SOLUTION *Section* 4.2 – Matrices and Linear Systems

Exercise

Write the given system in the form $\boldsymbol{x}' = \boldsymbol{P}(t)x + \boldsymbol{f}(t)$ $\quad x' = -3y, \quad y' = 3x$

Solution

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \boldsymbol{P}(t) = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} \quad \boldsymbol{f}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Exercise

Write the given system in the form $\boldsymbol{x}' = \boldsymbol{P}(t)x + \boldsymbol{f}(t)$ $\quad x' = 3x - 2y, \quad y' = 2x + y$

Solution

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \boldsymbol{P}(t) = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} \quad \boldsymbol{f}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Exercise

Write the given system in the form $\boldsymbol{x}' = \boldsymbol{P}(t)x + \boldsymbol{f}(t)$ $\quad x' = tx - e^t y + \cos t, \quad y' = e^{-t}x + t^2 y - \sin t$

Solution

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \boldsymbol{P}(t) = \begin{bmatrix} t & -e^t \\ e^{-t} & t^2 \end{bmatrix} \quad \boldsymbol{f}(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$$

Exercise

Write the given system in the form $\boldsymbol{x}' = \boldsymbol{P}(t)x + \boldsymbol{f}(t)$ $\quad x' = y + z, \quad y' = z + x, \quad z' = x + y$

Solution

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \boldsymbol{P}(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \boldsymbol{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise

Write the given system in the form $\boldsymbol{x}' = \boldsymbol{P}(t)x + \boldsymbol{f}(t)$ $\quad x' = 2x - 3y, \quad y' = x + y + 2z, \quad z' = 5y - 7z$

Solution

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \boldsymbol{P}(t) = \begin{bmatrix} 2 & -3 & 0 \\ 1 & 1 & 2 \\ 0 & 5 & -7 \end{bmatrix} \quad \boldsymbol{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise

Write the given system in the form $\boldsymbol{x}' = \boldsymbol{P}(t)x + \boldsymbol{f}(t)$

$$x' = 3x - 4y + z + t, \quad y' = x - 3z + t^2, \quad z' = 6y - 7z + t^3$$

Solution

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \boldsymbol{P}(t) = \begin{bmatrix} 3 & -4 & 1 \\ 1 & 0 & -3 \\ 0 & 6 & -7 \end{bmatrix} \quad \boldsymbol{f}(t) = \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}$$

Exercise

Write the given system in the form $\boldsymbol{x}' = \boldsymbol{P}(t)x + \boldsymbol{f}(t)$ $\quad x_1' = x_2, \quad x_2' = 2x_3, \quad x_3' = 3x_4, \quad x_4' = 4x_1$

Solution

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \boldsymbol{P}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{bmatrix} \quad \boldsymbol{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise

Write the given system in the form $\boldsymbol{x}' = \boldsymbol{P}(t)x + \boldsymbol{f}(t)$

$$x_1' = x_2 + x_3 + 1, \quad x_2' = x_3 + x_4 + t, \quad x_3' = x_1 + x_4 + t^2, \quad x_4' = 4x_1 + x_2 + t^3$$

Solution

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \boldsymbol{P}(t) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad \boldsymbol{f}(t) = \begin{bmatrix} 0 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

Exercise

$$\boldsymbol{x}' = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \boldsymbol{x}; \quad \vec{x}_1 = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}$$

a) Verify that the given vectors are solutions of the given system.

b) Use the Wronskian to show that they are linearly independent.

c) Write the general solution of the system.

Solution

a) $\vec{x}_1{}' = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix}' = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix}$ $\qquad$ $\boldsymbol{x}'\,\vec{x}_1 = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix} = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix} = \vec{x}_1{}'$ $\checkmark$

$\vec{x}_2{}' = \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}' = \begin{bmatrix} 2e^{2t} \\ -2e^{2t} \end{bmatrix}$ $\qquad$ $\boldsymbol{x}'\,\vec{x}_2 = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ -2e^{2t} \end{bmatrix} = \vec{x}_2{}'$ $\checkmark$

b) $W = \begin{vmatrix} 2e^t & e^{2t} \\ -3e^t & -e^{2t} \end{vmatrix} = e^{3t} \neq 0$

The solutions x_1 *and* x_2 are linearly independent.

c) $\boldsymbol{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 = C_1\begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix} + C_2\begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix} = \begin{pmatrix} 2C_1e^t + C_2e^{2t} \\ -3C_1e^t - C_2e^{2t} \end{pmatrix}$

