

Solution

Section 4.4 – Area and Lengths in Polar Coordinates

Exercise

Find the slopes of the curves at the given points. Sketch the curves along with their tangents at these points.

Cardioid $r = -1 + \cos \theta; \quad \theta = \pm \frac{\pi}{2}$

Solution

$$\theta = \frac{\pi}{2} \Rightarrow r = -1 + \cos \frac{\pi}{2} = -1 \rightarrow \left(-1, \frac{\pi}{2}\right)$$

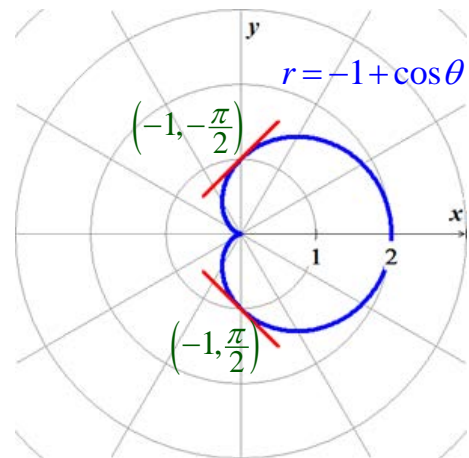
$$\theta = -\frac{\pi}{2} \Rightarrow r = -1 + \cos\left(-\frac{\pi}{2}\right) = -1 \rightarrow \left(-1, -\frac{\pi}{2}\right)$$

$$r' = \frac{dr}{d\theta} = -\sin \theta$$

$$\text{Slope} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta} = \frac{-\sin^2 \theta + r \cos \theta}{-\sin \theta \cos \theta - r \sin \theta}$$

$$\text{Slope} \bigg|_{\left(-1, \frac{\pi}{2}\right)} = \frac{-\sin^2 \frac{\pi}{2} + (-1) \cos \frac{\pi}{2}}{-\sin \frac{\pi}{2} \cos \frac{\pi}{2} + \sin \frac{\pi}{2}} = -1$$

$$\text{Slope} \bigg|_{\left(-1, -\frac{\pi}{2}\right)} = \frac{-\sin^2\left(-\frac{\pi}{2}\right) + (-1) \cos\left(-\frac{\pi}{2}\right)}{-\sin\left(-\frac{\pi}{2}\right) \cos\left(-\frac{\pi}{2}\right) + \sin\left(-\frac{\pi}{2}\right)} = 1$$



Exercise

Find the slopes of the curves at the given points. Sketch the curves along with their tangents at these points.

Cardioid $r = -1 + \sin \theta; \quad \theta = 0, \pi$

Solution

$$\theta = 0 \Rightarrow r = -1 + \sin 0 = -1 \rightarrow (-1, 0)$$

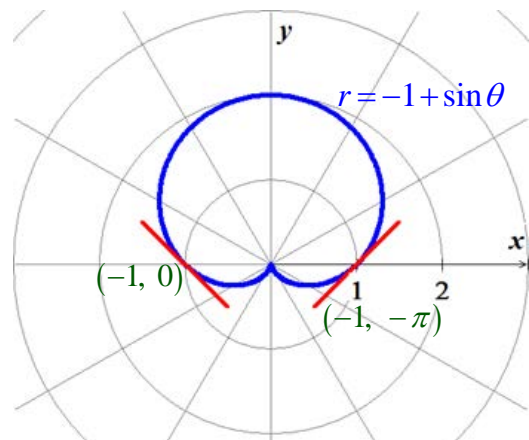
$$\theta = \pi \Rightarrow r = -1 + \sin \pi = -1 \rightarrow (-1, \pi)$$

$$r' = \frac{dr}{d\theta} = \cos \theta$$

$$\text{Slope} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta} = \frac{\cos \theta \sin \theta + r \cos \theta}{\cos^2 \theta - r \sin \theta}$$

$$\text{Slope} \bigg|_{(-1, 0)} = \frac{\cos(0) \sin(0) + (-1) \cos(0)}{\cos^2(0) - (-1) \sin(0)} = -1$$

$$\text{Slope} \bigg|_{(-1, \pi)} = \frac{\cos(\pi) \sin(\pi) + (-1) \cos(\pi)}{\cos^2(\pi) - (-1) \sin(\pi)} = 1$$



Exercise

Find the slopes of the curves at the given points. Sketch the curves along with their tangents at these points.
Four-leaved rose $r = \sin 2\theta$; $\theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$

Solution

$$\theta = -\frac{\pi}{4} \Rightarrow r = \sin\left(-\frac{\pi}{2}\right) = -1 \rightarrow \left(-1, -\frac{\pi}{4}\right)$$

$$\theta = \frac{\pi}{4} \Rightarrow r = \sin\left(\frac{\pi}{2}\right) = 1 \rightarrow \left(1, \frac{\pi}{4}\right)$$

$$\theta = -\frac{3\pi}{4} \Rightarrow r = \sin\left(-\frac{3\pi}{2}\right) = 1 \rightarrow \left(1, -\frac{3\pi}{4}\right)$$

$$\theta = \frac{3\pi}{4} \Rightarrow r = \sin\left(\frac{3\pi}{2}\right) = -1 \rightarrow \left(-1, \frac{3\pi}{4}\right)$$

$$r' = \frac{dr}{d\theta} = 2\cos 2\theta$$

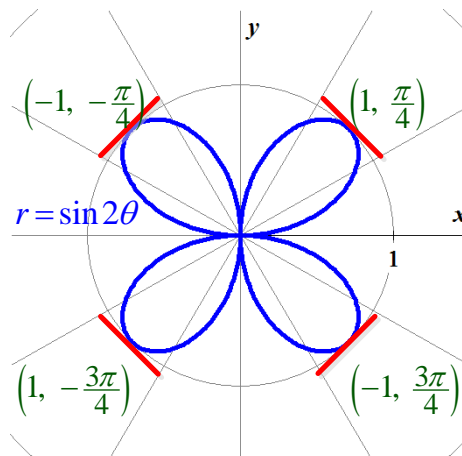
$$\text{Slope} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta} = \frac{2\cos 2\theta \sin \theta + r \cos \theta}{2\cos 2\theta \cos \theta - r \sin \theta}$$

$$\text{Slope} \bigg|_{\left(-1, -\frac{\pi}{4}\right)} = \frac{2\cos\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{4}\right) + (-1)\cos\left(-\frac{\pi}{4}\right)}{2\cos\left(-\frac{\pi}{2}\right)\cos\left(-\frac{\pi}{4}\right) - (-1)\sin\left(-\frac{\pi}{4}\right)} = 1$$

$$\text{Slope} \bigg|_{\left(1, \frac{\pi}{4}\right)} = \frac{2\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right) + (1)\cos\left(\frac{\pi}{4}\right)}{2\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right) - (1)\sin\left(\frac{\pi}{4}\right)} = -1$$

$$\text{Slope} \bigg|_{\left(1, -\frac{3\pi}{4}\right)} = \frac{2\cos\left(-\frac{3\pi}{2}\right)\sin\left(-\frac{3\pi}{4}\right) + (1)\cos\left(-\frac{3\pi}{4}\right)}{2\cos\left(-\frac{3\pi}{2}\right)\cos\left(-\frac{3\pi}{4}\right) - (1)\sin\left(-\frac{3\pi}{4}\right)} = 1$$

$$\text{Slope} \bigg|_{\left(-1, \frac{3\pi}{4}\right)} = \frac{2\cos\left(\frac{3\pi}{2}\right)\sin\left(\frac{3\pi}{4}\right) + (-1)\cos\left(\frac{3\pi}{4}\right)}{2\cos\left(\frac{3\pi}{2}\right)\cos\left(\frac{3\pi}{4}\right) - (-1)\sin\left(\frac{3\pi}{4}\right)} = -1$$



Exercise

Find the slopes of the curves at the given points. Sketch the curves along with their tangents at these points.
Four – leaved rose $r = \cos 2\theta$; $\theta = 0, \pm \frac{\pi}{2}, \pi$

Solution

$$\theta = 0 \Rightarrow r = \cos(0) = 1 \rightarrow (1, 0)$$

$$\theta = \frac{\pi}{2} \Rightarrow r = \cos(\pi) = -1 \rightarrow \left(-1, \frac{\pi}{2}\right)$$

$$\theta = -\frac{\pi}{2} \Rightarrow r = \cos(-\pi) = -1 \rightarrow \left(-1, -\frac{\pi}{2}\right)$$

$$\theta = \pi \Rightarrow r = \cos(2\pi) = 1 \rightarrow (1, \pi)$$

$$r' = \frac{dr}{d\theta} = -2\sin 2\theta$$

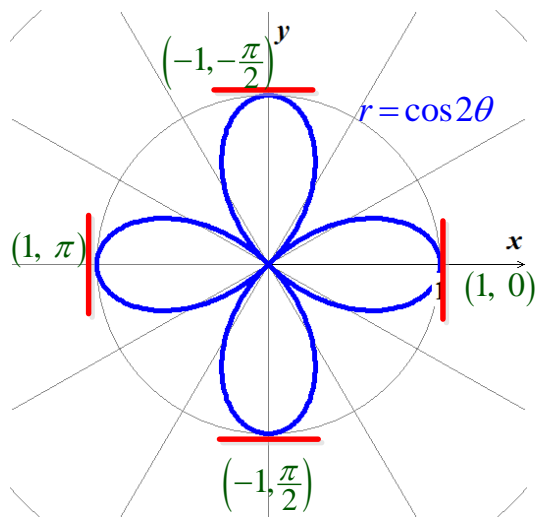
$$\text{Slope} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta} = \frac{-2\sin 2\theta \sin \theta + r \cos \theta}{-2\sin 2\theta \cos \theta - r \sin \theta}$$

$$\text{Slope} \Big|_{(1,0)} = \frac{-2\sin(0)\sin(0) + (1)\cos(0)}{-2\sin(0)\cos(0) - (1)\sin(0)} = \text{undefined}$$

$$\text{Slope} \Big|_{\left(-1, \frac{\pi}{2}\right)} = \frac{-2\sin(\pi)\sin\left(\frac{\pi}{2}\right) + (1)\cos\left(\frac{\pi}{2}\right)}{-2\sin(\pi)\cos\left(\frac{\pi}{2}\right) - (1)\sin\left(\frac{\pi}{2}\right)} = 0$$

$$\text{Slope} \Big|_{\left(-1, -\frac{\pi}{2}\right)} = \frac{-2\sin(-\pi)\sin\left(-\frac{\pi}{2}\right) + (1)\cos\left(-\frac{\pi}{2}\right)}{-2\sin(-\pi)\cos\left(-\frac{\pi}{2}\right) - (1)\sin\left(-\frac{\pi}{2}\right)} = 0$$

$$\text{Slope} \Big|_{(1,\pi)} = \frac{-2\sin(2\pi)\sin(\pi) + (1)\cos(\pi)}{-2\sin(2\pi)\cos(\pi) - (1)\sin(\pi)} = \text{undefined}$$

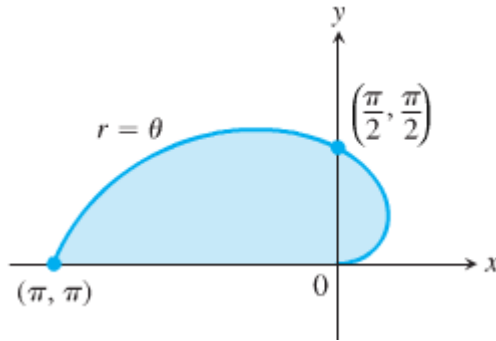


Exercise

Find the area of the region bounded by the spiral $r = \theta$ for $0 \leq \theta \leq \pi$

Solution

$$\begin{aligned} A &= \frac{1}{2} \int_0^{\pi} r^2 d\theta \\ &= \frac{1}{2} \int_0^{\pi} \theta^2 d\theta \\ &= \frac{1}{6} \theta^3 \Big|_0^{\pi} \\ &= \frac{\pi^3}{6} \end{aligned}$$

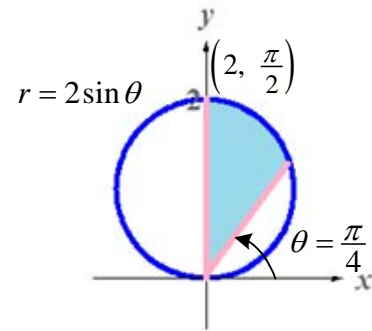


Exercise

Find the area of the region bounded by the circle $r = 2 \sin \theta$ for $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

Solution

$$\begin{aligned} A &= \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2 \sin \theta)^2 d\theta \\ &= 2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 \theta d\theta \\ &= 2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1 - \cos 2\theta}{2} d\theta \\ &= \left[\theta - \frac{1}{2} \sin 2\theta \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= \left(\frac{\pi}{2} - \frac{1}{2} \sin \pi \right) - \left(\frac{\pi}{4} - \frac{1}{2} \sin \frac{\pi}{2} \right) \\ &= \frac{\pi}{2} - \frac{\pi}{4} + \frac{1}{2} \left(-\frac{1}{2} \sin \pi \right) - \left(-\sin \frac{\pi}{2} \right) \\ &= \frac{\pi}{4} + \frac{1}{2} \end{aligned}$$



Exercise

Find the area of the region inside the oval limaçon $r = 4 + 2 \sin \theta$

Solution

$$\begin{aligned} A &= \frac{1}{2} \int_0^{2\pi} (4 + 2 \sin \theta)^2 d\theta \\ &= \int_0^{2\pi} \frac{1}{2} (16 + 16 \sin \theta + 4 \sin^2 \theta) d\theta \\ &= \int_0^{2\pi} \left(8 + 8 \sin \theta + 2 \frac{1 - \cos 2\theta}{2} \right) d\theta \\ &= \int_0^{2\pi} (8 + 8 \sin \theta + 1 - \cos 2\theta) d\theta \\ &= \int_0^{2\pi} (9 + 8 \sin \theta - \cos 2\theta) d\theta \\ &= \left[9\theta - 8 \cos \theta - \frac{1}{2} \sin 2\theta \right]_0^{2\pi} \\ &= 18\pi - 8 \cos 2\pi - \frac{1}{2} \sin 4\pi - \left(0 - 8 \cos 0 - \frac{1}{2} \sin 0 \right) \\ &= 18\pi - 8 + 8 \\ &= 18\pi \end{aligned}$$

Exercise

Find the area of the region inside the cardioid $r = a(1 + \cos \theta)$, $a > 0$

Solution

$$\begin{aligned} A &= \frac{1}{2} \int_0^{2\pi} a^2 (1 + \cos \theta)^2 d\theta \\ &= \frac{a^2}{2} \int_0^{2\pi} (1 + 2 \cos \theta + \cos^2 \theta) d\theta \\ &= \frac{a^2}{2} \int_0^{2\pi} \left(1 + 2 \cos \theta + \frac{1 + \cos 2\theta}{2} \right) d\theta \\ &= \frac{a^2}{2} \int_0^{2\pi} \left(\frac{3}{2} + 2 \cos \theta + \frac{1}{2} \cos 2\theta \right) d\theta \end{aligned}$$

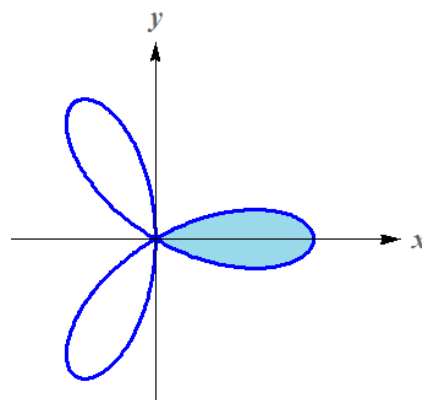
$$\begin{aligned}
&= \frac{a^2}{2} \left[\frac{3}{2} \theta + 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_0^{2\pi} \\
&= \frac{a^2}{2} (3\pi) \\
&= \frac{3}{2} \pi a^2
\end{aligned}$$

Exercise

Find the area of the region inside one leaf of the three-leaved rose $r = \cos 3\theta$

Solution

$$\begin{aligned}
A &= \frac{1}{2} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (\cos 3\theta)^2 d\theta \\
&= \frac{1}{2} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2 3\theta d\theta \\
&= \frac{1}{2} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1 + \cos 6\theta}{2} d\theta \\
&= \frac{1}{4} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (1 + \cos 6\theta) d\theta \\
&= \frac{1}{4} \left[\theta + \frac{1}{6} \sin 6\theta \right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \\
&= \frac{1}{4} \left[\left(\frac{\pi}{6} + \frac{1}{6} \sin \pi \right) - \left(-\frac{\pi}{6} + \frac{1}{6} \sin(-\pi) \right) \right] \\
&= \frac{1}{4} \left(\frac{\pi}{6} + \frac{\pi}{6} \right) \\
&= \frac{\pi}{12}
\end{aligned}$$



Exercise

Find the area of the region inside the six-leaved rose $r^2 = 2 \sin 3\theta$

Solution

$$A = \frac{1}{2} \int_0^{2\pi} r^2 d\theta$$

$$\begin{aligned}
&= (6)(2)\frac{1}{2} \int_0^{\pi/2} 2 \sin 3\theta \, d\theta \\
&= 12 \int_0^{\pi/2} \sin 3\theta \, d\theta \\
&= 12 \left[-\frac{1}{3} \cos 3\theta \right]_0^{\pi/2} \\
&= -4 \cos \frac{3\pi}{2} \\
&= \underline{4}
\end{aligned}$$

