Solution *Section* 4.1 – Introduction and Review of Power Series

Exercise

Determine the centre, radius, and interval of convergencae of the power series $\sum_{n=0}^{\infty} \frac{x^{2n}}{\sqrt{n+1}}$

Solution

$$R = \lim_{n\to\infty} \left| \frac{1}{\sqrt{n+2}} \cdot \sqrt{n+1} \right| = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}} = 1$$

The radius of convergence is 1.
The centre of convergence is 0.
The interval of convergence is (–1, 1).
The series does not converge at $x=-1$ or $x=1$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=0}^{\infty} 3n(x+1)^n$

Solution

$$R = \lim_{n\to\infty} \left| \frac{3n}{3(n+1)} \right| \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{3n}{3n}$$

$$\underline{=1|}$$

The radius of convergence is 1, and the centre of convergence is –1. $(x+1=0)$

$a-R < x < a+R \quad \Rightarrow \quad -1-1 < x < -1+1$

Therefore, the given series convergences absolutely on $(-2,\ 0)$

At $x=-2$, the series is $\sum_{n=0}^{\infty} 3n(-1)^n$ which diverges.

At $x=0$, the series is $\sum_{n=0}^{\infty} 3n(1)^n = \sum_{n=0}^{\infty} 3n$ which diverges.

Hence, the interval of convergence is $(-2,\ 0)$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} x^n$

Solution

$$R = \lim_{n\to\infty} \left| \frac{(n+1)^4 2^{2n+2}}{n^4 2^{2n}} \right| \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= 4 \lim_{n\to\infty} \left| \left(\frac{n+1}{n} \right)^4 \right|$$

$$= 4$$

The radius of convergence is 4, and the centre of convergence is 0.

$a - R < x < a + R \quad \Rightarrow \quad -4 < x < 4$, the given series convergences absolutely on $(-4,\ 4)$

At $x = -4$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} (-4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} (-1)^n 2^{2n} = \sum_{n=1}^{\infty} \frac{1}{n^4}$ which converges (p-series).

At $x = 4$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} (4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} 2^{2n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ which also converges.

Hence, the interval of convergence is $[-4,\ 4]$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{e^n}{n^3} (4-x)^n$

Solution

$$R = \lim_{n\to\infty} \left| \frac{e^n}{n^3} \cdot \frac{(n+1)^3}{e^{n+1}} \right| \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \frac{1}{e} \lim_{n\to\infty} \left| \left(\frac{n+1}{n} \right)^3 \right|$$

$$= \frac{1}{e}$$

The radius of convergence is $\frac{1}{e}$.

The centre of convergence is 4. $(4 - x = 0 \ \Rightarrow \ x = 4)$

$a - R < x < a + R \quad \Rightarrow \quad 4 - \frac{1}{e} < x < 4 + \frac{1}{e}$, which the given series convergences absolutely

At $x = 4 - \frac{1}{e}$, the series is $\sum_{n=1}^{\infty} \frac{e^n}{n^3} \left(\frac{1}{e} \right)^n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ which converges (p-series).

At $x=4+\frac{1}{e}$, the series is $\sum_{n=1}^{\infty}\frac{e^n}{n^3}\left(-\frac{1}{e}\right)^n=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$ which also converges (p-series).

Hence, the interval of convergence is $\left[4-\frac{1}{e},\ 4+\frac{1}{e}\right]$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=1}^{\infty}\frac{(4x-1)^n}{n^n}$

Solution

$$\sum_{n=1}^{\infty}\frac{(4x-1)^n}{n^n}=\sum_{n=1}^{\infty}\frac{4^n\left(x-\frac{1}{4}\right)^n}{n^n}$$

$$R=\lim_{n\to\infty}\left|\frac{4^n}{n^n}\cdot\frac{(n+1)^{n+1}}{4^{n+1}}\right| \qquad R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$=\frac{1}{4}\lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^n(n+1)\right|$$

$$=\infty$$

The radius of convergence is ∞ .

The centre of convergence is $x=\frac{1}{4}$.

The interval of convergence is the real line $(-\infty,\ \infty)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=0}^{\infty}\frac{1+5^n}{n!}x^n$

Solution

$$R=\lim_{n\to\infty}\left|\frac{1+5^n}{n!}\cdot\frac{(n+1)!}{1+5^{n+1}}\right| \qquad R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{1+5^n}{1+5^{n+1}}(n+1)\right|$$

$$=\infty$$

The radius of convergence is ∞ .
The centre of convergence is 0.
The interval of convergence is the real line $(-\infty,\ \infty)$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=e^{2x}, \quad a=0$

Solution

$$f(x)=e^{2x} \quad \rightarrow \quad f(0)=e^{2(0)}=1$$

$$f'(x)=2e^{2x} \quad \rightarrow \quad f'(0)=2$$

$$f''(x)=4e^{2x} \quad \rightarrow \quad f''(0)=4$$

$$f'''(x)=8e^{2x} \quad \rightarrow \quad f'''(0)=8$$

$$P_0(x)=f(0)=1$$

$$P_1(x)=f(0)+f'(0)(x-0)=1+2x$$

$$P_2(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2=1+2x+2x^2$$

$$P_3(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3$$

$$=1+2x+2x^2+\frac{4}{3}x^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\sin x, \quad a=0$

