Solution   *Section* 4.5 – Divergence and Curl

Exercise

Find the divergence of the following vector field $\boldsymbol{F} = \langle 2x,\ 4y,\ -3z \rangle$

Solution

$$div\boldsymbol{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(4y) + \frac{\partial}{\partial z}(-3z) \qquad div\boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= 2 + 4 - 3$$

$$\underline{= 3}$$

Exercise

Find the divergence of the following vector field $\boldsymbol{F} = \langle -2y,\ 3x,\ z \rangle$

Solution

$$div\boldsymbol{F} = \frac{\partial}{\partial x}(-2y) + \frac{\partial}{\partial y}(3x) + \frac{\partial}{\partial z}(z) \qquad div\boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$\underline{= 1}$$

Exercise

Find the divergence of the following vector field $\boldsymbol{F} = \left\langle x^2yz,\ -xy^2z,\ -xyz^2 \right\rangle$

Solution

$$div\boldsymbol{F} = \frac{\partial}{\partial x}\left(x^2yz\right) + \frac{\partial}{\partial y}\left(-xy^2z\right) + \frac{\partial}{\partial z}\left(-xyz^2\right) \qquad div\boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= 2xyz - 2xyz - 2xyz$$

$$\underline{= -2xyz}$$

Exercise

Find the divergence of the following vector field $\boldsymbol{F} = \left\langle x^2 - y^2,\ y^2 - z^2,\ z^2 - x^2 \right\rangle$

Solution

$$div\boldsymbol{F} = \frac{\partial}{\partial x}\left(x^2 - y^2\right) + \frac{\partial}{\partial y}\left(y^2 - z^2\right) + \frac{\partial}{\partial z}\left(z^2 - x^2\right) \qquad div\boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$\underline{= 2x + 2y + 2z}$$

Exercise

Find the divergence of the following vector field $\boldsymbol{F} = \left\langle e^{-x+y},\ e^{-y+z},\ e^{-z+x} \right\rangle$

Solution

$$div\boldsymbol{F} = \frac{\partial}{\partial x}\left(e^{-x+y}\right) + \frac{\partial}{\partial y}\left(e^{-y+z}\right) + \frac{\partial}{\partial z}\left(e^{-z+x}\right) \qquad div\boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= -e^{-x+y} - e^{-y+z} - e^{-z+x}$$

Exercise

Find the divergence of the following vector field $\boldsymbol{F} = \langle yz\cos x,\ xz\cos y,\ xy\cos z \rangle$

Solution

$$div\boldsymbol{F} = \frac{\partial}{\partial x}(yz\cos x) + \frac{\partial}{\partial y}(xz\cos y) + \frac{\partial}{\partial z}(xy\cos z)$$

$$= -yz\sin x - xz\sin y - xy\sin z$$

Exercise

Find the divergence of the following vector field $\boldsymbol{F} = \langle 12x,\ 4y,\ -3z \rangle$

Solution

$$div\boldsymbol{F} = \frac{\partial}{\partial x}(12x) + \frac{\partial}{\partial y}(4y) + \frac{\partial}{\partial z}(-3z) \qquad div\boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= 12 + 4 - 3$$

$$= 13$$

Exercise

Find the divergence of the following vector field $\boldsymbol{F} = \dfrac{\langle x,\ y,\ z \rangle}{1 + x^2 + y^2}$

Solution

$$div\boldsymbol{F} = \frac{\partial}{\partial x}\left(\frac{x}{1 + x^2 + y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{1 + x^2 + y^2}\right) + \frac{\partial}{\partial z}\left(\frac{z}{1 + x^2 + y^2}\right) \qquad div\boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= \frac{1 + x^2 + y^2 - 2x^2}{\left(1 + x^2 + y^2\right)^2} + \frac{1 + x^2 + y^2 - 2y^2}{\left(1 + x^2 + y^2\right)^2} + \frac{1}{\left(1 + x^2 + y^2\right)^2}$$

$$= \frac{3}{\left(1 + x^2 + y^2\right)^2}$$

Exercise

Calculate the divergence of the radial fields. $\quad \boldsymbol{F} = \dfrac{\langle x,\ y,\ z\rangle}{x^2+y^2+z^2} = \dfrac{\boldsymbol{r}}{|\boldsymbol{r}|^2}$

Express the result in terms of the position vector $\boldsymbol{r}$ and its length $|\boldsymbol{r}|$.

Solution

$$\nabla\cdot\boldsymbol{F} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right)+\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2+z^2}\right)+\frac{\partial}{\partial z}\left(\frac{z}{x^2+y^2+z^2}\right)$$

$$= \frac{-x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^2}+\frac{x^2-y^2+z^2}{\left(x^2+y^2+z^2\right)^2}+\frac{x^2+y^2-z^2}{\left(x^2+y^2+z^2\right)^2} \qquad \left(\frac{x}{x^2+y^2+z^2}\right)' = \frac{x^2+y^2+z^2-2x^2}{\left(x^2+y^2+z^2\right)^2}$$

$$= \frac{x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^2}$$

$$\boxed{= \frac{\boldsymbol{r}}{|\boldsymbol{r}|^2}}$$

Exercise

Calculate the divergence of the radial fields. $\quad \boldsymbol{F} = \langle x,\ y,\ z\rangle\left(x^2+y^2+z^2\right) = 5|\boldsymbol{r}|^2$

Express the result in terms of the position vector $\boldsymbol{r}$ and its length $|\boldsymbol{r}|$.

Solution

$$\nabla\cdot\boldsymbol{F} = \frac{\partial}{\partial x}\left(x\left(x^2+y^2+z^2\right)\right)+\frac{\partial}{\partial y}\left(y\left(x^2+y^2+z^2\right)\right)+\frac{\partial}{\partial z}\left(z\left(x^2+y^2+z^2\right)\right)$$

$$= \frac{\partial}{\partial x}\left(x^3+xy^2+xz^2\right)+\frac{\partial}{\partial y}\left(x^2y+y^3+yz^2\right)+\frac{\partial}{\partial z}\left(x^2z+y^2z+z^3\right)$$

$$= 3x^2+y^2+z^2+x^2+3y^2+z^2+x^2+y^2+3z^2$$

$$= 5\left(x^2+y^2+z^2\right)$$

$$\boxed{= 5|\boldsymbol{r}|^2}$$

Exercise

Calculate the divergence of the radial fields. $\quad \boldsymbol{F} = \dfrac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^{3/2}} = \dfrac{\boldsymbol{r}}{|\boldsymbol{r}|^3}$

Express the result in terms of the position vector $\boldsymbol{r}$ and its length $|\boldsymbol{r}|$.

Solution

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}} + \frac{\partial}{\partial y}\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}} + \frac{\partial}{\partial z}\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$= \frac{x^2+y^2+z^2-3x^2}{\left(x^2+y^2+z^2\right)^{5/2}} + \frac{x^2+y^2+z^2-3y^2}{\left(x^2+y^2+z^2\right)^{5/2}} + \frac{x^2+y^2+z^2-3z^2}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$= 0$$

Exercise

Calculate the divergence of the radial fields. $\boldsymbol{F} = \dfrac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^2} = \dfrac{\boldsymbol{r}}{|\boldsymbol{r}|^4}$

Express the result in terms of the position vector $\boldsymbol{r}$ and its length $|\boldsymbol{r}|$.