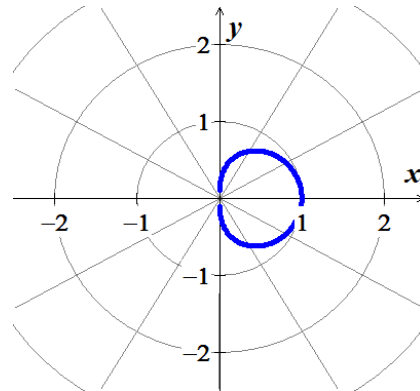
Exercise

Find the area of the region inside the curve $r = \sqrt{\cos \theta}$

Solution

$$r = \sqrt{\cos \theta} \geq 0 \Rightarrow \cos \theta \geq 0 \Rightarrow \underline{-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}}$$

$$\begin{aligned}
A &= \frac{1}{2} \int_0^{\pi/2} 2(\sqrt{\cos \theta})^2 \, d\theta & A &= \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta \\
&= \int_0^{\pi/2} \cos \theta \, d\theta \\
&= \sin \theta \Big|_0^{\pi/2} \\
&= \underline{1}
\end{aligned}$$



Exercise

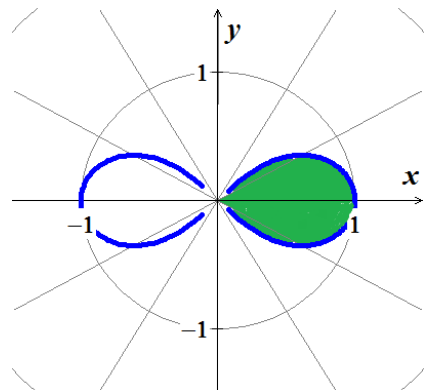
Find the area of the region inside the right lobe of $r = \sqrt{\cos 2\theta}$

Solution

$$r = \sqrt{\cos 2\theta} \geq 0 \Rightarrow \cos 2\theta \geq 0$$

$$\rightarrow -\frac{\pi}{2} \leq 2\theta \leq \frac{\pi}{2} \Rightarrow \underline{-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}}$$

$$\begin{aligned}
A &= \frac{1}{2} \int_0^{\pi/4} 2(\sqrt{\cos 2\theta})^2 \, d\theta & A &= \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta \\
&= \int_0^{\pi/4} \cos 2\theta \, d\theta
\end{aligned}$$



$$= \frac{1}{2} \sin 2\theta \Big|_0^{\pi/4}$$

$$= \frac{1}{2}$$

Exercise

Find the area of the region inside the cardioid $r = 4 + 4 \sin \theta$

Solution

$$A = \frac{1}{2} \int_0^{2\pi} (4 + 4 \sin \theta)^2 d\theta$$

$$= 8 \int_0^{2\pi} (1 + 2 \sin \theta + \sin^2 \theta) d\theta$$

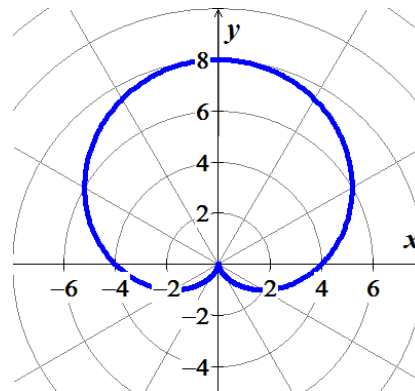
$$= 8 \int_0^{2\pi} \left(1 + 2 \sin \theta + \frac{1}{2} - \frac{1}{2} \cos 2\theta\right) d\theta$$

$$= 8 \left(\frac{3}{2} \theta - 2 \cos \theta - \frac{1}{4} \sin 2\theta \right) \Big|_0^{2\pi}$$

$$= 8(3\pi - 2 + 2)$$

$$= 24\pi$$

$$A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$$



Exercise

Find the area of the region inside the limaçon $r = 2 + \cos \theta$

Solution

$$A = \frac{1}{2} \int_0^{\pi} 2(2 + \cos \theta)^2 d\theta$$

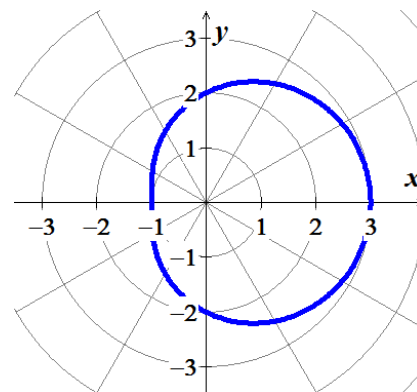
$$= \int_0^{\pi} (4 + 4 \cos \theta + \cos^2 \theta) d\theta$$

$$= \int_0^{\pi} \left(4 + 4 \cos \theta + \frac{1}{2} + \frac{1}{2} \cos 2\theta\right) d\theta$$

$$= \left(\frac{9}{2} \theta + 4 \sin \theta + \frac{1}{4} \sin 2\theta \right) \Big|_0^{\pi}$$

$$= \frac{9\pi}{2}$$

$$A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$$



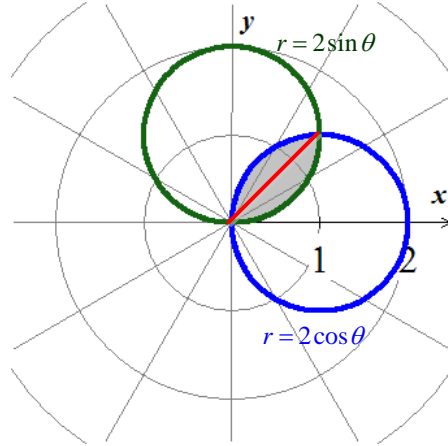
Exercise

Find the area of the region shared by the circles $r = 2\cos\theta$ and $r = 2\sin\theta$

Solution

$$r = 2\cos\theta = 2\sin\theta \Rightarrow \cos\theta = \sin\theta \rightarrow \theta = \frac{\pi}{4}$$

$$\begin{aligned} A &= 2 \int_0^{\pi/4} \frac{1}{2} (2\sin\theta)^2 d\theta \\ &= \int_0^{\pi/4} 4\sin^2\theta d\theta \\ &= \int_0^{\pi/4} 2(1 - \cos 2\theta) d\theta \\ &= [2\theta - \sin 2\theta]_0^{\pi/4} \\ &= 2\frac{\pi}{4} - \sin \frac{\pi}{2} - 0 \\ &= \frac{\pi}{2} - 1 \end{aligned}$$



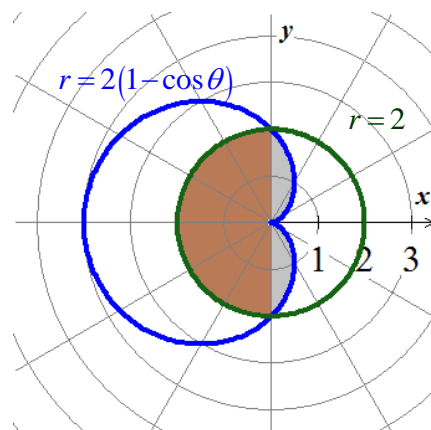
Exercise

Find the area of the region shared by the circle $r = 2$ and the cardioid $r = 2(1 - \cos\theta)$

Solution

$$r = 2 - 2\cos\theta = 2 \rightarrow \cos\theta = 0 \Rightarrow \theta = \pm \frac{\pi}{2}$$

$$\begin{aligned} A &= \frac{1}{2} \text{Area of the circle} + 2 \int_0^{\pi/2} \frac{1}{2} [2(1 - \cos\theta)]^2 d\theta \\ &= \frac{1}{2} \pi (2^2) + 4 \int_0^{\pi/2} (1 - 2\cos\theta + \cos^2\theta) d\theta \\ &= 2\pi + \int_0^{\pi/2} (4 - 8\cos\theta + 2 + 2\cos 2\theta) d\theta \\ &= 2\pi + \int_0^{\pi/2} (6 - 8\cos\theta + 2\cos 2\theta) d\theta \\ &= 2\pi + [6\theta - 8\sin\theta + \sin 2\theta]_0^{\pi/2} \\ &= 2\pi + 3\pi - 8 \\ &= 5\pi - 8 \end{aligned}$$



Exercise

Find the area of the region inside the circle $r = 6$ above the line $r = 3\csc\theta$

Solution

$$r = 3\csc\theta = \frac{3}{\sin\theta} = 6 \rightarrow \sin\theta = \frac{3}{6} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2} [6^2 - (3\csc\theta)^2] d\theta$$

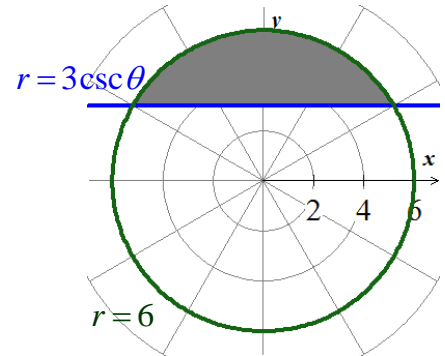
$$= \frac{1}{2} \int_{\pi/6}^{5\pi/6} [36 - 9\csc^2\theta] d\theta$$

$$= \frac{9}{2} [4\theta + \cot\theta]_{\pi/6}^{5\pi/6}$$

$$= \frac{9}{2} \left[\left(\frac{10\pi}{3} - \sqrt{3} \right) - \left(\frac{2\pi}{3} + \sqrt{3} \right) \right]$$

$$= \frac{9}{2} \left(\frac{8\pi}{3} - 2\sqrt{3} \right)$$

$$= 12\pi - 9\sqrt{3}$$



Exercise

Find the area of the region in the plane enclosed by the four-leaf rose $r = f(\theta) = 2\cos 2\theta$

Solution

The curve is symmetric about the x -axis:

$$r = 2\cos(-2\theta) = 2\cos 2\theta$$

$$(r, \theta) = (r, -\theta)$$

The curve is symmetric about the y -axis:

$$-r = 2\cos 2(\pi - \theta) = -2\cos 2\theta$$

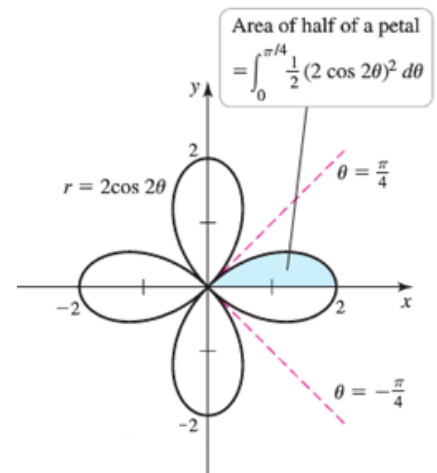
$$r = 2\cos 2\theta$$

$$(r, \theta) = (-r, \pi - \theta)$$

The graph of the rose appears to be symmetric about x - and y -axes.

$$A = 8 \int_0^{\pi/4} \frac{1}{2} r^2 d\theta = 4 \int_0^{2\pi} (2\cos 2\theta)^2 d\theta$$

$$= 16 \int_0^{\pi/4} \cos^2 2\theta d\theta$$



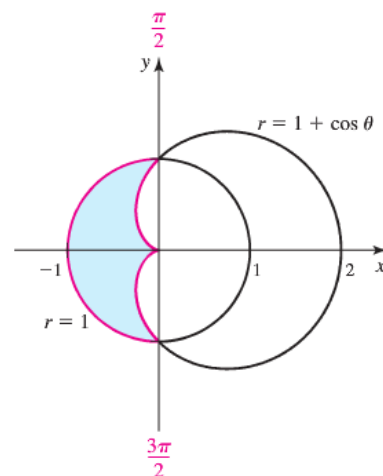
$$\begin{aligned}
&= 8 \int_0^{\pi/4} (1 + \cos 4\theta) d\theta \\
&= 8 \left(\theta + \frac{1}{4} \sin 4\theta \right) \Big|_0^{\pi/4} \\
&= 8 \left[\frac{\pi}{4} + 0 - (0 + 0) \right] \\
&= \underline{2\pi}
\end{aligned}$$

Exercise

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 + \cos \theta$

Solution

$$\begin{aligned}
A &= \int_{\pi/2}^{3\pi/2} \frac{1}{2} (r_2^2 - r_1^2) d\theta \\
&= 2 \int_{\pi/2}^{\pi} \frac{1}{2} (1^2 - (1 + \cos \theta)^2) d\theta \\
&= \int_{\pi/2}^{\pi} (1 - (1 + 2\cos \theta + \cos^2 \theta)) d\theta \\
&= \int_{\pi/2}^{\pi} \left(-2\cos \theta - \frac{1}{2} - \frac{1}{2} \cos 2\theta \right) d\theta \\
&= \left[-2\sin \theta - \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta \right]_{\pi/2}^{\pi} \\
&= -\frac{\pi}{2} + 2 + \frac{\pi}{4} \\
&= \underline{2 - \frac{\pi}{4}}
\end{aligned}$$

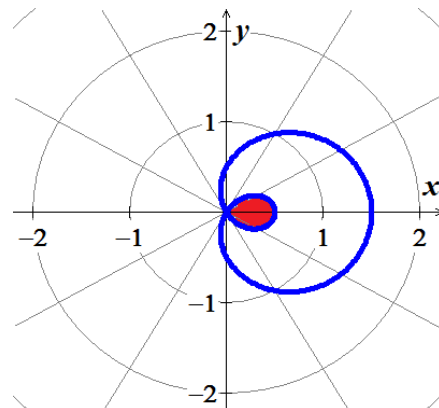


Exercise

Find the area of the region inside the inner loop $r = \cos \theta - \frac{1}{2}$

Solution

$$\begin{aligned}
A &= \frac{1}{2} \int_0^{\pi/3} 2 \left(\cos \theta - \frac{1}{2} \right)^2 d\theta \quad A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta \\
&= \int_0^{\pi/3} \left(\cos^2 \theta - \cos \theta + \frac{1}{4} \right) d\theta
\end{aligned}$$



$$\begin{aligned}
&= \int_0^{\pi/3} \left(\frac{1}{2} + \frac{1}{2} \cos 2\theta - \cos \theta + \frac{1}{4} \right) d\theta \\
&= \left(\frac{3}{4} \theta + \frac{1}{4} \sin 2\theta - \sin \theta \right) \Big|_0^{\pi/3} \\
&= \frac{\pi}{4} + \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{2} \\
&= \frac{\pi}{4} - \frac{3\sqrt{3}}{8}
\end{aligned}$$

Exercise

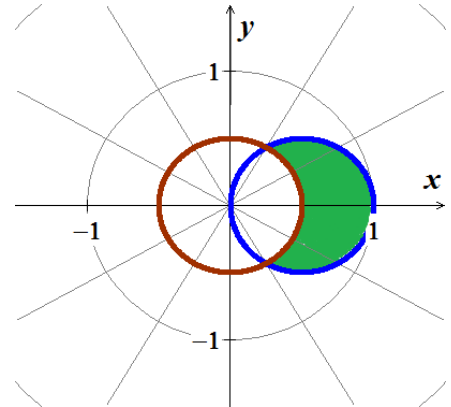
Find the area of the region outside the circle $r = \frac{1}{2}$ and inside the circle $r = \cos \theta$

Solution

$$r = \cos \theta = \frac{1}{2} \rightarrow \theta = \pm \frac{\pi}{3}$$

$$\begin{aligned}
A &= \frac{1}{2} \int_0^{\pi/3} 2 \left(\cos^2 \theta - \frac{1}{4} \right) d\theta \\
&= \int_0^{\pi/3} \left(\frac{1}{2} + \frac{1}{2} \cos 2\theta - \frac{1}{4} \right) d\theta \\
&= \left(\frac{1}{4} \theta + \frac{1}{4} \sin 2\theta \right) \Big|_0^{\pi/3} \\
&= \frac{\pi}{12} + \frac{\sqrt{3}}{8}
\end{aligned}$$

$$A = \frac{1}{2} \int_{\alpha}^{\beta} (r_2^2 - r_1^2) d\theta$$



Exercise

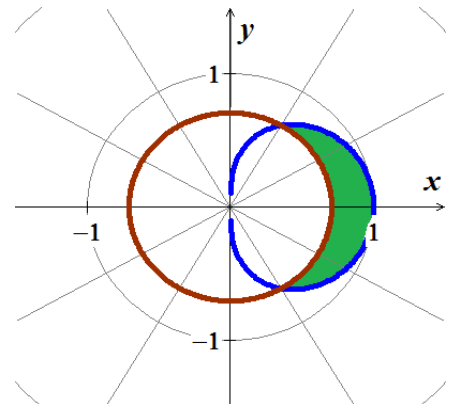
Find the area of the region outside the circle $r = \frac{1}{\sqrt{2}}$ and inside the curve $r = \sqrt{\cos \theta}$

