Solution Section 2.1 – Graphs and Level Curves

Exercise

Find the specific values for $f(x,y,z)=\dfrac{x-y}{y^2+z^2}$

a) $f(3,-1,2)$ b) $f\left(1,\ \frac{1}{2},\ -\frac{1}{4}\right)$ c) $f\left(0,\ -\frac{1}{3},\ 0\right)$ d) $f(2,\ 2,\ 100)$

Solution

a) $f(3,-1,2)=\dfrac{3-(-1)}{(-1)^2+2^2}\ \underline{=\frac{4}{5}|}$

b) $f\left(1,\ \frac{1}{2},\ -\frac{1}{4}\right)=\dfrac{1-\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)^2+\left(-\frac{1}{4}\right)^2}=\dfrac{\frac{1}{2}}{\frac{1}{4}+\frac{1}{16}}=\dfrac{\frac{1}{2}}{\frac{5}{16}}\ \underline{=\frac{8}{5}|}$

c) $f\left(0,\ -\frac{1}{3},\ 0\right)=\dfrac{0-\left(-\frac{1}{3}\right)}{\left(-\frac{1}{3}\right)^2+0^2}=\dfrac{\frac{1}{3}}{\frac{1}{9}}\ \underline{=3|}$

d) $f(2,\ 2,\ 100)=\dfrac{2-(2)}{(2)^2+100^2}\ \underline{=0|}$

Exercise

Find the specific values for $f(x,y,z)=\sqrt{49-x^2-y^2-z^2}$

a) $f(0,\ 0,\ 0)$ b) $f(2,\ -3,\ 6)$ c) $f(-1,\ 2,\ 3)$ d) $f\left(\frac{4}{\sqrt{2}},\ \frac{5}{\sqrt{2}},\ \frac{6}{\sqrt{2}}\right)$

Solution

a) $f(0,\ 0,\ 0)==\sqrt{49-0^2-0^2-0^2}\ \underline{=7|}$

b) $f(2,\ -3,\ 6)=\sqrt{49-2^2-(-3)^2-6^2}\ \underline{=0|}$

c) $f(-1,\ 2,\ 3)=\sqrt{49-(-1)^2-2^2-3^2}\ \underline{=\sqrt{35}|}$

d) $f\left(\frac{4}{\sqrt{2}},\ \frac{5}{\sqrt{2}},\ \frac{6}{\sqrt{2}}\right)=\sqrt{49-\left(\frac{4}{\sqrt{2}}\right)^2-\left(\frac{5}{\sqrt{2}}\right)^2-\left(\frac{6}{\sqrt{2}}\right)^2}=\sqrt{49-\frac{16}{2}-\frac{25}{2}-\frac{36}{2}}\ \underline{=\sqrt{\frac{21}{2}}|}$

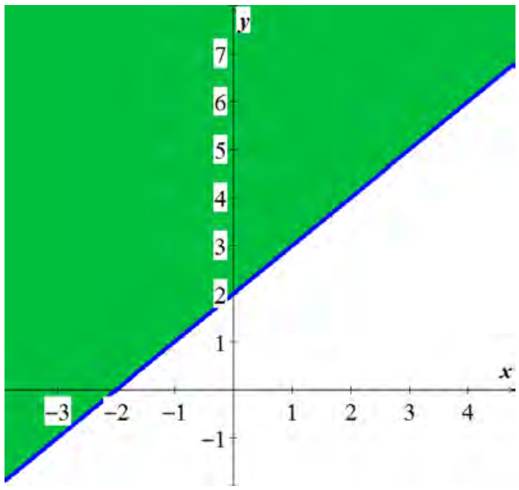
Exercise

Find and sketch the domain for each function $f(x,y)=\sqrt{y-x-2}$

Solution

$$y-x-2\ge 0 \quad \Rightarrow \quad y\ge x+2$$

$y=x+2$



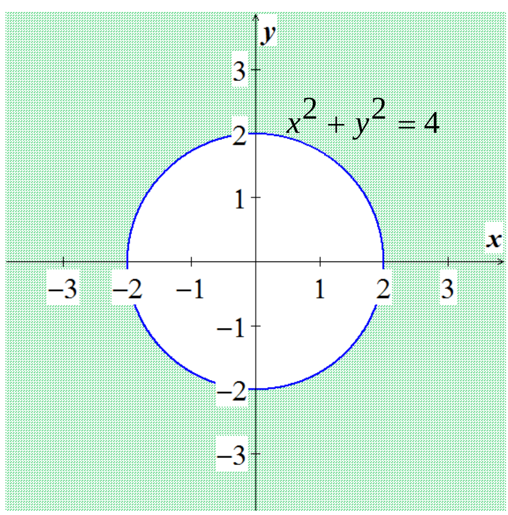
Exercise

Find and sketch the domain for each function $f(x,y)=\ln\left(x^2+y^2-4\right)$

Solution

$$x^2+y^2-4>0 \quad \Rightarrow \quad x^2+y^2>4$$

Domain: All points (x,y) outside the circle



Exercise

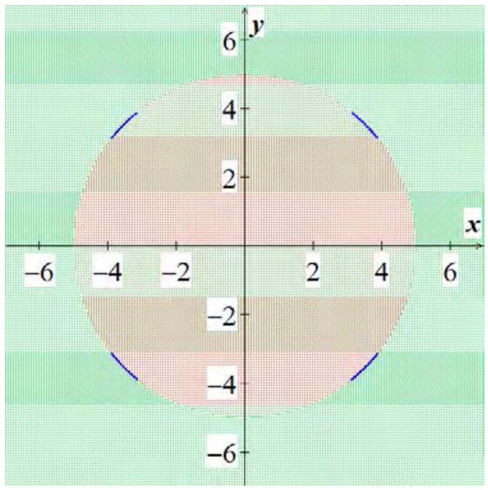
Find and sketch the domain for each function $f(x,y)=\dfrac{\sin(xy)}{x^2+y^2-25}$

Solution

$$x^2+y^2-25\neq 0 \quad\Rightarrow\quad x^2+y^2\neq 25$$

Domain: All points (x,y) not lying on the circle $x^2+y^2=25$

$x^2+y^2=25$



Exercise

Find and sketch the domain for each function $f(x,y)=\ln(xy+x-y-1)$

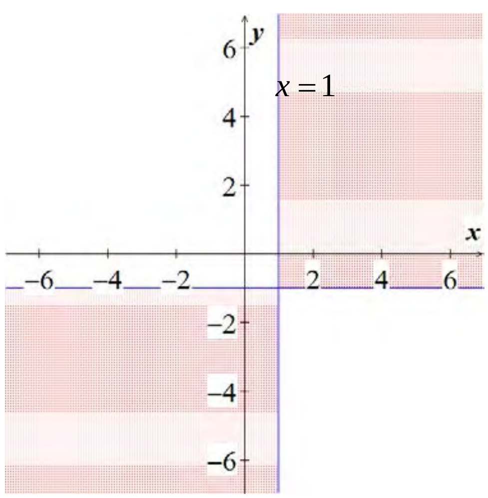
Solution

$$xy+x-y-1>0 \quad\Rightarrow\quad x(y+1)-(y+1)>0$$

$$(x-1)(y+1)>0$$

Domain: All points (x,y) satisfying $(x-1)(y+1)>0$

$y=-1$



Exercise

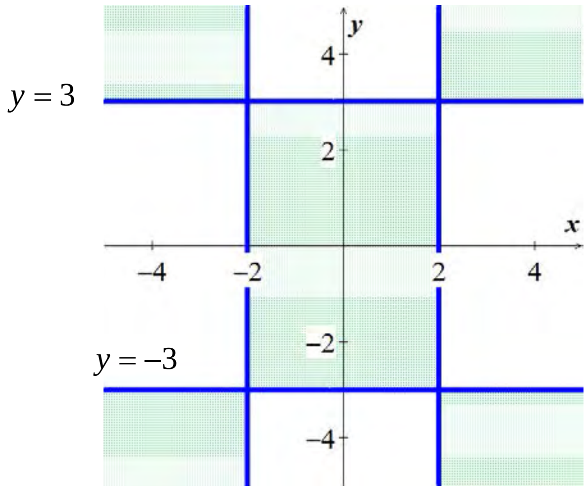
Find and sketch the domain for each function $f(x,y)=\sqrt{\left(x^2-4\right)\left(y^2-9\right)}$

Solution

$$\left(x^2-4\right)\left(y^2-9\right)\ge 0 \quad \Rightarrow \quad (x-2)(x+2)(y-3)(y+3)\ge 0$$

Domain: All points (x,y) satisfying $(x-2)(x+2)(y-3)(y+3)\ge 0$

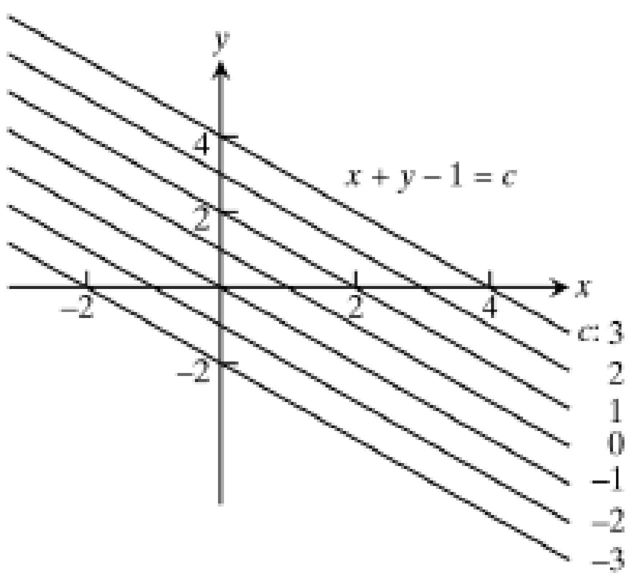
$x = 22$



Exercise

Find and sketch the level curves $f(x,y)=c$ on the same set of coordinate axes for the given values of c, we refer to these level curves as a contour map. $f(x,y)=x+y-1, \quad c=-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3$

Solution

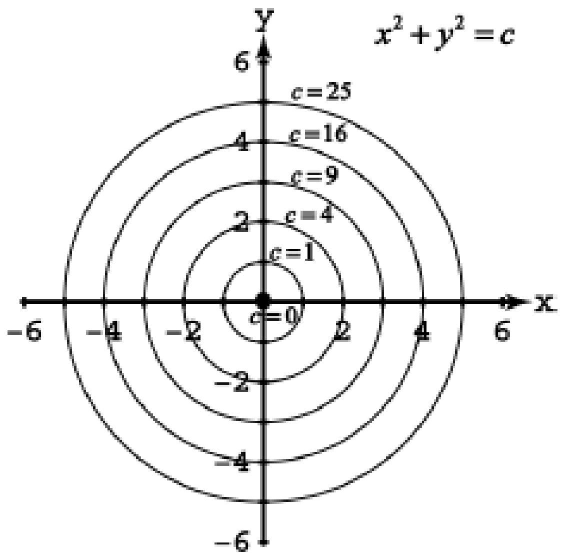


Exercise

Find and sketch the level curves $f(x,y)=c$ on the same set of coordinate axes for the given values of *c*, we refer to these level curves as a contour map.

$f(x,y)=x^2+y^2, \quad c=0,\ 1,\ 4,\ 9,\ 16,\ 25$

Solution



Exercise

For the function: $f(x,y)=4x^2+9y^2$:

a) Find the function's domain
b) Find the function's range
c) Find the function's level curves
d) Find the boundary of the function's domain
e) Determine if the domain is an open region, a closed region, or neither
f) Decide if the domain is bounded or unbounded

Solution

a) *Domain*: all points in the *xy*-plane

b) *Range*: $z \ge 0$

c) Level curves: For $f(x,y)=0 \quad \rightarrow \quad$ *Origin*

For $f(x,y)=c>0 \quad \rightarrow \quad$ *ellipses* with center (0, 0) and major and minor axes along the *x*- and *y*-axes, respectively

d) No boundary points

e) Both open and closed

f) Unbounded

Exercise

For the function: $f(x,y) = xy$:

a) Find the function's domain
b) Find the function's range
c) Find the function's level curves
d) Find the boundary of the function's domain
e) Determine if the domain is an open region, a closed region, or neither
f) Decide if the domain is bounded or unbounded

Solution

a) *Domain*: all points in the *xy*-plane

b) *Range*: $\mathbb{R}$

c) Level curves: Hyperbolas with the *x*- and *y*-axes as asymptotes when $f(x,y) \neq 0$ and the *x*- and *y*-axes when $f(x,y) = 0$

d) No boundary points

e) Both open and closed

f) Unbounded

Exercise

For the function: $f(x,y) = e^{-\left(x^2+y^2\right)}$

a) Find the function's domain
b) Find the function's range
c) Find the function's level curves
d) Find the boundary of the function's domain
e) Determine if the domain is an open region, a closed region, or neither
f) Decide if the domain is bounded or unbounded

Solution

a) *Domain*: all points in the *xy*-plane

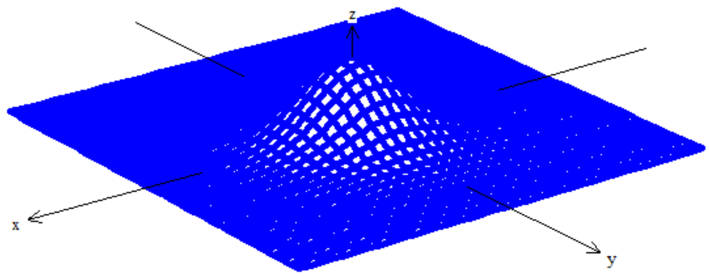
b) *Range*: $0 < z \le 1$

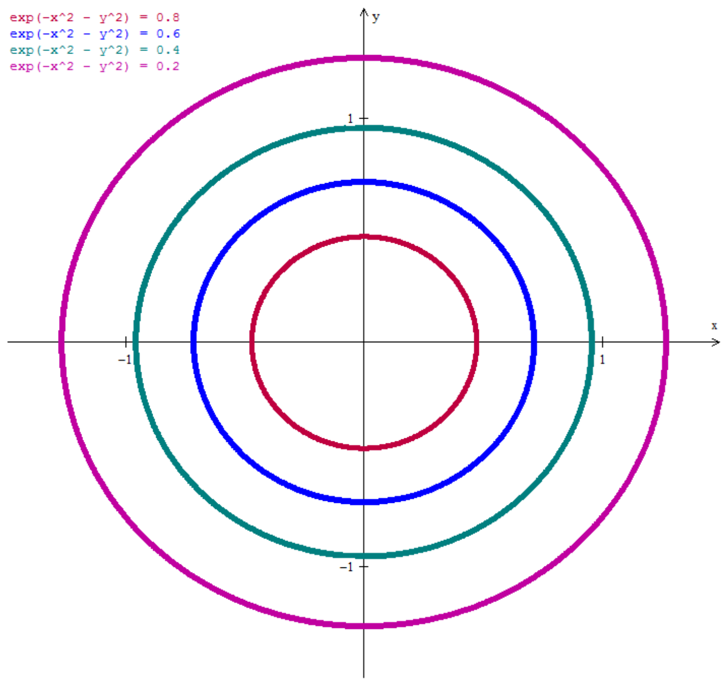
c) Level curves are the origin itself and the circles with center (0, 0) and radii $r > 0$

d) No boundary points

e) Both open and closed

f) Unbounded





Exercise

For the function: $f(x,y)=\ln\left(9-x^2-y^2\right)$

a) Find the function's domain
b) Find the function's range
c) Find the function's level curves
d) Find the boundary of the function's domain
e) Determine if the domain is an open region, a closed region, or neither
f) Decide if the domain is bounded or unbounded

Solution

$$9-x^2-y^2>0 \quad \to \quad x^2+y^2<9$$

a) *Domain*: all points inside the circle $x^2+y^2=9$

b) *Range*: $z<\ln 9$

c) Level curves are circles centered at the origin and radii $r<9$

d) Boundary: the circle $x^2+y^2=9$

e) Open

f) Bounded

Exercise

Find an equation for $f(x,y)=16-x^2-y^2$ and sketch the graph of the level curve of the function $f(x,y)$ that passes through the point $\left(2\sqrt{2},\ \sqrt{2}\right)$

Solution

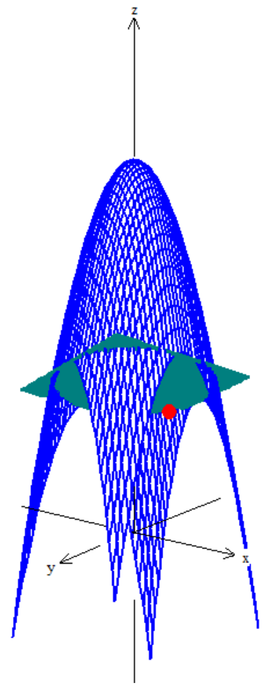
$$z=\left(16-x^2-y^2\right)_{\left(2\sqrt{2},\ \sqrt{2}\right)}$$

$$=16-\left(2\sqrt{2}\right)^2-\left(\sqrt{2}\right)^2$$

$$=6$$

$$6=16-x^2-y^2$$

$$x^2+y^2=10$$



Exercise

Find an equation for $f(x,y)=\dfrac{2y-x}{x+y+1}$ and sketch the graph of the level curve of the function $f(x,y)$ that passes through the point $(-1,\ 1)$

Solution

$$z=\left(\frac{2y-x}{x+y+1}\right)_{(-1,1)}$$

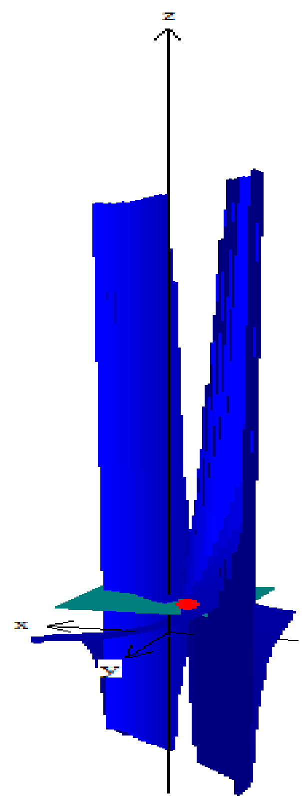
$$=\frac{2(1)-(-1)}{-1+1+1}$$

$$=3$$

$$3=\frac{2y-x}{x+y+1}$$

$$3x+3y+3=2y-x$$

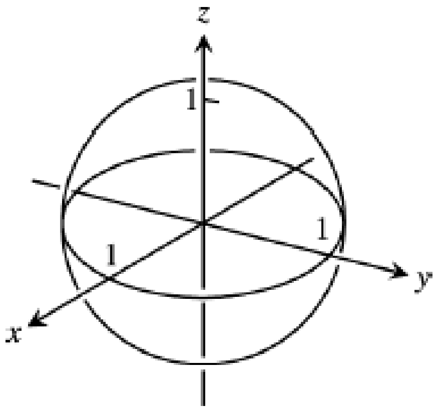
$$\boxed{y=-4x-3}$$



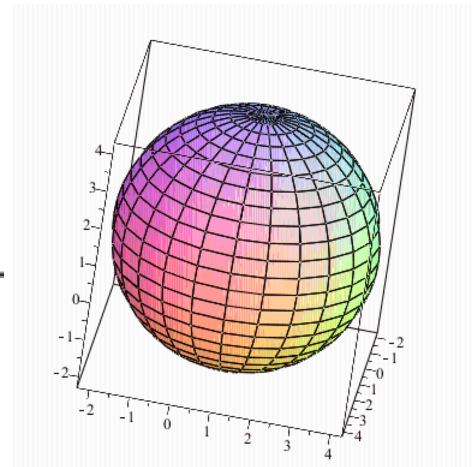
Exercise

Sketch a typical level surface for the function $f(x,y,z) = x^2 + y^2 + z^2$

Solution



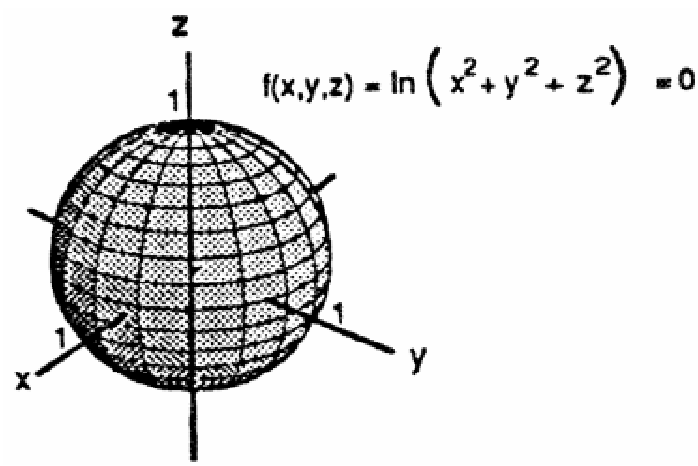
$f(x, y, z) = x^2 + y^2 + z^2 = 1$



Exercise

Sketch a typical level surface for the function $f(x,y,z) = \ln\left(x^2 + y^2 + z^2\right)$

