Exercise

Verify that $rank(A) = rank\left(A^T\right)$

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -\frac{6}{7} & -\frac{4}{7} \\ 0 & 1 & \frac{17}{7} & \frac{2}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad rank(A) = 2$$

$$A^T = \begin{bmatrix} 1 & -3 & -2 \\ 2 & 1 & 3 \\ 4 & 5 & 6 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad rank\left(A^T\right) = 2$$

$$rank(A) = rank\left(A^T\right) = \underline{2}|$$

Exercise

Find the rank and nullity of the matrix; then verify that the values obtained satisfy $rank(A) + N(A) = n$

a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

Solution

a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$

$rank(A) = 2; \quad nullity(A) = 1; \quad rank(A) + nullity(A) = 2 + 1 = 3 = n \quad \leftarrow number\ of\ columns$

b) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$rank(A) = 2; \quad nullity(A) = 1; \quad rank(A) + nullity(A) = 2 + 1 = 3 = n$

c) $\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$rank(A) = 2; \quad nullity(A) = 2; \quad rank(A) + nullity(A) = 2 + 2 = 4 = n$

d) $A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & 2 & \frac{4}{3} \\ 0 & 1 & 0 & 0 & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & -\frac{5}{12} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$rank(A) = 3$; $NS(A) = 2$; Number of column = 5; $rank(A) + NS(A) = 3 + 2 = 5 = n$

Exercise

If *A* is an $m \times n$ matrix, what is the largest possible value for its rank and the smallest possible value of the nullity of *A*.

Solution

The largest possible value for the rank of an $m \times n$ matrix:

- $n \quad if \quad m \geq n$ (when every column of the *rref* (A) contains a leading 1)
- $m \quad if \quad m < n$ (when every row of the *rref* (A) contains a leading 1)

The smallest possible value for the nullity of an $m \times n$ matrix:

- $0 \quad if \quad m \geq n$ (when every column of the *rref* (A) contains a leading 1)
- $n - m \quad if \quad m < n$ (when every row of the *rref* (A) contains a leading 1)

Exercise

Discuss how the rank of A varies with t.

$$a)\ A=\begin{bmatrix}1 & 1 & t\\ 1 & t & 1\\ t & 1 & 1\end{bmatrix} \qquad\qquad b)\ A=\begin{bmatrix}t & 3 & -1\\ 3 & 6 & -2\\ -1 & -3 & t\end{bmatrix}$$

Solution

a) $$\begin{vmatrix}1 & 1 & t\\ 1 & t & 1\\ t & 1 & 1\end{vmatrix}=t+t+t-t^3-1-1$$

$$=-t^3+3t-2=0$$

Solve for t: $\boxed{t=1,-2,-2}$

Therefore, $rank(A)=3$ for $\forall t-\{1,-2,-2\}$

If $t=1$, $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ $rank(A)=1$

If $t=-2$, $A=\begin{bmatrix}1 & 1 & -2\\ 1 & -2 & 1\\ -2 & 1 & 1\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$ $rank(A)=2$

b) $$\begin{vmatrix}t & 3 & -1\\ 3 & 6 & -2\\ -1 & -3 & t\end{vmatrix}=6t^2+6+9-6-6t-9t$$

$$=6t^2-15t+9=0$$

Solve for t: $\boxed{t=1,\ \frac{3}{2}}$

Therefore, $rank(A)=3$ for $\forall t-\left\{1,\ \frac{3}{2}\right\}$

If $t=1$, $A=\begin{bmatrix}1 & 3 & -1\\ 3 & 6 & -2\\ -1 & -3 & 1\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -\frac{1}{3}\\ 0 & 0 & 0\end{bmatrix}$ $rank(A)=2$

If $t=\frac{3}{2}$, $A=\begin{bmatrix}\frac{3}{2} & 3 & -1\\ 3 & 6 & -2\\ -1 & -3 & \frac{3}{2}\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -\frac{5}{6}\\ 0 & 0 & 0\end{bmatrix}$ $rank(A)=2$

Exercise

Are there values of r and s for which

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

Has rank 1? Has rank 2? If so, find those values.

Solution

Since the third column will always have a nonzero entry, the *rank* will never be 1. (row 1 and row 4 never have a nonzero entry).

If $r = 2$ and $s = 1$, that implies to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 3 \end{bmatrix} \to rank = 2$$

Exercise

Find the row reduced form $\boldsymbol{R}$ and the rank r of $\boldsymbol{A}$ (those depend on c).
Which are the pivot columns of $\boldsymbol{A}$? Which variables are free? What are the special solutions and the nullspace matrix $\boldsymbol{N}$ (always depending on c)?

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & c \end{bmatrix} \quad and \quad A = \begin{bmatrix} c & c \\ c & c \end{bmatrix}$$

Solution

a) $c \neq 4 \quad R = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$

$rank(A) = 2$, the pivot columns are 1 and 3, the second variable x_2 is free.

The special solution: $x_2 = 1$ which yields to $N = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$c = 4 \quad R = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$

$rank(A)=1$, the pivot column is column 1, the second and third variables x_2, x_3 are free.

The special solution goes into $N=\begin{pmatrix}-2 & -1\\ 1 & 0\\ 0 & 1\end{pmatrix}$

b) $c\neq 0 \qquad R=\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$,

$rank(A)=1$, the pivot column is the first column, the second variable x_2 is free.

The special solution: $x_2=1$ which yields to $N=\begin{pmatrix}-1\\ 1\end{pmatrix}$

$c=0 \qquad R=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$

$rank(A)=0$, the matrix has no pivot column, and both variables are free.

The special solution is: $N=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

Exercise

Find the row reduced form R and the rank r of A (those depend on c).
Which are the pivot columns of A? Which variables are free? What are the special solutions and the nullspace matrix N (always depending on c)?

$$A=\begin{bmatrix}1 & 1 & 2 & 2\\ 2 & 2 & 4 & 4\\ 1 & c & 2 & 2\end{bmatrix} \quad \textit{and} \quad A=\begin{bmatrix}1-c & 2\\ 0 & 2-c\end{bmatrix}$$

Solution

$$A=\begin{bmatrix}1 & 1 & 2 & 2\\ 2 & 2 & 4 & 4\\ 1 & c & 2 & 2\end{bmatrix} \xrightarrow[R_3-R_1]{R_2-2R_1} \begin{bmatrix}1 & 1 & 2 & 2\\ 0 & 0 & 0 & 0\\ 0 & c-1 & 0 & 0\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 1 & 2 & 2\\ 0 & c-1 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

a) If $c=1$, then

$$\begin{pmatrix}1 & 1 & 2 & 2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

This has only one pivot (first column) and 3 free variables x_2, x_3, x_4.

The nullspace matrix: $\begin{pmatrix}-1 & -2 & -2\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

b) If $c \neq 1$, then

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\frac{1}{c-1}R_2} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2-R_1} \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are two pivots $\left(C_1,\ C_2\right)$ and 2 free variables $x_3,\ x_4$

The nullspace matrix: $\begin{pmatrix} -2 & -2 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}$$

a) If $c = 1 \Rightarrow A = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1-2R_2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to 0a + b = 0 \Rightarrow b = 0$$

This has a single pivot in the second column and one free variable with the nullspace matrix $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

b) If $c = 2 \Rightarrow A = \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

$$\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to a - 2b = 0 \Rightarrow if\ \ b = 1 \quad a = 2$$

This has a single pivot in the first column with the nullspace matrix $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

c) Otherwise $c \neq 1,\ 2 \Rightarrow A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix} \xrightarrow{\frac{1}{1-c}R_1} \begin{bmatrix} 1 & \frac{2}{1-c} \\ 0 & 2-c \end{bmatrix}$

$$\begin{bmatrix} 1 & \frac{2}{1-c} \\ 0 & 2-c \end{bmatrix} \xrightarrow[\frac{1}{2-c}R_2]{} \begin{bmatrix} 1 & \frac{2}{1-c} \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1-\frac{2}{1-c}R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The result is the identity matrix with 2 pivots, which has (2 – 2) 0 null space.

