

Solution

Section 4.3 – Multiplicative Inverses of Matrices

Exercise

Show that B is Multiplicative inverse of A

$$A = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

Solution

$$\begin{aligned} AB &= \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \\ &= \begin{pmatrix} -6 & \\ & \end{pmatrix} \\ &\neq I \end{aligned}$$

B is not multiplicative inverse of A

Exercise

Show that B is Multiplicative inverse of A

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \quad \& \quad B = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

Solution

$$\begin{aligned} AB &= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I \end{aligned}$$

$\therefore B$ is Multiplicative inverse of A

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$

Solution

$$\begin{aligned} A^{-1} &= \frac{1}{-4+6} \begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix} \end{aligned}$$

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$

Solution

$$\begin{aligned} A^{-1} &= \frac{1}{10-10} \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix} \\ &= \frac{1}{0} \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix} \end{aligned}$$

\therefore Inverse *doesn't exist*

Exercise

Find the inverse of $A = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$

Solution

$$\begin{array}{l} \left[\begin{array}{cc|cc} -2 & 3 & 1 & 0 \\ -3 & 4 & 0 & 1 \end{array} \right] \quad -\frac{1}{2}R_1 \qquad \begin{array}{cccc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ -3 & 4 & 0 & 1 \end{array} \\ \\ \left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ -3 & 4 & 0 & 1 \end{array} \right] \quad R_2 + 3R_1 \qquad \begin{array}{cccc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array} \end{array}$$

$$\begin{array}{cc}
 \left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array} \right] & -2R_2 & \begin{array}{cccc} 0 & 1 & 3 & -2 \end{array} \\
 \\
 \left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 3 & -2 \end{array} \right] & R_1 + \frac{3}{2}R_2 & \begin{array}{cccc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & \frac{9}{2} & -3 \\ \hline 1 & 0 & 4 & -3 \end{array} \\
 \\
 \left[\begin{array}{cc|cc} 1 & 0 & 4 & -3 \\ 0 & 1 & 3 & -2 \end{array} \right] & & \\
 \\
 A^{-1} = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}
 \end{array}$$

Exercise

Find the inverse of $A = \begin{bmatrix} a & b \\ 3 & 3 \end{bmatrix}$

Solution

$$\begin{aligned}
 A^{-1} &= \frac{1}{3a-3b} \begin{bmatrix} 3 & -b \\ -3 & a \end{bmatrix} \\
 &= \begin{bmatrix} \frac{3}{3(a-b)} & \frac{-b}{3(a-b)} \\ \frac{-3}{3(a-b)} & \frac{a}{3(a-b)} \end{bmatrix} \\
 &= \begin{bmatrix} \frac{1}{a-b} & \frac{-b}{3(a-b)} \\ \frac{-1}{a-b} & \frac{a}{3(a-b)} \end{bmatrix}
 \end{aligned}$$

Exercise

Find the inverse of $A = \begin{bmatrix} -2 & a \\ 4 & a \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{-2a-4a} \begin{bmatrix} a & -a \\ -4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a}{-6a} & \frac{-a}{-6a} \\ \frac{-4}{-6a} & \frac{-2}{-6a} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{6} & \frac{1}{6} \\ \frac{2}{3a} & \frac{1}{3a} \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{bmatrix} 4 & 4 \\ b & a \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{4a-4b} \begin{bmatrix} a & -4 \\ -b & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a}{4(a-b)} & \frac{-4}{4(a-b)} \\ \frac{-b}{4(a-b)} & \frac{4}{4(a-b)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a}{4(a-b)} & \frac{-1}{a-b} \\ \frac{-b}{4(a-b)} & \frac{1}{a-b} \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{-1+4} \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$

Solution

$$\begin{aligned} A^{-1} &= \frac{1}{3} \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$

Solution

$$\begin{aligned} A^{-1} &= -\frac{1}{14} \begin{pmatrix} -1 & -4 \\ -3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{14} & \frac{2}{7} \\ \frac{3}{14} & -\frac{1}{7} \end{pmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

Solution

$$\begin{aligned} A^{-1} &= -\frac{1}{5} \begin{pmatrix} -1 & -3 \\ -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{11} \begin{pmatrix} 5 & -3 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{11} & -\frac{3}{11} \\ \frac{2}{11} & \frac{1}{11} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{\textcolor{red}{0}} \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$$

\therefore Inverse *doesn't exist*

Exercise

Find the inverse of $A = \begin{pmatrix} -6 & 9 \\ 2 & -3 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{\textcolor{red}{18-18}} \begin{pmatrix} & \\ & \end{pmatrix}$$

\therefore Inverse *doesn't exist*

Exercise

Find the inverse of $A = \begin{pmatrix} -2 & 7 \\ 0 & 2 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{4} \begin{pmatrix} 2 & -7 \\ 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{7}{4} \\ 0 & -\frac{1}{2} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$

Solution

$$A = \frac{1}{-16+16} \begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$$

\therefore Inverse *doesn't exist*

Exercise

Find the inverse of $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Solution

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 2 & 1 \\ a & a \end{pmatrix}$

Solution

$$\begin{aligned} A^{-1} &= \frac{1}{a} \begin{pmatrix} a & -1 \\ -a & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\frac{1}{a} \\ -1 & \frac{2}{a} \end{pmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{pmatrix} b & 3 \\ b & 2 \end{pmatrix}$

Solution

$$\begin{aligned} A^{-1} &= -\frac{1}{b} \begin{pmatrix} 2 & -3 \\ -b & b \end{pmatrix} \\ &= \begin{pmatrix} -\frac{2}{b} & \frac{3}{b} \\ 1 & -1 \end{pmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & a \\ 3 & a \end{pmatrix}$

Solution

$$\begin{aligned} A^{-1} &= -\frac{1}{2a} \begin{pmatrix} a & -a \\ -3 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2a} & -\frac{1}{2a} \end{pmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix}$

Solution

$$\begin{aligned} A^{-1} &= \frac{1}{a^2 - 4} \begin{pmatrix} a & -2 \\ -2 & a \end{pmatrix} \\ &= \begin{pmatrix} \frac{a}{a^2 - 4} & \frac{-2}{a^2 - 4} \\ \frac{-2}{a^2 - 4} & \frac{a}{a^2 - 4} \end{pmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{\textcolor{red}{0}} \begin{pmatrix} & \\ & \end{pmatrix}$$

\therefore Inverse *doesn't exist*

Exercise

Find the inverse of $A = \begin{pmatrix} -3 & \frac{1}{2} \\ 6 & -1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{\textcolor{red}{0}} \begin{pmatrix} \\ \\ \end{pmatrix}$$

