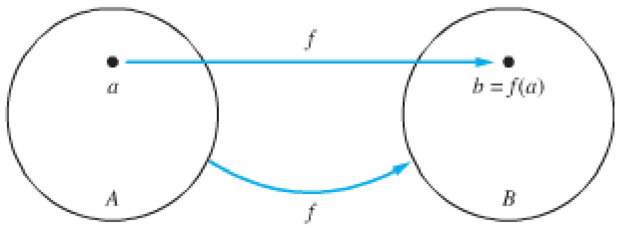
Section 1.9 – Functions

Definition

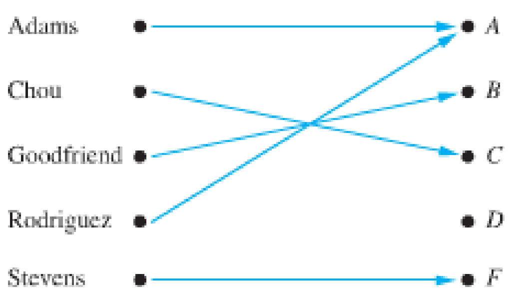
Let A and B be nonempty sets. A function f from A and B is an assignment of exactly one element of B to each of A. We write $f(a)=b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f:A\to B$



When we define a function we specify its domain, its codomain, and the mapping of elements of the domain to elements in the codomain. Two functions are ***equal*** when they have the same domain, have the same codomain, and map each element of their common domain to the same element in their common codomain.

Example

What are the domain, codomain, and range of the function that assigns grades to students shown below



Solution

The domain is the set ***G*** = {Adams, Chou, Goodfriend, Rodriguez, Stevens}

The codomain is the set {*A, B, C, D, F*}

The range of ***G*** is the set {*A, B, C, F*}

Example

Let $f:\mathbb{Z}\to\mathbb{Z}$ assign the square of an integer to this integer. Then $f(x)=x^2$, where the domain of f is the set of all integers, the codomain of f is the set of all integers, and the range of f is the set of all integers that are perfect squares, namely, {0, 1, 4, 9, . . .}

Definition

Let f_1 and f_2 be functions from *A* to **R**. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from *A* to **R** defined for all $x \in A$ by

$$\left(f_1 + f_2\right)(x) = f_1(x) + f_2(x)$$

$$\left(f_1 f_2\right)(x) = f_1(x)\, f_2(x)$$

Example

Let f_1 and f_2 be functions from **R** to **R** such that $f_1(x) = x^2$ and $f_2 = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

Solution

$$\left(f_1 + f_2\right)(x) = x^2 + \left(x - x^2\right) = x$$

$$\left(f_1 f_2\right)(x) = x^2\left(x - x^2\right) = x^3 - x^4$$

Definition

Let f be function from *A* to *B* and Let *S* be a subset of *A*. The ***image*** of *S* under the function f is the subset of *B* that consists of the images of the elements of *S*. We denote the image of *S* by $f(S)$, so

$$f(S) = \left\{ t \,\middle|\, \exists s \in S \ \left(t = f(s)\right) \right\}$$

We also use the shorthand $\left\{ f(S) \,\middle|\, s \in S \right\}$ to denote this set.

One-to-One and Onto Functions

Definition

A function f is said to be *one-to-one*, or an ***injunction***, if and only if $f(a) = f(b)$ implies that $a = b$ for all *a* and *b* in the domain of f. A function is said to be ***injective*** if it is one-to-one.

Note:

A function *f* is one-to-one (1 – 1) if different inputs have different outputs that is,

$$\text{if } \ a \neq b, \quad \text{then} \quad f(a) \neq f(b)$$

A function *f* is one-to-one (1 – 1) if different outputs the same, the inputs are the same – that is,

$$\text{if } \ f(a) = f(b), \quad \text{then } \ a = b$$

Remark

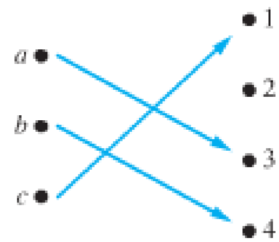
We can express that f is one-to-one using the qualifier as $\forall a \forall b\left(f(a) = f(b) \rightarrow a = b \right)$ or equivalently $\forall a \forall b\left(a \neq b \rightarrow f(a) \neq f(b)\right)$, where the universe of disclosure is the domain of the function.

Example

Determine whether the function f from {*a*, *b*, *c*, *d*} to {1, 2, 3, 4, 5} with $f(a)=4$, $f(b)=5$, $f(c)=1$, and $f(d)=3$ is one-to-one.

Solution

The function is one-to-one because *f* takes on different values of the four elements of its domain.



Example

Determine whether the function $f(x)=x^2$ from the set of integers to the set of integers is one-to-one.

Solution

The function is ***not*** one-to-one because $f(-1) = f(1) = 1$ but $1 \neq -1$

Example

Determine whether the function $f(x)=x+1$ from the set of real numbers to itself is one-to-one.

