Solution Section 4.1 – Antiderivatives, Substitution and General Power Rule

Exercise

Find indefinite integral $\int v^2 dv$

Solution

$$\int v^2 dv = \boxed{\frac{v^3}{3} + C}$$

Exercise

Find indefinite integral $\int x^{1/2} dx$

Solution

$$\int x^{1/2} dx = \boxed{\frac{2x^{3/2}}{3} + C}$$

Exercise

Find indefinite integral $\int e^{3t} dt$

Solution

$$\int e^{3t} dt = \boxed{\frac{1}{3}e^{3t} + C}$$

Exercise

Find indefinite integral $\int (6x^2 - 2e^x) dx$

Solution

$$\int (6x^2 - 2e^x) dx = \frac{6x^3}{3} - 2e^x + C$$

$$\boxed{= 2x^3 - 2e^x + C}$$

Exercise

Find indefinite integral $\int 4y^{-3}dy$

Solution

$$\int 4y^{-3}dy = 4\frac{y^{-2}}{-2} + C$$

$$\boxed{= -\frac{2}{y^2} + C}$$

Exercise

Find indefinite integral $\int (x^3 - 4x + 2)dx$

Solution

$$\int (x^3 - 4x + 2)dx = \frac{x^4}{4} - 4\frac{x^2}{2} + 2x + C$$

$$\boxed{= \frac{1}{4}x^4 - 2x^2 + 2x + C}$$

Exercise

Find indefinite integral $\int \left(3z^2 - 4z + 5\right)dz$

Solution

$$\int \left(3z^2 - 4z + 5\right)dz = 3\frac{z^3}{3} - 4\frac{z^2}{2} + 5z + C$$

$$\boxed{= z^3 - 2z^2 + 5z + C}$$

Exercise

Find indefinite integral $\int \left(x^2 - 1\right)^2 dx$

Solution

$$\int \left(x^2 - 1\right)^2 dx = \int \left(x^4 - 2x^2 + 1\right)dx \qquad (a-b)^2 = a^2 - 2ab + b^2$$

$$\boxed{= \frac{1}{5}x^5 - \frac{2}{3}x^3 + x + C}$$

Exercise

Find indefinite integral $\int \left(\sqrt[4]{x^3} + 1 \right) dx$

Solution

$$\int \left(\sqrt[4]{x^3} + 1 \right) dx = \int \left(x^{3/4} + 1 \right) dx$$

$$\boxed{= \frac{4}{7} x^{7/4} + x + C}$$

Exercise

Find indefinite integral $\int \sqrt{x}(x+1)dx$

Solution

$$\int \sqrt{x}(x+1)dx = \int x^{1/2}(x+1)dx$$

$$= \int \left(x^{3/2} + x^{1/2} \right) dx$$

$$\boxed{= \frac{2}{5} x^{5/2} + \frac{2}{3} x^{3/2} + C}$$

Exercise

Find the integral $\int (1+3t)t^2 dt$

Solution

$$\int (1+3t)t^2 dt = \int \left(t^2 + 3t^3 \right) dt$$

$$\boxed{= \frac{1}{3} t^3 + \frac{3}{4} t^4 + C}$$

Exercise

Find indefinite integral $\int \frac{x^2-5}{x^2}\,dx$

Solution

$$\int \frac{x^2-5}{x^2}\,dx = \int \left(1-\frac{5}{x^2}\right)dx$$

$$= \int \left(1-5x^{-2}\right)dx$$

$$= x+5x^{-1}+C$$

$$\boxed{= x+\frac{5}{x}+C}$$

Exercise

Find indefinite integral $\int (-40x+250)\,dx$

Solution

$$\int (-40x+250)\,dx \boxed{= -20x^2+250x+C}$$

Exercise

Find the integral $\int \frac{x+2}{\sqrt{x}}\,dx$

Solution

$$\int \frac{x+2}{\sqrt{x}}\,dx = \int \left[\frac{x}{x^{1/2}}+\frac{2}{x^{1/2}}\right]dx$$