Solution

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}\frac{x}{\left(x^2+y^2+z^2\right)^{2}} + \frac{\partial}{\partial y}\frac{y}{\left(x^2+y^2+z^2\right)^{2}} + \frac{\partial}{\partial z}\frac{z}{\left(x^2+y^2+z^2\right)^{2}}$$

$$= \frac{x^2+y^2+z^2-4x^2}{\left(x^2+y^2+z^2\right)^{3}} + \frac{x^2+y^2+z^2-4y^2}{\left(x^2+y^2+z^2\right)^{3}} + \frac{x^2+y^2+z^2-4z^2}{\left(x^2+y^2+z^2\right)^{3}}$$

$$= -\frac{x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^{3}}$$

$$= -\frac{1}{|\vec{r}|^4}$$

Exercise

Consider the following vector fields $\boldsymbol{F} = \langle x,\ x+y\rangle$, the circle C, and two points P and Q.

a) Without computing the divergence, does the graph suggest that the divergence is positive or negative at P and Q?

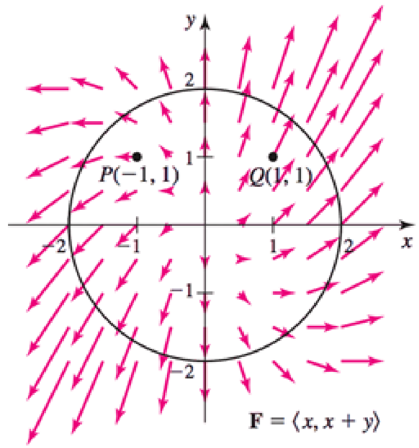
b) Compute the divergence and confirm your conjecture in part (a).

c) On what part of C is the flux outward? Inward?

d) Is the net outward flux across C positive or negative?vector

Solution

a) At both P and Q, the arrows going away from the point are larger in both number and magnitude than those going in, so we would expect the divergence to be positive at both points.



b) $\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(x+y) = 1+1 \underline{=2|} > 0$

It is positive everywhere.

c) The arrows all point roughly away from the origin, so we the flux is outward everywhere.

d) The net flux across C should be positive.

Exercise

Consider the following vector fields $\boldsymbol{F} = \left\langle x,\ y^2 \right\rangle$, the circle C, and two points P and Q.

a) Without computing the divergence, does the graph suggest that the divergence is positive or negative at P and Q?

b) Compute the divergence and confirm your conjecture in part (a).

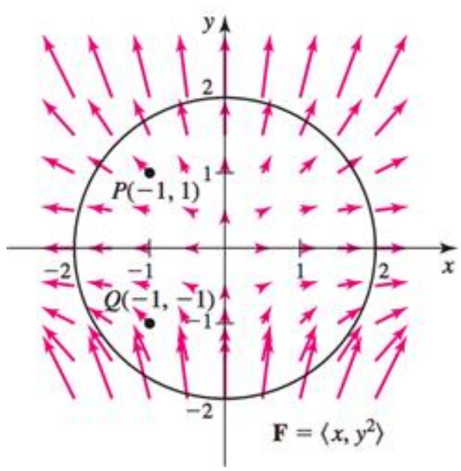
c) On what part of C is the flux outward? Inward?

d) Is the net outward flux across C positive or negative?

Solution

a) At P, the divergence should be positive.

At Q, the larger arrows point in towards Q, so the divergence should be negative.



b) $\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}\left(y^2\right) \underline{= 1+2y|}$

At $P=(-1,\ 1)\ \rightarrow\ \nabla\cdot\boldsymbol{F}\ \underline{=3}$ At $Q=(-1,\ -1)\ \rightarrow\ \nabla\cdot\boldsymbol{F}\ \underline{=-1}$

c) The flux is outward above the line $y=-1$; below this line, the flux is inward across *C*.

d) The size of the narrows pointing outward at the top of the circle seems to roughly equal those pointing inward at the bottom, so the remaining outward-pointing arrows result in a net positive flux across *C*.

Exercise

Consider the vector fields $\boldsymbol{F}=\langle 1,\ 0,\ 0\rangle\times\boldsymbol{r}$, where $\boldsymbol{r}=\langle x,\ y,\ z\rangle$

a) Compute the curl field and verify that it has the same direction as the axis of rotation

b) Compute the magnitude of the curl of the field

Solution

a) $\nabla\times\boldsymbol{F}=\nabla\times\left[\langle 1,\ 0,\ 0\rangle\times\langle x,\ y,\ z\rangle\right]$

$$=\nabla\times\begin{vmatrix}\hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}}\\ 1 & 0 & 0\\ x & y & z\end{vmatrix}$$

$$=\nabla\times\left(-z\,\hat{\boldsymbol{j}}+y\,\hat{\boldsymbol{k}}\right)=\begin{vmatrix}\hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & -z & y\end{vmatrix}$$

$$\nabla\times(f,g,h)=\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)\hat{\boldsymbol{i}}+\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)\hat{\boldsymbol{j}}+\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)\hat{\boldsymbol{k}}$$

$$=(1+1)\hat{\boldsymbol{i}}+(0-0)\hat{\boldsymbol{j}}+(0-0)\hat{\boldsymbol{k}}$$

$$\underline{=2\hat{\boldsymbol{i}}}$$

The curl is the same direction as the axis of rotation.

b) The magnitude of the curl is $\left|2\hat{\boldsymbol{i}}\right|\ \underline{=2}$

Exercise

Consider the vector fields $\boldsymbol{F}=\langle 1,\ -1,\ 0\rangle\times\boldsymbol{r}$, where $\boldsymbol{r}=\langle x,\ y,\ z\rangle$

a) Compute the curl field and verify that it has the same direction as the axis of rotation

b) Compute the magnitude of the curl of the field

Solution

a) $\nabla\times\boldsymbol{F}=\nabla\times\left[\langle 1,\ -1,\ 0\rangle\times\langle x,\ y,\ z\rangle\right]$

$$=\nabla\times\begin{vmatrix}\hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}}\\ 1 & -1 & 0\\ x & y & z\end{vmatrix}$$

$$= \nabla \times \left(-z\,\hat{\boldsymbol{i}} - z\,\hat{\boldsymbol{j}} + (x+y)\,\hat{\boldsymbol{k}} \right) \qquad \nabla \times (f, g, h) = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right)\hat{\boldsymbol{i}} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right)\hat{\boldsymbol{j}} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)\hat{\boldsymbol{k}}$$

$$= (1+1)\hat{\boldsymbol{i}} + (-1-1)\hat{\boldsymbol{j}} + (0-0)\hat{\boldsymbol{k}}$$

$$= 2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$$

The curl is the same direction as the axis of rotation.

b) The magnitude of the curl is $\left| 2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} \right| = 2\sqrt{2}$

Exercise

Consider the vector fields $\boldsymbol{F} = \langle 1,\ -1,\ 1 \rangle \times \boldsymbol{r}$, where $\boldsymbol{r} = \langle x,\ y,\ z \rangle$

a) Compute the curl field and verify that it has the same direction as the axis of rotation

b) Compute the magnitude of the curl of the field

Solution

a) $\nabla \times \boldsymbol{F} = \nabla \times \left[\langle 1,\ -1,\ 1 \rangle \times \langle x,\ y,\ z \rangle \right]$

$$= \nabla \times \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 1 & -1 & 1 \\ x & y & z \end{vmatrix}$$

$$= \nabla \times \left((-z-y)\,\hat{\boldsymbol{i}} + (x-z)\,\hat{\boldsymbol{j}} + (x+y)\,\hat{\boldsymbol{k}} \right) \qquad \nabla \times (f, g, h) = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right)\hat{\boldsymbol{i}} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right)\hat{\boldsymbol{j}} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)\hat{\boldsymbol{k}}$$

$$= (1+1)\hat{\boldsymbol{i}} + (-1-1)\hat{\boldsymbol{j}} + (1+1)\hat{\boldsymbol{k}}$$

$$= 2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$$

The curl is the same direction as the axis of rotation.

b) The magnitude of the curl is $\left| 2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}} \right| = 2\sqrt{3}$

