Solution Section 2.8 – Basic Properties of the Laplace Transform

Exercise

Find the Laplace transform and defined the time domain of $y(t)=t^2+4t+5$

Solution

$$\mathcal{L}\left(t^2+4t+5\right)(s)=\mathcal{L}\left(t^2\right)(s)+4\,\mathcal{L}(4t)(s)+5\,\mathcal{L}(1)(s)$$

$$=\frac{2!}{s^3}+4\frac{1}{s^2}+5\frac{1}{s}$$

$$=\frac{2}{s^3}+\frac{4}{s^2}+\frac{5}{s}$$

$$=\frac{2+4s+5s^2}{s^3} \qquad s>0$$

Exercise

Find the Laplace transform and defined the time domain of $y(t)=-2\cos t+4\sin 3t$

Solution

$$\mathcal{L}(-2\cos t+4\sin 3t)(s)=-2\,\mathcal{L}(\cos t)(s)+4\,\mathcal{L}(\sin 3t)(s)$$

$$=-2\frac{s}{s^2+1}+4\frac{3}{s^2+9}$$

$$=\frac{-2s\left(s^2+9\right)+12\left(s^2+1\right)}{\left(s^2+1\right)\left(s^2+9\right)}$$

$$=\frac{-2s^3-18s+12s^2+12}{\left(s^2+1\right)\left(s^2+9\right)}$$

$$=\frac{-2s^3+12s^2-18s+12}{\left(s^2+1\right)\left(s^2+9\right)} \qquad s>0$$

Exercise

Find the Laplace transform and defined the time domain of $y(t)=2\sin 3t+3\cos 5t$

Solution

$$\mathcal{L}(2\sin 3t+3\cos 5t)(s)=2\,\mathcal{L}(\sin 3t)(s)+3\,\mathcal{L}(\cos 5t)(s)$$

$$= 2\frac{3}{s^2+9} + 3\frac{s}{s^2+25}$$

$$= \frac{6s^2 + 150 + 3s^3 + 27s}{\left(s^2+9\right)\left(s^2+25\right)}$$

$$\boxed{= \frac{3s^3 + 6s^2 + 27s + 150}{\left(s^2+9\right)\left(s^2+25\right)}} \quad (s > 0)$$

Exercise

Find the Laplace transform and defined the time domain of $f(t) = 2t^4$

Solution

$$\mathcal{L}\left(2t^4\right)(s) = 2\mathcal{L}\left(t^4\right)(s)$$

$$= 2\frac{4!}{s^5}$$

$$\boxed{= \frac{48}{s^5}} \qquad s > 0$$

Exercise

Find the Laplace transform and defined the time domain of $f(t) = t^5$

Solution

$$\mathcal{L}\left(t^5\right)(s) \boxed{= \frac{5!}{s^5}}$$

Exercise

Find the Laplace transform of $f(t) = 4t - 10$

Solution

$$F(s) = \mathcal{L}\{4t - 10\}$$

$$= \frac{4}{s^2} - \frac{10}{s}$$

$$\boxed{= \frac{4 - 10s}{s^2}}$$

Exercise

Find the Laplace transform of $f(t) = 7t + 3$

Solution

$$F(s) = \mathcal{L}\{7t+3\}$$
$$= \frac{7}{s^2} + \frac{3}{s}$$
$$= \frac{7+3s}{s^2}$$

Exercise

Find the Laplace transform of $f(t) = 3t^4 - 2t^2 + 1$

Solution

$$F(s) = \mathcal{L}\left\{3t^4 - 2t^2 + 1\right\}$$
$$= \frac{72}{s^4} - \frac{4}{s^3} + \frac{1}{s}$$
$$= \frac{s^3 - 4s + 72}{s^4}$$

Exercise

Find the Laplace transform of $f(t) = (t+1)^3$

Solution

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{t^3 + 3t^2 + 3t + 1\right\}$$
$$F(s) = \frac{6}{s^4} + \frac{6}{s^3} + \frac{3}{s^2} + \frac{1}{s}$$
$$= \frac{s^3 + 3s^2 + 6s + 6}{s^4}$$

