Solution Section 4.2 – Line Integrals

Exercise

Evaluate $\int_C (x+y)\,ds$ where C is the straight-line segment $x=t,\ y=(1-t),\ z=0$ from (0, 1, 0) to $(1,\ 0,\ 0)$.

Solution

$$r(t) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = t\boldsymbol{i} + (1-t)\boldsymbol{j}$$

$$\frac{dr}{dt} = \boldsymbol{i} - \boldsymbol{j} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right| = \sqrt{1+1} = \sqrt{2}$$

$$\begin{aligned} x &= t \\ y &= 1-t \end{aligned} \quad \rightarrow \quad x + y = t + 1 - t = 1$$

$$\begin{aligned} \int_C f(x,y,z) &= \int_0^1 f(t,1-t,0)\left|\frac{dr}{dt}\right| dt \\ &= \int_0^1 (1)\sqrt{2}\,dt \\ &= \sqrt{2}\,[t]_0^1 \\ &= \boxed{\sqrt{2}} \end{aligned}$$

Exercise

Evaluate $\int_C (x-y+z-2)\,ds$ where C is the straight-line segment $x=t,\ y=(1-t),\ z=1$ from $(0,\ 1,\ 1)$ to $(1,\ 0,\ 1)$.

Solution

$$r(t) = t\boldsymbol{i} + (1-t)\boldsymbol{j} + \boldsymbol{k} \quad 0 \le t \le 1$$

$$\frac{dr}{dt} = \boldsymbol{i} - \boldsymbol{j} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right| = \sqrt{1+1} = \sqrt{2}$$

$$\begin{aligned} x &= t \\ y &= 1-t \\ z &= 1 \end{aligned} \quad \rightarrow \quad x - y + z - 2 = t - 1 + t + 1 - 2 = 2t - 2$$

$$\int_C f(x,y,z) = \int_0^1 (2t-2)\sqrt{2}dt$$

$$= \sqrt{2}\left[t^2 - 2t\right]_0^1$$

$$= \sqrt{2}(1-2)$$

$$\underline{\boxed{= -\sqrt{2}}}$$

Exercise

Evaluate $\int_C (xy + y + z)ds$ along the curve $\boldsymbol{r}(t) = 2t\boldsymbol{i} + t\boldsymbol{j} + (2-2t)\boldsymbol{k}, \quad 0 \le t \le 1$

Solution

$$\boldsymbol{r}(t) = 2t\boldsymbol{i} + t\boldsymbol{j} + (2-2t)\boldsymbol{k}, \quad 0 \le t \le 1$$

$$\frac{dr}{dt} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right| = \sqrt{4+1+4} = 3$$

$$\begin{aligned} x &= 2t \\ y &= t \quad \rightarrow \quad xy + y + z = 2t^2 + t + 2 - 2t = 2t^2 - t + 2 \\ z &= 2 - 2t \end{aligned}$$

$$\int_C (xy + y + z)ds = \int_0^1 \left(2t^2 - t + 2\right)(3)dt$$

$$= 3\left[\frac{2}{3}t^3 - \frac{1}{2}t^2 + 2t\right]_0^1$$

$$= 3\left(\frac{2}{3} - \frac{1}{2} + 2\right)$$

$$= 3\left(\frac{13}{6}\right)$$

$$\underline{\boxed{= \frac{13}{2}}}$$

Exercise

Find the integral of $f(x,y,z) = x + y + z$ over the straight line segment from $(1,\ 2,\ 3)$ to $(0,\ -1,\ 1)$

Solution

$$\begin{aligned}\boldsymbol{r}(t) &= (\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) + t\big((0-1)\boldsymbol{i} + (-1-2)\boldsymbol{j} + (1-3)\boldsymbol{k}\big)\\ &= (\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) + t(-\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k})\\ &= (1-t)\boldsymbol{i} + (2-3t)\boldsymbol{j} + (3-2t)\boldsymbol{k}, \quad 0 \le t \le 1\end{aligned}$$

$$\frac{dr}{dt} = -\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right| = \sqrt{1+9+4} = \sqrt{14}$$

$$\begin{aligned}&x = 1-t\\ &y = 2-3t \quad \to \quad x+y+z = 1-t+2-3t+3-2t = 6-6t\\ &z = 3-2t\end{aligned}$$

$$\begin{aligned}\int_C (x+y+z)\,ds &= \int_0^1 (6-6t)\left(\sqrt{14}\right)dt\\ &= \sqrt{14}\left[6t - 3t^2\right]_0^1\\ &= 3\sqrt{14}\end{aligned}$$

Exercise

Find the integral of $f(x,y,z) = \dfrac{\sqrt{3}}{x^2+y^2+z^2}$ over the curve $\boldsymbol{r}(t) = t\boldsymbol{i} + t\boldsymbol{j} + t\boldsymbol{k}, \quad 1 \le t \le \infty$

Solution

$$\boldsymbol{r}(t) = t\boldsymbol{i} + t\boldsymbol{j} + t\boldsymbol{k}, \quad 1 \le t \le \infty$$

$$\frac{dr}{dt} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right| = \sqrt{1+1+1} = \sqrt{3}$$

$$x^2+y^2+z^2 = t^2+t^2+t^2 = 3t^2$$

$$\begin{aligned}\int_C \frac{\sqrt{3}}{x^2+y^2+z^2}\,ds &= \int_1^\infty \frac{\sqrt{3}}{3t^2}\left(\sqrt{3}\right)dt\\ &= \left[-\frac{1}{t}\right]_1^\infty\\ &= -\left(\frac{1}{\infty} - 1\right)\\ &= 1\end{aligned}$$

Exercise

Evaluate $\int_C x\,ds$ where C is

a) The straight-line segment $x=t,\ \ y=\frac{t}{2}$, from (0, 0) to (4, 2).

b) The parabolic curve $x=t,\ \ y=t^2$, from (0, 0) to (2, 4).

Solution

a) $x=t \ \rightarrow\ \begin{cases} x=0 & t=0 \\ x=4 & t=4 \end{cases} \qquad t=2y \ \rightarrow\ \begin{cases} y=0 & t=0 \\ y=2 & t=4 \end{cases}$

$$\boldsymbol{r}(t)=t\boldsymbol{i}+\frac{t}{2}\boldsymbol{j},\quad 0\le t\le 4$$

$$\frac{dr}{dt}=\boldsymbol{i}+\frac{1}{2}\boldsymbol{j} \ \Rightarrow\ \left|\frac{dr}{dt}\right|=\sqrt{1+\frac{1}{4}}=\frac{\sqrt{5}}{2}$$

$$\int_C x\,ds=\int_0^4 t\frac{\sqrt{5}}{2}dt$$

$$=\frac{\sqrt{5}}{2}\left[\frac{1}{2}t^2\right]_0^4$$

$$=4\sqrt{5}$$

b) $x=t \ \rightarrow\ \begin{cases} x=0 & t=0 \\ x=2 & t=2 \end{cases} \qquad t=\sqrt{y} \ \rightarrow\ \begin{cases} y=0 & t=0 \\ y=4 & t=2 \end{cases}$

$$\boldsymbol{r}(t)=t\boldsymbol{i}+t^2\boldsymbol{j},\quad 0\le t\le 2$$

$$\frac{dr}{dt}=\boldsymbol{i}+2t\boldsymbol{j} \ \Rightarrow\ \left|\frac{dr}{dt}\right|=\sqrt{1+4t^2}$$

$$\int_C x\,ds=\int_0^2 t\sqrt{1+4t^2}\,dt \qquad\qquad d\left(1+4t^2\right)=8tdt$$

$$=\frac{1}{8}\int_0^2\left(1+4t^2\right)^{1/2}d\left(1+4t^2\right)$$

$$=\frac{1}{8}\left[\frac{2}{3}\left(1+4t^2\right)^{3/2}\right]_0^2$$

$$=\frac{1}{12}\left[(17)^{3/2}-1\right]$$

$$=\frac{1}{12}\left(17\sqrt{17}-1\right)$$

Exercise

Evaluate $\int_C \sqrt{x+2y}\ ds$ where C is

a) The straight-line segment $x=t,\ \ y=4t$, from (0, 0) to (1, 4).

b) $C_1 \cup C_2$: C_1 is the line segment (0, 0) to (1, 0) and C_2 is the line segment (1, 0) to (1, 2).

Solution

a) $x=t \quad \rightarrow \quad \begin{cases} x=0 & t=0 \\ x=1 & t=1 \end{cases} \qquad t=\frac{y}{4} \quad \rightarrow \quad \begin{cases} y=0 & t=0 \\ y=4 & t=1 \end{cases}$

$\boldsymbol{r}(t)=t\boldsymbol{i}+4t\boldsymbol{j}, \quad 0\le t\le 1$

$\frac{dr}{dt}=\boldsymbol{i}+4\boldsymbol{j} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right|=\sqrt{1+16}=\sqrt{17}$

$$\int_C \sqrt{x+2y}\ ds=\int_0^1 \sqrt{t+8t}\left(\sqrt{17}\right)dt$$

$$=\sqrt{17}\int_0^1 \sqrt{9t}\,dt$$

$$=3\sqrt{17}\left[\frac{2}{3}t^{3/2}\right]_0^1$$

$$=2\sqrt{17}$$

b) $C_1:\ \ \boldsymbol{r}(t)=(0\boldsymbol{i}+0\boldsymbol{j})+t(\boldsymbol{i}+0\boldsymbol{j})=t\boldsymbol{i} \quad 0\le t\le 1$

$\frac{dr}{dt}=\boldsymbol{i} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right|=1$

$C_2:\ \ \boldsymbol{r}(t)=(1\boldsymbol{i}+0\boldsymbol{j})+t\left((1-1)\boldsymbol{i}+(2-0)\boldsymbol{j}\right)$

$=\boldsymbol{i}+2t\boldsymbol{j} \qquad\qquad 0\le t\le 2$

$\frac{dr}{dt}=2\boldsymbol{j} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right|=2$

$$\int_C \sqrt{x+2y}\ ds=\int_0^1 \sqrt{t}\,(1)\,dt+\int_0^2 \sqrt{1+4t}\,(2)\,dt$$

$$=\left[\frac{2}{3}t^{3/2}\right]_0^1+\frac{1}{2}\int_0^2 (1+4t)^{1/2}\ d(1+4t)$$

$$=\frac{2}{3}+\frac{1}{3}\left[(1+4t)^{3/2}\right]_0^2$$

$$=\frac{2}{3}+\frac{1}{3}\left[(9)^{3/2}-1\right]$$

$$=\frac{2}{3}+\frac{1}{3}(26)$$

$$=\frac{28}{3}$$

Exercise

Find the line integral of $f(x,y)=\dfrac{\sqrt{y}}{x}$ along the curve $r(t)=t^3\boldsymbol{i}+t^4\boldsymbol{j},\quad \frac{1}{2}\le t\le 1$

Solution

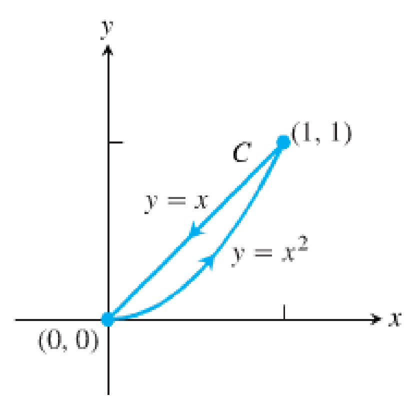
$$r(t)=t^3\boldsymbol{i}+t^4\boldsymbol{j},\quad \frac{1}{2}\le t\le 1$$

$$\frac{dr}{dt}=3t^2\boldsymbol{i}+4t^3\boldsymbol{j} \quad\Rightarrow\quad \left|\frac{dr}{dt}\right|=\sqrt{9t^4+16t^6}=t^2\sqrt{9+16t^2}$$

$$\begin{aligned}\int_C \frac{\sqrt{y}}{x}\,ds &= \int_{1/2}^{1}\frac{\sqrt{t^4}}{t^3}\left(t^2\sqrt{9+16t^2}\right)dt\\
&=\int_{1/2}^{1} t\left(9+16t^2\right)^{1/2}dt \qquad\qquad d\left(9+16t^2\right)=32tdt\\
&=\frac{1}{32}\int_{1/2}^{1}\left(9+16t^2\right)^{1/2}d\left(9+16t^2\right)\\
&=\frac{1}{32}\left(\frac{2}{3}\right)\left[\left(9+16t^2\right)^{3/2}\right]_{1/2}^{1}\\
&=\frac{1}{48}\left[(25)^{3/2}-(13)^{3/2}\right]\\
&=\frac{1}{48}\left(125-13\sqrt{13}\right)\end{aligned}$$

Exercise

Evaluate $\displaystyle\int_C\left(x+\sqrt{y}\right)ds$ where C is



Solution

$$C_1:\quad \boldsymbol{r}(t)=t\boldsymbol{i}+t^2\boldsymbol{j}\quad 0\le t\le 1$$

$$\frac{dr}{dt}=\boldsymbol{i}+2t\boldsymbol{j} \quad\Rightarrow\quad \left|\frac{dr}{dt}\right|=\sqrt{1+4t^2}$$

$$\begin{aligned}C_2:\quad \boldsymbol{r}(t)&=(1\boldsymbol{i}+1\boldsymbol{j})+t(-\boldsymbol{i}-\boldsymbol{j})\\
&=(1-t)\boldsymbol{i}+(1-t)\boldsymbol{j}\qquad 0\le t\le 1\end{aligned}$$

$$\frac{dr}{dt}=-\boldsymbol{i}-\boldsymbol{j} \quad\Rightarrow\quad \left|\frac{dr}{dt}\right|=\sqrt{2}$$

$$\int_C \left(x+\sqrt{y}\right)\,ds = \int_0^1 \left(t+\sqrt{t^2}\right)\left(\sqrt{1+4t^2}\right)dt + \int_0^1 \left(1-t+\sqrt{1-t}\right)\left(\sqrt{2}\right)dt$$

$$= \int_0^1 2t\left(\sqrt{1+4t^2}\right)dt - \sqrt{2}\int_0^1 \left((1-t)+\sqrt{1-t}\right)d(1-t)$$

$$= \frac{1}{4}\int_0^1 \left(1+4t^2\right)^{1/2} d\left(1+4t^2\right) - \sqrt{2}\left[\frac{1}{2}(1-t)^2+\frac{2}{3}(1-t)^{3/2}\right]_0^1$$

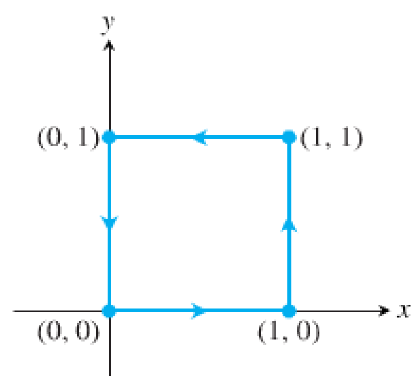
$$= \frac{1}{6}\left[\left(1+4t^2\right)^{3/2}\right]_0^1 - \sqrt{2}\left[-\frac{1}{2}-\frac{2}{3}\right]$$

$$= \frac{1}{6}\left[(5)^{3/2}-1\right]+\frac{7\sqrt{2}}{6}$$

$$= \boxed{\frac{5\sqrt{5}-1+7\sqrt{2}}{6}}$$

Exercise

Evaluate $\int_C \frac{1}{x^2+y^2+1}\,ds$ where C is



Solution

$C_1: \quad \boldsymbol{r}(t) = t\boldsymbol{i} \quad 0 \le t \le 1$

$$\frac{dr}{dt} = \boldsymbol{i} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right| = 1$$

$C_2: \quad \boldsymbol{r}(t) = \boldsymbol{i} + t\boldsymbol{j} \qquad 0 \le t \le 1$

$$\frac{dr}{dt} = \boldsymbol{j} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right| = 1$$

$C_3: \quad \boldsymbol{r}(t) = (1-t)\boldsymbol{i} + \boldsymbol{j} \quad 0 \le t \le 1$

$$\frac{dr}{dt} = -\boldsymbol{i} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right| = 1$$

$C_4: \quad \boldsymbol{r}(t) = (1-t)\boldsymbol{j} \qquad 0 \le t \le 1$

$$\frac{dr}{dt} = -\boldsymbol{j} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right| = 1$$

$$\int_C \frac{1}{x^2+y^2+1}\,ds = \int_0^1 \frac{1}{t^2+1}(1)\,dt + \int_0^1 \frac{1}{1+t^2+1}(1)\,dt$$

$$+\int_0^1 \frac{1}{(1-t)^2+1+1}(1)\,dt + \int_0^1 \frac{1}{(1-t)^2+1}(1)\,dt$$

$$= \int_0^1 \frac{1}{t^2+1}\,dt + \int_0^1 \frac{1}{t^2+2}\,dt - \int_0^1 \frac{1}{(1-t)^2+2}\,d(1-t) - \int_0^1 \frac{1}{(1-t)^2+1}\,d(1-t)$$

$$= \left[\tan^{-1} t\right]_0^1 + \frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{t}{\sqrt{2}}\right]_0^1 - \frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{1-t}{\sqrt{2}}\right]_0^1 - \left[\tan^{-1}(1-t)\right]_0^1$$

$$= \frac{\pi}{4} + \frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}} + \frac{\pi}{4}$$

$$= \frac{\pi}{2} + \frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

Exercise

Find the line integral of $f(x,y) = \frac{x^3}{y}$ over the curve $C: \quad y = \frac{x^2}{2}, \quad 0 \le x \le 2$

Solution

$$r(t) = x\boldsymbol{i} + y\boldsymbol{j} = x\boldsymbol{i} + \frac{1}{2}x^2\boldsymbol{j} \qquad 0 \le x \le 2$$

$$\frac{dr}{dt} = \boldsymbol{i} + xj \quad \Rightarrow \quad \left|\frac{dr}{dt}\right| = \sqrt{1+x^2}$$

$$\int_C f(x,y)\,ds = \int_C \frac{x^3}{y^2}\,ds$$

$$= \int_0^2 2x\sqrt{1+x^2}\,dx \qquad\qquad d\left(1+x^2\right) = 2x\,dx$$

$$= \int_0^2 \left(1+x^2\right)^{1/2} d\left(1+x^2\right)$$

$$= \frac{2}{3}\left[\left(1+x^2\right)^{3/2}\right]_0^2$$

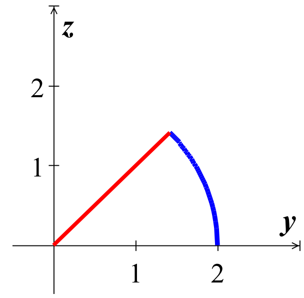
$$= \frac{2}{3}\left[(5)^{3/2} - 1\right]$$

$$= \frac{10\sqrt{5}-2}{3}$$

Exercise

Find the line integral of $f(x,y) = x^2 - y$ over the curve $C: \ x^2 + y^2 = 4$ in the first quadrant from (0, 2) to $\left(\sqrt{2},\ \sqrt{2}\right)$

Solution



$$x = r\cos t \quad y = r\sin t$$

$$r(t) = (2\sin t)\boldsymbol{i} + (2\cos t)\boldsymbol{j} \qquad 0 \le t \le \frac{\pi}{4}$$

$$\frac{dr}{dt} = (2\cos t)\boldsymbol{i} - (2\sin t)\boldsymbol{j} \quad \rightarrow \quad \left|\frac{dr}{dt}\right| = \sqrt{4\cos^2 t + 4\sin^2 t} = 2$$

$$f(x,y) = x^2 - y = 4\sin^2 t - 2\cos t$$

$$\int_C f(x,y)\,ds = \int_0^{\pi/4} \left(4\sin^2 t - 2\cos t\right)(2)\,dt \qquad\qquad \sin^2 t = \frac{1-\cos 2t}{2}$$

$$= 4\int_0^{\pi/4} (1 - \cos 2t - \cos t)\,dt$$

$$= 4\left[t - \frac{1}{2}\sin 2t - \sin t\right]_0^{\pi/4}$$

$$= 4\left(\frac{\pi}{4} - \frac{1}{2} - \frac{\sqrt{2}}{2}\right)$$

$$= 4\left(\frac{\pi}{4} - \frac{1+\sqrt{2}}{2}\right)$$

$$= \pi - 2\left(1+\sqrt{2}\right)$$

Solution Section 4.3 – Conservative Vector Fields

Exercise

Find the gradient field of the function $f(x,y,z)=\left(x^2+y^2+z^2\right)^{-1/2}$

Solution

$$\frac{\partial f}{\partial x}=-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2x)=-x\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\frac{\partial f}{\partial y}=-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2y)=-y\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\frac{\partial f}{\partial z}=-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2z)=-z\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\nabla f=-x\left(x^2+y^2+z^2\right)^{-3/2}\boldsymbol{i}-y\left(x^2+y^2+z^2\right)^{-3/2}\boldsymbol{j}-z\left(x^2+y^2+z^2\right)^{-3/2}\boldsymbol{k}$$

$$=\frac{-x\boldsymbol{i}-y\boldsymbol{j}-z\boldsymbol{k}}{\left(x^2+y^2+z^2\right)^{3/2}}$$

Exercise

Find the gradient field of the function $f(x,y,z)=\ln\sqrt{x^2+y^2+z^2}$

Solution

$$f(x,y,z)=\ln\sqrt{x^2+y^2+z^2}=\ln\left(x^2+y^2+z^2\right)^{1/2}=\frac{1}{2}\ln\left(x^2+y^2+z^2\right)$$

$$\frac{\partial f}{\partial x}=\frac{1}{2}\frac{2x}{x^2+y^2+z^2}=\frac{x}{x^2+y^2+z^2}$$

$$\frac{\partial f}{\partial y}=\frac{1}{2}\frac{2y}{x^2+y^2+z^2}=\frac{y}{x^2+y^2+z^2}$$

$$\frac{\partial f}{\partial z}=\frac{1}{2}\frac{2z}{x^2+y^2+z^2}=\frac{z}{x^2+y^2+z^2}$$

$$\nabla f=\frac{x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}}{x^2+y^2+z^2}$$

Exercise

Find the gradient field of the function $f(x,y,z) = e^z - \ln\left(x^2+y^2\right)$

Solution

$$\frac{\partial f}{\partial x} = -\frac{2x}{x^2+y^2} \qquad \frac{\partial f}{\partial y} = -\frac{2y}{x^2+y^2} \qquad \frac{\partial f}{\partial z} = e^z$$

$$\nabla f = -\frac{2x}{x^2+y^2}\boldsymbol{i} - \frac{2y}{x^2+y^2}\boldsymbol{j} + e^z\boldsymbol{k}$$

Exercise

Find the line integral of $\int_C (x-y)\,dx$ where $C:\ x=t,\quad y=2t+1,\quad \text{for } 0 \le t \le 3$

Solution

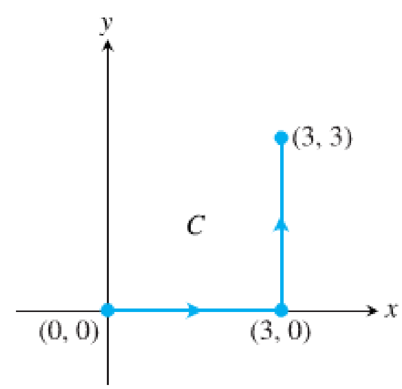
$x=t, \quad y=2t+1, \quad \text{for } 0 \le t \le 3$

$dx = dt$

$$\begin{aligned}\int_C (x-y)\,dx &= \int_0^3 \left(t-(2t+1)\right)dt \\ &= \int_0^3 (-t-1)\,dt \\ &= -\left[\frac{1}{2}t^2+t\right]_0^3 \\ &= -\left(\frac{9}{2}+3\right) \\ &= -\frac{15}{2}\end{aligned}$$

Exercise

Find the line integral of $\int_C \left(x^2+y^2\right)dy$ where C is



Solution

$C_1:\quad x=t, \quad y=0, \quad 0 \le t \le 3 \qquad \Rightarrow dy = 0$

$C_2:\quad x=3, \quad y=t, \quad 0 \le t \le 3 \qquad \Rightarrow dy = dt$

$$\int_C \left(x^2+y^2\right)dy = \int_{C_1} \left(x^2+y^2\right)dy + \int_{C_2} \left(x^2+y^2\right)dy$$

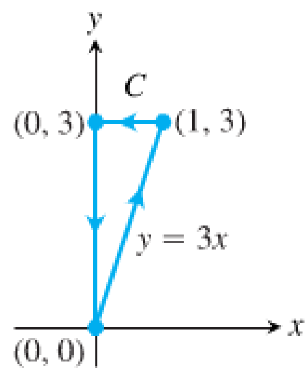
$$= \int_0^3 \left(t^2+0\right)(0) + \int_0^3 \left(9+t^2\right)dt$$

$$= \left[9t + \frac{1}{3}t^3 \right]_0^3$$

$$\underline{= 36}$$

Exercise

Find the line integral of $\int_C \sqrt{x+y}\ dx$ where C is



Solution

$C_1: \quad x=t, \quad y=3t, \quad 0\le t\le 1 \qquad \Rightarrow dx = dt$

$C_2: \quad x=1-t, \quad y=3, \quad 0\le t\le 1 \qquad \Rightarrow dx = -dt$

$C_3: \quad x=0, \quad y=3-t, \quad 0\le t\le 3 \qquad \Rightarrow dx = 0$

$$\int_C \sqrt{x+y}\ dx = \int_{C_1} \sqrt{x+y}\ dx + \int_{C_2} \sqrt{x+y}\ dx + \int_{C_3} \sqrt{x+y}\ dx$$

$$= \int_0^1 \sqrt{t+3t}dt + \int_0^1 \sqrt{1-t+3}(-dt) + \int_0^3 \sqrt{3-t}(0)$$

$$= \int_0^1 2\sqrt{t}\ dt + \int_0^1 \sqrt{4-t}\ d(4-t)$$

$$= 2\left[\frac{2}{3}t^{3/2}\right]_0^1 + \left[\frac{2}{3}(4-t)^{3/2}\right]_0^1$$

$$= \frac{4}{3} + \frac{2}{3}\left(3^{3/2} - 4^{3/2}\right)$$

$$= \frac{4}{3} + \frac{2}{3}\left(3\sqrt{3} - 8\right)$$

$$= \frac{4+6\sqrt{3}-16}{3}$$

$$= \frac{6\sqrt{3}-12}{3}$$

$$\underline{= 2\sqrt{3} - 4}$$

Exercise

Find the work done by the force field $\boldsymbol{F} = xy\boldsymbol{i} + y\boldsymbol{j} - yz\boldsymbol{k}$ over the curve $\boldsymbol{r}(t) = t\boldsymbol{i} + t^2\boldsymbol{j} + t\boldsymbol{k}, \quad 0 \le t \le 1$.

