intermediate value theorem, determine, if possible, whether the function has a real zero between a and b .

a) $f(x) = x^3 + x^2 - 6x$; $a = -4$, $b = -2$

b) $f(x) = x^3 + x^2 - 6x$; $a = -1$, $b = 3$

Solution

a) $f(x) = x^3 + x^2 - 6x$; $a = -4$, $b = -2$

$$f(-4) = (-4)^3 + (-4)^2 - 6(-4) \\ = -24$$

$$f(-2) = (-2)^3 + (-2)^2 - 6(-2) \\ = 8$$

$\therefore f(x)$ has a zero between -4 and -2 .

b) $f(x) = x^3 + x^2 - 6x$; $a = -1$, $b = 3$

$$f(-1) = (-1)^3 + (-1)^2 - 6(-1) = 6$$

$$f(3) = (3)^3 + (3)^2 - 6(3) = 18$$

Can't be determined.

The Rational Zeros *Theorem*

If the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ has integer coefficients, then

$$\text{possible rational zeros} = \frac{\text{possibilities for } a_0}{\text{possibilities for } a_n}$$

Example

Find all rational solutions of the equation: $3x^4 + 14x^3 + 14x^2 - 8x - 8 = 0$

Solution

$$\begin{aligned} \text{Possibilities: } \pm \left\{ \frac{8}{3} \right\} &= \pm \left\{ \frac{1, 2, 4, 8}{1, 3} \right\} \\ &= \pm \left\{ 1, 2, 4, 8, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3} \right\} \end{aligned}$$

The calculation will show that -2 is a zero.

$$\begin{array}{r|rrrrr} -2 & 3 & 14 & 14 & -8 & -8 \\ & & -6 & -16 & 4 & 8 \\ \hline -\frac{2}{3} & 3 & 8 & -2 & -4 & 0 \\ & & -2 & -4 & 4 & \\ \hline & 3 & 6 & -6 & 0 & \end{array} \rightarrow 3x^3 + 8x^2 - 2x - 4 \rightarrow \pm \left\{ \frac{4}{3} \right\} = \pm \left\{ 1, 2, 4, \frac{1}{3}, \frac{2}{3}, \frac{4}{3} \right\}$$

$$\rightarrow 3x^2 + 6x - 6 = 0 \Rightarrow x = -1 \pm \sqrt{3}$$

Hence, the polynomial has roots $x = -2, -\frac{2}{3}, -1 \pm \sqrt{3}$

Sketching

Example

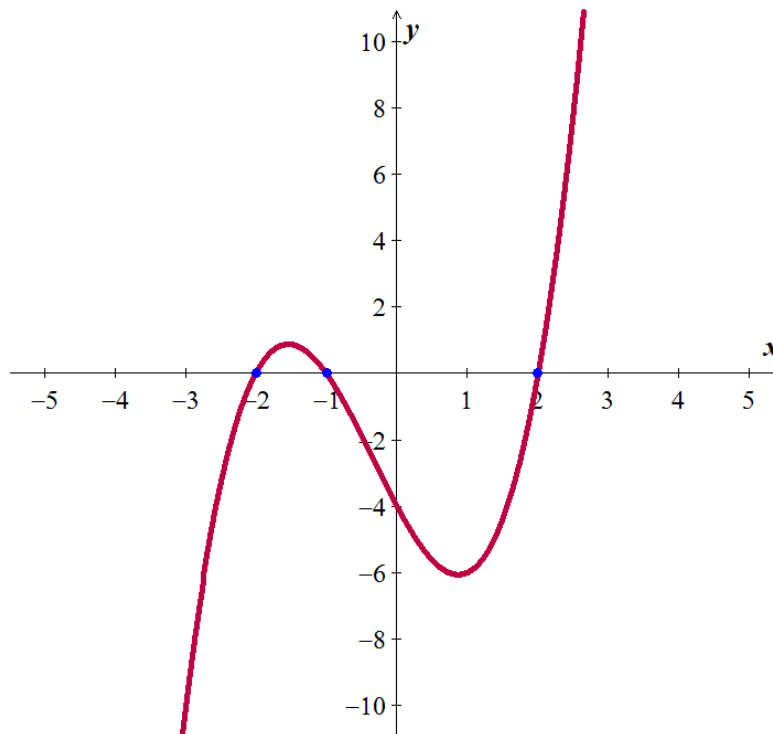
Let $f(x) = x^3 + x^2 - 4x - 4$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f .

