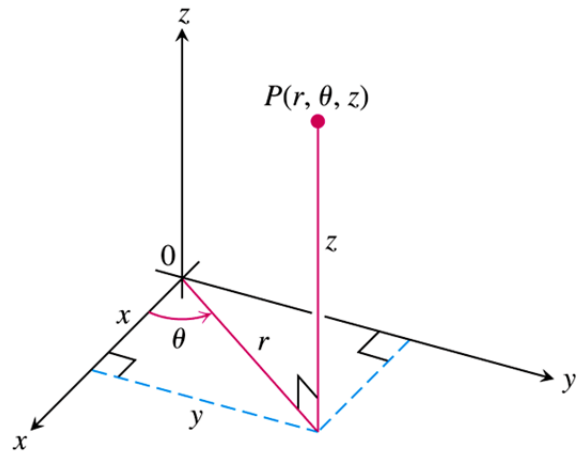
Section 3.5 – Triple Integrals in Cylindrical and Spherical Coordinates

Integration in Cylindrical Coordinates

Definition

Cylindrical coordinates represents a point P in space by ordered triples (r,θ,z) in which

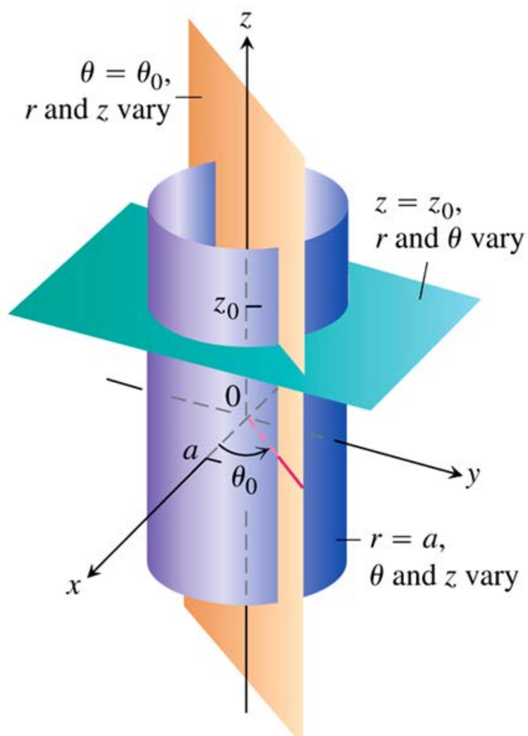
1. r *and* θ are polar coordinates for the vertical projection of P on the xy-plane
2. z is the rectangular vertical coordinate.



Equations Reating Rectangular (x,y,z) and Cylindrical (r,θ,z) Coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$



The triple integral of a function f over D is obtained by taking a limit of such Riemann sums with partitions whose norms approach zero:

$$\lim_{n\to\infty} S_n = \iiint\limits_D f dV = \iiint\limits_D f\ dz\ r dr d\theta$$

Example

Find the limits of integration in cylindrical coordinates for integrating a function $f(r,\theta,z)$ over the region *D* bounded below by the plane $z=0$, laterally by the circular cylinder $x^2+(y-1)^2=1$, and above by the paraboloid $z=x^2+y^2$.

Solution

Base of *D* is the region's projection *R* on the *xy*-plane.

The boundary of *R* is the circle $x^2+(y-1)^2=1$.

The polar coordinate equation is

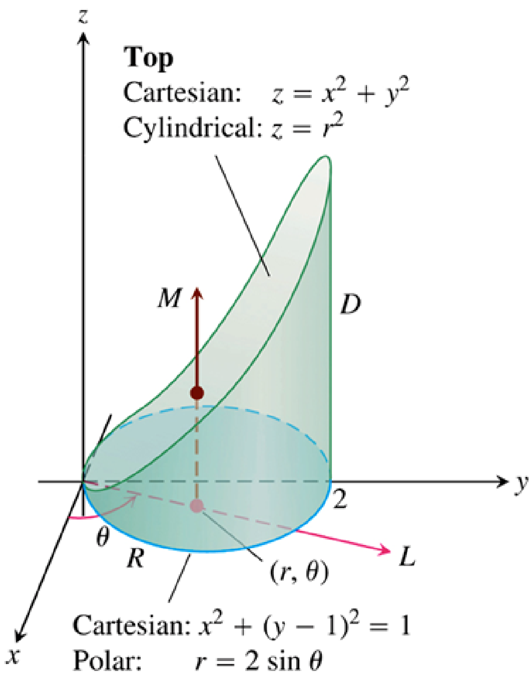
$$x^2+(y-1)^2=1$$

$$x^2+y^2-2y+1=1$$

$$r^2-2r\sin\theta=0$$

$$r(r-2\sin\theta)=0$$

$$r=2\sin\theta$$



z-limits: A line *M* through a typical point $(r,\ \theta)$ in *R* // *z*-axis enters *D* at $z=0$ and leaves at $z=x^2+y^2=r^2$

r-limits: starts at $r=0$ and ends at $r=2\sin\theta$

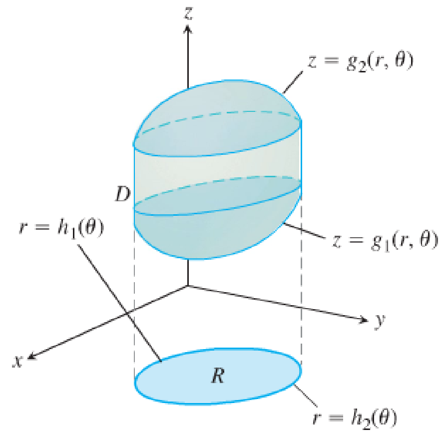
θ-limits: From $\theta=0$ to $\theta=\pi$

$$\iiint_D f\ dz\ rdrd\theta=\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2} f(r,\theta,z)dz\ r\ drd\theta$$

