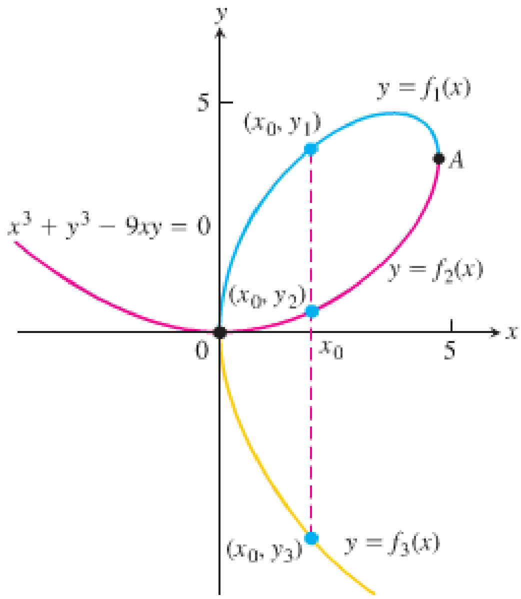
Section 2.7 – Implicit Differentiation

Definition

A relation $F(x, y) = 0$ is said to define the function $y = f(x)$ implicitly if, for x in the domain of f

$\rightarrow F(x, f(x)) = 0$

Example: $\quad x^3 + y^3 - 9xy = 0, \quad x^2 + y^2 = 25$

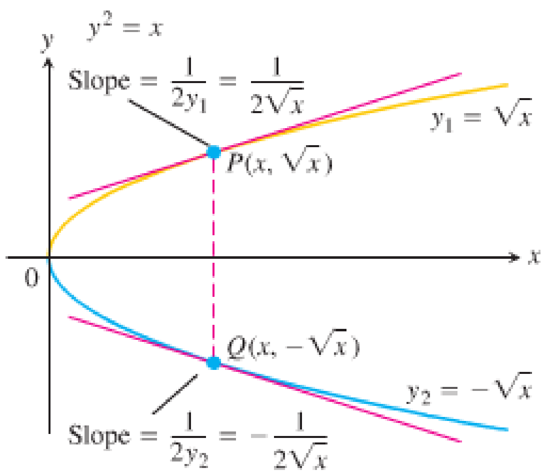


Implicitly Defined Functions

It is always assumed that the given equation determines y implicitly as a differentiable function of x so that $\frac{dy}{dx}$ exists.

Example

Find $\frac{dy}{dx}$ if $y^2 = x$



Solution

$$y^2 = x$$

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(x)$$

$$2y\frac{dy}{dx} = 1$$

$$\boxed{\frac{dy}{dx} = \frac{1}{2y}}$$

Example

Find the slope of the circle $x^2 + y^2 = 25$ at the point (3, –4).

Solution

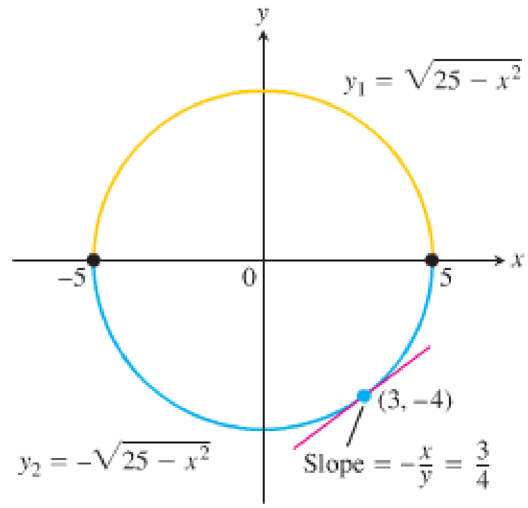
$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

The slope at (3, –4) is $\left.\frac{dy}{dx}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}$



Implicit Differentiation

✓ Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.

✓ Collect the terms with $\frac{dy}{dx}$ on one side of the equation and solve for $\frac{dy}{dx}$.