Exercise

Find the Laplace transform of $f(t) = (2t-1)^3$

Solution

$$F(s) = \mathcal{L}\left\{8t^3 - 12t^2 + 6t - 1\right\}$$
$$= \frac{48}{s^4} - \frac{24}{s^3} + \frac{6}{s^2} - \frac{1}{s}$$

$$=\boxed{\frac{48-24s+6s^2-s^3}{s^4}}$$

Exercise

Find the Laplace transform of $f(t)=(t-1)^4$

Solution

$$F(s)=\mathcal{L}\left\{t^4-4t^3+6t^2-4t+1\right\}(s)$$

$$=\frac{4!}{s^5}-\frac{24}{s^4}+\frac{12}{s^3}-\frac{4}{s^2}+\frac{1}{s}$$

$$=\boxed{\frac{s^4-4s^3+12s^2-24s+24}{s^5}}$$

Exercise

Find the Laplace transform of $f(t)=t^2+6t-3$

Solution

$$\mathcal{L}\{f(t)\}=\mathcal{L}\left\{t^2+6t-3\right\}$$

$$F(s)=\frac{2}{s^3}+\frac{6}{s^2}-\frac{3}{s}$$

$$=\boxed{\frac{2s^2+6s-3}{s^3}}$$

Exercise

Find the Laplace transform of $f(t)=-4t^2+16t+9$

Solution

$$\mathcal{L}\{f(t)\}=\mathcal{L}\left\{-4t^2+16t+9\right\}$$

$$F(s)=-\frac{8}{s^3}+\frac{16}{s^2}+\frac{9}{s}$$

$$=\boxed{\frac{9s^2+16s-8}{s^3}}$$

Exercise

Find the Laplace transform of $f(t) = 3t^2 - e^{2t}$

Solution

$$F(s) = \mathcal{L}\left\{3t^2 - e^{2t}\right\}(s)$$

$$= \frac{6}{s^3} - \frac{1}{s-2}$$

$$= \boxed{\frac{-s^3 + 6s - 12}{s^3(s-2)}}$$

Exercise

Find the Laplace transform of $f(t) = t^2 - e^{-9t} + 9$

Solution

$$F(s) = \mathcal{L}\left\{t^2 - e^{-9t} + 9\right\}(s)$$

$$= \frac{2}{s^3} - \frac{1}{s+9} + \frac{9}{s}$$

$$= \boxed{\frac{-s^3 + 9s^2 + 2s + 18}{s^3(s+9)}}$$

Exercise

Find the Laplace transform of $f(t) = 6e^{-3t} - t^2 + 2t - 8$

Solution

$$F(s) = \mathcal{L}\left\{6e^{-3t} - t^2 + 2t - 8\right\}(s)$$

$$= \frac{6}{s+3} - \frac{1}{s^3} + \frac{2}{s^2} - \frac{8}{s}$$

$$= \frac{6s^3 - s - 3 + 2s^2 + 2s - 8s^3 - 24s^2}{s^3(s+3)}$$

$$= \boxed{\frac{-2s^3 - 22s^2 + s - 3}{s^3(s+3)}}$$

Exercise

Find the Laplace transform of $f(t) = 5 - e^{2t} + 6t^2$

Solution

$$F(s) = \mathcal{L}\left\{5 - e^{2t} + 6t^2\right\}(s)$$

$$= \frac{5}{s} - \frac{1}{s-2} + \frac{12}{s^3}$$

$$= \frac{5s^2 - s^3 + 12s - 24}{s^3(s-2)}$$

Exercise

Find the Laplace transform of $f(t) = t^2 e^{2t}$

Solution

$$f(t) = e^{2t} \xrightarrow{\mathcal{L}} F(s) = \frac{1}{s-2}$$

$$\mathcal{L}\left\{t^2 e^{2t}\right\}(s) = (-1)^2 Y''(s)$$ *Using Derivative of a Laplace Transform Proposition*

$$= \frac{d}{ds}\left(\frac{-1}{(s-2)^2}\right)$$

$$= -\frac{(-1)2(s-2)}{(s-2)^4}$$

$$= \frac{2}{(s-2)^3}$$ ***OR*** *Using Laplace Transform table*

Exercise

Find the Laplace transform of $f(t) = e^{-2t}(2t+3)$

Solution

$$f(t) = 2t + 3 \xrightarrow{\mathcal{L}} F(s) = 2\frac{1}{s^2} + 3\frac{1}{s}$$

$$= \frac{2+3s}{s^2}$$

$$\mathcal{L}\left\{e^{-2t}(2t+3)\right\} = Y(s+2)$$

$$= \frac{2+3(s+2)}{(s+2)^2}$$

$$=\boxed{\dfrac{3s+8}{(s+2)^2}}$$

Exercise

Find the Laplace transform of $\quad f(t)=e^{-t}\left(t^2+3t+4\right)$

Solution

$$y(t)=t^2e^{-t}+3te^{-t}+4e^{-t}$$

$$Y(s)=\mathcal{L}\left(t^2e^{-t}\right)(s)+3\mathcal{L}\left(te^{-t}\right)(s)+4\mathcal{L}\left(e^{-t}\right)(s) \qquad\qquad \mathcal{L}\left(t^ne^{-at}\right)(s)=\frac{n!}{(s+a))^{n+1}}$$

