Solution   *Section* 2.7 – Implicit Differentiation

Exercise

Find $\frac{dy}{dx}$: $y^2 + x^2 - 2y - 4x = 4$

Solution

$$\frac{d}{dx}[y^2 + x^2 - 2y - 4x] = \frac{d}{dx}[4]$$

$$\frac{d}{dx}\left[y^2\right] + \frac{d}{dx}\left[x^2\right] - \frac{d}{dx}[2y] - \frac{d}{dx}[4x] = \frac{d}{dx}[4]$$

$$2y\frac{dy}{dx} + 2x - 2\frac{dy}{dx} - 4 = 0$$

$$2(y-1)\frac{dy}{dx} = 4 - 2x$$

$$(y-1)\frac{dy}{dx} = 2 - x$$

$$\boxed{\frac{dy}{dx} = \frac{2-x}{y-1}}$$

Exercise

Find $\frac{dy}{dx}$: $x^2y^2 - 2x = 3$

Solution

$$2xy^2 + 2x^2yy' - 2 = 0$$

$$2x^2yy' = 2 - 2xy^2$$

$$y' = \frac{2\left(1 - xy^2\right)}{2x^2y}$$

$$\boxed{\frac{dy}{dx} = \frac{1 - xy^2}{x^2y}}$$

Exercise

Find $\frac{dy}{dx}$: $x+\sqrt{x}\sqrt{y}=y^2$

Solution

$$\frac{d}{dx}\left(x+x^{1/2}y^{1/2}\right)=\frac{d}{dx}y^2$$

$$1+\frac{d}{dx}\left(x^{1/2}\right)y^{1/2}+x^{1/2}\frac{d}{dx}\left(y^{1/2}\right)=2y\frac{dy}{dx}$$

$$1+\frac{1}{2}x^{-1/2}y^{1/2}+x^{1/2}\frac{1}{2}y^{-1/2}\frac{dy}{dx}=2y\frac{dy}{dx}$$

$$1+\frac{y^{1/2}}{2x^{1/2}}+\frac{x^{1/2}}{2y^{1/2}}\frac{dy}{dx}=2y\frac{dy}{dx}$$

$$1+\frac{y^{1/2}}{2x^{1/2}}=2y\frac{dy}{dx}-\frac{x^{1/2}}{2y^{1/2}}\frac{dy}{dx}$$

$$\left(\frac{4y^{3/2}-x^{1/2}}{2y^{1/2}}\right)\frac{dy}{dx}=\frac{2x^{1/2}+y^{1/2}}{2x^{1/2}}$$

$$\frac{dy}{dx}=\frac{2x^{1/2}+y^{1/2}}{2x^{1/2}}\cdot\frac{2y^{1/2}}{4y^{3/2}-x^{1/2}}$$

$$=\frac{4x^{1/2}y^{1/2}+2y}{8x^{1/2}y^{3/2}-2x}$$ *Divide every term by 2*

$$=\frac{2x^{1/2}y^{1/2}+y}{4x^{1/2}y^{3/2}-x}$$

Exercise

Find $\frac{dy}{dx}$: $x^2y+xy^2=6$

Solution

$$\left(2xy+x^2\frac{dy}{dx}\right)+\left(y^2+2xy\frac{dy}{dx}\right)=0$$

$$x^2\frac{dy}{dx}+2xy\frac{dy}{dx}=-2xy-y^2$$

$$\left(x^2+2xy\right)\frac{dy}{dx}=-2xy-y^2$$

$$\frac{dy}{dx}=\frac{-2xy-y^2}{x^2+2xy}$$

Exercise

Find $\frac{dy}{dx}$: $\quad x^3 - xy + y^3 = 1$

Solution

$$3x^2 - \left(y + x\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0$$

$$3x^2 - y - x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$\left(3y^2 - x\right)\frac{dy}{dx} = y - 3x^2$$

$$\boxed{\frac{dy}{dx} = \frac{y - 3x^2}{3y^2 - x}}$$

Exercise

Find $\frac{dy}{dx}$: $\quad y^2 = \frac{x-1}{x+1}$

Solution

$$2yy' = \frac{1(x+1) - (1)(x-1)}{(x+1)^2}$$

$$2yy' = \frac{x+1-x+1}{(x+1)^2}$$

$$y' = \frac{2}{2y(x+1)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{y(x+1)^2}}$$

