

## Section 2.9 – Rank and the Fundamental Matrix Spaces

The **Reduced Row Echelon Form** (*rref*) is a matrix ( $R$ ) with each pivot column has only one nonzero entry (the pivots which is always 1).

$$R = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R = \text{rref}(A)$$

### Rank of a Matrix

The rank of a matrix  $A$  ( $m$  by  $n$ ) is the number of **nonzero rows** in the row-reduced echelon form of  $A$  (it is the number of pivot). The common dimension of the row space and column space of a matrix  $A$  is called the **rank** of  $A$  and is denoted by

$$\text{rank}(A) = r$$

#### Note:

The rank of a matrix is well defined due to the uniqueness of the row-reduced echelon form. No matter what sequence of elementary row operations is performed to put the given matrix in row-reduced echelon form; there will always be the same number of nonzero rows.

### Theorem

The row space and column space of a matrix  $A$  have the same dimension

The objective is to connect **rank** and **dimension**.

- The **rank** of a matrix is the number of pivots.
- The **dimension** of a subspace is the number of vectors in a basis.

✓ *A has full row rank if every row has a pivot:  $r = m$ . No zero in  $R$ .*

✓ *A has full column rank if every column has a pivot:  $r = n$ . No free variables.*

### Example

Find the rank of  $A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 1 & -3 & 0 & 5 \end{bmatrix}$

### Solution

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 1 & -3 & 0 & 5 \end{bmatrix} \quad R_3 - R_1$$

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & -2 & -2 & 4 \end{bmatrix} \quad \begin{array}{l} R_1 + R_2 \\ R_3 + 2R_2 \end{array}$$

$$R = \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix  $R$  has 2 nonzero rows, therefore the  $\text{rank}(A) = 2$

### Example

The columns of  $A$  are dependent.  $A\vec{x} = \vec{0}$  has a nonzero solution.

$$Ax = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

$$-3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The rank of  $A$  is only  $r = 2$ .

Independent columns would give full column rank  $r = n = 3$ .

✚ The columns of  $A$  are independent exactly when the rank is  $r = n$ . There are  $n$  pivots and no free variables. Only  $\vec{x} = \vec{0}$  is the nullspace.

### ***Example***

When all rows are multiplying of one pivot row, the rank is  $r = 1$ :

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix}, \quad \begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \quad [6]$$

### **Solution**

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix} \quad R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix} \quad 3R_2 - 5R_1$$

$$\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \quad \frac{1}{3}R_2$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} \quad 5R_2 - 2R_1 \quad \frac{1}{5}R_1 \quad \rightarrow \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The row-reduced echelon form  $R = rref(A)$ :

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad [1]$$

These matrices have only one pivot.

## Dimension *Theorem* for Matrices

If  $A$  is a matrix with  $n$  columns, then

$$\boxed{\text{rank}(A) + \text{nullity}(A) = n}$$

### *Theorem*

If  $A$  is an  $m \times n$  matrix, then

- $\text{rank}(A)$  = the number of leading variables in the general solution of  $A\vec{x} = \vec{0}$
- $\text{nullity}(A)$  = the number of parameters in the general solution of  $A\vec{x} = \vec{0}$

### *Theorem*

If  $A$  is any matrix, then  $\text{rank}(A) = \text{rank}(A^T)$

✚  $Ax = 0$  has  $n - r$  free variables and special solutions:  $n$  columns minus  $r$  pivot columns. The null matrix  $N$  has  $n - r$  columns (the special solutions).

✚ The particular solution solves:  $A\vec{x}_p = \vec{b}$

✚ Full column rank  $R = \begin{bmatrix} n \text{ by } n \text{ identity matrix} \\ m - n \text{ rows of zeros} \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$

The reduced row echelon form looks like:

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \quad \begin{array}{l} \text{r pivot rows} \\ \text{m - r zero rows} \end{array}$$

The pivot variables in the  $n - r$  special columns come by changing  $F$  to  $-F$ :

$$\text{Nullspace matrix: } N = \begin{pmatrix} -F \\ I \end{pmatrix} \quad \begin{array}{l} \text{r pivot variables} \\ \text{n - r free variables} \end{array}$$

➤ Every matrix  $A$  with **full column rank** ( $r = n$ ) has all these properties:

1. All columns of  $A$  are pivot columns
2. There are no free variables or special solutions.
3. The nullspace  $NS(A)$  contains only the zero vector  $\vec{x} = \vec{0}$
4. If  $A\vec{x} = \vec{b}$  has a solution (might not) then it has only one solution.