Exercise

If A has a rank r, then it has an r by r sub-matrix S that is invertible. Remove $m-r$ rows and $n-r$ columns to find an invertible sub-matrix S inside each A (you could keep the pivot rows and pivot columns of A).

$$A=\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 4\end{pmatrix} \quad A=\begin{pmatrix}1 & 2 & 3\\ 2 & 4 & 6\end{pmatrix} \quad A=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Solution

If a matrix ***A*** has rank r, then the

(dimension of the column space) = (dimension of the row space) = r

For the invertible sub-matrix S, we need to find r linearly independent rows and r linearly independent columns.

For matrix ***A***:

$$\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 4\end{pmatrix} \xrightarrow[R_2-R_1]{} \begin{pmatrix}1 & 2 & 3\\ 0 & 0 & 1\end{pmatrix} \xrightarrow{R_1-3R_2} \begin{pmatrix}1 & 2 & 0\\ 0 & 0 & 1\end{pmatrix}$$

The 1[st] and 3[rd] columns are linearly independent, and the 1[st] and 2[nd] rows are also linearly independent.

Rank (***A***) = 2.

The sub matrices are: $S_A=\begin{pmatrix}1 & 3\\ 1 & 4\end{pmatrix} \quad S_A=\begin{pmatrix}2 & 3\\ 2 & 4\end{pmatrix} \quad S_A=\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}$

For matrix ***B***:

$$\begin{pmatrix}1 & 2 & 3\\ 2 & 4 & 6\end{pmatrix} \xrightarrow[R_2-2R_1]{} \begin{pmatrix}1 & 2 & 3\\ 0 & 0 & 0\end{pmatrix}$$

Rank (***B***) = 1.

The submatrix is: $S_A=(1)$

For matrix ***C***:

$$C=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Rank (***C***) = 2.

The submatrix is by disregarding (deleting) 1[st] column and 2[nd] row: $S_A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

Exercise

Suppose that column 3 of 4 x 6 matrix is all zero. Then x_3 must be a ______ variable. Give one special solution for this matrix.

Solution

The x_3 must be a ***free variable***.

A special solution for this variable can be taken to be.

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Exercise

Fill in the missing numbers to make A rank1, rank 2, rank 3. (your solution should be 3 matrices)

$$A = \begin{pmatrix} & -3 & \\ 1 & 3 & -1 \\ & 9 & -3 \end{pmatrix}$$

Solution

$$A = \begin{pmatrix} a & -3 & b \\ 1 & 3 & -1 \\ c & 9 & -3 \end{pmatrix}$$

If rank (A) = 1, then we need the 1st and 3rd to be multiple of the 2nd row to get zero in these rows.

$$A = \begin{pmatrix} a & -3 & b \\ 1 & 3 & -1 \\ c & 9 & -3 \end{pmatrix} \begin{matrix} R_1 + R_2 \\ \\ R_3 - 3R_2 \end{matrix} \begin{pmatrix} a+1 & 0 & b-1 \\ 1 & 3 & -1 \\ c-3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} a+1=0 \\ b-1=0 \\ c-3=0 \end{cases} \rightarrow \begin{cases} a=-1 \\ b=1 \\ c=3 \end{cases}$$

$$A = \begin{pmatrix} -1 & -3 & 1 \\ 1 & 3 & -1 \\ 3 & 9 & -3 \end{pmatrix}$$

If rank $(A) = 2$, then we need the 1st ***or*** 3rd to be multiple of the 2nd row to get zero row.

$$A = \begin{pmatrix} a & -3 & b \\ 1 & 3 & -1 \\ c & 9 & -3 \end{pmatrix} \begin{matrix} R_1 + R_2 \\ \\ R_3 - 3R_2 \end{matrix} \begin{pmatrix} a+1 & 0 & b-1 \\ 1 & 3 & -1 \\ c-3 & 0 & 0 \end{pmatrix} \quad \boxed{c \ne 3}$$

$$A = \begin{pmatrix} -1 & -3 & 1 \\ 1 & 3 & -1 \\ 2 & 9 & -3 \end{pmatrix}$$

If rank $(A) = 3$ (full rank), then the appropriate to start using 0's or 1's to fill the blank.

$$A = \begin{pmatrix} 0 & -3 & 0 \\ 1 & 3 & -1 \\ 1 & 9 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -1 \\ 0 & -3 & 0 \\ 1 & 9 & -3 \end{pmatrix} \xrightarrow[R_3 - R_1]{} \begin{pmatrix} 1 & 3 & -1 \\ 0 & -3 & 0 \\ 0 & 6 & -2 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{3}R_2} \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 6 & -2 \end{pmatrix} \xrightarrow[R_1 - 6R_2]{R_1 - 3R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\xrightarrow[-\frac{1}{2}R_3]{} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1 + R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, it has rank 3.

Exercise

Fill out these matrices so that they have rank 1:

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & & \\ 4 & & \end{pmatrix} \qquad B = \begin{pmatrix} 2 & & \\ 1 & & \\ 2 & 6 & -3 \end{pmatrix} \qquad M = \begin{pmatrix} a & b \\ c & \end{pmatrix}$$

Solution

Rank = 1 means that all the rows are multiples of each other.

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & a & b \\ 4 & c & d \end{pmatrix} \xrightarrow[R_3 = 4R_1]{R_2 = 2R_1} \begin{matrix} a = 2(2) & b = 2(4) \\ c = 4(2) & d = 4(4) \end{matrix}$$

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & a & b \\ 1 & c & d \\ 2 & 6 & -3 \end{pmatrix} \xrightarrow[R_2 = \frac{1}{2}R_3]{R_1 = R_3} \begin{matrix} a = 6 & b = -3 \\ c = 3 & d = -\frac{3}{2} \end{matrix}$$

$$B = \begin{pmatrix} 2 & 6 & -3 \\ 1 & 3 & -\frac{3}{2} \\ 2 & 6 & -3 \end{pmatrix}$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow[R_2 = \frac{c}{a}R_1]{} d = \frac{c}{a}b \to M = \begin{pmatrix} a & b \\ c & \frac{bc}{a} \end{pmatrix}$$

Exercise

Suppose A and B are n by n matrices, and $AB = I$. Prove from $rank(AB) \le rank(A)$ that the $rank(A) = n$. So, A is invertible and B must be its two-sided inverse. Therefore $BA = I$ (which is not so obvious!).

Solution

Since A is n by n $\Rightarrow rank(A) \le n$

$$n = rank\left(I_n\right) = rank(AB) \le rank(A)$$

Exercise

Every m by n matrix of rank r reduces to (m by r) times (r by n):

A = (pivot columns of A) (first r rows of R) $= (COL)(ROW)^T$

Write the 3 by 4 matrix $A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{pmatrix}$ as the product of the 3 by 2 from the pivot columns and the 2 by 4 matrix from R.