Exercise

Find $\frac{dy}{dx}$: $\quad (3xy+7)^2 = 6y$

Solution

$$2(3xy+7)(3y+3xy') = 6y'$$

$$6(3xy+7)(y+xy') = 6y' \qquad \textit{Divide by 6 both sides}$$

$$(3xy+7)(y+xy') = y'$$

$$3xy^2 + 3x^2yy' + 7y + 7xy' = y'$$

$$3x^2yy' + 7xy' - y' = -3xy^2 - 7y$$

$$\left(3x^2y+7x-1\right)y' = -\left(3xy^2+7y\right)$$

$$\boxed{\frac{dy}{dx} = -\frac{3xy^2+7y}{3x^2y+7x-1}}$$

Exercise

Find $\frac{dy}{dx}$: $\quad xy = \cot(xy)$

Solution

$$y + xy' = -\csc^2(xy)\ \left(y + xy'\right)$$

$$y + xy' = -y\csc^2(xy) - x\csc^2(xy)\,y'$$

$$x\csc^2(xy)\,y' + xy' = -y\csc^2(xy) - y$$

$$x\left(\csc^2(xy)+1\right)y' = -y\left(\csc^2(xy)+1\right)$$

$$y' = -\frac{y\left(\csc^2(xy)+1\right)}{x\left(\csc^2(xy)+1\right)}$$

$$\boxed{\frac{dy}{dx} = -\frac{y}{x}}$$

Exercise

Find $\frac{dy}{dx}$: $\quad x + \tan(xy) = 0$

Solution

$$1 + \sec^2(xy)\left(y + xy'\right) = 0$$

$$1 + y\sec^2(xy) + x\sec^2(xy)\,y' = 0$$

$$x\sec^2(xy)\,y' = -y\sec^2(xy) - 1$$

$$y' = -\frac{y\sec^2(xy)}{x\sec^2(xy)} - \frac{1}{x\sec^2(xy)}$$

$$\frac{dy}{dx} = -\frac{y}{x} - \frac{\cos^2 x}{x}$$

$$\boxed{= \frac{-y-\cos^2 x}{x}}$$

Exercise

Find $\frac{dy}{dx}$: $\quad x\cos(2x+3y) = y\sin x$

Solution

$$\cos(2x+3y) - \sin(2x+3y)(2x+3y') = y'\sin x + y\cos x$$

$$\cos(2x+3y) - 2x\sin(2x+3y) - 3\sin(2x+3y)y' = y'\sin x + y\cos x$$

$$\cos(2x+3y) - 2x\sin(2x+3y) - y\cos x = y'\sin x + 3\sin(2x+3y)y'$$

$$\cos(2x+3y) - 2x\sin(2x+3y) - y\cos x = y'\left(\sin x + 3\sin(2x+3y)\right)$$

$$y' = \frac{\cos(2x+3y) - 2x\sin(2x+3y) - y\cos x}{\sin x + 3\sin(2x+3y)}$$

Exercise

Find $\frac{dy}{dx}$: $\quad y = \frac{e^y}{1+\sin x}$

Solution

$$y(1+\sin x) = e^y$$

$$y'(1+\sin x) + y\cos x = y'e^y$$

$$y'\left(e^y - 1 - \sin x\right) = y\cos x$$

$$\frac{dy}{dx} = \frac{y\cos x}{e^y - 1 - \sin x}$$

Exercise

Find $\frac{dy}{dx}$: $\quad \sin x\cos(y-1) = \frac{1}{2}$

Solution

$$\cos x\cos(y-1) - y'\sin x\sin(y-1) = 0$$

$$y'\sin x\sin(y-1) = \cos x\cos(y-1)$$

$$y' = \frac{\cos x\cos(y-1)}{\sin x\sin(y-1)}$$

$$\frac{dy}{dx} = \cot x\cot(y-1)$$

Exercise

Find $\frac{dy}{dx}$: $\quad y\sqrt{x^2+y^2}=15$

Solution

$$y'\sqrt{x^2+y^2}+\frac{1}{2}y\left(2x+2yy'\right)\left(x^2+y^2\right)^{-1/2}=0 \qquad \times\sqrt{x^2+y^2}$$

