Solution *Section* 4.5 – Diagonalization

Exercise

The Lucas numbers are like Fibonacci numbers except they start with $L_1 = 1$ *and* $L_2 = 3$. Following the rule $L_{k+2} = L_{k+1} + L_k$. The next Lucas numbers are 4, 7, 11, 18. Show that the Lucas number $L_{100} = \lambda_1^{100} + \lambda_2^{100}$.

Solution

Let $u_k = \begin{pmatrix} L_{k+1} \\ L_k \end{pmatrix}$

the rule $\begin{cases} L_{k+2} = L_{k+1} + L_k \\ L_{k+1} = L_{k+1} \end{cases}$

becomes $\vec{u}_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \vec{u}_k$.

$\Rightarrow A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} \\ &= -\lambda(1-\lambda) - 1 \\ &= \lambda^2 - \lambda - 1 \end{aligned}$$

The characteristic equation is $\lambda^2 - \lambda - 1 = 0$ and the solutions are

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618 \quad \textit{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -.618$$

For $\lambda_1 \quad \Rightarrow \quad (A - \lambda_1 I)\vec{v}_1 = 0$

$$\begin{pmatrix} 1-\lambda_1 & 1 \\ 1 & -\lambda_1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad x_1 - \lambda_1 y_1 = 0$$

$x_1 = \lambda_1 y_1$

$\Rightarrow \vec{v}_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$

For $\lambda_2 \quad \Rightarrow \quad (A - \lambda_2 I)\vec{v}_2 = 0$

$$\begin{pmatrix} 1-\lambda_2 & 1 \\ 1 & -\lambda_2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x_1 - \lambda_2 y_1 = 0$$

$$\underline{x_1 = \lambda_2 y_1}$$

$$\Rightarrow \underline{\vec{v}_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}}$$

The linear combination:

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{u}_1$$

$$= \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\lambda_1\vec{v}_1 + \lambda_2\vec{v}_2 = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1^2 + \lambda_2^2 \\ \lambda_1 + \lambda_2 \end{pmatrix}$$

$$= \begin{bmatrix} \text{trace of } A^2 \\ \text{trace of } A \end{bmatrix}$$

$$= \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

The solution:

$$\underline{\vec{u}_{100} = A^{99}\,\vec{u}_1}$$

$$L_{100} = c_1\lambda_1^{99} + c_2\lambda_2^{99}$$

$$\underline{= \lambda_1^{100} + \lambda_2^{100}}$$

Exercise

Find all eigenvector matrices S that diagonalize A (rank 1) to give $S^{-1}AS = \Lambda$:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

What is A^n? Which matrices B commute with A (so that $AB = BA$)

Solution

Since A has rank 1, its nullspace is a two-dimensional plane. Any vector with $x + y + z = 0$ solves $A\vec{v} = \vec{0}$. So $\lambda = 0$ is an eigenvalue with multiplicity 2. There are two independent eigenvectors. The other eigenvalues must be $\lambda = 3$ because the trace A is $1 + 1 + 1 = 3$.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^3 + 2 - 3(1-\lambda)$$

$$= -\lambda^3 + 3\lambda^2$$

The eigenvalues are $\underline{\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 3}$

For $\lambda_{1,2} = 0 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x + y + z = 0$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \& \quad V_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

For $\lambda_3 = 3 \quad \Rightarrow \quad (A - \lambda_3 I)\vec{v}_3 = 0$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow x_3 + y_3 - 2z_3$$

$$\Rightarrow \quad V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The possible matrices S:

$$S = \begin{pmatrix} x & X & c \\ y & Y & c \\ -x-y & -X-Y & c \end{pmatrix}$$

and

$$S^{-1}AS = \Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

where $c \neq 0$ *and* $xY \neq yX$.

The powers A^n come:

$$A^2 = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} = 3A$$

and

$$A^n = 3^{n-1}A$$

If $AB = BA$, all the column and row of $\boldsymbol{B}$ must be the same.

One possible $\boldsymbol{B}$ is $\boldsymbol{A}$ itself, since $AA = AA$, $\boldsymbol{B}$ is any linear combination of permutation matrices.

Exercise

Determine whether the matrix is diagonalizable $\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$

Solution

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)^2 = 0$$

The eigenvalues are: $\lambda_{1,2} = 2$

For $\lambda_{1,2} = 2 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x = 0$$

$$\Rightarrow \quad \boxed{V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$$S = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$$

$$\boxed{\det(S) = 0}$$

The inverse doesn't exist.

Therefore, the matrix A is not diagonalizable.

Exercise

Determine whether the matrix is diagonalizable $\begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$

Solution

$$A = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix}$$

$$= (-3-\lambda)(1-\lambda) + 4$$

$$= \lambda^2 + 2\lambda + 1 = 0$$

The only eigenvalue: $\boxed{\lambda_{1,2} = -1}$

For $\lambda_{1,2} = -1 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad -2x + 2y = 0$$

$$\boxed{x = y}$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad V_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$ (***linearly dependent***)

$$S = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$$\boxed{\det(S) = 0}$$

The inverse doesn't exist.

Therefore, the matrix A is not diagonalizable.

Exercise

Determine whether the matrix is diagonalizable $\begin{pmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{pmatrix}$

Solution

$$A = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 0 & 1 \\ -1 & 3-\lambda & 0 \\ -4 & 13 & -1-\lambda \end{vmatrix}$$

$$= (-1-\lambda)(3-\lambda)(-1-\lambda) - 13 + 4(3-\lambda)$$

$$= \left(1 + 2\lambda + \lambda^2\right)(3-\lambda) - 13 + 12 - 4\lambda$$

$$= 3 + 6\lambda + 3\lambda^2 - \lambda - 2\lambda^2 - \lambda^3 - 1 - 4\lambda$$

$$= -\lambda^3 + \lambda^2 + \lambda + 2 = 0$$

$$\begin{array}{c|cccc} 2 & -1 & 1 & 1 & 2 \\ & & -2 & -2 & -2 \\ \hline & -1 & -1 & -1 & 0 \end{array} \quad \to \quad \lambda^2 + \lambda + 1 = 0$$

The eigenvalues are: $\lambda_1 = 2, \quad \lambda_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \lambda_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

For $\lambda_1 = 2 \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -3 & 0 & 1 \\ -1 & 1 & 0 \\ -4 & 13 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \to \quad -3x_1 + z_1 = 0 \\ \to \quad -x_1 + y_1 = 0 \\ \to \quad -4x_1 + 13y_1 - 3z_1 = 0 \end{array}$$

$$\begin{cases} z_1 = 3x_1 \\ y_1 = x_1 \end{cases}$$

$$\Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

Exercise

Determine whether the matrix is diagonalizable $\begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Solution

Since the matrix is upper triangular with diagonal entries 2, 2, 3, and 3, the eigenvalues are 2 and 3 (each multiplicity of 2)

For $\lambda_1 = 2 \quad \Rightarrow \quad (A - 2I)V_1 = 0$

$$\begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{cases} -x_2 + x_4 = 0 \\ x_3 - x_4 = 0 \\ x_3 + 2x_4 = 0 \\ x_4 = 0 \end{cases}$$

$$\Rightarrow \quad \underline{x_2 = x_3 = x_4 = 0}$$

$V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ has dimension 1.

For $\lambda_2 = 3 \quad \Rightarrow \quad (A - 3I)V_2 = 0$

$$\begin{bmatrix} -1 & -1 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{cases} -x_1 - x_2 + x_4 = 0 \\ -x_2 + x_3 - x_4 = 0 \\ 2x_4 = 0 \end{cases}$$

$$\begin{cases} \boxed{x_1 = -x_2} \\ \boxed{x_3 = x_2} \\ \boxed{x_4 = 0} \end{cases}$$

$$V_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

The matrix is not diagonalizable since it has only 2 distinct eigenvectors. Note that showing that the geometric multiplicity of either eigenvalue is less than its algebraic multiplicity is sufficient to show that the matrix is not diagonalizable.

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix ***P*** that diagonalizes *A* and determine $P^{-1}AP$

$$A = \begin{pmatrix} -14 & 12 \\ -20 & 17 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} -14-\lambda & 12 \\ -20 & 17-\lambda \end{vmatrix}$$

$$= \lambda^2 - 3\lambda + 2 = 0$$

The eigenvalues are: $\boxed{\lambda_1 = 2 \quad \lambda_2 = 1}$

For $\lambda_1 = 1 \quad \Rightarrow \quad (A - I)V_1 = 0$

$$\begin{pmatrix} -15 & 12 \\ -20 & 16 \end{pmatrix} \begin{pmatrix} x_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -15x_1 + 12y_1 = 0 \\ -20x_1 + 16y_1 = 0 \end{cases} \to \boxed{5x_1 = 4y_1}$$

Therefore, the eigenvector: $\boxed{V_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}}$

For $\lambda_2 = 3 \quad \Rightarrow \quad (A - 3I)V_2 = 0$

$$\begin{pmatrix} -16 & 12 \\ -20 & 15 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -16x_2 + 12y_2 = 0 \\ -20x_2 + 15y_2 = 0 \end{cases} \rightarrow 4x_2 = 3y_2$$

Therefore, the eigenvector: $\underline{V_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}}$

The eigenvectors matrix form:

$$P = \begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 4 & -3 \\ -5 & 4 \end{pmatrix}$$

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} 4 & -3 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} -14 & 12 \\ -20 & 17 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 4 & -3 \\ -10 & 8 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \end{aligned}$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix P that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{vmatrix}$$

$$= \lambda^2 - 1 = 0$$

The eigenvalues are: $\underline{\lambda_1 = -1, \quad \lambda_2 = 1}$

For $\lambda_1 = -1 \quad \Rightarrow \quad (A + I)V_1 = 0$

$$\begin{pmatrix} 2 & 0 \\ 6 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow 2x_1 = 0$$

$$\Rightarrow \underline{x_1 = 0}$$

Therefore, the eigenvector: $\underline{V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

For $\lambda_2 = 1 \;\Rightarrow\; (A-I)V_2 = 0$

$$\begin{pmatrix} 0 & 0 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow\; 6x_2 - 2y_2 = 0$$

$$\rightarrow\; \underline{3x_2 = y_2}$$

Therefore, the eigenvector: $\underline{V_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}}$

The ***eigenvector matrix*** is given by:

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} -3 & 1 \\ 1 & 0 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -3 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \checkmark$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix ***P*** that diagonalizes *A* and determine $P^{-1}AP$

$$A = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}$$

Solution

Upper triangular; the eigenvalues are the main diagonal entries.

The eigenvalues are: $\underline{\lambda_{1,2} = 3}$

For $\lambda_{1,2} = 3 \;\Rightarrow\; (A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow\; 2y_1 = 0$$

$$\Rightarrow\; \underline{y_1 = 0}$$

Therefore, the eigenvector: $\boxed{V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

Since, the eigenvalues are repeated, then the matrix A is ***not*** diagonalizable.

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{vmatrix}$$

$$= 8 - 6\lambda + \lambda^2 + 1$$

$$= \lambda^2 - 6\lambda + 9 = 0$$

The eigenvalues are: $\boxed{\lambda_{1,2} = 3}$

For $\lambda_{1,2} = 3 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow x_1 + y_1 = 0$$

$$\Rightarrow \quad \boxed{x_2 = -y_1}$$

Therefore, the eigenvector: $\boxed{V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

Since, the eigenvalues are repeated, then the matrix A is ***not*** diagonalizable.