Exercise

$\boldsymbol{x}' = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}\boldsymbol{x};\quad \vec{x}_1 = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix},\quad \vec{x}_2 = \begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix},\quad \begin{cases} x_1(0) = 0 \\ x_2(0) = 5 \end{cases}$

a) Verify that the given vectors are solutions of the given system.
b) Use the Wronskian to show that they are linearly independent.
c) Write the general solution of the system.
d) Find the particular solution that satisfies the given initial conditions

Solution

a) $\vec{x}_1{}' = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix}' = \begin{bmatrix} 3e^{3t} \\ 9e^{3t} \end{bmatrix}$ $\qquad$ $\boldsymbol{x}'\cdot\vec{x}_1 = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix} = \begin{bmatrix} 3e^{3t} \\ 9e^{3t} \end{bmatrix} = \vec{x}_1{}'$ $\checkmark$

$\vec{x}_2{}' = \begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix}' = \begin{bmatrix} -4e^{-2t} \\ -2e^{-2t} \end{bmatrix}$ $\qquad$ $\boldsymbol{x}'\cdot\vec{x}_2 = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix} = \begin{bmatrix} -4e^{2t} \\ -2e^{2t} \end{bmatrix} = \vec{x}_2{}'$ $\checkmark$

b) $W = \begin{vmatrix} e^{3t} & 2e^{-2t} \\ 3e^{3t} & e^{-2t} \end{vmatrix} = -5e^t \neq 0$

The solutions x_1 *and* x_2 are linearly independent.

c) $\boldsymbol{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 = C_1\begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix} + C_2\begin{pmatrix} 2e^{-2t} \\ e^{-2t} \end{pmatrix} = \begin{pmatrix} C_1e^{3t} + 2C_2e^{-2t} \\ 3C_1e^{3t} + C_2e^{-2t} \end{pmatrix}$

d) $x_1 = C_1e^{3t} + 2C_2e^{-2t} \qquad x_2 = 3C_1e^{3t} + C_2e^{-2t}$

$$x_1(0) = C_1 + 2C_2 = 0 \qquad x_2(0) = 3C_1 + C_2 = 5$$

$$\Rightarrow \quad C_1 = 2 \quad C_2 = -1$$

$$\begin{cases} x_1 = 2e^{3t} - 2e^{-2t} \\ x_2 = 6e^{3t} - e^{-2t} \end{cases}$$

Exercise

$$\mathbf{x}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\mathbf{x}; \quad \vec{x}_1 = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = e^{-2t}\begin{bmatrix} 1 \\ 5 \end{bmatrix}, \quad \begin{cases} x_1(0) = 5 \\ x_2(0) = -3 \end{cases}$$

a) Verify that the given vectors are solutions of the given system.

b) Use the Wronskian to show that they are linearly independent.

c) Write the general solution of the system.

d) Find the particular solution that satisfies the given initial conditions

Solution

a) $\vec{x}_1{}' = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}' = \begin{bmatrix} 2e^{2t} \\ 2e^{2t} \end{bmatrix} \qquad \mathbf{x}' \cdot \vec{x}_1 = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ 2e^{2t} \end{bmatrix} = \vec{x}_1{}' \ \checkmark$

$\vec{x}_2{}' = \begin{bmatrix} e^{-2t} \\ 5e^{-2t} \end{bmatrix}' = \begin{bmatrix} -2e^{-2t} \\ -10e^{-2t} \end{bmatrix} \qquad \mathbf{x}' \cdot \vec{x}_2 = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} e^{-2t} \\ 5e^{-2t} \end{bmatrix} = \begin{bmatrix} -2e^{2t} \\ -10e^{2t} \end{bmatrix} = \vec{x}_2{}' \ \checkmark$

b) $W = \begin{vmatrix} e^{2t} & e^{-2t} \\ e^{2t} & 5e^{-2t} \end{vmatrix} = 4 \neq 0$

The solutions x_1 *and* x_2 are linearly independent.

c) $\mathbf{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 = C_1\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} + C_2\begin{pmatrix} e^{-2t} \\ 5e^{-2t} \end{pmatrix} = \begin{pmatrix} C_1e^{2t} + C_2e^{-2t} \\ C_1e^{2t} + 5C_2e^{-2t} \end{pmatrix}$

d) $x_1 = C_1e^{2t} + C_2e^{-2t} \qquad x_2 = C_1e^{2t} + 5C_2e^{-2t}$

$x_1(0) = C_1 + C_2 = 5 \qquad x_2(0) = C_1 + 5C_2 = -3$

$\Rightarrow \quad C_1 = 7 \quad C_2 = -2$

$$\begin{cases} x_1 = 7e^{2t} - 2e^{-2t} \\ x_2 = 7e^{2t} - 10e^{-2t} \end{cases}$$