Solution

$$A=\begin{pmatrix}1&1&2&4\\1&2&2&5\\1&3&2&6\end{pmatrix}\begin{matrix}\\R_2-R_1\\R_3-R_1\end{matrix}\begin{pmatrix}1&1&2&4\\0&1&0&1\\0&2&0&2\end{pmatrix}$$

$$\xrightarrow[R_3-2R_2]{}\begin{pmatrix}1&1&2&4\\0&1&0&1\\0&0&0&0\end{pmatrix}$$

$$\xrightarrow{R_1-R_2}\begin{pmatrix}1&0&2&3\\0&1&0&1\\0&0&0&0\end{pmatrix}$$

The pivots columns are the 1st and 2nd column.

$$A=\begin{pmatrix}1&1&2&4\\1&2&2&5\\1&3&2&6\end{pmatrix}=\begin{pmatrix}1&1\\1&2\\1&3\end{pmatrix}\begin{pmatrix}1&0&2&3\\0&1&0&1\end{pmatrix}$$

Exercise

Suppose R is m by n matrix of rank r, with pivot columns first: $\begin{bmatrix}I&F\\0&0\end{bmatrix}$

a) What are the shapes of those 4 blocks?

b) Find the right-inverse B with $RB = I$ if $r = m$.

c) Find the right-inverse C with $CR = I$ if $r = n$.

d) What is the reduced row echelon form of R^T (with shapes)?

e) What is the reduced row echelon form of R^TR (with shapes)?

Prove that R^TR has the same nullspace as R. Then show that A^TA always has the same nullspace as A (a value fact).

f) Suppose you allow elementary column operations on $\boldsymbol{A}$ as well as elementary row operations (which get to $\boldsymbol{R}$). What is the "row-and-column reduced form" for an m by n matrix of rank $\boldsymbol{r}$?

Solution

a) $\begin{bmatrix}I&F\\0&0\end{bmatrix}:\begin{bmatrix}r\times r&r\times(n-r)\\(m-r)\times r&(m-r)\times(n-r)\end{bmatrix}$

b) $R=\begin{bmatrix}I&F\end{bmatrix}$

$RB=I\Rightarrow\begin{bmatrix}I&F\end{bmatrix}B=I$

$$\begin{bmatrix}I&F\end{bmatrix}\begin{pmatrix}M\\N\end{pmatrix}=I$$

$$IM + FN = I \Rightarrow \begin{cases} M = I \\ N = 0 \end{cases} \rightarrow F : r \times (n-r)$$

$$B = \begin{bmatrix} I_{r\times r} \\ 0_{(n-r)\times r} \end{bmatrix}$$

c) $R = \begin{bmatrix} I & 0 \end{bmatrix}$

$$CR = I \Rightarrow C\begin{bmatrix} I & 0 \end{bmatrix} = I$$

$$C = \begin{bmatrix} I_{r\times r} & 0_{r\times(m-r)} \end{bmatrix}$$

d) $R^T = \begin{bmatrix} I_{r\times r} & 0_{(m-r)\times r} \\ F_{r\times(n-r)} & 0_{(m-r)\times(n-r)} \end{bmatrix} \Rightarrow rref\left(R^T\right) = \begin{bmatrix} I_{r\times r} & 0_{(m-r)\times r} \\ 0_{(n-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}$

e) $R^T R = \begin{bmatrix} I & 0 \\ F & 0 \end{bmatrix}\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & F \\ FI & 0 \end{bmatrix}$

$FI : r \times (n-r) \quad r \times r$, the inner is not equal but to make work, we can use the F transpose.

$(n-r)\times r \quad r \times r \Rightarrow F^T I = F^T$

$$R^T R = \begin{bmatrix} I & 0 \\ F & 0 \end{bmatrix}\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} I & F \\ F^T & 0 \end{bmatrix}$$

$$rref\left(R^T R\right) = \begin{bmatrix} I_{r\times r} & F_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix} = R$$

So, that $N(A) = N\left(rref(A)\right)$ for any matrix $\boldsymbol{A}$. So, $N(A) = N\left(R^T R\right)$

f) After getting to R we can use the column operations to get rid of F.

$$\begin{bmatrix} I_{r\times r} & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}$$

Exercise

True or False (check addition or give a counterexample)

a) The symmetric matrices in M $\left(with\ A^T = A\right)$ from a subspace.

b) The skew-symmetric matrices in M $\left(with\ A^T = -A\right)$ from a subspace.

c) The un-symmetric matrices in M $\left(with\ A^T \neq A\right)$ from a subspace.

d) Invertible matrices

e) Singular matrices

Solution

a) True: $A^T = A$ and $B^T = B$ lead to $(A+B)^T = A^T + B^T = A + B$

b) True: $A^T = -A$ and $B^T = -B$ lead to $(A+B)^T = A^T + B^T = -A - B = -(A+B)$

c) False: $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

d) False: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ are invertible matrices but $A + B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ is not invertible. $\therefore$ The zero matrix is not invertible but any linear subspace should contain the zero matrix. So, invertible matrices do not form a linear subspace.

e) False: $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ are singular matrices but $A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ is not singular.

Exercise

Let $A = \begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 2 & 4 & -3 & 7 & 0 \\ 3 & 6 & -5 & 10 & -2 \\ 5 & 10 & -9 & 16 & 0 \end{pmatrix}$

a) Reduce A to row-reduced echelon from.

b) What is the rank of A?

c) What are the pivots?

d) What are the free variables?

e) Find the special solutions. What is the nullspace $N(A)$?

f) Exhibit an r x r submatrix of A which is invertible, where $r = rank(A)$. (An r x r submatrix of A is obtained by keeping r rows and r columns of A)

Solution

a) $A=\begin{pmatrix}1&2&-2&3&0\\2&4&-3&7&0\\3&6&-5&10&-2\\5&10&-9&16&0\end{pmatrix}\begin{matrix}\\R_2-2R_1\\R_3-3R_1\\R_4-5R_1\end{matrix}\begin{pmatrix}1&2&-2&3&0\\0&0&1&1&0\\0&0&1&1&-2\\0&0&1&1&0\end{pmatrix}$

$$\begin{matrix}\\ \\R_3-R_2\\R_4-R_2\end{matrix}\begin{pmatrix}1&2&-2&3&0\\0&0&1&1&0\\0&0&0&0&-2\\0&0&0&0&0\end{pmatrix}$$

b) Rank(***A***) = 3

c) The pivots are x_1, x_3, x_5

d) The free variables are x_2, x_4

e) $\begin{pmatrix}1&2&-2&3&0\\0&0&1&1&0\\0&0&0&0&-2\\0&0&0&0&0\end{pmatrix}-\frac{1}{2}R3\begin{pmatrix}1&2&-2&3&0\\0&0&1&1&0\\0&0&0&0&1\\0&0&0&0&0\end{pmatrix}R_1+2R_2\begin{pmatrix}1&2&0&5&0\\0&0&1&1&0\\0&0&0&0&1\\0&0&0&0&0\end{pmatrix}$

Let $x=x_2s_2+x_4s_4$

$$Rx=\begin{pmatrix}1&2&0&5&0\\0&0&1&1&0\\0&0&0&0&1\\0&0&0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\qquad\rightarrow\begin{cases}x_1+2x_2+5x_4=0\\x_3+x_4=0\\x_5=0\end{cases}$$

1. Set $x_2=1,\quad x_4=0\rightarrow\begin{cases}x_1+2=0\Rightarrow x_1=-2\\x_3=0\\x_5=0\end{cases}$

The special solution: $s_2=(-2,\ 1,\ 0,\ 0,\ 0)$

2. Set $x_2=0,\quad x_4=1\rightarrow\begin{cases}x_1+5=0\Rightarrow x_1=-5\\x_3+1=0\Rightarrow x_3=-1\\x_5=0\end{cases}$

The special solution: $s_3=(-5,\ 0,\ -1,\ 1,\ 0)$

The nullspace is the set $\left\{ x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -5 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

f) The pivot rows and columns must be included in a submatrix. To do that, just take the rows and columns of ***A*** containing pivots, which are columns 1, 3, 5 and rows 1, 2, 3. That will give us a 3 by 3 submatrix. Therefore, this submatrix of ***A*** will be invertible.