Solution

The function is one-to-one because $x+1 \neq y+1$ when $x \neq y$

Definition

A function *f* whose domain and codomain are subsets of the set of real numbers is called ***increasing*** if $f(x) \leq f(y)$, and ***strictly increasing*** if $f(x) < f(y)$, whenever $x < y$ and *x* and *y* are in the domain of *f*. Similarly, *f* is called ***decreasing*** if $f(x) \geq f(y)$, and ***strictly increasing*** if $f(x) > f(y)$, whenever $x < y$ and *x* and *y* are in the domain of *f*. (The word ***strictly*** in this definition indicates a strict inequality.)

Definition

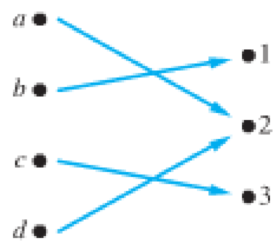
A function f from *A* to *B* is called ***onto***, or a ***surjection***, *iff* for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called ***surjective*** if it is onto.

Example

Let f be function from {*a, b, c, d*} to {1, 2, 3} defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Solution

Because all three elements of the codomain are images of elements in the domain, we see that f is onto.



Example

Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution

The function is ***not*** onto because there is no integer *x* with $x^2 = -1$.

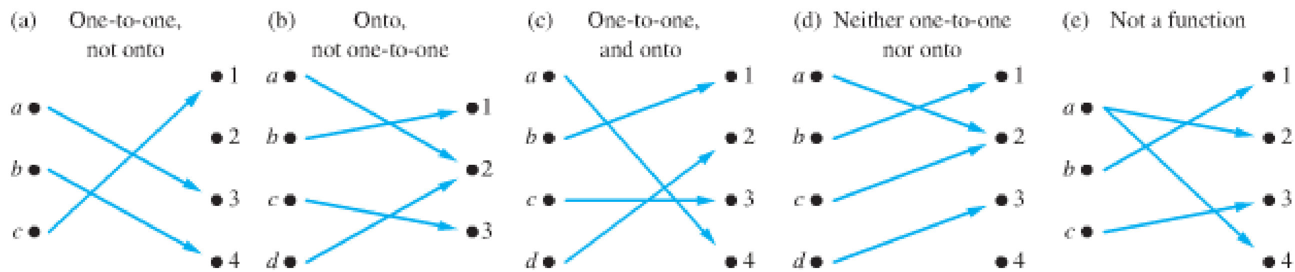
Example

Is the function $f(x) = x + 1$ from the set of integers to the set of integers onto?

Solution

The function is onto because for every integer *y* there is an integer *x* such that $f(x) = y$.

$f(x) = y$ *iff* $x + 1 = y$, which holds if and only if $x = y - 1$



Definition

The function f is *one-to-one correspondence*, or a ***bijection***, if it is both one-to-one and onto. We say also that such a function is ***bijective***.

Example

Let f be function from {*a, b, c, d*} to {1, 2, 3, 4} defined by $f(a)=4$, $f(b)=2$, $f(c)=1$, and $f(d)=3$. Is f a bijection?

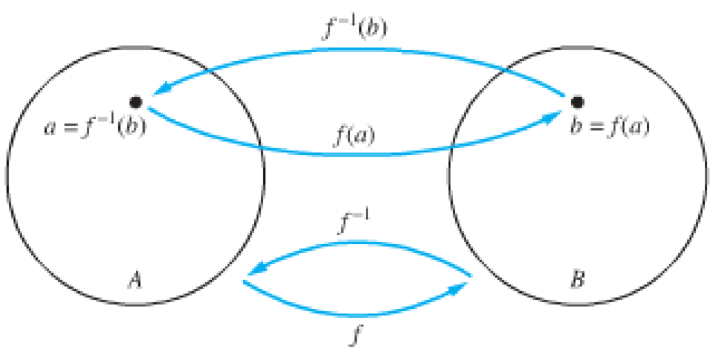
Solution

The function *f* is one-to-one and onto.
It is one-to-one because no two values in the domain are assigned the same function value.
It is onto because all four elements of the codomain are images of elements in the domain.
Hence, *f* is a bijection.

Inverse Functions and Compositions of Functions

Definition

Let f be a one-to-one correspondence from the set *A* to the set *B*. The inverse function of *f* is the function that assigns to an element b belonging to *B* the unique element a in *A* such that $f(a)=b$. The inverse function of *f* is denoted by f^{-1}. Hence, $f^{-1}(b)=a$ when $f(a)=b$



Example

Let f be function from {*a, b, c*} to {1, 2, 3} defined by $f(a)=2$, $f(b)=3$, and $f(c)=1$. Is f invertible, and if it is, what is its inverse?

Solution

The function f is invertible since it is a one-to-one.

The inverse function: $f^{-1}(1)=c$, and $f^{-1}(2)=a$, $f^{-1}(3)=b$

Example

Let $f: \mathbb{Z} \to \mathbb{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?

Solution

The function f is invertible since it is a one-to-one.

