6.3 Conservative Vector Fields

Vector Fields: $\vec{F} = M(x,y,z)\hat{i} + N\hat{j} + P\hat{k}$
(x,y,z)

Defn line integral of $\vec{F}$ along C

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int \vec{F}\cdot\frac{d\vec{r}}{ds}\,ds$$

$$= \int_C \vec{F}\cdot d\vec{r}$$

$$\vec{T} = \frac{d\vec{r}}{ds},\ \frac{\vec{v}}{|\vec{v}|}$$

$\int_C \vec{F}\cdot d\vec{r}$ $\quad \vec{F} = z\hat{i} + xy\hat{j} - y^2\hat{k}$

$0 \le t \le 1$ $\quad \vec{r}(t) = t^2\hat{i} + t\hat{j} + \sqrt{t}\,\hat{k}$

$$\vec{F} = \sqrt{t}\,\hat{i} + t^3\hat{j} - t^2\hat{k}$$

$$\frac{d\vec{r}}{dt} = 2t\,\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\hat{k}$$

$$\vec{F}\cdot\frac{d\vec{r}}{dt} = 2t^{3/2} + t^3 - \frac{1}{2}t^{3/2}$$

$$= t^3 + \frac{3}{2}t^{3/2}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \left(t^3 + \frac{3}{2}t^{3/2}\right)dt$$

$$= \frac{1}{4}t^4 + \frac{3}{5}t^{5/2}\Big|_0^1$$

$$= \frac{1}{4} + \frac{3}{5}$$

$$= \frac{17}{20}$$

Work Done

$$W = \int_C \vec{F} \cdot \vec{T}\, ds$$

$$= \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_a^b \left(\vec{F} \cdot \frac{d\vec{r}}{dt} \right) dt$$

$$= \int_a^b \left(M \frac{dx}{dt} + N \frac{dy}{dt} + P \frac{dz}{dt} \right) dt$$

$$= \int_C M\,dx + N\,dy + P\,dz$$

Ex Work? $\vec{F} = (y-x^2)\hat{i} + (z-y^2)\hat{j} + (x-z^2)\hat{k}$

$\vec{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k} \quad 0 \le t \le 1$

Orig → (1, 1, 1)

Soln

$$\vec{F} = (t^2 - t^2)\hat{i} + (t^3 - t^4)\hat{j} + (t - t^6)\hat{k}$$

$$= (t^3 - t^4)\hat{j} + (t - t^6)\hat{k}$$

$$\frac{d\vec{r}}{dt} = \hat{i} + 2t\,\hat{j} + 3t^2\,\hat{k}$$

$$\int_0^1 \vec{F} \cdot \frac{d\vec{r}}{dt}\,dt = \int_0^1 (2t^4 - 2t^5 + 3t^3 - 3t^8)\,dt$$

$$= \frac{2}{5}t^5 - \frac{1}{3}t^6 + \frac{3}{4}t^4 - \frac{1}{3}t^9 \Big|_0^1$$

$$= \frac{2}{5} - \frac{1}{3} + \frac{3}{4} - \frac{1}{3}$$

$$= \frac{24 - 40 + 45}{60}$$

$$= \boxed{\frac{29}{60}}$$

Ex W? $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$

$r(t) = \cos\pi t\,\hat{i} + t^2\hat{j} + \sin\pi t\,\hat{k}$

$0 \le t \le 1$

Soln

$\vec{F} = \cos\pi t\,\hat{i} + t^2\hat{j} + \sin\pi t\,\hat{k}$

$\frac{dr}{dt} = -\pi\sin\pi t\,\hat{i} + 2t\hat{j} + \pi\cos\pi t\,\hat{k}$

$W = \int_0^1 (-\pi\cos\pi t\sin\pi t + 2t^3 + \pi\cos\pi t\sin\pi t)\,dt$

$= \frac{1}{2}t^4 \Big|_0^1$

$= \frac{1}{2}$

Flow and Circulation

$\text{Flow} = \int_C \vec{F}\cdot\vec{T}\,ds$

$a \to b$.