$$= \int \frac{x}{x^{1/2}}\,dx + \int \frac{2}{x^{1/2}}\,dx$$

$$= \int x^{1/2}\,dx + 2\int x^{-1/2}\,dx$$

$$= \frac{x^{3/2}}{3/2} + 2\frac{x^{1/2}}{1/2} + C$$

$$\boxed{= \frac{2}{3}x^{3/2} + 4x^{1/2} + C}$$

Exercise

Evaluate: $\int\left(\frac{2}{\sqrt[3]{x}}-6\sqrt{x}\right)dx$

Solution

$$\int\left(\frac{2}{\sqrt[3]{x}}-6\sqrt{x}\right)dx=\int\left(2x^{-1/3}-6x^{1/2}\right)dx$$

$$=2\frac{3}{2}x^{2/3}-6\frac{2}{3}x^{3/2}+C$$

$$\boxed{=3x^{2/3}-4x^{3/2}+C}$$

Exercise

Find the integral $\int (x^2-1)^3(2x)dx$

Solution

$$u=x^2-1\Rightarrow du=2xdx$$

$$\int (x^2-1)^3(2x)dx=\int u^3du$$

$$=\frac{1}{4}u^4+C$$

$$\boxed{=\frac{1}{4}(x^2-1)^4+C}$$

Exercise

Find the integral $\int \frac{6x}{(1+x^2)^3}dx$

Solution

$$u=1+x^2\Rightarrow du=2xdx \qquad \Rightarrow\frac{1}{2x}du=dx$$

$$\int\frac{6x}{u^3}\frac{1}{2x}du=\int 3\frac{1}{u^3}du$$

$$=3\int u^{-3}du$$

$$=3\frac{u^{-2}}{-2}+C$$

$$=-\frac{3}{2}\left(1+x^2\right)^{-2}+C$$

$$\boxed{=-\frac{3}{2}\frac{1}{\left(1+x^2\right)^2}+C}$$

Exercise

Find the integral $\int u^3\sqrt{u^4+2}\ du$

Solution

$$Let\ \ x=u^4+2\ \Rightarrow dx=4u^3du$$

$$\Rightarrow \frac{1}{4u^3}dx=du$$

$$\int u^3\sqrt{u^4+2}\ du=\int u^3x^{1/2}\ \frac{1}{4u^3}dx$$

$$=\frac{1}{4}\int x^{1/2}\ dx$$

$$=\frac{1}{4}\frac{2}{3}x^{3/2}+C$$

$$\boxed{=\frac{1}{6}\left(u^4+2\right)^{3/2}+C}$$

Exercise

Find the integral $\int \frac{t+2t^2}{\sqrt{t}}\ dt$

Solution

$$\int \frac{t+2t^2}{\sqrt{t}}\ dt=\int\left(\frac{t}{t^{1/2}}+2\frac{t^2}{t^{1/2}}\right)\ dt$$

$$=\int\left(t^{1/2}+2t^{3/2}\right)\ dt$$

$$=\frac{2}{3}t^{3/2}+2\frac{2}{5}t^{5/2}+C$$

$$\boxed{=\frac{2}{3}t^{3/2}+\frac{4}{5}t^{5/2}+C}$$

Exercise

Find the integral $\int \left(1+\frac{1}{t}\right)^3 \frac{1}{t^2}\ dt$

Solution

$$u = 1 + t^{-1} \Rightarrow du = -t^{-2}dt$$

$$\Rightarrow -t^2 du = dt$$

$$\begin{aligned}\int \left(1+\frac{1}{t}\right)^3 \frac{1}{t^2}\ dt &= \int u^3 \frac{1}{t^2}\ (-t^2 du)\\ &= -\int u^3 du\\ &= -\frac{1}{4}u^4 + C\\ &= \boxed{-\frac{1}{4}\left(1+\frac{1}{t}\right)^4 + C}\end{aligned}$$