\therefore Inverse *doesn't exist*

Exercise

Find A^{-1} if $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \quad \begin{array}{cccccc} 2 & -2 & -1 & 0 & 1 & 0 \\ -2 & 0 & -2 & -2 & 0 & 0 \\ \hline 0 & -2 & -3 & -2 & 1 & 0 \end{array} \quad \begin{array}{cccccc} 3 & 0 & 0 & 0 & 0 & 1 \\ -3 & 0 & -3 & -3 & 0 & 0 \\ \hline 0 & 0 & -3 & -3 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array} \right] -\frac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array} \right] -\frac{1}{3}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array} \right] \begin{array}{l} R_1 - R_3 \\ R_2 - \frac{3}{2}R_3 \\ \end{array} \quad \begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & \frac{1}{3} \\ \hline 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{array} \quad \begin{array}{cccccc} 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & 0 & \frac{1}{2} \\ \hline 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} \textcolor{blue}{0} & \textcolor{blue}{0} & \textcolor{blue}{\frac{1}{3}} \\ \textcolor{blue}{-\frac{1}{2}} & \textcolor{blue}{-\frac{1}{2}} & \textcolor{blue}{\frac{1}{2}} \\ \textcolor{blue}{1} & \textcolor{blue}{0} & \textcolor{blue}{-\frac{1}{3}} \end{bmatrix}$$

Exercise

Find A^{-1} , where $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 + 2R_1 \end{array}$$

$$\begin{array}{cccccc} 3 & 5 & 3 & 0 & 1 & 0 \\ -3 & -6 & 3 & -3 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \end{array} \quad \begin{array}{cccccc} 2 & 4 & 3 & 0 & 0 & 1 \\ -2 & -4 & 2 & -2 & 0 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right] -R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right] R_1 - 2R_2$$

$$\begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 12 & -6 & 2 & 0 \\ 1 & 0 & 11 & -5 & 2 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right] \frac{1}{5}R_3$$

$$0 \quad 0 \quad 1 \quad -\frac{2}{5} \quad 0 \quad \frac{1}{5}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array} \right] \begin{array}{l} R_1 - 11R_3 \\ R_2 + 6R_3 \end{array}$$

$$\begin{array}{cccccc} 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 6 & -\frac{12}{5} & 0 & \frac{6}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \end{array} \quad \begin{array}{cccccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 0 & -11 & \frac{22}{5} & 0 & -\frac{11}{5} \\ 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} -\frac{3}{5} & 2 & -\frac{11}{5} \\ \frac{3}{5} & -1 & \frac{6}{5} \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{bmatrix}$$

Exercise

Find A^{-1} , where $A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2 + 2R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{cccccc} -2 & 0 & 1 & 0 & 1 & 0 \\ 2 & 4 & -2 & 2 & 0 & 0 \\ 0 & 4 & -1 & 2 & 1 & 0 \end{array} \quad \begin{array}{cccccc} 1 & -1 & 0 & 0 & 0 & 1 \\ -1 & -2 & 1 & -1 & 0 & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 4 & -1 & 2 & 1 & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array} \right] \frac{1}{4}R_2$$

$$0 \quad 1 \quad -\frac{1}{4} \quad \frac{1}{2} \quad \frac{1}{4} \quad 0$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array} \right] \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 3R_2 \end{array}$$

$$\begin{array}{cccccc} 0 & -3 & 1 & -1 & 0 & 1 \\ 0 & 3 & -\frac{3}{4} & \frac{3}{2} & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{array} \quad \begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & \frac{1}{2} & -1 & -\frac{1}{2} & 0 \\ 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{array} \right] 4R_3$$

$$0 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array} \right] \begin{array}{l} R_1 + \frac{1}{2}R_3 \\ R_2 + \frac{1}{4}R_3 \\ \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 1 & \frac{3}{2} & 2 \\ 1 & 0 & 0 & 1 & 1 & 2 \end{array} \quad \begin{array}{cccccc} 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

Exercise

Find A^{-1} , where $A = \begin{bmatrix} -2 & 5 & 3 \\ 4 & -1 & 3 \\ 7 & -2 & 5 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} -2 & 5 & 3 & 1 & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array} \right] \quad \frac{1}{-2}R_1 \quad \begin{array}{cccccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array} \right] \quad R_2 - 4R_1 \quad \begin{array}{cccccc} 4 & -1 & 3 & 0 & 1 & 0 \\ -4 & 10 & 6 & 2 & 0 & 0 \\ \hline 0 & 9 & 9 & 2 & 1 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array} \right] \quad R_3 - 7R_1 \quad \begin{array}{cccccc} 7 & -2 & 5 & 0 & 0 & 1 \\ -7 & \frac{35}{2} & \frac{21}{2} & \frac{7}{2} & 0 & 0 \\ \hline 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array} \right] \quad \frac{1}{9}R_2 \quad \begin{array}{cccccc} 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array} \right] \quad R_3 - \frac{31}{2}R_2 \quad \begin{array}{cccccc} 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \\ 0 & -\frac{31}{2} & -\frac{31}{2} & -\frac{31}{9} & -\frac{31}{18} & 0 \\ \hline 0 & 0 & 0 & \frac{1}{18} & -\frac{31}{18} & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{18} & -\frac{31}{18} & 1 \end{array} \right]$$

\therefore The inverse matrix *doesn't exist*

OR

$$\left[\begin{array}{ccc|ccc} -2 & 5 & 3 & 1 & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array} \right] \quad \begin{array}{l} R_2 + 2R_1 \\ -2R_3 + 7R_1 \end{array} \quad \begin{array}{cccccc} 4 & -1 & 3 & 0 & 1 & 0 \\ -4 & 10 & 6 & 2 & 0 & 0 \\ \hline 0 & 9 & 9 & 2 & 1 & 0 \end{array} \quad \begin{array}{cccccc} -14 & 4 & -10 & 0 & 0 & -2 \\ 14 & 35 & 21 & 7 & 0 & 0 \\ \hline 0 & 39 & 11 & 7 & 0 & -2 \end{array}$$

$$\begin{array}{l}
 \left[\begin{array}{ccc|ccc} -2 & 5 & 3 & 1 & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 0 & 39 & 11 & 7 & 0 & -2 \end{array} \right] \begin{array}{l} 9R_1 - 5R_2 \\ \\ 9R_3 - 39R_2 \end{array} \quad \begin{array}{cccccc} -18 & 45 & 27 & 9 & 0 & 0 \\ 0 & -45 & -45 & -10 & -5 & 0 \\ \hline -18 & 0 & -18 & -1 & -5 & 0 \\ \\ 0 & 351 & -99 & 63 & 0 & -18 \\ 0 & -351 & 99 & -78 & -39 & 0 \\ \hline 0 & 0 & 0 & -15 & -39 & -18 \end{array}
 \end{array}$$

$$\left[\begin{array}{ccc|ccc} -18 & 0 & -18 & -1 & -5 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 0 & 0 & 0 & -15 & -39 & -18 \end{array} \right]$$