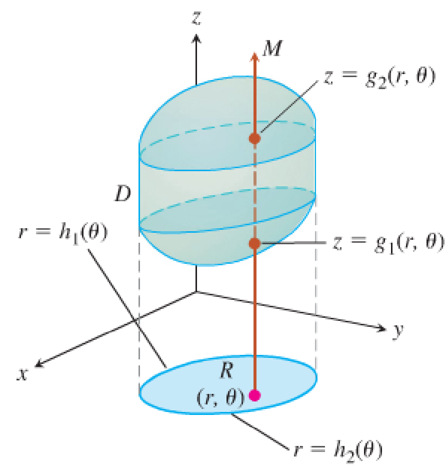
How to integrate in Cylindrical Coordinates

To evaluate $$\iiint\limits_D F(r,\ \theta,\ z)\,dV$$

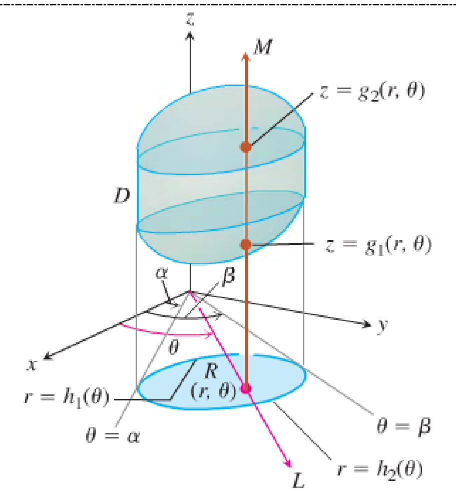
1. ***Sketch***: Sketch the region D along with its projection R on the xy-plane. Label the upper and lower bounding surfaces of D and R.



2. ***Find the z-limits of integration***: Draw a line M passing through (r,θ) in R // z-axis. As z increases, M enters D at $z = g_1(r,\theta)$ to $z = g_2(r,\theta)$.



3. ***Find the r-limits of integration***: Draw a line L passing through (r,θ) from the origin. From $r = h_1(\theta)$ to $r = h_2(\theta)$.



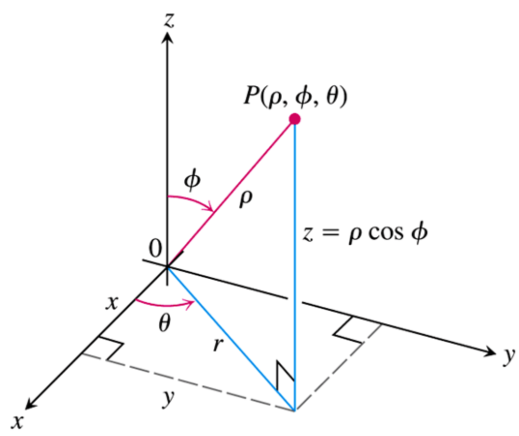
4. ***Find the θ-limits of integration***: As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = \alpha$ *and* $\theta = \beta$.

$$\iiint\limits_D F(r,\ \theta,\ z)\,dV = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r=h_2(\theta)}\int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} F(r,\theta,z)\,dz\ r\,dr\,d\theta$$

Definition

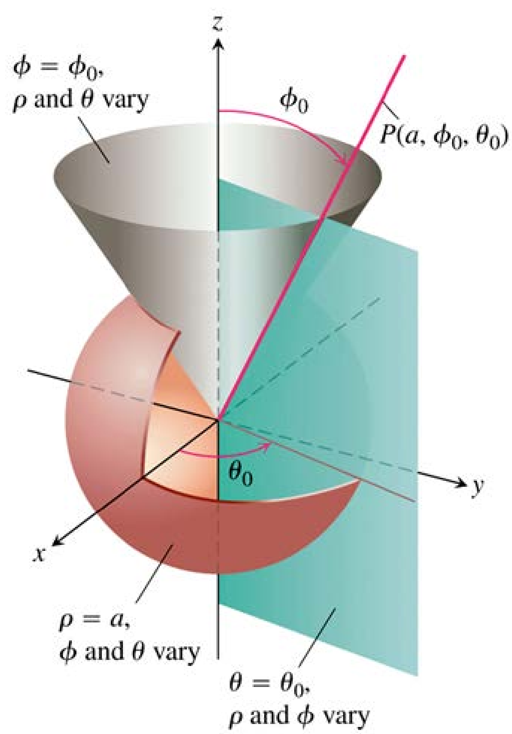
Spherical coordinates represent a point P in space by ordered triple $(\rho,\ \phi,\ \theta)$ in which

1. ρ is the distance from P to the origin.
2. ϕ is the angle $\overrightarrow{OP}$ makes with positive z-axis $(0 \le \phi \le \pi)$.
3. θ is the angle from the cylindrical coordinates $(0 \le \theta \le 2\pi)$



Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$r = \rho \sin\phi, \quad x = r\cos\theta = \rho \sin\phi \cos\theta,$$
$$z = \rho \cos\phi, \quad y = r\sin\theta = \rho \sin\phi \sin\theta,$$
$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$$