Exercise

Find the integral $\int (7-3x-3x^2)(2x+1)\ dx$

Solution

$$\begin{aligned}u = 7-3x-3x^2 &\Rightarrow du = (-3-6x^2)dx\\ &\Rightarrow du = -3(2x^2+1)dx\\ &\Rightarrow -\frac{1}{3}du = (2x^2+1)dx\end{aligned}$$

$$\begin{aligned}\int (7-3x-3x^2)(2x+1)\ dx &= \int u\left(-\frac{1}{3}\right) du\\ &= -\frac{1}{3}\int u\ du\\ &= -\frac{1}{6}u^2 + C\\ &= \boxed{-\frac{1}{6}\left(7-3x-3x^2\right)^2 + C}\end{aligned}$$

Exercise

Find the integral $\int \sqrt{x}\left(4-x^{3/2}\right)^2\ dx$

Solution

$$u=4-x^{3/2} \Rightarrow du=-\tfrac{3}{2}x^{1/2}dx$$

$$\Rightarrow -\tfrac{2}{3}du=\sqrt{x}dx$$

$$\begin{aligned}\int \sqrt{x}\left(4-x^{3/2}\right)^2\ dx &= \int u^2\left(-\tfrac{2}{3}\right)du \\ &= -\tfrac{2}{3}\int u^2 du \\ &= -\tfrac{2}{9}u^3+C \\ &= \boxed{-\tfrac{2}{9}\left(4-x^{3/2}\right)^3+C}\end{aligned}$$

Exercise

Find the integral $\int \frac{1}{\sqrt{x}+\sqrt{x+1}}dx$

Solution

$$\begin{aligned}\int \frac{1}{\sqrt{x}+\sqrt{x+1}}dx &= \int \frac{1}{\sqrt{x}+\sqrt{x+1}}\,\frac{\sqrt{x}-\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}}dx \\ &= \int \frac{\sqrt{x}-\sqrt{x+1}}{x-(x+1)}dx \\ &= \int \frac{\sqrt{x}-\sqrt{x+1}}{-1}dx \\ &= -\int \left(x^{1/2}-(x+1)^{1/2}\right)dx \\ &= -\left(\tfrac{2}{3}x^{3/2}-\tfrac{2}{3}(x+1)^{3/2}\right)+C \\ &= \boxed{\tfrac{2}{3}(x+1)^{3/2}-\tfrac{2}{3}x^{3/2}+C}\end{aligned}$$

Exercise

Find the integral $\int \sqrt{1-x}\ dx$

Solution

$$u = 1 - x$$

$$du = -dx \ \Rightarrow \ -du = dx$$

$$\int \sqrt{1-x}\ dx = \int \sqrt{u}\ (-du) \qquad \textit{Substitute for x and dx}$$

$$= -\int u^{1/2}\ du$$

$$= -\frac{u^{3/2}}{3/2} + C$$

$$\boxed{= -\frac{2}{3}(1-x)^{3/2} + C}$$

Exercise

Find the integral $\int x\sqrt{x^2+4}\ dx$

Solution

$$u = x^2 + 4 \ \Rightarrow du = 2xdx$$

$$xdx = \frac{1}{2}du$$

$$\int \sqrt{x^2+4}\ xdx = \int u^{1/2}\ \frac{1}{2}du$$

$$= \frac{1}{2}\frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{3}u^{3/2} + C$$

$$\boxed{= \frac{1}{3}(x^2+4)^{3/2} + C}$$

Exercise

Derive the position function if a ball is thrown upward with initial velocity of 32 ft per second from an initial height of 48 ft. When does the ball hit the ground? With what velocity does the ball hit the ground?

