Section 2.2 – Norm, Dot product, and distance in R^n

Norm of a Vector

The ***length*** (or ***norm***) of a vector $\vec{v}$ is the square root of $\vec{v}\cdot\vec{v}$

$$Length = \|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$$

$$= \sqrt{x^2 + y^2} \qquad \textbf{2-dimension}$$

$$= \sqrt{x^2 + y^2 + z^2} \qquad \textbf{3-dimension}$$

Definition

If $\vec{v} = \left(v_1,\ v_2,\ \ldots,\ v_n\right)$ is a vector in $\boldsymbol{R}^n$, then the norm of $\vec{v}$ (also called the length of $\vec{v}$ or the magnitude of $\vec{v}$) is denoted by $\|\vec{v}\|$, and is defined by the formula

$$\boxed{\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \ldots + v_n^2}}$$

Example

Find the length of the vector $\vec{v} = (1,\ 2,\ 3)$

Solution

$$\|\vec{v}\| = \sqrt{1^2 + 2^2 + 3^2}$$

$$= \sqrt{14}$$

Theorem

If $\vec{v}$ is a vector in $\boldsymbol{R}^n$, and if $\boldsymbol{k}$ is any scalar, then:

a) $\|\vec{v}\| \ge 0$

b) $\|\vec{v}\| = 0 \quad iff \quad \vec{v} = \vec{0}$

c) $\|k\vec{v}\| = |k| \cdot \|\vec{v}\|$

Unit Vectors

Definition

A ***unit vector*** $\vec{u}$ is a vector whose length equals to one. Then $\vec{u}\cdot\vec{u}=1$

Divide any nonzero vector $\vec{v}$ by its length. Then $\vec{u}=\frac{\vec{v}}{\|\vec{v}\|}$ is a unit vector in the same direction as v.

Example

Find the unit vector $\vec{u}$ that has the same direction as $\vec{v}=(2, 2, -1)$

Solution

$$\|\vec{v}\|=\sqrt{2^2+2^2+(-1)^2}$$
$$=3$$

$$\vec{u}=\frac{\vec{v}}{\|\vec{v}\|}$$
$$=\frac{1}{3}(2,\ 2,\ -1)$$
$$=\left(\frac{2}{3},\ \frac{2}{3},\ -\frac{1}{3}\right)$$

$$\|\vec{u}\|=\sqrt{\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^2+\left(\frac{-1}{3}\right)^2}$$
$$=\sqrt{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}}$$
$$=\sqrt{\frac{9}{9}}$$
$$=1 \quad \checkmark$$

Example of unit vectors

$$\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \hat{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

In $\boldsymbol{R}^3$

$$\hat{i} = (1,\ 0,\ 0) \quad \hat{j} = (0,\ 1,\ 0) \quad \textit{and} \quad \hat{k} = (0,\ 0,\ 1)$$

In general, these formulas can be defined as ***standard unit vector*** in $\boldsymbol{R}^n$

$$\hat{e}_1 = (1,\ 0,\ \cdots,\ 0), \quad \hat{e}_2 = (0,\ 1,\ \cdots,\ 0), \quad \ldots, \quad \hat{e}_n = (0,\ 0,\ \cdots,\ 1)$$

$$\begin{aligned} \vec{v} &= \left(v_1,\ v_2,\ \ldots,\ v_n\right) \\ &= v_1\hat{e}_1 + v_2\hat{e}_2 + \ldots + v_n\hat{e}_n \end{aligned}$$

Example

$$(7,\ 3,\ -4,\ 5) = 7\hat{e}_1 + 3\hat{e}_2 - 4\hat{e}_3 + 5\hat{e}_4$$

Distance in $\boldsymbol{R}^n$

In $\boldsymbol{R}^2$ $\quad d = \left\| \overrightarrow{P_1P_2} \right\| = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$

In $\boldsymbol{R}^3$ $\quad d\left(\vec{u},\ \vec{v}\right) = \left\| \overrightarrow{P_1P_2} \right\| = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}$

Definition

If $\vec{u} = \left(u_1,\ u_2,\ \ldots,\ u_n\right)$ and $\vec{v} = \left(v_1,\ v_2,\ \ldots,\ v_n\right)$ are points in $\boldsymbol{R}^n$, then we denote the distance between u and v by $d\left(\vec{u},\ \vec{v}\right)$ and define it to be

$$d\left(\vec{u},\ \vec{v}\right) = \left\| \vec{u} - \vec{v} \right\|$$

$$= \sqrt{\left(u_1 - v_1\right)^2 + \left(u_2 - v_2\right)^2 + \ldots + \left(u_n - v_n\right)^2}$$