Solution

$$r = \sqrt{\cos \theta} = \frac{1}{\sqrt{2}} \rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \pm \frac{\pi}{3}$$

$$\begin{aligned}
A &= \frac{1}{2} \int_0^{\pi/3} 2 \left(\cos \theta - \frac{1}{2} \right) d\theta \\
&= \left(\sin \theta - \frac{1}{2} \theta \right) \Big|_0^{\pi/3} \\
&= \frac{\sqrt{3}}{2} - \frac{\pi}{6} \text{ unit}^2
\end{aligned}$$

$$A = \frac{1}{2} \int_{\alpha}^{\beta} (r_2^2 - r_1^2) d\theta$$



Exercise

Find the area of the region inside the circle $r = \frac{1}{\sqrt{2}}$ in QI and inside the right lobe of $r = \sqrt{\cos 2\theta}$

Solution

$$r = \sqrt{\cos 2\theta} = \frac{1}{\sqrt{2}} \rightarrow \cos 2\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$$

$$\sqrt{\cos 2\theta} = 0 \rightarrow \cos 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

$$A = A_1 + A_2$$

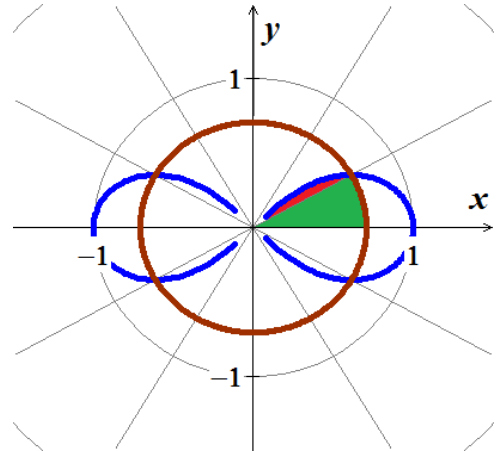
$$A = \frac{1}{2} \int_0^{\pi/6} \left(\frac{1}{\sqrt{2}} \right)^2 d\theta + \frac{1}{2} \int_{\pi/6}^{\pi/4} \left(\sqrt{\cos 2\theta} \right)^2 d\theta$$

$$= \frac{1}{2} \int_0^{\pi/6} \frac{1}{2} d\theta + \frac{1}{2} \int_{\pi/6}^{\pi/4} \cos 2\theta d\theta$$

$$= \frac{1}{4} \theta \Big|_0^{\pi/6} + \frac{1}{4} \sin 2\theta \Big|_{\pi/6}^{\pi/4}$$

$$= \frac{1}{2} \left(\frac{\sqrt{3}}{2} - \frac{\pi}{12} \right) + \frac{1}{4} \left(1 - \frac{\sqrt{3}}{2} \right)$$

$$= \frac{\pi}{24} + \frac{1}{4} - \frac{\sqrt{3}}{8} \text{ unit}^2$$



Exercise

Find the area of the region inside the rose $r = 4\sin 2\theta$ and inside the circle $r = 2$

Solution

$$r = 4\sin 2\theta = 2 \rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

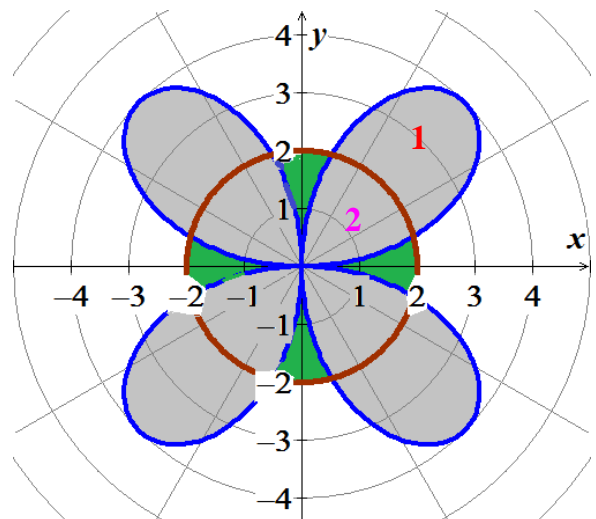
The area (1) inside one leaf but outside the circle is:

$$A = \frac{1}{2} \int_{\pi/12}^{5\pi/12} (16\sin^2 2\theta - 4) d\theta$$

$$= \frac{1}{2} \int_{\pi/12}^{5\pi/12} (8 - 8\cos 4\theta - 4) d\theta$$

$$= \int_{\pi/12}^{5\pi/12} (2 - 4\cos 4\theta) d\theta$$

$$= 2\theta - \sin 4\theta \Big|_{\pi/12}^{5\pi/12}$$



$$= \frac{5\pi}{6} + \frac{\sqrt{3}}{2} - \frac{\pi}{6} + \frac{\sqrt{3}}{2}$$

$$= \frac{2\pi}{3} + \sqrt{3} \Big|$$

Area inside one leaf (2) is:

$$A = \frac{1}{2} \int_0^{\pi/2} 16 \sin^2(2\theta) d\theta$$

$$= \int_0^{\pi/2} (4 - 4 \cos 4\theta) d\theta$$

$$= 4\theta - \sin 4\theta \Big|_0^{\pi/2}$$

$$= 2\pi \Big|$$

$$\text{Total Area} = 4 \left(2\pi - \frac{2\pi}{3} - \sqrt{3} \right) = \frac{16\pi}{3} - 4\sqrt{3} \text{ unit}^2 \Big|$$

Exercise

Find the area of the region inside the lemniscate $r^2 = 2 \sin 2\theta$ and outside the circle $r = 1$

Solution

$$r^2 = 2 \sin 2\theta = 1 \rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

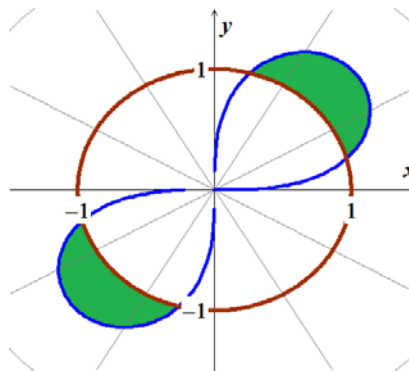
$$\Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12} \Big|$$

$$A = \frac{1}{2} \int_{\pi/12}^{5\pi/12} (2 \sin 2\theta - 1) d\theta$$

$$= -\cos 2\theta - \theta \Big|_{\pi/12}^{5\pi/12}$$

$$= \frac{\sqrt{3}}{2} - \frac{5\pi}{12} + \frac{\sqrt{3}}{2} + \frac{\pi}{12}$$

$$= \sqrt{3} - \frac{\pi}{3} \text{ unit}^2 \Big|$$



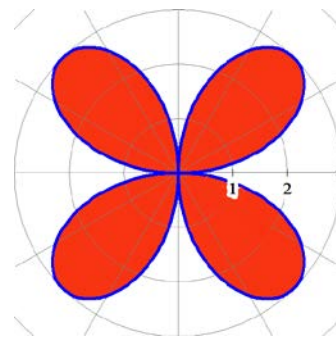
Exercise

Find the area of the region inside all the leaves of the rose $r = 3 \sin 2\theta$

Solution

$$A = \frac{1}{2} (8) \int_0^{\pi/4} 9 \sin^2 2\theta d\theta \quad (\text{Using symmetry } \frac{1}{2} - \text{leaf})$$

$$\begin{aligned}
&= 18 \int_0^{\pi/4} (1 - \cos 4\theta) d\theta \\
&= 18 \left(\theta - \frac{1}{4} \sin 4\theta \right) \Big|_0^{\pi/4} \\
&= 18 \left(\frac{\pi}{4} \right) \\
&= \underline{\underline{\frac{9\pi}{2} \text{ unit}^2}}
\end{aligned}$$

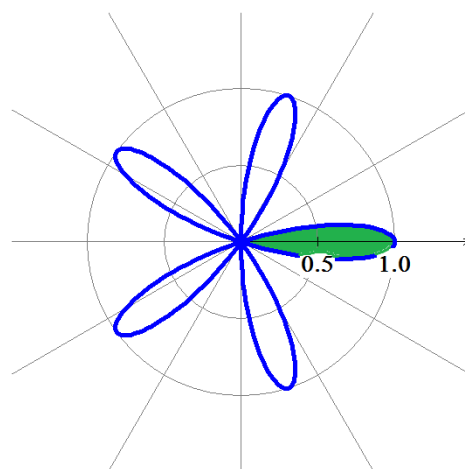


Exercise

Find the area of the region inside one leaf of the rose $r = \cos 5\theta$

Solution

$$\begin{aligned}
0 \leq 5\theta \leq \frac{\pi}{2} &\rightarrow 0 \leq \theta \leq \frac{\pi}{10} \\
A &= \frac{1}{2} (2) \int_0^{\pi/10} \cos^2 5\theta d\theta \quad (\text{Using symmetry } \frac{1}{2} - \text{leaf}) \\
&= \frac{1}{2} \int_0^{\pi/10} (1 + \cos 10\theta) d\theta \\
&= \frac{1}{2} \left(\theta + \frac{1}{10} \sin 10\theta \right) \Big|_0^{\pi/10} \\
&= \underline{\underline{\frac{\pi}{20} \text{ unit}^2}}
\end{aligned}$$

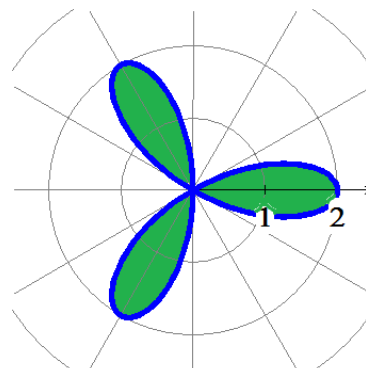


Exercise

Find the area of the region of a complete three-leaf rose $r = 2 \cos 3\theta$

Solution

$$\begin{aligned}
0 \leq 3\theta \leq \frac{\pi}{2} &\rightarrow 0 \leq \theta \leq \frac{\pi}{6} \\
A &= (6) \frac{1}{2} \int_0^{\pi/6} 4 \cos^2 3\theta d\theta \quad (\text{Using symmetry } \frac{1}{2} - \text{leaf}) \\
&= 6 \int_0^{\pi/6} (1 + \cos 6\theta) d\theta \\
&= 6 \left(\theta + \frac{1}{6} \sin 6\theta \right) \Big|_0^{\pi/6} \\
&= \underline{\underline{\pi \text{ unit}^2}}
\end{aligned}$$



Exercise

Find the area of the region inside the rose $r = 4 \cos 2\theta$ and outside the circle $r = 2$

Solution

$$r = 4 \cos 2\theta = 2 \rightarrow \cos 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{3}$$

$$0 \leq \theta \leq \frac{\pi}{6}$$

$$A = (8) \frac{1}{2} \int_0^{\pi/6} (16 \cos^2 2\theta - 4) d\theta \quad (\text{Using symmetry } \frac{1}{2} - \text{leaf})$$

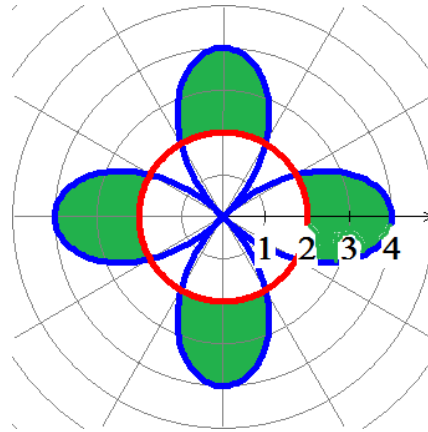
$$= 4 \int_0^{\pi/6} (8 + 8 \cos 4\theta - 4) d\theta$$

$$= 4 \int_0^{\pi/6} (4 + 8 \cos 4\theta) d\theta$$

$$= 4 \left(4\theta + 2 \sin 4\theta \right) \Big|_0^{\pi/6}$$

$$= 4 \left(\frac{2\pi}{3} + \sqrt{3} \right)$$

$$= \frac{8\pi}{3} + 4\sqrt{3} \text{ unit}^2$$



Exercise

Find the area of the region inside one leaf of $r = \cos 3\theta$

Solution

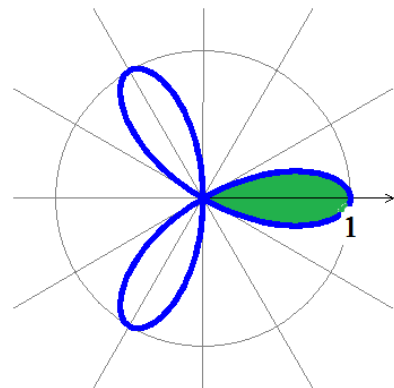
$$0 \leq 3\theta \leq \frac{\pi}{2} \rightarrow 0 \leq \theta \leq \frac{\pi}{6}$$

$$A = (2) \frac{1}{2} \int_0^{\pi/6} \cos^2 3\theta d\theta \quad (\text{Using symmetry } \frac{1}{2} - \text{leaf})$$

$$= \frac{1}{2} \int_0^{\pi/6} (1 + \cos 6\theta) d\theta$$

$$= \frac{1}{2} \left(\theta + \frac{1}{6} \sin 6\theta \right) \Big|_0^{\pi/6}$$

$$= \frac{\pi}{12} \text{ unit}^2$$

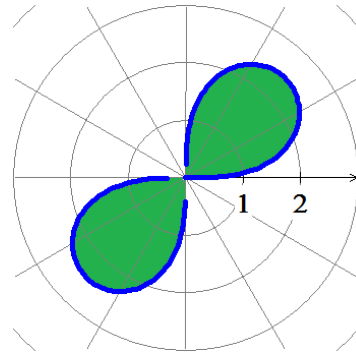


Exercise

Find the area of the region bounded by the lemniscate $r^2 = 6 \sin 2\theta$

Solution

$$\begin{aligned}
 A &= (2) \frac{1}{2} \int_0^{\pi/2} 6 \sin 2\theta \, d\theta \quad (\text{Using symmetry}) \\
 &= 6 \left(-\frac{1}{2} \cos 2\theta \right) \Big|_0^{\pi/2} \\
 &= \underline{6 \text{ unit}^2}
 \end{aligned}$$

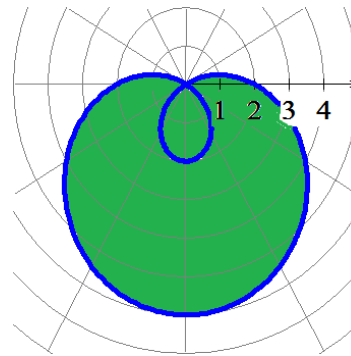


Exercise

Find the area of the region bounded by the limaçon $r = 2 - 4 \sin \theta$

Solution

$$\begin{aligned}
 2 - 4 \sin \theta &= 0 \rightarrow \sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6} \\
 A &= (2) \frac{1}{2} \int_{-\pi/2}^{\pi/6} (2 - 4 \sin \theta)^2 \, d\theta \\
 &= \int_{-\pi/2}^{\pi/6} (4 - 16 \sin \theta + 16 \sin^2 \theta) \, d\theta \\
 &= \int_{-\pi/2}^{\pi/6} (4 - 16 \sin \theta + 8 - 8 \cos 2\theta) \, d\theta \\
 &= (12\theta + 16 \cos \theta - 4 \sin 2\theta) \Big|_{-\pi/2}^{\pi/6} \\
 &= 2\pi + 8\sqrt{3} - 2\sqrt{3} + 6\pi \\
 &= \underline{8\pi + 6\sqrt{3} \text{ unit}^2}
 \end{aligned}$$