Solution

$$s(t) = -16t^2 + 32t + 48$$

$$s(0) = 48$$

$$s'(0) = 32$$

$$s''(t) = -32$$

$$s'(t) = \int -32dt$$

$$= -32t + C_1$$

$$s'(0) = -32(0) + C_1 = 32$$

$$\Rightarrow C_1 = 32$$

$$s'(t) = -32t + 32$$

$$s(t) = \int (-32t + 32)dt$$

$$= -32\frac{t^2}{2} + 32t + C_2$$

$$s(0) = -32\frac{0^2}{2} + 32(0) + C_2 = 48 \qquad \Rightarrow C_2 = 48$$

$$s(t) = -16t^2 + 32t + 48$$

$$s(t) = -16t^2 + 32t + 48 = 0$$

$$-t^2 + 2t + 3 = 0 \ \Rightarrow t = -1, \ t = 3$$

The ball hits the ground in 3 seconds

The velocity: $v(t) = s'(t) = -32t + 32$

$$v(t=3) = -32(3) + 32 \ \boxed{= -64 \ ft / \sec^2}$$

Exercise

Suppose a publishing company has found that the marginal cost at a level of production of x thousand books is given by

$$\frac{dC}{dx} = \frac{50}{\sqrt{x}}$$

And that the fixed cost (the cost before the first book can be produced) is a $25,000. Find the cost function $C(x)$.

Solution

$$\frac{dC}{dx} = \frac{50}{\sqrt{x}} = 50x^{-1/2}$$

$$dC = 50x^{-1/2}dx$$

$$\int dC = \int 50x^{-1/2}dx$$

$$C(x) = 50\frac{x^{1/2}}{1/2} + C$$

$$= 50(2)x^{1/2} + C$$

$$= 100\sqrt{x} + C$$

$$25000 = 100\sqrt{0} + C$$ Before the first ($x = 0$) costs 25,000

$$\boxed{25000 = C}$$

$$C(x) = 100\sqrt{x} + 25,000$$

Exercise

If the marginal cost of producing x units of a commodity is given by

$$C'(x) = 0.3x^2 + 2x$$

And the fixed cost is $2,000, find the cost function $C(x)$ and the cost of producing 20 units.

Solution

Given: $C(0) = 2,000$

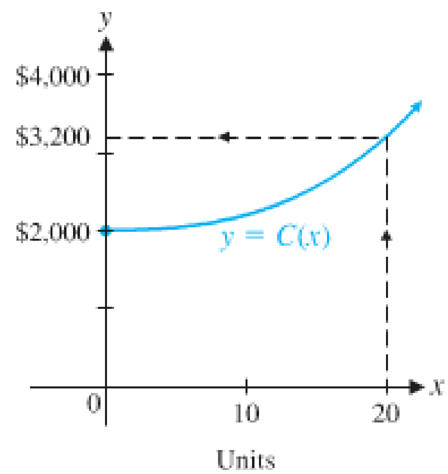
$$C(x) = \int \left(0.3x^2 + 2x\right)dx$$

$$= 0.1x^3 + x^2 + K$$

$$C(0) = 0.1(0)^3 + (0)^2 + K \quad \boxed{2,000 = K}$$

$$C(x) = 0.1x^3 + x^2 + 2,000$$

$$C(20) = 0.1(20)^3 + (20)^2 + 2,000 = \$3,200$$



Exercise

A satellite radio station is launching an aggressive advertising campaign in order to increase the number of daily listeners. The station currently has 27,000 daily listeners, and management expects the number of daily listeners, $S(t)$, to grow at the rate of

$$S'(t) = 60t^{1/2}$$

Listeners per day, where *t* is the number of days since the campaign began. How long should the campaign last if the station wants the number of daily listeners to grow to 41,000?

