Lecture One – Applications of Definite Integrals

Section 1.1 – Velocity and Net Change

Velocity, Position, and Displacement

Definitions

1. ***Position*** of an object at time *t*, denoted $s(t)$, is the location of the object relative to the origin.

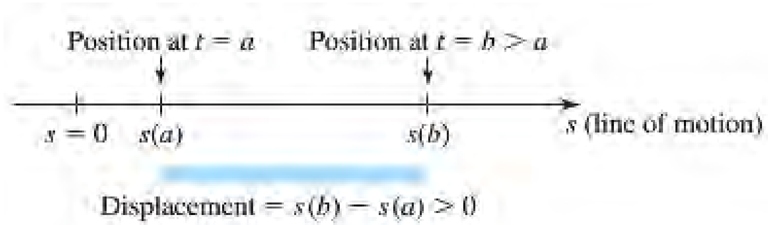
2. ***Velocity*** of an object at time t is $v(t)=s'(t)$

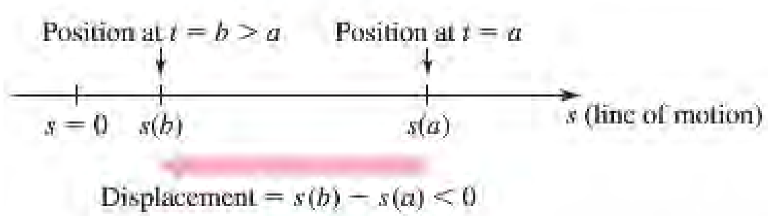
3. ***Displacement*** of the object between $t=a$ and $t=b>a$ is

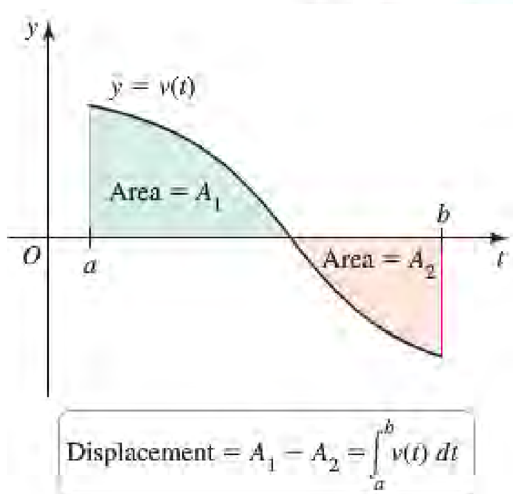
$$s(b)-s(a)=\int_a^b v(t)\,dt$$

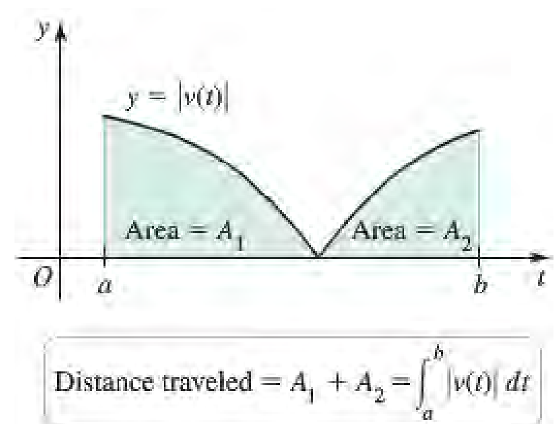
4. ***Distance traveled*** by the object between $t=a$ and $t=b>a$ is

$$\int_a^b |v(t)|\,dt$$ where $|v(t)|$ is the speed of the object at time *t*.









Example

A cyclist pedals along a straight road with velocity $v(t) = 2t^2 - 8t + 6 \ \left(mi/hr\right)$ for $0 \le t \le 3$, where *t* is measured in hours.

a) Graph the velocity function over the interval [0, 3]. Determine when the cyclist moves in the position direction and when she moves in the negative direction.

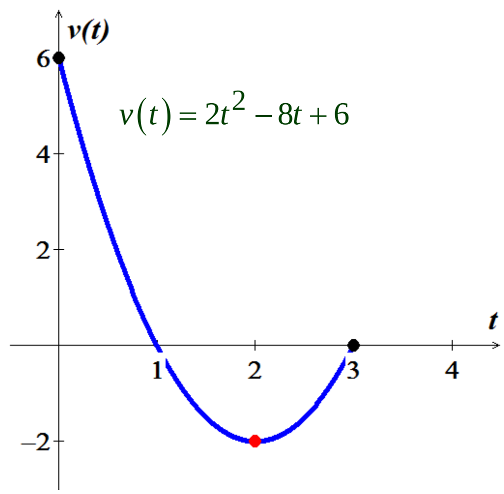
b) Find the displacement of the cyclist (in miles) on the time intervals [0, 1], [1, 3], and [0, 3]. Interpret these results.

c) Find the distance traveled over the interval [0, 3]

Solution

a) $v(t) = 2t^2 - 8t + 6 = 0$

$t = 1, \ 3$



The velocity is zero at $t = 1$ and $t = 3$.

The velocity is positive on $0 \le t < 1$, which means the cyclist moves in the positive *s* direction.

The velocity is negative on $1 < t < 3$, which means the cyclist moves in the negative *s* direction.

b) Displacement over [0, 1]

$$\begin{aligned} s(1) - s(0) &= \int_0^1 v(t)\,dt \\ &= \int_0^1 \left(2t^2 - 8t + 6\right)dt \\ &= \frac{2}{3}t^3 - 4t^2 + 6t \Big|_0^1 \\ &= \frac{2}{3} - 4 + 6 \\ &= \frac{8}{3} \end{aligned}$$

Displacement over [1, 3]

$$s(3)-s(1)=\int_1^3\left(2t^2-8t+6\right)dt$$

$$=\frac{2}{3}t^3-4t^2+6t\,\Big|_1^3$$

$$=18-36+18-\frac{2}{3}+4-6$$

$$\boxed{=-\frac{8}{3}}$$

Displacement over [0, 3]

$$s(3)-s(0)=\int_0^3\left(2t^2-8t+6\right)dt$$

$$=\frac{2}{3}t^3-4t^2+6t\,\Big|_0^3$$

$$=18-36+18$$

$$\boxed{=0}$$

The cyclist returns to the starting point after 3 hours.

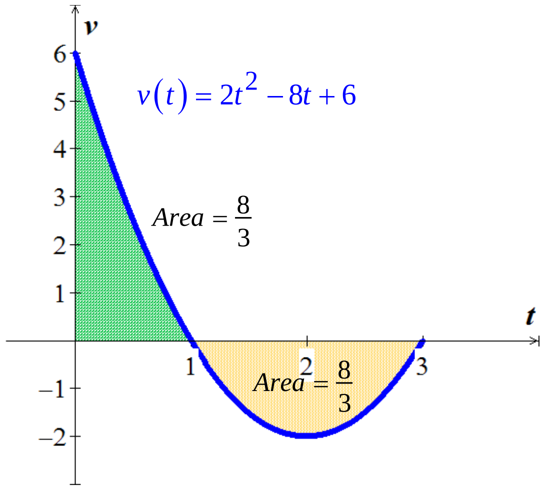
c) $$Distance=\int_0^3\left|v(t)\right|dt$$

$$=\int_0^1\left(2t^2-8t+6\right)dt-\int_1^3\left(2t^2-8t+6\right)dt$$

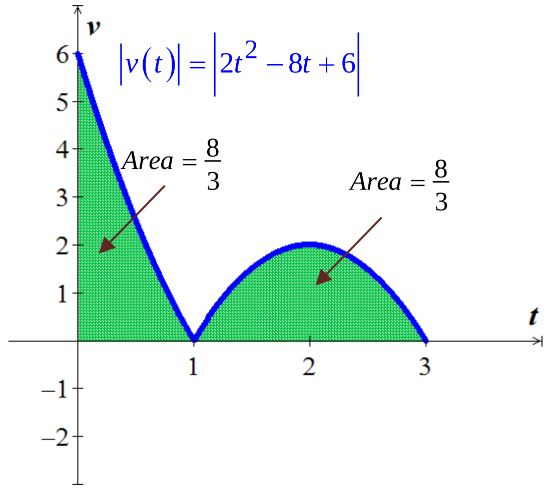
$$=\frac{2}{3}t^3-4t^2+6t\,\Big|_0^1-\left(\frac{2}{3}t^3-4t^2+6t\,\Big|_1^3\right.$$

$$=\frac{8}{3}+\frac{8}{3}$$

$$\boxed{=\frac{16}{3}}$$



Displacement from $t = 0$ to $t = 3$ is 0



$$Distance = \int_0^3 |v(t)|\,dt = \frac{8}{3} + \frac{8}{3} = \frac{16}{3}$$

Future Value of the Position Function

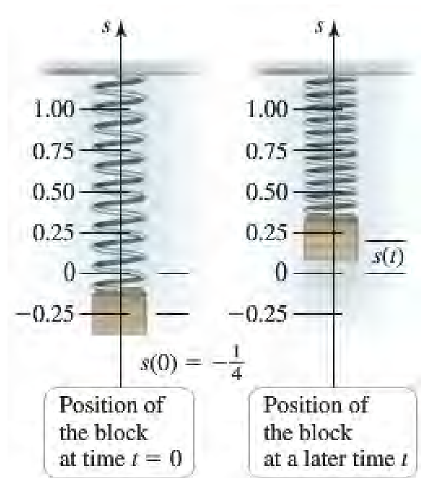
Theorem

Given the velocity $v(t)$ of an object moving along a line and its initial position $s(0)$, the position function of the object for future times $t \geq 0$ is

$$\underbrace{s(t)}_{\substack{\text{position}\\ \text{at time } t}} = \underbrace{s(0)}_{\substack{\text{initial}\\ \text{position}}} + \underbrace{\int_0^t v(x)\,dx}_{\substack{\text{displacement}\\ \text{over } [0,\, t]}}$$

Example

A block hangs at rest from a massless spring at the origin ($s = 0$). At $t = 0$, the block is pulled downward $\frac{1}{4}m$ to its initial position $s(0) = -\frac{1}{4}$ and released. Its velocity is given by $v(t) = \frac{1}{4}\sin t \ \ (m/s)$ for $t \geq 0$. Assume that the upward direction is positive.



a) Find the position of the block for $t \geq 0$

b) Graph the position function for $0 \leq t \leq 3\pi$.

c) When does the block move through the origin for the first time?

d) When does the block reach its highest point for the first time and what is its position at that time?

e) When does the block return to its lowest point?

Solution

a)

1st *method*

$$s(t) = \int v(t)\,dt$$

$$= \int \frac{1}{4}\sin t \ dt$$

$$= -\frac{1}{4}\cos t + C$$

Since $s(0) = -\frac{1}{4}$, then

$$-\frac{1}{4} = -\frac{1}{4}\cos(0) + C$$

$$\rightarrow \ C = 0$$

$$s(t) = -\frac{1}{4}\cos t$$

2nd *method*

$$s(t) = s(0) + \int_0^t v(x)\,dx$$

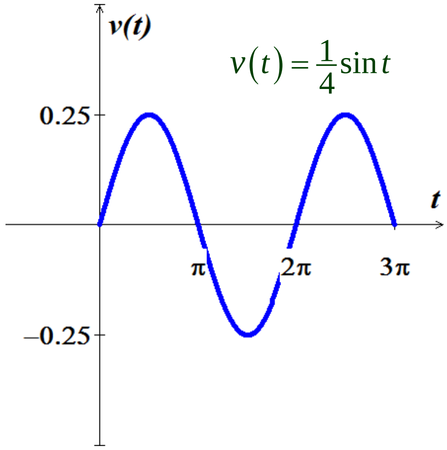
$$= -\frac{1}{4} + \int_0^t \frac{1}{4}\sin x \ dx$$

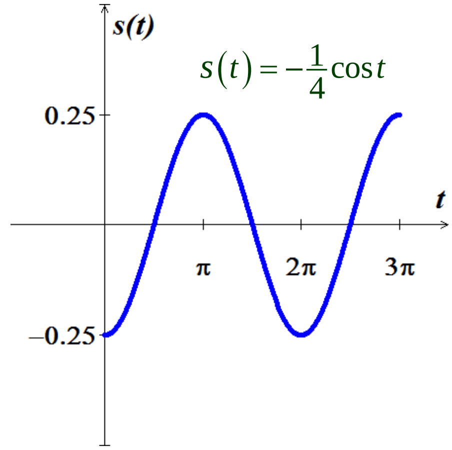
$$= -\frac{1}{4} - \frac{1}{4}\left[\cos x\right]_0^t$$

$$= -\frac{1}{4} - \frac{1}{4}(\cos t - 1)$$

$$= -\frac{1}{4}\cos t$$

b)





c) The block moves through the origin for the first time when $s=0$

$s(t)=-\frac{1}{4}\cos t=0$

$\rightarrow \quad \boxed{t=\frac{\pi}{2}}$

d) The block moves in the positive direction and reaches its high point for the first time when $t=\pi$

$s(\pi)=-\frac{1}{4}\cos\pi$

$\quad =\boxed{\frac{1}{4}\ m}$

e) The block returns to the lowest point at $t=2\pi$. This motion repeats every 2π seconds

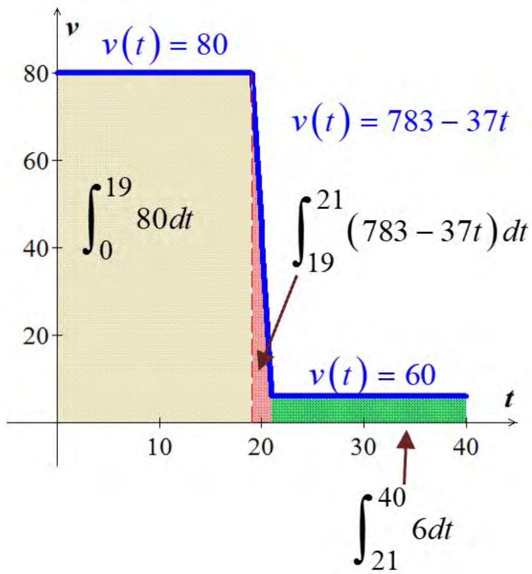
Example

Suppose a skydiver leaps from a hovering helicopter and fall in a straight line. He falls at a terminal velocity of 80 *m/s* for 19 *sec*, at which time he opens his parachute.
The velocity decreases linearly to 6 *m/s* over two-second period and then remains constant until he reaches the ground at *t* = 40 *s*. The motion is described by the velocity function

$$v(t)=\begin{cases} 80 & \text{if } 0\le t<19 \\ 783-37t & \text{if } 19\le t<21 \\ 6 & \text{if } 21\le t\le 40 \end{cases}$$

Determine the altitude from which the skydiver jumper.

Solution



$$d = \int_0^{40} |v(t)|\,dt$$

$$= \int_0^{19} 80dt + \int_{19}^{21} (783-37t)\,dt + \int_{21}^{40} 6dt$$

$$= 80t \Big|_0^{19} + \left(783\,t - \frac{37}{2}t^2 \Big|_{19}^{21} + 6t \Big|_{21}^{40}\right.$$

$$= 15.20 + 783(21) - \tfrac{1}{2}(37)(21)^2 - 783(19) + \tfrac{1}{2}(37)(19)^2 + 240 - 126$$

$$= 1{,}720 \quad m$$

The skydiver jumped from 1720 *m* above the ground.

Acceleration

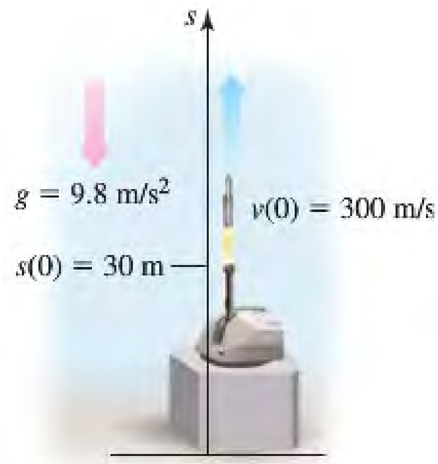
Theorem (velocity from Acceleration)

Given the acceleration $a(t)$ of an object moving along a line and its initial velocity $v(0)$, the velocity of the object for future times $t \ge 0$ is

$$v(t) = v(0) + \int_0^t a(x)\,dx$$

Example

An artillery shell is fired directly upward with an initial velocity of 300 *m/s* from a point 30 *m* above the ground. Assume that only the force of gravity acts on the shell and it produces an acceleration of 9.8 m/s^2 .



Find the velocity of the shell for $t \ge 0$

Solution

$$v(t) = v(0) + \int_0^t a(x)\,dx$$

$$= 300 + \int_0^t (-9.8)\,dx \qquad \textit{Upward}$$

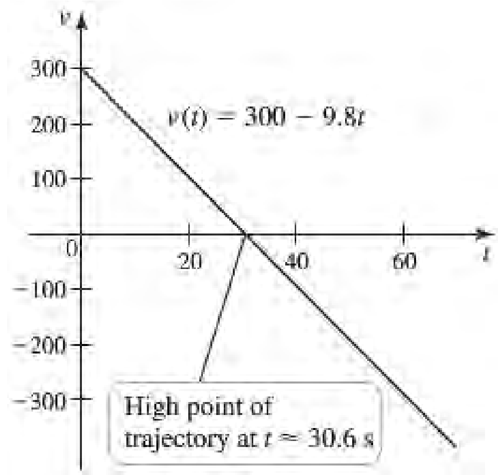
$$\underline{= 300 - 9.8t}$$

The velocity decreases from its initial value of 300 *m/s*, reaching zero at the high point of the trajectory when

$$v(t) = 300 - 9.8t = 0$$

$$t = \frac{300}{9.8}$$

$$= \frac{1500}{49} \approx 30.6 \text{ sec}$$



At this point the velocity becomes negative, and the shell begins its descent to Earth.

Net Change and Future Value

Theorem

Suppose a quantity Q changes over time at a known rate Q'. Then the net change in Q between $t=a$ and $t=b$ is

$$\int_a^b Q'(t)\,dt = Q(b) - Q(a) = \text{net change in } Q \text{ over } [a,\ b]$$

$$\int_0^t Q'(t)\,dt = Q(t) - Q(0)$$

Given the ***initial value*** $Q(0)$, the ***future value*** Q at future times $t \ge 0$ is

$$\underbrace{Q(t)}_{\substack{\text{future}\\ \text{value}}} = \underbrace{Q(0)}_{\substack{\text{initial}\\ \text{value}}} + \underbrace{\int_0^t Q'(t)\,dt}_{\substack{\text{net change}\\ \text{over } [0,\ t]}}$$

Velocity–Displacement Problems	***General Problems***
Position $s(t)$	Quantity $Q(t)$ (such as volume or population size)
Velocity: $s'(t)=v(t)$	Rate of change $Q'(t)$
Displacement: $s(b)-s(a)=\int_a^b v(t)\,dt$	Net change: $Q(b)-Q(a)=\int_a^b Q'(t)\,dt$
Future position: $s(t)=s(0)+\int_0^t v(x)\,dx$	Future value of Q: $Q(t)=Q(0)+\int_0^t Q'(x)\,dx$

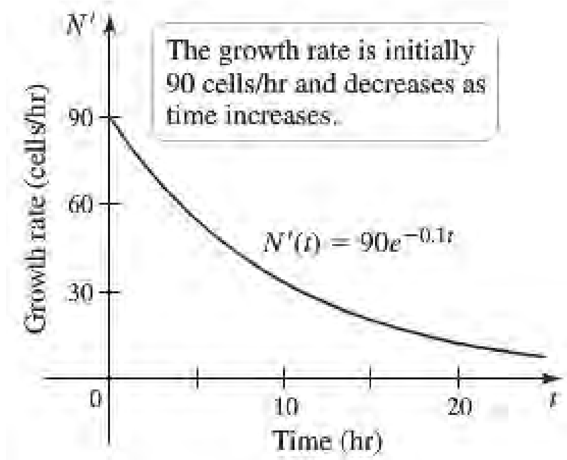
Example

A culture of cells in a lab has a population of 100 cells when nutrients are added at time $t=0$. Suppose the population $N(t)$ increases at a rate given by

$$N'(t) = 90e^{-0.1t} \quad \text{cells / hr}$$

Find $N(t)$ for $t \ge 0$

Solution



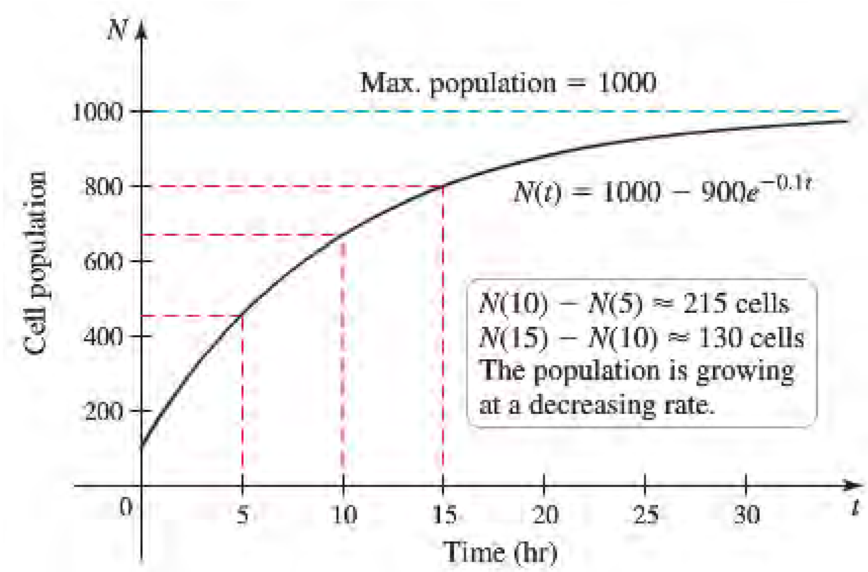
$$N(t) = N(0) + \int_0^t N'(x)\,dx$$

$$= 100 + \int_0^t 90e^{-0.1x}\,dx$$

$$= 100 - \frac{90}{0.1}\left(e^{-0.1x} \right|_0^t$$

$$= 100 - 900\left(e^{-0.1t} - 1\right)$$

$$\boxed{= 1000 - 900e^{-0.1t}}$$



The graph of the population function shows that the population increases, but at a decreasing rate. Note that the initial condition $N(0) = 100$ cells is satisfied and that population size approaches 1000 cells as $t \to \infty$.

Example

A book publisher estimates that the marginal cost of a particular title (in dollars/book) is given by

$$C'(x) = 12 - 0.0002x$$

Where $0 \le x \le 50,000$ is the number of books printed. What is the cost of producing the 12,001st through the 15,000 book?

Solution

$$\begin{aligned}
C(15,000) - C(12,000) &= \int_{12,000}^{15,000} C'(x)\,dx \\
&= \int_{12,000}^{15,000} (12 - 0.0002x)\,dx \\
&= 12x - 0.0001x^2 \Big|_{12,000}^{15,000} \\
&= 180\times 10^3 - 225\times 10^2 - 144\times 10^3 + 144\times 10^2 \\
&= \underline{\$27,900}
\end{aligned}$$

Exercises Section 1.1 – Velocity and Net Change

(1 – 3) Assume t is time measured in seconds and velocities have units of *m/s*.

a) Graph the velocity function over the given interval. Then determine when the motion is in the positive direction.
b) Find the displacement over the given interval.
c) Find the distance traveled over the given interval.

1. $v(t)=6-2t; \quad 0\le t\le 6$ **2.** $v(t)=10\sin 2t; \quad 0\le t\le 2\pi$ **3.** $v(t)=50e^{-2t}; \quad 0\le t\le 4$

(4 – 5) Consider an object moving along a line with the following velocities and initial positions

a) Graph the velocity function on the given interval. Then determine when the object is moving in the positive direction and when it is moving in the negative direction.
b) Determine the position function for $t\ge 0$ using both the antiderivative method and the Fundamental Theorem of Calculus. Check for agreement between the two methods.
c) Graph the position function on the given interval.

4. $v(t)=6-2t \quad on\ [0,\ 5] \quad s(0)=0$ **5.** $v(t)=9-t^2 \quad on\ [0,\ 4] \quad s(0)=-2$

(6 – 7) Find the position and velocity of an object moving along a straight line with the given acceleration, initial velocity, ansd initial position. Assume units of meters and seconds.

6. $a(t)=-9.8, \quad v(0)=20, \quad s(0)=0$ **7.** $a(t)=e^{-t}, \quad v(0)=60, \quad s(0)=40$

8. A mass hanging from a spring is set in motion and its ensuing velocity is given by $v(t)=2\pi\cos\pi t$ for $t\ge 0$. Assume that the position direction is upward and $s(0)=0$.

a) Determine the position function for $t\ge 0$.
b) Graph the position function on the interval [0, 3].
c) At what times does the mass reach its lowest point the first three times?
d) At what times does the mass reach its highest point the first three times?

9. The velocity of an airplane flying into a headwind is given by $v(t)=30\left(16-t^2\right)$ *mi/hr* for $0\le t\le 3\ hr$.

Assume that $s(0)=0$

a) Determine and graph the position function for $0\le t\le 3$.
b) How far does the airplane travel in the first 2 *hr*.?
c) How far has the airplane traveled at the instant its velocity reaches 400 *mi/hr*.?

10. A car slows down with an acceleration of $a(t)=-15\ ft/s^2$. Assume that $v(0)=60\ ft/s$ and $s(0)=0$

a) Determine and graph the position function for $t\ge 0$.
b) How far does the car travel in the time it takes to come to rest?

11. The owners of an oil reserve begin extracting oil at $t = 0$. Based on estimates of the reserves, suppose the projected extraction rate is given by $Q'(t) = 3t^2(40-t)^2$, where $0 \le t \le 40$, Q is measured in millions of barrels, and t is measured in years.

a) When does the peak extraction rate occur?

b) How much oil is extracted in the first 10, 20, and 30 years?

c) What is the total amount of oil extracted in 40 year?

d) Is one-fourth of the total oil extracted in the first one-fourth of the extraction period? Explain.

12. Starting with an initial value of $P(0) = 55$, the population of a prairie dog community grows at a rate of $P'(t) = 20 - \frac{t}{5}$ (in units of prairie *dogs/month*), for $0 \le t \le 200$.

a) What is the population 6 months later?

b) Find the population $P(t)$ for $0 \le t \le 200$.

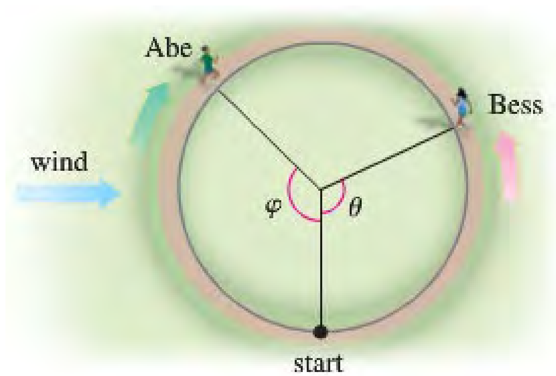
13. The population of a community of foxes is observed to fluctuate on a 10-year cycle due to variations in the availability of prey. When population measurements began (t = 0 years), the population was 35 foxes. The growth rate in units of *foxes/yr*. was observed to be

$$P'(t) = 5 + 10\sin\left(\frac{\pi t}{5}\right)$$

a) What is the population 15 years later? 35 years later?

b) Find the population $P(t)$ at any time $t \ge 0$.

14. A strong west wind blows across a circular running track. Abe and Bess start at the south end of the track and at the same time, Abe starts running clockwise and Bess starts running counterclockwise. Abe runs with a speed (in units of *mi/hr.*) given by $u(\varphi) = 3 - 2\cos\varphi$ and Bess runs with a speed given by $v(\theta) = 3 + 2\cos\theta$, where φ and θ are the central angles of the runners



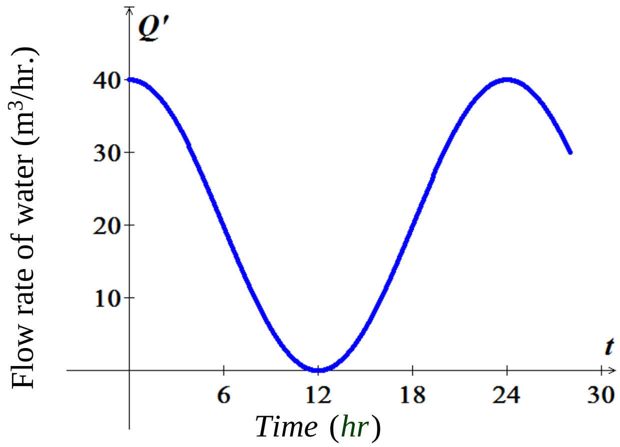
a) Graph the speed functions *u* and *v*, and explain why they describe the runners' speed (in light of the wind).

b) Which runner has the greater average speed for one lap?

c) If the track has a radius of $\frac{1}{10}$ *mi*, how long does it take each runner to complete one lap and who wins the race?

15. A reservoir with a capacity of 2500 m^3 is filled with a single inflow pipe. The reservoir is empty and the inflow pipe is opened at $t = 0$. Letting $Q(t)$ be the amount of water in the reservoir at time t, the flow rate of water into reservoir (in m^3/hr) oscillates on 24-*hr* cycle and is given by

$$Q'(t) = 20\left[1 + \cos\frac{\pi t}{12}\right]$$



a) How much water flows into the reservoir in the first 2 *hrs*.?

b) Find and graph the function that gives the amount of water in the reservoir over the interval $[0,\ t]$ where $t \ge 0$.

c) When is the reservoir full?

16. The velocity of an object moving along a line is given by $v(t) = 20\cos\pi t\ \ (ft/s)$. What is the displacement of the object after 1.5 *sec*?

17. A projectile is launched vertically from the ground at $t = 0$, and its velocity in flight (in *m/s*) is given by $v(t) = 20 - 10t$. Find the position, displacement, and distance traveled after t seconds, for $0 \le t \le 4$

18. At $t = 0$, a car begins decelerating from a velocity of 80 *ft/s* at a constant rate of $5\ ft/s^2$. Find its position function assuming $s(0) = 0$.

19. The acceleration of an object moving along a line is given by $a(t) = 2\sin\left(\frac{\pi t}{4}\right)$. The initial velocity and position are $v(0) = -\frac{8}{\pi}$ and $s(0) = 0$

a) Find the velocity and position for $t \ge 0$

b) What are the minimum and maximum values of s?

c) Find the average velocity and average position over the interval $[0,\ 8]$

20. Starting at the same point on a straight road, Anna and Benny begin running with velocities (in *mi/hr*) given by $v_A(t) = 2t + 1$ and $v_B(t) = 4 - t$, respectively.

a) Graph the velocity functions, for $0 \le t \le 4$.

b) If the runners run for 1 *hr*, who runs farther? Interpret your conclusion geometrically using the graph in part (*a*).

c) If the runners run for 6 *mi* who wins the race? Interpret your conclusion geometrically using the graph in part (*a*).

21. A small plane in flight consumes fuel at a rate (in *gal/min*) given by

$$R'(t) = \begin{cases} 4t^{1/3} & \text{if } \ 0 \le t \le 8 \ \ (\text{take} - \text{off}) \\ 2 & \text{if } \ t > 8 \ \ (\text{cruising}) \end{cases}$$

a) Find a function *R* that gives the total fuel consumed, for $0 \le t \le 8$

b) Find a function *R* that gives the total fuel consumed, for $t \ge 0$

c) If the fuel tank capacity is 150 *gal*, when does the fuel run out?

22. Water flows out of a tank at a rate $\left(in\ m^3 / hr\right)$ given by $V'(t) = \frac{15}{t+1}$. If the tank initially holds 75 m^3 of water, when will the tank be empty?

23. A projectile is fired upward, and its velocity in *m/s* is given by $v(t) = 200e^{-t/10}, \quad for \ \ t \ge 0$.

a) Graph the velocity function, for $t \ge 0$.

b) When does the velocity reach 50 *m/s*?

c) Find and graph the position function for the projectile for $t \ge 0$, assuming $s(0) = 0$.

d) Given unlimited time, can the projectile travel 2500 *m*? If so, at what time does the distance traveled equal 2500 *m*?

24. A projectile is fired upward, and its velocity in *m/s* is given by $v(t) = \frac{200}{\sqrt{t+1}}, \quad for \ \ t \ge 0$.

a) Graph the velocity function, for $t \ge 0$.

b) Find and graph the position function for the projectile for $t \ge 0$, assuming $s(0) = 0$.

c) Given unlimited time, can the projectile travel 2500 *m*? If so, at what time does the distance traveled equal 2500 *m*?

25. Jeff and Mel took a bike ride, both starting at the same time and position. Jeff started riding at 20 *mi/hr*, and his velocity decreased according to the function $v(t) = 20e^{-2t}, \quad for \ \ t \ge 0$. Mel started riding at 15 *mi/hr*, and her velocity decreased according to the function $u(t) = 15e^{-t}, \quad for \ \ t \ge 0$

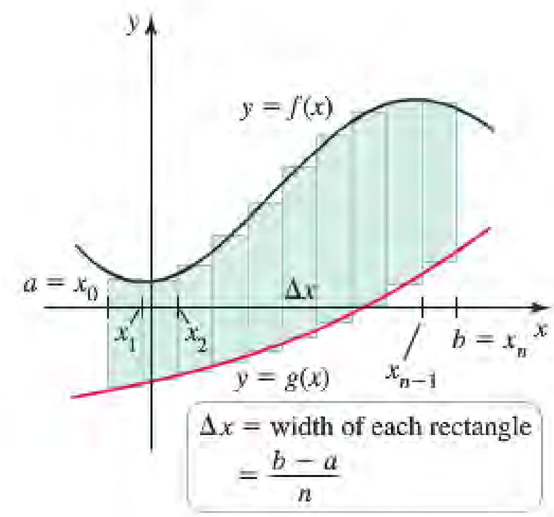
a) Find and graph the positions of Jeff and Mel.

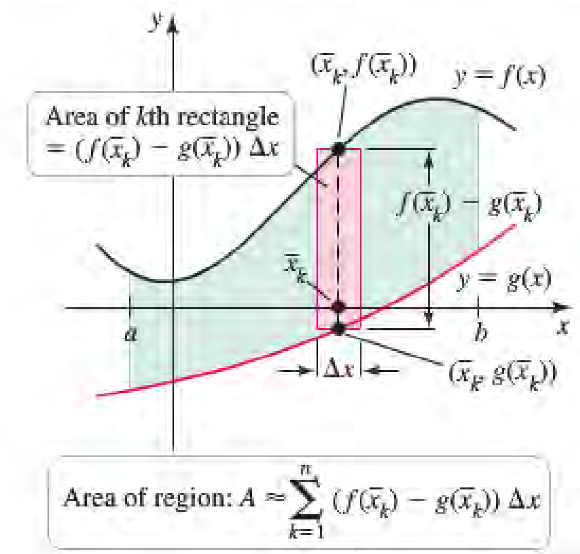
b) Find the times at which the riders have the same position at the same time.

c) Who ultimate took the lead and remained in the lead?

Section 1.2 – Region between Curves

Areas between Curves





Definition

If f and g are continuous with $f(x) \ge g(x)$ throughout [*a*, *b*], then the ***area of the region between the curves*** $y = f(x)$ *and* $y = g(x)$ **from *a* to *b*** is:

$$A = \int_a^b \left[f(x) - g(x) \right] dx$$

Example

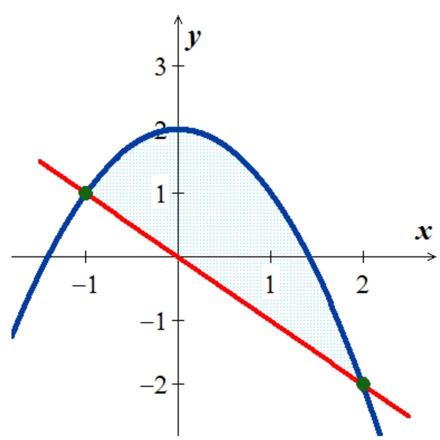
Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution

The limits of integrations are found by letting:

$$2 - x^2 = -x$$

$$x^2 - x - 2 = 0 \quad \rightarrow \quad \underline{x = -1,\ 2}\,|$$



$$A = \int_{-1}^{2} \left[f(x) - g(x) \right] dx$$

$$= \int_{-1}^{2} \left(2 - x^2 - (-x) \right) dx$$

$$= \int_{-1}^{2} \left(2 - x^2 + x \right) dx$$

$$= 2x - \frac{x^3}{3} + \frac{x^2}{2} \Big|_{-1}^{2}$$

$$= 4 - \frac{8}{3} + 2 - \left(-2 + \frac{1}{3} + \frac{1}{2}\right)$$

$$= 8 - \frac{8}{3} - \frac{1}{3} - \frac{1}{2}$$

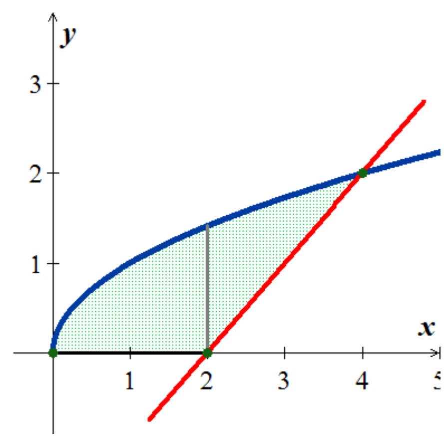
$$= 5 - \frac{1}{2}$$

$$\boxed{= \frac{9}{2} \quad unit^2}$$

Example

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below the x-axis and the line $y = x - 2$

Solution



$$\left(y = \sqrt{x}\right) \cap (y = 0) \quad \to \quad (0,\ 0)$$

$$\left(y = \sqrt{x}\right) \cap (y = x - 2) \quad \to \quad \sqrt{x} = x - 2$$

$$\left(\sqrt{x}\right)^2 = (x-2)^2$$

$$x = x^2 - 4x + 4$$

$$x^2 - 5x + 4 = 0$$

$$\to \quad x = \cancel{1},\ 4$$

$$(y = 0) \cap (y = x - 2) \quad \to \quad x = 2$$

$$\text{Total Area } = \int_0^2 \left(\sqrt{x} - 0\right) dx + \int_2^4 \left(\sqrt{x} - (-x + 2)\right) dx$$

$$= \frac{2}{3} x^{3/2} \Big|_0^2 + \left(\frac{2}{3} x^{3/2} - \frac{x^2}{2} + 2x \Big|_2^4\right.$$

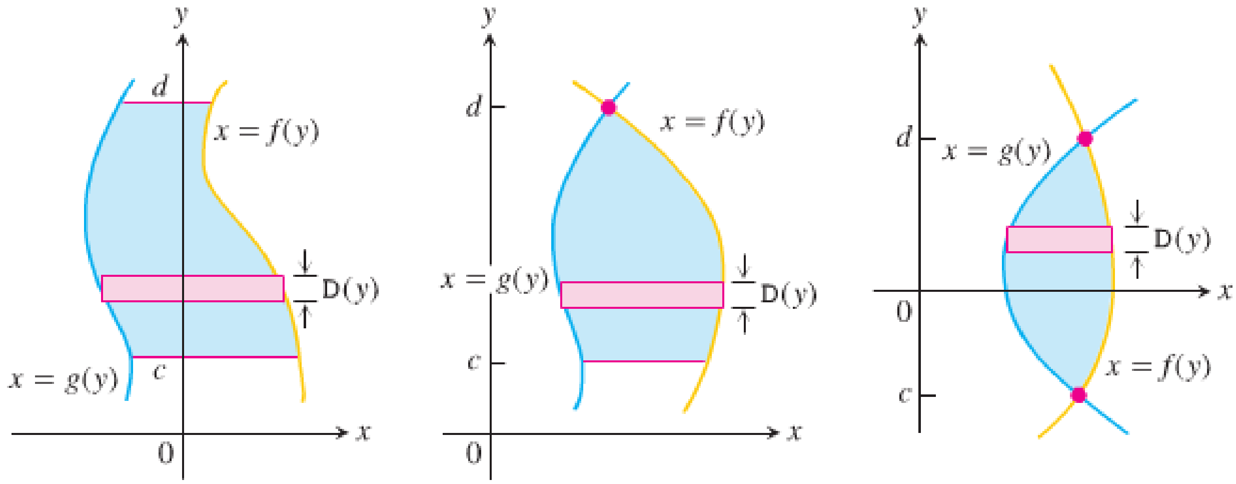
$$= \frac{2}{3}\left(2^{3/2}\right) - 0 + \left(\frac{2}{3} 4^{3/2} - \frac{4^2}{2} + 2(4)\right) - \left(\frac{2}{3} 2^{3/2} - \frac{2^2}{2} + 2(2)\right)$$

$$= \frac{2}{3}\left(2^{3/2}\right) + \frac{2}{3} 4^{3/2} - \frac{16}{2} + 8 - \frac{2}{3} 2^{3/2} + \frac{4}{2} - 4$$

$$= \frac{2}{3}(8) - 2$$

$$\boxed{= \frac{10}{3} \quad unit^2}$$

Integration with Respect to *y*

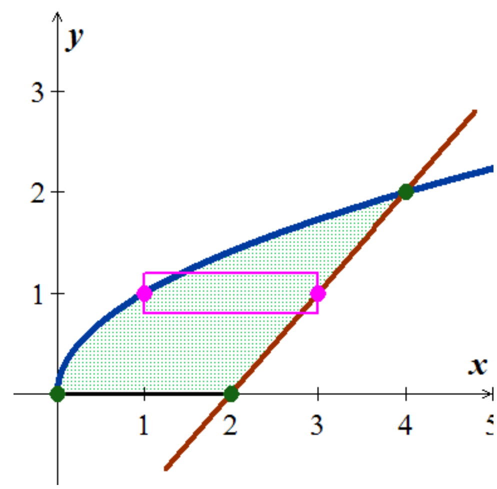


$$A=\int_c^d \left(f(y)-g(y)\right)dy$$ **(*From right hand to left* hand)**

Example

Find the area of the region by integrating with respect to y, in the first quadrant that is bounded above by $y=\sqrt{x}$ and below the *x*-axis and the line $y=x-2$.

Solution



$y=\sqrt{x} \quad\rightarrow\quad x=y^2$

$y=x-2 \quad\rightarrow\quad x=y+2$

$\left(x=y^2\right)\cap(y=0) \quad\rightarrow\quad (0,\ 0)$

$\left(x=y^2\right)\cap(x=y+2) \quad\rightarrow\quad y^2=y+2$

$y^2-y-2=0 \quad\rightarrow\quad y=-1,\ 2$

$(y=0)\cap(x=y+2) \quad\rightarrow\quad y=0$

$$A=\int_0^2\left(y+2-y^2\right)dy$$

$$=\frac{y^2}{2}+2y-\frac{y^3}{3}\Big|_0^2$$

$$=\frac{2^2}{2}+2(2)-\frac{2^3}{3}-0$$

$$=\frac{10}{3}\ unit^2$$

Exercises — *Section* 1.2 – Region between Curves

Find the area of the region bounded by the graphs of

1. $y = 2x - x^2$ and $y = -3$
2. $y = 7 - 2x^2$ and $y = x^2 + 4$
3. $y = x^4 - 4x^2 + 4$ and $y = x^2$
4. $x = 2y^2, \quad x = 0, \quad and \quad y = 3$
5. $x = y^3 - y^2 \quad and \quad x = 2y$
6. $4x^2 + y = 4 \quad and \quad x^4 - y = 1$
7. $y = \sin\frac{\pi x}{2} \quad and \quad y = x$
8. $y = 3 - x^2$ *and* $y = 2x$
9. $y = x^2 - x - 2$ *and* x-axis
10. $y = \sqrt{x}, \quad y = x\sqrt{x}$
11. $y = x^{1/2} \quad and \quad y = x^3$
12. $x + 4y^2 = 4, \quad x + y^4 = 1, \quad x \ge 0$
13. $y = 2\sin x, \quad y = \sin 2x, \quad 0 \le x \le \pi$
14. $y = x^2 + 1$ *and* $y = x$ for $0 \le x \le 2$
15. $y = x^2 - 2x$ *and* $y = x$ on [0, 4]
16. $x = 1, \quad x = 2, \quad y = x^3 + 2, \quad y = 0$
17. $y = x^2 - 18, \quad y = x - 6$
18. $y = -x^2 + 3x + 1, \quad y = -x + 1$
19. $y = x, \quad y = 2 - x, \quad y = 0$
20. $y = \frac{4}{x^2}, \quad y = 0, \quad x = 1, \quad x = 4$
21. $f(y) = y^2, \quad g(y) = y + 2$
22. $f(x) = 2^x, \quad g(x) = \frac{3}{2}x + 1$
23. $x = \sqrt[3]{y} \quad and \quad x = \sqrt[5]{y}$
24. $f(x) = x^3 + 2x^2 - 3x, \quad g(x) = x^2 + 3x$
25. $y = \sec^2 x, \quad y = \tan^2 x, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$
26. $f(x) = -x^2 + 1, \quad g(x) = 2x + 4, \quad x = -1, \ x = 2$
27. $f(x) = \sqrt{x} + 3, \quad g(x) = \frac{1}{2}x + 3$
28. $f(x) = \sqrt[3]{x - 1}, \quad g(x) = x - 1$
29. $f(y) = y(2 - y), \quad g(y) = -y$
30. $f(y) = \dfrac{y}{\sqrt{16 - y^2}}, \quad g(y) = 0, \quad y = 3$
31. $f(y) = y^2 + 1, \quad g(y) = 0, \quad y = -1, \quad y = 2$
32. $f(x) = \frac{10}{x}, \quad x = 0, \quad y = 2, \quad y = 10$
33. $g(x) = \frac{4}{2 - x}, \quad y = 4, \quad x = 0$
34. $f(x) = \cos x, \quad g(x) = 2 - \cos x, \quad 0 \le x \le 2\pi$
35. $f(x) = \sin x, \quad g(x) = \cos 2x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{6}$
36. $f(x) = 2\sin x, \quad g(x) = \tan x, \quad -\frac{\pi}{3} \le x \le \frac{\pi}{3}$
37. $f(x) = \sec\frac{\pi x}{4}\tan\frac{\pi x}{4}, \quad g(x) = \left(\sqrt{2} - 4\right)x + 4, \quad x = 0$
38. $f(x) = xe^{-x^2}, \quad y = 0, \quad 0 \le x \le 1$
39. $y = \sin x$ *and* $y = x \quad 0 \le x \le 2\pi$
40. $y = x^2, \ y = 2x^2 - 4x$ and $y = 0$
41. $y = 8\cos x, \quad y = \sec^2 x, \quad -\frac{\pi}{3} \le x \le \frac{\pi}{3}$
42. $y^2 = 4x + 4, \quad y = 4x - 16$
43. $x = 2y^2, \quad x = 0, \quad y = 3$
44. $x = y^3 \quad and \quad y = x$

45. Find the area of the region in the first quadrant bounded by $y = 4x$ *and* $y = x\sqrt{25 - x^2}$

46. Find the area of the region in the first quadrant bounded by the curve $\sqrt{x} + \sqrt{y} = 1$

47. Find the area of the region in the first quadrant bounded by $y = \frac{x}{6}$ and $y = 1 - \left|\frac{x}{2} - 1\right|$

48. Find the area of the region in the first quadrant bounded by $y = x^p$ and $y = \sqrt[p]{x}$ where $p = 100$ and $p = 1000$

49. Consider the functions $y = \frac{x^2}{a}$ and $y = \sqrt{\frac{x}{a}}$, where $a > 0$. Find $A(a)$, the area of the region between the curves.

50. Find the area between the curves $y = \ln x$ and $y = \ln 2x$ from $x = 1$ to $x = 5$.

51. Find the total area of the region enclosed by the curve $x = y^{2/3}$ and lines $x = y$ and $y = -1$.

52. Find the area of the "triangular region in the first quadrant bounded on the left by the *y-axis* and on the right by the curves $\sin x$ *and* $\cos x$.

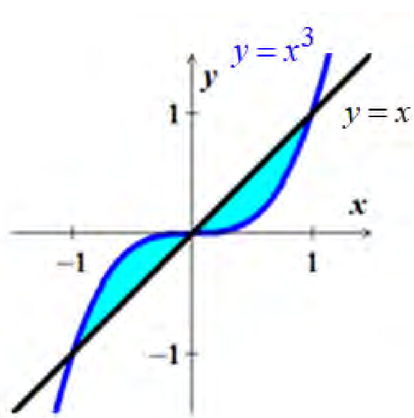
53. Find the area of the "triangular region in the first quadrant bounded above by the curve $y = e^{2x}$, below by the curve $y = e^x$, and on the right by the line $x = \ln 3$.