$$=\frac{2!}{(s+1)^3}+3\frac{1}{(s+1)^2}+4\frac{1}{s+1}$$

$$=\frac{2+3(s+1)+4(s+1)^2}{(s+1)^3}$$

$$=\frac{2+3s+3+4s^2+8s+4}{(s+1)^3}$$

$$=\boxed{\frac{4s^2+11s+9}{(s+1)^3}} \quad (s>0)$$

Exercise

Find the Laplace transform of $\quad f(t)=1+e^{4t}$

Solution

$$\mathcal{L}\{f(t)\}=\mathcal{L}\left\{1+e^{4t}\right\}$$

$$F(s)=\frac{1}{s}+\frac{1}{s-4}$$

$$=\boxed{\frac{2s-4}{s^2-4s}}$$

Exercise

Find the Laplace transform of $\;y(t)=e^{2t}\cos 2t$

Solution

$$f(t)=\cos 2t \xrightarrow{\mathcal{L}} F(s)=\frac{s}{s^2+4}$$

$$y(t) = e^{2t}\cos 2t \xrightarrow{\mathcal{L}} Y(s) = F(s-2)$$

$$\begin{aligned} Y(s) &= F(s-2) \\ &= \frac{s-2}{(s-2)^2+4} \\ &= \frac{s-2}{s^2-4s+8} \end{aligned}$$

Exercise

Find the Laplace transform of $\quad f(t) = t^3 - te^t + e^{4t}\cos t$

Solution

$$\mathcal{L}\left(e^{at}\cos\omega t\right) = \frac{s-a}{(s-a)^2+\omega^2} \qquad\qquad \mathcal{L}\left(t^n e^{-at}\right)(s) = \frac{n!}{(s+a)^{n+1}}$$

$$\begin{aligned} F(s) &= \mathcal{L}\left\{t^3 - te^t + e^{4t}\cos t\right\} \\ &= \frac{6}{s^4} - \frac{1}{(s-1)^2} + \frac{s-4}{(s-4)^2+1} \end{aligned}$$

Exercise

Find the Laplace transform of $\quad f(t) = t^2 - 3t - 2e^{-t}\sin 3t$

Solution

$$\mathcal{L}\left(e^{at}\sin\omega t\right) = \frac{\omega}{(s-a)^2+\omega^2}$$

$$\begin{aligned} F(s) &= \mathcal{L}\left\{t^2 - 3t - 2e^{-t}\sin 3t\right\} \\ &= \frac{6}{s^3} - \frac{3}{s^2} - \frac{6}{(s+1)^2+9} \end{aligned}$$

Exercise

Find the Laplace transform of $\quad f(t) = \sin^2 t$

Solution

$$\begin{aligned} F(s) &= \mathcal{L}\left\{\sin^2 t\right\} \\ &= \tfrac{1}{2}\mathcal{L}\{1 - \cos 2t\} \end{aligned}$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$= \boxed{\frac{4}{2s\left(s^2+4\right)}}$$

Exercise

Find the Laplace transform of $f(t) = e^{7t}\sin^2 t$

Solution

$$F(s) = \mathcal{L}\left\{e^{7t}\sin^2 t\right\}$$

$$= \frac{1}{2}\mathcal{L}\left\{e^{7t} - e^{7t}\cos 2t\right\}$$

$$\mathcal{L}\left(e^{at}\cos\omega t\right) = \frac{s-a}{(s-a)^2+\omega^2}$$

$$= \frac{1}{2}\left(\frac{1}{s-7} - \frac{s-7}{(s-7)^2+4}\right)$$

$$= \boxed{\frac{2}{(s-7)\left((s-7)^2+4\right)}}$$

Exercise

Find the Laplace transform of $f(t) = t\sin^2 t$

Solution

$$F(s) = \mathcal{L}\left\{t\sin^2 t\right\}$$

$$= \frac{1}{2}\mathcal{L}\{t - t\cos 2t\}$$

$$\mathcal{L}(t\cos\omega t) = \frac{s^2-\omega^2}{\left(s^2+\omega^2\right)^2}$$

$$= \boxed{\frac{1}{2}\frac{1}{s^2} - \frac{1}{2}\frac{s^2-4}{\left(s^2+4\right)^2}}$$