$y = x + 1 \quad \Rightarrow \quad x = y - 1$

$f^{-1}(y) = y - 1$

Example

Let $f: \mathbb{R} \to \mathbb{R}$ be such that $f(x) = x^2$ Is f invertible?

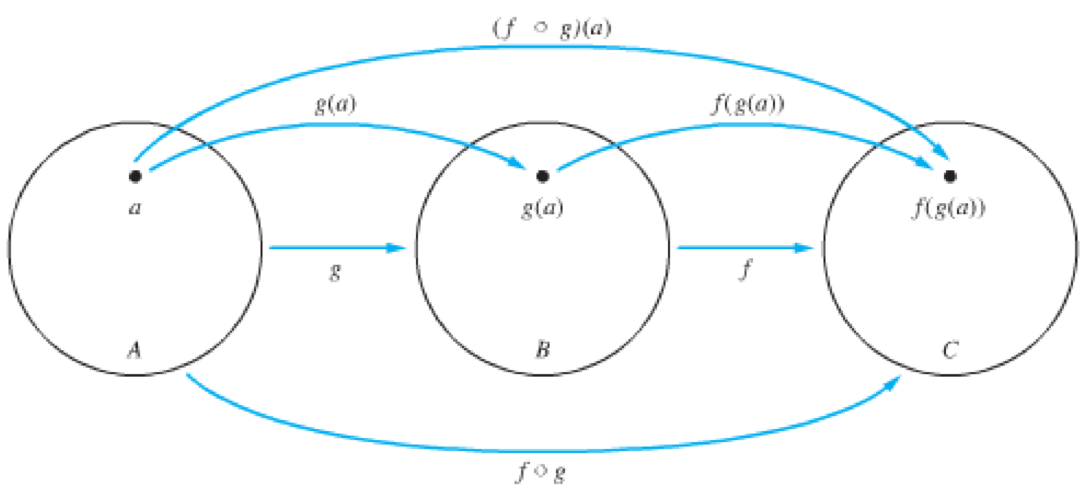
Solution

The function is ***not*** one-to-one. Hence, f is ***not*** invertible.

Definition

Let *g* be a function from the set *A* to the set *B* and let *f* be a function from the set *B* to the set *C*. The composition of the function *f* and *g*, denoted for all $a \in A$ by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a))$$



Example

Let *g* be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let *f* be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$. What is the composition of *f* and *g*, and what is the composition of *g* and *f*?

Solution

$(f \circ g)(a) = f(g(a)) = f(b) = 2$

$$(f \circ g)(b) = f(g(b)) = f(c) = 1$$
$$(f \circ g)(c) = f(g(c)) = f(a) = 3$$

$$(g \circ f)(a) = g(f(a)) = g(3) \quad \nexists .$$

Therefore; $g \circ f$ is not defined, because the range of *f* is not a subset of the domain of *g*.

Example

Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of *f* and *g*, and what is the composition of *g* and *f*?

Solution

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(3x+2) \\ &= 2(3x+2)+3 \\ &= \underline{6x+7}|\end{aligned}$$

$$(g \circ f)(x) = g(f(x)) = g(2x+3) = 3(2x+3)+2 = \underline{6x+11}| .$$

Exercises *Section* 1.9 – Functions

1. Why is f not a function from $\mathbb{R}$ to $\mathbb{R}$ if

a) $f(x)=\frac{1}{x}$?

b) $f(x)=\sqrt{x}$?

c) $f(x)=\pm\sqrt{x^2+1}$?

2. Determine whether f is a function from $\mathbb{Z}$ to $\mathbb{R}$ if

a) $f(x)=\pm x$?

b) $f(x)=\sqrt{x^2+1}$?

c) $f(x)=\dfrac{1}{x^2-4}$?

3. Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.

a) The function that assigns to each bit string the number of ones in the string minus the number of zeros in the string.

b) The function that assigns to each bit string twice the number of zeros in that string.

c) The function that assigns the number of bits over when a bit string is split into bytes (which are blocks of 8 bits).

4. Determine whether each of these functions from $\{a, b, c, d\}$ to itself is one-to-one and onto.

a) $f(a)=b,\ f(b)=a,\ f(c)=c,\ f(d)=d$

b) $f(a)=b,\ f(b)=b,\ f(c)=d,\ f(d)=c$

c) $f(a)=d,\ f(b)=b,\ f(c)=c,\ f(d)=d$

5. Determine whether the function $f:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ is onto if

a) $f(m,\ n)=m+n$

b) $f(m,\ n)=m^2+n^2$

c) $f(m,\ n)=m$

d) $f(m,\ n)=|n|$

e) $f(m,\ n)=m-n$

6. Determine whether each of these functions is a bijection from $\mathbb{R}\to\mathbb{R}$

a) $f(x)=2x+1$

b) $f(x)=x^2+1$

c) $f(x)=x^3$

d) $f(x)=\dfrac{x^2+1}{x^2+2}$

e) $f(x)=x^5+1$

7. Suppose that g is a function from A to B and f is a function from B to C.

a) Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.

b) Show that if both f and g are onto functions, then $f \circ g$ is also onto.