

Solution ***Section 2.1 – Vectors in 2-Space, 3-Space, and n -Space***

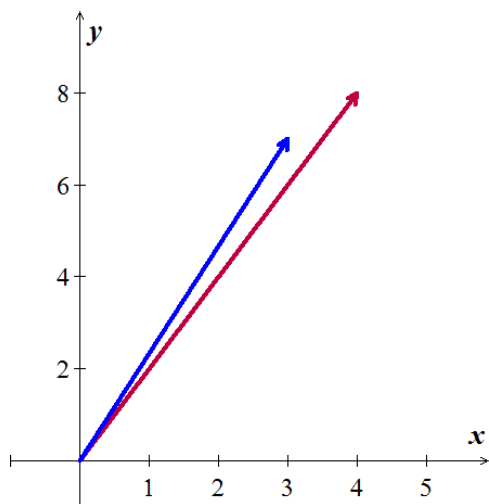
Exercise

Sketch the following vectors with initial points located at the origin

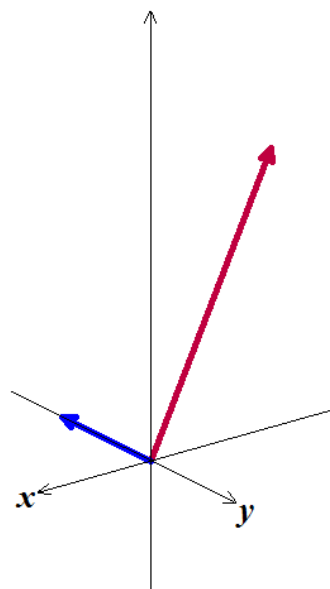
- a) $P_1(4, 8)$ $P_2(3, 7)$
- b) $P_1(-1, 0, 2)$ $P_2(0, -1, 0)$
- c) $P_1(3, -7, 2)$ $P_2(-2, 5, -4)$

Solution

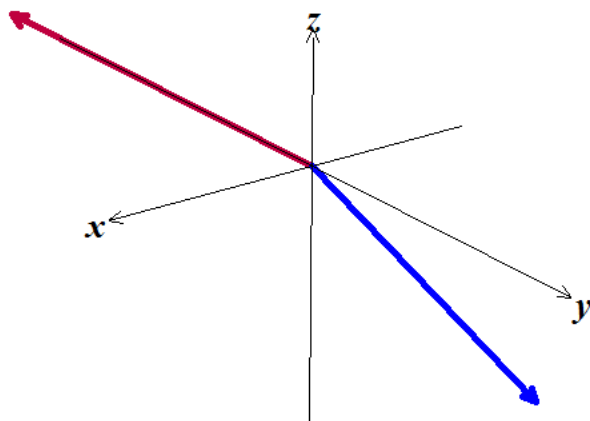
a)



b)



c)



Exercise

Find the components of the vector $\overrightarrow{P_1 P_2}$

a) $P_1(3, 5) \quad P_2(2, 8)$

b) $P_1(5, -2, 1) \quad P_2(2, 4, 2)$

c) $P_1(0, 0, 0) \quad P_2(-1, 6, 1)$

Solution

a) $\overrightarrow{P_1 P_2} = (2-3, 8-5)$
 $\quad \quad \quad = (-1, 3)$

b) $\overrightarrow{P_1 P_2} = (2-5, 4-(-2), 2-1)$
 $\quad \quad \quad = (-3, 6, 1)$

c) $\overrightarrow{P_1 P_2} = (-1-0, 6-0, 1-0)$
 $\quad \quad \quad = (-1, 6, 1)$

Exercise

Find the terminal point of the vector that is equivalent to $\vec{u} = (1, 2)$ and whose initial point is $A(1, 1)$

Solution

The terminal point: $B(b_1, b_2)$

$$(b_1 - 1, b_2 - 1) = (1, 2)$$

$$\begin{cases} b_1 - 1 = 1 & \Rightarrow b_1 = 2 \\ b_2 - 1 = 2 & \Rightarrow b_2 = 3 \end{cases}$$

