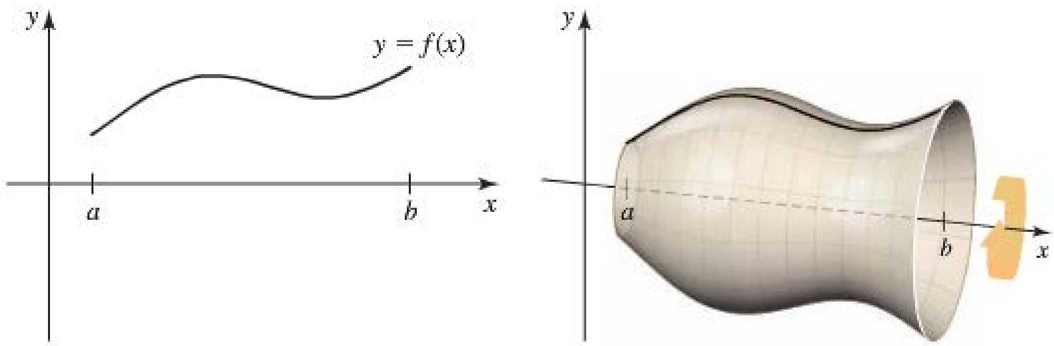
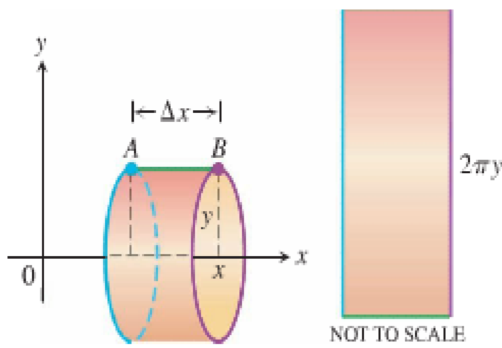
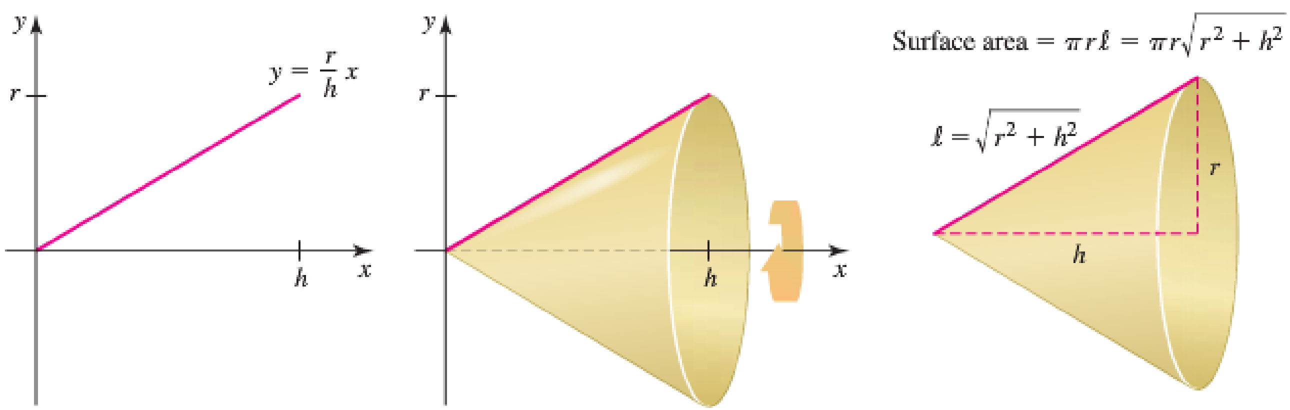
Section 1.6 – Surface Area

Consider a curve $y = f(x)$ on an interval $[a,\ b]$, where *f* is a nonnegative function with a continuous first derivative on $[a,\ b]$. Revolving the curve about the *x-axis* to generate a surface of revolution.