Exercise

Find the Laplace transform of $f(t) = \cos^3 t$

Solution

$$F(s) = \mathcal{L}\left\{\cos^3 t\right\}$$

$$= \frac{1}{2}\mathcal{L}\left\{\cos t\left(1+\cos 2t\right)\right\}$$

$$= \frac{1}{2}\mathcal{L}\left\{\cos t + \cos t \cos 2t\right\} \qquad \cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$$

$$= \frac{1}{2}\mathcal{L}\left\{\cos t + \frac{1}{2}\cos 3t + \frac{1}{2}\cos t\right\}$$

$$= \mathcal{L}\left\{\frac{3}{4}\cos t + \frac{1}{4}\cos 3t\right\} \qquad \mathcal{L}\left\{\cos \omega t\right\} = \frac{s}{s^2+\omega^2}$$

$$= \frac{3s}{4\left(s^2+1\right)} + \frac{s}{4\left(s^2+9\right)}$$

Exercise

Find the Laplace transform of $f(t) = te^{-t}\sin 2t$

Solution

$$F(s) = \mathcal{L}\left\{te^{-t}\sin 2t\right\} \qquad \mathcal{L}\left\{te^{-at}\sin \omega t\right\} = \frac{2\omega(s+a)}{\left((s+a)^2+\omega^2\right)^2}$$

$$= \frac{4(s+1)}{\left((s+1)^2+4\right)^2}$$

Exercise

Find the Laplace transform of $f(t) = e^{2t}\cos 5t$

Solution

$$F(s) = \mathcal{L}\left\{e^{2t}\cos 5t\right\} \qquad \mathcal{L}\left\{e^{-at}\cos \omega t\right\} = \frac{s+a}{(s+a)^2+\omega^2}$$

$$= \frac{s-2}{(s-2)^2+25}$$

Exercise

Find the Laplace transform of $f(t) = t^2 + e^t \sin 2t$

Solution

$$F(s) = \mathcal{L}\left\{t^2 + e^t\sin 2t\right\} \qquad \mathcal{L}\left\{e^{at}\sin \omega t\right\} = \frac{\omega}{(s-a)^2+\omega^2}$$

$$= \frac{2}{s^3} + \frac{2}{(s-1)^2+4}$$

Exercise

Find the Laplace transform of $f(t) = e^{-t}\cos 3t + e^{6t} - 1$

Solution

$$F(s) = \mathcal{L}\left\{e^{-t}\cos 3t + e^{6t} - 1\right\} \qquad \mathcal{L}\left\{e^{-at}\cos \omega t\right\} = \frac{s+a}{(s+a)^2+\omega^2}$$

$$= \frac{s+1}{(s+1)^2+9} + \frac{1}{s-6} + \frac{1}{s}$$

Exercise

Find the Laplace transform of $f(t) = e^{-2t}\sin 2t + t^2 e^{3t}$

Solution

$$\mathcal{L}\left\{e^{at}\sin \omega t\right\} = \frac{\omega}{(s-a)^2+\omega^2} \qquad \mathcal{L}\left(t^n e^{-at}\right)(s) = \frac{n!}{(s+a)^{n+1}}$$

$$F(s) = \mathcal{L}\left\{e^{-2t}\sin 2t + t^2 e^{3t}\right\}$$

$$= \frac{2}{(s+2)^2+4} + \frac{2}{(s-3)^3}$$

Exercise

Find the Laplace transform of $f(t) = 2t^2 e^{-2t} - t + \cos 4t$

Solution

$$F(s) = \mathcal{L}\left\{2t^2 e^{-2t} - t + \cos 4t\right\} \qquad \mathcal{L}\left(t^n e^{-at}\right)(s) = \frac{n!}{(s+a)^{n+1}}$$

$$= \frac{4}{(s+2)^3} - \frac{1}{s} - \frac{4}{s^2+4}$$

Exercise

Find the Laplace transform of $f(t) = t\sin 3t$

Solution

$$f(t) = \sin 3t \xrightarrow{\mathcal{L}} F(s) = \frac{3}{s^2+9}$$

$$\mathcal{L}\{t\sin 3t\}(s) = -Y'(s)$$ *Using Derivative of a Laplace Transform Proposition*

$$= -\frac{3(-2s)}{\left(s^2+9\right)^2}$$

$$= \boxed{\frac{6s}{\left(s^2+9\right)^2}}$$

Exercise

Find the Laplace transform of $f(t) = t^2 \cos 2t$

Solution

$$f(t) = \cos 2t \xrightarrow{\mathcal{L}} F(s) = \frac{s}{s^2+4}$$

$$\mathcal{L}\left\{t^2 \cos 2t\right\}(s) = (-1)^2 Y''(s)$$ ***Using Derivative of a Laplace Transform Proposition***

$$= \left(Y'(s)\right)'$$

$$= \frac{d}{ds}\left[\frac{(s^2+4)(1)-s(2s)}{\left(s^2+4\right)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{4-s^2}{\left(s^2+4\right)^2}\right]$$