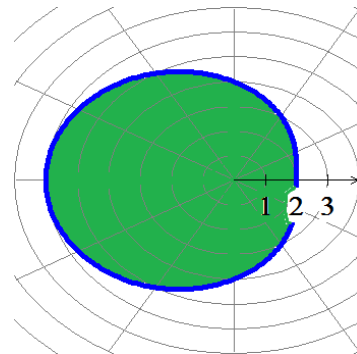


Exercise

Find the area of the region bounded by the limaçon $r = 4 - 2 \cos \theta$

Solution

$$\begin{aligned}
 A &= \frac{1}{2} \int_0^{2\pi} (4 - 2 \cos \theta)^2 \, d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} (16 - 16 \cos \theta + 4 \cos^2 \theta) \, d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} (16 - 16 \cos \theta + 2 + 2 \cos 2\theta) \, d\theta
 \end{aligned}$$



$$= \frac{1}{2} (18\theta - 16\sin\theta - \sin 2\theta) \Big|_0^{2\pi}$$

$$= 18\pi \text{ unit}^2$$

Exercise

Find the area of the given region inner loop of $r = 1 + 2\cos\theta$

Solution

$$r = 1 + 2\cos\theta = 0 \rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}$$

$$r = 1 + 2\cos\theta = -1 \rightarrow \cos\theta = -1 \Rightarrow \theta = \pi$$

$$A = 2 \cdot \frac{1}{2} \int_{2\pi/3}^{\pi} (1 + 2\cos\theta)^2 d\theta$$

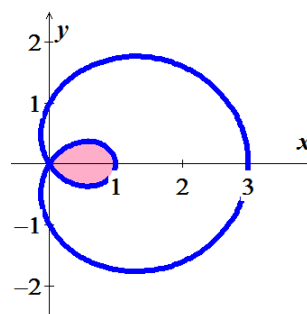
$$= \int_{2\pi/3}^{\pi} (1 + 4\cos\theta + 4\cos^2\theta) d\theta$$

$$= \int_{2\pi/3}^{\pi} (1 + 4\cos\theta + 2 + 2\cos 2\theta) d\theta$$

$$= 3\theta + 4\sin\theta + \sin 2\theta \Big|_{2\pi/3}^{\pi}$$

$$= 3\pi - 2\pi - 2\sqrt{3} + \frac{\sqrt{3}}{2}$$

$$= \pi - \frac{3\sqrt{3}}{2}$$



Exercise

Find the area of the given region Inner loop of $r = 2 - 4\cos\theta$

Solution

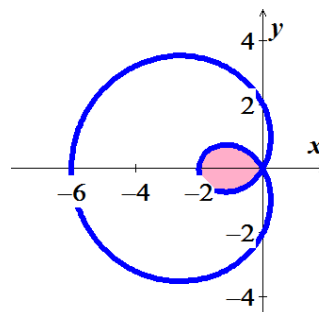
$$r = 2 - 4\cos\theta = 0 \rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

$$r = 2 - 4\cos\theta = -2 \rightarrow \cos\theta = 1 \Rightarrow \theta = 0$$

$$A = 2 \cdot \frac{1}{2} \int_0^{\pi/3} (2 - 4\cos\theta)^2 d\theta$$

$$= \int_0^{\pi/3} (4 - 16\cos\theta + 16\cos^2\theta) d\theta$$

$$= \int_0^{\pi/3} (4 - 16\cos\theta + 8 + 8\cos 2\theta) d\theta$$



$$\begin{aligned}
&= (12\theta - 16\sin\theta + 4\sin 2\theta) \Big|_0^{\pi/3} \\
&= 4\pi - 8\sqrt{3} + 2\sqrt{3} \\
&= \underline{4\pi - 6\sqrt{3}}
\end{aligned}$$

Exercise

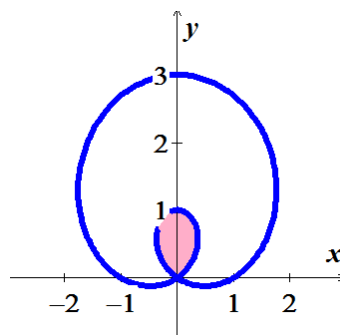
Find the area of the given region Inner loop of $r = 1 + 2\sin\theta$

Solution

$$r = 1 + 2\sin\theta = 0 \rightarrow \sin\theta = -\frac{1}{2} \Rightarrow \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\frac{7\pi}{6} \leq \theta \leq \frac{11\pi}{6}$$

$$\begin{aligned}
A &= \frac{1}{2} \int_{7\pi/6}^{11\pi/6} (1 + 2\sin\theta)^2 d\theta \\
&= \frac{1}{2} \int_{7\pi/6}^{11\pi/6} (1 + 4\sin\theta + 4\sin^2\theta) d\theta \\
&= \frac{1}{2} \int_{7\pi/6}^{11\pi/6} (1 + 4\sin\theta + 2 - 2\cos 2\theta) d\theta \\
&= \frac{1}{2} (3\theta - 4\cos\theta - \sin 2\theta) \Big|_{7\pi/6}^{11\pi/6} \\
&= \frac{1}{2} \left(\frac{11\pi}{2} - 2\sqrt{3} + \frac{\sqrt{3}}{2} - \frac{7\pi}{2} - 2\sqrt{3} + \frac{\sqrt{3}}{2} \right) \\
&= \underline{\frac{1}{2} (2\pi - 3\sqrt{3})}
\end{aligned}$$



Exercise

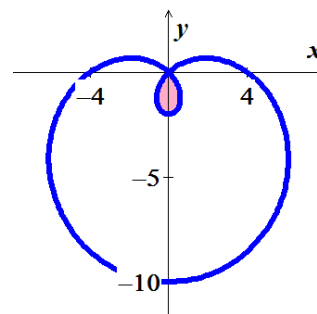
Find the area of the given region Inner loop of $r = 4 - 6\sin\theta$

Solution

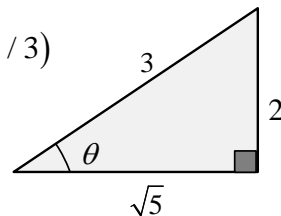
$$r = 4 - 6\sin\theta = 0 \rightarrow \sin\theta = \frac{2}{3} \Rightarrow \theta = \arcsin \frac{2}{3}$$

$$r = 4 - 6\sin\theta = -2 \rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$$

$$\begin{aligned}
A &= 2 \cdot \frac{1}{2} \int_{\arcsin 2/3}^{\pi/2} (4 - 6\sin\theta)^2 d\theta \\
&= \int_{\arcsin 2/3}^{\pi/2} (16 - 48\sin\theta + 36\sin^2\theta) d\theta
\end{aligned}$$



$$\begin{aligned}
&= \int_{\arcsin 2/3}^{\pi/2} (16 - 48 \sin \theta + 18 - 18 \cos 2\theta) d\theta \\
&= (34\theta + 48 \cos \theta - 9 \sin 2\theta) \Big|_{\arcsin 2/3}^{\pi/2} \\
&= 17\pi - 34 \arcsin \frac{2}{3} + 48 \cos(\arcsin 2/3) - 18 \sin(\arcsin 2/3) \cos(\arcsin 2/3) \\
&= 17\pi - 34 \arcsin \frac{2}{3} + 16\sqrt{5} - 4\sqrt{5} \\
&= \underline{17\pi - 34 \arcsin \frac{2}{3} + 12\sqrt{5}}
\end{aligned}$$



Exercise

Find the area of the given region between the loops $r = 1 + 2 \cos \theta$

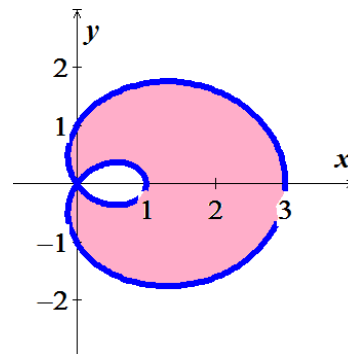
Solution

$$r = 1 + 2 \cos \theta = 0 \rightarrow \cos \theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}$$

$$r = 1 + 2 \cos \theta = -1 \rightarrow \cos \theta = -1 \Rightarrow \theta = \pi$$

Area of the inner loop:

$$\begin{aligned}
A_1 &= \int_{2\pi/3}^{\pi} (1 + 2 \cos \theta)^2 d\theta \\
&= \int_{2\pi/3}^{\pi} (1 + 4 \cos \theta + 2 + 2 \cos 2\theta) d\theta \\
&= (3\theta + 4 \sin \theta + \sin 2\theta) \Big|_{2\pi/3}^{\pi} \\
&= 3\pi - 2\pi - 2\sqrt{3} + \frac{\sqrt{3}}{2} \\
&= \underline{\pi - \frac{3\sqrt{3}}{2}}
\end{aligned}$$



Area of the outer loop:

$$\begin{aligned}
A_2 &= 2 \frac{1}{2} \int_0^{2\pi/3} (1 + 2 \cos \theta)^2 d\theta \\
&= \int_0^{2\pi/3} (1 + 4 \cos \theta + 2 + 2 \cos 2\theta) d\theta \\
&= (3\theta + 4 \sin \theta + \sin 2\theta) \Big|_0^{2\pi/3} \\
&= 2\pi + 2\sqrt{3} - \frac{\sqrt{3}}{2} \\
&= \underline{2\pi + \frac{3\sqrt{3}}{2}}
\end{aligned}$$

$$\begin{aligned}\text{Are between the loops: } &= 2\pi + \frac{3\sqrt{3}}{2} - \left(\pi - \frac{3\sqrt{3}}{2}\right) \\ &= \pi + 3\sqrt{3}\end{aligned}$$

Exercise

Find the area of the given region between the loops $r = 2(1 + 2\sin\theta)$

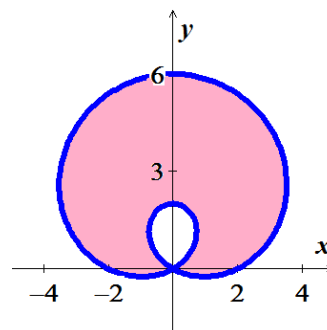
Solution

$$r = 1 + 2\sin\theta = 0 \rightarrow \sin\theta = -\frac{1}{2} \Rightarrow \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$r = 2(1 + 2\sin\theta) = -2 \rightarrow \sin\theta = -1 \Rightarrow \theta = \frac{3\pi}{2}$$

Area inside of the inner loop:

$$\begin{aligned}A_1 &= 2 \cdot \frac{1}{2} \int_{7\pi/6}^{3\pi/2} (2 + 4\sin\theta)^2 d\theta \\ &= 4 \int_{7\pi/6}^{3\pi/2} (1 + 4\sin\theta + 4\sin^2\theta) d\theta \\ &= 4 \int_{7\pi/6}^{3\pi/2} (1 + 4\sin\theta + 2 - 2\cos 2\theta) d\theta \\ &= 4(3\theta - 4\cos\theta - \sin 2\theta) \Big|_{7\pi/6}^{3\pi/2} \\ &= 4\left(\frac{9\pi}{2} - \frac{7\pi}{2} - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right) \\ &= 4\pi - 6\sqrt{3}\end{aligned}$$



Area inside of the outer loop:

$$r = 2(1 + 2\sin\theta) = 3 \rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$$

$$\begin{aligned}A_2 &= 2 \frac{1}{2} \int_{-\pi/6}^{\pi/2} 4(1 + 2\sin\theta)^2 d\theta \\ &= 4 \int_{-\pi/6}^{\pi/2} (1 + 4\sin\theta + 4\sin^2\theta) d\theta \\ &= 4 \int_{-\pi/6}^{\pi/2} (1 + 4\sin\theta + 2 - 2\cos 2\theta) d\theta \\ &= 4(3\theta - 4\cos\theta - \sin 2\theta) \Big|_{-\pi/6}^{\pi/2}\end{aligned}$$

$$\begin{aligned}
&= 4 \left(\frac{3\pi}{2} + \frac{\pi}{2} + 2\sqrt{3} - \frac{\sqrt{3}}{2} \right) \\
&= \underline{8\pi + 6\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
\text{The area between the loops} &= 8\pi + 6\sqrt{3} - (4\pi - 6\sqrt{3}) \\
&= \underline{4\pi + 12\sqrt{3}}
\end{aligned}$$

Exercise

Find the area of the given region between the loops $r = 3 - 6\sin\theta$

Solution

$$r = 3 - 6\sin\theta = 0 \rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$r = 3 - 6\sin\theta = -3 \rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$$

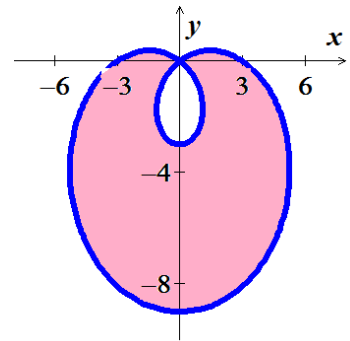
$$r = 3 - 6\sin\theta = 6 \rightarrow \sin\theta = -1 \Rightarrow \theta = \frac{3\pi}{2}$$

Area inside of the *inner* loop:

$$\begin{aligned}
A_1 &= 2 \cdot \frac{1}{2} \int_{\pi/6}^{\pi/2} 9(1 - 2\sin\theta)^2 d\theta \\
&= 9 \int_{\pi/6}^{\pi/2} (1 - 4\sin\theta + 4\sin^2\theta) d\theta \\
&= 9 \int_{\pi/6}^{\pi/2} (1 - 4\sin\theta + 2 - 2\cos 2\theta) d\theta \\
&= 9(3\theta + 4\cos\theta - \sin 2\theta) \Big|_{\pi/6}^{\pi/2} \\
&= 9 \left(\frac{3\pi}{2} - \frac{\pi}{2} - 2\sqrt{3} + \frac{\sqrt{3}}{2} \right) \\
&= \underline{9\pi - \frac{27\sqrt{3}}{2}}
\end{aligned}$$

Area inside of the *outer* loop:

$$\begin{aligned}
A_2 &= 2 \cdot \frac{1}{2} \int_{5\pi/6}^{3\pi/2} 9(1 - 2\sin\theta)^2 d\theta \\
&= 9 \int_{5\pi/6}^{3\pi/2} (1 - 4\sin\theta + 4\sin^2\theta) d\theta \\
&= 9 \int_{5\pi/6}^{3\pi/2} (1 - 4\sin\theta + 2 - 2\cos 2\theta) d\theta
\end{aligned}$$



$$\begin{aligned}
&= 9(3\theta + 4\cos\theta - \sin 2\theta) \Big|_{5\pi/6}^{3\pi/2} \\
&= 9\left(\frac{9\pi}{2} - \frac{5\pi}{2} + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right) \\
&= 18\pi + \frac{27\sqrt{3}}{2}
\end{aligned}$$

$$\begin{aligned}
\text{The area between the loops} &= 18\pi + \frac{27\sqrt{3}}{2} - \left(9\pi - \frac{27\sqrt{3}}{2}\right) \\
&= \underline{9\pi + 27\sqrt{3}}
\end{aligned}$$

Exercise

Find the area of the given region between the loops $r = \frac{1}{2} + \cos\theta$

Solution

$$r = \frac{1}{2} + \cos\theta = 0 \rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$r = \frac{1}{2} + \cos\theta = -\frac{1}{2} \rightarrow \cos\theta = -1 \Rightarrow \theta = \pi$$

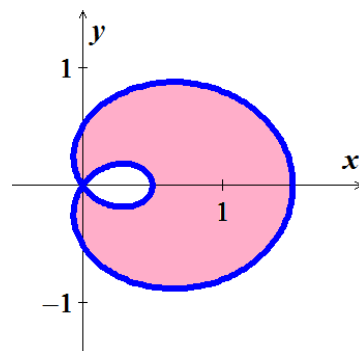
$$r = \frac{1}{2} + \cos\theta = \frac{3}{2} \rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

Area inside of the *inner* loop:

$$\begin{aligned}
A_1 &= 2 \cdot \frac{1}{2} \int_{2\pi/3}^{\pi} \left(\frac{1}{2} + \cos\theta\right)^2 d\theta \\
&= \int_{2\pi/3}^{\pi} \left(\frac{1}{4} + \cos\theta + \cos^2\theta\right) d\theta \\
&= \int_{2\pi/3}^{\pi} \left(\frac{1}{4} + \cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta \\
&= \left(\frac{3}{4}\theta + \sin\theta + \frac{1}{4}\sin 2\theta\right) \Big|_{2\pi/3}^{\pi} \\
&= \frac{3\pi}{4} - \frac{\pi}{2} - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{8} \\
&= \underline{\frac{\pi}{4} - \frac{3\sqrt{3}}{8}}
\end{aligned}$$

Area inside of the *outer* loop:

$$A_2 = 2 \cdot \frac{1}{2} \int_0^{2\pi/3} \left(\frac{1}{2} + \cos\theta\right)^2 d\theta$$



$$\begin{aligned}
&= \int_0^{2\pi/3} \left(\frac{1}{4} + \cos \theta + \cos^2 \theta \right) d\theta \\
&= \int_0^{2\pi/3} \left(\frac{1}{4} + \cos \theta + \frac{1}{2} + \frac{1}{2} \cos 2\theta \right) d\theta \\
&= \left(\frac{3}{4} \theta + \sin \theta + \frac{1}{4} \sin 2\theta \right) \Big|_0^{2\pi/3} \\
&= \frac{\pi}{2} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8} \\
&= \frac{\pi}{2} + \frac{3\sqrt{3}}{8}
\end{aligned}$$

$$\begin{aligned}
\text{The area between the loops} &= \frac{\pi}{2} + \frac{3\sqrt{3}}{8} - \left(\frac{\pi}{4} - \frac{3\sqrt{3}}{8} \right) \\
&= \frac{\pi}{4} + \frac{3\sqrt{3}}{4}
\end{aligned}$$

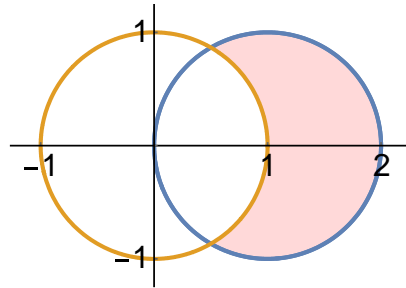
Exercise

Find the area of the given region inside $r = 2\cos \theta$ and outside $r = 1$

Solution

$$r = 2\cos \theta = 1 \rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

$$\begin{aligned}
A &= 2 \cdot \frac{1}{2} \int_0^{\pi/3} \left[(2\cos \theta)^2 - 1 \right] d\theta \\
&= \int_0^{\pi/3} (4\cos^2 \theta - 1) d\theta \\
&= \int_0^{\pi/3} (2 + 2\cos 2\theta - 1) d\theta \\
&= \theta + \sin 2\theta \Big|_0^{\pi/3} \\
&= \frac{\pi}{3} + \frac{\sqrt{3}}{2}
\end{aligned}$$