Exercise

$$\boldsymbol{x}' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\boldsymbol{x}; \quad \vec{x}_1 = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 8 \\ x_2(0) = 0 \end{cases}$$

a) Verify that the given vectors are solutions of the given system.
b) Use the Wronskian to show that they are linearly independent.
c) Write the general solution of the system.
d) Find the particular solution that satisfies the given initial conditions

Solution

a) $\vec{x}_1{}' = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}' = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix}$ $\qquad \boldsymbol{x}' \cdot \vec{x}_1 = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix} = \vec{x}_1{}'$ $\checkmark$

$\vec{x}_2{}' = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}' = \begin{bmatrix} -5e^{-2t} \\ -15e^{-2t} \end{bmatrix}$ $\qquad \boldsymbol{x}' \cdot \vec{x}_2 = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix} = \begin{bmatrix} -5e^{2t} \\ -15e^{2t} \end{bmatrix} = \vec{x}_2{}'$ $\checkmark$

b) $W = \begin{vmatrix} 3e^{2t} & e^{-5t} \\ 2e^{2t} & 3e^{-5t} \end{vmatrix} = 7e^{-3t} \neq 0$

The solutions x_1 *and* x_2 are linearly independent.

c) $\boldsymbol{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 = C_1\begin{pmatrix} 3e^{2t} \\ 2e^{2t} \end{pmatrix} + C_2\begin{pmatrix} e^{-5t} \\ 3e^{-5t} \end{pmatrix} = \begin{pmatrix} 3C_1e^{2t} + C_2e^{-5t} \\ 2C_1e^{2t} + 3C_2e^{-5t} \end{pmatrix}$

d) $x_1 = 3C_1e^{2t} + C_2e^{-5t} \qquad x_2 = 2C_1e^{2t} + 3C_2e^{-5t}$

$x_1(0) = 3C_1 + C_2 = 8 \qquad x_2(0) = 2C_1 + 3C_2 = 0 \quad \Rightarrow \quad \underline{C_1 = \frac{24}{7} \quad C_2 = -\frac{16}{7}}$

$$\begin{cases} x_1 = \frac{72}{7}e^{2t} - \frac{16}{7}e^{-2t} \\ x_2 = \frac{48}{7}e^{2t} - \frac{48}{7}e^{-2t} \end{cases}$$

Exercise

$$\boldsymbol{x}' = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\boldsymbol{x}; \quad \vec{x}_1 = \begin{bmatrix} 2e^{t} \\ 2e^{t} \\ e^{t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} -2e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 0 \\ x_2(0) = 0 \\ x_3(0) = 4 \end{cases}$$

a) Verify that the given vectors are solutions of the given system.
b) Use the Wronskian to show that they are linearly independent.
c) Write the general solution of the system.
d) Find the particular solution that satisfies the given initial conditions

Solution

a) $\vec{x}_1{}' = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}' = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}$ $\qquad$ $\mathbf{x}' \cdot \vec{x}_1 = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix} = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix} = \vec{x}_1{}'$ ✓

$\vec{x}_2{}' = \begin{bmatrix} -2e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}' = \begin{bmatrix} -6e^{3t} \\ 0 \\ 3e^{3t} \end{bmatrix}$ $\qquad$ $\mathbf{x}' \cdot \vec{x}_2 = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} -2e^{3t} \\ 0 \\ e^{3t} \end{bmatrix} = \begin{bmatrix} -6e^{3t} \\ 0 \\ 3e^{3t} \end{bmatrix} = \vec{x}_2{}'$ ✓

$\vec{x}_3{}' = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}' = \begin{bmatrix} 10e^{5t} \\ -10e^{5t} \\ 5e^{5t} \end{bmatrix}$ $\qquad$ $\mathbf{x}' \cdot \vec{x}_3 = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix} = \begin{bmatrix} 10e^{5t} \\ -10e^{5t} \\ 5e^{5t} \end{bmatrix} = \vec{x}_3{}'$ ✓

b) $W = \begin{vmatrix} 2e^t & -2e^{3t} & 2e^{5t} \\ 2e^t & 0 & -2e^{5t} \\ e^t & e^{3t} & e^{5t} \end{vmatrix} = 4e^{9t} + 4e^{9t} + 4e^{9t} + 4e^{9t} = 16e^{9t} \neq 0$

The solutions $x_1,\ x_2\ and\ x_3$ are linearly independent.

