Section 4.6 – Substitution Rule

Substitution: Running the Chain Rule Backwards

The Chain rule formula is: $$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}$$

We can see that $\frac{u^{n+1}}{n+1}$ is an antiderivative of the function $u^n \frac{du}{dx}$. Therefore, if we integrate both sides

$$\int u^n \frac{du}{dx} dx = \int \frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) dx \qquad \boxed{\int u^n du = \frac{u^{n+1}}{n+1} + C}$$

Example

Find the integral $\int \left(x^3 + x\right)^5 \left(3x^2 + 1\right) dx$

Solution

Let:

$u = x^3 + x \Rightarrow \quad du = \frac{du}{dx} dx = \left(3x^2 + 1\right) dx$

$$\int \left(x^3 + x\right)^5 \left(3x^2 + 1\right) dx = \int u^5 du$$

$$= \frac{u^6}{6} + C$$

$$= \frac{1}{6}\left(x^3 + x\right)^6 + C$$

$d\left(x^3 + x\right) = \left(3x^2 + 1\right) dx$

$$\int \left(x^3 + x\right)^5 \left(3x^2 + 1\right) dx = \int \left(x^3 + x\right)^5 d\left(x^3 + x\right)$$

$$= \frac{1}{6}\left(x^3 + x\right)^6 + C$$

Example

Find the integral $\int \sqrt{2x + 1} dx$

Solution

Let: $u = 2x + 1 \quad \Rightarrow \quad du = \frac{du}{dx} dx = 2dx$

$$dx = \frac{1}{2} du$$

$$\int \sqrt{2x + 1} dx = \frac{1}{2} \int u^{1/2} du$$

$$= \frac{1}{2} \frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{3}(2x + 1)^{3/2} + C$$

$d(2x + 1) = 2dx$

$$\int \sqrt{2x + 1}\, dx = \frac{1}{2} \int (2x + 1)^{1/2} d(2x + 1)$$

$$= \frac{1}{3}(2x + 1)^{3/2} + C$$

Theorem – The Substitution Rule

If $u = g(x)$ is differentiable function whose range is an interval *I*, and *f* is continuous on *I*, then

$$\int f\big((g(x))\big)g'(x)\,dx = \int f(u)\,du$$

Proof

By the Chain Rule, $F(g(x))$ is an antiderivative of $f(g(x))\cdot g'(x)$ whenever *F* is an antiderivative of *f*:

$$\begin{aligned}\frac{d}{dx}F(g(x)) &= F'(g(x))\cdot g'(x)\\ &= f(g(x))\cdot g'(x)\end{aligned}$$

If we make the substitution $u = g(x)$, then

$$\begin{aligned}\int f(g(x))g'(x)\,dx &= \int \frac{d}{dx}F(g(x))\,dx\\ &= F(g(x)) + C\\ &= F(u) + C\\ &= \int F'(u)\,du\\ &= \int f(u)\,du\end{aligned}$$

Integral of $\int \frac{1}{u}\,du$

If u is a differentiable function that is never zero $\quad \int \frac{1}{u}\,du = \ln|u| + C$

Example

Evaluate the integral $\int \tan x\,dx$

Solution

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx \qquad u = \cos x > 0 \quad \to du = -\sin x\,dx \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\begin{aligned}\int \tan x\,dx &= -\int \frac{d(\cos x)}{\cos x}\\ &= -\ln|\cos x| + C\end{aligned}$$

$$= \ln \frac{1}{|\cos x|} + C$$

$$\boxed{= \ln|\sec x| + C}$$

Example

Evaluate the integral $\displaystyle\int_0^2 \frac{2x}{x^2-5}dx$

Solution

$$u = x^2 - 5 \quad \to du = 2xdx \quad \begin{cases} x=0 & \to u=-5 \\ x=2 & \to u=-1 \end{cases}$$

$$\int_0^2 \frac{2x}{x^2-5}dx = \int_{-5}^{-1} \frac{du}{u}$$

$$= \ln|u| \Big|_{-5}^{-1}$$

$$= \ln|-1| - \ln|-5|$$

$$= \ln 1 - \ln 5$$

$$\boxed{= -\ln 5}$$

$$d\left(x^2-5\right) = 2xdx$$

$$\int_0^2 \frac{2x}{x^2-5}dx = \int_0^2 \frac{d\left(x^2-5\right)}{x^2-5}$$

$$= \ln\left|x^2-5\right| \Big|_0^2$$

$$= \ln|-1| - \ln|-5|$$

$$\boxed{= -\ln 5}$$

Example

Find the integral $\displaystyle\int \sec^2(5t+1)\cdot 5dt$

Solution

$$\int \sec^2(5t+1)\cdot 5dt = \int \sec^2(5t+1)\, d(5t+1)$$

$$\boxed{= \tan(5t+1) + C}$$

$$d(5t+1) = 5dt$$

$$\frac{d}{du}\tan u = \sec^2 u$$

The General Antiderivative of the Exponential Function

$$\int e^{u}du = e^{u} + C$$

Example

Evaluate the integral $\displaystyle\int_0^{\ln 2} e^{3x}dx$

Solution

$$u = 3x \quad du = 3dx \rightarrow dx = \tfrac{1}{3}dx \quad \begin{cases} x = 0 & \rightarrow u = 0 \\ x = \ln 2 & \rightarrow u = 3\ln 2 = \ln 2^3 = \ln 8 \end{cases}$$

$$\begin{aligned} \int_0^{\ln 2} e^{3x}dx &= \int_0^{\ln 8} e^{u} \cdot \tfrac{1}{3}du \\ &= \frac{1}{3}\int_0^{\ln 8} e^{u}du \\ &= \frac{1}{3}e^{u}\Big|_0^{\ln 8} \\ &= \frac{1}{3}\left(e^{\ln 8} - e^{0}\right) \\ &= \frac{1}{3}(8-1) \\ &= \frac{7}{3} \end{aligned}$$

