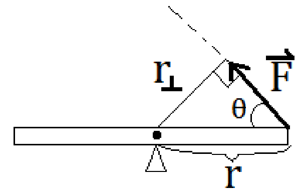
10.9 Torque

Torque (τ) is a physical quantity used as a measure of rotational effect of force. Rotational effect is proportional to the distance ($r_\perp$) between the point of rotation and the line of action of the force, and hence it is defined to be the product of the magnitude of the force & the perpendicular distance.

$\boxed{|\tau| = Fr_\perp}$

$|\tau| \rightarrow$ magnitude of torque
$F \rightarrow$ force(magnitude)
$r_\perp \rightarrow$ perpendicular distance between point of rotation & line of action of force.



If θ is the angle formed between the line joining the point of rotation with the point of application of force, and r is the distance between the point of rotation and the point of application of force, then $r_\perp = r\sin\theta$ and

$$\boxed{|\tau| = F\,r\sin\theta}$$

Torque is a vector quantity. Its direction is perpendicular to the plane determined by the position vector of the point of application of force with respect to the point of rotation and the force. The two possible directions are perpendicularly out of the plane and perpendicularly into the plane. A torque whose direction is perpendicularly out of the plane tends to cause a counterclockwise rotation and is taken to be positive. A torque whose direction is perpendicularly into the plane tends to cause a clockwise rotation and is taken to be negative. Unit of measure measurement of torque is Newton meter ($N \cdot m$).

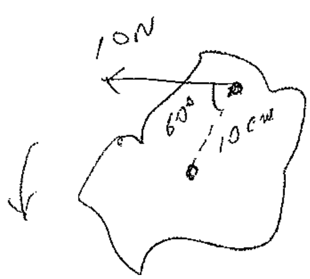
Example: Calculate the torque acting on the object in each of the following

a)

$r = 0.04\,m$
$F = 10N$
$\theta = 30°$

$|\tau| = Fr\sin\theta$
$= 10(.04)\sin 30 = 0.2\,N\cdot m$
Since this force tends to cause clockwise rotation, it is negative
$\tau = -0.2N\cdot m$

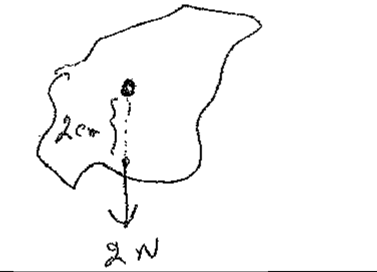
b)



$F = 10N$
$r = 0.1m$
$\theta = 60°$

$|\tau| = Fr\sin\theta = (10)(0.1)\sin 60$
$= 0.87\,N\cdot m$
Since it tends to cause a counterclockwise rotation, it is positive
$\tau = 0.87N\cdot m$

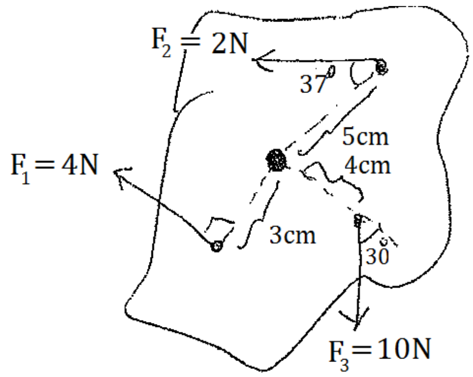
c)

	$F = 2N$ $r = 0.2m$ $\theta = 0$	$\lvert\tau\rvert = Fr\sin\theta = (0.2)(2)\sin 0 = 0$ No rotation!

If there are a number of forces acting on an object, the net torque is obtained by adding all the torques due to the individual forces vectorially (with appropriate signs)

$$\vec{\tau}_{net} = \vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 + \cdots$$

Example: Calculate the net torque acting on the object shown and determine whether it is rotating clockwise or counterclockwise.



$F_1 = 4N$ $\theta_1 = 90°$ $r_1 = 0.03m$	$F_2 = 2N$ $\theta_2 = 37°$ $r_2 = 0.05m$	$F_3 = 10N$ $\theta_3 = 30°\ (or\ 150°)$ $r_3 = 0.04m$

$$\tau_{net} = \tau_1 + \tau_2 + \tau_3$$

$\vec{\tau}_1$ & $\vec{\tau}_3$ produce clockwise rotation

$\vec{\tau}_2$ produces counterclockwise rotation

$\therefore\ \ \tau_{net} = -F_1 r_1 \sin\theta_1 + F_2 r_2 \sin\theta_2 - F_3 r_3 \sin\theta_3$

$= -4(.03)\sin 30 + 2(.05)\sin 37 - 10(0.4)\sin 30$

$= -.21 N\cdot m$

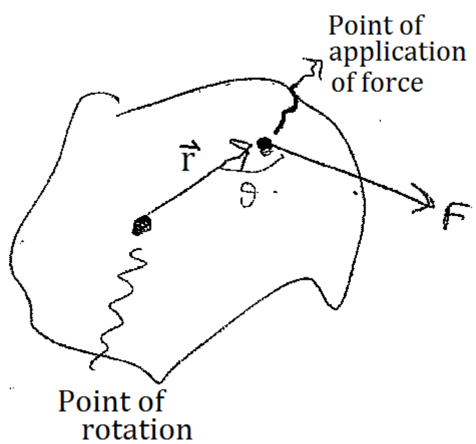
Since $\vec{\tau}_{net}$ is negative, it is rotating clockwise

11.3 Torque as a cross product

If $\vec{r}$ is the position vector of the point of application of force with respect to the point of rotation, then torque is equal to the cross product between the position vector and the force.