Solution

$$\begin{aligned}f(x) &= x^3 + x^2 - 4x - 4 \\&= x^2(x+1) - 4(x+1) \\&= (x+1)(x^2 - 4) \\&= (x+1)(x+2)(x-2)\end{aligned}$$

The zeros of $f(x)$ (x -intercepts) are: -2 , -1 , and 2

<i>Interval</i>	$-\infty$	-2	-1	0	2	∞
Sign of $f(x)$		−	+		−	+
Position		Below x-axis	Above x-axis		Below x-axis	Above x-axis



We can conclude from the chart and the graph that:

$$f(x) > 0 \quad \text{if } x \text{ is in } (-2, -1) \cup (2, \infty)$$

$$f(x) < 0 \quad \text{if } x \text{ is in } (-\infty, -2) \cup (-1, 2)$$

Example

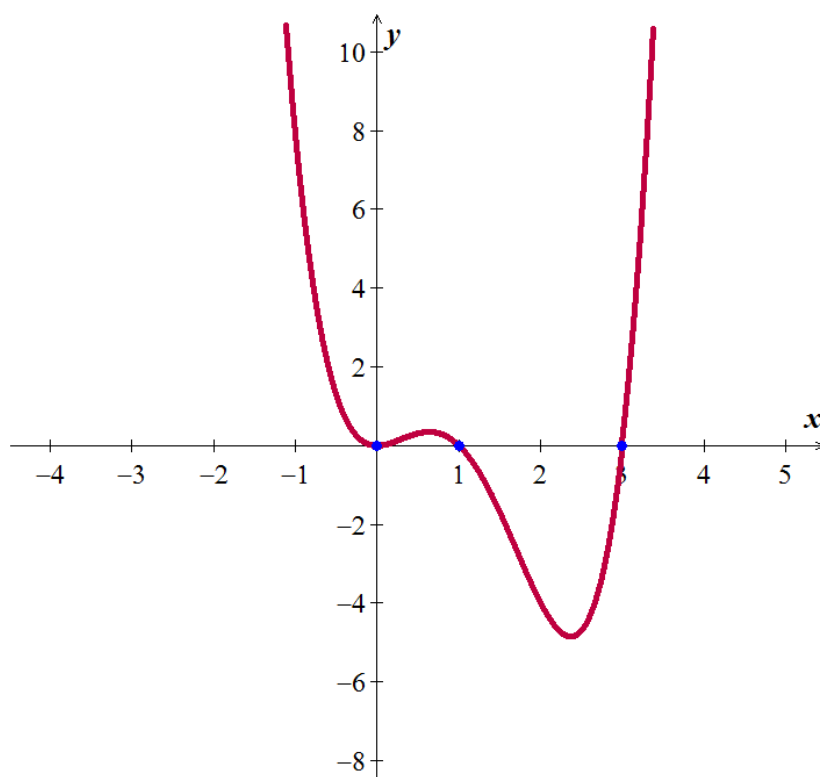
Let $f(x) = x^4 - 4x^3 + 3x^2$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f .