c) $\mathbf{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 + C_3\vec{x}_3 = C_1\begin{pmatrix} 2e^t \\ 2e^t \\ e^t \end{pmatrix} + C_2\begin{pmatrix} -2e^{3t} \\ 0 \\ e^{3t} \end{pmatrix} + C_3\begin{pmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{pmatrix}$

$$= \begin{pmatrix} 2C_1e^t - 2C_2e^{3t} + 2C_3e^{5t} \\ 2C_1e^t \qquad\qquad - 2C_3e^{5t} \\ C_1e^t + C_2e^{3t} + C_3e^{5t} \end{pmatrix}$$

d) $x_1 = 2C_1e^t - 2C_2e^{3t} + 2C_3e^{5t} \qquad x_2 = 2C_1e^t - 2C_3e^{5t} \qquad x_3 = C_1e^t + C_2e^{3t} + C_3e^{5t}$

$\begin{cases} x_1(0) = 2C_1 - 2C_2 + 2C_3 = 0 \\ x_2(0) = 2C_1 - 2C_3 = 0 \\ x_3(0) = C_1 + C_2 + C_3 = 4 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ 2 & 0 & -2 & 0 \\ 1 & 1 & 1 & 4 \end{array}\right] \quad \Rightarrow \underline{C_1 = 1 \quad C_2 = 2 \quad C_3 = 1|}$

$$\begin{cases} x_1(t) = 2e^t - 4e^{3t} + 2e^{5t} \\ x_2(t) = 2e^t - 2e^{5t} \\ x_3(t) = e^t + 2e^{3t} + e^{5t} \end{cases}$$

Exercise

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\mathbf{x}; \quad \vec{x}_1 = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 10 \\ x_2(0) = 12 \\ x_3(0) = -1 \end{cases}$$

a) Verify that the given vectors are solutions of the given system.

b) Use the Wronskian to show that they are linearly independent.

c) Write the general solution of the system.

d) Find the particular solution that satisfies the given initial conditions

Solution

a)

$$\vec{x}_1{}' = \begin{pmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix}' = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \\ 2e^{2t} \end{pmatrix} \qquad \mathbf{x}' \cdot \vec{x}_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \\ 2e^{2t} \end{pmatrix} = \vec{x}_1{}' \ \checkmark$$

$$\vec{x}_2{}' = \begin{pmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{pmatrix}' = \begin{pmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{pmatrix} \qquad \mathbf{x}' \cdot \vec{x}_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{pmatrix} = \vec{x}_2{}' \ \checkmark$$

$$\vec{x}_3{}' = \begin{pmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{pmatrix}' = \begin{pmatrix} 0 \\ -e^{-t} \\ e^{-t} \end{pmatrix} \qquad \mathbf{x}' \cdot \vec{x}_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{pmatrix} = \begin{pmatrix} 0 \\ -e^{-t} \\ e^{-t} \end{pmatrix} = \vec{x}_3{}' \ \checkmark$$

b) $W = \begin{vmatrix} e^{2t} & e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & -e^{-t} & -e^{-t} \end{vmatrix} = 3 \neq 0$ The solutions $x_1,\ x_2$ *and* x_3 are linearly independent.

c)

$$\mathbf{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 + C_3\vec{x}_3 = C_1\begin{pmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix} + C_2\begin{pmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{pmatrix} + C_3\begin{pmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{pmatrix} = \begin{pmatrix} C_1e^{2t} + C_2e^{-t} \\ C_1e^{2t} + C_3e^{-t} \\ C_1e^{2t} - C_2e^{-t} - C_3e^{-t} \end{pmatrix}$$

d) $x_1 = C_1e^{2t} + C_2e^{-t} \qquad x_2 = C_1e^{2t} + C_3e^{-t} \qquad x_3 = C_1e^{2t} - C_2e^{-t} - C_3e^{-t}$

$$\begin{cases} x_1(0) = C_1 + C_2 = 10 \\ x_2(0) = C_1 + C_3 = 12 \\ x_3(0) = C_1 - C_2 - C_3 = -1 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 1 & 0 & 1 & 12 \\ 1 & -1 & -1 & -1 \end{array}\right] \Rightarrow \underline{C_1 = 7 \quad C_2 = 3 \quad C_3 = 5}$$

$$\begin{cases} x_1(t) = 7e^{2t} + 3e^{-t} \\ x_2(t) = 7e^{2t} + 5e^{-t} \\ x_3(t) = 7e^{2t} - 8e^{-t} \end{cases}$$