\therefore The inverse matrix *doesn't exist*

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array} \right) R_2 + R_1 \quad \begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ \hline 0 & 4 & 4 & 1 & 1 & 0 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 4 & 4 & 1 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array} \right) \frac{1}{4}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} R_1 - R_2 \\ R_3 - 4R_2 \end{array} \quad \begin{array}{cccccc} 0 & 4 & 3 & 0 & 0 & 1 \\ 0 & -4 & -4 & -1 & -1 & 0 \\ \hline 0 & 0 & -1 & -1 & -1 & 1 \end{array} \quad \begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ \hline 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array} \right) -R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array} \right) \begin{array}{l} R_1 + R_3 \\ R_2 - R_3 \end{array} \quad \begin{array}{cccccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ \hline 1 & 0 & 0 & \frac{7}{4} & \frac{3}{4} & -1 \end{array} \quad \begin{array}{cccccc} 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \\ \hline 0 & 1 & 0 & -\frac{3}{4} & -\frac{3}{4} & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{4} & \frac{3}{4} & -1 \\ 0 & 1 & 0 & -\frac{3}{4} & -\frac{3}{4} & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} \frac{7}{4} & \frac{3}{4} & -1 \\ -\frac{3}{4} & -\frac{3}{4} & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ -2 & -3 & 0 & 0 & 0 & 1 \end{array} \right) \quad R_3 + 2R_1 \quad \begin{array}{cccccc} -2 & -3 & 0 & 0 & 0 & 1 \\ 2 & -2 & 2 & 2 & 0 & 0 \\ \hline 0 & -5 & 2 & 2 & 0 & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & -5 & 2 & 2 & 0 & 1 \end{array} \right) \quad -\frac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & -5 & 2 & 2 & 0 & 1 \end{array} \right) \quad \begin{array}{l} R_1 + R_2 \\ R_3 + 5R_2 \end{array} \quad \begin{array}{cccccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ \hline 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \end{array} \quad \begin{array}{cccccc} 0 & -5 & 2 & 2 & 0 & 1 \\ 0 & 5 & -\frac{5}{2} & 0 & -\frac{5}{2} & 0 \\ \hline 0 & 0 & -\frac{1}{2} & 2 & -\frac{5}{2} & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 2 & -\frac{5}{2} & 1 \end{array} \right) \quad -2R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -4 & 5 & -2 \end{array} \right) \quad \begin{array}{l} R_1 - \frac{1}{2}R_3 \\ R_2 + \frac{1}{2}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & -4 & 5 & -2 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_2 + R_1 \\ R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array} \right) \frac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array} \right) R_3 + R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array} \right) 2R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array} \right) \begin{array}{l} R_1 - 2R_3 \\ R_2 - \frac{5}{2}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 1 & 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & -1 \\ 3 & 1 & 2 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & -1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -4 & -3 & 1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{array} \right) -R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 3 & -1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{array} \right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 2R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -2 & 1 & 0 \\ 0 & 1 & 4 & 3 & -1 & 0 \\ 0 & 0 & 7 & 3 & -2 & 1 \end{array} \right) \frac{1}{7}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -2 & 1 & 0 \\ 0 & 1 & 4 & 3 & -1 & 0 \\ 0 & 0 & 1 & \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{array} \right) \begin{array}{l} R_1 + 3R_3 \\ R_2 - 4R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{7} & \frac{1}{7} & \frac{3}{7} \\ 0 & 1 & 0 & \frac{9}{7} & \frac{1}{7} & -\frac{4}{7} \\ 0 & 0 & 1 & \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} -\frac{5}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{9}{7} & \frac{1}{7} & -\frac{4}{7} \\ \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 3 & 3 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array} \right) \frac{1}{3}R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & -3 & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \end{array} \right) \begin{array}{l} R_1 - R_2 \\ R_3 + 3R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & \frac{7}{3} & -\frac{5}{3} & 3 & 1 \end{array} \right) \frac{3}{7}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 & -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{array} \right) \begin{array}{l} R_1 + \frac{1}{3}R_3 \\ R_2 - \frac{2}{3}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ 0 & 1 & 0 & \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ 0 & 0 & 1 & -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} -3 & 1 & -1 \\ 1 & -4 & -7 \\ 1 & 2 & 5 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} -3 & 1 & -1 & 1 & 0 & 0 \\ 1 & -4 & -7 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array} \right) \quad -\frac{1}{3}R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 1 & -4 & -7 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array} \right) \quad \begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{11}{3} & -\frac{22}{3} & \frac{1}{3} & 1 & 0 \\ 0 & \frac{7}{3} & \frac{14}{3} & \frac{1}{3} & 0 & 1 \end{array} \right) \quad -\frac{3}{11}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 2 & -\frac{1}{11} & -\frac{3}{11} & 0 \\ 0 & \frac{7}{3} & \frac{14}{3} & \frac{1}{3} & 0 & 1 \end{array} \right) \quad R_3 - \frac{7}{3}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 2 & -\frac{1}{11} & -\frac{3}{11} & 0 \\ 0 & 0 & 0 & -\frac{1}{11} & -\frac{3}{11} & 0 \end{array} \right)$$

\therefore Inverse ***does not exist***

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 1 & -3 \\ 2 & -4 & 1 \\ -5 & 7 & 1 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 2 & -4 & 1 & 0 & 1 & 0 \\ -5 & 7 & 1 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + 5R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & -6 & 7 & -2 & 1 & 0 \\ 0 & 12 & -14 & 5 & 0 & 1 \end{array} \right) -\frac{1}{6}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 12 & -14 & 5 & 0 & 1 \end{array} \right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 12R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

\therefore Inverse **does not exist**

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_3 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right) -R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right) R_1 - 2R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right) \frac{1}{5}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array} \right) \begin{array}{l} R_1 - 11R_3 \\ R_2 + 6R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} -\frac{3}{5} & 2 & -\frac{11}{5} \\ \frac{3}{5} & -1 & \frac{6}{5} \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} -2 & -3 & 4 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & -6 & 1 \\ -2 & -2 & 5 & 1 \end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc} -2 & -3 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & 5 & 1 & 0 & 0 & 0 & 1 \end{array} \right] -\frac{1}{2}R_1$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & 5 & 1 & 0 & 0 & 0 & 1 \end{array} \right] R_4 + 2R_1$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array} \right] R_4 - R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