Dot Product

If $\vec{u}$ and $\vec{v}$ are nonzero vectors in $\boldsymbol{R}^2$ or $\boldsymbol{R}^3$, and if θ is he angle between $\vec{u}$ and $\vec{v}$, then the ***dot product*** (also called the ***Euclidean inner product***) of $\vec{u}$ and $\vec{v}$ is denoted by $\vec{u} \bullet \vec{v}$ and is defined as

$$\vec{u} \bullet \vec{v} = \left\| \vec{u} \right\| \ \left\| \vec{v} \right\| \ \cos\theta$$

Cosine Formula

If $\vec{u}$ and $\vec{v}$ are nonzero vectors that implies

$$\Rightarrow \ \cos\theta = \frac{\vec{u} \bullet \vec{v}}{\left\| \vec{u} \right\| . \left\| \vec{v} \right\|}$$

Example

Find the dot product of the vectors $\vec{u} = \left(0,\ 0,\ 1\right)$ and $\vec{v} = \left(0,\ 2,\ 2\right)$ and have an angle of 45°.

Solution

$\left\| \vec{u} \right\| = 1$

$\left\| \vec{v} \right\| = \sqrt{0 + 2^2 + 2^2}$

$= \sqrt{8}$

$= 2\sqrt{2}$

$$\vec{u} \bullet \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$
$$= (1)\left(2\sqrt{2}\right)\cos 45^\circ$$
$$= \left(2\sqrt{2}\right)\frac{1}{\sqrt{2}}$$
$$\underline{= 2}$$

Component Form of the Dot Product

The ***dot product*** or ***inner product*** of $\vec{v} = \left(v_1,\ v_2\right)$ and $\vec{w} = \left(w_1,\ w_2\right)$ is the number

$$\vec{v} \bullet \vec{w} = v_1 w_1 + v_2 w_2$$

Example

Find the dot product of $\vec{v} = (4,\ 2)$ and $\vec{w} = (-1,\ 2)$

Solution

$$\vec{v} \bullet \vec{w} = 4.(-1) + 2(2)$$
$$\underline{= 0}$$

- ***For dot products, zero means that the 2 vectors are perpendicular*** (= 90°).

Example

Put a weight of 4 at the point $x = -1$ and weight of 2 at the point $x = 2$. The x-axis will balance on the center point $x = 0$.

Solution

The weight balance is $4(-1) + 2(2) = 0$ (*dot product*).

In 3-dimensionals the dot product:

$$\left(v_1,\ v_2,\ v_3\right) \bullet \left(w_1,\ w_2,\ w_3\right) = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Theorem

a) $\vec{u}\bullet\vec{v}=\vec{v}\bullet\vec{u}$

b) $\vec{u}\bullet\left(\vec{v}+\vec{w}\right)=\vec{u}\bullet\vec{v}+\vec{u}\bullet\vec{w}$

c) $\vec{u}\bullet\left(\vec{v}-\vec{w}\right)=\vec{u}\bullet\vec{v}-\vec{u}\bullet\vec{w}$

d) $\left(\vec{u}+\vec{v}\right)\bullet\vec{w}=\vec{u}\bullet\vec{w}+\vec{v}\bullet\vec{w}$

e) $\left(\vec{u}-\vec{v}\right)\bullet\vec{w}=\vec{u}\bullet\vec{w}-\vec{v}\bullet\vec{w}$

f) $k\left(\vec{u}\bullet\vec{v}\right)=\left(k\vec{u}\right)\bullet\vec{v}$

g) $k\left(\vec{u}\bullet\vec{v}\right)=\vec{u}\bullet\left(k\vec{v}\right)$

h) $\vec{v}\bullet\vec{v}\geq 0$ *and* $\vec{v}\bullet\vec{v}=0$ *iff* $\vec{v}=0$

i) $\vec{0}\bullet\vec{v}=\vec{v}\bullet\vec{0}=0$

Right Angles

The dot product is $\vec{v}\bullet\vec{w}=0$ when $\vec{v}$ is perpendicular to $\vec{w}$