### Example

Suppose  $A$  is a square invertible matrix,  $m = n = r$ . What are  $\vec{x}_p$  and  $\vec{x}_n$ ?

### Solution

The particular solution is the one and only solution  $A^{-1}\vec{b}$ .

There are no special solutions or free variables.  $R = I$  has no zero rows.

The only vector in the null space is  $\vec{x}_n = \vec{0}$ .

The complete solution is

$$\begin{aligned}\vec{x} &= \vec{x}_p + \vec{x}_n \\ &= A^{-1}\vec{b} + \vec{0} \\ &= \underline{A^{-1}\vec{b}}\end{aligned}$$

### Example

Compute  $N(A)$  for  $A: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  given by  $A = (x + y, x, 2x - y)$

### Solution

To find  $N(A)$ , we must solve the equation  $A(x, y) = (0, 0, 0)$

$$\begin{pmatrix} x + y \\ x \\ 2x - y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} x + y = 0 &\Rightarrow \boxed{y = 0} \\ \boxed{x = 0} \end{aligned}$$

Thus  $NS(A) = \{0\}$ , the set that consists solely of the zero vector.



If  $A\vec{x} = \vec{0}$  has more unknowns than equations (more columns than rows) then it has nonzero solutions. There must be free columns, without pivots.

## Definition

If  $W$  is a subspace of  $\mathbb{R}^n$  that are orthogonal to every vector in  $W$  is called orthogonal complement of  $W$  and is denoted by the symbol  $W^\perp$ .  $N(A)^\perp$  is exactly the row space  $C(A^T)$

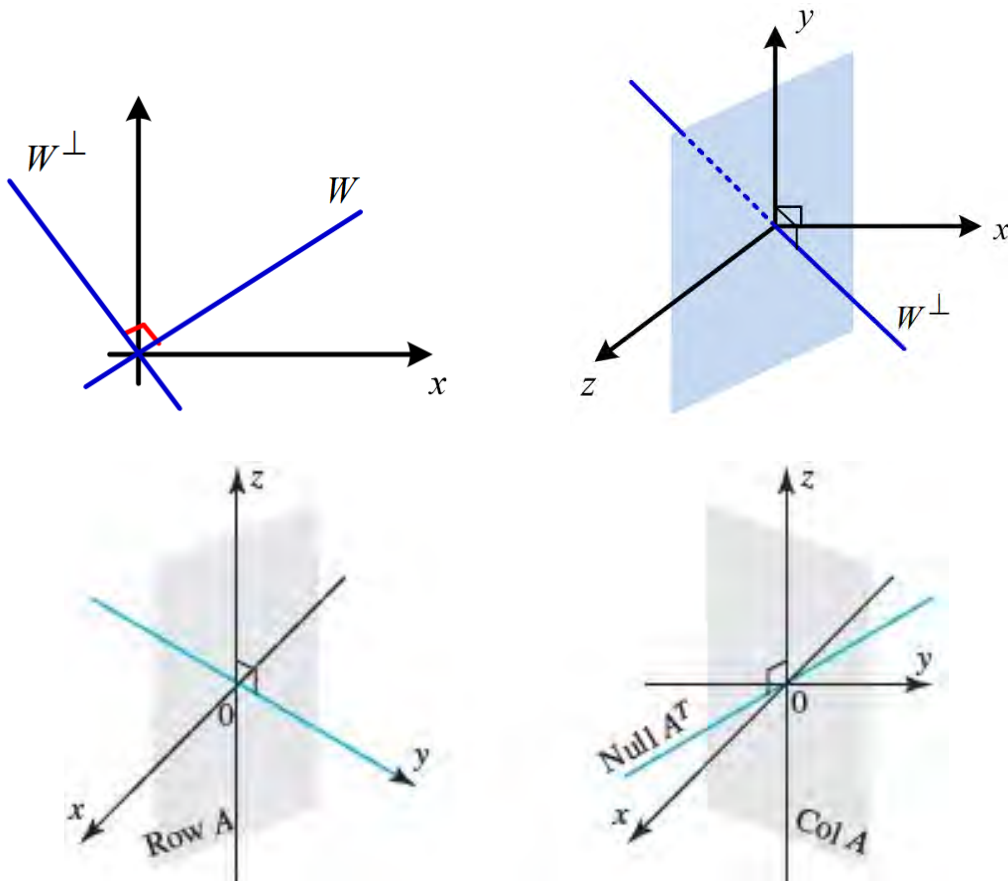
## Fundamental Theorem of Linear Algebra

