Solution   *Section* 2.8 – Lagrange Multipliers

Exercise

Find the points on the ellipse $x^2 + 2y^2 = 1$ where $f(x,y) = xy$ has its extreme values.

Solution

$$g(x,y) = x^2 + 2y^2 - 1$$

$$\nabla f = y\mathbf{i} + x\mathbf{j}, \quad \nabla g = 2x\mathbf{i} + 4y\mathbf{j}$$

$$\nabla f = \lambda \nabla g$$

$$y\mathbf{i} + x\mathbf{j} = 2\lambda x\mathbf{i} + 4\lambda y\mathbf{j}$$

$$y = 2\lambda x \qquad x = 4\lambda y$$

$$x = 8x\lambda^2$$

$$8x\lambda^2 - x = 0$$

$$x\left(8\lambda^2 - 1\right) = 0 \quad \Rightarrow \quad \begin{cases} \lambda^2 = \frac{1}{8} \to \lambda = \pm\frac{1}{2\sqrt{2}} \\ x = 0 \end{cases}$$

Case 1: If $x = 0 \Rightarrow y = 2\lambda x = 0$. But (0, 0) is not on the ellipse so $x \neq 0$

Case 2: If $x \neq 0$ *and* $\lambda = \pm\frac{\sqrt{2}}{4}$

$$\Rightarrow x = 4\lambda y = \pm\sqrt{2}y$$

$$\left(\pm\sqrt{2}y\right)^2 + 2y^2 = 1$$

$$2y^2 + 2y^2 = 1$$

$$y^2 = \frac{1}{4} \quad \Rightarrow \quad \boxed{y = \pm\frac{1}{2}}$$

$$x \pm \sqrt{2}y \quad \Rightarrow \quad \boxed{x = \pm\frac{\sqrt{2}}{2}}$$

$$f(x,y) = xy = \pm\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \pm\frac{\sqrt{2}}{4}$$

Therefore, f has extreme values at $\left(\pm\frac{\sqrt{2}}{2}, \ \pm\frac{1}{2}\right)$

The extreme values of f on the ellipse are $\pm\frac{\sqrt{2}}{4}$

Exercise

Find the extreme values of $f(x,y) = xy$ subject to the constraint $g(x,y) = x^2 + y^2 - 10 = 0$.

Solution

$$g(x,y) = x^2 + y^2 - 10$$

$$\nabla f = y\mathbf{i} + x\mathbf{j}, \quad \nabla g = 2x\mathbf{i} + 2y\mathbf{j}$$

$$\nabla f = \lambda \nabla g$$

$$y\mathbf{i} + x\mathbf{j} = 2\lambda x\mathbf{i} + 2\lambda y\mathbf{j}$$

$$y = 2\lambda x \qquad x = 2\lambda y = 4x\lambda^2$$

$$x\left(4\lambda^2 - 1\right) = 0 \quad \Rightarrow \quad \begin{cases} \lambda^2 = \frac{1}{4} \to \lambda = \pm\frac{1}{2} \\ x = 0 \end{cases}$$

Case 1: If $x = 0 \ \Rightarrow y = 2\lambda x = 0$. But (0, 0) is not on the circle so $x \neq 0$

Case 2: If $x \neq 0$ *and* $\lambda = \pm\frac{1}{2}$

$$\Rightarrow x = 2\lambda y = \pm y$$

$$g(x,y) = x^2 + y^2 - 10 = 0$$

$$(\pm y)^2 + y^2 = 10$$

$$2y^2 = 10$$

$$y^2 = 5 \quad \Rightarrow \quad \boxed{y = \pm\sqrt{5} = \pm x}$$

$$f(x,y) = xy = \pm\sqrt{5}\left(\sqrt{5}\right) = \pm 5$$

Therefore, f has extreme values at $\left(\pm\sqrt{5},\ \pm\sqrt{5}\right)$

The extreme values of f on the circle are ± 5

Exercise

Find the extreme values of $f(x,y) = x^3 + y^2$ on the circle $x^2 + y^2 = 1$

Solution

$$\nabla f = 3x^2\hat{i} + 2y\hat{j} \qquad \nabla f = f_x\hat{i} + f_y\hat{j}$$

$$g(x,y) = x^2 + y^2 - 1 \quad \to \quad \nabla g = 2x\hat{i} + 2y\hat{j}$$

$$3x^2\hat{i} + 2y\hat{j} = \lambda\left(2x\hat{i} + 2y\hat{j}\right) \qquad \nabla f = \lambda\nabla g$$

$$= 2\lambda x\hat{i} + 2\lambda y\hat{j}$$

$$\rightarrow \begin{cases} 3x^2 = 2\lambda x \\ 2y = 2\lambda y & \rightarrow \lambda = 1 \ \ or \ \ y = 0 \end{cases}$$

For $\lambda = 1$

$$3x^2 = 2x$$

$$x(3x-2) = 0 \quad \rightarrow \quad x = 0, \ \frac{2}{3}$$

$$y = \pm\sqrt{1-x^2}$$

$$\rightarrow \begin{cases} x = 0 & \rightarrow y = \pm 1 \\ x = \frac{2}{3} & \rightarrow y = \pm\sqrt{1-\frac{4}{9}} = \pm\frac{\sqrt{5}}{3} \end{cases}$$

$$(0, \ -1), \ (0, \ 1), \ \left(\frac{2}{3}, \ -\frac{\sqrt{5}}{3}\right) \ \ \& \ \ \left(\frac{2}{3}, \ \frac{\sqrt{5}}{3}\right)$$

For $y = 0$

$$x^2 = 1 \Rightarrow x = \pm 1$$

$$(\pm 1, \ 0)$$

$$f(x,y) = x^3 + y^2$$

$$f(-1, \ 0) = -1$$

$$f(1, \ 0) = 1$$

$$f(0, \ -1) = 1$$

$$f(0, \ 1) = 1$$

$$f\left(\frac{2}{3}, \ -\frac{\sqrt{5}}{3}\right) = \frac{8}{27} + \frac{5}{9} \ = \frac{23}{27}$$

$$f\left(\frac{2}{3}, \ \frac{\sqrt{5}}{3}\right) = \frac{8}{27} + \frac{5}{9} \ = \frac{23}{27}$$

Absolute Max. is 1 @ $(0, \ \pm 1), \ (1, \ 0)$

Absolute Min. is -1 @ $(-1, \ 0)$

Exercise

Find the extreme values of $f(x,y) = x^2 + y^2 - 3x - xy$ on the circle $x^2 + y^2 \le 9$

Solution

$$\nabla f = (2x - 3 - y)\hat{i} + (2y - x)\hat{j} \qquad \nabla f = f_x \hat{i} + f_y \hat{j}$$

$$g(x,y) = x^2 + y^2 - 9 \quad \rightarrow \quad \nabla g = 2x\hat{i} + 2y\hat{j}$$

$$(2x-3-y)\hat{i}+(2y-x)\hat{j}=2\lambda x\hat{i}+2\lambda y\hat{j} \qquad \nabla f=\lambda\nabla g$$

$$\rightarrow\begin{cases}2x-3-y=2x\lambda & \rightarrow\ 2x(1-\lambda)-y=3 \quad (1)\\ 2y-x=2y\lambda & \rightarrow\ 2y(1-\lambda)=x \quad (2)\end{cases}$$

$$(2)\rightarrow\ 1-\lambda=\frac{x}{2y}$$

$$(1)\rightarrow\ 2x\left(\frac{x}{2y}\right)-y=3$$

$$x^2-y^2=3y$$

$$x^2=y^2+3y$$

$$x^2+y^2=9$$

$$y^2+3y+y^2=9$$

$$2y^2+3y-9=0\ \rightarrow\ \boxed{y=-3,\ \frac{3}{2}}$$

For $y=-3$

$$x=\pm\sqrt{9-y^2}=0$$

$$\therefore (0,\ -3)$$

For $y=\frac{3}{2}$

$$x=\pm\sqrt{9-\frac{9}{4}}=\pm\frac{3\sqrt{3}}{2}$$

$$\therefore\left(\pm\frac{3\sqrt{3}}{2},\ \frac{3}{2}\right)$$

$$\begin{cases}f_x=2x-3-y=0 & \qquad 3y=3\rightarrow \boxed{y=1}\\ f_y=2y-x=0 & \rightarrow x=2y\end{cases}$$

$$\therefore (2,\ 1)\ \ \textbf{\textit{C.P.}}$$

$$f(0,\ -3)=9$$

$$f(2,\ 1)=-3$$

$$f\left(\frac{3\sqrt{3}}{2},\ \frac{3}{2}\right)=9-\frac{27\sqrt{3}}{4}=\frac{36-27\sqrt{3}}{4}$$

$$f\left(-\frac{3\sqrt{3}}{2},\ \frac{3}{2}\right)=9+\frac{27\sqrt{3}}{4}=\frac{36+27\sqrt{3}}{4}$$

Absolute Max. is $\frac{36+27\sqrt{3}}{4}$ @ $\left(-\frac{3\sqrt{3}}{2},\ \frac{3}{2}\right)$

Absolute Min. is -3 @ $(2,\ 1)$

Exercise

Find the maximum value of $f(x,y)=49-x^2-y^2$ on the line $x+3y=10$.

Solution

$$\nabla f=-2x\boldsymbol{i}-2y\boldsymbol{j},\quad \nabla g=\boldsymbol{i}+3\boldsymbol{j}$$

$$\nabla f=\lambda\nabla g \quad\to\quad -2x\boldsymbol{i}-2y\boldsymbol{j}=\lambda\boldsymbol{i}+3\lambda\boldsymbol{j}$$

$$-2x=\lambda \qquad -2y=3\lambda$$

$$x=-\frac{\lambda}{2} \qquad y=-\frac{3\lambda}{2}$$

$$x+3y=10$$

$$-\frac{\lambda}{2}+3\left(-\frac{3\lambda}{2}\right)=10$$

$$-5\lambda=10 \quad\Rightarrow\quad \boxed{\lambda=-2}$$

$$\underline{\left\lfloor x=-\frac{\lambda}{2}=1\right\rfloor} \quad \text{and} \quad \underline{\left\lfloor y=-\frac{3\lambda}{2}=3\right\rfloor}$$

$$f(x,y)=49-1^2-3^2=39$$

Therefore, f has extreme values at $(1,\ 3)$.

The extreme values of f is 39

Exercise

Find the points on the curve $x^2y=2$ nearest the origin.

Solution

Let $f(x,y)=x^2+y^2$, the square of the distance to the origin subject to the constraint

$$g(x,y)=x^2y-2=0$$

$$\nabla f=2x\boldsymbol{i}+2y\boldsymbol{j},\quad \nabla g=2xy\boldsymbol{i}+x^2\boldsymbol{j}$$

$$\nabla f=\lambda\nabla g \quad\to\quad 2x\boldsymbol{i}+2y\boldsymbol{j}=2xy\lambda\boldsymbol{i}+x^2\lambda\boldsymbol{j}$$

$$2x=2xy\lambda \qquad 2y=x^2\lambda$$

$$y=\frac{1}{\lambda} \qquad x^2=\frac{2y}{\lambda}=\frac{2}{\lambda^2}$$

$$x^2y-2=0$$

$$\left(\frac{2}{\lambda^2}\right)\left(\frac{1}{\lambda}\right)-2=0$$

$$\frac{2}{\lambda^3}=2 \quad\Rightarrow\quad \lambda^3=1\to\boxed{\lambda=1}$$

$\boxed{y=1}$ $x^2=2 \Rightarrow \boxed{x=\pm\sqrt{2}}$

$\therefore \left(\pm\sqrt{2},\ 1\right)$ are the points on the curve $x^2y=2$ nearest the origin.

Exercise

Use the method of Lagrange multipliers to find

a) The minimum value of $x+y$, subject to the constraints $xy=16, \quad x>0, \quad y>0$

b) The maximum value of xy, subject to the constraints $x+y=16$

Solution

a) $\nabla f=\boldsymbol{i}+\boldsymbol{j}, \quad \nabla g=y\boldsymbol{i}+x\boldsymbol{j}$

$$\nabla f=\lambda\nabla g \quad \rightarrow \quad \boldsymbol{i}+\boldsymbol{j}=y\lambda\boldsymbol{i}+x\lambda\boldsymbol{j}$$

$$1=y\lambda, \quad 1=x\lambda$$

$$y=\frac{1}{\lambda}, \quad x=\frac{1}{\lambda}$$

$$g(x,y)=xy-16=0$$

$$\frac{1}{\lambda^2}-16=0 \quad \Rightarrow \quad \lambda^2=\frac{1}{16} \rightarrow \boxed{\lambda=\pm\frac{1}{4}}$$

For $\lambda=-\frac{1}{4} \quad \rightarrow \quad \cancel{x=y=-4}$ *since* $x>0, \quad y>0$

For $\lambda=\frac{1}{4} \quad \rightarrow \quad \boxed{x=y=4}$

The minimum value is $\underline{\left|f=x+y=4+4 \ \overline{=8}\right|}$.

$xy=16, \quad x>0, \quad y>0$ is a branch of a hyperbola in the first quadrant with *x*- and *y*-axes as asymptotes.

The equations $x+y=c$ give a family of parallel lines with $m=-1$. Thus the minimum value of $\boldsymbol{c}$ occurs where $x+y=c$ is tangent to the hyperbola's branch.

b) $\nabla f=y\boldsymbol{i}+x\boldsymbol{j}, \quad \nabla g=\boldsymbol{i}+\boldsymbol{j}$

$$\nabla f=\lambda\nabla g \quad \rightarrow \quad y\boldsymbol{i}+x\boldsymbol{j}=\lambda\boldsymbol{i}+\lambda\boldsymbol{j}$$

$$y=\lambda, \quad x=\lambda$$

$$g(x,y)=x+y-16=0 \quad \rightarrow \quad 2\lambda=16 \Rightarrow \boxed{\lambda=8}$$

For $\lambda=8 \quad \rightarrow \quad \boxed{x=y=8}$

The maximum value is $\underline{\left|f=xy=8\times 8 \ \overline{=64}\right|}$.

The equations $xy = c, \quad x > 0,\ y > 0 \quad or \quad x < 0,\ y < 0$ give a family of hyperbolas in the first and third quadrants with x- and y-axes as asymptotes. Thus the maximum value of $\boldsymbol{c}$ occurs where $xy = c$ is tangent to the line $x + y = 16$.

Exercise

Find the radius and height of the open right circular cylinder of largest surface area that can be inscribed in a sphere of radius a. What is the largest surface area?

Solution

For a cylinder of radius r and height h, to maximize the surface area $S = 2\pi rh$ subject to the constraint $g(r,h) = r^2 + \left(\frac{h}{2}\right)^2 - a^2 = 0$

$$\nabla S = 2\pi h\mathbf{i} + 2\pi r\mathbf{j} \quad and \quad \nabla g = 2r\mathbf{i} + \frac{h}{2}\mathbf{j}$$

$$\nabla S = \lambda \nabla g \quad \to \quad 2\pi h\mathbf{i} + 2\pi r\mathbf{j} = 2r\lambda\mathbf{i} + \frac{h}{2}\lambda\mathbf{j}$$

$$2\pi h = 2r\lambda, \quad 2\pi r = \frac{h}{2}\lambda$$

$$\lambda = \frac{\pi h}{r} \quad \to 2\pi r = \frac{h}{2}\frac{\pi h}{r}$$

$$4r^2 = h^2 \quad \Rightarrow \quad h = 2r$$

$$r^2 + \left(\frac{h}{2}\right)^2 = a^2$$

$$r^2 + r^2 = a^2$$

$$2r^2 = a^2 \quad \to \quad \boxed{r = \frac{a}{\sqrt{2}}}$$

$$h = \frac{2a}{\sqrt{2}} = a\sqrt{2}$$

$$\begin{aligned} S &= 2\pi rh \\ &= 2\pi \frac{a}{\sqrt{2}} a\sqrt{2} \\ &= 2\pi a^2 \end{aligned}$$

Exercise

Use the method of Lagrange multipliers to find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ with sides parallel to the coordinate axes.

Solution

The area of a rectangle is $A(x,y)=(2x)(2y)=4xy$ subject to the constraint

$$g(x,y)=\frac{x^2}{16}+\frac{y^2}{9}-1=0.$$

$$\nabla A=4y\mathbf{i}+4x\mathbf{j} \quad and \quad \nabla g=\frac{1}{8}x\mathbf{i}+\frac{2}{9}y\mathbf{j}$$

$$\nabla A=\lambda\nabla g \quad \Rightarrow \quad 4y\mathbf{i}+4x\mathbf{j}=\frac{1}{8}x\lambda\mathbf{i}+\frac{2}{9}y\lambda\mathbf{j}$$

$$4y=\frac{1}{8}x\lambda \quad and \quad 4x=\frac{2}{9}y\lambda$$

$$\lambda=\frac{32y}{x} \quad \Rightarrow \quad 4x=\frac{2y}{9}\frac{32y}{x}\to x^2=\frac{64y^2}{36} \quad \boxed{x=\pm\frac{4}{3}y}$$

$$\frac{1}{16}\frac{16y^2}{9}+\frac{1}{9}y^2=1$$

$$\frac{2}{9}y^2=1 \quad \to \quad y^2=\frac{9}{2}\Rightarrow y=\pm\frac{3\sqrt{2}}{2}$$

Since x and y represents distance, then $y=\frac{3\sqrt{2}}{2} \quad \to \quad x=\frac{4}{3}\frac{3\sqrt{2}}{2}=2\sqrt{2}$

$\therefore$ The length is $2x=4\sqrt{2}$ and the width is $2y=3\sqrt{2}$

Exercise

Find the maximum and minimum values of x^2+y^2 subject to the constraint $x^2-2x+y^2-4y=0$

Solution

$$\nabla f=2x\mathbf{i}+2y\mathbf{j} \quad and \quad \nabla g=(2x-2)\mathbf{i}+(2y-4)\mathbf{j}$$

$$\nabla f=\lambda\nabla g \quad \Rightarrow \quad 2x\mathbf{i}+2y\mathbf{j}=(2x-2)\lambda\mathbf{i}+(2y-4)\lambda\mathbf{j}$$

$$\begin{array}{lll} 2x=2(x-1)\lambda & and & 2y=2(y-2)\lambda \\ x=x\lambda-\lambda & & y=y\lambda-2\lambda \\ x(\lambda-1)=\lambda & & y(\lambda-1)=2\lambda \\ x=\dfrac{\lambda}{\lambda-1} & & y=\dfrac{2\lambda}{\lambda-1}=2x \quad (\lambda\neq 1) \end{array}$$

$$x^2-2x+y^2-4y=0$$

$$x^2-2x+4x^2-8x=0$$

$$5x^2-10x=0$$

$$5x(x-2)=0 \quad \Rightarrow \quad x=0, \ 2$$

$$x=0 \quad y=2x=0\to(0,\ 0)$$

$$x=2 \quad y=2x=4\to(2,\ 4)$$

$\therefore$ $f(0,0)=0$ is the minimum value, and $f(2,4)=20$ is the maximum value.

Exercise

The temperature at a point (x,y) on a metal plate is $T(x,y)=4x^2-4xy+y^2$. An ant on the plate walks around the circle of radius 5 centered at the origin. What are the highest and lowest temperatures encountered by the ant?