Proof

Perpendicular vectors: $\|\vec{v}\|^2+\|\vec{w}\|^2=\|\vec{v}-\vec{w}\|^2$

Let $\vec{v}=\left(v_1,\ v_2\right)$ & $\vec{w}=\left(w_1,\ w_2\right)$

$$
\begin{aligned}
\|\vec{v}-\vec{w}\|^2 &= \left(v_1-w_1\right)^2+\left(v_2-w_2\right)^2 \\
&= v_1^2-2v_1w_1+w_1^2+v_2^2-2v_2w_2+w_2^2 \\
&= v_1^2+w_1^2+v_2^2+w_2^2-2\left(v_1w_1+v_2w_2\right) \qquad v_1w_1+v_2w_2=0 \quad \textit{dot product} \\
&= v_1^2+w_1^2+v_2^2+w_2^2 \\
&= v_1^2+v_2^2+w_1^2+w_2^2 \\
&= \|\vec{v}\|^2+\|\vec{w}\|^2
\end{aligned}
$$

If $\vec{u}$ and U are unit vectors, then $\vec{u} \bullet \vec{U} = \cos\theta$

Certainly,

$$\left|\vec{u} \bullet \vec{U}\right| \le 1$$

$$-1 \le \cos\theta \le 1$$

$$-1 \le dot\ product \le 1$$

Schwarz Inequality

If $\vec{v}$ and $\vec{w}$ are any vectors $\Rightarrow \|\vec{v} \bullet \vec{w}\| \le \|\vec{v}\|.\|\vec{w}\|$

Proof

The dot product of $\vec{v} = (a,\ b)$ and $\vec{w} = (b,\ a)$ is $2ab$ and both lengths are $\sqrt{a^2 + b^2}$.

Then, the Schwarz inequality says that: $2ab \le a^2 + b^2$

$$a^2 + b^2 - 2ab = (a-b)^2 \ge 0$$

$$a^2 + b^2 - 2ab \ge 0$$

$$a^2 + b^2 \ge 2ab$$

This proves the Schwarz inequality:

$$2ab \le a^2 + b^2$$

$$\Rightarrow \|\vec{v} \bullet \vec{w}\| \le \|\vec{v}\|.\|\vec{w}\|$$

Theorem – Parallelogram Equation for Vectors

If $\vec{u}$ and $\vec{v}$ are vectors in $\boldsymbol{R}^n$, then

$$\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right)$$

Proof

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) + (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) \\
&= \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v} + \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v} \\
&= 2(\vec{u}\cdot\vec{u}) + 2(\vec{v}\cdot\vec{v}) \\
&= 2\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right)
\end{aligned}$$

Theorem

If $\vec{u}$ and $\vec{v}$ are vectors in $\boldsymbol{R}^n$ with the Euclidean Inner product, then

$$\vec{u}\bullet\vec{v} = \frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2$$

Exercises Section 2.2 – Norm, Dot product, and distance in R^n

1. If $\|\vec{v}\| = 5$ and $\|\vec{w}\| = 3$, what are the smallest and largest possible values of $\|\vec{v} - \vec{w}\|$ and $\vec{v}.\vec{w}$?

2. If $\|\vec{v}\| = 7$ and $\|\vec{w}\| = 3$, what are the smallest and largest possible values of $\|\vec{v} + \vec{w}\|$ and $\vec{v} \bullet \vec{w}$?

3. Given that $\cos(\alpha) = \frac{\vec{v}_1}{\|\vec{v}\|}$ and $\sin(\alpha) = \frac{\vec{v}_2}{\|\vec{v}\|}$. Similarly, $\cos(\beta) =$ ______ and $\sin(\beta) =$ ______. The angle θ is $\beta - \alpha$. Substitute into the trigonometry formula $\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$ for $\cos(\beta - \alpha)$ to find $\cos\theta = \frac{\vec{v} \bullet \vec{w}}{\|\vec{v}\|.\|\vec{w}\|}$

4. Can three vectors in the xy plane have $\vec{u} \bullet \vec{v} < 0$, $\vec{v} \bullet \vec{w} < 0$ and $\vec{u} \bullet \vec{w} < 0$?

