Lecture One – First Order Equations

Section 1.1 – Introduction to Differential Equations

Basic Terminology

A differential equation is an equation that contains an unknown function together with one or more of its derivatives.

Examples

1. $y' = 3x + \sin x$

2. $xy'' + 2y' + 3y = 5x^4$

3. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} = 0$

4. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \quad (Laplace's\ equation)$

Type

If the unknown function depends on a single independent variable, then the equation is an ***ordinary differential equation*** (ODE); if the unknown function depends on more than one independent variable, then the equation is a ***partial differential equation*** (PDE).

Order

The ***order*** of a differential equation is the order of the highest derivative of the unknown function appearing in the equation.

Examples

1. $y' = 3y + \sin x \qquad order\ 1$

2. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} = 0 \qquad order\ \mathbf{3}$

3. $xy'' + 2y' + 3y = 5x^4 \qquad order\ \mathbf{2}$

4. $\dfrac{d^2y}{dx^2} - 3x\cos\left(\dfrac{dy}{dx}\right) - 5x = \dfrac{d^3}{dx^3}\left(e^{4x}\right) \qquad order\ \mathbf{2}$

Since the unknown function is $y(x)$ not in *x*.

Ordinary Differential Equations

Involve an unknown function of a single variable with one or more of its derivatives.

$$\frac{dy}{dt} = y - t$$

y: $y(t)$ is unknown function

t: independent variable

Some other example:

$$y' = y^2 - t$$

$$ty' = y$$

$$y' + 4y = e^{-3t}$$

$$yy'' + t^2 y = \cos t$$

$$y' = \cos(ty)$$

$\therefore$ The order of a differential equation is the order of the highest derivative that occurs in the equation.

y'' : *second order*

$\frac{\partial^2 \omega}{\partial t^2} = c^2 \frac{\partial^2 \omega}{\partial x^2}$ is not an ODE (ω is dependent on x and t)

This equation is called a ***partial differential equation***.

Definition

A first-order differential equation of the form $\frac{dy}{dt} = y' = f(t, y)$ is said to be in normal form.

$y^{(n)} = f\left(t, y, y', \ldots, y^{(n-1)}\right)$ is said to be in normal form.

f: is a given function of 2 variables t & y (***rate function***)

Solutions

A solution of a differential equation is a function defined on some domain D such that the equation reduces to an identity when the function is substituted into the equation.

A solution of the first-order, ordinary differential equation $f(t, y, y') = 0$ is a differentiable function $y(t)$ such that $f(t, y(t), y'(t)) = 0$ for all t in the interval where $y(t)$ is defined.

1. Can be found in explicit and implicit form by applying manipulation (integration)
2. No real solution.

Examples

1. $y' = 3x^2 + \sin x \quad \Rightarrow \quad y = x^3 - \cos x + C$

2. $y' = f(x) \quad \Rightarrow \quad y = \int f(x) + C = F(x) + C$

Example

Show that $y(t) = Ce^{-t^2}$ is a solution of the 1st order equation $y' = -2ty$

Solution

$$y(t) = Ce^{-t^2} \Rightarrow y' = -2tCe^{-t^2}$$

$$y' = -2tCe^{-t^2}$$

$$y' = -2t\, y(t) \qquad \text{True; it is a solution}$$

$y(t)$ is called the ***general solution***.

The solutions from the graph are called ***solution curves***.

Example

Is the function $y(t) = \cos t$ a solution to the differential equation $y' = 1 + y^2$

Solution

$$y' = -\sin t$$

$$y' = 1 + y^2 = -\sin t$$

$$1 + \cos^2 t \overset{?}{=} -\sin t \qquad \text{False; it is not a solution.}$$

Example

Find values of r such that $y(t)=e^{rt}$ is a solution of $y''-2y'-15y=0$

Solution

$$y'(t)=re^{rt}$$

$$y''=r^2e^{rt}$$

$$y''-2y'-15y=r^2e^{rt}-2re^{rt}-15e^{rt}=0$$

$$r^2-2r-15=0 \quad \Rightarrow \quad \underline{r=-3,\ 5}$$

n-Parameter Family of Solutions

To find a set of solutions of an n-th order differential equation we ***integrate*** n times, with each integration step producing an arbitrary constant of integration. Thus, "*in theory*", an n-th order differential equation has an ***n-parameter family of solutions***.