Solution

$$\begin{aligned} f(x) &= x^2(x^2 - 4x + 3) \\ &= x^2(x-1)(x-3) \end{aligned}$$

The zeros are: 0, 1, 3. Since the factor x^2 is always positive, it has no factor

$-\infty$	1	2	3	∞
+		-		+



$$f(x) > 0 \Rightarrow x \text{ is in } (-\infty, 0) \cup (0, 1) \cup (3, \infty)$$

$$f(x) < 0 \Rightarrow x \text{ is in } (1, 3)$$

Exercises Section 1.3 – Polynomial Functions & Graphs

(1 – 4) Find the quotient and remainder if $f(x)$ is divided by $p(x)$

1. $f(x) = 2x^4 - x^3 + 7x - 12$; $p(x) = x^2 - 3$ 3. $f(x) = 7x + 2$; $p(x) = 2x^2 - x - 4$
2. $f(x) = 3x^3 + 2x - 4$; $p(x) = 2x^2 + 1$ 4. $f(x) = 9x + 4$; $p(x) = 2x - 5$

(5 – 6) Use the remainder theorem to find $f(c)$

5. $f(x) = x^4 - 6x^2 + 4x - 8$; $c = -3$ 6. $f(x) = x^4 + 3x^2 - 12$; $c = -2$

7. Use the factor theorem to show that $x - c$ is a factor of $f(x)$: $f(x) = x^3 + x^2 - 2x + 12$; $c = -3$

(8 – 10) Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second

8. $2x^3 - 3x^2 + 4x - 5$; $x - 2$ 10. $9x^3 - 6x^2 + 3x - 4$; $x - \frac{1}{3}$
9. $5x^3 - 6x^2 + 15$; $x - 4$

(11 – 13) Use the synthetic division to find $f(c)$

11. $f(x) = 2x^3 + 3x^2 - 4x + 4$; $c = 3$ 13. $f(x) = x^3 - 3x^2 - 8$; $c = 1 + \sqrt{2}$
12. $f(x) = 8x^5 - 3x^2 + 7$; $c = \frac{1}{2}$

14. Use the synthetic division to show that c is a zero of $f(x)$: $f(x) = 3x^4 + 8x^3 - 2x^2 - 10x + 4$; $c = -2$

15. Use the synthetic division to show that c is a zero of $f(x)$: $f(x) = 27x^4 - 9x^3 + 3x^2 + 6x + 1$; $c = -\frac{1}{3}$

(16 – 18) Find all values of k such that $f(x)$ is divisible by the given linear polynomial:

16. $f(x) = kx^3 + x^2 + k^2x + 3k^2 + 11$; $x + 2$
17. $f(x) = x^3 + k^3x^2 + 2kx - 2k^4$; $x - 1.6$
18. $f(x) = k^2x^3 - 4kx + 3$; $x - 1$

(19 – 30) Find all solutions of the equation

19. $x^3 - x^2 - 10x - 8 = 0$ 23. $x^4 + 3x^3 - 30x^2 - 6x + 56 = 0$
20. $x^3 + x^2 - 14x - 24 = 0$ 24. $3x^5 - 10x^4 - 6x^3 + 24x^2 + 11x - 6 = 0$
21. $2x^3 - 3x^2 - 17x + 30 = 0$ 25. $6x^5 + 19x^4 + x^3 - 6x^2 = 0$
22. $12x^3 + 8x^2 - 3x - 2 = 0$ 26. $x^4 - x^3 - 9x^2 + 3x + 18 = 0$

$$27. \quad 2x^4 - 9x^3 + 9x^2 + x - 3 = 0$$

$$29. \quad 3x^3 - x^2 + 11x - 20 = 0$$

$$28. \quad 8x^3 + 18x^2 + 45x + 27 = 0$$

$$30. \quad 6x^4 + 5x^3 - 17x^2 - 6x = 0$$

31. If $f(x) = 3x^3 - kx^2 + x - 5k$, find a number k such that the graph of f contains the point $(-1, 4)$.

32. If $f(x) = kx^3 + x^2 - kx + 2$, find a number k such that the graph of f contains the point $(2, 12)$.

33. If one zero of $f(x) = x^3 - 2x^2 - 16x + 16k$ is 2, find two other zeros.

34. If one zero of $f(x) = x^3 - 3x^2 - kx + 12$ is -2 , find two other zeros.

35. Find a polynomial $f(x)$ of degree 3 that has the zeros $-1, 2, 3$; and satisfies the given condition:
 $f(-2) = 80$

36. Find a polynomial $f(x)$ of degree 3 that has the zeros $-2i, 2i, 3$; and satisfies the given condition:
 $f(1) = 20$

37. Find a polynomial $f(x)$ of degree 4 with leading coefficient 1 such that both -4 and 3 are zeros of multiplicity 2, and sketch the graph of f .