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

$$|\vec{\tau}| = |\vec{r}||\vec{F}|\sin\theta$$



Example: If a force is applied at the point $(2\hat{\imath} + 3\hat{\jmath})$m with respect to the point of rotation (i.e. point of rotation as origin) and the force is given as $\vec{F} = (-4\hat{\imath} + 6\hat{\jmath})N$, calculate the torque acting on it.

$\vec{\tau} = \vec{r} \times \vec{F}$

$\vec{r} = (2\hat{\imath} + 3\hat{\jmath})m$

$\vec{F} = (-4\hat{\imath} + 6\hat{\jmath})N$

$\vec{\tau} = (2\hat{\imath} + 3\hat{\jmath}) \times (-4\hat{\imath} + 6\hat{\jmath})$

$= \hat{k}[2(6) - (-4)(3)] N\cdot m$

$= 24\hat{k}\ N\cdot m$

Example: If the forces $\vec{F}_1 = (3\hat{\imath} + \hat{k})N, \vec{F}_2 = (-\hat{\imath} + 2\hat{\jmath} + 4\hat{k})N$, and $\vec{F}_3 = (10\hat{k})N$ are applied on an object at the points $(-2\hat{\imath})m$, $(3\hat{\imath}+4\hat{\jmath})m$, and $(2\hat{\imath}-6\hat{\jmath})m$ with respect to the point of rotation, calculate the net torque acting on the object.

$\vec{F}_1 = (3\hat{\imath}-\hat{k})N$ $\vec{r}_1 = -2\hat{\imath}\, m$	$\vec{F}_2 = (-\hat{\imath}+2\hat{\jmath}+4\hat{k})N$ $\vec{r}_2 = (3\hat{\imath}+4\hat{\jmath})m$	$\vec{F}_3 = 10\hat{k}\, N$ $\vec{r}_3 = (2\hat{\imath}-6\hat{\jmath})m$

$$\vec{\tau}_{net} = ??$$

$$\vec{\tau}_{net} = \vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3$$
$$= \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \vec{r}_3 \times \vec{F}_3$$

$$\vec{r}_1 \times \vec{F}_1 = (-2\hat{\imath}) \times (3\hat{\imath}-\hat{k}) = 2\hat{\jmath}\; N\cdot m$$

$$\vec{r}_2 \times \vec{F}_2 = (3\hat{\imath}+4\hat{\jmath}) \times (-\hat{\imath}+2\hat{\jmath}+4\hat{k})$$

$$= \begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 4 & 0 \\ -1 & 2 & 4 \end{matrix} = \hat{\imath}\begin{vmatrix} 4 & 0 \\ 2 & 4 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 3 & 0 \\ -1 & 4 \end{vmatrix} + \hat{k}\begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix}$$

$$= (16\hat{\imath} - 12\hat{\jmath} + 10\hat{k})N\cdot m$$

$$\vec{r}_3 \times \vec{F}_3 = (2\hat{\imath}-6\hat{\jmath}) \times (10\hat{k})$$
$$= (-20\hat{\jmath} - 60\hat{\imath})N\cdot m$$

$$\vec{\tau} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \vec{r}_3 \times \vec{F}_3$$
$$= (2\hat{\jmath}) + (16\hat{\imath} - 12\hat{\jmath} + 10\hat{k}) + (-60\hat{\imath} - 20\hat{\jmath})$$
$$= (-44\hat{\imath} - 30\hat{\jmath} + 10\hat{k})N\cdot m$$

Example: The position vector of a particle varies according to the equation $\vec{r} = 2t^3\hat{\imath} + 4t\hat{\jmath} + 8\hat{k}$ with respect to the point of rotation. If the mass of the particle is 2 kg, calculate the torque acting on it after 2 seconds.

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad \vec{F} = m\vec{a}$$
$$= m\frac{d\vec{v}}{dt} = m\frac{d^2\vec{r}}{dt^2}$$

$$\therefore \quad \vec{\tau} = m\vec{r} \times \frac{d^2\vec{r}}{dt^2}$$

$$\vec{\tau}(t=2) = m\vec{r}(t=2) \times \frac{d^2\vec{r}}{dt^2}(t=2)$$

$$m = 2kg$$

$$\vec{r} = 2t^3\hat{\imath} + 4t\hat{\jmath} + 8\hat{k}$$
$$\vec{r}(t=2) = 2(2)^3\hat{\imath} + 4(2)\hat{\jmath} + 8\hat{k}$$
$$\boxed{\vec{r} = (16\hat{\imath} + 8\hat{\jmath} + 8\hat{k})m} \Downarrow$$