The nullspace is the orthogonal complement of the row space (in  $\mathbb{R}^n$ ).

The left nullspace is the orthogonal complement of the column space (in  $\mathbb{R}^m$ ).

If  $W$  is a subspace of  $\mathbb{R}^n$

- $W^\perp$  is a subspace of  $\mathbb{R}^n$ .
- The only vector common to  $W$  and  $W^\perp$  is 0.
- The orthogonal complement of  $W^\perp$  is  $W$ .



## Left Nullspace

A matrix  $A^T$  has  $m$  columns and has  $r$  ranks, so the number of free columns of  $A^T$  must be  $m - r$ .

$$\dim N(A^T) = m - r$$

The left nullspace is the collection of vectors  $\bar{y}$  for which  $A^T \bar{y} = \vec{0}$ . Equivalently,  $\bar{y}^T A = \vec{0}$ , where  $\bar{y}$  and  $\vec{0}$  are row vectors. We can call “**left nullspace**” because  $\bar{y}^T$  is on the left of matrix  $A$  in that equation.

To find a basis for the left nullspace we reduce an augmented type of  $A$ .

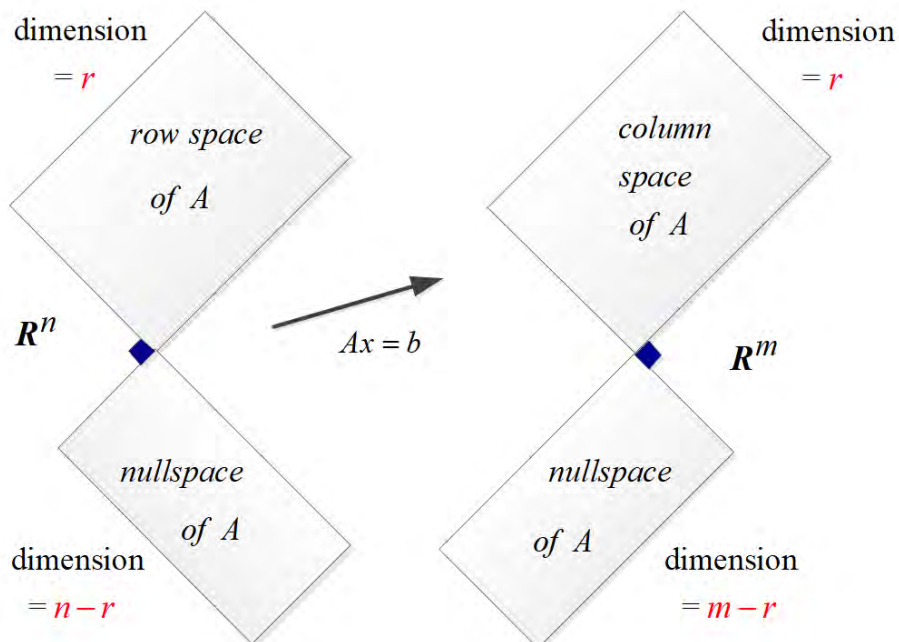
$$\left[ A_{m \times n} \mid I_{m \times m} \right] \rightarrow \left[ R_{m \times n} \mid E_{m \times m} \right]$$

Where matrix  $E$  can be found from  $EA = R$ .

If matrix  $A$  is a square matrix, then  $E = A^{-1}$ .