\therefore Inverse *does not exist*

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 1 & -14 & 7 & 38 \\ -1 & 2 & 1 & -2 \\ 1 & 2 & -1 & -6 \\ 1 & -2 & 3 & 6 \end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 2 & -1 & -6 & 0 & 0 & 1 & 0 \\ 1 & -2 & 3 & 6 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R_2 + R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ 0 & -12 & 8 & 36 & 1 & 1 & 0 & 0 \\ 0 & 16 & -8 & -44 & -1 & 0 & 1 & 0 \\ 0 & 12 & -4 & -32 & -1 & 0 & 0 & 1 \end{array} \right] -\frac{1}{12}R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 16 & -8 & -44 & -1 & 0 & 1 & 0 \\ 0 & 12 & -4 & -32 & -1 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R_1 + 14R_2 \\ R_3 - 16R_2 \\ R_4 - 12R_2 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -\frac{7}{3} & -4 & -\frac{1}{6} & -\frac{7}{6} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & \frac{8}{3} & 4 & \frac{1}{3} & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 & 1 & 0 & 1 \end{array} \right] \frac{3}{8}R_3$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -\frac{7}{3} & -4 & -\frac{1}{6} & -\frac{7}{6} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 4 & 4 & 0 & 1 & 0 & 1 \end{array} \right] \begin{array}{l} R_1 + \frac{7}{3}R_3 \\ R_2 + \frac{2}{3}R_3 \\ \\ R_4 - 4R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ 0 & 1 & 0 & -2 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & -2 & -\frac{1}{2} & -1 & -\frac{3}{2} & 1 \end{array} \right] -\frac{1}{2}R_4$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ 0 & 1 & 0 & -2 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{array} \right] \begin{array}{l} R_1 + \frac{1}{2}R_4 \\ R_2 + 2R_4 \\ R_3 - \frac{3}{2}R_4 \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{5}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{5}{4} & \frac{7}{4} & -1 \\ 0 & 0 & 1 & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{5}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{5}{4} & \frac{7}{4} & -1 \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{bmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 10 & 20 & -30 & 15 \\ 3 & -7 & 14 & -8 \\ -7 & -2 & -1 & 2 \\ 4 & 4 & -3 & 1 \end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc} 10 & 20 & -30 & 15 & 1 & 0 & 0 & 0 \\ 3 & -7 & 14 & -8 & 0 & 1 & 0 & 0 \\ -7 & -2 & -1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 4 & -3 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \quad \frac{1}{10}R_1$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0 \\ 3 & -7 & 14 & -8 & 0 & 1 & 0 & 0 \\ -7 & -2 & -1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 4 & -3 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 + 7R_1 \\ R_4 - 4R_1 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0 \\ 0 & -13 & 23 & -\frac{25}{2} & -\frac{3}{10} & 1 & 0 & 0 \\ 0 & 12 & -22 & \frac{25}{2} & \frac{7}{10} & 0 & 1 & 0 \\ 0 & -4 & 9 & -5 & -\frac{2}{5} & 0 & 0 & 1 \end{array} \right] \quad -\frac{1}{13}R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 12 & -22 & \frac{25}{2} & \frac{7}{10} & 0 & 1 & 0 \\ 0 & -4 & 9 & -5 & -\frac{2}{5} & 0 & 0 & 1 \end{array} \right] \quad \begin{array}{l} R_1 - 2R_2 \\ R_3 - 12R_2 \\ R_4 + 4R_2 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \frac{7}{13} & -\frac{11}{26} & \frac{7}{130} & \frac{2}{13} & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 0 & -\frac{10}{13} & \frac{25}{26} & \frac{11}{26} & \frac{12}{13} & 1 & 0 \\ 0 & 0 & \frac{25}{13} & -\frac{15}{13} & -\frac{4}{13} & -\frac{4}{13} & 0 & 1 \end{array} \right] \quad -\frac{13}{10}R_3$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \frac{7}{13} & -\frac{11}{26} & \frac{7}{130} & \frac{2}{13} & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 0 & 1 & -\frac{5}{4} & -\frac{11}{20} & -\frac{6}{5} & -\frac{13}{10} & 0 \\ 0 & 0 & \frac{25}{13} & -\frac{15}{13} & -\frac{4}{13} & -\frac{4}{13} & 0 & 1 \end{array} \right] \begin{array}{l} R_1 - \frac{7}{13}R_3 \\ R_2 + \frac{23}{13}R_3 \\ R_4 - \frac{25}{13}R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & -\frac{5}{4} & -\frac{19}{20} & -\frac{11}{5} & -\frac{23}{10} & 0 \\ 0 & 0 & 1 & -\frac{5}{4} & -\frac{11}{20} & -\frac{6}{5} & -\frac{13}{10} & 0 \\ 0 & 0 & 0 & \frac{5}{4} & \frac{3}{4} & 2 & \frac{5}{2} & 1 \end{array} \right] \begin{array}{l} R_2 + R_4 \\ R_3 + R_4 \end{array}.$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & \frac{5}{4} & \frac{3}{4} & 2 & \frac{5}{2} & 1 \end{array} \right] \frac{4}{5}R_4$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{array} \right] R_1 - \frac{1}{4}R_4$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{bmatrix}$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1} $A = \begin{bmatrix} x \end{bmatrix}$

Solution

For A^{-1} exists, $x \neq 0$

$$AA^{-1} = I$$

$$\begin{bmatrix} x \end{bmatrix} \begin{bmatrix} a \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

$$xa = 1$$

$$\underline{a = \frac{1}{x}}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{x} \end{bmatrix}$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1} $A = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$

Solution

For A^{-1} exists, $\underline{x, y \neq 0}$

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad AA^{-1} = I$$

$$\begin{cases} ax = 1 & bx = 0 \\ cy = 0 & dy = 1 \end{cases}$$

$$\begin{cases} a = \frac{1}{x} & b = 0 \\ c = 0 & d = \frac{1}{y} \end{cases}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{x} & 0 \\ 0 & \frac{1}{y} \end{bmatrix}$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1}

$$A = \begin{bmatrix} 0 & 0 & x \\ 0 & y & 0 \\ z & 0 & 0 \end{bmatrix}$$

Solution

For A^{-1} exists, $\underline{x, y, z \neq 0}$

$$\begin{pmatrix} 0 & 0 & x \\ 0 & y & 0 \\ z & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad AA^{-1} = I$$

$$\begin{pmatrix} xg & xh & xi \\ yd & ye & yf \\ za & zb & zc \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{cases} xg = 1 & xh = 0 & xi = 0 \\ yd = 0 & ye = 1 & yf = 0 \\ za = 0 & zb = 0 & zc = 1 \end{cases}$$

$$\begin{cases} g = \frac{1}{x} & h = 0 & i = 0 \\ d = 0 & e = \frac{1}{y} & f = 0 \\ a = 0 & b = 0 & c = \frac{1}{z} \end{cases}$$

$$A^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{z} \\ 0 & \frac{1}{y} & 0 \\ \frac{1}{z} & 0 & 0 \end{pmatrix}$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1}

$$A = \begin{bmatrix} x & 1 & 1 & 1 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{bmatrix}$$

