Lecture Three

SOLUTION

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Section 3.1 – **Inner Products**

Exercise

Let $\langle \vec{u},\ \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (1,\ 1)$, $\vec{v} = (3,\ 2)$, $\vec{w} = (0,\ -1)$, and $k = 3$. Compute the following.

a) $\langle \vec{u},\ \vec{v} \rangle$ c) $\langle \vec{u} + \vec{v},\ \vec{w} \rangle$ e) $d(\vec{u},\ \vec{v})$

b) $\langle k\vec{v},\ \vec{w} \rangle$ d) $\|\vec{v}\|$ f) $\|\vec{u} - k\vec{v}\|$

Solution

a) $\langle \vec{u},\ \vec{v} \rangle = 1(3) + 1(2)$

$= 5$

b) $\langle k\vec{v},\ \vec{w} \rangle = \langle 3\vec{v},\ \vec{w} \rangle$

$= 9 \cdot 0 + 6 \cdot (-1)$

$= -6$

c) $\langle \vec{u} + \vec{v},\ \vec{w} \rangle = \langle \vec{u},\ \vec{w} \rangle + \langle \vec{v},\ \vec{w} \rangle$

$= 1 \cdot 0 + 1 \cdot (-1) + 3 \cdot 0 + 2 \cdot (-1)$

$= -3$

d) $\|\vec{v}\| = \sqrt{\langle \vec{v},\ \vec{v} \rangle}$

$= \sqrt{3^2 + 2^2}$

$= \sqrt{13}$

e) $d(\vec{u},\ \vec{v}) = \|\vec{u} - \vec{v}\|$

$= \|(-2,\ -1)\|$

$= \sqrt{(-2)^2 + (-1)^2}$

$= \sqrt{5}$

f) $\|\vec{u} - k\vec{v}\| = \|(1,\ 1) - 3(3,\ 2)\|$

$= \|(-8,\ -5)\|$

$= \sqrt{(-8)^2 + (-5)^2}$

$$= \sqrt{89}$$

Exercise

Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (3, -2)$, $\vec{v} = (4, 5)$, $\vec{w} = (-1, 6)$, and $k = -4$. Verify the following.

a) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

c) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

d) $\langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle$

e) $\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$

Solution

a) $\langle \vec{u}, \vec{v} \rangle = 3 \cdot 4 + (-2) \cdot (5)$

$$= 2$$

$\langle \vec{v}, \vec{u} \rangle = 4 \cdot 3 + (5) \cdot (-2)$

$$= 2$$

$\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ ✓

b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle (7,3), (-1,6) \rangle$

$$= 7(-1) + 3(6)$$

$$= 11$$

$\langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle = (3)(-1) + (-2)(6) + (4)(-1) + (5)(6)$

$$= 11$$

$\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ ✓

c) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle (3, -2), (3, 11) \rangle$

$$= 3(3) + (-2)(11)$$

$$= -13$$

$\langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle = (3)(4) + (-2)(5) + (3)(-1) + (-2)(6)$

$$= -13$$

$\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$ ✓

d) $\langle k\vec{u}, \vec{v} \rangle = (-4 \cdot 3) \cdot 4 + ((-4)(-2)) \cdot (5)$

$$= -8$$

$k \langle \vec{u}, \vec{v} \rangle = (-4)(3 \cdot 4 + (-2) \cdot (5))$

$$= -8$$

$\langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle$ ✓

e) $\langle \vec{0}, \vec{v} \rangle = 0 \cdot 4 + 0 \cdot (5)$

$\underline{= 0}$

$\langle \vec{v}, \vec{0} \rangle = 4 \cdot 0 + (5) \cdot (0)$

$\underline{= 0}$

$\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$ ✓

Exercise

Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^3$, and let $\vec{u} = (1, 0, -2)$, $\vec{v} = (5, 1, 2)$, $\vec{w} = (5, 2, -1)$, and $k = 3$. Compute the following.

a) $\langle \vec{u}, \vec{v} \rangle$　　　　　　　c) $\langle \vec{u} + \vec{v}, \vec{w} \rangle$　　　　　　　e) $d(\vec{u}, \vec{v})$

b) $\langle k\vec{v}, \vec{w} \rangle$　　　　　　　d) $\|\vec{v}\|$　　　　　　　　　　f) $\|\vec{u} - k\vec{v}\|$

Solution

a) $\langle \vec{u}, \vec{v} \rangle = (1, 0, -2) \bullet (5, 1, 2)$

$= 1(5) + 0(1) - 2(2)$

$\underline{= 1}$

b) $\langle k\vec{v}, \vec{w} \rangle = \langle 3\vec{v}, \vec{w} \rangle$

$= (3(5, 1, 2)) \bullet (5, 2, -1)$

$= (15, 3, 6) \bullet (5, 2, -1)$

$= 15(5) + 3(2) + 6(-1)$

$\underline{= 75}$

c) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

$= (1, 0, -2) \bullet (5, 2, -1) + (5, 1, 2) \bullet (5, 2, -1)$

$= (5 + 0 + 2) + (25 + 2 - 2)$

$\underline{= 32}$

d) $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$

$= \sqrt{(5, 1, 2) \bullet (5, 1, 2)}$

$= \sqrt{25 + 1 + 4}$

$$= \sqrt{30}$$

e) $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$

$$= \|(1, 0, -2) - (5, 1, 2)\|$$

$$= \|(-4, -1, -4)\|$$

$$= \sqrt{16 + 1 + 16}$$

$$= \sqrt{33}$$

f) $\|\vec{u} - k\vec{v}\| = \|(1, 0, -2) - 3(5, 1, 2)\|$

$$= \|(-14, -3, -8)\|$$

$$= \sqrt{196 + 9 + 64}$$

$$= \sqrt{269}$$

Exercise

Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^3$, and let $\vec{u} = (2, 4, 3)$, $\vec{v} = (0, -3, -4)$, $\vec{w} = (6, 3, 1)$, and $k = 2$. Verify the following.

a) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ d) $\langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle$

b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ e) $\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$

c) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

Solution

a) $\langle \vec{u}, \vec{v} \rangle = (2, 4, 3) \cdot (0, -3, -4)$

$$= -12 - 12$$

$$= -24$$

$$\langle \vec{v}, \vec{u} \rangle = (0, -3, -4) \cdot (2, 4, 3)$$

$$= -12 - 12$$

$$= -24$$

$$\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle \quad \checkmark$$

b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle (2, 4, 3) + (0, -3, -4), (6, 3, 1) \rangle$

$$= \langle (2, 1, -1), (6, 3, 1) \rangle$$

$$= 12 + 3 - 1$$

$$= 14$$

$$\langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle = (2, 4, 3) \bullet (6, 3, 1) + (0, -3, -4) \bullet (6, 3, 1)$$
$$= (12 + 12 + 3) + (0 - 9 - 4)$$
$$\underline{= 14} \mid$$

$$\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \quad \checkmark$$

c) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle (2, 4, 3), \quad (0, -3, -4) + (6, 3, 1) \rangle$
$$= \langle (2, 4, 3), \quad (6, 0, -3) \rangle$$
$$= 12 + 0 - 9$$
$$\underline{= 3} \mid$$

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle = (2, 4, 3) \bullet (0, -3, -4) + (2, 4, 3) \bullet (6, 3, 1)$$
$$= (0 - 12 - 12) + (12 + 12 + 3)$$
$$\underline{= 3} \mid$$

$$\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle \quad \checkmark$$

d) $\langle k\vec{u}, \vec{v} \rangle = 2(2, 4, 3) \bullet (0, -3, -4)$
$$= (4, 8, 6) \bullet (0, -3, -4)$$
$$= 0 - 24 - 24$$
$$\underline{= -48} \mid$$

$$k \langle \vec{u}, \vec{v} \rangle = (2)\big((2, 4, 3) \bullet (0, -3, -4)\big)$$
$$= (2)(-12 - 12)$$
$$\underline{= -48} \mid$$

$$\langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle \quad \checkmark$$

e) $\langle \vec{0}, \vec{v} \rangle = (0, 0, 0) \bullet (0, -3, -4)$
$$= 0 + 0 + 0$$
$$\underline{= 0} \mid$$

$$\langle \vec{v}, \vec{0} \rangle = (0, -3, -4) \bullet (0, 0, 0)$$
$$= 0 + 0 + 0$$
$$\underline{= 0} \mid$$

$$\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0 \quad \checkmark$$

Exercise

Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (1, 1)$, $\vec{v} = (3, 2)$, $\vec{w} = (0, -1)$ and $k = 3$. Compute the following for the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 2u_1 v_1 + 3u_2 v_2$.

a) $\langle \vec{u}, \vec{v} \rangle$ c) $\langle \vec{u} + \vec{v}, \vec{w} \rangle$ e) $d(\vec{u}, \vec{v})$

b) $\langle k\vec{v}, \vec{w} \rangle$ d) $\|\vec{v}\|$ f) $\|\vec{u} - k\vec{v}\|$

Solution

a) $\langle \vec{u}, \vec{v} \rangle = 2(1)(3) + 3(1)(2)$

$\qquad = 12$

b) $\langle k\vec{v}, \vec{w} \rangle = 2(3 \cdot 3)(0) + 3(3 \cdot 2)(-1)$

$\qquad = -18$

c) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

$\qquad = 1 \cdot 0 + 1 \cdot (-1) + 3 \cdot 0 + 2 \cdot (-1)$

$\qquad = -3$

d) $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$

$\qquad = \sqrt{2(3)(3) + 3(2)(2)}$

$\qquad = \sqrt{30}$

e) $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$

$\qquad = \|(-2, -1)\|$

$\qquad = \sqrt{2(-2)(-2) + 3(-1)(-1)}$

$\qquad = \sqrt{11}$

f) $\|\vec{u} - k\vec{v}\| = \|(1, 1) - 3(3, 2)\|$

$\qquad = \|(-8, -5)\|$

$\qquad = \sqrt{2(-8)^2 + 3(-5)^2}$

$\qquad = \sqrt{203}$

Exercise

Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (3, -2)$, $\vec{v} = (4, 5)$, $\vec{w} = (-1, 6)$, and $k = -4$. Verify the following for the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 4u_1 v_1 + 5u_2 v_2$.

a) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

c) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

d) $\langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle$

e) $\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$

Solution

a) $\langle \vec{u}, \vec{v} \rangle = 4 \cdot 3 \cdot 4 + 5 \cdot (-2) \cdot (5)$

$\underline{\underline{= -2}}$

$\langle \vec{v}, \vec{u} \rangle = 4 \cdot 4 \cdot 3 + 5 \cdot (5) \cdot (-2)$

$\underline{\underline{= -2}}$

$\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ ✓

b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle (7, 3), (-1, 6) \rangle$

$= 4 \cdot 7(-1) + 5 \cdot 3(6)$

$\underline{\underline{= 62}}$

$\langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle = \left(4 \cdot (3)(-1) + 5 \cdot (-2)(6) \right) + \left(4 \cdot (4)(-1) + 5 \cdot (5)(6) \right)$

$= (-12 - 60) + (-16 + 150)$

$\underline{\underline{= 62}}$

$\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ ✓

c) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle (3, -2), (4, 5) + (-1, 6) \rangle$

$= \langle (3, -2), (3, 11) \rangle$

$= 4 \cdot (3)(3) + 5 \cdot (-2)(11)$

$\underline{\underline{= -74}}$

$\langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle = \left(4 \cdot (3)(4) + 5 \cdot (-2)(5) \right) + \left(4 \cdot (3)(-1) + 5 \cdot (-2)(6) \right)$

$= (48 - 50) + (-12 - 60)$

$\underline{\underline{= -74}}$

$\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$ ✓

d) $\langle k\vec{u}, \vec{v} \rangle = \langle -4(3, -2), (4, 5) \rangle$

$= \langle (-12, 8), (4, 5) \rangle$

$$= 4 \cdot (-12)(4) + 5 \cdot (8)(5)$$

$$= 8$$

$$k\langle \vec{u}, \vec{v} \rangle = (-4)\big(4 \cdot 3 \cdot 4 + 5 \cdot (-2) \cdot (5)\big)$$

$$= 8$$

$$\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle \checkmark$$

e) $\langle \vec{0}, \vec{v} \rangle = 4 \cdot (0)(4) + 5 \cdot (0)(5)$

$$= 0$$

$$\langle \vec{v}, \vec{0} \rangle = 4 \cdot (4)(0) + 5 \cdot (5)(0)$$

$$= 0$$

$$\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0 \quad \checkmark$$

Exercise

Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (-3, 2)$, $\vec{v} = (5, 4)$, $\vec{w} = (1, -6)$, and $k = 2$. Verify the following for the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 4u_1 v_1 + 5u_2 v_2$

a) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

c) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

d) $\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$

e) $\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$

Solution

a) $\langle \vec{u}, \vec{v} \rangle = 4(-3)(5) + 5(2)(4)$

$$= -60 + 40$$

$$= -20$$

$$\langle \vec{v}, \vec{u} \rangle = 4(5)(-3) + 5(4)(2)$$

$$= -60 + 40$$

$$= -20$$

$$\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle \quad \checkmark$$

b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle (-3, 2) + (5, 4), (1, -6) \rangle$

$$= \langle (2, 6), (1, -6) \rangle$$

$$= 4 \cdot (2)(1) + 5 \cdot (6)(-6)$$

$$= 8 - 180$$

$$\underline{= -172}\,|$$

$$\langle \vec{u},\ \vec{w}\rangle + \langle \vec{v},\ \vec{w}\rangle = \big(4\cdot(-3)(1)+5\cdot(2)(-6)\big)+\big(4\cdot(5)(1)+5\cdot(4)(-6)\big)$$

$$= (-12-60)+(20-120)$$

$$\underline{= -172}\,|$$

$$\langle \vec{u}+\vec{v},\ \vec{w}\rangle = \langle \vec{u},\ \vec{w}\rangle + \langle \vec{v},\ \vec{w}\rangle \quad \checkmark$$

c) $\langle \vec{u},\ \vec{v}+\vec{w}\rangle = \big\langle(-3,\ 2),\ (5,\ 4)+(1,\ -6)\big\rangle$

$$= \big\langle(-3,\ 2),\ (6,\ -2)\big\rangle$$

$$= 4\cdot(-3)(6)+5\cdot(2)(-2)$$

$$= -72-20$$

$$\underline{= -92}\,|$$

$$\langle \vec{u},\ \vec{v}\rangle + \langle \vec{u},\ \vec{w}\rangle = \big(4\cdot(-3)(5)+5\cdot(2)(4)\big)+\big(4\cdot(-3)(1)+5\cdot(2)(-6)\big)$$

$$= (-60+40)+(-12-60)$$

$$\underline{= -92}\,|$$

$$\langle \vec{u},\ \vec{v}+\vec{w}\rangle = \langle \vec{u},\ \vec{v}\rangle + \langle \vec{u},\ \vec{w}\rangle \quad \checkmark$$

d) $\langle k\vec{u},\ \vec{v}\rangle = \big\langle 2(-3,\ 2),\ (5,\ 4)\big\rangle$

$$= \big\langle(-6,\ 4),\ (5,\ 4)\big\rangle$$

$$= 4\cdot(-6)(5)+5\cdot(4)(4)$$

$$= -120+80$$

$$\underline{= -40}\,|$$

$$k\langle \vec{u},\ \vec{v}\rangle = 2\cdot\big(4\cdot(-3)(5)+5\cdot(2)(4)\big)$$

$$= 2\cdot(-60+40)$$

$$\underline{= -40}\,|$$

$$\langle k\vec{u},\ \vec{v}\rangle = k\langle \vec{u},\ \vec{v}\rangle \quad \checkmark$$

e) $\langle \vec{0},\ \vec{v}\rangle = 4\cdot(0)(5)+5\cdot(0)(4)$

$$\underline{= 0}\,|$$

$$\langle \vec{v},\ \vec{0}\rangle = 4\cdot(5)(0)+5\cdot(4)(0)$$

$$\underline{= 0}\,|$$

$$\langle \vec{0},\ \vec{v}\rangle = \langle \vec{v},\ \vec{0}\rangle = 0 \quad \checkmark$$

Exercise

Let $\langle \vec{u},\ \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^3$, and let $\vec{u} = (2,\ 1,\ -2)$, $\vec{v} = (-1,\ 3,\ 2)$,

$\vec{w} = (2,\ 1,\ 0)$ and $k = 2$.

Compute the following for the weighted Euclidean inner product $\langle \vec{u},\ \vec{v} \rangle = 2u_1v_1 + 3u_2v_2 + u_3v_3$.

a) $\langle \vec{u},\ \vec{v} \rangle$ c) $\langle \vec{u}+\vec{v},\ \vec{w} \rangle$ e) $d(\vec{u},\ \vec{v})$

b) $\langle k\vec{v},\ \vec{w} \rangle$ d) $\|\vec{v}\|$ f) $\|\vec{u}-k\vec{v}\|$

Solution

a) $\langle \vec{u},\ \vec{v} \rangle = 2\cdot(2)(-1)+3(1)(3)+(-2)(2)$

$\qquad = -4+9-4$

$\qquad \underline{= 1}$

b) $\langle k\vec{v},\ \vec{w} \rangle = \langle 2(-1,\ 3,\ 2),\ (2,\ 1,\ 0)\rangle$

$\qquad = \langle (-2,\ 6,\ 4),\ (2,\ 1,\ 0)\rangle$

$\qquad = 2\cdot(-2)(2)+3(6)(1)+(4)(0)$

$\qquad = -8+18$

$\qquad \underline{= 10}$

c) $\langle \vec{u}+\vec{v},\ \vec{w} \rangle = \langle (2,\ 1,\ -2)+(-1,\ 3,\ 2),\ (2,\ 1,\ 0)\rangle$

$\qquad = \langle (1,\ 4,\ 0),\ (2,\ 1,\ 0)\rangle$

$\qquad = 2\cdot(1)(2)+3(4)(1)+(0)(0)$

$\qquad \underline{= 14}$

d) $\|\vec{v}\| = \sqrt{\langle \vec{v},\ \vec{v}\rangle}$

$\qquad = \sqrt{2\cdot(-1)(-1)+3(3)(3)+(2)(2)}$

$\qquad = \sqrt{2+27+4}$

$\qquad \underline{= \sqrt{33}}$

e) $d(\vec{u},\ \vec{v}) = \|(\vec{u}-\vec{v})\|$

$\qquad = \|(2,\ 1,\ -2)-(-1,\ 3,\ 2)\|$

$\qquad = \|(3,\ -2,\ -4)\|$

$\qquad = \sqrt{2\cdot(3)(3)+3(-2)(-2)+(-4)(-4)}$

$\qquad = \sqrt{18+12+16}$

$\qquad \underline{= \sqrt{29}}$

f) $\|\vec{u} - k\vec{v}\| = \|(2,\ 1,\ -2) - 2(-1,\ 3,\ 2)\|$

$= \|(2,\ 1,\ -2) - (-2,\ 6,\ 4)\|$

$= \|\langle(4,\ -5,\ -6)\rangle\|$

$= \sqrt{2(16) + 3(25) + 36}$

$= \sqrt{32 + 75 + 36}$

$= \sqrt{143}$

Exercise

Let $\vec{u} = \left(u_1,\ u_2\right)$ and $\vec{v} = \left(v_1,\ v_2\right)$. Show that the following are inner product on $\mathbb{R}^2$ by verifying that the inner product axioms hold. $\langle \vec{u},\ \vec{v}\rangle = 3u_1 v_1 + 5u_2 v_2$

Solution

Let $\vec{w} = \left(w_1,\ w_2\right)$

Axiom 1: $\langle \vec{u},\ \vec{v}\rangle = 3u_1 v_1 + 5u_2 v_2$

$= 3v_1 u_1 + 5v_2 u_2$

$= \langle \vec{v},\ \vec{u}\rangle$ ✓

Axiom 2: $\langle \vec{u} + \vec{v},\ \vec{w}\rangle = 3\left(u_1 + v_1\right)w_1 + 5\left(u_2 + v_2\right)w_2$

$= 3\left(u_1 w_1 + v_1 w_1\right) + 5\left(u_2 w_2 + v_2 w_2\right)$

$= 3u_1 w_1 + 3v_1 w_1 + 5u_2 w_2 + 5v_2 w_2$

$= \left(3u_1 w_1 + 5u_2 w_2\right) + \left(3v_1 w_1 + 5v_2 w_2\right)$

$= \langle \vec{u},\ \vec{w}\rangle + \langle \vec{v},\ \vec{w}\rangle$ ✓

Axiom 3: $\langle k\vec{u},\ \vec{v}\rangle = 3\left(ku_1\right)v_1 + 5\left(ku_2\right)v_2$

$= k\left(3u_1 v_1 + 5u_2 v_2\right)$

$= k\langle \vec{u},\ \vec{v}\rangle$ ✓

Axiom 4: $\langle \vec{v},\ \vec{v}\rangle = 3v_1 v_1 + 5v_2 v_2$

$= 3v_1^2 + 5v_2^2 \geq 0$

$v_1 = v_2 = 0$ *iff* $\vec{v} = \vec{0}$ ✓

Exercise

Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$. Show that the following are inner product on $\mathbb{R}^3$ by verifying that the inner product axioms hold. $\langle \vec{u}, \vec{v} \rangle = 2u_1v_1 + 3u_2v_2 + 5u_3v_3$

Solution

Let $\vec{w} = (w_1, w_2, w_3)$

Axiom 1: $\langle \vec{u}, \vec{v} \rangle = 2u_1v_1 + 3u_2v_2 + 5u_3v_3$

$$= 2v_1u_1 + 3v_2u_2 + 5v_3u_3$$

$$= \langle \vec{v}, \vec{u} \rangle \qquad \checkmark$$

Axiom 2: $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle (u_1, u_2, u_3) + (v_1, v_2, v_3),\ (w_1, w_2, w_3) \rangle$

$$= \langle (u_1 + v_1, u_2 + v_2, u_3 + v_3),\ (w_1, w_2, w_3) \rangle$$

$$= 2(u_1 + v_1)w_1 + 3(u_2 + v_2)w_2 + 5(u_3 + v_3)w_3$$

$$= 2u_1w_1 + 2v_1w_1 + 3u_2w_2 + 3v_2w_2 + 5u_3w_3 + 5v_3w_3$$

$$= (2u_1w_1 + 3u_2w_2 + 5u_3w_3) + (2v_1w_1 + 3v_2w_2 + 5v_3w_3)$$

$$= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \quad \checkmark$$

Axiom 3: $\langle k\vec{u}, \vec{v} \rangle = \langle k(u_1, u_2, u_3),\ (v_1, v_2, v_3) \rangle$

$$= \langle (ku_1, ku_2, ku_3),\ (v_1, v_2, v_3) \rangle$$

$$= 2(ku_1)v_1 + 3(ku_2)v_2 + 5(ku_3)v_3$$

$$= k(2u_1v_1 + 3u_2v_2 + 5u_3v_3)$$

$$= k\langle \vec{u}, \vec{v} \rangle \qquad \checkmark$$

Axiom 4: $\langle \vec{v}, \vec{v} \rangle = 2v_1v_1 + 3v_2v_2 + 5v_3v_3$

$$= 2v_1^2 + 3v_2^2 + 5v_3^2 \geq 0$$

$$v_1 = v_2 = v_3 = 0 \quad \text{iff} \quad \vec{v} = \vec{0} \qquad \checkmark$$

Exercise

Show that the following identity holds for the vectors in any inner product space

$$\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$$

Solution

$$
\begin{aligned}
\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 &= \langle \vec{u} + \vec{v},\ \vec{u} + \vec{v} \rangle + \langle \vec{u} - \vec{v},\ \vec{u} - \vec{v} \rangle \\
&= \langle \vec{u},\ \vec{u} + \vec{v} \rangle + \langle \vec{v},\ \vec{u} + \vec{v} \rangle + \langle \vec{u},\ \vec{u} - \vec{v} \rangle - \langle \vec{v},\ \vec{u} - \vec{v} \rangle \\
&= \langle \vec{u},\ \vec{u} \rangle + \langle \vec{u},\ \vec{v} \rangle + \langle \vec{v},\ \vec{u} \rangle + \langle \vec{v},\ \vec{v} \rangle + \langle \vec{u},\ \vec{u} \rangle - \langle \vec{u},\ \vec{v} \rangle - \langle \vec{v},\ \vec{u} \rangle + \langle \vec{v},\ \vec{v} \rangle \\
&= 2\langle \vec{u},\ \vec{u} \rangle + 2\langle \vec{v},\ \vec{v} \rangle \\
&= 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2 \quad \checkmark
\end{aligned}
$$

Exercise

Show that the following identity holds for the vectors in any inner product space

$$\langle \vec{u},\ \vec{v} \rangle = \frac{1}{4}\left(\|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 \right)$$

Solution

$$
\begin{aligned}
\|\vec{u} + \vec{v}\|^2 &= \langle \vec{u} + \vec{v},\ \vec{u} + \vec{v} \rangle \\
&= \|\vec{u}\|^2 + 2\langle \vec{u},\ \vec{v} \rangle + \|\vec{v}\|^2
\end{aligned}
$$

$$
\begin{aligned}
\|\vec{u} - \vec{v}\|^2 &= \langle \vec{u} - \vec{v},\ \vec{u} - \vec{v} \rangle \\
&= \|\vec{u}\|^2 - 2\langle \vec{u},\ \vec{v} \rangle + \|\vec{v}\|^2
\end{aligned}
$$

$$
\begin{aligned}
\|\vec{u} + \vec{v}\|^2 &= \|\vec{u}\|^2 + 2\langle \vec{u},\ \vec{v} \rangle + \|\vec{v}\|^2 \\
- \quad \|\vec{u} - \vec{v}\|^2 &= \|\vec{u}\|^2 - 2\langle \vec{u},\ \vec{v} \rangle + \|\vec{v}\|^2 \\
\hline
\|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 &= 4\langle \vec{u},\ \vec{v} \rangle
\end{aligned}
$$

$$\langle \vec{u},\ \vec{v} \rangle = \frac{1}{4}\left(\|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 \right) \quad \checkmark$$

Exercise

Prove that $\|k\vec{v}\| = |k|\ \|\vec{v}\|$

Solution

$$\|k\vec{v}\|^2 = \langle k\vec{v}, \ \vec{v} \rangle$$

$$= k^2 \langle \vec{v}, \ \vec{v} \rangle$$

$$= k^2 \|\vec{v}\|^2$$

$$\|k\vec{v}\| = k \ \|\vec{v}\| \quad \checkmark$$

Solution

Section **3.2 – Angle and Orthogonality in Inner Product Spaces**

Exercise

Which of the following form orthonormal set?

$$\{(1,\ 0),\ (0,\ 2)\} \quad in\ \mathbb{R}^2$$

Solution

$$(1,\ 0) \bullet (0,\ 2) = 1(0) + 0(2)$$
$$= 0$$

Therefore, they are *orthonormal* set.

Exercise

Which of the following form orthonormal set?

$$\{(2,\ -4),\ (2,\ 1)\} \quad in\ \mathbb{R}^2$$

Solution

$$(2,\ -4) \bullet (2,\ 1) = 4 - 4$$
$$= 0$$

Therefore, they are *orthonormal* set.

Exercise

Which of the following form orthonormal set?

$$\left\{ \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}} \right),\ \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}} \right) \right\} \quad in\ \mathbb{R}^2$$

Solution

$$\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}} \right) \bullet \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}} \right)\frac{1}{\sqrt{2}}$$
$$= \frac{1}{2} - \frac{1}{2}$$
$$= 0$$

Therefore, they are *orthonormal* set.

Exercise

Which of the following form orthonormal set?

$$\left\{\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\} \quad in \ \mathbb{R}^2$$

Solution

$$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}$$

$$= -\frac{1}{2} - \frac{1}{2}$$

$$= -1 \neq 0$$

Therefore, they are *not orthonormal* set

Exercise

Which of the following form orthonormal set?

$$\left\{\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)\right\} \quad in \ \mathbb{R}^3$$

Solution

$$\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right) \cdot \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) = \frac{1}{\sqrt{6}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}}\left(-\frac{1}{\sqrt{2}}\right) + 0$$

$$= 0$$

Therefore, they are *orthonormal* set.

Exercise

Which of the following form orthonormal set?

$$\left\{(4, -1, 1), (-1, 0, 4), (-4, -17, -1)\right\} \quad in \ \mathbb{R}^3$$

Solution

$$(4, -1, 1) \cdot (-1, 0, 4) \cdot (-4, -17, -1) = 16 + 0 - 4$$

$$= 12 \neq 0$$

Therefore, they are *not orthonormal* set.

Exercise

Which of the following form orthonormal set?

$$\left\{\left(\frac{2}{3}, \ -\frac{2}{3}, \ \frac{1}{3}\right), \ \left(\frac{2}{3}, \ \frac{1}{3}, \ -\frac{2}{3}\right), \ \left(\frac{1}{3}, \ \frac{2}{3}, \ \frac{2}{3}\right)\right\} \quad in \ \mathbb{R}^3$$

Solution

$$\left(\frac{2}{3}, \ -\frac{2}{3}, \ \frac{1}{3}\right) \cdot \left(\frac{2}{3}, \ \frac{1}{3}, \ -\frac{2}{3}\right) \cdot \left(\frac{1}{3}, \ \frac{2}{3}, \ \frac{2}{3}\right) = \frac{2}{3}\frac{2}{3}\frac{1}{3} + \left(-\frac{2}{3}\right)\frac{1}{3}\frac{2}{3} + \frac{1}{3}\left(-\frac{2}{3}\right)\frac{2}{3}$$

$$= \frac{4}{27} - \frac{4}{27} - \frac{4}{27}$$

$$= -\frac{4}{27} \neq 0$$

Therefore, they are ***not orthonormal*** set.

Exercise

Which of the following form orthonormal set?

$$\left\{\left(\frac{1}{\sqrt{2}}, \ 0, \ \frac{1}{\sqrt{2}}\right), \ \left(\frac{1}{\sqrt{3}}, \ \frac{1}{\sqrt{3}}, \ -\frac{1}{\sqrt{3}}\right), \ \left(-\frac{1}{\sqrt{2}}, \ 0, \ \frac{1}{\sqrt{2}}\right)\right\} \quad in \ \mathbb{R}^3$$

Solution

$$\left(\frac{1}{\sqrt{2}}, \ 0, \ \frac{1}{\sqrt{2}}\right) \cdot \left(\frac{1}{\sqrt{3}}, \ \frac{1}{\sqrt{3}}, \ -\frac{1}{\sqrt{3}}\right) \cdot \left(-\frac{1}{\sqrt{2}}, \ 0, \ \frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{2}}\right) + 0 + \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{3}}\right)\frac{1}{\sqrt{2}}$$

$$= -\frac{1}{2}\frac{1}{\sqrt{3}} - \frac{1}{2}\frac{1}{\sqrt{3}}$$

$$= -\frac{1}{\sqrt{3}} \neq 0$$

Therefore, they are ***not orthonormal*** set.

Exercise

Which of the following form orthonormal set?

$$\left\{\left(\frac{1}{\sqrt{2}}, \ 0, \ 0, \ \frac{1}{\sqrt{2}}\right), \ \left(0, \ \frac{1}{\sqrt{2}}, \ \frac{1}{\sqrt{2}}, \ 0\right), \ \left(-\frac{1}{2}, \ \frac{1}{2}, \ -\frac{1}{2}, \ \frac{1}{2}\right)\right\} \quad in \ \mathbb{R}^4$$

Solution

$$\left(\frac{1}{\sqrt{2}}, \ 0, \ 0, \ \frac{1}{\sqrt{2}}\right) \cdot \left(0, \ \frac{1}{\sqrt{2}}, \ \frac{1}{\sqrt{2}}, \ 0\right) \cdot \left(-\frac{1}{2}, \ \frac{1}{2}, \ -\frac{1}{2}, \ \frac{1}{2}\right) = 0 + 0 + 0 + 0$$

$$= 0$$

Therefore, they are **orthonormal** set.

Exercise

Which of the following form orthonormal set?

$$\left\{ \left(\frac{1}{\sqrt{2}}, \ \frac{1}{\sqrt{2}}, \ 0, \ 0 \right), \ \left(0, \ 0, \ -\frac{1}{\sqrt{2}}, \ -\frac{1}{\sqrt{2}} \right), \ \left(-\frac{1}{3}, \ \frac{1}{3}, \ -\frac{1}{3}, \ \frac{1}{3} \right) \right\} \quad in \ \mathbb{R}^4$$

Solution

$$\left(\frac{1}{\sqrt{2}}, \ \frac{1}{\sqrt{2}}, \ 0, \ 0 \right) \cdot \left(0, \ 0, \ -\frac{1}{\sqrt{2}}, \ -\frac{1}{\sqrt{2}} \right) \cdot \left(-\frac{1}{3}, \ \frac{1}{3}, \ -\frac{1}{3}, \ \frac{1}{3} \right) = 0 + 0 + 0 + 0$$

$$= 0$$

Therefore, they are **orthonormal** set.

Exercise

Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.

$$\vec{u} = (1, \ -3), \quad \vec{v} = (2, \ 4)$$

Solution

$$\|\vec{u}\| = \sqrt{1^2 + (-3)^2}$$

$$= \sqrt{10}$$

$$\|\vec{v}\| = \sqrt{2^2 + 4^2}$$

$$= \sqrt{20}$$

$$\langle \vec{u}, \ \vec{v} \rangle = 1(2) + (-3)(4)$$

$$= -10$$

$$\cos \theta = \frac{-10}{\sqrt{10} \ \sqrt{20}} \qquad\qquad \cos \theta = \frac{\langle \vec{u}, \ \vec{v} \rangle}{\|\vec{u}\| \ \|\vec{v}\|}$$

$$= -\frac{10}{\sqrt{200}}$$

$$= -\frac{1}{\sqrt{2}}$$

Exercise

Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.

$$\vec{u} = (-1,\ 0);\quad \vec{v} = (3,\ 8)$$

Solution

$$\|\vec{u}\| = \sqrt{(-1)^2 + 0^2}$$

$$= 1$$

$$\|\vec{v}\| = \sqrt{3^2 + 8^2}$$

$$= \sqrt{73}$$

$$\langle \vec{u},\ \vec{v} \rangle = (-1)(3) + (0)(8)$$

$$= -3$$

$$\cos\theta = \frac{-3}{1\sqrt{73}} \qquad\qquad \cos\theta = \frac{\langle \vec{u},\ \vec{v} \rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$

$$= -\frac{3}{\sqrt{73}}$$

Exercise

Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.

$$\vec{u} = (-1,\ 5,\ 2);\quad \vec{v} = (2,\ 4,\ -9)$$

Solution

$$\|\vec{u}\| = \sqrt{(-1)^2 + 5^2 + 2^2}$$

$$= \sqrt{30}$$

$$\|\vec{v}\| = \sqrt{2^2 + 4^2 + (-9)^2}$$

$$= \sqrt{101}$$

$$\langle \vec{u},\ \vec{v} \rangle = (-1)(2) + (5)(4) + (2)(-9)$$

$$= 0$$

$$\cos\theta = 0 \qquad\qquad \cos\theta = \frac{\langle \vec{u},\ \vec{v} \rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$

Exercise

Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.

$$\vec{u} = (4,\ 1,\ 8);\quad \vec{v} = (1,\ 0,\ -3)$$

Solution

$$\|\vec{u}\| = \sqrt{4^2 + 1^2 + 8^2}$$

$$= 9$$

$$\|\vec{v}\| = \sqrt{1 + 0 + 9}$$

$$= \sqrt{10}$$

$$\langle \vec{u},\ \vec{v} \rangle = (4)(1) + (1)(0) + (8)(-3)$$

$$= -20$$

$$\cos\theta = -\frac{20}{9\sqrt{10}} \qquad\qquad \cos\theta = \frac{\langle \vec{u},\ \vec{v} \rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$

$$= -\frac{2\sqrt{10}}{9}$$

Exercise

Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.

$$\vec{u} = (1,\ 0,\ 1,\ 0);\quad \vec{v} = (-3,\ -3,\ -3,\ -3)$$

Solution

$$\|\vec{u}\| = \sqrt{2}$$

$$\|\vec{v}\| = \sqrt{9 + 9 + 9 + 9}$$

$$= 12$$

$$\langle \vec{u},\ \vec{v} \rangle = -3 + 0 - 3 + 0$$

$$= -6$$

$$\cos\theta = \frac{-6}{12\sqrt{2}} \qquad\qquad \cos\theta = \frac{\langle \vec{u},\ \vec{v} \rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$

$$= -\frac{\sqrt{2}}{4}$$

Exercise

Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.
$$\vec{u} = (2,\ 1,\ 7,\ -1); \quad \vec{v} = (4,\ 0,\ 0,\ 0)$$

Solution

$$\|\vec{u}\| = \sqrt{2^2 + 1^2 + 7^2 + (-1)^2}$$
$$= \sqrt{55}$$

$$\|\vec{v}\| = \sqrt{4^2 + 0}$$
$$= 4$$

$$\langle \vec{u},\ \vec{v} \rangle = (2)(4) + (1)(0) + (7)(0) + (-1)(0)$$
$$= 8$$

$$\cos\theta = \frac{8}{4\sqrt{55}} \qquad\qquad \cos\theta = \frac{\langle \vec{u},\ \vec{v} \rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$

$$= \frac{2}{\sqrt{55}}$$

Exercise

Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.
$$\vec{u} = (1,\ 3,\ -5,\ 4), \quad \vec{v} = (2,\ -4,\ 4,\ 1)$$

Solution

$$\|\vec{u}\| = \sqrt{1 + 9 + 25 + 16}$$
$$= \sqrt{51}$$

$$\|\vec{v}\| = \sqrt{4 + 16 + 16 + 1}$$
$$= \sqrt{37}$$

$$\langle \vec{u},\ \vec{v} \rangle = 2 - 12 - 20 + 4$$
$$= -26$$

$$\cos\theta = \frac{-26}{\sqrt{51}\sqrt{37}} \qquad\qquad \cos\theta = \frac{\langle \vec{u},\ \vec{v} \rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$

Exercise

Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.

$$\vec{u} = (1,\ 2,\ 3,\ 4), \quad \vec{v} = (-1,\ -2,\ -3,\ -4)$$

Solution

$$\|\vec{u}\| = \sqrt{1+4+9+16}$$

$$= \sqrt{30}$$

$$\|\vec{v}\| = \sqrt{1+4+9+16}$$

$$= \sqrt{30}$$

$$\langle \vec{u},\ \vec{v} \rangle = -1-4-9-16$$

$$= -30$$

$$\cos\theta = \frac{-30}{\sqrt{30}\sqrt{30}} \qquad\qquad \cos\theta = \frac{\langle \vec{u},\ \vec{v} \rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$

$$= -1$$

Exercise

Find the cosine of the angle between A and B.

$$A = \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$$

Solution

$$\|A\| = \sqrt{\langle A,\ A \rangle}$$

$$= \sqrt{2^2 + 6^2 + 1^2 + (-3)^2}$$

$$= \sqrt{50}$$

$$= 5\sqrt{2}$$

$$\|B\| = \sqrt{\langle B,\ B \rangle}$$

$$= \sqrt{9+4+1+0}$$

$$= \sqrt{14}$$

$$\langle A,\ B \rangle = 2(3) + 6(2) + 1(1) + (-3)(0)$$

$$= 19$$

$$\cos\theta = \frac{19}{5\sqrt{2}\sqrt{14}} \qquad\qquad \cos\theta = \frac{\langle A,\ B \rangle}{\|A\|\ \|B\|}$$

$$= \frac{19}{10\sqrt{7}}$$

Exercise

Find the cosine of the angle between **A** and **B**.

$$A = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$$

Solution

$$\|A\| = \sqrt{\langle A,\, A \rangle}$$

$$= \sqrt{2^2 + 4^2 + (-1)^2 + 3^2}$$

$$= \sqrt{30}$$

$$\|B\| = \sqrt{\langle B,\, B \rangle}$$

$$= \sqrt{(-3)^2 + 1^2 + 4^2 + 2^2}$$

$$= \sqrt{30}$$

$$\langle A,\, B \rangle = 2(-3) + 4(1) + (-1)(4) + 3(2)$$

$$= 0$$

$$\cos\theta = \frac{0}{30} \qquad \cos\theta = \frac{\langle A,\, B \rangle}{\|A\|\,\|B\|}$$

$$= 0$$

Exercise

Find the cosine of the angle between **A** and **B**.

$$A = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Solution

$$\|A\| = \sqrt{81 + 64 + 49 + 36 + 25 + 16} \qquad \|A\| = \sqrt{\langle A,\, A \rangle}$$

$$= \sqrt{271}$$

$$\|B\| = \sqrt{1 + 4 + 9 + 16 + 25 + 36} \qquad \|B\| = \sqrt{\langle B,\, B \rangle}$$

$$= \sqrt{91}$$

$$\langle A, B \rangle = 9 + 16 + 21 + 24 + 25 + 24$$

$$= 119$$

$$\cos \theta = \frac{119}{\sqrt{271}\sqrt{91}} \qquad \cos \theta = \frac{\langle A, B \rangle}{\|A\| \|B\|}$$

Exercise

Find the cosine of the angle between *A* and *B*.

$$A = \begin{pmatrix} 1 & -2 & 7 \\ 6 & -3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 3 \\ 6 & 5 & -4 \end{pmatrix}$$

Solution

$$\|A\| = \sqrt{1^2 + (-2)^2 + 7^2 + 6^2 + (-3)^2 + 4^2} \qquad \|A\| = \sqrt{\langle A, A \rangle}$$

$$= \sqrt{115}$$

$$\|B\| = \sqrt{1^2 + (-2)^2 + 3^2 + 6^2 + 5^2 + (-4)^2} \qquad \|B\| = \sqrt{\langle B, B \rangle}$$

$$= \sqrt{91}$$

$$\langle A, B \rangle = 1 + 4 + 21 + 36 - 15 - 16$$

$$= 31$$

$$\cos \theta = \frac{31}{\sqrt{115}\sqrt{91}} \qquad \cos \theta = \frac{\langle A, B \rangle}{\|A\| \|B\|}$$

Exercise

Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

$$\vec{u} = (-1, 3, 2), \quad \vec{v} = (4, 2, -1)$$

Solution

$$\langle \vec{u}, \vec{v} \rangle = (-1)(4) + 3(2) + 2(-1)$$

$$= 0$$

Therefore, the given vectors are orthogonal.

Exercise

Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

$$\vec{u} = (a, b), \quad \vec{v} = (-b, a)$$

Solution

$$\langle \vec{u}, \vec{v} \rangle = a(-b) + b(a)$$
$$\underline{= 0}$$

Therefore, the given vectors are orthogonal.

Exercise

Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

$$\vec{u} = (-2, -2, -2), \quad \vec{v} = (1, 1, 1)$$

Solution

$$\langle \vec{u}, \vec{v} \rangle = (-2)(1) + (-2)(1) + (-2)(1)$$
$$\underline{= -6}$$

Therefore, the given vectors are **not** orthogonal.

Exercise

Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

$$\vec{u} = (-4, 6, -10, 1), \quad \vec{v} = (2, 1, -2, 9)$$

Solution

$$\langle\langle \vec{u}, \vec{v} \rangle\rangle = (-4)(2) + (6)(1) + (-10)(-2) + (1)(9)$$
$$\underline{= 27} \quad \neq 0$$

Therefore, the given vectors are **not** orthogonal.

Exercise

Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

$$\vec{u} = (-4, 6, -10, 1), \quad \vec{v} = (2, 1, -2, 9)$$

Solution

$$\|\vec{u}\| = \sqrt{(-4)^2 + 6^2 + (-10)^2 + 1^2}$$

$$= \sqrt{153}$$

$$= 3\sqrt{17}$$

$$\|\vec{v}\| = \sqrt{2^2 + 1^2 + (-2)^2 + 9^2}$$

$$= \sqrt{90}$$

$$= 3\sqrt{10}$$

$$\langle \vec{u}, \vec{v} \rangle = (-4)(2) + 6(1) - 10(-2) + 1(9)$$

$$= 27$$

$$\cos\theta = \frac{27}{3\sqrt{17}\left(3\sqrt{10}\right)} \qquad\qquad \cos\theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \, \|\vec{v}\|}$$

$$= \frac{3}{\sqrt{170}}$$

The vectors $\vec{u}$ *and* $\vec{v}$ are **not** orthogonal with respect to the Euclidean

Exercise

Do there exist scalars k and l such that the vectors $\vec{u} = (2,\, k,\, 6)$, $\vec{v} = (l,\, 5,\, 3)$, *and* $\vec{w} = (1,\, 2,\, 3)$ are mutually orthogonal with respect to the Euclidean inner product?

Solution

$$\langle \vec{u}, \vec{w} \rangle = (2)(1) + (k)(2) + (6)(3)$$

$$= 20 + 2k = 0$$

$$\Rightarrow k = -10$$

$$\langle \vec{v}, \vec{w} \rangle = (l)(1) + (5)(2) + (3)(3)$$

$$= l + 19 = 0$$

$$\Rightarrow l = -19$$

$$\langle \vec{u}, \vec{v} \rangle = (2)(l) + (k)(5) + (6)(3)$$

$$= 2l + 5k + 18 = 0$$

$$2(-19) + 5(-10) + 18 = -70 \neq 0$$

Thus, there are no scalars such that the vectors are mutually orthogonal.

Exercise

Let $\mathbb{R}^3$ have the Euclidean inner product. For which values of k are $\vec{u}$ and $\vec{v}$ orthogonal?

a) $\vec{u} = (2, 1, 3), \quad \vec{v} = (1, 7, k)$

b) $\vec{u} = (k, k, 1), \quad \vec{v} = (k, 5, 6)$

Solution

a) $\langle \vec{u}, \vec{v} \rangle = (2)(1) + (1)(7) + (3)(k)$

$\qquad = 9 + 3k = 0$

$\vec{u}$ and $\vec{v}$ are orthogonal for $\underline{k = -3}$

b) $\langle \vec{u}, \vec{v} \rangle = (k)(k) + (k)(5) + (1)(6)$

$\qquad = k^2 + 5k + 6 = 0$

$\vec{u}$ and $\vec{v}$ are orthogonal for $\underline{k = -2, \ -3}$

Exercise

Let V be an inner product space. Show that if $\vec{u}$ and $\vec{v}$ are orthogonal unit vectors in V, then $\|\vec{u} - \vec{v}\| = \sqrt{2}$

Solution

$\|\vec{u} - \vec{v}\|^2 = \langle \vec{u} - \vec{v}, \vec{u} - \vec{v} \rangle$

$\qquad = \langle \vec{u}, \vec{u} - \vec{v} \rangle - \langle \vec{v}, \vec{u} - \vec{v} \rangle$

$\qquad = \langle \vec{u}, \vec{u} \rangle - \langle \vec{u}, \vec{v} \rangle - \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{v} \rangle$

$\qquad = \|\vec{u}\|^2 - 0 - 0 + \|\vec{v}\|^2 \qquad\qquad$ since $\vec{u}$ and $\vec{v}$ are orthogonal unit vectors

$\qquad = 1 + 1$

$\qquad = 2$

Thus $\underline{\|\vec{u} - \vec{v}\| = \sqrt{2}}$

Exercise

Let $\boldsymbol{S}$ be a subspace of $\mathbb{R}^n$. Explain what $\left(\boldsymbol{S}^\perp \right)^\perp = \boldsymbol{S}$ means and why it is true.

Solution

$\left(S^{\perp}\right)^{\perp}$ is the orthogonal complement of , $S^{\perp}$, which is itself the orthogonal complement of S, so

$\left(S^{\perp}\right)^{\perp} = S$ means that S is the orthogonal of its orthogonal complement.

We need to show that S is contained in $\left(S^{\perp}\right)^{\perp}$ and, conversely, that $\left(S^{\perp}\right)^{\perp}$ is contained in S to be true.

i. Suppose $\vec{v} \in S^{\perp}$ and $\vec{w} \in S^{\perp}$. Then $\langle \vec{v}, \vec{w} \rangle = 0$ by definition of $S^{\perp}$.

Thus, S is certainly contained is $\left(S^{\perp}\right)^{\perp}$ (which consists of all vectors in $\mathbb{R}^n$ which are orthogonal to $S^{\perp}$).

ii. Suppose $\vec{v} \in \left(S^{\perp}\right)^{\perp}$ (means $\vec{v}$ is orthogonal to all vectors in $S^{\perp}$); then we need to show that $\vec{v} \in S$.

Let assume $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ be a basis for S and let $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_q\}$ be a basis for $S^{\perp}$.

If $\vec{v} \notin S$, then $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p, \vec{v}\}$ is linearly independent set. Since each vector ifs that set is orthogonal to all of $S^{\perp}$, the set $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p, \vec{v}, \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_q\}$ is linearly independent.

Since there are $p + q + 1$ vectors in this set, this means that $p + q + 1 \le n \iff p + q \le n - 1$.

On the other hand, If A is the matrix whose i^{th} row is $\vec{u}_i^T$, then the row space of A is S and the nullspace of A is $S^{\perp}$.

Since S is p-dimensional, the rank of A is p, meaning that the dimension of $\text{nul}(A) = S^{\perp}$ is $q = n - p$. Therefore,

$$p + q = p + (n - p) = n$$

Which contradict the fact that $p + q \le n - 1$. From this, we see that, if $\vec{v} \in \left(S^{\perp}\right)^{\perp}$, it must be the case that $\vec{v} \in S$.

Exercise

The methane molecule CH_4 is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$ – (***note*** that all six edges have length $\sqrt{2}$, so the tetrahedron is regular). What is the cosine of the angle between the rays going from the center $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ to the vertices?

Solution
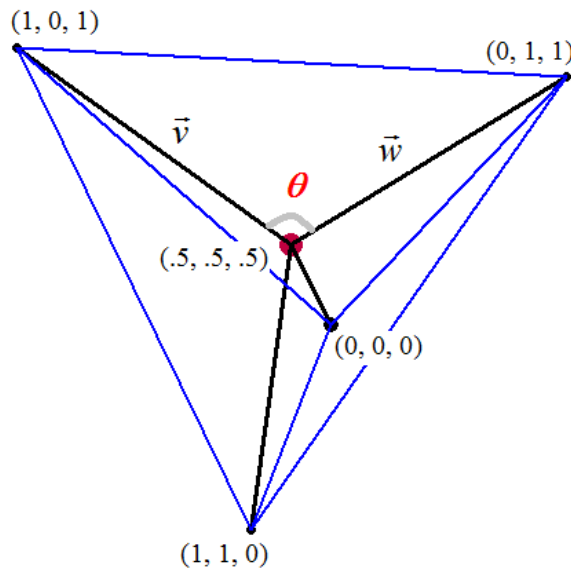
Let $\vec{v}$ be the vector of the segment $(1, 0, 1)$ and $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

Let be the vector of the segment $(0, 1, 1)$ and $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$$\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -\dfrac{1}{2} \\ \dfrac{1}{2} \\ \dfrac{1}{2} \end{pmatrix}$$

We have:

$$\cos \theta = \frac{\langle \vec{v}, \vec{w} \rangle}{\| \vec{v} \| \, \| \vec{w} \|}$$

$$= \frac{\left(\dfrac{1}{2}, -\dfrac{1}{2}, \dfrac{1}{2} \right) \cdot \left(-\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{2} \right)}{\sqrt{\dfrac{1}{4} + \dfrac{1}{4} + \dfrac{1}{4}} \, \sqrt{\dfrac{1}{4} + \dfrac{1}{4} + \dfrac{1}{4}}}$$

$$= \frac{-\dfrac{1}{4}}{\dfrac{3}{4}}$$

$$= -\frac{1}{3}$$

$$\theta \approx 109.47°$$

Exercise

Determine if the given vectors are orthogonal.

$$\vec{x}_1 = (1, \, 0, \, 1, \, 0), \quad \vec{x}_2 = (0, \, 1, \, 0, \, 1), \quad \vec{x}_3 = (1, \, 0, \, -1, \, 0), \quad \vec{x}_4 = (1, \, 1, \, -1, \, -1)$$

Solution

$$\vec{x}_1 \bullet \vec{x}_2 = (1, \, 0, \, 1, \, 0) \bullet (0, \, 1, \, 0, \, 1)$$

$$= 0$$

$$\vec{x}_1 \bullet \vec{x}_3 = (1, \, 0, \, 1, \, 0) \bullet (1, \, 0, \, -1, \, 0)$$

$$= 1 - 1$$

$$= 0$$

$$\vec{x}_1 \bullet \vec{x}_4 = (1, \, 0, \, 1, \, 0) \bullet (1, \, 1, \, -1, \, -1)$$

$$= 1 - 1$$

$$= 0$$

$$\vec{x}_2 \bullet \vec{x}_3 = (0, \, 1, \, 0, \, 1) \bullet (1, \, 0, \, -1, \, 0)$$

$$= 0$$

$$\vec{x}_2 \bullet \vec{x}_4 = (0, \, 1, \, 0, \, 1) \bullet (1, \, 1, \, -1, \, -1)$$

$$= 1 - 1$$

$$= 0 \ |$$

$$\vec{x}_3 \cdot \vec{x}_4 = (1,\ 0,\ -1,\ 0) \cdot (1,\ 1,\ -1,\ -1)$$

$$= 1 - 1$$

$$= 0 \ |$$

The given vectors are **orthogonal.**

Exercise

Is the set of vectors orthogonal with respect to the Euclidean inner:

$$\left\{ \left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}} \right),\ \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}} \right),\ \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}} \right) \right\}$$

Solution

$$\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}} \right) \cdot \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}} \right) = \frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}}$$

$$= 0 \ |$$

$$\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}} \right) \cdot \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}} \right) = -\frac{1}{2} + 0 + \frac{1}{2}$$

$$= 0 \ |$$

$$\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}} \right) \cdot \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}} \right) = -\frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}}$$

$$= -\frac{2}{\sqrt{6}} \neq 0 \ |$$

Therefore, the given vectors are **not** *orthogonal.*

$$\begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{vmatrix} = \frac{1}{2\sqrt{3}} + \frac{1}{2\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} \neq 0 \ |$$

Exercise

Is the set of vectors orthogonal with respect to the Euclidean inner:

$$\left\{ \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right) \right\}$$

Solution

$$\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right) \bullet \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right) = \frac{4}{9} - \frac{2}{9} - \frac{2}{9}$$

$$= 0$$

$$\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right) \bullet \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right) = \frac{2}{9} - \frac{4}{9} + \frac{2}{9}$$

$$= 0$$

$$\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right) \bullet \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right) = \frac{2}{9} + \frac{2}{9} - \frac{4}{9}$$

$$= 0$$

Therefore, the given vectors are *orthogonal*.

$$\begin{vmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{vmatrix} = \frac{4}{27} + \frac{4}{27} + \frac{4}{27} - \frac{1}{27} + \frac{8}{27} + \frac{8}{27}$$

$$= 0$$

Exercise

Is the set of vectors orthogonal with respect to the Euclidean inner:

$$\left\{ \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \right\}$$

Solution

$$\begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{vmatrix} = \frac{1}{2\sqrt{3}} + \frac{1}{2\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} \neq 0$$

Therefore, the given vectors are **not** *orthogonal*.

$$\left(\frac{1}{\sqrt{2}}, \, 0, \, \frac{1}{\sqrt{2}}\right) \bullet \left(\frac{1}{\sqrt{3}}, \, \frac{1}{\sqrt{3}}, \, -\frac{1}{\sqrt{3}}\right) \bullet \left(-\frac{1}{\sqrt{2}}, \, 0, \, \frac{1}{\sqrt{2}}\right) = -\frac{1}{2\sqrt{3}} - \frac{1}{2\sqrt{3}}$$

$$= -\frac{1}{\sqrt{3}} \quad \neq 0$$

Exercise

Which of the following sets of vectors are orthogonal with respect to the Euclidean inner

$$\left\{\left(\frac{1}{\sqrt{2}}, \, 0, \, 0, \, \frac{1}{\sqrt{2}}\right), \, \left(0, \, \frac{1}{\sqrt{2}}, \, \frac{1}{\sqrt{2}}, \, 0\right), \, \left(-\frac{1}{2}, \, \frac{1}{2}, \, -\frac{1}{2}, \, \frac{1}{2}\right)\right\}$$

Solution

$$\left(\frac{1}{\sqrt{2}}, \, 0, \, 0, \, \frac{1}{\sqrt{2}}\right) \bullet \left(0, \, \frac{1}{\sqrt{2}}, \, \frac{1}{\sqrt{2}}, \, 0\right) \bullet \left(-\frac{1}{2}, \, \frac{1}{2}, \, -\frac{1}{2}, \, \frac{1}{2}\right) = 0 + 0 + 0 + 0$$

$$= 0$$

Therefore, the given vectors are *orthogonal*.

Exercise

Consider vectors $\vec{u} = (2, \, 3, \, 5) \quad \vec{v} = (1, \, -4, \, 3) \quad in \quad \mathbb{R}^3$

    a) $\langle \vec{u}, \, \vec{v} \rangle$ *b)* $\|\vec{u}\|$ *c)* $\|\vec{v}\|$ *d)* Cosine between $\vec{u}$ *and* $\vec{v}$

Solution

    a) $\langle \vec{u}, \, \vec{v} \rangle = (2, \, 3, \, 5) \bullet (1, \, -4, \, 3)$

$$= 2 - 12 + 15$$

$$= 5$$

    b) $\|\vec{u}\| = \sqrt{4 + 9 + 25}$

$$= \sqrt{38}$$

    c) $\vec{v} = \sqrt{1 + 16 + 9}$

$$= \sqrt{26}$$

    d) $\cos\theta = \dfrac{5}{\sqrt{38}\,\sqrt{26}}$ $\cos\theta = \dfrac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$

$$= \frac{5}{2\sqrt{19}\,\sqrt{13}}$$

Exercise

Consider vectors $\vec{u} = (1, 1, 1)$ $\vec{v} = (1, 2, -3)$ *in* $\mathbb{R}^3$

    a) $\langle \vec{u}, \vec{v} \rangle$ *b)* $\|\vec{u}\|$ *c)* $\|\vec{v}\|$ *d)* Cosine θ between $\vec{u}$ and $\vec{v}$

Solution

    a) $\langle \vec{u}, \vec{v} \rangle = (1, 1, 1) \cdot (1, 2, -3)$

               $= 1 + 2 - 3$

               $\underline{= 0}$

    b) $\|\vec{u}\| = \sqrt{1 + 1 + 1}$

           $\underline{= \sqrt{3}}$

    c) $\|\vec{v}\| = \sqrt{1 + 4 + 9}$

           $\underline{= \sqrt{14}}$

    d) $\underline{\cos \theta = 0}$ $\cos \theta = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$

       $\vec{u}$ *and* $\vec{v}$ are orthogonal vectors.

