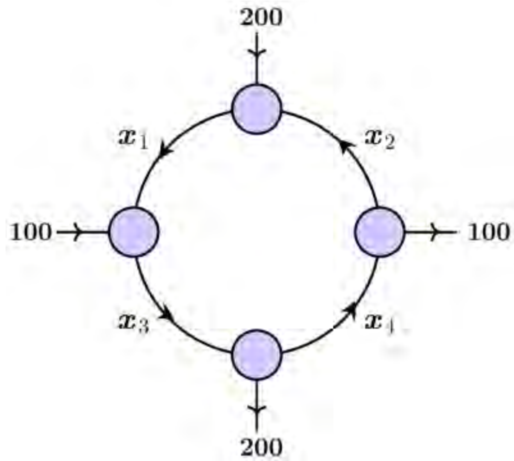
Solution   *Section* 1.8 – Applications

Exercise

The flow of traffic, in vehicles per hour, through a network of streets as is shown below



a) Solve this system for $x_i, \quad i = 1,\ 2,\ 3,\ 4.$

b) Find the traffic flow when $x_4 = 0$.

c) Find the traffic flow when $x_4 = 100$.

d) Find the traffic flow when $x_1 = 2x_2$.

Solution

a) $\begin{cases} x_1 + 100 = x_3 \\ x_2 + 200 = x_1 \\ x_2 + 100 = x_4 \\ x_4 + 200 = x_3 \end{cases}$

$$\begin{cases} -x_1 + x_3 = 100 \\ x_1 - x_2 = 200 \\ -x_2 + x_4 = 100 \\ x_3 - x_4 = 200 \end{cases}$$

$$\left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 100 \\ 1 & -1 & 0 & 0 & 200 \\ 0 & -1 & 0 & 1 & 100 \\ 0 & 0 & 1 & -1 & 200 \end{array}\right) \quad R_2 + R_1$$

$$\begin{vmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} = -1\begin{vmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$\left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 100 \\ 0 & -1 & 1 & 0 & 300 \\ 0 & -1 & 0 & 1 & 100 \\ 0 & 0 & 1 & -1 & 200 \end{array}\right) \quad R_3 - R_2$$

$$\left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 100 \\ 0 & -1 & 1 & 0 & 300 \\ 0 & 0 & -1 & 1 & -200 \\ 0 & 0 & 1 & -1 & 200 \end{array}\right) \quad R_4 + R_3$$

$$\left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 100 \\ 0 & -1 & 1 & 0 & 300 \\ 0 & 0 & -1 & 1 & -200 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \rightarrow -x_1 + x_3 = 100 \\ \rightarrow -x_2 + x_3 = 100 \\ \rightarrow -x_3 + x_4 = 100 \\ \\ \end{array}$$

Let x_4 be the free variable

$$\begin{cases} \quad \boxed{x_3 = x_4 + 200} \\ \quad \boxed{x_2 = x_4 - 100} \\ x_1 = 200 + x_2 \boxed{= x_4 + 100} \end{cases}$$

Solution: $\left(x_4 + 100, \quad x_4 - 100, \quad x_4 + 200, \quad x_4\right)$

OR

$$\begin{vmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} = -1 \begin{vmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= -1(1) - 1(-1)$$

$$= -1 + 1$$

$$\boxed{= 0}$$

$$\begin{cases} -x_1 + x_3 = 100 & \rightarrow \ x_1 = x_3 - 100 \boxed{= x_4 + 100} \\ x_1 - x_2 = 200 & \\ -x_2 + x_4 = 100 & \rightarrow \ \boxed{x_2 = x_4 - 100} \\ x_3 - x_4 = 200 & \rightarrow \ \boxed{x_3 = x_4 + 200} \end{cases}$$

b) The traffic flow when $x_4 = 0$ is:

$\therefore \ (100, \quad -100, \quad 200, \quad 0)$

c) The traffic flow when $x_4 = 100$ is:

$\therefore \ (200, \quad 0, \quad 300, \quad 100)$

d) The traffic flow when $x_1 = 2x_2$:

$x_4 + 100 = 2(x_4 - 100)$

$x_4 + 100 = 2x_4 - 200$

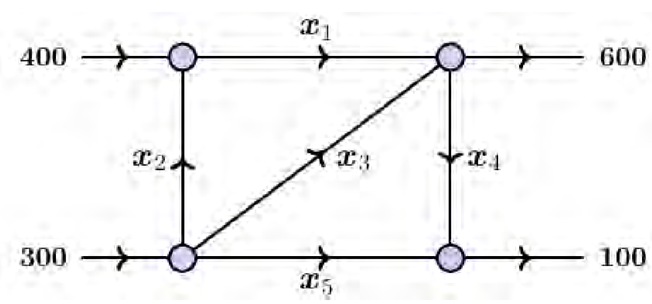
$x_4 = 300$

$\therefore \ (400, \quad 200, \quad 500, \quad 300)$

Exercise

Through a network, Express x_n 's in terms of the parameters ***s*** and ***t***.

Solution



$$\begin{cases} x_1 = x_2 + 400 \\ x_1 + x_3 = x_4 + 600 \\ x_4 + x_5 = 100 \\ x_2 + x_3 + x_5 = 300 \end{cases}$$

$$\begin{cases} x_1 - x_2 = 400 \\ x_2 + x_3 - x_4 = 600 \\ x_4 + x_5 = 100 \\ x_2 + x_3 + x_5 = 300 \end{cases}$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 1 & 0 & 1 & -1 & 0 & 600 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 1 & 1 & 0 & 1 & 300 \end{array}\right) \begin{array}{l} \\ R_2 - R_1 \\ \\ R_3 \leftrightarrow R_4 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 0 & 1 & 1 & -1 & 0 & 200 \\ 0 & 1 & 1 & 0 & 1 & 300 \\ 0 & 0 & 0 & 1 & 1 & 100 \end{array}\right) \begin{array}{l} \\ \\ R_3 - R_2 \\ \\ \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 0 & 1 & 1 & -1 & 0 & 200 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 100 \end{array}\right) \begin{array}{l} \\ \\ \\ R_4 - R_3 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 0 & 1 & 1 & -1 & 0 & 200 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{rcl} x_1 - x_2 = 400 & \rightarrow & x_1 = 400 + x_2 \\ x_2 + x_3 - x_4 = 200 & \rightarrow & x_2 = 200 - x_3 + x_4 \\ x_4 + x_5 = 100 & \rightarrow & \underline{x_4 = 100 - t} \end{array}$$

Let $x_5 = t$ & $x_3 = s$

$x_2 = 200 - s + 100 - t \ \underline{= 300 - s - t}$

$x_1 = 400 + 300 - s - t \ \underline{= 700 - s - t}$

Exercise

Water is flowing through a network of pipes. Express x_n 's in terms of the parameters ***s*** and ***t***.