Example

Find the integral $\displaystyle\int \cos(7\theta + 3)d\theta$

Solution

$$\int \cos(7\theta + 3)d\theta = \frac{1}{7}\int \cos(7\theta + 3)\ d(7\theta + 3) \qquad\qquad d(7\theta + 3) = 7d\theta$$

$$= \frac{1}{7}\sin(7\theta + 3) + C$$

Example

Find the integral $\displaystyle\int x^2 \sin\left(x^3\right)dx$

Solution

$$\int x^2 \sin\left(x^3\right) dx = \frac{1}{3}\int \sin\left(x^3\right) d\left(x^3\right)$$

$$\boxed{= -\frac{1}{3}\cos\left(x^3\right) + C}$$

Example

Find the integral $\int x\sqrt{2x+1}\ dx$

Solution

Let: $u = 2x+1 \quad \Rightarrow \quad du = 2dx$

$$dx = \frac{1}{2}du$$

$$u = 2x+1 \rightarrow 2x = u-1 \quad \Rightarrow \quad x = \frac{u-1}{2}$$

$$\int x\sqrt{2x+1}\ dx = \int \frac{1}{2}(u-1)\sqrt{u}\ \ \frac{1}{2}du$$

$$= \frac{1}{4}\int (u-1)u^{1/2}\ du$$

$$= \frac{1}{4}\int \left(u^{3/2} - u^{1/2}\right)\ du$$

$$= \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C$$

$$\boxed{= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C}$$

Example

Find the integral $\displaystyle\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$

Solution

Let: $u = z^2 + 1 \quad \Rightarrow \quad du = 2zdz$

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{du}{u^{1/3}}$$

$$= \int u^{-1/3}du$$

$$= \frac{3}{2}u^{2/3} + C$$

$$= \frac{3}{2}\left(z^2+1\right)^{2/3} + C$$

Or let: $u = \sqrt[3]{z^2+1} \quad \rightarrow \quad u^3 = z^2 + 1$

$3u^2du = 2zdz$

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2du}{u}$$

$$= 3\int u\,du$$

$$= 3\cdot\frac{u^2}{2} + C$$

$$= \frac{3}{2}\left(z^2+1\right)^{2/3} + C$$

Definition

If $a > 0$ and u is a differentiable of x, then a^u is a differentiable function of x and

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx} \quad \rightarrow \quad \int a^u du = \frac{a^u}{\ln a} + C$$

Example

- $\displaystyle\int 2^x dx = \frac{2^x}{\ln 2} + C$
- $\displaystyle\int 2^{\sin x}\cos x dx = \int 2^{\sin x} d(\sin x) = \frac{2^{\sin x}}{\ln 2} + C$
- $\displaystyle\int \frac{\log_2 x}{x}dx = \int \frac{1}{x}\frac{\ln x}{\ln 2}dx$

$$= \frac{1}{\ln 2}\int \frac{\ln x}{x}dx$$

$$= \frac{1}{\ln 2}\int \ln x\; d(\ln x)$$

$$= \frac{1}{\ln 2}\cdot\frac{1}{2}(\ln x)^2 + C$$

$$= \frac{(\ln x)^2}{2\ln 2} + C$$

Substitution Formula

Theorem

If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Proof

Let F denote any antiderivative of f. Then

$$\begin{aligned}\int_a^b f(g(x)) \cdot g'(x)\,dx &= F(g(x))\Big|_{x=a}^{x=b} \\ &= F(g(b)) - F(g(a)) \\ &= F(u)\Big|_{u=g(a)}^{u=g(b)} \\ &= \int_{g(a)}^{g(b)} f(u)\,du\end{aligned}$$

Example

Evaluate $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$

Solution

$$\begin{aligned}\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx &= \int_{-1}^{1} \left(x^3+1\right)^{1/2} d\left(x^3+1\right) \qquad d\left(x^3+1\right) = 3x^2dx \\ &= \frac{2}{3}\left(x^3+1\right)^{3/2}\Big|_{-1}^{1} \\ &= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] \\ &= \frac{2}{3}2^{3/2} \qquad 2^{5/2} = \sqrt{2^5} = 2^2\sqrt{2} \\ &= \boxed{\frac{4\sqrt{2}}{3}}\end{aligned}$$

Example

Evaluate $\displaystyle\int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta \, d\theta$

Solution

Let $u = \cot\theta \quad \Rightarrow \quad du = -\csc^2\theta d\theta \to -du = \csc^2\theta d\theta \qquad \to \begin{cases} \theta = \frac{\pi}{4} & u = \cot\frac{\pi}{4} = 1 \\ \theta = \frac{\pi}{2} & u = \cot\frac{\pi}{2} = 0 \end{cases}$

$$\begin{aligned}
\int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta \, d\theta &= \int_1^0 u \cdot (-du) \\
&= -\int_1^0 u \, du \\
&= -\left[\frac{u^2}{2}\right]_1^0 \\
&= -\left[\frac{0^2}{2} - \frac{1^2}{2}\right] \\
&= \boxed{\frac{1}{2}}
\end{aligned}$$