Exercise

$$\mathbf{x}' = \begin{bmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{bmatrix}\mathbf{x}; \quad \vec{x}_1 = \begin{bmatrix} e^{-t} \\ 0 \\ 0 \\ e^{-t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 0 \\ 0 \\ e^{-t} \\ 0 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{bmatrix}, \quad \vec{x}_4 = \begin{bmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{bmatrix}, \quad \begin{cases} x_1(0) = 1 \\ x_2(0) = 3 \\ x_3(0) = 4 \\ x_4(0) = 7 \end{cases}$$

a) Verify that the given vectors are solutions of the given system.

b) Use the Wronskian to show that they are linearly independent.

c) Write the general solution of the system.

d) Find the particular solution that satisfies the given initial conditions

Solution

a) $\vec{x}_1{}' = \begin{pmatrix} e^{-t} \\ 0 \\ 0 \\ e^{-t} \end{pmatrix}' = \begin{pmatrix} -e^{-t} \\ 0 \\ 0 \\ -e^{-t} \end{pmatrix}$ $\qquad$ $\mathbf{x}' \cdot \vec{x}_1 = \begin{pmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} e^{-t} \\ 0 \\ 0 \\ e^{-t} \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ 0 \\ 0 \\ -e^{-t} \end{pmatrix} = \vec{x}_1{}'$ ✓

$\vec{x}_2{}' = \begin{pmatrix} 0 \\ 0 \\ e^{-t} \\ 0 \end{pmatrix}' = \begin{pmatrix} 0 \\ 0 \\ -e^{-t} \\ 0 \end{pmatrix}$ $\qquad$ $\mathbf{x}' \cdot \vec{x}_2 = \begin{pmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ e^{-t} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -e^{-t} \\ 0 \end{pmatrix} = \vec{x}_2{}'$ ✓

$\vec{x}_3{}' = \begin{pmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{pmatrix}' = \begin{pmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{pmatrix}$ $\qquad$ $\mathbf{x}' \cdot \vec{x}_3 = \begin{pmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{pmatrix} = \begin{pmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{pmatrix} = \vec{x}_3{}'$ ✓

$\vec{x}_4{}' = \begin{pmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{pmatrix}' = \begin{pmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{pmatrix}$ $\qquad$ $\mathbf{x}' \cdot \vec{x}_3 = \begin{pmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{pmatrix} = \begin{pmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{pmatrix} = \vec{x}_3{}'$ ✓

b) $W=\begin{vmatrix} e^{-t} & 0 & 0 & e^{t} \\ 0 & 0 & e^{t} & 0 \\ 0 & e^{-t} & 0 & 3e^{t} \\ e^{-t} & 0 & -2e^{t} & 0 \end{vmatrix}=e^{-t}\begin{vmatrix} 0 & e^{t} & 0 \\ e^{-t} & 0 & 3e^{t} \\ 0 & -2e^{t} & 0 \end{vmatrix}-e^{t}\begin{vmatrix} 0 & 0 & e^{t} \\ 0 & e^{-t} & 0 \\ e^{-t} & 0 & -2e^{t} \end{vmatrix}=0-(-1)=1\neq 0$

The solutions $x_1,\ x_2\ \textit{and}\ x_3$ are linearly independent.

c) $\mathbf{x}(t)=C_1\vec{x}_1+C_2\vec{x}_2+C_3\vec{x}_3+C_4\vec{x}_4=C_1\begin{pmatrix} e^{-t} \\ 0 \\ 0 \\ e^{-t} \end{pmatrix}+C_2\begin{pmatrix} 0 \\ 0 \\ e^{-t} \\ 0 \end{pmatrix}+C_3\begin{pmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{pmatrix}+C_4\begin{pmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{pmatrix}$

$$=\begin{pmatrix} C_1e^{-t}+C_4e^{t} \\ C_3e^{t} \\ C_2e^{-t}+3C_4e^{t} \\ C_1e^{-t}-2C_3e^{t} \end{pmatrix}$$

d) $x_1(t)=C_1e^{-t}+C_4e^{t},\quad x_2(t)=C_3e^{t},\quad x_3(t)=C_2e^{-t}+3C_4e^{t},\quad x_4(t)=C_1e^{-t}-2C_3e^{t}$

$$\begin{cases} x_1(0)=C_1+C_4=1 \\ x_2(0)=C_3=3 \\ x_3(0)=C_2+3C_4=4 \\ x_4(0)=C_1-2C_3=7 \end{cases}\quad \Rightarrow\ \underline{C_1=13\quad C_2=40\quad C_3=3\quad C_4=-12|}$$

$$\begin{cases} x_1(t)=13e^{-t}-12e^{t} \\ x_2(t)=3e^{t} \\ x_3(t)=40e^{-t}-36e^{t} \\ x_3(t)=13e^{-t}-6e^{t} \end{cases}$$