Solution

$x_1 + x_3 = 900$

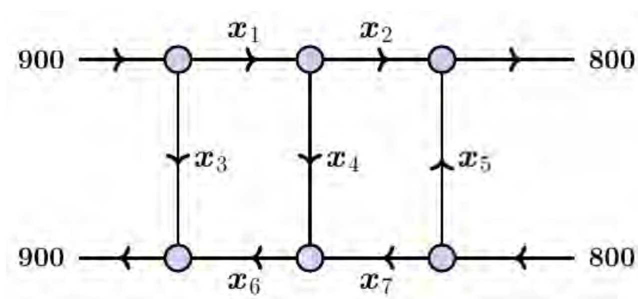
$x_1 = x_2 + x_4 \quad \rightarrow \quad x_1 - x_2 - x_4 = 0$

$x_2 + x_5 = 800$

$x_5 + x_7 = 800$

$x_6 = x_4 + x_7 \quad \rightarrow \quad x_4 - x_6 + x_7 = 0$

$x_3 + x_6 = 900$



$$\begin{array}{c} \left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 900 \end{array}\right] \end{array} \begin{array}{l} \\ R_2 - R_1 \\ \\ \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & -900 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 900 \end{array}\right] \begin{array}{l} \\ \\ R_3 + R_2 \\ R_6 \\ \\ R_4 \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & -900 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 900 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \end{array}\right] \begin{array}{l} \\ -R_2 \\ \\ R_4 + R_3 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 900 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 800 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ R_5 + R_4 \\ \\ \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 900 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ \\ R_6 - R_5 \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 900 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{rll} x_2 &= 900 - x_3 & (5) \\ x_2 &= 900 - x_3 - x_4 & (4) \\ x_3 &= 100 - x_4 + x_5 & (3) \\ -x_4 &= 800 - x_5 - x_6 & (2) \\ x_5 &= 800 - x_7 & (1) \end{array}$$

Let $x_6 = s$ & $x_7 = t$

$(1) \rightarrow \quad x_5 = 800 - t$

$(2) \rightarrow \quad x_4 = s - t$

$(3) \rightarrow \quad x_3 = 900 - s$

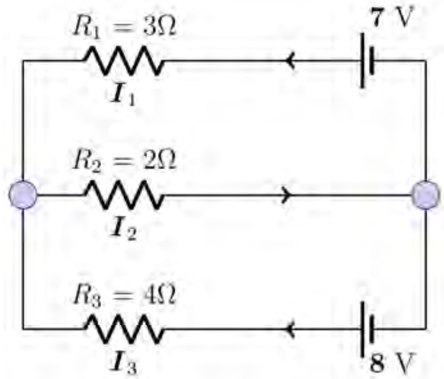
$(2) \rightarrow \quad x_2 = t$

$(1) \rightarrow \quad x_2 = s$

Solution: $\underline{(s,\ t,\ 900 - s,\ s - t,\ 800 - t,\ s,\ t)}$

Exercise

Determine the currents $I_1, I_2,$ *and* I_3 for the electrical network shown below



Solution

$$I_2 = I_1 + I_3$$
$$3I_1 + 2I_2 = 7$$
$$2I_2 + 4I_3 = 8$$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 3I_1 + 2I_2 \qquad = 7 \\ \qquad I_2 + 2I_3 = 4 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 3 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 13 \qquad D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 7 & 2 & 0 \\ 4 & 1 & 2 \end{vmatrix} = 13 \qquad D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 3 & 7 & 0 \\ 0 & 4 & 2 \end{vmatrix} = 26 \qquad D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 3 & 2 & 7 \\ 0 & 1 & 4 \end{vmatrix} = 13$$

$\underline{I_1 = 1\ A}\rvert$ $\underline{I_2 = 2\ A}\rvert$ $\underline{I_3 = 1\ A}\rvert$

OR

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 3 & 2 & 0 & 7 \\ 0 & 1 & 2 & 4 \end{array}\right) \quad R_2 - 3R_1$$

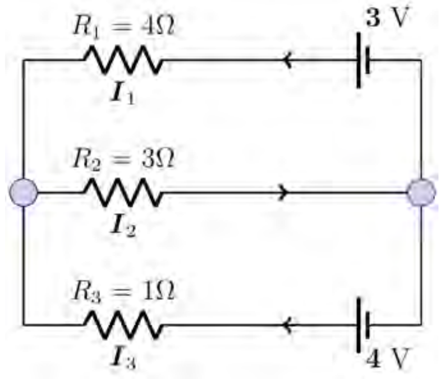
$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 5 & -3 & 7 \\ 0 & 1 & 2 & 4 \end{array}\right) \quad -5R_3 + R_2$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 5 & -3 & 7 \\ 0 & 0 & -13 & -13 \end{array}\right) \quad \begin{array}{l} I_1 = I_2 - I_3 \\ 5I_2 = 3I_3 + 7 \\ \underline{I_3 = 1}\rvert \end{array}$$

$\underline{I_2 = 2}\rvert$ $\underline{I_1 = 1}\rvert$

Exercise

Determine the currents $I_1, I_2,$ *and* I_3 for the electrical network shown below



Solution

$$I_2 = I_1 + I_3$$

$$4I_1 + 3I_2 = 3$$

$$3I_2 + I_3 = 4$$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 4I_1 + 3I_2 \qquad = 3 \\ \qquad 3I_2 + I_3 = 4 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 4 & 3 & 0 \\ 0 & 3 & 1 \end{vmatrix} = 19 \qquad D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 3 & 3 & 0 \\ 4 & 3 & 1 \end{vmatrix} = 0 \qquad D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 4 & 3 & 0 \\ 0 & 4 & 1 \end{vmatrix} = 19 \qquad D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 4 & 3 & 3 \\ 0 & 3 & 4 \end{vmatrix} = 19$$

$\underline{I_1 = 0\ A}$ $\underline{I_2 = 1\ A}$ $\underline{I_3 = 1\ A}$

OR

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 4 & 3 & 0 & 3 \\ 0 & 3 & 1 & 4 \end{array}\right) \quad R_2 - 4R_1$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 7 & -4 & 3 \\ 0 & 3 & 1 & 4 \end{array}\right) \quad 7R_3 - 3R_2$$

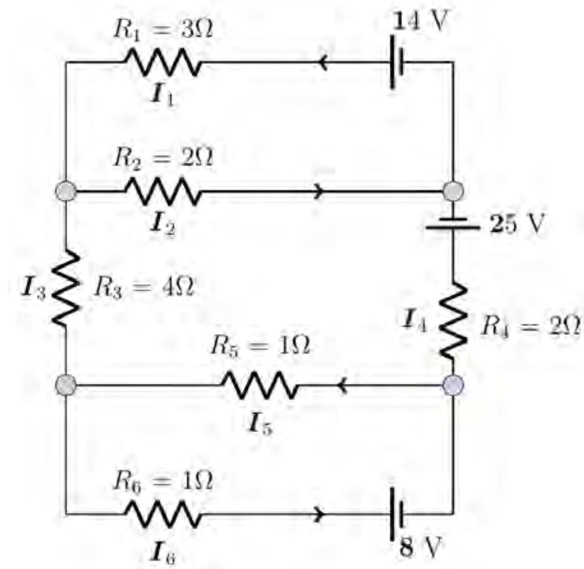
$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 7 & -4 & 3 \\ 0 & 0 & 19 & 19 \end{array}\right) \begin{array}{l} \rightarrow I_1 = I_2 - I_3 \quad (2) \\ \rightarrow 7I_2 = 4I_3 + 3 \quad (1) \end{array}$$

$\underline{I_3 = 1}$

$\underline{I_2 = 1}$ $\underline{I_1 = 0}$

Exercise

Determine the currents $I_1, I_2, I_3, I_4, I_5,$ *and* I_6 for the electrical network shown below



