

Solution

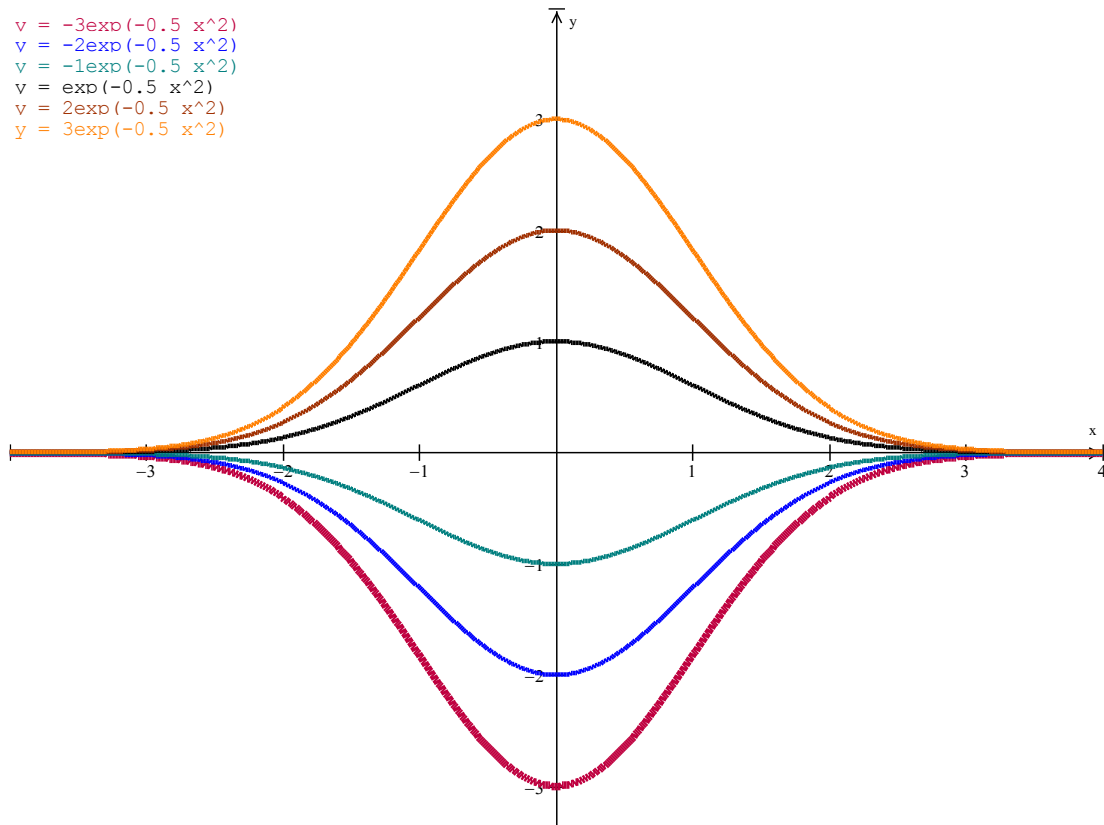
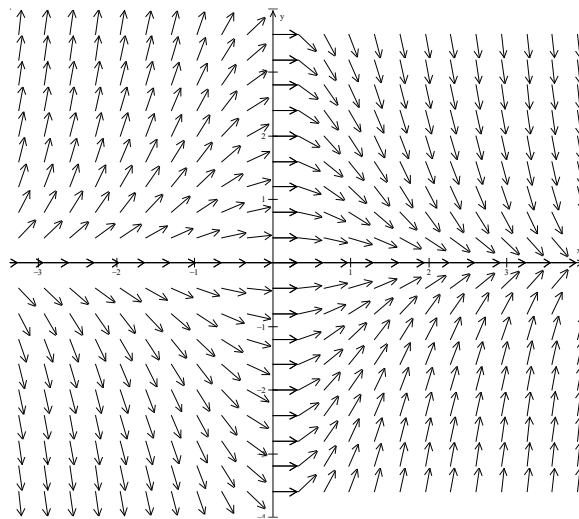
Section 1.1 – Differential Equations & Solutions

Exercise

Show that $y(t) = Ce^{-(1/2)t^2}$ is a solution of the 1st order equation $y' = -ty$ for $-3 \leq C \leq 3$

Solution

$$\begin{aligned} y' &= -\frac{1}{2} 2tCe^{-(1/2)t^2} \\ &= -tCe^{-(1/2)t^2} \\ &= \underline{-ty} \end{aligned}$$



Exercise

Show that $y(t) = \frac{4}{1 + Ce^{-4t}}$ is a solution of the 1st order equation $y' = y(4 - y)$