Exercise

Consider the vector fields $\boldsymbol{F} = \langle 1,\ -2,\ -3 \rangle \times \boldsymbol{r}$, where $\boldsymbol{r} = \langle x,\ y,\ z \rangle$

a) Compute the curl field and verify that it has the same direction as the axis of rotation

b) Compute the magnitude of the curl of the field

Solution

a) $\nabla \times \boldsymbol{F} = \nabla \times \left[\langle 1,\ -2,\ -3 \rangle \times \langle x,\ y,\ z \rangle \right]$

$$= \nabla \times \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 1 & -2 & -3 \\ x & y & z \end{vmatrix}$$

$$= \nabla \times \left((-2z+3y)\,\hat{\boldsymbol{i}} + (-3x-z)\,\hat{\boldsymbol{j}} + (y+2x)\,\hat{\boldsymbol{k}} \right)$$

$$= (1+1)\hat{\boldsymbol{i}} + (-2-2)\hat{\boldsymbol{j}} + (-3-3)\hat{\boldsymbol{k}}$$

$$\boxed{= 2\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}}}$$

The curl is the same direction as the axis of rotation.

b) The magnitude of the curl is

$$\left|2\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}}\right| \boxed{= 2\sqrt{14}}$$

Exercise

Compute the curl of the vector field $\vec{F} = \left\langle x^2 - y^2,\ xy,\ z \right\rangle$

Solution

$$curl\ \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & xy & z \end{vmatrix} \qquad curl\ \vec{F} = \begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(xy)\right)\hat{i} + \left(\frac{\partial}{\partial z}\left(x^2 - y^2\right) - \frac{\partial}{\partial x}(z)\right)\hat{j} + \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}\left(x^2 - y^2\right)\right)\hat{k}$$

$$= (0-0)\hat{i} + (0-0)\hat{j} + (y+2y)\hat{k}$$

$$\boxed{= 3y\,\hat{k}}$$

Exercise

Compute the curl of the vector field $\vec{F} = \left\langle 0,\ z^2 - y^2,\ -yz \right\rangle$

Solution

$$curl\ \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & z^2 - y^2 & -yz \end{vmatrix} \qquad curl\ \vec{F} = \begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(-yz) - \frac{\partial}{\partial z}\left(z^2 - y^2\right)\right)\hat{i} + \left(\frac{\partial}{\partial z}(0) - \frac{\partial}{\partial x}(-yz)\right)\hat{j} + \left(\frac{\partial}{\partial x}\left(z^2 - y^2\right) - \frac{\partial}{\partial y}(0)\right)\hat{k}$$

$$= (-z-2z)\hat{i} + (0-0)\hat{j} + (0-0)\hat{k}$$

$$\boxed{= -3z\,\hat{i}}$$

Exercise

Compute the curl of the vector field $\vec{F}=\left\langle z^2\sin y,\ xz^2\cos y,\ 2xz\sin y\right\rangle$

Solution

$$curl\ \vec{F}=\begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2\sin y & xz^2\cos y & 2xz\sin y \end{vmatrix} \qquad curl\ \vec{F}=\begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$=\left(2xz\cos y-2xz\cos y\right)\hat{\imath}+\left(2z\sin y-2z\sin y\right)\hat{\jmath}+\left(z^2\cos y-z^2\cos y\right)\hat{k}$$

$$\underline{=0}$$

Exercise

Compute the curl of the vector field $\vec{F}=\dfrac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^{3/2}}=\dfrac{\vec{r}}{|\vec{r}|^3}$

Solution

$$curl\ \vec{F}=\begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\left(x^2+y^2+z^2\right)^{3/2}} & \frac{y}{\left(x^2+y^2+z^2\right)^{3/2}} & \frac{z}{\left(x^2+y^2+z^2\right)^{3/2}} \end{vmatrix} \qquad curl\ \vec{F}=\begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$\left(U^nV^m\right)'=U^{n-1}V^{m-1}\left(nU'V+mUV'\right)$$

$$\left(z\left(x^2+y^2+z^2\right)^{-3/2}\right)'_y=(1)\left(x^2+y^2+z^2\right)^{-5/2}\left[(0)\left(x^2+y^2+z^2\right)-\frac{3}{2}z(2y)\right]$$

$$=\left(x^2+y^2+z^2\right)^{-5/2}(-3yz)$$

$$=\frac{1}{\left(x^2+y^2+z^2\right)^{5/2}}\left((-3yz+3yz)\hat{\boldsymbol{i}}+(-3xz+3xz)\hat{\boldsymbol{j}}+(-3xy+3xy)\hat{\boldsymbol{k}}\right)$$

$$\underline{=0}$$

Exercise

Compute the curl of the vector field $\vec{F} = \vec{r} = \langle x,\ y,\ z \rangle$

Solution

$$curl\ \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$curl\ \vec{F} = \begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= (0-0)\hat{\boldsymbol{i}} + (0-0)\hat{\boldsymbol{j}} + (0-0)\hat{\boldsymbol{k}}$$

$$= 0$$

Exercise

Compute the curl of the vector field $\vec{F} = \left\langle 3xz^3 e^{y^2},\ 2xz^3 e^{y^2},\ 3xz^2 e^{y^2} \right\rangle$

Solution

$$curl\ \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xz^3 e^{y^2} & 2xz^3 e^{y^2} & 3xz^2 e^{y^2} \end{vmatrix}$$

$$curl\ \vec{F} = \begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left(6xyz^3 e^{y^2} - 6xz^2 e^{y^2}\right)\hat{\boldsymbol{i}} + \left(9xz^2 e^{y^2} - 3z^2 e^{y^2}\right)\hat{\boldsymbol{j}} + \left(2z^3 e^{y^2} - 6xyz^3 e^{y^2}\right)\hat{\boldsymbol{k}}$$

$$= z^2 e^{y^2}\left[(6xyz - 6x)\hat{\boldsymbol{i}} + (9x-3)\hat{\boldsymbol{j}} + (2z - 6xyz)\hat{\boldsymbol{k}}\right]$$

Exercise

Compute the curl of the vector field $\vec{F} = \left\langle x^2 - z^2,\ 1,\ 2xz \right\rangle$

Solution

$$curl\ \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - z^2 & 1 & 2xz \end{vmatrix}$$

$$curl\ \vec{F} = \begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= (0-0)\hat{i} + (-2z - 2z)\hat{j} + (0-0)\hat{k}$$

$$= -4z\,\hat{\boldsymbol{j}}$$

Exercise

Compute the curl of the vector field $\vec{F} = \dfrac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^{1/2}} = \dfrac{\vec{r}}{|\vec{r}|}$

Solution

$$curl\ \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\left(x^2+y^2+z^2\right)^{1/2}} & \frac{y}{\left(x^2+y^2+z^2\right)^{1/2}} & \frac{z}{\left(x^2+y^2+z^2\right)^{1/2}} \end{vmatrix}$$

$$= \frac{1}{\left(x^2+y^2+z^2\right)^{3/2}}\Big((-yz+yz)\,\hat{i} + (-xz+xz)\,\hat{j} + (-xy+xy)\,\hat{k}\Big)$$

$$\underline{=0}$$

$$\left(U^n V^m\right)' = U^{n-1}V^{m-1}\left(nU'V + mUV'\right)$$

Exercise

Compute the divergence and curl of the following vector fields, state whether the field is *source-free* or *irrotational*. $\vec{F} = \langle yz,\ xz,\ xy\rangle$

Solution

$$div\vec{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy)$$

$$div\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$\underline{=0}$$

$$curl\ \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$curl\ \vec{F} = \begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz)\right)\hat{i} + \left(\frac{\partial}{\partial z}(yz) - \frac{\partial}{\partial x}(xy)\right)\hat{j} + \left(\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(yz)\right)\hat{k}$$

$$= (x-x)\,\hat{i} + (y-y)\,\hat{j} + (z-z)\,\hat{k}$$

$$\underline{=\vec{0}}$$

$\therefore$ The field is both source-free and irrotational.