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(2t^3\hat{\imath} + 4t\hat{\jmath} + 8\hat{k})$$
$$= 6t^2\hat{\imath} + 4\hat{\jmath} + 0$$

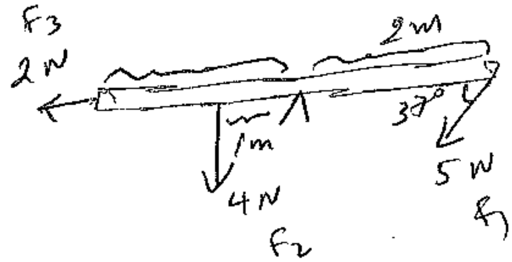
$$\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}(6t^2\hat{\imath} + 4\hat{\jmath})$$
$$= 12t\,\hat{\imath}$$
$$\frac{d^2\vec{r}}{dt^2}(t=2) = 12(2)\hat{\imath}$$

$$\boxed{\frac{d^2\vec{r}}{dt^2} = 24\hat{\imath}} \Longrightarrow\Longrightarrow$$

$$\therefore \quad \vec{\tau}(t=2) = m\vec{r}(t=2) \times \frac{d^2\vec{r}}{dt^2}(t=2)$$
$$= 2(16\hat{\imath} + 8\hat{\jmath} + 8\hat{k}) \times 24\hat{\imath}$$
$$(32\hat{\imath} + 16\hat{\jmath} + 16\hat{k}) \times 24\hat{\imath}$$
$$= -(24)(16)\hat{k} + (16)(24)\hat{\jmath}$$
$$= -388\hat{k} + 388\hat{\jmath}$$
$$= \underline{[388\hat{\jmath} - 388\hat{k}]N\cdot m}$$

Example: Find the net torque acting on the uniform lever shown about the pivot

$\vec{\tau}_{net} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \vec{r}_3 \times \vec{F}_3$



$\vec{r}_1 = 2\hat{\imath}$ $\vec{F}_1 = 5\cos(180+37)\,\hat{\imath} + 5\sin(180+37)\,\hat{\jmath}$	$\vec{r}_2 = -1\hat{\jmath}$ $\vec{F}_2 = -4\hat{\jmath}$	$\vec{r}_3 = -2\hat{\imath}$ $\vec{F}_3 = -2\hat{\imath}$

$$\begin{aligned}\vec{\tau}_{net} &= (2\hat{\imath}) \times (-4\hat{\imath} - 3\hat{\jmath}) + (-1\hat{\imath}) \times (-4\hat{\jmath}) + (-2\hat{\imath}) \times (-2\hat{\imath}) \\ &= -6\hat{k} + 4\hat{k} + 0 \\ &= (-2\hat{k})N \cdot m\end{aligned}$$

$-\hat{k}$ direction corresponds to a clockwise rotation

10.9.1 Relationship between torque and angular acceleration

Assuming an object is made up of small mass elements *dm* and that *dm* is at a perpendicular distance $r_\perp$ from the point of rotation, the torque acting on this element may be written as $d\tau = dF r_\perp$ where *dF* is the force acting on *dm*. Then the torque acting on the whole object is

$$\tau = \int d\tau = \int dF\, r_\perp$$

But from Newton's 2nd law $dF = (dm)a_t$ where a_t is the tangential acceleration

$$\therefore \quad \tau = \int r_\perp\, a_t\ dm$$

Also $a_t = r_\perp \alpha$ where α is the angular acceleration of the object

$$\therefore \quad \tau = \int r_\perp (r_\perp \alpha)\ dm$$

Angular acceleration α is the same for all the mass elements

$$\therefore \quad \tau = \alpha \int r_\perp^2 dm$$

But $\int r_\perp^2 dm$ is the moment of inertia, *I*, of the object

$$\therefore \quad \boxed{\tau = I\alpha}$$

Torque acting on an object about a given axis is equal to the moment of inertia about the axis multiplied by the angular acceleration of the object.

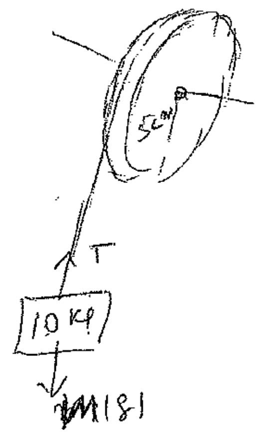
Example: The angular speed of a sphere of radius 2 cm & mass 0.5 kg rotating about an axis passing through its center increased from 2 rad/s to 14 rad/s in 3 seconds. Calculate the torque acting on it. (The moment of inertia of a sphere of radius *R* & mass *M* about an axis passing through its center is given by $I = \frac{2}{5}MR^2$).

$$\tau = I\alpha$$

$\omega_i = 2$ rad/s $\omega_f = 14$ rad/s $t = 2\ seconds$	$\alpha = \dfrac{\omega_f - \omega_i}{t}$ $\alpha = \dfrac{14-2}{3} = 4\dfrac{rad}{s^2}$	$M = 0.5kg$ $R = 0.02m$	$I = \dfrac{2}{5}MR^2$ $I = \dfrac{2}{5}(0.5)(0.02)^2 = 8 \times 10^{-5} kg \cdot m^2$

$$\tau = I\alpha$$
$$= (8 \times 10^{-5})(4) N \cdot m$$
$$\tau = 3.2 \times 10^{-4} N \cdot m$$

Example: A cylindrical wheel of radius 5 cm and mass 2 kg is rotating by a means of a hanging object of mass 10 kg as shown. Calculate the tension in the string and the acceleration of the hanging object. (Moment of inertia of a cylinder of mass M & radius R is given by $I = \frac{MR^2}{2}$).