Exercise

Consider vectors $\vec{u} = (1, 2, 5)$ $\vec{v} = (2, -3, 5)$ $\vec{w} = (4, 2, -3)$ *in* $\mathbb{R}^3$

    a) $\langle \vec{u}, \vec{v} \rangle$ *d)* $\|\vec{u}\|$ *g)* Cosine α between $\vec{u}$ and $\vec{v}$

    b) $\langle \vec{u}, \vec{w} \rangle$ *e)* $\|\vec{v}\|$ *h)* Cosine β between $\vec{u}$ and $\vec{w}$

                                                    i) Cosine θ between $\vec{v}$ and $\vec{w}$

    c) $\langle \vec{v}, \vec{w} \rangle$ *f)* $\|\vec{w}\|$ *j)* $(\vec{u} + \vec{v}) \cdot \vec{w}$

Solution

    a) $\langle \vec{u}, \vec{v} \rangle = (1, 2, 5) \cdot (2, -3, 5)$

              $= 2 - 6 + 25$

              $\underline{= 21}$

    b) $\langle \vec{u}, \vec{w} \rangle = (1, 2, 5) \cdot (4, 2, -3)$

              $= 4 + 4 - 15$

              $\underline{= -7}$

    c) $\langle \vec{v}, \vec{w} \rangle = (2, -3, 5) \cdot (4, 2, -3)$

              $= 8 - 6 - 15$

              $\underline{= -13}$

    d) $\|\vec{u}\| = \sqrt{1 + 4 + 25}$

$$= \sqrt{30}$$

e) $\|\vec{v}\| = \sqrt{4 + 9 + 25}$

$$= \sqrt{38}$$

f) $\|\vec{w}\| = \sqrt{16 + 4 + 9}$

$$= \sqrt{29}$$

g) $\cos\alpha = \dfrac{21}{\sqrt{30}\sqrt{38}}$ $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$

$$= \dfrac{21}{2\sqrt{114}}$$

h) $\cos\beta = \dfrac{-7}{\sqrt{30}\sqrt{29}}$ $\cos\theta = \dfrac{\vec{u} \cdot \vec{w}}{\|\vec{u}\|\,\|\vec{w}\|}$

$$= \dfrac{-7}{\sqrt{870}}$$

i) $\cos\theta = \dfrac{-13}{\sqrt{38}\sqrt{29}}$ $\cos\theta = \dfrac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}$

$$= \dfrac{-13}{\sqrt{1,102}}$$

j) $(\vec{u} + \vec{v}) \cdot \vec{w} = \big[(1,\ 2,\ 5) + (2,\ -3,\ 5)\big] \cdot (4,\ 2,\ -3)$

$$= (3,\ -1,\ 10) \cdot (4,\ 2,\ -3)$$

$$= 12 - 2 - 30$$

$$= -20$$

Exercise

Consider vectors $\vec{u} = (-1,\ -1,\ -1)$ $\vec{v} = (2,\ 2,\ 2)$ $\vec{w} = (4,\ 2,\ -3)$ *in* $\mathbb{R}^3$

a) $\langle \vec{u},\ \vec{v} \rangle$ d) $\|\vec{u}\|$ g) Cosine α between $\vec{u}$ *and* $\vec{v}$

b) $\langle \vec{u},\ \vec{w} \rangle$ e) $\|\vec{v}\|$ h) Cosine β between $\vec{u}$ *and* $\vec{w}$

c) $\langle \vec{v},\ \vec{w} \rangle$ f) $\|\vec{w}\|$ i) Cosine θ between $\vec{v}$ *and* $\vec{w}$

                                    j) $(\vec{u} + \vec{v}) \cdot \vec{w}$

Solution

a) $\langle \vec{u},\ \vec{v} \rangle = (-1,\ -1,\ -1) \cdot (2,\ 2,\ 2)$

$$= -2 - 2 - 2$$

$$= -6$$

b) $\langle \vec{u},\ \vec{w} \rangle = (-1,\ -1,\ -1) \cdot (4,\ 2,\ -3)$

$$= -4 - 2 + 3$$
$$= -3$$

c) $\langle \vec{v}, \vec{w} \rangle = (2,\ 2,\ 2) \bullet (4,\ 2,\ -3)$
$$= 8 + 4 - 6$$
$$= 6$$

d) $\|\vec{u}\| = \sqrt{1+1+1}$
$$= \sqrt{3}$$

e) $\|\vec{v}\| = \sqrt{4+4+4}$
$$= 2\sqrt{3}$$

f) $\|\vec{w}\| = \sqrt{16+4+9}$
$$= \sqrt{29}$$

g) $\cos\alpha = \dfrac{-6}{6}$ $\qquad\qquad\qquad \cos\theta = \dfrac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$
$$= -1$$

h) $\cos\beta = \dfrac{-3}{\sqrt{3}\sqrt{29}}$ $\qquad\qquad \cos\theta = \dfrac{\vec{u} \bullet \vec{w}}{\|\vec{u}\|\ \|\vec{w}\|}$
$$= -\dfrac{3}{\sqrt{87}}$$

i) $\cos\theta = \dfrac{6}{2\sqrt{3}\sqrt{29}}$ $\qquad\qquad \cos\theta = \dfrac{\vec{v} \bullet \vec{w}}{\|\vec{v}\|\ \|\vec{w}\|}$
$$= \dfrac{3}{\sqrt{87}}$$

j) $(\vec{u}+\vec{v}) \bullet \vec{w} = \left[(-1,\ -1,\ -1) + (2,\ 2,\ 2)\right] \bullet (4,\ 2,\ -3)$
$$= (1,\ 1,\ 1) \bullet (4,\ 2,\ -3)$$
$$= 4 + 2 - 3$$
$$= 3$$

Exercise

Consider vectors $\vec{u} = (-2,\ 0,\ 1,\ 3)$ $\vec{v} = (1,\ 1,\ 1,\ 1)$ $\vec{w} = (3,\ -1,\ 5,\ 2)$ *in* $\mathbb{R}^4$

a) $\langle \vec{u},\ \vec{v} \rangle$

b) $\langle \vec{u},\ \vec{w} \rangle$

c) $\langle \vec{v},\ \vec{w} \rangle$

d) $\|\vec{u}\|$

e) $\|\vec{v}\|$

f) $\|\vec{w}\|$

g) Cosine α between $\vec{u}$ and $\vec{v}$

h) Cosine β between $\vec{u}$ and $\vec{w}$

i) Cosine θ between $\vec{v}$ and $\vec{w}$

j) $(\vec{u} + \vec{v}) \cdot \vec{w}$

Solution

a) $\langle \vec{u},\ \vec{v} \rangle = (-2,\ 0,\ 1,\ 3) \cdot (1,\ 1,\ 1,\ 1)$

$\qquad = -2 + 0 + 1 + 3$

$\qquad \underline{= 2}$

b) $\langle \vec{u},\ \vec{w} \rangle = (-2,\ 0,\ 1,\ 3) \cdot (3,\ -1,\ 5,\ 2)$

$\qquad = -6 + 0 + 5 + 6$

$\qquad \underline{= 5}$

c) $\langle \vec{v},\ \vec{w} \rangle = (1,\ 1,\ 1,\ 1) \cdot (3,\ -1,\ 5,\ 2)$

$\qquad = 3 - 1 + 5 + 2$

$\qquad \underline{= 9}$

d) $\|\vec{u}\| = \sqrt{4 + 1 + 9}$

$\qquad \underline{= \sqrt{14}}$

e) $\|\vec{v}\| = \sqrt{1 + 1 + 1 + 1}$

$\qquad \underline{= 2}$

f) $\|\vec{w}\| = \sqrt{9 + 1 + 25 + 4}$

$\qquad \underline{= \sqrt{39}}$

g) $\cos \alpha = \dfrac{2}{2\sqrt{14}}$ $\qquad\qquad \cos \theta = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$

$\qquad \underline{= \dfrac{1}{\sqrt{14}}}$

h) $\cos \beta = \dfrac{5}{\sqrt{14}\sqrt{39}}$ $\qquad\quad \cos \theta = \dfrac{\vec{u} \cdot \vec{w}}{\|\vec{u}\|\,\|\vec{w}\|}$

$\qquad \underline{= \dfrac{5}{\sqrt{546}}}$

i) $\cos \theta = \dfrac{9}{2\sqrt{39}}$ $\qquad\qquad \cos \theta = \dfrac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}$

j) $(\vec{u} + \vec{v}) \cdot \vec{w} = \big[(-2,\ 0,\ 1,\ 3) + (1,\ 1,\ 1,\ 1) \big] \cdot (3,\ -1,\ 5,\ 2)$

$$= (-1,\ 1,\ 1,\ 3) \bullet (3,\ -1,\ 5,\ 2)$$

$$= -3 - 1 + 5 + 6$$

$$= 7$$

Exercise

Consider polynomial $f(t) = 3t - 5;\ \ g(t) = t^2\ \ \ in\ \ \mathbb{P}(t)$

    a) $\langle f, g \rangle$ *b)* $\|f\|$ *c)* $\|g\|$ *d)* Cosine between f *and* g

Solution

a) $\displaystyle \langle f, g \rangle = \int_0^1 f(t) g(t)\, dt$

$$= \int_0^1 (3t - 5) t^2\, dt$$

$$= \int_0^1 \left(3t^3 - 5t^2 \right) dt$$

$$= \frac{3}{4} t^4 - \frac{5}{3} t^3 \Big|_0^1$$

$$= \frac{3}{4} - \frac{5}{3}$$

$$= -\frac{11}{12}$$

b) $\displaystyle \langle f, f \rangle = \int_0^1 f(t) f(t)\, dt$

$$= \int_0^1 (3t - 5)^2\, dt$$

$$= \frac{1}{3} \int_0^1 (3t - 5)^2\, d(3t - 5)$$

$$= \frac{1}{9} (3t - 5)^3 \Big|_0^1$$

$$= \frac{1}{9} (8 - 125)$$

$$= 13$$

$$\|f\| = \sqrt{|\langle f, f \rangle|}$$

$$= \sqrt{13}$$

c) $\langle g, g \rangle = \displaystyle\int_0^1 g(t)g(t)\,dt$

$$= \int_0^1 t^4\,dt$$

$$= \frac{1}{5}t^5 \Big|_0^1$$

$$= \frac{1}{5}$$

$$\|g\| = \sqrt{|\langle g, g \rangle|}$$

$$= \frac{1}{\sqrt{5}}$$

d) $\cos\theta = \dfrac{-\dfrac{11}{12}}{\sqrt{13}\,\dfrac{\sqrt{5}}{5}}$ $\qquad\qquad$ $\cos\theta = \dfrac{f \cdot g}{\|f\|\,\|g\|}$

$$= -\frac{55}{12\sqrt{65}}$$

Exercise

Consider polynomial $f(t) = t + 2$; $g(t) = 3t - 2$; $h(t) = t^2 - 2t - 3$ $\;$ in $\;$ $\mathbb{P}(t)$

a) $\langle f, g \rangle$ $\qquad$ d) $\|f\|$ $\qquad$ g) Cosine α between f and g

b) $\langle f, h \rangle$ $\qquad$ e) $\|g\|$ $\qquad$ h) Cosine β between f and h

c) $\langle g, h \rangle$ $\qquad$ f) $\|h\|$ $\qquad$ i) Cosine θ between g and h

Solution

a) $\langle f, g \rangle = \displaystyle\int_0^1 (t+2)(3t-2)\,dt$ $\qquad\qquad$ $\langle f, g \rangle = \displaystyle\int_0^1 f(t)g(t)\,dt$

$$= \int_0^1 \left(3t^2 + 4t - 4\right)dt$$

$$= t^3 + 2t^2 - 4t \;\Big|_0^1$$

$$= 1 + 2 - 4$$

$$= -1$$

b) $\langle f, h \rangle = \displaystyle\int_0^1 (t+2)\left(t^2 - 2t - 3\right) dt$ $\qquad$ $\langle f, h \rangle = \displaystyle\int_0^1 f(t)h(t)\, dt$

$\qquad\qquad = \displaystyle\int_0^1 \left(t^3 - 7t - 6\right) dt$

$\qquad\qquad = \dfrac{1}{4}t^4 - \dfrac{7}{2}t^2 - 6t \;\Big|_0^1$

$\qquad\qquad = \dfrac{1}{4} - \dfrac{7}{2} - 6$

$\qquad\qquad \underline{= -\dfrac{37}{4}}$

c) $\langle g, h \rangle = \displaystyle\int_0^1 (3t-2)\left(t^2 - 2t - 3\right) dt$ $\qquad$ $\langle f, h \rangle = \displaystyle\int_0^1 g(t)h(t)\, dt$

$\qquad\qquad = \displaystyle\int_0^1 \left(3t^3 - 8t^2 - 5t + 6\right) dt$

$\qquad\qquad = \dfrac{3}{4}t^4 - \dfrac{8}{3}t^3 - \dfrac{5}{2}t^2 + 6t \;\Big|_0^1$

$\qquad\qquad = \dfrac{3}{4} - \dfrac{8}{3} - \dfrac{5}{2} + 6$

$\qquad\qquad \underline{= \dfrac{9}{4}}$

d) $\langle f, f \rangle = \displaystyle\int_0^1 (t+2)^2 \, dt$ $\qquad$ $\langle f, f \rangle = \displaystyle\int_0^1 f(t)f(t)\, dt$

$\qquad\qquad = \dfrac{1}{3}(t+2)^3 \;\Big|_0^1$

$\qquad\qquad = \dfrac{1}{3}(27 - 8)$

$\qquad\qquad \underline{= \dfrac{19}{3}}$

$\qquad \|f\| = \sqrt{|\langle f, f \rangle|}$

$\qquad\qquad \underline{= \sqrt{\dfrac{19}{3}}}$

e) $\langle g, g \rangle = \displaystyle\int_0^1 (3t-2)^2 \, dt$ $\qquad$ $\langle g, g \rangle = \displaystyle\int_0^1 g(t)g(t)\, dt$

$$= \frac{1}{3} \int_0^1 (3t-2)^2 \, d(3t-2)$$

$$= \frac{1}{9} (3t-2)^3 \Big|_0^1$$

$$= \frac{1}{9} (1+8)$$

$$= 1$$

$$\|g\| = \sqrt{|\langle g, g \rangle|}$$

$$= 1$$

f) $\quad \langle h, h \rangle = \int_0^1 \left(t^2 - 2t - 3\right)^2 dt \qquad\qquad \langle h, h \rangle = \int_0^1 h(t)h(t)\,dt$

$$= \int_0^1 \left(t^4 - 4t^3 - 2t^2 + 12t + 9\right) dt$$

$$= \left(\frac{1}{5}t^5 - t^4 - \frac{2}{3}t^3 + 6t^2 + 9t\right) \Big|_0^1$$

$$= \frac{1}{5} - 1 - \frac{2}{3} + 6 + 9$$

$$= \frac{203}{15}$$

$$\|h\| = \sqrt{|\langle h, h \rangle|}$$

$$= \sqrt{\frac{203}{15}}$$

g) $\quad \cos\alpha = \dfrac{-1}{\sqrt{\dfrac{19}{3}}} \qquad\qquad\qquad \cos\alpha = \dfrac{f \bullet g}{\|f\| \, \|g\|}$

$$= -\sqrt{\frac{3}{19}}$$

h) $\quad \cos\beta = -\dfrac{37}{4} \dfrac{1}{\sqrt{\dfrac{19}{3}} \sqrt{\dfrac{203}{15}}} \qquad\qquad \cos\beta = \dfrac{f \bullet g}{\|f\| \, \|g\|}$

$$= -\frac{111}{4} \sqrt{\frac{5}{3{,}857}}$$

i) $\cos\theta = \dfrac{9}{4}\dfrac{1}{\sqrt{\dfrac{203}{15}}}$

$\cos\theta = \dfrac{f \cdot g}{\|f\| \, \|g\|}$

$= \dfrac{9}{4}\sqrt{\dfrac{15}{203}}$

Exercise

Consider polynomial $f(x) = x^2 - 2x + 3$; $\ g(x) = 2x + 3$; $\ h(x) = x - 2$ $\ in \ \ \mathbb{P}(x)$

a) $\langle f, g \rangle$ d) $\|f\|$ g) Cosine α between f and g

b) $\langle f, h \rangle$ e) $\|g\|$ h) Cosine β between f and h

c) $\langle g, h \rangle$ f) $\|h\|$ i) Cosine θ between g and h

Solution

a) $\langle f, g \rangle = \displaystyle\int_0^1 \left(x^2 - 2x + 3\right)(2x + 3)\,dx$ $\langle f, g \rangle = \displaystyle\int_0^1 f(x)g(x)\,dx$

$= \displaystyle\int_0^1 \left(2x^3 - x^2 + 9\right)dx$

$= \dfrac{1}{2}x^4 - \dfrac{1}{3}x^3 + 9x \ \Big|_0^1$

$= \dfrac{1}{2} - \dfrac{1}{3} + 9$

$= \dfrac{55}{6}$

b) $\langle f, h \rangle = \displaystyle\int_0^1 \left(x^2 - 2x + 3\right)(x - 2)\,dx$ $\langle f, h \rangle = \displaystyle\int_0^1 f(x)h(x)\,dx$

$= \displaystyle\int_0^1 \left(x^3 - 4x^2 + 7x - 6\right)dx$

$= \dfrac{1}{4}x^4 - \dfrac{4}{3}x^3 + \dfrac{7}{2}x^2 - 6x \ \Big|_0^1$

$= \dfrac{1}{4} - \dfrac{4}{3} - \dfrac{7}{2} - 6$

$= \dfrac{3 - 16 - 42 + 72}{12}$

$= \dfrac{17}{12}$

c) $\langle g, h \rangle = \displaystyle\int_0^1 (2x+3)(x-2)\ dx$ $\qquad\qquad$ $\langle f, h \rangle = \displaystyle\int_0^1 g(x)h(x)dx$

$\qquad\quad = \displaystyle\int_0^1 \left(2x^2 - x - 6\right)dx$

$\qquad\quad = \dfrac{2}{3}x^3 - \dfrac{1}{2}x^2 - 6x \ \Big|_0^1$

$\qquad\quad = \dfrac{2}{3} - \dfrac{1}{2} - 6$

$\qquad\quad = -\dfrac{37}{6}$

d) $\langle f, f \rangle = \displaystyle\int_0^1 \left(x^2 - 2x + 3\right)^2 dx$ $\qquad\qquad$ $\langle f, f \rangle = \displaystyle\int_0^1 f(x)f(x)dx$

$\qquad\quad = \displaystyle\int_0^1 \left(x^4 - 8x^3 + 6x^2 - 12x + 3\right)dx$

$\qquad\quad = \dfrac{1}{5}x^5 - 2x^4 + 2x^3 - 6x^2 + 3x \ \Big|_0^1$

$\qquad\quad = \dfrac{1}{5} - 2 + 2 - 6 + 3$

$\qquad\quad = -\dfrac{14}{5}$

$\quad \|f\| = \sqrt{|\langle f, f \rangle|}$

$\qquad = \sqrt{\dfrac{14}{5}}$

e) $\langle g, g \rangle = \displaystyle\int_0^1 (2x+3)^2 dx$ $\qquad\qquad$ $\langle g, g \rangle = \displaystyle\int_0^1 g(x)g(x)dx$

$\qquad\quad = \dfrac{1}{2}\displaystyle\int_0^1 (2x+3)^2\, d(2x+3)$

$\qquad\quad = \dfrac{1}{6}(2x+3)^3 \ \Big|_0^1$

$\qquad\quad = \dfrac{1}{6}(125 + 27)$

$\qquad\quad = \dfrac{152}{6}$

$$= \frac{76}{3}$$

$$\|g\| = \sqrt{|\langle g, g \rangle|}$$

$$= \sqrt{\frac{76}{3}}$$

f) $\quad \langle h, h \rangle = \int_0^1 (x-2)^2 \, dx \qquad\qquad \langle h, h \rangle = \int_0^1 h(x)h(x)\,dx$

$$= \frac{1}{3}(x-2)^3 \Big|_0^1$$

$$= \frac{1}{3}(-1+8)$$

$$= \frac{7}{3}$$

$$\|h\| = \sqrt{|\langle h, h \rangle|}$$

$$= \sqrt{\frac{7}{3}}$$

g) $\quad \cos\alpha = \dfrac{\dfrac{55}{6}}{\sqrt{\dfrac{76}{3}}\sqrt{\dfrac{14}{5}}} \qquad\qquad \cos\alpha = \dfrac{f \cdot g}{\|f\|\,\|g\|}$

$$= \frac{55}{6}\sqrt{\frac{5}{14} \cdot \frac{3}{76}}$$

$$= \frac{55}{12}\sqrt{\frac{15}{266}}$$

h) $\quad \cos\beta = \dfrac{17}{12}\sqrt{\dfrac{3}{7} \cdot \dfrac{5}{14}} \qquad\qquad \cos\beta = \dfrac{f \cdot h}{\|f\|\,\|h\|}$

$$= \frac{17}{84}\sqrt{\frac{15}{2}}$$

i) $\quad \cos\theta = -\dfrac{37}{6}\sqrt{\dfrac{3}{76} \cdot \dfrac{3}{7}} \qquad\qquad \cos\theta = \dfrac{h \cdot g}{\|h\|\,\|g\|}$

$$= -\frac{37}{4}\sqrt{\frac{1}{133}}$$

Exercise

Suppose $\langle \vec{u}, \vec{v} \rangle = 3 + 2i$ in a complex inner product space V. Find:

a) $\langle (2 - 4i)\vec{u}, \vec{v} \rangle$　　　b) $\langle \vec{u}, (4 + 3i)\vec{v} \rangle$　　　c) $\langle (3 - 6i)\vec{u}, (5 - 2i)\vec{v} \rangle$　　　d) $\| \vec{u}, \vec{v} \|$

Solution

a) $\langle (2 - 4i)\vec{u}, \vec{v} \rangle = (2 - 4i)\langle \vec{u}, \vec{v} \rangle$

$\qquad\qquad = (2 - 4i)(3 + 2i)$

$\qquad\qquad = 6 + 4i - 12i + 8 \qquad\qquad i^2 = -1$

$\qquad\qquad \underline{= 14 - 8i}$

b) $\langle \vec{u}, (4 + 3i)\vec{v} \rangle = (4 + 3i)\langle \vec{u}, \vec{v} \rangle$

$\qquad\qquad = (4 + 3i)(3 + 2i)$

$\qquad\qquad = 12 + 8i + 9i - 6 \qquad\qquad i^2 = -1$

$\qquad\qquad \underline{= 14 - 8i}$

c) $\langle (3 - 6i)\vec{u}, (5 - 2i)\vec{v} \rangle = (3 - 6i)(5 - 2i)\langle \vec{u}, \vec{v} \rangle$

$\qquad\qquad\qquad = (15 - 36i - 12)(3 + 2i)$

$\qquad\qquad\qquad = (3 - 36i)(3 + 2i)$

$\qquad\qquad\qquad = 9 - 102i + 72$

$\qquad\qquad\qquad \underline{= 81 - 102i}$

d) $\| \vec{u}, \vec{v} \| = \sqrt{\langle \vec{u}, \vec{v} \rangle}$

$\qquad = \sqrt{9 + 4}$

$\qquad \underline{= \sqrt{13}}$

Exercise

Suppose $\langle \vec{u}, \vec{v} \rangle = 2 - 3i$ in a complex inner product space V. Find:

a) $\langle (2 + 2i)\vec{u}, \vec{v} \rangle$　　　b) $\langle \vec{u}, (3 - 4i)\vec{v} \rangle$　　　c) $\langle (1 + 3i)\vec{u}, (5 - 2i)\vec{v} \rangle$　　　d) $\| \vec{u}, \vec{v} \|$

Solution

a) $\langle (2 + 2i)\vec{u}, \vec{v} \rangle = (2 + 2i)\langle \vec{u}, \vec{v} \rangle$

$\qquad\qquad = (2 + 2i)(2 - 3i)$

$\qquad\qquad = 4 - 6i + 4i + 6$

$\qquad\qquad \underline{= 10 - 2i}$

b) $\langle \vec{u}, (3-4\,i)\vec{v}\rangle = (3-4\,i)\langle \vec{u}, \vec{v}\rangle$

$$= (3-4\,i)(2-3\,i)$$

$$= 4 - 9i - 8i - 12$$

$$\underline{= -8 - 17i}$$

c) $\langle (1+3\,i)\vec{u}, (5-2\,i)\vec{v}\rangle = (1+3\,i)\overline{(5-2\,i)}\langle \vec{u}, \vec{v}\rangle$

$$= (5 - 2i + 15i + 6)(2 - 3i)$$

$$= (11 + 13i)(2 - 3i)$$

$$= 22 - 33i + 26i + 39$$

$$\underline{= 61 - 7i}$$

d) $\|\vec{u}, \vec{v}\| = \sqrt{\langle \vec{u}, \vec{v}\rangle}$

$$= \sqrt{4+9}$$

$$\underline{= \sqrt{13}}$$

Exercise

Find the Fourier coefficient c and the projection $c\vec{v}$ of $\vec{u} = (3+4\,i,\ 2-3\,i)$ along $\vec{v} = (5+i,\ 2i)$ in $\mathbb{C}^2$

Solution

$$c = \frac{(3+4\,i)\overline{(5+i)} + (2-3\,i)\overline{(2i)}}{5^2 + 1^2 + 2^2} \qquad\qquad c = \frac{\langle \vec{u}, \vec{v}\rangle}{\langle \vec{v}, \vec{v}\rangle}$$

$$= \frac{(3+4\,i)(5-i) + (2-3\,i)(-2i)}{25 + 1 + 4}$$

$$= \frac{15 + 17i + 4 - 4i - 6}{30}$$

$$= \frac{13 + 13i}{30}$$

$$\underline{= \frac{13}{30} + \frac{13}{30}i}$$

$proj(\vec{u}, \vec{v}) = c\vec{v}$

$$= \left(\frac{13}{30} + \frac{13}{30}i\right)(5+i,\ 2i)$$

$$= \left(\frac{13}{6} + \frac{13}{6}i + \frac{13}{30}i - \frac{13}{30},\ \frac{13}{15}i - \frac{13}{15}\right)$$

$$= \left(\frac{52}{30} + \frac{78}{30}i,\ -\frac{13}{15} + \frac{13}{15}i\right)$$

$$= \left(\frac{26}{15} + \frac{39}{30}i, \ -\frac{13}{15} + \frac{13}{15}i \right)$$

Exercise

Suppose $\vec{v} = (1, \ 3, \ 5, \ 7)$. Find the projection of $\vec{v}$ onto W or find $\vec{w} \in W$ that minimizes $\| \vec{v} - \vec{w} \|$, where W is the subspace of $\mathbb{R}^4$ spanned by:

    a) $\ \vec{u}_1 = (1, \ 1, \ 1, \ 1) \quad and \quad \vec{u}_2 = (1, \ -3, \ 4, \ -2)$

    b) $\ \vec{v}_1 = (1, \ 1, \ 1, \ 1) \quad and \quad \vec{v}_2 = (1, \ 2, \ 3, \ 2)$

Solution

    a) $\ \vec{u}_1 \cdot \vec{u}_2 = (1, \ 1, \ 1, \ 1) \cdot (1, \ -3, \ 4, \ -2)$

$$= 1 - 3 + 4 - 2$$
$$= 0$$

Therefore, $\vec{u}_1$ *and* $\vec{u}_2$ are orthogonal.

$$
\begin{aligned}
c_1 &= \frac{\langle \vec{v}, \vec{u}_1 \rangle}{\langle \vec{u}_1, \vec{u}_1 \rangle} \\[2mm]
&= \frac{(1, \ 3, \ 5, \ 7) \cdot (1, \ 1, \ 1, \ 1)}{\| (1, \ 1, \ 1, \ 1) \|^2} \\[2mm]
&= \frac{1 + 3 + 5 + 7}{1 + 1 + 1 + 1} \\[2mm]
&= \frac{16}{4} \\[2mm]
&= 4
\end{aligned}
$$

$$
\begin{aligned}
c_2 &= \frac{\langle \vec{v}, \vec{u}_2 \rangle}{\langle \vec{u}_2, \vec{u}_2 \rangle} \\[2mm]
&= \frac{(1, \ 3, \ 5, \ 7) \cdot (1, \ -3, \ 4, \ -2)}{\| (1, \ -3, \ 4, \ -2) \|^2} \\[2mm]
&= \frac{1 - 9 + 20 - 14}{1 + 9 + 16 + 4} \\[2mm]
&= \frac{-2}{30} \\[2mm]
&= \frac{1}{15}
\end{aligned}
$$

$$w = proj(\vec{v}, \ W)$$

$$= c_1 \vec{u}_1 + c_2 \vec{u}_2$$

$$= 4(1,\ 1,\ 1,\ 1) + \frac{1}{15}(1,\ -3,\ 4,\ -2)$$

$$= \left(\frac{59}{15},\ \frac{63}{5},\ \frac{56}{15},\ \frac{62}{15}\right)$$

b) $\quad \vec{v}_1 \cdot \vec{v}_2 = (1,\ 1,\ 1,\ 1) \cdot (1,\ 2,\ 3,\ 2)$

$$= 1 + 2 + 3 + 2$$

$$= 8 \ \neq 0$$

Therefore, $\vec{v}_1 \ and \ \vec{v}_2$ are *not* orthogonal.

Applying Gram-Schmidt algorithm

$$\vec{w}_1 = \vec{v}_1 = (1,\ 1,\ 1,\ 1)$$

$$\vec{w}_2 = (1,\ 2,\ 3,\ 2) - \frac{(1,\ 2,\ 3,\ 2) \cdot (1,\ 1,\ 1,\ 1)}{4}(1,\ 1,\ 1,\ 1) \qquad\qquad \vec{w}_2 = \vec{v}_2 - \frac{\langle \vec{v},\ \vec{w}_1 \rangle}{\left\| \vec{w}_1 \right\|^2}\vec{w}_1$$

$$= (1,\ 2,\ 3,\ 2) - 2(1,\ 1,\ 1,\ 1)$$

$$= (-1,\ 0,\ 1,\ 0)$$

$$c_1 = \frac{(1,\ 3,\ 5,\ 7) \cdot (1,\ 1,\ 1,\ 1)}{\left\| (1,\ 1,\ 1,\ 1) \right\|^2} \qquad\qquad c_1 = \frac{\langle \vec{v},\ \vec{w}_1 \rangle}{\langle \vec{w}_1,\ \vec{w}_1 \rangle}$$

$$= \frac{1 + 3 + 5 + 7}{1 + 1 + 1 + 1}$$

$$= \frac{16}{4}$$

$$= 4$$

$$c_2 = \frac{(1,\ 3,\ 5,\ 7) \cdot (-1,\ 0,\ 1,\ 0)}{\left\| (-1,\ 0,\ 1,\ 0) \right\|^2} \qquad\qquad c_2 = \frac{\langle \vec{v},\ \vec{w}_2 \rangle}{\langle \vec{w}_2,\ \vec{w}_2 \rangle}$$

$$= \frac{-1 + 0 + 5 + 0}{2}$$

$$= -3$$

$$w = proj(\vec{v},\ W)$$

$$= c_1 \vec{w}_1 + c_2 \vec{w}_2$$

$$= 4(1,\ 1,\ 1,\ 1) - 3(-1,\ 0,\ 1,\ 0)$$

$$= (7,\ 4,\ 1,\ 4)$$

Exercise

Suppose $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ is an orthogonal set of vectors. Prove that (*Pythagoras*)

$$\left\| \vec{u}_1 + \vec{u}_2 + \ldots + \vec{u}_n \right\|^2 = \left\| \vec{u}_1 \right\|^2 + \left\| \vec{u}_2 \right\|^2 + \ldots + \left\| \vec{u}_n \right\|^2$$

Solution

$$\left\| \vec{u}_1 + \vec{u}_2 + \ldots + \vec{u}_n \right\|^2 = \left\langle \vec{u}_1 + \vec{u}_2 + \ldots + \vec{u}_n, \ \vec{u}_1 + \vec{u}_2 + \ldots + \vec{u}_n \right\rangle$$

$$= \left\langle \vec{u}_1, \vec{u}_1 \right\rangle + \left\langle \vec{u}_2, \vec{u}_2 \right\rangle + \ldots + \left\langle \vec{u}_n, \vec{u}_n \right\rangle$$

$$= \left\| \vec{u}_1 \right\|^2 + \left\| \vec{u}_2 \right\|^2 + \ldots + \left\| \vec{u}_n \right\|^2$$

Exercise

Suppose A is an orthogonal matrix. Show that $\left\langle \vec{u}A, \vec{v}A \right\rangle = \left\langle \vec{u}, \vec{v} \right\rangle$ for any $\vec{u}, \vec{v} \in V$

Solution

A is an orthogonal matrix $\Rightarrow AA^T = I$

And $\left\langle \vec{u}, \vec{v} \right\rangle = \vec{u}^T \vec{v}$

$$\left\langle \vec{u}A, \vec{v}A \right\rangle = \left(A\vec{u} \right)^T \left(A\vec{v} \right)$$

$$= \vec{u}^T \left(A^T A \right) \vec{v}$$

$$= \vec{u}^T I \ \vec{v}$$

$$= \vec{u}^T \vec{v}$$

$$= \left\langle \vec{u}, \vec{v} \right\rangle \ \checkmark$$

Exercise

Suppose A is an orthogonal matrix. Show that $\left\| \vec{u}A \right\| = \left\| \vec{u} \right\|$ for every $\vec{u} \in V$

Solution

A is an orthogonal matrix

$\Rightarrow AA^T = I$ and $\left\langle \vec{u}, \vec{u} \right\rangle = \vec{u}^T \vec{u}$

$$\left\| \vec{u}A \right\|^2 = \left\langle \vec{u}A, \vec{u}A \right\rangle$$

$$= \left(A\vec{u} \right)^T \left(A\vec{u} \right)$$

$$= \vec{u}^T \left(A^T A \right) \vec{u}$$

$$= \vec{u}^T I \, \vec{u}$$

$$= \vec{u}^T \vec{u}$$

$$= \langle \vec{u}, \vec{u} \rangle \; \checkmark$$

Exercise

Let V be an inner product space over $\mathbb{R}$ or $\mathbb{C}$. Show that

$$\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$$

If and only if

$$\|s\vec{u} + t\vec{v}\| = s\|\vec{u}\| + t\|\vec{v}\| \quad \text{for all} \quad s, \, t \geq 0$$

Solution

Suppose that $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$. For $s, \, t \geq 0$

$$\|s\vec{u} + t\vec{v}\|^2 = s^2 \|\vec{u}\|^2 + t^2 \|\vec{v}\|^2 + 2st \, \vec{u}\vec{v}$$

$$\leq s^2 \|\vec{u}\|^2 + t^2 \|\vec{v}\|^2$$

$$\leq s\|\vec{u}\| + t\|\vec{v}\|$$

$$\|s\vec{u} + t\vec{v}\| \leq s\|\vec{u}\| + t\|\vec{v}\|$$

$$\|s\vec{u} + t\vec{v}\| = \|s\vec{u} + t\vec{u} - t\vec{u} + t\vec{v}\|$$

$$= \|t\vec{u} + t\vec{v} - t\vec{u} + s\vec{u}\|$$

$$= \|t(\vec{u} + \vec{v}) - (t-s)\vec{u}\|$$

$$\geq \big| \, t\|\vec{u} + \vec{v}\| - (t-s)\|\vec{u}\| \, \big| \qquad\qquad \|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$$

$$= t\|\vec{u}\| + \|\vec{v}\| - t\|\vec{u}\| + s\|\vec{u}\|$$

$$= t\|\vec{v}\| + s\|\vec{u}\|$$

$$\begin{cases} \|s\vec{u} + t\vec{v}\| \leq s\|\vec{u}\| + t\|\vec{v}\| \\ \qquad and \\ \|s\vec{u} + t\vec{v}\| \geq s\|\vec{u}\| + t\|\vec{v}\| \end{cases} \Rightarrow \quad \|s\vec{u} + t\vec{v}\| = s\|\vec{u}\| + t\|\vec{v}\|$$

Exercise

Let V be an inner product vector space over $\mathbb{R}$.

a) If e_1, e_2, e_3 are three vectors in V with pairwise product negative, that is,

$$\langle e_1, e_2 \rangle < 0, \quad i, j = 1, 2, 3, \quad i \neq j$$

Show that e_1, e_2, e_3 are linearly independent.

b) Is it possible for three vectors on the xy–plane to have pairwise negative products?

c) Does part (a) remain valid when the word "negative: is replaced with positive?

d) Suppose $\vec{u}$, $\vec{v}$, *and* $\vec{w}$ are three–unit vectors in the xy–plane. What are the maximum and minimum values that

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle$$

Can attain? And when?

Solution

a) Suppose that e_1, e_2, e_3 are linearly dependent.

Then, assume that e_1, e_2, e_3 are unit vectors and that

$$e_3 = c_1 e_1 + c_2 e_2$$

Then

$$\langle e_1, e_3 \rangle = c_1 \langle e_1, e_1 \rangle + c_2 \langle e_1, e_2 \rangle \qquad\qquad \langle e_1, e_1 \rangle = 1$$

$$= c_1 + c_2 \langle e_1, e_2 \rangle < 0$$

$$c_1 < -c_2 \langle e_1, e_2 \rangle$$

$$\langle e_2, e_3 \rangle = c_1 \langle e_2, e_1 \rangle + c_2 \langle e_2, e_2 \rangle \qquad\qquad \langle e_2, e_2 \rangle = 1$$

$$= c_1 \langle e_2, e_1 \rangle + c_2 < 0$$

$$c_2 < -c_1 \langle e_2, e_1 \rangle$$

$$c_2 < -c_1 \langle e_2, e_1 \rangle$$

$$< -\left(-c_2 \langle e_1, e_2 \rangle \right) \langle e_2, e_1 \rangle$$

$$= c_2 \langle e_1, e_2 \rangle^2 \qquad\qquad \langle e_1, e_2 \rangle^2 > 1$$

$$c_2 < c_2 \quad \textit{Contradiction}$$

Therefore, e_1, e_2, e_3 are linearly independent.

b) To have all three vectors on the xy–plane which is in 2 dimensional.

Therefore, it is ***impossible*** for three to have pairwise negative products.

c) No

d) Given: $\vec{u}$, $\vec{v}$, *and* $\vec{w}$ are three–unit vectors in the xy–plane and

$$\left|\langle \vec{u},\ \vec{u} \rangle\right| = \left|\langle \vec{v},\ \vec{v} \rangle\right| = \left|\langle \vec{w},\ \vec{w} \rangle\right| = 1$$

$$\cos \alpha_1 = \frac{\langle \vec{u},\ \vec{v} \rangle}{\|\vec{u}\|\ \|\vec{v}\|} \quad \rightarrow \quad \cos \alpha_1 = \langle \vec{u},\ \vec{v} \rangle$$

$$\cos \alpha_2 = \frac{\langle \vec{v},\ \vec{w} \rangle}{\|\vec{v}\|\ \|\vec{w}\|} \quad \rightarrow \quad \cos \alpha_2 = \langle \vec{v},\ \vec{w} \rangle$$

$$\cos \alpha_3 = \frac{\langle \vec{u},\ \vec{w} \rangle}{\|\vec{u}\|\ \|\vec{w}\|} \quad \rightarrow \quad \cos \alpha_3 = \langle \vec{u},\ \vec{w} \rangle$$

$$\langle \vec{u},\ \vec{v} \rangle + \langle \vec{v},\ \vec{w} \rangle + \langle \vec{w},\ \vec{u} \rangle = \cos \alpha_1 + \cos \alpha_2 + \cos \alpha_3$$

Since $-1 \le \cos \theta \le 1$

$$\langle \vec{u},\ \vec{v} \rangle + \langle \vec{v},\ \vec{w} \rangle + \langle \vec{w},\ \vec{u} \rangle = 1 + 1 + 1$$

$$\underline{= 3}$$

Since the 3 vectors are unit vectors in the xy–plane and which it will divide the plane into a three equal angles $\alpha = \dfrac{2\pi}{3}$

$$\langle \vec{u},\ \vec{v} \rangle + \langle \vec{v},\ \vec{w} \rangle + \langle \vec{w},\ \vec{u} \rangle = \cos \frac{2\pi}{3} + \cos \frac{2\pi}{3} + \cos \frac{2\pi}{3}$$

$$= 3 \cos \frac{2\pi}{3}$$

$$= 3\left(-\frac{1}{2}\right)$$

$$\underline{= -\frac{3}{2}}$$

Therefore, the minimum $\langle \vec{u},\ \vec{v} \rangle + \langle \vec{v},\ \vec{w} \rangle + \langle \vec{w},\ \vec{u} \rangle = -\dfrac{3}{2}$

The maximum: $\langle \vec{u},\ \vec{v} \rangle + \langle \vec{v},\ \vec{w} \rangle + \langle \vec{w},\ \vec{u} \rangle = 3$

Solution

Section 3.3 – **Gram-Schmidt Process**

Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces $\mathbb{R}^m$.

$$\vec{u}_1 = (1, \ -3), \quad \vec{u}_2 = (2, \ 2)$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (1, \ -3)$$

$$\vec{q}_1 = \frac{(1, \ -3)}{\sqrt{1+9}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\| \vec{v}_1 \right\|}$$

$$= \left(\frac{1}{\sqrt{10}}, \ -\frac{3}{\sqrt{10}} \right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2, \ \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1$$

$$= (2, \ 2) - \frac{1}{10} \left[(2, \ 2) \bullet (1, \ -3) \right] (1, \ -3)$$

$$= (2, \ 2) + \frac{2}{5} (1, \ -3)$$

$$= (2, \ 2) + \left(\frac{2}{5}, \ -\frac{6}{5} \right)$$

$$= \left(\frac{12}{5}, \ \frac{4}{5} \right)$$

$$\left\| \vec{v}_2 \right\| = \sqrt{ \left(\frac{12}{5} \right)^2 + \left(\frac{4}{5} \right)^2 }$$

$$= \sqrt{ \frac{144}{25} + \frac{16}{25} }$$

$$= \sqrt{ \frac{160}{25} }$$

$$= \frac{4\sqrt{10}}{5}$$

$$\vec{q}_2 = \frac{5}{4\sqrt{10}} \left(\frac{12}{5}, \ \frac{4}{5} \right) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\| \vec{v}_2 \right\|}$$

$$= \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right)$$

The *orthogonal* basis: $\left\{ (1, -3), \left(\frac{12}{5}, \frac{4}{5} \right) \right\}$

The *orthonormal* basis: $\left\{ \left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}} \right), \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right) \right\}$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\vec{u}_1 = (1, 0), \quad \vec{u}_2 = (3, -5)$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (1, 0)$$

$$\vec{q}_1 = \frac{(1, 0)}{\sqrt{1+0}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\| \vec{v}_1 \right\|}$$

$$= (1, 0)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1$$

$$= (3, -5) - \left[(3, -5) \cdot (1, 0) \right](1, 0)$$

$$= (3, -5) - (3)(1, 0)$$

$$= (3, -5) - (3, 0)$$

$$= (0, -5)$$

$$\vec{q}_2 = \frac{1}{5}(0, -5) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\| \vec{v}_2 \right\|}$$

$$= (0, -1)$$

The *orthogonal* basis: $\{ (1, 0), (0, -5) \}$

The *orthonormal* basis: $\{ (1, 0), (0, -1) \}$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\vec{u}_1 = (1,\ 0,\ 0),\quad \vec{u}_2 = (3,\ 7,\ -2),\quad \vec{u}_3 = (0,\ 4,\ 1)$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (1,\ 0,\ 0)$$

$$\vec{q}_1 = \frac{(1,\ 0,\ 0)}{\sqrt{1+0+0}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= (1,\ 0,\ 0)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2,\ \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$= (3,\ 7,\ -2) - \left[(3,\ 7,\ -2)\bullet(1,\ 0,\ 0)\right](1,\ 0,\ 0)$$

$$= (3,7,-2) - 3(1,0,0)$$

$$= (0,\ 7,\ -2)$$

$$\vec{q}_2 = \frac{1}{\sqrt{53}}(0,\ 7,\ -2) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \left(0,\ \frac{7}{\sqrt{53}},\ -\frac{2}{\sqrt{53}}\right)$$

$$\vec{u}_3 \bullet \vec{v}_1 = (0,\ 4,\ 1)\bullet(1,\ 0,\ 0)$$

$$= 0$$

$$\vec{u}_3 \bullet \vec{v}_2 = (0,\ 4,\ 1)\bullet(0,\ 7,\ -2)$$

$$= 26$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\left\langle \vec{u}_3,\ \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1 - \frac{\left\langle \vec{u}_3,\ \vec{v}_2 \right\rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2 \quad`$$

$$= (0,\ 4,\ 1) - 0 - \left(\frac{26}{53}\right)(0,\ 7,\ -2)$$

$$= (0,\ 4,\ 1) - \left(0,\ \frac{182}{53},\ -\frac{52}{53}\right)$$

$$= \left(0,\ \frac{30}{53},\ \frac{105}{53}\right)$$

$$\vec{q}_3 = \frac{1}{\sqrt{\left(\frac{30}{53}\right)^2 + \left(\frac{105}{53}\right)^2}} \left(0, \frac{30}{53}, \frac{105}{53}\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{53}{\sqrt{11925}} \left(0, \frac{30}{53}, \frac{105}{53}\right)$$

$$= \frac{53}{15\sqrt{53}} \left(0, \frac{30}{53}, \frac{105}{53}\right)$$

$$= \left(0, \frac{2}{\sqrt{53}}, \frac{7}{\sqrt{53}}\right)$$

The *orthogonal* basis: $\left\{(1, 0, 0), (0, 7, -2), \left(0, \frac{30}{53}, \frac{105}{53}\right)\right\}$

The *orthonormal* basis: $\left\{(1, 0, 0), \left(0, \frac{7}{\sqrt{53}}, -\frac{2}{\sqrt{53}}\right), \left(0, \frac{2}{\sqrt{53}}, \frac{7}{\sqrt{53}}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\vec{u}_1 = (1, 1, 0, -1), \quad \vec{u}_2 = (1, 3, 0, 1), \quad \vec{u}_3 = (4, 2, 2, 0)$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (1, 1, 0, -1)$$

$$\vec{q}_1 = \frac{1}{\sqrt{3}}(1, 1, 0, -1) \qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (1, 3, 0, 1) - \frac{(1, 3, 0, 1) \cdot (1, 1, 0, -1)}{3}(1, 1, 0, -1)$$

$$= (1, 3, 0, 1) - (1, 1, 0, -1)$$

$$= (0, 2, 0, 2)$$

$$\vec{q}_2 = \frac{1}{2\sqrt{2}}(0, 2, 0, 2) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(0, \ \frac{1}{\sqrt{2}}, \ 0, \ \frac{1}{\sqrt{2}} \right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1 = \frac{(4, \ 2, \ 2, \ 0) \cdot (1, \ 1, \ 0, \ -1)}{3} (1, \ 1, \ 0, \ -1)$$

$$= (2, \ 2, \ 0, \ -2)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\left\| \vec{v}_2 \right\|^2} \vec{v}_2 = \frac{(4, \ 2, \ 2, \ 0) \cdot (0, \ 2, \ 0, \ 2)}{8} (0, \ 2, \ 0, \ 2)$$

$$= \frac{1}{2} (0, \ 2, \ 0, \ 2)$$

$$= (0, \ 1, \ 0, \ 1)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\left\| \vec{v}_2 \right\|^2} \vec{v}_2$$

$$= (4, \ 2, \ 2, \ 0) - (2, \ 2, \ 0, \ -2) - (0, \ 1, \ 0, \ 1)$$

$$= (2, \ -1, \ 2, \ 1)$$

$$\vec{q}_3 = \frac{1}{\sqrt{10}} (2, \ -1, \ 2, \ 1) \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\left\| \vec{v}_3 \right\|}$$

$$= \left(\frac{2}{\sqrt{10}}, \ -\frac{1}{\sqrt{10}}, \ \frac{2}{\sqrt{10}}, \ \frac{1}{\sqrt{10}} \right)$$

The *orthogonal* basis: $\left\{ (1, \ 1, \ 0, \ -1), \ (0, \ 2, \ 0, \ 2), \ (2, \ -1, \ 2, \ 1) \right\}$

The *orthogonal* basis:

$$\left\{ \left(\frac{1}{\sqrt{3}}, \ \frac{1}{\sqrt{3}}, \ 0, \ -\frac{1}{\sqrt{3}} \right), \ \left(0, \ \frac{1}{\sqrt{2}}, \ 0, \ \frac{1}{\sqrt{2}} \right), \ \left(\frac{2}{\sqrt{10}}, \ -\frac{1}{\sqrt{10}}, \ \frac{2}{\sqrt{10}}, \ \frac{1}{\sqrt{10}} \right) \right\}$$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\vec{u}_1 = (1, \ 1, \ 1, \ 1), \quad \vec{u}_2 = (1, \ 2, \ 4, \ 5), \quad \vec{u}_3 = (1, \ -3, \ -4, \ -2)$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (1, \ 1, \ 1, \ 1)$$

$$\vec{q}_1 = \frac{(1,\ 1,\ 1,\ 1)}{\sqrt{4}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\| \vec{v}_1 \right\|}$$

$$= \left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2} \right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2,\ \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1$$

$$= (1,\ 2,\ 4,\ 5) - \left(\frac{1}{4} \right) \left[(1,\ 2,\ 4,\ 5) \cdot (1,\ 1,\ 1,\ 1) \right] (1,\ 1,\ 1,\ 1)$$

$$= (1,\ 2,\ 4,\ 5) - \left(\frac{12}{4} \right) (1,\ 1,\ 1,\ 1)$$

$$= (1,\ 2,\ 4,\ 5) - (3,\ 3,\ 3,\ 3)$$

$$= (-2,\ -1,\ 1,\ 2)$$

$$\vec{q}_2 = \frac{1}{\sqrt{10}} (-2,\ -1,\ 1,\ 2) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\| \vec{v}_2 \right\|}$$

$$= \left(-\frac{2}{\sqrt{10}},\ -\frac{1}{\sqrt{10}},\ \frac{1}{\sqrt{10}},\ \frac{2}{\sqrt{10}} \right)$$

$$\vec{u}_3 \cdot \vec{v}_1 = (1,\ -3,\ -4,\ -2) \cdot (1,\ 1,\ 1,\ 1)$$

$$= -8$$

$$\vec{u}_3 \cdot \vec{v}_2 = (1,\ -3,\ -4,\ -2) \cdot (-2,\ -1,\ 1,\ 2)$$

$$= -7$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\left\langle \vec{u}_3,\ \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1 - \frac{\left\langle \vec{u}_3,\ \vec{v}_2 \right\rangle}{\left\| \vec{v}_2 \right\|^2} \vec{v}_2 \quad \text{`}$$

$$= (1,\ -3,\ -4,\ -2) + \frac{8}{4}(1,\ 1,\ 1,\ 1) + \frac{7}{10}(-2,\ -1,\ 1,\ 2)$$

$$= (1,\ -3,\ -4,\ -2) + (2,\ 2,\ 2,\ 2) + \left(-\frac{7}{5},\ -\frac{7}{10},\ \frac{7}{10},\ \frac{7}{5} \right)$$

$$= \left(\frac{8}{5},\ -\frac{17}{10},\ -\frac{13}{10},\ \frac{7}{5} \right)$$

$$\vec{q}_3 = \frac{1}{\sqrt{\dfrac{64}{25} + \dfrac{289}{100} + \dfrac{169}{100} + \dfrac{49}{25}}} \left(\frac{8}{5},\ -\frac{17}{10},\ -\frac{13}{10},\ \frac{7}{5} \right) \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\left\| \vec{v}_3 \right\|}$$

$$= \frac{10}{\sqrt{910}} \left(\frac{8}{5},\ -\frac{17}{10},\ -\frac{13}{10},\ \frac{7}{5} \right)$$

$$= \left(\frac{16}{\sqrt{910}}, \; -\frac{17}{\sqrt{910}}, \; -\frac{13}{\sqrt{910}}, \; \frac{14}{\sqrt{910}} \right)$$

The *orthogonal* basis: $\left\{ (1, 1, 1, 1), \; (-2, -1, 1, 2), \; \left(\frac{8}{5}, -\frac{17}{10}, -\frac{13}{10}, \frac{7}{5} \right) \right\}$

The orthonormal basis:

$$\left\{ \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), \; \left(-\frac{2}{\sqrt{10}}, -\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}} \right), \; \left(\frac{16}{\sqrt{910}}, -\frac{17}{\sqrt{910}}, -\frac{13}{\sqrt{910}}, \frac{14}{\sqrt{910}} \right) \right\}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\vec{u}_1 = (1, 1, 1, 1), \quad \vec{u}_2 = (1, 1, 2, 4), \quad \vec{u}_3 = (1, 2, -4, -3)$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (1, 1, 1, 1)$$

$$\vec{q}_1 = \frac{(1, 1, 1, 1)}{\sqrt{4}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\| \vec{v}_1 \right\|}$$

$$= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1$$

$$= (1, 1, 2, 4) - \frac{1}{4} \left[(1, 1, 2, 4) \bullet (1, 1, 1, 1) \right] (1, 1, 1, 1)$$

$$= (1, 1, 2, 4) - (2, 2, 2, 2)$$

$$= (-1, -1, 0, 2)$$

$$\vec{q}_2 = \frac{1}{\sqrt{1+1+4}} (-1, -1, 0, 2) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\| \vec{v}_2 \right\|}$$

$$= \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}} \right)$$

$$\vec{u}_3 \bullet \vec{v}_1 = (1, 2, -4, -3) \bullet (1, 1, 1, 1)$$

$$= -4$$

$$\vec{u}_3 \bullet \vec{v}_2 = (1, 2, -4, -3) \bullet (-1, -1, 0, 2)$$

$$= -9$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 \quad \text{`}$$

$$= (1, 2, -4, -3) + \frac{4}{4}(1, 1, 1, 1) + \frac{9}{6}(-1, -1, 0, 2)$$

$$= (1, 2, -4, -3) + (1, 1, 1, 1) + \left(-\frac{3}{2}, -\frac{3}{2}, 0, 3\right)$$

$$= \left(\frac{1}{2}, \frac{3}{2}, -3, 1\right)$$

$$\vec{q}_3 = \frac{1}{\sqrt{\frac{1}{4} + \frac{9}{4} + 9 + 1}}\left(\frac{1}{2}, \frac{3}{2}, -3, 1\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{2}{5\sqrt{2}}\left(\frac{1}{2}, \frac{3}{2}, -3, 1\right)$$

$$= \left(\frac{1}{5\sqrt{2}}, \frac{3}{5\sqrt{2}}, -\frac{6}{5\sqrt{2}}, \frac{2}{5\sqrt{2}}\right)$$

The *orthogonal* basis: $\left\{(1, 1, 1, 1), (-1, -1, 0, 2), \left(\frac{1}{2}, \frac{3}{2}, -3, 1\right)\right\}$

The *orthonormal* basis:

$$\left\{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}}\right), \left(\frac{1}{5\sqrt{2}}, \frac{3}{5\sqrt{2}}, -\frac{6}{5\sqrt{2}}, \frac{2}{5\sqrt{2}}\right)\right\}$$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\vec{u}_1 = (0, 2, 1, 0); \quad \vec{u}_2 = (1, -1, 0, 0); \quad \vec{u}_3 = (1, 2, 0, -1); \quad \vec{u}_4 = (1, 0, 0, 1)$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (0, 2, 1, 0)$$

$$\vec{q}_1 = \frac{(0, 2, 1, 0)}{\sqrt{4+1}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (1, -1, 0, 0) - \frac{1}{5}\left[(1, -1, 0, 0) \cdot (0, 2, 1, 0)\right](0, 2, 1, 0)$$

$$= (1, -1, 0, 0) + \frac{2}{5}(0, 2, 1, 0)$$

$$= (1, -1, 0, 0) + \left(0, \frac{4}{5}, \frac{2}{5}, 0\right)$$

$$= \underline{\left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)}$$

$$\vec{q}_2 = \frac{1}{\sqrt{1 + \frac{1}{25} + \frac{4}{25}}}\left(1, -\frac{1}{5}, \frac{2}{5}, 0\right) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{5}{\sqrt{30}}\left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)$$

$$= \underline{\left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0\right)}$$

$$u_3 \cdot v_1 = (1, 2, 0, -1) \cdot (0, 2, 1, 0)$$

$$= \underline{4}$$

$$u_3 \cdot v_2 = (1, 2, 0, -1) \cdot \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)$$

$$= 1 - \frac{2}{5}$$

$$= \underline{\frac{3}{5}}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 \ ,$$

$$= (1, 2, 0, -1) - \left(\frac{4}{5}\right)(0, 2, 1, 0) - \frac{1}{1 + \frac{1}{25} + \frac{4}{25}}\left(\frac{3}{5}\right)\left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)$$

$$= (1, 2, 0, -1) - \left(0, \frac{8}{5}, \frac{4}{5}, 0\right) - \left(\frac{3}{5}\right)\left(\frac{25}{30}\right)\left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)$$

$$= (1, 2, 0, -1) - \left(0, \frac{8}{5}, \frac{4}{5}, 0\right) - \left(\frac{1}{2}, -\frac{1}{10}, \frac{1}{5}, 0\right)$$

$$= \underline{\left(\frac{1}{2}, \frac{1}{2}, -1, -1\right)}$$

$$\vec{q}_3 = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + (-1)^2 + (-1)^2}}\left(\frac{1}{2}, \frac{1}{2}, -1, -1\right) \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\sqrt{\frac{5}{2}}}\left(\frac{1}{2}, \frac{1}{2}, -1, -1\right)$$

$$= \frac{\sqrt{2}}{\sqrt{5}}\left(\frac{1}{2}, \frac{1}{2}, -1, -1\right)$$

$$= \frac{2}{\sqrt{10}}\left(\frac{1}{2}, \frac{1}{2}, -1, -1\right)$$

$$= \left(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, -\frac{2}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right)$$

$$u_4 \cdot v_1 = (1, 0, 0, 1) \cdot (0, 2, 1, 0)$$

$$= 0$$

$$u_4 \cdot v_2 = (1, 0, 0, 1) \cdot \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)$$

$$= 1$$

$$u_4 \cdot v_3 = (1, 0, 0, 1) \cdot \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right)$$

$$= -\frac{1}{2}$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$

$$= (1, 2, 0, -1) - (0) - \left(\frac{25}{5}\right)\left(1, -\frac{1}{5}, \frac{2}{5}, 0\right) + \left(-\frac{1}{2}\right)\left(\frac{2}{5}\right)\left(\frac{1}{2}, \frac{1}{2}, -1, -1\right)$$

$$= (1, 2, 0, -1) - \left(\frac{5}{6}, -\frac{1}{6}, \frac{1}{3}, 0\right) + \left(\frac{1}{10}, \frac{1}{10}, -\frac{1}{5}, -\frac{1}{5}\right)$$

$$= \left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right)$$

$$\vec{q}_4 = \frac{1}{\sqrt{\left(\frac{4}{15}\right)^2 + \left(\frac{4}{15}\right)^2 + \left(-\frac{8}{15}\right)^2 + \left(\frac{4}{5}\right)^2}}\left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right) \qquad \vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

$$= \frac{1}{\sqrt{\frac{240}{225}}}\left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right)$$

$$= \frac{1}{\frac{4}{\sqrt{15}}}\left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right)$$

$$= \frac{15}{4\sqrt{15}}\left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right)$$

$$= \left(\frac{1}{\sqrt{15}}, \frac{1}{\sqrt{15}}, -\frac{2}{\sqrt{15}}, \frac{3}{\sqrt{15}}\right)$$

The *orthogonal* basis:

$$\left\{(0,\ 2,\ 1,\ 0),\ \left(1,\ -\frac{1}{5},\ \frac{2}{5},\ 0\right),\ \left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right),\ \left(\frac{4}{15},\ \frac{4}{15},\ -\frac{8}{15},\ \frac{4}{5}\right)\right\}$$

The *orthonormal* basis:

$$\left\{\begin{array}{l}\left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right),\ \left(\frac{5}{\sqrt{30}},\ -\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right),\\[3mm]\left(\frac{1}{\sqrt{10}},\ \frac{1}{\sqrt{10}},\ -\frac{2}{\sqrt{10}},\ -\frac{2}{\sqrt{10}}\right),\ \left(\frac{1}{\sqrt{15}},\ \frac{1}{\sqrt{15}},\ -\frac{2}{\sqrt{15}},\ \frac{3}{\sqrt{15}}\right)\end{array}\right\}$$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\{(3,\ 4),\ (1,\ 0)\}$$

Solution

Let $\quad \vec{u}_1 = (3,\ 4) \quad \vec{u}_2 = (1,\ 0)$

$\underline{\vec{v}_1 = \vec{u}_1 = (3,\ 4)}$

$\vec{q}_1 = \dfrac{(3,\ 4)}{\sqrt{9+16}}$ $\qquad\qquad\qquad \vec{q}_1 = \dfrac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$

$\underline{= \left(\dfrac{3}{5},\ \dfrac{4}{5}\right)}$

$\vec{v}_2 = \vec{u}_2 - \dfrac{\left\langle \vec{u}_2,\ \vec{v}_1\right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$

$\quad = (1,\ 0) - \dfrac{1}{25}\left[(1,\ 0)\bullet(3,\ 4)\right](3,\ 4)$

$\quad = (1,\ 0) - \left(\dfrac{9}{25},\ \dfrac{12}{25}\right)$

$\quad \underline{= \left(\dfrac{16}{25},\ -\dfrac{12}{25}\right)}$

$\vec{q}_2 = \dfrac{25}{20}\left(\dfrac{16}{25},\ -\dfrac{12}{25}\right)$ $\qquad\qquad \vec{q}_2 = \dfrac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$

$\quad \underline{= \left(\dfrac{4}{5},\ -\dfrac{3}{5}\right)}$

The *orthogonal* basis: $\quad \left\{(3,\ 4),\ \left(\dfrac{16}{25},\ -\dfrac{12}{25}\right)\right\}$

The *orthonormal* basis: $\left\{\left(\frac{3}{5}, \frac{4}{5}\right), \left(\frac{4}{5}, -\frac{3}{5}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\{(3, 0, -1), (8, 5, -6)\}$$

Solution

Let $\quad \vec{u}_1 = (3, 0, -1) \quad \vec{u}_2 = (8, 5, -6)$

$\vec{v}_1 = \vec{u}_1 = (3, 0, -1)$

$\vec{q}_1 = \dfrac{(3, 0, -1)}{\sqrt{9+1}} \qquad\qquad\qquad \vec{q}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$

$\quad = \left(\dfrac{3}{\sqrt{10}}, 0, -\dfrac{1}{\sqrt{10}}\right)$

$\vec{v}_2 = \vec{u}_2 - \dfrac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$

$\quad = (8, 5, -6) - \dfrac{1}{10}\left[(8, 5, -6) \cdot (3, 0, -1)\right](3, 0, -1)$

$\quad = (8, 5, -6) - (3)(3, 0, -1)$

$\quad = (8, 5, -6) - (9, 0, -3)$

$\quad = (-1, 5, -3)$

$\vec{q}_2 = \dfrac{1}{\sqrt{1+25+9}}(-1, 5, -3) \qquad\qquad \vec{q}_2 = \dfrac{\vec{v}_2}{\|\vec{v}_2\|}$

$\quad = \dfrac{1}{\sqrt{35}}(-1, 5, -3)$

$\quad = \left(-\dfrac{1}{\sqrt{35}}, \dfrac{5}{\sqrt{35}}, -\dfrac{3}{\sqrt{35}}\right)$

The *orthogonal* basis: $\quad \{(3, 0, -1), (-1, 5, -3)\}$

The *orthonormal* basis: $\left\{\left(\dfrac{3}{\sqrt{10}}, 0, -\dfrac{1}{\sqrt{10}}\right), \left(-\dfrac{1}{\sqrt{35}}, \dfrac{5}{\sqrt{35}}, -\dfrac{3}{\sqrt{35}}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\{(0,\ 4,\ 2),\ (5,\ 6,\ -7)\}$$

Solution

Let $\quad \vec{u}_1 = (0,\ 4,\ 2) \quad \vec{u}_2 = (5,\ 6,\ -7)$

$$\vec{v}_1 = \vec{u}_1 = (0,\ 4,\ 2)$$

$$\vec{q}_1 = \frac{(0,\ 4,\ 2)}{\sqrt{16+4}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2,\ \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$= (5,\ 6,\ -7) - \frac{1}{20}\left[(5,\ 6,\ -7) \bullet (0,\ 4,\ 2)\right](0,\ 4,\ 2)$$

$$= (5,\ 6,\ -7) - \left(\frac{10}{20}\right)(0,\ 4,\ 2)$$

$$= (5,\ 6,\ -7) - (0,\ 2,\ 1)$$

$$= (5,\ 4,\ -8)$$

$$\vec{q}_2 = \frac{1}{\sqrt{25+16+64}}(5,\ 4,\ -8) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \frac{1}{\sqrt{105}}(5,\ 4,\ -8)$$

$$= \left(\frac{5}{\sqrt{105}},\ \frac{4}{\sqrt{105}},\ -\frac{8}{\sqrt{105}}\right)$$

The *orthogonal* basis: $\quad \{(0,\ 4,\ 2),\ (5,\ 4,\ -8)\}$

The *orthonormal* basis: $\quad \left\{\left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right),\ \left(\frac{5}{\sqrt{105}},\ \frac{4}{\sqrt{105}},\ -\frac{8}{\sqrt{105}}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces $\mathbb{R}^m$.

$$\{(1,\ 1,\ 1),\ (-1,\ 1,\ 0),\ (1,\ 2,\ 1)\}$$

Solution

Let $\quad \vec{u}_1 = (1,\ 1,\ 1) \quad \vec{u}_2 = (-1,\ 1,\ 0) \quad \vec{u}_3 = (1,\ 2,\ 1)$

$$\vec{v}_1 = \vec{u}_1 = (1,\ 1,\ 1)$$

$$\vec{q}_1 = \frac{(1,\ 1,\ 1)}{\sqrt{1+1+1}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \frac{(1,\ 1,\ 1)}{\sqrt{3}}$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2,\ \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2} \vec{v}_1$$

$$= (-1,\ 1,\ 0) - \frac{1}{3}\left[(-1,\ 1,\ 0) \bullet (1,\ 1,\ 1)\right](1,\ 1,\ 1)$$

$$= (-1,\ 1,\ 0) - (0)(1,\ 1,\ 1)$$

$$= (-1,\ 1,\ 0)$$

$$\vec{q}_2 = \frac{1}{\sqrt{2}}(-1,\ 1,\ 0) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$\left\langle \vec{u}_3,\ \vec{v}_1 \right\rangle = (1,\ 2,\ 1) \bullet (1,\ 1,\ 1)$$

$$= 4$$

$$\left\langle \vec{u}_3,\ \vec{v}_2 \right\rangle = (1,\ 2,\ 1) \bullet (-1,\ 1,\ 0)$$

$$= 1$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\left\langle \vec{u}_3,\ \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2} \vec{v}_1 - \frac{\left\langle \vec{u}_3,\ \vec{v}_2 \right\rangle}{\left\|\vec{v}_2\right\|^2} \vec{v}_2$$

$$= (1,\ 2,\ 1) - \tfrac{4}{3}(1,\ 1,\ 1) - \tfrac{1}{2}(-1,\ 1,\ 0)$$

$$= (1,\ 2,\ 1) - \left(\tfrac{4}{3},\ \tfrac{4}{3},\ \tfrac{4}{3}\right) - \left(-\tfrac{1}{2},\ \tfrac{1}{2},\ 0\right)$$

$$= \left(\tfrac{1}{6},\ \tfrac{1}{6},\ -\tfrac{1}{3}\right)$$

$$\vec{q}_3 = \frac{1}{\sqrt{\tfrac{1}{36} + \tfrac{1}{36} + \tfrac{1}{9}}}\left(\tfrac{1}{6},\ \tfrac{1}{6},\ -\tfrac{1}{3}\right) \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{6}{\sqrt{6}}\left(\tfrac{1}{6},\ \tfrac{1}{6},\ -\tfrac{1}{3}\right)$$

$$= \left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right)$$

The *orthogonal* basis: $\left\{(1,\ 1,\ 1),\ (-1,\ 1,\ 0),\ \left(\tfrac{1}{6},\ \tfrac{1}{6},\ -\tfrac{1}{3}\right)\right\}$

The *orthonormal* basis: $\left\{\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right),\ \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right),\ \left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\{(1,\ 1,\ 1),\ (0,\ 1,\ 1),\ (0,\ 0,\ 1)\}$$

Solution

Let $\quad \vec{u}_1 = (1,\ 1,\ 1) \quad \vec{u}_2 = (0,\ 1,\ 1) \quad \vec{u}_3 = (0,\ 0,\ 1)$

$$\vec{v}_1 = \vec{u}_1 = (1,\ 1,\ 1)$$

$$\vec{q}_1 = \frac{(1,\ 1,\ 1)}{\sqrt{1+1+1}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ 1,\ 1)}{\sqrt{3}}$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (0, 1, 1) - \frac{1}{3}\left[(0, 1, 1) \cdot (1, 1, 1)\right](1, 1, 1)$$

$$= (0, 1, 1) - \left(\frac{2}{3}\right)(1, 1, 1)$$

$$= (0, 1, 1) - \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$= \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \Big|$$

$$\|\vec{v}_2\| = \sqrt{\left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2}$$

$$= \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{1}{9}}$$

$$= \sqrt{\frac{6}{9}}$$

$$= \frac{\sqrt{6}}{3} \Big|$$

$$\vec{q}_2 = \frac{3}{\sqrt{6}}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \Big|$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = (0, 0, 1) \cdot (1, 1, 1)$$

$$= 1 \Big|$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = (0, 0, 1) \cdot \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= \frac{1}{3} \Big|$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (0, 0, 1) - \frac{1}{3}(1, 1, 1) - \frac{1}{3}\left(\frac{9}{6}\right)\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= (0, 0, 1) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) - \left(-\frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right)$$

$$= \left(0, -\frac{1}{2}, \frac{1}{2}\right) \Big|$$

$$\vec{q}_3 = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}} \left(0, \ -\frac{1}{2}, \ \frac{1}{2}\right) \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \left(\sqrt{2}\right)\left(0, \ -\frac{1}{2}, \ \frac{1}{2}\right)$$

$$= \left(0, \ -\frac{\sqrt{2}}{2}, \ \frac{\sqrt{2}}{2}\right)$$

The *orthogonal* basis: $\left\{(1, \ 1, \ 1), \ \left(-\frac{2}{3}, \ \frac{1}{3}, \ \frac{1}{3}\right), \ \left(0, \ -\frac{1}{2}, \ \frac{1}{2}\right)\right\}$

The *orthonormal* basis: $\left\{\left(\frac{1}{\sqrt{3}}, \ \frac{1}{\sqrt{3}}, \ \frac{1}{\sqrt{3}}\right), \ \left(-\frac{2}{\sqrt{6}}, \ \frac{1}{\sqrt{6}}, \ \frac{1}{\sqrt{6}}\right), \ \left(0, \ -\frac{1}{\sqrt{2}}, \ \frac{1}{\sqrt{2}}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\{(1, \ 1, \ 1), \ (0, \ 2, \ 1), \ (1, \ 0, \ 3)\}$$

Solution

Let $\vec{u}_1 = (1, \ 1, \ 1) \quad \vec{u}_2 = (0, \ 2, \ 1) \quad \vec{u}_3 = (1, \ 0, \ 3)$

$$\vec{v}_1 = \vec{u}_1 = (1, \ 1, \ 1)$$

$$\vec{q}_1 = \frac{(1, \ 1, \ 1)}{\sqrt{1+1+1}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1, \ 1, \ 1)}{\sqrt{3}}$$

$$= \left(\frac{1}{\sqrt{3}}, \ \frac{1}{\sqrt{3}}, \ \frac{1}{\sqrt{3}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (0, \ 2, \ 1) - \frac{1}{3}\left[(0, \ 2, \ 1) \bullet (1, \ 1, \ 1)\right](1, \ 1, \ 1)$$

$$= (0, \ 2, \ 1) - \frac{3}{3}(1, \ 1, \ 1)$$

$$= (0, \ 2, \ 1) - (1, \ 1, \ 1)$$

$$= (-1, \ 1, \ 0)$$

$$\vec{q}_2 = \frac{1}{\sqrt{2}}(-1,\ 1,\ 0) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$\langle \vec{u}_3,\ \vec{v}_1 \rangle = (1,\ 0,\ 3) \bullet (1,\ 1,\ 1)$$

$$= 4$$

$$\langle \vec{u}_3,\ \vec{v}_2 \rangle = (1,\ 0,\ 3) \bullet (-1,\ 1,\ 0)$$

$$= -1$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3,\ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3,\ \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (1,\ 0,\ 3) - \frac{4}{3}(1,\ 1,\ 1) + \frac{1}{2}(-1,\ 1,\ 0)$$

$$= (1,\ 0,\ 3) - \left(\frac{4}{3},\ \frac{4}{3},\ \frac{4}{3}\right) + \left(-\frac{1}{2},\ \frac{1}{2},\ 0\right)$$

$$= \left(-\frac{5}{6},\ -\frac{5}{6},\ \frac{5}{3}\right)$$

$$\vec{q}_3 = \frac{1}{\sqrt{\frac{25}{36} + \frac{25}{36} + \frac{25}{9}}}\left(-\frac{5}{6},\ -\frac{5}{6},\ \frac{5}{3}\right) \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\frac{5}{6}\sqrt{6}}\left(-\frac{5}{6},\ -\frac{5}{6},\ \frac{5}{3}\right)$$

$$= \left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)$$

The *orthogonal* basis: $\left\{(1,\ 1,\ 1),\ (-1,\ 1,\ 0),\ \left(-\frac{5}{6},\ -\frac{5}{6},\ \frac{5}{3}\right)\right\}$

The *orthonormal* basis: $\left\{\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right),\ \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right),\ \left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\{(2,\ 2,\ 2),\ (1,\ 0,\ -1),\ (0,\ 3,\ 1)\}$$

Solution

Let $\vec{u}_1 = (2,\ 2,\ 2)\quad \vec{u}_2 = (1,\ 0,\ -1)\quad \vec{u}_3 = (0,\ 3,\ 1)$

$$\vec{v}_1 = \vec{u}_1 = (2,\ 2,\ 2)$$

$$\vec{q}_1 = \frac{(2,\ 2,\ 2)}{\sqrt{4+4+4}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \frac{(2,\ 2,\ 2)}{2\sqrt{3}}$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2,\ \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$= (1,\ 0,\ -1) - \frac{1}{12}\left[(1,\ 0,\ -1)\bullet(2,\ 2,\ 2)\right](2,\ 2,\ 2)$$

$$= (1,\ 0,\ -1) - \frac{0}{12}(2,\ 2,\ 2)$$

$$= (1,\ 0,\ -1)$$

$$\vec{q}_2 = \frac{(1,\ 0,\ -1)}{\sqrt{2}} \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \left(\frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right)$$

$$\left\langle \vec{u}_3,\ \vec{v}_1 \right\rangle = (0,\ 3,\ 1)\bullet(2,\ 2,\ 2)$$

$$= 8$$

$$\left\langle \vec{u}_3,\ \vec{v}_2 \right\rangle = (0,\ 3,\ 1)\bullet(1,\ 0,\ -1)$$

$$= -1$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\left\langle \vec{u}_3,\ \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1 - \frac{\left\langle \vec{u}_3,\ \vec{v}_2 \right\rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2$$

$$= (0,\ 3,\ 1) - \frac{8}{12}(2,\ 2,\ 2) + \frac{1}{2}(1,\ 0,\ -1)$$

$$= (0,\ 3,\ 1) - \left(\frac{4}{3},\ \frac{4}{3},\ \frac{4}{3}\right) + \left(\frac{1}{2},\ 0,\ -\frac{1}{2}\right)$$

$$\underline{= \left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right)}$$

$$\vec{q}_3 = \frac{1}{\sqrt{\left(-\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(-\frac{5}{6}\right)^2}}\left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right) \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\sqrt{\left(-\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(-\frac{5}{6}\right)^2}}\left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right)$$

$$= \frac{1}{\sqrt{\frac{25}{36} + \frac{25}{36} + \frac{25}{9}}}\left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right)$$

$$= \frac{1}{\frac{5}{6}\sqrt{6}}\left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right)$$

$$\underline{= \left(-\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}}\right)}$$