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & -3 & 0 \\ 3 & -5 & -2 \end{pmatrix}$$

Exercise

Let $A = \begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 2 & 1 & 0 & 0 & -15 \\ 6 & -1 & -8 & -1 & -47 \\ 0 & 2 & 4 & 3 & 16 \end{pmatrix}$

a) Reduce ***A*** to (ordinary) echelon from.
b) What the pivots?
c) What are the free variables?
d) Reduce ***A*** to row-reduced echelon form.
e) Find the special solutions. What is the nullspace $N(A)$?
f) What is the rank of ***A***?
g) Give the complete solution to $Ax = b,\ where\ \ b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Solution

a) $A = \begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 2 & 1 & 0 & 0 & -15 \\ 6 & -1 & -8 & -1 & -47 \\ 0 & 2 & 4 & 3 & 16 \end{pmatrix} \xrightarrow[R_3+6R_1]{R_2+2R_1} \begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 0 & 5 & 10 & 0 & -5 \\ 0 & 11 & 22 & -1 & -17 \\ 0 & 2 & 4 & 3 & 16 \end{pmatrix}$

$$\begin{array}{r} \\ \\ 5R_3-11R_2 \\ 5R_4-2R_2 \end{array}\begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 0 & 5 & 10 & 0 & -5 \\ 0 & 0 & 0 & -5 & -30 \\ 0 & 0 & 0 & 15 & 90 \end{pmatrix} \xrightarrow{R_4+3R_3} \begin{pmatrix} \boxed{-1} & 2 & 5 & 0 & 5 \\ 0 & \boxed{5} & 10 & 0 & -5 \\ 0 & 0 & 0 & \boxed{-5} & -30 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

b) The pivots are –1, 5, and –5 (Columns 1, 2, 4)

c) The free variables are 3rd and 5th $\left(x_3, x_5\right)$

d) $$\begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 0 & 5 & 10 & 0 & -5 \\ 0 & 0 & 0 & -5 & -30 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{c} -R_1 \\ \frac{1}{5}R_2 \\ -\frac{1}{5}R_3 \\ \\ \end{array} \begin{pmatrix} 1 & -2 & -5 & 0 & -5 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1+2R_2} \begin{pmatrix} 1 & 0 & -1 & 0 & -7 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 0 & -1 & 0 & -7 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

e) Let $x = x_3 s_1 + x_5 s_2$

$$Rx = \begin{pmatrix} 1 & 0 & -1 & 0 & -7 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \rightarrow \begin{cases} x_1 - x_3 - 7x_5 = 0 \\ x_2 + 2x_3 - x_5 = 0 \\ x_4 + 6x_5 = 0 \end{cases}$$

1. Set $x_3 = 1, \quad x_5 = 0 \rightarrow \begin{cases} x_1 - 1 = 0 \\ x_2 + 2 = 0 \\ x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = -2 \\ x_4 = 0 \end{cases}$

The special solution: $s_1 = (1, \ -2, \ 1, \ 0, \ 0)$

2. Set $x_3 = 0, \quad x_5 = 1 \rightarrow \begin{cases} x_1 - 7 = 0 \\ x_2 - 1 = 0 \\ x_4 + 6 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 7 \\ x_2 = 1 \\ x_4 = -6 \end{cases}$

The special solution: $s_2 = (7, \ 1, \ 0, \ -6, \ 1)$

The nullspace is the set $\left\{ x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 7 \\ 1 \\ 0 \\ -6 \\ 1 \end{pmatrix} \right\}$

f) Rank(***A***) = 3

g) $Ax = b, \text{ where } b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

The complete solution = (the particular solution) + (special solution)

$$x = x_p + x_n$$

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 7 \\ 1 \\ 0 \\ -6 \\ 1 \end{pmatrix}$$

Exercise

Let $A = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

a) Reduce A to row-reduced echelon form.
b) What is the rank of A?
c) What the pivots variables?
d) What are the free variables?
e) Find the special solutions.
f) What is the nullspace $N(A)$?

Solution

a) $$\begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} R_2 - R_1 \\ \\ R_3 - 2R_1 \end{matrix} \to \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & -1 & -4 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} -R_2 \\ R_3 - R_2 \\ R_4 - 2R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} R_5 - R_4 \to \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{aligned} x_1 &= -2x_2 - 3x_4 \\ x_3 &= -4x_4 \\ x_5 &= 0 \end{aligned}$$

b) Rank(***A***) = 3

c) The pivots variables are: x_1, x_3, x_5

d) The free variables are: x_2, x_4

e) Let $x = x_2 s_1 + x_4 s_2$

$$\begin{cases} x_1 = -2x_2 - 3x_4 \\ x_3 = \quad\quad\; -4x_4 \end{cases}$$

Set $x_2 = 1, \quad x_4 = 0$

The special solution: $s_1 = (-2,\ 1,\ 0,\ 0,\ 0)$

Set $x_2 = 0, \quad x_4 = 1$;

The special solution: $s_2 = (-3,\ 0,\ -4,\ 1,\ 0)$

f) The nullspace is the set $\left\{ x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix} \right\}$

$$N(A) = \begin{pmatrix} -2 & -3 \\ 1 & 0 \\ 0 & -4 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Exercise

Let $A = \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix}$

a) Reduce A to row-reduced echelon form.

b) What is the rank of A?

c) What the pivots?

d) What are the free variables?

e) Find the special solutions.

f) What is the nullspace $N(A)$?

Solution

a)

$$\begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix} \begin{array}{c} \\ 3R_2 - R_1 \\ 3R_3 - 2R_1 \\ R_4 - 2R_1 \end{array} \rightarrow \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 0 & 0 & -3 & -15 & -3 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & -1 & -5 & 0 \end{pmatrix} \begin{array}{c} \\ -R_2 \\ \\ 3R_4 + R_2 \end{array}$$

$$\begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 0 & 0 & 3 & 15 & 3 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \begin{array}{c} \frac{1}{3}R_1 \\ \frac{1}{3}R_2 \\ \frac{1}{3}R_3 \\ R_4 - R_3 \end{array} \rightarrow \begin{pmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{c} \\ R_2 - R_3 \\ \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{aligned} x_1 &= -7x_2 - 3x_4 \\ x_3 &= \qquad\quad -5x_4 \\ x_5 &= 0 \end{aligned}$$

b) Rank(***A***) = 3

c) The pivots variables are: x_1, x_3, x_5

d) The free variables are: x_2, x_4

e) Let $x = x_2 s_1 + x_4 s_2$

$$\begin{cases} x_1 = -7x_2 - 3x_4 \\ x_3 = \quad\quad -5x_4 \end{cases}$$

Set $x_2 = 1, \quad x_4 = 0$

The special solution: $s_1 = (-7,\ 1,\ 0,\ 0,\ 0)$

Set $x_2 = 0, \quad x_4 = 1$;

The special solution: $s_2 = (-3,\ 0,\ -5,\ 1,\ 0)$

f) The nullspace is the set $\left\{ x_2 \begin{pmatrix} -7 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ -5 \\ 1 \\ 0 \end{pmatrix} \right\}$

$$N(A) = \begin{pmatrix} -7 & -3 \\ 1 & 0 \\ 0 & -5 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Exercise

The 3 by 3 matrix A has rank 2.

$$Ax = b \quad is \quad \begin{aligned} x_1 + 2x_2 + 3x_3 + 5x_4 &= b_1 \\ 2x_1 + 4x_2 + 8x_3 + 12x_4 &= b_2 \\ 3x_1 + 6x_2 + 7x_3 + 13x_4 &= b_3 \end{aligned}$$

a) Reduce $[A \quad b]$ to $[U \quad c]$, so that $Ax = b$ becomes triangular system $Ux = c$.

b) Find the condition on $(b_1,\ b_2,\ b_3)$ for $Ax = b$ to have a solution

c) Describe the column space of A. Which plane in $\boldsymbol{R}^3$?

d) Describe the nullspace of A. Which special solutions in $\boldsymbol{R}^4$?

e) Find a particular solution to $Ax = (0,\ 6,\ -6)$ and then complete solution.