Solution

$$I_1 + I_3 = I_2 \quad \rightarrow \quad I_1 - I_2 + I_3 = 0$$

$$I_1 + I_4 = I_2 \quad \rightarrow \quad I_1 - I_2 + I_4 = 0$$

$$I_3 + I_6 = I_5 \quad \rightarrow \quad I_3 - I_5 + I_6 = 0$$

$$\begin{cases} 3I_1 + 2I_2 = 14 \\ 2I_2 + 4I_3 + I_5 + 2I_4 = 25 \\ I_5 + I_6 = 8 \end{cases}$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 & 14 \\ 0 & 2 & 4 & 2 & 1 & 0 & 25 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ \\ R_4 - 3R_1 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 2 & 4 & 2 & 1 & 0 & 25 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] \begin{array}{l} \\ R_4 \\ \\ R_2 \\ R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 2 & 4 & 2 & 1 & 0 & 25 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] \begin{array}{l} \\ \\ 5R_3 - 2R_2 \\ \\ R_5 + R_4 \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] \begin{array}{l} \\ \\ \\ 26R_4 + R_3 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & 0 & 36 & 5 & 0 & 97 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ 36R_5 - R_4 \\ \\ \end{array}$$

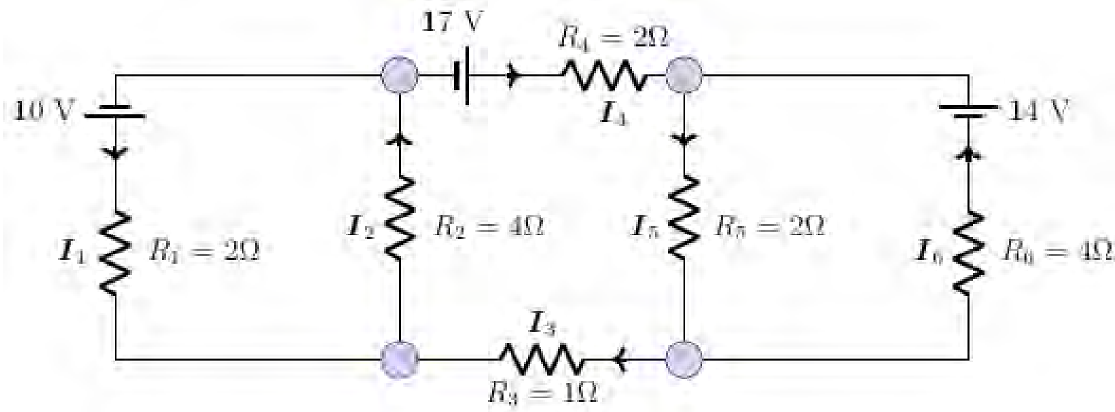
$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & 0 & 36 & 5 & 0 & 97 \\ 0 & 0 & 0 & 0 & -41 & 36 & -97 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ \\ 41R_6 + R_5 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & 0 & 36 & 5 & 0 & 97 \\ 0 & 0 & 0 & 0 & -41 & 36 & -97 \\ 0 & 0 & 0 & 0 & 0 & 77 & 231 \end{array}\right]$$

$$\begin{aligned} I_1 &= 4-2 & &\rightarrow \boxed{I_1 = 2} \\ 5I_2 &= 14+3(2) & &\rightarrow \boxed{I_2 = 4} \\ 26I_3 &= 97-10(2)-5(5) & &\rightarrow \boxed{I_3 = 2} \\ 36I_4 &= 97-5(5) & &\rightarrow \boxed{I_4 = 2} \\ -41I_5 &= -97-36(3) & &\rightarrow \boxed{I_5 = 5} \\ 77I_6 &= 231 & &\rightarrow \boxed{I_6 = 3} \end{aligned}$$

Exercise

Determine the currents $I_1, I_2, I_3, I_4, I_5,$ *and* I_6 for the electrical network shown below



Solution

$1 \rightarrow \quad I_1 + I_3 = I_2$

$2 \rightarrow \quad I_1 + I_4 = I_2$

$3 \rightarrow \quad I_3 + I_6 = I_5$

$4 \rightarrow \quad I_4 + I_6 = I_5$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ I_1 - I_2 + I_4 = 0 \\ I_3 - I_5 + I_6 = 0 \\ I_4 - I_5 + I_6 = 0 \\ 2I_1 + 4I_2 = 10 \\ 4I_2 + I_3 + 2I_4 + 2I_5 = 17 \\ 2I_5 + 4I_6 = 14 \end{cases}$$

$$\begin{array}{ll} \left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 5 \\ 0 & 4 & 1 & 2 & 2 & 0 & 17 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \end{array}\right) & \begin{array}{l} \\ R_2 - R_1 \\ \\ \\ R_5 - R_1 \\ \\ \\ \end{array} \end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 4 & 1 & 2 & 2 & 0 & 17 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \end{array}\right) \begin{array}{l} \\ \\ R_3 + R_2 \\ \\ \\ 3R_6 - 4R_5 \\ \\ \end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 7 & 6 & 6 & 0 & 31 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \end{array}\right) \begin{array}{l} \\ \\ \\ R_4 - R_3 \\ \\ \\ \\ \end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 7 & 6 & 6 & 0 & 31 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \end{array}\right) \begin{array}{l} \\ \\ \\ \\ \\ R_6 + 7R_2 \\ \\ \end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 13 & 6 & 0 & 31 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \\ \\ \\ \\ R_5 - 13R_4 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 19 & -13 & 31 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \\ \\ \\ \\ \\ 19R_6 - R_5 \\ \\ \end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 19 & -13 & 31 \\ 0 & 0 & 0 & 0 & 0 & 51 & 102 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{c} (4) \\ (3) \\ (2) \\ (1) \\ I_5 = \frac{1}{19}\left(31 + 13I_6\right) \\ \underline{I_6 = 2} \end{array}$$

$\underline{I_5 = 3}$

$(1) \rightarrow \quad \underline{I_4} = I_5 - I_6 \ \underline{= 1}$

$(2) \rightarrow \quad \underline{I_3} = I_4 \ \underline{= 1}$

$(3) \rightarrow \quad \underline{I_2} = \frac{1}{3}\left(I_3 + 5\right) \ \underline{= 2}$

$(4) \rightarrow \quad \underline{I_1} = I_2 - I_3 \ \underline{= 1}$

Exercise

Consider the invertible matrix: $\quad A = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{pmatrix}$

The message: ***ICEBERG DEAD AHEAD***

a) Write the uncoded row matrices 1×3 for the message.

b) Use the matrix A to encode the message.

c) Decode a message from part *b*) given the matrix A.

Solution

a)

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

$$\begin{array}{cccccc} I\ C\ E & B\ E\ R & G\ \_\ D & E\ A\ D & \_\ A\ H & E\ A\ D \\ [9\ \ 3\ \ 5] & [2\ \ 5\ \ 18] & [7\ \ 0\ \ 4] & [5\ \ 1\ \ 4] & [0\ \ 1\ \ 8] & [5\ \ 1\ \ 4] \end{array}$$

b) Let encode the message ***ICEBERG DEAD AHEAD***

$$\begin{bmatrix} 9 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 29 & 80 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & 18 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} -37 & 3 & 175 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} -5 & 6 & 42 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} -4 & 9 & 47 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 8 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} -21 & -5 & 65 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} -4 & 9 & 47 \end{bmatrix}$$

The sequence of coded row matrices is

$$\begin{bmatrix} 3 & 29 & 80 \end{bmatrix} \quad \begin{bmatrix} -37 & 3 & 175 \end{bmatrix} \quad \begin{bmatrix} -5 & 6 & 42 \end{bmatrix} \quad \begin{bmatrix} -4 & 9 & 47 \end{bmatrix} \quad \begin{bmatrix} -21 & -9 & 65 \end{bmatrix} \quad \begin{bmatrix} -4 & 9 & 47 \end{bmatrix}$$