Solution

$$f(x)=\sin x \quad \rightarrow \quad f(0)=0$$

$$f'(x)=\cos x \quad \rightarrow \quad f'(0)=1$$

$$f''(x)=-\sin x \quad \rightarrow \quad f''(0)=0$$

$$f'''(x)=-\cos x \quad \rightarrow \quad f'''(0)=-1$$

$$P_0(x)=f(0)=0$$

$$P_1(x)=f(0)+f'(0)(x-0)=x$$

$$P_2(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2=x$$

$$P_3(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3=x-\frac{1}{6}x^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\ln(1+x), \quad a=0$

Solution

$$f(x)=\ln(1+x) \quad \to \quad f(0)=0$$

$$f'(x)=\frac{1}{1+x} \quad \to \quad f'(0)=1$$

$$f''(x)=-\frac{1}{(1+x)^2} \quad \to \quad f''(0)=-1$$

$$f'''(x)=\frac{2}{(1+x)^3} \quad \to \quad f'''(0)=2$$

$$P_0(x)=f(0)=0$$

$$P_1(x)=f(0)+f'(0)(x-0)=x$$

$$P_2(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2=x-\frac{1}{2}x^2$$

$$P_3(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3=x-\frac{1}{2}x^2+\frac{1}{3}x^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\frac{1}{x+2}, \quad a=0$

Solution

$$f(x)=(x+2)^{-1} \quad \to \quad f(0)=\frac{1}{2}$$

$$f'(x)=-(x+2)^{-2} \quad \to \quad f'(0)=-\frac{1}{4}$$

$$f''(x)=2(x+2)^{-3} \quad \to \quad f''(0)=\frac{1}{4}$$

$$f'''(x)=-6(x+2)^{-4} \quad \to \quad f'''(0)=-\frac{3}{8}$$

$$P_0(x)=f(0)=\frac{1}{2}$$

$$P_1(x)=f(0)+f'(0)(x-0)=\frac{1}{2}-\frac{1}{4}x$$

$$P_2(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2=\frac{1}{2}-\frac{1}{4}x+\frac{1}{8}x^2$$

$$P_3(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3=\frac{1}{2}-\frac{1}{4}x+\frac{1}{8}x^2-\frac{1}{16}x^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\sqrt{1-x}, \quad a=0$

Solution

$$f(x)=(1-x)^{1/2} \quad \to \quad f(0)=1$$

$$f'(x)=-\tfrac{1}{2}(1-x)^{-1/2} \quad \to \quad f(0)=-\tfrac{1}{2}$$

$$f''(x)=-\tfrac{1}{4}(1-x)^{-3/2} \quad \to \quad f(0)=-\tfrac{1}{4}$$

$$f'''(x)=-\tfrac{3}{8}(1-x)^{-5/2} \quad \to \quad f(0)=-\tfrac{3}{8}$$

$$P_0(x)=f(0)=1$$

$$P_1(x)=f(0)+f'(0)(x-0)=1-\tfrac{1}{2}x$$

$$P_2(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2=1-\tfrac{1}{2}x-\tfrac{1}{8}x^2$$

$$P_3(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3=\boxed{1-\tfrac{1}{2}x-\tfrac{1}{8}x^2-\tfrac{1}{16}x^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=x^3, \quad a=1$

Solution

$$f(x)=x^3 \quad \to \quad f(1)=1$$

$$f'(x)=3x^2 \quad \to \quad f'(1)=3$$

$$f''(x)=6x \quad \to \quad f''(1)=6$$

$$f'''(x)=6 \quad \to \quad f'''(1)=6$$

$$P_0(x)\ \boxed{=1} \qquad P_0(x)=f(a)$$

$$P_1(x)\ \boxed{=1+3(x-1)} \qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)\ \boxed{=1+3(x-1)+3(x-1)^2} \qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)\ \boxed{=1+3(x-1)+3(x-1)^2+(x-1)^3} \qquad P_3(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=8\sqrt{x}, \quad a=1$

Solution

$$f(x)=8x^{1/2} \quad \to \quad f(1)=8$$

$$f'(x)=4x^{-1/2} \quad \to \quad f'(1)=4$$

$$f''(x)=-2x^{-3/2} \quad \to \quad f''(1)=-2$$

$$f'''(x)=3x^{-5/2} \quad \to \quad f'''(1)=3$$

$$P_0(x)=8 \qquad\qquad P_0(x)=f(a)$$

$$P_1(x)=8+4(x-1) \qquad\qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)=8+4(x-1)-(x-1)^2 \qquad\qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)=8+4(x-1)-(x-1)^2+3(x-1)^3 \qquad\qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\sin x, \quad a=\frac{\pi}{4}$

Solution

$$f(x)=\sin x \quad \to \quad f\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

$$f'(x)=\cos x \quad \to \quad f'\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

$$f''(x)=-\sin x \quad \to \quad f''\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$$

$$f'''(x)=-\cos x \quad \to \quad f'''\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$$

$$P_0(x)=\frac{\sqrt{2}}{2} \qquad\qquad P_0(x)=f(a)$$

$$P_1(x)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) \qquad\qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2 \qquad\qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2-\frac{\sqrt{2}}{12}\left(x-\frac{\pi}{4}\right)^3 \qquad\qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\cos x, \quad a=\frac{\pi}{6}$

