Solution   *Section* 2.1 – Integration by Parts

Exercise

Evaluate the integral $\int x \ln x \, dx$

Solution

Let: $u = \ln x \Rightarrow du = \frac{1}{x} dx$

$dv = xdx \Rightarrow v = \int dv = \int xdx = \frac{1}{2}x^2$

$$\int x \ln x dx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x^2 \frac{dx}{x}$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x \, dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$$

Exercise

Evaluate the integral $\int \ln x^2 \, dx$

Solution

$$\int \ln x^2 \, dx = 2\int \ln x \, dx \qquad u = \ln x \Rightarrow du = \frac{1}{x} dx \qquad v = \int dx = x$$

$$= 2\left[x \ln x - \int x \frac{1}{x} dx \right]$$

$$= 2\left[x \ln x - \int dx \right]$$

$$= 2(x \ln x - x) + C$$

$$= 2x(\ln x - 1) + C$$

Exercise

Evaluate the integral $\int \ln(3x) \, dx$

Solution

$$u = \ln 3x \Rightarrow du = \frac{3}{3x}dx = \frac{1}{x}dx$$

$$dv = dx \Rightarrow v = x$$

$$\int \ln(3x)\,dx = x\ln(3x) - \int x\frac{1}{x}dx$$

$$= x\ln(3x) - \int dx$$

$$= x\ln(3x) - x + C$$

$$= x\left(\ln(3x) - 1\right) + C$$

Exercise

Evaluate the integral $\int \frac{1}{x\ln x}dx$

Solution

$$\int \frac{1}{x\ln x}dx = \int \frac{1}{\ln x}d(\ln x)$$

$$= \ln|\ln x| + C$$

Exercise

Evaluate the integrals $\int x(\ln x)^2\,dx$

Solution

$$u = \ln x \quad \rightarrow \quad x = e^u$$

$$du = \frac{1}{x}dx \quad \Rightarrow \quad xdu = dx \rightarrow dx = e^u du$$

$$\int x(\ln x)^2\,dx = \int e^u u^2 e^u du$$

$$= \int u^2 e^{2u}\,du$$

$$= \frac{1}{2}u^2e^{2u} - \frac{1}{2}ue^{2u} + \frac{1}{4}e^{2u} + C$$

$$= \frac{1}{4}e^{2u}\left(2u^2 - 2u + 1\right) + C$$

$$= \frac{1}{4}x^2\left(2(\ln x)^2 - 2\ln x + 1\right) + C$$

		$\int e^{2u}du$
+	u^2	$\frac{1}{2}e^{2u}$
−	$2u$	$\frac{1}{4}e^{2u}$
+	2	$\frac{1}{8}e^{2u}$
−	0	

2^{nd} *Method*

$$u = \ln x \qquad dv = \int (x \ln x)\, dx$$

$$du = \frac{1}{x}dx \quad v = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2$$

$$\int x(\ln x)^2\, dx = (\ln x)\left(\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right) - \int \left(\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right)\frac{1}{x}dx$$

$$= \frac{1}{2}x^2 (\ln x)^2 - \frac{1}{4}x^2 \ln x - \int \left(\frac{1}{2}x \ln x - \frac{1}{4}x\right)dx$$

$$u = \ln x \Rightarrow du = \frac{1}{x}dx$$

$$dv = xdx \Rightarrow v = \frac{1}{2}x^2$$

$$\int x \ln xdx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x^2 \frac{dx}{x}$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int xdx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2$$

$$= \frac{1}{2}x^2 (\ln x)^2 - \frac{1}{4}x^2 \ln x - \left(\frac{1}{2}\left(\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right) - \frac{1}{8}x^2\right) + C$$

$$= \frac{1}{2}x^2 (\ln x)^2 - \frac{1}{4}x^2 \ln x - \frac{1}{4}x^2 \ln x + \frac{1}{8}x^2 + \frac{1}{8}x^2 + C$$

$$\boxed{= \frac{1}{2}x^2 (\ln x)^2 - \frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + C}$$

3^{nd} *Method*

$$u = (\ln x)^2 \qquad dv = \int x\, dx$$

$$du = 2(\ln x)\frac{1}{x}dx \qquad v = \frac{1}{2}x^2$$

$$\int x(\ln x)^2\, dx = \frac{1}{2}x^2 (\ln x)^2 - \int \frac{1}{2}x^2 (2\ln x)\frac{1}{x}dx$$

$$= \frac{1}{2}x^2 (\ln x)^2 - \int x \ln x\, dx$$

$$u = \ln x \Rightarrow du = \frac{1}{x}dx$$

$$dv = xdx \Rightarrow v = \frac{1}{2}x^2$$

$$\int x \ln x\, dx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x^2 \frac{dx}{x}$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\, dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2$$

$$= \frac{1}{2}x^2 (\ln x)^2 - \frac{1}{4}x^2 \ln x - \left(\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right) + C$$

$$= \frac{1}{2}x^2 (\ln x)^2 - \frac{1}{4}x^2 \ln x - \frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + C$$

$$= \frac{1}{2}x^2 (\ln x)^2 - \frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + C$$

Exercise

Evaluate the integral $\int x^2 (\ln x)^2 \, dx$

Solution

$$u = (\ln x)^2 \qquad v = \int x^2 dx$$

$$du = 2\frac{\ln x}{x} dx \qquad v = \frac{1}{3}x^3$$

$$\int x^2 (\ln x)^2 \, dx = \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{3}\int x^2 \ln x dx$$

$$u = \ln x \qquad v = \int x^2 dx$$

$$du = \frac{1}{x} dx \qquad v = \frac{1}{3}x^3$$

$$\int x^2 (\ln x)^2 \, dx = \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{3}\left(\frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx\right)$$

$$= \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{9}x^3 \ln x + \frac{2}{27}x^3 + C$$

$$= \frac{1}{27}x^3 \left(9\ln^2 x - 6\ln x + 2\right) + C$$

Or

Let $y = \ln x \quad \Rightarrow \quad x = e^y$

$$dx = e^y dy$$

$$\int x^2 (\ln x)^2 \, dx = \int \left(e^y\right)^2 y^2 e^y dy$$

$$= \int y^2 e^{3y}\, dy$$

		$\int e^{3y}\, dy$
$+$	y^2	$\frac{1}{3}e^{3y}$
$-$	$2y$	$\frac{1}{9}e^{3y}$
$+$	2	$\frac{1}{27}e^{3y}$

$$\int x^2 (\ln x)^2\, dx = e^{3y}\left(\frac{1}{3}y^2 - \frac{2}{9}y + \frac{2}{27}\right) + C$$

$$= x^3\left(\frac{1}{3}\ln^2 x - \frac{2}{9}\ln x + \frac{2}{27}\right) + C$$

$$= \frac{1}{27}x^3\left(9\ln^2 x - 6\ln x + 2\right) + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{(\ln x)^3}{x}\, dx$

Solution

$$\int \frac{(\ln x)^3}{x}\, dx = \int (\ln x)^3\, d(\ln x) \qquad d(\ln x) = \frac{dx}{x}$$

$$= \frac{1}{4}(\ln x)^4 + C$$

Exercise

Evaluate the integral $\displaystyle\int x^2 \ln x^3\, dx$

Solution

$$u = \ln x \qquad v = \int 3x^2 dx = x^3 \qquad \int u\,dv = uv - \int v\,du$$

$$du = \frac{1}{x}dx$$

$$\int x^2 \ln x^3\, dx = \int 3x^2 \ln x\, dx$$

$$= x^3 \ln x - \int x^2\, dx$$

$$\boxed{= x^3 \ln x - \frac{1}{3}x^3 + C}$$

Exercise

Evaluate the integral $\int \ln\left(x + x^2\right) dx$

Solution

Let: $u = \ln\left(x + x^2\right) \quad dv = dx$

$du = \frac{2x+1}{x+x^2}\,dx \qquad v = x$

$$\int \ln\left(x+x^2\right)dx = x\ln\left(x+x^2\right) - \int x\frac{2x+1}{x+x^2}\,dx$$

$$= x\ln\left(x+x^2\right) - \int \frac{2x+1}{x(1+x)}\,x\,dx$$

$$= x\ln\left(x+x^2\right) - \int \frac{2x+2-1}{1+x}\,dx$$

$$= x\ln\left(x+x^2\right) - \int \frac{2(x+1)-1}{x+1}\,dx$$

$$= x\ln\left(x+x^2\right) - \int \left(2 - \frac{1}{x+1}\right)dx$$

$$= x\ln\left(x+x^2\right) - \left(2x - \ln|x+1|\right) + C$$

$$\boxed{= x\ln\left(x+x^2\right) - 2x + \ln|x+1| + C}$$

Exercise

Evaluate the integral $\int x\ln(x+1)\,dx$

Solution

$u = \ln(x+1) \Rightarrow du = \frac{1}{x+1}dx$

$dv = x\,dx \Rightarrow v = \frac{1}{2}x^2$

$$\int x\ln(x+1)\,dx = \frac{1}{2}x^2\ln(x+1) - \frac{1}{2}\int \frac{x^2}{x+1}dx$$

$$= \frac{1}{2}x^2\ln(x+1) - \frac{1}{2}\int \left(x - 1 + \frac{1}{x+1}\right)dx$$

$$= \frac{1}{2}x^2\ln(x+1) - \frac{1}{2}\left(\frac{1}{2}x^2 - x + \ln(x+1)\right) + C$$

$$= \frac{1}{2}x^2\ln(x+1) - \frac{1}{4}x^2 + \frac{1}{2}x - \frac{1}{2}\ln(x+1) + C$$

$$= -\frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{2}\left(x^2 - 1\right)\ln(x+1) + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{(\ln x)^2}{x}dx$

Solution

$$\int \frac{(\ln x)^2}{x}dx = \int (\ln x)^2\, d(\ln x)$$

$$= \frac{1}{3}(\ln x)^3 + C$$

Exercise

Evaluate the integral $\displaystyle\int x^5 \ln 3x\, dx$

Solution

		$\int x^5\,dx$
+	$\ln 3x$	$\frac{1}{6}x^6$
−	$\frac{1}{x}$	$\int \frac{1}{6}x^6$

$$\int x^5 \ln 3x\, dx = \frac{1}{6}x^6 \ln 3x - \frac{1}{6}\int x^6 \frac{1}{x}dx$$

$$= \frac{1}{6}x^6 \ln 3x - \frac{1}{6}\int x^5 dx$$

$$= \frac{1}{6}x^6 \ln 3x - \frac{1}{36}x^6 + C$$

Exercise

Evaluate the integral $\displaystyle\int x^5 \ln x \, dx$

Solution

		$\int x^5 \, dx$
$+$	$\ln x$	$\frac{1}{6}x^6$
$-$	$\frac{1}{x}$	$\int \frac{1}{6}x^6$

$$\int x^5 \ln x \, dx = \frac{1}{6}x^6 \ln x - \frac{1}{6}\int x^6 \frac{1}{x} dx$$

$$= \frac{1}{6}x^6 \ln x - \frac{1}{6}\int x^5 \, dx$$

$$\boxed{= \frac{1}{6}x^6 \ln x - \frac{1}{36}x^6 + C}$$

Exercise

Evaluate the integral $\displaystyle\int \ln(x+1) dx$

Solution

		$\int dx$
$+$	$\ln(x+1)$	$\frac{1}{2}x$
$-$	$\frac{1}{x+1}$	$\frac{1}{2}\int x$

$$\int \ln(x+1) dx = \frac{1}{2}x \ln(x+1) - \frac{1}{2}\int \frac{x}{x+1} \, dx$$

$$= \frac{1}{2}x \ln(x+1) - \frac{1}{2}\int \left(1 - \frac{1}{x+1}\right) dx$$

$$= \frac{1}{2}x \ln(x+1) - \frac{1}{2}\left(x - \ln(x+1)\right) + C$$

$$= \frac{1}{2}x \ln(x+1) - \frac{1}{2}x + \frac{1}{2}\ln(x+1) + C$$

$$\boxed{= \frac{1}{2}(x+1)\ln(x+1) - \frac{1}{2}x + C}$$

Exercise

Evaluate the integral $\int \frac{\ln x}{x^{10}} dx$

Solution

		$\int x^{-10}\ dx$
$+$	$\ln x$	$-\frac{1}{9}x^{-9}$
$-$	$\frac{1}{x}$	$-\frac{1}{9}\int x^{-9}$

$$\int \frac{\ln x}{x^{10}} dx = -\frac{1}{9x^9}\ln x + \frac{1}{9}\int \frac{1}{x}x^{-9} dx$$

$$= -\frac{1}{9x^9}\ln x + \frac{1}{9}\int x^{-10} dx$$

$$\boxed{= -\frac{\ln x}{9x^9} - \frac{1}{81x^9} + C}$$

Exercise

Evaluate the integral $\int xe^{2x} dx$

Solution

Let: $u = x \Rightarrow du = dx$

$$dv = e^{2x}dx \Rightarrow v = \int dv = \int e^{2x}dx = \frac{1}{2}e^{2x}$$

$$\int udv = uv - \int vdu$$

$$\int x\, e^{2x}dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx$$

$$\boxed{= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C}$$

		$\int e^{2x}\ dx$
$+$	x	$\frac{1}{2}e^{2x}$
$-$	1	$\frac{1}{4}e^{2x}$

$$\boxed{\int xe^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C}$$

Exercise

Evaluate the integral $\int x^3 e^x dx$

Solution

		$\int e^x \, dx$
+	x^3	e^x
−	$3x^2$	e^x
+	$6x$	e^x
−	6	e^x

$$\int x^3 e^x dx = \boxed{e^x\left(x^3 - 3x^2 + 6x - 6\right) + C}$$

Or

Let: $u = x^3 \quad \Rightarrow du = 3x^2 dx$

$$dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$\int x^3 e^x dx = x^3 e^x - \int e^x 3x^2 dx$$

$$= x^3 e^x - 3\int e^x x^2 dx$$

Let: $u = x^2 \quad \Rightarrow du = 2xdx$

$$dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$\int e^x x^2 dx = x^2 e^x - 2\int xe^x dx$$

$$\int x^3 e^x dx = x^3 e^x - 3\left[x^2 e^x - 2\int xe^x dx\right]$$

$$= x^3 e^x - 3x^2 e^x + 6\int xe^x dx$$

Let: $u = x \quad \Rightarrow du = dx$

$$dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x$$

$$\int x^3 e^x dx = x^3 e^x - 3x^2 e^x + 6\left[xe^x - e^x\right] + C$$

$$= x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C$$

$$= e^x \left(x^3 - 3x^2 + 6x - 6\right) + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{2x}{e^x} dx$

