Solution — Section 4.2 – Arithmetic and Geometric Sequences

Exercise

Show that the sequence $-6,\ -2,\ 2,\ \dots,\ 4n-10,\ \dots$ is arithmetic, and find the common difference.

Solution

We to show that $a_{k+1}-a_k$ equals to a constant.

$$\begin{aligned} a_{k+1}-a_k &= 4(k+1)-10-(4k-10) \\ &= 4k+4-10-4k+10 \\ &= 4 \end{aligned}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $2,\ 6,\ 10,\ 14,\ \dots$

Solution

$d=6-2=4$

$$\begin{aligned} a_n &= 2+(n-1)4 \qquad\qquad a_n=a_1+(n-1)d \\ &= 2+4n-4 \\ &= 4n-2 \end{aligned}$$

$a_{10}=4(10)-2=38$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $3,\ 2.7,\ 2.4,\ 2.1,\ \dots$

Solution

$d=2.7-3=-0.3$

$$\begin{aligned} a_n &= 3+(n-1)(-0.3) \qquad\qquad a_n=a_1+(n-1)d \\ &= 3-0.3n+0.3 \\ &= 3.3-0.3n \end{aligned}$$

$a_{10}=3.3-0.3(10)=0.3$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $-6,\ -4.5,\ -3,\ -1.5,\ \dots$

Solution

$$d = -4.5 - (-6) = 1.5$$

$$a_n = -6 + (n-1)(1.5) \qquad a_n = a_1 + (n-1)d$$

$$= -6 + 1.5n - 1.5$$

$$\underline{= 1.5n - 7.5}$$

$$a_{10} = 1.5(10) - 7.5 \ \underline{= 7.5}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $\ln 3,\ \ln 9,\ \ln 27,\ \ln 81,\ \ldots$

Solution

$$\ln 3,\ \ln 3^2,\ \ln 3^3,\ \ln 3^4,\ \ldots$$

$$\ln 3,\ 2\ln 3,\ 3\ln 3,\ 4\ln 3,\ \ldots$$

$$d = 2\ln 3 - \ln 3 = \ln 3$$

$$a_n = \ln 3 + (n-1)\ln 3 \qquad a_n = a_1 + (n-1)d$$

$$= \ln 3 + n\ln 3 - \ln 3$$

$$\underline{= n\ln 3}$$

$$a_{10} = 10\ln 3 \ \underline{= \ln 3^{10}}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = 2, \quad d = 3$

Solution

$$a_n = 2 + 3(n-1) \qquad a_n = a_1 + (n-1)d$$

$$= 2 + 3n - 3$$

$$\underline{= 3n - 1}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = 5, \quad d = -3$

Solution

$$a_n = 5 + (n-1)(-3) \qquad a_n = a_1 + (n-1)d$$

$$= 5 - 3n + 3$$

$$\underline{= 8 - 3n}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = 1, \quad d = -\frac{1}{2}$

Solution

$$a_n = 1 + (n-1)\left(-\tfrac{1}{2}\right) \qquad\qquad a_n = a_1 + (n-1)d$$

$$= 1 - \tfrac{1}{2}n + \tfrac{1}{2}$$

$$\boxed{= \tfrac{3}{2} - \tfrac{1}{2}n}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = -2, \quad d = 4$

Solution

$$a_n = -2 + (n-1)(4) \qquad\qquad a_n = a_1 + (n-1)d$$

$$= -2 + 4n - 4$$

$$\boxed{= 4n - 6}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = \sqrt{2}, \quad d = \sqrt{2}$

Solution

$$a_n = \sqrt{2} + (n-1)\sqrt{2} \qquad\qquad a_n = a_1 + (n-1)d$$

$$= \sqrt{2} + \sqrt{2}n - \sqrt{2}$$

$$\boxed{= \sqrt{2}\, n}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = 0, \quad d = \pi$

Solution

$$a_n = 0 + (n-1)(\pi) \qquad\qquad a_n = a_1 + (n-1)d$$

$$\boxed{= \pi n - \pi}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = 13, \quad d = 4$

Solution

$$a_n = 13 + (n-1)(4) \qquad\qquad a_n = a_1 + (n-1)d$$

$$= 4n + 9$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = -40, \quad d = 5$

Solution

$$a_n = -40 + (n-1)(5) \qquad\qquad a_n = a_1 + (n-1)d$$

$$= 5n - 45$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = -32, \quad d = 4$

Solution

$$a_n = -32 + (n-1)(4) \qquad\qquad a_n = a_1 + (n-1)d$$

$$= 4n - 36$$

Exercise

Find the common difference for the arithmetic sequence with the specified terms: $a_4 = 14, \quad a_{11} = 35$

Solution

$$a_n = a_1 + (n-1)d$$

$$a_{11} = a_1 + 10d \quad \to \quad 35 = a_1 + 10d$$

$$a_4 = a_1 + 3d \quad \to \quad 14 = a_1 + 3d$$

$$21 = 7d \quad \to \boxed{d = 3}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: $a_{12}; \quad a_1 = 9.1, \quad a_2 = 7.5$

Solution

$$d = a_2 - a_1 = 7.5 - 9.1 = -1.6$$

$$a_n = a_1 + (n-1)d$$

$$a_{12} = 9.1 + (11)(-1.6) = -8.5$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_1; $a_8 = 47$, $a_9 = 53$

Solution

$$d = a_9 - a_8 = 53 - 47 = 6$$

$$a_8 = a_1 + (7)(6) \qquad\qquad a_n = a_1 + (n-1)d$$

$$\boxed{a_1 = 47 - 42 = 5}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_{10}; $a_2 = 1$, $a_{18} = 49$

Solution

$$a_2 = a_1 + d \Rightarrow a_1 = a_2 - d$$

$$a_{18} = a_1 + (17)d = a_2 - d + 17d = a_2 + 16d$$

$$49 = 1 + 16d \Rightarrow 16d = 48 \Rightarrow \boxed{d = \frac{48}{16} = 3}$$

$$a_1 = a_2 - d = 1 - 3 = -2$$

$$\boxed{a_{10} = -2 + 9(3) = 25}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_{10}; $a_8 = 8$, $a_{20} = 44$

Solution

$$\boxed{d = \frac{44-8}{20-8} = \frac{36}{12} = 3} \qquad\qquad d = \frac{a_y - a_x}{y - x}$$

$$a_8 = a_1 + (8-1)(3) \qquad\qquad a_n = a_1 + (n-1)d$$

$$8 = a_1 + 21$$

$$a_1 = -13$$

$$\boxed{a_{10} = -13 + 9(3) = 14}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_{12}; $a_8 = 4$, $a_{18} = -96$

Solution

$$\boxed{d = \frac{-96-4}{18-8} = \frac{-100}{10} = -10} \qquad\qquad d = \frac{a_y - a_x}{y - x}$$

$$a_8 = a_1 + (8-1)(-10) \qquad a_n = a_1 + (n-1)d$$

$$4 = a_1 - 70$$

$$a_1 = 74$$

$$\boxed{a_{12} = 74 + (11)(-10) = -36}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_8; $a_{15} = 0$, $a_{40} = -50$

Solution

$$\boxed{d = \frac{-50-0}{40-15} = \frac{-50}{25} = -2} \qquad d = \frac{a_y - a_x}{y - x}$$

$$a_{15} = a_1 + (15-1)(-2) = 0 \qquad a_n = a_1 + (n-1)d$$

$$a_1 = 28$$

$$\boxed{a_8 = 28 + (7)(-2) = 14}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_{20}; $a_9 = -5$, $a_{15} = 31$

Solution

$$\boxed{d = \frac{31+5}{15-9} = \frac{36}{6} = 6} \qquad d = \frac{a_y - a_x}{y - x}$$

$$a_9 = a_1 + (9-1)(6) \qquad a_n = a_1 + (n-1)d$$

$$-5 = a_1 + 42$$

$$a_1 = -47$$

$$\boxed{a_{20} = -47 + (19)(6) = 67}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_n; $a_8 = 8$, $a_{20} = 44$

Solution

$$\boxed{d = \frac{44-8}{20-8} = \frac{36}{12} = 3} \qquad d = \frac{a_y - a_x}{y - x}$$

$$a_8 = a_1 + 3(8-1) = 8 \qquad a_n = a_1 + (n-1)d$$

$$\boxed{a_1 = -13}$$

$$\boxed{a_n = -13 + 3(n-1) = 3n - 16}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_n; $a_8 = 4$, $a_{18} = -96$

Solution

$$d = \frac{-96-4}{18-8} = -10 \qquad d = \frac{a_y - a_x}{y-x}$$

$$a_8 = a_1 - 10(8-1) = 4 \qquad a_n = a_1 + (n-1)d$$

$$a_1 = 74$$

$$a_n = 74 - 10(n-1) = -10n + 84$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_n; $a_{14} = -1$, $a_{15} = 31$

