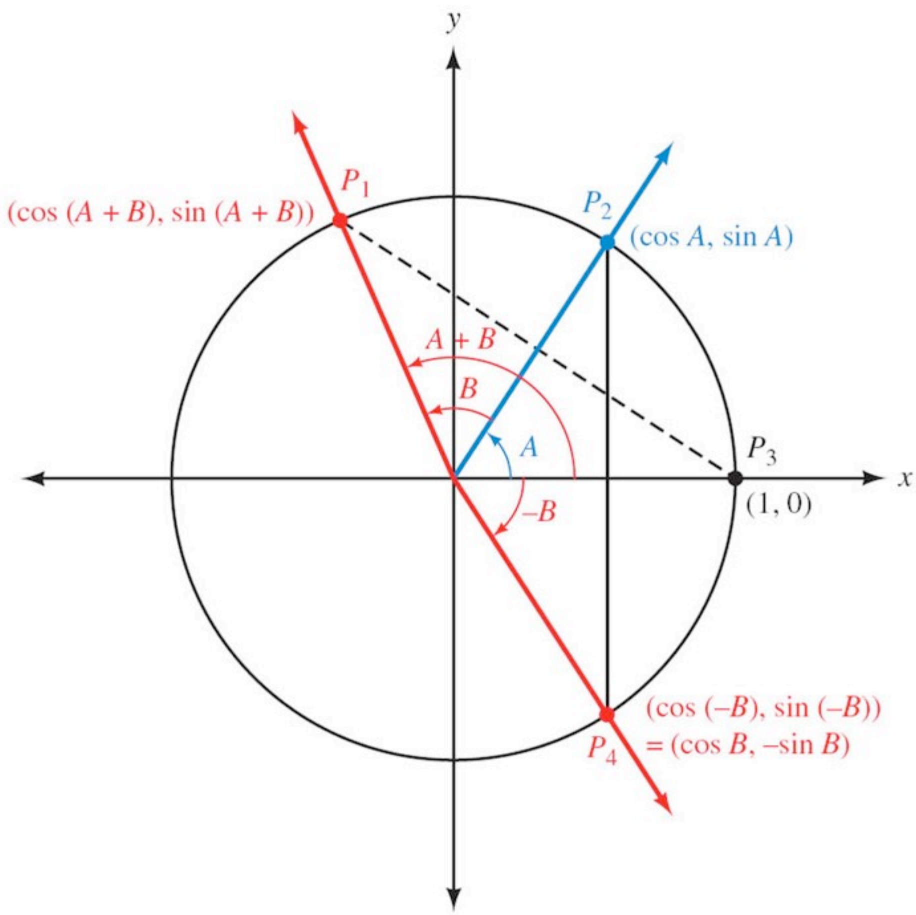
Section 7.2 – Sum and Difference Formulas



$$P_1P_3 = P_2P_4$$

$$\left(P_1P_3\right)^2 = \left(P_2P_4\right)^2$$

Distance between points

$$\left[\cos(A+B)-1\right]^2 + \left[\sin(A+B)-0\right]^2 = \left(\cos A - \cos B\right)^2 + \left(\sin A + \sin B\right)^2$$

$$\cos^2(A+B) - 2\cos(A+B) + 1 + \sin^2(A+B) = \left(\cos A - \cos B\right)^2 + \left(\sin A + \sin B\right)^2$$

$$1 - 2\cos(A+B) + 1 = \cos^2 A - 2\cos B\cos A + \cos^2 B + \sin^2 A + 2\sin B\sin A + \sin^2 B$$

$$2 - 2\cos(A+B) = \cos^2 A + \sin^2 A + \cos^2 B + \sin^2 B - 2\cos B\cos A + 2\sin B\sin A$$

$$2 - 2\cos(A+B) = 1 + 1 - 2\cos B\cos A + 2\sin B\sin A$$

$$2 - 2\cos(A+B) = 2 - 2\cos B\cos A + 2\sin B\sin A$$

$$-2\cos(A+B) = -2\cos B\cos A + 2\sin B\sin A$$

$$\cos(A+B) = \cos B\cos A - \sin B\sin A$$

$$\boxed{\begin{aligned} \cos(A+B) &= \cos A\cos B - \sin A\sin B \\ \cos(A-B) &= \cos A\cos B + \sin A\sin B \\ \sin(A+B) &= \sin A\cos B + \cos A\sin B \\ \sin(A-B) &= \sin A\cos B - \cos A\sin B \end{aligned}}$$

