

Solution **Section 3.1 – Inner Products**

Exercise

Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on \mathbb{R}^2 , and let $\vec{u} = (1, 1)$, $\vec{v} = (3, 2)$, $\vec{w} = (0, -1)$, and $k = 3$. Compute the following.

- | | | |
|--|---|-----------------------------|
| a) $\langle \vec{u}, \vec{v} \rangle$ | c) $\langle \vec{u} + \vec{v}, \vec{w} \rangle$ | e) $d(\vec{u}, \vec{v})$ |
| b) $\langle k\vec{v}, \vec{w} \rangle$ | d) $\ \vec{v}\ $ | f) $\ \vec{u} - k\vec{v}\ $ |

Solution

$$\begin{aligned} a) \quad \langle \vec{u}, \vec{v} \rangle &= 1(3) + 1(2) \\ &= 5 \end{aligned}$$

$$\begin{aligned} b) \quad \langle k\vec{v}, \vec{w} \rangle &= \langle 3\vec{v}, \vec{w} \rangle \\ &= 9 \cdot 0 + 6 \cdot (-1) \\ &= -6 \end{aligned}$$

$$\begin{aligned} c) \quad \langle \vec{u} + \vec{v}, \vec{w} \rangle &= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \\ &= 1 \cdot 0 + 1 \cdot (-1) + 3 \cdot 0 + 2 \cdot (-1) \\ &= -3 \end{aligned}$$

$$\begin{aligned} d) \quad \|\vec{v}\| &= \sqrt{\langle \vec{v}, \vec{v} \rangle} \\ &= \sqrt{3^2 + 2^2} \\ &= \sqrt{13} \end{aligned}$$

$$\begin{aligned} e) \quad d(\vec{u}, \vec{v}) &= \|\vec{u} - \vec{v}\| \\ &= \|(-2, -1)\| \\ &= \sqrt{(-2)^2 + (-1)^2} \\ &= \sqrt{5} \end{aligned}$$

$$\begin{aligned} f) \quad \|\vec{u} - k\vec{v}\| &= \|(1, 1) - 3(3, 2)\| \\ &= \|(-8, -5)\| \\ &= \sqrt{(-8)^2 + (-5)^2} \\ &= \sqrt{89} \end{aligned}$$

Exercise

Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on \mathbb{R}^2 , and let $\vec{u} = (1, 1)$, $\vec{v} = (3, 2)$, $\vec{w} = (0, -1)$ and $k = 3$. Compute the following for the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 2u_1v_1 + 3u_2v_2$.

a) $\langle \vec{u}, \vec{v} \rangle$

c) $\langle \vec{u} + \vec{v}, \vec{w} \rangle$

e) $d(\vec{u}, \vec{v})$

b) $\langle k\vec{v}, \vec{w} \rangle$

d) $\|\vec{v}\|$

f) $\|\vec{u} - k\vec{v}\|$

Solution

a) $\langle \vec{u}, \vec{v} \rangle = 2(1)(3) + 3(1)(2)$
 $\underline{= 12}$

b) $\langle k\vec{v}, \vec{w} \rangle = 2(3 \cdot 3)(0) + 3(3 \cdot 2)(-1)$
 $\underline{= -18}$

c) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$
 $= 1 \cdot 0 + 1 \cdot (-1) + 3 \cdot 0 + 2 \cdot (-1)$
 $\underline{= -3}$

d) $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$
 $= \sqrt{2(3)(3) + 3(2)(2)}$
 $\underline{= \sqrt{30}}$

e) $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$
 $= \|\langle (-2, -1) \rangle\|$
 $= \sqrt{2(-2)(-2) + 3(-1)(-1)}$
 $\underline{= \sqrt{11}}$

f) $\|\vec{u} - k\vec{v}\| = \|(1, 1) - 3(3, 2)\|$
 $= \|\langle (-8, -5) \rangle\|$
 $= \sqrt{2(-8)^2 + 3(-5)^2}$
 $\underline{= \sqrt{203}}$

Exercise

Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on \mathbb{R}^2 , and let $\vec{u} = (3, -2)$, $\vec{v} = (4, 5)$, $\vec{w} = (-1, 6)$, and $k = -4$. Verify the following.

$$a) \quad \langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$$

$$d) \quad \langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle$$

$$b) \quad \langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$$

$$e) \quad \langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$$

$$c) \quad \langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$$

Solution

$$a) \quad \langle \vec{u}, \vec{v} \rangle = 3 \cdot 4 + (-2) \cdot (5)$$

$$= 2$$

$$\langle \vec{v}, \vec{u} \rangle = 4 \cdot 3 + (5) \cdot (-2)$$

$$= 2$$

$$\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle \quad \checkmark$$

$$b) \quad \langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle (7, 3), (-1, 6) \rangle$$

$$= 7(-1) + 3(6)$$

$$= 11$$

$$\langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle = (3)(-1) + (-2)(6) + (4)(-1) + (5)(6)$$

$$= 11$$

$$\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \quad \checkmark$$

$$c) \quad \langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle (3, -2), (3, 11) \rangle$$

$$= 3(3) + (-2)(11)$$

$$= -13$$

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle = (3)(4) + (-2)(5) + (3)(-1) + (-2)(6)$$

$$= -13$$

$$\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle \quad \checkmark$$

$$d) \quad \langle k\vec{u}, \vec{v} \rangle = (-4 \cdot 3) \cdot 4 + ((-4)(-2)) \cdot (5)$$

$$= -8$$

$$k \langle \vec{u}, \vec{v} \rangle = (-4)(3 \cdot 4 + (-2) \cdot (5))$$

$$= -8$$

$$\langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle \quad \checkmark$$

$$\begin{aligned}
 e) \quad \langle \vec{0}, \vec{v} \rangle &= 0 \cdot 4 + 0 \cdot (5) \\
 &= 0 \\
 \langle \vec{v}, \vec{0} \rangle &= 4 \cdot 0 + (5) \cdot (0) \\
 &= 0 \\
 \langle \vec{0}, \vec{v} \rangle &= \langle \vec{v}, \vec{0} \rangle = 0 \quad \checkmark
 \end{aligned}$$

Exercise

Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on \mathbb{R}^2 , and let $\vec{u} = (3, -2)$, $\vec{v} = (4, 5)$, $\vec{w} = (-1, 6)$, and $k = -4$. Verify the following for the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 4u_1v_1 + 5u_2v_2$.

$$\begin{aligned}
 a) \quad \langle \vec{u}, \vec{v} \rangle &= \langle \vec{v}, \vec{u} \rangle & d) \quad \langle k\vec{u}, \vec{v} \rangle &= k \langle \vec{u}, \vec{v} \rangle \\
 b) \quad \langle \vec{u} + \vec{v}, \vec{w} \rangle &= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle & e) \quad \langle \vec{0}, \vec{v} \rangle &= \langle \vec{v}, \vec{0} \rangle = 0 \\
 c) \quad \langle \vec{u}, \vec{v} + \vec{w} \rangle &= \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle
 \end{aligned}$$

Solution

$$\begin{aligned}
 a) \quad \langle \vec{u}, \vec{v} \rangle &= 4 \cdot 3 \cdot 4 + 5 \cdot (-2) \cdot (5) \\
 &= -2 \\
 \langle \vec{v}, \vec{u} \rangle &= 4 \cdot 4 \cdot 3 + 5 \cdot (5) \cdot (-2) \\
 &= -2 \\
 \langle \vec{u}, \vec{v} \rangle &= \langle \vec{v}, \vec{u} \rangle \quad \checkmark \\
 b) \quad \langle \vec{u} + \vec{v}, \vec{w} \rangle &= \langle (7, 3), (-1, 6) \rangle \\
 &= 4 \cdot 7 \cdot (-1) + 5 \cdot 3 \cdot (6) \\
 &= 62 \\
 \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle &= 4 \cdot (3) \cdot (-1) + 5 \cdot (-2) \cdot (6) + 4 \cdot (4) \cdot (-1) + 5 \cdot (5) \cdot (6) \\
 &= 62 \\
 \langle \vec{u} + \vec{v}, \vec{w} \rangle &= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \quad \checkmark \\
 c) \quad \langle \vec{u}, \vec{v} + \vec{w} \rangle &= \langle (3, -2), (3, 11) \rangle \\
 &= 4 \cdot 3 \cdot (3) + 5 \cdot (-2) \cdot (11) \\
 &= -74 \\
 \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle &= 4 \cdot (3) \cdot (4) + 5 \cdot (-2) \cdot (5) + 4 \cdot (3) \cdot (-1) + 5 \cdot (-2) \cdot (6)
 \end{aligned}$$

$$\underline{= -74}$$

$$\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle \quad \checkmark$$

$$d) \quad \langle k\vec{u}, \vec{v} \rangle = 4 \cdot (-4 \cdot 3) \cdot 4 + 5 \cdot ((-4)(-2)) \cdot (5) \\ \underline{= 8}$$

$$k \langle \vec{u}, \vec{v} \rangle = (-4)(4 \cdot 3 \cdot 4 + 5 \cdot (-2) \cdot (5)) \\ \underline{= 8}$$

$$\langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle \quad \checkmark$$

$$e) \quad \langle \vec{0}, \vec{v} \rangle = 4 \cdot 0 \cdot 4 + 5 \cdot 0 \cdot (5) \\ \underline{= 0}$$

$$\langle \vec{v}, \vec{0} \rangle = 4 \cdot 4 \cdot 0 + 5 \cdot (5) \cdot (0) \\ \underline{= 0}$$

$$\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0 \quad \checkmark$$

Exercise

Let $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$. Show that the following are inner product on \mathbb{R}^2 by verifying that the inner product axioms hold. $\langle \vec{u}, \vec{v} \rangle = 3u_1v_1 + 5u_2v_2$

Solution

$$\text{Axiom 1: } \langle \vec{u}, \vec{v} \rangle = 3u_1v_1 + 5u_2v_2 \\ = 3v_1u_1 + 5v_2u_2 \\ = \langle \vec{v}, \vec{u} \rangle \quad \checkmark$$

$$\text{Axiom 2: } \langle \vec{u} + \vec{v}, \vec{w} \rangle = 3(u_1 + v_1)w_1 + 5(u_2 + v_2)w_2 \\ = 3(u_1w_1 + v_1w_1) + 5(u_2w_2 + v_2w_2) \\ = 3u_1w_1 + 3v_1w_1 + 5u_2w_2 + 5v_2w_2 \\ = (3u_1w_1 + 5u_2w_2) + (3v_1w_1 + 5v_2w_2) \\ = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \quad \checkmark$$

$$\text{Axiom 3: } \langle k\vec{u}, \vec{v} \rangle = 3(ku_1)v_1 + 5(ku_2)v_2 \\ = k(3u_1v_1 + 5u_2v_2)$$

$$= k \langle \vec{u}, \vec{v} \rangle \quad \checkmark$$

$$\text{Axiom 4: } \langle \vec{v}, \vec{v} \rangle = 3v_1v_1 + 5v_2v_2$$

$$= 3v_1^2 + 5v_2^2 \geq 0$$

$$v_1 = v_2 = 0 \quad \text{iff} \quad \vec{v} = \vec{0} \quad \checkmark$$

Exercise

Show that the following identity holds for the vectors in any inner product space

$$\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$$

Solution

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 &= \langle \vec{u} + \vec{v}, \vec{u} + \vec{v} \rangle + \langle \vec{u} - \vec{v}, \vec{u} - \vec{v} \rangle \\ &= \langle \vec{u}, \vec{u} + \vec{v} \rangle + \langle \vec{v}, \vec{u} + \vec{v} \rangle + \langle \vec{u}, \vec{u} - \vec{v} \rangle - \langle \vec{v}, \vec{u} - \vec{v} \rangle \\ &= \langle \vec{u}, \vec{u} \rangle + \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{u} \rangle + \langle \vec{v}, \vec{v} \rangle + \langle \vec{u}, \vec{u} \rangle - \langle \vec{u}, \vec{v} \rangle - \langle \vec{v}, \vec{u} \rangle + \langle \vec{v}, \vec{v} \rangle \\ &= 2\langle \vec{u}, \vec{u} \rangle + 2\langle \vec{v}, \vec{v} \rangle \\ &= 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2 \quad \checkmark \end{aligned}$$

Exercise

Show that the following identity holds for the vectors in any inner product space

$$\langle \vec{u}, \vec{v} \rangle = \frac{1}{4} \left(\|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 \right)$$

Solution

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 &= \langle \vec{u} + \vec{v}, \vec{u} + \vec{v} \rangle = \|\vec{u}\|^2 + 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2 \\ \|\vec{u} - \vec{v}\|^2 &= \langle \vec{u} - \vec{v}, \vec{u} - \vec{v} \rangle = \|\vec{u}\|^2 - 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2 \\ \|\vec{u} + \vec{v}\|^2 &= \|\vec{u}\|^2 + 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2 \\ - \|\vec{u} - \vec{v}\|^2 &= \|\vec{u}\|^2 - 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2 \\ \hline \|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 &= 4\langle \vec{u}, \vec{v} \rangle \\ \langle \vec{u}, \vec{v} \rangle &= \frac{1}{4} \left(\|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 \right) \quad \checkmark \end{aligned}$$

Exercise

Prove that $\|k\vec{v}\| = |k| \|\vec{v}\|$

Solution

$$\begin{aligned}\|k\vec{v}\|^2 &= \langle k\vec{v}, \vec{v} \rangle \\ &= k^2 \langle \vec{v}, \vec{v} \rangle \\ &= k^2 \|\vec{v}\|^2\end{aligned}$$

$$\|k\vec{v}\| = |k| \|\vec{v}\| \quad \checkmark$$

Solution **Section 3.2 – Angle and Orthogonality in Inner Product Spaces**

Exercise

Which of the following form orthonormal sets?

- a) $(1, 0), (0, 2)$ in \mathbb{R}^2
- b) $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in \mathbb{R}^2
- c) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in \mathbb{R}^2
- d) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ in \mathbb{R}^3
- e) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ in \mathbb{R}^3
- f) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ in \mathbb{R}^3

Solution

$$\begin{aligned} \text{a) } (1, 0) \cdot (0, 2) &= 1(0) + 0(2) \\ &= 0 \end{aligned}$$

They are **orthonormal** sets

$$\begin{aligned} \text{b) } \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) &= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right) \frac{1}{\sqrt{2}} \\ &= \frac{1}{2} - \frac{1}{2} \\ &= 0 \end{aligned}$$

They are **orthonormal** sets

$$\begin{aligned} \text{c) } \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) &= \left(-\frac{1}{\sqrt{2}}\right) \frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right) \frac{1}{\sqrt{2}} \\ &= -\frac{1}{2} - \frac{1}{2} \\ &= -1 \neq 0 \end{aligned}$$

They are **not orthonormal** sets

$$\text{d) } \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \cdot \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{3}} \left(-\frac{1}{\sqrt{2}} \right) + 0 + \frac{1}{\sqrt{2}} \left(-\frac{1}{\sqrt{3}} \right) \frac{1}{\sqrt{2}} \\
&= -\frac{1}{2} \frac{1}{\sqrt{3}} - \frac{1}{2} \frac{1}{\sqrt{3}} \\
&= -\frac{1}{\sqrt{3}} \neq 0
\end{aligned}$$

They are **not orthonormal** sets

$$\begin{aligned}
e) \quad &\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right) \cdot \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right) \cdot \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right) \\
&= \frac{2}{3} \frac{2}{3} \frac{1}{3} + \left(-\frac{2}{3} \right) \frac{1}{3} \frac{2}{3} + \frac{1}{3} \left(-\frac{2}{3} \right) \frac{2}{3} \\
&= \frac{4}{27} - \frac{4}{27} - \frac{4}{27} \\
&= -\frac{4}{27} \neq 0
\end{aligned}$$

They are **not orthonormal** sets

$$\begin{aligned}
f) \quad &\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \right) \cdot \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right) = \frac{1}{\sqrt{6}} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}} \left(-\frac{1}{\sqrt{2}} \right) + 0 \\
&= 0
\end{aligned}$$

They are **orthonormal** sets

Exercise

Find the cosine of the angle between \vec{u} and \vec{v} .

$$a) \quad \vec{u} = (1, -3), \quad \vec{v} = (2, 4)$$

$$e) \quad \vec{u} = (1, 0, 1, 0), \quad \vec{v} = (-3, -3, -3, -3)$$

$$b) \quad \vec{u} = (-1, 0), \quad \vec{v} = (3, 8)$$

$$f) \quad \vec{u} = (2, 1, 7, -1), \quad \vec{v} = (4, 0, 0, 0)$$

$$c) \quad \vec{u} = (-1, 5, 2), \quad \vec{v} = (2, 4, -9)$$

$$g) \quad \vec{u} = (1, 3, -5, 4), \quad \vec{v} = (2, -4, 4, 1)$$

$$d) \quad \vec{u} = (4, 1, 8), \quad \vec{v} = (1, 0, -3)$$

$$h) \quad \vec{u} = (1, 2, 3, 4), \quad \vec{v} = (-1, -2, -3, -4)$$

Solution

$$a) \quad \vec{u} = (1, -3), \quad \vec{v} = (2, 4)$$

$$\begin{aligned}
\|\vec{u}\| &= \sqrt{1^2 + (-3)^2} \\
&= \sqrt{10}
\end{aligned}$$

$$\begin{aligned}
\|\vec{v}\| &= \sqrt{2^2 + 4^2} \\
&= \sqrt{20}
\end{aligned}$$

$$\begin{aligned}\langle \vec{u}, \vec{v} \rangle &= 1(2) + (-3)(4) \\ &= -10\end{aligned}$$

$$\begin{aligned}\cos \theta &= \frac{-10}{\sqrt{10} \sqrt{20}} \\ &= -\frac{10}{\sqrt{200}} \\ &= -\frac{1}{\sqrt{2}}\end{aligned}$$

$$\cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

$$b) \quad \vec{u} = (-1, 0); \quad \vec{v} = (3, 8)$$

$$\begin{aligned}\|\vec{u}\| &= \sqrt{(-1)^2 + 0^2} \\ &= 1\end{aligned}$$

$$\begin{aligned}\|\vec{v}\| &= \sqrt{3^2 + 8^2} \\ &= \sqrt{73}\end{aligned}$$

$$\begin{aligned}\langle \vec{u}, \vec{v} \rangle &= (-1)(3) + (0)(8) \\ &= -3\end{aligned}$$

$$\begin{aligned}\cos \theta &= \frac{-3}{1\sqrt{73}} \\ &= -\frac{3}{\sqrt{73}}\end{aligned}$$

$$\cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

$$c) \quad \vec{u} = (-1, 5, 2); \quad \vec{v} = (2, 4, -9)$$

$$\begin{aligned}\|\vec{u}\| &= \sqrt{(-1)^2 + 5^2 + 2^2} \\ &= \sqrt{30}\end{aligned}$$

$$\begin{aligned}\|\vec{v}\| &= \sqrt{2^2 + 4^2 + (-9)^2} \\ &= \sqrt{101}\end{aligned}$$

$$\begin{aligned}\langle \vec{u}, \vec{v} \rangle &= (-1)(2) + (5)(4) + (2)(-9) \\ &= 0\end{aligned}$$

$$\cos \theta = 0$$

$$\cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

$$d) \quad \vec{u} = (4, 1, 8); \quad \vec{v} = (1, 0, -3)$$

$$\|\vec{u}\| = \sqrt{4^2 + 1^2 + 8^2}$$

$$\begin{aligned}
 &= 9 \mid \\
 \|\vec{v}\| &= \sqrt{1+0+9} \\
 &= \sqrt{10} \mid
 \end{aligned}$$

$$\begin{aligned}
 \langle \vec{u}, \vec{v} \rangle &= (4)(1) + (1)(0) + (8)(-3) \\
 &= -20 \mid
 \end{aligned}$$

$$\cos \theta = \frac{-\frac{20}{9\sqrt{10}}}{\left| \right|} \qquad \cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

$$e) \quad \vec{u} = (1, 0, 1, 0); \quad \vec{v} = (-3, -3, -3, -3)$$

$$\|\vec{u}\| = \sqrt{2} \mid$$

$$\begin{aligned}
 \|\vec{v}\| &= \sqrt{9+9+9+9} \\
 &= 12 \mid
 \end{aligned}$$

$$\begin{aligned}
 \langle \vec{u}, \vec{v} \rangle &= -3+0-3+0 \\
 &= -6 \mid
 \end{aligned}$$

$$\cos \theta = \frac{-6}{12\sqrt{2}} \qquad \cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

$$\frac{-\frac{1}{2\sqrt{2}}}{\left| \right|}$$

$$f) \quad \vec{u} = (2, 1, 7, -1); \quad \vec{v} = (4, 0, 0, 0)$$

$$\begin{aligned}
 \|\vec{u}\| &= \sqrt{2^2+1^2+7^2+(-1)^2} \\
 &= \sqrt{55} \mid
 \end{aligned}$$

$$\begin{aligned}
 \|\vec{v}\| &= \sqrt{4^2+0} \\
 &= 4 \mid
 \end{aligned}$$

$$\begin{aligned}
 \langle \vec{u}, \vec{v} \rangle &= (2)(4) + (1)(0) + (7)(0) + (-1)(0) \\
 &= 8 \mid
 \end{aligned}$$

$$\cos \theta = \frac{8}{4\sqrt{55}} \qquad \cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

$$\frac{\frac{2}{\sqrt{55}}}{\left| \right|}$$

$$g) \quad \vec{u} = (1, 3, -5, 4), \quad \vec{v} = (2, -4, 4, 1)$$

$$\|\vec{u}\| = \sqrt{1+9+25+16}$$

$$\begin{aligned}
&= \sqrt{51} \mid \\
\|\vec{v}\| &= \sqrt{4+16+16+1} \\
&= \sqrt{37} \mid \\
\langle \vec{u}, \vec{v} \rangle &= 2 - 12 - 20 + 4 \\
&= -26 \mid \\
\cos \theta &= \frac{-26}{\sqrt{51}\sqrt{37}} \mid \qquad \cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}
\end{aligned}$$

$$h) \quad \vec{u} = (1, 2, 3, 4), \quad \vec{v} = (-1, -2, -3, -4)$$

$$\begin{aligned}
\|\vec{u}\| &= \sqrt{1+4+9+16} \\
&= \sqrt{30} \mid \\
\|\vec{v}\| &= \sqrt{1+4+9+16} \\
&= \sqrt{30} \mid \\
\langle \vec{u}, \vec{v} \rangle &= -1 - 4 - 9 - 16 \\
&= -30 \mid \\
\cos \theta &= \frac{-30}{\sqrt{30}\sqrt{30}} \\
&= -1 \mid \qquad \cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}
\end{aligned}$$

Exercise

Find the cosine of the angle between A and B .

$$a) \quad A = \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$$

$$c) \quad A = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$b) \quad A = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$$

$$d) \quad A = \begin{pmatrix} 1 & -2 & 7 \\ 6 & -3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 3 \\ 6 & 5 & -4 \end{pmatrix}$$

Solution

$$\begin{aligned}
a) \quad A &= \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} \\
\|A\| &= \sqrt{\langle A, A \rangle} \\
&= \sqrt{2^2 + 6^2 + 1^2 + (-3)^2} \\
&= \sqrt{50}
\end{aligned}$$

$$= 5\sqrt{2} \mid$$

$$\begin{aligned}\|B\| &= \sqrt{\langle B, B \rangle} \\ &= \sqrt{9+4+1+0} \\ &= \sqrt{14} \mid\end{aligned}$$

$$\begin{aligned}\langle A, B \rangle &= 2(3) + 6(2) + 1(1) + (-3)(0) \\ &= 19 \mid\end{aligned}$$

$$\begin{aligned}\cos \theta &= \frac{19}{5\sqrt{2}\sqrt{14}} \\ &= \frac{19}{10\sqrt{7}} \mid\end{aligned}$$

$$\cos \theta = \frac{\langle A, B \rangle}{\|A\| \|B\|}$$

$$b) \quad A = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$$

$$\begin{aligned}\|A\| &= \sqrt{\langle A, A \rangle} \\ &= \sqrt{2^2 + 4^2 + (-1)^2 + 3^2} \\ &= \sqrt{30} \mid\end{aligned}$$

$$\begin{aligned}\|B\| &= \sqrt{\langle B, B \rangle} \\ &= \sqrt{(-3)^2 + 1^2 + 4^2 + 2^2} \\ &= \sqrt{30} \mid\end{aligned}$$

$$\begin{aligned}\langle A, B \rangle &= 2(-3) + 4(1) + (-1)(4) + 3(2) \\ &= 0 \mid\end{aligned}$$

$$\begin{aligned}\cos \theta &= \frac{0}{30} \\ &= 0 \mid\end{aligned}$$

$$\cos \theta = \frac{\langle A, B \rangle}{\|A\| \|B\|}$$

$$c) \quad A = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$\begin{aligned}\|A\| &= \sqrt{81+64+49+36+25+16} \\ &= \sqrt{271} \mid\end{aligned}$$

$$\|A\| = \sqrt{\langle A, A \rangle}$$

$$\begin{aligned}\|B\| &= \sqrt{1+4+9+16+25+36} \\ &= \sqrt{91} \mid\end{aligned}$$

$$\|B\| = \sqrt{\langle B, B \rangle}$$

$$\langle A, B \rangle = 9 + 16 + 21 + 24 + 25 + 24$$

$$= 119$$

$$\cos \theta = \frac{119}{\sqrt{271}\sqrt{91}}$$

$$\cos \theta = \frac{\langle A, B \rangle}{\|A\| \|B\|}$$

$$d) \quad A = \begin{pmatrix} 1 & -2 & 7 \\ 6 & -3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 3 \\ 6 & 5 & -4 \end{pmatrix}$$

$$\|A\| = \sqrt{1^2 + (-2)^2 + 7^2 + 6^2 + (-3)^2 + 4^2}$$

$$= \sqrt{115}$$

$$\|A\| = \sqrt{\langle A, A \rangle}$$

$$\|B\| = \sqrt{1^2 + (-2)^2 + 3^2 + 6^2 + 5^2 + (-4)^2}$$

$$= \sqrt{91}$$

$$\|B\| = \sqrt{\langle B, B \rangle}$$

$$\langle A, B \rangle = 1 + 4 + 21 + 36 - 15 - 16$$

$$= 31$$

$$\cos \theta = \frac{31}{\sqrt{115}\sqrt{91}}$$

$$\cos \theta = \frac{\langle A, B \rangle}{\|A\| \|B\|}$$

Exercise

Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

$$a) \quad \vec{u} = (-1, 3, 2), \quad \vec{v} = (4, 2, -1)$$

$$d) \quad \vec{u} = (-4, 6, -10, 1), \quad \vec{v} = (2, 1, -2, 9)$$

$$b) \quad \vec{u} = (a, b), \quad \vec{v} = (-b, a)$$

$$e) \quad \vec{u} = (-4, 6, -10, 1), \quad \vec{v} = (2, 1, -2, 9)$$

$$c) \quad \vec{u} = (-2, -2, -2), \quad \vec{v} = (1, 1, 1)$$

Solution

$$a) \quad \langle \vec{u}, \vec{v} \rangle = (-1)(4) + 3(2) + 2(-1)$$

$$= 0$$

Therefore, the given vectors are orthogonal.

$$b) \quad \langle \vec{u}, \vec{v} \rangle = a(-b) + b(a)$$

$$= 0$$

Therefore, the given vectors are orthogonal.

$$c) \quad \langle \vec{u}, \vec{v} \rangle = (-2)(1) + (-2)(1) + (-2)(1)$$

$$= -6$$

Therefore, the given vectors are **not** orthogonal.

$$\begin{aligned} d) \quad \langle \vec{u}, \vec{v} \rangle &= (-4)(2) + (6)(1) + (-10)(-2) + (1)(9) \\ &= 27 \quad | \quad \neq 0 \end{aligned}$$

Therefore, the given vectors are **not** orthogonal.

$$\begin{aligned} e) \quad \|\vec{u}\| &= \sqrt{(-4)^2 + 6^2 + (-10)^2 + 1^2} \\ &= \sqrt{153} \\ &= 3\sqrt{17} \quad | \end{aligned}$$

$$\begin{aligned} \|\vec{v}\| &= \sqrt{2^2 + 1^2 + (-2)^2 + 9^2} \\ &= \sqrt{90} \\ &= 3\sqrt{10} \quad | \end{aligned}$$

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= (-4)(2) + 6(1) - 10(-2) + 1(9) \\ &= 27 \quad | \end{aligned}$$

$$\begin{aligned} \cos \theta &= \frac{27}{3\sqrt{17}(3\sqrt{10})} & \cos \theta &= \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|} \\ &= \frac{3}{\sqrt{170}} \quad | \end{aligned}$$

The vectors \vec{u} and \vec{v} are **not** orthogonal with respect to the Euclidean

Exercise

Do there exist scalars k and l such that the vectors $\vec{u} = (2, k, 6)$, $\vec{v} = (l, 5, 3)$, and $\vec{w} = (1, 2, 3)$ are mutually orthogonal with respect to the Euclidean inner product?

Solution

$$\begin{aligned} \langle \vec{u}, \vec{w} \rangle &= (2)(1) + (k)(2) + (6)(3) \\ &= 20 + 2k = 0 \\ \Rightarrow k &= -10 \quad | \end{aligned}$$

$$\begin{aligned} \langle \vec{v}, \vec{w} \rangle &= (l)(1) + (5)(2) + (3)(3) \\ &= l + 19 = 0 \\ \Rightarrow l &= -19 \quad | \end{aligned}$$

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= (2)(l) + (k)(5) + (6)(3) \\ &= 2l + 5k + 18 = 0 \end{aligned}$$

$$2(-19) + 5(-10) + 18 = -70 \neq 0$$

Thus, there are no scalars such that the vectors are mutually orthogonal.