The *orthogonal* basis: $\left\{(2,\ 2,\ 2),\ (1,\ 0,\ -1),\ \left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right)\right\}$

The *orthonormal* basis: $\left\{\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right),\ \left(\frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right),\ \left(-\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\{(1,\ -1,\ 0),\ (0,\ 1,\ 0),\ (2,\ 3,\ 1)\}$$

Solution

Let $\vec{u}_1 = (1,\ -1,\ 0)$ $\vec{u}_2 = (0,\ 1,\ 0)$ $\vec{u}_3 = (2,\ 3,\ 1)$

$$\underline{\vec{v}_1 = \vec{u}_1 = (1,\ -1,\ 0)}$$

$$\vec{q}_1 = \frac{(1,\ -1,\ 0)}{\sqrt{2}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1, \, -1, \, 0)}{\sqrt{2}}$$

$$= \left(\frac{1}{\sqrt{2}}, \, -\frac{1}{\sqrt{2}}, \, 0 \right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \, \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1$$

$$= (0, \, 1, \, 0) - \frac{1}{2} \big[(0, \, 1, \, 0) \bullet (1, \, -1, \, 0) \big] (1, \, -1, \, 0)$$

$$= (0, \, 1, \, 0) + \frac{1}{2}(1, \, -1, \, 0)$$

$$= (0, \, 1, \, 0) + \left(\frac{1}{2}, \, -\frac{1}{2}, \, 0 \right)$$

$$= \left(\frac{1}{2}, \, \frac{1}{2}, \, 0 \right)$$

$$\vec{q}_2 = \frac{1}{\sqrt{\frac{1}{4} + \frac{1}{4}}} \left(\frac{1}{2}, \, \frac{1}{2}, \, 0 \right) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\| \vec{v}_2 \right\|}$$

$$= \left(\frac{1}{\sqrt{2}}, \, \frac{1}{\sqrt{2}}, \, 0 \right)$$

$$\langle \vec{u}_3, \, \vec{v}_1 \rangle = (2, \, 3, \, 1) \bullet (1, \, -1, \, 0)$$

$$= -1$$

$$\langle \vec{u}_3, \, \vec{v}_2 \rangle = (2, \, 3, \, 1) \bullet \left(\frac{1}{2}, \, \frac{1}{2}, \, 0 \right)$$

$$= \frac{5}{2}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \, \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \, \vec{v}_2 \rangle}{\left\| \vec{v}_2 \right\|^2} \vec{v}_2$$

$$= (2, \, 3, \, 1) + \frac{1}{2}(1, \, -1, \, 0) - \frac{5}{2}(2)\left(\frac{1}{2}, \, \frac{1}{2}, \, 0 \right)$$

$$= (2, \, 3, \, 1) + \left(\frac{1}{2}, \, -\frac{1}{2}, \, 0 \right) - \left(\frac{5}{2}, \, \frac{5}{2}, \, 0 \right)$$

$$= (0, \, 0, \, 1)$$

$$\vec{q}_3 = \frac{1}{1}(0, \, 0, \, 1) \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\left\| \vec{v}_3 \right\|}$$

$$= (0, \, 0, \, 1)$$

The *orthogonal* basis: $\left\{ (1, \ -1, \ 0), \ \left(\frac{1}{2}, \ \frac{1}{2}, \ 0 \right), \ (0, \ 0, \ 1) \right\}$

The *orthonormal* basis: $\left\{ \left(\frac{1}{\sqrt{2}}, \ -\frac{1}{\sqrt{2}}, \ 0 \right), \ \left(\frac{1}{\sqrt{2}}, \ \frac{1}{\sqrt{2}}, \ 0 \right), \ (0, \ 0, \ 1) \right\}$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\{(3, \ 0, \ 4), \ (-1, \ 0, \ 7), \ (2, \ 9, \ 11)\}$$

Solution

Let $\vec{u}_1 = (3, \ 0, \ 4)$ $\vec{u}_2 = (-1, \ 0, \ 7)$ $\vec{u}_3 = (2, \ 9, \ 11)$

$\vec{v}_1 = \vec{u}_1 = (3, \ 0, \ 4)$

$$\vec{q}_1 = \frac{(3, \ 0, \ 4)}{\sqrt{9+16}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\| \vec{v}_1 \right\|}$$

$$= \left(\frac{3}{5}, \ 0, \ \frac{4}{5} \right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2, \ \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1$$

$$= (-1, \ 0, \ 7) - \frac{1}{25} \left[(-1, \ 0, \ 7) \cdot (3, \ 0, \ 4) \right] (3, \ 0, \ 4)$$

$$= (-1, \ 0, \ 7) - \frac{25}{25} (3, \ 0, \ 4)$$

$$= (-1, \ 0, \ 7) - (3, \ 0, \ 4)$$

$$= (-4, \ 0, \ 3)$$

$$\vec{q}_2 = \frac{1}{\sqrt{16+9}} (-4, \ 0, \ 3) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\| \vec{v}_2 \right\|}$$

$$= \left(-\frac{4}{5}, \ 0, \ \frac{3}{5} \right)$$

$$\left\langle \vec{u}_3, \ \vec{v}_1 \right\rangle = (2, \ 9, \ 11) \cdot (3, \ 0, \ 4)$$

$$= 50$$

$$\left\langle \vec{u}_3, \ \vec{v}_2 \right\rangle = (2, \ 9, \ 11) \cdot (-4, \ 0, \ 3)$$

$$\underline{= 25}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (2, 9, 11) - \frac{50}{25}(3, 0, 4) - \frac{25}{25}(-4, 0, 3)$$

$$= (2, 9, 11) - (6, 0, 8) - (-4, 0, 3)$$

$$\underline{= (0, 9, 0)}$$

$$\vec{q}_3 = \frac{1}{9}(0, 9, 0) \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$\underline{= (0, 1, 0)}$$

The *orthogonal* basis: $\{(3, 0, 4), (-4, 0, 3), (0, 9, 0)\}$

The *orthonormal* basis: $\left\{ \left(\frac{3}{5}, 0, \frac{4}{5}\right), \left(-\frac{4}{5}, 0, \frac{3}{5}\right), (0, 1, 0) \right\}$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\{(2, 1, -1), (1, 2, 2), (2, -2, 1)\}$$

Solution

Let $\vec{u}_1 = (2, 1, -1)$ $\vec{u}_2 = (1, 2, 2)$ $\vec{u}_3 = (2, -2, 1)$

$$\vec{v}_1 = \vec{u}_1 = (2, 1, -1)$$

$$\vec{q}_1 = \frac{(2, 1, -1)}{\sqrt{4+1+1}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\underline{= \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|} \vec{v}_1$$

$$= (1, 2, 2) - \frac{1}{4+1+1}\left[(1, 2, 2) \bullet (2, 1, -1)\right](2, 1, -1)$$

$$= (1, 2, 2) - \frac{1}{6}(2)(2, 1, -1)$$

$$= (1,\ 2,\ 2) - \left(\frac{2}{3},\ \frac{1}{3},\ -\frac{1}{3} \right)$$

$$= \left(\frac{1}{3},\ \frac{5}{3},\ \frac{7}{3} \right)$$

$$\vec{q}_2 = \frac{1}{\sqrt{\frac{1}{9} + \frac{25}{9} + \frac{49}{9}}} \left(\frac{1}{3},\ \frac{5}{3},\ \frac{7}{3} \right) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{3}{5\sqrt{3}} \left(\frac{1}{3},\ \frac{5}{3},\ \frac{7}{3} \right)$$

$$= \left(\frac{1}{5\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{7}{5\sqrt{3}} \right)$$

$$\left\langle \vec{u}_3,\ \vec{v}_1 \right\rangle = (2,\ -2,\ 1) \bullet (2,\ 1,\ -1)$$

$$= 4 - 2 - 1$$

$$= 1$$

$$\left\langle \vec{u}_3,\ \vec{v}_2 \right\rangle = (2,\ -2,\ 1) \bullet \left(\frac{1}{3},\ \frac{5}{3},\ \frac{7}{3} \right)$$

$$= \frac{2}{3} - \frac{10}{3} + \frac{7}{3}$$

$$= -\frac{1}{3}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\left\langle \vec{u}_3,\ \vec{v}_1 \right\rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\left\langle \vec{u}_3,\ \vec{v}_2 \right\rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (2,\ -2,\ 1) - \frac{1}{6}(2,\ 1,\ -1) + \left(\frac{1}{3} \right)\left(\frac{9}{75} \right)\left(\frac{1}{3},\ \frac{5}{3},\ \frac{7}{3} \right)$$

$$= (2,\ -2,\ 1) - \left(\frac{1}{3},\ \frac{1}{6},\ -\frac{1}{6} \right) + \left(\frac{1}{75},\ \frac{1}{15},\ \frac{7}{75} \right)$$

$$= \left(\frac{42}{25},\ -\frac{63}{30},\ \frac{63}{50} \right)$$

$$\vec{q}_2 = \frac{1}{\sqrt{\frac{1{,}764}{625} + \frac{3{,}969}{900} + \frac{3{,}969}{2{,}500}}} \left(\frac{42}{25},\ -\frac{63}{30},\ \frac{63}{50} \right) \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\sqrt{\frac{63{,}504 + 99{,}225 + 35{,}721}{22{,}500}}} \left(\frac{42}{25},\ -\frac{63}{30},\ \frac{63}{50} \right)$$

$$= \frac{150}{\sqrt{198{,}450}} \left(\frac{42}{25},\ -\frac{63}{30},\ \frac{63}{50} \right)$$

$$= \frac{150}{315\sqrt{2}} \left(\frac{42}{25},\ -\frac{63}{30},\ \frac{63}{50} \right)$$

$$= \frac{10}{21\sqrt{2}}\left(\frac{42}{25}, \ -\frac{63}{30}, \ \frac{63}{50}\right)$$

$$= \left(\frac{4}{5\sqrt{2}}, \ -\frac{1}{\sqrt{2}}, \ \frac{3}{5\sqrt{2}}\right)$$

The *orthogonal* basis: $\left\{(2, 1, \ -1), \ \left(\frac{1}{3}, \ \frac{5}{3}, \ \frac{7}{3}\right), \ \left(\frac{42}{25}, \ -\frac{63}{30}, \ \frac{63}{50}\right)\right\}$

The *orthonormal* basis:

$$\left\{\left(\frac{2}{\sqrt{6}}, \ \frac{1}{\sqrt{6}}, \ -\frac{1}{\sqrt{6}}\right), \ \left(\frac{1}{5\sqrt{3}}, \ \frac{1}{\sqrt{3}}, \ \frac{7}{5\sqrt{3}}\right), \ \left(\frac{4}{5\sqrt{2}}, \ -\frac{1}{\sqrt{2}}, \ \frac{3}{5\sqrt{2}}\right)\right\}$$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\left\{(1, 1, 0), \ (1, 2, 0), \ (0, 1, 2)\right\}$$

Solution

Let $\vec{u}_1 = (1, 1, 0)$ $\vec{u}_2 = (1, 2, 0)$ $\vec{u}_3 = (0, 1, 2)$

$\vec{v}_1 = \vec{u}_1 = (1, 1, 0)$

$$\vec{q}_1 = \frac{(1, 1, 0)}{\sqrt{2}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \left(\frac{1}{\sqrt{2}}, \ \frac{1}{\sqrt{2}}, \ 0\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2, \ \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$= (1, 2, 0) - \frac{(1, 2, 0) \cdot (1, 1, 0)}{1+1+0}(1, 1, 0)$$

$$= (1, 2, 0) - \frac{3}{2}(1, 1, 0)$$

$$= \left(-\frac{1}{2}, \ \frac{1}{2}, \ 0\right)$$

$$\vec{q}_2 = \frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}}}\left(-\frac{1}{2}, \ \frac{1}{2}, \ 0\right) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \frac{2}{\sqrt{2}}\left(-\frac{1}{2}, \frac{1}{2}, 0\right)$$

$$= \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(0, 1, 2) \cdot (1, 1, 0)}{2}(1, 1, 0)$$

$$= \left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{1}{\frac{1}{4}+\frac{1}{4}}\left[(0, 1, 2) \cdot \left(-\frac{1}{2}, \frac{1}{2}, 0\right)\right]\left(-\frac{1}{2}, \frac{1}{2}, 0\right)$$

$$= 2\left(\frac{1}{2}\right)\left(-\frac{1}{2}, \frac{1}{2}, 1\right)$$

$$= \left(-\frac{1}{2}, \frac{1}{2}, 1\right)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (0, 1, 2) - \left(\frac{1}{2}, \frac{1}{2}, 0\right) - \left(-\frac{1}{2}, \frac{1}{2}, 1\right)$$

$$= (0, 0, 1)$$

$$\vec{q}_3 = (0, 0, 1) \qquad\qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

The *orthogonal* basis: $\left\{(1, 1, 0), \left(-\frac{1}{2}, \frac{1}{2}, 0\right), (0, 0, 1)\right\}$

The *orthogonal* basis: $\left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), (0, 0, 1)\right\}$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(1, \ -2, \ 2), \ (2, \ 2, \ 1), \ (2, \ -1, \ -2)\}$$

Solution

Let $\quad \vec{u}_1 = (1, \ -2, \ 2) \quad \vec{u}_2 = (2, \ 2, \ 1) \quad \vec{u}_3 = (2, \ -1, \ -2)$

$\vec{v}_1 = \vec{u}_1 = (1, \ -2, \ 2)$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (2, \ 2, \ 1) - \frac{(2, \ 2, \ 1) \bullet (1, \ -2, \ 2)}{9}(1, \ -2, \ 2)$$

$$= (2, \ 2, \ 1) - \frac{0}{9}(1, \ -2, \ 2)$$

$$= (2, \ 2, \ 1)$$

$$\frac{\langle \vec{u}_3, \ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(2, \ -1, \ -2) \bullet (1, \ -2, \ 2)}{9}(1, \ -2, \ 2)$$

$$= \frac{0}{9}(1, \ -2, \ 2)$$

$$= (0, \ 0, \ 0)$$

$$\frac{\langle \vec{u}_3, \ \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{1}{9}\left[(2, \ -1, \ -2) \bullet (2, \ 2, \ 1)\right](2, \ 2, \ 1)$$

$$= (0, \ 0, \ 0)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \ \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (2, \ -1, \ -2) - (0, \ 0, \ 0) - (0, \ 0, \ 0)$$

$$= (2, \ -1, \ -2)$$

$\vec{q}_1 = \frac{1}{3}(1, \ -2, \ 2) \qquad\qquad\qquad \vec{q}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$

$$= \left(\frac{1}{3}, \ -\frac{2}{3}, \ \frac{2}{3}\right)$$

$$\vec{q}_2 = \frac{1}{3}(2, \ 2, \ 1) \qquad\qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(\frac{2}{3}, \ \frac{2}{3}, \ \frac{1}{3}\right)$$

$$\vec{q}_3 = \frac{1}{3}(2, \ -1, \ -2) \qquad\qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \left(\frac{2}{3}, \ -\frac{1}{3}, \ -\frac{2}{3}\right)$$

The *orthogonal* basis: $\quad \{(1, \ -2, \ 2), \ (2, \ 2, \ 1), \ (2, \ -1, \ -2)\}$

The *orthogonal* basis:

$$\left\{\left(\frac{1}{3}, \ -\frac{2}{3}, \ \frac{2}{3}\right), \ \left(\frac{2}{3}, \ \frac{2}{3}, \ \frac{1}{3}\right), \ \left(\frac{2}{3}, \ -\frac{1}{3}, \ -\frac{2}{3}\right)\right\}$$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\{(1, \ 0, \ 0), \ (1, \ 1, \ 1), \ (1, \ 1, \ -1)\}$$

Solution

Let $\quad \vec{u}_1 = (1, \ 0, \ 0) \quad \vec{u}_2 = (1, \ 1, \ 1) \quad \vec{u}_3 = (1, \ 1, \ -1)$

$$\vec{v}_1 = \vec{u}_1 = (1, \ 0, \ 0)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (1, \ 1, \ 1) - \frac{(1, \ 1, \ 1) \bullet (1, \ 0, \ 0)}{1}(1, \ 0, \ 0)$$

$$= (1, \ 1, \ 1) - (1, \ 0, \ 0)$$

$$= (0, \ 1, \ 1)$$

$$\frac{\langle \vec{u}_3, \ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(1, \ 1, \ -1) \bullet (1, \ 0, \ 0)}{1}(1, \ 0, \ 0)$$

$$= (1, \ 0, \ 0)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{1}{1} \Big[(1,\ 1,\ -1) \bullet (0,\ 1,\ 1) \Big] (0,\ 1,\ 1)$$

$$= 0 (0,\ 1,\ 1)$$

$$\underline{= (0,\ 0,\ 0)} \Big|$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (1,\ 1,\ -1) - (1,\ 0,\ 0) - (0,\ 0,\ 0)$$

$$\underline{= (0,\ 1,\ -1)} \Big|$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{1} (1,\ 0,\ 0)$$

$$\underline{= (1,\ 0,\ 0)} \Big|$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{\sqrt{2}} (0,\ 1,\ 1)$$

$$\underline{= \left(0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}} \right)} \Big|$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\sqrt{2}} (0,\ 1,\ -1)$$

$$\underline{= \left(0,\ \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}} \right)} \Big|$$

The *orthogonal* basis: $\left\{ (1,\ 0,\ 0),\ (0,\ 1,\ 1),\ (0,\ 1,\ -1) \right\}$

The *orthogonal* basis:

$$\left\{ (1,\ 0,\ 0),\ \left(0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}} \right),\ \left(0,\ \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}} \right) \right\}$$

Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces $\mathbb{R}^m$.

$$\{(4, \ -3, \ 0), \ (1, \ 2, \ 0), \ (0, \ 0, \ 4)\}$$

Solution

Let $\quad \vec{u}_1 = (4, \ -3, \ 0) \quad \vec{u}_2 = (1, \ 2, \ 0) \quad \vec{u}_3 = (0, \ 0, \ 4)$

$\underline{\vec{v}_1 = \vec{u}_1 = (4, \ -3, \ 0)}$

$\vec{v}_2 = \vec{u}_2 - \dfrac{\langle \vec{u}_2, \ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$

$\qquad = (1, \ 2, \ 0) - \dfrac{(1, \ 2, \ 0) \bullet (4, \ -3, \ 0)}{25}(4, \ -3, \ 0)$

$\qquad = (1, \ 2, \ 0) + \dfrac{2}{25}(4, \ -3, \ 0)$

$\qquad \underline{= \left(\dfrac{33}{25}, \ \dfrac{44}{25}, \ 0\right)}$

$\dfrac{\langle \vec{u}_3, \ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \dfrac{(0, \ 0, \ 4) \bullet (4, \ -3, \ 0)}{25}(4, \ -3, \ 0)$

$\qquad \underline{= (0, \ 0, \ 0)}$

$\dfrac{\langle \vec{u}_3, \ \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \dfrac{225}{3,025}\left[(0, \ 0, \ 4) \bullet \left(\dfrac{33}{25}, \ \dfrac{44}{25}, \ 0\right)\right]\left(\dfrac{33}{25}, \ \dfrac{44}{25}, \ 0\right)$

$\qquad \underline{= (0, \ 0, \ 0)}$

$\vec{v}_3 = \vec{u}_3 - \dfrac{\langle \vec{u}_3, \ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \dfrac{\langle \vec{u}_3, \ \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$

$\qquad = (0, \ 0, \ 4) - (0, \ 0, \ 0) - (0, \ 0, \ 0)$

$\qquad \underline{= (0, \ 0, \ 4)}$

$\vec{q}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$

$\qquad = \dfrac{1}{\sqrt{16+9}}(4, \ -3, \ 0)$

$\qquad \underline{= \left(\dfrac{4}{5}, \ -\dfrac{3}{5}, \ 0\right)}$

$$\vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \frac{25}{\sqrt{3,025}}\left(\frac{33}{25}, \frac{44}{25}, 0\right)$$

$$= \frac{25}{55}\left(\frac{33}{25}, \frac{44}{25}, 0\right)$$

$$= \left(\frac{3}{5}, \frac{4}{5}, 0\right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$= \frac{1}{4}(0, 0, 4)$$

$$= (0, 0, 1)$$

The *orthogonal* basis: $\left\{(4, -3, 0), \left(\frac{33}{25}, \frac{44}{25}, 0\right), (0, 0, 4)\right\}$

The *orthogonal* basis: $\left\{\left(\frac{4}{5}, -\frac{3}{5}, 0\right), \left(\frac{3}{5}, \frac{4}{5}, 0\right), (0, 0, 1)\right\}$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\{(0, 1, 2), (2, 0, 0), (1, 1, 1)\}$$

Solution

Let $\vec{u}_1 = (0, 1, 2)$ $\vec{u}_2 = (2, 0, 0)$ $\vec{u}_3 = (1, 1, 1)$

$$\vec{v}_1 = \vec{u}_1 = (0, 1, 2)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$= (2, 0, 0) - \frac{(2, 0, 0) \cdot (0, 1, 2)}{5}(0, 1, 2)$$

$$= (2, 0, 0) + \frac{0}{5}(0, 1, 2)$$

$$= (2, 0, 0)$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(1, 1, 1) \bullet (0, 1, 2)}{5} (0, 1, 2)$$

$$= \frac{3}{5}(0, 1, 2)$$

$$= \left(0, \frac{3}{5}, \frac{6}{5} \right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{1}{4} \left[(1, 1, 1) \bullet (2, 0, 0) \right] (2, 0, 0)$$

$$= \frac{1}{2}(2, 0, 0)$$

$$= (1, 0, 0)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (1, 1, 1) - (1, 0, 0) - \left(0, \frac{3}{5}, \frac{6}{5} \right)$$

$$= \left(0, \frac{2}{5}, -\frac{1}{5} \right)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{5}}(0, 1, 2)$$

$$= \left(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{2}(2, 0, 0)$$

$$= (1, 0, 0)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{5}{\sqrt{5}} \left(0, \frac{2}{5}, -\frac{1}{5} \right)$$

$$= \left(0, \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right)$$

The *orthogonal* basis: $\left\{(0,\ 1,\ 2),\ (2,\ 0,\ 0),\ \left(0,\ \dfrac{2}{5},\ -\dfrac{1}{5}\right)\right\}$

The *orthogonal* basis: $\left\{\left(0,\ \dfrac{1}{\sqrt{5}},\ \dfrac{2}{\sqrt{5}}\right),\ (1,\ 0,\ 0),\ \left(0,\ \dfrac{2}{\sqrt{5}},\ -\dfrac{1}{\sqrt{5}}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$\{(0,\ 1,\ 1),\ (1,\ 1,\ 0),\ (1,\ 0,\ 1)\}$

Solution

Let $\quad \vec{u}_1 = (0,\ 1,\ 1) \quad \vec{u}_2 = (1,\ 1,\ 0) \quad \vec{u}_3 = (1,\ 0,\ 1)$

$\underline{\vec{v}_1 = \vec{u}_1 = (0,\ 1,\ 1)}$

$\vec{v}_2 = \vec{u}_2 - \dfrac{\langle \vec{u}_2,\ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$

$\qquad = (1,\ 1,\ 0) - \dfrac{(1,\ 1,\ 0) \bullet (0,\ 1,\ 1)}{2}(0,\ 1,\ 1)$

$\qquad = (1,\ 1,\ 0) - \dfrac{1}{2}(0,\ 1,\ 1)$

$\qquad \underline{= \left(1,\ \dfrac{1}{2},\ -\dfrac{1}{2}\right)}$

$\dfrac{\langle \vec{u}_3,\ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \dfrac{(1,\ 0,\ 1) \bullet (0,\ 1,\ 1)}{2}(0,\ 1,\ 1)$

$\qquad\qquad \underline{= \left(0,\ \dfrac{1}{2},\ \dfrac{1}{2}\right)}$

$\dfrac{\langle \vec{u}_3,\ \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \dfrac{4}{6}\left[(1,\ 0,\ 1) \bullet \left(1,\ \dfrac{1}{2},\ -\dfrac{1}{2}\right)\right]\left(1,\ \dfrac{1}{2},\ -\dfrac{1}{2}\right)$

$\qquad\qquad = \dfrac{1}{3}\left(1,\ \dfrac{1}{2},\ -\dfrac{1}{2}\right)$

$\qquad\qquad \underline{= \left(\dfrac{1}{3},\ \dfrac{1}{6},\ -\dfrac{1}{6}\right)}$

$\vec{v}_3 = \vec{u}_3 - \dfrac{\langle \vec{u}_3,\ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \dfrac{\langle \vec{u}_3,\ \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$

$$= (1,\ 0,\ 1) - \left(0,\ \tfrac{1}{2},\ \tfrac{1}{2}\right) - \left(\tfrac{1}{3},\ \tfrac{1}{6},\ -\tfrac{1}{6}\right)$$

$$= \left(\tfrac{2}{3},\ -\tfrac{2}{3},\ \tfrac{2}{3}\right)$$

$$\vec{q}_1 = \frac{1}{\sqrt{2}}(0,\ 1,\ 1) \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(0,\ \tfrac{1}{\sqrt{2}},\ \tfrac{1}{\sqrt{2}}\right)$$

$$\vec{q}_2 = \frac{2}{\sqrt{6}}\left(1,\ \tfrac{1}{2},\ -\tfrac{1}{2}\right) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(\tfrac{2}{\sqrt{6}},\ \tfrac{1}{\sqrt{6}},\ -\tfrac{1}{\sqrt{6}}\right)$$

$$\vec{q}_3 = \frac{3}{\sqrt{12}}\left(\tfrac{2}{3},\ -\tfrac{2}{3},\ \tfrac{2}{3}\right) \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \left(\tfrac{1}{\sqrt{3}},\ -\tfrac{1}{\sqrt{3}},\ \tfrac{1}{\sqrt{3}}\right)$$

The *orthogonal* basis: $\left\{(0,\ 1,\ 1),\ \left(1,\ \tfrac{1}{2},\ -\tfrac{1}{2}\right),\ \left(\tfrac{2}{3},\ -\tfrac{2}{3},\ \tfrac{2}{3}\right)\right\}$

The *orthogonal* basis: $\left\{\left(0,\ \tfrac{1}{\sqrt{2}},\ \tfrac{1}{\sqrt{2}}\right),\ \left(\tfrac{2}{\sqrt{6}},\ \tfrac{1}{\sqrt{6}},\ -\tfrac{1}{\sqrt{6}}\right),\ \left(\tfrac{1}{\sqrt{3}},\ -\tfrac{1}{\sqrt{3}},\ \tfrac{1}{\sqrt{3}}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\{(2,\ -1,\ 3),\ (3,\ 4,\ 1),\ (2,\ -3,\ 4)\}$$

Solution

Let $\vec{u}_1 = (2,\ -1,\ 3)$ $\vec{u}_2 = (3,\ 4,\ 1)$ $\vec{u}_3 = (2,\ -3,\ 4)$

$$\vec{v}_1 = \vec{u}_1 = (2,\ -1,\ 3)$$

$$\vec{q}_1 = \frac{1}{\sqrt{4+1+9}}(2,\ -1,\ 3) \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(\tfrac{2}{\sqrt{14}},\ -\tfrac{1}{\sqrt{14}},\ \tfrac{3}{\sqrt{14}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (3,\ 4,\ 1) - \frac{(3,\ 4,\ 1) \cdot (2,\ -1,\ 3)}{14}(2,\ -1,\ 3)$$

$$= (3,\ 4,\ 1) - \frac{5}{14}(2,\ -1,\ 3)$$

$$= (3,\ 4,\ 1) - \left(\frac{5}{7},\ -\frac{5}{14},\ \frac{15}{14}\right)$$

$$= \left(\frac{16}{7},\ \frac{61}{14},\ -\frac{1}{14}\right)$$

$$\vec{q}_2 = \frac{14}{\sqrt{32^2 + 61^2 + 1}}\left(\frac{16}{7},\ \frac{61}{14},\ -\frac{1}{14}\right) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{14}{\sqrt{4{,}746}}\left(\frac{16}{7},\ \frac{61}{14},\ -\frac{1}{14}\right)$$

$$= \left(\frac{32}{\sqrt{4{,}746}},\ \frac{61}{\sqrt{4{,}746}},\ -\frac{1}{\sqrt{4{,}746}}\right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(2,\ -3,\ 4) \cdot (2,\ -1,\ 3)}{14}(2,\ -1,\ 3)$$

$$= \frac{19}{14}(2,\ -1,\ 3)$$

$$= \left(\frac{19}{7},\ -\frac{19}{14},\ \frac{57}{14}\right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{196}{4{,}746}\left((2,\ -3,\ 4) \cdot \left(\frac{16}{7},\ \frac{61}{14},\ -\frac{1}{14}\right)\right)\left(\frac{16}{7},\ \frac{61}{14},\ -\frac{1}{14}\right)$$

$$= \frac{14}{339}\left(\frac{32}{7} - \frac{183}{14} - \frac{2}{7}\right)\left(\frac{16}{7},\ \frac{61}{14},\ -\frac{1}{14}\right)$$

$$= \frac{14}{339}\left(-\frac{123}{14}\right)\left(\frac{16}{7},\ \frac{61}{14},\ -\frac{1}{14}\right)$$

$$= -\frac{41}{113}\left(\frac{16}{7},\ \frac{61}{14},\ -\frac{1}{14}\right)$$

$$= \left(-\frac{656}{791},\ -\frac{2{,}501}{1{,}582},\ \frac{41}{1{,}582}\right)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (2, \ -3, \ 4) - \left(\frac{19}{7}, \ -\frac{19}{14}, \ \frac{57}{14}\right) - \left(-\frac{656}{791}, \ -\frac{2{,}501}{1{,}582}, \ \frac{41}{1{,}582}\right)$$

$$= \left(-\frac{5}{7}, \ -\frac{23}{14}, \ -\frac{1}{14}\right) - \left(-\frac{656}{791}, \ -\frac{2{,}501}{1{,}582}, \ \frac{41}{1{,}582}\right)$$

$$= \left(\frac{13}{113}, \ -\frac{7}{113}, \ -\frac{11}{113}\right) \Big|$$

$$\vec{q}_3 = \frac{113}{\sqrt{169 + 49 + 121}} \left(\frac{13}{113}, \ -\frac{7}{113}, \ -\frac{11}{113}\right) \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\sqrt{339}} (13, \ -7, \ -11)$$

$$= \left(\frac{13}{\sqrt{339}}, \ -\frac{7}{\sqrt{339}}, \ -\frac{11}{\sqrt{339}}\right) \Big|$$

The *orthogonal* basis: $\quad \left\{(2, \ -1, \ 3), \ \left(\frac{16}{7}, \ \frac{61}{14}, \ -\frac{1}{14}\right), \ \left(\frac{13}{113}, \ -\frac{7}{113}, \ -\frac{11}{113}\right)\right\}$

The *orthogonal* basis:

$$\left\{\left(\frac{2}{\sqrt{14}}, \ -\frac{1}{\sqrt{14}}, \ \frac{3}{\sqrt{14}}\right), \ \left(\frac{32}{\sqrt{4{,}746}}, \ \frac{61}{\sqrt{4{,}746}}, \ -\frac{1}{\sqrt{4{,}746}}\right), \ \left(\frac{13}{\sqrt{339}}, \ -\frac{7}{\sqrt{339}}, \ -\frac{11}{\sqrt{339}}\right)\right\}$$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\{(-3, \ 0, \ 4), \ (5, \ -1, \ 2), \ (1, \ 1, \ 3)\}$$

Solution

Let $\quad \vec{u}_1 = (-3, \ 0, \ 4) \quad \vec{u}_2 = (5, \ -1, \ 2) \quad \vec{u}_3 = (1, \ 1, \ 3)$

$$\vec{v}_1 = \vec{u}_1 = (-3, \ 0, \ 4) \Big|$$

$$\vec{q}_1 = \frac{(-3, \ 0, \ 4)}{\sqrt{9 + 16}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(-\frac{3}{5}, \ 0, \ \frac{4}{5}\right) \Big|$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (5, \ -1, \ 2) - \frac{(5, \ -1, \ 2) \cdot (-3, \ 0, \ 4)}{25}(-3, \ 0, \ 4)$$

$$= (5, \ -1, \ 2) + \frac{7}{25}(-3, \ 0, \ 4)$$

$$= (5, \ -1, \ 2) + \left(-\frac{21}{25}, \ 0, \ \frac{28}{25}\right)$$

$$= \left(\frac{104}{25}, \ -1, \ \frac{78}{25}\right)$$

$$\vec{q}_2 = \frac{25}{\sqrt{104^2 + 25^2 + 78^2}} \left(\frac{104}{25}, \ -1, \ \frac{78}{25}\right) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{\sqrt{17{,}525}} \ (104, \ -25, \ 78)$$

$$= \frac{1}{5\sqrt{701}} \ (104, \ -25, \ 78)$$

$$= \left(\frac{104}{5\sqrt{701}}, \ -\frac{5}{\sqrt{701}}, \ \frac{78}{5\sqrt{701}}\right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(1, \ 1, \ 3) \bullet (-3, \ 0, \ 4)}{25}(-3, \ 0, \ 4)$$

$$= \frac{9}{25}(-3, \ 0, \ 4)$$

$$= \left(-\frac{27}{25}, \ 0, \ \frac{36}{25}\right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{25^2}{17{,}525}\left((1, \ 1, \ 3) \bullet \left(\frac{104}{25}, \ -1, \ \frac{78}{25}\right)\right)\left(\frac{104}{25}, \ -1, \ \frac{78}{25}\right)$$

$$= \frac{25}{701}\left(\frac{104}{25} - 1 + \frac{234}{25}\right)\left(\frac{104}{25}, \ -1, \ \frac{78}{25}\right)$$

$$= \frac{1}{701}\left(\frac{313}{25}\right)(104, \ -25, \ 78)$$

$$= \frac{313}{17{,}525} \ (104, \ -25, \ 78)$$

$$= \left(\frac{32{,}552}{17{,}525}, \ -\frac{313}{701}, \ \frac{24{,}414}{17{,}525}\right)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (1, \ 1, \ 3) - \left(-\frac{27}{25}, \ 0, \ \frac{36}{25}\right) - \left(\frac{32{,}552}{17{,}525}, \ -\frac{313}{701}, \ \frac{24{,}414}{17{,}525}\right)$$

$$= \left(\frac{52}{25}, \ 1, \ \frac{39}{25}\right) - \left(\frac{32{,}552}{17{,}525}, \ -\frac{313}{701}, \ \frac{24{,}414}{17{,}525}\right)$$

$$= \left(\frac{156}{701}, \ \frac{1{,}014}{701}, \ \frac{117}{701} \right)$$

$$\vec{q}_3 = \frac{701}{\sqrt{156^2 + 1014^2 + 117^2}} \left(\frac{156}{701}, \ \frac{1{,}014}{701}, \ \frac{117}{701} \right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{701}{\sqrt{1{,}066{,}221}} \left(\frac{156}{701}, \ \frac{1{,}014}{701}, \ \frac{117}{701} \right)$$

$$= \frac{701}{39\sqrt{701}} \left(\frac{156}{701}, \ \frac{1{,}014}{701}, \ \frac{117}{701} \right)$$

$$= \left(\frac{4}{\sqrt{701}}, \ \frac{26}{\sqrt{701}}, \ \frac{3}{\sqrt{701}} \right)$$

The *orthogonal* basis: $\quad \left\{ (-3, \ 0, \ 4), \ \left(\frac{104}{25}, \ -1, \ \frac{78}{25} \right), \ \left(\frac{156}{701}, \ \frac{1{,}014}{701}, \ \frac{117}{701} \right) \right\}$

The *orthogonal* basis:

$$\left\{ \left(-\frac{3}{5}, \ 0, \ \frac{4}{5} \right), \ \left(\frac{104}{5\sqrt{701}}, \ -\frac{5}{\sqrt{701}}, \ \frac{78}{5\sqrt{701}} \right), \ \left(\frac{4}{\sqrt{701}}, \ \frac{26}{\sqrt{701}}, \ \frac{3}{\sqrt{701}} \right) \right\}$$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\{ (-2, \ 0, \ 1), \ (3, \ 2, \ 5), \ (6, -1, \ 1), \ (7, \ 0, \ -2) \}$$

Solution

Let $\quad \vec{u}_1 = (-2, \ 0, \ 1) \quad \vec{u}_2 = (3, \ 2, \ 5) \quad \vec{u}_3 = (6, -1, \ 1) \quad \vec{u}_4 = (7, \ 0, \ -2)$

$$\vec{v}_1 = \vec{u}_1 = (-2, \ 0, \ 1)$$

$$\vec{q}_1 = \frac{(-2, \ 0, \ 1)}{\sqrt{5}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(-\frac{2}{\sqrt{5}}, \ 0, \ \frac{1}{\sqrt{5}} \right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (3, \ 2, \ 5) - \frac{(3, \ 2, \ 5) \cdot (-2, \ 0, \ 1)}{5} (-2, \ 0, \ 1)$$

$$= (3,\ 2,\ 5) + \frac{1}{5}(-2,\ 0,\ 1)$$

$$= (3,\ 2,\ 5) + \left(-\frac{2}{5},\ 0,\ \frac{1}{5}\right)$$

$$= \left(\frac{13}{5},\ 2,\ \frac{26}{5}\right)$$

$$\vec{q}_2 = \frac{5}{\sqrt{13^2 + 10^2 + 26^2}}\left(\frac{13}{5},\ 2,\ \frac{26}{5}\right) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \frac{1}{\sqrt{945}}(13,\ 10,\ 26)$$

$$= \frac{1}{3\sqrt{105}}(13,\ 10,\ 26)$$

$$= \left(\frac{13}{3\sqrt{105}},\ -\frac{10}{3\sqrt{105}},\ \frac{26}{3\sqrt{105}}\right)$$

$$\frac{\langle \vec{u}_3,\ \vec{v}_1\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1 = \frac{(6,-1,\ 1)\cdot(-2,\ 0,\ 1)}{5}(-2,\ 0,\ 1)$$

$$= -\frac{11}{5}(-2,\ 0,\ 1)$$

$$= \left(\frac{22}{5},\ 0,\ -\frac{11}{5}\right)$$

$$\frac{\langle \vec{u}_3,\ \vec{v}_2\rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2 = \frac{25}{945}\left((6,-1,\ 1)\cdot\left(\frac{13}{5},\ 2,\ \frac{26}{5}\right)\right)\left(\frac{13}{5},\ 2,\ \frac{26}{5}\right)$$

$$= \frac{5}{189}\left(\frac{78}{5} - 2 + \frac{26}{5}\right)\left(\frac{13}{5},\ 2,\ \frac{26}{5}\right)$$

$$= \frac{94}{189}\left(\frac{13}{5},\ 2,\ \frac{26}{5}\right)$$

$$= \left(\frac{1,222}{945},\ -\frac{188}{189},\ \frac{2,444}{945}\right)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3,\ \vec{v}_1\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3,\ \vec{v}_2\rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2$$

$$= (6,-1,\ 1) - \left(\frac{22}{5},\ 0,\ -\frac{11}{5}\right) - \left(\frac{1,222}{945},\ -\frac{188}{189},\ \frac{2,444}{945}\right)$$

$$= \left(\frac{8}{5},\ -1,\ \frac{16}{5}\right) - \left(\frac{1,222}{945},\ -\frac{188}{189},\ \frac{2,444}{945}\right)$$

$$= \left(\frac{290}{945},\ -\frac{377}{189},\ \frac{580}{945}\right)$$

$$= \left(\frac{58}{189}, \ -\frac{377}{189}, \ \frac{116}{189} \right)$$

$$\vec{q}_3 = \frac{189}{\sqrt{58^2 + 377^2 + 116^2}} \left(\frac{58}{189}, \ -\frac{377}{189}, \ \frac{116}{189} \right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\sqrt{158,949}} (58, \ -377, \ 116)$$

$$= \frac{1}{87\sqrt{21}} (58, \ -377, \ 116)$$

$$= \left(\frac{2}{3\sqrt{21}}, \ -\frac{13}{3\sqrt{21}}, \ \frac{4}{3\sqrt{21}} \right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(7, \ 0, \ -2) \bullet (-2, \ 0, \ 1)}{5} (-2, \ 0, \ 1)$$

$$= -\frac{16}{5} (-2, \ 0, \ 1)$$

$$= \left(\frac{32}{5}, \ 0, \ -\frac{16}{5} \right)$$

$$\|\vec{v}_2\|^2 = \frac{13^2 + 10^2 + 26^2}{25}$$

$$= \frac{945}{25}$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{25}{945} \left[(7, \ 0, \ -2) \bullet \left(\frac{13}{5}, \ 2, \ \frac{26}{5} \right) \right] \left(\frac{13}{5}, \ 2, \ \frac{26}{5} \right)$$

$$= \frac{25}{945} \left(\frac{39}{5} \right) \left(\frac{13}{5}, \ \frac{10}{5}, \ \frac{26}{5} \right)$$

$$= \frac{13}{315} (13, \ 10, \ 26)$$

$$= \left(\frac{169}{315}, \ \frac{2}{63}, \ \frac{338}{315} \right)$$

$$\|\vec{v}_3\|^2 = \frac{58^2 + 377^2 + 116^2}{189^2}$$

$$= \frac{158,949}{35,721}$$

$$= \frac{841}{189}$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 = \frac{189}{841} \left[(7, 0, -2) \cdot \left(\frac{58}{189}, -\frac{377}{189}, \frac{116}{189} \right) \right] \left(\frac{58}{189}, -\frac{377}{189}, \frac{116}{189} \right)$$

$$= \frac{189}{841} \left(\frac{174}{189} \right) \left(\frac{58}{189}, -\frac{377}{189}, \frac{116}{189} \right)$$

$$= \frac{174}{841} \left(\frac{58}{189}, -\frac{377}{189}, \frac{116}{189} \right)$$

$$= \frac{58}{841} \left(\frac{58}{63}, -\frac{377}{63}, \frac{116}{63} \right)$$

$$= \left(\frac{3,364}{52,983}, -\frac{21,866}{52,983}, \frac{6,728}{52,983} \right)$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3$$

$$= (7, 0, -2) - \left(\frac{32}{5}, 0, -\frac{16}{5} \right) - \left(\frac{169}{315}, \frac{2}{63}, \frac{338}{315} \right) - \left(\frac{3,364}{52,983}, -\frac{21,866}{52,983}, \frac{6,728}{52,983} \right)$$

$$= \left(\frac{3}{5}, 0, \frac{6}{5} \right) - \left(\frac{3}{5}, 0, \frac{6}{5} \right)$$

$$= (0, 0, 0)$$

$$\vec{q}_4 = (0, 0, 0) \qquad\qquad \vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

The *orthogonal* basis:

$$\left\{ (-2, 0, 1), \left(\frac{13}{5}, 2, \frac{26}{5} \right), \left(\frac{58}{189}, -\frac{377}{189}, \frac{116}{189} \right), (0, 0, 0) \right\}$$

The *orthogonal* basis:

$$\left\{ \begin{array}{l} \left(-\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}} \right), \left(\frac{13}{3\sqrt{105}}, -\frac{10}{3\sqrt{105}}, \frac{26}{3\sqrt{105}} \right), \\ \left(\frac{2}{3\sqrt{21}}, -\frac{13}{3\sqrt{21}}, \frac{4}{3\sqrt{21}} \right), (0, 0, 0) \end{array} \right\}$$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\left\{ (1, 2, -2), (1, 0, -4), (5, 2, 0), (1, 1, -1) \right\}$$

Solution

Let $\vec{u}_1 = (1, 2, -2)$ $\vec{u}_2 = (1, 0, -4)$ $\vec{u}_3 = (5, 2, 0)$ $\vec{u}_4 = (1, 1, -1)$

$$\vec{v}_1 = \vec{u}_1 = (1,\ 2,\ -2)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2,\ \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1$$

$$= (1,\ 0,\ -4) - \frac{(1,\ 0,\ -4) \bullet (1,\ 2,\ -2)}{9}(1,\ 2,\ -2)$$

$$= (1,\ 0,\ -4) - (1,\ 2,\ -2)$$

$$= (0,\ -2,\ -2)$$

$$\frac{\langle \vec{u}_3,\ \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1 = \frac{(5,\ 2,\ 0) \bullet (1,\ 2,\ -2)}{9}(1,\ 2,\ -2)$$

$$= (1,\ 2,\ -2)$$

$$\frac{\langle \vec{u}_3,\ \vec{v}_2 \rangle}{\left\| \vec{v}_2 \right\|^2} \vec{v}_2 = \frac{1}{8}\left[(5,\ 2,\ 0) \bullet (0,\ -2,\ -2) \right](0,\ -2,\ -2)$$

$$= -\frac{1}{2}(0,\ -2,\ -2)$$

$$= (0,\ 1,\ 1)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3,\ \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3,\ \vec{v}_2 \rangle}{\left\| \vec{v}_2 \right\|^2} \vec{v}_2$$

$$= (5,\ 2,\ 0) - (1,\ 2,\ -2) - (0,\ 1,\ 1)$$

$$= (4,\ -1,\ 1)$$

$$\frac{\langle \vec{u}_4,\ \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1 = \frac{(1,\ 1,\ -1) \bullet (1,\ 2,\ -2)}{9}(1,\ 2,\ -2)$$

$$= \frac{5}{9}(1,\ 2,\ -2)$$

$$= \left(\frac{5}{9},\ \frac{10}{9},\ -\frac{10}{9} \right)$$

$$\frac{\langle \vec{u}_4,\ \vec{v}_2 \rangle}{\left\| \vec{v}_2 \right\|^2} \vec{v}_2 = \frac{1}{8}\left[(1,\ 1,\ -1) \bullet (0,\ -2,\ -2) \right](0,\ -2,\ -2)$$

$$= (0,\ 0,\ 0)$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\left\| \vec{v}_3 \right\|^2} \vec{v}_3 = \frac{1}{18}\big[(1,\ 1,\ -1) \bullet (4,\ -1,\ 1)\big](4,\ -1,\ 1)$$

$$= \frac{1}{9}(4,\ -1,\ 1)$$

$$= \left(\frac{4}{9},\ -\frac{1}{9},\ \frac{1}{9}\right)$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\left\| \vec{v}_2 \right\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\left\| \vec{v}_3 \right\|^2} \vec{v}_3$$

$$= (1,\ 1,\ -1) - \left(\frac{5}{9},\ \frac{10}{9},\ -\frac{10}{9}\right) - \left(\frac{4}{9},\ -\frac{1}{9},\ \frac{1}{9}\right)$$

$$= (0,\ 0,\ 0)$$

$$\vec{q}_1 = \frac{1}{3}(1,\ 2,\ -2) \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\| \vec{v}_1 \right\|}$$

$$= \left(\frac{1}{3},\ \frac{2}{3},\ -\frac{2}{3}\right)$$

$$\vec{q}_2 = \frac{1}{2\sqrt{2}}(0,\ -2,\ -2) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\| \vec{v}_2 \right\|}$$

$$= \left(0,\ -\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right)$$

$$\vec{q}_3 = \frac{1}{3\sqrt{2}}(4,\ -1,\ 1) \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\left\| \vec{v}_3 \right\|}$$

$$= \left(\frac{4}{3\sqrt{2}},\ -\frac{1}{3\sqrt{2}},\ \frac{1}{3\sqrt{2}}\right)$$

$$\vec{q}_4 = \frac{\vec{v}_4}{\left\| \vec{v}_4 \right\|}$$

$$= (0,\ 0,\ 0)$$

The *orthogonal* basis: $\left\{(1,\ 2,\ -2),\ (0,\ -2,\ -2),\ (4,\ -1,\ 1),\ (0,\ 0,\ 0)\right\}$

The *orthogonal* basis:

$$\left\{\left(\frac{1}{3},\ \frac{2}{3},\ -\frac{2}{3}\right),\ \left(0,\ -\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right),\ \left(\frac{4}{3\sqrt{2}},\ -\frac{1}{3\sqrt{2}},\ \frac{1}{3\sqrt{2}}\right),\ (0,\ 0,\ 0)\right\}$$

Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces $\mathbb{R}^m$.

$$\{(-3,\ 1,\ 2),\ (1,\ 1,\ 1),\ (2,\ 0,\ -1),\ (1,\ -3,\ 2)\}$$

Solution

Let $\vec{u}_1 = (-3,\ 1,\ 2)$ $\vec{u}_2 = (1,\ 1,\ 1)$ $\vec{u}_3 = (2,\ 0,\ -1)$ $\vec{u}_4 = (1,\ -3,\ 2)$

$\vec{v}_1 = \vec{u}_1 = (-3,\ 1,\ 2)$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2,\ \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1$$

$$= (1,\ 1,\ 1) - \frac{(1,\ 1,\ 1) \bullet (-3,\ 1,\ 2)}{14} (-3,\ 1,\ 2)$$

$$= (1,\ 1,\ 1) - \frac{0}{14} (1,\ 2,\ -2)$$

$$= (1,\ 1,\ 1)$$

$$\frac{\langle \vec{u}_3,\ \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1 = \frac{(2,\ 0,\ -1) \bullet (-3,\ 1,\ 2)}{14} (-3,\ 1,\ 2)$$

$$= -\frac{4}{7} (-3,\ 1,\ 2)$$

$$= \left(\frac{12}{7},\ -\frac{4}{7},\ -\frac{8}{7} \right)$$

$$\frac{\langle \vec{u}_3,\ \vec{v}_2 \rangle}{\left\| \vec{v}_2 \right\|^2} \vec{v}_2 = \frac{1}{3} \left[(2,\ 0,\ -1) \bullet (1,\ 1,\ 1) \right] (1,\ 1,\ 1)$$

$$= \frac{1}{3} (1,\ 1,\ 1)$$

$$= \left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3} \right)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3,\ \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3,\ \vec{v}_2 \rangle}{\left\| \vec{v}_2 \right\|^2} \vec{v}_2$$

$$= (2,\ 0,\ -1) - \left(\frac{12}{7},\ -\frac{4}{7},\ -\frac{8}{7} \right) - \left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3} \right)$$

$$= \left(-\frac{1}{21},\ \frac{5}{21},\ -\frac{4}{21} \right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(1, -3, 2) \bullet (-3, 1, 2)}{14}(-3, 1, 2)$$

$$= -\frac{1}{7}(-3, 1, 2)$$

$$= \left(\frac{3}{7}, -\frac{1}{7}, -\frac{2}{7} \right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{(1, -3, 2) \bullet (1, 1, 1)}{3}(1, 1, 1)$$

$$= (0, 0, 0)$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 = \frac{441}{42}\left[(1, -3, 2) \bullet \left(-\frac{1}{21}, \frac{5}{21}, -\frac{4}{21} \right) \right]\left(-\frac{1}{21}, \frac{5}{21}, -\frac{4}{21} \right)$$

$$= \frac{441}{42}\left(-\frac{24}{21} \right)\left(-\frac{1}{21}, \frac{5}{21}, -\frac{4}{21} \right)$$

$$= \left(\frac{4}{7}, -\frac{20}{7}, \frac{16}{7} \right)$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3$$

$$= (1, -3, 2) - \left(\frac{3}{7}, -\frac{1}{7}, -\frac{2}{7} \right) - (0, 0, 0) - \left(\frac{4}{7}, -\frac{20}{7}, \frac{16}{7} \right)$$

$$= (0, 0, 0)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{14}}(-3, 1, 2)$$

$$= \left(-\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}} \right)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{\sqrt{3}}(1, 1, 1)$$

$$= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{21}{\sqrt{42}}\left(-\frac{1}{21}, \ \frac{5}{21}, \ -\frac{4}{21}\right)$$

$$= \left(-\frac{1}{\sqrt{42}}, \ \frac{5}{\sqrt{42}}, \ -\frac{4}{\sqrt{42}}\right)$$

$$\vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

$$= (0, \ 0, \ 0)$$

The *orthogonal* basis: $\left\{(-3, 1, 2), \ (1, 1, 1), \ \left(-\frac{1}{21}, \ \frac{5}{21}, \ -\frac{4}{21}\right), \ (0, 0, 0)\right\}$

The *orthogonal* basis:

$$\left\{\left(-\frac{3}{\sqrt{14}}, \ \frac{1}{\sqrt{14}}, \ \frac{2}{\sqrt{14}}\right), \ \left(\frac{1}{\sqrt{3}}, \ \frac{1}{\sqrt{3}}, \ \frac{1}{\sqrt{3}}\right), \ \left(-\frac{1}{\sqrt{42}}, \ \frac{5}{\sqrt{42}}, \ -\frac{4}{\sqrt{42}}\right), \ (0, 0, 0)\right\}$$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\left\{(2, 1, 1), \ (0, 3, -1), \ (3, -4, -2), \ (-1, -1, 3)\right\}$$

Solution

Let $\vec{u}_1 = (2, 1, 1)$ $\vec{u}_2 = (0, 3, -1)$ $\vec{u}_3 = (3, -4, -2)$ $\vec{u}_4 = (-1, -1, 3)$

$$\vec{v}_1 = \vec{u}_1 = (2, 1, 1)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (0, 3, -1) - \frac{(0, 3, -1) \cdot (2, 1, 1)}{6}(2, 1, 1)$$

$$= (0, 3, -1) - \frac{1}{3}(2, 1, 1)$$

$$= \left(-\frac{2}{3}, \ \frac{8}{3}, \ -\frac{4}{3}\right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(3, -4, -2) \cdot (2, 1, 1)}{6} (2, 1, 1)$$

$$= (0, 0, 0)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{9}{84} \left[(3, -4, -2) \cdot \left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right) \right] \left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right)$$

$$= \frac{3}{28} (-10) \left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right)$$

$$= \left(\frac{5}{7}, -\frac{20}{7}, \frac{10}{7} \right)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (3, -4, -2) - (0, 0, 0) - \left(\frac{5}{7}, -\frac{20}{7}, \frac{10}{7} \right)$$

$$= \left(\frac{16}{7}, -\frac{8}{7}, -\frac{24}{7} \right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(-1, -1, 3) \cdot (2, 1, 1)}{6} (2, 1, 1)$$

$$= (0, 0, 0)$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{9}{84} \left[(-1, -1, 3) \cdot \left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right) \right] \left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right)$$

$$= \frac{3}{28} \left(-\frac{18}{3} \right) \left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right)$$

$$= \left(\frac{3}{7}, -\frac{12}{7}, \frac{6}{7} \right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 = \frac{49}{896} \left[(-1, -1, 3) \cdot \left(\frac{16}{7}, -\frac{8}{7}, -\frac{24}{7} \right) \right] \left(\frac{16}{7}, -\frac{8}{7}, -\frac{24}{7} \right)$$

$$= \frac{7}{128} \left(-\frac{80}{7} \right) \left(\frac{16}{7}, -\frac{8}{7}, -\frac{24}{7} \right)$$

$$= \left(-\frac{10}{7}, \frac{5}{7}, \frac{15}{7} \right)$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3$$

$$= (-1, -1, 3) - (0, 0, 0) - \left(\frac{3}{7}, -\frac{12}{7}, \frac{6}{7} \right) - \left(-\frac{10}{7}, \frac{5}{7}, \frac{15}{7} \right)$$

$$= \underline{(0, 0, 0)}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{6}} (2, 1, 1)$$

$$= \underline{\left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right)}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{3}{2\sqrt{21}} \left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right)$$

$$= \underline{\left(-\frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, -\frac{2}{\sqrt{21}} \right)}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{7}{8\sqrt{14}} \left(\frac{16}{7}, -\frac{8}{7}, -\frac{24}{7} \right)$$

$$= \underline{\left(\frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, -\frac{3}{\sqrt{14}} \right)}$$

$$\vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

$$= \underline{(0, 0, 0)}$$

The *orthogonal* basis: $\left\{ (2, 1, 1), \left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right), \left(\frac{16}{7}, -\frac{8}{7}, -\frac{24}{7} \right), (0, 0, 0) \right\}$

The *orthogonal* basis:

$$\left\{ \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right), \left(-\frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, -\frac{2}{\sqrt{21}} \right), \left(\frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, -\frac{3}{\sqrt{14}} \right), (0, 0, 0) \right\}$$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\{(1,\ 1,\ 1,\ 1),\ (0,\ 1,\ 1,\ 1),\ (0,\ 0,\ 1,\ 1)\}$$

Solution

Let $\quad \vec{u}_1 = (1,\ 1,\ 1,\ 1)\quad \vec{u}_2 = (0,\ 1,\ 1,\ 1)\quad \vec{u}_3 = (0,\ 0,\ 1,\ 1)$

$\vec{v}_1 = \vec{u}_1 = (1,\ 1,\ 1,\ 1)$

$\vec{q}_1 = \dfrac{(1,\ 1,\ 1,\ 1)}{2}$ $\qquad\qquad\qquad \vec{q}_1 = \dfrac{\vec{v}_1}{\left\| \vec{v}_1 \right\|}$

$\quad = \left(\dfrac{1}{2},\ \dfrac{1}{2},\ \dfrac{1}{2},\ \dfrac{1}{2} \right)$

$\vec{v}_2 = \vec{u}_2 - \dfrac{\left\langle \vec{u}_2,\ \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2}\vec{v}_1$

$\quad = (0,\ 1,\ 1,\ 1) - \dfrac{1}{4}\Big[(0,\ 1,\ 1,\ 1) \bullet (1,\ 1,\ 1,\ 1)\Big](1,\ 1,\ 1,\ 1)$

$\quad = (0,\ 1,\ 1,\ 1) - \dfrac{3}{4}(1,\ 1,\ 1,\ 1)$

$\quad = \left(-\dfrac{3}{4},\ \dfrac{1}{4},\ \dfrac{1}{4},\ \dfrac{1}{4} \right)$

$\vec{q}_2 = \dfrac{4}{\sqrt{9+1+1+1}}\left(-\dfrac{3}{4},\ \dfrac{1}{4},\ \dfrac{1}{4},\ \dfrac{1}{4} \right)$ $\qquad \vec{q}_2 = \dfrac{\vec{v}_2}{\left\| \vec{v}_2 \right\|}$

$\quad = \dfrac{2}{\sqrt{3}}\left(-\dfrac{3}{4},\ \dfrac{1}{4},\ \dfrac{1}{4},\ \dfrac{1}{4} \right)$

$\quad = \left(-\dfrac{3}{2\sqrt{3}},\ \dfrac{1}{2\sqrt{3}},\ \dfrac{1}{2\sqrt{3}},\ \dfrac{1}{2\sqrt{3}} \right)$

$\left\langle \vec{u}_3,\ \vec{v}_1 \right\rangle = (0,\ 0,\ 1,\ 1) \bullet (1,\ 1,\ 1,\ 1)$

$\qquad\quad = 2$

$\left\langle \vec{u}_3,\ \vec{v}_2 \right\rangle = (0,\ 0,\ 1,\ 1) \bullet \left(-\dfrac{3}{4},\ \dfrac{1}{4},\ \dfrac{1}{4},\ \dfrac{1}{4} \right)$

$\qquad\quad = \dfrac{1}{2}$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\left\|\vec{v}_1\right\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\left\|\vec{v}_2\right\|^2} \vec{v}_2$$

$$= (0, 0, 1, 1) - \frac{2}{4}(1, 1, 1, 1) - \left(\frac{1}{2}\right)\left(\frac{16}{12}\right)\left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$$

$$= (0, 0, 1, 1) - \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) - \left(-\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$$

$$\underline{= \left(0, -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)}$$

$$\vec{q}_2 = \frac{3}{\sqrt{4+1+1}}\left(0, -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$= \frac{3}{\sqrt{6}}\left(0, -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$\underline{= \left(0, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)}$$

The *orthogonal* basis: $\left\{(1, 1, 1, 1), \left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), \left(0, -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$

The *orthonormal* basis:

$$\left\{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(-\frac{3}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}\right), \left(0, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)\right\}$$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\{(1, 2, -1, 0), (2, 2, 0, 1), (1, 1, -1, 0)\}$$

Solution

Let $\vec{u}_1 = (1, 2, -1, 0)$ $\vec{u}_2 = (2, 2, 0, 1)$ $\vec{u}_3 = (1, 1, -1, 0)$

$$\underline{\vec{v}_1 = \vec{u}_1 = (1, 2, -1, 0)}$$

$$\vec{q}_1 = \frac{(1, 2, -1, 0)}{\sqrt{1+4+1}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$\underline{= \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0\right)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (2, 2, 0, 1) - \frac{1}{6}\big[(2, 2, 0, 1) \cdot (1, 2, -1, 0)\big](1, 2, -1, 0)$$

$$= (2, 2, 0, 1) - (1, 2, -1, 0)$$

$$= (1, 0, 1, 1)$$

$$\vec{q}_2 = \frac{1}{\sqrt{3}}(1, 0, 1, 1) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = (1, 1, -1, 0) \cdot (1, 2, -1, 0)$$

$$= 4$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = (1, 1, -1, 0) \cdot (1, 0, 1, 1)$$

$$= 0$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \, \grave{}$$

$$= (1, 1, -1, 0) - \frac{4}{6}(1, 2, -1, 0) - (0)(1, 0, 1, 1)$$

$$= (1, 1, -1, 0) - \left(\frac{2}{3}, \frac{4}{3}, -\frac{2}{3}, 0\right)$$

$$= \left(\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 0\right)$$

$$\vec{q}_2 = \frac{3}{\sqrt{3}}\left(\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 0\right) \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right)$$

The *orthogonal* basis: $\left\{(1, 2, -1, 0),\ (1, 0, 1, 1),\ \left(\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 0\right)\right\}$

The *orthonormal* basis:

$$\left\{\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0\right),\ \left(\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right),\ \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right)\right\}$$

Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis and an *orthonormal* basis for the subspaces $\mathbb{R}^m$.

$$\{(1,\ 1,\ 1,\ 1),\ (1,\ 2,\ 1,\ 0),\ (1,\ 3,\ 0,\ 0)\}$$

Solution

Let $\ \vec{u}_1 = (1,\ 1,\ 1,\ 1)\quad \vec{u}_2 = (1,\ 2,\ 1,\ 0)\quad \vec{u}_3 = (1,\ 3,\ 0,\ 0)$

$$\vec{v}_1 = \vec{u}_1 = (1,\ 1,\ 1,\ 1)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ 1,\ 1,\ 1)}{\sqrt{4}}$$

$$= \left(\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2,\ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (1,\ 2,\ 1,\ 0) - \frac{(1,\ 2,\ 1,\ 0)\cdot(1,\ 1,\ 1,\ 1)}{\|(1,\ 1,\ 1,\ 1)\|^2}(1,\ 1,\ 1,\ 1)$$

$$= (1,\ 2,\ 1,\ 0) - \frac{1+2+1}{4}(1,\ 1,\ 1,\ 1)$$

$$= (1,\ 2,\ 1,\ 0) - \frac{4}{4}(1,\ 1,\ 1,\ 1)$$

$$= (1,\ 2,\ 1,\ 0) - (1,\ 1,\ 1,\ 1)$$

$$= (0,\ 1,\ 0,\ -1)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{(0,\ 1,\ 0,\ -1)}{\sqrt{1+1}}$$

$$= \left(0,\ \tfrac{1}{\sqrt{2}},\ 0,\ -\tfrac{1}{\sqrt{2}}\right)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3,\ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3,\ \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (1,\ 3,\ 0,\ 0) - \frac{(1,\ 3,\ 0,\ 0) \bullet (1,\ 1,\ 1,\ 1)}{\|(1,\ 1,\ 1,\ 1)\|^2}(1,\ 1,\ 1,\ 1) - \frac{(1,\ 3,\ 0,\ 0) \bullet (0,\ 1,\ 0,\ -1)}{\|(0,\ 1,\ 0,\ -1)\|^2}(0,\ 1,\ 0,\ -1)$$

$$= (1,\ 3,\ 0,\ 0) - \frac{4}{4}(1,\ 1,\ 1,\ 1) - \frac{3}{2}(0,\ 1,\ 0,\ -1)$$

$$= (1,\ 3,\ 0,\ 0) - (1,\ 1,\ 1,\ 1) - \left(0,\ \frac{3}{2},\ 0,\ -\frac{3}{2}\right)$$

$$= \left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)$$

$$\vec{q}_3 = \frac{1}{\sqrt{\frac{1}{4}+1+\frac{1}{4}}}\left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right) \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{2}{\sqrt{6}}\left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)$$

$$= \left(0,\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

The *orthogonal* basis: $\left\{(1,\ 1,\ 1,\ 1),\ (0,\ 1,\ 0,\ -1),\ \left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)\right\}$

The *orthonormal* basis: $\left\{\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right),\ \left(0,\ \frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right),\ \left(0,\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\left\{(0,\ 2,\ -1,\ 1),\ (0,\ 0,\ 1,\ 1),\ (-2,\ 1,\ 1,\ -1)\right\}$$

Solution

Let $\quad \vec{u}_1 = (0,\ 2,\ -1,\ 1) \quad \vec{u}_2 = (0,\ 0,\ 1,\ 1) \quad \vec{u}_3 = (-2,\ 1,\ 1,\ -1)$

$$\vec{v}_1 = \vec{u}_1 = (0,\ 2,\ -1,\ 1)$$

$$\vec{q}_1 = \frac{(0,\ 2,\ -1,\ 1)}{\sqrt{6}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (0, 0, 1, 1) - \frac{1}{6} \Big[(0, 0, 1, 1) \cdot (0, 2, -1, 1) \Big] (0, 2, -1, 1)$$

$$= (0, 0, 1, 1) - \frac{1}{6}(0)(0, 2, -1, 1)$$

$$= (0, 0, 1, 1) \;\big|$$

$$\vec{q}_2 = \frac{(0, 0, 1, 1)}{\sqrt{2}} \qquad\qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \big|$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = (-2, 1, 1, -1) \cdot (0, 2, -1, 1)$$

$$= 0 \;\big|$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = (-2, 1, 1, -1) \cdot (0, 0, 1, 1)$$

$$= 0 \;\big|$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (-2, 1, 1, -1) - 0 - 0$$

$$= (-2, 1, 1, -1) \;\big|$$

$$\vec{q}_3 = \frac{(-2, 1, 1, -1)}{\sqrt{(-2)^2 + 1^2 + 1^2 + (-1)^2}} \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{(-2, 1, 1, -1)}{\sqrt{7}}$$

$$= \left(-\frac{2}{\sqrt{7}}, \frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}, -\frac{1}{\sqrt{7}} \right) \big|$$