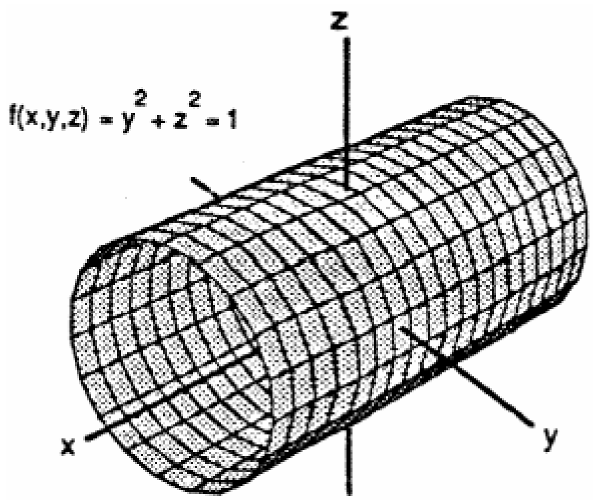
Solution



Exercise

Sketch a typical level surface for the function $f(x,y,z) = y^2 + z^2$

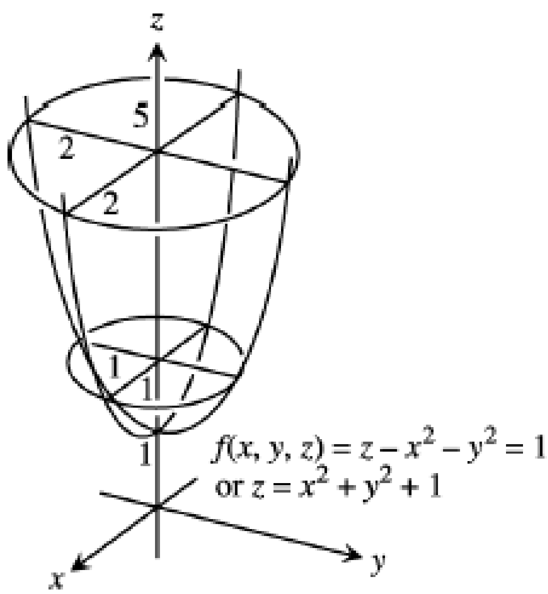
Solution



Exercise

Sketch a typical level surface for the function $f(x,y,z) = z - x^2 - y^2$

Solution



Solution   *Section* 2.2 – Limits and Continuity

Exercise

Find the limits $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2-y^2+5}{x^2+y^2+2}$

Solution

$$\lim_{(x,y)\to(0,0)} \frac{3x^2-y^2+5}{x^2+y^2+2} = \frac{3(0)^2-(0)^2+5}{(0)^2+(0)^2+2} = \frac{5}{2}$$

Exercise

Find the limits $\lim\limits_{(x,y)\to(0,4)} \dfrac{x}{\sqrt{y}}$

Solution

$$\lim_{(x,y)\to(0,4)} \frac{x}{\sqrt{y}} = \frac{0}{\sqrt{4}} = 0$$

Exercise

Find the limits $\lim\limits_{(x,y)\to(3,4)} \sqrt{x^2+y^2-1}$

Solution

$$\lim_{(x,y)\to(3,4)} \sqrt{x^2+y^2-1} = \sqrt{3^2+4^2-1} = \sqrt{24} = 2\sqrt{6}$$

Exercise

Find the limits $\lim\limits_{(x,y)\to(0,0)} \cos\dfrac{x^2+y^3}{x+y+1}$

Solution

$$\lim_{(x,y)\to(0,0)} \cos\frac{x^2+y^3}{x+y+1} = \cos\frac{0^2+0^3}{0+0+1} = \cos 0 = 1$$

Exercise

Find the limits $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^y \sin x}{x}$

Solution

$$\lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x} = e^0 \cdot \lim_{(x,y)\to(0,0)} \frac{\sin x}{x} = 1(1) \underline{=1|}$$

Exercise

Find the limits $\lim\limits_{(x,y)\to\left(\frac{\pi}{2},0\right)} \dfrac{\cos y + 1}{y - \sin x}$

Solution

$$\lim_{(x,y)\to\left(\frac{\pi}{2},0\right)} \frac{\cos y + 1}{y - \sin x} = \frac{\cos 0 + 1}{0 - \sin \frac{\pi}{2}} = \frac{1+1}{-1} \underline{=-2|}$$

Exercise

Find the limits $\lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq y}} \dfrac{x^2 - 2xy + y^2}{x - y}$

Solution

$$\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2 - 2xy + y^2}{x - y} = \frac{1^2 - 2(1)(1) + 1^2}{1-1} = \frac{0}{0}$$

$$\begin{aligned}
\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2 - 2xy + y^2}{x - y} &= \lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{(x-y)^2}{x-y} \\
&= \lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} (x-y) \\
&= 1-1 \\
&\underline{= 0|}
\end{aligned}$$

Exercise

Find the limits $\lim\limits_{\substack{(x,y)\to(1,1) \\ x\neq y}} \dfrac{x^2-y^2}{x-y}$

Solution

$$\lim\limits_{\substack{(x,y)\to(1,1) \\ x\neq y}} \frac{x^2-y^2}{x-y} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$\begin{aligned}\lim\limits_{\substack{(x,y)\to(1,1) \\ x\neq y}} \frac{x^2-y^2}{x-y} &= \lim\limits_{\substack{(x,y)\to(1,1) \\ x\neq y}} \frac{(x-y)(x+y)}{x-y} \\ &= \lim\limits_{\substack{(x,y)\to(1,1) \\ x\neq y}} (x+y) \\ &= 1+1 \\ &= 2\end{aligned}$$

Exercise

Find the limits $\lim\limits_{\substack{(x,y)\to(2,-4) \\ x\neq x^2,\ y\neq -4}} \dfrac{y+4}{x^2y-xy+4x^2-4x}$

Solution

$$\begin{aligned}\lim\limits_{\substack{(x,y)\to(2,-4) \\ x\neq x^2,\ y\neq -4}} \frac{y+4}{x^2y-xy+4x^2-4x} &= \lim\limits_{\substack{(x,y)\to(2,-4) \\ x\neq x^2,\ y\neq -4}} \frac{y+4}{y\left(x^2-x\right)+4\left(x^2-x\right)} \\ &= \lim\limits_{\substack{(x,y)\to(2,-4) \\ x\neq x^2,\ y\neq -4}} \frac{y+4}{\left(x^2-x\right)(y+4)} \\ &= \lim\limits_{\substack{(x,y)\to(2,-4) \\ x\neq x^2,\ y\neq -4}} \frac{1}{x(x-1)} \\ &= \frac{1}{2(2-1)} \\ &= \frac{1}{2}\end{aligned}$$

Exercise

Find the limits $\lim\limits_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \dfrac{\sqrt{x}-\sqrt{y+1}}{x-y-1}$

Solution

$$\lim\limits_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1} = \lim\limits_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{\left(\sqrt{x}-\sqrt{y+1}\right)\left(\sqrt{x}+\sqrt{y+1}\right)}$$

$$= \lim\limits_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{1}{\sqrt{x}+\sqrt{y+1}}$$

$$= \frac{1}{\sqrt{4}+\sqrt{3+1}} = \frac{1}{2+2}$$

$$= \frac{1}{4}$$

Exercise

Find the limits $\lim\limits_{(x,y)\to(1,-1)} \dfrac{x^3+y^3}{x+y}$

Solution

$$\lim\limits_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$\lim\limits_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y} = \lim\limits_{(x,y)\to(1,-1)} \frac{(x+y)\left(x^2-xy+y^2\right)}{x+y}$$

$$= \lim\limits_{(x,y)\to(1,-1)} \left(x^2-xy+y^2\right)$$

$$= 1^2-(1)(-1)+(-1)^2$$

$$= 3$$

Exercise

Find the limits $\lim\limits_{(x,y)\to(2,2)} \dfrac{x-y}{x^4-y^4}$

Solution

$$\lim\limits_{(x,y)\to(2,2)} \frac{x-y}{x^4-y^4} = \lim\limits_{(x,y)\to(2,2)} \frac{x-y}{\left(x^2-y^2\right)\left(x^2+y^2\right)}$$

$$= \lim\limits_{(x,y)\to(2,2)} \frac{x-y}{(x-y)(x+y)\left(x^2+y^2\right)}$$

$$= \lim_{(x,y)\to(2,2)} \frac{1}{(x+y)\left(x^2+y^2\right)}$$

$$= \frac{1}{(2+2)\left(2^2+2^2\right)}$$

$$\underline{= \frac{1}{32}}$$

Exercise

Find the limits $\displaystyle \lim_{P\to(1,3,4)} \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$

Solution

$$\lim_{P\to(1,3,4)} \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \frac{1}{1}+\frac{1}{3}+\frac{1}{4} \;\underline{= \frac{19}{12}}$$

Exercise

Find the limits $\displaystyle \lim_{P\to(1,-1,-1)} \frac{2xy+yz}{x^2+z^2}$

Solution

$$\lim_{P\to(1,-1,-1)} \frac{2xy+yz}{x^2+z^2} = \frac{2(1)(-1)+(-1)(-1)}{1^2+(-1)^2} \;\underline{= -\frac{1}{2}}$$

Exercise

Find the limits $\displaystyle \lim_{P\to(\pi,0,2)} ze^{-2y}\cos 2x$

Solution

$$\lim_{P\to(\pi,0,2)} ze^{-2y}\cos 2x = 2e^{-2(0)}\cos 2\pi \;\underline{= 2}$$

Exercise

Find the limits $\displaystyle \lim_{P\to(2,-3,6)} \ln\sqrt{x^2+y^2+z^2}$

Solution

$$\lim_{P\to(2,-3,6)} \ln\sqrt{x^2+y^2+z^2} = \ln\sqrt{4+9+36} = \ln\sqrt{49} \;\underline{= \ln 7}$$

Exercise

At what points (x, y, z) in space are the functions continuous $f(x,y,z)=x^2+y^2-2z^2$

Solution

All (x,y,z)

Exercise

At what points (x, y, z) in space are the functions continuous $f(x,y,z)=\sqrt{x^2+y^2-1}$

Solution

$x^2+y^2-1\ge 0 \quad\rightarrow\quad x^2+y^2\ge 1$. All (x,y,z) except the interior of the cylinder $x^2+y^2=1$

Exercise

At what points (x, y, z) in space are the functions continuous $f(x,y,z)=\ln(xyz)$

Solution

All (x,y,z) so that $xyz>0$

Exercise

At what points (x, y, z) in space are the functions continuous $f(x,y,z)=e^{x+y}\cos z$

Solution

All (x,y,z)

Exercise

At what points (x, y, z) in space are the functions continuous $h(x,y,z)=\dfrac{1}{|y|+|z|}$

Solution

All (x,y,z) except $(x,\ 0,\ 0)$

Exercise

At what points (x, y, z) in space are the functions continuous $h(x,y,z)=\dfrac{1}{z-\sqrt{x^2+y^2}}$

Solution

All (x,y,z) except $z\ne\sqrt{x^2+y^2}$

Solution　　*Section* 2.3 – Partial Derivatives

Exercise

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$　　$f(x,y)=2x^2-3y-4$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(2x^2-3y-4\right)\ \underline{=4x}$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(2x^2-3y-4\right)\ \underline{=-3}$$

Exercise

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$　　$f(x,y)=x^2-xy+y^2$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(x^2-xy+y^2\right)\ \underline{=2x-y}$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(x^2-xy+y^2\right)\ \underline{=-x+2y}$$

Exercise

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$　　$f(x,y)=5xy-7x^2-y^2+3x-6y+2$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(5xy-7x^2-y^2+3x-6y+2\right)\ \underline{=5y-14x+3}$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(5xy-7x^2-y^2+3x-6y+2\right)\ \underline{=5x-2y-6}$$

Exercise

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$　　$f(x,y)=(xy-1)^2$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}(xy-1)^2\ \underline{=2y(xy-1)}$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(xy-1)^2\ \boxed{=2x(xy-1)}$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\qquad f(x,y)=\left(x^3+\frac{y}{2}\right)^{2/3}$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(x^3+\frac{y}{2}\right)^{2/3}=\frac{2}{3}\left(x^3+\frac{y}{2}\right)^{-1/3}\left(3x^2\right)\ \boxed{=\frac{2x^2}{\sqrt[3]{x^3+\frac{y}{2}}}}$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(x^3+\frac{y}{2}\right)^{2/3}=\frac{2}{3}\left(x^3+\frac{y}{2}\right)^{-1/3}\left(\frac{1}{2}\right)\ \boxed{=\frac{1}{3\sqrt[3]{x^3+\frac{y}{2}}}}$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\qquad f(x,y)=\frac{1}{x+y}$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(\frac{1}{x+y}\right)=-\frac{1}{(x+y)^2}\frac{\partial}{\partial x}(x+y)\ \boxed{=-\frac{1}{(x+y)^2}}$$

$$\frac{\partial}{dx}\left(\frac{1}{u}\right)=-\frac{u'}{u^2}$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(\frac{1}{x+y}\right)\ \boxed{=-\frac{1}{(x+y)^2}}$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\qquad f(x,y)=\frac{x}{x^2+y^2}$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)=\frac{1\left(x^2+y^2\right)-x(2x)}{\left(x^2+y^2\right)^2}\ \boxed{=\frac{y^2-x^2}{\left(x^2+y^2\right)^2}}$$

$$\frac{\partial}{dx}\left(\frac{u}{v}\right)=\frac{u'v-v'u}{v^2}$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)=\frac{(0)\left(x^2+y^2\right)-x(2y)}{\left(x^2+y^2\right)^2}\ \boxed{=-\frac{2xy}{\left(x^2+y^2\right)^2}}$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\qquad f(x,y)=\tan^{-1}\frac{y}{x}$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(\tan^{-1}\frac{y}{x}\right)=\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$=-\frac{y}{1+\frac{y^2}{x^2}}\left(\frac{1}{x^2}\right)$$

$$=-\frac{y}{\frac{x^2+y^2}{x^2}}\left(\frac{1}{x^2}\right)$$

$$=-\frac{y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(\tan^{-1}\frac{y}{x}\right)=\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$=\frac{1}{\frac{x^2+y^2}{x^2}}\left(\frac{1}{x}\right)$$

$$=\frac{x}{x^2+y^2}$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\qquad f(x,y)=e^{-x}\sin(x+y)$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(e^{-x}\sin(x+y)\right)$$

$$=\sin(x+y)\frac{\partial}{\partial x}\left(e^{-x}\right)+e^{-x}\frac{\partial}{\partial x}\left(\sin(x+y)\right)$$

$$=-e^{-x}\sin(x+y)+e^{-x}\cos(x+y)$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(e^{-x}\sin(x+y)\right)$$

$$=e^{-x}\cos(x+y)$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x,y)=e^{xy}\ln y$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(e^{xy}\ln y\right)\ \underline{=ye^{xy}\ln y|}$$

$$\begin{aligned}\frac{\partial f}{\partial y}&=\frac{\partial}{\partial y}\left(e^{xy}\ln y\right)\\&=\ln y\frac{\partial}{\partial y}\left(e^{xy}\right)+e^{xy}\frac{\partial}{\partial y}(\ln y)\\&=xe^{xy}\ln y+\frac{1}{y}e^{xy}\end{aligned}$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x,y)=\sin^2(x-3y)$

Solution

$$\begin{aligned}\frac{\partial f}{\partial x}&=\frac{\partial}{\partial x}\left(\sin^2(x-3y)\right)\\&=2\sin(x-3y)\frac{\partial}{\partial x}\sin(x-3y)\\&=2\sin(x-3y)\cos(x-3y)\frac{\partial}{\partial x}(x-3y)\\&=2\sin(x-3y)\cos(x-3y)\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y}&=\frac{\partial}{\partial y}\left(\sin^2(x-3y)\right)\\&=2\sin(x-3y)\frac{\partial}{\partial y}\sin(x-3y)\\&=2\sin(x-3y)\cos(x-3y)\frac{\partial}{\partial y}(x-3y)\\&=-6\sin(x-3y)\cos(x-3y)\end{aligned}$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x,y)=\cos^2\left(3x-y^2\right)$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(\cos^2\left(3x-y^2\right)\right)$$

$$= 2\cos\left(3x - y^2\right)\frac{\partial}{\partial x}\left(\cos\left(3x - y^2\right)\right)$$

$$= -2\cos\left(3x - y^2\right)\sin\left(3x - y^2\right)\frac{\partial}{\partial x}\left(3x - y^2\right)$$

$$\underline{= -6\cos\left(3x - y^2\right)\sin\left(3x - y^2\right)}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\cos^2\left(3x - y^2\right)\right)$$

$$= 2\cos\left(3x - y^2\right)\frac{\partial}{\partial y}\left(\cos\left(3x - y^2\right)\right)$$

$$= -2\cos\left(3x - y^2\right)\sin\left(3x - y^2\right)\frac{\partial}{\partial y}\left(3x - y^2\right)$$

$$\underline{= 4y\cos\left(3x - y^2\right)\sin\left(3x - y^2\right)}$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x, y) = x^y$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^y\right) \underline{= yx^{y-1}}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^y\right) \underline{= x^y \ln x}$$

Exercise

Find f_x, f_y, *and* f_z $\quad f(x, y, z) = 1 + xy^2 - 2z^2$

Solution

$$\underline{f_x = y^2} \qquad \underline{f_y = 2xy} \qquad \underline{f_z = -4z}$$

Exercise

Find f_x, f_y, *and* f_z $\quad f(x, y, z) = xy + yz + xz$

Solution

$$\underline{f_x = y + z} \qquad \underline{f_y = x + y} \qquad \underline{f_z = y + x}$$

Exercise

Find f_x, f_y, *and* $f_z \quad f(x,y,z) = x - \sqrt{y^2 + z^2}$

Solution

$$f_x = \underline{1}$$

$$\begin{aligned} f_y &= -\frac{1}{2}\left(y^2 + z^2\right)^{-1/2} \frac{\partial}{\partial y}\left(y^2 + z^2\right) \\ &= -\frac{1}{2}\left(y^2 + z^2\right)^{-1/2} (2y) \\ &= -\frac{y}{\sqrt{y^2 + z^2}} \end{aligned}$$

$$\begin{aligned} f_z &= -\frac{1}{2}\left(y^2 + z^2\right)^{-1/2} \frac{\partial}{\partial z}\left(y^2 + z^2\right) \\ &= -\frac{1}{2}\left(y^2 + z^2\right)^{-1/2} (2z) \\ &= -\frac{z}{\sqrt{y^2 + z^2}} \end{aligned}$$

Exercise

Find f_x, f_y, *and* $f_z \quad f(x,y,z) = \left(x^2 + y^2 + z^2\right)^{-1/2}$

Solution

$$f_x = -\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2}(2x) = -x\left(x^2 + y^2 + z^2\right)^{-3/2}$$

$$f_y = -\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2}(2y) = -y\left(x^2 + y^2 + z^2\right)^{-3/2}$$

$$f_z = -\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2}(2z) = -z\left(x^2 + y^2 + z^2\right)^{-3/2}$$