Exercise

Compute the divergence and curl of the following vector fields, state whether the field is *source-free* or *irrotational*. $\vec{F} = \vec{r}\,|\vec{r}| = \langle x,\ y,\ z\rangle\sqrt{x^2+y^2+z^2}$

Solution

$$\frac{\partial}{\partial x}\left(x\sqrt{x^2+y^2+z^2}\right) = \frac{x^2+y^2+z^2+x^2}{\sqrt{x^2+y^2+z^2}} \qquad \left(U^m V^n\right)' = U^{m-1}V^{n-1}\left(mU'V+nUV'\right)$$

$$= \frac{|\vec{r}|^2+x^2}{|\vec{r}|}$$

$$\frac{\partial}{\partial y}\left(y\sqrt{x^2+y^2+z^2}\right) = \frac{x^2+y^2+z^2+y^2}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{|\vec{r}|^2+y^2}{|\vec{r}|}$$

$$\frac{\partial}{\partial z}\left(z\sqrt{x^2+y^2+z^2}\right) = \frac{x^2+y^2+z^2+z^2}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{|\vec{r}|^2+z^2}{|\vec{r}|}$$

$$div\vec{F} = \frac{|\vec{r}|^2+x^2}{|\vec{r}|}+\frac{|\vec{r}|^2+y^2}{|\vec{r}|}+\frac{|\vec{r}|^2+z^2}{|\vec{r}|} \qquad div\vec{F} = \frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}$$

$$= \frac{3|\vec{r}|^2+x^2+y^2+z^2}{|\vec{r}|}$$

$$= \frac{4|\vec{r}|^2}{|\vec{r}|}$$

$$\boxed{= 4|\vec{r}|}$$

$$curl\ \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x|\vec{r}| & y|\vec{r}| & z|\vec{r}| \end{vmatrix} \qquad curl\ \vec{F} = \begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}\left(z|\vec{r}|\right) - \frac{\partial}{\partial z}\left(y|\vec{r}|\right)\right)\hat{i} + \left(\frac{\partial}{\partial z}\left(x|\vec{r}|\right) - \frac{\partial}{\partial x}\left(z|\vec{r}|\right)\right)\hat{j} + \left(\frac{\partial}{\partial x}\left(y|\vec{r}|\right) - \frac{\partial}{\partial y}\left(x|\vec{r}|\right)\right)\hat{k}$$

$$= \left(\frac{yz}{|\vec{r}|} - \frac{yz}{|\vec{r}|}\right)\hat{i} + \left(\frac{xz}{|\vec{r}|} - \frac{zx}{|\vec{r}|}\right)\hat{j} + \left(\frac{yx}{|\vec{r}|} - \frac{xy}{|\vec{r}|}\right)\hat{k}$$

$$= \vec{0}$$

$\therefore$ The field is irrotational but not a source-free.

Exercise

Compute the divergence and curl of the following vector fields, state whether the field is *source-free* or *irrotational*. $\vec{F} = \langle \sin xy,\ \cos yz,\ \sin xz \rangle$

Solution

$$div\vec{F} = \frac{\partial}{\partial x}(\sin xy) + \frac{\partial}{\partial y}(\cos yz) + \frac{\partial}{\partial z}(\sin xz) \qquad div\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= y\cos xy - z\sin yz + x\cos xz$$

$$curl\ \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin xy & \cos yz & \sin xz \end{vmatrix} \qquad curl\ \vec{F} = \begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(\sin xz) - \frac{\partial}{\partial z}(\cos yz)\right)\hat{i} + \left(\frac{\partial}{\partial z}(\sin xy) - \frac{\partial}{\partial x}(\sin xz)\right)\hat{j} + \left(\frac{\partial}{\partial x}(\cos yz) - \frac{\partial}{\partial y}(\sin xy)\right)\hat{k}$$

$$= (0 + y\,\text{sinyz})\hat{i} + (0 - z\cos xz)\hat{j} + (0 - x\cos xy)\hat{k}$$

$$= y\,\text{sinyz}\,\hat{i} - z\cos xz\hat{j} - x\cos xy\hat{k}$$

$\therefore$ The field is neither source-free nor irrotational.

Exercise

Compute the divergence and curl of the following vector fields, state whether the field is *source-free* or *irrotational*. $\vec{F} = \left\langle 2xy + z^4,\ x^2,\ 4xz^3 \right\rangle$

Solution

$$div\vec{F} = \frac{\partial}{\partial x}\left(2xy + z^4\right) + \frac{\partial}{\partial y}\left(x^2\right) + \frac{\partial}{\partial z}\left(4xz^3\right) \qquad div\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= 2y + 12xz^2$$

$$curl\ \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^4 & x^2 & 4xz^3 \end{vmatrix} \qquad curl\ \vec{F} = \begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}\left(4xz^3\right) - \frac{\partial}{\partial z}\left(x^2\right)\right)\hat{i} + \left(\frac{\partial}{\partial z}\left(2xy+z^4\right) - \frac{\partial}{\partial x}\left(4xz^3\right)\right)\hat{j} + \left(\frac{\partial}{\partial x}\left(x^2\right) - \frac{\partial}{\partial y}\left(2xy+z^4\right)\right)\hat{k}$$

$$= (0)\hat{i} + \left(4z^3 - 4z^3\right)\hat{j} + (2x-2x)\hat{k}$$

$$= \vec{0}$$

∴ The field is irrotational but not a source-free.

Exercise

Let $\vec{F} = \langle z,\ x,\ -y\rangle$

a) What are the components of curl $\vec{F}$ in the directions $\vec{n} = \langle 1,\ 0,\ 0\rangle$ and $\vec{n} = \left\langle 0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right\rangle$

b) In what direction is the scalar component of curl $\vec{F}$ a maximum?

Solution

a) $curl\ \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & -y \end{vmatrix} \qquad curl\ \vec{F} = \begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$

$$= -\hat{i} + \hat{j} + \hat{k}$$

Scalar component in the direction of $\vec{n} = \langle 1,\ 0,\ 0\rangle$

$$\langle -1,\ 1,\ 1\rangle \bullet \langle 1,\ 0,\ 0\rangle = -1$$

Scalar component in the direction of $\vec{n} = \left\langle 0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right\rangle$

$$\langle -1,\ 1,\ 1\rangle \bullet \left\langle 0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right\rangle = 0$$

b) The scalar component of the curl is a maximum in the direction of the curl, in the direction $\langle -1,\ 1,\ 1\rangle$ whose unit direction vector is $\frac{1}{\sqrt{3}}\langle -1,\ 1,\ 1\rangle$

Exercise

Let $\vec{F} = \langle z,\ 0,\ -y \rangle$

a) What are the components of curl $\vec{F}$ in the directions $\vec{n} = \langle 1,\ 0,\ 0 \rangle$ and $\vec{n} = \langle 1,\ -1,\ 1 \rangle$

b) In what direction $\vec{n}$ is $\left(curl\ \vec{F} \right) \bullet \vec{n}$ a maximum?