The *orthogonal* basis: $\quad \left\{ (0, 2, -1, 1), (0, 0, 1, 1), (-2, 1, 1, -1) \right\}$

The *orthonormal* basis:

$$\left\{ \left(0, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right), \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \left(-\frac{2}{\sqrt{7}}, \frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}, -\frac{1}{\sqrt{7}} \right) \right\}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\left\{(-1,\ 3,\ 1,\ 1),\ (6,\ -8,\ -2,\ -4),\ (6,\ 3,\ 6,\ -3)\right\}$$

Solution

Let $\vec{u}_1 = (-1,\ 3,\ 1,\ 1)$ $\vec{u}_2 = (6,\ -8,\ -2,\ -4)$ $\vec{u}_3 = (6,\ 3,\ 6,\ -3)$

$\vec{v}_1 = \vec{u}_1 = (-1,\ 3,\ 1,\ 1)$

$$\vec{q}_1 = \frac{(-1,\ 3,\ 1,\ 1)}{\sqrt{1+9+1+1}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \left(-\frac{1}{2\sqrt{3}},\ \frac{3}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2,\ \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2} \vec{v}_1$$

$$= (6,\ -8,\ -2,\ -4) - \frac{1}{12}\left[(6,\ -8,\ -2,\ -4)\bullet(-1,\ 3,\ 1,\ 1)\right](-1,\ 3,\ 1,\ 1)$$

$$= (6,\ -8,\ -2,\ -4) - \frac{1}{12}(-36)(-1,\ 3,\ 1,\ 1)$$

$$= (6,\ -8,\ -2,\ -4) + (-3,\ 9,\ 3,\ 3)$$

$$= (3,\ 1,\ 1,\ -1)$$

$$\vec{q}_2 = \frac{1}{\sqrt{12}}(3,\ 1,\ 1,\ -1) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \left(\frac{3}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}},\ \frac{1}{2\sqrt{3}},\ -\frac{1}{2\sqrt{3}}\right)$$

$$\left\langle \vec{u}_3,\ \vec{v}_1 \right\rangle = (6,\ 3,\ 6,\ -3)\bullet(-1,\ 3,\ 1,\ 1)$$

$$= 6$$

$$\left\langle \vec{u}_3,\ \vec{v}_2 \right\rangle = (6,\ 3,\ 6,\ -3)\bullet(3,\ 1,\ 1,\ -1)$$

$$= 30$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\left\langle \vec{u}_3,\ \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2} \vec{v}_1 - \frac{\left\langle \vec{u}_3,\ \vec{v}_2 \right\rangle}{\left\|\vec{v}_2\right\|^2} \vec{v}_2 \ `$$

$$= (6, 3, 6, -3) - \frac{6}{12}(-1, 3, 1, 1) - \frac{30}{12}(3, 1, 1, -1)$$

$$= (6, 3, 6, -3) - \left(-\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right) - \left(\frac{15}{2}, \frac{5}{2}, \frac{5}{2}, -\frac{5}{2}\right)$$

$$\underline{= (-1, -1, 3, -1)}$$

$$\vec{q}_3 = \frac{1}{2\sqrt{3}}(-1, -1, 3, -1) \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$\underline{= \left(-\frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, \frac{3}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}\right)}$$

The *orthogonal* basis: $\quad \{(-1, 3, 1, 1), (3, 1, 1, -1), (-1, -1, 3, -1)\}$

The *orthonormal* basis:

$$\left\{ \left(-\frac{1}{2\sqrt{3}}, \frac{3}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}\right), \left(\frac{3}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}\right), \right.$$
$$\left. \left(-\frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, \frac{3}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}\right) \right\}$$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\{(0, 0, 2, 2), (3, 3, 0, 0), (1, 1, 0, -1)\}$$

Solution

Let $\quad \vec{u}_1 = (0, 0, 2, 2) \quad \vec{u}_2 = (3, 3, 0, 0) \quad \vec{u}_3 = (1, 1, 0, -1)$

$$\underline{\vec{v}_1 = \vec{u}_1 = (0, 0, 2, 2)}$$

$$\vec{q}_1 = \frac{(0, 0, 2, 2)}{\sqrt{4+4}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(0, 0, 2, 2)}{2\sqrt{2}}$$

$$\underline{= \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (3,\ 3,\ 0,\ 0) - \frac{1}{8}\Big[(3,\ 3,\ 0,\ 0) \cdot (0,\ 0,\ 2,\ 2)\Big](0,\ 0,\ 2,\ 2)$$

$$= (3,\ 3,\ 0,\ 0) - \frac{1}{8}(0)\ (0,\ 0,\ 2,\ 2)$$

$$= (3,\ 3,\ 0,\ 0)$$

$$\vec{q}_2 = \frac{1}{3\sqrt{2}}(3,\ 3,\ 0,\ 0) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = (1,\ 1,\ 0,\ -1) \cdot (0,\ 0,\ 2,\ 2)$$

$$= -2$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = (1,\ 1,\ 0,\ -1) \cdot (3,\ 3,\ 0,\ 0)$$

$$= 6$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \grave{}$$

$$= (1,\ 1,\ 0,\ -1) + \frac{2}{8}(0,\ 0,\ 2,\ 2) - \frac{6}{18}(3,\ 3,\ 0,\ 0)$$

$$= (1,\ 1,\ 0,\ -1) + \left(0,\ 0,\ \frac{1}{2},\ \frac{1}{2}\right) - (1,\ 1,\ 0,\ 0)$$

$$= \left(0,\ 0,\ \frac{1}{2},\ -\frac{1}{2}\right)$$

$$\vec{q}_3 = \frac{1}{\sqrt{\frac{1}{4} + \frac{1}{4}}}\left(0,\ 0,\ \frac{1}{2},\ -\frac{1}{2}\right) \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{2}{\sqrt{2}}\left(0,\ 0,\ \frac{1}{2},\ -\frac{1}{2}\right)$$

$$= \left(0,\ 0,\ \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right)$$

The *orthogonal* basis: $\left\{(0,\ 0,\ 2,\ 2),\ (3,\ 3,\ 0,\ 0),\ \left(0,\ 0,\ \frac{1}{2},\ -\frac{1}{2}\right)\right\}$

The *orthonormal* basis: $\left\{\left(0,\ 0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right),\ \left(0,\ 0,\ \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right)\right\}$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(1,\ 0,\ 0,\ 0),\quad (1,\ 1,\ 0,\ 0),\quad (1,\ 1,\ 1,\ 0),\quad (1,\ 1,\ 1,\ 1)\}$$

Solution

Let $\vec{u}_1 = (1,\ 0,\ 0,\ 0)$ $\vec{u}_2 = (1,\ 1,\ 0,\ 0)$ $\vec{u}_3 = (1,\ 1,\ 1,\ 0)$ $\vec{u}_4 = (1,\ 1,\ 1,\ 1)$

$\underline{\vec{v}_1 = \vec{u}_1 = (1,\ 0,\ 0,\ 0)}$

$$\vec{q}_1 = \frac{(1,\ 0,\ 0,\ 0)}{1} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\underline{= (1,\ 0,\ 0,\ 0)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2,\ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (1,\ 1,\ 0,\ 0) - \big[(1,\ 1,\ 0,\ 0) \bullet (1,\ 0,\ 0,\ 0)\big] (1,\ 0,\ 0,\ 0)$$

$$= (1,\ 1,\ 0,\ 0) - (1,\ 0,\ 0,\ 0)$$

$$\underline{= (0,\ 1,\ 0,\ 0)}$$

$$\vec{q}_2 = \frac{1}{1}(0,\ 1,\ 0,\ 0) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$\underline{= (0,\ 1,\ 0,\ 0)}$$

$$\langle \vec{u}_3,\ \vec{v}_1 \rangle = (1,\ 1,\ 1,\ 0) \bullet (1,\ 0,\ 0,\ 0)$$

$$\underline{= 1}$$

$$\langle \vec{u}_3,\ \vec{v}_2 \rangle = (1,\ 1,\ 1,\ 0) \bullet (0,\ 1,\ 0,\ 0)$$

$$\underline{= 1}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3,\ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3,\ \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \quad \text{`}$$

$$= (1,\ 1,\ 1,\ 0) - \frac{1}{1}(1,\ 0,\ 0,\ 0) - \frac{1}{1}(0,\ 1,\ 0,\ 0)$$

$$= (1,\ 1,\ 1,\ 0) - (1,\ 0,\ 0,\ 0) - (0\ 1,\ 0,\ 0)$$

$$\underline{= (0,\ 0,\ 1,\ 0)}$$

$$\vec{q}_3 = \frac{1}{1}(0,\ 0,\ 1,\ 0) \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= (0,\ 0,\ 1,\ 0)$$

$$\frac{\langle \vec{u}_4,\ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(1,\ 1,\ 1,\ 1)\bullet(1,\ 0,\ 0,\ 0)}{1}(1,\ 0,\ 0,\ 0)$$

$$= (1,\ 0,\ 0,\ 0)$$

$$\frac{\langle \vec{u}_4,\ \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \big[(1,\ 1,\ 1,\ 1)\bullet(0,\ 1,\ 0,\ 0)\big](0,\ 1,\ 0,\ 0)$$

$$= (0,\ 1,\ 0,\ 0)$$

$$\frac{\langle \vec{u}_4,\ \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3 = \big[(1,\ 1,\ 1,\ 1)\bullet(0,\ 0,\ 1,\ 0)\big](0,\ 0,\ 1,\ 0)$$

$$= (0,\ 0,\ 1,\ 0)$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4,\ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4,\ \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4,\ \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$

$$= (1,\ 1,\ 1,\ 1) - (1,\ 0,\ 0,\ 0) - (0,\ 1,\ 0,\ 0) - (0,\ 0,\ 1,\ 0)$$

$$= (0,\ 0,\ 0,\ 1)$$

$$\vec{q}_4 = (0,\ 0,\ 0,\ 1) \qquad\qquad \vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

The *orthogonal* basis: $\quad \{(1,\ 0,\ 0,\ 0),\ (0,\ 1,\ 0,\ 0),\ (0,\ 0,\ 1,\ 0),\ (0,\ 0,\ 0,\ 1)\}$

The *orthonormal* basis: $\quad \{(1,\ 0,\ 0,\ 0),\ (0,\ 1,\ 0,\ 0),\ (0,\ 0,\ 1,\ 0),\ (0,\ 0,\ 0,\ 1)\}$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces $\mathbb{R}^m$.

$$\{(3,\ 8,\ 7,\ -3),\ (1,\ 5,\ 3,\ -1),\ (2,\ -1,\ 2,\ 6),\ (1,\ 4,\ 0,\ 3)\}$$

Solution

Let $\quad \vec{u}_1 = (3,\ 8,\ 7,\ -3)\quad \vec{u}_2 = (1,\ 5,\ 3,\ -1)\quad \vec{u}_3 = (2,\ -1,\ 2,\ 6)\quad \vec{u}_4 = (1,\ 4,\ 0,\ 3)$

$\vec{v}_1 = \vec{u}_1 = (3,\ 8,\ 7,\ -3)$

$$\vec{q}_1 = \frac{(3,\ 8,\ 7,\ -3)}{\sqrt{9+64+49+9}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \frac{(3,\ 8,\ 7,\ -3)}{\sqrt{131}}$$

$$= \left(\frac{3}{\sqrt{131}},\ \frac{8}{\sqrt{131}},\ \frac{7}{\sqrt{131}},\ -\frac{3}{\sqrt{131}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2,\ \vec{v}_1\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$= (1,\ 5,\ 3,\ -1) - \frac{1}{131}\left[(1,\ 5,\ 3,\ -1)\cdot(3,\ 8,\ 7,\ -3)\right](3,\ 8,\ 7,\ -3)$$

$$= (1,\ 5,\ 3,\ -1) - \frac{67}{131}(3,\ 8,\ 7,\ -3)$$

$$= (1,\ 5,\ 3,\ -1) - \left(\frac{201}{131},\ \frac{536}{131},\ \frac{469}{131},\ -\frac{201}{131}\right)$$

$$= \left(-\frac{70}{131},\ \frac{119}{131},\ -\frac{76}{131},\ \frac{70}{131}\right)$$

$$\vec{q}_2 = \frac{131}{\sqrt{70^2 + 119^2 + 76^2 + 70^2}}\left(-\frac{70}{131},\ \frac{119}{131},\ -\frac{76}{131},\ \frac{70}{131}\right) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \frac{131}{\sqrt{29,737}}\left(-\frac{70}{131},\ \frac{119}{131},\ -\frac{76}{131},\ \frac{70}{131}\right)$$

$$= \left(-\frac{70}{\sqrt{29,737}},\ \frac{119}{\sqrt{29,737}},\ -\frac{76}{\sqrt{29,737}},\ \frac{70}{\sqrt{29,737}}\right)$$

$$\langle \vec{u}_3,\ \vec{v}_1\rangle = (2,\ -1,\ 2,\ 6)\cdot(3,\ 8,\ 7,\ -3)$$

$$= -6$$

$$\langle \vec{u}_3,\ \vec{v}_2\rangle = (2,\ -1,\ 2,\ 6)\cdot\left(-\frac{70}{131},\ \frac{119}{131},\ -\frac{76}{131},\ \frac{70}{131}\right)$$

$$= \frac{-140 - 119 - 152 + 420}{131}$$

$$= \underline{\frac{9}{131}}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 ,$$

$$= (2, -1, 2, 6) + \frac{6}{131}(3, 8, 7, -3) - \frac{9}{131}\left(\frac{131^2}{29,737}\right)\left(-\frac{70}{131}, \frac{119}{131}, -\frac{76}{131}, \frac{70}{131}\right)$$

$$= (2, -1, 2, 6) + \left(\frac{18}{131}, \frac{48}{131}, \frac{42}{131}, -\frac{18}{131}\right) - \frac{9}{131}\left(\frac{131}{227}\right)\left(-\frac{70}{131}, \frac{119}{131}, -\frac{76}{131}, \frac{70}{131}\right)$$

$$= \left(\frac{280}{131}, -\frac{83}{131}, \frac{304}{131}, \frac{768}{131}\right) - \left(-\frac{630}{29,737}, \frac{1,071}{29,737}, -\frac{684}{29,737}, \frac{630}{29,737}\right)$$

$$= \left(\frac{64,190}{29,737}, -\frac{19,912}{29,737}, \frac{69,692}{29,737}, \frac{173,706}{29,737}\right)$$

$$= \underline{\left(\frac{490}{227}, -\frac{152}{227}, \frac{532}{227}, \frac{1,326}{227}\right)}$$

$$\vec{q}_3 = \frac{227}{\sqrt{490^2 + 152^2 + 532^2 + 1,326^2}}\left(\frac{490}{227}, -\frac{152}{227}, \frac{532}{227}, \frac{1,326}{227}\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{227}{\sqrt{2,304,504}}\left(\frac{490}{227}, -\frac{152}{227}, \frac{532}{227}, \frac{1,326}{227}\right)$$

$$= \frac{1}{6\sqrt{64,014}}(490, -152, 532, 1,326)$$

$$= \underline{\left(\frac{245}{3\sqrt{64,014}}, -\frac{76}{3\sqrt{64,014}}, \frac{266}{3\sqrt{64,014}}, \frac{221}{\sqrt{64,014}}\right)}$$

$$\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(1, 4, 0, 3) \cdot (3, 8, 7, -3)}{131}(3, 8, 7, -3)$$

$$= \frac{26}{131}(3, 8, 7, -3)$$

$$= \underline{\left(\frac{78}{131}, \frac{208}{131}, \frac{182}{131}, -\frac{78}{131}\right)}$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{131^2}{29,737}\left[(1, 4, 0, 3) \cdot \left(-\frac{70}{131}, \frac{119}{131}, -\frac{76}{131}, \frac{70}{131}\right)\right]\left(-\frac{70}{131}, \frac{119}{131}, -\frac{76}{131}, \frac{70}{131}\right)$$

$$= \frac{131}{29,737}\left(\frac{-70 + 476 + 210}{131}\right)(-70, 119, -76, 70)$$

$$= \frac{616}{29,737} \left(-70,\ 119,\ -76,\ 70\right)$$

$$= \left(-\frac{43,120}{29,737},\ \frac{73,304}{29,737},\ -\frac{46,816}{29,737},\ \frac{43,120}{29,737}\right)$$

$$\frac{\langle \vec{u}_4,\ \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3 = \frac{227^2}{2,304,504} \left[\left(1,\ 4,\ 0,\ 3\right) \cdot \left(\frac{490}{227},\ -\frac{152}{227},\ \frac{532}{227},\ \frac{1,326}{227}\right)\right] \left(\frac{490}{227},\ -\frac{152}{227},\ \frac{532}{227},\ \frac{1,326}{227}\right)$$

$$= \frac{227}{2,304,504} \left(\frac{490 - 608 + 3,978}{227}\right) \left(490,\ -152,\ 532,\ 1,326\right)$$

$$= \frac{3,860}{2,304,504} \left(490,\ -152,\ 532,\ 1,326\right)$$

$$= \frac{965}{576,126} \left(490,\ -152,\ 532,\ 1,326\right)$$

$$= \left(\frac{472,850}{576,126},\ -\frac{146,680}{576,126},\ \frac{513,380}{576,126},\ \frac{1,279,590}{576,126}\right)$$

$$= \left(\frac{236,425}{288,063},\ -\frac{73,340}{288,063},\ \frac{256,690}{288,063},\ \frac{213,265}{96,021}\right)$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4,\ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4,\ \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4,\ \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$

$$= \left(1,\ 4,\ 0,\ 3\right) - \left(\frac{78}{131},\ \frac{208}{131},\ \frac{182}{131},\ -\frac{78}{131}\right) - \left(-\frac{43,120}{29,737},\ \frac{73,304}{29,737},\ -\frac{46,816}{29,737},\ \frac{43,120}{29,737}\right)$$

$$- \left(\frac{236,425}{288,063},\ -\frac{73,340}{288,063},\ \frac{256,690}{288,063},\ \frac{213,265}{96,021}\right)$$

$$= \left(\frac{53}{131},\ \frac{316}{131},\ -\frac{182}{131},\ \frac{471}{131}\right) - \left(-\frac{43,120}{29,737},\ \frac{73,304}{29,737},\ -\frac{46,816}{29,737},\ \frac{43,120}{29,737}\right)$$

$$- \left(\frac{236,425}{288,063},\ -\frac{73,340}{288,063},\ \frac{256,690}{288,063},\ \frac{213,265}{96,021}\right)$$

$$= \left(\frac{55,151}{29,737},\ -\frac{1,572}{29,737},\ \frac{5,502}{29,737},\ \frac{63,797}{29,737}\right) - \left(\frac{236,425}{288,063},\ -\frac{73,340}{288,063},\ \frac{256,690}{288,063},\ \frac{213,265}{96,021}\right)$$

$$= \left(\frac{421}{227},\ -\frac{12}{227},\ \frac{42}{227},\ \frac{487}{227}\right) - \left(\frac{236,425}{288,063},\ -\frac{73,340}{288,063},\ \frac{256,690}{288,063},\ \frac{213,265}{96,021}\right)$$

$$= \left(\frac{297,824}{288,063},\ \frac{58,112}{288,063},\ -\frac{203,392}{288,063},\ -\frac{7,264}{96,021}\right)$$

$$= \left(\frac{1,312}{1,269}, \ \frac{256}{1,269}, \ -\frac{896}{1,269}, \ -\frac{32}{423} \right)$$

$$\| \vec{v}_4 \| = \frac{1}{1269} \sqrt{ (1,312)^2 + (256)^2 + (896)^2 + (96)^2 }$$

$$= \frac{\sqrt{2,598,912}}{1269}$$

$$= \frac{96 \sqrt{282}}{1269}$$

$$\vec{q}_4 = \frac{1269}{96 \sqrt{282}} \left(\frac{1,312}{1,269}, \ \frac{256}{1,269}, \ -\frac{896}{1,269}, \ -\frac{32}{423} \right) \qquad \vec{q}_4 = \frac{\vec{v}_4}{\| \vec{v}_4 \|}$$

$$= \left(\frac{41}{3 \sqrt{282}}, \ \frac{8}{3 \sqrt{282}}, \ -\frac{28}{3 \sqrt{282}}, \ -\frac{1}{\sqrt{282}} \right)$$

The *orthogonal* basis:

$$\left\{ \begin{array}{l} (3, \ 8, \ 7, \ -3), \ \left(-\frac{70}{131}, \ \frac{119}{131}, \ -\frac{76}{131}, \ \frac{70}{131} \right), \ \left(\frac{490}{227}, \ -\frac{152}{227}, \ \frac{532}{227}, \ \frac{1,326}{227} \right), \\[2mm] \left(\frac{1,312}{1,269}, \ \frac{256}{1,269}, \ -\frac{896}{1,269}, \ -\frac{32}{423} \right) \end{array} \right\}$$

The *orthonormal* basis:

$$\left\{ \begin{array}{l} \left(\frac{3}{\sqrt{131}}, \ \frac{8}{\sqrt{131}}, \ \frac{7}{\sqrt{131}}, \ -\frac{3}{\sqrt{131}} \right), \ \left(-\frac{70}{\sqrt{29,737}}, \ \frac{119}{\sqrt{29,737}}, \ -\frac{76}{\sqrt{29,737}}, \ \frac{70}{\sqrt{29,737}} \right), \\[2mm] \left(\frac{245}{3 \sqrt{64,014}}, \ -\frac{76}{3 \sqrt{64,014}}, \ \frac{266}{3 \sqrt{64,014}}, \ \frac{221}{\sqrt{64,014}} \right), \\[2mm] \left(\frac{41}{3 \sqrt{282}}, \ \frac{8}{3 \sqrt{282}}, \ -\frac{28}{3 \sqrt{282}}, \ -\frac{1}{\sqrt{282}} \right) \end{array} \right\}$$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\left\{ (0, \ 3, \ -3, \ -6), \ (-2, \ 0, \ 0, \ -6), \ (0, \ -4, \ -2, \ -2), \ (0, \ -8, \ 4, \ -4) \right\}$$

Solution

Let

$$\vec{u}_1 = (0, \ 3, \ -3, \ -6) \quad \vec{u}_2 = (-2, \ 0, \ 0, \ -6) \quad \vec{u}_3 = (0, \ -4, \ -2, \ -2) \quad \vec{u}_4 = (0, \ -8, \ 4, \ -4)$$

$$\vec{v}_1 = \vec{u}_1 = (0,\ 3,\ -3,\ -6)$$

$$\vec{q}_1 = \frac{(0,\ 3,\ -3,\ -6)}{\sqrt{9+9+36}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(0,\ 3,\ -3,\ -6)}{3\sqrt{6}}$$

$$= \left(0,\ \frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2,\ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (-2,\ 0,\ 0,\ -6) - \frac{1}{54}\big[(-2,\ 0,\ 0,\ -6)\cdot(0,\ 3,\ -3,\ -6)\big](0,\ 3,\ -3,\ -6)$$

$$= (-2,\ 0,\ 0,\ -6) - \frac{2}{3}(0,\ 3,\ -3,\ -6)$$

$$= (-2,\ 0,\ 0,\ -6) - (0,\ 2,\ -2,\ -4)$$

$$= (-2,\ -2,\ 2,\ -2)$$

$$\vec{q}_2 = \frac{1}{\sqrt{4+4+4+4}}(-2,\ -2,\ 2,\ -2) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{4}(-2,\ -2,\ 2,\ -2)$$

$$= \left(-\frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2}\right)$$

$$\langle \vec{u}_3,\ \vec{v}_1 \rangle = (0,\ -4,\ -2,\ -2)\cdot(0,\ 3,\ -3,\ -6)$$

$$= 6$$

$$\langle \vec{u}_3,\ \vec{v}_2 \rangle = (0,\ -4,\ -2,\ -2)\cdot(-2,\ -2,\ 2,\ -2)$$

$$= 8$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3,\ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3,\ \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 \ ^\backprime$$

$$= (0,\ -4,\ -2,\ -2) - \frac{6}{54}(0,\ 3,\ -3,\ -6) - \frac{8}{16}(-2,\ -2,\ 2,\ -2)$$

$$= (0,\ -4,\ -2,\ -2) - \left(0,\ \frac{1}{3},\ -\frac{1}{3},\ -\frac{2}{3}\right) - (-1,\ -1,\ 1,\ -1)$$

$$= \left(1, \ -\frac{10}{3}, \ -\frac{8}{3}, \ -\frac{1}{3}\right)$$

$$\vec{q}_3 = \frac{3}{\sqrt{9+100+64+1}}\left(1, \ -\frac{10}{3}, \ -\frac{8}{3}, \ -\frac{1}{3}\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{3}{\sqrt{174}}\left(1, \ -\frac{10}{3}, \ -\frac{8}{3}, \ -\frac{1}{3}\right)$$

$$= \left(\frac{3}{\sqrt{174}}, \ -\frac{10}{\sqrt{174}}, \ -\frac{8}{\sqrt{174}}, \ -\frac{1}{\sqrt{174}}\right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(0, \ -8, \ 4, \ -4) \bullet (0, \ 3, \ -3, \ -6)}{54}(0, \ 3, \ -3, \ -6)$$

$$= -\frac{12}{54}(0, \ 3, \ -3, \ -6)$$

$$= \left(0, \ -\frac{2}{3}, \ \frac{2}{3}, \ \frac{4}{3}\right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{1}{16}\left[(0, \ -8, \ 4, \ -4) \bullet (-2, \ -2, \ 2, \ -2)\right](-2, \ -2, \ 2, \ -2)$$

$$= \frac{32}{16}(-2, \ -2, \ 2, \ -2)$$

$$= (-4, \ -4, \ 4, \ -4)$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3 = \frac{9}{174}\left[(0, \ -8, \ 4, \ -4) \bullet \left(1, \ -\frac{10}{3}, \ -\frac{8}{3}, \ -\frac{1}{3}\right)\right]\left(1, \ -\frac{10}{3}, \ -\frac{8}{3}, \ -\frac{1}{3}\right)$$

$$= \frac{9}{174}\left(\frac{80-32+4}{3}\right)\left(1, \ -\frac{10}{3}, \ -\frac{8}{3}, \ -\frac{1}{3}\right)$$

$$= \frac{9}{174}\left(\frac{52}{3}\right)\left(1, \ -\frac{10}{3}, \ -\frac{8}{3}, \ -\frac{1}{3}\right)$$

$$= \frac{78}{87}\left(1, \ -\frac{10}{3}, \ -\frac{8}{3}, \ -\frac{1}{3}\right)$$

$$= \left(\frac{26}{29}, \ -\frac{260}{87}, \ -\frac{208}{87}, \ -\frac{26}{87}\right)$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$

$$= (0, \ -8, \ 4, \ -4) - \left(0, \ -\frac{2}{3}, \ \frac{2}{3}, \ \frac{4}{3}\right) - (-4, \ -4, \ 4, \ -4) - \left(\frac{26}{29}, \ -\frac{260}{87}, \ -\frac{208}{87}, \ -\frac{26}{87}\right)$$

$$= (4, \ -4, \ 0, \ 0) + \left(-\frac{26}{29}, \ \frac{318}{87}, \ \frac{150}{87}, \ -\frac{30}{29} \right)$$

$$= \left(\frac{90}{29}, \ -\frac{10}{29}, \ \frac{50}{29}, \ -\frac{30}{29} \right)$$

$$\| \vec{v}_4 \| = \sqrt{ \left(\frac{90}{29} \right)^2 + \left(\frac{10}{29} \right)^2 + \left(\frac{50}{29} \right)^2 + \left(\frac{90}{87} \right)^2 }$$

$$= \sqrt{ \frac{8,100}{841} + \frac{100}{841} + \frac{2,500}{841} + \frac{8,100}{7,569} }$$

$$= 10 \sqrt{ \frac{729 + 9 + 225 + 81}{7,569} }$$

$$= \frac{10}{87} \sqrt{1,044}$$

$$= \frac{20}{29} \sqrt{29}$$

$$\vec{q}_4 = \frac{29}{20} \frac{1}{\sqrt{29}} \left(\frac{90}{29}, \ -\frac{10}{29}, \ \frac{50}{29}, \ -\frac{90}{87} \right) \qquad\qquad \vec{q}_4 = \frac{\vec{v}_4}{\| \vec{v}_4 \|}$$

$$= \left(\frac{9}{2\sqrt{29}}, \ -\frac{1}{2\sqrt{29}}, \ \frac{5}{2\sqrt{29}}, \ -\frac{3}{2\sqrt{29}} \right)$$

The *orthogonal* basis:

$$\left\{ \begin{matrix} (0, \ 3, \ -3, \ -6), \ (-2, \ -2, \ 2, \ -2), \ \left(1, \ -\frac{10}{3}, \ -\frac{8}{3}, \ -\frac{1}{3} \right), \\ \left(\frac{90}{29}, \ -\frac{10}{29}, \ \frac{50}{29}, \ -\frac{30}{29} \right) \end{matrix} \right\}$$

The *orthonormal* basis:

$$\left\{ \begin{matrix} \left(0, \ \frac{1}{\sqrt{6}}, \ -\frac{1}{\sqrt{6}}, \ -\frac{2}{\sqrt{6}} \right), \ \left(-\frac{1}{2}, \ -\frac{1}{2}, \ \frac{1}{2}, \ -\frac{1}{2} \right), \\ \left(\frac{3}{\sqrt{174}}, \ -\frac{10}{\sqrt{174}}, \ -\frac{8}{\sqrt{174}}, \ -\frac{1}{\sqrt{174}} \right), \ \left(\frac{9}{2\sqrt{29}}, \ -\frac{1}{2\sqrt{29}}, \ \frac{5}{2\sqrt{29}}, \ -\frac{3}{2\sqrt{29}} \right) \end{matrix} \right\}$$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\{ (3, \ 0, \ -3, \ 6), \ (0, \ 2, \ 3, \ 1), \ (0, \ -2, \ -2, \ 0), \ (-2, \ 1, \ 2, \ 1) \}$$

Solution

Let $\quad \vec{u}_1 = (3, \ 0, \ -3, \ 6) \quad \vec{u}_2 = (0, \ 2, \ 3, \ 1) \quad \vec{u}_3 = (0, \ -2, \ -2, \ 0) \quad \vec{u}_4 = (-2, \ 1, \ 2, \ 1)$

$$\vec{v}_1 = \vec{u}_1 = (3,\ 0,\ -3,\ 6)$$

$$\vec{q}_1 = \frac{(3,\ 0,\ -3,\ 6)}{\sqrt{9+9+36}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \frac{(3,\ 0,\ -3,\ 6)}{3\sqrt{6}}$$

$$= \left(\frac{1}{\sqrt{6}},\ 0,\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2,\ \vec{v}_1 \rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$= (0,\ 2,\ 3,\ 1) - \frac{1}{54}\Big[(0,\ 2,\ 3,\ 1)\bullet(3,\ 0,\ -3,\ 6)\Big](3,\ 0,\ -3,\ 6)$$

$$= (0,\ 2,\ 3,\ 1) + \frac{1}{18}\ (3,\ 0,\ -3,\ 6)$$

$$= (0,\ 2,\ 3,\ 1) + \left(\frac{1}{6},\ 0,\ -\frac{1}{6},\ \frac{1}{3}\right)$$

$$= \left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right)$$

$$\vec{q}_2 = \frac{1}{\sqrt{\frac{1}{36}+4+\frac{289}{36}+\frac{16}{9}}}\ \left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \frac{1}{\sqrt{\frac{498}{36}}}\ \left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right)$$

$$= \frac{6}{\sqrt{498}}\ \left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right)$$

$$= \left(\frac{1}{\sqrt{498}},\ \frac{12}{\sqrt{498}},\ \frac{17}{\sqrt{498}},\ \frac{8}{\sqrt{498}}\right)$$

$$\frac{\langle \vec{u}_3,\ \vec{v}_1 \rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1 = \frac{1}{54}\Big[(0,\ -2,\ -2,\ 0)\bullet(3,\ 0,\ -3,\ 6)\Big](3,\ 0,\ -3,\ 6)$$

$$= \frac{6}{54}\ (3,\ 0,\ -3,\ 6)$$

$$= \left(\frac{1}{3},\ 0,\ -\frac{1}{3},\ \frac{2}{3}\right)$$

$$\frac{\langle \vec{u}_3,\ \vec{v}_2 \rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2 = \frac{36}{498}\Big[(0,\ -2,\ -2,\ 0)\bullet\left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right)\Big]\left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right)$$

$$= \frac{6}{83}\left(-\frac{29}{3}\right)\left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right)$$

$$= -\frac{58}{83}\left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right)$$

$$= \left(-\frac{29}{249},\ -\frac{116}{83},\ -\frac{493}{249},\ -\frac{232}{249}\right)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3,\ \vec{v}_1 \rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3,\ \vec{v}_2 \rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2 \ ,$$

$$= (0,\ -2,\ -2,\ 0) - \left(\frac{1}{3},\ 0,\ -\frac{1}{3},\ \frac{2}{3}\right) - \left(-\frac{29}{249},\ -\frac{116}{83},\ -\frac{493}{249},\ -\frac{232}{249}\right)$$

$$= \left(-\frac{1}{3},\ -2,\ -\frac{5}{3},\ -\frac{2}{3}\right) - \left(-\frac{29}{249},\ -\frac{116}{83},\ -\frac{493}{249},\ -\frac{232}{249}\right)$$

$$= \left(-\frac{18}{83},\ -\frac{50}{83},\ \frac{26}{83},\ \frac{22}{83}\right)$$

$$\vec{q}_3 = \frac{83}{\sqrt{324 + 2500 + 676 + 484}}\left(-\frac{18}{83},\ -\frac{50}{83},\ \frac{26}{83},\ \frac{22}{83}\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$= \frac{1}{\sqrt{3,984}}(-18,\ -50,\ 26,\ 22)$$

$$= \frac{1}{4\sqrt{249}}(-18,\ -50,\ 26,\ 22)$$

$$= \left(-\frac{9}{2\sqrt{249}},\ -\frac{25}{2\sqrt{249}},\ \frac{13}{2\sqrt{249}},\ \frac{11}{2\sqrt{249}}\right)$$

$$\frac{\langle \vec{u}_4,\ \vec{v}_1 \rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1 = \frac{(-2,\ 1,\ 2,\ 1)\bullet(3,\ 0,\ -3,\ 6)}{54}(3,\ 0,\ -3,\ 6)$$

$$= -\frac{6}{54}(3,\ 0,\ -3,\ 6)$$

$$= -\frac{1}{9}(3,\ 0,\ -3,\ 6)$$

$$= \left(-\frac{1}{3},\ 0,\ \frac{1}{3},\ -\frac{2}{3}\right)$$

$$\frac{\langle \vec{u}_4,\ \vec{v}_2 \rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2 = \frac{36}{498}\left[(-2,\ 1,\ 2,\ 1)\bullet\left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right)\right]\left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right)$$

$$= \frac{6}{83}\left(-\frac{1}{3} + 2 + \frac{17}{3} + \frac{4}{3}\right)\left(\frac{1}{6},\ 2,\ \frac{17}{6},\ \frac{4}{3}\right)$$

$$= \frac{6}{83} \left(\frac{26}{3} \right) \left(\frac{1}{6}, \ 2, \ \frac{17}{6}, \ \frac{4}{3} \right)$$

$$= \frac{52}{83} \left(\frac{1}{6}, \ 2, \ \frac{17}{6}, \ \frac{4}{3} \right)$$

$$= \left(\frac{26}{249}, \ \frac{104}{83}, \ \frac{442}{249}, \ \frac{208}{249} \right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\left\| \vec{v}_3 \right\|^2} \vec{v}_3 = \frac{83^2}{3{,}984} \left[(-2, \ 1, \ 2, \ 1) \cdot \left(-\frac{18}{83}, \ -\frac{50}{83}, \ \frac{26}{83}, \ \frac{22}{83} \right) \right] \left(-\frac{18}{83}, \ -\frac{50}{83}, \ \frac{26}{83}, \ \frac{22}{83} \right)$$

$$= \frac{83}{3{,}984} \ \frac{36 - 50 + 52 + 22}{83} \ (-18, \ -50, \ 26, \ 22)$$

$$= \frac{60}{3{,}984} \ (-18, \ -50, \ 26, \ 22)$$

$$= \frac{15}{996} \ (-18, \ -50, \ 26, \ 22)$$

$$= \frac{5}{332} \ (-18, \ -50, \ 26, \ 22)$$

$$= \left(-\frac{45}{166}, \ -\frac{125}{166}, \ \frac{65}{166}, \ \frac{55}{166} \right)$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\left\| \vec{v}_2 \right\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\left\| \vec{v}_3 \right\|^2} \vec{v}_3$$

$$= (-2, \ 1, \ 2, \ 1) - \left(-\frac{1}{3}, \ 0, \ \frac{1}{3}, \ -\frac{2}{3} \right) - \left(\frac{26}{249}, \ \frac{104}{83}, \ \frac{442}{249}, \ \frac{208}{249} \right) - \left(-\frac{45}{166}, \ -\frac{125}{166}, \ \frac{65}{166}, \ \frac{55}{166} \right)$$

$$= \left(-\frac{5}{3}, \ 1, \ \frac{5}{3}, \ \frac{5}{3} \right) - \left(\frac{26}{249}, \ \frac{104}{83}, \ \frac{442}{249}, \ \frac{208}{249} \right) + \left(\frac{45}{166}, \ \frac{125}{166}, \ -\frac{65}{166}, \ -\frac{55}{166} \right)$$

$$= \left(-\frac{441}{249}, \ -\frac{21}{83}, \ -\frac{27}{249}, \ \frac{207}{249} \right) + \left(\frac{45}{166}, \ \frac{125}{166}, \ -\frac{65}{166}, \ -\frac{55}{166} \right)$$

$$= \left(-\frac{147}{83}, \ -\frac{21}{83}, \ -\frac{9}{83}, \ \frac{69}{83} \right) + \left(\frac{45}{166}, \ \frac{125}{166}, \ -\frac{65}{166}, \ -\frac{55}{166} \right)$$

$$= \left(-\frac{249}{166}, \ \frac{83}{166}, \ -\frac{83}{166}, \ \frac{83}{166} \right)$$

$$= \left(-\frac{3}{2}, \ \frac{1}{2}, \ -\frac{1}{2}, \ \frac{1}{2} \right)$$

$$\vec{q}_4 = \frac{1}{\sqrt{\frac{9}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}}} \left(-\frac{3}{2}, \ \frac{1}{2}, \ -\frac{1}{2}, \ \frac{1}{2} \right) \qquad\qquad \vec{q}_4 = \frac{\vec{v}_4}{\left\| \vec{v}_4 \right\|}$$

$$= \frac{2}{\sqrt{12}} \left(-\frac{3}{2}, \ \frac{1}{2}, \ -\frac{1}{2}, \ \frac{1}{2} \right)$$

$$= \frac{1}{\sqrt{3}} \left(-\frac{3}{2}, \ \frac{1}{2}, \ -\frac{1}{2}, \ \frac{1}{2} \right)$$

$$= \left(-\frac{3}{2\sqrt{3}}, \ \frac{1}{2\sqrt{3}}, \ -\frac{1}{2\sqrt{3}}, \ \frac{1}{2\sqrt{3}} \right)$$

The *orthogonal* basis:

$$\left\{ \begin{array}{c} (3, \ 0, \ -3, \ 6), \ \left(\frac{1}{6}, \ 2, \ \frac{17}{6}, \ \frac{4}{3} \right), \ \left(-\frac{18}{83}, \ -\frac{50}{83}, \ \frac{26}{83}, \ \frac{22}{83} \right), \\ \left(-\frac{3}{2}, \ \frac{1}{2}, \ -\frac{1}{2}, \ \frac{1}{2} \right) \end{array} \right\}$$

The *orthonormal* basis:

$$\left\{ \begin{array}{c} \left(\frac{1}{\sqrt{6}}, \ 0, \ -\frac{1}{\sqrt{6}}, \ \frac{2}{\sqrt{6}} \right), \ \left(\frac{1}{\sqrt{498}}, \ \frac{12}{\sqrt{498}}, \ \frac{17}{\sqrt{498}}, \ \frac{8}{\sqrt{498}} \right), \\ \left(-\frac{9}{2\sqrt{249}}, \ -\frac{25}{2\sqrt{249}}, \ \frac{13}{2\sqrt{249}}, \ \frac{11}{2\sqrt{249}} \right), \ \left(-\frac{3}{2\sqrt{3}}, \ \frac{1}{2\sqrt{3}}, \ -\frac{1}{2\sqrt{3}}, \ \frac{1}{2\sqrt{3}} \right) \end{array} \right\}$$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis and an **orthonormal** basis for the subspaces $\mathbb{R}^m$.

$$\left\{ (3, \ 4, \ 0, \ 0), \ (-1, \ 1, \ 0, \ 0), \ (2, \ 1, \ 0, \ -1), \ (0, \ 1, \ 1, \ 0) \right\}$$

Solution

Let $\quad \vec{u}_1 = (3, \ 4, \ 0, \ 0) \quad \vec{u}_2 = (-1, \ 1, \ 0, \ 0) \quad \vec{u}_3 = (2, \ 1, \ 0, \ -1) \quad \vec{u}_4 = (0, \ 1, \ 1, \ 0)$

$$\vec{v}_1 = \vec{u}_1 = (3, \ 4, \ 0, \ 0)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (-1, \ 1, \ 0, \ 0) - \frac{(-1, \ 1, \ 0, \ 0) \bullet (3, \ 4, \ 0, \ 0)}{25} (3, \ 4, \ 0, \ 0)$$

$$= (-1, \ 1, \ 0, \ 0) - \frac{1}{25} (3, \ 4, \ 0, \ 0)$$

$$= \left(-\frac{28}{25}, \ \frac{21}{25}, \ 0, \ 0 \right)$$

$$\frac{\langle \vec{u}_3, \ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(2, \ 1, \ 0, \ -1) \bullet (3, \ 4, \ 0, \ 0)}{25} (3, \ 4, \ 0, \ 0)$$

$$= \frac{10}{25} (3, \ 4, \ 0, \ 0)$$

$$= \left(\frac{6}{5}, \ \frac{8}{5}, \ 0, \ 0 \right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\left\| \vec{v}_2 \right\|^2} \vec{v}_2 = \frac{625}{1,225} \left[(2, \ 1, \ 0, \ -1) \bullet \left(-\frac{28}{25}, \ \frac{21}{25}, \ 0, \ 0 \right) \right] \left(-\frac{28}{25}, \ \frac{21}{25}, \ 0, \ 0 \right)$$

$$= \frac{25}{49} \left(-\frac{35}{25} \right) \left(-\frac{28}{25}, \ \frac{21}{25}, \ 0, \ 0 \right)$$

$$= \left(\frac{4}{5}, \ -\frac{3}{5}, \ 0, \ 0 \right)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\left\| \vec{v}_2 \right\|^2} \vec{v}_2$$

$$= (2, \ 1, \ 0, \ -1) - \left(\frac{6}{5}, \ \frac{8}{5}, \ 0, \ 0 \right) - \left(\frac{4}{5}, \ -\frac{3}{5}, \ 0, \ 0 \right)$$

$$= (0, \ 0, \ 0, \ -1)$$

$$\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1 = \frac{(0, \ 1, \ 1, \ 0) \bullet (3, \ 4, \ 0, \ 0)}{25} (3, \ 4, \ 0, \ 0)$$

$$= \frac{4}{25} (3, \ 4, \ 0, \ 0)$$

$$= \left(\frac{12}{25}, \ \frac{16}{25}, \ 0, \ 0 \right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\left\| \vec{v}_2 \right\|^2} \vec{v}_2 = \frac{25}{49} \left[(0, \ 1, \ 1, \ 0) \bullet \left(-\frac{28}{25}, \ \frac{21}{25}, \ 0, \ 0 \right) \right] \left(-\frac{28}{25}, \ \frac{21}{25}, \ 0, \ 0 \right)$$

$$= \frac{25}{49} \left(\frac{21}{25} \right) \left(-\frac{28}{25}, \ \frac{21}{25}, \ 0, \ 0 \right)$$

$$= \left(-\frac{12}{25}, \ \frac{9}{25}, \ 0, \ 0 \right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\left\| \vec{v}_3 \right\|^2} \vec{v}_3 = \left[(0, \ 1, \ 1, \ 0) \bullet (0, \ 0, \ 0, \ -1) \right] (0, \ 0, \ 0, \ -1)$$

$$= (0, \ 0, \ 0, \ 0)$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\left\| \vec{v}_2 \right\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\left\| \vec{v}_3 \right\|^2} \vec{v}_3$$

$$= (0, \ 1, \ 1, \ 0) - \left(\frac{12}{25}, \ \frac{16}{25}, \ 0, \ 0 \right) - \left(-\frac{12}{25}, \ \frac{9}{25}, \ 0, \ 0 \right) - (0, \ 0, \ 0, \ 0)$$

$$= (0, 0, 1, 0)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{5}(3, 4, 0, 0)$$

$$= \left(\frac{3}{5}, \frac{4}{5}, 0, 0\right)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{25}{35}\left(-\frac{28}{25}, \frac{21}{25}, 0, 0\right)$$

$$= \left(-\frac{4}{5}, \frac{3}{5}, 0, 0\right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= (0, 0, 0, -1)$$

$$\vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

$$= (0, 0, 1, 0)$$

The *orthogonal* basis:

$$\left\{(3, 4, 0, 0), \left(-\frac{28}{25}, \frac{21}{25}, 0, 0\right), (0, 0, 0, -1), (0, 0, 1, 0)\right\}$$

The *orthonormal* basis:

$$\left\{\left(\frac{3}{5}, \frac{4}{5}, 0, 0\right), \left(-\frac{4}{5}, \frac{3}{5}, 0, 0\right), (0, 0, 0, -1), (0, 0, 1, 0)\right\}$$

Exercise

Find the *QR*–decomposition of $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

Solution

Since $\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5 \neq 0$, The matrix is invertible

$$\vec{u}_1 (1, 2), \quad \vec{u}_2 = (-1, 3)$$

$$\vec{v}_1 = \vec{u}_1 = (1,\ 2)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ 2)}{\sqrt{1^2 + 2^2}}$$

$$= \frac{(1,\ 2)}{\sqrt{5}}$$

$$= \left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}} \right)$$

$$\vec{v}_2 = \vec{u}_2 - \left(\vec{u}_2 \bullet \vec{v}_1 \right) \vec{v}_1$$

$$= (-1,\ 3) - \left[(-1,\ 3) \bullet \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) \right] \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)$$

$$= (-1,\ 3) - \left(\frac{5}{\sqrt{5}} \right) \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)$$

$$= (-1,\ 3) - (1,\ 2)$$

$$= (-2,\ 1)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{(-2,\ 1)}{\sqrt{(-2)^2 + 1^2}}$$

$$= \left(-\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}} \right)$$

$$\left\langle \vec{u}_1,\ \vec{q}_1 \right\rangle = (1,\ 2) \bullet \left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}} \right)$$

$$= \frac{5}{\sqrt{5}}$$

$$= \sqrt{5}$$

$$\left\langle \vec{u}_2,\ \vec{q}_1 \right\rangle = (-1,\ 3) \bullet \left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}} \right)$$

$$= \frac{5}{\sqrt{5}}$$

$$= \sqrt{5}$$

$$\left\langle \vec{u}_2, \vec{q}_2 \right\rangle = (-1,\ 3) \cdot \left(-\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}} \right)$$

$$= \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{5}}$$

$$= \sqrt{5}$$

$$R = \begin{bmatrix} \left\langle \vec{u}_1, \vec{q}_1 \right\rangle & \left\langle \vec{u}_2, \vec{q}_1 \right\rangle \\ 0 & \left\langle \vec{u}_2, \vec{q}_2 \right\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}$$

The *QR*-decomposition of the matrix is

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}$$

Exercise

Find the *QR*–decomposition of $\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix}$

Solution

The column vectors of are: $\vec{u}_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$$\vec{v}_1 = \vec{u}_1 = (3,\ -4)$$

$$\vec{q}_1 = \frac{(3,\ -4)}{\sqrt{9+16}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\| \vec{v}_1 \right\|}$$

$$= \left(\frac{3}{5},\ -\frac{4}{5} \right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1$$

$$= (5,\ 0) - \frac{(5,\ 0) \cdot (3,\ -4)}{25}(3,\ -4)$$

$$= (5,\ 0) - \frac{15}{25}(3,\ -4)$$

$$= (5,\ 0) - \frac{3}{5}(3,\ -4)$$

$$= (5,\ 0) - \left(\frac{9}{5},\ -\frac{12}{5}\right)$$

$$= \left(\frac{16}{5},\ \frac{12}{5}\right)$$

$$\vec{q}_2 = \frac{1}{\sqrt{\frac{256}{25} + \frac{144}{25}}}\left(\frac{16}{5},\ \frac{12}{5}\right) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{\sqrt{\frac{400}{25}}}\left(\frac{16}{5},\ \frac{12}{5}\right)$$

$$= \frac{1}{\sqrt{16}}\left(\frac{16}{5},\ \frac{12}{5}\right)$$

$$= \frac{1}{4}\left(\frac{16}{5},\ \frac{12}{5}\right)$$

$$= \left(\frac{4}{5},\ \frac{3}{5}\right)$$

$$R = \begin{bmatrix} \langle \vec{u}_1,\ \vec{q}_1 \rangle & \langle \vec{u}_2,\ \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2,\ \vec{q}_2 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} 3\left(\frac{3}{5}\right) - 4\left(-\frac{4}{5}\right) & 5\left(\frac{3}{5}\right) + 0\left(-\frac{4}{5}\right) \\ 0 & 5\left(\frac{4}{5}\right) - 0\left(\frac{3}{5}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 \\ 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \cdot \begin{bmatrix} 5 & 3 \\ 0 & 4 \end{bmatrix}$$

$$A \quad = \quad Q \quad\quad R$$

Exercise

Find the *QR*–decomposition of $\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$

Solution

The column vectors of are: $\vec{u}_1 = (1,\ -2) \quad \vec{u}_2 = (3,\ 4)$

$$\vec{v}_1 = \vec{u}_1 = (1,\ -2)$$

$$\vec{q}_1 = \frac{(1, -2)}{\sqrt{1+4}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (3, 4) - \frac{(3, 4) \cdot (1, -2)}{5}(1, -2)$$

$$= (3, 4) + \frac{5}{5}(1, -2)$$

$$= (3, 4) + (1, -2)$$

$$= (4, 2)$$

$$\vec{q}_2 = \frac{1}{\sqrt{16+4}}(4, 2) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{2\sqrt{5}}(4, 2)$$

$$= \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right)$$

$$R = \begin{bmatrix} 1\left(\frac{1}{\sqrt{5}}\right) - 2\left(-\frac{2}{\sqrt{5}}\right) & 3\left(\frac{1}{\sqrt{5}}\right) + 4\left(-\frac{2}{\sqrt{5}}\right) \\ 0 & 3\left(\frac{2}{\sqrt{5}}\right) + 4\left(\frac{1}{\sqrt{5}}\right) \end{bmatrix} \qquad R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix}$$

$$= \begin{pmatrix} \frac{5}{\sqrt{5}} & -\frac{5}{\sqrt{5}} \\ 0 & \frac{10}{\sqrt{5}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{5} & -\sqrt{5} \\ 0 & 2\sqrt{5} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{5} & -\sqrt{5} \\ 0 & 2\sqrt{5} \end{pmatrix}$$

$$\boldsymbol{A} \quad = \quad \boldsymbol{Q} \qquad \boldsymbol{R}$$

Exercise

Find the QR–decomposition of $\begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}$

Solution

The column vectors of are: $\vec{u}_1 = (1,\ -1)$ $\vec{u}_2 = (-2,\ 4)$

$\vec{v}_1 = \vec{u}_1 = (1,\ -1)$

$$\vec{q}_1 = \frac{(1,\ -1)}{\sqrt{2}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}} \right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2,\ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (-2,\ 4) - \frac{(-2,\ 4)\cdot(1,\ -1)}{2}(1,\ -1)$$

$$= (-2,\ 4) + \frac{6}{2}(1,\ -1)$$

$$= (-2,\ 4) + (3,\ -3)$$

$$= (1,\ 1)$$

$$\vec{q}_2 = \frac{1}{\sqrt{2}}(1,\ 1) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}} \right)$$

$$R = \begin{bmatrix} \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) & -2\left(\frac{1}{\sqrt{2}}\right) + 4\left(-\frac{1}{\sqrt{2}}\right) \\ 0 & -2\left(\frac{1}{\sqrt{2}}\right) + 4\left(\frac{1}{\sqrt{2}}\right) \end{bmatrix} \qquad R = \begin{bmatrix} \langle \vec{u}_1,\ \vec{q}_1 \rangle & \langle \vec{u}_2,\ \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2,\ \vec{q}_2 \rangle \end{bmatrix}$$

$$= \begin{pmatrix} \frac{2}{\sqrt{2}} & -\frac{6}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{2} & -3\sqrt{2} \\ 0 & \sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} & -3\sqrt{2} \\ 0 & \sqrt{2} \end{pmatrix}$$

$$\boldsymbol{A} \qquad = \qquad \boldsymbol{Q} \qquad\qquad \boldsymbol{R}$$

Exercise

Find the QR–decomposition of $\begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix}$

Solution

The column vectors of are: $\vec{u}_1 = (-1,\ 2) \quad \vec{u}_2 = (1,\ -4)$

$\underline{\vec{v}_1 = \vec{u}_1 = (-1,\ 2)}$

$\vec{q}_1 = \dfrac{(-1,\ 2)}{\sqrt{5}}$ $\qquad\qquad$ $\vec{q}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$

$\underline{= \left(-\dfrac{1}{\sqrt{5}},\ \dfrac{2}{\sqrt{5}} \right)}$

$\vec{v}_2 = \vec{u}_2 - \dfrac{\langle \vec{u}_2,\ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$

$= (1,\ -4) - \dfrac{(1,\ -4) \cdot (-1,\ 2)}{5} (-1,\ 2)$

$= (1,\ -4) + \dfrac{9}{5} (-1,\ 2)$

$= (1,\ -4) + \left(-\dfrac{9}{5},\ \dfrac{18}{5} \right)$

$\underline{= \left(-\dfrac{4}{5},\ -\dfrac{2}{5} \right)}$

$\vec{q}_2 = \dfrac{5}{\sqrt{16+4}} \left(-\dfrac{4}{5},\ -\dfrac{2}{5} \right)$ $\qquad\qquad$ $\vec{q}_2 = \dfrac{\vec{v}_2}{\|\vec{v}_2\|}$

$= \dfrac{5}{2\sqrt{5}} \left(-\dfrac{4}{5},\ -\dfrac{2}{5} \right)$

$\underline{= \left(-\dfrac{2}{\sqrt{5}},\ -\dfrac{1}{\sqrt{5}} \right)}$

$$R = \begin{bmatrix} \frac{1}{\sqrt{5}} + 2\left(\frac{2}{\sqrt{5}}\right) & \left(-\frac{1}{\sqrt{5}}\right) - 4\left(\frac{2}{\sqrt{5}}\right) \\ 0 & \left(-\frac{2}{\sqrt{5}}\right) - 4\left(-\frac{1}{\sqrt{5}}\right) \end{bmatrix}$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix}$$

$$= \begin{pmatrix} \frac{5}{\sqrt{5}} & -\frac{9}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \cdot \begin{pmatrix} \frac{5}{\sqrt{5}} & -\frac{9}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$\quad A \quad = \quad Q \quad\quad R$$

Exercise

Find the *QR*–decomposition of $\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}$

Solution

The column vectors of are: $\vec{u}_1 = (1, -2) \quad \vec{u}_2 = (-3, 4)$

$\underline{\vec{v}_1 = \vec{u}_1 = (1, -2)}$

$\vec{q}_1 = \frac{(1, -2)}{\sqrt{5}}$ $\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$

$\underline{= \left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)}$

$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$

$\quad = (-3, 4) - \frac{(-3, 4) \cdot (1, -2)}{5}(1, -2)$

$\quad = (-3, 4) + \frac{11}{5}(1, -2)$

$\underline{= \left(-\frac{4}{5}, -\frac{2}{5}\right)}$

$\vec{q}_2 = \frac{5}{\sqrt{16+4}}\left(-\frac{4}{5}, -\frac{2}{5}\right)$ $\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$

$$= \frac{5}{2\sqrt{5}}\left(-\frac{4}{5},\ -\frac{2}{5}\right)$$

$$\underline{= \left(-\frac{2}{\sqrt{5}},\ -\frac{1}{\sqrt{5}}\right)}$$

$$R = \begin{bmatrix} \frac{1}{\sqrt{5}} - 2\left(-\frac{2}{\sqrt{5}}\right) & -3\left(\frac{1}{\sqrt{5}}\right) + 4\left(-\frac{2}{\sqrt{5}}\right) \\ 0 & -3\left(-\frac{2}{\sqrt{5}}\right) + 4\left(-\frac{1}{\sqrt{5}}\right) \end{bmatrix} \qquad R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix}$$

$$= \begin{pmatrix} \frac{5}{\sqrt{5}} & -\frac{11}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \cdot \begin{pmatrix} \frac{5}{\sqrt{5}} & -\frac{11}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$A \quad = \quad Q \quad\quad R$$

Exercise

Find the *QR*–decomposition of $\begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix}$

Solution

The column vectors of are: $\vec{u}_1 = (-2,\ 4)$ $\vec{u}_2 = (6,\ -8)$

$$\underline{\vec{v}_1 = \vec{u}_1 = (-2,\ 4)}$$

$$\vec{q}_1 = \frac{(-2,\ 4)}{\sqrt{20}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\underline{= \left(-\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2,\ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (6,\ -8) - \frac{(6,\ -8) \cdot (-2,\ 4)}{20}(-2,\ 4)$$

$$= (6,\ -8) + \frac{44}{20}(-2,\ 4)$$

$$= (6, \ -8) + \frac{11}{5}(-2, \ 4)$$

$$= \left(\frac{8}{5}, \ \frac{4}{5}\right)$$

$$\vec{q}_2 = \frac{5}{\sqrt{64+16}}\left(\frac{8}{5}, \ \frac{4}{5}\right) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{5}{4\sqrt{5}}\left(\frac{8}{5}, \ \frac{4}{5}\right)$$

$$= \left(\frac{2}{\sqrt{5}}, \ \frac{1}{\sqrt{5}}\right)$$

$$R = \begin{bmatrix} -2\left(-\frac{1}{\sqrt{5}}\right)+4\left(\frac{2}{\sqrt{5}}\right) & 6\left(-\frac{1}{\sqrt{5}}\right)-8\left(\frac{2}{\sqrt{5}}\right) \\ 0 & 6\left(\frac{2}{\sqrt{5}}\right)-8\left(\frac{1}{\sqrt{5}}\right) \end{bmatrix} \qquad R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix}$$

$$= \begin{pmatrix} \dfrac{10}{\sqrt{5}} & -\dfrac{22}{\sqrt{5}} \\ 0 & \dfrac{4}{\sqrt{5}} \end{pmatrix}$$

$$\begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{\sqrt{5}} & \dfrac{2}{\sqrt{5}} \\ \dfrac{2}{\sqrt{5}} & \dfrac{1}{\sqrt{5}} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{10}{\sqrt{5}} & -\dfrac{22}{\sqrt{5}} \\ 0 & \dfrac{4}{\sqrt{5}} \end{pmatrix}$$

$$A \qquad = \qquad Q \qquad\qquad R$$

Exercise

Find the QR–decomposition of $\qquad \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}$

Solution

The column vectors of are: $\vec{u}_1 = (-1, \ 1) \quad \vec{u}_2 = (2, \ 0)$

$$\vec{v}_1 = \vec{u}_1 = (-1, \ 1)$$

$$\vec{q}_1 = \frac{(-1, \ 1)}{\sqrt{2}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(-\frac{1}{\sqrt{2}}, \ \frac{1}{\sqrt{2}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1$$

$$= (2, 0) - \frac{(2, 0) \cdot (-1, 1)}{2} (-1, 1)$$

$$= (2, 0) + (-1, 1)$$

$$= (1, 1)$$

$$\vec{q}_2 = \frac{1}{\sqrt{2}} (1, 1) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\| \vec{v}_2 \right\|}$$

$$= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$R = \begin{bmatrix} -\left(-\dfrac{1}{\sqrt{2}}\right) + \left(\dfrac{1}{\sqrt{2}}\right) & 2\left(-\dfrac{1}{\sqrt{2}}\right) + 0 \\[2mm] 0 & 2\left(\dfrac{1}{\sqrt{2}}\right) + 0 \end{bmatrix} \qquad R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\[2mm] 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix}$$

$$= \begin{pmatrix} \dfrac{2}{\sqrt{2}} & -\dfrac{2}{\sqrt{2}} \\[3mm] 0 & \dfrac{2}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\[3mm] \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{2}{\sqrt{2}} & -\dfrac{2}{\sqrt{2}} \\[3mm] 0 & \dfrac{2}{\sqrt{2}} \end{pmatrix}$$

$$\quad A \qquad = \qquad Q \qquad\qquad R$$

Exercise

Find the QR–decomposition of $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$

Solution

Since the column vectors $\vec{u}_1 = (1, 0, 1)$, $\vec{u}_2 = (2, 1, 4)$ are linearly independent, so has a QR–decomposition.

$$\vec{v}_1 = \vec{u}_1 = (1, 0, 1)$$

$$\vec{q}_1 = \frac{(1,\ 0,\ 1)}{\sqrt{1^2 + 0 + 1^2}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ 0,\ 1)}{\sqrt{2}}$$

$$= \left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \left\langle \vec{u}_2,\ \vec{v}_1 \right\rangle \vec{v}_1$$

$$= (2,\ 1,\ 4) - \left[(2,\ 1,\ 4) \bullet \left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\right]\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$= (2,\ 1,\ 4) - \left(\frac{6}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$= (2,\ 1,\ 4) - (3,\ 0,\ 3)$$

$$= (-1,\ 1,\ 1)$$

$$\vec{q}_2 = \frac{(-1,\ 1,\ 1)}{\sqrt{(-1)^2 + 1^2 + 1^2}} \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(-\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\left\langle \vec{u}_1,\ \vec{q}_1 \right\rangle = (1,\ 0,\ 1) \bullet \left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$= \frac{2}{\sqrt{2}}$$

$$= \sqrt{2}$$

$$\left\langle \vec{u}_2,\ \vec{q}_1 \right\rangle = (2,\ 1,\ 4) \bullet \left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$= \frac{2}{\sqrt{2}} + \frac{4}{\sqrt{2}}$$

$$= 3\sqrt{2}$$

$$\left\langle \vec{u}_2,\ \vec{q}_2 \right\rangle = (2,\ 1,\ 4) \bullet \left(-\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= -\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{4}{\sqrt{3}}$$

$$= \frac{3}{\sqrt{3}}$$

$$= \sqrt{3}$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$$

The QR–decomposition of the matrix is

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{3}} \\ 0 & \dfrac{1}{\sqrt{3}} \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$$

$$\quad A \quad = \quad\quad Q \quad\quad\quad\quad R$$

Exercise

Find the QR–decomposition of $\quad \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{pmatrix}$

Solution

Since $\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{vmatrix} = -4 \neq 0$,

The matrix is invertible, so it has a QR-decomposition.

$$\vec{u}_1 = (1, 1, 0), \quad \vec{u}_2 = (2, 1, 3), \quad \vec{u}_3 = (1, 1, 1)$$

$$\underline{\vec{v}_1 = \vec{u}_1 = (1, 1, 0)}$$

$$\vec{q}_1 = \frac{(1, 1, 0)}{\sqrt{1+1+0}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1, 1, 0)}{\sqrt{2}}$$

$$\underline{= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (2, 1, 3) - \frac{1}{2} \left[(2, 1, 3) \bullet (1, 1, 0) \right] (1, 1, 0)$$

$$= (2, 1, 3) - \frac{3}{2} (1, 1, 0)$$

$$= (2, 1, 3) - \left(\frac{3}{2}, \frac{3}{2}, 0 \right)$$

$$= \left(\frac{1}{2}, -\frac{1}{2}, 3 \right)$$

$$\vec{q}_2 = \frac{\left(\frac{1}{2}, -\frac{1}{2}, 3 \right)}{\sqrt{\frac{1}{4} + \frac{1}{4} + 9}} \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{2}{\sqrt{38}} \left(\frac{1}{2}, -\frac{1}{2}, 3 \right)$$

$$= \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}} \right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(1, 1, 1) \bullet (1, 1, 0)}{2} (1, 1, 0)$$

$$= (1, 1, 0)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{4}{38} \left((1, 1, 1) \bullet \left(\frac{1}{2}, -\frac{1}{2}, 3 \right) \right) \left(\frac{1}{2}, -\frac{1}{2}, 3 \right)$$

$$= \frac{6}{19} \left(\frac{1}{2}, -\frac{1}{2}, 3 \right)$$

$$= \left(\frac{3}{19}, -\frac{3}{19}, \frac{18}{19} \right)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (1, 1, 1) - (1, 1, 0) - \left(\frac{3}{19}, -\frac{3}{19}, \frac{18}{19} \right)$$

$$= (0, 0, 1) - \left(\frac{3}{19}, -\frac{3}{19}, \frac{18}{19} \right)$$

$$= \left(-\frac{3}{19}, \frac{3}{19}, \frac{1}{19} \right)$$

$$\vec{q}_3 = \frac{1}{\sqrt{\left(-\frac{3}{19}\right)^2 + \left(\frac{3}{19}\right)^2 + \left(\frac{1}{19}\right)^2}} \left(-\frac{3}{19}, \frac{3}{19}, \frac{1}{19}\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{19}{\sqrt{19}} \left(-\frac{3}{19}, \frac{3}{19}, \frac{1}{19}\right)$$

$$= \left(-\frac{3}{\sqrt{19}}, \frac{3}{\sqrt{19}}, \frac{1}{\sqrt{19}}\right)$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix}$$

$$\langle \vec{u}_1, \vec{q}_1 \rangle = (1, 1, 0) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$= \frac{2}{\sqrt{2}}$$

$$= \sqrt{2}$$

$$\langle \vec{u}_2, \vec{q}_1 \rangle = (2, 1, 3) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$= \frac{3}{\sqrt{2}}$$

$$\langle \vec{u}_2, \vec{q}_2 \rangle = (2, 1, 3) \cdot \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}}\right)$$

$$= \frac{2 - 1 + 18}{\sqrt{38}}$$

$$= \frac{19}{\sqrt{38}}$$

$$= \frac{19}{\sqrt{2}\sqrt{19}}$$

$$= \frac{\sqrt{19}}{\sqrt{2}}$$

$$\langle \vec{u}_3, \vec{q}_1 \rangle = (1, 1, 1) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$= \frac{2}{\sqrt{2}}$$

$$= \sqrt{2}$$

$$\langle \vec{u}_3, \vec{q}_2 \rangle = (1,\ 1,\ 1) \cdot \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}} \right)$$

$$= \frac{1-1+6}{\sqrt{38}}$$

$$= \frac{6}{\sqrt{2}\sqrt{19}} \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{3\sqrt{2}}{\sqrt{19}}$$

$$\langle \vec{u}_3, \vec{q}_3 \rangle = (1,\ 1,\ 1) \cdot \left(-\frac{3}{\sqrt{19}}, \frac{3}{\sqrt{19}}, \frac{1}{\sqrt{19}} \right)$$

$$= \frac{-3+3+1}{\sqrt{19}}$$

$$= \frac{1}{\sqrt{19}}$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle & \langle \vec{u}_3, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle & \langle \vec{u}_3, \vec{q}_2 \rangle \\ 0 & 0 & \langle \vec{u}_3, \vec{q}_3 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix}$$

The QR–decomposition of the matrix is

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix}$$

$$\ \ A \quad = \qquad \qquad Q \qquad \qquad \qquad R$$

Exercise

Find the QR–decomposition of $\quad\begin{pmatrix} 7 & 1 & 2 \\ 2 & -1 & 3 \\ -3 & 0 & 4 \end{pmatrix}$

Solution

Since $\begin{vmatrix} 7 & 1 & 2 \\ 2 & -1 & 3 \\ -3 & 0 & 4 \end{vmatrix} = -51 \neq 0$,

The matrix is invertible, so it has a QR-decomposition.

$$\vec{u}_1 = (7,\ 2,\ -3),\quad \vec{u}_2 = (1,\ -1,\ 0),\quad \vec{u}_3 = (2,\ 3,\ 4)$$

$$\underline{\vec{v}_1 = \vec{u}_1 = (7,\ 2,\ -3)}$$

$$\vec{q}_1 = \frac{(7,\ 2,\ -3)}{\sqrt{49+4+9}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(7,\ 2,\ -3)}{\sqrt{62}}$$

$$\underline{= \left(\frac{7}{\sqrt{62}},\ \frac{2}{\sqrt{62}},\ -\frac{3}{\sqrt{62}} \right)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2,\ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (1,\ -1,\ 0) - \frac{1}{62}\left[(1,\ -1,\ 0) \bullet (7,\ 2,\ -3) \right](7,\ 2,\ -3)$$

$$= (1,\ -1,\ 0) - \frac{5}{62}(7,\ 2,\ -3)$$

$$= (1,\ -1,\ 0) - \left(\frac{35}{62},\ \frac{5}{31},\ -\frac{15}{62} \right)$$

$$\underline{= \left(\frac{27}{62},\ -\frac{36}{31},\ \frac{15}{62} \right)}$$

$$\vec{q}_2 = \frac{62}{\sqrt{27^2 + 72^2 + 15^2}}\left(\frac{27}{62},\ -\frac{36}{31},\ \frac{15}{62} \right) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{62}{\sqrt{6,138}}\left(\frac{27}{62},\ -\frac{36}{31},\ \frac{15}{62} \right)$$

$$= \frac{62}{3\sqrt{682}}\left(\frac{27}{62},\ -\frac{36}{31},\ \frac{15}{62} \right)$$

$$\underline{= \left(\frac{9}{\sqrt{682}},\ -\frac{24}{\sqrt{682}},\ \frac{5}{\sqrt{682}} \right)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(2, 3, 4) \bullet (7, 2, -3)}{62} (7, 2, -3)$$

$$= \frac{8}{62} (7, 2, -3)$$

$$= \frac{4}{31} (7, 2, -3)$$

$$= \left(\frac{28}{31}, \frac{8}{31}, -\frac{12}{31} \right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{3,844}{6,138} \left((2, 3, 4) \bullet \left(\frac{27}{62}, -\frac{36}{31}, \frac{15}{62} \right) \right) \left(\frac{27}{62}, -\frac{36}{31}, \frac{15}{62} \right)$$

$$= \frac{62}{6,138} \left(\frac{54}{62} - \frac{108}{31} + \frac{60}{62} \right) (27, -72, 15)$$

$$= -\frac{102}{6,138} (27, -72, 15)$$

$$= -\frac{17}{1,023} (27, -72, 15)$$

$$= \left(-\frac{153}{341}, \frac{408}{341}, -\frac{85}{341} \right)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (2, 3, 4) - \left(\frac{28}{31}, \frac{8}{31}, -\frac{12}{31} \right) - \left(-\frac{153}{341}, \frac{408}{341}, -\frac{85}{341} \right)$$

$$= \left(\frac{34}{31}, \frac{85}{31}, \frac{136}{31} \right) - \left(-\frac{153}{341}, \frac{408}{341}, -\frac{85}{341} \right)$$

$$= \left(\frac{17}{11}, \frac{17}{11}, \frac{51}{11} \right)$$

$$\vec{q}_3 = \frac{11}{\sqrt{17^2 + 17^2 + 51^2}} \left(\frac{17}{11}, \frac{17}{11}, \frac{51}{11} \right) \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\sqrt{3,179}} (17, 17, 51)$$

$$= \frac{1}{17\sqrt{11}} (17, 17, 51)$$

$$= \left(\frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}} \right)$$

$$Q = \begin{bmatrix} \dfrac{7}{\sqrt{62}} & \dfrac{3}{\sqrt{682}} & \dfrac{1}{\sqrt{11}} \\[3mm] \dfrac{2}{\sqrt{62}} & -\dfrac{24}{\sqrt{682}} & \dfrac{1}{\sqrt{11}} \\[3mm] -\dfrac{3}{\sqrt{62}} & \dfrac{5}{\sqrt{19}} & \dfrac{3}{\sqrt{11}} \end{bmatrix}$$

$$\left\langle \vec{u}_1, \vec{q}_1 \right\rangle = (7,\ 2,\ -3) \cdot \left(\frac{7}{\sqrt{62}},\ \frac{2}{\sqrt{62}},\ -\frac{3}{\sqrt{62}} \right)$$

$$= \frac{49 + 4 + 9}{\sqrt{62}}$$

$$= \frac{62}{\sqrt{62}}$$

$$\left\langle \vec{u}_2, \vec{q}_1 \right\rangle = (1,\ -1,\ 0) \cdot \left(\frac{7}{\sqrt{62}},\ \frac{2}{\sqrt{62}},\ -\frac{3}{\sqrt{62}} \right)$$

$$= \frac{5}{\sqrt{62}}$$

$$\left\langle \vec{u}_2, \vec{q}_2 \right\rangle = (1,\ -1,\ 0) \cdot \left(\frac{9}{\sqrt{682}},\ -\frac{24}{\sqrt{682}},\ \frac{5}{\sqrt{682}} \right)$$

$$= \frac{33}{\sqrt{682}}$$

$$\left\langle \vec{u}_3, \vec{q}_1 \right\rangle = (2,\ 3,\ 4) \cdot \left(\frac{7}{\sqrt{62}},\ \frac{2}{\sqrt{62}},\ -\frac{3}{\sqrt{62}} \right)$$

$$= \frac{14 + 6 - 12}{\sqrt{62}}$$

$$= \frac{8}{\sqrt{62}}$$

$$\left\langle \vec{u}_3, \vec{q}_2 \right\rangle = (2,\ 3,\ 4) \cdot \left(\frac{9}{\sqrt{682}},\ -\frac{24}{\sqrt{682}},\ \frac{5}{\sqrt{682}} \right)$$

$$= \frac{18 - 72 + 20}{\sqrt{682}}$$

$$= -\frac{34}{\sqrt{682}}$$

$$\left\langle \vec{u}_3, \vec{q}_3 \right\rangle = (2,\ 3,\ 4) \cdot \left(\frac{1}{\sqrt{11}},\ \frac{1}{\sqrt{11}},\ \frac{3}{\sqrt{11}} \right)$$

$$= \frac{2 + 3 + 12}{\sqrt{11}}$$

$$= \frac{17}{\sqrt{11}}$$

$$R = \begin{bmatrix} \dfrac{62}{\sqrt{62}} & \dfrac{5}{\sqrt{62}} & \dfrac{8}{\sqrt{62}} \\ 0 & \dfrac{33}{\sqrt{682}} & -\dfrac{34}{\sqrt{682}} \\ 0 & 0 & \dfrac{17}{\sqrt{11}} \end{bmatrix}$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle & \langle \vec{u}_3, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle & \langle \vec{u}_3, \vec{q}_2 \rangle \\ 0 & 0 & \langle \vec{u}_3, \vec{q}_3 \rangle \end{bmatrix}$$

The QR–decomposition of the matrix is

$$\begin{pmatrix} 7 & 1 & 2 \\ 2 & -1 & 3 \\ -3 & 0 & 4 \end{pmatrix} = \begin{pmatrix} \dfrac{7}{\sqrt{62}} & \dfrac{9}{\sqrt{682}} & \dfrac{1}{\sqrt{11}} \\ \dfrac{2}{\sqrt{62}} & -\dfrac{24}{\sqrt{682}} & \dfrac{1}{\sqrt{11}} \\ -\dfrac{3}{\sqrt{62}} & \dfrac{5}{\sqrt{682}} & \dfrac{3}{\sqrt{11}} \end{pmatrix} \begin{pmatrix} \dfrac{62}{\sqrt{62}} & \dfrac{5}{\sqrt{62}} & \dfrac{8}{\sqrt{62}} \\ 0 & \dfrac{33}{\sqrt{682}} & -\dfrac{34}{\sqrt{682}} \\ 0 & 0 & \dfrac{17}{\sqrt{11}} \end{pmatrix}$$

$$A \qquad = \qquad Q \qquad\qquad R$$

Exercise

Find the QR–decomposition of $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$

Solution

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \quad \begin{array}{l} R_2 + R_1 \\ R_3 - R_1 \\ R_4 + R_1 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \quad R_4 - R_2$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix is linearly dependent, so ***doesn't*** have a QR–decomposition.