Rotating Cylinder

Force acting:

Tension in the string (T)

The perpendicular distance between the tension and the axis is the radius, R.

$\tau = TR$ But also $\tau = I\alpha$ and $\alpha = \frac{a}{R}$

$$\tau = I\frac{a}{R}$$

$$\tau = TR = I\frac{a}{R}$$
$$TR = I\frac{a}{R}$$

$$I = \frac{MR^2}{2} \qquad M = 2\ kg,\quad R = 0.05m$$
$$I = \frac{2(0.05)^2}{2} = \boxed{25 \times 10^{-4} kg \cdot m^2}$$

$$T(0.05) = .0025\frac{a}{.05}$$

$$\Rightarrow \boxed{T = a} \ldots\ldots (1)$$

Hanging Object

Forces acting:

Tension in string (T) and its weight ($m|g|$)

Applying Newton's 2nd law for linear motion

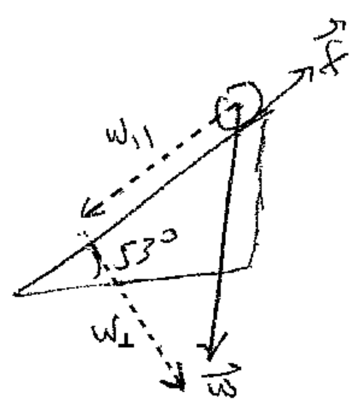
$$m|g| - T = ma \qquad m = 10\ kg$$
$$10(10) - T = 10a$$
$$\boxed{100 - T = 10a} \ldots\ldots (2)$$

But for equation (1) $T = a$

$$\therefore\ 100 - a = 10a$$
$$100 = 11a$$
$$a = 9.1\ \text{m}/s^2$$

$$T = a = \underline{9.1N}$$

Example: A sphere of mass 10 kg is rolling down a 53° inclined plane. Use equations of linear and rotational motion to calculate its acceleration and the force of friction.



The forces acting on the sphere are its weight($w = m|g|$) and friction.

$$w_{||} - f = ma \qquad m = 10\ kg$$
$$w_{||} = m|g|\sin 53°$$
$$60 - f = 10a \ldots\ldots (1) \qquad = (10)(10)(.6) = 60\ N$$

The weight does not contribute to the rotational motion because it acts at the center of mass. Only friction to rotation (torque).

$$\tau = fR = I\alpha$$

$$I = \frac{2}{5}MR^2$$ Where R is the radius of the sphere

$$fR = \left(\frac{2}{5}MR^2\right)\alpha \quad \& \quad \alpha = \frac{a}{R}$$

$$\therefore \; fR = \left(\frac{2}{5}MR^2\right)\left(\frac{a}{R}\right)$$

$$f = \frac{2}{5}ma = \frac{2}{5}(10)a$$

$$f = 4a \ldots\ldots (2)$$

Substituting eq(2) in eq(1)

$$60 - f = 10a \qquad f = 4a$$

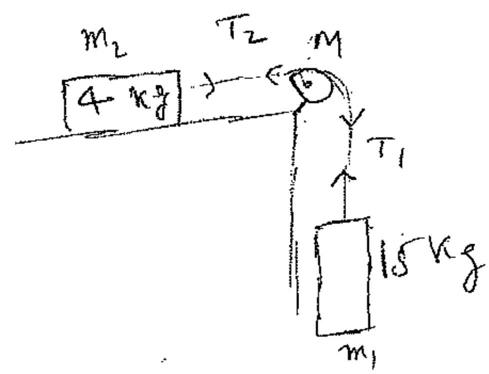
$$60 - 4a = 10a$$

$$60 = 14a$$

$$a = \frac{60}{14} = \boxed{\left(\frac{30}{7}\right)\frac{m}{s^2}}$$

$$\& \;\; f = 4a = 4\left(\frac{30}{7}\right) = \boxed{\frac{120}{7}N}$$

Example: Consider the system shown where a 15 kg and a 4 kg objects are connected by a string via a pulley. The pulley is a disc of radius 2cm and mass 0.2 kg. (Assume there is no friction) Calculate the tensions T_1 & T_2 and the acceleration of the system.