Solution

$$g(x,y)=x^2+y^2-25=0$$

$$\nabla T=(8x-4y)\boldsymbol{i}+(-4x+2y)\boldsymbol{j} \quad \text{and} \quad \nabla g=2x\boldsymbol{i}+2y\boldsymbol{j}$$

$$\nabla T=\lambda\nabla g \quad \to \quad (8x-4y)\boldsymbol{i}+(-4x+2y)\boldsymbol{j}=2x\lambda\boldsymbol{i}+2y\lambda\boldsymbol{j}$$

$$8x-4y=2x\lambda, \quad -4x+2y=2y\lambda$$

$$4x-2y=x\lambda, \quad y-y\lambda=2x \to y=\frac{2x}{1-\lambda} \quad (\lambda\neq 1)$$

$$4x-2\frac{2x}{1-\lambda}=x\lambda$$

$$4x-\frac{4x}{1-\lambda}-x\lambda=0$$

$$x\left(4-4\lambda-4-\lambda+\lambda^2\right)=0$$

$$x\left(\lambda^2-5\lambda\right)=0 \quad \Rightarrow \quad \boxed{x=0}, \quad \boxed{\lambda=0,\ 5}$$

Case 1: $x=0$ $\left\lfloor y=\frac{2x}{1-\lambda}=0\right\rfloor$, but (0, 0) is not on the circle $x^2+y^2=25$

Case 2: $\lambda=0 \quad y=2x$

$$\Rightarrow x^2+(2x)^2=25 \quad \to 5x^2=25 \quad \boxed{x=\pm\sqrt{5}} \quad \left(\sqrt{5},2\sqrt{5}\right) \quad \left(-\sqrt{5},-2\sqrt{5}\right)$$

Case 3: $\lambda=5 \quad y=-\frac{x}{2}$

$$\Rightarrow x^2+\frac{x^2}{4}=25 \quad \to 5x^2=100 \quad \boxed{x=\pm 2\sqrt{5}} \quad \left(2\sqrt{5},-\sqrt{5}\right) \quad \left(-2\sqrt{5},\sqrt{5}\right)$$

$$T\left(\sqrt{5},2\sqrt{5}\right)=4\left(\sqrt{5}\right)^2-4\left(\sqrt{5}\right)\left(2\sqrt{5}\right)+\left(2\sqrt{5}\right)^2=0^\circ$$

$$T\left(-\sqrt{5},-2\sqrt{5}\right)=4\left(-\sqrt{5}\right)^2-4\left(-\sqrt{5}\right)\left(-2\sqrt{5}\right)+\left(-2\sqrt{5}\right)^2=0^\circ$$

$$T\left(2\sqrt{5},-\sqrt{5}\right)=4\left(2\sqrt{5}\right)^2-4\left(2\sqrt{5}\right)\left(-\sqrt{5}\right)+\left(-\sqrt{5}\right)^2=125^\circ$$

$$T\left(-2\sqrt{5},\sqrt{5}\right)=4\left(-2\sqrt{5}\right)^2-4\left(-2\sqrt{5}\right)\left(\sqrt{5}\right)+\left(\sqrt{5}\right)^2=125^\circ$$

$\therefore$ The minimum temperature is 0° at $\left(\sqrt{5},2\sqrt{5}\right)$ $\left(-\sqrt{5},-2\sqrt{5}\right)$

The maximum temperature is 125° at $\left(2\sqrt{5},-\sqrt{5}\right)$ $\left(-2\sqrt{5},\sqrt{5}\right)$

Exercise

Your firm has been asked to design a storage tank for liquid petroleum gas. The customer's specifications call for a cylindrical tank with hemispherical ends, and the tank is to hold $8000\ m^3$ of gas. He customer also wants to use the smallest amount of material possible in building the tank. What radius and height do you recommend for the cylindrical portion of the tank?

Solution

The surface area is: $S = 4\pi r^2 + 2\pi rh$ subject to the constraint $V = \frac{4}{3}\pi r^3 + \pi r^2 h = 8000$.

$$\nabla S = (8\pi r + 2\pi h)\boldsymbol{i} + 2\pi r\boldsymbol{j} \quad \text{and} \quad \nabla V = \left(4\pi r^2 + 2\pi rh\right)\boldsymbol{i} + \pi r^2\boldsymbol{j}$$

$$\nabla S = \lambda\nabla V \quad \to \quad (8\pi r + 2\pi h)\boldsymbol{i} + 2\pi r\boldsymbol{j} = \left(4\pi r^2 + 2\pi rh\right)\lambda\boldsymbol{i} + \pi r^2\lambda\boldsymbol{j}$$

$$8\pi r + 2\pi h = 2\pi r(2r+h)\lambda \quad \text{and} \quad 2\pi r = \pi r^2\lambda$$

$$4r + h = r(2r+h)\lambda \qquad r^2\lambda - 2r = 0 \to r(\lambda r - 2) = 0$$

$$r = 0 \quad \text{and} \quad \lambda = \frac{2}{r} \ (r \neq 0)$$

$$4r + h = r(2r+h)\frac{2}{r} \quad \to \quad 4r + h = 4r + 2h \ \Rightarrow \boxed{h = 0}$$

The tank is a sphere, there is no cylindrical part, and

$$\frac{4}{3}\pi r^3 + \pi r^2(0) = 8000$$

$$r^3 = \frac{6000}{\pi}$$

$$r = 10\left(\frac{6}{\pi}\right)^{1/3} \approx 12.4$$

Exercise

A closed rectangular box is to have volume $V\ cm^3$. The cost of the material used in the box is a $a\ cents/cm^2$ for top and bottom, $b\ cents/cm^2$ for front and back, and $c\ cents/cm^2$ for the remaining sides. What dimensions minimize the total cost of materials?

Solution

The cost is given by: $f(x,y,z) = 2axy + 2bxz + 2cyz$

Subject to the constraint $xyz = V$

$$\nabla f = (2ay + 2bz)\hat{i} + (2ax + 2cz)\hat{j} + (2bx + 2cy)\hat{k}$$

$$g(x,y,z) = xyz - V$$

$$\nabla g = yz\hat{i} + xz\hat{j} + xy\hat{k}$$

$$(2ay + 2bz)\hat{i} + (2ax + 2cz)\hat{j} + (2bx + 2cy)\hat{k} = \lambda\left(yz\hat{i} + xz\hat{j} + xy\hat{k}\right)$$

$$\begin{matrix} x \\ -y \\ \\ \end{matrix}\begin{cases} 2ay + 2bz = yz\lambda \\ 2ax + 2cz = xz\lambda \\ 2bx + 2cy = xy\lambda \end{cases} \qquad \begin{array}{r} 2axy + 2xbz = xyz\lambda \\ -2axy - 2cyz = -xyz\lambda \\ \hline 2bxz - 2cyz = 0 \end{array} \quad \to bx = cy$$

$$\boxed{y = \frac{b}{c}x}$$

$$\begin{matrix} x \\ \\ -z \end{matrix}\begin{cases} 2ay + 2bz = yz\lambda \\ \\ 2bx + 2cy = xy\lambda \end{cases} \qquad \begin{array}{r} 2axy + 2bxz = xyz\lambda \\ -2bxz - 2cyz = -xyz\lambda \\ \hline 2axy - 2cyz = 0 \end{array} \quad \to \ ax = cz$$

$$\boxed{z = \frac{a}{c}x}$$

$$\begin{aligned} V &= xyz \\ &= x\left(\frac{b}{c}x\right)\left(\frac{a}{c}x\right) \\ &= \frac{ab}{c^2}x^3 \end{aligned}$$

$$x^3 = \frac{c^2}{ab}V$$

$$x = \left(\frac{c^2}{ab}V\right)^{1/3} \qquad (width)$$

$$y = \frac{b}{c}\left(\frac{c^2}{ab}V\right)^{1/3} = \left(\frac{b^2}{ac}V\right)^{1/3} \qquad (depth)$$

$$z = \frac{a}{c}\left(\frac{c^2}{ab}V\right)^{1/3} = \left(\frac{a^2}{bc}V\right)^{1/3} \qquad (height)$$

Exercise

Find the extreme values of $f(x,y,z) = x(y+z)$ on the curve of intersection of the right circular cylinder $x^2 + y^2 = 1$ and the hyperbolic cylinder $xz = 1$.

Solution

$$g(x,y,z) = x^2 + y^2 - 1$$

$$h(x,y,z) = xz - 1$$

$$\nabla f = (y+z)\hat{i} + x\hat{j} + x\hat{k}$$

$$\nabla g = 2x\hat{i} + 2y\hat{j}$$

$$\nabla h = z\hat{i} + x\hat{k}$$

$$(y+z)\hat{i} + x\hat{j} + x\hat{k} = \lambda\left(2x\hat{i} + 2y\hat{j}\right) + \mu\left(z\hat{i} + x\hat{k}\right) \qquad \nabla f = \lambda\nabla g + \mu\nabla h$$

$$\begin{cases} y + z = 2\lambda x + \mu z \\ \quad x = 2\lambda y \\ \quad x = \mu x \qquad \rightarrow\ x = 0\ \ or\ \ \mu = 1 \end{cases}$$

For $x = 0$ $\Rightarrow$ impossible since $xz = 1$

For $\mu = 1$

$$\begin{cases} y + z = 2\lambda x + z & \rightarrow\ y = 2\lambda x \\ \quad x = 2\lambda y \end{cases}$$

$$y = 2\lambda(2\lambda y)$$

$$= 4\lambda^2 y$$

$$y\left(4\lambda^2 - 1\right) = 0 \rightarrow \begin{cases} y = 0 \\ \lambda = \pm\frac{1}{2} \end{cases}$$

If $y = 0$ $\rightarrow x^2 = 1$ $\Rightarrow x = \pm 1$

$xz = 1$ $\Rightarrow$ $(1,\ 0,\ 1)$ $(-1,\ 0,\ -1)$

If $\lambda = -\frac{1}{2}$ $\rightarrow y = -x$

$$x^2 + y^2 = 2x^2 = 1 \ \rightarrow\ \boxed{x = \pm\frac{1}{\sqrt{2}}}$$

$$y = \mp\frac{1}{\sqrt{2}}$$

$$z = \frac{1}{x} = \pm\sqrt{2}$$

$$\Rightarrow \left(-\frac{1}{\sqrt{2}},\ \pm\frac{1}{\sqrt{2}},\ -\sqrt{2}\right)\ \&\ \left(\frac{1}{\sqrt{2}},\ \pm\frac{1}{\sqrt{2}},\ \sqrt{2}\right)$$

$$f(1,\ 0,\ 1) = 1$$

$$f(-1,\ 0,\ -1) = 1$$

$$f\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ -\sqrt{2}\right) = \frac{1}{2}$$

$$f\left(-\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ -\sqrt{2}\right) = \frac{3}{2}$$

$$f\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ \sqrt{2}\right) = \frac{1}{2}$$

$$f\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ \sqrt{2}\right)=\frac{3}{2}$$

Abs. Max is $\frac{3}{2}$ @ $f\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ \sqrt{2}\right)$ & $f\left(-\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ -\sqrt{2}\right)$

Abs. Min is $\frac{1}{2}$ @ $f\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ \sqrt{2}\right)$ & $f\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ -\sqrt{2}\right)$

Exercise

Find the point closest to the origin on the curve of intersection of the plane $x+y+z=1$ and the cone $z^2=2x^2+2y^2$

Solution

$$f(x,y,z)=x^2+y^2+z^2 \quad \to \nabla f=2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$g(x,y,z)=x+y+z-1 \quad \to \nabla g=\hat{i}+\hat{j}+\hat{k}$$

$$h(x,y,z)=2x^2+2y^2-z^2 \quad \to \nabla h=4x\hat{i}+4y\hat{j}-2z\hat{k}$$

$$2x\hat{i}+2y\hat{j}+2z\hat{k}=\lambda\left(\hat{i}+\hat{j}+\hat{k}\right)+\mu\left(4x\hat{i}+4y\hat{j}-2z\hat{k}\right) \qquad \nabla f=\lambda\nabla g+\mu\nabla h$$

$$=(\lambda+4\mu x)\hat{i}+(\lambda+4\mu y)\hat{j}+(\lambda-2\mu z)\hat{k}$$

$$\begin{cases} 2x=\lambda+4\mu x & \to \lambda=2x(1-2\mu) \\ 2y=\lambda+4\mu y & \to \lambda=2y(1-2\mu) \\ 2z=\lambda-2\mu z & \to \lambda=2z(1-\mu) \end{cases}$$

$$\lambda=2x(1-2\mu)=2y(1-2\mu)=2z(1-\mu)$$

$$x(1-2\mu)=y(1-2\mu) \quad \to \begin{cases} x=y \\ 1-2\mu=0 \end{cases}$$

If $x=y \ \to \ z^2=2x^2+2x^2=4x^2$

$\underline{z=\pm 2x}$

$$x+y+z=1 \ \to \begin{cases} x+x+2x=1 \Rightarrow \underline{x=\frac{1}{4}} \\ x+x-2x\neq 1 \end{cases}$$

$\therefore \ \left(\frac{1}{4},\ \frac{1}{4},\ \frac{1}{2}\right)$

If $\mu=\frac{1}{2} \ \to \ \lambda=0=2z\left(1-\frac{1}{2}\right)$

$\underline{z=0}$

$\to \ z^2=2x^2+2y^2=0$

$$x^2 + y^2 = 0 \;\rightarrow\; \underline{x = y = 0|}$$

But $x + y + z = 1$, therefore $x = y = z \neq 0$

$\therefore$ The point $\left(\frac{1}{4},\ \frac{1}{4},\ \frac{1}{2}\right)$ on the curve of intersection is closest to the origin.

Exercise

Find the point on the plane $x + 2y + 3z = 13$ closest to the point $(1,\ 1,\ 1)$

Solution

Let $f(x, y, z) = (x-1)^2 + (y-1)^2 + (z-1)^2$ (be the square of the distance from $(1,1,1)$)

$\nabla f = 2(x-1)\boldsymbol{i} + 2(y-1)\boldsymbol{j} + 2(z-1)\boldsymbol{k}$ *and* $\nabla g = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$

$\nabla f = \lambda \nabla g \;\Rightarrow\; 2(x-1)\boldsymbol{i} + 2(y-1)\boldsymbol{j} + 2(z-1)\boldsymbol{k} = \lambda\boldsymbol{i} + 2\lambda\boldsymbol{j} + 3\lambda\boldsymbol{k}$

$$\rightarrow \begin{cases} 2(x-1) = \lambda \rightarrow x = \frac{\lambda}{2} + 1 \\ 2(y-1) = 2\lambda \rightarrow y = \lambda + 1 \\ 2(z-1) = 3\lambda \rightarrow z = \frac{3\lambda}{2} + 1 \end{cases} \rightarrow \frac{\lambda}{2} + 1 + 2(\lambda + 1) + 3\left(\frac{3\lambda}{2} + 1\right) = 13$$

$$\frac{\lambda}{2} + 1 + 2\lambda + 2 + \frac{9\lambda}{2} + 3 = 13$$

$$7\lambda = 7 \;\Rightarrow\; \boxed{\lambda = 1}$$

$$x = \frac{\lambda}{2} + 1 = \frac{3}{2}, \quad y = \lambda + 1 = 2, \quad z = \frac{3\lambda}{2} + 1 = \frac{5}{2}$$

$\therefore$ The point $\left(\frac{3}{2},\ 2,\ \frac{5}{2}\right)$ is closet.

Exercise

Find the point on the sphere $x^2 + y^2 + z^2 = 4$ farthest from the point $(1,\ -1,\ 1)$

Solution

Let $f(x, y, z) = (x-1)^2 + (y+1)^2 + (z-1)^2$ (be the square of the distance from $(1,-1,1)$)

$\nabla f = 2(x-1)\boldsymbol{i} + 2(y+1)\boldsymbol{j} + 2(z-1)\boldsymbol{k}$ *and* $\nabla g = 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k}$

$\nabla f = \lambda \nabla g \;\Rightarrow\; 2(x-1)\boldsymbol{i} + 2(y+1)\boldsymbol{j} + 2(z-1)\boldsymbol{k} = 2x\lambda\boldsymbol{i} + 2y\lambda\boldsymbol{j} + 2z\lambda\boldsymbol{k}$

$$\rightarrow\begin{cases} x-1=x\lambda \rightarrow x=\dfrac{1}{1-\lambda} \\ y+1=y\lambda \rightarrow y=-\dfrac{1}{1-\lambda} \\ z-1=z\lambda \rightarrow z=\dfrac{1}{1-\lambda}\end{cases} \rightarrow \left(\frac{1}{1-\lambda}\right)^2+\left(-\frac{1}{1-\lambda}\right)^2+\left(\frac{1}{1-\lambda}\right)^2=4$$

$$3\left(\frac{1}{1-\lambda}\right)^2=4 \quad \rightarrow \left(\frac{1}{1-\lambda}\right)^2=\frac{4}{3} \quad \rightarrow \frac{1}{1-\lambda}=\pm\frac{2}{\sqrt{3}}$$

$$x=\pm\frac{2}{\sqrt{3}}, \quad y=\mp\frac{2}{\sqrt{3}}, \quad z=\pm\frac{2}{\sqrt{3}}$$

$$\left(\frac{2}{\sqrt{3}},\ -\frac{2}{\sqrt{3}},\ \frac{2}{\sqrt{3}}\right) \quad \text{and} \quad \left(-\frac{2}{\sqrt{3}},\ \frac{2}{\sqrt{3}},\ -\frac{2}{\sqrt{3}}\right)$$

$$f\left(\frac{2}{\sqrt{3}},-\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}}\right)=\left(\frac{2}{\sqrt{3}}-1\right)^2+\left(-\frac{2}{\sqrt{3}}+1\right)^2+\left(\frac{2}{\sqrt{3}}-1\right)^2\approx 0.72$$

$$f\left(-\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}},-\frac{2}{\sqrt{3}}\right)=\left(-\frac{2}{\sqrt{3}}-1\right)^2+\left(\frac{2}{\sqrt{3}}+1\right)^2+\left(-\frac{2}{\sqrt{3}}-1\right)^2\approx 13.928$$

$\therefore$ The largest value of f occurs at $\left(-\frac{2}{\sqrt{3}},\ \frac{2}{\sqrt{3}},\ -\frac{2}{\sqrt{3}}\right)$ on the sphere.

Exercise

Find the minimum distance from the surface $x^2-y^2-z^2=1$ to the origin

Solution

Let $f(x,y,z)=x^2+y^2+z^2$ (be the square of the distance from origin)

$\nabla f=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}$ *and* $\nabla g=2x\mathbf{i}-2y\mathbf{j}-2z\mathbf{k}$

$\nabla f=\lambda\nabla g \quad \Rightarrow \quad 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}=2x\lambda\mathbf{i}-2y\lambda\mathbf{j}-2z\lambda\mathbf{k}$

$$\rightarrow\begin{cases} 2x=2x\lambda \\ 2y=-2y\lambda \\ 2z=-2z\lambda\end{cases} \rightarrow \quad \lambda=1 \text{ or } x=0$$

Case 1: $\lambda=1 \quad \rightarrow\begin{cases} 2y=-2y\lambda & \boxed{y=0} \\ 2z=-2z\lambda & \boxed{z=0}\end{cases} \quad x^2-y^2-z^2=1\Rightarrow \boxed{x=\pm 1}$

Case 2: $x=0 \quad \rightarrow -y^2-z^2=1$ *No solution*

$\therefore$ The points on the unit circle $y^2+z^2=1$ are the points on the surface $x^2-y^2-z^2=1$ closest to the origin.

Exercise

Find the maximum and minimum values of $f(x,y,z) = x - 2y + 5z$ on the sphere $x^2 + y^2 + z^2 = 30$

Solution

$$\nabla f = \boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k} \quad \textit{and} \quad \nabla g = 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k}$$

$$\nabla f = \lambda \nabla g \quad \Rightarrow \quad \boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k} = 2x\lambda\boldsymbol{i} + 2y\lambda\boldsymbol{j} + 2z\lambda\boldsymbol{k}$$

$$\rightarrow \begin{cases} 2x\lambda = 1 \\ 2y\lambda = -2 \\ 2z\lambda = 5 \end{cases} \rightarrow \begin{matrix} x = \frac{1}{2\lambda} \\ y = -\frac{1}{\lambda} \\ z = \frac{5}{2\lambda} \end{matrix}$$

$$\left(\frac{1}{2\lambda}\right)^2 + \left(-\frac{1}{\lambda}\right)^2 + \left(\frac{5}{2\lambda}\right)^2 = 30$$

$$\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{25}{4\lambda^2} = 30$$

$$\frac{30}{4\lambda^2} = 30 \quad \rightarrow \quad \lambda^2 = \frac{1}{4} \Rightarrow \boxed{\lambda = \pm\frac{1}{2}}$$

$$\lambda = \frac{1}{2} \quad \Rightarrow \quad \boxed{x = 1,\ y = -2,\ z = 5}$$

$$\lambda = -\frac{1}{2} \quad \Rightarrow \quad \boxed{x = -1,\ y = 2,\ z = -5}$$

$$f(1,-2,5) = 1 + 4 + 25 = 30$$

$$f(-1,2,-5) = -1 - 4 - 25 = -30$$

$\therefore$ The maximum value $f(1,-2,5) = 30$ and the minimum is $f(-1,2,-5) = -30$

Exercise

Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.

Solution

$$f(x,y,z) = x^2 + y^2 + z^2 \quad \textit{and} \quad g(x,y,z) = x + y + z - 9 = 0$$

$$\nabla f = 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k} \quad \textit{and} \quad \nabla g = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$$

$$\nabla f = \lambda \nabla g \quad \Rightarrow \quad 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k} = \lambda\boldsymbol{i} + \lambda\boldsymbol{j} + \lambda\boldsymbol{k}$$

$$\rightarrow \begin{cases} 2x = \lambda \\ 2y = \lambda \\ 2z = \lambda \end{cases} \rightarrow \quad x = y = z = \frac{1}{2\lambda}$$

$$\frac{1}{2\lambda} + \frac{1}{2\lambda} + \frac{1}{2\lambda} = 9$$

$$\frac{3}{2\lambda}=9 \quad \Rightarrow \quad \lambda=\frac{1}{6}$$

$$x=y=z=\frac{1}{2\frac{1}{6}} \ \underline{=3|}$$

Exercise

A space probe in the shape of the ellipsoid $4x^2+y^2+4z^2=16$ enters Earth's atmosphere and its surface begins to heat. After 1 hour, the temperature at the point (x,y,z) on the probe's surface is $T(x,y,z)=8x^2+4yz-16z+600$. Find the hottest point on the probe's surface.