Exercise

Find the *QR*–decomposition of
$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{matrix} R_2 - R_1 \\ \\ R_4 - R_3 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ \\ R_3 - R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ \\ R_3 - R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The matrix has a *QR*-decomposition.

$$\vec{u}_1 = (1, 1, 1, 1), \quad \vec{u}_2 = (0, 1, 1, 1), \quad \vec{u}_3 = (0, 0, 1, 1)$$

$$\vec{v}_1 = \vec{u}_1 = (1, 1, 1, 1) \, \Big|$$

$$\vec{q}_1 = \frac{(1, 1, 1, 1)}{\sqrt{4}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \Big|$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (0, 1, 1, 1) - \frac{1}{4} \left[(0, 1, 1, 1) \cdot (1, 1, 1, 1) \right] (1, 1, 1, 1)$$

$$= (0, 1, 1, 1) - \left(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4} \right)$$

$$= \left(-\frac{3}{4}, \ \frac{1}{4}, \ \frac{1}{4}, \ \frac{1}{4} \right)$$

$$\vec{q}_2 = \frac{4}{\sqrt{9+1+1+1}} \left(-\frac{3}{4}, \ \frac{1}{4}, \ \frac{1}{4}, \ \frac{1}{4} \right) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\| \vec{v}_2 \right\|}$$

$$= \frac{2}{\sqrt{3}} \left(-\frac{3}{4}, \ \frac{1}{4}, \ \frac{1}{4}, \ \frac{1}{4} \right)$$

$$= \left(-\frac{3}{2\sqrt{3}}, \ \frac{1}{2\sqrt{3}}, \ \frac{1}{2\sqrt{3}}, \ \frac{1}{2\sqrt{3}} \right)$$

$$\frac{\left\langle \vec{u}_3, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1 = \frac{(0, \ 0, \ 1, \ 1) \bullet (1, \ 1, \ 1, \ 1)}{4} \ (1, \ 1, \ 1, \ 1)$$

$$= \frac{1}{2} \ (1, \ 1, \ 1, \ 1)$$

$$= \left(\frac{1}{2}, \ \frac{1}{2}, \ \frac{1}{2}, \ \frac{1}{2} \right)$$

$$\frac{\left\langle \vec{u}_3, \vec{v}_2 \right\rangle}{\left\| \vec{v}_2 \right\|^2} \vec{v}_2 = \frac{16}{12} \left((0, \ 0, \ 1, \ 1) \bullet \left(-\frac{3}{4}, \ \frac{1}{4}, \ \frac{1}{4}, \ \frac{1}{4} \right) \right) \left(-\frac{3}{4}, \ \frac{1}{4}, \ \frac{1}{4}, \ \frac{1}{4} \right)$$

$$= \frac{4}{3} \left(\frac{1}{2} \right) \left(-\frac{3}{4}, \ \frac{1}{4}, \ \frac{1}{4}, \ \frac{1}{4} \right)$$

$$= \left(-\frac{1}{2}, \ \frac{1}{6}, \ \frac{1}{6}, \ \frac{1}{6} \right)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\left\langle \vec{u}_3, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1 - \frac{\left\langle \vec{u}_3, \vec{v}_2 \right\rangle}{\left\| \vec{v}_2 \right\|^2} \vec{v}_2$$

$$= (0, \ 0, \ 1, \ 1) - \left(\frac{1}{2}, \ \frac{1}{2}, \ \frac{1}{2}, \ \frac{1}{2} \right) - \left(-\frac{1}{2}, \ \frac{1}{6}, \ \frac{1}{6}, \ \frac{1}{6} \right)$$

$$= \left(0, \ -\frac{2}{3}, \ \frac{1}{3}, \ \frac{1}{3} \right)$$

$$\vec{q}_3 = \frac{3}{\sqrt{6}} \left(0, \ -\frac{2}{3}, \ \frac{1}{3}, \ \frac{1}{3} \right) \qquad\qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\left\| \vec{v}_3 \right\|}$$

$$= \left(0, \ -\frac{2}{\sqrt{6}}, \ \frac{1}{\sqrt{6}}, \ \frac{1}{\sqrt{6}} \right)$$

$$Q = \begin{pmatrix} \dfrac{1}{2} & -\dfrac{3}{2\sqrt{3}} & 0 \\[3mm] \dfrac{1}{2} & \dfrac{1}{2\sqrt{3}} & -\dfrac{2}{\sqrt{6}} \\[3mm] \dfrac{1}{2} & \dfrac{1}{2\sqrt{3}} & \dfrac{1}{\sqrt{6}} \\[3mm] \dfrac{1}{2} & \dfrac{1}{2\sqrt{3}} & \dfrac{1}{\sqrt{6}} \end{pmatrix}$$

$$\langle \vec{u}_1, \vec{q}_1 \rangle = (1,\ 1,\ 1,\ 1) \cdot \left(\dfrac{1}{2},\ \dfrac{1}{2},\ \dfrac{1}{2},\ \dfrac{1}{2} \right)$$
$$= 2$$

$$\langle \vec{u}_2, \vec{q}_1 \rangle = (0,\ 1,\ 1,\ 1) \cdot \left(\dfrac{1}{2},\ \dfrac{1}{2},\ \dfrac{1}{2},\ \dfrac{1}{2} \right)$$
$$= \dfrac{3}{2}$$

$$\langle \vec{u}_2, \vec{q}_2 \rangle = (0,\ 1,\ 1,\ 1) \cdot \left(-\dfrac{3}{2\sqrt{3}},\ \dfrac{1}{2\sqrt{3}},\ \dfrac{1}{2\sqrt{3}},\ \dfrac{1}{2\sqrt{3}} \right)$$
$$= \dfrac{3}{2\sqrt{3}}$$

$$\langle \vec{u}_3, \vec{q}_1 \rangle = (0,\ 0,\ 1,\ 1) \cdot \left(\dfrac{1}{2},\ \dfrac{1}{2},\ \dfrac{1}{2},\ \dfrac{1}{2} \right)$$
$$= 1$$

$$\langle \vec{u}_3, \vec{q}_2 \rangle = (0,\ 0,\ 1,\ 1) \cdot \left(-\dfrac{3}{2\sqrt{3}},\ \dfrac{1}{2\sqrt{3}},\ \dfrac{1}{2\sqrt{3}},\ \dfrac{1}{2\sqrt{3}} \right)$$
$$= \dfrac{1}{\sqrt{3}}$$

$$\langle \vec{u}_3, \vec{q}_3 \rangle = (0,\ 0,\ 1,\ 1) \cdot \left(0,\ -\dfrac{2}{\sqrt{6}},\ \dfrac{1}{\sqrt{6}},\ \dfrac{1}{\sqrt{6}} \right)$$
$$= \dfrac{2}{\sqrt{6}}$$

$$R = \begin{pmatrix} 2 & \dfrac{3}{2} & 1 \\[3mm] 0 & \dfrac{3}{2\sqrt{3}} & \dfrac{1}{\sqrt{3}} \\[3mm] 0 & 0 & \dfrac{2}{\sqrt{6}} \end{pmatrix} \qquad R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle & \langle \vec{u}_3, \vec{q}_1 \rangle \\[3mm] 0 & \langle \vec{u}_2, \vec{q}_2 \rangle & \langle \vec{u}_3, \vec{q}_2 \rangle \\[3mm] 0 & 0 & \langle \vec{u}_3, \vec{q}_3 \rangle \end{bmatrix}$$

The QR–decomposition of the matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{2} & -\dfrac{3}{2\sqrt{3}} & 0 \\ \dfrac{1}{2} & \dfrac{1}{2\sqrt{3}} & -\dfrac{2}{\sqrt{6}} \\ \dfrac{1}{2} & \dfrac{1}{2\sqrt{3}} & \dfrac{1}{\sqrt{6}} \\ \dfrac{1}{2} & \dfrac{1}{2\sqrt{3}} & \dfrac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 2 & \dfrac{3}{2} & 1 \\ 0 & \dfrac{3}{2\sqrt{3}} & \dfrac{1}{\sqrt{3}} \\ 0 & 0 & \dfrac{2}{\sqrt{6}} \end{pmatrix}$$

$$\quad\ \ A \qquad = \qquad\qquad\ Q \qquad\qquad\qquad R$$

Exercise

Find the *QR*–decomposition of $\begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{pmatrix}$

Solution

The matrix has a *QR*-decomposition.

$$\vec{u}_1 = (3,\ 1,\ -1,\ 3),\quad \vec{u}_2 = (-5,\ 1,\ 5,\ -7),\quad \vec{u}_3 = (1,\ 1,\ -2,\ 8)$$

$$\vec{v}_1 = \vec{u}_1 = (3,\ 1,\ -1,\ 3)$$

$$\vec{q}_1 = \frac{(3,\ 1,\ -1,\ 3)}{\sqrt{20}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(\frac{3}{2\sqrt{5}},\ \frac{1}{2\sqrt{5}},\ -\frac{1}{2\sqrt{5}},\ \frac{3}{2\sqrt{5}} \right)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2,\ \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (-5,\ 1,\ 5,\ -7) - \frac{1}{20} \left[(-5,\ 1,\ 5,\ -7) \cdot (3,\ 1,\ -1,\ 3) \right] (3,\ 1,\ -1,\ 3)$$

$$= (-5,\ 1,\ 5,\ -7) + 2\,(3,\ 1,\ -1,\ 3)$$

$$= (1,\ 3,\ 3,\ -1)$$

$$\vec{q}_2 = \frac{1}{\sqrt{20}}(1,\ 3,\ 3,\ -1) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \left(\frac{1}{2\sqrt{5}},\ \frac{3}{2\sqrt{5}},\ \frac{3}{2\sqrt{5}},\ -\frac{1}{2\sqrt{5}} \right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(1, 1, -2, 8) \bullet (3, 1, -1, 3)}{20} (3, 1, -1, 3)$$

$$= \frac{3}{2} (3, 1, -1, 3)$$

$$= \left(\frac{9}{2}, \frac{3}{2}, -\frac{3}{2}, \frac{9}{2} \right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{(1, 1, -2, 8) \bullet (1, 3, 3, -1)}{20} (1, 3, 3, -1)$$

$$= -\frac{1}{2} (1, 3, 3, -1)$$

$$= \left(-\frac{1}{2}, -\frac{3}{2}, -\frac{3}{2}, \frac{1}{2} \right)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (1, 1, -2, 8) - \left(\frac{9}{2}, \frac{3}{2}, -\frac{3}{2}, \frac{9}{2} \right) - \left(-\frac{1}{2}, -\frac{3}{2}, -\frac{3}{2}, \frac{1}{2} \right)$$

$$= (-3, 1, 1, 3)$$

$$\vec{q}_3 = \frac{1}{2\sqrt{5}} (-3, 1, 1, 3) \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \left(-\frac{3}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}, \frac{3}{2\sqrt{5}} \right)$$

$$Q = \begin{pmatrix} \frac{3}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} & -\frac{3}{2\sqrt{5}} \\ \frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} \\ \frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} \\ \frac{3}{2\sqrt{5}} & -\frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} \end{pmatrix}$$

$$\langle \vec{u}_1, \vec{q}_1 \rangle = (3, 1, -1, 3) \bullet \left(\frac{3}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}, -\frac{1}{2\sqrt{5}}, \frac{3}{2\sqrt{5}} \right)$$

$$= \frac{10}{\sqrt{5}}$$

$$\langle \vec{u}_2, \vec{q}_1 \rangle = (-5, 1, 5, -7) \bullet \left(\frac{3}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}, -\frac{1}{2\sqrt{5}}, \frac{3}{2\sqrt{5}} \right)$$

$$= -\frac{20}{\sqrt{5}}$$

$$\langle \vec{u}_2, \vec{q}_2 \rangle = (-5, 1, 5, -7) \cdot \left(\frac{1}{2\sqrt{5}}, \frac{3}{2\sqrt{5}}, \frac{3}{2\sqrt{5}}, -\frac{1}{2\sqrt{5}} \right)$$

$$= \frac{10}{\sqrt{5}}$$

$$\langle \vec{u}_3, \vec{q}_1 \rangle = (1, 1, -2, 8) \cdot \left(\frac{3}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}, -\frac{1}{2\sqrt{5}}, \frac{3}{2\sqrt{5}} \right)$$

$$= \frac{15}{\sqrt{5}}$$

$$\langle \vec{u}_3, \vec{q}_2 \rangle = (1, 1, -2, 8) \cdot \left(\frac{1}{2\sqrt{5}}, \frac{3}{2\sqrt{5}}, \frac{3}{2\sqrt{5}}, -\frac{1}{2\sqrt{5}} \right)$$

$$= -\frac{5}{\sqrt{5}}$$

$$\langle \vec{u}_3, \vec{q}_3 \rangle = (1, 1, -2, 8) \cdot \left(-\frac{3}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}, \frac{3}{2\sqrt{5}} \right)$$

$$= \frac{10}{\sqrt{5}}$$

$$R = \begin{pmatrix} \frac{10}{\sqrt{5}} & -\frac{20}{\sqrt{5}} & \frac{15}{\sqrt{5}} \\ 0 & \frac{10}{\sqrt{5}} & -\frac{5}{\sqrt{5}} \\ 0 & 0 & \frac{10}{\sqrt{5}} \end{pmatrix} \qquad R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle & \langle \vec{u}_3, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle & \langle \vec{u}_3, \vec{q}_2 \rangle \\ 0 & 0 & \langle \vec{u}_3, \vec{q}_3 \rangle \end{bmatrix}$$

The QR–decomposition of the matrix is

$$\begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{pmatrix} = \begin{pmatrix} \frac{3}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} & -\frac{3}{2\sqrt{5}} \\ \frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} \\ \frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} \\ \frac{3}{2\sqrt{5}} & -\frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{10}{\sqrt{5}} & -\frac{20}{\sqrt{5}} & \frac{15}{\sqrt{5}} \\ 0 & \frac{10}{\sqrt{5}} & -\frac{5}{\sqrt{5}} \\ 0 & 0 & \frac{10}{\sqrt{5}} \end{pmatrix}$$

$$\quad A \qquad\quad = \qquad\qquad Q \qquad\qquad\qquad R$$

Exercise

Find the QR–decomposition of
$\begin{pmatrix} -1 & 2 & 2 \\ 2 & 5 & -1 \\ 2 & 8 & 0 \\ -3 & 3 & 5 \end{pmatrix}$

Solution

$$\begin{pmatrix} 1 & -2 & -2 \\ 2 & 5 & -1 \\ 2 & 8 & 0 \\ -3 & 3 & 5 \end{pmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \\ R_4 + 3R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & -2 & -2 \\ 0 & 9 & 3 \\ 0 & 12 & 4 \\ 0 & -3 & -1 \end{pmatrix} \begin{matrix} \\ \frac{1}{3}R_2 \\ \frac{1}{4}R_3 \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & -2 & -2 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \\ 0 & -3 & -1 \end{pmatrix} \begin{matrix} \\ \\ R_3 - R_2 \\ R_4 + R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & -2 & -2 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The matrix is linearly dependent, so ***doesn't*** have a QR–decomposition.

Exercise

Verify that the Cauchy-Schwarz inequality holds for the given vectors using the Euclidean inner product

$$\vec{u} = (0, \ -2, \ 2, \ 1), \quad \vec{v} = (-1, \ -1, \ 1, \ 1)$$

Solution

$$\langle \vec{u}, \ \vec{v} \rangle = 0 - 2(-1) + 2(1) + 1(1)$$
$$\underline{= 5 \,|}$$

$$\underline{\|\langle \vec{u}, \ \vec{v} \rangle\| = \sqrt{5} \,|}$$

$$\|\vec{u}\| \, \|\vec{v}\| = \sqrt{0+4+4+1} \, \sqrt{1+1+1+1}$$
$$= \sqrt{9}\sqrt{4}$$
$$= 6$$

$$\sqrt{5} < 6 \quad \Rightarrow \quad \left\|\langle \vec{u}, \, \vec{v} \rangle\right\| \leq \|\vec{u}\| \, \|\vec{v}\|$$

Exercise

Apply the Gram-Schmidt *orthonormalization* process in $C^0[-1, \, 1]$ spanned by the functions, using the inner product $\qquad f_1(x) = x+2, \quad f_2(x) = x^2 - 3x + 4$

Solution

Let $\quad \vec{u}_1 = f_1 = x+2, \quad \vec{u}_2 = f_2 = x^2 - 3x + 4$

$$\vec{v}_1 = \vec{u}_1 = x+2$$

$$\langle \vec{v}_1, \, \vec{v}_1 \rangle = \int_{-1}^{1} (x+2)^2 \, dx$$
$$= \tfrac{1}{3}(x+2)^3 \Big|_{-1}^{1}$$
$$= \tfrac{1}{3}(27-1)$$
$$= \frac{26}{3}$$

$$\langle \vec{u}_2, \, \vec{v}_1 \rangle = \int_{-1}^{1} \left(x^2 - 3x + 4\right)(x+2) \, dx$$
$$= \int_{-1}^{1} \left(x^3 - x^2 - 2x + 8\right) dx$$
$$= \left(\tfrac{1}{4}x^4 - \tfrac{1}{3}x^3 - x^2 + 8x\right)\Big|_{-1}^{1}$$
$$= \tfrac{1}{4} - \tfrac{1}{3} - 1 + 8 - \tfrac{1}{4} - \tfrac{1}{3} + 1 + 8$$
$$= \frac{46}{3}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$\vec{v}_2 = x^2 - 3x + 4 - \frac{46}{3}\left(\frac{3}{26}\right)(x+2)$$

$$= x^2 - 3x + 4 - \frac{23}{13}x - \frac{46}{13}$$

$$= x^2 - \frac{62}{13}x + \frac{6}{13}$$

The *orthogonal* basis is $\left\{x+2, \quad x^2 - \frac{62}{13}x + \frac{6}{13}\right\}$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^{1}\left(x^2 - \frac{62}{13}x + \frac{6}{13}\right)^2 dx$$

$$= \frac{1}{169}\int_{-1}^{1}\left(13x^2 - 62x + 6\right)^2 dx$$

$$= \frac{1}{169}\int_{-1}^{1}\left(169x^4 + 3{,}844x^2 + 36 - 1{,}612x^3 + 156x^2 - 744x\right) dx$$

$$= \frac{1}{169}\left(\frac{169}{5}x^5 + \frac{4{,}000}{3}x^3 + 36x - 403x^4 - 372x^2\right)\Big|_{-1}^{1}$$

$$= \frac{1}{169}\left(\frac{169}{5} + \frac{4{,}000}{3} + 36 - 403 - 372 + \frac{169}{5} + \frac{4{,}000}{3} + 36 + 403 + 372\right)$$

$$= \frac{1}{169}\left(\frac{338}{5} + \frac{8{,}000}{3} + 72\right)$$

$$= \frac{3{,}238}{195}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{\sqrt{3}}{\sqrt{26}}(x+2)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \sqrt{\frac{195}{3238}}\left(x^2 - \frac{62}{13}x + \frac{6}{13}\right)$$

The *orthonormal* basis is $\left\{\frac{\sqrt{3}}{\sqrt{26}}(x+2), \quad \sqrt{\frac{195}{3238}}\left(x^2 - \frac{62}{13}x + \frac{6}{13}\right)\right\}$

Exercise

Apply the Gram-Schmidt *orthonormalization* process in $C^0[-1, 1]$ spanned by the functions, using

the inner product $\qquad f_1(x) = 1 + 3x^2, \quad f_2(x) = x - x^2$

Solution

Let $\quad \vec{u}_1 = 1 + 3x^2, \quad \vec{u}_2 = x - x^2$

$\underline{\vec{v}_1 = \vec{u}_1 = 1 + 3x^2}$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^{1} \left(1 + 3x^2\right)^2 dx$$

$$= 2 \int_{0}^{1} \left(1 + 6x^2 + 9x^4\right) dx$$

$$= 2\left(x + 2x^3 + \frac{9}{5}x^5\right) \bigg|_{0}^{1}$$

$$= 2\left(1 + 2 + \frac{9}{5}\right)$$

$$\underline{= \frac{48}{5}}$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_{-1}^{1} \left(1 + 3x^2\right)\left(x - x^2\right) dx$$

$$= \int_{-1}^{1} \left(x - x^2 + 3x^3 - 3x^4\right) dx$$

$$= \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{3}{4}x^4 - \frac{3}{5}x^5 \bigg|_{-1}^{1}$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{3}{4} - \frac{3}{5} - \left(\frac{1}{2} + \frac{1}{3} + \frac{3}{4} + \frac{3}{5}\right)$$

$$\underline{= -\frac{28}{15}}$$

$$\vec{v}_2 = x - x^2 - \frac{5}{48}\left(-\frac{28}{15}\right)\left(1 + 3x^2\right) \qquad\qquad \vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$= x - x^2 + \frac{7}{36} + \frac{7}{12}x^2$$

$$= -\frac{5}{12}x^2 + x + \frac{7}{36}$$

The *orthogonal* basis is $\left\{ 1 + 3x^2, \quad -\frac{5}{12}x^2 + x + \frac{7}{36} \right\}$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^{1} \left(-\frac{5}{12}x^2 + x + \frac{7}{36} \right)^2 \, dx$$

$$= \frac{1}{1,296} \int_{-1}^{1} \left(-15x^2 + 36x + 7 \right)^2 \, dx$$

$$= \frac{1}{1,296} \int_{-1}^{1} \left(225x^4 - 1,080x^3 - 210x^2 + 504x + 49 \right) \, dx$$

$$= \frac{1}{1,296} \left(45x^5 - 270x^4 - 70x^3 + 252x^2 + 49x \, \Big|_{-1}^{1} \right)$$

$$= \frac{1}{648} \left(-270x^4 + 252x^2 \, \Big|_{0}^{1} \right)$$

$$= \frac{1}{648} \left(-270 + 252 \right)$$

$$= \frac{12}{648}$$

$$= \frac{1}{54}$$

$$\vec{q}_1 = \frac{\sqrt{5}}{\sqrt{48}} \left(1 + 3x^2 \right) \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{\sqrt{15}}{12} \left(1 + 3x^2 \right)$$

$$\vec{q}_2 = 54 \left(-\frac{4}{12}x^2 + x + \frac{7}{36} \right) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= -18x^2 + 54x + \frac{63}{2}$$

The *orthonormal* basis is $\left\{ \frac{\sqrt{15}}{12} \left(1 + 3x^2 \right), \quad -18x^2 + 54x + \frac{63}{2} \right\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1, 1]$ spanned by the functions, using

the inner product $\quad f_1(x) = 5x - 3, \quad f_2(x) = x^3 - x^2$

Solution

Let $\quad \vec{u}_1 = 5x - 3, \quad \vec{u}_2 = x^3 - x^2$

$\underline{\vec{v}_1 = \vec{u}_1 = 5x - 3}$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^{1} (5x - 3)^2 \, dx$$

$$= \int_{-1}^{1} \left(25x^2 - 30x + 9\right) dx$$

$$= \left(\frac{25}{3}x^3 - 15x^2 + 9x \right) \Big|_{-1}^{1}$$

$$= \frac{25}{3} - 15 + 9 + \frac{25}{3} + 15 + 9$$

$$= \underline{\frac{104}{3}}$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_{-1}^{1} \left(x^3 - x^2\right)(5x - 3) \, dx$$

$$= \int_{-1}^{1} \left(5x^4 - 8x^3 - 3x^2\right) + dx$$

$$= x^5 - 2x^4 + x^3 \Big|_{-1}^{1}$$

$$= \underline{4}$$

$$\vec{v}_2 = x^3 - x^2 - \frac{3}{104}(4)\left(1 + 3x^2\right) \qquad\qquad \vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1$$

$$= x^3 - x^2 - \frac{3}{26}\left(1 + 3x^2\right)$$

$$= \underline{x^3 - \frac{35}{26}x^2 - \frac{3}{26}}$$

The *orthogonal* basis is $\left\{ 1 + 3x^2, \quad x^3 - \frac{35}{26}x^2 - \frac{3}{26} \right\}$

$$\left\langle \vec{v}_2, \vec{v}_2 \right\rangle = \int_{-1}^{1} \left(x^3 - \frac{35}{26}x^2 - \frac{3}{26} \right)^2 dx$$

$$= \frac{1}{676} \int_{-1}^{1} \left(26x^3 - 35x^2 - 3 \right)^2 dx$$

$$= \frac{1}{676} \int_{-1}^{1} \left(676x^6 - 1{,}820x^5 + 1{,}225x^4 - 156x^3 + 210x^2 + 9 \right) dx$$

$$= \frac{1}{676} \left(\frac{676}{7}x^7 - \frac{910}{3}x^6 + 245x^5 - 39x^4 + 70x^3 + 9x \right) \Big|_{-1}^{1}$$

$$= \frac{2}{676} \left(\frac{676}{7} + 245 + 70 + 9 \right)$$

$$= \frac{1}{338} \left(\frac{2{,}944}{7} \right)$$

$$\underline{= \frac{1{,}472}{1{,}183}}$$

$$\underline{\vec{q}_1 = \frac{3}{104}\left(1 + 3x^2 \right)} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\underline{\vec{q}_2 = \frac{1{,}472}{1{,}183}\left(x^3 - \frac{35}{26}x^2 - \frac{3}{26} \right)} \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

The *orthonormal* basis is $\left\{ \frac{3}{104}\left(1 + 3x^2 \right), \ \frac{1{,}472}{1{,}183}\left(x^3 - \frac{35}{26}x^2 - \frac{3}{26} \right) \right\}$

Exercise

Apply the Gram-Schmidt **orthonormalization** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $\qquad f_1(x) = 1, \quad f_2(x) = 2x - 1$

Solution

Let $\quad \vec{u}_1 = 1, \quad \vec{u}_2 = 2x - 1$

$$\underline{\vec{v}_1 = \vec{u}_1 = 1}$$

$$\left\langle \vec{v}_1, \vec{v}_1 \right\rangle = \int_{-1}^{1} (1)^2 dx$$

$$= x \Big|_{-1}^{1}$$

$$\underline{\underline{= 2}}$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_{-1}^{1} (2x - 1)(1) \, dx$$

$$= x^2 - x \Big|_{-1}^{1}$$

$$\underline{\underline{= -2}}$$

$$\vec{v}_2 = 2x - 1 - \frac{1}{2}(-2)(1) \qquad\qquad \vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$\underline{= 2x}$$

The *orthogonal* basis is $\{ 1, \quad 2x \}$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^{1} (2x)^2 \, dx$$

$$= \int_{-1}^{1} 4x^2 \, dx$$

$$= \frac{4}{3} x^3 \Big|_{-1}^{1}$$

$$\underline{= \frac{8}{3}}$$

$$\underline{\vec{q}_1 = \frac{1}{\sqrt{2}}} \qquad\qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\underline{\vec{q}_2 = \frac{3}{\sqrt{2}} x} \qquad\qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

The *orthonormal* basis is $\left\{ \frac{1}{2}, \quad \frac{3}{4} x \right\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $\quad f_1(x) = e^x, \quad f_2(x) = x$

Solution

Let $\quad \vec{u}_1 = e^x, \quad \vec{u}_2 = x$

$\underline{\vec{v}_1 = \vec{u}_1 = e^x}\ |$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^{1} e^{2x} dx$$

$$= \frac{1}{2} e^x \Big|_{-1}^{1}$$

$$\underline{= \frac{1}{2}\left(e - e^{-1}\right)}\ |$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_{-1}^{1} x\, e^x\, dx$$

$$= e^x (x-1) \Big|_{-1}^{1}$$

$$= (0)e + 2e$$

$$\underline{= 2e}\ |$$

$$\vec{v}_2 = x - \frac{2}{e - e^{-1}}(2e)\left(e^x\right) \qquad\qquad \vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$\underline{= x - \frac{4e^2}{e^2 - 1} e^x}\ |$$

The orthogonal basis is $\left\{ e^x, \quad x - \dfrac{4e^2}{e^2 - 1} e^x \right\}$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^{1} \left(x - \frac{4e^2}{e^2 - 1} e^x \right)^2 dx$$

$$= \int_{-1}^{1} \left(x^2 - \frac{8e^2}{e^2 - 1} x e^x + \left(\frac{4e^2}{e^2 - 1}\right)^2 e^{2x} \right) dx$$

$$= \frac{1}{3}x^3 - \frac{8e^2}{e^2-1}(x-1)e^x + \frac{8e^4}{\left(e^2-1\right)^2}e^{2x} \Bigg|_{-1}^{1}$$

$$= \frac{1}{3} + \frac{8e^4}{\left(e^2-1\right)^2}e^2 + \frac{1}{3} + \frac{16e^2}{e^2-1}e^{-1} + \frac{8e^4}{\left(e^2-1\right)^2}e^{-2}$$

$$= \frac{2}{3} + \frac{8e^6}{\left(e^2-1\right)^2} + \frac{16e}{e^2-1} + \frac{8e^2}{\left(e^2-1\right)^2}$$

$$= \frac{1}{\left(e^2-1\right)^2}\left(\frac{2}{3}e^4 - \frac{4}{3}e^2 + \frac{4}{3} + 8e^6 + 16e^3 - 16e + 8e^2\right)$$

$$= \frac{1}{\left(e^2-1\right)^2}\left(8e^6 + \frac{2}{3}e^4 + 16e^3 + \frac{20}{3}e^2 - 16e + \frac{4}{3}\right)$$

$$\vec{q}_1 = \sqrt{\frac{2}{e-e^{-1}}}\,e^x \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\vec{q}_2 = \frac{e^2-1}{8e^6 + \frac{2}{3}e^4 + 16e^3 + \frac{20}{3}e^2 - 16e + \frac{4}{3}}\left(x - \frac{4e^2}{e^2-1}e^x\right) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

The *orthonormal* basis is

$$\left\{ \sqrt{\frac{2}{e-e^{-1}}}\,e^x,\ \frac{e^2-1}{8e^6 + \frac{2}{3}e^4 + 16e^3 + \frac{20}{3}e^2 - 16e + \frac{4}{3}}\left(x - \frac{4e^2}{e^2-1}e^x\right)\right\}$$

Exercise

Apply the Gram-Schmidt *orthonormalization* process in $C^0[-1, 1]$ spanned by the functions, using the inner product

$$f_1(x) = x, \quad f_2(x) = x^3, \quad f_3(x) = x^5$$

Solution

Let $\quad \vec{u}_1 = f_1 = x, \quad \vec{u}_2 = f_2 = x^3, \quad \vec{u}_3 = f_3 = x^5$

$$\vec{v}_1 = \vec{u}_1 = x$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^{1} x^2 dx$$

$$= \frac{1}{3}x^3 \Big|_{-1}^{1}$$

$$= \frac{2}{3}$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_{-1}^{1} x^4 \, dx$$

$$= \frac{1}{5}x^5 \Big|_{-1}^{1}$$

$$= \frac{2}{5}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= x^3 - \frac{2}{5}\left(\frac{3}{2}\right)(x)$$

$$= x^3 - \frac{3}{5}x$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^{1} \left(x^3 - \frac{3}{5}x\right)^2 dx$$

$$= \int_{-1}^{1} \left(x^6 - \frac{6}{5}x^4 + \frac{9}{25}x^2\right) dx$$

$$= \left(\frac{1}{7}x^7 - \frac{6}{25}x^5 + \frac{3}{25}x^3\right)\Big|_{-1}^{1}$$

$$= 2\left(\frac{1}{7} - \frac{6}{25} + \frac{3}{25}\right)$$

$$= \frac{8}{175}$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_{-1}^{1} x^6 \, dx$$

$$= \frac{1}{7}x^7 \Big|_{-1}^{1}$$

$$= \frac{2}{7}$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^{1} x^5 \left(x^3 - \frac{3}{5}x\right) dx$$

$$= \int_{-1}^{1} \left(x^8 - \frac{3}{5}x^6 \right) dx$$

$$= \left(\frac{1}{9}x^9 - \frac{3}{35}x^7 \right) \Big|_{-1}^{1}$$

$$= 2\left(\frac{1}{9} - \frac{3}{35} \right)$$

$$= \underline{\frac{16}{315}}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3 , \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3 , \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= x^5 - \frac{16}{315}\left(\frac{175}{8} \right)\left(x^3 - \frac{3}{5}x \right) - \frac{2}{7}\left(\frac{3}{2} \right)x$$

$$= x^5 - \frac{70}{63}\left(x^3 - \frac{3}{5}x \right) - \frac{3}{7}x$$

$$= x^5 - \frac{70}{63}x^3 + \frac{14}{21}x - \frac{3}{7}x$$

$$= \underline{x^5 - \frac{70}{63}x^3 + \frac{5}{21}x}$$

The orthogonal basis is $\left\{ x, \quad x^3 - \frac{3}{5}x , \quad x^5 - \frac{70}{63}x^3 + \frac{5}{21}x \right\}$

$$\langle \vec{v}_3 , \vec{v}_3 \rangle = \int_{-1}^{1} \left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x \right)^2 dx$$

$$= \int_{-1}^{1} \frac{1}{3,969}\left(63x^5 - 70x^3 + 15x \right)^2 dx$$

$$= \frac{1}{3,969} \int_{-1}^{1} \left(3,969x^{10} - 8,820x^8 + 1,890x^6 - 2,100x^4 + 4,900x^6 + 225x^2 \right) dx$$

$$= \frac{1}{3,969}\left(\frac{3,969}{11}x^{11} - 980x^9 + 970x^7 - 420x^5 + 75x^3 \right) \Big|_{-1}^{1}$$

$$= \frac{2}{3,969}\left(\frac{3,969}{11} - 980 + 970 - 420 + 75 \right)$$

$$= \frac{2}{3,969}\left(\frac{3,969}{11} - 355 \right)$$

$$= \frac{2}{3,969}\left(\frac{64}{11} \right)$$

$$= \underline{\frac{128}{43,659}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{x}{\sqrt{2/3}}$$

$$= \frac{\sqrt{3}}{\sqrt{2}} x$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \sqrt{\frac{175}{8}} \left(x^3 - \frac{3}{5} x \right)$$

$$= \frac{5\sqrt{7}}{2\sqrt{2}} \left(x^3 - \frac{3}{5} x \right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \sqrt{\frac{43,659}{128}} \left(x^5 - \frac{70}{63} x^3 + \frac{5}{21} x \right)$$

$$= \frac{63\sqrt{11}}{8\sqrt{2}} \left(x^5 - \frac{70}{63} x^3 + \frac{5}{21} x \right)$$

The *orthonormal* basis is $\left\{ \frac{\sqrt{3}}{\sqrt{2}} x, \quad \frac{5}{2}\sqrt{\frac{7}{2}} \left(x^3 - \frac{3}{5} x \right), \quad \frac{63}{8}\sqrt{\frac{11}{2}} \left(x^5 - \frac{70}{63} x^3 + \frac{5}{21} x \right) \right\}$

Exercise

Apply the Gram-Schmidt *orthonormalization* process in $C^0[-1, 1]$ spanned by the functions, using

the inner product $f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x) = \frac{1}{2}(3x^2 - 1)$

Solution

Let $\vec{u}_1 = f_1 = 1, \quad \vec{u}_2 = f_2 = x, \quad \vec{u}_3 = f_3 = \frac{3}{2} x^2 - \frac{1}{2}$

$$\vec{v}_1 = \vec{u}_1 = 1$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^{1} 1 \, dx$$

$$= x \Big|_{-1}^{1}$$

$$\underline{\underline{= 2}}\ \big|$$

$$\left\langle \vec{u}_2, \vec{v}_1 \right\rangle = \int_{-1}^{1} x\ dx$$

$$= \frac{1}{2} x^2 \ \bigg|_{-1}^{1}$$

$$\underline{\underline{= 0}}\ \big|$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1$$

$$= x - \frac{0}{2}(1)$$

$$\underline{= x}\ \big|$$

$$\left\langle \vec{v}_2, \vec{v}_2 \right\rangle = \int_{-1}^{1} x^2\ dx$$

$$= \frac{1}{3} x^3 \ \bigg|_{-1}^{1}$$

$$\underline{\underline{= \frac{2}{3}}}\ \big|$$

$$\left\langle \vec{u}_3, \vec{v}_1 \right\rangle = \frac{1}{2} \int_{-1}^{1} \left(3x^2 - 1 \right) dx$$

$$= \frac{1}{2} \left(x^3 - x \right) \ \bigg|_{-1}^{1}$$

$$\underline{\underline{= 0}}\ \big|$$

$$\left\langle \vec{u}_3, \vec{v}_2 \right\rangle = \frac{1}{2} \int_{-1}^{1} x \left(3x^2 - 1 \right) dx$$

$$= \frac{1}{2} \int_{-1}^{1} \left(3x^3 - x \right) dx$$

$$= \frac{1}{2} \left(\frac{3}{4} x^4 - \frac{1}{2} x^2 \right) \ \bigg|_{-1}^{1}$$

$$\underline{\underline{= 0}}\ \big|$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= \frac{3}{2}x^2 - \frac{1}{2} - \frac{0}{1}(1) - \frac{0}{\frac{2}{3}}(x)$$

$$\underline{= \frac{1}{2}\left(3x^2 - 1\right)}$$

The *orthogonal* basis is $\left\{ 1, \ x, \ \frac{1}{2}\left(3x^2 - 1\right) \right\}$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \frac{1}{4} \int_{-1}^{1} \left(3x^2 - 1\right)^2 dx$$

$$= \frac{1}{4} \int_{-1}^{1} \left(9x^4 - 6x^2 + 1\right) dx$$

$$= \frac{1}{4}\left(\frac{9}{5}x^5 - 2x^3 + x \Big|_{-1}^{1}\right)$$

$$= \frac{1}{2}\left(\frac{9}{5} - 2 + 1\right)$$

$$\underline{= \frac{2}{5}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\underline{= \frac{1}{\sqrt{2}}}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$\underline{= \sqrt{\frac{3}{2}} \, x}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\sqrt{\frac{2}{5}}} \frac{1}{2}\left(3x^2 - 1\right)$$

$$\underline{= \frac{1}{2}\sqrt{\frac{5}{2}}\left(3x^2 - 1\right)}$$

The *orthonormal* basis is $\left\{ \frac{1}{\sqrt{2}}, \ \sqrt{\frac{3}{2}}x, \ \frac{1}{2}\sqrt{\frac{5}{2}}\left(3x^2 - 1\right) \right\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $\qquad f_1(x) = 5, \quad f_2(x) = x^2 - 6x, \quad f_3(x) = (3-x)^2$

<u>***Solution***</u>

Let $\quad \vec{u}_1 = 5, \quad \vec{u}_2 = x^2 - 6x, \quad \vec{u}_3 = x^2 - 6x + 9$

$$\underline{\vec{v}_1 = \vec{u}_1 = 5}$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^{1} 5 \, dx$$

$$= 5x \Big|_{-1}^{1}$$

$$\underline{\underline{= 10}}$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = 5 \int_{-1}^{1} \left(x^2 - 6x \right) dx$$

$$= 5\left(\frac{1}{3}x^3 - 3x^2 \right) \Big|_{-1}^{1}$$

$$\underline{= \frac{10}{3}}$$

$$\vec{v}_2 = x^2 - 6x + 9 - \frac{1}{100}\left(\frac{10}{3} \right)(5) \qquad\qquad \vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1$$

$$= x^2 - 6x + 9 - \frac{1}{6}$$

$$\underline{= x^2 - 6x + \frac{53}{6}}$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \frac{1}{36} \int_{-1}^{1} \left(6x^2 - 36x + 53 \right)^2 dx$$

$$= \frac{1}{36} \int_{-1}^{1} \left(36x^4 - 432x^3 + 1{,}932x^2 - 3{,}816x + 2{,}809 \right) dx$$

$$= \frac{1}{36}\left(\frac{36}{5}x^5 - 108^4 + 644x^3 - 1{,}908x^2 + 2{,}809x \right) \Big|_{-1}^{1}$$

$$= \frac{1}{18}\left(\frac{36}{5}x^5 + 644x^3 + 2{,}809x\right)\bigg|_0^1$$

$$= \frac{1}{18}\left(\frac{36}{5} + 644 + 2{,}809\right)$$

$$= \frac{17{,}301}{90}$$

$$= \frac{5{,}767}{30}$$

$$\left\langle \vec{u}_3, \vec{v}_1 \right\rangle = 5\int_{-1}^{1}\left(x^2 - 6x + 9\right)dx$$

$$= 5\left(\frac{1}{3}x^3 - 3x^2 + 9x\right)\bigg|_{-1}^1$$

$$= 10\left(\frac{1}{3}x^3 + 9x\right)\bigg|_0^1$$

$$= 10\left(\frac{1}{3} + 9\right)$$

$$= \frac{280}{3}$$

$$\left\langle \vec{u}_3, \vec{v}_2 \right\rangle = \int_{-1}^{1}\left(x^2 - 6x + 9\right)\left(x^2 - 6x + \frac{53}{6}\right)dx$$

$$= \int_{-1}^{1}\left(x^4 - 12x^3 + \frac{323}{6}x^2 - 107x + \frac{159}{2}\right)dx$$

$$= \left(\frac{1}{5}x^5 - 3x^4 + \frac{323}{18}x^3 - \frac{107}{2}x^2 + \frac{159}{2}x\right)\bigg|_{-1}^1$$

$$= 2\left(\frac{1}{5}x^5 + \frac{323}{18}x^3 + \frac{159}{2}x\right)\bigg|_0^1$$

$$= 2\left(\frac{1}{5} + \frac{323}{18} + \frac{159}{2}\right)$$

$$= 2\left(\frac{18 + 1{,}615 + 7{,}155}{90}\right)$$

$$= \frac{8{,}788}{45}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\left\langle \vec{u}_3, \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1 - \frac{\left\langle \vec{u}_3, \vec{v}_2 \right\rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2$$

$$= x^2 - 6x + 9 - \left(\frac{280}{3}\right)\left(\frac{1}{10}\right)(5) - \left(\frac{8{,}788}{45}\right)\left(\frac{30}{5{,}767}\right)\left(x^2 - 6x + \frac{53}{6}\right)$$

$$= x^2 - 6x + 9 - \frac{140}{3} - \frac{17,576}{17,301}\left(x^2 - 6x + \frac{53}{6}\right)$$

$$= x^2 - 6x - \frac{113}{3} - \frac{17,576}{17,301}x^2 - \frac{35,152}{5,767}x + \frac{465,764}{51,903}$$

$$= -\frac{275}{17,301}x^2 - \frac{69,754}{5,767}x - \frac{1,489,249}{51,903}$$

The *orthogonal* basis is $\left\{ 5, \ x^2 - 6x + \frac{53}{6}, \ -\frac{275}{17,301}x^2 - \frac{69,754}{5,767}x - \frac{1,489,249}{51,903} \right\}$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_{-1}^{1}\left(-\frac{275}{17,301}x^2 - \frac{69,754}{5,767}x - \frac{1,489,249}{51,903}\right)^2 dx$$

$$= \frac{4,700,107,122,016}{2,693,921,409}$$

$$\vec{q}_1 = \frac{5}{\sqrt{10}} \qquad\qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\vec{q}_2 = \sqrt{\frac{60}{5,767}}\left(x^2 - 6x + \frac{53}{6}\right) \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$\vec{q}_3 = \frac{2,693,921,409}{4,700,107,122,016}\left(-\frac{275}{17,301}x^2 - \frac{69,754}{5,767}x - \frac{1,489,249}{51,903}\right) \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\sqrt{\frac{2}{5}}}\frac{1}{2}\left(3x^2 - 1\right)$$

$$= \frac{1}{2}\sqrt{\frac{5}{2}}\left(3x^2 - 1\right)$$

The *orthonormal* basis is

$$\left\{ \frac{5}{\sqrt{10}}, \ \sqrt{\frac{60}{5,767}}\left(x^2 - 6x + \frac{53}{6}\right), \ \frac{2,693,921,409}{4,700,107,122,016}\left(-\frac{275}{17,301}x^2 - \frac{69,754}{5,767}x - \frac{1,489,249}{51,903}\right) \right\}$$

Exercise

Apply the Gram-Schmidt **orthonormalization** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $f_1(x) = 1, \quad f_2(x) = \sin \pi x, \quad f_3(x) = \cos \pi x$

<u>Solution</u>

Let $\quad \vec{u}_1 = 1, \quad \vec{u}_2 = \sin \pi x, \quad \vec{u}_3 = \cos \pi x$

$$\vec{v}_1 = \vec{u}_1 = 1$$

$$\left\langle \vec{v}_1, \vec{v}_1 \right\rangle = \int_{-1}^{1} 1 \, dx$$

$$= x \Big|_{-1}^{1}$$

$$= 2$$

$$\left\langle \vec{u}_2, \vec{v}_1 \right\rangle = \int_{-1}^{1} \sin \pi x \, dx$$

$$= -\frac{1}{\pi} \cos \pi x \Big|_{-1}^{1}$$

$$= 0$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2, \vec{v}_1 \right\rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= \sin \pi x$$

$$\left\langle \vec{v}_2, \vec{v}_2 \right\rangle = \int_{-1}^{1} \sin^2 \pi x \, dx$$

$$= \frac{1}{2} \int_{-1}^{1} (1 - \cos 2\pi x) \, dx$$

$$= \frac{1}{2} \left(x - \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^{1}$$

$$= 1$$

$$\left\langle \vec{u}_3, \vec{v}_1 \right\rangle = \int_{-1}^{1} \cos \pi x \, dx$$

$$= \frac{1}{\pi} \sin \pi x \Big|_{-1}^{1}$$

$$= 0$$

$$\left\langle \vec{u}_3, \vec{v}_2 \right\rangle = \int_{-1}^{1} \cos \pi x \sin \pi x \, dx$$

$$= \frac{1}{2} \int_{-1}^{1} \sin 2\pi x \ dx$$

$$= -\frac{1}{4\pi} \cos 2\pi x \ \Big|_{-1}^{1}$$

$$\underline{= 0} \ |$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= \cos \pi x - \frac{0}{} - \frac{0}{}$$

$$\underline{= \cos \pi x} \ |$$

The *orthogonal* basis is $\left\{ 1, \ \sin \pi x - \frac{1}{\pi}, \ \cos \pi x \right\}$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_{-1}^{1} \cos^2 \pi x \ dx$$

$$= \frac{1}{2} \int_{-1}^{1} (1 + \cos 2\pi x) \ dx$$

$$= \frac{1}{2} \left(x + \frac{1}{2\pi} \sin 2\pi x \ \Big|_{-1}^{1} \right)$$

$$\underline{= 1} \ |$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\underline{= \frac{1}{\sqrt{2}}} \ |$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$\underline{= \sin \pi x} \ |$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$\underline{= \cos \pi x} \ |$$

The *orthonormal* basis is $\left\{ \frac{1}{\sqrt{2}}, \ \sin \pi x, \ \cos \pi x \right\}$

Exercise

Apply the Gram-Schmidt *orthonormalization* process in $C^0[-1, 1]$ spanned by the functions, using

the inner product $f_1(x) = 1, \quad f_2(x) = \sin x, \quad f_3(x) = \sin 2x$

Solution

Let $\vec{u}_1 = 1, \quad \vec{u}_2 = \sin x, \quad \vec{u}_3 = \sin 2x$

$$\underline{\vec{v}_1 = \vec{u}_1 = 1}$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^{1} 1\, dx$$

$$= x \Big|_{-1}^{1}$$

$$\underline{\underline{= 2}}$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_{-1}^{1} \sin x\, dx$$

$$= -\cos x \Big|_{-1}^{1}$$

$$= -\left(\cos 1 - \cos(-1)\right)$$

$$= -\left(\cos 1 - \cos 1\right)$$

$$\underline{\underline{= 0}}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= \sin x - (0)$$

$$\underline{\underline{= \sin x}}$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^{1} \sin^2 x\, dx$$

$$= \frac{1}{2} \int_{-1}^{1} (1 - \cos 2x)\, dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) \Big|_{-1}^{1}$$

$$= \tfrac{1}{2}\left(1 - \tfrac{1}{2}\sin 2 + 1 - \tfrac{1}{2}\sin 2\right)$$

$$= 1 - \tfrac{1}{2}\sin 2$$

$$\underline{= 1 - \sin(1)\cos(1)} \mid$$

$$\left\langle \vec{u}_3, \vec{v}_1 \right\rangle = \int_{-1}^{1} \sin 2x \, dx$$

$$= -\tfrac{1}{2}\cos 2x \Big|_{-1}^{1}$$

$$= -\tfrac{1}{2}\left(\cos 2 - \cos 2\right)$$

$$\underline{= 0} \mid$$

$$\left\langle \vec{u}_3, \vec{v}_2 \right\rangle = \int_{-1}^{1} \sin 2x \, \sin x \, dx$$

$$= 2\int_{-1}^{1} \sin x \, \cos x \, \sin x \, dx$$

$$= 2\int_{-1}^{1} \sin^2 x \, d(\sin x)$$

$$= \tfrac{2}{3}\sin^3 x \Big|_{-1}^{1}$$

$$= \tfrac{2}{3}\left(\sin^3 1 + \sin^3 1\right)$$

$$\underline{= \tfrac{4}{3}\sin^3 1} \mid$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\left\langle \vec{u}_3, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2}\vec{v}_1 - \frac{\left\langle \vec{u}_3, \vec{v}_2 \right\rangle}{\left\| \vec{v}_2 \right\|^2}\vec{v}_2$$

$$= \sin 2x - (0) - \tfrac{4}{3}\sin^3(1)\frac{1}{1 - \sin(1)\cos(1)}\sin x$$

$$\underline{= \sin 2x - \frac{4}{3}\frac{\sin^3(1)}{1 - \sin(1)\cos(1)}\sin x} \mid$$

The *orthogonal* basis is $\left\{ 1, \quad \sin x, \quad \sin 2x - \dfrac{4}{3}\dfrac{\sin^3(1)}{1 - \sin(1)\cos(1)}\sin x \right\}$

$$\left\langle \vec{v}_3, \vec{v}_3 \right\rangle = \int_{-1}^{1} \left(\sin 2x - \frac{4\sin^3(1)}{3-3\sin(1)\cos(1)} \sin x \right)^2 dx$$

$$= \int_{-1}^{1} \left(\sin^2 2x - \frac{8\sin^3(1)}{3-3\sin(1)\cos(1)} \sin 2x \sin x + \frac{16\sin^6(1)}{\left(3-3\sin(1)\cos(1)\right)^2} \sin^2 x \right) dx$$

$$= \int_{-1}^{1} \left(\frac{1}{2} - \frac{1}{2}\cos 4x \right) dx - \frac{16\sin^3(1)}{3-3\sin(1)\cos(1)} \int_{-1}^{1} \sin^2 x \, d(\sin x)$$

$$+ \frac{8\sin^6(1)}{\left(3-3\sin(1)\cos(1)\right)^2} \int_{-1}^{1} (1-\cos 2x) \, dx$$

$$= \frac{1}{2}x - \frac{1}{8}\sin 4x - \frac{16\sin^3(1)}{9-9\sin(1)\cos(1)} \sin^3 x + \frac{8\sin^6(1)}{\left(3-3\sin(1)\cos(1)\right)^2}\left(x - \frac{1}{2}\sin 2x\right) \Big|_{-1}^{1}$$

$$= \frac{1}{2} - \frac{1}{8}\sin 4 - \frac{16\sin^6 1}{9-9\sin 1\cos 1} + \frac{8\sin^6 1}{\left(3-3\sin 1\cos 1\right)^2}\left(1 - \frac{1}{2}\sin 2\right)$$

$$- \left(-\frac{1}{2} + \frac{1}{8}\sin 4 + \frac{16\sin^6 1}{9-9\sin 1\cos 1} + \frac{8\sin^6 1}{\left(3-3\sin 1\cos 1\right)^2}\left(-1 + \frac{1}{2}\sin 2\right) \right)$$

$$= \frac{1}{2} - \frac{1}{8}\sin 4 - \frac{16\sin^6 1}{9-9\sin 1\cos 1} + \frac{8\sin^6 1}{\left(3-3\sin 1\cos 1\right)^2} - \frac{4\sin^6 1\sin 2}{\left(3-3\sin 1\cos 1\right)^2}$$

$$+ \frac{1}{2} - \frac{1}{8}\sin 4 - \frac{16\sin^6 1}{9-9\sin 1\cos 1} + \frac{8\sin^6 1}{\left(3-3\sin 1\cos 1\right)^2} - \frac{4\sin^6 1\sin 2}{\left(3-3\sin 1\cos 1\right)^2}$$

$$= 1 - \frac{1}{4}\sin 4 - \frac{32\sin^6 1}{9-9\sin 1\cos 1} + \frac{16\sin^6 1}{\left(3-3\sin 1\cos 1\right)^2} - \frac{8\sin^6 1\sin 2}{\left(3-3\sin 1\cos 1\right)^2}$$

$$\boxed{\vec{q}_1 = \frac{1}{\sqrt{2}}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\boxed{\vec{q}_2 = \frac{\sin x}{\sqrt{1-\sin(1)\cos(1)}}} \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$\vec{q}_3 = \frac{\sin 2x - \frac{4}{3}\frac{\sin^3(1)}{1-\sin(1)\cos(1)} \sin x}{\sqrt{1 - \frac{1}{4}\sin 4 - \frac{32\sin^6 1}{9-9\sin 1\cos 1} + \frac{16\sin^6 1}{\left(3-3\sin 1\cos 1\right)^2} - \frac{8\sin^6 1\sin 2}{\left(3-3\sin 1\cos 1\right)^2}}} \qquad\qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

The *orthonormal* basis is

$$\left\{ \frac{1}{\sqrt{2}}, \ \frac{\sin x}{\sqrt{1-\sin(1)\cos(1)}}, \right.$$

$$\left. \frac{\sin 2x - \frac{4}{3}\frac{\sin^3(1)}{1-\sin(1)\cos(1)}\sin x}{\sqrt{1-\frac{1}{4}\sin 4 - \frac{32\sin^6 1}{9-9\sin 1\cos 1} + \frac{16\sin^6 1}{(3-3\sin 1\cos 1)^2} - \frac{8\ \sin^6 1\ \sin 2}{(3-3\sin 1\cos 1)^2}}} \right\}$$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1, 1]$ spanned by the functions, using

the inner product $f_1(x) = 6, \quad f_2(x) = 3\sin^2 x, \quad f_3(x) = 2\cos^2 x$