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix}$$

$$= 2 - 3\lambda + \lambda^2 - 12$$

$$= \lambda^2 - 3\lambda - 10 = 0$$

The eigenvalues are: $\boxed{\lambda_1 = -2, \quad \lambda_2 = 5}$

For $\lambda_1 = -2 \;\;\Rightarrow\;\; \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 3 & 3 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to 3x_1 + 3y_1 = 0$$

$$\Rightarrow \boxed{x_1 = -y_1}$$

Therefore, the eigenvector: $\boxed{V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

For $\lambda_2 = 5 \;\;\Rightarrow\;\; \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -4 & 3 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to 4x_2 - 3y_2 = 0$$

$$\to \boxed{4x_2 = 3y_2}$$

Therefore, the eigenvector: $\boxed{V_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}}$

The ***eigenvector matrix*** is given by:

$$P = \begin{pmatrix} -1 & 3 \\ 1 & 4 \end{pmatrix}$$

$$P^{-1} = \frac{1}{-7}\begin{pmatrix} 4 & -3 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{7} & \frac{3}{7} \\ \frac{1}{7} & \frac{1}{7} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -\frac{4}{7} & \frac{3}{7} \\ \frac{1}{7} & \frac{1}{7} \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{7} & \frac{3}{7} \\ \frac{1}{7} & \frac{1}{7} \end{pmatrix}\begin{pmatrix} 2 & 15 \\ -2 & 20 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix} \qquad \checkmark$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

$$A=\begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 3 & 2 \end{pmatrix}$$

Solution

$$\det(A-\lambda I)=\begin{vmatrix} -\lambda & 1 & 1 \\ 2 & 1-\lambda & 2 \\ 3 & 3 & 2-\lambda \end{vmatrix}$$

$$=-2\lambda+3\lambda^2-\lambda^3+6+6-3+3\lambda+6\lambda-4+2\lambda$$

$$\boxed{=-\lambda^3+3\lambda^2+9\lambda+5=0}$$

$\lambda=-1$

$$\begin{array}{r|rrrr} -1 & -1 & 3 & 9 & 5 \\ & & 1 & -4 & -5 \\ \hline & -1 & 4 & 5 & 0 \end{array} \quad \rightarrow \quad \boxed{-\lambda^2+4\lambda+5=0}$$

The eigenvalues are: $\boxed{\lambda_{1,2}=-1, \quad \lambda_3=5}$

For $\lambda_{1,2}=-1 \quad \Rightarrow \quad (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow x_1+y_1+z_1=0 \quad (1)$$

Let $z_1=0 \quad (1)\rightarrow \quad \boxed{x_1=-y_1}$

Therefore, the eigenvector: $\boxed{V_1=\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}$

Let $y_1=0 \quad (1)\rightarrow \quad \boxed{x_1=-z_1}$

Therefore, the eigenvector: $\boxed{V_2=\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$

For $\lambda_3 = 5 \quad \Rightarrow \quad (A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{pmatrix} \begin{matrix} \\ 5R_2 + 2R_1 \\ 5R_2 + 3R_1 \end{matrix}$$

$$\begin{pmatrix} -5 & 1 & 1 \\ 0 & -18 & 12 \\ 0 & 18 & -12 \end{pmatrix} \begin{matrix} \\ \frac{1}{6}R_2 \\ R_3 + R_2 \end{matrix}$$

$$\begin{pmatrix} -5 & 1 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} 3R_1 + R_2 \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} -15 & 0 & 5 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \frac{1}{5}R_1 \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} -3 & 0 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \rightarrow & -3x_3 + z_3 = 0 \\ \rightarrow & -3y_3 + 2z_3 = 0 \\ & \end{matrix}$$

$$\begin{cases} 3x_3 = z_3 \\ 3y_3 = 2z_3 \end{cases}$$

Therefore, the eigenvector: $V_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

The ***eigenvector matrix*** is given by:

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array} \right) \quad R_2 + R_1$$

$$\left(\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 3 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} -R_1 + R_2 \\ \\ R_3 + R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 6 & 1 & 1 & 1 \end{array}\right) \begin{array}{l} 3R_1 - R_3 \\ 2R_2 - R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 3 & 0 & 0 & -1 & 2 & -1 \\ 0 & -2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 6 & 1 & 1 & 1 \end{array}\right) \begin{array}{l} \frac{1}{3}R_1 \\ -\frac{1}{2}R_2 \\ \frac{1}{6}R_3 \end{array}$$

$$P^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 3 & 2 \end{pmatrix} \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 1 & 1 & 5 \\ -1 & 0 & 10 \\ 0 & -1 & 15 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \checkmark$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)^3 + 1 + 1 - 3(3-\lambda)$$

$$= 27 - 27\lambda + 9\lambda^2 - \lambda^3 - 7 + 3\lambda$$

$$= -\lambda^3 + 9\lambda^2 - 24\lambda + 20 = 0$$

$\lambda = 2$

$$\begin{array}{r|rrrr} 2 & -1 & 9 & -24 & 20 \\ & & -2 & 14 & -20 \\ \hline & -1 & 7 & -10 & 0 \end{array} \quad \rightarrow \quad -\lambda^2 + 7\lambda - 10 = 0$$

The eigenvalues are: $\lambda_{1,2} = 2, \quad \lambda_3 = 5$

For $\lambda_{1,2} = 2 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow x_1 + y_1 + z_1 = 0 \quad (1)$$

Let $z_1 = 0 \quad (1) \rightarrow \quad x_1 = -y_1$

Therefore, the eigenvector: $V_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Let $y_1 = 0 \quad (1) \rightarrow \quad x_1 = -z_1$

Therefore, the eigenvector: $V_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

For $\lambda_3 = 5 \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \quad \begin{matrix} \\ 2R_2 + R_1 \\ 2R_3 + R_1 \end{matrix}$$

$$\begin{pmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix} \quad \begin{matrix} 3R_1 + R_2 \\ \\ R_3 + R_2 \end{matrix}$$

$$\begin{pmatrix} -6 & 0 & 6 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} -\frac{1}{6}R_1 \\ -\frac{1}{3}R_2 \\ \\ \end{matrix}$$

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \rightarrow & -x_3 + z_3 = 0 \\ \rightarrow & -y_3 + z_3 = 0 \\ & \end{matrix}$$

$$\begin{cases} x_3 = z_3 \\ y_3 = z_3 \end{cases}$$

Therefore, the eigenvector: $V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

The ***eigenvector matrix*** is given by:

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{matrix} \\ R_2 + R_1 \\ \\ \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{matrix} -R_1 + R_2 \\ \\ R_3 + R_2 \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{array}\right) \quad \begin{array}{l} 3R_1 - R_3 \\ 3R_2 - 2R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 3 & 0 & 0 & -1 & 2 & -1 \\ 0 & -3 & 0 & 1 & 1 & -2 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{array}\right) \quad \begin{array}{c} \frac{1}{3}R_1 \\ -\frac{1}{3}R_2 \\ \frac{1}{3}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} -2 & -2 & 5 \\ 2 & 0 & 5 \\ 0 & 2 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} \quad \checkmark$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

$$A=\begin{pmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{pmatrix}$$

Solution

$$\det(A-\lambda I)=\begin{vmatrix} 2-\lambda & 2 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -2 & 2-\lambda \end{vmatrix}$$

$$=(3-\lambda)(2-\lambda)^2+2+2-3+\lambda-4(2-\lambda)$$

$$=(3-\lambda)\left(4-4\lambda+\lambda^2\right)+1+\lambda-8+4\lambda$$

$$=12-16\lambda+7\lambda^2-\lambda^3+5\lambda-7$$

$$\underline{=-\lambda^3+7\lambda^2-11\lambda+5=0}$$

$\lambda=1$

$$\begin{array}{c|cccc} 1 & -1 & 7 & -11 & 5 \\ & & -1 & 6 & -5 \\ \hline & -1 & 6 & -5 & 0 \end{array} \quad \rightarrow \quad \underline{-\lambda^2+6\lambda-5=0}$$

The eigenvalues are: $\underline{\lambda_{1,2}=1, \quad \lambda_3=5}$

For $\lambda_{1,2}=1 \quad \Rightarrow \quad (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ -1 & -2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ -1 & -2 & 1 \end{pmatrix} \quad \begin{matrix} R_2-R_1 \text{ ,,,,,,,,} \\ R_3+R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x_1+2y_1-z_1=0$$

Let $z_1=0 \quad (1)\rightarrow \quad \underline{x_1=-2y_1}$

Therefore, the eigenvector: $V_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

Let $y_1 = 0$ $(1) \rightarrow$ $x_1 = z_1$

Therefore, the eigenvector: $V_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

For $\lambda_3 = 5 \Rightarrow (A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} -3 & 2 & -1 \\ 1 & -2 & -1 \\ -1 & -2 & -3 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 2 & -1 \\ 1 & -2 & -1 \\ -1 & -2 & -3 \end{pmatrix} \quad \begin{array}{l} \\ 3R_2 + R_1 \\ -3R_3 + R_1 \end{array}$$

$$\begin{pmatrix} -3 & 2 & -1 \\ 0 & -4 & -4 \\ 0 & 8 & 8 \end{pmatrix} \quad \begin{array}{l} 2R_1 + R_2 \\ \\ R_3 + 2R_2 \end{array}$$

$$\begin{pmatrix} -6 & 0 & -6 \\ 0 & -4 & -4 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} -\frac{1}{2}R_1 \\ -\frac{1}{4}R_2 \\ \\ \end{array}$$

$$\begin{pmatrix} 3 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{array}{l} \rightarrow \quad 3x_3 + 3z_3 = 0 \\ \rightarrow \quad y_3 + z_3 = 0 \end{array}$$

$$\begin{cases} x_3 = -z_3 \\ y_3 = -z_3 \end{cases}$$

Therefore, the eigenvector: $V_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

The ***eigenvector matrix*** is given by:

$$P = \begin{pmatrix} -2 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} -2 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} \\ 2R_2 + R_1 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} -2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 - R_2 \\ \\ R_3 - R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} -2 & 0 & 2 & 0 & -2 & 0 \\ 0 & 1 & -3 & 1 & 2 & 0 \\ 0 & 0 & 4 & -1 & -2 & 1 \end{array}\right) \quad \begin{array}{l} -2R_1 + R_3 \\ 4R_2 + 3R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 4 & 0 & 0 & -1 & 2 & 1 \\ 0 & 4 & 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & -1 & -2 & 1 \end{array}\right) \quad \begin{array}{l} \frac{1}{4}R_1 \\ \frac{1}{4}R_2 \\ \frac{1}{4}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & -1 & -\frac{3}{2} \\ -\frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{pmatrix} \begin{pmatrix} -2 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} -2 & 1 & -5 \\ 1 & 0 & -5 \\ 0 & 1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad \checkmark$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

$$A=\begin{pmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

Solution

$$\det(A-\lambda I)=\begin{vmatrix} 2-\lambda & 0 & -2 \\ 1 & 3-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix}$$

$$=(2-\lambda)(3-\lambda)^2=0$$

The eigenvalues are: $\lambda_1=2,\quad \lambda_{2,3}=3$

For $\lambda_1=2 \;\Rightarrow\; (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} 0 & 0 & -2 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow x_1+y_1+2z_1=0 \quad (1)$$

$$\rightarrow z_1=0$$

$(1) \rightarrow \; x_1=-y_1$

Therefore, the eigenvector: $V_1=\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda_{2,3}=3 \;\Rightarrow\; (A-3I)V_2=0$

$$\begin{pmatrix} -1 & 0 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow x_2+2z_2=0$$