Example

Find a spherical coordinate equation for the sphere $x^2+y^2+(z-1)^2=1$

Solution

$$x^2+y^2+(z-1)^2=1$$

$$\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta+(\rho\cos\phi-1)^2=1$$

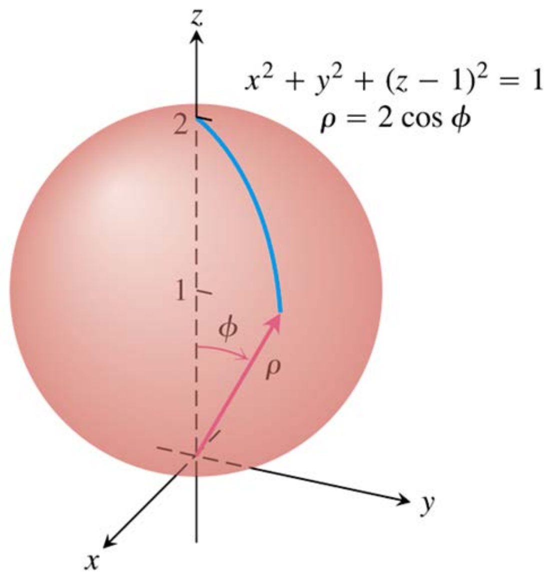
$$\rho^2\sin^2\phi\left(\cos^2\theta+\sin^2\theta\right)+\rho^2\cos^2\phi-2\rho\cos\phi+1=1 \qquad \cos^2\theta+\sin^2\theta=1$$

$$\rho^2\left(\sin^2\phi+\cos^2\phi\right)-2\rho\cos\phi=0$$

$$\rho^2-2\rho\cos\phi=0$$

$$\rho(\rho-2\cos\phi)=0 \qquad \rho>0$$

$$\boxed{\rho=2\cos\phi}$$



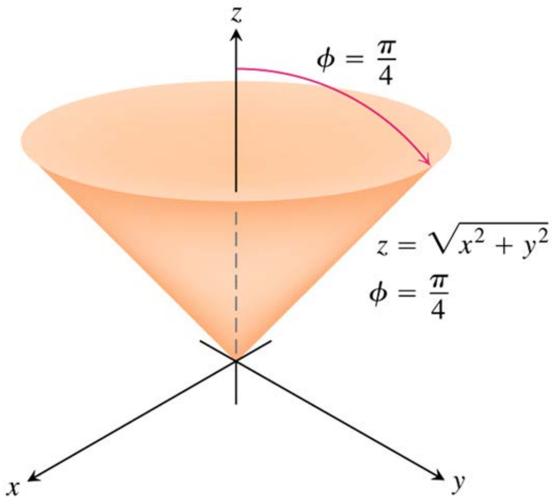
The angle ϕ varies from 0 to the north pole of the sphere to $\frac{\pi}{2}$ at the south pole; the angle θ doesn't appear in the expression for ρ, reflecting the symmetry about the z-axis.

Example

Find a spherical coordinate equation for the sphere $z = \sqrt{x^2 + y^2}$

Solution

The cone is symmetric with respect to the *z*-axis and cuts the first quadrant of the yz-plane along the line $z = y$. The angle between the cone and the positive *z*-axis is therefore $\frac{\pi}{4}$ rad. The cone consists of the points whose spherical coordinates have $\phi = \frac{\pi}{4}$.



$$z = \sqrt{x^2 + y^2}$$

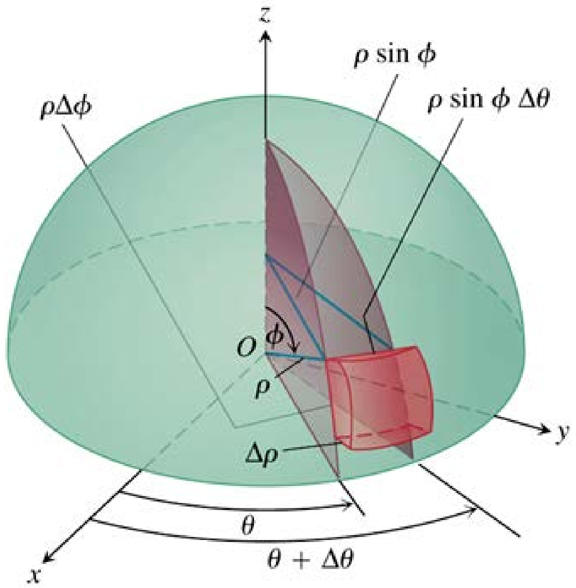
$$\rho \cos\phi = \sqrt{\rho^2 \sin^2 \phi}$$

$$\rho \cos\phi = \rho \sin\phi$$

$$\cos\phi = \sin\phi \quad \rightarrow \quad \boxed{\phi = \frac{\pi}{4}}$$

Volume Differential in Spherical Coordinates

$$dV = \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta$$



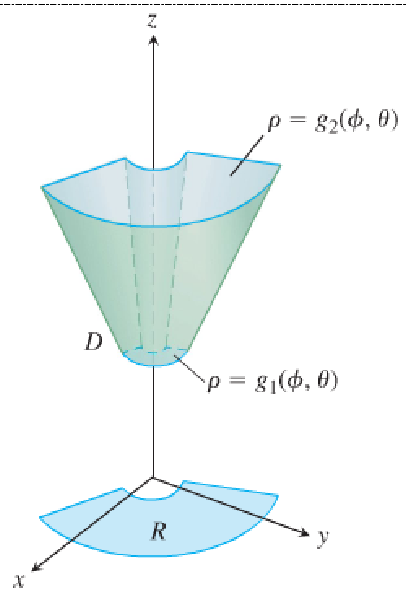
$$dV = d\rho \cdot \rho d\phi \cdot \rho \sin\phi d\theta = \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta$$