The cryptogram:

$$3 \;\; 29 \;\; 80 \;\; -37 \;\; 3 \;\; 175 \;\; -5 \;\; 6 \;\; 42 \;\; -4 \;\; 9 \;\; 47 \;\; -21 \;\; -9 \;\; 65 \;\; -4 \;\; 9 \;\; 47$$

c) To decode a message given the matrix *A*.

$$|A| = \begin{vmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{vmatrix} = 1$$

$$A^{-1} = \begin{bmatrix} 67 & -18 & 4 \\ -48 & 13 & -3 \\ 15 & -4 & 1 \end{bmatrix}$$

With the cryptogram:

$$\begin{bmatrix} 3 & 29 & 80 \end{bmatrix} \quad \begin{bmatrix} -37 & 3 & 175 \end{bmatrix} \quad \begin{bmatrix} -5 & 6 & 42 \end{bmatrix} \quad \begin{bmatrix} -4 & 9 & 47 \end{bmatrix} \quad \begin{bmatrix} -21 & -9 & 65 \end{bmatrix} \quad \begin{bmatrix} -4 & 9 & 47 \end{bmatrix}$$

$$\begin{bmatrix}3 & 29 & 80\end{bmatrix}\begin{bmatrix}67 & -18 & 4\\ -48 & 13 & -3\\ 15 & -4 & 1\end{bmatrix}=\begin{bmatrix}9 & 3 & 5\end{bmatrix}$$

$$\begin{bmatrix}-37 & 3 & 175\end{bmatrix}\begin{bmatrix}67 & -18 & 4\\ -48 & 13 & -3\\ 15 & -4 & 1\end{bmatrix}=\begin{bmatrix}2 & 5 & 18\end{bmatrix}$$

$$\begin{bmatrix}-5 & 6 & 42\end{bmatrix}\begin{bmatrix}67 & -18 & 4\\ -48 & 13 & -3\\ 15 & -4 & 1\end{bmatrix}=\begin{bmatrix}7 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}-4 & 9 & 47\end{bmatrix}\begin{bmatrix}67 & -18 & 4\\ -48 & 13 & -3\\ 15 & -4 & 1\end{bmatrix}=\begin{bmatrix}5 & 1 & 4\end{bmatrix}$$

$$\begin{bmatrix}-21 & -9 & 65\end{bmatrix}\begin{bmatrix}67 & -18 & 4\\ -48 & 13 & -3\\ 15 & -4 & 1\end{bmatrix}=\begin{bmatrix}0 & 1 & 8\end{bmatrix}$$

$$\begin{bmatrix}-4 & 9 & 47\end{bmatrix}\begin{bmatrix}67 & -18 & 4\\ -48 & 13 & -3\\ 15 & -4 & 1\end{bmatrix}=\begin{bmatrix}5 & 1 & 4\end{bmatrix}$$

The message is:

9	3	5	2	5	18	7	0	1	5	1	4	0	1	8	5	1	4
I	*C*	*E*	*B*	*E*	*R*	*G*	_	*D*	*E*	*A*	*D*	_	*A*	*H*	*E*	*A*	*D*

Exercise

You want to send the message: ***LINEAR ALGEBRA*** with a key word ***MATH***

a) Write the matrix *A*.

b) Write the uncoded row matrices 1×2 for the message.

c) Use the matrix *A* to encode the message.

d) Decode a message from part *b*) given the matrix *A*.

Solution

a)

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

$M \quad A \quad T \quad H$

13 1 20 8

$$A = \begin{pmatrix} 13 & 1 \\ 20 & 8 \end{pmatrix}$$

b)

$L \quad I \quad N \quad E \quad A \quad R \quad \_ \quad A \quad L \quad G \quad E \quad B \quad R \quad A$

12 9 14 5 1 18 0 1 12 7 5 2 18 1

$$[12 \quad 9] \quad [14 \quad 5] \quad [1 \quad 18] \quad [0 \quad 1] \quad [12 \quad 7] \quad [5 \quad 2] \quad [18 \quad 1]$$

c) Encoding the message

$$[12 \quad 9]\begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = [336 \quad 84]$$

$$[14 \quad 5]\begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = [282 \quad 54]$$

$$[1 \quad 18]\begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = [373 \quad 145]$$

$$[0 \quad 1]\begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = [20 \quad 8]$$

$$[12 \quad 7]\begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = [296 \quad 68]$$

$$[5 \quad 2]\begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = [105 \quad 21]$$

$$[18 \quad 1]\begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = [254 \quad 26]$$

The cryptogram:

336 84 282 54 373 145 20 8 296 68 105 21 254 26

d) To decode a message given the matrix *A*.

$$A = \begin{pmatrix} 13 & 1 \\ 20 & 8 \end{pmatrix}$$

$$A^{-1} = \frac{1}{84}\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{pmatrix}$$

With the cryptogram:

$$[336 \quad 84] \quad [282 \quad 54] \quad [373 \quad 145] \quad [20 \quad 8] \quad [296 \quad 68] \quad [105 \quad 21] \quad [254 \quad 26]$$

$$[336 \quad 84] \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = [12 \quad 9]$$

$$[282 \quad 54] \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = [14 \quad 5]$$

$$[373 \quad 145] \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = [1 \quad 18]$$

$$[20 \quad 8] \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = [0 \quad 1]$$

$$[296 \quad 68] \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = [12 \quad 7]$$

$$[105 \quad 21] \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = [5 \quad 2]$$

$$[254 \quad 26] \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = [18 \quad 1]$$

12	9	14	5	1	18	0	1	12	7	5	2	18	1
L	*I*	*N*	*E*	*A*	*R*	_	*A*	*L*	*G*	*E*	*B*	*R*	*A*

The message is: ***Linear Algebra***

Exercise

You want to send the message: ***CRYPTOGRAPHY IS A METHOD OF PROTECTING INFORMATIONS*** with a key word ***CODE***

a) Write the matrix *A*.

b) Write the uncoded row matrices 1×2 for the message.

c) Use the matrix *A* to encode the message.

d) Decode a message from part *b*) given the matrix *A*.

Solution

a)

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

$$\begin{matrix} C & O & D & E \\ 3 & 15 & 4 & 5 \end{matrix}$$

$$A = \begin{pmatrix} 3 & 15 \\ 4 & 5 \end{pmatrix}$$

b)

$$\begin{matrix} C & R & Y & P & T & O & G & R & A & P & H & Y & \_ & I & S & \_ & A & \_ \\ 3 & 18 & 25 & 16 & 20 & 15 & 7 & 18 & 1 & 16 & 8 & 25 & 0 & 9 & 19 & 0 & 1 & 0 \end{matrix}$$

$$\begin{matrix} M & E & T & H & O & D & \_ & O & F & \_ & P & R & O & T & E & C & T & I & N & G \\ 13 & 5 & 20 & 8 & 15 & 4 & 0 & 15 & 6 & 0 & 16 & 18 & 15 & 20 & 5 & 3 & 20 & 9 & 14 & 7 \end{matrix}$$

$$\begin{matrix} \_ & I & N & F & O & R & M & A & T & I & O & N & S & \_ \\ 0 & 9 & 14 & 6 & 15 & 18 & 13 & 1 & 20 & 9 & 15 & 14 & 19 & 0 \end{matrix}$$

$$[3 \quad 18] \quad [25 \quad 16] \quad [20 \quad 15] \quad [7 \quad 18] \quad [1 \quad 16] \quad [8 \quad 25] \quad [0 \quad 9] \quad [19 \quad 0]$$