Solution

		$\int e^{-x}\ dx$
+	$2x$	$-e^{-x}$
−	2	e^{-x}

$$\int \frac{2x}{e^x} dx = -e^{-x}(2x+2) + C$$

Or

$u = 2x \Rightarrow du = 2dx$

$dv = e^{-x} dx \Rightarrow v = -e^{-x}$

$$\int \frac{2x}{e^x} dx = 2x(-e^{-x}) - \int -e^{-x} 2dx$$

$$= -2xe^{-x} + 2\int e^{-x} dx$$

$$= -2xe^{-x} - 2e^{-x} + C$$

$$= -2e^{-x}(x+1) + C$$

$$= -\frac{2(x+1)}{e^x} + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^3 e^{x^2}}{\left(x^2+1\right)^2}\,dx$

Solution

Let: $u = x^2 e^{x^2} \quad \Rightarrow du = \left(2xe^{x^2} + 2xx^2 e^{x^2}\right)dx$

$$du = 2xe^{x^2}\left(1+x^2\right)dx$$

$$dv = x\left(x^2+1\right)^{-2}dx \qquad \Rightarrow v = \int x(x^2+1)^{-2}dx$$

$$= \frac{1}{2}\int (x^2+1)^{-2}\,d(x^2+1)$$

$$= \frac{(x^2+1)^{-1}}{-1}$$

$$= -\frac{1}{2(x^2+1)}$$

$$\int \frac{x^3 e^{x^2}}{(x^2+1)^2}dx = x^2 e^{x^2}\left(-\frac{1}{2(x^2+1)}\right) - \int -\frac{1}{2(x^2+1)}2xe^{x^2}(x^2+1)dx$$

$$= -\frac{x^2 e^{x^2}}{2(x^2+1)} + \int xe^{x^2}dx$$

$$\int \frac{x^3 e^{x^2}}{\left(x^2+1\right)^2}dx = -\frac{x^2 e^{x^2}}{2(x^2+1)} + \frac{1}{2}\int e^{x^2}\,d\left(x^2\right)$$

$$= -\frac{x^2 e^{x^2}}{2(x^2+1)} + \frac{1}{2}e^{x^2} + C$$

$$= \frac{1}{2}e^{x^2}\left(-\frac{x^2}{(x^2+1)} + 1\right) + C$$

$$= \frac{1}{2}e^{x^2}\left(\frac{-x^2+x^2+1}{(x^2+1)}\right) + C$$

$$= \frac{e^{x^2}}{2(x^2+1)} + C$$

Exercise

Evaluate the integral $\int x^2 e^{-3x}dx$

Solution

		$\int e^{-3x}$
+	x^2	$-\frac{1}{3}e^{-3x}$
−	$2x$	$\frac{1}{9}e^{-3x}$
+	2	$-\frac{1}{27}e^{-3x}$

$$\boxed{\int x^2 e^{-3x}dx = -\frac{1}{3}x^2 e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + C}$$

$u = x^2 \Rightarrow du = 2xdx$

$dv = e^{-3x}dx \Rightarrow v = -\frac{1}{3}e^{-3x}$

$$\int x^2 e^{-3x}dx = -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\int xe^{-3x}dx$$

$u = x \Rightarrow du = dx$

$dv = e^{-3x}dx \Rightarrow v = -\frac{1}{3}e^{-3x}$

$$\int x^2 e^{-3x}dx = -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\left[-\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}dx\right]$$

$$= -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\left(-\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}\right) + C$$

$$= -\frac{1}{3}x^2 e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + C$$

$$\boxed{= -\frac{9x^2 + 6x + 2}{27}e^{-3x} + C}$$

Exercise

Evaluate the integral $\int \left(x^2 - 2x + 1\right) e^{2x} dx$

Solution

		$\int e^{2x}$
+	$x^2 - 2x + 1$	$\frac{1}{2}e^{2x}$
−	$2x - 2$	$\frac{1}{4}e^{2x}$
+	2	$\frac{1}{8}e^{2x}$

$$\int \left(x^2 - 2x + 1\right) e^{2x}\, dx = \frac{1}{2}\left(x^2 - 2x + 1\right)e^{2x} - \frac{1}{4}(2x - 2)e^{2x} + \frac{1}{8}(2)e^{2x} + C$$

$$= \left(\frac{1}{2}x^2 - x + \frac{1}{2} - \frac{1}{2}x + \frac{1}{2} + \frac{1}{4}\right)e^{2x} + C$$

$$\boxed{= \left(\frac{1}{2}x^2 - \frac{3}{2}x + \frac{5}{4}\right)e^{2x} + C}$$

Exercise

Evaluate the integral $\int x^5 e^{x^3} dx$

Solution

Let:

$$u = x^3 \qquad dv = x^2 e^{x^3} dx = \frac{1}{3} d\left(e^{x^3}\right)$$

$$d\left(e^{x^3}\right) = 3x^2 e^{x^3} dx$$

$$du = 3x^2 dx \qquad v = \frac{1}{3}e^{x^3}$$

$$\int x^5 e^{x^3} dx = x^3 \frac{1}{3}e^{x^3} - \int \frac{1}{3}e^{x^3} 3x^2 dx \qquad d\left(e^{x^3}\right) = 3x^2 e^{x^3} dx \qquad \int u\,dv = uv - \int v\,du$$

$$= \frac{1}{3}x^3 e^{x^3} - \frac{1}{3}\int d\left(e^{x^3}\right)$$

$$\boxed{= \frac{1}{3}x^3 e^{x^3} - \frac{1}{3}e^{x^3} + C}$$

Exercise

Evaluate the integral $\displaystyle\int xe^{-4x}\,dx$

Solution

		$\int e^{-4x}\,dx$
+	x	$-\frac{1}{4}e^{-4x}$
−	1	$\frac{1}{16}e^{-4x}$

$$\int xe^{-4x}\,dx = \boxed{\left(-\frac{x}{4}-\frac{1}{16}\right)e^{-4x}+C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{xe^{2x}}{(2x+1)^2}dx$

Solution

$$u = xe^{2x} \quad \to \quad du = (2x+1)e^{2x}dx$$

$$\begin{aligned} v &= \int \frac{dx}{(2x+1)^2} \\ &= \frac{1}{2}\int \frac{1}{(2x+1)^2}d(2x+1) \\ &= \boxed{-\frac{1}{2}\frac{1}{2x+1}} \end{aligned}$$

$$\begin{aligned} \int \frac{xe^{2x}}{(2x+1)^2}\,dx &= -\frac{xe^{2x}}{4x+2}+\frac{1}{2}\int e^{2x}dx \\ &= \boxed{-\frac{x}{4x+2}e^{2x}+\frac{1}{4}e^{2x}+C} \end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{5x}{e^{2x}}\,dx$

Solution

		$\int e^{-2x}\,dx$
+	$5x$	$-\frac{1}{2}e^{-2x}$
−	5	$\frac{1}{4}e^{-2x}$

$$\int \frac{5x}{e^{2x}}\,dx = \int 5xe^{-2x}\,dx$$

$$= \left(-\frac{5}{2}x - \frac{5}{4}\right)e^{-2x} + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx$

Solution

$$\int \frac{e^{1/x}}{x^2}\,dx = -\int e^{1/x}\,d\left(\frac{1}{x}\right)$$

$$= -e^{1/x} + C$$

Exercise

Evaluate the integral $\displaystyle\int x^2 e^{4x}\,dx$

Solution

$$\int x^2 e^{4x}\,dx = \left(\frac{1}{4}x^2 - \frac{1}{8}x + \frac{1}{32}\right)e^{4x} + C$$

$$\int x^n e^{ax}\,dx = e^{ax}\sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!a^{k+1}} x^{n-k}$$

Exercise

Evaluate the integral $\displaystyle\int x^3 e^{-3x}\,dx$

Solution

$$\int x^3 e^{-3x}\,dx = \left(-\frac{1}{3}x^3 + \frac{1}{3}x^2 - \frac{2}{9}x + \frac{2}{27}\right)e^{-3x} + C$$

$$\int x^n e^{ax}\,dx = e^{ax}\sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!a^{k+1}} x^{n-k}$$

Exercise

Evaluate the integral $\int x^4 e^x dx$

Solution

		$\int e^x\, dy$
+	x^4	e^x
−	$4x^3$	e^x
+	$12x^2$	e^x
−	$24x$	e^x
+	24	e^x

$$\int x^4 e^x dx = \left(x^4 + 4x^3 + 12x^2 + 24x + 24\right)e^x + C$$

Exercise

Evaluate the integral $\int x^3 e^{4x} dx$

Solution

$$\int x^3 e^{4x} dx = e^{4x}\left(\frac{1}{4}x^3 - \frac{3}{16}x^2 + \frac{3}{32}x - \frac{3}{128}\right) + C$$

$$\int x^n e^{ax}\, dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k n!}{(n-k)!a^{k+1}}x^{n-k}$$

Exercise

Evaluate the integral $\int (x+1)^2 e^x dx$

Solution

		$\int e^x\, dx$
+	$(x+1)^2$	e^x
−	$2(x+1)$	e^x
+	2	e^x

$$\int (x+1)^2 e^x dx = e^x\left[(x+1)^2 - 2(x+1) + 2\right] + C$$

$$= e^x\left(x^2+2x+1-2x-2+2\right)+C$$

$$\boxed{= e^x\left(x^2+1\right)+C}$$

Exercise

Evaluate the integral $\displaystyle\int 2xe^{3x}\,dx$

Solution

		$\int e^{3x}\,dx$
$+$	$2x$	$\frac{1}{3}e^{3x}$
$-$	2	$\frac{1}{9}e^{3x}$

$$\int 2xe^{3x}\,dx = e^{3x}\left(\frac{2}{3}x-\frac{2}{9}\right)+C$$

$$\boxed{=\frac{2}{9}e^{3x}(3x-1)+C}$$

Exercise

Evaluate the integral $\displaystyle\int x^2\sin x\,dx$

Solution

		$\int \sin x$
x^2	$(+)$	$-\cos x$
$2x$	$(-)$	$-\sin x$
2	$(+)$	$\cos x$
0		

$$\boxed{\int x^2\sin x\,dx = -x^2\cos x+2x\sin x+2\cos x+C}$$

Exercise

Evaluate the integral $\int \theta \cos \pi\theta \, d\theta$

Solution

Let:

$u = \theta \quad \to \quad du = d\theta$

$$v = \int \cos \pi\theta d\theta$$

$$= \frac{1}{\pi} \sin \pi\theta$$

$$\int \theta \cos \pi\theta \, d\theta = \frac{\theta}{\pi} \sin \pi\theta - \int \frac{1}{\pi} \sin \pi\theta \, d\theta$$

$$= \frac{\theta}{\pi} \sin \pi\theta + \frac{1}{\pi}\frac{1}{\pi} \cos \pi\theta + C$$

$$= \frac{\theta}{\pi} \sin \pi\theta + \frac{1}{\pi^2} \cos \pi\theta + C$$

Exercise

Evaluate the integral $\int 4x \sec^2 2x \, dx$

Solution

Let: $u = 4x \quad \to \quad du = 4$

$$v = \int \sec^2 2x \, dx$$

$$= \frac{1}{2} \tan 2x$$

$$\int 4x \sec^2 2x \, dx = 2x \tan 2x - \int 4\left(\frac{1}{2} \tan 2x\right) dx$$

$$= 2x \tan 2x - 2\frac{1}{2} \ln \left|\sec 2x\right| + C$$

$$= 2x \tan 2x - \ln \left|\sec 2x\right| + C$$

Exercise

Evaluate the integral $\int x^3 \sin x \, dx$

Solution

		$\int \sin x$
$+$	x^3	$-\cos x$
$-$	$3x^2$	$-\sin x$
$+$	$6x$	$\cos x$
$-$	6	$\sin x$

$$\boxed{\int x^3 \sin x\, dx = -x^3 \cos x + 3x^2 \sin x + 6x\cos x - 6\sin x + C}$$