Solution

Let $\vec{u}_1 = 6, \quad \vec{u}_2 = 3\sin^2 x, \quad \vec{u}_3 = 2\cos^2 x$

$\underline{\vec{v}_1 = \vec{u}_1 = 6}$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^{1} 6\, dx$$

$$\underline{\underline{= 12}}$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = 18 \int_{-1}^{1} \sin^2 x \, dx$$

$$= 9 \int_{-1}^{1} (1 - \cos 2x)\, dx$$

$$= 9\left(x - \frac{1}{2}\sin 2x \right) \Big|_{-1}^{1}$$

$$= 9\left(1 - \frac{1}{2}\sin 2 + 1 - \frac{1}{2}\sin 2 \right)$$

$$\underline{\underline{= 18 - 9\sin 2}}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2}\vec{v}_1$$

$$= 3\sin^2 x - \frac{1}{12}(18 - 9\sin 2)(6)$$

$$= 3\sin^2 x + \frac{9}{2}\sin 2 - 9$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^{1} \left(3\sin^2 x + \frac{9}{2}\sin 2 - 9 \right)^2 dx$$

$$= \int_{-1}^{1} \left(9\sin^4 x + 6\left(\frac{9}{2}\sin 2 - 9\right)\sin^2 x + \left(\frac{9}{2}\sin 2 - 9\right)^2 \right) dx$$

$$= \int_{-1}^{1} \left(9\sin^4 x + 6\left(\frac{9}{2}\sin 2 - 9\right)\sin^2 x + \left(\frac{9}{2}\sin 2 - 9\right)^2 \right) dx$$

$$9\int_{-1}^{1} \sin^4 x \, dx = \frac{9}{4}\int_{-1}^{1} (1 - \cos 2x)^2 \, dx$$

$$= \frac{9}{4}\int_{-1}^{1} \left(1 - 2\cos 2x + \cos^2 2x \right) dx$$

$$= \frac{9}{4}\int_{-1}^{1} \left(1 - 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x \right) dx$$

$$= \frac{9}{4}\int_{-1}^{1} \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x \right) dx$$

$$= \frac{9}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x \right) \bigg|_{-1}^{1}$$

$$= \frac{9}{4}\left(3 - 2\sin 2 + \frac{1}{4}\sin 4 \right)$$

$$6\left(\frac{9}{2}\sin 2 - 9\right)\int_{-1}^{1} \sin^2 x \, dx = \left(\frac{27}{2}\sin 2 - 27\right)\int_{-1}^{1} (1 - \cos 2x) \, dx$$

$$= \frac{27}{2}(\sin 2 - 2)\left(x - \frac{1}{2}\sin 2x \right) \bigg|_{-1}^{1}$$

$$= 27(\sin 2 - 2)\left(1 - \frac{1}{2}\sin 2 \right)$$

$$\int_{-1}^{1} \left(\frac{9}{2}\sin 2 - 9\right)^2 dx = 2\left(\frac{9}{2}\sin 2 - 9\right)^2$$

$$= \frac{27}{4} - \frac{9}{2}\sin 2 + \frac{9}{16}\sin 4 + 27(\sin 2 - 2)\left(1 - \frac{1}{2}\sin 2 \right) + 2\left(\frac{9}{2}\sin 2 - 9\right)^2$$

$$= \frac{27}{4} - \frac{9}{2}\sin 2 + \frac{9}{16}\sin 4 + 54\sin 2 - \frac{27}{2}\sin^2 2 - 54 + \frac{81}{2}\sin^2 2 - 162\sin 2 + 162$$

$$= \frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2 \,|$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = 12 \int_{-1}^{1} \cos^2 x \, dx$$

$$= 6 \int_{-1}^{1} (1 + \cos 2x) \, dx$$

$$= 12 \left(x + \frac{1}{2}\sin 2x \right) \Big|_0^1$$

$$= 12 + 6\sin 2 \,|$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^{1} \left(3\sin^2 x + \frac{9}{2}\sin 2 - 9 \right)\left(2\cos^2 x \right) dx$$

$$= 3 \int_{-1}^{1} \left(2\sin^2 x \cos^2 x + (3\sin 2 - 6)\cos^2 x \right) dx$$

$$\int_{-1}^{1} 2\sin^2 x \cos^2 x \, dx = \frac{1}{2} \int_{-1}^{1} (1 - \cos 2x)(1 + \cos 2x) \, dx$$

$$= \frac{1}{2} \int_{-1}^{1} \left(1 - \cos^2 2x \right) dx$$

$$= \frac{1}{2} \int_{-1}^{1} \left(\frac{1}{2} - \frac{1}{2}\cos 4x \right) dx$$

$$= \frac{1}{2} \left(x - \frac{1}{4}\sin 4x \right) \Big|_0^1$$

$$= \frac{1}{2}(1 - \sin 4)$$

$$(3\sin 2 - 6)\int_{-1}^{1} \cos^2 x \, dx = \frac{1}{2}(3\sin 2 - 6)\int_{-1}^{1} (1 + \cos 2x) \, dx$$

$$= (3\sin 2 - 6)\left(x + \frac{1}{2}\sin 2x \right) \Big|_0^1$$

$$= (3\sin 2 - 6)\left(1 + \frac{1}{2}\sin 2 \right)$$

$$= \frac{3}{2}\sin^2 2 - 6$$

$$= \frac{1}{2} - \frac{1}{2}\sin 4 + \frac{3}{2}\sin^2 2 - 6$$

$$= \frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2} \mid$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= 2\cos^2 x - \left(\frac{1}{12}\right)(12 + 6\sin 2)(6) - \frac{\left(3\sin^2 x + \frac{9}{2}\sin 2 - 9\right)\left(\frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2}\right)}{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2}$$

$$= 2\cos^2 x - 6 - 3\sin 2 - \frac{\left(\frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2}\right)}{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2}\left(3\sin^2 x + \frac{9}{2}\sin 2 - 9\right) \mid$$

The *orthogonal* basis is

$$\left\{ \begin{array}{l} 6, \quad 3\sin^2 x + \frac{9}{2}\sin 2 - 9 \ , \\[2mm] 2\cos^2 x - 6 - 3\sin 2 - \dfrac{\left(\frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2}\right)}{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2}\left(3\sin^2 x + \frac{9}{2}\sin 2 - 9\right) \end{array} \right\}$$

$$\vec{v}_3 = 2\cos^2 x - \frac{3\left(\frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2}\right)}{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2}\sin^2 x$$

$$- \left(\frac{\left(\frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2}\right)\left(\frac{9}{2}\sin 2 - 9\right)}{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2} + 6 + 3\sin 2 \right)$$

$$2\int_{-1}^{1}\cos^2 x \, dx = \int_{-1}^{1}(1 + \cos 2x) \, dx$$

$$= x + \frac{1}{2}\sin 2x \Big|_{-1}^{1}$$

$$= 2 + \sin 2 \mid$$

$$\int_{-1}^{1}\sin^2 x \, dx = \frac{1}{2}\int_{-1}^{1}(1 - \cos 2x) \, dx$$

$$= \left(x - \frac{1}{2}\sin 2x \; \Big|_0^1 \right.$$

$$= 1 - \frac{1}{2}\sin 2$$

$$\int_{-1}^{1} dx = 2$$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = 2 + \sin 2 - \frac{3\left(\frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2} \right)\left(1 - \frac{1}{2}\sin 2 \right)}{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2}$$

$$- 2\left(\frac{\left(\frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2} \right)\left(\frac{9}{2}\sin 2 - 9 \right)}{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2} + 6 + 3\sin 2 \right)$$

$$\vec{q}_1 = \frac{6}{\sqrt{12}} \qquad\qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{3}{\sqrt{3}}$$

$$\vec{q}_2 = \frac{3\sin^2 x + \frac{9}{2}\sin 2 - 9}{\sqrt{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2}} \qquad\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{3\sin^2 x + \frac{9}{2}\sin 2 - 9}{\frac{1}{4}\sqrt{1,836 - 1,800\sin 2 + 9\sin 4 + 432\sin^2 2}}$$

$$= \frac{12\sin^2 x + 18\sin 2 - 36}{\sqrt{1,836 - 1,800\sin 2 + 9\sin 4 + 432\sin^2 2}}$$

$$\vec{q}_3 = \frac{2\cos^2 x - 6 - 3\sin 2 - \frac{\left(\frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2} \right)}{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2}\left(3\sin^2 x + \frac{9}{2}\sin 2 - 9 \right)}{\sqrt{\langle \vec{v}_3, \vec{v}_3 \rangle}} \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

The *orthonormal* basis is

$$\left\{ \frac{3}{\sqrt{3}}, \frac{12\sin^2 x + 18\sin 2 - 36}{\sqrt{1,836 - 1,800\sin 2 + 9\sin 4 + 432\sin^2 2}}, \right.$$

$$\left. \frac{2\cos^2 x - 6 - 3\sin 2 - \dfrac{\left(\frac{3}{2}\sin^2 2 - \frac{1}{2}\sin 4 - \frac{11}{2}\right)}{\frac{459}{4} - \frac{225}{2}\sin 2 + \frac{9}{16}\sin 4 + \frac{54}{2}\sin^2 2}\left(3\sin^2 x + \frac{9}{2}\sin 2 - 9\right)}{\sqrt{\left\langle \vec{v}_3, \vec{v}_3 \right\rangle}} \right\}$$

Exercise

Apply the Gram-Schmidt **orthonormalization** process in $C^0[-1, 1]$ spanned by the functions, using

the inner product $f_1(x) = \cos 2x$, $f_2(x) = \sin^2 x$, $f_3(x) = \cos^2 x$

Solution

Let $\vec{u}_1 = \cos 2x$, $\vec{u}_2 = \sin^2 x$, $\vec{u}_3 = \cos^2 x$

$\underline{\vec{v}_1 = \vec{u}_1 = \cos 2x}$

$$\left\langle \vec{v}_1, \vec{v}_1 \right\rangle = \int_{-1}^{1} \cos 2x \, dx$$

$$= \frac{1}{2}\sin 2x \Big|_{-1}^{1}$$

$$\underline{= \sin 2}$$

$$\left\langle \vec{u}_2, \vec{v}_1 \right\rangle = \int_{-1}^{1} \sin^2 x \cos 2x \, dx$$

$$= \frac{1}{2}\int_{-1}^{1} (1 - \cos 2x)\cos 2x \, dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(\cos 2x - \cos^2 2x\right) dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(\cos 2x - \frac{1}{2} - \frac{1}{2}\cos 4x\right) dx$$

$$= \left(\frac{1}{2}\sin 2x - \frac{1}{2}x - \frac{1}{8}\sin 8x\right)\Big|_{0}^{1}$$

$$\underline{= \frac{1}{2}\sin 2 - \frac{1}{2} - \frac{1}{8}\sin 8}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= \sin^2 x - \left(\frac{1}{\sin 2}\right)\left(\frac{1}{2}\sin 2 - \frac{1}{2} - \frac{1}{8}\sin 8\right)(\cos 2x)$$

$$= \sin^2 x - \left(\frac{4\sin 2 - 4 - \sin 8}{8\sin 2}\right)\cos 2x$$

$$= \frac{1}{2} - \left(\frac{1}{2} + \frac{4\sin 2 - \sin 8 - 4}{8\sin 2}\right)\cos 2x$$

$$\underline{= \frac{1}{2} - \left(\frac{8\sin 2 - \sin 8 - 4}{8\sin 2}\right)\cos 2x}$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^{1} \left(\sin^2 x - \left(\frac{4\sin 2 - 4 - \sin 8}{8\sin 2}\right)\cos 2x\right)^2 dx$$

$$= \int_{-1}^{1} \left(\sin^4 x - \left(\frac{4\sin 2 - 4 - \sin 8}{4\sin 2}\right)\cos 2x \sin^2 x + \frac{1}{64}\left(\frac{4\sin 2 - 4 - \sin 8}{\sin 2}\right)^2 \cos^2 2x\right) dx$$

$$\int_{-1}^{1} \sin^4 x \, dx = \frac{1}{4}\int_{-1}^{1}(1 - \cos 2x)^2 \, dx$$

$$= \frac{1}{4}\int_{-1}^{1}\left(1 - 2\cos 2x + \cos^2 2x\right) dx$$

$$= \frac{1}{4}\int_{-1}^{1}\left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right)\Big|_{-1}^{1}$$

$$\underline{= \frac{1}{4}\left(3 - 2\sin 2 + \frac{1}{4}\sin 4\right)}$$

$$\int_{-1}^{1}\cos 2x \sin^2 x \, dx = \frac{1}{2}\int_{-1}^{1}\cos 2x \left(1 - \cos 2x\right) dx$$

$$= \frac{1}{2}\int_{-1}^{1}\left(\cos 2x - \cos^2 2x\right) dx$$

$$= \frac{1}{2}\int_{-1}^{1}\left(\cos 2x - \frac{1}{2} - \frac{1}{2}\cos 4x\right) dx$$

$$= \left(\frac{1}{2}\sin 2x - \frac{1}{2}x - \frac{1}{8}\sin 8x \right) \Big|_0^1$$

$$= \frac{1}{2}\sin 2 - \frac{1}{2} - \frac{1}{8}\sin 8$$

$$\int_{-1}^{1} \cos^2 2x \, dx = \frac{1}{2} \int_{-1}^{1} (1 + \cos 4x) \, dx$$

$$= \left(x + \frac{1}{4}\sin 4x \right) \Big|_0^1$$

$$= 1 + \frac{1}{4}\sin 4$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \frac{1}{4}\left(3 - 2\sin 2 + \frac{1}{4}\sin 4 \right) - \left(\frac{4\sin 2 - 4 - \sin 8}{4 \sin 2} \right)\left(\frac{1}{2}\sin 2 - \frac{1}{2} - \frac{1}{8}\sin 8 \right)$$

$$+ \left(\frac{4\sin 2 - 4 - \sin 8}{8 \sin 2} \right)^2 \left(1 + \frac{1}{4}\sin 4 \right)$$

$$= \frac{3}{4} - \frac{1}{2}\sin 2 + \frac{1}{16}\sin 4 - \frac{(4\sin 2 - 4 - \sin 8)^2}{32 \sin 2} + \frac{(4 + \sin 4)(4\sin 2 - 4 - \sin 8)^2}{256 \sin^2 2}$$

$$= \frac{3}{4} - \frac{1}{2}\sin 2 + \frac{1}{16}\sin 4 + \frac{(4 + \sin 4)(4\sin 2 - 4 - \sin 8)^2 - 8 \sin 2 (4\sin 2 - 4 - \sin 8)^2}{256 \sin^2 2}$$

$$= \frac{3}{4} - \frac{1}{2}\sin 2 + \frac{1}{16}\sin 4 + \frac{(4 - 7\sin 2)(4\sin 2 - 4 - \sin 8)^2}{256 \sin^2 2}$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_{-1}^{1} \cos^2 x \cos 2x \, dx$$

$$= \frac{1}{2} \int_{-1}^{1} (1 + \cos 2x) \cos 2x \, dx$$

$$= \frac{1}{2} \int_{-1}^{1} \left(\cos 2x + \cos^2 2x \right) dx$$

$$= \frac{1}{2} \int_{-1}^{1} \left(\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x \right) dx$$

$$= \left(\frac{1}{2}\sin 2x + \frac{1}{2}x + \frac{1}{8}\sin 8x \right) \Big|_0^1$$

$$= \frac{1}{2}\sin 2 + \frac{1}{2} + \frac{1}{8}\sin 8$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^{1} \left(3\sin^2 x + \frac{9}{2}\sin 2 - 9 \right)\left(2\cos^2 x \right) dx$$

$$= 3 \int_{-1}^{1} \left(2\sin^2 x \cos^2 x + (3\sin 2 - 6)\cos^2 x \right) dx$$

$$6 \int_{-1}^{1} \sin^2 x \cos^2 x \, dx = \frac{3}{2} \int_{-1}^{1} (1 - \cos 2x)(1 + \cos 2x) \, dx$$

$$= \frac{3}{2} \int_{-1}^{1} \left(1 - \cos^2 2x \right) dx$$

$$= \frac{3}{2} \int_{-1}^{1} \left(\frac{1}{2} - \frac{1}{2}\cos 4x \right) dx$$

$$= \frac{3}{2}\left(x - \frac{1}{4}\sin 4x \right)\Big|_{0}^{1}$$

$$= \underline{\frac{3}{2}(1 - \sin 4)}$$

$$(9\sin 2 - 18) \int_{-1}^{1} \cos^2 x \, dx = \frac{1}{2}(9\sin 2 - 18) \int_{-1}^{1} (1 + \cos 2x) \, dx$$

$$= (9\sin 2 - 18)\left(x + \frac{1}{2}\sin 2x \right)\Big|_{0}^{1}$$

$$= (9\sin 2 - 18)\left(1 + \frac{1}{2}\sin 2 \right)$$

$$= \underline{\frac{9}{2}\sin^2 2 - 18}$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \frac{3}{2} - \frac{3}{2}\sin 4 + \frac{9}{2}\sin^2 2 - 18$$

$$= \underline{\frac{9}{2}\sin^2 2 - \frac{3}{2}\sin 4 - \frac{33}{2}}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= \cos^2 x - \frac{4\sin 2 + 4 + \sin 8}{8 \sin 2}\cos 2x$$

$$- \frac{72 \sin^2 2 - 24 \sin 4 - 264}{6 - 8\sin 2 + \sin 4 + \dfrac{(4 - 7\sin 2)(4\sin 2 - 4 - \sin 8)^2}{16 \sin^2 2}}\left(\sin^2 x - \left(\frac{4\sin 2 - 4 - \sin 8}{8 \sin 2} \right)\cos 2x \right)$$

$$= \frac{1}{2} + \frac{1}{2}\cos 2x - \frac{4\sin 2 + 4 + \sin 8}{8 \sin 2}\cos 2x$$

$$- \frac{72 \sin^2 2 - 24 \sin 4 - 264}{6 - 8\sin 2 + \sin 4 + \dfrac{(4 - 7\sin 2)(4\sin 2 - 4 - \sin 8)^2}{16 \sin^2 2}} \left(\frac{1}{2} - \frac{1}{2}\cos 2x - \left(\frac{4\sin 2 - 4 - \sin 8}{8 \sin 2} \right) \cos 2x \right)$$

$$= \frac{1}{2} - \frac{4 + \sin 8}{8 \sin 2}\cos 2x$$

$$- \frac{128 \sin^2 2 \left(9 \sin^2 2 - 3 \sin 4 - 33 \right)}{16 \left(6 - 8\sin 2 + \sin 4 \right) \sin^2 2 + (4 - 7\sin 2)(4\sin 2 - 4 - \sin 8)^2} \left(\frac{1}{2} - \left(\frac{8\sin 2 - 4 - \sin 8}{8 \sin 2} \right) \cos 2x \right)$$

The *orthogonal* basis is

$$\left\{ \cos 2x, \quad \frac{1}{2} - \left(\frac{8\sin 2 - \sin 8 - 4}{8 \sin 2} \right) \cos 2x, \right\}$$

$$\langle \vec{v}_3, \vec{v}_3 \rangle =$$

$$\underline{\vec{q}_1 = \frac{\cos 2x}{\sin 2}} \qquad\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\vec{q}_2 = \frac{1}{\sqrt{\dfrac{3}{4} - \dfrac{1}{2}\sin 2 + \dfrac{1}{16}\sin 4 + \dfrac{(4 - 7\sin 2)(4\sin 2 - 4 - \sin 8)^2}{256 \sin^2 2}}} \left(\frac{1}{2} - \left(\frac{8\sin 2 - \sin 8 - 4}{8 \sin 2} \right) \cos 2x \right)$$

$$\vec{q}_3 = \frac{\dfrac{1}{2} - \dfrac{4 + \sin 8}{8 \sin 2}\cos 2x - \dfrac{128 \sin^2 2 \left(9 \sin^2 2 - 3 \sin 4 - 33 \right)}{16 \left(6 - 8\sin 2 + \sin 4 \right) \sin^2 2 + (4 - 7\sin 2)(4\sin 2 - 4 - \sin 8)^2} \left(\frac{1}{2} - \left(\frac{8\sin 2 - 4 - \sin 8}{8 \sin 2} \right) \cos 2x \right)}{\sqrt{\langle \vec{v}_3, \vec{v}_3 \rangle}}$$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $f_1(x) = \sin \pi x, \quad f_2(x) = \sin 2\pi x, \quad f_3(x) = \sin 3\pi x$

Solution

Let $\vec{u}_1 = \sin \pi x, \quad \vec{u}_2 = \sin 2\pi x, \quad \vec{u}_3 = \sin 3\pi x$

$\underline{\vec{v}_1 = \vec{u}_1 = \sin \pi x}$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^{1} \sin^2 \pi x \; dx$$

$$= \frac{1}{2} \int_{-1}^{1} (1 - \cos 2\pi x) \; dx$$

$$= \frac{1}{2} \left(x - \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^{1}$$

$$\underline{= 1}$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_{-1}^{1} \sin \pi x \sin 2\pi x \; dx \qquad\qquad \sin a \sin b = \frac{1}{2} \left[\cos(a+b) - \cos(a-b) \right]$$

$$= \frac{1}{2} \int_{-1}^{1} \left(\cos 3\pi x - \cos(-\pi x) \right) \; dx$$

$$= \frac{1}{2} \int_{-1}^{1} \left(\cos 3\pi x - \cos \pi x \right) \; dx$$

$$= \frac{1}{2} \left(\frac{1}{3\pi} \sin 3\pi x - \frac{1}{\pi} \sin \pi x \right) \Big|_{-1}^{1}$$

$$\underline{= 0}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\| \vec{v}_1 \|^2} \vec{v}_1$$

$$\underline{= \sin 2\pi x}$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^{1} \sin^2 2\pi x \; dx$$

$$= \frac{1}{2} \int_{-1}^{1} (1 - \cos 4\pi x) \ dx$$

$$= \frac{1}{2} \left(x - \frac{1}{4\pi} \sin 4\pi x \right) \Big|_{-1}^{1}$$

$$\underline{= 1} \ \vert$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_{-1}^{1} \sin \pi x \sin 3\pi x \ dx \qquad\qquad \sin a \sin b = \frac{1}{2} [\cos(a+b) - \cos(a-b)]$$

$$= \frac{1}{2} \int_{-1}^{1} (\cos 4\pi x - \cos(-2\pi x)) \ dx$$

$$= \frac{1}{2} \int_{-1}^{1} (\cos 4\pi x - \cos 2\pi x) \ dx$$

$$= \frac{1}{2} \left(\frac{1}{4\pi} \sin 4\pi x - \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^{1}$$

$$\underline{= 0} \ \vert$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^{1} \sin 3\pi x \sin 2\pi x \ dx \qquad\qquad \sin a \sin b = \frac{1}{2} [\cos(a+b) - \cos(a-b)]$$

$$= \frac{1}{2} \int_{-1}^{1} (\cos 5\pi x - \cos \pi x) \ dx$$

$$= \frac{1}{2} \left(\frac{1}{5\pi} \sin 5\pi x - \frac{1}{\pi} \sin \pi x \right) \Big|_{-1}^{1}$$

$$\underline{= 0} \ \vert$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$\underline{= \sin 3\pi x} \ \vert$$

The *orthogonal* basis is $\{ \sin \pi x, \ \sin 2\pi x, \ \sin 3\pi x \}$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_{-1}^{1} \sin^2 3\pi x \ dx$$

$$= \frac{1}{2} \int_{-1}^{1} (1 - \cos 6\pi x) \ dx$$

$$= \frac{1}{2}\left(x - \frac{1}{6\pi}\sin 6\pi x\right)\Big|_{-1}^{1}$$

$$= 1$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \sin \pi x$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \sin 2\pi x$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \sin 3\pi x$$

The *orthonormal* basis is $\{\sin \pi x, \ \sin 2\pi x, \ \sin 3\pi x\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1, 1]$ spanned by the functions, using

the inner product $f_1(x) = \cos \pi x, \quad f_2(x) = \cos 2\pi x, \quad f_3(x) = \cos 3\pi x$

Solution

Let $\quad \vec{u}_1 = f_1 = \cos \pi x, \quad \vec{u}_2 = f_2 = \cos 2\pi x, \quad \vec{u}_3 = f_3 = \cos 3\pi x$

$$\underline{\vec{v}_1 = \vec{u}_1 = \cos \pi x}$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^{1} \cos^2 \pi x \, dx$$

$$= \frac{1}{2}\int_{-1}^{1}(1 + \cos 2\pi x)\, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{2\pi}\sin 2\pi x\right)\Big|_{-1}^{1}$$

$$= 1$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_{-1}^{1}\cos 2\pi x \cos \pi x \, dx \qquad\qquad \cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$$

$$= \frac{1}{2} \int_{-1}^{1} \left(\cos 3\pi x + \cos \pi x \right) dx$$

$$= \frac{1}{2} \left(\frac{1}{3\pi} \sin 3\pi x + \frac{1}{\pi} \sin \pi x \right) \Big|_{-1}^{1}$$

$$\underline{= 0}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$\underline{= \cos 2\pi x}$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^{1} \cos^2 2\pi x \, dx$$

$$= \frac{1}{2} \int_{-1}^{1} \left(1 + \cos 4\pi x \right) dx$$

$$= \frac{1}{2} \left(x + \frac{1}{4\pi} \sin 4\pi x \right) \Big|_{-1}^{1}$$

$$\underline{= 1}$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_{-1}^{1} \cos 3\pi x \cos \pi x \, dx \qquad\qquad \cos a \cos b = \frac{1}{2} \left[\cos(a+b) + \cos(a-b) \right]$$

$$= \frac{1}{2} \int_{-1}^{1} \left(\cos 4\pi x + \cos 2\pi x \right) dx$$

$$= \frac{1}{2} \left(\frac{1}{4\pi} \sin 4\pi x + \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^{1}$$

$$\underline{= 0}$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^{1} \cos 3\pi x \cos 2\pi x \, dx \qquad\qquad \cos a \cos b = \frac{1}{2} \left[\cos(a+b) + \cos(a-b) \right]$$

$$= \frac{1}{2} \int_{-1}^{1} \left(\cos 5\pi x + \cos \pi x \right) dx$$

$$= \frac{1}{2} \left(\frac{1}{5\pi} \sin 5\pi x + \frac{1}{\pi} \sin \pi x \right) \Big|_{-1}^{1}$$

$$\underline{= 0}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= \cos 3\pi x$$

The *orthogonal* basis is $\{\cos \pi x, \ \cos 2\pi x, \ \cos 3\pi x\}$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_{-1}^{1} \cos^2 3\pi x \ dx$$

$$= \frac{1}{2} \int_{-1}^{1} (1 + \cos 6\pi x) \ dx$$

$$= \frac{1}{2} \left(x + \frac{1}{6\pi} \sin 6\pi x \right) \Big|_{-1}^{1}$$

$$= 1$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \cos \pi x$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \cos 2\pi x$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \cos 3\pi x$$

The *orthonormal* basis is $\{\cos \pi x, \ \cos 2\pi x, \ \cos 3\pi x\}$

Exercise

For $\mathbb{P}_3[x]$, define the inner product over $\mathbb{R}$ as

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) \ dx$$

a) If $f(x) = 1$ is a unit vector in $\mathbb{P}_3[x]$?

b) Find an orthonormal basis for the subspace spanned by x *and* x^2.

c) Complete the basis in part (*b*) to an orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product.

d) Is

$$[f, g] = \int_0^1 f(x) g(x) \, dx$$

Also, an inner product for $\mathbb{P}_3[x]$

e) Find a pair of vectors $\vec{v}$ and $\vec{w}$ such that

$$\langle \vec{v}, \vec{w} \rangle = 0 \quad but \quad [\vec{v}, \vec{w}] \neq 0$$

f) Is the basis found in part *(c)* are orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product in part *(d)*?

Solution

a) $f(x) = 1$

$$\langle f, f \rangle = \int_{-1}^1 f(x) f(x) \, dx$$

$$= \int_{-1}^1 dx$$

$$= x \Big|_{-1}^1$$

$$= 1 + 1$$

$$\underline{= 2 \neq 1}$$

Therefore, when $f(x) = 1$ is **not** a unit vector in $\mathbb{P}_3[x]$

b) Let $\vec{u}_1 = f = x, \quad \vec{u}_2 = g = x^2$

$$\underline{\vec{v}_1 = \vec{u}_1 = x}$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^1 x^2 \, dx$$

$$= \frac{1}{3} x^3 \Big|_{-1}^1$$

$$= \frac{1}{3}(1+1)$$

$$\underline{= \frac{2}{3}}$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_{-1}^1 x^2(x) \, dx$$

$$= \int_{-1}^{1} x^3 \, dx$$

$$= \frac{1}{4} x^4 \Big|_{-1}^{1}$$

$$= \frac{1}{4}(1-1)$$

$$\underline{\underline{= 0}} \,|$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2 , \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1$$

$$= x^2 - \frac{0}{*}(\;)$$

$$\underline{\underline{= x^2}} \,|$$

$$\langle \vec{v}_2 , \vec{v}_2 \rangle = \int_{-1}^{1} x^4 \, dx$$

$$= \frac{1}{5} x^5 \Big|_{-1}^{1}$$

$$= \frac{1}{5}(1+1)$$

$$\underline{\underline{= \frac{2}{5}}} \,|$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\left\| \vec{v}_1 \right\|}$$

$$= \frac{x}{\sqrt{\frac{2}{3}}}$$

$$\underline{\underline{= \sqrt{\frac{3}{2}} \, x}} \,|$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\left\| \vec{v}_2 \right\|}$$

$$= \frac{x^2}{\sqrt{\frac{2}{5}}}$$

$$\underline{\underline{= \sqrt{\frac{5}{2}} \, x^2}} \,|$$

The orthonormal basis is $\left\{ \sqrt{\frac{3}{2}} \, x, \; \sqrt{\frac{5}{2}} \, x^2 \right\}$

c) Since $\vec{u}_1 = x$, $\vec{u}_2 = x^2$ in $\mathbb{P}_3[x]$

Then, let $\vec{u}_3 = 1$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_{-1}^{1} (1)(x)\, dx$$

$$= \int_{-1}^{1} x\, dx$$

$$= \frac{1}{2}x^2 \Big|_{-1}^{1}$$

$$= \frac{1}{2}(1-1)$$

$$= \underline{0}$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^{1} (1)\left(x^2\right)\, dx$$

$$= \int_{-1}^{1} x^2\, dx$$

$$= \frac{1}{3}x^3 \Big|_{-1}^{1}$$

$$= \frac{1}{3}(1+1)$$

$$= \underline{\frac{2}{3}}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\left\| \vec{v}_2 \right\|^2}\vec{v}_2$$

$$= 1 - \frac{0}{\frac{2}{3}}(x) - \frac{\frac{2}{3}}{\frac{2}{5}}\left(x^2\right)$$

$$= \underline{1 - \frac{5}{3}x^2}$$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_{-1}^{1} \left(1 - \frac{5}{3}x^2\right)^2\, dx$$

$$= \int_{-1}^{1} \left(1 - \frac{10}{3}x^2 + \frac{25}{9}x^4\right)\, dx$$

$$= \left(x - \frac{10}{9}x^3 + \frac{5}{9}x^5 \right) \Big|_{-1}^{1}$$

$$= 2\left(1 - \frac{10}{9} + \frac{5}{9} \right)$$

$$= 2\left(\frac{9-5}{9} \right)$$

$$\underline{= \frac{8}{9}}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \left(\sqrt{\frac{9}{8}} \right)\left(1 - \frac{5}{3}x^2 \right)$$

$$= \frac{3}{2\sqrt{2}}\left(1 - \frac{5}{3}x^2 \right)$$

$$\underline{= \frac{3}{2\sqrt{2}} - \frac{5}{2\sqrt{2}}\,x^2}$$

The orthonormal basis is $\left\{ \sqrt{\frac{3}{2}}\,x, \ \sqrt{\frac{5}{2}}\,x^2, \ \frac{3}{2\sqrt{2}} - \frac{5}{2\sqrt{2}}\,x^2 \right\}$

d) $[f,\,g] = \displaystyle\int_0^1 f(x)g(x)\,dx$

Let $\quad \vec{u}_1 = 1, \quad \vec{u}_2 = x, \quad \vec{u}_3 = x^2$

$$\underline{\vec{v}_1 = \vec{u}_1 = 1}$$

$$\langle \vec{v}_1,\, \vec{v}_1 \rangle = \int_0^1 1\,dx$$

$$= x \Big|_0^1$$

$$\underline{= 1}$$

$$\langle \vec{u}_2,\, \vec{v}_1 \rangle = \int_0^1 x(1)\,dx$$

$$= \int_0^1 x\,dx$$

$$= \frac{1}{2}x^2 \Big|_0^1$$

$$= \frac{1}{2} \Big|$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= x - \frac{\frac{1}{2}}{1}(1)$$

$$\underline{= x - \frac{1}{2}} \Big|$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_0^1 \left(x - \frac{1}{2}\right)^2 dx$$

$$= \int_0^1 \left(x - \frac{1}{2}\right)^2 d\left(x - \frac{1}{2}\right)$$

$$= \frac{1}{3}\left(x - \frac{1}{2}\right)^3 \Big|_0^1$$

$$= \frac{1}{3}\left(\left(\frac{1}{2}\right)^3 - \left(-\frac{1}{2}\right)^3\right)$$

$$= \frac{1}{3}\left(\frac{1}{8} + \frac{1}{8}\right)$$

$$\underline{= \frac{1}{12}} \Big|$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_0^1 \left(x^2\right)(1) \, dx$$

$$= \int_0^1 x^2 \, dx$$

$$= \frac{1}{3}x^3 \Big|_0^1$$

$$\underline{= \frac{1}{3}} \Big|$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_0^1 \left(x^2\right)\left(x - \frac{1}{2}\right) dx$$

$$= \int_0^1 \left(x^3 - \frac{1}{2}x^2\right) dx$$

$$= \left(\frac{1}{4}x^4 - \frac{1}{6}x^3 \right) \Big|_0^1$$

$$= \frac{1}{4} - \frac{1}{6}$$

$$= \frac{1}{12} \Big|$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\left\| \vec{v}_2 \right\|^2} \vec{v}_2$$

$$= x^2 - \frac{\frac{1}{3}}{1}(1) - \frac{\frac{1}{12}}{\frac{1}{12}}\left(x - \frac{1}{2} \right)$$

$$= x^2 - \frac{1}{3} - x + \frac{1}{2}$$

$$= x^2 - x + \frac{1}{6} \Big|$$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_0^1 \left(x^2 - x + \frac{1}{6} \right)^2 \, dx$$

$$= \int_0^1 \left(x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36} \right) \, dx$$

$$= \left(\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{4}{9}x^3 - \frac{1}{6}x^2 + \frac{1}{36}x \right) \Big|_0^1$$

$$= \frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36}$$

$$= \frac{2-5}{10} + \frac{16-6+1}{36}$$

$$= -\frac{3}{10} + \frac{11}{36}$$

$$= \frac{-108+110}{360}$$

$$= \frac{1}{180} \Big|$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\left\| \vec{v}_1 \right\|}$$

$$= 1 \Big|$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\left\| \vec{v}_2 \right\|}$$

$$= \frac{x - \frac{1}{2}}{\sqrt{\frac{1}{12}}}$$

$$\underline{= 2\sqrt{3}\left(x - \frac{1}{2}\right)}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \left(\sqrt{180}\right)\left(x^2 - x + \frac{1}{6}\right)$$

$$= \left(6\sqrt{5}\right)\left(x^2 - x + \frac{1}{6}\right)$$

$$\underline{= 6\sqrt{5}\,x^2 - 6\sqrt{5}\,x + \sqrt{5}}$$

The orthonormal basis is $\quad \left\{\, 1,\ 2\sqrt{3}\left(x - \frac{1}{2}\right),\ \sqrt{5}\left(6x^2 - 6x + 1\right)\,\right\}$

Therefore, $[f,\, g] = \displaystyle\int_0^1 f(x)\,g(x)\,dx$ is an inner product for $\mathbb{P}_3[x]$

e) Let assume: $\quad \vec{v} = 1 \quad and \quad \vec{w} = x$

$$\langle \vec{v},\ \vec{w} \rangle = \int_{-1}^1 1(x)\,dx$$

$$= \int_{-1}^1 x\,dx$$

$$= \frac{1}{2}x^2 \,\Big|_{-1}^1$$

$$= \frac{1}{2}(1 - 1)$$

$$\underline{= 0}\ \ \checkmark$$

$$[\vec{v},\ \vec{w}] = \int_0^1 1(x)\,dx$$

$$= \frac{1}{2}x^2 \,\Big|_0^1$$

$$\underline{= \frac{1}{2}\ \ne 0}\ \ \checkmark$$

f) The orthonormal basis in part *(c)* $\left\{ \sqrt{\frac{3}{2}}\, x, \ \sqrt{\frac{5}{2}}\, x^2, \ \frac{3}{2\sqrt{2}} - \frac{5}{2\sqrt{2}}\, x^2 \right\}$ are ***not*** the same

as

the orthonormal basis in part *(d)* $\left\{ 1, \ 2\sqrt{3}\left(x - \frac{1}{2}\right), \ \sqrt{5}\left(6x^2 - 6x + 1\right) \right\}$

Solution

Section **3.4 – Orthogonal Matrices**

Exercise

Show that the matrix is orthogonal $\quad A = \begin{pmatrix} \dfrac{3}{5} & \dfrac{4}{5} \\ \dfrac{4}{5} & -\dfrac{3}{5} \end{pmatrix}$

Solution

$$AA^T = \begin{pmatrix} \dfrac{3}{5} & \dfrac{4}{5} \\ \dfrac{4}{5} & -\dfrac{3}{5} \end{pmatrix} \begin{pmatrix} \dfrac{3}{5} & \dfrac{4}{5} \\ \dfrac{4}{5} & -\dfrac{3}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underline{= I} \quad \sqrt{}$$

$$A^T A = \begin{pmatrix} \dfrac{3}{5} & \dfrac{4}{5} \\ \dfrac{4}{5} & -\dfrac{3}{5} \end{pmatrix} \begin{pmatrix} \dfrac{3}{5} & \dfrac{4}{5} \\ \dfrac{4}{5} & -\dfrac{3}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underline{= I} \quad \sqrt{}$$

$$AA^T = A^T A = I$$

$\therefore$ ***A*** is orthogonal

Exercise

Show that the matrix is orthogonal $\quad A = \begin{pmatrix} \dfrac{1}{\sqrt{10}} & \dfrac{3}{\sqrt{10}} \\ -\dfrac{3}{\sqrt{10}} & \dfrac{1}{\sqrt{10}} \end{pmatrix}$

Solution

$$AA^T = \begin{pmatrix} \dfrac{1}{\sqrt{10}} & \dfrac{3}{\sqrt{10}} \\ -\dfrac{3}{\sqrt{10}} & \dfrac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \dfrac{1}{\sqrt{10}} & -\dfrac{3}{\sqrt{10}} \\ \dfrac{3}{\sqrt{10}} & \dfrac{1}{\sqrt{10}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I \quad \checkmark$$

$$A^T A = \begin{pmatrix} \dfrac{1}{\sqrt{10}} & -\dfrac{3}{\sqrt{10}} \\ \dfrac{3}{\sqrt{10}} & \dfrac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \dfrac{1}{\sqrt{10}} & \dfrac{3}{\sqrt{10}} \\ -\dfrac{3}{\sqrt{10}} & \dfrac{1}{\sqrt{10}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I \quad \checkmark$$

$$AA^T = A^T A = I$$

$\therefore$ A is orthogonal.

Exercise

Show that the matrix is orthogonal $\quad A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \dfrac{1}{\sqrt{2}} & 0 & \dfrac{1}{\sqrt{2}} \end{pmatrix}$

Solution

$$AA^T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \dfrac{1}{\sqrt{2}} & 0 & \dfrac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 & \dfrac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \dfrac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= I \quad \checkmark$$

$$A^T A = \begin{pmatrix} 0 & 1 & \dfrac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \dfrac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \dfrac{1}{\sqrt{2}} & 0 & \dfrac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= I \quad \checkmark$$

$$AA^T = A^T A = I$$

$\therefore$ **A** is orthogonal.

Exercise

Show that the matrix is orthogonal $\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{7}{\sqrt{53}} & \dfrac{2}{\sqrt{53}} \\ 0 & -\dfrac{2}{\sqrt{53}} & \dfrac{7}{\sqrt{53}} \end{pmatrix}$

Solution

$$AA^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{7}{\sqrt{53}} & \dfrac{2}{\sqrt{53}} \\ 0 & -\dfrac{2}{\sqrt{53}} & \dfrac{7}{\sqrt{53}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{7}{\sqrt{53}} & -\dfrac{2}{\sqrt{53}} \\ 0 & \dfrac{2}{\sqrt{53}} & \dfrac{7}{\sqrt{53}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{= I} \mid \quad \checkmark$$

$$A^T A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{7}{\sqrt{53}} & -\dfrac{2}{\sqrt{53}} \\ 0 & \dfrac{2}{\sqrt{53}} & \dfrac{7}{\sqrt{53}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{7}{\sqrt{53}} & \dfrac{2}{\sqrt{53}} \\ 0 & -\dfrac{2}{\sqrt{53}} & \dfrac{7}{\sqrt{53}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{= I} \mid \quad \checkmark$$

$$AA^T = A^T A = I$$

$\therefore$ **A** is orthogonal.

Exercise

Show that the matrix is orthogonal $\quad A = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$

Solution

$$AA^T = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix} \begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

$$A^T A = \begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix} \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

$$AA^T = A^T A = I$$

$\therefore$ *A* is orthogonal.

Exercise

Show that the matrix is orthogonal $\quad A = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$

Solution

$$AA^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{= I} \mid \ \checkmark$$

$$A^T A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{= I} \mid \ \checkmark$$

$$AA^T = A^T A = I$$

$\therefore$ **A** is orthogonal.

Exercise

Show that the matrix is orthogonal $A = \begin{pmatrix} 0 & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{15}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{15}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & -\frac{2}{\sqrt{10}} & -\frac{2}{\sqrt{15}} \\ 0 & 0 & -\frac{2}{\sqrt{10}} & \frac{3}{\sqrt{15}} \end{pmatrix}$

Solution

$$AA^T = \begin{pmatrix} 0 & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{15}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{15}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & -\frac{2}{\sqrt{10}} & -\frac{2}{\sqrt{15}} \\ 0 & 0 & -\frac{2}{\sqrt{10}} & \frac{3}{\sqrt{15}} \end{pmatrix} \begin{pmatrix} 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{5}{\sqrt{30}} & -\frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & 0 \\ \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & -\frac{2}{\sqrt{10}} & -\frac{2}{\sqrt{10}} \\ \frac{1}{\sqrt{15}} & \frac{1}{\sqrt{15}} & -\frac{2}{\sqrt{15}} & \frac{3}{\sqrt{15}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= I \quad \checkmark$$

$$A^T A = \begin{pmatrix} 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{5}{\sqrt{30}} & -\frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & 0 \\ \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & -\frac{2}{\sqrt{10}} & -\frac{2}{\sqrt{10}} \\ \frac{1}{\sqrt{15}} & \frac{1}{\sqrt{15}} & -\frac{2}{\sqrt{15}} & \frac{3}{\sqrt{15}} \end{pmatrix} \begin{pmatrix} 0 & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{15}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{15}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & -\frac{2}{\sqrt{10}} & -\frac{2}{\sqrt{15}} \\ 0 & 0 & -\frac{2}{\sqrt{10}} & \frac{3}{\sqrt{15}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= I \quad \checkmark$$

$$AA^T = A^T A = I$$

$\therefore$ **A** is orthogonal.

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{= I} \mid \checkmark$$

$$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ is orthogonal with inverse } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underline{= I} \mid \checkmark$$

$$\therefore \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ is orthogonal with inverse } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Solution

$$\det\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \underline{= 2 \neq \pm 1} \mid$$

$$\therefore \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \text{ is } \textit{not} \text{ orthogonal}$$

Inverse $\begin{pmatrix} \dfrac{1}{2} & -\dfrac{1}{2} \\ \dfrac{1}{2} & \dfrac{1}{2} \end{pmatrix}$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix}$$

Solution

$\begin{vmatrix} 4 & 1 \\ 3 & -1 \end{vmatrix} = -7 \neq \pm 1$

$\therefore \begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix}$ is *not* orthogonal.

Inverse $\begin{pmatrix} \dfrac{1}{7} & \dfrac{1}{7} \\ \dfrac{3}{7} & -\dfrac{4}{7} \end{pmatrix}$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \dfrac{3}{5} & \dfrac{4}{5} \\ \dfrac{4}{5} & -\dfrac{3}{5} \end{pmatrix}$$

Solution

$\begin{vmatrix} \dfrac{3}{5} & \dfrac{4}{5} \\ \dfrac{4}{5} & -\dfrac{3}{5} \end{vmatrix} = -\dfrac{9}{25} - \dfrac{16}{25}$

$= -1 \quad \checkmark$

$\therefore \begin{pmatrix} \dfrac{3}{5} & \dfrac{4}{5} \\ \dfrac{4}{5} & -\dfrac{3}{5} \end{pmatrix}$ is orthogonal.

$$\begin{pmatrix} \dfrac{3}{5} & \dfrac{4}{5} \\ \dfrac{4}{5} & -\dfrac{3}{5} \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{3}{5} & \dfrac{4}{5} \\ \dfrac{4}{5} & -\dfrac{3}{5} \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \dfrac{2}{\sqrt{6}} & 0 \\ 0 & \dfrac{3}{\sqrt{6}} \end{pmatrix}$$

Solution

$$\begin{vmatrix} \dfrac{2}{\sqrt{6}} & 0 \\ 0 & \dfrac{3}{\sqrt{6}} \end{vmatrix} = 1$$

$$\therefore \begin{pmatrix} \dfrac{2}{\sqrt{6}} & 0 \\ 0 & \dfrac{3}{\sqrt{6}} \end{pmatrix}$$

Is orthogonal.

$$\begin{pmatrix} \dfrac{2}{\sqrt{6}} & 0 \\ 0 & \dfrac{3}{\sqrt{6}} \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{2}{\sqrt{6}} & 0 \\ 0 & \dfrac{3}{\sqrt{6}} \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{bmatrix}$$

Solution

$$\begin{bmatrix} \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{bmatrix}^{T} = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I \mid \ \checkmark$$

$$\therefore \begin{bmatrix} \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{bmatrix} \text{ is orthogonal with inverse } \begin{pmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{pmatrix}$$

(It is a standard matrix for a rotation of 45°)

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Solution

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^{T} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\therefore \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \text{ is orthogonal with inverse } \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Solution

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}^{T} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\therefore \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \text{ is orthogonal with an inverse } \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 7 \\ 1 & 3 & -5 \\ -1 & 4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & & \\ & & \\ & & \end{pmatrix} \neq I$$

$$\therefore \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix} \text{ is } not \text{ orthogonal.}$$

$$\begin{vmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{vmatrix} = 78$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}^{-1} = \frac{1}{78}\begin{pmatrix} 26 & 3 & 7 \\ 26 & 9 & -5 \\ -26 & 12 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} & \frac{1}{26} & \frac{7}{78} \\ \frac{1}{3} & \frac{3}{26} & -\frac{5}{78} \\ -\frac{1}{3} & \frac{2}{13} & \frac{1}{39} \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Solution

$$
\begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}^T = \begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}
$$

$$
= \begin{pmatrix} \frac{3}{2} & & \\ & & \\ & & \end{pmatrix} \neq I
$$

Or $\left\| r_1 \right\| = \sqrt{0 + 1^2 + \left(\frac{1}{\sqrt{2}} \right)^2} = \sqrt{\frac{3}{2}} \neq 1$

$\therefore A$ is **not** orthogonal.

$$
\begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}^{-1} = -\sqrt{2} \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -1 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & \sqrt{2} \end{pmatrix}
$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$
\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}
$$

Solution

$$
\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}
$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\therefore \begin{bmatrix} -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{3}} \\ 0 & -\dfrac{2}{\sqrt{6}} & \dfrac{1}{\sqrt{3}} \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{3}} \end{bmatrix} \text{ is orthogonal with inverse } \begin{bmatrix} -\dfrac{1}{\sqrt{2}} & 0 & \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{6}} & -\dfrac{2}{\sqrt{6}} & \dfrac{1}{\sqrt{6}} \\ \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} \end{bmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\ -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution

$$\begin{vmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\ -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1\left(\dfrac{1}{2} + \dfrac{1}{2} \right)$$

$$\underline{= 1} \quad \checkmark$$

$\therefore$ The matrix is orthogonal.

$$\begin{pmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\ -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\ -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\ -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\ -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{T} = \begin{pmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\ -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} & 0 \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix}$$

Solution

$$\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix}^T = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{= I} \quad \checkmark$$

∴ The matrix is orthogonal.

$$\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{5\sqrt{3}} & \frac{4}{5\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{7}{5\sqrt{3}} & \frac{3}{5\sqrt{2}} \end{pmatrix}$$

Solution

$$\begin{pmatrix} \dfrac{2}{\sqrt{6}} & \dfrac{1}{5\sqrt{3}} & \dfrac{4}{5\sqrt{2}} \\ \dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{6}} & \dfrac{7}{5\sqrt{3}} & \dfrac{3}{5\sqrt{2}} \end{pmatrix} \begin{pmatrix} \dfrac{2}{\sqrt{6}} & \dfrac{1}{5\sqrt{3}} & \dfrac{4}{5\sqrt{2}} \\ \dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{6}} & \dfrac{7}{5\sqrt{3}} & \dfrac{3}{5\sqrt{2}} \end{pmatrix}^{T} = \begin{pmatrix} \dfrac{2}{\sqrt{6}} & \dfrac{1}{5\sqrt{3}} & \dfrac{4}{5\sqrt{2}} \\ \dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{6}} & \dfrac{7}{5\sqrt{3}} & \dfrac{3}{5\sqrt{2}} \end{pmatrix} \begin{pmatrix} \dfrac{2}{\sqrt{6}} & \dfrac{1}{\sqrt{6}} & -\dfrac{1}{\sqrt{6}} \\ \dfrac{1}{5\sqrt{3}} & \dfrac{1}{\sqrt{3}} & \dfrac{7}{5\sqrt{3}} \\ \dfrac{4}{5\sqrt{2}} & -\dfrac{1}{\sqrt{2}} & \dfrac{3}{5\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= I \quad \checkmark$$

∴ The matrix is orthogonal.

$$\begin{pmatrix} \dfrac{2}{\sqrt{6}} & \dfrac{1}{5\sqrt{3}} & \dfrac{4}{5\sqrt{2}} \\ \dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{6}} & \dfrac{7}{5\sqrt{3}} & \dfrac{3}{5\sqrt{2}} \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{2}{\sqrt{6}} & \dfrac{1}{5\sqrt{3}} & \dfrac{4}{5\sqrt{2}} \\ \dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{6}} & \dfrac{7}{5\sqrt{3}} & \dfrac{3}{5\sqrt{2}} \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{6}} \\ \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{6}} \\ \dfrac{1}{\sqrt{3}} & 0 & \dfrac{2}{\sqrt{6}} \end{pmatrix}$$

Solution

$$\begin{vmatrix} \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{6}} \\ \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{6}} \\ \dfrac{1}{\sqrt{3}} & 0 & \dfrac{2}{\sqrt{6}} \end{vmatrix} = \dfrac{1}{3} + \dfrac{1}{6} + \dfrac{1}{6} + \dfrac{1}{3}$$

$$= 1 \quad \checkmark$$

∴ The matrix is orthogonal.

$$\begin{pmatrix} \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{6}} \\[2ex] \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{6}} \\[2ex] \dfrac{1}{\sqrt{3}} & 0 & \dfrac{2}{\sqrt{6}} \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{6}} \\[2ex] \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{6}} \\[2ex] \dfrac{1}{\sqrt{3}} & 0 & \dfrac{2}{\sqrt{6}} \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} 0 & 1 & 0 \\[2ex] \dfrac{1}{\sqrt{5}} & 0 & \dfrac{2}{\sqrt{5}} \\[2ex] \dfrac{2}{\sqrt{5}} & 0 & -\dfrac{1}{\sqrt{5}} \end{pmatrix}$$

Solution

$$\begin{pmatrix} 0 & 1 & 0 \\[2ex] \dfrac{1}{\sqrt{5}} & 0 & \dfrac{2}{\sqrt{5}} \\[2ex] \dfrac{2}{\sqrt{5}} & 0 & -\dfrac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\[2ex] \dfrac{1}{\sqrt{5}} & 0 & \dfrac{2}{\sqrt{5}} \\[2ex] \dfrac{2}{\sqrt{5}} & 0 & -\dfrac{1}{\sqrt{5}} \end{pmatrix}^{T} = \begin{pmatrix} 0 & 1 & 0 \\[2ex] \dfrac{1}{\sqrt{5}} & 0 & \dfrac{2}{\sqrt{5}} \\[2ex] \dfrac{2}{\sqrt{5}} & 0 & -\dfrac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 0 & \dfrac{1}{\sqrt{5}} & \dfrac{2}{\sqrt{5}} \\[2ex] 1 & 0 & 0 \\[2ex] 0 & \dfrac{2}{\sqrt{5}} & -\dfrac{1}{\sqrt{5}} \end{pmatrix}$$

$$= I \; \checkmark$$

∴ The matrix is orthogonal.

$$\begin{pmatrix} 0 & 1 & 0 \\[2ex] \dfrac{1}{\sqrt{5}} & 0 & \dfrac{2}{\sqrt{5}} \\[2ex] \dfrac{2}{\sqrt{5}} & 0 & -\dfrac{1}{\sqrt{5}} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\[2ex] \dfrac{1}{\sqrt{5}} & 0 & \dfrac{2}{\sqrt{5}} \\[2ex] \dfrac{2}{\sqrt{5}} & 0 & -\dfrac{1}{\sqrt{5}} \end{pmatrix}$$

$$\begin{vmatrix} 0 & 1 & 0 \\[2ex] \dfrac{1}{\sqrt{5}} & 0 & \dfrac{2}{\sqrt{5}} \\[2ex] \dfrac{2}{\sqrt{5}} & 0 & -\dfrac{1}{\sqrt{5}} \end{vmatrix} = -\left(-\dfrac{1}{5} - \dfrac{4}{5}\right)$$

$$= 1 \rfloor \quad \checkmark$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} -\dfrac{3}{\sqrt{14}} & \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{42}} \\[2mm] \dfrac{1}{\sqrt{14}} & \dfrac{1}{\sqrt{3}} & \dfrac{5}{\sqrt{42}} \\[2mm] \dfrac{2}{\sqrt{14}} & \dfrac{1}{\sqrt{3}} & -\dfrac{4}{\sqrt{42}} \end{pmatrix}$$

Solution

$$\begin{pmatrix} -\dfrac{3}{\sqrt{14}} & \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{42}} \\[2mm] \dfrac{1}{\sqrt{14}} & \dfrac{1}{\sqrt{3}} & \dfrac{5}{\sqrt{42}} \\[2mm] \dfrac{2}{\sqrt{14}} & \dfrac{1}{\sqrt{3}} & -\dfrac{4}{\sqrt{42}} \end{pmatrix} \begin{pmatrix} -\dfrac{3}{\sqrt{14}} & \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{42}} \\[2mm] \dfrac{1}{\sqrt{14}} & \dfrac{1}{\sqrt{3}} & \dfrac{5}{\sqrt{42}} \\[2mm] \dfrac{2}{\sqrt{14}} & \dfrac{1}{\sqrt{3}} & -\dfrac{4}{\sqrt{42}} \end{pmatrix}^T = \begin{pmatrix} -\dfrac{3}{\sqrt{14}} & \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{42}} \\[2mm] \dfrac{1}{\sqrt{14}} & \dfrac{1}{\sqrt{3}} & \dfrac{5}{\sqrt{42}} \\[2mm] \dfrac{2}{\sqrt{14}} & \dfrac{1}{\sqrt{3}} & -\dfrac{4}{\sqrt{42}} \end{pmatrix} \begin{pmatrix} -\dfrac{3}{\sqrt{14}} & \dfrac{1}{\sqrt{14}} & \dfrac{2}{\sqrt{14}} \\[2mm] \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} \\[2mm] -\dfrac{1}{\sqrt{42}} & \dfrac{5}{\sqrt{42}} & -\dfrac{4}{\sqrt{42}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= I \rfloor \quad \checkmark$$

∴ The matrix is orthogonal.

$$\begin{pmatrix} -\dfrac{3}{\sqrt{14}} & \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{42}} \\[2mm] \dfrac{1}{\sqrt{14}} & \dfrac{1}{\sqrt{3}} & \dfrac{5}{\sqrt{42}} \\[2mm] \dfrac{2}{\sqrt{14}} & \dfrac{1}{\sqrt{3}} & -\dfrac{4}{\sqrt{42}} \end{pmatrix}^{-1} = \begin{pmatrix} -\dfrac{3}{\sqrt{14}} & \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{42}} \\[2mm] \dfrac{1}{\sqrt{14}} & \dfrac{1}{\sqrt{3}} & \dfrac{5}{\sqrt{42}} \\[2mm] \dfrac{2}{\sqrt{14}} & \dfrac{1}{\sqrt{3}} & -\dfrac{4}{\sqrt{42}} \end{pmatrix}$$

$$\begin{vmatrix} -\dfrac{3}{\sqrt{14}} & \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{42}} \\[2mm] \dfrac{1}{\sqrt{14}} & \dfrac{1}{\sqrt{3}} & \dfrac{5}{\sqrt{42}} \\[2mm] \dfrac{2}{\sqrt{14}} & \dfrac{1}{\sqrt{3}} & -\dfrac{4}{\sqrt{42}} \end{vmatrix} = \dfrac{12 + 10 - 1 + 2 + 15 + 4}{\sqrt{14}\,\sqrt{3}\,\sqrt{42}}$$

$$= \dfrac{42}{42}$$

$$= 1 \mid \ \checkmark$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

Solution

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I \mid \ \checkmark$$

∴ The matrix is orthogonal &

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \dfrac{7}{\sqrt{62}} & \dfrac{9}{\sqrt{682}} & \dfrac{1}{\sqrt{11}} \\[3mm] \dfrac{2}{\sqrt{62}} & -\dfrac{24}{\sqrt{682}} & \dfrac{1}{\sqrt{11}} \\[3mm] -\dfrac{3}{\sqrt{62}} & \dfrac{5}{\sqrt{682}} & \dfrac{3}{\sqrt{11}} \end{pmatrix}$$

Solution

$$\begin{pmatrix} \dfrac{7}{\sqrt{62}} & \dfrac{9}{\sqrt{682}} & \dfrac{1}{\sqrt{11}} \\[3mm] \dfrac{2}{\sqrt{62}} & -\dfrac{24}{\sqrt{682}} & \dfrac{1}{\sqrt{11}} \\[3mm] -\dfrac{3}{\sqrt{62}} & \dfrac{5}{\sqrt{682}} & \dfrac{3}{\sqrt{11}} \end{pmatrix} \begin{pmatrix} \dfrac{7}{\sqrt{62}} & \dfrac{9}{\sqrt{682}} & \dfrac{1}{\sqrt{11}} \\[3mm] \dfrac{2}{\sqrt{62}} & -\dfrac{24}{\sqrt{682}} & \dfrac{1}{\sqrt{11}} \\[3mm] -\dfrac{3}{\sqrt{62}} & \dfrac{5}{\sqrt{682}} & \dfrac{3}{\sqrt{11}} \end{pmatrix}^{T} = \begin{pmatrix} \dfrac{7}{\sqrt{62}} & \dfrac{9}{\sqrt{682}} & \dfrac{1}{\sqrt{11}} \\[3mm] \dfrac{2}{\sqrt{62}} & -\dfrac{24}{\sqrt{682}} & \dfrac{1}{\sqrt{11}} \\[3mm] -\dfrac{3}{\sqrt{62}} & \dfrac{5}{\sqrt{682}} & \dfrac{3}{\sqrt{11}} \end{pmatrix} \begin{pmatrix} \dfrac{7}{\sqrt{62}} & \dfrac{2}{\sqrt{62}} & -\dfrac{3}{\sqrt{62}} \\[3mm] \dfrac{9}{\sqrt{682}} & -\dfrac{24}{\sqrt{682}} & \dfrac{5}{\sqrt{682}} \\[3mm] \dfrac{1}{\sqrt{11}} & \dfrac{1}{\sqrt{11}} & \dfrac{3}{\sqrt{11}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{= I} \quad \checkmark$$

∴ The matrix is orthogonal &

$$\begin{pmatrix} \dfrac{7}{\sqrt{62}} & \dfrac{9}{\sqrt{682}} & \dfrac{1}{\sqrt{11}} \\[3mm] \dfrac{2}{\sqrt{62}} & -\dfrac{24}{\sqrt{682}} & \dfrac{1}{\sqrt{11}} \\[3mm] -\dfrac{3}{\sqrt{62}} & \dfrac{5}{\sqrt{682}} & \dfrac{3}{\sqrt{11}} \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{7}{\sqrt{62}} & \dfrac{9}{\sqrt{682}} & \dfrac{1}{\sqrt{11}} \\[3mm] \dfrac{2}{\sqrt{62}} & -\dfrac{24}{\sqrt{682}} & \dfrac{1}{\sqrt{11}} \\[3mm] -\dfrac{3}{\sqrt{62}} & \dfrac{5}{\sqrt{682}} & \dfrac{3}{\sqrt{11}} \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} \dfrac{1}{2} & \dfrac{1}{2} & \dfrac{1}{2} & \dfrac{1}{2} \\[3mm] \dfrac{1}{2} & -\dfrac{5}{6} & \dfrac{1}{6} & \dfrac{1}{6} \\[3mm] \dfrac{1}{2} & \dfrac{1}{6} & \dfrac{1}{6} & -\dfrac{5}{6} \\[3mm] \dfrac{1}{2} & \dfrac{1}{6} & -\dfrac{5}{6} & \dfrac{1}{6} \end{bmatrix}$$

Solution

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{= I} \quad \checkmark$$

∴ The matrix is orthogonal &

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{bmatrix}$$

Solution

$$\|r_2\| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{1}{3} + \frac{1}{4}}$$

$$= \sqrt{\frac{7}{12}} \neq 1$$

Or

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix}^T =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{5}{6} & & \\ & & & \\ & & & \end{pmatrix} \neq I$$

∴ The matrix is **not** orthogonal

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} 0 & -\frac{1}{2} & \frac{3}{\sqrt{174}} & \frac{9}{2\sqrt{29}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{2} & -\frac{10}{\sqrt{174}} & -\frac{1}{2\sqrt{29}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{2} & -\frac{8}{\sqrt{174}} & \frac{5}{2\sqrt{29}} \\ -\frac{2}{\sqrt{6}} & -\frac{1}{2} & -\frac{1}{\sqrt{174}} & -\frac{3}{2\sqrt{29}} \end{pmatrix}$$

Solution

$$\begin{pmatrix} 0 & -\frac{1}{2} & \frac{3}{\sqrt{174}} & \frac{9}{2\sqrt{29}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{2} & -\frac{10}{\sqrt{174}} & -\frac{1}{2\sqrt{29}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{2} & -\frac{8}{\sqrt{174}} & \frac{5}{2\sqrt{29}} \\ -\frac{2}{\sqrt{6}} & -\frac{1}{2} & -\frac{1}{\sqrt{174}} & -\frac{3}{2\sqrt{29}} \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{2} & \frac{3}{\sqrt{174}} & \frac{9}{2\sqrt{29}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{2} & -\frac{10}{\sqrt{174}} & -\frac{1}{2\sqrt{29}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{2} & -\frac{8}{\sqrt{174}} & \frac{5}{2\sqrt{29}} \\ -\frac{2}{\sqrt{6}} & -\frac{1}{2} & -\frac{1}{\sqrt{174}} & -\frac{3}{2\sqrt{29}} \end{pmatrix}^T$$

$$= \begin{pmatrix} 0 & -\dfrac{1}{2} & \dfrac{3}{\sqrt{174}} & \dfrac{9}{2\sqrt{29}} \\ \dfrac{1}{\sqrt{6}} & -\dfrac{1}{2} & -\dfrac{10}{\sqrt{174}} & -\dfrac{1}{2\sqrt{29}} \\ -\dfrac{1}{\sqrt{6}} & \dfrac{1}{2} & -\dfrac{8}{\sqrt{174}} & \dfrac{5}{2\sqrt{29}} \\ -\dfrac{2}{\sqrt{6}} & -\dfrac{1}{2} & -\dfrac{1}{\sqrt{174}} & -\dfrac{3}{2\sqrt{29}} \end{pmatrix} \begin{pmatrix} 0 & \dfrac{1}{\sqrt{6}} & -\dfrac{1}{\sqrt{6}} & -\dfrac{1}{\sqrt{6}} \\ -\dfrac{1}{2} & -\dfrac{1}{2} & \dfrac{1}{2} & -\dfrac{1}{2} \\ \dfrac{3}{\sqrt{174}} & -\dfrac{10}{\sqrt{174}} & -\dfrac{8}{\sqrt{174}} & -\dfrac{1}{\sqrt{174}} \\ \dfrac{9}{2\sqrt{29}} & -\dfrac{1}{2\sqrt{29}} & \dfrac{5}{2\sqrt{29}} & -\dfrac{3}{2\sqrt{29}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{= I} \mid \checkmark$$

∴ The matrix is orthogonal &

$$\begin{pmatrix} 0 & -\dfrac{1}{2} & \dfrac{3}{\sqrt{174}} & \dfrac{9}{2\sqrt{29}} \\ \dfrac{1}{\sqrt{6}} & -\dfrac{1}{2} & -\dfrac{10}{\sqrt{174}} & -\dfrac{1}{2\sqrt{29}} \\ -\dfrac{1}{\sqrt{6}} & \dfrac{1}{2} & -\dfrac{8}{\sqrt{174}} & \dfrac{5}{2\sqrt{29}} \\ -\dfrac{2}{\sqrt{6}} & -\dfrac{1}{2} & -\dfrac{1}{\sqrt{174}} & -\dfrac{3}{2\sqrt{29}} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -\dfrac{1}{2} & \dfrac{3}{\sqrt{174}} & \dfrac{9}{2\sqrt{29}} \\ \dfrac{1}{\sqrt{6}} & -\dfrac{1}{2} & -\dfrac{10}{\sqrt{174}} & -\dfrac{1}{2\sqrt{29}} \\ -\dfrac{1}{\sqrt{6}} & \dfrac{1}{2} & -\dfrac{8}{\sqrt{174}} & \dfrac{5}{2\sqrt{29}} \\ -\dfrac{2}{\sqrt{6}} & -\dfrac{1}{2} & -\dfrac{1}{\sqrt{174}} & -\dfrac{3}{2\sqrt{29}} \end{pmatrix}$$

Exercise

Determine if the given matrix is orthogonal. If it is, find its inverse

$$\begin{bmatrix} \dfrac{1}{9} & \dfrac{4}{5} & \dfrac{3}{7} \\ \dfrac{4}{9} & \dfrac{3}{5} & -\dfrac{2}{7} \\ \dfrac{8}{9} & -\dfrac{2}{5} & \dfrac{3}{7} \end{bmatrix}$$

<u>*Solution*</u>

$$\vec{q}_1 = \begin{bmatrix} \dfrac{1}{9} & \dfrac{4}{9} & \dfrac{8}{9} \end{bmatrix}^T \qquad \vec{q}_2 = \begin{bmatrix} \dfrac{4}{5} & \dfrac{3}{5} & -\dfrac{2}{5} \end{bmatrix}^T \qquad \vec{q}_3 = \begin{bmatrix} \dfrac{3}{7} & -\dfrac{2}{7} & \dfrac{3}{7} \end{bmatrix}^T$$

$$\vec{q}_1 \cdot \vec{q}_2 = \dfrac{4}{45} + \dfrac{12}{45} - \dfrac{16}{45}$$

$$\underline{= 0} \mid$$

$$\vec{q}_1 \cdot \vec{q}_3 = \dfrac{3}{63} - \dfrac{8}{63} + \dfrac{24}{63}$$

$$= \frac{19}{63} \neq 0 \;\rvert$$

$$\vec{q}_2 \cdot \vec{q}_3 = \frac{12}{35} - \frac{6}{35} + \frac{6}{35}$$

$$= \frac{12}{35} \neq 0 \;\rvert$$

The given matrix is **not** orthogonal

Exercise

Find a last column so that the resulting matrix is orthogonal

$$\begin{bmatrix} \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{6}} & \cdots \\[2mm] \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{6}} & \cdots \\[2mm] -\dfrac{1}{\sqrt{3}} & \dfrac{2}{\sqrt{6}} & \cdots \end{bmatrix}$$

Solution

$$\vec{q}_1 = \begin{bmatrix} \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{3}} \end{bmatrix}^T$$

$$\left\| \vec{q}_1 \right\| = \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}}$$

$$= 1 \;\rvert$$

$$\vec{q}_2 = \begin{bmatrix} \dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{6}} & \dfrac{2}{\sqrt{6}} \end{bmatrix}^T$$

$$\left\| \vec{q}_2 \right\| = \sqrt{\frac{1}{6} + \frac{1}{6} + \frac{4}{6}}$$

$$= 1 \;\rvert$$

Let $\vec{q}_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$

$$\vec{q}_1 \cdot \vec{q}_3 = \frac{1}{\sqrt{3}} x + \frac{1}{\sqrt{3}} y - \frac{1}{\sqrt{3}} z = 0 \quad \rightarrow \quad x + y - z = 0$$

$$\vec{q}_2 \cdot \vec{q}_3 = \frac{1}{\sqrt{6}} x + \frac{1}{\sqrt{6}} y - \frac{2}{\sqrt{6}} z = 0 \quad \rightarrow \quad x + y - 2z = 0$$

$$\begin{cases} x + y - z = 0 \\ x + y - 2z = 0 \end{cases} \rightarrow \quad \underline{z = 0} \;\rvert \quad and \quad x + y = 0 \implies \underline{x = -y} \;\rvert$$

$$x^2 + y^2 = 1$$

$$2x^2 = 1$$

$$x = \pm \frac{1}{\sqrt{2}}$$

$$\vec{q}_3 = \left[-\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad 0 \right]^T$$

