Section 2.2 – Rules of Differentiation

Notations for the Derivative

The derivative of $y = f(x)$ may be written in any of the following ways:

1st *derivative*	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}[f(x)]$	$D_x[y]$

Derivative of a *constant Function*

If f has the constant value $f(x) = c$

$$\frac{d}{dx}[c] = f'(c) = 0 \qquad c \text{ is constant}$$

Proof

Let $f(x) = c$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{c-c}{h}$$

$$= 0$$

So, $\frac{d}{dx}[c] = 0$

Example

Find the derivative

a) $f(x) = 9$

$f' = 0$

b) $h(t) = \pi$

$D_t[h(t)] = 0$

Power Rule

$$f(x) = x^n \quad \Rightarrow \quad f'(x) = nx^{n-1}$$ ***n*** is any real number

Proof

Let $f(x) = x^n$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + h^n - x^n}{h}$$

$$= \lim_{h\to 0} nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h + \ldots + h^{n-1}$$

$$= nx^{n-1}$$

Example

Find the derivative of: $a)\ x^3 \quad b)\ x^{2/3} \quad c)\ \frac{1}{x^4} \quad d)\ x^{\sqrt{2}} \quad e)\ \sqrt{x^{2+\pi}}$

Solution

a) $y = x^3$

$$\frac{dy}{dx} = 3x^{3-1}$$

$$\boxed{= 3x^2}$$

b) $y = x^{2/3}$

$$y' = \frac{2}{3}x^{2/3-1}$$

$$\boxed{= \frac{2}{3}x^{-1/3}}$$

c) $y = \frac{1}{x^4} = x^{-4}$

$$y' = -4x^{-4-1}$$

$$= -4x^{-5}$$

$$\boxed{= -\frac{4}{x^5}}$$

d) $D_x\left(x^{\sqrt{2}}\right) = \boxed{\sqrt{2}x^{\sqrt{2}-1}}$

e) $y = \left(x^{2+\pi}\right)^{1/2} = x^{(2+\pi)/2}$

$$y' = \left(\frac{2+\pi}{2}\right)x^{1+\pi/2-1}$$

$$\boxed{= \tfrac{1}{2}(2+\pi)\sqrt{x^{\pi}}}$$

Derivative Constant Multiple Rule

If f is a differentiable function of x, and c is a real number (constant), then $\frac{d}{dx}(cf) = c\frac{df}{dx}$

In particular, if n is any real number, then $\frac{d}{dx}\left(cx^n\right) = cnx^{n-1}$

Proof

$$\frac{d}{dx}(cf) = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}$$

$$= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad \textit{Factor } c$$

$$= c\frac{df}{dx}$$

Example

If $y = 8x^4$, find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx} = 8\left(4x^3\right) \ \boxed{= 32x^3}$$

Example

If $y = -\frac{3}{4}x^{12}$, find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx} = -\frac{3}{4}\left(12x^{11}\right)$$

$$\boxed{= -9x^{11}}$$

Sum or Difference Rule

The derivative of the sum or difference of two differentiable functions is the sum or difference of their derivatives.

$$\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx} \qquad\qquad \frac{d}{dx}(u-v)=\frac{du}{dx}-\frac{dv}{dx}$$
$$=u'+v' \qquad\qquad\qquad\qquad =u'-v'$$

Proof

$$f(x)=u(x)+v(x)$$

$$\begin{aligned}\frac{d}{dx}\left[u(x)+v(x)\right] &= \lim_{h\to 0}\frac{\left[u(x+h)+v(x+h)\right]-\left[u(x)+v(x)\right]}{h}\\ &= \lim_{h\to 0}\frac{u(x+h)+v(x+h)-u(x)-v(x)}{h}\\ &= \lim_{h\to 0}\left[\frac{u(x+h)-u(x)}{h}+\frac{v(x+h)-v(x)}{h}\right]\\ &= \lim_{h\to 0}\left[\frac{u(x+h)-u(x)}{h}\right]+\lim_{h\to 0}\left[\frac{v(x+h)-v(x)}{h}\right]\\ &= \frac{du}{dx}+\frac{dv}{dx}\end{aligned}$$

Example

Find the derivative of the polynomial $\qquad y=x^3+\frac{4}{3}x^2-5x+1$

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}x^3+\frac{d}{dx}\left(\frac{4}{3}x^2\right)-\frac{d}{dx}(5x)+\frac{d}{dx}(1)\\ &= 3x^2+\frac{8}{3}x-5+0\\ &= \boxed{3x^2+\frac{8}{3}x-5}\end{aligned}$$

Example

Find the derivative of $\qquad y=x^{5/2}+x^3+\frac{1}{2}x^2+4$

Solution

$$\boxed{y'=\frac{5}{2}x^{3/2}+3x^2+x}$$

Example

Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

<u>**Solution**</u>

$$y' = 4x^3 - 4x$$

$$y' = 0 \quad \Rightarrow \quad 4x^3 - 4x = 0$$

$$4x\left(x^2 - 1\right) = 0$$

$$\underline{x = 0, \ \pm 1}|$$

The curve has horizontal tangents at $x = 0,\ 1,\ and\ \ -1$.