$$= \frac{-2s\left(s^2+4\right)^2 - \left(4-s^2\right)(2)(2s)\left(s^2+4\right)}{\left(s^2+4\right)^4}$$

$$= \left(s^2+4\right)\frac{-2s\left(s^2+4\right) - 4s\left(4-s^2\right)}{\left(s^2+4\right)^4}$$

$$= \frac{-2s^3-8s-16s+4s^3}{\left(s^2+4\right)^3}$$

$$= \boxed{\frac{2s^3-24s}{\left(s^2+4\right)^3}}$$

Exercise

Find the Laplace transform of $f(t) = \left(1 + e^{-t}\right)^2$

Solution

$$F(s) = \mathcal{L}\left\{1 + 2e^{-t} + e^{-2t}\right\}$$

$$= \frac{1}{s} + \frac{2}{s+1} + \frac{1}{s+2}$$

$$= \frac{s^2 + 3s + 2 + 2s^2 + 4s + s^2 + s}{s(s+1)(s+2)}$$

$$= \frac{4s^2 + 8s + 2}{s(s+1)(s+2)}$$

Exercise

Find the Laplace transform of $f(t) = \left(1 + e^{2t}\right)^2$

Solution

$$F(s) = \mathcal{L}\left\{1 + 2e^{2t} + e^{4t}\right\}$$

$$= \frac{1}{s} + \frac{2}{s-2} + \frac{1}{s-4}$$

$$= \frac{4s^2 - 16s + 8}{s(s-2)(s-4)}$$

Exercise

Find the Laplace transform of $f(t) = \left(e^{t} - e^{-t}\right)^2$

Solution

$$F(s) = \mathcal{L}\left\{e^{2t} - 2 + e^{-2t}\right\}$$

$$= \frac{1}{s-2} - \frac{2}{s} + \frac{1}{s+2}$$

$$= \frac{s^2 + 2s - s^2 + 8 + s^2 - 2s}{s\left(s^2 - 4\right)}$$

$$= \frac{s^2 + 8}{s\left(s^2 - 4\right)}$$

Exercise

Find the Laplace transform of $f(t) = 4t^2 - 5\sin 3t$

Solution

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{4t^2 - 5\sin 3t\right\}$$

$$F(s) = \frac{2}{s^3} - \frac{15}{s^2+9}$$

$$= \frac{-15s^3 + 2s^2 + 18}{s^5 + 9s^3}$$

Exercise

Find the Laplace transform of $f(t) = \cos 5t + \sin 2t$

Solution

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos 5t + \sin 2t\}$$

$$F(s) = \frac{s}{s^2+25} + \frac{2}{s^2+4}$$

$$= \frac{s^3 + 2s^2 + 4s + 50}{\left(s^2+4\right)\left(s^2+25\right)}$$

Exercise

Find the Laplace transform of $f(t) = e^{3t}\sin 6t - t^3 + e^t$

Solution

$$F(s) = \mathcal{L}\left\{e^{3t}\sin 6t - t^3 + e^t\right\}$$

$$\mathcal{L}\left(e^{at}\sin \omega t\right) = \frac{\omega}{(s-a)^2 + \omega^2}$$

$$= \frac{6}{(s-3)^2 + 36} - \frac{6}{s^4} + \frac{1}{s-1}$$

Exercise

Find the Laplace transform of $f(t) = t^4 + t^2 - t + \sin\sqrt{2}t$

Solution

$$F(s) = \mathcal{L}\left\{t^4 + t^2 - t + \sin\sqrt{2}t\right\}$$

$$\mathcal{L}\left(e^{at}\sin \omega t\right) = \frac{\omega}{(s-a)^2 + \omega^2}$$

$$= \boxed{\frac{24}{s^5} + \frac{2}{s^3} - \frac{1}{s^2} + \frac{\sqrt{2}}{s^2+2}}$$

Exercise

Find the Laplace transform of $\quad f(t) = t^4 e^{5t} - e^t \cos\sqrt{7}t$

Solution

$$\mathcal{L}\left(t^n e^{-at}\right)(s) = \frac{n!}{(s+a)^{n+1}} \qquad \mathcal{L}\left(e^{at}\cos\omega t\right) = \frac{s-a}{(s-a)^2+\omega^2}$$

$$F(s) = \mathcal{L}\left\{t^4 e^{5t} - e^t \cos\sqrt{7}t\right\}$$

$$= \boxed{\frac{24}{(s-5)^5} - \frac{s-1}{(s-1)^2+7}}$$

Exercise

Find the Laplace transform of $\quad f(t) = e^{-2t}\cos\sqrt{3}t - t^2 e^{-2t}$

Solution

$$\mathcal{L}\left(t^n e^{-at}\right)(s) = \frac{n!}{(s+a)^{n+1}} \qquad \mathcal{L}\left(e^{at}\cos\omega t\right) = \frac{s-a}{(s-a)^2+\omega^2}$$