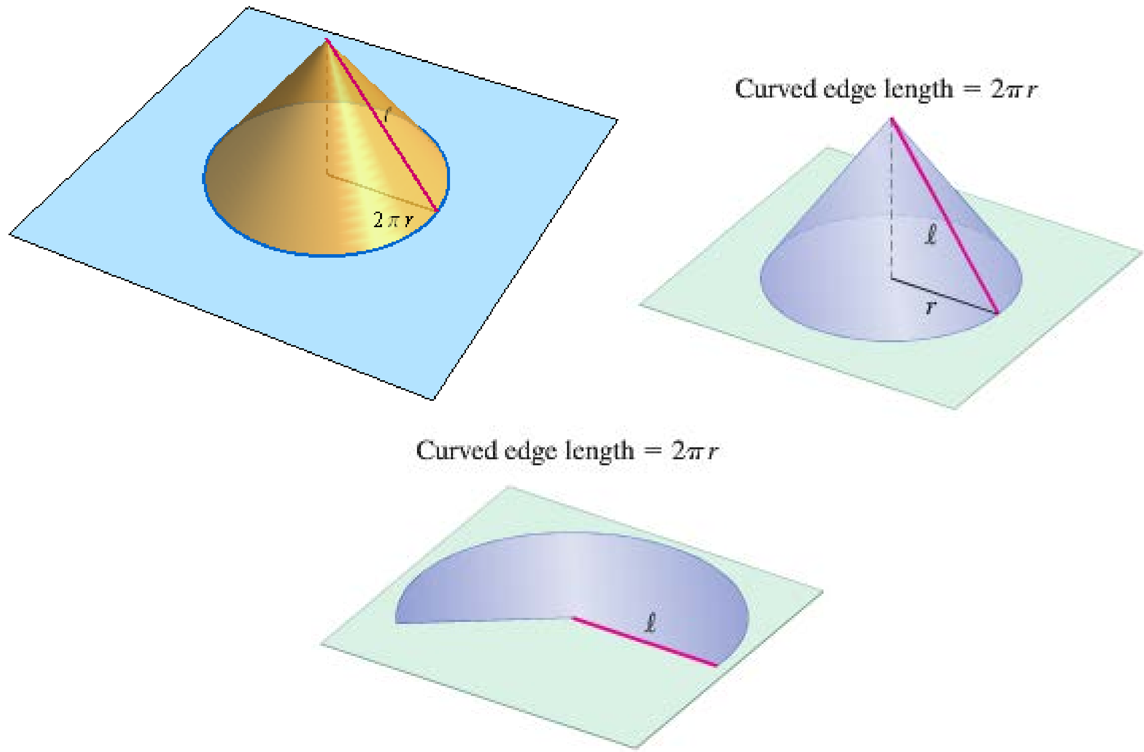


Consider the graph of $f(x) = \frac{r}{h}x$ on the interval $[0,\ h]$, where $h > 0$ and $r > 0$. When this line segment is revolved about the *x-axis*, it generates the surface of a cone of radius *r* and height *h*,

