Section 3.8 – Taylor and Maclaurin Series

The sum of a power series:

$$f(x)=\sum_{n=0}^{\infty} a_n (x-a)^n$$

$$= a_0 + a_1 (x-a) + a_2 (x-a)^2 + \cdots + a_n (x-a)^n + \cdots$$

$$f'(x) = a_1 + 2a_2 (x-a) + 3a_3 (x-a)^2 + \cdots + na_n (x-a)^{n-1} + \cdots$$

$$f''(x) = 2a_2 + 2\cdot 3a_3 (x-a) + 3\cdot 4a_4 (x-a)^2 + \cdots + (n-1)\cdot na_n (x-a)^{n-2} + \cdots$$

$$f'''(x) = 1\cdot 2\cdot 3a_3 + 2\cdot 3\cdot 4a_4 (x-a) + +3\cdot 4\cdot 5a_5 (x-a)^2 \cdots + (n-2)\cdot(n-1)\cdot na_n (x-a)^{n-3} + \cdots$$

$$f^{(n)}(x) = n!a_n + \textit{a sum of terms with } (x-a) \textit{ as a factor}$$

In general: $\boxed{f^{(n)}(x) = n!a_n} \quad \Rightarrow \quad a_n = \dfrac{f^{(n)}(a)}{n!}$

If f has a series representation, then the series must be

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Taylor and Maclaurin Series

Definitions

Let f be a function with derivatives of all orders throughout some interval containing $\boldsymbol{a}$ as an interior point. Then the ***Taylor series generated by*** f at $x=a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

The ***Maclaurin series generated by*** f is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

The Taylor series generated by f at $x=0$.

Example

Find the Taylor series generated by $f(x)=\frac{1}{x}$ *at* $a=2$. Where, if anywhere, does the series converges to $\frac{1}{x}$.

Solution

$$f(x)=x^{-1}$$

$$f'(x)=-x^{-2}$$

$$f''(x)=2!x^{-3}$$

$$f'''(x)=-3!x^{-4}$$

$$f^{(n)}(x)=(-1)^n n!x^{-(n+1)}$$

$$f(2)=2^{-1}=\frac{1}{2}, \quad f'(2)=-\frac{1}{2^2}, \quad \frac{f''(2)}{2!}=2^{-3}=\frac{(-1)^3}{2^3}, \cdots \quad \frac{f^{(n)}(2)}{n!}=\frac{(-1)^n}{2^{n+1}}$$

The Taylor series is:

$$f(2)+f'(2)(x-2)+\frac{f''(2)}{2!}(x-2)^2+\cdots+\frac{f^{(n)}(2)}{n!}(x-2)^n$$

$$=\frac{1}{2}-\frac{(x-2)}{2^2}+\frac{(x-2)^2}{2^3}+\cdots+(-1)^n\frac{(x-2)^n}{2^{n+1}}+\cdots$$

Taylor Polynomials

Definition

Let f be a function with derivatives of order *k* for $k=1, 2, \ldots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through *N*, the Taylor polynomial of order *n* generated by f at $x=a$ is the polynomial

$$P_n(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n$$

Example

Find the Taylor series and the Taylor polynomials generated by $f(x)=e^x$ at $x=0$

Solution

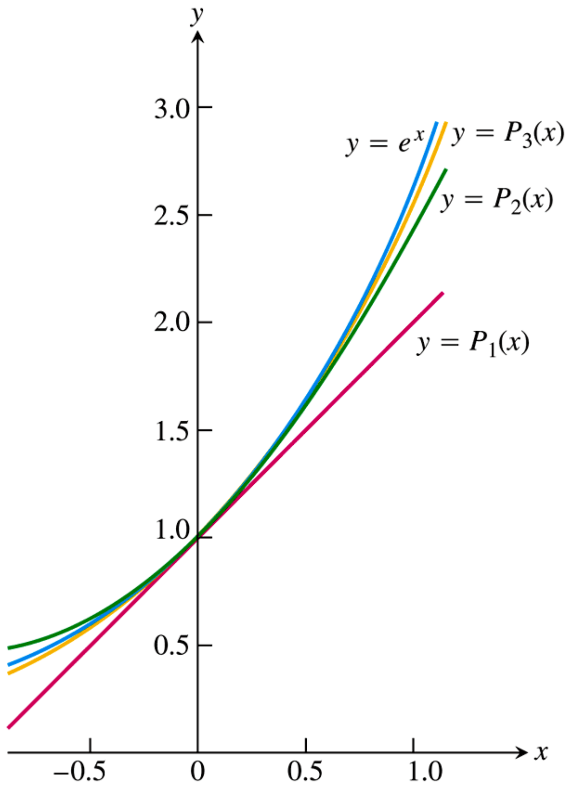
$$f^{(n)}(x)=e^x \quad \to \quad f^{(n)}(0)=1$$