Exercise

Let \mathbb{R}^3 have the Euclidean inner product. For which values of k are \vec{u} and \vec{v} orthogonal?

$$a) \quad \vec{u} = (2, 1, 3), \quad \vec{v} = (1, 7, k)$$

$$b) \quad \vec{u} = (k, k, 1), \quad \vec{v} = (k, 5, 6)$$

Solution

$$\begin{aligned} a) \quad \langle \vec{u}, \vec{v} \rangle &= (2)(1) + (1)(7) + (3)(k) \\ &= 9 + 3k = 0 \end{aligned}$$

\vec{u} and \vec{v} are orthogonal for $\underline{k = -3}$

$$\begin{aligned} b) \quad \langle \vec{u}, \vec{v} \rangle &= (k)(k) + (k)(5) + (1)(6) \\ &= k^2 + 5k + 6 = 0 \end{aligned}$$

\vec{u} and \vec{v} are orthogonal for $\underline{k = -2, -3}$

Exercise

Let V be an inner product space. Show that if \vec{u} and \vec{v} are orthogonal unit vectors in V , then $\|\vec{u} - \vec{v}\| = \sqrt{2}$

Solution

$$\begin{aligned} \|\vec{u} - \vec{v}\|^2 &= \langle \vec{u} - \vec{v}, \vec{u} - \vec{v} \rangle \\ &= \langle \vec{u}, \vec{u} - \vec{v} \rangle - \langle \vec{v}, \vec{u} - \vec{v} \rangle \\ &= \langle \vec{u}, \vec{u} \rangle - \langle \vec{u}, \vec{v} \rangle - \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{v} \rangle \\ &= \|\vec{u}\|^2 - 0 - 0 + \|\vec{v}\|^2 \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

since \vec{u} and \vec{v} are orthogonal unit vectors

Thus $\underline{\|\vec{u} - \vec{v}\| = \sqrt{2}}$

Exercise

Let \mathcal{S} be a subspace of \mathbb{R}^n . Explain what $(\mathcal{S}^\perp)^\perp = \mathcal{S}$ means and why it is true.

Solution

$(\mathcal{S}^\perp)^\perp$ is the orthogonal complement of \mathcal{S}^\perp , which is itself the orthogonal complement of \mathcal{S} , so $(\mathcal{S}^\perp)^\perp = \mathcal{S}$ means that \mathcal{S} is the orthogonal of its orthogonal complement.

We need to show that \mathcal{S} is contained in $(\mathcal{S}^\perp)^\perp$ and, conversely, that $(\mathcal{S}^\perp)^\perp$ is contained in \mathcal{S} to be true.

i. Suppose $\vec{v} \in \mathcal{S}^\perp$ and $\vec{w} \in \mathcal{S}^\perp$. Then $\langle \vec{v}, \vec{w} \rangle = 0$ by definition of \mathcal{S}^\perp .

Thus, \mathcal{S} is certainly contained in $(\mathcal{S}^\perp)^\perp$ (which consists of all vectors in \mathbb{R}^n which are orthogonal to \mathcal{S}^\perp).

ii. Suppose $\vec{v} \in (\mathcal{S}^\perp)^\perp$ (means \vec{v} is orthogonal to all vectors in \mathcal{S}^\perp); then we need to show that $\vec{v} \in \mathcal{S}$.

Let assume $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p\}$ be a basis for \mathcal{S} and let $\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_q\}$ be a basis for \mathcal{S}^\perp . If

$\vec{v} \notin \mathcal{S}$, then $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p, \vec{v}\}$ is linearly independent set. Since each vector in that set is orthogonal to all of \mathcal{S}^\perp , the set $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p, \vec{v}, \vec{w}_1, \vec{w}_2, \dots, \vec{w}_q\}$ is linearly independent.

Since there are $p + q + 1$ vectors in this set, this means that $p + q + 1 \leq n \Leftrightarrow p + q \leq n - 1$.

On the other hand, If A is the matrix whose i^{th} row is \vec{u}_i^T , then the row space of A is \mathcal{S} and the nullspace of A is \mathcal{S}^\perp .

Since \mathcal{S} is p -dimensional, the rank of A is p , meaning that the dimension of $\text{nul}(A) = \mathcal{S}^\perp$ is $q = n - p$. Therefore,

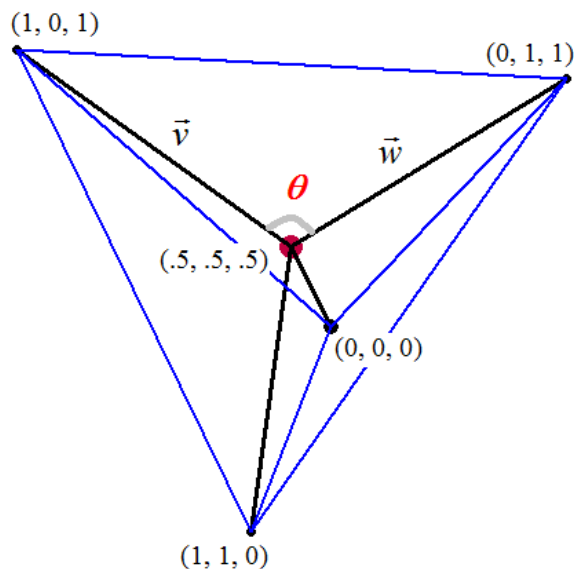
$$p + q = p + (n - p) = n$$

Which contradict the fact that $p + q \leq n - 1$. From this, we see that, if $\vec{v} \in (\mathcal{S}^\perp)^\perp$, it must be the case that $\vec{v} \in \mathcal{S}$.

Exercise

The methane molecule CH_4 is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$ – (**note** that all six edges have length $\sqrt{2}$, so the tetrahedron is regular). What is the cosine of the angle between the rays going from the center $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ to the vertices?

Solution



Let \vec{v} be the vector of the segment $(1, 0, 1)$ and $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$$\begin{aligned}\vec{v} &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}\end{aligned}$$

Let be the vector of the segment $(0, 1, 1)$ and $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$$\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

We have:

$$\begin{aligned} \cos \theta &= \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \|\vec{w}\|} \\ &= \frac{\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) \cdot \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}{\sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}} \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}}} \\ &= \frac{-\frac{1}{4}}{\frac{3}{4}} \\ &= -\frac{1}{3} \end{aligned}$$

$$\theta \approx 109.47^\circ$$

Exercise

Determine if the given vectors are orthogonal.

$$\vec{x}_1 = (1, 0, 1, 0), \quad \vec{x}_2 = (0, 1, 0, 1), \quad \vec{x}_3 = (1, 0, -1, 0), \quad \vec{x}_4 = (1, 1, -1, -1)$$

Solution

$$\begin{aligned} \vec{x}_1 \cdot \vec{x}_2 &= (1, 0, 1, 0) \cdot (0, 1, 0, 1) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \vec{x}_1 \cdot \vec{x}_3 &= (1, 0, 1, 0) \cdot (1, 0, -1, 0) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \vec{x}_1 \cdot \vec{x}_4 &= (1, 0, 1, 0) \cdot (1, 1, -1, -1) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \vec{x}_2 \cdot \vec{x}_3 &= (0, 1, 0, 1) \cdot (1, 0, -1, 0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \vec{x}_2 \cdot \vec{x}_4 &= (0, 1, 0, 1) \cdot (1, 1, -1, -1) \\ &= 1 - 1 \end{aligned}$$

$$\underline{=0}$$

$$\vec{x}_3 \cdot \vec{x}_4 = (1, 0, -1, 0) \cdot (1, 1, -1, -1)$$

$$= 1 - 1$$

$$\underline{=0}$$

The given vectors are **orthogonal**.

Exercise

Which of the following sets of vectors are orthogonal with respect to the Euclidean inner

$$a) \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right) \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)$$

$$b) \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right) \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right) \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right)$$

Solution

$$a) \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right) = \frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}}$$

$$\underline{=0}$$

$$\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \cdot \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) = -\frac{1}{2} + 0 + \frac{1}{2}$$

$$\underline{=0}$$

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right) \cdot \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) = -\frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}}$$

$$\underline{= -\frac{2}{\sqrt{6}} \neq 0}$$

Therefore, the given vectors are **not** orthogonal.

$$b) \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right) \cdot \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right) = \frac{4}{9} - \frac{2}{9} - \frac{2}{9}$$

$$\underline{=0}$$

$$\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right) \cdot \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right) = \frac{2}{9} - \frac{4}{9} + \frac{2}{9}$$

$$\underline{=0}$$

$$\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right) \cdot \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right) = \frac{2}{9} + \frac{2}{9} - \frac{4}{9}$$

$$\underline{=0}$$

Therefore, the given vectors are orthogonal.

Exercise

Consider vectors $\vec{u} = (2, 3, 5)$ $\vec{v} = (1, -4, 3)$ in \mathbb{R}^3

- a) $\langle \vec{u}, \vec{v} \rangle$ b) $\|\vec{u}\|$ c) $\|\vec{v}\|$ d) Cosine between \vec{u} and \vec{v}

Solution

$$\begin{aligned} a) \quad \langle \vec{u}, \vec{v} \rangle &= (2, 3, 5) \cdot (1, -4, 3) \\ &= 2 - 12 + 15 \\ &= 5 \end{aligned}$$

$$\begin{aligned} b) \quad \|\vec{u}\| &= \sqrt{4 + 9 + 25} \\ &= \sqrt{38} \end{aligned}$$

$$\begin{aligned} c) \quad \|\vec{v}\| &= \sqrt{1 + 16 + 9} \\ &= \sqrt{26} \end{aligned}$$

$$d) \quad \cos \theta = \frac{5}{\sqrt{38}\sqrt{26}} \quad \cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

Exercise

Consider vectors $\vec{u} = (1, 1, 1)$ $\vec{v} = (1, 2, -3)$ in \mathbb{R}^3

- a) $\langle \vec{u}, \vec{v} \rangle$ b) $\|\vec{u}\|$ c) $\|\vec{v}\|$ d) Cosine θ between \vec{u} and \vec{v}

Solution

$$\begin{aligned} a) \quad \langle \vec{u}, \vec{v} \rangle &= (1, 1, 1) \cdot (1, 2, -3) \\ &= 1 + 2 - 3 \\ &= 0 \end{aligned}$$

$$\begin{aligned} b) \quad \|\vec{u}\| &= \sqrt{1 + 1 + 1} \\ &= \sqrt{3} \end{aligned}$$

$$\begin{aligned} c) \quad \|\vec{v}\| &= \sqrt{1 + 4 + 9} \\ &= \sqrt{14} \end{aligned}$$

$$d) \quad \cos \theta = 0 \quad \cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

\vec{u} and \vec{v} are orthogonal vectors.

Exercise

Consider vectors $\vec{u} = (1, 2, 5)$ $\vec{v} = (2, -3, 5)$ $\vec{w} = (4, 2, -3)$ in \mathbb{R}^3

- | | | |
|---------------------------------------|------------------|--|
| a) $\langle \vec{u}, \vec{v} \rangle$ | d) $\ \vec{u}\ $ | g) Cosine α between \vec{u} and \vec{v} |
| b) $\langle \vec{u}, \vec{w} \rangle$ | e) $\ \vec{v}\ $ | h) Cosine β between \vec{u} and \vec{w} |
| c) $\langle \vec{v}, \vec{w} \rangle$ | f) $\ \vec{w}\ $ | i) Cosine θ between \vec{v} and \vec{w} |
| | | j) $(\vec{u} + \vec{v}) \cdot \vec{w}$ |

Solution

$$\begin{aligned} \text{a) } \langle \vec{u}, \vec{v} \rangle &= (1, 2, 5) \cdot (2, -3, 5) \\ &= 2 - 6 + 25 \\ &= \underline{21} \end{aligned}$$

$$\begin{aligned} \text{b) } \langle \vec{u}, \vec{w} \rangle &= (1, 2, 5) \cdot (4, 2, -3) \\ &= 4 + 4 - 15 \\ &= \underline{-7} \end{aligned}$$

$$\begin{aligned} \text{c) } \langle \vec{v}, \vec{w} \rangle &= (2, -3, 5) \cdot (4, 2, -3) \\ &= 8 - 6 - 15 \\ &= \underline{-13} \end{aligned}$$

$$\begin{aligned} \text{d) } \|\vec{u}\| &= \sqrt{1 + 4 + 25} \\ &= \underline{\sqrt{30}} \end{aligned}$$

$$\begin{aligned} \text{e) } \|\vec{v}\| &= \sqrt{4 + 9 + 25} \\ &= \underline{\sqrt{38}} \end{aligned}$$

$$\begin{aligned} \text{f) } \|\vec{w}\| &= \sqrt{16 + 4 + 9} \\ &= \underline{\sqrt{29}} \end{aligned}$$

$$\text{g) } \cos \alpha = \frac{21}{\sqrt{30}\sqrt{38}} \quad \cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

$$\text{h) } \cos \beta = \frac{-7}{\sqrt{30}\sqrt{29}} \quad \cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

$$\text{i) } \cos \theta = \frac{-13}{\sqrt{38}\sqrt{29}} \quad \cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

$$\begin{aligned} \text{j) } (\vec{u} + \vec{v}) \cdot \vec{w} &= [(1, 2, 5) + (2, -3, 5)] \cdot (4, 2, -3) \\ &= (3, -1, 10) \cdot (4, 2, -3) \\ &= 12 - 2 - 30 \\ &= \underline{-20} \end{aligned}$$

Exercise

Consider polynomial $f(t) = 3t - 5$; $g(t) = t^2$ in $\mathbb{P}(t)$

a) $\langle f, g \rangle$

b) $\|f\|$

c) $\|g\|$

d) Cosine between f and g

Solution

$$\begin{aligned} \text{a) } \langle f, g \rangle &= \int_0^1 f(t)g(t)dt \\ &= \int_0^1 (3t-5)t^2 dt \\ &= \int_0^1 (3t^3 - 5t^2) dt \\ &= \left. \frac{3}{4}t^4 - \frac{5}{3}t^3 \right|_0^1 \\ &= \frac{3}{4} - \frac{5}{3} \\ &= \underline{\underline{-\frac{11}{12}}} \end{aligned}$$

$$\begin{aligned} \text{b) } \langle f, f \rangle &= \int_0^1 f(t)f(t)dt \\ &= \int_0^1 (3t-5)^2 dt \\ &= \frac{1}{3} \int_0^1 (3t-5)^2 d(3t-5) \\ &= \left. \frac{1}{9}(3t-5)^3 \right|_0^1 \\ &= \frac{1}{9}(8-125) \\ &= \underline{\underline{13}} \end{aligned}$$

$$\begin{aligned} \|f\| &= \sqrt{\langle f, f \rangle} \\ &= \underline{\underline{\sqrt{13}}} \end{aligned}$$

$$\text{c) } \langle g, g \rangle = \int_0^1 g(t)g(t)dt$$

$$\begin{aligned}
&= \int_0^1 t^4 dt \\
&= \frac{1}{5} t^5 \Big|_0^1 \\
&= \frac{1}{5} \Big| \\
\|g\| &= \sqrt{|\langle g, g \rangle|} \\
&= \frac{1}{\sqrt{5}} \Big|
\end{aligned}$$

$$\begin{aligned}
d) \quad \cos \theta &= \frac{-\frac{11}{12}}{\sqrt{13} \frac{\sqrt{5}}{5}} \\
&= \frac{-55}{12\sqrt{65}} \Big|
\end{aligned}
\qquad
\cos \theta = \frac{f \cdot g}{\|f\| \|g\|}$$

Exercise

Consider polynomial $f(t) = t + 2$; $g(t) = 3t - 2$; $h(t) = t^2 - 2t - 3$ in $\mathbb{P}(t)$

- | | | |
|---------------------------|------------|--|
| a) $\langle f, g \rangle$ | d) $\ f\ $ | g) Cosine α between f and g |
| b) $\langle f, h \rangle$ | e) $\ g\ $ | h) Cosine β between f and h |
| c) $\langle g, h \rangle$ | f) $\ h\ $ | i) Cosine θ between g and h |

Solution

$$\begin{aligned}
a) \quad \langle f, g \rangle &= \int_0^1 (t+2)(3t-2) dt \\
&= \int_0^1 (3t^2 + 4t - 4) dt \\
&= t^3 + 2t^2 - 4t \Big|_0^1 \\
&= 1 + 2 - 4 \\
&= -1 \Big|
\end{aligned}
\qquad
\langle f, g \rangle = \int_0^1 f(t)g(t) dt$$

$$b) \quad \langle f, h \rangle = \int_0^1 (t+2)(t^2 - 2t - 3) dt \qquad \langle f, h \rangle = \int_0^1 f(t)h(t) dt$$

$$\begin{aligned}
&= \int_0^1 (t^3 - 7t - 6) dt \\
&= \left. \frac{1}{4}t^4 - \frac{7}{2}t^2 - 6t \right|_0^1 \\
&= \frac{1}{4} - \frac{7}{2} - 6 \\
&= \underline{-\frac{37}{4}}
\end{aligned}$$

$$\begin{aligned}
c) \quad \langle g, h \rangle &= \int_0^1 (3t-2)(t^2-2t-3) dt \\
&= \int_0^1 (3t^3 - 8t^2 - 5t + 6) dt \\
&= \left. \frac{3}{4}t^4 - \frac{8}{3}t^3 - \frac{5}{2}t^2 + 6t \right|_0^1 \\
&= \frac{3}{4} - \frac{8}{3} - \frac{5}{2} + 6 \\
&= \underline{\frac{9}{4}}
\end{aligned}$$

$$\begin{aligned}
d) \quad \langle f, f \rangle &= \int_0^1 (t+2)^2 dt \\
&= \left. \frac{1}{3}(t+2)^3 \right|_0^1 \\
&= \frac{1}{3}(27-8) \\
&= \underline{\frac{19}{3}}
\end{aligned}$$

$$\begin{aligned}
\|f\| &= \sqrt{\langle f, f \rangle} \\
&= \underline{\sqrt{\frac{19}{3}}}
\end{aligned}$$

$$\begin{aligned}
e) \quad \langle g, g \rangle &= \int_0^1 (3t-2)^2 dt \\
&= \frac{1}{3} \int_0^1 (3t-2)^2 d(3t-2) \\
&= \left. \frac{1}{9}(3t-2)^3 \right|_0^1
\end{aligned}$$

$$\langle f, h \rangle = \int_0^1 g(t)h(t)dt$$

$$\langle f, f \rangle = \int_0^1 f(t)f(t)dt$$

$$\langle g, g \rangle = \int_0^1 g(t)g(t)dt$$

$$= \frac{1}{9}(1+8)$$

$$= 1$$

$$\|g\| = \sqrt{\langle g, g \rangle}$$

$$= 1$$

$$f) \quad \langle g, g \rangle = \int_0^1 (t^2 - 2t - 3)^2 dt$$

$$\langle h, h \rangle = \int_0^1 h(t)h(t)dt$$

$$= \int_0^1 (t^4 - 4t^3 - 2t^2 + 12t + 9) dt$$

$$= \left(\frac{1}{5}t^5 - t^4 - \frac{2}{3}t^3 + 6t^2 + 9t \right) \Big|_0^1$$

$$= \frac{1}{5} - 1 - \frac{2}{3} + 6 + 9$$

$$= \frac{203}{15}$$

$$\|h\| = \sqrt{\langle h, h \rangle}$$

$$= \sqrt{\frac{203}{15}}$$

$$g) \quad \cos \alpha = \frac{-1}{\sqrt{\frac{19}{3}}}$$

$$\cos \alpha = \frac{f \cdot g}{\|f\| \|g\|}$$

$$= -\sqrt{\frac{3}{19}}$$

$$h) \quad \cos \beta = -\frac{37}{4} \frac{1}{\sqrt{\frac{19}{3}} \sqrt{\frac{203}{15}}}$$

$$\cos \beta = \frac{f \cdot g}{\|f\| \|g\|}$$

$$= -\frac{111}{4} \sqrt{\frac{5}{3,857}}$$

$$i) \quad \cos \theta = \frac{9}{4} \frac{1}{\sqrt{\frac{203}{15}}}$$

$$\cos \theta = \frac{f \cdot g}{\|f\| \|g\|}$$

$$= \frac{9}{4} \sqrt{\frac{15}{203}}$$

Exercise

Suppose $\langle \vec{u}, \vec{v} \rangle = 3 + 2i$ in a complex inner product space V . Find:

$$a) \langle (2-4i)\vec{u}, \vec{v} \rangle \quad b) \langle \vec{u}, (4+3i)\vec{v} \rangle \quad c) \langle (3-6i)\vec{u}, (5-2i)\vec{v} \rangle \quad d) \|\vec{u}, \vec{v}\|$$

Solution

$$\begin{aligned} a) \quad \langle (2-4i)\vec{u}, \vec{v} \rangle &= (2-4i)\langle \vec{u}, \vec{v} \rangle \\ &= (2-4i)(3+2i) \\ &= 6+4i-12i+8 \quad i^2 = -1 \\ &= \underline{14-8i} \end{aligned}$$

$$\begin{aligned} b) \quad \langle \vec{u}, (4+3i)\vec{v} \rangle &= (4+3i)\langle \vec{u}, \vec{v} \rangle \\ &= (4+3i)(3+2i) \\ &= 12+8i+9i-6 \quad i^2 = -1 \\ &= \underline{14-8i} \end{aligned}$$

$$\begin{aligned} c) \quad \langle (3-6i)\vec{u}, (5-2i)\vec{v} \rangle &= (3-6i)(5-2i)\langle \vec{u}, \vec{v} \rangle \\ &= (15-36i-12)(3+2i) \\ &= (3-36i)(3+2i) \\ &= 9-102i+72 \\ &= \underline{81-102i} \end{aligned}$$

$$\begin{aligned} d) \quad \|\vec{u}, \vec{v}\| &= \sqrt{\langle \vec{u}, \vec{v} \rangle} \\ &= \sqrt{9+4} \\ &= \underline{\sqrt{13}} \end{aligned}$$

Exercise

Find the Fourier coefficient c and the projection $c\vec{v}$ of $\vec{u} = (3+4i, 2-3i)$ along $\vec{v} = (5+i, 2i)$ in \mathbb{C}^2

Solution

$$\begin{aligned} c &= \frac{(3+4i)(\overline{5+i}) + (2-3i)(\overline{2i})}{5^2 + 1^2 + 2^2} & c &= \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \\ &= \frac{(3+4i)(5-i) + (2-3i)(-2i)}{25+1+4} \\ &= \frac{15+17i+4-4i-6}{30} \\ &= \underline{\frac{13+13i}{30}} \end{aligned}$$

$$= \frac{13}{30} + \frac{13}{30}i \quad \Big|$$

$$\text{proj}(\vec{u}, \vec{v}) = c\vec{v}$$

$$\begin{aligned} &= \left(\frac{13}{30} + \frac{13}{30}i\right)(5 + i, 2i) \\ &= \left(\frac{13}{6} + \frac{13}{6}i + \frac{13}{30}i - \frac{13}{30}, \frac{13}{15}i - \frac{13}{15}\right) \\ &= \left(\frac{52}{30} + \frac{78}{30}i, -\frac{13}{15} + \frac{13}{15}i\right) \\ &= \left(\frac{26}{15} + \frac{39}{30}i, -\frac{13}{15} + \frac{13}{15}i\right) \quad \Big| \end{aligned}$$

Exercise

Suppose $\vec{v} = (1, 3, 5, 7)$. Find the projection of \vec{v} onto W or find $\vec{w} \in W$ that minimizes $\|\vec{v} - \vec{w}\|$, where

W is the subspace of \mathbb{R}^4 spanned by:

- a) $\vec{u}_1 = (1, 1, 1, 1)$ and $\vec{u}_2 = (1, -3, 4, -2)$
 b) $\vec{v}_1 = (1, 1, 1, 1)$ and $\vec{v}_2 = (1, 2, 3, 2)$

Solution

$$\begin{aligned} \text{a) } \vec{u}_1 \cdot \vec{u}_2 &= (1, 1, 1, 1) \cdot (1, -3, 4, -2) \\ &= 1 - 3 + 4 - 2 \\ &= 0 \quad \Big| \end{aligned}$$

Therefore, \vec{u}_1 and \vec{u}_2 are orthogonal.

$$\begin{aligned} c_1 &= \frac{\langle \vec{v}, \vec{u}_1 \rangle}{\langle \vec{u}_1, \vec{u}_1 \rangle} \\ &= \frac{(1, 3, 5, 7) \cdot (1, 1, 1, 1)}{\|(1, 1, 1, 1)\|^2} \\ &= \frac{1+3+5+7}{1+1+1+1} \\ &= \frac{16}{4} \\ &= 4 \quad \Big| \end{aligned}$$

$$c_2 = \frac{\langle \vec{v}, \vec{u}_2 \rangle}{\langle \vec{u}_2, \vec{u}_2 \rangle}$$

$$\begin{aligned}
&= \frac{(1, 3, 5, 7) \cdot (1, -3, 4, -2)}{\|(1, -3, 4, -2)\|^2} \\
&= \frac{1-9+20-14}{1+9+16+4} \\
&= \frac{-2}{30} \\
&= \frac{1}{15} \Big|
\end{aligned}$$

$$\begin{aligned}
w &= \text{proj}(\vec{v}, W) \\
&= c_1 \vec{u}_1 + c_2 \vec{u}_2 \\
&= 4(1, 1, 1, 1) + \frac{1}{15}(1, -3, 4, -2) \\
&= \left(\frac{59}{15}, \frac{63}{5}, \frac{56}{15}, \frac{62}{15} \right) \Big|
\end{aligned}$$

$$\begin{aligned}
b) \quad \vec{v}_1 \cdot \vec{v}_2 &= (1, 1, 1, 1) \cdot (1, 2, 3, 2) \\
&= 1 + 2 + 3 + 2 \\
&= 8 \neq 0 \Big|
\end{aligned}$$

Therefore, \vec{v}_1 and \vec{v}_2 are *not* orthogonal.

Applying Gram-Schmidt algorithm

$$\vec{w}_1 = \vec{v}_1 = (1, 1, 1, 1) \Big|$$

$$\begin{aligned}
\vec{w}_2 &= (1, 2, 3, 2) - \frac{(1, 2, 3, 2) \cdot (1, 1, 1, 1)}{4}(1, 1, 1, 1) & \vec{w}_2 &= \vec{v}_2 - \frac{\langle \vec{v}, \vec{w}_1 \rangle}{\|\vec{w}_1\|^2} \vec{w}_1 \\
&= (1, 2, 3, 2) - 2(1, 1, 1, 1) \\
&= (-1, 0, 1, 0) \Big|
\end{aligned}$$

$$\begin{aligned}
c_1 &= \frac{(1, 3, 5, 7) \cdot (1, 1, 1, 1)}{\|(1, 1, 1, 1)\|^2} & c_1 &= \frac{\langle \vec{v}, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle} \\
&= \frac{1+3+5+7}{1+1+1+1} \\
&= \frac{16}{4} \\
&= 4 \Big|
\end{aligned}$$

$$\begin{aligned}
c_2 &= \frac{(1, 3, 5, 7) \cdot (-1, 0, 1, 0)}{\|(-1, 0, 1, 0)\|^2} & c_2 &= \frac{\langle \vec{v}, \vec{w}_2 \rangle}{\langle \vec{w}_2, \vec{w}_2 \rangle}
\end{aligned}$$

$$= \frac{-1+0+5+0}{2}$$

$$= -3 \mid$$

$$w = \text{proj}(\vec{v}, W)$$

$$= c_1 \vec{w}_1 + c_2 \vec{w}_2$$

$$= 4(1, 1, 1, 1) - 3(-1, 0, 1, 0)$$

$$= (7, 4, 1, 4) \mid$$

Exercise

Suppose $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ is an orthogonal set of vectors. Prove that (Pythagoras)

$$\|\vec{u}_1 + \vec{u}_2 + \dots + \vec{u}_n\|^2 = \|\vec{u}_1\|^2 + \|\vec{u}_2\|^2 + \dots + \|\vec{u}_n\|^2$$

Solution

$$\begin{aligned} \|\vec{u}_1 + \vec{u}_2 + \dots + \vec{u}_n\|^2 &= \langle \vec{u}_1 + \vec{u}_2 + \dots + \vec{u}_n, \vec{u}_1 + \vec{u}_2 + \dots + \vec{u}_n \rangle \\ &= \langle \vec{u}_1, \vec{u}_1 \rangle + \langle \vec{u}_2, \vec{u}_2 \rangle + \dots + \langle \vec{u}_n, \vec{u}_n \rangle \\ &= \|\vec{u}_1\|^2 + \|\vec{u}_2\|^2 + \dots + \|\vec{u}_n\|^2 \end{aligned}$$

Exercise

Suppose A is an orthogonal matrix. Show that $\langle \vec{u}A, \vec{v}A \rangle = \langle \vec{u}, \vec{v} \rangle$ for any $\vec{u}, \vec{v} \in V$

Solution

$$A \text{ is an orthogonal matrix} \Rightarrow AA^T = I$$

$$\text{And } \langle \vec{u}, \vec{v} \rangle = \vec{u}^T \vec{v}$$

$$\langle \vec{u}A, \vec{v}A \rangle = (\vec{u}A)^T (\vec{v}A)$$

$$= \vec{u}^T (A^T A) \vec{v}$$

$$= \vec{u}^T I \vec{v}$$

$$= \vec{u}^T \vec{v}$$

$$= \langle \vec{u}, \vec{v} \rangle \quad \checkmark$$

Exercise

Suppose A is an orthogonal matrix. Show that $\|\vec{u}A\| = \|\vec{u}\|$ for every $\vec{u} \in V$

Solution

A is an orthogonal matrix

$$\Rightarrow AA^T = I \text{ and } \langle \vec{u}, \vec{u} \rangle = \vec{u}^T \vec{u}$$

$$\begin{aligned}\|\vec{u}A\|^2 &= \langle \vec{u}A, \vec{u}A \rangle \\ &= (A\vec{u})^T (A\vec{u}) \\ &= \vec{u}^T (A^T A) \vec{u} \\ &= \vec{u}^T I \vec{u} \\ &= \vec{u}^T \vec{u} \\ &= \langle \vec{u}, \vec{u} \rangle \quad \checkmark\end{aligned}$$

Exercise

Let V be an inner product space over \mathbb{R} or \mathbb{C} . Show that

$$\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$$

If and only if

$$\|s\vec{u} + t\vec{v}\| = s\|\vec{u}\| + t\|\vec{v}\| \quad \text{for all } s, t \geq 0$$

Solution

Suppose that $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$. For $s, t \geq 0$

$$\begin{aligned}\|s\vec{u} + t\vec{v}\|^2 &= s^2 \|\vec{u}\|^2 + t^2 \|\vec{v}\|^2 + 2st \vec{u}\vec{v} \\ &\leq s^2 \|\vec{u}\|^2 + t^2 \|\vec{v}\|^2 \\ &\leq s\|\vec{u}\| + t\|\vec{v}\|\end{aligned}$$

$$\|s\vec{u} + t\vec{v}\| \leq s\|\vec{u}\| + t\|\vec{v}\|$$

$$\|s\vec{u} + t\vec{v}\| = \|s\vec{u} + t\vec{u} - t\vec{u} + t\vec{v}\|$$

$$= \|t\vec{u} + t\vec{v} - t\vec{u} + s\vec{u}\|$$

$$= \|t(\vec{u} + \vec{v}) - (t-s)\vec{u}\|$$

$$\geq |t\|\vec{u} + \vec{v}\| - (t-s)\|\vec{u}\||$$

$$= t\|\vec{u}\| + \|\vec{v}\| - t\|\vec{u}\| + s\|\vec{u}\|$$

$$\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$$

$$= t\|\vec{v}\| + s\|\vec{u}\|$$

$$\begin{cases} \|s\vec{u} + t\vec{v}\| \leq s\|\vec{u}\| + t\|\vec{v}\| \\ \text{and} \\ \|s\vec{u} + t\vec{v}\| \geq s\|\vec{u}\| + t\|\vec{v}\| \end{cases} \Rightarrow \|s\vec{u} + t\vec{v}\| = s\|\vec{u}\| + t\|\vec{v}\|$$

Exercise

Let V be an inner product vector space over \mathbb{R} .