Example

Evaluate the integral $\displaystyle\int_0^{\pi/6} \tan 2x dx$

Solution

$$\begin{aligned}
\int_0^{\pi/6} \tan 2x dx &= \int_0^{\pi/6} \tan u \cdot \left(\frac{du}{2}\right) \qquad u = 2x \quad \to du = 2dx \quad \Rightarrow dx = \frac{du}{2} \\
&= \frac{1}{2}\int_0^{\pi/6} \tan u \cdot du \\
&= \frac{1}{2}\ln\left|\sec 2x\right|_0^{\pi/6} \\
&= \frac{1}{2}\left[\ln\left|\sec 2\frac{\pi}{6}\right| - \ln\left|\sec 0\right|\right] \\
&= \frac{1}{2}(\ln 2 - \ln 1) \\
&= \boxed{\frac{1}{2}\ln 2}
\end{aligned}$$

Integrals of $\sin^2 x$ *and* $\cos^2 x$

Example

Find the integral $\int \sin^2 x\, dx$

Solution

$$\int \sin^2 x\, dx = \frac{1}{2}\int (1-\cos 2x)\, dx \qquad\qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + C$$

$$\boxed{= \frac{1}{2}x - \frac{1}{4}\sin 2x + C}$$

Example

Find the integral $\int \cos^2 x\, dx$

Solution

$$\int \cos^2 x\, dx = \int \frac{1+\cos 2x}{2}\, dx \qquad\qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right] + C$$

$$\boxed{= \frac{x}{2} + \frac{\sin 2x}{4} + C}$$

Integration Formulas

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C \qquad \left(\textit{Valid for } u^2 < a^2\right)$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \qquad \left(\textit{Valid for all } u\right)$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C \qquad \left(\textit{Valid for } |u| > a > 0\right)$$

Exercises *Section* 4.6 – Substitution Rule

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

28. $\int 2(2x+4)^5\,dx, \quad u = 2x+4$

29. $\int \frac{4x^3}{(x4+1)^2}\,dx, \quad u = x^4+1$

30. $\int x\sin\left(2x^2\right)dx, \quad u = 2x^2$

31. $\int 12\left(y^4+4y^2+1\right)^2\left(y^3+2y\right)dx, \quad u = y^4+4y^2+1$

32. $\int \csc^2 2\theta\cot 2\theta\,d\theta \quad \to \begin{cases} a)\ U\sin g\ \ u = \cot 2\theta \\ b)\ U\sin g\ \ u = \csc 2\theta \end{cases}$