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \end{pmatrix}$$

Exercise

Find a last column so that the resulting matrix is orthogonal

$$\begin{pmatrix} \frac{3}{5} & 0 & \cdots \\ \frac{4}{5} & 0 & \cdots \\ 0 & 1 & \cdots \end{pmatrix}$$

Solution

$$\vec{q}_1 = \left[\frac{3}{5} \quad \frac{4}{5} \quad 0 \right]^T$$

$$\left\| \vec{q}_1 \right\| = \sqrt{\frac{9}{25} + \frac{16}{25}}$$

$$= 1$$

$$\vec{q}_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

$$\left\| \vec{q}_2 \right\| = 1$$

Let $\vec{q}_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$

$$\vec{q}_1 \cdot \vec{q}_3 = \frac{3}{5}x + \frac{4}{5}y - 0z = 0$$

$$\frac{3}{5}x + \frac{4}{5}y = 0$$

$$3x + 4y = 0$$

$$\vec{q}_2 \cdot \vec{q}_3 = 0 \cdot x + 0 \cdot y + 1 \cdot z = 0$$

$$z = 0$$

From $3x + 4y = 0$

$$x = -\frac{4}{3}y$$

$$x^2 + y^2 + z^2 = 1$$

$$\frac{16}{9}y^2 + y^2 = 1$$

$$\frac{25}{9}y^2 = 1$$

$$y = \pm\frac{3}{5}$$

$$y = \frac{3}{5} \implies x = -\frac{4}{5}$$

Then $\vec{q}_3 = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} & 0 \end{bmatrix}^T$

$$\begin{pmatrix} \frac{3}{5} & 0 & -\frac{4}{5} \\ \frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \end{pmatrix}$$

$$\vec{q}_1 \cdot \vec{q}_3 = \frac{3}{5}\left(-\frac{4}{5}\right) + \frac{4}{5}\left(\frac{3}{5}\right) = 0 \quad \checkmark$$

Exercise

Find a last column so that the resulting matrix is orthogonal

$$\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & \cdots \\ 0 & 0 & \cdots \\ \frac{4}{5} & \frac{3}{5} & \cdots \end{pmatrix}$$

Solution

$$\vec{q}_1 = \begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}^T$$

$$\|\vec{q}_1\| = \sqrt{\frac{9}{25} + \frac{16}{25}}$$

$$= 1$$

$$\vec{q}_2 = \begin{bmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}^T$$

$$\|\vec{q}_2\| = \sqrt{\frac{9}{25} + \frac{16}{25}}$$

$\underline{=1}$ |

Let $\vec{q}_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$

$\vec{q}_1 \cdot \vec{q}_3 = \frac{3}{5}x - 0y + \frac{4}{5}z = 0$

$\qquad \frac{3}{5}x + \frac{4}{5}z = 0$

$\qquad \underline{3x + 4z = 0}$ |

$\vec{q}_2 \cdot \vec{q}_3 = \left(-\frac{4}{5}\right) \cdot x + 0 \cdot y + \left(\frac{3}{5}\right) \cdot z = 0$

$\qquad -\frac{4}{5}x + \frac{3}{5}z = 0$

$\qquad \underline{-4x + 3z = 0}$ |

$\begin{cases} 3x + 4z = 0 \\ -4x + 3z = 0 \end{cases}$

$x = \dfrac{\begin{vmatrix} 0 & 4 \\ 0 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ -4 & 3 \end{vmatrix}}$

$\quad = \dfrac{0}{25}$

$\quad \underline{= 0}$ |

$\Rightarrow \quad \underline{z = 0}$ |

$\qquad x^2 + y^2 + z^2 = 1$

$\qquad y^2 = 1$

$\qquad \underline{y = \pm 1}$ |

Then $\vec{q}_3 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$

$\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix}$

$\underline{\vec{q}_1 \cdot \vec{q}_3 = \frac{3}{5}(0) + 0 + \frac{4}{5}(0) = 0}$ | $\checkmark$

Exercise

Find a last column so that the resulting matrix is orthogonal

$$\begin{pmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & \cdots \\[2mm] -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & \cdots \\[2mm] 0 & 0 & \cdots \end{pmatrix}$$

Solution

$$\vec{q}_1 = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \end{bmatrix}^T$$

$$\left\| \vec{q}_1 \right\| = \sqrt{\dfrac{1}{2} + \dfrac{1}{2}}$$

$$= 1$$

$$\vec{q}_2 = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \end{bmatrix}^T$$

$$\left\| \vec{q}_2 \right\| = \sqrt{\dfrac{1}{2} + \dfrac{1}{2}}$$

$$= 1$$

Let $\vec{q}_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$

$$\vec{q}_1 \cdot \vec{q}_3 = \dfrac{1}{\sqrt{2}} x - \dfrac{1}{\sqrt{2}} y + 0 \cdot z = 0$$

$$\dfrac{1}{\sqrt{2}} x - \dfrac{1}{\sqrt{2}} y = 0$$

$$x = y$$

$$\vec{q}_2 \cdot \vec{q}_3 = \dfrac{1}{\sqrt{2}} x - \dfrac{1}{\sqrt{2}} y + 0 \cdot z = 0$$

$$\dfrac{1}{\sqrt{2}} x + \dfrac{1}{\sqrt{2}} y = 0$$

$$x = -y$$

$$x = \pm y \quad \Rightarrow \quad x = y = 0$$

$$x^2 + y^2 + z^2 = 1$$

$$z^2 = 1$$

$$z = \pm 1$$

Then $\vec{q}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$

$$\begin{pmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\[2mm] -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\[2mm] 0 & 0 & 1 \end{pmatrix}$$

Exercise

Find a last column so that the resulting matrix is orthogonal

$$\begin{pmatrix} \dfrac{2}{7} & \dfrac{3}{7} & \cdots \\[2mm] \dfrac{6}{7} & \dfrac{2}{7} & \cdots \\[2mm] -\dfrac{3}{7} & \dfrac{6}{7} & \cdots \end{pmatrix}$$

Solution

$$\vec{q}_1 = \begin{bmatrix} \dfrac{2}{7} & \dfrac{6}{7} & -\dfrac{3}{7} \end{bmatrix}^T$$

$$\left\| \vec{q}_1 \right\| = \sqrt{\dfrac{4}{49} + \dfrac{36}{49} + \dfrac{9}{49}}$$

$$\underline{= 1}$$

$$\vec{q}_2 = \begin{bmatrix} \dfrac{3}{7} & \dfrac{2}{7} & \dfrac{6}{7} \end{bmatrix}^T$$

$$\left\| \vec{q}_2 \right\| = \sqrt{\dfrac{9}{49} + \dfrac{4}{49} + \dfrac{36}{49}}$$

$$\underline{= 1}$$

Let $\vec{q}_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$

$$\vec{q}_1 \cdot \vec{q}_3 = \dfrac{2}{7}x + \dfrac{6}{7}y - \dfrac{3}{7}\cdot z = 0$$

$$\underline{2x + 6y - 3z = 0}$$

$$\vec{q}_2 \cdot \vec{q}_3 = \dfrac{3}{7}x + \dfrac{2}{7}y + \dfrac{6}{7}\cdot z = 0$$

$$\underline{3x + 2y + 6z = 0}$$

$$2 \left. + \begin{cases} 2x + 6y - 3z = 0 \\ 3x + 2y + 6z = 0 \end{cases} \right.$$

$$\overline{\quad 7x + 14y = 0 \quad}$$

$$x = -2y$$

$$z = \frac{2}{3}x + 2y$$

$$= -\frac{4}{3}y + 2y$$

$$= \frac{2}{3}y$$

$$x^2 + y^2 + z^2 = 1$$

$$4y^2 + y^2 + \frac{4}{9}y^2 = 1$$

$$\left(\frac{36 + 9 + 4}{9}\right)y^2 = 1$$

$$y^2 = \frac{9}{49}$$

$$y = \pm\frac{3}{7}$$

If $y = \frac{3}{7} \implies x = -\frac{6}{7}, \quad z = \frac{2}{7}$

Or

$$y = -\frac{3}{7} \implies x = \frac{6}{7}, \quad z = -\frac{2}{7}$$

Then $\vec{q}_3 = \begin{bmatrix} -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \end{bmatrix}^T$

$$\begin{pmatrix} \frac{2}{7} & \frac{3}{7} & -\frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & \frac{3}{7} \\ -\frac{3}{7} & \frac{6}{7} & \frac{2}{7} \end{pmatrix} \quad or \quad \begin{pmatrix} \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \\ -\frac{3}{7} & \frac{6}{7} & -\frac{2}{7} \end{pmatrix}$$

Exercise

Find a last column so that the resulting matrix is orthogonal:

$$\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \cdots \\ \frac{2}{3} & \frac{1}{3} & \cdots \\ -\frac{2}{3} & \frac{2}{3} & \cdots \end{pmatrix}$$

Solution

$$\vec{q}_1 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}^T$$

$$\|\vec{q}_1\| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}}$$

$$= 1$$

$$\vec{q}_2 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}^T$$

$$\|\vec{q}_2\| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}}$$

$$= 1$$

Let $\vec{q}_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$

$$\vec{q}_1 \cdot \vec{q}_3 = \frac{1}{3}x + \frac{2}{3}y - \frac{2}{3} \cdot z = 0$$

$$x + 2y - 2z = 0$$

$$\vec{q}_2 \cdot \vec{q}_3 = \frac{2}{3}x + \frac{1}{3}y + \frac{2}{3} \cdot z = 0$$

$$2x + y + 2z = 0$$

$$+ \begin{cases} x + 2y - 2z = 0 \\ 2x + y + 2z = 0 \end{cases}$$

$$\overline{\qquad 3x + 3y = 0 \qquad}$$

$$x = -y$$

$$z = \frac{1}{2}x + y$$

$$= -\frac{1}{2}y + y$$

$$= \frac{1}{2}y$$

$$x^2 + y^2 + z^2 = 1$$

$$y^2 + y^2 + \frac{1}{4}y^2 = 1$$

$$\left(\frac{9}{4}\right)y^2 = 1$$

$$y = \pm\frac{2}{3}$$

If $y = \frac{2}{3} \Rightarrow x = -\frac{2}{3}, \quad z = \frac{1}{3}$

Or

$$y = -\frac{2}{3} \Rightarrow x = \frac{2}{3}, \quad z = -\frac{1}{3}$$

$$\vec{q}_3 = \begin{bmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}^T \quad or \quad \vec{q}_3 = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}^T$$

$$\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \quad or \quad \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

Exercise

Find a last column so that the resulting matrix is orthogonal:

$$\begin{pmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & \cdots \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} & \cdots \\ -\frac{1}{3} & 0 & \cdots \end{pmatrix}$$

Solution

$$\vec{q}_1 = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}^T$$

$$\left\| \vec{q}_1 \right\| = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}}$$

$$\underline{\underline{= 1}}$$

$$\vec{q}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}^T$$

$$\left\| \vec{q}_2 \right\| = \sqrt{\frac{1}{2} + \frac{1}{2}}$$

$$\underline{\underline{= 1}}$$

Let $\vec{q}_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$

$$\vec{q}_1 \cdot \vec{q}_3 = \frac{2}{3}x + \frac{2}{3}y - \frac{1}{3} \cdot z = 0$$

$$\underline{2x + 2y - z = 0}$$

$$\vec{q}_2 \cdot \vec{q}_3 = \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y + 0 \cdot z = 0$$

$$\underline{y = x}$$

$$2x + 2y - z = 0$$

$$2x + 2x - z = 0$$

$$\underline{z = 4x}$$

$$x^2 + y^2 + z^2 = 1$$

$$x^2 + x^2 + 16x^2 = 1$$

$$x^2 = \frac{1}{18}$$

$$x = \pm\frac{1}{3\sqrt{2}}$$

If $x = \dfrac{1}{3\sqrt{2}} \quad \Rightarrow \quad y = \dfrac{1}{3\sqrt{2}}, \quad z = \dfrac{4}{3\sqrt{2}}$

Or

$$x = -\frac{1}{3\sqrt{2}} \quad \Rightarrow \quad y = -\frac{1}{3\sqrt{2}}, \quad z = -\frac{4}{3\sqrt{2}}$$

$$\vec{q}_3 = \begin{bmatrix} \dfrac{1}{3\sqrt{2}} & \dfrac{1}{3\sqrt{2}} & \dfrac{4}{3\sqrt{2}} \end{bmatrix}^T \quad or \quad \vec{q}_3 = \begin{bmatrix} -\dfrac{1}{3\sqrt{2}} & -\dfrac{1}{3\sqrt{2}} & -\dfrac{4}{3\sqrt{2}} \end{bmatrix}^T$$

For $\vec{q}_3 = \begin{bmatrix} \dfrac{1}{3\sqrt{2}} & \dfrac{1}{3\sqrt{2}} & \dfrac{4}{3\sqrt{2}} \end{bmatrix}^T$

$$\vec{q}_1 \bullet \vec{q}_3 = \frac{2}{3}\cdot\frac{1}{3\sqrt{2}} + \frac{2}{3}\cdot\frac{1}{3\sqrt{2}} - \frac{1}{3}\cdot\frac{4}{3\sqrt{2}} = 0 \quad \checkmark$$

$$\vec{q}_2 \bullet \vec{q}_3 = \frac{1}{\sqrt{2}}\cdot\frac{1}{3\sqrt{2}} - \frac{1}{\sqrt{2}}\cdot\frac{1}{3\sqrt{2}} + 0\cdot\frac{4}{3\sqrt{2}} = 0 \quad \checkmark$$

For $\vec{q}_3 = \begin{bmatrix} -\dfrac{1}{3\sqrt{2}} & -\dfrac{1}{3\sqrt{2}} & -\dfrac{4}{3\sqrt{2}} \end{bmatrix}^T$

$$\vec{q}_1 \bullet \vec{q}_3 = -\frac{2}{3}\cdot\frac{1}{3\sqrt{2}} - \frac{2}{3}\cdot\frac{1}{3\sqrt{2}} + \frac{1}{3}\cdot\frac{4}{3\sqrt{2}} = 0 \quad \checkmark$$

$$\vec{q}_2 \bullet \vec{q}_3 = \frac{1}{\sqrt{2}}\cdot\frac{1}{3\sqrt{2}} - \frac{1}{\sqrt{2}}\cdot\frac{1}{3\sqrt{2}} + 0\cdot\frac{4}{3\sqrt{2}} = 0 \quad \checkmark$$

$$\begin{pmatrix} \dfrac{2}{3} & \dfrac{1}{\sqrt{2}} & \dfrac{1}{3\sqrt{2}} \\[2mm] \dfrac{2}{3} & -\dfrac{1}{\sqrt{2}} & \dfrac{1}{3\sqrt{2}} \\[2mm] -\dfrac{1}{3} & 0 & \dfrac{4}{3\sqrt{2}} \end{pmatrix} \quad or \quad \begin{pmatrix} \dfrac{2}{3} & \dfrac{1}{\sqrt{2}} & -\dfrac{1}{3\sqrt{2}} \\[2mm] \dfrac{2}{3} & -\dfrac{1}{\sqrt{2}} & -\dfrac{1}{3\sqrt{2}} \\[2mm] -\dfrac{1}{3} & 0 & -\dfrac{4}{3\sqrt{2}} \end{pmatrix}$$

Exercise

Find a last column so that the resulting matrix is orthogonal:

$$\begin{pmatrix} \dfrac{2}{\sqrt{5}} & \dfrac{3}{\sqrt{70}} & \cdots \\[2ex] -\dfrac{1}{\sqrt{5}} & \dfrac{6}{\sqrt{70}} & \cdots \\[2ex] 0 & -\dfrac{5}{\sqrt{70}} & \cdots \end{pmatrix}$$

Solution

$$\vec{q}_1 = \begin{bmatrix} \dfrac{2}{\sqrt{5}} & -\dfrac{1}{\sqrt{5}} & 0 \end{bmatrix}^T$$

$$\left\| \vec{q}_1 \right\| = \sqrt{\dfrac{4}{5} + \dfrac{1}{5}}$$

$$\underline{\underline{= 1}}$$

$$\vec{q}_2 = \begin{bmatrix} \dfrac{3}{\sqrt{70}} & \dfrac{6}{\sqrt{70}} & -\dfrac{5}{\sqrt{70}} \end{bmatrix}^T$$

$$\left\| \vec{q}_2 \right\| = \sqrt{\dfrac{9}{70} + \dfrac{36}{70} + \dfrac{25}{70}}$$

$$\underline{\underline{= 1}}$$

Let $\vec{q}_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$

$$\vec{q}_1 \cdot \vec{q}_3 = \dfrac{2}{\sqrt{5}} x - \dfrac{1}{\sqrt{5}} y + 0 \cdot z = 0$$

$$\underline{y = 2x}$$

$$\vec{q}_2 \cdot \vec{q}_3 = \dfrac{3}{\sqrt{70}} x + \dfrac{6}{\sqrt{70}} y - \dfrac{5}{\sqrt{70}} \cdot z = 0$$

$$\underline{3x + 6y - 5z = 0}$$

$$3x + 12x - 5z = 0$$

$$5z = 15x$$

$$\underline{z = 3x}$$

$$x^2 + y^2 + z^2 = 1$$

$$x^2 + 4x^2 + 9x^2 = 1$$

$$x^2 = \dfrac{1}{14}$$

$$x = \pm \dfrac{1}{\sqrt{14}}$$

If $x = \dfrac{1}{\sqrt{14}} \implies y = \dfrac{2}{\sqrt{14}}, \quad z = \dfrac{3}{\sqrt{14}}$

Or

$x = -\dfrac{1}{\sqrt{14}} \implies y = -\dfrac{2}{\sqrt{14}}, \quad z = -\dfrac{3}{\sqrt{14}}$

$\vec{q}_3 = \begin{bmatrix} \dfrac{1}{\sqrt{14}} & \dfrac{2}{\sqrt{14}} & \dfrac{3}{\sqrt{14}} \end{bmatrix}^T$ $\quad or \quad$ $\vec{q}_3 = \begin{bmatrix} -\dfrac{1}{\sqrt{14}} & -\dfrac{2}{\sqrt{14}} & -\dfrac{3}{\sqrt{14}} \end{bmatrix}^T$

$$\begin{pmatrix} \dfrac{2}{\sqrt{5}} & \dfrac{3}{\sqrt{70}} & \dfrac{1}{\sqrt{14}} \\[3mm] -\dfrac{1}{\sqrt{5}} & \dfrac{6}{\sqrt{70}} & \dfrac{2}{\sqrt{14}} \\[3mm] 0 & -\dfrac{5}{\sqrt{70}} & \dfrac{3}{\sqrt{14}} \end{pmatrix}$$

Exercise

Prove that if A is orthogonal, then A^T is orthogonal.

Solution

Since A is orthogonal then $A^T = A^{-1}$ and $A = \left(A^T\right)^T$

Then $\left(A^T\right)^T A^T = A A^T = I \implies A^T$ is orthogonal.

Another word, since A is orthogonal, then both column and row vectors of A form an orthonormal set.

A^T is just A with its row and column vectors are swapped.

The column vectors of A^T (which are the row vectors of A) and row vectors of A^T (which are the column vectors of A) form orthonormal sets, therefore A^T is orthogonal

Exercise

Prove that if A is orthogonal, then A^{-1} is orthogonal

Solution

Since A is orthogonal then $A^T = A^{-1}$ and $A = \left(A^{-1}\right)^{-1}$

$\left(A^{-1}\right)^{-1} = \left(A^T\right)^{-1} \qquad A^T = A^{-1}$

$\qquad\qquad = \left(A^{-1}\right)^T$

$\therefore A^{-1}$ is orthogonal.

Exercise

Prove that if A and B are orthogonal, then AB is orthogonal.

Solution

Since A is orthogonal then $A^T = A^{-1}$

and B is orthogonal then $B^T = B^{-1}$

$$(AB)^T = B^T A^T$$
$$= B^{-1} A^{-1}$$
$$= (AB)^{-1}$$

$\therefore$ AB is orthogonal.

Exercise

Let Q be an $n \times n$ orthogonal matrix, and let A be an $n \times n$ matrix.

Show that $\det\left(QAQ^T\right) = \det(A)$

Solution

$$\det\left(QAQ^T\right) = \det(Q)\det(A)\det\left(Q^T\right)$$
$$= \det(A)\det\left(QQ^T\right) \qquad \text{Since } Q \text{ is an orthogonal matrix } \det\left(QQ^T\right) = \det(I)$$
$$= \det(A)\det(I)$$
$$= \det(A) \quad \checkmark$$

Exercise

Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$

a) Is matrix A an orthogonal matrix?
b) Let B be the matrix obtained by normalizing each row of A, find B.
c) Is B an orthogonal matrix?
d) Are the columns of B orthogonal?

Solution

a) $\quad AA^T = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 7 \\ 1 & 3 & -5 \\ -1 & 4 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 3 & & \\ & & \\ & & \end{pmatrix} \neq I$$

$\therefore \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$ is **not** orthogonal.

b) $\|(1, 1, -1)\| = \sqrt{1+1+1}$

$$= \sqrt{3}$$

$\|(1, 3, 4)\| = \sqrt{1+9+16}$

$$= \sqrt{26}$$

$\|(7, -5, 2)\| = \sqrt{49+25+4}$

$$= \sqrt{78}$$

$$B = \begin{pmatrix} \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{3}} \\ \dfrac{1}{\sqrt{26}} & \dfrac{3}{\sqrt{26}} & \dfrac{4}{\sqrt{26}} \\ \dfrac{7}{\sqrt{78}} & -\dfrac{5}{\sqrt{78}} & \dfrac{2}{\sqrt{78}} \end{pmatrix}$$

c) *Yes*, since the rows are orthogonal with unit vectors.

$$BB^T = \begin{pmatrix} \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{3}} \\ \dfrac{1}{\sqrt{26}} & \dfrac{3}{\sqrt{26}} & \dfrac{4}{\sqrt{26}} \\ \dfrac{7}{\sqrt{78}} & -\dfrac{5}{\sqrt{78}} & \dfrac{2}{\sqrt{78}} \end{pmatrix} \begin{pmatrix} \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{26}} & \dfrac{7}{\sqrt{78}} \\ \dfrac{1}{\sqrt{3}} & \dfrac{3}{\sqrt{26}} & -\dfrac{5}{\sqrt{78}} \\ -\dfrac{1}{\sqrt{3}} & \dfrac{4}{\sqrt{26}} & \dfrac{2}{\sqrt{78}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

d) *Yes*, since the rows of B form an orthonormal set of vectors. Then, the column of B must form an orthonormal set.

$$\left\|\left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{26}}, \dfrac{7}{\sqrt{78}}\right)\right\| = \sqrt{\dfrac{1}{3} + \dfrac{1}{26} + \dfrac{49}{78}}$$

$$= \sqrt{\dfrac{26+3+49}{78}}$$

$$= \sqrt{\frac{78}{78}}$$

$$= 1$$

$$\left\| \left(\frac{1}{\sqrt{3}}, \ \frac{3}{\sqrt{26}}, \ -\frac{5}{\sqrt{78}} \right) \right\| = \sqrt{\frac{1}{3} + \frac{9}{26} + \frac{25}{78}}$$

$$= \sqrt{\frac{26 + 27 + 25}{78}}$$

$$= \sqrt{\frac{78}{78}}$$

$$= 1$$

$$\left\| \left(-\frac{1}{\sqrt{3}}, \ \frac{4}{\sqrt{26}}, \ \frac{2}{\sqrt{78}} \right) \right\| = \sqrt{\frac{1}{3} + \frac{16}{26} + \frac{4}{78}}$$

$$= \sqrt{\frac{78}{78}}$$

$$= 1$$

Solution   *Section* **3.5 – Least Squares Analysis**

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(0,\ 2),\ (1,\ 2),\ (2,\ 0)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points.

Then

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

where $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} m \\ b \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

The normal equation formula: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$m = \frac{\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 3 & 3 \end{vmatrix}}$$

$$= \frac{-6}{6}$$

$$\underline{= -1}$$

$$b = \frac{\begin{vmatrix} 5 & 2 \\ 3 & 4 \end{vmatrix}}{6}$$

$$\underline{= \frac{7}{3}}$$

Thus, $\underline{y = -x + \dfrac{7}{3}}$
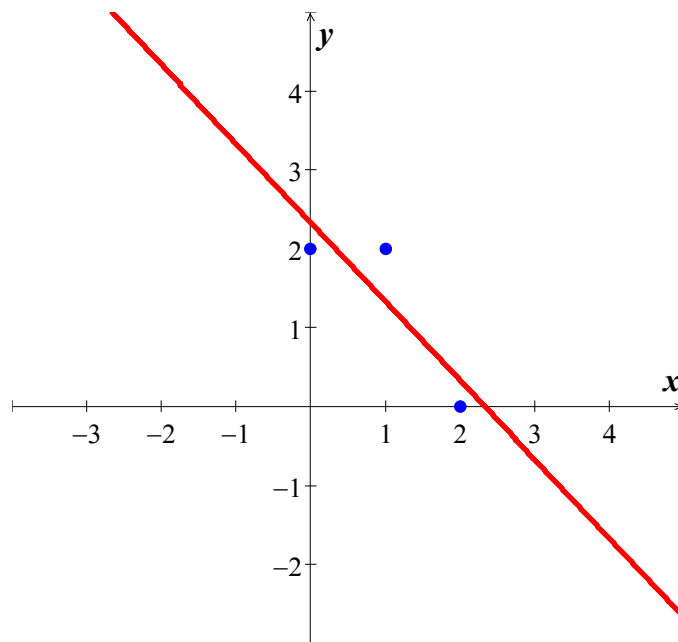
$$A\vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ \frac{7}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{7}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{7}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$$

Error: $\left\| \vec{y} - A\vec{x} \right\| = \sqrt{\dfrac{1}{9} + \dfrac{4}{9} + \dfrac{1}{9}}$

$$= \frac{\sqrt{6}}{3}$$

$$\approx 0.8164966$$

****

The **second** *order* equation: $y = -x^2 + x + 2$

Error = 0.00000

****

The **third order** equation: $y = -.1250x^3 - 0.1250x^2 + 0.25x + 2$

Error = 2.01556
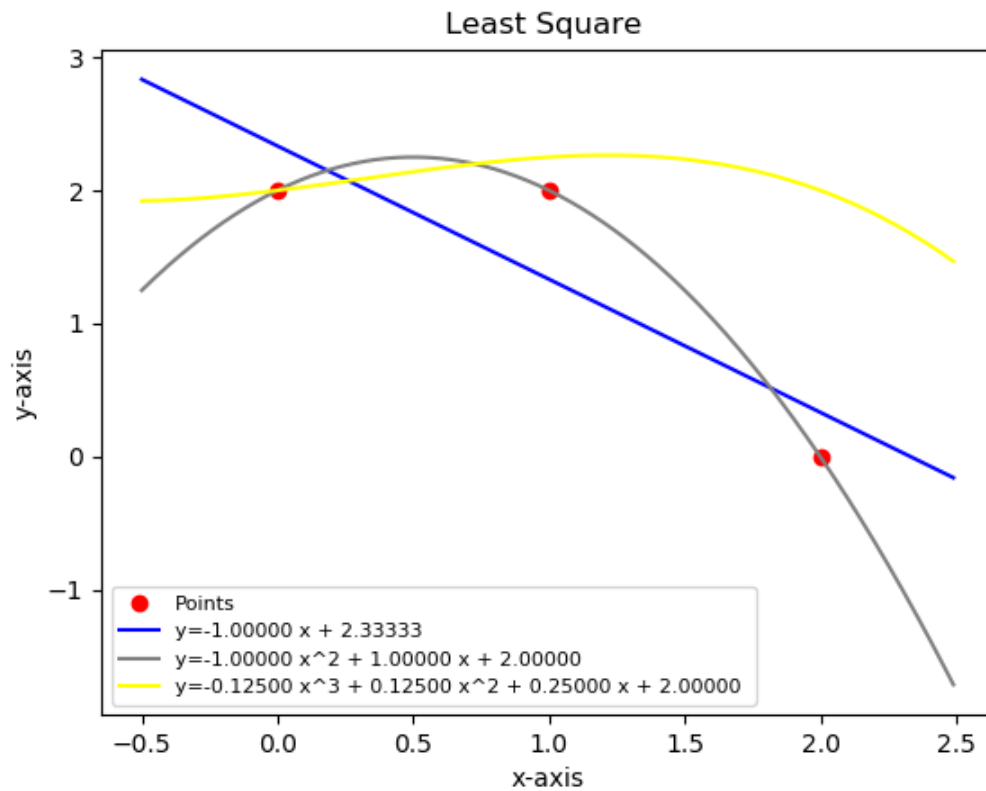
Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(0,\ 0),\ \ (1,\ 1),\ \ (2,\ 4)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points.

Then

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$$

where $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} m \\ b \end{bmatrix}$ $\quad \vec{y} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$

The normal equation formula: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$

$$m = \frac{\begin{vmatrix} 9 & 3 \\ 5 & 3 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 3 & 3 \end{vmatrix}}$$

$$= \frac{12}{6}$$

$$\underline{= 2}$$

$$b = \frac{\begin{vmatrix} 5 & 9 \\ 3 & 5 \end{vmatrix}}{6}$$

$$\underline{= -\frac{1}{3}}$$

Thus, $\underline{y = 2x - \frac{1}{3}}$

$$A\vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -\frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} \\ \frac{5}{3} \\ \frac{11}{3} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} -\frac{1}{3} \\ \frac{5}{3} \\ \frac{11}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

Error: $\| \vec{y} - A\vec{x} \| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{1}{9}}$

$$\underline{= \frac{\sqrt{6}}{3}}$$

$$\underline{\approx 0.8164966}$$

****
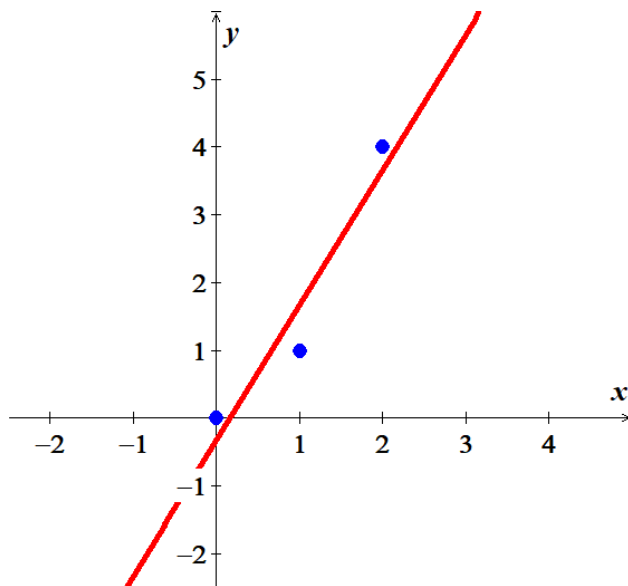
The **second** order equation: $\quad y = -x^2$

Error = 0.00000

****

The **third order** equation: $y = 2.78x^3 - 1.469x^2 + 1.3125x$

Error = 15.08776
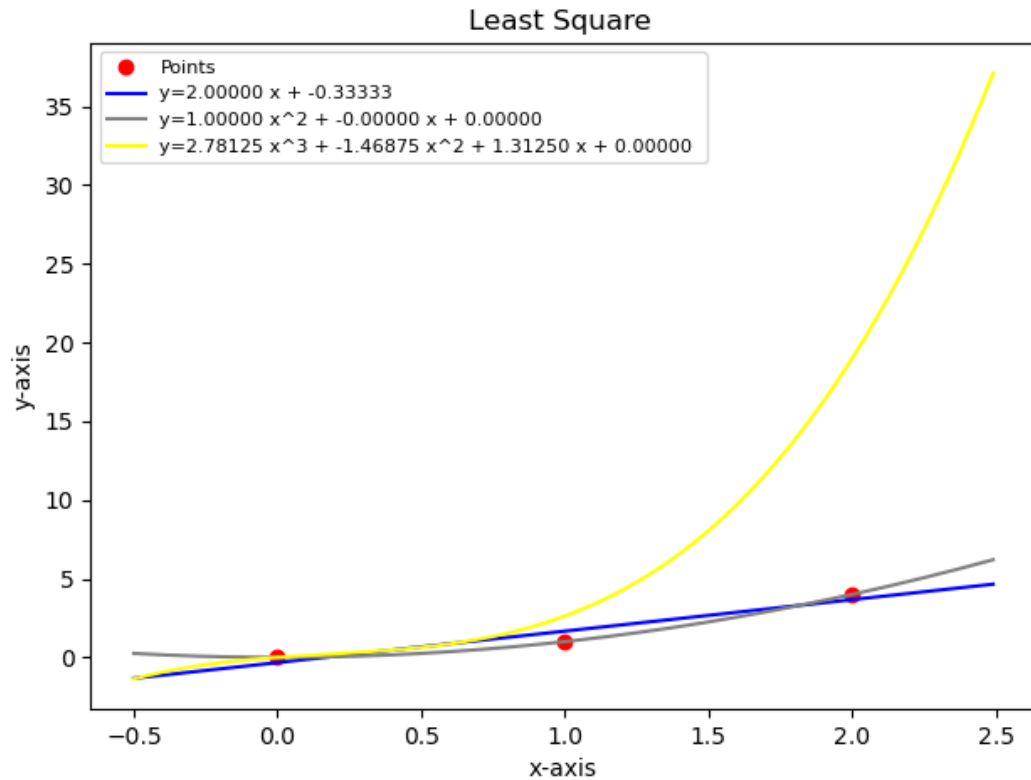
Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(0,\ 0),\ \ (2,\ 1),\ \ (4,\ 1)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then,

$$\begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

where $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 4 & 1 \end{pmatrix}$ $\vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$ $\vec{y} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

The normal equation formula: $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 0 & 2 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 20 & 6 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$m = \frac{\begin{vmatrix} 6 & 6 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} 20 & 6 \\ 6 & 3 \end{vmatrix}}$$

$$= \frac{6}{24}$$

$$= \frac{1}{4}$$

$$b = \frac{4}{24}$$

$$= \frac{1}{6}$$

Thus, $y = \frac{1}{4}x + \frac{1}{6}$

$$A\vec{x} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{6} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{6} \\ \frac{2}{3} \\ \frac{7}{6} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{6} \\ \frac{2}{3} \\ \frac{7}{6} \end{pmatrix}$$

$$= \begin{pmatrix} -\dfrac{1}{6} \\ \dfrac{1}{3} \\ -\dfrac{1}{6} \end{pmatrix}$$

Error: $\quad \|\vec{y} - A\vec{x}\| = \sqrt{\dfrac{1}{36} + \dfrac{1}{9} + \dfrac{1}{36}}$

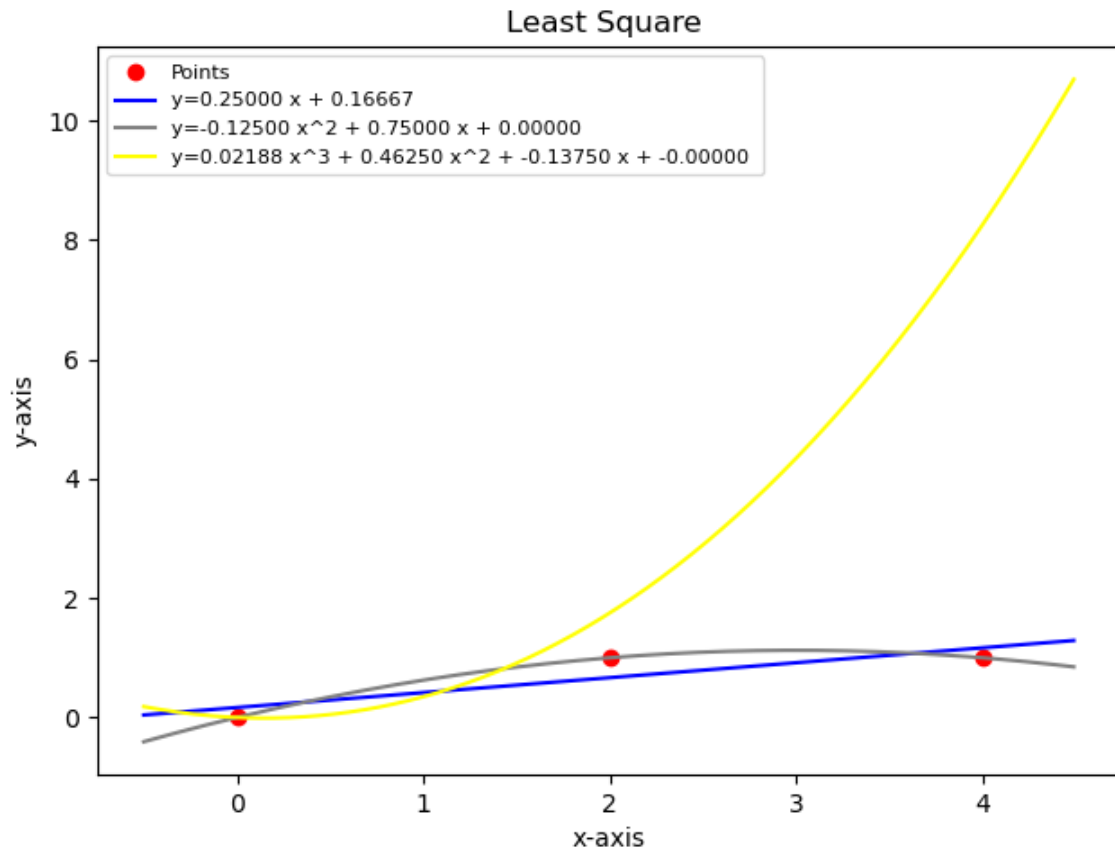$$= \dfrac{\sqrt{6}}{6}$$

$$\approx 0.40825$$

****

The **second** order equation: $\quad y = -0.125x^2 + .75x$

Error $= 0.00000$

****

The **third** order equation: $y = 0.01288x^3 + .4625x^2 - .1375x$

Error $= 7.28869$

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-1,\ -1),\ (1,\ 0),\ (2,\ 4)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then,

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$$

where $A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$ $\vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$ $\vec{y} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$

The normal equation formula: $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \end{pmatrix}$$

$$m = \frac{21}{14}$$

$$= \underline{\frac{3}{2}}$$

$$b = \frac{0}{14}$$

$$= \underline{0}$$

Thus, $\underline{y = \dfrac{3}{2} x}$

$$A\vec{x} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ 3 \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 1 \end{pmatrix}$$

Error: $\quad \|\vec{y} - A\vec{x}\| = \sqrt{\frac{1}{4} + \frac{9}{4} + 1}$

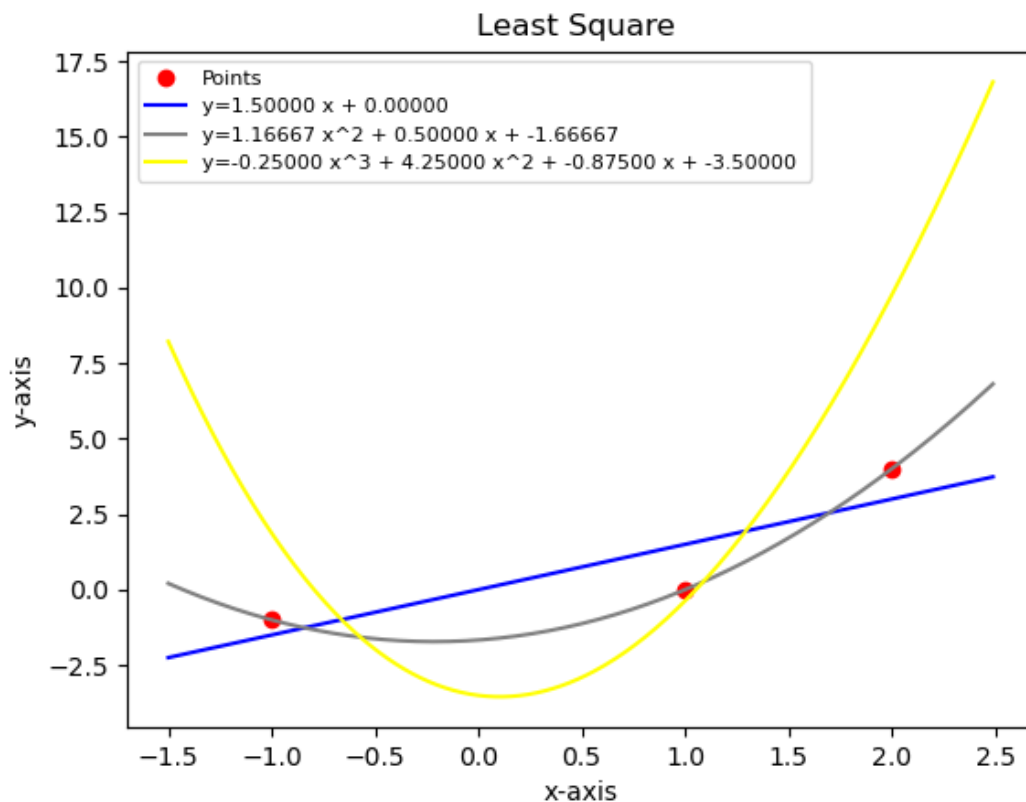$$= \frac{\sqrt{14}}{2}$$

$$\approx 1.87083$$

****

The **second** *order* equation: $\qquad y = 1.1667x^2 + .5x - 1.667$

Error = 0.00000

****

The **third order** equation: $\qquad y = -.25x^3 + 4.25x^2 - .875x - 3.5$

Error = 6.43962

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(0,\,1),\,(1,\,1),\,(2,\,2),\,(3,\,2)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then,

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$$

where $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$ $\vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$ $\vec{y} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$

The normal equation formula: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 6 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 11 \\ 6 \end{pmatrix}$$

$$m = \frac{8}{20}$$

$$= \frac{2}{5}$$

$$b = \frac{18}{20}$$

$$= \frac{9}{10}$$

Thus, $y = \frac{2}{5}x + \frac{9}{10}$

$$A\vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{5} \\ \frac{9}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{9}{10} \\[2mm] \dfrac{13}{10} \\[2mm] \dfrac{17}{10} \\[2mm] \dfrac{21}{10} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} \dfrac{9}{10} \\[2mm] \dfrac{13}{10} \\[2mm] \dfrac{17}{10} \\[2mm] \dfrac{21}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1}{10} \\[2mm] -\dfrac{3}{10} \\[2mm] \dfrac{3}{10} \\[2mm] -\dfrac{1}{10} \end{pmatrix}$$

E*rror*: $\quad \|\vec{y} - A\vec{x}\| = \sqrt{\dfrac{1}{100} + \dfrac{9}{100} + \dfrac{9}{100} + \dfrac{1}{100}}$

$$= \dfrac{2\sqrt{5}}{10}$$

$$\approx 0.44721$$

****

The *second order* equation: $\quad y = 0x^2 + .4x + .9$

E*rror* = 0. 44721

****

The *third order* equation: $\quad y = -.333x^3 + 1.5x^2 - 1.1667x + 1$

E*rror* = 0.00000
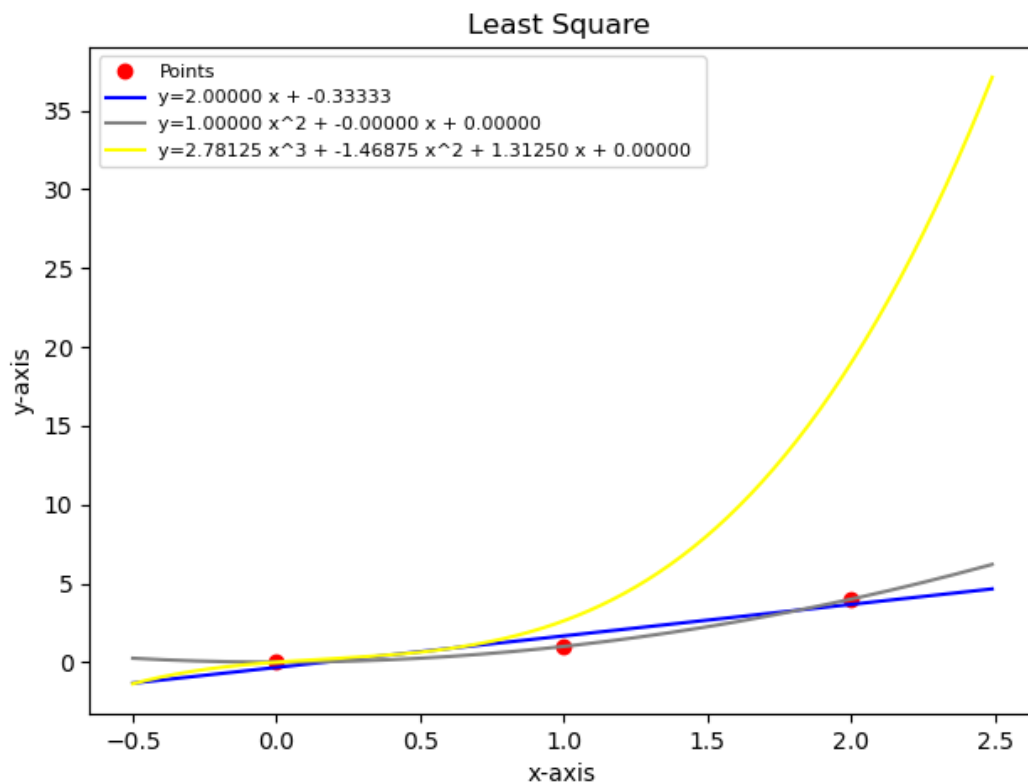
Least Square

Legend:
- Points
- y=2.00000 x + -0.33333
- y=1.00000 x^2 + -0.00000 x + 0.00000
- y=2.78125 x^3 + -1.46875 x^2 + 1.31250 x + 0.00000

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(2, 1), (5, 2), (7, 3), (8, 3)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then,

$$\begin{pmatrix} 2 & 1 \\ 5 & 1 \\ 7 & 1 \\ 8 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}$$

where $A = \begin{pmatrix} 2 & 1 \\ 5 & 1 \\ 7 & 1 \\ 8 & 1 \end{pmatrix}$ $\vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$ $\vec{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}$

The normal equation formula: $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 2 & 5 & 7 & 8 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 5 & 1 \\ 7 & 1 \\ 8 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 2 & 5 & 7 & 8 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 142 & 22 \\ 22 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 57 \\ 9 \end{pmatrix}$$

$$m = \frac{30}{84}$$

$$= \frac{5}{14}$$

$$b = \frac{24}{84}$$

$$= \frac{4}{14}$$

Thus, $y = \dfrac{5}{14}x + \dfrac{4}{14}$

$$A\vec{x} = \begin{pmatrix} 2 & 1 \\ 5 & 1 \\ 7 & 1 \\ 8 & 1 \end{pmatrix} \begin{pmatrix} \frac{5}{14} \\ \frac{4}{14} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ \frac{29}{14} \\ \frac{39}{14} \\ \frac{22}{7} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ \frac{29}{14} \\ \frac{39}{14} \\ \frac{22}{7} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -\frac{1}{14} \\ \frac{3}{14} \\ -\frac{1}{7} \end{pmatrix}$$

Error: $\quad \|\vec{y} - A\vec{x}\| = \sqrt{\dfrac{1}{196} + \dfrac{9}{196} + \dfrac{1}{49}}$

$$= \frac{\sqrt{14}}{14}$$

$$\approx 0.26726$$

****

The **second** order equation: $y = -.00758x^2 + .43182x + .14394$

Error = 0. 26112

****

The **third order** equation: $y = -.0333x^3 + .5x^2 - 1.86667x + 3$

Error = 0.00000
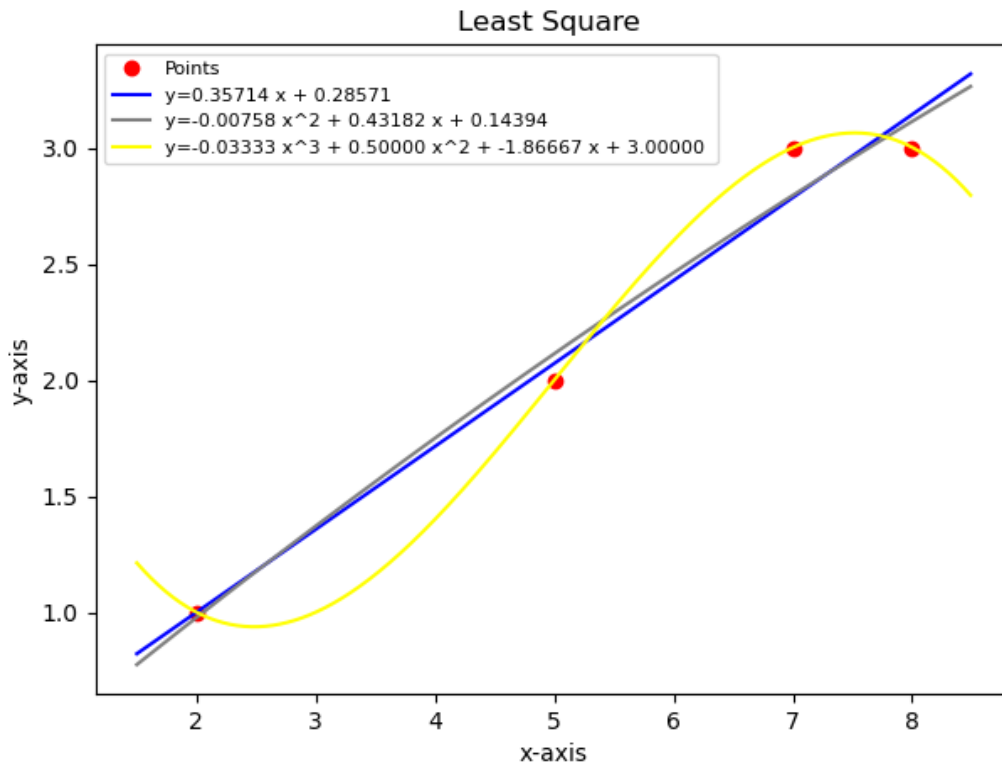
Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-2, \ -2), \ (-1, \ 0), \ (0, \ -2), \ (1, \ 0)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then,

$$\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

where $A = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ $\vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$ $\vec{y} = \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix}$

The normal equation formula: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} -2 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -2 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

$m = \dfrac{8}{20}$

$= \dfrac{2}{5}$

$b = -\dfrac{16}{20}$

$= -\dfrac{4}{5}$

Thus, $y = \dfrac{2}{5}x - \dfrac{4}{5}$

$$A\vec{x} = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{5} \\ -\frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{8}{5} \\ -\frac{6}{5} \\ -\frac{4}{5} \\ -\frac{2}{5} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} -\frac{8}{5} \\ -\frac{6}{5} \\ -\frac{4}{5} \\ -\frac{2}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -\dfrac{2}{5} \\[1.5ex] \dfrac{6}{5} \\[1.5ex] -\dfrac{6}{5} \\[1.5ex] \dfrac{2}{5} \end{pmatrix}$$

Error: $\quad \left\| \vec{y} - A\vec{x} \right\| = \sqrt{\dfrac{4}{25} + \dfrac{36}{25} + \dfrac{36}{25} + \dfrac{4}{25}}$
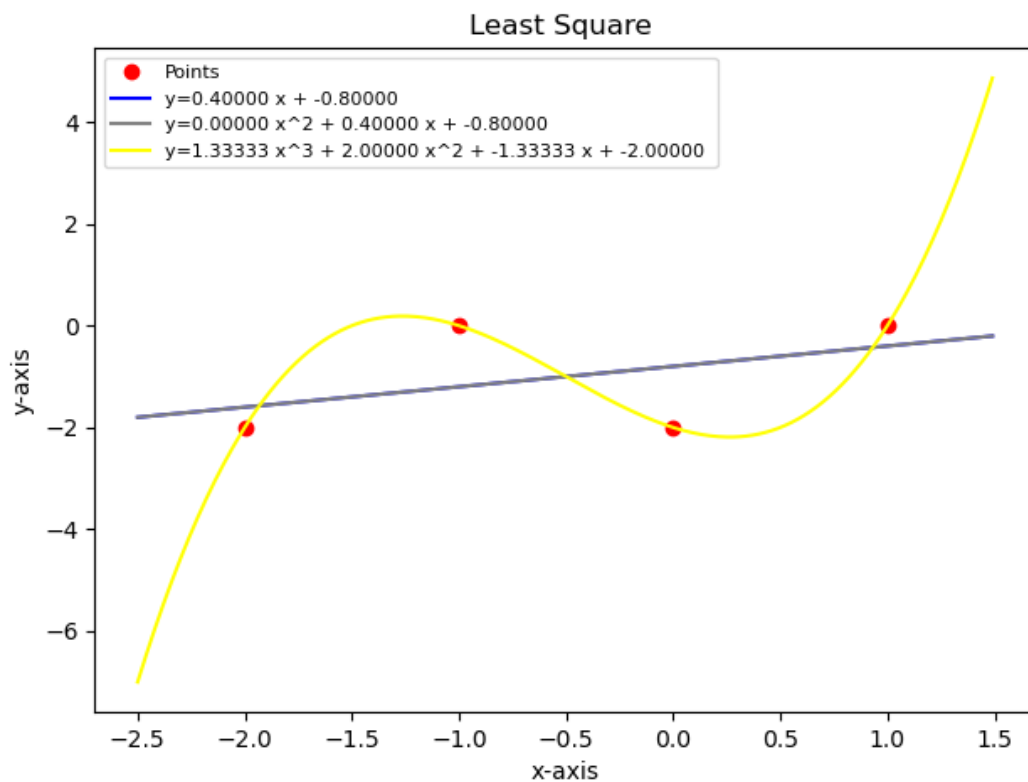
$$= \dfrac{4\sqrt{5}}{5}$$

$$\approx 1.78885$$

****

The **second** order equation: $\quad y = 0x^2 + .4x + .8$

Error = 1.78885

****

The **third order** equation: $\quad y = 1.333x^3 + 2x^2 - 1.333x - 2$

Error = 0.00000

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-2,\ 0),\ (-1,\ 1),\ (0,\ 1),\ (1,\ 2)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then,

$$\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

where $A = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$

The normal equation formula: $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} -2 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -2 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$m = \frac{12}{20}$$

$$= \frac{3}{5}$$

$$b = \frac{26}{20}$$

$$= \frac{13}{10}$$

Thus, $y = \dfrac{3}{5}x + \dfrac{13}{10}$ y = 0.60000 x + 1.3000

$$A\vec{x} = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{5} \\ \frac{13}{10} \end{pmatrix}$$

$$= \begin{pmatrix} -\dfrac{1}{10} \\[6pt] \dfrac{7}{10} \\[6pt] \dfrac{13}{10} \\[6pt] \dfrac{19}{10} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} -\dfrac{1}{10} \\[6pt] \dfrac{7}{10} \\[6pt] \dfrac{13}{10} \\[6pt] \dfrac{19}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1}{10} \\[6pt] \dfrac{3}{10} \\[6pt] -\dfrac{3}{10} \\[6pt] \dfrac{1}{10} \end{pmatrix}$$

E*rror*: $\qquad \|\vec{y} - A\vec{x}\| = \sqrt{\dfrac{1}{100} + \dfrac{9}{100} + \dfrac{9}{100} + \dfrac{1}{100}}$

$$= \dfrac{\sqrt{5}}{5}$$

$$\approx 0.44721$$

****

The *second* *order* equation: $\qquad y = 0x^2 + .6x + 1.3$

Error = 0.44721

****

The ***third order*** equation: $\qquad y = 0.333x^3 + .5x^2 + 1.6667x + 1$

Error = 0.00000
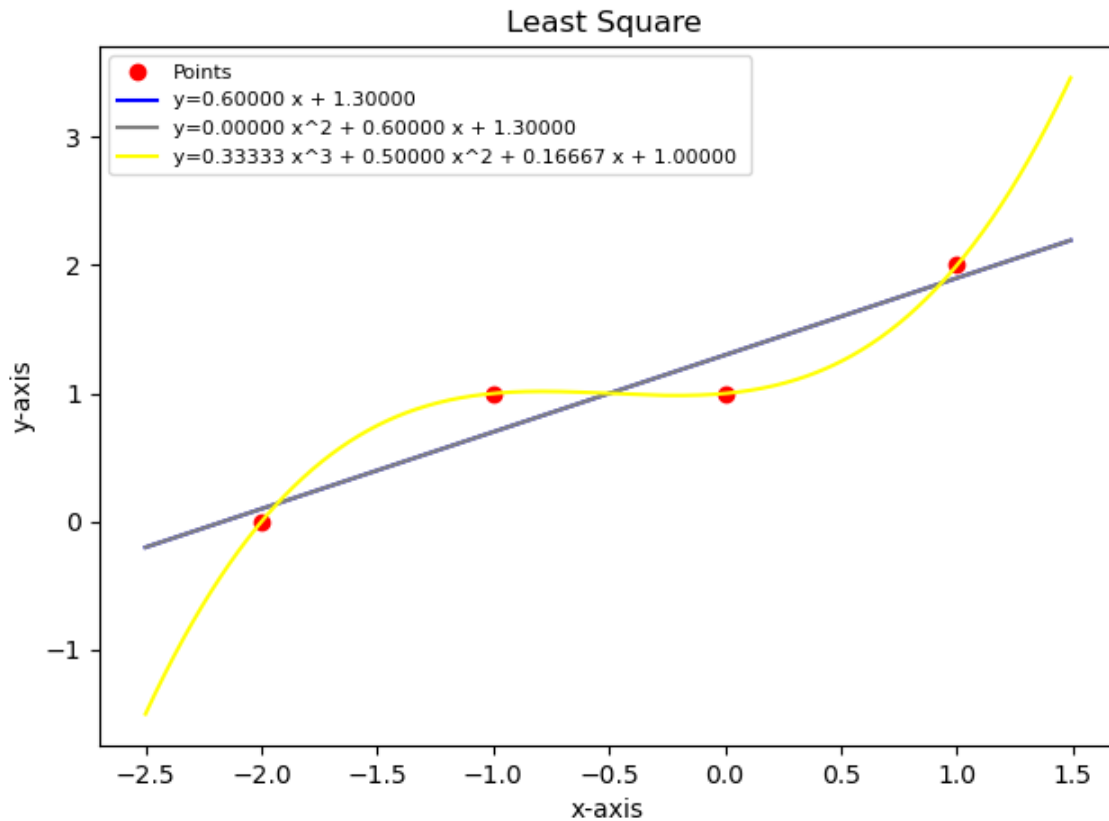
Least Square

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(0,\ 1),\ (1,\ 3),\ (2,\ 4),\ (3,\ 4)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

where $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$

The normal equation: $A^T A \vec{x} = A^T \vec{y}$
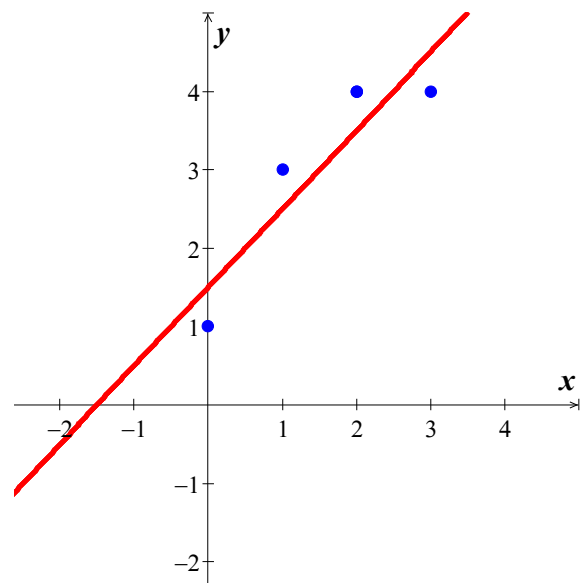
$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 6 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 23 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{20} \begin{pmatrix} 4 & -6 \\ -6 & 14 \end{pmatrix} \begin{pmatrix} 23 \\ 12 \end{pmatrix} \qquad \textcolor{teal}{X = A^{-1}B}$$

$$= \frac{1}{20} \begin{pmatrix} 20 \\ 30 \end{pmatrix}$$

$$= \textcolor{blue}{\begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}}$$

We have: $m = 1$ *and* $b = \dfrac{3}{2}$.

