Section 1.3 – Rational Functions

A function f is a ***rational function*** if $f(x)=\dfrac{g(x)}{h(x)}$,

Where $g(x)$ and $h(x)$ are polynomials. The domain of f consists of all real numbers ***except*** the zeros of the denominator $h(x)$.

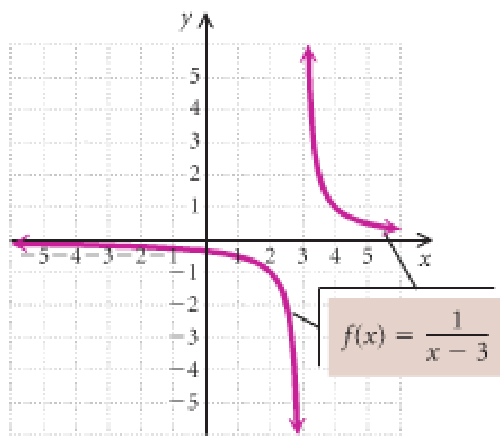
Notation	*Terminology*
$x \to a^-$	x approaches a from the left (through values ***less*** than a)
$x \to a^+$	x approaches a from the right (through values ***greater*** than a)
$f(x) \to \infty$	$f(x)$ increases without bound (can be made as large positive as desired)
$f(x) \to -\infty$	$f(x)$ decreases without bound (can be made as large negative as desired)

The Domain of a Rational Function

Example

Consider: $f(x)=\dfrac{1}{x-3}$

Find the domain and graph *f*.



Solution

$x-3=0 \quad \Rightarrow \quad \boxed{x=3}$

Thus the domain is: $\{x \mid x \neq 3\}$ *or* $(-\infty,\ 3) \cup (3,\ \infty)$

Function	*Domain*	
$f(x)=\dfrac{1}{x}$	$\{x \mid x \neq 0\}$	$(-\infty,\ 0) \cup (0,\ \infty)$
$f(x)=\dfrac{1}{x^2}$	$\{x \mid x \neq 0\}$	$(-\infty,\ 0) \cup (0,\ \infty)$
$f(x)=\dfrac{x-3}{x^2+x-2}$	$\{x \mid x \neq -2 \text{ and } x \neq 1\}$	$(-\infty,\ -2) \cup (-2,\ 1) \cup (1,\ \infty)$
$f(x)=\dfrac{2x+5}{2x-6}=\dfrac{2x+5}{2(x-3)}$	$\{x \mid x \neq 3\}$	$(-\infty,\ 3) \cup (3,\ \infty)$

Asymptotes

Vertical Asymptote (*VA*) - ***Think Domain***

The line $x = a$ is a ***vertical asymptote*** for the graph of a function f if

$$f(x) \to \infty \quad or \quad f(x) \to -\infty$$

As ***x*** approaches ***a*** from either the left or the right

Horizontal Asymptote (*HA*)

The line $y = c$ is a ***horizontal asymptote*** for the graph of a function f if

$$f(x) \to c \quad as \quad x \to -\infty \quad or \quad x \to -\infty$$

Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0} = \dfrac{a_n x^n}{b_m x^m}$ be a rational function.

1. If the degree of numerator is less than of denominator $(n < m) \Rightarrow$ y = 0

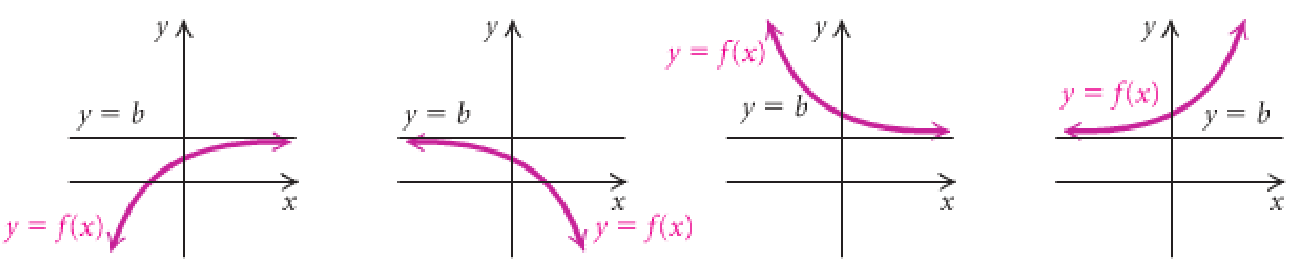
   $$y = \frac{2x+1}{4x^2+5} \quad \Rightarrow \boxed{y = 0}$$