Circulation $A = B$

Ex $\vec{F} = x\hat{i} + z\hat{j} + y\hat{k}$ Flow?

$r(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$

$0 \le t \le \pi/2$

Soln

$\vec{F} = \cos t\,\hat{i} + t\,\hat{j} + \sin t\,\hat{k}$

$\frac{d\vec{r}}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}$

$$\text{Flow} = \int_0^{\pi/2} (-\underbrace{\sin t\cos t}_{\frac{1}{2}\sin 2t} + t\cos t + \sin t)\,dt$$

	$\int \cos t$
$+\,t$	$\sin t$
$-\,1$	$-\cos t$

$$= +\frac{1}{4}\cos 2t + t\sin t + \cos t - \cos t \Big|_0^{\pi/2}$$

$$= -\frac{1}{4} + \frac{\pi}{2} - \frac{1}{4}$$

$$= \frac{\pi}{2} - \frac{1}{2}$$

Ex) $\vec{F} = (x-y)\,\hat{i} + x\,\hat{j}$ $\quad 0 \le t \le 2\pi$

$\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$

Cir (circulation)

Soln

$$\vec{F} = (\cos t - \sin t)\,\hat{i} + \cos t\,\hat{j}$$

$$\frac{d\vec{r}}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j}$$

$$Ci = \int_0^{2\pi} \left(-\cos t \sin t + \overbrace{\sin^2 t + \cos^2 t}^{=1}\right) dt$$

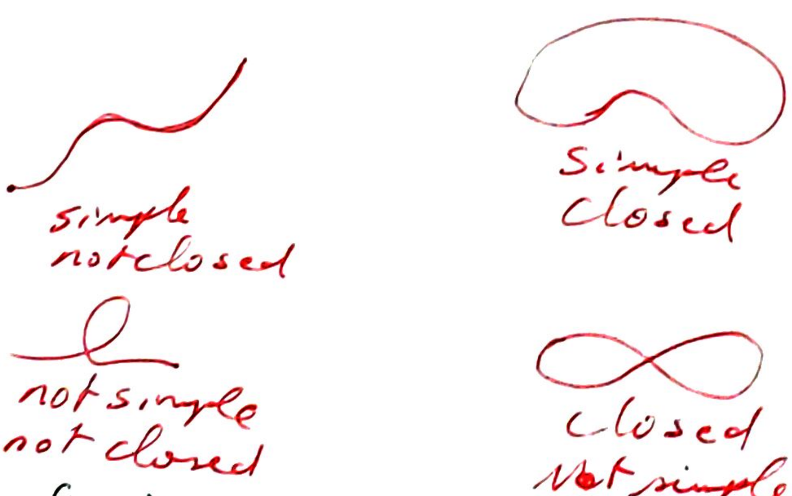
$$= \int_0^{2\pi} \left(1 - \frac{1}{2}\sin 2t\right) dt$$

$$= t + \frac{1}{4}\cos 2t \Big|_0^{2\pi}$$

$$= 2\pi + \frac{1}{4} - \frac{1}{4}$$

$$= \boxed{2\pi}$$

Flux



$$\text{Flux} = \int_C \vec{F} \cdot \vec{n}\, ds$$

$$\vec{F}\cdot\vec{n} = M(x,y)\frac{dy}{ds} - N(x,y)\frac{dx}{ds}$$

$$\text{Flux} = \oint_C M\,dy - N\,dx$$

Ex $\vec{F} = (x-y)\,\hat{i} + x\,\hat{j}$

C: $x^2 + y^2 = 1$ Flux ?