$x_2=-2z_2$

$\forall\, y_2 \in \mathbb{R}$

Therefore, the eigenvector: $V_2=\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ *and* $V_3=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

The ***eigenvector matrix*** is given by:

$$P=\begin{pmatrix}-1 & -2 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc}-1 & -2 & 0 & 1 & 0 & 0\\ 1 & 0 & 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & 0 & 1\end{array}\right) \quad R_2+R_1$$

$$\left(\begin{array}{ccc|ccc}-1 & -2 & 0 & 1 & 0 & 0\\ 0 & -2 & 1 & 1 & 1 & 0\\ 0 & 1 & 0 & 0 & 0 & 1\end{array}\right) \quad \begin{array}{l}-R_1+R_2\\ \\ 2R_3+R_2\end{array}$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & 1 & 0 & 1 & 0\\ 0 & -2 & 1 & 1 & 1 & 0\\ 0 & 0 & 1 & 1 & 1 & 2\end{array}\right) \quad \begin{array}{l}R_1-R_3\\ R_2-R_3\\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & 0 & -1 & 0 & -2\\ 0 & -2 & 0 & 0 & 0 & -2\\ 0 & 0 & 1 & 1 & 1 & 2\end{array}\right) \quad -\tfrac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & 0 & -1 & 0 & -2\\ 0 & 1 & 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 1 & 1 & 2\end{array}\right)$$

$$P^{-1}=\begin{pmatrix}-1 & 0 & -2\\ 0 & 0 & 1\\ 1 & 1 & 2\end{pmatrix}$$

$$P^{-1}AP=\begin{pmatrix}-1 & 0 & -2\\ 0 & 0 & 1\\ 1 & 1 & 2\end{pmatrix}\begin{pmatrix}2 & 0 & -2\\ 1 & 3 & 2\\ 0 & 0 & 3\end{pmatrix}\begin{pmatrix}-1 & -2 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

$$=\begin{pmatrix}-2 & 0 & -4\\ 0 & 0 & 3\\ 3 & 3 & 6\end{pmatrix}\begin{pmatrix}-1 & -2 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

$$=\begin{pmatrix}2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{pmatrix} \quad \checkmark$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix ***P*** that diagonalizes *A* and determine $P^{-1}AP$

$$A=\begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

Solution

$$\det(A-\lambda I)=\begin{vmatrix} -\lambda & -1 & -1 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix}$$

$$=\lambda^2(2-\lambda)+1+1-2+\lambda-2\lambda$$

$$=-\lambda^3+2\lambda^2-\lambda$$

$$=-\lambda\left(\lambda^2-2\lambda+1\right)=0$$

The eigenvalues are: $\lambda_1=0, \quad \lambda_{2,3}=1$

For $\lambda_1=0 \quad \Rightarrow \quad \left(A-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{array}{ll} \rightarrow \ -y_1-z_1=0 & (1) \\ \rightarrow \ x_1+2y_1+z_1=0 & (2) \\ \rightarrow \ -x_1-y_1=0 & (3) \end{array}$$

$(1) \rightarrow \quad z_1=-y_1$

$(3) \rightarrow \quad x_1=-y_1$

Therefore, the eigenvector: $V_1=\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

For $\lambda_{2,3}=1 \quad \Rightarrow \quad (A-I)V_2=0$

$$\begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \ x_2+y_2+z_2=0$$

Let $z_2=0 \quad \rightarrow \quad x_2=-y_2$

Therefore, the eigenvector: $V_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Let $y_2 = 0 \rightarrow x_2 = -z_2$

Therefore, the eigenvector: $V_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

The ***eigenvector matrix*** is given by:

$$P = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 + R_1 \\ R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 + R_3 \\ R_2 + R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right) \quad R_3 - R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{array}\right) \quad \begin{array}{l} R_1 - R_3 \\ R_2 - R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \checkmark$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix ***P*** that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & -2 \\ 1 & 3 & 1 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & -3 \\ 2 & 5-\lambda & -2 \\ 1 & 3 & 1-\lambda \end{vmatrix}$$

$$= \left(1 - 2\lambda + \lambda^2\right)(5 - \lambda) - 4 - 18 + 15 - 3\lambda + 6 - 6\lambda - 4 + 4\lambda$$

$$= 5 - 11\lambda + 7\lambda^2 - \lambda^3 - 5\lambda - 5$$

$$= -\lambda^3 + 7\lambda^2 - 16\lambda$$

$$= -\lambda\left(\lambda^2 - 7\lambda + 16\right) = 0$$

$$\lambda = \frac{7 \pm \sqrt{49 - 64}}{2}$$

$$= \frac{7}{2} \pm i\frac{\sqrt{15}}{2}$$

The eigenvalues are: $\lambda_1 = 0, \quad \lambda_{2,3} = \frac{7}{2} \pm i\frac{\sqrt{15}}{2}$

For $\lambda_1 = 0 \quad \Rightarrow \quad \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & -2 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & -2 \\ 1 & 3 & 1 \end{pmatrix} \quad \begin{matrix} R_2 - 2R_1 \\ R_2 - R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{pmatrix} \quad \begin{matrix} R_1 - 2R_2 \\ \\ R_3 - R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & -11 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \begin{matrix} \to & x_1 - 11z_1 = 0 \\ \to & y_1 + 4z_1 = 0 \\ \to & \end{matrix}$$

$$\begin{cases} x_1 = 11z_1 \\ y_1 = -4z_1 \end{cases}$$

Therefore, the eigenvector: $V_1 = \begin{pmatrix} 11 \\ -4 \\ 1 \end{pmatrix}$

Since the only real eigenvalue $\lambda = 0$ which has only a one-dimensional eigenspace.

Therefore, the given matrix A is ***not diagonalizable*** over real numbers.

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{pmatrix}$$

Solution

Since the given matrix is a lower triangular, then

The eigenvalues are: $\lambda_{1,2,3} = 2$

For $\lambda_1 = 2 \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \\ \to & 2x = 0 \\ \to & 2x + 2y = 0 \end{matrix}$$

$$\begin{cases} \boxed{x = y = 0} \\ \forall\ z \in \mathbb{R} \end{cases}$$

Therefore, the eigenvector: $\boxed{V_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$

Since the eigenvalue $(\lambda = 2)$ has only one–dimensional eigenspace.

Therefore, the given matrix A is ***not diagonalizable*** over real numbers.

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -2 & -2 \\ 3 & -3-\lambda & -2 \\ 2 & -2 & -2-\lambda \end{vmatrix}$$

$$= \left(4 - \lambda^2\right)(3 + \lambda) + 8 + 12 - 12 - 4\lambda - 8 + 4\lambda - 12 - 6\lambda$$

$$= 12 + 4\lambda - 3\lambda^2 - \lambda^3 - 6\lambda - 12$$

$$= -\lambda^3 - 3\lambda^2 - 2\lambda$$

$$\boxed{= -\lambda\left(\lambda^2 + 3\lambda + 2\right) = 0}$$

The eigenvalues are: $\boxed{\lambda_1 = -2, \quad \lambda_2 = -1 \quad \lambda_3 = 0}$

For $\lambda_1 = -2 \quad \Rightarrow \quad \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 4 & -2 & -2 \\ 3 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -2 & -2 \\ 3 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix} \begin{matrix} \\ 4R_2 - 3R_1 \\ 2R_3 - R_1 \end{matrix}$$

$$\begin{pmatrix} 4 & -2 & -2 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix} \begin{array}{l} R_1 + R_2 \\ \\ R_3 + R_2 \end{array}$$

$$\begin{pmatrix} 4 & 0 & -4 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \to \quad 4x_1 - 4z_1 = 0 \\ \to \quad 2y_1 - 2z_1 = 0 \\ \end{array}$$

$$\begin{cases} x_1 = z_1 \\ y_1 = z_1 \end{cases}$$

Therefore, the eigenvector: $\underline{V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$

For $\lambda_2 = -1 \quad \Rightarrow \quad (A + I)V_2 = 0$

$$\begin{pmatrix} 3 & -2 & -2 \\ 3 & -2 & -2 \\ 2 & -2 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -2 & -2 \\ 3 & -2 & -2 \\ 2 & -2 & -1 \end{pmatrix} \begin{array}{l} \\ R_2 - R_1 \\ 3R_3 - 2R_1 \end{array}$$

$$\begin{pmatrix} 3 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{array}{l} R_1 - R_3 \\ \\ \\ \end{array}$$

$$\begin{pmatrix} 3 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \to \quad 3x_2 - 3z_2 = 0 \\ \\ \to \quad -2y_2 + z_2 = 0 \end{array}$$

$$\to \begin{cases} x_2 = z_2 \\ 2y_2 = z_2 \end{cases}$$

Therefore, the eigenvector: $\underline{V_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}}$

For $\lambda_3 = 0 \quad \Rightarrow \quad (A)V_3 = 0$

$$\begin{pmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{pmatrix} \quad \begin{matrix} 2R_2 - 3R_1 \\ R_3 - R_1 \end{matrix}$$

$$\begin{pmatrix} 2 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \rightarrow & 2x_3 - 2y_3 - 2z_3 = 0 \\ \rightarrow & 2z_3 = 0 \end{matrix}$$

$$\begin{cases} \boxed{x_3 = y_3} \\ \boxed{z_3 = 0} \end{cases}$$

Therefore, the eigenvector: $\boxed{V_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$

The ***eigenvector matrix*** is given by:

$$P = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right) \quad R_1 + 2R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 2 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right) \quad R_1 + R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & 1 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} \\ -R_2 \\ -R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} -2 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -2 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -4 & -2 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \checkmark$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix ***P*** that diagonalizes *A* and determine $P^{-1}AP$

$$A = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix}$$

$$= (3-\lambda)^2(5-\lambda) - 4(5-\lambda)$$

$$= (5-\lambda)\left(\lambda^2 - 6\lambda + 9 - 4\right)$$

$$= (5-\lambda)\left(\lambda^2 - 6\lambda + 5\right)$$

$$= (5-\lambda)(\lambda-5)(\lambda-1) = 0$$

The eigenvalues are: $\boxed{\lambda_1 = 1, \quad \lambda_{2,3} = 5}$

For $\lambda_1 = 1 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \rightarrow 2x_1 - 2y_1 = 0 \\ \\ \rightarrow 4z_1 = 0 \end{matrix}$$

$$\begin{cases} \boxed{x_1 = y_1} \\ \boxed{z_1 = 0} \end{cases}$$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$$

For $\lambda_{2,3} = 5 \quad \Rightarrow \quad (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow -2x_2 - 2y_2 = 0$$

$$\Rightarrow \boxed{x_2 = -y_2}$$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad and \quad V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

The ***eigenvector matrix*** is:

$$P = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \quad R_2 - R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \quad 2R_1 + R_2$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \frac{1}{2}R_1 \\ \frac{1}{2}R_2 \\ \ \end{array}$$

$$P^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -5 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} \quad \checkmark$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & -2 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)^3 = 0$$

The eigenvalues are: $\lambda_{1,2,3} = 3$

For $\lambda = 3 \quad \Rightarrow \quad (A - \lambda I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \ 2z = 0$$

$$\Rightarrow \ \underline{z = 0}$$

$$\Rightarrow \ V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad V_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The ***eigenvector matrix*** is:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Since the one eigenvector has no–dimensional eigenspace.