## The Four Fundamental Subspaces

1. The **row space** is  $C(A^T)$ , a subspace of  $\mathbb{R}^n$ .
2. The **column space** is  $C(A)$ , a subspace of  $\mathbb{R}^m$ .
3. The **null space** is  $N(A)$ , a subspace of  $\mathbb{R}^n$ .
4. The **left null space** is  $N(A^T)$ , a subspace of  $\mathbb{R}^m$ .



Two pairs of orthogonal subspaces.

For an  $m \times n$  matrix of rank  $r$ :

<i>Fundamental Space</i>	<i>Subspace of</i>	<i>Dimension</i>
Nullspace	$\mathbb{R}^n$	$n - r$
Column Space	$\mathbb{R}^m$	$r$
Row space	$\mathbb{R}^n$	$r$
Left nullspace	$\mathbb{R}^m$	$m - r$

### Example

Find a basis for each of the four subspaces associated with matrix  $A$ :

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix}$$

### Solution

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix} \quad R_2 - 2R_1$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \quad x_1 = -2x_2 - 4x_3 \quad \leftarrow \text{Row space}$$

1. Basis for **row space**:  $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

2. Basis of the **column spaces**:  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\text{Rank}(A) = 1$$

$$\text{Dimension of } A = 1$$

The pivots variables are:  $x_1$

The free variables are:  $x_2, x_3$

$$\text{Set } x_2 = 1 \quad x_3 = 0$$

$$\text{The special solution: } s_1 = (-2, 1, 0)$$

$$\text{Set } x_2 = 0 \quad x_3 = 1$$

$$\text{The special solution: } s_2 = (-4, 0, 1)$$



3. Basis of the **Null space**:  $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$

$$A^T = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 8 \end{pmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \end{array}$$

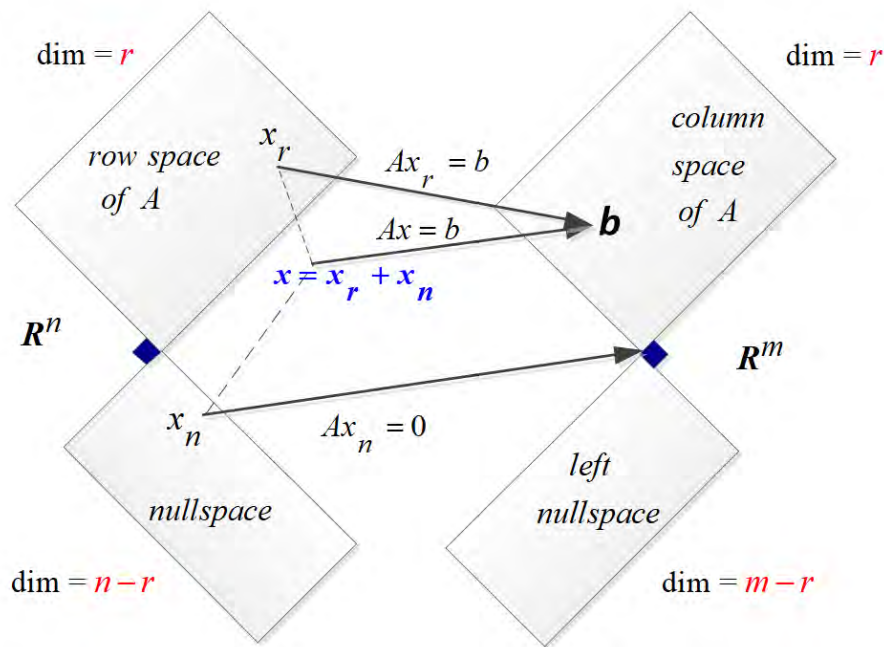
$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad y_1 = -2y_2$$

$$\text{Set } y_2 = 1 \Rightarrow s^*_1 = (-2, 1)$$

4. Basis of the **Left Nullspace**:  $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

## Combining Bases from Subspaces

- Any  $n$  linearly independent vectors in  $\mathbb{R}^n$  must span  $\mathbb{R}^n$ . They are basis. Any  $n$  vectors that span  $\mathbb{R}^n$  must be independent. They are a basis.
- If the  $n$  columns of  $A$  are independent, they span  $\mathbb{R}^n$ , So  $A\vec{x} = \vec{b}$  is solvable,
- If the  $n$  columns span  $\mathbb{R}^n$ , they are independent. So  $A\vec{x} = \vec{b}$  has only one solution.