Solution

a) $curl\ \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & 0 & -y \end{vmatrix}$

$$curl\ \vec{F} = \begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$= -\hat{\imath} + \hat{\jmath}$

Scalar component in the direction of $\vec{n} = \langle 1,\ 0,\ 0 \rangle$

$$\langle -1,\ 1,\ 0 \rangle \bullet \langle 1,\ 0,\ 0 \rangle \ \underline{= -1}$$

Scalar component in the direction of $\vec{n} = \langle 1,\ -1,\ 1 \rangle$

$$\langle -1,\ 1,\ 0 \rangle \bullet \langle 1,\ -1,\ 1 \rangle \ \underline{= -2}$$

b) The component of the curl is a maximum in the direction of the curl, in the direction $\langle -1,\ 1,\ 0 \rangle$ whose unit direction vector is $\frac{1}{\sqrt{2}} \langle -1,\ 1,\ 0 \rangle$

Exercise

Within the cube $\{(x,\ y,\ z):\ -1 \le x \le 1,\ \ -1 \le y \le 1,\ \ -1 \le z \le 1\}$, where does $div\vec{F}$ have the greatest magnitude when $\vec{F} = \left\langle x^2 - y^2,\ \ xy^2 z,\ \ 2xz \right\rangle$

Solution

$$div\vec{F} = \frac{\partial}{\partial x}\left(x^2 - y^2\right) + \frac{\partial}{\partial y}\left(xy^2 z\right) + \frac{\partial}{\partial z}(2xz)$$

$$div\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= 2x + 2xyz + 2x$$

$$\underline{= 4x + 2xyz}$$

For the greatest magnitude, x and y & z have the same sign

$(-1,\ -1,\ -1)$ $\quad \left|div\vec{F}\right| = |-4 - 2| \ \underline{= 6}$

$(-1,\ 1,\ 1)$ $\quad \left|div\vec{F}\right| = |-4 - 2| \ \underline{= 6}$

$(1,\ 1,\ 1)$ $\quad \left|div\vec{F}\right| = |4 + 2| \ \underline{= 6}$

$(1,\ -1,\ -1)\quad \left|div\vec{F}\right| = |4+2|\ \underline{= 6}$

$div\vec{F}$ have the greatest magnitude of 6.

Exercise

Show that the general rotation field $\boldsymbol{F} = \boldsymbol{a} \times \boldsymbol{r}$, where $\boldsymbol{a}$ is a nonzero constant vector and $\boldsymbol{r} = \langle x,\ y,\ z \rangle$, has zero divergence.

Solution

Let $\boldsymbol{a} = \langle a_1,\ a_2,\ a_3 \rangle$

$$\boldsymbol{F} = \boldsymbol{a} \times \boldsymbol{r} = \langle a_1,\ a_2,\ a_3 \rangle \times \langle x,\ y,\ z \rangle$$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= \left(a_2 z - a_3 y\right)\hat{\boldsymbol{i}} + \left(a_3 x - a_1 z\right)\hat{\boldsymbol{j}} + \left(a_1 y - a_2 x\right)\hat{\boldsymbol{k}}$$

$$\nabla \times \boldsymbol{F} = \nabla \times \langle a_2 z - a_3 y,\ a_3 x - a_1 z,\ a_1 y - a_2 x \rangle \qquad \nabla \times (f,\ g,\ h) = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{\boldsymbol{i}} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\hat{\boldsymbol{j}} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{\boldsymbol{k}}$$

$$= \left(a_1 - a_1\right)\hat{\boldsymbol{i}} + \left(a_2 - a_2\right)\hat{\boldsymbol{j}} + \left(a_3 - a_3\right)\hat{\boldsymbol{k}}$$

$$\underline{= 0}$$

Exercise

Let $\boldsymbol{a} = \langle 0,\ 1,\ 0 \rangle$, $\boldsymbol{r} = \langle x,\ y,\ z \rangle$ and consider the rotation field $\boldsymbol{F} = \boldsymbol{a} \times \boldsymbol{r}$. Use the right-hand rule for cross product to find the direction of $\boldsymbol{F}$ at the points $(0,\ 1,\ 1)$, $(1,\ 1,\ 0)$, $(0,\ 1,\ -1)$, and $(-1,\ 1,\ 0)$

Solution

$$\langle 0,\ 1,\ 0 \rangle \times \langle 0,\ 1,\ 1 \rangle = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \langle 1,\ 0,\ 0 \rangle;$$

$\boldsymbol{F}$ points in the positive x-direction

$$\langle 0,\ 1,\ 0 \rangle \times \langle 1,\ 1,\ 0 \rangle = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = \langle 0,\ 0,\ -1 \rangle;$$

$\boldsymbol{F}$ points in the negative z-direction

$$\langle 0,\ 1,\ 0\rangle \times \langle 0,\ 1,\ -1\rangle = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \langle -1,\ 0,\ 0\rangle;$$ $\boldsymbol{F}$ points in the negative x-direction

$$\langle 0,\ 1,\ 0\rangle \times \langle -1,\ 1,\ 0\rangle = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{vmatrix} = \langle 0,\ 0,\ 1\rangle;$$ $\boldsymbol{F}$ points in the positive z-direction

Exercise

Find the exact points on the circle $x^2 + y^2 = 2$ at which the field $\boldsymbol{F} = \langle f,\ g\rangle = \left\langle x^2,\ y\right\rangle$ switches from pointing inward to outward on the circle, or vice versa.

Solution

The field switches from inward-pointing to outward-pointing at points where it is tangent to the circle $x^2 + y^2 = 2$, where it is orthogonal to the normal to the circle.

The normal to the circle at (x, y) is a multiple of (x, y), so we want to find $x,\ y$ so that

$$\langle x,\ y\rangle \cdot \left\langle x^2,\ y\right\rangle = x^3 + y^2 = 0$$

$$x^2 + \left(-x^3\right) = 2$$

$$x^3 - x^2 + 2 = 0 \xrightarrow{\ solutions\ } x = -1,\ 1 \pm i$$

The solutions are: $\boxed{x = -1 \ \ \rightarrow y = \pm 1}$

Exercise

Suppose a solid object in $\mathbb{R}^3$ has a temperature distribution given by $T(x,\ y,\ z)$. The heat flow vector field in the object is $\boldsymbol{F} = -k\nabla T$, where the conductivity $k > 0$ is a property of the material. Note that the heat flow vector points in the direction opposite to that of the gradient, which is the direction of greatest temperature decrease. The divergence of the heat flow vector is $\boldsymbol{F} = -k\nabla \cdot \nabla T = -k\nabla^2 T$ (the Laplacian of T). Compute the heat flow vector field and its divergence for the following temperature distribution.

a) $T(x,\ y,\ z) = 100\, e^{-\sqrt{x^2+y^2+z^2}}$

b) $T(x,\ y,\ z) = 100\, e^{-x^2+y^2+z^2}$

c) $T(x,\ y,\ z) = 100\left(1 + \sqrt{x^2 + y^2 + z^2}\right)$

Solution

$$\textbf{\textit{a)}}\quad T(x, y, z) = 100\, e^{-\sqrt{x^2+y^2+z^2}}$$

$$\boldsymbol{F} = -k\nabla T = -100\, k\ \nabla\, e^{-\sqrt{x^2+y^2+z^2}} \qquad \nabla T = \left(\frac{\partial}{\partial x}T\right)\hat{\boldsymbol{i}} + \left(\frac{\partial}{\partial y}T\right)\hat{\boldsymbol{j}} + \left(\frac{\partial}{\partial z}T\right)\hat{\boldsymbol{k}}$$

$$= \frac{100\, k e^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\ \langle x, y, z\rangle$$

$$= 100\, k\left[\frac{\partial}{\partial x}\left(\frac{xe^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial y}\left(\frac{ye^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial z}\left(\frac{ze^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\right)\right]$$

$$\frac{\partial}{\partial x}\left(\frac{xe^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\right) = e^{-\sqrt{x^2+y^2+z^2}}\ \frac{\left(1-x^2\left(x^2+y^2+z^2\right)^{-1/2}\right)\sqrt{x^2+y^2+z^2} - x^2\left(x^2+y^2+z^2\right)^{-1/2}}{x^2+y^2+z^2}$$