2. If the degree of numerator is equal of denominator $(n = m) \Rightarrow y = \dfrac{a_n}{b_m}$

   $$y = \frac{2x^2+1}{4x^2+5} \quad \Rightarrow \boxed{y = \frac{2}{4} = \frac{1}{2}}$$

3. If the degree of numerator is greater than of denominator $(n > m) \Rightarrow$ No horizontal asymptote

   $$y = \frac{2x^3+1}{4x^2+5} \quad \Rightarrow No \quad HA$$



$f(x) \to b$ as $x \to \infty$ $\quad$ $f(x) \to b$ as $x \to -\infty$ $\quad$ $f(x) \to b$ as $x \to \infty$ $\quad$ $f(x) \to b$ as $x \to -\infty$

Example

Find the vertical asymptote of $f(x)=\dfrac{1}{x-2}$, and sketch the graph.

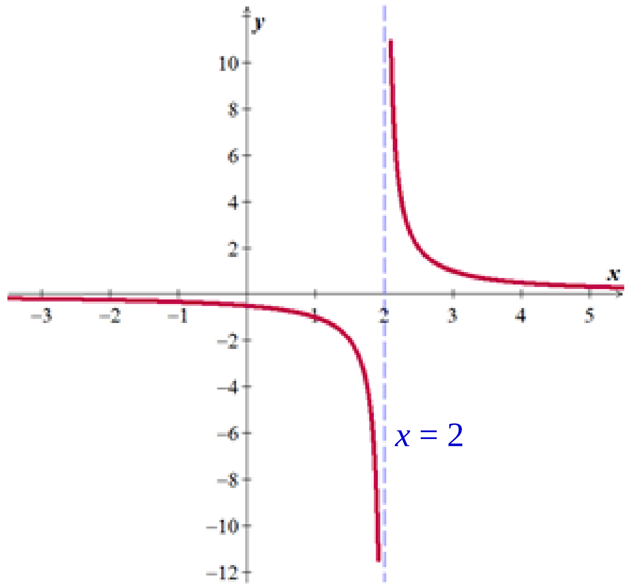
Solution

VA: $x=2$

HA: $y=0$

$f(x)\to\infty \quad as \quad x\to 2^+$

$f(x)\to-\infty \quad as \quad x\to 2^-$



Hole

Example

Sketch the graph of g if $g(x)=\dfrac{3x^2+x-4}{2x^2-7x+5}$

Solution

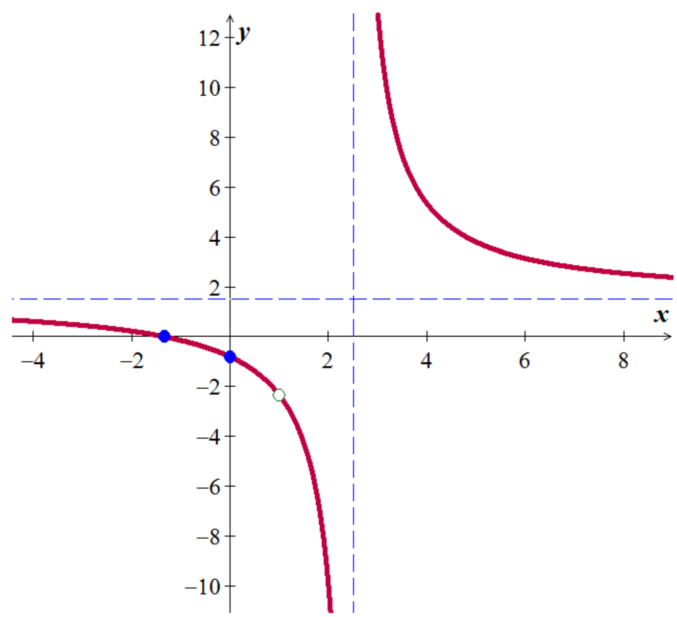
$$g(x)=\frac{(3x+4)(x-1)}{(2x-5)(x-1)}=\frac{3x+4}{2x-5}=f(x)$$