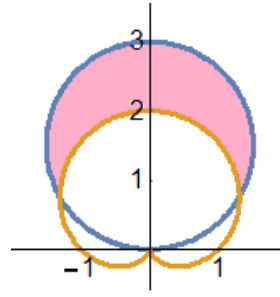
Exercise

Find the area of the given region inside $r = 3\sin \theta$ and outside $r = 1 + \sin \theta$

Solution

$$r = 3\sin \theta = 1 + \sin \theta \rightarrow \sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$$

$$\begin{aligned}
A &= 2 \cdot \frac{1}{2} \int_{\pi/6}^{\pi/2} \left[(3\sin\theta)^2 - (1 + \sin\theta)^2 \right] d\theta \\
&= \int_{\pi/6}^{\pi/2} (9\sin^2\theta - 1 - 2\sin\theta - \sin^2\theta) d\theta \\
&= \int_{\pi/6}^{\pi/2} (4 - 4\cos 2\theta - 1 - 2\sin\theta) d\theta \\
&= 3\theta - 2\sin 2\theta + 2\cos\theta \Big|_{\pi/6}^{\pi/2} \\
&= \frac{3\pi}{2} - \frac{\pi}{2} + \sqrt{3} - \sqrt{3} \\
&= \pi
\end{aligned}$$



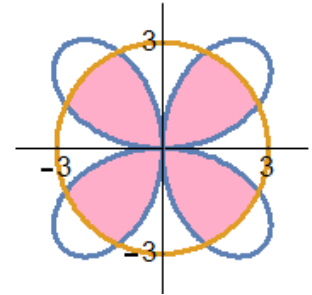
Exercise

Find the area of the given region common interior of $r = 4\sin 2\theta$ and $r = 2$

Solution

$$r = 4\sin 2\theta = 2 \rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6} \rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

$$\begin{aligned}
A &= 4 \cdot \frac{1}{2} \int_0^{\pi/12} (4\sin 2\theta)^2 d\theta + 4 \cdot \frac{1}{2} \int_{\pi/12}^{5\pi/12} (2)^2 d\theta + 4 \cdot \frac{1}{2} \int_{5\pi/12}^{\pi/2} (4\sin 2\theta)^2 d\theta \\
&= 32 \int_0^{\pi/12} \sin^2 2\theta d\theta + 8\theta \Big|_{\pi/12}^{5\pi/12} + 32 \int_{5\pi/12}^{\pi/2} \sin^2 2\theta d\theta \\
&= 16 \int_0^{\pi/12} (1 - \cos 4\theta) d\theta + 8\left(\frac{5\pi}{12} - \frac{\pi}{12}\right) + 16 \int_{5\pi/12}^{\pi/2} (1 - \cos 4\theta) d\theta \\
&= \frac{8\pi}{3} + 16\left(\theta - \frac{1}{4}\sin 4\theta\right) \Big|_0^{\pi/12} + 16\left(\theta - \frac{1}{4}\sin 4\theta\right) \Big|_{5\pi/12}^{\pi/2} \\
&= \frac{8\pi}{3} + 16\left(\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right) + 16\left(\frac{\pi}{2} - \frac{5\pi}{12} - \frac{\sqrt{3}}{8}\right) \\
&= \frac{8\pi}{3} + \frac{4\pi}{3} - 2\sqrt{3} + \frac{4\pi}{3} - 2\sqrt{3} \\
&= \frac{16\pi}{3} - 4\sqrt{3}
\end{aligned}$$



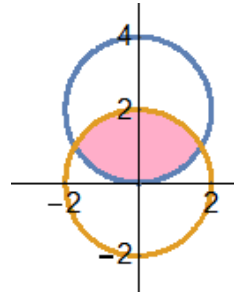
Exercise

Find the area of the given region common interior of $r = 4\sin\theta$ and $r = 2$

Solution

$$r = 4\sin\theta = 2 \rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$$

$$\begin{aligned} A &= 2 \cdot \frac{1}{2} \int_0^{\pi/6} 16\sin^2\theta \, d\theta + 2 \cdot \frac{1}{2} \int_{\pi/6}^{\pi/2} 4 \, d\theta \\ &= 8 \int_0^{\pi/6} (1 - \cos 2\theta) \, d\theta + 4\theta \Big|_{\pi/6}^{\pi/2} \\ &= 8 \left(\theta - \frac{1}{2} \sin 2\theta \right) \Big|_0^{\pi/6} + 4 \left(\frac{\pi}{2} - \frac{\pi}{6} \right) \\ &= 8 \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4} \right) + \frac{4\pi}{3} \\ &= \frac{8\pi}{3} - 2\sqrt{3} \end{aligned}$$



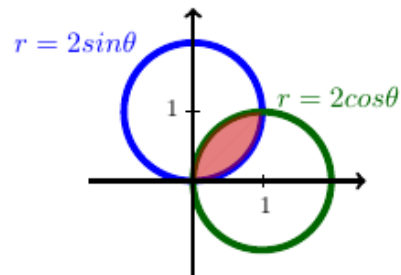
Exercise

Find the area of the given region common interior of $r = 2\cos\theta$ and $r = 2\sin\theta$

Solution

$$r = 2\cos\theta = 2\sin\theta \rightarrow \theta = \frac{\pi}{4}$$

$$\begin{aligned} A &= 2 \cdot \frac{1}{2} \int_{\pi/4}^{\pi/2} 4\cos^2\theta \, d\theta \\ &= 2 \int_{\pi/4}^{\pi/2} (1 + \cos 2\theta) \, d\theta \\ &= 2 \left(\theta + \frac{1}{2} \sin 2\theta \right) \Big|_{\pi/4}^{\pi/2} \\ &= 2 \left(\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2} \right) \\ &= \frac{\pi}{2} - 1 \end{aligned}$$



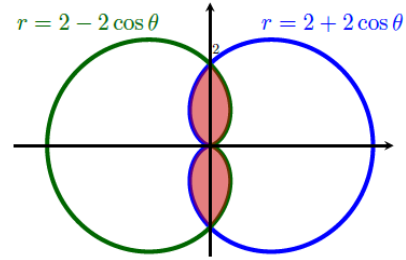
Exercise

Find the area of the given region common interior of $r = 2(1 + \cos \theta)$ and $r = 2(1 - \cos \theta)$

Solution

$$r = 2(1 + \cos \theta) = 2(1 - \cos \theta) \rightarrow \cos \theta = -\cos \theta \Rightarrow \theta = \frac{\pi}{2}$$

$$\begin{aligned} A &= 4 \cdot \frac{1}{2} \int_0^{\pi/2} 4(1 - \cos \theta)^2 d\theta \\ &= 8 \int_0^{\pi/2} (1 - 2\cos \theta + \cos^2 \theta) d\theta \\ &= 8 \int_0^{\pi/2} \left(1 - 2\cos \theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta \\ &= 8 \left(\frac{3}{2}\theta - 2\sin \theta + \frac{1}{4}\sin 2\theta \right) \Big|_0^{\pi/2} \\ &= 8 \left(\frac{3\pi}{4} - 2 \right) \\ &= \underline{6\pi - 16} \end{aligned}$$



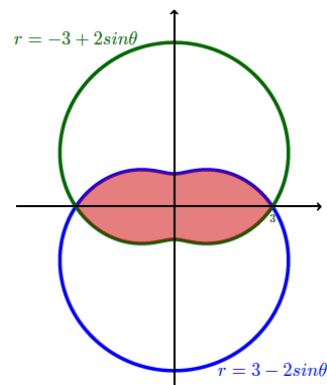
Exercise

Find the area of the given region common interior of $r = 3 - 2\sin \theta$ and $r = -3 + 2\sin \theta$

Solution

$$r = 3 - 2\sin \theta = -3 + 2\sin \theta \rightarrow 4\sin \theta = 6 \quad \text{X}$$

$$\begin{aligned} A &= 4 \cdot \frac{1}{2} \int_0^{\pi/2} (3 - 2\sin \theta)^2 d\theta \\ &= 2 \int_0^{\pi/2} (9 - 12\sin \theta + 4\sin^2 \theta) d\theta \\ &= 2 \int_0^{\pi/2} (9 - 12\sin \theta + 2 - 2\cos 2\theta) d\theta \\ &= 2 (11\theta + 12\cos \theta - \sin 2\theta) \Big|_0^{\pi/2} \\ &= 2 \left(\frac{11\pi}{2} - 12 \right) \\ &= \underline{11\pi - 24} \end{aligned}$$



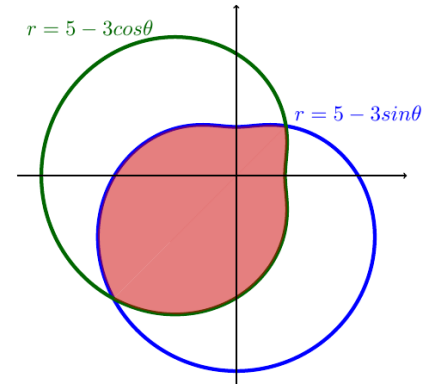
Exercise

Find the area of the given region common interior of $r = 5 - 3\sin\theta$ and $r = 5 - 3\cos\theta$

Solution

$$r = 5 - 3\sin\theta = 5 - 3\cos\theta \rightarrow \sin\theta = \cos\theta \Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$\begin{aligned} A &= 2 \cdot \frac{1}{2} \int_{\pi/4}^{5\pi/4} (5 - 3\sin\theta)^2 d\theta \\ &= \int_{\pi/4}^{5\pi/4} (25 - 30\sin\theta + 9\sin^2\theta) d\theta \\ &= \int_{\pi/4}^{5\pi/4} \left(25 - 30\sin\theta + \frac{9}{2} - \frac{9}{2}\cos 2\theta \right) d\theta \\ &= \frac{59}{2}\theta + 30\cos\theta - \frac{9}{4}\sin 2\theta \Big|_{\pi/4}^{5\pi/4} \\ &= 5 \cdot \frac{59\pi}{8} - 15\sqrt{2} - \frac{9}{4} - \frac{59\pi}{8} - 15\sqrt{2} + \frac{9}{4} \\ &= \frac{59\pi}{2} - 30\sqrt{2} \end{aligned}$$

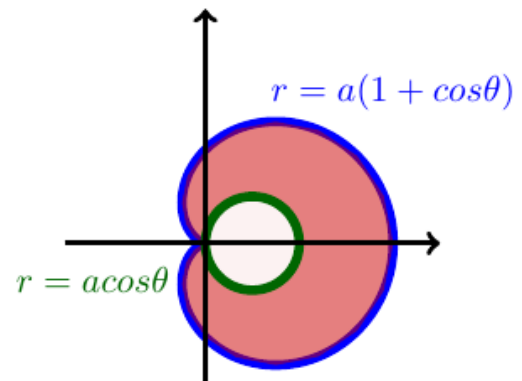


Exercise

Find the area of the region inside $r = a(1 + \cos\theta)$ and outside $r = a\cos\theta$

Solution

$$\begin{aligned} A &= 2 \cdot \frac{1}{2} \int_0^\pi a^2 (1 + \cos\theta)^2 d\theta - (\text{area of a circle}) \\ &= a^2 \int_0^\pi (1 + 2\cos\theta + \cos^2\theta) d\theta - \pi \left(\frac{a}{2} \right)^2 \\ &= a^2 \int_0^\pi \left(1 + 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta \right) d\theta - \frac{\pi a^2}{4} \\ &= a^2 \left(\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta \right) \Big|_0^\pi - \frac{\pi a^2}{4} \\ &= a^2 \left(\frac{3\pi}{2} \right) - \frac{\pi a^2}{4} \\ &= \frac{5\pi a^2}{4} \end{aligned}$$



Exercise

Find the area of the region inside $r = 2a \cos \theta$ and outside $r = a$

Solution

$$A = \text{Area}(2a \cos \theta) - \text{Area of sector} - 2 \times \text{Area}(\text{between } r = 2a \cos \theta \text{ \& lines})$$

$$= \pi a^2 - \frac{\pi}{3} a^2 - 2 \cdot \frac{1}{2} \int_{\pi/3}^{\pi/2} (2a \cos \theta)^2 d\theta$$

$$= \frac{2\pi a^2}{3} - 2a^2 \int_{\pi/3}^{\pi/2} (1 + \cos 2\theta) d\theta$$

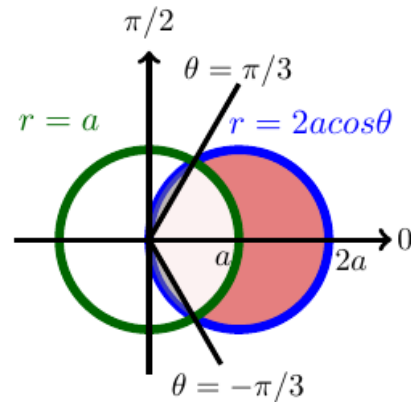
$$= \frac{2\pi a^2}{3} - 2a^2 \left(\theta + \frac{1}{2} \sin 2\theta \right) \Big|_{\pi/3}^{\pi/2}$$

$$= \frac{2\pi a^2}{3} - 2a^2 \left(\frac{\pi}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{4} \right)$$

$$= \frac{2\pi a^2}{3} - \frac{\pi a^2}{3} + \frac{\sqrt{3} a^2}{2}$$

$$= \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \right) a^2$$

$$= \left(\frac{2\pi + 3\sqrt{3}}{6} \right) a^2$$



Exercise

Find the area of the region common interior of $r = a(1 + \cos \theta)$ and $r = a \sin \theta$

Solution

$$A = \frac{1}{2} \int_0^{\pi/2} a^2 \sin^2 \theta d\theta + \frac{1}{2} \int_{\pi/2}^{\pi} a^2 (1 + \cos \theta)^2 d\theta$$

$$= \frac{a^2}{4} \int_0^{\pi/2} (1 - \cos 2\theta) d\theta + \frac{a^2}{2} \int_{\pi/2}^{\pi} \left(1 + 2\cos \theta + \cos^2 \theta \right) d\theta$$

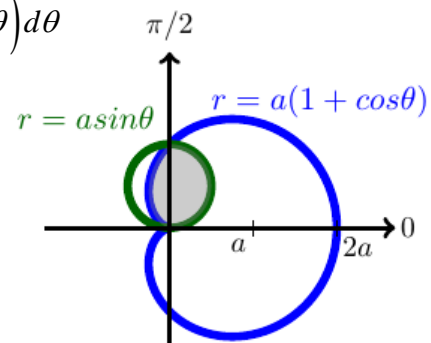
$$= \frac{a^2}{4} \left(\theta - \frac{1}{2} \sin 2\theta \right) \Big|_0^{\pi/2} + \frac{a^2}{2} \int_{\pi/2}^{\pi} \left(1 + 2\cos \theta + \frac{1}{2} + \frac{1}{2} \cos 2\theta \right) d\theta$$

$$= \frac{\pi a^2}{8} + \frac{a^2}{2} \left(\frac{3}{2} \theta + 2 \sin \theta + \frac{1}{4} \sin 2\theta \right) \Big|_{\pi/2}^{\pi}$$

$$= \frac{\pi a^2}{8} + \frac{a^2}{2} \left(\frac{3\pi}{2} - \frac{3\pi}{4} - 2 \right)$$

$$= \frac{\pi a^2}{8} + \frac{3\pi a^2}{8} - a^2$$

$$= (\pi - 2) \frac{a^2}{2}$$



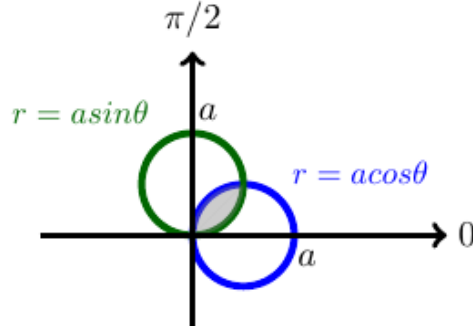
Exercise

Find the area of the region common interior of $r = a \cos \theta$ and $r = a \sin \theta$, where $a > 0$

Solution

$$r = a \cos \theta = a \sin \theta \rightarrow \theta = \frac{\pi}{4}$$

$$\begin{aligned} A &= 2 \cdot \frac{1}{2} \int_0^{\pi/4} a^2 \sin^2 \theta \, d\theta \\ &= \frac{a^2}{2} \int_0^{\pi/4} (1 - \cos 2\theta) \, d\theta \\ &= \frac{a^2}{2} \left(\theta - \frac{1}{2} \sin 2\theta \right) \Big|_0^{\pi/4} \\ &= \frac{a^2}{2} \left(\frac{\pi}{4} - \frac{1}{2} \right) \\ &= \boxed{(\pi - 2) \frac{a^2}{8}} \end{aligned}$$