The corresponding points on the curve are; (0, 2), (1, 1) and (–1, 1)

Second– and *Higher*–Order Derivatives

Notation for Higher-Order Derivatives							
1.	***1st derivative***	y'	***y*** prime	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}[f(x)]$	$D_x[y]$
2.	***2***						
3.	***3rd derivative***	y'''	***y*** triple prime	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}[f(x)]$	$D_x^3[y]$
4.	***4th derivative***	$y^{(4)}$		$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}[f(x)]$	$D_x^4[y]$
5.	***nth derivative***	$y^{(n)}$		$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	$\frac{d^n}{dx^n}[f(x)]$	$D_x^n[y]$

Example

Find the first four derivatives of $y = x^3 - 3x^2 + 2$

<u>**Solution**</u>

$$y' = 3x^2 - 6x$$

$$y'' = 6x - 6$$

$$y''' = 6$$

$$y^{(4)} = 0$$

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad \Rightarrow \boxed{f^{(n)}(x) = n! a_n}$$

Exercises Section 2.2 – Rules of Differentiation

Find the derivative of each function

1. $y = \frac{1}{x^3}$
2. $D_x\left(x^{4/3}\right)$
3. $y = \sqrt{z}$
4. $D_t(-8t)$
5. $y = \frac{9}{4x^2}$
6. $y = 6x^3 + 15x^2$
7. $y = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$
8. $p(t) = 12t^4 - 6\sqrt{t} + \frac{5}{t}$
9. $f(x) = \frac{x^3 + 3\sqrt{x}}{x}$
10. $y = \frac{x^3 - 4x}{\sqrt{x}}$
11. $f(x) = \left(4x^2 - 3x\right)^2$
12. $y = 3x\left(2x^2 + 5x\right)$
13. $y = 3\left(2x^2 + 5x\right)$
14. $y = (3x-2)(2x+3)$
15. $y = \frac{x^2 + 4x}{5}$
16. $y = \frac{3x^4}{5}$
17. $g(s) = \frac{s^2 - 2s + 5}{\sqrt{s}}$
18. $f(x) = \frac{x+1}{\sqrt{x}}$
19. $f(x) = 4x^{5/3} + 6x^{-3/2} - 11x$
20. $f(x) = \frac{2}{3}x^3 + \pi x^2 + 7x + 1$
21. $f(x) = \frac{x^5 - x^3}{15}$
22. $f(x) = x^{1/3} + 2x^{1/4} - 3x^{1/5}$
23. $f(t) = 3\sqrt[3]{t^2} - \frac{2}{\sqrt{t^3}}$
24. $f(t) = \sqrt{t}\left(5 - t - \frac{1}{3}t^2\right)$
25. $f(x) = \frac{3}{5}x^{5/3} + \frac{5}{3}x^{-3/5}$
26. $f(x) = x^{23} - x^{-23}$

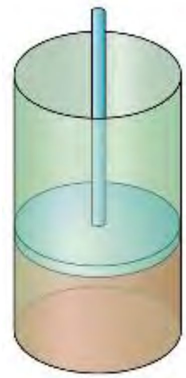
Find the ***first*** and ***second*** derivatives

27. $y = -x^3 + 3$
28. $y = 3x^7 - 7x^3 + 21x^2$
29. $y = 6x^2 - 10x - \frac{1}{x}$
30. $f(x) = \frac{1}{2}x^4 + \pi x^3 - 7x + 1$
31. $y = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$
32. $y = (2x-3)(1-5x)$

Find the derivatives

33. $f(x) = 3x^4 - 6x^3 + \frac{x^2}{8} + 5,\quad f^{(4)}(x)$
34. $f(x) = 3x^4 - 6x^3 + \frac{x^2}{8} + 5,\quad f^{(5)}(x)$
35. $f(x) = 2x^6 + 4x^4 - x + 2,\quad f^{(6)}(x)$
36. $f(x) = 4x^5 + 4x^4 + x^2 - 2,\quad f^{(5)}(x)$
37. $f(x) = 4x^5 + 4x^4 + x^2 - 2,\quad f^{(6)}(x)$
38. $f(x) = 4x^4 - 2x^3 + x + 2,\quad f^{(4)}(x)$

39. Find an equation for the line perpendicular to the tangent to the curve $y = x^3 - 4x + 1$ at the point $(2,\ 1)$.



40. If gas in a cylinder is maintained at a constant temperature T, the pressure P is related to the volume V by a formula of the form

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2}$$

In which a, b, n, and R are constants. Find $\frac{dP}{dV}$

41. Show that if $(a,\ f(a))$ is any point on the graph of $f(x) = x^2$, then the slope of the tangent line at that point is $m = 2a$

42. Show that if $(a,\ f(a))$ is any point on the graph of $f(x) = bx^2 + cx + d$, then the slope of the tangent line at that point is $m = 2ab + c$

43. Let $f(x) = x^2$

a) Show that $\dfrac{f(x) - f(y)}{x - y} = f'\left(\dfrac{x+y}{2}\right)$, for all $x \neq y$

b) Is this property true for $f(x) = ax^2$, where a is a nonzero real number?

c) Give a geometrical interpretation of this property.

d) Is this property true for $f(x) = ax^3$?