- a) If e_1, e_2, e_3 are three vectors in V with pairwise product negative, that is,

$$\langle e_i, e_j \rangle < 0, \quad i, j = 1, 2, 3, \quad i \neq j$$

Show that e_1, e_2, e_3 are linearly independent.

- b) Is it possible for three vectors on the xy -plane to have pairwise negative products?
c) Does part (a) remain valid when the word “negative: is replaced with positive?”
d) Suppose \vec{u}, \vec{v} , and \vec{w} are three-unit vectors in the xy -plane. What are the maximum and minimum values that

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle$$

Can attain? And when?

Solution

- a) Suppose that e_1, e_2, e_3 are linearly dependent.

Then, assume that e_1, e_2, e_3 are unit vectors and that

$$e_3 = c_1 e_1 + c_2 e_2$$

Then

$$\langle e_1, e_3 \rangle = c_1 \langle e_1, e_1 \rangle + c_2 \langle e_1, e_2 \rangle \quad \langle e_1, e_1 \rangle = 1$$

$$= c_1 + c_2 \langle e_1, e_2 \rangle < 0$$

$$c_1 < -c_2 \langle e_1, e_2 \rangle$$

$$\langle e_2, e_3 \rangle = c_1 \langle e_2, e_1 \rangle + c_2 \langle e_2, e_2 \rangle \quad \langle e_2, e_2 \rangle = 1$$

$$= c_1 \langle e_2, e_1 \rangle + c_2 < 0$$

$$c_2 < -c_1 \langle e_2, e_1 \rangle$$

$$c_2 < -c_1 \langle e_2, e_1 \rangle$$

$$< -(-c_2 \langle e_1, e_2 \rangle) \langle e_2, e_1 \rangle$$

$$= c_2 \langle e_1, e_2 \rangle^2 \quad \langle e_1, e_2 \rangle^2 > 1$$

$$c_2 < c_2 \quad \text{Contradiction}$$

Therefore, e_1, e_2, e_3 are linearly independent.

b) To have all three vectors on the xy -plane which is in 2 dimensional.
Therefore, it is **impossible** for three to have pairwise negative products.

c) No

d) Given: \vec{u}, \vec{v} , and \vec{w} are three-unit vectors in the xy -plane and

$$|\langle \vec{u}, \vec{u} \rangle| = |\langle \vec{v}, \vec{v} \rangle| = |\langle \vec{w}, \vec{w} \rangle| = 1$$

$$\cos \alpha_1 = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|} \rightarrow \cos \alpha_1 = \langle \vec{u}, \vec{v} \rangle$$

$$\cos \alpha_2 = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \|\vec{w}\|} \rightarrow \cos \alpha_2 = \langle \vec{v}, \vec{w} \rangle$$

$$\cos \alpha_3 = \frac{\langle \vec{u}, \vec{w} \rangle}{\|\vec{u}\| \|\vec{w}\|} \rightarrow \cos \alpha_3 = \langle \vec{u}, \vec{w} \rangle$$

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle = \cos \alpha_1 + \cos \alpha_2 + \cos \alpha_3$$

Since $-1 \leq \cos \theta \leq 1$

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle = 1 + 1 + 1$$

$$\underline{= 3}$$

Since the 3 vectors are unit vectors in the xy -plane and which it will divide the plane into a three equal angles $\alpha = \frac{2\pi}{3}$

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle = \cos \frac{2\pi}{3} + \cos \frac{2\pi}{3} + \cos \frac{2\pi}{3}$$

$$= 3 \cos \frac{2\pi}{3}$$

$$= 3 \left(-\frac{1}{2} \right)$$

$$\underline{= -\frac{3}{2}}$$

Therefore, the minimum $\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle = -\frac{3}{2}$

The maximum: $\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle = 3$

Solution **Section 3.3 – Gram-Schmidt Process**

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of \mathbb{R}^m .

$$\vec{u}_1 = (1, -3), \quad \vec{u}_2 = (2, 2)$$

Solution

$$\begin{aligned}\vec{v}_1 &= \frac{\vec{u}_1}{\|\vec{u}_1\|} \\ &= \frac{(1, -3)}{\sqrt{1^2 + (-3)^2}} \\ &= \frac{(1, -3)}{\sqrt{10}} \\ &= \left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}} \right)\end{aligned}$$

$$\begin{aligned}\vec{w}_2 &= \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1) \vec{u}_1 \\ &= (2, 2) - \left[(2, 2) \cdot \left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}} \right) \right] \left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}} \right) \\ &= (2, 2) - \left[\frac{2}{\sqrt{10}} - \frac{6}{\sqrt{10}} \right] \left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}} \right) \\ &= (2, 2) - \left[-\frac{4}{\sqrt{10}} \right] \left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}} \right) \\ &= (2, 2) - \left(-\frac{4}{10}, \frac{12}{10} \right) \\ &= (2, 2) - \left(-\frac{2}{5}, \frac{6}{5} \right) \\ &= \left(\frac{12}{5}, \frac{4}{5} \right)\end{aligned}$$

$$\begin{aligned}\|\vec{w}_2\| &= \sqrt{\left(\frac{12}{5} \right)^2 + \left(\frac{4}{5} \right)^2} \\ &= \sqrt{\frac{144}{25} + \frac{16}{25}} \\ &= \sqrt{\frac{160}{25}} \\ &= \frac{4\sqrt{10}}{5}\end{aligned}$$

$$\begin{aligned}
 \vec{v}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\
 &= \frac{4\sqrt{10}}{5} \left(\frac{12}{5}, \frac{4}{5} \right) \\
 &= \left(\frac{48\sqrt{10}}{25}, \frac{16\sqrt{10}}{25} \right)
 \end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of \mathbb{R}^m .

$$\vec{u}_1 = (1, 0), \quad \vec{u}_2 = (3, -5)$$

Solution

$$\begin{aligned}
 \vec{v}_1 &= \frac{\vec{u}_1}{\|\vec{u}_1\|} \\
 &= \frac{(1, 0)}{\sqrt{1^2 + 0^2}} \\
 &= (1, 0)
 \end{aligned}$$

$$\begin{aligned}
 \vec{w}_2 &= \vec{u}_2 - (\vec{u}_2 \cdot \vec{v}_1) \vec{v}_1 \\
 &= (0, -5) \\
 &= (3, -5) - [(3, -5) \cdot (1, 0)](1, 0) \\
 &= (3, -5) - [3](1, 0) \\
 &= (3, -5) - (3, 0) \\
 &= (0, -5)
 \end{aligned}$$

$$\begin{aligned}
 \|\vec{w}_2\| &= \sqrt{0^2 + (-5)^2} \\
 &= 5
 \end{aligned}$$

$$\begin{aligned}
 \vec{v}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\
 &= \frac{1}{5}(0, -5) \\
 &= (0, -1)
 \end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of \mathbb{R}^m .

$$\{(1, 1, 1), (-1, 1, 0), (1, 2, 1)\}$$

Solution

$$\begin{aligned}\vec{u}_1 &= \frac{(1, 1, 1)}{\sqrt{1^2+1^2+1^2}} \\ &= \frac{(1, 1, 1)}{\sqrt{3}}\end{aligned}$$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

$$\begin{aligned}\vec{w}_2 &= \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1) \vec{u}_1 \\ &= (-1, 1, 0) - \left[(-1, 1, 0) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \right] \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\ &= (-1, 1, 0) - \left[-\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + 0 \right] \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\ &= (-1, 1, 0) - 0 \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\ &= (-1, 1, 0)\end{aligned}$$

$$\begin{aligned}\|\vec{w}_2\| &= \sqrt{(-1)^2 + 1^2} \\ &= \sqrt{2}\end{aligned}$$

$$\vec{u}_2 = \frac{(-1, 1, 0)}{\sqrt{2}}$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)$$

$$\begin{aligned}\vec{v}_3 \cdot \vec{u}_1 &= (1, 2, 1) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\ &= \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} \\ &= \frac{3}{\sqrt{3}} \frac{\sqrt{3}}{\sqrt{3}} \\ &= \sqrt{3}\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 \cdot \vec{u}_2 &= (1, 2, 1) \cdot \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \\
&= -\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} + 0 \\
&= \frac{1}{\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
\vec{w}_3 &= \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2 \\
&= (1, 2, 1) - \sqrt{3}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) - \sqrt{2}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \\
&= (1, 2, 1) - (1, 1, 1) - (-1, 1, 0) \\
&= (1, 0, 0)
\end{aligned}$$

$$\begin{aligned}
\vec{u}_3 &= \frac{(1, 0, 0)}{\sqrt{1}} \\
&= (1, 0, 0)
\end{aligned}
\qquad
\vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of \mathbf{R}^m .

$$\{(1, 1, 1), (0, 1, 1), (0, 0, 1)\}$$

Solution

$$\begin{aligned}
\vec{u}_1 &= \frac{(1, 1, 1)}{\sqrt{1+1+1}} \\
&= \frac{(1, 1, 1)}{\sqrt{3}} \\
&= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)
\end{aligned}
\qquad
\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\begin{aligned}
\vec{w}_2 &= (0, 1, 1) - \left[(0, 1, 1) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right] \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\
&= (0, 1, 1) - \left[\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right] \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\
&= (0, 1, 1) - \left[\frac{2}{\sqrt{3}}\right] \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\
&= (0, 1, 1) - \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)
\end{aligned}
\qquad
\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

$$= \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right) \Big|$$

$$\begin{aligned} \|\vec{w}_2\| &= \sqrt{\left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} \\ &= \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{1}{9}} \\ &= \sqrt{\frac{6}{9}} \\ &= \frac{\sqrt{6}}{3} \Big| \end{aligned}$$

$$\vec{u}_2 = \frac{\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)}{\frac{\sqrt{6}}{3}}$$

$$= \frac{3}{\sqrt{6}} \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right)$$

$$= \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right) \Big|$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$\begin{aligned} \vec{v}_3 \cdot \vec{u}_1 &= (0, 0, 1) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\ &= \frac{1}{\sqrt{3}} \Big| \end{aligned}$$

$$\begin{aligned} \vec{v}_3 \cdot \vec{u}_2 &= (0, 0, 1) \cdot \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right) \\ &= \frac{1}{\sqrt{6}} \Big| \end{aligned}$$

$$\begin{aligned} \vec{w}_3 &= (0, 0, 1) - \frac{1}{\sqrt{3}} \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) - \frac{1}{\sqrt{6}} \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right) & \vec{w}_3 &= \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1) \vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2) \vec{u}_2 \\ &= (0, 0, 1) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) - \left(-\frac{1}{3}, \frac{1}{6}, \frac{1}{6} \right) \\ &= \left(0, -\frac{1}{2}, \frac{1}{2} \right) \Big| \end{aligned}$$

$$\begin{aligned} \vec{u}_3 &= \frac{\left(0, -\frac{1}{2}, \frac{1}{2} \right)}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}} \\ &= \frac{\left(0, -\frac{1}{2}, \frac{1}{2} \right)}{\sqrt{\frac{1}{2}}} \end{aligned}$$

$$\vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$\begin{aligned}
&= \frac{\left(0, -\frac{1}{2}, \frac{1}{2}\right)}{\frac{1}{\sqrt{2}}} \\
&= \sqrt{2} \left(0, -\frac{1}{2}, \frac{1}{2}\right) \\
&= \left(0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \Big|
\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of \mathbb{R}^m .

$$\{(1, 1, 1), (0, 2, 1), (1, 0, 3)\}$$

Solution

$$\begin{aligned}
\vec{u}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
&= \frac{(1, 1, 1)}{\sqrt{1+1+1}} \\
&= \frac{(1, 1, 1)}{\sqrt{3}} \\
&= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{w}_2 &= \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1) \vec{u}_1 \\
&= (0, 2, 1) - \left[(0, 2, 1) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \right] \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\
&= (0, 2, 1) - \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\
&= (0, 2, 1) - \frac{3}{\sqrt{3}} \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\
&= (0, 2, 1) - \sqrt{3} \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\
&= (0, 2, 1) - (1, 1, 1) \\
&= (-1, 1, 0) \Big|
\end{aligned}$$

$$\begin{aligned}
\|\vec{w}_2\| &= \sqrt{(-1)^2 + (1)^2 + (0)^2} \\
&= \sqrt{2} \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{u}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\
&= \frac{(-1, 1, 0)}{\sqrt{2}} \\
&= \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 \cdot \vec{u}_1 &= (1, 0, 3) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\
&= \frac{1}{\sqrt{3}} + \frac{3}{\sqrt{3}} \\
&= \frac{4}{\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 \cdot \vec{u}_2 &= (1, 0, 3) \cdot \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\
&= -\frac{1}{\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
\vec{w}_3 &= \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1) \vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2) \vec{u}_2 \\
&= (1, 0, 3) - \frac{4}{\sqrt{3}} \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) + \frac{1}{\sqrt{2}} \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\
&= (1, 0, 3) - \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3} \right) + \left(-\frac{1}{2}, \frac{1}{2}, 0 \right) \\
&= \left(-\frac{5}{6}, -\frac{5}{6}, \frac{5}{3} \right)
\end{aligned}$$

$$\begin{aligned}
\vec{u}_3 &= \frac{\vec{w}_3}{\|\vec{w}_3\|} \\
&= \frac{1}{\sqrt{\left(-\frac{5}{6}\right)^2 + \left(-\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2}} \left(-\frac{5}{6}, -\frac{5}{6}, \frac{5}{3} \right) \\
&= \frac{1}{\sqrt{\frac{25}{36} + \frac{25}{36} + \frac{25}{9}}} \left(-\frac{5}{6}, -\frac{5}{6}, \frac{5}{3} \right) \\
&= \frac{1}{\frac{5}{6}\sqrt{6}} \left(-\frac{5}{6}, -\frac{5}{6}, \frac{5}{3} \right) \\
&= \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right)
\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of \mathbb{R}^m .

$$\{(2, 2, 2), (1, 0, -1), (0, 3, 1)\}$$

Solution

$$\begin{aligned}\vec{u}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{(2, 2, 2)}{\sqrt{2^2+2^2+2^2}} \\ &= \frac{(2, 2, 2)}{\sqrt{12}} \\ &= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \end{aligned}$$

$$\begin{aligned}\vec{w}_3 &= \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1) \vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2) \vec{u}_2 \\ &= (1, 0, -1) - \left[(1, 0, -1) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \right] \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\ &= (1, 0, -1) - \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} \right) \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\ &= (0, 2, 1) - (0) \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\ &= (1, 0, -1) \end{aligned}$$

$$\begin{aligned}\vec{u}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\ &= \frac{(1, 0, -1)}{\sqrt{2}} \\ &= \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right) \end{aligned}$$

$$\begin{aligned}\vec{v}_3 \cdot \vec{u}_1 &= (0, 3, 1) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\ &= \frac{3}{\sqrt{3}} + \frac{1}{\sqrt{3}} \\ &= \frac{4}{\sqrt{3}} \end{aligned}$$

$$\vec{v}_3 \cdot \vec{u}_2 = (0, 3, 1) \cdot \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right)$$

$$\left| -\frac{1}{\sqrt{2}} \right|$$

$$\begin{aligned}\vec{w}_3 &= \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1) \vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2) \vec{u}_2 \\ &= (0, 3, 1) - \frac{4}{\sqrt{3}} \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) + \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right) \\ &= (0, 3, 1) - \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3} \right) + \left(\frac{1}{2}, 0, -\frac{1}{2} \right) \\ &= \left(-\frac{5}{6}, \frac{5}{3}, -\frac{5}{6} \right) \end{aligned}$$

$$\begin{aligned}\vec{u}_3 &= \frac{\vec{w}_3}{\|\vec{w}_3\|} \\ &= \frac{1}{\sqrt{\left(-\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(-\frac{5}{6}\right)^2}} \left(-\frac{5}{6}, \frac{5}{3}, -\frac{5}{6} \right) \\ &= \frac{1}{\sqrt{\frac{25}{36} + \frac{25}{9} + \frac{25}{36}}} \left(-\frac{5}{6}, \frac{5}{3}, -\frac{5}{6} \right) \\ &= \frac{1}{\frac{5}{6}\sqrt{6}} \left(-\frac{5}{6}, \frac{5}{3}, -\frac{5}{6} \right) \\ &= \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right) \end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of \mathbb{R}^m .

$$\{(1, -1, 0), (0, 1, 0), (2, 3, 1)\}$$

Solution

$$\begin{aligned}\vec{u}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{(1, -1, 0)}{\sqrt{2}} \\ &= \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right) \end{aligned}$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1) \vec{u}_1$$

$$\begin{aligned}
&= (0, 1, 0) - \left[(0, 1, 0) \cdot \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right) \right] \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right) \\
&= (0, 1, 0) + \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right) \\
&= (0, 1, 0) + \left(\frac{1}{2}, -\frac{1}{2}, 0 \right) \\
&= \left(\frac{1}{2}, \frac{1}{2}, 0 \right) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{u}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\
&= \frac{1}{\sqrt{\frac{1}{4} + \frac{1}{4}}} \left(\frac{1}{2}, \frac{1}{2}, 0 \right) \\
&= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 \cdot \vec{u}_1 &= (2, 3, 1) \cdot \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right) \\
&= \frac{2}{\sqrt{2}} - \frac{3}{\sqrt{2}} \\
&= -\frac{1}{\sqrt{2}} \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 \cdot \vec{u}_2 &= (2, 3, 1) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\
&= \frac{4}{\sqrt{2}} \\
&= 2\sqrt{2} \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{w}_3 &= \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1) \vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2) \vec{u}_2 \\
&= (2, 3, 1) + \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right) - 2\sqrt{2} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\
&= (2, 3, 1) + \left(\frac{1}{2}, -\frac{1}{2}, 0 \right) - (2, 2, 0) \\
&= \left(\frac{1}{2}, \frac{1}{2}, 1 \right) \Big|
\end{aligned}$$

$$\vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{\frac{1}{4} + \frac{1}{4} + 1}} \left(\frac{1}{2}, \frac{1}{2}, 1 \right) \\
&= \frac{2}{\sqrt{6}} \left(\frac{1}{2}, \frac{1}{2}, 1 \right) \\
&= \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right)
\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of \mathbb{R}^m .

$$\{(3, 0, 4), (-1, 0, 7), (2, 9, 11)\}$$

Solution

$$\begin{aligned}
\vec{u}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
&= \frac{(3, 0, 4)}{\sqrt{9+16}} \\
&= \left(\frac{3}{5}, 0, \frac{4}{5} \right)
\end{aligned}$$

$$\begin{aligned}
\vec{w}_2 &= \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1) \vec{u}_1 \\
&= (-1, 0, 7) - \left[(-1, 0, 7) \cdot \left(\frac{3}{5}, 0, \frac{4}{5} \right) \right] \left(\frac{3}{5}, 0, \frac{4}{5} \right) \\
&= (-1, 0, 7) - \left(-\frac{3}{5} + \frac{28}{5} \right) \left(\frac{3}{5}, 0, \frac{4}{5} \right) \\
&= (-1, 0, 7) - 5 \left(\frac{3}{5}, 0, \frac{4}{5} \right) \\
&= (-1, 0, 7) - (3, 0, 4) \\
&= (-4, 0, 3)
\end{aligned}$$

$$\begin{aligned}
\vec{u}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\
&= \frac{1}{\sqrt{16+9}} (-4, 0, 3) \\
&= \left(-\frac{4}{5}, 0, \frac{3}{5} \right)
\end{aligned}$$

$$\vec{v}_3 \cdot \vec{u}_1 = (2, 9, 11) \cdot \left(\frac{3}{5}, 0, \frac{4}{5} \right)$$

$$= \frac{6}{5} + \frac{44}{5}$$

$$= 10 \quad |$$

$$\vec{v}_3 \cdot \vec{u}_2 = (2, 9, 11) \cdot \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$$

$$= -\frac{8}{5} + \frac{33}{5}$$

$$= 5 \quad |$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1) \vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2) \vec{u}_2$$

$$\vec{w}_3 = (2, 9, 11) - 10\left(\frac{3}{5}, 0, \frac{4}{5}\right) - 5\left(-\frac{4}{5}, 0, \frac{3}{5}\right)$$

$$= (0, 9, 0) \quad |$$

$$\vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$= \frac{1}{9}(0, 9, 0)$$

$$= (0, 1, 0) \quad |$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of \mathbb{R}^m .

$$\{(1, 1, 1, 1), (1, 2, 1, 0), (1, 3, 0, 0)\}$$

Solution

$$\vec{v}_1 = \vec{u}_1$$

$$= (1, 1, 1, 1) \quad |$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1, 1, 1, 1)}{\sqrt{4}}$$

$$= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \quad |$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$\begin{aligned}
&= (1, 2, 1, 0) - \frac{(1, 2, 1, 0) \cdot (1, 1, 1, 1)}{\|(1, 1, 1, 1)\|^2} (1, 1, 1, 1) \\
&= (1, 2, 1, 0) - \frac{1+2+1}{4} (1, 1, 1, 1) \\
&= (1, 2, 1, 0) - \frac{4}{4} (1, 1, 1, 1) \\
&= (1, 2, 1, 0) - (1, 1, 1, 1) \\
&= \underline{(0, 1, 0, -1)} \quad |
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \frac{(0, 1, 0, -1)}{\sqrt{1+1}} \\
&= \underline{\left(0, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)} \quad |
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
&= (1, 3, 0, 0) - \frac{(1, 3, 0, 0) \cdot (1, 1, 1, 1)}{\|(1, 1, 1, 1)\|^2} (1, 1, 1, 1) - \frac{(1, 3, 0, 0) \cdot (0, 1, 0, -1)}{\|(0, 1, 0, -1)\|^2} (0, 1, 0, -1) \\
&= (1, 3, 0, 0) - \frac{4}{4} (1, 1, 1, 1) - \frac{3}{2} (0, 1, 0, -1) \\
&= (1, 3, 0, 0) - (1, 1, 1, 1) - \left(0, \frac{3}{2}, 0, -\frac{3}{2}\right) \\
&= \underline{\left(0, \frac{1}{2}, -1, \frac{1}{2}\right)} \quad |
\end{aligned}$$

$$\begin{aligned}
\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
&= \frac{\left(0, \frac{1}{2}, -1, \frac{1}{2}\right)}{\sqrt{\frac{1}{4}+1+\frac{1}{4}}} \\
&= \frac{\left(0, \frac{1}{2}, -1, \frac{1}{2}\right)}{\frac{\sqrt{6}}{2}} \\
&= \frac{2}{\sqrt{6}} \left(0, \frac{1}{2}, -1, \frac{1}{2}\right) \\
&= \underline{\left(0, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)} \quad |
\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of \mathbb{R}^m .

$$\{(0, 2, -1, 1), (0, 0, 1, 1), (-2, 1, 1, -1)\}$$

Solution

$$\begin{aligned}\vec{u}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{(0, 2, -1, 1)}{\sqrt{6}} \\ &= \left(0, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \end{aligned}$$

$$\begin{aligned}\vec{w}_2 &= \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1) \vec{u}_1 \\ &= (0, 0, 1, 1) - \left[(0, 0, 1, 1) \cdot \left(0, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \right] \left(0, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \\ &= (0, 0, 1, 1) - \left[-\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}} \right] \left(0, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \\ &= (0, 0, 1, 1) - [0] \left(0, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \\ &= (0, 0, 1, 1) \end{aligned}$$

$$\begin{aligned}\vec{u}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\ &= \frac{(0, 0, 1, 1)}{\sqrt{2}} \\ &= \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \end{aligned}$$

$$\begin{aligned}\vec{v}_3 \cdot \vec{u}_1 &= (-2, 1, 1, -1) \cdot \left(0, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \\ &= \frac{2}{\sqrt{6}} - \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} \\ &= 0 \end{aligned}$$

$$\begin{aligned}\vec{v}_3 \cdot \vec{u}_2 &= (-2, 1, 1, -1) \cdot \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \\ &= 0 \end{aligned}$$

$$\begin{aligned}
 \vec{w}_3 &= \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1) \vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2) \vec{u}_2 \\
 &= (-2, 1, 1, -1) - 0 - 0 \\
 &= (-2, 1, 1, -1) \quad |
 \end{aligned}$$

$$\begin{aligned}
 \vec{u}_3 &= \frac{\vec{w}_3}{\|\vec{w}_3\|} \\
 &= \frac{(-2, 1, 1, -1)}{\sqrt{(-2)^2 + 1^2 + 1^2 + (-1)^2}} \\
 &= \frac{(-2, 1, 1, -1)}{\sqrt{7}} \\
 &= \left(-\frac{2}{\sqrt{7}}, \frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}, -\frac{1}{\sqrt{7}} \right) \quad |
 \end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of \mathbb{R}^m .

$$\vec{u}_1 = (1, 0, 0), \quad \vec{u}_2 = (3, 7, -2), \quad \vec{u}_3 = (0, 4, 1)$$

Solution

$$\begin{aligned}
 \vec{v}_1 &= \frac{\vec{u}_1}{\|\vec{u}_1\|} \\
 &= \frac{(1, 0, 0)}{\sqrt{1^2 + 0^2 + 0^2}} \\
 &= (1, 0, 0) \quad |
 \end{aligned}$$

$$\begin{aligned}
 \vec{w}_2 &= \vec{u}_2 - (\vec{u}_2 \cdot \vec{v}_1) \vec{v}_1 \\
 &= (3, 7, -2) - [(3, 7, -2) \cdot (1, 0, 0)](1, 0, 0) \\
 &= (3, 7, -2) - 3(1, 0, 0) \\
 &= (0, 7, -2) \quad |
 \end{aligned}$$

$$\begin{aligned}
 \vec{v}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\
 &= \frac{1}{\sqrt{53}}(0, 7, -2) \\
 &= \left(0, \frac{7}{\sqrt{53}}, -\frac{2}{\sqrt{53}} \right) \quad |
 \end{aligned}$$

$$\begin{aligned}\vec{u}_3 \cdot \vec{v}_1 &= (0, 4, 1) \cdot (1, 0, 0) \\ &= 0\end{aligned}$$

$$\begin{aligned}\vec{u}_3 \cdot \vec{v}_2 &= (0, 4, 1) \cdot \left(0, \frac{7}{\sqrt{53}}, -\frac{2}{\sqrt{53}}\right) \\ &= \frac{26}{\sqrt{53}}\end{aligned}$$

$$\begin{aligned}\vec{w}_3 &= \vec{u}_3 - (\vec{u}_3 \cdot \vec{v}_1)\vec{v}_1 - (\vec{u}_3 \cdot \vec{v}_2)\vec{v}_2 \\ &= (0, 4, 1) - 0 - \frac{26}{\sqrt{53}}\left(0, \frac{7}{\sqrt{53}}, -\frac{2}{\sqrt{53}}\right) \\ &= (0, 4, 1) - 0 - \frac{26}{53}\left(0, 7, -2\right) \\ &= (0, 4, 1) - \left(0, \frac{182}{53}, -\frac{52}{53}\right) \\ &= \left(0, \frac{30}{53}, \frac{105}{53}\right)\end{aligned}$$

$$\begin{aligned}\vec{v}_3 &= \frac{\vec{w}_3}{\|\vec{w}_3\|} \\ &= \frac{\left(0, \frac{30}{53}, \frac{105}{53}\right)}{\sqrt{\left(\frac{30}{53}\right)^2 + \left(\frac{105}{53}\right)^2}} \\ &= \frac{53}{\sqrt{11925}}\left(0, \frac{30}{53}, \frac{105}{53}\right) \\ &= \frac{53}{15\sqrt{53}}\left(0, \frac{30}{53}, \frac{105}{53}\right) \\ &= \left(0, \frac{2}{\sqrt{15}}, \frac{7}{\sqrt{15}}\right)\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an *orthonormal* basis for the subspaces of \mathbb{R}^m .