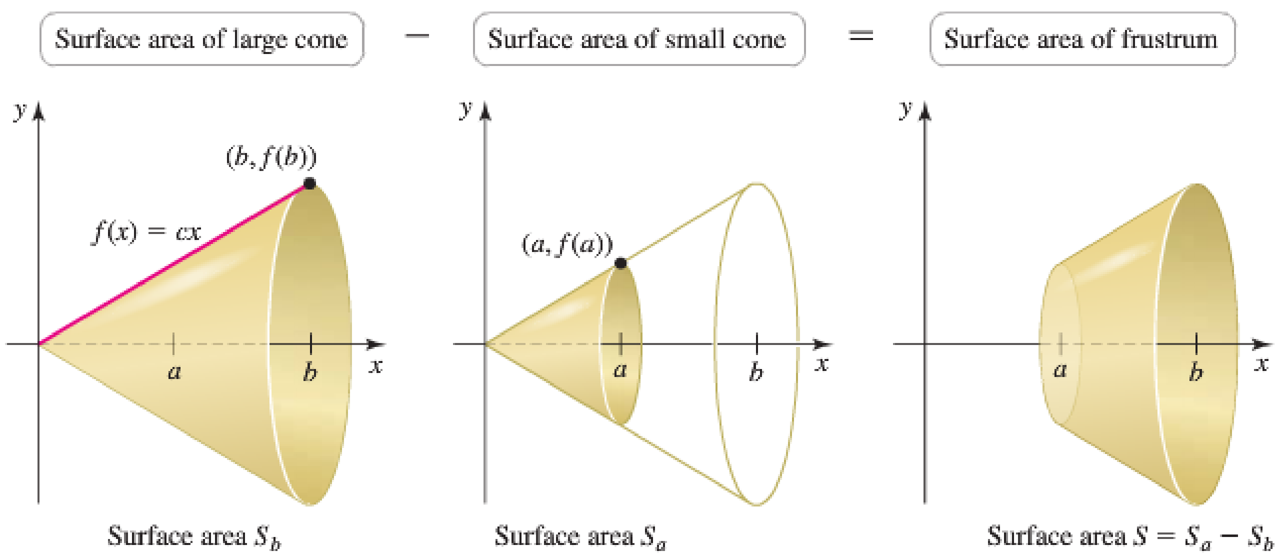


The surface area of a right circular cone, excluding the base, is $\pi r\sqrt{r^2 + h^2} = \pi r\ell$

One way to derive the formula for the surface area of a cone to cut the cone on a line from its base to its vertex. When the cone is unfolded it forms a sector of a circular disk of radius ℓ. So the area of the sector, which is also the surface area of the cone, is $\pi\ell^2\frac{r}{\ell} = \pi r\ell$



ℓ
$2\pi r$
Curved edge length = $2\pi r$
ℓ
r
Curved edge length = $2\pi r$
ℓ



Surface area of large cone − Surface area of small cone = Surface area of frustrum
y
$(b, f(b))$
$f(x) = cx$
a
b
x
Surface area S_b
y
$(a, f(a))$
a
b
x
Surface area S_a
y
a
b
x
Surface area $S = S_a - S_b$

Definition

If the function $f(x) \ge 0$ is continuously differentiable on [*a*, *b*], the area of the surface generated by revolving the graph of $y = f(x)$ about the *x-axis* is

$$S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_a^b f(x)\sqrt{1+\left(f'(x)\right)^2}\, dx$$

Example

Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}, \quad 1 \le x \le 3$, about the *x-axis*.

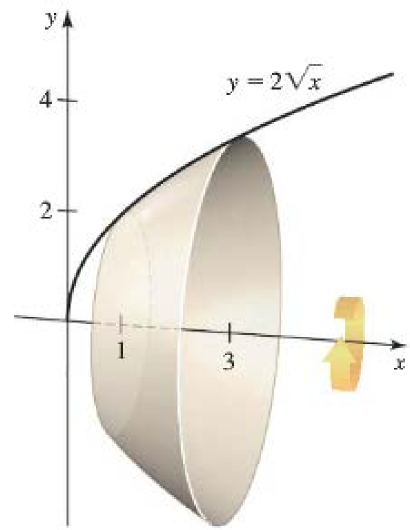
Solution

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}}, \quad a = 1, \quad b = 3$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\frac{1}{\sqrt{x}}\right)^2}$$

$$= \sqrt{1+\frac{1}{x}}$$

$$= \sqrt{\frac{x+1}{x}}$$



$$S = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 4\pi \int_1^3 \left(\sqrt{x}\right)\frac{\sqrt{x+1}}{\sqrt{x}}\, dx$$

$$= 4\pi \int_1^3 (x+1)^{1/2}\, dx$$

$$= \frac{8\pi}{3}(x+1)^{3/2}\Big|_1^3$$

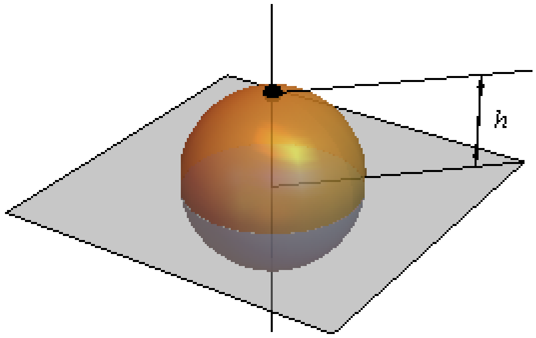
$$= \frac{8\pi}{3}\left(4^{3/2} - 2^{3/2}\right)$$