(38 – 43) Find the zeros of the following functions and state the multiplicity of each zero

$$38. \quad f(x) = x^2(3x + 2)(2x - 5)^3$$

$$41. \quad f(x) = (6x^2 + 7x - 5)^4(4x^2 - 1)^2$$

$$39. \quad f(x) = 4x^5 + 12x^4 + 9x^3$$

$$42. \quad f(x) = x^4 + 7x^2 - 144$$

$$40. \quad f(x) = (x^2 + x - 12)^3(x^2 - 9)^2$$

$$43. \quad f(x) = x^4 + 21x^2 - 100$$

(44 – 102) Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f

$$44. \quad f(x) = x^4 - 4x^2$$

$$51. \quad f(x) = x^3 + 2x^2 - 5x - 6$$

$$45. \quad f(x) = x^4 + 3x^3 - 4x^2$$

$$52. \quad f(x) = x^3 + 8x^2 + 11x - 20$$

$$46. \quad f(x) = x^3 + 2x^2 - 4x - 8$$

$$53. \quad f(x) = x^4 + x^2 - 2$$

$$47. \quad f(x) = x^3 - 3x^2 - 9x + 27$$

$$54. \quad f(x) = x^4 - x^3 - 6x^2 + 4x + 8$$

$$48. \quad f(x) = -x^4 + 12x^2 - 27$$

$$55. \quad f(x) = 4x^5 - 8x^4 - x + 2$$

$$49. \quad f(x) = x^2(x + 2)(x - 1)^2(x - 2)$$

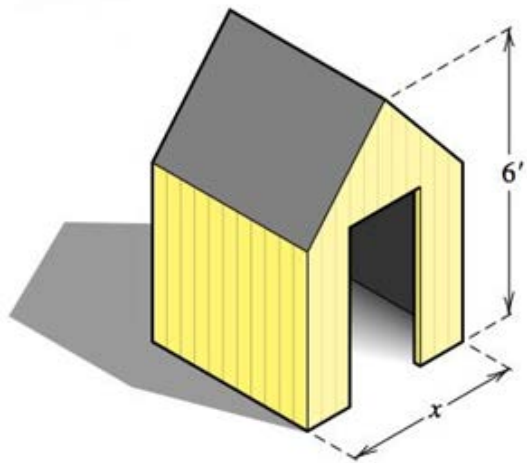
$$56. \quad f(x) = 2x^4 - x^3 - 5x^2 + 2x + 2$$