Example

Find the exact value for $\cos 75°$

Solution

$$\begin{aligned} \cos 75° &= \cos(45° + 30°) \\ &= \cos 45° \cos 30° - \sin 45° \sin 30° \\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} \\ &= \frac{\sqrt{6}-\sqrt{2}}{4} \end{aligned}$$

Example

Show that $\cos(x + 2\pi) = \cos x$

Solution

$$\begin{aligned} \cos(x+2\pi) &= \cos x \cos 2\pi - \sin x \sin 2\pi \\ &= \cos x \cdot (1) - \sin x \cdot (0) \\ &= \cos x \quad \checkmark \end{aligned}$$

Example

Simplify: $\cos 3x \cos 2x - \sin 3x \sin 2x$

Solution

$$\begin{aligned} \cos 3x \cos 2x - \sin 3x \sin 2x &= \cos(3x + 2x) \\ &= \cos 5x \end{aligned}$$

Example

Show that $\cos(90° - A) = \sin A$

Solution

$$\cos(90° - A) = \cos 90° \cos A + \sin 90° \sin A$$
$$= 0 \cdot \cos A + 1 \cdot \sin A$$
$$= \sin A \quad \checkmark$$

Example

Find the exact value of $\sin\frac{\pi}{12}$

Solution

$$\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$
$$= \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}$$
$$= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} - \frac{1}{2}\frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

Example

Find the exact value of $\cos 15°$

Solution

$$\cos 15° = \cos\left(45° - 30°\right)$$
$$= \cos\left(45°\right)\cos\left(30°\right) + \sin\left(45°\right)\sin\left(30°\right)$$
$$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2}$$
$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$
$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

Example

If $\sin A = \frac{3}{5}$ with A in QI, and $\cos B = -\frac{5}{13}$ with B in QIII, find $\sin(A+B)$, $\cos(A+B)$, and $\tan(A+B)$

Solution

$\sin A = \frac{3}{5} \to A \in QI$

$\cos A = \frac{4}{5}$

$\cos B = -\frac{5}{13} \to B \in QIII$

$\sin B = -\frac{12}{13}$

$$\begin{aligned}\sin(A+B) &= \sin A\cos B + \cos A\sin B \\ &= \frac{3}{5}\left(-\frac{5}{13}\right) + \frac{4}{5}\left(-\frac{12}{13}\right) \\ &= -\frac{15}{65} - \frac{48}{65} \\ &= -\frac{63}{65}\end{aligned}$$

$$\begin{aligned}\cos(A+B) &= \cos A\cos B - \sin A\sin B \\ &= \frac{4}{5}\left(-\frac{5}{13}\right) - \frac{3}{5}\left(-\frac{12}{13}\right) \\ &= -\frac{20}{65} + \frac{36}{65} \\ &= \frac{16}{65}\end{aligned}$$

$$\begin{aligned}\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} \\ &= -\frac{63}{16}\end{aligned}$$

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

$$= \frac{\frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B - \sin A\sin B}{\cos A\cos B}}$$

$$= \frac{\frac{\sin A\cos B}{\cos A\cos B} + \frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B} - \frac{\sin A\sin B}{\cos A\cos B}}$$

$$= \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A}{\cos A}\frac{\sin B}{\cos B}}$$

$$= \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\boxed{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}}$$

$$\boxed{\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}}$$

Example

If $\sin A = \frac{3}{5}$ with A in Q*I*, and $\cos B = -\frac{5}{13}$ with B in Q*III*, find $\tan(A+B)$