Solution

a) $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2-2b_1 \\ 0 & 0 & -2 & -2 & b_3-3b_1 \end{array}\right]$

$$\xrightarrow[R_3+R_2]{} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & b_3+b_2-5b_1 \end{array}\right]$$

b) The last equation $b_3+b_2-5b_1=0$ shows the solvability condition.

c) (***i***) The column space is the plane containing all combinations of the pivot columns: 1st (1, 2, 3) and 3rd (3, 8, 7).

(***ii***) The column space contains all vectors with $b_3+b_2-5b_1=0$. That makes $Ax=b$ solvable, so ***b*** is in the column space. All columns of A pass this test $b_3+b_2-5b_1=0$. This is the equation for the plane in (***i***).

d) The special solutions have free variables:

$$\begin{cases} x_1+2x_2+3x_3+5x_4=0 \\ 2x_3+2x_4=0 \end{cases} \Rightarrow \begin{cases} x_1=-2x_2-2x_4 \\ x_3=-x_4 \end{cases}$$

$$x_2=1,\ x_4=0 \Rightarrow \begin{cases} x_1=-2 \\ x_3=0 \end{cases} \rightarrow s_1=\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$x_2=0,\ x_4=1 \Rightarrow \begin{cases} x_1=-2 \\ x_3=-1 \end{cases} \rightarrow s_2=\begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

The nullspace $N(A)$ in $\boldsymbol{R}^4$ contains all $x_n=x_2\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}+x_4\begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

e) One particular solution x_p has free variables = zero.

$$\begin{cases} x_1+2x_2+3x_3+5x_4=0 \\ 2x_3+2x_4=6 \end{cases} \Rightarrow \begin{cases} x_1=-2x_2-3x_3-5x_4 \\ x_3=3-x_4 \end{cases} \rightarrow\Rightarrow \begin{cases} x_1=-2x_2-9-2x_4 \\ x_3=3-x_4 \end{cases}$$

$$x_2=x_4=0 \Rightarrow \begin{cases} x_1=-9 \\ x_3=3 \end{cases}$$

$$x_p = \begin{pmatrix} -9 \\ 0 \\ -3 \\ 0 \end{pmatrix}$$

The complete solution to $Ax = (0,\ 6,\ -6)$ is $x = x_p + all\ x_n$

Exercise

Find the special solutions and describe the complete solution to $Ax = 0$ for

A_1 = 3 by 4 zero matrix $\quad A_2 = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \quad A_3 = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$

Which are the pivot columns? Which are the free variables? What is the R (Reduced Row Echelon matrix) in each case?

Solution

$A_1 x = 0$ has 4 solutions. They are the columns $s_1,\ s_2,\ s_3,\ s_4$ of the identity matrix (4 by 4).

The Nullspace is of $\boldsymbol{R}^4$.

The complete solution: $x = c_1 s_1 + c_2 s_2 + c_3 s_3 + c_4 s_4$ in $\boldsymbol{R}^4$.

There are no pivot columns; all variables are free; the reduced R is the same zero matrix as A_1.

$A_2 x = 0$

$$\Rightarrow A_2 x = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 + 2x_2 = 0$$

The vector solution: $s = (-2,\ 1)$, The first column of A_2 is its pivot column, and x_2 is the free variable.

$$\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \xrightarrow[R_2 - R_1]{} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

$$R_3 = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

All variables are free. There are three special solutions to $A_3 x = 0$

$$s_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad s_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad s_3 = \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The complete solution: $x = c_1 s_1 + c_2 s_2 + c_3 s_3$.

Exercise

Create a 3 by 4 matrix whose special solutions to $Ax = 0$ are s_1 *and* s_2 :

$$s_1 = \begin{pmatrix} -3 \\ 2 \\ 0 \\ 0 \end{pmatrix} \quad and \quad s_2 = \begin{pmatrix} -2 \\ 0 \\ -6 \\ 1 \end{pmatrix}$$

You could create the matrix A in row reduced form R. Then describe all possible matrices A with the required Nullspace $N(A) =$ all combinations of s_1 *and* s_2 .

Solution

We can write the solution:

$x = x_2 s_1 + x_4 s_2$

$$x_2 \begin{pmatrix} -3 \\ 2 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -6 \\ 1 \end{pmatrix} = \begin{pmatrix} -3x_2 - 2x_4 \\ 2x_2 \\ -6x_4 \\ x_4 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3x_2 - 2x_4 \\ 2x_2 \\ -6x_4 \\ x_4 \end{pmatrix} \rightarrow \begin{cases} x_1 = -3x_2 - 2x_4 \\ \\ x_3 = -6x_4 \end{cases}$$

$$\rightarrow \begin{cases} x_1 + 3x_2 + 2x_4 = 0 \\ \\ x_3 + 6x_4 = 0 \end{cases} \Rightarrow \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The entries 3, 2, 6 are the negatives of -3, -2, -6 in the special solutions.

Every 3 by 4 matrix has at least one special solution. These A's have two.

Exercise

The plane $x-3y-z=12$ is parallel to the plane $x-3y-z=0$. One particular point on this plane is $(12,\ 0,\ 0)$. All points on the plane have the form (fill the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \ \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} \ \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} \ \\ 0 \\ 1 \end{bmatrix}$$

Solution

$$x-3y-z=12 \Rightarrow x=3y+z+12$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} 12+3y+z \\ y \\ z \end{pmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} z \\ 0 \\ 1 \end{bmatrix}$$

Exercise

Construct a matrix whose column space contains (1, 1, 5) and (0, 3, 1) and whose Nullspace contains $(1,\ 1,\ 2)$.

Solution

$$A = \begin{pmatrix} 1 & 0 & a \\ 1 & 3 & b \\ 5 & 1 & c \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & a \\ 1 & 3 & b \\ 5 & 1 & c \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+2a \\ 1+3+2b \\ 5+1+2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 1+2a=0 \\ 4+2b=0 \\ 6+2c=0 \end{cases} \rightarrow \begin{cases} 2a=-1 \\ 2b=-4 \\ 2c=-6 \end{cases} \rightarrow \begin{cases} a=-\frac{1}{2} \\ b=-2 \\ c=-3 \end{cases}$$

$$A = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{pmatrix}$$

Exercise

Construct a matrix whose column space contains (1, 1, 0) and (0, 1, 1) and whose Nullspace contains (1, 0, 1) and (0, 0, 1).

Solution

It is impossible. Matrix $\boldsymbol{A}$ must be 3 by 3. Since the nullspace is supposed to contain two independent vectors, $\boldsymbol{A}$ can have at most $3-2=1$ pivots. Since the column space supposes to contain two independent vectors. A must has at least 2 pivots. These conditions can't both be met.

Exercise

Construct a matrix whose column space contains (1, 1, 1) and whose Nullspace contains $(1,\ 1,\ 1,\ 1)$.