g has a hole at $x=1\to f(1)=-\frac{7}{3}$

VA: $x=\frac{5}{2}$

HA: $y=0$

Hole: $\left(1,\ -\frac{7}{3}\right)$



Slant or *Oblique* Asymptotes

When the degree of the numerator is one greater than the degree of the numerator, the graph has a slant or oblique asymptote and it is a line $y = ax + b, \quad a \neq 0$. To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

$$y = \frac{3x^2 - 1}{x + 2}$$

$$\begin{array}{r|l} & \quad 3x - 6 \\ x + 2 & 3x^2 + 0x - 1 \\ & \underline{3x^2 + 6x} \\ & \qquad -6x - 1 \\ & \qquad \underline{-6x - 12} \\ & \qquad\qquad R = 11 \end{array}$$

$$y = \frac{3x^2 - 1}{x + 2} = (3x - 6) + \frac{11}{x + 2}$$

The ***oblique asymptote*** is the line ***y* = 3*x* - 6**

Example

Find all the asymptotes and sketch the graph of f if $f(x) = \frac{x^2 - 9}{2x - 4}$

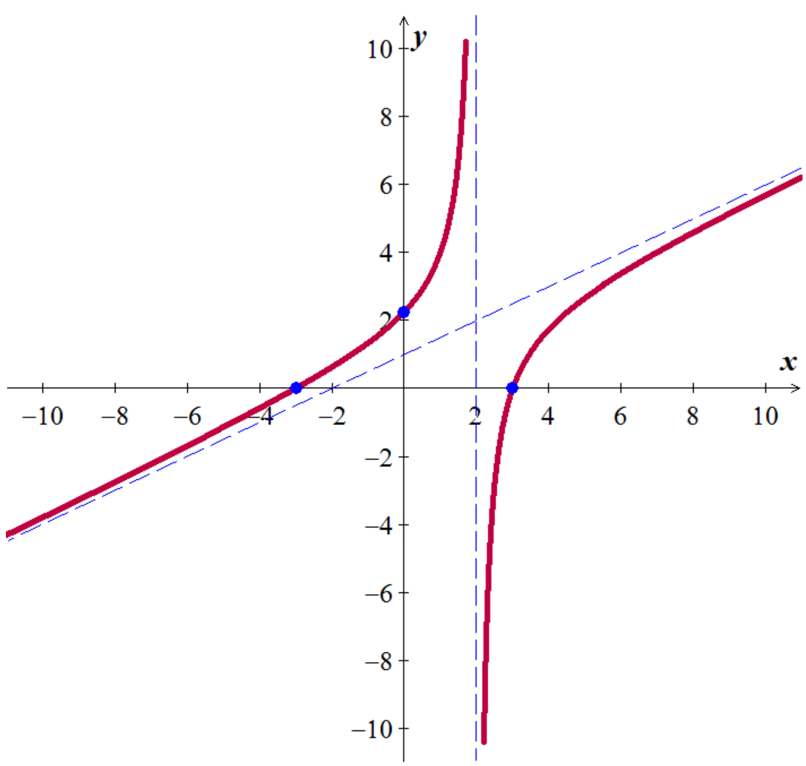
Solution

$$\begin{array}{r|l} & \quad \frac{1}{2}x + 1 \\ 2x - 4 & x^2 \qquad - 9 \\ & \underline{x^2 - 2x} \\ & \qquad 2x - 9 \\ & \qquad \underline{2x - 4} \\ & \qquad\quad \boxed{-5} \end{array}$$

$$f(x) = \frac{x^2 - 9}{2x - 4} = \left(\frac{1}{2}x + 1\right) - \frac{5}{2x - 4}$$