Solution

For A^{-1} exists, $\underline{x, y, z, w \neq 0}$

$$\begin{pmatrix} x & 1 & 1 & 1 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad AA^{-1} = I$$

$$\begin{cases} xa_{11} + xa_{21} + xa_{31} + xa_{41} = 1 \\ xa_{12} + xa_{22} + xa_{32} + xa_{42} = 0 \\ xa_{13} + xa_{23} + xa_{33} + xa_{43} = 0 \\ xa_{14} + xa_{24} + xa_{34} + xa_{44} = 0 \end{cases}$$

$$\begin{cases} ya_{21} = 0 & \underline{a_{21} = 0} \\ ya_{22} = 1 & \underline{a_{22} = \frac{1}{y}} \\ ya_{23} = 0 & \underline{a_{23} = 0} \\ ya_{24} = 0 & \underline{a_{24} = 0} \end{cases}$$

$$\begin{cases} za_{31} = 0 & \underline{a_{31} = 0} \\ za_{32} = 0 & \underline{a_{32} = 0} \\ za_{33} = 1 & \underline{a_{33} = \frac{1}{z}} \\ za_{34} = 0 & \underline{a_{34} = 0} \end{cases}$$

$$\begin{cases} wa_{41} = 0 & \underline{a_{41} = 0} \\ wa_{42} = 0 & \underline{a_{42} = 0} \\ wa_{43} = 0 & \underline{a_{43} = 0} \\ wa_{44} = 1 & \underline{a_{44} = \frac{1}{w}} \end{cases}$$

$$\rightarrow \begin{cases} xa_{11} = 1 & \underline{a_{11} = \frac{1}{x}} \\ xa_{12} + \frac{x}{y} = 0 & \underline{a_{12} = -\frac{1}{y}} \\ xa_{13} + \frac{x}{z} = 0 & \underline{a_{13} = -\frac{1}{z}} \\ xa_{14} + \frac{x}{w} = 0 & \underline{a_{14} = -\frac{1}{w}} \end{cases}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{x} & -\frac{1}{y} & -\frac{1}{z} & -\frac{1}{w} \\ 0 & \frac{1}{y} & 0 & 0 \\ 0 & 0 & \frac{1}{z} & 0 \\ 0 & 0 & 0 & \frac{1}{w} \end{pmatrix}$$

Exercise

Solve the system using A^{-1} $\begin{cases} x + 2z = 6 \\ -x + 2y + 3z = -5 \\ x - y = 6 \end{cases}$ Given $A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

Solution

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3(6) - 2(-5) - 4(6) \\ 3(6) - 2(-5) - 5(6) \\ -1(6) + 1(-5) + 2(6) \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

Solution: $(4, -2, 1)$

Exercise

Solve the system using A^{-1} $\begin{cases} x + 2y + 5z = 2 \\ 2x + 3y + 8z = 3 \\ -x + y + 2z = 3 \end{cases}$

a) Write the linear system as a matrix equation in the form $AX = B$

b) Solve the system using the inverse that is given for the coefficient matrix

the inverse of $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{bmatrix}$ is $\begin{bmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{bmatrix}$

Solution

$$a) \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$$

$$b) \begin{bmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$$

Exercise

Solve the system using A^{-1}
$$\begin{cases} x - y + z = 8 \\ 2y - z = -7 \\ 2x + 3y = 1 \end{cases}$$

- a) Write the linear system as a matrix equation in the form $AX = B$
- b) Solve the system using the inverse that is given for the coefficient matrix

$$\text{the inverse is } \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

Solution

$$a) \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 1 \end{bmatrix}$$

$$b) \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ -7 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system
$$\begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$$

Solution

$$A = \begin{pmatrix} 3 & 2 \\ 2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} -4 \\ -5 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{7} \begin{pmatrix} -1 & -2 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{pmatrix} \begin{pmatrix} -4 \\ 5 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{6}{7} \\ -\frac{23}{7} \end{pmatrix}$$

$$\therefore \text{Solution: } \left(\frac{6}{7}, -\frac{23}{7} \right)$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$$

Solution

$$A = \begin{pmatrix} 2 & 5 \\ 5 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{29} \begin{pmatrix} -2 & -5 \\ -5 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{29} & \frac{5}{29} \\ \frac{5}{29} & -\frac{2}{29} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{2}{29} & \frac{5}{29} \\ \frac{5}{29} & -\frac{2}{29} \end{pmatrix} \begin{pmatrix} 7 \\ -3 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{29} \\ -\frac{41}{7} \end{pmatrix}$$

$$\therefore \text{Solution: } \left(-\frac{1}{29}, -\frac{41}{29} \right)$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system
$$\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$$

Solution

$$A = \begin{pmatrix} 4 & -7 \\ 2 & 5 \end{pmatrix} \quad B = \begin{pmatrix} -16 \\ 9 \end{pmatrix}$$

$$\begin{aligned} A^{-1} &= \frac{1}{34} \begin{pmatrix} 5 & 7 \\ -2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} \frac{5}{34} & \frac{7}{34} \\ -\frac{1}{17} & \frac{2}{17} \end{pmatrix} \end{aligned}$$

$$X = \begin{pmatrix} \frac{5}{34} & \frac{7}{34} \\ -\frac{1}{17} & \frac{2}{17} \end{pmatrix} \begin{pmatrix} -16 \\ 9 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 2 \end{pmatrix}$$

$$\therefore \text{Solution: } \left(-\frac{1}{2}, 2\right)$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system
$$\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$$

Solution

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$\begin{aligned} A^{-1} &= -1 \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix} \end{aligned}$$

$$X = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

$$\therefore \text{Solution: } (-2, 5)$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} 3x + 4y = 2 \\ 2x + 5y = -1 \end{cases}$$

Solution

$$A = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{7} \begin{pmatrix} 5 & -4 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{7} & -\frac{4}{7} \\ -\frac{2}{7} & \frac{3}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{5}{7} & -\frac{4}{7} \\ -\frac{2}{7} & \frac{3}{7} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\therefore \text{Solution: } \underline{(2, -1)}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} 5x - 2y = 4 \\ -10x + 4y = 7 \end{cases}$$