$$= \frac{8\pi}{3}\left(8 - 2\sqrt{2}\right)$$

$$= \frac{16\pi}{3}\left(4 - \sqrt{2}\right)\ unit^2$$

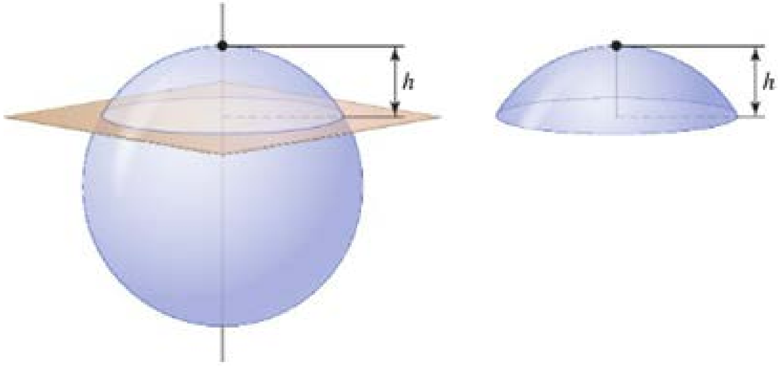
Example

A spherical cap is produced when a sphere of radius a is sliced by a horizontal plane that is a vertical distance h below the north pole of the sphere, where $0 \le h \le 2a$. We take the spherical cap to be that part of the sphere above the plane, so that h is the depth of the cap. Show that the area of a spherical cap of depth h cut from sphere of radius a is $2\pi ah$.



Solution

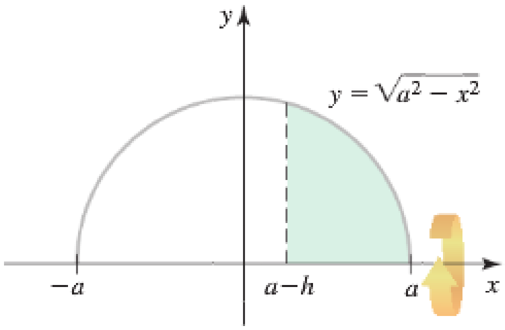
To generate the spherical surface, we revolved the curve $f(x) = \sqrt{a^2 - x^2}$ on the interval $[-a,\ a]$ about the $x\text{-}axis$.



The spherical cap of height h corresponds to that part of the sphere on the interval $[-a+h,\ a]$ for $0 \le h \le 2a$

$$f'(x) = -x\left(a^2 - x^2\right)^{-1/2}$$

$$1 + f'(x)^2 = 1 + \frac{x^2}{a^2 - x^2} = \frac{a^2}{a^2 - x^2}$$



$$S = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\,dx$$

$$= 2\pi \int_{a-h}^{a} \sqrt{a^2 - x^2}\,\frac{a}{\sqrt{a^2 - x^2}}\,dx$$

$$= 2\pi \int_{a-h}^{a} a\,dx$$

$$= 2\pi a x\Big|_{a-h}^{a}$$

$$= 2\pi ah \quad unit^2$$

Surface Area for revolution about the y-axis

If $x = g(y) \ge 0$ is continuously differentiable on [*c*, *d*], the area of the surface generated by revolving the graph of $x = g(y)$ about the *y*-axis is

$$S = \int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y)\sqrt{1+\left(g'(y)\right)^2}\, dy$$

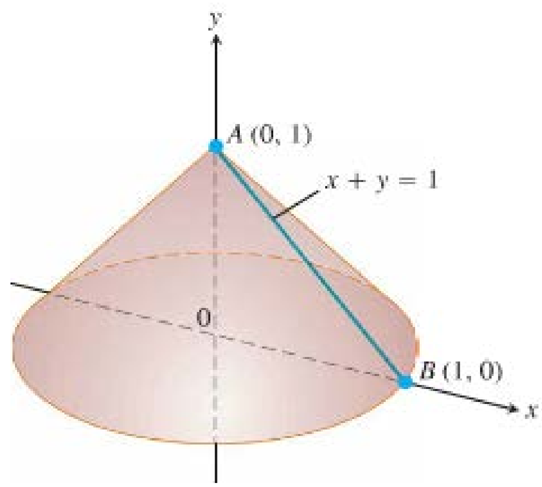
Example

The line segment $x = 1 - y, \quad 0 \le y \le 1$, is revolved about the *y*-axis to generate the cone. Find its lateral surface area (which excludes the base area)