Example

Find $\frac{dy}{dx}$ if $y^2 = x^2 + \sin xy$

Solution

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(\sin xy)$$

$$2y\frac{dy}{dx} = 2x + \cos(xy)\frac{d}{dx}(xy)$$

$$2y\frac{dy}{dx} = 2x + \cos(xy)\left(y + x\frac{dy}{dx}\right)$$

$$2y\frac{dy}{dx} = 2x + y\cos(xy) + x\cos(xy)\frac{dy}{dx}$$

$$2y\frac{dy}{dx} - x\cos(xy)\frac{dy}{dx} = 2x + y\cos(xy)$$

$$(2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy$$

$$\boxed{\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}}$$

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(\sin xy)$$

Let $y' = \frac{dy}{dx}$

$$2yy' = 2x + \cos(xy)(xy)'$$

$$2yy' = 2x + \cos(xy)(y + xy')$$

$$2yy' = 2x + y\cos(xy) + x\cos(xy)y'$$

$$2yy' - x\cos(xy)y' = 2x + y\cos(xy)$$

$$(2y - x\cos xy)y' = 2x + y\cos xy$$

$$\boxed{y' = \frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}}$$

Example

Find $\quad \dfrac{d^2y}{dx^2}$ if $2x^3-3y^2=8$

Solution

Let $y'=\dfrac{dy}{dx}$

$$\frac{d}{dy}\left(2x^3-3y^2\right)=\frac{d}{dy}(8)$$

$$6x^2-6yy'=0$$

$$6x^2=6yy'$$

$$y'=\frac{6x^2}{6y}=\frac{x^2}{y}$$

$$y''=\left(\frac{x^2}{y}\right)' \qquad\qquad \begin{array}{ll} u=x^2 & v=y \\ u'=2x & v'=y' \end{array}$$

$$y''=\frac{2xy-x^2y'}{y^2}$$

$$=\frac{2xy-x^2\frac{x^2}{y}}{y^2}$$

$$=\frac{\frac{2xy^2-x^4}{y}}{y^2}$$

$$=\frac{2xy^2-x^4}{y^3}$$

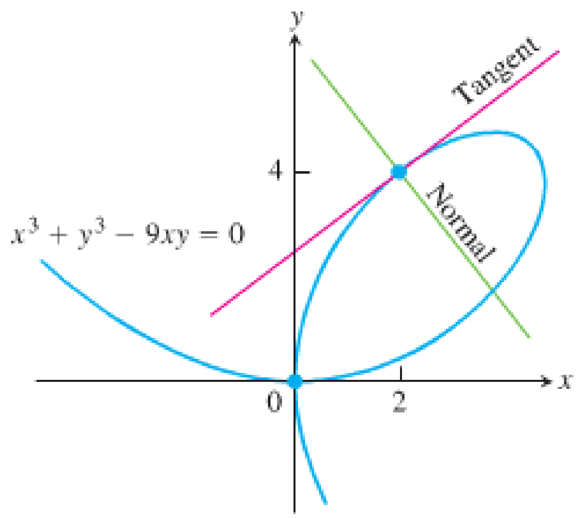
Normal Lines

The ***normal*** is the line ***perpendicular*** to the ***tangent*** of the profile curve at the point of entry.

Example

Show that the point (2, 4) lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve there.

Solution



$$\frac{d}{dx}\left(x^3\right)+\frac{d}{dx}\left(y^3\right)-\frac{d}{dx}(9xy)=0$$

$$3x^2+3y^2y'-9\left(y+xy'\right)=0$$

$$3x^2+3y^2y'-9y-9xy'=0$$

$$3\left(y^2-3x\right)y'=9y-3x^2$$

$$y'=\frac{3\left(3y-x^2\right)}{3\left(y^2-3x\right)}$$

$$=\boxed{\frac{3y-x^2}{y^2-3x}}$$

The slope: $y'\Big|_{(2,4)}=\frac{3(4)-(2)^2}{(4)^2-3(2)}=\frac{8}{10}=\boxed{\frac{4}{5}}$