Evaluate the integrals

33. $\int \frac{1}{\sqrt{5s+4}}\,ds$

34. $\int \theta\sqrt[4]{1-\theta^2}\,d\theta$

35. $\int \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2}\,dx$

36. $\int \tan^2 x\sec^2 x\,dx$

37. $\int \sin^5\frac{x}{3}\cos\frac{x}{3}\,dx$

38. $\int \tan^7\frac{x}{2}\sec^2\frac{x}{2}\,dx$

39. $\int r^4\left(7-\frac{r^5}{10}\right)^3 dr$

40. $\int x^{1/2}\sin\left(x^{3/2}+1\right)dx$

41. $\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv$

42. $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt$

43. $\int \frac{\sec z\ \tan z}{\sqrt{\sec z}}\,dz$

44. $\int \frac{1}{\sqrt{t}}\cos\left(\sqrt{t}+3\right)dt$

45. $\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\,d\theta$

46. $\int t^3\left(1+t^4\right)^3 dt$

47. $\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}\,dx$

48. $\int x^3\sqrt{x^2+1}\,dx$

49. $\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}\,d\theta$

50. $\int 2x\sqrt{x^2-2}\,dx$

51. $\int \frac{x}{\left(x^2-4\right)^3}\,dx$

52. $\int x^3\left(3x^4+1\right)^2 dx$

53. $\int 2\left(3x^4+1\right)^2 dx$

54. $\int 5x\sqrt{x^2-1}\,dx$

55. $\int \left(x^2-1\right)^3(2x)dx$

56. $\int \frac{6x}{\left(1+x^2\right)^3}\,dx$

57. $\displaystyle\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}\,dr$

58. $\displaystyle\int u^3\sqrt{u^4+2}\;du$

59. $\displaystyle\int \frac{t+2t^2}{\sqrt{t}}\;dt$

60. $\displaystyle\int \left(1+\frac{1}{t}\right)^3 \frac{1}{t^2}\;dt$

61. $\displaystyle\int \left(7-3x-3x^2\right)(2x+1)\;dx$

62. $\displaystyle\int \sqrt{x}\left(4-x^{3/2}\right)^2\;dx$

63. $\displaystyle\int \frac{1}{\sqrt{x}+\sqrt{x+1}}\,dx$

64. $\displaystyle\int \sqrt{1-x}\;dx$

65. $\displaystyle\int x\sqrt{x^2+4}\;dx$

66. $\displaystyle\int \sin^2\left(\theta+\frac{\pi}{6}\right)d\theta$

67. $\displaystyle\int \cos^2(8\theta)\,d\theta$

68. $\displaystyle\int \sin^2(2\theta)\,d\theta$

69. $\displaystyle\int 8\cos^4 2\pi x\;dx$

70. $\displaystyle\int \sec x dx$

71. $\displaystyle\int \frac{dx}{\sqrt{1-4x^2}}$

72. $\displaystyle\int \frac{dx}{\sqrt{3-4x^2}}$

73. $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-6}}$

74. $\displaystyle\int \frac{dx}{\sqrt{4x-x^2}}$

75. $\displaystyle\int \frac{dx}{4x^2+4x+2}$

76. $\displaystyle\int \frac{1}{6x-5}\,dx$

77. $\displaystyle\int \frac{x^2+2x+3}{x^3+3x^2+9x+1}\,dx$

78. $\displaystyle\int \frac{1}{x(\ln x)^2}\,dx$

79. $\displaystyle\int \frac{x-3}{x+3}\,dx$

80. $\displaystyle\int \frac{3x}{x^2+4}\,dx$

81. $\displaystyle\int \frac{dx}{2\sqrt{x}+2x}$

82. $\displaystyle\int \frac{\sec x dx}{\sqrt{\ln(\sec x+\tan x)}}$

83. $\displaystyle\int 8e^{(x+1)}dx$

84. $\displaystyle\int 4xe^{x^2}\,dx$

85. $\displaystyle\int (2x+1)e^{x^2+x}dx$

86. $\displaystyle\int \frac{e^{-\sqrt{r}}}{\sqrt{r}}\,dr$

87. $\displaystyle\int t^3 e^{t^4}\,dt$

88. $\displaystyle\int e^{\sec \pi t}\sec\pi\;\;\tan\pi t\;dt$

89. $\displaystyle\int (2x+1)e^{x^2+x}dx$

90. $\displaystyle\int \frac{dx}{1+e^x}$

91. $\displaystyle\int \frac{e^x}{1+e^x}\,dx$

92. $\displaystyle\int \frac{2}{e^{-x}+1}\,dx$

93. $\displaystyle\int \frac{1}{x^3}e^{1/4x^2}dx$

94. $\displaystyle\int \frac{e^{1/\sqrt{x}}}{x^{3/2}}\,dx$

95. $\displaystyle\int \frac{-e^{3x}}{2-e^{3x}}\,dx$

96. $\displaystyle\int \frac{7e^{7x}}{3+e^{7x}}dx$

97. $\displaystyle\int \frac{2(e^x-e^{-x})}{(e^x+e^{-x})^2}\,dx$

98. $\displaystyle\int \frac{3^x}{3-3^x}dx$

99. $\displaystyle\int \frac{x\;2^{x^2}}{1+2^{x^2}}dx$

100. $\int (6x+e^x)\sqrt{3x^2+e^x}\,dx$

101. $\int \frac{dx}{x\left(\log_8 x\right)^2}$

102. $\int \frac{dx}{x\sqrt{25x^2-2}}$

103. $\int \frac{6dr}{\sqrt{4-(r+1)^2}}$

104. $\int \frac{dx}{2+(x-1)^2}$

105. $\int \frac{\sec^2 y\, dy}{\sqrt{1-\tan^2 y}}$

106. $\int \frac{dx}{\sqrt{-x^2+4x-3}}$

107. $\int \frac{dx}{\sqrt{2x-x^2}}$

108. $\int \frac{x-2}{x^2-6x+10}dx$

109. $\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$

110. $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$

111. $\int \frac{e^{\cos^{-1}x}dx}{\sqrt{1-x^2}}$

112. $\int \frac{\left(\sin^{-1}x\right)^2 dx}{\sqrt{1-x^2}}$

113. $\int \frac{dy}{\left(\sin^{-1}y\right)\sqrt{1+y^2}}$

114. $\int \frac{1}{\sqrt{x}(x+1)\left(\left(\tan^{-1}\sqrt{x}\right)^2+9\right)}dx$