Solution

The matrix needs to be 3 by 4 matrix.

$$\begin{pmatrix} 1 & a & b & c \\ 1 & d & e & f \\ 1 & g & h & i \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 1+a+b+c=0 \\ 1+d+e+f=0 \\ 1+g+h+i=0 \end{cases} \Rightarrow \begin{cases} a+b+c=-1 \\ d+e+f=-1 \\ g+h+i=-1 \end{cases} \to \begin{cases} if \quad b=c=0 \quad a=-1 \\ if \quad d=f=0 \quad e=-1 \\ if \quad g=h=0 \quad i=-1 \end{cases}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

Exercise

How is the Nullspace $N(C)$ related to the spaces N(A) and N(B), if $C=\begin{bmatrix} A \\ B \end{bmatrix}$?

Solution

$$Cx = \begin{bmatrix} Ax \\ Bx \end{bmatrix} = 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If and only if Ax =0 and Bx = 0.

$$N(C) = N(A) \cap N(B)$$

Exercise

Why does no 3 by 3 matrix have a nullspace that equals its column space?

Solution

If nullspace = column space, then $n - r = r$ (there are r pivots).
For $n = 3 \Rightarrow 3 = 2\boldsymbol{r}$ is impossible.

Exercise

If $AB = 0$ then the column space B is contained in the _______ of A. Give an example of A and B.

Solution

If $AB = 0$ then the column space B is contained in the ***nullspace*** of A.

Example: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

Exercise

True or false (with reason if true or example to show it is false)

a) A square matrix has no free variables.

b) An invertible matrix has no free variables.

c) An m by n matrix has no more than $\boldsymbol{n}$ pivot variables.

d) An m by n matrix has no more than $\boldsymbol{m}$ pivot variables.

Solution

a) False. Any matrix with fewer than full number of pivots will. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

b) True. Since it is invertible, we will get the full number of pivots. The nullspace has dimension, so we have 0 free variables.

c) True, the number of pivot variables is the dimension of the nullspace, which is at most the number of columns. The nullspace dimension + column space dimension = number of columns.

d) True, in reduced echelon matrix the pivot columns are all 0 except for a single 1, and there are only up to m vectors of this type.

Exercise

Suppose an m by n matrix has r pivots. The number of special solutions is ______.

The Nullspace contains only $x = 0$ when $r =$ _______.

The column space is all of $\boldsymbol{R}^m$ when $r =$ ______.

Solution

Suppose an m by n matrix has r pivots. The number of special solutions is _ $\boldsymbol{n-r}$_.

The Nullspace contains only $x = 0$ when $r =$ _ $\boldsymbol{n}$ _.

The column space is all of $\boldsymbol{R}^m$ when $r =$ _$\boldsymbol{m}$ _.

Exercise

Find the complete solution in the form $x_p + x_n$ to these full rank system:

$$a)\ \ x+y+z=4 \qquad\qquad b)\ \ \begin{matrix} x+y+z=4 \\ x-y+z=4 \end{matrix}$$

Solution

a) $x+y+z=4$

The equivalent matrix is given by: $\begin{cases} Ax=4 \\ A=\begin{pmatrix}1 & 1 & 1\end{pmatrix} \end{cases}$

The complete solution in the form $x = x_p + x_n$

x_n is the homogeneous solution to $Ax_n = 0$

Size of $\boldsymbol{A}$ is $m = 1$ and $n = 3$, rank($\boldsymbol{A}$) $= r = 1$

$$Ax_n = 0 \Rightarrow \begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}x_1 \\ x_2 \\ x_3\end{pmatrix} = \begin{pmatrix}0 \\ 0 \\ 0\end{pmatrix} \Rightarrow x_1+x_2+x_3=0 \Rightarrow \boxed{x_1=-x_2-x_3}$$

Set $x_2 = 1, \quad x_3 = 0$ The special solution: $s_1 = (-1,\ 1,\ 0)$

Set $x_2 = 0, \quad x_3 = 1$ The special solution: $s_2 = (-1,\ 0,\ 1)$

The nullspace is the set $\left\{ x_2\begin{pmatrix}-1 \\ 1 \\ 0\end{pmatrix} + x_3\begin{pmatrix}-1 \\ 0 \\ 1\end{pmatrix} \right\}$

$x = 4-y-z \ \Rightarrow x_1 = 4-x_2-x_3$

Set $x_2 = 0, \quad x_3 = 0$ that implies to the particular solution: $x_p = \begin{pmatrix}4 \\ 0 \\ 0\end{pmatrix}$

The complete solution in the form $x = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Note: that the null space of A is spanned by the two linearly independent vectors $(-1,\ 1,\ 0)^T$ *and* $(-1,\ 0,\ 1)^T$

b) $\begin{aligned} x+y+z&=4 \\ x-y+z&=4 \end{aligned}$

The equivalent matrix is given by: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$ and $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -1 & 1 & 4 \end{array}\right] \xrightarrow[R_2-R_1]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & 0 \end{array}\right] \xrightarrow[-\frac{1}{2}R_2]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_1-R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

The pivots are x_1, x_2 ; The free variable is x_3

Rank r = 2, n = 2, m = 3. The nullspace has dimension $m - r$ = 1.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 + x_3 = 0 \to x_1 = -x_3 \\ \qquad\qquad x_2 = 0 \end{cases}$$

If $x_3 = 1 \Rightarrow x_1 = -1$ The special solution: $s_1 = (-1,\ 0,\ 1)$

The nullspace is the set $\left\{ x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

Set $x_3 = 0$ that implies $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 = 4 \\ x_2 = 0 \end{cases}$

Then the particular solution: $x_p = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$

The complete solution in the form $x = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Exercise

Find the complete solution in the form $x_p + x_n$ to the system: $\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$

Solution

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{cccc|c} 1 & 3 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 2 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The pivots are x_1, x_3; The free variables are x_2, x_4

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 + 3x_2 = 0 \\ x_3 + 2x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -3x_2 \\ x_3 = -2x_4 \end{cases}$$

1. Set $x_2 = 1, \quad x_4 = 0$ The special solution: $s_1 = (-3,\ 1,\ 0,\ 0)$

2. Set $x_2 = 0, \quad x_4 = 1$ The special solution: $s_2 = (0,\ 0,\ -2,\ 1)$

The special solution: $x_n = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 + 3(0) = \frac{1}{2} \\ x_3 + 2(0) = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} x_1 = \frac{1}{2} \\ x_3 = \frac{1}{2} \end{cases}$$

Then the particular solution: $x_p = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$

The complete solution in the form $x = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}$

Exercise

If $\boldsymbol{A}$ is 3 x 7 matrix, its largest possible rank is ________. In this case, there is a pivot in every ________ of $\boldsymbol{U}$ and $\boldsymbol{R}$, the solution to $Ax = b$ _________ (always exists or is unique), and the column space of $\boldsymbol{A}$ is _________. Construct an example of such a matrix $\boldsymbol{A}$.

Solution

If $\boldsymbol{A}$ is 3 x 7 matrix, its largest possible rank is **3**. In this case, there is a pivot in every ***row*** of $\boldsymbol{U}$ and R, the solution to $Ax = b$ ***always exists***, and the column space of $\boldsymbol{A}$ is $\boldsymbol{R}^3$.

$$A = \begin{pmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{pmatrix}$$

$rank(A) \le 3$, that implies that you have 3 pivots (1 each row)

$$A = \begin{pmatrix} 1 & 0 & 0 & * & * & * & * \\ 0 & 1 & 0 & * & * & * & * \\ 0 & 0 & 1 & * & * & * & * \end{pmatrix} \rightarrow A = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 5 & 6 & 7 & 8 \\ 0 & 0 & 1 & 9 & 10 & 11 & 12 \end{pmatrix}$$

Exercise

If $\boldsymbol{A}$ is 6 x 3 matrix, its largest possible rank is ________. In this case, there is a pivot in every ________ of $\boldsymbol{U}$ and R, the solution to $Ax = b$ _________ (always exists or is unique), and the nullspace of $\boldsymbol{A}$ is _________. Construct an example of such a matrix $\boldsymbol{A}$.