The terminal point: $B(2, 3)$

Exercise

Find the initial point of the vector that is equivalent to $\vec{u} = (1, 1, 3)$ and whose terminal point is $B(-1, -1, 2)$

Solution

The initial point: $A(x, y, z)$

$$(-1-x, -1-y, 2-z) = (1, 1, 3)$$

$$\begin{cases} -1 - x = 1 & \Rightarrow x = -2 \\ -1 - y = 1 & \Rightarrow y = -2 \\ 2 - z = 3 & \Rightarrow z = -1 \end{cases}$$

The initial point: $A(-2, -2, -1)$

Exercise

Find a nonzero vector \vec{u} with initial point $P(-1, 3, -5)$ such that

- a) \vec{u} has the same direction as $\vec{v} = (6, 7, -3)$
- b) \vec{u} is oppositely directed as $\vec{v} = (6, 7, -3)$

Solution

- a) \vec{u} has the same direction as \vec{v}

$$\vec{u} = \vec{v} = (6, 7, -3)$$

The initial point $P(-1, 3, -5)$ then the terminal point:

$$(-1+6, 3+7, -5-3) = \underline{(5, 10, -8)}$$

- b) \vec{u} is oppositely directed as $\vec{v} = (6, 7, -3)$

$$\vec{u} = -\vec{v} = (-6, -7, 3)$$

The initial point $P(-1, 3, -5)$ then the terminal point:

$$(-1-6, 3-7, -5+3) = \underline{(-7, -4, -2)}$$

Exercise

Let $\vec{u} = (-3, 1, 2)$, $\vec{v} = (4, 0, -8)$, and $\vec{w} = (6, -1, -4)$. Find the components

- a) $\vec{v} - \vec{w}$
- b) $6\vec{u} + 2\vec{v}$
- c) $5(\vec{v} - 4\vec{u})$
- d) $-3(\vec{v} - 8\vec{w})$
- e) $(2\vec{u} - 7\vec{w}) - (8\vec{v} + \vec{u})$
- f) $-\vec{u} + (\vec{v} - 4\vec{w})$

Solution

$$\begin{aligned} \text{a) } \vec{v} - \vec{w} &= (4-6, 0-(-1), -8-(-4)) \\ &= \underline{(-2, 1, -4)} \end{aligned}$$

$$\begin{aligned} \text{b) } 6\vec{u} + 2\vec{v} &= (-18, 6, 12) + (8, 0, -16) \\ &= \underline{(-10, 6, -4)} \end{aligned}$$

$$\begin{aligned} \text{c) } 5(\vec{v} - 4\vec{u}) &= 5(4-(-12), 0-4, -8-8) \\ &= 5(16, -4, -16) \end{aligned}$$

$$= \underline{(80, -20, -80)}$$

$$\begin{aligned} d) \quad -3(\vec{v} - 8\vec{w}) &= -3(4 - 48, 0 - (-8), -8 - (-32)) \\ &= -3(-44, 8, 24) \\ &= \underline{(32, -24, -72)} \end{aligned}$$

$$\begin{aligned} e) \quad (2\vec{u} - 7\vec{w}) - (8\vec{v} + \vec{u}) &= [(-6, 2, 4) - (42, -7, -28)] - [(32, 0, -64) + (-3, 1, 2)] \\ &= (-48, 9, 32) - (29, 1, -62) \\ &= \underline{(-77, 8, 94)} \end{aligned}$$

$$\begin{aligned} f) \quad -u + (v - 4w) &= (3, -1, -2) + [(4, 0, -8) - (24, -4, -16)] \\ &= (3, -1, -2) + (-20, 4, 8) \\ &= \underline{(-17, 3, 6)} \end{aligned}$$

Exercise

Let $\vec{u} = (2, 1, 0, 1, -1)$ and $\vec{v} = (-2, 3, 1, 0, 2)$. Find scalars a and b so that $a\vec{u} + b\vec{v} = (-8, 8, 3, -1, 7)$