Solution

$$\nabla T=16x\boldsymbol{i}+4z\boldsymbol{j}+(4y-16)\boldsymbol{k} \quad \text{and} \quad \nabla g=8x\boldsymbol{i}+2y\boldsymbol{j}+8z\boldsymbol{k}$$

$$\nabla f=\lambda\nabla g \quad \Rightarrow \quad 16x\boldsymbol{i}+4z\boldsymbol{j}+(4y-16)\boldsymbol{k}=8x\lambda\boldsymbol{i}+2y\lambda\boldsymbol{j}+8z\lambda\boldsymbol{k}$$

$$\to \begin{cases} 16x=8x\lambda \\ 4z=2y\lambda \\ 4y-16=8z\lambda \end{cases} \to \quad \lambda=2 \ \text{or} \ x=0$$

Case 1: $\lambda=2 \quad \to \begin{cases} 2z=y\lambda & \to 2z=2y \Rightarrow z=y \\ y-4=2z\lambda & y-4=2y(2) \end{cases} \quad 3y=-4 \to \boxed{y=-\frac{4}{3}=z}$

$$4x^2+\frac{16}{9}+4\left(\frac{16}{9}\right)=16 \quad \to x^2=\frac{16}{9} \quad \boxed{x=\pm\frac{4}{3}}$$

Case 2: $x=0 \quad \to \lambda=\frac{2z}{y} \quad \Rightarrow y-4=2z\frac{2z}{y}$

$$y^2-4y=4z^2$$

$$4x^2+y^2+4z^2=16 \quad \to \quad y^2+y^2-4y=16$$

$$2y^2-4y-16=0 \quad \to \quad \boxed{y=4, \ -2}$$

$$\begin{cases} y=4 & \to 4z^2=4^2-16=0 \Rightarrow \boxed{z=0} \\ y=-2 & \to 4z^2=(-2)^2+8=13 \Rightarrow \boxed{z=\pm\sqrt{3}} \end{cases}$$

$$T\left(-\frac{4}{3},-\frac{4}{3},-\frac{4}{3}\right)=8\left(-\frac{4}{3}\right)^2+4\left(-\frac{4}{3}\right)\left(-\frac{4}{3}\right)-16\left(-\frac{4}{3}\right)+600$$

$$\underline{\approx 642.667^\circ|}$$

$$T\left(\frac{4}{3},-\frac{4}{3},-\frac{4}{3}\right)=8\left(\frac{4}{3}\right)^2+4\left(-\frac{4}{3}\right)\left(-\frac{4}{3}\right)-16\left(-\frac{4}{3}\right)+600$$

$$\underline{\approx 642.667^\circ|}$$

$$T(0,4,0) = 0 + 0 - 0 + 600$$
$$\approx 600°$$
$$T\left(0,-2,-\sqrt{3}\right) = 0 + 4(-2)\left(-\sqrt{3}\right) - 16\left(-\sqrt{3}\right) + 600$$
$$\approx 641.6°$$
$$T\left(0,-2,\sqrt{3}\right) = 0 + 4(-2)\left(\sqrt{3}\right) - 16\left(\sqrt{3}\right) + 600$$
$$\approx 558.43°$$

$\therefore \left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$ are the hottest points on the space probe.

Exercise

Find the extreme values of $f(x,y,z) = xyz$

Subject to the constraint $\begin{cases} x + y + z = 32 \\ x - y + z = 0 \end{cases}$

Solution

$$f(x,y,z) = xyz \qquad g_1(x,y,z) = x + y + z - 32 = 0 \qquad g_2(x,y,z) = x - y + z = 0$$
$$\nabla f = yz\hat{i} + xz\hat{j} + xy\hat{k} \qquad \nabla g_1 = \hat{i} + \hat{j} + \hat{k} \qquad \nabla g_2 = \hat{i} - \hat{j} + \hat{k}$$

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

$$yz\hat{i} + xz\hat{j} + xy\hat{k} = \lambda\hat{i} + \lambda\hat{j} + \lambda\hat{k} + \mu\hat{i} + \mu\hat{j} + \mu\hat{k}$$

$$\begin{cases} yz = \lambda + \mu & (1) \\ xz = \lambda - \mu & (2) \\ xy = \lambda + \mu & (3) \end{cases} \Rightarrow \begin{cases} (1)+(2) \to & 2\lambda = yz + zx \\ (3)+(2) \to & 2\lambda = xy + zx \end{cases} \to 2\lambda = yz + zx = xy + zx$$

$$yz = xy \quad \Rightarrow \quad y = 0 \quad or \quad x = z \quad (y \neq 0)$$

Case 1:

$$\text{If } \ y = 0 \ \Rightarrow \begin{cases} g_1(x,y,z) = x + z - 32 = 0 \\ g_2(x,y,z) = x + z = 0 \qquad \to x = -z \end{cases} \qquad \cancel{-32 = 0}$$

Case 2:

$$\text{If } \ x = z \ \Rightarrow \begin{cases} g_1(x,y,z) = 2x + y - 32 = 0 \\ g_2(x,y,z) = 2x - y = 0 \qquad \to y = 2x \end{cases} \qquad 4x = 32 \to \boxed{x = 8 = z} \quad \boxed{y = 16}$$

$$f(x,y,z) = xyz = (8)(16)(8) = 1024$$

The extreme point is $(8,\ 16,\ 8)$ with a value of 1024.

Exercise

Find the extreme values of $f(x,y,z) = x^2 + y^2 + z^2$

Subject to the constraint $\begin{cases} x + 2z = 6 \\ x + y = 12 \end{cases}$

Solution

$f(x,y,z) = x^2 + y^2 + z^2$ | $g_1(x,y,z) = x + 2z - 6 = 0$ | $g_2(x,y,z) = x + y - 12 = 0$

$\nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$ | $\nabla g_1 = \hat{i} + 2\hat{k}$ | $\nabla g_2 = \hat{i} + \hat{j}$

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

$$2x\hat{i} + 2y\hat{j} + 2z\hat{k} = \lambda\hat{i} + 2\lambda\hat{k} + \mu\hat{i} + \mu\hat{j}$$

$$\begin{cases} 2x = \lambda + \mu & (1) \\ 2y = \mu & (2) \\ 2z = 2\lambda & (3) \end{cases} \quad \Rightarrow \; 2x = z + 2y \; \Rightarrow \; z = 2x - 2y$$

$$\begin{cases} x + 2z = 6 \\ x + y = 12 \end{cases} \quad \begin{cases} x + 4x - 4y = 6 \\ x + y = 12 \end{cases} \quad \Rightarrow \quad \begin{cases} 5x - 4y = 6 \\ x + y = 12 \end{cases} \quad x = 6, \quad y = 6 \; \Rightarrow \; z = 0$$

$$f(x,y,z) = x^2 + y^2 + z^2 \; \underline{= 72}$$

The extreme point is $(6, \; 6, \; 0)$ with a value of 72.

Exercise

What point on the plane $x + y + 4z = 8$ is closest to the origin? Give an argument showing you have found an absolute minimum of the distance function.

Solution

It suffices to minimize the function $f(x,y,z) = x^2 + y^2 + z^2$

$$x + y + 4z = 8 \; \rightarrow \; x = 8 - y - 4z$$

$$f(y,z) = (8 - y - 4z)^2 + y^2 + z^2$$

$$\begin{aligned} f_y &= -2(8 - y - 4z) + 2y \\ &= 4y + 8z - 16 = 0 \end{aligned}$$

$$\begin{aligned} f_z &= -8(8 - y - 4z) + 2z \\ &= 8y + 34z - 64 = 0 \end{aligned}$$

$$\begin{cases} y + 2z = 4 \\ 4y + 17z = 32 \end{cases} \quad \Delta = \begin{vmatrix} 1 & 2 \\ 4 & 17 \end{vmatrix} = 9 \quad \Delta_y = \begin{vmatrix} 4 & 2 \\ 32 & 17 \end{vmatrix} = 4$$

$$y=\frac{4}{9} \quad \to z=\frac{16}{9}$$

$$x=8-\frac{4}{9}-\frac{64}{9} \ \ \boxed{=\frac{4}{9}}$$

$\therefore$ The closest point is $\left(\frac{4}{9},\ \frac{4}{9},\ \frac{16}{9}\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$f(x,\ y)=2x+y+10 \quad \textit{subject to} \quad 2(x-1)^2+4(y-1)^2=1$

Solution

$$f(x,\ y)=2x+y+10$$

$$\nabla f=2\hat{i}+\hat{j} \qquad\qquad \nabla f=f_x\hat{i}+f_y\hat{j}$$

$$g(x,\ y)=2(x-1)^2+4(y-1)^2-1$$

$$\nabla g=4(x-1)\hat{i}+8(y-1)\hat{j}$$

$$2\hat{i}+\hat{j}=\lambda\left(4(x-1)\hat{i}+8(y-1)\hat{j}\right) \qquad\qquad \nabla f=\lambda\nabla g$$

$$\begin{matrix}\hat{i}\\ \hat{j}\end{matrix}\ \begin{cases}2=4\lambda(x-1)\\ 1=8\lambda(y-1)\end{cases} \quad \to \begin{cases}x-1=\frac{1}{2\lambda}\\ y-1=\frac{1}{8\lambda}\end{cases}$$

$$2(x-1)^2+4(y-1)^2=1$$

$$2\left(\frac{1}{2\lambda}\right)^2+4\left(\frac{1}{8\lambda}\right)^2=1$$

$$\frac{1}{2\lambda^2}+\frac{1}{16\lambda^2}=1$$

$$\frac{9}{16\lambda^2}=1$$

$$\lambda^2=\frac{9}{16} \quad \to \quad \boxed{\lambda=\pm\frac{3}{4}}$$

For $\lambda=-\frac{3}{4}$

$$\begin{cases}x=-\frac{1}{2}\frac{4}{3}+1=\frac{1}{3}\\ y=-\frac{1}{8}\frac{4}{3}+1=\frac{5}{6}\end{cases} \qquad \left(\frac{1}{3},\ \frac{5}{6}\right)$$

For $\lambda=\frac{3}{4}$

$$\begin{cases} x = \frac{1}{2}\frac{4}{3} + 1 = \frac{1}{3} \\ y = \frac{1}{8}\frac{4}{3} + 1 = \frac{5}{6} \end{cases} \qquad \left(\frac{5}{3},\ \frac{7}{6}\right)$$

$$f\left(\frac{1}{3},\ \frac{5}{6}\right) = \frac{2}{3} + \frac{5}{6} + 10 = \frac{23}{2}$$

$$f\left(\frac{5}{3},\ \frac{7}{6}\right) = \frac{10}{3} + \frac{7}{6} + 10 = \frac{29}{2}$$

Maximum is $\frac{29}{2}$ @ $\left(\frac{5}{3},\ \frac{7}{6}\right)$

Minimum is $\frac{23}{2}$ @ $\left(\frac{1}{3},\ \frac{5}{6}\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$f(x,\ y) = x^2y^2 \quad$ *subject to* $\quad 2x^2 + y^2 = 1$

Solution

$$\nabla f = 2xy^2\hat{i} + 2x^2y\hat{j} \qquad \nabla f = f_x\hat{i} + f_y\hat{j}$$

$$g(x,\ y) = 2x^2 + y^2 - 1$$

$$\nabla g = 4x\hat{i} + 2y\hat{j}$$

$$2xy^2\hat{i} + 2x^2y\hat{j} = \lambda\left(4x\hat{i} + 2y\hat{j}\right) \qquad \nabla f = \lambda \nabla g$$

$$\begin{matrix} \hat{i} \\ \hat{j} \end{matrix} \begin{cases} 2xy^2 = 4\lambda x \\ 2x^2y = 2\lambda y \end{cases}$$

$$2x\left(y^2 - 2\lambda\right) = 0 \quad \rightarrow \quad x = 0,\ y^2 = 2\lambda$$

$$2y\left(x^2 - \lambda\right) = 0 \quad \rightarrow \quad y = 0,\ \ x^2 = \lambda$$

$(0,\ 0)$

$$2\lambda + 2\lambda = 1 \quad \rightarrow \quad \lambda = \frac{1}{4}$$

For $\lambda = \frac{1}{4}$

$$\begin{cases} x^2 = \frac{1}{4} \rightarrow x = \pm\frac{1}{2} \\ y^2 = \frac{1}{2} \rightarrow y = \pm\frac{1}{\sqrt{2}} \end{cases} \rightarrow \quad \pm\left(\frac{1}{2},\ \frac{\sqrt{2}}{2}\right)$$

$$f\left(\pm\frac{1}{2},\ \pm\frac{\sqrt{2}}{2}\right)=\left(\pm\frac{1}{2}\right)^2\left(\pm\frac{\sqrt{2}}{2}\right)^2 \quad \boxed{=\frac{1}{8}}$$

$$\boxed{f(0,\ \pm 1)=0}$$

$$\boxed{f\left(\pm\frac{\sqrt{2}}{2},\ 0\right)=0}$$

Maximum is $\frac{1}{8}$ @ $\left(\pm\frac{1}{2},\ \pm\frac{\sqrt{2}}{2}\right)$

Minimum is 0 @ $(0,\ \pm 1)$ & $\left(\pm\frac{\sqrt{2}}{2},\ 0\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$$f(x,\ y)=x+2y \quad \textit{subject to} \quad x^2+y^2=4$$

Solution

$$\nabla f=\hat{i}+2\hat{j} \qquad\qquad \nabla f=f_x\,\hat{i}+f_y\,\hat{j}$$

$$g(x,\ y)=x^2+y^2-4$$

$$\nabla g=2x\hat{i}+2y\hat{j}$$

$$\hat{i}+2\hat{j}=\lambda\left(2x\hat{i}+2y\hat{j}\right) \qquad\qquad \nabla f=\lambda\nabla g$$

$$\begin{array}{l} \hat{i} \\ \hat{j} \end{array}\begin{cases} 1=2\lambda x & \rightarrow 2\lambda=\frac{1}{x} \\ 2=2\lambda y & \rightarrow 2\lambda=\frac{2}{y} \end{cases}$$

$$2\lambda=\frac{1}{x}=\frac{2}{y} \quad\Rightarrow\quad y=2x$$

$$x^2+(2x)^2=4$$

$$5x^2=4 \quad\rightarrow\quad \boxed{x=\pm\frac{2}{\sqrt{5}}} \Rightarrow \boxed{y=\pm\frac{4}{\sqrt{5}}}$$

The points: $\pm\left(\frac{2}{\sqrt{5}},\ \frac{4}{\sqrt{5}}\right)$

$$f\left(\frac{2}{\sqrt{5}},\ \frac{4}{\sqrt{5}}\right)=\frac{2}{\sqrt{5}}+\frac{8}{\sqrt{5}}$$

$$=\frac{10}{\sqrt{5}}$$

$$\underline{= 2\sqrt{5}}\,\big|$$

$$f\left(-\frac{2}{\sqrt{5}},\ -\frac{4}{\sqrt{5}}\right) = -\frac{2}{\sqrt{5}} - \frac{8}{\sqrt{5}}$$

$$= -\frac{10}{\sqrt{5}}$$

$$\underline{= -2\sqrt{5}}\,\big|$$

Maximum is $2\sqrt{5}$ @ $\left(\frac{2}{\sqrt{5}},\ \frac{4}{\sqrt{5}}\right)$

Minimum is $-2\sqrt{5}$ @ $\left(-\frac{2}{\sqrt{5}},\ -\frac{4}{\sqrt{5}}\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$$f(x, y) = xy^2 \quad \textit{subject to} \quad x^2 + y^2 = 1$$

Solution

$\nabla f = y^2\hat{i} + 2xy\hat{j}$ $\qquad \nabla f = f_x\,\hat{i} + f_y\,\hat{j}$

$g(x, y) = x^2 + y^2 - 1$

$\nabla g = 2x\hat{i} + 2y\hat{j}$

$y^2\hat{i} + 2xy\hat{j} = \lambda\left(2x\hat{i} + 2y\hat{j}\right)$ $\qquad \nabla f = \lambda \nabla g$

$$\begin{matrix} \hat{i} \\ \hat{j} \end{matrix} \begin{cases} y^2 = 2\lambda x & (1) \\ 2xy = 2\lambda y & \rightarrow y = 0,\ x = \lambda \end{cases}$$

For $y = 0 = \lambda$

$x^2 + 0 = 1 \quad \rightarrow \quad \underline{x = \pm 1}|$

The points: $(\pm 1,\ 0)$

For $\lambda = x$

$(1) \rightarrow y^2 = 2x^2$

$x^2 + 2x^2 = 1$

$3x^2 = 1 \quad \rightarrow \quad \underline{x = \pm\frac{1}{\sqrt{3}}}|$

$y^2 = 2\left(\frac{1}{\sqrt{3}}\right)^2 \rightarrow \lfloor y = \pm\sqrt{\frac{2}{3}} \underline{= \pm\frac{\sqrt{6}}{3}}|$

The points: $\left(\pm\frac{\sqrt{3}}{3},\ \pm\frac{\sqrt{6}}{3}\right)$

	$f(x, y) = xy^2$
$(1,\ 0)$	0
$(-1,\ 0)$	0
$\left(\frac{\sqrt{3}}{3},\ \frac{\sqrt{6}}{3}\right)$	$\frac{\sqrt{3}}{3}\frac{2}{3} = \frac{2\sqrt{3}}{9}$
$\left(\frac{\sqrt{3}}{3},\ -\frac{\sqrt{6}}{3}\right)$	$\frac{\sqrt{3}}{3}\frac{2}{3} = \frac{2\sqrt{3}}{9}$
$\left(-\frac{\sqrt{3}}{3},\ \frac{\sqrt{6}}{3}\right)$	$-\frac{\sqrt{3}}{3}\frac{2}{3} = -\frac{2\sqrt{3}}{9}$
$\left(-\frac{\sqrt{3}}{3},\ -\frac{\sqrt{6}}{3}\right)$	$-\frac{\sqrt{3}}{3}\frac{2}{3} = -\frac{2\sqrt{3}}{9}$

Maximum is $\frac{2\sqrt{3}}{9}$ @ $\left(\frac{\sqrt{3}}{3},\ \pm\frac{\sqrt{6}}{3}\right)$

Minimum is $-\frac{2\sqrt{3}}{9}$ @ $\left(\pm\frac{\sqrt{3}}{3},\ \frac{\sqrt{6}}{3}\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$$f(x, y) = x + y \quad \textit{subject to} \quad x^2 - xy + y^2 = 1$$

Solution

$\nabla f = \hat{i} + \hat{j}$ $\qquad\qquad$ $\nabla f = f_x\,\hat{i} + f_y\,\hat{j}$

$g(x, y) = x^2 + y^2 - xy - 1$

$\nabla g = (2x - y)\,\hat{i} + (2y - x)\,\hat{j}$

$\hat{i} + \hat{j} = \lambda(2x - y)\,\hat{i} + (2y - x)\,\hat{j}$ $\qquad\qquad$ $\nabla f = \lambda \nabla g$

$$\begin{matrix}\hat{i} \\ \hat{j}\end{matrix}\ \begin{cases} 1 = \lambda(2x - y) & (1) \\ 1 = \lambda(2y - x) & (2) \end{cases}$$

$$\lambda = \frac{1}{2x - y} = \frac{1}{2y - x}$$

$$2y - x = 2x - y \quad \to \quad \underline{y = x|}$$

$x^2 - xy + y^2 = 1$

$x^2 - x^2 + x^2 = 1$

$x^2 = 1 \quad \rightarrow \quad \underline{x = \pm 1 = y}|$

The points: $\pm(1,\ 1)$

	$f(x,\ y) = x + y$
$(1,\ 1)$	2
$(-1,\ -1)$	-2

Maximum is 2 @ $(1,\ 1)$

Minimum is -2 @ $(-1,\ -1)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$$f(x,\ y) = x^2 + y^2 \quad \textit{subject to} \quad 2x^2 + 3xy + 2y^2 = 7$$

Solution

$\nabla f = 2x\hat{i} + 2y\hat{j}$ $\qquad\qquad$ $\nabla f = f_x \hat{i} + f_y \hat{j}$