Solution

$$f(x)=\cos x \quad \to \quad f\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$$

$$f'(x)=-\sin x \quad \to \quad f'\left(\frac{\pi}{6}\right)=-\frac{1}{2}$$

$$f''(x)=-\cos x \quad \to \quad f''\left(\frac{\pi}{6}\right)=-\frac{\sqrt{3}}{2}$$

$$f'''(x)=\sin x \quad \to \quad f'''\left(\frac{\pi}{6}\right)=\frac{1}{2}$$

$$P_0(x)=\frac{\sqrt{3}}{2} \qquad\qquad P_0(x)=f(a)$$

$$P_1(x)=\frac{\sqrt{2}}{2}-\frac{1}{2}\left(x-\frac{\pi}{6}\right) \qquad\qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)=\frac{\sqrt{2}}{2}-\frac{1}{2}\left(x-\frac{\pi}{6}\right)-\frac{\sqrt{3}}{4}\left(x-\frac{\pi}{6}\right)^2 \qquad\qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)=\frac{\sqrt{3}}{2}-\frac{1}{2}\left(x-\frac{\pi}{6}\right)-\frac{\sqrt{3}}{4}\left(x-\frac{\pi}{6}\right)^2+\frac{1}{12}\left(x-\frac{\pi}{6}\right)^3 \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\sqrt{x}, \quad a=9$

Solution

$$f(x)=x^{1/2} \quad \to \quad f(9)=3$$

$$f'(x)=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}} \quad \to \quad f'(9)=\frac{1}{6}$$

$$f''(x)=-\frac{1}{4}x^{-3/2}=-\frac{1}{4x\sqrt{x}} \quad \to \quad f''(9)=-\frac{1}{4\times 3^3}$$

$$f'''(x)=\frac{3}{8}x^{-5/2}=\frac{3}{8x^2\sqrt{x}} \quad \to \quad f'''(9)=\frac{1}{2^3\times 3^4}$$

$$P_0(x)=3 \qquad\qquad P_0(x)=f(a)$$

$$P_1(x)=3+\frac{1}{6}(x-9) \qquad\qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)=3+\frac{1}{2\cdot 3}(x-9)-\frac{1}{2^3\cdot 3^3}(x-9)^2 \qquad\qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)=3+\frac{1}{2\cdot 3}(x-9)-\frac{1}{2^2\cdot 3^3}(x-9)^2+\frac{1}{2^4\cdot 3^5}(x-9)^3 \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\sqrt[3]{x}, \quad a=8$

Solution

$$f(x)=x^{1/3} \quad\to\quad f(8)=2$$

$$f'(x)=\frac{1}{3}x^{-2/3}=\frac{1}{3x^{2/3}} \quad\to\quad f'(8)=\frac{1}{2^2\times 3}$$

$$f''(x)=-\frac{2}{9}x^{-5/3}=-\frac{2}{3^2x^{5/3}} \quad\to\quad f''(8)=-\frac{1}{3^2\times 2^4}$$

$$f'''(x)=\frac{10}{3^3}x^{-8/3}=\frac{2\cdot 5}{3^3x^{8/3}} \quad\to\quad f'''(8)=\frac{5}{2^7\times 3^3}$$

$$P_0(x)\ \underline{=2} \qquad\qquad P_0(x)=f(a)$$

$$P_1(x)\ \underline{=2+\frac{1}{2^2\cdot 3}(x-8)} \qquad\qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)\ \underline{=2+\frac{1}{2^2\cdot 3}(x-8)-\frac{1}{2^5\cdot 3^2}(x-8)^2} \qquad\qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)\ \underline{=2+\frac{1}{2^2\cdot 3}(x-8)-\frac{1}{2^5\cdot 3^2}(x-8)^2+\frac{1}{2^8\cdot 3^4}(x-8)^3} \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\ln x, \quad a=e$

Solution

$$f(x)=\ln x \quad\to\quad f(e)=1$$

$$f'(x)=\frac{1}{x} \quad\to\quad f'(e)=\frac{1}{e}$$

$$f''(x)=-\frac{1}{x^2} \quad\to\quad f''(e)=-\frac{1}{e^2}$$

$$f'''(x)=\frac{2}{x^3} \quad\to\quad f'''(e)=\frac{2}{e^3}$$

$$P_0(x)\ \underline{=1} \qquad\qquad P_0(x)=f(a)$$

$$P_1(x)\ \underline{=1+\frac{1}{e}(x-e)} \qquad\qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)\ \underline{=1+\frac{1}{e}(x-e)-\frac{1}{2e^2}(x-e)^2} \qquad\qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)\ \underline{=1+\frac{1}{e}(x-e)-\frac{1}{2e^2}(x-e)^2+\frac{1}{3e^3}(x-e)^3} \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\sqrt[4]{x}, \quad a=8$