$$[1 \quad 0] \quad [13 \quad 5] \quad [20 \quad 8] \quad [15 \quad 4] \quad [0 \quad 15] \quad [6 \quad 0] \quad [16 \quad 18] \quad [15 \quad 20]$$

$$[5 \quad 3] \quad [20 \quad 9] \quad [14 \quad 7] \quad [0 \quad 9] \quad [14 \quad 6] \quad [15 \quad 18] \quad [13 \quad 1] \quad [20 \quad 9]$$

$$[15 \quad 14] \quad [19 \quad 0]$$

c) Encoding the message

$$[3 \quad 18]\begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = [81 \quad 135]$$

$$\begin{bmatrix}25 & 16\end{bmatrix}\begin{bmatrix}3 & 15\\4 & 5\end{bmatrix}=\begin{bmatrix}139 & 455\end{bmatrix}$$

$$\begin{bmatrix}20 & 15\end{bmatrix}\begin{bmatrix}3 & 15\\4 & 5\end{bmatrix}=\begin{bmatrix}120 & 375\end{bmatrix}$$

$$\begin{bmatrix}7 & 18\end{bmatrix}\begin{bmatrix}3 & 15\\4 & 5\end{bmatrix}=\begin{bmatrix}93 & 195\end{bmatrix}$$

$$\begin{bmatrix}1 & 16\end{bmatrix}\begin{bmatrix}3 & 15\\4 & 5\end{bmatrix}=\begin{bmatrix}67 & 95\end{bmatrix}$$

$$\begin{bmatrix}8 & 25\end{bmatrix}\begin{bmatrix}3 & 15\\4 & 5\end{bmatrix}=\begin{bmatrix}124 & 245\end{bmatrix}$$

$$\begin{bmatrix}0 & 9\end{bmatrix}\begin{bmatrix}3 & 15\\4 & 5\end{bmatrix}=\begin{bmatrix}36 & 45\end{bmatrix}$$

$$\begin{bmatrix}19 & 0\end{bmatrix}\begin{bmatrix}3 & 15\\4 & 5\end{bmatrix}=\begin{bmatrix}57 & 285\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}3 & 15\\4 & 5\end{bmatrix}=\begin{bmatrix}3 & 15\end{bmatrix}$$

$$\begin{bmatrix}13 & 5\end{bmatrix}\begin{bmatrix}3 & 15\\4 & 5\end{bmatrix}=\begin{bmatrix}59 & 220\end{bmatrix}$$

$$\begin{bmatrix}20 & 8\end{bmatrix}\begin{bmatrix}3 & 15\\4 & 5\end{bmatrix}=\begin{bmatrix}92 & 340\end{bmatrix}$$

$$\begin{bmatrix}15 & 4\end{bmatrix}\begin{bmatrix}3 & 15\\4 & 5\end{bmatrix}=\begin{bmatrix}61 & 245\end{bmatrix}$$

$$\begin{bmatrix}0 & 15\end{bmatrix}\begin{bmatrix}3 & 15\\4 & 5\end{bmatrix}=\begin{bmatrix}60 & 75\end{bmatrix}$$

$$\begin{bmatrix}6 & 0\end{bmatrix}\begin{bmatrix}3 & 15\\4 & 5\end{bmatrix}=\begin{bmatrix}18 & 90\end{bmatrix}$$

$$\begin{bmatrix}16 & 18\end{bmatrix}\begin{bmatrix}3 & 15\\4 & 5\end{bmatrix}=\begin{bmatrix}120 & 330\end{bmatrix}$$

$$\begin{bmatrix}15 & 20\end{bmatrix}\begin{bmatrix}3 & 15\\4 & 5\end{bmatrix}=\begin{bmatrix}125 & 325\end{bmatrix}$$

$$\begin{bmatrix}5 & 3\end{bmatrix}\begin{bmatrix}3 & 15\\4 & 5\end{bmatrix}=\begin{bmatrix}27 & 90\end{bmatrix}$$

$$[20 \quad 9]\begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = [96 \quad 345]$$

$$[14 \quad 7]\begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = [70 \quad 245]$$

$$[0 \quad 9]\begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = [36 \quad 45]$$

$$[14 \quad 6]\begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = [66 \quad 240]$$

$$[15 \quad 18]\begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = [117 \quad 315]$$

$$[13 \quad 1]\begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = [43 \quad 200]$$

$$[20 \quad 9]\begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = [96 \quad 345]$$

$$[15 \quad 14]\begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = [101 \quad 295]$$

$$[19 \quad 0]\begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = [57 \quad 285]$$

The cryptogram:

81 135 139 455 120 375 93 195 67 95 124 245 36 45 57 285

3 15 59 220 92 340 61 245 60 75 18 90 120 330 125 325

27 90 96 345 70 245 36 45 66 240 117 315 43 200 96 345

101 295 57 285

d) To decode a message given the matrix A.

$$A = \begin{pmatrix} 3 & 15 \\ 4 & 5 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{45}\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix}$$

$$-\frac{1}{45}[81 \quad 135]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-135 \quad -810]$$

$$= [3 \quad 18]$$

$$-\frac{1}{45}\begin{bmatrix}139 & 455\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-1{,}125 & -720\end{bmatrix}$$
$$=\begin{bmatrix}25 & 16\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}120 & 375\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-900 & -675\end{bmatrix}$$
$$=\begin{bmatrix}20 & 15\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}93 & 195\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-315 & -810\end{bmatrix}$$
$$=\begin{bmatrix}7 & 18\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}67 & 95\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-45 & -720\end{bmatrix}$$
$$=\begin{bmatrix}1 & 16\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}124 & 245\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-360 & -1{,}125\end{bmatrix}$$
$$=\begin{bmatrix}8 & 25\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}36 & 45\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}0 & -405\end{bmatrix}$$
$$=\begin{bmatrix}0 & 9\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}57 & 285\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-855 & 0\end{bmatrix}$$
$$=\begin{bmatrix}19 & 0\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}3 & 15\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-45 & 0\end{bmatrix}$$
$$=\begin{bmatrix}1 & 0\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}59 & 220\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-585 & -225\end{bmatrix}$$
$$=\begin{bmatrix}13 & 5\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}92 & 340\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-900 & -360\end{bmatrix}$$
$$=\begin{bmatrix}20 & 8\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}61 & 245\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-675 & -180\end{bmatrix}$$
$$=\begin{bmatrix}15 & 4\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}60 & 75\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}0 & -675\end{bmatrix}$$
$$=\begin{bmatrix}0 & 15\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}18 & 90\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-270 & 0\end{bmatrix}$$
$$=\begin{bmatrix}6 & 0\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}120 & 330\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-720 & -810\end{bmatrix}$$
$$=\begin{bmatrix}16 & 18\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}125 & 325\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-675 & -900\end{bmatrix}$$
$$=\begin{bmatrix}15 & 20\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}27 & 90\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-225 & -135\end{bmatrix}$$
$$=\begin{bmatrix}5 & 3\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}96 & 345\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-900 & -405\end{bmatrix}$$
$$=\begin{bmatrix}20 & 9\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}70 & 245\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-630 & -315\end{bmatrix}$$
$$=\begin{bmatrix}14 & 7\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}36 & 45\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}0 & -405\end{bmatrix}$$
$$=\begin{bmatrix}0 & 9\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}66 & 240\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-630 & -270\end{bmatrix}$$
$$=\begin{bmatrix}14 & 6\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}117 & 315\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-675 & -810\end{bmatrix}$$
$$=\begin{bmatrix}15 & 18\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}43 & 200\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-585 & -45\end{bmatrix}$$