$g(x,\ y) = 2x^2 + 3xy + 2y^2 - 7$

$\nabla g = (4x + 3y)\hat{i} + (3x + 4y)\hat{j}$

$2x\hat{i} + 2y\hat{j} = \lambda\left((4x + 3y)\hat{i} + (3x + 4y)\hat{j}\right)$ $\qquad\qquad$ $\nabla f = \lambda \nabla g$

$$\begin{matrix} \hat{i} \\ \hat{j} \end{matrix} \begin{cases} 2x = \lambda(4x + 3y) \\ 2y = \lambda(3x + 4y) \end{cases} \rightarrow \lambda = \frac{2x}{4x + 3y} = \frac{2y}{3x + 4y}$$

$3x^2 + 4xy = 4xy + 3y^2$

$x^2 = y^2 \quad \rightarrow \quad x = \pm y$

For $x = y$

$2x^2 + 3xy + 2y^2 = 7$

$2y^2 + 3y^2 + 2y^2 = 7$

$7y^2 = 7$

$y^2 = 1 \quad \rightarrow \quad \underline{y = \pm 1 = x}|$

The points: $(\pm 1,\ \pm 1)$

For $x = -y$

$$2y^2 - 3y^2 + 2y^2 = 7$$

$$y^2 = 7 \quad \to \quad \underline{y = \pm\sqrt{7} = -x|}$$

The points: $\left(\sqrt{7},\ -\sqrt{7}\right)$ & $\left(-\sqrt{7},\ \sqrt{7}\right)$

	$f(x, y) = x^2 + y^2$
$(1,\ 1)$	2
$(-1,\ -1)$	2
$\left(-\sqrt{7},\ \sqrt{7}\right)$	$7 + 7 = 14$
$\left(\sqrt{7},\ -\sqrt{7}\right)$	$7 + 7 = 14$

Maximum is 14 @ $\left(\sqrt{7},\ -\sqrt{7}\right)$ & $\left(-\sqrt{7},\ \sqrt{7}\right)$

Minimum is 2 @ $(1,\ 1),\ (-1,\ -1)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$$f(x, y) = xy \quad \textit{subject to} \quad x^2 + y^2 - xy = 9$$

Solution

$$\nabla f = y\hat{i} + x\hat{j} \qquad\qquad \nabla f = f_x\,\hat{i} + f_y\,\hat{j}$$

$$g(x, y) = x^2 + y^2 - xy - 9$$

$$\nabla g = (2x - y)\,\hat{i} + (2y - x)\,\hat{j}$$

$$y\hat{i} + x\hat{j} = \lambda\left((2x - y)\,\hat{i} + (2y - x)\,\hat{j}\right) \qquad\qquad \nabla f = \lambda \nabla g$$

$$\begin{matrix} \hat{i} \\ \hat{j} \end{matrix} \begin{cases} y = \lambda(2x - y) \\ x = \lambda(2y - x) \end{cases} \quad \to \quad \lambda = \frac{y}{2x - y} = \frac{x}{2y - x}$$

$$2y^2 - xy = 2x^2 - xy$$

$$x^2 = y^2 \quad \to \quad x = \pm y$$

For $x = y$

$$x^2 + y^2 - xy = 9$$

$$y^2 + y^2 - y^2 = 9$$

$$y^2 = 9 \quad \to \quad \underline{y = \pm 3 = x|}$$

The points: $(3,\ 3),\ (-3,\ -3)$

For $x=-y$

$$x^2+y^2-xy=9$$

$$y^2+y^2+y^2=9$$

$$y^2=3 \quad \to \quad \underline{y=\pm\sqrt{3}=-x|}$$

The points: $\left(\sqrt{3},\ -\sqrt{3}\right)$ & $\left(-\sqrt{3},\ \sqrt{3}\right)$

	$f(x,\ y)=xy$
$(3,\ 3)$	9
$(-3,\ -3)$	9
$\left(\sqrt{3},\ -\sqrt{3}\right)$	-3
$\left(-\sqrt{3},\ \sqrt{3}\right)$	-3

Maximum is 9 @ $(3,\ 3)$, $(-3,\ -3)$

Minimum is -3 @ $\left(\sqrt{3},\ -\sqrt{3}\right)$ & $\left(-\sqrt{3},\ \sqrt{3}\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$$f(x,\ y)=x-y \quad \textit{subject to} \quad x^2-3xy+y^2=20$$

Solution

$$\nabla f=\hat{i}-\hat{j} \qquad\qquad \nabla f=f_x\,\hat{i}+f_y\,\hat{j}$$

$$g(x,\ y)=x^2-3xy+y^2-20$$

$$\nabla g=(2x-3y)\,\hat{i}+(2y-3x)\,\hat{j}$$

$$\hat{i}-\hat{j}=\lambda\left((2x-3y)\,\hat{i}+(2y-3x)\,\hat{j}\right) \qquad\qquad \nabla f=\lambda\nabla g$$

$$\begin{matrix}\hat{i}\\ \hat{j}\end{matrix}\begin{cases}1=\lambda(2x-3y)\\ -1=\lambda(2y-3x)\end{cases} \quad \to \quad \lambda=\frac{1}{2x-3y}=-\frac{1}{2y-3x}$$

$$2y-3x=-2x+3y$$

$$\underline{-x=y\,|}$$

For $x=-y$

$$x^2-3xy+y^2=20$$

$$y^2+3y^2+y^2=20$$

$y^2 = 4 \quad \to \quad \underline{y = \pm 2 = -x}$

The points: $(2,\ -2)$ & $(-2,\ 2)$

	$f(x, y) = x - y$
$(2,\ -2)$	4
$(-2,\ 2)$	-4

Maximum is 4 @ $(2,\ -2)$

Minimum is -4 @ $(-2,\ 2)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$$f(x, y) = e^{2xy} \quad \textit{subject to} \quad x^2 + y^2 = 16$$

Solution

$\nabla f = 2ye^{2xy}\hat{i} + 2xe^{2xy}\hat{j} \qquad \nabla f = f_x\hat{i} + f_y\hat{j}$

$g(x, y) = x^2 + y^2 - 16$

$\nabla g = 2x\hat{i} + 2y\hat{j}$

$2ye^{2xy}\hat{i} + 2xe^{2xy}\hat{j} = \lambda\left(2x\hat{i} + 2y\hat{j}\right) \qquad \nabla f = \lambda \nabla g$

$$\begin{matrix} \hat{i} \\ \hat{j} \end{matrix} \begin{cases} 2ye^{2xy} = 2\lambda x \\ 2xe^{2xy} = 2\lambda y \end{cases} \to \lambda = \frac{ye^{2xy}}{x} = \frac{xe^{2xy}}{y}$$

$x^2 = y^2 \ \to \ x = \pm y$

For $x = y$

$x^2 + y^2 = 16$

$y^2 + y^2 = 16$

$y^2 = 8 \quad \to \quad \underline{y = \pm 2\sqrt{2} = x}$

The points: $\left(2\sqrt{2},\ 2\sqrt{2}\right)$ & $\left(-2\sqrt{2},\ -2\sqrt{2}\right)$

For $x = -y$

$x^2 + y^2 = 16$

$y^2 + y^2 = 16$

$y^2 = 8 \quad \to \quad \underline{y = \pm 2\sqrt{2} = -x}$

The points: $\left(2\sqrt{2},\ -2\sqrt{2}\right)$ & $\left(-2\sqrt{2},\ 2\sqrt{2}\right)$

	$f(x,y)=e^{2xy}$
$\left(2\sqrt{2},\ 2\sqrt{2}\right)$	e^{16}
$\left(-2\sqrt{2},\ -2\sqrt{2}\right)$	e^{16}
$\left(2\sqrt{2},\ -2\sqrt{2}\right)$	e^{-16}
$\left(-2\sqrt{2},\ 2\sqrt{2}\right)$	e^{-16}

Maximum is e^{16} @ $\left(2\sqrt{2},\ 2\sqrt{2}\right)$ & $\left(-2\sqrt{2},\ -2\sqrt{2}\right)$

Minimum is e^{-16} @ $\left(2\sqrt{2},\ -2\sqrt{2}\right)$ & $\left(-2\sqrt{2},\ 2\sqrt{2}\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$$f(x,y)=x^2+y^2 \quad \textit{subject to} \quad x^6+y^6=1$$

Solution

$$\nabla f=2x\hat{i}+2y\hat{j} \qquad\qquad \nabla f=f_x\hat{i}+f_y\hat{j}$$

$$g(x,y)=x^6+y^6-1$$

$$\nabla g=6x^5\hat{i}+6y^5\hat{j}$$

$$2x\hat{i}+2y\hat{j}=\lambda\left(6x^5\hat{i}+6y^5\hat{j}\right) \qquad\qquad \nabla f=\lambda\nabla g$$

$$\begin{matrix}\hat{i}\\ \hat{j}\end{matrix}\ \begin{cases}x=3\lambda x^5\\ y=3\lambda y^5\end{cases} \ \to\ \lambda=\frac{1}{x^4}=\frac{1}{y^4} \quad or \quad x=0 \quad or \quad y=0$$

$$x^4=y^4 \ \to\ x=\pm y$$

For $x=y$

$$x^6+y^6=1$$

$$y^6=\frac{1}{2} \ \to\ \boxed{y=\pm\frac{1}{\sqrt[6]{2}}=x}$$

The points: $\left(\frac{1}{\sqrt[6]{2}},\ \frac{1}{\sqrt[6]{2}}\right)$ & $\left(-\frac{1}{\sqrt[6]{2}},\ -\frac{1}{\sqrt[6]{2}}\right)$

For $x=-y$

$$y^6 = \frac{1}{2} \quad \rightarrow \quad \boxed{y = \pm\frac{1}{\sqrt[6]{2}} = -x}$$

The points: $\left(-\frac{1}{\sqrt[6]{2}},\ \frac{1}{\sqrt[6]{2}}\right)$ & $\left(\frac{1}{\sqrt[6]{2}},\ -\frac{1}{\sqrt[6]{2}}\right)$

For $x = 0$

$$y^6 = 1 \quad \rightarrow \quad \boxed{y = \pm 1}$$

The points: $(0,\ \pm 1)$

For $y = 0$

$$x^6 = 1 \quad \rightarrow \quad \boxed{x = \pm 1}$$

The points: $(\pm 1,\ 0)$

	$f(x,\ y) = x^2 + y^2$
$\left(\frac{1}{\sqrt[6]{2}},\ \frac{1}{\sqrt[6]{2}}\right)$	$\frac{1}{2^{1/3}} + \frac{1}{2^{1/3}} = \frac{2}{2^{1/3}} = 2^{2/3}$
$\left(-\frac{1}{\sqrt[6]{2}},\ -\frac{1}{\sqrt[6]{2}}\right)$	$2^{2/3}$
$\left(-\frac{1}{\sqrt[6]{2}},\ \frac{1}{\sqrt[6]{2}}\right)$	$2^{2/3}$
$\left(\frac{1}{\sqrt[6]{2}},\ -\frac{1}{\sqrt[6]{2}}\right)$	$2^{2/3}$
$(0,\ \pm 1)$	1
$(\pm 1,\ 0)$	1

Maximum is $2^{2/3}$ @ $\left(\frac{1}{\sqrt[6]{2}},\ \frac{1}{\sqrt[6]{2}}\right),\ \left(-\frac{1}{\sqrt[6]{2}},\ -\frac{1}{\sqrt[6]{2}}\right),\ \left(-\frac{1}{\sqrt[6]{2}},\ \frac{1}{\sqrt[6]{2}}\right)$ & $\left(\frac{1}{\sqrt[6]{2}},\ -\frac{1}{\sqrt[6]{2}}\right)$

Minimum is 1 @ $(0,\ \pm 1)$ & $(\pm 1,\ 0)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$$f(x,\ y) = y^2 - 4x^2 \quad \textit{subject to} \quad x^2 + 2y^2 = 4$$

Solution

$$\nabla f = -8x\hat{i} + 2y\hat{j} \qquad\qquad \nabla f = f_x\,\hat{i} + f_y\,\hat{j}$$

$$g(x,\ y) = x^2 + 2y^2 - 4$$

$\nabla g = 2x\hat{i} + 4y\hat{j}$

$-8x\hat{i} + 2y\hat{j} = \lambda\left(2x\hat{i} + 4y\hat{j}\right)$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \nabla f = \lambda \nabla g$

$$\begin{array}{l} \hat{i} \\ \hat{j} \end{array} \begin{cases} -4x = \lambda x & \rightarrow x = 0, \quad \lambda = -4 \ (x \neq 0) \\ y = 2\lambda y & \rightarrow y = 0, \quad \lambda = \frac{1}{2} \ (y \neq 0) \end{cases}$$

$x^2 + 2y^2 = 4$

For $x = 0$

$y^2 = 2 \quad \rightarrow \quad \underline{y = \pm\sqrt{2}|}$

The points: $\left(0, \ \pm\sqrt{2}\right)$

For $y = 0$

$x^2 = 4 \quad \rightarrow \quad \underline{x = \pm 2|}$

The points: $(\pm 2, \ 0)$

For $\lambda = -4 = \frac{1}{2}$ contradiction

	$f(x, y) = y^2 - 4x^2$
$\left(0, \ \pm\sqrt{2}\right)$	2
$(\pm 2, \ 0)$	-16

Maximum is 2 @ $\left(0, \ \pm\sqrt{2}\right)$

Minimum is -16 @ $(\pm 2, \ 0)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$$f(x, y) = xy + x + y \quad \textit{subject to} \quad x^2y^2 = 4$$

Solution

$\nabla f = (y+1)\hat{i} + (x+1)\hat{j}$ $\quad\quad\quad\quad \nabla f = f_x\,\hat{i} + f_y\,\hat{j}$

$g(x, y) = x^2y^2 - 4$

$\nabla g = 2xy^2\hat{i} + 2yx^2\hat{j}$

$(y+1)\hat{i} + (x+1)\hat{j} = \lambda\left(2xy^2\hat{i} + 2yx^2\hat{j}\right)$ $\quad\quad\quad\quad\quad\quad \nabla f = \lambda \nabla g$

$$\begin{array}{l} \hat{i} \\ \hat{j} \end{array} \begin{cases} y+1 = 2\lambda xy^2 \\ x+1 = 2\lambda yx^2 \end{cases} \rightarrow \lambda = \frac{y+1}{2xy^2} = \frac{x+1}{2yx^2}$$

$$2x^2y^2 + 2yx^2 = 2x^2y^2 + 2xy^2$$

$$x^2y = xy^2 \quad \rightarrow \quad \begin{cases} x = 0 \\ y = 0 \\ x = y \end{cases}$$

$$x^2y^2 = 4$$

For $x = 0$ & $y = 0$

$$0 = 4 \quad \rightarrow \quad \textit{Impossible}$$

For $x = y$

$$y^4 = 4 \quad \rightarrow \quad \underline{y = \pm\sqrt{2} = x|}$$

The points: $\left(\pm\sqrt{2},\ \pm\sqrt{2}\right)$

	$f(x, y) = xy + x + y$
$\left(\sqrt{2},\ \sqrt{2}\right)$	$2 + 2\sqrt{2}$
$\left(-\sqrt{2},\ -\sqrt{2}\right)$	$2 - 2\sqrt{2}$

Maximum is $2 + 2\sqrt{2}$ @ $\left(\sqrt{2},\ \sqrt{2}\right)$

Minimum is $2 - 2\sqrt{2}$ @ $\left(-\sqrt{2},\ -\sqrt{2}\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$$f(x, y, z) = x + 3y - z \quad \textit{subject to} \quad x^2 + y^2 + z^2 = 4$$

Solution

$$\nabla f = \hat{i} + 3\hat{j} - \hat{k} \qquad\qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$g(x, y) = x^2 + y^2 + z^2 - 4$$

$$\nabla g = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\hat{i} + 3\hat{j} - \hat{k} = \lambda\left(2x\hat{i} + 2y\hat{j} + 2z\hat{k}\right) \qquad\qquad \nabla f = \lambda \nabla g$$

$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \begin{cases} 1 = 2\lambda x \\ 3 = 2\lambda y \\ -1 = 2\lambda z \end{cases} \rightarrow \lambda = \frac{1}{2x} = \frac{3}{2y} = -\frac{1}{2z}$$

$$y = 3x, \quad y = -3z, \quad x = -z \quad x,\ y,\ z \neq 0$$

$$x^2 + y^2 + z^2 = 4$$

For $y = 3x$ & $z = -x$

$$x^2 + 9x^2 + x^2 = 4$$

$$11x^2 = 4 \quad \rightarrow \quad \boxed{x = \pm \frac{2}{\sqrt{11}}}$$

$$y = \pm \frac{6}{\sqrt{11}} \quad z = \mp \frac{2}{\sqrt{11}}$$

The points: $\left(\pm \frac{2}{\sqrt{11}},\ \pm \frac{6}{\sqrt{11}},\ \mp \frac{2}{\sqrt{11}} \right)$

	$f(x,\ y,\ z) = x + 3y - z$
$\left(\frac{2}{\sqrt{11}},\ \frac{6}{\sqrt{11}},\ -\frac{2}{\sqrt{11}} \right)$	$\frac{2}{\sqrt{11}} + \frac{18}{\sqrt{11}} + \frac{2}{\sqrt{11}} = \frac{22}{\sqrt{11}} = 2\sqrt{11}$
$\left(-\frac{2}{\sqrt{11}},\ -\frac{6}{\sqrt{11}},\ \frac{2}{\sqrt{11}} \right)$	$-\frac{2}{\sqrt{11}} - \frac{18}{\sqrt{11}} - \frac{2}{\sqrt{11}} = -2\sqrt{11}$

Maximum is $2\sqrt{11}$ @ $\left(\frac{2}{\sqrt{11}},\ \frac{6}{\sqrt{11}},\ -\frac{2}{\sqrt{11}} \right)$

Minimum is $-2\sqrt{11}$ @ $\left(-\frac{2}{\sqrt{11}},\ -\frac{6}{\sqrt{11}},\ \frac{2}{\sqrt{11}} \right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$$f(x,\ y,\ z) = xyz \quad \textit{subject to} \quad x^2 + 2y^2 + 4z^2 = 9$$

Solution

$$\nabla f = yz\hat{i} + xz\hat{j} + xy\hat{k} \qquad\qquad \nabla f = f_x\,\hat{i} + f_y\,\hat{j} + f_z\,\hat{k}$$

$$g(x,\ y) = x^2 + 2y^2 + 4z^2 - 9$$

$$\nabla g = 2x\hat{i} + 4y\hat{j} + 8z\hat{k}$$

$$yz\hat{i} + xz\hat{j} + xy\hat{k} = \lambda\left(2x\hat{i} + 4y\hat{j} + 8z\hat{k}\right) \qquad\qquad \nabla f = \lambda \nabla g$$

$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \begin{cases} yz = 2\lambda x \\ xz = 4\lambda y \\ xy = 8\lambda z \end{cases} \rightarrow \ \lambda = \frac{yz}{2x} = \frac{xz}{4y} = \frac{xy}{8z}$$

$$2y^2 = x^2, \quad 2z^2 = y^2, \quad 4z^2 = x^2 \qquad x,\ y,\ z \neq 0$$

$$x^2 + 2y^2 + 4z^2 = 9$$

For $y^2 = 2z^2$ & $x^2 = 4z^2$

$$4z^2 + 4z^2 + 4z^2 = 9$$

$$12z^2 = 9 \quad \rightarrow \quad \boxed{z = \pm\frac{3}{2\sqrt{3}} = \pm\frac{\sqrt{3}}{2}}$$

$$y^2 = \frac{9}{6} \quad \rightarrow y = \pm\frac{3}{\sqrt{6}} = \pm\frac{\sqrt{6}}{2}$$

$$x^2 = 3 \quad \rightarrow x = \pm\sqrt{3}$$

The points: $\left(\pm\frac{\sqrt{3}}{2},\ \pm\frac{\sqrt{6}}{2},\ \pm\sqrt{3}\right)$

	$f(x,\ y,\ z) = xyz$
$\left(\frac{\sqrt{3}}{2},\ \frac{\sqrt{6}}{2},\ \sqrt{3}\right)$	$\frac{\sqrt{54}}{4} = \frac{3\sqrt{6}}{4}$
$\left(-\frac{\sqrt{3}}{2},\ -\frac{\sqrt{6}}{2},\ -\sqrt{3}\right)$	$-\frac{3\sqrt{6}}{4}$

Maximum is $\frac{3\sqrt{6}}{4}$ @ $\left(\frac{\sqrt{3}}{2},\ \frac{\sqrt{6}}{2},\ \sqrt{3}\right)$

