SOLUTION Section 3.5 – The Ratio and Root Tests

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

Solution

$$\lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{n!}{2^n}} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}$$

$$= \lim_{n\to\infty} \frac{2}{(n+1)}$$

$$= 0 < 1$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}}{n3^{n-1}}$

Solution

$$\lim_{n\to\infty} \frac{2^{n+2}}{(n+1)3^n} \cdot \frac{n3^{n-1}}{2^{n+1}} = \lim_{n\to\infty} \frac{2n}{3(n+1)}$$

$$= \lim_{n\to\infty} \frac{2n}{3n}$$

$$= \frac{2}{3} < 1$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$

Solution

$$\lim_{n\to\infty} \frac{3^{n+3}}{\ln(n+1)} \cdot \frac{\ln n}{3^{n+2}} = \lim_{n\to\infty} \frac{3\ln n}{\ln(n+1)}$$

$$= 3 \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)}$$

$$= 3 \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}}$$

$$= 3 \lim_{n\to\infty} \frac{n+1}{n}$$

$$\underline{= 3 > 1}$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2(n+2)!}{n!3^{2n}}$

Solution

$$\lim_{n\to\infty} \frac{(n+1)^2(n+3)!}{(n+1)!\ 3^{2(n+1)}} \cdot \frac{n!\ 3^{2n}}{n^2(n+2)!} = \lim_{n\to\infty} \frac{(n+1)^2(n+3)}{n^2(n+1)\ \cdot 3^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)(n+3)}{n^2\ \cdot 3^2}$$

$$= \lim_{n\to\infty} \frac{n^2+4n+3}{9n^2}$$

$$\underline{= \frac{1}{9} < 1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{n5^n}{(2n+3)\ln(n+1)}$

Solution

$$\lim_{n\to\infty} \frac{(n+1)\cdot 5^{n+1}}{(2n+5)\ln(n+2)} \cdot \frac{(2n+3)\ln(n+1)}{n\cdot 5^n} = \lim_{n\to\infty} \frac{5(n+1)(2n+3)\ln(n+1)}{n(2n+5)\ln(n+2)}$$

$$= \lim_{n\to\infty} \frac{10n^2+10n+6}{2n^2+5n} \cdot \lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n+2)}$$

$$= 5 \cdot \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n+2}}$$

$$= 5 \cdot \lim_{n \to \infty} \frac{n+2}{n+1}$$

$$\underline{= 5 > 1}$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{99^n}{n!}$

Solution

$$\rho = \lim_{n \to \infty} \frac{99^{n+1}}{(n+1)!} \Big/ \frac{99^n}{n!} \qquad \rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n \to \infty} \frac{99}{n+1}$$

$$\underline{= 0 < 1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n^5}{2^n}$

Solution

$$\rho = \lim_{n \to \infty} \frac{(n+1)^5}{2^{n+1}} \Big/ \frac{n^5}{2^n} \qquad \rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n \to \infty} \frac{1}{2} \left(\frac{n+1}{n} \right)^5$$

$$\underline{= \frac{1}{2} < 1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \Big/ \frac{n!}{n^n}$$ $$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$= \frac{1}{e} < 1$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(2(n+1))!}{((n+1)!)^2} \Big/ \frac{(2n)!}{(n!)^2}$$ $$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{(2(n+1))!}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{4n^2}{n^2}$$

$$= 4 > 1$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{1}{5^{n+1}} \cdot \frac{5^n}{1} \qquad\qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \frac{1}{5} < 1$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

Solution

$$\rho = \lim_{n\to\infty} \frac{1}{(n+1)!} \cdot \frac{n!}{1} \qquad\qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{1}{n+1}$$

$$= 0 < 1$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{3^n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!} \qquad\qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{n+1}{3}$$

$$= \infty$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

Solution

$$\rho = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \qquad\qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{2}{n+1}$$

$$\boxed{= 0 < 1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} n\left(\frac{6}{5}\right)^n$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)\left(\frac{6}{5}\right)^{n+1}}{n\left(\frac{6}{5}\right)^n} \qquad\qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{n+1}{n}\left(\frac{6}{5}\right)$$

$$\boxed{= \frac{6}{5} > 1}$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} n\left(\frac{7}{8}\right)^n$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)\left(\frac{7}{8}\right)^{n+1}}{n\left(\frac{7}{8}\right)^n} \qquad\qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{n+1}{n}\left(\frac{7}{8}\right)$$

$$\boxed{= \frac{7}{8} < 1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n}{4^n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{n+1}{4^{n+1}} \cdot \frac{4^n}{n}$$

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \frac{n+1}{4n}$$

$$= \frac{1}{4} < 1$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{5^n}{n^4}$

