Solution   Section 4.6 – Exponential Growth and Decay

Exercise

Suppose that \$10,000 is invested at interest rate of 5.4% per year, compounded continuously.

a) Find the exponential growth function

b) What will the balance be after, 1 yr 10 yrs?

c) What will the balance be after, 1 yr 10 yrs?

Solution

a) Find the exponential growth function

$$P(t) = 10000e^{0.054t}$$

b) What will the balance be after, 1 yr 10 yr?

$$P(t=1) = 10000e^{0.054(1)} \approx \$10,555$$

$$P(t=10) = 10000e^{0.054(10)} \approx \$17,160$$

c) What is the doubling time?

$$T = \frac{\ln 2}{k} = \frac{\ln 2}{0.054} \approx 12.8\ yr$$

Exercise

In 1990, the population of Africa was 643 million and by 2000 it had grown to 813 million

a. Use the exponential growth function $A(t) = A_0 e^{kt}$, in which t is the number of years after 1990, to find the exponential growth function that models data

b. By which year will Africa's population reach 2000 million, or two billion?

Solution

a. $A(t) = A_0 e^{kt}$ From 1990 to 2000, is 10 years

$$813 = 643e^{k(10)}$$

$$\frac{813}{643} = e^{10k}$$

$$\ln\frac{813}{643} = \ln e^{10k}$$

$$\ln\frac{813}{643} = 10k$$

$$\frac{1}{10}\ln\frac{813}{643} = k \qquad k \approx 0.023 \qquad \Rightarrow A(t) = 643e^{0.023t}$$

b. $2000 = 643e^{0.023t}$

$$\frac{2000}{643} = e^{0.023t}$$

$$\ln\frac{2000}{643} = \ln e^{0.023t}$$

$$\ln\frac{2000}{643} = 0.023t$$

$$\frac{\ln\frac{2000}{643}}{0.023} = t \qquad \Rightarrow t \approx 49 \quad \rightarrow Year:\ 2039$$

Exercise

The radioactive element carbon-14 has a half-life of 5750 yr. The percentage of carbon-14 present in the remains of organic matter can be used to determine the age of that organic matter. Archaeologists discovered that the linen wrapping from one of the Dead Sea Scrolls had lost 22.3% of its carbon-14 at the time it was found. How old was the linen wrapping?

Solution

When t = 5750 (half-life) → P(t) will be half P_0 → $P(t) = \frac{1}{2}P_0$

$$P(t) = P_0 e^{-kt}$$

$$\frac{1}{2}P_0 = P_0 e^{-k(5750)}$$

$$\frac{1}{2} = e^{-k(5750)}$$

$$\ln\frac{1}{2} = \ln e^{-k(5750)}$$

$$\ln\frac{1}{2} = -k(5750)$$

$$k = -\frac{\ln\frac{1}{2}}{5750} \approx 0.00012$$

$$P(t) = P_0 e^{-(0.00012)t}$$

Lost 22.3% ⇒ 100 – 22.3 = 77.7% left from it is original.

$$0.777P_0 = P_0 e^{-0.00012t}$$

$$0.777 = e^{-0.00012t}$$

$$ln 0.777 = ln e^{-0.00012t}$$

$$ln 0.777 = -0.00012t$$

$$-0.00012t = ln 0.777$$

$$t = \frac{\ln 0.777}{-0.00012} \approx 2103$$

Exercise

Suppose that \$2000 is invested at interest rate $\boldsymbol{k}$, compounded continuously, and grows to \$2983.65 in 5 yrs.

a) What is the interest rate?

b) Find the exponential growth function

c) What will the balance be after 10 yrs?

d) After how long will the \$2000 have doubled?

Solution

a) $P(t) = P_0 e^{kt}$

$$P(t=5) = P_0 e^{k5} = 2983.65$$

$$2000e^{k5} = 2983.65$$

$$e^{k5} = \frac{2983.65}{2000}$$

$$\ln e^{k5} = \ln\left(\frac{2983.65}{2000}\right)$$

$$5k \ln e = \ln\left(\frac{2983.65}{2000}\right)$$

$$k = \frac{1}{5}\ln\left(\frac{2983.65}{2000}\right) \approx 0.08$$

or k = 8%

b) $P(t) = 2000e^{0.08t}$

c) $P(t=10) = 2000e^{0.08(10)} \approx \$4451.08$

d) $T = \frac{\ln 2}{k} = \frac{\ln 2}{0.08} \approx 8.7\ yr$

Exercise

In 2005, the population of China was about 1.306 billion, and the exponential growth rate was 0.6% per year.

a) Find the exponential growth function
b) Estimate the population in 2008
c) After how long will the population be double what it was in 2005?

Solution

a) In 2005 $\Rightarrow$ t = 0

$$k = \frac{0.6}{100} = 0.006$$

P_0: 1.306

$$P(t) = 1.306e^{0.006t}$$

b) $P(t = 3) = 1.306e^{0.006(3)} \approx 1.33$

c) $2(1.306) = 1.306e^{0.006t}$

$$2 = e^{0.006t}$$

$$e^{0.006t} = 2$$

$$\ln e^{0.006t} = \ln 2$$

$$0.006t = \ln 2$$

$$t = \frac{\ln 2}{0.006} \approx 116$$

Exercise

How long will it take for the money in an account that is compounded continuously at 3% interest to double?

Solution

$$T = \frac{\ln 2}{k}$$

$$= \frac{\ln 2}{0.03}$$

$$\approx 23\ yr$$

Exercise

If 600 g of radioactive substance are present initially and 3 *yr* later only 300 g remain, how much of the substance will be present after 6 *yr*?

Solution

$$y(t) = y_0 e^{kt}$$

$$y(t) = 600e^{kt}$$

When $t = 3 \to y = 300$

$$300 = 600e^{k(3)}$$

$$\frac{300}{600} = e^{3k}$$

$$\ln\frac{300}{600} = \ln e^{3k} \qquad \ln e = 1$$

$$3k = \ln\frac{300}{600}$$

$$k = \frac{1}{3}\ln\frac{300}{600} \qquad \ln(.5)/3$$

$$\approx -.231$$

$$y(t) = 600e^{-.231t}$$

$$y(6) = 600e^{-.231(6)} \qquad 600e \wedge (-.231 * 6)$$

$$\approx 150\ g$$

Exercise

The population of an endangered species of bird was 4200 in 1990. Thirteen years later, in 2003, the bird population declined to 3000. The population of the birds is decreasing exponentially according to the function $A(t) = 4200e^{kt}$ where *A* is the bird population *t* years after 1990. Use this information to find the value of ***k***.

Solution

$$3000 = 4200e^{k(13)}$$

$$\frac{3000}{4200} = e^{13k}$$

$$\ln\frac{3000}{4200} = \ln e^{13k}$$

$$\ln\frac{3000}{4200} = 13k$$

$$k = \frac{\ln\left(\frac{3000}{4200}\right)}{13} \approx -0.26$$