Exercise

Find $f_x,\ f_y,\ and\ f_z \quad f(x,y,z)=\sec^{-1}(x+yz)$

Solution

$$f_x=\frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}\frac{\partial}{\partial x}(x+yz)\ \underline{=\frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}}$$

$$f_y=\frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}\frac{\partial}{\partial y}(x+yz)\ \underline{=\frac{z}{|x+yz|\sqrt{(x+yz)^2-1}}}$$

$$f_z=\frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}\frac{\partial}{\partial z}(x+yz)\ \underline{=\frac{y}{|x+yz|\sqrt{(x+yz)^2-1}}}$$

Exercise

Find $f_x,\ f_y,\ and\ f_z \quad f(x,y,z)=\ln(x+2y+3z)$

Solution

$$f_x=\frac{1}{x+2y+3z}\cdot\frac{\partial}{\partial x}(x+2y+3z)\ \underline{=\frac{1}{x+2y+3z}}$$

$$f_y=\frac{1}{x+2y+3z}\cdot\frac{\partial}{\partial y}(x+2y+3z)\ \underline{=\frac{2}{x+2y+3z}}$$

$$f_z=\frac{1}{x+2y+3z}\cdot\frac{\partial}{\partial z}(x+2y+3z)\ \underline{=\frac{3}{x+2y+3z}}$$

Exercise

Find $f_x,\ f_y,\ and\ f_z \quad f(x,y,z)=e^{-\left(x^2+y^2+z^2\right)}$

Solution

$$f_x=e^{-\left(x^2+y^2+z^2\right)}\frac{\partial}{\partial x}\left(-\left(x^2+y^2+z^2\right)\right)\ \underline{=-2xe^{-\left(x^2+y^2+z^2\right)}}$$

$$f_y=e^{-\left(x^2+y^2+z^2\right)}\frac{\partial}{\partial y}\left(-\left(x^2+y^2+z^2\right)\right)\ \underline{=-2ye^{-\left(x^2+y^2+z^2\right)}}$$

$$f_z=e^{-\left(x^2+y^2+z^2\right)}\frac{\partial}{\partial z}\left(-\left(x^2+y^2+z^2\right)\right)\ \underline{=-2ze^{-\left(x^2+y^2+z^2\right)}}$$

Exercise

Find f_x, f_y, *and* f_z $\quad f(x,y,z)=\tanh(x+2y+3z)$

Solution

$$f_x=\boxed{\operatorname{sech}^2(x+2y+3z)}$$

$$f_y=\boxed{2\operatorname{sech}^2(x+2y+3z)}$$

$$f_z=\boxed{3\operatorname{sech}^2(x+2y+3z)}$$

Exercise

Find f_x, f_y, *and* f_z $\quad f(x,y,z)=\sinh\left(xy-z^2\right)$

Solution

$$f_x=\cosh\left(xy-z^2\right)\frac{\partial}{\partial x}\left(xy-z^2\right)=\boxed{y\cosh\left(xy-z^2\right)}$$

$$f_y=\boxed{x\cosh\left(xy-z^2\right)}$$

$$f_z=\boxed{-2z\cosh\left(xy-z^2\right)}$$

Exercise

Find all the second-order partial derivatives of $\quad f(x,y)=x+y+xy$

Solution

$$\frac{\partial f}{\partial x}=1+y \qquad \frac{\partial f}{\partial y}=1+x \qquad \frac{\partial^2 f}{\partial x\partial y}=1$$

$$\frac{\partial^2 f}{\partial x^2}=0 \qquad \frac{\partial^2 f}{\partial y^2}=0 \qquad \frac{\partial^2 f}{\partial y\partial x}=1$$

Exercise

Find all the second-order partial derivatives of $\quad f(x,y)=\sin xy$

Solution

$$\boxed{\frac{\partial f}{\partial x}=y\cos xy} \qquad \boxed{\frac{\partial^2 f}{\partial x^2}=-y^2\sin xy} \qquad \boxed{\frac{\partial^2 f}{\partial y\partial x}=\cos xy-xy\sin xy}$$

$$\boxed{\frac{\partial f}{\partial y}=x\cos xy} \qquad \boxed{\frac{\partial^2 f}{\partial y^2}=-x^2\sin xy} \qquad \boxed{\frac{\partial^2 f}{\partial x\partial y}=\cos xy-xy\sin xy}$$

Exercise

Find all the second-order partial derivatives of $g(x,y)=x^2y+\cos y+y\sin x$

Solution

$$\frac{\partial g}{\partial x}=2xy+y\cos x \qquad \frac{\partial^2 g}{\partial x^2}=2y-y\sin x \qquad \frac{\partial^2 g}{\partial y\partial x}=2x+\cos x$$

$$\frac{\partial g}{\partial y}=x^2-\sin y+\sin x \qquad \frac{\partial^2 g}{\partial y^2}=-\cos y \qquad \frac{\partial^2 g}{\partial x\partial y}=2x+\cos x$$

Exercise

Find all the second-order partial derivatives of $r(x,y)=\ln(x+y)$

Solution

$$\frac{\partial r}{\partial x}=\frac{1}{x+y} \qquad \frac{\partial^2 r}{\partial x^2}=-\frac{1}{(x+y)^2} \qquad \frac{\partial^2 r}{\partial y\partial x}=-\frac{1}{(x+y)^2}$$

$$\frac{\partial r}{\partial y}=\frac{1}{x+y} \qquad \frac{\partial^2 r}{\partial y^2}=-\frac{1}{(x+y)^2} \qquad \frac{\partial^2 r}{\partial x\partial y}=-\frac{1}{(x+y)^2}$$

Exercise

Find all the second-order partial derivatives of $w=x^2\tan(xy)$

Solution

$$\frac{\partial w}{\partial x}=2x\tan(xy)+x^2y\sec^2(xy)$$

$$\begin{aligned}\frac{\partial^2 w}{\partial x^2}&=2\tan(xy)+2xy\sec^2(xy)+2xy\sec^2(xy)+2x^2y\sec(xy)\frac{\partial}{\partial x}\sec(xy)\\&=2\tan(xy)+4xy\sec^2(xy)+2x^2y\sec(xy)\sec(xy)\tan(xy)\frac{\partial}{\partial x}(xy)\\&=\underline{2\tan(xy)+4xy\sec^2(xy)+2x^2y^2\sec^2(xy)\tan(xy)}\end{aligned}$$

$$\frac{\partial w}{\partial y}=\underline{x^3\sec^2(xy)}$$

$$\begin{aligned}\frac{\partial^2 w}{\partial y^2}&=2x^3\sec(xy)\left[x\sec(xy)\tan(xy)\right]\\&=\underline{2x^4\sec^2(xy)\tan(xy)}\end{aligned}$$

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y} = 3x^2 \sec^2(xy) + x^3 \left(2\sec(xy)\sec(xy)\tan(xy)\cdot y\right)$$

$$= \underline{3x^2 \sec^2(xy) + 2x^3 y \sec^2(xy)\tan(xy)}$$

Exercise

Find all the second-order partial derivatives of $w = ye^{x^2-y}$

Solution

$$\frac{\partial w}{\partial x} = \underline{2xye^{x^2-y}}$$

$$\frac{\partial^2 w}{\partial x^2} = 2ye^{x^2-y} + 4x^2ye^{x^2-y} = \underline{2ye^{x^2-y}\left(1+2x^2\right)}$$

$$\frac{\partial w}{\partial y} = e^{x^2-y} - ye^{x^2-y} = \underline{e^{x^2-y}(1-y)}$$

$$\frac{\partial^2 w}{\partial y^2} = -e^{x^2-y}(1-y) - e^{x^2-y}$$

$$= e^{x^2-y}(-1+y-1)$$

$$= \underline{e^{x^2-y}(y-2)}$$

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y}$$

$$= 2xe^{x^2-y} - 2xye^{x^2-y}$$

$$= \underline{2xe^{x^2-y}(1-y)}$$

Exercise

Let $f(x,y) = 2x + 3y - 4$. Find the slope of the line tangent to this surface at the point $(2,\ -1)$ and lying in the **a**. plane $x = 2$ **b**. plane $y = -1$.

Solution

a) In the plane $x = 2$; $\quad m = f_y\Big|_{(2,-1)} = \underline{3}$

b) In the plane $y = -1$; $\quad m = f_z\Big|_{(2,-1)} = \underline{2}$

Exercise

Let $w = f(x, y, z)$ be a function of three independent variables and writs the formal definition of the partial derivative $\frac{\partial f}{\partial y}$ at (x_0, y_0, z_0). Use this definition to find $\frac{\partial f}{\partial y}$ at $(-1,\ 0,\ 3)$ for $f(x, y, z) = -2xy^2 + yz^2$.

Solution

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x_0, y_0 + h, z_0) - f(x_0, y_0, z_0)}{h}$$

$$\begin{aligned} f_y(-1,0,3) &= \lim_{h \to 0} \frac{f(-1, 0+h, 3) - f(-1,0,3)}{h} \\ &= \lim_{h \to 0} \frac{-2(-1)h^2 + h(3)^2 - (0+0)}{h} \\ &= \lim_{h \to 0} \frac{2h^2 + 9h}{h} \\ &= \lim_{h \to 0} (2h + 9) \\ &= 9 \end{aligned}$$

Exercise

Find the value of $\frac{\partial x}{\partial z}$ at the point $(1, -1, -3)$ if the equation $xz + y \ln x - x^2 + 4 = 0$ defines x as a function of the two independent variables y and z and the partial derivative exists.

Solution

$$\frac{\partial x}{\partial z} z + x + y\left(\frac{1}{x}\right)\frac{\partial x}{\partial z} - 2x\frac{\partial x}{\partial z} = 0$$

$$\left(z + \frac{y}{x} - 2x\right)\frac{\partial x}{\partial z} = -x$$

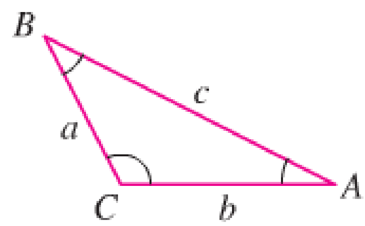
$$\Rightarrow \frac{\partial x}{\partial z} = -\frac{x}{z + \frac{y}{x} - 2x}$$

$$\left.\frac{\partial x}{\partial z}\right|_{(1,-1,-3)} = -\frac{1}{-3 + \frac{-1}{1} - 2} = \frac{1}{6}$$

Exercise

Express A implicitly as a function of a, b, and c and calculate $\frac{\partial A}{\partial a}$ and $\frac{\partial A}{\partial b}$.

Solution



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\frac{\partial}{\partial a}\left(a^2 = b^2 + c^2 - 2bc\cos A\right)$$

$$2a = (2bc\sin A)\frac{\partial A}{\partial a} \quad \rightarrow \boxed{\frac{\partial A}{\partial a} = \frac{a}{bc\sin A}}$$

$$\frac{\partial}{\partial b}\left(a^2 = b^2 + c^2 - 2bc\cos A\right)$$

$$0 = 2b - 2c\cos A + 2bc\sin A\left(\frac{\partial A}{\partial b}\right)$$

$$\left(\frac{\partial A}{\partial b}\right) = \frac{c\cos A - b}{bc\sin A}$$

Exercise

An important partial differential equation that describes the distribution of heat in a region at time t can be represented by the one-dimensional heat equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$$

Show that $u(x,t) = \sin(\alpha x)\cdot e^{-\beta t}$ satisfies the heat equation for constants α and β. What is the relationship between α and β for this function to be a solution?

Solution

$$u_t = -\beta\sin(\alpha x)\cdot e^{-\beta t}$$

$$u_x = \alpha\cos(\alpha x)\cdot e^{-\beta t}$$

$$u_{xx} = -\alpha^2\sin(\alpha x)\cdot e^{-\beta t}$$

$$\text{For } \frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} \quad \rightarrow \quad u_t = u_{xx}$$

$$-\beta\sin(\alpha x)\cdot e^{-\beta t} = -\alpha^2\sin(\alpha x)\cdot e^{-\beta t}$$

$$\Rightarrow \boxed{\beta = \alpha^2}$$

Solution   *Section* 2.4 – Chain Rule

Exercise

Express $\frac{dw}{dt}$ as a function of t, then evaluate $\frac{dw}{dt}$ at the given value of t.

$w = x^2 + y^2, \quad x = \cos t, \quad y = \sin t, \quad t = \pi$

Solution

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= \frac{\partial}{\partial x}\left(x^2 + y^2\right)\frac{d}{dt}(\cos t) + \frac{\partial f}{\partial y}\left(x^2 + y^2\right)\frac{d}{dt}(\sin t) \\ &= 2x(-\sin t) + 2y\cos t \\ &= -2(\cos t)\sin t + 2(\sin t)\cos t \\ &= 0\end{aligned}$$

$$\begin{aligned}w &= x^2 + y^2 \\ &= \cos^2 t + \sin^2 t \\ &= 1\end{aligned}$$

$$\frac{dw}{dt} = 0$$

$$\frac{dw}{dt}(t = \pi) = 0$$

Exercise

Express $\frac{dw}{dt}$ as a function of t, then evaluate $\frac{dw}{dt}$ at the given value of t.

$w = x^2 + y^2, \quad x = \cos t + \sin t, \quad y = \cos t - \sin t, \quad t = 0$

Solution

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= \frac{\partial}{\partial x}\left(x^2 + y^2\right)\frac{d}{dt}(\cos t + \sin t) + \frac{\partial f}{\partial y}\left(x^2 + y^2\right)\frac{d}{dt}(\cos t - \sin t) \\ &= (2x)(-\sin t + \cos t) + (2y)(-\sin t - \cos t) \\ &= 2(\cos t + \sin t)(\cos t - \sin t) - 2(\cos t - \sin t)(\sin t + \cos t) \\ &= 0\end{aligned}$$

$$\frac{dw}{dt}(t = 0) = 0$$

Exercise

Express $\frac{dw}{dt}$ as a function of t, then evaluate $\frac{dw}{dt}$ at the given value of t.

$w = \ln\left(x^2 + y^2 + z^2\right), \quad x = \cos t, \quad y = \sin t, \quad z = 4\sqrt{t}, \quad t = 3$

Solution

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \\
&= \frac{2x}{x^2+y^2+z^2}(-\sin t) + \frac{2y}{x^2+y^2+z^2}(\cos t) + \frac{2z}{x^2+y^2+z^2}\left(2\frac{1}{\sqrt{t}}\right) \\
&= \frac{-2\cos t\sin t + 2\sin t\cos t + 4\left(4\sqrt{t}\right)\left(t^{-1/2}\right)}{\cos^2 t + \sin^2 t + 16t} \\
&= \frac{16}{1+16t}
\end{aligned}$$

$$w = \ln\left(x^2+y^2+z^2\right) = \ln\left(\cos^2 t + \sin^2 t + 16t\right) = \ln(1+16t)$$

$$\frac{dw}{dt} = \frac{16}{1+16t}$$

$$\frac{dw}{dt}(3) = \frac{16}{1+16(3)} = \frac{16}{49}$$

Exercise

Express $\frac{dw}{dt}$ as a function of t, then evaluate $\frac{dw}{dt}$ at the given value of t.

$w = z - \sin xy, \quad x = t, \quad y = \ln t, \quad z = e^{t-1}, \quad t = 1$

Solution

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \\
&= (-y\cos xy)(1) + (-x\cos xy)\left(\tfrac{1}{t}\right) + (1)\left(e^{t-1}\right) \\
&= -(\ln t)\cos(t\ln t) - \cos(t\ln t) + e^{t-1} \\
&= -(\ln t + 1)\cos(t\ln t) + e^{t-1}
\end{aligned}$$

$$\begin{aligned}
w &= z - \sin xy \\
&= e^{t-1} - \sin(t\ln t)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial w}{\partial t} &= e^{t-1} - \cos(t\ln t)\left[\ln t + t\left(\tfrac{1}{t}\right)\right] \\
&= e^{t-1} - (\ln t + 1)\cos(t\ln t)
\end{aligned}$$

$$\frac{\partial w}{\partial t}(1) = -(\ln 1 + 1)\cos(1\ln 1) + e^{1-1} = -1\cos 0 + 1 = 0$$

Exercise

Express $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ as functions of u and v if $z = 4e^x \ln y, \quad x = \ln(u\cos v), \quad y = u\sin v$, then evaluate $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ at the point $(u,v) = \left(2,\ \frac{\pi}{4}\right)$.

Solution

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{dx}{du} + \frac{\partial z}{\partial y}\frac{dy}{du} \\
&= \left(4e^x \ln y\right)\left(\frac{\cos v}{u\cos v}\right) + \left(4\frac{e^x}{y}\right)(\sin v) \\
&= 4e^x\left(\frac{\ln y}{u} + \frac{\sin v}{y}\right) \\
&= 4e^{\ln(u\cos v)}\left(\frac{\ln(u\sin v)}{u} + \frac{\sin v}{u\sin v}\right) \\
&= 4(u\cos v)\left(\frac{\ln(u\sin v)}{u} + \frac{1}{u}\right) \\
&= 4\cos v\ln(u\sin v) + 4\cos v
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{dx}{dv} + \frac{\partial z}{\partial y}\frac{dy}{dv} \\
&= \left(4e^x \ln y\right)\left(\frac{-u\sin v}{u\cos v}\right) + \left(4\frac{e^x}{y}\right)(u\cos v) \\
&= 4e^{\ln(u\cos v)}\left[\frac{-\ln(u\sin v)(u\sin v)}{u\cos v} + \frac{u\cos v}{u\sin v}\right] \\
&= 4u\cos v\left(\frac{-u\sin^2 v\cdot\ln(u\sin v) + u\cos^2 v}{u\cos v\sin v}\right) \\
&= 4\left(\frac{-u\sin^2 v\cdot\ln(u\sin v) + u\cos^2 v}{\sin v}\right) \\
&= -4u\sin v\cdot\ln(u\sin v) + 4u\frac{\cos^2 v}{\sin u}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial u}\left(2,\ \frac{\pi}{4}\right) &= 4\cos\frac{\pi}{4}\ln\left(2\sin\frac{\pi}{4}\right) + 4\cos\frac{\pi}{4} \\
&= 2\sqrt{2}\ln\sqrt{2} + 2\sqrt{2} \\
&= 2\sqrt{2}\left(\tfrac{1}{2}\ln 2 + 1\right) \\
&= \sqrt{2}(\ln 2 + 2)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial v}\left(2,\ \frac{\pi}{4}\right) &= -8\sin\left(\tfrac{\pi}{4}\right)\cdot\ln\left(2\sin\left(\tfrac{\pi}{4}\right)\right) + 8\frac{\cos^2\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)} \\
&= -4\sqrt{2}\ln\left(\sqrt{2}\right) + 8\cdot\tfrac{1}{2}\cdot\sqrt{2} \\
&= -2\sqrt{2}\ln 2 + 4\sqrt{2}
\end{aligned}$$

Exercise

Express $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ as functions of u and v if $w = xy + yz + xz, \quad x = u + v, \quad y = u - v, \quad z = uv$, then evaluate $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ at the point $(u, v) = \left(\frac{1}{2},\ 1\right)$.

Solution

$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{dx}{du} + \frac{\partial w}{\partial y}\frac{dy}{du} + \frac{\partial w}{\partial z}\frac{dz}{du}$ $= (y+z)(1) + (x+z)(1) + (y+x)(v)$ $= y + z + x + z + (y+x)(v)$ $= y + x + 2z + yv + xv$ $= u - v + u + v + 2uv + uv - v^2 + uv + v^2$ $\underline{= 2u + 4uv}$	$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{dx}{dv} + \frac{\partial w}{\partial y}\frac{dy}{dv} + \frac{\partial w}{\partial z}\frac{dz}{dv}$ $= (y+z)(1) + (x+z)(-1) + (y+x)(u)$ $= y + z - x - z + yu + xu$ $= y - x + yu + xu$ $= u - v - u - v + u^2 - uv + u^2 + uv$ $\underline{= -2v + 2u^2}$
$\frac{\partial w}{\partial u}\left(\frac{1}{2},\ 1\right) = 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right)(1) = \underline{3}$	$\frac{\partial w}{\partial v}\left(\frac{1}{2},\ 1\right) = -2(1) + 2\left(\frac{1}{2}\right)^2 = \underline{-\frac{3}{2}}$

Exercise

Express $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ as functions of x, y and z if $u = e^{qr}\sin^{-1} p, \quad p = \sin x, \quad q = z^2 \ln y, \quad r = \frac{1}{z}$, then evaluate $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ at the point $(x, y, z) = \left(\frac{\pi}{4},\ \frac{1}{2},\ -\frac{1}{2}\right)$.