Solution

$$y' = \frac{d}{dt} \left(\frac{4}{1 + Ce^{-4t}} \right)$$

$$= \frac{-4(Ce^{-4t})'}{(1 + Ce^{-4t})^2}$$

$$= \frac{16Ce^{-4t}}{(1 + Ce^{-4t})^2}$$

$$= \frac{A}{1 + Ce^{-4t}} + \frac{B}{(1 + Ce^{-4t})^2}$$

$$= \frac{A + ACe^{-4t} + B}{(1 + Ce^{-4t})^2}$$

$$\Rightarrow \begin{cases} A = 16 \\ A + B = 0 \rightarrow B = -16 \end{cases}$$

$$= \frac{16}{1 + Ce^{-4t}} - \frac{16}{(1 + Ce^{-4t})^2}$$

$$= \frac{4}{1 + Ce^{-4t}} \left(4 - \frac{4}{1 + Ce^{-4t}} \right)$$

$$= y(4 - y)$$

$$y(4 - y) =$$

$$= \frac{4}{1 + Ce^{-4t}} \left[4 - \frac{4}{1 + Ce^{-4t}} \right]$$

$$= \frac{4}{1 + Ce^{-4t}} \left[\frac{4 + 4Ce^{-4t} - 4}{1 + Ce^{-4t}} \right]$$

$$= \frac{4}{1 + Ce^{-4t}} \left[\frac{4Ce^{-4t}}{1 + Ce^{-4t}} \right]$$

$$= \frac{16Ce^{-4t}}{(1 + Ce^{-4t})^2}$$

Exercise

Show that $y(x) = x^{-3/2}$ is a solution of $4x^2y'' + 12xy' + 3y = 0$ for $x > 0$

Solution

$$y(x) = x^{-3/2}$$

$$y' = -\frac{3}{2}x^{-5/2}$$

$$y'' = \frac{15}{4}x^{-7/2}$$

$$4x^2y'' + 12xy' + 3y = 0$$

$$4x^2\left(\frac{15}{4}x^{-7/2}\right) + 12x\left(-\frac{3}{2}x^{-5/2}\right) + 3x^{-3/2} \stackrel{?}{=} 0$$

$$15x^{-3/2} - 18x^{-3/2} + 3x^{-3/2} \stackrel{?}{=} 0$$

$$0 = 0 \quad \checkmark$$

$$\therefore y(x) = x^{-3/2} \text{ is a solution of } 4x^2y'' + 12xy' + 3y = 0$$

Exercise

A general solution may fail to produce all solutions of a differential equation $y(t) = \frac{4}{1 + Ce^{-4t}}$. Show that

$y = 0$ is a solution of the differential equation, but no value of C in the given general solution will produce this solution.

Solution

$$y(t) = 0 \Rightarrow y' = 0$$

$$y(4 - y) = 0(4 - 0) = 0$$

Exercise

Use the given general solution to find a solution of the differential equation having the given initial condition. $ty' + y = t^2$, $y(t) = \frac{1}{3}t^2 + \frac{C}{t}$, $y(1) = 2$

Solution

$$y(1) = 2$$

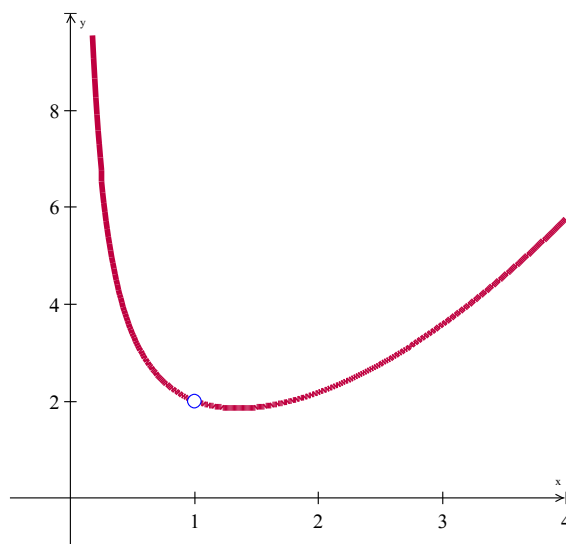
$$y(1) = \frac{1}{3}(1)^2 + \frac{C}{1}$$

$$2 = \frac{1}{3} + C$$

$$C = 2 - \frac{1}{3} = \frac{5}{3}$$

$$y(t) = \frac{1}{3}t^2 + \frac{5}{3t}$$

The interval of existence is $(0, \infty)$



Exercise

Show that $y(t) = 2t - 2 + Ce^{-t}$ is a solution of the 1st order equation $y' + y = 2t$ for $-3 \leq C \leq 3$