Solution

If $\boldsymbol{A}$ is 6 x 3 matrix, its largest possible rank is **3**. In this case, there is a pivot in every ***column*** of $\boldsymbol{U}$ and R, the solution to $Ax = b$ ***is unique***, and the column space of $\boldsymbol{A}$ is $\{\mathbf{0}\}$.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Exercise

Find the rank of A, A^TA and AA^T for $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix}$

Solution

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix}_{R_3 + R_1} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 3 \end{pmatrix}_{\frac{1}{2}R_2} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 3 \end{pmatrix}_{R_3 - 3R_2} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \boxed{rank(A) = 2}$$

$$A^TA = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 9 \end{pmatrix} \xrightarrow[2R_2 + R_1]{} \begin{pmatrix} 2 & -1 \\ 0 & 17 \end{pmatrix} \quad \Rightarrow \quad \boxed{rank\left(A^TA\right) = 2}$$

$$AA^T = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 4 \\ 1 & 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 4 \\ 1 & 4 & 5 \end{pmatrix} \xrightarrow[2R_3 - R_1]{R_2 - R_1} \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 6 & 9 \end{pmatrix} \xrightarrow[R_3 - 3R_2]{} \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \boxed{rank\left(AA^T\right) = 2}$$

$\therefore rank(A) = rank\left(A^TA\right) = rank\left(AA^T\right)$ for any matrix, $\boldsymbol{A}$.

Exercise

Explain why these are all false:

a) The complete solution is any linear combination of x_p *and* x_n.

b) A system $Ax = b$ has at most one particular solution.

c) The solution x_p with all free variables zero is the shortest solution (minimum length $\|x\|$). Find a 2 by 2 counterexample.

d) If A is invertible there is no solution x_n in the null space.

Solution

a) The coefficient of x_p must be one.

b) If $x_n \in N(A)$ is the nullspace of $\boldsymbol{A}$ and x_p is one particular solution, then x_p *and* x_n is also a particular solution.

c) If $\boldsymbol{A}$ is a 2 by 2 matrix of rank 1, then the solution to $Ax = b$ form a line parallel to the line that the nullspace. The line $x + y = 1$ gives such an example.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad x_p = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Then $\left\| x_p \right\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{2\frac{1}{4}} = \frac{1}{\sqrt{2}} < 1$ while the particular solutions having some coordinate equal to zero are (1, 0) and (0, 1) and they both have $\left\| x_p \right\| = 1$

d) There is always $x_n = 0$

Exercise

Write down all known relation between r and m and n if $Ax = b$ has

a) No solution for some $\boldsymbol{b}$.

b) Infinitely many solutions for every $\boldsymbol{b}$.

c) Exactly one solution for some $\boldsymbol{b}$, no solution for another $\boldsymbol{b}$.

d) Exactly one solution for every $\boldsymbol{b}$.

Solution

a) The system has less than full row rank: $r < m$.

b) The system has full row rank and less than full column rank: $m = r < n$.

c) The system has full column rank and less than full row rank: $n = r < m$.

d) The system has full row and column rank (it is invertible): $m = r = n$.

Exercise

Find a basis for its row space, find a basis for its column space, and determine its rank

$$a)\ \begin{bmatrix} 0 & 2 & -3 & 4 & 1 & 2 & 1 & 7 \\ 0 & 0 & 3 & -2 & 0 & 4 & -5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 6 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad b)\ \begin{bmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{bmatrix}$$

Solution

a) Row Space: every row

Column Space: $\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\3\\0\\0\\0\end{bmatrix}, \begin{bmatrix}2\\4\\6\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-5\\-2\\2\\0\end{bmatrix}$

Rank = 4

b) $\begin{bmatrix}3 & 2 & -1\\6 & 3 & 5\\-3 & -1 & -6\\0 & -1 & 7\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 0 & \frac{13}{3}\\0 & 1 & -7\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$

Row Space: $\begin{bmatrix}3 & 2 & -1\end{bmatrix}, \begin{bmatrix}6 & 3 & 5\end{bmatrix}$

Column Space: $\begin{bmatrix}3\\6\\-3\\0\end{bmatrix}, \begin{bmatrix}2\\3\\-1\\-1\end{bmatrix}$

Rank = 2

Exercise

Find a basis for the row space, find a basis for the null space, find $\dim RS$, find $\dim NS$, and verify $\dim RS + \dim NS = n$

$$\begin{bmatrix}1 & -2 & 4 & 1\\3 & 1 & -3 & -1\\5 & -3 & 5 & 1\end{bmatrix}$$

Solution

$$\begin{bmatrix}1 & -2 & 4 & 1\\3 & 1 & -3 & -1\\5 & -3 & 5 & 1\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 0 & -\frac{2}{7} & -\frac{1}{7}\\0 & 1 & -\frac{15}{7} & -\frac{4}{7}\\0 & 0 & 0 & 0\end{bmatrix}$$

Row Space: $\begin{bmatrix}1 & -2 & 4 & 1\end{bmatrix}, \begin{bmatrix}3 & 1 & -3 & -1\end{bmatrix}$

Column Space: $\begin{bmatrix}1\\3\\5\end{bmatrix}, \begin{bmatrix}-2\\1\\-3\end{bmatrix}$

$\dim RS = 2$

$\dim NS = 2$

$2 + 2 = 2 \quad \Rightarrow \quad \dim RS + \dim NS = n$

Exercise

Determine if $\boldsymbol{b}$ lies in the column space of the given matrix. If it does, express $\boldsymbol{b}$ as linear combination of the column.

$$\begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

Solution

$$\left[\begin{array}{cc|c} 2 & -3 & 4 \\ -4 & 6 & -6 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$\boldsymbol{b}$ does not lie in the column space

Exercise

Find the transition matrix from B to C and find $[\boldsymbol{x}]_c$

a) $B = \{(3, 1), (-1, -2)\}, \quad C = \{(1, -3), (5, 0)\}, \quad [x]_B = [-1 \quad -2]^T$

b) $B = \{(1, 1, 1), (-2, -1, 0), (2, 1, 2)\}, \quad C = \{(-6, -2, 1), (-1, 1, 5), (-1, -1, 1)\},$

$[\boldsymbol{x}]_B = [-3 \quad 2 \quad 4]^T$

Solution

a) $$\left[\begin{array}{cc|cc} 1 & 5 & 3 & -1 \\ -3 & 0 & 1 & -2 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{array}\right]$$

$$[\boldsymbol{x}]_c = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

b) $$\left[\begin{array}{ccc|ccc} -6 & -1 & -1 & 1 & -2 & 2 \\ -2 & 1 & -1 & 1 & -1 & 1 \\ 1 & 5 & 1 & 1 & 0 & 2 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{13} & \frac{4}{13} & -\frac{6}{13} \\ 0 & 1 & 0 & \frac{4}{13} & -\frac{3}{26} & \frac{11}{26} \\ 0 & 0 & 1 & -\frac{5}{13} & \frac{7}{26} & \frac{9}{26} \end{array}\right]$$

$$[\boldsymbol{x}]_c = \begin{bmatrix} -\frac{2}{13} & \frac{4}{13} & -\frac{6}{13} \\ \frac{4}{13} & -\frac{3}{26} & \frac{11}{26} \\ -\frac{5}{13} & \frac{7}{26} & \frac{9}{26} \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -\frac{10}{13} \\ \frac{17}{13} \\ \frac{35}{13} \end{bmatrix}$$

Exercise

Does A and A^T have the same number of pivots.