Solution

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{dp}{dx} + \frac{\partial u}{\partial q}\frac{dq}{dx} + \frac{\partial u}{\partial r}\frac{dr}{dx}$$

$$= \left(\frac{e^{qr}}{\sqrt{1-p^2}}\right)(\cos x) + \left(re^{qr}\sin^{-1} p\right)(0) + \left(qe^{qr}\sin^{-1} p\right)(0)$$

$$= \frac{e^{z\ln y}\cos x}{\sqrt{1-\sin^2 x}}$$

$$= e^{\ln y^z}\frac{\cos x}{|\cos x|}$$

$$\underline{= y^z} \quad \text{if} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$\frac{\partial u}{\partial x}\left(\frac{\pi}{4},\ \frac{1}{2},\ -\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{-1/2} = 2^{1/2} = \underline{\sqrt{2}}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial p}\frac{dp}{dy}+\frac{\partial u}{\partial q}\frac{dq}{dy}+\frac{\partial u}{\partial r}\frac{dr}{dy}$$

$$=\left(\frac{e^{qr}}{\sqrt{1-p^2}}\right)(0)+\left(re^{qr}\sin^{-1}p\right)\left(\frac{z^2}{y}\right)+\left(qe^{qr}\sin^{-1}p\right)(0)$$

$$=\frac{z^2}{y}\frac{1}{z}e^{z\ln y}\sin^{-1}(\sin x)$$

$$=\frac{z}{y}e^{\ln y^z}(x)$$

$$=\frac{xz}{y}y^z$$

$$\boxed{=xzy^{z-1}}$$

$$\frac{\partial u}{\partial y}\left(\frac{\pi}{4},\ \frac{1}{2},\ -\frac{1}{2}\right)=\left(\frac{\pi}{4}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)^{-1/2-1}$$

$$=-\left(\frac{\pi}{8}\right)2^{3/2}$$

$$=\boxed{-\frac{\pi\sqrt{2}}{4}}$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial p}\frac{dp}{dz}+\frac{\partial u}{\partial q}\frac{dq}{dz}+\frac{\partial u}{\partial r}\frac{dr}{dz}$$

$$=\left(\frac{e^{qr}}{\sqrt{1-p^2}}\right)(0)+\left(re^{qr}\sin^{-1}p\right)(2z\ln y)+\left(qe^{qr}\sin^{-1}p\right)\left(-\frac{1}{z^2}\right)$$

$$=2z\ln y\left(\frac{1}{z}y^z\sin^{-1}(\sin x)\right)-\frac{1}{z^2}\left(z^2(\ln y)y^z\sin^{-1}(\sin x)\right)$$

$$=2xy^z\ln y-xy^z\ln y$$

$$\boxed{=xy^z\ln y}$$

$$\frac{\partial u}{\partial z}\left(\frac{\pi}{4},\ \frac{1}{2},\ -\frac{1}{2}\right)=\left(\frac{\pi}{4}\right)\left(\frac{1}{2}\right)^{-1/2}\ln\left(\frac{1}{2}\right)$$

$$=\left(\frac{\pi}{4}\right)\left(\sqrt{2}\right)(-\ln 2)$$

$$=\boxed{-\frac{\pi\sqrt{2}}{4}\ln 2}$$

Exercise

Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z^3 - xy + yz + y^3 - 2 = 0$ at the point $(1,\ 1,\ 1)$

Solution

$$F(x,y,z) = z^3 - xy + yz + y^3 - 2$$

$$F_x = -y, \quad F_y = -x + z + 3y^2, \quad F_z = 3z^2 + y$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-y}{3z^2 + y} = \underline{\frac{y}{3z^2 + y}}$$

$$\frac{\partial z}{\partial x}(1,\ 1,\ 1) = \frac{1}{3(1)^2 + 1} = \underline{\frac{1}{4}}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-x + z + 3y^2}{3z^2 + y} = \underline{\frac{x - z - 3y^2}{3z^2 + y}}$$

$$\frac{\partial z}{\partial y}(1,\ 1,\ 1) = \frac{1 - 1 - 3(1)^2}{3(1)^2 + 1} = \underline{-\frac{3}{4}}$$

Exercise

Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $\sin(x+y) + \sin(y+z) + \sin(x+z) = 0$ at the point $(\pi,\ \pi,\ \pi)$

Solution

$$F(x,y,z) = \sin(x+y) + \sin(y+z) + \sin(x+z)$$

$$F_x = \cos(x+y) + \cos(x+z)$$

$$F_y = \cos(x+y) + \cos(y+z)$$

$$F_z = \cos(y+z) + \cos(x+z)$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\cos(x+y) + \cos(x+z)}{\cos(y+z) + \cos(x+z)}$$

$$\frac{\partial z}{\partial x}(\pi,\ \pi,\ \pi) = -\frac{\cos(2\pi) + \cos(2\pi)}{\cos(2\pi) + \cos(2\pi)} = \underline{-1}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\cos(x+y) + \cos(y+z)}{\cos(y+z) + \cos(x+z)}$$

$$\frac{\partial z}{\partial x}(\pi,\ \pi,\ \pi) = -\frac{\cos(2\pi) + \cos(2\pi)}{\cos(2\pi) + \cos(2\pi)} = \underline{-1}$$

Exercise

Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $xe^{y}+ye^{z}+2\ln x-2-3\ln 2=0$ at the point $(1,\ \ln 2,\ \ln 3)$

Solution

$$F(x,y,z)=xe^{y}+ye^{z}+2\ln x-2-3\ln 2$$

$$F_x=e^{y}+\frac{2}{x} \qquad F_y=xe^{y}+e^{z} \qquad F_z=ye^{z}$$

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{e^{y}+\frac{2}{x}}{ye^{z}}=-\frac{xe^{y}+2}{xye^{z}}$$

$$\frac{\partial z}{\partial x}(1,\ln 2,\ln 3)=-\frac{(1)e^{\ln 2}+2}{\ln 2e^{\ln 3}}=-\frac{2+2}{3\ln 2}=\underline{-\frac{4}{3\ln 2}}$$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{xe^{y}+e^{z}}{ye^{z}}$$

$$\frac{\partial z}{\partial x}(1,\ln 2,\ln 3)=-\frac{e^{\ln 2}+e^{\ln 3}}{\ln 2e^{\ln 3}}=-\frac{2+3}{3\ln 2}=\underline{-\frac{5}{3\ln 2}}$$

Exercise

Find $\frac{\partial w}{\partial r}$ when $r=1,\ s=-1$ if $w=(x+y+z)^2, \quad x=r-s, \quad y=\cos(r+s), \quad z=\sin(r+s)$

Solution

$$\begin{aligned}\frac{\partial w}{\partial r}&=\frac{\partial w}{\partial x}\frac{dx}{dr}+\frac{\partial w}{\partial y}\frac{dy}{dr}+\frac{\partial w}{\partial z}\frac{dz}{dr}\\&=2(x+y+z)(1)+2(x+y+z)(-\sin(r+s))+2(x+y+z)(\cos(r+s))\\&=2(x+y+z)[1-\sin(r+s)+\cos(r+s)]\\&=2(r-s+\cos(r+s)+\sin(r+s))(1-\sin(r+s)+\cos(r+s))\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial r}(1,-1)&=2(1-(-1)+\cos(1-1)+\sin(1-1))(1-\sin(1-1)+\cos(1-1))\\&=2(1+1+1+0)(1-0+1)\\&=2(3)(2)\\&\underline{=12}\end{aligned}$$

Exercise

Find $\frac{\partial z}{\partial u}$ when $u=0,\ v=1$ if $z=\sin xy+x\sin y, \quad x=u^2+v^2, \quad y=uv$

Solution

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{dx}{du}+\frac{\partial z}{\partial y}\frac{dy}{du} \\
&= (y\cos x+\sin y)(2u)+(x\cos xy+x\cos y)(v) \\
&= 2u\left(uv\cos\left(u^2+v^2\right)+\sin uv\right)+v\left(\left(u^2+v^2\right)\cos\left(u^3v+uv^3\right)+\left(u^2+v^2\right)\cos uv\right) \\
&= 2u\left(uv\cos\left(u^2+v^2\right)+\sin uv\right)+v\left(u^2+v^2\right)\left(\cos\left(u^3v+uv^3\right)+\cos uv\right)
\end{aligned}$$

$$\begin{aligned}
\left.\frac{\partial z}{\partial u}\right|_{u=0,\ v=1} &= 2(0)\big(0\cos(1)+\sin 0\big)+1(1)\big(\cos(0)+\cos 0\big) \\
&= 0+1(1+1) \\
&= 2
\end{aligned}$$

Exercise

Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ when $u=\ln 2,\ v=1$ if $z=5\tan^{-1}x, \quad x=e^u+\ln v$

Solution

$$\frac{\partial z}{\partial u}=\frac{dz}{dx}\frac{\partial x}{\partial u}=\left(\frac{5}{1+x^2}\right)e^u=\left(\frac{5}{1+\left(e^u+\ln v\right)^2}\right)e^u$$

$$\begin{aligned}
\left.\frac{\partial z}{\partial u}\right|_{u=\ln 2,\ v=1} &= \left(\frac{5}{1+\left(e^{\ln 2}+\ln 1\right)^2}\right)e^{\ln 2} \\
&= \left(\frac{5}{1+(2+0)^2}\right)(2) \\
&= 2\left(\frac{5}{5}\right) \\
&= 2
\end{aligned}$$

Exercise

Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ when $u=1,\ v=-2$ if $z=\ln q,\quad q=\sqrt{v+3}\ \tan^{-1}u$

Solution

$$\frac{\partial z}{\partial u}=\frac{dz}{dq}\frac{\partial q}{\partial u}$$

$$=\left(\frac{1}{q}\right)\left(\sqrt{v+3}\frac{1}{1+u^2}\right)$$

$$=\frac{1}{\sqrt{v+3}\ \tan^{-1}u}\cdot\frac{\sqrt{v+3}}{1+u^2}$$

$$=\frac{1}{\left(1+u^2\right)\tan^{-1}u}$$

$$\left.\frac{\partial z}{\partial u}\right|_{u=1,\ v=-2}=\frac{1}{\left(1+1^2\right)\tan^{-1}1}=\frac{1}{2\cdot\frac{\pi}{4}}=\underline{\frac{2}{\pi}}$$

$$\frac{\partial z}{\partial v}=\frac{dz}{dq}\frac{\partial q}{\partial v}$$

$$=\left(\frac{1}{q}\right)\left(\frac{1}{2\sqrt{v+3}}\tan^{-1}u\right)$$

$$=\left(\frac{1}{\sqrt{v+3}\ \tan^{-1}u}\right)\left(\frac{\tan^{-1}u}{2\sqrt{v+3}}\right)$$

$$=\frac{1}{2(v+3)}$$

$$\left.\frac{\partial z}{\partial v}\right|_{u=1,\ v=-2}=\frac{1}{2(-2+3)}=\underline{\frac{1}{2}}$$

Exercise

Assume that $w=f\left(s^3+t^2\right)$ and $f'(x)=e^x$. Find $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$

Solution

$$w=f\left(s^3+t^2\right)=f(x)\quad\rightarrow\quad x=s^3+t^2$$

$$\frac{\partial w}{\partial t}=\frac{dw}{dx}\frac{\partial x}{\partial t}=f'(x)\cdot 2t=2te^x=\underline{2te^{s^3+t^2}}$$

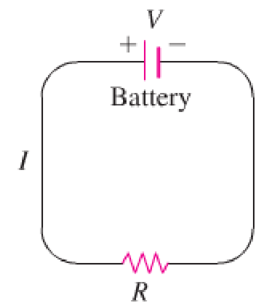
$$\frac{\partial w}{\partial s}=\frac{dw}{dx}\frac{\partial x}{\partial s}=\left(e^x\right)\left(3s^2\right)=\underline{3s^2e^{s^3+t^2}}$$

Exercise

The voltage V in a circuit that satisfies the law $V = IR$ is slowly dropping as the battery wears out. At the same time, the resistance R is increasing as the resistor heats up. Use the equation

$$\frac{dV}{dt} = \frac{\partial V}{\partial I}\frac{dI}{dt} + \frac{\partial V}{\partial R}\frac{dR}{dt}$$

To find how the current is changing at the instant when $R = 600\ \Omega$, $I = 0.04A$, $\frac{dR}{dt} = 0.5\ ohm/\sec$, and $\frac{dV}{dt} = -0.01\ volt/\sec$



Solution

$$V = IR \quad \to \quad \frac{\partial V}{\partial I} = R, \quad \frac{\partial V}{\partial R} = I$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial I}\frac{dI}{dt} + \frac{\partial V}{\partial R}\frac{dR}{dt}$$

$$-0.01 = (600)\frac{dI}{dt} + (0.04)(0.5)$$

$$-0.02 - 0.01 = 600\frac{dI}{dt}$$

$$\frac{dI}{dt} = \underline{-0.00005\ amps/\sec}$$

Exercise

The lengths a, b, and c of the edges of a rectangular box are changing with time. At the instant in question, $a = 1\ m$, $b = 2\ m$, $c = 3\ m$, $\frac{da}{dt} = \frac{db}{dt} = 1\ m/\sec$, and $\frac{dc}{dt} = -3\ m/\sec$. At what rates the box's volume V and surface area S changing at that instant? Are the box's interior diagonals increasing in length or decreasing?

Solution

$$V = abc \quad \Rightarrow \quad \frac{\partial V}{\partial t} = \frac{\partial V}{\partial a}\frac{da}{dt} + \frac{\partial V}{\partial b}\frac{db}{dt} + \frac{\partial V}{\partial c}\frac{dc}{dt}$$

$$\frac{\partial V}{\partial t} = (bc)\frac{da}{dt} + (ac)\frac{db}{dt} + (ab)\frac{dc}{dt}$$

$$= (2m)(3m)(1\ m/\sec) + (1m)(3m)(1\ m/\sec) + (1m)(2m)(-3\ m/\sec)$$

$$\underline{= 3\ m^3/\sec}$$

Exercise

Let $T = f(x,y)$ be the temperature at the point (x,y) on the circle $x=\cos t, \quad y=\sin t, \ 0 \le t \le 2\pi$ and suppose that

$$\frac{\partial T}{\partial x} = 8x - 4y, \quad \frac{\partial T}{\partial y} = 8y - 4x$$

a) Find where the maximum and minimum temperatures on the circle occur by examining the derivatives $\frac{dT}{dt}$ and $\frac{d^2T}{dt^2}$.

b) Suppose that $T = 4x^2 - 4xy + 4y^2$. Find the maximum and minimum values of T on the circle.

Solution

a) $\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$

$$\begin{aligned}
&= (8x-4y)(-\sin t) + (8y-4x)(\cos t) \\
&= (8\cos t - 4\sin t)(-\sin t) + (8\sin t - 4\cos t)(\cos t) \\
&= -8\cos t\sin t + 4\sin^2 t + 8\cos t\sin t - 4\cos^2 t \\
&\underline{= 4\sin^2 t - 4\cos^2 t}
\end{aligned}$$

$$\frac{dT}{dt} = 0 \quad \Rightarrow \quad 4\sin^2 t - 4\cos^2 t = 0$$

$$\begin{aligned}
&\sin^2 t = \cos^2 t \\
&\sin t = \pm\cos t \\
&\underline{t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}} \qquad \text{on the interval } 0 \le t \le 2\pi
\end{aligned}$$

$$\frac{d^2T}{dt^2} = 8\sin t\cos t + 8\cos t\sin t = \underline{16\sin t\cos t}$$

$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{\pi}{4}} = 16\sin\frac{\pi}{4}\cos\frac{\pi}{4} > 0 \quad \Rightarrow$ T has a minimum at $(x,\ y) = \left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$

$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{3\pi}{4}} = 16\sin\frac{3\pi}{4}\cos\frac{3\pi}{4} < 0 \Rightarrow$ T has a maximum at $(x,\ y) = \left(-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$

$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{5\pi}{4}} = 16\sin\frac{5\pi}{4}\cos\frac{5\pi}{4} > 0 \Rightarrow$ T has a minimum at $(x,\ y) = \left(-\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\right)$

$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{7\pi}{4}} = 16\sin\frac{7\pi}{4}\cos\frac{7\pi}{4} < 0 \quad \Rightarrow$ T has a maximum at $(x,\ y) = \left(\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\right)$

b) $T = 4x^2 - 4xy + 4y^2$

$$T\left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)=4\left(\frac{\sqrt{2}}{2}\right)^2-4\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)+4\left(\frac{\sqrt{2}}{2}\right)^2=2-2+2=\underline{2|}$$

$$T\left(-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)=4\left(-\frac{\sqrt{2}}{2}\right)^2-4\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)+4\left(\frac{\sqrt{2}}{2}\right)^2=2+2+2=\underline{6|}$$

$$T\left(-\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\right)=4\left(-\frac{\sqrt{2}}{2}\right)^2-4\left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)+4\left(-\frac{\sqrt{2}}{2}\right)^2=2-2+2=\underline{2|}$$

$$T\left(\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\right)=4\left(\frac{\sqrt{2}}{2}\right)^2-4\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)+4\left(-\frac{\sqrt{2}}{2}\right)^2=2+2+2=\underline{6|}$$

The maximum value is 6 at $\left(-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$ and $\left(\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\right)$

The minimum value is 2 at $\left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$ and $\left(-\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\right)$

Exercise

Evaluate $\frac{dy}{dx}$: $\quad x^2-2y^2-1=0$

Solution

$$F(x,\ y)=x^2-2y^2-1$$

$$\frac{dy}{dx}=-\frac{F_x}{F_y}$$

$$=-\frac{2x}{-4y}$$

$$\boxed{=\frac{x}{2y}}$$

Exercise

Evaluate $\frac{dy}{dx}$: $\quad x^3+3xy^2-y^5=0$

Solution

$$F(x,\ y)=x^3+3xy^2-y^5$$

$$\frac{dy}{dx}=-\frac{F_x}{F_y}$$

$$\boxed{=-\frac{3x^2+3y^2}{6xy-5y^4}}$$

Exercise

Evaluate $\frac{dy}{dx}$: $\quad 2\sin xy = 1$

Solution

$$F(x,\ y) = 2\sin xy - 1$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= -\frac{2y\cos xy}{2x\cos xy}$$

$$\boxed{= -\frac{y}{x}}$$

Exercise

Evaluate $\frac{dy}{dx}$: $\quad ye^{xy} - 2 = 0$

Solution

$$F(x,\ y) = ye^{xy} - 2$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= -\frac{y^2e^{xy}}{e^{xy} + xye^{xy}}$$

$$\boxed{= -\frac{y^2}{1 + xy}}$$

Exercise

Evaluate $\frac{dy}{dx}$: $\quad \sqrt{x^2 + 2xy + y^4} = 3$

Solution

$$F(x,\ y) = \sqrt{x^2 + 2xy + y^4} - 3$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} \quad = -\frac{\frac{1}{2}(2x + 2y)\left(x^2 + 2xy + y^4\right)^{-1/2}}{\frac{1}{2}\left(2x + 4y^3\right)\left(x^2 + 2xy + y^4\right)^{-1/2}}$$

$$\boxed{= -\frac{x + y}{x + 2y^3}}$$

Exercise

Evaluate $\dfrac{dy}{dx}$: $\qquad y\ln\left(x^2+y^2+4\right)=3$

Solution

$$F(x,\ y)=y\ln\left(x^2+y^2+4\right)-3$$

$$\frac{dy}{dx}=-\frac{F_x}{F_y}\ =-\frac{\frac{2xy}{x^2+y^2+4}}{\ln\left(x^2+y^2+4\right)+\frac{2y^2}{x^2+y^2+4}}$$

$$=-\frac{2xy}{2y^2+\left(x^2+y^2+4\right)\ln\left(x^2+y^2+4\right)}$$

Exercise

Find $\dfrac{dz}{dx}$ and $\dfrac{dz}{dy}$ at the given point. $\qquad z^3-xy+yz+y^3-2=0;\quad (1,\ 1,\ 1)$