5 kg object

Forces Acting:

Weight $(m_1|g|)$

$$-T_1 + m_1|g| = m_1 a$$

$$-T_1 + 15(10) = 15a$$

$$-T_1 + 150 = 15a \ldots\ldots (1)$$

4 kg object

Forces Acting:

Tension T_2(Also normal force and its weight which cancel each other)

$\therefore T_2 = m_2 a = 4a \ldots\ldots (2)$

Pulley(rotation)

Forces Acting:

Tensions T_1 & T_2

The perpendicular distance between T_1, T_2, and the axis of rotation of the pulley is the radius*(R)* of the pulley

$$\tau = T_1 R - T_2 R = I\alpha$$

For a disk

$$I = \frac{MR^2}{2} \quad \& \quad \alpha = \frac{a}{R}$$

$$(T_1 - T_2)R = \frac{MR^2}{2} \cdot \frac{a}{R}$$

$$T_1 - T_2 = \frac{M}{2}a$$

$M = 0.2\,kg$ (mass of pulley)

$$T_1 - T_2 = 0.1a \ldots\ldots (3)$$

Substituting T_2 in (3) from (2)

$$T_1 - T_2 = 0.1a \quad but \quad T_2 = 4a$$

$$T_1 - 4a = 0.1a$$

$$T_1 = 4.1a \ldots\ldots (4)$$

Substituting for T_1 in eq(1) from eq(4)

$$-T_1 + 50 = 15a \quad but \;\; T_1 = 4.1a$$
$$-4.1a + 50 = 15a$$
$$19.1a = 50$$
$$a = \frac{50}{19.1} \approx \boxed{2.6\, m/s^2}$$

eq(4) $\Rightarrow T_2 = 4a$

$= 4(2.6) = \boxed{10.4N = T_2}$

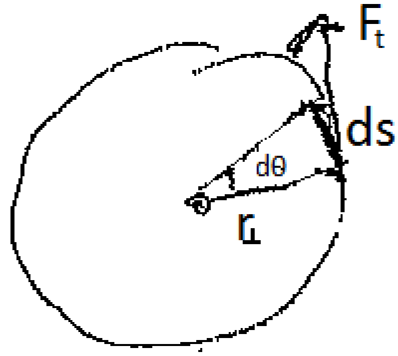
$T_1 = 4.1a$

$= (4.1)(2.6) \approx \boxed{10.7N = T_1}$

10.9.2 Work Done by torque

Consider a tangential force F_t acting at a perpendicular distance $r_\perp$ from the axis of rotation. The work done in displacing the particle by an arc length ds is given by

$$dw = F_t ds \quad but \quad \tau = F_t r_\perp \quad \therefore F_t = \frac{\tau}{r_\perp}$$
$$= \frac{\tau}{r_\perp} ds = \tau\left(\frac{ds}{r_\perp}\right) \quad but \quad \frac{ds}{r_\perp} = d\theta$$
$$\therefore \quad dw = \tau d\theta \quad = \quad \boxed{w = \int \tau d\theta}$$

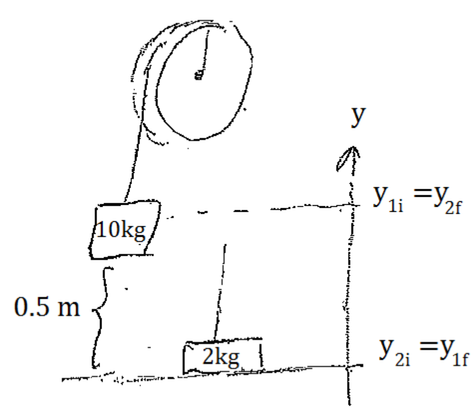


10.9.3 Work kinetic energy theorem in terms of rotational variables

The net work done is the work done by the net torque.

$$\therefore \;\; w_{net} = \int \tau_{net} d\theta \quad but \quad \tau_{net} = I\alpha$$
$$\therefore \;\; w_{net} = \int I\alpha\, d\theta \quad but \quad \alpha = \frac{d\omega}{dt}$$
$$\Rightarrow w_{net} = \int I\frac{d\omega}{dt} d\theta = \int Idw\frac{d\theta}{dt} \quad but\, \frac{d\theta}{dt} = \omega$$
$$\therefore \;\; w_{net} = \int I\omega\, d\omega = I\int_{\omega_i}^{\omega_f} I\omega\, d\omega$$
$$= I\left(\frac{\omega_f^2}{2} - \frac{\omega_i^2}{2}\right)$$
$$\therefore \boxed{\begin{aligned} w_{net} &= \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 \\ &= KE_{rot}^f - KE_{rot}^i = \Delta KE \end{aligned}}$$

Net work done is equal to the change in kinetic energy



Example: Consider the pulley system shown. The pulley is a cylindrical object of radius 12 cm and mass 2 kg. Intially the 2kg object is at the ground and the 10kg object is .5m above the ground. If the 10kg object is released from 0.5 rest, calculate its speed by the time it reaches the ground.

The only external forces (neglecting friction) acting on the system are the weights of the hanging objects. Therefore since gravity is conservative mechanical energy is conserved. Note: the kinetic energy of the system also includes the rotational kinetic energy of the pulley.

Let m_1 & m_2 be the masses of the 10 kg & 2k objects, respectively. Let *v* represent the speeds of the hanging object and ω the angular speed of the pulley.

