

Solution **Section 1.8 – Set Operations**

Exercise

Let A be the set of students who live within one mile of school and let B be the set of students who walk to classes. Describe the students in each of these sets

- a) $A \cap B$
- b) $A \cup B$
- c) $A - B$
- d) $B - A$

Solution

- a) The set of students who live one mile of school and walk to classes.
- b) The set of students who live one mile of school or walk to classes.
- c) The set of students who live one mile of school but not walk to class.
- d) The set of students who live more than one mile from school but nevertheless walk to class.

Exercise

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$

- a) $A \cup B$
- b) $A \cap B$
- c) $A - B$
- d) $B - A$

Solution

- a) $\{0, 1, 2, 3, 4, 5, 6\}$
- b) $\{3\}$
- c) $\{1, 2, 4, 5\}$
- d) $\{0, 6\}$

Exercise

Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$

- a) $A \cup B$
- b) $A \cap B$
- c) $A - B$
- d) $B - A$

Solution

- a) $\{a, b, c, d, e, f, g, h\} = B$

$$b) \{a, b, c, d, e\} = A$$

c) \emptyset , since there are no elements in A that are not in B .

$$d) \{f, g, h\}$$

Exercise

Prove the domination laws by showing that

$$a) A \cup U = U$$

$$b) A \cap U = A$$

$$c) A \cup \emptyset = A$$

$$d) A \cap \emptyset = \emptyset$$

Solution

$$\begin{aligned} a) \quad A \cup U &= \{x \mid x \in A \vee x \in U\} \\ &= \{x \mid x \in A \vee \mathbf{T}\} \\ &= \{x \mid \mathbf{T}\} \\ &= U \end{aligned}$$

$$\begin{aligned} b) \quad A \cap U &= \{x \mid x \in A \wedge x \in U\} \\ &= \{x \mid x \in A \wedge \mathbf{T}\} \\ &= \{x \mid x \in A\} \\ &= A \end{aligned}$$

$$\begin{aligned} c) \quad A \cup \emptyset &= \{x \mid x \in A \vee x \in \emptyset\} \\ &= \{x \mid x \in A \vee \mathbf{F}\} \\ &= \{x \mid x \in A\} \\ &= A \end{aligned}$$

$$\begin{aligned} d) \quad A \cap \emptyset &= \{x \mid x \in A \wedge x \in \emptyset\} \\ &= \{x \mid x \in A \wedge \mathbf{F}\} \\ &= \{x \mid \mathbf{F}\} \\ &= \emptyset \end{aligned}$$

Exercise

Prove the complement laws by showing that

$$a) A \cup \bar{A} = U$$

$$b) A \cap \bar{A} = \emptyset$$

Solution

$$\begin{aligned}
a) \quad A \cup \bar{A} &= \{x \mid x \in A \vee x \in \bar{A}\} \\
&= \{x \mid x \in A \vee x \notin A\} \\
&= \{x \mid \mathbf{T}\} \\
&= U
\end{aligned}$$

$$\begin{aligned}
b) \quad A \cap \bar{A} &= \{x \mid x \in A \wedge x \in \bar{A}\} \\
&= \{x \mid x \in A \wedge x \notin A\} \\
&= \{x \mid \mathbf{F}\} \\
&= \emptyset
\end{aligned}$$

Exercise

Show that

$$a) \quad A - \emptyset = A$$

$$b) \quad \emptyset - A = \emptyset$$

Solution

$$\begin{aligned}
a) \quad A - \emptyset &= \{x \mid x \in A \wedge x \notin \emptyset\} \\
&= \{x \mid x \in A \wedge \mathbf{T}\} \\
&= \{x \mid x \in A\} \\
&= A
\end{aligned}$$

$$\begin{aligned}
b) \quad \emptyset - A &= \{x \mid x \in \emptyset \wedge x \notin A\} \\
&= \{x \mid \mathbf{F} \wedge x \notin A\} \\
&= \{x \mid \mathbf{F}\} \\
&= \emptyset
\end{aligned}$$

Exercise

Prove the absorption law by showing that if A and B are sets, then

$$a) \quad A \cap (A \cup B) = A$$

$$b) \quad A \cup (A \cap B) = A$$

Solution

a) Suppose $x \in A \cap (A \cup B)$, then $x \in A$ and $x \in A \cup B$ by the definition of intersection. We have $x \in A$ and in the latter case $x \in A$ or $x \in B$ by the definition of union. Since both of these are true, $x \in A \cup B$ by the definition of intersection, and we have shown that the right-hand side is a subset of the left-hand side.

b) Suppose $x \in A \cup (A \cap B) \Rightarrow x \in A$ *or* $x \in (A \cap B)$ by definition of union.

$$x \in A \text{ *or* } (x \in A \text{ *and* } x \in B)$$

By the definition of the intersection, in any event, $x \in A$. Therefore, $x \in A \cup (A \cap B)$ as well.

That proves that the right-hand side is a subset of the left-hand side.

Exercise

Show that if A , B , and C are sets, then $\overline{A \cap B \cap C} = \bar{A} \cup \bar{B} \cup \bar{C}$

Solution

Suppose $x \in \overline{A \cap B \cap C}$, then $x \notin A \cap B \cap C$, which means that x fails to be in at least one of these three sets. In other words, $x \notin A$ *or* $x \notin B$ *or* $x \notin C$. This is equivalent to saying that $x \in \bar{A}$ *or* $x \in \bar{B}$ *or* $x \in \bar{C}$. Therefore $x \in \bar{A} \cup \bar{B} \cup \bar{C}$.