- When the orthogonal complement of a subspace  $S$  is defined to be the subspace whose vectors pairs to zero with the vectors in  $S$ . The larger the  $S$  is, the more restriction  $S^\perp$  has, and hence the smaller  $S^\perp$  is.

### ***Theorem – Equivalent Statements***

If  $A$  is an  $n \times n$  matrix, then the following statements are equivalent.

- a)  $A$  is invertible
- b)  $A\vec{x} = \vec{0}$  has only the trivial solution
- c) The reduced row echelon form of  $A$  is  $I_n$
- d)  $A$  is expressible as a product of elementary matrices
- e)  $A\vec{x} = \vec{b}$  is consistent for every  $n \times 1$  matrix  $\vec{b}$
- f)  $A\vec{x} = \vec{b}$  has exactly one solution for every  $n \times 1$  matrix  $\vec{b}$
- g)  $\det(A) \neq 0$
- h) The column vectors of  $A$  are linearly independent
- i) The row vectors of  $A$  are linearly independent
- j) The column vectors of  $A$  span  $\mathbb{R}^n$
- k) The row vectors of  $A$  span  $\mathbb{R}^n$
- l) The column vectors of  $A$  form a basis for  $\mathbb{R}^n$
- m) The row vectors of  $A$  form a basis for  $\mathbb{R}^n$
- n)  $A$  has a rank  $n$ .
- o)  $A$  has nullity 0.
- p) The orthogonal complement of the null space of  $A$  is  $\mathbb{R}^n$
- q) The orthogonal complement of the row space of  $A$  is  $\{0\}$

## Exercises

### Section 2.9 – Rank and the Fundamental Matrix Spaces

1. Verify that  $\text{rank}(A) = \text{rank}(A^T)$

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix}$$

2. Find the rank and nullity of the matrix; then verify that the values obtained satisfy  $\text{rank}(A) + N(A) = n$

a)  $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

b)  $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

c)  $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

d)  $A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

3. If  $A$  is an  $m \times n$  matrix, what is the largest possible value for its rank and the smallest possible value of the nullity of  $A$ .
4. Discuss how the rank of  $A$  varies with  $t$ .

a)  $A = \begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix}$

b)  $A = \begin{bmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{bmatrix}$

5. Are there values of  $r$  and  $s$  for which

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

Has rank 1? Has rank 2? If so, find those values.

6. Find the row reduced form  $R$  and the rank  $r$  of  $A$  (those depend on  $c$ ).

Which are the pivot columns of  $A$ ? Which variables are free? What are the special solutions and the nullspace matrix  $N$  (always depending on  $c$ )?

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & c \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} c & c \\ c & c \end{bmatrix}$$

7. Find the row reduced form  $R$  and the rank  $r$  of  $A$  (those depend on  $c$ ).

Which are the pivot columns of  $A$ ? Which variables are free? What are the special solutions and the nullspace matrix  $N$  (always depending on  $c$ )?

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}$$

8. If  $A$  has a rank  $r$ , then it has an  $r$  by  $r$  sub-matrix  $S$  that is invertible. Remove  $m - r$  rows and  $n - r$  columns to find an invertible sub-matrix  $S$  inside each  $A$  (you could keep the pivot rows and pivot columns of  $A$ ).

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

9. Suppose that column 3 of  $4 \times 6$  matrix is all zero. Then  $x_3$  must be a \_\_\_\_\_ variable. Give one special solution for this matrix.

10. Fill in the missing numbers to make  $A$  rank 1, rank 2, rank 3. (your solution should be 3 matrices)

$$A = \begin{pmatrix} & -3 & \\ 1 & 3 & -1 \\ & 9 & -3 \end{pmatrix}$$

11. Fill out these matrices so that they have rank 1:

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & & \\ 4 & & \end{pmatrix} \quad B = \begin{pmatrix} 2 & & \\ 1 & & \\ 2 & 6 & -3 \end{pmatrix} \quad M = \begin{pmatrix} a & b \\ c & \end{pmatrix}$$

12. Suppose  $A$  and  $B$  are  $n$  by  $n$  matrices, and  $AB = I$ . Prove from  $\text{rank}(AB) \leq \text{rank}(A)$  that the  $\text{rank}(A) = n$ . So  $A$  is invertible and  $B$  must be its two-sided inverse. Therefore  $BA = I$  (which is not so obvious!).