How to integrate in Spherical Coordinates

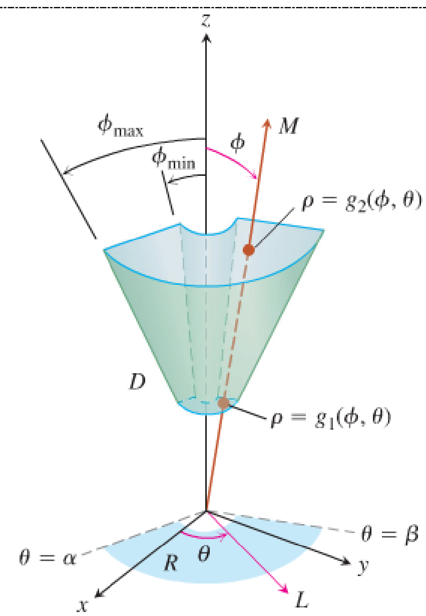
To evaluate $$\iiint_D F(\rho,\ \phi,\ \theta)\,dV$$

1. ***Sketch***: Sketch the region D along its projection R on the xy-plane. Label the surface that bound of D.

2. ***Find the ρ-limits of integration***: Draw a ray M from the origin through D making an angle ϕ with the positive z-axis. Also draw the projection of M on the xy-plane (call the projection L). The ray L makes an angle θ with the positive x-axis. As ρ increases, M enters D at $\rho = g_1(\phi,\theta)$ to $\rho = g_2(\phi,\theta)$.



3. ***Find the ϕ-limits of integration***: For the given θ, the angle ϕ that M makes with the z-axis runs $\phi = \phi_{\min}$ to $\phi = \phi_{\max}$.



5. ***Find the θ-limits of integration***: As L sweeps over R as θ runs from α to β.

$$\iiint_D f(\rho,\ \phi,\ \theta)\,dV = \int_{\theta=\alpha}^{\theta=\beta}\int_{\phi=\phi_{\min}}^{\phi=\phi_{\max}}\int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} f(\rho,\ \phi,\ \theta)\rho^2 \sin\phi\ d\rho\, d\phi\, d\theta$$

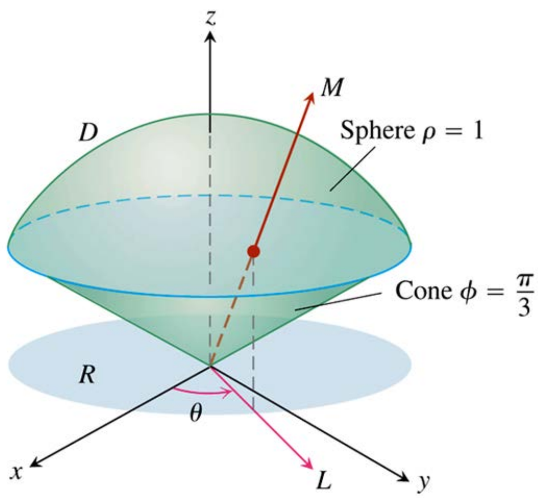
Example

Find the volume of the "ice cream cone" D cut from the solid sphere $\rho \le 1$ by the cone $\phi = \frac{\pi}{3}$

Solution

$$f(\rho, \phi, \theta) = 1$$

$$V = \iiint\limits_D \rho^2 \sin\phi \, d\rho d\phi d\theta$$



ρ-limits: Draw a ray M from the origin through D making an angle ϕ with the positive z-axis. And L, the projection of M on the xy-plane, along with the angle θ that L makes with the positive x-axis. Ray M enters D form $\rho = 0$ to $\rho = 1$

ϕ-limits: The cone $\phi = \frac{\pi}{3}$ makes with the positive z-axis. $0 \le \phi \le \frac{\pi}{3}$

θ-limits: $0 \le \theta \le 2\pi$

$$V = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi \, d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \left[\frac{1}{3}\rho^3\right]_0^1 \sin\phi \, d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \frac{1}{3} \sin\phi \, d\phi d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi} \left[\cos\phi\right]_0^{\pi/3} d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi} \left(\frac{1}{2} - 1\right) d\theta$$

$$= \frac{1}{6}\int_0^{2\pi} d\theta$$

$$= \frac{1}{6}\theta \Big|_0^{2\pi}$$

$$= \frac{1}{6}(2\pi - 0)$$

$$= \frac{\pi}{3} \quad unit^3$$

Coordinate Conversion Formulas

Cylindrical to Rectangular	***Spherical to Rectangular***	***Spherical to Cylindrical***
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\ dy\ dz \\ &= dz\ rdr\ d\theta \\ &= \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta \end{aligned}$$

Exercises Section 3.5 – Triple Integrals in Cylindrical and Spherical Coordinates

Evaluate the cylindrical coordinate integral

1. $\displaystyle\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\ r dr\ d\theta$