5. Find the norm of $\vec{v}$, a unit vector that has the same direction as $\vec{v}$, and a unit vector that is oppositely directed.

a) $\vec{v} = (4, -3)$

b) $\vec{v} = (1, -1, 2)$

c) $\vec{v} = (-2, 3, 3, -1)$

6. Evaluate the given expression with $\vec{u} = (2, -2, 3)$, $\vec{v} = (1, -3, 4)$, and $\vec{w} = (3, 6, -4)$

a) $\|\vec{u} + \vec{v}\|$

b) $\|-2\vec{u} + 2\vec{v}\|$

c) $\|3\vec{u} - 5\vec{v} + \vec{w}\|$

d) $\|3\vec{v}\| - 3\|\vec{v}\|$

e) $\|\vec{u}\| + \|-2\vec{v}\| + \|-3\vec{w}\|$

7. Let $\boldsymbol{v} = (1, 1, 2, -3, 1)$. Find all scalars k such that $\|k\vec{v}\| = 5$

8. Find $\vec{u} \bullet \vec{v}$, $\vec{u} \bullet \vec{u}$, *and* $\vec{v} \bullet \vec{v}$

a) $\vec{u} = (3, 1, 4)$, $\vec{v} = (2, 2, -4)$

b) $\vec{u} = (1, 1, 4, 6)$, $\vec{v} = (2, -2, 3, -2)$

c) $\vec{u} = (2, -1, 1, 0, -2)$, $\vec{v} = (1, 2, 2, 2, 1)$

9. Find the Euclidean distance between $\vec{u}$ and $\vec{v}$, then find the angle between them

a) $\vec{u} = (3, 3, 3)$, $\vec{v} = (1, 0, 4)$

b) $\vec{u} = (1, 2, -3, 0)$, $\vec{v} = (5, 1, 2, -2)$

c) $\vec{u} = (0, 1, 1, 1, 2)$, $\vec{v} = (2, 1, 0, -1, 3)$

10. Find a unit vector that has the same direction as the given vector

a) $(-4, -3)$

b) $\left(-3,\ 2,\ \sqrt{3}\right)$

c) $(1, 2, 3, 4, 5)$

11. Find a unit vector that is oppositely to the given vector

a) $(-12, -5)$ *b)* $(3, -3, 3)$ *c)* $\left(-3,\ 1,\ \sqrt{6},\ 3\right)$

12. Verify that the Cauchy-Schwarz inequality holds

a) $\vec{u} = (-3, 1, 0),\ \vec{v} = (2, -1, 3)$

b) $\vec{u} = (0, 2, 2, 1),\ \vec{v} = (1, 1, 1, 1)$

c) $\vec{u} = (1, 3, 5, 2, 0, 1),\ \vec{v} = (0, 2, 4, 1, 3, 5)$

13. Find $\vec{u} \bullet \vec{v}$ and then the angle θ between $\vec{u}$ and $\vec{v}$ $\quad \vec{u} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$

14. Find the norm: $\|\vec{u}\| + \|\vec{v}\|,\ \|\vec{u} + \vec{v}\|$ for $\vec{u} = (3,\ -1,\ -2,\ 1,\ 4)\quad \vec{v} = (1,\ 1,\ 1,\ 1,\ 1)$

15. Find all numbers *r* such that: $\|r(1,\ 0,\ -3,\ -1,\ 4,\ 1)\| = 1$

16. Find the distance between $P_1(7, -5, 1)$ and $P_2(-7, -2, -1)$

17. Given $\vec{u} = (1, -5, 4),\ \vec{v} = (3, 3, 3)$

a) Find $\vec{u} \bullet \vec{v}$

b) Find the cosine of the angle θ between $\vec{u}$ and $\vec{v}$.

18. Let $\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ *and* $\vec{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. Find $\left\| \frac{1}{\|2\vec{u} + \vec{v}\|}(2\vec{u} + \vec{v}) \right\|$

19. Let $\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ *and* $\vec{v} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$. Find $\left\| \frac{1}{\|\vec{u} - \vec{v}\|}(\vec{u} - \vec{v}) \right\|$

20. Let $\vec{u} = \begin{pmatrix} 18 \\ 6 \end{pmatrix}$ *and* $\vec{v} = \begin{pmatrix} -11 \\ 12 \end{pmatrix}$. Find $\left\| \frac{1}{\|5\vec{u} + 3\vec{v}\|}(5\vec{u} + 3\vec{v}) \right\|$

21. Let $\vec{u} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$. Calculate the following:

a) $\vec{u} + \vec{v}$ *b)* $2\vec{u} + 3\vec{v}$ *c)* $\vec{v} + (2\vec{u} - 3\vec{v})$ *d)* $\|\vec{u}\|$ *e)* $\|\vec{v}\|$ *f)* *unit vector of* $\vec{v}$

22. Let $\vec{u} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 4 \\ 3 \\ 1 \end{pmatrix}$. Calculate the following:

a) $\vec{u} - \vec{v}$ *b)* $3\vec{u} - 2\vec{v}$ *c)* $2(\vec{u} - \vec{v}) + 3\vec{u}$ *d)* $\|\vec{u}\|$ *e) unit vector of* $\vec{v}$