Solution

$$Lateral\ Surface\ Area = \frac{base\ circumference}{2} \times slant\ height\ = \pi\sqrt{2}$$

$$x = 1 - y \quad \frac{dx}{dy} = -1, \quad c = 0, \quad d = 1$$



$$S = \int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_0^1 2\pi(1-y)\sqrt{1+(-1)^2}\, dy$$

$$= 2\pi\int_0^1 (1-y)\sqrt{2}\, dy$$

$$= 2\pi\sqrt{2}\left[y - \frac{y^2}{2}\right]_0^1$$

$$= 2\pi\sqrt{2}\left(1 - \frac{1}{2}\right)$$

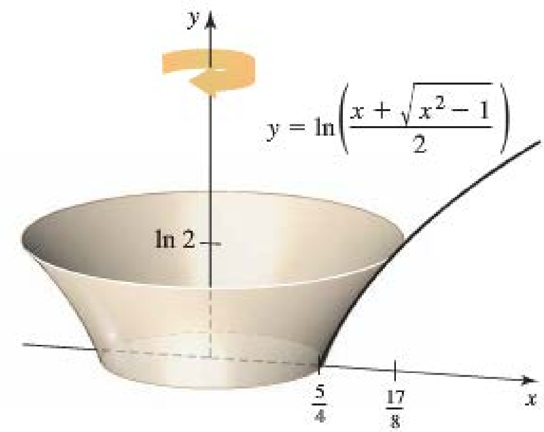
$$\boxed{= \pi\sqrt{2}\ \ unit^2}$$

Example

Consider the function $y = \ln\left(\dfrac{x+\sqrt{x^2-1}}{2}\right)$

Find the area of the surface generated when the part of the curve between the points $\left(\frac{5}{4},\ 0\right)$ and $\left(\frac{17}{8},\ \ln 2\right)$ is revolved about *y-axis*.

Solution



$$y = \ln\left(\frac{x+\sqrt{x^2-1}}{2}\right) \quad \rightarrow \quad e^y = \frac{x+\sqrt{x^2-1}}{2}$$

$$\left(2e^y - x\right)^2 = \left(\sqrt{x^2-1}\right)^2$$

$$4e^{2y} - 4xe^y + x^2 = x^2 - 1$$

$$4xe^y = 4e^{2y} + 1 \quad \Rightarrow \quad x = e^y + \frac{1}{4}e^{-y} = g(y)$$

$$g'(y) = e^y - \frac{1}{4}e^{-y}$$

$$\sqrt{1+g'(y)^2} = \sqrt{1+\left(e^y - \frac{1}{4}e^{-y}\right)^2}$$

$$= \sqrt{1+e^{2y} - \frac{1}{2} + \frac{1}{16}e^{-2y}}$$

$$= \sqrt{e^{2y} + \frac{1}{2} + \frac{1}{16}e^{-2y}}$$

$$= \sqrt{\left(e^y + \frac{1}{4}e^{-y}\right)^2}$$

$$= e^y + \frac{1}{4}e^{-y}$$

$$S = 2\pi\int_0^{\ln 2}\left(e^y + \frac{1}{4}e^{-y}\right)^2 dy$$

$$= 2\pi\int_0^{\ln 2}\left(e^{2y} + \frac{1}{2} + \frac{1}{16}e^{-2y}\right)dy$$

$$= 2\pi\left(\frac{1}{2}e^{2y} + \frac{y}{2} - \frac{1}{32}e^{-2y}\right)\Bigg|_0^{\ln 2}$$

$$= 2\pi\left(2 + \frac{\ln 2}{2} - \frac{1}{128} - \frac{1}{2} + \frac{1}{32}\right)$$

$$= \pi\left(\frac{195}{64} + \ln 2\right)\ unit^2$$

Exercises *Section* 1.6 – Surface Area

1. Find the lateral (side) surface area of the cone generated by revolving the line segment $y = \frac{x}{2}, \quad 0 \le x \le 4$, about the *x*-axis. Check your answer with the geometry formula

$$\textit{Lateral surface area} = \frac{1}{2} \times \textit{base circumference} \times \textit{slant height}$$

2. Find the lateral surface area of the cone generated by revolving the line segment $y = \frac{x}{2}, \quad 0 \le x \le 4$, about the *y*-axis. Check your answer with the geometry formula

$$\textit{Lateral surface area} = \frac{1}{2} \times \textit{base circumference} \times \textit{slant height}$$

3. Find the lateral surface area of the cone frustum generated by revolving the line segment $y = \frac{x}{2} + \frac{1}{2}, \quad 1 \le x \le 3$, about the *x*-axis. Check your answer with the geometry formula