Solution

$$d = \frac{31+1}{15-14} = 32 \qquad d = \frac{a_y - a_x}{y-x}$$

$$a_{14} = a_1 + 32(14-1) = -1 \qquad a_n = a_1 + (n-1)d$$

$$a_1 = -417$$

$$a_n = -417 + 32(n-1) = 32n + 449$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_n; $a_9 = -5$, $a_{15} = 31$

Solution

$$d = \frac{31+5}{15-9} = 6 \qquad d = \frac{a_y - a_x}{y-x}$$

$$a_9 = a_1 + 6(9-1) = -5 \qquad a_n = a_1 + (n-1)d$$

$$a_1 = -53$$

$$a_n = -53 + 6(n-1) = 6n - 59$$

Exercise

Find the sum S_n of the arithmetic sequence that satisfies the conditions: $a_1 = 40$, $d = -3$, $n = 30$

Solution

$$S_n = \frac{30}{2}\left[2(40) + (30-1)(-3)\right] \qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$= -105$$

Exercise

Find the sum S_n of the arithmetic sequence that satisfies the conditions: $a_7 = \frac{7}{3}, \quad d = -\frac{2}{3}, \quad n = 15$

Solution

$$a_7 = a_1 + (6)\left(-\frac{2}{3}\right) = \frac{7}{3}$$

$$a_1 = \frac{7}{3} + 4 = \frac{19}{3}$$

$$S_n = \frac{15}{2}\left[2\left(\frac{19}{3}\right) + (15-1)\left(-\frac{2}{3}\right)\right] \qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$= 25$$

Exercise

Find the number of integers between 32 and 390 that are divisible by 6, find their sum

Solution

Number of terms: $n = \frac{390-36}{6} + 1 = 60$

$$S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$= \frac{60}{2}(36 + 390)$$

$$= 12780$$

Exercise

Find the number of terms in the arithmetic sequence with the given conditions:

$a_1 = -2, \quad d = \frac{1}{4}, \quad S = 21$

Solution

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$21 = \frac{n}{2}\left[2(-2) + (n-1)\frac{1}{4}\right]$$

$$21 = -2n + \frac{1}{8}n(n-1)$$

$$(8)21 = -2n(8) + \frac{1}{8}n(n-1)(8)$$

$$168 = -16n + \left(n^2 - n\right)$$

$$0 = n^2 - 17n - 168$$

$$\boxed{n = 24} \qquad n = -7$$

Exercise

Express the sum in terms of summation notation and find the sum $2+11+20+\ldots+16{,}058$.

Solution

Difference in terms: $d = 11 - 2 = 9$

Number of terms: $n = \dfrac{16058-2}{9}+1 = 1785$

$a_n = 2+(n-1)(9) = 2+9n-9 = 9n-7$ $\qquad a_n = a_1 + (n-1)d$

Hence the *n*th term is: $\displaystyle\sum_{n=1}^{1785}(9n-7)$

$S_{1785} = \frac{1789}{2}(2+16058)$ $\qquad S_n = \frac{n}{2}(a_1 + a_n)$

$= 14{,}333{,}550$

Exercise

Express the sum in terms of summation notation and find the sum $60+64+68+72+\cdots+120$.

Solution

Difference in terms: $d = 64 - 60 = 4$

Number of terms: $n = \dfrac{120-60}{4}+1 = 16$ $\qquad n = \dfrac{a_n - a_1}{d}+1$

$a_n = 60+(n-1)(4) = 4n-54$

Hence the *n*th term is: $\displaystyle\sum_{n=1}^{16}(4n-54)$

$S = \frac{16}{2}(60+120)$ $\qquad S_n = \frac{n}{2}(a_1 + a_n)$

$= 1440$

Exercise

Find each arithmetic sum $\quad 1+3+5+\cdots+(2n-1)$

Solution

Difference in terms: $d = 3 - 1 = 2$

Number of terms: $n = \dfrac{(2n-1)-1}{2}+1$ $\qquad n = \dfrac{a_n - a_1}{d}+1$

$= \dfrac{2n-2}{2}+1$

$= n-1+1$

$= n$

$$S = \frac{n}{2}\left(1+(2n-1)\right) \qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$= \frac{n}{2}(2n)$$

$$\underline{= n^2}$$

Exercise

Find each arithmetic sum $\quad 2+4+6+\cdots+2n$

Solution

Difference in terms: $d = 4 - 2 = 2$

Number of terms: $n = \frac{2n-2}{2}+1 \qquad n = \frac{a_n - a_1}{d}+1$

$$= n-1+1$$

$$\underline{= n}$$

$$S = \frac{n}{2}(2+2n) \qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$= n(n+1)$$

$$\underline{= n^2 + n}$$

Exercise

Find each arithmetic sum $\quad 2+5+8+\cdots+41$

Solution

Difference in terms: $d = 5 - 2 = 3$

Number of terms: $n = \frac{41-2}{3}+1 = 14 \qquad n = \frac{a_n - a_1}{d}+1$

$$S = \frac{14}{2}(2+41) \qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$\underline{= 301}$$

Exercise

Find each arithmetic sum $\quad 7+12+17+\cdots+(2+5n)$

Solution

Difference in terms: $d = 12 - 7 = 5$

Number of terms: $n = \frac{2+5n-7}{5}+1 \qquad n = \frac{a_n - a_1}{d}+1$

$$= \frac{5n-5}{5}+1$$

$$= \frac{5n}{5} - \frac{5}{5}+1$$

$$\boxed{= n}$$

$$S = \frac{n}{2}(7 + 2 + 5n) \qquad S_n = \frac{n}{2}(a_1 + a_n)$$

$$\boxed{= \frac{n}{2}(9 + 5n)}$$

Exercise

Find each arithmetic sum $73 + 78 + 83 + 88 + \cdots + 558$

Solution

Difference in terms: $d = 78 - 73 = 5$

Number of terms: $n = \frac{558 - 73}{5} + 1 = \boxed{98} \qquad n = \frac{a_n - a_1}{d} + 1$

$$S = \frac{98}{2}(73 + 558) \qquad S_n = \frac{n}{2}(a_1 + a_n)$$

$$\boxed{= 30{,}919}$$

Exercise

Find each arithmetic sum $7 + 1 - 5 - 11 - \cdots - 299$

Solution

Difference in terms: $d = 1 - 7 = -6$

Number of terms: $n = \frac{-299 - 7}{-6} + 1 = \boxed{52} \qquad n = \frac{a_n - a_1}{d} + 1$

$$S = \frac{52}{2}(7 - 299) \qquad S_n = \frac{n}{2}(a_1 + a_n)$$

$$\boxed{= -7592}$$

Exercise

Find each arithmetic sum $-1 + 2 + 7 + \cdots + (4n - 5)$

Solution

$$S = \frac{n}{2}(-1 + 4n - 5) \qquad S_n = \frac{n}{2}(a_1 + a_n)$$

$$\boxed{= n(2n - 3)}$$

Exercise

Find each arithmetic sum $5 + 9 + 13 + \cdots + 49$

Solution

Difference in terms: $d = 9 - 5 = 4$

Number of terms: $n = \frac{49-5+4}{4}\ \underline{=12|}$

$n = \frac{a_n - a_1 + d}{d}$

$S = \frac{12}{2}(5+49)$

$S_n = \frac{n}{2}\left(a_1 + a_n\right)$

$\underline{=324|}$

Exercise

Find each arithmetic sum $2+4+6+\cdots+70$

Solution

Difference in terms: $d = 4-2 = 2$

Number of terms: $n = \frac{70-2+2}{2}\ \underline{=35|}$

$n = \frac{a_n - a_1 + d}{d}$

$S = \frac{35}{2}(70+2)$

$S_n = \frac{n}{2}\left(a_1 + a_n\right)$

$\underline{=1,260|}$

Exercise

Find each arithmetic sum $1+3+5+\cdots+59$

Solution

Difference in terms: $d = 3-1 = 2$

Number of terms: $n = \frac{59-1+2}{2}\ \underline{=30|}$

$n = \frac{a_n - a_1 + d}{d}$

$S = \frac{30}{2}(59+1)$

$S_n = \frac{n}{2}\left(a_1 + a_n\right)$

$\underline{=900|}$

Exercise

Find each arithmetic sum $4+4.5+5+5.5+\cdots+100$

Solution

Difference in terms: $d = 4.5-4 = 0.5$

Number of terms: $n = \frac{100-4+0.5}{0.5}\ \underline{=193|}$

$n = \frac{a_n - a_1 + d}{d}$

$S = \frac{193}{2}(4+100)$

$S_n = \frac{n}{2}\left(a_1 + a_n\right)$

$\underline{=10,036|}$

Exercise

Find each arithmetic sum $8+8\frac{1}{4}+8\frac{1}{2}+8\frac{3}{4}+9+\cdots+50$

Solution

Difference in terms: $d=8\frac{1}{4}-8=\frac{1}{4}$

Number of terms: $n=\frac{50-8+0.25}{0.25}\ \underline{=169|}$ $\qquad n=\frac{a_n-a_1+d}{d}$

$S=\frac{169}{2}(8+50)$ $\qquad S_n=\frac{n}{2}\left(a_1+a_n\right)$

$\underline{=4,901|}$

Exercise

Show that the given sequence is geometric, and find the common ratio

$$5,\ -\frac{5}{4},\ \frac{5}{16},\ \ldots,\ 5\left(-\frac{1}{4}\right)^{n-1},\ \ldots$$

Solution

To be geometric, we must show that $\frac{a_{k+1}}{a_k}=r$ is equal to some constant, which is the common ratio.