But $\omega = \frac{v}{R}$ where *R* is the radius of the pulley

Let *M* be the mass of the pulley

$$ME_{if} = ME_f$$

$$\frac{1}{2}m_1v_i^2 + m_1|g|y_{1i} + \frac{1}{2}m_2v_i^2 + m_2|g|y_{2i} + \frac{1}{2}I\omega_i^2 = \frac{1}{2}m_1v_f^2$$
$$+m_1|g|y_{1f} + \frac{1}{2}m_2v_f^2 + m_2|g|y_{2f} + \frac{1}{2}I\omega_f^2$$

$v_i = \omega_i = 0$ (because released from rest)

$y_{2i} = y_{1f} = 0$ (ground level)

$$m_1|g|y_{1i} = \frac{1}{2}m_1v_f^2 + \frac{1}{2}m_2v_f^2 + m_2|g|y_{2f} + \frac{1}{2}I\omega_f^2$$

since $y_{2f} = y_{1i}$

$$(m_1 - m_2)|g|y_{1i} = \frac{1}{2}(m_1 + m_2)v_f^2 + \frac{1}{2}I\omega_f^2$$

but $\omega_f = \dfrac{v_f}{R}$

$$\therefore \quad (m_1 - m_2)|g|y_{1i} = \frac{1}{2}(m_1 + m_2)v_f^2 + \frac{1}{2}I\frac{v_f^2}{R^2}$$

And for a cylinder $I = \frac{MR^2}{2}$

$$(m_1 - m_2)|g|y_{1i} = \frac{1}{2}(m_1 + m_2)v_f^2 + \frac{1}{2}\left(\frac{MR^2}{2}\right)\frac{v_f^2}{R^2}$$

$$(m_1 - m_2)|g|y_{1i} = \frac{1}{2}(m_1 + m_2)v_f^2 + \frac{1}{4}Mv_f^2$$

$$m_1 = 10kg \quad m_2 = 2kg \quad M = 2kg \quad y_{1i} = 0.5\,m$$

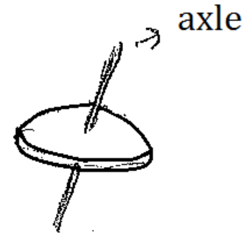
$$(10 - 2)(10)(0.5) = \frac{1}{2}(10 + 2)v_f^2 + \frac{1}{4}(2)v_f^2$$

$$40 = 6v_f^2 + \frac{v_f^2}{2} = \frac{13}{2}v_f^2$$

$$v_f^2 = \frac{80}{13} \Rightarrow \boxed{v_f = \sqrt{\frac{80}{13}}\ m/s}$$

Example: A disc of radius 0.2 m & mass 2 kg is rotating about an axle as shown. Its angular speed decreased from 20 rad/s to 10 rad/s in 5 seconds because of the force of friction between the disc and the axle.

a) Calculate the work done by friction.
Solution: From the work-kinetic energy theorem



$$w_f = \Delta KE_{rot} = \frac{1}{2}I\omega_f^2 = \frac{1}{2}I\omega_i^2 \qquad I = \frac{MR^2}{2} = \frac{(2)(0.2)^2}{2}$$
$$= \frac{1}{2}(.04)(10)^2\frac{1}{2}(.04)(20)^2 \qquad = .04\ kg\cdot m^2$$

$$\boxed{w_f = -6J}$$

b) If the radius of the axle is 0.01, calculate the force of friction

$$w_f = \tau_f \Delta\theta \Rightarrow -(f \cdot R_{axle})\Delta\theta = -6$$
$$\Rightarrow \ f(.01)(75) = 6$$
$$\Rightarrow \boxed{f = 8N}$$

$$\Delta\theta = \left(\frac{\omega_i + \omega_f}{2}\right)t = \left(\frac{20+10}{2}\right)5$$
$$= 75\, rad$$

10.4 Angular Momentum

Angular momentum is a physical quantity used as a measure of rotational motion. If the mass of a particle is m, the position vector of the particle with respect to the point of rotation is $\vec{r}$ and the velocity of the particle is $\vec{v}$, then its angular momentum $\vec{L}$, is defined as

$$\vec{L} = m\vec{r} \times \vec{v} = \vec{r} \times m\vec{v}$$

But $m\vec{v}$ is the linear momentum of $\vec{P}$

$$\therefore \quad \boxed{\vec{L} = \vec{r} \times \vec{P}}$$

Angular momentum is equal to the cross product between the position vector and the linear momentum.