Exercise

Find the length of the spiral $r = \theta^2$, $0 \leq \theta \leq \sqrt{5}$

Solution

$$r = \theta^2 \Rightarrow \frac{dr}{d\theta} = 2\theta$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{\theta^4 + 4\theta^2} = |\theta| \sqrt{\theta^2 + 4}$$

$$L = \int_0^{\sqrt{5}} \theta \sqrt{\theta^2 + 4} \, d\theta$$

$$= \frac{1}{2} \int_4^9 u^{1/2} \, du$$

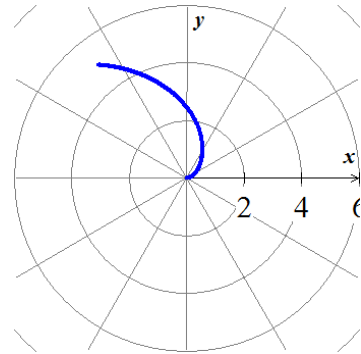
$$= \frac{1}{2} \left[\frac{2}{3} u^{3/2} \right]_4^9$$

$$= \frac{1}{9} (9^{3/2} - 4^{3/2})$$

$$= \boxed{\frac{19}{3} \text{ unit}}$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

$$u = \theta^2 + 4 \rightarrow du = 2\theta d\theta \Rightarrow \boxed{\theta d\theta = \frac{1}{2} du} \quad \begin{cases} \theta = \sqrt{5} & u = 9 \\ \theta = 0 & u = 4 \end{cases}$$



Exercise

Find the length of the spiral $r = \frac{e^\theta}{\sqrt{2}}$, $0 \leq \theta \leq \pi$

Solution

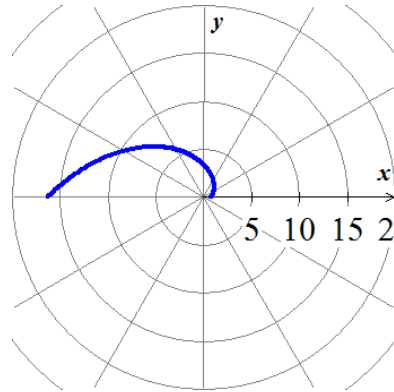
$$r = \frac{e^\theta}{\sqrt{2}} \Rightarrow \frac{dr}{d\theta} = \frac{1}{\sqrt{2}} e^\theta$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{\frac{1}{2}e^{2\theta} + \frac{1}{2}e^{2\theta}} = \sqrt{e^{2\theta}} = e^\theta$$

$$L = \int_0^\pi e^\theta d\theta \qquad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

$$= \left[e^\theta \right]_0^\pi$$

$$= e^\pi - 1$$



Exercise

Find the length of the curve $r = a \sin^2\left(\frac{\theta}{2}\right)$, $0 \leq \theta \leq \pi$, $a > 0$

Solution

$$r = a \sin^2\left(\frac{\theta}{2}\right) \Rightarrow \frac{dr}{d\theta} = a \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right)$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2 \sin^4\left(\frac{\theta}{2}\right) + a^2 \sin^2\left(\frac{\theta}{2}\right) \cos^2\left(\frac{\theta}{2}\right)}$$

$$= a \sin\left(\frac{\theta}{2}\right) \sqrt{\sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right)}$$

$$\sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right) = 1$$

$$= a \left| \sin\left(\frac{\theta}{2}\right) \right|$$

$$L = \int_0^\pi a \sin\left(\frac{\theta}{2}\right) d\theta$$

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

$$= \left[-2a \cos\left(\frac{\theta}{2}\right) \right]_0^\pi$$

$$= -2a \left(\cos\left(\frac{\pi}{2}\right) - \cos 0 \right)$$

$$= 2a$$

Exercise

Find the length of the parabolic segment $r = \frac{6}{1 + \cos \theta}$, $0 \leq \theta \leq \frac{\pi}{2}$

Solution

$$r = \frac{6}{1 + \cos \theta} \Rightarrow \frac{dr}{d\theta} = \frac{6 \sin \theta}{(1 + \cos \theta)^2}$$

$$\begin{aligned} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} &= \sqrt{\frac{36}{(1 + \cos \theta)^2} + \frac{36 \sin^2 \theta}{(1 + \cos \theta)^4}} \\ &= \frac{6}{|1 + \cos \theta|} \sqrt{1 + \frac{\sin^2 \theta}{(1 + \cos \theta)^2}} \\ &= \frac{6}{|1 + \cos \theta|} \sqrt{\frac{1 + 2 \cos \theta + \cos^2 \theta + \sin^2 \theta}{(1 + \cos \theta)^2}} \\ &= \frac{6}{(1 + \cos \theta)^2} \sqrt{2 + 2 \cos \theta} \\ &= \frac{6\sqrt{2}}{(1 + \cos \theta)^2} (1 + \cos \theta)^{1/2} \\ &= \frac{6\sqrt{2}}{(1 + \cos \theta)^{3/2}} \end{aligned}$$

$$L = \int_0^{\pi/2} \frac{6\sqrt{2}}{(1 + \cos \theta)^{3/2}} d\theta$$

$$= 6\sqrt{2} \int_0^{\pi/2} \frac{d\theta}{\left(2 \cos^2 \frac{\theta}{2}\right)^{3/2}}$$

$$= 6\sqrt{2} \int_0^{\pi/2} \frac{d\theta}{2^{3/2} \cos^3 \frac{\theta}{2}}$$

$$= \frac{6}{2} \int_0^{\pi/2} \sec^3 \frac{\theta}{2} d\theta$$

$$= 3 \int_0^{\pi/2} \sec^3 \frac{\theta}{2} d\theta$$

$$= 6 \int_0^{\pi/4} \sec^3 u du$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

$$u = \frac{\theta}{2} \rightarrow du = \frac{1}{2} d\theta \quad \begin{cases} \theta = \frac{\pi}{2} & u = \frac{\pi}{4} \\ \theta = 0 & u = 0 \end{cases}$$

$$\begin{aligned}
&= 6 \left(\left[\frac{1}{2} \sec u \tan u \right]_0^{\pi/4} + \frac{1}{2} \int_0^{\pi/4} \sec u \, du \right) \\
&= 6 \left(\frac{1}{2} (\sqrt{2}(1) - 0) + \frac{1}{2} \left[\ln |\sec u + \tan u| \right]_0^{\pi/4} \right) \\
&= 6 \left(\frac{\sqrt{2}}{2} + \frac{1}{2} (\ln |\sqrt{2} + 1| - \ln 1) \right) \\
&= \underline{3\sqrt{2} + 3\ln(\sqrt{2} + 1) \text{ unit}}
\end{aligned}$$

Exercise

Find the length of the curve $r = \cos^3\left(\frac{\theta}{3}\right)$, $0 \leq \theta \leq \frac{\pi}{4}$

Solution

$$r = \cos^3\left(\frac{\theta}{3}\right) \Rightarrow \frac{dr}{d\theta} = -\cos^2\left(\frac{\theta}{3}\right) \sin\left(\frac{\theta}{3}\right)$$

$$\begin{aligned}
\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} &= \sqrt{\cos^6\left(\frac{\theta}{3}\right) + \cos^4\left(\frac{\theta}{3}\right) \sin^2\left(\frac{\theta}{3}\right)} \\
&= \left| \cos^2\left(\frac{\theta}{3}\right) \right| \sqrt{\cos^2\left(\frac{\theta}{3}\right) + \sin^2\left(\frac{\theta}{3}\right)} \\
&= \cos^2\left(\frac{\theta}{3}\right)
\end{aligned}$$

$$L = \int_0^{\pi/4} \cos^2\left(\frac{\theta}{3}\right) d\theta \qquad L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

$$= \frac{1}{2} \int_0^{\pi/4} \left(1 + \cos \frac{2\theta}{3}\right) d\theta$$

$$= \frac{1}{2} \left[\theta + \frac{3}{2} \sin \frac{2\theta}{3} \right]_0^{\pi/4}$$

$$= \frac{1}{2} \left(\frac{\pi}{4} + \frac{3}{2} \sin \frac{\pi}{6} - 0 \right)$$

$$= \frac{1}{2} \left(\frac{\pi}{4} + \frac{3}{4} \right)$$

$$= \underline{\frac{\pi}{8} + \frac{3}{8} \text{ unit}}$$

Exercise

Find the length of the curve $r = \sqrt{1 + \sin 2\theta}$, $0 \leq \theta \leq \pi\sqrt{2}$

Solution

$$r = \sqrt{1 + \sin 2\theta} \Rightarrow \frac{dr}{d\theta} = \frac{1}{2}(1 + \sin 2\theta)^{-1/2} (2 \cos 2\theta) = \cos 2\theta (1 + \sin 2\theta)^{-1/2}$$

$$\begin{aligned}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} &= \sqrt{1 + \sin 2\theta + \cos^2 2\theta (1 + \sin 2\theta)^{-1}} \\&= \sqrt{1 + \sin 2\theta + \frac{\cos^2 2\theta}{1 + \sin 2\theta}} \\&= \sqrt{\frac{1 + 2\sin 2\theta + \sin^2 2\theta + \cos^2 2\theta}{1 + \sin 2\theta}} \\&= \sqrt{\frac{2 + 2\sin 2\theta}{1 + \sin 2\theta}} \\&= \sqrt{\frac{2(1 + \sin 2\theta)}{1 + \sin 2\theta}} \\&= \sqrt{2}\end{aligned}$$

$$\begin{aligned}L &= \int_0^{\pi\sqrt{2}} \sqrt{2} \, d\theta \\&= \sqrt{2}\theta \Big|_0^{\pi\sqrt{2}} \\&= \sqrt{2}(\pi\sqrt{2} - 0) \\&= \underline{2\pi \text{ unit}}\end{aligned}$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

Exercise

Find the length of $r = 8$ $0 \leq \theta \leq 2\pi$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{64 + 0} = 8$$

$$\begin{aligned}L &= \int_0^{2\pi} 8 \, d\theta \\&= \underline{16\pi \text{ unit}}\end{aligned}$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

Exercise

Find the length of $r = a \quad 0 \leq \theta \leq 2\pi$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2 + 0} = \underline{a}$$

$$\begin{aligned} L &= \int_0^{2\pi} a \, d\theta \\ &= \underline{2\pi a \text{ unit}} \end{aligned}$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

Exercise

Find the length of $r = 4 \sin \theta \quad 0 \leq \theta \leq \pi$

Solution

$$\begin{aligned} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} &= \sqrt{16 \sin^2 \theta + 16 \cos^2 \theta} \\ &= 4\sqrt{\sin^2 \theta + \cos^2 \theta} \\ &= \underline{4} \end{aligned}$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\begin{aligned} L &= \int_0^{\pi} 4 \, d\theta \\ &= \underline{4\pi \text{ unit}} \end{aligned}$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

Exercise

Find the length of $r = 2a \cos \theta \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{4a^2 \cos^2 \theta + 4a^2 \sin^2 \theta} = \underline{2a}$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\begin{aligned} L &= 2a \int_{-\pi/2}^{\pi/2} d\theta \\ &= 2a \left(\frac{\pi}{2} + \frac{\pi}{2} \right) \\ &= \underline{2\pi a \text{ unit}} \end{aligned}$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

Exercise

Find the length of $r = 1 + \sin \theta$ $0 \leq \theta \leq 2\pi$

Solution

$$\begin{aligned}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} &= \sqrt{(1 + \sin \theta)^2 + \cos^2 \theta} \\ &= \sqrt{1 + 2\sin \theta + \sin^2 \theta + \cos^2 \theta} \\ &= \sqrt{2 + 2\sin \theta}\end{aligned}$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\begin{aligned}L &= \sqrt{2} \int_0^{2\pi} \sqrt{1 + \sin \theta} \, d\theta \\ &= \sqrt{2} \int_0^{2\pi} \sqrt{1 + \sin \theta} \frac{\sqrt{1 - \sin \theta}}{\sqrt{1 - \sin \theta}} \, d\theta \\ &= 2\sqrt{2} \int_{\pi/2}^{3\pi/2} \frac{-\cos \theta}{\sqrt{1 - \sin \theta}} \, d\theta \\ &= 2\sqrt{2} \int_{\pi/2}^{3\pi/2} (1 - \sin \theta)^{-1/2} \, d(1 - \sin \theta) \\ &= 4\sqrt{2} \sqrt{1 - \sin \theta} \Big|_{\pi/2}^{3\pi/2} \\ &= 4\sqrt{2} (\sqrt{2} - 0) \\ &= 8 \text{ unit}\end{aligned}$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

$$\cos \theta = \pm \sqrt{1 - \sin^2 \theta}$$

$$\frac{\pi}{2} < \theta < \frac{3\pi}{2} \rightarrow \cos \theta < 0$$

Exercise

Find the length of $r = 8(1 + \cos \theta)$ $0 \leq \theta \leq 2\pi$

Solution

$$\begin{aligned}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} &= \sqrt{64(1 + \cos \theta)^2 + 64\sin^2 \theta} \\ &= 8\sqrt{1 + 2\cos \theta + \cos^2 \theta + \sin^2 \theta} \\ &= 8\sqrt{2 + 2\cos \theta}\end{aligned}$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\begin{aligned}L &= 8\sqrt{2} \int_0^{2\pi} \sqrt{1 + \cos \theta} \, d\theta \\ &= 8\sqrt{2} \int_0^{2\pi} \sqrt{1 + \cos \theta} \frac{\sqrt{1 - \cos \theta}}{\sqrt{1 - \cos \theta}} \, d\theta\end{aligned}$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

$$\sin \theta = \pm \sqrt{1 - \cos^2 \theta}$$

$$= 16\sqrt{2} \int_0^{\pi} \frac{\sin \theta}{\sqrt{1 - \cos \theta}} d\theta$$

$$0 < \theta < \pi \rightarrow \sin \theta > 0$$

$$= 16\sqrt{2} \int_0^{\pi} (1 - \cos \theta)^{-1/2} d(1 - \cos \theta)$$

$$= 32\sqrt{2} \sqrt{1 - \cos \theta} \Big|_0^{\pi}$$

$$= 32\sqrt{2} (\sqrt{2} - 0)$$

$$= 64 \text{ unit}$$

Exercise

Find the length of $r = 2\theta$ $0 \leq \theta \leq \frac{\pi}{2}$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{4\theta^2 + 4} = 2\sqrt{1 + \theta^2}$$

$$L = 2 \int_0^{\pi/2} \sqrt{1 + \theta^2} d\theta$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

$$= 2 \int_0^{\pi/2} \sec^3 \alpha d\alpha$$

$$\theta = \tan \alpha \quad \sqrt{\theta^2 + 1} = \sec \alpha$$

$$d\theta = \sec^2 \alpha d\alpha$$

$$= 2 \left[\frac{1}{2} \sec \alpha \tan \alpha + \frac{1}{2} \ln |\sec \alpha + \tan \alpha| \right]_0^{\pi/2}$$

$$\int \sec^3 x dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x|$$

$$= 2 \left[\frac{1}{2} \theta \sqrt{1 + \theta^2} + \frac{1}{2} \ln |\sqrt{1 + \theta^2} + \theta| \right]_0^{\pi/2}$$

$$= \frac{\pi}{2} \sqrt{1 + \frac{\pi^2}{4}} + \ln \left(\sqrt{1 + \frac{\pi^2}{4}} + \frac{\pi}{2} \right) \text{ unit} \quad \approx 4.158$$

Exercise

Find the length of $r = \sec \theta$ $0 \leq \theta \leq \frac{\pi}{3}$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{\sec^2 \theta + \sec^2 \theta \tan^2 \theta}$$

$$= \sec \theta \sqrt{1 + \tan^2 \theta}$$

$$= \sec^2 \theta$$

$$\begin{aligned}
 L &= \int_0^{\pi/3} \sec^2 \theta \, d\theta \\
 &= \tan \theta \Big|_0^{\pi/3} \\
 &= \sqrt{3} \text{ unit}
 \end{aligned}$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

Exercise

Find the length of $r = \frac{1}{\theta}$ $\pi \leq \theta \leq 2\pi$

Solution

$$\begin{aligned}
 \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} &= \sqrt{\frac{1}{\theta^2} + \frac{1}{\theta^4}} \\
 &= \frac{1}{\theta^2} \sqrt{\theta^2 + 1}
 \end{aligned}$$

$$\begin{aligned}
 L &= \int_{\pi}^{2\pi} \frac{1}{\theta^2} \sqrt{\theta^2 + 1} \, d\theta \\
 &= \sinh^{-1} \theta - \frac{\sqrt{1 + \theta^2}}{\theta} \Big|_{\pi}^{2\pi} \\
 &= \sinh^{-1} 2\pi - \frac{\sqrt{1 + 4\pi^2}}{2\pi} - \sinh^{-1} \pi + \frac{\sqrt{1 + \pi^2}}{\pi} \\
 &= 2.5376 - 1.01259 - 1.8623 + 1.04944 \approx 0.71215
 \end{aligned}$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

Exercise

Find the length of $r = e^{\theta}$ $0 \leq \theta \leq \pi$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{e^{2\theta} + e^{2\theta}} = \sqrt{2}e^{\theta}$$

$$\begin{aligned}
 L &= \sqrt{2} \int_0^{\pi} e^{\theta} \, d\theta \\
 &= \sqrt{2} (e^{\pi} - 1) \text{ unit}
 \end{aligned}$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