The common ratio: $r=\frac{a_{k+1}}{a_k}=\frac{a_2}{a_1}=\frac{-\frac{5}{4}}{5}\ \underline{=-\frac{1}{4}|}$

Exercise

Find the ***n***th term, the fifth term, and the eighth term of the geometric sequence $8,\ 4,\ 2,\ 1,\ \ldots$

Solution

Given: $a_1=8,\quad r=\frac{4}{8}=\frac{1}{2}$

$a_n=a_1r^{n-1}=8\left(\frac{1}{2}\right)^{n-1}$

$=2^3\left(2^{-1}\right)^{n-1}$

$=2^3 2^{-n+1}$

$\underline{=2^{4-n}|}$

$\boxed{a_5=2^{4-5}=2^{-1}=\frac{1}{2}}$

$\boxed{a_8=2^{4-8}=2^{-4}=\frac{1}{16}}$

Exercise

Find the $\boldsymbol{n}$th term, the fifth term, and the eighth term of the geometric sequence

$$300, \ -30, \ 3, \ -0.3, \ \dots$$

Solution

Given: $a_1 = 300, \quad r = \frac{-30}{300} = -0.1$

$$a_n = a_1 r^{n-1} = 300(-0.1)^{n-1}$$

$$3\left(10^2\right)\left(-10^{-1}\right)^{n-1} = 3(10)^2(-10)^{-n+1} = 3(-10)^{-n+3}$$

$$a_5 = 300(-0.1)^{5-1} = 300\left(-10^{-1}\right)^4 \ \underline{= 0.03}$$

$$a_8 = 3(-10)^{-8+3} = 3(-10)^{-5} \ \underline{= -.00003}$$

Exercise

Find the $\boldsymbol{n}$th term, the fifth term, and the eighth term of the geometric sequence $1, \ -\sqrt{3}, \ 3, \ -3\sqrt{3}, \ \dots$

Solution

Given: $a_1 = 1, \quad r = \frac{a_2}{a_1} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$

$$a_n = a_1 r^{n-1} = 1\left(-\sqrt{3}\right)^{n-1} = \left(-\sqrt{3}\right)^{n-1}$$

$$a_5 = 1\left(-\sqrt{3}\right)^{5-1} \ \underline{= 9}$$

$$a_8 = 1\left(-\sqrt{3}\right)^{8-1} = \left(-\sqrt{3}\right)^7 \ \underline{= -27\sqrt{3}}$$

Exercise

Find the $\boldsymbol{n}$th term, the fifth term, and the eighth term of the geometric sequence $4, \ -6, \ 9, \ -13.5, \ \dots$

Solution

Given: $a_1 = 4, \quad r = \frac{a_2}{a_1} = \frac{-6}{4} = -\frac{3}{2}$

$$a_n = a_1 r^{n-1} = 4\left(-\frac{3}{2}\right)^{n-1}$$

$$a_5 = 4\left(-\frac{3}{2}\right)^{5-1} = 4\left(-\frac{3}{2}\right)^4 = 4\left(\frac{3^4}{2^4}\right) \ \underline{= \frac{81}{4} \approx 20.25}$$

$$a_8 = 4\left(-\frac{3}{2}\right)^7 = -4\left(\frac{3^7}{2^7}\right) \ \underline{= -\frac{2187}{32} \approx -68.34375}$$

Exercise

Find the ***n***th term, the fifth term, and the eighth term of the geometric sequence $1,\ -x^2,\ x^4,\ -x^6,\ \ldots$

Solution

Given: $a_1 = 1, \quad r = \frac{a_2}{a_1} = \frac{-x^2}{1} = -x^2$

$$a_n = a_1 r^{n-1} = \left(-x^2\right)^{n-1}$$

$$\boxed{a_5} = \left(-x^2\right)^4 \ \boxed{= x^8}$$

$$\boxed{a_8} = \left(-x^2\right)^7 \ \boxed{= -x^{14}}$$

Exercise

Find the ***n***th term, the fifth term, and the eighth term of the geometric sequence

$$10,\ 10^{2x-1},\ 10^{4x-3},\ 10^{6x-5},\ \ldots$$

Solution

Given: $a_1 = 10, \quad r = \frac{a_2}{a_1} = \frac{10^{2x-1}}{10} = 10^{2x-1-1} = 10^{2x-2}$

$$a_n = a_1 r^{n-1} = 10\left(10^{2x-2}\right)^{n-1} = 10\left(10^{(2x-2)(n-1)}\right) = 10\left(10^{(2x-2)n-2x+2}\right)$$

$$= 10^{2nx-2n-2x+2+1}$$

$$\boxed{= 10^{2(n-1)x-2n+3}}$$

$$\boxed{a_5} = 10^{2(5-1)x-2(5)+3} \boxed{= 10^{8x-7}}$$

$$\boxed{a_8} = 10^{2(8-1)x-2(8)+3} \boxed{= 10^{14x-13}}$$

Exercise

Find the ***n***th term, the fifth term, and the eighth term of the geometric sequence $a_1 = 2, \quad r = 3$

Solution

Given: $a_1 = 10, \quad r = 3$

$$a_n = \boxed{2 \cdot 3^{n-1}} \qquad\qquad a_n = a_1 r^{n-1}$$

$$a_5 = 2 \cdot 3^4 = \boxed{162} \qquad\qquad a_8 = 2 \cdot 3^7 = \boxed{4374}$$

Exercise

Find the **n**th term, the fifth term, and the eighth term of the geometric sequence $a_1 = 1, \quad r = -\frac{1}{2}$

Solution

Given: $a_1 = 1, \quad r = -\frac{1}{2}$

$a_n = \boxed{\left(-\frac{1}{2}\right)^{n-1}}$ $\qquad a_n = a_1 r^{n-1}$

$a_5 = \left(-\frac{1}{2}\right)^4 = \boxed{\frac{1}{16}}$ $\qquad a_8 = \left(-\frac{1}{2}\right)^7 = \boxed{-\frac{1}{128}}$

Exercise

Find the **n**th term, the fifth term, and the eighth term of the geometric sequence $a_1 = -2, \quad r = 4$

Solution

Given: $a_1 = -2, \quad r = 4$

$a_n \boxed{= -2\cdot(4)^{n-1}}$ $\qquad a_n = a_1 r^{n-1}$

$a_5 = \left(-\frac{1}{2}\right)^4 = \boxed{\frac{1}{16}}$ $\qquad a_8 = \left(-\frac{1}{2}\right)^7 = \boxed{-\frac{1}{128}}$

Exercise

Find the **n**th term, the fifth term, and the eighth term of the geometric sequence $a_1 = \sqrt{2}, \quad r = \sqrt{2}$

Solution

Given: $a_1 = \sqrt{2}, \quad r = \sqrt{2}$

$a_n = \sqrt{2}\left(\sqrt{2}\right)^{n-1} \boxed{= \left(\sqrt{2}\right)^n}$ $\qquad a_n = a_1 r^{n-1}$

$a_5 = \left(\sqrt{2}\right)^5 = \boxed{4\sqrt{2}}$ $\qquad a_8 = \left(\sqrt{2}\right)^8 = \boxed{16}$

Exercise

Find the **n**th term, the fifth term, and the eighth term of the geometric sequence $a_1 = 0, \quad r = \pi$

Solution

Given: $a_1 = 0, \quad r = \pi$

$a_n = 0(\pi)^{n-1} \boxed{= 0}$ $\qquad a_n = a_1 r^{n-1}$

$a_5 = 0^5 = \boxed{0}$ $\qquad a_8 = 0^8 = \boxed{0}$

Exercise

Find the ***n***th term, the fifth term, and the eighth term of the geometric sequence $\{s_n\} = \{3^n\}$

