Section 3.6 – Integrals for Mass Calculations

Mass and Moment Calculations

We treat coil springs and wires as masses distributed along smooth curves in space. The distribution is described by a continuous density function $\delta(x,y,z)$ representing mass per unit length. When a curve C is parametrized by $\boldsymbol{r}(t)=x(t)\boldsymbol{i}+y(t)\boldsymbol{j}+z(t)\boldsymbol{k},\quad a\le t\le b$, the density is the function $\delta(x(t),y(t),z(t))$, and then the arc length differential is given by

$$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\ dt$$

The formula of mass is

$$M=m=\int_a^b \delta(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\ dt$$

Mass and moment formulas for coil springs, wires, and thin rods lying along a smooth curve C in space

Mass: $m=\int_C \delta ds$ $\quad$ *$\delta=\delta(x,y,z)$ is the density at (x,y,z)*

First moments about the coordinates planes:

$$M_{yz}=\int_C x\delta ds,\quad M_{xz}=\int_C y\delta ds,\quad M_{xy}=\int_C z\delta ds$$

Coordinates of the center of mass:

$$\overline{x}=\frac{M_{yz}}{M},\quad \overline{y}=\frac{M_{xz}}{M},\quad \overline{z}=\frac{M_{xy}}{M}$$

Moments of inertia about axes and other lines:

$$I_x=\int_C\left(y^2+z^2\right)\delta ds,\quad I_y=\int_C\left(x^2+z^2\right)\delta ds,\quad I_z=\int_C\left(x^2+y^2\right)\delta ds$$

$$I_L=\int_C r^2\delta ds$$

$r(x,y,z)=$ distance from the point (x,y,z) to the line L

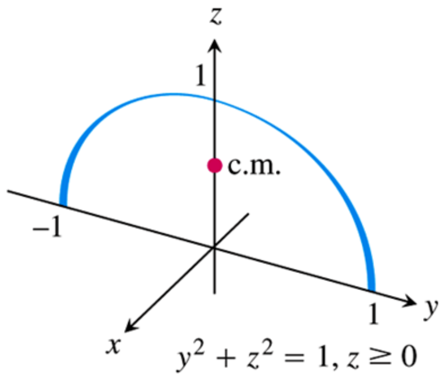
Example

A slender metal arch, denser at the bottom than top, lies along the semicircle $z^2+y^2=1,\ \ z\ge 0$, in the *yz*-plane. Find the center of the arch's mass if the density at the point (x,y,z) on the arch is $\delta(x,y,z)=2-z$

Solution

$\overline{x}=0\ \ and\ \ \overline{y}=0$, because the arch lies in the *yz*-plane with its mass distributed symmetrically about the *z*-axis.

$$\boldsymbol{r}(t)=(\cos t)\,\boldsymbol{j}+(\sin t)\,\boldsymbol{k},\quad 0\le t\le\pi$$



$$\begin{aligned}|v(t)|&=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\\&=\sqrt{(0)^2+(-\sin t)^2+(\cos t)^2}\\&=\sqrt{\sin^2 t+\cos^2 t}\\&=1\end{aligned}$$

$$\Rightarrow ds=|v|\,dt=dt$$

$$\begin{aligned}m&=\int_0^{\pi}(2-z)\ dt\\&=\int_0^{\pi}(2-\sin t)\ dt\\&=\left[2t+\cos t\right]_0^{\pi}\\&=2\pi+\cos\pi-\cos 0\\&=2\pi-2\end{aligned}$$

$$\begin{aligned}M_{xy}&=\int_C z\delta ds\\&=\int_C z(2-z)ds\\&=\int_0^{\pi}(\sin t)(2-\sin t)\ dt\\&=\int_0^{\pi}\left(2\sin t-\sin^2 t\right)\ dt\\&=\left[-2\cos t-\frac{t}{2}+\frac{\sin 2t}{4}\right]_0^{\pi}\end{aligned}$$