Solution

$$\boldsymbol{r}(t) = t\boldsymbol{i} + t^2\boldsymbol{j} + t\boldsymbol{k}, \quad 0 \le t \le 1$$

$$\Rightarrow \frac{d\boldsymbol{r}}{dt} = \boldsymbol{i} + 2t\boldsymbol{j} + \boldsymbol{k}$$

$$\begin{aligned}\boldsymbol{F} &= xy\boldsymbol{i} + y\boldsymbol{j} - yz\boldsymbol{k} \\ &= t^3\boldsymbol{i} + t^2\boldsymbol{j} - t^3\boldsymbol{k}\end{aligned}$$

$$\begin{aligned}F \cdot \frac{d\boldsymbol{r}}{dt} &= \left(t^3\boldsymbol{i} + t^2\boldsymbol{j} - t^3\boldsymbol{k}\right) \cdot \left(\boldsymbol{i} + 2t\boldsymbol{j} + \boldsymbol{k}\right) \\ &= t^3 + 2t^3 - t^3 \\ &= 2t^3\end{aligned}$$

$$\begin{aligned}Work &= \int_0^1 F \cdot \frac{d\boldsymbol{r}}{dt}\, dt \\ &= \int_0^1 2t^3 dt \\ &= \left[\frac{1}{2}t^4\right]_0^1 \\ &= \frac{1}{2}\end{aligned}$$

Exercise

Find the work done by the force field $\boldsymbol{F} = 2y\boldsymbol{i} + 3x\boldsymbol{j} + (x + y)\boldsymbol{k}$ over the curve $\boldsymbol{r}(t) = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} + \frac{t}{6}\boldsymbol{k}, \quad 0 \le t \le 2\pi$.

Solution

$$\begin{aligned}\boldsymbol{F} &= 2y\boldsymbol{i} + 3x\boldsymbol{j} + (x + y)\boldsymbol{k} \\ &= (2\sin t)\boldsymbol{i} + (3\cos t)\boldsymbol{j} + (\cos t + \sin t)\boldsymbol{k}\end{aligned}$$

$$\frac{d\boldsymbol{r}}{dt} = (-\sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j} + \frac{1}{6}\boldsymbol{k}$$

$$\begin{aligned}F \cdot \frac{d\boldsymbol{r}}{dt} &= \left((2\sin t)\boldsymbol{i} + (3\cos t)\boldsymbol{j} + (\cos t + \sin t)\boldsymbol{k}\right) \cdot \left((-\sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j} + \frac{1}{6}\boldsymbol{k}\right) \\ &= -2\sin^2 t + 3\cos^2 t + \frac{1}{6}\cos t + \frac{1}{6}\sin t \\ &= -2\left(\frac{1 - \cos 2t}{2}\right) + 3\left(\frac{1 + \cos 2t}{2}\right) + \frac{1}{6}\cos t + \frac{1}{6}\sin t\end{aligned}$$

$$= \cos 2t - 1 + \frac{3}{2} + \frac{3}{2}\cos 2t + \frac{1}{6}\cos t + \frac{1}{6}\sin t$$

$$= \frac{1}{2} + \frac{5}{2}\cos 2t + \frac{1}{6}\cos t + \frac{1}{6}\sin t$$

$$Work = \int_0^{2\pi} F \bullet \frac{d\boldsymbol{r}}{dt}\,dt$$

$$= \int_0^{2\pi} \left(\frac{1}{2} + \frac{5}{2}\cos 2t + \frac{1}{6}\cos t + \frac{1}{6}\sin t\right)dt$$

$$= \left[\frac{1}{2}t + \frac{5}{4}\sin 2t + \frac{1}{6}\sin t - \frac{1}{6}\cos t\right]_0^{2\pi}$$

$$= \left(\pi - \frac{1}{6}\right) - \left(-\frac{1}{6}\right)$$

$$\underline{= \pi}$$

Exercise

Find the work done by the force field $\boldsymbol{F} = z\boldsymbol{i} + x\boldsymbol{j} + y\boldsymbol{k}$ over the curve $\boldsymbol{r}(t) = (\sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j} + t\boldsymbol{k}, \quad 0 \le t \le 2\pi$.

Solution

$$\boldsymbol{F} = z\boldsymbol{i} + x\boldsymbol{j} + y\boldsymbol{k}$$

$$= t\boldsymbol{i} + (\sin t)\boldsymbol{j} + (\cos t)\boldsymbol{k}$$

$$\frac{d\boldsymbol{r}}{dt} = (\cos t)\boldsymbol{i} + (-\sin t)\boldsymbol{j} + \boldsymbol{k}$$

$$F \bullet \frac{d\boldsymbol{r}}{dt} = \left(t\boldsymbol{i} + (\sin t)\boldsymbol{j} + (\cos t)\boldsymbol{k}\right) \bullet \left((\cos t)\boldsymbol{i} + (-\sin t)\boldsymbol{j} + \boldsymbol{k}\right)$$

$$= t\cos t - \sin^2 t + \cos t$$

$$= t\cos t - \frac{1}{2} + \frac{1}{2}\cos 2t + \cos t$$

$$Work = \int_0^{2\pi} F \bullet \frac{d\boldsymbol{r}}{dt}\,dt$$

$$= \int_0^{2\pi} \left(t\cos t - \frac{1}{2} + \frac{1}{2}\cos 2t + \cos t\right)dt$$

$$= \left[t\sin t + \cos t - \frac{1}{2}t + \frac{1}{4}\sin 2t + \sin t\right]_0^{2\pi}$$

$$= (1 - \pi) - (1)$$

$$\underline{= -\pi}$$

		$\int \cos t$
+	t	$\sin t$
−	1	$-\cos t$

Exercise

Evaluate $\int_C \boldsymbol{F}\bullet\boldsymbol{T}ds$ for the vector field $\boldsymbol{F} = x^2\boldsymbol{i} - y\boldsymbol{j}$ along the curve $x = y^2$ from $(4,\ 2)$ to $(1,\ -1)$

Solution

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} = y^2\boldsymbol{i} + y\boldsymbol{j} \qquad -1 \le y \le 2$$

$$\boldsymbol{F} = x^2\boldsymbol{i} - y\boldsymbol{j} = y^4\boldsymbol{i} - y\boldsymbol{j}$$

$$\frac{d\boldsymbol{r}}{dy} = 2y\boldsymbol{i} + \boldsymbol{j}$$

$$F\bullet\frac{d\boldsymbol{r}}{dy} = \left(y^4\boldsymbol{i} - y\boldsymbol{j}\right)\cdot\left(2y\boldsymbol{i} + \boldsymbol{j}\right) = 2y^5 - y$$

$$\begin{aligned}
\int_C \boldsymbol{F}\bullet\boldsymbol{T}ds &= \int_2^{-1} F\bullet\frac{d\boldsymbol{r}}{dy}dy \\
&= \int_2^{-1}\left(2y^5 - y\right)dy \\
&= \left[\frac{1}{3}y^6 - \frac{1}{2}y^2\right]_2^{-1} \\
&= \left(\frac{1}{3} - \frac{1}{2}\right) - \left(\frac{64}{3} - 2\right) \\
&= -\frac{39}{2}
\end{aligned}$$

Exercise

Find the circulation and flux of the fields $\boldsymbol{F}_1 = x\boldsymbol{i} + y\boldsymbol{j}$ *and* $\boldsymbol{F}_2 = -y\boldsymbol{i} + x\boldsymbol{j}$ around and across each of the following curves.

a) The circle $\boldsymbol{r}(t) = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j}, \quad 0 \le t \le 2\pi$

b) The ellipse $\boldsymbol{r}(t) = (\cos t)\boldsymbol{i} + (4\sin t)\boldsymbol{j}, \quad 0 \le t \le 2\pi$

Solution

a) $\boldsymbol{r}(t) = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j}, \quad 0 \le t \le 2\pi$

$$\frac{d\boldsymbol{r}}{dt} = (-\sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j}$$

$$\boldsymbol{F}_1 = x\boldsymbol{i} + y\boldsymbol{j} = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j}$$

$$\begin{aligned}
F_1\bullet\frac{d\boldsymbol{r}}{dt} &= \left((\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j}\right)\cdot\left((-\sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j}\right) \\
&= -\cos t\sin t + \sin t\cos t \\
&= 0
\end{aligned}$$

$$F_2 = -y\boldsymbol{i} + x\boldsymbol{j} = -(\sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j}$$

$$F_2 \bullet \frac{d\boldsymbol{r}}{dt} = \left(-(\sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j}\right) \cdot \left((-\sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j}\right)$$
$$= \sin^2 t + \cos^2 t$$
$$\underline{= 1}$$

$$Cir_1 = \int_0^{2\pi} \left(F_1 \bullet \frac{d\boldsymbol{r}}{dt}\right) dt = \int_0^{2\pi} 0 dt = \underline{0}$$

$$Cir_2 = \int_0^{2\pi} \left(F_2 \bullet \frac{d\boldsymbol{r}}{dt}\right) dt = \int_0^{2\pi} dt = \underline{2\pi}$$

$$dx = -\sin t\ dt, \quad dy = \cos t\ dt$$
$$M_1 = x = \cos t, \quad N_1 = y = \sin t$$
$$M_2 = -y = -\sin t, \quad N_2 = x = \cos t$$

$$Flux_1 = \int_C M_1 dy - N_1 dx = \int_0^{2\pi} \left(\cos^2 t + \sin^2 t\right) dt = \int_0^{2\pi} dt = \underline{2\pi}$$

$$Flux_2 = \int_C M_2 dy - N_2 dx = \int_0^{2\pi} (-\sin t \cos t + \sin t \cos t) dt = \int_0^{2\pi} (0) dt = \underline{0}$$

b) $\boldsymbol{r}(t) = (\cos t)\boldsymbol{i} + (4\sin t)\boldsymbol{j}, \quad 0 \le t \le 2\pi$

$$\frac{d\boldsymbol{r}}{dt} = (-\sin t)\boldsymbol{i} + (4\cos t)\boldsymbol{j}$$

$$F_1 = x\boldsymbol{i} + y\boldsymbol{j} = (\cos t)\boldsymbol{i} + (4\sin t)\boldsymbol{j}$$

$$F_1 \bullet \frac{d\boldsymbol{r}}{dt} = \left((\cos t)\boldsymbol{i} + (4\sin t)\boldsymbol{j}\right) \cdot \left((-\sin t)\boldsymbol{i} + (4\cos t)\boldsymbol{j}\right)$$
$$= -\cos t \sin t + 16 \sin t \cos t$$
$$= 15 \sin t \cos t$$

$$F_2 = -y\boldsymbol{i} + x\boldsymbol{j} = -(4\sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j}$$

$$F_2 \bullet \frac{d\boldsymbol{r}}{dt} = \left((-4\sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j}\right) \cdot \left((-\sin t)\boldsymbol{i} + (4\cos t)\boldsymbol{j}\right) = 4\sin^2 t + 4\cos^2 t = 4$$

$$Cir_1 = \int_0^{2\pi} \left(F_1 \bullet \frac{d\boldsymbol{r}}{dt}\right) dt$$

$$= \int_0^{2\pi} 15\sin t\cos t dt \qquad\qquad d(\sin t) = \cos t\ dt$$

$$= 15\int_0^{2\pi} \sin t\ d(\sin t)$$

$$= \frac{15}{2}\left[\sin^2 t\right]_0^{2\pi}$$

$$= \frac{15}{2}(1-1)$$

$$\underline{= 0}$$

$$Cir_2 = \int_0^{2\pi}\left(F_2 \bullet \frac{d\boldsymbol{r}}{dt}\right)dt = \int_0^{2\pi} 4dt = [4t]_0^{2\pi} = \underline{8\pi}$$

$$dx = -\sin t\ dt, \quad dy = 4\cos t\ dt$$

$$M_1 = x = \cos t, \quad N_1 = y = 4\sin t$$

$$M_2 = -y = -4\sin t, \quad N_2 = x = \cos t$$

$$Flux_1 = \int_C M_1 dy - N_1 dx$$

$$= \int_0^{2\pi}\left(4\cos^2 t + 4\sin^2 t\right)dt$$

$$= 4\int_0^{2\pi} dt$$

$$= \underline{8\pi}$$

$$Flux_2 = \int_C M_2 dy - N_2 dx$$

$$= -15\int_0^{2\pi}(\sin t\cos t)dt$$

$$= -15\int_0^{2\pi}\sin t\ d(\sin t)$$

$$= -15\left[\frac{1}{2}\sin^2 t\right]_0^{2\pi}$$

$$\underline{= 0}$$

Exercise

Find the circulation and flux of the fields $\boldsymbol{F}_1 = 2x\boldsymbol{i} - 3y\boldsymbol{j}$ *and* $\boldsymbol{F}_2 = 2x\boldsymbol{i} + (x-y)\boldsymbol{j}$ across the circle $\boldsymbol{r}(t) = (a\cos t)\boldsymbol{i} + (a\sin t)\boldsymbol{j}, \quad 0 \le t \le 2\pi$

Solution

$$\frac{d\boldsymbol{r}}{dt} = (-a\sin t)\boldsymbol{i} + (a\cos t)\boldsymbol{j}$$

$$\boldsymbol{F}_1 = 2x\boldsymbol{i} - 3y\boldsymbol{j} = (2a\cos t)\boldsymbol{i} - (3a\sin t)\boldsymbol{j}$$

$$F_1 \bullet \frac{d\boldsymbol{r}}{dt} = \left((2a\cos t)\boldsymbol{i} - (3a\sin t)\boldsymbol{j}\right)\cdot\left((-a\sin t)\boldsymbol{i} + (a\cos t)\boldsymbol{j}\right) = -5a^2\cos t\sin t$$

$$\boldsymbol{F}_2 = 2x\boldsymbol{i} + (x - y)\boldsymbol{j} = (2a\cos t)\boldsymbol{i} + a(\cos t - \sin t)\boldsymbol{j}$$

$$\begin{aligned} F_2 \bullet \frac{d\boldsymbol{r}}{dt} &= \left((2a\cos t)\boldsymbol{i} + a(\cos t - \sin t)\boldsymbol{j}\right)\cdot\left((-a\sin t)\boldsymbol{i} + (a\cos t)\boldsymbol{j}\right) \\ &= -2a^2\cos t\sin t + a^2\cos^2 t - a^2\cos t\sin t \\ &= a^2\left(\cos^2 t - 3\cos t\sin t\right) \end{aligned}$$

$Cir_1 = \int_0^{2\pi}\left(F_1 \bullet \frac{d\boldsymbol{r}}{dt}\right)dt$	$Cir_2 = \int_0^{2\pi}\left(F_2 \bullet \frac{d\boldsymbol{r}}{dt}\right)dt$
$= -5a^2\int_0^{2\pi}\sin t\cos t dt$	$= a^2\int_0^{2\pi}\left(\cos^2 t - 3\cos t\sin t\right)dt$
$= -5a^2\int_0^{2\pi}\sin t\ d(\sin t)$	$= a^2\left[\int_0^{2\pi}\left(\frac{1}{2} + \frac{1}{2}\cos 2t\right)dt - 3\int_0^{2\pi}(\sin t)d(\sin t)\right]$
$= -5a^2\left[\sin^2 t\right]_0^{2\pi}$	$= a^2\left[\frac{1}{2}t + \frac{1}{4}\sin 2t - 0\right]_0^{2\pi}$
$= 0$	$= \pi a^2$

$$\begin{aligned} &dx = -a\sin t\ dt, \quad dy = a\cos t\ dt \\ &M_1 = 2x = 2a\cos t, \quad N_1 = -3y = -3a\sin t \\ &M_2 = 2a\cos t, \quad N_2 = a\cos t - a\sin t \end{aligned}$$

$$\begin{aligned} Flux_1 &= \int_C M_1 dy - N_1 dx \\ &= \int_0^{2\pi}\left(2a^2\cos^2 t - 3a^2\sin^2 t\right)dt \\ &= a^2\int_0^{2\pi}\left(1 + \cos 2t - \frac{3}{2} + \frac{3}{2}\cos 2t\right)dt \\ &= a^2\int_0^{2\pi}\left(\frac{5}{2}\cos 2t - \frac{1}{2}\right)dt \\ &= a^2\left[\frac{5}{4}\sin 2t - \frac{1}{2}t\right]_0^{2\pi} \\ &= a^2\left[0 - \frac{1}{2}(2\pi)\right] \\ &= -\pi a^2 \end{aligned}$$

$\cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t, \quad \sin^2 t = \frac{1}{2} - \frac{1}{2}\cos 2t$

$$Flux_2 = \int_C M_2\,dy - N_2\,dx$$

$$= \int_0^{2\pi} \left(2a^2\cos^2 t - a^2\sin^2 t + a^2\cos t\sin t\right)dt$$

$$= a^2\left[\int_0^{2\pi}\left(1+\cos 2t - \frac{1}{2} + \frac{1}{2}\cos 2t\right)dt + \int_0^{2\pi}(\sin t)\,d(\sin t)\right]$$

$$= a^2\left[\frac{1}{2}t + \frac{3}{4}\sin 2t + \frac{1}{2}\sin^2 t\right]_0^{2\pi}$$

$$= a^2\frac{1}{2}(2\pi)$$

$$= \pi a^2$$

Exercise

Find a field $\boldsymbol{F} = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$ in the *xy*-plane with the property that at each point $(x,y) \neq (0,0)$, $\boldsymbol{F}$ points toward the origin and $|\boldsymbol{F}|$ is

a) The distance from (x, y) to the origin

b) Inversely proportional to the distance from (x, y) to the origin. (The field is undefined at (0, 0).)

Solution

a) The slope of the line through the origin and a point (x,y) is: $m = \frac{y}{x}$

The vector parallel to the line is given by: $\boldsymbol{v} = x\boldsymbol{i} + y\boldsymbol{j}$

Pointing away from the origin: $\boldsymbol{F} = -\frac{\boldsymbol{v}}{|\boldsymbol{v}|} = -\frac{x\boldsymbol{i}+y\boldsymbol{j}}{\sqrt{x^2+y^2}}$ is the unit vector pointing toward the origin.

$$|\boldsymbol{F}| = \sqrt{x^2+y^2}$$

$$\boldsymbol{F} = \sqrt{x^2+y^2}\left(-\frac{x\boldsymbol{i}+y\boldsymbol{j}}{\sqrt{x^2+y^2}}\right) = -x\boldsymbol{i} - y\boldsymbol{j}$$

b) $|\boldsymbol{F}| = \frac{C}{\sqrt{x^2+y^2}}, \quad C \neq 0$

$$\boldsymbol{F} = \frac{C}{\sqrt{x^2+y^2}}\left(-\frac{x\boldsymbol{i}+y\boldsymbol{j}}{\sqrt{x^2+y^2}}\right) = -C\left(\frac{x\boldsymbol{i}+y\boldsymbol{j}}{x^2+y^2}\right)$$

Exercise

A fluid's velocity field is $\boldsymbol{F} = -4xy\boldsymbol{i} + 8y\boldsymbol{j} + 2\boldsymbol{k}$. Find the flow along the curve

$\boldsymbol{r}(t) = t\boldsymbol{i} + t^2\boldsymbol{j} + \boldsymbol{k}, \quad 0 \le t \le 2$

Solution

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{i} + 2t\boldsymbol{j}$$

$$\boldsymbol{F} = -4xy\boldsymbol{i} + 8y\boldsymbol{j} + 2\boldsymbol{k} = -4t^3\boldsymbol{i} + 8t^2\boldsymbol{j} + 2\boldsymbol{k}$$

$$F \bullet \frac{d\boldsymbol{r}}{dt} = \left(-4t^3\boldsymbol{i} + 8t^2\boldsymbol{j} + 2\boldsymbol{k}\right)\cdot\left(\boldsymbol{i} + 2t\boldsymbol{j}\right) = -4t^3 + 16t^3 = 12t^3$$

$$Flow = \int_R F \bullet \frac{d\boldsymbol{r}}{dt}\,dt = \int_0^2 12t^3\,dt = \left[3t^4\right]_0^2 \underline{\underline{=48}}$$

Exercise

A fluid's velocity field is $\boldsymbol{F} = x^2\boldsymbol{i} + yz\boldsymbol{j} + y^2\boldsymbol{k}$. Find the flow along the curve

$\boldsymbol{r}(t) = 3t\boldsymbol{j} + 4t\boldsymbol{k}, \quad 0 \le t \le 1$

Solution

$$\frac{d\boldsymbol{r}}{dt} = 3\boldsymbol{i} + 4\boldsymbol{j}$$

$$\boldsymbol{F} = x^2\boldsymbol{i} + yz\boldsymbol{j} + y^2\boldsymbol{k} = 12t^2\boldsymbol{j} + 9t^2\boldsymbol{k}$$

$$F \bullet \frac{d\boldsymbol{r}}{dt} = \left(12t^2\boldsymbol{j} + 9t^2\boldsymbol{k}\right)\cdot\left(3\boldsymbol{i} + 4\boldsymbol{j}\right) = 36t^2 + 36t^2 = 72t^2$$

$$Flow = \int_C F \bullet \frac{d\boldsymbol{r}}{dt}\,dt = \int_0^1 72t^2\,dt = \left[24t^3\right]_0^1 \underline{\underline{=24}}$$

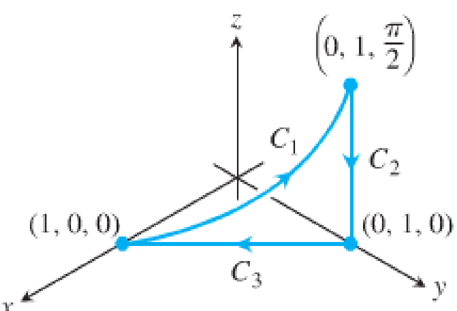
Exercise

Find the circulation of $\boldsymbol{F} = 2x\boldsymbol{i} + 2z\boldsymbol{j} + 2y\boldsymbol{k}$ around the closed path consisting of the following three curves traversed in the direction of increasing t.

$C_1: \quad \boldsymbol{r}(t) = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} + t\boldsymbol{k}, \quad 0 \le t \le \frac{\pi}{2}$

$C_2: \quad \boldsymbol{r}(t) = \boldsymbol{j} + \frac{\pi}{2}(1-t)\boldsymbol{k}, \quad 0 \le t \le 1$

$C_3: \quad \boldsymbol{r}(t) = t\boldsymbol{i} + (1-t)\boldsymbol{j}, \quad 0 \le t \le 1$



Solution

$C_1: \quad \boldsymbol{r}(t) = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} + t\boldsymbol{k}, \quad 0 \le t \le \frac{\pi}{2}$

$$\frac{d\boldsymbol{r}}{dt} = (-\sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j} + \boldsymbol{k}$$

$$\boldsymbol{F} = 2x\boldsymbol{i} + 2z\boldsymbol{j} + 2y\boldsymbol{k} = (2\cos t)\boldsymbol{i} + 2t\boldsymbol{j} + (2\sin t)\boldsymbol{k}$$

$$\begin{aligned} F \bullet \frac{d\boldsymbol{r}}{dt} &= ((2\cos t)\boldsymbol{i} + 2t\boldsymbol{j} + (2\sin t)\boldsymbol{k}) \cdot ((-\sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j} + \boldsymbol{k}) \\ &= -2\sin t\cos t + 2t\cos t + 2\sin t \\ &= -\sin 2t + 2t\cos t + 2\sin t \end{aligned}$$

$$\begin{aligned} Flow_1 &= \int_0^{\pi/2} (-\sin 2t + 2t\cos t + 2\sin t)\,dt \\ &= \left[\frac{1}{2}\cos 2t + 2t\sin t + 2\cos t - 2\cos t\right]_0^{\pi/2} \\ &= \left[\frac{1}{2}\cos 2t + 2t\sin t\right]_0^{\pi/2} \\ &= \left(-\frac{1}{2} + 2\frac{\pi}{2}\right) - \left(\frac{1}{2}\right) \\ &= \pi - 1 \end{aligned}$$

		$\int \cos t$
+	t	$\sin t$
−	1	$-\cos t$

$C_2: \quad \boldsymbol{r}(t) = \boldsymbol{j} + \frac{\pi}{2}(1-t)\boldsymbol{k}, \quad 0 \le t \le 1$

$$\frac{d\boldsymbol{r}}{dt} = -\frac{\pi}{2}\boldsymbol{k}$$

$$\boldsymbol{F} = 2x\boldsymbol{i} + 2z\boldsymbol{j} + 2y\boldsymbol{k} = \pi(1-t)\boldsymbol{j} + 2\boldsymbol{k}$$

$$F \bullet \frac{d\boldsymbol{r}}{dt} = (\pi(1-t)\boldsymbol{j} + 2\boldsymbol{k}) \cdot \left(-\frac{\pi}{2}\boldsymbol{k}\right) = -\pi$$

$$\begin{aligned} Flow_2 &= \int_0^1 (-\pi)\,dt \\ &= -\pi[t]_0^1 \\ &= -\pi \end{aligned}$$

$C_3: \quad \boldsymbol{r}(t) = t\boldsymbol{i} + (1-t)\boldsymbol{j}, \quad 0 \le t \le 1$

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{i} - \boldsymbol{j}$$

$$\boldsymbol{F} = 2x\boldsymbol{i} + 2z\boldsymbol{j} + 2y\boldsymbol{k} = 2t\boldsymbol{i} + 2(1-t)\boldsymbol{k}$$

$$F \bullet \frac{d\boldsymbol{r}}{dt} = (2t\boldsymbol{i} + 2(1-t)\boldsymbol{k}) \cdot (\boldsymbol{i} - \boldsymbol{j}) = 2t$$

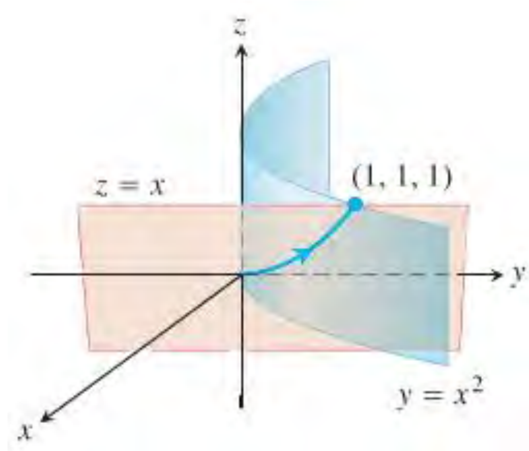
$$\begin{aligned} Flow_3 &= \int_0^1 (2t)\,dt \\ &= \left[t^2\right]_0^1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} Circulation &= Flow_1 + Flow_2 + Flow_3 \\ &= \pi - 1 - \pi + 1 \\ &= 0 \end{aligned}$$

Exercise

The field $\boldsymbol{F} = xy\boldsymbol{i} + y\boldsymbol{j} - yz\boldsymbol{k}$ is the velocity field of a flow in space. Find the flow from $(0,\ 0,\ 0)$ to $(1,\ 1,\ 1)$ along the curve of intersection of the cylinder $y = x^2$ and the plane $z = x$. (***Hint***: Use $t = x$ as the parameter.)