Solution

$a_5 \ \boxed{= 3^5} \quad a_8 \ \boxed{= 3^8}$

Exercise

Find the ***n***th term, the fifth term, and the eighth term of the geometric sequence $\{s_n\} = \{(-5)^n\}$

Solution

$a_5 = (-5)^5 \ \boxed{= -5^5} \quad a_8 = (-5)^8 \ \boxed{= 5^8}$

Exercise

Find the ***n***th term, the fifth term, and the eighth term of the geometric sequence $\{s_n\} = \left\{-3\left(\frac{1}{2}\right)^n\right\}$

Solution

$a_5 = -3\left(\frac{1}{2}\right)^5 \ \boxed{= -\frac{3}{32}} \quad a_8 = -3\left(\frac{1}{2}\right)^8 \ \boxed{= -\frac{3}{256}}$

Exercise

Find the ***n***th term, the fifth term, and the eighth term of the geometric sequence $\{u_n\} = \left\{\frac{3^{n-1}}{2^n}\right\}$

Solution

$a_5 = \frac{3^4}{2^5} \ \boxed{= \frac{81}{32}} \quad a_8 = \frac{3^7}{2^8} \ \boxed{= \frac{3^7}{256}}$

Exercise

Find the ***n***th term, the fifth term, and the eighth term of the geometric sequence $\{u_n\} = \left\{\frac{2^n}{3^{n-1}}\right\}$

Solution

$a_5 = \frac{2^5}{3^4} \ \boxed{= \frac{32}{81}} \quad a_8 = \frac{2^8}{3^7} \ \boxed{= \frac{256}{3^7}}$

Exercise

Find all possible values of ***r*** for a geometric sequence with the two given terms $a_4 = 3, \quad a_6 = 9$

Solution

$$\frac{a_6}{a_4} = \frac{9}{3} = 3$$

$$\frac{a_6}{a_4} = \frac{a_1 r^5}{a_1 r^3} = r^2 = 3 \Rightarrow \boxed{r = \pm\sqrt{3}}$$

Exercise

Find the sixth term of the geometric sequence whose first two terms are 4 and 6

Solution

Given: $a_1 = 4, \quad a_2 = 6$

$$r = \frac{a_2}{a_1} = \frac{6}{4} = \frac{3}{2}$$

$$a_6 = a_1 r^{n-1} = 4\left(\frac{3}{2}\right)^5 \; \underline{= \frac{243}{8}}$$

Exercise

Given a geometric sequence with $a_4 = 4, \quad a_7 = 12$, find r and a_{10}

Solution

$$r = \left(\frac{12}{4}\right)^{1/(7-4)} = 3^{1/3} \quad \Rightarrow \boxed{r = \sqrt[3]{3}}$$

$$r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$a_4 = a_1 r^{n-1} \Rightarrow a_1 = \frac{a_4}{r^3} = \frac{4}{3}$$

$$a_{10} = a_1 r^{n-1} = \frac{4}{3}\left(\sqrt[3]{3}\right)^9 \; \underline{= 36}$$

Exercise

Find the specified term of the geometric sequence $\quad a_6; \quad a_1 = 4, \quad a_2 = 6$

Solution

$$r = \left(\frac{6}{4}\right)^{1/(2-1)} = \frac{3}{2}$$

$$r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$a_6 = 4\left(\frac{3}{2}\right)^5 \; \underline{= \frac{3^5}{8}}$$

$$a_n = a_1 r^{n-1}$$

Exercise

Find the specified term of the geometric sequence $\quad a_7; \quad a_2 = 3, \quad a_3 = -\sqrt{3}$

Solution

$$r = \left(\frac{-\sqrt{3}}{3}\right)^{1/(3-2)} = -\frac{\sqrt{3}}{3} \qquad\qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$a_2 = a_1\left(-\frac{\sqrt{3}}{3}\right)^1 = 3 \qquad\qquad a_n = a_1 r^{n-1}$$

$$a_1 = -\frac{9}{\sqrt{3}} = -3\sqrt{3}$$

$$a_7 = -3\sqrt{3}\left(-\frac{\sqrt{3}}{3}\right)^6 = -3\sqrt{3}\,\frac{3^3}{3^6} \boxed{= -\frac{\sqrt{3}}{9}}$$

Exercise

Find the specified term of the geometric sequence $\quad a_6; \quad a_2 = 3, \quad a_3 = -\sqrt{2}$

Solution

$$r = \left(\frac{-\sqrt{2}}{3}\right)^{1/(3-2)} = -\frac{\sqrt{2}}{3} \qquad\qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$a_2 = a_1\left(-\frac{\sqrt{2}}{3}\right)^1 = 3 \qquad\qquad a_n = a_1 r^{n-1}$$

$$a_1 = -\frac{9}{\sqrt{2}}$$

$$a_6 = -\frac{9}{\sqrt{2}}\left(-\frac{\sqrt{2}}{3}\right)^5 = 9\frac{\sqrt{2}^4}{3^5} \boxed{= \frac{4}{27}}$$

Exercise

Find the specified term of the geometric sequence $\quad a_5; \quad a_1 = 4, \quad a_2 = 7$

Solution

$$r = \frac{7}{4} \qquad\qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$a_5 = 4\left(\frac{7}{4}\right)^4 \boxed{= \frac{7^4}{64}} \qquad\qquad a_n = a_1 r^{n-1}$$

Exercise

Find the specified term of the geometric sequence $a_9;\quad a_2 = 3,\quad a_5 = -81$

Solution

$$r = \left(\frac{-81}{3}\right)^{1/(5-2)} = (-27)^{1/3} = -3 \qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$a_2 = a_1(-3)^3 = 3 \qquad a_n = a_1 r^{n-1}$$

$$a_1 = -\tfrac{1}{9}$$

$$a_9 = -\tfrac{1}{9}(-3)^8 \underline{= -3^6}$$

Exercise

Find the specified term of the geometric sequence $a_7;\quad a_1 = -4,\quad a_3 = -1$

Solution

$$r = \left(\frac{-1}{-4}\right)^{1/(3-1)} = \left(\frac{1}{4}\right)^{1/2} = \frac{1}{2} \qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$a_7 = -4\left(\frac{1}{2}\right)^6 \underline{= -\frac{1}{16}} \qquad a_n = a_1 r^{n-1}$$

Exercise

Find the specified term of the geometric sequence $a_8;\quad a_2 = 3,\quad a_4 = 6$

Solution

$$r = \left(\frac{-81}{3}\right)^{1/(5-2)} = (-27)^{1/3} = -3 \qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$a_2 = a_1(-3)^3 = 3 \qquad a_n = a_1 r^{n-1}$$

$$a_1 = -\tfrac{1}{9}$$

$$a_8 = -\tfrac{1}{9}(-3)^7 \underline{= 3^5}$$

Exercise

Express the sum in terms of summation notation: $4+11+18+25+32$. (Answers are not unique)

Solution

$$n = 5 \qquad d = 11 - 4 = 7$$

$$a_n = 4 + (n-1)7 \qquad a_n = a_1 + (n-1)d$$

$$= 4 + 7n - 7$$
$$= 7n - 3$$

$$4 + 11 + 18 + 25 + 32 = \sum_{n=1}^{5} (7n - 3)$$

Exercise

Express the sum in terms of summation notation: $4 + 11 + 18 + \ldots + 466$. (Answers are not unique)

Solution

Difference in terms: $d = 11 - 4 = 7$

Number of terms: $n = \frac{466 - 4}{7} + 1 = 67$

$$a_n = 4 + (n-1)7 \qquad\qquad a_n = a_1 + (n-1)d$$
$$= 4 + 7n - 7$$
$$= 7n - 3$$

$$4 + 11 + 18 + \ldots + 466 = \sum_{n=1}^{67} (7n - 3)$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*) $2 + 4 + 8 + 16 + 32 + 64 + 128$

Solution

$$2 + 4 + 8 + 16 + 32 + 64 + 128 = 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7$$
$$= \sum_{n=1}^{7} 2^n$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*) $2 - 4 + 8 - 16 + 32 - 64$

Solution

$$r = \frac{-4}{2} = -2 \qquad\qquad r = \frac{a_2}{a_1}$$

$$a_n = 2(-2)^{n-1} = (-1)^{n-1} 2^n \qquad\qquad a_n = a_1 r^{n-1}$$

$$2 - 4 + 8 - 16 + 32 - 64 = \sum_{n=1}^{6} (-1)^{n-1} 2^n$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*) $3+8+13+18+23$