13. Every  $m$  by  $n$  matrix of rank  $r$  reduces to  $(m \text{ by } r)$  times  $(r \text{ by } n)$ :

$$A = (\text{pivot columns of } A) (\text{first } r \text{ rows of } R) = (COL)(ROW)^T$$

Write the 3 by 4 matrix  $A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{pmatrix}$  as the product of the 3 by 2 from the pivot columns and the 2 by 4 matrix from  $R$ .

14. Suppose  $R$  is  $m$  by  $n$  matrix of rank  $r$ , with pivot columns first:  $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

- What are the shapes of those 4 blocks?
- Find the right-inverse  $B$  with  $RB = I$  if  $r = m$ .
- Find the right-inverse  $C$  with  $CR = I$  if  $r = n$ .
- What is the reduced row echelon form of  $R^T$  (with shapes)?
- What is the reduced row echelon form of  $R^T R$  (with shapes)?

Prove that  $R^T R$  has the same nullspace as  $R$ . Then show that  $A^T A$  always has the same nullspace as  $A$  (a value fact).

- Suppose you allow elementary column operations on  $A$  as well as elementary row operations (which get to  $R$ ). What is the “row-and-column reduced form” for an  $m$  by  $n$  matrix of rank  $r$ ?

15. True or False (check addition or give a counterexample)

- The symmetric matrices in  $M$  (with  $A^T = A$ ) form a subspace.
- The skew-symmetric matrices in  $M$  (with  $A^T = -A$ ) form a subspace.
- The un-symmetric matrices in  $M$  (with  $A^T \neq A$ ) form a subspace.
- Invertible matrices
- Singular matrices

16. Let  $A = \begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 2 & 4 & -3 & 7 & 0 \\ 3 & 6 & -5 & 10 & -2 \\ 5 & 10 & -9 & 16 & 0 \end{pmatrix}$

- Reduce  $A$  to row-reduced echelon form.
- What is the rank of  $A$ ?
- What are the pivots?
- What are the free variables?
- Find the special solutions. What is the nullspace  $N(A)$ ?
- Exhibit an  $r \times r$  submatrix of  $A$  which is invertible, where  $r = \text{rank}(A)$ . (An  $r \times r$  submatrix of  $A$  is obtained by keeping  $r$  rows and  $r$  columns of  $A$ )

17. Let  $A = \begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 2 & 1 & 0 & 0 & -15 \\ 6 & -1 & -8 & -1 & -47 \\ 0 & 2 & 4 & 3 & 16 \end{pmatrix}$

- Reduce  $A$  to row-reduced echelon form.
- What is the rank of  $A$ ?
- What the pivots?
- What are the free variables?
- Find the special solutions. What is the nullspace  $N(A)$ ?

f) Give the complete solution to  $Ax = b$ , where  $b = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

18. Let  $A = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

- Reduce  $A$  to row-reduced echelon form.
- What is the rank of  $A$ ?
- What the pivots?
- What are the free variables?
- Find the special solutions.
- What is the nullspace  $N(A)$ ?

19. Let  $A = \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix}$

- Reduce  $A$  to row-reduced echelon form.
- What is the rank of  $A$ ?
- What the pivots?
- What are the free variables?
- Find the special solutions.
- What is the nullspace  $N(A)$ ?