Exercise

Evaluate the integral $\int \left(x^3 - 2x\right)\sin 2x\; dx$

Solution

		$\int \sin 2x\; dx$
$+$	$x^3 - 2x$	$-\frac{1}{2}\cos 2x$
$-$	$3x^2 - 2$	$-\frac{1}{4}\sin 2x$
$+$	$6x$	$\frac{1}{8}\cos 2x$
$-$	6	$\frac{1}{16}\sin 2x$

$$\begin{aligned}\int \left(x^3 - 2x\right)\sin 2x\; dx &= -\frac{1}{2}\left(x^3 - 2x\right)\cos 2x + \frac{1}{4}\left(3x^2 - 2\right)\sin 2x + \frac{3}{4}x\cos 2x - \frac{3}{8}\sin 2x + C \\ &= \left(-\frac{1}{2}x^3 + x + \frac{3}{4}x\right)\cos 2x + \left(\frac{3}{4}x^2 - \frac{1}{2} - \frac{3}{8}\right)\sin 2x + C \\ &= \boxed{\left(-\frac{1}{2}x^3 + \frac{7}{4}x\right)\cos 2x + \left(\frac{3}{4}x^2 - \frac{7}{8}\right)\sin 2x + C}\end{aligned}$$

Exercise

Evaluate the integral $\int x^2 \sin 2x\, dx$

Solution

		$\int \sin 2x dx$
+	x^2	$-\frac{1}{2}\cos 2x$
−	$2x$	$-\frac{1}{4}\sin 2x$
+	2	$\frac{1}{8}\cos 2x$

$$\int x^2 \sin 2x\, dx = -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x + \frac{1}{4}\cos 2x + C$$

$$= -\frac{1}{4}\left(2x^2 - 1\right)\cos 2x + \frac{1}{2}x \sin 2x + C$$

Exercise

Evaluate the integral $\int x^2 \sin(1-x)\, dx$

Solution

		$\int \sin(1-x) dx$
+	x^2	$\cos(1-x)$
−	$2x$	$-\sin(1-x)$
+	2	$\cos(1-x)$

$$\int x^2 \sin(1-x)\, dx = x^2 \cos(1-x) + 2x \sin(1-x) + 2\cos(1-x) + C$$

$$= \left(x^2 + 2\right)\cos(1-x) + 2x \sin(1-x) + C$$

Exercise

Evaluate the integral $\int x \sin x \cos x\, dx$

Solution

		$\int \sin 2x\, dx$
$+$	x	$-\frac{1}{2}\cos 2x$
$-$	1	$-\frac{1}{4}\sin 2x$

$$\int x\sin x\cos x\, dx = \frac{1}{2}\int x\sin 2x\, dx$$

$$= \frac{1}{2}\left(-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right)$$

$$= -\frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x + C$$

Exercise

Evaluate the integral $\int x\cos x\, dx$

Solution

		$\int \cos x$
$+$	x	$\sin x$
$-$	1	$-\cos x$

$$\int x\cos x\, dx = x\sin x + \cos x + C$$

Exercise

Evaluate the integral $\int x\csc x\cot x\, dx$

Solution

$u = x \quad \rightarrow \quad du = dx$

$dv = \csc x\cot x\, dx \quad \rightarrow \quad v = -\csc x$

$$\int x\csc x\cot x\, dx = -x\csc x + \int \csc x\, dx$$

$$= -x\csc x - \ln\left|\csc x + \cot x\right| + C$$

Exercise

Evaluate the integral $\int x^2 \cos x \, dx$

Solution

		$\int \cos x$
+	x^2	$\sin x$
–	$2x$	$-\cos x$
+	2	$-\sin x$

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x\cos x - 2\sin x + C$$

Exercise

Evaluate the integral $\int x^3 \cos 2x \, dx$

Solution

		$\int \cos 2x$
+	x^3	$\frac{1}{2}\sin 2x$
–	$3x^2$	$-\frac{1}{4}\cos 2x$
+	$6x$	$-\frac{1}{8}\sin 2x$
–	6	$\frac{1}{16}\cos 2x$

$$\int x^3 \cos 2x \, dx = \frac{1}{2}x^3 \sin 2x + \frac{3}{4}x^2 \cos 2x - \frac{3}{4}x\sin 2x - \frac{3}{8}\cos 2x + C$$

$$= \left(\frac{1}{2}x^3 - \frac{3}{4}x\right)\sin 2x + \left(\frac{3}{4}x^2 - \frac{3}{8}\right)\cos 2x + C$$

Exercise

Evaluate the integral $\int \frac{\cos\sqrt{x}}{\sqrt{x}} \, dx$

Solution

$$d\left(\sqrt{x}\right)=\frac{1}{2\sqrt{x}}\,dx$$

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx=2\int \cos\sqrt{x}\;d\left(\sqrt{x}\right)$$

$$=2\sin\sqrt{x}+C$$

Exercise

Evaluate the integral $\displaystyle\int x\sinh x\,dx$

Solution

		$\int \sinh x\,dx$
+	x	$\cosh x$
−	1	$\sinh x$

$$\int x\sinh x\,dx=x\cosh x-\sinh x+C$$

Exercise

Evaluate the integral $\displaystyle\int x^2\cosh x\,dx$

Solution

		$\int \cosh x$
+	x^2	$\sinh x$
−	$2x$	$\cosh x$
+	2	$\sinh x$

$$\int x^2\cosh x\,dx=x^2\sinh x-2x\cosh x+2\sinh x+C$$

$$=\left(x^2+2\right)\sinh x-2x\cosh x+C$$

Exercise

Evaluate the integral $\displaystyle\int e^{2x}\cos 3x\,dx$

Solution

		$\displaystyle\int \cos 3x\ dx$
+	e^{2x}	$\frac{1}{3}\sin 3x$
−	$2e^{2x}$	$-\frac{1}{9}\cos 3x$
+	$4e^{2x}$	$-\frac{1}{9}\displaystyle\int \cos 3x\ dx$

$$\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x - \frac{4}{9}\int e^{2x}\cos 3x\,dx$$

$$\left(1+\frac{4}{9}\right)\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x$$

$$\frac{13}{9}\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x$$

$$\int e^{2x}\cos 3x\,dx = \frac{e^{2x}}{13}(3\sin 3x + 2\cos 3x) + C$$

Exercise

Evaluate the integral $\displaystyle\int e^{-3x}\sin 5x\ dx$

Solution

		$\displaystyle\int \sin 5x$
+	e^{-3x}	$-\frac{1}{5}\cos 5x$
−	$-3e^{-3x}$	$-\frac{1}{25}\sin 5x$
+	$9e^{-3x}$	$-\displaystyle\int \frac{1}{25}\sin 5x$

$$\int e^{-3x}\sin 5x\ dx = -\frac{1}{5}e^{-3x}\cos 5x - \frac{3}{25}e^{-3x}\sin 5x - \frac{9}{25}\int e^{-3x}\sin 5x\ dx$$

$$\left(1+\frac{9}{25}\right)\int e^{-3x}\sin 5x\,dx = -\frac{1}{25}(5\cos 5x+3\sin 5x)e^{-3x}$$

$$\frac{34}{25}\int e^{-3x}\sin 5x\,dx = -\frac{1}{25}(5\cos 5x+3\sin 5x)e^{-3x}$$

$$\int e^{-3x}\sin 5x\,dx = \boxed{-\frac{1}{34}(5\cos 5x+3\sin 5x)e^{-3x}+C}$$

Exercise

Evaluate the integral $\int e^{-x}\sin 4x\,dx$

Solution

		$\int \sin 4x\,dx$
+	e^{-x}	$-\frac{1}{4}\cos 4x$
−	$-e^{-x}$	$-\frac{1}{16}\sin 4x$
+	e^{-x}	$-\frac{1}{16}\int \sin 4x\,dx$

$$\int e^{-x}\sin 4x\,dx = -\frac{1}{4}e^{-x}\cos 4x - \frac{1}{16}e^{-x}\sin 4x - \frac{1}{16}\int e^{-x}\sin 4x\,dx$$

$$\left(1+\frac{1}{16}\right)\int e^{-x}\sin 4x\,dx = -\frac{1}{4}e^{-x}\cos 4x - \frac{1}{16}e^{-x}\sin 4x$$

$$\frac{17}{16}\int e^{-x}\sin 4x\,dx = -\frac{1}{16}e^{-x}(4\cos 4x+\sin 4x)$$

$$\int e^{-x}\sin 4x\,dx = \boxed{-\frac{e^{-x}}{17}(4\cos 4x+\sin 4x)+C}$$

Exercise

Evaluate the integral $\int e^{-2\theta}\sin 6\theta\,d\theta$

Solution

		$\int \sin 6\theta \, d\theta$
+	$e^{-2\theta}$	$-\frac{1}{6}\cos 6\theta$
−	$-2e^{-2\theta}$	$-\frac{1}{36}\sin 6\theta$
+	$4e^{-2\theta}$	$-\frac{1}{36}\int \sin 6\theta \, d\theta$

$$\int e^{-2\theta} \sin 6\theta \, d\theta = -\frac{1}{6}e^{-2\theta}\cos 6\theta - \frac{1}{18}e^{-2\theta}\sin 6\theta - \frac{1}{9}\int e^{-2\theta}\sin 6\theta \, d\theta$$

$$\left(1+\frac{1}{9}\right)\int e^{-2\theta}\sin 6\theta \, d\theta = -\frac{1}{18}e^{-2\theta}\left(3\cos 6\theta + \sin 6\theta\right)$$

$$\frac{10}{9}\int e^{-2\theta}\sin 6\theta \, d\theta = -\frac{1}{18}e^{-2\theta}\left(3\cos 6\theta + \sin 6\theta\right)$$

$$\int e^{-2\theta}\sin 6\theta \, d\theta = -\frac{e^{-2\theta}}{20}\left(3\cos 6\theta + \sin 6\theta\right) + C$$

Exercise

Evaluate the integral $\int e^{-3x} \sin 4x \, dx$

Solution

		$\int \sin 4x$
+	e^{-3x}	$-\frac{1}{4}\cos 4x$
−	$-3e^{-3x}$	$-\frac{1}{16}\sin 4x$
+	$9e^{-3x}$	$-\frac{1}{16}\int \sin 4x$

$$\int e^{-3x}\sin 4x \, dx = -\frac{1}{4}e^{-3x}\cos 4x - \frac{3}{16}e^{-3x}\sin 4x - \frac{9}{16}\int e^{-3x}\sin 4x \, dx$$

$$\left(1+\frac{9}{16}\right)\int e^{-3x}\sin 4x \, dx = -\frac{1}{16}\left(4\cos 4x + 3\sin 4x\right)e^{-3x}$$

$$\frac{25}{16}\int e^{-3x}\sin 4x \, dx = -\frac{1}{16}\left(4\cos 4x + 3\sin 4x\right)e^{-3x}$$

$$\int e^{-3x}\sin 4x \, dx = -\frac{1}{25}\left(4\cos 4x + 3\sin 4x\right)e^{-3x} + C$$

Exercise

Evaluate the integral $\int e^{4x} \cos 2x \, dx$

Solution

		$\int \cos 2x \, dx$
$+$	e^{4x}	$\frac{1}{2}\sin 2x$
$-$	$4e^{4x}$	$-\frac{1}{4}\cos 2x$
$+$	$16e^{4x}$	$-\frac{1}{4}\int \cos 2x$

$$\int e^{4x} \cos 2x \, dx = \frac{1}{2}e^{4x} \sin 2x + e^{4x} \cos 2x - 4\int e^{4x} \cos 2x \, dx$$

$$5\int e^{4x} \cos 2x \, dx = \frac{1}{2}(\sin 2x + 2\cos 2x)e^{4x}$$

$$\int e^{4x} \cos 2x \, dx = \frac{1}{10}(\sin 2x + 2\cos 2x)e^{4x} + C$$

Exercise

Evaluate the integral $\int e^{3x} \cos 3x \, dx$

Solution

		$\int \cos 3x$
$+$	e^{3x}	$\frac{1}{3}\sin 3x$
$-$	$3e^{3x}$	$-\frac{1}{9}\cos 3x$
$+$	$9e^{3x}$	$-\frac{1}{9}\int \cos 3x$

$$\int e^{3x} \cos 3x \, dx = \frac{1}{3}e^{3x} \sin 3x + \frac{1}{3}e^{3x} \cos 3x - \int e^{3x} \cos 3x \, dx$$

$$2\int e^{3x} \cos 3x \, dx = \frac{1}{3}(\sin 3x + \cos 3x)e^{3x}$$

$$\int e^{3x} \cos 3x \, dx = \frac{1}{6}(\sin 3x + \cos 3x)e^{3x} + C$$

Exercise

Evaluate the integral $\int e^{3x} \cos 2x\, dx$

Solution

		$\int \cos 2x$
+	e^{3x}	$\frac{1}{2}\sin 2x$
−	$3e^{3x}$	$-\frac{1}{4}\cos 2x$
+	$9e^{3x}$	$-\frac{1}{4}\int \cos 2x$

$$\int e^{3x} \cos 2x\, dx = e^{3x}\left(\frac{1}{2}\sin 2x + \frac{3}{4}\cos 2x\right) - \frac{9}{4}\int e^{3x}\cos 2x\, dx$$

$$\left(1+\frac{9}{4}\right)\int e^{3x} \cos 2x\, dx = \frac{1}{4}e^{3x}\left(2\sin 2x + 3\cos 2x\right)$$

$$\frac{13}{4}\int e^{3x} \cos 2x\, dx = \frac{1}{4}e^{3x}\left(2\sin 2x + 3\cos 2x\right)$$

$$\int e^{3x} \cos 2x\, dx = \frac{1}{13}e^{3x}\left(2\sin 2x + 3\cos 2x\right) + C$$