Solution

True

The number of pivots of A is its column rank, r. We know that the column rank of A equals the row rank of A, which is the column rank of A^T.

Hence, A^T must have the same number of pivots as A.

Exercise

Let $A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *where* $b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

a) What is the rank of A?

b) What is the dimension of A?

c) What are the pivots variables?

d) What are the free variables?

e) Find the special (homogeneous) solutions.

f) What is the nullspace $N(A)$?

g) Find the particular solution to $Ax = b$

h) Give the complete solution.

Solution

a) Rank(***A***) = 2

b) Dimension of A = 2

c) The pivots variables are: x_1, x_3

d) The free variables are: x_2, x_4

e) Let $x = x_2 s_1 + x_4 s_2$

$$A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{cases} x_1 = -3x_2 - 2x_4 \\ x_3 = \qquad\quad -4x_4 \end{cases}$$

Set $x_2 = 1, \quad x_4 = 0$

The special solution: $s_1 = (-3,\ 1,\ 0,\ 0)$

Set $x_2 = 0, \quad x_4 = 1$;

The special solution: $s_2 = (-2,\ 0,\ -4,\ 1)$

f) The nullspace is the set $\left\{ x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix} \right\}$

$$N(A) = \begin{pmatrix} -3 & -2 \\ 1 & 0 \\ 0 & -4 \\ 0 & 1 \end{pmatrix}$$

g) $x_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

h) $x = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Exercise

Let $A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ *where* $b = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

a) What is the rank of A?

b) What is the dimension of A?

c) What are the pivots variables?

d) What are the free variables?

e) Find the special (homogeneous) solutions.

f) What is the nullspace $N(A)$?

g) Find the particular solution to $Ax = b$

h) Give the complete solution.

Solution

a) Rank(***A***) = 3

b) Dimension of A = 1

c) The pivots variables are: x_1, x_2, x_4

d) The free variables are: x_3

e) Let $x = x_3 s_1$

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{cases} x_1 = -2x_3 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$$

The special solution: $s_1 = (-2,\ 0,\ 1,\ 0)$

f) $N(A) = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

g) $x_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

h) $x = x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Exercise

Let $A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *where* $b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

a) What is the rank of A?
b) What is the dimension of A?
c) What are the pivots variables?
d) What are the free variables?
e) Find the special (homogeneous) solutions.
f) What is the nullspace $N(A)$?
g) Find the particular solution to $Ax = b$
h) Give the complete solution.

Solution

$$\begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ -\frac{1}{3}R_2 \\ \\ \end{matrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -\frac{1}{3} & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & \frac{2}{3} & -4 \\ 0 & 1 & -\frac{1}{3} & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} x_1 = -\frac{2}{3}x_3 + 4x_4 \\ x_2 = \frac{1}{3}x_3 - 4x_4 \end{cases}$$

a) Rank($\boldsymbol{A}$) = 2

b) Dimension of A = 2

c) The pivots variables are: $x_1,\ x_2$

d) The free variables are: $x_3,\ x_4$

e) Let $x = x_3 s_1 + x_4 s_2$

Set $x_3 = 1,\quad x_4 = 0$

The special solution: $s_1 = \left(-\frac{2}{3},\ \frac{1}{3},\ 1,\ 0\right)$

Set $x_3 = 0,\quad x_4 = 1$;

The special solution: $s_2 = (4,\ -4,\ 0,\ 1)$

f) The nullspace is the set $\left\{ x_3 \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 4 \\ -4 \\ 0 \\ 1 \end{pmatrix} \right\}$

$$N(A) = \begin{pmatrix} -\frac{2}{3} & 4 \\ \frac{1}{3} & -4 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

g) $x_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

h) $x = x_3\begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ 1 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} 4 \\ -4 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Exercise

Let $\quad A = \begin{pmatrix} 1 & 0 & 0 & \frac{13}{11} \\ 0 & 1 & 0 & -\frac{17}{11} \\ 0 & 0 & 1 & \frac{6}{11} \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad$ *where* $\; b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

a) What is the rank of A?
b) What is the dimension of A?
c) What are the pivots variables?
d) What are the free variables?
e) Find the special (homogeneous) solutions.
f) What is the nullspace $N(A)$?
g) Find the particular solution to $Ax = b$
h) Give the complete solution.

Solution

a) Rank($\boldsymbol{A}$) = 3

b) Dimension of A = 1

c) The pivots variables are: $x_1,\ x_2,\ x_3$

d) The free variables are: x_4

e) Let $x = x_4 s_1$

$$A = \begin{pmatrix} 1 & 0 & 0 & \frac{13}{11} \\ 0 & 1 & 0 & -\frac{17}{11} \\ 0 & 0 & 1 & \frac{6}{11} \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{cases} x_1 = -\frac{13}{11} \\ x_2 = \frac{17}{11} \\ x_3 = -\frac{6}{11} \end{cases}$$

Set $x_4 = 1$

The special solution: $s_1 = \left(-\frac{13}{11},\ \frac{17}{11},\ -\frac{6}{11},\ 1\right)$

f) $N(A) = \begin{pmatrix} -\frac{13}{11} \\ \frac{17}{11} \\ -\frac{6}{11} \\ 1 \end{pmatrix}$

g) $x_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

h) $x = x_4 \begin{pmatrix} -\frac{13}{11} \\ \frac{17}{11} \\ -\frac{6}{11} \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Exercise

Let $A = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *where* $b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

a) What is the rank of A?

b) What is the dimension of A?

c) What are the pivots variables?

d) What are the free variables?

e) Find the special (homogeneous) solutions.

f) What is the nullspace $N(A)$?

g) Find the particular solution to $Ax = b$

h) Give the complete solution.

Solution

a) Rank($\boldsymbol{A}$) = 3

b) Dimension of A = 1

c) The pivots variables are: x_1, x_2, x_3

d) The free variables are: x_4

e) Let $x = x_4 s_1$

$$A = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} x_1 = \frac{1}{3} \\ x_2 = 1 \\ x_3 = \frac{1}{3} \end{cases}$$

Set $x_4 = 1$

The special solution: $s_1 = \left(\frac{1}{3}, 1, \frac{1}{3}, 1\right)$

f) $N(A) = \begin{pmatrix} \frac{1}{3} \\ 1 \\ \frac{1}{3} \\ 1 \end{pmatrix}$

g) $x_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

h) $x = x_4 \begin{pmatrix} \frac{1}{3} \\ 1 \\ \frac{1}{3} \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Exercise

Let $A = \begin{pmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ *where* $b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

a) What is the rank of A?

b) What is the dimension of A?

c) What are the pivots variables?

d) What are the free variables?

e) Find the special (homogeneous) solutions.

f) What is the nullspace $N(A)$?

g) Find the particular solution to $Ax = b$

h) Give the complete solution.

Solution

a) Rank($\boldsymbol{A}$) = 3

b) Dimension of A = 2

c) The pivots variables are: x_1, x_3, x_5

d) The free variables are: x_2, x_4

e) Let $x = x_2 s_1 + x_4 s_2$

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} x_1 = -2x_2 + x_4 \\ x_3 = \quad -x_4 \\ x_5 = 0 \end{cases}$$

Set $x_2 = 1 \quad x_4 = 0$

The special solution: $s_1 = (-2,\ 1,\ 0,\ 0,\ 0)$

Set $x_2 = 0 \quad x_4 = 1$

The special solution: $s_2 = (1,\ 0,\ -1,\ 1,\ 0)$

f) $$N(A) = \begin{pmatrix} -2 & 1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

g) $$x_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

h) $$x = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$