Solution

$$y' + y = (2t - 2 + Ce^{-t})' + 2t - 2 + Ce^{-t}$$

$$= 2 - Ce^{-t} + 2t - 2 + Ce^{-t}$$

$$= 2t \quad \checkmark$$

Exercise

Use the given general solution to find a solution of the differential equation having the given initial condition. $y' + 4y = \cos t$, $y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t + Ce^{-4t}$, $y(0) = -1$

Solution

$$y(0) = \frac{4}{17}\cos(0) + \frac{1}{17}\sin(0) + Ce^{-4(0)}$$

$$-1 = \frac{4}{17} + C$$

$$C = -1 - \frac{4}{17} = -\frac{21}{17}$$

$$y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t - \frac{21}{17}e^{-4t}$$

Exercise

Use the given general solution to find a solution of the differential equation having the given initial condition. $ty' + (t+1)y = 2te^{-t}$, $y(t) = e^{-t}\left(t + \frac{C}{t}\right)$, $y(1) = \frac{1}{e}$

Solution

$$y(1) = \frac{1}{e} = e^{-1}$$

$$y(1) = e^{-1}\left(1 + \frac{C}{1}\right)$$

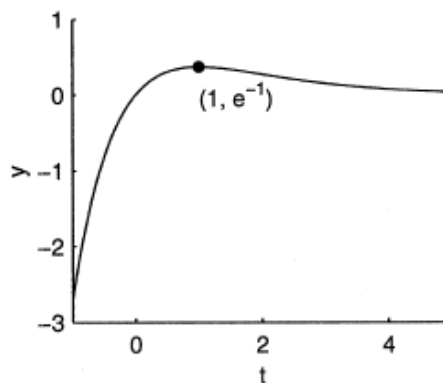
$$e^{-1} = e^{-1}(1 + C)$$

$$1 = 1 + C$$

Hence, $C = 0$

The solution is: $y(t) = te^{-t}$

This function is defined and differentiable on the whole real line. Hence, the interval of existence is the whole real line.



Exercise

Use the given general solution to find a solution of the differential equation having the given initial condition. $y' = y(2 + y)$, $y(t) = \frac{2}{-1 + Ce^{-2t}}$, $y(0) = -3$

Solution

$$y(0) = \frac{2}{-1 + Ce^{-2(0)}}$$

$$-3 = \frac{2}{-1 + C}$$

$$3 - 3C = 2$$

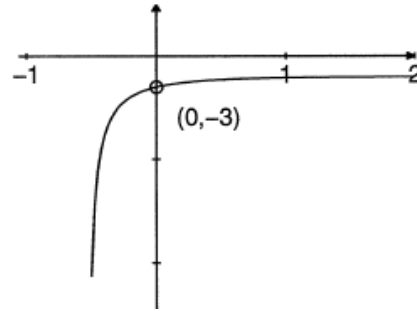
$$-3C = -1$$

$$\boxed{C = \frac{1}{3}}$$

The solution is:

$$y(t) = \frac{2}{-1 + \frac{1}{3}e^{-2t}}$$

$$= \frac{6}{-3 + e^{-2t}}$$



Exercise

Find the values of m so that the function $y = e^{mx}$ is a solution of the given differential equation

a) $y' + 2y = 0$

c) $y'' - 5y' + 6y = 0$

b) $5y' - 2y = 0$

d) $2y'' + 7y' - 4y = 0$

Solution

$$y = e^{mx} \Rightarrow y' = me^{mx} \Rightarrow y'' = m^2 e^{mx}$$

a) $y' + 2y = 0$

$$me^{mx} + 2e^{mx} = 0 \Rightarrow (m + 2)e^{mx} = 0$$

$$\boxed{m = -2}$$

b) $5y' - 2y = 0$

$$5me^{mx} - 2e^{mx} = 0 \Rightarrow (5m - 2)e^{mx} = 0$$

$$\boxed{m = \frac{2}{5}}$$

c) $y'' - 5y' + 6y = 0$

$$m^2 e^{mx} - 5me^{mx} + 6e^{mx} = 0 \Rightarrow (m^2 - 5m + 6)e^{mx} = 0$$

$$\boxed{m = 2, 3}$$

d) $2y'' + 7y' - 4y = 0$

$$2m^2 e^{mx} + 7me^{mx} - 4e^{mx} = 0 \Rightarrow (2m^2 + 7m - 4)e^{mx} = 0$$

$$\boxed{m = \frac{1}{2}, -4}$$

Exercise

Let $x = c_1 \cos t + c_2 \sin t$ is 2-parameter family solutions of the second order differential equation of $x'' + x = 0$. Find a solution of the second-order consisting of this differential equation and the given initial conditions.