Solution



Let $x = t \quad \Rightarrow \quad y = x^2 = t^2$

$z = x = t$

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = t\boldsymbol{i} + t^2\boldsymbol{j} + t\boldsymbol{k} \qquad 0 \le t \le 1$$

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{i} + 2t\boldsymbol{j} + \boldsymbol{k}$$

$$\boldsymbol{F} = xy\boldsymbol{i} + y\boldsymbol{j} - yz\boldsymbol{k} = t^3\boldsymbol{i} + t^2\boldsymbol{j} - t^3\boldsymbol{k}$$

$$F \bullet \frac{d\boldsymbol{r}}{dt} = \left(t^3\boldsymbol{i} + t^2\boldsymbol{j} - t^3\boldsymbol{k}\right) \cdot \left(\boldsymbol{i} + 2t\boldsymbol{j} + \boldsymbol{k}\right)$$

$$= t^3 + 2t^3 - t^3 = 2t^3$$

$$Flow = \int_0^1 \left(2t^3\right)dt = \left[\frac{1}{2}t^4\right]_0^1 \ \boxed{= \frac{1}{2}}$$

Solution Section 4.4 – Green's Theorem

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\boldsymbol{F} = (x-y)\boldsymbol{i} + (y-x)\boldsymbol{j}$ and curve C is the square bounded by $x=0,\ x=1,\ y=0,\ y=1$

Solution

$$M = x - y \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = -1$$

$$N = y - x \quad \Rightarrow \quad \frac{\partial N}{\partial x} = -1, \quad \frac{\partial N}{\partial y} = 1$$

$$\begin{aligned}
Flux &= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy \\
&= \iint_R (1+1)\, dxdy \\
&= 2\int_0^1 \int_0^1 dxdy \\
&= 2\int_0^1 dy \\
&= 2
\end{aligned}$$

$$Circulation = \iint_R \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dxdy = \int_0^1 \int_0^1 \left(-1-(-1)\right) dxdy = 0$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\boldsymbol{F} = \left(x^2+4y\right)\boldsymbol{i} + \left(x+y^2\right)\boldsymbol{j}$ and curve C is the square bounded by $x=0,\ x=1,\ y=0,\ y=1$

Solution

$$M = x^2 + 4y \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 2x, \quad \frac{\partial M}{\partial y} = 4$$

$$N = x + y^2 \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 1, \quad \frac{\partial N}{\partial y} = 2y$$

$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_0^1 (2x + 2y)\, dxdy$$

$$= \int_0^1 \left[x^2 + 2yx \right]_0^1 dy$$

$$= \int_0^1 (1 + 2y)\, dy$$

$$= \left[y + y^2 \right]_0^1$$

$$= 2$$

$$Circulation = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_0^1 (1 - 4)\, dxdy$$

$$= -3 \int_0^1 dy$$

$$= -3$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\boldsymbol{F} = (x + y)\boldsymbol{i} - \left(x^2 + y^2\right)\boldsymbol{j}$ and curve C is the triangle bounded by $y = 0,\ x = 1,\ y = x$

Solution

$$M = x + y \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = 1$$

$$N = -\left(x^2 + y^2\right) \quad \Rightarrow \quad \frac{\partial N}{\partial x} = -2x, \quad \frac{\partial N}{\partial y} = -2y$$

$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_0^x (1 - 2y)\, dydx$$

$$= \int_0^1 \left[y - y^2 \right]_0^x dx$$

$$= \int_0^1 \left(x - x^2 \right) dx$$

$$= \left[\frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6}$$

$$Circulation = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_0^x (-2x - 1)\, dydx$$

$$= \int_0^1 \left[-2xy - y \right]_0^x dx$$

$$= \int_0^1 \left(-2x^2 - x \right) dx$$

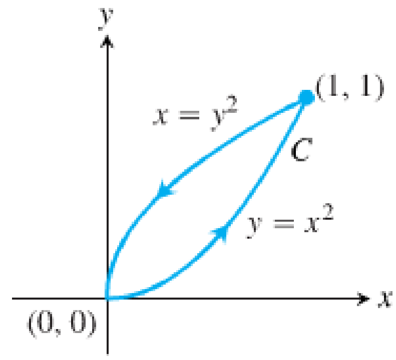
$$= \left[-\frac{2}{3}x^3 - \frac{1}{2}x^2 \right]_0^1$$

$$= -\frac{2}{3} - \frac{1}{2}$$

$$= -\frac{7}{6}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\boldsymbol{F} = \left(xy + y^2\right)\boldsymbol{i} + (x - y)\boldsymbol{j}$ and curve C



Solution

$$M = xy + y^2 \quad \Rightarrow \quad \frac{\partial M}{\partial x} = y, \quad \frac{\partial M}{\partial y} = x + 2y$$

$$N = x - y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 1, \quad \frac{\partial N}{\partial y} = -1$$

$$Flux = \iint\limits_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (y - 1)\, dydx$$

$$= \int_0^1 \left[\frac{1}{2} y^2 - y \right]_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 \left(\frac{1}{2} x - \sqrt{x} - \left(\frac{1}{2} x^4 - x^2 \right) \right) dx$$

$$= \int_0^1 \left(\frac{1}{2} x - x^{1/2} - \frac{1}{2} x^4 + x^2 \right) dx$$

$$= \left[\frac{1}{4} x^2 - \frac{2}{3} x^{3/2} - \frac{1}{10} x^5 + \frac{1}{3} x^3 \right]_0^1$$

$$= \frac{1}{4} - \frac{2}{3} - \frac{1}{10} + \frac{1}{3}$$

$$= -\frac{11}{60}$$

$$Circulation = \iint\limits_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (1 - x - 2y)\, dydx$$

$$= \int_0^1 \left[y - xy - y^2 \right]_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 \left(\sqrt{x} - x\sqrt{x} - x - x^2 + x^3 + x^4 \right) dx$$

$$=\left[\frac{2}{3}x^{3/2}-\frac{2}{5}x^{5/2}-\frac{1}{2}x^2-\frac{1}{3}x^3+\frac{1}{4}x^4+\frac{1}{5}x^5\right]_0^1$$

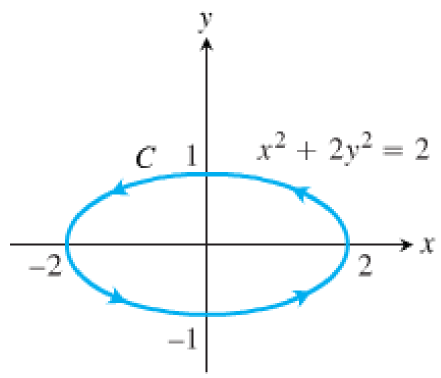
$$=\frac{2}{3}-\frac{2}{5}-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$$

$$=-\frac{7}{60}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\boldsymbol{F}=(x+3y)\boldsymbol{i}+(2x-y)\boldsymbol{j}$ and curve C

Solution



$$M=x+3y \quad \Rightarrow \quad \frac{\partial M}{\partial x}=1, \quad \frac{\partial M}{\partial y}=3$$

$$N=2x-y \quad \Rightarrow \quad \frac{\partial N}{\partial x}=2, \quad \frac{\partial N}{\partial y}=-1$$

$$Flux=\iint_R\left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right)dxdy$$

$$=\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{\left(2-x^2\right)/2}}^{\sqrt{\left(2-x^2\right)/2}}(1-1)\,dydx$$

$$=0$$

$$Circulation=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$$

$$=\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{\left(2-x^2\right)/2}}^{\sqrt{\left(2-x^2\right)/2}}(2-3)\,dydx$$

$$=-\int_{-\sqrt{2}}^{\sqrt{2}}\left(\sqrt{\frac{2-x^2}{2}}+\sqrt{\frac{2-x^2}{2}}\right)dx$$

$$=-\frac{2}{\sqrt{2}}\int_{-\sqrt{2}}^{\sqrt{2}}\left(\sqrt{2-x^2}\right)dx \qquad \int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$=-\frac{2}{\sqrt{2}}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{-\sqrt{2}}^{\sqrt{2}}$$

$$= -\frac{2}{\sqrt{2}}\left[0+\sin^{-1}\frac{\sqrt{2}}{\sqrt{2}}-\left(0+\sin^{-1}\frac{-\sqrt{2}}{\sqrt{2}}\right)\right]$$

$$= -\frac{2}{\sqrt{2}}\left[\frac{\pi}{2}+\frac{\pi}{2}\right]$$

$$= -\frac{2\pi}{\sqrt{2}}$$

$$\boxed{= -\pi\sqrt{2}}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\boldsymbol{F} = \left(x+e^{x}\sin y\right)\boldsymbol{i}+\left(x+e^{x}\cos y\right)\boldsymbol{j}$ and curve C is the right-hand loop of the lemniscate $r^2=\cos 2\theta$

Solution

$$M = x+e^{x}\sin y \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 1+e^{x}\sin y, \quad \frac{\partial M}{\partial y} = e^{x}\cos y$$

$$N = x+e^{x}\cos y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 1+e^{x}\cos y, \quad \frac{\partial N}{\partial y} = -e^{x}\sin y$$

$$Flux = \iint_R \left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right)dxdy$$

$$= \iint_R \left(1+e^{x}\sin y-e^{x}\sin y\right)dxdy$$

$$= \iint_R dxdy$$

$$= \int_{-\pi/4}^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} r\ drd\theta$$

$$= \int_{-\pi/4}^{\pi/4}\left[\frac{1}{2}r^2\right]_0^{\sqrt{\cos 2\theta}} d\theta$$

$$= \frac{1}{2}\int_{-\pi/4}^{\pi/4}\cos 2\theta d\theta$$

$$= \frac{1}{4}\left[\sin 2\theta\right]_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{4}(1-(-1))$$

$$\boxed{= \frac{1}{2}}$$

$$Circulation = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$$

$$= \iint_R \left(1+e^{x}\cos y-e^{x}\cos y\right)dxdy$$

$$= \iint_R dxdy$$

$$= \int_{-\pi/4}^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} r\ drd\theta$$

$$\boxed{= \frac{1}{2}}$$

Exercise

Find the outward flux for the field $\boldsymbol{F} = \left(3xy - \dfrac{x}{1+y^2}\right)\boldsymbol{i} + \left(e^x + \tan^{-1} y\right)\boldsymbol{j}$ across the cardioid $r = a(1+\cos\theta),\ \ a > 0$

Solution

$$M = 3xy - \frac{x}{1+y^2} \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 3y - \frac{1}{1+y^2}$$

$$N = e^x + \tan^{-1} y \quad \Rightarrow \quad \frac{\partial N}{\partial y} = \frac{1}{1+y^2}$$

$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$= \iint_R \left(3y - \frac{1}{1+y^2} + \frac{1}{1+y^2}\right) dxdy$$

$$= \iint_R 3y\ dxdy$$

$$= 3\int_0^{2\pi} \int_0^{a(1+\cos\theta)} (r\sin\theta)\ rdrd\theta$$

$$= 3\int_0^{2\pi} \frac{1}{3}\sin\theta \left[r^3\right]_0^{a(1+\cos\theta)} d\theta$$

$$= a^3 \int_0^{2\pi} \sin\theta(1+\cos\theta)^3\, d\theta \qquad\qquad d(1+\cos\theta) = -\sin\theta d\theta$$

$$= -a^3 \int_0^{2\pi} (1+\cos\theta)^3\, d(1+\cos\theta)$$

$$= -\frac{1}{4}a^3\left[(1+\cos\theta)^4\right]_0^{2\pi}$$

$$= -\frac{1}{4}a^3\left(2^4 - 2^4\right)$$

$$= 0$$

Exercise

Find the work done by $\boldsymbol{F} = 2xy^3\boldsymbol{i} + 4x^2y^2\boldsymbol{j}$ in moving a particle once counterclockwise around the curve *C*: The boundary of the triangular region in the first quadrant enclosed by the *x*-axis, the line $x = 1$ and the curve $y = x^3$

Solution

$$M = 2xy^3 \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 6xy^2$$

$$N = 4x^2y^2 \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 8xy^2$$

$$Work = \iint\limits_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy \qquad\qquad Work = \oint\limits_C Mdx + Ndy$$

$$= \int_0^1 \int_0^{x^3} \left(8xy^2 - 6xy^2 \right) dydx$$

$$= \int_0^1 \int_0^{x^3} \left(2xy^2 \right) dydx$$

$$= \int_0^1 \left[\frac{2}{3} xy^3 \right]_0^{x^3} dx$$

$$= \frac{2}{3} \int_0^1 x^{10} dx$$

$$= \left[\frac{2}{33} x^{11} \right]_0^1$$

$$= \frac{2}{33}$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint\limits_C \left(y^2 dx + x^2 dy \right)$ *C*: The triangle bounded by

$x = 0,\ x + y = 1,\ y = 0$

Solution

$$M = y^2 \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 2y$$

$$N = x^2 \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 2x$$

$$\oint_C \left(y^2dx + x^2dy\right) = \int_0^1 \int_0^{1-x} (2x - 2y)\,dydx$$

$$= \int_0^1 \left[2xy - y^2\right]_0^{1-x} dx$$

$$= \int_0^1 \left[2x(1-x) - (1-x)^2\right] dx$$

$$= \int_0^1 \left(2x - 2x^2 - 1 + 2x - x^2\right) dx$$

$$= \int_0^1 \left(-3x^2 + 4x - 1\right) dx$$

$$= \left[-x^3 + 2x^2 - x\right]_0^1$$

$$= -1 + 2 - 1$$

$$\underline{= 0}$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint_C (3ydx + 2xdy)$ C: The boundary of $0 \le x \le \pi, \quad 0 \le y \le \sin x$

Solution

$$M = 3y \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 3$$

$$N = 2x \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 2$$

$$\oint_C (3ydx + 2xdy) = \int_0^{\pi} \int_0^{\sin x} (2 - 3)\,dydx$$

$$= -\int_0^{\pi} [y]_0^{\sin x}\, dx$$

$$= -\int_0^{\pi} \sin x\; dx$$

$$= [\cos x]_0^{\pi}$$

$$\underline{= -2}$$

Exercise

Evaluate $\int_C (x-y)\,dx+(x+y)\,dy$ counterclockwise around the triangle with vertices (0, 0), $(1,\ 0)$ and $(0,\ 1)$

Solution

Along $(0,0)\to(1,0)$: $\quad \boldsymbol{r}(t)=t\boldsymbol{i}, \quad 0\le t\le 1$

$$\boldsymbol{F}=(x-y)\boldsymbol{i}+(x+y)\boldsymbol{j}=t\boldsymbol{i}+t\boldsymbol{j}$$

$$\frac{d\boldsymbol{r}}{dt}=\boldsymbol{i}$$

$$F\bullet\frac{d\boldsymbol{r}}{dt}=(t\boldsymbol{i}+t\boldsymbol{j})\cdot(\boldsymbol{i})=t$$

Along $(1,0)\to(0,1)$: $\quad \boldsymbol{r}(t)=(1-t)\boldsymbol{i}+t\boldsymbol{j}, \quad 0\le t\le 1$

$$\boldsymbol{F}=(x-y)\boldsymbol{i}+(x+y)\boldsymbol{j}=(1-2t)\boldsymbol{i}+\boldsymbol{j}$$

$$\frac{d\boldsymbol{r}}{dt}=-\boldsymbol{i}+\boldsymbol{j}$$

$$F\bullet\frac{d\boldsymbol{r}}{dt}=\big((1-2t)\boldsymbol{i}+\boldsymbol{j}\big)\cdot(-\boldsymbol{i}+\boldsymbol{j})=-1+2t+1=2t$$

Along $(0,1)\to(0,0)$: $\quad \boldsymbol{r}(t)=(1-t)\boldsymbol{j}, \quad 0\le t\le 1$

$$\boldsymbol{F}=(x-y)\boldsymbol{i}+(x+y)\boldsymbol{j}=(t-1)\boldsymbol{i}+(1-t)\boldsymbol{j}$$

$$\frac{d\boldsymbol{r}}{dt}=-\boldsymbol{j}$$

$$F\bullet\frac{d\boldsymbol{r}}{dt}=\big((t-1)\boldsymbol{i}+(1-t)\boldsymbol{j}\big)\cdot(-\boldsymbol{j})=t-1$$

$$\begin{aligned}\int_C (x-y)\,dx+(x+y)\,dy &= \int_0^1 t\,dt+\int_0^1 2t\,dt+\int_0^1 (t-1)\,dt\\ &=\int_0^1 (t+2t+t-1)\,dt\\ &=\int_0^1 (4t-1)\,dt\\ &=\left[2t^2-t\right]_0^1\\ &=2-1\\ &=1\end{aligned}$$

Solution Section 4.5 – Divergence and Curl

Exercise

Find the divergence of the following vector fields $\boldsymbol{F} = \langle 2x,\ 4y,\ -3z \rangle$

Solution

$$div\boldsymbol{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(4y) + \frac{\partial}{\partial z}(-3z)$$

$$= 2 + 4 - 3$$

$$= 3$$

$$div\boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

Exercise

Find the divergence of the following vector fields $\boldsymbol{F} = \langle -2y,\ 3x,\ z \rangle$

Solution

$$div\boldsymbol{F} = \frac{\partial}{\partial x}(-2y) + \frac{\partial}{\partial y}(3x) + \frac{\partial}{\partial z}(z)$$

$$= 1$$

$$div\boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

Exercise

Find the divergence of the following vector fields $\boldsymbol{F} = \left\langle x^2yz,\ -xy^2z,\ -xyz^2 \right\rangle$

Solution

$$div\boldsymbol{F} = \frac{\partial}{\partial x}\left(x^2yz\right) + \frac{\partial}{\partial y}\left(-xy^2z\right) + \frac{\partial}{\partial z}\left(-xyz^2\right)$$

$$= 2xyz - 2xyz - 2xyz$$

$$= -2xyz$$

$$div\boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

Exercise

Find the divergence of the following vector fields $\boldsymbol{F} = \left\langle x^2 - y^2,\ y^2 - z^2,\ z^2 - x^2 \right\rangle$

Solution

$$div\boldsymbol{F} = \frac{\partial}{\partial x}\left(x^2 - y^2\right) + \frac{\partial}{\partial y}\left(y^2 - z^2\right) + \frac{\partial}{\partial z}\left(z^2 - x^2\right)$$

$$= 2x + 2y + 2z$$

$$div\boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

Exercise

Find the divergence of the following vector fields $\boldsymbol{F} = \left\langle e^{-x+y},\ e^{-y+z},\ e^{-z+x} \right\rangle$

Solution

$$divF = \frac{\partial}{\partial x}\left(e^{-x+y}\right) + \frac{\partial}{\partial y}\left(e^{-y+z}\right) + \frac{\partial}{\partial z}\left(e^{-z+x}\right) \qquad div\boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= -e^{-x+y} - e^{-y+z} - e^{-z+x}$$

Exercise

Find the divergence of the following vector fields $\boldsymbol{F} = \langle yz\cos x,\ xz\cos y,\ xy\cos z \rangle$

Solution

$$div\boldsymbol{F} = \frac{\partial}{\partial x}(yz\cos x) + \frac{\partial}{\partial y}(xz\cos y) + \frac{\partial}{\partial z}(xy\cos z)$$

$$= -yz\sin x - xz\sin y - xy\sin z$$

Exercise

Calculate the divergence of the radial fields. $\boldsymbol{F} = \dfrac{\langle x,\ y,\ z \rangle}{x^2+y^2+z^2} = \dfrac{\boldsymbol{r}}{|\boldsymbol{r}|^2}$

Express the result in terms of the position vector $\boldsymbol{r}$ and its length $|\boldsymbol{r}|$.

Solution

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2+z^2}\right) + \frac{\partial}{\partial z}\left(\frac{z}{x^2+y^2+z^2}\right)$$

$$= \frac{-x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^2} + \frac{x^2-y^2+z^2}{\left(x^2+y^2+z^2\right)^2} + \frac{x^2+y^2-z^2}{\left(x^2+y^2+z^2\right)^2} \qquad \left(\frac{x}{x^2+y^2+z^2}\right)' = \frac{x^2+y^2+z^2-2x^2}{\left(x^2+y^2+z^2\right)^2}$$

$$= \frac{x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^2}$$

$$= \frac{\boldsymbol{r}}{|\boldsymbol{r}|^2}$$

Exercise

Calculate the divergence of the radial fields. $\boldsymbol{F} = \langle x,\, y,\, z\rangle\left(x^2+y^2+z^2\right) = 5|\boldsymbol{r}|^2$

Express the result in terms of the position vector $\boldsymbol{r}$ and its length $|\boldsymbol{r}|$.

Solution

$$\begin{aligned}
\nabla\cdot\boldsymbol{F} &= \frac{\partial}{\partial x}\left(x\left(x^2+y^2+z^2\right)\right)+\frac{\partial}{\partial y}\left(y\left(x^2+y^2+z^2\right)\right)+\frac{\partial}{\partial z}\left(z\left(x^2+y^2+z^2\right)\right)\\
&= \frac{\partial}{\partial x}\left(x^3+xy^2+xz^2\right)+\frac{\partial}{\partial y}\left(x^2y+y^3+yz^2\right)+\frac{\partial}{\partial z}\left(x^2z+y^2z+z^3\right)\\
&= 3x^2+y^2+z^2+x^2+3y^2+z^2+x^2+y^2+3z^2\\
&= 5\left(x^2+y^2+z^2\right)\\
&= 5|\boldsymbol{r}|^2
\end{aligned}$$

Exercise

Calculate the divergence of the radial fields. $\boldsymbol{F} = \dfrac{\langle x,\, y,\, z\rangle}{\left(x^2+y^2+z^2\right)^{3/2}} = \dfrac{\boldsymbol{r}}{|\boldsymbol{r}|^3}$

Express the result in terms of the position vector $\boldsymbol{r}$ and its length $|\boldsymbol{r}|$.

Solution

$$\begin{aligned}
\nabla\cdot\boldsymbol{F} &= \frac{\partial}{\partial x}\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}+\frac{\partial}{\partial y}\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}+\frac{\partial}{\partial z}\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}}\\
&= \frac{x^2+y^2+z^2-3x^2}{\left(x^2+y^2+z^2\right)^{5/2}}+\frac{x^2+y^2+z^2-3y^2}{\left(x^2+y^2+z^2\right)^{5/2}}+\frac{x^2+y^2+z^2-3z^2}{\left(x^2+y^2+z^2\right)^{5/2}}\\
&= 0
\end{aligned}$$

Exercise

Consider the following vector fields $\boldsymbol{F} = \langle x,\, x+y\rangle$, the circle C, and two points P and Q.

a) Without computing the divergence, does the graph suggest that the divergence is positive or negative at P and Q?

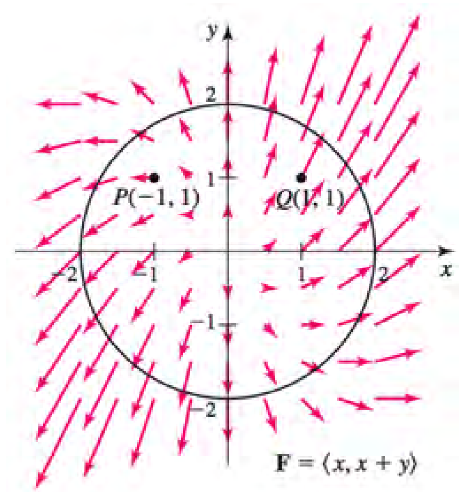
b) Compute the divergence and confirm your conjecture in part (*a*).

c) On what part of C is the flux outward? Inward?

d) Is the net outward flux across C positive or negative?vector

Solution

a) At both P and Q, the arrows going away from the point are larger in both number and magnitude than those going in, so we would expect the divergence to be positive at both points.



b) $\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(x+y) = 1+1 \underline{=2} > 0$

It is positive everywhere.

c) The arrows all point roughly away from the origin, so we the flux is outward everywhere.

d) The net flux across C should be positive.

Exercise

Consider the following vector fields $\boldsymbol{F} = \left\langle x,\ y^2 \right\rangle$, the circle C, and two points P and Q.

a) Without computing the divergence, does the graph suggest that the divergence is positive or negative at P and Q?

b) Compute the divergence and confirm your conjecture in part (*a*).

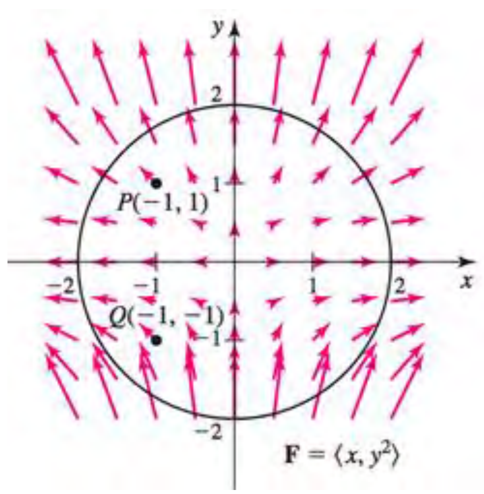
c) On what part of C is the flux outward? Inward?

d) Is the net outward flux across C positive or negative?

Solution

a) At P, the divergence should be positive.

At Q, the larger arrows point in towards Q, so the divergence should be negative.



b) $\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}\left(y^2\right) \underline{= 1 + 2y}$

At $P = (-1,\ 1) \quad \rightarrow \quad \nabla \cdot \boldsymbol{F} \underline{=3}$ $\qquad$ At $Q = (-1,\ -1) \quad \rightarrow \quad \nabla \cdot \boldsymbol{F} \underline{=-1}$

c) The flux is outward above the line $y=-1$; below this line, the flux is inward across *C*.

d) The size of the narrows pointing outward at the top of the circle seems to roughly equal those pointing inward at the bottom, so the remaining outward-pointing arrows result in a net positive flux across *C*.