$$= -2(-1) - \frac{\pi}{2} + 2$$

$$= 4 - \frac{\pi}{2}$$

$$\boxed{= \frac{8-\pi}{2}}$$

$$\bar{z} = \frac{M_{xy}}{m}$$

$$= \frac{8-\pi}{2} \cdot \frac{1}{2\pi - 2}$$

$$= \frac{8-\pi}{4\pi - 4}$$

$$\boxed{\approx 0.57}$$

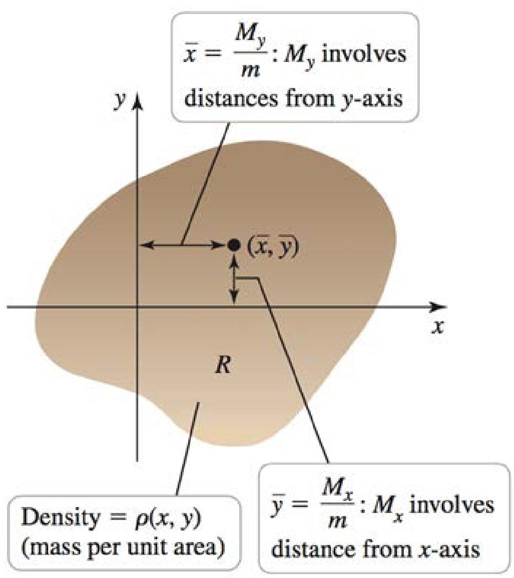
The center mass is $(0,\ 0,\ 0.57)$

Two-Dimensional Objects

Definition

Let ρ be an integrable area density function defined over a closed bounded region $\boldsymbol{R}$ in $\mathbb{R}^2$. The coordinates of the center of mass of the object represented by $\boldsymbol{R}$ are:

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_R x\rho(x,y)\,dA \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_R y\rho(x,y)\,dA$$



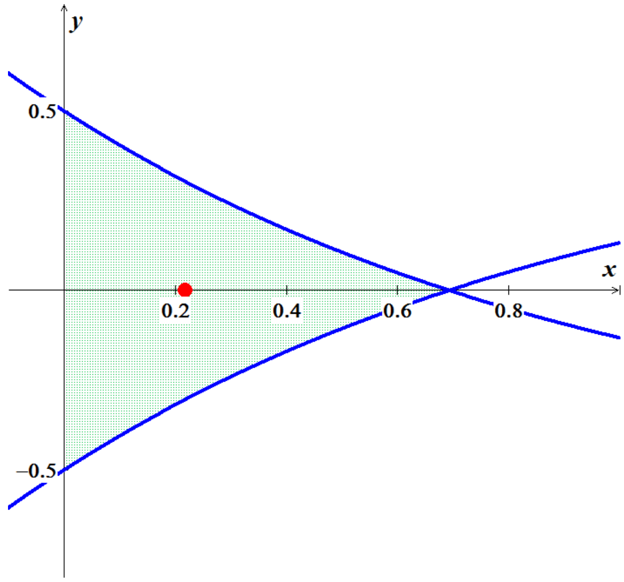
Where $m = \iint_R \rho(x,y)\,dA$ is the mass, and M_y and M_x are the moments with respect to the y-axis and x-axis, respectively. If ρ is constant, the center of mass is called the ***centroid*** and is independent of the density,

Example

Find the centroid (center of mass) of the constant density, dart-shaped region bounded by the *y*-axis and the curves $y = e^{-x} - \frac{1}{2}$ *and* $y = \frac{1}{2} - e^{-x}$

Solution

Assume: $\rho = 1$



$$
\begin{aligned}
m &= \int_0^{\ln 2} \int_{\frac{1}{2}-e^{-x}}^{e^{-x}-\frac{1}{2}} 1\,dydx \\
&= \int_0^{\ln 2} y \Big|_{\frac{1}{2}-e^{-x}}^{e^{-x}-\frac{1}{2}} dx \\
&= \int_0^{\ln 2} \left(2e^{-x} - 1\right) dx \\
&= \left[-2e^{-x} - x\right]_0^{\ln 2} \\
&= -2e^{-\ln 2} - \ln 2 + 2 \\
&= -2\left(\frac{1}{2}\right) - \ln 2 + 2 \\
&= 1 - \ln 2 \quad \approx 0.307
\end{aligned}
$$