$$50. \quad f(x) = 2x^3 + 11x^2 - 7x - 6$$

$$57. \quad f(x) = x^3 - x^2 - 10x - 8$$

58. $f(x) = x^3 + x^2 - 14x - 24$
59. $f(x) = 2x^3 - 3x^2 - 17x + 30$
60. $f(x) = 12x^3 + 8x^2 - 3x - 2$
61. $f(x) = x^3 + x^2 - 6x - 8$
62. $f(x) = x^3 - 19x - 30$
63. $f(x) = 2x^3 + x^2 - 25x + 12$
64. $f(x) = 3x^3 + 11x^2 - 6x - 8$
65. $f(x) = 2x^3 + 9x^2 - 2x - 9$
66. $f(x) = x^3 + 3x^2 - 6x - 8$
67. $f(x) = 3x^3 - x^2 - 6x + 2$
68. $f(x) = x^3 - 8x^2 + 8x + 24$
69. $f(x) = x^3 - 7x^2 - 7x + 69$
70. $f(x) = x^3 - 3x - 2$
71. $f(x) = x^3 - 2x + 1$
72. $f(x) = x^3 - 2x^2 - 11x + 12$
73. $f(x) = x^3 - 2x^2 - 7x - 4$
74. $f(x) = x^3 - 10x - 12$
75. $f(x) = x^3 - 5x^2 + 17x - 13$
76. $f(x) = 6x^3 + 25x^2 - 24x + 5$
77. $f(x) = 8x^3 + 18x^2 + 45x + 27$
78. $f(x) = 3x^3 - x^2 + 11x - 20$
79. $f(x) = x^4 - x^3 - 9x^2 + 3x + 18$
80. $f(x) = 2x^4 - 9x^3 + 9x^2 + x - 3$
81. $f(x) = 6x^4 + 5x^3 - 17x^2 - 6x$
82. $f(x) = x^4 - 2x^2 - 16x - 15$
83. $f(x) = x^4 - 2x^3 - 5x^2 + 8x + 4$
84. $f(x) = 2x^4 - 17x^3 + 4x^2 + 35x - 24$
85. $f(x) = x^4 + x^3 - 3x^2 - 5x - 2$
86. $f(x) = 6x^4 - 17x^3 - 11x^2 + 42x$
87. $f(x) = x^4 - 5x^2 - 2x$
88. $f(x) = 3x^4 - 4x^3 - 11x^2 + 16x - 4$
89. $f(x) = 6x^4 + 23x^3 + 19x^2 - 8x - 4$
90. $f(x) = 4x^4 - 12x^3 + 3x^2 + 12x - 7$
91. $f(x) = 2x^4 - 9x^3 - 2x^2 + 27x - 12$
92. $f(x) = 2x^4 - 19x^3 + 51x^2 - 31x + 5$
93. $f(x) = 4x^4 - 35x^3 + 71x^2 - 4x - 6$
94. $f(x) = 2x^4 + 3x^3 - 4x^2 - 3x + 2$
95. $f(x) = x^4 + 3x^3 - 30x^2 - 6x + 56$
96. $f(x) = 3x^5 - 10x^4 - 6x^3 + 24x^2 + 11x - 6$
97. $f(x) = 6x^5 + 19x^4 + x^3 - 6x^2$
98. $f(x) = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$
99. $f(x) = x^5 - x^4 - 7x^3 + 7x^2 + 12x - 12$
100. $f(x) = x^5 - 2x^3 - 8x$
101. $f(x) = x^5 - 7x^4 + 19x^3 - 37x^2 + 60x - 36$
102. $f(x) = 3x^6 - 10x^5 - 29x^4 + 34x^3 + 50x^2 - 24x - 24$

- 103.** A storage shelter is to be constructed in the shape of a cube with a triangular prism forming the roof. The length x of a side of the cube is yet to be determined.

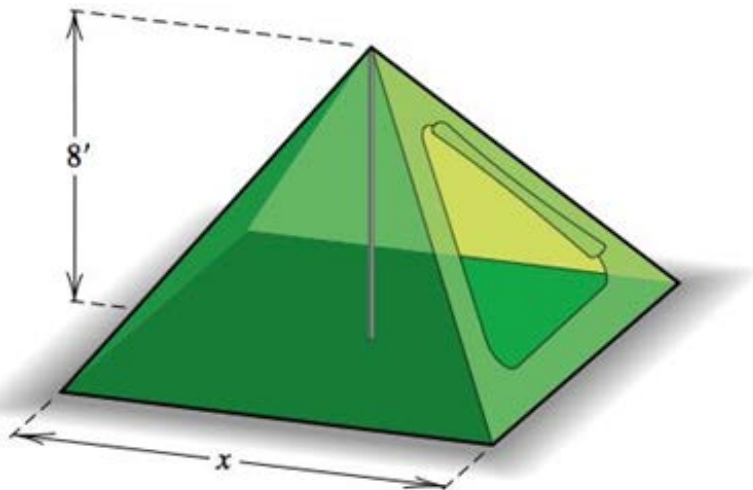


- a) If the total height of the structure is 6 feet, show that its volume V is given by

$$V = x^3 + \frac{1}{2}x^2(6 - x)$$

- b) Determine x so that the volume is 80 ft^3

- 104.** A canvas camping tent is to be constructed in the shape of a pyramid with a square base. An 8-foot pole will form the center support. Find the length x of a side of the base so that the total amount of canvas needed for the sides and bottom is 384 ft^2