115. $\int 2x\left(x^2+1\right)^4 dx$

116. $\int 8x\cos\left(4x^2+3\right)dx$

117. $\int \sin^3 x\cos x\, dx$

118. $\int (6x+1)\sqrt{3x^2+x}\, dx$

119. $\int 2x\left(x^2-1\right)^{99} dx$

120. $\int xe^{x^2}\, dx$

121. $\int \frac{2x^2}{\sqrt{1-4x^3}}dx$

122. $\int \frac{\left(\sqrt{x}+1\right)^4}{2\sqrt{x}}dx$

123. $\int \left(x^2+x\right)^{10}(2x+1)dx$

124. $\int \frac{dx}{10x-3}$

125. $\int x^3\left(x^4+16\right)^6 dx$

126. $\int \sin^{10}\theta\, \cos\theta\, d\theta$

127. $\int \frac{dx}{\sqrt{1-9x^2}}$

128. $\int x^9 \sin x^{10} dx$

129. $\int \left(x^6 - 3x^2\right)^4 \left(x^5 - x\right) dx$

130. $\int \frac{x}{x-2}\, dx$

131. $\int \frac{dx}{1+4x^2}$

132. $\int \frac{3}{1+25y^2}\, dy$

133. $\int \frac{2}{x\sqrt{4x^2-1}}\, dx \quad \left(x > \frac{1}{2}\right)$

134. $\int \frac{8x+6}{2x^2+3x}\, dx$

135. $\int \frac{x}{\sqrt{x-4}}\, dx$

136. $\int \frac{x^2}{(x+1)^4}\, dx$

137. $\int \frac{x}{\sqrt[3]{x+4}}\, dx$

138. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$

139. $\int x\, \sqrt[3]{2x+1}\, dx$

140. $\int (x+1)\sqrt{3x+2}\, dx$

141. $\int \sin^2 x\, dx$

142. $\int \sin^2\left(\theta + \frac{\pi}{6}\right) d\theta$

143. $\int x \cos^2\left(x^2\right) dx$

144. $\int \sec 4x\, \tan 4x\, dx$

145. $\int \sec^2 10x\, dx$

146. $\int \left(\sin^5 x + 3\sin^3 x - \sin x\right) \cos x\, dx$

147. $\int \frac{\csc^2 x}{\cot^3 x}\, dx$

148. $\int \left(x^{3/2} + 8\right)^5 \sqrt{x}\, dx$

149. $\int \sin x\, \sec^8 x\, dx$

150. $\int \frac{e^{2x}}{e^{2x}+1}\, dx$

151. $\int \sec^3\theta\, \tan\theta\, d\theta$

152. $\int x\, \sin^4 x^2\, \cos x^2\, dx$

153. $\int \frac{dx}{\sqrt{1+\sqrt{1+x}}}$

154. $\int \tan^{10} 4x\, \sec^2 4x\, dx$

155. $\int \frac{x^2}{x^3+27}\, dx$

156. $\int y^2 \left(3y^3 + 1\right)^4 dy$

157. $\int x \sin x^2 \cos^8 x^2\, dx$

158. $\int \frac{\sin 2x}{1+\cos^2 x}\, dx$

159. $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$

160. $\int \frac{dx}{\left(\tan^{-1} x\right)\left(1+x^2\right)}$

161. $\int \frac{\left(\tan^{-1} x\right)^5}{1+x^2}\,dx$

162. $\int \frac{1}{x^2}\sin\frac{1}{x}\,dx$

163. Evaluate the integral $\int \frac{18\tan^2 x\sec^2 x}{\left(2+\tan^3 x\right)^2}\,dx$

a) $u = \tan x$, followed by $v = u^3$ *then by* $w = 2+v$

b) $u = \tan^3 x$, followed by $v = 2+u$

c) $u = 2+\tan^3 x$

Evaluate the integrals

164. $\int_0^1 (2t+3)^3\,dt$

165. $\int_0^2 \sqrt{4-x^2}\,dx$

166. $\int_0^3 \sqrt{y+1}\,dy$

167. $\int_{-1}^1 r\sqrt{1-r^2}\,dr$

168. $\int_0^{\pi/4} \tan x\sec^2 x\,dx$

169. $\int_{2\pi}^{3\pi} 3\cos^2 x\sin x\,dx$

170. $\int_0^1 t^3\left(1+t^4\right)^3\,dt$

171. $\int_0^1 \frac{r}{\left(4+r^2\right)^2}\,dr$

172. $\int_0^1 \frac{10\sqrt{v}}{\left(1+v^{3/2}\right)^2}\,dv$

173. $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$

174. $\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$

175. $\int_0^{\pi/6} (1-\cos 3t)\sin 3t\,dt$

176. $\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$

177. $\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$

178. $\int_{-\pi/2}^{0} \frac{\sin w}{(3+2\cos w)^2}\,dw$

179. $\displaystyle\int_0^1 \sqrt{t^5+2t}\,\left(5t^4+2\right)dt$

180. $\displaystyle\int_1^4 \frac{dy}{2\sqrt{y}\left(1+\sqrt{y}\right)^2}$

181. $\displaystyle\int_0^1 \left(4y-y^2+4y^3+1\right)^{-2/3}\left(12y^2-2y+4\right)dy$

182. $\displaystyle\int_0^5 |x-2|\,dx$

183. $\displaystyle\int_0^{\pi/2} e^{\sin x}\cos x\,dx$

184. $\displaystyle\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$

185. $\displaystyle\int_0^{\pi/3} \frac{4\sin\theta}{1-4\cos\theta}\,d\theta$

186. $\displaystyle\int_1^2 \frac{2\ln x}{x}\,dx$

187. $\displaystyle\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$

188. $\displaystyle\int_0^{\pi/2} \tan\frac{x}{2}\,dx$

189. $\displaystyle\int_{\pi/4}^{\pi/2} \cot x\,dx$

190. $\displaystyle\int_{-\ln 2}^{0} e^{-x}dx$

191. $\displaystyle\int_{\pi/4}^{\pi/2} \left(1+e^{\cot\theta}\right)\csc^2\theta\,d\theta$

192. $\displaystyle\int_0^{\sqrt{\ln\pi}} 2x\,e^{x^2}\cos\left(e^{x^2}\right)dx$

193. $\displaystyle\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx$

194. $\displaystyle\int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan t}\sec^2 t\,dt$

195. $\displaystyle\int_1^e x^{(\ln 2)-1}dx$

196. $\displaystyle\int_0^9 \frac{2\log_{10}(x+1)}{x+1}\,dx$

197. $\displaystyle\int_1^e \frac{2\ln 10\log_{10} x}{x}\,dx$

198. $\displaystyle\int_1^{e^x} \frac{1}{t}\,dt$

199. $\displaystyle\frac{1}{\ln a}\int_1^x \frac{1}{t}\,dt \quad x>0$

200. $\displaystyle\int_0^{3\sqrt{2}/4} \frac{dx}{\sqrt{9-4x^2}}$

201. $\displaystyle\int_{\pi/6}^{\pi 4} \frac{\csc^2 x dx}{1+(\cot x)^2}$

202. $\displaystyle\int_1^{e^{\pi/4}} \frac{4dt}{t\left(1+\ln^2 t\right)}$

203. $\displaystyle\int_{1/2}^{1} \frac{6dx}{\sqrt{-4x^2+4x+3}}$

204. $\displaystyle\int_{2/\sqrt{3}}^{2} \frac{\cos\left(\sec^{-1} x\right)dx}{x\sqrt{x^2-1}}$