2. $\displaystyle\int_0^{2\pi}\int_0^{\theta/(2\pi)}\int_r^{3+24r^2} dz\ r dr\ d\theta$

3. $\displaystyle\int_0^{\pi}\int_0^{\theta/\pi}\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} z dz\ r dr\ d\theta$

4. $\displaystyle\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2}\left(r^2\sin^2\theta+z^2\right)dz\ r dr\ d\theta$

Evaluate the integral

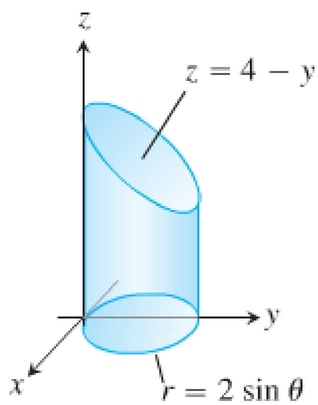
5. $\displaystyle\int_0^{2\pi}\int_0^3\int_0^{z/3} r^3 dr\ dz\ d\theta$

6. $\displaystyle\int_0^1\int_0^{\sqrt{z}}\int_0^{2\pi}\left(r^2\cos^2\theta+z^2\right)r\ d\theta\ dr dz$

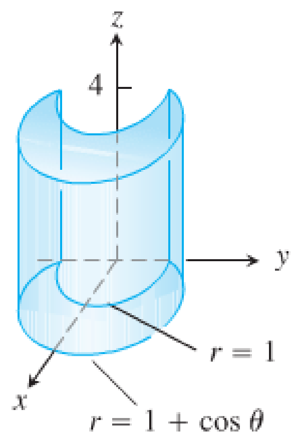
7. $\displaystyle\int_0^2\int_{r-2}^{\sqrt{4-r^2}}\int_0^{2\pi}(r\sin\theta+1)r\ d\theta\ dz\ dr$

8. Convert the integral $\displaystyle\int_{-1}^1\int_0^{\sqrt{1-y^2}}\int_0^x\left(x^2+y^2\right)dzdxdy$ to an equivalent integral in cylindrical coordinates and evaluate the result.

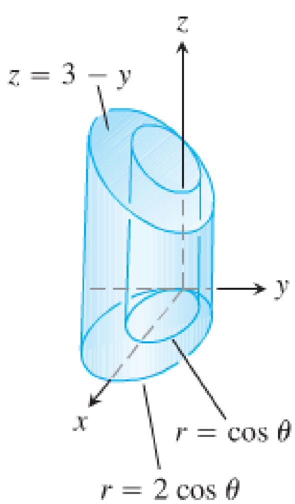
9. Set up the iterated integral for evaluating $\displaystyle\iiint_D f(r,\theta,z)\,dzdrd\theta$ over the region D that is the right circular cylinder whose base is the circle $r=2\sin\theta$ in the xy-plane and whose top lies in the plane $z=4-y$



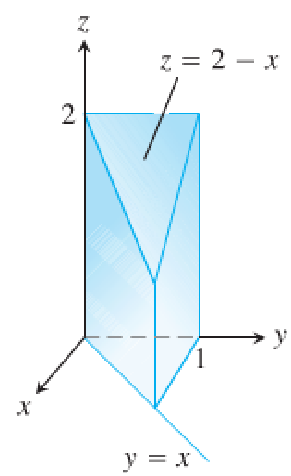
10. Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\,dz\,dr\,d\theta$ over the region D which is the solid right cylinder whose base is the region in the *xy*-plane that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ and whose top lies in the plane $z = 4$



11. Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\,dz\,dr\,d\theta$ over the region D which is the solid right cylinder whose base is the region between the circles $r = \cos\theta$ and $r = 2\cos\theta$ and whose top lies in the plane $z = 3 - y$



12. Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\,dz\,dr\,d\theta$ over the region D which is the prism whose base is the triangle in the *xy*-plane bounded by the *y*-axis and the lines $y = x$ and $y = 1$ and whose top lies in the plane $z = 2 - x$



13. Evaluate the spherical coordinate integral $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

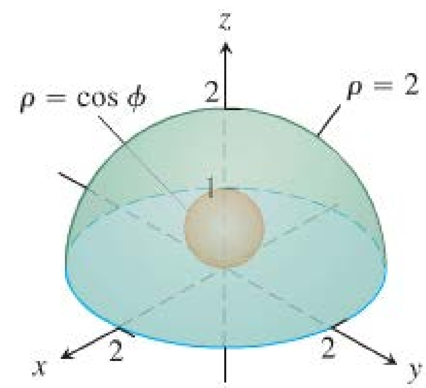
14. Evaluate the spherical coordinate integral $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} (\rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

15. Evaluate the spherical coordinate integral $\displaystyle\int_0^{3\pi/2}\int_0^{\pi}\int_0^{1} 5\rho^3 \sin^3\phi \, d\rho \, d\phi \, d\theta$

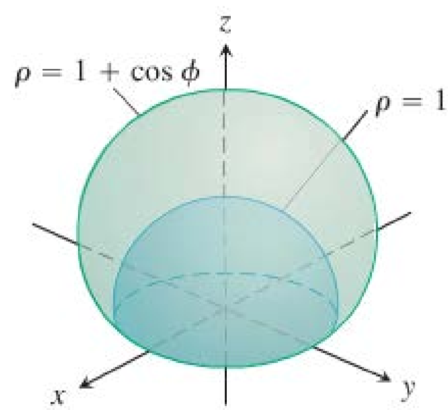
16. Evaluate the integral $\displaystyle\int_0^{2}\int_{-\pi}^{0}\int_{\pi/4}^{\pi/2} \rho^3 \sin 2\phi \, d\phi \, d\theta \, d\rho$