- 105.** Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T(k) = \frac{1}{6}(k^3 + 3k^2 + 2k)$$



Where k is the number of levels in the pyramid. If 220 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

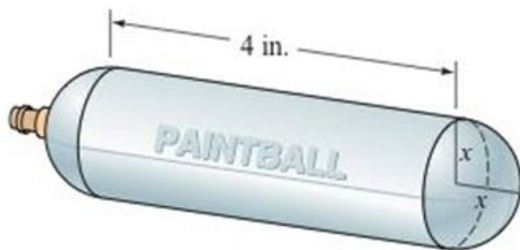
- 106.** Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T(k) = \frac{1}{6}(2k^3 + 3k^2 + k)$$



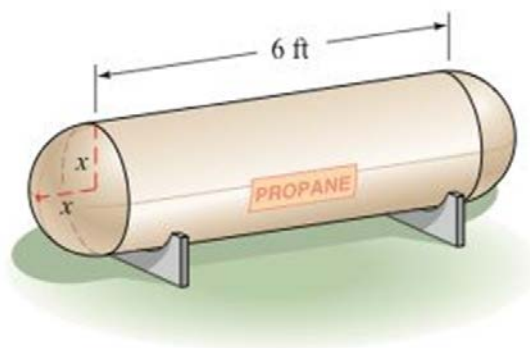
Where k is the number of levels in the pyramid. If 140 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

- 107.** A carbon dioxide cartridge for a paintball rifle has the shape of a right circular cylinder with a hemisphere at each end. The cylinder is 4 *inches* long, and the volume of the cartridge is $2\pi \text{ in}^3$.

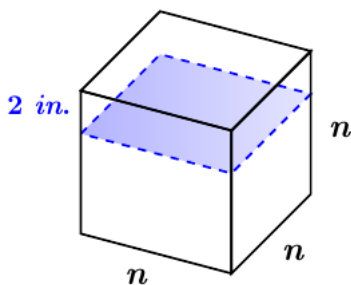


The common interior radius of the cylinder and the hemispheres is denoted by x . Estimate the length of the radius x .

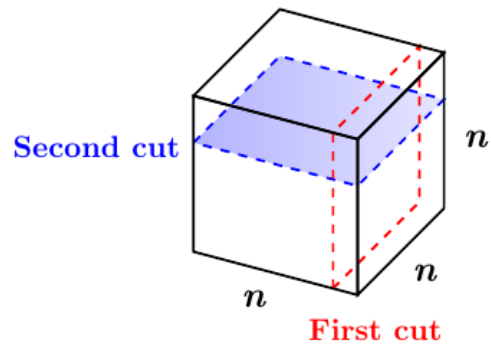
- 108.** A propane tank has the shape of a circular cylinder with a hemisphere at each end. The cylinder is 6 *feet* long and the volume of the tank is $9\pi \text{ ft}^3$. Find the length of the radius x .



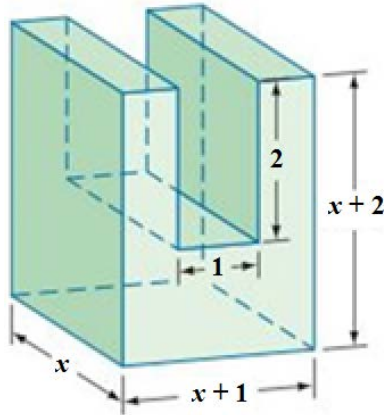
- 109.** A cube measures n inches on each edge. If a slice 2 *inches* thick is cut from one face of the cube, the resulting solid has a volume of 567 in^3 . Find n .