$$y'\left(x^2+y^2\right)+y\left(x+yy'\right)=0$$

$$y'\left(x^2+y^2\right)+y^2y'=-xy$$

$$y'\left(x^2+2y^2\right)=-xy$$

$$\boxed{\frac{dy}{dx}=-\frac{xy}{x^2+2y^2}}$$

Exercise

Find $\frac{dr}{d\theta}$ $\quad r-2\sqrt{\theta}=\frac{3}{2}\theta^{2/3}+\frac{4}{3}\theta^{3/4}$

Solution

$$r-2\theta^{1/2}=\frac{3}{2}\theta^{2/3}+\frac{4}{3}\theta^{3/4}$$

$$\frac{dr}{d\theta}-2\frac{1}{2}\theta^{-1/2}=\frac{3}{2}\frac{2}{3}\theta^{-1/3}+\frac{4}{3}\frac{3}{4}\theta^{-1/4}$$

$$\boxed{\frac{dr}{d\theta}=\theta^{-1/3}+\theta^{-1/4}+\theta^{-1/2}}$$

Exercise

Find $\frac{dr}{d\theta}$ $\quad \sin(r\theta)=\frac{1}{2}$

Solution

$$\cos(r\theta)\left(\theta\frac{dr}{d\theta}+r\right)=0$$

$$\theta\frac{dr}{d\theta}+r=0 \qquad \cos(r\theta)\neq 0$$

$$\boxed{\frac{dr}{d\theta}=-\frac{r}{\theta}} \qquad \cos(r\theta)\neq 0$$

Exercise

Find $\dfrac{d^2y}{dx^2}$ $\quad x^{2/3}+y^{2/3}=1$

Solution

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}y'=0$$

Multiply all terms by $\frac{3}{2}$

$$x^{-1/3}+y^{-1/3}y'=0$$

$$y^{-1/3}y'=-x^{-1/3}$$

$$y'=-\frac{x^{-1/3}}{y^{-1/3}}$$

$$=-\frac{y^{1/3}}{x^{1/3}}=-\left(\frac{y}{x}\right)^{1/3}$$

$$y''=-\frac{1}{3}\left(\frac{y}{x}\right)^{-2/3}\left(\frac{xy'-y}{x^2}\right)$$

$$=-\frac{1}{3}\left(\frac{x}{y}\right)^{2/3}\left(\frac{-x\left(\frac{y}{x}\right)^{1/3}-y}{x^2}\right)=\frac{1}{3}\left(\frac{x^{4/3}y^{1/3}}{y^{2/3}x^2}+\frac{x^{2/3}y}{y^{2/3}x^2}\right)$$

$$=\frac{1}{3}\left(\frac{x}{y}\right)^{2/3}\left(\frac{x\frac{y^{1/3}}{x^{1/3}}+y}{x^2}\right)$$

$$=\frac{1}{3}\frac{x^{2/3}}{y^{2/3}}\frac{x^{2/3}y^{1/3}+y}{x^2}$$

$$\boxed{=\frac{1}{3}\left(\frac{1}{y^{1/3}x^{2/3}}+\frac{y^{1/3}}{x^{4/3}}\right)}$$