$$P_n(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots$$

$$=1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots+\frac{1}{n!}x^n+\cdots$$

$$=\sum_{k=0}^{\infty}\frac{1}{k!}x^k$$

This is also the Maclaurin series of e^x



The Taylor polynomial of order n at $x=0$ is

$$P_n(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots+\frac{x^n}{n!}$$

Example

Find the Taylor series and the Taylor polynomials generated by $f(x)=\cos x$ at $x=0$

Solution

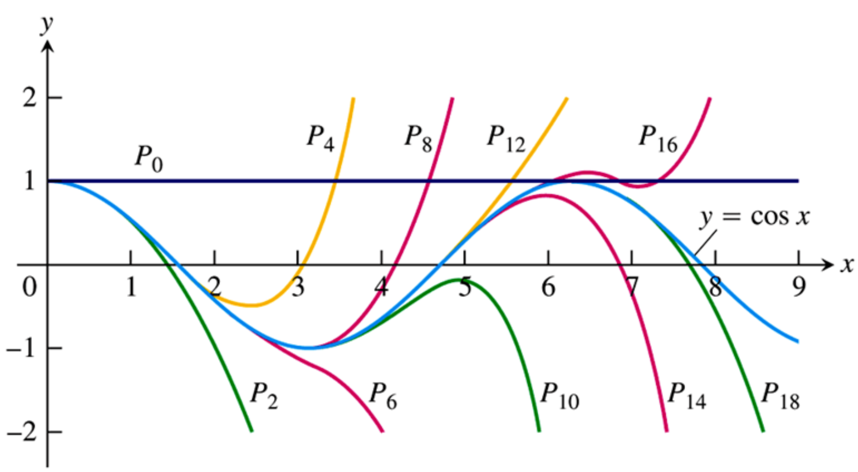
$$
\begin{array}{ll}
f(x)=\cos x, & f'(x)=-\sin x, \\
f''(x)=-\cos x, & f'''(x)=\sin x, \\
\vdots & \vdots \\
f^{(2n)}(x)=(-1)^n\cos x, & f^{(2n+1)}(x)=(-1)^{2n+1}\sin x
\end{array}
$$

$$f^{(2n)}(0)=(-1)^n, \qquad f^{(2n+1)}(0)=0$$

The Taylor series generated by f at $x=0$ is

$$
\begin{aligned}
f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots \\
&=1+0\cdot x-\frac{x^2}{2!}+0\cdot x^3+\frac{x^4}{4!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots \\
&=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots \\
&=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}
\end{aligned}
$$

$$P_{2n}(x)=P_{2n+1}(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}$$



Example

Find the Taylor series for $\cos x$ about $\frac{\pi}{3}$. Where is the series valid?

Solution

$$\begin{aligned}\cos x &= \cos\left(x-\frac{\pi}{3}+\frac{\pi}{3}\right)\\ &= \cos\left(x-\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)-\sin\left(x-\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)\\ &= \frac{1}{2}\left[1-\frac{1}{2!}\left(x-\frac{\pi}{3}\right)^2+\frac{1}{4!}\left(x-\frac{\pi}{3}\right)^4-\cdots\right]-\frac{\sqrt{3}}{2}\left[\left(x-\frac{\pi}{3}\right)-\frac{1}{3!}\left(x-\frac{\pi}{3}\right)^3+\frac{1}{5!}\left(x-\frac{\pi}{3}\right)^5-\cdots\right]\\ &= \frac{1}{2}-\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)-\frac{1}{2}\frac{1}{2!}\left(x-\frac{\pi}{3}\right)^2-\frac{\sqrt{3}}{2}\frac{1}{3!}\left(x-\frac{\pi}{3}\right)^3+\frac{1}{2}\frac{1}{4!}\left(x-\frac{\pi}{3}\right)^4+\frac{1}{5!}\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)^5-\cdots\end{aligned}$$

This series representation is valid for all x.