Solution

$$f(x)=x^{1/4} \quad \rightarrow \quad f(8)=\sqrt[4]{8}$$

$$f'(x)=\frac{1}{4}x^{-3/4}=\frac{1}{4x^{3/4}} \quad \rightarrow \quad f'\left(8=2^3\right)=\frac{1}{2^2\times 2^{9/4}}=\frac{1}{2^4\sqrt[4]{2}}$$

$$f''(x)=-\frac{3}{16}x^{-7/4}=-\frac{3}{2^4x^{7/4}} \quad \rightarrow \quad f''(8)=-\frac{3}{2^4\,2^{21/4}}=-\frac{3}{2^9\sqrt[4]{2}}$$

$$f'''(x)=\frac{21}{2^6}x^{-11/4}=\frac{21}{2^6x^{11/4}} \quad \rightarrow \quad f'''(8)=\frac{21}{2^6\times 2^{33/4}}=\frac{21}{2^{14}\sqrt[4]{2}}$$

$$P_0(x)=\boxed{\sqrt[4]{8}} \qquad P_0(x)=f(a)$$

$$P_1(x)=\boxed{\sqrt[4]{8}+\frac{1}{2^4.\sqrt[4]{2}}(x-8)} \qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)=\boxed{\sqrt[4]{8}+\frac{1}{2^4.\sqrt[4]{2}}(x-8)-\frac{3}{2^{10}.\sqrt[4]{2}}(x-8)^2} \qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)=\boxed{\sqrt[4]{8}+\frac{1}{2^4.\sqrt[4]{2}}(x-8)-\frac{3}{2^{10}.\sqrt[4]{2}}(x-8)^2+\frac{7}{2^{15}.\sqrt[4]{2}}(x-8)^3} \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\tan^{-1}x+x^2+1, \quad a=1$

Solution

$$f(x)=\tan^{-1}x+x^2+1 \quad \rightarrow \quad f(1)=\frac{\pi}{4}+2$$

$$f'(x)=\frac{1}{x^2+1}+2x \quad \rightarrow \quad f'(1)=\frac{5}{2}$$

$$f''(x)=-\frac{2x}{\left(x^2+1\right)^2}+2 \quad \rightarrow \quad f''(1)=-\frac{1}{2}+2=\frac{3}{2}$$

$$f'''(x)=-\frac{2x^2+2-8x^2}{\left(x^2+1\right)^3}=-\frac{2-2x^2}{\left(x^2+1\right)^3} \quad \rightarrow \quad f'''(1)=0 \qquad \left(U^nV^m\right)'=U^{n-1}V^{m-1}\left(nU'V+mUV'\right)$$

$$P_0(x)=\boxed{\frac{\pi}{4}+2} \qquad P_0(x)=f(a)$$

$$P_1(x)=\boxed{\frac{\pi}{4}+2+\frac{5}{2}(x-1)} \qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$P_2(x)=\boxed{\frac{\pi}{4}+2+\frac{5}{2}(x-1)-\frac{3}{4}(x-1)^2} \qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x)=\boxed{\frac{\pi}{4}+2+\frac{5}{2}(x-1)-\frac{3}{4}(x-1)^2} \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=e^x, \quad a=\ln 2$

Solution

$$f(x)=e^x \quad \to \quad f(\ln 2)=2$$

$$f'(x)=e^x \quad \to \quad f'(\ln 2)=2$$

$$f''(x)=e^x \quad \to \quad f''(\ln 2)=2$$

$$f'''(x)=e^x \quad \to \quad f'''(\ln 2)=2$$

$P_0(x)=2$ $\qquad P_0(x)=f(a)$

$P_1(x)=2+2(x-\ln 2)$ $\qquad P_1(x)=f(a)+f'(a)(x-a)$

$P_2(x)=2+2(x-\ln 2)+(x-\ln 2)^2$ $\qquad P_2(x)=f(a)+f'(a)(x-a)+\dfrac{f''(a)}{2!}(x-a)^2$

$P_3(x)=2+2(x-\ln 2)+(x-\ln 2)^2+\frac{1}{3}(x-\ln 2)^3$ $\qquad P_3(x)=P_2(x)+\dfrac{f'''(a)}{3!}(x-a)^3$

Exercise

Find the *n*th Maclaurin polynomial for the function $\quad f(x)=e^{4x}, \quad n=4$

Solution

$$f(x)=e^{4x} \quad \to \quad f(0)=1$$

$$f'(x)=4e^{4x} \quad \to \quad f'(0)=4$$

$$f''(x)=16e^{4x} \quad \to \quad f''(0)=16$$

$$f'''(x)=64e^{4x} \quad \to \quad f'''(0)=64$$

$$f^{(4)}(x)=256e^{4x} \quad \to \quad f^{(4)}(0)=256$$

$$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$

$$P_4(x)=1+4x+8x^2+\frac{32}{3}x^3+\frac{32}{3}x^4$$

Exercise

Find the *n*th Maclaurin polynomial for the function $\quad f(x)=e^{-x}, \quad n=5$

Solution

$$f(x)=e^{-x} \quad \to \quad f(0)=1$$

$$f'(x)=-e^{-x} \quad \to \quad f'(0)=-1$$

$$f''(x) = e^{-x} \quad \to \quad f''(0) = 1$$

$$f'''(x) = -e^{-x} \quad \to \quad f'''(0) = -1$$

$$f^{(4)}(x) = e^{-x} \quad \to \quad f^{(4)}(0) = 1$$

$$f^{(5)}(x) = -e^{-x} \quad \to \quad f^{(5)}(0) = -1$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$\underline{P_5(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5}$$

Exercise

Find the *n*th Maclaurin polynomial for the function $f(x) = e^{-x/2}, \quad n = 4$

Solution

$$f(x) = e^{-x/2} \quad \to \quad f(0) = 1$$

$$f'(x) = -\frac{1}{2}e^{-x/2} \quad \to \quad f'(0) = -\frac{1}{2}$$

$$f''(x) = \frac{1}{4}e^{-x/2} \quad \to \quad f''(0) = \frac{1}{4}$$

$$f'''(x) = -\frac{1}{8}e^{-x/2} \quad \to \quad f'''(0) = -\frac{1}{8}$$

$$f^{(4)}(x) = \frac{1}{16}e^{-x/2} \quad \to \quad f^{(4)}(0) = \frac{1}{16}$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\underline{P_4(x) = 1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4}$$