54. Find the area of the triangular region bounded on the left by $x + y = 2$, on the right by $y = x^2$, and above by $y = 2$

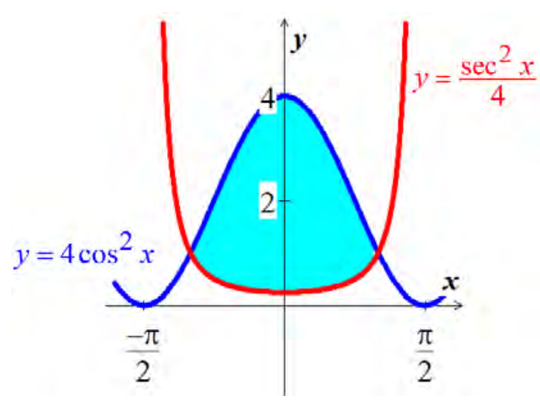
55. Find the extreme values of $f(x) = x^3 - 3x^2$ and find the area of the region enclosed by the graph of *f* and the *x*-axis.

(56 – 59) Determine the area of the shaded region in the following

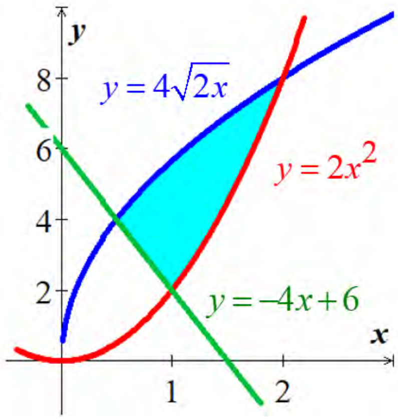
56.



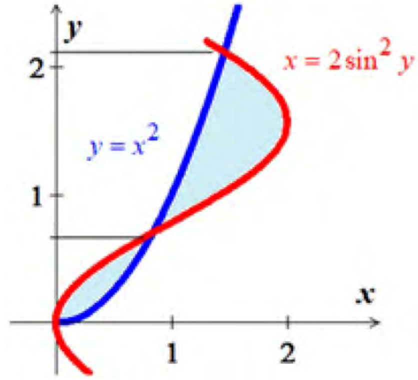
57.



58.

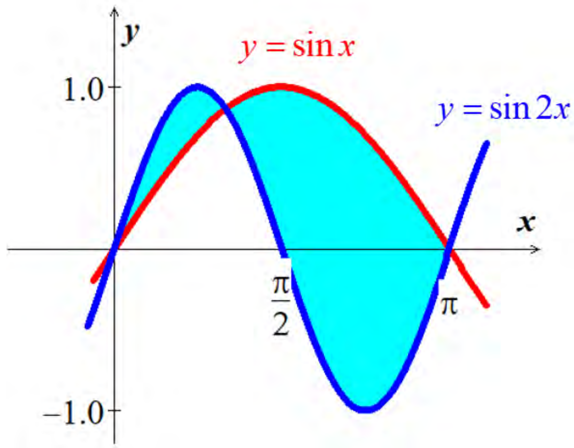


59.

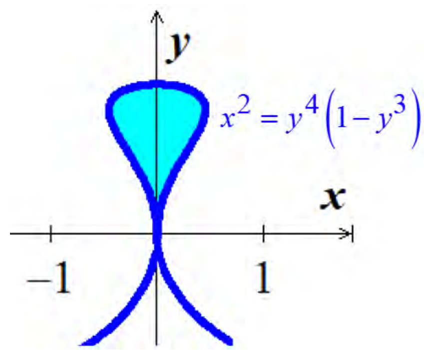


(**60 – 71**) Determine the area of the shaded regions

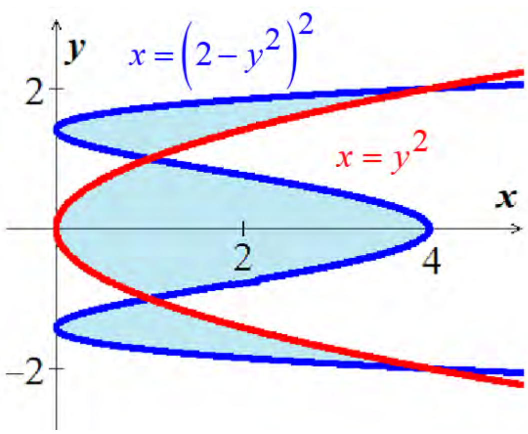
60. $y = \sin x$ *and* $y = \sin 2x$, *for* $0 \le x \le \pi$



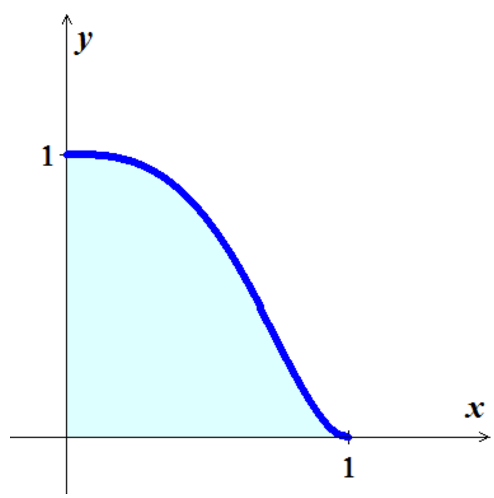
61. Bounded by $x^2 = y^4\left(1 - y^3\right)$



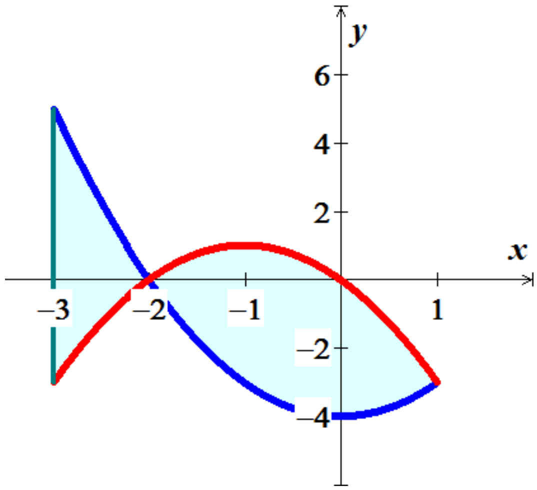
62. bounded by $x = y^2$ and $x = \left(2 - y^2\right)^2$



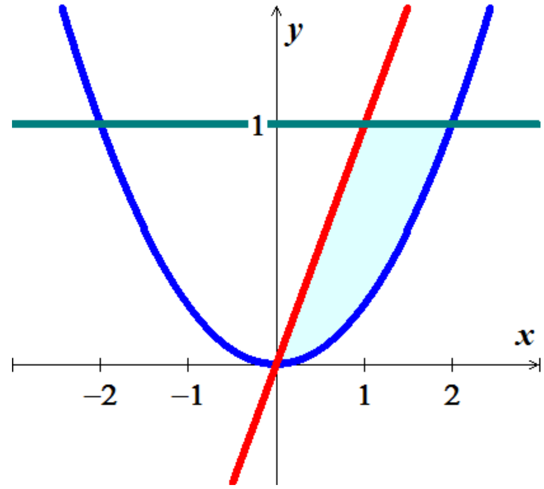
63. $x^3 + \sqrt{y} = 1, \quad x = 0, \quad y = 0, \quad 0 \le x \le 1$



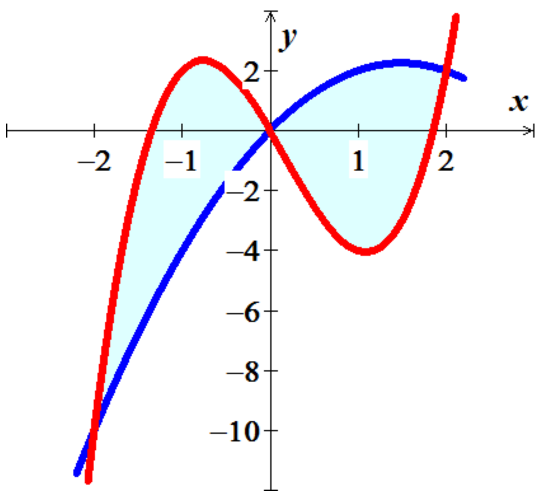
64. $y = x^2 - 4, \quad y = -x^2 - 2x, \quad -3 \le x \le 1$



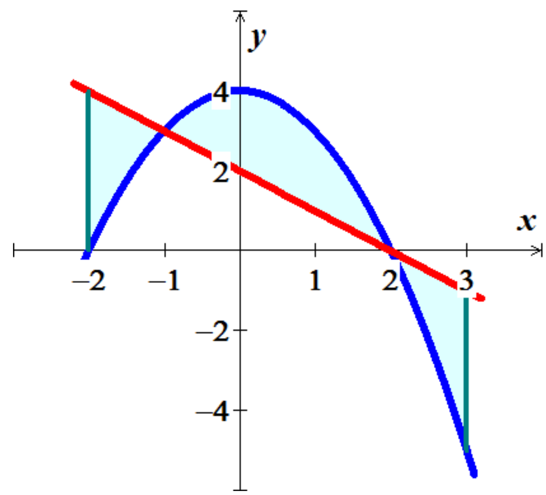
65. $y = \frac{1}{4}x^2, \quad y = x, \quad y = 1$



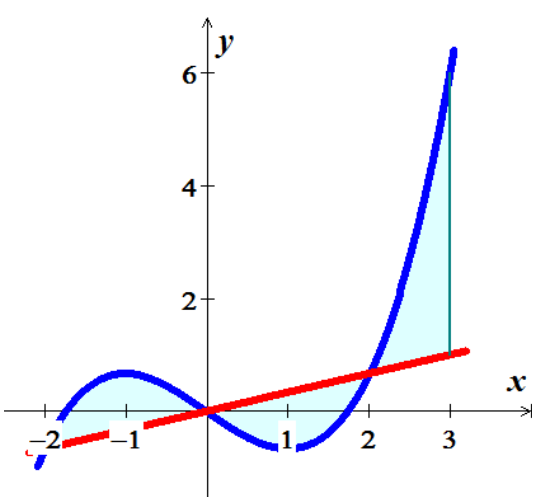
66. $y = -x^2 + 3x, \quad y = 2x^3 - x^2 - 5x, \quad -2 \le x \le 2$



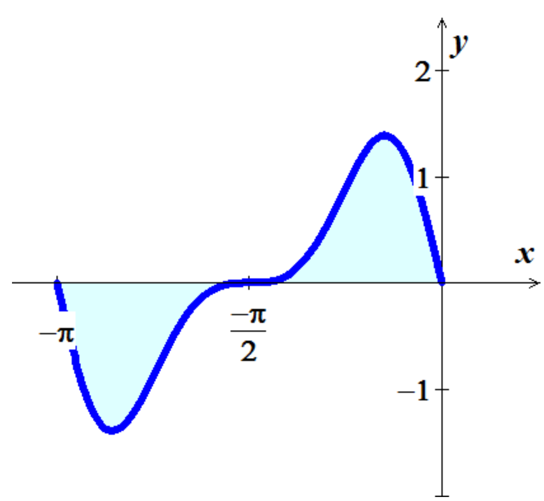
67. $y = 4 - x^2, \quad y = -x + 2, \quad -2 \le x \le 3$



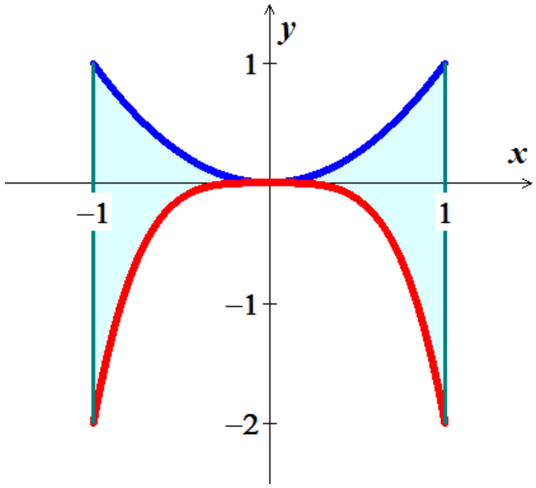
68. $y = \frac{1}{3}x^3 - x, \quad y = \frac{1}{3}x, \quad -2 \le x \le 3$



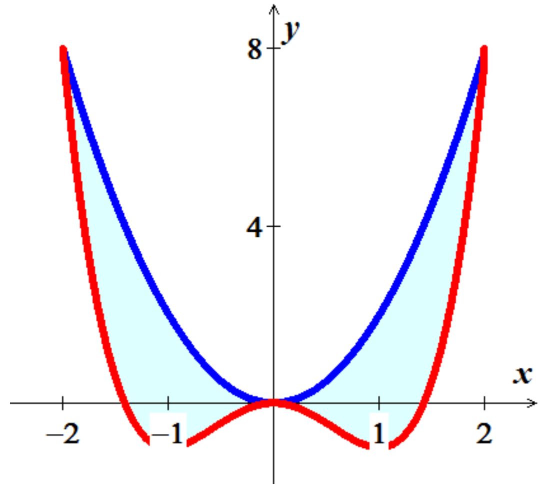
69. $y = \frac{\pi}{2}\cos x \ \sin\left(\pi + \pi \sin x\right) \quad -\pi \le x \le 0$



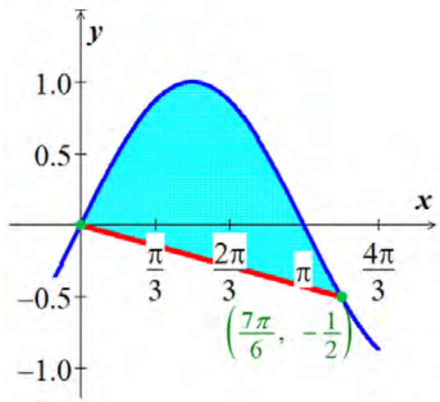
70. $y = x^2, \quad y = -2x^4, \quad -1 \le x \le 1$



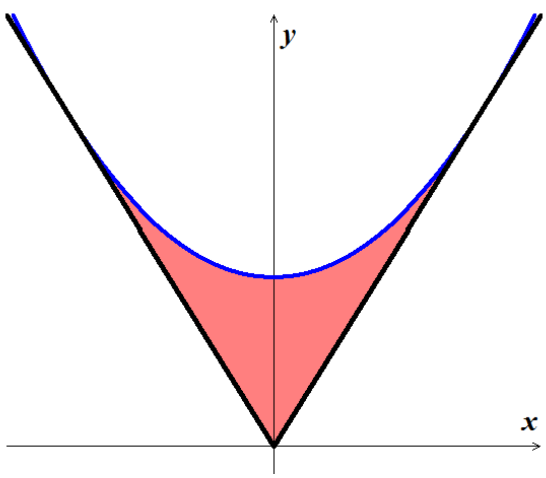
71. $y = 2x^2, \quad y = x^4 - 2x^2, \quad -2 \le x \le 2$



72. Find the area between the graph of $y = \sin x$ and the line segment joining the points $(0,\ 0)$ and $\left(\frac{7\pi}{6},\ -\frac{1}{2}\right)$.



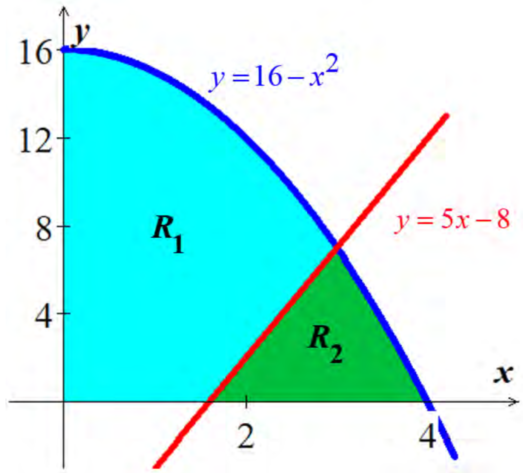
73. The surface of a machine part is the region between the graphs of $y_1 = |x|$ and $y_2 = 0.08x^2 + k$



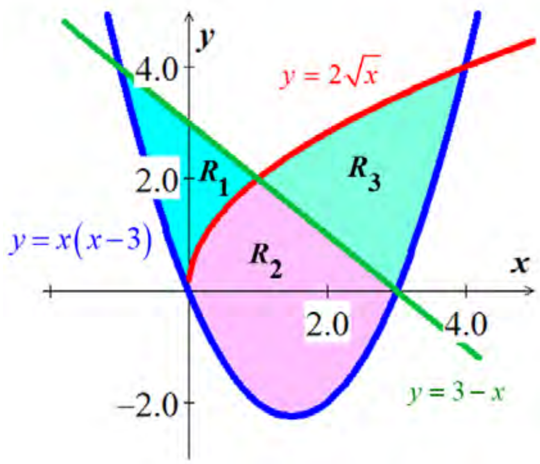
a) Find k where the parabola is tangent to the graph of y_1

b) Find the area of the surface of the machine part.

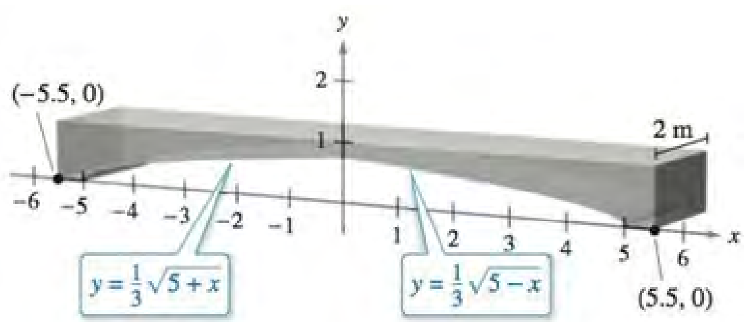
74. Find the area of the regions R_1 and R_2 (separately) shown in the figure, which are formed by the graphs of $y = 16 - x^2$ and $y = 5x - 8$



75. Find the area of the regions R_1, R_2 and R_3 (separately) shown in the figure, which are formed by the graphs of $y = 2\sqrt{x}$, $y = 3 - x$, and $y = x(x-3)$



76. Concrete sections for a new building have the dimensions (in meters) and shape shown in figure



a) Find the area of the face of the section superimposed on the rectangularcoordinate system.

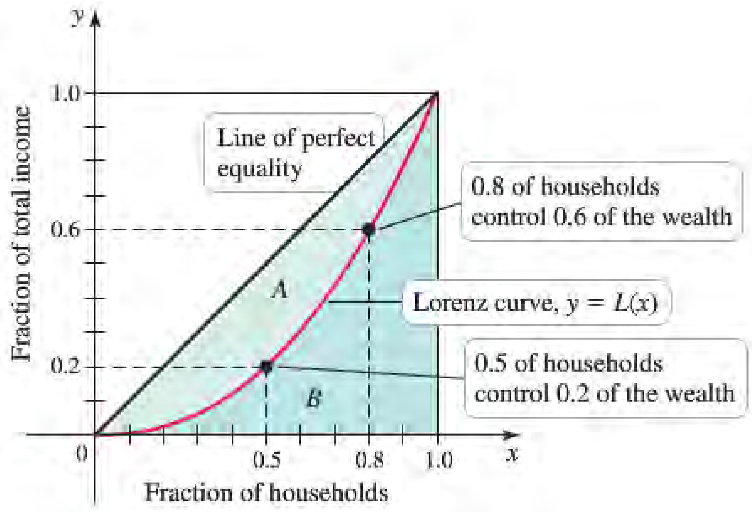
b) Find the volume of concrete in one of the sections by multiplying the area in part (*a*) by 2 *meters*.

c) One cubic meter of concrete weighs 5,000 *pounds*. Find the weight of the section.

77. A Lorenz curve is given by $y = L(x)$, where $0 \le x \le 1$ represents the lowest fraction of the population of a society in terms of wealth and $0 \le y \le 1$ represents the fraction of the total wealth that is owned by that fraction of the society. For example, the Lorenz curve in the figure shows that $L(0.5) = 0.2$, which means that the lowest 0.5 (50%) of the society owns 0.2 (20%) of the wealth.

a) A Lorenz curve $y = L(x)$ is accompanied by the line $y = x$, called the ***line of perfect equality***. Explain why this line is given the name.

b) Explain why a Lorenz curve satisfies the conditions $L(0) = 0,\ L(1) = 1,\ and\ L'(x) \ge 0 \quad on\ [0,\ 1]$



c) Graph the Lorenz curves $L(x) = x^p$ corresponding to *p* = 1.1, 1.5, 2, 3, 4. Which value of *p* corresponds to the ***most*** equitable distribution of wealth (closest to the line of perfect equality)? Which value of *p* corresponds to the ***least*** equitable distribution of wealth? Explain.

d) The information in the Lorenz curve is often summarized in a single measure called the ***Gini index***, which is defined as follows. Let *A* be the area of the region between $y = x$ and $y = L(x)$ and Let *B* be the area of the region between $y = L(x)$ and the *x*-axis. Then the Gini index is $G = \dfrac{A}{A+B}$.

Show that $G = 2A\ = 1 - 2\displaystyle\int_0^1 L(x)dx$.

e) Compute the Gini index for the cases $L(x) = x^p$ and *p* = 1.1, 1.5, 2, 3, 4.

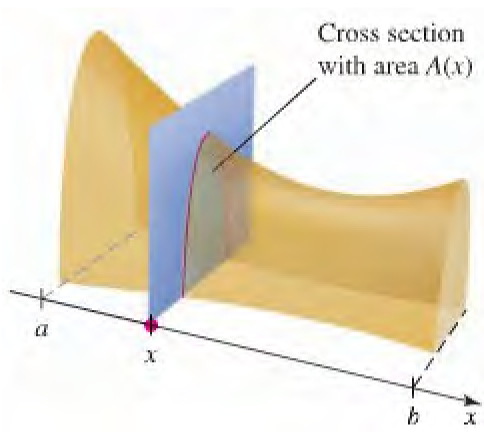
f) What is the smallest interval [*a*, *b*] on which values of the Gini index lie, for $L(x) = x^p$ with $p \ge 1$? Which endpoints of [*a*, *b*] correspond to the least and most equitable distribution of wealth?

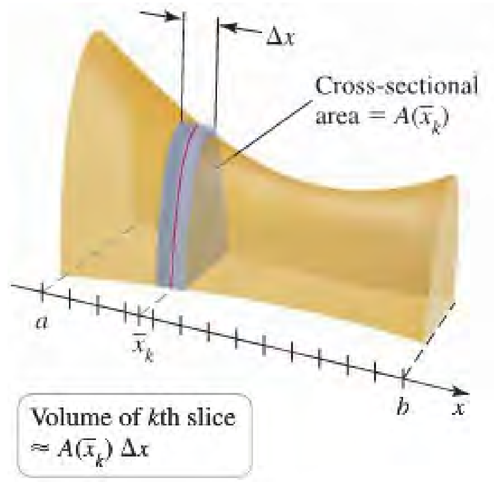
g) Consider the Lorenz curve described by $L(x) = \dfrac{5x^2}{6} + \dfrac{x}{6}$. Show that it satisfies the conditions $L(0) = 0,\ L(1) = 1,\ and\ L'(x) \ge 0 \quad on\ [0,\ 1]$. Find the Gini index for this function.

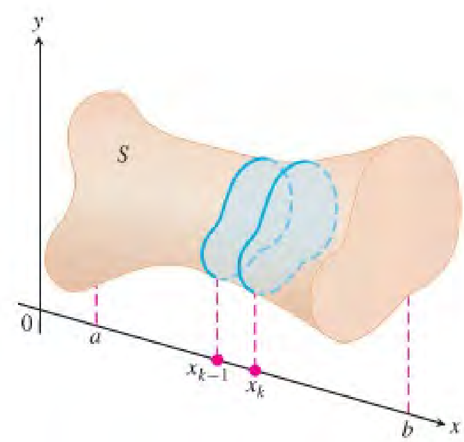
Section 1.3 – Volumes by Slicing

If we want to find a volume of a solid S and if the cylindrical solid has known base area A and height h, then the volume of the cylindrical solid is

$$Volume = area \times height = A \bullet h$$



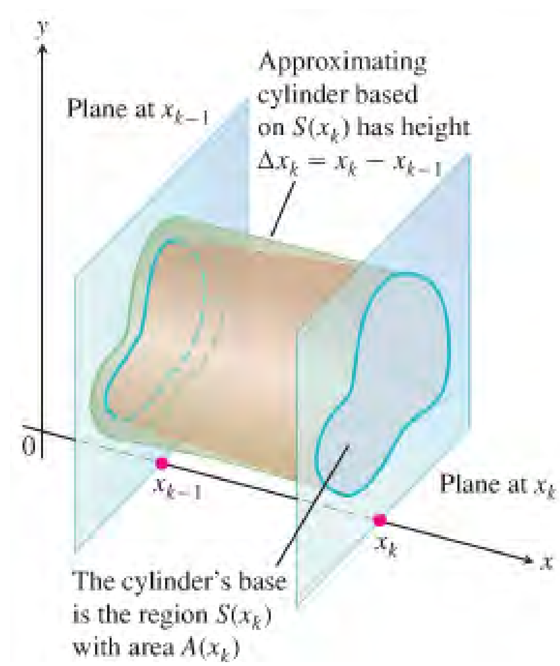




Definition

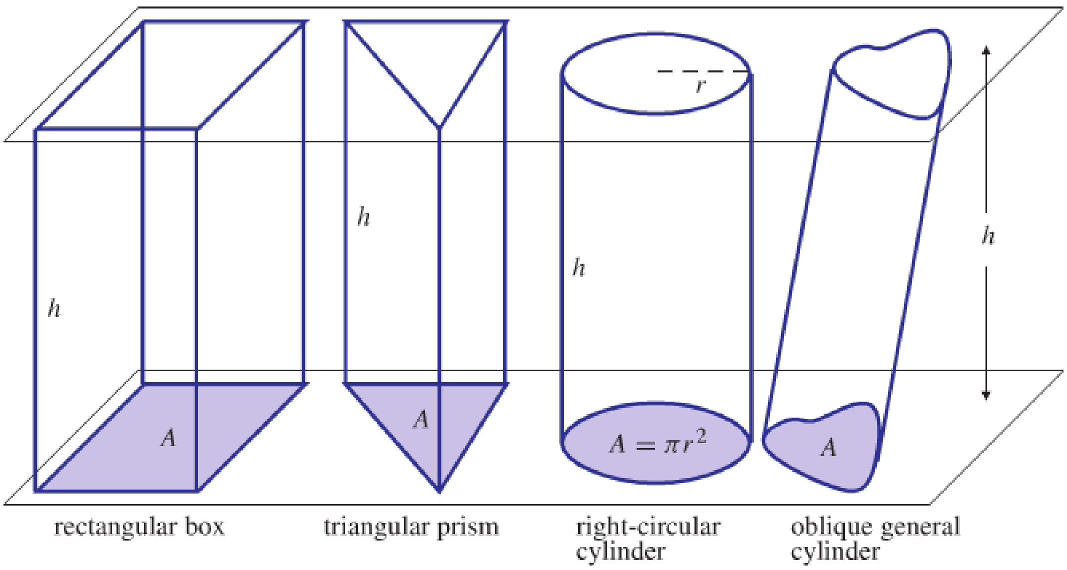
The volume of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b

$$V = \int_a^b A(x)\,dx$$



Calculating the volume of a solid

1. Sketch the solid and a typical cross-section
2. Find a formula for $A(x)$, the area of a typical cross-section
3. Find the limits of integration
4. Integrate $A(x)$ to find the volume



Example

A pyramid 3*m* high has a square base that is 3*m* on a side. The cross-section of the pyramid perpendicular to the altitude ***x*** *m* down from the vertex is a square ***x*** *m* on a side. Find the volume of the pyramid.

Solution

The area of the square is given by the formula:

$A(x) = x^2$

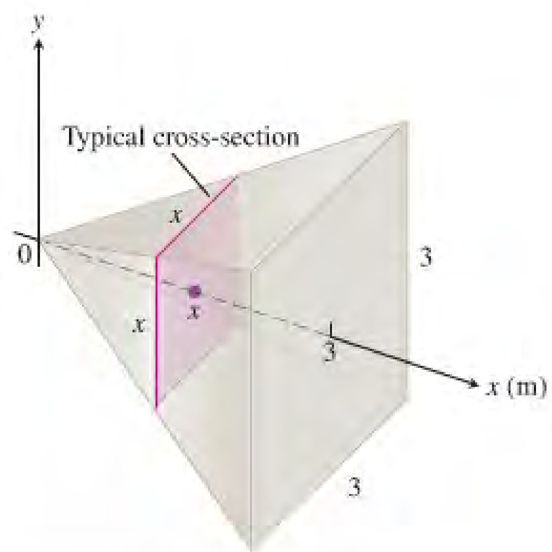
The volume: $V = \int_0^3 A(x)\,dx$

$= \int_0^3 x^2 dx$

$= \frac{1}{3}x^3 \Big|_0^3$

$= \frac{1}{3}\left(3^3 - 0\right)$

$\underline{= 9 \;\; m^3}$



Example

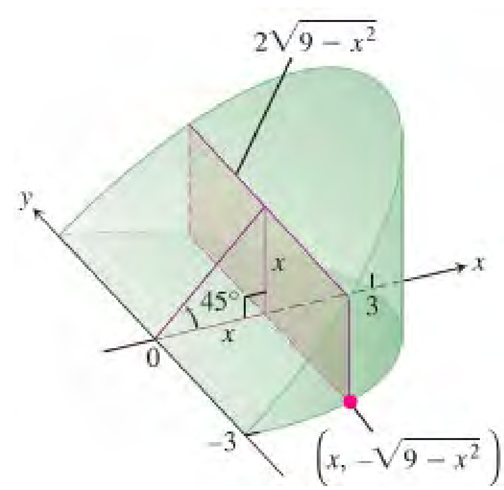
A curved wedge is cut from a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second place crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

Solution

The base of the cylinder is a circle $x^2+y^2=9$.

Since the second plane cut the base at the cylinder at the center, therefore, the base of the wedge is semi-circle. $y=\pm\sqrt{9-x^2}=radius$.

When we slice the wedge by a plane perpendicular to the axis of the cylinder, we obtained a cross-section at *x* which is a rectangle of height *x*.



The area of this cross-section is: $A(x)=height\times width$

$$=x\left(2\sqrt{9-x^2}\right)$$

$$=2x\sqrt{9-x^2}$$

The rectangles run from $x = 0$ to $x = 3$, so

$$V=\int_0^3 A(x)dx$$

$$=\int_0^3 2x\sqrt{9-x^2}dx \qquad or \;\; u=9-x^2 \;\; \rightarrow \;\; du=-2xdx$$

$$=-\int_0^3\left(9-x^2\right)^{1/2}d\left(9-x^2\right)$$

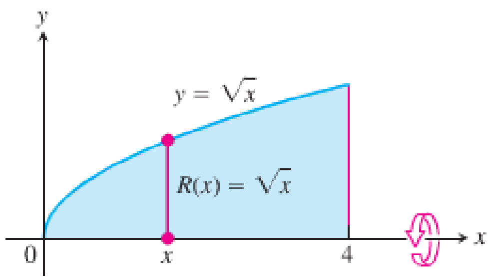
$$=-\frac{2}{3}\left(9-x^2\right)^{3/2}\Big|_0^3$$

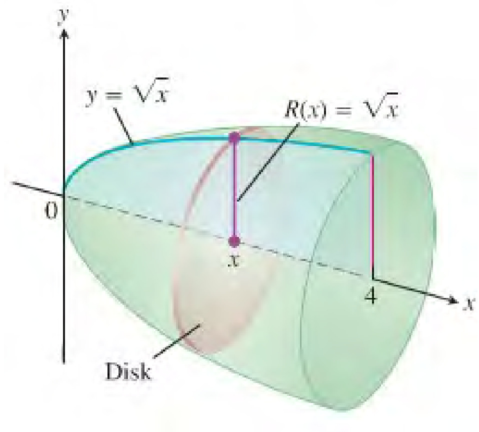
$$=-\frac{2}{3}\left[\left(9-3^2\right)^{3/2}-\left(9-0^2\right)^{3/2}\right]$$

$$\underline{=18 \;\; unit^3}$$

Solids of Revolution: The Disk Method

The solid generated by rotating (or revolving) a plane region about an axis in its plane is called a ***solid of revolution***.





The cross-sectional area $A(x)$ is the area of a disk of radius $R(x)$, the distance of the planar region's boundary from the axis of revolution. The area then

$$A(x) = \pi(radius)^2 = \pi[R(x)]^2$$

And the volume
$$V = \int_a^b A(x)dx$$
$$= \int_a^b \pi[R(x)]^2 dx$$

Example

The region between the curve $y = \sqrt{x}, \quad 0 \le x \le 4$, and the *x*-axis is revolved about the *x*-axis to generate a solid. Find its volume.

Solution

$$V = \int_a^b \pi[R(x)]^2 dx$$
$$= \pi \int_0^4 (\sqrt{x})^2 dx$$
$$= \pi \int_0^4 x\, dx$$

$$= \pi \left. \frac{x^2}{2} \right|_0^4$$

$$= \frac{\pi}{2}\left(4^2 - 0\right)$$

$$\underline{= 8\pi \ \ unit^3 \,|}$$

Example

The circle $x^2 + y^2 = a^2$ is rotated about *x*-axis to generate a sphere. Find its volume

Solution

$$A(x) = \pi y^2 = \pi\left(a^2 - x^2\right)$$

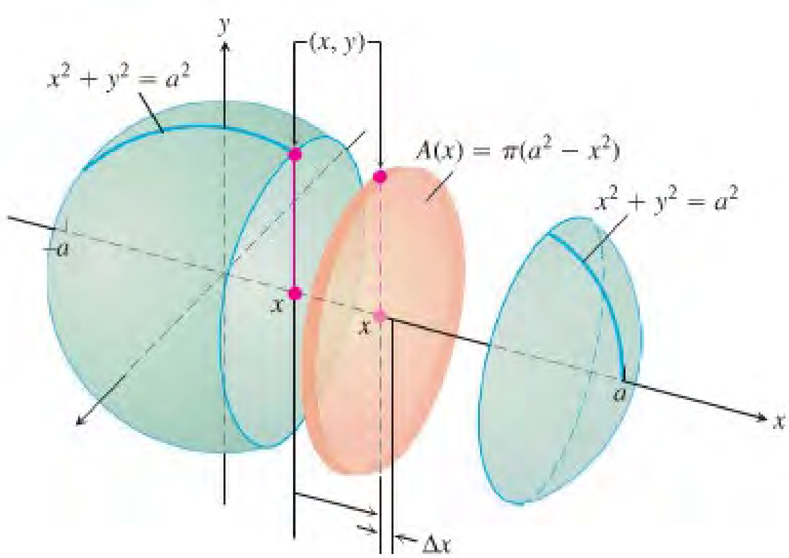
$$V = \int_{-a}^{a} \pi\left(a^2 - x^2\right) dx$$

$$= 2\pi \left. \left(a^2 x - \frac{1}{3}x^3\right. \right|_0^a$$

$$= 2\pi\left(a^2(a) - \frac{1}{3}a^3\right)$$

$$= 2\pi\left(\frac{2}{3}a^3\right)$$

$$\underline{= \frac{4}{3}\pi a^3 \ \ unit^3 \,|}$$



Volume by Disks for Rotation about the *y*-axis

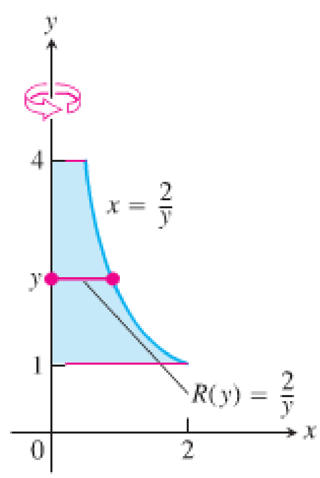
$$V = \int_c^d A(y)\,dy$$

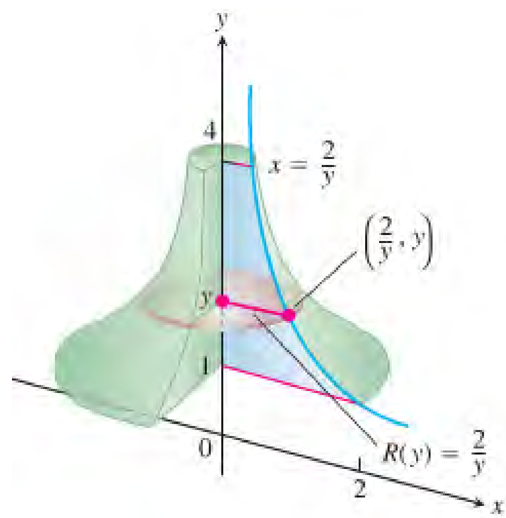
$$= \int_c^d \pi\left[R(y)\right]^2 dy$$

Example

Find the volume of the solid generated by revolving the region between the *y*-axis and the curve $x = \frac{2}{y}, \quad 1 \le y \le 4$ about the *y*-axis.

Solution





$$V = \int_1^4 \pi\left[R(y)\right]^2 dy$$

$$= \pi \int_1^4 \left(\frac{2}{y}\right)^2 dy$$

$$= \pi \int_1^4 \frac{4}{y^2} dy$$

$$= 4\pi\left(-\frac{1}{y}\right)\Big|_1^4$$

$$= 4\pi\left[-\frac{1}{4} - \left(-\frac{1}{1}\right)\right]$$

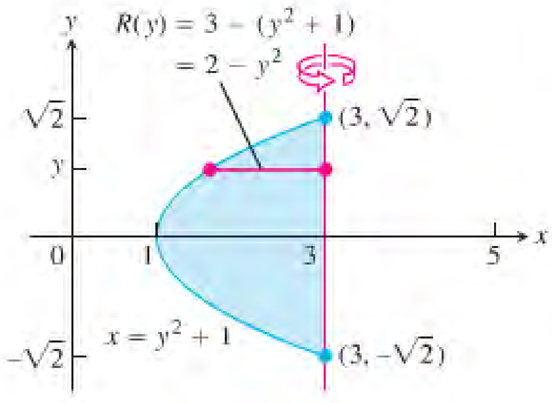
$$= 4\pi\left(\frac{3}{4}\right)$$

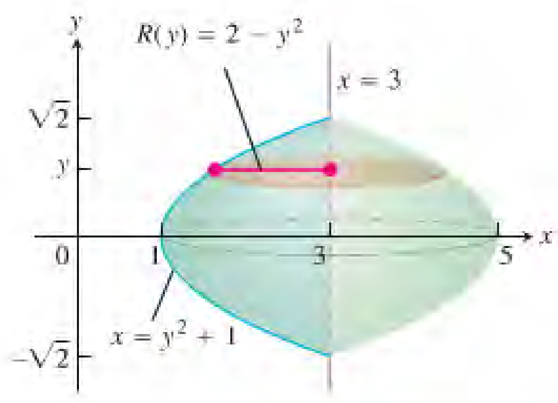
$$= 3\pi \ \ unit^3$$

Example

Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

Solution





$x = y^2 + 1 \quad \to \quad y^2 = x - 1$

When $x = 3 \quad \Rightarrow \quad y^2 = 2 \to y = \pm\sqrt{2}$

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi\left[R(y)\right]^2 dy$$

$$= \pi \int_{-\sqrt{2}}^{\sqrt{2}} \left[3 - \left(y^2 + 1\right)\right]^2 dy$$

$$= \pi \int_{-\sqrt{2}}^{\sqrt{2}} \left(2 - y^2\right)^2 dy$$

$$= \pi \int_{-\sqrt{2}}^{\sqrt{2}} \left(4 - 4y^2 + y^4\right) dy$$ ***Even Function***

$$= 2\pi \left(4y - \frac{4}{3}y^3 + \frac{1}{5}y^5 \right|_0^{\sqrt{2}}$$

$$= 2\pi\left[4\left(\sqrt{2}\right) - \frac{4}{3}\left(\sqrt{2}\right)^3 + \frac{1}{5}\left(\sqrt{2}\right)^5\right]$$

$$= 2\pi\left(4\sqrt{2} - \frac{8}{3}\sqrt{2} + \frac{4}{5}\sqrt{2}\right)$$

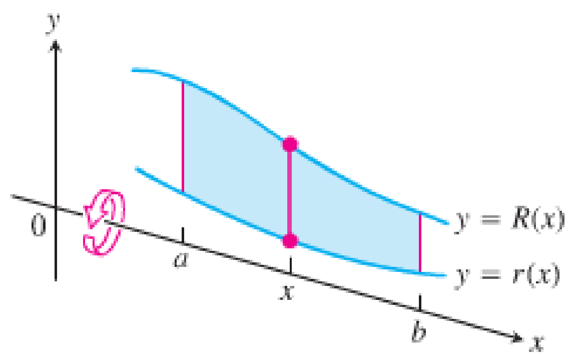
$$\underline{= \frac{64\pi\sqrt{2}}{15} \quad unit^3}$$

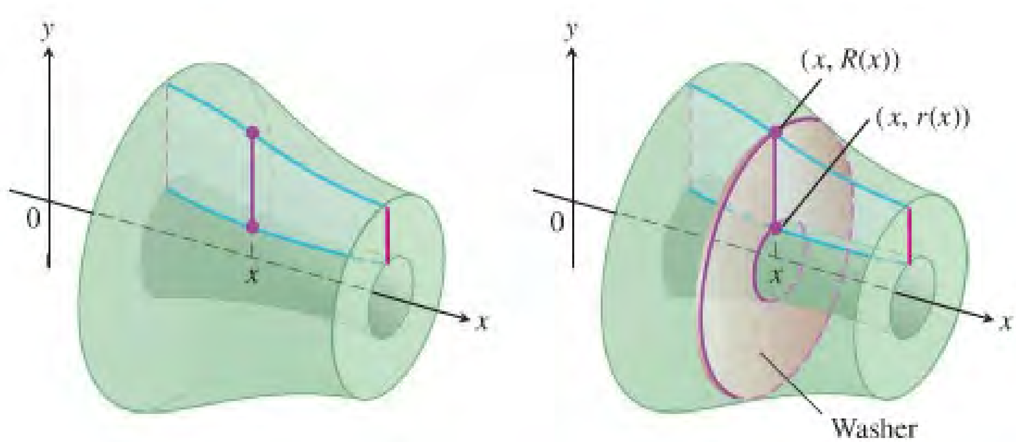
Solids of Revolution: The *Washer Method*

If the region we revolved to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it. The cross-sections perpendicular to the axis of revolution are ***washers*** instead of disks.
The dimensions of a typical washer are:

Outer radius: $R(x)$

Inner radius: $r(x)$





The washer's area is:

$$A(x) = \pi\left[R(x)\right]^2 - \pi\left[r(x)\right]^2 = \pi\left(\left[R(x)\right]^2 - \left[r(x)\right]^2\right)$$

The washer's volume:

$$V = \int_a^b A(x)\,dx$$

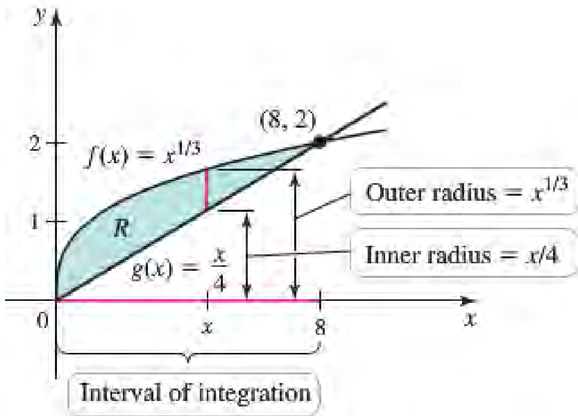
$$= \int_a^b \pi\left(\left[R(x)\right]^2 - \left[r(x)\right]^2\right)dx$$

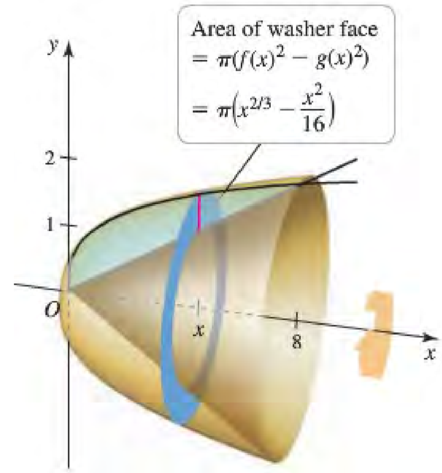
Example

The R be the region in the first quadrant bounded by the graphs of $x = y^3$ and $x = 4y$. Which is the greater, the volume of the solid generated when R is revolved about the x-axis or the y-axis?

Solution

About the ***x-axis***





$$x = y^3 = \left(\frac{x}{4}\right)^3$$

$$4^3 x - x^3 = 0$$

$$x\left(64 - x^2\right) = 0$$

$$\Rightarrow \quad \underline{x = 0,\ 8}\ ,\quad \cancel{-8}\ (\notin QI)$$

$$V = \pi \int_0^8 \left(f(x)^2 - g(x)^2\right) dx$$

$$= \pi \int_0^8 \left(x^{2/3} - \frac{x^2}{16}\right) dx$$

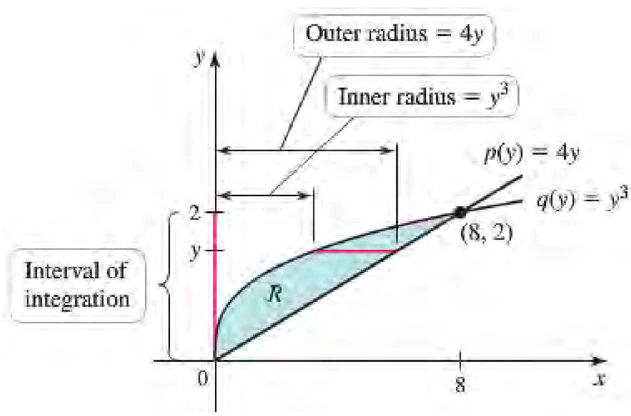
$$= \pi \left(\frac{3}{5}x^{5/3} - \frac{1}{48}x^3\right\vert_0^8$$

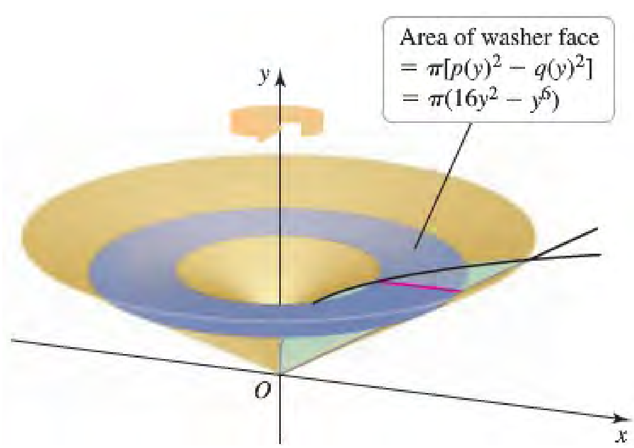
$$= \pi\left(\frac{96}{5} - \frac{32}{3}\right)$$

$$= \frac{128\pi}{15}\ \ unit^3$$

$$\approx 26.81\ \ \text{unit}^3$$

About the ***y-axis***





$$x = y^3 = 4y$$

$$y\left(y^2 - 4\right) = 0$$

$$\underline{y = 0,\ 2}\ ,\quad \cancel{-2}\ (\notin QI)$$

$$V=\pi\int_0^2\left(p(y)^2-q(y)^2\right)dy$$

$$=\pi\int_0^2\left(16y^2-y^6\right)dy$$

$$=\pi\left(\frac{16}{3}y^3-\frac{1}{7}y^7\right.\Bigg|_0^2$$

$$=\pi\left(\frac{128}{3}-\frac{128}{7}\right)$$

$$\boxed{=\frac{512\pi}{12}\ \ unit^3}$$

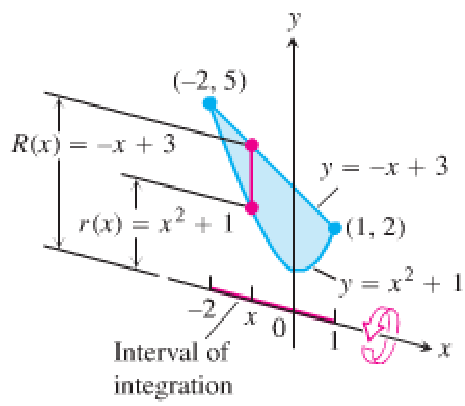
$$\boxed{\approx 76.60\ \ unit^3}$$

The region that is revolving about the ***y-axis*** produces a solid of greater volume.