Exercise

Consider the vector fields $\boldsymbol{F}=\langle 1,\ 0,\ 0\rangle\times\boldsymbol{r}$, where $\boldsymbol{r}=\langle x,\ y,\ z\rangle$

a) Compute the curl field and verify that it has the same direction as the axis of rotation

b) Compute the magnitude of the curl of the field

Solution

a)
$$\begin{aligned}\nabla\times\boldsymbol{F}&=\nabla\times\left[\langle 1,\ 0,\ 0\rangle\times\langle x,\ y,\ z\rangle\right]\\&=\nabla\times\begin{vmatrix}\hat{\boldsymbol{i}}&\hat{\boldsymbol{j}}&\hat{\boldsymbol{k}}\\1&0&0\\x&y&z\end{vmatrix}\\&=\nabla\times\left(-z\,\hat{\boldsymbol{j}}+y\,\hat{\boldsymbol{k}}\right)=\begin{vmatrix}\hat{\boldsymbol{i}}&\hat{\boldsymbol{j}}&\hat{\boldsymbol{k}}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\0&-z&y\end{vmatrix}\\&=(1+1)\hat{\boldsymbol{i}}+(0-0)\hat{\boldsymbol{j}}+(0-0)\hat{\boldsymbol{k}}\\&=2\hat{\boldsymbol{i}}\end{aligned}$$

$$\nabla\times(f,g,h)=\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)\hat{\boldsymbol{i}}+\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)\hat{\boldsymbol{j}}+\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)\hat{\boldsymbol{k}}$$

The curl is the same direction as the axis of rotation.

b) The magnitude of the curl is $\left|2\hat{\boldsymbol{i}}\right|=2$

Exercise

Consider the vector fields $\boldsymbol{F}=\langle 1,\ -1,\ 0\rangle\times\boldsymbol{r}$, where $\boldsymbol{r}=\langle x,\ y,\ z\rangle$

a) Compute the curl field and verify that it has the same direction as the axis of rotation

b) Compute the magnitude of the curl of the field

Solution

a)
$$\begin{aligned}\nabla\times\boldsymbol{F}&=\nabla\times\left[\langle 1,\ -1,\ 0\rangle\times\langle x,\ y,\ z\rangle\right]\\&=\nabla\times\begin{vmatrix}\hat{\boldsymbol{i}}&\hat{\boldsymbol{j}}&\hat{\boldsymbol{k}}\\1&-1&0\\x&y&z\end{vmatrix}\\&=\nabla\times\left(-z\,\hat{\boldsymbol{i}}-z\,\hat{\boldsymbol{j}}+(x+y)\,\hat{\boldsymbol{k}}\right)\\&=(1+1)\hat{\boldsymbol{i}}+(-1-1)\hat{\boldsymbol{j}}+(0-0)\hat{\boldsymbol{k}}\end{aligned}$$

$$\nabla\times(f,g,h)=\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)\hat{\boldsymbol{i}}+\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)\hat{\boldsymbol{j}}+\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)\hat{\boldsymbol{k}}$$

$$\boxed{= 2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}}$$

The curl is the same direction as the axis of rotation.

b) The magnitude of the curl is $\left|2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right| \ \boxed{= 2\sqrt{2}}$

Exercise

Consider the vector fields $\boldsymbol{F} = \langle 1, \ -1, \ 1 \rangle \times \boldsymbol{r}$, where $\boldsymbol{r} = \langle x, \ y, \ z \rangle$

a) Compute the curl field and verify that it has the same direction as the axis of rotation

b) Compute the magnitude of the curl of the field

Solution

a) $\nabla \times \boldsymbol{F} = \nabla \times \left[\langle 1, \ -1, \ 1 \rangle \times \langle x, \ y, \ z \rangle \right]$

$$= \nabla \times \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 1 & -1 & 1 \\ x & y & z \end{vmatrix}$$

$$= \nabla \times \left((-z - y)\ \hat{\boldsymbol{i}} + (x - z)\ \hat{\boldsymbol{j}} + (x + y)\ \hat{\boldsymbol{k}} \right) \quad \nabla \times (f, g, h) = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right)\hat{\boldsymbol{i}} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right)\hat{\boldsymbol{j}} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)\hat{\boldsymbol{k}}$$

$$= (1 + 1)\hat{\boldsymbol{i}} + (-1 - 1)\hat{\boldsymbol{j}} + (1 + 1)\hat{\boldsymbol{k}}$$

$$\boxed{= 2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}}$$

The curl is the same direction as the axis of rotation.

b) The magnitude of the curl is $\left|2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}\right| \ \boxed{= 2\sqrt{3}}$

Exercise

Consider the vector fields $\boldsymbol{F} = \langle 1, \ -2, \ -3 \rangle \times \boldsymbol{r}$, where $\boldsymbol{r} = \langle x, \ y, \ z \rangle$

a) Compute the curl field and verify that it has the same direction as the axis of rotation

b) Compute the magnitude of the curl of the field

Solution

a) $\nabla \times \boldsymbol{F} = \nabla \times \left[\langle 1, \ -2, \ -3 \rangle \times \langle x, \ y, \ z \rangle \right]$

$$= \nabla \times \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 1 & -2 & -3 \\ x & y & z \end{vmatrix}$$

$$= \nabla \times \left((-2z + 3y)\ \hat{\boldsymbol{i}} + (-3x - z)\ \hat{\boldsymbol{j}} + (y + 2x)\ \hat{\boldsymbol{k}} \right)$$

$$= (1 + 1)\hat{\boldsymbol{i}} + (-2 - 2)\hat{\boldsymbol{j}} + (-3 - 3)\hat{\boldsymbol{k}}$$

$$\boxed{= 2\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}}}$$

The curl is the same direction as the axis of rotation.

b) The magnitude of the curl is $\left|2\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}}\right| \underline{= 2\sqrt{14}|}$

Exercise

Compute the curl of the vector fields $\quad \boldsymbol{F} = \left\langle x^2 - y^2,\ xy,\ z \right\rangle$

Solution

$$\nabla \times \boldsymbol{F} = \nabla \times \left\langle x^2 - y^2,\ xy,\ z \right\rangle \qquad \nabla \times (f, g, h) = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{\boldsymbol{i}} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\hat{\boldsymbol{j}} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{\boldsymbol{k}}$$

$$= (0 - 0)\hat{\boldsymbol{i}} + (0 - 0)\hat{\boldsymbol{j}} + (y + 2y)\hat{\boldsymbol{k}}$$

$$\underline{= 3y\,\hat{\boldsymbol{k}}|}$$

Exercise

Compute the curl of the vector fields $\quad \boldsymbol{F} = \left\langle 0,\ z^2 - y^2,\ -yz \right\rangle$

Solution

$$\nabla \times \boldsymbol{F} = \nabla \times \left\langle 0,\ z^2 - y^2,\ -yz \right\rangle \qquad \nabla \times (f, g, h) = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{\boldsymbol{i}} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\hat{\boldsymbol{j}} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{\boldsymbol{k}}$$

$$= (-z - 2z)\hat{\boldsymbol{i}} + (0 - 0)\hat{\boldsymbol{j}} + (0 - 0)\hat{\boldsymbol{k}}$$

$$\underline{= -3z\,\hat{\boldsymbol{i}}|}$$

Exercise

Compute the curl of the vector fields $\quad \boldsymbol{F} = \left\langle z^2 \sin y,\ \ xz^2 \cos y,\ 2xz \sin y \right\rangle$

Solution

$$\nabla \times \boldsymbol{F} = \nabla \times \left\langle z^2 \sin y,\ \ xz^2 \cos y,\ 2xz \sin y \right\rangle \qquad \nabla \times (f, g, h) = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{\boldsymbol{i}} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\hat{\boldsymbol{j}} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{\boldsymbol{k}}$$

$$= (2xz \cos y - 2xz \cos y)\hat{\boldsymbol{i}} + (2z \sin y - 2z \sin y)\hat{\boldsymbol{j}} + \left(z^2 \cos y - z^2 \cos y\right)\hat{\boldsymbol{k}}$$

$$\underline{= 0|}$$

Exercise

Compute the curl of the vector fields $\quad \boldsymbol{F} = \boldsymbol{r} = \langle x,\ y,\ z \rangle$

Solution

$$\nabla \times \boldsymbol{F} = \nabla \times \langle x,\ \ y,\ z \rangle \qquad \nabla \times (f, g, h) = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{\boldsymbol{i}} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\hat{\boldsymbol{j}} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{\boldsymbol{k}}$$

$$=(0-0)\hat{\boldsymbol{i}}+(0-0)\hat{\boldsymbol{j}}+(0-0)\hat{\boldsymbol{k}}$$

$$\underline{=0}$$

Exercise

Compute the curl of the vector fields $\boldsymbol{F}=\dfrac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^{3/2}}=\dfrac{\boldsymbol{r}}{|\boldsymbol{r}|^3}$

Solution

$$\nabla\times\boldsymbol{F}=\nabla\times\frac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\nabla\times(f,g,h)=\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)\hat{\boldsymbol{i}}+\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)\hat{\boldsymbol{j}}+\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)\hat{\boldsymbol{k}}$$

$$\left(U^nV^m\right)'=U^{n-1}V^{m-1}\left(nU'V+mUV'\right)$$

$$\left(z\left(x^2+y^2+z^2\right)^{-3/2}\right)'_y=(1)\left(x^2+y^2+z^2\right)^{-5/2}\left[(0)\left(x^2+y^2+z^2\right)-\tfrac{3}{2}z(2y)\right]$$

$$=\left(x^2+y^2+z^2\right)^{-5/2}(-3yz)$$

$$=\frac{1}{\left(x^2+y^2+z^2\right)^{5/2}}\left((-3yz+3yz)\hat{\boldsymbol{i}}+(-3xz+3xz)\hat{\boldsymbol{j}}+(-3xy+3xy)\hat{\boldsymbol{k}}\right)$$

$$\underline{=0}$$

Exercise

Compute the curl of the vector fields $\boldsymbol{F}=\left\langle 3xz^3e^{y^2},\ 2xz^3e^{y^2},\ 3xz^2e^{y^2}\right\rangle$

Solution

$$\nabla\times\boldsymbol{F}=\nabla\times\left\langle 3xz^3e^{y^2},\ 2xz^3e^{y^2},\ 3xz^2e^{y^2}\right\rangle$$

$$\nabla\times(f,g,h)=\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)\hat{\boldsymbol{i}}+\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)\hat{\boldsymbol{j}}+\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)\hat{\boldsymbol{k}}$$

$$=\left(6xyz^3e^{y^2}-6xz^2e^{y^2}\right)\hat{\boldsymbol{i}}+\left(9xz^2e^{y^2}-3z^2e^{y^2}\right)\hat{\boldsymbol{j}}+\left(2z^3e^{y^2}-6xyz^3e^{y^2}\right)\hat{\boldsymbol{k}}$$

$$\underline{=z^2e^{y^2}\left[(6xyz-6x)\hat{\boldsymbol{i}}+(9x-3)\hat{\boldsymbol{j}}+(2z-6xyz)\hat{\boldsymbol{k}}\right]}$$

Exercise

Show that the general rotation field $\boldsymbol{F}=\boldsymbol{a}\times\boldsymbol{r}$, where $\boldsymbol{a}$ is a nonzero constant vector and $\boldsymbol{r}=\langle x,\ y,\ z\rangle$, has zero divergence.

Solution

Let $\boldsymbol{a}=\langle a_1,\ a_2,\ a_3\rangle$

$$\boldsymbol{F}=\boldsymbol{a}\times\boldsymbol{r}=\langle a_1,\ a_2,\ a_3\rangle\times\langle x,\ y,\ z\rangle$$

$$=\begin{vmatrix}\hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}}\\ a_1 & a_2 & a_3\\ x & y & z\end{vmatrix}$$

$$=\left(a_2z-a_3y\right)\hat{\boldsymbol{i}}+\left(a_3x-a_1z\right)\hat{\boldsymbol{j}}+\left(a_1y-a_2x\right)\hat{\boldsymbol{k}}$$

$$\nabla\times\boldsymbol{F}=\nabla\times\left\langle a_2z-a_3y,\ a_3x-a_1z,\ a_1y-a_2x\right\rangle$$

$$\nabla\times(f,g,h)=\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)\hat{\boldsymbol{i}}+\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)\hat{\boldsymbol{j}}+\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)\hat{\boldsymbol{k}}$$

$$=\left(a_1-a_1\right)\hat{\boldsymbol{i}}+\left(a_2-a_2\right)\hat{\boldsymbol{j}}+\left(a_3-a_3\right)\hat{\boldsymbol{k}}$$

$$=0$$

Exercise

Let $\boldsymbol{a}=\langle 0,\ 1,\ 0\rangle$, $\boldsymbol{r}=\langle x,\ y,\ z\rangle$ and consider the rotation field $\boldsymbol{F}=\boldsymbol{a}\times\boldsymbol{r}$. Use the right-hand rule for cross product to find the direction of $\boldsymbol{F}$ at the points $(0,\ 1,\ 1)$, $(1,\ 1,\ 0)$, $(0,\ 1,\ -1)$, and $(-1,\ 1,\ 0)$

Solution

$$\langle 0,\ 1,\ 0\rangle\times\langle 0,\ 1,\ 1\rangle=\begin{vmatrix}\hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}}\\ 0 & 1 & 0\\ 0 & 1 & 1\end{vmatrix}=\langle 1,\ 0,\ 0\rangle;$$

$\boldsymbol{F}$ points in the positive x-direction

$$\langle 0,\ 1,\ 0\rangle\times\langle 1,\ 1,\ 0\rangle=\begin{vmatrix}\hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}}\\ 0 & 1 & 0\\ 1 & 1 & 0\end{vmatrix}=\langle 0,\ 0,\ -1\rangle;$$

$\boldsymbol{F}$ points in the negative z-direction

$$\langle 0,\ 1,\ 0\rangle\times\langle 0,\ 1,\ -1\rangle=\begin{vmatrix}\hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}}\\ 0 & 1 & 0\\ 0 & 1 & -1\end{vmatrix}=\langle -1,\ 0,\ 0\rangle;$$

$\boldsymbol{F}$ points in the negative x-direction

$$\langle 0,\ 1,\ 0\rangle\times\langle -1,\ 1,\ 0\rangle=\begin{vmatrix}\hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}}\\ 0 & 1 & 0\\ -1 & 1 & 0\end{vmatrix}=\langle 0,\ 0,\ 1\rangle;$$

$\boldsymbol{F}$ points in the positive z-direction

Exercise

Find the exact points on the circle $x^2 + y^2 = 2$ at which the field $\boldsymbol{F} = \langle f, g \rangle = \left\langle x^2, y \right\rangle$ switches from pointing inward to outward on the circle, or vice versa.

Solution

The field switches from inward-pointing to outward-pointing at points where it is tangent to the circle $x^2 + y^2 = 2$, where it is orthogonal to the normal to the circle.

The normal to the circle at (x, y) is a multiple of (x, y), so we want to find x, y so that

$$\langle x, y \rangle \cdot \left\langle x^2, y \right\rangle = x^3 + y^2 = 0$$

$$x^2 + \left(-x^3\right) = 2$$

$$x^3 - x^2 + 2 = 0 \xrightarrow{\quad solutions \quad} x = -1, \ 1 \pm i$$

The solutions are: $x = -1 \quad \rightarrow y = \pm 1$

Exercise

Suppose a solid object in $\mathbb{R}^3$ has a temperature distribution given by $T(x, y, z)$. The heat flow vector field in the object is $\boldsymbol{F} = -k\nabla T$, where the conductivity $k > 0$ is a property of the material. Note that the heat flow vector points in the direction opposite to that of the gradient, which is the direction of greatest temperature decrease. The divergence of the heat flow vector is $\boldsymbol{F} = -k\nabla \cdot \nabla T = -k\nabla^2 T$ (the Laplacian of T). Compute the heat flow vector field and its divergence for the following temperature distribution.

a) $T(x, y, z) = 100\, e^{-\sqrt{x^2+y^2+z^2}}$

b) $T(x, y, z) = 100\, e^{-x^2+y^2+z^2}$

c) $T(x, y, z) = 100\left(1 + \sqrt{x^2 + y^2 + z^2}\right)$

Solution

a) $T(x, y, z) = 100\, e^{-\sqrt{x^2+y^2+z^2}}$

$$\boldsymbol{F} = -k\nabla T = -100\, k\ \nabla\, e^{-\sqrt{x^2+y^2+z^2}}$$

$$\nabla T = \left(\frac{\partial}{\partial x} T\right)\hat{\boldsymbol{i}} + \left(\frac{\partial}{\partial y} T\right)\hat{\boldsymbol{j}} + \left(\frac{\partial}{\partial z} T\right)\hat{\boldsymbol{k}}$$

$$= \frac{100\, k e^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2 + y^2 + z^2}} \langle x, y, z \rangle$$

$$= 100\, k\left[\frac{\partial}{\partial x}\left(\frac{x e^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y e^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial z}\left(\frac{z e^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\right)\right]$$

$$\frac{\partial}{\partial x}\left(\frac{xe^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\right)=e^{-\sqrt{x^2+y^2+z^2}}\frac{\left(1-x^2\left(x^2+y^2+z^2\right)^{-1/2}\right)\sqrt{x^2+y^2+z^2}-x^2\left(x^2+y^2+z^2\right)^{-1/2}}{x^2+y^2+z^2}$$

$$=e^{-\sqrt{x^2+y^2+z^2}}\frac{x^2+y^2+z^2-x^2\left(x^2+y^2+z^2\right)^{1/2}-x^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$=\frac{y^2+z^2-x^2\left(x^2+y^2+z^2\right)^{1/2}}{\left(x^2+y^2+z^2\right)^{3/2}}e^{-\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial}{\partial y}\left(\frac{ye^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\right)=e^{-\sqrt{x^2+y^2+z^2}}\frac{\left(1-y^2\left(x^2+y^2+z^2\right)^{-1/2}\right)\sqrt{x^2+y^2+z^2}-y^2\left(x^2+y^2+z^2\right)^{-1/2}}{x^2+y^2+z^2}$$

$$=\frac{x^2+z^2-y^2\left(x^2+y^2+z^2\right)^{1/2}}{\left(x^2+y^2+z^2\right)^{3/2}}e^{-\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial}{\partial z}\left(\frac{ye^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\right)=e^{-\sqrt{x^2+y^2+z^2}}\frac{\left(1-z^2\left(x^2+y^2+z^2\right)^{-1/2}\right)\sqrt{x^2+y^2+z^2}-z^2\left(x^2+y^2+z^2\right)^{-1/2}}{x^2+y^2+z^2}$$

$$=\frac{x^2+y^2-z^2\left(x^2+y^2+z^2\right)^{1/2}}{\left(x^2+y^2+z^2\right)^{3/2}}e^{-\sqrt{x^2+y^2+z^2}}$$

$$\boldsymbol{F}=100\,k\frac{e^{-\sqrt{x^2+y^2+z^2}}}{\left(x^2+y^2+z^2\right)^{3/2}}\left(2\left(x^2+y^2+z^2\right)-\left(x^2+y^2+z^2\right)\left(x^2+y^2+z^2\right)^{1/2}\right)$$

$$=\frac{100\ k\ e^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\left(2-\sqrt{x^2+y^2+z^2}\right)$$

b) $T(x,\ y,\ z)=100\,e^{-x^2+y^2+z^2}$

$$\boldsymbol{F}=-k\nabla T=-100\,k\ \nabla\ e^{-x^2+y^2+z^2}$$

$$\nabla\ e^{-x^2+y^2+z^2}=e^{-x^2+y^2+z^2}\left\langle -2x,\ 2y,\ 2z\right\rangle$$

$$=-200\,k\ e^{-x^2+y^2+z^2}\ \left\langle -x,\ y,\ z\right\rangle$$

$$=-200\,k\left[\frac{\partial}{\partial x}\left(-xe^{-x^2+y^2+z^2}\right)+\frac{\partial}{\partial y}\left(ye^{-x^2+y^2+z^2}\right)+\frac{\partial}{\partial z}\left(ze^{-x^2+y^2+z^2}\right)\right]$$

$$=-200\,ke^{-x^2+y^2+z^2}\left(-1+2x^2+1+2y^2+1+2z^2\right)$$

$$= -200\, ke^{-x^2+y^2+z^2}\left(1+2x^2+2y^2+2z^2\right)$$

c) $T(x,\, y,\, z) = 100\left(1+\sqrt{x^2+y^2+z^2}\right)$

$$\boldsymbol{F} = -k\nabla T = -100\, k\ \nabla\left(1+\sqrt{x^2+y^2+z^2}\right)$$

$$= -100\, k\left[\left(x^2+y^2+z^2\right)^{-1/2}\langle x,\, y,\, z\rangle\right]$$

$$= -100\, k\left[\frac{\partial}{\partial x}\left(x\left(x^2+y^2+z^2\right)^{-1/2}\right)+\frac{\partial}{\partial y}\left(y\left(x^2+y^2+z^2\right)^{-1/2}\right)+\frac{\partial}{\partial z}\left(z\left(x^2+y^2+z^2\right)^{-1/2}\right)\right]$$

$$= -100\, k\left(x^2+y^2+z^2\right)^{-1/2}\left[3-\left(x^2+y^2+z^2\right)\left(x^2+y^2+z^2\right)^{-1}\right]$$

$$= -100\, k\left(x^2+y^2+z^2\right)^{-1/2}\left[3-1\right]$$

$$= \frac{-200\, k}{\sqrt{x^2+y^2+z^2}}$$

Solution Section 4.6 – Surfaces and Area

Exercise

Find a parametrization of the surface: The paraboloid $z = x^2 + y^2,\ z \le 4$

Solution

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$z = x^2 + y^2 = r^2 \qquad z \le 4 \ \rightarrow\ r^2 \le 4 \Rightarrow 0 \le r \le 2$$

Then: $\boldsymbol{r}(r,\ \theta) = (r\cos\theta)\boldsymbol{i} + (r\sin\theta)\boldsymbol{j} + r^2\boldsymbol{k} \quad 0 \le r \le 2, \quad 0 \le \theta \le 2\pi$

Exercise

Find a parametrization of the surface: The portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 4$

Solution

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$z = 2\sqrt{x^2 + y^2} = 2r \qquad \begin{aligned} z = 2 &\rightarrow r = 1 \\ z = 4 &\rightarrow r = 2 \end{aligned}$$

Then: $\boldsymbol{r}(r,\ \theta) = (r\cos\theta)\boldsymbol{i} + (r\sin\theta)\boldsymbol{j} + 2r\boldsymbol{k} \quad 1 \le r \le 2, \quad 0 \le \theta \le 2\pi$

Exercise

Find a parametrization of the surface cut from the sphere $x^2 + y^2 + z^2 = 8$ by the plane $z = -2$

Solution

$$x^2 + y^2 + z^2 = 8 = \rho^2 \ \rightarrow\ \rho = 2\sqrt{2}$$

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$

$$x = 2\sqrt{2}\sin\phi\cos\theta, \quad y = 2\sqrt{2}\sin\phi\sin\theta, \quad z = 2\sqrt{2}\cos\phi$$

$$z = -2 \ \Rightarrow\ 2\sqrt{2}\cos\phi = -2 \rightarrow \cos\phi = -\frac{1}{\sqrt{2}} \ \rightarrow \boxed{\phi = \frac{3\pi}{4}}$$

$$z = 2\sqrt{2} \ \Rightarrow\ 2\sqrt{2}\cos\phi = 2\sqrt{2} \rightarrow \cos\phi = 1 \ \rightarrow \boxed{\phi = 0}$$

Then: $\boldsymbol{r}(\phi,\ \theta) = \left(2\sqrt{2}\sin\phi\cos\theta\right)\boldsymbol{i} + \left(2\sqrt{2}\sin\phi\sin\theta\right)\boldsymbol{j} + \left(2\sqrt{2}\cos\phi\right)\boldsymbol{k}$

$$0 \le \phi \le \frac{3\pi}{4}, \quad 0 \le \theta \le 2\pi$$

Exercise

Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral of the portion of the plane $y+2z=2$ inside the cylinder $x^2+y^2=1$

Solution

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$y+2z=2 \quad \rightarrow \quad z=\frac{2-y}{2}=\frac{2-r\sin\theta}{2}$$

Then: $\boldsymbol{r}(r,\ \theta)=(r\cos\theta)\boldsymbol{i}+(r\sin\theta)\boldsymbol{j}+\left(\frac{2-r\sin\theta}{2}\right)\boldsymbol{k} \qquad 0\le r\le 1, \quad 0\le\theta\le 2\pi$

$$\boldsymbol{r}_r=(\cos\theta)\boldsymbol{i}+(\sin\theta)\boldsymbol{j}-\left(\frac{\sin\theta}{2}\right)\boldsymbol{k}$$

$$\boldsymbol{r}_\theta=(-r\sin\theta)\boldsymbol{i}+(r\cos\theta)\boldsymbol{j}-\left(\frac{r\cos\theta}{2}\right)\boldsymbol{k}$$

$$\boldsymbol{r}_r\times\boldsymbol{r}_\theta=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \cos\theta & \sin\theta & -\frac{1}{2}\sin\theta\\ -r\sin\theta & r\cos\theta & -\frac{1}{2}r\cos\theta\end{vmatrix}$$

$$=\left(-\frac{1}{2}r\cos\theta\sin\theta+\frac{1}{2}r\cos\theta\sin\theta\right)\boldsymbol{i}-\left(-\frac{1}{2}r\cos^2\theta-\frac{1}{2}r\sin^2\theta\right)\boldsymbol{j}$$

$$+\left(r\cos^2\theta+r\sin^2\theta\right)\boldsymbol{k}$$

$$=\frac{1}{2}r\boldsymbol{j}+r\boldsymbol{k}$$

$$\left|\boldsymbol{r}_r\times\boldsymbol{r}_\theta\right|=\sqrt{\frac{r^2}{4}+r^2}=\frac{\sqrt{5}}{2}r$$

$$A=\int_0^{2\pi}\int_0^1\frac{\sqrt{5}}{2}r\ drd\theta$$

$$=\frac{\sqrt{5}}{4}\int_0^{2\pi}\left[r^2\right]_0^1 d\theta$$

$$=\frac{\sqrt{5}}{4}\int_0^{2\pi}d\theta$$

$$=\frac{\sqrt{5}}{4}(2\pi)$$

$$\boxed{=\frac{\pi\sqrt{5}}{2}}$$

Exercise

Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral of the portion of the cone $z=\dfrac{\sqrt{x^2+y^2}}{3}$ between the planes $z=1$ and $z=\frac{4}{3}$