Solution

$$F(x,\ y,\ z)=z^3-xy+yz+y^3-2$$

$$F_x=-y,\quad F_y=-x+z+3y^2,\quad \text{and}\quad F_z=3z^2+y\Big|_{(1,1,1)}=4\neq 0$$

$$\frac{dz}{dx}=-\frac{F_x}{F_z}=-\frac{-y}{3z^2+y}\qquad\qquad \frac{dz}{dy}=-\frac{F_y}{F_z}=-\frac{e^{xz}-z\sin y}{2z+xye^{xz}+\cos y}$$

$$\frac{dz}{dx}\Big|_{(1,1,1)}=-\frac{-1}{4}=\frac{1}{4}\qquad\qquad \frac{dz}{dz}\Big|_{(1,1,1)}=-\frac{3}{4}$$

Exercise

Find $\dfrac{dz}{dx}$ and $\dfrac{dz}{dy}$ at the given point. $\qquad \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}-1=0;\quad (2,\ 3,\ 6)$

Solution

$$F(x,\ y,\ z)=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1$$

$$F_x=-\frac{1}{x^2}\Big|_{(2,3,6)}=-\frac{1}{4},\quad F_y=-\frac{1}{y^2}\Big|_{(2,3,6)}=-\frac{1}{9},\quad \text{and}\quad F_z=-\frac{1}{z^2}\Big|_{(2,3,6)}=-\frac{1}{36}\neq 0$$

$$\frac{dz}{dx}\Big|_{(2,3,6)}=-\frac{F_x}{F_z}\Big|_{(2,3,6)}=-\frac{-\frac{1}{4}}{-\frac{1}{36}}=-9\qquad\qquad \frac{dz}{dy}\Big|_{(2,3,6)}=-\frac{F_y}{F_z}\Big|_{(2,3,6)}=-\frac{-\frac{1}{9}}{-\frac{1}{36}}=-4$$

Exercise

Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ at the given point. $\sin(x+y)+\sin(y+z)+\sin(x+z)=0; \quad (\pi,\ \pi,\ \pi)$

Solution

$$F(x,\ y,\ z)=\sin(x+y)+\sin(y+z)+\sin(x+z)$$

$$F_x=\cos(x+y)+\cos(x+z)\Big|_{(\pi,\pi,\pi)}=\cos 2\pi+\cos 2\pi\ \underline{=2|}$$

$$F_y=\cos(x+y)+\cos(y+z)\Big|_{(\pi,\pi,\pi)}=\cos 2\pi+\cos 2\pi\ \underline{=2|}$$

$$F_z=\cos(y+z)+\cos(x+z)\Big|_{(\pi,\pi,\pi)}=\cos 2\pi+\cos 2\pi\ \underline{=2|}\neq 0$$

$$\frac{dz}{dx}\Big|_{(\pi,\pi,\pi)}=-\frac{F_x}{F_z}\Big|_{(\pi,\pi,\pi)}=-\frac{2}{2}\ \underline{=-1|} \qquad \frac{dz}{dy}\Big|_{(\pi,\pi,\pi)}=-\frac{F_y}{F_z}\Big|_{(\pi,\pi,\pi)}=-\frac{2}{2}\ \underline{=-1|}$$

Exercise

Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ at the given point. $xe^y+ye^z+2\ln x-2-3\ln 2=0; \quad (1,\ \ln 2,\ \ln 3)$

Solution

$$F(x,\ y,\ z)=xe^y+ye^z+2\ln x-2-3\ln 2$$

$$F_x=e^y+\frac{2}{x}\Big|_{(1,\ln 2,\ln 3)}=2+2\ \underline{=4|}$$

$$F_y=xe^y+e^z\Big|_{(1,\ln 2,\ln 3)}=e^{\ln 2}+e^{\ln 3}=2+3\ \underline{=5|}$$

$$F_z=ye^z\Big|_{(1,\ln 2,\ln 3)}=\ln 2e^{\ln 3}\ \underline{=3\ln 2|}\neq 0$$

$$\frac{dz}{dx}\Big|_{(1,\ln 2,\ln 3)}=-\frac{F_x}{F_z}\Big|_{(1,\ln 2,\ln 3)}\ \underline{=-\frac{4}{3\ln 2}|}$$

$$\frac{dz}{dy}\Big|_{(1,\ln 2,\ln 3)}=-\frac{F_y}{F_z}\Big|_{(1,\ln 2,\ln 3)}\ \underline{=-\frac{5}{3\ln 2}|}$$

Solution Section 2.5 – Directional Derivatives and the Gradient

Exercise

Find the gradient of the function at the given point. Then sketch the gradient together with the level curve that passes through the point $f(x,y) = y - x, \quad (2,\ 1)$

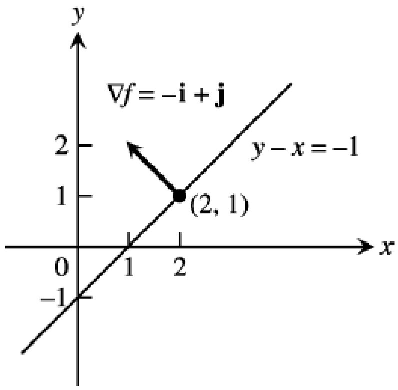
Solution

$$\frac{\partial f}{\partial x} = -1, \quad \frac{\partial f}{\partial y} = 1$$

$$\nabla f = f_x \boldsymbol{i} + f_y \boldsymbol{j} = -\boldsymbol{i} + \boldsymbol{j}$$

$$f(2,\ 1) = 1 - 2 = \underline{-1}$$

$-1 = y - x$ is the level curve



Exercise

Find the gradient of the function at the given point. Then sketch the gradient together with the level curve that passes through the point

$$f(x,y) = \ln\left(x^2 + y^2\right), \quad (1,\ 1)$$

Solution

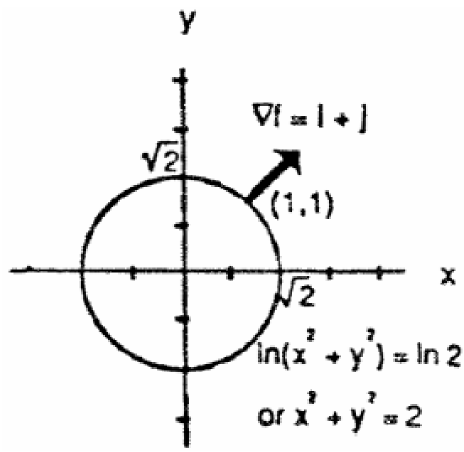
$$\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2}, \quad \left.\frac{\partial f}{\partial x}\right|_{(1,1)} = \frac{2}{1+1} = 1$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2}, \quad \left.\frac{\partial f}{\partial y}\right|_{(1,1)} = \frac{2}{1+1} = 1$$

$$\nabla f = f_x \boldsymbol{i} + f_y \boldsymbol{j} = \boldsymbol{i} + \boldsymbol{j}$$

$$f(1,\ 1) = \underline{\ln 2}$$

$\ln 2 = \ln\left(x^2 + y^2\right) \quad \rightarrow \quad x^2 + y^2 = 2$ is the level curve



Exercise

Find the gradient of the function at the given point. Then sketch the gradient together with the level curve that passes through the point $f(x,y) = \sqrt{2x + 3y}, \quad (-1,\ 2)$

Solution

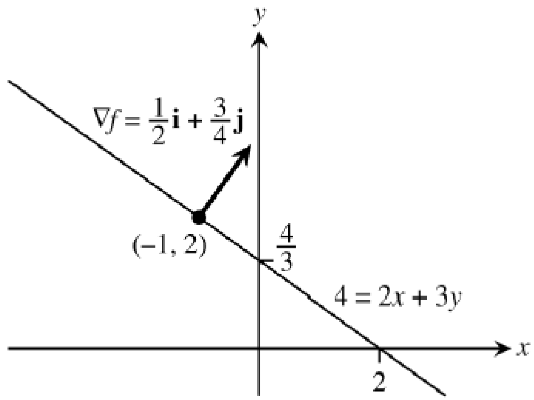
$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{2x+3y}}, \quad \left.\frac{\partial f}{\partial x}\right|_{(-1,2)} = \frac{1}{\sqrt{-2+6}} = \frac{1}{2}$$

$$\frac{\partial f}{\partial y} = \frac{3}{2\sqrt{2x+3y}}, \quad \left.\frac{\partial f}{\partial y}\right|_{(-1,2)} = \frac{3}{2\sqrt{-2+6}} = \frac{3}{4}$$

$$\nabla f = \frac{1}{2}\boldsymbol{i} + \frac{3}{4}\boldsymbol{j}$$

$$f(-1,\ 2) = \sqrt{2(-1)+3(2)} = \underline{2|}$$

$2x + 3y = 4$ is the level curve



Exercise

Find ∇f at the given point $\quad f(x,y,z) = x^2 + y^2 - 2z^2 + z\ln x, \quad (1,\ 1,\ 1)$

Solution

$$\frac{\partial f}{\partial x} = 2x + \frac{z}{x}, \quad \left.\frac{\partial f}{\partial x}\right|_{(1,1,1)} = 2 + \frac{1}{1} = 3$$

$$\frac{\partial f}{\partial y} = 2y, \quad \left.\frac{\partial f}{\partial y}\right|_{(1,1,1)} = 2$$

$$\frac{\partial f}{\partial z} = -4z + \ln x, \quad \left.\frac{\partial f}{\partial z}\right|_{(1,1,1)} = -4 + \ln 1 = -4$$

$$\underline{\nabla f = 3\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}|}$$

Exercise

Find ∇f at the given point $\quad f(x,y,z) = 2x^3 - 3\left(x^2 + y^2\right)z + \tan^{-1} xz, \quad (1,\ 1,\ 1)$

Solution

$$\frac{\partial f}{\partial x} = 6x^2 - 6xz + \frac{z}{1+x^2z^2}, \quad \left.\frac{\partial f}{\partial x}\right|_{(1,1,1)} = 6 - 6 + \frac{1}{1+1} = \frac{1}{2}$$

$$\frac{\partial f}{\partial y} = -6yz, \quad \left.\frac{\partial f}{\partial y}\right|_{(1,1,1)} = -6$$

$$\frac{\partial f}{\partial z} = -3\left(x^2 + y^2\right) + \frac{z}{1+x^2z^2}, \quad \left.\frac{\partial f}{\partial z}\right|_{(1,1,1)} = -3(2) + \frac{1}{2} = -\frac{11}{2}$$

$$\underline{\nabla f = \frac{1}{2}\boldsymbol{i} - 6\boldsymbol{j} - \frac{11}{2}\boldsymbol{k}|}$$

Exercise

Find ∇f at the given point $f(x,y,z)=e^{x+y}\cos z+(y+1)\sin^{-1}x,\quad \left(0,\ 0,\ \frac{\pi}{6}\right)$

Solution

$$\frac{\partial f}{\partial x}=e^{x+y}\cos z+\frac{y+1}{\sqrt{1-x^2}}, \quad \to \left.\frac{\partial f}{\partial x}\right|_{\left(0,0,\frac{\pi}{6}\right)}=e^0\cos\frac{\pi}{6}+\frac{0+1}{\sqrt{1-0}}=\frac{\sqrt{3}}{2}+1$$

$$\frac{\partial f}{\partial y}=e^{x+y}\cos z+\sin^{-1}x, \quad \to \left.\frac{\partial f}{\partial y}\right|_{\left(0,0,\frac{\pi}{6}\right)}=e^0\cos\frac{\pi}{6}+0=\frac{\sqrt{3}}{2}$$

$$\frac{\partial f}{\partial z}=-e^{x+y}\sin z, \quad \to \left.\frac{\partial f}{\partial z}\right|_{\left(0,0,\frac{\pi}{6}\right)}=-e^0\sin\frac{\pi}{6}=-\frac{1}{2}$$

$$\nabla f=\left(\frac{\sqrt{3}}{2}+1\right)\boldsymbol{i}+\frac{\sqrt{3}}{2}\boldsymbol{j}-\frac{1}{2}\boldsymbol{k}$$

Exercise

Find the derivative of the function $f(x,y)=2xy-3y^2$ at $P_0(5,\ 5)$ in the direction of $\boldsymbol{v}=4\boldsymbol{i}+3\boldsymbol{j}$

Solution

$$\boldsymbol{u}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}=\frac{4\boldsymbol{i}+3\boldsymbol{j}}{\sqrt{16+9}}=\frac{4}{5}\boldsymbol{i}+\frac{3}{5}\boldsymbol{j}$$

$$f_x=2y \quad\Rightarrow\quad f_x(5,5)=10$$

$$f_y=2x-6y \quad\Rightarrow\quad f_y(5,5)=10-30=-20$$

$$\nabla f=10\boldsymbol{i}-20\boldsymbol{j}$$

$$\begin{aligned}\left(D_{\boldsymbol{u}}f\right)_{P_0}&=\nabla f\bullet\boldsymbol{u}\\&=(10\boldsymbol{i}-20\boldsymbol{j})\cdot\left(\frac{4}{5}\boldsymbol{i}+\frac{3}{5}\boldsymbol{j}\right)\\&=10\left(\frac{4}{5}\right)-20\left(\frac{3}{5}\right)\\&=8-12\\&=-4\end{aligned}$$

Exercise

Find the derivative of the function $f(x,y)=\dfrac{x-y}{xy+2}$ at $P_0(1,\ -1)$ in the direction of $\boldsymbol{v}=12\boldsymbol{i}+5\boldsymbol{j}$

Solution

$$\boldsymbol{u}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}=\frac{12\boldsymbol{i}+5\boldsymbol{j}}{\sqrt{144+25}}=\frac{12}{13}\boldsymbol{i}+\frac{5}{13}\boldsymbol{j}$$

$$f_x=\frac{xy+2-y(x-y)}{(xy+2)^2}=\frac{xy+2-xy+y^2}{(xy+2)^2}=\frac{2+y^2}{(xy+2)^2}\quad\Rightarrow\quad f_x(1,-1)=\frac{2+1}{(-1+2)^2}=3$$

$$f_y=\frac{-xy-2-x(x-y)}{(xy+2)^2}=\frac{-2-x^2}{(xy+2)^2}\quad\Rightarrow\quad f_y(1,-1)=\frac{-2-1}{(-1+2)^2}=-3$$

$$\nabla f=3\boldsymbol{i}-3\boldsymbol{j}$$

$$\begin{aligned}\left(D_{\boldsymbol{u}}f\right)_{P_0}&=\nabla f\bullet\boldsymbol{u}\\&=(3\boldsymbol{i}-3\boldsymbol{j})\cdot\left(\frac{12}{13}\boldsymbol{i}+\frac{5}{13}\boldsymbol{j}\right)\\&=\frac{36}{13}-\frac{15}{13}\\&=\frac{21}{13}\end{aligned}$$

Exercise

Find the derivative of the function $h(x,y)=\tan^{-1}\left(\dfrac{y}{x}\right)+\sqrt{3}\sin^{-1}\left(\dfrac{xy}{2}\right)$ at $P_0(1,\ 1)$ in the direction of $\boldsymbol{v}=3\boldsymbol{i}-2\boldsymbol{j}$

Solution

$$\boldsymbol{u}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}=\frac{3\boldsymbol{i}-2\boldsymbol{j}}{\sqrt{9+4}}=\frac{3}{\sqrt{13}}\boldsymbol{i}-\frac{2}{\sqrt{13}}\boldsymbol{j}$$

$$h_x=\frac{-\frac{y}{x^2}}{\left(\frac{y}{x}\right)^2+1}+\sqrt{3}\frac{\frac{y}{2}}{\sqrt{1-\left(\frac{x^2y^2}{4}\right)}}\quad\Rightarrow\quad h_x(1,1)=\frac{-1}{1+1}+\sqrt{3}\frac{\frac{1}{2}}{\sqrt{1-\frac{1}{4}}}=-\frac{1}{2}+\sqrt{3}\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\frac{1}{2}$$

$$h_y=\frac{\frac{1}{x}}{\left(\frac{y}{x}\right)^2+1}+\sqrt{3}\frac{\frac{x}{2}}{\sqrt{1-\frac{x^2y^2}{4}}}\quad\Rightarrow\quad h_y(1,1)=\frac{1}{2}+\sqrt{3}\frac{\frac{1}{2}}{\sqrt{1-\frac{1}{4}}}=\frac{3}{2}$$

$$\nabla h=\frac{1}{2}\boldsymbol{i}+\frac{3}{2}\boldsymbol{j}$$

$$\left(D_{\boldsymbol{u}}h\right)_{P_0}=\nabla h\bullet\boldsymbol{u}$$

$$= \left(\tfrac{1}{2}\boldsymbol{i} + \tfrac{3}{2}\boldsymbol{j}\right)\cdot\left(\frac{3}{\sqrt{13}}\boldsymbol{i} - \frac{2}{\sqrt{13}}\boldsymbol{j}\right)$$

$$= \frac{3}{2\sqrt{13}} - \frac{3}{\sqrt{13}}$$

$$= -\frac{3}{2\sqrt{13}}$$

Exercise

Find the derivative of the function $f(x,y,z) = xy + yz + zx$ at $P_0(1,\ -1,\ 2)$ in the direction of $\boldsymbol{v} = 3\boldsymbol{i} + 6\boldsymbol{j} - 2\boldsymbol{k}$

Solution

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{3\boldsymbol{i} + 6\boldsymbol{j} - 2\boldsymbol{k}}{\sqrt{9+36+4}} = \tfrac{3}{7}\boldsymbol{i} + \tfrac{6}{7}\boldsymbol{j} - \tfrac{2}{7}\boldsymbol{k}$$

$$f_x = y + z \quad\Rightarrow\quad f_x(1,-1,2) = -1 + 2 = 1$$

$$f_y = x + z \quad\Rightarrow\quad f_y(1,-1,2) = 1 + 2 = 3$$

$$f_z = y + x \quad\Rightarrow\quad f_z(1,-1,2) = -1 + 1 = 0$$

$$\nabla f = \boldsymbol{i} + 3\boldsymbol{j}$$

$$\left(D_{\boldsymbol{u}} f\right)_{P_0} = \nabla f \bullet \boldsymbol{u}$$

$$= (\boldsymbol{i} + 3\boldsymbol{j})\cdot\left(\tfrac{3}{7}\boldsymbol{i} + \tfrac{6}{7}\boldsymbol{j} - \tfrac{2}{7}\boldsymbol{k}\right)$$

$$= \frac{3}{7} + \frac{18}{7}$$

$$= 3$$

Exercise

Find the derivative of the function $g(x,y,z) = 3e^x \cos yz$ at $P_0(0,\ 0,\ 0)$ in the direction of $\boldsymbol{v} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$

Solution

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}}{\sqrt{4+1+4}} = \tfrac{2}{3}\boldsymbol{i} + \tfrac{1}{3}\boldsymbol{j} - \tfrac{2}{3}\boldsymbol{k} \quad g_y = -3ze^x \sin yz \quad\Rightarrow\quad g_y(0,0,0) = -3(0)e^0 \sin 0 = 0$$

$$g_x = 3e^x \cos yz \quad\Rightarrow\quad g_x(0,0,0) = 3e^0 \cos(0) = 3$$

$$g_z = -3ye^x \sin yz \quad\Rightarrow\quad g_z(0,0,0) = -3(0)e^0 \sin 0 = 0$$

$$\nabla g = 3\boldsymbol{i}$$

$$\left(D_{\boldsymbol{u}} g\right)_{P_0} = \nabla g \bullet \boldsymbol{u}$$

$$= (3\boldsymbol{i}) \cdot \left(\frac{2}{3}\boldsymbol{i} + \frac{1}{3}\boldsymbol{j} - \frac{2}{3}\boldsymbol{k}\right)$$

$$= 2$$

Exercise

Find the derivative of the function $h(x, y, z) = \cos xy + e^{yz} + \ln zx$ at $P_0\left(1,\ 0,\ \frac{1}{2}\right)$ in the direction of $\boldsymbol{v} = \boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$

Solution

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}}{\sqrt{1+4+4}} = \frac{1}{3}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} + \frac{2}{3}\boldsymbol{k}$$

$$h_x = -y\sin xy + \frac{1}{x} \quad \Rightarrow \quad h_x\left(1, 0, \frac{1}{2}\right) = -(0)\sin(0) + \frac{1}{1} = 1$$

$$h_y = -x\sin xy + ze^{yz} \quad \Rightarrow \quad h_y\left(1, 0, \frac{1}{2}\right) = -(1)\sin 0 + \frac{1}{2}e^0 = \frac{1}{2}$$

$$h_z = ye^{yz} + \frac{1}{z} \quad \Rightarrow \quad h_z\left(1, 0, \frac{1}{2}\right) = 0e^0 + \frac{1}{\frac{1}{2}} = 2$$

$$\nabla h = \boldsymbol{i} + \frac{1}{2}\boldsymbol{j} + 2\boldsymbol{k}$$

$$\left(D_{\boldsymbol{u}} h\right)_{P_0} = \nabla h \bullet \boldsymbol{u}$$

$$= \left(\boldsymbol{i} + \frac{1}{2}\boldsymbol{j} + 2\boldsymbol{k}\right) \cdot \left(\frac{1}{3}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} + \frac{2}{3}\boldsymbol{k}\right)$$

$$= \frac{1}{3} + \frac{1}{3} + \frac{4}{3}$$

$$= 2$$

Exercise

Find the directions in which the function $f(x, y) = x^2 + xy + y^2$ increase and decrease most rapidly at $P_0(-1,\ 1)$. Then find the derivatives of the function in these directions.