Exercise

Evaluate the integral $\int e^{x} \sin x\, dx$

Solution

		$\int \sin x$
+	e^{x}	$-\cos x$
−	e^{x}	$-\sin x$
+	e^{x}	$-\int \sin x$

$$\int e^{x} \sin x\, dx = e^{x}\left(-\cos x + \sin x\right) - \int e^{x}\sin x\, dx$$

$$2\int e^{x} \sin x\, dx = e^{x}\left(\sin x - \cos x\right)$$

$$\int e^x \sin x\, dx = \boxed{\frac{1}{2}e^x(\sin x - \cos x) + C}$$

Exercise

Evaluate the integral $\displaystyle\int e^{-2x} \sin 3x dx$

Solution

		$\int \sin 3x$
+	e^{-2x}	$-\frac{1}{3}\cos 3x$
−	$-2e^{-2x}$	$-\frac{1}{9}\sin 3x$
+	$4e^{-2x}$	$-\frac{1}{9}\int \sin 3x$

$$\int e^{-2x} \sin 3x\, dx = e^{-2x}\left(-\frac{1}{3}\cos 3x - \frac{2}{9}\sin 3x\right) - \frac{4}{9}\int e^{-2x}\sin 3x\, dx$$

$$\left(1+\frac{4}{9}\right)\int e^{-2x}\sin 3x\, dx = -\frac{1}{9}e^{-2x}(3\cos 3x + 2\sin 3x)$$

$$\frac{13}{9}\int e^{-2x}\sin 3x\, dx = -\frac{1}{9}e^{-2x}(3\cos 3x + 2\sin 3x)$$

$$\int e^{-2x}\sin 3x\, dx = \boxed{-\frac{1}{13}e^{-2x}(3\cos 3x + 2\sin 3x) + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x}{\sqrt{x-1}}dx$

Solution

Let: $u = x \Rightarrow du = dx$

$$dv = \frac{dx}{\sqrt{x-1}} \Rightarrow v = \int (x-1)^{-1/2}d(x-1)$$

$$= \frac{(x-1)^{1/2}}{1/2}$$

$$= 2(x-1)^{1/2}$$

$$\int \frac{x}{\sqrt{x-1}}dx = 2x\sqrt{x-1} - 2\int (x-1)^{1/2}dx$$

$$= 2x\sqrt{x-1} - 2\frac{(x-1)^{3/2}}{3/2} + C$$

$$= 2x\sqrt{x-1} - \frac{4}{3}(x-1)\sqrt{x-1} + C$$

$$= \sqrt{x-1}\left[2x - \frac{4}{3}x + \frac{4}{3}\right] + C$$

$$= \sqrt{x-1}\left[\frac{6x-4x+4}{3}\right] + C$$

$$= \sqrt{x-1}\left[\frac{2x+4}{3}\right] + C$$

$$= \frac{2}{3}\sqrt{x-1}(x+2) + C$$

Or

Let: $u = x-1 \quad \Rightarrow \quad x = u+1$

$du = dx$

$$\int \frac{x}{\sqrt{x-1}}dx = \int (u+1)u^{-1/2}du$$

$$= \int \left(u^{1/2} + u^{-1/2}\right)du$$

$$= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + C$$

$$= (x-1)^{1/2}\left(\frac{2}{3}x - \frac{2}{3} + 2\right) + C$$

$$= \sqrt{x-1}\left[\frac{2x+4}{3}\right] + C$$

$$= \frac{2}{3}\sqrt{x-1}(x+2) + C$$

Exercise

Evaluate the integral $\int x\sqrt{x-5}\ dx$

Solution

Let $u = \sqrt{x-5}$

$u^2 = x-5 \Rightarrow x = u^2 + 5$

$$2udu = dx$$

$$\int x\sqrt{x-5}\,dx = \int \left(u^2+5\right)u\left(2u\,du\right)$$

$$= \int \left(2u^4+10u^2\right)du$$

$$\boxed{= \frac{2}{5}u^5 + \frac{10}{3}u^3 + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x}{\sqrt{6x+1}}\,dx$

Solution

$$u = x \quad \to \quad du = dx$$

$$dv = (6x+1)^{-1/2}\,dx$$

$$= \frac{1}{6}(6x+1)^{-1/2}\,d(6x+1)$$

$$v = \frac{1}{3}(6x+1)^{1/2}$$

$$\int \frac{x}{\sqrt{6x+1}}\,dx = \frac{1}{3}x\sqrt{6x+1} - \frac{1}{3}\int (6x+1)^{1/2}\,dx$$

$$= \frac{1}{3}x\sqrt{6x+1} - \frac{1}{18}\int (6x+1)^{1/2}\,d(6x+1)$$

$$\boxed{= \frac{1}{3}x\sqrt{6x+1} - \frac{1}{27}(6x+1)^{3/2} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x}{2\sqrt{x+2}}\,dx$

Solution

		$\int (x+2)^{-1/2}\,d(x+2)$
+	x	$2(x+2)^{1/2}$
−	1	$\frac{4}{3}(x+2)^{3/2}$

$$\int \frac{x}{2\sqrt{x+2}}\,dx = \frac{1}{2}\int x(x+2)^{-1/2}\,dx$$

$$= \frac{1}{2}\left[2x\sqrt{x+2} - \frac{4}{3}(x+2)^{3/2}\right] + C$$

$$= \frac{1}{3}\sqrt{x+2}\,(3x - 2(x+2)) + C$$

$$= \frac{1}{3}\sqrt{x+2}\,(3x - 2x - 4) + C$$

$$= \frac{1}{3}\sqrt{x+2}\,(x-4) + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{2x^2 - 3x}{(x-1)^3}\,dx$

Solution

		$\int (x-1)^{-3}\,d(x-1)$
+	$2x^2 - 3x$	$-\frac{1}{2}(x-1)^{-2}$
−	$4x - 3$	$\frac{1}{2}(x-1)^{-1}$
+	4	$\frac{1}{2}\ln\lvert x-1\rvert$

$$\int \frac{2x^2 - 3x}{(x-1)^3}\,dx = -\frac{1}{2}\frac{2x^2 - 3x}{(x-1)^2} - \frac{1}{2}\frac{4x-3}{x-1} + 2\ln\lvert x-1\rvert + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^2 + 3x + 4}{\sqrt[3]{2x+1}}\,dx$

Solution

		$\frac{1}{2}\int (2x+1)^{-1/3}\,d(2x+1)$
+	$x^2 + 3x + x$	$\frac{3}{4}(2x+1)^{2/3}$
−	$2x + 3$	$\frac{1}{2}\frac{9}{20}(2x+1)^{5/3}$
+	2	$\frac{1}{2}\frac{27}{320}(2x+1)^{8/3}$

$$\int \frac{x^2+3x+4}{\sqrt[3]{2x+1}}dx = \frac{3}{4}\left(x^2+3x+4\right)(2x+1)^{2/3} - \frac{9}{40}(2x+3)(2x+1)^{5/3} + \frac{27}{320}(2x+1)^{8/3} + C$$

Exercise

Evaluate the integral $\int \frac{x}{\sqrt{x+1}}dx$

Solution

		$\int (x+1)^{-1/2}\ dx$
+	x	$2(x+1)^{1/2}$
−	1	$\frac{4}{3}(x+1)^{3/2}$

$$\int \frac{x}{\sqrt{x+1}}dx = 2x(x+1)^{1/2} - \frac{4}{3}(x+1)^{3/2} + C$$

Exercise

Evaluate the integral $\int \frac{x^5}{\sqrt{1-2x^3}}dx$

Solution

		$\int x^2\left(1-2x^3\right)^{-1/2}\ dx = -\frac{1}{6}\int \left(1-2x^3\right)^{-1/2}\ d\left(1-2x^3\right)$
+	x^3	$-\frac{1}{3}\left(1-2x^3\right)^{1/2}$
−	$3x^2$	$\int -\frac{1}{3}\left(1-2x^3\right)^{1/2}$

$$\begin{aligned}\int \frac{x^5}{\sqrt{1-2x^3}}dx &= -\frac{1}{3}x^3\sqrt{1-2x^3} + \int x^2\left(1-2x^3\right)^{1/2}\ dx \\ &= -\frac{1}{3}x^3\sqrt{1-2x^3} - \frac{1}{6}\int \left(1-2x^3\right)^{1/2}\ d\left(1-2x^3\right) \\ &= -\frac{1}{3}x^3\sqrt{1-2x^3} - \frac{1}{9}\left(1-2x^3\right)^{3/2} + C\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int x\sqrt{1-3x}\ dx$

Solution

		$\int (1-3x)^{1/2}\,dx = -\frac{1}{3}\int (1-3x)^{1/2}\,d(1-3x)$
+	x	$-\frac{2}{9}(1-3x)^{3/2}$
−	1	$-\left(-\frac{1}{3}\right)\frac{2}{9}\frac{2}{5}(1-3x)^{5/2}$

$$\int x\sqrt{1-3x}\ dx = -\frac{2x}{9}(1-3x)^{3/2} - \frac{4}{135}(1-3x)^{5/2} + C$$

Exercise

Evaluate the integral $\displaystyle\int \sin(\ln x)\ dx$

Solution

		$\int dx$
+	$\sin(\ln x)$	x
−	$\frac{\cos(\ln x)}{x}$	$\int x\,dx$

$$\int \sin(\ln x)\ dx = x\sin(\ln x) - \int x\frac{\cos(\ln x)}{x}dx$$

$$= x\sin(\ln x) - \int \cos(\ln x)\ dx$$

		$\int dx$
+	$\cos(\ln x)$	x
−	$-\frac{\sin(\ln x)}{x}$	$\int x\,dx$

$$= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\ dx$$

$$2\int \sin(\ln x)\ dx = x\sin(\ln x) - x\cos(\ln x)$$

$$\int \sin(\ln x)\, dx = \boxed{\frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x) + C}$$

Exercise

Evaluate the integral $\int \tan^{-1} y\, dy$

Solution

Let: $u = \tan^{-1} y \quad dv = dy$

$du = \dfrac{dy}{1+y^2} \quad v = y$

$$\int \tan^{-1} y\, dy = y\tan^{-1} y - \int \frac{y}{1+y^2}dy \qquad d\left(1+y^2\right) = 2ydy \quad \rightarrow \quad \tfrac{1}{2}d\left(1+y^2\right) = ydy$$

$$= y\tan^{-1} y - \frac{1}{2}\int \frac{1}{1+y^2}\, d\left(1+y^2\right)$$

$$= y\tan^{-1} y - \frac{1}{2}\ln\left(1+y^2\right) + C$$

$$\boxed{= y\tan^{-1} y - \ln\sqrt{1+y^2} + C}$$

Exercise

Evaluate the integral $\int \sin^{-1} y\, dy$

Solution

Let: $u = \sin^{-1} y \quad dv = dy$

$du = \dfrac{dy}{\sqrt{1-y^2}} \quad v = y$

$$\int \sin^{-1} y\, dy = y\sin^{-1} y - \int \frac{y}{\sqrt{1-y^2}}dy \qquad d\left(1-y^2\right) = -2ydy \quad \rightarrow \quad -\tfrac{1}{2}d\left(1-y^2\right) = ydy$$

$$= y\sin^{-1} y + \frac{1}{2}\int \left(1-y^2\right)^{-1/2} d\left(1-y^2\right)$$

$$= y\sin^{-1} y + \frac{1}{2}(2)\left(1-y^2\right)^{1/2} + C$$

$$\boxed{= y\sin^{-1} y + \sqrt{1-y^2} + C}$$

Or

		$\int dy$
+	$\sin^{-1} y$	y
−	$\frac{1}{\sqrt{1-y^2}}$	$\int y$

$$\int \sin^{-1} y\, dy = y\sin^{-1} y - \int \frac{y}{\sqrt{1-y^2}}\, dy$$

$$= y\sin^{-1} y + \frac{1}{2}\int \left(1-y^2\right)^{-1/2} d\left(1-y^2\right)$$