$$\vec{u}_1 = (1, 1, 1, 1), \quad \vec{u}_2 = (1, 2, 4, 5), \quad \vec{u}_3 = (1, -3, -4, -2)$$

Solution

$$\begin{aligned}\vec{v}_1 &= \frac{\vec{u}_1}{\|\vec{u}_1\|} \\ &= \frac{(1, 1, 1, 1)}{\sqrt{4}} \\ &= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \end{aligned}$$

$$\begin{aligned}\vec{w}_2 &= \vec{u}_2 - (\vec{u}_2 \cdot \vec{v}_1) \vec{v}_1 \\ &= (1, 2, 4, 5) - \left[(1, 2, 4, 5) \cdot \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \right] \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \\ &= (1, 2, 4, 5) - 6 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \\ &= (1, 2, 4, 5) - (3, 3, 3, 3) \\ &= (-2, -1, 1, 2) \end{aligned}$$

$$\begin{aligned}\vec{v}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\ &= \frac{1}{\sqrt{10}} (-2, -1, 1, 2) \\ &= \left(-\frac{2}{\sqrt{10}}, -\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}} \right) \end{aligned}$$

$$\begin{aligned}\vec{u}_3 \cdot \vec{v}_1 &= (1, -3, -4, -2) \cdot \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \\ &= \frac{1-3-4-2}{2} \\ &= -4 \end{aligned}$$

$$\begin{aligned}\vec{u}_3 \cdot \vec{v}_2 &= (1, -3, -4, -2) \cdot \left(-\frac{2}{\sqrt{10}}, -\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}} \right) \\ &= \frac{-2+3-4-4}{\sqrt{10}} \\ &= -\frac{7}{\sqrt{10}} \end{aligned}$$

$$\vec{w}_3 = \vec{u}_3 - (\vec{u}_3 \cdot \vec{v}_1) \vec{v}_1 - (\vec{u}_3 \cdot \vec{v}_2) \vec{v}_2$$

$$\begin{aligned}
&= (1, -3, -4, -2) + 4\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + \frac{7}{\sqrt{10}}\left(-\frac{2}{\sqrt{10}}, -\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right) \\
&= (1, -3, -4, -2) + (2, 2, 2, 2) + \left(-\frac{7}{5}, -\frac{7}{10}, \frac{7}{10}, \frac{7}{5}\right) \\
&= \left(\frac{8}{5}, -\frac{17}{10}, -\frac{27}{10}, \frac{7}{5}\right) \Big| \\
\vec{v}_3 &= \frac{\vec{w}_3}{\|\vec{w}_3\|} \\
&= \frac{1}{\sqrt{\frac{64}{25} + \frac{289}{100} + \frac{289}{100} + \frac{49}{25}}}\left(\frac{8}{5}, -\frac{17}{10}, -\frac{27}{10}, \frac{7}{5}\right) \\
&= \frac{1}{\sqrt{\frac{1030}{100}}}\left(\frac{8}{5}, -\frac{17}{10}, -\frac{27}{10}, \frac{7}{5}\right) \\
&= \left(\frac{16}{\sqrt{1030}}, -\frac{17}{\sqrt{1030}}, -\frac{27}{\sqrt{1030}}, \frac{14}{\sqrt{1030}}\right) \Big|
\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an *orthonormal* basis for the subspaces of \mathbb{R}^m .

$$\vec{u}_1 = (1, 1, 1, 1), \quad \vec{u}_2 = (1, 1, 2, 4), \quad \vec{u}_3 = (1, 2, -4, -3)$$

Solution

$$\begin{aligned}
\vec{v}_1 &= \frac{\vec{u}_1}{\|\vec{u}_1\|} \\
&= \frac{(1, 1, 1, 1)}{\sqrt{4}} \\
&= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{w}_2 &= \vec{u}_2 - (\vec{u}_2 \cdot \vec{v}_1) \vec{v}_1 \\
&= (1, 1, 2, 4) - \left[(1, 1, 2, 4) \cdot \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right] \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \\
&= (1, 1, 2, 4) - 4\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \\
&= (1, 1, 2, 4) - (2, 2, 2, 2) \\
&= (-1, -1, 0, 2) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{v}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\
&= \frac{1}{\sqrt{1+1+4}}(-1, -1, 0, 2) \\
&= \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}} \right)
\end{aligned}$$

$$\begin{aligned}
\vec{u}_3 \cdot \vec{v}_1 &= (1, 2, -4, -3) \cdot \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \\
&= \frac{1+2-4-3}{2} \\
&= -2
\end{aligned}$$

$$\begin{aligned}
\vec{u}_3 \cdot \vec{v}_2 &= (1, 2, -4, -3) \cdot \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}} \right) \\
&= \frac{-1-2-6}{\sqrt{6}} \\
&= -\frac{9}{\sqrt{6}}
\end{aligned}$$

$$\begin{aligned}
\vec{w}_3 &= \vec{u}_3 - (\vec{u}_3 \cdot \vec{v}_1) \vec{v}_1 - (\vec{u}_3 \cdot \vec{v}_2) \vec{v}_2 \\
&= (1, 2, -4, -3) + 2 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) + \frac{9}{\sqrt{6}} \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}} \right) \\
&= (1, 2, -4, -3) + (1, 1, 1, 1) + \left(-\frac{3}{2}, -\frac{3}{2}, 0, 3 \right) \\
&= \left(\frac{1}{2}, \frac{3}{2}, -3, 1 \right)
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \frac{\vec{w}_3}{\|\vec{w}_3\|} \\
&= \frac{1}{\sqrt{\frac{1}{4} + \frac{9}{4} + 9 + 1}} \left(\frac{1}{2}, \frac{3}{2}, -3, 1 \right) \\
&= \frac{2}{5\sqrt{2}} \left(\frac{1}{2}, \frac{3}{2}, -3, 1 \right) \\
&= \left(\frac{1}{5\sqrt{2}}, \frac{3}{5\sqrt{2}}, -\frac{6}{5\sqrt{2}}, \frac{2}{5\sqrt{2}} \right)
\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an *orthonormal* basis for the subspaces of \mathbb{R}^m .

$$\vec{u}_1 = (0, 2, 1, 0); \quad \vec{u}_2 = (1, -1, 0, 0); \quad \vec{u}_3 = (1, 2, 0, -1); \quad \vec{u}_4 = (1, 0, 0, 1)$$

Solution

$$\begin{aligned}\vec{v}_1 &= \frac{\vec{u}_1}{\|\vec{u}_1\|} \\ &= \frac{(0, 2, 1, 0)}{\sqrt{5}} \\ &= \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right) \end{aligned}$$

$$\begin{aligned}\vec{w}_2 &= \vec{u}_2 - (\vec{u}_2 \cdot \vec{v}_1) \vec{v}_1 \\ &= (1, -1, 0, 0) - \left[(1, -1, 0, 0) \cdot \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right) \right] \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right) \\ &= (1, -1, 0, 0) - \left(-\frac{2}{\sqrt{5}}\right) \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right) \\ &= \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right) \end{aligned}$$

$$\begin{aligned}\vec{v}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\ &= \frac{1}{\sqrt{1 + \frac{1}{25} + \frac{4}{25} + 0}} \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right) \\ &= \frac{5}{\sqrt{30}} \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right) \\ &= \left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0\right) \end{aligned}$$

$$\begin{aligned}u_3 \cdot v_1 &= (1, 2, 0, -1) \cdot \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right) \\ &= \frac{4}{\sqrt{5}} \end{aligned}$$

$$\begin{aligned}u_3 \cdot v_2 &= (1, 2, 0, -1) \cdot \left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0\right) \\ &= \frac{5}{\sqrt{30}} - \frac{2}{\sqrt{30}} \end{aligned}$$

$$= \frac{3}{\sqrt{30}} \Big|$$

$$\begin{aligned}\vec{w}_3 &= \vec{u}_3 - (\vec{u}_3 \cdot \vec{v}_1) \vec{v}_1 - (\vec{u}_3 \cdot \vec{v}_2) \vec{v}_2 \\ &= (1, 2, 0, -1) - \left(\frac{4}{\sqrt{5}}\right) \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right) - \left(\frac{3}{\sqrt{30}}\right) \left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0\right) \\ &= (1, 2, 0, -1) - \left(0, \frac{8}{5}, \frac{4}{5}, 0\right) - \left(\frac{1}{2}, -\frac{1}{10}, \frac{1}{5}, 0\right) \\ &= \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right) \Big| \end{aligned}$$

$$\begin{aligned}\vec{v}_3 &= \frac{\vec{w}_3}{\|\vec{w}_3\|} \\ &= \frac{\left(\frac{1}{2}, \frac{1}{2}, -1, -1\right)}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + (-1)^2 + (-1)^2}} \\ &= \frac{1}{\sqrt{5}} \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right) \\ &= \frac{\sqrt{2}}{\sqrt{5}} \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right) = \frac{2}{\sqrt{10}} \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right) \\ &= \left(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, -\frac{2}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right) \Big| \end{aligned}$$

$$\begin{aligned}u_4 \cdot v_1 &= (1, 0, 0, 1) \cdot \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right) \\ &= 0 \Big| \end{aligned}$$

$$\begin{aligned}u_4 \cdot v_2 &= (1, 0, 0, 1) \cdot \left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0\right) \\ &= \frac{5}{\sqrt{30}} \Big| \end{aligned}$$

$$\begin{aligned}u_4 \cdot v_3 &= (1, 0, 0, 1) \cdot \left(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, -\frac{2}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right) \\ &= -\frac{1}{\sqrt{10}} \Big| \end{aligned}$$

$$\begin{aligned}\vec{w}_4 &= \vec{u}_4 - (\vec{u}_4 \cdot \vec{v}_1) \vec{v}_1 - (\vec{u}_4 \cdot \vec{v}_2) \vec{v}_2 - (\vec{u}_4 \cdot \vec{v}_3) \vec{v}_3 \\ &= (1, 2, 0, -1) - (0) - \left(\frac{5}{\sqrt{30}}\right) \left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0\right) + \left(\frac{1}{\sqrt{10}}\right) \left(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, -\frac{2}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right) \end{aligned}$$

$$\begin{aligned}
&= (1, 2, 0, -1) - \left(\frac{5}{6}, -\frac{1}{6}, \frac{1}{3}, 0\right) + \left(\frac{1}{10}, \frac{1}{10}, -\frac{1}{5}, -\frac{1}{5}\right) \\
&= \left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right) \Big| \\
\vec{v}_4 &= \frac{\vec{w}_4}{\|\vec{w}_4\|} \\
&= \frac{\left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right)}{\sqrt{\left(\frac{4}{15}\right)^2 + \left(\frac{4}{15}\right)^2 + \left(-\frac{8}{15}\right)^2 + \left(\frac{4}{5}\right)^2}} \\
&= \frac{1}{\sqrt{\frac{240}{225}}} \left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right) \\
&= \frac{1}{\frac{4}{\sqrt{15}}} \left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right) \\
&= \frac{15}{4\sqrt{15}} \left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right) \\
&= \left(\frac{1}{\sqrt{15}}, \frac{1}{\sqrt{15}}, -\frac{2}{\sqrt{15}}, \frac{3}{\sqrt{15}}\right) \Big|
\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis for the subspaces of \mathbb{R}^m .

$$\{(1, 1, 0), (1, 2, 0), (0, 1, 2)\}$$

Solution

$$\vec{v}_1 = (1, 1, 0) \Big|$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
&= (0, 2, 1) - \frac{(1, 2, 0) \cdot (1, 1, 0)}{1+1+0} (1, 1, 0) \\
&= (0, 2, 1) - \frac{3}{2} (1, 1, 0) \\
&= \left(-\frac{3}{2}, \frac{1}{2}, 1\right) \Big|
\end{aligned}$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(0, 1, 2) \cdot (1, 1, 0)}{2} (1, 1, 0)$$

$$= \left(\frac{1}{2}, \frac{1}{2}, 0 \right) \Big|$$

$$\begin{aligned} \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{1}{\frac{9}{4} + \frac{1}{4} + 1} (0, 1, 2) \cdot \left(-\frac{3}{2}, \frac{1}{2}, 1 \right) \left(-\frac{3}{2}, \frac{1}{2}, 1 \right) \\ &= \frac{2}{7} \frac{5}{2} \left(-\frac{3}{2}, \frac{1}{2}, 1 \right) \\ &= \left(-\frac{15}{14}, \frac{5}{14}, \frac{5}{7} \right) \Big| \end{aligned}$$

$$\begin{aligned} \vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= (0, 1, 2) - \left(\frac{1}{2}, \frac{1}{2}, 0 \right) - \left(-\frac{15}{14}, \frac{5}{14}, \frac{5}{7} \right) \\ &= \left(\frac{4}{7}, \frac{1}{7}, \frac{9}{7} \right) \Big| \end{aligned}$$

$$\begin{aligned} \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{1}{\sqrt{2}} (1, 1, 0) \\ &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \Big| \end{aligned}$$

$$\begin{aligned} \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{1}{\sqrt{\frac{9}{4} + \frac{1}{4} + 1}} \left(-\frac{3}{2}, \frac{1}{2}, 1 \right) \\ &= \frac{2}{\sqrt{14}} \left(-\frac{3}{2}, \frac{1}{2}, 1 \right) \\ &= \left(-\frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}} \right) \Big| \end{aligned}$$

$$\begin{aligned} \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= \frac{1}{\sqrt{\frac{16}{49} + \frac{1}{49} + \frac{81}{49}}} \left(\frac{4}{7}, \frac{1}{7}, \frac{9}{7} \right) \\ &= \frac{7}{\sqrt{98}} \left(\frac{4}{7}, \frac{1}{7}, \frac{9}{7} \right) \end{aligned}$$

$$= \frac{7}{7\sqrt{2}} \left(\frac{4}{7}, \frac{1}{7}, \frac{9}{7} \right)$$

$$= \left(\frac{4}{7\sqrt{2}}, \frac{1}{7\sqrt{2}}, \frac{9}{7\sqrt{2}} \right) \Big|$$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis for the subspaces of \mathbb{R}^m .

$$\{(1, -2, 2), (2, 2, 1), (2, -1, -2)\}$$

Solution

$$\underline{\vec{v}_1 = \vec{u}_1 = (1, -2, 2) \Big|}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (2, 2, 1) - \frac{(2, 2, 1) \cdot (1, -2, 2)}{9} (1, -2, 2)$$

$$= (2, 2, 1) - \frac{0}{9} (1, -2, 2)$$

$$= (2, 2, 1) \Big|$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(2, -1, -2) \cdot (1, -2, 2)}{9} (1, -2, 2)$$

$$= \frac{0}{9} (1, -2, 2)$$

$$= (0, 0, 0) \Big|$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{1}{9} [(2, -1, -2) \cdot (2, 2, 1)] (2, 2, 1)$$

$$= (0, 0, 0) \Big|$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (2, -1, -2) - (0, 0, 0) - (0, 0, 0)$$

$$= (2, -1, -2) \Big|$$

$$\begin{aligned}\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{1}{3}(1, -2, 2) \\ &= \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right) \mid\end{aligned}$$

$$\begin{aligned}\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{1}{3}(2, 2, 1) \\ &= \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right) \mid\end{aligned}$$

$$\begin{aligned}\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= \frac{1}{3}(2, -1, -2) \\ &= \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right) \mid\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis for the subspaces of \mathbb{R}^m .

$$\{(1, 0, 0), (1, 1, 1), (1, 1, -1)\}$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (1, 0, 0) \mid$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (1, 1, 1) - \frac{(1, 1, 1) \cdot (1, 0, 0)}{1} (1, 0, 0) \\ &= (1, 1, 1) - (1, 0, 0) \\ &= (0, 1, 1) \mid\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(1, 1, -1) \cdot (1, 0, 0)}{1} (1, 0, 0) \\ &= (1, 0, 0) \mid\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{1}{1} [(1, 1, -1) \cdot (0, 1, 1)] (0, 1, 1) \\ &= 0(0, 1, 1) \\ &= \underline{(0, 0, 0)}\end{aligned}$$

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= (1, 1, -1) - (1, 0, 0) - (0, 0, 0) \\ &= \underline{(0, 1, -1)}\end{aligned}$$

$$\begin{aligned}\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{1}{1} (1, 0, 0) \\ &= \underline{(1, 0, 0)}\end{aligned}$$

$$\begin{aligned}\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{1}{\sqrt{2}} (0, 1, 1) \\ &= \underline{\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}\end{aligned}$$

$$\begin{aligned}\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= \frac{1}{\sqrt{2}} (0, 1, -1) \\ &= \underline{\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)}\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis for the subspaces of \mathbb{R}^m .

$$\{(4, -3, 0), (1, 2, 0), (0, 0, 4)\}$$

Solution

$$\underline{\vec{v}_1 = \vec{u}_1 = (4, -3, 0)}$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
&= (1, 2, 0) - \frac{(1, 2, 0) \cdot (4, -3, 0)}{25} (4, -3, 0) \\
&= (1, 2, 0) + \frac{2}{25} (4, -3, 0) \\
&= \left(\frac{33}{25}, \frac{44}{25}, 0 \right) \Big|
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(0, 0, 4) \cdot (4, -3, 0)}{25} (4, -3, 0) \\
&= (0, 0, 0) \Big|
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{225}{3,025} \left[(0, 0, 4) \cdot \left(\frac{33}{25}, \frac{44}{25}, 0 \right) \right] \left(\frac{33}{25}, \frac{44}{25}, 0 \right) \\
&= (0, 0, 0) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
&= (0, 0, 4) - (0, 0, 0) - (0, 0, 0) \\
&= (0, 0, 4) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
&= \frac{1}{\sqrt{16+9}} (4, -3, 0) \\
&= \left(\frac{4}{5}, -\frac{3}{5}, 0 \right) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \frac{25}{\sqrt{3,025}} \left(\frac{33}{25}, \frac{44}{25}, 0 \right) \\
&= \frac{25}{55} \left(\frac{33}{25}, \frac{44}{25}, 0 \right) \\
&= \left(\frac{3}{5}, \frac{4}{5}, 0 \right) \Big|
\end{aligned}$$

$$\begin{aligned}
 \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
 &= \frac{1}{4}(0, 0, 4) \\
 &= \underline{(0, 0, 1)}
 \end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis for the subspaces of \mathbb{R}^m .

$$\{(0, 1, 2), (2, 0, 0), (1, 1, 1)\}$$

Solution

$$\underline{\vec{v}_1 = \vec{u}_1 = (0, 1, 2)}$$

$$\begin{aligned}
 \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
 &= (2, 0, 0) - \frac{(2, 0, 0) \cdot (0, 1, 2)}{5} (0, 1, 2) \\
 &= (2, 0, 0) + \frac{0}{5} (0, 1, 2) \\
 &= \underline{(2, 0, 0)}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(1, 1, 1) \cdot (0, 1, 2)}{5} (0, 1, 2) \\
 &= \frac{3}{5} (0, 1, 2) \\
 &= \underline{\left(0, \frac{3}{5}, \frac{6}{5}\right)}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{1}{4} [(1, 1, 1) \cdot (2, 0, 0)] (2, 0, 0) \\
 &= \frac{1}{2} (2, 0, 0) \\
 &= \underline{(1, 0, 0)}
 \end{aligned}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (1, 1, 1) - (1, 0, 0) - \left(0, \frac{3}{5}, \frac{6}{5}\right)$$

$$= \left(0, \frac{2}{5}, -\frac{1}{5}\right) \Big|$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{5}}(0, 1, 2)$$

$$= \left(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \Big|$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{2}(2, 0, 0)$$

$$= (1, 0, 0) \Big|$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{5}{\sqrt{5}}\left(0, \frac{2}{5}, -\frac{1}{5}\right)$$

$$= \left(0, \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right) \Big|$$

Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis for the subspaces of \mathbb{R}^m .

$$\{(0, 1, 1), (1, 1, 0), (1, 0, 1)\}$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (0, 1, 1) \Big|$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (1, 1, 0) - \frac{(1, 1, 0) \cdot (0, 1, 1)}{2} (0, 1, 1)$$

$$= (1, 1, 0) - \frac{1}{2} (0, 1, 1)$$

$$= \left(1, \frac{1}{2}, -\frac{1}{2}\right) \Big|$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(1, 0, 1) \cdot (0, 1, 1)}{2} (0, 1, 1)$$

$$= \left(0, \frac{1}{2}, \frac{1}{2} \right) \Big|$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{4}{6} \left[(1, 0, 1) \cdot \left(1, \frac{1}{2}, -\frac{1}{2} \right) \right] \left(1, \frac{1}{2}, -\frac{1}{2} \right)$$

$$= \frac{1}{3} \left(1, \frac{1}{2}, -\frac{1}{2} \right)$$

$$= \left(\frac{1}{3}, \frac{1}{6}, -\frac{1}{6} \right) \Big|$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (1, 0, 1) - \left(0, \frac{1}{2}, \frac{1}{2} \right) - \left(\frac{1}{3}, \frac{1}{6}, -\frac{1}{6} \right)$$

$$= \left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3} \right) \Big|$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{2}} (0, 1, 1)$$

$$= \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \Big|$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{2}{\sqrt{6}} \left(1, \frac{1}{2}, -\frac{1}{2} \right)$$

$$= \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right) \Big|$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{3}{\sqrt{12}} \left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3} \right)$$

$$= \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \Big|$$

Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis for the subspaces of \mathbb{R}^m .

$$\{(1, 2, -2), (1, 0, -4), (5, 2, 0), (1, 1, -1)\}$$

Solution

$$\underline{\vec{v}_1 = \vec{u}_1 = (1, 2, -2) \mid}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (1, 0, -4) - \frac{(1, 0, -4) \cdot (1, 2, -2)}{9} (1, 2, -2) \\ &= (1, 0, -4) - (1, 2, -2) \\ &= (0, -2, -2) \mid\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(5, 2, 0) \cdot (1, 2, -2)}{9} (1, 2, -2) \\ &= (1, 2, -2) \mid\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{1}{8} [(5, 2, 0) \cdot (0, -2, -2)] (0, -2, -2) \\ &= -\frac{1}{2} (0, -2, -2) \\ &= (0, 1, 1) \mid\end{aligned}$$

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= (5, 2, 0) - (1, 2, -2) - (0, 1, 1) \\ &= (4, -1, 1) \mid\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(1, 1, -1) \cdot (1, 2, -2)}{9} (1, 2, -2) \\ &= \frac{5}{9} (1, 2, -2) \\ &= \left(\frac{5}{9}, \frac{10}{9}, -\frac{10}{9} \right) \mid\end{aligned}$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{1}{8}[(1, 1, -1) \cdot (0, -2, -2)](0, -2, -2)$$

$$= \underline{(0, 0, 0) \mid}$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 = \frac{1}{18}[(1, 1, -1) \cdot (4, -1, 1)](4, -1, 1)$$

$$= \frac{1}{9}(4, -1, 1)$$

$$= \underline{\left(\frac{4}{9}, -\frac{1}{9}, \frac{1}{9}\right) \mid}$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3$$

$$= (1, 1, -1) - \left(\frac{5}{9}, \frac{10}{9}, -\frac{10}{9}\right) - \left(\frac{4}{9}, -\frac{1}{9}, \frac{1}{9}\right)$$

$$= \underline{(0, 0, 0) \mid}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{3}(1, 2, -2)$$

$$= \underline{\left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right) \mid}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{2\sqrt{2}}(0, -2, -2)$$

$$= \underline{\left(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \mid}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{3\sqrt{2}}(4, -1, 1)$$

$$= \underline{\left(\frac{4}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}\right) \mid}$$

$$\vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

$$= \underline{(0, 0, 0) \mid}$$

Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis for the subspaces of \mathbb{R}^m .

$$\{(-3, 1, 2), (1, 1, 1), (2, 0, -1), (1, -3, 2)\}$$

Solution

$$\underline{\vec{v}_1 = \vec{u}_1 = (-3, 1, 2)}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (1, 1, 1) - \frac{(1, 1, 1) \cdot (-3, 1, 2)}{14} (-3, 1, 2) \\ &= (1, 1, 1) - \frac{0}{14} (1, 2, -2) \\ &= (1, 1, 1)\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(2, 0, -1) \cdot (-3, 1, 2)}{14} (-3, 1, 2) \\ &= -\frac{4}{7} (-3, 1, 2) \\ &= \left(\frac{12}{7}, -\frac{4}{7}, -\frac{8}{7} \right)\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{1}{3} [(2, 0, -1) \cdot (1, 1, 1)] (1, 1, 1) \\ &= \frac{1}{3} (1, 1, 1) \\ &= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)\end{aligned}$$

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= (2, 0, -1) - \left(\frac{12}{7}, -\frac{4}{7}, -\frac{8}{7} \right) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \\ &= \left(-\frac{1}{21}, \frac{5}{21}, -\frac{4}{21} \right)\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(1, -3, 2) \cdot (-3, 1, 2)}{14} (-3, 1, 2) \\ &= -\frac{1}{7} (-3, 1, 2)\end{aligned}$$

$$=\left(\frac{3}{7}, -\frac{1}{7}, -\frac{2}{7}\right) \Big|$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{(1, -3, 2) \cdot (1, 1, 1)}{3} (1, 1, 1)$$

$$=(0, 0, 0) \Big|$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 = \frac{441}{42} \left[(1, -3, 2) \cdot \left(-\frac{1}{21}, \frac{5}{21}, -\frac{4}{21}\right) \right] \left(-\frac{1}{21}, \frac{5}{21}, -\frac{4}{21}\right)$$

$$= \frac{441}{42} \left(-\frac{24}{21}\right) \left(-\frac{1}{21}, \frac{5}{21}, -\frac{4}{21}\right)$$

$$=\left(\frac{4}{7}, -\frac{20}{7}, \frac{16}{7}\right) \Big|$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3$$

$$=(1, -3, 2) - \left(\frac{3}{7}, -\frac{1}{7}, -\frac{2}{7}\right) - (0, 0, 0) - \left(\frac{4}{7}, -\frac{20}{7}, \frac{16}{7}\right)$$

$$=(0, 0, 0) \Big|$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{14}} (-3, 1, 2)$$

$$=\left(-\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right) \Big|$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{\sqrt{3}} (1, 1, 1)$$

$$=\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \Big|$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{21}{\sqrt{42}} \left(-\frac{1}{21}, \frac{5}{21}, -\frac{4}{21}\right)$$

$$=\left(-\frac{1}{\sqrt{42}}, \frac{5}{\sqrt{42}}, -\frac{4}{\sqrt{42}}\right) \Big|$$

$$\begin{aligned}\vec{q}_4 &= \frac{\vec{v}_4}{\|\vec{v}_4\|} \\ &= \underline{(0, 0, 0)}\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis for the subspaces of \mathbb{R}^m .

$$\{(2, 1, 1), (0, 3, -1), (3, -4, -2), (-1, -1, 3)\}$$

Solution

$$\underline{\vec{v}_1 = \vec{u}_1 = (2, 1, 1)}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (0, 3, -1) - \frac{(0, 3, -1) \cdot (2, 1, 1)}{6} (2, 1, 1) \\ &= (0, 3, -1) - \frac{1}{3} (2, 1, 1) \\ &= \underline{\left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3}\right)}\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(3, -4, -2) \cdot (2, 1, 1)}{6} (2, 1, 1) \\ &= \underline{(0, 0, 0)}\end{aligned}$$

$$\begin{aligned}\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{9}{84} \left[(3, -4, -2) \cdot \left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3}\right) \right] \left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3}\right) \\ &= \frac{3}{28} (-10) \left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3}\right) \\ &= \underline{\left(\frac{5}{7}, -\frac{20}{7}, \frac{10}{7}\right)}\end{aligned}$$

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= (3, -4, -2) - (0, 0, 0) - \left(\frac{5}{7}, -\frac{20}{7}, \frac{10}{7}\right) \\ &= \underline{\left(\frac{16}{7}, -\frac{8}{7}, -\frac{24}{7}\right)}\end{aligned}$$

$$\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(-1, -1, 3) \cdot (2, 1, 1)}{6} (2, 1, 1)$$

$$= (0, 0, 0) \mid$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{9}{84} \left[(-1, -1, 3) \cdot \left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right) \right] \left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right)$$

$$= \frac{3}{28} \left(-\frac{18}{3} \right) \left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right)$$

$$= \left(\frac{3}{7}, -\frac{12}{7}, \frac{6}{7} \right) \mid$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 = \frac{49}{896} \left[(-1, -1, 3) \cdot \left(\frac{16}{7}, -\frac{8}{7}, -\frac{24}{7} \right) \right] \left(\frac{16}{7}, -\frac{8}{7}, -\frac{24}{7} \right)$$

$$= \frac{7}{128} \left(-\frac{80}{7} \right) \left(\frac{16}{7}, -\frac{8}{7}, -\frac{24}{7} \right)$$

$$= \left(-\frac{10}{7}, \frac{5}{7}, \frac{15}{7} \right) \mid$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3$$

$$= (-1, -1, 3) - (0, 0, 0) - \left(\frac{3}{7}, -\frac{12}{7}, \frac{6}{7} \right) - \left(-\frac{10}{7}, \frac{5}{7}, \frac{15}{7} \right)$$

$$= (0, 0, 0) \mid$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{6}} (2, 1, 1)$$

$$= \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right) \mid$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{3}{2\sqrt{21}} \left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right)$$

$$= \left(-\frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, -\frac{2}{\sqrt{21}} \right) \mid$$

$$\begin{aligned}
\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
&= \frac{7}{8\sqrt{14}} \left(\frac{16}{7}, -\frac{8}{7}, -\frac{24}{7} \right) \\
&= \left(\frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, -\frac{3}{\sqrt{14}} \right) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{q}_4 &= \frac{\vec{v}_4}{\|\vec{v}_4\|} \\
&= (0, 0, 0) \Big|
\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an *orthogonal* basis for the subspaces of \mathbb{R}^m .