Minimum is $-\frac{3\sqrt{6}}{4}$ @ $\left(-\frac{\sqrt{3}}{2},\ -\frac{\sqrt{6}}{2},\ -\sqrt{3}\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$$f(x,\ y,\ z) = x \quad \textit{subject to} \quad x^2 + y^2 + z^2 - z = 1$$

Solution

$$\nabla f = \hat{i} \qquad\qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$g(x,\ y) = x^2 + y^2 + z^2 - z - 1$$

$$\nabla g = 2x\hat{i} + 2y\hat{j} + (2z - 1)\hat{k}$$

$$\hat{i} = \lambda\left(2x\hat{i} + 2y\hat{j} + (2z-1)\hat{k}\right) \qquad\qquad \nabla f = \lambda \nabla g$$

$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \begin{cases} 1 = 2\lambda x \\ 0 = 2\lambda y \\ 0 = \lambda(2z-1) \end{cases} \rightarrow \lambda = \frac{1}{2x}, \quad y = 0 \text{ or } z = \frac{1}{2}$$

$$x^2 + y^2 + z^2 - z = 1$$

For $y = 0$ & $z = \frac{1}{2}$

$$x^2 + \frac{1}{4} - \frac{1}{2} = 1$$

$$x^2 = \frac{5}{4} \rightarrow \underline{x = \pm\frac{\sqrt{5}}{2}}$$

The points: $\left(\pm\frac{\sqrt{5}}{2}, 0, \frac{1}{2}\right)$

	$f(x, y, z) = x$
$\left(\frac{\sqrt{5}}{2}, 0, \frac{1}{2}\right)$	$\frac{\sqrt{5}}{2}$
$\left(-\frac{\sqrt{5}}{2}, 0, \frac{1}{2}\right)$	$-\frac{\sqrt{5}}{2}$

Maximum is $\frac{\sqrt{5}}{2}$ @ $\left(\frac{\sqrt{5}}{2}, 0, \frac{1}{2}\right)$

Minimum is $-\frac{\sqrt{5}}{2}$ @ $\left(-\frac{\sqrt{5}}{2}, 0, \frac{1}{2}\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$$f(x, y, z) = x - z \quad \text{subject to} \quad x^2 + y^2 + z^2 - y = 2$$

Solution

$$\nabla f = \hat{i} - \hat{k} \qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$g(x, y) = x^2 + y^2 + z^2 - y - 2$$

$$\nabla g = 2x\hat{i} + (2y-1)\hat{j} + 2z\hat{k}$$

$$\hat{i} - \hat{k} = \lambda\left(2x\hat{i} + (2y-1)\hat{j} + 2z\hat{k}\right) \qquad \nabla f = \lambda\nabla g$$

$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \begin{cases} 1 = 2\lambda x \\ 0 = \lambda(2y-1) \\ -1 = 2\lambda z \end{cases} \rightarrow \lambda = \frac{1}{2x} = -\frac{1}{2z}, \quad y = \frac{1}{2}$$

$\boxed{z = -x}$

$x^2 + y^2 + z^2 - y = 2$

For $y = \frac{1}{2}$ & $z = -x$

$$x^2 + \frac{1}{4} + x^2 - \frac{1}{2} = 2$$

$$2x^2 = \frac{9}{4} \quad \rightarrow \quad \boxed{x = \pm\frac{3}{2\sqrt{2}} = \pm\frac{3\sqrt{2}}{4} = -z}$$

The points: $\left(\pm\frac{3\sqrt{2}}{4},\ \frac{1}{2},\ \mp\frac{3\sqrt{2}}{4}\right)$

	$f(x,\ y,\ z) = x - z$
$\left(\frac{3\sqrt{2}}{4},\ \frac{1}{2},\ -\frac{3\sqrt{2}}{4}\right)$	$\frac{3\sqrt{2}}{2}$
$\left(-\frac{3\sqrt{2}}{4},\ \frac{1}{2},\ \frac{3\sqrt{2}}{4}\right)$	$-\frac{3\sqrt{2}}{2}$

Maximum is $\frac{3\sqrt{2}}{2}$ @ $\left(\frac{3\sqrt{2}}{4},\ \frac{1}{2},\ -\frac{3\sqrt{2}}{4}\right)$

Minimum is $-\frac{3\sqrt{2}}{2}$ @ $\left(-\frac{3\sqrt{2}}{4},\ \frac{1}{2},\ \frac{3\sqrt{2}}{4}\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$$f(x,\ y,\ z) = x^2 + y^2 + z^2 \quad \textit{subject to} \quad x^2 + y^2 + z^2 - 4xy = 1$$

Solution

$\nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$ $\qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$

$g(x,\ y) = x^2 + y^2 + z^2 - 4xy - 1$

$\nabla g = (2x - 4y)\hat{i} + (2y - 4x)\hat{j} + 2z\hat{k}$

$2x\hat{i} + 2y\hat{j} + 2z\hat{k} = \lambda\left((2x - 4y)\hat{i} + (2y - 4x)\hat{j} + 2z\hat{k}\right)$ $\qquad \nabla f = \lambda \nabla g$

$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \begin{cases} 2x = \lambda(2x - 4y) \\ 2y = \lambda(2y - 4x) \\ \quad 2z = 2\lambda z & \rightarrow z = 0 \text{ or } \lambda = 1 \end{cases} \quad \rightarrow \lambda = \frac{x}{x - 2y} = \frac{y}{y - 2x}$$

$x^2 + y^2 + z^2 - 4xy = 1$

For $z = 0$

$$2z = 2z \;\rightarrow\; z = \pm 1$$

$$xy - 2x^2 = xy - 2y^2 \;\;\Rightarrow\;\; x^2 = y^2 \rightarrow \boxed{x = \pm y}$$

For $x = y$

$$x^2 + x^2 - 4x^2 = 1$$

$$-2x^2 = 1 \;\rightarrow x^2 = -\frac{1}{2} \quad \textit{impossible}$$

For $x = -y$

$$x^2 + x^2 + 4x^2 = 1$$

$$6x^2 = 1 \;\rightarrow \boxed{x = \pm\frac{1}{\sqrt{6}} = -y}$$

The points: $\left(\pm\frac{\sqrt{6}}{6},\ \mp\frac{\sqrt{6}}{6},\ 0\right)$

For $\lambda = 1$

$$\lambda = \frac{x}{x-2y} = \frac{y}{y-2x} = 1$$

$$x - 2y = x \;\rightarrow\; y = 0$$

$$y = y - 2x \;\rightarrow\; x = 0$$

$$z^2 = 1 \;\rightarrow\; z = \pm 1$$

The points: $(0,\ 0,\ \pm 1)$

	$f(x,\ y,\ z) = x^2 + y^2 + z^2$
$\left(\frac{\sqrt{6}}{6},\ -\frac{\sqrt{6}}{6},\ 0\right)$	$\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$
$\left(-\frac{\sqrt{6}}{6},\ \frac{\sqrt{6}}{6},\ 0\right)$	$\frac{1}{3}$
$(0,\ 0,\ \pm 1)$	1

Maximum is 1 @ $(0,\ 0,\ \pm 1)$

Minimum is $\frac{1}{3}$ @ $\left(-\frac{\sqrt{6}}{6},\ \frac{\sqrt{6}}{6},\ 0\right)$ & $\left(\frac{\sqrt{6}}{6},\ -\frac{\sqrt{6}}{6},\ 0\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$$f(x, y, z) = x + y + z \quad \textit{subject to} \quad x^2 + y^2 + z^2 - 2x - 2y = 1$$

Solution

$$\nabla f = \hat{i} + \hat{j} + \hat{k} \qquad\qquad \nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$$

$$g(x, y) = x^2 + y^2 + z^2 - 2x - 2y - 1$$

$$\nabla g = (2x-2)\hat{i} + (2y-2)\hat{j} + 2z\hat{k}$$

$$\hat{i} + \hat{j} + \hat{k} = \lambda\left((2x-2)\hat{i} + (2y-2)\hat{j} + 2z\hat{k}\right) \qquad\qquad \nabla f = \lambda \nabla g$$

$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \begin{cases} 1 = \lambda(2x-2) \\ 1 = \lambda(2y-2) \\ \quad 1 = 2\lambda z \end{cases} \rightarrow \lambda = \frac{1}{2x-2} = \frac{1}{2y-2} = \frac{1}{2z}$$

$$\frac{1}{x-1} = \frac{1}{y-1} = \frac{1}{z}$$

$$x = y, \quad z = y - 1 = x - 1, \quad x, y \neq 1, \quad z \neq 0$$

$$x^2 + y^2 + z^2 - 2x - 2y = 1$$

For $x = y = z + 1$

$$(z+1)^2 + (z+1)^2 + z^2 - 2(z+1) - 2(z+1) = 1$$

$$2z^2 + 4z + 2 + z^2 - 4z - 4 = 1$$

$$3z^2 = 3 \quad \rightarrow \quad \underline{z = \pm 1}$$

$$z = -1 \quad \rightarrow \quad x = y = 0$$

$$z = 1 \quad \rightarrow \quad x = y = 2$$

The points: $(0, 0, -1)$ & $(2, 2, 1)$

	$f(x, y, z) = x + y + z$
$(0, 0, -1)$	-1
$(2, 2, 1)$	5

Maximum is 5 @ $(2, 2, 1)$

Minimum is -1 @ $(0, 0, -1)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$$f(x, y, z) = 2x + z^2 \quad \textit{subject to} \quad x^2 + y^2 + 2z^2 = 25$$

Solution

$$\nabla f = 2\hat{i} + 2z\hat{k} \qquad\qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$g(x, y) = x^2 + y^2 + 2z^2 - 25$$

$$\nabla g = 2x\hat{i} + 2y\hat{j} + 4z\hat{k}$$

$$2\hat{i} + 2z\hat{k} = \lambda\left(2x\hat{i} + 2y\hat{j} + 4z\hat{k}\right) \qquad\qquad \nabla f = \lambda \nabla g$$

$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \begin{cases} 2 = 2\lambda x & \to \lambda = \frac{1}{x} \\ 0 = 2\lambda y & \to y = 0 \\ 2z = 4\lambda z & \to \lambda = \frac{1}{2} \text{ or } z = 0 \end{cases}$$

$$x^2 + y^2 + 2z^2 = 25$$

For $\lambda = \frac{1}{2}$ $\quad y = 0$

$$x = \frac{1}{\lambda} = 2$$

$$4 + 2z^2 = 25$$

$$2z^2 = 21 \quad \to \quad z = \pm\sqrt{\frac{21}{2}}$$

The points: $\left(2,\ 0,\ \pm\sqrt{\frac{21}{2}}\right)$

For $z = 0$ $\quad y = 0$

$$x^2 = 25 \quad \to \quad x = \pm 5$$

The points: $(\pm 5,\ 0,\ 0)$

	$f(x, y, z) = 2x + z^2$
$\left(2,\ 0,\ \pm\sqrt{\frac{21}{2}}\right)$	$4 + \frac{21}{2} = \frac{29}{2}$
$(5,\ 0,\ 0)$	10
$(-5,\ 0,\ 0)$	-10

Maximum is $\frac{29}{2}$ @ $\left(2,\ 0,\ \pm\sqrt{\frac{21}{2}}\right)$

Minimum is -10 @ $(-5,\ 0,\ 0)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$$f(x, y, z) = x^2 + y^2 - z \quad \textit{subject to} \quad z = 2x^2y^2 + 1$$

Solution

$\nabla f = 2x\hat{i} + 2y\hat{j} - \hat{k}$ $\qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$

$g(x, y) = 2x^2y^2 + 1 - z$

$\nabla g = 4xy^2\hat{i} + 4x^2y\hat{j} - \hat{k}$

$2x\hat{i} + 2y\hat{j} - \hat{k} = \lambda\left(4xy^2\hat{i} + 4x^2y\hat{j} - \hat{k}\right)$ $\qquad \nabla f = \lambda \nabla g$

$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \begin{cases} 2x = 4\lambda xy^2 & \to x = 0, \ \lambda = \dfrac{1}{2y^2} \\ 2y = 4\lambda x^2 y & \to y = 0, \ \lambda = \dfrac{1}{2x^2} \\ -1 = -\lambda & \to \lambda = 1 \end{cases}$$

$z = 2x^2y^2 + 1$

For $x = 0 \quad y = 0$

$\underline{z = 1}|$

The points: $(0, \ 0, \ 1)$

For $\lambda = 1$

$\lambda = \dfrac{1}{2y^2} = \dfrac{1}{2x^2} = 1$

$x^2 = \dfrac{1}{2} \quad \to \quad \underline{x = \pm\dfrac{1}{\sqrt{2}}}|$

$y^2 = \dfrac{1}{2} \quad \to \quad \underline{y = \pm\dfrac{1}{\sqrt{2}}}|$

$\lfloor z = 2\dfrac{1}{2}\dfrac{1}{2} + 1 = \underline{\dfrac{3}{2}}|$

The points: $\left(\pm\dfrac{1}{\sqrt{2}}, \ \pm\dfrac{1}{\sqrt{2}}, \ \dfrac{3}{2}\right)$

	$f(x, y, z) = x^2 + y^2 - z$
$(0, \ 0, \ 1)$	-1
$\left(\pm\dfrac{1}{\sqrt{2}}, \ \pm\dfrac{1}{\sqrt{2}}, \ \dfrac{3}{2}\right)$	$\dfrac{1}{2} + \dfrac{1}{2} - \dfrac{3}{2} = -\dfrac{1}{2}$

Maximum is $-\frac{1}{2}$ @ $\left(-\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ \frac{3}{2}\right), \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ \frac{3}{2}\right), \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ \frac{3}{2}\right), \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ \frac{3}{2}\right)$

Minimum is -1 @ $(0,\ 0,\ 1)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$$f(x,\ y,\ z) = \sqrt{xyz} \quad \textit{subject to} \quad x + y + z = 1 \quad \textit{with} \ \ x \ge 0,\ y \ge 0,\ z \ge 0$$

Solution

$$\nabla f = \frac{1}{2}\frac{yz}{\sqrt{xyz}}\hat{i} + \frac{1}{2}\frac{xz}{\sqrt{xyz}}\hat{j} + \frac{1}{2}\frac{xy}{\sqrt{xyz}}\hat{k} \qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$g(x,\ y) = x + y + z - 1 = 0$$

$$\nabla g = \hat{i} + \hat{j} + \hat{k}$$

$$\frac{1}{2}\frac{yz}{\sqrt{xyz}}\hat{i} + \frac{1}{2}\frac{xz}{\sqrt{xyz}}\hat{j} + \frac{1}{2}\frac{xy}{\sqrt{xyz}}\hat{k} = \lambda\left(\hat{i} + \hat{j} + \hat{k}\right) \qquad \nabla f = \lambda\nabla g$$

$$\begin{array}{c} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \begin{cases} \lambda = \frac{1}{2}\frac{yz}{\sqrt{xyz}} \\ \lambda = \frac{1}{2}\frac{xz}{\sqrt{xyz}} \\ \lambda = \frac{1}{2}\frac{xy}{\sqrt{xyz}} \end{cases} \rightarrow \lambda = \frac{1}{2}\frac{yz}{\sqrt{xyz}} = \frac{1}{2}\frac{xz}{\sqrt{xyz}} = \frac{1}{2}\frac{xy}{\sqrt{xyz}}$$

$$xy = xz = yz \quad x,\ y,\ z \ \neq 0$$

$$\underline{x = y = z|}$$

$$\begin{cases} xy = xz & \rightarrow x = 0, \quad y = z \\ xy = yz & \rightarrow y = 0, \quad x = z \\ xz = yz & \rightarrow z = 0, \quad x = y \end{cases}$$

$$x + y + z = 1$$

For $x = y = z$

$$3z = 1 \quad \rightarrow \underline{z = \frac{1}{3} = y = x|}$$

The point: $\left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$

For $x = y = 0$

$$\underline{z = 1|}$$

The point: $(0,\ 0,\ 1)$

$x = z = 0 \rightarrow \quad (0,\ 1,\ 0)$

$y = z = 0 \rightarrow \quad (1,\ 0,\ 0)$

	$f(x,\ y,\ z) = \sqrt{xyz}$
$(0,\ 0,\ 1)$	0
$(0,\ 1,\ 0)$	0
$(1,\ 0,\ 0)$	0
$\left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$	$\frac{1}{3\sqrt{3}}$

Maximum is $\frac{1}{3\sqrt{3}}$ @ $\left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$

Minimum is 0 @ $(0,\ 0,\ 1),\ (0,\ 1,\ 0),\ (1,\ 0,\ 0)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$$f(x,\ y,\ z) = x^2 + y^2 + z^2 \quad \textit{subject to} \quad x^2 + y^2 + z^2 - 4xy = 1$$

Solution

$\nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$ $\qquad\qquad \nabla f = f_x\,\hat{i} + f_y\,\hat{j} + f_z\,\hat{k}$

$g(x,\ y) = x^2 + y^2 + z^2 - 4xy - 1$

$\nabla g = (2x - 4y)\,\hat{i} + (2y - 4x)\,\hat{j} + 2z\hat{k}$

$2x\hat{i} + 2y\hat{j} + 2z\hat{k} = \lambda\left((2x - 4y)\,\hat{i} + (2y - 4x)\,\hat{j} + 2z\hat{k}\right)$ $\qquad\qquad \nabla f = \lambda \nabla g$

$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \begin{cases} 2x = 2\lambda(x - 2y) & \rightarrow \lambda = \dfrac{x}{x - 2y} \\ 2y = 2\lambda(y - 2x) & \rightarrow \lambda = \dfrac{y}{y - 2x} \\ \quad 2z = 2\lambda z & \rightarrow z = 0, \quad \lambda = 1 \end{cases}$$

$\lambda = \dfrac{x}{x - 2y} = \dfrac{y}{y - 2x}$

$xy - 2x^2 = xy - 2y^2$

$x^2 = y^2 \quad \rightarrow \quad x = \pm y$

$x^2 + y^2 + z^2 - 4xy = 1$

For $z = 0$

$x = y$

$$x^2 + x^2 - 4x^2 = 1 \quad \to \quad -2x^2 = 1 \quad (\textit{impossible})$$

$x = -y$

$$x^2 + x^2 + 4x^2 = 1 \quad \to x^2 = \frac{1}{6} \quad \Rightarrow \underline{x = \pm\frac{1}{\sqrt{6}} = -y}$$

The point: $\left(\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0\right), \left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ 0\right)$

For $\lambda = 1$

$$\frac{x}{x-2y} = 1 \ \to \ \underline{y = 0}$$

$$\frac{y}{y-2x} = 1 \ \to \ \underline{x = 0}$$

$$z^2 = 1 \ \to \ \underline{z = \pm 1}$$

The point: $(0,\ 0,\ \pm 1)$

	$f(x, y, z) = x^2 + y^2 + z^2$
$\left(\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0\right)$	$\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$
$\left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ 0\right)$	$\frac{1}{3}$
$(0,\ 0,\ \pm 1)$	1

Maximum is 1 @ $(0,\ 0,\ \pm 1)$

Minimum is $\frac{1}{3}$ @ $\left(\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0\right), \left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ 0\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$f(x, y, z) = x + 2y - z$ *subject to* $x^2 + y^2 + z^2 = 1$

Solution

$$\nabla f = \hat{i} + 2\hat{j} - \hat{k} \qquad \nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$$

$$g(x, y) = x^2 + y^2 + z^2 - 1$$

$$\nabla g = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\hat{i} + 2\hat{j} - \hat{k} = \lambda\left(2x\hat{i} + 2y\hat{j} + 2z\hat{k}\right) \qquad \nabla f = \lambda \nabla g$$

$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \begin{cases} 2\lambda x = 1 & \to x = \dfrac{1}{2\lambda} \\ 2\lambda y = 2 & \to y = \dfrac{1}{\lambda} \\ 2\lambda z = -1 & \to z = -\dfrac{1}{2\lambda} \end{cases}$$

$$\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{4\lambda^2} = 1$$

$$\frac{6}{4\lambda^2} = 1 \quad \to \quad \lambda^2 = \frac{6}{4}$$

$$\boxed{\lambda = \pm \frac{\sqrt{6}}{2}}$$

For $\lambda = -\dfrac{\sqrt{6}}{2}$

$$\begin{cases} x = -\dfrac{\sqrt{6}}{6} \\ y = -\dfrac{\sqrt{6}}{3} \\ z = \dfrac{\sqrt{6}}{6} \end{cases} \quad \to \quad \left(-\frac{\sqrt{6}}{6},\ -\frac{\sqrt{6}}{3},\ \frac{\sqrt{6}}{6} \right)$$