$$F(s) = \mathcal{L}\left\{e^{-2t}\cos\sqrt{3}t - t^2 e^{-2t}\right\}$$

$$= \boxed{\frac{s+2}{(s+2)^2+3} - \frac{2}{(s+2)^3}}$$

Exercise

Find the Laplace transform of $\quad f(t) = 6e^{-5t} + e^{3t} + 5t^3 - 9$

Solution

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{6e^{-5t} + e^{3t} + 5t^3 - 9\right\}$$

$$\boxed{F(s) = \frac{6}{s+5} + \frac{1}{s-3} + \frac{5}{s^4} - \frac{9}{s}}$$

Exercise

Find the Laplace transform of $f(t) = 4\cos 4t - 9\sin 4t + 2\cos 10t$

Solution

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{4\cos 4t - 9\sin 4t + 2\cos 10t\}$$

$$F(s) = 4\frac{s}{s^2+4^2} - 9\frac{4}{s^2+4^2} + 2\frac{s}{s^2+10^2}$$

$$= \frac{4s}{s^2+16} - \frac{36}{s^2+16} + \frac{2s}{s^2+100}$$

Exercise

Find the Laplace transform of $f(t) = 3\sinh 2t + 3\sin 2t$

Solution

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{3\sinh 2t + 3\sin 2t\}$$

$$F(s) = 3\frac{2}{s^2-2^2} + 3\frac{2}{s^2+2^2}$$

$$= \frac{6}{s^2-4} + \frac{6}{s^2+4}$$

Exercise

Find the Laplace transform of $f(t) = e^{3t} + \cos 6t - e^{3t}\cos 6t$

Solution

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{e^{3t} + \cos 6t - e^{3t}\cos 6t\right\}$$

$$F(s) = \frac{1}{s-3} + \frac{s}{s^2+36} - \frac{s-3}{(s-3)^2+36}$$

Exercise

Find the Laplace transform of $f(t) = t\cosh 3t$

Solution

$$f(t) = \cosh 3t \xrightarrow{\mathcal{L}} Y(s) = \frac{s}{s^2-9}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\cosh 3t\}$$

$$F(s) = -Y'(s)$$

$$= -\frac{s^2 - 9 - 2s^2}{\left(s^2 - 9\right)^2}$$

$$= \boxed{\frac{s^2 + 9}{\left(s^2 - 9\right)^2}}$$

Exercise

Find the Laplace transform of $f(t) = t^2 \sin 2t$

Solution

$$f(t) = \sin 2t \xrightarrow{\mathcal{L}} Y(s) = \frac{2}{s^2 + 4}$$

$$Y'(s) = -\frac{4s}{\left(s^2 + 4\right)^2}$$

$$Y''(s) = -4\frac{s^2 + 4 - 4s^2}{\left(s^2 + 4\right)^3} = \frac{12s^2 - 16}{\left(s^2 + 4\right)^3} \qquad \left(U^m V^n\right)' = U^{m-1}V^{n-1}\left(mU'V + nUV'\right)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{t^2 \sin 2t\right\}$$