Exercise

Find the *n*th Maclaurin polynomial for the function $f(x) = e^{x/3}, \quad n = 4$

Solution

$$f(x) = e^{x/3} \quad \to \quad f(0) = 1$$

$$f'(x) = \frac{1}{3}e^{x/3} \quad \to \quad f'(0) = \frac{1}{3}$$

$$f''(x) = \frac{1}{9}e^{x/3} \quad \to \quad f''(0) = \frac{1}{9}$$

$$f'''(x) = \frac{1}{27}e^{x/3} \quad \to \quad f'''(0) = \frac{1}{27}$$

$$f^{(4)}(x) = \frac{1}{81}e^{x/3} \quad \to \quad f^{(4)}(0) = \frac{1}{81}$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\boxed{P_4(x) = 1 + \frac{1}{3}x + \frac{1}{18}x^2 + \frac{1}{162}x^3 + \frac{1}{1944}x^4}$$

Exercise

Find the *n*th Maclaurin polynomial for the function $\quad f(x) = \sin x, \quad n = 5$

Solution

$$f(x) = \sin x \quad \to \quad f(0) = 0$$

$$f'(x) = \cos x \quad \to \quad f'(0) = 1$$

$$f''(x) = -\sin x \quad \to \quad f''(0) = 0$$

$$f'''(x) = -\cos x \quad \to \quad f'''(0) = -1$$

$$f^{(4)}(x) = \sin x \quad \to \quad f^{(4)}(0) = 0$$

$$f^{(5)}(x) = \cos x \quad \to \quad f^{(5)}(0) = 1$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$\boxed{P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5}$$

Exercise

Find the *n*th Maclaurin polynomial for the function $\quad f(x) = \cos \pi x, \quad n = 4$

Solution

$$f(x) = \cos \pi x \quad \to \quad f(0) = 1$$

$$f'(x) = -\pi \sin \pi x \quad \to \quad f'(0) = 0$$

$$f''(x) = -\pi^2 \cos \pi x \quad \to \quad f''(0) = -\pi^2$$

$$f'''(x) = \pi^3 \sin \pi x \quad \to \quad f'''(0) = 0$$

$$f^{(4)}(x) = \pi^4 \cos \pi x \quad \to \quad f^{(4)}(0) = \pi^4$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\boxed{P_4(x) = 1 - \frac{\pi^2}{2}x^2 + \frac{\pi^4}{24}x^4}$$

Exercise

Find the *n*th Maclaurin polynomial for the function $f(x)=xe^{x}, \quad n=4$

Solution

$$f(x)=xe^{x} \quad \rightarrow \quad f(0)=0$$

$$f'(x)=e^{x}+xe^{x} \quad \rightarrow \quad f'(0)=1$$

$$f''(x)=2e^{x}+xe^{x} \quad \rightarrow \quad f''(0)=2$$

$$f'''(x)=3e^{x}+xe^{x} \quad \rightarrow \quad f'''(0)=3$$

$$f^{(4)}(x)=4e^{x}+xe^{x} \quad \rightarrow \quad f^{(4)}(0)=4$$

$$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$

$$P_4(x)=x+x^2+\frac{1}{2}x^3+\frac{1}{6}x^4$$

Exercise

Find the *n*th Maclaurin polynomial for the function $f(x)=x^2e^{-x}, \quad n=4$

Solution

$$f(x)=x^2e^{-x} \quad \rightarrow \quad f(0)=0$$

$$f'(x)=2xe^{-x}-x^2e^{-x} \quad \rightarrow \quad f'(0)=0$$

$$f''(x)=2e^{-x}-4xe^{x}+x^2e^{-x} \quad \rightarrow \quad f''(0)=2$$

$$f'''(x)=-6e^{-x}+6xe^{-x}-x^2e^{-x} \quad \rightarrow \quad f'''(0)=-6$$

$$f^{(4)}(x)=12e^{-x}-8xe^{-x}+x^2e^{-x} \quad \rightarrow \quad f^{(4)}(0)=12$$

$$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$

$$P_4(x)=x^2-x^3+\frac{1}{2}x^4$$

Exercise

Find the *n*th Maclaurin polynomial for the function $f(x)=\frac{1}{x+1}, \quad n=5$

Solution

$$f(x)=\frac{1}{x+1} \quad \rightarrow \quad f(0)=1$$

$$f'(x)=-(x+1)^{-2} \quad \rightarrow \quad f'(0)=-1$$

$$f''(x)=2(x+1)^{-3} \quad \rightarrow \quad f''(0)=2$$

$$f'''(x) = -6(x+1)^{-4} \quad \to \quad f'''(0) = -6$$

$$f^{(4)}(x) = 24(x+1)^{-5} \quad \to \quad f^{(4)}(0) = 24$$

$$f^{(5)}(x) = -120(x+1)^{-6} \quad \to \quad f^{(5)}(0) = -120$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$\boxed{P_5(x) = 1 - x + x^2 - x^3 + x^4 - x^5}$$

Exercise

Find the nth Maclaurin polynomial for the function $f(x) = \dfrac{x}{x+1}, \quad n = 4$