Solution

$$A = \begin{pmatrix} 5 & -2 \\ -10 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\textcolor{red}{0}} \begin{pmatrix} 4 & 2 \\ 10 & 5 \end{pmatrix}$$

Inverse matrix *doesn't exist*.

$$\begin{cases} 5x - 2y = 4 \\ \textcolor{red}{-\frac{1}{2}} \begin{cases} 5x - 2y = -\frac{7}{2} \end{cases} \end{cases}$$

$$\textcolor{red}{4} \neq \textcolor{red}{-\frac{7}{2}}$$

\therefore *No Solution*

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system
$$\begin{cases} x - 4y = -8 \\ 5x - 20y = -40 \end{cases}$$

Solution

$$A = \begin{pmatrix} 1 & -4 \\ 5 & -20 \end{pmatrix} \quad B = \begin{pmatrix} -8 \\ -40 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\textcolor{red}{0}} \begin{pmatrix} & \\ & \end{pmatrix}$$

Inverse matrix *doesn't exist*.

$$\textcolor{red}{\frac{1}{5}} \begin{cases} x - 4y = -8 \\ x - 4y = -8 \end{cases}$$

$$\therefore \text{Solution: } \underline{(4y - 8, y)}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system
$$\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$$

Solution

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{3} \begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$\textcolor{green}{X = A^{-1}B}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \textcolor{blue}{2} \\ \textcolor{blue}{-1} \end{pmatrix}$$

$$\therefore \text{Solution: } \underline{(\textcolor{blue}{2}, \textcolor{blue}{-1})}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system
$$\begin{cases} 2x + 10y = -14 \\ 7x - 2y = -16 \end{cases}$$

Solution

$$A = \begin{pmatrix} 2 & 10 \\ 7 & -2 \end{pmatrix} \quad B = \begin{pmatrix} -14 \\ -16 \end{pmatrix}$$

$$\begin{aligned} A^{-1} &= -\frac{1}{74} \begin{pmatrix} -2 & -10 \\ -7 & 2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{37} & \frac{5}{37} \\ \frac{7}{74} & -\frac{1}{37} \end{pmatrix} \end{aligned}$$

$$X = \begin{pmatrix} \frac{1}{37} & \frac{5}{37} \\ \frac{7}{74} & -\frac{1}{37} \end{pmatrix} \begin{pmatrix} -14 \\ -16 \end{pmatrix} \quad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{94}{37} \\ -\frac{33}{37} \end{pmatrix}$$

$$\therefore \text{Solution: } \left(-\frac{94}{37}, -\frac{33}{37} \right)$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system
$$\begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$$

Solution

$$A = \begin{pmatrix} 4 & -3 \\ -3 & 9 \end{pmatrix} \quad B = \begin{pmatrix} 24 \\ -1 \end{pmatrix}$$

$$\begin{aligned} A^{-1} &= \frac{1}{27} \begin{pmatrix} 9 & 3 \\ 3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} & \frac{1}{9} \\ \frac{1}{9} & \frac{4}{27} \end{pmatrix} \end{aligned}$$

$$X = \begin{pmatrix} \frac{1}{3} & \frac{1}{9} \\ \frac{1}{9} & \frac{4}{27} \end{pmatrix} \begin{pmatrix} 24 \\ -1 \end{pmatrix} \quad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{71}{9} \\ \frac{68}{27} \end{pmatrix}$$

$$\therefore \text{Solution: } \underline{\left(\frac{71}{9}, \frac{68}{27} \right)}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system $\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$

Solution

$$A = \begin{pmatrix} 4 & 2 \\ 3 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 12 \\ 16 \end{pmatrix}$$

$$\begin{aligned} A^{-1} &= -\frac{1}{14} \begin{pmatrix} -2 & -2 \\ -3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{3}{14} & -\frac{2}{7} \end{pmatrix} \end{aligned}$$

$$X = \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{3}{14} & -\frac{2}{7} \end{pmatrix} \begin{pmatrix} 12 \\ 16 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$$\therefore \text{Solution: } \underline{(4, -2)}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system $\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & 2 \\ 4 & -2 \end{pmatrix} \quad B = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{10} \begin{pmatrix} -2 & -2 \\ -4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{10} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{10} \end{pmatrix} \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore \text{Solution: } \underline{(1, -1)}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} x - 2y = 5 \\ -10x + 2y = 4 \end{cases}$$

Solution

$$\frac{1}{2} \begin{cases} x - 2y = 5 \\ -5x + y = 2 \end{cases}$$

$$A = \begin{pmatrix} 1 & -2 \\ -5 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{9} \begin{pmatrix} 1 & 2 \\ 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{9} & -\frac{2}{9} \\ -\frac{5}{9} & -\frac{1}{9} \end{pmatrix}$$

$$X = \begin{pmatrix} -\frac{1}{9} & -\frac{2}{9} \\ -\frac{5}{9} & -\frac{1}{9} \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

$$\therefore \text{Solution: } \underline{(-1, -3)}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} 12x + 15y = -27 \\ 30x - 15y = -15 \end{cases}$$

Solution

$$\frac{1}{3} \rightarrow \begin{cases} 4x + 5y = -9 \\ 2x - y = -1 \end{cases}$$

$$\frac{1}{15}$$

$$A = \begin{pmatrix} 4 & 5 \\ 2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} -9 \\ -1 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{14} \begin{pmatrix} -1 & -5 \\ -2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{14} & \frac{5}{14} \\ \frac{1}{7} & -\frac{2}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{14} & \frac{5}{14} \\ \frac{1}{7} & -\frac{2}{7} \end{pmatrix} \begin{pmatrix} -9 \\ -1 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\therefore \text{Solution: } \underline{(-1, -1)}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$$

Solution

$$\frac{1}{4} \rightarrow \begin{cases} x - y = -3 \\ x + y = -5 \end{cases}$$

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$$

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -3 \\ -5 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$$

$$\therefore \text{Solution: } \underline{(-4, -1)}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} -2x + 3y = 4 \\ -3x + 4y = 5 \end{cases}$$

Solution

$$A = \begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}$$

$$X = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\therefore \text{Solution: } \underline{(1, 2)}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} x - 2y = 6 \\ 4x + 3y = 2 \end{cases}$$

Solution

$$A = \begin{pmatrix} 1 & -2 \\ 4 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{11} \begin{pmatrix} 3 & 2 \\ -4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{11} & \frac{2}{11} \\ -\frac{4}{11} & \frac{1}{11} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{3}{11} & \frac{2}{11} \\ -\frac{4}{11} & \frac{1}{11} \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$\therefore \text{Solution: } \underline{(2, -2)}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} 2x - 3y = 7 \\ 4x + y = -7 \end{cases}$$