Example

The region bounded by the curve $y=x^2+1$ and the line $y=-x+3$ is revolved about the x-axis to generate a solid.

Find the volume of the solid.



Solution

$$y=x^2+1=-x+3$$

$$x^2+x-2=0 \qquad\qquad \textit{Solve for x}$$

$$\boxed{x=-2,\ 1}$$

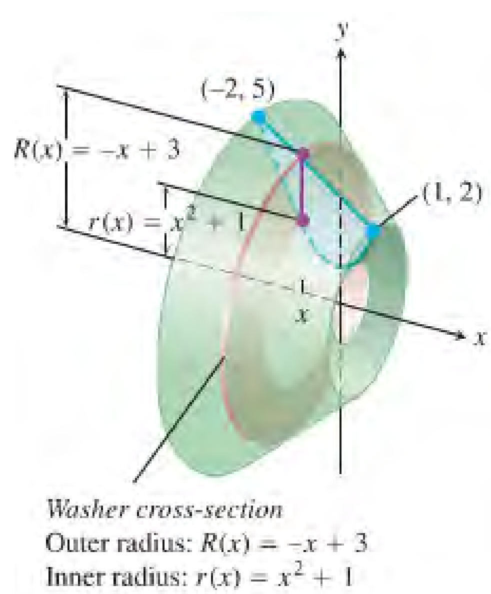
$$V=\int_{-2}^{1}\pi\left((-x+3)^2-\left(x^2+1\right)^2\right)dx$$

$$=\pi\int_{-2}^{1}\left(x^2-6x+9-x^4-2x^2-1\right)dx$$

$$=\pi\int_{-2}^{1}\left(-x^4-x^2-6x+8\right)dx$$

$$=\pi\left(-\frac{x^5}{5}-\frac{x^3}{3}-3x^2+8x\right.\Bigg|_{-2}^{1}$$

$$=\pi\left[\left(-\frac{1}{5}-\frac{1}{3}-3+8\right)-\left(-\frac{(-2)^5}{5}-\frac{(-2)^3}{3}-3(-2)^2+8(-2)\right)\right]$$



$$= \pi\left(5 - \frac{8}{15} - \frac{32}{5} - \frac{8}{3} + 28\right)$$

$$= \pi\left(33 - \frac{144}{15}\right)$$

$$= \pi\left(\frac{351}{15}\right)$$

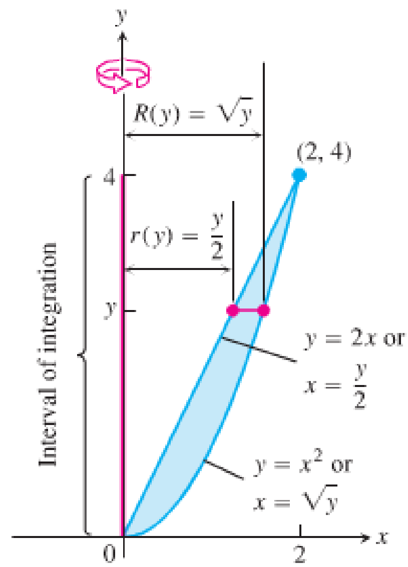
$$\boxed{= \frac{117\pi}{5} \quad unit^3}$$

Example

The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid.

Find the volume of the solid.

Solution



$y = x^2 \quad \rightarrow \quad x = \sqrt{y} = R(y)$

$y = 2x \quad \rightarrow \quad x = \frac{1}{2}y = r(y)$

$\sqrt{y} = \frac{1}{2}y \quad \rightarrow \quad 4y = y^2$

$y^2 - 4y = 0$

$\boxed{y = 0,\ 4}$

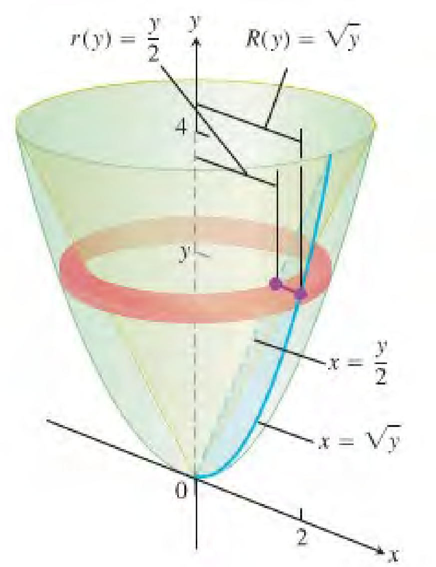
$$V = \int_0^4 \pi\left(\left(\sqrt{y}\right)^2 - \left(\frac{y}{2}\right)^2\right)dy$$

$$= \pi\int_0^4 \left(y - \frac{1}{4}y^2\right)dy$$

$$= \pi\left(\frac{y^2}{2} - \frac{1}{12}y^3 \right|_0^4$$

$$= \pi\left(\frac{4^2}{2} - \frac{1}{12}(4)^3\right)$$

$$\boxed{= \frac{8\pi}{3} \quad unit^3}$$



Example

The region R is bounded by the graphs of $f(x)=\sqrt{x}$ and $g(x)=x^2$ between $x=0$ and $x=1$. What is the volume of the solid that results when R is revolved about the x-axis?

Solution

$$V=\pi\int_0^1\left(f(x)^2-g(x)^2\right)dx$$

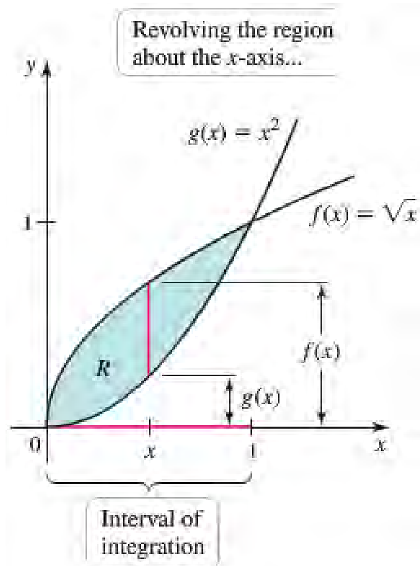
$$=\pi\int_0^1\left(\left(\sqrt{x}\right)^2-\left(x^2\right)^2\right)dx$$

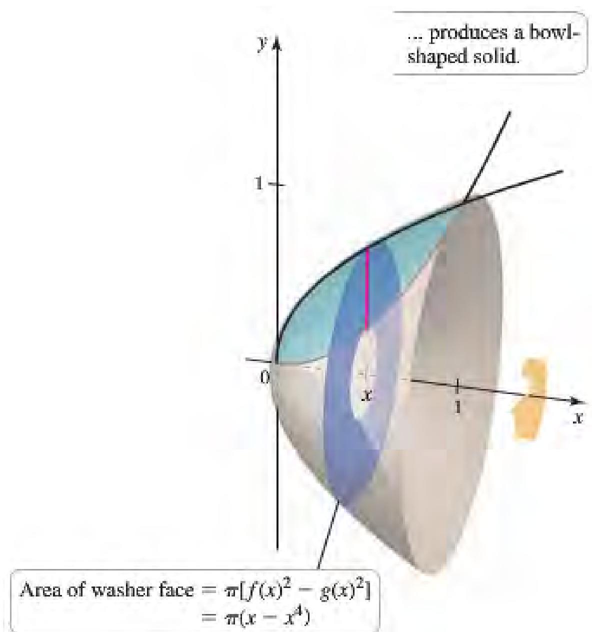
$$=\pi\int_0^1\left(x-x^4\right)dx$$

$$=\pi\left(\frac{x^2}{2}-\frac{x^5}{5}\right)\Big|_0^1$$

$$=\pi\left(\frac{1}{2}-\frac{1}{5}\right)$$

$$=\frac{3\pi}{10}\ unit^3$$

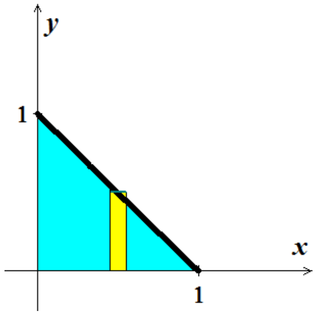




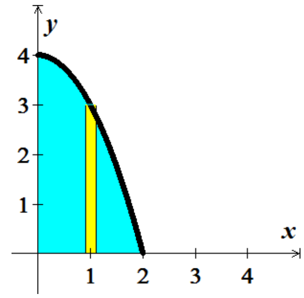
Exercises Section 1.3 – Volume by Slicing

Find the volume of the solid formed by revolving the region about the *x-axis*

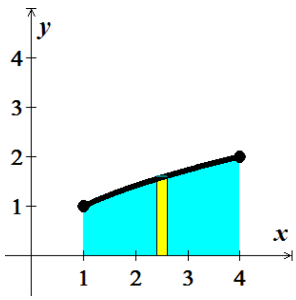
1. $y = -x + 1$



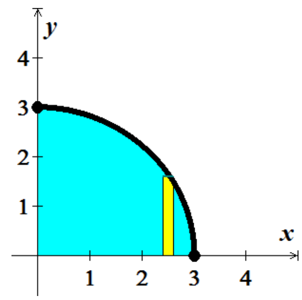
2. $y = 4 - x^2$



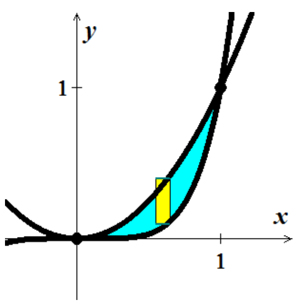
3. $y = \sqrt{x}$



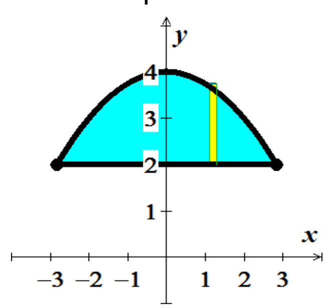
4. $y = \sqrt{9 - x^2}$



5. $y = x^2, \quad y = x^5$

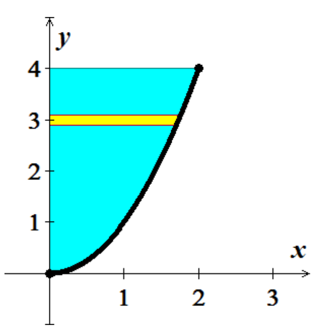


6. $y = 2, \quad y = 4 - \frac{x^2}{4}$

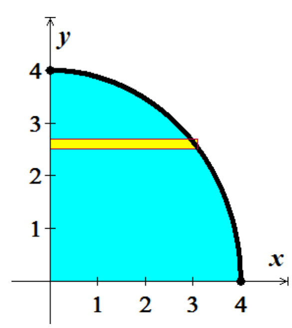


Find the volume of the solid formed by revolving the region about the *y-axis*

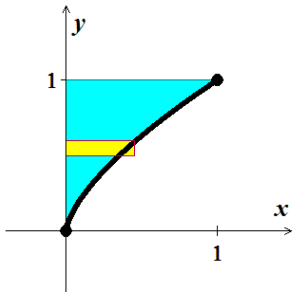
7. $y = x^2$



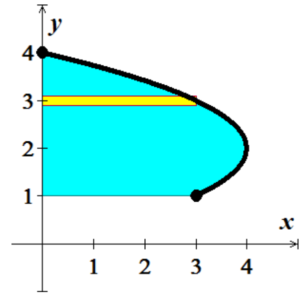
8. $y = \sqrt{16 - x^2}$



9. $y = x^{2/3}$



10. $x = -y^2 + 4y$



11. Find the volumes of the solids generated by revolving the region by the graphs of the equations about the given lines. $y = \sqrt{x}, \quad y = 0, \quad x = 3$

a) the *x-axis* *b)* the *y-axis* *c)* the line $x = 3$ *d)* the line $x = 6$

12. Find the volumes of the solids generated by revolving the region by the graphs of the equations about the given lines. $y = 2x^2, \quad y = 0, \quad x = 2$

a) the *x-axis* *b)* the *y-axis* *c)* the line $y = 8$ *d)* the line $x = 2$

13. Find the volumes of the solids generated by revolving the region by the graphs of the equations about the given lines. $y = x^2, \quad y = 4x - x^2$

a) the *x-axis* *b)* the line $y = 6$

14. Find the volumes of the solids generated by revolving the region by the graphs of the equations about the given lines. $y = -x^2 + 2x + 4, \quad y = 4 - x$

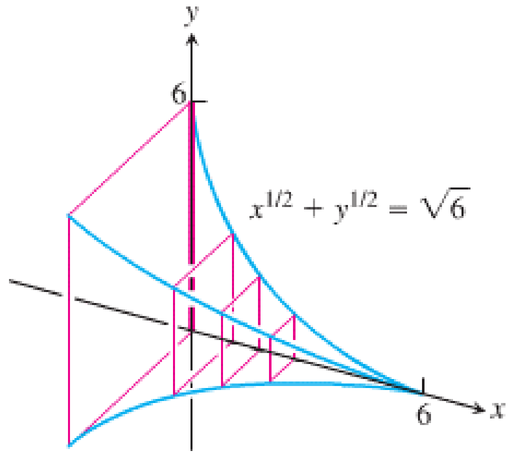
a) the *x-axis* *b)* the line $y = 1$

15. The solid lies between planes perpendicular to the *x*-axis at *x* = 0 and *x* = 4. The cross-sections perpendicular to the axis on the interval $0 \le x \le 4$ are squares whose diagonals run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$. Find the volume of the solid.

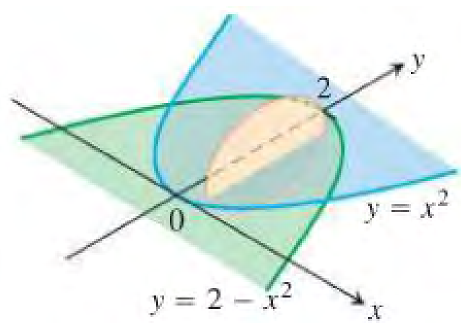
16. The solid lies between planes perpendicular to the *x*-axis at *x* = 0 and *x* = 1. The cross-sections perpendicular to the *x*-axis between these planes are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = \sqrt{x}$. Find the volume of the solid.

17. The base of the solid is the region in the first quadrant between the line $y = x$ and the parabola $y = 2\sqrt{x}$. The cross-sections of the solid perpendicular to the *x*-axis are equilateral triangles whose bases stretch from the line to the curve. Find the volume of the solid.

18. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 6$. The cross-sections between these planes are squares whose bases run from the x-axis up to the curve $x^{1/2} + y^{1/2} = \sqrt{6}$. Find the volume of the solid.

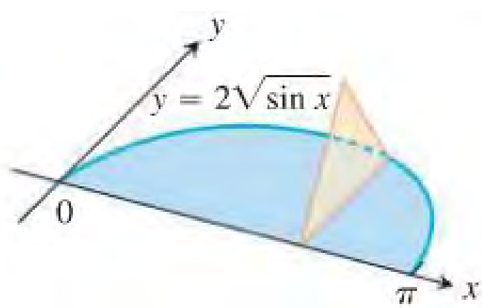


19. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis are circular whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$. Find the volume of the solid.



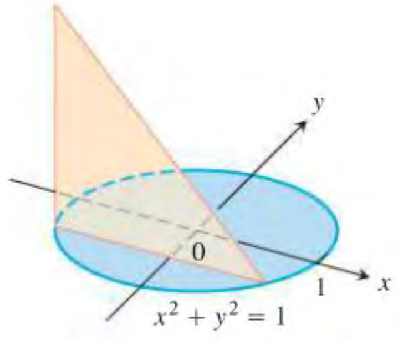
20. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the axis between these planes are squares whose bases run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$. Find the volume of the solid.

21. The base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x-axis. The cross-sections perpendicular to the x-axis are

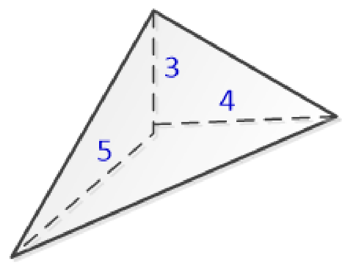


a) Equilateral triangles with bases running from the x-axis to the curve as shown
b) Squares with bases running from the x-axis to the curve. Find the volume of the solid.

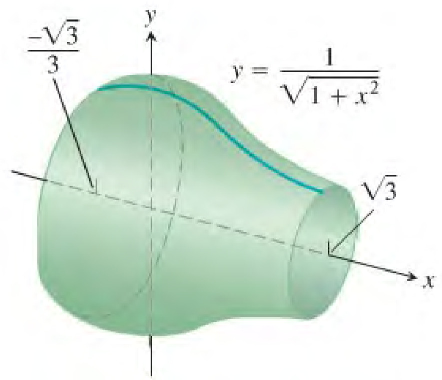
22. The base of the solid is the disk $x^2 + y^2 \le 1$. The cross-sections by planes perpendicular to the y-axis between $y = -1$ and $y = 1$ are isosceles right triangles with one leg in the disk.



23. Find the volume of the given tetrahedron. (***Hint***: Consider slices perpendicular to one of the labeled edges)



24. Find the volume of the solid of revolution



25. Find the volume of the solid generated by revolving the region bounded by $y = x^2$ and the lines $y = 0, \quad x = 2$ about the x-axis.

26. Find the volume of the solid generated by revolving the region bounded by $y = x - x^2$ and the line $y = 0$ about the x-axis.

27. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{\cos x}$ and the lines $0 \le x \le \frac{\pi}{2}, \quad y = 0, \quad x = 0$ about the x-axis.

28. Find the volume of the solid generated by revolving the region bounded by $y = \sec x$ and the lines $y = 0, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$ about the x-axis.

29. Find the volume of the solid generated by revolving the region bounded by $x = \sqrt{5}\, y^2$ and the lines $x = 0, \quad y = -1, \quad y = 1$ about the *y*-axis.

30. Find the volume of the solid generated by revolving the region bounded by $y = 2\sqrt{x}$ and the lines $y = 2, \quad x = 0$ about the *x*-axis.

31. Find the volume of the solid generated by revolving the region bounded by $y = \sec x, \quad y = \tan x$ and the lines $x = 0, \quad x = 1$ about the *x*-axis.

32. Find the volume of the solid generated by revolving the region bounded by $x = \sqrt{2 \sin 2y}$ and the lines $0 \le y \le \frac{\pi}{2}, \quad x = 0$ about the *y*-axis.

33. What is the volume of the solid whose base is the region in the first quadrant bounded by $y = \sqrt{x}$, $y = 2 - x$, and the *x*-axis, and whose cross sections perpendicular to the base and parallel to the *y*-axis are squares?

34. What is the volume of the solid whose base is the region in the first quadrant bounded by $y = \sqrt{x}$, $y = 2 - x$, and the *x*-axis, and whose cross sections perpendicular to the base and parallel to the *y*-axis are semicircles?

35. What is the volume of the solid whose base is the region in the first quadrant bounded by $y = \sqrt{x}$, $y = 2 - x$, and the *y*-axis, and whose cross sections perpendicular to the base and parallel to the *x*-axis are square?

36. The region bounded by the curves $y = -x^2 + 2x + 2$ and $y = 2x^2 - 4x + 2$ is revolved about the *x*-axis. What is the volume of the solid that is generated?

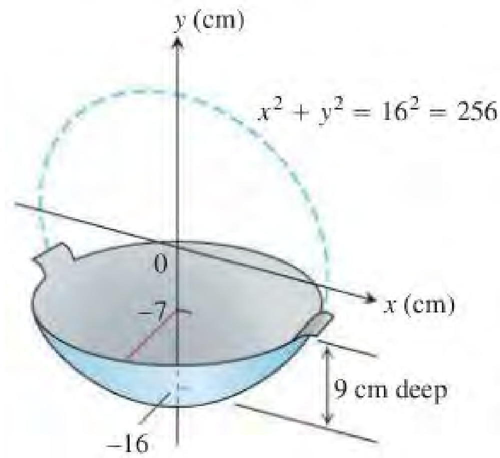
37. The region bounded by the curves $y = 2e^{-x}$, $y = e^x$, and the *y*-axis is revolved about the *x*-axis. What is the volume of the solid that is generated?

38. The region bounded by the curves $y = \sec x$, $y = 2$, for $0 \le x \le \frac{\pi}{3}$ is revolved around the *x*-axis. What is the volume of the solid that is generated?

39. The region bounded by the graph $y = (x - 2)^2$ and $y = 4$ is revolved about the line $y = 4$. What is the volume of the resulting solid?

40. Find the volume of a solid ball having radius *a*.

41. You are designing a wok frying pan that will be shaped like a spherical bowl with handles. A bit of experimentation at home persuades you that you can get one that hold about 3 *L* if you make it 9 *cm* deep and give the sphere a radius of 16 *cm*. To be sure, you picture the wok as a solid of revolution, and calculate its volume with an integral. To the nearest cubic centimeter, what volume do you really get? $\left(1\,L = 1{,}000\ cm^3\right)$



42. A curved wedge is cut from a circular cylinder of radius 3 by two planes. On plane is perpendicular to the axis of the cylinder. The second place crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

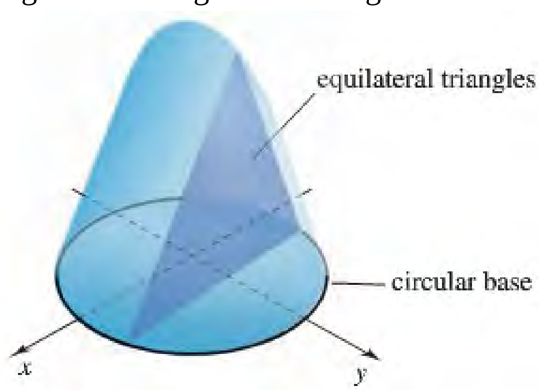
43. Find the volume of the solid generated by revolving the region bounded by $y=\sqrt{x}$ and the lines $y=1$, $x = 4$ about the line $y = 1$.

44. Let R be the region bounded by $y=\sin^{-1}x, \quad x=0, \quad y=\frac{\pi}{4}$. Use the disk method to find the volume of the solid generated when R is revolved about y-axis.

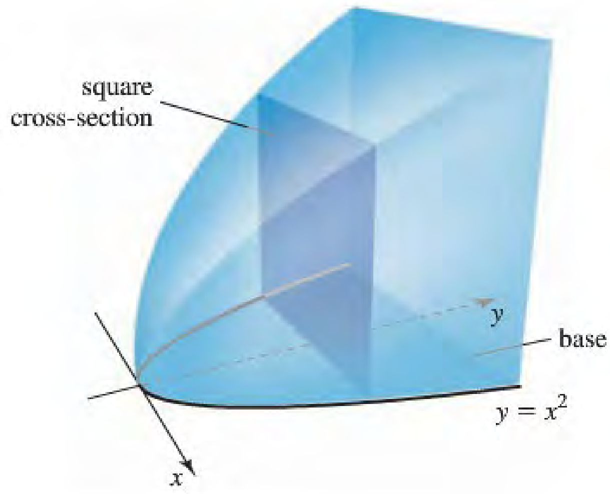
45. Find the volume of the solid of revolution bounded by $y=\frac{\ln x}{\sqrt{x}}, \quad y=0, \quad and \quad x=2$ revolved about the x-axis. Sketch the region

46. Find the volume of the solid of revolution bounded by $y=e^{-x}, \quad y=e^{x}, \quad x=0, \quad x=\ln 4$ revolved about the x-axis. Sketch the region

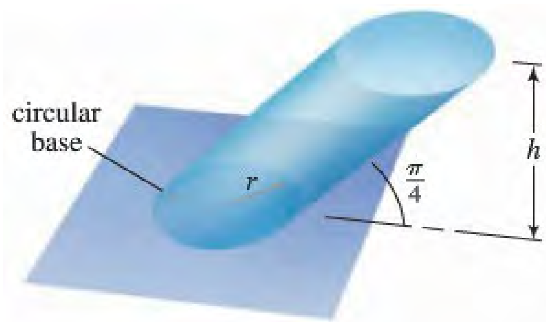
47. The solid with a circular base of radius 5 whose cross sections perpendicular to the base and parallel to the x-axis are equilateral triangles. Use the general slicing method to find the volume of the solid.



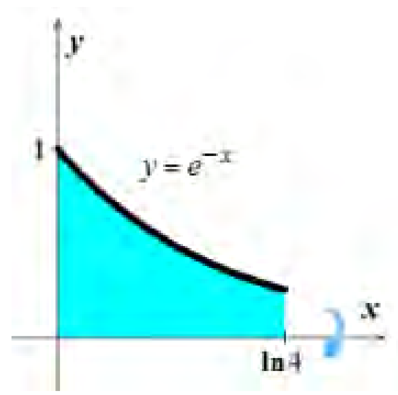
48. The solid whose base is the region bounded by $y = x^2$ and the line $y = 1$ and whose cross sections perpendicular to the base and parallel to the x-axis squares. Use the general slicing method to find the volume of the solid.



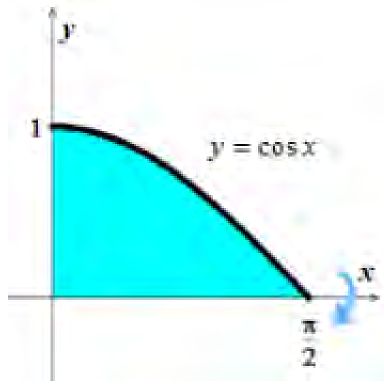
49. A circular cylinder of radius r and height h whose curved surface is at an angle of $\frac{\pi}{4}$ *rad* . Use the general slicing method to find the volume of the solid



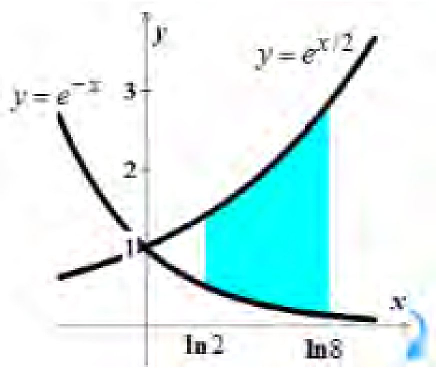
50. Let R be the region bounded by $y = e^{-x}$, $y = 0$, $x = 0$, $x = \ln 4$. Use the disk method to find the volume of the solid generated when R is revolved about x-axis.



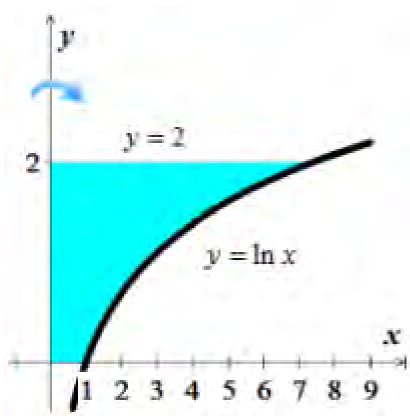
51. Let R be the region bounded by $y = \cos x, \quad y = 0, \quad x = 0$. Use the disk method to find the volume of the solid generated when R is revolved about x-axis,



52. Let R be the region bounded by $y = e^{x/2}, \quad y = e^{-x}, \quad x = \ln 2, \quad x = \ln 8$. Use the disk method to find the volume of the solid generated when R is revolved about x-axis.



53. Let R be the region bounded by $y = 0, \quad y = \ln x, \quad y = 2, \quad x = 0$. Use the disk method to find the volume of the solid generated when R is revolved about y-axis.



Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $y = 4$

54. $y = x, \quad y = 3, \quad x = 0$

55. $y = \frac{1}{2}x^3, \quad y = 4, \quad x = 0$

56. $y = \frac{3}{1+x}, \quad y = 0, \quad x = 0, \quad x = 3$

57. $y = \sec x, \quad y = 0, \quad 0 \le x \le \frac{\pi}{3}$

Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $x = 5$

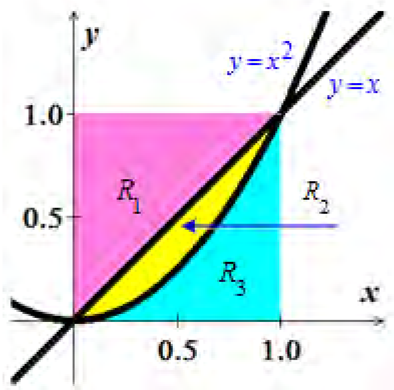
58. $y = x, \quad y = 0, \quad y = 4, \quad x = 5$

59. $y = 3 - x, \quad y = 0, \quad y = 2, \quad x = 0$

60. $x = y^2, \quad x = 4$

61. $xy = 3, \quad y = 1, \quad y = 4, \quad x = 5$

62. Find the volume generated by rotating the given region $y = x^2$ *and* $y = x$ about the specified line.



a) R_1 *about* $x = 0$

b) R_1 *about* $x = 1$

c) R_2 *about* $y = 0$

d) R_2 *about* $y = 1$

e) R_3 *about* $x = 0$

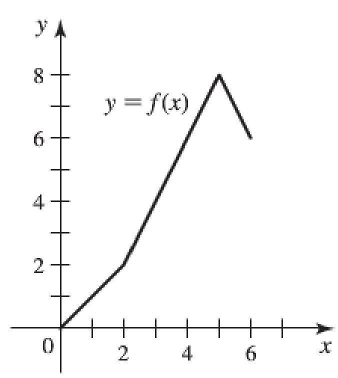
f) R_3 *about* $x = 1$

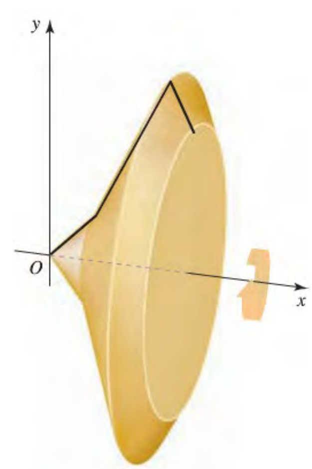
g) R_2 *about* $x = 0$

h) R_2 *about* $x = 1$

63. Let $f(x) = \begin{cases} x & \text{if} \quad 0 \le x \le 2 \\ 2x - 2 & \text{if} \quad 2 < x \le 5 \\ -2x + 18 & \text{if} \quad 5 < x \le 6 \end{cases}$

Find the volume of the solid formed when the region bounded by the graph of *f*, the *x*-axis, and the line $x = 6$ is revolved about the *x*-axis



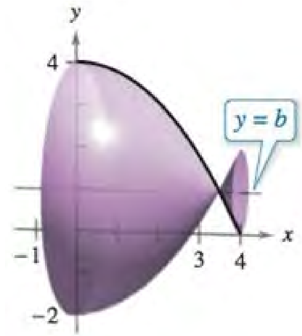


64. Consider the solid formed by revolving the region bounded by $y=\sqrt{x}$, $y=0$, and $x=4$ about the *x-axis*

a) Find the value of x in the interval $[0,\ 4]$ that divides the solids into two parts of equal volume.

b) Find the values of x in the interval $[0,\ 4]$ that divide the solids into three parts of equal volume.

65. The arc of $y=4-\frac{1}{4}x^2$ on the interval $[0,\ 4]$ is revolved about the line $y=b$

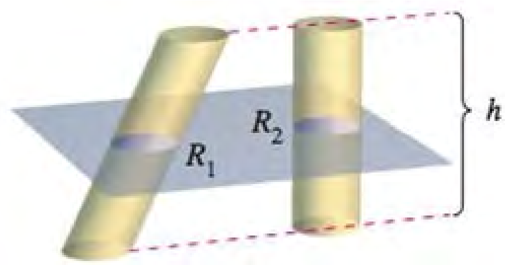


a) Find the volume of the resulting solid as a function of b.

b) Graph the function in part (*a*), and approximate the value of b that minimizes the volume of the solid.

c) Find the value of b that minimizes the volume of the solid, and compare the result with the answer in part (*b*).

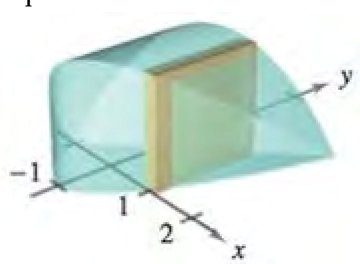
66. Prove that if two solids have equal altitudes and all plane sections parallel to their bases and at equal distances from their bases have equal areas, then the solids hace the same volume.



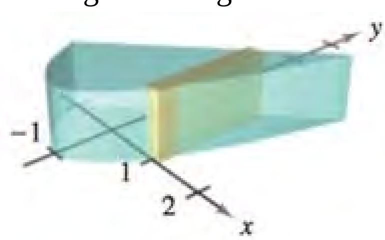
Area of R_1 = *Area of* R_2

67. Find the volumes of the solids whose bases are bounded by the graph of $y=x+1$ and $y=x^2-1$, with the indicated cross sections taken perpendicular to the *x-axis*

a) Squares

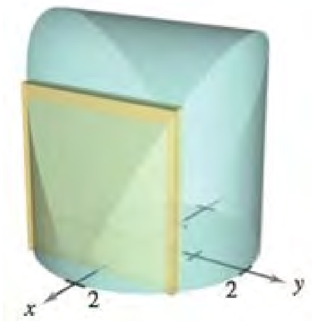


b) Rectangles of height 1

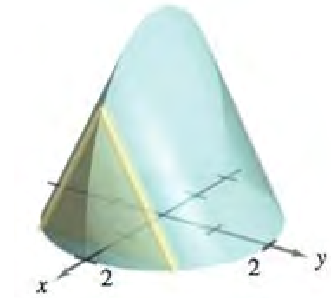


68. Find the volumes of the solids whose bases are bounded by the circle $x^2 + y^2 = 4$, with the indicated cross sections taken perpendicular to the *x-axis*

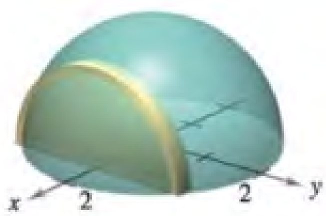
a) Squares



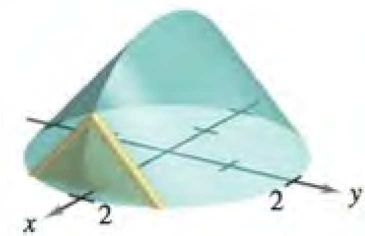
b) Equilateral triangles



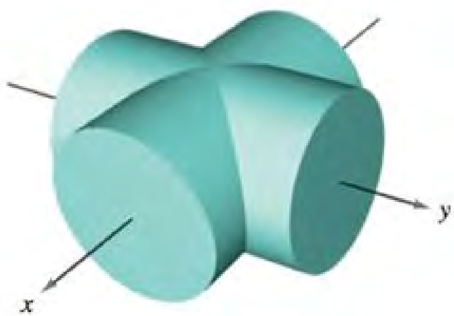
c) Semicircles



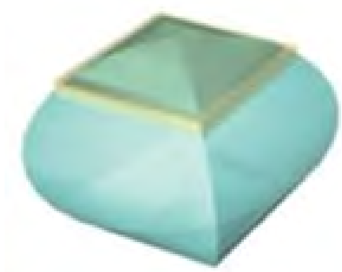
d) Iscosceles right triangles



69. Find the volume of the solid of intersection (the solid common to both) of the two right circular cylinders of radius *r* whose axes meet at right angles.



Two intersecting cylinders

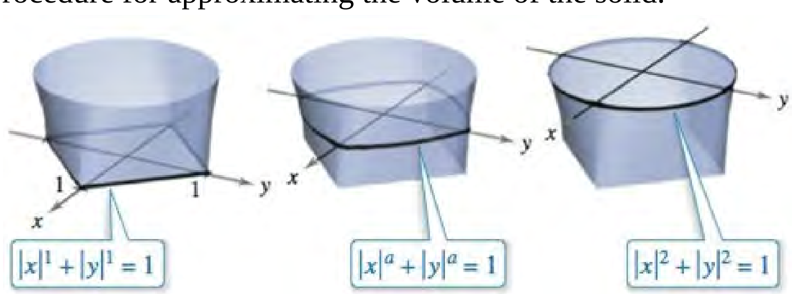


Solid of intersection

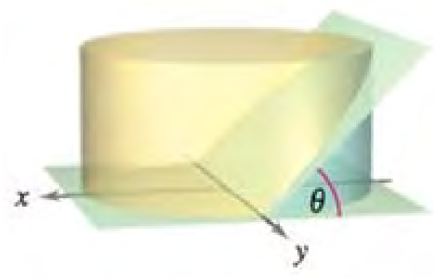
70. The solid shown in the figure has cross sections bounded by the graph $|x|^a + |y|^a = 1$ where $1 \le a \le 2$.

a) Describe the cross section when $a = 1$ and $a = 2$.

b) Describe a procedure for approximating the volume of the solid.



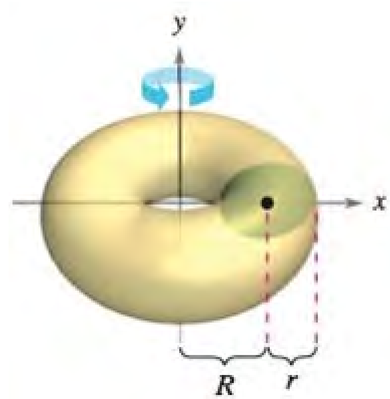
71. Two planes cut a right circular cylinder to form a wedge. One plane is perpendicular to the axis of the cylinder and the second makes an angle of θ degrees with the first.



a) Find the volume of the wedge if $\theta = 45°$.

b) Find the volume of the wedge for an arbitrary angle θ. Assuming that the cylinder has sufficient length, how does the volume of the wedge change as θ increases from 0° to 90°?

72. For the given torus (donut).

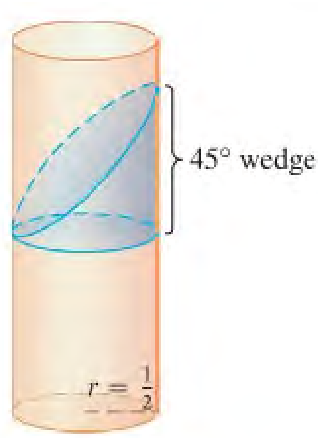


a) Show that the volume of the torus is given by the integral

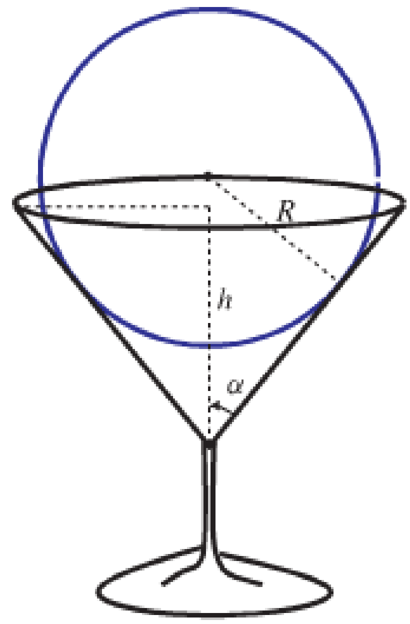
$$8\pi R\int_0^r \sqrt{r^2 - y^2}\,dy \quad \textit{where} \quad R > r > 0$$

b) Find the volume of the torus

73. Consider a right-circular cylinder of diameter 1. Form a wedge by making one slice parallel to the base of the cylinder completely through the cylinder, and another slice at an angle of 45° to the first slice and intersecting the first slice at the opposite edge of the cylinder. Find the volume of the wedge.

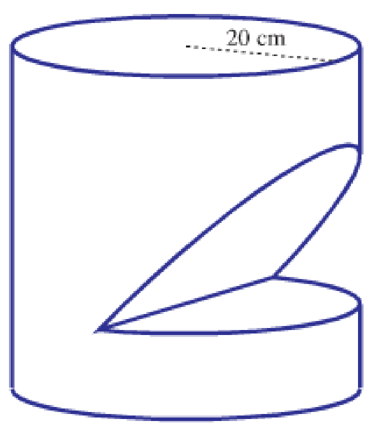


74. A martini glass in the shape of a right-circular cone of height h and semi-vertical angle α is filled with liquid. Slowly a ball is lowered into the glass, displacing liquid and causing it to overflow.



Find the radius R of the ball that causes the greatest volume of liquid to overflow out the glass.

75. A 45° notch is cut to the center of a cylindrical log having radius 20 *cm*. One plane face the notch is perpendicular to the axis of the log. What volume of wood was removed from the log by cutting the notch?



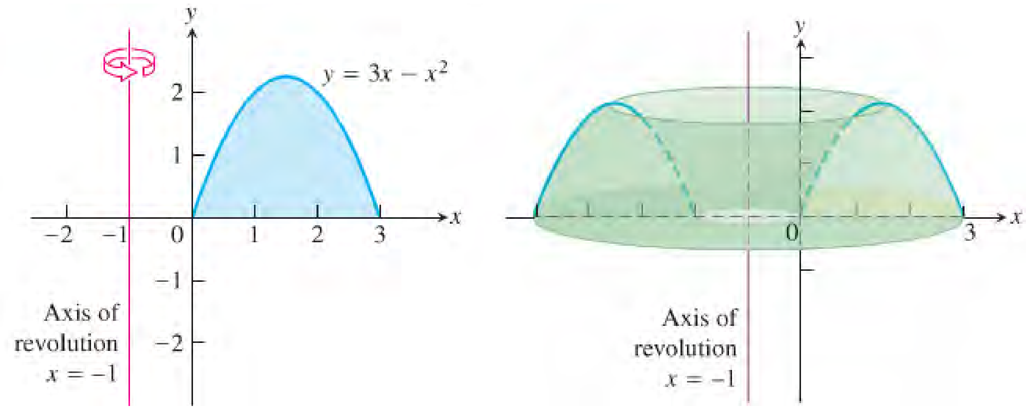
Section 1.4 – Volume by Shells

Slicing with Cylinders

Example

The region enclosed by the *x*-axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the vertical line $x = -1$ to generate a solid. Find the volume of the solid

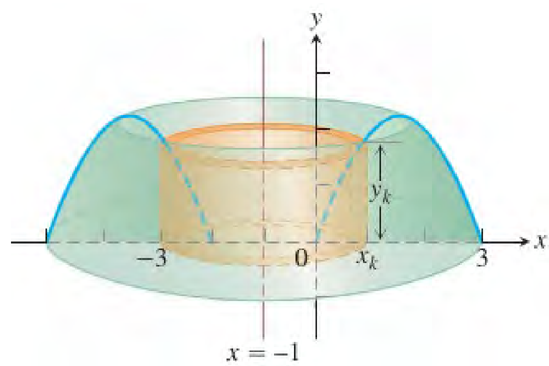
Solution

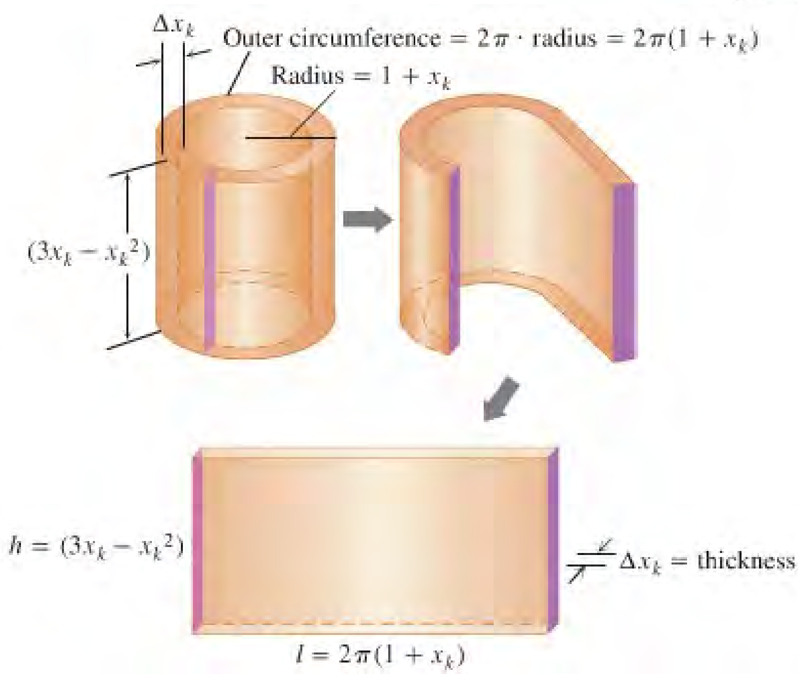


If we rotate a vertical strip of thickness Δx, this rotation produces a cylindrical shell of height y_k above a point x_k within the base of the vertical strip.

$$\Delta V_k = \mathit{circumference} \times \mathit{height} \times \mathit{thickness}$$

$$= 2\pi\left(1 + x_k\right)\cdot\left(3x_k - x_k^2\right)\cdot\Delta x_k$$





The Riemann sum:

$$\sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} 2\pi\left(1+x_k\right)\cdot\left(3x_k - x_k^2\right)\cdot\Delta x_k$$

Taking the limit as the thickness $\Delta x_k \to 0$ and $n \to \infty$ gives the volume integral

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi\left(1+x_k\right)\cdot\left(3x_k - x_k^2\right)\cdot\Delta x_k$$

$$= \int_0^3 2\pi\left(x+1\right)\left(3x-x^2\right)dx$$

$$= 2\pi\int_0^3 \left(3x^2+3x-x^2-x^3\right)dx$$

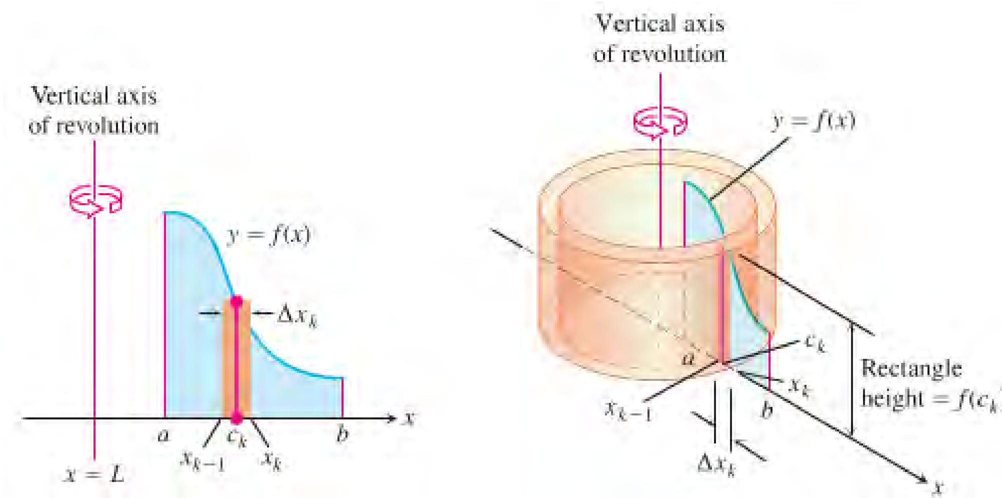
$$= 2\pi\int_0^3 \left(2x^2+3x-x^3\right)dx$$

$$= 2\pi\left(\frac{2}{3}x^3+\frac{3}{2}x^2-\frac{1}{4}x^4 \,\right|_0^3$$

$$= 2\pi\left(\frac{2}{3}(3)^3+\frac{3}{2}(3)^2-\frac{1}{4}(3)^4\right)$$

$$= \frac{45\pi}{2} \quad unit^3$$

Shell Method

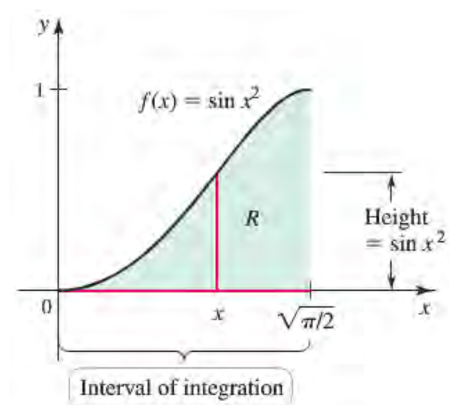


$$V = 2\pi \int_a^b \begin{pmatrix} shell \\ radius \end{pmatrix} \begin{pmatrix} shell \\ height \end{pmatrix} dx$$

Example

Let R be the region bounded by the graph of $f(x) = \sin x^2$, the x-axis, and the vertical line $x = \sqrt{\frac{\pi}{2}}$. Find the volume of the solid generated when R is revolved about the y-axis.