Solution

$$\left.\begin{array}{l} f_x = 2x + y \quad \Rightarrow \quad f_x(-1, 1) = 2(-1) + 1 = -1 \\ f_y = x + 2y \quad \Rightarrow \quad f_y(-1, 1) = (-1) + 2(1) = 1 \end{array}\right\} \to \quad \nabla f = -\boldsymbol{i} + \boldsymbol{j}$$

$$\boldsymbol{u}=\frac{\nabla f}{|\nabla f|}=\frac{-\boldsymbol{i}+\boldsymbol{j}}{\sqrt{1+1}}=-\frac{1}{\sqrt{2}}\boldsymbol{i}+\frac{1}{\sqrt{2}}\boldsymbol{j}$$

f *increases* most rapidly in the direction $\boldsymbol{u}=-\frac{1}{\sqrt{2}}\boldsymbol{i}+\frac{1}{\sqrt{2}}\boldsymbol{j}$

f *decreases* most rapidly in the direction $-\boldsymbol{u}=\frac{1}{\sqrt{2}}\boldsymbol{i}-\frac{1}{\sqrt{2}}\boldsymbol{j}$

$$\begin{aligned}\left(D_{\boldsymbol{u}}f\right)_{P_0}&=\nabla f\bullet\boldsymbol{u}\\&=(-\boldsymbol{i}+\boldsymbol{j})\cdot\left(-\frac{1}{\sqrt{2}}\boldsymbol{i}+\frac{1}{\sqrt{2}}\boldsymbol{j}\right)\\&=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\\&=\sqrt{2}\end{aligned}$$

$$\left(D_{-\boldsymbol{u}}f\right)_{P_0}=-\sqrt{2}$$

Exercise

Find the directions in which the function $f(x,y)=x^2y+e^{xy}\sin y$ increase and decrease most rapidly at $P_0(1,\ 0)$. Then find the derivatives of the function in these directions.

Solution

$$\left.\begin{aligned}f_x&=2xy+ye^{xy}\sin y &\Rightarrow\quad f_x(1,0)&=2(1)(0)+0e^0=0\\f_y&=x^2+xe^{xy}\sin y+e^{xy}\cos y &\Rightarrow\quad f_y(1,0)&=1^2+0+1=2\end{aligned}\right\}\rightarrow\ \nabla f=2\boldsymbol{j}$$

$$\boldsymbol{u}=\frac{\nabla f}{|\nabla f|}=\boldsymbol{j}$$

f *increases* most rapidly in the direction $\boldsymbol{u}=\boldsymbol{j}$

f *decreases* most rapidly in the direction $-\boldsymbol{u}=-\boldsymbol{j}$

$$\left(D_{\boldsymbol{u}}f\right)_{P_0}=\nabla f\bullet\boldsymbol{u}=2$$

$$\left(D_{-\boldsymbol{u}}f\right)_{P_0}=-2$$

Exercise

Find the directions in which the function $g(x,y,z)=xe^{y}+z^{2}$ increase and decrease most rapidly at $P_0\left(1,\ \ln 2,\ \frac{1}{2}\right)$. Then find the derivatives of the function in these directions.

Solution

$$g_x=e^{y} \quad\Rightarrow\quad g_x\left(1,\ln 2,\tfrac{1}{2}\right)=e^{\ln 2}=2$$

$$g_y=xe^{y} \quad\Rightarrow\quad g_y\left(1,\ln 2,\tfrac{1}{2}\right)=e^{\ln 2}=2 \quad\to\quad \nabla g=2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$$

$$g_z=2z \quad\Rightarrow\quad g_z\left(1,\ln 2,\tfrac{1}{2}\right)=2\left(\tfrac{1}{2}\right)=1$$

$$\boldsymbol{u}=\frac{\nabla g}{|\nabla g|}=\frac{2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}}{\sqrt{4+4+1}}=\frac{2}{3}\boldsymbol{i}+\frac{2}{3}\boldsymbol{j}+\frac{1}{3}\boldsymbol{k}$$

g *increases* most rapidly in the direction $\boldsymbol{u}=\frac{2}{3}\boldsymbol{i}+\frac{2}{3}\boldsymbol{j}+\frac{1}{3}\boldsymbol{k}$

g *decreases* most rapidly in the direction $-\boldsymbol{u}=-\frac{2}{3}\boldsymbol{i}-\frac{2}{3}\boldsymbol{j}-\frac{1}{3}\boldsymbol{k}$

$$\begin{aligned}\left(D_{\boldsymbol{u}}g\right)_{P_0}&=\nabla g\bullet\boldsymbol{u}\\&=(2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k})\left(\tfrac{2}{3}\boldsymbol{i}+\tfrac{2}{3}\boldsymbol{j}+\tfrac{1}{3}\boldsymbol{k}\right)\\&=\frac{4}{3}+\frac{4}{3}+\frac{1}{3}\\&=3\end{aligned}$$

$$\left(D_{-\boldsymbol{u}}g\right)_{P_0}=-3$$

Exercise

Find the directions in which the function $h(x,y,z)=\ln\left(x^{2}+y^{2}-1\right)+y+6z$ increase and decrease most rapidly at $P_0\,(1,\ 1,\ 0)$. Then find the derivatives of the function in these directions.

Solution

$$h_x=\frac{2x}{x^2+y^2-1} \quad\Rightarrow\quad h_x(1,1,0)=\frac{2}{1+1-1}=2$$

$$h_y=\frac{2y}{x^2+y^2-1}+1 \quad\Rightarrow\quad h_y(1,1,0)=\frac{2}{1+1-1}+1=3 \quad\to\quad \nabla h=2\boldsymbol{i}+3\boldsymbol{j}+6\boldsymbol{k}$$

$$h_z=6 \quad\Rightarrow\quad h_z(1,1,0)=6$$

$$\boldsymbol{u}=\frac{\nabla h}{|\nabla h|}=\frac{2\boldsymbol{i}+3\boldsymbol{j}+6\boldsymbol{k}}{\sqrt{4+9+36}}=\frac{2}{7}\boldsymbol{i}+\frac{3}{7}\boldsymbol{j}+\frac{6}{7}\boldsymbol{k}$$

h *increases* most rapidly in the direction $\boldsymbol{u}=\frac{2}{7}\boldsymbol{i}+\frac{3}{7}\boldsymbol{j}+\frac{6}{7}\boldsymbol{k}$

h *decreases* most rapidly in the direction $-\boldsymbol{u}=-\frac{2}{7}\boldsymbol{i}-\frac{3}{7}\boldsymbol{j}-\frac{6}{7}\boldsymbol{k}$

$$\left(D_{\boldsymbol{u}}h\right)_{P_0}=\nabla h\bullet\boldsymbol{u}=(2\boldsymbol{i}+3\boldsymbol{j}+6\boldsymbol{k})\left(\frac{2}{7}\boldsymbol{i}+\frac{3}{7}\boldsymbol{j}+\frac{6}{7}\boldsymbol{k}\right)=\frac{4}{7}+\frac{9}{7}+\frac{36}{7}=7$$

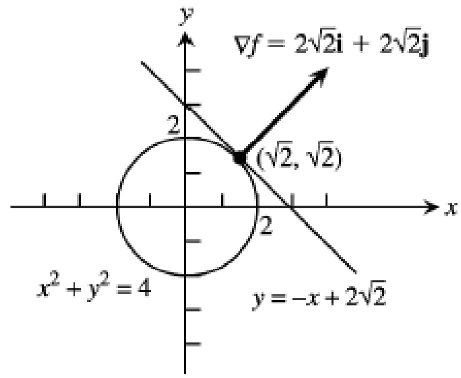
$$\left(D_{-\boldsymbol{u}}h\right)_{P_0}=-7$$

Exercise

Sketch the curve $x^2+y^2=4;\ \ (f(x,y)=c)$ together with ∇f and the tangent line at the point $\left(\sqrt{2},\ \sqrt{2}\right)$. Then write an equation for the tangent line.

Solution

$$f_x=2x \quad\Rightarrow\quad f_x\left(\sqrt{2},\sqrt{2}\right)=2\sqrt{2}$$
$$f_y=2y \quad\Rightarrow\quad f_y\left(\sqrt{2},\sqrt{2}\right)=2\sqrt{2}$$
$$\rightarrow\quad \nabla f=2\sqrt{2}\boldsymbol{i}+2\sqrt{2}\boldsymbol{j}$$



Tangent line: $2\sqrt{2}\left(x-\sqrt{2}\right)+2\sqrt{2}\left(y-\sqrt{2}\right)=0$

$$2\sqrt{2}x-4+2\sqrt{2}y-4=0$$

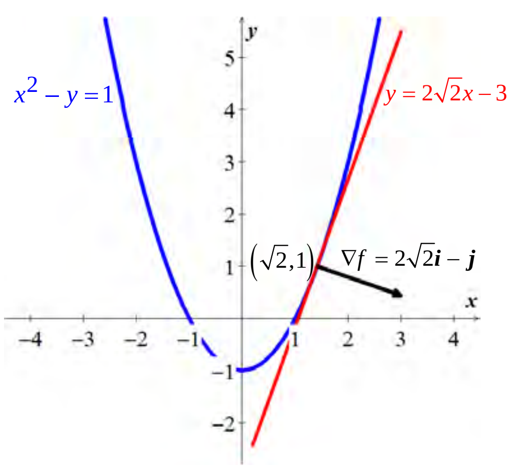
$$2\sqrt{2}x+2\sqrt{2}y=8$$

$$\boxed{\sqrt{2}x+\sqrt{2}y=4}$$

Exercise

Sketch the curve $x^2-y=1;\ \ (f(x,y)=c)$ together with ∇f and the tangent line at the point $\left(\sqrt{2},\ 1\right)$. Then write an equation for the tangent line.

Solution



$$f_x=2x \quad\Rightarrow\quad f_x\left(\sqrt{2},1\right)=2\sqrt{2}$$
$$f_y=-1 \quad\Rightarrow\quad f_y\left(\sqrt{2},1\right)=-1$$
$$\rightarrow\quad \nabla f=2\sqrt{2}\boldsymbol{i}-\boldsymbol{j}$$

Tangent line: $2\sqrt{2}\left(x-\sqrt{2}\right)-(y-1)=0$

$$2\sqrt{2}x-4-y+1=0$$

$$\boxed{y=2\sqrt{2}x-3}$$

Exercise

Sketch the curve $x^2 - xy + y^2 = 7$; $\left(f(x,y) = c\right)$ together with ∇f and the tangent line at the point $(-1,\ 2)$. Then write an equation for the tangent line.

Solution

$$f_x = 2x - y \quad \Rightarrow \quad f_x(-1,2) = -4$$

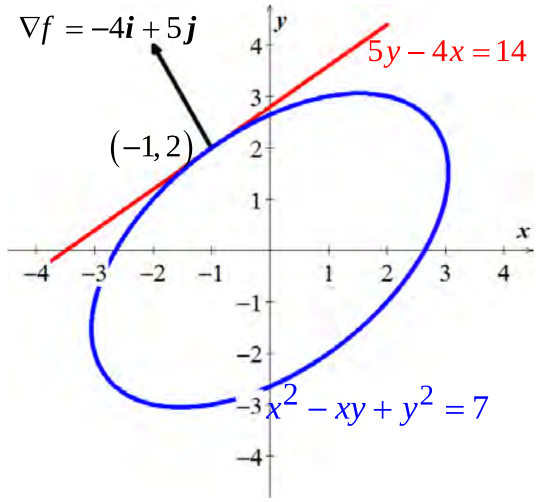
$$f_y = -x + 2y \quad \Rightarrow \quad f_y(-1,2) = 5$$

$$\rightarrow \nabla f = -4\boldsymbol{i} + 5\boldsymbol{j}$$

Tangent line: $-4(x+1) + 5(y-2) = 0$

$$-4x + 5y - 14 = 0$$

$$\boxed{5y - 4x = 14}$$



Exercise

In what direction is the derivative of $f(x,y) = xy + y^2$ at $P(3,\ 2)$ equal to zero?

Solution

$$\left.\begin{aligned} f_x &= y \\ f_y &= x + 2y \end{aligned}\right. \quad \rightarrow \quad \nabla f = x\boldsymbol{i} + (x + 2y)\boldsymbol{j}$$

$$\nabla f(3,2) = 2\boldsymbol{i} + 7\boldsymbol{j}$$

A vector is orthogonal to ∇f is $v = 7\boldsymbol{i} - 2\boldsymbol{j}$

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{7\boldsymbol{i} - 2\boldsymbol{j}}{\sqrt{49+4}} = \frac{7}{\sqrt{53}}\boldsymbol{i} - \frac{2}{\sqrt{53}}\boldsymbol{j}$$

$$-\boldsymbol{u} = -\frac{7}{\sqrt{53}}\boldsymbol{i} + \frac{2}{\sqrt{53}}\boldsymbol{j}$$

$\boldsymbol{u}$ and $-\boldsymbol{u}$ are the directions where the derivatives is zero.

Solution Section 2.6 – Tangent Planes and Linear Approximation

Exercise

Find the tangent plane and normal line of the surface $x^2+y^2+z^2=3$ at the point $P_0(1,\ 1,\ 1)$

Solution

$$f(x,y,z)=x^2+y^2+z^2$$

$$f_x=2x,\quad f_y=2y,\quad f_z=2z$$

$$\nabla f=2x\boldsymbol{i}+2y\boldsymbol{j}+2z\boldsymbol{k} \quad\Rightarrow\quad \nabla f(1,1,1)=2\boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k}$$

Tangent Line: $f_x(P_0)(x-x_0)+f_y(P_0)(y-y_0)+f_z(P_0)(z-z_0)=0$

$$2(x-1)+2(y-1)+2(z-1)=0$$

$$2x+2y+2z=6$$

$$\boxed{x+y+z=3}$$

Normal Line: $x=x_0+f_x(P_0)t,\quad y=y_0+f_y(P_0)t,\quad z=z_0+f_z(P_0)t$

$$\boxed{x=1+2t,\quad y=1+2t,\quad z=1+2t}$$

Exercise

Find the tangent plane and normal line of the surface $x^2+2xy-y^2+z^2=7$ at the point $P_0(1,\ -1,\ 3)$

Solution

$$f(x,y,z)=x^2+2xy-y^2+z^2 \quad\to\quad f_x=2x+2y,\quad f_y=2x-2y,\quad f_z=2z$$

$$\nabla f=(2x+2y)\boldsymbol{i}+(2x-2y)\boldsymbol{j}+2z\boldsymbol{k} \quad\Rightarrow\quad \nabla f(1,-1,3)=4\boldsymbol{j}+6\boldsymbol{k}$$

Tangent Line: $f_x(P_0)(x-x_0)+f_y(P_0)(y-y_0)+f_z(P_0)(z-z_0)=0$

$$0(x-1)+4(y+1)+6(z-3)=0$$

$$4y+6z=14$$

$$\boxed{2y+3z=7}$$

Normal Line: $x=x_0+f_x(P_0)t,\quad y=y_0+f_y(P_0)t,\quad z=z_0+f_z(P_0)t$

$$\boxed{x=1,\quad y=-1+4t,\quad z=3+6t}$$

Exercise

Find the tangent plane and normal line of the surface $\cos \pi x - x^2 y + e^{xz} + yz = 4$ at the point $P_0(0,\ 1,\ 2)$

Solution

$$f(x,y,z) = \cos \pi x - x^2 y + e^{xz} + yz$$

$$\rightarrow \quad f_x = -\pi \sin \pi x - 2xy + ze^{xz}, \quad f_y = -x^2 + z, \quad f_z = xe^{xz} + y$$

$$\nabla f = \left(-\pi \sin \pi x - 2xy + ze^{xz}\right)\boldsymbol{i} + \left(z - x^2\right)\boldsymbol{j} + \left(xe^{xz} + y\right)\boldsymbol{k} \quad \Rightarrow \quad \nabla f(0,1,2) = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$$

Tangent Line: $f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$

$$2(x-0) + 2(y-1) + (z-2) = 0$$

$$\boxed{2x + 2y + z - 4 = 0}$$

Normal Line: $x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t$

$$\boxed{x = 2t, \quad y = 1 + 2t, \quad z = 2 + t}$$

Exercise

Find the tangent plane and normal line of the surface $x^2 - xy - y^2 - z = 0$ at the point $P_0(1,\ 1,\ -1)$

Solution

$$f(x,y,z) = x^2 - xy - y^2 - z$$

$$\rightarrow \quad f_x = 2x - y, \quad f_y = -x - 2y, \quad f_z = -1$$

$$\nabla f = (2x - y)\boldsymbol{i} - (x + 2y)\boldsymbol{j} - \boldsymbol{k} \quad \Rightarrow \quad \nabla f(1,1,-1) = \boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}$$

Tangent Line: $f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$

$$(x-1) - 3(y-1) - (z+1) = 0$$

$$\boxed{x - 3y - z + 1 = 0}$$

Normal Line: $x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t$

$$\boxed{x = 1 + t, \quad y = 1 - 3t, \quad z = -1 - t}$$

Exercise

Find the tangent plane and normal line of the surface $x^2+y^2-2xy-x+3y-z=-4$ at the point $P_0(2,\ -3,\ 18)$

Solution

$$f(x,y,z)=x^2+y^2-2xy-x+3y-z$$

$$\rightarrow\quad f_x=2x-2y-1,\quad f_y=2y-2x+3,\quad f_z=-1$$

$$\nabla f=(2x-2y-1)\boldsymbol{i}-(2y-2x+3)\boldsymbol{j}-\boldsymbol{k}\quad\Rightarrow\quad \nabla f(2,-3,18)=9\boldsymbol{i}-7\boldsymbol{j}-\boldsymbol{k}$$

Tangent Line: $f_x(P_0)(x-x_0)+f_y(P_0)(y-y_0)+f_z(P_0)(z-z_0)=0$

$$9(x-2)-7(y+3)-(z-18)=0$$

$$\boxed{9x-7y-z=21}$$

Normal Line: $x=x_0+f_x(P_0)t,\quad y=y_0+f_y(P_0)t,\quad z=z_0+f_z(P_0)t$

$$\boxed{x=2+9t,\quad y=-3-7t,\quad z=18-t}$$

Exercise

Find an equation for the plane that is tangent to the surface $z=\ln\left(x^2+y^2\right)$ at the point $(1,\ 0,\ 0)$