For $\lambda = \dfrac{\sqrt{6}}{2}$

$$\begin{cases} x = \dfrac{\sqrt{6}}{6} \\ y = \dfrac{\sqrt{6}}{3} \\ z = -\dfrac{\sqrt{6}}{6} \end{cases} \quad \to \quad \left(\frac{\sqrt{6}}{6},\ \frac{\sqrt{6}}{3},\ -\frac{\sqrt{6}}{6} \right)$$

$$f\left(-\frac{\sqrt{6}}{6},\ -\frac{\sqrt{6}}{3},\ \frac{\sqrt{6}}{6} \right) = -\frac{\sqrt{6}}{6} - \frac{2\sqrt{6}}{3} - \frac{\sqrt{6}}{6} \ \boxed{= -\sqrt{6}}$$

$$f\left(\frac{\sqrt{6}}{6},\ \frac{\sqrt{6}}{3},\ -\frac{\sqrt{6}}{6} \right) = \frac{\sqrt{6}}{6} + \frac{2\sqrt{6}}{3} + \frac{\sqrt{6}}{6} \ \boxed{= \sqrt{6}}$$

Maximum is $\sqrt{6}$ @ $\left(\dfrac{\sqrt{6}}{6},\ \dfrac{\sqrt{6}}{3},\ -\dfrac{\sqrt{6}}{6} \right)$

Minimum is $-\sqrt{6}$ @ $\left(-\dfrac{\sqrt{6}}{6},\ -\dfrac{\sqrt{6}}{3},\ \dfrac{\sqrt{6}}{6} \right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$f(x, y, z) = x^2y^2z \quad$ *subject to* $\quad 2x^2 + y^2 + z^2 = 25$

Solution

$$\nabla f = 2xy^2z\hat{i} + 2x^2yz\hat{j} + x^2y^2\hat{k} \qquad\qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$g(x, y) = 2x^2 + y^2 + z^2 - 25$$

$$\nabla g = 4x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$2xy^2z\hat{i} + 2x^2yz\hat{j} + x^2y^2\hat{k} = \lambda\left(4x\hat{i} + 2y\hat{j} + 2z\hat{k}\right) \qquad\qquad \nabla f = \lambda\nabla g$$

$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array}\begin{cases} 4\lambda x = 2xy^2z & \rightarrow x = 0, \ \ \lambda = \frac{1}{2}y^2z \\ 2\lambda y = 2x^2yz & \rightarrow y = 0, \ \ \lambda = x^2z \\ 2\lambda z = x^2y^2 & \rightarrow \ \lambda = \dfrac{x^2y^2}{2z} \end{cases}$$

$$\therefore \ (0, \ 0, \ 0)$$

$$\lambda = \frac{1}{2}y^2z = x^2z \ \rightarrow \ y^2 = 2x^2$$

$$\lambda = \frac{1}{2}y^2z = \frac{x^2y^2}{2z} \ \rightarrow \ z^2 = x^2 = \frac{1}{2}y^2$$

$$2x^2 + y^2 + z^2 = 25$$

$$y^2 + y^2 + \frac{1}{2}y^2 = 25$$

$$\frac{5}{2}y^2 = 25 \ \rightarrow \ y = \pm 5\sqrt{\frac{2}{5}} \ \underline{= \pm\sqrt{10}}$$

$$x^2 = \frac{1}{2}y^2 = 5 \ \rightarrow \ \underline{x = z = \pm\sqrt{5}}$$

$$\therefore \ \left(\pm\sqrt{5}, \ \pm\sqrt{10}, \ \pm\sqrt{5}\right)$$

$$f(x, y, z) = x^2y^2z$$

$$f\left(\pm\sqrt{5}, \ \pm\sqrt{10}, \ \sqrt{5}\right) = (5)(10)\sqrt{5} \ \underline{= 50\sqrt{5}}$$

$$f\left(\pm\sqrt{5}, \ \pm\sqrt{10}, \ -\sqrt{5}\right) = (5)(10)\left(-\sqrt{5}\right) \ \underline{= -50\sqrt{5}}$$

Maximum is $50\sqrt{5}$ @ $\left(\pm\sqrt{5}, \ \pm\sqrt{10}, \ \sqrt{5}\right)$

Minimum is $-50\sqrt{5}$ @ $\left(\pm\sqrt{5}, \ \pm\sqrt{10}, \ -\sqrt{5}\right)$

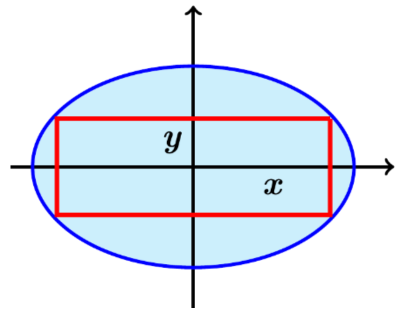
Exercise

Use Lagrange multipliers to find the dimensions of the rectangle with the maximum perimeter that can be inscribed with sides parallel to the coordinate axes in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution

Let $(x,\ y)$ be the corner of the rectangle in QI.

Perimeter: $P=4(x+y)$



$f(x,\ y)=x+y$ subject to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\nabla f=\hat{i}+\hat{j}$ $\qquad\qquad$ $\nabla f=f_x\hat{i}+f_y\hat{j}$

$g(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$

$\nabla g=\frac{2x}{a^2}\hat{i}+\frac{2y}{b^2}\hat{j}$

$\hat{i}+\hat{j}=\lambda\left(\frac{2x}{a^2}\hat{i}+\frac{2y}{b^2}\hat{j}\right)$ $\qquad\qquad$ $\nabla f=\lambda\nabla g$

$$\begin{matrix}\hat{i}\\ \hat{j}\end{matrix}\begin{cases}\frac{2x}{a^2}\lambda=1 & \rightarrow\ 2\lambda=\frac{a^2}{x}\\ \frac{2y}{b^2}\lambda=1 & \rightarrow\ 2\lambda=\frac{b^2}{y}\end{cases}$$

$\frac{a^2}{x}=\frac{b^2}{y}\quad\rightarrow\quad y=\frac{b^2}{a^2}x$

$\frac{x^2}{a^2}+\frac{1}{b^2}\frac{b^4}{a^4}x^2=1$

$\left(\frac{1}{a^2}+\frac{b^2}{a^4}\right)x^2=1$

$x^2=\frac{a^4}{a^2+b^2}\quad\Rightarrow\quad x=\frac{a^2}{\sqrt{a^2+b^2}}$ $\quad\in QI$

$\rightarrow\ y=\frac{a^2}{\sqrt{a^2+b^2}}$

Dimension of rectangle with greatest perimeter are $\frac{a^2}{\sqrt{a^2+b^2}}$ *by* $\frac{b^2}{\sqrt{a^2+b^2}}$

Exercise

Use Lagrange multipliers to find the dimensions of the right circular cylinder of minimum surface area (including the circular ends) with a volume of $32\pi\ in^3$

Solution

$V = 32\pi\ in^3$

Circular cylinder: Let r: radius h: height

Surface area: $A = 2\pi r^2 + 2\pi rh$

Volume: $V = \pi r^2 h$

Suffices to Minimize $f(r,\ h) = \pi r^2 + \pi rh$ subject to $\pi r^2 h = 32$

$$f(r,\ h) = \pi r^2 + \pi rh$$

$$\nabla f = (2r + h)\hat{i} + r\hat{j}$$

$$g(r,\ h) = \pi r^2 h - 32$$

$$\nabla g = 2rh\hat{i} + r^2\hat{j}$$

$$(2r + h)\hat{i} + r\hat{j} = 2rh\lambda\hat{i} + r^2\lambda\hat{j} \qquad \nabla f = \lambda\nabla g$$

$$\begin{cases} 2\lambda rh = 2r + h \\ \lambda r^2 = r \qquad \rightarrow \boxed{r = \frac{1}{\lambda}} \end{cases}$$

$$2\lambda\frac{1}{\lambda}h = 2\frac{1}{\lambda} + h$$

$$\boxed{h = \frac{2}{\lambda}}$$

$$\pi r^2 h = 32$$

$$\pi\frac{2}{\lambda^3} = 32$$

$$\lambda^3 = \frac{\pi}{16} \quad \rightarrow \quad \boxed{\lambda = \frac{1}{2}\sqrt[3]{\frac{\pi}{2}}}$$

$$\boxed{r = 2\sqrt[3]{\frac{2}{\pi}}\ in} \quad \boxed{h = 4\sqrt[3]{\frac{2}{\pi}}\ in}$$

Exercise

Find the point(s) on the cone $z^2 - x^2 - y^2 = 0$ that are closest to the point $(1,\ 3,\ 1)$. Give an argument showing you have found an absolute minimum of the distance function.

Solution

$$f(x, y, z) = (x-1)^2 + (y-3)^2 + (z-1)^2$$

Subject to $g(x,y,z) = z^2 - x^2 - y^2$

$\nabla f = 2(x-1)\hat{i} + 2(y-3)\hat{j} + 2(z-1)\hat{k}$

$\nabla g = -2x\hat{i} - 2y\hat{j} + 2z\hat{k}$

$\nabla f = \lambda \nabla g$

$$2(x-1) = -2\lambda x \quad \Rightarrow \quad x-1 = -\lambda x \ \rightarrow \ x = \frac{1}{1+\lambda}$$

$$2(y-3) = -2\lambda y \quad \Rightarrow \quad y-3 = -\lambda y \ \rightarrow \ x = \frac{3}{1+\lambda}$$

$$2(z-1) = 2\lambda z \quad \Rightarrow \quad z-1 = \lambda z \ \rightarrow \ z = \frac{1}{1-\lambda}$$

$$z^2 - x^2 - y^2 = 0$$

$$\frac{1}{(1-\lambda)^2} - \frac{1}{(1+\lambda)^2} - \frac{9}{(1+\lambda)^2} = 0$$

$$\frac{1}{(1-\lambda)^2} - \frac{10}{(1+\lambda)^2} = 0$$

$$1 + 2\lambda + \lambda^2 - 10 + 20\lambda - 10\lambda^2 = 0$$

$$-9\lambda^2 + 22\lambda - 9 = 0 \quad \rightarrow \quad \lambda = \frac{-22 \pm \sqrt{160}}{-18} = \frac{11 \mp 2\sqrt{10}}{9}$$

For $\lambda = \frac{11 - 2\sqrt{10}}{9}$

$$x = \frac{9}{9 + 11 - 2\sqrt{10}} = \frac{1}{2}\frac{9}{10 - \sqrt{10}}\frac{10+\sqrt{10}}{10+\sqrt{10}} = \frac{1}{20}\left(10 + \sqrt{10}\right)$$

$$y = \frac{27}{9 + 11 - 2\sqrt{10}} = \frac{1}{2}\frac{27}{10 - \sqrt{10}}\frac{10+\sqrt{10}}{10+\sqrt{10}} = \frac{3}{20}\left(10 + \sqrt{10}\right)$$

$$z = \frac{9}{9 - 11 + 2\sqrt{10}} = -\frac{1}{2}\frac{9}{1 - \sqrt{10}}\frac{1+\sqrt{10}}{1+\sqrt{10}} = \frac{1}{2}\left(1 + \sqrt{10}\right)$$

$$\therefore \left(\frac{1}{2} + \frac{\sqrt{10}}{20},\ \frac{3}{2} + \frac{3\sqrt{10}}{20},\ \frac{1}{2} + \frac{\sqrt{10}}{2}\right)$$

For $\lambda = \frac{11 + 2\sqrt{10}}{9}$

$$x = \frac{9}{9 + 11 + 2\sqrt{10}} = \frac{1}{2}\frac{9}{10 + \sqrt{10}}\frac{10-\sqrt{10}}{10-\sqrt{10}} = \frac{1}{20}\left(10 - \sqrt{10}\right)$$

$$y = \frac{27}{9 + 11 + 2\sqrt{10}} = \frac{1}{2}\frac{27}{10 + \sqrt{10}}\frac{10-\sqrt{10}}{10-\sqrt{10}} = \frac{3}{20}\left(10 - \sqrt{10}\right)$$

$$z = \frac{9}{9 - 11 - 2\sqrt{10}} = -\frac{1}{2}\frac{9}{1 + \sqrt{10}}\frac{1-\sqrt{10}}{1-\sqrt{10}} = \frac{1}{2}\left(1 - \sqrt{10}\right)$$

$$\therefore\ \left(\frac{1}{2}-\frac{\sqrt{10}}{20},\ \ \frac{3}{2}-\frac{3\sqrt{10}}{20},\ \ \frac{1}{2}-\frac{\sqrt{10}}{2}\right)$$

Therefore, there are 3 solutions to the Lagrange conditions:

$$\left(\frac{1}{2}\pm\frac{\sqrt{10}}{20},\ \ \frac{3}{2}\pm\frac{3\sqrt{10}}{20},\ \ \frac{1}{2}\pm\frac{\sqrt{10}}{2}\right)\ \&\ (0,\ 0,\ 0)$$

$$f(x,y,z)=(x-1)^2+(y-3)^2+(z-1)^2$$

$$f(0,\ 0,\ 0)\ \underline{=11\,|}$$

$$\begin{aligned}
f\left(\frac{1}{2}+\frac{\sqrt{10}}{20},\ \frac{3}{2}+\frac{3\sqrt{10}}{20},\ \frac{1}{2}+\frac{\sqrt{10}}{2}\right)&=\left(\frac{1}{2}+\frac{\sqrt{10}}{20}-1\right)^2+\left(\frac{3}{2}+\frac{3\sqrt{10}}{20}-3\right)^2+\left(\frac{1}{2}+\frac{\sqrt{10}}{2}-1\right)^2\\
&=\left(\frac{\sqrt{10}}{20}-\frac{1}{2}\right)^2+\left(\frac{3\sqrt{10}}{20}-\frac{3}{2}\right)^2+\left(\frac{\sqrt{10}}{2}-\frac{1}{2}\right)^2\\
&=\frac{1}{400}\left(\sqrt{10}-10\right)^2+\frac{9}{400}\left(\sqrt{10}-10\right)^2+\frac{1}{4}\left(\sqrt{10}-1\right)^2\\
&=\frac{1}{40}\left(110-20\sqrt{10}\right)+\frac{1}{4}\left(11-2\sqrt{10}\right)\\
&=\frac{11}{4}-\frac{1}{2}\sqrt{10}+\frac{11}{4}-\frac{1}{2}\sqrt{10}\\
&=\boxed{\frac{11}{2}-\sqrt{10}}
\end{aligned}$$

$$\begin{aligned}
f\left(\frac{1}{2}-\frac{\sqrt{10}}{20},\ \frac{3}{2}-\frac{3\sqrt{10}}{20},\ \frac{1}{2}-\frac{\sqrt{10}}{2}\right)&=\left(\frac{1}{2}-\frac{\sqrt{10}}{20}-1\right)^2+\left(\frac{3}{2}-\frac{3\sqrt{10}}{20}-3\right)^2+\left(\frac{1}{2}-\frac{\sqrt{10}}{2}-1\right)^2\\
&=\left(-\frac{\sqrt{10}}{20}-\frac{1}{2}\right)^2+\left(-\frac{3\sqrt{10}}{20}-\frac{3}{2}\right)^2+\left(-\frac{\sqrt{10}}{2}-\frac{1}{2}\right)^2\\
&=\frac{1}{400}\left(\sqrt{10}+10\right)^2+\frac{9}{400}\left(\sqrt{10}+10\right)^2+\frac{1}{4}\left(\sqrt{10}+1\right)^2\\
&=\frac{1}{40}\left(110+20\sqrt{10}\right)+\frac{1}{4}\left(11+2\sqrt{10}\right)\\
&=\frac{11}{4}+\frac{1}{2}\sqrt{10}+\frac{11}{4}+\frac{1}{2}\sqrt{10}\\
&=\boxed{\frac{11}{2}+\sqrt{10}}
\end{aligned}$$

The closest point is $\left(\frac{1}{2}+\frac{\sqrt{10}}{20},\ \ \frac{3}{2}+\frac{3\sqrt{10}}{20},\ \ \frac{1}{2}+\frac{\sqrt{10}}{2}\right)$

Exercise

Let $P_0(a,\ b,\ c)$ be a fixed point in $\mathbb{R}^3$ and let $d(x,\ y,\ z)$ be the distance between P_0 and a variable point $P(x,\ y,\ z)$.

a) Compute $\nabla d(x,\ y,\ z)$

b) Show that $\nabla d(x,\ y,\ z)$ points in the direction from P_0 to P and has magnitude 1 for all $(x,\ y,\ z)$.

c) Describe the level surfaces of d and give the direction of $\nabla d(x,\ y,\ z)$ relative to the level surfaces of d.

d) Discuss $\lim\limits_{P\to P_0} \nabla d(x,\ y,\ z)$

Solution

$$d(x,y,z) = \left|\overrightarrow{PP_0}\right|$$
$$= \sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$$

a) $\nabla d(x,y,z) = \dfrac{1}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}\langle x-a,\ \ y-b,\ \ z-c\rangle$

b) $\nabla d(x,y,z) = \dfrac{1}{d(x,y,z)}\langle x-a,\ \ y-b,\ \ z-c\rangle$

$$= \frac{1}{\left|\overrightarrow{PP_0}\right|}\overrightarrow{PP_0}$$

$$= d(x,y,z)$$

$\therefore\ \ \nabla d(x,y,z)$ is a unit vector.

c) The level surfaces of d are spheres centered at $a,\ b,\ c$ and $\nabla d(x,y,z)$ is $\perp$ to these spheres, pointing outwards.

d) $\lim\limits_{P\to P_0} \nabla d(x,y,z) = \nexists$

Because of $P \to P_0$ in the direction of a unit vector $\vec{u}$.

$\nabla d(x,y,z) = \pm\vec{u}$, but the limit must be the same in all directions.

Exercise

A shipping company requires that the sum of length plus girth of rectangular boxes must not exceed 108 *in*. Find the dimensions of the box with maximum volume that meets this condition. (the girth is the perimeter of the smallest base of the box).

Solution

Let x, y, and z represent the length, width, and height.

The girth is: $= 2y + 2z(= P)$

Volume: $V = xyz$

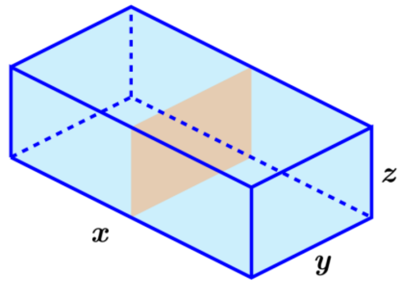
We want to maximize the volume of the box satisfying: $\quad x+2y+2z=108$

$f(x, y, z) = xyz \quad subject\ to \quad g(x, y) = x+2y+2z-108$

$\nabla f = yz\hat{i} + xz\hat{j} + xy\hat{k} \qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$

$\nabla g = \hat{i} + 2\hat{j} + 2\hat{k}$

$yz\hat{i} + xz\hat{j} + xy\hat{k} = \lambda\left(\hat{i} + 2\hat{j} + 2\hat{k}\right) \qquad \nabla f = \lambda \nabla g$



$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \begin{cases} yz = \lambda \\ xz = 2\lambda & \rightarrow \lambda = yz = \dfrac{xz}{2} = \dfrac{xy}{2} \quad x, y, z \neq 0 \\ xy = 2\lambda \end{cases}$$

$$\begin{cases} yz = \dfrac{1}{2}xz & \rightarrow y = \dfrac{1}{2}x \\ yz = \dfrac{1}{2}xy & \rightarrow z = \dfrac{1}{2}x \\ xz = xy & \rightarrow y = z \end{cases}$$

$x+2y+2z=108$

For $y = \frac{1}{2}x \quad z = \frac{1}{2}x$

$x + x + x = 108$

$3x = 108 \quad \rightarrow \quad \underline{x = 36}$

$\underline{y = 18 \quad z = 18}$

For $\quad x = 0 \quad y = 0$

$2z = 108 \quad \rightarrow \quad \underline{z = 54}$

For $\quad x = 0 \quad z = 0$

$2y = 108 \quad \rightarrow \quad \underline{y = 54}$

For $\quad y = 0 \quad z = 0$

$\underline{x = 108}$

$\therefore$ The critical points are: $(0, 0, 0), (0, 54, 0), (0, 0, 54), (108, 0, 0), (18, 18, 36)$

$\underline{V(0, 0, 0) = 0}$

$\underline{V(0, 54, 0) = 0}$

$\underline{V(0, 0, 54) = 0}$

$\underline{V(108, 0, 0) = 0}$

$\underline{V} = 36(18)(18)$

$\underline{= 11,664}$

The dimensions of the package are: $\boldsymbol{x} = 36$ *in*., $\boldsymbol{y} = 18$ *in*, $\boldsymbol{z} = 18$ *in*.

The maximum volume is $11,664\ \ in^3$

Exercise

Find the rectangular box with a volume of 16 ft^3 that has minimum surface area.