Solution



$$V = 2\pi \int_a^b \begin{pmatrix} shell \\ radius \end{pmatrix} \begin{pmatrix} shell \\ height \end{pmatrix} dx$$

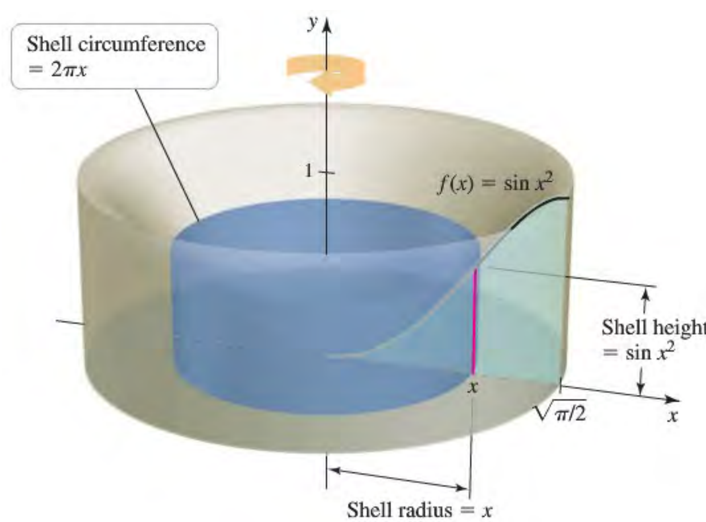
$$= 2\pi \int_0^{\sqrt{\pi/2}} x \sin x^2 \, dx$$

$$= \pi \int_0^{\sqrt{\pi/2}} \sin x^2 \, d\left(x^2\right)$$

$$= -\pi \cos\left(x^2\right) \Big|_0^{\sqrt{\pi/2}}$$

$$= -\pi \left(\cos\left(\frac{\pi}{2}\right) - \cos 0\right)$$

$$= \pi \quad unit^3$$



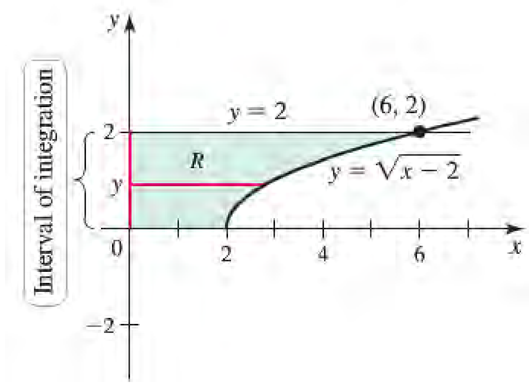
Example

Let R be the region in the first region bounded by the graph $y = \sqrt{x-2}$ and the line $y = 2$.

a) Find the volume of the solid generated when R is revolved about the x-axis.

b) Find the volume of the solid generated when R is revolved about the line $y = -2$.

Solution



a) $y = \sqrt{x-2} \;\rightarrow\; y^2 = x - 2 \;\Rightarrow\; x = y^2 + 2$

$0 \le y \le 2$

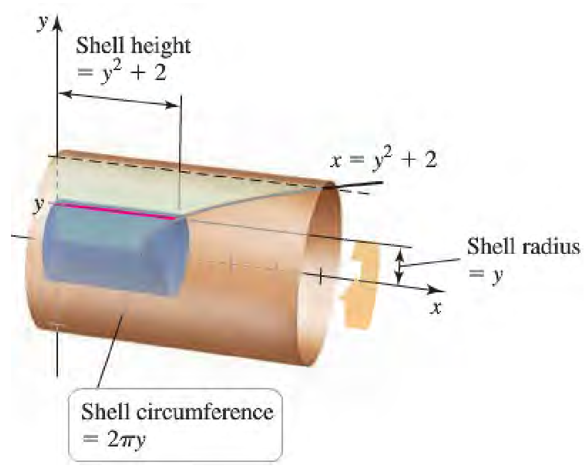
$$V = 2\pi \int_c^d \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dy$$

$$= 2\pi \int_0^2 y\left(y^2 + 2\right) dy$$

$$= 2\pi \int_0^2 \left(y^3 + 2y\right) dy$$

$$= 2\pi \left(\frac{y^4}{4} + y^2 \right) \Big|_0^2$$

$$\underline{= 16\pi \quad unit^3}$$



b) Revolved R about the line $y = -2$.

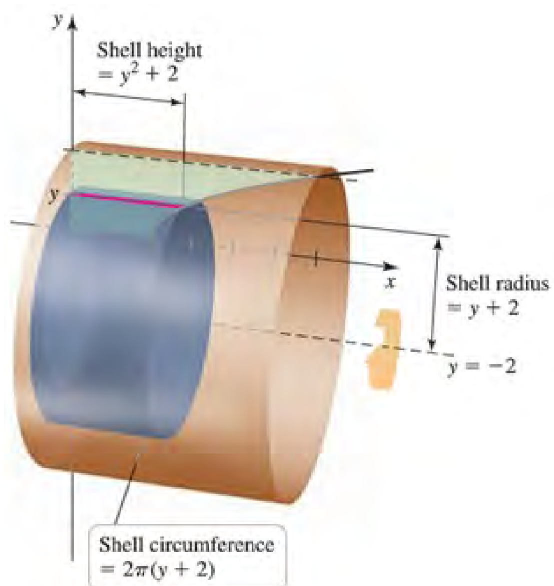
$$V = 2\pi \int_c^d \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dy$$

$$= 2\pi \int_0^2 (y+2)\left(y^2 + 2\right) dy$$

$$= 2\pi \left(\frac{1}{4}y^4 + \frac{2}{3}y^3 + y^2 + 4y \right) \Big|_0^2$$

$$= 2\pi \left(4 + \frac{16}{3} + 4 + 8 \right)$$

$$\underline{= \frac{128\pi}{3} \quad unit^3}$$



Example

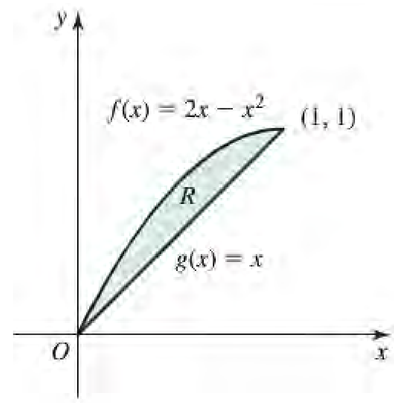
The region R is bounded by the graphs of $f(x) = 2x - x^2$ and $g(x) = x$ on the interval $[0, 1]$.
Use the washer method and the shell method to find the volume of the solid formed when R is revolved about the x-axis.

Solution

$$f(x) = g(x)$$

$$2x - x^2 = x$$

$$x^2 - x = 0 \Rightarrow \boxed{x = 0,\ 1}$$



Washer Method:

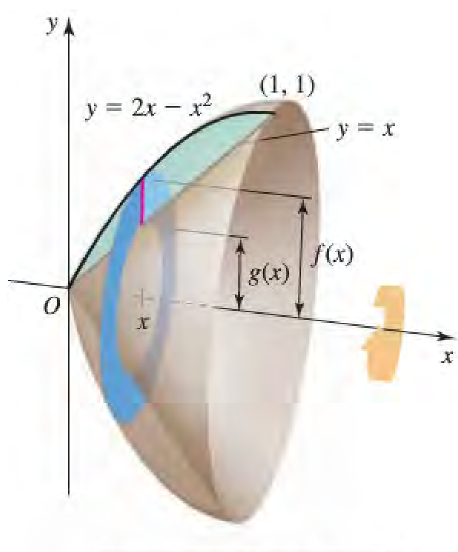
$$V = \pi \int_0^1 \left[\left(2x - x^2\right)^2 - x^2 \right] dx$$

$$= \pi \int_0^1 \left(3x^2 - 4x^3 + x^4\right) dx$$

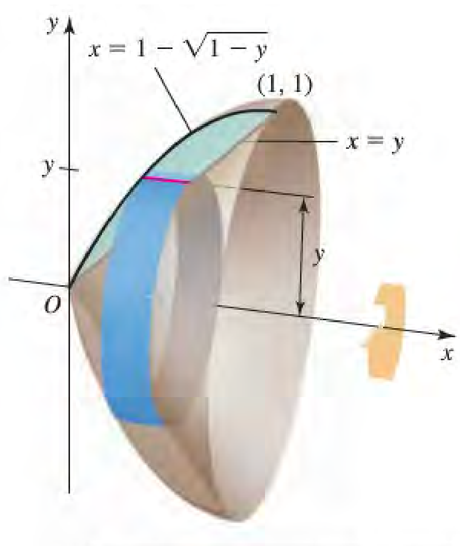
$$= \pi \left(x^3 - x^4 + \frac{x^5}{5} \right. \Big|_0^1$$

$$= \pi\left(1 - 1 + \frac{1}{5}\right)$$

$$\boxed{= \frac{\pi}{5} \quad unit^3}$$



(Outer radius)$^2 = (2x - x^2)^2$
(Inner radius)$^2 = x^2$



Shell height $= y - (1 - \sqrt{1 - y})$
Shell radius $= y$

***Shell Method*:**

$x = y$

$y = 2x - x^2$

$x^2 - 2x + y = 0$

$$x = \frac{2 \pm \sqrt{4-4y}}{2}$$

$$= \begin{cases} 1 - \sqrt{1-y} \\ \cancel{1 + \sqrt{1-y}} \end{cases}$$

$x = 0 \quad \to y = 0$

$x = 1 \quad \to y = 1$

$$V = 2\pi \int_0^1 y \left[y - \left(1 - \sqrt{1-y} \right) \right] dy$$

$$= 2\pi \int_0^1 y \left(y - 1 + \sqrt{1-y} \right) dy$$

$$= 2\pi \int_0^1 \left(y^2 - y + y(1-y)^{1/2} \right) dy$$

$$= 2\pi \left(\frac{1}{3}y^3 - \frac{1}{2}y^2 + \frac{2}{5}(1-y)^{5/2} - \frac{2}{3}(1-y)^{3/2} \right. \Big|_0^1$$

$$= 2\pi \left(\frac{1}{3} - \frac{1}{2} - \left(\frac{2}{5} - \frac{2}{3} \right) \right)$$

$$= 2\pi \left(-\frac{1}{6} + \frac{4}{15} \right)$$

$$= 2\pi \left(\frac{9}{90} \right)$$

$$= \frac{\pi}{5} \quad unit^3$$

Let $u = 1 - y \quad \to \quad y = 1 - u$

$dy = -du$

$$\int y(1-y)^{1/2} dy = -\int (1-u)u^{1/2} du$$

$$= -\int \left(u^{1/2} - u^{3/2} \right) du$$

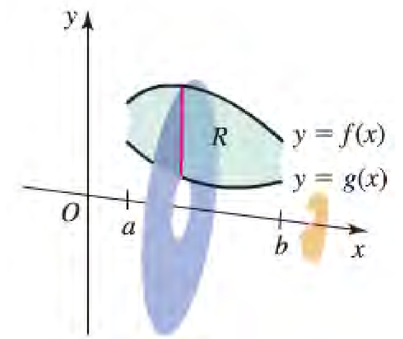
$$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}$$

$$= \frac{2}{5}(1-y)^{5/2} - \frac{2}{3}(1-y)^{3/2}$$

Summary of the Shell Method

1. **1.** Draw the region and sketch a line segment across it parallel to the axis of revolution. Label the segment's height or length (*shell height*) and distance from the axis of revolution (*shell radius*)
2. **2.** Find the limits of integration for the thickness variable.
3. **3.** Integrate the product 2π (*shell radius*) (*shell height*) with respect to the thickness variable (x or y) to find the volume

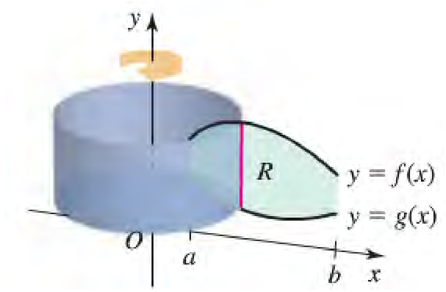
Integration With respect to x



Disk/washer method about the x-axis

Disks/washers are ***perpendicular*** to the x-axis

$$V = \pi \int_a^b \left(f(x)^2 - g(x)^2 \right) dx$$

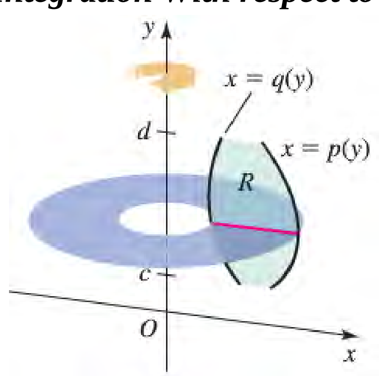


Shell method about the y-axis

Shells are ***parallel*** to the y-axis

$$V = 2\pi \int_a^b x \left(f(x) - g(x) \right) dx$$

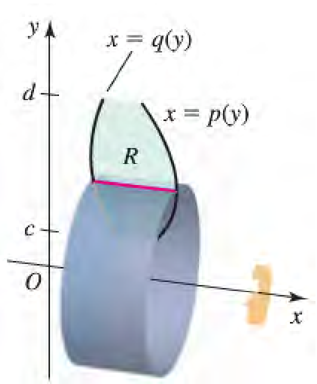
Integration With respect to y



Disk/washer method about the y-axis

Disks/washers are ***perpendicular*** to the y-axis

$$V = \pi \int_c^d \left(p(y)^2 - q(y)^2 \right) dy$$



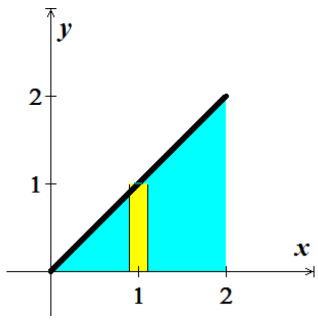
Shells are ***parallel*** to the x-axis

$$V = 2\pi \int_c^d y \left(p(y) - q(y) \right) dy$$

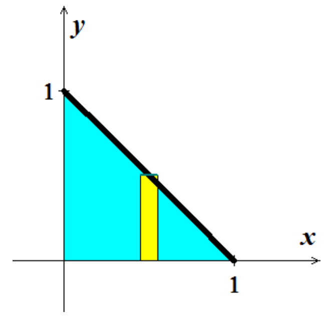
Exercises Section 1.4 – Volume by Shells

(1 – 13) Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

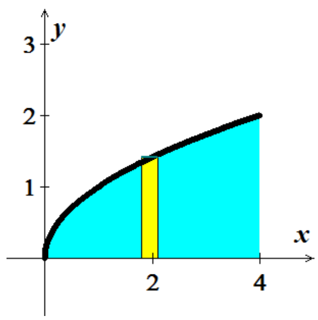
1. $y = x$



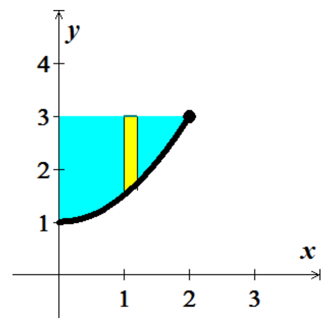
2. $y = 1 - x$



3. $y = \sqrt{x}$



4. $y = \frac{1}{2}x^2 + 1$



5. $y = \frac{1}{4}x^2, \quad y = 0, \quad x = 4$

6. $y = \frac{1}{2}x^3, \quad y = 0, \quad x = 3$

7. $y = x^2, \quad y = 4x - x^2$

8. $y = 9 - x^2, \quad y = 0$

9. $y = 4x - x^2, \quad x = 0, \quad y = 4$

10. $y = x^{3/2}, \quad y = 8, \quad x = 0$

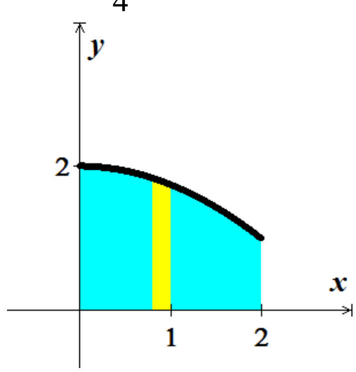
11. $y = \sqrt{x-2}, \quad y = 0, \quad x = 4$

12. $y = -x^2 + 1, \quad y = 0$

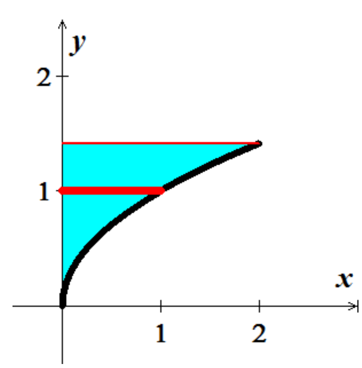
13. $y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad y = 0, \quad x = 0, \quad x = 1$

(14 – 15) Use the shell method to find the volume of the solid generated by revolving the shaded region about the indicated axis

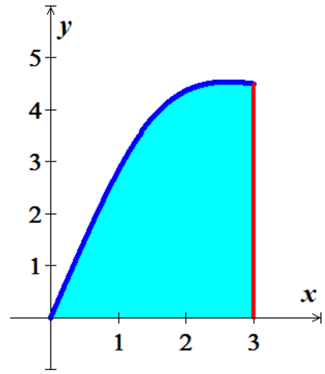
14. $y = 2 - \frac{1}{4}x^2$



15. $x = y^2$



16. Use the shell method to find the volume of the solid generated by revolving the shaded region about the y-axis



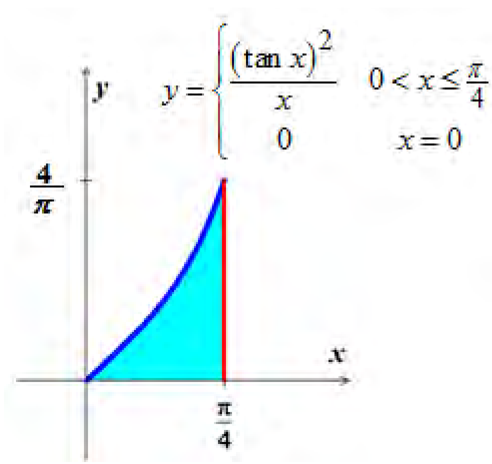
$$y = \frac{9x}{\sqrt{x^3+9}}$$

17. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = x^2, \quad y = 2 - x, \quad x = 0, \quad \textit{for } x \geq 0$ about the y-axis.

18. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = 2 - x^2, \quad y = x^2, \quad x = 0$ about the y-axis.

19. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = \frac{3}{2\sqrt{x}}, \quad y = 0, \quad x = 1, \quad x = 4$ about the y-axis.

20. Let $g(x) = \begin{cases} \frac{(\tan x)^2}{x} & 0 < x \leq \frac{\pi}{4} \\ 0 & x = 0 \end{cases}$



a) Show that $x \cdot g(x) = (\tan x)^2, \quad 0 \leq x \leq \frac{\pi}{4}$

b) Find the volume of the solid generated by revolving the shaded region about the y-axis.

21. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $x = \sqrt{y}, \quad x = -y, \quad y = 2$ about the x-axis.

22. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $x = y^2, \quad x = -y, \quad y = 2, \quad y \geq 0$ about the x-axis.

23. Compute the volume of the solid generated by revolving the region bounded by the lines $y = x \quad and \quad y = x^2$ about each coordinate axis using

a) The *shell* method

b) The *washer* method

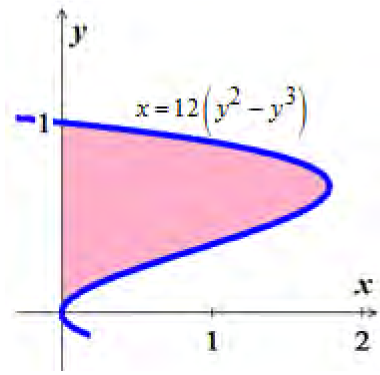
24. Use the shell method to find the volumes of the solids generated by revolving the shaded regions about the *indicated* axes.

a) The x-axis

b) The line $y = 1$

c) The line $y = \frac{8}{5}$

d) The line $y = -\frac{2}{5}$



25. Use the *washer* method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = \sqrt{x}, \quad y = 2, \quad x = 0$ about

a) the x-axis

b) the y-axis

c) the line $x = 4$

d) the line $y = 1$

26. Find the volume of the solid generated by revolving the region bounded by $y = \frac{4}{x^3}$ and the lines $x = 1, \quad and \quad y = \frac{1}{2}$ about

a) the x-axis;

b) the y-axis;

c) the line $x = 2$;

d) the line $y = 4$.

27. The region in the first quadrant that is bounded by the curve $y = \frac{1}{\sqrt{x}}$, on the left by the line $x = \frac{1}{4}$, and below by the line $y = 1$ is revolved about the y-axis to generate a solid. Find the volume of the solid by

a) The *shell* method

b) The *washer* method

28. The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ to generate a solid. Find the volume of the solid.

a) revolved about the x-axis

b) revolved about the y-axis

29. Find the volume of the solid generated by revolving the region bounded by $y = \sin x$ and the lines $x = 0, \quad x = \pi, \quad and \quad y = 2$ about the line $y = 2$.

30. A cylinder hole with radius r is drilled symmetrically through the center of a sphere with radius R, where $r \le R$. What is the volume of the remaining material?

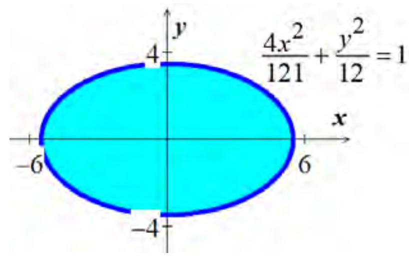
31. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y=\sqrt{x}, \quad y=2-x, \quad y=0$ about the x-axis.

32. Find the volume of the region bounded by $y=\frac{\ln x}{x^2}, \quad y=0, \quad x=1, \quad and\ x=3$ revolved about the *y-axis*

33. Find the volume of the region bounded by $y=\frac{e^x}{x}, \quad y=0, \quad x=1, \quad and\ x=2$ revolved about the *y-axis*

34. Find the volume of the region bounded by $y^2=\ln x, \quad y^2=\ln x^3, \quad and\ y=2$ revolved about the *x-axis*

35. The profile of a football resembles the ellipse. Find the football's volume to the nearest *cubic inch*.



(**36 – 38**) Find the volume using both the *disk/washer* and *shell* methods of

36. $y=(x-2)^3-2, \quad x=0, \quad y=25;$ revolved about the *y-axis*

37. $y=\sqrt{\ln x}, \quad y=\sqrt{\ln x^2}, \quad y=1;$ revolved about the *x-axis*

38. $y=\frac{6}{x+3}, \quad y=2-x;$ revolved about the *x-axis*

(**39 – 42**) Use the shell method to find the volume of the solid generated by the revolving the plane region about the given line

39. $y=2x-x^2, \quad y=0,$ *about the line* $x=4$

40. $y=\sqrt{x}, \quad y=0, \quad x=4,$ *about the line* $x=6$

41. $y=x^2, \quad y=4x-x^2,$ *about the line* $x=4$

42. $y=\frac{1}{3}x^3, \quad y=6x-x^2,$ *about the line* $x=3$

(**43 – 44**) Use the disk method or shell method to find the volume of the solid generated vy revolving the region bounded by the graph of the equations about the given lines.

43. $y=x^3, \quad y=0, \quad x=2$

a) *the x-axis* *b)* *the y-axis* *c)* *the line* $x=4$

44. $y = \dfrac{10}{x^2}, \quad y = 0, \quad x = 1, \quad x = 5$

a) *the x-axis* *b)* *the y-axis* *c)* *the line* $y = 10$

45. Let V_1 and V_2 be the volumes of the solids that result when the plane region bounded by $y = \dfrac{1}{x}$, $y = 0$, $x = \dfrac{1}{4}$, and $x = c$ $\left(\textit{where } c > \dfrac{1}{4}\right)$ is revolved about the *x-axis* and the *y-axis*, respectively. Find the value of c for which $V_1 = V_2$

46. The region bounded by $y = r^2 - x^2$, $y = 0$, and $x = 0$ is revolved about the *y-axis* to form a paraboloid. A hole, centered along the axis of revolution, is drilled through this solid. The hole has a radius k, $0 < k < r$. Find the volume of the resulting ring

a) By integrating with respect to x.

b) By integrating with respect to y.

47. The region R in the first quadrant bounded by the parabola $y = 4 - x^2$ and the coordinate axes is revolved about the y-axis to produce a dome-shaped solid. Find the volume of the solid in the following ways.

a) Apply the disk method and integrate with respect to y.

b) Apply the shell method and integrate with respect to x.

48. The region bounded by the curves $y = 1 + \sqrt{x}$, $y = 1 - \sqrt{x}$, and the line $x = 1$ is revolved about the y-axis. Find the volume of the resulting solid by

a) Integrating with respect to x and

b) Integrating with respect to y.

49. The region bounded by the graphs of $x = 0$, $x = \sqrt{\ln y}$, and $x = \sqrt{2 - \ln y}$ in the first quadrant is revolved about the y-axis. What is the volume of the resulting solid?

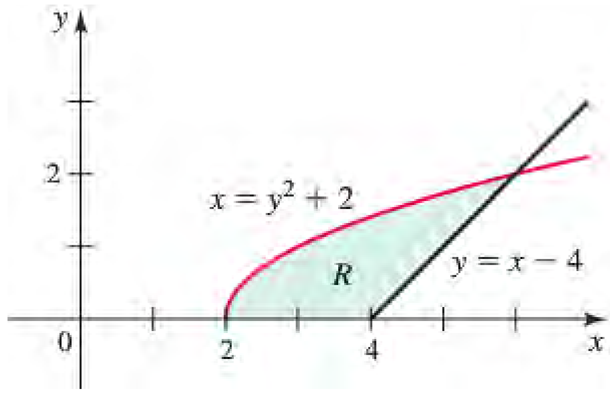
50. The region bounded by $y = \left(1 - x^2\right)^{-1/2}$ and the x-axis over the interval $\left[0, \dfrac{\sqrt{3}}{2}\right]$ is revolved about the y-axis. What is the volume of the solid that is generated?

51. The region bounded by the graph $y = 4 - x^2$ and the x-axis over the interval $[-2, 2]$ is revolved about the line $x = -2$. What is the volume of the solid that is generated?

52. The region bounded by the graph $y = 6x$ and $y = x^2 + 5$ is revolved about the line $y = -1$ and the line $x = -1$. Find the volumes of the resulting solids. Which one is greater?

53. The region bounded by the graph $y = 2x$, $y = 6 - x$ and $y = 0$ is revolved about the line $y = -2$ and the line $x = -2$. Find the volumes of the resulting solids. Which one is greater?

54. The region R is bounded by the curves $x = y^2 + 2$, $y = x - 4$, *and* $y = 0$



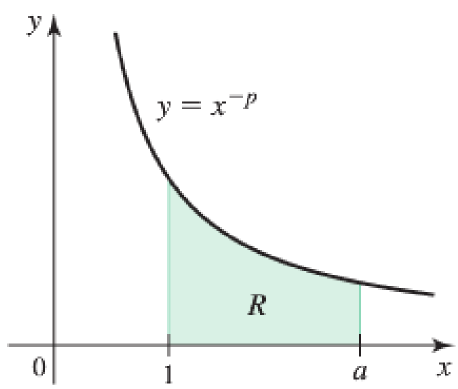
a) Write a single integral that gives the area of *R*.

b) Write a single integral that gives the volume of the solid generated when *R* is revolved about the *x*-axis.

c) Write a single integral that gives the volume of the solid generated when *R* is revolved about the *y*-axis.

d) Suppose *S* is a solid whose base is *R* and whose cross sections perpendicular to *R* and parallel to the *x*-axis are semicircles. Write a single integral that gives the volume of *S*.

55. The region R is bounded by $y = \dfrac{1}{x^p}$ and the *x*-axis on the interval $[1,\ a]$, where $p > 0$ and $a > 1$.



Let V_x and V_y be the volumes of the solids generated when R is revolved about the *x*- and *y*-axes, respectively.

a) With $a = 2$ and $p = 1$, which is greater, V_x or V_y?

b) With $a = 4$ and $p = 3$, which is greater, V_x or V_y?

c) Find a general expression for V_x in terms of *a* and *p*. Note that $p = \dfrac{1}{2}$ is a special case, what is V_x when $p = \dfrac{1}{2}$?

d) Find a general expression for V_y in terms of a and p. Note that $p = 2$ is a special case, what is V_y when $p = 2$?

e) Explain how parts (*c*) and (*d*) demonstrate that $\lim_{h \to 0} \frac{a^h - 1}{h} = \ln a$

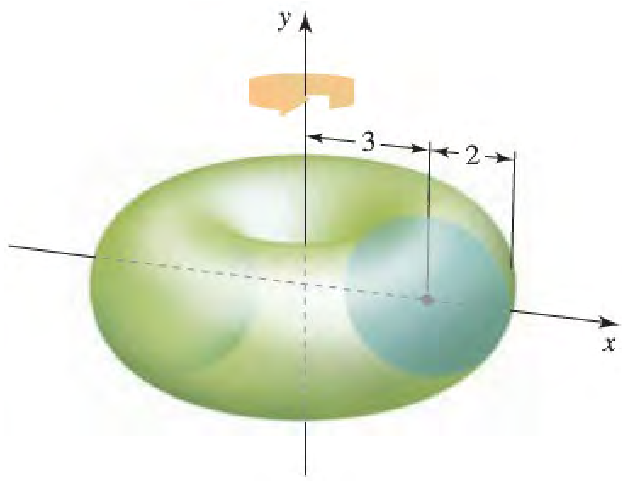
f) Find any values of a and p for which $V_x > V_y$

56. Let R be the region bounded by the graph of $f(x) = cx(1-x)$ and the x-*axis* on $[0, 1]$. Find the positive value of c such that the volume of the solid generated by revolving R about the x-*axis* equals the volume of the solid generated by revolving R about the y-*axis*.

57. Find the volume of the torus (doughnut formed when the circle of radius 2 centered at (3, 0) is revolved about the y-axis.

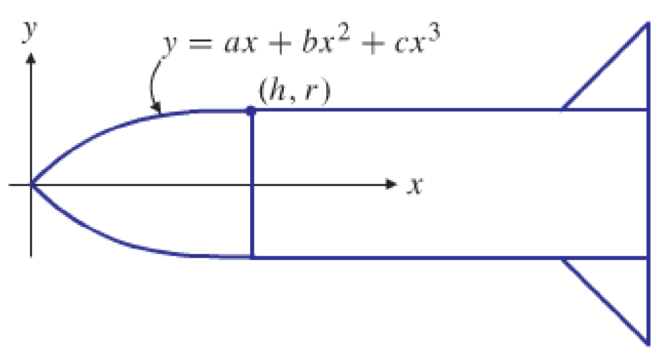
a) Use geometry to evaluate the integral

b) Use Shell method (use integral table)



58. The nose of a rocket is a solid of revolution of base radius r and height h that must join smoothly to the cylindrical body of the rocket. Taking the origin at the tip of the nose and the x-axis along the central axis of the rocket, various nose shapes can be obtained by revolving the cubic curve about x-axis.

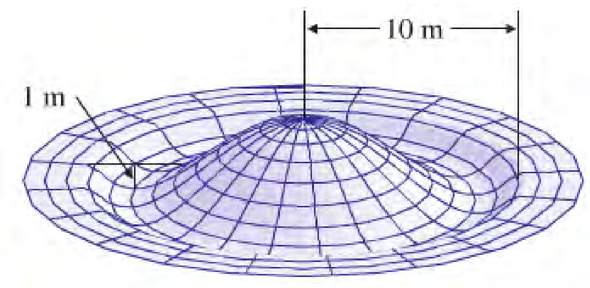
$$y = f(x) = ax + bx^2 + cx^3$$



The cubic curve must have slope 0 at $x = h$, and its slope must be positive for $0 < x < h$. Find the particular cubic curve that maximizes the volume of the nose. Also show that his choice of the cubic makes the slope $\frac{dy}{dx}$ at the origin as large as possible and, hence, corresponds to the bluntest nose.

59. A landscaper wants to create on level ground a ring-shaped pool having an outside radius of 10 *m* and a maximum depth of 1 *m* surrounding a hill that will be built up using all the earth excavated from the pool. She decided to use a fourth-degree polynomial to determine the cross-sectional shape of the hill and pool bottom: at distance ***r*** *m* from the center of the development the height above or below normal ground level will be

$$h(r) = a\left(r^2 - 100\right)\left(r^2 - k^2\right) \quad m$$



For some $a > 0$, where k is the inner radius of the pool.

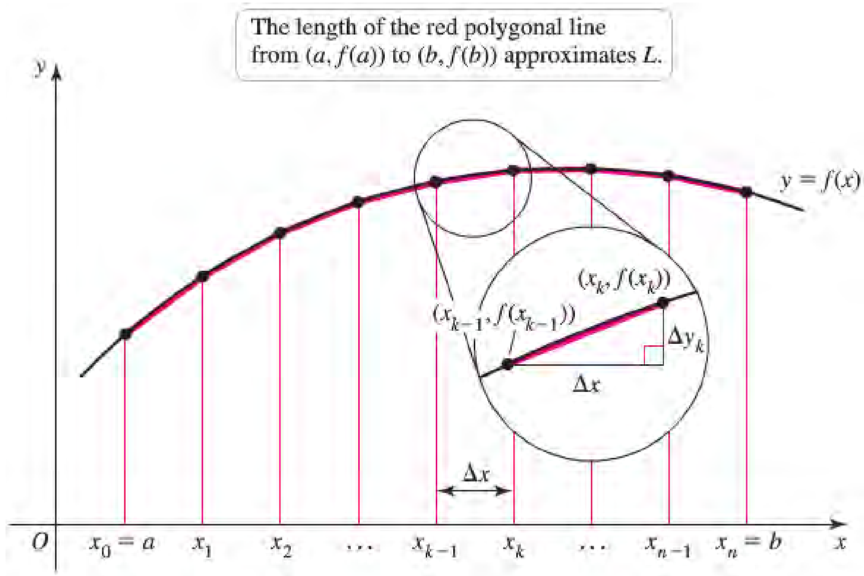
Find k and a so that the requirements given above are all satisfied.

How much earth must be moved from the pool to build the hill?

Section 1.5 – Length of Curves

Length of a curve $y = f(x)$

We assume that f has a continuous derivative at every point of $[a, b]$. Such function is called ***smooth***, and its graph is a ***smooth curve*** because it doesn't have any breaks, corners, or cusps.



Definition

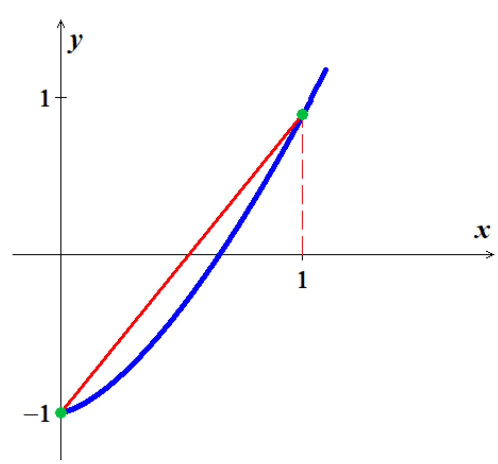
If f' is continuous on $[a, b]$, then the length (arc length) of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + \left[f'(x) \right]^2} \, dx$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \, dx$$

Example

Find the length of the curve $y = \frac{4\sqrt{2}}{3} x^{3/2} - 1, \quad 0 \le x \le 1$



Solution

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \frac{3}{2} x^{1/2}$$

$$= 2\sqrt{2} x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2}x^{1/2}\right)^2$$

$$= 8x$$

$$L = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \int_0^1 (1+8x)^{1/2}\ dx \qquad \text{or}\ \ u = 1+8x \quad du = 8dx \to dx = \frac{du}{8}$$

$$= \frac{1}{8}\int_0^1 (1+8x)^{1/2}\ d(1+8x)$$

$$= \frac{1}{8}\left(\frac{2}{3}(1+8x)^{3/2}\right|_0^1$$

$$= \frac{1}{12}\left[\left(1+8(1)\right)^{3/2} - \left(1+8(0)\right)^{3/2}\right]$$

$$= \frac{1}{12}\left((9)^{3/2} - (1)^{3/2}\right)$$

$$= \frac{1}{12}[27-1]$$

$$= \frac{1}{12}(26)$$

$$\boxed{= \frac{13}{6}\quad unit}$$

$$\boxed{\approx 2.17\quad unit}$$

***Length* of a curve** $y = f(x)$:

If $f(x) = ax^m + bx^n$, then

$$\boldsymbol{L} = \int_c^d \sqrt{1 + \left(\frac{df}{dx}\right)^2}\ dx$$

$$= \left(ax^m - bx^n \right|_c^d$$

Iff $f(x)$ satisfies these 2 conditions:

1. $m + n = 2$

2. $abmn = -\frac{1}{4}$

Proof

$$f'(x) = max^{m-1} + nbx^{n-1}$$

$$1 + (f')^2 = 1 + \left(max^{m-1} + nbx^{n-1}\right)^2$$

$$= 1 + m^2a^2x^{2m-2} + 2abmnx^{m+n-2} + n^2b^2x^{2n-2}$$

We need to combined to a perfect square $\qquad a^2 - 2ab + b^2 = (a-b)^2$

- If $x^{m+n-2} = 1 = x^0 \quad \rightarrow \quad \boxed{m + n = 2}$

$$= m^2a^2x^{2m-2} + (1 + 2abmn) + n^2b^2x^{2n-2} \qquad a^2 - 2ab + b^2 = (a-b)^2$$

- Let $1 + 2abmn = -2abmn \quad \rightarrow \quad \boxed{abmn = -\frac{1}{4}}$

$$= m^2a^2x^{2m-2} - 2abmn + n^2b^2x^{2n-2} \qquad x^{2(m+n-2)} = 1$$

$$= \left(max^{m-1} - nbx^{n-1}\right)^2$$

$$\boldsymbol{L} = \int_c^d \sqrt{\left(max^{m-1} - nbx^{n-1}\right)^2}\ dx$$

$$= \int_c^d \left(max^{m-1} - nbx^{n-1}\right) dx$$

$$= \left(ax^m - bx^n \right|_c^d \quad \checkmark$$

Example

Find the length of the graph of $f(x)=\frac{x^3}{12}+\frac{1}{x}, \quad 1\le x\le 4$

Solution

$a=\frac{1}{12}, \quad m=3, \quad b=1, \quad n=-1$

1. $m+n=3-1=2$ ✓

2. $abmn=\frac{1}{12}(1)(3)(-1)=-\frac{1}{4}$ ✓

$$L=\left(\frac{x^3}{12}-\frac{1}{x}\right|_1^4$$

$$=\left(\frac{4^3}{12}-\frac{1}{4}\right)-\left(\frac{1}{12}-\frac{1}{1}\right)$$

$$=\frac{72}{12}$$

$$=6 \quad unit$$

***Discontinuities* in $\frac{dy}{dx}$**

Formula for the length of $x=g(y), \quad c\le y\le d$

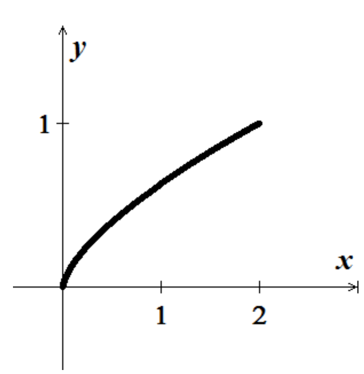
If g' is continuous on $[c, d]$, the length of the curve $x=g(y)$ from the point $A=(g(c),c)$ to the point $B=(g(d),d)$ is the value of the integral

$$L=\int_c^d \sqrt{1+\left[g'(y)\right]^2}\,dy \quad = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

Example

Find the length of the curve $y=\left(\frac{x}{2}\right)^{2/3}$ from $x = 0$ to $x = 2$.

Solution



$$\frac{dy}{dx}=\frac{2}{3}\left(\frac{x}{2}\right)^{-1/3}\left(\frac{1}{2}\right)$$

$$=\frac{1}{3}\left(\frac{2}{x}\right)^{1/3} \qquad \boxed{x\ne 0} \ (\textbf{\textit{CP}})$$

$$y=\left(\frac{x}{2}\right)^{2/3}$$ ***Raised both sides to the power 3/2***

$$y^{3/2} = \frac{x}{2}$$

$$x = 2y^{3/2}$$

$$\frac{dx}{dy} = 2\left(\frac{3}{2}\right)y^{1/2}$$

$$= 3y^{1/2}$$

$$\rightarrow \begin{cases} x = 0 & \Rightarrow y = 0 \\ x = 2 & \Rightarrow y = \left(\frac{2}{2}\right)^{2/3} = 1 \end{cases}$$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_0^1 \sqrt{1 + \left(3y^{1/2}\right)^2}\, dy$$

$$= \int_0^1 \sqrt{1 + 9y}\, dy$$

$$= \int_0^1 (1 + 9y)^{1/2}\, dy$$

$$= \frac{1}{9}\frac{2}{3}\,(1 + 9y)^{3/2}\Big|_0^1$$

$$= \frac{2}{27}\left[(1 + 9)^{3/2} - (1 + 0)^{3/2}\right]$$

$$= \frac{2}{27}\left(10^{3/2} - 1\right)$$

$$= \frac{2}{27}\left(10\sqrt{10} - 1\right) \quad unit$$

$$\approx 2.27 \ \ unit$$

If $f(x) = ae^{mx} + be^{nx}$, then

$$\boldsymbol{L} = \int_c^d \sqrt{1+\left(\frac{df}{dx}\right)^2}\ dx$$

$$= \left(ae^{mx} - be^{nx}\right|_c^d$$

Iff $f(x)$ satisfies these 2 conditions:

1. $m = -n$
2. $abmn = -\frac{1}{4}$

Proof

$$f'(x) = ame^{mx} + bne^{nx}$$

$$1 + (f')^2 = 1 + \left(ame^{mx} + bne^{nx}\right)^2$$

$$= 1 + m^2a^2e^{2mx} + 2abmne^{(m+n)x} + n^2b^2e^{2nx}$$

- If $e^{(m+n)x} = 1 = e^{(x=0)} \quad \rightarrow \quad \boxed{m-n}$

$$= m^2a^2x^{2m-2} + (1 + 2abmn) + n^2b^2x^{2n-2} \qquad a^2 - 2ab + b^2 = (a-b)^2$$

- Let $\quad 1 + 2abmn = -2abmn \quad \rightarrow \quad \boxed{abmn = -\frac{1}{4}}$

$$= m^2a^2e^{2mx} - 2abmne^{(m+n)x} + n^2b^2e^{2nx} \qquad x^{2(m+n-2)} = 1$$

$$= \left(ame^{mx} - bne^{nx}\right)^2$$

$$\left(ame^{mx} - bne^{nx}\right)^2 = m^2a^2e^{2mx} - 2abmne^{(m+n)x} + n^2b^2e^{2nx}$$

$$\boldsymbol{L} = \int_c^d \sqrt{\left(ame^{mx} - bne^{nx}\right)^2}\ dx$$

$$= \int_c^d \left(ame^{mx} - bne^{nx}\right)\ dx$$

$$= \left(ame^{mx} - bne^{nx}\right|_c^d \quad \checkmark$$

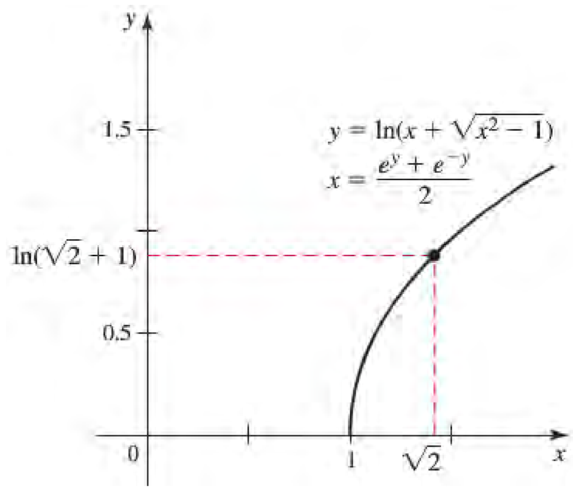
Example

Find the arc length function for the curve $f(x)=\ln\left(x+\sqrt{x^2-1}\right)$ on the interval $\left[1,\ \sqrt{2}\right]$

Solution

$$f'(x)=\frac{1+x\left(x^2-1\right)^{-1/2}}{x+\sqrt{x^2-1}}$$

$$=\frac{\sqrt{x^2-1}+x}{x\sqrt{x^2-1}+x^2-1}$$



$$y=\ln\left(x+\sqrt{x^2-1}\right)$$

$$x+\sqrt{x^2-1}=e^y$$

$$\left(\sqrt{x^2-1}\right)^2=\left(e^y-x\right)^2$$

$$x^2-1=e^{2y}-2xe^y+x^2$$

$$2xe^y=e^{2y}+1$$

$$x=\frac{e^{2y}+1}{2e^y}\left(\frac{e^y}{e^y}\right)$$

$$=\frac{e^y+e^{-y}}{2}$$

$$y=\ln\left(x+\sqrt{x^2-1}\right)$$

$$\boxed{x=\frac{e^y+e^{-y}}{2}=g(y)}$$

$$x=1 \quad\to\quad y=0$$

$$x=\sqrt{2} \quad\to\quad y=\ln\left(\sqrt{2}+1\right)$$

$$a=\frac{1}{2},\quad m=1,\quad b=\frac{1}{2},\quad n=-1$$

1. $m=-n$ ✓

2. $abmn=-\frac{1}{4}$ ✓

$$L=\frac{1}{2}\left(e^y-e^{-y}\right.\Big|_0^{\ln\left(\sqrt{2}+1\right)}$$

$$=\frac{1}{2}\left(\sqrt{2}+1-\frac{1}{\sqrt{2}+1}-1+1\right)$$

$$=\frac{1}{2}\left(\frac{3+2\sqrt{2}-1}{\sqrt{2}+1}\right)$$

$$=\frac{1}{2}\left(\frac{2+2\sqrt{2}}{\sqrt{2}+1}\right)$$

$$\underline{=1 \quad unit}$$

OR

$$L=\int_0^{\ln\left(\sqrt{2}+1\right)}\sqrt{1+g'(y)^2}\;dy$$

$$=\int_0^{\ln\left(\sqrt{2}+1\right)}\sqrt{1+\left(\frac{e^y-e^{-y}}{2}\right)^2}\;dy$$

$$=\frac{1}{2}\int_0^{\ln\left(\sqrt{2}+1\right)}\sqrt{4+e^{2y}-2+e^{-2y}}\;dy$$

$$=\frac{1}{2}\int_0^{\ln\left(\sqrt{2}+1\right)}\sqrt{e^{2y}+2+e^{-2y}}\;dy$$

$$=\frac{1}{2}\int_0^{\ln\left(\sqrt{2}+1\right)}\sqrt{\left(e^y+e^{-y}\right)^2}\;dy$$

$$=\frac{1}{2}\int_0^{\ln\left(\sqrt{2}+1\right)}\left(e^y+e^{-y}\right)\;dy$$

$$=\frac{1}{2}\left(e^y-e^{-y}\right)\Big|_0^{\ln\left(\sqrt{2}+1\right)}$$

$$=\frac{1}{2}\left(\sqrt{2}+1-\frac{1}{\sqrt{2}+1}-1+1\right)$$

$$=\frac{1}{2}\left(\frac{3+2\sqrt{2}-1}{\sqrt{2}+1}\right)$$

$$=\frac{1}{2}\left(\frac{2+2\sqrt{2}}{\sqrt{2}+1}\right)$$

$$\underline{=1 \quad unit}$$

The differential Formula for Arc length

If $y = f(x)$ and if f' is continuous on [a, b], then by the Fundamental Theorem of Calculus, we can define a new function

$$s(x) = \int_a^x \sqrt{1+\left[f'(t)\right]^2}\, dt$$

$$\frac{ds}{dx} = \sqrt{1+\left[f'(t)\right]^2}$$
$$= \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$
$$= \sqrt{(dx)^2+(dx)^2\frac{(dy)^2}{(dx)^2}}$$

$$\boxed{ds = \sqrt{dx^2+dy^2}}$$

Example

Find the arc length function for the curve $f(x) = \frac{x^3}{12}+\frac{1}{x}$ taking $A = \left(1,\ \frac{13}{12}\right)$ as the starting point

Solution

$$1+\left[f'(x)\right]^2 = \left(\frac{x^2}{4}+\frac{1}{x^2}\right)^2$$

$$s(x) = \int_1^x \sqrt{1+\left[f'(t)\right]^2}\, dt$$
$$= \int_1^x \left(\frac{t^2}{4}+\frac{1}{t^2}\right) dt$$
$$= \left(\frac{t^3}{12}-\frac{1}{t}\right|_1^x$$
$$= \left(\frac{x^3}{12}-\frac{1}{x}\right)-\left(\frac{1}{12}-1\right)$$
$$= \frac{x^3}{12}-\frac{1}{x}+\frac{11}{12}$$

$$s(4) = \frac{4^3}{12} - \frac{1}{4} + \frac{11}{12}$$

$$\underline{= 6 \quad unit}$$

Exercises *Section* 1.5 – Length of Curves

(1 – 30) Find the length of the curve of

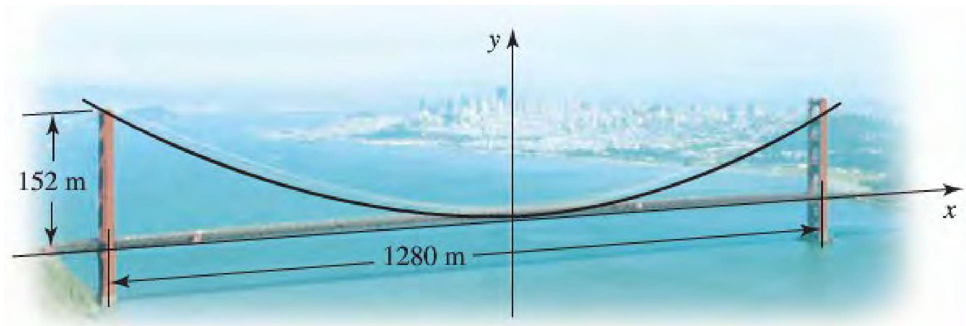
1. $y=\frac{1}{3}\left(x^2+2\right)^{3/2}$ from $x = 0$ to $x = 3$
2. $y=(x)^{3/2}$ from $x = 0$ to $x = 4$
3. $x=\frac{y^{3/2}}{3}-y^{1/2}$ from $y = 1$ to $y = 9$
4. $x=\frac{y^3}{6}+\frac{1}{2y}$ from $y = 2$ to $y = 3$
5. $f(x)=x^3+\frac{1}{12x}$ *for* $\frac{1}{2}\le x\le 2$
6. $f(x)=\frac{1}{5}x^5+\frac{1}{12x^3}$ $1\le x\le 2$
7. $y=\frac{1}{3}x^{1/2}-x^{3/2}, \quad 0\le x\le \frac{1}{3}$
8. $y=\frac{1}{3}x^3+\frac{1}{4x}, \quad 1\le x\le 2$
9. $y=2e^x+\frac{1}{8}e^{-x}$ $0\le x\le \ln 2$
10. $y=e^{2x}+\frac{1}{16}e^{-2x}$ $0\le x\le \ln 3$
11. $y=\ln(\cos x)$ $0\le x\le \frac{\pi}{4}$
12. $f(y)=2e^{\sqrt{2}y}+\frac{1}{16}e^{-\sqrt{2}y}$ $0\le y\le \frac{\ln 2}{\sqrt{2}}$
13. $y=\frac{x^3}{3}+x^2+x+1+\frac{1}{4x+4}$ $0\le x\le 2$
14. $y=\frac{x^3}{3}+x^2+x+1+\frac{1}{4x+4}$ $0\le x\le 4$
15. $y=\ln\left(e^x-1\right)-\ln\left(e^x+1\right)$ $\ln 2\le x\le \ln 3$
16. $f(x)=\frac{2}{3}x^{3/2}-\frac{1}{2}x^{1/2}$ $1\le x\le 4$
17. $f(x)=x^3+\frac{1}{12x}$ $1\le x\le 4$
18. $f(x)=\frac{1}{8}x^4+\frac{1}{4x^2}$ $1\le x\le 10$
19. $f(x)=\frac{1}{4}x^4+\frac{1}{8x^2}$ $3\le x\le 8$
20. $f(x)=\frac{1}{10}x^5+\frac{1}{6x^3}$ $1\le x\le 7$
21. $f(x)=\frac{3}{10}x^{1/3}-\frac{3}{2}x^{5/3}$ $0\le x\le 12$
22. $f(x)=x^{1/2}-\frac{1}{3}x^{3/2}$ $2\le x\le 9$
23. $y=x^{1/2}-\frac{1}{3}x^{3/2}$ $1\le x\le 4$
24. $x=y^{2/3}, \quad 1\le y\le 8$
25. $y=2x+4$ $-2\le x\le 2$
26. $y=\frac{x^3}{6}+\frac{1}{2x}$ $x\in[1,\ 2]$
27. $f(x)=x^{1/2}-\frac{1}{3}x^{3/2}$ $1\le x\le 3$
28. $y=\frac{3}{4}x^{4/3}-\frac{3}{8}x^{2/3}+5, \quad 1\le x\le 8$
29. $y=\ln x-\frac{1}{8}x^2; \quad 1\le x\le 2$
30. $y=\frac{1}{2}x^2-\frac{1}{4}\ln x; \quad 1\le x\le 3$

31. Find the length of the curve $y=\int_{-2}^{x}\sqrt{2t^4-1}\,dt$ $\quad -2\le x\le -1$

32. Find the length of the curve $x=\int_{0}^{y}\sqrt{\sec^4 t-1}\,dt$ $\quad -\frac{\pi}{4}\le y\le \frac{\pi}{4}$

33. Find the length of the curve $y = 3 - 2x \quad 0 \le x \le 2$. Check your answer by finding the length of the segment as the hypotenuse of a right triangle.