Solution

$x=r\cos\theta, \quad y=r\sin\theta$

$z=\dfrac{\sqrt{x^2+y^2}}{3}=\dfrac{r}{3}$ $\qquad$ $z=1\to r=3$

$z=\frac{4}{3}\to r=4$

Then: $\boldsymbol{r}(r,\ \theta)=(r\cos\theta)\boldsymbol{i}+(r\sin\theta)\boldsymbol{j}+\left(\frac{r}{3}\right)\boldsymbol{k}$ $\qquad 3\le r\le 4, \quad 0\le\theta\le 2\pi$

$\boldsymbol{r}_r=(\cos\theta)\boldsymbol{i}+(\sin\theta)\boldsymbol{j}+\frac{1}{3}\boldsymbol{k}$

$\boldsymbol{r}_\theta=(-r\sin\theta)\boldsymbol{i}+(r\cos\theta)\boldsymbol{j}$

$$\boldsymbol{r}_r\times\boldsymbol{r}_\theta=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \cos\theta & \sin\theta & \frac{1}{3}\\ -r\sin\theta & r\cos\theta & 0\end{vmatrix}$$

$$=\left(0-\tfrac{1}{3}r\cos\theta\right)\boldsymbol{i}-\left(0+\tfrac{1}{3}r\sin\theta\right)\boldsymbol{j}+\left(r\cos^2\theta+r\sin^2\theta\right)\boldsymbol{k}$$

$$=\left(-\tfrac{1}{3}r\cos\theta\right)\boldsymbol{i}-\left(\tfrac{1}{3}r\sin\theta\right)\boldsymbol{j}+r\boldsymbol{k}$$

$$\left|\boldsymbol{r}_r\times\boldsymbol{r}_\theta\right|=\sqrt{\tfrac{1}{9}r^2\cos^2\theta+\tfrac{1}{9}r^2\sin^2\theta+r^2}=\sqrt{\tfrac{1}{9}r^2+r^2}=\frac{\sqrt{10}}{3}r$$

$$A=\int_0^{2\pi}\int_3^4\frac{\sqrt{10}}{3}r\ drd\theta$$

$$=\frac{\sqrt{10}}{6}\int_0^{2\pi}\left[r^2\right]_3^4\ d\theta$$

$$=\frac{\sqrt{10}}{6}(16-9)\int_0^{2\pi}\ d\theta$$

$$=\frac{7\sqrt{10}}{6}(2\pi)$$

$$=\frac{7\pi\sqrt{10}}{3}$$

Exercise

Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral of the portion of the cylinder $x^2 + z^2 = 10$ between the planes $y = -1$ and $y = 1$

Solution

$$x = u\cos v, \quad z = u\sin v$$

$$x^2 + z^2 = 10 = u^2\cos^2 v + u^2\sin^2 v$$

$$u^2 = 10 \to u = \sqrt{10}$$

Then: $\boldsymbol{r}(y,\, v) = (u\cos v)\boldsymbol{i} + y\boldsymbol{j} + (u\sin v)\boldsymbol{k}$

$$= \left(\sqrt{10}\cos v\right)\boldsymbol{i} + y\boldsymbol{j} + \left(\sqrt{10}\sin v\right)\boldsymbol{k}$$

$$\boldsymbol{r}_y = \boldsymbol{j}$$

$$\boldsymbol{r}_v = \left(-\sqrt{10}\sin v\right)\boldsymbol{i} + \left(\sqrt{10}\cos v\right)\boldsymbol{k}$$

$$\boldsymbol{r}_y \times \boldsymbol{r}_v = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 1 & 0 \\ -\sqrt{10}\sin v & 0 & \sqrt{10}\cos v \end{vmatrix}$$

$$= \left(\sqrt{10}\cos v\right)\boldsymbol{i} + \left(\sqrt{10}\sin v\right)\boldsymbol{k}$$

$$\left|\boldsymbol{r}_r \times \boldsymbol{r}_\theta\right| = \sqrt{10\cos^2 v + 10\sin^2 v} = \sqrt{10}$$

$$A = \int_0^{2\pi}\int_{-1}^{1} \sqrt{10}\ dydv$$

$$= \sqrt{10}\int_0^{2\pi} [y]_{-1}^{1}\ dv$$

$$= 2\sqrt{10}\int_0^{2\pi} dv$$

$$\boxed{= 4\pi\sqrt{10}}$$

Exercise

Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral of the portion of the cap cut from the paraboloid $z = x^2 + y^2$ between the planes $z = 1$ and $z = 4$

Solution

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$z = x^2 + y^2 = r^2 \qquad \begin{matrix} z = 1 \to r = 1 \\ z = 4 \to r = 2 \end{matrix}$$

Then: $\boldsymbol{r}(r,\ \theta) = (r\cos\theta)\boldsymbol{i} + (r\sin\theta)\boldsymbol{j} + r^2\boldsymbol{k} \qquad 1 \le r \le 2, \quad 0 \le \theta \le 2\pi$

$$\boldsymbol{r}_r = (\cos\theta)\boldsymbol{i} + (\sin\theta)\boldsymbol{j} + 2r\boldsymbol{k}$$

$$\boldsymbol{r}_\theta = (-r\sin\theta)\boldsymbol{i} + (r\cos\theta)\boldsymbol{j}$$

$$\boldsymbol{r}_r \times \boldsymbol{r}_\theta = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \left(0 - 2r^2\cos\theta\right)\boldsymbol{i} - \left(0 + 2r^2\sin\theta\right)\boldsymbol{j} + \left(r\cos^2\theta + r\sin^2\theta\right)\boldsymbol{k}$$

$$= \left(-2r^2\cos\theta\right)\boldsymbol{i} - \left(2r^2\sin\theta\right)\boldsymbol{j} + r\boldsymbol{k}$$

$$\left|\boldsymbol{r}_r \times \boldsymbol{r}_\theta\right| = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2}$$

$$= \sqrt{4r^4 + r^2}$$

$$= r\sqrt{4r^2 + 1}$$

$$A = \int_0^{2\pi}\int_1^2 r\sqrt{4r^2+1}\ drd\theta \qquad d\left(4r^2+1\right) = 8rdr$$

$$= \frac{1}{8}\int_0^{2\pi}\int_1^2 \left(4r^2+1\right)^{1/2}\ d\left(4r^2+1\right)d\theta$$

$$= \frac{1}{12}\int_0^{2\pi}\left[\left(4r^2+1\right)^{3/2}\right]_1^2\ d\theta$$

$$= \frac{17^{3/2} - 5^{3/2}}{12}\int_0^{2\pi} d\theta$$

$$= \frac{17\sqrt{17} - 5\sqrt{5}}{12}\left[\theta\right]_0^{2\pi}$$

$$= \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$$

Exercise

Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 2$.

Solution

$$\boldsymbol{p} = \boldsymbol{k}, \quad \nabla f = 2x\boldsymbol{i} + 2y\boldsymbol{j} - \boldsymbol{k}$$

$$|\nabla f| = \sqrt{(2x)^2 + (2y)^2 + 1} = \sqrt{4x^2 + 4y^2 + 1}$$

$$|\nabla f \bullet \boldsymbol{p}| = 1$$

$$z = 2 \quad \Rightarrow \quad x^2 + y^2 = 2$$

$$x = r\cos\theta, \quad y = r\sin\theta \qquad\qquad r^2 = x^2 + y^2 = 2 \Rightarrow r = \sqrt{2}$$

$$\begin{aligned}
Surface\ area &= \iint_R \frac{|\nabla F|}{|\nabla F \cdot p|} dA \\
&= \iint_R \sqrt{4x^2 + 4y^2 + 1}\ dxdy \\
&= \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + 1}\ rdrd\theta \\
&= \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2 + 1}\ rdrd\theta \qquad\qquad d\left(4r^2 + 1\right) = 8rdr \\
&= \frac{1}{8} \int_0^{2\pi} \int_0^{\sqrt{2}} \left(4r^2 + 1\right)^{1/2}\ d\left(4r^2 + 1\right) d\theta \\
&= \frac{1}{12} \int_0^{2\pi} \left[\left(4r^2 + 1\right)^{3/2}\right]_0^{\sqrt{2}}\ d\theta \\
&= \frac{27 - 1}{12} \int_0^{2\pi} d\theta \\
&= \frac{13}{6}[\theta]_0^{2\pi} \\
&= \boxed{\frac{13\pi}{3}}
\end{aligned}$$

Exercise

Find the area of the portion of the surface $x^2 - 2z = 0$ that lies above the triangle bounded by the lines $x = \sqrt{3}$, $y = 0$, and $y = x$ in the *xy*-plane.

Solution

$$\boldsymbol{p} = \boldsymbol{k}$$

$$\nabla f = 2x\boldsymbol{i} - 2\boldsymbol{k}$$

$$|\nabla f| = \sqrt{4x^2 + 4} = 2\sqrt{x^2 + 1}$$

$$|\nabla f \bullet \boldsymbol{p}| = |(2x\boldsymbol{i} - 2\boldsymbol{k}) \bullet (\boldsymbol{k})| = 2$$

$$Surface\ area = \iint_R \frac{|\nabla F|}{|\nabla F \cdot p|} dA$$

$$= \int_0^{\sqrt{3}} \int_0^x \frac{2\sqrt{x^2 + 1}}{2} dydx$$

$$= \int_0^{\sqrt{3}} \left[y\sqrt{x^2 + 1} \right]_0^x dx$$

$$= \int_0^{\sqrt{3}} x\sqrt{x^2 + 1} dx \qquad d\left(x^2 + 1\right) = 2xdx$$

$$= \frac{1}{2} \int_0^{\sqrt{3}} \left(x^2 + 1\right)^{1/2} d\left(x^2 + 1\right)$$

$$= \frac{1}{2} \left[\frac{2}{3} \left(x^2 + 1\right)^{3/2} \right]_0^{\sqrt{3}}$$

$$= \frac{1}{3}\left(4^{3/2} - 1\right)$$

$$= \frac{1}{3}(8 - 1)$$

$$= \frac{7}{3}$$

Exercise

Find the area of the cap cut from the sphere $x^2+y^2+z^2=2$ by the cone $z=\sqrt{x^2+y^2}$.

Solution

$$\boldsymbol{p}=\boldsymbol{k} \qquad \nabla f=2x\boldsymbol{i}+2y\boldsymbol{i}+2z\boldsymbol{k}$$

$$|\nabla f|=\sqrt{4x^2+4y^2+4z^2}=2\sqrt{x^2+y^2+z^2}=2\sqrt{2}$$

$$|\nabla f\bullet \boldsymbol{p}|=|(2x\boldsymbol{i}+2y\boldsymbol{i}+2z\boldsymbol{k})\bullet(\boldsymbol{k})|=2z$$

$$z=\sqrt{x^2+y^2} \quad \to \quad x^2+y^2+z^2=z^2+z^2=2z^2=2 \Rightarrow z=1$$

$$x^2+y^2+z^2=2 \to z=\sqrt{2-\left(x^2+y^2\right)}$$

$$\begin{aligned}
Surface\ area &= \iint_R \frac{|\nabla F|}{|\nabla F\cdot p|}dA \\
&= \iint_R \frac{2\sqrt{2}}{2z}dydx \\
&= \sqrt{2}\iint_R \frac{1}{\sqrt{2-\left(x^2+y^2\right)}}dydx \\
&= \sqrt{2}\int_0^{2\pi}\int_0^1 \frac{1}{\sqrt{2-r^2}}rdrd\theta \\
&= -\frac{\sqrt{2}}{2}\int_0^{2\pi}\int_0^1 \left(2-r^2\right)^{-1/2}d\left(2-r^2\right)d\theta \\
&= -\frac{\sqrt{2}}{2}\int_0^{2\pi}\left[2\left(2-r^2\right)^{1/2}\right]_0^1 d\theta \\
&= -\sqrt{2}\int_0^{2\pi}\left(1-\sqrt{2}\right)d\theta \\
&= \sqrt{2}\left(\sqrt{2}-1\right)[\theta]_0^{2\pi} \\
&= 2\pi\left(2-\sqrt{2}\right)
\end{aligned}$$

Exercise

Find the area of the ellipse cut from the plane $z = cx$ (c a constant) by the cylinder $x^2 + y^2 = 1$.

Solution

$$cx - z = 0$$

$$\boldsymbol{p} = \boldsymbol{k} \qquad \nabla f = c\boldsymbol{i} - \boldsymbol{k}$$

$$|\nabla f| = \sqrt{c^2 + 1}$$

$$|\nabla f \bullet \boldsymbol{p}| = |(c\boldsymbol{i} - \boldsymbol{k}) \bullet (\boldsymbol{k})| = 1$$

$$\begin{aligned}
Surface\ area &= \iint_R \frac{|\nabla F|}{|\nabla F \cdot p|} dA \\
&= \iint_R \sqrt{c^2 + 1}\ dxdy \\
&= \int_0^{2\pi} \int_0^1 \sqrt{c^2 + 1}\ rdrd\theta \\
&= \frac{1}{2}\sqrt{c^2 + 1} \int_0^{2\pi} \left[r^2 \right]_0^1 d\theta \\
&= \frac{1}{2}\sqrt{c^2 + 1} \int_0^{2\pi} d\theta \\
&= \frac{1}{2}\sqrt{c^2 + 1}[\theta]_0^{2\pi} \\
&= \pi\sqrt{c^2 + 1}
\end{aligned}$$

Exercise

Find the area of the surface cut from the nose of the paraboloid $x = 1 - y^2 - z^2$ by yz-plane.

Solution

$$f_y(y,z) = -2y, \quad f_z(y,z) = -2z \qquad \sqrt{f_y^2 + f_z^2 + 1} = \sqrt{4y^2 + 4z^2 + 1}$$

$$\begin{aligned}
Area &= \iint_R \sqrt{4y^2 + 4z^2 + 1}\ dydz \\
&= \int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1}\ rdrd\theta \qquad d\left(4r^2 + 1\right) = 8rdr
\end{aligned}$$

$$=\frac{1}{8}\int_0^{2\pi}\int_0^1\left(4r^2+1\right)^{1/2}\ d\left(4r^2+1\right)d\theta$$

$$=\frac{1}{12}\int_0^{2\pi}\left[\left(4r^2+1\right)^{3/2}\right]_0^1 d\theta$$

$$=\frac{1}{12}\left(5^{3/2}-1\right)\int_0^{2\pi} d\theta$$

$$=\frac{5\sqrt{5}-1}{12}\left[\theta\right]_0^{2\pi}$$

$$=\frac{\pi}{6}\left(5\sqrt{5}-1\right)$$

Exercise

Find the area of the surface in the first octant cut from the cylinder $y=\frac{2}{3}z^{3/2}$ by the planes $x=1$ and $y=\frac{16}{3}$

Solution

$$y=\frac{2}{3}z^{3/2},\quad f_x(x,z)=0,\quad f_z(x,z)=z^{1/2}$$

$$\sqrt{f_x^2+f_z^2+1}=\sqrt{z+1}$$

$$y=\frac{2}{3}z^{3/2}=\frac{16}{3}\Rightarrow z^{3/2}=8\rightarrow \underline{z=8^{2/3}=4}$$

$$Area=\int_0^4\int_0^1\sqrt{z+1}\ dxdz$$

$$=\int_0^4\left[x\sqrt{z+1}\right]_0^1 dz \qquad\qquad d(z+1)=dz$$

$$=\int_0^4(z+1)^{1/2}d(z+1)$$

$$=\left[(z+1)^{3/2}\right]_0^4$$

$$=\frac{2}{3}\left(5^{3/2}-1\right)$$

$$=\frac{2}{3}\left(5\sqrt{5}-1\right)$$

Solution

Exercise

Verify that the line integral and the surface integral of Stokes' Theorem are equal for the following vector fields, surfaces S, and closed curves C. Assume that C has counterclockwise orientation and S has a consistent orientation.

$\boldsymbol{F} = \langle y,\ -x,\ 10\rangle$; S is the upper half of the sphere $x^2+y^2+z^2=1$ and C is the circle $x^2+y^2=1$ in the xy-plane

Solution

$$\boldsymbol{F} = \langle y,\ -x,\ 10\rangle = \langle \sin t,\ -\cos t,\ 10\rangle$$

$$x^2+y^2=1=r^2 \ \Rightarrow\ \boldsymbol{r}(t)=\langle \cos t,\ \sin t,\ 0\rangle \ \Rightarrow\ \boldsymbol{r}'(t)=\langle -\sin t,\ \cos t,\ 0\rangle$$

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_R \langle \sin t,\ -\cos t,\ 10\rangle\cdot\langle -\sin t,\ \cos t,\ 0\rangle dA$$

$$= \int_0^{2\pi}\left(-\sin^2 t-\cos^2 t\right)dt \qquad \sin^2 t+\cos^2 t=1$$

$$= -\int_0^{2\pi} dt$$

$$\underline{= -2\pi}$$

$$\nabla\times\boldsymbol{F} = \nabla\times\langle y,\ -x,\ 10\rangle$$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 10 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(10)+\frac{\partial}{\partial z}(x)\right)\hat{\boldsymbol{i}} + \left(\frac{\partial}{\partial z}(y)-\frac{\partial}{\partial x}(10)\right)\hat{\boldsymbol{j}} + \left(\frac{\partial}{\partial x}(-x)-\frac{\partial}{\partial y}(y)\right)\hat{\boldsymbol{k}}$$

$$\underline{= \langle 0,\ 0,\ -2\rangle}$$

$$\iint_S (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\, dS = \iint_R \langle 0,\ 0,\ -2\rangle\cdot\left\langle \frac{x}{z},\ \frac{y}{z},\ 1\right\rangle dA$$

$$= \int_0^{2\pi} d\theta \int_0^1 -2r\,dr$$

$$= -(2\pi)\left[r^2\right]_0^1$$

$$\underline{= -2\pi}$$

Or

Using the standard parametrization of the sphere

$\rightarrow \boldsymbol{n} = \left\langle \sin^2\phi\cos\theta,\ \sin^2\phi\cos\theta,\ \cos\phi\sin\phi \right\rangle$

$$\iint_S (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\, dS = \iint_R \langle 0,\ 0,\ -2\rangle\cdot\left\langle \sin^2\phi\cos\theta,\ \sin^2\phi\cos\theta,\ \cos\phi\sin\phi \right\rangle dA$$

$$= \int_0^{2\pi}\int_0^{\pi/2} (-2\cos\phi\sin\phi)\,d\phi d\theta$$

$$= -\int_0^{2\pi} d\theta \int_0^{\pi/2} \sin 2\phi d\phi$$

$$= -(2\pi)\left[-\tfrac{1}{2}\cos 2\phi\right]_0^{\pi/2}$$

$$= -2\pi$$

Exercise

Verify that the line integral and the surface integral of Stokes' Theorem are equal for the following vector fields, surfaces *S*, and closed curves *C*. Assume that *C* has counterclockwise orientation and *S* has a consistent orientation.

$\boldsymbol{F} = \langle 0,\ -x,\ y\rangle$; *S* is the upper half of the sphere $x^2+y^2+z^2=4$ and *C* is the circle $x^2+y^2=4$ in the *xy*-plane

Solution

$x^2+y^2=4=r^2 \Rightarrow \boldsymbol{r}(t)=\langle 2\cos t,\ 2\sin t,\ 0\rangle \Rightarrow \boldsymbol{r}'(t)=\langle -2\sin t,\ 2\cos t,\ 0\rangle$

$\boldsymbol{F} = \langle 0,\ -x,\ y\rangle = \langle 0,\ -2\cos t,\ 2\sin t\rangle$

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_R \langle 0,\ -2\cos t,\ 2\sin t\rangle\cdot\langle -2\sin t,\ 2\cos t,\ 0\rangle dA$$

$$= \int_0^{2\pi}\left(-4\cos^2 t\right)dt$$

$$= -2\int_0^{2\pi}(1+\cos 2t)\,dt$$

$$= -2\left[t+\tfrac{1}{2}\sin 2t\right]_0^{2\pi}$$

$$= -4\pi$$

$\nabla\times\boldsymbol{F} = \nabla\times\langle 0,\ -x,\ y\rangle$

$$=\begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -x & y \end{vmatrix}$$

$$=\left(\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(x)\right)\hat{\boldsymbol{i}}+\left(\frac{\partial}{\partial z}(0)-\frac{\partial}{\partial x}(y)\right)\hat{\boldsymbol{j}}+\left(\frac{\partial}{\partial x}(-x)-\frac{\partial}{\partial y}(0)\right)\hat{\boldsymbol{k}}$$

$$=\langle 1,\ 0,\ -1\rangle$$

$$\iint_S (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\,dS=\iint_R \langle 1,\ 0,\ -1\rangle\cdot\left\langle \frac{x}{z},\ \frac{y}{z},\ 1\right\rangle dA$$

$$=\iint_R \left(\frac{x}{z}-1\right)dA$$

$$=\int_0^{2\pi}\int_0^2 \left(\frac{r\cos\theta}{\sqrt{4-r^2}}-1\right)rdrd\theta$$

$$=\int_0^{2\pi}\int_0^2 \left(\frac{r^2\cos\theta}{\sqrt{4-r^2}}-r\right)drd\theta \qquad \int\frac{x^2}{\sqrt{a^2-x^2}}dx=\frac{a^2}{2}\sin^{-1}\frac{x}{a}-\frac{x}{2}\sqrt{a^2-x^2}$$

$$=\int_0^{2\pi}\left(\left(2\sin^{-1}\left(\frac{r}{2}\right)-\frac{r}{2}\sqrt{4-r^2}\right)\cos\theta-\frac{1}{2}r^2\right)\Bigg|_0^2 d\theta$$

$$=\int_0^{2\pi}(\pi\cos\theta-2)\,d\theta$$

$$=\left[\pi\sin\theta-2\theta\right]_0^{2\pi}$$

$$=-4\pi$$

Exercise

Verify that the line integral and the surface integral of Stokes' Theorem are equal for the following vector fields, surfaces *S*, and closed curves *C*. Assume that *C* has counterclockwise orientation and *S* has a consistent orientation.

$\boldsymbol{F}=\langle x,\ y,\ z\rangle$; *S* is the paraboloid $z=8-x^2-y^2$ for $0\le z\le 8$ and *C* is the circle $x^2+y^2=8$ in the *xy*-plane

Solution

$$x^2+y^2=8=r^2 \ \Rightarrow\ \boldsymbol{r}(t)=\left\langle 2\sqrt{2}\cos t,\ 2\sqrt{2}\sin t,\ 0\right\rangle \ \Rightarrow\ \boldsymbol{r}'(t)=\left\langle -2\sqrt{2}\sin t,\ 2\sqrt{2}\cos t,\ 0\right\rangle$$

$$\boldsymbol{F}=\langle x,\ y,\ z\rangle=\left\langle 2\sqrt{2}\cos t,\ 2\sqrt{2}\sin t,\ 0\right\rangle$$

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_R \left\langle 2\sqrt{2}\cos t,\ 2\sqrt{2}\sin t,\ 0\right\rangle \cdot \left\langle -2\sqrt{2}\sin t,\ 2\sqrt{2}\cos t,\ 0\right\rangle dA$$

$$= \int_0^{2\pi} \left(-8\cos t\sin t + 8\cos t\sin t\right)dt$$

$$\underline{= 0|}$$

Surface integral: $\iint_S (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\ dS \ \underline{= 0|}$

Exercise

Verify that the line integral and the surface integral of Stokes' Theorem are equal for the following vector fields, surfaces *S*, and closed curves *C*. Assume that *C* has counterclockwise orientation and *S* has a consistent orientation.

$\boldsymbol{F} = \langle 2z,\ -4x,\ 3y\rangle$; *S* is the cap of the sphere $x^2+y^2+z^2=169$ above the plane $z=12$ and *C* is the boundary of *S*.

Solution

$x^2+y^2+12^2=169 \ \rightarrow\ x^2+y^2=25$ is the intersection of the sphere with the plane $z=12$.

$\boldsymbol{r}(t) = \langle 5\cos t,\ 5\sin t,\ 0\rangle \ \Rightarrow\ \boldsymbol{r}'(t) = \langle -5\sin t,\ 5\cos t,\ 0\rangle$

$\boldsymbol{F} = \langle 2z,\ -4x,\ 3y\rangle = \langle 2(12),\ -4\times 5\cos t,\ 3\times 5\sin t\rangle = \langle 24,\ -20\cos t,\ 15\sin t\rangle$

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_R \langle 24,\ -20\cos t,\ 15\sin t\rangle \cdot \langle -5\sin t,\ 5\cos t,\ 0\rangle dA$$

$$= \int_0^{2\pi}\left(-120\sin t - 100\cos^2 t\right)dt$$

$$= 10\int_0^{2\pi}\left(-12\sin t - 5 - 5\cos 2t\right)dt$$

$$= 10\left[12\cos t - 5t - \frac{5}{2}\sin 2t\right]_0^{2\pi}$$

$$= 10(12 - 10\pi - 12)$$

$$\underline{= -100\pi|}$$

$\nabla\times\boldsymbol{F} = \nabla\times\langle 2z,\ -4x,\ 3y\rangle$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & -4x & 3y \end{vmatrix}$$

$$= (3+0)\hat{\boldsymbol{i}} + (2-0)\hat{\boldsymbol{j}} + (-4-0)\hat{\boldsymbol{k}}$$

$$\underline{= \langle 3,\ 2,\ -4 \rangle}|$$

$$\iint\limits_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, dS = \iint\limits_R \langle 3,\ 2,\ -4 \rangle \cdot \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle dA$$

$$= \iint\limits_R \left(\frac{3x}{z} + \frac{2y}{z} - 4 \right) dA$$

$$= \int_0^{2\pi} \int_0^5 \left(\frac{3r\cos\theta}{\sqrt{169-r^2}} + \frac{2r\sin\theta}{\sqrt{169-r^2}} - 4 \right) r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^5 \left(\frac{3r^2\cos\theta}{\sqrt{169-r^2}} + \frac{2r^2\sin\theta}{\sqrt{169-r^2}} - 4r \right) dr\,d\theta$$

$$\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{x}{2}\sqrt{a^2-x^2}$$

$$= \int_0^{2\pi} \left(\begin{array}{l} 3\left(\frac{169}{2}\sin^{-1}\left(\frac{r}{13}\right) - \frac{r}{2}\sqrt{169-r^2} \right)\cos\theta \\ \quad + 2\left(\frac{169}{2}\sin^{-1}\left(\frac{r}{13}\right) - \frac{r}{2}\sqrt{169-r^2} \right)\sin\theta - 2r^2 \end{array} \right)_0^5 d\theta$$

$$= \int_0^{2\pi} \left(\left(\frac{507}{2}\sin^{-1}\left(\frac{5}{13}\right) - 90 \right)\cos\theta + \left(169\sin^{-1}\left(\frac{5}{13}\right) - 60 \right)\sin\theta - 50 \right) d\theta$$

$$= \left[\left(\frac{507}{2}\sin^{-1}\left(\frac{5}{13}\right) - 90 \right)\sin\theta - \left(169\sin^{-1}\left(\frac{5}{13}\right) - 60 \right)\cos\theta - 50\theta \right]_0^{2\pi}$$

$$= -\left(169\sin^{-1}\left(\frac{5}{13}\right) - 60 \right) - 100\pi + \left(169\sin^{-1}\left(\frac{5}{13}\right) - 60 \right)$$

$$\underline{= -100\pi}|$$

Exercise

Verify that the line integral and the surface integral of Stokes' Theorem are equal for the following vector fields, surfaces *S*, and closed curves *C*. Assume that *C* has counterclockwise orientation and *S* has a consistent orientation.