Therefore, the given matrix A is ***not diagonalizable***

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix ***P*** that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{pmatrix} 19-\lambda & -9 & -6 \\ 25 & -11-\lambda & -9 \\ 17 & -9 & -4-\lambda \end{pmatrix}$$

$$= (19-\lambda)(-11-\lambda)(-4-\lambda) + 1,377 + 1,350 - 102(11+\lambda) - 81(19-\lambda) - 225(4+\lambda)$$

$$= \left(209 + 8\lambda - \lambda^2\right)(4+\lambda) + 2,727 - 1,122 - 102\lambda - 1,539 + 81\lambda - 900 - 225\lambda$$

$$= 836 + 241\lambda + 4\lambda^2 - \lambda^3 - 834 - 246\lambda$$

$$\underline{= -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0}$$

$$\begin{array}{r|rrrr} 1 & -1 & 4 & -5 & 2 \\ & & -1 & 3 & -2 \\ \hline & -1 & 3 & -2 & 0 \end{array} \quad \rightarrow \ \underline{-\lambda^2 + 3\lambda - 2 = 0}$$

The eigenvalues are: $\underline{\lambda_{1,2} = 1,\ 1 \quad \lambda_3 = 2}$

For $\lambda_{1,2} = 1 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 18 & -9 & -6 \\ 25 & -12 & -9 \\ 17 & -9 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \rightarrow & 18x - 9y - 6z = 0 \\ \rightarrow & 25x - 12y - 9z = 0 \\ \rightarrow & 17x - 9y - 5z = 0 \end{matrix}$$

$$\begin{pmatrix} 6 & -3 & -2 \\ 25 & -12 & -9 \\ 17 & -9 & -5 \end{pmatrix} \quad \begin{matrix} 6R_2 - 25R_1 \\ 6R_3 - 17R_1 \end{matrix}$$

$$\begin{pmatrix} 6 & -3 & -2 \\ 0 & 3 & -4 \\ 0 & -3 & 4 \end{pmatrix} \quad \begin{matrix} R_1 + R_2 \\ \\ R_3 + R_2 \end{matrix}$$

$$\begin{pmatrix} 6 & 0 & -6 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \rightarrow & 6x - 6z = 0 \\ \rightarrow & 3y - 4z = 0 \end{matrix}$$

$$\begin{cases} \underline{x = z} \\ \underline{3y = 4z} \end{cases}$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} 1 \\ \frac{4}{3} \\ 1 \end{pmatrix}$$

For $\lambda_3 = 2 \quad \Rightarrow \quad (A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} 17 & -9 & -6 \\ 25 & -13 & -9 \\ 17 & -9 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \rightarrow & 17x - 9y - 6z = 0 \\ \rightarrow & 25x - 13y - 9z = 0 \\ \rightarrow & 17x - 9y - 6z = 0 \end{matrix}$$

$$\begin{pmatrix} 17 & -9 & -6 \\ 25 & -12 & -9 \\ 17 & -9 & -5 \end{pmatrix} \quad \begin{matrix} 17R_2 - 25R_1 \\ R_3 - R_1 \end{matrix}$$

$$\begin{pmatrix} 17 & -9 & -6 \\ 0 & 21 & -3 \\ 0 & 0 & 1 \end{pmatrix} \quad 7R_1 + 3R_2$$

$$\begin{pmatrix} 119 & 0 & -6 \\ 0 & 21 & -51 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \rightarrow & 119x - 6z = 0 \\ \rightarrow & 21y - 51z = 0 \\ \rightarrow & \underline{z = 0} \end{matrix}$$

$\underline{x = y = z = 0}$

$$\Rightarrow \quad \boxed{V_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}$$

The ***eigenvector matrix*** is:

$$P = \begin{pmatrix} 3 & 1 & 0 \\ 4 & \frac{4}{3} & 0 \\ 3 & 1 & 0 \end{pmatrix}$$

P^{-1} doesn't exist, one column with zero entries.

Since the eigenvalue $(\lambda = 2)$ has no–dimensional eigenspace.

Therefore, the given matrix A is ***not diagonalizable*** (repeated eigenvalues)

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Solution

Since the matrix A is a lower triangular, then the eigenvalues are the entries values of the main diagonal.

The eigenvalues are: $\boxed{\lambda_{1,2} = -2, \quad \lambda_{3,4} = 3}$

For $\lambda_{1,2} = -2 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \qquad \begin{array}{l} \to \ 5v_3 = 0 \\ \to \ v_3 + 5v_4 = 0 \end{array}$$

$$\boxed{v_3 = v_4 = 0}$$

$$\Rightarrow \quad \underline{V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad and \quad V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}$$

For $\lambda_{3,4} = 3 \quad \Rightarrow \quad (A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} -5 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \quad \begin{matrix} \rightarrow & -5v_1 = 0 \\ \rightarrow & -5v_2 = 0 \\ & \\ \rightarrow & \underline{v_3 = 0} \end{matrix}$$

$\underline{v_1 = v_2 = v_3 = 0}$

$$\Rightarrow \quad \underline{V_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad and \quad V_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}$$

The ***eigenvector matrix*** is:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

P^{-1} doesn't exist, one row with zero entries

Since the 2 eigenvectors have only the same one–dimensional eigenspace.

Therefore, the given matrix A is ***not diagonalizable***

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

$$A=\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Solution

Since the matrix A is an upper triangular, then the eigenvalues are: $\underline{\lambda_{1,2}=-2 \quad \lambda_{3,4}=3}$

For $\lambda=-2 \quad \Rightarrow \quad (A+2I)V_1=0$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -5 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{cases} 5x_3-5x_4=0 \\ 5x_3=0 \\ 5x_4=0 \end{cases} \quad \Rightarrow \quad x_3=x_4=0$$

$$\Rightarrow \quad V_1=\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \textit{and} \quad V_2=\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda=3 \quad \Rightarrow \quad (A-3I)V_2=0$

$$\begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & -5 & 5 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} \rightarrow & -5x_1=0 \\ \rightarrow & -5x_2+5x_3-5x_4=0 \end{matrix}$$

$$\Rightarrow \begin{cases} \underline{x_1=0} \\ \underline{x_2=x_3-x_4} \end{cases}$$

$$V_3=\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \textit{and} \quad V_4=\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

The ***eigenvector matrix*** is:

$$P=\begin{pmatrix}1&0&0&0\\0&1&1&-1\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$\left(\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&1&1&-1&0&1&0&0\\0&0&1&0&0&0&1&0\\0&0&0&1&0&0&0&1\end{array}\right) \qquad R_2-R_3$$

$$\left(\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&1&0&-1&0&1&-1&0\\0&0&1&0&0&0&1&0\\0&0&0&1&0&0&0&1\end{array}\right) \qquad R_2+R_4$$

$$\left(\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&1&0&0&0&1&-1&1\\0&0&1&0&0&0&1&0\\0&0&0&1&0&0&0&1\end{array}\right)$$

$$P^{-1}=\begin{pmatrix}1&0&0&0\\0&1&-1&1\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$P^{-1}AP=\begin{pmatrix}1&0&0&0\\0&1&-1&1\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}-2&0&0&0\\0&-2&5&-5\\0&0&3&0\\0&0&0&3\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&1&-1\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}-2&0&0&0\\0&-2&2&-2\\0&0&3&0\\0&0&0&3\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&1&-1\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}-2&0&0&0\\0&-2&0&0\\0&0&3&0\\0&0&0&3\end{pmatrix} \qquad \checkmark$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix ***P*** that diagonalizes *A* and determine $P^{-1}AP$

$$A=\begin{pmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Solution

Since the matrix *A* is an upper triangular, then the eigenvalues are: $\underline{\lambda_{1,2}=2 \quad \lambda_3=3 \quad \lambda_4=5}$

$$A-\lambda I=\begin{pmatrix} 5-\lambda & -3 & 0 & 9 \\ 0 & 3-\lambda & 1 & -2 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{pmatrix}$$

For $\lambda=2 \quad \Rightarrow \quad (A-2I)V_1=0$

$$\begin{pmatrix} 3 & -3 & 0 & 9 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -3 & 0 & 9 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad R_1+3R_2$$

$$\begin{pmatrix} 3 & 0 & 3 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\to\begin{cases} 3x_1+3x_3+3x_4=0 \\ x_2+x_3-2x_4=0 \end{cases}$$

$$\begin{cases} \underline{x_1=-x_3-x_4} \\ \underline{x_2=-x_3+2x_4} \end{cases}$$

$$\Rightarrow \quad \boxed{V_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \quad and \quad V_2 = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}}$$

For $\lambda = 3 \quad \Rightarrow \quad (A-3I)V_3 = 0$

$$\begin{pmatrix} 2 & -3 & 0 & 9 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} 2x_1 - 3x_2 + 9x_4 = 0 & (1) \\ x_3 - 2x_4 = 0 \\ \boxed{x_3 = 0} \\ \boxed{x_4 = 0} \end{cases}$$

$$(1) \rightarrow \quad \boxed{2x_1 = 3x_2}$$

$$\Rightarrow \quad \boxed{V_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}}$$

For $\lambda = 5 \quad \Rightarrow \quad (A-5I)V_4 = 0$

$$\begin{pmatrix} 0 & -3 & 0 & 9 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} -3x_2 + 9x_4 = 0 & (1) \\ -2x_2 + x_3 - 2x_4 = 0 \\ \boxed{x_3 = 0} \\ \boxed{x_4 = 0} \end{cases}$$

$$(1) \rightarrow \quad \boxed{x_2 = 0}$$

$$\Rightarrow \quad V_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The ***eigenvector matrix*** is:

$$P = \begin{pmatrix} -1 & -1 & 3 & 1 \\ -1 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|cccc} -1 & -1 & 3 & 1 & 1 & 0 & 0 & 0 \\ -1 & 2 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 - R_1 \\ R_3 + R_1 \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|cccc} -1 & -1 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} 3R_1 + R_2 \\ \\ 3R_3 + R_2 \\ 3R_4 - R_2 \end{array}$$

$$\left(\begin{array}{cccc|cccc} -3 & 0 & 8 & 2 & 2 & 1 & 0 & 0 \\ 0 & 3 & -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 8 & 2 & 2 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 & 0 & 3 \end{array}\right) \begin{array}{l} R_1 - R_3 \\ 8R_3 + R_3 \\ \\ 8R_4 - R_3 \end{array}$$

$$\left(\begin{array}{cccc|cccc} -3 & 0 & 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 24 & 0 & -6 & -6 & 9 & 3 & 0 \\ 0 & 0 & 8 & 2 & 2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 6 & 6 & -9 & -3 & 24 \end{array}\right) \begin{array}{l} \\ R_2 + R_4 \\ 3R_3 - R_4 \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|cccc} -3 & 0 & 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 24 & 0 & 0 & 0 & 0 & 0 & 24 \\ 0 & 0 & 24 & 0 & 0 & 12 & 12 & -24 \\ 0 & 0 & 0 & 6 & 6 & -9 & -3 & 24 \end{array}\right) \begin{array}{l} -\frac{1}{3}R_1 \\ \frac{1}{24}R_2 \\ \frac{1}{24}R_3 \\ \frac{1}{6}R_4 \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -1 \\ 0 & 0 & 0 & 1 & 1 & -\frac{3}{2} & -\frac{1}{2} & 4 \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \\ 1 & -\frac{3}{2} & -\frac{1}{2} & 4 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \\ 1 & -\frac{3}{2} & -\frac{1}{2} & 4 \end{pmatrix}\begin{pmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & -1 & 3 & 1 \\ -1 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \\ 1 & -\frac{3}{2} & -\frac{1}{2} & 4 \end{pmatrix}\begin{pmatrix} -2 & -2 & 9 & 5 \\ -2 & 4 & 6 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} \quad \checkmark$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

$$A=\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}$$

Solution

Since the matrix A is a lower triangular, then the eigenvalues are: $\underline{\lambda_{1,2}=2 \quad \lambda_{3,4}=3}$

$$A-\lambda I=\begin{pmatrix} 3-\lambda & 0 & 0 & 0 \\ 0 & 2-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 1 & 0 & 0 & 3-\lambda \end{pmatrix}$$

For $\lambda_{1,2}=2 \quad \Rightarrow \quad (A-2I)V_1=0$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \to \quad \underline{x_1=0} \\ \\ \\ \to \quad x_1=x_4=0 \end{matrix}$$

$$\Rightarrow \quad \underline{x_1=x_4=0}$$

$$\Rightarrow \quad V_1=\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad V_2=\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda_{3,4}=3 \quad \Rightarrow \quad (A-3I)V_3=0$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \underline{x_1=x_2=x_3=0}$$

$$\Rightarrow \quad V_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad V_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Since the eigenvalue $(\lambda = 3)$ has only one–dimensional eigenspace.