34. The profile of the cables on a suspension bridge may be modeled by a parabola. The central span of the Golden Gate Bridge is 1280 *m* long and 152 *m* high. The parabola $y = 0.00037x^2$ gives a good fit to the shape of the cables, where $|x| \le 640$, and *x* and *y* are measured in meters. Approximate the length of the cables that stretch between the tops of the two towers.

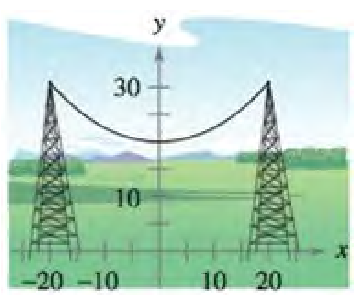


35. Find a curve through the origin in the *xy*-plane whose length from *x* = 0 to *x* = 1 is

$$L = \int_0^1 \sqrt{1 + \tfrac{1}{4}e^x}\, dx$$

36. Confirm that the circumference of a circle of radius *a* is $2\pi a$

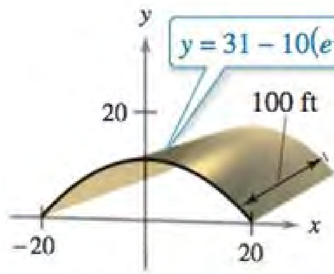
37. Electrical wires suspended between two towers form a caternary modeled by the equation



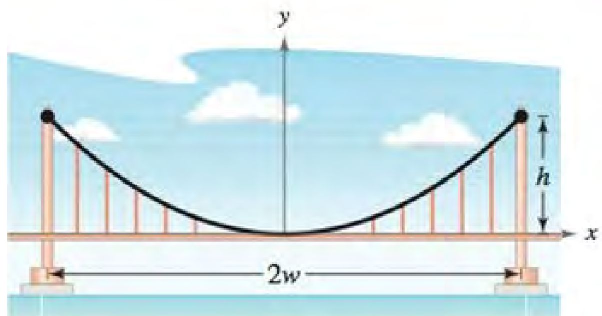
$$y = 20\cosh\frac{x}{20}, \quad -20 \le x \le 20$$

Where *x* and *y* are measured in meters. The towers are 40 meters apart. Find the length of the suspended cable.

38. A barn is 100 feet long and 40 feet wide. A cross section of the roof is the inverted caternary $y = 31 - 10\left(e^{x/20} + e^{-x/20}\right)$. Find the number of square feet of roofing on the barn.



39. A cable for a suspension bridge has the shape of a parabola with equation $y = kx^2$. Let h represent the height of the cable from it lowest point to its highest point and let $2w$ represent the total span of the bridge.



Show that the length C of the cable is given by $C = 2\int_0^w \sqrt{1 + \frac{4h^2}{w^4}x^2}\, dx$

40. Find the total length of the graph of the astroid $x^{2/3} + y^{2/3} = 4$

41. Find the arc length from $(0,\ 3)$ clockwise to $(2,\ \sqrt{5})$ along the circle $x^2 + y^2 = 9$

42. Find the arc length from $(-3,\ 4)$ clockwise to $(4,\ 3)$ along the circle $x^2 + y^2 = 25$. Show that the result is one-fourth the circumference of the circle.

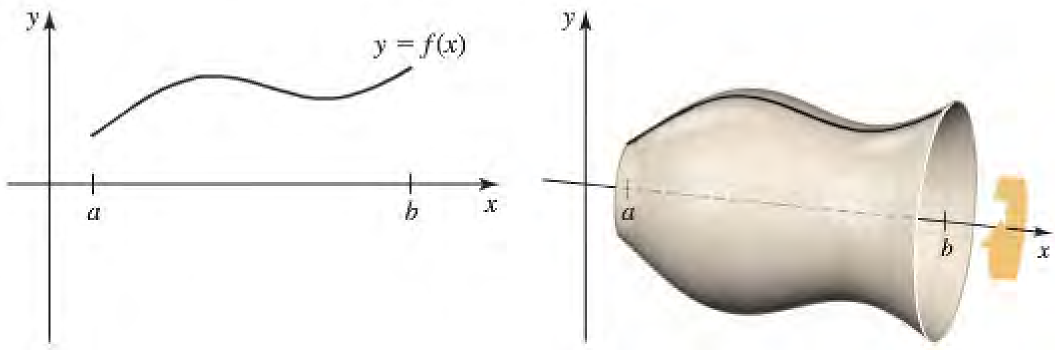
43. $y = \ln x$ between $x = 1$ and $x = b > 1$ that

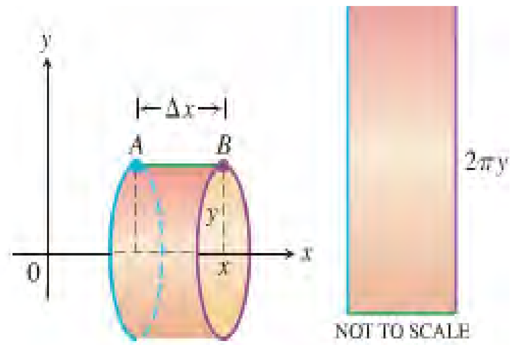
$$\int \frac{\sqrt{x^2 + a^2}}{x}dx = \sqrt{x^2 + a^2} - a\ln\left(\frac{a + \sqrt{x^2 + a^2}}{x}\right) + C$$

Use any means to approximate the value of b for which the curve has length 2.

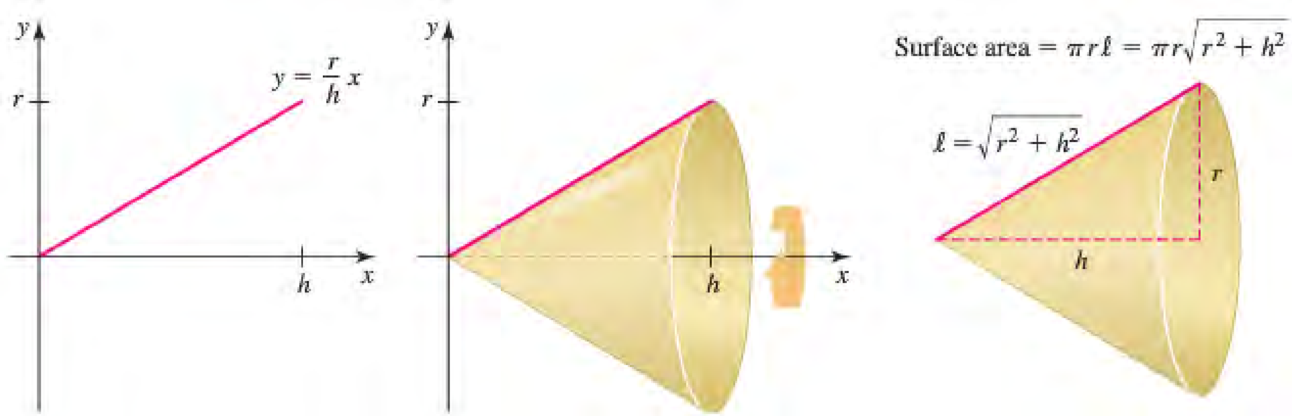
Section 1.6 – Surface Area

Consider a curve $y = f(x)$ on an interval $[a, b]$, where *f* is a nonnegative function with a continuous first derivative on $[a, b]$. Revolving the curve about the *x-axis* to generate a surface of revolution.



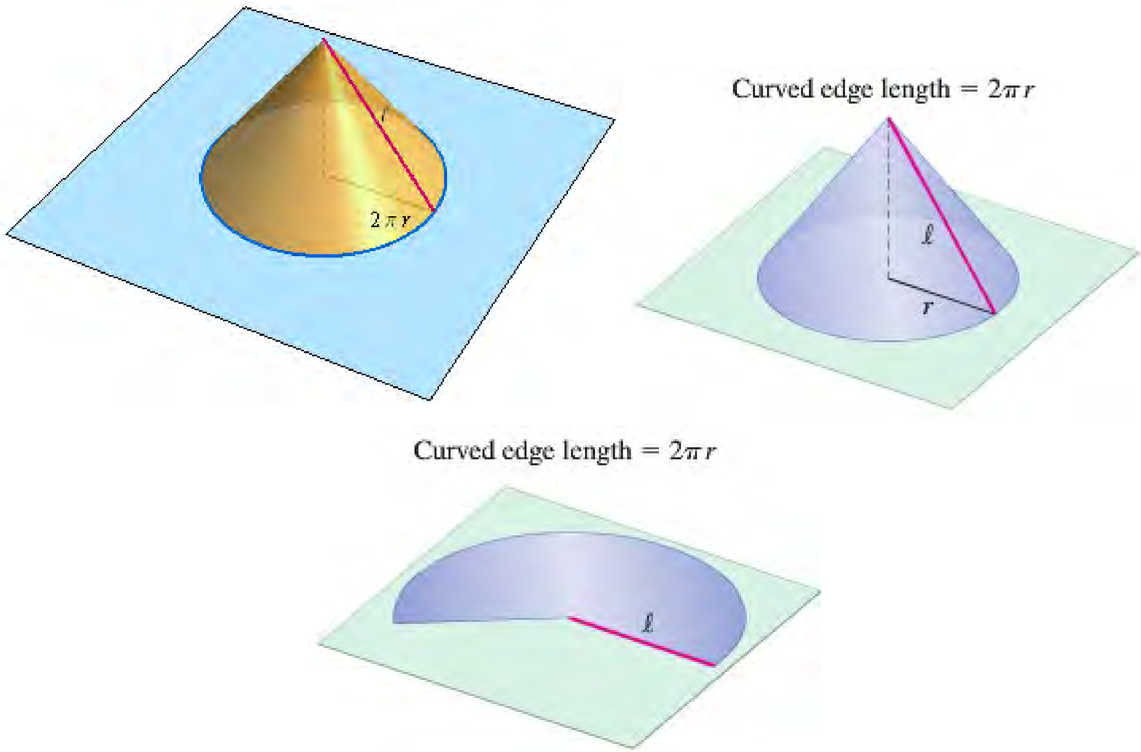


Consider the graph of $f(x) = \frac{r}{h}x$ on the interval $[0, h]$, where $h > 0$ and $r > 0$. When this line segment is revolved about the *x-axis*, it generates the surface of a cone of radius *r* and height *h*,

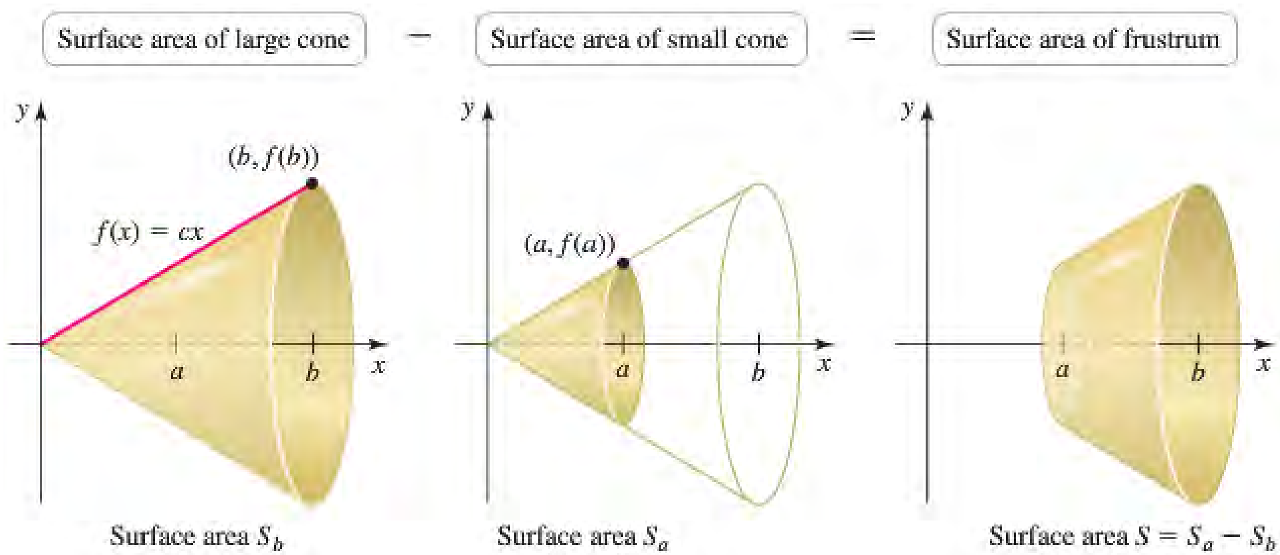


The surface area of a right circular cone, excluding the base, is $\pi r\sqrt{r^2 + h^2} = \pi r\ell$

One way to derive the formula for the surface area of a cone to cut the cone on a line from its base to its vertex. When the cone is unfolded it forms a sector of a circular disk of radius ℓ. So the area of the sector, which is also the surface area of the cone, is $\pi\ell^2 \frac{r}{\ell} = \pi r\ell$



2πr
Curved edge length = 2πr
ℓ
r
Curved edge length = 2πr
ℓ



Surface area of large cone
−
Surface area of small cone
=
Surface area of frustrum
y
(b, f(b))
f(x) = cx
a
b
x
Surface area S_b
y
(a, f(a))
a
b
x
Surface area S_a
y
a
b
x
Surface area $S = S_a - S_b$

Definition

If the function $f(x) \geq 0$ is continuously differentiable on [*a*, *b*], the area of the surface generated by revolving the graph of $y = f(x)$ about the *x-axis* is

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_a^b f(x) \sqrt{1 + \left(f'(x)\right)^2}\, dx$$

Example

Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}, \quad 1 \leq x \leq 3$, about the *x-axis* .

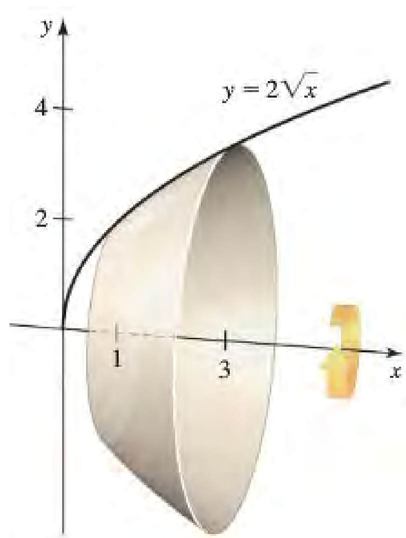
Solution

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}}, \quad a = 1, \quad b = 3$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}$$

$$= \sqrt{1 + \frac{1}{x}}$$

$$= \sqrt{\frac{x+1}{x}}$$



$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 4\pi \int_1^3 \left(\sqrt{x}\right) \frac{\sqrt{x+1}}{\sqrt{x}}\, dx$$

$$= 4\pi \int_1^3 (x+1)^{1/2}\, dx$$

$$= \frac{8\pi}{3} (x+1)^{3/2} \Big|_1^3$$

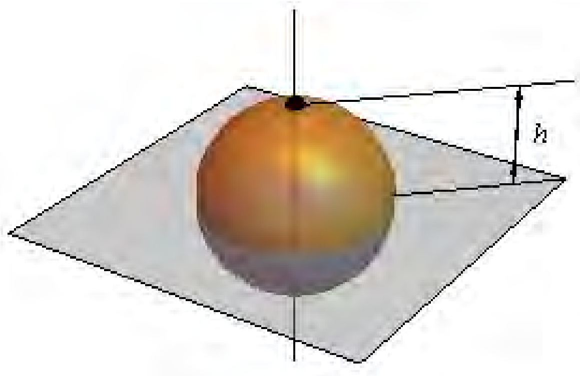
$$= \frac{8\pi}{3}\left(4^{3/2} - 2^{3/2}\right)$$

$$= \frac{8\pi}{3}\left(8 - 2\sqrt{2}\right)$$

$$\boxed{= \frac{16\pi}{3}\left(4 - \sqrt{2}\right) \quad unit^2}$$

Example

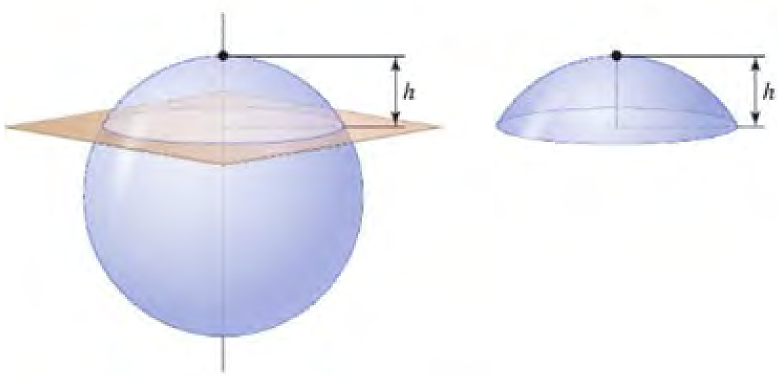
A spherical cap is produced when a sphere of radius a is sliced by a horizontal plane that is a vertical distance h below the north pole of the sphere, where $0 \le h \le 2a$. We take the spherical cap to be that part of the sphere above the plane, so that h is the depth of the cap.



Show that the area of a spherical cap of depth h cut from sphere of radius a is $2\pi ah$.

Solution

To generate the spherical surface, we revolved the curve $f(x) = \sqrt{a^2 - x^2}$ on the interval $[-a,\ a]$ about the $x\text{-}axis$.

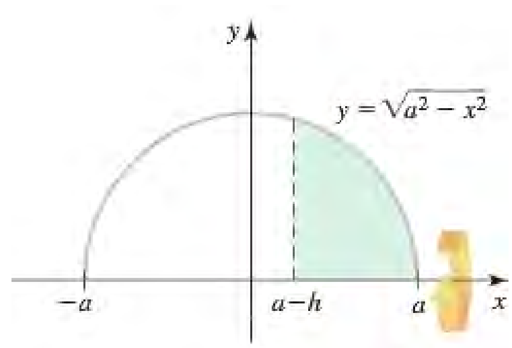


The spherical cap of height h corresponds to that part of the sphere on the interval $[-a+h,\ a]$ for $0 \le h \le 2a$

$$f'(x) = -x\left(a^2 - x^2\right)^{-1/2}$$

$$1 + f'(x)^2 = 1 + \frac{x^2}{a^2 - x^2}$$

$$= \frac{a^2}{a^2 - x^2}$$



$$S = 2\pi \int_a^b f(x)\,\sqrt{1 + f'(x)^2}\,dx$$

$$= 2\pi \int_{a-h}^{a} \sqrt{a^2 - x^2}\,\frac{a}{\sqrt{a^2 - x^2}}\,dx$$

$$= 2\pi \int_{a-h}^{a} a\,dx$$

$$= 2\pi a x \Big|_{a-h}^{a}$$

$$\boxed{= 2\pi a h \quad unit^2}$$

Surface Area for revolution about the y-axis

If $x = g(y) \geq 0$ is continuously differentiable on [*c*, *d*], the area of the surface generated by revolving the graph of $x = g(y)$ about the *y*-axis is

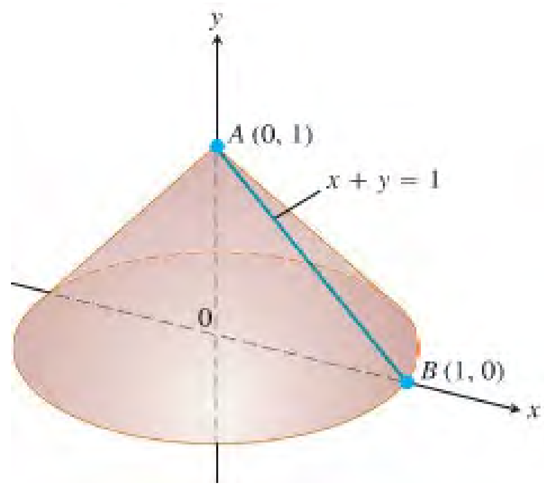
$$S = 2\pi \int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

$$= 2\pi \int_c^d g(y)\sqrt{1+\left(g'(y)\right)^2}\ dy$$

Example

The line segment $x = 1 - y, \quad 0 \leq y \leq 1$, is revolved about the *y*-axis to generate the cone. Find its lateral surface area (which excludes the base area)

Solution



$$Lateral\ Surface\ Area = \frac{base\ circumference}{2} \times slant\ height$$

$$= \pi\sqrt{2}$$

$$x = 1 - y \quad \frac{dx}{dy} = -1, \quad c = 0, \quad d = 1$$

$$S = \int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

$$= \int_0^1 2\pi(1-y)\sqrt{1+(-1)^2}\ dy$$

$$= 2\pi \int_0^1 (1-y)\sqrt{2}\ dy$$

$$= 2\pi\sqrt{2}\left(y - \frac{y^2}{2}\right)\Big|_0^1$$

$$= 2\pi\sqrt{2}\left(1 - \frac{1}{2}\right)$$

$$= \pi\sqrt{2} \quad unit^2$$

Formula

Surface **of a curve** $y = f(x)$ is given by the formula:

$$S = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx$$

If $f(x) = ax^m + bx^n$, then

$$\sqrt{1+(f'(x))^2} = \overline{f'(x)}$$

$\overline{f'(x)}$: is the conjugate of $f'(x)$

Iff $f(x)$ satisfies these 2 conditions:

1. $m+n=2$
2. $abmn = -\frac{1}{4}$

Proof

$$f'(x) = max^{m-1} + nbx^{n-1}$$

$$1+(f')^2 = 1+\left(max^{m-1}+nbx^{n-1}\right)^2$$

$$= 1+m^2a^2x^{2m-2} + 2abmnx^{m+n-2} + n^2b^2x^{2n-2}$$

We need to combined to a perfect square $a^2-2ab+b^2 = (a-b)^2$

➢ If $x^{m+n-2} = 1 = x^0 \rightarrow \boxed{m+n=2}$

$$= m^2a^2x^{2m-2} + (1+2abmn) + n^2b^2x^{2n-2}$$ $a^2-2ab+b^2=(a-b)^2$

➢ Let $1+2abmn = -2abmn \rightarrow \boxed{abmn = -\frac{1}{4}}$

$$= m^2a^2x^{2m-2} - 2abmn + n^2b^2x^{2n-2}$$ $x^{2(m+n-2)} = 1$

$$= \left(max^{m-1} - nbx^{n-1}\right)^2$$

$$\sqrt{\left(max^{m-1}-nbx^{n-1}\right)^2} = max^{m-1} - nbx^{n-1}$$ ✓

$$f'(x) = max^{m-1} + nbx^{n-1}$$

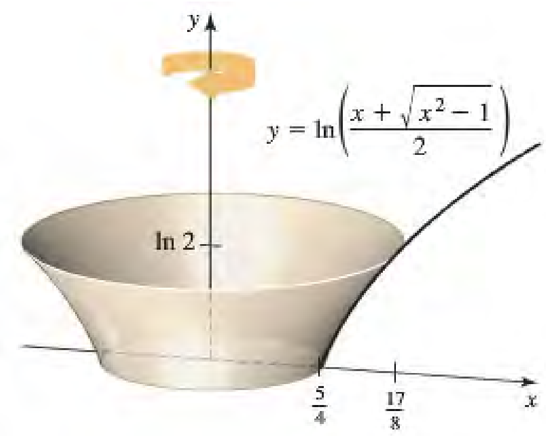
$$\sqrt{1+(f'(x))^2} = max^{m-1} - nbx^{n-1} = \overline{f'(x)}$$

Example

Consider the function $y=\ln\left(\dfrac{x+\sqrt{x^2-1}}{2}\right)$

Find the area of the surface generated when the part of the curve between the points $\left(\frac{5}{4},\ 0\right)$ and $\left(\frac{17}{8},\ \ln 2\right)$ is revolved about *y-axis*.

Solution



$$y=\ln\left(\frac{x+\sqrt{x^2-1}}{2}\right)$$

$$e^y=\frac{x+\sqrt{x^2-1}}{2}$$

$$\left(2e^y-x\right)^2=\left(\sqrt{x^2-1}\right)^2$$

$$4e^{2y}-4xe^y+x^2=x^2-1$$

$$4xe^y=4e^{2y}+1$$

$$x=e^y+\frac{1}{4}e^{-y}=g(y)$$

$$\boxed{g'(y)=e^y-\frac{1}{4}e^{-y}}$$

$$a=\frac{1}{2},\quad m=1,\quad b=\frac{1}{2},\quad n=-1$$

1. $m=-n$ ✓

2. $abmn=-\frac{1}{4}$ ✓

$$S=2\pi\int_0^{\ln 2}\left(e^y+\frac{1}{4}e^{-y}\right)\left(e^y+\frac{1}{4}e^{-y}\right)dy$$

$$=2\pi\int_0^{\ln 2}\left(e^{2y}+\frac{1}{2}+\frac{1}{16}e^{-2y}\right)dy$$

$$=2\pi\left(\frac{1}{2}e^{2y}+\frac{y}{2}-\frac{1}{32}e^{-2y}\right)\Bigg|_0^{\ln 2}$$

$$=2\pi\left(2+\frac{\ln 2}{2}-\frac{1}{128}-\frac{1}{2}+\frac{1}{32}\right)$$

$$\boxed{=\pi\left(\frac{195}{64}+\ln 2\right)\quad unit^2}$$

OR — · · — · · — · · — ·

$$\sqrt{1+g'(y)^2} = \sqrt{1+\left(e^y - \frac{1}{4}e^{-y}\right)^2}$$

$$= \sqrt{1+e^{2y} - \frac{1}{2} + \frac{1}{16}e^{-2y}}$$

$$= \sqrt{e^{2y} + \frac{1}{2} + \frac{1}{16}e^{-2y}}$$

$$= \sqrt{\left(e^y + \frac{1}{4}e^{-y}\right)^2}$$

$$= e^y + \frac{1}{4}e^{-y}$$

$$S = 2\pi \int_0^{\ln 2} \left(e^y + \frac{1}{4}e^{-y}\right)^2 \, dy$$

$$= 2\pi \int_0^{\ln 2} \left(e^{2y} + \frac{1}{2} + \frac{1}{16}e^{-2y}\right) dy$$

$$= 2\pi \left(\frac{1}{2}e^{2y} + \frac{y}{2} - \frac{1}{32}e^{-2y}\right)\Bigg|_0^{\ln 2}$$

$$= 2\pi \left(2 + \frac{\ln 2}{2} - \frac{1}{128} - \frac{1}{2} + \frac{1}{32}\right)$$

$$= \pi \left(\frac{195}{64} + \ln 2\right) \quad unit^2$$

Exercises *Section* 1.6 – Surface Area

1. Find the lateral (side) surface area of the cone generated by revolving the line segment $y=\frac{x}{2},\quad 0\le x\le 4$, about the *x*-axis. Check your answer with the geometry formula

$$\textit{Lateral surface area}=\frac{1}{2}\times \textit{base circumference}\times \textit{slant height}$$

2. Find the lateral surface area of the cone generated by revolving the line segment $y=\frac{x}{2},\quad 0\le x\le 4$, about the *y*-axis. Check your answer with the geometry formula

$$\textit{Lateral surface area}=\frac{1}{2}\times \textit{base circumference}\times \textit{slant height}$$

3. Find the lateral surface area of the cone frustum generated by revolving the line segment $y=\frac{x}{2}+\frac{1}{2},\quad 1\le x\le 3$, about the *x*-axis. Check your answer with the geometry formula

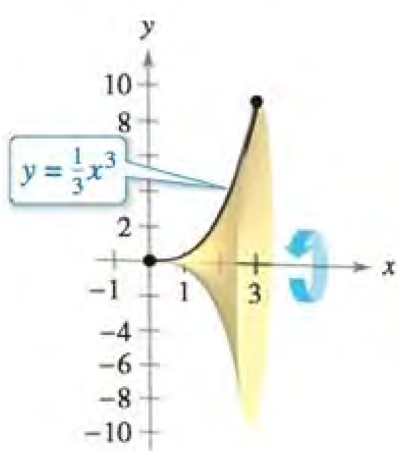
$$\textit{Frustum surface area}=\pi\left(r_1+r_2\right)\times \textit{slant height}$$

4. Find the lateral surface area of the cone frustum generated by revolving the line segment $y=\frac{x}{2}+\frac{1}{2},\quad 1\le x\le 3$, about the *y*-axis. Check your answer with the geometry formula

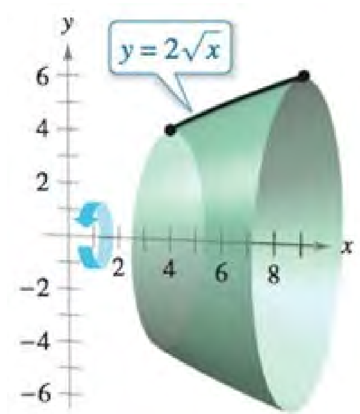
$$\textit{Frustum surface area}=\pi\left(r_1+r_2\right)\times \textit{slant height}$$

(5 – 22) Find the area of the surface generated by revolving the curve about the *x-axis*

5.



6.



7. $y=\frac{x^3}{9},\quad 0\le x\le 2$

8. $y=\sqrt{x+1},\quad 1\le x\le 5$

9. $y=\sqrt{2x-x^2},\quad 0.5\le x\le 1.5$

10. $y=3x+4,\quad 0\le x\le 6$

11. $y=12-3x,\quad 1\le x\le 3$

12. $y=x^{3/2}-\frac{1}{3}x^{1/2},\quad 1\le x\le 2$

13. $y=\sqrt{4x+6},\quad 0\le x\le 5$

14. $y=\frac{1}{4}\left(e^{2x}+e^{-2x}\right),\quad -2\le x\le 2$

15. $y=\frac{1}{8}x^4+\frac{1}{4x^2},\quad 1\le x\le 2$

16. $y = 8\sqrt{x}, \quad 9 \le x \le 20$

17. $y = x^3, \quad 0 \le x \le 1$

18. $y = \frac{1}{3}x^3 + \frac{1}{4x}, \quad \frac{1}{2} \le x \le 2$

19. $y = \sqrt{5x - x^2}, \quad 1 \le x \le 4$

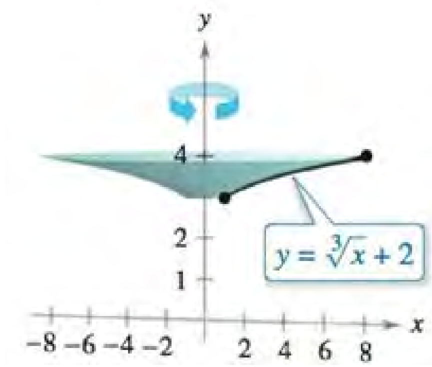
20. $y = \frac{1}{6}x^3 + \frac{1}{2x}, \quad 1 \le x \le 2$

21. $y = \sqrt{4 - x^2}, \quad -1 \le x \le 1$

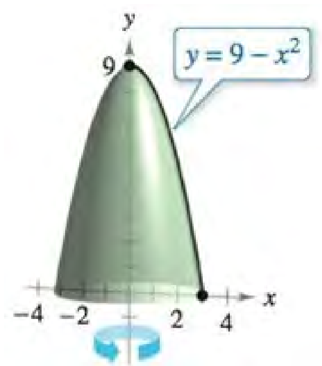
22. $y = \sqrt{9 - x^2}, \quad -2 \le x \le 2$

(23 – 29) Find the area of the surface generated by revolving the curve about the y-*axis*

23.



24.



25. $y = (3x)^{1/3}; \quad 0 \le x \le \frac{8}{3}$

26. $x = \sqrt{12y - y^2}; \quad 2 \le y \le 10$

27. $x = 4y^{3/2} - \frac{1}{12}y^{1/2}; \quad 1 \le y \le 4$

28. $y = 1 - \frac{1}{4}x^2, \quad 0 \le x \le 2$

29. $y = \frac{1}{2}x + 3, \quad 1 \le x \le 5$

30. A right circular cone is generated by revolving the region bounded by $y = \frac{3}{4}x$, $y = 3$, and $x = 0$ about the y-*axis*. Find the lateral surface area of the cone.

31. A right circular cone is generated by revolving the region bounded by $y = \frac{h}{r}x$, $y = h$, and $x = 0$ about the y-*axis*. Verify that the lateral surface area of the cone is $S = \pi r\sqrt{r^2 + h^2}$

32. Find the area of the zone of a sphere formed by revolving the graph of $y = \sqrt{9 - x^2}, \quad 0 \le x \le 2$, about the y-*axis*

33. Find the area of the zone of a sphere formed by revolving the graph of $y = \sqrt{r^2 - x^2}, \quad 0 \le x \le a$, about the y-*axis*. Assume that $a < r$.

34. Find the area of the surface generated by part of the curve $y = 4x - 1$ between the points $(1, 3)$ and $(4, 15)$ *about y-axis*

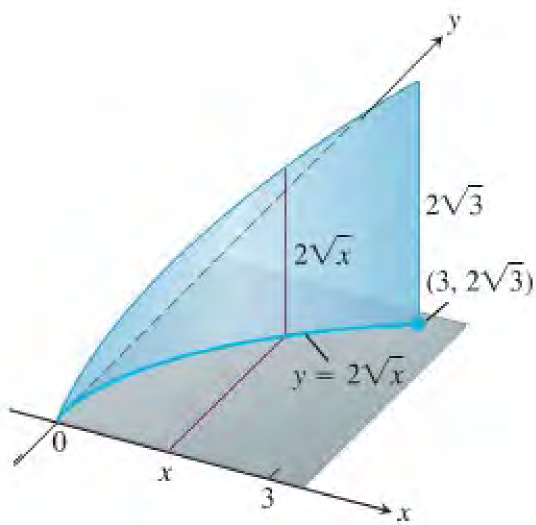
35. Find the area of the surface generated by part of the curve $y=\frac{1}{2}\ln\left(2x+\sqrt{4x^2-1}\right)$ between the points $\left(\frac{1}{2},\ 0\right)$ and $\left(\frac{17}{16},\ \ln 2\right)$ *about y-axis*

36. Find the area of the surface generated by $y=1+\sqrt{1-x^2}$ between the points $(1,\ 1)$ and $\left(\frac{\sqrt{3}}{1},\ \frac{3}{2}\right)$ *about y-axis*

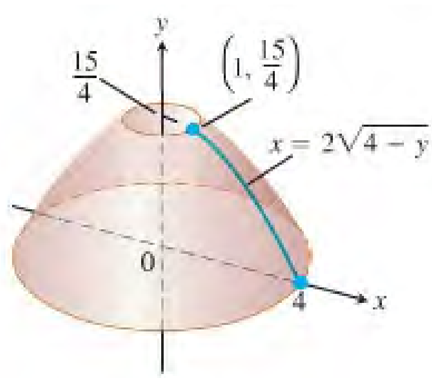
37. Find the area of the surface generated by $y=\frac{1}{3}x^3, \quad 0\le x\le 1, \quad x-axis$

38. Find the area of the surface generated by $x=\sqrt{4y-y^2}, \quad 1\le y\le 2; \quad y-axis$

39. At points on the curve $y=2\sqrt{x}$, line segments of length $h=y$ are drawn perpendicular to the xy-plane. Find the area of the surface formed by these perpendiculars from $(0,\ 0)$ *to* $\left(3,\ 2\sqrt{3}\right)$

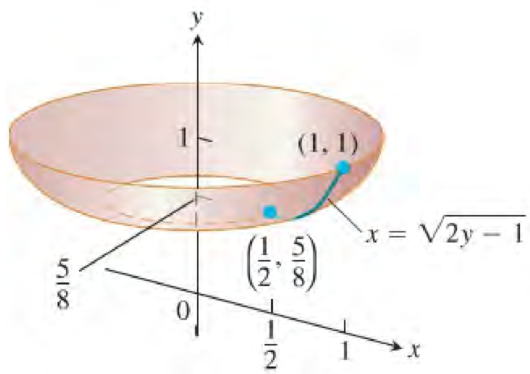


40. Find the area of the surface generated by $x=2\sqrt{4-y} \quad 0\le y\le \frac{15}{4}, \quad y-axis$

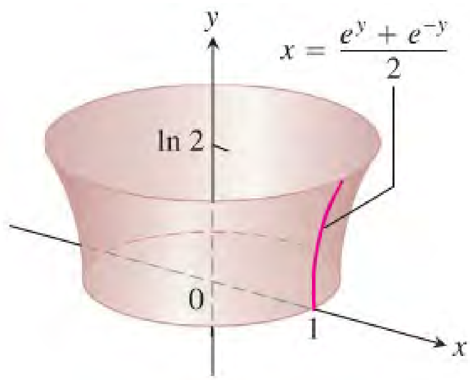


41. $y=\frac{1}{3}\left(x^2+2\right)^{3/2}$, $0 \le x \le \sqrt{2}$; $y-axis$ (Hint: Express $ds=\sqrt{dx^2+dy^2}$ in terms of dy, and evaluate the integral $S=\int 2\pi y\, ds$ with appropriate limits.)

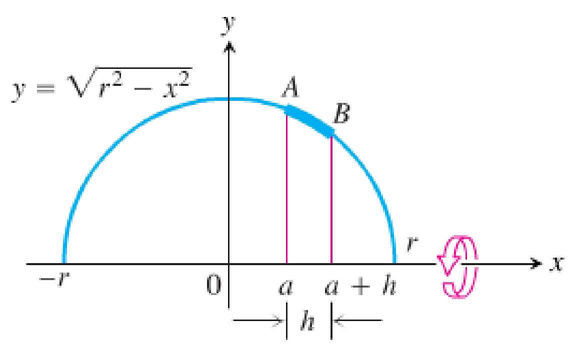
42. Find the area of the surface generated by $x=\sqrt{2y-1}$ $\frac{5}{8} \le y \le 1$, $y-axis$



43. Find the area of the surface generated by revolving the curve $x=\frac{1}{2}\left(e^y+e^{-y}\right)$, $0 \le y \le \ln 2$, about y-axis



44. Did you know that if you can cut a spherical loaf of bread into slices of equal width, each slice will have the same amount of crust? To see why, suppose the semicircle $y=\sqrt{r^2-x^2}$ shown here is revolved about the *x*-axis to generate a sphere. Let *AB* be an arc of the semicircle that lies above an interval of length *h* on the *x*-axis. Show that the area swept out by *AB* does not depend on the location of the interval. (It does depend on the length of the interval.)



45. The curved surface of a funnel is generated by revolving the graph of $y = f(x) = x^3 + \frac{1}{12x}$ on the interval $[1,\ 2]$ about the x-$axis$. Approximately what volume of paint is needed to cover the outside of the funnel with a layer of paint 0.05 *cm* thick? Assume that *x* and *y* measured in centimeters.

46. When the circle $x^2 + (y-a)^2 = r^2$ on the interval $[-r,\ r]$ is revolved about the x-$axis$, the result is the surface of a torus, where $0 < r < a$. Show that the surface area of the torus is $S = 4\pi^2 ar$.

47. A 1.5–*mm* layer of paint is applied to one side. Find the approximate volume of paint needed of the spherical zone generated when the curve $y = \sqrt{8x - x^2}$ on the interval $[1,\ 7]$ is revolved about the x-$axis$. Assume *x* and *y* are in *meters*.

48. A 1.5–*mm* layer of paint is applied to one side. Find the approximate volume of paint needed of the spherical zone generated when the upper portion of the circle $x^2 + y^2 = 100$ on the interval $[-8,\ 8]$ is revolved about the x-$axis$. Assume *x* and *y* are in *meters*.

49. Find the surface area of a cone (excluding the base) with radius 4 and height 8 using integration and a surface area integral.

50. Let $f(x) = \frac{1}{3}x^3$ and let *R* be the region bounded by the graph of *f* and the x-$axis$ on the interval $[0,\ 2]$

a) Find the area of the surface generated when the graph of *f* on $[0,\ 2]$ is revolved about the x-$axis$.

b) Find the volume of the solid generated when *R* is revolved about the y-$axis$.

c) Find the volume of the solid generated when *R* is revolved about the x-$axis$.

51. Let $f(x) = \sqrt{3x - x^2}$ and let *R* be the region bounded by the graph of *f* and the x-$axis$ on the interval $[0,\ 3]$

a) Find the area of the surface generated when the graph of *f* on $[0,\ 3]$ is revolved about the x-$axis$.

b) Find the volume of the solid generated when *R* is revolved about the x-$axis$.

52. Let $f(x) = \frac{1}{2}x^4 + \frac{1}{16x^2}$ and let *R* be the region bounded by the graph of *f* and the x-$axis$ on the interval $[1,\ 2]$

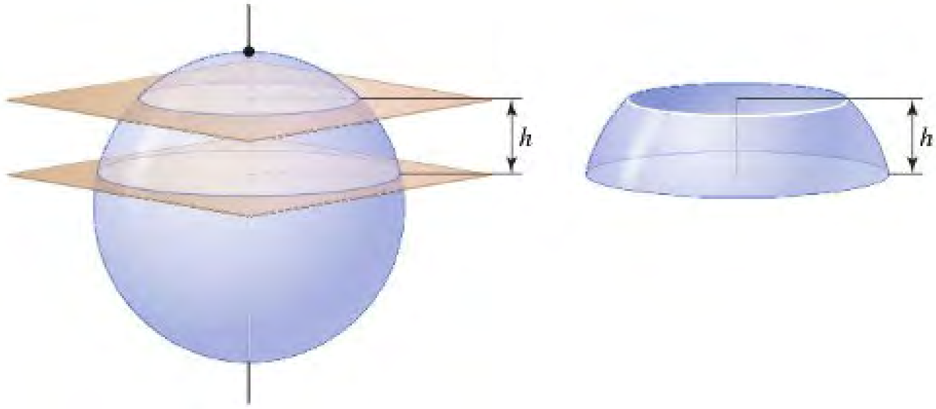
a) Find the area of the surface generated when the graph of *f* on $[1,\ 2]$ is revolved about the x-$axis$.

b) Find the length of the curve $y = f(x)$ on $[1,\ 2]$

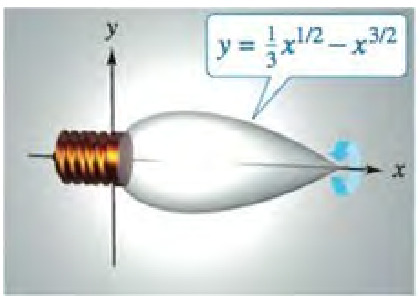
c) Find the volume of the solid generated when *R* is revolved about the y-$axis$.

d) Find the volume of the solid generated when *R* is revolved about the x-$axis$.

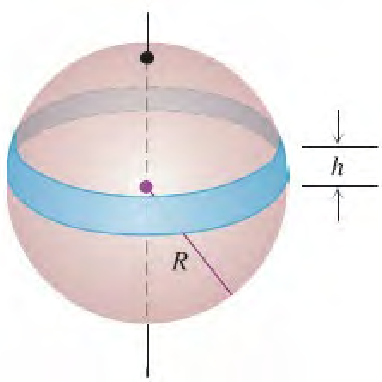
53. Suppose a sphere of radius r is sliced by two horizontal planes h units apart. Show that the surface area of the resulting zone on the sphere is $2\pi h$, independent of the location of the cutting planes.