Example

Find the Taylor series for $\ln x$ in powers of $x-2$. Where does the series converge to $\ln x$?

Solution

Let $t = \frac{x-2}{2}$, then

$$\begin{aligned}
\ln x &= \ln\left(2+(x-2)\right) \\
&= \ln\left[2\left(1+\frac{x-2}{2}\right)\right] \\
&= \ln 2 + \ln(1+t)
\end{aligned}$$

$$f(t) = \ln(1+t) \qquad f(0) = \ln(1) = 0$$

$$f'(t) = \frac{1}{1+t} \qquad f'(0) = 1$$

$$f''(t) = \frac{-1}{(1+t)^2} \qquad f''(0) = -1$$

$$f'''(t) = \frac{2}{(1+t)^3} \qquad f'''(0) = 2$$

$$f^{(4)}(t) = \frac{-6}{(1+t)^4} \qquad f'''(0) = -6$$

$$f^{(n)}(t) = (-1)^{n+1}\frac{(n-1)!}{(1+t)^n}$$

$$\begin{aligned}
\ln(1+t) &= f(0) + f'(0)t + \frac{f''(0)}{2!}t^2 + \frac{f'''(0)}{3!}t^3 + \frac{f^{(IV)}(0)}{4!}t^4 + \cdots \\
&= t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots
\end{aligned}$$

$$\begin{aligned}
\ln x &= \ln 2 + \ln(1+t) \\
&= \ln 2 + t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \\
&= \ln 2 + \frac{x-2}{2} - \frac{(x-2)^2}{2\times 2^2} + \frac{(x-2)^3}{3\times 2^3} - \frac{(x-2)^4}{4\times 2^4} + \cdots \\
&= \ln 2 + \sum_{n=1}^{\in} \frac{(-1)^{n-1}}{n\times 2^n}(x-2)^n
\end{aligned}$$

Since the series for $\ln(1+t)$ is valid for $-1 < t \le 1$, this series for $\ln x$ is valid for $-1 < \frac{x-2}{2} \le 1$