Example

Solve the differential equation: $y'''-12x+6e^{2x}=0$

Solution

$$y'''=12x-6e^{2x}$$

$$\int y'''dx=\int\left(12x-6e^{2x}\right)dx$$

$$y''=6x^2-3e^{2x}+C_1$$

$$\int y''dx=\int\left(6x^2-3e^{2x}+C_1\right)dx$$

$$y'=2x^3-\frac{3}{2}e^{2x}+C_1x+C_2$$

$$\int y'dx=\int\left(2x^3-\frac{3}{2}e^{2x}+C_1x+C_2\right)dx$$

$$\underline{y(x)=\frac{1}{2}x^4-\frac{3}{2}e^{2x}+\frac{1}{2}C_1x^2+C_2x+C_3}$$

General Solution/Singular Solutions

An "*n-parameter family of solutions*" is also called the ***general solution***.

Solutions of an n-th order differential equation which are not included in n-parameter family of solutions are called ***singular solutions***.

Example

Given the differential equation $y' = (4x+2)(y-2)^{1/3}$ has a general solution $(y-2)^{2/3} = \frac{4}{3}x^2 + \frac{4}{3}x + C$.

Is the singular solution $y \equiv 2$?

Solution

$$\frac{2}{3}(y-2)^{-1/3}\frac{dy}{dx} = \frac{8}{3}x + \frac{4}{3}$$

$$\frac{3}{2}(y-2)^{1/3} \quad \left[\frac{2}{3}\frac{1}{(y-2)^{1/3}}\frac{dy}{dx} = \frac{8}{3}x + \frac{4}{3}\right]$$

$$\frac{dy}{dx} = (4x+2)(y-2)^{1/3} \qquad y \neq 2$$

$y = 2$ is not a part of the general solution.

Particular Solutions

In many applications of integrations, we are given enough information to determine a particular solution. To do this, we need to know the value of *F(x)* for one value of *x*. This information is called an initial condition.

Example

Solve the differential equation: $y' = te^t$ that satisfies $y(0) = 2$

Solution

$$y = \int te^t dt$$

$$y = te^t - e^t + C$$

$$y(0) = (0)e^0 - e^0 + C = 2$$

$$-1 + C = 2$$

$$C = 3$$

$$\boxed{y(t) = e^t(t-1) + 3}$$

	$\int e^t dt$
t	e^t
1	e^t

Example

Solve the differential equation: $y' = \frac{1}{x}$ that satisfies $y(1) = 3$

Solution

$$y = \int \frac{1}{x} dx$$

$$= \ln|x| + C$$

$$y(1) = \ln|1| + C = 3$$

$$\boxed{C = 3}$$

$$y(x) = \ln x + 3 \qquad \textit{with } x > 0$$

n–th Order initial-Value Problems

Example

Find a solution of $y' = 3x^2 + 2x + 1$ which passes through the point $(-2,\ 4)$

Solution

$$y = \int \left(3x^2 + 2x + 1\right) dx$$

$$= x^3 + x^2 + x + C$$

$$4 = (-2)^3 + (-2)^2 - 2 + C$$

$$4 = -8 + 4 - 2 + C \quad \Rightarrow \quad C = 10$$

$$y = x^3 + x^2 + x + 10$$

Example

$y = C_1 \cos 3x + C_2 \sin 3x$ is the general solution of $y'' + 9y = 0$.

a) Find a solution which satisfies $y(0) = 3$

b) Find a solution which satisfies $y(0) = 4, \quad y(\pi) = 4$

c) Find a solution which satisfies $y\left(\frac{\pi}{4}\right) = 1, \quad y'\left(\frac{\pi}{4}\right) = 2$

Solution

a) $3 = C_1 \cos(0) + C_2 \sin(0) \quad \Rightarrow \quad \underline{C_1 = 3}$

$\underline{y = 3\cos 3x + C_2 \sin 3x}$ *for any* C_2

b) $4 = C_1 \cos(0) + C_2 \sin(0) \quad \Rightarrow \quad \underline{C_1 = 4}$

$4 = C_1 \cos(3\pi) + C_2 \sin(3\pi) \quad \Rightarrow \quad \underline{C_1 = -4}$

$\therefore$ No Solution

c) $1 = C_1 \cos\left(\frac{3\pi}{4}\right) + C_2 \sin\left(\frac{3\pi}{4}\right) \quad \Rightarrow \quad -\frac{1}{\sqrt{2}} C_1 + \frac{1}{\sqrt{2}} C_2 = 1 \to \boxed{-C_1 + C_2 = \sqrt{2}} \ (1)$