$$\boxed{= y\sin^{-1} y + \sqrt{1-y^2} + C}$$

Exercise

Evaluate the integral $\int x \tan^{-1} x\, dx$

Solution

$u = \tan^{-1} x \qquad v = \int x dx$

$du = \frac{dx}{x^2+1} \qquad v = \frac{1}{2}x^2$

$$\int x\tan^{-1} x\, dx = \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \frac{x^2}{x^2+1} dx$$

$$= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \left(1 - \frac{1}{x^2+1}\right) dx$$

$$= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\left(x - \tan^{-1} x\right) + C$$

$$= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + C$$

$$\boxed{= \frac{1}{2}\left(x^2+1\right)\tan^{-1} x - \frac{1}{2}x + C}$$

Exercise

Evaluate the integral $\int \sinh^{-1} x \, dx$

Solution

		$\int dx$
+	$\sinh^{-1} x$	x
−	$\frac{1}{\sqrt{x^2+1}}$	$\int x \, dx$

$$\int \sinh^{-1} x \, dx = x \sinh^{-1} x - \int \frac{x}{\sqrt{x^2+1}} dx$$

$$= x \sinh^{-1} x - \frac{1}{2} \int \left(x^2+1\right)^{-1/2} d\left(x^2+1\right)$$

$$\boxed{= x \sinh^{-1} x - \sqrt{x^2+1} + C}$$

Exercise

Evaluate the integral $\int \tan^{-1} 3x \, dx$

Solution

		$\int dx$
+	$\tan^{-1} 3x$	x
−	$\frac{3}{9x^2+1}$	$\int x \, dx$

$$\int \tan^{-1} 3x \, dx = x \tan^{-1} 3x - \int \frac{3x}{9x^2+1} dx$$

$$= x \tan^{-1} 3x - \frac{1}{6} \int \frac{1}{9x^2+1} d\left(9x^2+1\right)$$

$$\boxed{= x \tan^{-1} 3x - \frac{1}{6} \ln\left(9x^2+1\right) + C}$$

Exercise

Evaluate the integral $\displaystyle\int \cos^{-1}\left(\frac{x}{2}\right) dx$

Solution

		$\int dx$
+	$\cos^{-1}\left(\frac{x}{2}\right)$	x
−	$\dfrac{\frac{1}{2}}{\sqrt{1-\frac{1}{4}x^2}}$	$\int x\, dx$

$$\frac{\frac{1}{2}}{\sqrt{1-\frac{1}{4}x^2}} = \frac{\frac{1}{2}}{\sqrt{\frac{4-x^2}{4}}}$$

$$= \frac{1}{\sqrt{4-x^2}}$$

$$\int \cos^{-1}\left(\frac{x}{2}\right) dx = x\cos^{-1}\left(\frac{x}{2}\right) - \int \frac{x}{\sqrt{4-x^2}} dx$$

$$= x\cos^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}\int \left(4-x^2\right)^{-1/2} d\left(4-x^2\right)$$

$$= x\cos^{-1}\left(\frac{x}{2}\right) + \sqrt{4-x^2} + C$$

Exercise

Evaluate the integral $\displaystyle\int x\sec^{-1} x\, dx \quad x \ge 1$

Solution

		$\int x\, dx$
+	$\sec^{-1} x$	$\frac{1}{2}x^2$
−	$\dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{1}{2}\int x^2$

$$\int x\sec^{-1}x\,dx = \frac{1}{2}x^2\sin^{-1}x - \frac{1}{2}\int \frac{1}{x\sqrt{x^2-1}}x^2dx$$

$$= \frac{1}{2}x^2\sin^{-1}x - \frac{1}{2}\int x\left(x^2-1\right)^{-1/2}dx$$

$$= \frac{1}{2}x^2\sin^{-1}x - \frac{1}{4}\int \left(x^2-1\right)^{-1/2}d\left(x^2-1\right)$$

$$= \frac{1}{2}x^2\sin^{-1}x - \frac{1}{2}\sqrt{x^2-1} + C$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^{0} 2x^2\sqrt{x+1}\,dx$

Solution

		$\int (x+1)^{1/2}d(x+1)$
$+$	$2x^2$	$\frac{2}{3}(x+1)^{3/2}$
$-$	$4x$	$\frac{4}{15}(x+1)^{5/2}$
$+$	4	$\frac{8}{105}(x+1)^{7/2}$

$$\int_{-1}^{0} 2x^2\sqrt{x+1}\,dx = \frac{4}{3}x^2(x+1)^{3/2} - \frac{16}{15}x(x+1)^{5/2} + \frac{32}{105}(x+1)^{7/2}\Big|_{-1}^{0}$$

$$= \frac{32}{105}$$

Exercise

Evaluate the integral $\displaystyle\int_{0}^{1/\sqrt{2}} x\tan^{-1}x^2\,dx$

Solution

$$\int_{0}^{1/\sqrt{2}} x\tan^{-1}x^2\,dx = \frac{1}{2}\int_{0}^{1/\sqrt{2}} \tan^{-1}x^2\,d\left(x^2\right)$$

$$= \frac{1}{2}\int_0^{1/\sqrt{2}} \tan^{-1} y\, dy \qquad \left(Let\ \ y = x^2\right)$$

		$\int dy$
+	$\tan^{-1} y$	y
−	$\frac{1}{1+y^2}$	$\int y\, dy$

$$\int_0^{1/\sqrt{2}} x\tan^{-1} x^2\, dx = \frac{1}{2}\left[y\tan^{-1} y \Big|_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} \frac{y}{1+y^2} dy \right]$$

$$= \frac{1}{2}x^2 \tan^{-1} x^2 \Big|_0^{1/\sqrt{2}} - \frac{1}{4}\int_0^{1/\sqrt{2}} \frac{1}{1+y^2} d\left(1+y^2\right)$$

$$= \frac{1}{4}\tan^{-1}\frac{1}{2} - \frac{1}{4}\ln\left(1+x^4\right) \Big|_0^{1/\sqrt{2}}$$

$$= \frac{1}{4}\tan^{-1}\frac{1}{2} - \frac{1}{4}\ln\left(1+\frac{1}{4}\right)$$

$$= \frac{1}{4}\tan^{-1}\frac{1}{2} - \frac{1}{4}\ln\frac{5}{4}$$

Exercise

Evaluate the integral $\displaystyle\int_1^e x^2 \ln x\, dx$

Solution

		$\int x^2\, dx$
+	$\ln x$	$\frac{1}{3}x^3$
−	$\frac{1}{x}$	$\frac{1}{3}\int x^3 dx$

$$\int_1^e x^2 \ln x\, dx = \frac{1}{3}x^3 \ln x \Big|_1^e - \frac{1}{3}\int_1^e \frac{1}{x}x^3\, dx$$

$$= \frac{1}{3}\left(e^3 - 0\right) - \frac{1}{3}\int_1^e x^2\, dx$$

$$= \frac{1}{3}e^3 - \frac{1}{9}\left(x^3 \right|_1^e$$

$$= \frac{1}{3}e^3 - \frac{1}{9}\left(e^3 - 1\right)$$

$$= \frac{1}{3}e^3 - \frac{1}{9}e^3 + \frac{1}{9}$$

$$\underline{= \frac{1}{9}\left(2e^3 + 1\right)}$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^{\ln 2} \frac{3t}{e^t}\, dt$

Solution

		$\int e^{-t}$
+	t	$-e^{-t}$
−	1	e^{-t}

$$\int_{-1}^{\ln 2} \frac{3t}{e^t}\, dt = 3e^{-t}(-t-1) \Big|_{-1}^{\ln 2}$$

$$= -3\left(e^{-\ln 2}(\ln 2 + 1) - e(0)\right)$$

$$\underline{= -\frac{3}{2}(\ln 2 + 1)}$$

Exercise

Evaluate the integral $\displaystyle\int_{\pi}^{2\pi} \cot\frac{x}{3}\, dx$

Solution

$$\int_{\pi}^{2\pi} \cot\frac{x}{3}\, dx = \int_{\pi}^{2\pi} \frac{\cos\frac{x}{3}}{\sin\frac{x}{3}}\, dx$$

$$= 3\int_{\pi}^{2\pi} \frac{1}{\sin\frac{x}{3}}\, d\left(\sin\frac{x}{3}\right)$$

$$= 3\ln\left|\sin\frac{x}{3}\right| \Big|_{\pi}^{2\pi}$$

$$= 3\left(\ln\left|\sin\frac{2\pi}{3}\right| - \ln\left|\sin\frac{\pi}{3}\right|\right)$$

$$= 3\left(\ln\frac{\sqrt{3}}{2} - \ln\frac{\sqrt{3}}{2}\right)$$

$$= 0$$

Exercise

Evaluate the integral $\int_0^{1/\sqrt{2}} 2x\sin^{-1}\left(x^2\right)dx$

Solution

$u = \sin^{-1}\left(x^2\right) \qquad dv = 2xdx$

$du = \frac{2x}{\sqrt{1-x^4}}dx \qquad v = x^2$

$$\int udv = uv - \int vdu$$

$$\int_0^{1/\sqrt{2}} 2x\sin^{-1}\left(x^2\right)dx = \left[x^2\sin^{-1}\left(x^2\right)\right]_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} x^2\frac{2x}{\sqrt{1-x^4}}dx \qquad d\left(1-x^4\right) = -4x^3dx$$

$$= \left(\left(\frac{1}{\sqrt{2}}\right)^2\sin^{-1}\left(\left(\frac{1}{\sqrt{2}}\right)^2\right) - 0\right) + \int_0^{1/\sqrt{2}}\frac{d\left(1-x^4\right)}{2\sqrt{1-x^4}}$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) + \left[\sqrt{1-x^4}\right]_0^{1/\sqrt{2}}$$

$$= \frac{1}{2}\frac{\pi}{6} + \left(\sqrt{1-\frac{1}{4}} - 1\right)$$

$$= \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

$$= \frac{1}{12}\left(\pi + 6\sqrt{3} - 12\right)$$

Exercise

Evaluate the integral $\int_1^e x^3\ln xdx$

Solution

$u = \ln x$

$du = \frac{1}{x}dx \qquad v = \int x^3dx = \frac{1}{4}x^4$

$$\int_1^e x^3 \ln x dx = \frac{1}{4}\left(x^4 \ln x \right|_1^e - \frac{1}{4}\int_1^e x^4 \frac{dx}{x}$$

$$= \frac{1}{4}\left(e^4 \ln e - 1^4 \ln 1\right) - \frac{1}{4}\int_1^e x^3 dx$$

$$= \frac{e^4}{4} - \frac{1}{16}\left(x^4 \right|_1^e$$

$$= \frac{e^4}{4} - \frac{1}{16}\left(e^4 - 1\right)$$

$$= \frac{4}{4}\frac{e^4}{4} - \frac{1}{16}e^4 + \frac{1}{16}$$

$$= \frac{3e^4+1}{16}$$

Exercise

Evaluate the integral $\int_0^1 x\sqrt{1-x}dx$

Solution

$u = x \quad \rightarrow \quad du = dx$

$v = -\int (1-x)^{1/2} d(1-x)$

$= -\frac{2}{3}(1-x)^{2/3}$

$$\int_0^1 x\sqrt{1-x}dx = -\frac{2}{3}\left(x(1-x)^{2/3} \right|_0^1 - \int_0^1 -\frac{2}{3}(1-x)^{2/3} dx \qquad \int udv = uv - \int vdu$$

$$= 0 - \frac{2}{3}\int_0^1 (1-x)^{2/3} d(1-x)$$

$$= -\frac{4}{15}(1-x)^{5/3} \Big|_0^1$$

$$= \frac{4}{15}$$

		$-\int (1-x)^{1/2}\ d(1-x)$
+	x	$-\frac{2}{3}(1-x)^{3/2}$

−	1	$\frac{4}{15}(1-x)^{5/2}$

$$\int_0^1 x\sqrt{1-x}\,dx = -\frac{2}{3}x(1-x)^{3/2} - \frac{4}{15}(1-x)^{5/2}\Big|_0^1$$

$$= \frac{4}{15}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/3} x\tan^2 x\,dx$