$$\vec{u}_1 = (1, 1, 0, -1), \quad \vec{u}_2 = (1, 3, 0, 1), \quad \vec{u}_3 = (4, 2, 2, 0)$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (1, 1, 0, -1) \Big|$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
&= (1, 3, 0, 1) - \frac{(1, 3, 0, 1) \cdot (1, 1, 0, -1)}{3} (1, 1, 0, -1) \\
&= (1, 3, 0, 1) - (1, 1, 0, -1) \\
&= (0, 2, 0, 2) \Big|
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(4, 2, 2, 0) \cdot (1, 1, 0, -1)}{3} (1, 1, 0, -1) \\
&= (2, 2, 0, -2) \Big|
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{(4, 2, 2, 0) \cdot (0, 2, 0, 2)}{8} (0, 2, 0, 2) \\
&= \frac{1}{2} (0, 2, 0, 2) \\
&= (0, 1, 0, 1) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
&= (4, 2, 2, 0) - (2, 2, 0, -2) - (0, 1, 0, 1) \\
&= (2, -1, 2, 1) \quad |
\end{aligned}$$

$$\begin{aligned}
\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
&= \frac{1}{\sqrt{3}}(1, 1, 0, -1) \\
&= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}} \right) \quad |
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \frac{1}{2\sqrt{2}}(0, 2, 0, 2) \\
&= \left(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \quad |
\end{aligned}$$

$$\begin{aligned}
\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
&= \frac{1}{\sqrt{10}}(2, -1, 2, 1) \\
&= \left(\frac{2}{\sqrt{10}}, -\frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right) \quad |
\end{aligned}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of \mathbb{R}^m .

$$\vec{u}_1 = (1, 1, 1, 1), \quad \vec{u}_2 = (1, 1, 2, 4), \quad \vec{u}_3 = (1, 2, -4, -3)$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (1, 1, 1, 1) \quad |$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
&= (1, 1, 2, 4) - \frac{(1, 1, 2, 4) \cdot (1, 1, 1, 1)}{4} (1, 1, 1, 1)
\end{aligned}$$

$$= (1, 1, 2, 4) - 2(1, 1, 1, 1)$$

$$= \underline{(-1, -1, 0, 2)}$$

$$\begin{aligned} \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(1, 2, -4, -3) \cdot (1, 1, 1, 1)}{4} (1, 1, 1, 1) \\ &= \underline{(-1, -1, -1, -1)} \end{aligned}$$

$$\begin{aligned} \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{(1, 2, -4, -3) \cdot (-1, -1, 0, 2)}{6} (-1, -1, 0, 2) \\ &= -\frac{3}{2}(-1, -1, 0, 2) \\ &= \underline{\left(\frac{3}{2}, \frac{3}{2}, 0, -3\right)} \end{aligned}$$

$$\begin{aligned} \vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= (1, 2, -4, -3) - (-1, -1, -1, -1) - \left(\frac{3}{2}, \frac{3}{2}, 0, -3\right) \\ &= \underline{\left(\frac{1}{2}, \frac{3}{2}, -3, 1\right)} \end{aligned}$$

$$\begin{aligned} \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{1}{2}(1, 1, 1, 1) \\ &= \underline{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)} \end{aligned}$$

$$\begin{aligned} \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{1}{\sqrt{6}}(-1, -1, 0, 2) \\ &= \underline{\left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}}\right)} \end{aligned}$$

$$\begin{aligned} \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= \frac{2}{\sqrt{50}}\left(\frac{1}{2}, \frac{3}{2}, -3, 1\right) \end{aligned}$$

$$= \frac{2}{5\sqrt{2}} \left(\frac{1}{2}, \frac{3}{2}, -3, 1 \right)$$

$$= \left(\frac{1}{5\sqrt{2}}, \frac{3}{5\sqrt{2}}, -\frac{6}{5\sqrt{2}}, \frac{2}{5\sqrt{2}} \right) \Big|$$

Exercise

Use the Gram-Schmidt process to find an **orthogonal** basis for the subspaces of \mathbb{R}^m .

$$\{(3, 4, 0, 0), (-1, 1, 0, 0), (2, 1, 0, -1), (0, 1, 1, 0)\}$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (3, 4, 0, 0) \Big|$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$= (-1, 1, 0, 0) - \frac{(-1, 1, 0, 0) \cdot (3, 4, 0, 0)}{25} (3, 4, 0, 0)$$

$$= (-1, 1, 0, 0) - \frac{1}{25} (3, 4, 0, 0)$$

$$= \left(-\frac{28}{25}, \frac{21}{25}, 0, 0 \right) \Big|$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{(2, 1, 0, -1) \cdot (3, 4, 0, 0)}{25} (3, 4, 0, 0)$$

$$= \frac{10}{25} (3, 4, 0, 0)$$

$$= \left(\frac{6}{5}, \frac{8}{5}, 0, 0 \right) \Big|$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{625}{1,225} \left[(2, 1, 0, -1) \cdot \left(-\frac{28}{25}, \frac{21}{25}, 0, 0 \right) \right] \left(-\frac{28}{25}, \frac{21}{25}, 0, 0 \right)$$

$$= \frac{25}{49} \left(-\frac{35}{25} \right) \left(-\frac{28}{25}, \frac{21}{25}, 0, 0 \right)$$

$$= \left(\frac{4}{5}, -\frac{3}{5}, 0, 0 \right) \Big|$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (2, 1, 0, -1) - \left(\frac{6}{5}, \frac{8}{5}, 0, 0 \right) - \left(\frac{4}{5}, -\frac{3}{5}, 0, 0 \right)$$

$$= \underline{(0, 0, 0, -1)}$$

$$\begin{aligned} \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 &= \frac{(0, 1, 1, 0) \cdot (3, 4, 0, 0)}{25} (3, 4, 0, 0) \\ &= \frac{4}{25} (3, 4, 0, 0) \\ &= \underline{\left(\frac{12}{25}, \frac{16}{25}, 0, 0 \right)} \end{aligned}$$

$$\begin{aligned} \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 &= \frac{25}{49} \left[(0, 1, 1, 0) \cdot \left(-\frac{28}{25}, \frac{21}{25}, 0, 0 \right) \right] \left(-\frac{28}{25}, \frac{21}{25}, 0, 0 \right) \\ &= \frac{25}{49} \left(\frac{21}{25} \right) \left(-\frac{28}{25}, \frac{21}{25}, 0, 0 \right) \\ &= \underline{\left(-\frac{12}{25}, \frac{9}{25}, 0, 0 \right)} \end{aligned}$$

$$\begin{aligned} \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 &= [(0, 1, 1, 0) \cdot (0, 0, 0, -1)] (0, 0, 0, -1) \\ &= \underline{(0, 0, 0, 0)} \end{aligned}$$

$$\begin{aligned} \vec{v}_4 &= \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3 \\ &= (0, 1, 1, 0) - \left(\frac{12}{25}, \frac{16}{25}, 0, 0 \right) - \left(-\frac{12}{25}, \frac{9}{25}, 0, 0 \right) - (0, 0, 0, 0) \\ &= \underline{(0, 0, 1, 0)} \end{aligned}$$

$$\begin{aligned} \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{1}{5} (3, 4, 0, 0) \\ &= \underline{\left(\frac{3}{5}, \frac{4}{5}, 0, 0 \right)} \end{aligned}$$

$$\begin{aligned} \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{25}{35} \left(-\frac{28}{25}, \frac{21}{25}, 0, 0 \right) \\ &= \underline{\left(-\frac{4}{5}, \frac{3}{5}, 0, 0 \right)} \end{aligned}$$

$$\begin{aligned}\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= (0, 0, 0, -1) \end{aligned}$$

$$\begin{aligned}\vec{q}_4 &= \frac{\vec{v}_4}{\|\vec{v}_4\|} \\ &= (0, 0, 1, 0) \end{aligned}$$

Exercise

Find the QR -decomposition of

$$a) \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

$$c) \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$$

$$e) \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$b) \begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix}$$

$$d) \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$$

Solution

a) Since $\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5 \neq 0$, The matrix is invertible

$$\vec{u}_1(1, 2), \quad \vec{u}_2 = (-1, 3)$$

$$\vec{v}_1 = \vec{u}_1 = (1, 2)$$

$$\begin{aligned}\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{(1, 2)}{\sqrt{1^2 + 2^2}} \\ &= \frac{(1, 2)}{\sqrt{5}} \\ &= \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) \end{aligned}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - (\vec{u}_2 \cdot \vec{v}_1) \vec{v}_1 \\ &= (-1, 3) - \left[(-1, 3) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) \right] \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)\end{aligned}$$

$$\begin{aligned}
&= (-1, 3) - \left(\frac{5}{\sqrt{5}}\right) \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \\
&= (-1, 3) - (1, 2) \\
&= \underline{(-2, 1)}
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \frac{(-2, 1)}{\sqrt{(-2)^2 + 1^2}} \\
&= \underline{\left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)}
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_1, \vec{q}_1 \rangle &= (1, 2) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \\
&= \frac{5}{\sqrt{5}} \\
&= \underline{\sqrt{5}}
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_2, \vec{q}_1 \rangle &= (-1, 3) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \\
&= \frac{5}{\sqrt{5}} \\
&= \underline{\sqrt{5}}
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_2, \vec{q}_2 \rangle &= (-1, 3) \cdot \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \\
&= \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{5}} \\
&= \underline{\sqrt{5}}
\end{aligned}$$

$$\begin{aligned}
R &= \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}
\end{aligned}$$

The QR -decomposition of the matrix is

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}$$

b) The column vectors of are: $\vec{u}_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$$\vec{v}_1 = \vec{u}_1 = (3, -4)$$

$$\begin{aligned} \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{(3, -4)}{\sqrt{9+16}} \\ &= \left(\frac{3}{5}, -\frac{4}{5}\right) \end{aligned}$$

$$\begin{aligned} \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= (5, 0) - \frac{(5, 0) \cdot (3, -4)}{25} (3, -4) \\ &= (5, 0) - \frac{15}{25} (3, -4) \\ &= (5, 0) - \frac{3}{5} (3, -4) \\ &= (5, 0) - \left(\frac{9}{5}, -\frac{12}{5}\right) \\ &= \left(\frac{16}{5}, \frac{12}{5}\right) \end{aligned}$$

$$\begin{aligned} \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{\left(\frac{16}{5}, \frac{12}{5}\right)}{\sqrt{\frac{256}{25} + \frac{144}{25}}} \\ &= \frac{1}{\sqrt{\frac{400}{25}}} \left(\frac{16}{5}, \frac{12}{5}\right) \\ &= \frac{1}{\sqrt{16}} \left(\frac{16}{5}, \frac{12}{5}\right) \\ &= \frac{1}{4} \left(\frac{16}{5}, \frac{12}{5}\right) \\ &= \left(\frac{4}{5}, \frac{3}{5}\right) \end{aligned}$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} 3\left(\frac{3}{5}\right) - 4\left(-\frac{4}{5}\right) & 5\left(\frac{3}{5}\right) + 0\left(-\frac{4}{5}\right) \\ 0 & 5\left(\frac{4}{5}\right) - 0\left(\frac{3}{5}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 \\ 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \cdot \begin{bmatrix} 5 & 3 \\ 0 & 4 \end{bmatrix}$$

$$\mathbf{A} = \mathbf{Q} \mathbf{R}$$

c) Since the column vectors $\vec{u}_1 (1, 0, 1)$, $\vec{u}_2 = (2, 1, 4)$ are linearly independent, so has a QR -decomposition.

$$\underline{\vec{v}_1 = \vec{u}_1 = (1, 0, 1) \mid}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1, 0, 1)}{\sqrt{1^2 + 0 + 1^2}}$$

$$= \frac{(1, 0, 1)}{\sqrt{2}}$$

$$= \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \mid$$

$$\vec{v}_2 = \vec{u}_2 - \langle \vec{u}_2, \vec{v}_1 \rangle \vec{v}_1$$

$$= (2, 1, 4) - \left[(2, 1, 4) \cdot \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \right] \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)$$

$$= (2, 1, 4) - \left(\frac{6}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)$$

$$= (2, 1, 4) - (3, 0, 3)$$

$$= (-1, 1, 1) \mid$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{(-1, 1, 1)}{\sqrt{(-1)^2 + 1^2 + 1^2}}$$

$$= \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \Big|$$

$$\begin{aligned} \langle \vec{u}_1, \vec{q}_1 \rangle &= (1, 0, 1) \cdot \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \\ &= \frac{2}{\sqrt{2}} \\ &= \sqrt{2} \Big| \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{q}_1 \rangle &= (2, 1, 4) \cdot \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \\ &= \frac{2}{\sqrt{2}} + \frac{4}{\sqrt{2}} \\ &= 3\sqrt{2} \Big| \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{q}_2 \rangle &= (2, 1, 4) \cdot \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\ &= -\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{4}{\sqrt{3}} \\ &= \frac{3}{\sqrt{3}} \\ &= \sqrt{3} \Big| \end{aligned}$$

$$\begin{aligned} R &= \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix} \end{aligned}$$

The QR -decomposition of the matrix is

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$$

$\mathbf{A} = \mathbf{Q} \mathbf{R}$

d) Since $\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{vmatrix} = -4 \neq 0,$

The matrix is invertible, so it has a QR -decomposition.

$$\vec{u}_1 = (1, 1, 0), \quad \vec{u}_2 = (2, 1, 3), \quad \vec{u}_3 = (1, 1, 1)$$

$$\vec{v}_1 = \vec{u}_1 = (1, 1, 0)$$

$$\begin{aligned} \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{(1, 1, 0)}{\sqrt{1^2 + 1^2 + 0}} \\ &= \frac{(1, 1, 0)}{\sqrt{2}} \\ &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \end{aligned}$$

$$\begin{aligned} \vec{v}_2 &= \vec{u}_2 - (\vec{u}_2 \cdot \vec{v}_1) \vec{v}_1 \\ &= (2, 1, 3) - \left[(2, 1, 3) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \right] \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\ &= (2, 1, 3) - \frac{3}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\ &= (2, 1, 3) - \left(\frac{3}{2}, \frac{3}{2}, 0 \right) \\ &= \left(\frac{1}{2}, -\frac{1}{2}, 3 \right) \end{aligned}$$

$$\begin{aligned} \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{\left(\frac{1}{2}, -\frac{1}{2}, 3 \right)}{\sqrt{\left(\frac{1}{2} \right)^2 + \left(-\frac{1}{2} \right)^2 + 3^2}} \\ &= \frac{\left(\frac{1}{2}, -\frac{1}{2}, 3 \right)}{\sqrt{\frac{19}{2}}} \\ &= \frac{\sqrt{2}}{\sqrt{19}} \left(\frac{1}{2}, -\frac{1}{2}, 3 \right) \\ &= \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}} \right) \end{aligned}$$

$$\vec{v}_3 = \vec{u}_3 - (\vec{u}_3 \cdot \vec{v}_1) \vec{v}_1 - (\vec{u}_3 \cdot \vec{v}_2) \vec{v}_2$$

$$\begin{aligned}
&= (1,1,1) - \left[(1,1,1) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \right] \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\
&\quad - \left[(1,1,1) \cdot \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}} \right) \right] \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}} \right) \\
&= (1,1,1) - \frac{2}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) - \frac{6}{\sqrt{38}} \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}} \right) \\
&= (1,1,1) - (1,1,0) - \left(\frac{3}{19}, -\frac{3}{19}, \frac{18}{19} \right) \\
&= \left(-\frac{3}{19}, \frac{3}{19}, \frac{1}{19} \right) \Big|
\end{aligned}$$

$$\begin{aligned}
\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
&= \frac{\left(-\frac{3}{19}, \frac{3}{19}, \frac{1}{19} \right)}{\sqrt{\left(-\frac{3}{19} \right)^2 + \left(\frac{3}{19} \right)^2 + \left(\frac{1}{19} \right)^2}} \\
&= \frac{19}{\sqrt{19}} \left(-\frac{3}{19}, \frac{3}{19}, \frac{1}{19} \right) \\
&= \left(-\frac{3}{\sqrt{19}}, \frac{3}{\sqrt{19}}, \frac{1}{\sqrt{19}} \right) \Big|
\end{aligned}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix}$$

$$\begin{aligned}
\langle \vec{u}_1, \vec{q}_1 \rangle &= (1, 1, 0) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\
&= \frac{2}{\sqrt{2}} \\
&= \sqrt{2} \Big|
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_2, \vec{q}_1 \rangle &= (2, 1, 3) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\
&= \frac{3}{\sqrt{2}} \Big|
\end{aligned}$$

$$\langle \vec{u}_2, \vec{q}_2 \rangle = (2, 1, 3) \cdot \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}} \right)$$

$$\begin{aligned}
&= \frac{2-1+18}{\sqrt{38}} \\
&= \frac{19}{\sqrt{38}} \\
&= \frac{19}{\sqrt{2}\sqrt{19}} \\
&= \frac{\sqrt{19}}{\sqrt{2}} \Big|
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{q}_1 \rangle &= (1, 1, 1) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\
&= \frac{2}{\sqrt{2}} \\
&= \sqrt{2} \Big|
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{q}_2 \rangle &= (1, 1, 1) \cdot \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}} \right) \\
&= \frac{1-1+6}{\sqrt{38}} \\
&= \frac{6}{\sqrt{2}\sqrt{19}} \frac{\sqrt{2}}{\sqrt{2}} \\
&= \frac{3\sqrt{2}}{\sqrt{19}} \Big|
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{q}_3 \rangle &= (1, 1, 1) \cdot \left(-\frac{3}{\sqrt{19}}, \frac{3}{\sqrt{19}}, \frac{1}{\sqrt{19}} \right) \\
&= \frac{-3+3+1}{\sqrt{19}} \\
&= \frac{1}{\sqrt{19}} \Big|
\end{aligned}$$

$$\begin{aligned}
R &= \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle & \langle \vec{u}_3, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle & \langle \vec{u}_3, \vec{q}_2 \rangle \\ 0 & 0 & \langle \vec{u}_3, \vec{q}_3 \rangle \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix}
\end{aligned}$$

The QR -decomposition of the matrix is

$$\begin{matrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix} \\ \mathbf{A} \end{matrix} = \begin{matrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix} \\ \mathbf{Q} \end{matrix} \begin{matrix} \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix} \\ \mathbf{R} \end{matrix}$$

$$e) \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \quad \begin{matrix} R_2 + R_1 \\ R_3 - R_1 \\ R_4 + R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \quad R_4 - R_2$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix is linearly dependent, so *doesn't* have a QR -decomposition.

Exercise

Verify that the Cauchy-Schwarz inequality holds for the given vectors using the Euclidean inner product

$$\vec{u} = (0, -2, 2, 1), \quad \vec{v} = (-1, -1, 1, 1)$$

Solution

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= 0 - 2(-1) + 2(1) + 1(1) \\ &= 5 \end{aligned}$$

$$\|\langle \vec{u}, \vec{v} \rangle\| = \sqrt{5}$$

$$\begin{aligned} \|\vec{u}\| \|\vec{v}\| &= \sqrt{0+4+4+1} \sqrt{1+1+1+1} \\ &= \sqrt{9} \sqrt{4} \\ &= 6 \end{aligned}$$

$$\sqrt{5} < 6 \Rightarrow \|\langle \vec{u}, \vec{v} \rangle\| \leq \|\vec{u}\| \|\vec{v}\|$$

Exercise

Apply the Gram-Schmidt *orthonormalization* process in $C^0[-1, 1]$ spanned by the functions, using the inner product $f_1(x) = x + 2$, $f_2(x) = x^2 - 3x + 4$

Solution

$$\text{Let } \vec{u}_1 = f_1 = x + 2, \quad \vec{u}_2 = f_2 = x^2 - 3x + 4$$

$$\underline{\vec{v}_1 = \vec{u}_1 = x + 2}$$

$$\begin{aligned} \langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^1 (x+2)^2 dx \\ &= \frac{1}{3}(x+2)^3 \Big|_{-1}^1 \\ &= \frac{1}{3}(27-1) \\ &= \frac{26}{3} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^1 (x^2 - 3x + 4)(x+2) dx \\ &= \int_{-1}^1 (x^3 - x^2 - 2x + 8) dx \\ &= \left(\frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 + 8x \right) \Big|_{-1}^1 \\ &= \frac{1}{4} - \frac{1}{3} - 1 + 8 - \frac{1}{4} - \frac{1}{3} + 1 + 8 \\ &= \frac{46}{3} \end{aligned}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$\vec{v}_2 = x^2 - 3x + 4 - \frac{46}{3} \left(\frac{3}{26} \right) (x+2)$$

$$= x^2 - 3x + 4 - \frac{23}{13}x - \frac{46}{13}$$

$$\underline{= x^2 - \frac{62}{13}x + \frac{6}{13}}$$

The orthogonal basis is $\left\{ x+2, \quad x^2 - \frac{62}{13}x + \frac{6}{13} \right\}$

$$\begin{aligned}
\langle \vec{v}_2, \vec{v}_2 \rangle &= \int_{-1}^1 \left(x^2 - \frac{62}{13}x + \frac{6}{13} \right)^2 dx \\
&= \frac{1}{169} \int_{-1}^1 \left(13x^2 - 62x + 6 \right)^2 dx \\
&= \frac{1}{169} \int_{-1}^1 \left(169x^4 + 3,844x^2 + 36 - 1,612x^3 + 156x^2 - 744x \right) dx \\
&= \frac{1}{169} \left(\frac{169}{5}x^5 + \frac{4,000}{3}x^3 + 36x - 403x^4 - 372x^2 \right) \Big|_{-1}^1 \\
&= \frac{1}{169} \left(\frac{169}{5} + \frac{4,000}{3} + 36 - 403 - 372 + \frac{169}{5} + \frac{4,000}{3} + 36 + 403 + 372 \right) \\
&= \frac{1}{169} \left(\frac{338}{5} + \frac{8,000}{3} + 72 \right) \\
&= \frac{3,238}{195}
\end{aligned}$$

$$\begin{aligned}
\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
&= \frac{\sqrt{3}}{\sqrt{26}}(x+2)
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \frac{\sqrt{195}}{\sqrt{3238}} \left(x^2 - \frac{62}{13}x + \frac{6}{13} \right)
\end{aligned}$$

The orthonormal basis is $\left\{ \frac{\sqrt{3}}{\sqrt{26}}(x+2), \frac{\sqrt{195}}{\sqrt{3238}} \left(x^2 - \frac{62}{13}x + \frac{6}{13} \right) \right\}$

Exercise

Apply the Gram-Schmidt *orthonormalization* process in $C^0[-1, 1]$ spanned by the functions, using the inner product $f_1(x) = x$, $f_2(x) = x^3$, $f_3(x) = x^5$

Solution

$$\text{Let } \vec{u}_1 = f_1 = x, \quad \vec{u}_2 = f_2 = x^3, \quad \vec{u}_3 = f_3 = x^5$$

$$\underline{\vec{v}_1 = \vec{u}_1 = x}$$

$$\begin{aligned} \langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^1 x^2 dx \\ &= \frac{1}{3} x^3 \Big|_{-1}^1 \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^1 x^4 dx \\ &= \frac{1}{5} x^5 \Big|_{-1}^1 \\ &= \frac{2}{5} \end{aligned}$$

$$\begin{aligned} \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= x^3 - \frac{2}{5} \left(\frac{3}{2} \right) (x) \\ &= x^3 - \frac{3}{5} x \end{aligned}$$

$$\begin{aligned} \langle \vec{v}_2, \vec{v}_2 \rangle &= \int_{-1}^1 \left(x^3 - \frac{3}{5} x \right)^2 dx \\ &= \int_{-1}^1 \left(x^6 - \frac{6}{5} x^4 + \frac{9}{25} x^2 \right) dx \\ &= \left(\frac{1}{7} x^7 - \frac{6}{25} x^5 + \frac{3}{25} x^3 \right) \Big|_{-1}^1 \\ &= 2 \left(\frac{1}{7} - \frac{6}{25} + \frac{3}{25} \right) \end{aligned}$$

$$= \frac{8}{175} \Big|$$

$$\begin{aligned} \langle \vec{u}_3, \vec{v}_1 \rangle &= \int_{-1}^1 x^6 \, dx \\ &= \frac{1}{7} x^7 \Big|_{-1}^1 \\ &= \frac{2}{7} \Big| \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_3, \vec{v}_2 \rangle &= \int_{-1}^1 x^5 \left(x^3 - \frac{3}{5}x \right) \, dx \\ &= \int_{-1}^1 \left(x^8 - \frac{3}{5}x^6 \right) \, dx \\ &= \left(\frac{1}{9}x^9 - \frac{3}{35}x^7 \right) \Big|_{-1}^1 \\ &= 2 \left(\frac{1}{9} - \frac{3}{35} \right) \\ &= \frac{16}{315} \Big| \end{aligned}$$

$$\begin{aligned} \vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= x^5 - \frac{16}{315} \left(\frac{175}{8} \right) \left(x^3 - \frac{3}{5}x \right) - \frac{2}{7} \left(\frac{3}{2} \right) x \\ &= x^5 - \frac{70}{63} \left(x^3 - \frac{3}{5}x \right) - \frac{3}{7}x \\ &= x^5 - \frac{70}{63}x^3 + \frac{14}{21}x - \frac{3}{7}x \\ &= x^5 - \frac{70}{63}x^3 + \frac{5}{21}x \Big| \end{aligned}$$

The orthogonal basis is $\left\{ x, \, x^3 - \frac{3}{5}x, \, x^5 - \frac{70}{63}x^3 + \frac{5}{21}x \right\}$

$$\begin{aligned} \langle \vec{v}_3, \vec{v}_3 \rangle &= \int_{-1}^1 \left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x \right)^2 \, dx \\ &= \int_{-1}^1 \frac{1}{3,969} \left(63x^5 - 70x^3 + 15x \right)^2 \, dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3,969} \int_{-1}^1 \left(3,969x^{10} - 8,820x^8 + 1,890x^6 - 2,100x^4 + 4,900x^6 + 225x^2 \right) dx \\
&= \frac{1}{3,969} \left(\frac{3,969}{11} x^{11} - 980x^9 + 970x^7 - 420x^5 + 75x^3 \right) \Big|_{-1}^1 \\
&= \frac{2}{3,969} \left(\frac{3,969}{11} - 980 + 970 - 420 + 75 \right) \\
&= \frac{2}{3,969} \left(\frac{3,969}{11} - 355 \right) \\
&= \frac{2}{3,969} \left(\frac{64}{11} \right) \\
&= \frac{128}{43,659}
\end{aligned}$$

$$\begin{aligned}
\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
&= \frac{x}{\sqrt{2/3}} \\
&= \frac{\sqrt{3}}{\sqrt{2}} x
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \sqrt{\frac{175}{8}} \left(x^3 - \frac{3}{5}x \right) \\
&= \frac{5\sqrt{7}}{2\sqrt{2}} \left(x^3 - \frac{3}{5}x \right)
\end{aligned}$$

$$\begin{aligned}
\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
&= \sqrt{\frac{43,659}{128}} \left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x \right) \\
&= \frac{63\sqrt{11}}{8\sqrt{2}} \left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x \right)
\end{aligned}$$

The orthonormal basis is $\left\{ \frac{\sqrt{3}}{\sqrt{2}}x, \frac{5}{2}\sqrt{\frac{7}{2}}\left(x^3 - \frac{3}{5}x\right), \frac{63}{8}\sqrt{\frac{11}{2}}\left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x\right) \right\}$

Exercise

Apply the Gram-Schmidt **orthonormalization** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $f_1(x) = 1, f_2(x) = x, f_3(x) = \frac{1}{2}(3x^2 - 1)$

Solution

$$\text{Let } \vec{u}_1 = f_1 = 1, \quad \vec{u}_2 = f_2 = x, \quad \vec{u}_3 = f_3 = \frac{3}{2}x^2 - \frac{1}{2}$$

$$\underline{\vec{v}_1 = \vec{u}_1 = 1}$$

$$\begin{aligned} \langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^1 1 \, dx \\ &= x \Big|_{-1}^1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^1 x \, dx \\ &= \frac{1}{2}x^2 \Big|_{-1}^1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= x - \frac{0}{2}(1) \\ &= x \end{aligned}$$

$$\begin{aligned} \langle \vec{v}_2, \vec{v}_2 \rangle &= \int_{-1}^1 x^2 \, dx \\ &= \frac{1}{3}x^3 \Big|_{-1}^1 \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_3, \vec{v}_1 \rangle &= \frac{1}{2} \int_{-1}^1 (3x^2 - 1) \, dx \\ &= \frac{1}{2} (x^3 - x) \Big|_{-1}^1 \end{aligned}$$

$$\underline{=0}$$

$$\begin{aligned}\langle \vec{u}_3, \vec{v}_2 \rangle &= \frac{1}{2} \int_{-1}^1 x(3x^2 - 1) \, dx \\ &= \frac{1}{2} \int_{-1}^1 (3x^3 - x) \, dx \\ &= \frac{1}{2} \left(\frac{3}{4}x^4 - \frac{1}{2}x^2 \right) \Big|_{-1}^1 \\ &\underline{=0}\end{aligned}$$