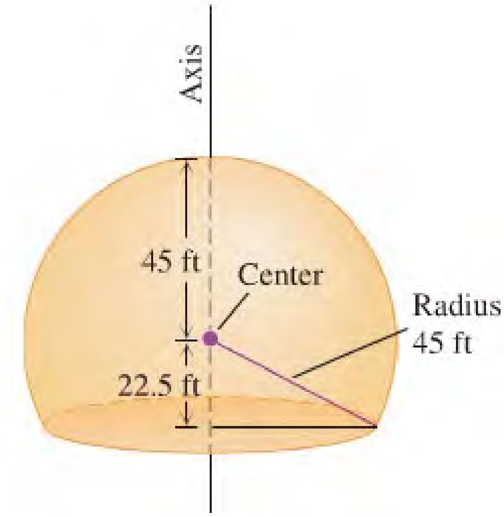
54. An ornamental light bulb is designed by revolving the graph of $y = \frac{1}{3}x^{1/2} - x^{3/2}, \quad 0 \le x \le \frac{1}{3}$ about the x-*axis*, where x and y are mesured in *feet*. Find the surface area of the bulb and use the result to approximate the amount of glass needed to make the bulb.
(Assume that the glass is 0.015 *inch* thick)



55. The shaded band is cut from a sphere of radius R by parallel planes h units apart. Show that the surface area of the band is $2\pi Rh$



56. A drawing of a 90-*ft* dome is used by the National Weather Service. How much outside surface is there to paint (not counting the bottom)?



Section 1.7 – Physical Applications

Density and Mass

Density is the concentration of mass in an object and is usually measured in units of mass per volume. An object with uniform density satisfies the basic relationship

$$mass = density \cdot volume$$

When density of an object varies, this formula no longer holds, and we must appeal to calculus.

Definition

Suppose a thin bar or wire can be represented as a line segment on the interval $a \le x \le b$ with a density function ρ (with units of mass per length). The mass of the object is

$$m = \int_a^b \rho(x)\, dx$$

Example

A thin 2-*m* bar, represented by the interval $0 \le x \le 2$, is made of any alloy whose density in units of *kg/m* is given by $\rho(x) = 1 + x^2$. What is the mass of the bar?

Solution

$$\begin{aligned} m &= \int_0^2 \left(1 + x^2\right) dx \\ &= x + \frac{1}{3}x^3 \Big|_0^2 \\ &= 2 + \frac{8}{3} \\ &= \boxed{\frac{14}{3}\ kg} \end{aligned}$$

Work Done By a Constant Force

When a body moves a distance *d* along a straight line as a result of being acted on by a force of constant magnitude ***F*** in the direction of motion, we define the ***work*** *W* done by the force on the body with the formula

$$Work = force \cdot distance$$

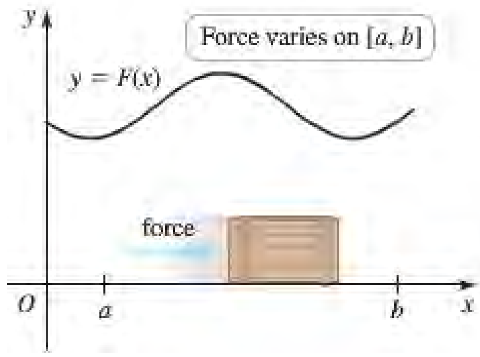
$$W = Fd \quad \textbf{(\textit{Constant-force formula for work})}$$

The unit of work is a newton-meter (***N.m***), also called ***joule***.

Definition

The work done by a variable force $F(x)$ in the direction of motion along the *x*-axis from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\,dx$$

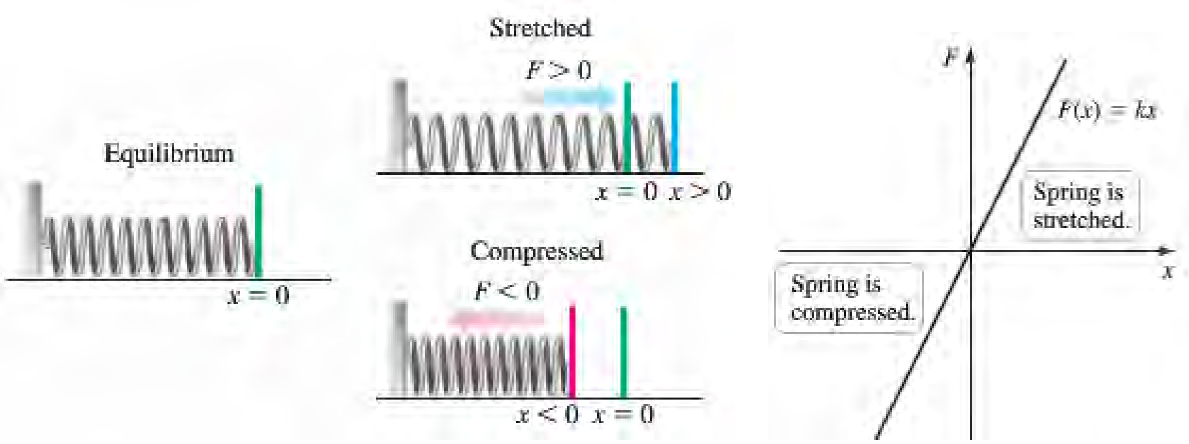


Hooke's Law for Springs: *F = kx*

Hooke's Law says that the force required to hold a stretched or compressed spring *x* units from its natural (unstressed) length is proportional to *x*. In symbols

$$F = kx$$

The constant *k*, measured in force units per unit length, is a characteristic of the spring, called ***the force constant*** (or ***spring constant***) of the spring.



- To stretch the spring to a position $x > 0$, a force $F > 0$ (in the positive direction) is required.
- To compress the spring to a position $x < 0$, a force $F < 0$ (in the negative direction) is required.

Example

Find the work required to compress a spring from its natural length of 1 *ft* to a length of 0.75 *ft* if the force constant is $k = 16$ *lb./ft*.

Solution

$$F = kx = 16x \qquad F(0) = 16 \cdot 0 = 0 \ lb$$

$$F(0.25) = 16 \cdot (0.25) = 4 \ lb$$

$$W = \int_a^b F(x)dx$$

$$= \int_0^{0.25} 16x \ dx$$

$$= 8x^2 \Big|_0^{0.25}$$

$$= 8\left(0.25^2 - 0\right)$$

$$= 0.5 \quad ft - lb$$

Example

A spring has a natural length of 1 *m*. A force of 24 *N* holds the spring stretched to a total length of 1.8 *m*.

a) Find the force constant *k*.

b) How much work will it take to stretch the spring 2 *m* beyond its natural length?

c) How far will a 45-*N* force stretch the spring?

Solution

a) $F = kx$

$$24 = k(1.8 - 1)$$

$$24 = k(0.8)$$

$$k = \frac{24}{0.8}$$

$$= 30 \quad N/m$$

b) $F(x) = 30x$

$$W = \int_0^2 30x \ dx$$

$$= 15x^2 \Big|_0^2$$

$$= 15\left(2^2 - 0\right)$$

$$\boxed{= 60 \quad J}$$

c) $45 = 30x$

$$x = \frac{45}{30}$$

$$\boxed{= \frac{3}{2} \quad m}$$

Example

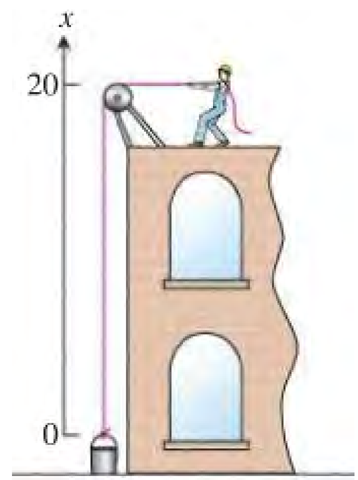
A 5-*lb* bucket is lifted from the ground into the air by pulling in 20 *feet* of rope at a constant speed. The rope weighs 0.08 *lb/ft*. How much work was spent lifting the bucket and rope?

Solution

Work done on lifting the bucket only is:

weight x *distance* = 5 (20) = 100 *ft-lb*

Work on the rope:



$$W = \int_0^{20} 0.08(20 - x)\,dx$$

$$= \frac{8}{100}\left(20x - \frac{x^2}{2}\right)\Big|_0^{20}$$

$$= \frac{2}{25}\left[\left(20(20) - \frac{20^2}{2}\right) - 0\right]$$

$$= \frac{2}{25}(400 - 200)$$

$$= \frac{2}{25}(200)$$

$$\boxed{= 16 \quad ft - lb}$$

The total work for the bucket and the rope combined is:

$100 + 16$ $\boxed{= 116 \quad ft - lb}$

Lifting

Another common work problem arises when the motion is vertical and the force is the gravitational force.

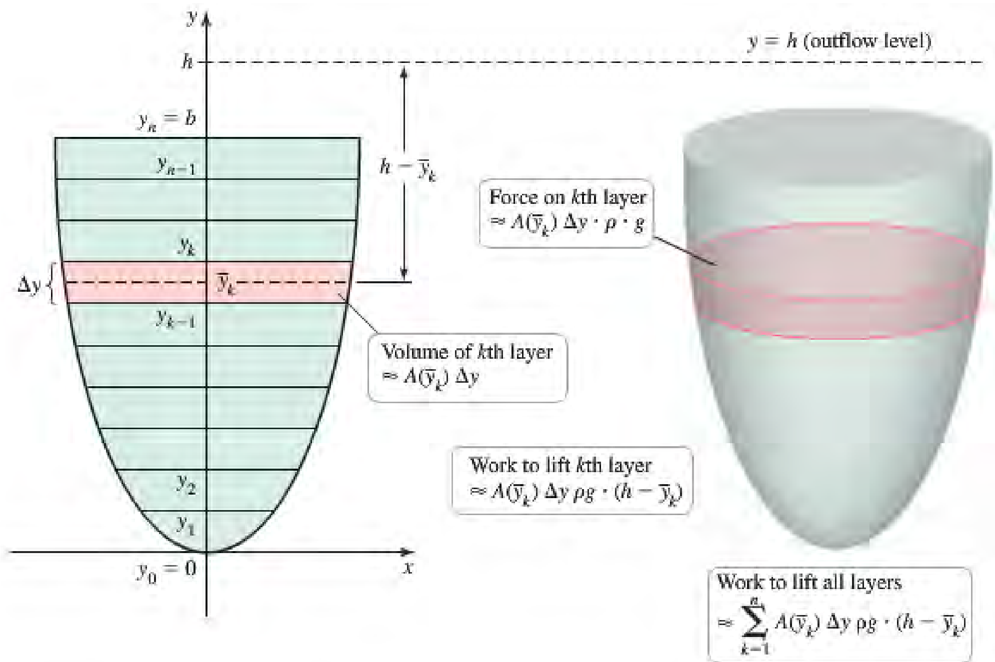
The gravitational force exerted on an object with a mass of m is $F = mg$, where $g \approx 9.8\ m/s^2 \approx 32.2\ ft/s^2$ is the acceleration due the gravity near the surface of Earth.

The work in joules required to lift an object of mass m a vertical distance of y meters is

$$work = force \cdot distance = mgy$$

This type of problem leads to 3 key observation to the solution:

- ✓ Water from different levels of the tank is lifted different vertical distances, requiring different amounts of work,
- ✓ Water from the same horizontal plane is lifted the same distance, requiring the same amount of work.
- ✓ A volume V of water has mass ρV, where $\rho = 1\ g/cm^3 = 1000\ kg/m^3$ is the density of water.



$$F_k = mg$$

$$\approx \overbrace{\underbrace{A(\bar{y}_k)\Delta y}_{volume} \cdot \rho}^{mass} \cdot g$$

$$W_k = \underbrace{A(\bar{y}_k)\Delta y \rho\, g}_{force} \cdot \underbrace{(h - \bar{y}_i)}_{distance}$$

$$W \approx \sum_{k=1}^{n} W_k$$

$$= \sum_{k=1}^{n} \rho\, gA\left(\bar{y}_k\right)\left(h - \bar{y}_k\right)\Delta y$$

$$W = \lim_{n\to\infty} \sum_{k=1}^{n} \rho\, gA\left(\bar{y}_k\right)\left(h - \bar{y}_k\right)\Delta y$$

$$= \int_0^b \rho\, gA(y)\underbrace{(h - y)}_{D(y)}dy$$

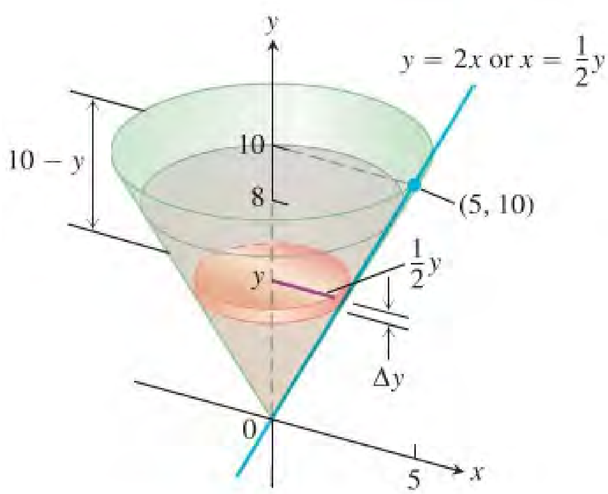
Solving Lifting Problems

1. Draw a y-*axis* in the vertical direction (parallel to gravity) and choose a convenient origin. Assume the interval $[a,\ b]$ corresponds to the vertical extent of the fluid.
2. For $a \le y \le b$, find the cross-sectional area $A(y)$ of the horizontal slices and the distance $D(y)$ the slices must be lifted.
3. The work required to lift the water is

$$W = \rho\, g \int_a^b A(y)D(y)dy$$

Example

The conical tank is filled to within 2 *feet* of the top with olive oil weighing 57 lb/ft^3. How much work does it take to pump the oil to the rim of the tank?



Solution

The volume of a slab between the planes y and Δy:

$\Delta V = \pi(radius)^2 (thickness)$

$$= \pi\left(\tfrac{1}{2}y\right)^2 \Delta y$$

$$\underline{= \tfrac{\pi}{4} y^2 \Delta y \quad ft^3 \,\big|}$$

The force $F(y)$ required to lift this slab is equal to its weight

$$F(y) = 57\Delta V$$

$$= 57\tfrac{\pi}{4} y^2 \Delta y$$

Distance to lift to the level of the rim of the cone is about $(10 - y)$ *feet*, so the work done lifting the slab

$$W = \frac{57\pi}{4}\int_0^8 y^2(10 - y)\,dy$$

$$= \frac{57\pi}{4}\int_0^8 \left(10y^2 - y^3\right)dy$$

$$= \frac{57\pi}{4}\left(\tfrac{10}{3}y^3 - \tfrac{1}{4}y^4 \right|_0^8$$

$$= \frac{57\pi}{4}\left[\left(\tfrac{10}{3}(8)^3 - \tfrac{1}{4}(8)^4\right) - 0\right]$$

$$= \frac{57\pi}{4}\left(\frac{5120}{3} - 1,024\right)$$

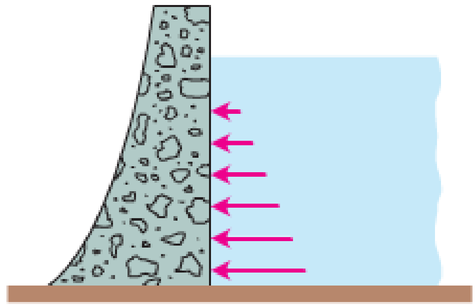
$$= \frac{57\pi}{4}\left(\frac{2,048}{3}\right)$$

$$\underline{= 9,728\pi \quad ft - lb \,\big|}$$

$$\underline{\approx 30,561 \quad ft - lb \,\big|}$$

Pressure and Force

Dams are built thicker at the bottom than at the top because the pressure against them increases with depth



Pressure is a force per unit area, measured in units such as N/m^2.

For example, the pressure of the atmosphere on the surface of Earth is about 14 lb/in^2 $\left(\approx 100 \text{ kilopascals, or } 10^5\ N/m^2\right)$

Another example, if you stood on the bottom of a swimming pool, you would feel pressure due to the weight (force) of the column of water above your head. If your head is flat and has surface area $A\ m^2$ and it is h meters below the surface, then the column of water above your head has volume $Ah\ m^3$. That column of water exerts a force:

$$F = mass \cdot acceleration = \underbrace{volume \cdot density}_{mass} \cdot g = Ah\rho g$$

Where ρ is the density of water

g is the acceleration due to gravity.

Therefore, the pressure on your head is the force divided by the surface area of your head

$$pressure = \frac{force}{A} = \frac{Ah\rho g}{A} = \rho g h$$

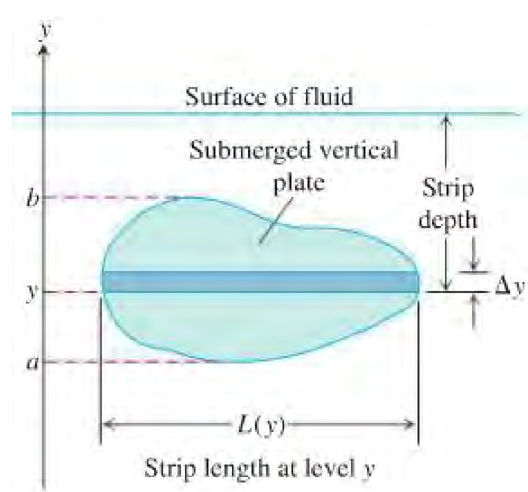
This pressure is called ***hydrostatic pressure*** (meaning the pressure of water at rest), and it has the following important property: it has the same magnitude in all directions.

Suppose that a plate submerge vertically in fluid of weight-density w runs from $y = a$ to $y = b$ on the y-axis.

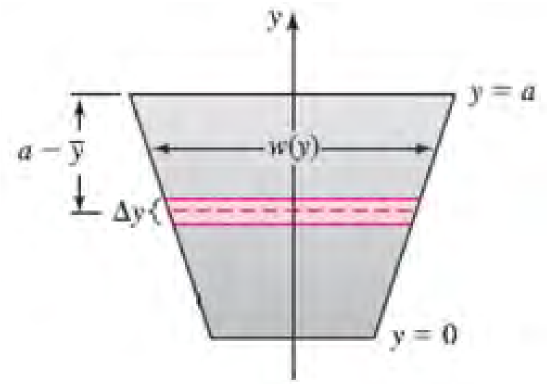
Let $L(y)$ be the ***length*** (or ***width***) of the horizontal strip measured from left to right along the surface of the plate at level y.

Then the force exerted by the fluid against one side of the plate is

$$F = \int_a^b w \cdot (strip\ depth) \cdot L(y)\,dy$$



Solving Force / Pressure Problems



1. Draw a *y-axis* on the face of the dam in the vertical direction and choose a convenient origin (often taken to be the base of the dam).
2. Find the width function $w(y)$ for each value of y on the face of the dam
3. If the base of the dam is at $y = 0$ and the top of the dam is at $y = a$, then the total force on the dam is

$$F = \int_0^a \rho g \underbrace{(a-y)}_{depth} \underbrace{w(y)}_{width} dy$$

Example

A flat isosceles right-triangular plate with base 6 *feet* and height 3 *feet* is submerged vertically, base up, 2 *feet* below the surface of a swimming pool. Find the force exerted by the water against on side of the plate.

(Freshwater Weight density: 62.4 $lb/\ ft^3$)

Solution

The width of a thin strip at level y is: $\quad L(y) = 2x = 2y$

The depth of the strip beneath the surface is: $(5-y)$

$$F = \int_a^b w \cdot (strip\ depth) \cdot L(y)\, dy$$

$$= 62.4 \int_0^3 (5-y)\cdot(2y)\,dy$$

$$= 124.8 \int_0^3 \left(5y - y^2\right) dy$$

$$= \frac{1,248}{10}\left(\frac{5}{2}y^2 - \frac{1}{3}y^3\right)\Big|_0^3$$

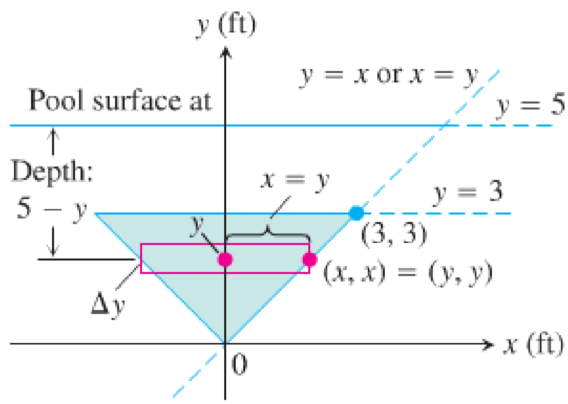
$$= \frac{624}{5}\left[\left(\frac{5}{2}(3)^2 - \frac{1}{3}(3)^3\right) - 0\right]$$

$$= \frac{624}{5}\left(\frac{45}{2} - 9\right)$$

$$= \frac{624}{5}\left(\frac{27}{2}\right)$$

$$= \boxed{\frac{8,424}{5}\ lb}$$

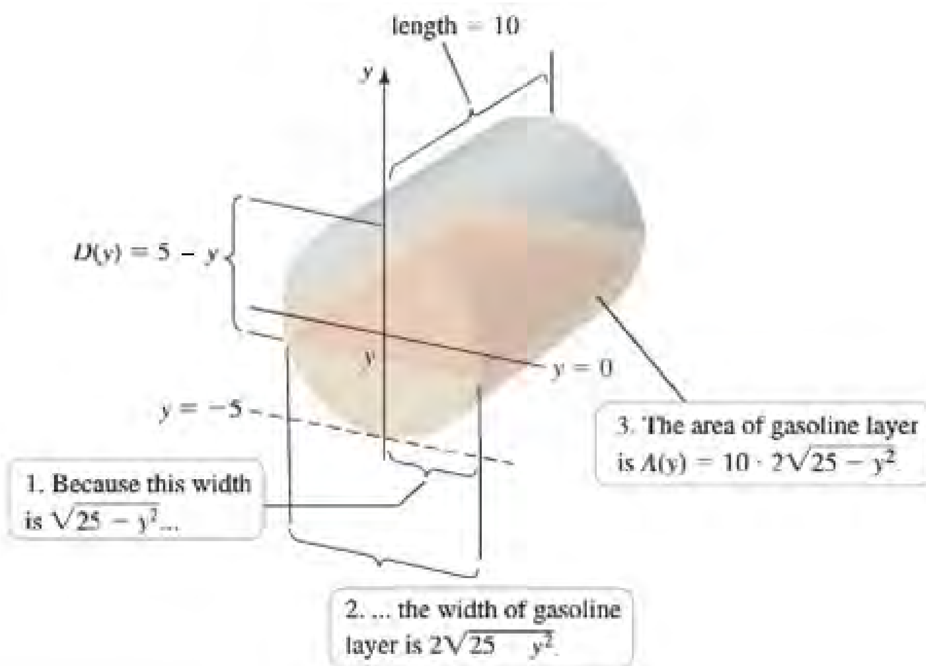
$$= \boxed{1684.8\ \ lb}$$



Example

A cylindrical tank with a length of 10 *m* and radius of 5 *m* is on its side and half-full of gasoline. How much work is required to empty the tank through an outlet pipe at the top of the tank?

The density of gasoline is $\rho \approx 737\ kg / m^3$.

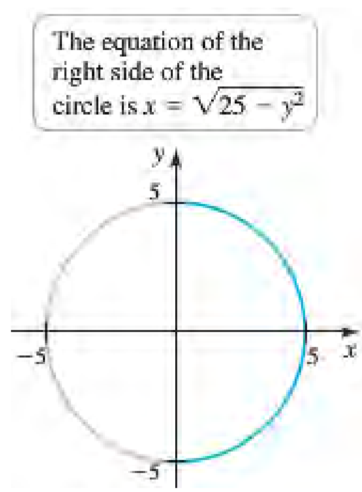


Solution

$$x^2 + y^2 = 5^2$$

$$x = \pm\sqrt{25 - y^2}$$

$$A(y) = 2(10)\sqrt{25 - x^2}$$



$$W = 737(9.8)\int_{-5}^{0} 20\sqrt{25 - y^2}\,(5 - y)\,dy$$

$$= 144,452 \underbrace{\int_{-5}^{0} 5\sqrt{25 - y^2}\,dy}_{\text{area of } \frac{1}{4} \text{ circle}} - 144,452\int_{-5}^{0} y\sqrt{25 - y^2}\,dy$$

$$= 144,452\left(5\cdot\frac{25\pi}{4} + \frac{1}{2}\int_{-5}^{0}\sqrt{25 - y^2}\,d\left(25 - y^2\right)\right)$$

$$= 144,452\left(\frac{125\pi}{4} + \frac{1}{3}\left(25 - y^2\right)^{3/2}\Big|_{-5}^{0}\right)$$

$$= 144,452\left(\frac{125\pi}{4} + \frac{125}{3}\right)$$

$$= 18,056,500\left(\frac{3\pi + 4}{12}\right)$$

$$\approx 20.2 \times 10^{6} \quad joules$$

Material	Weight Density	
	(kg/m^3)	(lb/ft^3)
Aluminum	2700	169
Copper	8940	558
Freshwater	1000	62.4
Gasoline	720	42 – 45
Gold	19320	1206
Iron	7870	491
Lead	11.34×10^3	708
Magnesium	1740	109
Mercury	13546	849
Milk	1030	64.5
Molasses	1600	100
Olive Oil	913	57
Platinum	21.45×10^3	1340
Seawater	1030	64

Exercises *Section* 1.7 – Physical Applications

(**1 – 8**) Find the mass of a thin bar with the given density function

1. $\rho(x) = 1 + \sin x; \quad 0 \le x \le \pi$

2. $\rho(x) = 1 + x^3; \quad 0 \le x \le 1$

3. $\rho(x) = 2 - \frac{x}{2}; \quad 0 \le x \le 2$

4. $\rho(x) = 5e^{-2x}; \quad 0 \le x \le 4$

5. $\rho(x) = x\sqrt{2 - x^2}; \quad 0 \le x \le 1$

6. $\rho(x) = \begin{cases} 1 & if \ \ 0 \le x \le 2 \\ 2 & if \ \ 2 < x \le 3 \end{cases}$

7. $\rho(x) = \begin{cases} 1 & if \ \ 0 \le x \le 2 \\ 1+x & if \ \ 2 < x \le 4 \end{cases}$

8. $\rho(x) = \begin{cases} x^2 & if \ \ 0 \le x \le 1 \\ x(2-x) & if \ \ 1 < x \le 2 \end{cases}$

9. Find the mass of a bar on the interval $0 \le x \le 9$ with a density (in *g/cm*) given by $\rho(x) = 3 + 2\sqrt{x}$

10. Find the mass of a 3-*m* bar on the interval $0 \le x \le 3$ with a density (in *g/m*) given by $\rho(x) = 150e^{-x/3}$

11. Find the mass of a bar on the interval $0 \le x \le 6$ with a density

$$\rho(x) = \begin{cases} 1 & if \quad 0 \le x < 2 \\ 2 & if \quad 2 \le x < 4 \\ 4 & if \quad 4 \le x \le 6 \end{cases}$$

12. It takes 50 *J* of work to stretch a spring 0.2 *m* from its equilibrium position. How much work is needed to stretch it an additional 0.5 *m*?

13. It takes 50 *N* of force to stretch a spring 0.2 *m* from its equilibrium position. How much work is needed to stretch it an additional 0.5 *m*?

14. A cylindrical water tank has a height of 6 *m* and a radius of 4 *m*. how much work is required to empty the full tank by pumping the water to an outflow pipe at the top of the tank?

15. Find the total force on the face of a semicircular dam with a radius of 20 *m* when its reservoir is full of water. The diameter of the semicircle is the top of the dam.

16. A rock climber is about to haul up 100 *N* (about 22.5 *lb*.) of equipment that has been hanging beneath her on 40 *m* rope that weighs 0.8 *N/m*. How much work will it take? (*Hint*: Solve for the rope and equipment separately, then add)

17. A 2-*oz* tennis ball was served at 160 *ft/sec*. How much work was done on the ball to make it go this fast? (to find the ball's mass from its weight, express the weight in pounds and divide by 32 ft/sec^2, the acceleration of gravity.)

18. How many foot-pounds of work does it take to throw a baseball 90 *mph*? A baseball weights 5 *oz*.

19. A 1.6-*oz* golf ball is driven off the tee at a speed of 280 *ft/sec* . How many foot-pounds of work are done on the ball getting it into the air?

20. You drove an 800-*gal* tank truck of water from the base of a mountain to the summit and discovered on arrival that the tank was only half full. You started with a full tank, climbed at a steady rate, and accomplished the 4750-*ft* elevation change in 50 *min*. Assuming that the water leaked out at a steady rate, how much work was spent in carrying water to the top? Do not count the work done in getting yourself and the truck there. Water weighs 8 *lb/gal*.

21. A force of 200 *N* will stretch a garage door spring 0.8 *m* beyond its unstressed strength. How far will a 300-*N* force stretch the spring? How much work does it take to stretch the spring this far from its unstressed length?

22. A heavy-duty shock absorber is compressed 2 *cm* from its equilibrium position by a mass of 500 *kg*. How much work is required tocompress the shock absorber 4 *cm* from its equilibrium position? (A mass of 500 *kg* exerts a force (in newtons) of 500 *g*)

23. A spring has a restoring force given by $F(x)=25x$. Let $W(x)$ be the work required to stretch the spring from its equilibrium position $(x=0)$ to a variable distance *x*. Graph the work function. Compare the work required to stretch the spring *x* units from equilibrium to the work required to compress the spring *x* units from equilibrium.

24. A swimming pool has the shape of a box with a base that measures 25 *m* by 15 *m* and a depth of 2.5 *m*. How much work is required to pump the water out of the pool when it is full?

25. Find the fluid force on a rectangular metal sheet measuring 3 *feet* by 4 *feet* that is submerged in 6 *feet* of water.

26. It took 1800 *J* of work to stretch a spring from its natural length of 2 *m* to a length of 5 *m*. Find the spring's force constant.

27. How much work is required to move am object from $x=1$ to $x=5$ (measired in meters) in the presence.

28. How much work is required to move am object from $x=0$ to $x=3$ (measired in meters) with a force (in *N*) is given by $F(x)=\dfrac{2}{x^2}$ acting along the x-*axis* .

29. A force of 200 *N* will stretch a garage door spring 0.8-*m* beyond its unstressed length.

a) How far will a 300-*N*-force stretch the spring?
b) How much work does it take to stretch the spring this far?

30. A spring on a horizontal surface can be stretched and held 0.5 *m* from its equilibrium position with a force of 50 *N*.

a) How much work is done in stretching the spring 1.5 *m* from its equilibrium position?

b) How much work is done in compressing the spring 0.5 *m* from its equilibrium position?

31. Suppose a force of 10 *N* is required to stretch a spring 0.1 *m* from its equilibrium position and hold it in that position.

a) Assuming that the spring obeys Hooke's law, find the spring constant *k*.

b) How much work is needed to ***compress*** the spring 0.5 *m* from its equilibrium position?

c) How much work is needed to ***stretch*** the spring 0.25 *m* from its equilibrium position?

d) How much additional work is required to stretch the spring 0.25 *m* if it has already been stretched 0.1 *m* from its equilibrium position?

32. A spring has a natural length of 10 *in*. An 800-*lb* force stretches the spring to 14 *in*.

a) Find the force constant.

b) How much work is done in stretching the spring from 10 *in* to 12 *in*?

c) How far beyond its natural length will a 1600-*lb* force stretch the spring?

33. It takes a force of 21,714 *lb*. to compress a coil spring assembly on a Transit Authority subway car from its free height of 8 *in*. to its fully compressed height of 5 *in*.

a) What is the assembly's force constant?

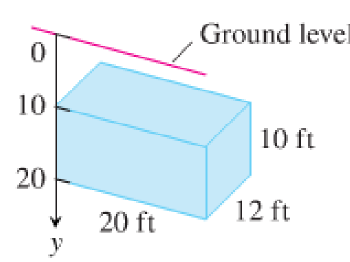
b) How much work does it take to compress the assembly the first half inch? The second half inch? Answer to the nearest *in-lb*.

34. A bag of sand originally weighing 144 *lb* was lifted a constant rate. As it rose, sand also leaked out at a constant rate. The sand was half gone by the time the bag had been lifted 10 18 *ft*. How much work was done lifting the sand this far? (Neglect the weight of the bag and lifting equipment.)

35. A mountain climber is about to haul up a 50 *m* length of hanging rope. How much work will it take if the rope weighs 0.624 *N/m*?

36. An electric elevator with a motor at the top has a multistrand cable weighing 4.5 *lb/ft*. When the car is at the first floor, 180 *ft* of cable are paid out, and effectively 0 *ft* are out when the car is at the top floor. How much work does the motor do just lifting the cable when it takes the car from the first floor to the top?

37. The rectangular cistern (storage tank for rainwater) shown has its top 10 *ft* below ground level. The cistern, currently full, is to be emptied for inspection by pumping its contents to ground level.



a) How much work will it take to empty the cistern?

b) How long will it take a 1/-*hp* pump, rated at 275 *ft-lb/sec*, to pump the tank dry?

c) How long will it take the pump in part (b) to empty the tank hallway? (It will be less than half the time required to empty the tank completely) *d)* What are the answers to parts (a) through (c) in a location where water weighs 62.6 lb/ft^3? 62.59 lb/ft^3?

38. When a particle of mass m is at $(x, 0)$, it is attracted toward the origin with a force whose magnitude is $\frac{k}{x^2}$. If the particle starts from rest at $x = b$ and is acted on by no other forces, find the work done on it by the time reaches $x = a$, $0 < a < b$.

39. The strength of Earth's gravitation field varies with the distance r from Earth's center, and the magnitude of the gravitational force experienced by a satellite of mass m during and after launch is

$$F(r) = \frac{mMG}{r^2}$$

Here, $M = 5.975 \times 10^{24}\ kg$ is Earth's mass, $G = 6.6720 \times 10^{-11}\ N \cdot m^2 kg^{-2}$ is the universal gravitational constant, and r is measured in meters. The work it takes to lift a 1000-*kg* satellite from Earth's surface to a circular orbit 35,780 *km* above Earth's center is therefore given by the integral

$$W = \int_{6,370,000}^{35,780,000} \frac{1000MG}{r^2}\, dr \quad \textit{joules}$$

Evaluate the integral. The lower limit of integration is Earth's radius in meters at the launch site. (This calculation does not take into account energy spend lifting the launch vehicle or energy spent bringing the satellite to orbit velocity.)

40. You drove an 800-*gal* truck from the base of Mt. Washington to the summit and discovered on arrival that the tank was only half full. You started with a full tank, climbed at a steady rate, and accomplished the 4750-*ft* elevation change in 50 minutes.

Assuming that the water leaked out at a steady rate, how much work was spent in carrying the water to the top? Do not count the work done in getting yourself and the truck there. Water weighs 8-*lb./gal*.

41. A cylindrical water tank has height 8 *m* and radius 2 *m*

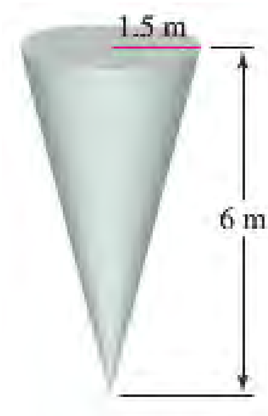
a) If the tank is full of water, how much work is required to pump the water to the level of the top of the tank and out of the tank?

b) Is it true it takes half as much work to pump the water out of the tank when it is half full as when it is full? Explain.

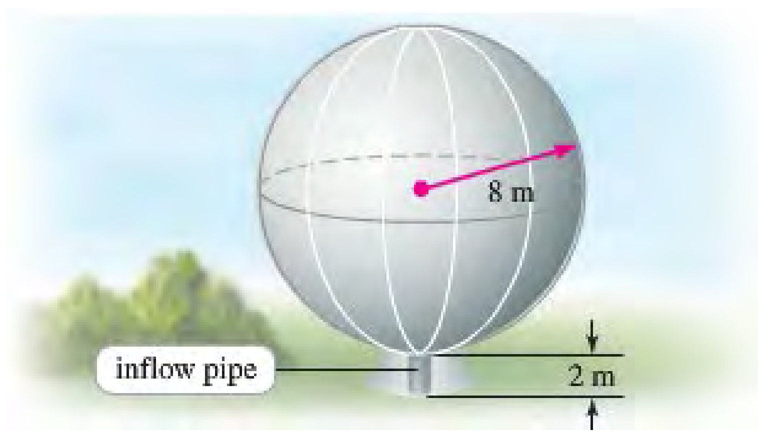
42. A water tank is shaped like an inverted cone with height 6 *m* and base radius 1.5 *m*.

a) If the tank is full of water, how much work is required to pump the water to the level of the top of the tank and out of the tank?

b) Is it true it takes half as much work to pump the water out of the tank when it is half full as when it is full? Explain



43. A spherical water tank with an inner radius of 8 *m* has its lowest point 2 *m* above the ground. It is filled by a pipe that feed the tank at its lowest point.4



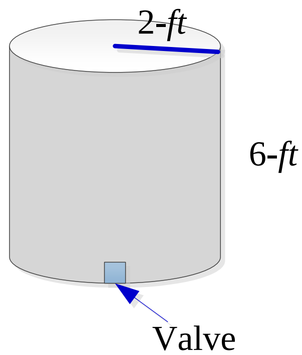
a) Neglecting the volume of the inflow pipe, how much work is required to fill the tank if it is initially empty?

b) Now assume that the inflow pipe feeds the tank at the top of the tank. Neglecting the volume of the inflow pipe, how much work is required to fill the tank if it is initially empty?

44. A large vertical dam in the shape of a symmetric trapezoid has a height of 30 *m*. a width of 20 *m*. at its base, and a width of 40 *m*. at the top. What is the total force on the face of the dam when the reservoir is full? $\left(\rho = 1000\frac{kg}{m^3},\ g = 9.8\frac{m}{s^2}\right)$

45. A vertical gate in a dam has the shape of an isosceles trapezoid 8 *feet* across the top and 6 *feet* across the bottom. With a height of 5 *feet*. What is the fluid force on the gate if the top of the gate is 4 *feet* below the surface of the water?

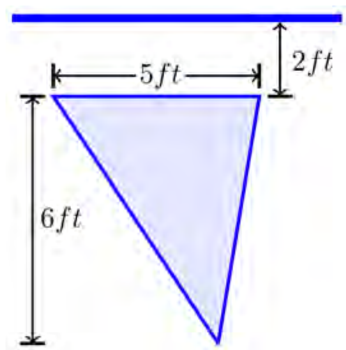
46. Pumping water from a lake 15-*feet* below the bottom of the tank can fill the cylindrical tank shown here.



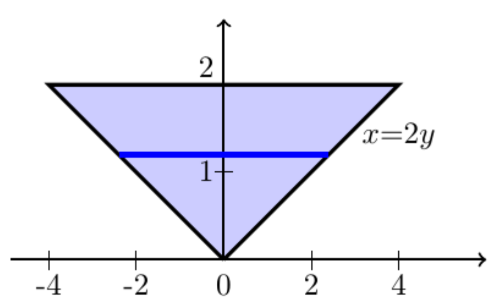
There are two ways to go about it. One is to pump the water through a hose attached to a valve in the bottom of the tank. The other is to attach the hose to the rim of the tank and let the water pour in. Which way will be faster? Give reason for answer.

47. A tank truck hauls milk in a 6-*feet* diameter horizontal right circular cylindrical tank. How must force does the milk exert on each end of the tank when the tank is half full?

48. A triangular plate, base 5 *feet*, height 6 *feet*, is submerged in water, vertex down, plane vertical, and 2 *feet* below the surface. Find the total force on one face of the plate.



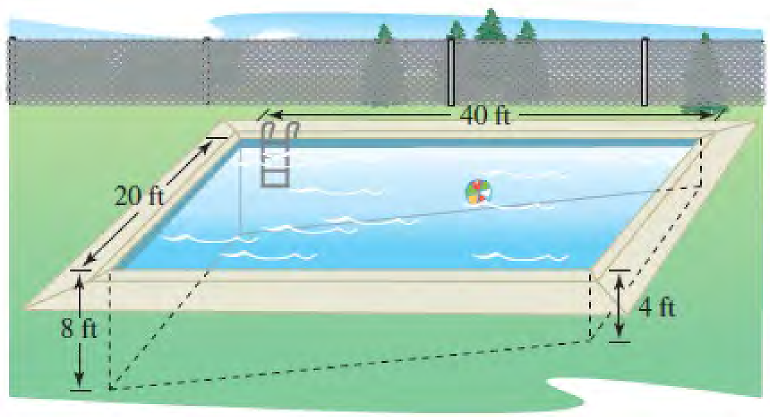
49. The vertical triangular plate shown here is the end plate of a trough full of water. What is the fluid force against the plate?



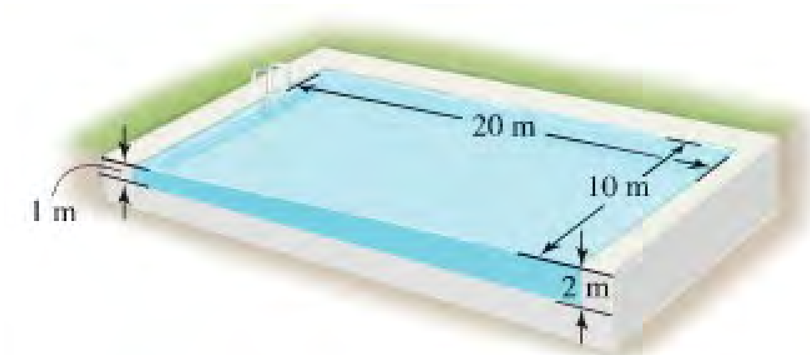
50. A cylindrical gasoline tank is placed so that the axis of the cylinder is horizontal. Find the fluid force on a circular end of the tank if the tank is ***half full*** assuming that the diameter is 3 *feet* and the gasoline weighs 42 *pounds* per *cubic foot*.

51. A cylindrical gasoline tank is placed so that the axis of the cylinder is horizontal. Find the fluid force on a circular end of the tank if the tank is ***full*** assuming that the diameter is 3 *feet* and the gasoline weighs 42 *pounds* per *cubic foot*.

52. A swimming pool is 20 *feet* wide, 40 *feet* long, 4 *feet* deep at one end, and 8 *feet* deep at the other end. The bottom is an inclined plane. Find the fluid force on each vertical wall.



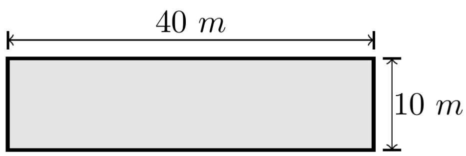
53. A swimming pool is 20 *m* leg long and 10 *m* wide, with a bottom that slopes uniformly from a depth of 1 *m* at one end to a depth of 2 *m* at the other end.



Assuming the pool is full, how much work is required to pump the water to a level 0.2 *m* above the top of the pool?

(**54 – 63**) Find the total force on the face of the given dam

54.



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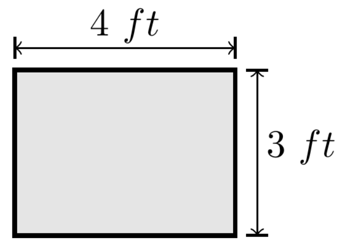

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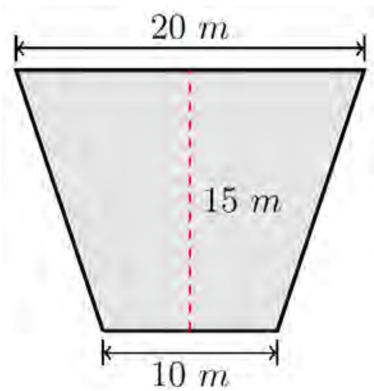

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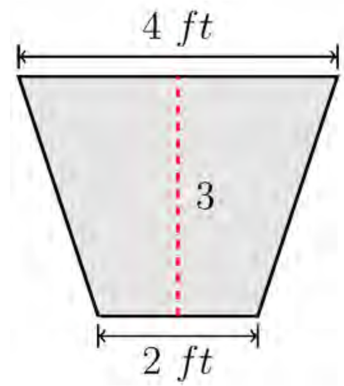

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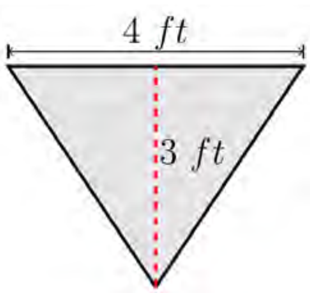
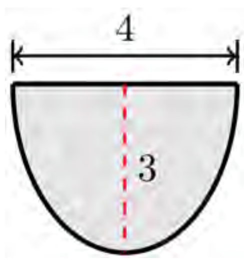


59. $y=-\frac{1}{2}\sqrt{36-9x^2}$ (*ft*)



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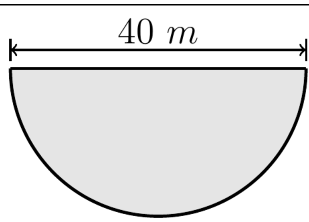

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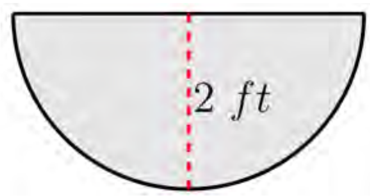

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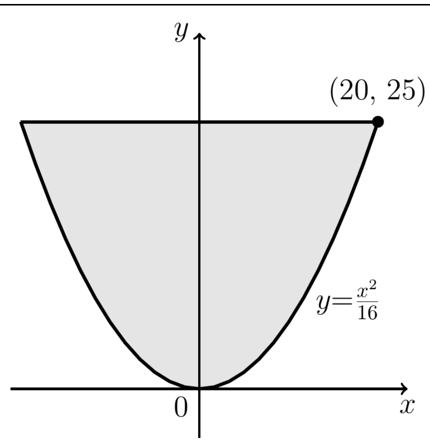

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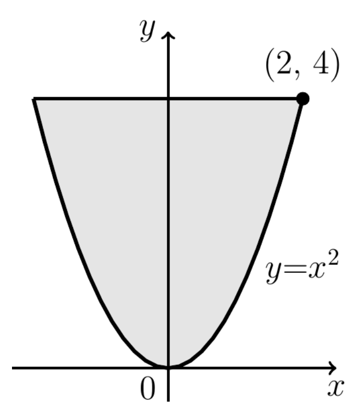
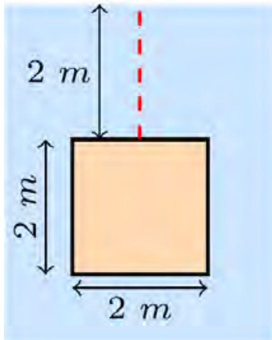


(**64 – 71**) Find the fluid force on the vertical plate submerged in water

64.



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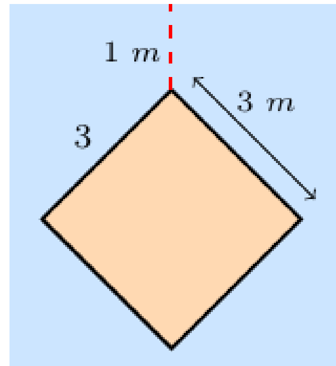

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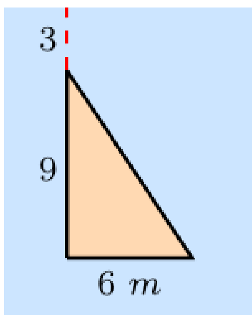

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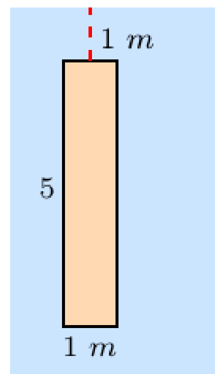

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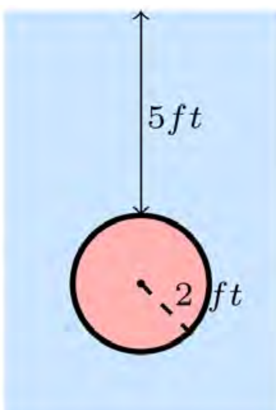

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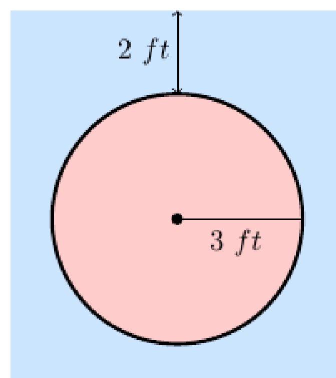

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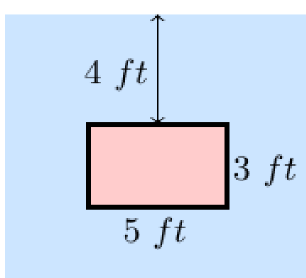

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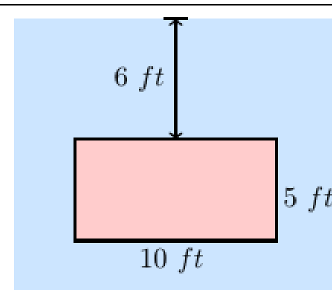


72. A rectangular plate of height ***h*** *feet* and base ***b*** *feet* is submerged vertically in a tank of fluid that weighs ***w*** *pounds per cubic foot*. The center is ***k*** *feet* below the surface of the fluid, where $h \le \frac{k}{2}$. Show that the fluid force on the surface of the plate is $F = wkhb$.