20. The 3 by 3 matrix  $A$  has rank 2.

$$A\vec{x} = \vec{b} \quad \text{is} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 + 5x_4 &= b_1 \\ 2x_1 + 4x_2 + 8x_3 + 12x_4 &= b_2 \\ 3x_1 + 6x_2 + 7x_3 + 13x_4 &= b_3 \end{aligned}$$

- Reduce  $\begin{bmatrix} A & \vec{b} \end{bmatrix}$  to  $\begin{bmatrix} U & \vec{c} \end{bmatrix}$ , so that  $A\vec{x} = \vec{b}$  becomes triangular system  $U\vec{x} = \vec{c}$ .
- Find the condition on  $(b_1, b_2, b_3)$  for  $A\vec{x} = \vec{b}$  to have a solution
- Describe the column space of  $A$ . Which plane in  $\mathbb{R}^3$ ?
- Describe the nullspace of  $A$ . Which special solutions in  $\mathbb{R}^4$ ?
- Find a particular solution to  $A\vec{x} = (0, 6, -6)$  and then complete solution.

21. Find the special solutions and describe the complete solution to  $A\vec{x} = \vec{0}$  for

$$A_1 = 3 \text{ by } 4 \text{ zero matrix} \quad A_2 = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \quad A_3 = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$$

Which are the pivot columns? Which are the free variables? What is the  $R$  (Reduced Row Echelon matrix) in each case?

22. Create a 3 by 4 matrix whose special solutions to  $A\vec{x} = \vec{0}$  are  $\vec{s}_1$  and  $\vec{s}_2$ :

$$\vec{s}_1 = \begin{pmatrix} -3 \\ 2 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{s}_2 = \begin{pmatrix} -2 \\ 0 \\ -6 \\ 1 \end{pmatrix}$$

You could create the matrix  $A$  in row reduced form  $R$ . Then describe all possible matrices  $A$  with the required Nullspace  $N(A) = \text{all combinations of } \vec{s}_1 \text{ and } \vec{s}_2$ .

23. The plane  $x - 3y - z = 12$  is parallel to the plane  $x - 3y - z = 0$ . One particular point on this plane is  $(12, 0, 0)$ . All points on the plane have the form (fill the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \phantom{0} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \phantom{0} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \phantom{0} \\ 0 \\ 1 \end{bmatrix}$$

- Construct a matrix whose column space contains  $(1, 1, 5)$  and  $(0, 3, 1)$  and whose Nullspace contains  $(1, 1, 2)$ .
- Construct a matrix whose column space contains  $(1, 1, 0)$  and  $(0, 1, 1)$  and whose Nullspace contains  $(1, 0, 1)$  and  $(0, 0, 1)$ .



26. Construct a matrix whose column space contains  $(1, 1, 1)$  and whose Nullspace contains  $(1, 1, 1, 1)$ .
27. How is the Nullspace  $N(C)$  related to the spaces  $N(A)$  and  $N(B)$ , if  $C = \begin{bmatrix} A \\ B \end{bmatrix}$ ?
28. Why does no 3 by 3 matrix have a nullspace that equals its column space?
29. If  $AB = 0$  then the column space  $B$  is contained in the \_\_\_\_\_ of  $A$ . Give an example of  $A$  and  $B$ .
30. True or false (with reason if true or example to show it is false)
- A square matrix has no free variables.
  - An invertible matrix has no free variables.
  - An  $m$  by  $n$  matrix has no more than  $n$  pivot variables.
  - An  $m$  by  $n$  matrix has no more than  $m$  pivot variables.
31. Suppose an  $m$  by  $n$  matrix has  $r$  pivots. The number of special solutions is \_\_\_\_\_.  
 The Nullspace contains only  $x = 0$  when  $r =$  \_\_\_\_\_.  
 The column space is all of  $\mathbb{R}^m$  when  $r =$  \_\_\_\_\_.
32. Find the complete solution in the form  $\vec{x}_p + \vec{x}_n$  to these full rank system:
- $x + y + z = 4$
  - $\begin{cases} x + y + z = 4 \\ x - y + z = 4 \end{cases}$
33. Find the complete solution in the form  $\vec{x}_p + \vec{x}_n$  to the system:
- $$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$
34. If  $A$  is 3 x 7 matrix, its largest possible rank is \_\_\_\_\_. In this case, there is a pivot in every \_\_\_\_\_ of  $U$  and  $R$ , the solution to  $A\vec{x} = \vec{b}$  \_\_\_\_\_ (always exists or is unique), and the column space of  $A$  is \_\_\_\_\_. Construct an example of such a matrix  $A$ .
35. If  $A$  is 6 x 3 matrix, its largest possible rank is \_\_\_\_\_. In this case, there is a pivot in every \_\_\_\_\_ of  $U$  and  $R$ , the solution to  $A\vec{x} = \vec{b}$  \_\_\_\_\_ (always exists or is unique), and the nullspace of  $A$  is \_\_\_\_\_. Construct an example of such a matrix  $A$ .
36. Find the rank of  $A$ ,  $A^T A$  and  $AA^T$  for  $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix}$