Solution

$u = x \quad \rightarrow \quad du = dx$

$$v = \int \tan^2 x\,dx$$

$$= \int \frac{\sin^2 x}{\cos^2 x}dx$$

$$= \int \frac{1-\cos^2 x}{\cos^2 x}dx$$

$$= \int \left(\frac{1}{\cos^2 x} - 1\right)dx$$

$$= \int \left(\sec^2 x - 1\right)dx$$

$$= \tan x - x$$

$$\int_0^{\pi/3} x\tan^2 x\,dx = \left(x\left(\tan x - x\right)\right)\Big|_0^{\pi/3} - \int_0^{\pi/3}\left(\tan x - x\right)dx \qquad \int udv = uv - \int vdu$$

$$= \frac{\pi}{3}\left(\tan\frac{\pi}{3} - \frac{\pi}{3}\right) - 0 + \left(\ln\left|\cos x\right| + \frac{x^2}{2}\right|_0^{\pi/3}$$

$$= \frac{\pi}{3}\left(\sqrt{3} - \frac{\pi}{3}\right) + \ln\left|\cos\frac{\pi}{3}\right| + \frac{1}{2}\left(\frac{\pi}{3}\right)^2 - \ln|1|$$

$$= \frac{\pi}{3}\sqrt{3} - \frac{\pi^2}{9} + \ln\left|\frac{1}{2}\right| + \frac{\pi^2}{18}$$

$$= \frac{\pi}{3}\sqrt{3} - \ln 2 - \frac{\pi^2}{18}$$

Exercise

Evaluate the integral $\int_0^{\pi} x\sin x\, dx$

Solution

		$\int \sin x\, dx$
$+$	x	$-\cos x$
$-$	1	$-\sin x$

$$\int_0^{\pi} x\sin x\, dx = -x\cos x + \sin x \Big|_0^{\pi}$$

$$= \pi$$

Exercise

Evaluate the integral $\int_1^{e} \ln 2x\, dx$

Solution

$$\int_1^{e} \ln 2x\, dx = \frac{1}{2}\int_1^{e} \ln 2x\, d(2x)$$

$$\int \ln x\, dx = x\ln x - x$$

$$= x\ln 2x - x \Big|_1^{e}$$

$$= e\ln 2e - e - \ln 2 + 1$$

$$= e(\ln 2 + \ln e) - e - \ln 2 + 1$$

$$= e\ln 2 - \ln 2 + 1$$

$$= (e-1)\ln 2 + 1$$

Exercise

Evaluate the integral $\int_0^{\pi/2} x\cos 2x\, dx$

Solution

		$\int \cos 2x\, dx$
$+$	x	$\frac{1}{2}\sin 2x$
$-$	1	$-\frac{1}{4}\cos 2x$

$$\int_0^{\pi/2} x\cos 2x\,dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x \Big|_0^{\pi/2}$$

$$= -\frac{1}{4} - \frac{1}{4}$$

$$\boxed{= -\frac{1}{2}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\ln 2} xe^x\,dx$

Solution

		$\int e^x\,dx$
+	x	e^x
−	1	e^x

$$\int_0^{\ln 2} xe^x\,dx = e^x(x-1)\Big|_0^{\ln 2}$$

$$= 2(\ln 2 - 1) + 1$$

$$\boxed{= 2\ln 2 - 1}$$

Exercise

Evaluate the integral $\displaystyle\int_1^{e^2} x^2 \ln x\,dx$

Solution

$$\int x^2 \ln x\,dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx$$

$u = \ln x$

$du = \frac{1}{x}dx \quad v = \int x^2 dx = \frac{1}{3}x^3$

$$\int_1^{e^2} x^2 \ln x\,dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 \Big|_1^{e^2}$$

$$= \frac{2}{3}e^6 - \frac{1}{9}e^6 + \frac{1}{9}$$

$$\boxed{= \frac{5}{9}e^6 + \frac{1}{9}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^3 xe^{x/2}dx$

Solution

$$\int x^n e^{ax}\, dx = e^{ax} \sum_{k=0}^{n} \frac{(-1)^k\, n!}{(n-k)!a^{k+1}} x^{n-k}$$

$$\int_0^3 xe^{x/2}dx = (2x-4)e^{x/2} \Big|_0^3$$

$$\boxed{= 2e^{3/2} + 4}$$

Exercise

Evaluate the integral $\displaystyle\int_0^2 x^2 e^{-2x}dx$

Solution

$$\int_0^2 x^2 e^{-2x}\, dx = \left(-\frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{4}\right)e^{-2x} \Big|_0^2$$

$$\int x^n e^{ax}\, dx = e^{ax} \sum_{k=0}^{n} \frac{(-1)^k\, n!}{(n-k)!a^{k+1}} x^{n-k}$$

$$= \left(-2 + 1 - \frac{1}{4}\right)e^{-4} + \frac{1}{4}$$

$$\boxed{= \frac{1}{4} - \frac{5}{4}e^{-4}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/4} x\cos 2x\, dx$

Solution

		$\int \cos 2x\, dx$
+	x	$\frac{1}{2}\sin 2x$
−	1	$-\frac{1}{4}\cos 2x$

$$\int_0^{\pi/4} x\cos 2x\, dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x \Big|_0^{\pi/4}$$

$$\boxed{= \frac{\pi}{8} - \frac{1}{4}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi} x \sin 2x \, dx$

Solution

		$\int \sin 2x \, dx$
$+$	x	$-\frac{1}{2}\cos 2x$
$-$	1	$-\frac{1}{4}\sin 2x$

$$\int_0^{\pi} x \sin 2x \, dx = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x \,\Big|_0^{\pi}$$

$$= -\frac{\pi}{2}$$

Exercise

Evaluate the integral $\displaystyle\int_1^{4} e^{\sqrt{x}} \, dx$

Solution

$$u = \sqrt{x} \quad \rightarrow \quad u^2 = x$$

$$2u \, du = dx$$

$$\int_1^4 e^{\sqrt{x}} \, dx = 2\int_1^4 ue^u \, du$$

		$\int e^u \, du$
$+$	u	e^u
$-$	1	e^u

$$\int_1^4 e^{\sqrt{x}} \, dx = 2e^u (u-1) \,\Big|_1^4$$

$$= 2e^{\sqrt{x}}\left(\sqrt{x}-1\right) \,\Big|_1^4$$

$$= 2\left[e^2(2-1) - e(1-1)\right]$$

$$= 2e^2$$

Exercise

Use integration by parts to establish the reduction formula

$$\int x^n \sin x\, dx = -x^n \cos x + n\int x^{n-1} \cos x\, dx$$

Solution

$u = x^n \qquad\qquad dv = \sin x dx$

$du = nx^{n-1}dx \qquad v = -\cos x$

$$\int x^n \sin x\, dx = -x^n \cos x + n\int x^{n-1} \cos x\, dx \quad \checkmark$$

Exercise

Use integration by parts to establish the reduction formula

$$\int x^n e^{ax} dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a}\int x^{n-1} e^{ax} dx, \quad a \neq 0$$

Solution

$u = x^n \qquad\qquad dv = e^{ax} dx$

$du = nx^{n-1}dx \qquad v = \frac{1}{a} e^{ax}$

$$\int x^n e^{ax} dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a}\int x^{n-1} e^{ax} dx, \quad a \neq 0 \quad \checkmark$$

Exercise

Use integration by parts to establish the reduction formula

$$\int (\ln x)^n\, dx = x(\ln x)^n - n\int (\ln x)^{n-1}\, dx$$

Solution

$u = (\ln x)^n \qquad\qquad dv = \int dx$

$du = n(\ln x)^{n-1} \frac{1}{x} dx \qquad v = x$

$$\int (\ln x)^n\, dx = x(\ln x)^n - n\int (\ln x)^{n-1}\, dx \quad \checkmark$$

Exercise

Use integration by parts to establish the reduction formula

$$\int_a^b \left(\int_x^b f(t)\,dt \right) dx = \int_a^b (x-a) f(x)\,dx$$

Solution

$$u = x - a \qquad dv = f(x)\,dx$$

$$du = dx \qquad v = \int_b^x f(t)\,dt$$

$$\int_a^b (x-a) f(x)\,dx = \left[(x-a) \int_b^x f(t)\,dt \right]_a^b - \int_a^b \left(\int_b^x f(t)\,dt \right) dx$$

$$= (b-a) \int_b^b f(t)\,dt - (a-a) \int_b^a f(t)\,dt - \int_a^b \left(-\int_x^b f(t)\,dt \right) dx$$

$$= \int_a^b \left(\int_x^b f(t)\,dt \right) dx \quad \checkmark \qquad \int_b^b f(t)\,dt = 0, \quad a - a = 0$$

Exercise

Use integration by parts to establish the reduction formula

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2} x \sqrt{1-x^2} + \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}}\,dx$$

Solution

$$u = \sqrt{1-x^2} \qquad dv = dx$$

$$du = \frac{-x}{\sqrt{1-x^2}}\,dx \qquad v = x$$

$$\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}}\,dx$$

$$= x\sqrt{1-x^2} - \int \left(\frac{1-x^2-1}{\sqrt{1-x^2}} \right) dx$$

$$= x\sqrt{1-x^2} - \int \left(\frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \right) dx$$

$$= x\sqrt{1-x^2} - \int \sqrt{1-x^2}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\int \sqrt{1-x^2}\,dx + \int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$2\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\,dx \quad \checkmark$$

Exercise

Find the indefinite integral: $\int 5x^n \ln ax\,dx \quad a \ne 0,\ n \ne -1$

Solution

$$u = \ln ax \qquad dv = x^n dx$$

$$du = \frac{a}{ax}dx = \frac{dx}{x} \qquad v = \frac{x^{n+1}}{n+1}$$

$$\int 5x^n \ln ax\,dx = 5\left[\frac{x^{n+1}}{n+1}\ln ax - \frac{1}{n+1}\int \frac{x^{n+1}}{x}dx\right]$$

$$= 5\left[\frac{x^{n+1}}{n+1}\ln ax - \frac{1}{n+1}\int x^n dx\right]$$

$$= 5\left[\frac{x^{n+1}}{n+1}\ln ax - \frac{1}{n+1}\frac{x^{n+1}}{n+1}\right] + C$$

$$= \frac{5x^{n+1}}{n+1}\left(\ln ax - \frac{1}{n+1}\right) + C$$

Exercise

Find the volume of the solid generated by the region bounded by $f(x) = x\ln x$, and the $x-axis$ on $\left[1,\ e^2\right]$ is revolved about the $y-axis$.

Solution

Using ***Disk*** Method:

$$V = \pi \int_1^{e^2} (x \ln x)^2 \, dx$$

$$= \pi \int_1^{e^2} x^2 \ln^2 x \, dx$$

Let $y = \ln x \quad \Rightarrow \quad x = e^y$

$$dx = e^y dy$$

$$\int x^2 (\ln x)^2 \, dx = \int \left(e^y\right)^2 y^2 e^y dy$$

$$= \int y^2 e^{3y} \, dy$$

		$\int e^{3y} \, dy$
+	y^2	$\frac{1}{3}e^{3y}$
−	$2y$	$\frac{1}{9}e^{3y}$
+	2	$\frac{1}{27}e^{3y}$

$$V = \pi e^{3y} \left(\frac{1}{3}y^2 - \frac{2}{9}y + \frac{2}{27}\right. \Bigg|_1^{e^2}$$

$$= \pi \left(x^3 \left(\frac{1}{3}\ln^2 x - \frac{2}{9}\ln x + \frac{2}{27}\right)\right. \Bigg|_1^{e^2}$$

$$= \pi \left(e^2\right)^3 \left(\frac{1}{3}\left(\ln e^2\right)^2 - \frac{2}{9}\ln e^2 + \frac{2}{27}\right) - \pi\left(\frac{2}{27}\right)$$

$$= \pi e^6 \left(\frac{4}{3} - \frac{4}{9} + \frac{2}{27}\right) - \frac{2}{27}\pi$$

$$= \pi e^6 \left(\frac{36 - 12 + 2}{27}\right) - \frac{2\pi}{27}$$

$$= \pi e^6 \left(\frac{26}{27}\right) - \frac{2\pi}{27}$$

$$\boxed{= \frac{2\pi}{27}\left(13e^2 - 1\right) \quad unit^3}$$

Exercise

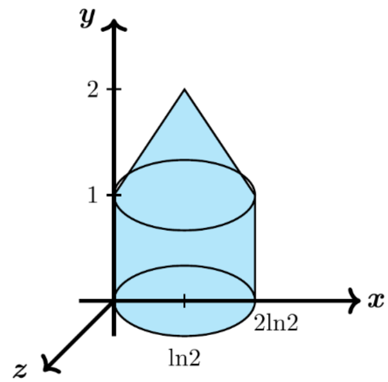
Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the cure $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$

Solution

$$V = 2\pi \int_0^{\ln 2} (\ln 2 - x) e^x dx$$

$$= 2\pi \int_0^{\ln 2} \left(\ln 2 e^x - x e^x\right) dx$$

$$= 2\pi \ln 2 \left(e^x \Big|_0^{\ln 2} - 2\pi \int_0^{\ln 2} x e^x dx\right.$$



		$\int e^x\, dx$
+	x	e^x
−	1	e^x

$$= 2\pi \ln 2\left(e^{\ln 2} - e^0\right) - 2\pi\left(x e^x - e^x \Big|_0^{\ln 2}\right.$$

$$= 2\pi \ln 2(2-1) - 2\pi\left[\ln 2 e^{\ln 2} - e^{\ln 2} - (0-1)\right]$$

$$= 2\pi \ln 2 - 2\pi[2\ln 2 - 2 + 1]$$

$$= 2\pi \ln 2 - 4\pi \ln 2 + 2\pi$$

$$= -2\pi \ln 2 + 2\pi$$

$$= 2\pi(1 - \ln 2) \quad unit^3$$

Exercise

Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate aces, the cure $y = e^{-x}$, and the line $x = 1$, about