$$F(s) = (-1)^2 Y''(s)$$

$$= \boxed{\frac{12s^2 - 16}{\left(s^2 + 4\right)^3}}$$

Exercise

Find the Laplace transform of $f(t) = \sinh kt$

Solution

$$F(s) = \mathcal{L}\{\sinh kt\}$$

$$= \frac{1}{2}\mathcal{L}\left\{e^{kt} - e^{-kt}\right\}$$

$$= \frac{1}{2}\left(\frac{1}{s-k} - \frac{1}{s+k}\right)$$

$$= \boxed{\frac{k}{s^2 - k^2}}$$

Exercise

Find the Laplace transform of $f(t)=\cosh kt$

Solution

$$F(s)=\mathcal{L}\{\cosh kt\}$$

$$=\frac{1}{2}\mathcal{L}\left\{e^{kt}+e^{-kt}\right\}$$

$$=\frac{1}{2}\left(\frac{1}{s-k}+\frac{1}{s+k}\right)$$

$$=\frac{s}{s^2-k^2}$$

Exercise

Find the Laplace transform of $f(t)=e^{t}\sinh kt$

Solution

$$F(s)=\mathcal{L}\left\{e^{t}\sinh kt\right\}$$

$$=\frac{1}{2}\mathcal{L}\left\{e^{t}\left(e^{kt}-e^{-kt}\right)\right\}$$

$$=\frac{1}{2}\mathcal{L}\left\{e^{(k+1)t}-e^{-(k-1)t}\right\}$$

$$=\frac{1}{2}\left(\frac{1}{s-(k+1)}-\frac{1}{s+(k-1)}\right)$$

Exercise

Find the Laplace transform of $f(t)=e^{-t}\cosh kt$

Solution

$$F(s)=\mathcal{L}\left\{e^{-t}\cosh kt\right\}$$

$$=\frac{1}{2}\mathcal{L}\left\{e^{-t}\left(e^{kt}+e^{-kt}\right)\right\}$$

$$=\frac{1}{2}\mathcal{L}\left\{e^{(k-1)t}+e^{-(k-1)t}\right\}$$

$$=\frac{1}{2}\left(\frac{1}{s-(k-1)}+\frac{1}{s+(k-1)}\right)$$

Exercise

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of y.

$$y' + 2y = t\sin t, \quad with \quad y(0) = 1$$

Solution

Let $Y(s) = \mathcal{L}(y)(s)$, then

Left side;

$$\begin{aligned}\mathcal{L}(y' + 2y)(s) &= s\mathcal{L}(y)(s) - y(0) + 2\mathcal{L}(y)(s)\\ &= sY(s) - 1 + 2Y(s)\\ &= (s+2)Y(s) - 1\end{aligned}$$

Right side; $\quad f(t) = \sin t \xrightarrow{\mathcal{L}} F(s) = \dfrac{1}{s^2+1}$

$\mathcal{L}\{t\sin t\}(s) = -F'(s)$ *Using Derivative of a Laplace Transform Proposition*

$$= \frac{2s}{\left(s^2+1\right)^2}$$

$$(s+2)Y(s) - 1 = \frac{2s}{\left(s^2+1\right)^2}$$

$$(s+2)Y(s) = \frac{2s}{\left(s^2+1\right)^2} + 1$$

$$\boxed{Y(s) = \frac{2s}{(s+2)\left(s^2+1\right)^2} + \frac{1}{s+2}}$$

Exercise

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of y.

$$y' - y = t^2 e^{-2t}, \quad with \quad y(0) = 0$$

Solution

Let $Y(s) = \mathcal{L}(y)(s)$, then

Left side; $\quad \mathcal{L}(y' + 2y)(s) = s\mathcal{L}(y)(s) - y(0) + 2\mathcal{L}(y)(s)$

$$= sY(s) - Y(s)$$

$$= (s-1)Y(s)$$

Right side; $f(t)=e^{2t} \xrightarrow{\mathcal{L}} F(s)=\dfrac{1}{s-2}$

$\mathcal{L}\left\{t^2e^{2t}\right\}(s)=(-1)^2 Y''(s)$ *Using Laplace Transform table*

$$=\frac{2}{(s-2)^3}$$

$$(s-1)Y(s)=\frac{2}{(s-2)^3}$$

$$\boxed{Y(s)=\frac{2}{(s-1)(s-2)^3}}$$

Exercise

Transform the initial value problem into an algebraic equation involving $\mathcal{L}y)$. Solve the resulting equation for the Laplace transform of y.

$$y''+y'+2y=e^{-t}\cos 2t, \quad with \quad y(0)=1 \;\; and \quad y'(0)=-1$$

Solution

$$\mathcal{L}(y''+y'+2y)(s)=\mathcal{L}\left(e^{-t}\cos 2t\right)$$

$$s^2\mathcal{L}(y)(s)-sy(0)-y'(0)+s\,\mathcal{L}(y)(s)-y(0)+2\,\mathcal{L}(y)(s)=\frac{s+1}{(s+1)^2+4}$$

$$s^2Y(s)-s+1+sY(s)-1+2Y(s)=\frac{s+1}{(s+1)^2+4}$$

$$\left(s^2+s+2\right)Y(s)-s=\frac{s+1}{s^2+2s+1+4}$$

$$\left(s^2+s+2\right)Y(s)=\frac{s+1}{s^2+2s+5}+s$$

$$Y(s)=\frac{s+1}{\left(s^2+2s+5\right)\left(s^2+s+2\right)}+\frac{s}{s^2+s+2}$$

$$=\frac{s+1+s\left(s^2+2s+5\right)}{\left(s^2+2s+5\right)\left(s^2+s+2\right)}$$

$$=\frac{s+1+s^3+2s^2+5s}{\left(s^2+2s+5\right)\left(s^2+s+2\right)}$$

$$\boxed{=\frac{s^3+2s^2+6s+1}{\left(s^2+2s+5\right)\left(s^2+s+2\right)}}$$

Exercise

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of y.