73. A circular plate of radius ***r*** *feet* is submerged vertically in a tank of fluid that weighs ***w*** *pounds per cubic foot*. The center of the circle is $k\ (k > r)$ *feet* below the surface of the fluid. Show that the fluid force on the surface of the plate is $F = \pi wkr^2$.

74. A large building shaped like a box is 50 *m* high with a face that is 80 *m* wide. A strong wind blows directly at the face of the building, exerting a pressure of 150 N/m^2 at the ground and increasing with height according to $P(y) = 150 + 2y$, where *y* is the height above the ground. Calculate the total force on the building, which is a meaure of the resistance that must be included in the design of the building.

75. Adiving pool that is 4 *m* deep full of water has a viewing window on one of its vertical walls. Find the force of the a square window, 0.5 *m* on side, with the lower edge of the window on the bottom of the pool.

76. Adiving pool that is 4 *m* deep full of water has a viewing window on one of its vertical walls. Find the force of the a square window, 0.5 *m* on side, with the lower edge of the window 1 *m* from the bottom of the pool.

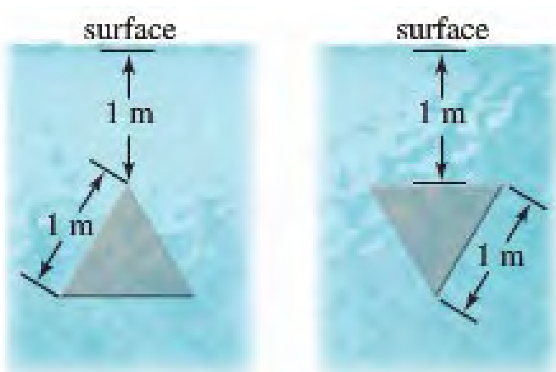
77. Adiving pool that is 4 *m* deep full of water has a viewing window on one of its vertical walls. Find the force of the a circle window, with a radius of 0.5 *m*, tangent to the bottom of the pool.

78. A rigid body with a mass of 2 *kg* moves along a line due to a force that produces a position function $x(t) = 4t^2$, where *x* is measured in *meters* and *t* is measured in *seconds*. Find the work done during the first 5 *sec*. in two ways.

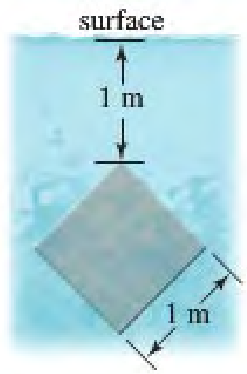
a) Note that $x''(t) = 8$; then use Newton's second law, $\left(F = ma = mx''(t)\right)$ to evaluate the work integral $W = \int_{x_0}^{x_f} F(x)\,dx$, where x_0 and x_f are the initial and final positions, repectively.

b) Change variables in the work integral and integrate with respect to *t*.

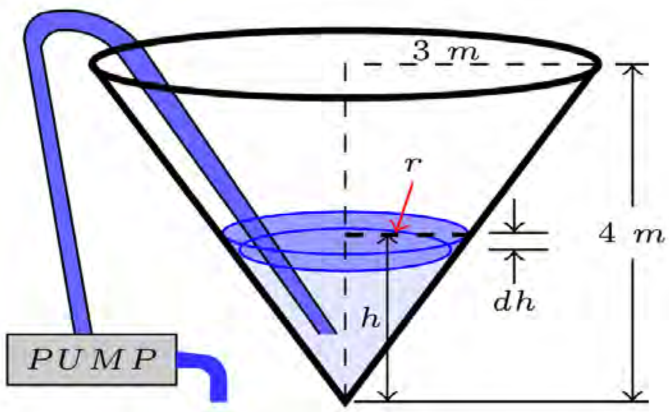
79. A plate shaped like an equilateral triangle 1 *m* on a side is placed on a vetical wall 1 *m* below the surface of a pool filled with water. On which plate in the figure is the force is greater



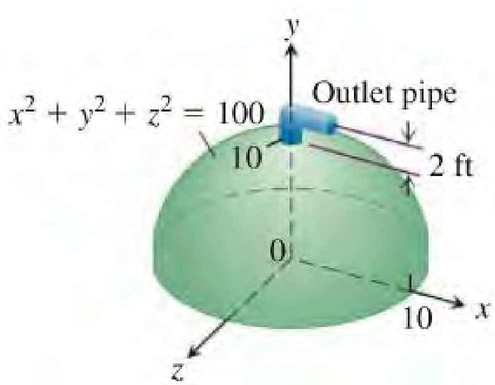
80. A square plate 1 *m* on a side is placed on a vetical wall 1 *m* below the surface of a pool filled with water. On which plate in the figure is the force is greater



81. Water fills a tank in the shape of a right-circular cone with top radius 3 *m* and depth 4 *m*. How much work must be done (against gravity) to pump all the water out of the tank over the top edge of the tank?

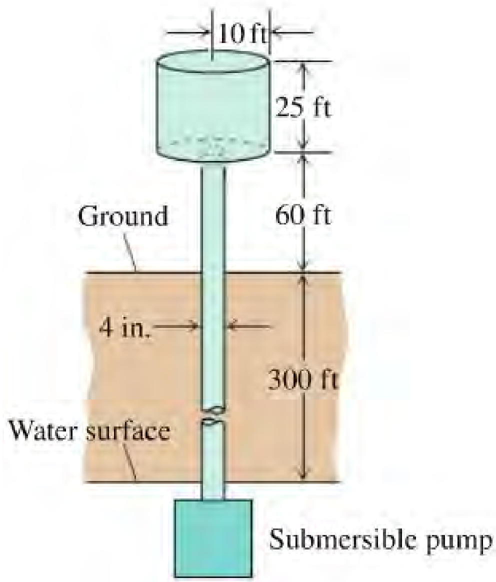


82. You are in charge of the evacuation and repair of the storage tank.



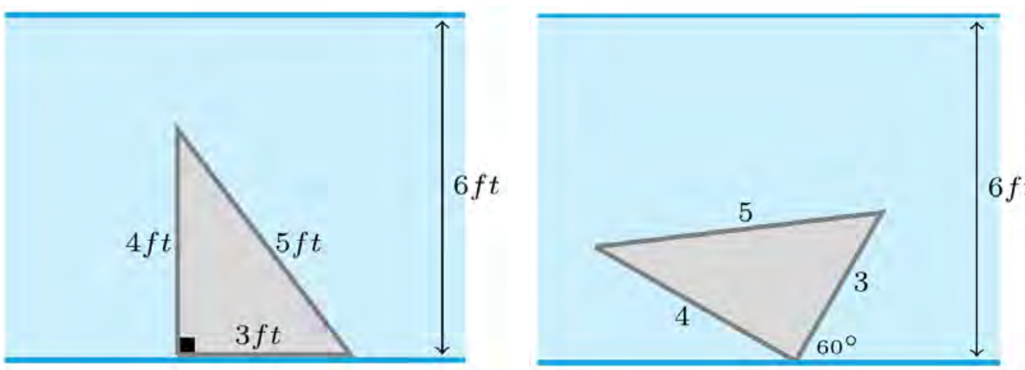
The tank is a hemisphere of radius 10 *feet* and is full of benzene weighing 56 lb/ft^3. A firm you contacted says it can empty the tank for $\frac{1}{2}$ ¢ per foot-pound of work. Find the work required to empty the tank by pumping the benzene to an outlet 2 *feet* above the top of the tank. If you have $5,000 budget for the job, can you afford to hire the firm?

83. You decided to drill a well to increase a water supply. You have determined that a water tower will be necessary to provide the pressure needed for distribution



The water is to be pumped from a 300-*ft* well through a vertical 4-*in*. pipe into the base of a cylindrical tank 20 *feet* in diameter and 25 *feet* high. The base of the tank will be 60 *feet* above ground. The pump is a 3-*hp* pump, rated at $1,650\ ft \cdot lb/sec$. How long will it take to fill the tank the first time? (Include the time it takes to fill the pipe). Assume that water weighs $62.4\ lb/ft^3$.

84. Calculate the fluid force on one side of a right-triangular plate with edges 3 *feet*, 4 *feet*, and 5 *feet* if the plate sits at the bottom of the pool filled with water to a depth of 6 *feet* on its 3-*feet* edge and titlted at $60°$ to the bottom of the pool.

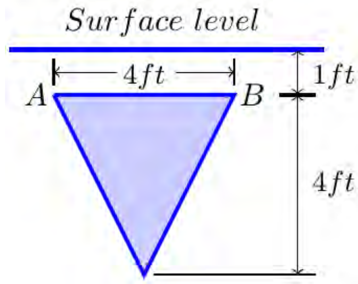


85. Two electrons *r meters* apart repel each other with a force of $F = \dfrac{23 \times 10^{29}}{r^2}$ *newtons*

a) Suppose one electron is held fixed at the point $(1,\ 0)$ on the *x*-axis (units in *meters*). How much work does it take to move a second electron along *x*-axis from the point $(-1,\ 0)$ to the origin?

b) Suppose one electron is held fixed at the point $(-1,\ 0)$ and $(1,\ 0)$. How much work does it take to move a third electron along *x*-axis from the point $(5,\ 0)$ to $(3,\ 0)$?

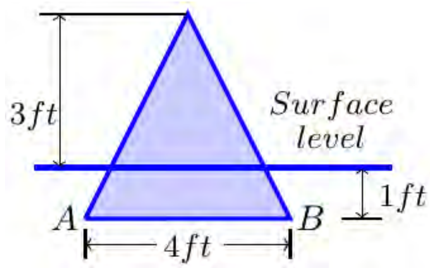
86. The isosceles triangular plate is submerged vertically 1 *feet* below the surface of a freshwater lake.



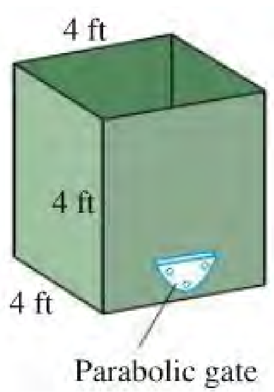
a) Find the fluid force against one face of the plate.

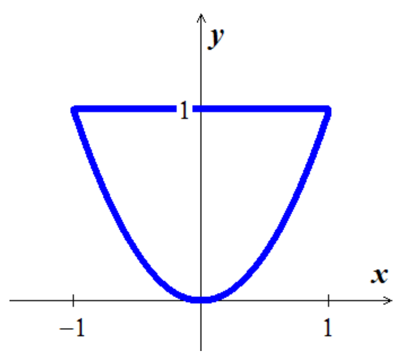
b) What would be the fluid force on one side of the plate if the water were seawater instead of freshwater?

87. The isosceles triangular plate is submerged vertically 3 *feet* above the surface of a freshwater lake. What force does the water exert on one face of the plate now?



88. The cubical metal tank has parabolic gate held in place by bolts and designed to withstand a fluid force of 160 *lb*. without rupturing. The liquid you plan to store has a weight-density of 50 lb/ft^3.

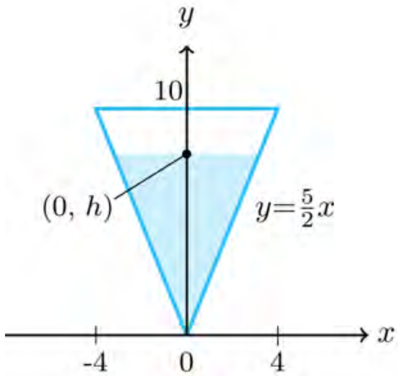


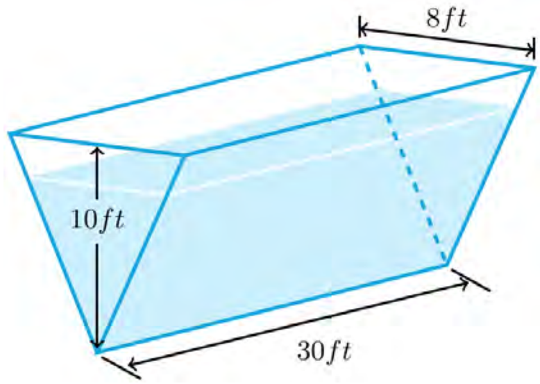


a) What is the fluid force on the gate when the liquid is 2 feet deep?

b) What is the maximum height to which the container can be filled without exceeding the gate's design limitation?

89. The end plates of the trough were designed to withstand a fluid force of 6,667 *lb*.

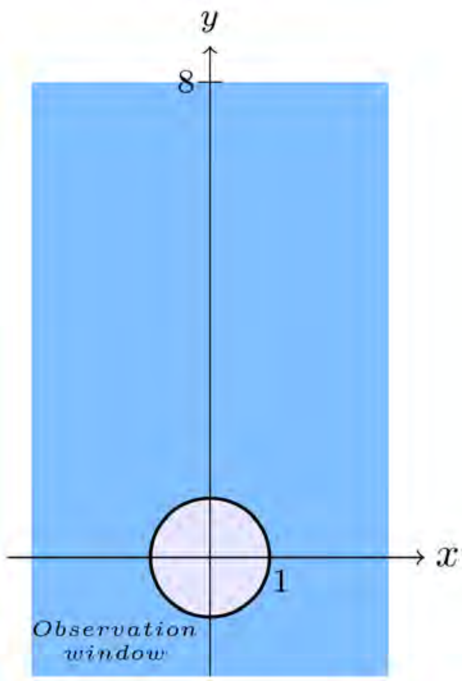




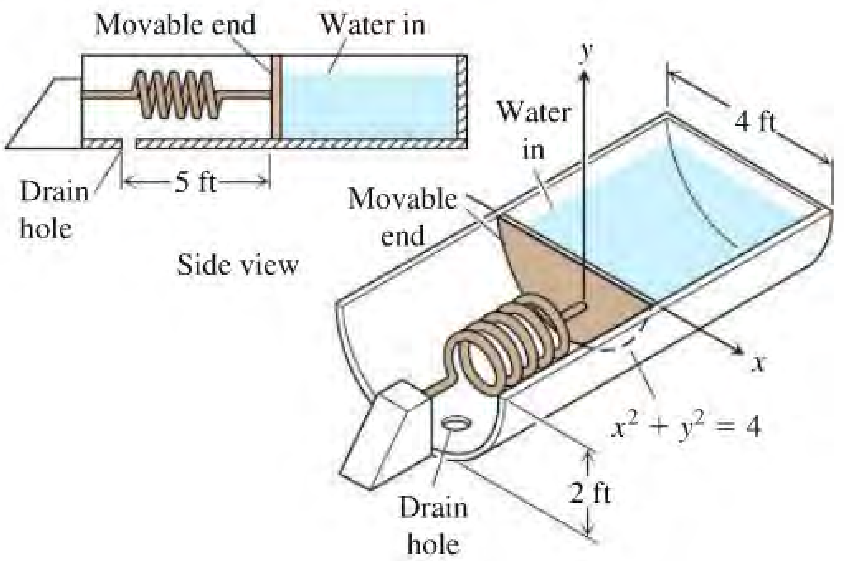
a) What is the value of *h*?

b) How many cubic feet of water can the tank hold without exceeding this limitation?

90. A circular observation window on a marine science ship has a radius of 1 *foot*, and the center of the window is 8 *feet* below water level. What is the fluid force on the window?



91. Water pours into the tank at the rate of $4\ ft^3/min$. The tank's cross-sections are 4-*ft*-diameter semicircles. One end of the tank is movable, but moving it to increase the volume compresses a spring. The spring constant is $k = 100\ lb/ft$. If the end of the tank moves 5 feet against the spring, the water will drain out of a safety hole in the bottom at the rate of $5\ ft^3/min$. Will the movable end reach the hole before the tank overflows?



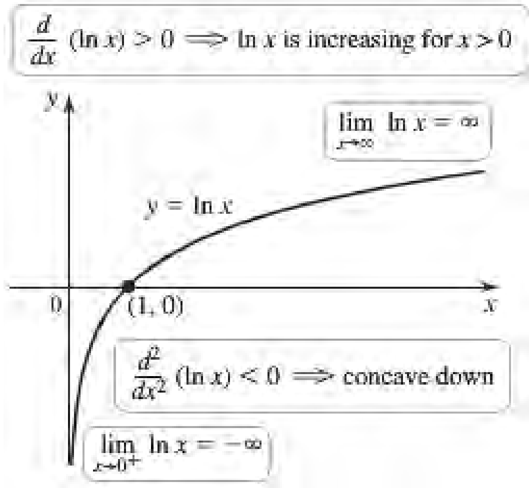
Section 1.8 – Exponential Models

Review

Definition

The ***number e*** is that number in the domain of the ***natural logarithm*** satisfying

$$\ln e = 1 \quad and \quad \int_1^e \frac{1}{t}dt = 1$$



The ***natural logarithm*** of a number $x > 0$, denoted by $\ln x$, is defined as

$$\ln x = \int_1^x \frac{1}{t}\,dt$$

Example

Evaluate $\displaystyle\int_0^4 \frac{x}{x^2+9}\,dx$

Solution

$$\int_0^4 \frac{x}{x^2+9}\,dx = \frac{1}{2}\int_0^4 \frac{1}{x^2+9}\,d\left(x^2+9\right)$$

$$= \frac{1}{2}\ln\left(x^2+9\right)\Big|_0^4$$

$$= \frac{1}{2}(\ln 25 - \ln 9)$$

$$= \frac{1}{2}(2\ln 5 - 2\ln 3)$$

$$\boxed{= \ln\frac{5}{3}}$$

The inverse of ln*x* and the Number *e*

The function $\ln x$, being *increasing* function of *x*. *Domain* (0, ∞) and *range* (–∞, ∞)

The inverse function $\ln^{-1} x$ with *Domain* (–∞, ∞) and *range* (0, ∞)

The function $\ln^{-1} x$ is usually denoted as $\exp x$ $\left(e^x\right)$

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \quad (all\ x > 0) \qquad\qquad \ln\left(e^x\right) = x \quad (all\ x)$$

The Derivative and Integral of e^x

The natural exponential function is differentiable because it is the inverse of a differentiable function whose derivative is never zero.

$$\ln\left(e^x\right) = x \qquad \textit{Inverse relationship}$$

$$\frac{d}{dx}\ln\left(e^x\right) = 1 \qquad \textit{Differentiate both sides.}$$

$$\frac{1}{e^x}\frac{d}{dx}\left(e^x\right) = 1 \qquad \frac{d}{dx}\ln u = \frac{1}{u}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}e^x = e^x$$

Theorem

For real numbers *x*,

$$\frac{d}{dx}\left(e^{u(x)}\right) = u'(x)e^{u(x)} \quad \text{and} \quad \int e^x dx = e^x + C$$

Example

Evaluate $\int \frac{e^x}{1+e^x}dx$

Solution

$$\int \frac{e^x}{1+e^x}dx = \int \frac{1}{1+e^x}d\left(1+e^x\right)$$

$$= \ln\left(1+e^x\right) + C$$

Definition

If $a > 0$ and u is a differentiable of x, then a^u is a differentiable function of x and

$$\frac{d}{dx}a^u = a^u \ln a\frac{du}{dx} \quad \& \quad \frac{d}{dx}\left(\log_a u\right) = \frac{1}{u}\cdot\frac{1}{\ln a}\frac{du}{dx}$$

$$\int a^u du = \frac{a^u}{\ln a} + C$$

Example

Evaluate $\displaystyle\int x\,3^{x^2}\,dx$

Solution

$$\int x\,3^{x^2}\,dx = \frac{1}{2}\int 3^{x^2}\,d\left(x^2\right)$$

$$= \frac{1}{2}\frac{1}{\ln 3}3^{x^2} + C$$

Example

Evaluate $\displaystyle\int_1^4 \frac{6^{-\sqrt{x}}}{\sqrt{x}}\,dx$

Solution

$$\int_1^4 \frac{6^{-\sqrt{x}}}{\sqrt{x}}\,dx = -2\int_1^4 6^{-\sqrt{x}}\,d\left(-\sqrt{x}\right) \qquad d\left(-\sqrt{x}\right) = -\frac{1}{2\sqrt{x}}dx$$

$$= -\frac{2}{\ln 6}\,6^{-\sqrt{x}}\Big|_1^4$$

$$= -\frac{2}{\ln 6}\left(\frac{1}{36} - \frac{1}{6}\right)$$

$$= \frac{5}{18\ln 6}$$

Power Rule – *Definition*

For any $x > 0$ and for any real number n, $\qquad x^n = e^{n\ln x}$

Example

Evaluate the derivative $f(x) = x^{2x}$

Solution

$$\frac{d}{dx}\left(x^{2x}\right) = \frac{d}{dx}\left(e^{2x\ln x}\right)$$

$$= e^{2x\ln x}\left(2x\ln x\right)'$$

$$= 2e^{2x\ln x}\left(\ln x + 1\right)$$

$$= 2x^{2x}\left(\ln x + 1\right)$$

Exponential Models

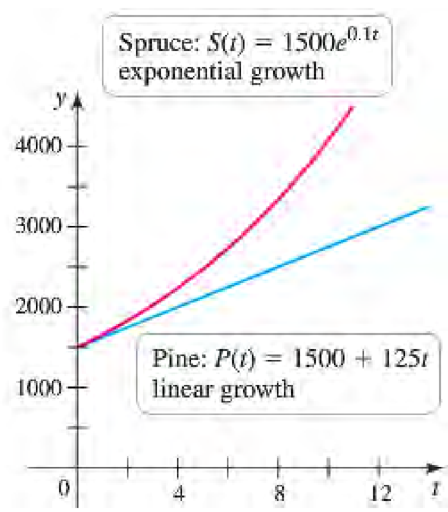
Exponential Growth Functions

Exponential growth is described by functions of the form $y(t) = y_0(t)e^{kt}$. The ***initial value*** of y at $t = 0$ is $y(0) = y_0$ and the ***rate constant*** $k > 0$ determines the rate of the growth. Exponential growth is characterized by a constant relative growth rate.

Example

Suppose the population of the town of Pine is given by $P(t) = 1500 + 125t$, while the population of the town of Spruce is given by $S(t) = 1500e^{0.1t}$, where $t \ge 0$ is measured in years. Find the growth rate and the relative growth rate of each town.

Solution



$$\frac{dP}{dt} = 125$$

$$\frac{dS}{dt} = 150e^{0.1t}$$

The relative growth rate of Pine is

$$\frac{1}{P}\frac{dP}{dt} = \frac{125}{1500 + 125t},$$

which decreases in time.

The relative growth rate of Spruce is

$$\frac{1}{S}\frac{dS}{dt} = \frac{150e^{0.1t}}{1500e^{0.1t}}$$

$$= 0.1 \quad \text{Contant for all times}$$

The linear population function has a constant absolute growth rate and the exponential population function has a constant relative growth rate.

Definition

The quantity described by the function $y(t) = y_0 e^{kt}$ *for* $k > 0$, has a constant doubling time of $T_2 = \frac{\ln 2}{k}$, with the same units as *t*.

Formula
To find either k or T:

$$A = A_0 e^{kt} \quad \Rightarrow \quad \boxed{kT = \ln \frac{A}{A_0}}$$

Proof

$$A = A_0 e^{kt}$$

$$\frac{A}{A_0} = e^{kt}$$

$$\ln \frac{A}{A_0} = \ln e^{kt}$$

$$\boxed{\ln \frac{A}{A_0} = kt} \quad \checkmark$$

Example
Human population growth rates vary geographically and fluctuate over time. The overall growth rate for world population peaked at an annual rate of 2.1% per year in the 1960s. Assume a world population of 6.0 billion in 1999 $(t = 0)$ and 6.9 billion in 2009 $(t = 10)$

a) Find an exponential growth function for the world population that fits the two data points.

b) Find the doubling time for the world population using the model in part (a).

c) Find the (absolute) growth rate $y'(t)$ and graph it, for $0 \le t \le 50$.

d) How fast was the population growing in 2014 $(t = 15)$?

Solution

Given: $y(0) = 6, \quad y(10) = 6.9$

a) $k = \frac{1}{T} \ln \left(\frac{y}{y_0} \right)$

$= \frac{1}{10} \ln \frac{6.9}{6}$

$\boxed{\approx 0.014}$

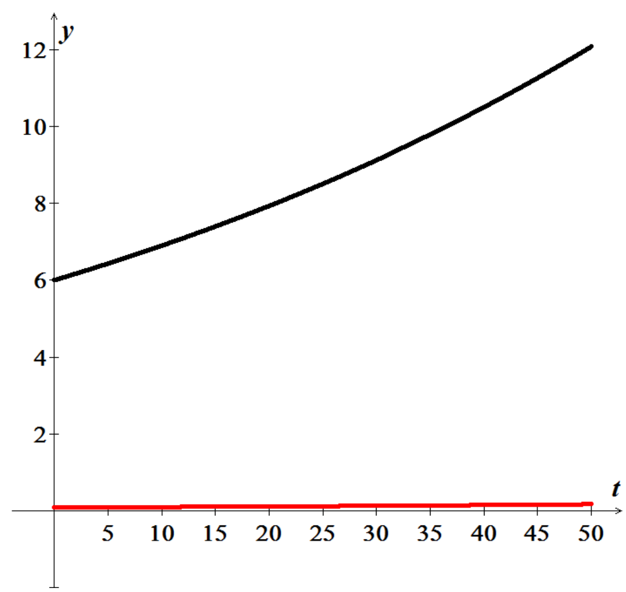
The growth function is: $y(t) = 6e^{0.014t}$

b) $T_2 = \frac{\ln 2}{0.014}$ $\qquad T = \frac{\ln 2}{k}$

$\boxed{\approx 50 \ \textit{years}}$

c) $y'(t) = 0.084e^{0.014t}$ $\quad$ (*billion of people /year*)

The growth rate itself increases exponentially



d) $y'(t=15) = 0.084e^{0.014(15)}$

$\approx 0.104 \;\; bil/yr$

Financial Model

The balance in the account increases exponentially at a rate that can be determined from the advertised ***annual percentage yield*** (or **APY**) of the account.

Effective Rate

The ***effective rate*** corresponding to a started rate of interest r compounded m times per year is

$$r_e = \left(1+\frac{r}{m}\right)^m - 1$$

APY is also referred to as ***effective rate*** or true interest rate.

Example

The APY of a savings account is the percentage increase in the balance over the course of a year. Suppose you deposit \$500 in a savings account that has an APY of 6.18% per year. Assume that the interest rate remains constant and that no additional deposits or withdrawals are made. How long will it take the balance to reach \$2500?

Solution

In one year the balance: $y(1) = (1+.0618)\, y_0 = 1.0618 y_0$

$$k = \frac{1}{T}\ln\left(\frac{y(1)}{y_0}\right)$$

$$= \ln 1.0618$$

$$\approx 0.05997$$

$$y(t) = 500e^{0.05997t}$$

$$T = \frac{1}{k}\ln\left(\frac{y}{y_0}\right)$$

$$= \frac{1}{0.05997}\ln\left(\frac{2500}{500}\right)$$

$$\approx 26.8 \;\; yrs$$

Resource Consumption

The rate at which energy is conssumed is called ***power***.
The basic unit power is the ***watt*** (**W**).
The basic unit energy is the ***joule*** (**J**).

$$1\,W = 1\,J/s$$

$$\text{Total energy used } = \int_a^b E'(t) = \int_a^b P(t)\,dt$$

$E(t)$: the total energy used

$P(t)$: Power is the rate at which energy used

Example

At the beginning of 2010, the rate energy consumption for the city of Denver was 7,000 megawatts (*MW*), where $1\,MW = 10^6\,W$. That rate is expected to increase at an annual growth rate of 2% per year.

a) Find the function that gives the power or rate of energy consumption for all times after the beginning of 2010.

b) Find the total amount of energy used during 2014.

c) Find the function that gives the total (cumulative) amount of energy used by the city between 2010 and any time $t \ge 0$.

Solution

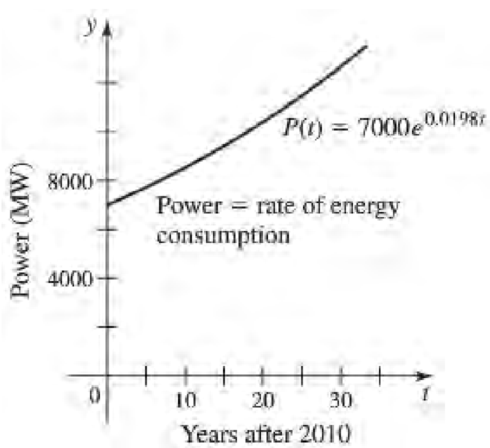
a) Let $t \ge 0$, be the number of years after the brginning of 2010.

$$k = \frac{1}{T}\ln\left(\frac{P(1)}{P_0}\right)$$

$$= \ln 1.02$$

$$\approx 0.0198$$

$$P(t) = 7,000e^{0.0198t}, \quad t \ge 0$$



b) Entire year 2014 $\rightarrow$ $4 \le t \le 5$

$$\textit{Total energy} = \int_4^5 P(t)\,dt$$

$$= \int_4^5 7,000\,e^{0.0198t}\,dt$$

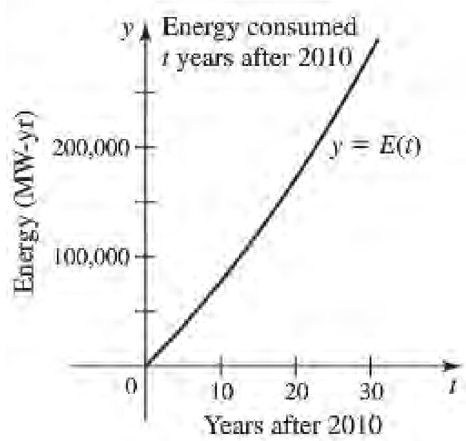
$$= \frac{7000}{0.0198}\,e^{0.0198t}\Big|_4^5$$

$$\approx 7652 \quad MW - yr$$

$$\approx 7652\ (MW \cdot yr) \times 8760 \frac{hr}{yr}$$

$$\boxed{\approx 6.7 \times 10^7 \quad MWh}$$

c) The total (cumulative) amount of energy used $t \ge 0$ is given by



$$E(t) = E(0) + \int_0^t E'(s)\,ds$$

$$= E(0) + \int_0^t P(s)\,ds$$

$$= 0 + \int_0^t 7000\, e^{0.0198s}\, ds$$

$$\boxed{\approx 353,535\left(e^{0.0198t} - 1\right)}$$

The total amount of energy consumed increases expotentially.

Exponential Decay Function

Exponential decay is described by functions of the form $y(t) = y_0 e^{-kt}$.

Rate constant: $k > 0$.

Initial value: y_0

Half-life is $T_{1/2} = \dfrac{\ln 2}{k}$

Example

Researchers determine that a fossilized bone has 30% of the C-14 of a live bone. Estimate the age of the bone. Assume a half-life for C-14 of ~5730 yrs.

Solution

$$k = \frac{\ln 2}{T_{1/2}}$$

$$= \frac{\ln 2}{5730}$$

$$\boxed{\approx 0.000121}$$

$$T = \frac{\ln \frac{y}{y_0}}{k}$$

$$= \frac{\ln 0.3}{-0.000121}$$

$$\boxed{\approx 9950\ yrs}$$

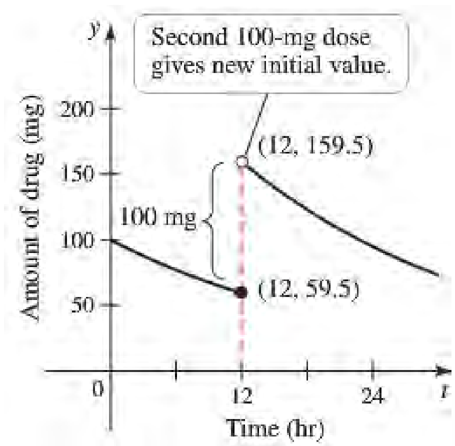
Example

An exponential decay function $y(t) = y_0 e^{-kt}$ models he amount of drug in the blood *t hr* after an initial dose of $y_0 = 100\ mg$ is administred. Assume the half-life of the drug is 16 hours.

a) Find the exponential decay function that governs the amount of drug in the blood.

b) How much time is required for the drug to reach 1% of the initial dose (1 *mg*)?

c) If a second 100-*mg* dose is given 12 *hr* after the first dose, how much time is required for the drug level to reach 1 *mg*?



Solution

a) $T_{1/2} = \frac{\ln 2}{k}$

$$= \frac{\ln 2}{16}$$

$$\boxed{\approx 0.0433}$$

$$\therefore\ \boxed{y(t) = 100e^{-0.0433t}}$$

b) $T = \dfrac{\ln \frac{1}{100}}{-0.0433}$

$$\boxed{\approx 106\ \ hrs}$$

It takes more than 4 days for the drug to be reduced to 1% of the initial dose.

c) $y(t = 12) = 100e^{-0.0433(12)}$

$$\boxed{\approx 59.5\ \ mg}$$

The second 100-*mg* dose given after 12 *hr* increases the amount of drug to 159.5 *mg* (new initial value)

$$\rightarrow\ y(t) = 159.5\, e^{-0.0433t}$$

The amount of drug reaches 1 mg in

$$t = \frac{\ln \frac{1}{159.5}}{-0.0433}$$

$$\boxed{\approx 117.1\ \ hrs}$$

Approximately 117 *hr* after the second dose (or 129 *hr* after the first dose), the amount of drug reaches 1 *mg*.

Exercises Section 1.8 – Exponential Models

(**1 – 26**) Find the derivative of

1. $y=\ln\left(\frac{\sqrt{\sin\theta\cos\theta}}{1+2\ln\theta}\right)$

2. $f(x)=e^{\left(4\sqrt{x}+x^2\right)}$

3. $f(t)=\ln\left(3te^{-t}\right)$

4. $f(x)=\dfrac{e^{\sqrt{x}}}{\ln\left(\sqrt{x}+1\right)}$

5. $y=\sqrt{\dfrac{(x+1)^{10}}{(2x+1)^5}}$

6. $f(x)=(2x)^{4x}$

7. $f(x)=2^{x^2}$

8. $h(y)=y^{\sin y}$

9. $f(x)=x^{\pi}$

10. $h(t)=(\sin t)^{\sqrt{t}}$

11. $p(x)=x^{-\ln x}$

12. $f(x)=x^{2x}$

13. $f(x)=x^{\tan x}$

14. $f(x)=x^{e}+e^{x}$

15. $f(x)=x^{x^{10}}$

16. $f(x)=\left(1+\frac{4}{x}\right)^{x}$

17. $f(x)=\cos\left(x^{2\sin x}\right)$

18. $f(x)=\ln(\ln x)$

19. $f(x)=\ln\left(\cos^2 x\right)$

20. $f(x)=\dfrac{\ln x}{\ln x+1}$

21. $f(x)=\dfrac{\ln x}{x}$

22. $f(x)=\dfrac{\tan^{10}x}{(5x+3)^6}$

23. $f(x)=\dfrac{(x+1)^{3/2}(x-4)^{5/2}}{(5x+3)^{2/3}}$

24. $f(x)=\dfrac{x^8\cos^3 x}{\sqrt{x-1}}$

25. $f(x)=(\sin x)^{\tan x}$

26. $f(x)=\left(1+\frac{1}{x}\right)^{2x}$

(**27 – 55**) Evaluate the integral

27. $\displaystyle\int\frac{2y}{y^2-25}dy$

28. $\displaystyle\int\frac{\sec y\tan y}{2+\sec y}dy$

29. $\displaystyle\int\frac{5}{e^{-5x}+7}dx$

30. $\displaystyle\int\frac{e^{2x}}{4+e^{2x}}dx$

31. $\displaystyle\int\frac{dx}{x\ln x\ln(\ln x)}$

32. $\displaystyle\int\frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

33. $\displaystyle\int\frac{e^{\sin x}}{\sec x}dx$

34. $\displaystyle\int\frac{e^x+e^{-x}}{e^x-e^{-x}}dx$

35. $\displaystyle\int\frac{4^{\cot x}}{\sin^2 x}dx$

36. $\displaystyle\int\frac{4x^2+2x+4}{x+1}dx$

37. $\displaystyle\int\frac{e^x}{4e^x+6}dx$

38. $\displaystyle\int\frac{x+4}{x^2+8x+25}dx$

39. $\displaystyle\int\frac{e^{2x}}{\sqrt{e^{2x}+4}}dx$

40. $\displaystyle\int\frac{x^2}{2x^3+1}dx$

41. $\displaystyle\int\frac{\sec^2 x}{\tan x}dx$

42. $\displaystyle\int_{e^2}^{e^8}\frac{dx}{x\ln x}$

43. $\displaystyle\int_1^4\frac{10^{\sqrt{x}}}{\sqrt{x}}\,dx$

44. $\int_{\ln 4}^{\ln 9} e^{x/2} dx$

45. $\int_{0}^{3} \frac{2x-1}{x+1} dx$

46. $\int_{e}^{e^2} \frac{dx}{x \ln^3 x}$

47. $\int_{0}^{\pi/2} \frac{\sin x}{1+\cos x} dx$

48. $\int_{3}^{4} \frac{dx}{2x \ln x \ln^3(\ln x)}$

49. $\int_{e^2}^{e^3} \frac{dx}{x \ln x \ln^2(\ln x)}$

50. $\int_{0}^{1} \frac{y \ln^4\left(y^2+1\right)}{y^2+1} dy$

51. $\int_{\ln 2}^{\ln 3} \frac{e^x + e^{-x}}{e^{2x} - 2 + e^{-2x}} dx$

52. $\int_{-2}^{2} \frac{e^{z/2}}{e^{z/2}+1} dz$

53. $\int_{0}^{\pi/2} 4^{\sin x} \cos x \, dx$

54. $\int_{1/3}^{1/2} \frac{10^{1/p}}{p^2} dp$

55. $\int_{1}^{2} (1+\ln x) x^x dx$

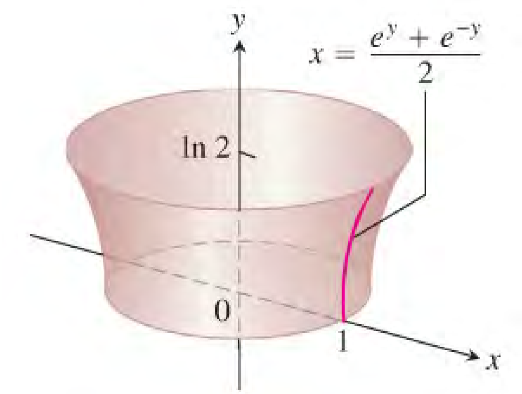
56. Find a curve through the origin in the *xy*-plane whose length from $x = 0$ to $x = 1$ is

$$L = \int_0^1 \sqrt{1+\tfrac{1}{4}e^x} \; dx$$

57. Find the length of the curve $y = \ln\left(e^x - 1\right) - \ln\left(e^x + 1\right)$ *from* $x = \ln 2$ *to* $x = \ln 3$

58. Find the length of the curve $y = \ln(\cos x)$ *from* $x = 0$ *to* $x = \frac{\pi}{4}$

59. Find the area of the surface generated by revolving the curve $x = \frac{1}{2}\left(e^y + e^{-y}\right), \quad 0 \le y \le \ln 2,$ about *y*-axis



60. The population of a town with a 2010 population of 90,000 grows at a rate of 2.4% /yr. In what year will the population coudle its initial value (to 180,000)?

61. How long will it take an initial deposit of \$1500 to increase in value to \$2500 in a saving account with an APY of 3.1%? Assume the interest rate reamins constant and no additional deposits or withdrawals are made.

62. The number of cells in a tumor doubles every 6 weeks starting with 8 cells. After how many weeks doses the tumor have 1500 cells?

63. According to the 2010 census, the U.S. population was 309 million with an estimated growth rate of 0.8% /*yr*.

a) Based on these figures, find the doubling time and project the population in 2050.

b) Suppose the actual growth rate is just 0.2 percentage point lower than 0.8% /*yr* (0.6%). What are the resulting doubling time and projected 2050 population? Repeat these calculstions assuming the growth rate is 0.2 percentage point higher than 0.8% /*yr*.

c) Comment on th sensitivity of these projections to the growth rate.

64. The homicide rate decreases at a rate of 3% per *year* in a city that had 800 homicides per *year* in 2010. At this rate, when will the homicide rate reach 600 homicides/yr?

65. A drug is eliminated from the body at a rate of 15% /*hr*. after how many hours does the amount of drug reach 10% of the initial dose?

66. The mass of radioactive material in a sample has decreased by 30% since the decay began. Assuming a half-life of 1500 *years*, how long ago did the decay begin?

67. Growing from an initial population of 150,000 at a constant annual growth rate of 4%/*yr*., how long will it take a city to reach a population of 1 *million*?

68. A savings account advertises an annual percentage yield (APY) of 5.4%, which means that the balance in the account increases at an annual growth rate of 5.4%/*yr*.

a) Find the balance in the account for $t \geq 0$ with an initial deposit of \$1500, assuming the APY remains fixed and no additional deposits or withdrawals are made.

b) What is the doubling time of the balance?

c) After how many years does the balance reach \$5,000?

69. A large die-casting machine used to make automobile engine blocks is purchased for \$2.5 *million*. For tax purposes, the value of the machine can be depreciated by 6.8% of its current value each year.

a) What is the value of the machine after 10 *years*?

b) After how many years is the value of the machine 10% of its original value?

70. Roughly 12,000 Americans are diagmosed with thyroid cancer every year, which accounts for 1% of all cancer cases. It occurs in women three times as frequency as in men. Fortunately, thyroid cancer can be treated successfully in many cases with radioactive iodine, or I-131. This unstable form of iodine has a half-life of 8 days and is given in small doses meansured in millicuries.

a) Suppose a patient is given an initial dose of 100 millicuries. Find the function that gives the amount of I-131 in the body after $t \geq 0$ days.

b) How long does it take the amount of I-131 to reach 10% of the initial dose?

c) Finding the initial dose to give a particular patient is a critical calculation. How does the time reach 10% of the initial dose change if the initial dose is increased by 5%?

71. City ***A*** has a current population of 500,000 people and grows at a rate of 3% /*yr*. City ***B*** has a cuurent population of 300,000 and grows at a rate of 5%/*yr*.

a) When will the cities have the same population?

b) Suppose City ***C*** has a current population of $y_0 < 500{,}000$ and a growth rate of $p > 3\% / yr$. What is the relationship between y_0 and p such that the Cities ***A*** and ***C*** have the same population in 10 years?

72. Suppose the acceleration of an object moving along a line is given by $a(t) = -kv(t)$, where k is a positive constant and v is the object's velocity. Assume that the initial velocity and position are given by $v(0) = 10$ and $s(0) = 0$, respectively.

a) Use $a(t) = v'(t)$ to find the velocity of the object as a function of time.

b) Use $v(t) = s'(t)$ to find the position of the object as a function of time.

c) Use the fact that $\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt}$ (by the *Chain Rule*) to find the velocity as a function of position.

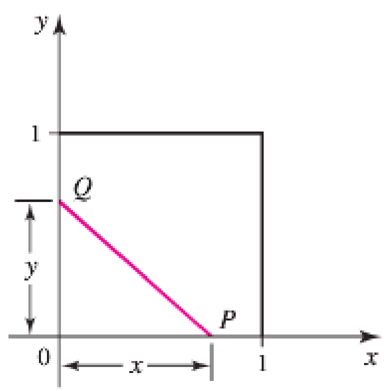
73. On the first day of the year $(t = 0)$, a city uses electricity at a rate of 2000 MW. That rate is projected to increase at a rate of 1.3% per *year*.

a) Based on these figures, find an exponential growth function for the power (rate of electricity use) for the city.

b) Find the total energy (in MW-*yr*) used by the city over four full years beginning at $t = 0$

c) Find a function that gives the total energy used (in MW-*yr*) between $t = 0$ and any future time $t > 0$

74. Two points P and Q are chosen randomly, one on each of two adjacent sides of a unit square.

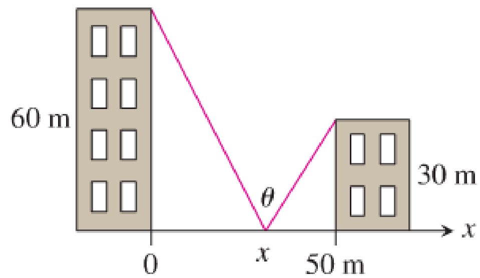


What is the probability that the area of the triangle formed by the sides of the square and the line segment PQ is less than one-fourth the area of the square? Begin by showing that x and y must satisfy $xy < \frac{1}{2}$ in order for the area consition to be met. Then argue that the required probability is

$$\frac{1}{2} + \int_{1/2}^{1} \frac{dx}{2x}$$ and evaluate the integral.

75. You are under contract to build a solar station at ground level on the east-west line between the two buildings. How far from the taller building should you place the station to maximize the number of hours it will be in the sun on a day when passes directly overhead? Begin by observing that

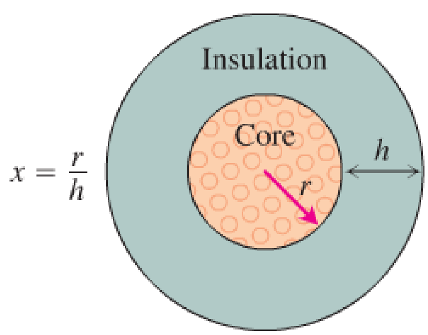
$$\theta = \pi - \cot^{-1}\left(\frac{x}{60}\right) - \cot^{-1}\left(\frac{50-x}{30}\right)$$



Then find the value of x that maximizes θ.

76. A round underwater transmission cable consists of a core of copper wires surrounded by nonconducting insulation. If x denotes the ratio of the radius of the core to the thickness of the insulation, it is known that the speed of the transmission signal is given by the equation $v = x^2 \ln\left(\frac{1}{x}\right)$.

If the radius of the core is 1 *cm*, what insulation thickness h will allow the greatest transmission speed?



77. A commonly used distribution in probability and statistics is the log-normal distribution. (If the logarithm of a variable has a normal distribution, then the variable itself has a log-normal distribution.) the distribution function is

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{\ln^2 x}{2\sigma^2}}, \quad \textit{for} \quad x > 0$$

Where $\ln x$ has zero mean and standard deviation $\sigma > 0$.

a) Graph f for $\sigma = \frac{1}{2}$, 1, *and* 2. Based on your graphs, does $\lim\limits_{x \to 0^+} f(x)$ appear to exist?

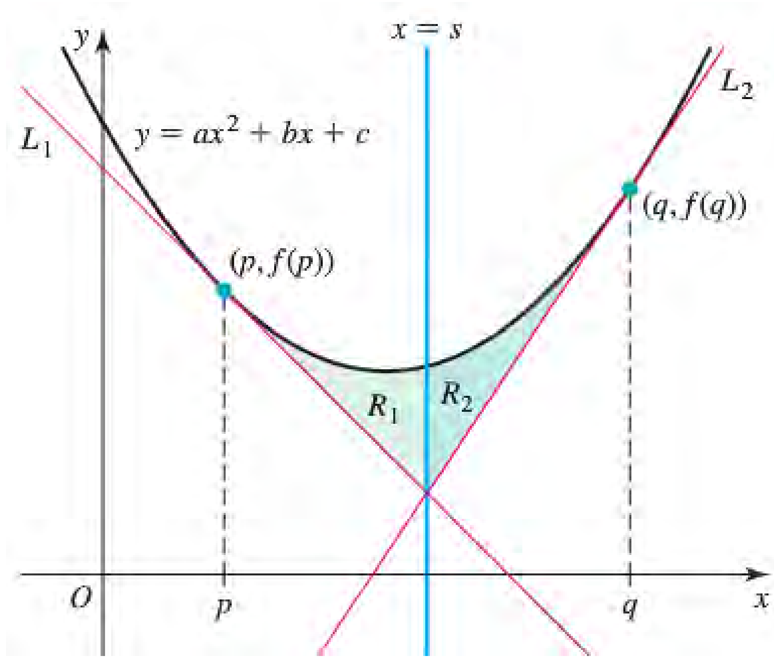
b) Evaluate $\lim\limits_{x \to 0^+} f(x)$. (*Hint*: Let $x = e^y$)

c) Show that f has a single local maximum at $x^* = e^{-\sigma^2}$

d) Evaluate $f\left(x^*\right)$ and express the result as a function of σ.

e) For what value of $\sigma > 0$ in part (*d*) does $f\left(x^*\right)$ have a minimum?