Solution

$$A = \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 7 \\ -7 \end{pmatrix}$$

$$A^{-1} = \frac{1}{14} \begin{pmatrix} 1 & 3 \\ -4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{14} & \frac{3}{14} \\ -\frac{2}{7} & \frac{1}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{14} & \frac{3}{14} \\ -\frac{2}{7} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} 7 \\ -7 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

$$\therefore \text{Solution: } \underline{(-1, -3)}$$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$$

Solution

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array} \right) -R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & -1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array} \right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 2R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & 3 & 2 & -1 & 0 \\ 0 & 0 & 6 & 3 & -2 & 1 \end{array} \right) \frac{1}{6}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & 3 & 2 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{3} & \frac{1}{6} \end{array} \right) \begin{array}{l} R_1 + 2R_3 \\ R_2 - 3R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{3} & \frac{1}{6} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

∴ Solution: (1, 2, -1)

Exercise

Use the **inverse** of the coefficient matrix to solve the linear system

$$\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$$

Solution

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & -1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -1 \\ 9 \\ 9 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 & 1 \end{array} \right) \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_2 - 3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 0 & -4 & 4 & -3 & 0 & 1 \end{array} \right) \quad -R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & -4 & 4 & -3 & 0 & 1 \end{array} \right) \quad \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 4R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & -8 & 5 & -4 & 1 \end{array} \right) \quad -\frac{1}{8}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{array} \right) \quad \begin{array}{l} R_1 - 2R_3 \\ R_2 + 3R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{8} & \frac{1}{2} & -\frac{3}{8} \\ 0 & 0 & 1 & -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & -\frac{3}{8} \\ -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & -\frac{3}{8} \\ -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{pmatrix} \begin{pmatrix} -1 \\ 9 \\ 9 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$$

∴ Solution: (2, 1, 4)

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$$

Solution

$$A = \begin{pmatrix} 1 & 5 & -1 \\ 0 & 3 & -1 \\ -3 & 6 & 2 \end{pmatrix} \quad B = \begin{pmatrix} -4 \\ -1 \\ 11 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 5 & -1 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ -3 & 6 & 2 & 0 & 0 & 1 \end{array} \right) \quad R_3 + 3R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 5 & -1 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 0 & 21 & -1 & 3 & 0 & 1 \end{array} \right) \quad \frac{1}{3}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 5 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 21 & -1 & 3 & 0 & 1 \end{array} \right) \quad \begin{array}{l} R_1 - 5R_2 \\ R_3 - 21R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & 1 & -\frac{5}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 6 & 3 & -7 & 1 \end{array} \right) \frac{1}{6}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & 1 & -\frac{5}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{array} \right) \begin{array}{l} R_1 - \frac{2}{3}R_3 \\ R_2 + \frac{1}{3}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & \frac{8}{9} & -\frac{1}{9} \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{1}{18} & \frac{1}{18} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{8}{9} & -\frac{1}{9} \\ \frac{1}{6} & -\frac{1}{18} & \frac{1}{18} \\ \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{2}{3} & \frac{8}{9} & -\frac{1}{9} \\ \frac{1}{6} & -\frac{1}{18} & \frac{1}{18} \\ \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} -4 \\ -1 \\ 11 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

\therefore Solution: $(-3, 0, 1)$

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$$

Solution

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -3 & 2 \\ 3 & -1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 14 \\ 10 \\ 9 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 2 & -3 & 2 & 0 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & -9 & -6 & -2 & 1 & 0 \\ 0 & -10 & -11 & -3 & 0 & 1 \end{array} \right) -\frac{1}{9}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{9} & -\frac{1}{9} & 0 \\ 0 & -10 & -11 & -3 & 0 & 1 \end{array} \right) \begin{array}{l} R_1 - 3R_2 \\ \\ R_3 + 10R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{9} & -\frac{1}{9} & 0 \\ 0 & 0 & -\frac{13}{3} & -\frac{7}{9} & -\frac{10}{9} & 1 \end{array} \right) -\frac{3}{13}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{9} & -\frac{1}{9} & 0 \\ 0 & 0 & 1 & \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{array} \right) \begin{array}{l} R_2 - 2R_3 \\ R_2 - \frac{2}{3}R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{39} & -\frac{7}{39} & \frac{6}{13} \\ 0 & 1 & 0 & \frac{4}{39} & -\frac{11}{39} & \frac{2}{13} \\ 0 & 0 & 1 & \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} -\frac{1}{39} & -\frac{7}{39} & \frac{6}{13} \\ \frac{4}{39} & -\frac{11}{39} & \frac{2}{13} \\ \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{pmatrix}$$

$$X = \begin{pmatrix} -\frac{1}{39} & -\frac{7}{39} & \frac{6}{13} \\ \frac{4}{39} & -\frac{11}{39} & \frac{2}{13} \\ \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{pmatrix} \begin{pmatrix} 14 \\ 10 \\ 9 \end{pmatrix} \quad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

∴ **Solution:** (2, 0, 3)

Exercise

Use the **inverse** of the coefficient matrix to solve the linear system

$$\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & -3 & 2 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -10 & 4 & -3 & 1 & 0 \\ 0 & -11 & 4 & -2 & 0 & 1 \end{array} \right) \begin{array}{l} 5R_1 + 2R_2 \\ \\ 10R_3 - 11R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 5 & 0 & 3 & -1 & 2 & 0 \\ 0 & -10 & 4 & -3 & 1 & 0 \\ 0 & 0 & -4 & 13 & -11 & 10 \end{array} \right) \begin{array}{l} 4R_1 + 3R_3 \\ R_2 + R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 20 & 0 & 0 & 35 & -25 & 30 \\ 0 & -10 & 0 & 10 & -10 & 10 \\ 0 & 0 & -4 & 13 & -11 & 10 \end{array} \right) \begin{array}{l} \frac{1}{20}R_1 \\ -\frac{1}{10}R_2 \\ -\frac{1}{4}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{4} & -\frac{5}{4} & \frac{3}{2} \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -\frac{13}{4} & \frac{11}{4} & -\frac{5}{2} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} \frac{7}{4} & -\frac{5}{4} & \frac{3}{2} \\ -1 & 1 & -1 \\ -\frac{13}{4} & \frac{11}{4} & -\frac{5}{2} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{7}{4} & -\frac{5}{4} & \frac{3}{2} \\ -1 & 1 & -1 \\ -\frac{13}{4} & \frac{11}{4} & -\frac{5}{2} \end{pmatrix} \begin{pmatrix} 20 \\ 8 \\ -16 \end{pmatrix}$$

$$X = A^{-1}B$$

$$= \begin{pmatrix} 35 - 10 - 24 \\ -20 + 8 + 16 \\ -65 + 22 + 40 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