$$
\begin{aligned}
M_y &= \int_0^{\ln 2} \int_{\frac{1}{2}-e^{-x}}^{e^{-x}-\frac{1}{2}} x\,dydx \\
&= \int_0^{\ln 2} xy \Big|_{\frac{1}{2}-e^{-x}}^{e^{-x}-\frac{1}{2}} dx \\
&= \int_0^{\ln 2} x\left(e^{-x} - \frac{1}{2} - \frac{1}{2} + e^{-x}\right) dx \\
&= \int_0^{\ln 2} x\left(2e^{-x} - 1\right) dx \\
&= \left[-2xe^{-x} - 2e^{-x} - \frac{1}{2}x^2\right]_0^{\ln 2} \\
&= -2(\ln 2)\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right) - \frac{1}{2}(\ln 2)^2 + 0 + 2 + 0 \\
&= 1 - \ln 2 - \frac{1}{2}(\ln 2)^2 \quad \approx 0.067
\end{aligned}
$$

$$\int x e^{ax} dx = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right)$$

		$\int e^{-x}$
+	x	$-e^{-x}$
−	1	e^{-x}

$$\bar{x} = \frac{M_y}{m} = \frac{0.067}{0.307} \approx 0.217$$

The center of mass is located approximately at $(0.217,\ 0)$

Three-Dimensional Objects

Definition

Let ρ be an integrable area density function defined over a closed bounded region ***D*** in $\mathbb{R}^3$. The coordinates of the center of mass of the object represented by ***D*** are:

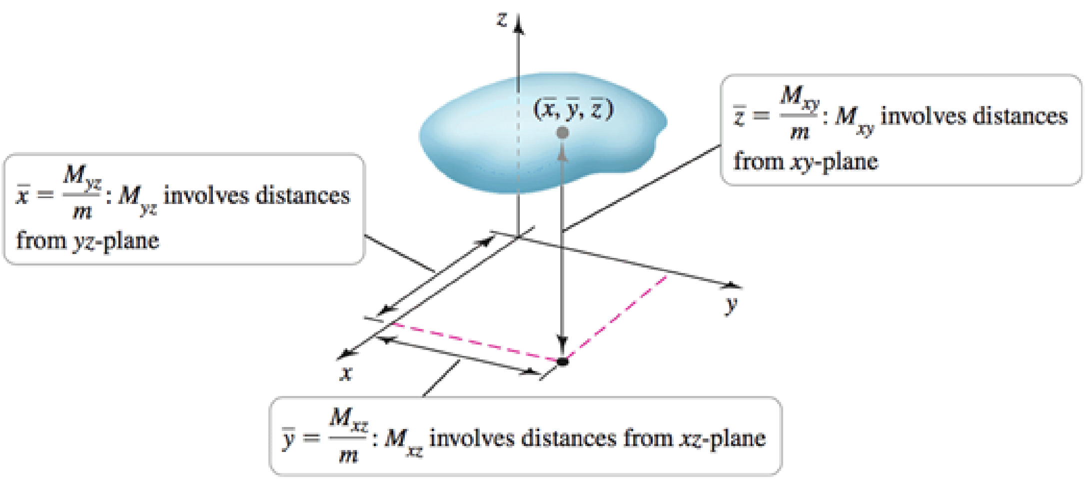
$$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m}\iiint_D x\,\rho(x,y,z)\,dV$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{1}{m}\iiint_D y\,\rho(x,y,z)\,dV$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m}\iiint_D z\,\rho(x,y,z)\,dV$$

Where $m = \iiint_D \rho(x,y,z)\,dV$ is the mass.

M_{yz}, M_{xz}, *and* M_{xy} are the moments with respect to the coordinates planes.



Example

Find the center of mass of the constant density solid cone ***D*** bounded by the surface

$$z = 4 - \sqrt{x^2 + y^2} \quad and \quad z = 0$$

Solution

The one is symmetric about the z-axis and has uniform density, the center of mass lies on the z-axis, that is, $\overline{x} = 0$ *and* $\overline{y} = 0$.

The disk has a radius of 4 and centered at the origin. Therefore, the cone has height 4 and radius 4; by the volume formula is $\frac{1}{3}\pi hr^2 = \frac{1}{3}\pi 4\left(4^2\right) = \frac{64\pi}{3}$.