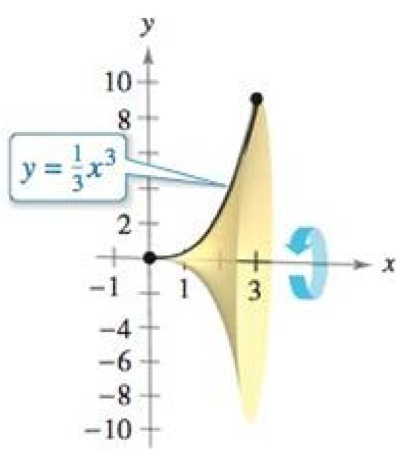
$$\textit{Frustum surface area} = \pi\left(r_1 + r_2\right) \times \textit{slant height}$$

4. Find the lateral surface area of the cone frustum generated by revolving the line segment $y = \frac{x}{2} + \frac{1}{2}, \quad 1 \le x \le 3$, about the *y*-axis. Check your answer with the geometry formula

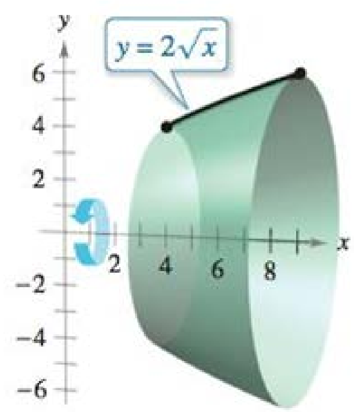
$$\textit{Frustum surface area} = \pi\left(r_1 + r_2\right) \times \textit{slant height}$$

Find the area of the surface generated by revolving the curve about the *x-axis*

5.



6.



7. $y = \frac{x^3}{9}, \quad 0 \le x \le 2$

8. $y = \sqrt{x+1}, \quad 1 \le x \le 5$

9. $y = \sqrt{2x - x^2}, \quad 0.5 \le x \le 1.5$

10. $y = 3x + 4, \quad 0 \le x \le 6$

11. $y = 12 - 3x, \quad 1 \le x \le 3$

12. $y = x^{3/2} - \frac{1}{3}x^{1/2}, \quad 1 \le x \le 2$

13. $y = \sqrt{4x+6}, \quad 0 \le x \le 5$

14. $y = \frac{1}{4}\left(e^{2x} + e^{-2x}\right), \quad -2 \le x \le 2$

15. $y = \frac{1}{8}x^4 + \frac{1}{4x^2}, \quad 1 \le x \le 2$

16. $y = 8\sqrt{x}, \quad 9 \le x \le 20$

17. $y = x^3, \quad 0 \le x \le 1$

18. $y = \frac{1}{3}x^3 + \frac{1}{4x}, \quad \frac{1}{2} \le x \le 2$

19. $y = \sqrt{5x - x^2}, \quad 1 \le x \le 4$

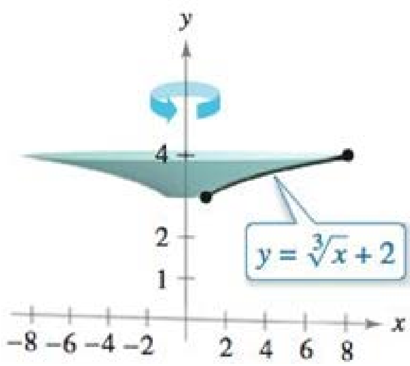
20. $y = \frac{1}{6}x^3 + \frac{1}{2x}, \quad 1 \le x \le 2$

21. $y = \sqrt{4 - x^2}, \quad -1 \le x \le 1$

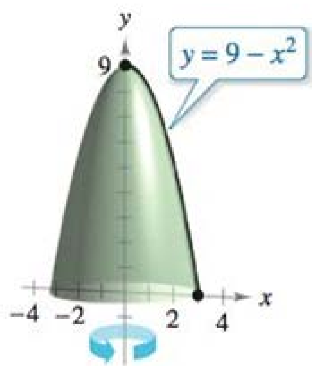
22. $y = \sqrt{9 - x^2}, \quad -2 \le x \le 2$

Find the area of the surface generated by revolving the curve about the *y-axis*

23.



24.