Exercise

Find $\dfrac{d^2y}{dx^2}$ $\quad 2\sqrt{y}=x-y$

Solution

$$2\frac{1}{2}y^{-1/2}y'=1-y'$$

$$2\frac{1}{2}y^{-1/2}y' + y' = 1$$

$$\left(y^{-1/2}+1\right)y' = 1 \Rightarrow \boxed{y' = \frac{1}{y^{-1/2}+1}}$$

$$y' = \frac{1}{y^{-1/2}+1} = \frac{1}{\frac{1}{\sqrt{y}}+1} = \frac{\sqrt{y}}{1+\sqrt{y}}$$

$$\left(y^{-1/2}+1\right)y'' + \left(-\frac{1}{2}y^{-3/2}y'\right)y' = 0$$

$$\left(y^{-1/2}+1\right)y'' - \frac{1}{2}y^{-3/2}\left(y'\right)^2 = 0$$

$$\left(y^{-1/2}+1\right)y'' = \frac{1}{2}y^{-3/2}\left(\frac{1}{y^{-1/2}+1}\right)^2$$

$$y'' = \frac{1}{2}y^{-3/2}\frac{1}{\left(y^{-1/2}+1\right)^2}\frac{1}{y^{-1/2}+1}$$

$$= \frac{1}{2}y^{-3/2}\frac{1}{\left(\frac{1}{\sqrt{y}}+1\right)^3}$$

$$= \frac{1}{2}y^{-3/2}\frac{1}{\left(\frac{1+\sqrt{y}}{\sqrt{y}}\right)^3}$$

$$= \frac{1}{2}y^{-3/2}\frac{1}{\frac{\left(1+\sqrt{y}\right)^3}{\left(y^{1/2}\right)^3}}$$

$$= \frac{1}{2}y^{-3/2}\frac{y^{3/2}}{\left(1+\sqrt{y}\right)^3}$$

$$= \frac{1}{2\left(1+\sqrt{y}\right)^3}$$

Exercise

If $x^3 + y^3 = 16$, find the value of $\frac{d^2y}{dx^2}$ at the point (2, 2).

Solution

$$3x^2 + 3y^2y' = 0$$

$$3y^2y' = -3x^2$$

$$y^2y' = -x^2$$

$$2yy'y' + y^2y'' = -2x$$

$$y^2 y'' = -2x - 2y(y')^2$$

$$y^2 y'' = -2x - 2y\left(\frac{-x^2}{y^2}\right)^2$$

$$y^2 y'' = -2x - 2\frac{x^4}{y^3}$$

$$y'' = -2\frac{x}{y^2} - 2\frac{x^4}{y^5}$$

$$= \frac{-2xy^3 - 2x^4}{y^5}$$

$$y''\Big|_{(2,2)} = \frac{-2(2)2^3 - 2(2)^4}{2^5}$$

$$= \frac{-2^5 - 2^5}{2^5}$$

$$= -2$$

Exercise

Find dy/dx: $x^2 - xy + y^2 = 4$ and evaluate the derivative at the given point $(0,-2)$

Solution

$$2x - (y + xy') + 2yy' = 0$$

$$-y - xy' + 2yy' = -2x$$

$$(2y - x)y' = y - 2x$$

$$\boxed{\frac{dy}{dx} = \frac{y-2x}{2y-x}}$$

$$@(0,-2) \to \frac{dy}{dx} = \frac{-2-2(0)}{2(-2)-(0)}$$

$$= \frac{-2}{-4}$$

$$= \frac{1}{2}$$

Exercise

Find the slope of the curve $\left(x^2+y^2\right)^2=(x-y)^2$ at the point (–2, 1) and (–2, –1)

Solution

1 and –1

Exercise

Find the slope of the tangent line to the circle $x^2-9y^2=16$ at the point (5, 1)

Solution

$$2x-18y\frac{dy}{dx}=0$$

$$-18y\frac{dy}{dx}=-2x$$

$$\frac{dy}{dx}=\frac{-2x}{-18y}=\frac{x}{9y}$$

$$@\ (5,\ 1)\ \rightarrow\ \frac{dy}{dx}=\frac{5}{9(1)}\ \underline{=\frac{5}{9}|}$$

Exercise

Find the slope of the tangent line to the circle $x^2+y^2=25$ at the point (3, –4)

