Solution   *Section* 1.5 – Continuity

Exercise

Given the graphed function $f(x)$

a) Does $f(-1)$ exist?

b) Does $\lim\limits_{x\to -1^+} f(x)$ exist?

c) Does $\lim\limits_{x\to -1^+} f(x) = f(-1)$?

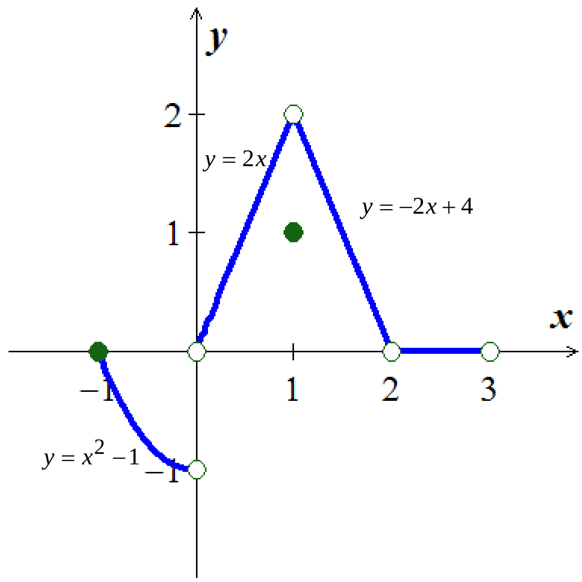
d) Is f continuous at $x=-1$?

e) Does $f(1)$ exist?

f) Does $\lim\limits_{x\to 1} f(x)$ exist?

g) Does $\lim\limits_{x\to 1} f(x) = f(1)$?

h) Is f continuous at $x=1$?



Solution

a) Yes $f(-1)=0$

b) Yes, $\lim\limits_{x\to -1^+} f(x) = 0$

c) Yes

d) Yes

e) Yes, $f(1)=1$

f) Yes, $\lim\limits_{x\to 1} f(x) = 2$

g) No

h) No

Exercise

At what points is the function $y=\dfrac{1}{x-2}-3x$ continuous?

Solution

The function is continuous everywhere except when $x-2=0 \Rightarrow x=2$

Exercise

At what points is the function $y=\dfrac{x+3}{x^2-3x-10}$ continuous?

Solution

The function is continuous everywhere except when $x^2-3x-10=0 \Rightarrow x=-2,\ 5$

Exercise

At what points is the function $y=|x-1|+\sin x$ continuous?

Solution

The function is continuous everywhere

Exercise

At what points is the function $y=\frac{x+2}{\cos x}$ continuous?

Solution

The function is continuous everywhere except when $\cos x=0 \Rightarrow x=\frac{\pi}{2}+n\pi, \quad n\in\mathbb{Z}$

Exercise

At what points is the function $y=\tan\frac{\pi x}{2}$ continuous?

Solution

The function is continuous everywhere except when $x=2n-1, \quad n\in\mathbb{Z}$

Exercise

At what points is the function $y=\frac{x\tan x}{x^2+1}$ continuous?

Solution

The function is continuous everywhere except when $x=(2n-1)\frac{\pi}{2}, \quad n\in\mathbb{Z}$

Exercise

At what points is the function $y=\frac{\sqrt{x^4+1}}{1+\sin^2 x}$ continuous?

Solution

The function is continuous everywhere

Exercise

At what points is the function $y=\sqrt{2x+3}$ continuous?

Solution

The function is continuous on the interval $2x+3\ge 0 \quad \to \quad x\ge -\frac{3}{2} \Rightarrow \left[-\frac{3}{2},\ \infty\right)$, and discontinuous when $x<-\frac{3}{2}$

Exercise

At what points is the function $y=\sqrt[4]{3x-1}$ continuous?

Solution

The function is continuous on the interval $3x-1\ge 0 \to \left[\frac{1}{3},\ \infty\right)$, and discontinuous when $x<\frac{1}{3}$

Exercise

At what points is the function $y=(2-x)^{1/5}$ continuous?