Solution

$$\rho = \lim_{n\to\infty} \frac{5^{n+1}}{(n+1)^4} \cdot \frac{n^4}{5^n}$$

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} 5\left(\frac{n}{n+1}\right)^4$$

$$= 5 > 1$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \frac{1}{3}\left(\frac{n+1}{n}\right)^3$$

$$= \frac{1}{3} < 1$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$

Solution

$$\rho = \lim_{n\to\infty} \frac{n+3}{(n+1)(n+2)} \cdot \frac{n(n+1)}{(n+2)} \qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{n(n+3)}{(n+2)^2}$$

$$\underline{=1}$$

$$\lim_{n\to\infty} \left|a_n\right| = \lim_{n\to\infty} \frac{(n+2)}{n(n+1)}$$

$$\underline{=0}$$

Therefore; the given series ***converges*** ***Conditionally*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$

Solution

$$\rho = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{2}{n+1}$$

$$\underline{=0<1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(\frac{3}{2}\right)^n}{n^2}$

Solution

$$\rho = \lim_{n\to\infty} \frac{\left(\frac{3}{2}\right)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{\left(\frac{3}{2}\right)^n} \qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{3}{2}\left(\frac{n}{n+1}\right)^2$$

$$= \boxed{\frac{3}{2} > 1}$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n3^n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)3^{n+1}} \cdot \frac{n3^n}{n!} \qquad\qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{n}{3}$$

$$= \boxed{\infty}$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use the Ratio Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(2n+2)!}{(n+1)^5} \cdot \frac{n^5}{(2n)!} \qquad\qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} (2n+1)(2n+2)\left(\frac{n}{n+1}\right)^5$$

$$= \boxed{\infty}$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\frac{4^n}{(3n)^n}} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{4}{3n}\right)^n}$$

$$= \lim_{n\to\infty} \frac{4}{3n}$$

$$\underline{= 0 < 1}$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{4n+3}{3n-5}\right)^{n}$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{4n+3}{3n-5}\right)^{n}} = \lim_{n\to\infty}\left(\frac{4n+3}{3n-5}\right)$$

$$\underline{= \frac{4}{3} > 1}$$

Therefore; the given series ***diverges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\left(\ln\left(e^{2}+\frac{1}{n}\right)\right)^{n+1}$

Solution

$$\lim_{n\to\infty} \sum_{n=1}^{\infty} \sqrt[n]{\left(\ln\left(e^{2}+\frac{1}{n}\right)\right)^{n+1}} = \lim_{n\to\infty} \sum_{n=1}^{\infty}\left(\ln\left(e^{2}+\frac{1}{n}\right)\right)^{1+\frac{1}{n}}$$

$$= \ln\left(e^{2}\right)$$

$$\underline{= 2 > 1}$$

Therefore; the given series ***diverges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\sin^{n}\left(\frac{1}{\sqrt{n}}\right)$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\sin^{n}\left(\frac{1}{\sqrt{n}}\right)} = \lim_{n\to\infty} \sin\left(\frac{1}{\sqrt{n}}\right)$$

$$= \sin(0)$$

$$= 0 < 1$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^{n^2}$

Solution

$$\lim_{n\to\infty}\sqrt[n]{\left(1-\frac{1}{n}\right)^{n^2}} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n}$$

$$= e^{-1} < 1$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\frac{e^{2n}}{n^n}$

Solution

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{e^{2n}}{n^n}\right|} = \lim_{n\to\infty}\frac{e^2}{n}$$

$$= 0 < 1$$

Therefore; the given series ***converges absolutely*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{5^n}$

Solution

$$\lim_{n\to\infty}\sqrt[n]{\frac{1}{5^n}} = \frac{1}{5} < 1$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{n^n}$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n^n}\right|} = \lim_{n\to\infty} \frac{1}{n}$$

$$\underline{= 0 < 1|}$$

Therefore; the given series ***converges*** *absolutely* by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n}{2n+1}\right)^n\right|} = \lim_{n\to\infty} \frac{n}{2n+1}$$

$$\underline{= \frac{1}{2} < 1|}$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{2n}{n+1}\right)^n\right|} = \lim_{n\to\infty} \frac{2n}{n+1}$$

$$\underline{= 2 > 1|}$$

Therefore; the given series ***diverges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \left(\frac{3n+2}{n+3}\right)^n$

Solution

$$\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{3n+2}{n+3}\right)^n\right|} = \lim_{n\to\infty}\frac{3n+2}{n+3}$$

$$\boxed{=3>1}$$

Therefore; the given series ***diverges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n-2}{5n+1}\right)^n$

Solution

$$\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{n-2}{5n+1}\right)^n\right|} = \lim_{n\to\infty}\left|\frac{n-2}{5n+1}\right|$$

$$\boxed{=\frac{1}{5}<1}$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{(\ln n)^n}$

Solution

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{(-1)^n}{(\ln n)^n}\right|} = \lim_{n\to\infty}\left|\frac{1}{\ln n}\right|$$

$$\boxed{=0<1}$$

Therefore; the given series ***converges*** by the Root Test.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{-3n}{2n+1}\right)^{3n}$

Solution

$$\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{-3n}{2n+1}\right)^{3n}\right|} = \lim_{n\to\infty}\left|\left(\frac{-3n}{2n+1}\right)^3\right|$$

$$=\left(\frac{3}{2}\right)^3$$

$$\boxed{=\frac{27}{8}>1}$$

Therefore; the given series ***diverges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\left(2\sqrt[n]{n}+1\right)^n$