Solution

$$\frac{d}{dx}\left[x^2+y^2\right]=\frac{d}{dx}[25]$$

$$2x+2y\frac{dy}{dx}=0$$

$$2y\frac{dy}{dx}=-2x\quad\rightarrow\quad\frac{dy}{dx}=-\frac{x}{y}$$

Slope: $\frac{dy}{dx}=-\frac{3}{-4}\ \underline{=\frac{3}{4}|}$

Exercise

Find an equation of the line tangent to the following curves at the given point

$$y=3x^3+\sin x;\quad (0,\ 0)$$

Solution

$$m=y'=9x^2+\cos x\Big|_{(0,\ 0)}$$

$$= 1$$

$$y = x$$ $$y = m\left(x - x_1\right) + y_1$$

Exercise

Find an equation of the line tangent to the following curves at the given point

$$y = \frac{4x}{x^2 + 3}; \quad (3,\ 1)$$

Solution

$$m = y' = \frac{4x^2 + 12 - 8x^2}{\left(x^2 + 3\right)^2}$$

$$= \left.\frac{12 - 4x^2}{\left(x^2 + 3\right)^2}\right|_{(3,\ 1)}$$

$$= \frac{-24}{144}$$

$$= -\frac{1}{6}$$

$$y = -\frac{1}{6}(x - 3) + 1 \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$= -\frac{1}{6}x + \frac{3}{2}$$

Exercise

Find an equation of the line tangent to the following curves at the given point

$$y + \sqrt{xy} = 6; \quad (1,\ 4)$$

Solution

$$\left. y' + \frac{1}{2}\left(y + xy'\right)\frac{1}{\sqrt{xy}} = 0 \right|_{(1,\ 4)}$$

$$y' + \frac{1}{2}\left(4 + y'\right)\frac{1}{2} = 0$$

$$y' + \frac{1}{4}y' = -1$$

$$\frac{5}{4}y' = -1$$

$$m = y' = -\frac{4}{5}$$

$$y = -\frac{4}{5}(x - 1) + 4 \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$= -\frac{4}{5}x + \frac{24}{5}$$

Exercise

Find an equation of the line tangent to the following curves at the given point

$$x^2y + y^3 = 75; \quad (4,\ 3)$$

Solution

$$2xy + x^2y' + 3y^2y' = 0\Big|_{(4,\ 3)}$$

$$(16 + 27)y' = -24$$

$$\boxed{y' = -\frac{24}{43} = m}$$

$$y = -\frac{24}{43}(x-4) + 3 \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$\boxed{= -\frac{24}{43}x + \frac{225}{43}}$$

Exercise

Find the equation of the tangent line to the circle $x^3 + y^3 = 9xy$ at the point (2, 4)

Solution

$$3x^2 + 3y^2y' = 9y + 9xy'$$

$$3y^2y' - 9xy' = 9y - 3x^2$$

$$\left(3y^2 - 9x\right)y' = 9y - 3x^2$$

$$y' = \frac{3\left(3y - x^2\right)}{3\left(y^2 - 3x\right)}$$

$$= \frac{3y - x^2}{y^2 - 3x}$$

$$m\Big|_{(2,4)} = \frac{3(4) - 2^2}{4^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}$$

$$y = \frac{4}{5}(x-2) + 4 \quad \Rightarrow \quad y = \frac{4}{5}x - \frac{8}{5} + 4 \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$\boxed{y = \frac{4}{5}x + \frac{12}{5}}$$

Exercise

Find the lines that are (***a***) tangent and (***b***) normal to the curve $x^2+xy-y^2=1$ at the point (2, 3).

Solution

$$2x+y+xy'-2yy'=0$$

$$(x-2y)y'=-2x-y$$

$$y'=\frac{-2x-y}{x-2y}=\frac{2x+y}{2y-x}$$

a) $\text{tangent slope}=y'\Big|_{(2,3)}=\frac{2(2)+3}{2(3)-2}=\frac{7}{4}$

$$y=\frac{7}{4}(x-2)+3 \qquad\qquad y=m\left(x-x_1\right)+y_1$$

$$y=\frac{7}{4}x-\frac{7}{2}+3$$

$$\boxed{y=\frac{7}{4}x-\frac{1}{2}}$$

b) $\text{normal slope}=-\frac{4}{7}$

$$y=-\frac{4}{7}(x-2)+3 \qquad\qquad y=m\left(x-x_1\right)+y_1$$

$$y=\frac{4}{7}x-\frac{8}{7}+3$$

$$\boxed{y=-\frac{4}{7}x+\frac{29}{7}}$$

Exercise

Find the lines that are (***a***) tangent and (***b***) normal to the curve $6x^2+3xy+2y^2+17y-6=0$ at the point (–1, 0).