Solution

The function is continuous everywhere $\forall x$

Exercise

Find $\lim\limits_{x\to\pi}\sin(x-\sin x)$, then is the function continuous at the point being approached?

Solution

$$\begin{aligned}\lim_{x\to\pi}\sin(x-\sin x)&=\sin(\pi-\sin\pi)\\&=\sin(\pi-0)\\&=\sin(\pi)\\&=0\end{aligned}$$

The function is continuous at $x=\pi$

Exercise

Find $\lim\limits_{x\to 0}\tan\left(\frac{\pi}{4}\cos\left(\sin x^{1/3}\right)\right)$, then is the function continuous at the point being approached?

Solution

$$\begin{aligned}\lim_{x\to 0}\tan\left(\frac{\pi}{4}\cos\left(\sin x^{1/3}\right)\right)&=\tan\left(\frac{\pi}{4}\cos\left(\sin(0)^{1/3}\right)\right)\\&=\tan\left(\frac{\pi}{4}\cos(0)\right)\\&=\tan\left(\frac{\pi}{4}\right)\\&=1\end{aligned}$$

The function is continuous at $x=0$

Exercise

Find $\lim\limits_{t\to 0}\cos\left(\frac{\pi}{\sqrt{19-3\sec 2t}}\right)$, then is the function continuous at the point being approached?

Solution

$$\begin{aligned}\lim_{t\to 0}\cos\left(\frac{\pi}{\sqrt{19-3\sec 2t}}\right)&=\cos\left(\frac{\pi}{\sqrt{19-3\sec 2(0)}}\right)\\&=\cos\left(\frac{\pi}{\sqrt{19-3}}\right)\\&=\cos\left(\frac{\pi}{\sqrt{16}}\right)\end{aligned}$$

$$= \cos\left(\frac{\pi}{4}\right)$$

$$\boxed{= \frac{\sqrt{2}}{2}} \qquad \text{The function is continuous at } t = 0$$

Exercise

Explain why the equation $\cos x = x$ has at least one solution.

Solution

$$\cos x - x = 0$$

$$\begin{cases} if \quad x = -\frac{\pi}{2} & \to \cos\left(-\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right) > 0 \\ if \quad x = \frac{\pi}{2} & \to \cos\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\right) < 0 \end{cases} \Rightarrow \cos x - x = 0 \text{ for some } x \text{ between } -\frac{\pi}{2} \text{ and } \frac{\pi}{2}$$

According to the Intermediate Value Theorem, and the function $\cos x = x$ is continuous and has at least one solution.

Exercise

Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval [–4, 4]

Solution

$$f(-4) = (-4)^3 - 15(-4) + 1 = -3$$

$$f(-2) = (-2)^3 - 15(-2) + 1 = 23$$

$$f(-1) = (-1)^3 - 15(-1) + 1 = 15$$

$$f(1) = (1)^3 - 15(1) + 1 = -13$$

$$f(4) = (4)^3 - 15(4) + 1 = 5$$

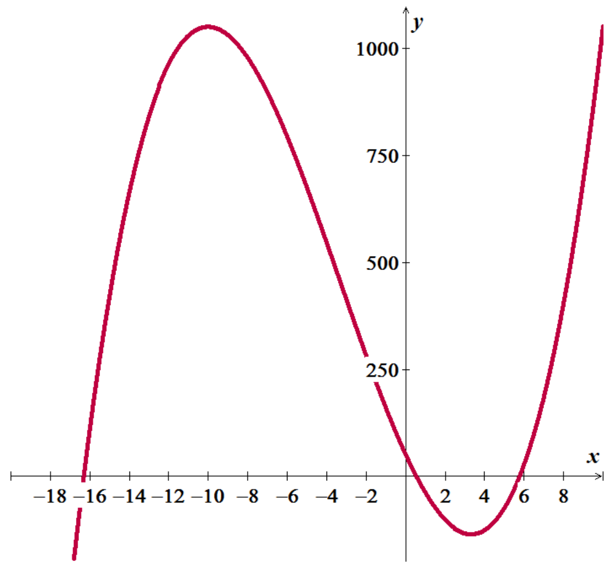
By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-4 < x < -1$, $-1 < x < 1$, and $1 < x < 4$. Thus, $x^3 - 15x + 1 = 0$ has three solutions in [–4, 4]. Since the polynomial of degree 3 can have at most 3 solutions, these are the solutions.