Solution

$d = 8-3 = 5$ $\quad\quad d = a_2 - a_1$

$a_n = 3+5(n-1) = 5n-2$ $\quad\quad a_n = a_1 + (n-1)d$

$$3+8+13+18+23 = \sum_{n=1}^{5}(5n-2)$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*) $256+192+144+108+\cdots$

Solution

$r = \frac{192}{256} = \frac{3}{4}$ $\quad\quad r = \frac{a_2}{a_1}$

$a_n = 256\left(\frac{3}{4}\right)^{n-1}$ $\quad\quad a_n = a_1 r^{n-1}$

$$256+192+144+108+\cdots = \sum_{n=1}^{\infty} 256\left(\frac{3}{4}\right)^{n-1}$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*): $\frac{5}{13}+\frac{10}{11}+\frac{15}{9}+\frac{20}{7}$

Solution

Number of terms: $n = 4$

Numerators : $5,10,15,20$ *common difference* 5

Denominators : $13,11,9,7$ *common difference* -2

Using the formula for nth term $a_n = a_1 + (n-1)d$:

Numerator: $a_n = 5+(n-1)5 = 5+5n-5 \underline{= 5n}$

Denominator: $a_n = 13+(n-1)(-2) = 13-2n+2 \underline{= 15-2n}$

Hence the nth term is: $\frac{5}{13}+\frac{10}{11}+\frac{15}{9}+\frac{20}{7} = \sum_{n=1}^{4}\frac{5n}{15-2n}$

Exercise

Express the sum in terms of summation notation (Answers are not unique.) $\frac{1}{4}-\frac{1}{12}+\frac{1}{36}-\frac{1}{108}$

Solution

$$\frac{1}{4}-\frac{1}{12}+\frac{1}{36}-\frac{1}{108}=\frac{1}{4}-\frac{1}{4}\frac{1}{3^1}+\frac{1}{4}\frac{1}{3^2}-\frac{1}{4}\frac{1}{3^3}$$

$$=\sum_{n=1}^{4}(-1)^{n+1}\frac{1}{4}\left(\frac{1}{3}\right)^{n-1}$$

Exercise

Express the sum in terms of summation notation (Answers are not unique.) $3+\frac{3}{5}+\frac{3}{25}+\frac{3}{125}+\frac{3}{625}$

Solution

$$3+\frac{3}{5}+\frac{3}{25}+\frac{3}{125}+\frac{3}{625}=\frac{3}{5^0}+\frac{3}{5^1}+\frac{3}{5^2}+\frac{3}{5^3}+\frac{3}{5^4}$$

$$=\sum_{n=0}^{4}\frac{3}{5^n}$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*): $\frac{3}{7}+\frac{6}{11}+\frac{9}{15}+\frac{12}{19}+\frac{15}{23}+\frac{18}{27}$

Solution

Numerators : 3, 6, 9, 12, 15,18 *common difference* 3

Denominators : 7, 11,15, 19, 13, 27 *common difference* 4

Numerator: $a_n=3+3(n-1)=3n$

Denominator: $a_n=7+4(n-1)=4n+3$

$$a_n=a_1+(n-1)d$$

$$\frac{3}{7}+\frac{6}{11}+\frac{9}{15}+\frac{12}{19}+\frac{15}{23}+\frac{18}{27}=\sum_{n=1}^{6}\frac{3n}{4n+3}$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*.) $\frac{x}{3}+\frac{x^2}{9}+\frac{x^3}{27}+\cdots,\quad |x|<3$

Solution

$$\frac{x}{3}+\frac{x^2}{9}+\frac{x^3}{27}+\cdots=\frac{x}{3}+\frac{x^2}{3^2}+\frac{x^3}{3^3}+\cdots=\sum_{n=1}^{\infty}\left(\frac{x}{3}\right)^n$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique.*) $2x+4x^2+8x^3+\cdots, \quad |x|<\frac{1}{2}$

Solution

$$2x+4x^2+8x^3+\cdots=2x+(2x)^2+(2x)^3+\cdots=\sum_{n=1}^{\infty}(2x)^n$$

Exercise

Find the sum of the infinite geometric series if it exists: $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$

Solution

$a_1=1, \quad r=-\frac{1}{2}$

$S=\dfrac{1}{1+\frac{1}{2}}$ $\qquad S=\dfrac{a_1}{1-r}$

$=\dfrac{1}{\frac{3}{2}}$

$=\boxed{\dfrac{2}{3}}$

Exercise

Find the sum of the infinite geometric series if it exists: $1.5+0.015+0.00015+\ldots$

Solution

$a_1=0.015, \; a_2=.00015, \quad r=\frac{.00015}{.015}=.01$

$S=1.5+\dfrac{a_1}{1-r}$

$=1.5+\dfrac{.015}{1-.01}$

$=\dfrac{15}{10}+\dfrac{.015}{.99}$

$=\dfrac{15}{10}+\dfrac{15}{990}$

$=\dfrac{15}{10}+\dfrac{15}{990}$

$=\dfrac{1500}{990}$

$=\boxed{\dfrac{50}{33}}$

Exercise

Find the sum of the infinite geometric series if it exists: $\sqrt{2}-2+\sqrt{8}-4+\ldots$

Solution

$$a_1=\sqrt{2},\ a_2=-2, \quad r=\frac{-2}{\sqrt{2}}=-\sqrt{2}$$

$$|r|=\sqrt{2}>1 \Rightarrow \text{The sum doesn't exist.}$$

Exercise

Find the sum of the infinite geometric series if it exists: $256+192+144+108+\ldots$

Solution

$$a_1=256, a_2=192, \quad r=\frac{192}{256}=\frac{3}{4}$$

$$S=\frac{256}{1-.75}\ \underline{=1024}$$

$$S=\frac{a_1}{1-r}$$

Exercise

Find the sum of the infinite geometric series if it exists: $\frac{1}{4}+\frac{2}{4}+\frac{2^2}{4}+\frac{2^3}{4}+\ldots+\frac{2^{n-1}}{4}$

Solution

$$r=\frac{\frac{2}{4}}{\frac{1}{4}}=2$$

$$S_n=a_1\frac{1-r^n}{1-r}$$

$$S_n=\frac{1}{4}\left(\frac{1-2^n}{1-2}\right)$$

$$\underline{=-\frac{1}{4}\left(1-2^n\right)}$$

Exercise

Find the sum of the infinite geometric series if it exists: $\frac{3}{9}+\frac{3^2}{9}+\frac{3^3}{9}+\ldots+\frac{3^n}{9}$

Solution

$$r=\frac{\frac{3^2}{9}}{\frac{3}{9}}=3$$

$$S_n=\frac{3}{9}\left(\frac{1-3^n}{1-3}\right)$$

$$S_n=a_1\frac{1-r^n}{1-r}$$

$$\underline{=-\frac{1}{6}\left(1-3^n\right)}$$

Exercise

Find the sum of the infinite geometric series if it exists: $-1-2-4-8-\cdots-2^{n-1}$

Solution

$$r=\frac{-2}{-1}=2$$

$$S_n=-1\left(\frac{1-2^n}{1-2}\right) \qquad S_n=a_1\frac{1-r^n}{1-r}$$

$$=1-2^n$$

Exercise

Find the sum of the infinite geometric series if it exists: $2+\frac{6}{5}+\frac{18}{25}+\ldots+2\left(\frac{3}{5}\right)^{n-1}$

Solution

$$r=\frac{\frac{6}{5}}{2}=\frac{3}{5}<1$$

$$S_n=2\cdot\frac{1-\left(\frac{3}{5}\right)^n}{1-\frac{3}{5}} \qquad S_n=a_1\frac{1-r^n}{1-r}$$

$$=2\cdot\frac{1-\left(\frac{3}{5}\right)^n}{\frac{2}{5}}$$

$$=5\left(1-\left(\frac{3}{5}\right)^n\right)$$

Exercise

Find the sum of the infinite geometric series if it exists: $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots$

Solution

$$r=\frac{1}{3}<1$$

$$S=\frac{1}{1-\frac{1}{3}} \qquad S=\frac{a_1}{1-r}$$

$=\frac{3}{2}$ The series ***converges***

Exercise

Find the sum of the infinite geometric series if it exists: $2+\frac{4}{3}+\frac{8}{9}+\frac{16}{27}+\cdots$

Solution

$$a_1=2 \quad r=\frac{\frac{4}{3}}{2}=\frac{2}{3}<1$$

$$S=\frac{2}{1-\frac{2}{3}}$$

$$S=\frac{a_1}{1-r}$$

$$\underline{=6|}$$ The series ***converges***

Exercise

Find the sum of the infinite geometric series if it exists: $2-\frac{1}{2}+\frac{1}{8}-\frac{1}{32}+\cdots$