$$=\begin{bmatrix}13 & 1\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}96 & 345\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-900 & -405\end{bmatrix}$$

$$=\begin{bmatrix}20 & 9\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}101 & 295\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-675 & -630\end{bmatrix}$$

$$=\begin{bmatrix}15 & 14\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}57 & 285\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-855 & 0\end{bmatrix}$$

$$=\begin{bmatrix}19 & 0\end{bmatrix}$$

3	18	25	16	20	15	7	18	1	16	8	25	0	9	19	0	1	0	13	5
C	*R*	*Y*	*P*	*T*	*O*	*G*	*R*	*A*	*P*	*H*	*Y*	_	*I*	*S*	_	*A*	_	*M*	*E*
20	8	15	4	0	15	6	0	16	18	15	20	5	3	20	9	14	7	0	9
T	*H*	*O*	*D*	_	*O*	*F*	_	*P*	*R*	*O*	*T*	*E*	*C*	*T*	*I*	*N*	*G*	_	*I*
14	6	15	18	16	1	20	9	15	14	19	0								
N	*F*	*O*	*R*	*M*	*A*	*T*	*I*	*O*	*N*	*S*	_								

The message is: ***Cryptography is a Method of Protecting Informations***

Exercise

Write the matrix A with a key word ***MATH,*** then decode the cryptogram

117 9 456 132 386 62 260 104 413 161 104 8

Solution

M	*A*	*T*	*H*
13	1	20	8

$$A=\begin{pmatrix}13 & 1\\ 20 & 8\end{pmatrix}$$

$$A^{-1}=\frac{1}{84}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix}$$

With the cryptogram:

$$\begin{bmatrix}117 & 9\end{bmatrix}\ \begin{bmatrix}456 & 132\end{bmatrix}\ \begin{bmatrix}386 & 62\end{bmatrix}\ \begin{bmatrix}260 & 104\end{bmatrix}\ \begin{bmatrix}413 & 161\end{bmatrix}\ \begin{bmatrix}104 & 8\end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix}117 & 9\end{bmatrix}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix} = \frac{1}{84}\begin{bmatrix}756 & 0\end{bmatrix}$$
$$= \begin{bmatrix}9 & 0\end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix}456 & 132\end{bmatrix}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix} = \frac{1}{84}\begin{bmatrix}1,008 & 1,260\end{bmatrix}$$
$$= \begin{bmatrix}12 & 15\end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix}386 & 62\end{bmatrix}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix} = \frac{1}{84}\begin{bmatrix}1,848 & 420\end{bmatrix}$$
$$= \begin{bmatrix}22 & 5\end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix}260 & 104\end{bmatrix}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix} = \frac{1}{84}\begin{bmatrix}0 & 1,092\end{bmatrix}$$
$$= \begin{bmatrix}0 & 13\end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix}413 & 161\end{bmatrix}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix} = \frac{1}{84}\begin{bmatrix}84 & 1,680\end{bmatrix}$$
$$= \begin{bmatrix}1 & 20\end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix}104 & 8\end{bmatrix}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix} = \frac{1}{84}\begin{bmatrix}672 & 0\end{bmatrix}$$
$$= \begin{bmatrix}8 & 0\end{bmatrix}$$

$$\begin{array}{cccccccccccc} 9 & 0 & 12 & 15 & 22 & 5 & 0 & 13 & 1 & 20 & 8 & 0 \\ I & - & L & O & V & E & - & M & A & T & H & - \end{array}$$

The message is: ***I love math***

Exercise

Write the matrix A with a key word ***MATH,*** then decode the cryptogram

438 150 145 37 240 96 635 191 445 157 260 104 413 161 104 8

Solution

$$\begin{array}{cccc} M & A & T & H \\ 13 & 1 & 20 & 8 \end{array}$$

$$A = \begin{pmatrix}13 & 1\\ 20 & 8\end{pmatrix}$$

$$A^{-1} = \frac{1}{84}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix}$$

With the cryptogram:

$$[438 \quad 150] \quad [145 \quad 37] \quad [240 \quad 96] \quad [635 \quad 191] \quad [445 \quad 157]$$

$$[260 \quad 104] \quad [413 \quad 161] \quad [104 \quad 8]$$

$$\frac{1}{84}[438 \quad 150]\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} = \frac{1}{84}[504 \quad 1{,}512]$$

$$= [6 \quad 18]$$

$$\frac{1}{84}[145 \quad 37]\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} = \frac{1}{84}[420 \quad 336]$$

$$= [5 \quad 4]$$

$$\frac{1}{84}[240 \quad 96]\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} = \frac{1}{84}[0 \quad 1{,}008]$$

$$= [0 \quad 12]$$

$$\frac{1}{84}[635 \quad 191]\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} = \frac{1}{84}[1{,}260 \quad 1{,}848]$$

$$= [15 \quad 22]$$

$$\frac{1}{84}[445 \quad 157]\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} = \frac{1}{84}[420 \quad 1{,}596]$$

$$= [5 \quad 19]$$

$$\frac{1}{84}[260 \quad 104]\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} = \frac{1}{84}[0 \quad 1{,}092]$$

$$= [0 \quad 13]$$

$$\frac{1}{84}[413 \quad 161]\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} = \frac{1}{84}[84 \quad 1{,}680]$$

$$= [1 \quad 20]$$

$$\frac{1}{84}[104 \quad 8]\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} = \frac{1}{84}[672 \quad 0]$$

$$= [8 \quad 0]$$

6	18	5	4	0	12	15	22	5	19	0	13	1	20	8	0
F	*R*	*E*	*D*	–	*L*	*O*	*V*	*E*	*S*	–	*M*	*A*	*T*	*H*	–

The message is: ***Fred loves math***

Exercise

Consider the invertible matrix: $A = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix}$

Decode the cryptogram

$$1 \quad -5 \quad 11 \quad 19 \quad -25 \quad -45 \quad 11 \quad -16 \quad -28 \quad 20 \quad -29 \quad -27$$
$$12 \quad -12 \quad -53 \quad 40 \quad -61 \quad -35 \quad 8 \quad -17 \quad 7$$

Solution

$|A| = 1$

$$A^{-1} = \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix}$$

With the cryptogram:

$$\begin{bmatrix}1 & -5 & 11\end{bmatrix} \; \begin{bmatrix}19 & -25 & -45\end{bmatrix} \; \begin{bmatrix}11 & -16 & -28\end{bmatrix} \; \begin{bmatrix}20 & -29 & -27\end{bmatrix}$$
$$\begin{bmatrix}12 & -12 & -53\end{bmatrix} \; \begin{bmatrix}40 & -61 & -35\end{bmatrix} \; \begin{bmatrix}8 & -17 & 7\end{bmatrix}$$

$$\begin{bmatrix}1 & -5 & 11\end{bmatrix}\begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix} = \begin{bmatrix}4 & 9 & 6\end{bmatrix}$$

$$\begin{bmatrix}19 & -25 & -45\end{bmatrix}\begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix} = \begin{bmatrix}6 & 5 & 18\end{bmatrix}$$

$$\begin{bmatrix}11 & -16 & -28\end{bmatrix}\begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix} = \begin{bmatrix}5 & 14 & 20\end{bmatrix}$$

$$\begin{bmatrix}20 & -29 & -27\end{bmatrix}\begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix} = \begin{bmatrix}9 & 1 & 12\end{bmatrix}$$

$$\begin{bmatrix}12 & -12 & -53\end{bmatrix}\begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix} = \begin{bmatrix}0 & 5 & 17\end{bmatrix}$$

$$\begin{bmatrix}40 & -61 & -35\end{bmatrix}\begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix} = \begin{bmatrix}21 & 1 & 20\end{bmatrix}$$

$$\begin{bmatrix} 8 & -17 & 7 \end{bmatrix} \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix} = \begin{bmatrix} 9 & 15 & 14 \end{bmatrix}$$

4	9	6	6	5	18	5	14	20	9	1	12	0	5	17	21	1	20	9	15	14
D	*I*	*F*	*F*	*E*	*R*	*E*	*N*	*T*	*I*	*A*	*L*	_	*E*	*Q*	*U*	*A*	*T*	*I*	*O*	*N*

The message is: ***Differential Equation***.