The tangent at (2, 4) is the line passes thru (2, 4) with slope $\frac{4}{5}$

$$y=\frac{4}{5}(x-2)+4 \qquad\qquad y=m\left(x-x_1\right)+y_1$$

$$y=\frac{4}{5}x-\frac{8}{5}+4$$

$$\boxed{y=\frac{4}{5}x+\frac{12}{5}}$$

The Normal to the curve at (2, 4) is the line perpendicular to the tangent there thru (2, 4) with slope $-\frac{5}{4}$

$$y=-\frac{5}{4}(x-2)+4 \qquad\qquad y=m\left(x-x_1\right)+y_1$$

$$y=-\frac{5}{4}x+\frac{5}{2}+4$$

$$\boxed{y=-\frac{5}{4}x+\frac{13}{2}}$$

Exercises Section 2.7 – Implicit Differentiation

Find $\frac{dy}{dx}$

1. $y^2 + x^2 - 2y - 4x = 4$

2. $x^2y^2 - 2x = 3$

3. $x + \sqrt{x}\sqrt{y} = y^2$

4. $x^2y + xy^2 = 6$

5. $x^3 - xy + y^3 = 1$

6. $y^2 = \frac{x-1}{x+1}$

7. $(3xy+7)^2 = 6y$

8. $xy = \cot(xy)$

9. $x + \tan(xy) = 0$

10. $x\cos(2x+3y) = y\sin x$

11. $y = \frac{e^y}{1+\sin x}$

12. $\sin x\cos(y-1) = \frac{1}{2}$

13. $y\sqrt{x^2+y^2} = 15$

Find $\frac{dr}{d\theta}$

14. $r - 2\sqrt{\theta} = \frac{3}{2}\theta^{2/3} + \frac{4}{3}\theta^{3/4}$

15. $\sin(r\theta) = \frac{1}{2}$

Find $\frac{d^2y}{dx^2}$

16. $x^{2/3} + y^{2/3} = 1$

17. $2\sqrt{y} = x - y$

18. If $x^3 + y^3 = 16$, find the value of $\frac{d^2y}{dx^2}$ at the point (2, 2).

19. Find dy/dx: $x^2 - xy + y^2 = 4$ and evaluate the derivative at the given point $(0,-2)$

20. Find the slope of the curve $\left(x^2+y^2\right)^2 = (x-y)^2$ at the point (–2, 1) and (–2, –1)

21. Find the slope of the tangent line to the circle $x^2 - 9y^2 = 16$ at the point (5, 1)

Find an equation of the line tangent to the following curves at the given point

22. $y = 3x^3 + \sin x;\quad (0,\ 0)$

23. $y = \frac{4x}{x^2+3};\quad (3,\ 1)$

24. $y + \sqrt{xy} = 6;\quad (1,\ 4)$

25. $x^2y + y^3 = 75;\quad (4,\ 3)$

26. $x^3 + y^3 = 9xy;\quad (2,\ 4)$

27. Find the lines that are (***a***) tangent and (***b***) normal to the curve $x^2 + xy - y^2 = 1$ at the point (2, 3).

28. Find the lines that are (***a***) tangent and (***b***) normal to the curve $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$ at the point (–1, 0).

29. Find the lines that are (***a***) tangent and (***b***) normal to the curve $x^2\cos^2 y - \sin y = 0$ at the point $(0,\ \pi)$.

30. Suppose that *x* and *y* are both functions of *t*, which can be considered to represent time, and that *x* and *y* are related by the equation

$$xy^2 + y = x^2 + 17$$

Suppose further that when *x* = 2 and *y* = 3, then $\frac{dx}{dt} = 13$. Find the value of the $\frac{dy}{dt}$ at that moment.

31. A cone-shaped icicle is dripping from the roof. The radius of the icicle is decreasing at a rate of 0.2 *cm* per hour, while the length is increasing at a rate of 0.8 *cm* per *hour*. If the icicle is currently 4 *cm* in radius and 20 *cm* long, is the volume of the icicle increasing or decreasing and at what rate?