Solution

$$\lim_{n\to\infty}\sqrt[n]{\left|\left(2\sqrt[n]{n}+1\right)^n\right|}=\lim_{n\to\infty}\left|\left(2\sqrt[n]{n}+1\right)\right|$$

Let $x=\lim\limits_{n\to\infty}\sqrt[n]{n} \quad\Rightarrow\quad \ln x=\lim\limits_{n\to\infty}\ln\sqrt[n]{n}$

$$\begin{aligned}\ln x&=\lim_{n\to\infty}\frac{\ln n}{n}\\&=\lim_{n\to\infty}\frac{1/n}{1}\\&=0\end{aligned}$$

$$x=e^0=1=\lim_{n\to\infty}\sqrt[n]{n}$$

$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{\left|\left(2\sqrt[n]{n}+1\right)^n\right|}&=2(1)+1\\&=3>1\end{aligned}$$

Therefore; the given series ***diverges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\displaystyle\sum_{n=0}^{\infty}e^{-3n}$

Solution

$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{\left|e^{-3n}\right|}&=\lim_{n\to\infty}\left|e^{-3}\right|\\&=\frac{1}{e^3}<1\end{aligned}$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n}{3^n}$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n}{3^n}\right|} = \lim_{n\to\infty} \left|\frac{\sqrt[n]{n}}{3}\right|$$

$$\text{Let } x = \sqrt[n]{n} \quad \Rightarrow \quad \ln x = \ln \sqrt[n]{n} = \frac{\ln}{n}$$

$$\begin{aligned} \lim_{n\to\infty} \ln x &= \lim_{n\to\infty} \frac{\ln}{n} \\ &= \lim_{n\to\infty} \frac{1/n}{1} \\ &= 0 \end{aligned}$$

$$x = e^0 = 1 = \lim_{n\to\infty} \sqrt[n]{n}$$

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n}{3^n}\right|} = \frac{1}{3} < 1$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \left(\frac{n}{500}\right)^n$

Solution

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n}{500}\right)^n\right|} &= \lim_{n\to\infty} \left|\frac{n}{500}\right| \\ &= \infty > 1 \end{aligned}$$

Therefore; the given series ***diverges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$

Solution

$$\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{1}{n}-\frac{1}{n^2}\right)^n\right|}=\lim_{n\to\infty}\left|\frac{1}{n}-\frac{1}{n^2}\right|$$

$$\underline{=0<1}$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty}\left(\frac{\ln n}{n}\right)^n$

Solution

$$\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{\ln n}{n}\right)^n\right|}=\lim_{n\to\infty}\left|\frac{\ln n}{n}\right|$$

$$\underline{=0<1}$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use the Root Test to determine if the series converges or diverges. $\sum_{n=2}^{\infty}\frac{n}{(\ln n)^n}$

Solution

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{n}{(\ln n)^n}\right|}=\lim_{n\to\infty}\left|\frac{n^{1/n}}{\ln n}\right|$$

$$\underline{=0<1}$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty}\frac{n^{\sqrt{2}}}{2^n}$

Solution

$$\lim_{n\to\infty}\frac{(n+1)^{\sqrt{2}}}{2^{n+1}}\cdot\frac{2^n}{n^{\sqrt{2}}}=\lim_{n\to\infty}\frac{1}{2}\cdot\frac{(n+1)^{\sqrt{2}}}{n^{\sqrt{2}}}$$

$$=\frac{1}{2}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{\sqrt{2}}$$

$$=\boxed{\frac{1}{2}<1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)^2}{e^{n+1}} \cdot \frac{e^n}{n^2} \qquad\qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{1}{e} \cdot \left(\frac{n+1}{n}\right)^2$$

$$=\boxed{\frac{1}{e}<1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} \qquad\qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{n+1}{10}$$

$$=\boxed{\infty > 1}$$

Therefore; the given series ***converges*** by the Ratio Test.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$

Solution

$$\lim_{n\to\infty} \sqrt[n]{\frac{(\ln n)^n}{n^n}} = \lim_{n\to\infty} \frac{\ln n}{n} \qquad\qquad \textit{Hopital Rule}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1}$$

$$= \lim_{n\to\infty} \frac{1}{n}$$

$$\boxed{= 0 < 1}$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{n2^n(n+1)!}{3^n n!}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)2^{n+1}(n+2)!}{3^{n+1}(n+1)!} \cdot \frac{3^n n!}{n2^n(n+1)!} \qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{2}{3} \frac{(n+1)(n+2)}{n(n+1)}$$

$$= \lim_{n\to\infty} \frac{2}{3}\left(\frac{n+2}{n}\right)$$

$$\boxed{= \frac{2}{3} < 1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

Solution

$$\rho = \lim_{n\to\infty} \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} \qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n+2)}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{2(2n+1)(n+1)}$$

$$= \lim_{n\to\infty} \frac{n+1}{4n+2}$$

$= \frac{1}{4} < 1$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

Solution

Using comparison method:

$$\frac{1}{n} = \frac{n^2}{n^3} < \frac{n^2+1}{n^3+1}$$

Since $\frac{n^2+1}{n^3+1} > \frac{1}{n}$

Therefore; the given series ***diverges*** by the *Comparison Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \left|\sin\frac{1}{n^2}\right|$