Solution

$$\begin{aligned} a\vec{u} + b\vec{v} &= a(2, 1, 0, 1, -1) + b(-2, 3, 1, 0, 2) \\ &= (a - 2b, a + 3b, b, a, -a + 2b) \\ &= \underline{(-8, 8, 3, -1, 7)} \end{aligned}$$

$$\begin{cases} a - 2b = -8 \\ a + 3b = 8 \\ b = 3 \\ a = -1 \\ -a + 2b = 7 \end{cases}$$

$$\rightarrow a = -1 \quad b = 3 \quad \text{Unique solution}$$

Exercise

Find all scalars c_1 , c_2 , and c_3 such that $c_1(1, 2, 0) + c_2(2, 1, 1) + c_3(0, 3, 1) = (0, 0, 0)$

Solution

$$\begin{aligned} c_1(1, 2, 0) + c_2(2, 1, 1) + c_3(0, 3, 1) &= (c_1 + 2c_2, 2c_1 + c_2 + 3c_3, c_2 + c_3) \\ &= (0, 0, 0) \end{aligned}$$

$$\begin{cases} c_1 + 2c_2 = 0 \\ 2c_1 + c_2 + 3c_3 = 0 \\ c_2 + c_3 = 0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right] \quad R_2 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right] \quad -\frac{1}{3}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right] \quad \begin{array}{l} R_1 - 2R_2 \\ R_3 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array} \right] \quad \frac{1}{2}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \quad R_2 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$\underline{c_1 = c_2 = c_3 = 0}$$

Exercise

Find the distance between the given points $[5 \ 1 \ 8 \ -1 \ 2 \ 9]$, $[4 \ 1 \ 4 \ 3 \ 2 \ 8]$

Solution

$$\begin{aligned} d &= \sqrt{(4-5)^2 + (1-1)^2 + (4-8)^2 + (3+1)^2 + (2-2)^2 + (8-9)^2} \\ &= \sqrt{1+0+16+16+0+1} \\ &= \underline{\underline{\sqrt{34}}} \end{aligned}$$

Exercise

Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar

multiplication operation on $\vec{u} = (u_1, u_2)$ $\vec{v} = (v_1, v_2)$

$$\vec{u} + \vec{v} = (u_1 + v_1 + 1, u_2 + v_2 + 1) \quad k\vec{u} = (ku_1, ku_2)$$

- a) Compute $\vec{u} + \vec{v}$ and $k\vec{u}$ for $\vec{u} = (0, 4)$, $\vec{v} = (1, -3)$, and $k = 2$.
- b) Show that $(0, 0) \neq \vec{0}$.
- c) Show that $(-1, -1) = \vec{0}$.
- d) Show that $\vec{u} + (-\vec{u}) = \vec{0}$ for $\vec{u} = (u_1, u_2)$
- e) Find two vector space axioms that fail to hold.

Solution

$$\begin{aligned} \text{a)} \quad \vec{u} + \vec{v} &= (0+1+1, 4-3+1) \\ &= (2, 2) \end{aligned}$$

$$\begin{aligned} k\vec{u} &= (ku_1, ku_2) \\ &= (2(0), 2(4)) \\ &= (0, 8) \end{aligned}$$

$$\begin{aligned} \text{b)} \quad (0, 0) + (u_1, u_2) &= (0+u_1+1, 0+u_2+1) \\ &= (u_1+1, u_2+1) \\ &\neq (u_1, u_2) \end{aligned}$$

Therefore $(0, 0)$ is not the zero vector $\vec{0}$ required (by Axiom).

$$\begin{aligned} \text{c)} \quad (-1, -1) + (u_1, u_2) &= (-1+u_1+1, -1+u_2+1) \\ &= (u_1, u_2) \\ (u_1, u_2) + (-1, -1) &= (u_1-1+1, u_2-1+1) \\ &= (u_1, u_2) \end{aligned}$$

Therefore $(-1, -1) = \vec{0}$ holds.