$\boldsymbol{F} = \langle y-z,\ z-x,\ x-y\rangle$; *S* is the cap of the sphere $x^2+y^2+z^2=16$ above the plane $z=\sqrt{7}$ and *C* is the boundary of *S*.

Solution

$x^2+y^2+7=16 \quad \to \quad x^2+y^2=9$ is the intersection of the sphere with the plane $z=\sqrt{7}$.

$\boldsymbol{r}(t)=\langle 3\cos t,\ 3\sin t,\ 0\rangle \ \Rightarrow\ \boldsymbol{r}'(t)=\langle -3\sin t,\ 3\cos t,\ 0\rangle$

$\boldsymbol{F}=\langle y-z,\ z-x,\ x-y\rangle=\left\langle 3\sin t-\sqrt{7},\ \sqrt{7}-3\cos t,\ 3\cos t-3\sin t\right\rangle$

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}=\iint_R \left\langle 3\sin t-\sqrt{7},\ \sqrt{7}-3\cos t,\ 3\cos t-3\sin t\right\rangle\cdot\langle -3\sin t,\ 3\cos t,\ 0\rangle\, dA$$

$$=\int_0^{2\pi}\left(-9\sin^2 t+3\sqrt{7}\sin t+3\sqrt{7}\cos t-9\cos^2 t\right)dt \qquad \sin^2 t+\cos^2 t=1$$

$$=\int_0^{2\pi}\left(3\sqrt{7}\sin t+3\sqrt{7}\cos t-9\right)dt$$

$$=\left[-3\sqrt{7}\cos t+3\sqrt{7}\sin t-9t\right]_0^{2\pi}$$

$$=-3\sqrt{7}-18\pi+3\sqrt{7}$$

$$\underline{=-18\pi}$$

$$\nabla\times\boldsymbol{F}=\nabla\times\langle y-z,\ z-x,\ x-y\rangle=\begin{vmatrix}\hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y-z & z-x & x-y\end{vmatrix}\ \underline{=\langle -2,\ -2,\ -2\rangle}$$

$$\iint_S (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\, dS=\iint_R \langle -2,\ -2,\ -2\rangle\cdot\left\langle \frac{x}{z},\ \frac{y}{z},\ 1\right\rangle dA$$

$$=\iint_R\left(-2\frac{x}{z}-2\frac{y}{z}-2\right)dA$$

$$=-2\int_0^{2\pi}\int_0^3\left(\frac{r\cos\theta}{\sqrt{16-r^2}}+\frac{r\sin\theta}{\sqrt{16-r^2}}+1\right)r\,dr\,d\theta$$

$$=-2\int_0^{2\pi}\int_0^3\left(\frac{r^2\cos\theta}{\sqrt{16-r^2}}+\frac{r^2\sin\theta}{\sqrt{16-r^2}}+r\right)dr\,d\theta$$

$$\int \frac{x^2}{\sqrt{a^2-x^2}}dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{x}{2}\sqrt{a^2-x^2}$$

$$= -2\int_0^{2\pi}\left(\begin{array}{l}\left(\frac{16}{2}\sin^{-1}\left(\frac{r}{4}\right)-\frac{r}{2}\sqrt{16-r^2}\right)\cos\theta \\ \quad +\left(\frac{16}{2}\sin^{-1}\left(\frac{r}{4}\right)-\frac{r}{2}\sqrt{16-r^2}\right)\sin\theta+\frac{1}{2}r^2\end{array}\right)\Bigg|_0^3 d\theta$$

$$= -2\int_0^{2\pi}\left(\left(8\sin^{-1}\left(\frac{3}{4}\right)-\frac{3\sqrt{7}}{2}\right)\cos\theta+\left(8\sin^{-1}\left(\frac{3}{4}\right)-\frac{3\sqrt{7}}{2}\right)\sin\theta+\frac{9}{2}\right)d\theta$$

$$= -2\left[-\left(8\sin^{-1}\left(\frac{3}{4}\right)-\frac{3\sqrt{7}}{2}\right)\sin\theta+\left(8\sin^{-1}\left(\frac{3}{4}\right)-\frac{3\sqrt{7}}{2}\right)\cos\theta+\frac{9}{2}\theta\right]_0^{2\pi}$$

$$= -2\left[8\sin^{-1}\left(\frac{3}{4}\right)-\frac{3\sqrt{7}}{2}-9\pi-8\sin^{-1}\left(\frac{3}{4}\right)+\frac{3\sqrt{7}}{2}\right]$$

$$\underline{= -18\pi}$$

Exercise

Evaluate the line integral $\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}$ by evaluating the surface integral in Stokes' Theorem with an appropriate choice of S. Assume that C has a counterclockwise orientation,

$\boldsymbol{F} = \langle 2y,\ -z,\ x\rangle$; C is the circle $x^2+y^2=12$ in the plane $z=0$.

Solution

$$\nabla\times\boldsymbol{F} = \nabla\times\langle 2y,\ -z,\ x\rangle = \begin{vmatrix}\hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2y & -z & x\end{vmatrix} \underline{= \langle 1,\ -1,\ -2\rangle}$$

$z=0 \quad (0x+0y) \quad \rightarrow \quad \boldsymbol{n} = \langle 0,\ 0,\ 1\rangle$

$$\iint_S (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\, dS = \iint_R \langle 1,\ -1,\ -2\rangle\cdot\langle 0,\ 0,\ 1\rangle dA$$

$$= \iint_R (-2)\, dA$$

$$= -2\int_0^{2\pi} d\theta\int_0^{2\sqrt{3}} r\,dr$$

$$= -2(2\pi)\left[\frac{1}{2}r^2\right]_0^{2\sqrt{3}}$$

$$\underline{= -24\pi}$$

Exercise

Evaluate the line integral $\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}$ by evaluating the surface integral in Stokes' Theorem with an appropriate choice of *S*. Assume that *C* has a counterclockwise orientation,

$$\boldsymbol{F} = \langle y,\ xz,\ -y\rangle;\ C \text{ is the ellipse } x^2 + \frac{y^2}{4} = 1 \text{ in the plane } z = 1.$$

Solution

$$\nabla\times\boldsymbol{F} = \nabla\times\langle y,\ xz,\ -y\rangle = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & -y \end{vmatrix} \underline{= \langle -1-x,\ 0,\ z-1\rangle}$$

$$z = 1\ \ (+0x + 0y) \quad\to\quad \boldsymbol{n} = \langle 0,\ 0,\ 1\rangle$$

$$\iint_S (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\ dS = \iint_R \langle -1-x,\ 0,\ z-1\rangle\cdot\langle 0,\ 0,\ 1\rangle dA$$

$$= \iint_R (z-1)dA \qquad\qquad \text{Because } z = 1$$

$$= \iint_R (0)dA$$

$$\underline{= 0}$$

Exercise

Evaluate the line integral $\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}$ by evaluating the surface integral in Stokes' Theorem with an appropriate choice of *S*. Assume that *C* has a counterclockwise orientation,

$$\boldsymbol{F} = \left\langle x^2 - z^2,\ y,\ 2xz\right\rangle;\ C \text{ is the boundary of the plane } z = 4 - x - y \text{ in the plane first octant.}$$

Solution

$$\nabla\times\boldsymbol{F} = \nabla\times\left\langle x^2 - z^2,\ y,\ 2xz\right\rangle = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - z^2 & y & 2xz \end{vmatrix} \underline{= \langle 0,\ -4z,\ 0\rangle}$$

$$x + y + z = 4 \quad\to\quad \boldsymbol{n} = \langle 1,\ 1,\ 1\rangle$$

$$\iint_S (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\ dS = \iint_R \langle 0,\ -4z,\ 0\rangle\cdot\langle 1,\ 1,\ 1\rangle dA$$

$$= \iint_R (-4z)\,dA$$

$$= -4\int_0^4 \int_0^{4-x} (4 - x - y)\,dxdy$$

$$= -4\int_0^4 \left(4y - xy - \frac{1}{2}y^2\right)\Big|_0^{4-x} dx$$

$$= -4\int_0^4 \left(16 - 4x - 4x + x^2 - \frac{1}{2}\left(16 - 8x + x^2\right)\right)dx$$

$$= -4\int_0^4 \left(\frac{1}{2}x^2 - 4x + 8\right)dx$$

$$= -4\left[\frac{1}{6}x^3 - 2x^2 + 8x\right]_0^4$$

$$= -4\left(\frac{32}{3} - 32 + 32\right)$$

$$\boxed{= -\frac{128}{3}}$$

Exercise

Evaluate the line integral $\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}$ by evaluating the surface integral in Stokes' Theorem with an appropriate choice of S. Assume that C has a counterclockwise orientation,

$\boldsymbol{F} = \left\langle y^2,\ -z^2,\ x\right\rangle$; C is the circle $\boldsymbol{r}(t) = \langle 3\cos t,\ 4\cos t,\ 5\sin t\rangle$ for $0 \le t \le 2\pi$.

Solution

$$\nabla\times\boldsymbol{F} = \nabla\times\left\langle y^2,\ -z^2,\ x\right\rangle = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & -z^2 & x \end{vmatrix} \boxed{= \langle -2z,\ -1,\ -2y\rangle}$$

S is the disk $\boldsymbol{t} = \langle 3r\cos t,\ 4r\cos t,\ 5r\sin t\rangle$

$\boldsymbol{t}_r = \langle 3\cos t,\ 4\cos t,\ 5\sin t\rangle$ & $\boldsymbol{t}_t = \langle -3r\sin t,\ -4r\sin t,\ 5r\cos t\rangle$

$$\boldsymbol{n} = \boldsymbol{t}_r \times \boldsymbol{t}_t = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 3\cos t & 4\cos t & 5\sin t \\ -3r\sin t & -4r\sin t & 5r\cos t \end{vmatrix} \boxed{= \langle 20r,\ -15r,\ 0\rangle}$$

$$\iint_S (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\,dS = \iint_R \langle -2z,\ -1,\ -2y\rangle\cdot\langle 20r,\ -15r,\ 0\rangle\,dA$$

$$= \int_0^{2\pi}\int_0^1 (-40rz + 15r)\,dr\,dt$$

$$= \int_0^{2\pi}\int_0^1 (-200r\sin t + 15r)\,dr\,dt$$

$$= \int_0^{2\pi}\left(-100r^2\sin t + \frac{15}{2}r^2\right)\Big|_0^1\,dt$$

$$= \int_0^{2\pi}\left(-100\sin t + \frac{15}{2}\right)dt$$

$$= \left[100\cos t + \frac{15}{2}t\right]_0^{2\pi}$$

$$= 100 + 15\pi - 100$$

$$\underline{= 15\pi}$$

Exercise

Evaluate the line integral $\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}$ by evaluating the surface integral in Stokes' Theorem with an appropriate choice of S. Assume that C has a counterclockwise orientation,

$\boldsymbol{F} = \left\langle 2xy\sin z,\ x^2\sin z,\ x^2y\cos z\right\rangle$; C is the boundary of the plane $z = 8 - 2x - 4y$ in the first octant.

Solution

$$\nabla\times\boldsymbol{F} = \nabla\times\left\langle 2xy\sin z,\ x^2\sin z,\ x^2y\cos z\right\rangle = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy\sin z & x^2\sin z & x^2y\cos z \end{vmatrix}$$

$$= \left\langle x^2\cos z - x^2\cos z,\ 2xy\cos z - 2xy\cos z,\ 2x\sin z - 2x\sin z\right\rangle$$

$$\underline{= \langle 0,\ 0,\ 0\rangle}$$

$$\iint_S (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\,dS\ \underline{= 0}$$

Exercise

Evaluate the line integral in Stokes' Theorem to evaluate the surface integral $\iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, dS$.

Assume that $\boldsymbol{n}$ points in an upward direction,

$\boldsymbol{F} = \langle x,\ y,\ z \rangle$; S is the upper half of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$

Solution

$$\nabla \times \boldsymbol{F} = \nabla \times \langle x,\ y,\ z \rangle = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \langle 0,\ 0,\ 0 \rangle$$

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, dS = 0$$

Let $z = 0 \quad \rightarrow \quad \frac{x^2}{4} + \frac{y^2}{9} = 1$

$\boldsymbol{r}(t) = \langle 2\cos t,\ 3\sin t,\ 0 \rangle \ \Rightarrow \ \boldsymbol{r}'(t) = \langle -2\sin t,\ 3\cos t,\ 0 \rangle$

$\boldsymbol{F} = \langle x,\ y,\ z \rangle = \langle 2\cos t,\ 3\sin t,\ 0 \rangle$

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_R \langle 2\cos t,\ 3\sin t,\ 0 \rangle \cdot \langle -2\sin t,\ 3\cos t,\ 0 \rangle dA$$

$$= \int_0^{2\pi} (-4\cos t \sin t + 9\sin t \cos t)\, dt$$

$$= \int_0^{2\pi} (5\sin t \cos t)\, dt$$

$$= \frac{5}{2} \int_0^{2\pi} \sin 2t\ dt$$

$$= \frac{5}{4} \left[-\cos 2t \right]_0^{2\pi}$$

$$= \frac{5}{2}(-1+1)$$

$$= 0$$

Exercise

Evaluate the line integral in Stokes' Theorem to evaluate the surface integral $\iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, dS$.

Assume that $\boldsymbol{n}$ points in an upward direction,

$\boldsymbol{F} = \langle 2y,\ -z,\ x-y-z \rangle$; S is the cap of the sphere $x^2+y^2+z^2=25 \quad for\ 3 \le x \le 5$

Solution

The boundary of the surface is the intersection of the plane $x=3$ and $x^2+y^2+z^2=25$

At $x=3 \quad \to \quad y^2+z^2=16$

$\boldsymbol{r}(t) = \langle 3,\ 4\cos t,\ 4\sin t \rangle \ \Rightarrow\ \boldsymbol{r}'(t) = \langle 0,\ -4\sin t,\ 4\cos t \rangle$

$\boldsymbol{F} = \langle 2y,\ -z,\ x-y-z \rangle = \langle 8\cos t,\ -4\sin t,\ 3-4\cos t-4\sin t \rangle$

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_R \langle 8\cos t,\ -4\sin t,\ 3-4\cos t-4\sin t \rangle \cdot \langle 0,\ -4\sin t,\ 4\cos t \rangle dA$$

$$= \int_0^{2\pi} \left(16\sin^2 t + 12\cos t - 16\cos^2 t - 16\sin t\cos t\right) dt \qquad \cos 2t = \cos^2 t - \sin^2 t$$

$$= \int_0^{2\pi} (12\cos t - 16\cos 2t - 8\sin 2t)\, dt$$

$$= \left[12\sin t - 8\sin 2t + 4\cos 2t\right]_0^{2\pi}$$

$$= (0-8+4-0+8-4)$$

$$= 0$$

$$\nabla \times \boldsymbol{F} = \nabla \times \langle x,\ y,\ z \rangle = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & -z & x-y-z \end{vmatrix} = \langle -1+1,\ 0-1,\ 0-2 \rangle = \langle 0,\ -1,\ -2 \rangle$$

$x=3 \quad \to \quad \boldsymbol{n} = \langle 3,\ 0,\ 0 \rangle$

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, dS = \iint_R \langle 0,\ -1,\ -2 \rangle \cdot \langle 3,\ 0,\ 0 \rangle dA$$

$$= 0$$

Exercise

Evaluate the line integral in Stokes' Theorem to evaluate the surface integral $\iint_S (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\,dS$.

Assume that $\boldsymbol{n}$ points in an upward direction,

$\boldsymbol{F} = \langle x+y,\ y+z,\ x+z\rangle$; S is the tilted disk enclosed $\boldsymbol{r}(t) = \langle \cos t,\ 2\sin t,\ \sqrt{3}\cos t\rangle$

Solution

$$\boldsymbol{r}(t) = \langle \cos t,\ 2\sin t,\ \sqrt{3}\cos t\rangle \Rightarrow \boldsymbol{r}'(t) = \langle -\sin t,\ 2\cos t,\ -\sqrt{3}\sin t\rangle$$

$$\boldsymbol{F} = \langle x+y,\ y+z,\ x+z\rangle = \langle \cos t + 2\sin t,\ 2\sin t + \sqrt{3}\cos t,\ \cos t + \sqrt{3}\cos t\rangle$$

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_R \langle \cos t + 2\sin t,\ 2\sin t + \sqrt{3}\cos t,\ \cos t + \sqrt{3}\cos t\rangle \cdot \langle -\sin t,\ 2\cos t,\ -\sqrt{3}\sin t\rangle dA$$

$$= \int_0^{2\pi} \left(-\cos t\sin t - 2\sin^2 t + 4\cos t\sin t + 2\sqrt{3}\cos^2 t - \sqrt{3}\sin t\cos t - 3\cos t\sin t\right) dt$$

$$= \int_0^{2\pi} \left(-2\sin^2 t + 2\sqrt{3}\cos^2 t - \sqrt{3}\sin t\cos t\right) dt$$

$$= \int_0^{2\pi} \left(-2\left(\frac{1-\cos 2t}{2}\right) + 2\sqrt{3}\left(\frac{1+\cos 2t}{2}\right) - \frac{\sqrt{3}}{2}\sin 2t\right) dt$$

$$= \int_0^{2\pi} \left(-1 + \cos 2t + \sqrt{3} + \sqrt{3}\cos 2t - \frac{\sqrt{3}}{2}\sin 2t\right) dt$$

$$= \left(\left(\sqrt{3}-1\right)t + \frac{1}{2}\sin 2t + \frac{\sqrt{3}}{2}\sin 2t + \frac{\sqrt{3}}{4}\cos 2t\right)\Big|_0^{2\pi}$$

$$= \left(\sqrt{3}-1\right)(2\pi) + \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}$$

$$= 2\pi\left(\sqrt{3}-1\right)$$

$$\nabla\times\boldsymbol{F} = \nabla\times\langle x,\ y,\ z\rangle = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & y+z & x+z \end{vmatrix} = \langle -1,\ -1,\ -1\rangle$$

S is the disk $\boldsymbol{t} = \langle r\cos t,\ 2r\sin t,\ \sqrt{3}r\cos t\rangle$

$$\boldsymbol{t}_r = \langle \cos t,\ 2\sin t,\ \sqrt{3}\cos t\rangle \quad \& \quad \boldsymbol{t}_t = \langle -r\sin t,\ 2r\cos t,\ -r\sqrt{3}\sin t\rangle$$

$$\boldsymbol{n} = \boldsymbol{t}_r\times\boldsymbol{t}_t = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \cos t & 2\sin t & \sqrt{3}\cos t \\ -r\sin t & 2r\cos t & -r\sqrt{3}\sin t \end{vmatrix} = \langle -2r\sqrt{3},\ 0,\ 2r\rangle$$

$$\iint_S (\nabla\times \boldsymbol{F})\cdot \boldsymbol{n}\, dS = \iint_R \langle -1,\ 0,\ -1\rangle \cdot \left\langle -2r\sqrt{3},\ -r\sqrt{3},\ 2r\right\rangle dA$$

$$= \int_0^{2\pi}\int_0^1 \left(2r\sqrt{3}-2r\right) dr\, dt \qquad\qquad = \int_0^{2\pi} dt \int_0^1 \left(2r\sqrt{3}-2r\right) dr$$

$$= (2\pi)\left[\sqrt{3}\, r^2 - r^2\right]_0^1$$

$$= 2\pi\left(\sqrt{3}-1\right)$$

Exercise

Use Stokes' Theorem and a surface integral to find the circulation on C of the vector field $\boldsymbol{F} = \langle -y,\ x,\ 0\rangle$ as a function of φ. For what value of φ is the circulation a maximum?

Solution

$$\nabla\times \boldsymbol{F} = \nabla\times\langle x,\ y,\ z\rangle = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \langle 0,\ 0,\ 2\rangle$$

$$\boldsymbol{t} = \langle r\cos\varphi\cos t,\ r\sin t,\ r\sin\varphi\cos t\rangle$$

$$\boldsymbol{t}_r = \langle \cos\varphi\cos t,\ \sin t,\ \sin\varphi\cos t\rangle \quad \& \quad \boldsymbol{t}_t = \langle -r\cos\varphi\sin t,\ r\cos t,\ -r\sin\varphi\sin t\rangle$$

$$\boldsymbol{n} = \boldsymbol{t}_r\times \boldsymbol{t}_t = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \cos\varphi\cos t & \sin t & \sin\varphi\cos t \\ -r\cos\varphi\sin t & r\cos t & -r\sin\varphi\sin t \end{vmatrix}$$

$$= \left\langle -r\sin\varphi\sin^2 t - r\sin\varphi\cos^2 t,\ 0,\ r\cos\varphi\cos^2 t + r\cos\varphi\sin^2 t\right\rangle$$

$$= \langle -r\sin\varphi,\ 0,\ r\cos\varphi\rangle$$

$$\iint_S (\nabla\times \boldsymbol{F})\cdot \boldsymbol{n}\, dS = \iint_R \langle 0,\ 0,\ 2\rangle\cdot\langle -r\sin\varphi,\ 0,\ r\cos\varphi\rangle dA$$

$$= \int_0^{2\pi}\int_0^1 (2r\cos\varphi)\, dr\, dt$$

$$= (2\pi)\left[r^2\cos\varphi\right]_0^1$$

$$= 2\pi\cos\varphi$$

The maximum value of the circulation when $\cos\varphi = 1 \quad \Rightarrow \quad \varphi = 0$ which is 2π

Exercise

A circle C in the plane $x+y+z=8$ has a radius of 4 and center (2, 3, 3). Evaluate $\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}$ for $\boldsymbol{F}=\langle 0,\ -z,\ 2y\rangle$ where C has a counterclockwise orientation when viewed from above. Does the circulation depend on the radius of the circle? Does it depend on the location of the center of the circle?

Solution

$$\nabla\times\boldsymbol{F}=\nabla\times\langle 0,\ -z,\ 2y\rangle=\begin{vmatrix}\hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & -z & 2y\end{vmatrix}=\langle 3,\ 0,\ 0\rangle$$

$$x+y+z=8 \quad\to\quad \boldsymbol{n}=\langle 1,\ 1,\ 1\rangle$$

$$\iint_S (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\,dS=\iint_R \langle 3,\ 0,\ 0\rangle\cdot\langle 1,\ 1,\ 1\rangle\,dA$$

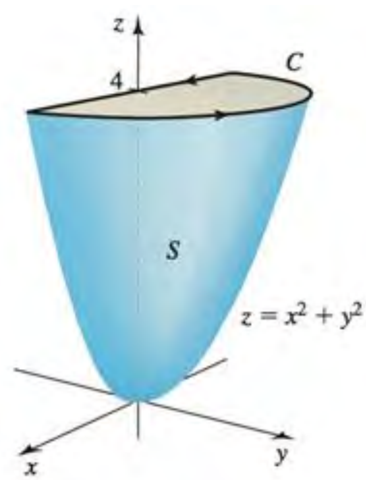
$$=\int_0^{2\pi}\int_0^4 (3)\ r dr\ dt$$

$$=(2\pi)\left[\frac{3}{2}r^2\right]_0^4$$

$$=48\pi$$

Exercise

Begin with the paraboloid $z=x^2+y^2$, for $0\le z\le 4$, and slice it with the plane $y=0$. Let S be the surface that remains for $y\ge 0$ (including the planar surface in the xz-plane). Let C be the semicircle and line segment that bound the cap of S in the plane $z=4$ with counterclockwise orientation. Let $\boldsymbol{F}=\langle 2z+y,\ 2x+z,\ 2y+x\rangle$



a) Describe the direction of the vectors normal to the surface that are consistent with the orientation of C.

b) Evaluate $\iint_S (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\,dS$

c) Evaluate $\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}$ and check for argument with part (*b*).

Solution

a) The normal vector point toward the z-axis on the curved surface of S and in the direction $\langle 0,\ 1,\ 0\rangle$ on the flat surface of S.

b) $\nabla \times \boldsymbol{F} = \nabla \times \langle 2z+y,\ 2x+z,\ 2y+x \rangle = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z+y & 2x+z & 2y+x \end{vmatrix} \underline{= \langle 1,\ 1,\ 1 \rangle}$

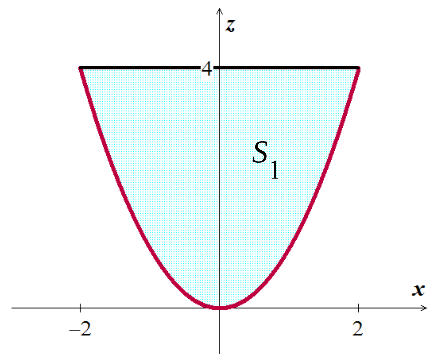
The planar surface in the *xz*-plane, then let S_1 be the surface parameterized by $\langle x,\ 0,\ z \rangle$.