Solution

$$z=f(x,y)=\ln\left(x^2+y^2\right)$$

$$f_x=\frac{2x}{x^2+y^2}\quad\rightarrow f_x(1,0)=2$$

$$f_y=\frac{2x}{x^2+y^2}\quad\rightarrow f_y(1,0)=0$$

Tangent Line: $2(x-1)-(y-0)-z=0$ $\qquad f_x(P_0)(x-x_0)+f_y(P_0)(y-y_0)-(z-z_0)=0$

$$\boxed{2x-z-2=0}$$

Exercise

Find an equation for the plane that is tangent to the surface $z=e^{-x^2-y^2}$ at the point $(0,\ 0,\ 1)$

Solution

$$z=f(x,y)=e^{-x^2-y^2}$$

$$f_x = -2xe^{-x^2-y^2} \quad \to f_x(0,0) = 0$$

$$f_y = -2ye^{-x^2-y^2} \quad \to f_y(0,0) = 0$$

Tangent Line: $-(z-1) = 0$ $\qquad f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) - (z-z_0) = 0$

$$\boxed{z = 1}$$

Exercise

Find an equation for the plane that is tangent to the surface $z = \sqrt{y-x}$ at the point $(1,\ 2,\ 1)$

Solution

$$z = f(x,y) = \sqrt{y-x}$$

$$f_x = -\frac{1}{2}(y-x)^{-1/2} \quad \to f_x(1,2) = -\frac{1}{2}$$

$$f_y = \frac{1}{2}(y-x)^{-1/2} \quad \to f_y(1,2) = \frac{1}{2}$$

Tangent Line: $-\frac{1}{2}(x-1) + \frac{1}{2}(y-2) - (z-1) = 0$

$$-\frac{1}{2}x + \frac{1}{2}y - z + \frac{1}{2} = 0$$

$$\boxed{x - y + 2z - 1 = 0}$$

Exercise

Find parametric equation for the line tangent to the curve of intersection of the surfaces $x + y^2 + 2z = 4, \quad x = 1$ at the point $(1,\ 1,\ 1)$

Solution

$$f_x = 1, \quad f_y = 2y, \quad f_z = 2$$

$$\nabla f = \boldsymbol{i} + 2y\boldsymbol{j} + 2\boldsymbol{k} \quad \Rightarrow \quad \nabla f(1,1,1) = \boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$$

$$\nabla g = \boldsymbol{i}$$

$$v = \nabla f \times \nabla g$$

$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 2 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= 2\boldsymbol{j} - 2\boldsymbol{k}$$

Tangent Line: $\boxed{x = 1, \quad y = 1 + 2t, \quad z = 1 - 2t}$

Exercise

Find parametric equation for the line tangent to the curve of intersection of the surfaces

$xyz = 1, \quad x^2 + 2y^2 + 3z^2 = 6$ at the point $(1,\ 1,\ 1)$

Solution

$$f_x = yz, \quad f_y = xz, \quad f_z = xy$$

$$\nabla f = yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k} \quad \Rightarrow \quad \nabla f(1,1,1) = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$$

$$g_x = 2x, \quad g_y = 4y, \quad g_z = 6z$$

$$\nabla g = 2x\boldsymbol{i} + 4y\boldsymbol{j} + 6y\boldsymbol{k} \quad \Rightarrow \quad \nabla g(1,1,1) = 2\boldsymbol{i} + 4\boldsymbol{j} + 6\boldsymbol{k}$$

$$v = \nabla f \times \nabla g$$

$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 1 \\ 2 & 4 & 6 \end{vmatrix}$$

$$= 2\boldsymbol{j} - 4\boldsymbol{j} + 2\boldsymbol{k}$$

Tangent Line: $\boxed{x = 1 + 2t, \quad y = 1 - 4t, \quad z = 1 + 2t}$

Exercise

Find parametric equation for the line tangent to the curve of intersection of the surfaces

$x^3 + 3x^2y^2 + y^3 + 4xy - z^2 = 0, \quad x^2 + y^2 + z^2 = 11$ at the point $(1,\ 1,\ 3)$

Solution

$$f_x = 3x^2 + 6xy^2 + 4y \quad \to f_x(1,1,3) = 13$$

$$f_y = 6x^2y + 3y^2 + 4x \quad \to f_y(1,1,3) = 13$$

$$f_z = -2z \quad \to f_z(1,1,3) = -6$$

$$\nabla f(1,1,3) = 13\boldsymbol{i} + 13\boldsymbol{j} - 6\boldsymbol{k}$$

$$g_x = 2x, \quad g_y = 2y, \quad g_z = 2z$$

$$\nabla g = 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k} \quad \Rightarrow \quad \nabla g(1,1,3) = 2\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}$$

$$v = \nabla f \times \nabla g$$

$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 13 & 13 & -6 \\ 2 & 2 & 6 \end{vmatrix}$$

$$= 90\boldsymbol{j} - 90\boldsymbol{j}$$

Tangent Line: $\boxed{x = 1 + 90t, \quad y = 1 - 90t, \quad z = 3}$

Exercise

Find parametric equation for the line tangent to the curve of intersection of the surfaces $x^2+y^2=4,\quad x^2+y^2-z=0$ at the point $\left(\sqrt{2},\ \sqrt{2},\ 4\right)$

Solution

$$f_x=2x,\quad f_y=2u,\quad f_z=0$$

$$\nabla f=2x\boldsymbol{i}+2y\boldsymbol{j}\quad\Rightarrow\quad\nabla f\left(\sqrt{2},\sqrt{2},4\right)=2\sqrt{2}\boldsymbol{i}+2\sqrt{2}\boldsymbol{j}$$

$$g_x=2x,\quad g_y=2y,\quad g_z=-1$$

$$\nabla g=2x\boldsymbol{i}+2y\boldsymbol{j}-\boldsymbol{k}\quad\Rightarrow\quad\nabla g\left(\sqrt{2},\sqrt{2},4\right)=2\sqrt{2}\boldsymbol{i}+2\sqrt{2}\boldsymbol{j}-\boldsymbol{k}$$

$$v=\nabla f\times\nabla g$$

$$=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\2\sqrt{2}&2\sqrt{2}&0\\2\sqrt{2}&2\sqrt{2}&-1\end{vmatrix}$$

$$=-2\sqrt{2}\boldsymbol{j}+2\sqrt{2}\boldsymbol{j}$$

Tangent Line: $\boxed{x=\sqrt{2}-2\sqrt{2}t,\quad y=\sqrt{2}+2\sqrt{2}t,\quad z=4}$

Exercise

By about how much will $f(x,y,z)=\ln\sqrt{x^2+y^2+z^2}$ change if the point $P(x,y,z)$ moves from $P_0(3,\ 4,\ 12)$ a distance of $ds=0.1$ unit in the direction of $3\boldsymbol{i}+6\boldsymbol{j}-2\boldsymbol{k}$?

Solution

$$f(x,y,z)=\ln\sqrt{x^2+y^2+z^2}=\frac{1}{2}\ln\left(x^2+y^2+z^2\right)\qquad\qquad(\ln u)'=\frac{u'}{u}$$

$$f_x=\frac{x}{x^2+y^2+z^2}\quad\Rightarrow\quad f_x(3,4,12)=\frac{3}{9+16+144}=\frac{3}{169}$$

$$f_y=\frac{y}{x^2+y^2+z^2}\quad\Rightarrow\quad f_y(3,4,12)=\frac{4}{9+16+144}=\frac{4}{169}\quad\to\quad\nabla f=\frac{3}{169}\boldsymbol{i}+\frac{4}{169}\boldsymbol{j}+\frac{12}{169}\boldsymbol{k}$$

$$f_z=\frac{z}{x^2+y^2+z^2}\quad\Rightarrow\quad f_z(3,4,12)=\frac{12}{9+16+144}=\frac{12}{169}$$

$$\boldsymbol{u}=\frac{v}{|v|}=\frac{3\boldsymbol{i}+6\boldsymbol{j}-2\boldsymbol{k}}{\sqrt{9+36+4}}=\frac{3}{7}\boldsymbol{i}+\frac{6}{7}\boldsymbol{j}-\frac{2}{7}\boldsymbol{k}$$

$$\nabla f\cdot\boldsymbol{u}=\left(\frac{3}{169}\boldsymbol{i}+\frac{4}{169}\boldsymbol{j}+\frac{12}{169}\boldsymbol{k}\right)\cdot\left(\frac{3}{7}\boldsymbol{i}+\frac{6}{7}\boldsymbol{j}-\frac{2}{7}\boldsymbol{k}\right)=\frac{9}{1183}$$

$$df=(\nabla f\cdot\boldsymbol{u})ds=\frac{9}{1183}(0.1)\approx\underline{0.0008}$$

Exercise

By about how much will $f(x,y,z)=e^x\cos yz$ change if the point $P(x,y,z)$ moves from origin a distance of $ds=0.1$ unit in the direction of $2\boldsymbol{i}+2\boldsymbol{j}-2\boldsymbol{k}$?

Solution

$$f_x=e^x\cos yz \quad \Rightarrow \quad f_x(0,0,0)=1$$

$$f_y=-ze^x\sin yz \quad \Rightarrow \quad f_y(0,0,0)=0 \quad \to \quad \nabla f=\boldsymbol{i}$$

$$f_z=-ze^x\sin yz \quad \Rightarrow \quad f_z(0,0,0)=0$$

$$\boldsymbol{u}=\frac{v}{|v|}=\frac{2\boldsymbol{i}+2\boldsymbol{j}-2\boldsymbol{k}}{\sqrt{4+4+4}}=\frac{2}{2\sqrt{3}}\boldsymbol{i}+\frac{2}{2\sqrt{3}}\boldsymbol{j}-\frac{2}{2\sqrt{3}}\boldsymbol{k}=\frac{1}{\sqrt{3}}\boldsymbol{i}+\frac{1}{\sqrt{3}}\boldsymbol{j}-\frac{1}{\sqrt{3}}\boldsymbol{k}$$

$$\nabla f\cdot\boldsymbol{u}=(\boldsymbol{i})\cdot\left(\frac{1}{\sqrt{3}}\boldsymbol{i}+\frac{1}{\sqrt{3}}\boldsymbol{j}-\frac{1}{\sqrt{3}}\boldsymbol{k}\right)=\frac{1}{\sqrt{3}}$$

$$df=(\nabla f\cdot\boldsymbol{u})ds=\frac{1}{\sqrt{3}}(0.1)\approx \underline{0.0577}$$

Exercise

Find the linearization $L(x,y)$ of $f(x,y)=x^2+y^2+1$ at the point (0, 0) and (1, 1)

Solution

$$f(0,0)=1$$

$$f_x=2x \quad \Rightarrow \quad f_x(0,0)=0$$

$$f_y=2y \quad \Rightarrow \quad f_y(0,0)=0$$

$$L(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

$$L(x,y)=1+0(x-0)+0(y-0)\ \underline{=1}$$

$$f(1,1)=3$$

$$f_x=2x \quad \Rightarrow \quad f_x(1,1)=2$$

$$f_y=2y \quad \Rightarrow \quad f_y(1,1)=2$$

$$L(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

$$L(x,y)=3+2(x-1)+2(y-1)$$

$$\underline{=2x+2y-1}$$

Exercise

Find the linearization $L(x,y)$ of $f(x,y)=(x+y+2)^2$ at the point (0, 0) and (1, 2)

Solution

$f(0,0)=4$

$f_x=2(x+y+2) \Rightarrow f_x(0,0)=4$

$f_y=2(x+y+2) \Rightarrow f_y(0,0)=4$

$L(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$

$L(x,y)=4+4(x-0)+4(y-0)$

$\quad =4+4x+4y$

$f(1,2)=(1+2+2)^2=25$

$f_x=2(x+y+2) \Rightarrow f_x(1,2)=10$

$f_y=2(x+y+2) \Rightarrow f_y(1,2)=10$

$L(x,y)=25+10(x-1)+10(y-2)$

$\quad =10x+10y-5$

Exercise

Find the linearization $L(x,y)$ of $f(x,y)=x^3y^4$ at the point (1, 1) and (0, 0)

Solution

$f(1,1)=1$

$f_x=3x^2y^4 \Rightarrow f_x(1,1)=3$

$f_y=4x^2y^3 \Rightarrow f_y(1,1)=4$

$L(x,y)=1+3(x-1)+4(y-1)$

$\quad =3x+4y-6$

$f(0,0)=0$

$f_x=3x^2y^4 \Rightarrow f_x(0,0)=0$

$f_y=4x^2y^3 \Rightarrow f_y(0,0)=0$

$L(x,y)=0+0(x-0)+0(y-0)\ =0$

Exercise

Find the linearization $L(x,y)$ of $f(x,y)=e^{2y-x}$ at the point (0, 0) and (1, 2)

Solution

$$f(0,0)=e^0=1$$

$$f_x=-e^{2y-x} \quad \Rightarrow \quad f_x(0,0)=-1$$

$$f_y=2e^{2y-x} \quad \Rightarrow \quad f_y(0,0)=2$$

$$L(x,y)=f\left(x_0,y_0\right)+f_x\left(x_0,y_0\right)\left(x-x_0\right)+f_y\left(x_0,y_0\right)\left(y-y_0\right)$$

$$L(x,y)=1-1(x-0)+2(y-0)$$

$$\underline{=1-x+2y}$$

$$f(1,2)=e^3$$

$$f_x=-e^{2y-x} \quad \Rightarrow \quad f_x(1,2)=-e^3$$

$$f_y=2e^{2y-x} \quad \Rightarrow \quad f_y(0,0)=2e^3$$

$$L(x,y)=e^3-e^3(x-1)+2e^3(y-2)$$

$$\underline{=-e^3x+2e^3y-2e^3}$$

Exercise

Find the linearization $L(x,y,z)$ of $f(x,y,z)=x^2+y^2+z^2$ at the point (1, 1, 1)

Solution

$$f(1,1,1)=3$$

$$f_x=2x \quad \Rightarrow \quad f_x(1,1,1)=2$$

$$f_y=2y \quad \Rightarrow \quad f_y(1,1,1)=2$$

$$f_z=2z \quad \Rightarrow \quad f_z(1,1,1)=2$$

$$L(x,y)=f\left(x_0,y_0\right)+f_x\left(x_0,y_0\right)\left(x-x_0\right)+f_y\left(x_0,y_0\right)\left(y-y_0\right)$$

$$L(x,y)=3+2(x-1)+2(y-1)+2(z-1)$$

$$\underline{=2x+2y+2z-3}$$

Exercise

Find the linearization $L(x, y, z)$ of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point (1, 2, 2)

Solution

$$f(1,1,1) = \sqrt{1+4+4} = 3$$

$$f_x = \frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \quad \Rightarrow \quad f_x(1,2,2) = \frac{1}{3}$$

$$f_y = \frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-1/2}(2y) = \frac{y}{\sqrt{x^2 + y^2 + z^2}} \quad \Rightarrow \quad f_y(1,2,2) = \frac{2}{3}$$

$$f_z = \frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-1/2}(2z) = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \quad \Rightarrow \quad f_z(1,2,2) = \frac{2}{3}$$

$$L(x,y) = f\left(x_0, y_0\right) + f_x\left(x_0, y_0\right)\left(x - x_0\right) + f_y\left(x_0, y_0\right)\left(y - y_0\right)$$

$$L(x,y) = 3 + \frac{1}{3}(x-1) + \frac{2}{3}(y-2) + \frac{2}{3}(z-2)$$

$$= \frac{1}{3}x + \frac{2}{3}y + \frac{2}{3}z$$

Exercise

Find the linearization $L(x, y, z)$ of $f(x, y, z) = \dfrac{\sin xy}{z}$ at the point $\left(\frac{\pi}{2}, 1, 1\right)$

Solution

$$f\left(\frac{\pi}{2}, 1, 1\right) = \frac{\sin\frac{\pi}{2}}{1} = 1$$

$$f_x = \frac{y\cos xy}{z} \quad \Rightarrow \quad f_x\left(\frac{\pi}{2}, 1, 1\right) = 0$$

$$f_y = \frac{x\cos xy}{z} \quad \Rightarrow \quad f_y\left(\frac{\pi}{2}, 1, 1\right) = 0$$

$$f_z = -\frac{\sin xy}{z^2} \quad \Rightarrow \quad f_z\left(\frac{\pi}{2}, 1, 1\right) = -1$$

$$L(x,y) = f\left(x_0, y_0\right) + f_x\left(x_0, y_0\right)\left(x - x_0\right) + f_y\left(x_0, y_0\right)\left(y - y_0\right)$$

$$L(x,y) = 1 + 0\left(x - \frac{\pi}{2}\right) + 0(y-1) - 1(z-1)$$

$$= 2 - z$$

Exercise

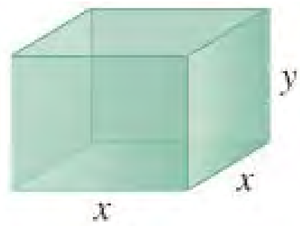
Find the linearization $L(x,y,z)$ of $f(x,y,z)=e^x+\cos(y+z)$ at the point $\left(0,\ \frac{\pi}{4},\ \frac{\pi}{4}\right)$

Consider a closed rectangular box with a square base. If x is measured with error at most 2% and y is measured with error at most 3% use a differential to estimate the corresponding percentage error in computing the box's

a) Surface area

b) Volume

Solution



Given: $\frac{dx}{x}\le 0.02,\quad \frac{dy}{y}\le 0.03$

a) $S=2(xx+xy+xy)=2x^2+4xy$

$$\begin{aligned}
dS &= (4x+4y)dx+4xdy \\
&= (4x+4y)\left(x\frac{dx}{x}\right)+4xy\frac{dy}{y} \\
&= \left(4x^2+4xy\right)\frac{dx}{x}+4xy\frac{dy}{y} \\
&\le \left(4x^2+4xy\right)(0.02)+4xy(0.03) \\
&= 0.02\left(4x^2\right)+0.02(4xy)+0.03(4xy) \\
&= 0.04\left(2x^2\right)+0.05(4xy) \\
&\le 0.05\left(2x^2\right)+0.05(4xy) \\
&= 0.05\left(2x^2+4xy\right) \\
&= 0.05\ S
\end{aligned}$$

b) $V=x^2y$

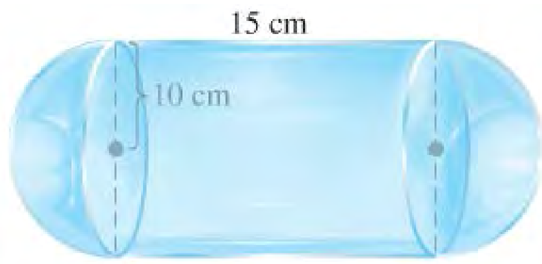
$$\begin{aligned}
dV &= 2xydx+x^2dy \\
&= 2x^2y\frac{dx}{x}+x^2y\frac{dy}{y} \\
&\le 2x^2y(0.02)+x^2y(.03) \\
&= .07\left(x^2y\right) \\
&= .07\ V
\end{aligned}$$

Exercise

Consider a closed container in the shape of a cylinder of radius 10 *cm* and height 15 *cm* with a hemisphere on each end.

The container is coated with a layer of ice $\frac{1}{2}$ cm thick. Use a differential to estimate the total volume of ice. (*Hint*: assume *r* is radius with $dr = \frac{1}{2}$ and *h* is height with $dh = 0$)

Solution



$$V = \frac{4\pi}{3}r^3 + \pi r^2 h$$

$$\begin{aligned} dV &= 4\pi r^2 dr + 2\pi rh dr + \pi r^2 dh \\ &= \left(4\pi r^2 + 2\pi rh\right)dr + \pi r^2 dh \\ &= \left(4\pi(10)^2 + 2\pi(10)(15)\right)\left(\tfrac{1}{2}\right) + \pi(10)^2(0) \\ &= 350\pi\ cm^3 \end{aligned}$$

Exercise

A standard 12-fl-oz can of soda is essentially a cylinder of radius $r = 1\ in$ and height $h = 5\ in.$

a) At these dimensions, how sensitive is the can's volume to a small change in radius versus a small change in height?

b) Could you design a soda can that appears to hold more soda but in fact holds the same 12-fl-oz? What might its dimensions be? (There is more than one correct answer.)