25. $y = (3x)^{1/3}; \quad 0 \le x \le \frac{8}{3}$

26. $x = \sqrt{12y - y^2}; \quad 2 \le y \le 10$

27. $x = 4y^{3/2} - \frac{1}{12}y^{1/2}; \quad 1 \le y \le 4$

28. $y = 1 - \frac{1}{4}x^2, \quad 0 \le x \le 2$

29. $y = \frac{1}{2}x + 3, \quad 1 \le x \le 5$

30. A right circular cone is generated by revolving the region bounded by $y = \frac{3}{4}x$, $y = 3$, and $x = 0$ about the *y-axis*. Find the lateral surface area of the cone.

31. A right circular cone is generated by revolving the region bounded by $y = \frac{h}{r}x$, $y = h$, and $x = 0$ about the *y-axis*. Verify that the lateral surface area of the cone is $S = \pi r\sqrt{r^2 + h^2}$

32. Find the area of the zone of a sphere formed by revolving the graph of $y = \sqrt{9 - x^2}, \quad 0 \le x \le 2$, about the *y-axis*

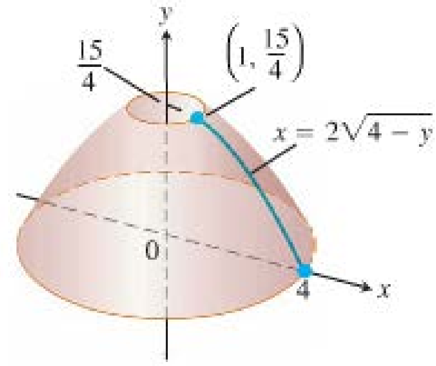
33. Find the area of the zone of a sphere formed by revolving the graph of $y = \sqrt{r^2 - x^2}, \quad 0 \le x \le a$, about the *y-axis*. Assume that $a < r$.

34. Part of the curve $y = 4x - 1$ between the points $(1,\ 3)$ and $(4,\ 15)$ *about y-axis*

35. Part of the curve $y = \frac{1}{2}\ln\left(2x + \sqrt{4x^2 - 1}\right)$ between the points $\left(\frac{1}{2},\ 0\right)$ and $\left(\frac{17}{16},\ \ln 2\right)$ *about y-axis*

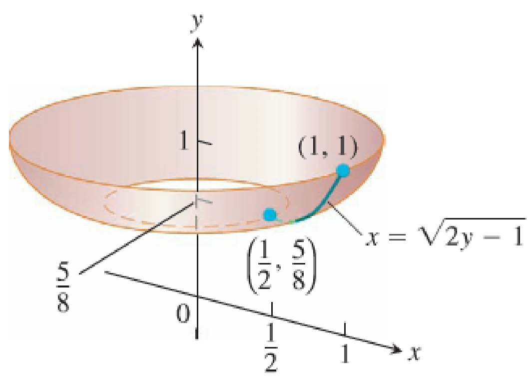
36. $y = 1 + \sqrt{1 - x^2}$ between the points $(1, 1)$ and $\left(\frac{\sqrt{3}}{1}, \frac{3}{2}\right)$ *about y-axis*

37. Find the area of the surface generated by $x = 2\sqrt{4 - y}$ $\quad 0 \le y \le \frac{15}{4}, \quad y - axis$

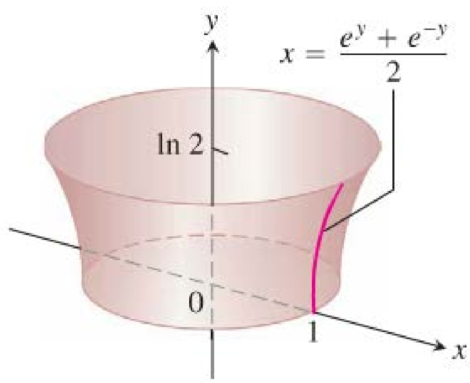


38. $y = \frac{1}{3}\left(x^2 + 2\right)^{3/2}, \quad 0 \le x \le \sqrt{2}; \quad y - axis$ (Hint: Express $ds = \sqrt{dx^2 + dy^2}$ in terms of dy, and evaluate the integral $S = \displaystyle\int 2\pi y \; ds$ with appropriate limits.)