78. Let $f(x) = ax^2 + bx + c$ be an arbitrary quadratic function and choose two points $x = p$ and $x = q$. Let L_1 be the line tangent to the graph of f at the point $\left(p, f(p)\right)$ and let L_2 be the line tangent to the graph at the point $\left(q, f(q)\right)$. Let $x = s$ be the vertical line through the intersection point of L_1 and L_2. Finally, let R_1 be the region bounded by $y = f(x)$, L_1, and the vertical line $x = s$, and let R_2 be the region bounded by $y = f(x)$, L_2, and the vertical line $x = s$.

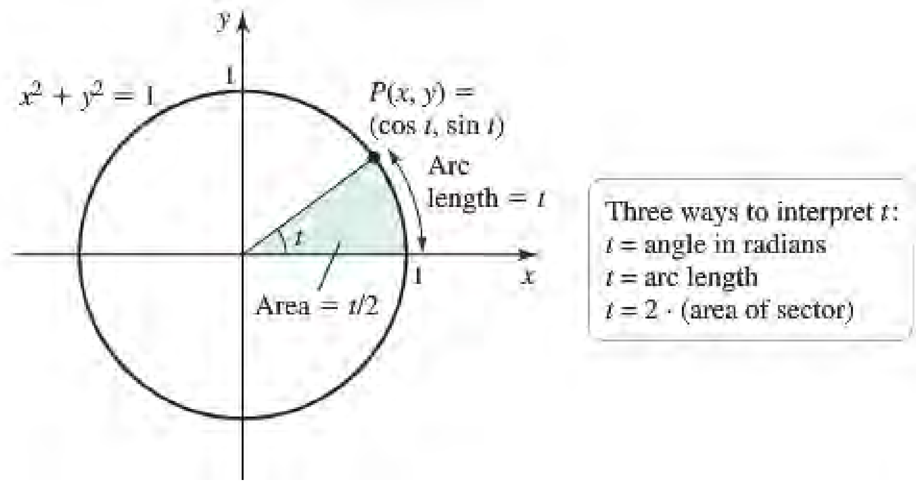


Prove that the area of R_1 equals the area of R_2

Section 1.9 – Hyperbolic Functions

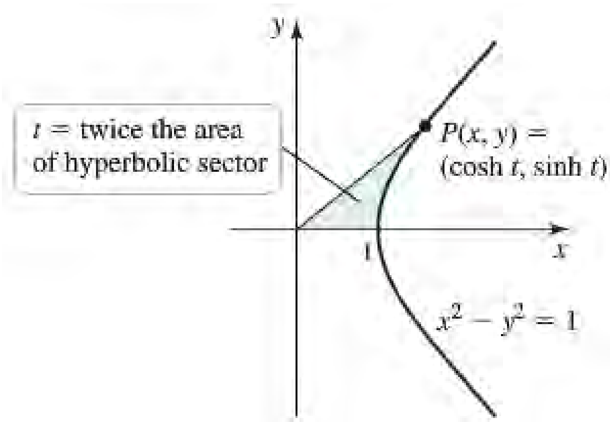
Relationship Between Trigonometric and Hyperbolic Functions

The trigonometric functions are based on relationships involving a circle, also known as ***circular*** functions. Specifically, $\cos t$ and $\sin t$ are equal to *x*- and *y*-coordinates, respectively, of the point $P(x, y)$ on the unit circle that corresponds to an angle of *t* radians.



Observe that *t* is twice the area of the circular sector.

The ***hyperbolic cosine*** and ***hyperbolic sine*** are defined in analogous fashion using the hyperbola $x^2 - y^2 = 1$ instead the circle $x^2 + y^2 = 1$.



Consider the region bouunded by the x-axis, the right branch of the unit hyperbola $x^2 - y^2 = 1$, and a line segment from the origin to a point $P(x, y)$ on the hyperbola; let *t* equal twice the area of this region.

The hyperbolic functions are formed by taking combinations of the two exponential functions e^x and e^{-x}

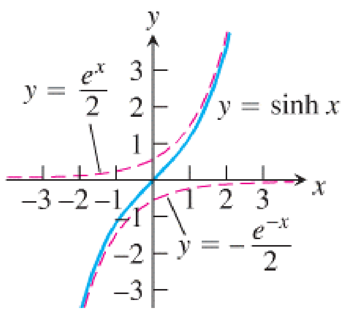
Definitions, Identities, and Graphs of the Hyperbolic Functions

The hyperbolic sine and hyperbolic cosine functions are defined by the equations

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \textit{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

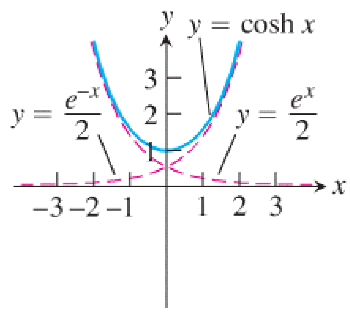
We pronounce: sinh x as “cinch x”, rhyming with “pinch x”

cosh x as “kosh x”, rhyming with “gosh x”



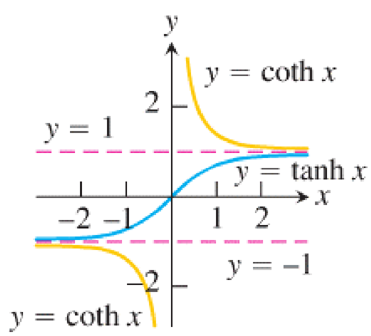
Hyperbolic sine

$$\sinh x = \frac{e^x - e^{-x}}{2}$$



Hyperbolic cosine

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

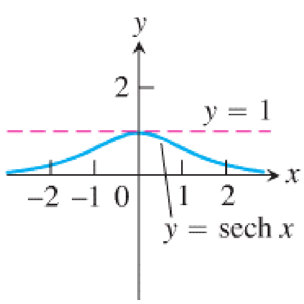


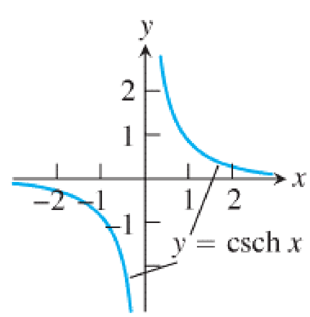
Hyperbolic tangent

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$





Hyperbolic secant

$$\text{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cosecant

$$\text{csch}\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Example

Derive identity $\sinh 2x = 2\sinh x \cosh x$

Solution

$$2\sinh x\cosh x = 2\left(\frac{e^{x}-e^{-x}}{2}\right)\left(\frac{e^{x}+e^{-x}}{2}\right)$$

$$=\left(\frac{e^{2x}-e^{-2x}}{2}\right)$$

$$\underline{= \sinh 2x|} \quad \checkmark$$

Example

Use the fundamental identity $\cosh^2 x - \sinh^2 x = 1$ to prove that $1-\tanh^2 x = \operatorname{sech}^2 x$

Solution

$$\frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

$$\underline{1-\tanh^2 x = \operatorname{sech}^2 x|} \quad \checkmark$$

Circular Functions: $\cosh^2 u - \sinh^2 u = 1$

Identities	*Derivatives*	*Integral*
$\cosh^2 u - \sinh^2 u = 1$	$\frac{d}{dx}(\sinh u) = u' \cosh u$	$\int \sinh u \, du = \cosh u + C$
$\sinh 2x = 2\sinh x \cosh x$	$\frac{d}{dx}(\cosh u) = u' \sinh u$	$\int \cosh u \, du = \sinh u + C$
$\cosh 2x = \cosh^2 x + \sinh^2 x$	$\frac{d}{dx}(\tanh u) = u' \operatorname{sech}^2 u$	$\int \operatorname{sech}^2 u \, du = \tanh u + C$
$\cosh^2 x = \frac{\cosh 2x + 1}{2}$	$\frac{d}{dx}(\coth u) = -u' \operatorname{csch}^2 u$	$\int \operatorname{csch}^2 u \, du = -\coth u + C$
$\sinh^2 x = \frac{\cosh 2x - 1}{2}$	$\frac{d}{dx}(\operatorname{sech} u) = -u' \operatorname{sech} u \ \tanh u$	$\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$
$\tanh^2 x = 1 - \operatorname{sech}^2 x$	$\frac{d}{dx}(\operatorname{csch} u) = -u' \operatorname{csch} u \ \coth u$	$\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$
$\coth^2 x = 1 + \operatorname{csch}^2 x$		

Example

a) $\frac{d}{dt}\left(\tanh\sqrt{1+t^2}\right) = \operatorname{sech}^2\sqrt{1+t^2} \cdot \frac{d}{dt}\left(\sqrt{1+t^2}\right)$

$$= \frac{t}{\sqrt{1+t^2}} \operatorname{sech}^2\sqrt{1+t^2}$$

b) $\frac{d^2}{dx^2}(\operatorname{sech} 3x) = \frac{d}{dx}(-3 \operatorname{sech} 3x \tanh 3x)$

$$= 9 \operatorname{sech} 3x \tanh^2 3x - 9 \operatorname{sech}^3 3x$$

$$= 9 \operatorname{sech} 3x \left(\tanh^2 3x - \operatorname{sech}^2 3x\right)$$

c) $\int \coth 5x dx = \int \frac{\cosh 5x}{\sinh 5x} dx$ $\qquad d(\sinh 5x) = 5\cosh 5x dx$

$$= \frac{1}{5}\int \frac{d(\sinh 5x)}{\sinh 5x}$$

$$= \frac{1}{5}\ln\left|\sinh 5x\right| + C$$

d) $$\int_0^1 \sinh^2 x\,dx = \int_0^1 \frac{\cosh 2x - 1}{2}\,dx$$

$$= \frac{1}{2}\int_0^1 (\cosh 2x - 1)\,dx$$

$$= \frac{1}{2}\left(\frac{1}{2}\sinh 2x - x\right|_0^1$$

$$= \frac{1}{2}\left[\left(\frac{1}{2}\sinh 2 - 1\right) - \left(\frac{1}{2}\sinh 0 - 0\right)\right]$$

$$= \frac{1}{2}\left(\frac{1}{2}\sinh 2 - 1\right)$$

$$\approx 0.40672$$

e) $$\int_0^{\ln 2} 4e^x \sinh x\,dx = \int_0^{\ln 2} 4\,e^x\,\frac{e^x - e^{-x}}{2}\,dx$$

$$= 2\int_0^{\ln 2} \left(e^{2x} - 1\right)dx$$

$$= 2\left(\frac{1}{2}e^{2x} - x\right|_0^{\ln 2}$$

$$= 2\left[\left(\frac{1}{2}e^{2\ln 2} - \ln 2\right) - \left(\frac{1}{2}e^0 - 0\right)\right]$$

$$= 2\left(\frac{1}{2}e^{\ln 2^2} - \ln 2 - \frac{1}{2}\right)$$

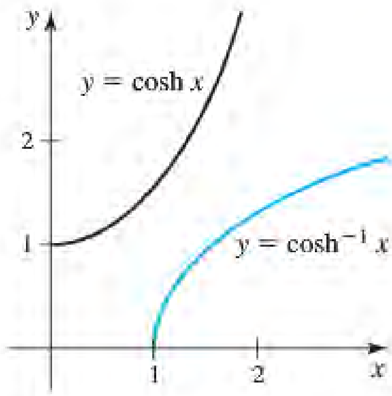
$$= 4 - 2\ln 2 - 1$$

$$\approx 1.6137$$

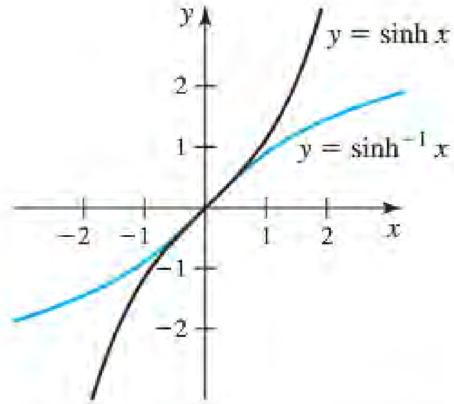
f) $$\int x\coth\left(x^2\right)dx = \frac{1}{2}\int \coth\left(x^2\right)d\left(x^2\right)$$

$$= \frac{1}{2}\ln\left(\sinh x^2\right) + C$$

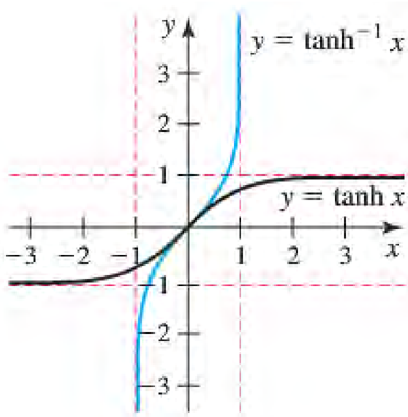
Inverse Hyperbolic Functions



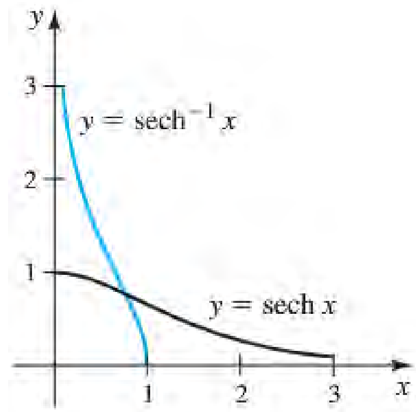
$y = \cosh^{-1} x \Leftrightarrow x = \cosh y$
for $x \geq 1$ and $0 \leq y < \infty$



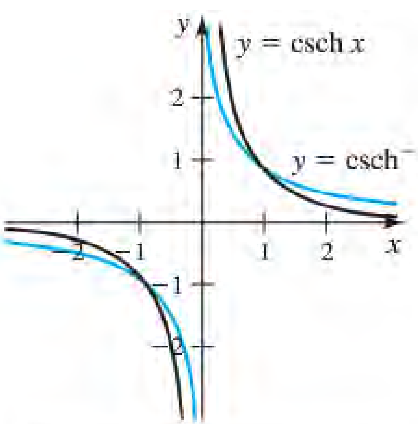
$y = \sinh^{-1} x \Leftrightarrow x = \sinh y$
for $-\infty < x < \infty$ and $-\infty < y < \infty$



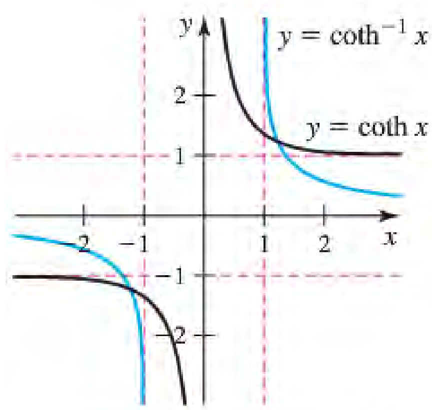
$y = \tanh^{-1} x \Leftrightarrow x = \tanh y$
for $-1 < x < 1$ and $-\infty < y < \infty$



$y = \operatorname{sech}^{-1} x \Leftrightarrow x = \operatorname{sech} y$
for $0 < x \leq 1$ and $0 \leq y < \infty$



$y = \operatorname{csch}^{-1} x \Leftrightarrow x = \operatorname{csch} y$
for $x \neq 0$ and $y \neq 0$



$y = \coth^{-1} x \Leftrightarrow x = \coth y$
for $|x| > 1$ and $y \neq 0$

$$\operatorname{sech}\left(\cosh^{-1}\left(\frac{1}{x}\right)\right)=\frac{1}{\cosh\left(\cosh^{-1}\left(\frac{1}{x}\right)\right)}$$

$$=\frac{1}{\frac{1}{x}}$$

$$\underline{=x\,|}$$

Identities	***Derivatives***	***Integral***
$\operatorname{sech}^{-1}x=\cosh^{-1}\frac{1}{x}$	$\frac{d}{dx}\left(\sinh^{-1}u\right)=\frac{u'}{\sqrt{1+u^2}}$	$\int\frac{du}{\sqrt{a^2+u^2}}=\sinh^{-1}\left(\frac{u}{a}\right)+C,\quad a>0$
$\operatorname{csch}^{-1}x=\sinh^{-1}\frac{1}{x}$	$\frac{d}{dx}\left(\cosh^{-1}u\right)=\frac{u'}{\sqrt{u^2-1}}$	$\int\frac{du}{\sqrt{u^2-a^2}}=\cosh^{-1}\left(\frac{u}{a}\right)+C,\quad u>a>0$
$\coth^{-1}x=\tanh^{-1}\frac{1}{x}$	$\frac{d}{dx}\left(\tanh^{-1}u\right)=\frac{u'}{1-u^2}$	$\int\frac{du}{a^2-u^2}=\frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right)+C,\quad u^2<a^2$
	$\frac{d}{dx}\left(\coth^{-1}u\right)=\frac{u'}{1-u^2}$	$\int\frac{du}{a^2-u^2}=\frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right)+C,\quad u^2>a^2$
	$\frac{d}{dx}\left(\operatorname{sech}^{-1}u\right)=-\frac{u'}{u\sqrt{1-u^2}}$	$\int\frac{du}{u\sqrt{a^2-u^2}}=-\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right)+C,\quad 0<u<a$
	$\frac{d}{dx}\left(\operatorname{csch}^{-1}u\right)=-\frac{u'}{\lvert u\rvert\sqrt{1+u^2}}$	$\int\frac{du}{u\sqrt{a^2+u^2}}=-\frac{1}{a}\operatorname{csch}^{-1}\left\lvert\frac{u}{a}\right\rvert+C,\quad u\neq 0, a>0$
		$\int\frac{du}{\sqrt{u^2\pm a^2}}=\ln\left(u+\sqrt{u^2\pm a^2}\right)+C$
		$\int\frac{du}{a^2-u^2}=\frac{1}{2a}\ln\left\lvert\frac{a+u}{a-u}\right\rvert+C$
		$\int\frac{du}{u\sqrt{a^2\pm u^2}}=-\frac{1}{a}\ln\frac{a+\sqrt{a^2\pm u^2}}{\lvert u\rvert}+C$

Example

Show that if u is a differentiable function of x whose values are greater than 1, then

$$\frac{d}{dx}\left(\cosh^{-1}u\right)=\frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$$

Solution

$$\left(f^{-1}\right)'(x)=\frac{1}{f'\left(f^{-1}(x)\right)}$$

$$=\frac{1}{\sinh\left(\cosh^{-1}x\right)}$$

$$\cosh^2 u-\sinh^2 u=1 \Rightarrow \sinh u=\sqrt{\cosh^2 u-1}$$

$$=\frac{1}{\sqrt{\cosh^2\left(\cosh^{-1}x\right)-1}}$$

$$\cosh\left(\cosh^{-1}x\right)=x$$

$$=\frac{1}{\sqrt{x^2-1}}$$

$$\boxed{\frac{d}{dx}\left(\cosh^{-1}u\right)=\frac{1}{\sqrt{u^2-1}}\frac{du}{dx}}$$

Example

Evaluate $\displaystyle\int_0^1\frac{2dx}{\sqrt{3+4x^2}}$

Solution

$$a=\sqrt{3},\quad u=2x \quad \rightarrow du=2dx$$

$$\int_0^1\frac{2dx}{\sqrt{3+4x^2}}=\int_0^1\frac{du}{\sqrt{a^2+u^2}}$$

$$=\sinh^{-1}\left(\frac{u}{a}\right)\Big|_0^1$$

$$=\sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\Big|_0^1$$

$$=\sinh^{-1}\left(\frac{2}{\sqrt{3}}\right)-\sinh^{-1}(0)$$

$$\boxed{=\sinh^{-1}\left(\frac{2}{\sqrt{3}}\right)}$$

$$\boxed{\approx 0.98665}$$

Example

Find the points at which the curves $y = \cosh x$ and $y = \frac{5}{3}$ intersect.

Solution

$$\cosh x = \frac{5}{3}$$

$$\cosh^{-1}(\cosh x) = \cosh^{-1}\left(\frac{5}{3}\right)$$

$$|x| = \ln\left(\frac{5}{3} + \sqrt{\left(\frac{5}{3}\right)^2 + 1}\right)$$

$$= \ln\left(\frac{5}{3} + \frac{4}{3}\right)$$

$$\underline{x = \pm \ln 3}$$

The points of intersection lie on the line $y = \frac{5}{3}$, are $\left(-\ln 3,\ \frac{5}{3}\right)$ *and* $\left(\ln 3,\ \frac{5}{3}\right)$

Example

Find the derivative $y = \tanh^{-1} 3x$

Solution

$$\boxed{y' = \frac{3}{1 - 9x^2}}$$

Example

Find the derivative $y = x^2 \sinh^{-1} x$

Solution

$$\boxed{y' = 2x \sinh^{-1} x + x^2 \frac{1}{\sqrt{x^2 + 1}}}$$

Example

Evaluate $\displaystyle\int_0^3 \frac{dx}{\sqrt{x^2 + 16}}$

Solution

$$\int_0^3 \frac{dx}{\sqrt{x^2 + 16}} = \sinh^{-1} \frac{x}{4} \Big|_0^3$$

$$= \sinh^{-1}\frac{3}{4} - \sinh^{1} 0$$

$$\boxed{= \sinh^{-1}\frac{3}{4}} \quad \boxed{\approx 0.639}$$

Example

Evaluate $\displaystyle\int_{9}^{25} \frac{dx}{\sqrt{x}(4-x)}$

Solution

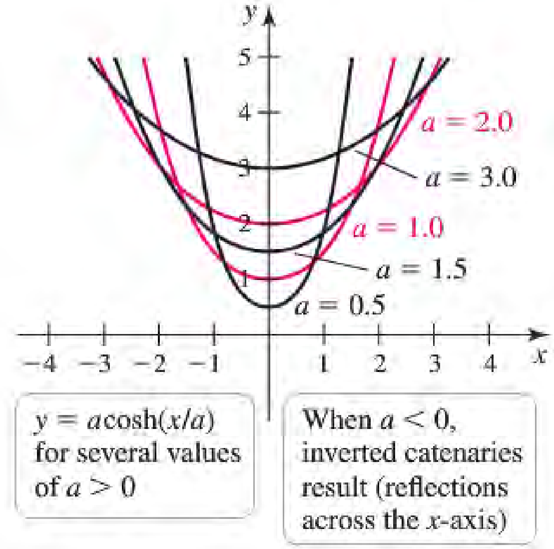
$$\int_{9}^{25} \frac{dx}{\sqrt{x}(4-x)} = 2\int_{9}^{25} \frac{du}{4-u^2} \qquad\qquad u = \sqrt{x} \to u^2 = x \quad du = \frac{1}{2}\frac{1}{\sqrt{x}}dx$$

$$= 2\frac{1}{2}\coth^{-1}\frac{\sqrt{x}}{2}\Big|_{9}^{25}$$

$$\boxed{= \coth^{-1}\frac{5}{2} - \coth^{-1}\frac{3}{2}}$$

Applications of Hyperbolic Functions

The Catenary: When a free-hanging rope or flexible cable supporting only its own weight is attached to 2 points of equal height, it takes the shape of a curve known as a ***catenary***.



Example

A climber anchors a rope at 2 points of equal height, separated by a distance of 100 *feet*. in order to perform a Tyrolean traverse. The rope follows the catenary $f(x)=200\cosh\frac{x}{200}$ over the interval $[-50,\ 50]$. Find the length of the rope between the two anchor points.

Solution

$f'(x)=\sinh\frac{x}{200}$

$\sqrt{1+f'(x)^2}=\sqrt{1+\sinh^2\frac{x}{200}}$ $\qquad\cosh^2 u-\sinh^2 u=1$

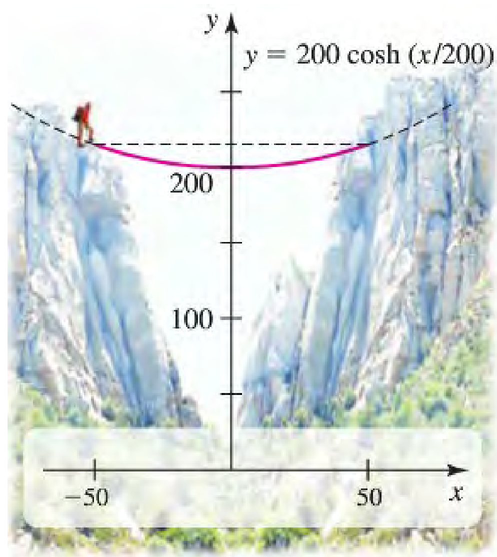
$\qquad=\cosh\frac{x}{200}$

$L=\int_{-50}^{50}\cosh\frac{x}{200}dx$ $\qquad L=\int_a^b\sqrt{1+f'(x)^2}\,dx$

$\quad=2(200)\sinh\frac{x}{200}\Big|_0^{50}$

$\quad=400\sinh\frac{1}{4}$

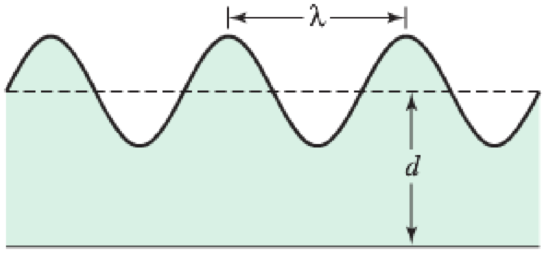
$\quad\approx 101\ ft$



Example

The velocity *v* (in *m/s*) of an idealized surface wave traveling on the ocean is modeled by the equation

$$v=\sqrt{\frac{g\lambda}{2\pi}\tanh\left(\frac{2\pi d}{\lambda}\right)}$$



Where $g=9.8\ m/s^2$ is the acceleration due thegravity, λ is the wavelength measured in meters from crest to crest, and *d* is the depth of the undisturbed water, also measured in *meters*.

a) A sea kayaker observes several waves that pass neneath her hayak, and she estimates that $\lambda=12\ m$ and $v=4\ m/s$. How deep is the water in which she is kayaking?

b) The deep-water equation for wave velocity is $v=\sqrt{\frac{g\lambda}{2\pi}}$, which is an approximation to the velocity formula given above. Waves are said to be in deep water if the depth-to-wavelength ration d/λ is greater than $\frac{1}{2}$. Explain why $v=\sqrt{\frac{g\lambda}{2\pi}}$ is a good approximation when $\frac{d}{\lambda}>\frac{1}{2}$.

Solution

a) ***Given***: $\lambda=12\ m$, $v=4\ m/s$

$$4=\sqrt{\frac{9.8(12)}{2\pi}\tanh\left(\frac{2\pi d}{12}\right)}$$

$$16=\frac{117.6}{2\pi}\tanh\left(\frac{\pi d}{6}\right)$$

$$\frac{32\pi}{117.6}=\tanh\left(\frac{\pi d}{6}\right)$$

$$\frac{\pi d}{6}=\tanh^{-1}\left(\frac{32\pi}{117.6}\right)$$

$$d=\frac{6}{\pi}\tanh^{-1}\left(\frac{32\pi}{117.6}\right)$$

$$\approx 2.4\quad m$$

Therefore, the kayaker is in water that is about 2.4 *m* deep.

b) Since $\frac{d}{dx}\tanh x=\operatorname{sech}^2 x>0$, then $\tanh x$ is an incrasing function whose values approaches 1 as $x\to\infty$.

Also when $\frac{d}{\lambda}=\frac{1}{2}$, $\tanh\left(\frac{2\pi d}{\lambda}\right)=\tanh\pi\approx 0.996$, which is nearly equal to 1.

These facts imply that whenever $\frac{d}{\lambda} > \frac{1}{2}$, we can replace $\tanh\left(\frac{2\pi d}{\lambda}\right)$ with 1 in the velocity formula, resulting in the deep-water velocity function $v = \sqrt{\frac{g\lambda}{2\pi}}$.

$$y = \sinh^{-1} x$$

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - e^{-y}$$

$$2xe^y = e^y e^y - e^{-y} e^y$$

$$e^{2y} - 2xe^y - 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

$$y = \ln\left(x \pm \sqrt{x^2 + 1} \right)$$

Since $x - \sqrt{x^2 + 1} < 0$ (***impossible***)

$$\therefore\ y = \ln\left(x + \sqrt{x^2 + 1} \right)$$

$$y = \cosh^{-1} x$$

$$x = \cosh y = \frac{e^y + e^{-y}}{2}$$

$$2x = e^y + e^{-y}$$

$$2xe^y = e^y e^y + e^{-y} e^y$$

$$e^{2y} - 2xe^y + 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

$$y = \ln\left(x \pm \sqrt{x^2 - 1} \right)$$

Inverse Hyperbolic Functions	
$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1} \right)$	$(-\infty,\ \infty)$
$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1} \right)$	$[1,\ \infty)$
$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x}$	$(-1,\ 1)$
$\coth^{-1} x = \frac{1}{2} \ln \frac{x+1}{x-1}$	$(-\infty,\ -1) \cup (1,\ \infty)$
$\operatorname{sech}^{-1} x = \ln \frac{1+\sqrt{1-x^2}}{x}$	$(0,\ 1]$
$\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{\lvert x \rvert} \right)$	$(-\infty,\ 0) \cup (0,\ \infty)$

Exercises Section 1.9 – Hyperbolic Functions

1. Rewrite the expression $\cosh 3x - \sinh 3x$ in terms of exponentials and simplify the results as much as you can.

2. Rewrite the expression $\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x)$ in terms of exponentials and simplify the results as much as you can.

3. Prove the identities

a) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

b) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

(4 – 39) Find the derivative of

4. $y = \frac{1}{2}\sinh(2x+1)$

5. $y = 2\sqrt{t}\tanh\sqrt{t}$

6. $y = \ln(\cosh z)$

7. $y = \operatorname{csch}\theta(1 - \ln \operatorname{csch}\theta)$

8. $y = \ln \sinh v - \frac{1}{2}\coth^2 v$

9. $y = \left(x^2+1\right)\operatorname{sech}(\ln x)$

10. $y = \left(4x^2-1\right)\operatorname{csch}(\ln 2x)$

11. $y = \cosh^{-1} 2\sqrt{x+1}$

12. $y = \left(\theta^2 + 2\theta\right)\tanh^{-1}(\theta+1)$

13. $y = (1-t)\coth^{-1}\sqrt{t}$

14. $y = \ln x + \sqrt{1-x^2}\ \operatorname{sech}^{-1} x$

15. $y = \operatorname{csch}^{-1}\left(\frac{1}{2}\right)^{\theta}$

16. $y = \cosh^{-1}(\sec x)$

17. $y = -\sinh^3 4x$

18. $y = \sqrt{\coth 3x}$

19. $y = \dfrac{x}{\operatorname{csch} x}$

20. $y = \tanh^2 x$

21. $y = \ln(\operatorname{sech} 2x)$

22. $y = x^2 \cosh^2 3x$

23. $f(t) = 2\tanh^{-1}\sqrt{t}$

24. $f(x) = \sinh^{-1} x^2$

25. $f(x) = \operatorname{csc}h^{-1}\left(\frac{2}{x}\right)$

26. $f(x) = x\sinh^{-1} x - \sqrt{x^2+1}$

27. $f(x) = \sinh^{-1}(\tan x)$

28. $y = 6\sinh\frac{x}{3}$

29. $y = \ln(\sinh x)$

30. $y = x^2 \tanh x$

31. $y = x^2 \tanh\frac{1}{x}$

32. $y = x\ \operatorname{sech} x$

33. $y = \sinh^{-1}\sqrt{x}$

34. $y = (1-t)\tanh^{-1} t$

35. $f(x) = \sinh\left(x^2-3\right)$

36. $f(x) = x\sinh x - \cosh x$

37. $f(x) = \ln(\sinh x)$

38. $f(x) = \ln\left(\tanh \frac{x}{2}\right)$

39. $f(x) = \arctan(\sinh x)$

(**40 – 41**) Compute the following derivatives

40. $\dfrac{d^5}{dx^5}(\cosh x)$

41. $\dfrac{d^6}{dx^6}(\sinh x)$

42. Verify the integration $\displaystyle\int x\,\text{sech}^{-1}x\,dx = \frac{x^2}{2}\text{sech}^{-1}x - \frac{1}{2}\sqrt{1-x^2} + C$

43. Verify the integration $\displaystyle\int \tanh^{-1}x\,dx = x\tanh^{-1}x + \frac{1}{2}\ln\left(1-x^2\right) + C$

(**44 – 101**) Evaluate the integral

44. $\displaystyle\int \sinh 2x\,dx$

45. $\displaystyle\int 4\cosh(3x - \ln 2)\,dx$

46. $\displaystyle\int \tanh\frac{x}{7}\,dx$

47. $\displaystyle\int \coth\frac{\theta}{\sqrt{3}}\,d\theta$

48. $\displaystyle\int \text{csch}^2(5-x)\,dx$

49. $\displaystyle\int \frac{\text{sech}\sqrt{t}\,\tanh\sqrt{t}}{\sqrt{t}}\,dt$

50. $\displaystyle\int \frac{\text{csch}(\ln t)\,\coth(\ln t)}{t}\,dt$

51. $\displaystyle\int \frac{\sinh x}{1+\cosh x}\,dx$

52. $\displaystyle\int \text{sech}^2 x\,\tanh x\,dx$

53. $\displaystyle\int \coth^2 x\,\text{csch}^2 x\,dx$

54. $\displaystyle\int \tanh^2 x\,dx$

55. $\displaystyle\int \frac{\sinh(\ln x)}{x}\,dx$

56. $\displaystyle\int \frac{dx}{8-x^2}$ $\quad x > 2\sqrt{2}$

57. $\displaystyle\int \frac{dx}{\sqrt{x^2-16}}$

58. $\displaystyle\int \sinh\frac{x}{5}\,dx$

59. $\displaystyle\int 6\cosh\left(\frac{x}{2} - \ln 3\right)dx$

60. $\displaystyle\int \text{sech}^2(2x-1)\,dx$

61. $\displaystyle\int \text{sech}^2\left(x - \frac{1}{2}\right)dx$

62. $\displaystyle\int \frac{\cosh x}{\sinh x}\,dx$

63. $\displaystyle\int x\,\mathrm{csch}^2\frac{x^2}{2}\;dx$

64. $\displaystyle\int \mathrm{sech}^3 x\;\tanh x\;dx$

65. $\displaystyle\int \frac{\mathrm{csch}\left(\frac{1}{x}\right)\coth\left(\frac{1}{x}\right)}{x^2}\;dx$

66. $\displaystyle\int \frac{\cosh x}{\sqrt{9-\sinh^2 x}}\;dx$

67. $\displaystyle\int \frac{x}{x^4+1}\;dx$

68. $\displaystyle\int \frac{2}{x\sqrt{1+4x^2}}\;dx$

69. $\displaystyle\int \frac{dx}{\sqrt{x^2-9}}$

70. $\displaystyle\int \cosh 2x\sinh^2 2x\;dx$

71. $\displaystyle\int \frac{1}{\sqrt{9x^2+25}}\,dx$

72. $\displaystyle\int \frac{1}{\sqrt{49-4x^2}}\,dx$

73. $\displaystyle\int \frac{e^x}{\sqrt{e^{2x}-16}}\,dx$

74. $\displaystyle\int \frac{e^x}{16-e^{2x}}\,dx$

75. $\displaystyle\int \frac{\mathrm{sech}^2 x}{2+\tanh x}\,dx$

76. $\displaystyle\int_{\ln 2}^{\ln 3} \coth x\;dx$

77. $\displaystyle\int_0^1 \frac{x^2}{9-x^6}\,dx$

78. $\displaystyle\int_0^{\ln 2} \tanh x\;dx$

79. $\displaystyle\int_0^1 \cosh^2 x\;dx$

80. $\displaystyle\int_0^4 \frac{1}{25-x^2}\;dx$

81. $\displaystyle\int_0^4 \frac{1}{\sqrt{25-x^2}}\;dx$

82. $\displaystyle\int_0^{\frac{\sqrt{2}}{4}} \frac{2}{\sqrt{1-4x^2}}\;dx$

83. $\displaystyle\int_0^{\ln 2} 2e^{-x}\cosh x\;dx$

84. $\displaystyle\int_0^{\ln 2} 2e^{x}\sinh x\;dx$

85. $\displaystyle\int_0^1 \cosh^3 3x\;\sinh 3x\;dx$

86. $\displaystyle\int_0^4 \frac{\mathrm{sech}^2\sqrt{x}}{\sqrt{x}}\,dx$

87. $\displaystyle\int_{\ln 2}^{\ln 3} \mathrm{csch} x\;dx$

88. $\displaystyle\int_{\ln 2}^{\ln 4} \coth x\;dx$

89. $\displaystyle\int_0^{\pi/2} 2\sinh(\sin\theta)\cos\theta\,d\theta$

90. $\displaystyle\int_1^2 \frac{8\cosh\sqrt{x}}{\sqrt{x}}\,dx$

91. $\displaystyle\int_{-\ln 2}^0 \cosh^2\left(\frac{x}{2}\right)dx$

92. $\displaystyle\int_0^{\ln 2} 4e^{-\theta}\sinh\theta\,d\theta$

93. $\displaystyle\int_1^{e^2} \frac{dx}{x\sqrt{\ln^2 x+1}}$

94. $\displaystyle\int_{1/8}^1 \frac{dx}{x\sqrt{1+x^{2/3}}}$

95. $\displaystyle\int_4^6 \frac{1}{\sqrt{x^2-9}}\,dx$

96. $\displaystyle\int_0^1 \frac{1}{\sqrt{16x^2+1}}\,dx$

97. $\displaystyle\int_1^3 \frac{1}{x\sqrt{4+x^2}}\,dx$

98. $\displaystyle\int_3^7 \frac{1}{\sqrt{x^2-4}}\,dx$

99. $\displaystyle\int_{-1}^1 \frac{1}{16-9x^2}\,dx$

100. $\displaystyle\int_0^1 \frac{1}{\sqrt{25x^2+1}}\,dx$

101. $\displaystyle\int_0^1 x^2\left(\operatorname{sech} x^3\right)^2 dx$

102. Derive the formula $\sinh^{-1} x = \ln\left(x+\sqrt{x^2+1}\right)$ for all real *x*. Explain in your derivation why the plus sign is used with the square root instead of the minus sign.

103. Find the linear approximation to $f(x) = \cosh x$ at $a = \ln 3$ and then use it to approximate the value of $\cosh 1$.

(**104 – 117**) Evaluate the limit:

104. $\displaystyle\lim_{x\to\infty} (\tanh x)^x$

105. $\displaystyle\lim_{x\to\infty} \tanh x$

106. $\displaystyle\lim_{x\to-\infty} \tanh x$

107. $\displaystyle\lim_{x\to\infty} \coth x$

108. $\displaystyle\lim_{x\to0^-} \coth x$

109. $\displaystyle\lim_{x\to0^+} \coth x$

110. $\displaystyle\lim_{x\to\infty} \operatorname{sech} x$

111. $\displaystyle\lim_{x\to\infty} \operatorname{csch} x$

112. $\displaystyle\lim_{x\to-\infty} \operatorname{csch} x$

113. $\displaystyle\lim_{x\to\infty} \sinh x$

114. $\displaystyle\lim_{x\to\infty} \frac{\sinh x}{e^x}$

115. $\displaystyle\lim_{x\to\infty} \frac{\sinh x}{x}$

116. $\displaystyle\lim_{x\to\infty} \cosh x$

117. $\displaystyle\lim_{x\to\infty} \frac{\cosh x}{e^x}$

118. Show that $\frac{d}{dx}\sqrt[4]{\frac{1+\tanh x}{1-\tanh x}}=\frac{1}{2}e^{x/2}$

119. Show that $\frac{d}{dx}\arctan\left(\tanh x\right)=\operatorname{sech} 2x$

120. Find the area of the region bounded by $y=\operatorname{sech} x$, $x=1$, and the unit circle

121. Find the area of the region bounded by the curves $f\left(x\right)=8\operatorname{sech}^2 x$ and $g\left(x\right)=\cosh x$

(**122 – 126**) Find the area of the region bounded by the given:

122. $y=\operatorname{sech}\frac{x}{2}, \quad -4\le x\le 4$

123. $y=\tanh 2x, \quad 0\le x\le 2$

124. $y=\sinh 3x, \quad 0\le x\le 1$

125. $y=\frac{5x}{\sqrt{x^4+1}}, \quad 0\le x\le 2$

126. $y=\frac{6}{\sqrt{x^2-4}}, \quad 3\le x\le 5$

(**127 – 128**) Find the length of the curve

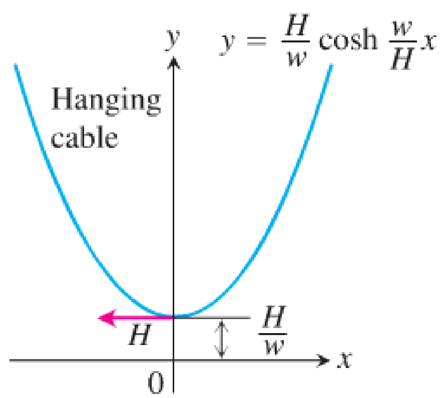
127. $y=\cosh^{-1} x \quad \sqrt{2}\le x\le\sqrt{5}$

128. $y=3+\frac{1}{2}\cosh 2x, \quad 0\le x\le 1$

129. A region in the first quadrant is bounded above the curve $y = \cosh x$, below by the curve $y = \sinh x$, and on the left and right by the y-axis and the line $x = 2$, respectively. Find the volume of the solid generated by revolving the region about the x-axis.

130. Imagine a cable, like a telephone line or TV cable, strung from one support to another and hanging freely. The cable's weight per unit length is a constant w and the horizontal tension at its lowest point is a vector of length H. If we choose a coordinate system for the plane of the cable in which the x-axis is horizontal, the force of gravity is straight down, the positive y-axis points straight up, and the lowest point of the cable lies at the point $y=\frac{H}{w}$ on the y-axis, then it can be shown that the cable lies along the graph of the hyperbolic cosine

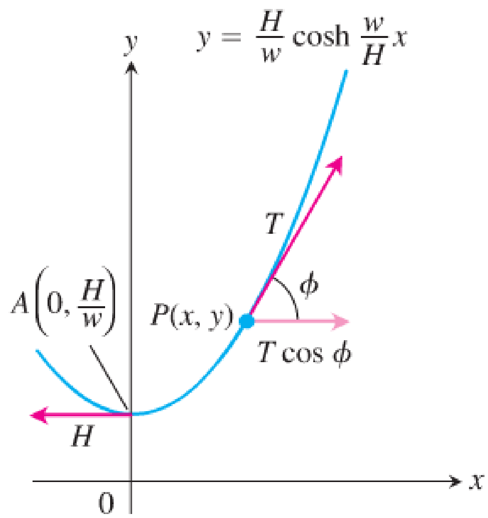
$$y=\frac{H}{w}\cosh\left(\frac{w}{H}x\right)$$



Such a curve is sometimes called a ***chain curve*** or a ***catenary***, the latter deriving from the Latin *catena*, meaning "*chain*".

a) Let $P(x, y)$ denote an arbitrary point on the cable. The next accompanying displays the tension H at the lowest point A. Show that the cable's slope at P is

$$\tan\phi = \frac{dy}{dx} = \sinh\left(\frac{w}{H}x\right)$$



b) Using the result in part (*a*) and the fact that the horizontal tension at P must equal H (the cable is not moving), show that $T = wy$. Hence, the magnitude of the tension at $P(x, y)$ is exactly equal to the weight of y units of cable.

c) The length of arc AP is $s = \frac{1}{a}\sinh ax$, where $a = \frac{w}{H}$. Show that the coordinates of P may be expressed in terms of s as

$$x = \frac{1}{a}\sinh^{-1} as, \quad y = \sqrt{s^2 + \frac{1}{a^2}}$$

131. The portion of the curve $y = \frac{17}{15} - \cosh x$ that lies above the x-axis forms a catenary arch. Find the average height of the arch above the x-axis.

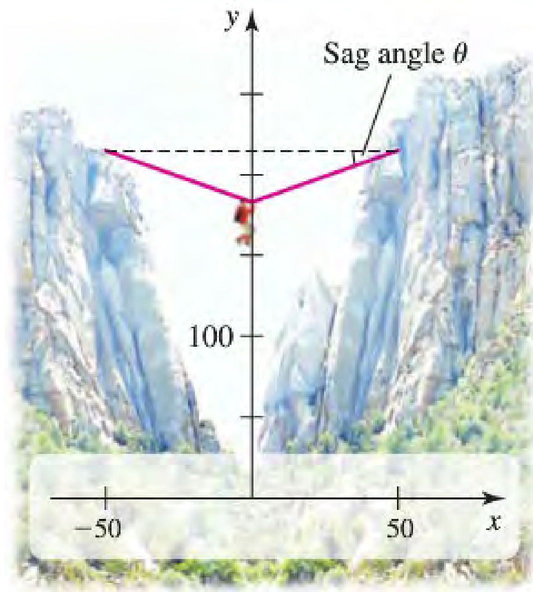
132. A power line is attached at the same height to two utility poles that are separated by a distance of 100 *feet*; the power line follows the curve $f(x) = a\cosh\left(\frac{x}{a}\right)$. Use the following steps to find the value of a that produces a sag of 10 *feet*. midway between the poles. Use the coordinate system that places the poles at $x = \pm 50$

a) Show that a satisfies the equation $\cosh\left(\frac{50}{a}\right) - 1 = \frac{10}{a}$

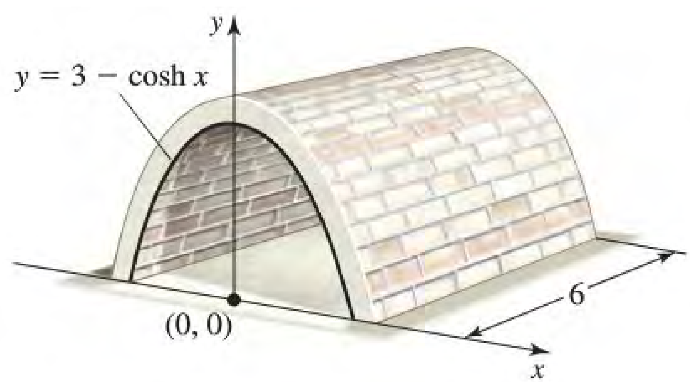
b) Let $t = \frac{10}{a}$, confirm that the equation in part (*a*) reduces to $\cosh 5t - 1 = t$, and solve for ***t*** using a graphing utility. (2 decimal places)

c) Use the answer in part (*b*) to find a and then compute the length of the power line.

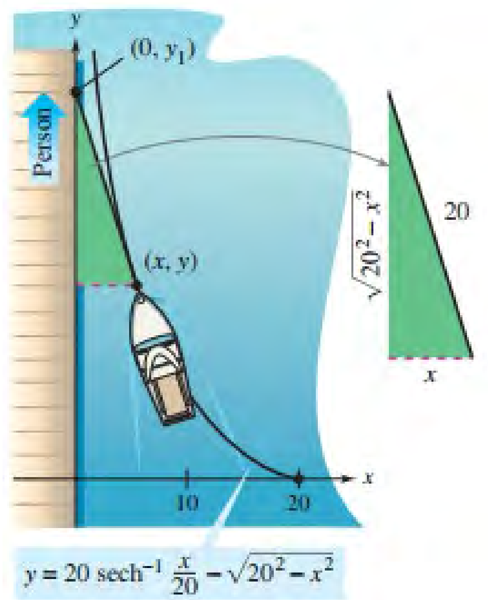
133. Imagine a climber clipping onto the rope and pulling hinself to the rope's midpoint. Because the rope is supporting the weight of the climber, it no longer takes the shape of the catenary $y = 200\cosh\left(\frac{x}{200}\right)$. Instead, the rope (nearly) forms two sides of an isosceles triangle. Compute the sag angle illustrated in the figure, assuming that the rope does not stretch when weighted. Assume the length of the rope is 101 *feet*.



134. Find the volume interior to the inverted catenary kiln (an oven used to fire pottery).



135. A person is holding a rope that is tied to a boat. As the person walks along the dock, the boat travels along a ***tractrix***, given by the equation



$$y = a\,\text{sech}^{-1}\frac{x}{a} - \sqrt{a^2 - x^2}$$

Where a is the length of the rope.

If $a = 20$ *feet*, find the distance the person must walk to bring the boat 5 *feet* from the dock.