$$\begin{aligned} \text{d)} \quad \text{Let } \vec{u} &= (u_1, u_2) \text{ \&} \\ -\vec{u} &= (-2-u_1, -2-u_2) \\ \vec{u} + (-\vec{u}) &= (u_1 + (-2-u_1)+1, u_2 + (-2-u_2)+1) \end{aligned}$$

$$= (-1, -1)$$

$$= \underline{\vec{0}}$$

$$\vec{u} + (-\vec{u}) = 0 \text{ holds}$$

e) Axiom 7: $k(\vec{u} + \vec{v}) \stackrel{?}{=} k\vec{u} + k\vec{v}$

$$\begin{aligned} k(\vec{u} + \vec{v}) &= k(u_1 + v_1 + 1, u_2 + v_2 + 1) \\ &= (ku_1 + kv_1 + k, ku_2 + kv_2 + k) \end{aligned}$$

$$\begin{aligned} k\vec{u} + k\vec{v} &= (ku_1, ku_2) + (kv_1, kv_2) \\ &= (ku_1 + kv_1 + 1, ku_2 + kv_2 + 1) \end{aligned}$$

Therefore, $k(\vec{u} + \vec{v}) \neq k\vec{u} + k\vec{v}$; Axiom 7 fails to hold

Axiom 8: $(k + m)\vec{u} \stackrel{?}{=} k\vec{u} + m\vec{u}$

$$\begin{aligned} (k + m)\vec{u} &= ((k + m)u_1, (k + m)u_2) \\ &= (ku_1 + mu_1, ku_2 + mu_2) \end{aligned}$$

$$\begin{aligned} k\vec{u} + m\vec{u} &= (ku_1, ku_2) + (mu_1, mu_2) \\ &= (ku_1 + mu_1 + 1, ku_2 + mu_2 + 1) \end{aligned}$$

Therefore, $(k + m)\vec{u} \neq k\vec{u} + m\vec{u}$; Axiom 8 fails to hold

Exercise

Find \vec{w} given that $10\vec{u} + 3\vec{w} = 4\vec{v} - 2\vec{w}$, $\vec{u} = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -20 \\ 5 \end{pmatrix}$.

Solution

$$-10\vec{u} + 10\vec{u} + 3\vec{w} + 2\vec{w} = -10\vec{u} + 4\vec{v} - 2\vec{w} + 2\vec{w}$$

$$5\vec{w} = -10\vec{u} + 4\vec{v}$$

$$\vec{w} = -2\vec{u} + \frac{4}{5}\vec{v}$$

$$= -2\begin{pmatrix} 1 \\ -6 \end{pmatrix} + \frac{4}{5}\begin{pmatrix} -20 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 12 \end{pmatrix} + \begin{pmatrix} -16 \\ 4 \end{pmatrix}$$

$$\underline{= \begin{pmatrix} -18 \\ 16 \end{pmatrix} \Big|}$$

Exercise

Find \vec{w} given that $\vec{u} + 3\vec{v} - 2\vec{w} = 5\vec{u} + \vec{v} - 4\vec{w}$, $\vec{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

Solution

$$\vec{u} - \vec{u} + 3\vec{v} - 3\vec{v} - 2\vec{w} + 4\vec{w} = 5\vec{u} - \vec{u} + \vec{v} - 3\vec{v} - 4\vec{w} + 4\vec{w}$$

$$2\vec{w} = 4\vec{u} - 2\vec{v}$$

$$\vec{w} = 2\vec{u} - \vec{v}$$

$$= 2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -2 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$\underline{= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Big|}$$

Exercise

Find \vec{w} given that $2\vec{u} + \vec{v} - 3\vec{w} = 5\vec{u} + 7\vec{v} + 3\vec{w}$, $\vec{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

Solution

$$2\vec{u} - 2\vec{u} + \vec{v} - \vec{v} - 3\vec{w} - 3\vec{w} = 5\vec{u} - 2\vec{u} + 7\vec{v} - \vec{v} + 3\vec{w} - 3\vec{w}$$

$$-6\vec{w} = 3\vec{u} + 6\vec{v}$$

$$\vec{w} = -\frac{1}{2}\vec{u} - \vec{v}$$

$$= -\frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$\underline{= \begin{pmatrix} \frac{3}{2} \\ -\frac{5}{2} \end{pmatrix} \Big|}$$