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= \frac{3}{2}x^2 - \frac{1}{2} - \frac{0}{1}(1) - \frac{0}{\frac{2}{3}}(x) \\ &\underline{= \frac{1}{2}(3x^2 - 1)}\end{aligned}$$

The orthogonal basis is $\left\{ 1, \, x, \, \frac{1}{2}(3x^2 - 1) \right\}$

$$\begin{aligned}\langle \vec{v}_3, \vec{v}_3 \rangle &= \frac{1}{4} \int_{-1}^1 (3x^2 - 1)^2 \, dx \\ &= \frac{1}{4} \int_{-1}^1 (9x^4 - 6x^2 + 1) \, dx \\ &= \frac{1}{4} \left(\frac{9}{5}x^5 - 2x^3 + x \right) \Big|_{-1}^1 \\ &= \frac{1}{2} \left(\frac{9}{5} - 2 + 1 \right) \\ &\underline{= \frac{2}{5}}\end{aligned}$$

$$\begin{aligned}\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &\underline{= \frac{1}{\sqrt{2}}}\end{aligned}$$

$$\begin{aligned}\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &\underline{= \sqrt{\frac{3}{2}} \, x}\end{aligned}$$

$$\begin{aligned}
 \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
 &= \frac{1}{\sqrt{\frac{2}{5}}} \frac{1}{2} (3x^2 - 1) \\
 &= \frac{1}{2} \sqrt{\frac{5}{2}} (3x^2 - 1)
 \end{aligned}$$

The orthonormal basis is $\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \frac{1}{2}\sqrt{\frac{5}{2}}(3x^2 - 1) \right\}$

Exercise

Apply the Gram-Schmidt **orthonormalization** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $f_1(x) = 1, f_2(x) = \sin \pi x, f_3(x) = \cos \pi x$

Solution

Let $\vec{u}_1 = f_1 = 1, \vec{u}_2 = f_2 = \sin \pi x, \vec{u}_3 = f_3 = \cos \pi x$

$$\vec{v}_1 = \vec{u}_1 = 1$$

$$\begin{aligned}
 \langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^1 1 dx \\
 &= x \Big|_{-1}^1 \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^1 \sin \pi x \, dx \\
 &= -\frac{1}{\pi} \cos \pi x \Big|_{-1}^1 \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
 &= \sin \pi x
 \end{aligned}$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^1 \sin^2 \pi x \, dx$$

$$\begin{aligned}
&= \frac{1}{2} \int_{-1}^1 (1 - \cos 2\pi x) \, dx \\
&= \frac{1}{2} \left(x - \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^1 \\
&= \underline{1} \mid
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{v}_1 \rangle &= \int_{-1}^1 \cos \pi x \, dx \\
&= \frac{1}{\pi} \sin \pi x \Big|_{-1}^1 \\
&= \underline{0} \mid
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{v}_2 \rangle &= \int_{-1}^1 \cos \pi x \sin \pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 \sin 2\pi x \, dx \\
&= -\frac{1}{4\pi} \cos 2\pi x \Big|_{-1}^1 \\
&= \underline{0} \mid
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
&= \cos \pi x - \underline{0} - \underline{0} \\
&= \underline{\cos \pi x} \mid
\end{aligned}$$

The orthogonal basis is $\left\{ \underline{1}, \sin \pi x - \frac{1}{\pi}, \underline{\cos \pi x} \right\}$

$$\begin{aligned}
\langle \vec{v}_3, \vec{v}_3 \rangle &= \int_{-1}^1 \cos^2 \pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 (1 + \cos 2\pi x) \, dx \\
&= \frac{1}{2} \left(x + \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^1 \\
&= \underline{1} \mid
\end{aligned}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{2}}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \sin \pi x$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \cos \pi x$$

The orthonormal basis is $\left\{ \frac{1}{\sqrt{2}}, \sin \pi x, \cos \pi x \right\}$

Exercise

Apply the Gram-Schmidt **orthonormalization** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $f_1(x) = \sin \pi x$, $f_2(x) = \sin 2\pi x$, $f_3(x) = \sin 3\pi x$

Solution

Let $\vec{u}_1 = f_1 = \sin \pi x$, $\vec{u}_2 = f_2 = \sin 2\pi x$, $\vec{u}_3 = f_3 = \sin 3\pi x$

$$\vec{v}_1 = \vec{u}_1 = \sin \pi x$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^1 \sin^2 \pi x \, dx$$

$$= \frac{1}{2} \int_{-1}^1 (1 - \cos 2\pi x) \, dx$$

$$= \frac{1}{2} \left(x - \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^1$$

$$= 1$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_{-1}^1 \sin \pi x \sin 2\pi x \, dx$$

$$= \frac{1}{2} \int_{-1}^1 (\cos 3\pi x - \cos(-\pi x)) \, dx$$

$$\sin a \sin b = \frac{1}{2} [\cos(a+b) - \cos(a-b)]$$

$$\begin{aligned}
&= \frac{1}{2} \int_{-1}^1 (\cos 3\pi x - \cos \pi x) \, dx \\
&= \frac{1}{2} \left(\frac{1}{3\pi} \sin 3\pi x - \frac{1}{\pi} \sin \pi x \right) \Big|_{-1}^1 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
&= \sin 2\pi x
\end{aligned}$$

$$\begin{aligned}
\langle \vec{v}_2, \vec{v}_2 \rangle &= \int_{-1}^1 \sin^2 2\pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 (1 - \cos 4\pi x) \, dx \\
&= \frac{1}{2} \left(x - \frac{1}{4\pi} \sin 4\pi x \right) \Big|_{-1}^1 \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{v}_1 \rangle &= \int_{-1}^1 \sin \pi x \sin 3\pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 (\cos 4\pi x - \cos(-2\pi x)) \, dx \\
&= \frac{1}{2} \int_{-1}^1 (\cos 4\pi x - \cos 2\pi x) \, dx \\
&= \frac{1}{2} \left(\frac{1}{4\pi} \sin 4\pi x - \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^1 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{v}_2 \rangle &= \int_{-1}^1 \sin 3\pi x \sin 2\pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 (\cos 5\pi x - \cos \pi x) \, dx \\
&= \frac{1}{2} \left(\frac{1}{5\pi} \sin 5\pi x - \frac{1}{\pi} \sin \pi x \right) \Big|_{-1}^1 \\
&= 0
\end{aligned}$$

$$\sin a \sin b = \frac{1}{2} [\cos(a+b) - \cos(a-b)]$$

$$\sin a \sin b = \frac{1}{2} [\cos(a+b) - \cos(a-b)]$$

$$\begin{aligned}\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\ &= \sin 3\pi x \quad | \end{aligned}$$

The orthogonal basis is $\{\sin \pi x, \sin 2\pi x, \sin 3\pi x\}$

$$\begin{aligned}\langle \vec{v}_3, \vec{v}_3 \rangle &= \int_{-1}^1 \sin^2 3\pi x \, dx \\ &= \frac{1}{2} \int_{-1}^1 (1 - \cos 6\pi x) \, dx \\ &= \frac{1}{2} \left(x - \frac{1}{6\pi} \sin 6\pi x \right) \Big|_{-1}^1 \\ &= 1 \quad | \end{aligned}$$

$$\begin{aligned}\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \sin \pi x \quad | \end{aligned}$$

$$\begin{aligned}\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \sin 2\pi x \quad | \end{aligned}$$

$$\begin{aligned}\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= \sin 3\pi x \quad | \end{aligned}$$

The orthonormal basis is $\{\sin \pi x, \sin 2\pi x, \sin 3\pi x\}$

Exercise

Apply the Gram-Schmidt **orthonormalization** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $f_1(x) = \cos \pi x$, $f_2(x) = \cos 2\pi x$, $f_3(x) = \cos 3\pi x$

Solution

Let $\vec{u}_1 = f_1 = \cos \pi x$, $\vec{u}_2 = f_2 = \cos 2\pi x$, $\vec{u}_3 = f_3 = \cos 3\pi x$

$$\vec{v}_1 = \vec{u}_1 = \cos \pi x \quad |$$

$$\begin{aligned}
\langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^1 \cos^2 \pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 (1 + \cos 2\pi x) \, dx \\
&= \frac{1}{2} \left(x + \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^1 \\
&= \underline{1}
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^1 \cos 2\pi x \cos \pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 (\cos 3\pi x + \cos \pi x) \, dx \\
&= \frac{1}{2} \left(\frac{1}{3\pi} \sin 3\pi x + \frac{1}{\pi} \sin \pi x \right) \Big|_{-1}^1 \\
&= \underline{0}
\end{aligned}$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
&= \underline{\cos 2\pi x}
\end{aligned}$$

$$\begin{aligned}
\langle \vec{v}_2, \vec{v}_2 \rangle &= \int_{-1}^1 \cos^2 2\pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 (1 + \cos 4\pi x) \, dx \\
&= \frac{1}{2} \left(x + \frac{1}{4\pi} \sin 4\pi x \right) \Big|_{-1}^1 \\
&= \underline{1}
\end{aligned}$$

$$\begin{aligned}
\langle \vec{u}_3, \vec{v}_1 \rangle &= \int_{-1}^1 \cos 3\pi x \cos \pi x \, dx \\
&= \frac{1}{2} \int_{-1}^1 (\cos 4\pi x + \cos 2\pi x) \, dx \\
&= \frac{1}{2} \left(\frac{1}{4\pi} \sin 4\pi x + \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^1
\end{aligned}$$

$$\cos a \cos b = \frac{1}{2} [\cos(a+b) + \cos(a-b)]$$

$$\cos a \cos b = \frac{1}{2} [\cos(a+b) + \cos(a-b)]$$

$$\underline{= 0}$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^1 \cos 3\pi x \cos 2\pi x \, dx$$

$$\cos a \cos b = \frac{1}{2} [\cos(a+b) + \cos(a-b)]$$

$$= \frac{1}{2} \int_{-1}^1 (\cos 5\pi x + \cos \pi x) \, dx$$

$$= \frac{1}{2} \left(\frac{1}{5\pi} \sin 5\pi x + \frac{1}{\pi} \sin \pi x \right) \Big|_{-1}^1$$

$$\underline{= 0}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$\underline{= \cos 3\pi x}$$

The orthogonal basis is $\{\cos \pi x, \cos 2\pi x, \cos 3\pi x\}$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_{-1}^1 \cos^2 3\pi x \, dx$$

$$= \frac{1}{2} \int_{-1}^1 (1 + \cos 6\pi x) \, dx$$

$$= \frac{1}{2} \left(x + \frac{1}{6\pi} \sin 6\pi x \right) \Big|_{-1}^1$$

$$\underline{= 1}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\underline{= \cos \pi x}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$\underline{= \cos 2\pi x}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$\underline{= \cos 3\pi x}$$

The orthonormal basis is $\{\cos \pi x, \cos 2\pi x, \cos 3\pi x\}$

Exercise

For $\mathbb{P}_3[x]$, define the inner product over \mathbb{R} as

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x) dx$$

a) If $f(x) = 1$ is a unit vector in $\mathbb{P}_3[x]$?

b) Find an orthonormal basis for the subspace spanned by x and x^2 .

c) Complete the basis in part (b) to an orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product.

d) Is

$$[f, g] = \int_0^1 f(x)g(x) dx$$

Also, an inner product for $\mathbb{P}_3[x]$

e) Find a pair of vectors \vec{v} and \vec{w} such that

$$\langle \vec{v}, \vec{w} \rangle = 0 \quad \text{but} \quad [\vec{v}, \vec{w}] \neq 0$$

f) Is the basis found in part (c) an orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product in part (d)?

Solution

a) $f(x) = 1$

$$\begin{aligned} \langle f, f \rangle &= \int_{-1}^1 f(x)f(x) dx \\ &= \int_{-1}^1 dx \\ &= x \Big|_{-1}^1 \\ &= 1 + 1 \\ &= 2 \neq 1 \end{aligned}$$

Therefore, when $f(x) = 1$ is **not** a unit vector in $\mathbb{P}_3[x]$

b) Let $\vec{u}_1 = f = x$, $\vec{u}_2 = g = x^2$

$$\underline{\vec{v}_1 = \vec{u}_1 = x}$$

$$\begin{aligned}
 \langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^1 x^2 \, dx \\
 &= \frac{1}{3} x^3 \Big|_{-1}^1 \\
 &= \frac{1}{3} (1+1) \\
 &= \frac{2}{3} \Big|
 \end{aligned}$$

$$\begin{aligned}
 \langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^1 x^2(x) \, dx \\
 &= \int_{-1}^1 x^3 \, dx \\
 &= \frac{1}{4} x^4 \Big|_{-1}^1 \\
 &= \frac{1}{4} (1-1) \\
 &= 0 \Big|
 \end{aligned}$$

$$\begin{aligned}
 \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\
 &= x^2 - \frac{0}{*} (\) \\
 &= x^2 \Big|
 \end{aligned}$$

$$\begin{aligned}
 \langle \vec{v}_2, \vec{v}_2 \rangle &= \int_{-1}^1 x^4 \, dx \\
 &= \frac{1}{5} x^5 \Big|_{-1}^1 \\
 &= \frac{1}{5} (1+1) \\
 &= \frac{2}{5} \Big|
 \end{aligned}$$

$$\begin{aligned}
 \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
 &= \frac{x}{\sqrt{\frac{2}{3}}}
 \end{aligned}$$

$$\left| = \sqrt{\frac{3}{2}} x \right|$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{x^2}{\sqrt{\frac{2}{5}}}$$

$$\left| = \sqrt{\frac{5}{2}} x^2 \right|$$

The orthonormal basis is $\left\{ \sqrt{\frac{3}{2}} x, \sqrt{\frac{5}{2}} x^2 \right\}$

c) Since $\vec{u}_1 = x$, $\vec{u}_2 = x^2$ in $\mathcal{P}_3[x]$

Then, let $\vec{u}_3 = 1$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_{-1}^1 (1)(x) dx$$

$$= \int_{-1}^1 x dx$$

$$= \frac{1}{2} x^2 \Big|_{-1}^1$$

$$= \frac{1}{2} (1 - 1)$$

$$= 0$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^1 (1)(x^2) dx$$

$$= \int_{-1}^1 x^2 dx$$

$$= \frac{1}{3} x^3 \Big|_{-1}^1$$

$$= \frac{1}{3} (1 + 1)$$

$$= \frac{2}{3}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= 1 - \frac{0}{\frac{2}{3}}(x) - \frac{\frac{2}{3}}{\frac{2}{5}}(x^2)$$

$$\underline{= 1 - \frac{5}{3}x^2}$$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_{-1}^1 \left(1 - \frac{5}{3}x^2\right)^2 dx$$

$$= \int_{-1}^1 \left(1 - \frac{10}{3}x^2 + \frac{25}{9}x^4\right) dx$$

$$= \left(x - \frac{10}{9}x^3 + \frac{5}{9}x^5\right) \Big|_{-1}^1$$

$$= 2\left(1 - \frac{10}{9} + \frac{5}{9}\right)$$

$$= 2\left(\frac{9-5}{9}\right)$$

$$\underline{= \frac{8}{9}}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \left(\sqrt{\frac{9}{8}}\right)\left(1 - \frac{5}{3}x^2\right)$$

$$= \frac{3}{2\sqrt{2}}\left(1 - \frac{5}{3}x^2\right)$$

$$\underline{= \frac{3}{2\sqrt{2}} - \frac{5}{2\sqrt{2}}x^2}$$

The orthonormal basis is $\left\{ \sqrt{\frac{3}{2}}x, \sqrt{\frac{5}{2}}x^2, \frac{3}{2\sqrt{2}} - \frac{5}{2\sqrt{2}}x^2 \right\}$

d) $[f, g] = \int_0^1 f(x)g(x) dx$

Let $\vec{u}_1 = 1, \vec{u}_2 = x, \vec{u}_3 = x^2$

$$\underline{\vec{v}_1 = \vec{u}_1 = 1}$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_0^1 1 dx$$

$$=x \Big|_0^1$$

$$\underline{=1}$$

$$\left\langle \vec{u}_2,\vec{v}_1\right\rangle =\int_0^1x(1)\;dx$$

$$=\int_0^1x\;dx$$

$$=\frac{1}{2}x^2\Big|_0^1$$

$$\underline{=\frac{1}{2}}$$

$$\vec{v}_2=\vec{u}_2-\frac{\left\langle \vec{u}_2,\vec{v}_1\right\rangle }{\left\| \vec{v}_1\right\| ^2}\vec{v}_1$$

$$=x-\frac{1}{2}(1)$$

$$\underline{=x-\frac{1}{2}}$$

$$\left\langle \vec{v}_2,\vec{v}_2\right\rangle =\int_0^1\left(x-\frac{1}{2}\right)^2\;dx$$

$$=\int_0^1\left(x-\frac{1}{2}\right)^2\;d\left(x-\frac{1}{2}\right)$$

$$=\frac{1}{3}\left(x-\frac{1}{2}\right)^3\Big|_0^1$$

$$=\frac{1}{3}\left(\left(\frac{1}{2}\right)^3-\left(-\frac{1}{2}\right)^3\right)$$

$$=\frac{1}{3}\left(\frac{1}{8}+\frac{1}{8}\right)$$

$$\underline{=\frac{1}{12}}$$

$$\left\langle \vec{u}_3,\vec{v}_1\right\rangle =\int_0^1\left(x^2\right)(1)\;dx$$

$$=\int_0^1x^2\;dx$$

$$= \frac{1}{3} x^3 \Big|_0^1$$

$$= \frac{1}{3} \Big|$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_0^1 \left(x^2 \right) \left(x - \frac{1}{2} \right) dx$$

$$= \int_0^1 \left(x^3 - \frac{1}{2} x^2 \right) dx$$

$$= \left(\frac{1}{4} x^4 - \frac{1}{6} x^3 \right) \Big|_0^1$$

$$= \frac{1}{4} - \frac{1}{6}$$

$$= \frac{1}{12} \Big|$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= x^2 - \frac{\frac{1}{3}}{1} (1) - \frac{\frac{1}{12}}{\frac{1}{12}} \left(x - \frac{1}{2} \right)$$

$$= x^2 - \frac{1}{3} - x + \frac{1}{2}$$

$$= x^2 - x + \frac{1}{6} \Big|$$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_0^1 \left(x^2 - x + \frac{1}{6} \right)^2 dx$$

$$= \int_0^1 \left(x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36} \right) dx$$

$$= \left(\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{4}{9}x^3 - \frac{1}{6}x^2 + \frac{1}{36}x \right) \Big|_0^1$$

$$= \frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36}$$

$$= \frac{2-5}{10} + \frac{16-6+1}{36}$$

$$= -\frac{3}{10} + \frac{11}{36}$$

$$= \frac{-108+110}{360}$$

$$= \frac{1}{180} \Big|$$

$$\begin{aligned} \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= 1 \Big| \end{aligned}$$

$$\begin{aligned} \vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{x - \frac{1}{2}}{\sqrt{\frac{1}{12}}} \\ &= 2\sqrt{3} \left(x - \frac{1}{2} \right) \Big| \end{aligned}$$

$$\begin{aligned} \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= (\sqrt{180}) \left(x^2 - x + \frac{1}{6} \right) \\ &= (6\sqrt{5}) \left(x^2 - x + \frac{1}{6} \right) \\ &= 6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5} \Big| \end{aligned}$$

The orthonormal basis is $\left\{ 1, 2\sqrt{3} \left(x - \frac{1}{2} \right), \sqrt{5} (6x^2 - 6x + 1) \right\}$

Therefore, $[f, g] = \int_0^1 f(x)g(x) dx$ is an inner product for $\mathbb{P}_3[x]$

e) Let assume: $\vec{v} = 1$ and $\vec{w} = x$

$$\begin{aligned} \langle \vec{v}, \vec{w} \rangle &= \int_{-1}^1 1(x) dx \\ &= \int_{-1}^1 x dx \\ &= \frac{1}{2}x^2 \Big|_{-1}^1 \\ &= \frac{1}{2}(1-1) \\ &= 0 \Big| \quad \checkmark \end{aligned}$$

$$\begin{aligned}
 [\vec{v}, \vec{w}] &= \int_0^1 1(x) \, dx \\
 &= \frac{1}{2} x^2 \Big|_0^1 \\
 &= \frac{1}{2} \neq 0 \quad \checkmark
 \end{aligned}$$

f) The orthonormal basis in part (c) $\left\{ \sqrt{\frac{3}{2}} x, \sqrt{\frac{5}{2}} x^2, \frac{3}{2\sqrt{2}} - \frac{5}{2\sqrt{2}} x^2 \right\}$ are ***not*** the same as
 the orthonormal basis in part (d) $\left\{ 1, 2\sqrt{3} \left(x - \frac{1}{2} \right), \sqrt{5} (6x^2 - 6x + 1) \right\}$

Solution **Section 3.4 – Orthogonal Matrices**

Exercise

Show that the matrix is orthogonal $A = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$

Solution

$$\begin{aligned} AA^T &= \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix} \begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \end{aligned}$$

$$\begin{aligned} A^T A &= \begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix} \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \end{aligned}$$

$$AA^T = A^T A = I$$

∴ A is an orthogonal

Exercise

Show that the matrix is orthogonal $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

Solution

$$\begin{aligned}
 AA^T &= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I
 \end{aligned}$$

$$\begin{aligned}
 A^T A &= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I
 \end{aligned}$$

$$AA^T = A^T A = I$$

$\therefore A$ is an orthogonal.

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution

$$\begin{aligned}
 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I
 \end{aligned}$$

$$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ is orthogonal with inverse } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Solution

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\therefore \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ is orthogonal with inverse } \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ (It is a standard matrix for a rotation of } 45^\circ \text{)}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

Solution

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}^T = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \\ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\therefore \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \text{ is orthogonal with inverse } \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$$

Solution

$$\begin{aligned} \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}^T &= \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \end{aligned}$$

$$\therefore \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \text{ is orthogonal with an inverse } \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$$

Solution

$$\begin{aligned} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}^T &= \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 7 \\ 1 & 3 & -5 \\ -1 & 4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 3 & & \\ & & \\ & & \end{pmatrix} \neq I \end{aligned}$$

$$\therefore \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix} \text{ is not an orthogonal}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Solution

$$\begin{aligned} \begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}^T &= \begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{2} \\ \\ \end{pmatrix} \neq I \end{aligned}$$

$$\text{Or } \|r_1\| = \sqrt{0+1^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{3}{2}} \neq 1 \quad \therefore A \text{ is not orthogonal}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Solution

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\therefore \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \text{ is orthogonal with inverse } \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

Solution

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\therefore \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \text{ is orthogonal with inverse } \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

Solution

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\therefore \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix} \text{ is orthogonal with inverse } \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{bmatrix}$$

Solution

$$\begin{aligned}
\|r_2\| &= \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{2}\right)^2} \\
&= \sqrt{\frac{1}{3} + \frac{1}{4}} \\
&= \sqrt{\frac{7}{12}} \neq 1
\end{aligned}$$

Or

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{5}{6} & & \\ & & & \end{pmatrix} \neq I
\end{aligned}$$

∴ The matrix is **not** an orthogonal

Exercise

Find a last column so that the resulting matrix is orthogonal

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \dots \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \dots \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & \dots \end{bmatrix}$$

Solution

$$\vec{q}_1 = \left[\frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{3}} \quad -\frac{1}{\sqrt{3}} \right]^T$$

$$\begin{aligned}
\|\vec{q}_1\| &= \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} \\
&= 1
\end{aligned}$$

$$\vec{q}_2 = \left[\frac{1}{\sqrt{6}} \quad \frac{1}{\sqrt{6}} \quad \frac{2}{\sqrt{6}} \right]^T$$

$$\begin{aligned}\|\vec{q}_2\| &= \sqrt{\frac{1}{6} + \frac{1}{6} + \frac{4}{6}} \\ &= 1\end{aligned}$$

$$\text{Let } \vec{q}_3 = [x \ y \ z]^T$$

$$\vec{q}_1 \cdot \vec{q}_3 = \frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y - \frac{1}{\sqrt{3}}z = 0 \rightarrow x + y - z = 0$$

$$\vec{q}_2 \cdot \vec{q}_3 = \frac{1}{\sqrt{6}}x + \frac{1}{\sqrt{6}}y - \frac{2}{\sqrt{6}}z = 0 \rightarrow x + y - 2z = 0$$

$$\begin{cases} x + y - z = 0 \\ x + y - 2z = 0 \end{cases} \rightarrow \underline{z = 0} \quad \text{and} \quad x + y = 0 \Rightarrow \underline{x = -y}$$

$$\vec{q}_3 = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}^T$$

Exercise

Determine if the given matrix is orthogonal. If it is, find its inverse

$$\begin{bmatrix} \frac{1}{9} & \frac{4}{5} & \frac{3}{7} \\ \frac{4}{9} & \frac{3}{5} & -\frac{2}{7} \\ \frac{8}{9} & -\frac{2}{5} & \frac{3}{7} \end{bmatrix}$$

Solution

$$\vec{q}_1 = \begin{bmatrix} \frac{1}{9} & \frac{4}{9} & \frac{8}{9} \end{bmatrix}^T \quad \vec{q}_2 = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} & -\frac{2}{5} \end{bmatrix}^T \quad \vec{q}_3 = \begin{bmatrix} \frac{3}{7} & -\frac{2}{7} & \frac{3}{7} \end{bmatrix}^T$$

$$\begin{aligned}\vec{q}_1 \cdot \vec{q}_2 &= \frac{4}{45} + \frac{12}{45} - \frac{16}{45} \\ &= 0\end{aligned}$$

$$\begin{aligned}\vec{q}_1 \cdot \vec{q}_3 &= \frac{3}{63} - \frac{8}{63} + \frac{24}{63} \\ &= \frac{19}{63} \neq 0\end{aligned}$$

$$\begin{aligned}\vec{q}_2 \cdot \vec{q}_3 &= \frac{12}{35} - \frac{6}{35} + \frac{6}{35} \\ &= \frac{12}{35} \neq 0\end{aligned}$$

The given matrix is **not** orthogonal

Exercise

Prove that if A is orthogonal, then A^T is orthogonal.

Solution

Since A is orthogonal then $A^T = A^{-1}$ and $A = (A^T)^T$

Then $(A^T)^T A^T = AA^T = I \Rightarrow A^T$ is orthogonal

Another word, since A is orthogonal, then both column and row vectors of A form an orthonormal set.

A^T is just A with its row and column vectors are swapped.

The column vectors of A^T (which are the row vectors of A) and row vectors of A^T (which are the column vectors of A) form orthonormal sets, therefore A^T is orthogonal

Exercise

Prove that if A is orthogonal, then A^{-1} is orthogonal

Solution

Since A is orthogonal then $A^T = A^{-1}$ and $A = (A^{-1})^{-1}$

$$\begin{aligned} (A^{-1})^{-1} &= (A^T)^{-1} & A^T &= A^{-1} \\ &= (A^{-1})^T \end{aligned}$$

$\therefore A^{-1}$ is orthogonal

Exercise

Prove that if A and B are orthogonal, then AB is orthogonal

Solution

Since A is orthogonal then $A^T = A^{-1}$

and B is orthogonal then $B^T = B^{-1}$

$$\begin{aligned} (AB)^T &= B^T A^T \\ &= B^{-1} A^{-1} \\ &= (AB)^{-1} \end{aligned}$$

$\therefore AB$ is orthogonal

Exercise

Let Q be an $n \times n$ orthogonal matrix, and let A be an $n \times n$ matrix.

Show that $\det(QAQ^T) = \det(A)$

Solution

$$\begin{aligned}\det(QAQ^T) &= \det(Q)\det(A)\det(Q^T) \\ &= \det(A)\det(QQ^T) \quad \text{Since } Q \text{ is an orthogonal matrix } \det(QQ^T) = \det(I) \\ &= \det(A)\det(I) \\ &= \det(A) \quad \checkmark\end{aligned}$$

Exercise

$$\text{Let } A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$$

- a) Is matrix A an orthogonal matrix?
- b) Let B be the matrix obtained by normalizing each row of A , find B .
- c) Is B an orthogonal matrix?
- d) Are the columns of B orthogonal?