205. $\displaystyle\int_{0}^{3} \frac{x}{\sqrt{25-x^2}}\,dx$

206. $\displaystyle\int_{0}^{\pi} \sin^2 5\theta\; d\theta$

207. $\displaystyle\int_{0}^{\pi} \left(1-\cos^2 3\theta\right)\; d\theta$

208. $\displaystyle\int_{2}^{3} \frac{x^2+2x-2}{x^3+3x^2-6x}\,dx$

209. $\displaystyle\int_{0}^{\ln 2} \frac{e^x}{1+e^{2x}}\,dx$

210. $\displaystyle\int_{1}^{3} x\,\sqrt[3]{x^2-1}\;dx$

211. $\displaystyle\int_{0}^{2} (x+3)^3\,dx$

212. $\displaystyle\int_{-2}^{2} e^{4x+8}dx$

213. $\displaystyle\int_{0}^{1} \sqrt{x}\left(\sqrt{x}+1\right)dx$

214. $\displaystyle\int_{0}^{1} \frac{dx}{\sqrt{4-x^2}}$

215. $\displaystyle\int_{0}^{2} \frac{2x}{\left(x^2+1\right)^2}\,dx$

216. $\displaystyle\int_{0}^{\pi/2} \sin^2\theta\;\cos\theta\; d\theta$

217. $\displaystyle\int_{0}^{\pi/4} \frac{\sin x}{\cos^2 x}\,dx$

218. $\displaystyle\int_{1/3}^{1/\sqrt{3}} \frac{4}{9x^2+1}\,dx$

219. $\displaystyle\int_{0}^{\ln 4} \frac{e^x}{3+2e^x}\,dx$

220. $\displaystyle\int_{-\pi}^{\pi} \cos^2 x\; dx$

221. $\displaystyle\int_{0}^{\pi/4} \cos^2 8\theta\; d\theta$

222. $\displaystyle\int_{-\pi/4}^{\pi/4} \sin^2 2\theta\; d\theta$

223. $\displaystyle\int_{0}^{\pi/6} \frac{\sin 2x}{\sin^2 x+2}\,dx$

224. $\displaystyle\int_{0}^{\pi/2} \sin^4\theta\; d\theta$

225. $\displaystyle\int_{0}^{1} x\sqrt{1-x^2}\;dx$

226. $\displaystyle\int_{0}^{1/4} \frac{x}{\sqrt{1-16x^2}}\,dx$

227. $\displaystyle\int_{2}^{3} \frac{x}{\sqrt[3]{x^2-1}}\,dx$

228. $\displaystyle\int_{0}^{6/5} \frac{dx}{25x^2+36}$

229. $\displaystyle\int_{0}^{2} x^3\,\sqrt{16-x^4}\;dx$

230. $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^2 x}\,dx$

231. $\displaystyle\int_{-1}^{1} (x-1)\left(x^2-2x\right)^7 dx$

232. $\displaystyle\int_{-\pi}^{0} \frac{\sin x}{2+\cos x}\,dx$

233. $\displaystyle\int_{0}^{1} \frac{(x+1)(x+2)}{2x^3+9x^2+12x+36}\,dx$

234. $\displaystyle\int_{1}^{2} \frac{4}{9x^2+6x+1}\,dx$

235. $\displaystyle\int_{0}^{\pi/4} e^{\sin^2 x} \sin 2x\,dx$

236. $\displaystyle\int_{0}^{1} x\sqrt{x+a}\;dx \quad (a>0)$

237. $\displaystyle\int_{0}^{1} x\sqrt[p]{x+a}\;dx \quad (a>0)$

238. $\displaystyle\int_{0}^{1} x\sqrt{1-\sqrt{x}}\;dx$

239. $\displaystyle\int_{0}^{1} \sqrt{x-x\sqrt{x}}\;dx$

240. $\displaystyle\int_{0}^{\pi/2} \frac{\cos\theta\sin\theta}{\sqrt{\cos^2\theta+16}}\,d\theta$

241. $\displaystyle\int_{\frac{2}{5\sqrt{3}}}^{\frac{2}{5}} \frac{dx}{x\sqrt{25x^2-1}}$

242. $\displaystyle\int_{0}^{4} \frac{x}{\sqrt{9+x^2}}\,dx$

243. $\displaystyle\int_{0}^{\pi/4} \frac{\sin\theta}{\cos^3\theta}\,d\theta$

244. $\displaystyle\int_{0}^{1} 2x\left(4-x^2\right)\,dx$

245. $\displaystyle\int_{0}^{3} \frac{x^2+1}{\sqrt{x^3+3x+4}}\,dx$

246. $\displaystyle\int_{0}^{4} \frac{x}{x^2+1}\,dx$

247. $\displaystyle\int_{1}^{e^2} \frac{\ln x}{x}\,dx$

248. $\displaystyle\int_{0}^{3} \frac{x^2+1}{\sqrt{x^3+3x+4}}\,dx$

249. $\displaystyle\int_{-\pi/4}^{\pi/4} \sin^2 2\theta\;d\theta$

250. $\displaystyle\int_{0}^{1} \left(y^3+6y^2-12y+9\right)^{-1/2}\left(y^2+4y-4\right)dy$

251. $\displaystyle\int_{-1}^{2} x^2 e^{x^3+1}\,dx$

252. $\displaystyle\int_{0}^{2} x^2 e^{x^3}\,dx$

253. $\displaystyle\int_{0}^{4} \frac{x}{x^2+1}\,dx$

Solve the initial value problem

254. $\frac{dy}{dt}=e^{t}\sin\left(e^{t}-2\right), \quad y(\ln 2)=0$

255. $\frac{dy}{dt}=e^{-t}\sec^{2}\left(\pi e^{-t}\right), \quad y(\ln 4)=\frac{2}{\pi}$

256. Verify the integration formula: $\displaystyle\int \frac{\tan^{-1}x}{x^{2}}dx=\ln x-\frac{1}{2}\ln\left(1+x^{2}\right)-\frac{\tan^{-1}x}{x}+C$

257. Verify the integration formula: $\displaystyle\int \ln\left(a^{2}+x^{2}\right)dx=x\ln\left(a^{2}+x^{2}\right)-2x+2a\tan^{-1}\frac{x}{a}+C$

258. Find the area of the region bounded by the graphs of $x=3\sin y\sqrt{\cos y}$, *and* $x=0$, $0\le y\le\frac{\pi}{2}$

259. Find the area of the region bounded by the graph of $f(x)=\frac{x}{\sqrt{x^{2}-9}}$ *on* $3\le x\le 4$

260. Find the area of the region bounded by the graph of $f(x)=\frac{x}{\sqrt{x^{2}-9}}$ and the *x-axis* between $x=4$ and $x=5$.