Solution

For $x \ge 0 \quad \Rightarrow \quad \sin x \le x$

$$\left|\sin\frac{1}{n^2}\right| = \sin\frac{1}{n^2} \le \frac{1}{n^2}$$

Therefore; the given series ***converges*** by *Comparison Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=8}^{\infty} \frac{1}{\pi^n+5}$

Solution

The given series converges by comparison with $\sum_{n=8}^{\infty} \left(\frac{1}{\pi}\right)^n$

Since $0 < \frac{1}{\pi^n+5} < \frac{1}{\pi^n}$

Therefore; the given series ***converges*** by *Comparison Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^3}$

Solution

Since $(\ln n)^3 < n \quad \Rightarrow \quad \frac{1}{(\ln n)^3} > \frac{1}{n}$

Therefore; the given series ***diverges*** by *Comparison Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{\pi^n - n^\pi}$

Solution

$$a_n = \frac{1}{\pi^n - n^\pi} \quad \Rightarrow \quad b_n = \frac{1}{\pi^n} = \left(\frac{1}{\pi}\right)^n$$

$$\sum_{n=1}^{\infty} \frac{1}{\pi^n} \text{ converges geometric since } |r| = \frac{1}{\pi} < 1$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\pi^n - n^\pi} \cdot \frac{1}{\frac{1}{\pi^n}}$$

$$= \lim_{n\to\infty} \frac{1}{1 - \frac{n^\pi}{\pi^n}}$$

$$= 1$$

Therefore; the given series ***converges*** by *Comparison Test* with geometric series

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=0}^{\infty} \frac{1+n}{2+n}$

Solution

$$\lim_{n\to\infty} \frac{1+n}{2+n} = 1 > 0$$

Therefore; the given series ***diverges*** by the divergence series.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\frac{1+n^{4/3}}{2+n^{5/3}}$

Solution

Let $b_n = \dfrac{1}{n^{1/3}}$

$$\lim_{x\to\infty}\frac{1+n^{4/3}}{2+n^{5/3}}\Big/\frac{1}{n^{1/3}} = \lim_{x\to\infty}\frac{n^{1/3}+n^{5/3}}{2+n^{5/3}}$$

$$= \lim_{x\to\infty}\frac{n^{5/3}}{n^{5/3}}$$

$$\underline{=1}$$

Therefore; the given series ***diverges*** to infinity by *Comparison Test* with divergent ***p****-series*

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{1+n\sqrt{n}}$

Solution

$$\lim_{n\to\infty}\frac{n^2}{1+n\sqrt{n}} = \lim_{n\to\infty}\frac{n^2}{n^{3/2}}$$

$$\underline{=\infty}$$

Therefore; the given series ***diverges*** to infinity.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln n(\ln\ln n)^2}$

Solution

$$\int_2^{\infty}\frac{1}{x\ln x(\ln\ln x)^2}dx = \int_2^{\infty}\frac{d(\ln\ln x)}{(\ln\ln x)^2}$$

$$= -\frac{1}{\ln(\ln x)}\Big|_2^{\infty}$$

$$\underline{=\frac{1}{\ln(\ln 2)}<\infty}$$

Therefore; the given series ***converges*** by *Integral Test*

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=3}^{\infty} \frac{1}{n \ln n \sqrt{\ln \ln n}}$

Solution

$$\int_3^{\infty} \frac{1}{x \ln x \sqrt{\ln \ln x}} dx = \int_3^{\infty} \frac{d(\ln \ln x)}{(\ln \ln x)^{1/2}}$$

$$= 2\sqrt{\ln(\ln x)} \Big|_3^{\infty}$$

$$= \infty$$

Therefore; the given series ***diverges*** by *Integral Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1+(-1)^n}{\sqrt{n}}$

Solution

$$\sum_{n=1}^{\infty} \frac{1+(-1)^n}{\sqrt{n}} = 0 + \frac{2}{\sqrt{2}} + 0 + \frac{2}{\sqrt{4}} + 0 + \frac{2}{\sqrt{6}}$$

$$= 2\sum_{k=1}^{\infty} \frac{1}{\sqrt{2k}}$$

$$= \sqrt{2}\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

$$= \sqrt{2}\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$$

Therefore; the given series ***diverges*** to infinity by p-series $\left(p = \frac{1}{2} < 1\right)$