Solution

$$a_1=2 \quad r=-\frac{1}{4},\ |r|<1$$

$$S=\frac{2}{1+\frac{1}{4}}$$

$$S=\frac{a_1}{1-r}$$

$$\underline{=\frac{8}{5}|}$$ The series ***converges***

Exercise

Find the sum of the infinite geometric series if it exists: $1-\frac{3}{4}+\frac{9}{16}-\frac{27}{64}+\cdots$

Solution

$$a_1=1 \quad r=-\frac{3}{4},\ |r|<1$$

$$S=\frac{1}{1+\frac{3}{4}}$$

$$S=\frac{a_1}{1-r}$$

$$\underline{=\frac{4}{7}|}$$ The series ***converges***

Exercise

Find the sum of the infinite geometric series if it exists: $9+12+16+\frac{64}{3}+\cdots$

Solution

$a_1=9 \quad r=\frac{4}{3}>1$ The series ***diverges***

Exercise

Find the sum of the infinite geometric series if it exists: $8+12+18+27+\cdots$

Solution

$a_1 = 8 \quad r = \frac{3}{2} > 1$ The series ***diverges***

Exercise

Find the sum of the infinite geometric series if it exists: $6+2+\frac{2}{3}+\frac{2}{9}+\cdots$

Solution

$a_1 = 6 \quad r = \frac{1}{3}, \ |r| < 1$

$$S = \frac{6}{1-\frac{1}{3}}$$

$$S = \frac{a_1}{1-r}$$

$$= \frac{6}{\frac{2}{3}}$$

$$\boxed{= 9}$$ The series ***converges***

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{20}(3k-5)$

Solution

$a_1 = 3(1)-5 = -2 \quad and \quad a_{20} = 3(20)-5 = 55$

$$\sum_{k=1}^{20}(3k-5) = \frac{20}{2}(-2+55)$$

$$S_n = \frac{n}{2}\left(a_1+a_n\right)$$

$$\boxed{= 530}$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{18}\left(\frac{1}{2}k+7\right)$

Solution

$a_1 = \frac{1}{2}(1)+7 = \frac{15}{2} \quad and \quad a_{18} = \frac{1}{2}(18)+7 = 16$

$$\sum_{k=1}^{18}\left(\frac{1}{2}k+7\right) = \frac{18}{2}\left(\frac{15}{2}+16\right)$$

$$S_n = \frac{n}{2}\left(a_1+a_n\right)$$

$$\boxed{= \frac{423}{2}}$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{80}(2k-5)$

Solution

$$a_1 = 2(1)-5 = -3 \quad \textit{and} \quad a_{80} = 2(80)-5 = 155$$

$$\sum_{k=1}^{80}(2k-5) = \frac{80}{2}(-3+155) \qquad\qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$= 40(152)$$

$$\underline{= 6080}$$

Exercise

Find the sum: $\displaystyle\sum_{n=1}^{90}(3-2n)$

Solution

$$a_1 = 3-2(1) = 1 \quad \textit{and} \quad a_{90} = 3-2(90) = -177$$

$$\sum_{n=1}^{80}(3-2n) = \frac{90}{2}(1-177) \qquad\qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$= 45(-176)$$

$$\underline{= -7920}$$

Exercise

Find the sum: $\displaystyle\sum_{n=1}^{100}\left(6-\frac{1}{2}n\right)$

Solution

$$a_1 = 6-\frac{1}{2}(1) = \frac{11}{2} \quad \textit{and} \quad a_{100} = 6-\frac{1}{2}(100) = -44$$

$$\sum_{n=1}^{100}\left(6-\frac{1}{2}n\right) = \frac{100}{2}\left(\frac{11}{2}-44\right) \qquad\qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$= 50\left(-\frac{77}{2}\right)$$

$$\underline{= -1925}$$

Exercise

Find the sum: $\sum_{n=1}^{80}\left(\frac{1}{3}n+\frac{1}{2}\right)$

Solution

$$a_1=\frac{1}{3}(1)+\frac{1}{2}=\frac{5}{6} \quad \textit{and} \quad a_{80}=\frac{1}{3}(80)+\frac{1}{2}=\frac{163}{6}$$

$$\sum_{n=1}^{80}\left(\frac{1}{3}n+\frac{1}{2}\right)=\frac{80}{2}\left(\frac{5}{6}+\frac{163}{6}\right) \qquad S_n=\frac{n}{2}\left(a_1+a_n\right)$$

$$=40\left(\frac{168}{6}\right)$$

$$=1,120$$

Exercise

Find the sum: $\sum_{k=1}^{10}3^k$

Solution

$$\sum_{k=1}^{10}3^k=3\frac{1-3^{10}}{1-3} \qquad S_n=a_1\frac{1-r^n}{1-r}$$

$$=3\frac{-59048}{-2}$$

$$=88,572$$

Exercise

Find the sum: $\sum_{k=1}^{9}\left(-\sqrt{5}\right)^k$

Solution

$$\begin{cases} a_1=-\sqrt{5} \\ a_2=\left(-\sqrt{5}\right)^2=5 \end{cases} \Rightarrow r=\frac{a_2}{a_1}=\frac{5}{-\sqrt{5}}=-\sqrt{5} \qquad S_n=a_1\frac{1-r^n}{1-r}$$

$$\sum_{k=1}^{9}\left(-\sqrt{5}\right)^k=\left(-\sqrt{5}\right)\frac{1-\left(-\sqrt{5}\right)^9}{1-\left(-\sqrt{5}\right)}$$

$$=\frac{\left(-\sqrt{5}\right)\left(1+625\sqrt{5}\right)}{1+\sqrt{5}}\frac{1-\sqrt{5}}{1-\sqrt{5}}$$

$$=\frac{3124\sqrt{5}-3120}{-4}$$

$$=780-781\sqrt{5}$$

Exercise

Find the sum: $\sum_{k=0}^{9}\left(-\frac{1}{2}\right)^{k+1}$

Solution

$$\sum_{k=0}^{9}\left(-\frac{1}{2}\right)^{k+1}=\left(-\frac{1}{2}\right)\frac{1-\left(-\frac{1}{2}\right)^{10}}{1+\frac{1}{2}}$$

$$S_n=a_1\frac{1-r^n}{1-r}$$

$$=-\frac{1}{2}\frac{1-\frac{1}{2^{10}}}{\frac{3}{2}}$$

$$=-\frac{\frac{1024-1}{1024}}{3}$$

$$=-\frac{1023}{3072}$$

$$\boxed{=-\frac{341}{1024}}$$

Exercise

Find the sum : $\sum_{n=1}^{\infty}2\left(\frac{2}{3}\right)^{n-1}$

Solution

$$\sum_{n=1}^{\infty}2\left(\frac{2}{3}\right)^{n-1}=\frac{2}{1-\frac{2}{3}}$$

$$S=\frac{a_1}{1-r}$$

$$=\frac{2}{\frac{1}{3}}$$

$\boxed{=6}$, the series ***converges***

Exercise

Find the sum: $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n}$

Solution

$$\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n}=\frac{2}{3}\frac{1}{1-\frac{2}{3}}$$

$|r|=\frac{2}{3}<1$ $\qquad$ $S=\frac{a_1}{1-r}$

$$=\frac{2}{3}(3)$$

$\boxed{=2}$, the series ***converges***

Exercise

Find the sum: $\displaystyle\sum_{n=1}^{\infty} 3\left(\frac{3}{2}\right)^{n}$

Solution

Since $|r| = \frac{3}{2} > 1$, the series ***diverges***

Exercise

Find the sum: $\displaystyle\sum_{n=1}^{\infty} 5\left(\frac{1}{4}\right)^{n-1}$

Solution

$$\sum_{n=1}^{\infty} 5\left(\frac{1}{4}\right)^{n-1} = \frac{5}{1-\frac{1}{4}} \qquad a_1 = 5 \quad |r| = \frac{1}{4} < 1 \qquad S = \frac{a_1}{1-r}$$

$$= \frac{20}{3}$$

The series ***converges***

Exercise

Find the sum: $\displaystyle\sum_{n=1}^{\infty} 8\left(\frac{1}{3}\right)^{n-1}$

Solution

$$\sum_{n=1}^{\infty} 8\left(\frac{1}{3}\right)^{n-1} = \frac{8}{1-\frac{1}{3}} \qquad a_1 = 8 \quad |r| = \frac{1}{3} < 1 \qquad S = \frac{a_1}{1-r}$$

$$= 12$$

The series ***converges***

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2}\cdot 3^{k-1}$

Solution

Since $|r| = 3 > 1$, the series ***diverges***

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{\infty} 6\left(-\frac{2}{3}\right)^{k-1}$