Conversely, if $x \in \bar{A} \cup \bar{B} \cup \bar{C}$, then $x \in \bar{A}$ *or* $x \in \bar{B}$ *or* $x \in \bar{C}$. This means $x \notin A$ *or* $x \notin B$ *or* $x \notin C$, so x cannot be in the intersection of A , B , and C . Since $x \notin A \cap B \cap C$, we conclude that $x \in \overline{A \cap B \cap C}$, as desired.

Or

A	B	C	$A \cap B \cap C$	$\overline{A \cap B \cap C}$	\bar{A}	\bar{B}	\bar{C}	$\bar{A} \cup \bar{B} \cup \bar{C}$
1	1	1	1	0	0	0	0	0
1	1	0	0	1	0	0	1	1
1	0	1	0	1	0	1	0	1
1	0	0	0	1	0	1	1	1
0	1	1	0	1	1	0	0	1
0	1	0	0	1	1	0	1	1
0	0	1	0	1	1	1	0	1
0	0	0	0	1	1	1	1	1

Exercise

Let A and B be sets. Show that

- a) $(A \cap B) \subseteq A$
- b) $A \subseteq (A \cup B)$
- c) $(A - B) \subseteq A$
- d) $A \cap (B - A) = \emptyset$
- e) $A \cup (B - A) = A \cup B$

Solution

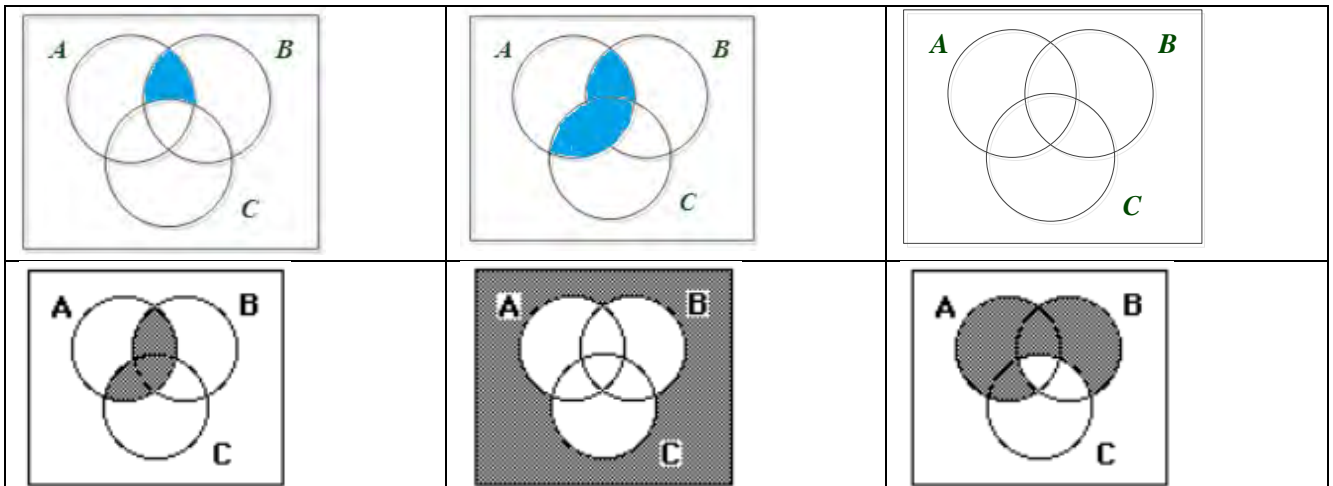
- a) If x is in $A \cap B$, then, by definition of intersection, it is in A .
- b) If x is in A , then perforce, by definition of union, it is in $A \cup B$.
- c) If x is in $A - B$, then perforce, by definition of difference, it is in A .
- d) Is $x \in A$ then $x \notin B - A$. Therefore there can be no elements in $A \cap (B - A)$, so $A \cap (B - A) = \emptyset$.
- e) The left-hand side consists of elements of either A or B or both. This is precisely the definition of the right-hand side.

Exercise

Draw the Venn diagrams for each of these combinations of the sets A , B , and C .

- a) $A \cap (B - C)$
- b) $(A \cap B) \cup (A \cap C)$
- c) $(A \cap \bar{B}) \cup (A \cap \bar{C})$
- d) $\bar{A} \cap \bar{B} \cap \bar{C}$
- e) $(A - B) \cup (A - C) \cup (B - C)$

Solution



Exercise

Show that $A \oplus B = (A \cup B) - (A \cap B)$

Solution

This is just a restatement of the definition. An element is in $(A \cup B) - (A \cap B)$ if it is in the union that in either A or B , but not in the intersection (i.e., not in both A and B)

Exercise

Show that $A \oplus B = (A - B) \cup (B - A)$

Solution

There are two ways that an item can be in either A or B but not both. It can be in A but not B (which is equivalent to saying that it is in $A - B$), or it can be in B but not A (which is equivalent to saying that it is in $B - A$).

Thus an element is in $A \oplus B$ if and only if it is in $(A - B) \cup (B - A)$.