$y' = -3C_1 \sin 3x + 3C_2 \cos 3x$

$2 = -3C_1 \sin\left(\frac{3\pi}{4}\right) + 3C_2 \cos\left(\frac{3\pi}{4}\right) \quad \Rightarrow \quad -\frac{3}{\sqrt{2}} C_1 - \frac{3}{\sqrt{2}} C_2 = 2 \to \boxed{-3C_1 - 3C_2 = 2\sqrt{2}} \ (2)$

$$\begin{cases} -3C_1 + 3C_2 = 3\sqrt{2} \\ -3C_1 - 3C_2 = 2\sqrt{2} \end{cases} \to \quad C_1 = -\frac{5\sqrt{2}}{6} \quad C_2 = \sqrt{2} - \frac{5\sqrt{2}}{6} = \frac{\sqrt{2}}{6}$$

$\underline{y = -\frac{5\sqrt{2}}{6} \cos 3x + \frac{\sqrt{2}}{6} \sin 3x}$

Example

Suppose a ball thrown into the air with initial velocity $v_0 = 20\ ft/\sec$. Assuming the ball thrown from a height of $x_0 = 6\ ft$, how long does it take for the ball to hit the ground?

Solution

$$\frac{dv}{dt} = -g \Rightarrow dv = -gdt$$

$$v(t) = -gt + C_1$$

$$v(t = 0) = -g(0) + C_1 = 20$$

$$C_1 = 20$$

$$v(t) = -32t + 20$$

$$\frac{dx}{dt} = v \Rightarrow dx = vdt$$

$$\int dx = \int vdt$$

$$x(t) = \int (-32t + 20)dt$$

$$= -16t^2 + 20t + C_2$$

$$x(t = 0) = -16(0)^2 + 20(0) + C_2 = 6$$

$$C_2 = 6$$

$$x(t) = -16t^2 + 20t + 6$$

Exercises *Section* 1.1 – Introduction to Differential Equations

1. Show that $y(t) = Ce^{-(1/2)t^2}$ is a solution of the 1st order equation $y' = -ty$ *for* $-3 \le C \le 3$

2. Show that $y(t) = \dfrac{4}{1 + Ce^{-4t}}$ is a solution of the 1st order equation $y' = y(4 - y)$

3. Show that $y(x) = x^{-3/2}$ is a solution of $4x^2 y'' + 12xy' + 3y = 0$ for $x > 0$

4. A general solution may fail to produce all solutions of a differential equation $y(t) = \dfrac{4}{1 + Ce^{-4t}}$. Show that $y = 0$ is a solution of the differential equation, but no value of *C* in the given general solution will produce this solution.

5. Use the given general solution to find a solution of the differential equation having the given initial condition. $ty' + y = t^2, \quad y(t) = \frac{1}{3}t^2 + \frac{C}{t}, \qquad y(1) = 2$

6. Show that $y(t) = 2t - 2 + Ce^{-t}$ is a solution of the 1st order equation $y' + y = 2t$ *for* $-3 \le C \le 3$

7. Use the given general solution to find a solution of the differential equation having the given initial condition. $y' + 4y = \cos t, \quad y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t + Ce^{-4t}, \qquad y(0) = -1$

8. Use the given general solution to find a solution of the differential equation having the given initial condition. $ty' + (t+1)y = 2te^{-t}, \quad y(t) = e^{-t}\left(t + \frac{C}{t}\right), \quad y(1) = \frac{1}{e}$

9. Use the given general solution to find a solution of the differential equation having the given initial condition. $y' = y(2 + y), \quad y(t) = \dfrac{2}{-1 + Ce^{-2t}}, \quad y(0) = -3$

10. Find the values of ***m*** so that the function $y = e^{mx}$ is a solution of the given differential equation

a) $y' + 2y = 0$

b) $5y' - 2y = 0$

c) $y'' - 5y' + 6y = 0$

d) $2y'' + 7y' - 4y = 0$

11. Let $x = c_1 \cos t + c_2 \sin t$ is 2-parameter family solutions of the second order differential equation of $x'' + x = 0$. Find a solution of the second-order consisting of this differential equation and the given initial conditions.

a) $x(0) = -1, \quad x'(0) = 8$

b) $x\left(\frac{\pi}{2}\right) = 0, \quad x'\left(\frac{\pi}{2}\right) = 1$

c) $x\left(\frac{\pi}{6}\right) = \frac{1}{2}, \quad x'\left(\frac{\pi}{6}\right) = 0$

d) $x\left(\frac{\pi}{4}\right) = \sqrt{2}, \quad x'\left(\frac{\pi}{4}\right) = 2\sqrt{2}$

12. Find values of *r* such that $y(x) = x^r$ is a solution of $x^2 y'' - 4xy' + 6y = 0$

Solve the differential equation:

13. $y' = 3x^2 - 2x + 4$

14. $y'' = 2x + \sin 2x$

15. Given the differential equation $x^2 y'' - 2xy' + 2y = 4x^3$, is the given equation a solution?

a) $y = 2x^3 + x^2$

b) $y = 2x + x^2$