VA	$x = 2$
HA	*n/a*
OA	$y = \frac{1}{2}x + 1$

x	***y***
0	2.25
±3	0



Example

Find all asymptotes for the graph of f , if it exists

a) $f(x)=\dfrac{3x-1}{x^2-x-6}$ *b)* $f(x)=\dfrac{5x^2+1}{3x^2-4}$ *c)* $f(x)=\dfrac{2x^4-3x^2+5}{x^2+1}$

Solution

a) $f(x)=\dfrac{3x-1}{x^2-x-6}$

VA: $x=-2,\quad x=3$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

b) $f(x)=\dfrac{5x^2+1}{3x^2-4}$

$$3x^2-4=0 \ \rightarrow\ 3x^2=4 \ \rightarrow\ x^2=\frac{4}{3} \ \rightarrow\ \boxed{x=\pm\frac{2}{\sqrt{3}}}$$

VA: $x=\pm\dfrac{2}{\sqrt{3}}$ ***HA***: $y=\dfrac{5}{3}$

Hole: n/a ***Oblique asymptote***: n/a

c) $f(x)=\dfrac{2x^4-3x^2+5}{x^2+1}$

VA: n/a ***HA***: n/a

Hole: n/a ***Oblique asymptote***: $y=2x^2-5$

$$\begin{array}{r|l} & \quad 2x^2-5 \\ \hline x^2+1 & 2x^4-3x^2+5 \\ & \underline{-2x^4-2x^2} \\ & \qquad -5x^2+5 \end{array}$$