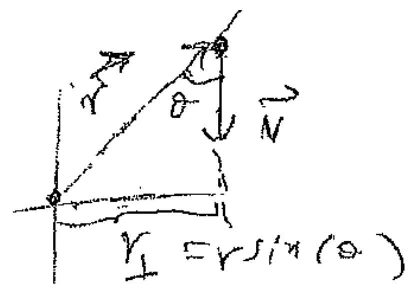
Magnitude $\left(\left|\vec{L}\right|\right)$

$\left|\vec{L}\right| = |\vec{r}| \cdot \left|\vec{P}\right| \sin\theta$

Where θ is angle between $\vec{r}$ & $\vec{P}$

And $\left|\vec{P}\right| = mv \quad |\vec{r}| = r$

$\therefore \quad \left|\vec{L}\right| = L = r\, m\, v \sin\theta = rsin\theta\, mv$



But $r\sin\theta = r_\perp$ perpendicular distance between point of rotation and the line of action of the velocity

$$\therefore \ L = r_\perp mv \quad also \quad v = r_\perp \omega$$
$$\therefore L = mr_\perp(r_\perp \omega)$$
$$L = mr_\perp^2 \omega$$
$$\text{But } mr_\perp^2 = I$$

$\therefore \ \boxed{L = I\omega}$ Angular momentum about a given axis of rotation is equal to the product of its moment of inertia about the axis and its angular speed

Angular Momentum in terms of position vector

$$\vec{L} = \vec{r} \times \vec{P} = \vec{r} \times m\vec{v} = m(\vec{r} \times \vec{v})$$
$$\text{but } \vec{v} = \frac{d\vec{r}}{dt}$$
$$\therefore \quad \boxed{L = m\, r \times \frac{d\vec{r}}{dt}}$$

Example: A particle of mass 3 kg is moving with a velocity $[3\hat{\imath} - 6\hat{\jmath}]$m/s. Calculate its angular momentum when its location with respect to the point of rotation is $(4{,}6)m$

$m = 3kg$
$\vec{v} = [3\hat{\imath} - 6\hat{\jmath}]$m/s
$\vec{r} = [4\hat{\imath} + 6\hat{\jmath}]m$
$\vec{L} = ??$

$$\vec{L} = m\,\vec{r} \times \vec{v}$$
$$= 3(4\hat{\imath} + 6\hat{\jmath}) \times (3\hat{\imath} - 6\hat{\jmath})$$
$$= 3\left[-24\hat{k} - 18\hat{k}\right]$$
$$= \boxed{-126\hat{k}\ kg \cdot \frac{m^2}{s^2}}$$

Example: The position vector of a particle of mass 0.1 kg varies with time according to the equation $\vec{r} = 4t^2\hat{\imath} - 2t\hat{\jmath}$. Calculate its angular moment time after 3 seconds

$m = 0.1\ kg$ $\vec{r}(t)$ $= 4t^2\hat{\imath} - 2t\hat{\jmath}$ $\vec{L}(t = 3) = ??$	$\vec{L} = m\,\vec{r} \times \frac{d\vec{r}}{dt}$	$\frac{d\vec{r}}{dt} = \frac{d}{dt}(4t^2\hat{\imath} - 2t\hat{\jmath})$ $= 8t\hat{\imath} - 2\hat{\jmath}$

$$\vec{L}(t = 3) = m\,\vec{r}(t = 3) \times \frac{d\vec{r}}{dt}(t = 3)$$
$$= 0.1[4(3)^2\hat{\imath} - 2(3)\hat{\jmath}] \times [8(3)\hat{\imath} - 2\hat{\jmath}]$$
$$= 0.1[36\hat{\imath} - 6\hat{\jmath}] \times [24\hat{\imath} - 2\hat{\jmath}]$$
$$= 0.1\left[-72\hat{k} + 144\hat{k}\right]$$
$$= 7.2\hat{k}\ kg \cdot \frac{m^2}{s^2}$$

Example: Calculate the angular momentum of a sphere of radius 4 cm and mass 4 kg ($I_{sphere}\frac{2}{5}MR^2$ about an axis passing through its center)

a) When it is rotating about an axis passing through its center with an angular velocity of 5 rad/s.

$$L = I\omega$$

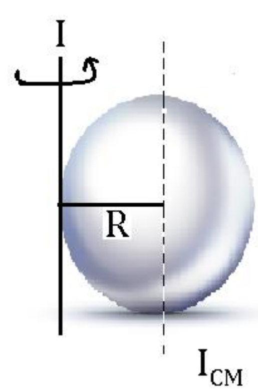
$$I_{CM} = \frac{2}{5}MR^2$$
$$= \frac{2}{5}(4)(.04)^2$$
$$I_{CM} = \frac{128}{5}kg \cdot m^2$$

$M = 4\ kg$
$R = 0.04\ m$
$\omega = 5$ rad/s

$$L = I_{CM}\omega$$
$$= \frac{128}{5} \cdot 5 = \boxed{125\ kg \cdot \frac{m^2}{s^2}}$$

b) When it is rotating about an axis tangent to the sphere with an angular speed of 2 rad/s.
From the parallel axis theorem

$I = I_{CM} + Md^2$ Where d is perpendicular distance between this axis & a parallel axis through the center of mass. In this case $d = R \rightarrow$ radius



$$I = I_{CM} + MR^2 \quad \text{and} \quad I_{CM} = \frac{2}{5}MR^2$$
$$I = \frac{2}{5}MR^2 + MR^2$$
$$= \frac{7}{5}MR^2$$
$$L = I\omega = \left(\frac{7}{5}MR^2\right)\omega$$

$M = 4\ kg$
$R = .04\ m$

$$= \frac{7}{5}(4)(.04)^2(2)$$
$$= \boxed{\frac{896}{5}\ kg \cdot \frac{m^2}{s^2}}$$

$\omega = 2$ rad/s

10.5 Conservation of angular momentum

Relationship between torque and angular momentum

$$\tau = I\alpha \quad \text{but} \quad \alpha = \frac{d\omega}{dt}$$
$$\tau = I\frac{d\omega}{dt} = \frac{d(I\omega)}{dt} \quad \text{but} \quad I\omega = L$$
$$\therefore \quad \boxed{\tau = \frac{dL}{dt}}$$

Torque is equal to the rate of change
of angular momentum with time.