37. Explain why these are all false:

- a) The complete solution is any linear combination of  $\vec{x}_p$  and  $\vec{x}_n$ .
- b) A system  $A\vec{x} = \vec{b}$  has at most one particular solution.
- c) The solution  $\vec{x}_p$  with all free variables zero is the shortest solution (minimum length  $\|\vec{x}\|$ ). Find a 2 by 2 counterexample.
- d) If  $A$  is invertible there is no solution  $\vec{x}_n$  in the null space.

38. Write down all known relation between  $r$  and  $m$  and  $n$  if  $A\vec{x} = \vec{b}$  has

- a) No solution for some  $\vec{b}$ .
- b) Infinitely many solutions for every  $\vec{b}$ .
- c) Exactly one solution for some  $\vec{b}$ , no solution for other  $\vec{b}$ .
- d) Exactly one solution for every  $\vec{b}$ .

39. Find a basis for its row space, find a basis for its column space, and determine its rank

$$a) \begin{bmatrix} 0 & 2 & -3 & 4 & 1 & 2 & 1 & 7 \\ 0 & 0 & 3 & -2 & 0 & 4 & -5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 6 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad b) \begin{bmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{bmatrix}$$

40. Find a basis for the row space, find a basis for the null space, find  $\dim RS$ , find  $\dim NS$ , and verify  $\dim RS + \dim NS = n$

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ 3 & 1 & -3 & -1 \\ 5 & -3 & 5 & 1 \end{bmatrix}$$

41. Determine if  $\vec{b}$  lies in the column space of the given matrix. If it does, express  $\vec{b}$  as linear combination of the column.

$$\begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

42. Find the transition matrix from  $B$  to  $C$  and find  $[\vec{x}]_C$

$$a) B = \{(3, 1), (-1, -2)\}, \quad C = \{(1, -3), (5, 0)\}, \quad [x]_B = [-1 \quad -2]^T$$

$$b) B = \{(1, 1, 1), (-2, -1, 0), (2, 1, 2)\}, \quad C = \{(-6, -2, 1), (-1, 1, 5), (-1, -1, 1)\},$$

$$[\vec{x}]_B = [-3 \quad 2 \quad 4]^T$$

43. Does  $A$  and  $A^T$  have the same number of pivots.

(44 – 49) For the given matrix  $A$ , which is given in row reduction echelon form

- a) What is the rank of  $A$ ?
- b) What is the dimension of  $A$ ?
- c) What are the pivots?
- d) What are the free variables?
- e) Find the special (homogeneous) solutions.
- f) What is the nullspace  $N(A)$ ?
- g) Find the particular solution  $Ax = b$
- h) Give the complete solution.

$$44. \quad A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{where } b = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$45. \quad A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{where } b = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$46. \quad A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{where } b = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$47. \quad A = \begin{pmatrix} 1 & 0 & 0 & \frac{13}{11} \\ 0 & 1 & 0 & -\frac{17}{11} \\ 0 & 0 & 1 & \frac{6}{11} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{where } b = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$48. \quad A = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{where } b = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$49. \quad A = \begin{pmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{where } b = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

(50 – 55) Find a basis for each of the four subspaces associated with each given matrix

$$50. \quad A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{pmatrix}$$

$$53. \quad D = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix}$$

$$51. \quad B = \begin{pmatrix} 1 & 3 & 0 & 5 \\ 2 & 6 & 1 & 16 \\ 5 & 15 & 0 & 25 \end{pmatrix}$$

$$54. \quad M = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 3 & 1 & -3 & -1 \\ 5 & -3 & 5 & 1 \end{pmatrix}$$

$$52. \quad C = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$55. \quad N = \begin{pmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{pmatrix}$$