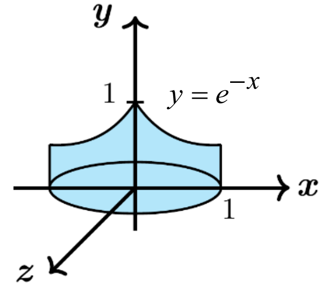
a) the line $y - axis$

b) the line $x = 1$

Solution

a) $V = 2\pi \int_0^1 x e^{-x}\, dx$

		$\int e^{-x}\,dx$
$(+)$	x	$-e^{-x}$
$(-)$	1	e^{-x}



$$= 2\pi \left(-xe^{-x} - e^{-x} \right|_0^1$$

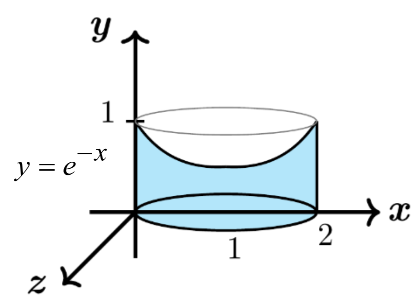
$$= 2\pi\left(-e^{-1} - e^{-1} + 0 + 1\right)$$

$$= 2\pi\left(-\frac{1}{e} - \frac{1}{e} + 1\right)$$

$$= 2\pi\left(-\frac{2}{e} + 1\right)$$

$$= 2\pi - \frac{4\pi}{e} \quad unit^3$$

b) $V = 2\pi \int_0^1 (1-x)e^{-x}dx$



$$= 2\pi\left(\int_0^1 e^{-x}dx - \int_0^1 xe^{-x}dx\right)$$

$$= 2\pi \left(-e^{-x} - \left(-xe^{-x} - e^{-x}\right) \right|_0^1$$

$$= 2\pi \left(e^{-x} + xe^{-x} - e^{-x} \right|_0^1$$

$$= 2\pi \left(xe^{-x} \right|_0^1$$

$$= 2\pi\left(e^{-1}\right)$$

$$= \frac{2\pi}{e} \quad unit^3$$

Exercise

Find the volume of the solid that is generated by the region bounded by $f(x) = e^{-x}$, $x = \ln 2$, and the coordinate axes is revolved about the *y-axis*.

Solution

Using ***Shells*** *Method*:

$$V = 2\pi \int_0^{\ln 2} x\, e^{-x}\ dx \qquad\qquad V = \int_a^b 2\pi (radius)(height)\, dx$$

		$\int e^{-x}\ dx$
+	x	$-e^{-x}$
−	1	e^{-x}

$$= 2\pi \left(e^{-x}(-x-1) \right) \Big|_0^{\ln 2}$$

$$= 2\pi \left(e^{-\ln 2}(-\ln 2 - 1) + 1 \right)$$

$$= 2\pi \left(\tfrac{1}{2}(-\ln 2 - 1) + 1 \right)$$

$$= 2\pi \left(-\tfrac{1}{2}\ln 2 + \tfrac{1}{2} \right)$$

$$= \pi (1 - \ln 2) \quad unit^3$$

Exercise

Find the volume of the solid that is generated by the region bounded by $f(x) = e^{-x}$, and the $x-axis$ on $[1,\ \ln 2]$ is revolved about the line $x = \ln 2$.

Solution

Using ***Shells*** *Method*:

$$V = 2\pi \int_0^{\ln 2} (\ln 2 - x) e^{-x}\ dx \qquad\qquad V = \int_a^b 2\pi (radius)(height)\, dx$$

$$= 2\pi \ln 2 \int_0^{\ln 2} e^{-x}\ dx - 2\pi \int_0^{\ln 2} xe^{-x}\ dx$$

		$\int e^{-x}\ dx$
+	x	$-e^{-x}$
−	1	e^{-x}

$$= 2\pi \left(-(\ln 2) e^{-x} - (-x-1) e^{-x} \right) \Big|_0^{\ln 2}$$

$$= 2\pi \left(-(\ln 2) e^{-\ln 2} + (\ln 2 + 1) e^{-\ln 2} - \ln 2 - 1 \right)$$

$$= 2\pi\left(-\frac{1}{2}\ln 2 + \frac{1}{2}(\ln 2 + 1) + \ln 2 - 1\right)$$

$$= 2\pi\left(-\frac{1}{2}\ln 2 + \frac{1}{2}\ln 2 + \frac{1}{2} + \ln 2 - 1\right)$$

$$= 2\pi\left(\ln 2 - \frac{1}{2}\right)$$

$$= \pi(2\ln 2 - 1)$$

$$= \pi(\ln 4 - 1) \quad unit^3$$

Exercise

Find the volume of the solid that is generated by the region bounded by $f(x) = \sin x$, and the $x\text{-}axis$ on $[0, \pi]$ is revolved about the $y\text{-}axis$.

Solution

Using ***Shells*** *Method*:

$$V = 2\pi \int_0^{\pi} x \sin x\, dx \qquad\qquad V = \int_a^b 2\pi(radius)(height)\,dx$$

		$\int \sin x\, dx$
+	x	$-\cos x$
–	1	$-\sin x$

$$= 2\pi\left(-x\cos x + \sin x\right)\Big|_0^{\pi}$$

$$= 2\pi^2 \quad unit^3$$

Exercise

Find the area of the region generated when the region bounded by $y = \sin x$ and $y = \sin^{-1} x$ on the interval $\left[0, \frac{1}{2}\right]$.

Solution

$$A = \int_0^{1/2}\left(\sin^{-1} x - \sin x\right)dx \qquad u = \sin^{-1} x \quad du = \frac{dx}{\sqrt{1-x^2}} \qquad v = \int dx = x$$

$$= x\sin^{-1} x\Big|_0^{1/2} - \int_0^{1/2} \frac{x\,dx}{\sqrt{1-x^2}} + \cos x\Big|_0^{1/2}$$

$$= x\sin^{-1}x + \cos x \Big|_0^{1/2} + \frac{1}{2}\int_0^{1/2}\left(1-x^2\right)^{-1/2} d\left(1-x^2\right)$$

$$= x\sin^{-1}x + \cos x + \left(1-x^2\right)^{1/2} \Big|_0^{1/2}$$

$$= \frac{1}{2}\sin^{-1}\frac{1}{2} + \cos\frac{1}{2} + \left(1-\frac{1}{4}\right)^{1/2} - 1 - 1$$

$$= \frac{\pi}{12} + \cos\frac{1}{2} + \frac{\sqrt{3}}{2} - 2 \quad unit^2$$

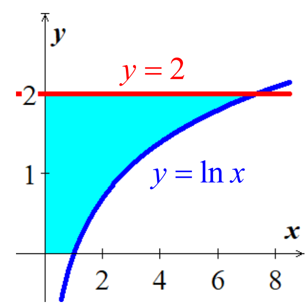
Exercise

Determine the area of the shaded region bounded by $y = \ln x, \quad y = 2, \quad y = 0, \quad and\ x = 0$

Solution

$y = \ln x = 0 \quad \rightarrow \quad x = 1$

$y = \ln x = 2 \quad \rightarrow \quad x = e^2$



$$A = 1\times 2 + \int_1^2 (2 - \ln x)\,dx$$

$$= 2 + \left(2x - x\ln x + x\right. \Big|_1^2$$

$$= 2 + 4 - 2\ln 2 + 2 - 2 - 1$$

$$= 5 - 2\ln 2 \quad unit^2$$

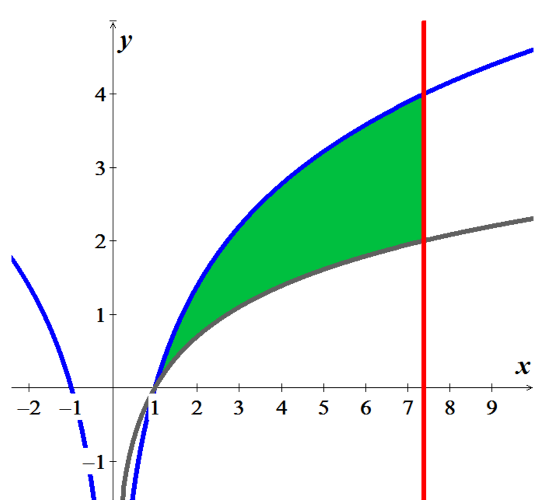
Exercise

Find the area between the curves $y = \ln x^2, \quad y = \ln x, \quad and\ \ x = e^2$

Solution

$y = \ln x^2 = \ln x \quad with \quad x > 0$

$x^2 = x \quad \Rightarrow \quad x = 1$



$$A = \int_1^{e^2}\left(\ln x^2 - \ln x\right)dx$$

$$= \int_1^{e^2}(2\ln x - \ln x)\,dx$$

$$= \int_1^{e^2} \ln x \, dx$$

$$u = \ln x \qquad dv = dx$$

$$du = \frac{1}{x}dx \quad v = \int dx = x$$

$$\int \ln x dx = x \ln x - \int x \frac{1}{x} dx$$

$$= x \ln x - \int dx$$

$$\boxed{= x \ln x - x}$$

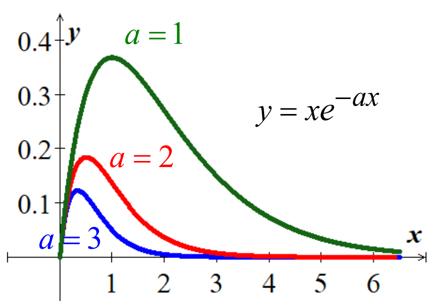
$$= x \ln x - x \Big|_1^{e^2}$$

$$= e^2 \ln e^2 - e^2 + 1$$

$$\boxed{= e^2 + 1 \quad unit^2}$$

Exercise

The curves $y = xe^{-ax}$ are shown in the figure for $a = 1,\ 2,\ and\ 3$.



a) Find the area of the region bounded by $y = xe^{-x}$ and the *x*-axis on the interval [0, 4].

b) Find the area of the region bounded by $y = xe^{-ax}$ and the *x*-axis on the interval [0, 4] where $a > 0$

c) Find the area of the region bounded by $y = xe^{-ax}$ and the *x*-axis on the interval [0, *b*]. Because this area depends on *a* and *b*, we call it $A(a,\ b)$ where $a > 0$ and $b > 0$.

d) Use part (*c*) to show that $A(1,\ \ln b) = 4A\left(2,\ \frac{1}{2}\ln b\right)$

e) Does this pattern continue? Is it true that $A(1,\ \ln b) = a^2 A\left(a,\ \frac{1}{a}\ln b\right)$

Solution

a) $\displaystyle\int_0^4 xe^{-x}dx = e^{-x}(-x-1)\Big|_0^4$

		$\int e^{-x}\,dx$
+	x	$-e^{-x}$
−	1	e^{-x}

$= e^{-4}(-5)-(-1)$

$= 1-\dfrac{5}{e^4}$ $unit^2$

b) $\displaystyle\int_0^4 xe^{-ax}\,dx = e^{-ax}\left(-\frac{1}{a}x-\frac{1}{a^2}\right)\Big|_0^4$

		$\int e^{-ax}\,dx$
+	x	$-\frac{1}{a}e^{-ax}$
−	1	$\frac{1}{a^2}e^{-ax}$

$= e^{-4a}\left(-\dfrac{4}{a}-\dfrac{1}{a^2}\right)-\left(-\dfrac{1}{a^2}\right)$

$= \dfrac{1}{a^2}-e^{-4a}\left(\dfrac{4a+1}{a^2}\right)$

$= \dfrac{1}{a^2}\left(1-\dfrac{4a+1}{e^{-4a}}\right)$ $unit^2$

c) $\displaystyle\int_0^b xe^{-ax}\,dx = e^{-ax}\left(-\frac{1}{a}x-\frac{1}{a^2}\right)\Big|_0^b$

$= e^{-ab}\left(-\dfrac{b}{a}-\dfrac{1}{a^2}\right)-\left(-\dfrac{1}{a^2}\right)$

$= \dfrac{1}{a^2}-e^{-ab}\left(\dfrac{ab+1}{a^2}\right)$

$= \dfrac{1}{a^2}\left(1-\dfrac{ab+1}{e^{ab}}\right)$ $unit^2$

d) $A(a,b)=\dfrac{1}{a^2}\left(1-\dfrac{ab+1}{e^{ab}}\right)$

$A(1,\ \ln b)=1-\dfrac{\ln b+1}{e^{\ln b}}$

$=1-\dfrac{\ln b+1}{b}$

$A\left(2,\ \dfrac{1}{2}\ln b\right)=\dfrac{1}{4}\left(1-\dfrac{\ln b+1}{e^{\ln b}}\right)$

$$=\frac{1}{4}\left(1-\frac{\ln b+1}{b}\right)$$

$$=\frac{1}{4}A(1,\ \ln b)$$

$$\therefore\quad \boxed{A(1,\ \ln b)=4A\left(2,\ \tfrac{1}{2}\ln b\right)}$$

e) $$A\left(a,\ \frac{1}{a}\ln b\right)=\frac{1}{a^2}\left(1-\frac{\ln b+1}{e^{\ln b}}\right)$$

$$=\frac{1}{a^2}\left(1-\frac{\ln b+1}{b}\right)$$

$$=\frac{1}{a^2}A(1,\ \ln b)$$

Yes, there is a pattern: $\boxed{A(1,\ \ln b)=a^2A\left(a,\ \frac{1}{a}\ln b\right)}$

Exercise

Suppose a mass on a spring that is slowed by friction has the position function $s(t)=e^{-t}\sin t$

a) Graph the position function. At what times does the oscillator pass through the position $s=0$?