Solution

$$f(x) = \frac{x}{x+1} = 1 - \frac{1}{x+1} \quad \to \quad f(0) = 0$$

$$f'(x) = (x+1)^{-2} \quad \to \quad f'(0) = 1$$

$$f''(x) = -2(x+1)^{-3} \quad \to \quad f''(0) = -2$$

$$f'''(x) = 6(x+1)^{-4} \quad \to \quad f'''(0) = 6$$

$$f^{(4)}(x) = -24(x+1)^{-5} \quad \to \quad f^{(4)}(0) = -24$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\boxed{P_4(x) = x - x^2 + x^3 - x^4}$$

Exercise

Find the nth Maclaurin polynomial for the function $f(x) = \sec x, \quad n = 2$

Solution

$$f(x) = \sec x \quad \to \quad f(0) = 1$$

$$f'(x) = \sec x \tan x \quad \to \quad f'(0) = 0$$

$$f''(x) = \sec x \tan^2 x + \sec^3 x \quad \to \quad f''(0) = 1$$

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$

$$\boxed{P_2(x) = 1 + \frac{1}{2}x^2}$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x)=\tan x, \quad n=3$

Solution

$$f(x)=\tan x \quad \to \quad f(0)=0$$

$$f'(x)=\sec^2 x \quad \to \quad f'(0)=1$$

$$f''(x)=2\sec^2 x\tan x \quad \to \quad f''(0)=0$$

$$f'''(x)=4\sec^2 x\tan^2 x+2\sec^4 x \quad \to \quad f'''(0)=2$$

$$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3$$

$$\boxed{P_4(x)=x+\frac{1}{3}x^3}$$

Exercise

Find the Maclaurin series for: $\quad xe^x$

Solution

$$f(x)=xe^x \quad \to \quad f(0)=0$$

$$f'(x)=e^x+xe^x \quad \to \quad f'(0)=1$$

$$f''(x)=2e^x+xe^x \quad \to \quad f''(0)=2$$

$$f'''(x)=3e^x+xe^x \quad \to \quad f'''(0)=3$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$f^{(n)}(x)=ne^x+xe^x \quad \to \quad f^{(n)}(0)=n$$

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots$$

$$xe^x=x+x^2+\frac{1}{2}x^3+\cdots=\sum_{n=0}^{\infty}\frac{1}{(n-1)!}x^n$$

Exercise

Find the Maclaurin series for: $\quad 5\cos\pi x$

Solution

$$f(x)=5\cos\pi x \quad \to \quad f(0)=5$$

$$f'(x)=-5\pi\sin\pi x \quad \to \quad f'(0)=0$$

$$f''(x) = -5\pi^2 \cos \pi x \quad \rightarrow \quad f''(0) = -5\pi^2$$

$$f'''(x) = 5\pi^3 \sin \pi x \quad \rightarrow \quad f'''(0) = 0$$

$$5\cos \pi x = 5 - \frac{5\pi^2 x^2}{2!} + \frac{5\pi^4 x^4}{4!} - \frac{5\pi^6 x^6}{6!} + \cdots = 5\sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!}$$

Exercise

Find the Maclaurin series for: $\dfrac{x^2}{x+1}$

Solution

$$f(x) = \frac{x^2}{x+1} \quad \rightarrow \quad f(0) = 0$$

$$f'(x) = \frac{2x(x+1) - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2} \quad \rightarrow \quad f'(0) = 0$$

$$f''(x) = \frac{(2x+2)(x+1)^2 - 2(x+1)\left(x^2 + 2x\right)}{(x+1)^4}$$

$$= \frac{2x^2 + 4x + 2 - 2x^2 - 4x}{(x+1)^3}$$

$$= \frac{2}{(x+1)^3} \quad \rightarrow \quad f''(0) = 2$$

$$f'''(x) = \frac{-6}{(x+1)^4} \quad \rightarrow \quad f'''(0) = -6$$

$$\frac{x^2}{x+1} = \frac{2}{2!}x^2 - \frac{6x^3}{3!} + \frac{24x^4}{4!} - \cdots = x^2 - x^3 + x^4 - \cdots = \sum_{n=2}^{\infty} (-1)^n x^n$$

Exercise

Find the Maclaurin series for: e^{3x+1}

Solution

$$e^{3x+1} = e \cdot e^{3x}$$

$$= e\left(\sum_{n=0}^{\infty} \frac{(3x)^n}{n!}\right)$$

$$=\boxed{\sum_{n=0}^{\infty}\frac{e3^n x^n}{n!}} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\cos\left(2x^3\right)$

Solution

$$\cos\left(2x^3\right)=1-\frac{\left(2x^3\right)^2}{2!}+\frac{\left(2x^3\right)^4}{4!}-\frac{\left(2x^3\right)^6}{6!}+\cdots$$

$$=1-\frac{2^2x^3}{2!}+\frac{2^4x^{12}}{4!}-\frac{2^6x^{18}}{6!}+\cdots$$

$$=\boxed{\sum_{n=0}^{\infty}\frac{(-1)^n 4^n}{(2n)!}x^{6n}} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\cos(2x-\pi)$

Solution

$$\cos(2x-\pi)=\cos(2x)\cos\pi+\sin(2x)\sin\pi$$

$$=-\cos(2x)$$

$$=-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(2x)^{2n}$$

$$=\boxed{\sum_{n=0}^{\infty}\frac{(-1)^{n+1}4^n}{(2n)!}x^{2n}} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $x^2\sin\left(\frac{x}{3}\right)$