Solution

$$\begin{aligned}a) \quad AA^T &= \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 7 \\ 1 & 3 & -5 \\ -1 & 4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 3 & & \\ & & \\ & & \end{pmatrix} \neq I\end{aligned}$$

$$\therefore \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix} \text{ is } \textbf{not} \text{ an orthogonal}$$

$$\begin{aligned}b) \quad \|(1, 1, -1)\| &= \sqrt{1+1+1} \\ &= \sqrt{3}\end{aligned}$$

$$\begin{aligned}\|(1, 3, 4)\| &= \sqrt{1+9+16} \\ &= \sqrt{26}\end{aligned}$$

$$\begin{aligned}\|(7, -5, 2)\| &= \sqrt{49 + 25 + 4} \\ &= \sqrt{78}\end{aligned}$$

$$B = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{26}} & \frac{3}{\sqrt{26}} & \frac{4}{\sqrt{26}} \\ \frac{7}{\sqrt{78}} & -\frac{5}{\sqrt{78}} & \frac{2}{\sqrt{78}} \end{pmatrix}$$

c) Yes, since the rows are orthogonal with unit vectors.

$$\begin{aligned}BB^T &= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{26}} & \frac{3}{\sqrt{26}} & \frac{4}{\sqrt{26}} \\ \frac{7}{\sqrt{78}} & -\frac{5}{\sqrt{78}} & \frac{2}{\sqrt{78}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{26}} & \frac{7}{\sqrt{78}} \\ \frac{1}{\sqrt{3}} & \frac{3}{\sqrt{26}} & -\frac{5}{\sqrt{78}} \\ -\frac{1}{\sqrt{3}} & \frac{4}{\sqrt{26}} & \frac{2}{\sqrt{78}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I\end{aligned}$$

d) Yes, since the rows of B form an orthonormal set of vectors. Then, the column of B must form an orthonormal set.

$$\begin{aligned}\left\|\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{26}}, \frac{7}{\sqrt{78}}\right)\right\| &= \sqrt{\frac{1}{3} + \frac{1}{26} + \frac{49}{78}} \\ &= \sqrt{\frac{26 + 3 + 49}{78}} \\ &= \sqrt{\frac{78}{78}} \\ &= 1\end{aligned}$$

$$\begin{aligned}\left\|\left(\frac{1}{\sqrt{3}}, \frac{3}{\sqrt{26}}, -\frac{5}{\sqrt{78}}\right)\right\| &= \sqrt{\frac{1}{3} + \frac{9}{26} + \frac{25}{78}} \\ &= \sqrt{\frac{26 + 27 + 25}{78}} \\ &= \sqrt{\frac{78}{78}} \\ &= 1\end{aligned}$$

$$\left\|\left(-\frac{1}{\sqrt{3}}, \frac{4}{\sqrt{26}}, \frac{2}{\sqrt{78}}\right)\right\| = \sqrt{\frac{1}{3} + \frac{16}{26} + \frac{4}{78}}$$

$$= \sqrt{\frac{78}{78}}$$

$$= \underline{1}$$

Solution **Section 3.5 – Least Squares Analysis**

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(0, 2), (1, 2), (2, 0)\}$$

Solution

$$\{(0, 2), (1, 2), (2, 0)\}$$

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\text{where } A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} m \\ b \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

The normal equation formula: $A^T A \vec{x} = A^T \vec{y}$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$m = \frac{\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 3 & 3 \end{vmatrix}}$$

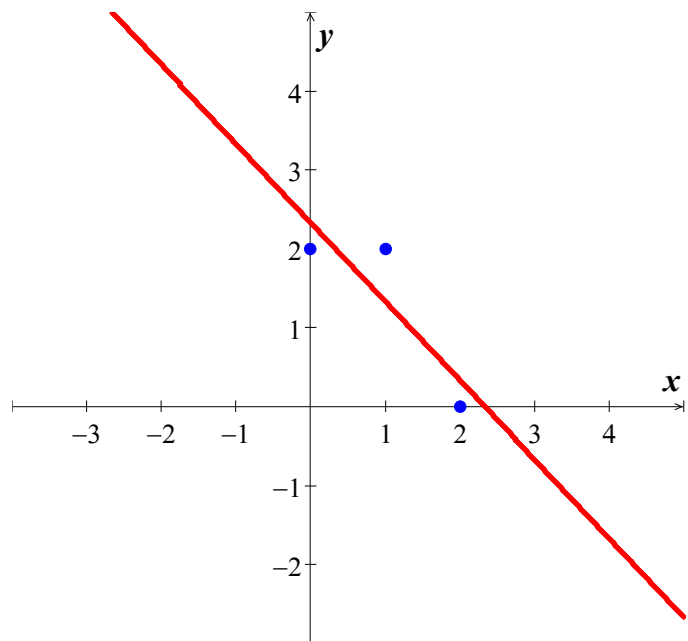
$$= \frac{-6}{6}$$

$$= -1$$

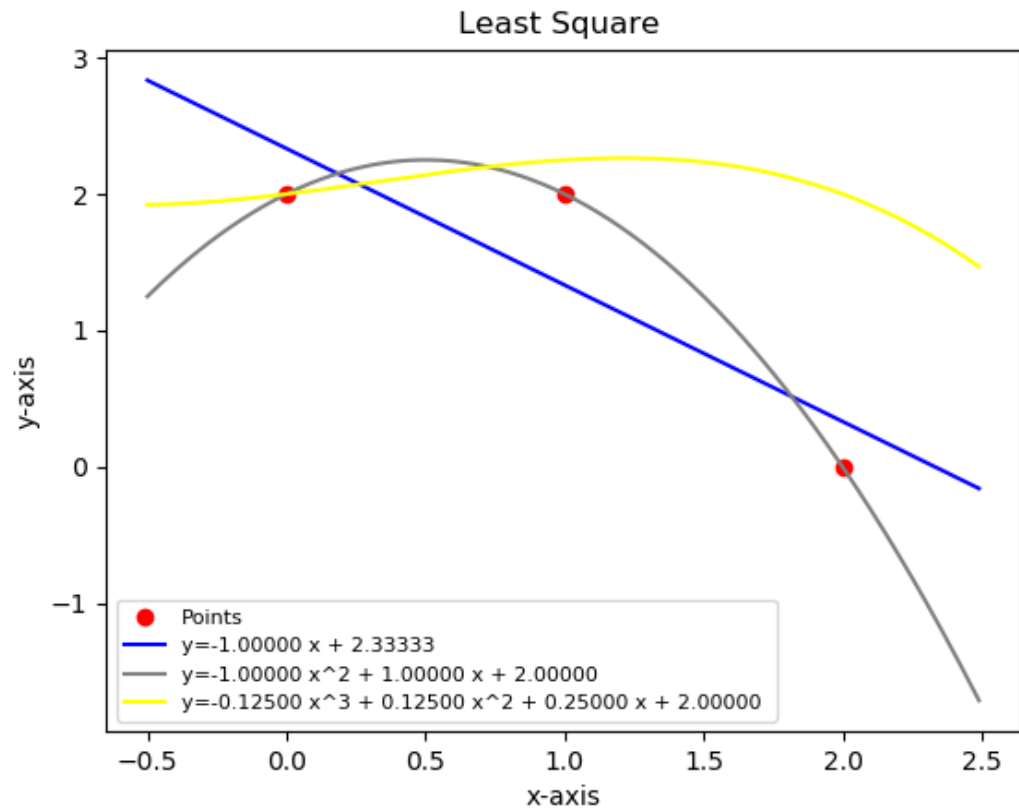
$$b = \frac{\begin{vmatrix} 5 & 2 \\ 3 & 4 \end{vmatrix}}{6}$$

$$= \frac{7}{3}$$

$$\text{Thus, } y = -x + \frac{7}{3}$$



$$\begin{aligned}
 A\vec{x} &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ \frac{7}{3} \end{pmatrix} \\
 &= \begin{pmatrix} \frac{7}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{pmatrix} \\
 \vec{y} - A\vec{x} &= \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{7}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}
 \end{aligned}$$



Error:

$$\begin{aligned}
 \|\vec{y} - A\vec{x}\| &= \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{1}{9}} \\
 &= \frac{\sqrt{6}}{3} \\
 &\approx 0.8164966
 \end{aligned}$$

The **second order** equation:

$$y = -x^2 + x + 2$$

Error = 0.00000

The **third order** equation:

$$y = -0.1250x^3 - 0.1250x^2 + 0.25x + 2$$

Error = 2.01556

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(1, 5), (2, 4), (3, 1), (4, 1), (5, -1)\}$$

Solution

$$\{(1, 5), (2, 4), (3, 1), (4, 1), (5, -1)\}$$

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{where } A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} m \\ b \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

The normal equation: $A^T A \vec{x} = A^T \vec{y}$

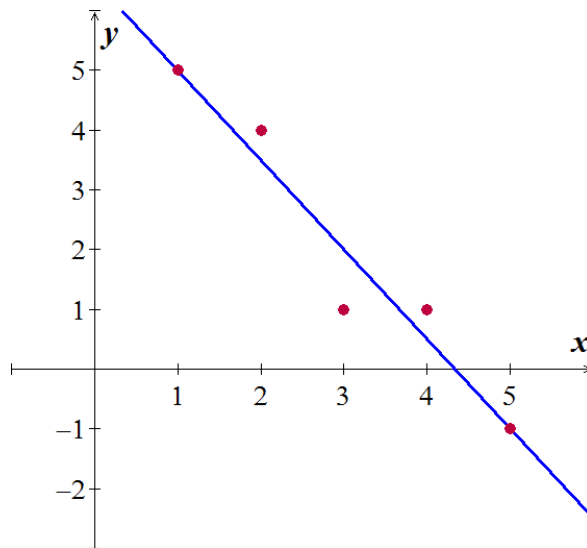
$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 55 & 15 \\ 15 & 5 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \end{bmatrix}$$

$$m = \frac{\begin{vmatrix} 15 & 15 \\ 10 & 5 \end{vmatrix}}{\begin{vmatrix} 55 & 15 \\ 15 & 5 \end{vmatrix}} = \frac{-75}{50} = -\frac{3}{2}$$

$$b = \frac{\begin{vmatrix} 55 & 15 \\ 15 & 10 \end{vmatrix}}{50} = \frac{325}{50} = \frac{13}{2}$$

$$\text{Thus, } \underline{y = -\frac{3}{2}x + \frac{13}{2}} \quad \text{or} \quad \underline{y = -1.5x + 6.5}$$

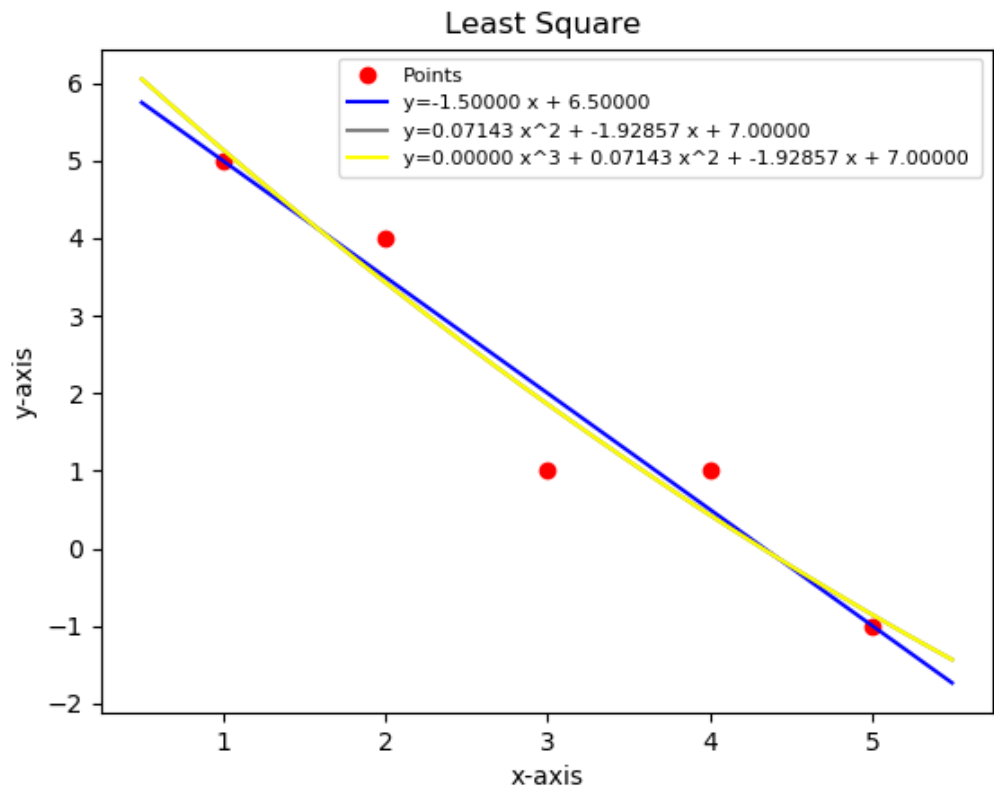


$$A\vec{x} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ \frac{13}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ \frac{7}{2} \\ 2 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ \frac{7}{2} \\ 2 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \frac{1}{2} \\ -1 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$



Error:

$$\begin{aligned} \|\vec{y} - A\vec{x}\| &= \sqrt{\frac{1}{4} + 1 + \frac{1}{4}} \\ &= \frac{\sqrt{6}}{2} \\ &\approx 1.224745 \end{aligned}$$

The **second order** equation:

$$y = 0.07143x^2 - 1.92857x + 7$$

Error = 1.19523

The **third order** equation:

$$y = 0.0x^3 + 0.07143x^2 - 1.92857x + 7$$

Error = 1.19523

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(0, 1), (1, 3), (2, 4), (3, 4)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

The normal equation: $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

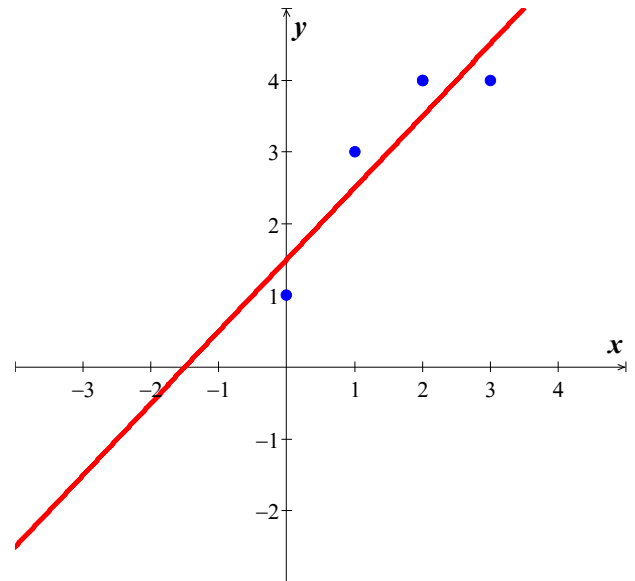
$$\begin{pmatrix} 14 & 6 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 23 \\ 12 \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} m \\ b \end{pmatrix} &= \frac{1}{20} \begin{pmatrix} 4 & -6 \\ -6 & 14 \end{pmatrix} \begin{pmatrix} 23 \\ 12 \end{pmatrix} & X = A^{-1}B \\ &= \frac{1}{20} \begin{pmatrix} 20 \\ 30 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix} \end{aligned}$$

We have: $m = 1$ and $b = \frac{3}{2}$.

Thus, $y = x + \frac{3}{2}$

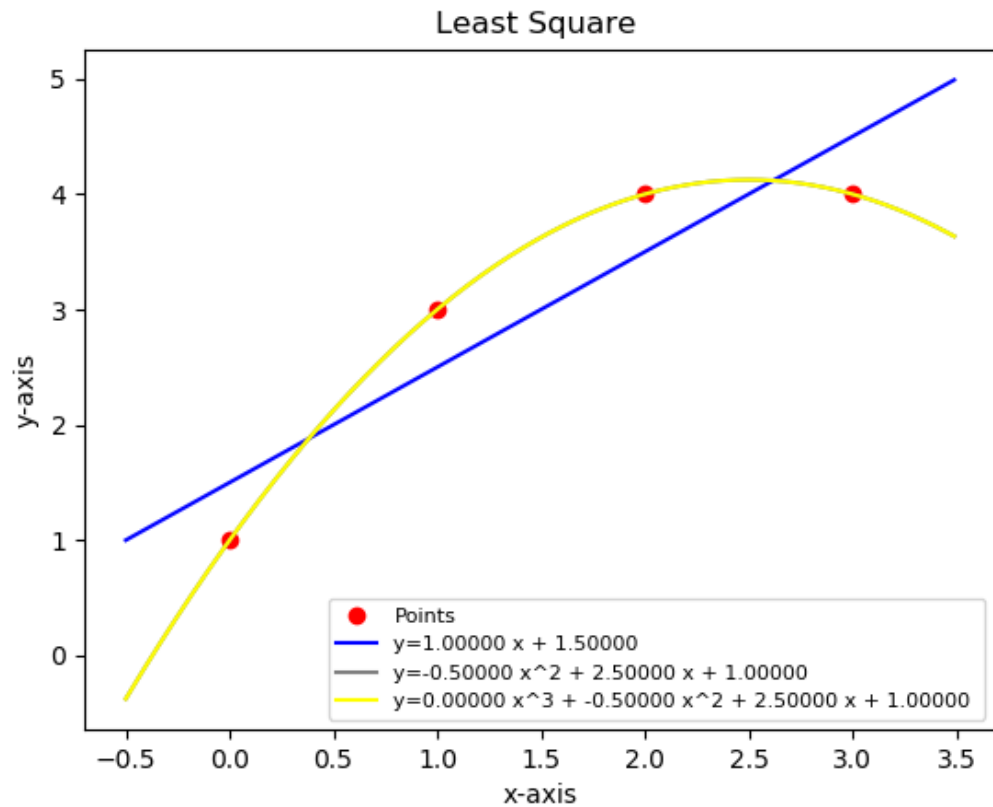
$$A\vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$$



$$= \begin{pmatrix} \frac{3}{2} \\ \frac{5}{2} \\ \frac{7}{2} \\ \frac{9}{2} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} \frac{3}{2} \\ \frac{5}{2} \\ \frac{7}{2} \\ \frac{9}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$



Error:

$$\|\vec{y} - A\vec{x}\| = \sqrt{4\left(\frac{1}{4}\right)}$$

$$= 1$$

The **second order** equation:

$$y = -0.50x^2 + 2.5x + 1.0$$

Error = 0.00000

The **third order** equation:

$$y = 0.0x^3 - 0.5x^2 + 2.5x + 1$$

Error = 0.00000

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-2, 0), (-1, 0), (0, 1), (1, 3), (2, 5)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

The normal equation: $A^T A \vec{x} = A^T \vec{y}$

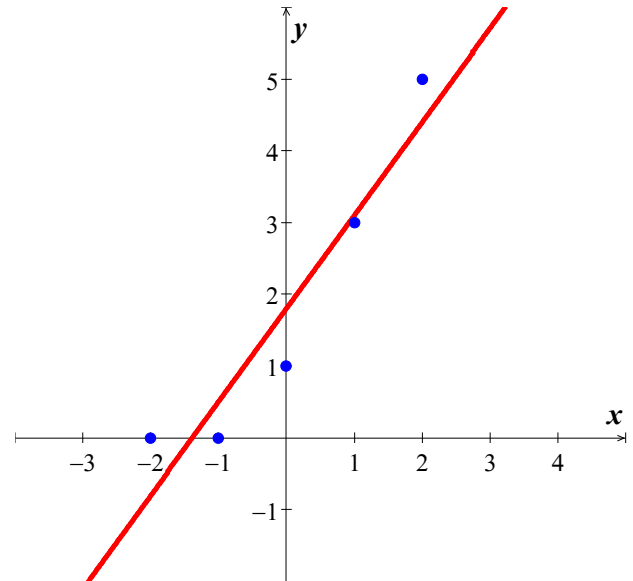
$$\begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 13 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{50} \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} 13 \\ 9 \end{pmatrix} = \begin{pmatrix} \frac{13}{10} \\ \frac{9}{5} \end{pmatrix}$$

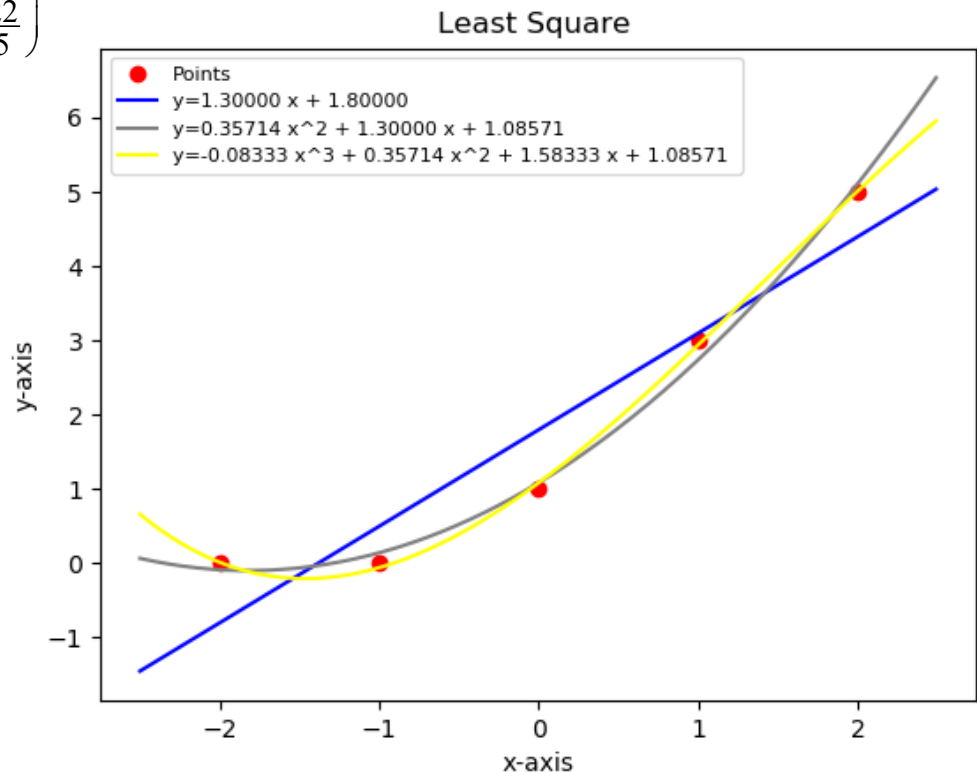
We have: $m = 1.3$ and $b = 1.8$

Thus, $y = \frac{13}{10}x + \frac{9}{5}$



$$A\vec{x} = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{13}{10} \\ \frac{9}{5} \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} \\ \frac{1}{2} \\ \frac{9}{5} \\ \frac{31}{10} \\ \frac{22}{5} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix} - \begin{pmatrix} -\frac{4}{5} \\ \frac{1}{2} \\ \frac{9}{5} \\ \frac{31}{10} \\ \frac{22}{5} \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ -\frac{1}{2} \\ -\frac{4}{5} \\ -\frac{1}{10} \\ \frac{3}{5} \end{pmatrix}$$



Error: $\|\vec{y} - A\vec{x}\| = \sqrt{\frac{16}{25} + \frac{1}{4} + \frac{16}{25} + \frac{1}{100} + \frac{9}{25}}$

$$= \sqrt{\frac{41}{25} + \frac{26}{100}}$$

$$= \frac{\sqrt{190}}{10}$$

$$\approx 1.37840$$

The *second order* equation:

$$y = 0.35714x^2 + 1.30x + 1.08571$$

Error = 0.33806

The *third order* equation:

$$y = -0.08333x^3 + 0.35714x^2 + 1.58333x + 1.08571$$

Error = 0.11952

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(2, 3), (3, 2), (5, 1), (6, 0)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

The normal equation: $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 2 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 74 & 16 \\ 16 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 17 \\ 6 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 74 & 16 \\ 16 & 4 \end{vmatrix} = 40 \quad \Delta_m = \begin{vmatrix} 17 & 16 \\ 6 & 4 \end{vmatrix} = -28 \quad \Delta_b = \begin{vmatrix} 74 & 17 \\ 16 & 6 \end{vmatrix} = 172$$

$$m = -\frac{28}{40} = -\frac{7}{10}$$

$$b = \frac{172}{40} = \frac{43}{10}$$

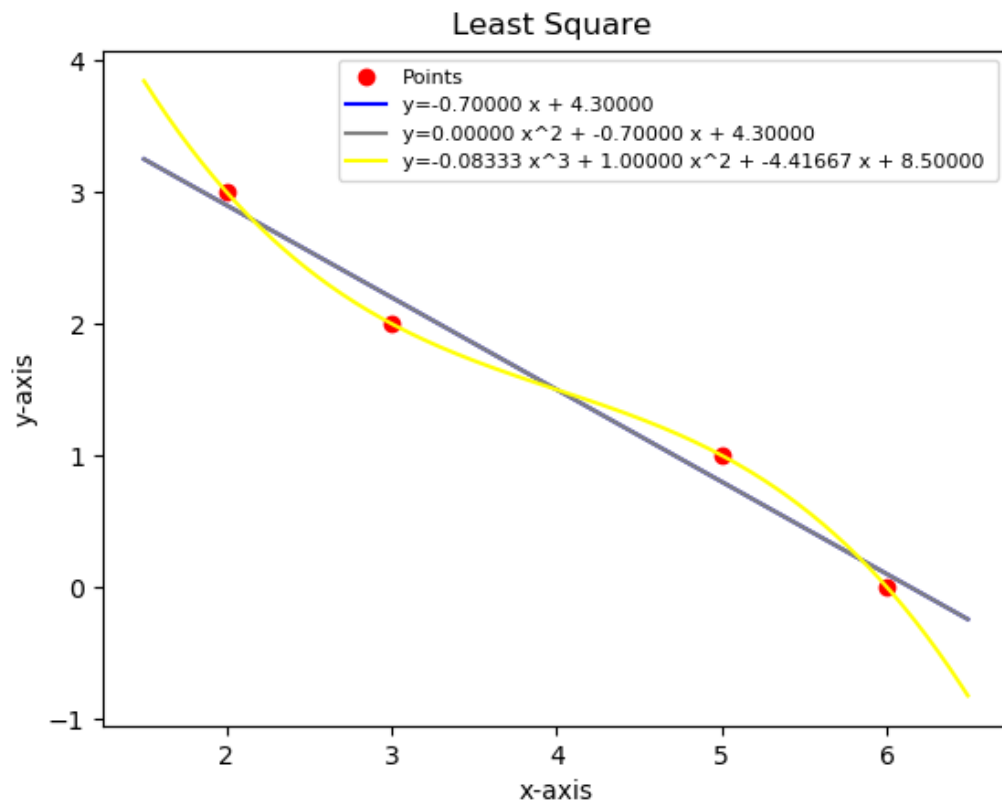
$$\text{Thus, } y = -\frac{7}{10}x + \frac{43}{10}$$

$$A\vec{x} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} -\frac{7}{10} \\ \frac{43}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{29}{10} \\ \frac{22}{10} \\ \frac{8}{10} \\ \frac{1}{10} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{29}{10} \\ \frac{22}{10} \\ \frac{8}{10} \\ \frac{1}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{10} \\ -\frac{1}{5} \\ \frac{1}{5} \\ -\frac{1}{10} \end{pmatrix}$$



Error:

$$\|\vec{y} - A\vec{x}\| = \sqrt{2\left(\frac{1}{100} + \frac{1}{25}\right)}$$

$$= \frac{\sqrt{10}}{10}$$

$$= 0.31623$$

The **second order** equation:

$$y = 0.0x^2 - 0.7x + 4.3$$

$$\text{Error} = 0.31623$$

The **third order** equation:

$$y = -0.08333x^3 + x^2 - 4.41667x + 8.5$$

$$\text{Error} = 0.00000$$

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-1, 0), (0, 1), (1, 2), (2, 4)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$$

The normal equation: $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 10 \\ 7 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 6 & 2 \\ 2 & 4 \end{vmatrix} = 20 \quad \Delta_m = \begin{vmatrix} 10 & 2 \\ 7 & 4 \end{vmatrix} = 26 \quad \Delta_b = \begin{vmatrix} 6 & 10 \\ 2 & 7 \end{vmatrix} = 22$$

$$m = \frac{26}{20} = \frac{13}{10}$$

$$b = \frac{22}{20} = \frac{11}{10}$$

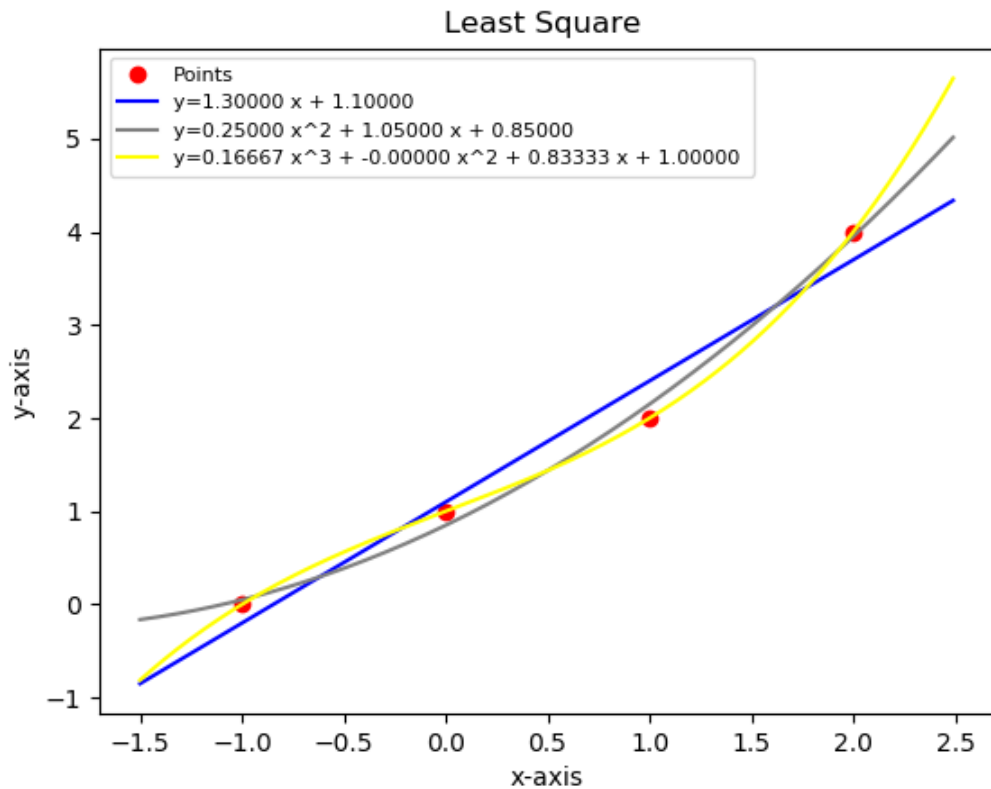
$$\text{Thus, } y = \frac{13}{10}x + \frac{11}{10}$$

$$A\vec{x} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{13}{10} \\ \frac{11}{10} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{5} \\ \frac{11}{10} \\ \frac{12}{5} \\ \frac{37}{10} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} -\frac{1}{5} \\ \frac{11}{10} \\ \frac{12}{5} \\ \frac{37}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} \\ -\frac{1}{10} \\ -\frac{2}{5} \\ \frac{3}{10} \end{pmatrix}$$