If the net torque acting on an object is zero, then

$$\tau = \frac{dL}{dt} = 0$$
$$dL = 0$$
$$\int_{L_i}^{L_f} dL = L_f - L_i = 0$$
$$L_f = L_i$$

If $\tau_{net} = 0$, then $L_f = L_i$

Principle of conservation of angular momentum: states that if the net torque acting on an object is zero, then its angular momentum is conserved.

If $\tau_{net} = 0$, then $L_f = L_i$

Or $\boxed{I_f\omega_f = I_i\omega_i}$

Example: A skater of moment of inertia $5\ kg \cdot m^2$ is revolving with an angular speed of 3 rad/s. Calculate her new angular speed if she collapses her hands to reduce her moment of inertia to $4\ kg \cdot m^2$.

$I_i = 5\ kg\ m^2$

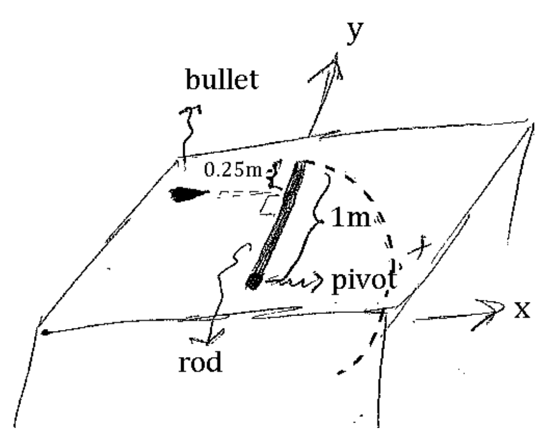
$\omega_i = 3$ rad/s

$I_f = 4\ kg\ m^2$

$\omega = ??$

$$I_f\omega_f = I_i\omega_i$$
$$(5)(3) = (4)\omega_f$$
$$\boxed{w_f = \frac{15}{4}\ rad/s}$$

Example: A rod of length 1m is lying on a frictionless table. One of its ends is pivoted so that it can be rotated freely. A bullet is fired to the rod perpendicularly as shown 0.25 m away from the free end with a speed of 500 m/s. The masses of the bullet and the rod are 0.4 kg and 5 kg, respectively.



If the bullet is embedded in the rod, calculate the angular speed of the bullet-rod system after the collision.

Solution

Let 'b' represent bullet and 'r' represent rod.

$m_b = 0.4\ kg$ $\quad\quad \omega_f = ??$

$m_r = 5\ kg$

$v_b = 500$ m/s

$L_r = 1\ m$

Because of conservation of angular momentum

$$\vec{L}_i = \vec{L}_f$$

Before the collision, only the bullet has angular momentum because the rod is at rest.

$$\vec{L}_i = \vec{L}_i^b$$

$$\vec{L}_i^b = m_b \vec{r}_b \times \vec{v}_b$$

Taking our origin to be the pivot

$\vec{r}_b = (1 - 0.25)m\ \hat{j}$(on the point of impact)

$= 0.75m\ \hat{j}$

$\vec{v}_b = 500$ m/s $\hat{\imath}$

$\vec{L}_i^b = 0.4(0.75\hat{j} \times 500\hat{\imath}) = -150\ \hat{k}$

After the collision the bullet-rod system will rotate about the pivot

$\vec{L}_f = I_f \omega_f (-\hat{k})$ (negative because they will rotate in a clockwise direction)

$I_f = I_{bf} + I_{rf}$

$I_{bf} = m_b r_{\perp b}^2$(treating the bullet like a particle)

$$(0.4)(1 - 0.25)^2$$

$$= 0.4(.75)^2 = \boxed{0.225\ kg \cdot m^2 = I_{bf}}$$

$$I_{rf} = \frac{1}{12} m_r L_r^2 + m_r \left(\frac{L_r}{2}\right)^2$$

(parallel axis theorem-the distance b/n pivot & CM is $\frac{L_r}{2}$)

$$I_{rf} = \frac{1}{12}(5)(1)^2 + 5\left(\frac{1}{2}\right)^2$$

$$\boxed{I_{rf} = 1.667\ kg \cdot m^2}$$

$$I_f = I_{bf} + I_{rf}$$

$$0.225 + 1.667 = 1.892\ kg \cdot m^2 \Longrightarrow$$

$$\vec{L}_i = \vec{L}_f$$

$$150 = 1.892\omega_f$$

$$\boxed{\omega_f = 79.281 \text{ rad/s}}$$