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n!}{n^2 e^n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^2 e^{n+1}} \cdot \frac{n^2 e^n}{n!}$$

$$= \lim_{n\to\infty} \frac{1}{e} \cdot \frac{n^2}{n+1}$$

$$\underline{= \infty}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!6^n}{(3n)!}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(2(n+1))!6^{n+1}}{(3(n+1))!} \cdot \frac{(3n)!}{(2n)!6^n}$$

$$= 6 \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(3n+1)(3n+2)(3n+3)}$$

$$= \lim_{n\to\infty} \frac{4n^2}{27n^3}$$

$$\underline{= 0}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{3^n \ln n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{\sqrt{n+1}}{3^{n+1} \ln(n+1)} \cdot \frac{3^n \ln n}{\sqrt{n}}$$

$$= \frac{1}{3} \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} \frac{\ln n}{\ln(n+1)}$$

$$= \frac{1}{3} \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}} \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)}$$

$$\underline{= \frac{1}{3} < 1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum\limits_{n=0}^{\infty} \frac{n^{100} 2^n}{\sqrt{n!}}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)^{100} 2^{n+1}}{\sqrt{(n+1)!}} \cdot \frac{\sqrt{n!}}{n^{100} 2^n}$$

$$= 2 \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{100} \frac{1}{\sqrt{n+1}}$$

$$\underline{= 0 \,|}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum\limits_{n=1}^{\infty} \frac{1+n!}{(1+n)!}$

Solution

$$1 + n! > n!$$

$$\frac{1+n!}{(1+n)!} > \frac{n!}{(1+n)!} = \frac{1}{n+1}$$

Therefore; the given series ***diverges*** by *Comparison Test* with the Harmonic

Exercise

Use any method to determine if the series converges or diverges. $\sum\limits_{n=1}^{\infty} \frac{2^n}{3^n - n^3}$

Solution

$$\rho = \lim_{n\to\infty} \frac{2^{n+1}}{3^{n+1} - (n+1)^3} \cdot \frac{3^n - n^3}{2^n}$$

$$= \frac{2}{3} \lim_{n\to\infty} \frac{3^n - n^3}{3^n - \frac{1}{3}(n+1)^3}$$

$$= \frac{2}{3} \lim_{n\to\infty} \frac{1 - \frac{n^3}{3^n}}{1 - \frac{(n+1)^3}{3^{n+1}}}$$

$$= \boxed{\frac{2}{3} < 1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{\pi^n n!}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{\pi^{n+1}(n+1)!} \cdot \frac{\pi^n n!}{n^n}$$

$$= \frac{1}{\pi} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$= \frac{1}{\pi} \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

$$= \boxed{\frac{e}{\pi} < 1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

Solution

$$\rho = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}$$

$$= \lim_{n\to\infty} \frac{2}{n+1}$$

$$= \boxed{0 < 1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)^2\, 2^{n+2}}{3^{n+1}} \cdot \frac{3^n}{n^2\, 2^{n+1}}$$

$$= \lim_{n\to\infty} \frac{2}{3}\left(\frac{n+1}{n}\right)^2$$

$$\boxed{= \frac{2}{3} < 1}$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}$$

$$= \lim_{n\to\infty} \frac{n+1}{n+1}\left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

$$\boxed{= e > 1}$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{100}{n}$

Solution

$$\sum_{n=1}^{\infty} \frac{100}{n} = 100 \sum_{n=1}^{\infty} \frac{1}{n}$$

Therefore; the given series ***diverges*** by *harmonic series*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$

Solution

$$\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}} = 1\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

Therefore; the given series ***converges*** by ***p****-series* $\left(p = \frac{3}{2} > 1\right)$

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \left(\frac{2\pi}{3}\right)^n$

Solution

$$|r| = \frac{2\pi}{3} > 1$$

Therefore; the given series ***diverges*** by *Geometric series*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{5n}{2n-1}$

Solution

$$\lim_{n\to\infty} \frac{5n}{2n-1} = \frac{5}{2} \neq 0$$

Therefore; the given series ***diverges*** by n*th*-Term Test.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$

Solution

$$\lim_{n\to\infty} \frac{n}{2n^2+1} = \frac{1}{2} > 0$$

Therefore; the given series ***diverges*** by *harmonic series*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} (-1)^n \frac{3^{n-2}}{2^n}$

Solution

$$\sum_{n=1}^{\infty} (-1)^n \frac{3^{n-2}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{9}\left(-\frac{3}{2}\right)^n$$

$$|r| = \frac{3}{2} > 1$$

Therefore; the given series ***diverges*** by *Geometric series*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{10}{3\sqrt{n^3}}$

Solution

$$\sum_{n=1}^{\infty} \frac{10}{3\sqrt{n^3}} = \frac{10}{3}\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

Therefore; the given series ***converges*** by ***p**-series* $\left(p = \frac{3}{2} > 1\right)$

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{10n+3}{n2^n}$

Solution

$$b_n = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$$

$$\lim_{n\to\infty} \frac{10n+3}{n2^n} \cdot \frac{2^n}{1} = \lim_{n\to\infty} \frac{10n+3}{n}$$

$$= 10 \qquad \textit{converges}$$

Therefore; the given series ***converges*** by Limit Comparison Test with Geometric series $\left(|r| = \frac{1}{2} < 1\right)$