∴ Solution: (1, 4, -3)

Exercise

Use the **inverse** of the coefficient matrix to solve the linear system

$$\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$$

Solution

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 2 & -3 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 17 \\ 7 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 2 & -3 & 2 & 0 & 0 & 1 \end{array} \right) \quad R_3 - 2R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & -7 & 0 & -2 & 0 & 1 \end{array} \right) \quad \frac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -7 & 0 & -2 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 7R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{7}{2} & -2 & \frac{7}{2} & 1 \end{array}\right) -\frac{2}{7}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{4}{7} & -1 & -\frac{2}{7} \end{array}\right) \begin{array}{l} R_1 - 2R_3 \\ R_2 + \frac{1}{2}R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{7} & 1 & \frac{4}{7} \\ 0 & 1 & 0 & \frac{2}{7} & 0 & -\frac{1}{7} \\ 0 & 0 & 1 & \frac{4}{7} & -1 & -\frac{2}{7} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{1}{7} & 1 & \frac{4}{7} \\ \frac{2}{7} & 0 & -\frac{1}{7} \\ \frac{4}{7} & -1 & -\frac{2}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} -\frac{1}{7} & 1 & \frac{4}{7} \\ \frac{2}{7} & 0 & -\frac{1}{7} \\ \frac{4}{7} & -1 & -\frac{2}{7} \end{pmatrix} \begin{pmatrix} 17 \\ 7 \\ -1 \end{pmatrix} \quad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$$

∴ **Solution:** (4, 5, 3)

Exercise

Use the *inverse* of the coefficient matrix to solve the linear system

$$\begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$$

Solution

$$A = \begin{pmatrix} -2 & 6 & 7 \\ -4 & 5 & 3 \\ -6 & 3 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 3 \\ 7 \\ -4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} -2 & 6 & 7 & 1 & 0 & 0 \\ -4 & 5 & 3 & 0 & 1 & 0 \\ -6 & 3 & 5 & 0 & 0 & 1 \end{array} \right) \quad -\frac{1}{2}R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & -3 & -\frac{7}{2} & -\frac{1}{2} & 0 & 0 \\ -4 & 5 & 3 & 0 & 1 & 0 \\ -6 & 3 & 5 & 0 & 0 & 1 \end{array} \right) \quad \begin{array}{l} R_2 + 4R_1 \\ R_3 + 6R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -3 & -\frac{7}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & -7 & -11 & -2 & 1 & 0 \\ 0 & -15 & -16 & -3 & 0 & 1 \end{array} \right) \quad -\frac{1}{7}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & -3 & -\frac{7}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{11}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & -15 & -16 & -3 & 0 & 1 \end{array} \right) \quad \begin{array}{l} R_1 + 3R_2 \\ R_3 + 15R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{17}{14} & \frac{5}{14} & -\frac{3}{7} & 0 \\ 0 & 1 & \frac{11}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & \frac{53}{7} & \frac{9}{7} & -\frac{15}{7} & 1 \end{array} \right) \quad \frac{7}{53}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{17}{14} & \frac{5}{14} & -\frac{3}{7} & 0 \\ 0 & 1 & \frac{11}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & 1 & \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{array} \right) \quad \begin{array}{l} R_1 - \frac{17}{14}R_3 \\ R_2 - \frac{11}{7}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{8}{53} & -\frac{9}{106} & -\frac{17}{106} \\ 0 & 1 & 0 & \frac{1}{53} & \frac{16}{53} & -\frac{11}{53} \\ 0 & 0 & 1 & \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} \frac{8}{53} & -\frac{9}{106} & -\frac{7}{106} \\ \frac{1}{53} & \frac{16}{53} & -\frac{11}{53} \\ \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{8}{53} & -\frac{9}{106} & -\frac{7}{106} \\ \frac{1}{53} & \frac{16}{53} & -\frac{11}{53} \\ \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{pmatrix} \begin{pmatrix} 3 \\ 7 \\ -4 \end{pmatrix} \quad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 3 \\ -2 \end{pmatrix}$$

$$\therefore \text{Solution: } \left(\frac{1}{2}, 3, -2 \right)$$

Exercise

Use the **inverse** of the coefficient matrix to solve the linear system

$$\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$$

Solution

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -3 & 4 \\ 4 & -2 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 1 & 1 & 0 & 0 \\ 3 & -3 & 4 & 0 & 1 & 0 \\ 4 & -2 & 3 & 0 & 0 & 1 \end{array} \right) \quad \begin{array}{l} \\ 2R_2 - 3R_1 \\ 2R_3 - 4R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 1 & 1 & 0 & 0 \\ 0 & -3 & 5 & -3 & 2 & 0 \\ 0 & 0 & 2 & -4 & 0 & 2 \end{array} \right) \quad \begin{array}{l} 3R_1 - R_2 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 6 & 0 & -2 & 6 & -2 & 0 \\ 0 & -3 & 5 & -3 & 2 & 0 \\ 0 & 0 & 2 & -4 & 0 & 2 \end{array} \right) \quad \begin{array}{l} R_1 + R_3 \\ 2R_2 - 5R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 6 & 0 & 0 & 2 & -2 & 2 \\ 0 & -6 & 0 & 14 & 4 & -10 \\ 0 & 0 & 2 & -4 & 0 & 2 \end{array} \right) \quad \begin{array}{l} \frac{1}{6}R_1 \\ -\frac{1}{6}R_2 \\ \frac{1}{2}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{7}{6} & -\frac{2}{3} & \frac{5}{3} \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{7}{6} & -\frac{2}{3} & \frac{5}{3} \\ -2 & 0 & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{7}{6} & -\frac{2}{3} & \frac{5}{3} \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

∴ Solution: (0, 1, 2)

Exercise

Use the **inverse** of the coefficient matrix to solve the linear system

$$\begin{cases} x - 2y - z = 2 \\ 2x - y + z = 4 \\ -x + y + z = 4 \end{cases}$$

Solution

$$A = \begin{pmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 3 & -2 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{array}\right) \frac{1}{3}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 + 2R_2 \\ R_3 + R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 1 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & 1 \end{array}\right) \begin{array}{l} R_1 - R_3 \\ R_2 - R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{3} & \frac{1}{3} & -1 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & -1 \\ -1 & 0 & -1 \\ \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & -1 \\ -1 & 0 & -1 \\ \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} \quad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \\ 6 \end{pmatrix}$$

∴ Solution: $(-4, -6, 6)$