Therefore, the matrix A is *not diagonalizable*.

Exercise

The 4 by 4 triangular Pascal matrix P_L and its inverse (alternating diagonals) are

$$P_L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \quad \text{and} \quad P_L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

Check that P_L and P_L^{-1} have the same eigenvalues. Find a diagonal matrix D with alternating signs that gives $P_L^{-1} = D^{-1}P_L D$, so P_L is similar to P_L^{-1}. Show that $P_L D$ with columns of alternating signs is its own inverse.

Since P_L and P_L^{-1} are similar they have the same Jordan form J. Find J by checking the number of independent eigenvectors of P_L with $\lambda = 1$.

Solution

The triangular matrices P_L and P_L^{-1} both have $\lambda = 1, 1, 1, 1$ on their main diagonals. Choose D with alternating 1 and -1 on its diagonal. D equals D^{-1}:

$$D^{-1}P_L D = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -2 & -1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}$$

$$= P_L^{-1}$$

Check:

Changing signs in rows 1 and 3 of P_L, and columns 1 and 3, produces the four negative entries in P_L^{-1}. Multiply row i by $(-1)^i$ and column j by $(-1)^j$, which gives the alternating diagonals.

Then $P_L D = pascal(n,\ 1)$ has columns with alternating signs and equals its own inverse!

$$\begin{aligned}(P_L D)(P_L D) &= P_L D^{-1} P_L D \\ &= P_L P_L^{-1} \\ &= I\end{aligned}$$

P_L has only one line of eigenvectors $x = (0,\ 0,\ 0,\ x_4)$ with λ = 1. The rank of $P_L - I$ is certainly 3. So its Jordan form J has only one block (also with λ = 1):

P_L and P_L^{-1} are somehow similar to Jordan's $J = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Exercise

These Jordan matrices have eigenvalues 0, 0, 0, 0. They have two eigenvectors (one from each block). But the block sizes don't match and they are not similar:

$$J = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{and} \quad K = \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

For any matrix M compare JM with MK. If they are equal show that M is not invertible. Then $M^{-1}JM = K$ is Impossible; J is not similar to K.

Solution

Let $M = (m_{ij})$, then

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix}$$

$$JM = \begin{pmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad and \quad MK = \begin{pmatrix} 0 & m_{11} & m_{12} & 0 \\ 0 & m_{21} & m_{22} & 0 \\ 0 & m_{31} & m_{32} & 0 \\ 0 & m_{41} & m_{42} & 0 \end{pmatrix}$$

If $JM = MK$ then $m_{21} = m_{22} = m_{24} = m_{41} = m_{42} = m_{44} = 0$

Which in particular means that the second row is either a multiple of the fourth row, or the fourth row is all 0's. In either of these cases M is not invertible.

Suppose that J were similar to K. Then there would be some invertible matrix M such that $MK = JM$. But we just showed that in this case M is never invertible (contradiction). Thus, J is not similar to K.

Exercise

If $\boldsymbol{x}$ is in the nullspace of A show that $M^{-1}\boldsymbol{x}$ is in the nullspace of $M^{-1}AM$.

The nullspaces of A and $M^{-1}AM$ have the same (vectors) (basis) (dimension)

Solution

$$x \in N(A) \Rightarrow Ax = 0 \Rightarrow M^{-1}AM\left(M^{-1}x\right) = 0 \Rightarrow M^{-1}x \in N\left(M^{-1}AM\right)$$

$$x \in N\left(M^{-1}AM\right) \Rightarrow M^{-1}AMx = 0 \Rightarrow AMx = 0 \Rightarrow Mx \in N\left(A\right)$$

So, any vector in $N(A)$ *resp.* $N\left(M^{-1}AM\right)$ is a linear combination of those in $N\left(M^{-1}AM\right)$ *resp.* $N(A)$, hence is contained in it. That is, the two vector spaces consist of the same vectors.

Exercise

Prove that A^T is always similar to A (λ′s are the same):

a) For one Jordan block J_i, find M_i so that $M_i^{-1}J_iM_i = J_i^T$.

b) For any J with blocks J_i, build M_0 from blocks so that $M_0^{-1}JM_0 = J^T$.

c) For any $A = MJM^{-1}$: Show that A^T is similar to J^T and so to J and so to A.

Solution

a) For one Jordan block J_i, then

$$\begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ \iddots & & & \\ 1 & & & \end{pmatrix}\begin{pmatrix} \lambda & 1 & 0 & & 0 \\ & \lambda & 1 & 0 & \\ & & \lambda & & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}\begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ \iddots & & & \\ 1 & & & \end{pmatrix} = \begin{pmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ 0 & 1 & \lambda & & \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix}$$

So, J is similar to J^T

b) For any J with block J_i, that satisfies $J_i^T = M_i^{-1}J_iM_i$

Let M_0 be the block-diagonal matrix consisting of the $M_i^{'}s$ along the diagonal. Then

$$M_0^{-1}JM_0 = \begin{pmatrix} M_1^{-1} & & & \\ & M_2^{-1} & & \\ & & \ddots & \\ & & & M_n^{-1} \end{pmatrix}\begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_n \end{pmatrix}\begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_n \end{pmatrix}$$

$$= \begin{pmatrix} M_1^{-1}J_1M_1 & & & \\ & M_2^{-1}J_2M_2 & & \\ & & \ddots & \\ & & & M_n^{-1}J_nM_n \end{pmatrix}$$

$$= \begin{pmatrix} J_1^T & & & \\ & J_2^T & & \\ & & \ddots & \\ & & & J_n^T \end{pmatrix}$$

$$\boxed{= J^T}$$

c) $A^T = \left(MJM^{-1}\right)^T = \left(M^{-1}\right)^T J^T M^T = \left(M^T\right)^{-1} J^T \left(M^T\right)$

So A^T is similar to J^T, which is similar to J, which is similar to A, Thus any matrix is similar to its transpose.

Exercise

Why are these statements all true?

a) If A is similar to B then A^2 is similar to B^2.

b) A^2 and B^2 can be similar when A and B are not similar.

c) $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ is similar to $\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}$

d) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ is not similar to $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

e) If we exchange rows 1 and 2 of A, and then exchange columns 1 and 2 the eigenvalues stay the same. In this case M =?

Solution

a) If A is similar to B then $A = M^{-1}BM$ for some M. Then $A^2 = M^{-1}B^2M$, so A^2 is similar to B^2.

b) Let $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $A^2 = B^2$ so they are similar but A is not similar to B because nothing but zero matrix.

c) $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

d) They are not similar because the first matrix has a plane of eigenvectors for the eigenvalues 3, while the second only has a line.

e) In order to exchange two rows of A we multiply on the left by

$$M = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

In order to exchange two columns, we multiply on the right by the same M. As $M = M^{-1}$ the new matrix is similar to the old one, so the eigenvalues stay the same.

Exercise

If an n x n matrix A has all eigenvalues $\lambda = 0$ prove that A^n is the zero matrix.

Solution

Suppose that the Jordan Block has a size of $\boldsymbol{i}$ with eigenvalue 0. Then J^2 will have a diagonal of 1's two diagonals above the main diagonal and zeroes elsewhere. J^3 will have a diagonal of 1's three diagonals above the main diagonal and zeroes elsewhere. Therefore $J^i = 0$, since there is no diagonal $\boldsymbol{i}$ diagonals above the main diagonal. If A has all eigenvalues $\lambda = 0$ then A is similar to some matrix with Jordan block $J_1, \ \ldots, J_k$ with each J_i of size n_i and $\sum_{i=1}^{k} n_i = n$.

Each Jordan block will have eigenvalue of 0, so that $J_i^{n_i} = 0$, and thus $J_i^n = 0$

As A^n is similar to a block-diagonal matrix with blocks $J_1^n, \ J_2^n, \ \ldots, J_k^n$ and each of these is 0 we know that $A^n = 0$.

Another way, if A has all eigenvalues 0 this means that the characteristic polynomial of A must be x^n, as this is the only polynomial of degree n all of whose roots are 0. Thus $A^n = 0$ by the Cayley-Hamilton theorem.

Exercise

If A is similar to A^{-1}, must all the eigenvalues equal to 1 or -1?.

Solution

No

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Thus $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ is similar to $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}^{-1}$

Exercise

Determine whether the *two matrices* are similar matrices $A=\begin{pmatrix}1&1\\1&2\end{pmatrix}$ $B=\begin{pmatrix}1&0\\3&-2\end{pmatrix}$

Solution

$$|A|=\begin{vmatrix}1&1\\1&2\end{vmatrix}\ \underline{=1}$$

$$|B|=\begin{vmatrix}1&0\\3&-2\end{vmatrix}\ \underline{=-2}$$

$|A|\neq|B|$; therefore, A and B are ***not*** similar

Exercise

Determine whether the *two matrices* are similar matrices $A=\begin{pmatrix}4&-1\\2&4\end{pmatrix}$ $B=\begin{pmatrix}4&1\\2&4\end{pmatrix}$

Solution

$$|A|=\begin{vmatrix}4&-1\\2&4\end{vmatrix}\ \underline{=18}$$

$$|B|=\begin{vmatrix}4&1\\2&4\end{vmatrix}\ \underline{=14}$$

$|A|\neq|B|$; therefore, A and B are ***not*** similar

Exercise

Determine whether the *two matrices* are similar matrices $A=\begin{pmatrix}1&2&3\\0&1&2\\0&0&1\end{pmatrix}$ $B=\begin{pmatrix}1&2&0\\\frac{1}{2}&1&0\\0&0&1\end{pmatrix}$

Solution

$$|A|=\begin{vmatrix}1&2&3\\0&1&2\\0&0&1\end{vmatrix}\ \underline{=1}$$

$$|B|=\begin{vmatrix}1&2&0\\\frac{1}{2}&1&0\\0&0&1\end{vmatrix}\ \underline{=0}$$

$|A|\neq|B|$; therefore, A and B are ***not*** similar

Exercise

Determine whether the *two matrices* are similar matrices $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}1&2&3\\0&4&5\\0&0&6\end{pmatrix}$