23. Let $\vec{u} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \\ -1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \\ -1 \end{pmatrix}$. Calculate the following:

a) $\vec{v} - \vec{u}$ *b)* $\vec{u} + 3\vec{v}$ *c)* $3(\vec{u} + \vec{v}) - 3\vec{u}$ *d)* $\|\vec{v}\|$ *e) unit vector of* $\vec{v}$

24. Let $\vec{u} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$. Calculate the following:

a) $\vec{u} \bullet \vec{v}$ *b)* $\vec{u} \bullet (\vec{v} + \vec{w})$ *c)* $(\vec{u} + 2\vec{v}) \bullet \vec{w}$ *d)* $\|(\vec{w} \bullet \vec{v})\vec{u}\|$

25. Let $\vec{u} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -2 \\ 5 \\ 2 \\ -6 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 4 \\ -1 \\ 0 \\ -2 \end{pmatrix}$. Calculate the following:

a) $\vec{u} \bullet \vec{v}$ *b)* $\vec{u} \bullet (\vec{v} + \vec{w})$ *c)* $(\vec{u} + \vec{v}) \bullet (\vec{u} - \vec{v})$ *d)* $\|(\vec{w} \bullet \vec{v})\vec{u}\|$

26. Let $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$. Calculate the following:

a) $\vec{u} \bullet (\vec{v} + \vec{w})$ *b)* $\|(\vec{w} \bullet \vec{v})\vec{u}\|$ *c)* $(\vec{u} \bullet \vec{w})\vec{v} + (\vec{v} \bullet \vec{w})\vec{u}$ *d)* $(\vec{u} + 2\vec{v}) \bullet (\vec{u} - \vec{v})$

27. Suppose $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^n$ such that $\vec{u} \bullet \vec{v} = 2$, $\vec{u} \bullet \vec{w} = -3$, and $\vec{v} \bullet \vec{w} = 5$. If possible, calculate the following values:

a) $\vec{u} \bullet (\vec{v} + \vec{w})$
b) $(\vec{u} + \vec{v}) \bullet \vec{w}$
c) $\vec{u} \bullet (2\vec{v} - \vec{w})$
d) $\vec{w} \bullet (2\vec{v} - 4\vec{u})$
e) $(\vec{u} + \vec{v}) \bullet (\vec{v} + \vec{w})$
f) $\vec{w} \bullet (5\vec{v} + \pi\vec{u})$
g) $\vec{w} \bullet ((\vec{u} \bullet \vec{w})\vec{u})$
h) $\vec{u} \bullet ((\vec{u} \bullet \vec{v})\vec{v} + (\vec{u} \bullet \vec{w})\vec{w})$

28. You are in an airplane flying from Chicago to Boston for a job interview. The compass in the cockpit of the plane shows that your plane is pointed due East, and the airspeed indicator on the plane shows that the plane is traveling through the air at 400 *mph*. there is a crosswind that affects your plane however, and the crosswind is blowing due South at 40 *mph*.
Given the crosswind you wonder; relative to the ground, in what direction are you really flying and how fast are you really traveling?

29. A jet airliner, flying due east at 500 *mph* in still air, encounters a 70-*mph* tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What speed and direction should the jetliner have in order for the resultant vector to be 500 *mph* due east?

30. A jet airliner, flying due east at 500 *mph* in still air, encounters a 70-*mph* tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What are they?

31. A bird flies from its nest 5 *km* in the direction 60° north east, where it stops to rest on a tree. It then flies 10 *km* in the direction due southeast and lands atop a telephone pole. Place an *xy*-coordinate system so that the origin is the bird's nest, the *x*-axis points east, and the *y*-axis points north.

a) At what point is the tree located?

b) At what point is the telephone pole?

32. Prove $\vec{u} \bullet \vec{u} = \|\vec{u}\|^2 \ \geq 0$

33. Prove, for any vectors and $\vec{v}$ in $\mathbb{R}^2$ and any scalars c and d,

$$\left(c\vec{u} + d\vec{v}\right) \bullet \left(c\vec{u} + d\vec{v}\right) = c^2 \|\vec{u}\|^2 + 2cd\vec{u} \bullet \vec{v} + d^2 \|\vec{v}\|^2$$

34. Prove $\vec{u} \bullet \left(\vec{v} + \vec{w}\right) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$

35. Prove Minkowski theorem: $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$