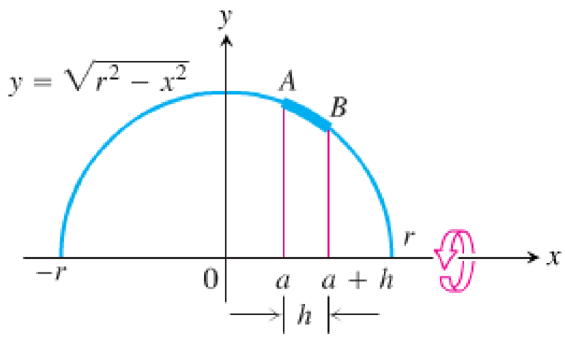
39. Find the area of the surface generated by $x = \sqrt{2y - 1}$ $\quad \frac{5}{8} \le y \le 1, \quad y - axis$



40. Find the area of the surface generated by revolving the curve $x = \frac{1}{2}\left(e^y + e^{-y}\right), \quad 0 \le y \le \ln 2,$ about y-axis



41. Did you know that if you can cut a spherical loaf of bread into slices of equal width, each slice will have the same amount of crust? To see why, suppose the semicircle $y=\sqrt{r^2-x^2}$ shown here is revolved about the *x*-axis to generate a sphere. Let *AB* be an arc of the semicircle that lies above an interval of length *h* on the *x*-axis. Show that the area swept out by *AB* does not depend on the location of the interval. (It does depend on the length of the interval.)



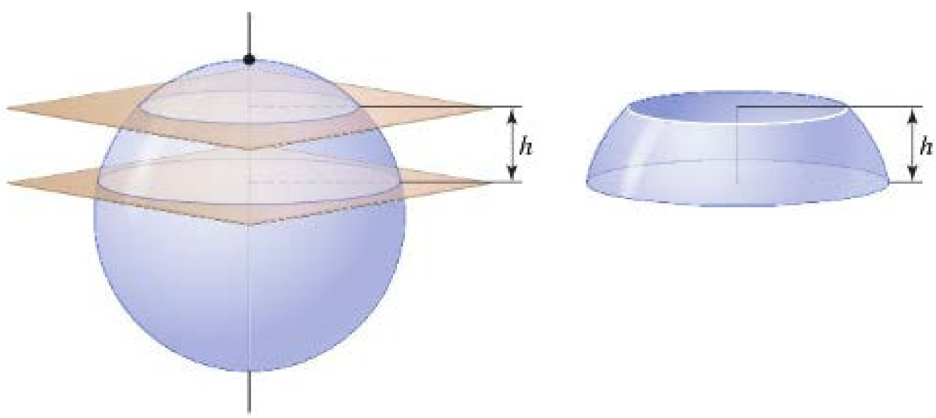
42. The curved surface of a funnel is generated by revolving the graph of $y=f(x)=x^3+\frac{1}{12x}$ on the interval $[1,\ 2]$ about the x-*axis*. Approximately what volume of paint is needed to cover the outside of the funnel with a layer of paint 0.05 *cm* thick? Assume that *x* and *y* measured in centimeters.

43. When the circle $x^2+(y-a)^2=r^2$ on the interval $[-r,\ r]$ is revolved about the x-*axis*, the result is the surface of a torus, where $0<r<a$. Show that the surface area of the torus is $S=4\pi^2 ar$.

44. A 1.5–*mm* layer of paint is applied to one side. Find the approximate volume of paint needed of the spherical zone generated when the curve $y=\sqrt{8x-x^2}$ on the interval $[1,\ 7]$ is revolved about the x-*axis*. Assume *x* and *y* are in *meters*.

45. A 1.5–*mm* layer of paint is applied to one side. Find the approximate volume of paint needed of the spherical zone generated when the upper portion of the circle $x^2+y^2=100$ on the interval $[-8,\ 8]$ is revolved about the x-*axis*. Assume *x* and *y* are in *meters*.

46. Suppose a sphere of radius *r* is sliced by two horizontal planes *h* units apart. Show that the surface area of the resulting zone on the sphere is $2\pi h$, independent of the location of the cutting planes.



47. An ornamental light bulb is designed by revolving the graph of $y = \frac{1}{3}x^{1/2} - x^{3/2}, \quad 0 \le x \le \frac{1}{3}$ about the *x-axis* , where *x* and *y* are mesured in *feet*. Find the surface area of the bulb and use the result to approximate the amount of glass needed to make the bulb.
(Assume that the glass is 0.015 *inch* thick)