Exercise

Show that the equation has three solutions in the given interval $x^3 + 10x^2 - 100x + 50 = 0; \quad (-20, 10)$

Solution

x	y
–19	–1299
–18	–742
–17	–273
–16	114
–15	425
–14	666
–13	962
–12	1029
–10	1050
–9	1031
–8	978
–7	897
–6	794
–5	675
–4	546
–3	413
–2	282
–1	159
0	50
1	–39
2	–102
3	–133
4	–126
5	–75
6	26



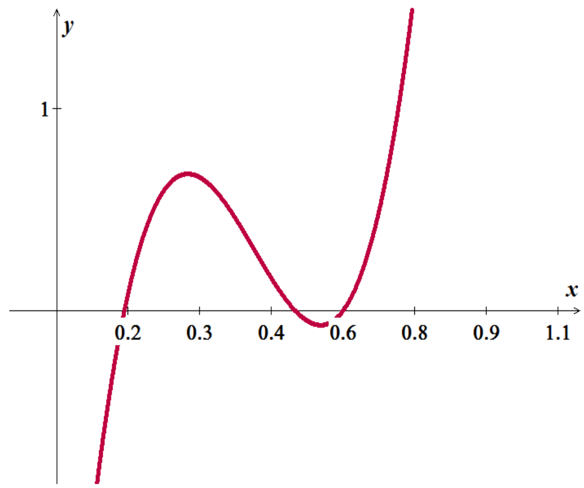
By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-17 < x < -16$, $0 < x < 1$, and $5 < x < 6$.

Exercise

Show that the equation has three solutions in the given interval $70x^3 - 87x^2 + 32x - 3 = 0; \quad (0,\ 1)$

Solution

x	y
.05	–1.6
.1	–0.6
.15	0.08
.2	.48
.25	.656
.3	.66
.35	.543
.4	.36
.45	.161
.5	0
.55	–.07
.6	0
.65	.266
.7	.78
.75	1.6
.8	2.76
.85	4.33
.9	6.36
.95	8.9



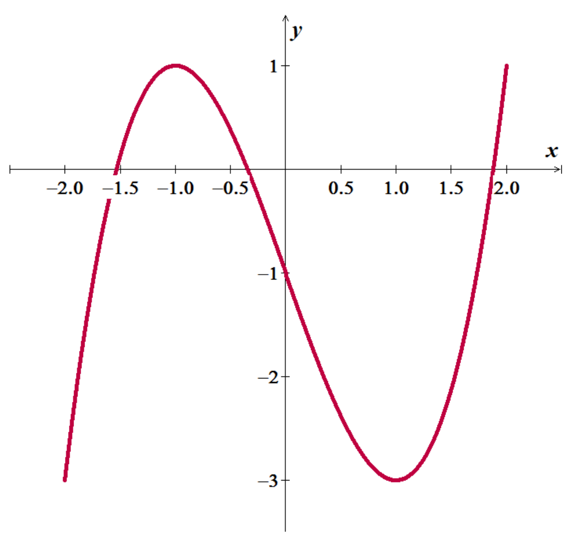
By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $0.1 < x < 0.15$, $0.5 < x < 0.55$, and $0.55 < x < 0.6$.

