Section 3.4 – Permutations and Combinations

Permutation

A permutation of a set of distinct objects is an arrangement of the objects is a ***specific Order Without*** repetition. An ordered arrangement of r elements of a set is called an ***r -permutation***.

Theorem

If n is a positive integer and r is an integer with $1 \le r \le n$, then there are

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$$

r –permutation of a set with n distinct elements.

Proof

Use the product rule. The first element can be chosen in n ways. The second in $n - 1$ ways, and so on until there are $(n - (r - 1))$ ways to choose the last element.

Corollary

If n and r are integers with $1 \le r \le n$, then

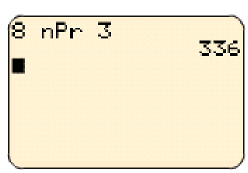
$$P_{n,r} = \frac{n!}{(n-r)!}$$

Example

In mid 2007, eight candidates sought the Democratic nomination for president. In how many ways could voters rank their first, second, and third choices?

<u>Solution</u>

$P_{8,3} = 336$ $\qquad$ $8\ Math \rightarrow Prob \rightarrow (nPr)\ 3$



Example

How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution

$$P(100,3) = 100 \cdot 99 \cdot 98 \ \underline{= 970,200|}$$

Example

Suppose that there are eight runners in a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?

Solution

There are: $P(8,3) = 8 \cdot 7 \cdot 6 \ \underline{= 336 \ \ ways|}$

Example

Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Solution

The first city is chosen, and the rest are ordered arbitrarily. Hence the orders are:

$$7! = \underline{5040|}$$

If she wants to find the tour with the shortest path that visits all the cities, she must consider 5040 paths!

Example

How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

Solution

We solve this problem by counting the permutations of six objects, *ABC*, *D*, *E*, *F*, *G*, and *H*.

$$6! = \underline{720|}$$

Combination

Definition

An *r-combination* of elements of a set is an unordered selection of *r* elements from the set. Thus, an *r*-combination is simply a subset of the set with *r* elements

Combination of a set of n distinct objects taken r @ a time ***without*** repetition is an r element subset of the set of n objects.

The arrangement of the elements *doesn't matter*.

$$C_{n,r} = \binom{n}{r} = \frac{P_{n,r}}{r!} = \frac{n!}{r!(n-r)!}$$

n Math $\rightarrow$ Prob $\rightarrow$ 3(nCr) r

Example

Let S be the set $\{a, b, c, d\}$. Then $\{a, c, d\}$ is a 3-combination from S. It is the same as $\{d, c, a\}$ since the order listed does not matter.

Solution

$C(4,2) = 6$ because the 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$.

Theorem

The number of *r*-combinations of a set with *n* elements, where $n \ge r \ge 0$, equals

$$C(n,\ r) = \frac{n!}{r!(n-r)!}$$

Proof

By the product rule $P(n, r) = C(n, r) \cdot P(r, r)$. Therefore,

$$C(n,\ r) = \frac{P(n,r)}{P(r,r)} = \frac{\frac{n!}{(n-r)!}}{\frac{r!}{(r-r)!}} = \frac{n!}{r!(n-r)!}$$

Example

How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a deck of 52 cards?

Solution

Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$C(52,\ 5) = \frac{52!}{5!(52-5)!} = 2,598,960$$

Corollary

Let n and r be nonnegative integers with $r \le n$. Then $C(n, r) = C(n, n - r)$.

Proof

From Theorem 2, it follows that $C(n,\ r) = \dfrac{n!}{r!(n-r)!}$

and $C(n,\ n-r) = \dfrac{n!}{(n-r)!\left[n-(n-r)!\right]} = \dfrac{n!}{(n-r)!r!}$

Hence, $C(n, r) = C(n, n - r)$.

Definition

A *combinatorial proof* of an identity is a proof that uses one of the following methods.

- A *double counting proof* uses counting arguments to prove that both sides of an identity count the same objects, but in different ways.
- A *bijective proof* shows that there is a bijection between the sets of objects counted by the two sides of the identity.

Example

How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?

Solution

The number of combinations is $C(10,\ 5) = \dfrac{10!}{5!(10-5)!} \ \underline{= 252|}$

Example

A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission?

Solution

The number of possible crews is $C(30,\ 6) = \dfrac{30!}{6!24!} \ \underline{= 593{,}775|}$

Example

How many bits strings of length n contain exactly r 1s?

Solution

The positions of r 1s in a bit string of length n form an r-combination of the set $\{1, 2, 3, \ldots, n\}$.

There are $C(n,\ r)$.

Example

Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty member from the mathematics department and four from the computer science department?

Solution

$$C(9,\ 3)\cdot C(11,\ 4) = \frac{9!}{3!6!}\cdot\frac{11!}{4!7!}$$

$$= 27,720$$

Exercises *Section* 3.4 – Permutations and Combinations

1. Decide whether the situation involves ***permutations*** or ***combinations***

a) A batting order for 9 players for a baseball game

b) An arrangement of 8 people for a picture

c) A committee of 7 delegates chosen from a class of 30 students to bring a petition to the administration

d) A selection of a chairman and a secretary from a committee of 14 people

e) A sample of 5 items taken from 71 items on an assembly line

f) A blend of 3 spices taken from 7 spices on a spice rack

g) From the 7 male and 10 female sales representatives for an insurance company, team of 8 will be selected to attend a national conference on insurance fraud.

h) Marbles are being drawn without replacement from a bag containing 15 marbles.

i) The new university president named 3 new officers a vice-president of finance, a vice-president of academic affairs, and a vice-president of student affairs.

j) A student checked out 4 novels from the library to read over the holiday.

k) A father ordered an ice cream cone (chocolate, vanilla, or strawberry) for each of his 4 children.

