$$A = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 \\ 9 & -3-\lambda \end{vmatrix}$$

$$= \lambda^2$$

Characteristic eqn : $\lambda^2 = 0$

Eigen values : $\boxed{\lambda_{1,2} = 0}$

For $\lambda_1 = 0 \Rightarrow (A - \lambda_1 I) V_1 = 0$

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow 3x_1 - y_1 = 0$$

$$3x_1 = y_1$$

$$\boxed{V_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}}$$

$(A - \lambda I) V_2 = V_1$

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad 3x_2 - y_2 = 1$$

$$3x_2 = 1 + y_2$$

let $x_2 = 1 \rightarrow y_2 = 2$ $\qquad \boxed{V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$

$$A = \begin{pmatrix} 4 & 5 \\ -2 & 6 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 5 \\ -2 & 6-\lambda \end{vmatrix}$$

$$= \lambda^2 - 10\lambda + 34 = 0$$

$$\lambda_{1,2} = \frac{10 \pm i(6)}{2}$$

$5 \pm 3i$

$$= \boxed{5 \pm 3i}$$

For $\lambda_1 = 5 - 3i \Rightarrow (A - \lambda_1 I) V_1 = 0$

$$\begin{pmatrix} -1+3i & 5 \\ -2 & 1+3i \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(-1+3i)x_1 = -5y_1$$

$$V_1 = \begin{pmatrix} -5 \\ -1+3i \end{pmatrix} \qquad V_2 = \begin{pmatrix} -5 \\ -1-3i \end{pmatrix}$$

$$A = \begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & -4 \\ 2 & -1-\lambda \end{vmatrix}$$

$$= \lambda^2 - 4\lambda + 3 = 0$$

$$\lambda_{1,2} = 1, 3$$

For $\lambda_1 = 1 \Rightarrow \quad (A - \lambda_1 I) V_1 = 0$

$$\begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2x_1 = 2y_1$$

$$V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 3$: $\quad (A - \lambda_2 I) V_2 = 0$

$$\begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_2 = 2y_2$$

$$V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

4.5 Diagonalization

$$D = \Lambda = P^{-1}AP$$

$$= \begin{bmatrix} \lambda_1 & & \phi \\ & \lambda_2 & \\ \phi & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

where P is eigenvectors matrix

Ex

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad \begin{pmatrix} .5 & .5 \\ .5 & .5 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} \frac{1}{2} - \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} - \lambda \end{vmatrix}$$

$$= \lambda^2 - \lambda = 0$$

$$\lambda_{1,2} = 0, 1$$

For $\lambda_1 = 0 \Rightarrow (A - \lambda_1 I) V_1 = 0$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \frac{1}{2} x_1 = -\frac{1}{2} y_1$$

$$V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 1 \rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad x_2 = y_2$$

$$\boxed{V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$$P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$P^{-1} = -\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \checkmark$$

$$= D.$$

P diagonalize A.

Ex/

$$A = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} .8-\lambda & .3 \\ .2 & .7-\lambda \end{vmatrix}$$

$$= \lambda^2 - 1.5\lambda + .5 = 0$$

$$\lambda_1 = 1 \qquad \lambda_2 = \frac{1}{2}$$

For $\lambda_1 = 1 \Rightarrow (A - \lambda_1 I) V_1 = 0$

$$\begin{pmatrix} -.2 & .3 \\ .2 & -.3 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$.2x_1 = .3y_1$$

$$2x_1 = 3y_1 \Rightarrow V_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

For $\lambda_2 = \frac{1}{2} \Rightarrow (A - \lambda_2 I) V_2 = 0$

$$\begin{pmatrix} .3 & .3 \\ .2 & .2 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_2 = -y_2$$

$$V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix} \rightarrow P^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 3 & -\frac{1}{2} \\ 2 & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$= D. \quad \checkmark$$

it's diagonalizable

$$\boxed{D = P^{-1}AP} \qquad A \overset{?}{=} PDP^{-1}$$

$$PDP^{-1} = \underbrace{PP^{-1}}\, A \,\underbrace{PP^{-1}}$$

$$= I A I$$

$$= A \checkmark$$

λ_i are distinct $\Rightarrow$ diagonalizable

$$A = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -3-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix}$$

$$= \lambda^2 + 2\lambda + 1 = 0$$

$\lambda_{1,2} = -1$

For $\lambda_1 = -1 \Rightarrow (A - \lambda_1 I) V_1 = 0$

$$\begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow x_1 = y_1$$

$V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad V_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

For V_2: $\quad (A - \lambda_1 I) V_2 = V_1$

$$\begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$-2x_2 + 2y_2 = 1$$

$y_2 = \frac{1}{2} + 2x_2$

$$V_2 = \begin{pmatrix} 2 \\ \frac{5}{2} \end{pmatrix}$$

$P = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ $\qquad |P| = 0$ ✓

Not diag.

$$P_1 = \begin{pmatrix} 1 & 2 \\ 1 & 5/2 \end{pmatrix} \qquad P_1^{-1} = 2 \begin{pmatrix} 5/2 & -2 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -4 \\ -2 & 2 \end{pmatrix}$$

$$P_1 D P_1^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & \frac{5}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & -4 \\ -2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 1 & \frac{5}{2} \end{pmatrix} \begin{pmatrix} 5 & -4 \\ -2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$f A$

$$V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad V_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\det = 0 \checkmark$$

$$A - \lambda_1 I \qquad (A - 2I)\, V_1$$

$\lambda_A \lambda_B \neq \lambda_{AB}$

$$A^2 = (PDP^{-1})(PDP^{-1})$$
$$= PD(P^{-1}P)DP^{-1}$$
$$= PDIDP^{-1}$$
$$= PD^2P^{-1}$$

$$A = PDP^{-1}$$
$$A^n = PD^nP^{-1}$$

Ex

A^k? $A = \begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}$

$$|A-\lambda I| = \begin{vmatrix}1-\lambda & 1\\ 0 & 2-\lambda\end{vmatrix}$$

$$= \lambda^2 - 3\lambda + 2 = 0$$

$$\lambda_{1,2} = 1, 2$$

For $\lambda_1 = 1 \Rightarrow (A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ y_1\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \qquad y_1 = 0$$

$$V_1 = \begin{pmatrix}1\\ 0\end{pmatrix}$$

For $\lambda_2 = 2 \Rightarrow (A - 2I)V_2 = 0$

$$\Rightarrow \begin{pmatrix}-1 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_2\\ y_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \qquad x_2 = y_2$$

$$V_2 = \begin{pmatrix}1\\ 1\end{pmatrix}$$

$$P = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \qquad P^{-1} = \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}$$

$$D^k = \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}^k = \begin{pmatrix}1 & 0\\ 0 & 2^k\end{pmatrix}$$

$$A^k = P D^k P^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2^k \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{pmatrix}$$