17. Evaluate the integral $\displaystyle\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\csc\phi}^{2} 5\rho^4 \sin^3\phi \, d\rho \, d\theta \, d\phi$

18. Find the volume of the solid between the sphere $\rho = \cos\phi$ and the hemisphere $\rho = 2,\ z \ge 0$

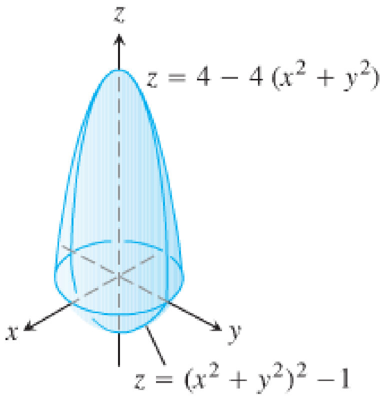


19. Find the volume of the solid bounded below by the hemisphere $\rho = 1,\ z \ge 0$, and above the cardioid of revolution $\rho = 1 + \cos\phi$

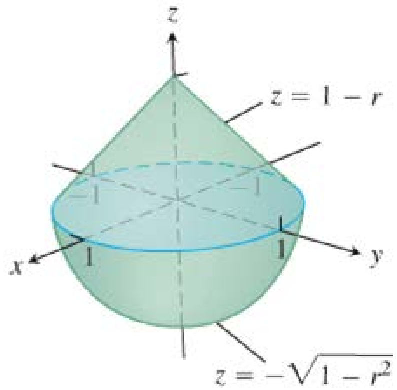


20. Find the volume of the solid

a)



b)



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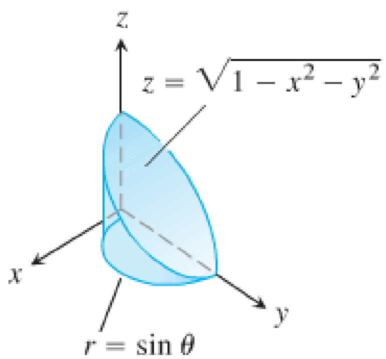

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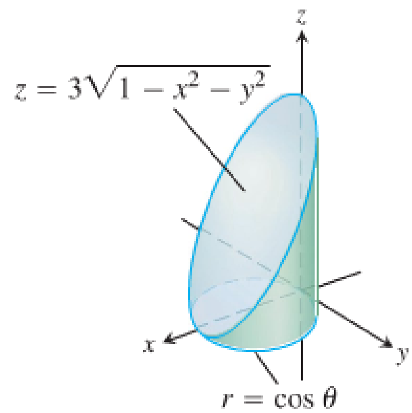
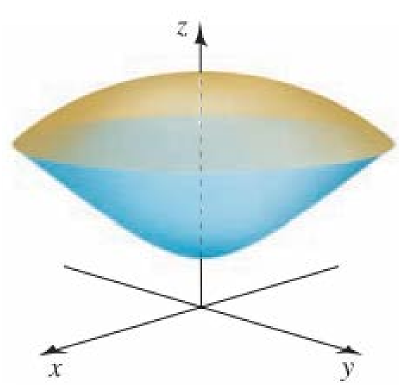


21. Find the volume of the smaller region cut from the solid sphere $\rho \le 2$ by the plane $z = 1$

22. Find the volume of the region bounded below by the paraboloid $z = x^2 + y^2$, laterally by the cylinder $x^2 + y^2 = 1$, and above by the paraboloid $z = x^2 + y^2 + 1$

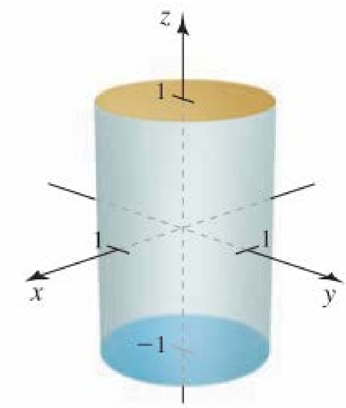
23. Find the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$

24. Find the volume of the solid between the sphere $x^2 + y^2 + z^2 = 19$ and the hyperboloid $z^2 - x^2 - y^2 = 1$ for $z > 0$



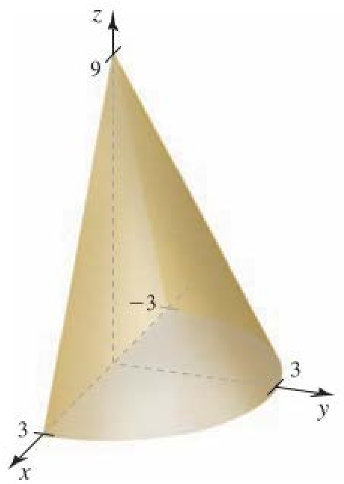
25. Evaluate the integral in cylindrical coordinates

$$\int_0^{2\pi}\int_0^1\int_{-1}^1 r\, dzdrd\theta$$



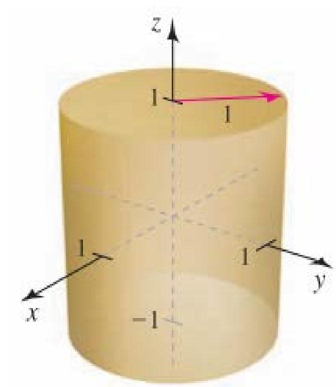
26. Evaluate the integral in cylindrical coordinates

$$\int_0^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_0^{9-3\sqrt{x^2+y^2}} dzdxdy$$