Where, since $y=0$, $z = x^2 + 0^2 \quad \Rightarrow \quad x^2 \le z \le 4$ and $z = 4 = x^2 \quad \Rightarrow \quad -2 \le x \le 0$

$\boldsymbol{t} = \langle x,\ 0,\ z \rangle$

$\boldsymbol{t}_x = \langle 1,\ 0,\ 0 \rangle \quad \& \quad \boldsymbol{t}_z = \langle 0,\ 0,\ 1 \rangle$

$\boldsymbol{n} = \boldsymbol{t}_x \times \boldsymbol{t}_z = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \underline{= \langle 0,\ -1,\ 0 \rangle}$



$$\iint_{S_1} (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\ dS = \iint_{S_1} \langle 1,\ 1,\ 1 \rangle \cdot \langle 0,\ -1,\ 0 \rangle dS$$

$$= \int_{-2}^{2} \int_{x^2}^{4} (-1)\ dz\ dx$$

$$= -\int_{-2}^{2} z \Big|_{x^2}^{4}\ dx$$

$$= -\int_{-2}^{2} \left(4 - x^2\right) dx$$

$$= -\left(4x - \frac{1}{3}x^3\right)\Big|_{-2}^{2}$$

$$= -\left(8 - \frac{8}{3} + 8 - \frac{8}{3}\right)$$

$$\underline{= -\frac{32}{3}}$$

Let S_2 be the surface of the half of the paraboloid for $y \ge 0$, parametrized as

$\boldsymbol{t} = \left\langle r\cos\phi,\ r\sin\phi,\ r^2 \right\rangle; \quad 0 \le r \le 2; \quad -\pi \le \phi \le 0$

$\boldsymbol{t}_r = \langle \cos\phi,\ \sin\phi,\ 2r \rangle \quad \boldsymbol{t}_\phi = \langle -r\sin\phi,\ r\cos\phi,\ 0 \rangle$

$\boldsymbol{n} = \boldsymbol{t}_r \times \boldsymbol{t}_\phi = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \cos\phi & \sin\phi & 2r \\ -r\sin\phi & r\cos\phi & 0 \end{vmatrix} \underline{= \left\langle -2r^2\cos\phi,\ -2r^2\sin\phi,\ r \right\rangle}$

$$\iint_{S_2} (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\ dS = \iint_{S_2} \langle 1,\ 1,\ 1 \rangle \cdot \left\langle -2r^2\cos\phi,\ -2r^2\sin\phi,\ r \right\rangle dS$$

$$= \int_{-\pi}^{0} \int_{0}^{2} \left(-2r^2 \cos\phi - 2r^2 \sin\phi + r\right) dr\, d\phi$$

$$= \int_{-\pi}^{0} \left(-\frac{2}{3} r^3 \cos\phi - \frac{2}{3} r^3 \sin\phi + \frac{1}{2} r^2\right)_{0}^{2} d\phi$$

$$= \int_{-\pi}^{0} \left(-\frac{16}{3} \cos\phi - \frac{16}{3} \sin\phi + 2\right) d\phi$$

$$= \left(-\frac{16}{3} \sin\phi + \frac{16}{3} \cos\phi + 2\phi\right)_{-\pi}^{0}$$

$$= \frac{16}{3} + \frac{16}{3} + 2\pi$$

$$= \frac{32}{3} + 2\pi$$

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, dS = \iint_{S_1} (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, dS + \iint_{S_2} (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, dS$$

$$= -\frac{32}{3} + \frac{32}{3} + 2\pi$$

$$= 2\pi$$

c) $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \oint_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r}_1 + \oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r}_2$

$\boldsymbol{F} = \langle 2z + y,\ 2x + z,\ 2y + x \rangle$

$C_1 :\ \boldsymbol{r}_1 = \langle t,\ 0,\ 4 \rangle = \langle x,\ y,\ z \rangle$ *for* $-2 \le t \le 2$

$\boldsymbol{r}_1' = \langle 1,\ 0,\ 0 \rangle$

$C_2 :\ \boldsymbol{r}_2 = \langle 2\cos t,\ 2\sin t,\ 4 \rangle = \langle x,\ y,\ z \rangle$ *for* $-\pi \le t \le 0$

$\boldsymbol{r}_2' = \langle -2\sin t,\ 2\cos t,\ 0 \rangle$

$$\oint_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r}_1 = -\int_{-2}^{2} \langle 2z + y,\ 2x + z,\ 2y + x \rangle \cdot \langle 1,\ 0,\ 0 \rangle\, dt$$

$$= -\int_{-2}^{2} (2z + y)\, dt$$

$$= -\int_{-2}^{2} (2(4) + 0)\, dt$$

$$= -\int_{-2}^{2} (8)\, dt$$

$$= -8t \Big|_{-2}^{2}$$

$$= -32$$

$$\oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r}_2 = \int_{-\pi}^{0} \langle 2z + y,\ 2x + z,\ 2y + x \rangle \cdot \langle -2\sin t,\ 2\cos t,\ 0 \rangle dt$$

$$= \int_{-\pi}^{0} \langle 8 + 2\sin t,\ 4\cos t + 4,\ 4\sin t + 2\cos t \rangle \cdot \langle -2\sin t,\ 2\cos t,\ 0 \rangle dt$$

$$= \int_{-\pi}^{0} \left(-16\sin t - 4\sin^2 t + 8\cos^2 t + 8\cos t \right) dt \qquad \sin^2 t = 1 - \cos^2 t$$

$$= \int_{-\pi}^{0} \left(-16\sin t - 4\left(1 - \cos^2 t\right) + 8\cos^2 t + 8\cos t \right) dt$$

$$= \int_{-\pi}^{0} \left(-16\sin t - 4 + 12\cos^2 t + 8\cos t \right) dt \qquad \cos^2 t = \frac{1 + \cos 2t}{2}$$

$$= \int_{-\pi}^{0} \left(-16\sin t + 2 + 6\cos 2t + 8\cos t \right) dt$$

$$= \left[16\cos t + 2t + 3\sin 2t + 8\sin t \right]_{-\pi}^{0}$$

$$= 32 + 2\pi$$

$$\oint_{C} \boldsymbol{F} \cdot d\boldsymbol{r} = \oint_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r}_1 + \oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r}_2$$

$$= -32 + 32 + 2\pi$$

$$= 2\pi$$

Exercise

The French Physicist André-Marie Ampère discovered that an electrical current I in a wire produces a magnetic field B. A special case of Ampère's Law relates the current to the magnetic field through the equation $\oint_C \boldsymbol{B} \cdot d\boldsymbol{r} = \mu I$, where C is any closed curve through which the wire passes and μ is a physical constant. Assume that the current I is given in terms of the current density $\boldsymbol{J}$ as $I = \iint_S \boldsymbol{J} \cdot \boldsymbol{n}\ dS$, where S is an oriented surface with C as a boundary. Use Stokes' Theorem to show that an equivalent form of Ampère's Law is $\nabla \times \boldsymbol{B} = \mu \boldsymbol{J}$.

Solution

$$\iint_S (\nabla \times B) \cdot \boldsymbol{n}\ dS = \oint_C \boldsymbol{B} \cdot d\boldsymbol{r} = \mu I = \mu \iint_S \boldsymbol{J} \cdot \boldsymbol{n}\ dS$$

$$\iint_S (\nabla \times B) \cdot \boldsymbol{n}\ dS - \mu \iint_S \boldsymbol{J} \cdot \boldsymbol{n}\ dS = 0$$

Thus $$\iint_S \left[(\nabla\times B)-\mu \boldsymbol{J}\right]\cdot \boldsymbol{n}\; dS = 0$$

For all surfaces S bounded by any given closed curve C.

It is clear that given the freedom to choose C and S, that it follows that the integrand is identically zero, i.e. that for any surface S, $\left((\nabla\times B)-\mu \boldsymbol{J}\right)\cdot \boldsymbol{n} = 0$.

From this, it is easy to see that we must have $(\nabla\times B)=\mu \boldsymbol{J}$, since we are free to make normal vector point in any direction at any given point by choosing S appropriately.

Exercise

Let S be the paraboloid $z=a\left(1-x^2-y^2\right)$, for $z\geq 0$, where $a>0$ is a real number. Let $\boldsymbol{F}=\langle x-y,\ y+z,\ z-x\rangle$. For what value(s) of a (if any) does $\iint_S (\nabla\times \boldsymbol{F})\cdot \boldsymbol{n}\; dS$ have its maximum value?

Solution

$$For\quad z=a\left(1-x^2-y^2\right)=0 \quad\Rightarrow\quad x^2+y^2=1$$

$$\boldsymbol{r}=\langle \cos t,\ \sin t,\ 0\rangle \quad\Rightarrow\quad \boldsymbol{r}'=\langle -\sin t,\ \cos t,\ 0\rangle$$

$$\boldsymbol{F}=\langle x-y,\ y+z,\ z-x\rangle=\langle \cos t-\sin t,\ \sin t,\ -\cos t\rangle$$

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^{2\pi}\langle \cos t-\sin t,\ \sin t,\ -\cos t\rangle\cdot\langle -\sin t,\ \cos t,\ 0\rangle\; dt$$

$$=\int_0^{2\pi}\left(-\cos t\sin t+\sin^2 t+\cos t\sin t\right)\; dt$$

$$=\int_0^{2\pi}\sin^2 t\; dt$$

$$=\frac{1}{2}\int_0^{2\pi}(1-\cos 2t)\; dt$$

$$=\frac{1}{2}\left[t-\frac{1}{2}\sin 2t\right]_0^{2\pi}$$

$$=\pi$$

$\therefore$ The integral is independent of a.

Exercise

The goal is to evaluate $A = \iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, dS$, where $\boldsymbol{F} = \langle yz,\ -xz,\ xy \rangle$ and S ids the surface of the upper half of the ellipsoid $x^2 + y^2 + 8z^2 = 1 \ \ (z \ge 0)$

a) Evaluate a surface integral over a more convenient surface to find the value of A.

b) Evaluate A using a line integral.

Solution

a) The boundary of this surface is the circle $x^2 + y^2 = 0$ at $z = 0$

$$\nabla \times \boldsymbol{F} = \nabla \times \langle yz,\ -xz,\ xy \rangle = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -xz & xy \end{vmatrix} = \langle 2x,\ 0,\ -2z \rangle$$

$$\nabla \times \boldsymbol{F} \Big|_{z=0} = \langle 2x,\ 0,\ 0 \rangle$$

At $z = 0 \ \rightarrow \ \boldsymbol{n} = \langle 0,\ 0,\ 1 \rangle$

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, dS = \iint_S \langle 2x,\ 0,\ 0 \rangle \cdot \langle 0,\ 0,\ 1 \rangle\, dS$$

$$= \iint_S (0)\, dS$$

$$= 0$$

b) With the parameterization of the boundary circle and $z = 0$, we have

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^{2\pi} \langle 0,\ 0,\ 1 \rangle \cdot \langle -\sin t,\ \cos t,\ 0 \rangle\, dt$$

$$= \int_0^{2\pi} 0\, dt$$

$$= 0$$

Solution   *Section* 4.8 – Divergence Theorem

Exercise

Evaluate both integrals of the Divergence Theorem for the following vector field and region. Check for agreement. $\boldsymbol{F} = \langle 2x,\ 3y,\ 4z\rangle \quad D = \left\{(x,y,z):\ x^2+y^2+z^2 \le 4\right\}$

Solution

$$\nabla\cdot\boldsymbol{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(4z) \ \underline{=9|}$$

$$\iiint\limits_D \nabla\cdot\boldsymbol{F}\ dV = \iiint\limits_D (9)\ dV$$

$$= 9\cdot volume(D) \qquad\qquad x^2+y^2+z^2 = 4 = r^2$$

$$= 9\cdot\frac{4\pi}{3}2^3 \qquad\qquad volume(D) = \frac{4\pi r^3}{3}$$

$$\underline{= 96\pi|}$$

$$\boldsymbol{r} = \langle x,\ y,\ z\rangle = \langle 2\sin\phi\cos\theta,\ 2\sin\phi\sin\theta,\ 2\cos\phi\rangle \qquad R = \left\{(\phi,\ \theta):\ 0\le\phi\le\pi,\ 0\le\theta\le 2\pi\right\}$$

$$t_\phi = \langle 2\cos\phi\cos\theta,\ 2\cos\phi\sin\theta,\ -2\sin\phi\rangle$$

$$t_\theta = \langle -2\sin\phi\sin\theta,\ 2\sin\phi\cos\theta,\ 0\rangle$$

$$t_\phi\times t_\theta = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \left(4\sin^2\phi\cos\theta\right)\hat{i} + \left(4\sin^2\phi\sin\theta\right)\hat{\boldsymbol{j}} + \left(4\sin\phi\cos\phi\cos^2\theta + 4\sin\phi\cos\phi\sin^2\theta\right)\hat{\boldsymbol{k}}$$

$$= \left\langle 4\sin^2\phi\cos\theta,\ 4\sin^2\phi\sin\theta,\ 4\sin^2\phi\cos\phi\right\rangle$$

$$\boldsymbol{F} = \langle 2x,\ 3y,\ 4z\rangle = \langle 2(2\sin\phi\cos\theta),\ 3(2\sin\phi\sin\theta),\ 4(2\cos\phi)\rangle$$

$$\boldsymbol{F}\cdot\left(T_\phi\times T_\theta\right) = \langle 4\sin\phi\cos\theta,\ 6\sin\phi\sin\theta,\ 8\cos\phi\rangle\cdot\left\langle 4\sin^2\phi\cos\theta,\ 4\sin^2\phi\sin\theta,\ 4\sin^2\phi\cos\phi\right\rangle$$

$$\underline{= 16\sin^3\phi\cos^2\theta + 24\sin^3\phi\sin^2\theta + 32\sin\phi\cos^2\phi|}$$

$$\iint\limits_S \boldsymbol{F}\cdot\boldsymbol{n}\ dS = \iint\limits_R \nabla\boldsymbol{F}\cdot\left(T_\phi\times T_\theta\right)\ dA$$

$$= 8\int_0^{2\pi}\int_0^{\pi}\left(2\cos^2\theta\sin^3\phi + 3\sin^2\theta\sin^3\phi + 4\sin\phi\cos^2\phi\right)\ d\phi d\theta$$

$$= 8\int_0^{2\pi}\int_0^{\pi}\left(\left(2\cos^2\theta + 3\sin^2\theta\right)\sin^2\phi + 4\cos^2\phi\right)\sin\phi\ d\phi d\theta$$

$$= -8\int_0^{2\pi}\int_0^{\pi}\left[\left(2\cos^2\theta+3\sin^2\theta\right)\left(1-\cos^2\phi\right)+4\cos^2\phi\right]\,d(\cos\phi)\,d\theta$$

$$= -8\int_0^{2\pi}\left[\left(2\cos^2\theta+3\left(1-\cos^2\theta\right)\right)\left(\cos\phi-\tfrac{1}{3}\cos^3\phi\right)+\tfrac{4}{3}\cos^3\phi\right]_0^{\pi}\,d\theta$$

$$= -8(2)\int_0^{2\pi}\left[\left(3-\cos^2\theta\right)\left(-\tfrac{2}{3}\right)-\tfrac{4}{3}\right]\,d\theta$$

$$= -16\int_0^{2\pi}\left[\tfrac{2}{3}\left(\frac{1+\cos 2\theta}{2}\right)-\tfrac{10}{3}\right]\,d\theta$$

$$= -\tfrac{16}{3}\int_0^{2\pi}\left[1+\cos 2\theta-10\right]\,d\theta$$

$$= -\tfrac{16}{3}\left[\tfrac{1}{2}\sin 2\theta-9\theta\right]_0^{2\pi}$$

$$= -\tfrac{16}{3}(-18\pi)$$

$$\underline{=96\pi}$$

Exercise

Evaluate both integrals of the Divergence Theorem for the following vector field and region. Check for agreement. $\boldsymbol{F}=\langle -x,\ -y,\ -z\rangle$ $\quad D=\{(x,y,z):\ |x|\le 1,\ |y|\le 1,\ |z|\le 1\}$

Solution

$$\nabla\cdot\boldsymbol{F}=\frac{\partial}{\partial x}(-x)+\frac{\partial}{\partial y}(-y)+\frac{\partial}{\partial z}(-z)\ \underline{=-3}$$

$$\iiint_D \nabla\cdot\boldsymbol{F}\ dV=\iiint_D(-3)\ dV$$

$$= -3\cdot volume(D) \qquad volume(D)=2\cdot 2\cdot 2 \quad (\text{cube of distance between } -1\ \&\ 1=2)$$

$$= -3\cdot(2)^3$$

$$\underline{=-24}$$

Since the surface has a form of cube, therefore we have 6 surfaces

$S_1: x=-1 \ \rightarrow\ \boldsymbol{n}=\langle -1,\ 0,\ 0\rangle \qquad \boldsymbol{F}\cdot\boldsymbol{n}_1=\langle 1,\ -y,\ -z\rangle\cdot\langle -1,\ 0,\ 0\rangle=-1$

$S_2: x=1 \ \rightarrow\ \boldsymbol{n}=\langle 1,\ 0,\ 0\rangle \qquad \boldsymbol{F}\cdot\boldsymbol{n}_2=\langle -1,\ -y,\ -z\rangle\cdot\langle 1,\ 0,\ 0\rangle=-1$

$S_3: y=-1 \ \rightarrow\ \boldsymbol{n}=\langle 0,\ -1,\ 0\rangle \qquad \boldsymbol{F}\cdot\boldsymbol{n}_3=\langle -x,\ 1,\ -z\rangle\cdot\langle 0,\ -1,\ 0\rangle=-1$

$S_4: y=1 \ \rightarrow\ \boldsymbol{n}=\langle 0,\ 1,\ 0\rangle \qquad \boldsymbol{F}\cdot\boldsymbol{n}_4=\langle -x,\ -1,\ -z\rangle\cdot\langle 0,\ 1,\ 0\rangle=-1$

$S_5 : z = -1 \quad \rightarrow \quad \boldsymbol{n} = \langle 0,\ 0,\ -1 \rangle \qquad \boldsymbol{F} \cdot \boldsymbol{n}_5 = \langle -x,\ -y,\ 1 \rangle \cdot \langle 0,\ 0,\ -1 \rangle = -1$

$S_6 : z = 1 \quad \rightarrow \quad \boldsymbol{n} = \langle 0,\ 0,\ 1 \rangle \qquad \boldsymbol{F} \cdot \boldsymbol{n}_6 = \langle -x,\ -y,\ -1 \rangle \cdot \langle 0,\ 0,\ 1 \rangle = -1$

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\ dS = \sum_{k=1}^{6} \iint_{S_k} \boldsymbol{F} \cdot \boldsymbol{n}_k\ dS$$

$$= 6 \int_{-1}^{1} dz \int_{-1}^{1} dy \int_{-1}^{0} (-1)\ dx$$

$$= 6 [z]_{-1}^{1} [y]_{-1}^{1} [-x]_{-1}^{0}$$

$$= 6(2)(2)(-1)$$

$$\underline{= -24}$$

Exercise

Evaluate both integrals of the Divergence Theorem for the following vector field and region. Check for agreement. $\quad \boldsymbol{F} = \langle z - y,\ x,\ -x \rangle \quad D = \left\{ (x, y, z) :\ \frac{x^2}{4} + \frac{y^2}{8} + \frac{z^2}{12} \le 1 \right\}$

Solution

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(z - y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(-x)\ \underline{= 0}$$

$$\iiint_D \nabla \cdot \boldsymbol{F}\ dV = \iiint_D (0)\ dV$$

$$\underline{= 0}$$

$$\frac{x^2}{4} + \frac{y^2}{8} + \frac{z^2}{12} = 1 \quad \rightarrow \quad x^2 = 4,\ y^2 = 8,\ z^2 = 12$$

$$\boldsymbol{t} = \left\langle 2\sin u \cos v,\ 2\sqrt{2}\ \sin u \sin v,\ 2\sqrt{3}\ \cos u \right\rangle$$

$$\boldsymbol{t}_u = \left\langle 2\cos u \cos v,\ 2\sqrt{2}\ \cos u \sin v,\ -2\sqrt{3}\ \sin u \right\rangle \quad \boldsymbol{t}_v = \left\langle -2\sin u \sin v,\ 2\sqrt{2}\ \sin u \cos v,\ 0 \right\rangle$$

$$\boldsymbol{n} = \boldsymbol{t}_u \times \boldsymbol{t}_v = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 2\cos u \cos v & 2\sqrt{2}\ \cos u \sin v & -2\sqrt{3}\ \sin u \\ -2\sin u \sin v & 2\sqrt{2}\ \sin u \cos v & 0 \end{vmatrix}$$

$$= \left\langle 4\sqrt{6}\ \sin^2 u \cos v,\ 4\sqrt{3}\ \sin^2 u \sin v,\ 4\sqrt{2}\ \sin u \cos u \cos^2 v + 4\sqrt{2}\ \sin u \cos u \sin^2 v \right\rangle$$

$$\underline{= \left\langle 4\sqrt{6}\ \sin^2 u \cos v,\ 4\sqrt{3}\ \sin^2 u \sin v,\ 4\sqrt{2}\ \sin u \cos u \right\rangle}$$

$$\boldsymbol{F} = \langle z-y,\ x,\ -x\rangle = \left\langle 2\sqrt{3}\cos u - 2\sqrt{2}\sin u\sin v,\ 2\sin u\cos v,\ -2\sin u\cos v\right\rangle$$

$$\begin{aligned}\boldsymbol{F}\cdot\boldsymbol{n} &= \left\langle 2\sqrt{3}\cos u - 2\sqrt{2}\sin u\sin v,\ 2\sin u\cos v,\ -2\sin u\cos v\right\rangle \\ &\qquad \cdot\left\langle 4\sqrt{6}\ \sin^2 u\cos v,\ 4\sqrt{3}\ \sin^2 u\sin v,\ 4\sqrt{2}\ \sin u\cos u\right\rangle \\ &= 24\sqrt{2}\cos u\sin^2 u\cos v - 16\sqrt{3}\sin^3 u\cos v\sin v + 8\sqrt{3}\sin^3 u\sin v\cos v - 8\sqrt{2}\sin^2 u\cos u\cos v \\ &= 16\sqrt{2}\cos u\sin^2 u\cos v - 8\sqrt{3}\sin^3 u\sin v\cos v\end{aligned}$$

$$\begin{aligned}\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\ dS &= \int_0^{2\pi}\int_0^{\pi}\left(16\sqrt{2}\cos u\sin^2 u\cos v - 8\sqrt{3}\sin^3 u\sin v\cos v\right)\ dudv \\ &= 8\left[\int_0^{2\pi}\int_0^{\pi}\left(2\sqrt{2}\sin^2 u\cos v\right)\ d(\sin u) + \int_0^{\pi}\sqrt{3}\left(1-\cos^2 u\right)\sin v\cos v\ d(\cos u)\right]\ dv \\ &= 8\left[\int_0^{2\pi}\frac{2\sqrt{2}}{3}\cos v\left[\sin^2 u\right]_0^{\pi} + \sqrt{3}\sin v\cos v\left(\cos u - \tfrac{1}{3}\cos^3 u\right)_0^{\pi}\right]\ dv \\ &= 8\int_0^{2\pi}\left(\frac{2\sqrt{2}}{3}\cos v(0) + \sqrt{3}\sin v\cos v\left(-2+\tfrac{2}{3}\right)\right)\ dv \\ &= -\frac{64\sqrt{3}}{3}\int_0^{2\pi}\sin v\ d(\sin v) \\ &= -\frac{64\sqrt{3}}{3}\left(\tfrac{1}{2}\sin v\right)_0^{2\pi} \\ &= 0\end{aligned}$$

Exercise

Evaluate both integrals of the Divergence Theorem for the following vector field and region. Check for agreement. $\boldsymbol{F} = \left\langle x^2,\ y^2,\ z^2\right\rangle \quad D = \left\{(x,y,z):\ |x|\le 1,\ |y|\le 2,\ |z|\le 3\right\}$

Solution

$$\nabla\cdot\boldsymbol{F} = \frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}\left(y^2\right) + \frac{\partial}{\partial z}\left(z^2\right) = 2x + 2y + 2z$$

$$\begin{aligned}\iiint_D \nabla\cdot\boldsymbol{F}\ dV &= 2\int_{-3}^{3}\int_{-2}^{2}\int_{-1}^{1}(x+y+z)\ dxdydz \\ &= 2\int_{-3}^{3}\int_{-2}^{2}\int_{-1}^{1}(x+y+z)\ dxdydz\end{aligned}$$

$$= 2\int_{-3}^{3}\int_{-2}^{2}\left(\frac{1}{2}x^2 + yx + zx\right)_{-1}^{1}\, dydz$$

$$= 2\int_{-3}^{3}\int_{-2}^{2}(2y+2z)\, dydz$$

$$= 2\int_{-3}^{3}\left(y^2 + 2zy\right)_{-2}^{2}\, dz$$

$$= 2\int_{-3}^{3}(8z)\, dz$$

$$= 2\left[4z^2\right]_{-3}^{3}$$

$$\underline{= 0}$$

$S_1 : x = -1 \quad \rightarrow \quad \boldsymbol{n} = \langle -1,\ 0,\ 0\rangle \qquad \boldsymbol{F}\cdot\boldsymbol{n}_1 = \left\langle x^2,\ y^2,\ z^2\right\rangle\cdot\langle -1,\ 0,\ 0\rangle = -x^2\Big|_{x=-1} = -1$

$S_2 : x = 1 \quad \rightarrow \quad \boldsymbol{n} = \langle 1,\ 0,\ 0\rangle \qquad \boldsymbol{F}\cdot\boldsymbol{n}_2 = \left\langle x^2,\ y^2,\ z^2\right\rangle\cdot\langle 1,\ 0,\ 0\rangle = x^2\Big|_{x=1} = 1$

$S_3 : y = -2 \quad \rightarrow \quad \boldsymbol{n} = \langle 0,\ -1,\ 0\rangle \qquad \boldsymbol{F}\cdot\boldsymbol{n}_3 = \left\langle x^2,\ y^2,\ z^2\right\rangle\cdot\langle 0,\ -1,\ 0\rangle = -y^2\Big|_{y=-2} = -4$

$S_4 : y = 2 \quad \rightarrow \quad \boldsymbol{n} = \langle 0,\ 1,\ 0\rangle \qquad \boldsymbol{F}\cdot\boldsymbol{n}_4 = \left\langle x^2,\ y^2,\ z^2\right\rangle\cdot\langle 0,\ 1,\ 0\rangle = y^2\Big|_{y=2} = 4$

$S_5 : z = -3 \quad \rightarrow \quad \boldsymbol{n} = \langle 0,\ 0,\ -1\rangle \qquad \boldsymbol{F}\cdot\boldsymbol{n}_5 = \left\langle x^2,\ y^2,\ z^2\right\rangle\cdot\langle 0,\ 0,\ -1\rangle = -z^2\Big|_{z=-3} = -9$

$S_6 : z = 3 \quad \rightarrow \quad \boldsymbol{n} = \langle 0,\ 0,\ 1\rangle \qquad \boldsymbol{F}\cdot\boldsymbol{n}_6 = \left\langle x^2,\ y^2,\ z^2\right\rangle\cdot\langle 0,\ 0,\ 1\rangle = z^2\Big|_{z=3} = 9$

$$\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\, dS = \sum_{k=1}^{6}\iint_{S_k}\boldsymbol{F}\cdot\boldsymbol{n}_k\, dS$$

$$= \iint_S (-1+1-4+4-9+9)\,dS$$

$$= \iint_S (0)\,dS$$

$$\underline{= 0}$$

Exercise

Find the net outward flux of the field $\boldsymbol{F} = \langle 2z - y,\ x,\ -2x \rangle$ across the sphere of radius 1 centered at the origin.