261. Find the area of the region bounded by the graph of $f(x)=x\ \sin x^{2}$ and the *x-axis* between $x=0$ and $x=\sqrt{\pi}$.

262. Find the area of the region bounded by the graph of $f(\theta)=\cos\theta\ \sin\theta$ and the *θ-axis* between $\theta=0$ and $\theta=\frac{\pi}{2}$.

263. Find the area of the region bounded by the graph of $f(x)=(x-4)^{4}$ and the *x-axis* between $x=2$ and $x=6$.

264. Perhaps the simplest change of variables is the shift or translation given by $u=x+c$, where c is a real number.

a) Prove that shifting a function does not change the net area under the curve, in the sense that

$$\int_{a}^{b} f(x+c)dx=\int_{a+c}^{b+c} f(u)du$$

b) Draw a picture to illustrate this change of variables in the case that $f(x)=\sin x$, $a=0$, $b=\pi$, and $c=\frac{\pi}{2}$

265. Another change of variables that can be interpreted geometrically is the scaling $u = cx$, where c is a real number. Prove and interpret the fact that

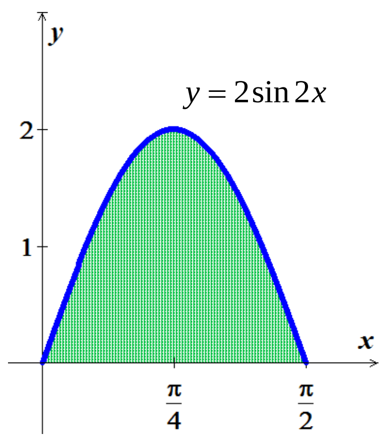
$$\int_a^b f(cx)\,dx = \frac{1}{c}\int_{ac}^{bc} f(u)\,du$$

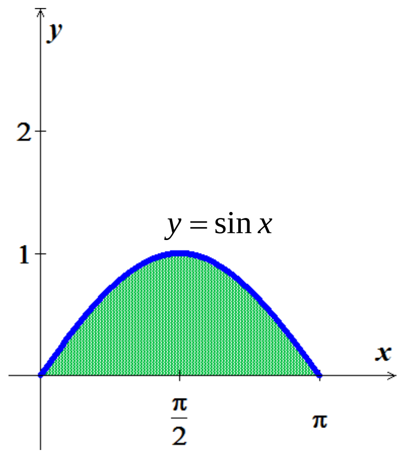
Draw a picture to illustrate this change of variables in the case that $f(x) = \sin x$, $a = 0$, $b = \pi$, and $c = \frac{1}{2}$

266. The function f satisfies the equation $3x^4 - 48 = \int_2^x f(t)\,dt$. Find f and check your answer by substitution.

267. Assume f' is continuous on $[2,\ 4]$, $\int_1^2 f'(2x)\,dx = 10$, and $f(2) = 4$. Evaluate $f(4)$.

268. The area of the shaded region under the curve $y = 2\sin 2x$ in

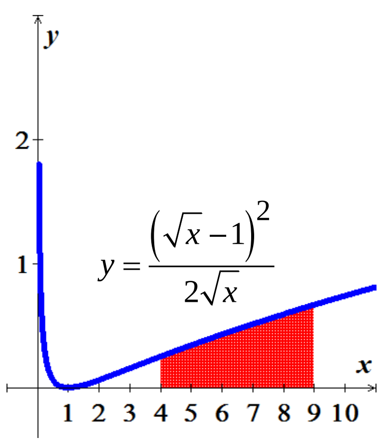


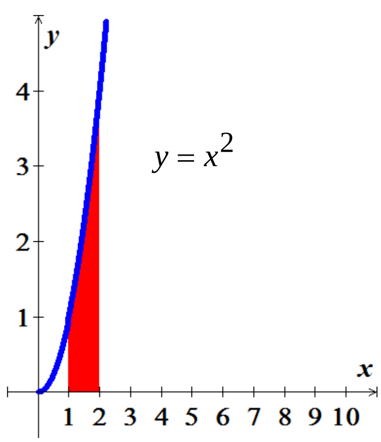


a) Equals the area on the shaded region under the curve $y = \sin x$

b) Explain why this is true without computation areas.

269. The area of the shaded region under the curve $y = \dfrac{\left(\sqrt{x}-1\right)^2}{2\sqrt{x}}$ on the interval $[4,\ 9]$





a) Equals the area on the shaded region under the curve $y = x^2$ on the interval $[1,\ 2]$

b) Explain why this is true without computation areas.

270. The family of parabolas $y = \frac{1}{a} - \frac{x^2}{a^3}$, where $a > 0$, has the property that for $x \geq 0$, the *x*-intercept is $(a,\ 0)$ and the *y*-intercept is $\left(0,\ \frac{1}{a}\right)$. Let $A(a)$ be the area of the region in the first quadrant bounded by the parabola and the *x-axis*. Find $A(a)$ and determine whether it is increasing, decreasing, or constant function of a.