Solution

$$x^2\sin\left(\frac{x}{3}\right)=x^2\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{x}{3}\right)^{2n+1}}{(2n+1)!}$$

$$=x^2\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{3^{2n+1}(2n+1)!}$$

$$=\boxed{\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+3}}{3^{2n+1}(2n+1)!}} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\cos^2\left(\frac{x}{2}\right)$

Solution

$$\cos^2\left(\frac{x}{2}\right) = \frac{1}{2}(1+\cos x)$$

$$= \frac{1}{2}\left(1+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}\right)$$

$$= \frac{1}{2}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$

$$=\boxed{1+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\sin x\cos x$

Solution

$$\sin x\cos x = \frac{1}{2}\sin(2x)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2x)^{2n+1}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}2^{2n+1}x^{2n+1}$$

$$=\boxed{\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}}{(2n+1)!}x^{2n+1}} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\tan^{-1}\left(5x^2\right)$

Solution

$$\tan^{-1}\left(5x^2\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(5x^2\right)^{2n+1} \qquad 5x^2\le 1 \ \to\ x^2\le\frac{1}{5} \ \Rightarrow\ -\frac{1}{\sqrt{5}}\le x\le\frac{1}{\sqrt{5}}$$

$$=\boxed{\sum_{n=0}^{\infty}\frac{(-1)^n 5^{2n+1}}{2n+1}x^{4n+2}} \quad \left(for\quad -\frac{1}{\sqrt{5}}\le x\le\frac{1}{\sqrt{5}}\right)$$

Exercise

Find the Maclaurin series for: $\ln\left(2+x^2\right)$

Solution

$$\ln\left(2+x^2\right)=\ln 2\left(1+\frac{x^2}{2}\right)$$

$$=\ln 2+\ln\left(1+\frac{x^2}{2}\right)$$

$$=\ln 2+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{x^2}{2}\right)^n \qquad \frac{x^2}{2}\le 1 \ \to\ x^2\le 2 \ \Rightarrow\ -\sqrt{2}\le x\le\sqrt{2}$$

$$\boxed{=\ln 2+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}\frac{x^{2n}}{2^n}} \quad \left(for\quad -\sqrt{2}\le x\le\sqrt{2}\right)$$

Exercise

Find the Maclaurin series for: $\dfrac{1+x^3}{1+x^2}$

Solution

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n=1-x+x^2-x^3+\cdots$$

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+\cdots$$

$$\frac{1+x^3}{1+x^2}=\left(1+x^3\right)\left(1-x^2+x^4-x^6+\cdots\right)$$

$$=1-x^2+x^4-x^6+\cdots+x^3-x^5+x^7-x^9+\cdots$$

$$=1-x^2+x^3+x^4-x^5-x^6+x^7+x^8-x^9-\cdots$$

$$\boxed{=1-x^2+\sum_{n=2}^{\infty}(-1)^n\left(x^{2n-1}+x^{2n}\right)} \quad \left(for\quad |x|<1\right)$$

Exercise

Find the Maclaurin series for: $\ln\frac{1+x}{1-x}$

Solution

$$\ln\frac{1+x}{1-x} = \ln(1+x) - \ln(1-x) \qquad = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots\right) = 2x + 2\frac{x^3}{3} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$= \sum_{n=0}^{\infty} \left((-1)^n + 1\right) \frac{x^{n+1}}{n+1}$$

$$= 2\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1} \qquad (-1 < x < 1)$$

Exercise

Find the Maclaurin series for: $\dfrac{e^{2x^2}-1}{x^2}$

Solution

$$\frac{e^{2x^2}-1}{x^2} = \frac{1}{x^2}\left(e^{2x^2} - 1\right)$$

$$= \frac{1}{x^2}\left(1 + 2x^2 + \frac{\left(2x^2\right)^2}{2!} + \frac{\left(2x^2\right)^3}{3!} + \cdots - 1\right)$$

$$= \frac{1}{x^2}\left(2x^2 + \frac{2^2 x^4}{2!} + \frac{2^3 x^6}{3!} + \cdots\right)$$

$$= 2 + \frac{2^2 x^2}{2!} + \frac{2^3 x^4}{3!} + \frac{2^4 x^6}{4!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{2^{n+1}}{(n+1)!} x^{2n} \qquad (\text{for all } \ x \neq 0)$$

Exercise

Find the Maclaurin series for: $\cosh x - \cos x$

Solution

$$\cosh x-\cos x=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$

$$1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)=2\frac{x^2}{2!}+2\frac{x^6}{6!}+\cdots$$

$$=\sum_{n=0}^{\infty}\left(1-(-1)^n\right)\frac{x^{2n}}{(2n)!}$$

$$\boxed{=2\sum_{n=0}^{\infty}\frac{x^{4n+2}}{(4n+2)!}}\quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\sinh x-\sin x$

Solution

$$\sinh x-\sin x=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

$$=\sum_{n=0}^{\infty}\left(1-(-1)^n\right)\frac{x^{2n+1}}{(2n+1)!}$$

$$\boxed{=2\sum_{n=0}^{\infty}\frac{x^{4n+3}}{(4n+3)!}}\quad (\text{for all } x)$$