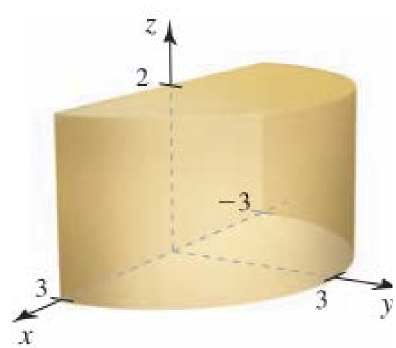
27. Evaluate the integral in cylindrical coordinates

$$\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{-1}^1 \left(x^2+y^2\right)^{3/2}\, dzdxdy$$

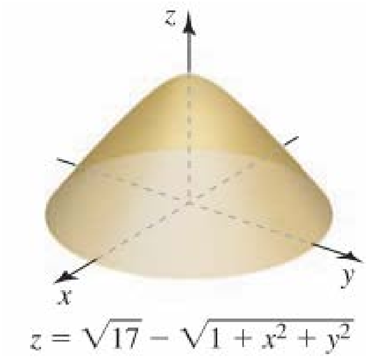


28. Evaluate the integral in cylindrical coordinates

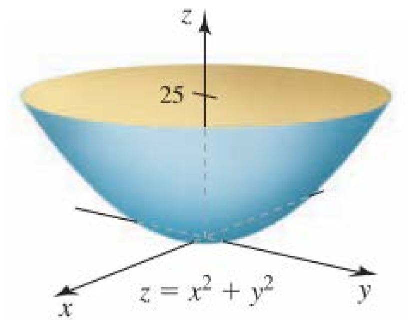
$$\int_{-3}^3\int_0^{\sqrt{9-x^2}}\int_0^2 \frac{1}{1+x^2+y^2}\, dzdydx$$



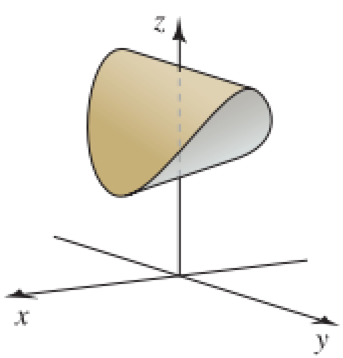
29. Evaluate the integral in cylindrical coordinates to find the volume of the solid bounded by the plane $z = 0$ and the hyperboloid $z = \sqrt{17} - \sqrt{1 + x^2 + y^2}$



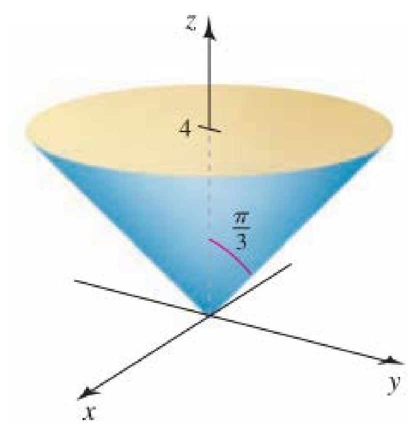
30. Evaluate the integral in cylindrical coordinates to find the volume of the solid bounded by the plane $z = 25$ and the paraboloid $z = x^2 + y^2$



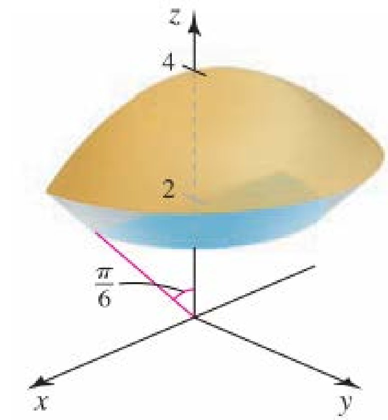
31. Evaluate the integral in cylindrical coordinates to find the volume of the solid bounded by the parabolic cylinders $z = y^2 + 1$ and $z = 2 - x^2$



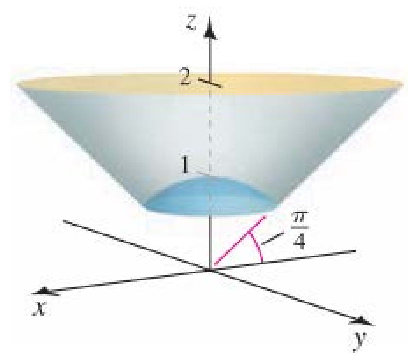
32. Evaluate the integral $\displaystyle\int_0^{2\pi} \int_0^{\pi/3} \int_0^{4\sec\varphi} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$



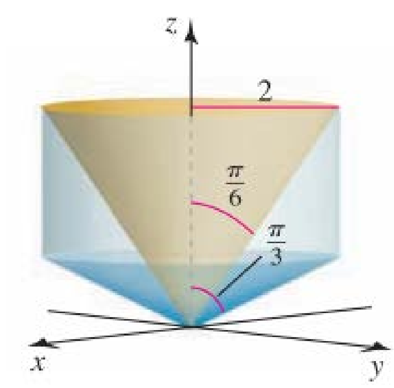
33. Evaluate the integral $\int_0^{\pi}\int_0^{\pi/6}\int_{2\sec\varphi}^{4} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$