Exercise

Find the length of $r = 5 \cos \theta \quad \frac{\pi}{2} \leq \theta \leq \pi$

Solution

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{25 \cos^2 \theta + 25 \sin^2 \theta} = 5$$

$$\begin{aligned} L &= \int_{\pi/2}^{\pi} 5 \, d\theta \\ &= 5\theta \Big|_{\pi/2}^{\pi} \\ &= \frac{5\pi}{2} \text{ unit} \end{aligned}$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

Exercise

Find the length of $r = 3(1 - \cos \theta) \quad 0 \leq \theta \leq \pi$

Solution

$$\begin{aligned} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} &= \sqrt{9(1 - \cos \theta)^2 + 9 \sin^2 \theta} \\ &= 3\sqrt{1 - 2\cos \theta + \cos^2 \theta + \sin^2 \theta} \\ &= 3\sqrt{2 - 2\cos \theta} \end{aligned}$$

$$L = 3 \int_0^{\pi} \sqrt{4 \sin^2 \theta} \, d\theta$$

$$\begin{aligned} L &= 6 \int_0^{\pi} \sin \theta \, d\theta \\ &= -6 \cos \theta \Big|_0^{\pi} \\ &= -6(1 - 1) \\ &= 12 \text{ unit} \end{aligned}$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

Exercise

Find the surface area bounded by $r = 6 \cos \theta \quad 0 \leq \theta \leq \frac{\pi}{2}$ revolving about Polar axis

Solution

$$\sqrt{r^2 + (r')^2} = \sqrt{36 \cos^2 \theta + 36 \sin^2 \theta} = 6$$

$$\begin{aligned}
S &= 2\pi \int_0^{\pi/2} 6 \cos \theta \sin \theta (6) \, d\theta \\
&= 36\pi \int_0^{\pi/2} \sin 2\theta \, d\theta \\
&= -18\pi \cos 2\theta \Big|_0^{\pi/2} \\
&= -18\pi(-1-1) \\
&= \underline{36\pi \text{ unit}}
\end{aligned}$$

$$S = 2\pi \int_{\alpha}^{\beta} f(\theta) \sin \theta \sqrt{(f(\theta))^2 + (f'(\theta))^2} \, d\theta$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

Exercise

Find the surface area bounded by $r = a \cos \theta$ $0 \leq \theta \leq \frac{\pi}{2}$ revolving about $\theta = \frac{\pi}{2}$

Solution

$$\sqrt{r^2 + (r')^2} = \sqrt{a^2 \cos^2 \theta + a^2 \sin^2 \theta} = \underline{a}$$

$$\begin{aligned}
S &= 2\pi \int_0^{\pi/2} a^2 \cos^2 \theta \, d\theta \\
&= a^2 \pi \int_0^{\pi/2} (1 + \cos 2\theta) \, d\theta \\
&= \pi a^2 \left(\theta + \frac{1}{2} \sin 2\theta \right) \Big|_0^{\pi/2} \\
&= \pi a^2 \left(\frac{\pi}{2} \right) \\
&= \underline{\frac{1}{2} \pi^2 a^2 \text{ unit}}
\end{aligned}$$

$$S = 2\pi \int_{\alpha}^{\beta} f(\theta) \cos \theta \sqrt{(f(\theta))^2 + (f'(\theta))^2} \, d\theta$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

Exercise

Find the surface area bounded by $r = e^{a\theta}$ $0 \leq \theta \leq \frac{\pi}{2}$ revolving about $\theta = \frac{\pi}{2}$

Solution

$$\sqrt{r^2 + (r')^2} = \sqrt{e^{2a\theta} + a^2 e^{2a\theta}} = \underline{e^{a\theta} \sqrt{1+a^2}}$$

$$\begin{aligned}
S &= 2\pi \sqrt{1+a^2} \int_0^{\pi/2} e^{a\theta} \cos \theta (e^{a\theta}) \, d\theta \\
&= 2\pi \sqrt{1+a^2} \int_0^{\pi/2} e^{2a\theta} \cos \theta \, d\theta
\end{aligned}$$

$$S = 2\pi \int_{\alpha}^{\beta} f(\theta) \cos \theta \sqrt{(f(\theta))^2 + (f'(\theta))^2} \, d\theta$$

$$\begin{aligned}
\int e^{2a\theta} \cos \theta \, d\theta &= e^{2a\theta} \sin \theta + 2ae^{2a\theta} \cos \theta - 4a^2 \int e^{2a\theta} \cos \theta \, d\theta \\
(1 + 4a^2) \int e^{2a\theta} \cos \theta \, d\theta &= e^{2a\theta} (\sin \theta + 2a \cos \theta) \\
&= \frac{2\pi\sqrt{1+a^2}}{1+4a^2} e^{2a\theta} (\sin \theta + 2a \cos \theta) \Big|_0^{\pi/2} \\
&= \frac{2\pi\sqrt{1+a^2}}{1+4a^2} (e^{a\pi} - 2a)
\end{aligned}$$

		$\int \cos \theta$
+	$e^{2a\theta}$	$\sin \theta$
-	$2ae^{2a\theta}$	$-\cos \theta$
+	$4a^2 e^{2a\theta}$	$-\int \cos \theta$

Exercise

Find the area surface bounded by $r = a(1 + \cos \theta)$ $0 \leq \theta \leq \pi$ revolving about polar axis

Solution

$$\begin{aligned}
\sqrt{r^2 + (r')^2} &= \sqrt{a^2(1 + \cos \theta)^2 + a^2 \sin^2 \theta} \\
&= a \sqrt{(1 + 2\cos \theta + \cos^2 \theta) + \sin^2 \theta} \\
&= a \sqrt{2 + 2\cos \theta}
\end{aligned}$$

$$\begin{aligned}
S &= 2a^2\pi\sqrt{2} \int_0^{\pi/2} (1 + \cos \theta) \sin \theta (\sqrt{1 + \cos \theta}) \, d\theta & S &= 2\pi \int_{\alpha}^{\beta} f(\theta) \sin \theta \sqrt{(f(\theta))^2 + (f'(\theta))^2} \, d\theta \\
&= -2a^2\pi\sqrt{2} \int_0^{\pi/2} (1 + \cos \theta)^{3/2} \, d(1 + \cos \theta) \\
&= -\frac{4\sqrt{2}}{5} a^2\pi (1 + \cos \theta)^{5/2} \Big|_0^{\pi/2} \\
&= -\frac{4\sqrt{2}}{5} a^2\pi (1 - 1 - 2^{5/2}) \\
&= \frac{4\sqrt{2}}{5} a^2\pi (4\sqrt{2}) \\
&= \frac{32}{5} \pi a^2
\end{aligned}$$

Exercise

Find the surface area bounded by $r = 1 + 4\cos \theta$ $0 \leq \theta \leq \frac{\pi}{2}$ revolving about Polar axis

Solution

$$\sqrt{r^2 + (r')^2} = \sqrt{(1 + 4\cos \theta)^2 + 16\sin^2 \theta}$$

$$\begin{aligned}
&= \sqrt{1 + 8\cos\theta + 16\cos^2\theta + 16\sin^2\theta} \\
&= \sqrt{17 + 8\cos\theta}
\end{aligned}$$

$$\begin{aligned}
S &= 2\pi \int_0^{\pi/2} (1 + 4\cos\theta) \sin\theta \left(\sqrt{17 + 8\cos\theta} \right) d\theta & S &= 2\pi \int_{\alpha}^{\beta} f(\theta) \sin\theta \sqrt{(f(\theta))^2 + (f'(\theta))^2} d\theta \\
&= 2\pi \int_0^{\pi/2} \sin\theta (17 + 8\cos\theta)^{1/2} d\theta + 8\pi \int_0^{\pi/2} \cos\theta \sin\theta (17 + 8\cos\theta)^{1/2} d\theta \\
&= -\frac{\pi}{4} \int_0^{\pi/2} (17 + 8\cos\theta)^{1/2} d(17 + 8\cos\theta) + 8\pi \int_0^{\pi/2} \cos\theta \sin\theta (17 + 8\cos\theta)^{1/2} d\theta \\
&\quad -\frac{\pi}{4} \int_0^{\pi/2} (17 + 8\cos\theta)^{1/2} d(17 + 8\cos\theta) = -\frac{\pi}{6} (17 + 8\cos\theta)^{3/2} \Big|_0^{\pi/2} \\
&\quad = -\frac{\pi}{6} (17\sqrt{17} - 125)
\end{aligned}$$

$$\begin{aligned}
&8\pi \int_0^{\pi/2} \cos\theta \sin\theta (17 + 8\cos\theta)^{1/2} d\theta & u &= 17 + 8\cos\theta & \cos\theta &= \frac{1}{8}(u - 17) \\
&\quad & du &= -8\sin\theta d\theta \\
&= -\frac{\pi}{8} \int_0^{\pi/2} (u - 17) u^{1/2} du \\
&= -\frac{\pi}{8} \int_0^{\pi/2} \left(u^{3/2} - 17u^{1/2} \right) du \\
&= -\frac{\pi}{8} \left(\frac{2}{5} (17 + 8\cos\theta)^{5/2} - \frac{34}{3} (17 + 8\cos\theta)^{3/2} \right) \Big|_0^{\pi/2} \\
&= -\frac{\pi}{8} \left(\frac{578}{5} \sqrt{17} - \frac{578\sqrt{17}}{3} - 1,250 + \frac{4,250}{3} \right) \\
&= -\frac{\pi}{8} \left(-\frac{1,156\sqrt{17}}{15} + \frac{500}{3} \right) \\
&= -\frac{17\sqrt{17}\pi}{6} + \frac{125\pi}{6} + \frac{289\sqrt{17}\pi}{30} - \frac{125\pi}{6} \\
&= \frac{34\pi\sqrt{17}}{5}
\end{aligned}$$

Exercise

Find the surface area bounded by $r = 2\sin\theta$ $0 \leq \theta \leq \frac{\pi}{2}$ revolving about $\theta = \frac{\pi}{2}$

Solution

$$\sqrt{r^2 + (r')^2} = \sqrt{4\sin^2\theta + 4\cos^2\theta} = 2$$

$$\begin{aligned}
S &= 2\pi \int_0^{\pi/2} 4\sin\theta \cos\theta \, d\theta \\
&= 4\pi \int_0^{\pi/2} \sin 2\theta \, d\theta \\
&= -2\pi \cos 2\theta \Big|_0^{\pi/2} \\
&= -2\pi(-1-1) \\
&= \underline{4\pi \text{ unit}}
\end{aligned}$$

$$S = 2\pi \int_{\alpha}^{\beta} f(\theta) \cos\theta \sqrt{(f(\theta))^2 + (f'(\theta))^2} \, d\theta$$

$$\sin 2\theta = 2\sin\theta \cos\theta$$

Exercise

Find the surface area of the torus generated by revolving the circle given by $r = 2$ about the line $r = 5\sec\theta$

Solution

$$\sqrt{r^2 + (r')^2} = \sqrt{4 + 0} = 2$$

$$\begin{aligned}
S &= 4\pi \int_0^{2\pi} \left(\frac{5}{\cos\theta} - 2\right) \cos\theta \, d\theta \\
&= 4\pi \int_0^{2\pi} (5 - 2\cos\theta) \, d\theta \\
&= 4\pi(5\theta - 2\sin\theta) \Big|_0^{2\pi} \\
&= 4\pi(10\pi) \\
&= \underline{40\pi^2}
\end{aligned}$$

$$S = 2\pi \int_{\alpha}^{\beta} (r_2 - r) \cos\theta \sqrt{r^2 + (r')^2} \, d\theta$$

Exercise

Find the surface area of the torus generated by revolving the circle given by $r = a$ about the line $r = b\sec\theta$, where $0 < a < b$

Solution

$$\sqrt{r^2 + (r')^2} = \sqrt{a^2 + 0} = a$$

$$\begin{aligned}
S &= 2\pi a \int_0^{2\pi} \left(\frac{b}{\cos\theta} - a\right) \cos\theta \, d\theta \\
&= 2\pi a \int_0^{2\pi} (b - a\cos\theta) \, d\theta
\end{aligned}$$

$$S = 2\pi \int_{\alpha}^{\beta} (r_2 - r) \cos\theta \sqrt{r^2 + (r')^2} \, d\theta$$

$$\begin{aligned}
 &= 2\pi a(b\theta - a\sin\theta) \Big|_0^{2\pi} \\
 &= 2\pi a(2b\pi) \\
 &= \underline{4\pi^2 ab}
 \end{aligned}$$

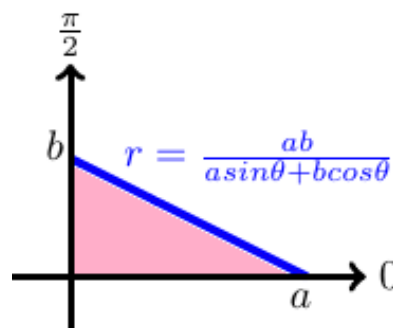
Exercise

Let a and b be positive constants. Find the area of the region in the first quadrant bounded by the graph of the polar equation

$$r = \frac{ab}{a\sin\theta + b\cos\theta}, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

Solution

$$\begin{aligned}
 r(a\sin\theta + b\cos\theta) &= ab \\
 ar\sin\theta + br\cos\theta &= ab \\
 ax + by &= ab \quad (\text{line segment}) \\
 x = 0 &\rightarrow y = b \\
 y = 0 &\rightarrow x = a \\
 \text{Area} &= \underline{\frac{1}{2}ab}
 \end{aligned}$$

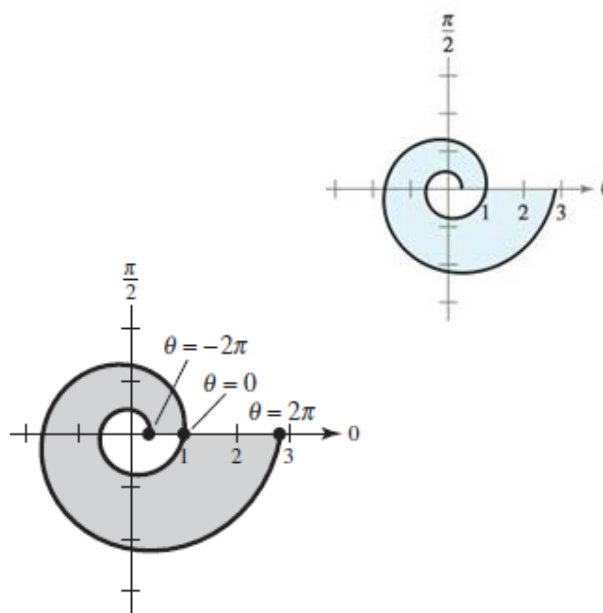


Exercise

The curve represented by the equation $r = ae^{b\theta}$, where a and b are constants, is called a logarithmic spiral. The figure shows the graph of $r = e^{\theta/6}$, $-2\pi \leq \theta \leq 2\pi$. Find the area of the shaded region.

Solution

$$\begin{aligned}
 r &= e^{\theta/6} \\
 A &= \frac{1}{2} \int_0^{2\pi} (e^{\theta/6})^2 d\theta - \frac{1}{2} \int_{-2\pi}^0 (e^{\theta/6})^2 d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} e^{\theta/3} d\theta - \frac{1}{2} \int_{-2\pi}^0 e^{\theta/3} d\theta \\
 &= \frac{3}{2} e^{\theta/3} \Big|_0^{2\pi} - \frac{3}{2} e^{\theta/3} \Big|_{-2\pi}^0 \\
 &= \frac{3}{2} (e^{2\pi/3} - 1) - \frac{3}{2} (1 - e^{-2\pi/3}) \\
 &= \underline{\frac{3}{2} (e^{2\pi/3} - 2 + e^{-2\pi/3})} \approx 9.3655
 \end{aligned}$$



Exercise

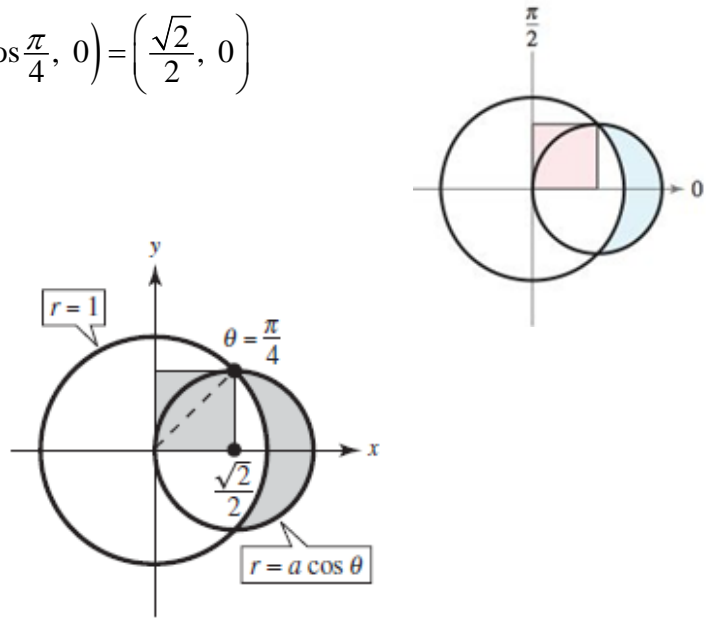
The larger circle in the figure is the graph of $r = 1$.

Find the polar equation of the smaller circle such that the shaded regions are equal.