Example

Sketch the graph of f if $f(x) = \dfrac{3x+4}{2x-5}$

Solution

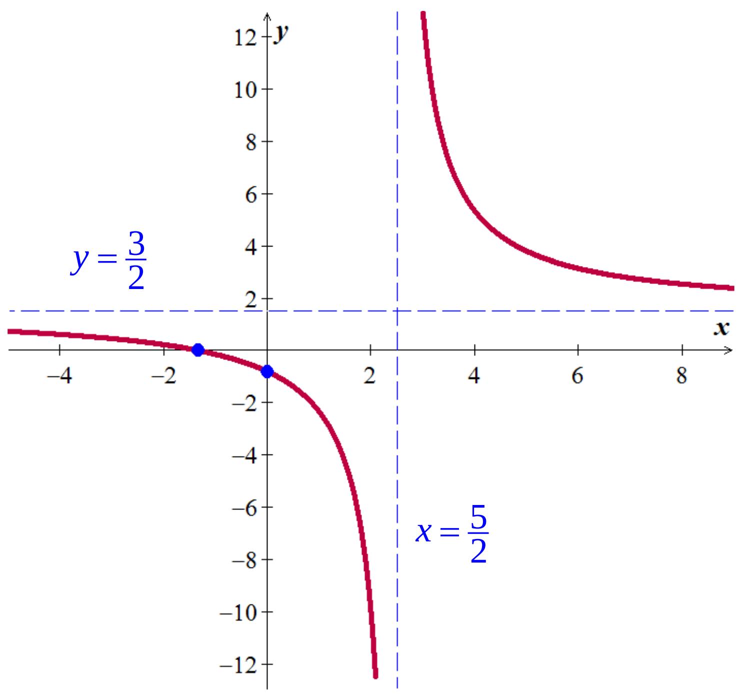
VA: $x = \frac{5}{2}$

HA: $y = -\frac{5}{3}$

Hole: n/a

Oblique asymptote: n/a

x	y
0	$-\frac{4}{5}$
$-\frac{4}{3}$	0
4	5.3



Example

Sketch the graph of f if $f(x)=\dfrac{x^2}{x^2-x-2}$

Solution

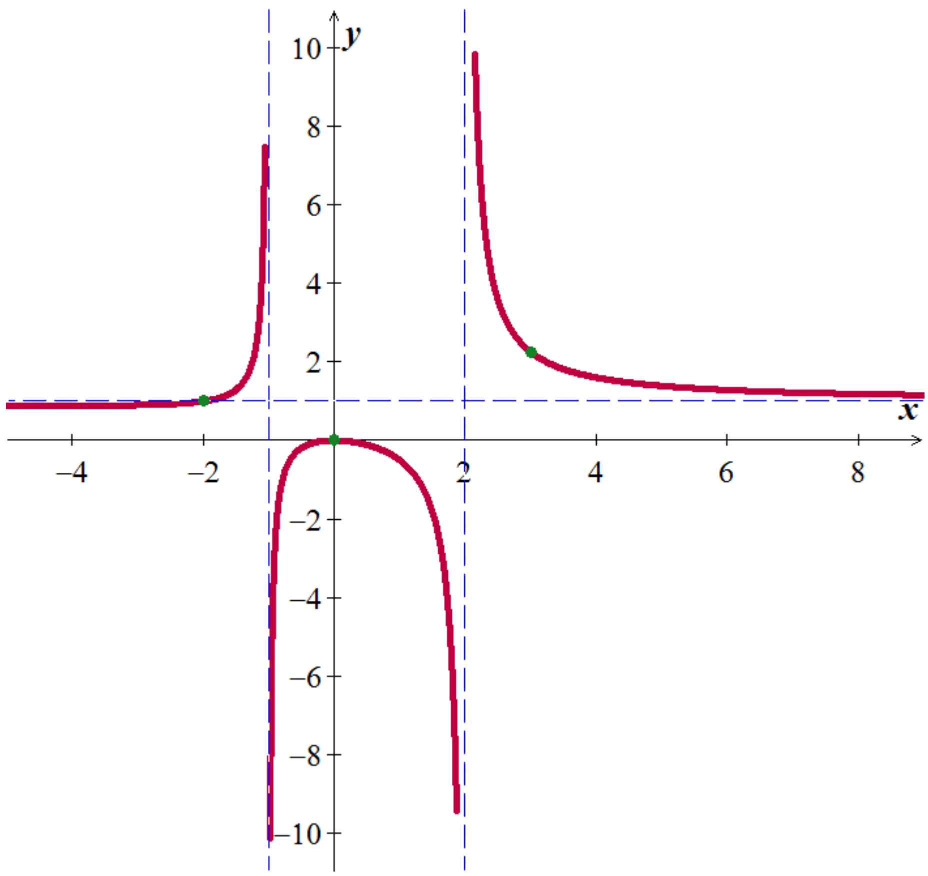
VA: $x=-1,\ 2$

HA: $y=1$

Hole: n/a

Oblique asymptote: n/a

x	y
0	0
–4	0.88
–2	1
3	$\frac{9}{4}$



Example

Sketch the graph of f if $f(x) = \dfrac{x-1}{x^2 - x - 6}$

Solution

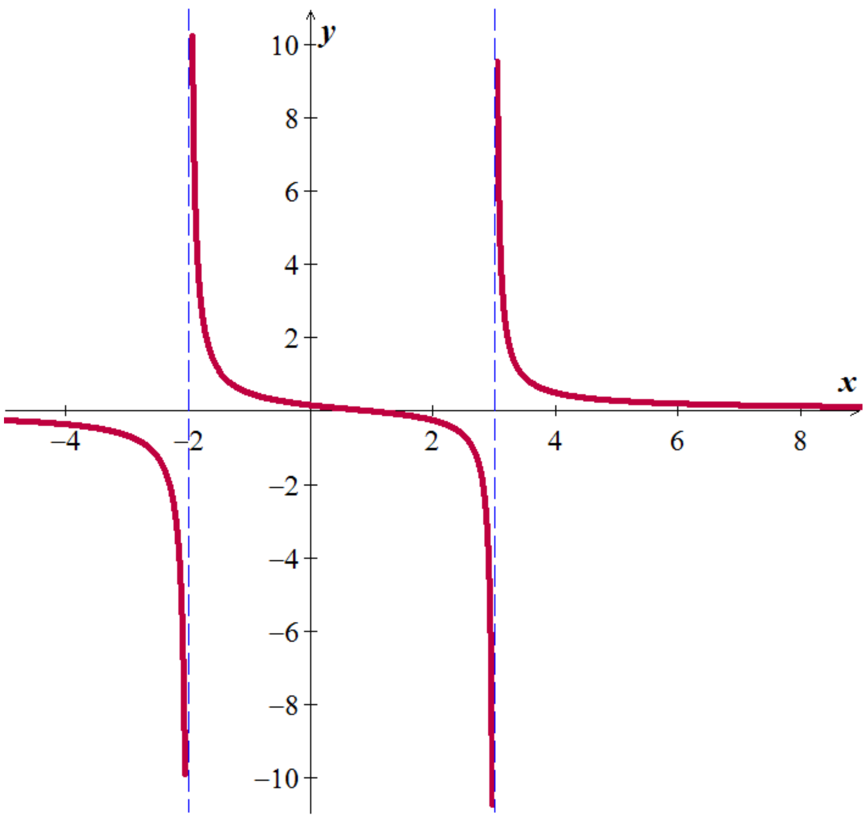
VA: $x = -2,\ 3$

HA: $y = 0$

Hole: n/a

Oblique asymptote: n/a

x	y
-4	$-.36$
-3	$-.67$
0	$\frac{1}{6}$
1	0
4	.5
5	$\frac{2}{7}$



Exercises *Section* 1.3 – Rational Functions

(1 – 21) Determine all asymptotes of the function

1. $y = \dfrac{3x}{1-x}$

2. $y = \dfrac{x^2}{x^2+9}$

3. $y = \dfrac{x-2}{x^2-4x+3}$

4. $y = \dfrac{3}{x-5}$

5. $y = \dfrac{x^3-1}{x^2+1}$

6. $y = \dfrac{3x^2-27}{(x+3)(2x+1)}$

7. $y = \dfrac{x^3+3x^2-2}{x^2-4}$

8. $y = \dfrac{x-3}{x^2-9}$

9. $y = \dfrac{6}{\sqrt{x^2-4x}}$

10. $y = \dfrac{5x-1}{1-3x}$

11. $f(x) = \dfrac{2x-11}{x^2+2x-8}$

12. $f(x) = \dfrac{x^2-4x}{x^3-x}$

13. $f(x) = \dfrac{x-2}{x^3-5x}$

14. $f(x) = \dfrac{4x}{x^2+10x}$

15. $f(x) = \dfrac{3-x}{(x-4)(x+6)}$