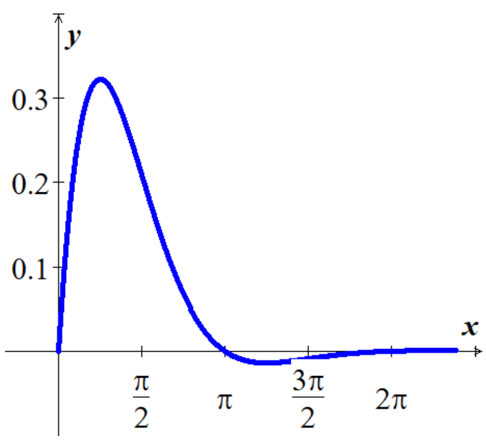
b) Find the average value of the position on the interval $[0,\ \pi]$.

c) Generalize part (*b*) and find the average value of the position on the interval $[n\pi,\ (n+1)\pi]$, for $n=0,\ 1,\ 2,\ \ldots$

Solution

a) $s(t)=e^{-t}\sin t=0$

$\sin t=0\ \rightarrow\ \boxed{t=n\pi}$



b) $$\int e^{-t}\sin t\ dt=-e^{-t}(\cos t+\sin t)-\int e^{-t}\sin t\ dt$$

		$\int \sin t\ dt$
+	e^{-t}	$-\cos t$
−	$-e^{-t}$	$-\sin t$
+	e^{-t}	$-\int \sin t\ dt$

$$2\int e^{-t}\sin t\ dt=-e^{-t}(\cos t+\sin t)$$

$$Average = \frac{1}{\pi}\int_0^{\pi} e^{-t}\sin t\ dt$$

$$= -\frac{1}{2\pi}e^{-t}\left(\cos t - \sin t\right)\Big|_0^{\pi}$$

$$= -\frac{1}{2\pi}\left(-e^{-\pi} - 1\right)$$

$$= \frac{1}{2\pi}\left(e^{-\pi} + 1\right)$$

c) $$Average = \frac{1}{\pi}\int_{n\pi}^{(n+1)\pi} e^{-t}\sin t\ dt$$

$$= -\frac{1}{2\pi}e^{-t}\left(\cos t - \sin t\right)\Big|_{n\pi}^{(n+1)\pi}$$

$$= -\frac{1}{2\pi}\left(e^{-(n+1)\pi}\left(\cos\left((n+1)\pi\right) - \sin\left((n+1)\pi\right)\right) - e^{-n\pi}\left(\cos n\pi - \sin n\pi\right)\right)$$

$$= -\frac{1}{2\pi}\left(e^{-(n+1)\pi}\cos\left((n+1)\pi\right) - e^{-n\pi}\cos n\pi\right)$$

$$= \frac{e^{-n\pi}}{2\pi}\left(\cos n\pi - e^{-\pi}\cos(n+1)\pi\right)$$

$$= \frac{e^{-n\pi}}{2\pi}\left((-1)^n - e^{-\pi}(-1)^{n+1}\right)$$

$$= (-1)^n\frac{e^{-n\pi}}{2\pi}\left(1 + e^{-\pi}\right)$$

Exercise

Given the region bounded by the graphs of $y = x\sin x, \quad y = 0, \quad x = 0, \quad x = \pi$, find

a) The area of the region.

b) The volume of the solid generated by revolving the region about the *x-axis*

c) The volume of the solid generated by revolving the region about the *y-axis*

d) The centroid of the region

Solution

a) $$A = \int_0^{\pi} x\ \sin x\ dx$$

$$= -x\cos x + \sin x\Big|_0^{\pi}$$

$$= \pi \quad unit^2$$

		$\int \sin x\, dx$
+	x	$-\cos x$
–	1	$-\sin x$

b) $V = \pi \int_0^{\pi} (x \sin x)^2 \, dx$

$= \pi \int_0^{\pi} x^2 \sin^2 x \, dx$

$= \frac{\pi}{2} \int_0^{\pi} x^2 (1 - \cos 2x) \, dx$

$= \frac{\pi}{2} \int_0^{\pi} \left(x^2 - x^2 \cos 2x\right) \, dx$

		$\int \cos 2x \, dx$
+	x^2	$\frac{1}{2}\sin 2x$
–	$2x$	$-\frac{1}{4}\cos 2x$
+	2	$-\frac{1}{8}\sin 2x$

$= \frac{\pi}{2} \left(\frac{1}{3}x^3 - \frac{1}{2}x^2 \sin 2x - \frac{1}{2}x \cos 2x + \frac{1}{4}\sin 2x \right|_0^{\pi}$

$= \frac{\pi}{2}\left(\frac{1}{3}\pi^3 - \frac{\pi}{2}\right)$

$= \frac{\pi^4}{6} - \frac{\pi^2}{4} \quad unit^3$

c) $V = 2\pi \int_0^{\pi} x(x \sin x) \, dx$

$= 2\pi \int_0^{\pi} \left(x^2 \sin x\right) \, dx$

		$\int \sin x \, dx$
+	x^2	$-\cos x$
–	$2x$	$-\sin x$
+	2	$\cos x$

$= 2\pi \left(-x^2 \cos x + 2x \sin x + 2\cos x \right|_0^{\pi}$

$= 2\pi\left(\pi^2 - 2 - 2\right)$

$$\underline{= 2\pi^3 - 8\pi \quad unit^3}$$

d) $m = \int_0^{\pi} x\sin x \, dx$ From (***a***)

$$= -x\cos x + \sin x \Big|_0^{\pi}$$

$$\underline{= \pi}$$

$$M_x = \frac{1}{2}\int_0^{\pi} (x\sin x)^2 \, dx \qquad \text{From } (\boldsymbol{b})$$

$$\underline{= \frac{1}{2}\left(\frac{\pi^3}{6} - \frac{\pi}{4}\right)}$$

$$M_y = \int_0^{\pi} x(x\sin x) \, dx \qquad \text{From } (\boldsymbol{c})$$

$$= \frac{2\pi^3 - 8\pi}{2\pi}$$

$$\underline{= \pi^2 - 4}$$

$$\overline{x} = \frac{M_y}{m} = \frac{\pi^2 - 4}{\pi} \qquad \approx 1.8684$$

$$\overline{y} = \frac{M_x}{m} = \frac{1}{\pi}\left(\frac{\pi^3}{12} - \frac{\pi}{8}\right)$$

$$= \frac{\pi^2}{12} - \frac{1}{8} \qquad \approx 0.6975$$

$\therefore$ The centre of mass: $\left(\frac{\pi^2 - 4}{\pi}, \ \frac{\pi^2}{12} - \frac{1}{8}\right)$

Exercise

The region R is bounded by the curve $y = \ln x$ and the x-*axis* on the interval $[1, e]$. Find the volume of the solid that is generated when R is revolved in the following ways

a) About the x-*axis*

b) About the y-*axis*

c) About the line $x = 1$

d) About the line $y = 1$

Solution

a) About the x-*axis*

$$V = \pi \int_1^e (\ln x)^2\, dx$$

Let $y = \ln x \quad \Rightarrow \quad x = e^y$

$$dx = e^y dy$$

$$V = \pi \int_1^e y^2\, e^y\, dy$$

		$\int e^y\, dy$
+	y^2	e^y
−	$2y$	e^y
+	2	e^y

$$= \pi\left(y^2 - 2y + 2\right)e^y \Big|_1^e$$

$$= \pi x\left((\ln x)^2 - 2\ln x + 2\right) \Big|_1^e$$

$$= \pi\left(e(1 - 2 + 2) - 2\right)$$

$$= \pi(e - 2)\ \ unit^3$$

b) About the *y-axis*

$$V = 2\pi \int_1^e x \ln x\, dx$$

		$\int x\, dx$
+	$\ln x$	$\frac{1}{2}x^2$
−	$\frac{1}{x}$	$\int \frac{1}{2}x^2$

$$= 2\pi\left[\frac{1}{2}x^2 \ln x - \int_1^e \frac{1}{2}x^2 \frac{1}{x}\, dx\right]$$

$$= 2\pi\left[\frac{1}{2}x^2 \ln x - \frac{1}{2}\int_1^e x\, dx\right]$$

$$= \pi \left(x^2 \ln x - \frac{1}{2} x^2 \right|_1^e$$

$$= \pi \left(e^2 \ln e - \frac{1}{2} e^2 + \frac{1}{2} \right)$$

$$= \frac{\pi}{2} \left(e^2 + 1 \right) \ \ unit^3$$

c) About the line $x = 1$

$$V = 2\pi \int_1^e (x-1) \ln x \, dx$$

		$\int (x-1)\,dx$
+	$\ln x$	$\frac{1}{2}x^2 - x$
−	$\frac{1}{x}$	$\int \left(\frac{1}{2}x^2 - x\right)$

$$= 2\pi \left[\left(\frac{1}{2}x^2 - x \right) \ln x - \int_1^e \left(\frac{1}{2}x^2 - x \right) \frac{1}{x} \, dx \right]$$

$$= 2\pi \left(\frac{1}{2}x^2 - x \right) \ln x \ - \ 2\pi \int_1^e \left(\frac{1}{2}x - 1 \right) \, dx$$

$$= 2\pi \left(\left(\frac{1}{2}x^2 - x \right) \ln x - \left(\frac{1}{4}x^2 - x \right) \right|_1^e$$

$$= 2\pi \left(\frac{1}{2}e^2 - e - \frac{1}{4}e^2 + e + \frac{1}{4} - 1 \right)$$

$$= 2\pi \left(\frac{1}{4}e^2 - \frac{3}{4} \right)$$

$$= \frac{\pi}{2} \left(e^2 - 3 \right) \ \ unit^3$$

d) About the line $y = 1$

$$V = \pi \int_1^e \left(1 - (1 - \ln x)^2 \right) \, dx$$

$$= \pi \int_1^e \left(1 - 1 + 2\ln x - (\ln x)^2 \right) \, dx$$

$$= \pi \int_1^e \left(2\ln x - (\ln x)^2 \right) \, dx$$

Let $y = \ln x \quad \Rightarrow \quad x = e^y$

$$dx = e^y dy$$

		$\int e^y \, dy$
+	y^2	e^y
−	$2y$	e^y
+	2	e^y

$$\int (\ln x)^2 \, dx = \left(y^2 - 2y + 2\right)e^y$$

$$= x\left((\ln x)^2 - 2\ln x + 2\right)$$

		$\int 2dx$
+	$\ln x$	$2x$
−	$\frac{1}{x}$	$\int 2x$

$$\int 2\ln x \, dx = 2x\ln x - \int 2x\frac{1}{x}dx$$

$$= 2x\ln x - 2\int dx$$

$$= 2x\ln x - 2x$$

$$V = \pi \int_1^e \left(2\ln x - (\ln x)^2\right) dx$$

$$= \pi\left(2x\ln x - 2x - x\left((\ln x)^2 - 2\ln x + 2\right)\right) \Big|_1^e$$

$$= \pi\left(2x\ln x - 2x - x(\ln x)^2 + 2x\ln x - 2x\right) \Big|_1^e$$

$$= \pi\left(4x\ln x - 4x - x(\ln x)^2\right) \Big|_1^e$$

$$= \pi(4e - 4e - e + 4)$$

$$= \pi(4 - e) \;\; unit^3$$

Exercise

A string stretched between the two points $(0,\ 0)$ and $(2,\ 0)$ is plucked by displacing the string h units at its midpoint. The motion of the string is modeled by a ***Fourier Sine series*** whose coefficients are given by

$$b_n = h\int_0^1 x\sin\frac{n\pi x}{2}dx + h\int_1^2 (-x+2)\sin\frac{n\pi x}{2}dx$$

Find b_n

Solution

$$b_n = h\int_0^1 x\sin\frac{n\pi x}{2}dx + h\int_1^2 (-x+2)\sin\frac{n\pi x}{2}dx$$

		$\int \sin\frac{n\pi x}{2}dx$	
$+$	x	$-\frac{2}{n\pi}\cos\frac{n\pi x}{2}$	$-x+2$
$-$	1	$-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}$	-1

$$= h\left(-\frac{2}{n\pi}x\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\Big|_0^1 + h\left(-\frac{2}{n\pi}(2-x)\cos\frac{n\pi x}{2} - \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\Big|_1^2$$

$$= h\left(-\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right) + h\left(-\frac{4}{n^2\pi^2}\sin n\pi + \frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right)$$

$$= h\left(\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\sin n\pi + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right) \qquad \left(\cos\frac{n\pi}{2} = 0 \quad \sin n\pi = 0\right)$$

$$= \frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= (-1)^n\frac{8h}{n^2\pi^2}$$