The cone has a constant density, so we assume that $\rho = 1$ and its mass is $\boxed{m = \frac{64\pi}{3}}$

$$z = 4 - \sqrt{x^2 + y^2} = 4 - r$$

$$\begin{aligned} M_{xy} &= \int_0^{2\pi}\int_0^4\int_0^{4-r} z\, dz\, rdrd\theta \\ &= \int_0^{2\pi}\int_0^4 r\left[\frac{1}{2}z^2\right]_0^{4-r} drd\theta \\ &= \frac{1}{2}\int_0^{2\pi}\int_0^4 r(4-r)^2\, drd\theta \\ &= \frac{1}{2}\int_0^{2\pi}\int_0^4 \left(16r - 8r^2 + r^3\right) drd\theta \\ &= \frac{1}{2}\left[8r^2 - \frac{8}{3}r^3 + \frac{1}{4}r^4\right]_0^4 \left[\theta\right]_0^{2\pi} \\ &= \frac{1}{2}\left(128 - \frac{512}{3} + 64\right)(2\pi) \\ &= \boxed{\frac{64\pi}{3}} \end{aligned}$$

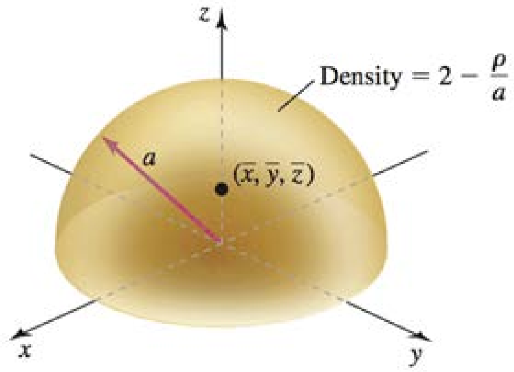
$$\overline{z} = \frac{M_{xy}}{m} = \frac{64\pi/3}{64\pi/3} \boxed{= 1}$$

∴ The center of mass is located at $(0,\ 0,\ 1)$

Example

Find the center of mass of the interior of the hemisphere ***D*** of a radius a with its base on the *xy*-plane. The density of the objects is $f(\rho,\ \phi,\ \theta) = 2 - \frac{\rho}{a}$ (heavy near the center and light near the outer surface.)

Solution



The one is symmetric about the *z*-axis and has uniform density, the center of mass lies on the *z*-axis, that is, $\bar{x} = 0$ *and* $\bar{y} = 0$.

$$
\begin{aligned}
m &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a}\left(2-\frac{\rho}{a}\right)\rho^2\sin\phi\ d\rho\ d\phi\ d\theta \\
&= \int_0^{2\pi} d\theta \int_0^{\pi/2}\sin\phi d\phi \int_0^{a}\left(2\rho^2 - \frac{1}{a}\rho^3\right)\ d\rho \\
&= \theta\Big|_0^{2\pi}\ \ -\cos\phi\Big|_0^{\pi/2}\ \left[\frac{2}{3}\rho^3 - \frac{1}{4a}\rho^4\right]_0^a \\
&= (2\pi)(1)\left(\frac{2}{3}a^3 - \frac{1}{4}a^3\right) \\
&= \frac{5\pi}{6}a^3
\end{aligned}
$$

In spherical coordinates $z = \rho\cos\phi$.

$$
\begin{aligned}
M_{xy} &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a}\rho\cos\phi\left(2-\frac{\rho}{a}\right)\rho^2\sin\phi\ d\rho\ d\phi\ d\theta \\
&= \int_0^{2\pi} d\theta\ \int_0^{\pi/2}\frac{1}{2}\sin 2\phi d\phi\ \int_0^{a}\left(2\rho^3 - \frac{1}{a}\rho^4\right)d\rho \\
&= \theta\Big|_0^{2\pi}\ \left[-\frac{1}{4}\cos 2\phi\right|_0^{\pi/2}\ \left[\frac{1}{2}\rho^4 - \frac{1}{5a}\rho^5\right]_0^a \\
&= -\frac{1}{4}(2\pi)(-2)\left(\frac{1}{2}a^4 - \frac{1}{5}a^4\right) \\
&= \frac{3\pi}{10}a^4
\end{aligned}
$$

$$M_{xy} = \iiint_D z\ \rho(x,y,z)dV$$

$$2\sin\phi\cos\phi = \sin 2\phi$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{\frac{3\pi a^4}{10}}{\frac{5\pi a^3}{6}} = \frac{9a}{25} \qquad = 0.36a$$

However, the center of mass of a uniform-density hemisphere solid of radius a is $\frac{3a}{8} = 0.375a$ units above the base. In this particular case, the variable density shifts the center of mass.