Exercise

Determine the key word, then decode the given cryptogram

6 18 5 4 15 13 1 20 8

102 649 238 57 324 112 128 622 207

180 613 290 102 360 259 151 580 297

Hint: First row is the key

Solution

The key word from the first row is

6	18	5	4	15	13	1	20	8
f	*r*	*e*	*d*	*o*	*m*	*a*	*t*	*h*

Since it has 9 numbers, then the matrix is $9 = 3^2$ which is 3×3

$$A = \begin{pmatrix} 6 & 18 & 5 \\ 4 & 15 & 13 \\ 1 & 20 & 8 \end{pmatrix}$$

$$|A| = -857$$

$$A^{-1} = -\frac{1}{857} \begin{pmatrix} -140 & -44 & 159 \\ -19 & 43 & -58 \\ 65 & -102 & 18 \end{pmatrix}$$

$$= \frac{1}{857} \begin{pmatrix} 140 & 44 & -159 \\ 19 & -43 & 58 \\ -65 & 102 & -18 \end{pmatrix}$$

With the cryptogram:

$$\begin{bmatrix} 102 & 649 & 238 \end{bmatrix} \quad \begin{bmatrix} 57 & 324 & 112 \end{bmatrix} \quad \begin{bmatrix} 128 & 622 & 207 \end{bmatrix}$$

$$\begin{bmatrix} 180 & 613 & 290 \end{bmatrix} \quad \begin{bmatrix} 102 & 360 & 259 \end{bmatrix} \quad \begin{bmatrix} 151 & 580 & 297 \end{bmatrix}$$

$$\frac{1}{857}\begin{bmatrix}102 & 649 & 238\end{bmatrix}\begin{pmatrix}140 & 44 & -159\\ 19 & -43 & 58\\ -65 & 102 & -18\end{pmatrix}=\frac{1}{857}\begin{bmatrix}11{,}141 & 857 & 17{,}140\end{bmatrix}$$

$$=\begin{bmatrix}13 & 1 & 20\end{bmatrix}$$

$$\frac{1}{857}\begin{bmatrix}57 & 324 & 112\end{bmatrix}\begin{pmatrix}140 & 44 & -159\\ 19 & -43 & 58\\ -65 & 102 & -18\end{pmatrix}=\frac{1}{857}\begin{bmatrix}6{,}856 & 0 & 7{,}713\end{bmatrix}$$

$$=\begin{bmatrix}8 & 0 & 9\end{bmatrix}$$

$$\frac{1}{857}\begin{bmatrix}128 & 622 & 207\end{bmatrix}\begin{pmatrix}140 & 44 & -159\\ 19 & -43 & 58\\ -65 & 102 & -18\end{pmatrix}=\frac{1}{857}\begin{bmatrix}16{,}283 & 0 & 11{,}998\end{bmatrix}$$

$$=\begin{bmatrix}19 & 0 & 14\end{bmatrix}$$

$$\frac{1}{857}\begin{bmatrix}180 & 613 & 290\end{bmatrix}\begin{pmatrix}140 & 44 & -159\\ 19 & -43 & 58\\ -65 & 102 & -18\end{pmatrix}=\frac{1}{857}\begin{bmatrix}17{,}997 & 11{,}141 & 1{,}714\end{bmatrix}$$

$$=\begin{bmatrix}21 & 13 & 2\end{bmatrix}$$

$$\frac{1}{857}\begin{bmatrix}102 & 360 & 259\end{bmatrix}\begin{pmatrix}140 & 44 & -159\\ 19 & -43 & 58\\ -65 & 102 & -18\end{pmatrix}=\frac{1}{857}\begin{bmatrix}4{,}285 & 15{,}426 & 0\end{bmatrix}$$

$$=\begin{bmatrix}5 & 18 & 0\end{bmatrix}$$

$$\frac{1}{857}\begin{bmatrix}151 & 580 & 297\end{bmatrix}\begin{pmatrix}140 & 44 & -159\\ 19 & -43 & 58\\ -65 & 102 & -18\end{pmatrix}=\frac{1}{857}\begin{bmatrix}12{,}855 & 11{,}998 & 4{,}285\end{bmatrix}$$

$$=\begin{bmatrix}15 & 14 & 5\end{bmatrix}$$

$$\begin{array}{cccccccccccccccccc}13 & 1 & 20 & 8 & 0 & 9 & 19 & 0 & 14 & 21 & 13 & 2 & 5 & 18 & 0 & 15 & 14 & 5\\ M & A & T & H & - & I & S & - & N & U & M & B & E & R & - & O & N & E\end{array}$$

The message is: ***Math is number one***

Exercise

Determine the key word, then decode the given cryptogram

5	17	21	1	20	9	15	14	19
259	863	783	77	378	357	301	448	565
106	266	318	325	365	485	301	522	653
326	653	738	10	566	495	115	640	555
290	791	762	115	474	507	119	332	279
305	454	513	339	645	611	226	341	426
260	338	368	406	657	830	270	649	590
110	337	418	74	318	330	261	561	469
114	426	390	160	543	372	89	535	441
323	842	783	97	344	245	84	601	444
424	851	944	175	262	339	379	698	755
226	341	426	37	454	217	156	694	536

Solution

The key word from the first row, because all the numbers are between 0 and 26, alphabetic letter.

Since it has 9 numbers, then the matrix is $9 = 3^2$ which is 3×3.

Therefore,

$$A = \begin{pmatrix} 5 & 17 & 21 \\ 1 & 20 & 9 \\ 15 & 14 & 19 \end{pmatrix}$$

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

The key word is:

5	17	21	1	20	9	15	14	19
E	*Q*	*U*	*A*	*T*	*I*	*O*	*N*	*S*

$$A = \begin{pmatrix} 5 & 17 & 21 \\ 1 & 20 & 9 \\ 15 & 14 & 19 \end{pmatrix}$$

$$|A| = -2,764$$

$$A^{-1} = -\frac{1}{2,764}\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix}$$

With the cryptogram:

$[259 \quad 863 \quad 783]$ $[77 \quad 378 \quad 357]$ $[301 \quad 448 \quad 565]$ $[106 \quad 266 \quad 318]$

$[325 \quad 365 \quad 485]$ $[301 \quad 522 \quad 653]$ $[326 \quad 653 \quad 738]$ $[103 \quad 566 \quad 495]$

$[115 \quad 640 \quad 555]$ $[290 \quad 791 \quad 762]$ $[115 \quad 474 \quad 507]$ $[119 \quad 332 \quad 279]$

$[305 \quad 454 \quad 513]$ $[339 \quad 645 \quad 611]$ $[226 \quad 341 \quad 426]$ $[260 \quad 338 \quad 368]$