Sol'n $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} \quad 0 \le t \le 2\pi$

$$M = x - y \qquad\qquad N = x$$
$$= \cos t - \sin t \qquad\qquad = \cos t$$

$$dy = d(\sin t) = \cos t\,dt$$

$$dx = d(\cos t) = -\sin t\,dt.$$

$$\text{Flux} = \oint_C M\,dy - N\,dx$$
$$= \int_0^{2\pi}\left[(\cos t - \sin t)\cos t\,dt - \cos t(-\sin t)\,dt\right]$$
$$= \int_0^{2\pi} \cos^2 t\,dt$$
$$= \frac{1}{2}\int_0^{2\pi}(1+\cos 2t)\,dt$$
$$= \frac{1}{2}\left(t + \frac{1}{2}\sin 2t\right)\Big|_0^{2\pi}$$
$$= \frac{1}{2}(2\pi)$$
$$= \boxed{\pi}$$

4.4 Green's Theorem

Defn divergence

$$\vec{F} = M\hat{i} + N\hat{j}$$

$$\operatorname{div}\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \begin{cases} > 0 & \text{expanding} \\ < 0 & \text{compressing} \end{cases}$$

Ex $\vec{F} = cx\hat{i} + cy\hat{j}$

$$\operatorname{div}\vec{F} = c + c$$
$$= 2c \rightarrow \begin{cases} c < 0 & \text{compressing} \\ c > 0 & \text{gas is uniform expansion.} \end{cases}$$

$\vec{F} = -cy\hat{i} + cx\hat{j}$

$\operatorname{div}\vec{F} = 0 + 0$
$= 0$

gas is neither expanding nor compressing

- $\vec{F} = y\hat{i}$ $\operatorname{div}\vec{F} = 0$

- $\vec{F}(x,y) = \dfrac{-y}{x^2+y^2}\hat{i} + \dfrac{x}{x^2+y^2}\hat{j}$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right)\hat{i} + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\hat{j} \quad \left(\frac{1}{u}\right)'$$
$$= +\frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2}$$
$$\boxed{= 0}$$

Curl

$$\text{Curl } \vec{F} \cdot \hat{k} > 0 \quad \text{ccw}$$
$$< 0 \quad \text{cw}$$

Defn circulation density $\vec{F} = M\hat{i} + N\hat{j}$
@ (x, y)

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

This expression is also called $\hat{k}$-component of the curl : $(\text{Curl } \vec{F}) \cdot \hat{k}$

a) $\vec{F} = cx\hat{i} + cy\hat{j}$

$$(\text{curl } \vec{F}) \cdot \hat{k} = \frac{\partial}{\partial x}(cy) - \frac{\partial}{\partial y}(cx)$$
$$= 0$$

b) $\vec{F} = -cy\hat{i} + cx\hat{j}$

$$(\text{curl } \vec{F}) \cdot \hat{k} = \frac{\partial}{\partial x}(cx) - \frac{\partial}{\partial y}(-cy)$$
$$= c + c$$
$$= 2c \quad \begin{cases} c > 0 & \text{rotation ccw} \\ c < 0 & \text{" cw} \end{cases}$$

c) $\vec{F} = y\hat{i}$

$$(\text{curl } \vec{F}) \cdot \hat{k} = \frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(y)$$
$$= -1$$

d) $(\text{curl}\,\vec{F})\cdot\hat{k} = \dfrac{\partial}{\partial x}\left(\dfrac{x}{x^2+y^2}\right) - \dfrac{\partial}{\partial y}\left(\dfrac{-y}{x^2+y^2}\right)$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$= 0$$

Green Theorem

divergence
(div)
flux

$$\oint_C \vec{F}\cdot\vec{N}\,ds = \oint_C M\,dy - N\,dx$$

$$= \iint_R \left(\frac{\partial M}{\partial x} + N_y\right) dx\,dy$$

circulation
(cir)
curl

$$\oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dx\,dy$$

Ex $\vec{F}(x,y) = (x-y)\hat{i} + x\hat{j}$

$r(t) = \cos t\,\hat{i} + \sin t\,\hat{j} \qquad 0 \le t \le 2\pi$

$M = x - y$
$= \cos t - \sin t$

$N = x$
$= \cos t.$

$dx = d(\cos t)$
$= -\sin t\,dt$

$dy = d(\sin t)$
$= \cos t\,dt$

$\frac{\partial M}{\partial x} = 1 \qquad M_y = -1 \qquad N_x = 1 \qquad N_y = 0$