Error:

$$\|\vec{y} - A\vec{x}\| = \sqrt{\frac{1}{25} + \frac{1}{100} + \frac{4}{25} + \frac{9}{100}}$$

$$= \sqrt{\frac{4+1+16+9}{100}}$$

$$= \frac{\sqrt{30}}{10}$$

$$= 0.54772$$

The **second order** equation:

$$y = 0.25x^2 + 1.05x + 0.85$$

$$\text{Error} = 0.22361$$

The **third order** equation:

$$y = 0.16667x^3 + 0.82222x + 1$$

$$\text{Error} = 0.00000$$

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(1, 0), (2, 1), (4, 2), (5, 3)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

The normal equation: $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 1 & 2 & 4 & 5 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 46 & 12 \\ 12 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 25 \\ 6 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 46 & 12 \\ 12 & 4 \end{vmatrix} = 40 \quad \Delta_m = \begin{vmatrix} 25 & 12 \\ 6 & 4 \end{vmatrix} = 28 \quad \Delta_b = \begin{vmatrix} 46 & 25 \\ 12 & 6 \end{vmatrix} = -24$$

$$m = \frac{28}{40} = \frac{7}{10}$$

$$b = -\frac{24}{40} = -\frac{3}{5}$$

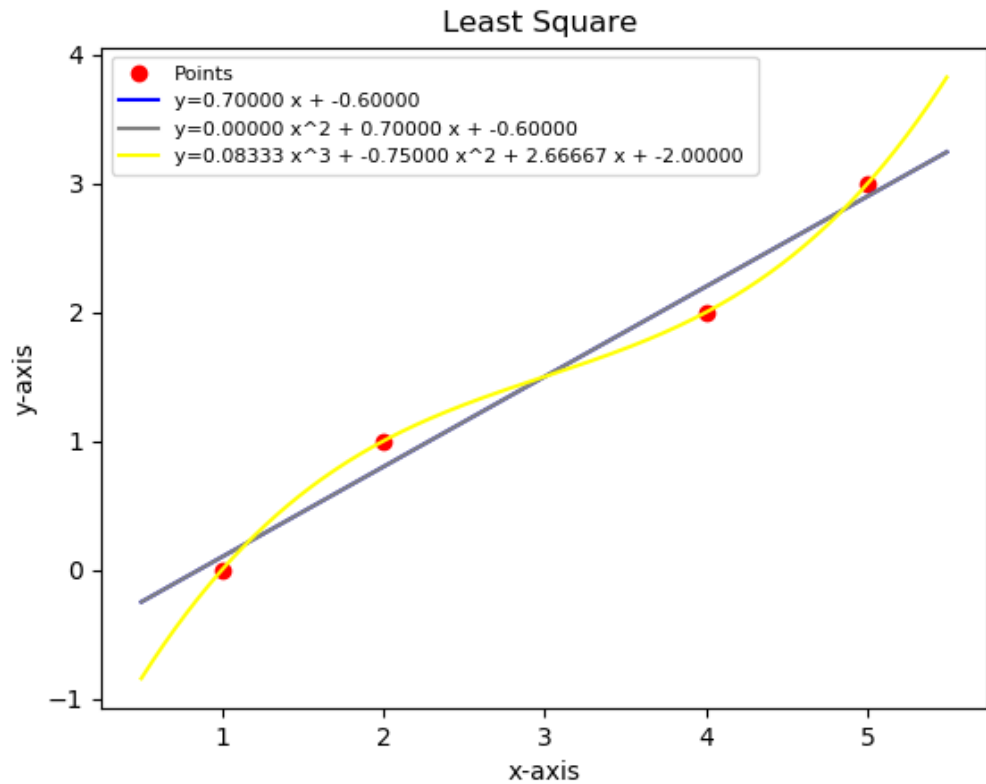
$$\text{Thus, } y = \frac{7}{10}x - \frac{3}{5}$$

$$A\vec{x} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} \frac{7}{10} \\ -\frac{3}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{10} \\ \frac{4}{5} \\ \frac{11}{5} \\ \frac{29}{10} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} \frac{1}{10} \\ \frac{4}{5} \\ \frac{11}{5} \\ \frac{29}{10} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{10} \\ \frac{1}{5} \\ -\frac{1}{5} \\ \frac{1}{10} \end{pmatrix}$$



Error:

$$\|\vec{y} - A\vec{x}\| = \sqrt{2\left(\frac{1}{100} + \frac{1}{25}\right)}$$

$$= \frac{\sqrt{10}}{10}$$

$$= 0.31623$$

The **second order** equation:

$$y = 0.0x^2 + 0.7x - .6$$

$$\text{Error} = 0.31623$$

The **third order** equation:

$$y = 0.08333x^3 - 0.75x^2 + 2.66667x - 2$$

$$\text{Error} = 0.00000$$

Exercise

Find the orthogonal projection of the vector \vec{u} on the subspace of \mathbb{R}^4 spanned by the vectors

$$\vec{u} = (-3, -3, 8, 9); \quad \vec{v}_1 = (3, 1, 0, 1), \quad \vec{v}_2 = (1, 2, 1, 1), \quad \vec{v}_3 = (-1, 0, 2, -1)$$

Solution

$$\text{Let } A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{pmatrix}$$

$$A^T \vec{u} = \begin{pmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ -3 \\ 8 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 8 \\ 10 \end{pmatrix}$$

The normal solution is $A^T A \vec{x} = A^T \vec{u}$

$$\begin{pmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \\ 10 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{vmatrix} = 134$$

$$\Delta_1 = \begin{vmatrix} -3 & 6 & -4 \\ 8 & 7 & 0 \\ 10 & 0 & 6 \end{vmatrix} = -134$$

$$\Delta_2 = \begin{vmatrix} 11 & -3 & -4 \\ 6 & 8 & 0 \\ -4 & 10 & 6 \end{vmatrix} = 268$$

$$\underline{x_1 = \frac{-134}{134} = -1} \quad \underline{x_2 = \frac{268}{134} = 2} \quad \underline{x_3 = \frac{134}{134} = 1}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

So $\text{proj}_W \vec{u} = A\vec{x}$

$$= \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 3 \\ 4 \\ 0 \end{pmatrix}$$

$$\underline{\text{proj}_W \vec{u} = (-2, 3, 4, 0)}$$

Exercise

Find the orthogonal projection of the vector \vec{u} on the subspace of \mathbb{R}^4 spanned by the vectors

$$\vec{u} = (6, 3, 9, 6); \quad \vec{v}_1 = (2, 1, 1, 1), \quad \vec{v}_2 = (1, 0, 1, 1), \quad \vec{v}_3 = (-2, -1, 0, -1)$$

Solution

$$\text{Let } A = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{pmatrix}$$

$$A^T \vec{u} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 9 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 30 \\ 21 \\ -21 \end{pmatrix}$$

The normal solution is $A^T A\vec{x} = A^T \vec{u}$

$$\begin{pmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 30 \\ 21 \\ -21 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{vmatrix} = 3$$

$$\Delta_1 = \begin{vmatrix} 30 & 4 & -6 \\ 21 & 3 & -3 \\ -21 & -3 & 6 \end{vmatrix} = 18$$

$$\Delta_2 = \begin{vmatrix} 7 & 30 & -6 \\ 4 & 21 & -3 \\ -6 & -21 & 6 \end{vmatrix} = 9$$

$$\underline{x_1 = \frac{18}{3} = 6 \quad x_2 = \frac{9}{3} = 3 \quad x_3 = 4}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$$

$$\text{So } \text{proj}_W \vec{u} = A\vec{x}$$

$$= \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 2 \\ 9 \\ 5 \end{pmatrix}$$

$$\underline{\text{proj}_W \vec{v} = (7, 2, 9, 5)}$$

Exercise

Find the orthogonal projection of the vector \vec{u} on the subspace of \mathbb{R}^4 spanned by the vectors

$$\vec{u} = (-2, 0, 2, 4); \quad \vec{v}_1 = (1, 1, 3, 0), \quad \vec{v}_2 = (-2, -1, -2, 1), \quad \vec{v}_3 = (-3, -1, 1, 3)$$

Solution

$$\text{Let } A = \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 1 & 1 & 3 & 0 \\ -2 & -1 & -2 & 1 \\ -3 & -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{pmatrix}$$

$$A^T \vec{u} = \begin{pmatrix} 1 & 1 & 3 & 0 \\ -2 & -1 & -2 & 1 \\ -3 & -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 4 \\ 20 \end{pmatrix}$$

The normal solution is $A^T A \vec{x} = A^T \vec{u}$

$$\begin{pmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 20 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{vmatrix} = 10 \quad \Delta_1 = \begin{vmatrix} 4 & -9 & -1 \\ 4 & 10 & 8 \\ 20 & 8 & 20 \end{vmatrix} = -8 \quad \Delta_2 = \begin{vmatrix} 11 & 4 & -1 \\ -9 & 4 & 8 \\ -1 & 20 & 20 \end{vmatrix} = -16$$

$$\underline{x_1 = \frac{-8}{10} = -\frac{4}{5}} \quad \underline{x_2 = \frac{-16}{10} = -\frac{8}{5}} \quad \underline{x_3 = \frac{8}{5}}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} \\ -\frac{8}{5} \\ \frac{8}{5} \end{pmatrix}$$

$$proj_W \vec{u} = A \vec{x}$$

$$= \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} -\frac{4}{5} \\ -\frac{8}{5} \\ \frac{8}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{12}{5} \\ -\frac{4}{5} \\ \frac{12}{5} \\ \frac{16}{5} \end{pmatrix}$$

$$\underline{\text{proj}_W \vec{u} = \left(-\frac{12}{5}, -\frac{4}{5}, \frac{12}{5}, \frac{16}{5} \right)}$$

Exercise

Find the standard matrix for the orthogonal projection P of \mathbb{R}^2 on the line passes through the origin and makes an angle θ with the positive x -axis.

Solution

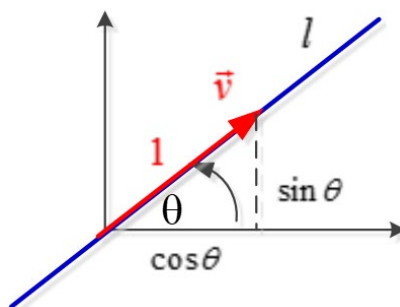
Since the line l is 2-dimensional, then we can take $\vec{v} = (\cos \theta, \sin \theta)$ as a basis for this subspace

$$A = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$

$$[P] = A^T A$$

$$= [\cos \theta \quad \sin \theta] \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}$$



Exercise

Hooke's law in physics states that the length x of a uniform spring is a linear function of the force y applied to it. If we express the relationship as $y = mx + b$, then the coefficient m is called the spring constant.

Suppose a particular unstretched spring has a measured length of 6.1 inches.(i.e., $x = 6.1$ when $y = 0$).

Forces of 2 pounds, 4 pounds, and 6 pounds are then applied to the spring, and the corresponding lengths are found to be 7.6 inches, 8.7 inches, and 10.4 inches. Find the spring constant.

Solution

$$M = \begin{pmatrix} 6.1 & 1 \\ 7.6 & 1 \\ 8.7 & 1 \\ 10.4 & 1 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \end{pmatrix}$$

The normal equation: $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 6.1 & 7.6 & 8.7 & 10.4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6.1 & 1 \\ 7.6 & 1 \\ 8.7 & 1 \\ 10.4 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 6.1 & 7.6 & 8.7 & 10.4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \end{pmatrix}$$

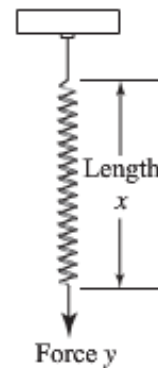
$$\begin{pmatrix} 278.82 & 32.8 \\ 32.8 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 112.4 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{39.44} \begin{pmatrix} 4 & -32.8 \\ -32.8 & 278.2 \end{pmatrix} \begin{pmatrix} 112.4 \\ 12 \end{pmatrix}$$

$$= \frac{1}{39.44} \begin{pmatrix} 56 \\ -348.32 \end{pmatrix}$$

$$= \begin{pmatrix} 1.4 \\ -8.8 \end{pmatrix}$$

Thus, the estimated value of the spring constant is ≈ 1.4 pounds



Exercise

Prove:

If A has a linearly independent column vectors, and if \vec{b} is orthogonal to the column space of A , then the least squares solution of $A\vec{x} = \vec{b}$ is $\vec{x} = \vec{0}$.

Solution

If A has linearly independent column vectors, then $A^T A$ is invertible and the least squares solution of $A\vec{x} = \vec{b}$ is the solution of $A^T A \vec{x} = A^T \vec{b}$, but since \vec{b} is orthogonal to the column space of A .

$A^T \vec{b} = 0$, so \vec{x} is a solution of $A^T A \vec{x} = 0$.

Thus $\vec{x} = \vec{0}$ since $A^T A$ is invertible.

Exercise

Let A be an $m \times n$ matrix with linearly independent row vectors. Find a standard matrix for the orthogonal projection of \mathbb{R}^n onto the row space of A .

Solution

A^T will have linearly independent column vectors, and the column space A^T is the row space of A .

Thus, the standard matrix for the orthogonal projection of \mathbb{R}^n onto the row space of A is

$$\begin{aligned}
 [P] &= A^T \left[\begin{pmatrix} A^T \\ A^T \end{pmatrix} \right]^{-1} \begin{pmatrix} A^T \end{pmatrix}^T \\
 &= A^T (AA^T)^{-1} A
 \end{aligned}$$

Exercise

Let W be the line with parametric equations $x = 2t, \quad y = -t, \quad z = 4t$

- Find a basis for W .
- Find the standard matrix for the orthogonal projection on W .
- Use the matrix in part (b) to find the orthogonal projection of a point $P_0(x_0, y_0, z_0)$ on W .
- Find the distance between the point $P_0(2, 1, -3)$ and the line W .

Solution

a) $W = \text{span}\{(2, -1, 4)\}$

So that the vector $(2, -1, 4)$ forms a basis for W (linear independence)

b) Let $A = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$

$$[P] = A(A^T A)^{-1} A^T$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \left(\begin{bmatrix} 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} [21]^{-1} \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \frac{1}{21} \begin{bmatrix} 4 & -2 & 8 \\ -2 & 1 & -4 \\ 8 & -4 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix}$$

$$c) \begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} \frac{4}{21}x_0 - \frac{2}{21}y_0 + \frac{8}{21}z_0 \\ -\frac{2}{21}x_0 + y_0 - \frac{4}{21}z_0 \\ \frac{8}{21}x_0 - \frac{4}{21}y_0 + \frac{16}{21}z_0 \end{bmatrix}$$

$$d) \begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -\frac{6}{7} \\ \frac{3}{7} \\ -\frac{12}{7} \end{bmatrix}$$

The distance between P_0 and W equals to the distance between P_0 and its projection on W .

The distance between $(2, 1, -3)$ and $\left(-\frac{6}{7}, \frac{3}{7}, -\frac{12}{7}\right)$ is

$$\begin{aligned} d &= \sqrt{\left(2 + \frac{6}{7}\right)^2 + \left(1 - \frac{3}{7}\right)^2 + \left(-3 + \frac{12}{7}\right)^2} \\ &= \sqrt{\frac{400}{49} + \frac{16}{49} + \frac{81}{49}} \\ &= \frac{\sqrt{497}}{7} \end{aligned}$$

Exercise

In R^3 , consider the line l given by the equations $x = t, y = t, z = t$

And the line m given by the equations $x = s, y = 2s - 1, z = 1$

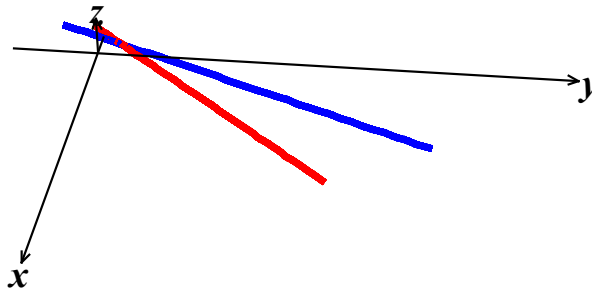
Let P be the point on l , and let Q be a point on m . Find the values of t and s that minimize the distance between the lines by minimizing the squared distance $\|P - Q\|^2$

Solution

When $t = 1 \Rightarrow$ Let $P = (1, 1, 1)$ is on line l

When $s = 1 \Rightarrow$ Let $Q = (1, 1, 1)$ is on line m

$$\|P - Q\| = \sqrt{(1-1)^2 + (1-1)^2 + (1-1)^2} = 0 \geq 0$$



Thus, these are the values $P = (1, 1, 1)$ and $Q = (1, 1, 1)$ are the values for $s = t = 1$ that minimize the distance between the lines.

Exercise

Determine whether the statement is true or false,

- If A is an $m \times n$ matrix, then $A^T A$ is a square matrix.
- If $A^T A$ is invertible, then A is invertible.
- If A is invertible, then $A^T A$ is invertible.
- If $A\vec{x} = \vec{b}$ is a consistent linear system, then $A^T A\vec{x} = A^T \vec{b}$ is also consistent.
- If $A\vec{x} = \vec{b}$ is an inconsistent linear system, then $A^T A\vec{x} = A^T \vec{b}$ is also inconsistent.
- Every linear system has a least squares solution.
- Every linear system has a unique least squares solution.
- If A is an $m \times n$ matrix with linearly independent columns and \vec{b} is in R^m , then $A\vec{x} = \vec{b}$ has a unique least squares solution.

Solution

- True;** $A^T A$ is an $n \times n$ matrix
- False;** only square matrix has inverses, but $A^T A$ can be invertible when A is not square matrix.
- True;** if A is invertible, so is A^T , so the product $A^T A$ is also invertible
- True**
- False;** the system $A^T A\vec{x} = A^T \vec{b}$ may be consistent
- True**
- False;** the least squares solution may involve a parameter
- True;** if A has linearly independent column vectors; then $A^T A$ is invertible, so $A^T A\vec{x} = A^T \vec{b}$ has a unique solution

Exercise

A certain experiment produces the data $\{(1, 1.8), (2, 2.7), (3, 3.4), (4, 3.8), (5, 3.9)\}$.

Find the function that it will fit these data in the form of $y = \beta_1 x + \beta_2 x^2$

Solution

Given: the equation $y = \beta_1 x + \beta_2 x^2$ that best fits the given points. Then

$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{pmatrix}$$

The normal equation formula: $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \end{pmatrix} \begin{pmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{pmatrix}$$

$$\begin{pmatrix} 55 & 225 \\ 225 & 979 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 52.1 \\ 201.5 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 55 & 225 \\ 225 & 979 \end{vmatrix} = 3,220 \quad \Delta\beta_1 = \begin{vmatrix} 52.1 & 225 \\ 201.5 & 979 \end{vmatrix} = 5,668.4 \quad \Delta\beta_2 = \begin{vmatrix} 55 & 52.1 \\ 225 & 201.5 \end{vmatrix} = -640$$

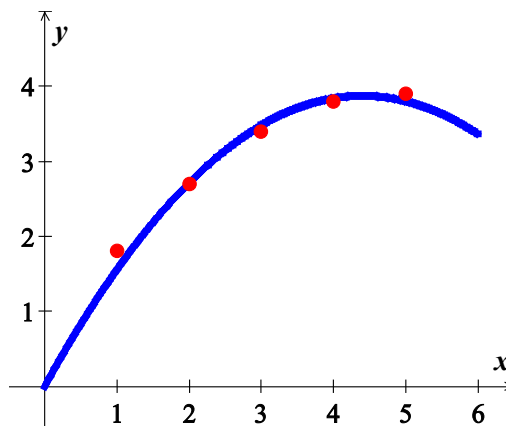
$$\beta_1 = \frac{5,668.4}{3,220}$$

$$\approx 1.76$$

$$\beta_2 = -\frac{640}{3,220}$$

$$\approx -0.199$$

$$y = 1.76x - .2x^2$$



Exercise

According to Kepler's first law, a comet should have an ellipse, parabolic, or hyperbolic orbit (with gravitational attractions from the planets ignored). In suitable polar coordinates, the position (r, ν) of a comet satisfies an equation of the form

$$r = \beta + e(r \cdot \cos \nu)$$

Where β is a constant and e is the eccentricity of the orbit, with $0 \leq e < 1$ for an ellipse, $e = 1$ for a parabolic, and $e > 1$ for a hyperbola.

Suppose observations of a newly discovered comet provide the data below.

ν	.88	1.10	1.42	1.77	2.14
r	3.00	2.30	1.65	1.25	1.01

Determine the type of orbit, and predict where the orbit will be when $\nu = 4.6$ (*radians*)?

Solution

Given: the equation in the form $r = \beta + e(r \cdot \cos \nu)$

$$3 = \beta + e(3 \cdot \cos(.88)) = \beta + 1.911e$$

$$2.3 = \beta + e(2.3 \cos(1.1)) = \beta + 1.043e$$

$$1.65 = \beta + e(1.65 \cos(1.42)) = \beta + .248e$$

$$1.25 = \beta + e(1.25 \cos(1.77)) = \beta - .247e$$

$$1.01 = \beta + e(1.01 \cos(2.14)) = \beta - .544e$$

$$\begin{pmatrix} 1 & 1.911 \\ 1 & 1.043 \\ 1 & .248 \\ 1 & -.247 \\ 1 & -.544 \end{pmatrix} \begin{pmatrix} \beta \\ e \end{pmatrix} = \begin{pmatrix} 3 \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} 1 & 1.911 \\ 1 & 1.043 \\ 1 & .248 \\ 1 & -.247 \\ 1 & -.544 \end{pmatrix} \quad \vec{v} = \begin{pmatrix} \beta \\ e \end{pmatrix} \quad \vec{r} = \begin{pmatrix} 3 \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{pmatrix}$$

The normal equation formula: $A^T A \vec{v} = A^T \vec{r}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1.911 & 1.043 & .248 & -.247 & -.544 \end{pmatrix} \begin{pmatrix} 1 & 1.911 \\ 1 & 1.043 \\ 1 & .248 \\ 1 & -.247 \\ 1 & -.544 \end{pmatrix} \begin{pmatrix} \beta \\ e \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1.911 & 1.043 & .248 & -.247 & -.544 \end{pmatrix} \begin{pmatrix} 3 \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 2.411 \\ 2.411 & 5.158 \end{pmatrix} \begin{pmatrix} \beta \\ e \end{pmatrix} = \begin{pmatrix} 9.21 \\ 7.683 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 5 & 2.411 \\ 2.411 & 5.158 \end{vmatrix} = 19.98 \quad \Delta_{\beta} = \begin{vmatrix} 9.21 & 2.411 \\ 7.683 & 5.158 \end{vmatrix} = 28.98 \quad \Delta_e = \begin{vmatrix} 5 & 9.21 \\ 2.411 & 7.683 \end{vmatrix} = 16.21$$

$$\beta = \frac{28.98}{19.98}$$

$$\approx 1.45$$

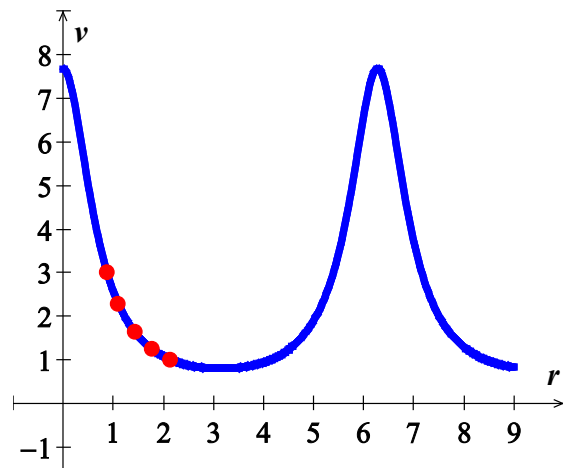
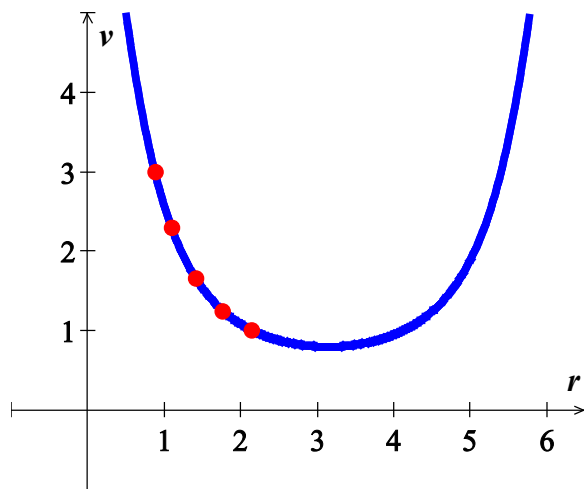
$$e = \frac{16.21}{19.98}$$

$$\approx 0.811 < 1$$

Therefore, the orbit is an *ellipse* type since $e \approx 0.811 < 1$

Since $r = \beta + e(r \cdot \cos v)$

$$\text{Then, } r(v) = \frac{1.45}{1 - 0.811 \cdot \cos v}$$



$$r(4.6) = \frac{1.45}{1 - 0.811 \cdot \cos 4.6}$$

$$\approx 1.329$$

Exercise

To measure the takeoff performance of an airplane, the horizontal position of the plane was measured every second, from $t = 0$ to $t = 12$

The position (in *feet*) were:

0, 8.8, 29.9, 62.0, 104.7, 159.1, 222.0, 294.5, 380.4, 471.1, 571.7, 686.8, and 809.2

- Find the least square cubic curve $y = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$ for these data.
- Estimate the velocity of the plane when $t = 4.5$ *sec*, using the result from part (a).

Solution

Given: the equation is in form $y = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \\ 1 & 6 & 36 & 216 \\ 1 & 7 & 49 & 343 \\ 1 & 8 & 64 & 512 \\ 1 & 9 & 81 & 729 \\ 1 & 10 & 100 & 1000 \\ 1 & 11 & 121 & 1331 \\ 1 & 12 & 144 & 1728 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 8.8 \\ 29.9 \\ 62 \\ 104.7 \\ 159.1 \\ 222.0 \\ 294.5 \\ 380.4 \\ 471.1 \\ 571.7 \\ 686.8 \\ 809.2 \end{pmatrix}$$

$\mathbf{A} \quad \vec{t} = \vec{y}$

The normal equation formula: $A^T A \vec{t} = A^T \vec{y}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \\ 0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 & 81 & 100 & 121 & 144 & \\ 0 & 1 & 8 & 27 & 64 & 125 & 216 & 343 & 512 & 729 & 1000 & 1331 & 1728 & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \\ 1 & 6 & 36 & 216 \\ 1 & 7 & 49 & 343 \\ 1 & 8 & 64 & 512 \\ 1 & 9 & 81 & 729 \\ 1 & 10 & 100 & 1000 \\ 1 & 11 & 121 & 1331 \\ 1 & 12 & 144 & 1728 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \\ 0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 & 81 & 100 & 121 & 144 & \\ 0 & 1 & 8 & 27 & 64 & 125 & 216 & 343 & 512 & 729 & 1000 & 1331 & 1728 & \end{pmatrix} \begin{pmatrix} 0 \\ 8.8 \\ 29.9 \\ 62 \\ 104.7 \\ 159.1 \\ 222.0 \\ 294.5 \\ 380.4 \\ 471.1 \\ 571.7 \\ 686.8 \\ 809.2 \end{pmatrix}$$

$$\begin{pmatrix} 13 & 78 & 650 & 6,084 \\ 78 & 650 & 6,084 & 60,710 \\ 650 & 6,084 & 60,710 & 630,708 \\ 6,084 & 60,710 & 630,708 & 6,735,950 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 3,800.2 \\ 35,127.7 \\ 348,063.9 \\ 3,599,800.9 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 13 & 78 & 650 & 6,084 \\ 78 & 650 & 6,084 & 60,710 \\ 650 & 6,084 & 60,710 & 630,708 \\ 6,084 & 60,710 & 630,708 & 6,735,950 \end{vmatrix} = 97,538,785,344$$

$$\Delta_0 = \begin{vmatrix} 3800.2 & 78 & 650 & 6,084 \\ 35,127.7 & 650 & 6,084 & 60,710 \\ 348,063.9 & 6,084 & 60,710 & 630,708 \\ 3,599,800.9 & 60,710 & 630,708 & 6,735,950 \end{vmatrix} = -83,470,691,303.8916$$

Or I use my program to find the values

```
rref = (Matrix([
  [1, 0, 0, 0, -0.855769230765803],
  [0, 1, 0, 0, 4.70248501498163],
  [0, 0, 1, 0, 5.55536963037029],
  [0, 0, 0, 1, -0.0273601398601744]]))
```

$$\beta_0 \approx -0.855769$$

$$\beta_1 \approx 4.702485$$

$$\beta_2 \approx 5.55537$$

$$\beta_3 \approx -0.02736$$

$$y(t) = -0.855769 + 4.702485t + 5.55537t^2 - 0.02736t^3$$

$$\text{Error} = 3.9734$$