Solution

$$\textit{Given}: \ S(0) = 27,000$$

$$S(t) = \int 60 \cdot t^{1/2} dt$$

$$= 40t^{3/2} + C$$

$$S(0) = 40(0)^{3/2} + C$$

$$\boxed{27,000 = C}$$

$$S(t) = 40t^{3/2} + 27,000$$

$$S(t) = 40t^{3/2} + 27,000 = 41,000$$

$$40t^{3/2} = 14,000$$

$$t^{3/2} = 350 \quad \rightarrow \quad t = (350)^{2/3} \approx 49.66$$

The advertising campaign should last approximately 50 days.

Exercise

In 2007, U.S. consumption of renewable energy was 6.8 quadrillion Btu (or 6.8×10^{15} Btu). Since the 1960s, consumption has been growing at a rate (in quadrillion Btu's per year) given by

$$f'(t) = 0.004t + 0.062$$

Where *t* is years after 1960. Find $f(t)$ and estimate U.S. consumption of renewable energy in 2020.

Solution

From 1960 to 2007 $\Rightarrow$ 47 years

Given: $f(47) = 6.8$ quadrillion Btu

$$f(t) = \int (0.004t + 0.062) dt$$

$$= 0.002t^2 + 0.062t + C$$

$$6.8 = 0.002(47)^2 + 0.062(47) + C \quad \rightarrow \quad C = -0.532$$

$f(t) = 0.002t^2 + 0.062t - 0.532$

In 2020, $t = 60$

$f(60) = 0.002(60)^2 + 0.062(60) - 0.532 \ \underline{= 10.4 \ \ \textit{quadrillion Btu}}$

Exercise

The graph of the marginal revenue function from the sale of x sports watches is given in the figure.

a) Using the graph shown, describe the shape of the graph of the revenue function $R(x)$ as x increases from 0 to 1.000.

b) Find the equation of the marginal revenue function. (linear function)

c) Find the equation of the revenue function that satisfies $R(0) = 0$. Graph the revenue function over the interval [0, 1,000]. Check the shape of the graph relative to the analysis in part (*a*).

d) Find the price-demand equation and determine the price when the demand is 700 units.

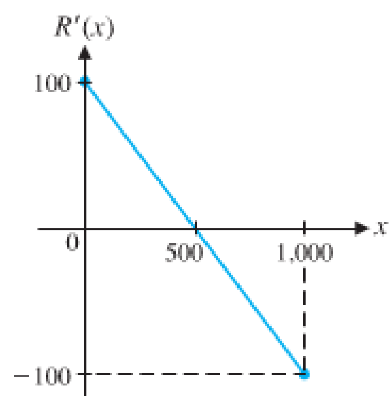
Solution

a) $R'(x) > 0 \quad for \quad 0 < x < 500 \qquad R(x)$ is increasing

$R'(x) < 0 \quad for \quad 500 < x < 1{,}000 \qquad R(x)$ is decreasing

Therefore, the graph of $R(x)$ has a local maximum at $x = 500$

b) The equation of $R(x)$ is $\dfrac{R'(x) - 100}{0 - 100} = \dfrac{x - 0}{500 - 0}$



$R'(x) - 100 = (-100)\dfrac{x}{500}$

$R'(x) = -\dfrac{1}{5}x + 100$

$\underline{R'(x) = 100 - 0.2x}$

c) $R(x) = \displaystyle\int R'(x)\,dx = \int (100 - 0.2x)\,dx$

$\qquad = 100x - 0.1x^2 + C$

Given: $R(0) = 0$

$R(0) = 100(0) - 0.1(0)^2 + C \ \rightarrow \ \boxed{C = 0}$

$\underline{R(x) = 100x - 0.1x^2}$

d) $p(x) = \dfrac{R(x)}{x} = \underline{100 - 0.1x}$

$p(700) = 100 - 0.1(700) = 30$

So the price is \$30 per sports watch when the demand is 700.

Exercise

The rate of change of the monthly sales of a newly released football game is given by

$$S'(t) = 500t^{1/4} \qquad S(0) = 0$$

Where t is the number of months since the game was released and $S(t)$ is the number of games sold each month. Find $S(t)$. When will monthly sales reach 20,000 games?