Solution

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(2z - y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(-2x) \ \underline{= 0}$$

So by the Divergence Theorem, the net outward flux is zero since the volume integral of $\nabla \cdot \boldsymbol{F}$ is zero.

Exercise

Find the net outward flux of the field $\boldsymbol{F} = \langle bz - cy,\ cx - az,\ ay - bx \rangle$ across any smooth closed surface $\mathbb{R}^3$, where *a*, *b*, and *c* are constants.

Solution

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(bz - cy) + \frac{\partial}{\partial y}(cx - az) + \frac{\partial}{\partial z}(ay - bx) \ \underline{= 0}$$

So by the Divergence Theorem, the net outward flux is zero since the volume integral of $\nabla \cdot \boldsymbol{F}$ is zero.

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *S*.

$\boldsymbol{F} = \langle x,\ -2y,\ 3z \rangle$; *S* is the sphere $\left\{(x, y, z):\ x^2 + y^2 + z^2 = 6\right\}$

Solution

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-2y) + \frac{\partial}{\partial z}(3z) = 1 - 2 + 3 \ \underline{= 2}$$

The sphere has a radius $\sqrt{6}$, therefore the volume of the sphere is $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi 6\sqrt{6} = 8\pi\sqrt{6}$

$$\begin{aligned} \iiint_D \nabla \cdot \boldsymbol{F}\ dV &= 2 \cdot (\textit{Volume of sphere}) \\ &= 2\left(8\pi\sqrt{6}\right) \\ &\underline{= 16\pi\sqrt{6}} \end{aligned}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *S*.

$\boldsymbol{F} = \left\langle x^2,\ 2xz,\ y^2 \right\rangle$; *S* is surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$

Solution

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}(2xz) + \frac{\partial}{\partial z}\left(y^2\right) \underline{= 2x}$$

$$\begin{aligned}\iiint_D \nabla \cdot \boldsymbol{F}\ dV &= \iiint_D 2x\ dV \\ &= \int_0^1 \int_0^1 \int_0^1 (2x)\,dxdydz \\ &= \int_0^1 dz \int_0^1 dy\ \left[x^2\right]_0^1 \\ &= z\Big|_0^1\ \ y\Big|_0^1\ \ (1) \\ &\underline{= 1}\end{aligned}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *S*.

$\boldsymbol{F} = \langle x,\ 2y,\ z \rangle$; *S* is boundary of the tetrahedron in the first octant formed by the plane $x + y + z = 1$

Solution

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(z) \underline{= 4}$$

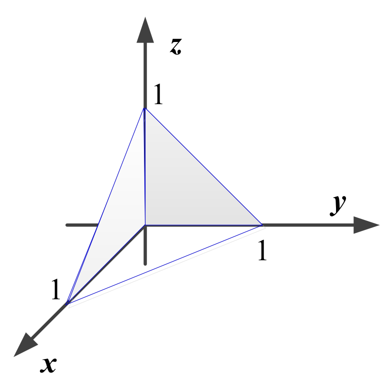
So by the Divergence Theorem, the net outward flux is 4 times the volume of the tetrahedron.

Volume of the tetrahedron $= \frac{1}{3}(\textit{area of the base})(\textit{height})$

$$\textit{area of the base} = \tfrac{1}{2}(x)(y) = \tfrac{1}{2}(1)(1) = \tfrac{1}{2}$$

$$V = \tfrac{1}{3}(\textit{area of the base})(\textit{height}) = \tfrac{1}{3}\left(\tfrac{1}{2}\right)(1) = \tfrac{1}{6}$$

$$\iiint_D \nabla \cdot \boldsymbol{F}\ dV = 4\left(\tfrac{1}{6}\right) \underline{= \tfrac{2}{3}}$$



Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *S*.

$\boldsymbol{F} = \left\langle x^2,\ y^2,\ z^2 \right\rangle$; *S* is the sphere $\left\{(x, y, z):\ x^2 + y^2 + z^2 = 25\right\}$

Solution

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}\left(y^2\right) + \frac{\partial}{\partial z}\left(z^2\right) \underline{= 2(x + y + z)}$$

So by the Divergence Theorem, the net outward flux is

$$\iiint_D \nabla \cdot \boldsymbol{F}\ dV = 2\iiint_D (x + y + z)\ dV$$

$$= 2\int_0^{2\pi}\int_0^{\pi}\int_0^{5} 5r\left(\sin\varphi\cos\theta + \sin\varphi\sin\theta + \cos\varphi\right)\, dr d\varphi d\theta$$

$$= 5\left[r^2\right]_0^5 \int_0^{2\pi}\int_0^{\pi} \left(\sin\varphi\cos\theta + \sin\varphi\sin\theta + \cos\varphi\right)\, d\varphi d\theta$$

$$= 125\int_0^{2\pi} \left(-\cos\varphi\cos\theta - \cos\varphi\sin\theta + \sin\varphi\right)\Big|_0^{\pi}\ d\theta$$

$$= 125\int_0^{2\pi} 2(\cos\theta + \sin\theta)\ d\theta$$

$$= 250\ \left[\sin\theta - \cos\theta\right]_0^{2\pi}$$

$$= 250\ (0)$$

$$\underline{= 0}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *S*.

$\boldsymbol{F} = \langle x,\ y,\ z\rangle$; *S* is the surface of the paraboloid $z = 4 - x^2 - y^2$, for $z \ge 0$, plus its base in the *xy*-plane

Solution

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \underline{= 3}$$

So by the Divergence Theorem, the net outward flux is

$$\iiint_D \nabla \cdot \boldsymbol{F}\ dV = \int_0^{2\pi}\int_0^{2} (3) rz\ dr d\theta$$

$$= 3\int_0^{2\pi} d\theta \int_0^2 r\left(4-r^2\right)dr$$

$$= 3(2\pi)\left[2r^2 - \frac{1}{4}r^4\right]_0^2$$

$$= 6\pi(8-4)$$

$$\underline{= 24\pi|}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *S*.

$\boldsymbol{F} = \langle x,\ y,\ z\rangle$; *S* is the surface of the cone $z^2 = x^2 + y^2$, for $0 \le z \le 4$, plus its top surface in the plane $z = 4$

Solution

$$\nabla\cdot\boldsymbol{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \ \underline{= 3|}$$

Volume of a cone $= \frac{1}{3}(\mathit{area\ of\ the\ base})(\mathit{height}) = \frac{1}{3}\left(\pi r^2\right)(4) = \frac{4\pi}{3}(16) = \frac{64\pi}{3}$

So by the Divergence Theorem, the net outward flux is

$$\iiint_D \nabla\cdot\boldsymbol{F}\ dV = (3)\left(\frac{64\pi}{3}\right) \ \underline{= 64\pi|}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *D*.

$\boldsymbol{F} = \langle z-x,\ x-y,\ 2y-z\rangle$; *D* is the region between the spheres of radius 2 and 4 centered at origin.

Solution

$$\nabla\cdot\boldsymbol{F} = \frac{\partial}{\partial x}(z-x) + \frac{\partial}{\partial y}(x-y) + \frac{\partial}{\partial z}(2y-z) \ \underline{= -3|}$$

Volume between 2 spheres $= \frac{4}{3}\pi\left(4^3 - 2^3\right) = \frac{224}{3}\pi$

$$\iiint_D \nabla\cdot\boldsymbol{F}\ dV = (-3)\left(\frac{224\pi}{3}\right) \ \underline{= -224\pi|}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *D*.

$\boldsymbol{F} = \boldsymbol{r}|\boldsymbol{r}| = \langle x,\ y,\ z\rangle\sqrt{x^2+y^2+z^2}$; *D* is the region between the spheres of radius 1 and 2 centered at origin.

Solution

$$\left(U^n V^m\right)' = U^{n-1}V^{m-1}\left(nU'V + mUV'\right)$$

$$\frac{\partial}{\partial x}\left(x\left(x^2+y^2+z^2\right)^{1/2}\right) = \left(x^2+y^2+z^2\right)^{-1/2}\left[x^2+y^2+z^2+\tfrac{1}{2}(2x)(x)\right]$$

$$= \left(2x^2+y^2+z^2\right)\left(x^2+y^2+z^2\right)^{-1/2}$$

$$\nabla\cdot\boldsymbol{F} = \frac{\partial}{\partial x}\left(x\left(x^2+y^2+z^2\right)^{1/2}\right) + \frac{\partial}{\partial y}\left(y\left(x^2+y^2+z^2\right)^{1/2}\right) + \frac{\partial}{\partial z}\left(z\left(x^2+y^2+z^2\right)^{1/2}\right)$$

$$= \left(x^2+y^2+z^2\right)^{-1/2}\left(2x^2+y^2+z^2+x^2+2y^2+z^2+x^2+y^2+2z^2\right)$$

$$= 4\left(x^2+y^2+z^2\right)^{-1/2}\left(x^2+y^2+z^2\right)$$

$$= 4\sqrt{x^2+y^2+z^2} \quad \underline{= 4|\boldsymbol{r}|}$$

$$\iiint_D \nabla\cdot\boldsymbol{F}\ dV = \iiint_D 4\sqrt{x^2+y^2+z^2}\ dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{r}(4\rho)\rho^2\sin\varphi\ d\rho\, d\varphi\, d\theta$$

$$= 4\int_0^{2\pi} d\theta \int_0^{\pi}\sin\varphi d\varphi \int_0^{r}\rho^3\ d\rho$$

$$= 4(2\pi)\left[-\cos\varphi\right]_0^1 \left[\tfrac{1}{4}\rho^4\right]_0^r$$

$$= 4(2\pi)(2)\left(\tfrac{1}{4}r^4\right)$$

$$= 4\pi r^4$$

$$= 4\pi\left(2^4-1^4\right)$$

$$\underline{= 60\pi}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *D*.

$\boldsymbol{F} = \frac{\boldsymbol{r}}{|\boldsymbol{r}|} = \frac{\langle x,\ y,\ z\rangle}{\sqrt{x^2+y^2+z^2}}$; *D* is the region between the spheres of radius 1 and 2 centered at origin.

Solution

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\left(x^2+y^2+z^2\right)^{1/2} - x^2\left(x^2+y^2+z^2\right)^{-1/2}}{x^2+y^2+z^2} = \frac{y^2+z^2}{\left(x^2+y^2+z^2\right)^2}$$

$$\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\left(x^2+y^2+z^2\right)^{1/2} - y^2\left(x^2+y^2+z^2\right)^{-1/2}}{x^2+y^2+z^2} = \frac{x^2+z^2}{\left(x^2+y^2+z^2\right)^2}$$

$$\frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\left(x^2+y^2+z^2\right)^{1/2} - z^2\left(x^2+y^2+z^2\right)^{-1/2}}{x^2+y^2+z^2} = \frac{x^2+y^2}{\left(x^2+y^2+z^2\right)^2}$$

$$\nabla\cdot\boldsymbol{F} = \frac{y^2+z^2+x^2+z^2+x^2+y^2}{\left(x^2+y^2+z^2\right)^2} = \frac{2\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^2}$$

$$= \frac{2}{x^2+y^2+z^2} = \frac{2}{|\boldsymbol{r}|^2}$$

$$\iiint_D \nabla\cdot\boldsymbol{F}\ dV = 2\iiint_D \frac{1}{x^2+y^2+z^2}\ dV$$

$$= 2\int_0^{2\pi}\int_0^{\pi}\int_0^{r}\left(\frac{1}{\rho}\right)\rho^2\sin\varphi\ d\rho\ d\varphi\ d\theta$$

$$= 2\int_0^{2\pi} d\theta\ \int_0^{\pi}\sin\varphi d\varphi\ \int_0^{r}\rho\ d\rho$$

$$= 2(2\pi)\left[-\cos\varphi\right]_0^1\ \left[\frac{1}{2}\rho^2\right]_0^r$$

$$= 2(2\pi)(2)\left(\frac{1}{2}r^2\right)$$

$$= 4\pi r^2$$

$$= 4\pi\left(2^2-1^2\right)$$

$$= 12\pi$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *D*.

$\boldsymbol{F} = \langle z-y,\ x-z,\ 2y-x \rangle;\quad D = \{(x,y,z):\ 1 \le |x| \le 3,\ 1 \le |y| \le 3,\ 1 \le |z| \le 3\}$ is the region between two cubes

Solution

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(z-y) + \frac{\partial}{\partial y}(x-z) + \frac{\partial}{\partial z}(2y-x) \ \underline{=0}$$

Therefore by the Divergence Theorem, the net outward flux is ***zero***.

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *D*.

$\boldsymbol{F} = \langle x,\ 2y,\ 3z \rangle$; *D* is the region between the cylinders $x^2+y^2=1$ and $x^2+y^2=4$ for $0 \le z \le 8$

Solution

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z) \ \underline{=6}$$

Volume of the sphere $x^2+y^2=4$

$$\begin{aligned} V &= \int_0^8 \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\, dx\, dz \\ &= \int_0^8 dz \int_{-2}^{2} y \Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx \\ &= (8)\int_{-2}^{2} 2\sqrt{4-x^2}\, dx \\ &= 16\left[\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2}\right]_{-2}^{2} \\ &= 16\left[2\left(\frac{\pi}{2}\right) - 2\left(-\frac{\pi}{2}\right)\right] \\ &\underline{=32\pi} \qquad \textbf{Or } V_1 = z\left(\pi r^2\right) = 8(4\pi) = 32\pi \end{aligned}$$

Volume of the sphere $x^2+y^2=1$

$$\begin{aligned} V &= \int_0^8 \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\, dx\, dz \\ &= \int_0^8 dz \int_{-1}^{1} y \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx \\ &= (8)\int_{-1}^{1} 2\sqrt{1-x^2}\, dx \\ &= 16\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}\frac{x}{1}\right]_{-1}^{1} \\ &= 16\left[\frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\left(-\frac{\pi}{2}\right)\right] \\ &\underline{=8\pi} \qquad \textbf{Or } V_2 = z\left(\pi r^2\right) = 8(\pi) = 8\pi \end{aligned}$$

$$\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

Therefore, the net outward flux is $= 6(32\pi - 8\pi)\ \underline{=144\pi}$

Exercise

Compute the outward flux of the following vector field across the given surface

$\boldsymbol{F} = \left\langle x^2 e^y \cos z,\ -4xe^y \cos z,\ 2xe^y \sin z \right\rangle$; S is the boundary of the ellipsoid $\frac{x^2}{4} + y^2 + z^2 = 1$

Solution

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}\left(x^2 e^y \cos z\right) + \frac{\partial}{\partial y}\left(-4xe^y \cos z\right) + \frac{\partial}{\partial z}\left(2xe^y \sin z\right)$$

$$= 2xe^y \cos z - 4xe^y \cos z + 2xe^y \cos z$$

$$= 0$$

Therefore by the Divergence Theorem, the net outward flux is ***zero***.

Exercise

Compute the outward flux of the following vector field across the given surface $\boldsymbol{F} = \langle -yz,\ xz,\ 1\rangle$; S is the boundary of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{4} + z^2 = 1$

Solution

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(-yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(1) = 0$$

Therefore by the Divergence Theorem, the net outward flux is ***zero***.

Exercise

Compute the outward flux of the following vector field across the given surface

$\boldsymbol{F} = \langle x\sin y,\ -\cos y,\ z\sin y\rangle$; S is the boundary of the region bounded by the planes $x = 1$, $y = 0$, $y = \frac{\pi}{2}$, $z = 0$, and $z = x$

Solution

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(x\sin y) + \frac{\partial}{\partial y}(-\cos y) + \frac{\partial}{\partial z}(z\sin y)$$

$$= \sin y + \sin y + \sin y$$

$$= 3\sin y$$

$$\iiint_D \nabla \cdot \boldsymbol{F}\ dV = \int_0^{\pi/2} \int_0^1 \int_0^x 3\sin y\ dzdxdy$$

$$= \int_0^{\pi/2} \sin y\ dy\ \int_0^1 3z\Big|_0^x\ dx$$

$$= -\cos y \Big|_0^{\pi/2} \int_0^1 3x\, dx$$

$$= \frac{3}{2} x^2 \Big|_0^1$$

$$= \frac{3}{2}$$

Exercise

The electric field due to a point charge Q is $\boldsymbol{E} = \frac{Q}{4\pi\varepsilon_0} \frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}$, where $\boldsymbol{r} = \langle x, y, z \rangle$ and ε_0 is a constant

a) Show that the flux of the field across a sphere of radius a centered at the origin is

$$\iint_S \boldsymbol{E} \cdot \boldsymbol{n}\, dS = \frac{Q}{\varepsilon_0}$$

b) Let S be the boundary of the origin between two spheres centered of radius a and b with $a < b$. Use the Divergence Theorem to show that the net outward flux across S is zero.

c) Suppose there is a distribution of charge within a region D. Let $q(x, y, z)$ be the charge density (charge per unit volume). Interpret the statement that

$$\iint_S \boldsymbol{E} \cdot \boldsymbol{n}\, dS = \frac{1}{\varepsilon_0} \iiint_D q(x, y, z)\, dV$$

d) Assuming $\boldsymbol{E}$ satisfies the conditions of the Divergence Theorem, conclude from part (c) that

$$\nabla \cdot \boldsymbol{E} = \frac{q}{\varepsilon_0}$$

e) Because the electric force is conservative, it has a potential function ϕ. From part (d) conclude that $\nabla^2 \phi = \nabla \cdot \nabla \phi = \frac{q}{\varepsilon_0}$

Solution

a) $\boldsymbol{F} \cdot \boldsymbol{n} = \frac{\boldsymbol{r}}{|\boldsymbol{r}|^3} \cdot \frac{\boldsymbol{r}}{|\boldsymbol{r}|} = \frac{|\boldsymbol{r}|^2}{|\boldsymbol{r}|^4} = |\boldsymbol{r}|^{-2}$

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS = |\boldsymbol{r}|^{-2}$$

Area of sphere $= |\boldsymbol{r}|^{-2} \left(4\pi r^2\right) = |a|^{-2} \left(4\pi a^2\right) = 4\pi$

$$\iint_S \boldsymbol{E} \cdot \boldsymbol{n}\, dS = \frac{Q}{4\pi\varepsilon_0} (4\pi) = \frac{Q}{\varepsilon_0}$$

b) The net outward flux across S is the difference of the fluxes across the inner and outer sphere; by part (a), these are equal (independent of the radius), so the net flux across S is zero.

c) The left-hand side is the flux across the boundary of *D*, while the right-hand side is the sum of the charge densities at each point of *D*.

$$\iint_S \boldsymbol{E}\cdot\boldsymbol{n}\, dS = \frac{1}{\varepsilon_0}\iiint_D q(x,\ y,\ z)\ dV = \frac{Q}{\varepsilon_0}$$

$$\rightarrow\ \iiint_D q(x,\ y,\ z)\ dV = Q$$

The statement says that the flux across the boundary, up to multiplication by a constant, is the sum of the charge densities in the region.

d) $\frac{1}{\varepsilon_0}\iiint_D q(x,\ y,\ z)\ dV = \iint_S \boldsymbol{E}\cdot\boldsymbol{n}\ dS = \iiint_D \nabla\cdot\boldsymbol{E}\ dV$

This holds for all regions *D*.

Therefore; that implies that $\nabla\cdot\boldsymbol{E} = \dfrac{q(x,\ y,\ z)}{\varepsilon_0}$

e) $\nabla^2\phi = \nabla\cdot\nabla\phi = \nabla\cdot\boldsymbol{E} = \dfrac{q}{\varepsilon_0}$

Exercise

Fourier's Law of heat transfer (or heat conduction) states that the heat flow vector $\boldsymbol{F}$ at a point is proportional to the negative gradient of the temperature that is, $\boldsymbol{F} = -k\nabla T$, which means that heat energy flows from hot regions to cold region. The constant $k>0$ is called the *conductivity*, which has metric units of $J\,/\,m\text{-}s\text{-}K$. A temperature function for a region *D* is given. Find the net outward heat flux $\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\ dS = -k\iint_S \nabla T\cdot\boldsymbol{n}\ dS$ across the boundary *S* of *D*. In some cases it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume $k=1$.

$$T(x,\ y,\ z) = 100 + x + 2y + z;\quad D = \{(x,y,z):\ 0\le x\le 1,\ 0\le y\le 1,\ 0\le z\le 1\}$$

Solution

$$\nabla T = T_x\hat{\boldsymbol{i}} + T_y\hat{\boldsymbol{j}} + T_z\hat{\boldsymbol{k}}$$

$$= \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$$

$$\boldsymbol{F} = -k\nabla T = k\langle -1,\ -2,\ -1\rangle$$

$$\nabla\cdot\boldsymbol{F} = \frac{\partial}{\partial x}(-k) + \frac{\partial}{\partial y}(-2k) + \frac{\partial}{\partial z}(-k)\ \underline{=0|}$$

Therefore, the heat flux is ***zero***.

Exercise

Fourier's Law of heat transfer (or heat conduction) states that the heat flow vector $\boldsymbol{F}$ at a point is proportional to the negative gradient of the temperature that is, $\boldsymbol{F} = -k\nabla T$, which means that heat energy flows from hot regions to cold region. The constant $k > 0$ is called the *conductivity*, which has metric units of $J / m\text{-}s\text{-}K$. A temperature function for a region D is given.

Find the net outward heat flux $\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS = -k \iint_S \nabla T \cdot \boldsymbol{n}\, dS$ across the boundary S of D. In some cases it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume $k = 1$.

$$T(x, y, z) = 100 + x^2 + y^2 + z^2; \quad D = \{(x, y, z): \ 0 \le x \le 1, \ 0 \le y \le 1, \ 0 \le z \le 1\}$$

Solution

$$\nabla T = T_x \hat{\boldsymbol{i}} + T_y \hat{\boldsymbol{j}} + T_z \hat{\boldsymbol{k}} \ = 2x\hat{\boldsymbol{i}} + 2y\hat{\boldsymbol{j}} + 2z\hat{\boldsymbol{k}}$$

$$\boldsymbol{F} = -\nabla T = \langle -2x, \ -2y, \ -2z \rangle$$

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(-2x) + \frac{\partial}{\partial y}(-2y) + \frac{\partial}{\partial z}(-2z) \ \underline{= -6}$$

Therefore, the heat flux is **–6** times the volume of the region (or **–6**).

Exercise

Fourier's Law of heat transfer (or heat conduction) states that the heat flow vector $\boldsymbol{F}$ at a point is proportional to the negative gradient of the temperature that is, $\boldsymbol{F} = -k\nabla T$, which means that heat energy flows from hot regions to cold region. The constant $k > 0$ is called the *conductivity*, which has metric units of $J / m\text{-}s\text{-}K$. A temperature function for a region D is given.

Find the net outward heat flux $\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS = -k \iint_S \nabla T \cdot \boldsymbol{n}\, dS$ across the boundary S of D. In some cases it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume $k = 1$.

$$T(x, y, z) = 100 + e^{-z}; \quad D = \{(x, y, z): \ 0 \le x \le 1, \ 0 \le y \le 1, \ 0 \le z \le 1\}$$

Solution

$$\nabla T = T_x \hat{\boldsymbol{i}} + T_y \hat{\boldsymbol{j}} + T_z \hat{\boldsymbol{k}} \ = -e^{-z}\hat{\boldsymbol{k}}$$

$$\boldsymbol{F} = -\nabla T = \left\langle 0, \ 0, \ e^{-z} \right\rangle$$

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}\left(e^{-z}\right) \ \underline{= -e^{-z}}$$

Therefore, the heat flux is

$$\int_0^1 \int_0^1 \int_0^1 \left(-e^{-z}\right) dxdydz = \int_0^1 \left(-e^{-z}\right) dz \ \int_0^1 dy \ \int_0^1 dx$$

$$= e^{-z} \Big|_0^1 \ (1)(1)$$

$$\underline{= e^{-1} - 1}$$

Exercise

Fourier's Law of heat transfer (or heat conduction) states that the heat flow vector $\boldsymbol{F}$ at a point is proportional to the negative gradient of the temperature that is, $\boldsymbol{F} = -k\nabla T$, which means that heat energy flows from hot regions to cold region. The constant $k > 0$ is called the *conductivity*, which has metric units of $J / m\text{-}s\text{-}K$. A temperature function for a region D is given.

Find the net outward heat flux $\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS = -k \iint_S \nabla T \cdot \boldsymbol{n}\, dS$ across the boundary S of D. In some cases it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume $k = 1$.

$T(x, y, z) = 100e^{-x^2-y^2-z^2}$; D is the sphere of radius a centered at the origin.

Solution

$$\nabla T = T_x \hat{\boldsymbol{i}} + T_y \hat{\boldsymbol{j}} + T_z \hat{\boldsymbol{k}}$$

$$= -200xe^{-x^2-y^2-z^2}\hat{\boldsymbol{i}} - 200ye^{-x^2-y^2-z^2}\hat{\boldsymbol{j}} - 200ze^{-x^2-y^2-z^2}\hat{\boldsymbol{k}}$$

$$\boldsymbol{F} = -\nabla T = 200\left\langle xe^{-x^2-y^2-z^2},\ ye^{-x^2-y^2-z^2},\ ze^{-x^2-y^2-z^2} \right\rangle$$

$$\nabla \cdot \boldsymbol{F} = 200\frac{\partial}{\partial x}\left(xe^{-x^2-y^2-z^2} \right) + 200\frac{\partial}{\partial y}\left(ye^{-x^2-y^2-z^2} \right) + 200\frac{\partial}{\partial z}\left(ze^{-x^2-y^2-z^2} \right)$$

$$= 200e^{-x^2-y^2-z^2}\left(1 - 2x^2 + 1 - 2y^2 + 1 - 2z^2\right)$$

$$= 200e^{-x^2-y^2-z^2}\left(3 - 2x^2 - 2y^2 - 2z^2\right)$$

$$\iiint_D \nabla \cdot \boldsymbol{F}\, dV = \iiint_D 200e^{-x^2-y^2-z^2}\left(3 - 2x^2 - 2y^2 - 2z^2\right) dV$$

$$= 200\int_0^{2\pi}\int_0^{\pi}\int_0^{a} e^{-\rho^2}\left(3 - 2\rho^2\right)\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= 200\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\, d\varphi \int_0^{a}\left(3\rho^2 - 2\rho^4\right)e^{-\rho^2}\, d\rho$$

$$= 200(2\pi)\left[-\cos\varphi\right]_0^{\pi}\left(\rho^3 e^{-\rho^2}\right)_0^{a} \qquad \left(3\rho^2 - 2\rho^4\right)e^{-\rho^2} = \left(\rho^3 e^{-\rho^2}\right)'$$

$$= 800\pi a^3\, e^{-a^2}$$