$$\oint_C M\,dy - N\,dx = \int_0^{2\pi} (\cos t - \sin t)\cos t\,dt - \cos t(-\sin t)\,dt$$

$$= \int_0^{2\pi} \cos^2 t\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} (1 + \cos 2t)\,dt$$

$$= \frac{1}{2}\left(t + \frac{1}{2}\sin 2t\right)\Big|_0^{2\pi}$$

$$= \pi$$

$$\iint_R (M_x + N_y)\,dx\,dy = \iint_R (1+0)\,dx\,dy$$

$$= \iint_R dx\,dy \quad \Big\}\ \text{Area of a circle } \pi r^2$$

$$= \pi$$

$$\oint_C M\,dx + N\,dy = \int_0^{2\pi} (\cos t - \sin t)(-\sin t\,dt) + (\cos t)(\cos t\,dt)$$

$$= \int_0^{2\pi} (-\cos t \sin t + \sin^2 t + \cos^2 t)\,dt$$

$$= \int_0^{2\pi} \left(1 - \frac{1}{2}\sin 2t\right) dt$$

$$= t + \frac{1}{4}\cos 2t \Big|_0^{2\pi}$$

$$= \underline{2\pi}$$

$$\iint_R (N_x - M_y)\,dx\,dy = \iint_R (1+1)\,dx\,dy$$

$$= 2\iint_R dx\,dy \longrightarrow (\pi)$$

$$= \underline{2\pi}$$

EX

$$\oint_C \overset{M}{xy\,dy} - \overset{N}{y^2\,dx}$$

C: "square" Q I → $x = 1$, $y = 1$

Soln $\quad M = xy \qquad N = y^2$

$$\oint_C xy\,dy - y^2\,dx = \iint_R (M_x + N_y)\,dx\,dy$$

$$= \int_0^1 dx \int_0^1 (\underbrace{y + 2y}_{3y})\,dy$$

$$= \frac{3}{2} y^2 \Big|_0^1$$

$$= \frac{3}{2}$$

R

$M = -y^2 \qquad N = xy$

$$\oint_C xy\,dy - y^2\,dx = \iint_R (N_x - M_y)\,dx\,dy$$

$$= \int_0^1 dx \int_0^1 (y + 2y)\,dy$$

$$= \frac{3}{2} y^2$$

$$= \frac{3}{2}$$

Ex outward flux $\vec{F} = x\hat{i} + y^2\hat{j}$

$x = \pm 1, \; y = \pm 1$

$M = x \quad N = y^2$

$$\text{flux} = \iint_R (M_x + N_y)\,dx\,dy$$

$$= \int_{-1}^{1} dx \int_{-1}^{1} (1 + 2y)\,dy$$

$$= 2\left(y + y^2\right)\Big|_{-1}^{1}$$

$$= 2(2 - 0)$$

$$= 4$$

$\frac{\#3}{4.4}$ $\vec{F} = (x+y)\hat{i} - (x^2+y^2)\hat{j}$

$y=0, \ x=1, \ y=x$

$M = x+y \qquad N = -x^2-y^2$

$M_x = 1 \qquad N_x = -2x$

$M_y = 1 \qquad N_y = -2y$

$$\text{Flux} = \iint_R (M_x + N_y)\,dx\,dy$$

$$= \int_0^1 \int_0^x (1-2y)\,dy\,dx$$

$$= \int_0^1 (y - y^2\Big|_0^x \,dx$$

$$= \int_0^1 (x - x^2)\,dx$$

$$= \frac{1}{2}x^2 - \frac{1}{3}x^3\Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \boxed{\frac{1}{6}}$$

$$\text{Ci} = \iint_R (N_x - M_y)\,dx\,dy$$

$$= \int_0^1 \int_0^x (-2x-1)\,dy\,dx$$

$$= \int_0^1 (-2xy - y\Big|_0^x \,dx$$

$$= \int_0^1 (-2x^2 - x)\,dx$$

(ii) $= -\frac{2}{3}x^3 - \frac{1}{2}x^2 \Big|_0^1$

$$= -\frac{2}{3} - \frac{1}{2}$$

$$= \frac{-7}{6}$$