Solution

$$S(t) = \int S'(t)dt = \int 500t^{1/4}dt$$

$$= 500\left(\frac{4}{5}\right)t^{5/4} + C$$

$$= 400t^{5/4} + C$$

Given: $S(0) = 0$

$$0 = 400(0)^{5/4} + C \quad \rightarrow \quad \boxed{C = 0}$$

$$S(t) = 400t^{5/4}$$

$$S(t) = 400t^{5/4} = 20,000$$

$$t^{5/4} = 50$$

$$t = 50^{4/5} \approx \underline{23 \;\; months}$$

Exercise

If the rate of labor is given by: $\quad g(x) = 2{,}000x^{-1/3}$

And if the first 8 control units require 12,000 labor-hours, how many labor-hours, $L(x)$, will be required for the first x control units? The first 27 control units?

Solution

$$L(x) = \int g(x)dx = \int 2{,}000x^{-1/3}dx$$

$$= 2{,}000\left(\frac{3}{2}\right)x^{2/3} + C$$

$$= 3{,}000x^{2/3} + C$$

Given: $L(8) = 12{,}000$

$$L(8) = 3{,}000(8)^{2/3} + C$$

$$C = 12{,}000 - 3{,}000(8)^{2/3} \;\; \underline{= 0}$$

$$L(x) = 3{,}000x^{2/3}$$

$L(27) = 3{,}000(27)^{2/3} \;\; \underline{= 27{,}000}$ ***labor hours***

Exercise

The area *A* of a healing wound changes at a rate given approximately by

$$\frac{dA}{dt} = -4t^{-3} \qquad 1 \le t \le 10$$

Where *t* is time in days and $A(1) = 2\ cm^2$. What will the area of the wound be in 10 days?

Solution

$$A = \int -4t^{-3}dt$$

$$= -4\frac{t^{-2}}{-2} + C$$

$$= 2t^{-2} + C$$

Given: $A(1) = 2$

$$2 = 2(1)^{-2} + C \quad \to \quad 2 = 2 + C \quad \boxed{C = 0}$$

$$A(t) = 2t^{-2}$$

$$A(10) = 2(10)^{-2} = 0.02\ cm^2$$

Exercise

The marginal revenue (in thousands of dollars) from the sale of *x* gadgets is given by the following function $R'(x) = 4x\left(x^2 + 27{,}000\right)^{-2/3}$

a) Find the total revenue function if the revenue from 115 gadgets is \$55,581.

b) How many gadgets must be sold for a revenue of at least \$50,000.

Solution

a) $R(x) = \int R'(x)\,dx$

$$= \int 4x\left(x^2 + 27{,}000\right)^{-2/3} dx \qquad u = x^2 + 27{,}000 \quad du = 2xdx \to 4xdx = 2du$$

$$= \int u^{-2/3}(2du)$$

$$= 2\left(3u^{1/3}\right) + C$$

$$= 6\left(x^2 + 27{,}000\right)^{1/3} + C$$

$R(x = 115) = 55.581$

$$6\left(115^2+27,000\right)^{1/3}+C=55.581$$

$$C=55.581-6\left(115^2+27,000\right)^{1/3}$$

$$\boxed{C\approx -150}$$

$$R(x)=6\left(x^2+27,000\right)^{1/3}-150$$

b) $R(x)=6\left(x^2+27,000\right)^{1/3}-150=50$

$$6\left(x^2+27,000\right)^{1/3}=200$$

$$\left(x^2+27,000\right)^{1/3}=\frac{200}{6}$$

$$x^2+27,000=\left(\frac{200}{6}\right)^3$$

$$x^2=\left(\frac{200}{6}\right)^3-27,000=10037$$

$$x=\sqrt{\left(\frac{200}{6}\right)^3-27,000}\approx 100.2$$

101 gadgets must be sold to generate a revenue of at least $50,000.