34. Evaluate the integral $\int_0^{2\pi}\int_0^{\pi/4}\int_1^{2\sec\varphi} \left(\rho^{-3}\right)\rho^2 \sin\varphi \, d\rho d\varphi d\theta$

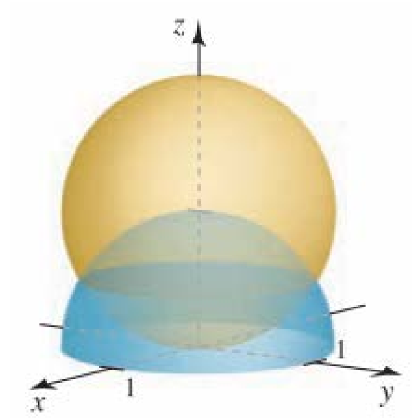


35. Evaluate the integral $\int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_0^{2\csc\varphi} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$



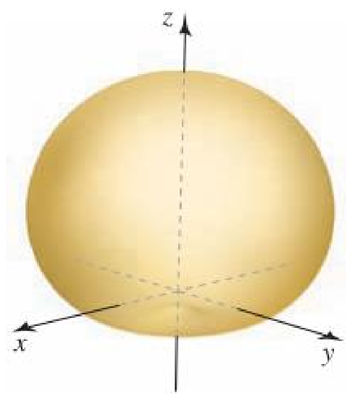
36. Use the spherical coordinates to find the volume of a ball of radius $a > 0$

37. Use the spherical coordinates to find the volume of the solid bounded by the sphere $\rho = 2\cos\varphi$ and the hemisphere $\rho = 1,\ z \ge 0$

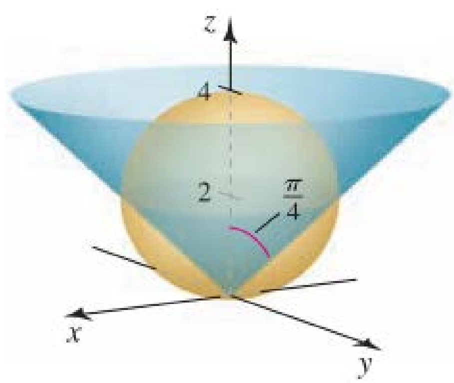


38. Use the spherical coordinates to find the volume of the solid of revolution

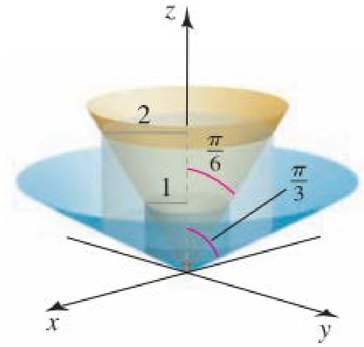
$D = \{(\rho, \varphi, \theta):\ 0 \le \rho \le 1 + \cos\varphi,\ 0 \le \varphi \le \pi,\ 0 \le \theta \le 2\pi\}$



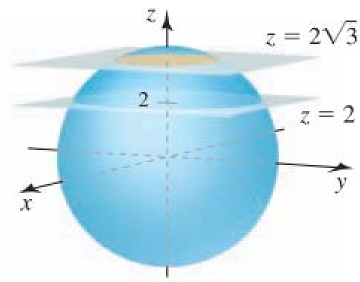
39. Use the spherical coordinates to find the volume of the solid outside the cone $\varphi = \dfrac{\pi}{4}$ and inside the sphere $\rho = 4\cos\varphi$



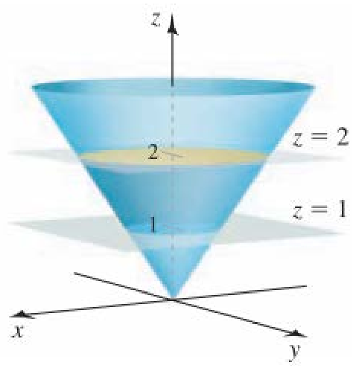
40. Use the spherical coordinates to find the volume of the solid bounded by the cylinders $r = 1$ and $r = 2$, and the cone $\varphi = \frac{\pi}{6}$ and $\varphi = \frac{\pi}{3}$



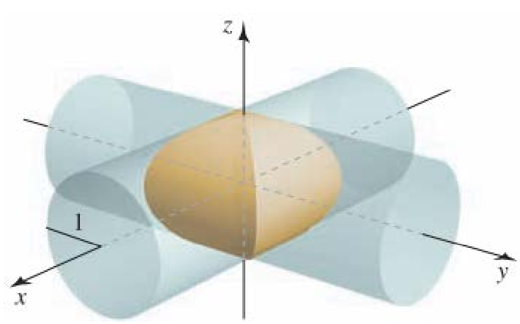
41. Use the spherical coordinates to find the volume of the ball $\rho \le 4$ that lies between the planes $z = 2$ and $z = 2\sqrt{3}$



42. Use the spherical coordinates to find the volume of the solid inside the cone $z = \left(x^2 + y^2\right)^{1/2}$ that lies between the planes $z = 1$ and $z = 2$



43. The *x*- and *y*-axes from the axes of two right circular cylinders with radius 1.



Find the volume of the solid that is common to the two cylinders.