Solution

$$\tan A = \frac{3/5}{4/5} \qquad \tan A = \frac{\sin A}{\cos A}$$

$$= \frac{3}{4}$$

$$\tan B = \frac{-12/13}{-5/13} \qquad \tan B = \frac{\sin B}{\cos B}$$

$$= \frac{12}{5}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$= \frac{\frac{3}{4} + \frac{12}{5}}{1 - \frac{3}{4}\frac{12}{5}}$$

$$=\frac{\frac{15}{20}+\frac{48}{20}}{1-\frac{36}{20}}$$

$$=\frac{\frac{63}{20}}{-\frac{16}{20}}$$

$$\boxed{=-\frac{63}{16}}$$

Example

Establish the identity: $\dfrac{\cos(x-y)}{\sin x \sin y}=\cot x \cot y+1$

Solution

$$\frac{\cos(x-y)}{\sin x \sin y}=\frac{\cos x\cos y+\sin x\sin y}{\sin x \sin y}$$

$$=\frac{\cos x\cos y}{\sin x \sin y}+\frac{\sin x\sin y}{\sin x \sin y}$$

$$\boxed{=\cot x\cot y+1} \quad \checkmark$$

Example

Establish the identity: $\cot(x+y)=\dfrac{\cot x\cot y-1}{\cot x+\cot y}$

Solution

$$\cot(x+y)=\frac{\cos(x+y)}{\sin(x+y)}$$

$$=\frac{\cos x\cos y-\sin x\sin y}{\sin x\cos y+\cos x\sin y}$$

$$=\frac{\frac{\cos x\cos y}{\sin x\sin y}-\frac{\sin x\sin y}{\sin x\sin y}}{\frac{\sin x\cos y}{\sin x\sin y}+\frac{\cos x\sin y}{\sin x\sin y}}$$

$$\boxed{=\frac{\cot x\cot y-1}{\cot x+\cot y}} \quad \checkmark$$

Example

Establish the identity: $\sec(x-y)=\dfrac{\cos x\cos y-\sin x\sin y}{\cos^2 x-\sin^2 y}$

Solution

$$\sec(x-y)=\frac{1}{\cos(x-y)}\frac{\cos(x+y)}{\cos(x+y)}$$

$$=\frac{\cos x\cos y-\sin x\sin y}{(\cos x\cos y+\sin x\sin y)(\cos x\cos y-\sin x\sin y)}$$

$$=\frac{\cos x\cos y-\sin x\sin y}{\cos^2 x\cos^2 y-\sin^2 x\sin^2 y}$$

$$=\frac{\cos x\cos y-\sin x\sin y}{\cos^2 x\cos^2 y-\sin^2 x\sin^2 y}$$

$$=\frac{\cos x\cos y-\sin x\sin y}{\cos^2 x\left(1-\sin^2 y\right)-\left(1-\cos^2 x\right)\sin^2 y}$$

$$=\frac{\cos x\cos y-\sin x\sin y}{\cos^2 x-\cos^2 x\sin^2 y-\sin^2 y+\cos^2 x\sin^2 y}$$

$$=\frac{\cos x\cos y-\sin x\sin y}{\cos^2 x-\sin^2 y}\quad \checkmark$$

Exercises Section 7.2 – Sum and Difference Formulas

1. Write the expression as a single trigonometric function $\sin 8x \cos x - \cos 8x \sin x$

2. Show that $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$

3. If $\sin A = \frac{4}{5}\ (A \in QII)$, and $\cos B = -\frac{5}{13}\ (B \in QIII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$
d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

4. If $\sin A = \frac{3}{5}\ (A \in QII)$, and $\cos B = -\frac{12}{13}\ (B \in QIII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$
d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

5. If $\sin A = \frac{1}{\sqrt{5}}\ (A \in QI)$, and $\tan B = \frac{3}{4}\ (B \in QI)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$
d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