Solution

Let x, y, and z represent the length, width, and height (positive values)

Volume: $V = xyz = 16$

We want to minimum surface area: $2xy + 2yz + 2xz$

$$f(x,\ y,\ z) = 2xy + 2yz + 2xz \quad \textit{subject to} \quad g(x,\ y) = xyz - 16$$

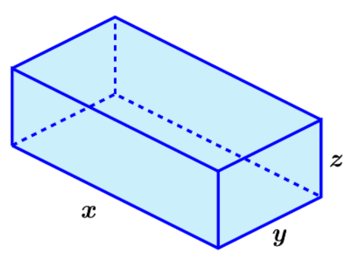
$$\nabla f = 2(y+z)\hat{i} + 2(x+z)\hat{j} + 2(x+y)\hat{k} \qquad \nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$$

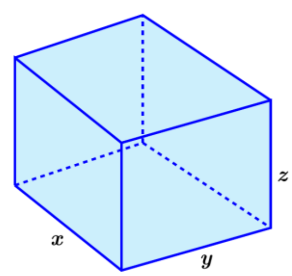
$$\nabla g = yz\hat{i} + xz\hat{j} + xy\hat{k}$$

$$2(y+z)\hat{i} + 2(x+z)\hat{j} + 2(x+y)\hat{k} = \lambda\left(yz\hat{i} + xz\hat{j} + xy\hat{k}\right) \qquad \nabla f = \lambda \nabla g$$

$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \begin{cases} 2(y+z) = \lambda yz \\ 2(x+z) = \lambda xz \\ 2(x+y) = \lambda xy \end{cases} \rightarrow \lambda = \frac{2(y+z)}{yz} = \frac{2(x+z)}{xz} = \frac{2(x+y)}{xy} \quad x,\ y,\ z \neq 0$$

$$\begin{cases} xy + xz = xy + yz & \rightarrow x = y \\ xy + xz = xz + yz & \rightarrow x = z \\ xy + yz = xz + yz & \rightarrow y = z \end{cases}$$





$xyz = 16$

For $x = y = z$

$$x^3 = 16 = 2^4 \quad \rightarrow \quad \underline{x = 2\sqrt[3]{2} = y = z}$$

The box is a cube shape box with length of $2\sqrt[3]{2}$

Exercise

Find the minimum and maximum distances between the ellipse $x^2 + xy + 2y^2 = 1$ and the origin.

Solution

$$f(x,\ y) = x^2 + y^2 \quad \textit{subject to} \quad x^2 + xy + y^2 = 1$$

$$\nabla f = 2x\hat{i} + 2y\hat{j} \qquad \nabla f = f_x \hat{i} + f_y \hat{j}$$

$$g(x,\ y) = x^2 + xy + 2y^2 - 1 = 0$$

$$\nabla g = (2x+y)\,\hat{i} + (4y+x)\,\hat{j}$$

$$2x\hat{i} + 2y\hat{j} = \lambda\left((2x+y)\,\hat{i} + (4y+x)\,\hat{j}\right) \qquad \nabla f = \lambda \nabla g$$

$$\begin{matrix} \hat{i} \\ \hat{j} \end{matrix} \begin{cases} 2x = \lambda(2x+y) \\ 2y = \lambda(4y+x) \end{cases} \rightarrow\ \lambda = \frac{2x}{2x+y} = \frac{2y}{4y+x} \quad x,\ y \neq 0$$

$$8xy + 2x^2 = 4xy + 2y^2$$

$$x^2 + 2xy - y^2 = 0$$

$$\begin{cases} x^2 + xy + 2y^2 = 1 \\ -x^2 - 2xy + y^2 = 0 \end{cases} \rightarrow 3y^2 - xy = 1$$

$$\Rightarrow\quad x = \frac{3y^2 - 1}{y}$$

$$\left(\frac{3y^2 - 1}{y}\right)^2 + 3y^2 - 1 + 2y^2 = 1$$

$$\frac{9y^4 - 6y^2 + 1}{y^2} + 5y^2 - 2 = 0$$

$$14y^4 - 8y^2 + 1 = 0$$

$$y^2 = \frac{8 \pm \sqrt{8}}{28} = \frac{4 \pm \sqrt{2}}{14} \quad \rightarrow \quad \begin{cases} y = \pm\sqrt{\frac{4-\sqrt{2}}{14}} \approx \pm.429766 \\ y = \pm\sqrt{\frac{4+\sqrt{2}}{14}} \approx \pm.621876 \end{cases}$$

$$\begin{cases} y = \sqrt{\frac{4-\sqrt{2}}{14}} & \rightarrow x = \dfrac{3\frac{4-\sqrt{2}}{14} - 1}{\sqrt{\frac{4-\sqrt{2}}{14}}} = \dfrac{\left(-2-3\sqrt{2}\right)\sqrt{14}}{14\sqrt{4-\sqrt{2}}} \approx -1.0375475 \\ y = -\sqrt{\frac{4-\sqrt{2}}{14}} & \rightarrow x = \dfrac{3\frac{4-\sqrt{2}}{14} - 1}{-\sqrt{\frac{4-\sqrt{2}}{14}}} = \dfrac{\left(2+3\sqrt{2}\right)\sqrt{14}}{14\sqrt{4-\sqrt{2}}} \approx 1.0375475 \end{cases}$$

$$\begin{cases} y = \sqrt{\frac{4+\sqrt{2}}{14}} & \rightarrow x = \dfrac{3\frac{4+\sqrt{2}}{14} - 1}{\sqrt{\frac{4+\sqrt{2}}{14}}} = \dfrac{\left(-2+3\sqrt{2}\right)\sqrt{14}}{14\sqrt{4+\sqrt{2}}} \approx .2575894 \\ y = -\sqrt{\frac{4+\sqrt{2}}{14}} & \rightarrow x = \dfrac{3\frac{4+\sqrt{2}}{14} - 1}{-\sqrt{\frac{4+\sqrt{2}}{14}}} = \dfrac{\left(2-3\sqrt{2}\right)\sqrt{14}}{14\sqrt{4+\sqrt{2}}} \approx -.2575894 \end{cases}$$

$$f(1.0375475,\ .429766) = (1.0375475)^2 + (.429766)^2$$

$$\approx 1.2612 \qquad \textbf{\textit{Maximum}} \text{ value}$$

$$f(.2575894,\ .621876) = (.2575894)^2 + (.621876)^2$$

$$\approx 0.453 \qquad \textbf{\textit{Minimum}} \text{ value}$$

Exercise

Find the dimensions of the rectangle of maximum area with sides parallel to the coordinate axes that can be inscribed in the ellipse $4x^2 + 16y^2 = 16$

Solution

$$f(x,\ y) = xy \quad \textit{subject to} \quad 4x^2 + 16y^2 = 16$$

$$\nabla f = y\hat{i} + x\hat{j}$$

$$g(x,\ y) = 4x^2 + 16y^2 - 16$$

$$\nabla g = 8x\hat{i} + 32y\hat{j}$$

$$y\hat{i} + x\hat{j} = \lambda\left(8x\hat{i} + 32y\hat{j}\right)$$

$$\begin{matrix} \hat{i} \\ \hat{j} \end{matrix} \begin{cases} y = 8\lambda x \\ x = 32\lambda y \end{cases} \quad \to \lambda = \frac{y}{8x} = \frac{x}{32y} \quad x,\ y \neq 0$$

$$32y^2 = 8x^2 \quad \to \quad x = \pm 2y$$

$$4x^2 + 16y^2 = 16$$

For $x = 2y$ (only positive since its length)

$$16y^2 + 16y^2 = 16$$

$$y^2 = \frac{1}{2} \quad \to \quad y = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$x = \sqrt{2}$$

$$f\left(\sqrt{2},\ \frac{\sqrt{2}}{2}\right) = 1$$

Maximum area of the rectangle is:

$$Area = 4 \times f\left(\sqrt{2},\ \frac{\sqrt{2}}{2}\right)$$

$$= 4\ \ unit^2$$

Exercise

Find the dimensions of the rectangle of maximum perimeter with sides parallel to the coordinate axes that can be inscribed in the ellipse $2x^2 + 4y^2 = 3$

Solution

$$Perimeter = 4(x + y)$$

$$f(x, y) = x + y \quad subject\ to \quad 2x^2 + 4y^2 = 3$$

$$\nabla f = \hat{i} + \hat{j}$$

$$g(x, y) = 2x^2 + 4y^2 - 3$$

$$\nabla g = 4x\hat{i} + 8y\hat{j}$$

$$\hat{i} + \hat{j} = \lambda\left(4x\hat{i} + 8y\hat{j}\right)$$

$$\begin{matrix} \hat{i} \\ \hat{j} \end{matrix} \begin{cases} 1 = 4\lambda x \\ 1 = 8\lambda y \end{cases} \rightarrow \lambda = \frac{1}{4x} = \frac{1}{8y} \quad x, y > 0$$

$$8y = 4x \quad \rightarrow \quad \underline{x = 2y}$$

$$2x^2 + 4y^2 = 3$$

For $x = 2y$

$$8y^2 + 4y^2 = 3$$

$$y^2 = \frac{1}{4} \quad \rightarrow \quad \underline{y = \frac{1}{2}}$$

$$\underline{x = 1}$$

Dimensions of the rectangle of maximum perimeter is: 2×1

Exercise

Find the point on the plane $2x + 3y + 6z - 10 = 0$ *closest to the point* $(-2, 5, 1)$

Solution

$$f(x, y, z) = (x+2)^2 + (y-5)^2 + (z-1)^2 \quad subject\ to \quad 2x + 3y + 6z - 10 = 0$$

$$\nabla f = 2(x+2)\hat{i} + 2(y-5)\hat{j} + 2(z-1)\hat{k}$$

$$g(x, y, z) = 2x + 3y + 6z - 10$$

$$\nabla g = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

$$2(x+2)\hat{i} + 2(y-5)\hat{j} + 2(z-1)\hat{k} = \lambda\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right)$$

$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \begin{cases} 2(x+2)=2\lambda \\ 2(y-5)=3\lambda \\ 2(z-1)=6\lambda \end{cases} \rightarrow \lambda = x+2=\frac{2}{3}(y-5)=\frac{1}{3}(z-1)$$

$$\begin{cases} x=\lambda-2 \\ y=\frac{3}{2}\lambda+5 \\ z=3\lambda+1 \end{cases}$$

$$2x+3y+6z-10=0$$

$$2\lambda-4+\frac{9}{2}\lambda+15+18\lambda+6-10=0$$

$$\frac{49}{2}\lambda=-7 \rightarrow \underline{\lambda=-\frac{2}{7}}$$

$$\begin{cases} x=-\frac{2}{7}-2=-\frac{16}{7} \\ y=-\frac{3}{7}+5=\frac{32}{7} \\ z=-\frac{6}{7}+1=\frac{1}{7} \end{cases}$$

The closest point is $\left(-\frac{16}{7},\ \frac{32}{7},\ \frac{1}{7}\right)$

$$f\left(-\frac{16}{7},\ \frac{32}{7},\ \frac{1}{7}\right)=\left(-\frac{16}{7}+2\right)^2+\left(\frac{32}{7}-5\right)^2+\left(\frac{1}{7}-1\right)^2$$

$$=\frac{4}{49}+\frac{9}{49}+\frac{36}{49}$$

$$\underline{=1}$$

The distance is 1.

Exercise

Find the point on the surface $4x+y-1=0$ *closest to the point* $(1,\ 2,\ -3)$

Solution

$$f(x,\ y,\ z)=(x-1)^2+(y-2)^2+(z+3)^2 \quad \textit{subject to} \quad 4x+y-1=0$$

$$\nabla f=2(x-1)\hat{i}+2(y-2)\hat{j}+2(z+3)\hat{k}$$

$$g(x,\ y,\ z)=4x+y-1$$

$$\nabla g=4\hat{i}+\hat{j}$$

$$2(x-1)\hat{i}+2(y-2)\hat{j}+2(z+3)\hat{k}=\lambda\left(4\hat{i}+\hat{j}\right)$$

$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \begin{cases} 2(x-1) = 4\lambda \\ 2(y-2) = \lambda & \to \lambda = \frac{1}{2}(x-1) = 2(y-2) \\ 2(z+3) = 0 & \underline{z=-3|} \end{cases}$$

$$\begin{cases} x = 2\lambda + 1 \\ y = \frac{1}{2}\lambda + 2 \end{cases}$$

$4x + y - 1 = 0$

$8\lambda + 4 + \frac{1}{2}\lambda + 2 - 1 = 0$

$\frac{17}{2}\lambda = -5 \ \to \ \underline{\lambda = -\frac{10}{17}|}$

$$\begin{cases} x = -\frac{20}{17} + 1 = -\frac{3}{17} \\ y = -\frac{5}{17} + 2 = \frac{29}{17} \end{cases}$$

The closest point is $\left(-\frac{3}{17},\ \frac{29}{17},\ -3\right)$

Exercise

Find the points on the cone $z^2 = x^2 + y^2$ *closest to the point* $(1,\ 2,\ 0)$

Solution

$f(x,\ y,\ z) = (x-1)^2 + (y-2)^2 + z^2 \quad$ *subject to* $\quad z^2 = x^2 + y^2$

$\nabla f = 2(x-1)\hat{i} + 2(y-2)\hat{j} + 2z\hat{k}$

$g(x,\ y,\ z) = x^2 + y^2 - z^2$

$\nabla g = 2x\hat{i} + 2y\hat{j} - 2z\hat{k}$

$2(x-1)\hat{i} + 2(y-2)\hat{j} + 2z\hat{k} = \lambda\left(2x\hat{i} + 2y\hat{j} - 2z\hat{k}\right)$

$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \begin{cases} x - 1 = \lambda x \\ y - 2 = \lambda y & \to \lambda = \frac{x-1}{x} = \frac{y-2}{y} \\ z = -\lambda z & z = 0, \quad \lambda = -1 \end{cases}$$

$xy - y = xy - 2x \quad \to \quad y = 2x$

$z^2 = x^2 + y^2$

For $z = 0$

$x^2 + y^2 = 0 \ \to \ \underline{x = y = 0|}$

For $\lambda = -1$

$$\begin{cases} \dfrac{x-1}{x}=-1 & \to x-1=-x \quad \Rightarrow \quad \underline{x=\dfrac{1}{2}} \\ \dfrac{y-2}{y}=-1 & \to y-2=-y \quad \Rightarrow \quad \underline{y=1} \end{cases}$$

$$z^2=\frac{1}{4}+1=\frac{5}{4}$$

$$z=\pm\frac{\sqrt{5}}{2}$$

The closest point is $\left(\frac{1}{2},\ 1,\ \pm\frac{\sqrt{5}}{2}\right)$

Exercise

Find the minimum and maximum distances between the sphere

$$x^2+y^2+z^2=9 \quad \textit{closest to the point} \quad (2,\ 3,\ 4)$$

Solution

$$f(x,\ y,\ z)=(x-2)^2+(y-3)^2+(z-4)^2 \quad \textit{subject to} \quad x^2+y^2+z^2=9$$

$$\nabla f=2(x-2)\hat{i}+2(y-3)\hat{j}+2(z-4)\hat{k}$$

$$g(x,\ y,\ z)=x^2+y^2+z^2-9$$

$$\nabla g=2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$2(x-2)\hat{i}+2(y-3)\hat{j}+2(z-4)\hat{k}=\lambda\left(2x\hat{i}+2y\hat{j}+2z\hat{k}\right)$$

$$\begin{matrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{matrix}\begin{cases} x-2=\lambda x \\ y-3=\lambda y & \to \lambda=\dfrac{x-2}{x}=\dfrac{y-3}{y}=\dfrac{z-4}{z} \\ z-4=\lambda z \end{cases}$$

$$\begin{cases} x\lambda=x-2 & \to x=\dfrac{2}{1-\lambda} \\ y\lambda=y-3 & \to y=\dfrac{3}{1-\lambda} \\ z\lambda=z-4 & \to z=\dfrac{4}{1-\lambda} \end{cases}$$

$$x^2+y^2+z^2-9=0$$

$$\left(\frac{2}{1-\lambda}\right)^2+\left(\frac{3}{1-\lambda}\right)^2+\left(\frac{4}{1-\lambda}\right)^2-9=0$$

$$\frac{29}{(1-\lambda)^2}-9=0$$

$$\frac{29-9+18\lambda-9\lambda^2}{(1-\lambda)^2}=0$$

$$9\lambda^2-18\lambda-20=0 \quad \to \quad \lambda=\frac{18\pm\sqrt{1044}}{18}=\boxed{\frac{3\pm\sqrt{29}}{3}}$$

For $\lambda=\frac{3-\sqrt{29}}{3}=1-\frac{\sqrt{29}}{3}$

$$\begin{cases} x=\dfrac{2}{1-1+\frac{\sqrt{29}}{3}}=\dfrac{6}{\sqrt{29}} \\ y=\dfrac{3}{1-1+\frac{\sqrt{29}}{3}}=\dfrac{9}{\sqrt{29}} \\ z=\dfrac{4}{1-1+\frac{\sqrt{29}}{3}}=\dfrac{12}{\sqrt{29}} \end{cases}$$

The points are: $\pm\left(\frac{6}{\sqrt{29}},\ \frac{9}{\sqrt{29}},\ \frac{12}{\sqrt{29}}\right)$

$$\begin{aligned} f\left(\frac{6}{\sqrt{29}},\ \frac{9}{\sqrt{29}},\ \frac{12}{\sqrt{29}}\right) &= \left(\frac{6}{\sqrt{29}}-2\right)^2+\left(\frac{9}{\sqrt{29}}-3\right)^2+\left(\frac{12}{\sqrt{29}}-4\right)^2 \\ &= \frac{36-24\sqrt{29}+116}{29}+\frac{81-54\sqrt{29}+261}{29}+\frac{144-96\sqrt{29}+464}{29} \\ &= \frac{1,102-174\sqrt{29}}{29} \\ &= \boxed{38-6\sqrt{29}} \end{aligned}$$

$$\begin{aligned} f\left(-\frac{6}{\sqrt{29}},\ -\frac{9}{\sqrt{29}},\ -\frac{12}{\sqrt{29}}\right) &= \left(-\frac{6}{\sqrt{29}}-2\right)^2+\left(-\frac{9}{\sqrt{29}}-3\right)^2+\left(-\frac{12}{\sqrt{29}}-4\right)^2 \\ &= \frac{36+24\sqrt{29}+116}{29}+\frac{81+54\sqrt{29}+261}{29}+\frac{144+96\sqrt{29}+464}{29} \\ &= \boxed{38+6\sqrt{29}} \end{aligned}$$

$$\begin{aligned} \sqrt{38-6\sqrt{29}} &= \sqrt{9-2\left(3\sqrt{29}\right)+29} \\ &= \sqrt{\left(\sqrt{29}-3\right)^2} \\ &= \sqrt{29}-3 \end{aligned}$$

∴ ***Minimum*** distance: $\sqrt{29}-3$

∴ ***Maximum*** distance: $\sqrt{29}+3$

Exercise

Find the maximum value of $x_1 + x_2 + x_3 + x_4$ *subject to* $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 16$

Solution

$$f\left(x_1, x_2, x_3, x_4\right) = x_1 + x_2 + x_3 + x_4$$

$$\nabla f = \langle 1,\ 1,\ 1,\ 1\rangle$$

$$g\left(x_1, x_2, x_3, x_4\right) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - 16$$

$$\nabla g = \left\langle 2x_1,\ 2x_2,\ 2x_3,\ 2x_4 \right\rangle$$

$$\langle 1,\ 1,\ 1,\ 1\rangle = \lambda\left\langle 2x_1,\ 2x_2,\ 2x_3,\ 2x_4 \right\rangle$$

$$\begin{cases} 2\lambda x_1 = 1 \\ 2\lambda x_2 = 1 \\ 2\lambda x_3 = 1 \\ 2\lambda x_4 = 1 \end{cases} \rightarrow \quad \lambda = \frac{1}{2x_1} = \frac{1}{2x_2} = \frac{1}{2x_2} = \frac{1}{2x_4}$$

$$x_1 = x_2 = x_3 = x_4$$

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 16$$

$$4x_1^2 = 16 \quad \rightarrow \quad \underline{x_1 = \pm 2 = x_2 = x_3 = x_4}|$$

$$f\left(2,\ 2,\ 2,\ 2\right) = 8 \qquad \textbf{\textit{Maximum}} \text{ value}$$

$$f\left(-2,\ -2,\ -2,\ -2\right) = -8 \qquad \textbf{\textit{Minimum}} \text{ value}$$

Exercise

Find the maximum value of $x_1 + x_2 + \cdots + x_n$ *subject to* $x_1^2 + x_2^2 + \cdots + x_n^2 = c^2$