Exercises Section 3.6 – Integrals for Mass Calculations

Find the mass and center of mass of the thin rods with the following density functions.

1. $\rho(x) = 1 + \sin x \quad for \ \ 0 \le x \le \pi$

2. $\rho(x) = 1 + x^3 \quad for \ \ 0 \le x \le 1$

3. $\rho(x) = 2 - \frac{x^2}{16} \quad for \ \ 0 \le x \le 4$

4. $\rho(x) = 2 + \cos x \quad for \ \ 0 \le x \le \pi$

5. $\rho(x) = \begin{cases} x^2 & if \ \ 0 \le x \le 1 \\ 2x - x^2 & if \ \ 1 \le x \le 2 \end{cases}$

Find the mass and centroid (center of mass) of the following thin plates, assuming a constant density. Sketch the region corresponding to the plate and indicate the location of the center of mass. Use symmetry when possible to simplify your work.

6. The region bounded by $y = \sin x$ and $y = 1 - \sin x$ between $x = \frac{\pi}{4} \quad and \quad x = \frac{3\pi}{4}$

7. The region bounded by $y = 1 - |x|$ and the x-axis

8. The region bounded by $y = e^x, \quad y = e^{-x}, \quad x = 0, \quad and \quad x = \ln 2$

9. The region bounded by $y = \ln x, \quad x - axis, \quad and \quad x = e$

10. The region bounded by $x^2 + y^2 = 1 \quad and \quad x^2 + y^2 = 9, \quad for \ \ y \ge 0$

Find the coordinates of the center of mass of the following plane regions with variable density. Describe the distribution of mass in the region

11. $R = \{(x, y): \ 0 \le x \le 4, \ 0 \le y \le 2\}; \quad \rho(x, y) = 1 + \frac{x}{2}$

12. The triangular plate in the first quadrant bounded by $x + y = 4$ with $\rho(x, y) = 1 + x + y$

13. The upper half $(y \ge 0)$ of the disk bounded by the circle $x^2 + y^2 = 4$ with $\rho(x, y) = 1 + \frac{y}{2}$

14. The upper half $(y \ge 0)$ of the disk bounded by the ellipse $x^2 + 9y^2 = 9$ with $\rho(x, y) = 1 + y$

Find the center of mass of the following solids, assuming a constant density of 1. Sketch the region and indicate the location of the centroid. Use symmetry when possible and choose a convenient coordinate system.

15. The upper half of the ball $x^2 + y^2 + z^2 \le 16 \quad (for \ z \ge 0)$

16. The region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 25$

17. The tetrahedron in the first octant bounded by $z = 1 - x - y$ and the coordinate planes

18. The solid bounded by the cone $z = 16 - r$ and the plane $z = 0$

19. Consider the thin constant-density plate $\{(r,\ \theta):\ a \le r \le 1,\ 0 \le \theta \le \pi\}$ bounded by two semicircles and the *x*-axis.

a) Find the graph the *y*-coordinate of the center of mass of the plate as a function of a.

b) For what value of a is the center of mass on the edge of the plate?

20. Consider the thin constant-density plate $\left\{(\rho,\ \phi,\ \theta):\ 0 < a \le \rho \le 1,\ 0 \le \phi \le \frac{\pi}{2},\ 0 \le \theta \le 2\pi\right\}$ bounded by two hemispheres and the *xy*-axis.

a) Find the graph the *z*-coordinate of the center of mass of the plate as a function of a.

b) For what value of a is the center of mass on the edge of the solid?

21. A cylindrical soda can has a radius of 4 *cm* and a height of 12 *cm*. When the can is full of soda, the center of mass of the contents of the can is 6 cm above the base on the axis of the can (halfway along the axis of the can). As the can is drained, the center of mass descends for a while. However, when the can is empty (filled only with air), the center of mass is once again 6 *cm* above the base on the axis of the can. Find the depth of soda in the can for which the center of mass is at its lowest point. Neglect the mass of the can, and assume the density of the soda is $1\ g\,/\,cm^3$ and the density of air is $0.001\ g\,/\,cm^3$.