Solution

$$|A|=\begin{vmatrix}1&0&0\\0&4&0\\0&0&6\end{vmatrix}$$

$$= 24$$

$$|B|=\begin{vmatrix}1&2&3\\0&4&5\\0&0&6\end{vmatrix}$$

$$= 24$$

$$|A|=|B|$$

Therefore, A and B are similar

Exercise

Determine whether the *two matrices* are similar matrices $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}4&0&0\\7&1&0\\8&9&6\end{pmatrix}$

Solution

$$|A|=\begin{vmatrix}1&0&0\\0&4&0\\0&0&6\end{vmatrix}$$

$$= 24$$

$$|B|=\begin{vmatrix}4&0&0\\7&1&0\\8&9&6\end{vmatrix}$$

$$= 24$$

$$|A|=|B|$$

Therefore, A and B are similar

Exercise

Determine whether the *two matrices* are similar matrices $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix} \quad B=\begin{pmatrix}4&7&0\\0&1&0\\8&9&6\end{pmatrix}$

Solution

$$|A|=\begin{vmatrix}1&0&0\\0&4&0\\0&0&6\end{vmatrix}$$

$$=24$$

$$|B|=\begin{vmatrix}4&7&0\\0&1&0\\8&9&6\end{vmatrix}$$

$$=24$$

$$|A|=|B|$$

Therefore, A and B are similar

Exercise

Determine whether the *two matrices* are similar matrices $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix} \quad B=\begin{pmatrix}1&2&0\\3&4&5\\0&7&6\end{pmatrix}$

Solution

$$|A|=\begin{vmatrix}1&0&0\\0&4&0\\0&0&6\end{vmatrix}$$

$$=24$$

$$|B|=\begin{vmatrix}1&2&0\\3&4&5\\0&7&6\end{vmatrix}$$

$$=-47$$

$$|A|\neq|B|$$

Therefore, A and B are ***not*** similar

Exercise

Determine whether the *two matrices* are similar matrices $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}1&2&0\\3&4&0\\0&5&6\end{pmatrix}$

Solution

$$|A|=\begin{vmatrix}1&0&0\\0&4&0\\0&0&6\end{vmatrix}$$

$$\underline{=24}$$

$$|B|=\begin{vmatrix}1&2&0\\3&4&0\\0&5&6\end{vmatrix}$$

$$\underline{=-12}$$

$$|A|\neq|B|$$

Therefore, A and B are ***not*** similar

Exercise

Determine whether the *two matrices* are similar matrices $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}1&0&0\\2&5&1\\3&1&5\end{pmatrix}$

Solution

$$|A|=\begin{vmatrix}1&0&0\\0&4&0\\0&0&6\end{vmatrix}$$

$$\underline{=24}$$

$$|B|=\begin{vmatrix}1&0&0\\2&5&1\\3&1&5\end{vmatrix}$$

$$\underline{=24}$$

$$|A|=|B|$$

Therefore, A and B are similar

Exercise

Prove that two similar matrices have the same determinant. Explain geometrically why this is reasonable.

Solution

Suppose that $A = PBP^{-1}$

Then $\det(A) = \det\left(PBP^{-1}\right)$ $\qquad |AB| = |A||B|$

$$= \det(P)\cdot\det(B)\cdot\det\left(P^{-1}\right)$$

$$= \det(B)\cdot\det(P)\cdot\det\left(P^{-1}\right)$$

$$= \det(B)\cdot\det\left(PP^{-1}\right)$$

$$= \det(B)\cdot\det(I)$$

$$= \det(B)$$

Geometric Explanation: The determinant tells us what Factor area changes when using a linear transformation. This "factor" doesn't care about the particular basis you use.

Exercise

Prove that two similar matrices have the same characteristic polynomial and thus the same eigenvalues. Explain geometrically why this is reasonable.

Solution

Suppose that $A = PBP^{-1}$

Then the characteristic polynomial is equal to $\det(A - \lambda I)$.

$$A - \lambda I = PBP^{-1} - \lambda\left(PIP^{-1}\right)$$

$$= P(B - \lambda I)P^{-1}$$

$$\det(A - \lambda I) = \det\left(P(B - \lambda I)P^{-1}\right)$$

$$= \det(P)\cdot\det(B - \lambda I)\cdot\det\left(P^{-1}\right)$$

$$= \det(B - \lambda I)\cdot\det(P)\cdot\det\left(P^{-1}\right)$$

$$= \det(B - \lambda I)\cdot\det\left(PP^{-1}\right) \qquad \det\left(PP^{-1}\right) = \det(I) = 1$$

$$= \det(B - \lambda I)$$

Geometric Explanation: At least in terms of the eigenvalues, these values are numbers λ such that there exists a vector $\vec{v} \neq 0$ such that the linear transformation T satisfies $T(\vec{v}) = \lambda\vec{v}$.

Exercise

Suppose that A is a matrix. Suppose that the linear transformation associated to A has two linearly independent eigenvectors. Prove that A is similar to a diagonal matrix.

Solution

Let T be the linear transformation associated with A. Consider the basis $\vec{v}_1, \vec{v}_2$ of the 2 linearly independent eigenvectors of A where λ_1, λ_2 the eigenvalues associated with. Then,

$$T(\vec{v}_1) = \lambda_1 \vec{v}_1 \quad \textit{and} \quad T(\vec{v}_2) = \lambda_2 \vec{v}_2$$

Let T be a matrix with respect to the basis $\vec{v}_1, \vec{v}_2$, then we obtain the matrix $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

This completes the proof because A is similar to this diagonal matrix by definition.

Exercise

Prove that if A is a 2×2 matrix that has two distinct eigenvalues, then A is similar to a diagonal matrix.

Solution

Suppose A has 2 distinct eigenvalues λ_1, λ_2.

Let $\vec{v}_1 \neq 0$ be an eigenvector for λ_1.

Suppose that $\vec{v}_1, \vec{v}_2$ are not linearly independent, thus they are scalar multiples of each other.

So, there exists $c \neq 0$ such that $c\vec{v}_1 = \vec{v}_2$. Then

$$\begin{aligned}
\lambda_2 \vec{v}_2 &= A\vec{v}_2 \\
&= A(c\vec{v}_1) \\
&= c(A\vec{v}_1) \\
&= c\lambda_1 \vec{v}_1 \\
&= \lambda_1 c\vec{v}_1 && c\vec{v}_1 = \vec{v}_2 \\
&= \lambda_1 \vec{v}_2
\end{aligned}$$

So, that $\lambda_2 \vec{v}_2 - \lambda_1 \vec{v}_2 = 0 \quad \Rightarrow \quad (\lambda_2 - \lambda_1)\vec{v}_2 = 0$

But then $\lambda_2 = \lambda_1$ which contradicts the initial assumption.

Thus $\vec{v}_1, \vec{v}_2$ are linearly independent then $T(\vec{v}_1) = \lambda_1 \vec{v}_1 \quad \textit{and} \quad T(\vec{v}_2) = \lambda_2 \vec{v}_2$

Let T be a matrix with respect to the basis $\vec{v}_1, \vec{v}_2$, then we obtain the matrix $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

This completes the proof because A is similar to this diagonal matrix by definition.

Exercise

Suppose that the characteristic polynomial of a matrix has a double root. Is it true that the matrix has two linearly independent eigenvectors? Consider the example $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Is it true that matrices with equal characteristic polynomial are necessarily similar?

Solution

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 0 \\ 1 & -\lambda \end{vmatrix}$$

$$= \lambda^2 = 0$$

The characteristic polynomial: $p(x) = x^2$ which has a double root $(\textit{eigenvalue}:\ \lambda = 0)$.

$$(A - \lambda I)V = AV$$

$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \ x = 0$$

Therefore, the eigenvectors are vectors of the form $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ which can transform to $\begin{pmatrix} x \\ 0 \end{pmatrix}$

Thus, matrices whose characteristic polynomials have a double root do not necessarily have 2 linear independent.

Let $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then the characteristic polynomial: $p(x) = x^2$ which has a double root $(\textit{eigenvalue}:\ \lambda = 0)$. But they are not similar. The eigenvector is the $\vec{0}$ vector.

The linear transformation associated to the second matrix send every vector to $\vec{0}$. Thus the 2 matrices can't represent the same linear transformation.

Thus, matrices with equal characteristic polynomial are not necessarily similar.

Exercise

Show that the given matrix is not diagonalizable. $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

Solution

$$\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = 0$$

Since the determinant is 0, the inverse doesn't exist.

Therefore, the matrix is not diagonalizable

Exercise

Determine if the given matrix is diagonalizable. If, so, find matrices S and $\Lambda(D)$ such that the given matrix equals $S\Lambda S^{-1}$

a) $\begin{bmatrix} 3 & 3 \\ 4 & 2 \end{bmatrix}$ ***b)*** $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & -1 \end{bmatrix}$

Solution

a) $\begin{vmatrix} 3-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 12$

$$= \lambda^2 - 5\lambda - 6 = 0$$

The eigenvalues $\lambda_1 = -1,\ \lambda_2 = 6$

For $\lambda = -1 \Rightarrow (A+I)V_1 = 0$

$$\begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow 4x_1 + 3y_1 = 0$$

$$\Rightarrow 4x_1 = -3y_1$$

Therefore, the eigenvector: $V_1 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

For $\lambda = 6 \Rightarrow (A-6I)V_2 = 0$

$$\begin{pmatrix} -3 & 3 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow -3x_2 + 3y_2 = 0$$

$$\Rightarrow x_2 = y_2$$

Therefore, the eigenvector: $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$S = \begin{pmatrix} -3 & 1 \\ 4 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} -3 & 1 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{array}\right) \quad 3R_2 + 4R_1$$

$$\left(\begin{array}{cc|cc} -3 & 1 & 1 & 0 \\ 0 & 7 & 4 & 3 \end{array}\right) \quad 7R_1 - R_2$$

$$\left(\begin{array}{cc|cc} -21 & 0 & 3 & -3 \\ 0 & 7 & 4 & 3 \end{array}\right) \quad \begin{array}{c} -\frac{1}{21}R_1 \\ R_2 \end{array}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{7} & \frac{1}{7} \\ 0 & 1 & \frac{4}{7} & \frac{3}{7} \end{array}\right)$$

$$S^{-1} = \begin{pmatrix} -\frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{3}{7} \end{pmatrix}$$

and $\Lambda = \begin{pmatrix} -1 & 0 \\ 0 & 6 \end{pmatrix}$

$$\begin{aligned} S\Lambda S^{-1} &= \begin{pmatrix} -3 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} -\frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{3}{7} \end{pmatrix} \\ &= \begin{pmatrix} 3 & 6 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} -\frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{3}{7} \end{pmatrix} \\ &= \begin{pmatrix} 3 & 3 \\ 4 & 2 \end{pmatrix} \\ &= A \end{aligned}$$

b) $$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & -1-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(-1-\lambda) + (2-\lambda)$$

$$\begin{aligned} &= (2-\lambda)\big((1-\lambda)(-1-\lambda)+1\big) \\ &= (2-\lambda)\big(-1+\lambda^2+1\big) \\ &= (2-\lambda)\lambda^2 = 0 \end{aligned}$$

The eigenvalues $\lambda_1 = 2,\ \lambda_{2,3} = 0$

The given matrix is not diagonalizable, since the eigenvalues are not distinct.

Exercise

A is a 5×5 matrix with *two* eigenvalues. One eigenspace is *three*–dimensional, and the other eigenspace is *two*–dimensional. Is *A* diagonalizable? Why?

Solution

Since 5×5 matrix *A* has two eigenvalues with one of the eigenvalues has three linearly independent eigenvectors in the *three*–dimensional and the other eigenvalue has two linearly independent eigenvectors in the *two*–dimensional.