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{4n^2-1}$

Solution

$$\lim_{n\to\infty} \frac{2^n}{4n^2-1} = \lim_{n\to\infty} \frac{2^n(\ln 2)}{8n}$$

$$= \lim_{n\to\infty} \frac{2^n(\ln 2)^2}{8}$$

$$= \infty$$

Therefore; the given series ***diverges*** by nth-Term Test.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{3^n}$

Solution

$$\left|\frac{\cos n}{3^n}\right| \le \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$$

Therefore; the given series ***converges*** by *Direct Comparison Test* with Geometric series $\left(|r| = \frac{1}{3} < 1\right)$

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n\,7^n}$

Solution

$$\rho = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)7^{n+1}} \cdot \frac{n7^n}{n!} \qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{1}{7}n$$

$$= \infty$$

Therefore; the given series ***diverges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

Solution

$$\frac{\ln n}{n^2} \le \frac{1}{n^{3/2}}$$

Therefore; the given series ***converges*** by *Comparison Test* with ***p**-series* $\left(p = \frac{3}{2} > 1\right)$

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$

Solution

Let $\quad f(x) = \dfrac{1}{x(\ln x)^2}$

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \int_2^{\infty} \frac{1}{(\ln x)^2} d(\ln x)$$

$$= -\frac{1}{\ln x} \Big|_2^{\infty}$$

$$= -\left(0 - \frac{1}{\ln 2}\right)$$

$$= \frac{1}{\ln 2}$$

Therefore; the given series ***converges*** by *Integral Test*

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{\ln(k+1)}\right)^k$

Solution

$$\lim_{k\to\infty} \sqrt[k]{\left(\frac{1}{\ln(k+1)}\right)^k} = \lim_{k\to\infty} \frac{1}{\ln(k+1)}$$

$$= \frac{1}{\infty}$$

$$= 0$$

Therefore; the given series ***converges*** by *Root Test*

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=2}^{\infty} \frac{1}{k^2 (\ln k)^2}$

Solution

$$k \ln k > k$$

$$(k \ln k)^2 > k^2$$

$$\frac{1}{(k \ln k)^2} < \frac{1}{k^2}$$

$\sum \frac{1}{k^2}$ converges by ***p**-series* $(p = 2 > 1)$

Therefore; the given series also ***converges*** by *Comparison Test*

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=3}^{\infty} \frac{1}{\ln k}$

Solution

Let $\quad b_k = \frac{1}{k}$

$$a_k = \frac{1}{\ln k}$$

$$\begin{aligned}
\lim_{k\to\infty} \frac{a_k}{b_k} &= \lim_{k\to\infty} \frac{1}{\ln k} \cdot \frac{k}{1} \\
&= \lim_{k\to\infty} \frac{k}{\ln k} = \frac{\infty}{\infty} \\
&= \lim_{k\to\infty} \frac{1}{\frac{1}{k}} \\
&= \lim_{k\to\infty} k \\
&= \infty
\end{aligned}$$

Therefore; the given series also ***diverges*** by *Limit Comparison Test*

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=2}^{\infty} \frac{5 \ln k}{k}$

Solution

$$\int_2^\infty \frac{5\ln x}{x}\,dx = 5\int_2^\infty \ln x\; d(\ln x)$$

$$= \frac{5}{2}(\ln x)^2 \Big|_2^\infty$$

$$= \frac{5}{2}\left(\infty - (\ln 2)^2\right)$$

$$\underline{= \infty\,|}$$

Therefore; the given series ***diverges*** by *Integral Test*

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=1}^{\infty} \ln\left(\frac{k+2}{k+1}\right)$

Solution

$$\sum_{k=1}^{\infty} \ln\left(\frac{k+2}{k+1}\right) = \sum_{k=1}^{\infty} \left(\ln(k+2) - \ln(k+1)\right) \qquad \textit{Telescopic series}$$

$$= (\ln 3 - \ln 2) + (\ln 4 - \ln 3) + (\ln 5 - \ln 4) + \cdots + \left(\ln(n+2) - \ln(n+1)\right)$$

$$= \ln(n+2) - \ln 2$$

$$\lim_{k\to\infty} \ln\left(\frac{k+2}{k+1}\right) = \lim_{k\to\infty} \left(\ln(k+2) - \ln 2\right)$$

$$\underline{= \infty\,|}$$

Therefore; the given series ***diverges***.

Exercise

Use any method to determine if the series converges or diverges. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2 \ln k}$

Solution

$k^2 \ln k > k^2$

$\dfrac{1}{k^2 \ln k} < \dfrac{1}{k^2}$

$\sum \dfrac{1}{k^2}$ converges by ***p**-series* $(p = 2 > 1)$

Therefore; the given series also ***converges*** by *Comparison Test.*

Exercise

Use any method to determine if the series converges or diverges. $\sum_{k=2}^{\infty} \frac{1}{k^{\ln k}}$

Solution

$\ln k > 2$ For large k.