Solution

$$\sum_{k=1}^{\infty} 6\left(-\frac{2}{3}\right)^{k-1} = \frac{6}{1+\frac{2}{3}} \qquad a_1 = 6 \quad |r| = \frac{2}{3} < 1 \qquad S = \frac{a_1}{1-r}$$

$$= \frac{18}{5}$$

The series ***converges***

Exercise

Find the sum: $\sum_{k=1}^{\infty} 4\left(-\frac{1}{2}\right)^{k-1}$

Solution

$$\sum_{k=1}^{\infty} 4\left(-\frac{1}{2}\right)^{k-1} = \frac{4}{1+\frac{1}{2}} \qquad a_1 = 4 \quad |r| = \frac{1}{2} < 1 \qquad S = \frac{a_1}{1-r}$$

$$= \frac{8}{3}$$

The series ***converges***

Exercise

Find the sum: $\sum_{k=8}^{14} \left(3^{k-7} + 2j^2\right)$

Solution

$$a_n = 3^{n-7} \rightarrow a_1 = 3^{-6}; \ r = 3 \quad \& \quad n = 14 - 8 + 1 = 7$$

$$\sum_{k=8}^{14} \left(3^{k-7} + 2j^2\right) = \sum_{k=8}^{14} 3^{k-7} + 2\sum_{k=8}^{14} j^2$$

$$= 3^{-6} \cdot \frac{1-3^7}{1-3} + 2(7)j^2$$

$$= -\frac{1}{2}\left(\frac{1-3^7}{3^6}\right) + 14j^2$$

$$= -\frac{1}{2}\left(\frac{-2,186}{729}\right) + 14j^2$$

$$= \frac{1,093}{729} + 14j^2$$

Exercise

Find the sum: $14, \ 16, \ 18, \ 20, \ \cdots$

Solution

$$n = 120; \quad a_1 = 14, \quad d = 16 - 14 = 2$$

$$S_{120} = \frac{120}{2}\left[2(14) + 2(120-1)\right] \qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$= 60(48 + 238)$$

$$\boxed{= 17,160}$$

Exercise

Find the sum of the first 46 terms of $2, \ -1, \ -4, \ -7, \ \cdots$

Solution

$$n = 46; \quad a_1 = 2, \quad d = -1 - 2 = -3$$

$$S_{46} = \frac{46}{2}\left[2(2) - 3(46-1)\right] \qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$= 23(4 - 135)$$

$$\boxed{= -3,013}$$

Exercise

Find the rational number represented by the repeating decimal $0.\overline{23}$

Solution

$$0.\overline{23} = 0.23 + 0.0023 + .000023 + \dots \qquad a_1 = 0.23, \quad r = \frac{.0023}{.23} = 0.01$$

$$S = \frac{0.23}{1 - 0.01} \qquad S = \frac{a_1}{1-r}$$

$$= \frac{0.23}{0.99}$$

$$\boxed{= \frac{23}{99}}$$

Exercise

Find the rational number represented by the repeating decimal $0.0\overline{71}$

Solution

$$0.0\overline{71} = 0.071 + 0.00071 + .0000071 + \dots \qquad a_1 = 0.071, \quad r = \frac{.00071}{.071} = 0.01$$

$$S = \frac{0.071}{1 - 0.01} \qquad S = \frac{a_1}{1-r}$$

$$= \frac{0.071}{0.990}$$

$$\boxed{= \frac{71}{990}}$$

Exercise

Find the rational number represented by the repeating decimal $2.4\overline{17}$

Solution

$$2.4\overline{17} = 2.4 + 0.017 + 0.00017 + .0000017 + \ldots$$

$$a_1 = 0.017, \quad r = \frac{.00017}{.017} = 0.01$$

$$S = 2.4 + \frac{0.017}{1-0.01} \qquad\qquad S = \frac{a_1}{1-r}$$

$$= \frac{24}{10} + \frac{0.017}{0.990}$$

$$= \frac{24}{10} + \frac{17}{990}$$

$$= \frac{240+17}{990}$$

$$= \frac{2,393}{990}$$

Exercise

Find the rational number represented by the repeating decimal $10.\overline{5}$

Solution

$$10.\overline{5} = 10 + 0.5 + 0.05 + .005 + \ldots$$

$$a_1 = 0.5, \quad r = \frac{0.05}{0.5} = 0.1$$

$$S = 10 + \frac{0.5}{1-0.1} \qquad\qquad S = \frac{a_1}{1-r}$$

$$= 10 + \frac{0.5}{0.9}$$

$$= 10 + \frac{5}{9}$$

$$= \frac{95}{9}$$

Exercise

Find the rational number represented by the repeating decimal $5.\overline{146}$

Solution

$$5.\overline{146} = 5 + 0.146 + 0.000146 + .000000146 + \ldots$$

$$a_1 = 0.146, \quad r = \frac{0.000146}{0.146} = 0.001$$

$$S = 5 + \frac{0.146}{1-0.001} \qquad\qquad S = \frac{a_1}{1-r}$$

$$= 5 + \frac{0.146}{0.999}$$

$$= 5 + \frac{146}{999}$$

$$\boxed{= \frac{5,141}{999}}$$

Exercise

Find the rational number represented by the repeating decimal $3.2\overline{394}$

Solution

$$3.2\overline{394} = 3.2 + 0.0394 + 0.0000394 + \dots$$

$$a_1 = 0.0394, \quad r = \frac{0.0000394}{0.0394} = 0.001$$

$$S = 3.2 + \frac{0.0394}{1 - 0.001} \qquad\qquad S = \frac{a_1}{1-r}$$

$$= \frac{32}{10} + \frac{0.0394}{0.9990}$$

$$= \frac{32}{10} + \frac{394}{9990}$$

$$= \frac{31968 + 394}{9990}$$

$$= \frac{32,362}{9,990}$$

$$\boxed{= \frac{16,181}{4,995}}$$

Exercise

Find the rational number represented by the repeating decimal $1.\overline{6124}$

Solution

$$1.\overline{6124} = 1 + 0.6124 + 0.00006124 + \dots$$

$$a_1 = 0.6124, \quad r = \frac{0.00006124}{0.6124} = 0.0001$$

$$S = 1 + \frac{0.6124}{1 - 0.0001} \qquad\qquad S = \frac{a_1}{1-r}$$

$$= 1 + \frac{0.6124}{0.9999}$$

$$= 1 + \frac{6124}{9999}$$

$$\boxed{= \frac{16,123}{9,999}}$$

Exercise

Find x so that $x+3$, $2x+1$, and $5x+2$ are consecutive terms of an arithmetic sequence.

Solution

$$d = 2x+1-(x+3) \underline{= x-2|}$$

$$d = 5x+2-(2x+1) \underline{= 3x+1|}$$

$$d = 3x+1 = x-2$$

$$2x = -3 \quad \rightarrow \quad \underline{x = -\frac{3}{2}|}$$

Exercise

Find x so that $2x$, $3x+2$, and $5x+3$ are consecutive terms of an arithmetic sequence.

Solution

$$d = 3x+2-2x \underline{= x+2|}$$

$$d = 5x+3-(3x+2) \underline{= 2x+1|}$$

$$d = 2x+1 = x+2 \quad \rightarrow \quad \underline{x = 1|}$$

Exercise

Find x so that x, $x+2$, and $x+3$ are consecutive terms of a geometric sequence.

Solution

$$r = \frac{x+2}{x}, \quad r = \frac{x+3}{x+2}$$

$$r = \frac{x+2}{x} = \frac{x+3}{x+2}$$

$$(x+2)^2 = x^2+3x$$

$$x^2+4x+4-x^2-3x = 0$$

$$x+4 = 0 \quad \rightarrow \quad \underline{x = -4|}$$

Exercise

Find x so that $x-1$, x and $x+2$ are consecutive terms of a geometric sequence.

Solution

$$r = \frac{x}{x-1} = \frac{x+2}{x}$$

$$x^2 = x^2+x-2$$

$$x-2 = 0 \quad \rightarrow \quad \underline{x = 2|}$$

Exercise

How many terms must be added in an arithmetic sequence whose first term is 11 and whose common difference is 3 to obtain a sum of 1092?

Solution

Given: $a_1 = 11; \quad d = 3; \quad S = 1092$

$1092 = \frac{n}{2}\left(22 + 3(n-1)\right)$ $\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

$n(3n+19) = 2184$

$3n^2 + 19n - 2184 = 0$ $\qquad n = \frac{-19 \pm \sqrt{361 + 26208}}{6} = \frac{-19 \pm 163}{6}$

$\underline{n = 24}|$ & $\cancel{n = -\frac{91}{3}}$

Exercise

How many terms must be added in an arithmetic sequence whose first term is 78 and whose common difference is –4 to obtain a sum of 702?