Therefore, since all the ***five*** eigenvectors are linearly independent eigenvectors, that implies that the 5×5 matrix *A* is diagonalizable.

Exercise

A is a 3×3 matrix with *two* eigenvalues. Each eigenspace is *one*–dimensional. Is *A* diagonalizable? Why?

Solution

The given 3×3 matrix *A* has two eigenvalues that implies one of the eigenvalues is repeated value. Since the eigenvectors are in *one*–dimensional, the repeated eigenvalue will result with two eigenvectors linearly dependent.

Therefore, the given 3×3 matrix *A* is ***not*** diagonalizable

Exercise

A is a 4×4 matrix with *three* eigenvalues. One eigenspace is *one*–dimensional, and one of the other eigenspace is *two*–dimensional. Is it possible that *A* is *not* diagonalizable? Justify your answer?

Solution

The given 4×4 matrix *A* has three eigenvalues that implies one of the eigenvalues is repeated value.

However, one eigenspace is *one*–dimensional, and one of the other eigenspace is *two*–dimensional which include that these two eigenvectors are linearly independent.

Since, the other two distinct eigenvalues will result to the linearly independent eigenvectors.

That implies that all the eigenvectors are linearly independent.

Therefore, the given 4×4 matrix *A* is diagonalizable.

Exercise

A is a 7×7 matrix with *three* eigenvalues. One eigenspace is *two*–dimensional, and one of the other eigenspace is *three*–dimensional. Is it possible that *A* is *not* diagonalizable? Justify your answer?

Solution

The given 7×7 matrix *A* has three eigenvalues which results to 7 eigenvalues.

Since, one eigenspace is *two*–dimensional, and one of the other eigenspace is *three*–dimensional that will result to 5 linearly independent eigenvectors for that two eigenvalues.

If the third eigenvalue is repeated with *one*–dimensional, it will result to linearly dependent eigenvectors.

Therefore, the given 7×7 matrix *A* is ***not*** diagonalizable

Exercise

Show that if *A* is diagonalizable and invertible, then so is A^{-1}.

Solution

Since *A* is invertible, then:

$$AA^{-1} = A^{-1}A = I$$

And *A* is diagonalizable:

$$A = PDP^{-1}$$

$$A = PDP^{-1}$$

$$\begin{aligned} A^{-1} &= \left(PDP^{-1}\right)^{-1} \\ &= \left(P^{-1}\right)^{-1} D^{-1}P^{-1} \\ &= PD^{-1}P^{-1} \end{aligned}$$

Since *D* is diagonal then D^{-1} is diagonal matrix.

Therefore, A^{-1} is diagonalizable

Exercise

Show that if *A* has *n* linearly independent eigenvectors, then so does A^{T}.

Solution

If *A* has *n* linearly independent eigenvectors, then *A* is diagonalizable.

By the diagonalizable theorem $A = PDP^{-1}$

$$A^T = \left(PDP^{-1}\right)^T$$

$$= \left(P^{-1}\right)^T D^T P^T \qquad \text{Since } D \text{ is diagonal then } D^T = D$$

$$= \left(P^T\right)^{-1} DP^T \qquad \text{Assume that } Q = \left(P^T\right)^{-1}$$

$$= QDQ^{-1}$$

Therefore, A^T is diagonalizable with the columns Q are n linearly independent eigenvectors

Exercise

A factorization $A = PDP^{-1}$ is not unique. Demonstrate this for the matrix $A = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix}$. With $D_1 = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$, find a matrix P_1 such that $A = P_1 D_1 P_1^{-1}$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 7-\lambda & 2 \\ -4 & 1-\lambda \end{vmatrix}$$

$$= 7 - 8\lambda + \lambda^2 + 8$$

$$= \lambda^2 - 8\lambda + 15 = 0$$

The eigenvalues are: $\lambda_1 = 3, \quad \lambda_2 = 5$

For $\lambda_1 = 3 \quad \Rightarrow \quad (A - 3I)V_1 = 0$

$$\begin{pmatrix} 4 & 2 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow 4x_1 + 2y_1 = 0$$

$$\Rightarrow 2x_1 = -y_1$$

Therefore, the eigenvector: $V_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

For $\lambda_2 = 5 \quad \Rightarrow \quad (A - 5I)V_2 = 0$

$$\begin{pmatrix} 2 & 2 \\ -4 & -4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow 2x_2 + 2y_2 = 0$$

$$\Rightarrow x_2 = -y_2$$

Therefore, the eigenvector: $V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

The ***eigenvector matrix*** is given by: $P = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$

Which implies to: $P_1 = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$

$$P_1^{-1} = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$$

$$\begin{aligned} P_1 D_1 P_1^{-1} &= \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -3 & -5 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix} \\ &= A \quad \checkmark \end{aligned}$$

However, if we multiply the eigenvector V_1 with 2, it will result $V_1 = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ that implies to:

$$P_2 = \begin{pmatrix} -2 & -1 \\ 4 & 1 \end{pmatrix}$$

$$\begin{aligned} P_2^{-1} &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -4 & -2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -2 & -1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} P_2 D_2 P_2^{-1} &= \begin{pmatrix} -2 & -1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -6 & -5 \\ 12 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix} \\ &= A \quad \checkmark \end{aligned}$$

$$A = PDP^{-1} = P_1 D_1 P_1^{-1} = P_2 D_2 P_2^{-1}$$

Therefore, that is shows that matrix A has many different factorizations.

Exercise

Construct a nonzero 2×2 matrix that is invertible but not diagonalizable.

Solution

For a 2×2 invertible matrix A, the eigenvalues must be nonzero and determinant of A is not equal to zero.

Let assume $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$

$\det(A) = ac \neq 0$

Matrix A is invertible

Since the matrix A is an upper triangular then the eigenvalues are the main diagonal entries

$\lambda_1 = a \quad \& \quad \lambda_2 = c$

For the matrix A to be not diagonalizable when the eigenvectors are linearly dependents or in one–dimensional.

If we have a repeated eigenvalue that it will result in *one*–dimensional, that it will result that $a = c$.

$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$

$\lambda_{1,2} = a$

For $\lambda_1 = a \quad \Rightarrow \quad (A - aI)V_1 = 0$

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad by_1 = 0$$

$\Rightarrow \quad y_1 = 0$

The eigenvectors are: $V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \& \quad V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Therefore, the matrix A to be not diagonalizable since the eigenvectors are linearly dependent in *one*–dimensional

Example: $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

Exercise

Construct a nonzero 2×2 matrix that is diagonalizable but not invertible.

Solution

Any 2×2 matrix with 2 distinct eigenvalues is diagonalizable.

Any 2×2 matrix is not invertible when determinant is zero, or either one row or one column is equal to zero.
If one of the eigenvalues is zero, then the matrix is not invertible.

Let assume $A=\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ $\quad a,\ b\ \neq 0$

The eigenvalues are: $\lambda_1 = a$ & $\lambda_2 = 0$

For $\lambda_1 = a \quad \Rightarrow \quad (A-aI)V_1 = 0$

$$\begin{pmatrix} 0 & b \\ 0 & -a \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad by_1 = 0$$

$$\Rightarrow \quad \underline{y_1 = 0}$$

The eigenvectors are: $V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

For $\lambda_2 = 0 \quad \Rightarrow \quad (A)V_2 = 0$

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad ax_1 + by_1 = 0$$

$$\Rightarrow \quad \underline{ax_1 = -by_1}$$

The eigenvectors are: $V_1 = \begin{pmatrix} -b \\ a \end{pmatrix}$

The ***eigenvector matrix*** is given by:

$$P=\begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix}$$

$$P^{-1}=\frac{1}{a}\begin{pmatrix} a & -b \\ 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & -\frac{b}{a} \\ 0 & \frac{1}{a} \end{pmatrix}$$

$$P^{-1}AP=\begin{pmatrix} 1 & -\frac{b}{a} \\ 0 & \frac{1}{a} \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix}$$

$$=\begin{pmatrix} 1 & -\frac{b}{a} \\ 0 & \frac{1}{a} \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \qquad \checkmark$$

Therefore, the result proves that is diagonalizable but not invertible

More Example: $\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$

Exercise

What are the matrices that are similar to themselves only?

Solution

Any matrix to be similar to itself if only if the similar formula

$A = P^{-1}AP$

$PA = PP^{-1}AP$

$PA = IAP$

$PA = AP$

One of the matrices that are similar is a scalars matrices (cI).

Exercise

For any scalars a, b, and c, show that

$$A = \begin{pmatrix} b & c & a \\ c & a & b \\ a & b & c \end{pmatrix}, \quad B = \begin{pmatrix} c & a & b \\ a & b & c \\ b & c & a \end{pmatrix}, \quad C = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$$

are similar.

Moreover, if $BC = CB$, then A has two zero eigenvalues.

Solution

$$\det(A) = \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix}$$

$$\underline{= 3abc - a^2 - b^2 - c^2 \,|}$$

$$\det(B) = \begin{vmatrix} c & a & b \\ a & b & c \\ b & c & a \end{vmatrix}$$

$$\underline{= 3abc - a^2 - b^2 - c^2 \,|}$$

$$\det(C) = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$\underline{= 3abc - a^2 - b^2 - c^2}\;\rfloor$$

Since $\det(A) = \det(B) = \det(C)$, then the matrices *A*, *B*, and *C* are similars.

$$|A - \lambda I| = \begin{vmatrix} b-\lambda & c & a \\ c & a-\lambda & b \\ a & b & c-\lambda \end{vmatrix}$$

$$= (b-\lambda)(a-\lambda)(c-\lambda) + 2abc - a^2(a-\lambda) - b^2(b-\lambda) - c^2(c-\lambda)$$

$$= abc - bc\lambda - ac\lambda + c\lambda^2 - ab\lambda + b\lambda^2 + a\lambda^2 - \lambda^3 + 2abc - a^3 + a^2\lambda - b^3 + b^2\lambda - c^3 + c^2\lambda$$

$$= -\lambda^3 + (c+b+a)\lambda^2 + \left(a^2 + b^2 + c^2 - bc - ac - ab\right)\lambda - a^3 - b^3 - c^3 + 3abc$$

Given that $BC = CB$

$$BC = \begin{pmatrix} c & a & b \\ a & b & c \\ b & c & a \end{pmatrix}\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$$

$$= \begin{pmatrix} ab+bc+ac & ab+bc+ac & a^2+b^2+c^2 \\ a^2+b^2+c^2 & ab+bc+ac & ab+bc+ac \\ ab+bc+ac & a^2+b^2+c^2 & ab+bc+ac \end{pmatrix}$$

$$CB = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}\begin{pmatrix} c & a & b \\ a & b & c \\ b & c & a \end{pmatrix}$$

$$= \begin{pmatrix} ab+bc+ac & a^2+b^2+c^2 & ab+bc+ac \\ ab+bc+ac & ab+bc+ac & a^2+b^2+c^2 \\ a^2+b^2+c^2 & ab+bc+ac & ab+bc+ac \end{pmatrix}$$