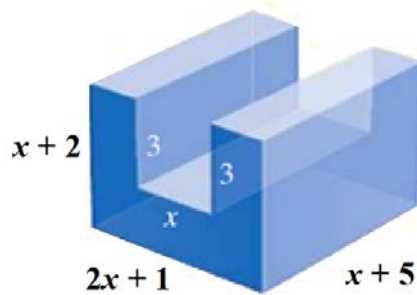
- 110.** A cube measures n inches on each edge. If a slice 1 *inch* thick is cut from one face of the cube and then a slice 3 inches thick is cut from another face of the cube, the resulting solid has a volume of 1560 in^3 . Find the dimensions of the original cube.



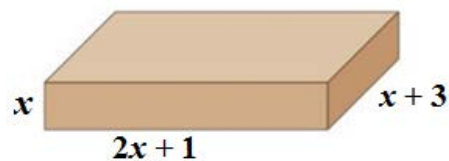
111. For what value of x will the volume of the following solid be 112 in^3



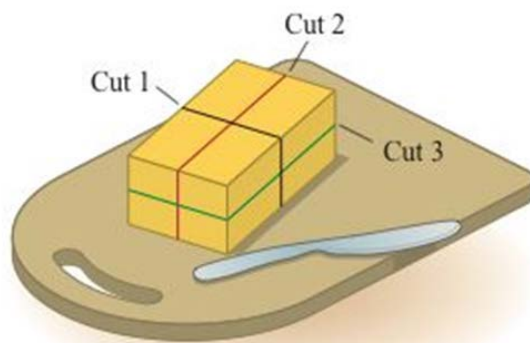
112. For what value of x will the volume of the following solid be 208 in^3



113. The length of rectangular box is 1 inch more than twice the height of the box, and the width is 3 inches more than the height. If the volume of the box is 126 in^3 , find the dimensions of the box.



- 114.** One straight cut through a thick piece of cheese produces two pieces. Two straight cuts can produce a maximum of four pieces. Two straight cuts can produce a maximum of four pieces. Three straight cuts can produce a maximum of eight pieces.

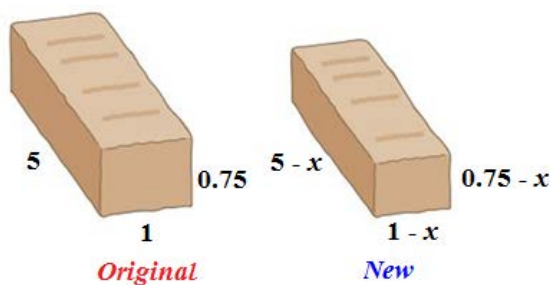


You might be inclined to think that every additional cut doubles number of pieces. However, for four straight cuts, you get a maximum of 15 pieces. The maximum number of pieces P that can be produced by n straight cuts is given by

$$P(n) = \frac{n^3 + 5n + 6}{6}$$

- a) Determine number of pieces that can be produced by five straight cuts.
 b) What is the fewest number of straight cuts that are needed to produce 64 pieces?
- 115.** The number of ways one can select three cards from a group of n cards (the order of the selection matters), where $n \geq 3$, is given by $P(n) = n^3 - 3n^2 + 2n$. For a certain card trick, a magician has determined that there are exactly 504 ways to choose three cards from a given group. How many cards are in the group?

- 116.** A nutrition bar in the shape of a rectangular solid measure 0.75 in. by 1 in. by 5 inches.



To reduce costs, the manufacturer has decided to decrease each of the dimensions of the nutrition bar by x inches, what value of x will produce a new bar with a volume that is 0.75 in^3 less than the present bar's volume.

- 117.** A rectangular box is square on two ends and has length plus girth of 81 *inches*. (Girth: distance *around* the box). Determine the possible lengths l ($l > w$) of the box if its volume is 4900 in^3 .