Solution

Given: $a_1 = 78; \quad d = -4; \quad S = 702$

$702 = \frac{n}{2}\left(2(78) - 4(n-1)\right)$ $\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

$n(160 - 4n) = 1404$

$-4n^2 + 160n - 1404 = 0$ $\qquad n = \frac{-160 \pm \sqrt{25,600 - 22464}}{-8} = \frac{160 \pm 56}{8}$

$\underline{n = 13}|$ & $\underline{n = 27}|$

Exercise

The first ten rows of seating in a certain section of a stadium have 30 seats, 32 seats, 34 seats, and so on. The eleventh through the twentieth rows each contain 50 seats. Find the total number of seats in the section.

Solution

Given: $a_1 = 30; \quad d = 2$

$S = S_{10} + 50(20 - 11 + 1)$ $\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

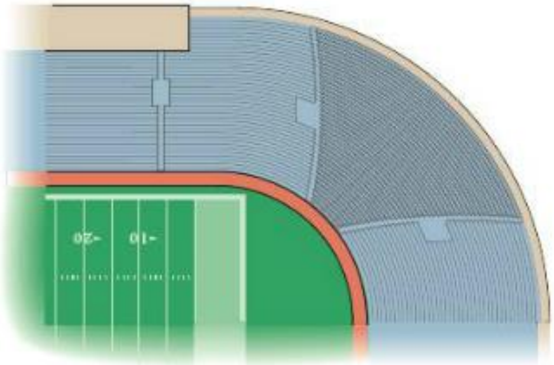
$= \frac{10}{2}\left(2(30) + 2(9)\right) + 50(10)$

$= 5(78) + 500$

$\underline{= 890 \ \ seats}|$

Exercise

The corner section of a football stadium has 15 seats in the first row and 40 rows in all. Each successive row contains two additional seats. How many seats are in this section?



Solution

Given: $a_1 = 15;\quad d = 2;\quad n = 40$

$$S_{40} = \frac{40}{2}\left(30 + 2(40-1)\right) \qquad\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$= 20(30+78)$$

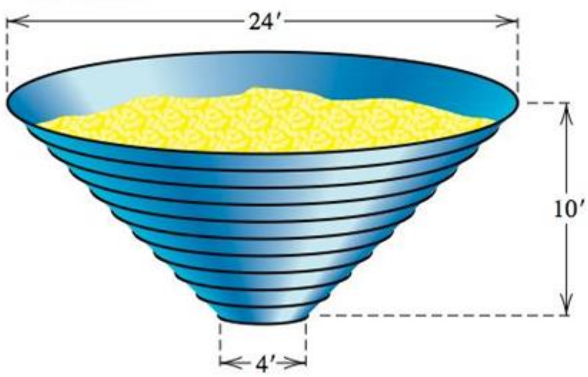
$$= 20(108)$$

$$\underline{= 2{,}160}|$$

The corner section has 2,160 seats.

Exercise

A gain bin is to be constructed in the shape of a frustum of a cone.



The bin is to be 10 *feet* tall with 11 metal rings positioned uniformly around it, from the 4-foot opening at the bottom to the 24-foot opening at the top. Find the total length of metal needed to make the rings.

Solution

The circumference of each ring is πD.

$a_1 = 4\pi;\quad a_{11} = 24\pi$

$$24 = 4 + (11-1)d \;\rightarrow\; 10d = 20 \;\Rightarrow\; \underline{d = 2}| \qquad\qquad a_n = a_1 + (n-1)d$$

$$S_{11} = \frac{11}{2}(4\pi + 24\pi)\; \underline{= 154\pi \;\; ft}| \qquad\qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

Exercise

A bicycle rider coasts downhill, traveling 4 *feet* the first second. In each succeeding second, the rider travels 5 *feet* farther than in the preceding second. If the rider reaches the bottom of the hill in 11 *seconds*, find the total distance traveled.

Solution

Given: $a_1 = 4\ ft \quad \& \quad d = 5\ ft$

$S_{11} = \frac{11}{2}(8 + 5(10))\ \underline{= 319\ ft}|$ $\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

∴ the total distance traveled 319 *feet*.

Exercise

A contest will have five each prizes totaling \$5,000, and there will be a \$100 difference between successive prices. Find the first prize.

Solution

Given: $n = 5 \quad S_5 = 5000 \quad d = -100$

$5,000 = \frac{5}{2}\left[2a_1 + 4(-100)\right]$ $\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

$2,000 = 2a_1 - 400$

$\underline{a_1 = \$1,200}|$

Exercise

A Company is to distribute \$46,000 in bonuses to its top ten salespeople. The tenth salesperson on the list will receive \$1,000, and the difference in bonus money between successively ranked salesperson is to be constant. Find the bonus for each salesperson.

Solution

Given: $n = 10 \quad S_{10} = 46,000 \quad a_{10} = 1,000$

$46,000 = \frac{10}{2}\left(a_1 + 1000\right)$ $\qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$

$9,200 = a_1 + 1000$

$a_1 = 8,200$

$|\underline{d} = \frac{1,000 - 8,200}{9}\ \underline{= -800}|$ $\qquad a_n = a_1 + (n-1)d$

\$8,200 \$7,400 \$6,600 \$5,800 \$5,000 \$4,200 \$3,400 \$2,600 \$1,800 \$1,000

Exercise

Assuming air resistance is negligible, a small object that is dropped from a hot air balloon falls 16 *feet* during the first second, 48 *feet* during the second second, 80 *feet* during the third second, 112 *feet* during the fourth second, and so on. Find an expression for the distance the object falls in n seconds.

Solution

Given the sequence: 16, 48, 80, 112, $\cdots$

This is an arithmetic sequence with: $a_1 = 16 \quad \& \quad d = 48 - 16 = 32$

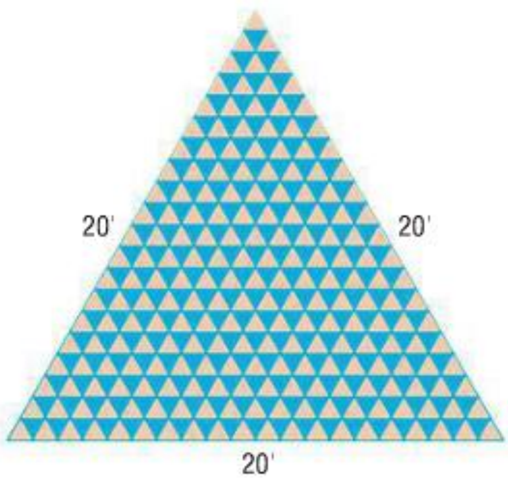
$$S_n = \frac{n}{2}\left(32 + 32(n-1)\right) \qquad\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$= \frac{n}{2}(32n)$$

$$= 16n^2$$

Exercise

A mosaic is designed in the shape of an equilateral triangle, 20 *feet* on each side. Each tile in the mosaic is in the shape of an equilateral triangle, 12 *inches* to a side. The tiles are to alternate in color as shown below.



How many tiles of each color will be required?

Solution

Bottom row has 20 lighter colored tiles.

Top row has 1 lighter colored tile.

The number decreases by 1 as we move up the triangle.

$\therefore$ This is an arithmetic sequence with: $a_1 = 20; \quad d = -1; \quad n = 20$

$$S_{20} = \frac{20}{2}\left(40 + (-1)(20-1)\right) \qquad\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$= 10(40 - 19)$$

$$= 10(21)$$

$$= 210$$

$\therefore$ There are 210 lighter colored tiles.

Bottom row has 19 darker colored tiles.
Top row has 1 darker colored tile.
∴ This is an arithmetic sequence with: $a_1 = 1;\quad d = -1;\quad n = 19$

$$S_{19} = \frac{19}{2}\left(2(19) + (-1)(19-1)\right) \qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$= \frac{19}{2}(38-18)$$

$$\underline{= 190}$$

∴ There are 190 darker colored tiles.

Exercise

A brick staircase has a total of 30 steps. The bottom step requires 100 bricks. Each successive step requires two fewer bricks than the prior step.

a) How many bricks are required for the top step?

b) How many bricks are required to build the staircase?

Solution

a) ***Given***: $n = 30 \quad a_1 = 100 \quad d = -2$

$$a_n = 100 - 2(n-1) \ \underline{= -2n + 102} \qquad a_n = a_1 + (n-1)d$$

$$a_{30} = 102 - 60 \ \underline{= 42}$$

b) $S_{30} = 15(100+42) \ \underline{= 2,130} \qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$

It required 2130 bricks to build the staircase.