Since $BC = CB$, then

$$a^2 + b^2 + c^2 = ab + bc + ac$$

$$a^2 + b^2 + c^2 - (ab + bc + ac) = 0$$

$$\det(A - \lambda I) = -\lambda^3 + (c+b+a)\lambda^2 + \left(a^2 + b^2 + c^2 - bc - ac - ab\right)\lambda - a^3 - b^3 - c^3 + 3abc$$

$$= -\lambda^3 + (c+b+a)\lambda^2 - a^3 - b^3 - c^3 + 3abc$$

$$\left(a^2+b^2+c^2-ab-bc-ac\right)(a+b+c)=0(a+b+c)$$

$$a^3+ab^2+ac^2-a^2b-abc-a^2c+a^2b+b^3+bc^2-ab^2-b^2c-abc$$
$$+a^2c+b^2c+c^3-abc-bc^2-ac^2=0$$

$$a^3+b^3+c^3-3abc=0$$

So,

$$\det(A-\lambda I)=-\lambda^3+(c+b+a)\lambda^2$$
$$=-\lambda^2(\lambda-(c+b+a))$$

The eigenvalues are: $\boxed{\lambda_{1,2}=0 \quad \& \quad \lambda_3=a+b+c}$

Since $BC=CB$, then A has ***two zero*** eigenvalues

Exercise

For positive integer $k\geq 2$, compute $\begin{pmatrix}2 & 1\\ 2 & 3\end{pmatrix}^k$

Solution

Let $A=\begin{pmatrix}2 & 1\\ 2 & 3\end{pmatrix}$

$$\det(A-\lambda I)=\begin{vmatrix}2-\lambda & 1\\ 2 & 3-\lambda\end{vmatrix}$$
$$=6-5\lambda+\lambda^2-2$$
$$=\lambda^2-5\lambda+4=0$$

The eigenvalues are: $\boxed{\lambda_1=1 \quad \& \quad \lambda_2=4}$

For $\lambda_1=1 \Rightarrow \left(A-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \quad \to x+y=0$$

$$\Rightarrow \boxed{x=-y}$$

$$\Rightarrow \boxed{V_1=\begin{pmatrix}-1\\ 1\end{pmatrix}}$$

For $\lambda_2=4 \Rightarrow \left(A-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad -2x + y = 0$$

$$\Rightarrow \underline{2x = y}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

The eigenvectors matrix:

$$P = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$P^{-1} = -\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \quad \Rightarrow \quad D^k = \begin{pmatrix} 1^k & 0 \\ 0 & 4^k \end{pmatrix}$$

$$A^k = PD^kP^{-1}$$

$$= \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4^k \end{pmatrix}\begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 4^k \\ 1 & 2\left(4^k\right) \end{pmatrix}\begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2^{2k} \\ 1 & 2\left(2^{2k}\right) \end{pmatrix}\begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2^{2k} \\ 1 & 2^{2k+1} \end{pmatrix}\begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{3} + \frac{2^{2k}}{3} & -\frac{1}{3} + \frac{2^{2k}}{3} \\ -\frac{2}{3} + \frac{2^{2k+1}}{3} & \frac{1}{3} + \frac{2^{2k+1}}{3} \end{pmatrix}$$

Exercise

For positive integer $k \geq 2$, compute $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^k$

Solution

Let $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

Since it is an upper triangular, then

The eigenvalues are: $\lambda_{1,2} = \lambda$

For $\lambda_1 = \lambda \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to y = 0$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Since the eigenvalues are repeated and the eigenvectors are one-dimensional, therefore the matrix is not diagonalizable.

To compute $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^k$

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix}$$

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^3 = \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{pmatrix}$$

$$\vdots \quad \vdots$$

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^k = \begin{pmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{pmatrix}$$

Exercise

For positive integer $k \geq 2$, compute $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^k$

Solution

Since the eigenvalues $\left(\lambda_{1,2,3} = 0\right)$ are repeated then it is not diagonalizable, which it will result the matrix doesn't have linearly independent eigenvectors.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore;

If $k = 2 \quad \Rightarrow \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^k = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Otherwise $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Exercise

For positive integer $k \ge 2$, compute $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^k$

Solution

Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix}$$

$$= -\lambda^3 + 1 = 0$$

The eigenvalues are: $\lambda_{1,2,3} = -1$

For $\lambda_1 = -1 \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{array}{l} \to \quad -x + y = 0 \\ \to \quad -y + z = 0 \\ \to \quad x - z = 0 \end{array}$$

$$\Rightarrow x = y = z$$

$$\Rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad V_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The given matrix is not diagonalizable, since the matrix doesn't have linearly independent eigenvectors.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{=I}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\underline{=A}$$

When

$$k=3m+1 \quad \Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^k = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = A$$

$$k=3m+2 \quad \Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^k = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$k=3m+3 \quad \Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^k = I_{3\times 3}$$

Exercise

Let $A=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Show that A^k is similar to A fro every positive integer k. It is true more generally for any matrix with all eigenvalues equal to 1.

Solution

Since it is an upper triangular, then

The eigenvalues are: $\underline{\lambda_{1,2}=1}$

For $\lambda_1=1 \quad \Rightarrow \quad (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow y=0$$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

Since the eigenvalues are repeated and the eigenvectors are one-dimensional which are not linearly independent, therefore the matrix is not diagonalizable.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

$$\vdots \qquad \vdots$$

$$A^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

Since it is an upper triangular, then

The eigenvalues are: $\boxed{\lambda_{1,2} = 1}$

Therefore, A^k is similar to A fro every positive integer k.

Let A be $n \times n$ matrix with upper triangular and one's in the main diagonal, which implies that all eigenvalues equal to 1. If we use Jordan block, then each A^k block is similar to A.

Exercise

Can a matrix be similar to two different diagonal matrices?

Solution

The matrix can be similar to two different diagonal matrices as long the size is greater or equal to 3. And they the same eigenvalues by changing the entries in the main diagonal.

Example:

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \quad B = \begin{pmatrix} c & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \quad C = \begin{pmatrix} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & a \end{pmatrix}$$

Exercise

Prove that if A is diagonalizable, then A^T is diagonalizable.

Solution

If A is diagonalizable, then by the diagonalizable theorem

$$A = PDP^{-1}$$

$$\begin{aligned} A^T &= \left(PDP^{-1}\right)^T \\ &= \left(P^{-1}\right)^T D^T P^T && \text{Since } D \text{ is diagonal then } D^T = D \\ &= \left(P^T\right)^{-1} DP^T && \text{Assume that } Q = \left(P^T\right)^{-1} \\ &= QDQ^{-1} \end{aligned}$$

Therefore, A^T is diagonalizable with the columns Q are n linearly independent eigenvectors

Exercise

Prove that if the eigenvalues of a diagonalizable matrix A are all ± 1, then the matrix is equal to its inverse.

Solution

Since the matrix A is diagonalizable with eigenvalues are ± 1, then the diagonal matrix D has ± 1 entries along the main diagonal.

So, $D = D^{-1}$

Matrix A is diagonalizable that implies to $A = PDP^{-1}$

$$\begin{aligned} A^{-1} &= \left(PDP^{-1}\right)^{-1} \\ &= \left(P^{-1}\right)^{-1} D^{-1} P^{-1} \\ &= PDP^{-1} \\ &= A \quad \checkmark \end{aligned}$$

Therefore, the matrix is equal to its inverse

Exercise

Prove that if A is diagonalizable with n real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $|A| = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$

Solution

If A is diagonalizable with n real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and D is diagonal with the eigenvalues as entries, then

$$D = \begin{bmatrix} \lambda_1 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

$$\underline{|D| = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n}$$

$$D = P^{-1}AP$$

$$\left|P^{-1}AP\right| = |D|$$

$$|A| = \left|P^{-1}AP\right|$$

$$= \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$$

Exercise

If x is a real number, then we can define e^x by the series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

In similar way, If X is a square matrix, then we can define e^X by the series

$$e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \frac{1}{4!}X^4 + \cdots$$

Evaluate e^X, where X is the indicated square matrix.

a) $X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

b) $X = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

c) $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

d) $X = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

Solution

a) $X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$$e^X = I + I + \frac{1}{2!}I^2 + \frac{1}{3!}I^3 + \frac{1}{4!}I^4 + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots & 0 \\ 0 & 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots \end{pmatrix}$$

$$= \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$$

Where, $e^1 = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+ \cdots$

b) $X = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

The eigenvalues are: $\underline{\lambda_{1,2} = 0,\ 1}|$

For $\lambda_1 = 0 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = 0$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}|$$

For $\lambda_2 = 1 \quad \Rightarrow \quad (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = y$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}|$$

The eigenvectors matrix:

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad D^k = \begin{pmatrix} 0 & 0 \\ 0 & 1^k \end{pmatrix}$$

$$X^k = PD^kP^{-1}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \frac{1}{4!}X^4 + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{4!}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{4!}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots & 0 \\ 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$$

Given that: $e^1 = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+ \cdots$

c) $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$X^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$X^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\vdots \quad \vdots \quad \vdots$$

$$X^{2k+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$X^{2k} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \frac{1}{4!}X^4 + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{4!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \cdots$$

$$=\begin{pmatrix} 1+\frac{1}{2!}+\frac{1}{4!}+\cdots & 1+\frac{1}{3!}+\frac{1}{5!}+\cdots \\ 1+\frac{1}{3!}+\frac{1}{5!}+\cdots & 1+\frac{1}{2!}+\frac{1}{4!}+\cdots \end{pmatrix}$$

$$=\begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{e+e^{-1}}{2} & \frac{e-e^{-1}}{2} \\ \frac{e-e^{-1}}{2} & \frac{e+e^{-1}}{2} \end{pmatrix}$$

d) $X=\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

$$|X-\lambda I|=\begin{vmatrix} 2-\lambda & 0 \\ 0 & -2-\lambda \end{vmatrix}$$

$$=(2-\lambda)(-2-\lambda)=0$$

The eigenvalues are: $\lambda_{1,2}=-2,\ 2$

For $\lambda_1=-2 \Rightarrow (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow x=0$$

$$\Rightarrow V_1=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For $\lambda_2=2 \Rightarrow (A-\lambda_2 I)V_2=0$

$$\begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow y=0$$

$$\Rightarrow V_2=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The eigenvectors matrix:

$$P=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$P^{-1}=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \quad \Rightarrow \quad D^k = \begin{pmatrix} (-2)^k & 0 \\ 0 & 2^k \end{pmatrix}$$

$$\begin{aligned} X^k &= PD^kP^{-1} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (-2)^k & 0 \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2^k \\ (-2)^k & \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2^k & 0 \\ 0 & (-2)^k \end{pmatrix} \end{aligned}$$

$$\begin{aligned} e^X &= I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \frac{1}{4!}X^4 + \cdots \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 2^2 & 0 \\ 0 & (-2)^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 2^3 & 0 \\ 0 & (-2)^3 \end{pmatrix} + \frac{1}{4!}\begin{pmatrix} 2^4 & 0 \\ 0 & (-2)^4 \end{pmatrix} + \cdots \\ &= \begin{pmatrix} 1 + 2 + \frac{1}{2!}2^2 + \frac{1}{3!}2^3 + \frac{1}{4!}2^4 + \cdots & 0 \\ 0 & 1 + (-2) + \frac{1}{2!}(-2)^2 + \frac{1}{3!}(-2)^3 + \frac{1}{4!}(-2)^4 + \cdots \end{pmatrix} \\ &= \begin{pmatrix} e^2 & 0 \\ 0 & e^{-2} \end{pmatrix} \end{aligned}$$

Where, $e^2 = 1 + 2 + \frac{1}{2!}2^2 + \frac{1}{3!}2^3 + \frac{1}{4!}2^4 + \cdots$

$$e^{-2} = 1 + (-2) + \frac{1}{2!}(-2)^2 + \frac{1}{3!}(-2)^3 + \frac{1}{4!}(-2)^4 + \cdots$$