$k^{\ln k} > k^2$

$\frac{1}{k^{\ln k}} < \frac{1}{k^2}$

$\sum \frac{1}{k^2}$ converges by ***p**-series* $(p = 2 > 1)$

Therefore; the given series also ***converges*** by *Comparison Test*

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Solution

Let $a_n = \frac{n!}{n^n}$

$$\rho = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \qquad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty} \frac{n+1}{n+1}\left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-n}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$$

$$= \lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^n\right)^{-1} \qquad \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

$$= e^{-1}$$

$$= \frac{1}{e} < 1$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\frac{1+\sqrt{2}}{2}+\frac{1+\sqrt{3}}{4}+\frac{1+\sqrt{4}}{8}+\cdots$

Solution

$$\frac{1+\sqrt{2}}{2}+\frac{1+\sqrt{3}}{4}+\frac{1+\sqrt{4}}{8}+\cdots=\sum_{n=1}^{\infty}\frac{1+\sqrt{n+1}}{2^n}$$

Let $\quad a_n=\frac{1+\sqrt{n+1}}{2^n}$

$$\rho=\lim_{n\to\infty}\frac{1+\sqrt{n+2}}{2^{n+1}}\cdot\frac{2^n}{1+\sqrt{n+1}} \qquad \rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$=\frac{1}{2}\lim_{n\to\infty}\frac{1+\sqrt{n+2}}{1+\sqrt{n+1}}$$

$$=\frac{1}{2}\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}}$$

$$=\frac{1}{2}<1$$

Therefore; the given series ***converges*** by the *Ratio Test*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty}\frac{(-3)^n}{3\cdot5\cdot7\cdots(2n+1)}$

Solution

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{(-3)^{n+2}}{3\cdot5\cdot7\cdots(2n+1)\cdot(2n+3)}\cdot\frac{3\cdot5\cdot7\cdots(2n+1)}{(-3)^n}$$

$$=\lim_{n\to\infty}\frac{9}{2n+3}$$

$$=0$$

Therefore; the given series ***converges*** by the Ratio Test.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{18^n (2n-1) n!}$

Solution

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)}{18^{n+1}(2n-1)(2n+1)(n+1)!} \cdot \frac{18^n (2n-1) n!}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$$

$$= \lim_{n\to\infty} \frac{1}{18} \frac{2n+3}{(2n+1)(n+1)}$$

$$= 0$$

Therefore; the given series ***converges*** by the Ratio Test.

Exercise

Use the integral test to show that $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges. Show that the sum s of the series is less than $\frac{\pi}{2}$

Solution

$$\int_0^\infty f(x)\,dx = \int_0^\infty \frac{1}{1+x^2}\,dx$$

$$= \tan^{-1} x \Big|_0^\infty$$

$$= \frac{\pi}{2}$$

Therefore; the given series ***converges*** by the *Integral Test* and its sum is less than $\frac{\pi}{2}$

Exercise

Use the root test to show that $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n^n}$ converges

Solution

$$\lim_{n\to\infty} \sqrt[n]{\frac{2^{n+1}}{n^n}} = \lim_{n\to\infty} \frac{2^{(n+1)/n}}{n}$$

$$= \lim_{n\to\infty} \frac{2 \times 2^{1/n}}{n}$$

$$= 0$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Use the root test to test that $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ converges

Solution

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^{n^2/n}$$

$$= \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^{\frac{1}{n}}$$

$$\boxed{=\frac{1}{e}<1}$$

Therefore; the given series ***converges*** by the *Root Test*.

Exercise

Try to use the ratio test to determine whether $\sum_{n=1}^{\infty} \frac{2^{2n}(n!)^2}{(2n)!}$ converges. What happen?

Now observe that $\frac{2^{2n}(n!)^2}{(2n)!} = \frac{\left[2n(2n-2)(2n-4)\ \cdots\ 6\times 4\times 2\right]^2}{2n(2n-1)(2n-2)\ \cdots\ 3\times 2\times 1}$

$$= \frac{2n}{2n-1}\times\frac{2n-2}{2n-3}\times\frac{4}{3}\times\frac{2}{1}$$

Does the given series converges? Why or why not?

Solution

$$\lim_{n\to\infty} \frac{2^{2n+2}((n+1)!)^2}{(2n+2)!}\cdot\frac{(2n)!}{2^{2n}(n!)^2} = \lim_{n\to\infty} \frac{2^2(n+1)^2}{(2n+2)(2n+1)}$$

$$= \lim_{n\to\infty} \frac{4n^2}{4n^2}$$

$$\boxed{=1}$$

Therefore; the ratio test provides no information

However from the given:

$$\frac{2^{2n}(n!)^2}{(2n)!} = \frac{\left[2n(2n-2)(2n-4)\ \cdots\ 6\times 4\times 2\right]^2}{2n(2n-1)(2n-2)\ \cdots\ 3\times 2\times 1}$$

$$=\frac{2n}{2n-1}\times\frac{2n-2}{2n-3}\times\frac{4}{3}\times\frac{2}{1}\ \underline{\geq 1}$$

Therefore; the given series ***diverges*** to infinity.