Exercise

Show that the equation has three solutions in the given interval $x^3 - 3x - 1 = 0; \quad [-2,\ 2]$

Solution

x	y
–2	–3.0
–1.75	–1.109
–1.5	0.125
–1.25	0.797
–1.0	1
–0.75	0.828
–0.5	0.375
–0.25	–0.266
0	–1.0
0.25	–1.73
0.5	–2.375
0.75	–2.828
1.0	–3.0
1.25	–2.797
1.5	–2.12
1.75	–0.89
2.	1.0



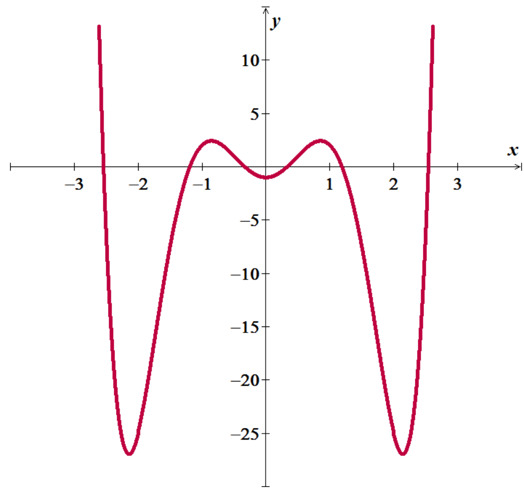
By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-1.75 < x < -1.5$, $-0.5 < x < -0.25$, and $1.75 < x < 2$.

Exercise

Show that the equation has six solutions in the given interval $x^6 - 8x^4 + 10x^2 - 1 = 0; \quad [-3,\ 3]$

Solution

x	y
–3.0	170.0
–2.5	–6.86
–2.0	–25.0
–1.5	–7.61
–1.0	2.0
–0.5	1.02
0.0	–1.0
0.5	1.01
1.0	2.0
1.5	–7.6
2.0	–25.0
2.5	–6.86
3.0	170.0



By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-3.0 < x < -2.5$, $-1.5 < x < -1.0$, $-0.5 \le x \le 0$, $-0.0 \le x \le 0.5$, $1.0 \le x \le 1.5$ and $2.5 < x < 3.0$.

Exercise

If functions $f(x)$ and $g(x)$ are continuous for $0 \le x \le 1$, could $\dfrac{f(x)}{g(x)}$ possibly be discontinuous at a point of [0, 1]? Give reason for your answer.

Solution

Yes, if we can get a value of $g(x)$ is between [0, 1], $x = \frac{1}{2} \quad \Rightarrow g(x) = 2x - 1$ and $f(x) = x$.

Then $\dfrac{f(x)}{g(x)} = \dfrac{x}{2x-1} \quad \Rightarrow \dfrac{f(x)}{g(x)}$ is discontinuous at $x = \frac{1}{2}$

Exercise

Suppose that a function f is continuous on the closed interval [0, 1] and that $0 \le f(x) \le 1$ for every x in [0, 1]. Show that there must exist a number c in [0, 1] such that $f(c) = c$ (**c** is called a ***fixed point*** of f).

Solution

Let $f(x) = x \Rightarrow f(0) = 0 \quad or \quad f(1) = 1$. In these cases, c = 0 or c =1.

Let $f(0) = a > 0 \quad and \quad f(1) = b < 1$ because $0 \le f(x) \le 1$.

Define $g(x) = f(x) - x \Rightarrow g$ is continuous on [0, 1]. $\Rightarrow \begin{cases} g(0) = f(0) - 0 = a > 0 \\ g(1) = f(1) - 1 = b - 1 < 0 \end{cases}$

By the Intermediate Value Theorem there is a number c in [0, 1] such that

$g(c) = 0 \Rightarrow f(c) - c = 0 \Rightarrow f(c) = c$

Exercise

Use the Intermediate Value Theorem to show that the equation $x^5 + 7x + 5 = 0$ has a solution in the interval $(-1,\ 0)$.

Solution

$f(-1) = -1 - 7 + 5 = -3 < 0$

$f(0) = 5 > 0$

By Intermediate value theorem, the function has a solution in $(-1,\ 0)$

Exercise

The amount of an antibiotic (in *mg*) in the blood *t* hours after an intravenous line is opened is given by

$$m(t) = 100\left(e^{-0.1t} - e^{-0.3t}\right)$$

a) Use the Intermediate Value Theorem to show that the amount of drug is 30 *mg* at some time in the interval $[0,\ 5]$ and again at some time in the interval $[5,\ 15]$

b) Estimate the times at which $m = 30\ mg$

c) Is the amount of drug in the blood ever 50 *mg*?