$$= e^{-\sqrt{x^2+y^2+z^2}}\ \frac{x^2+y^2+z^2-x^2\left(x^2+y^2+z^2\right)^{1/2}-x^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$= \frac{y^2+z^2-x^2\left(x^2+y^2+z^2\right)^{1/2}}{\left(x^2+y^2+z^2\right)^{3/2}}\, e^{-\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial}{\partial y}\left(\frac{ye^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\right) = e^{-\sqrt{x^2+y^2+z^2}}\ \frac{\left(1-y^2\left(x^2+y^2+z^2\right)^{-1/2}\right)\sqrt{x^2+y^2+z^2} - y^2\left(x^2+y^2+z^2\right)^{-1/2}}{x^2+y^2+z^2}$$

$$= \frac{x^2+z^2-y^2\left(x^2+y^2+z^2\right)^{1/2}}{\left(x^2+y^2+z^2\right)^{3/2}}\, e^{-\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial}{\partial z}\left(\frac{ye^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\right) = e^{-\sqrt{x^2+y^2+z^2}}\ \frac{\left(1-z^2\left(x^2+y^2+z^2\right)^{-1/2}\right)\sqrt{x^2+y^2+z^2} - z^2\left(x^2+y^2+z^2\right)^{-1/2}}{x^2+y^2+z^2}$$

$$= \frac{x^2+y^2-z^2\left(x^2+y^2+z^2\right)^{1/2}}{\left(x^2+y^2+z^2\right)^{3/2}}\, e^{-\sqrt{x^2+y^2+z^2}}$$

$$\boldsymbol{F} = 100\, k\, \frac{e^{-\sqrt{x^2+y^2+z^2}}}{\left(x^2+y^2+z^2\right)^{3/2}}\left(2\left(x^2+y^2+z^2\right) - \left(x^2+y^2+z^2\right)\left(x^2+y^2+z^2\right)^{1/2}\right)$$

$$\boxed{= \frac{100\, k\, e^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\left(2-\sqrt{x^2+y^2+z^2}\right)}$$

b) $T(x, y, z) = 100\, e^{-x^2+y^2+z^2}$

$$\boldsymbol{F} = -k\nabla T = -100\, k\ \nabla\, e^{-x^2+y^2+z^2} \qquad\qquad \nabla\, e^{-x^2+y^2+z^2} = e^{-x^2+y^2+z^2}\langle -2x,\ 2y,\ 2z\rangle$$

$$= -200\, k\, e^{-x^2+y^2+z^2}\ \langle -x,\ y,\ z\rangle$$

$$= -200\, k\left[\frac{\partial}{\partial x}\left(-xe^{-x^2+y^2+z^2}\right) + \frac{\partial}{\partial y}\left(ye^{-x^2+y^2+z^2}\right) + \frac{\partial}{\partial z}\left(ze^{-x^2+y^2+z^2}\right)\right]$$

$$= -200\, ke^{-x^2+y^2+z^2}\left(-1+2x^2+1+2y^2+1+2z^2\right)$$

$$= -200\, ke^{-x^2+y^2+z^2}\left(1+2x^2+2y^2+2z^2\right)$$

c) $T(x, y, z) = 100\left(1+\sqrt{x^2+y^2+z^2}\right)$

$$\boldsymbol{F} = -k\nabla T = -100\, k\ \nabla\left(1+\sqrt{x^2+y^2+z^2}\right)$$

$$= -100\, k\left[\left(x^2+y^2+z^2\right)^{-1/2}\langle x,\ y,\ z\rangle\right]$$

$$= -100\, k\left[\frac{\partial}{\partial x}\left(x\left(x^2+y^2+z^2\right)^{-1/2}\right) + \frac{\partial}{\partial y}\left(y\left(x^2+y^2+z^2\right)^{-1/2}\right) + \frac{\partial}{\partial z}\left(z\left(x^2+y^2+z^2\right)^{-1/2}\right)\right]$$

$$= -100\, k\left(x^2+y^2+z^2\right)^{-1/2}\left[3-\left(x^2+y^2+z^2\right)\left(x^2+y^2+z^2\right)^{-1}\right]$$

$$= -100\, k\left(x^2+y^2+z^2\right)^{-1/2}\left[3-1\right]$$

$$= \frac{-200\, k}{\sqrt{x^2+y^2+z^2}}$$

Exercise

Consider the rotational velocity field $\vec{v} = \langle -2y,\ 2z,\ 0\rangle$

a) If a paddle is placed in the *xy*-plane with its axis normal to this plane, what is its angular speed?
b) If a paddle is placed in the *xz*-plane with its axis normal to this plane, what is its angular speed?
c) If a paddle is placed in the *yz*-plane with its axis normal to this plane, what is its angular speed?

Solution

$$curl\ \vec{v} = \nabla\times\vec{v}$$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2y & 2z & 0 \end{vmatrix}$$

$$= \langle -2,\ 0,\ 2\rangle$$

a) In *xy*-plane with its axis normal to this plane, then the wheel is placed with its axis in the direction of *z*-axis $\langle 0,\ 0,\ 1\rangle$, so the component of velocity in that direction is

$$\langle -2,\ 0,\ 2\rangle \cdot \langle 0,\ 0,\ 1\rangle \underline{=2}$$

its angular speed: $\omega = \frac{1}{2}\cdot 2 \underline{=1}$

b) In *xz*-plane with its axis normal to this plane, then the wheel is placed with its axis in the direction of *y*-axis $\langle 0,\ 1,\ 0\rangle$, so the component of velocity in that direction is

$$\langle -2,\ 0,\ 2\rangle \cdot \langle 0,\ 1,\ 0\rangle \underline{=0}$$

The wheel ***does not*** turn.

c) In *yz*-plane with its axis normal to this plane, then the wheel is placed with its axis in the direction of *x*-axis $\langle 1,\ 0,\ 0\rangle$, so the component of velocity in that direction is

$$\langle -2,\ 0,\ 2\rangle \cdot \langle 1,\ 0,\ 0\rangle \underline{=-2}$$

its angular speed: $\omega = \frac{1}{2}\cdot |-2| \underline{=1}$

Exercise

Consider the rotational velocity field $\vec{v} = \langle 0,\ 10z,\ -10y\rangle$. If a paddle wheel is placed in the plane $x+y+z=1$ with its axis normal to this plane, how fast does the paddle wheel spin (revolutions per unit time)?

Solution

$$curl\ \vec{v} = \nabla \times \vec{v}$$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 10z & -10y \end{vmatrix}$$

$$= \langle -20,\ 0,\ 0\rangle$$

Since the wheel is placed in the plane $x+y+z=1$ with its axis normal to this plane, then must point in the direction $\langle 1,\ 1,\ 1\rangle$

The unit vector: $\frac{1}{\sqrt{3}}\langle 1,\ 1,\ 1\rangle$

$$\frac{1}{\sqrt{3}}\langle 1,\ 1,\ 1\rangle \cdot \langle -20,\ 0,\ 0\rangle = -\frac{20}{\sqrt{3}}$$

The component of velocity along that direction is $-\frac{20}{\sqrt{3}}$

The angular velocity $\omega = \frac{1}{2}\left|-\frac{20}{\sqrt{3}}\right| \underline{= \frac{10}{\sqrt{3}}}$

The paddle wheel spin: $\frac{10}{\sqrt{3}}\frac{1}{2\pi} = \boxed{\frac{5}{\pi\sqrt{3}} \quad \text{rev/time}}$