16. $f(x) = \dfrac{x^3}{2x^3-x^2-3x}$

17. $f(x) = \dfrac{3x^2+5}{4x^2-3}$

18. $f(x) = \dfrac{x+6}{x^3+2x^2}$

19. $f(x) = \dfrac{x^2+4x-1}{x+3}$

20. $f(x) = \dfrac{x^2-6x}{x-5}$

21. $f(x) = \dfrac{x^3-x^2+x-4}{x^2+2x-1}$

(22 – 53) Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

22. $f(x) = \dfrac{-3x}{x+2}$

23. $f(x) = \dfrac{x+1}{x^2+2x-3}$

24. $f(x) = \dfrac{2x^2-2x-4}{x^2+x-12}$

25. $f(x) = \dfrac{-2x^2+10x-12}{x^2+x}$

26. $f(x) = \dfrac{x^2-x-6}{x+1}$

27. $f(x) = \dfrac{x^3+1}{x-2}$

28. $f(x) = \dfrac{2x^2+x-6}{x^2+3x+2}$

29. $f(x) = \dfrac{x-1}{1-x^2}$

30. $f(x) = \dfrac{x^2+x-2}{x+2}$

31. $f(x) = \dfrac{x^3-2x^2-4x+8}{x-2}$

32. $f(x) = \dfrac{2x^2-3x-1}{x-2}$

33. $f(x) = \dfrac{2x+3}{3x^2+7x-6}$

34. $f(x) = \dfrac{x^2-1}{x^2+x-6}$

35. $f(x) = \dfrac{-2x^2-x+15}{x^2-x-12}$

36. $f(x) = \dfrac{1}{x-3}$

37. $f(x) = \dfrac{-2}{x+3}$

38. $f(x) = \dfrac{x}{x+2}$

39. $f(x) = \dfrac{x-5}{x+4}$

40. $f(x) = \dfrac{2x^2-2}{x^2-9}$

41. $f(x) = \dfrac{x^2-3}{x^2+4}$

42. $f(x) = \dfrac{x^2+4}{x^2-3}$

43. $f\left(x\right)=\dfrac{x^2}{x^2-6x+9}$

44. $f\left(x\right)=\dfrac{x^2+x+4}{x^2+2x-1}$

45. $f\left(x\right)=\dfrac{2x^2+14}{x^2-6x+5}$

46. $f\left(x\right)=\dfrac{x^2-4x-5}{2x+5}$

47. $f\left(x\right)=\dfrac{x-3}{x^2-3x+2}$