271. Consider the right triangle with vertices $(0,\ 0)$, $(0,\ b)$, and $(a,\ 0)$, where $a > 0$ and $b > 0$. Show that the average vertical distance from points on the *x-axis* to the hypotenuse is $\frac{b}{2}$, for all $a > 0$.

272. Consider the integral $I = \int \sin^2 x\ \cos^2 x\ dx$

a) Find *I* using the identity $\sin 2x = 2\sin x \cos x$

b) Find *I* using the identity $\cos^2 x = 1 - \sin^2 x$

c) Confirm that the results in part (*a*) and (*b*) are consistent and compare the work involved in each method.

273. Let $H(x) = \int_0^x \sqrt{4 - t^2}\, dt$, for $-2 \leq x \leq 2$.

a) Evaluate $H(0)$

b) Evaluate $H'(1)$

c) Evaluate $H'(2)$

d) Use geometry to evaluate $H(2)$

e) Find the value of *s* such that $H(x) = sH(-x)$

Evaluate the limits

274. $\lim\limits_{x \to 2} \dfrac{\int_2^x e^{t^2}\, dt}{x - 2}$

275. $\lim\limits_{x \to 1} \dfrac{\int_1^{x^2} e^{t^3}\, dt}{x - 1}$

276. Prove that for nonzero constants *a* and *b*, $\int \frac{dx}{a^2 x^2 + b^2} = \frac{1}{ab} \tan^{-1}\left(\frac{ax}{b}\right) + C$

277. Let $a > 0$ be a real number and consider the family of functions $f(x) = \sin ax$ on the interval $\left[0, \frac{\pi}{a}\right]$.

a) Graph f, for a = 1, 2, 3.

b) Let $g(a)$ be the area of the region bounded by the graph of f and the *x-axis* on the interval $\left[0, \frac{\pi}{a}\right]$. Graph g for $0 < a < \infty$. Is g an increasing function, a decreasing function, or neither?

278. Explain why if a function u satisfies the equation $u(x) + 2\int_0^x u(t)\,dt = 10$, then it also satisfies the equation $u'(x) + 2u(x) = 0$. Is it true that is u satisfies the second equation, then it satisfies the first equation?

279. Let $f(x) = \int_0^x (t-1)^{15}(t-2)^9\,dt$

a) Find the interval on which f is increasing and the intervals on which f is decreasing.

b) Find the intervals on which f is concave up and the intervals on which f is concave down.

c) For what values of x does f have local minima? Local maxima?

d) Where are the inflection points of f ?

280. A company is considering a new manufacturing process in one of its plants. The new process provides substantial initial savings, with the savings declining with time t (in years) according to the rate-of-savings function

$$S'(t) = 100 - t^2$$

where $S'(t)$ is in thousands of dollars per year. At the same time, the cost of operating the new process increases with time t (in years), according to the rate-of-cost function (in thousands of dollars per year)

$$C'(t) = t^2 + \tfrac{14}{3}t$$

a) For how many years will the company realize savings?

b) What will be the net total savings during this period?

281. Find the producers' surplus if the supply function for pork bellies is given by

$$S(x) = x^{5/2} + 2x^{3/2} + 50$$

Assume supply and demand are in equilibrium at $x = 16$.

282. An object moves along a line with a velocity in m/s given by $v(t) = 8\cos\left(\frac{\pi t}{6}\right)$. Its initial position is $s(0) = 0$.

a) Graph the velocity function.

b) The position of the object is given by $s(t) = \int_0^t v(y)\,dy$, for $t \geq 0$. Find the position function, for $t \geq 0$.

c) What is the period of the motion – that is, starting at any point, how long does it take the object to return to that position?

283. The population of a culture of bacteria has a growth rate given by $p'(t) = \dfrac{200}{(t+1)^r}$ bacteria per hour, for $t \geq 0$, where $r > 1$ is a real number. It is shown that the increase in the population over time interval $[0,\ t]$ is given by $\int_0^t p'(s)\,ds$. (note that the growth rate decreases in time, reflecting competition for space and food.)

a) Using the population model with $r = 2$, what is the increase in the population over the time interval $0 \leq t \leq 4$?

b) Using the population model with $r = 3$, what is the increase in the population over the time interval $0 \leq t \leq 6$?

c) Let ΔP be the increase in the population over a fixed time interval $[0,\ T]$. For fixed *T*, does ΔP increase or decrease with the parameter *r*? Explain.

d) A lab technician measures an increase in the population of 350 bacteria over the 10-hr period $[0,\ 10]$. Estimate the value of *r* that best fits this data point.

e) Use the population model in part (*b*) to find the increase in population over time interval $[0,\ T]$, for any $T > 0$. If the culture is allowed to grow indefinitely $(T \to \infty)$, does the bacteria population increase without bound? Or does it approach a finite limit?

284. Consider the function $f(x) = x^2 - 5x + 4$ and the area function $A(x) = \int_0^x f(t)\,dt$.

a) Graph f on the interval [0, 6].

b) Compute and graph *A* on the interval [0, 6].

c) Show that the local extrema of *A* occur at the zeros of f.

d) Give a geometric and analytical explanation for the observation in part (c).

e) Find the approximate zeros of *A*, other than 0, and call them x_1 and x_2.

f) Find *b* such that the area bounded by the graph of f and the *x*-axis on the interval $[0,\ x_1]$ equals the area bounded by the graph of f and the *x*-axis on the interval $[x_1,\ b]$.

g) If f is an integrable function and $A(x) = \int_0^x f(t)\,dt$, is it always true that the local extrema of *A* occur at the zeros of f? Explain