Exercise

The potential function for the gravitational force field due to a mass M at the origin acting on a mass m is $\phi = \frac{GMm}{|\vec{r}|}$, where $\vec{r} = \langle x,\ y,\ z\rangle$ is the position vector of the mass m and G is the gravitational constant.

a) Compute the gravitational force field $\vec{F} = -\nabla\phi$

b) Show that the field is irrotational; that is $\nabla \times \vec{F} = \vec{0}$

Solution

a)

$$\begin{aligned}
\vec{F} &= -\nabla\phi \\
&= -\nabla\left(\frac{GMm}{\sqrt{x^2+y^2+z^2}}\right) \\
&= -GMm\left\langle \frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+y^2+z^2}},\ \ \frac{\partial}{\partial y}\frac{1}{\sqrt{x^2+y^2+z^2}},\ \ \frac{\partial}{\partial z}\frac{1}{\sqrt{x^2+y^2+z^2}}\right\rangle \\
&= -\frac{GMm}{\left(x^2+y^2+z^2\right)^{3/2}}\langle -x,\ \ -y,\ \ -z\rangle \\
&= \frac{GMm}{\left(x^2+y^2+z^2\right)^{3/2}}\langle x,\ \ y,\ \ z\rangle \\
&\boxed{= GMm\ \frac{\vec{r}}{|\vec{r}|^3}}
\end{aligned}$$

b)

$$\nabla \times \vec{F} = GMm\begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{|\vec{r}|^3} & \frac{y}{|\vec{r}|^3} & \frac{z}{|\vec{r}|^3} \end{vmatrix}$$

$$\frac{\partial}{\partial y}\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}} = \frac{-3xy}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial z}\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}} = \frac{-3xz}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial x}\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}=\frac{-3xy}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial z}\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}=\frac{-3yz}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial y}\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}=\frac{-3xy}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial z}\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}=\frac{-3yz}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\nabla\times\vec{F}=GMm\left((-3yz+3yz)\hat{i}+(-3xz+3xz)\hat{j}+(-3xy+3xy)\hat{k}\right)$$

$$\boxed{=\vec{0}}$$

Exercise

The potential function for the force field due to a charge q at the origin is $\phi=\frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}|}$, where $\vec{r}=\langle x,\ y,\ z\rangle$ is the position vector of the mass m and G is the gravitational constant.

a) Compute the force field $\vec{F}=-\nabla\phi$

b) Show that the field is irrotational; that is $\nabla\times\vec{F}=\vec{0}$

Solution

a) $\vec{F}=-\nabla\phi$

$$=-\frac{q}{4\pi\varepsilon_0}\nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$=-\frac{q}{4\pi\varepsilon_0}\left\langle\frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+y^2+z^2}},\ \frac{\partial}{\partial y}\frac{1}{\sqrt{x^2+y^2+z^2}},\ \frac{\partial}{\partial z}\frac{1}{\sqrt{x^2+y^2+z^2}}\right\rangle$$

$$=-\frac{q}{4\pi\varepsilon_0}\frac{1}{\left(x^2+y^2+z^2\right)^{3/2}}\langle -x,\ -y,\ -z\rangle$$

$$=\frac{q}{4\pi\varepsilon_0}\frac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\boxed{=\frac{q}{4\pi\varepsilon_0}\frac{\vec{r}}{|\vec{r}|^3}}$$

b) $\nabla \times \vec{F} = \dfrac{q}{4\pi\varepsilon_0} \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{|\vec{r}|^3} & \frac{y}{|\vec{r}|^3} & \frac{z}{|\vec{r}|^3} \end{vmatrix}$

$$\frac{\partial}{\partial y} \frac{x}{\left(x^2+y^2+z^2\right)^{3/2}} = \frac{-3xy}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial z} \frac{x}{\left(x^2+y^2+z^2\right)^{3/2}} = \frac{-3xz}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial x} \frac{y}{\left(x^2+y^2+z^2\right)^{3/2}} = \frac{-3xy}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial z} \frac{y}{\left(x^2+y^2+z^2\right)^{3/2}} = \frac{-3yz}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial y} \frac{x}{\left(x^2+y^2+z^2\right)^{3/2}} = \frac{-3xy}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial z} \frac{y}{\left(x^2+y^2+z^2\right)^{3/2}} = \frac{-3yz}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\nabla \times \vec{F} = \frac{q}{4\pi\varepsilon_0}\Big((-3yz+3yz)\,\hat{i} + (-3xz+3xz)\,\hat{j} + (-3xy+3xy)\,\hat{k} \Big)$$

$$= \vec{0}$$

Exercise

The Navier-Stokes equation is the fundamental equation of fluid dynamics that models the motion of water in everything from bathtubs to oceans. In one of its many forms (incompressible, viscous flow), the equation is

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \bullet \nabla\right)\vec{V}\right) = -\nabla p + \mu\left(\nabla \bullet \nabla\right)\vec{V}$$

In this notation $\vec{V} = \langle u,\ v,\ w \rangle$ is the three-dimensional velocity field, p is the (scalar) pressure, ρ is the constant density of the fluid, and μ is the constant viscosity. Write out the three component equations of this vector equation.

Solution

$$\frac{\partial\vec{V}}{\partial t}=\left\langle\frac{\partial u}{\partial t},\ \frac{\partial v}{\partial t},\ \frac{\partial w}{\partial t}\right\rangle$$

$$\left(\vec{V}\bullet\nabla\right)\vec{V}=\left(\langle u,\ v,\ w\rangle\bullet\left\langle\frac{\partial}{\partial t},\ \frac{\partial}{\partial t},\ \frac{\partial}{\partial t}\right\rangle\right)\langle u,\ v,\ w\rangle$$

$$=\left(u\frac{\partial}{\partial t}+v\frac{\partial}{\partial t}+w\frac{\partial}{\partial t}\right)\langle u,\ v,\ w\rangle$$

$$=\left(u+v+w\right)\frac{\partial}{\partial t}\langle u,\ v,\ w\rangle$$

$$=\left\langle\left(u+v+w\right)\frac{\partial u}{\partial t},\quad\left(u+v+w\right)\frac{\partial v}{\partial t},\quad\left(u+v+w\right)\frac{\partial w}{\partial t}\right\rangle$$

$$\frac{\partial\vec{V}}{\partial t}+\left(\vec{V}\bullet\nabla\right)\vec{V}=\left\langle\frac{\partial u}{\partial t},\ \frac{\partial v}{\partial t},\ \frac{\partial w}{\partial t}\right\rangle+\left\langle\left(u+v+w\right)\frac{\partial u}{\partial t},\quad\left(u+v+w\right)\frac{\partial v}{\partial t},\quad\left(u+v+w\right)\frac{\partial w}{\partial t}\right\rangle$$

$$=\left\langle\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z},\quad\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z},\quad\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right\rangle$$

$$\nabla p=\left\langle\frac{\partial p}{\partial t},\ \frac{\partial p}{\partial t},\ \frac{\partial p}{\partial t}\right\rangle$$

$$\left(\nabla\bullet\nabla\right)\vec{V}=\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial t^2}\right)\langle u,\ v,\ w\rangle$$

$$=\left\langle\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2},\quad\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2},\quad\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right\rangle$$

$$\rho\left(\frac{\partial\vec{V}}{\partial t}+\left(\vec{V}\bullet\nabla\right)\vec{V}\right)=-\nabla p+\mu\left(\nabla\bullet\nabla\right)\vec{V}$$

$$\rho\left\langle\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z},\quad\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z},\quad\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right\rangle=$$

$$-\left\langle\frac{\partial p}{\partial t},\ \frac{\partial p}{\partial t},\ \frac{\partial p}{\partial t}\right\rangle+\mu\left\langle\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2},\quad\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2},\quad\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right\rangle$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