48. $f\left(x\right)=\dfrac{x^2+2}{x^2+3x+2}$

49. $f\left(x\right)=\dfrac{x-2}{x^2-3x+2}$

50. $f\left(x\right)=\dfrac{x^2+x}{x+1}$

51. $f\left(x\right)=\dfrac{x^2-2x}{x-2}$

52. $f\left(x\right)=\dfrac{x^2-3x}{x+3}$

53. $f(x)=\dfrac{x^3+3x^2-4x+6}{x+2}$

(**54 – 59**) Find an equation of a rational function f that satisfies the given conditions

54. $\begin{cases} \textit{vertical asymptote}: \ x=4 \\ \textit{horizontal asymptote}: \ y=-1 \\ x-\textit{intercept}: \ 3 \end{cases}$

55. $\begin{cases} \textit{vertical asymptote}: \ x=-4, x=5 \\ \textit{horizontal asymptote}: \ y=\dfrac{3}{2} \\ x-\textit{intercept}: \ -2 \end{cases}$

56. $\begin{cases} \textit{vertical asymptote}: \ x=5 \\ \textit{horizontal asymptote}: \ y=-1 \\ x-\textit{intercept}: \ 2 \end{cases}$

57. $\begin{cases} \textit{vertical asymptote}: \ x=-2, \ x=0 \\ \textit{horizontal asymptote}: \ y=0 \\ x-\textit{intercept}: \ 2, \quad f\left(3\right)=1 \end{cases}$

58. $\begin{cases} \textit{vertical asymptote}: \ x=-3, \ \ x=1 \\ \textit{horizontal asymptote}: \ y=0 \\ x-\textit{intercept}: \ -1, \ f(0)=-2 \\ \textit{hole at} \ \ x=2 \end{cases}$

59. $\begin{cases} \textit{vertical asymptote}: \ x=-1, \ \ x=3 \\ \textit{horizontal asymptote}: \ y=2 \\ x-\textit{intercept}: \ -2, \ \ 1 \\ \textit{hole}: \ \ x=0 \end{cases}$