Exercise

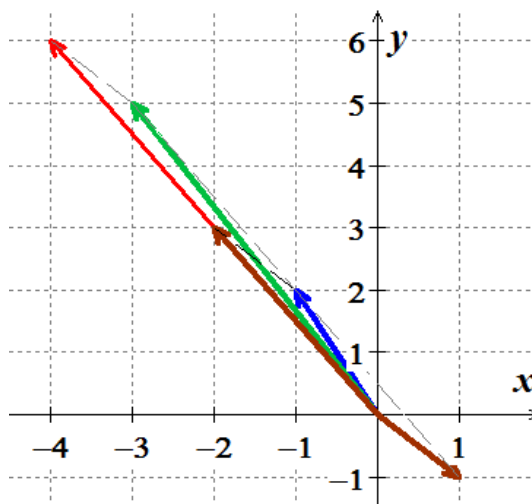
Draw \vec{u} , \vec{v} , $\vec{u} + \vec{v}$, and $\vec{u} + 2\vec{v}$

$$\vec{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \vec{v} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

Solution

$$\begin{aligned} \vec{u} + \vec{v} &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \vec{u} + 2\vec{v} &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} -2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -4 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 5 \end{pmatrix} \end{aligned}$$



Exercise

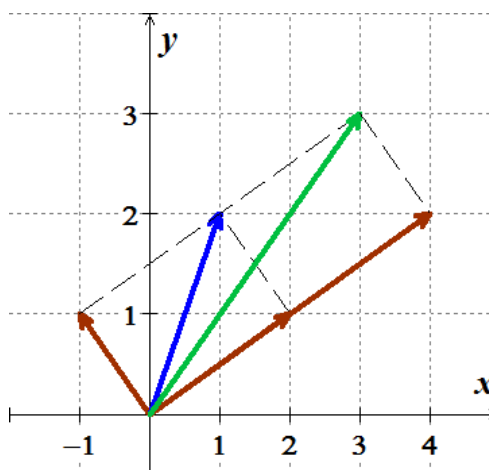
Draw \vec{u} , \vec{v} , $\vec{u} + \vec{v}$, and $\vec{u} + 2\vec{v}$

$$\vec{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Solution

$$\begin{aligned} \vec{u} + \vec{v} &= \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \vec{u} + 2\vec{v} &= \begin{pmatrix} -1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 3 \end{pmatrix} \end{aligned}$$



Exercise

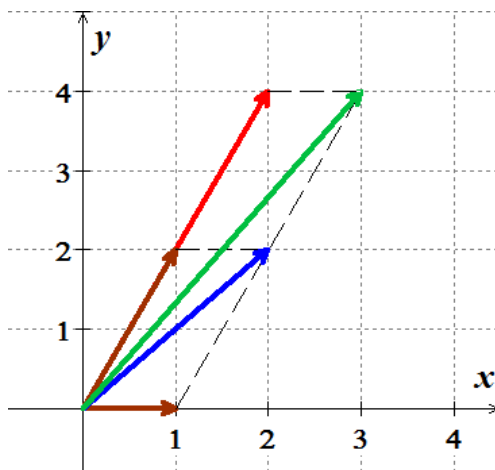
Draw \vec{u} , \vec{v} , $\vec{u} + \vec{v}$, and $\vec{u} + 2\vec{v}$

$$\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Solution

$$\begin{aligned} \vec{u} + \vec{v} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \vec{u} + 2\vec{v} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 4 \end{pmatrix} \end{aligned}$$



Exercise

Draw \vec{u} , \vec{v} , $\vec{u} + \vec{v}$, and $\vec{u} + 2\vec{v}$

$$\vec{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Solution

$$\begin{aligned} \vec{u} + \vec{v} &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \vec{u} + 2\vec{v} &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \end{aligned}$$