Exercise

Suppose $a_n > 0$ and $\frac{a_{n+1}}{a_n} > \frac{n}{n+1}$ for all n. Show that $\sum_{n=1}^{\infty} a_n$ diverges.

$\left(a_n \geq \frac{K}{n} \ \text{for some constant } K\right)$

Solution

If $a_n > 0$ and $\frac{a_{n+1}}{a_n} > \frac{n}{n+1}$ for all n.

Then, by using induction

$$\frac{a_2}{a_1} > \frac{1}{2} \quad\Rightarrow\quad a_2 > \frac{1}{2}a_1$$

$$\frac{a_3}{a_2} > \frac{2}{3} \quad\Rightarrow\quad a_3 > \frac{2}{3}a_2 > \frac{1}{3}a_1$$

$$\frac{a_4}{a_3} > \frac{3}{4} \quad\Rightarrow\quad a_4 > \frac{3}{4}a_3 > \frac{1}{4}a_1$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\frac{a_n}{a_{n-1}} > \frac{n-1}{n} \quad\Rightarrow\quad a_n > \frac{1}{n}a_{n-1}$$

Therefore; the given series ***diverges*** by comparison with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$

Exercise

Working in the early 1600s, the mathematicians Wallis, Pascal, and Fermat were calculating the area of the region under the curve $y = x^p$ between $x = 0$ and $x = 1$, where p is the positive integer. Using arguments that predated the Fundamental Theorem of Calculus, they were able to prove that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{k}{n}\right)^p = \frac{1}{p+1}$$

Use Riemann sums and integrals to verify this limit.

Solution

The sum on the left is simply the left Riemann sum over n equal intervals between 0 and 1 for $f(x) = x^p$.

The limit of the sum is:

$$\int_0^1 x^p\, dx = \frac{1}{p+1} x^{p+1} \Big|_0^1$$

$$\boxed{= \frac{1}{p+1}} \qquad (p > 0)$$

Exercise

Complete the following steps to find the values of $p > 0$ for which the series $\sum_{k=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2k-1)}{p^k k!}$ converges

a) Use the Ratio Test to show that $\sum_{k=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2k-1)}{p^k k!}$ converges for $p > 2$.

b) Use Stirling's formula, $k! = \sqrt{2\pi k}\; k^k e^{-k}$ for large k, to determine whether the series converges when $p = 2$.

$$\left(\textit{Hint}: \; 1\cdot 3\cdot 5\cdots(2k-1) = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2k-1)2k}{2\cdot 4\cdot 6\cdots 2k} \right)$$

Solution

a) Using the Ratio Test

$$\frac{a_{k+1}}{a_k} = \frac{1\cdot 3\cdot 5\cdots(2k-1)\cdot(2(k+1)-1)}{p^{k+1}(k+1)!} \cdot \frac{p^k k!}{1\cdot 3\cdot 5\cdots(2k-1)}$$

$$= \frac{2k+1}{(k+1)p}$$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{2k+1}{(k+1)p}$$

$$\boxed{= \frac{2}{p}}$$

Therefore; the given series converges for $p > 2$.

b) When $p = 2$

Given: $1\cdot 3\cdot 5\cdots(2k-1) = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2k-1)2k}{2\cdot 4\cdot 6\cdots 2k} = \frac{(2k)!}{2\cdot 4\cdot 6\cdots 2k}$

$$\sum_{k=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^k k!} = \sum_{k=1}^{\infty} \frac{(2k)!}{2^k k!(2\cdot 4\cdot 6\cdots 2k)}$$

$$= \sum_{k=1}^{\infty} \frac{(2k)!}{\left(2^k\right)^2 (k!)^2}$$

Given: $k! = \sqrt{2\pi k}\ k^k e^{-k}$

$$\rightarrow\ (2k)! = \sqrt{2\pi(2k)}\ (2k)^{2k}\ e^{-2k}$$

$$= 2\sqrt{\pi}\sqrt{k}\ \left(2^k\right)^2 \left(k^k\right)^2 e^{-2k}$$

$$\sum_{k=1}^{\infty} \frac{(2k)!}{\left(2^k\right)^2 (k!)^2} = \sum_{k=1}^{\infty} \frac{2\sqrt{\pi}\sqrt{k}\ \left(2^k\right)^2 \left(k^k\right)^2 e^{-2k}}{\left(2^k\right)^2 \left(\sqrt{2\pi k}\ k^k e^{-k}\right)^2}$$

$$= \sum_{k=1}^{\infty} \frac{2\sqrt{\pi}\sqrt{k}}{\left(\sqrt{2\pi k}\ \right)^2}$$

$$= \sum_{k=1}^{\infty} \frac{1}{\sqrt{\pi}\sqrt{k}}$$

$$= \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{1}{k^{1/2}} \quad \left(p = \frac{1}{2} < 1\right)$$

Therefore; the given series diverges for $p = 2$ by the *Limit Comparison Test* with ***p***-series $\left(p = \frac{1}{2} < 1\right)$