Exercise

Finding Taylor and Maclaurin Series generated by f *at* $x = a$: $\quad f(x)=x^3-2x+4,\quad a=2$

Solution

$f(x)=x^3-2x+4 \quad\rightarrow\quad f(2)=8$

$f'(x)=3x^2-2 \quad\rightarrow\quad f'(2)=10$

$f''(x)=6x \quad\rightarrow\quad f''(2)=12$

$f'''(x)=6 \quad\rightarrow\quad f'''(2)=6$

$f^{(n)}(x)=0 \quad\rightarrow\quad f^{(n)}(2)=0 \quad (n>3)$

$$P_n(x)=f(2)+f'(2)(x-2)+\frac{f''(2)}{2!}(x-2)^2+\frac{f'''(2)}{3!}(x-2)^3+\cdots$$

$$\boxed{x^3-2x+4=8+10(x-2)+6(x-2)^2+(x-2)^3}$$

Exercise

Finding Taylor and Maclaurin Series generated by f *at* $x = a$: $f(x)=2x^3+x^2+3x-8, \quad a=1$

Solution

$$f(x)=2x^3+x^2+3x-8 \quad \to \quad f(1)=-2$$

$$f'(x)=6x^2+2x+3 \quad \to \quad f'(1)=11$$

$$f''(x)=12x+2 \quad \to \quad f''(1)=14$$

$$f'''(x)=12 \quad \to \quad f'''(1)=12$$

$$f^{(n)}(x)=0 \quad \to \quad f^{(n)}(1)=0 \quad (n\ge 4)$$

$$P_n(x)=f(1)+f'(1)(x-1)+\frac{f''(1)}{2!}(x-1)^2+\frac{f'''(1)}{3!}(x-1)^3+\cdots$$

$$2x^3+x^2+3x-8=-2+11(x-1)+7(x-1)^2+2(x-1)^3$$

Exercise

Finding Taylor and Maclaurin Series generated by f *at* $x = a$:

$$f(x)=3x^5-x^4+2x^3+x^2-2, \quad a=-1$$

Solution

$$f(x)=3x^5-x^4+2x^3+x^2-2 \quad \to \quad f(-1)=-7$$

$$f'(x)=15x^4-4x^3+6x^2+2x \quad \to \quad f'(-1)=23$$

$$f''(x)=60x^3-12x^2+12x+2 \quad \to \quad f''(-1)=-82$$

$$f'''(x)=180x^2-24x+12 \quad \to \quad f'''(-1)=216$$

$$f^{(4)}(x)=360x-24 \quad \to \quad f^{(4)}(-1)=-384$$

$$f^{(5)}(x)=360 \quad \to \quad f^{(5)}(-1)=360$$

$$f^{(n)}(x)=0 \quad \to \quad f^{(n)}(-1)=0 \quad (n\ge 6)$$

$$P_n(x)=f(-1)+f'(-1)(x+1)+\frac{f''(-1)}{2!}(x+1)^2+\frac{f'''(-1)}{3!}(x+1)^3+\frac{f^{(4)}(-1)}{4!}(x+1)^2+\frac{f^{(4)}(-1)}{5!}(x+1)^3$$

$$3x^5-x^4+2x^3+x^2-2=-7+23(x+1)-\frac{82}{2!}(x+1)^2+\frac{216}{3!}(x+1)^3-\frac{384}{4!}(x+1)^4+\frac{360}{5!}(x+1)^3$$

$$=-7+23(x+1)-41(x+1)^2+36(x+1)^3-16(x+1)^4+3(x+1)^3$$

Exercise

Finding Taylor and Maclaurin Series generated by f *at* $x = a$: $\quad f(x)=\cos\left(2x+\frac{\pi}{2}\right), \quad a=\frac{\pi}{4}$

Solution

$$f(x)=\cos\left(2x+\frac{\pi}{2}\right) \quad\to\quad f\left(\frac{\pi}{4}\right)=-1$$

$$f'(x)=-2\sin\left(2x+\frac{\pi}{2}\right) \quad\to\quad f'\left(\frac{\pi}{4}\right)=0$$

$$f''(x)=-4\cos\left(2x+\frac{\pi}{2}\right) \quad\to\quad f''\left(\frac{\pi}{4}\right)=4$$

$$f'''(x)=8\sin\left(2x+\frac{\pi}{2}\right) \quad\to\quad f'''\left(\frac{\pi}{4}\right)=0$$

$$f^{(4)}(x)=16\cos\left(2x+\frac{\pi}{2}\right) \quad\to\quad f^{(4)}\left(\frac{\pi}{4}\right)=-16$$

$$f^{(5)}(x)=-32\sin\left(2x+\frac{\pi}{2}\right) \quad\to\quad f^{(5)}\left(\frac{\pi}{4}\right)=0$$

$$\to\quad f^{(2n)}\left(\frac{\pi}{4}\right)=(-1)^n 2^{2n}$$

$$\begin{aligned}\cos\left(2x+\frac{\pi}{2}\right)&=-1+\frac{4}{2!}\left(x-\frac{\pi}{4}\right)^2-\frac{16}{4!}\left(x-\frac{\pi}{4}\right)^4+\cdots\\&=-1+2\left(x-\frac{\pi}{4}\right)^2-\frac{2}{3}\left(x-\frac{\pi}{4}\right)^4+\cdots\\&=\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}}{(2n)!}\left(x-\frac{\pi}{4}\right)^{2n}\end{aligned}$$