Solution

Small circle: $r = a \cos \theta$ with center at $\left(1 \cos \frac{\pi}{4}, 0\right) = \left(\frac{\sqrt{2}}{2}, 0\right)$

$$\begin{aligned}
 A &= 2 \frac{1}{2} \int_0^{\pi/4} \left[(a \cos \theta)^2 - 1 \right] d\theta \\
 &= \int_0^{\pi/4} (a^2 \cos^2 \theta - 1) d\theta \\
 &= \int_0^{\pi/4} \left(\frac{a^2}{2} (1 + \cos 2\theta) - 1 \right) d\theta \\
 &= \frac{a^2}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) - \theta \Big|_0^{\pi/4} \\
 &= \frac{a^2}{2} \left(\frac{\pi}{4} + \frac{1}{2} \right) - \frac{\pi}{4}
 \end{aligned}$$



Exercise

Find equations of the circles in the figure.

Determine whether the combined area of the circles is greater than or less than the area of the region inside the square but outside the circles.

Solution

$$Area(\triangle ABC) = \frac{1}{2}bh = \frac{1}{2}(2)(1) = 1$$

The radius of a circle inscribed in the triangle ABC is

For the bigger circle, the radius is:

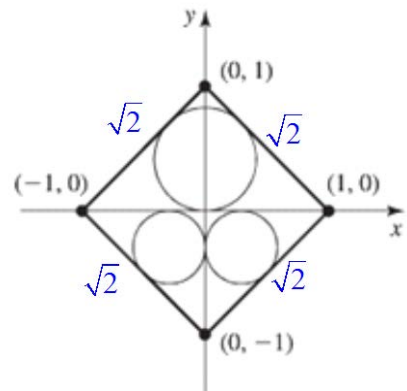
$$R = \frac{Area}{\frac{1}{2} perimeter} = \frac{2}{2 + \sqrt{2} + \sqrt{2}} = \frac{1}{1 + \sqrt{2}}$$

$$Area(\triangle AOD) = Area(\triangle COD) = \frac{1}{2}(1)(1) = \frac{1}{2}$$

The radius of the small circle inscribed in the triangle COD & AOD is

$$R_s = \frac{Area}{\frac{1}{2}(1+1+\sqrt{2})} = \frac{1}{2+\sqrt{2}}$$

The area inside the 3 circles is:



$$\begin{aligned}
\text{Area} &= \pi \left(\frac{1}{1+\sqrt{2}} \right)^2 + 2\pi \left(\frac{1}{2+\sqrt{2}} \right)^2 \\
&= \frac{\pi}{(1+\sqrt{2})^2} + \frac{2\pi}{(2+\sqrt{2})^2} \\
&\approx 1.078
\end{aligned}$$

The area of the square is $= (\sqrt{2})^2 = 2$

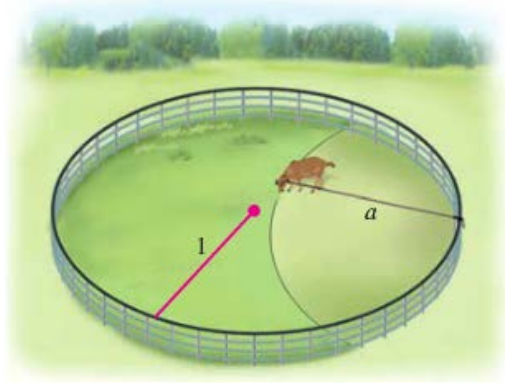
The area outside the circle but inside the square is $\approx 2 - 1.078 = 0.922$

Therefore, the area inside the circles is more than outside the circles but inside the square.

Exercise

A circular corral of unit radius is enclosed by a fence. A goat inside the corral is tied to the fence with a rope of length $0 \leq a \leq 2$.

What is the area of the region (inside the corral) that the goat can graze? Check your answer with the special cases $a = 0$ and $a = 2$



Solution

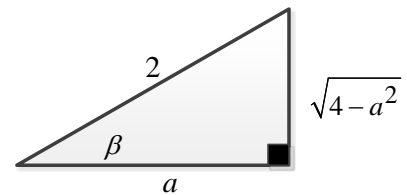
Suppose that the goat is tethered at the origin, and that the center of the corral is $(1, \pi)$.

The circle that the goat can graze is $r = a$, and the corral is given by $r = -2\cos\theta$.

The intersection occurs for $\theta = \cos^{-1}\left(-\frac{a}{2}\right)$

The area grazed by the goat is twice the area of the sector of the circle $r = a$ between $\cos^{-1}\left(-\frac{a}{2}\right)$ and π , plus twice the area of the circle $r = -2\cos\theta$ between $\frac{\pi}{2}$ and $\cos^{-1}\left(-\frac{a}{2}\right)$.

$$\begin{aligned}
A &= \int_{\cos^{-1}\left(-\frac{a}{2}\right)}^{\pi} a^2 d\theta + \int_{\pi/2}^{\cos^{-1}\left(-\frac{a}{2}\right)} 4\cos^2\theta d\theta \\
&= a^2\theta \Big|_{\cos^{-1}\left(-\frac{a}{2}\right)}^{\pi} + 2 \int_{\pi/2}^{\cos^{-1}\left(-\frac{a}{2}\right)} (1 + \cos 2\theta) d\theta \\
&= a^2 \left(\pi - \cos^{-1}\left(-\frac{a}{2}\right) \right) + (2\theta + \sin 2\theta) \Big|_{\pi/2}^{\cos^{-1}\left(-\frac{a}{2}\right)} \\
&= a^2 \left(\pi - \cos^{-1}\left(-\frac{a}{2}\right) \right) + 2\cos^{-1}\left(-\frac{a}{2}\right) + \sin\left(2\cos^{-1}\left(-\frac{a}{2}\right)\right) - \pi \\
&= a^2 \left(\pi - \cos^{-1}\left(-\frac{a}{2}\right) \right) + 2\cos^{-1}\left(-\frac{a}{2}\right) - \frac{1}{2}a\sqrt{4-a^2} - \pi
\end{aligned}$$



$$\sin 2\beta = 2 \sin \beta \cos \beta$$

$$\sin 2\beta = 2 \frac{\sqrt{4-a^2}}{2} \frac{a}{2}$$

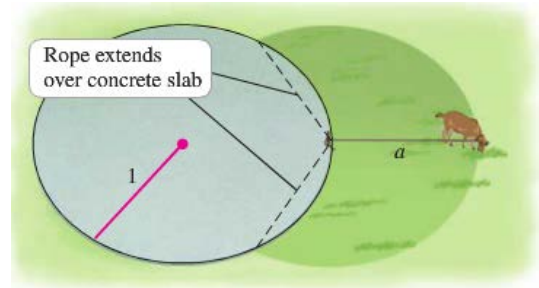
Case $a = 0$: $A = \pi - \pi = \underline{0}$

Case $a = 2$: $A = 4(\pi - \pi) + 2\pi - \pi = \underline{\pi}$

Exercise

A circular concrete slab of unit radius is surrounded by grass. A goat is tied to the edge of the slab with a rope of length $0 \leq a \leq 2$.

What is the area of the grassy region that the goat can graze? Note that the rope can extend over the concrete slab. Check your answer with the special cases $a = 0$ and $a = 2$

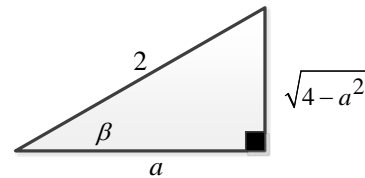


Solution

$$\begin{aligned} A &= \int_{\cos^{-1}\left(\frac{a}{2}\right)}^{\pi} a^2 d\theta + \int_{\pi/2}^{\cos^{-1}\left(\frac{a}{2}\right)} 4\cos^2 \theta d\theta \\ &= a^2 \left(\pi - \cos^{-1}\left(\frac{a}{2}\right) \right) + (2\theta + \sin 2\theta) \Big|_{\pi/2}^{\cos^{-1}\left(\frac{a}{2}\right)} \\ &= a^2 \pi - a^2 \cos^{-1}\left(\frac{a}{2}\right) + 2\cos^{-1}\left(\frac{a}{2}\right) + \sin\left(2\cos^{-1}\left(\frac{a}{2}\right)\right) - \pi \\ &= \pi(a^2 - 1) + (2 - a^2)\cos^{-1}\left(\frac{a}{2}\right) + \frac{1}{2}a\sqrt{4 - a^2} \end{aligned}$$

$$\sin 2\beta = 2 \sin \beta \cos \beta$$

$$\sin 2\beta = 2 \frac{\sqrt{4 - a^2}}{2} \frac{a}{2}$$



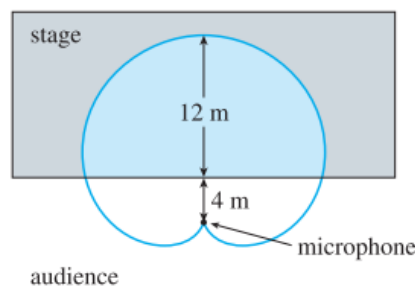
Case $a = 0$: $A = -\pi + 2\frac{\pi}{2} = \underline{0}$

Case $a = 2$: $A = \underline{3\pi}$

Exercise

When recording live performance, sound engineers often use a microphone with a cardioid pickup pattern because it suppresses noise from the audience. Suppose the microphone is placed 4 m from the front of the stage and the boundary of the optimal pickup region is given by the cardioid $r = 8 + 8\sin \theta$, where r is measured in meters and the microphone is at the pole.

The musicians want to know the area they will have on stage within the optimal pickup range of the microphone. Answer their question.



Solution

At $y = 4 = r \sin \theta$, the line represents the front stage with angle $\theta = \alpha$. $\Leftrightarrow r = \frac{4}{\sin \theta}$

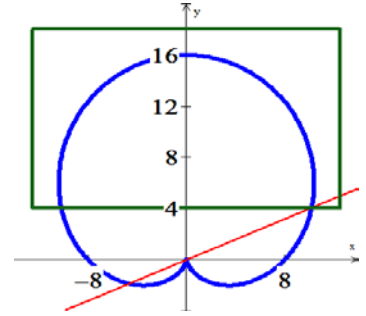
The line intersects the curve:

$$r = 8 + 8 \sin \theta = \frac{4}{\sin \theta}$$

$$2 \sin^2 \theta + 2 \sin \theta - 1 = 0$$

$$\sin \theta = \frac{-2 \pm \sqrt{12}}{4} = \frac{-1 \pm \sqrt{3}}{2} \Rightarrow \sin \theta = \frac{-1 + \sqrt{3}}{2}$$

$$\theta = \alpha = \sin^{-1} \frac{\sqrt{3}-1}{2}$$



$$A = 2 \int_{\alpha}^{\pi/2} \frac{1}{2} \left[(8 + 8 \sin \theta)^2 - \left(\frac{4}{\sin \theta} \right)^2 \right] d\theta$$

$$= \int_{\alpha}^{\pi/2} (64 + 128 \sin \theta + 64 \sin^2 \theta - 16 \csc^2 \theta) d\theta$$

$$= 16 \int_{\alpha}^{\pi/2} (4 + 8 \sin \theta + 2 - 2 \cos 2\theta - \csc^2 \theta) d\theta$$

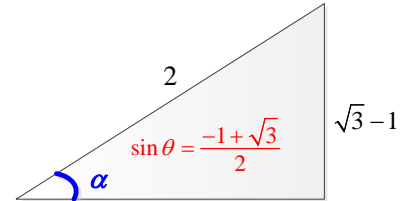
$$= 16 (6\theta - 8 \cos \theta - \sin 2\theta - \cot \theta) \Big|_{\alpha}^{\pi/2}$$

$$= 16 (3\pi - 6\alpha + 8 \cos \alpha + \sin 2\alpha + \cot \alpha)$$

$$x^2 + (\sqrt{3} - 1)^2 = 4 \rightarrow x = \sqrt{4 - 3 + 2\sqrt{3} - 1} = \sqrt{2\sqrt{3}} = \sqrt{\sqrt{4}\sqrt{3}} = (\sqrt{12})^{1/2} = \sqrt[4]{12}$$

$$= 16 \left(3\pi - 6 \sin^{-1} \frac{\sqrt{3}-1}{2} + 4\sqrt[4]{12} + \frac{\sqrt[4]{12}\sqrt{3}-1}{2} + \frac{\sqrt{3}-1}{\sqrt[4]{12}} \right)$$

$$\approx 204.16 \text{ m}^2$$



Exercise

The curve given by the parametric equations

$$x(t) = \frac{1-t^2}{1+t^2} \quad \text{and} \quad y(t) = \frac{t(1-t^2)}{1+t^2}$$

- Find the rectangular equation of the strophoid.
- Find a polar equation of the strophoid.
- Sketch a graph of the strophoid.
- Find the equations of the two tangent lines at the origin.
- Find the points on the graph at which the tangent lines are horizontal.

Solution

$$a) \quad x^2(t) = \frac{(1-t^2)^2}{(1+t^2)^2} \quad y^2(t) = \frac{t^2(1-t^2)^2}{(1+t^2)^2}$$

$$\begin{aligned} \frac{1-x}{1+x} &= \frac{1 - \frac{1-t^2}{1+t^2}}{1 + \frac{1-t^2}{1+t^2}} \\ &= \frac{1+t^2 - 1+t^2}{1+t^2 + 1-t^2} \\ &= \frac{2t^2}{2} \\ &= t^2 \end{aligned}$$

$$\begin{aligned} y^2(t) &= t^2 \left(\frac{1-t^2}{1+t^2} \right)^2 \\ &= \left(\frac{1-x}{1+x} \right) x^2 \end{aligned}$$

$$b) \quad y^2 = \left(\frac{1-x}{1+x} \right) x^2$$

$$r^2 \sin^2 \theta = r^2 \cos^2 \theta \left(\frac{1-r \cos \theta}{1+r \cos \theta} \right) \quad (r \neq 0)$$

$$\sin^2 \theta + r \cos \theta \sin^2 \theta = \cos^2 \theta - r \cos^3 \theta$$

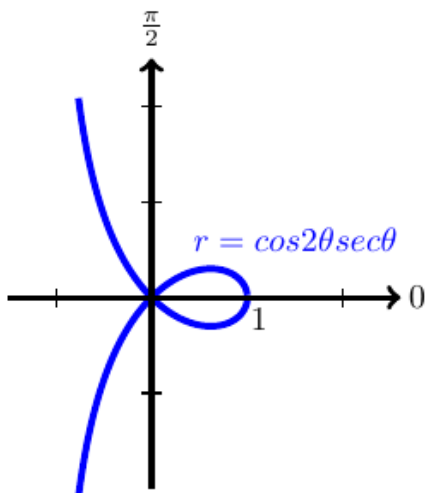
$$r \cos \theta \sin^2 \theta + r \cos^3 \theta = \cos^2 \theta - \sin^2 \theta$$

$$r \cos \theta (\sin^2 \theta + \cos^2 \theta) = \cos 2\theta$$

$$r \cos \theta = \cos 2\theta$$

$$r = \cos 2\theta \sec \theta$$

c)



$$d) \quad r = \cos 2\theta \sec \theta = 0 \rightarrow \cos 2\theta = 0$$

$$2\theta = \frac{\pi}{2}, \frac{3\pi}{2} \rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

The tangent lines to curve at origin at the origin:

$$\underline{y = x} \quad \text{and} \quad \underline{y = -x}$$

$$\begin{aligned} e) \quad y' &= \frac{(1-3t^2)(1+t^2) - 2t(t-t^3)}{(1+t^2)^2} \\ &= \frac{1-2t^2-3t^4-2t^2+2t^4}{(1+t^2)^2} \\ &= \frac{1-4t^2-t^4}{(1+t^2)^2} = 0 \end{aligned}$$

$$t^4 + 4t^2 - 1 = 0 \rightarrow \underline{t^2 = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}}$$

$$\begin{aligned} x &= \frac{1-t^2}{1+t^2} = \frac{1-(-2+\sqrt{5})}{1+(-2+\sqrt{5})} \\ &= \frac{3-\sqrt{5}}{-1+\sqrt{5}} \frac{-1-\sqrt{5}}{-1-\sqrt{5}} \\ &= \frac{-3-2\sqrt{5}+5}{-4} \\ &= \underline{\underline{\frac{\sqrt{5}-1}{2}}} \end{aligned}$$

$$\begin{aligned} y &= \pm x \sqrt{\frac{1-x}{1+x}} \\ &= \pm \frac{\sqrt{5}-1}{2} \sqrt{\frac{1-\frac{\sqrt{5}-1}{2}}{1+\frac{\sqrt{5}-1}{2}}} \\ &= \pm \frac{\sqrt{5}-1}{2} \sqrt{\frac{3-\sqrt{5}}{\sqrt{5}+1}} \\ &= \pm \frac{\sqrt{5}-1}{2} \sqrt{\frac{4\sqrt{5}-8}{4}} \\ &= \pm \frac{\sqrt{5}-1}{2} \sqrt{\sqrt{5}-2} \\ &= \underline{\underline{\left(\frac{\sqrt{5}-1}{2}, \pm \frac{\sqrt{5}-1}{2} \sqrt{\sqrt{5}-2} \right)}} \end{aligned}$$