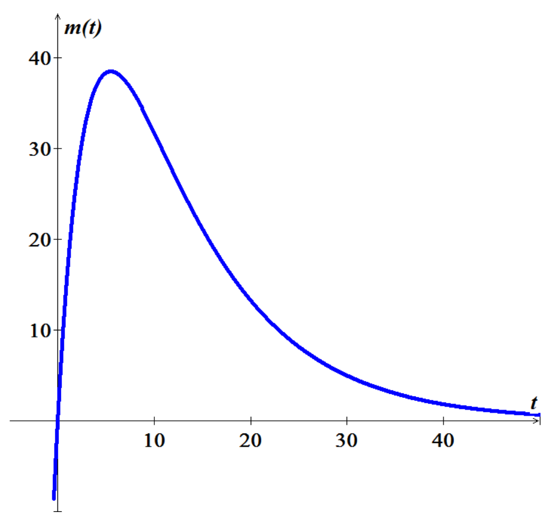
Solution

a) $m(0) = 100(1-1) = 0$

$m(5) \approx 38.34 > 30$

$m(15) \approx 21.2 < 30$

30 is an intermediate value between for both $[0,\ 5]$ and $[5,\ 15]$.



b) $m(t) = 100\left(e^{-0.1t} - e^{-0.3t}\right) = 30$

$$e^{-0.1t} - e^{-0.3t} = 0.3 \xrightarrow{\text{software}} \begin{cases} \boxed{t_1 \approx 2.4} \\ \boxed{t_2 \approx 10.8} \end{cases}$$

c) No, peak is 38.5 (using the graph)

Exercise

Determine whether the following functions are continuous at a. $\quad f(x) = \dfrac{1}{x-5}; \quad a = 5$

Solution

$f(5)\ \nexists$

The function is continuous everywhere except @ $x = 5$

Exercise

Determine whether the following functions are continuous at a. $h(x)=\sqrt{x^2-9}$; $a=3$

Solution

$\lim\limits_{x\to 3^-} h(x)\ \nexists$ $\quad\therefore\ h$ is discontinuous @ 3

Exercise

Determine whether the following functions are continuous at a. $g(x)=\begin{cases}\dfrac{x^2-16}{x-4} & if \quad x\neq 4\\ 9 & if \quad x=4\end{cases}$; $a=4$

Solution

$$\lim_{x\to 4} g(x)=\lim_{x\to 4}\frac{(x-4)(x+4)}{x-4}=\lim_{x\to 4}(x+4)=8\neq 9=g(4)$$

$\therefore\ g$ is discontinuous @ 4

Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $\quad f(x)=\sqrt{x^2-5}$

Solution

$\sqrt{x^2-5\geq 0} \quad\Rightarrow\quad x\leq -5$ & $x\geq 5$

The function is continuous at -5 to the left and right of $x=5$

Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $\quad f(x)=e^{\sqrt{x-2}}$

Solution

The function is continuous at and to the right of $x=2$

Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $\quad f(x)=\dfrac{2x}{x^3-25x}$

Solution

The function is continuous everywhere except at $x=0,\ \pm 5$

The function is continuous to the left of -5, then to the right of -5 to the left of 0, then to the right of 0 thru the left of 5 then to the tight of 5.

Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $\quad f(x)=\cos e^{x}$

Solution

The function is continuous everywhere.

Exercise

Let $g(x)=\begin{cases} 5x-2 & \text{if} \quad x<1 \\ a & \text{if} \quad x=1 \\ ax^2+bx & \text{if} \quad x>1 \end{cases}$

Determine values of the constants a and b for which $g(x)$ is continuous at $x=1$

Solution

$$\lim_{x\to 1^-} g(x)=g(1)=5-2=\underline{3=a}$$

$$\lim_{x\to 1^-} g(x)=g(1)=a+b=3+b=3 \quad \Rightarrow \quad \underline{b=0}$$