$$y' - 5y = e^{-2t}, \quad with \quad y(0) = 1$$

Solution

$$\mathcal{L}(y' - 5y)(s) = \mathcal{L}\left(e^{-2t}\right)(s)$$

$$\mathcal{L}(y')(s) - 5\,\mathcal{L}(y)(s) = \frac{1}{s+2}$$

$$s\mathcal{L}(y)(s) - y(0) - 5\,\mathcal{L}(y)(s) = \frac{1}{s+2}$$

Let $Y(s) = \mathcal{L}(y)(s)$, then

$$sY(s) - 1 - 5Y(s) = \frac{1}{s+2}$$

$$(s-5)Y(s) = \frac{1}{s+2} + 1$$

$$Y(s) = \frac{1}{(s-5)(s+2)} + \frac{1}{(s-5)}$$

$$= \frac{1+s+2}{(s-5)(s+2)}$$

$$\boxed{= \frac{s+3}{(s-5)(s+2)}}$$

Exercise

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of y.

$$y' - 4y = \cos 2t, \quad with \quad y(0) = -2$$

Solution

$$\mathcal{L}(y' - 4y)(s) = \mathcal{L}(\cos 2t)(s)$$

$$\mathcal{L}(y')(s) - 4\,\mathcal{L}(y)(s) = \frac{s}{s^2+4}$$

$$s\,\mathcal{L}(y)(s) - y(0) - 4\,\mathcal{L}(y)(s) = \frac{s}{s^2+4}$$

Let $Y(s) = \mathcal{L}(y)(s)$, then

$$sY(s) + 2 - 4Y(s) = \frac{s}{s^2+4}$$

$$(s-4)Y(s) = \frac{s}{s^2+4} - 2$$

$$Y(s) = \frac{s}{(s-4)\left(s^2+4\right)} - \frac{2}{s-4}$$

$$= \frac{s-2s^2-8}{(s-4)\left(s^2+4\right)}$$

$$= \frac{-2s^2+s-8}{(s-4)\left(s^2+4\right)}$$

Exercise

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of *y*.

$$y'' + 2y' + 2y = \cos 2t; \quad with \quad y(0) = 1 \ \ and \quad y'(0) = 0$$

Solution

$$\mathcal{L}\left(y'' + 2y' + 2y\right)(s) = \mathcal{L}\left(\cos 2t\right)(s)$$

Let $Y(s) = \mathcal{L}(y)(s)$, then

$$s^2Y(s) - sy(0) - y'(0) + 2\left(sY(s) - y(0)\right) + 2Y(s) = \frac{s}{s^2+4}$$

$$s^2Y(s) - s + 2sY(s) - 2 + 2Y(s) = \frac{s}{s^2+4}$$

$$\left(s^2+2s+2\right)Y(s) = \frac{s}{s^2+4} + s + 2$$

$$= \frac{s+s^3+2s^2+4s+8}{s^2+4}$$

$$= \frac{s^3+2s^2+5s+8}{s^2+4}$$

$$Y(s) = \frac{s^3+2s^2+5s+8}{\left(s^2+4\right)\left(s^2+2s+2\right)}$$

Exercise

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of *y*.

$$y'' + 3y' + 5y = t + e^{-t}; \quad with \quad y(0) = -1 \ \ and \quad y'(0) = 0$$

Solution

$$\mathcal{L}\left(y'' + 3y' + 5y\right)(s) = \mathcal{L}(t)(s) + \mathcal{L}\left(e^{-t}\right)(s)$$

$$s^2Y(s) - sy(0) - y'(0) + 3\big(sY(s) - y(0)\big) + 5Y(s) = \frac{1}{s^2} + \frac{1}{s+1}$$

$$s^2Y(s) + s + 3\big(sY(s) + 1\big) + 5Y(s) = \frac{s+1+s^2}{s^2(s+1)}$$

$$s^2Y(s) + s + 3sY(s) + 3 + 5Y(s) = \frac{s+1+s^2}{s^2(s+1)}$$

$$\left(s^2 + 3s + 5\right)Y(s) = \frac{s+1+s^2}{s^2(s+1)} - s - 3$$

$$= \frac{s+1+s^2 - s^2(s+1)(s+3)}{s^2(s+1)}$$

$$= \frac{s+1+s^2 - s^2\left(s^2 + 4s + 3\right)}{s^2(s+1)}$$

$$= \frac{s+1+s^2 - s^4 + 4s^3 + 3s^2}{s^2(s+1)}$$

$$= \frac{-s^4 + 4s^3 + 4s^2 + s + 1}{s^2(s+1)}$$

$$\boxed{Y(s) = \frac{-s^4 + 4s^3 + 4s^2 + s + 1}{s^2(s+1)\left(s^2 + 3s + 5\right)}}$$