Thus, $\textcolor{blue}{y = x + \dfrac{3}{2}}$

$$A\vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2} \\ \frac{5}{2} \\ \frac{7}{2} \\ \frac{9}{2} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} \frac{3}{2} \\ \frac{5}{2} \\ \frac{7}{2} \\ \frac{9}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -\dfrac{1}{2} \\[6pt] \dfrac{1}{2} \\[6pt] \dfrac{1}{2} \\[6pt] -\dfrac{1}{2} \end{pmatrix}$$

Error:

$$\left\| \vec{y} - A\vec{x} \right\| = \sqrt{4\left(\dfrac{1}{4}\right)}$$
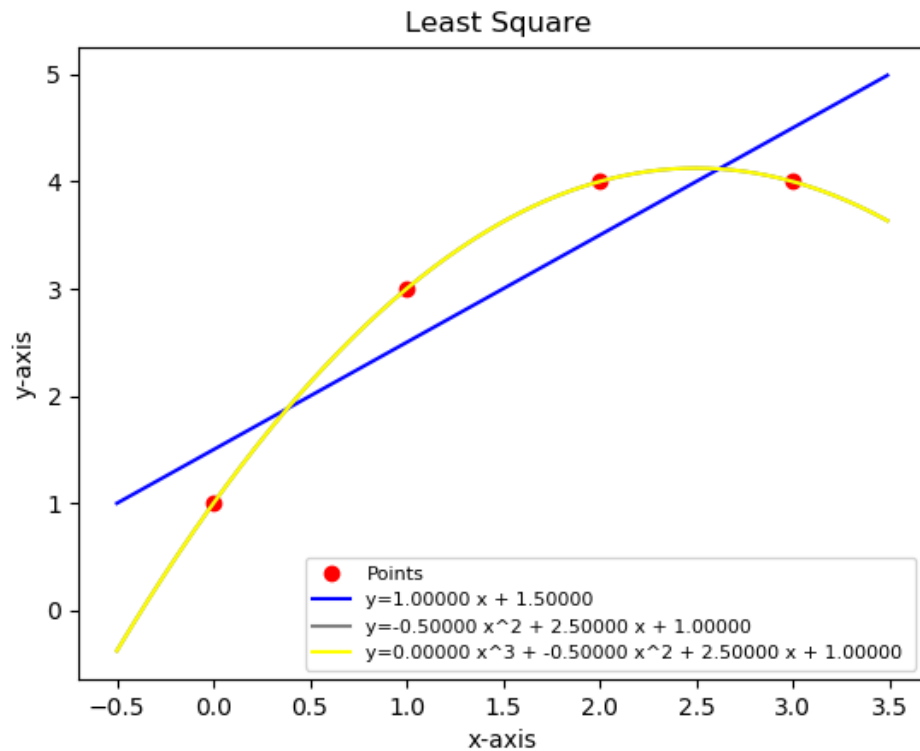
$$= 1$$

****

The **second** *order* equation: $\qquad y = -0.50x^2 + 2.5x + 1.0$

Error = 0.00000

****

The **third** *order* equation: $\qquad y = 0.0x^3 - 0.5x^2 + 2.5x + 1$

Error = 0.00000

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(2,\ 3),\ (3,\ 2),\ (5,\ 1),\ (6,\ 0)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

where $A = \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix}$ $\vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$ $\vec{y} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$

The normal equation: $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 2 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 74 & 16 \\ 16 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 17 \\ 6 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 74 & 16 \\ 16 & 4 \end{vmatrix} = 40 \quad \Delta_m = \begin{vmatrix} 17 & 16 \\ 6 & 4 \end{vmatrix} = -28 \quad \Delta_b = \begin{vmatrix} 74 & 17 \\ 16 & 6 \end{vmatrix} = 172$$

$$\underline{m = -\frac{28}{40} = -\frac{7}{10}}$$

$$\underline{b = \frac{172}{40} = \frac{43}{10}}$$

Thus, $\underline{y = -\frac{7}{10}x + \frac{43}{10}}$

$$A\vec{x} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} -\frac{7}{10} \\ \frac{43}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{29}{10} \\ \dfrac{22}{10} \\ \dfrac{8}{10} \\ \dfrac{1}{10} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \dfrac{29}{10} \\ \dfrac{22}{10} \\ \dfrac{8}{10} \\ \dfrac{1}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1}{10} \\ -\dfrac{1}{5} \\ \dfrac{1}{5} \\ -\dfrac{1}{10} \end{pmatrix}$$

E*rror*:

$$\left\| \vec{y} - A\vec{x} \right\| = \sqrt{2\left(\dfrac{1}{100} + \dfrac{1}{25} \right)}$$

$$= \dfrac{\sqrt{10}}{10}$$

$$= 0.31623$$

****
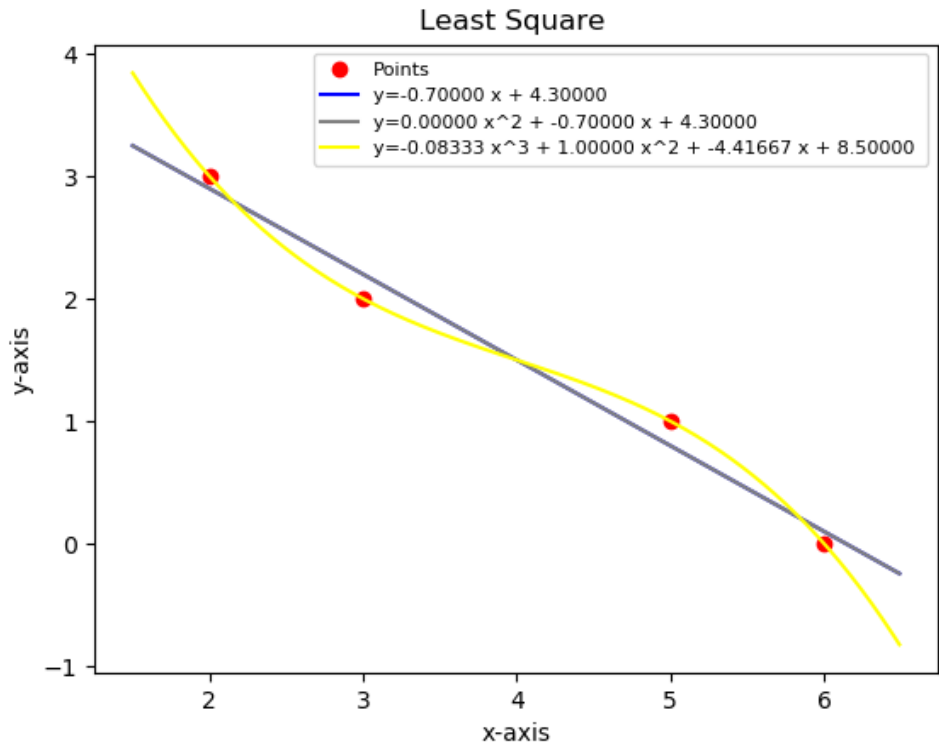
The *second order* equation: $y = 0.0x^2 - 0.7x + 4.3$

Error = 0.31623

****

The *third order* equation: $y = -0.08333x^3 + x^2 - 4.41667x + 8.5$

Error = 0.00000

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-1,\ 0),\ (0,\ 1),\ (1,\ 2),\ (2,\ 4)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$$

where $A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$

The normal equation: $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 10 \\ 7 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 6 & 2 \\ 2 & 4 \end{vmatrix} = 20 \quad \Delta_m = \begin{vmatrix} 10 & 2 \\ 7 & 4 \end{vmatrix} = 26 \quad \Delta_b = \begin{vmatrix} 6 & 10 \\ 2 & 7 \end{vmatrix} = 22$$

$$m = \frac{26}{20} = \frac{13}{10}$$

$$b = \frac{22}{20} = \frac{11}{10}$$

Thus, $y = \frac{13}{10}x + \frac{11}{10}$

$$A\vec{x} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{13}{10} \\ \frac{11}{10} \end{pmatrix}$$

$$= \begin{pmatrix} -\dfrac{1}{5} \\[8pt] \dfrac{11}{10} \\[8pt] \dfrac{12}{5} \\[8pt] \dfrac{37}{10} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} -\dfrac{1}{5} \\[8pt] \dfrac{11}{10} \\[8pt] \dfrac{12}{5} \\[8pt] \dfrac{37}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1}{5} \\[8pt] -\dfrac{1}{10} \\[8pt] -\dfrac{2}{5} \\[8pt] \dfrac{3}{10} \end{pmatrix}$$

E*rror*:

$$\left\| \vec{y} - A\vec{x} \right\| = \sqrt{\dfrac{1}{25} + \dfrac{1}{100} + \dfrac{4}{25} + \dfrac{9}{100}}$$

$$= \sqrt{\dfrac{4 + 1 + 16 + 9}{100}}$$

$$= \dfrac{\sqrt{30}}{10}$$

$$= 0.54772$$

****

The **second order** equation: $\qquad y = 0.25x^2 + 1.05x + 0.85$

Error $= 0.22361$

****

The **third order** equation: $\qquad y = 0.16667x^3 + 0.82222x + 1$

Error $= 0.00000$
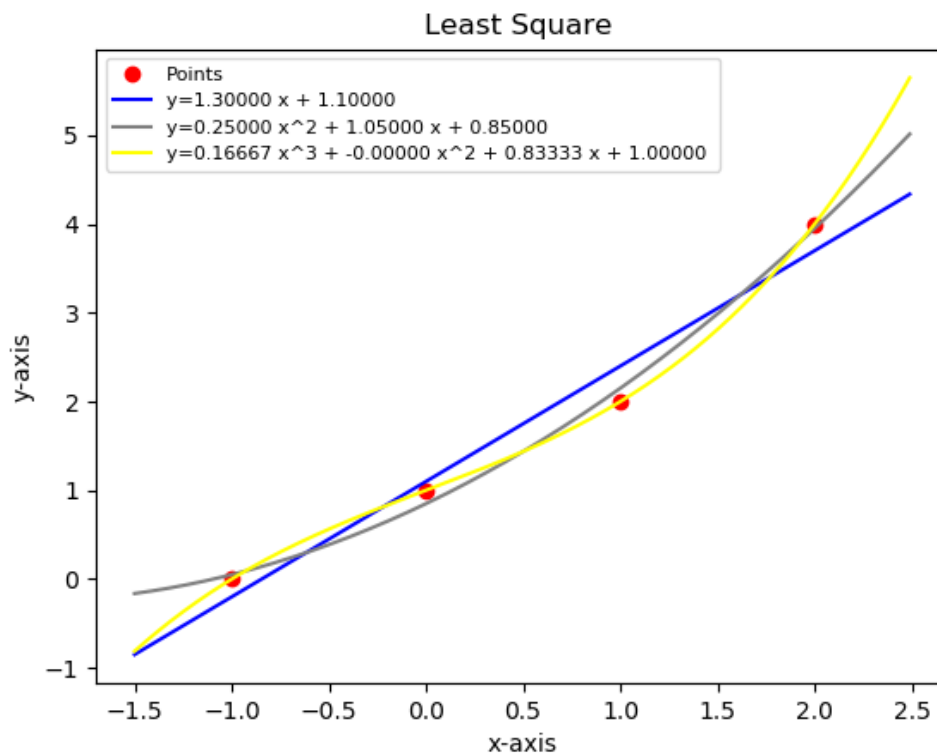
Least Square

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(1,\ 0),\ (2,\ 1),\ (4,\ 2),\ (5,\ 3)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

where $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix}$ $\vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$ $\vec{y} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$

The normal equation: $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 1 & 2 & 4 & 5 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 46 & 12 \\ 12 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 25 \\ 6 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 46 & 12 \\ 12 & 4 \end{vmatrix} = 40 \quad \Delta_m = \begin{vmatrix} 25 & 12 \\ 6 & 4 \end{vmatrix} = 28 \quad \Delta_b = \begin{vmatrix} 46 & 25 \\ 12 & 6 \end{vmatrix} = -24$$

$$m = \frac{28}{40} = \frac{7}{10}$$

$$b = -\frac{24}{40} = -\frac{3}{5}$$

Thus, $y = \dfrac{7}{10}x - \dfrac{3}{5}$

$$A\vec{x} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{7}{10} \\ -\dfrac{3}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1}{10} \\ \dfrac{4}{5} \\ \dfrac{11}{5} \\ \dfrac{29}{10} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} \dfrac{1}{10} \\ \dfrac{4}{5} \\ \dfrac{11}{5} \\ \dfrac{29}{10} \end{pmatrix}$$

$$= \begin{pmatrix} -\dfrac{1}{10} \\ \dfrac{1}{5} \\ -\dfrac{1}{5} \\ \dfrac{1}{10} \end{pmatrix}$$

E*rror*:

$$\left\| \vec{y} - A\vec{x} \right\| = \sqrt{2 \left(\frac{1}{100} + \frac{1}{25} \right)}$$

$$= \frac{\sqrt{10}}{10}$$

$$= 0.31623$$

****

The **second order** equation:

$$y = 0.0x^2 + 0.7x - .6$$

Error = 0.31623

****

The **third order** equation:

$$y = 0.08333x^3 - 0.75x^2 + 2.66667x - 2$$

Error = 0.00000
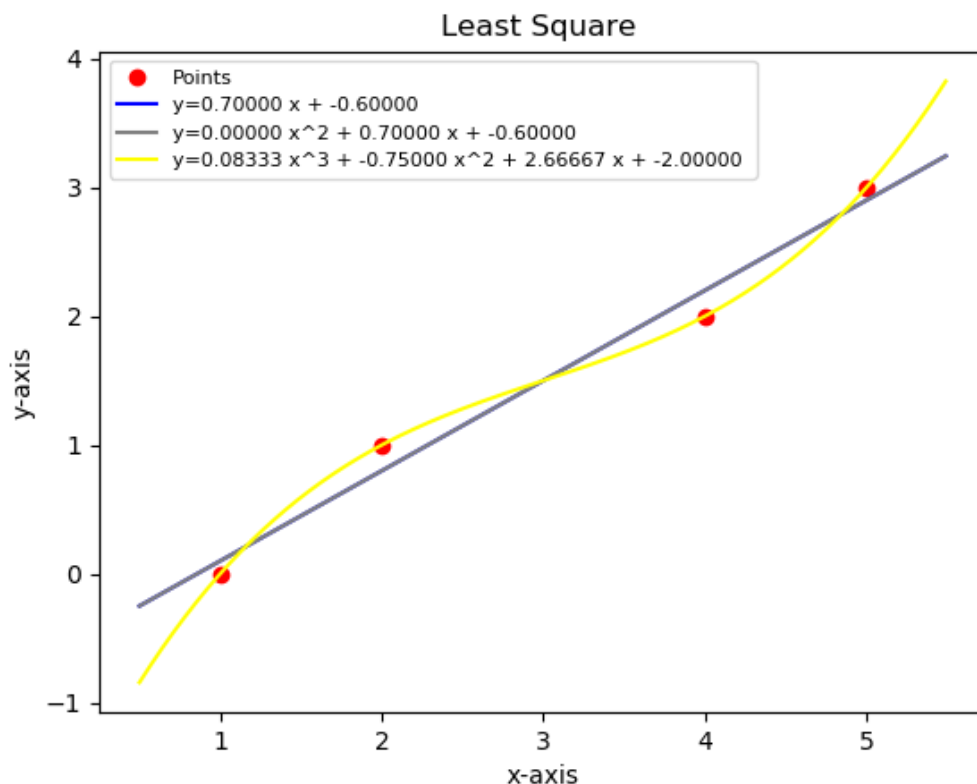
Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(1, 5), (2, 4), (3, 1), (4, 1), (5, -1)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix} \qquad \text{where} \quad A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} m \\ b \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

The normal equation: $A^T A \vec{x} = A^T \vec{y}$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 55 & 15 \\ 15 & 5 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \end{bmatrix}$$

$m = \dfrac{\begin{vmatrix} 15 & 15 \\ 10 & 5 \end{vmatrix}}{\begin{vmatrix} 55 & 15 \\ 15 & 5 \end{vmatrix}}$

$= \dfrac{-75}{50}$

$\underline{= -\dfrac{3}{2}}$

$b = \dfrac{\begin{vmatrix} 55 & 15 \\ 15 & 10 \end{vmatrix}}{50}$

$= \dfrac{325}{50}$

$\underline{= \dfrac{13}{2}}$
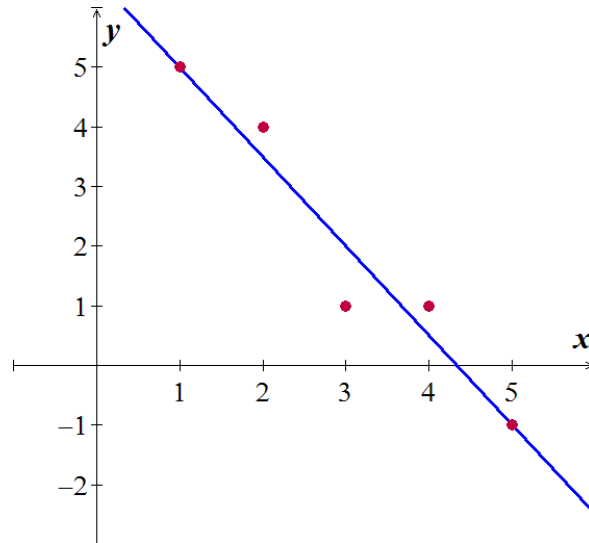
Thus, $\underline{y = -\dfrac{3}{2}x + \dfrac{13}{2}}$ *or* $\underline{y = -1.5x + 6.5}$

$$A\vec{x} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} -\dfrac{3}{2} \\ \dfrac{13}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ \dfrac{7}{2} \\ 2 \\ \dfrac{1}{2} \\ -1 \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ \dfrac{7}{2} \\ 2 \\ \dfrac{1}{2} \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \dfrac{1}{2} \\ -1 \\ \dfrac{1}{2} \\ 0 \end{pmatrix}$$

Error:

$$\|\vec{y} - A\vec{x}\| = \sqrt{\dfrac{1}{4} + 1 + \dfrac{1}{4}}$$

$$= \dfrac{\sqrt{6}}{2}$$

$$\approx 1.224745$$

****

The **second order** equation: $\qquad y = 0.07143x^2 - 1.92857x + 7$

Error = 1.19523

****

The **third order** equation: $\qquad y = 0.0x^3 + 0.07143x^2 - 1.92857x + 7$
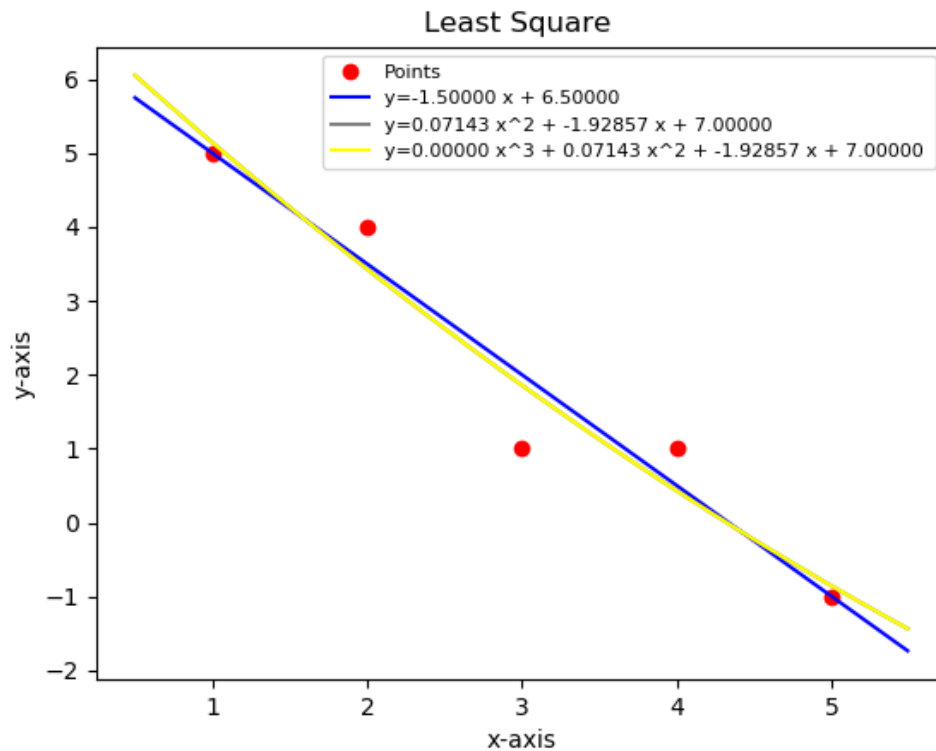
Error = 1.19523

Least Square

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-2,\ 0),\ (-1,\ 0),\ (0,\ 1),\ (1,\ 3),\ (2,\ 5)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

where $A = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$

The normal equation: $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 13 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{50} \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} 13 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{13}{10} \\ \frac{9}{5} \end{pmatrix}$$

We have: $m = 1.3$ *and* $b = 1.8$

Thus, $y = \dfrac{13}{10}x + \dfrac{9}{5}$

$$A\vec{x} = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{13}{10} \\ \frac{9}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{5} \\ \frac{1}{2} \\ \frac{9}{5} \\ \frac{31}{10} \\ \frac{22}{5} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix} - \begin{pmatrix} -\frac{4}{5} \\ \frac{1}{2} \\ \frac{9}{5} \\ \frac{31}{10} \\ \frac{22}{5} \end{pmatrix}$$
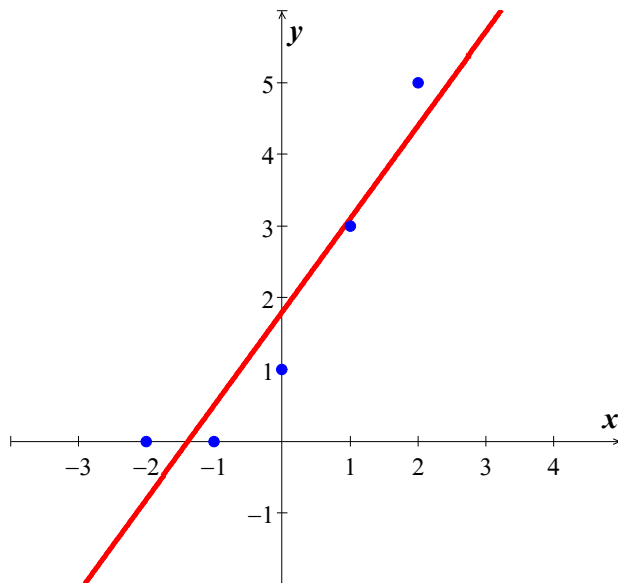
$$= \begin{pmatrix} \frac{4}{5} \\[6pt] -\frac{1}{2} \\[6pt] -\frac{4}{5} \\[6pt] -\frac{1}{10} \\[6pt] \frac{3}{5} \end{pmatrix}$$

Error: $\quad \left\| \vec{y} - A\vec{x} \right\| = \sqrt{\dfrac{16}{25} + \dfrac{1}{4} + \dfrac{16}{25} + \dfrac{1}{100} + \dfrac{9}{25}}$

$$= \sqrt{\dfrac{41}{25} + \dfrac{26}{100}}$$

$$= \dfrac{\sqrt{190}}{10}$$
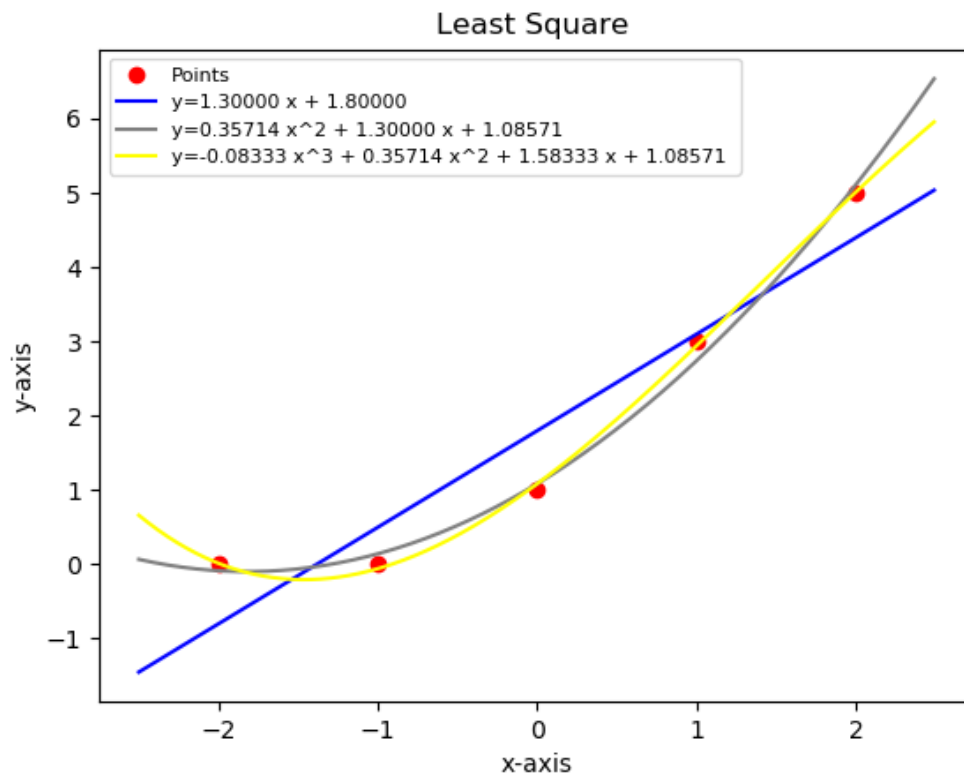
$$\approx 1.37840$$

****

The **second order** equation: $\qquad y = 0.35714x^2 + 1.30x + 1.08571$

Error = 0.33806

****

The **third order** equation: $\qquad y = -0.08333x^3 + 0.35714x^2 + 1.58333x + 1.08571$

Error = 0.11952

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-5,\ 10),\quad (-1,\ 8),\quad (3,\ 6),\quad (7,\ 4),\quad (5,\ 5)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} -5 & 1 \\ -1 & 1 \\ 3 & 1 \\ 7 & 1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 5 \end{pmatrix} \qquad \text{where} \quad A = \begin{pmatrix} -5 & 1 \\ -1 & 1 \\ 3 & 1 \\ 7 & 1 \\ 5 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 5 \end{pmatrix}$$

The normal equation: $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} -5 & -1 & 3 & 7 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -5 & 1 \\ -1 & 1 \\ 3 & 1 \\ 7 & 1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -5 & -1 & 3 & 7 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 109 & 9 \\ 9 & 5 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 13 \\ 33 \end{pmatrix}$$

$$m = \frac{232}{464}$$

$$\boxed{= -\frac{1}{2}}$$

$$b = \frac{3{,}480}{464}$$

$$\boxed{= \frac{15}{2}}$$

Thus, $\boxed{y = -\frac{1}{2}x + \frac{15}{2}}$

$$A\vec{x} = \begin{pmatrix} -5 & 1 \\ -1 & 1 \\ 3 & 1 \\ 7 & 1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} -\dfrac{1}{2} \\ \dfrac{15}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 5 \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 10 \\ 8 \\ 6 \\ 4 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Error: $\left\| \vec{y} - A\vec{x} \right\| = 0$
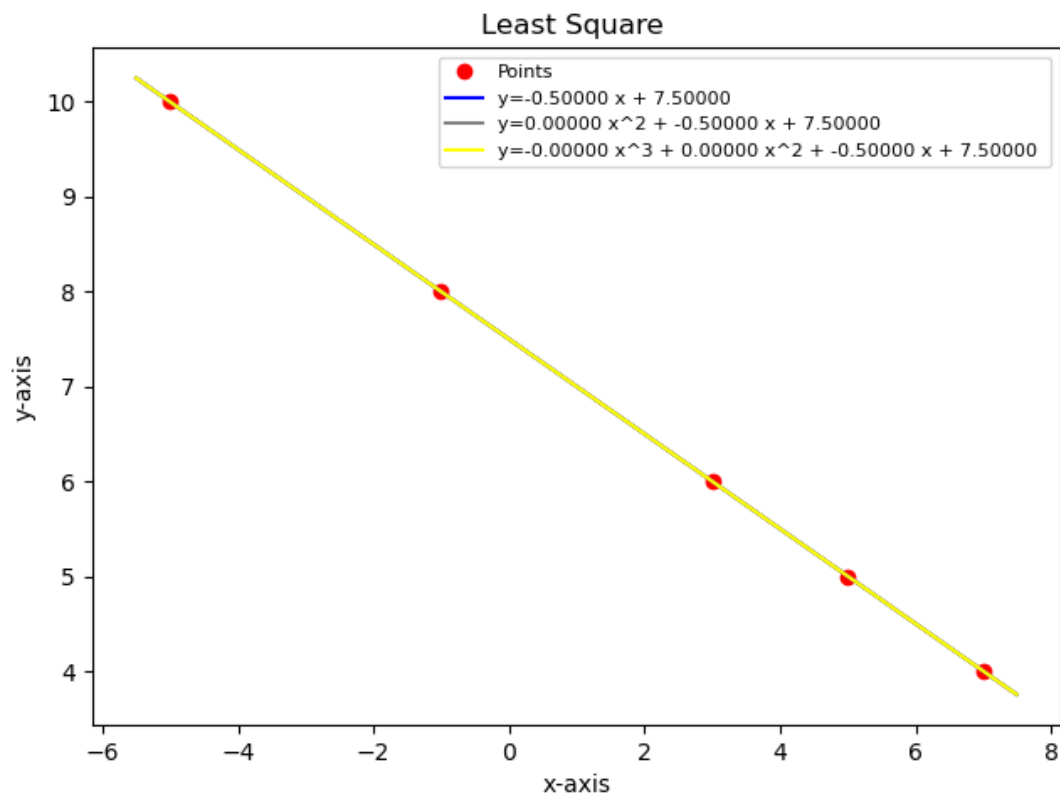
Least Square

Points
y=-0.50000 x + 7.50000
y=0.00000 x^2 + -0.50000 x + 7.50000
y=-0.00000 x^3 + 0.00000 x^2 + -0.50000 x + 7.50000

Exercise

Find the orthogonal projection of the vector $\vec{u}$ on the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{u} = (-3, \ -3, \ 8, \ 9); \quad \vec{v}_1 = (3, \ 1, \ 0, \ 1), \quad \vec{v}_2 = (1, \ 2, \ 1, \ 1), \quad \vec{v}_3 = (-1, \ 0, \ 2, \ -1)$$

Solution

Let $A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{pmatrix}$$

$$A^T \vec{u} = \begin{pmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ -3 \\ 8 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 8 \\ 10 \end{pmatrix}$$

The normal solution is $A^T A \vec{x} = A^T \vec{u}$

$$\begin{pmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \\ 10 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{vmatrix} = 134 \qquad \Delta_1 = \begin{vmatrix} -3 & 6 & -4 \\ 8 & 7 & 0 \\ 10 & 0 & 6 \end{vmatrix} = -134 \qquad \Delta_2 = \begin{vmatrix} 11 & -3 & -4 \\ 6 & 8 & 0 \\ -4 & 10 & 6 \end{vmatrix} = 268$$

$$x_1 = \frac{-134}{134} = -1 \qquad x_2 = \frac{268}{134} = 2 \qquad x_3 = \frac{134}{134} = 1$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

So $proj_W \vec{u} = A\vec{x}$

$$= \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 3 \\ 4 \\ 0 \end{pmatrix}$$

$$\underline{proj_W \vec{u} = (-2,\ 3,\ 4,\ 0)}$$

Exercise

Find the orthogonal projection of the vector $\vec{u}$ on the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{u} = (6,\ 3,\ 9,\ 6);\quad \vec{v}_1 = (2,\ 1,\ 1,\ 1),\quad \vec{v}_2 = (1,\ 0,\ 1,\ 1),\quad \vec{v}_3 = (-2,\ -1,\ 0,\ -1)$$

Solution

Let $A = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{pmatrix}$$

$$A^T \vec{u} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 9 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 30 \\ 21 \\ -21 \end{pmatrix}$$

The normal solution is $A^T A \vec{x} = A^T \vec{u}$

$$\begin{pmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 30 \\ 21 \\ -21 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{vmatrix} = 3 \qquad \Delta_1 = \begin{vmatrix} 30 & 4 & -6 \\ 21 & 3 & -3 \\ -21 & -3 & 6 \end{vmatrix} = 18 \qquad \Delta_2 = \begin{vmatrix} 7 & 30 & -6 \\ 4 & 21 & -3 \\ -6 & -21 & 6 \end{vmatrix} = 9$$

$$\underline{x_1 = \frac{18}{3} = 6} \qquad \underline{x_2 = \frac{9}{3} = 3} \qquad \underline{x_3 = 4}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$$

So $proj_W \vec{u} = A\vec{x}$

$$= \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 2 \\ 9 \\ 5 \end{pmatrix}$$

$$\underline{proj_W \vec{v} = (7, \ 2, \ 9, \ 5)}$$

Exercise

Find the orthogonal projection of the vector $\vec{u}$ on the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{u} = (-2, \ 0, \ 2, \ 4); \quad \vec{v}_1 = (1, \ 1, \ 3, \ 0), \quad \vec{v}_2 = (-2, \ -1, \ -2, \ 1), \quad \vec{v}_3 = (-3, \ -1, \ 1, \ 3)$$

Solution

Let $A = \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

$A^T A = \begin{pmatrix} 1 & 1 & 3 & 0 \\ -2 & -1 & -2 & 1 \\ -3 & -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

$= \begin{pmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{pmatrix}$

$A^T \vec{u} = \begin{pmatrix} 1 & 1 & 3 & 0 \\ -2 & -1 & -2 & 1 \\ -3 & -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 2 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ 4 \\ 20 \end{pmatrix}$

The normal solution is $A^T A \vec{x} = A^T \vec{u}$

$\begin{pmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 20 \end{pmatrix}$

$\Delta = \begin{vmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{vmatrix} = 10 \qquad \Delta_1 = \begin{vmatrix} 4 & -9 & -1 \\ 4 & 10 & 8 \\ 20 & 8 & 20 \end{vmatrix} = -8 \qquad \Delta_2 = \begin{vmatrix} 11 & 4 & -1 \\ -9 & 4 & 8 \\ -1 & 20 & 20 \end{vmatrix} = -16$

$x_1 = \dfrac{-8}{10} = -\dfrac{4}{5} \qquad x_2 = \dfrac{-16}{10} = -\dfrac{8}{5} \qquad x_3 = \dfrac{8}{5}$

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\dfrac{4}{5} \\ -\dfrac{8}{5} \\ \dfrac{8}{5} \end{pmatrix}$

$proj_W \vec{u} = A\vec{x}$

$$= \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} -\frac{4}{5} \\ -\frac{8}{5} \\ \frac{8}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{12}{5} \\ -\frac{4}{5} \\ \frac{12}{5} \\ \frac{16}{5} \end{pmatrix}$$

$$\underline{proj_W \vec{u} = \left(-\frac{12}{5}, \ -\frac{4}{5}, \ \frac{12}{5}, \ \frac{16}{5} \right)}$$

Exercise

For the given matrix A and $\vec{y}$

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 2 \\ 0 \\ 11 \end{pmatrix}$$

a) Find the least-squares solution of the linear system $A\vec{x} = \vec{y}$
b) Find the orthogonal projection of $\vec{y}$ on the column space of A
c) Find the **error vector** and the **error**

Solution

a) Let $\vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$

$A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 11 \end{pmatrix}$$

$$\begin{pmatrix} 17 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 19 \\ 11 \end{pmatrix}$$

$m = \dfrac{84}{84}$

$\underline{\underline{= 1}}$

$b = \dfrac{168}{84}$

$= 2$

Thus $y = x + 2$

b) The orthogonal projection of $\vec{y}$ on the column space of A

$$A\vec{x} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix}$$

c) $\vec{y} - A\vec{x} = \begin{pmatrix} 2 \\ 0 \\ 11 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix}$

$$= \begin{pmatrix} -2 \\ -4 \\ 8 \end{pmatrix}$$
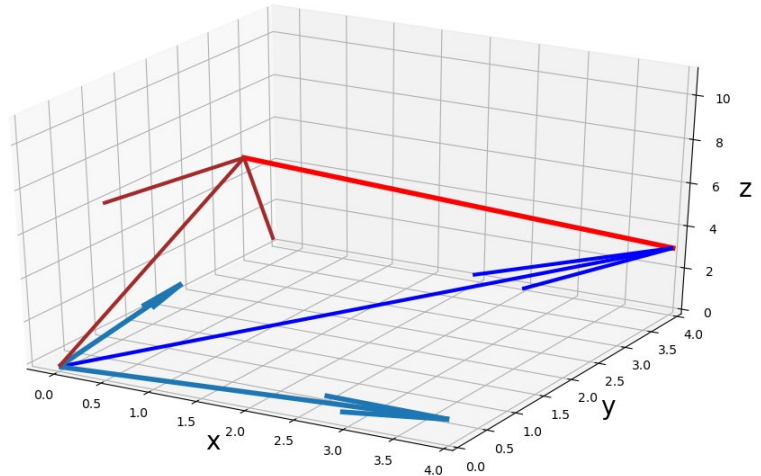
The **error**: $\|\vec{y} - A\vec{x}\| = \sqrt{4 + 16 + 64}$

$$= 2\sqrt{21}$$

$$\approx 9.16515$$

Exercise

For the given matrix A and $\vec{y}$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -3 \\ -1 & 3 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

a) Find the least-squares solution of the linear system $A\vec{x} = \vec{y}$

b) Find the orthogonal projection of $\vec{y}$ on the column space of A

c) Find the **error vector** and the **error**

Solution

a) Let $\vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$

$A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -3 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -7 \\ -7 & 22 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \end{pmatrix}$$

$$\underline{m = \frac{165}{83}}$$

$$\underline{b = \frac{94}{83}}$$

Thus $\underline{y = \dfrac{165}{83}x + \dfrac{94}{83}}$

b) The orthogonal projection of $\vec{y}$ on the column space of A

$$A\vec{x} = \begin{pmatrix} 1 & 2 \\ 2 & -3 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} \dfrac{165}{83} \\ \dfrac{94}{83} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{353}{83} \\ \dfrac{48}{83} \\ \dfrac{117}{83} \end{pmatrix}$$

c) $\vec{y} - A\vec{x} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} \dfrac{353}{83} \\ \dfrac{48}{83} \\ \dfrac{117}{83} \end{pmatrix}$

$$= \begin{pmatrix} -\dfrac{21}{83} \\ -\dfrac{35}{83} \\ \dfrac{49}{83} \end{pmatrix}$$
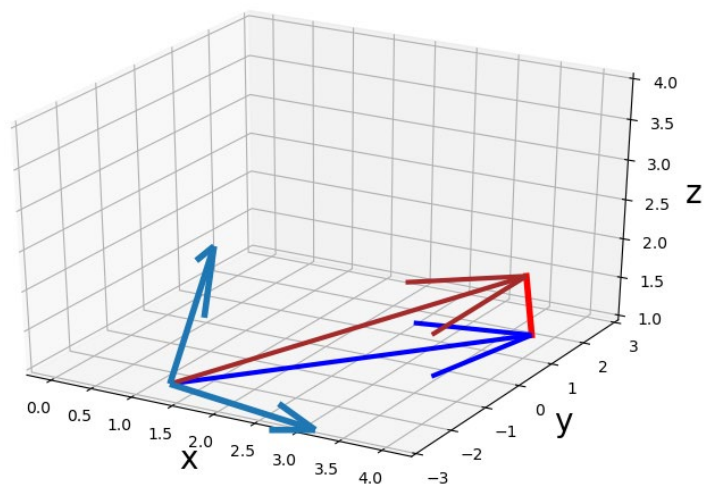
The **error**: $\left\| \vec{y} - A\vec{x} \right\| = \dfrac{1}{83}\sqrt{441 + 1,225 + 2,401}$

$$= \frac{\sqrt{4,067}}{83}$$

$$= \frac{7\sqrt{83}}{83}$$

$$\underline{\approx 0.76835}$$

Exercise

For the given matrix A and $\vec{y}$

$$A = \begin{pmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} -5 \\ 8 \\ 1 \end{pmatrix}$$

a) Find the least-squares solution of the linear system $A\vec{x} = \vec{y}$

b) Find the orthogonal projection of $\vec{y}$ on the column space of A

c) Find the **error vector** and the **error**

Solution

a) Let $\vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$

$A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} -5 \\ 8 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 12 & 8 \\ 8 & 10 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -24 \\ -2 \end{pmatrix}$$

$m = -\dfrac{224}{56}$

$\underline{= -4}$

$b = \dfrac{168}{56}$

$\underline{= 3}$

Thus $\underline{y = -4x + 3}$

b) The orthogonal projection of $\vec{y}$ on the column space of A

$$A\vec{x} = \begin{pmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 8 \\ 1 \end{pmatrix}$$

c) $\vec{y} - A\vec{x} = \begin{pmatrix} -5 \\ 8 \\ 1 \end{pmatrix} - \begin{pmatrix} -5 \\ 8 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The **error**: $\left\| \vec{y} - A\vec{x} \right\| = 0$
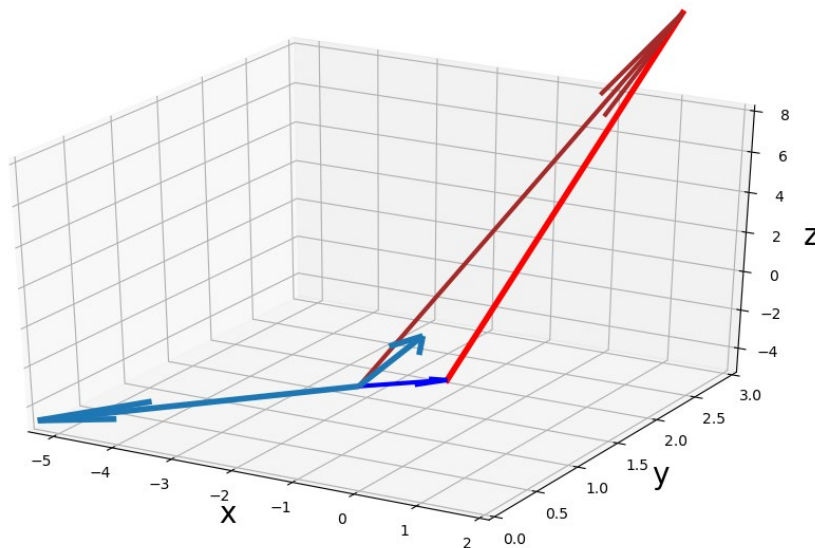
Exercise

For the given matrix A and $\vec{y}$

$$A = \begin{pmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{pmatrix} \qquad \vec{y} = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 6 \end{pmatrix}$$

a) Find the least-squares solution of the linear system $A\vec{x} = \vec{y}$

b) Find the orthogonal projection of $\vec{y}$ on the column space of A

c) Find the **error vector** and the **error**

Solution

a) Let $\vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$

$A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -6 & -2 & 1 & 7 \end{pmatrix} \begin{pmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -6 & -2 & 1 & 7 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 0 \\ 0 & 90 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 8 \\ 45 \end{pmatrix}$$

$$m = \frac{720}{360}$$

$$\underline{= 2}$$

$$b = \frac{180}{360}$$

$$\underline{= \frac{1}{2}}$$

Thus $\underline{y = 2x + \frac{1}{2}}$

b) The orthogonal projection of $\vec{y}$ on the column space of A

$$A\vec{x} = \begin{pmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{pmatrix}\begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 1 \\ 3 \\ \frac{11}{2} \end{pmatrix}$$

c) $\vec{y} - A\vec{x} = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 6 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 3 \\ \frac{11}{2} \end{pmatrix}$

$$= \begin{pmatrix} 0 \\ 1 \\ -2 \\ \frac{1}{2} \end{pmatrix}$$
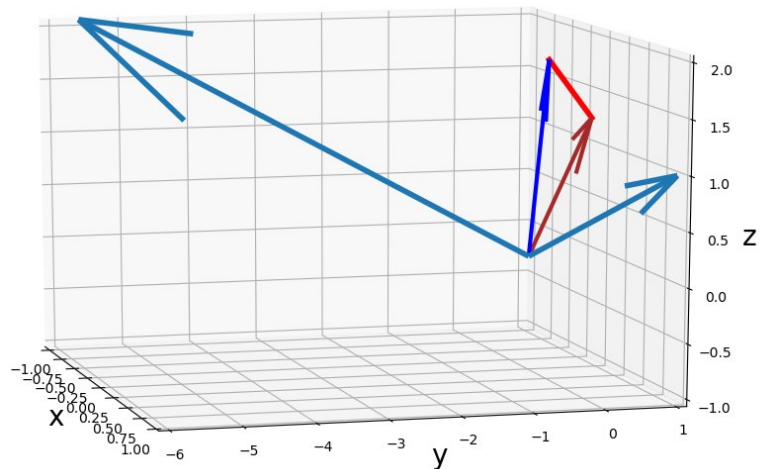
The **error**: $\quad \|\vec{y} - A\vec{x}\| = \sqrt{1 + 4 + \frac{1}{4}}$

$$\underline{= \frac{\sqrt{21}}{2}}$$

$$\underline{\approx 2.2913}$$

Exercise

For the given matrix A and $\vec{y}$

$$A = \begin{pmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix}$$

a) Find the least-squares solution of the linear system $A\vec{x} = \vec{y}$

b) Find the orthogonal projection of $\vec{y}$ on the column space of A

c) Find the **error vector** and the **error**

Solution

a) Let $\vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$

$A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 6 \\ 6 & 42 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$$

$m = \dfrac{288}{216}$

$= \dfrac{4}{3}$

$b = -\dfrac{72}{216}$

$= -\dfrac{1}{3}$

Thus $y = \dfrac{4}{3}x - \dfrac{1}{3}$

b) The orthogonal projection of $\vec{y}$ on the column space of A

$$A\vec{x} = \begin{pmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} \dfrac{4}{3} \\ -\dfrac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -2 \\ -1 \\ 1 \end{pmatrix}$$

c) $\quad \vec{y} - A\vec{x} = \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ -1 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 \\ 3 \\ -3 \\ 1 \end{pmatrix}$$

The **error**: $\quad \left\| \vec{y} - A\vec{x} \right\| = \sqrt{1 + 9 + 9 + 1}$

$$= 2\sqrt{5} \qquad\qquad \approx 4.47214$$

Exercise

For the given matrix A and $\vec{y}$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \qquad \vec{y} = \begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix}$$

a) Find the least-squares solution of the linear system $A\vec{x} = \vec{y}$

b) Find the orthogonal projection of $\vec{y}$ on the column space of A

c) Find the **error vector** and the **error.**

a) Let $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 14 \\ 4 \\ 10 \end{pmatrix}$$

$$\begin{cases} 4a + 2b + 2c = 14 \\ 2a + 2b = 4 \\ 2a + 2c = 10 \end{cases}$$

$$\begin{cases} 2a + b + c = 7 \\ a + b = 2 \\ a + c = 5 \end{cases}$$

$$\Delta = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 0 \qquad \Delta_c = \begin{vmatrix} 2 & 1 & 7 \\ 1 & 1 & 2 \\ 1 & 0 & 5 \end{vmatrix} = 0$$

$$\begin{cases} b = 2 - a \\ c = 5 - a \end{cases}$$

Assume $a = 1 \;\rightarrow\; b = 1 \quad c = 4$

Thus $x + y + 4z = 0$

b) The orthogonal projection of $\vec{y}$ on the column space of A

$$A\vec{x} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 2 \\ 5 \\ 5 \end{pmatrix}$$

c) $\vec{y} - A\vec{x} = \begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 5 \\ 5 \end{pmatrix}$

$$= \begin{pmatrix} -1 \\ 1 \\ 3 \\ -3 \end{pmatrix}$$

The **error**: $\quad \|\vec{y} - A\vec{x}\| = \sqrt{1 + 1 + 9 + 9}$

$$= 2\sqrt{5}$$

$$\approx 4.47214$$

Exercise

For the given matrix A and $\vec{y}$

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix} \qquad \vec{y} = \begin{pmatrix} 3 \\ 5 \\ 7 \\ -3 \end{pmatrix}$$

a) Find the least-squares solution of the linear system $A\vec{x} = \vec{y}$

b) Find the orthogonal projection of $\vec{y}$ on the column space of A

c) Find the **error vector** and the **error.**

Solution

a) Let $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$A^T A \vec{x} = A^T \vec{y}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 5 & 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 5 & 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 7 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 8 & 10 \\ 8 & 20 & 26 \\ 10 & 26 & 38 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \\ 20 \end{pmatrix}$$

$$\begin{cases} 2a + 4b + 5c = 6 \\ 4a + 10b + 13c = 6 \\ 5a + 13b + 19c = 10 \end{cases}$$

$$\Delta = \begin{vmatrix} 2 & 4 & 5 \\ 4 & 10 & 13 \\ 5 & 13 & 19 \end{vmatrix} = 8 \qquad \Delta_a = \begin{vmatrix} 6 & 4 & 5 \\ 6 & 10 & 13 \\ 10 & 13 & 19 \end{vmatrix} = 80$$

$$\Delta_b = \begin{vmatrix} 2 & 6 & 5 \\ 4 & 6 & 13 \\ 5 & 10 & 19 \end{vmatrix} = -48 \qquad \Delta_c = \begin{vmatrix} 2 & 4 & 6 \\ 4 & 10 & 6 \\ 5 & 13 & 10 \end{vmatrix} = 16$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 10 \\ -6 \\ 2 \end{pmatrix}$$

Thus $\underline{10x - 6y + 2z = 0}$

b) The orthogonal projection of $\vec{y}$ on the column space of A

$$A\vec{x} = \begin{pmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix} \begin{pmatrix} 10 \\ -6 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 4 \\ 8 \\ -2 \end{pmatrix}$$

c) $\vec{y} - A\vec{x} = \begin{pmatrix} 3 \\ 5 \\ 7 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 8 \\ -2 \end{pmatrix}$

$$= \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

The **error**: $\underline{\left\| \vec{y} - A\vec{x} \right\| = 2}$

Exercise

For the given matrix A and $\vec{y}$

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \qquad \vec{y} = \begin{pmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{pmatrix}$$

a) Find the least-squares solution of the linear system $A\vec{x} = \vec{y}$

b) Find the orthogonal projection of $\vec{y}$ on the column space of A

c) Find the **error vector** and the **error.**

Solution

d) Let $\vec{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

$A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 4 & 2 & 2 \\ 4 & 4 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 2 \\ 6 \end{pmatrix}$$

$$\begin{cases} 3a + 2b + c + d = 2 \\ 2a + 2b + d = 1 \\ a + c = 1 \\ a + b + d = 3 \end{cases}$$

$$\Delta = \begin{vmatrix} 3 & 2 & 1 & 1 \\ 2 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix} = 0 \qquad \Delta_d = \begin{vmatrix} 3 & 2 & 1 & 2 \\ 2 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 3 \end{vmatrix} = 0$$

$(3) \rightarrow \underline{c = 1 - a}$

$\begin{cases} 2a + 2b + d = 1 \\ a + b + d = 3 \end{cases}$

$\overline{a + b = -2}$

$\underline{b = -2 - a}$

$\underline{d = 5}$

Assume $\underline{a = 0 \rightarrow b = -2 \quad c = 1}$

$-2y + z + 5 = 0$

Therefore, $\underline{-2y + z = -5}$

e) The orthogonal projection of $\vec{y}$ on the column space of A

$$A\vec{x} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \\ 1 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ -2 \\ 1 \\ 1 \\ 3 \\ 3 \end{pmatrix}$$

f) $\vec{y} - A\vec{x} = \begin{pmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \\ 1 \\ 1 \\ 3 \\ 3 \end{pmatrix}$

$$= \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 2 \\ -2 \end{pmatrix}$$

The **error**: $\|\vec{y} - A\vec{x}\| = \sqrt{1+1+1+1+4+4}$

$$= \sqrt{12}$$
$$= 2\sqrt{3}$$
$$\approx 3.4641$$

Exercise

Find the standard matrix for the orthogonal projection P of $\mathbb{R}^2$ on the line passes through the origin and makes an angle θ with the positive x-axis.
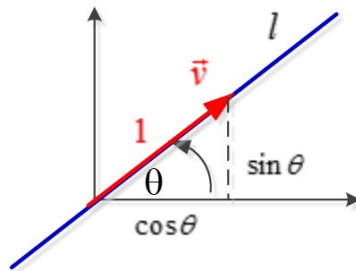
Solution

Since the line l in 2-dimensional, than we can take $\vec{v} = (\cos\theta,\ \sin\theta)$ as a basis for this subspace

$$A = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$[P] = A^T A$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

Exercise

Hooke's law in physics states that the length x of a uniform spring is a linear function of the force y applied to it. If we express the relationship as $y = mx + b$, then the coefficient m is called the spring constant.

Suppose a particular unstretched spring has a measured length of 6.1 *inches*.(i.e., $x = 6.1$ when $y = 0$). Forces of 2 pounds, 4 pounds, and 6 pounds are then applied to the spring, and the corresponding lengths are found to be 7.6 inches, 8.7 inches, and 10.4 inches. Find the spring constant.

Solution

$$M = \begin{pmatrix} 6.1 & 1 \\ 7.6 & 1 \\ 8.7 & 1 \\ 10.4 & 1 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \end{pmatrix}$$

The normal equation: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 6.1 & 7.6 & 8.7 & 10.4 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 6.1 & 1 \\ 7.6 & 1 \\ 8.7 & 1 \\ 10.4 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 6.1 & 7.6 & 8.7 & 10.4 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 278.82 & 32.8 \\ 32.8 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 112.4 \\ 12 \end{pmatrix}$$
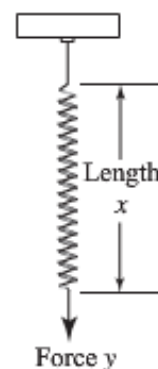
$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{39.44}\begin{pmatrix} 4 & -32.8 \\ -32.8 & 278.2 \end{pmatrix}\begin{pmatrix} 112.4 \\ 12 \end{pmatrix}$$

$$= \frac{1}{39.44}\begin{pmatrix} 56 \\ -348.32 \end{pmatrix}$$

$$= \begin{pmatrix} 1.4 \\ -8.8 \end{pmatrix}$$

$$y = 1.4x - 8.8$$

Length

x

Force y

Thus, the estimated value of the spring constant is ≈ 1.4 *pounds*

Exercise

Prove:

If A has a linearly independent column vectors, and if $\vec{b}$ is orthogonal to the column space of A, then the least squares solution of $A\vec{x} = \vec{b}$ is $\vec{x} = \vec{0}$.

Solution

If A has linearly independent column vectors, then $A^T A$ is invertible and the least squares solution of $A\vec{x} = \vec{b}$ is the solution of $A^T A\vec{x} = A^T \vec{b}$, but since $\vec{b}$ is orthogonal to the column space of A.

$A^T \vec{b} = 0$, so $\vec{x}$ is a solution of $A^T A\vec{x} = 0$.

Thus $\vec{x} = \vec{0}$ since $A^T A$ is invertible.

Exercise

Let A be an $m \times n$ matrix with linearly independent row vectors. Find a standard matrix for the orthogonal projection of $\mathbb{R}^n$ onto the row space of A.

Solution

A^T will have linearly independent column vectors, and the column space A^T is the row space of A. Thus, the standard matrix for the orthogonal projection of $\mathbb{R}^n$ onto the row space of A is

$$[P] = A^T \left[\left(A^T \right)^T A^T \right]^{-1} \left(A^T \right)^T$$

$$= A^T \left(A A^T \right)^{-1} A$$

Exercise

Let W be the line with parametric equations $x = 2t, \quad y = -t, \quad z = 4t$

a) Find a basis for W.

b) Find the standard matrix for the orthogonal projection on W.

c) Use the matrix in part (b) to find the orthogonal projection of a point $P_0\left(x_0, y_0, z_0\right)$ on W.

d) Find the distance between the point $P_0\left(2, 1, -3\right)$ and the line W.

Solution

a) $W = span\left\{(2, -1, 4)\right\}$

So that the vector $(2, -1, 4)$ forms a basis for W (linear independence)

b) Let $A = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$

$$[P] = A\left(A^T A\right)^{-1} A^T$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \left(\begin{bmatrix} 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} [21]^{-1} \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \frac{1}{21} \begin{bmatrix} 4 & -2 & 8 \\ -2 & 1 & -4 \\ 8 & -4 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{4}{21} & -\dfrac{2}{21} & \dfrac{8}{21} \\[2mm] -\dfrac{2}{21} & 1 & -\dfrac{4}{21} \\[2mm] \dfrac{8}{21} & -\dfrac{4}{21} & \dfrac{16}{21} \end{bmatrix}$$

c) $\begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} \frac{4}{21}x_0 - \frac{2}{21}y_0 + \frac{8}{21}z_0 \\ -\frac{2}{21}x_0 + y_0 - \frac{4}{21}z_0 \\ \frac{8}{21}x_0 - \frac{4}{21}y_0 + \frac{16}{21}z_0 \end{bmatrix}$

d) $\begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -\frac{6}{7} \\ \frac{3}{7} \\ -\frac{12}{7} \end{bmatrix}$

The distance between P_0 and W equals to the distance between P_0 and its projection on W.

The distance between $(2,\ 1,\ -3)$ and $\left(-\frac{6}{7},\ \frac{3}{7},\ -\frac{12}{7}\right)$ is

$$d = \sqrt{\left(2+\frac{6}{7}\right)^2 + \left(1-\frac{3}{7}\right)^2 + \left(-3+\frac{12}{7}\right)^2}$$

$$= \sqrt{\frac{400}{49} + \frac{16}{49} + \frac{81}{49}}$$

$$= \frac{\sqrt{497}}{7}$$

Exercise

In R^3, consider the line l given by the equations $\quad x = t,\quad y = t,\quad z = t$

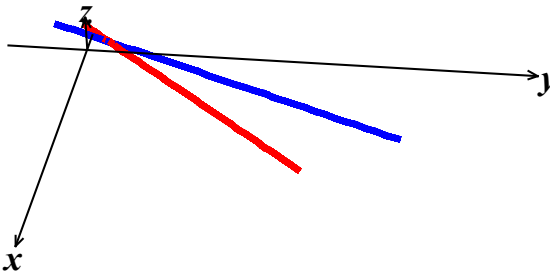And the line m given by the equations $\quad x = s,\quad y = 2s-1,\quad z = 1$

Let P be the point on l, and let Q be a point on m. Find the values of t and s that minimize the distance between the lines by minimizing the squared distance $\lVert P - Q \rVert^2$

Solution

When $t = 1 \implies$ Let $P = (1,\ 1,\ 1)$ is on line l

When $s = 1 \implies$ Let $Q = (1,\ 1,\ 1)$ is on line m

$\lVert P - Q \rVert = \sqrt{(1-1)^2 + (1-1)^2 + (1-1)^2} = 0 \geq 0$

Thus, these are the values $P = (1, 1, 1)$ and $Q = (1, 1, 1)$ are the values for $s = t = 1$ that minimize the distance between the lines.

Exercise

Determine whether the statement is true or false,

a) If A is an $m \times n$ matrix, then $A^T A$ is a square matrix.

b) If $A^T A$ is invertible, then A is invertible.

c) If A is invertible, then $A^T A$ is invertible.

d) If $A\vec{x} = \vec{b}$ is a consistent linear system, then $A^T A\vec{x} = A^T \vec{b}$ is also consistent.

e) If $A\vec{x} = \vec{b}$ is an inconsistent linear system, then $A^T A\vec{x} = A^T \vec{b}$ is also inconsistent.

f) Every linear system has a least squares solution.

g) Every linear system has a unique least squares solution.

h) If A is an $m \times n$ matrix with linearly independent columns and $\vec{b}$ is in R^m, then $A\vec{x} = \vec{b}$ has a unique least squares solution.

Solution

a) **True**; $A^T A$ is an $n \times n$ matrix.

b) **False**; only square matrix has inverses, but $A^T A$ can be invertible when A is not square matrix.

c) **True**; if A is invertible, so is A^T, so the product $A^T A$ is also invertible.

d) **True**

e) **False**; the system $A^T A\vec{x} = A^T \vec{b}$ may be consistent.

f) **True**

g) **False**; the least squares solution may involve a parameter.

h) **True**; if A has linearly independent column vectors; then $A^T A$ is invertible, so $A^T A\vec{x} = A^T \vec{b}$ has a unique solution.

Exercise

A certain experiment produces the data $\{(1,\ 1.8),\ (2,\ 2.7),\ (3,\ 3.4),\ (4,\ 3.8),\ (5,\ 3.9)\}$.

Find the function that it will fit these data in the form of $y = \beta_1 x + \beta_2 x^2$

Solution

Given: the equation $y = \beta_1 x + \beta_2 x^2$ that best fits the given points. Then

$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{pmatrix}$$

where $A = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{pmatrix}$ $\vec{x} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ $\vec{y} = \begin{pmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{pmatrix}$

The normal equation formula: $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \end{pmatrix} \begin{pmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{pmatrix}$$

$$\begin{pmatrix} 55 & 225 \\ 225 & 979 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 52.1 \\ 201.5 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 55 & 225 \\ 225 & 979 \end{vmatrix} = 3{,}220 \qquad \Delta_{\beta_1} = \begin{vmatrix} 52.1 & 225 \\ 201.5 & 979 \end{vmatrix} = 5{,}668.4 \qquad \Delta_{\beta_2} = \begin{vmatrix} 55 & 52.1 \\ 225 & 201.5 \end{vmatrix} = -640$$

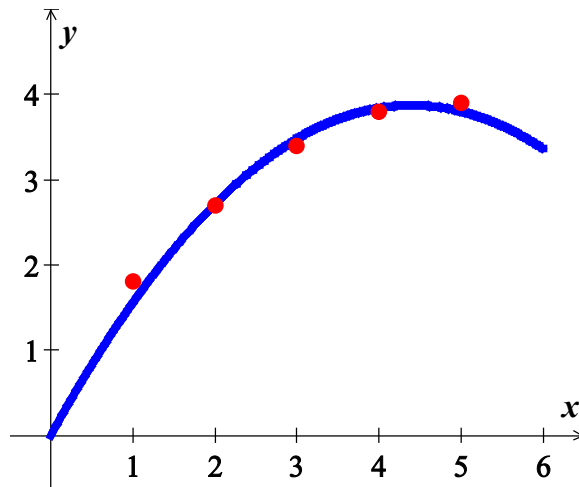$$\beta_1 = \frac{5{,}668.4}{3{,}220}$$

$$\approx 1.76$$

$$\beta_2 = -\frac{640}{3{,}220}$$

$$\approx -0.199$$

$$y = 1.76x - .2x^2$$

Exercise

According to Kepler's first law, a comet should have an ellipse, parabolic, or hyperbolic orbit (with gravitational attractions from the planets ignored). In suitable polar coordinates, the position $(r,\ \upsilon)$ of a comet satisfies an equation of the form

$$r = \beta + e(r \cdot \cos\upsilon)$$

Where β is a constant and e is the eccentricity of the orbit, with $0 \le e < 1$ for an ellipse, $e = 1$ for a parabolic, and $e > 1$ for a hyperbola.

Suppose observations of a newly discovered comet provide the data below.

υ	.88	1.10	1.42	1.77	2.14
r	3.00	2.30	1.65	1.25	1.01

Determine the type of orbit, and predict where the orbit will be when $\upsilon = 4.6\ (radians)$?

Solution

Given: the equation in the form $r = \beta + e(r \cdot \cos\upsilon)$

$$3 = \beta + e\big(3 \cdot \cos(.88)\big) = \beta + 1.911\,e$$
$$2.3 = \beta + e\big(2.3\cos(1.1)\big) = \beta + 1.043\,e$$
$$1.65 = \beta + e\big(1.65\cos(1.42)\big) = \beta + .248\,e$$
$$1.25 = \beta + e\big(1.25\cos(1.77)\big) = \beta - .247e$$
$$1.01 = \beta + e\big(1.01\cos(2.14)\big) = \beta - .544\,e$$

$$\begin{pmatrix} 1 & 1.911 \\ 1 & 1.043 \\ 1 & .248 \\ 1 & -.247 \\ 1 & -.544 \end{pmatrix} \begin{pmatrix} \beta \\ e \end{pmatrix} = \begin{pmatrix} 3 \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{pmatrix}$$

where $A = \begin{pmatrix} 1 & 1.911 \\ 1 & 1.043 \\ 1 & .248 \\ 1 & -.247 \\ 1 & -.544 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} \beta \\ e \end{pmatrix}$ $\vec{r} = \begin{pmatrix} 3 \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{pmatrix}$

The normal equation formula: $A^T A \vec{v} = A^T \vec{r}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1.911 & 1.043 & .248 & -.247 & -.544 \end{pmatrix} \begin{pmatrix} 1 & 1.911 \\ 1 & 1.043 \\ 1 & .248 \\ 1 & -.247 \\ 1 & -.544 \end{pmatrix} \begin{pmatrix} \beta \\ e \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1.911 & 1.043 & .248 & -.247 & -.544 \end{pmatrix} \begin{pmatrix} 3 \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 2.411 \\ 2.411 & 5.158 \end{pmatrix} \begin{pmatrix} \beta \\ e \end{pmatrix} = \begin{pmatrix} 9.21 \\ 7.683 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 5 & 2.411 \\ 2.411 & 5.158 \end{vmatrix} = 19.98 \qquad \Delta_\beta = \begin{vmatrix} 9.21 & 2.411 \\ 7.683 & 5.158 \end{vmatrix} = 28.98 \qquad \Delta_e = \begin{vmatrix} 5 & 9.21 \\ 2.411 & 7.683 \end{vmatrix} = 16.21$$

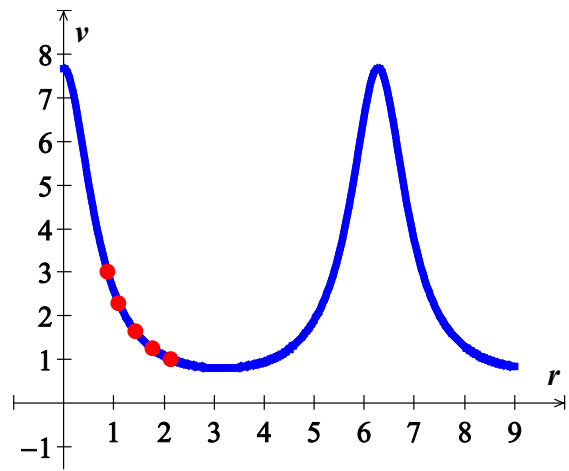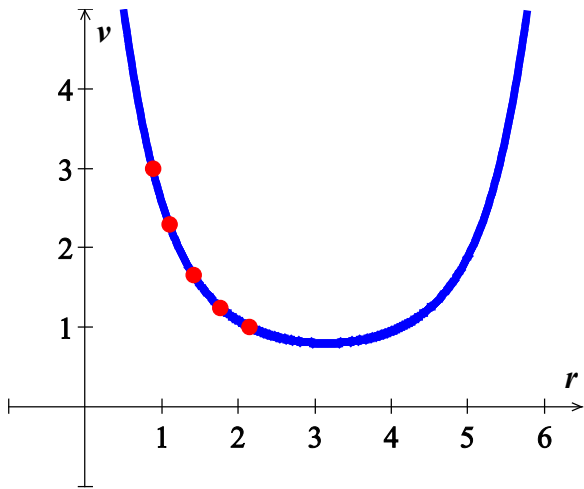$\beta = \dfrac{28.98}{19.98}$

≈ 1.45

$e = \dfrac{16.21}{19.98}$

$\approx 0.811 \quad < 1$

Therefore, the orbit is an **ellipse** type since $e \approx 0.811 \quad < 1$

Since $r = \beta + e(r \cdot \cos v)$

Then, $r(v) = \dfrac{1.45}{1 - 0.811 \cdot \cos v}$

$$r(4.6) = \frac{1.45}{1 - 0.811 \cdot \cos 4.6}$$

$$\approx 1.329$$

Exercise

To measure the takeoff performance of an airplane, the horizontal position of the plane was measured every second, from $t = 0$ to $t = 12$

The position (in *feet*) were:

0, 8.8, 29.9, 62.0, 104.7, 159.1, 222.0, 294.5, 380.4, 471.1, 571.7, 686.8, *and* 809.2

a) Find the least square cubic curve $y = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$ for these data.

b) Estimate the velocity of the plane when $t = 4.5 \; sec$, using the result from part (*a*).

Solution

Given: the equation is in form $y = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
1 & 2 & 4 & 8 \\
1 & 3 & 9 & 27 \\
1 & 4 & 16 & 64 \\
1 & 5 & 25 & 125 \\
1 & 6 & 36 & 216 \\
1 & 7 & 49 & 343 \\
1 & 8 & 64 & 512 \\
1 & 9 & 81 & 729 \\
1 & 10 & 100 & 1000 \\
1 & 11 & 121 & 1331 \\
1 & 12 & 144 & 1728
\end{pmatrix}
\begin{pmatrix}
\beta_0 \\
\beta_1 \\
\beta_2 \\
\beta_3
\end{pmatrix}
=
\begin{pmatrix}
0 \\
8.8 \\
29.9 \\
62 \\
104.7 \\
159.1 \\
222.0 \\
294.5 \\
380.4 \\
471.1 \\
571.7 \\
686.8 \\
809.2
\end{pmatrix}
$$

$$\quad\quad A \quad\quad\quad\quad \vec{t} \;=\; \vec{y}$$

The normal equation formula: $A^T A\,\vec{t} = A^T \vec{y}$

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 & 81 & 100 & 121 & 144 \\
0 & 1 & 8 & 27 & 64 & 125 & 216 & 343 & 512 & 729 & 1000 & 1331 & 1728
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
1 & 2 & 4 & 8 \\
1 & 3 & 9 & 27 \\
1 & 4 & 16 & 64 \\
1 & 5 & 25 & 125 \\
1 & 6 & 36 & 216 \\
1 & 7 & 49 & 343 \\
1 & 8 & 64 & 512 \\
1 & 9 & 81 & 729 \\
1 & 10 & 100 & 1000 \\
1 & 11 & 121 & 1331 \\
1 & 12 & 144 & 1728
\end{pmatrix}
\begin{pmatrix}
\beta_0 \\
\beta_1 \\
\beta_2 \\
\beta_3
\end{pmatrix}
$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 & 81 & 100 & 121 & 144 \\ 0 & 1 & 8 & 27 & 64 & 125 & 216 & 343 & 512 & 729 & 1000 & 1331 & 1728 \end{pmatrix} \begin{pmatrix} 0 \\ 8.8 \\ 29.9 \\ 62 \\ 104.7 \\ 159.1 \\ 222.0 \\ 294.5 \\ 380.4 \\ 471.1 \\ 571.7 \\ 686.8 \\ 809.2 \end{pmatrix}$$

$$\begin{pmatrix} 13 & 78 & 650 & 6,084 \\ 78 & 650 & 6,084 & 60,710 \\ 650 & 6,084 & 60,710 & 630,708 \\ 6,084 & 60,710 & 630,708 & 6,735,950 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 3,800.2 \\ 35,127.7 \\ 348,063.9 \\ 3,599,800.9 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 13 & 78 & 650 & 6,084 \\ 78 & 650 & 6,084 & 60,710 \\ 650 & 6,084 & 60,710 & 630,708 \\ 6,084 & 60,710 & 630,708 & 6,735,950 \end{vmatrix} = 97,538,785,344$$

$$\Delta_0 = \begin{vmatrix} 3800.2 & 78 & 650 & 6,084 \\ 35,127.7 & 650 & 6,084 & 60,710 \\ 348,063.9 & 6,084 & 60,710 & 630,708 \\ 3,599,800.9 & 60,710 & 630,708 & 6,735,950 \end{vmatrix} = -83,470,691,303.8916$$

Or I use my program to find the values

rref = (Matrix([
[1, 0, 0, 0, -0.855769230765803],
[0, 1, 0, 0, 4.70248501498163],
[0, 0, 1, 0, 5.55536963037029],
[0, 0, 0, 1, -0.0273601398601744]]))
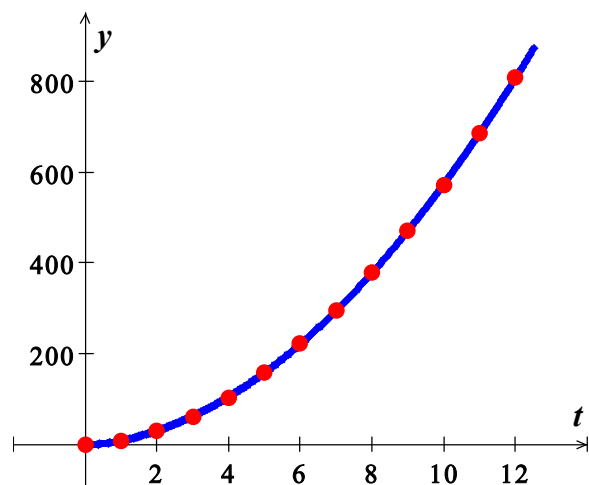
$$\beta_0 \approx -0.855769$$

$$\beta_1 \approx 4.702485$$

$$\beta_2 \approx 5.55537$$

$$\beta_3 \approx -0.02736$$

$$y(t) = -0.855769 + 4.702485\,t + 5.55537\,t^2 - 0.02736\,t^3$$

Error $= 3.9734$