Solution

$$12x+3y+3xy'+4yy'+17y'=0$$

$$(3x+4y+17)y'=-12x-3y$$

$$y'=\frac{-12x-3y}{3x+4y+17}$$

a) $\text{tangent slope}=y'\Big|_{(-1,0)}=\frac{-12(-1)-3(0)}{3(-1)+4(0)+17}=\boxed{\frac{6}{7}}$

$$y=\frac{6}{7}(x+1) \Rightarrow \boxed{y=\frac{6}{7}x+\frac{6}{7}} \qquad\qquad y=m\left(x-x_1\right)+y_1$$

b) $\text{normal slope}=-\frac{7}{6}$

$$y=-\frac{7}{6}(x+1) \quad\Rightarrow\quad \boxed{y=-\frac{7}{6}x-\frac{7}{6}}$$

$$y=m\left(x-x_1\right)+y_1$$

Exercise

Find the lines that are (***a***) tangent and (***b***) normal to the curve $x^2\cos^2 y-\sin y=0$ at the point (0, π).

Solution

$$2x\cos^2 y+x^2\left(2\cos y(-\sin y)y'\right)-(\cos y)y'=0$$

$$\left(-2x^2\cos y\sin y-\cos y\right)y'=-2x\cos^2 y$$

$$y'=\frac{-2x\cos^2 y}{-\left(2x^2\sin y+1\right)\cos y}=\frac{2x\cos y}{2x^2\sin y+1}$$

a) $\text{tangent slope}=y'\Big|_{(0,\pi)}=\frac{2(0)\cos(\pi)}{2(0)^2\sin(\pi)+1}=\underline{0|}$

$$y-\pi=0(x-0) \quad\Rightarrow\quad \boxed{y=\pi}$$

b) $\text{normal slope}=0$

$$\Rightarrow\quad \boxed{x=0}$$

Exercise

Suppose that x and y are both functions of t, which can be considered to represent time, and that x and y are related by the equation $xy^2+y=x^2+17$

Suppose further that when $x = 2$ and $y = 3$, then $\frac{dx}{dt}=13$. Find the value of the $\frac{dy}{dt}$ at that moment.

Solution

$$y^2\frac{dx}{dt}+2xy\frac{dy}{dt}+\frac{dy}{dt}=2x\frac{dx}{dt}$$

$$3^2(13)+2(2)(3)\frac{dy}{dt}+\frac{dy}{dt}=2(2)(13)$$

$$117+12\frac{dy}{dt}+\frac{dy}{dt}=52$$

$$13\frac{dy}{dt}=-65$$

$$\frac{dy}{dt}=\frac{-65}{13}$$

$$\underline{=-5|}$$

Exercise

A cone-shaped icicle is dripping from the roof. The radius of the icicle is decreasing at a rate of 0.2 *cm* per hour, while the length is increasing at a rate of 0.8 *cm* per *hour*. If the icicle is currently 4 *cm* in radius and 20 *cm* long, is the volume of the icicle increasing or decreasing and at what rate?

Solution

The volume of the cone is given by the formula: $V = \frac{1}{3}\pi r^2 h$.

$$\frac{dV}{dt} = \frac{1}{3}\pi\left[2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right]$$

Given the values:

$$\frac{dr}{dt} = -0.2 \qquad \frac{dh}{dt} = 0.8 \qquad r = 4 \qquad h = 20$$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left[2(4)(20)(-0.2) + 4^2(0.8)\right]$$

$$= -20$$

The volume is decreasing at a rate of 20 cm^3 per *hour*.