$-2 < x-2 \le 2 \quad \rightarrow \quad \underline{0 < x \le 4}$

Exercises   *Section* 3.8 – Taylor and Maclaurin Series

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a

1. $f(x)=e^{2x}, \quad a=0$

2. $f(x)=\sin x, \quad a=0$

3. $f(x)=\ln(1+x), \quad a=0$

4. $f(x)=\frac{1}{x+2}, \quad a=0$

5. $f(x)=\sqrt{1-x}, \quad a=0$

6. $f(x)=x^3, \quad a=1$

7. $f(x)=8\sqrt{x}, \quad a=1$

8. $f(x)=\sin x, \quad a=\frac{\pi}{4}$

9. $f(x)=\cos x, \quad a=\frac{\pi}{6}$

10. $f(x)=\sqrt{x}, \quad a=9$

11. $f(x)=\sqrt[3]{x}, \quad a=8$

12. $f(x)=\ln x, \quad a=e$

13. $f(x)=\sqrt[4]{x}, \quad a=8$

14. $f(x)=\tan^{-1}x+x^2+1, \quad a=1$

15. $f(x)=e^{x}, \quad a=\ln 2$

Find the nth Maclaurin polynomial for the function

16. $f(x)=e^{4x}, \quad n=4$

17. $f(x)=e^{-x}, \quad n=5$

18. $f(x)=e^{-x/2}, \quad n=4$

19. $f(x)=e^{x/3}, \quad n=4$

20. $f(x)=\sin x, \quad n=5$

21. $f(x)=\cos \pi x, \quad n=4$

22. $f(x)=xe^{x}, \quad n=4$

23. $f(x)=x^2e^{-x}, \quad n=4$

24. $f(x)=\frac{1}{x+1}, \quad n=5$

25. $f(x)=\frac{x}{x+1}, \quad n=4$

26. $f(x)=\sec x, \quad n=2$

27. $f(x)=\tan x, \quad n=3$

Find the Maclaurin series for

28. xe^{x}

29. $5\cos \pi x$

30. $\frac{x^2}{x+1}$

31. e^{3x+1}

32. $\cos\left(2x^3\right)$

33. $\cos(2x-\pi)$

34. $x^2\sin\left(\frac{x}{3}\right)$

35. $\cos^2\left(\frac{x}{2}\right)$

36. $\sin x\cos x$

37. $\tan^{-1}\left(5x^2\right)$

38. $\ln\left(2+x^2\right)$

39. $\frac{1+x^3}{1+x^2}$

40. $\ln\frac{1+x}{1-x}$

41. $\frac{e^{2x^2}-1}{x^2}$

42. $\cosh x-\cos x$

43. $\sinh x-\sin x$

Finding Taylor and Maclaurin Series generated by f *at* $x=a$

44. $f(x)=x^3-2x+4, \quad a=2$

45. $f(x)=2x^3+x^2+3x-8, \quad a=1$

46. $f(x)=3x^5-x^4+2x^3+x^2-2, \quad a=-1$

47. $f(x)=\cos\left(2x+\frac{\pi}{2}\right), \quad a=\frac{\pi}{4}$

Find the Taylor series of the functions, where is each series representation valid?

48. $f(x) = e^{-2x}$ *about* -1

49. $f(x) = \sin x$ *about* $\frac{\pi}{2}$

50. $f(x) = \ln x$ *in powers of* $x-3$

51. $f(x) = \ln(2+x)$ *in powers of* $x-2$

52. $f(x) = e^{2x+3}$ *in powers of* $x+1$

53. $f(x) = \sin x - \cos x$ *about* $\frac{\pi}{4}$

54. $f(x) = \cos^2 x$ *about* $\frac{\pi}{8}$

55. $f(x) = \frac{x}{1+x}$ *in powers of* $x-1$

56. $f(x) = xe^{x}$ *in powers of* $x+2$

Find the *n*th Taylor polynomial centered at *c* for the function

57. $f(x) = \frac{2}{x}, \quad n=3, \quad c=1$

58. $f(x) = \frac{1}{x^2}, \quad n=4, \quad c=2$

59. $f(x) = \sqrt{x}, \quad n=3, \quad c=4$

60. $f(x) = \sqrt[3]{x}, \quad n=3, \quad c=8$

61. $f(x) = \ln x, \quad n=4, \quad c=2$

62. $f(x) = x^2 \cos x, \quad n=2, \quad c=\pi$

Find the sums of the series

63. $1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots$

64. $1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \frac{x^8}{9!} + \cdots$

65. $x^3 - \frac{x^9}{3! \times 4} + \frac{x^{15}}{5! \times 16} - \frac{x^{21}}{7! \times 64} + \frac{x^{27}}{9! \times 256} - \cdots$

66. The limit $\lim\limits_{n\to\infty} \frac{n!}{\sqrt{2\pi}\; n^{n+1/2} e^{-n}} = 1$ that is the relative error in the approximation

$n! \approx \sqrt{2\pi}\; n^{n+1/2} e^{-n}$

Approaches zero as n increases. That is $n!$ grows at a rate comparable to $\sqrt{2\pi}\; n^{n+1/2} e^{-n}$. This result, known as Stirling's Formula, is often very useful in applied mathemmatics and statistics. Prove it by carrying out the following steps.

a) Use the identity $\ln(n!) = \sum_{j=1}^{n} \ln j$ and the increasing nature of ln to show that if $n \geq 1$,

$$\int_0^n \ln x \, dx < \ln(n!) < \int_1^{n+1} \ln x \, dx$$

And hence that $n \ln n - n < \ln(n!) < (n+1)\ln(n+1) - n$

b) If $c_n = \ln(n!) - \left(n + \frac{1}{2}\right)\ln n + n$, show that

$$c_n - c_{n+1} = \left(n + \frac{1}{2}\right)\ln\left(\frac{n+1}{n}\right) - 1$$

$$= \left(n + \frac{1}{2}\right) \ln \left(\frac{1 + \frac{1}{2n+1}}{1 - \frac{1}{2n+1}} \right) - 1$$

c) Use the Maclaurin series for $\ln \frac{1+t}{1-t}$ to show that

$$0 < c_n - c_{n+1} < \frac{1}{3} \left(\frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^3} + \cdots \right)$$

$$= \frac{1}{12} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

and therefore that $\{c_n\}$ is decreasing and $\left\{c_n - \frac{1}{12n}\right\}$ is increasing. Hence conclude that $\lim_{n\to\infty} c_n = c$ exists, and that

$$\lim_{n\to\infty} \frac{n!}{n^{n+1/2} e^{-n}} = \lim_{n\to\infty} e^{c_n} = e^c$$