a) $x(0) = -1, \quad x'(0) = 8$

c) $x\left(\frac{\pi}{6}\right) = \frac{1}{2}, \quad x'\left(\frac{\pi}{6}\right) = 0$

b) $x\left(\frac{\pi}{2}\right) = 0, \quad x'\left(\frac{\pi}{2}\right) = 1$

d) $x\left(\frac{\pi}{4}\right) = \sqrt{2}, \quad x'\left(\frac{\pi}{4}\right) = 2\sqrt{2}$

Solution

$$x = c_1 \cos t + c_2 \sin t \Rightarrow x' = -c_1 \sin t + c_2 \cos t$$

a) $x(0) = -1 \Rightarrow \underline{-1 = c_1}$

$$x'(0) = 8 \Rightarrow \underline{8 = c_2}$$

b) $x\left(\frac{\pi}{2}\right) = 0 \Rightarrow \underline{0 = c_2}$

$$x'\left(\frac{\pi}{2}\right) = 1 \Rightarrow \underline{-1 = c_1}$$

c) $x\left(\frac{\pi}{6}\right) = \frac{1}{2} \Rightarrow \frac{\sqrt{3}}{2}c_1 + \frac{1}{2}c_2 = \frac{1}{2} \rightarrow \sqrt{3}c_1 + c_2 = 1$

$$x'\left(\frac{\pi}{6}\right) = 0 \Rightarrow -\frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2 = 0 \rightarrow -c_1 + \sqrt{3}c_2 = 0$$

$$c_1 = \underline{\frac{\sqrt{3}}{4}}, \quad c_2 = \underline{\frac{1}{4}}$$

d) $x\left(\frac{\pi}{4}\right) = \sqrt{2} \Rightarrow \frac{\sqrt{2}}{2}c_1 + \frac{\sqrt{2}}{2}c_2 = \sqrt{2} \rightarrow c_1 + c_2 = 2$

$$x'\left(\frac{\pi}{4}\right) = 2\sqrt{2} \Rightarrow \frac{\sqrt{2}}{2}c_1 + \frac{\sqrt{2}}{2}c_2 = 2\sqrt{2} \rightarrow -c_1 + c_2 = 4$$

$$\underline{c_1 = -1} \quad \underline{c_2 = 3}$$

Exercise

Find values of r such that $y(x) = x^r$ is a solution of $x^2 y'' - 4xy' + 6y = 0$

Solution

$$y(x) = x^r \Rightarrow y' = rx^{r-1}$$

$$y'' = r(r-1)x^{r-2}$$

$$x^2 r(r-1)x^{r-2} - 4rx^{r-1} + 6x^r = 0$$

$$r(r-1)x^r - 4rx^r + 6x^r = 0$$

$$r^2 - r - 4r + 6 = 0 \quad \text{since } x^r \neq 0$$

$$r^2 - 5r + 6 = 0 \rightarrow \underline{r = 3, 2}$$

Exercise

Solve the differential equation $y' = 3x^2 - 2x + 4$

Solution

$$\begin{aligned} y(x) &= \int (3x^2 - 2x + 4) dx \\ &= \underline{x^3 - x^2 + 4x + C} \end{aligned}$$

Exercise

Solve the differential equation $y'' = 2x + \sin 2x$

Solution

$$\begin{aligned} y' &= \int (2x + \sin 2x) dx \\ &= x^2 - \frac{1}{2} \cos 2x + C_1 \\ y &= \int \left(x^2 - \frac{1}{2} \cos 2x + C_1 \right) dx \\ &= \underline{\frac{1}{3} x^3 - \frac{1}{4} \sin 2x + C_1 x + C_2} \end{aligned}$$

Exercise

Given the differential equation $x^2 y'' - 2xy' + 2y = 4x^3$, is the given equation a solution to?

Solution

$$a) \quad y = 2x^3 + x^2$$

$$y' = 6x^2 + 2x$$

$$y'' = 12x + 2$$

$$x^2 y'' - 2xy' + 2y \stackrel{?}{=} 4x^3$$

$$\begin{aligned} x^2 (12x + 2) - 2x (6x^2 + 2x) + 2 (2x^3 + x^2) &= 12x^3 + 2x^2 - 12x^3 - 4x^2 + 4x^3 + 2x^2 \\ &= \underline{4x^3} \quad \checkmark \end{aligned}$$

$$y = 2x^3 + x^2 \text{ is a solution.}$$

$$b) \quad y = 2x + x^2$$

$$y' = 2 + 2x$$

$$y'' = 2$$

$$x^2 y'' - 2xy' + 2y \stackrel{?}{=} 4x^3$$

$$x^2(2) - 2x(2 + 2x) + 2(2x + x^2) = 2x^2 - 4x - 4x^2 + 4x + 2x^2$$

$$= 0 \neq 4x^3$$

$y = 2x + x^2$ is **not** a solution.