2. How many different permutations are the of the set $\{a, b, c, d, e, f, g\}$?

3. How many permutations of $\{a, b, c, d, e, f, g\}$ end with a?

4. Find the number of 5-permutations of a set with nine elements

5. In how many different orders can five runners finish a race if no ties are allowed?

6. A coin flipped eight times where each flip comes up either heads or tails. How many possible outcomes

a) Are there in total?

b) Contain exactly three heads?

c) Contain at least three heads?

d) Contain the same number of heads and tails?

7. A coin flipped 10 times where each flip comes up either heads or tails. How many possible outcomes

a) Are there in total?

b) Contain exactly two heads?

c) Contain at least three heads?

d) Contain the same number of heads and tails?

8. How many bit strings of length 12 contain

a) Exactly three 1s?

b) At most three 1s?

c) At least three 1s?

d) An equal number of 0s and 1s?

9. A group contains n men and n women. How many ways are there to arrange these people in a row if the men and women alternate?

10. In how many ways can a set of two positive integers less than 100 be chosen?

11. In how many ways can a set of five letters be selected from the English alphabet?

12. How many subsets with an odd number of elements does a set with 10 elements have?

13. How many subsets with more than two elements does a set with 100 elements have?

14. How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other?

15. How many ways are there for six men and 10 women to stand in a line so that no two men stand next to each other?

16. A professor writes 40 discrete mathematics true/false questions. Of the statements in these questions. 17 are true. If the questions can be positioned in any order, how many different answer keys are possible?

17. Thirteen people on a softball team show up for a game.

a) How many ways are there to choose 10 players to take the field?

b) How many ways are there to assign the 10 positions by selecting players from the 13 people who show up?

c) Of the 13 people who show up, there are three women. How many ways are there to choose 10 players to take the field if at least one of these players must be a woman?

18. A club has 25 members

a) How many ways are there to choose four members of the club to serve on an executive committee?

b) How many ways are there to choose a president, vice president, secretary, and treasurer of the club, where no person can hold more than one office?

19. How many 4-permutations of the positive integers not exceeding 100 contain three consecutive integers, k, $k + 1$, $k + 2$, in the order

a) Where these consecutive integers can perhaps be separated by other integers in the permutation?

b) Where they are in consecutive positions in the permutation?

20. The English alphabet contains 21 constants and five vowels. How many strings of six lowercase letters of the English alphabet contain

a) Exactly one vowel?

b) Exactly two vowels?

c) At least one vowel?

d) At least two vowels?

21. Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have

a) The same number of men and women?

b) More women than men?

22. How many bit strings contain exactly eight 0s and 10 1s if every 0 must be immediately followed by a 1?

23. How many bit strings contain exactly five 0s and 14 1s if every 0 must be immediately followed by two 1s?

24. A concert to raise money for an economics prize is to consist of 5 works; 2 overtures, 2 sonatas, and a piano concerto.

a) In how many ways can the program be arranged?

b) In how many ways can the program be arranged if an overture must come first?

25. A zydeco band from Louisiana will play 5 traditional and 3 original Cajun compositions at a concert. In how many ways can they arrange the program if

a) The begin with a traditional piece?

b) An original piece will be played last?

26. In an election with 3 candidates for one office and 6 candidates for another office, how many different ballots may be printed?

27. A business school gives courses in typing, shorthand, transcription, business English, technical writing, and accounting. In how many ways can a student arrange a schedule if 3 courses are taken? assume that the order in which courses are schedules matters.

28. If your college offers 400 courses, 25 of which are in mathematics, and your counselor arranges your schedule of 4 courses by random selection, how many schedules are possible that do not include a math course? Assume that the order in which courses are scheduled matters.

29. A baseball team has 19 players. How many 9-player batting orders are possible?

30. A chapter of union Local 715 has 35 members. In how many different ways can the chapter select a president, a vice-president, a treasurer, and a secretary?

31. An economics club has 31 members.

a) If a committee of 4 is to be selected, in how many ways can the selection be made?

b) In how many ways can a committee of at least 1 and at most 3 be selected?

32. In a club with 9 male and 11 female members, how many 5-member committees can be chosen that have

a) All men?

b) All women?

c) 3 men and 2 women?

33. In a club with 9 male and 11 female members, how many 5-member committees can be selected that have

a) At least 4 women?

b) No more than 2 men?

34. In a game of musical chairs, 12 children will sit in 11 chairs arranged in a row (one will be left out). In how many ways can this happen, if we count rearrangements of the children in the chairs as different outcomes?

35. A group of 3 students is to be selected from a group of 14 students to take part in a class in cell biology.

a) In how many ways can this be done?

b) In how many ways can the group who will not take part be chosen?

36. Marbles are being drawn without replacement from a bag containing 16 marbles.

a) How many samples of 2 marbles can be drawn?

b) How many samples of 2 marbles can be drawn?

c) If the bag contains 3 yellow, 4 white, and 9 blue marbles, how many samples of 2 marbles can be drawn in which both marbles are blue?

37. A bag contains 5 black, 1 red, and 3 yellow jelly beans; you take 3 at random. How many samples are possible in which the jelly beans are

a) All black?

b) All red?

c) All yellow?

d) 2 black and 1 red?

e) 2 black and 1 yellow?

f) 2 yellow and 1 black?

g) 2 red and 1 yellow?