6. If $\sin A = \frac{3}{5}\ (A \in QII)$, and $\cos B = \frac{12}{13}\ (B \in QIV)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$
d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

7. If $\sin A = \frac{7}{25}\ (A \in QII)$, and $\cos B = -\frac{8}{17}\ (B \in QIII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$
d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

8. If $\cos A = -\frac{4}{5}\ (A \in QII)$, and $\sin B = \frac{24}{25}\ (B \in QII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$
d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

9. If $\cos A = \frac{15}{17}\ (A \in QI)$, and $\cos B = -\frac{12}{13}\ (B \in QII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$
d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

10. If $\sin A = -\frac{3}{5}\ (A \in QIV)$, and $\sin B = \frac{7}{25}\ (B \in QII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$
d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

11. If $\sec A = \sqrt{5}$ with A in Q*I*, and $\sec B = \sqrt{10}$ with B in Q*I*, find $\sec(A+B)$

(12–30) Prove the identity

12. $\dfrac{\sin(A-B)}{\cos A \cos B} = \tan A - \tan B$

13. $\sec(A+B) = \dfrac{\cos(A-B)}{\cos^2 A - \sin^2 B}$

14. $\dfrac{\cos 4\alpha}{\sin \alpha} - \dfrac{\sin 4\alpha}{\cos \alpha} = \dfrac{\cos 5\alpha}{\sin \alpha \cos \alpha}$

15. $\dfrac{\cos(x+y)}{\cos(x-y)} = \dfrac{\cot y - \tan x}{\cot y + \tan x}$

16. $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\cot y + \cot x}{\cot y - \cot x}$

17. $\dfrac{\sin(x-y)}{\sin x \cos y} = 1 - \cot x \tan y$

18. $\dfrac{\sin(x-y)}{\sin x \sin y} = \cot y - \cot x$

19. $\dfrac{\cos(x+y)}{\cos x \sin y} = \cot y - \tan x$

20. $\dfrac{\sin(x+y)}{\cos(x-y)} = \dfrac{1+\cot x \tan y}{\cot x + \tan y}$

21. $\sin\left(\frac{\pi}{4}+x\right) + \sin\left(\frac{\pi}{4}-x\right) = \sqrt{2}\cos x$

22. $\cos(A+B) + \cos(A-B) = 2\cos A \cos B$

23. $\sin(x-y) - \sin(y-x) = 2\sin x \cos y - 2\cos x \sin y$

24. $\cos(x-y) + \cos(y-x) = 2\cos x \cos y + 2\sin x \sin y$

25. $\tan(x+y)\tan(x-y) = \dfrac{\tan^2 x - \tan^2 y}{1-\tan^2 x \tan^2 y}$

26. $\dfrac{\cos(\alpha-\beta)}{\sin(\alpha+\beta)} = \dfrac{1+\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$

27. $\sec(x+y) = \dfrac{\cos x \cos y + \sin x \sin y}{\cos^2 x - \sin^2 y}$

28. $\csc(x-y) = \dfrac{\sin x \cos y + \cos x \sin y}{\sin^2 x - \sin^2 y}$

29. $\tan(x+y) + \tan(x-y) = \dfrac{2\tan x}{\cos^2 y\left(1-\tan^2 x \tan^2 y\right)}$

30. $\dfrac{\cos(x-y)}{\cos(x+y)} = \dfrac{1+\tan x \tan y}{1-\tan x \tan y}$

31. Common household current is called ***alternating current*** because the current alternates direction within the wires. The voltage V in a typical 115-*volt* outlet can be expressed by the function $V(t) = 163\sin\omega t$ where ω is the angular speed (in *radians* per *second*) of the rotating generator at the electrical plant, and t is time measured in seconds.

a) It is essential for electric generators to rotate at precisely 60 cycles per second so household appliances and computers will function properly. Determine ω for these electric generators.

b) Determine a value of ϕ so that the graph of $V(t) = 163\cos(\omega t - \phi)$ is the same as the graph of $V(t) = 163\sin\omega t$