Solution

$$f\left(x_1, x_2, \ldots, x_n\right) = x_1 + x_2 + \ldots + x_n$$

$$\nabla f = \langle 1,\ 1,\ \ldots,\ 1\rangle$$

$$g\left(x_1, x_2, \ldots, x_n\right) = x_1^2 + x_2^2 + \cdots + x_n^2 - c^2$$

$$\nabla g = \left\langle 2x_1,\ 2x_2, \cdots,\ 2x_n \right\rangle$$

$$\langle 1,\ 1,\ \ldots,\ 1\rangle = \lambda\left\langle 2x_1,\ 2x_2,\ \ldots,\ 2x_n \right\rangle$$

$$\begin{cases} 2\lambda x_1 = 1 \\ 2\lambda x_2 = 1 \\ \vdots \quad \vdots \\ 2\lambda x_n = 1 \end{cases} \rightarrow \lambda = \frac{1}{2x_1} = \frac{1}{2x_2} = \dots = \frac{1}{2x_n}$$

$$x_1 = x_2 = \cdots = x_n$$

$$x_1^2 + x_2^2 + \cdots + x_n^2 = c^2$$

$$nx_1^2 = c^2 \quad \rightarrow \quad \boxed{x_1 = \pm\frac{c}{\sqrt{n}} = x_2 = \dots = x_n}$$

$$f\left(\frac{c}{\sqrt{n}},\ \frac{c}{\sqrt{n}},\ \cdots,\ \frac{c}{\sqrt{n}}\right) = \frac{nc}{\sqrt{n}}$$

$$\boxed{= c\sqrt{n}}$$ ***Maximum*** value

Exercise

Find the maximum value of $a_1x_1 + a_2x_2 + \cdots + a_nx_n$ *subject to* $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$ for the given positive real numbers $a_1, a_2, \cdots, a_n$

Solution

$$f\left(x_1, x_2, \dots, x_n\right) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

$$\nabla f = \left\langle a_1,\ a_2,\ \dots,\ a_n \right\rangle$$

$$g\left(x_1, x_2, \dots, x_n\right) = x_1^2 + x_2^2 + \cdots + x_n^2 - 1$$

$$\nabla g = \left\langle 2x_1,\ 2x_2, \cdots,\ 2x_n \right\rangle$$

$$\left\langle a_1,\ a_2,\ \dots,\ a_n \right\rangle = \lambda\left\langle 2x_1,\ 2x_2,\ \dots,\ 2x_n \right\rangle$$

$$\begin{cases} 2\lambda x_1 = a_1 \\ 2\lambda x_2 = a_2 \\ \vdots \quad \vdots \\ 2\lambda x_n = a_n \end{cases} \rightarrow \lambda = \frac{a_1}{2x_1} = \frac{a_2}{2x_2} = \dots = \frac{a_n}{2x_n}$$

$$\frac{a_1}{2x_1} = \frac{a_2}{2x_2} = \dots = \frac{a_n}{2x_n}$$

$$x_2 = \frac{a_2}{a_1}x_1, \quad x_1 = \frac{a_3}{a_1}x_1, \quad \dots, \quad x_n = \frac{a_n}{a_1}x_1$$

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1$$

$$x_1^2 + \frac{a_2^2}{a_1^2}x_1^2 + \frac{a_3^2}{a_1^2}x_1^2 + \ldots + \frac{a_n^2}{a_1^2}x_1^2 = 1$$

$$a_1^2x_1^2 + a_2^2x_1^2 + a_3^2x_1^2 + \ldots + a_n^2x_1^2 = a_1^2$$

$$\left(a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2\right)x_1^2 = a_1^2$$

$$x_1^2 = \frac{a_1^2}{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2}$$

$$x_1 = \pm\frac{a_1}{\sqrt{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2}}$$

$$x_2 = \pm\frac{a_2}{\sqrt{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2}}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$x_n = \pm\frac{a_n}{\sqrt{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2}}$$

The negative values of x's will give us the minimum

$$f\left(x_1, x_2, \ldots, x_n\right) = \frac{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2}{\sqrt{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2}}$$

$$= \sqrt{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2}$$ ***Maximum*** value

Exercise

The planes $x + 2z = 12$ and $x + y = 6$ intersect in a line L. Find the point on L nearest the origin.

Solution

$f(x, y, z) = x^2 + y^2 + z^2$ *subject to* $g(x, y, z) = x + 2z - 12$ $h(x, y, z) = x + y - 6$

$$\nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$g(x, y, z) = x + 2z - 12 \quad \to \quad \nabla g = \hat{i} + 2\hat{k}$$

$$h(x, y, z) = x + y - 6 \quad \to \quad \nabla h = \hat{i} + \hat{j}$$

$$2x\hat{i} + 2y\hat{j} + 2z\hat{k} = \lambda\left(\hat{i} + 2\hat{k}\right) + \mu\left(\hat{i} + \hat{j}\right) \qquad \nabla f = \lambda \nabla g + \mu \nabla h$$

$$\begin{cases} 2x = \lambda + \mu \\ 2y = \mu \\ 2z = 2\lambda \quad \to z = \lambda \end{cases}$$

$$2x = z + 2y \ \to \ x = \frac{1}{2}z + y$$

$$x + 2z = 12 \ \to \ \frac{1}{2}z + y + 2z = 12$$

$$2y + 5z = 24$$

$$x + y = 6 \quad \& \quad 2y + \frac{1}{2}z = 6$$

$$4y + z = 12$$

$$\begin{cases} 2y + 5z = 24 \\ 4y + z = 12 \end{cases} \qquad \Delta = \begin{vmatrix} 2 & 5 \\ 4 & 1 \end{vmatrix} = -18 \quad \Delta_y = \begin{vmatrix} 24 & 5 \\ 12 & 1 \end{vmatrix} = -36 \quad \Delta_z = \begin{vmatrix} 2 & 24 \\ 4 & 12 \end{vmatrix} = -72$$

$$\underline{y = 2, \quad z = 4, \quad \to x = 4}$$

The point $(4,\ 2,\ 4)$ is the pint on the line closet to the rogin.

Exercise

Find the maximum and minimum values of

$$f(x,\ y,\ z) = xyz \quad \text{subject to} \quad x^2 + y^2 = 4 \ \text{ and } \ x + y + z = 1$$

Solution

$$\nabla f = yz\hat{i} + xz\hat{j} + xy\hat{k}$$

$$g(x,\ y,\ z) = x^2 + y^2 - 4 \ \to \ \nabla g = 2x\hat{i} + 2y\hat{j}$$

$$h(x,\ y,\ z) = x + y + z - 1 \ \to \ \nabla h = \hat{i} + \hat{j} + \hat{k}$$

$$yz\hat{i} + xz\hat{j} + xy\hat{k} = \lambda\left(2x\hat{i} + 2y\hat{j}\right) + \mu\left(\hat{i} + \hat{j} + \hat{k}\right) \qquad \nabla f = \lambda \nabla g + \mu \nabla h$$

$$\begin{cases} yz = 2x\lambda + \mu & \to yz = 2x\lambda + xy \\ xz = 2\lambda y + \mu & \to xz = 2\lambda y + xy \\ xy = \mu \end{cases}$$

$$xy = yz - 2x\lambda = xz - 2\lambda y$$

$$2\lambda(y - x) = z(x - y)$$

$$z = -2\lambda \quad \text{or} \quad y = x$$

For $y = x$

$$x^2 + y^2 = 4 \ \to \ 2x^2 = 4 \ \Rightarrow \ \underline{x = \pm\sqrt{2} = y}$$

For $y = x = \sqrt{2}$

$$x + y + z = 1 \quad \rightarrow \quad \underline{z = 1 - 2\sqrt{2}|}$$

For $y = x = -\sqrt{2}$

$$\underline{z = 1 + 2\sqrt{2}|}$$

$\therefore$ Points: $\left(\sqrt{2},\ \sqrt{2},\ 1 - 2\sqrt{2}\right)$ $\left(-\sqrt{2},\ -\sqrt{2},\ 1 + 2\sqrt{2}\right)$

For $z = -2\lambda$

$$\mu = xy = yz + xz$$

$$z = \frac{xy}{x + y}$$

$$x + y + \frac{xy}{x + y} = 1$$

$$x^2 + y^2 + 3xy = x + y$$

$$4 + 3xy = x + y$$

$$4 + (3x - 1)\, y - x = 0$$

$$\left((3x - 1)\sqrt{4 - x^2}\right)^2 = (x - 4)^2$$

$$\left(9x^2 - 6x + 1\right)\left(4 - x^2\right) = x^2 - 8x + 16$$

$$36x^2 - 9x^4 - 24x + 6x^3 + 4 - x^2 - x^2 + 8x - 16 = 0$$

$$9x^4 - 6x^3 - 34x^2 + 16x + 12 = 0$$

Using Maple: evalf (solve ($9x^4 - 6x^3 - 34x^2 + 16x + 12$, x))

$$\underline{x \approx -1.78, \ -0.42, \ .912, \ 1.955 \,|}$$

x	y	$z = 1 - x - y$
-1.78	$.912$	1.868
	$-.912$	3.692
$-.42$	1.955	$-.535$
	-1.955	3.375
$.912$	1.78	-1.692
	-1.78	1.868
1.955	$.42$	-1.375
	$-.42$	$-.535$

	$f(x,\ y,\ z) = xyz$
$\left(\sqrt{2},\ \sqrt{2},\ 1 - 2\sqrt{2}\right)$	$2 - 4\sqrt{2} \approx -3.657$
$\left(-\sqrt{2},\ -\sqrt{2},\ 1 + 2\sqrt{2}\right)$	$2 + 4\sqrt{2} \approx 7.657$

$(-1.78,\ .912,\ 1.868)$	-3.032
$(-1.78,\ -.912,\ 3.692)$	5.99
$(-.42,\ 1.955,\ -.535)$	0.439
$(-.42,\ -1.955,\ 3.375)$	2.77
$(.912,\ 1.78,\ -1.692)$	-2.747
$(.912,\ -1.78,\ 1.868)$	-3.032
$(1.955,\ .42,\ -1.375)$	-1.129
$(1.955,\ -.42,\ -.535)$	0.439

Minimum value of $2-4\sqrt{2}$ @ $\left(\sqrt{2},\ \sqrt{2},\ 1-2\sqrt{2}\right)$

Maximum value of $2+4\sqrt{2}$ @ $\left(-\sqrt{2},\ -\sqrt{2},\ 1+2\sqrt{2}\right)$

Exercise

The paraboloid $z=x^2+2y^2+1$ and the plane $x-y+2z=4$ intersect in a curve *C*. Find the points on *C* that have minimum and maximum distance from the origin.

Solution

$$f(x,\ y,\ z)=x^2+y^2+z^2 \quad \textit{subject to} \quad g(x,\ y,\ z)=x^2+2y^2-z+1 \quad h(x,\ y,\ z)=x-y+2z-4$$

$$\nabla f=2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$g(x,\ y,\ z)=x^2+2y^2-z+1 \quad \rightarrow \quad \nabla g=2x\hat{i}+4y\hat{j}-\hat{k}$$

$$h(x,\ y,\ z)=x-y+2z-4 \quad \rightarrow \quad \nabla h=\hat{i}-\hat{j}+2\hat{k}$$

$$2x\hat{i}+2y\hat{j}+2z\hat{k}=\lambda\left(2x\hat{i}+4y\hat{j}-\hat{k}\right)+\mu\left(\hat{i}-\hat{j}+2\hat{k}\right) \qquad \nabla f=\lambda\nabla g+\mu\nabla h$$

$$\begin{cases} 2x=2\lambda x+\mu & \rightarrow x=\dfrac{\mu}{2-2\lambda} & (1) \\ 2y=4\lambda y-\mu & \rightarrow y=\dfrac{\mu}{4\lambda-2} & (2) \\ 2z=-\lambda+2\mu & \rightarrow z=-\dfrac{1}{2}\lambda+\mu & (3) \end{cases}$$

(1) & (2) $\rightarrow$ $\mu=2x-2\lambda x=4\lambda y-2y$

$$(1-\lambda)x=(2\lambda-1)y$$

$$y=\frac{1-\lambda}{2\lambda-1}x$$

(1) & (3) $\rightarrow$ $\mu=2x-2\lambda x=z+\frac{1}{2}\lambda$

$$\begin{cases} 2\lambda x + \mu = 2x \\ 4\lambda y - \mu = 2y \end{cases} \rightarrow \quad 2\lambda(x+2y) = 2(x+y) \quad \Rightarrow \boxed{\lambda = \frac{x+y}{x+2y}}$$

$$\mu = 2x\left(1 - \frac{x+y}{x+2y}\right)$$

$$= \boxed{\frac{2xy}{x+2y}}$$

$$(3) \rightarrow z = -\frac{1}{2}\frac{x+y}{x+2y} + \frac{2xy}{x+2y}$$

$$= \frac{4xy - x - y}{2(x+2y)}$$

$$x - y + 2z = 4$$

$$x - y + \frac{4xy - x - y}{x+2y} = 4$$

$$x^2 + xy - 2y^2 + 4xy - x - y = 4x + 8y$$

$$x^2 - 2y^2 + 5xy - 5x - 9y = 0$$

$$x^2 + 2y^2 - z + 1 = 0$$

$$z = x^2 + 2y^2 + 1$$

$$z = 2 - \frac{1}{2}x + \frac{1}{2}y = x^2 + 2y^2 + 1$$

$$2x^2 + x + 4y^2 - y - 2 = 0$$

$$x = \frac{-1 \pm \sqrt{-32y^2 + 8y + 17}}{4}$$

$$\begin{array}{l} {\color{red}-2\times} \begin{cases} x^2 - 2y^2 + 5xy - 5x - 9y = 0 \\ 2x^2 + 4y^2 + x - y - 2 = 0 \end{cases} \\ \hline 8y^2 - 10xy + 11x + 17y - 2 = 0 \end{array}$$

$$x = \frac{8y^2 + 17y - 2}{10y - 11} = \frac{-1 \pm \sqrt{-32y^2 + 8y + 17}}{4}$$

$$32y^2 + 68y - 8 = -10y + 11 \pm (10y - 11)\sqrt{-32y^2 + 8y + 17}$$

$$32y^2 + 78y - 19 = \pm(10y - 11)\sqrt{-32y^2 + 8y + 17}$$

Using Maple: Ploy1:= $32y^2 + 78y - 19 = (10y - 11)\sqrt{-32y^2 + 8y + 17}$;

fsolve(Poly1)

Using Maple: Ploy2:= $32y^2 + 78y - 19 = -(10y - 11)\sqrt{-32y^2 + 8y + 17}$;

fsolve(Poly2)

$$\begin{cases} y \approx -0.3917 & \rightarrow\ x \approx 0.4982 \quad z \approx 1.9468 \\ y \approx 0.4613 & \rightarrow\ x \approx -1.1814 \quad z \approx 2.36 \end{cases}$$

The points on C are: $(0.4982,\ -0.3917,\ 1.9468)\quad (-1.1814,\ 0.4613,\ 2.36)$

Minimum distance from the origin: $\sqrt{(-.3917)^2 + .4982^2 + 1.9468^2}\ \underline{\approx 2.04735}$

Maximum distance from the origin: $\sqrt{1.1814^2 + .4613^2 + 2.36^2}\ \underline{\approx 2.3235}$

Exercise

Find the maximum and minimum values of $f(x,\ y,\ z) = x^2 + y^2 + z^2$ on the curve on which the cone $z^2 = 4x^2 + 4y^2$ and the plane $2x + 4z = 5$ intersect.

Solution

$f(x,\ y,\ z) = x^2 + y^2 + z^2 \quad \textit{subject to} \quad g(x,\ y,\ z) = 4x^2 + 4y^2 - z^2 \quad h(x,\ y,\ z) = 2x + 4z - 5$

$$\nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$
$$\nabla g = 8x\hat{i} + 8y\hat{j} - 2z\hat{k}$$

$$\nabla h = 2\hat{i} + 4\hat{k}$$

$$2x\hat{i} + 2y\hat{j} + 2z\hat{k} = \lambda\left(8x\hat{i} + 8y\hat{j} - 2z\hat{k}\right) + \mu\left(2\hat{i} + 4\hat{k}\right) \qquad \nabla f = \lambda \nabla g + \mu \nabla h$$

$$\begin{cases} 2x = 8\lambda x + 2\mu & (1) \\ 2y = 8\lambda y & \rightarrow y = 0 \ \ or\ \ \lambda = \frac{1}{4} \\ 2z = -2\lambda z + 4\mu & (2) \end{cases}$$

For $y = 0$

$$z^2 = 4x^2 + 4y^2 \quad \rightarrow \quad z = \pm 2x$$

For $z = -2x$

$$2x - 8x = 5 \ \rightarrow\ x = -\frac{5}{6}$$

$$\rightarrow\ z = \frac{5}{3}$$

Point: $\left(-\frac{5}{6},\ 0,\ \frac{5}{3}\right)$

For $z = 2x$

$$2x + 8x = 5 \ \rightarrow\ x = \frac{1}{2}$$

$$\rightarrow\ z = 1$$

Point: $\left(\frac{1}{2},\ 0,\ 1\right)$

For $\lambda = \frac{1}{4}$

$(1) \rightarrow \quad x = x + \mu \quad \Rightarrow \quad \boxed{\mu = 0}$

$(2) \quad \rightarrow \quad z = -\frac{1}{4}z \quad \Rightarrow \quad z = 0$

$z^2 = 4x^2 + 4y^2 = 0 \quad$ (impossible)

There are no solutions to the Lagrange conditions.

	$f(x, y, z) = x^2 + y^2 + z^2$
$\left(-\frac{5}{6}, 0, \frac{5}{3}\right)$	$\frac{25}{36} + \frac{25}{9} = \frac{125}{36}$
$\left(\frac{1}{2}, 0, 1\right)$	$\frac{5}{4}$

The maximum value of f is $\frac{125}{36}$ @ $\left(-\frac{5}{6}, 0, \frac{5}{3}\right)$

The minimum value of f is $\frac{5}{4}$ @ $\left(\frac{1}{2}, 0, 1\right)$

Exercise

The temperature of points on a elliptical plate $x^2 + y^2 + xy \le 1$ is given by $T(x, y) = 25\left(x^2 + y^2\right)$. Find the hottest and coldest temperatures on the edge of the elliptical plate.

Solution

$T(x, y) = 25x^2 + 25y^2 \quad$ *subject to* $\quad g(x, y) = x^2 + y^2 + xy - 1$

$\nabla T = 50x\hat{i} + 50y\hat{j}$

$\nabla g = (2x + y)\hat{i} + (2y + x)\hat{j}$

$50x\hat{i} + 50y\hat{j} = \lambda\left((2x + y)\hat{i} + (2y + x)\hat{j}\right) \qquad \nabla f = \lambda \nabla g$

$$\begin{cases} 50x = (2x + y)\lambda & \rightarrow \lambda = \dfrac{50x}{2x + y} \\ 50y = (2y + x)\lambda & \rightarrow \lambda = \dfrac{50y}{2y + x} \end{cases}$$

$\frac{50x}{2x + y} = \frac{50y}{2y + x}$

$2xy + x^2 = 2xy + y^2$

$x = \pm y$

$x^2 + y^2 + xy = 1$

For $x = -y$

$$x^2 + x^2 - x^2 = 1$$

$$x^2 = 1 \quad \rightarrow \quad \underline{x = \pm 1 = -y}|$$

Points: $(1, \ -1)$ $(-1, \ 1)$

For $x = y$

$$3x^2 = 1 \quad \rightarrow \quad \underline{x = \pm \frac{1}{\sqrt{3}} = y}|$$

Points: $\left(\frac{1}{\sqrt{3}}, \ \frac{1}{\sqrt{3}}\right)$ $\left(-\frac{1}{\sqrt{3}}, \ -\frac{1}{\sqrt{3}}\right)$

	$T(x, y) = 25\left(x^2 + y^2\right)$
$(1, \ -1)$	50
$(-1, \ 1)$	50
$\left(\frac{1}{\sqrt{3}}, \ \frac{1}{\sqrt{3}}\right)$	$25\left(\frac{2}{3}\right) = \frac{50}{3}$
$\left(-\frac{1}{\sqrt{3}}, \ -\frac{1}{\sqrt{3}}\right)$	$\frac{50}{3}$

The hottest temperature on the edge of the plate is 50 @ $(1, \ -1)$ $(-1, \ 1)$

The coldest temperature on the edge of the plate is $\frac{50}{3}$ @ $\left(\frac{1}{\sqrt{3}}, \ \frac{1}{\sqrt{3}}\right)$ $\left(-\frac{1}{\sqrt{3}}, \ -\frac{1}{\sqrt{3}}\right)$