Exercise

One of Maxwell's equations for electromagnetic waves is $\nabla \times \vec{B} = C\dfrac{\partial \vec{E}}{\partial t}$, where $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, and C is a constant.

a) Show that the fields $\vec{E}(z,\ t) = A\sin(kz - \omega t)\,\hat{i}$ $\quad\quad$ $\vec{B}(z,\ t) = A\sin(kz - \omega t)\,\hat{j}$

Satisfy the equation for constants A, k, and ω, provided $\omega = \dfrac{k}{C}$

b) Make a rough sketch showing the directions of $\vec{E}$ and $\vec{B}$

Solution

a)

$$\nabla \times \vec{B} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & A\sin(kz - \omega t) & 0 \end{vmatrix}$$

$$= -\frac{\partial}{\partial z}\left(A\sin(kz - \omega t)\right)\hat{i} + 0\,\hat{j} + \frac{\partial}{\partial x}\left(A\sin(kz - \omega t)\right)\hat{k}$$

$$\boxed{= -Ak\sin(kz - \omega t)\,\hat{i}}$$

$$C\frac{\partial \vec{E}}{\partial t} = C\frac{\partial}{\partial t}\left(A\sin(kz - \omega t)\,\hat{i}\right)$$

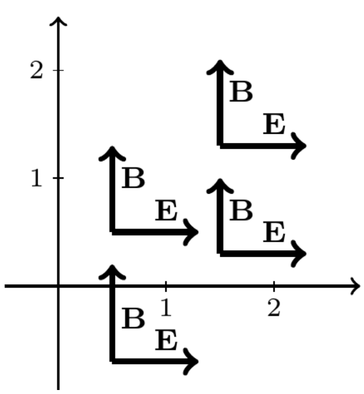
$$\boxed{= -AC\omega\sin(kz - \omega t)\,\hat{i}}$$

$$\nabla \times \vec{B} = C\frac{\partial \vec{E}}{\partial t}$$

$$-Ak\sin(kz - \omega t)\,\hat{i} = -AC\omega\sin(kz - \omega t)\,\hat{i}$$

$$\Rightarrow\ k = \omega C\ \rightarrow\ \boxed{\omega = \frac{k}{C}}$$

b)



Exercise

Prove that for a real number p, with $\vec{r} = \langle x,\ y,\ z \rangle$, $\nabla \bullet \dfrac{\langle x,\ y,\ z \rangle}{|\vec{r}|^p} = \dfrac{3-p}{|\vec{r}|^p}$

Solution

$$\nabla \bullet \frac{\langle x,\ y,\ z \rangle}{|\vec{r}|^p} = \left\langle \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z} \right\rangle \bullet \frac{\langle x,\ y,\ z \rangle}{\left(x^2+y^2+z^2\right)^{p/2}}$$

$$= \frac{\partial}{\partial x} \frac{x}{\left(x^2+y^2+z^2\right)^{p/2}} + \frac{\partial}{\partial y} \frac{y}{\left(x^2+y^2+z^2\right)^{p/2}} + \frac{\partial}{\partial z} \frac{z}{\left(x^2+y^2+z^2\right)^{p/2}}$$

$$= \frac{x^2+y^2+z^2-px^2}{\left(x^2+y^2+z^2\right)^{1+p/2}} + \frac{x^2+y^2+z^2-py^2}{\left(x^2+y^2+z^2\right)^{1+p/2}} + \frac{x^2+y^2+z^2-pz^2}{\left(x^2+y^2+z^2\right)^{1+p/2}}$$

$$= \frac{3\left(x^2+y^2+z^2\right) - p\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{1+p/2}}$$

$$= \frac{(3-p)\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{1+p/2}}$$

$$= \frac{3-p}{\left(x^2+y^2+z^2\right)^{p/2}}$$

$$= \frac{3-p}{|\vec{r}|^p} \quad \checkmark$$

Exercise

Prove that for a real number p, with $\vec{r} = \langle x,\ y,\ z \rangle$, $\nabla\left(\dfrac{1}{|\vec{r}|^p}\right) = \dfrac{-p\vec{r}}{|\vec{r}|^{p+2}}$

Solution

$$\nabla\left(\frac{1}{|\vec{r}|^p}\right) = \nabla\left(\frac{1}{\left(x^2+y^2+z^2\right)^{p/2}}\right)$$

$$= \left\langle \frac{\partial}{\partial x} \frac{1}{\left(x^2+y^2+z^2\right)^{p/2}},\ \frac{\partial}{\partial y} \frac{1}{\left(x^2+y^2+z^2\right)^{p/2}},\ \frac{\partial}{\partial z} \frac{1}{\left(x^2+y^2+z^2\right)^{p/2}} \right\rangle$$

$$= \frac{1}{\left(x^2+y^2+z^2\right)^{1+p/2}} \langle -px,\ \ -py,\ \ -pz \rangle$$

$$= \frac{-p}{\left(x^2+y^2+z^2\right)^{\frac{2+p}{2}}} \langle x,\ y,\ z \rangle$$

$$\boxed{= \frac{-p\vec{r}}{|\vec{r}|^{p+2}}} \quad \checkmark$$

Exercise

Prove that for a real number p, with $\vec{r} = \langle x,\ y,\ z \rangle$, $\nabla \bullet \nabla \left(\frac{1}{|\vec{r}|^p} \right) = \frac{p(p-1)}{|\vec{r}|^{p+2}}$

Solution

$$\nabla \left(\frac{1}{|\vec{r}|^p} \right) = \nabla \left(\frac{1}{\left(x^2+y^2+z^2\right)^{p/2}} \right)$$

$$= \left\langle \frac{\partial}{\partial x} \frac{1}{\left(x^2+y^2+z^2\right)^{p/2}},\ \ \frac{\partial}{\partial y} \frac{1}{\left(x^2+y^2+z^2\right)^{p/2}},\ \ \frac{\partial}{\partial z} \frac{1}{\left(x^2+y^2+z^2\right)^{p/2}} \right\rangle$$

$$= \frac{1}{\left(x^2+y^2+z^2\right)^{1+p/2}} \langle -px,\ \ -py,\ \ -pz \rangle$$

$$= \frac{-p}{\left(x^2+y^2+z^2\right)^{\frac{2+p}{2}}} \langle x,\ y,\ z \rangle$$

$$= -\frac{p\vec{r}}{|\vec{r}|^{p+2}}$$

$$\nabla \bullet \nabla \left(\frac{1}{|\vec{r}|^{p+2}} \right) = \left\langle \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z} \right\rangle \bullet \frac{-p\langle x,\ y,\ z \rangle}{\left(x^2+y^2+z^2\right)^{\frac{p+2}{2}}}$$

$$= -p \frac{\partial}{\partial x} \frac{x}{\left(x^2+y^2+z^2\right)^{\frac{p+2}{2}}} - p \frac{\partial}{\partial y} \frac{y}{\left(x^2+y^2+z^2\right)^{\frac{p+2}{2}}} - p \frac{\partial}{\partial z} \frac{z}{\left(x^2+y^2+z^2\right)^{\frac{p+2}{2}}}$$

$$= -\frac{p}{\left(x^2+y^2+z^2\right)^{1+\frac{p+2}{2}}}\left(3\left(x^2+y^2+z^2\right)-(p+2)x^2-(p+2)y^2-(p+2)z^2\right)$$

$$= -\frac{p\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{1+\frac{p+2}{2}}}\left(3-(p+2)\right)$$

$$= \frac{p(p-1)}{\left(x^2+y^2+z^2\right)^{\frac{p+2}{2}}}$$

$$\boxed{= \frac{p(p-1)}{|\vec{r}|^{p+2}}} \quad \checkmark$$