$[406 \quad 657 \quad 830]$ $[270 \quad 649 \quad 590]$ $[110 \quad 337 \quad 418]$ $[74 \quad 318 \quad 330]$

$[261 \quad 561 \quad 469]$ $[114 \quad 426 \quad 390]$ $[160 \quad 543 \quad 372]$ $[89 \quad 535 \quad 441]$

$[323 \quad 842 \quad 783]$ $[97 \quad 344 \quad 245]$ $[84 \quad 601 \quad 444]$ $[424 \quad 851 \quad 944]$

$[175 \quad 262 \quad 339]$ $[379 \quad 698 \quad 755]$ $[226 \quad 341 \quad 426]$ $[37 \quad 454 \quad 217]$

$[156 \quad 694 \quad 536]$

To decode a message given the matrix *A*.

$$-\frac{1}{2,764}\begin{bmatrix}259 & 863 & 783\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-58,044 & -52,516 & -24,876\end{bmatrix}$$

$$= \begin{bmatrix}21 & 19 & 9\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}77 & 378 & 357\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-38,696 & -19,348 & 0\end{bmatrix}$$

$$= \begin{bmatrix}14 & 7 & 0\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}301 & 448 & 565\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-33,168 & -2,764 & -44,224\end{bmatrix}$$

$$= \begin{bmatrix}12 & 1 & 16\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}106 & 266 & 318\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-33,168 & -2,764 & -8,292\end{bmatrix}$$

$$= \begin{bmatrix}12 & 1 & 3\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}325 & 365 & 485\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-13,820 & 0 & -55,280\end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 20 \end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix} 301 & 522 & 653 \end{bmatrix}\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix} -49{,}752 & -2{,}764 & -38{,}696 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 1 & 14 \end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix} 326 & 653 & 738 \end{bmatrix}\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix} -52{,}516 & -16{,}584 & -41{,}460 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 6 & 15 \end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix} 103 & 566 & 495 \end{bmatrix}\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix} -49{,}752 & -35{,}932 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 13 & 0 \end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix} 115 & 640 & 555 \end{bmatrix}\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix} -55{,}280 & -41{,}460 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 15 & 0 \end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix} 290 & 791 & 762 \end{bmatrix}\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix} -52{,}516 & -41{,}460 & -33{,}168 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 15 & 12 \end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix} 115 & 474 & 507 \end{bmatrix}\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix} -60{,}808 & -13{,}820 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 5 & 0 \end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix} 119 & 332 & 279 \end{bmatrix}\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix} -11{,}056 & -24{,}876 & -16{,}584 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 9 & 6 \end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix} 305 & 454 & 513 \end{bmatrix}\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix} -16{,}584 & -13{,}820 & -49{,}752 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 5 & 18 \end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}339 & 645 & 611\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-13,820 & -38,696 & -55,280\end{bmatrix}$$
$$= \begin{bmatrix}5 & 14 & 20\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}226 & 341 & 426\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-24,876 & -2,764 & -33,168\end{bmatrix}$$
$$= \begin{bmatrix}9 & 1 & 12\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}260 & 338 & 368\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}0 & 13,820 & -46,988\end{bmatrix}$$
$$= \begin{bmatrix}0 & 5 & 17\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}406 & 657 & 830\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-58,044 & -2,764 & -55,280\end{bmatrix}$$
$$= \begin{bmatrix}21 & 1 & 20\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}270 & 649 & 590\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-24,876 & -41,460 & -38,696\end{bmatrix}$$
$$= \begin{bmatrix}9 & 15 & 14\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}110 & 337 & 418\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-52,516 & 0 & -2,764\end{bmatrix}$$
$$= \begin{bmatrix}19 & 0 & 1\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}74 & 318 & 330\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-38,696 & -11,056 & 0\end{bmatrix}$$
$$= \begin{bmatrix}14 & 4 & 0\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}261 & 561 & 469\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-2,764 & -44,224 & -44,224\end{bmatrix}$$
$$= \begin{bmatrix}1 & 16 & 16\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}114 & 426 & 390\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-33,168 & -24,876 & -8,292\end{bmatrix}$$
$$=\begin{bmatrix}12 & 9 & 3\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}160 & 543 & 372\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-2,764 & -55,280 & -24.876\end{bmatrix}$$
$$=\begin{bmatrix}1 & 20 & 9\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}89 & 535 & 441\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-41,460 & -38,696 & 0\end{bmatrix}$$
$$=\begin{bmatrix}15 & 14 & 0\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}323 & 842 & 783\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-44,224 & -49,752 & -41,460\end{bmatrix}$$
$$=\begin{bmatrix}16 & 18 & 15\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}97 & 344 & 245\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-5,528 & -33,168 & -13,820\end{bmatrix}$$
$$=\begin{bmatrix}2 & 12 & 5\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}84 & 601 & 444\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-35,932 & -52,516 & 0\end{bmatrix}$$
$$=\begin{bmatrix}13 & 19 & 0\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}424 & 851 & 944\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-63,572 & -24,876 & -55,280\end{bmatrix}$$
$$=\begin{bmatrix}23 & 9 & 20\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}175 & 262 & 339\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-22,112 & 0 & -24,876\end{bmatrix}$$
$$=\begin{bmatrix}8 & 0 & 9\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}379 & 698 & 755\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-38,696 & -24,876 & -55,280\end{bmatrix}$$

$$= \begin{bmatrix}14 & 9 & 20\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}226 & 341 & 426\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-24,876 & -2,764 & -33,168\end{bmatrix}$$

$$= \begin{bmatrix}9 & 1 & 12\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}37 & 454 & 217\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}0 & -60,808 & -2,764\end{bmatrix}$$

$$= \begin{bmatrix}0 & 22 & 1\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}156 & 694 & 536\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-33,168 & -58,044 & -13,820\end{bmatrix}$$

$$= \begin{bmatrix}12 & 21 & 5\end{bmatrix}$$

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

21	19	9	14	7	0	12	1	16	12	1	3	5	0	20	18	1	14	19	6	15	
U	*S*	*I*	*N*	*G*	–	*L*	*A*	*P*	*L*	*A*	*C*	*E*	–	*T*	*R*	*A*	*N*	*S*	*F*	*O*	
18	13	0	20	15	0	19	15	12	22	5	0	4	9	6	6	5	18	5	14	20	
R	*M*	–	*T*	*O*	–	*S*	*O*	*L*	*V*	*E*	–	*D*	*I*	*F*	*F*	*E*	*R*	*E*	*N*	*T*	
9	1	12	0	5	17	21	1	20	9	15	14	19	0	1	14	4	0	1	16	16	
I	*A*	*L*	–	*E*	*Q*	*U*	*A*	*T*	*I*	*O*	*N*	*S*	–	*A*	*N*	*D*	–	*A*	*P*	*P*	
12	9	3	1	20	9	15	14	0	16	18	15	2	12	5	13	19	0	23	9	20	
L	*I*	*C*	*A*	*T*	*I*	*O*	*N*	–	*P*	*R*	*O*	*B*	*L*	*E*	*M*	*S*	–	*W*	*I*	*T*	
8	0	9	14	9	20	9	1	12	0	22	1	12	21	5							
H	–	*I*	*N*	*I*	*T*	*I*	*A*	*L*	–	*V*	*A*	*L*	*U*	*E*							

The message is:

Using Laplace Transform to Solve Differential Equations and Application Problems with Initial Value