Solution

Given: $r = 1\ in \quad h = 5\ in.$

a) $V = \pi r^2 h \quad \Rightarrow \quad dV = 2\pi rh dr + \pi r^2 dh$

$$\begin{aligned} dV &= 10\pi dr + \pi dh \\ &= \pi(10dr + dh) \end{aligned}$$

The volume is about 10 times more sensitive to a change in *r*.

b) $dV = 0 \quad \Rightarrow \quad 2\pi rh dr + \pi r^2 dh = 0$

$$2h dr + r dh = 0$$

$$10dr + dh = 0 \quad \Rightarrow \quad dr = -\frac{1}{10}dh$$

Assume $dh = 1.5$, then $dr = -.15$

$$2h(-0.15) + r(1.5) = 0$$

$r = 0.85\ in \quad h = 6.5\ in.$ is one solution for $\Delta V \approx dV = 0$

Solution   *Section* 2.7 – Maximum/Minimum Problems

Exercise

Find all the local maxima, local minima, and saddle points of the function

$f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$

Solution

$$f_x = 2x + y + 3 = 0 \qquad f_y = x + 2y - 3 = 0$$

$$\begin{cases} 2x + y = -3 \\ x + 2y = 3 \end{cases} \rightarrow \quad x = -3 \quad y = 3$$ Therefore, the critical point is $(-3,\ 3)$

$$f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = 1$$

$$f_{xx} f_{yy} - f_{xy}^2 = (2)(2) - 1^2 = 3 > 0 \quad \textit{and} \quad f_{xx} = 2 > 0$$

The function *f* has a local minimum at $(-3,\ 3)$ and the value is

$$f(-3,\ 3) = (-3)^2 + (-3)(3) + 3^2 + 3(-3) - 3(3) + 4 \ \underline{= -5}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$f(x,y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$

Solution

$$f_x = 2y - 10x + 4 = 0 \qquad f_y = 2x - 4y + 4 = 0$$

$$\begin{cases} -5x + y = -2 \\ x - 2y = -2 \end{cases} \rightarrow \quad x = \frac{2}{3} \quad y = \frac{4}{3}$$ Therefore, the critical point is $\left(\frac{2}{3},\ \frac{4}{3}\right)$

$$f_{xx}\Big|_{\left(\frac{2}{3},\ \frac{4}{3}\right)} = -10 \quad f_{yy}\Big|_{\left(\frac{2}{3},\ \frac{4}{3}\right)} = -4 \quad f_{xy}\Big|_{\left(\frac{2}{3},\ \frac{4}{3}\right)} = 2$$

$$f_{xx} f_{yy} - f_{xy}^2 = (-10)(-4) - 2^2 = 36 > 0 \quad \textit{and} \quad f_{xx} = -10 < 0$$

The function f has a local maximum at $\left(\frac{2}{3},\ \frac{4}{3}\right)$ and the value is

$$f\left(\frac{2}{3}, \frac{4}{3}\right) = 2\left(\frac{2}{3}\right)\left(\frac{4}{3}\right) - 5\left(\frac{2}{3}\right)^2 - 2\left(\frac{4}{3}\right)^2 + 4\left(\frac{2}{3}\right) + 4\left(\frac{4}{3}\right) - 4 \ \underline{= 0}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$f(x,y) = x^2 - 4xy + y^2 + 6y + 2$

Solution

$$f_x = 2x - 4y = 0 \qquad f_y = -4x + 2y + 6 = 0$$

$$\begin{cases} x - 2y = 0 \\ -2x + y = -3 \end{cases} \rightarrow \ x = 2 \quad y = 1 \quad \text{Therefore, the critical point is } (2,\ 1)$$

$$f_{xx}\Big|_{(2,1)} = 2, \quad f_{yy}\Big|_{(2,1)} = 2, \quad f_{xy}\Big|_{(2,1)} = -4$$

$$f_{xx}f_{yy} - f_{xy}^2 = (2)(2) - (-4)^2 = -12 < 0 \quad \Rightarrow \quad \textit{Saddle point}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$f(x,y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$

Solution

$$f_x = 4x + 3y - 5 = 0 \qquad f_y = 3x + 8y + 2 = 0$$

$$\begin{cases} 4x + 3y = 5 \\ 3x + 8y = -2 \end{cases} \rightarrow \ x = 2 \quad y = -1 \quad \text{Therefore, the critical point is } (2,\ -1)$$

$$f_{xx}\Big|_{(2,-1)} = 4, \quad f_{yy}\Big|_{(2,-1)} = 8, \quad f_{xy}\Big|_{(2,-1)} = 3$$

$$f_{xx}f_{yy} - f_{xy}^2 = (4)(8) - 3^2 = 23 > 0 \quad \textit{and} \quad f_{xx} = 4 > 0$$

The function f has a local minimum at $(2,\ -1)$ and the value is

$$f(2,-1) = 2(2)^2 + 3(2)(-1) + 4(-1)^2 - 5(2) + 2(-1) \ \underline{= -6}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$f(x,y) = x^2 - y^2 - 2x + 4y + 6$

Solution

$$f_x = 2x - 2 = 0 \qquad f_y = -2y + 4 = 0$$

$$\begin{cases} 2x = 2 \\ 2y = 4 \end{cases} \rightarrow \quad x = 1 \quad y = 2$$

Therefore, the critical point is $(1,\ 2)$

$$f_{xx}\Big|_{(1,2)} = 2, \quad f_{yy}\Big|_{(1,2)} = -2, \quad f_{xy}\Big|_{(1,2)} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = (2)(-2) - 0^2 = -4 < 0 \quad \rightarrow \quad \textit{Saddle Point}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$$f(x, y) = \sqrt{56x^2 - 8y^2 - 16x - 31} + 1 - 8x$$

Solution

$$f_x = \frac{1}{2}\frac{112x - 16}{\sqrt{56x^2 - 8y^2 - 16x - 31}} - 8 = 0 \qquad f_y = \frac{1}{2}\frac{-16y}{\sqrt{56x^2 - 8y^2 - 16x - 31}} = 0$$

$$\begin{cases} 56x - 8 = 8\sqrt{56x^2 - 8y^2 - 16x - 31} \\ -8y = 0 \end{cases} \rightarrow \quad \begin{array}{l} x = \frac{16}{7} \quad \cancel{x = -2} \\ y = 0 \end{array}$$

Therefore, the critical point is $\left(\frac{16}{7},\ 0\right)$

$$f_{xx}\Big|_{\left(\frac{16}{7},\ 0\right)} = \frac{56\sqrt{56x^2 - 8y^2 - 16x - 31} - (56x - 8)(56x - 8)\left(56x^2 - 8y^2 - 16x - 31\right)^{-1/2}}{56x^2 - 8y^2 - 16x - 31} = -\frac{8}{15}$$

$$f_{yy}\Big|_{\left(\frac{16}{7},\ 0\right)} = \frac{-8\sqrt{56x^2 - 8y^2 - 16x - 31} - (-8y)\left(56x^2 - 8y^2 - 16x - 31\right)^{-1/2}(-8y)}{56x^2 - 8y^2 - 16x - 31} = -\frac{8}{15}$$

$$f_{xy}\Big|_{\left(\frac{16}{7},\ 0\right)} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = \left(-\frac{8}{15}\right)\left(-\frac{8}{15}\right) - 0 = \frac{34}{225} > 0 \quad \textit{and} \quad f_{xx} = -\frac{8}{15} < 0$$

The function f has a local maximum at $\left(\frac{16}{7},\ 0\right)$ and the value is

$$f\left(\frac{16}{7}, 0\right) = \sqrt{56\left(\frac{16}{7}\right)^2 - 8(0)^2 - 16\left(\frac{16}{7}\right) - 31} + 1 - 8\left(\frac{16}{7}\right) = -\frac{16}{7}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y)=1-\sqrt[3]{x^2+y^2}$

Solution

$$f_x=-\frac{1}{3}2x\left(x^2+y^2\right)^{-2/3}=\frac{-2x}{3\left(x^2+y^2\right)^{2/3}}=0$$

$$f_y=-\frac{1}{3}2y\left(x^2+y^2\right)^{-2/3}=\frac{-2y}{3\left(x^2+y^2\right)^{2/3}}=0$$

There are no solutions to the system $f_x(x,y)=0$ *and* $f_y(x,y)=0$, however, this occurs when $x=0 \quad y=0$. The critical point is $(0,\ 0)$

We cannot use the second derivative test, but this is the only possible local maximum, local minimum, or saddle point. $f(x,y)$ has a local maximum of $f(0,0)=1$ since

$$f(x,y)=1-\sqrt[3]{x^2+y^2}\le 1 \quad \forall(x,y)-\{(0,0)\}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y)=x^3+y^3+3x^2-3y^2-8$

Solution

$$f_x=3x^2+6x=0 \qquad f_y=3y^2-6y=0$$

$$\begin{cases}3x(x+2)=0\\ 3y(y-2)=0\end{cases} \rightarrow \quad \begin{array}{l}x=0,\ -2\\ y=0,\ 2\end{array}$$

Therefore, the critical point is $(0,0)$, $(0,2)$, $(-2,0)$, and $(-2,2)$

$$f_{xx}=6x+6, \quad f_{yy}=6y-6, \quad f_{xy}=0$$

For $(0,0)$ $\quad f_{xx}\big|_{(0,0)}=6, \quad f_{yy}\big|_{(0,0)}=-6, \quad f_{xy}\big|_{(0,0)}=0$

$$f_{xx}f_{yy}-f_{xy}^2=(6)(-6)-0^2=-36<0 \quad\Rightarrow\quad \textit{Saddle Point}$$

For $(0,2)$ $\quad f_{xx}\big|_{(0,2)}=6, \quad f_{yy}\big|_{(0,2)}=6, \quad f_{xy}\big|_{(0,2)}=0$

$$f_{xx}f_{yy}-f_{xy}^2=(6)(6)-0^2=36>0 \quad \textit{and} \quad f_{xx}>0$$

The function f has a local minimum at $(0,2)$ and the value is $f(0,2) = -12$

For $(-2,0)$ $\quad f_{xx}\big|_{(-2,0)} = -6, \quad f_{yy}\big|_{(-2,0)} = -6, \quad f_{xy}\big|_{(-2,0)} = 0$

$$f_{xx} f_{yy} - f_{xy}^2 = (-6)(-6) - 0^2 = 36 > 0 \quad and \quad f_{xx} < 0$$

The function f has a local maximum at $(-2,0)$ and the value is $f(-2,0) = -4$

For $(-2,2)$ $\quad f_{xx}\big|_{(-2,2)} = 6, \quad f_{yy}\big|_{(-2,2)} = 6, \quad f_{xy}\big|_{(-2,2)} = 0$

$$f_{xx} f_{yy} - f_{xy}^2 = (-6)(6) - 0^2 = -36 < 0 \quad \Rightarrow \quad \textit{Saddle Point}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y) = 4xy - x^4 - y^4$

Solution

$$f_x = 4y - 4x^3 = 0 \qquad f_y = 4x - 4y^3 = 0$$

$$\begin{cases} y - x^3 = 0 \\ x - y^3 = 0 \end{cases} \Rightarrow x = y \to x - x^3 = 0 \to x\left(1 - x^2\right) = 0 \quad \to x = 0, \ \pm 1$$

Therefore, the critical point is $(0,0)$, $(1,1)$, and $(-1,-1)$

$$f_{xx} = -12x^2, \quad f_{yy} = -12y^2, \quad f_{xy} = 4$$

For $(0,0)$ $\quad f_{xx}\big|_{(0,0)} = 0, \quad f_{yy}\big|_{(0,0)} = 0, \quad f_{xy}\big|_{(0,0)} = 4$

$$f_{xx} f_{yy} - f_{xy}^2 = 0 - 4^2 = -16 < 0 \quad \Rightarrow \quad \textit{Saddle Point}$$

For $(1,1)$ $\quad f_{xx}\big|_{(1,1)} = -12, \quad f_{yy}\big|_{(1,1)} = -12, \quad f_{xy}\big|_{(1,1)} = 4$

$$f_{xx} f_{yy} - f_{xy}^2 = (-12)(-12) - 4^2 = 128 > 0 \quad and \quad f_{xx} < 0$$

The function has a local maximum at $(1,1)$ and the value is $f(1,1) = 2$

For $(-1,-1)$ $\quad f_{xx}\big|_{(-1,-1)} = -12, \quad f_{yy}\big|_{(-1,-1)} = -12, \quad f_{xy}\big|_{(-1,-1)} = 0$

$$f_{xx} f_{yy} - f_{xy}^2 = (-12)(-12) - 4^2 = 128 > 0 \quad and \quad f_{xx} < 0 \ f$$

The function f has a local maximum at $(-1,-1)$ and the value is $f(-1,-1) = 2$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y)=\dfrac{1}{x^2+y^2-1}$

Solution

$$f_x=\frac{-2x}{\left(x^2+y^2-1\right)^2}=0 \qquad f_y=\frac{-2y}{\left(x^2+y^2-1\right)^2}=0$$

$\Rightarrow x=y=0$ Therefore, the critical point is $(0,0)$

$$f_{xx}=\frac{-2\left(x^2+y^2-1\right)^2-(-2x)(4x)\left(x^2+y^2-1\right)}{\left(x^2+y^2-1\right)^4}=\frac{-2x^2-2y^2+2+8x^2}{\left(x^2+y^2-1\right)^3}=\frac{6x^2-2y^2+2}{\left(x^2+y^2-1\right)^3}$$

$$f_{yy}=\frac{-2\left(x^2+y^2-1\right)^2-(-2y)(4y)\left(x^2+y^2-1\right)}{\left(x^2+y^2-1\right)^4}=\frac{-2x^2-2y^2+2+8y^2}{\left(x^2+y^2-1\right)^3}=\frac{-2x^2+6y^2+2}{\left(x^2+y^2-1\right)^3}$$

$$f_{xy}=\frac{-2x(4y)\left(x^2+y^2-1\right)}{\left(x^2+y^2-1\right)^4}=\frac{-8xy}{\left(x^2+y^2-1\right)^3}$$

$$f_{xx}\Big|_{(0,0)}=-2, \quad f_{yy}\Big|_{(0,0)}=-2, \quad f_{xy}\Big|_{(0,0)}=0$$

$$f_{xx}f_{yy}-f_{xy}^2=(-2)(-2)-0^2=4>0 \quad \text{and} \quad f_{xx}<0$$

The function f has a local maximum at $(0,0)$ and the value is $f(0,0)=-1$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y)=\dfrac{1}{x}+xy+\dfrac{1}{y}$

Solution

$$f_x=-\frac{1}{x^2}+y=0 \qquad f_y=x-\frac{1}{y^2}=0$$

$$\to\begin{cases} y=\dfrac{1}{x^2} & (x\neq 0)\\ x=\dfrac{1}{y^2} & (y\neq 0)\end{cases} \quad x=x^4 \quad \Rightarrow x=1=y$$

Therefore, the critical point is $(1,1)$

$$f_{xx}\Big|_{(1,1)}=\left(\frac{2}{x^3}\right)\Big|_{(1,1)}=2, \quad f_{yy}\Big|_{(1,1)}=\left(\frac{2}{y^3}\right)\Big|_{(1,1)}=-2, \quad f_{xy}\Big|_{(1,1)}=(1)\Big|_{(1,1)}=1$$

$$f_{xx}f_{yy}-f_{xy}^2=(2)(2)-1^2=3>0 \quad \text{and} \quad f_{xx}>0$$

The function f has a local minimum at $(1,1)$ and the value is $f(1,1)=3$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y) = y\sin x$

Solution

$$f_x = y\cos x = 0 \qquad f_y = \sin x = 0$$

$$\rightarrow \begin{cases} y\cos x = 0 \\ \sin x = 0 \end{cases} \quad x = n\pi \quad y = 0 \quad \text{Therefore, the critical point is } (n\pi, 0)$$

$$f_{xx}\Big|_{(n\pi,0)} = -y\sin x\Big|_{(n\pi,0)} = 0, \quad f_{yy}\Big|_{(n\pi,0)} = 0, \quad f_{xy}\Big|_{(n\pi,0)} = \cos x\Big|_{(n\pi,0)} = \pm 1$$

If n is even: $\quad f_{xx}f_{yy} - f_{xy}^2 = 0 - 1^2 = -1 < 0 \quad \Rightarrow \quad$ *Saddle Point*

If n is odd: $\quad f_{xx}f_{yy} - f_{xy}^2 = 0 - (-1)^2 = -1 < 0 \quad \Rightarrow \quad$ *Saddle Point*

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y) = e^{2x}\cos y$

Solution

$$f_x = 2e^{2x}\cos y = 0 \qquad f_y = -e^{2x}\sin y = 0$$

Since $e^{2x} \neq 0 \quad \forall x$, the functions $\cos y$ and $\sin y$ cannot equal to zero for the same y.

$\therefore$ No critical points $\Rightarrow$ no extrema and no saddle points.

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y) = e^y - ye^x$

Solution

$$f_x = -ye^x = 0 \qquad f_y = e^y - e^x = 0$$

$$\rightarrow \begin{cases} -ye^x = 0 \\ e^y - e^x = 0 \end{cases} \quad y = 0 \quad e^x = e^y = 1 = e^0 \Rightarrow x = 0 \qquad \therefore \text{ The critical point is } (0,0)$$

$$f_{xx}\Big|_{(0,0)} = -ye^x\Big|_{(0,0)} = 0, \quad f_{yy}\Big|_{(0,0)} = e^y = 1, \quad f_{xy}\Big|_{(0,0)} = -e^x\Big|_{(0,0)} = -1$$

$$f_{xx}f_{yy} - f_{xy}^2 = 0(1) - (-1)^2 = -1 < 0 \quad \Rightarrow \quad \textit{Saddle Point}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y)=e^{-y}\left(x^2+y^2\right)$

Solution

$$f_x=2xe^{-y}=0 \qquad f_y=-e^{-y}\left(x^2+y^2\right)+2ye^{-y}=e^{-y}\left(2y-x^2-y^2\right)=0$$

$$\to\begin{cases} 2xe^{-y}=0 & \to \boxed{x=0} \\ e^{-y}\left(2y-x^2-y^2\right)=0 \end{cases} \quad 2y-x^2-y^2=0\to y(2-y=0) \quad \boxed{y=0,\ 2}$$ $\therefore$ The critical point is $(0,0)$ and $(0,2)$

$$f_{xx}=2e^{-y}$$

$$f_{yy}=-e^{-y}\left(2y-x^2-y^2\right)+e^{-y}(2-2y)=e^{-y}\left(2-4y+x^2+y^2\right)$$

$$f_{xy}=-2xye^{-y}$$

For $(0,0)$ $\quad f_{xx}\big|_{(0,0)}=2,\quad f_{yy}\big|_{(0,0)}=2,\quad f_{xy}\big|_{(0,0)}=0$

$$f_{xx}f_{yy}-f_{xy}^2=(2)(2)-0^2=4>0 \quad \textit{and} \quad f_{xx}>0$$

The function f has a local minimum at $(0,0)$ and the value is $f(0,0)\ \underline{=0}$

For $(0,2)$ $\quad f_{xx}\big|_{(0,2)}=\frac{2}{e^2},\quad f_{yy}\big|_{(0,2)}=-\frac{2}{e^2},\quad f_{xy}\big|_{(0,2)}=0$

$$f_{xx}f_{yy}-f_{xy}^2=\frac{2}{e^2}\left(-\frac{2}{e^2}\right)-0^2=-\frac{4}{e^4}<0 \quad\Rightarrow\quad \textit{Saddle Point}$$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x,y)=2\ln x+\ln y-4x-y$

Solution

$$f_x=\frac{2}{x}-4=0 \qquad f_y=\frac{1}{y}-1=0$$

$$\to\begin{cases} 2=4x \\ 1=y \end{cases} \quad x=\frac{1}{2} \qquad \therefore \text{ The critical point is } \left(\tfrac{1}{2},1\right)$$

$$f_{xx}\Big|_{\left(\frac{1}{2},1\right)}=\left(-\frac{2}{x^2}\right)\Big|_{\left(\frac{1}{2},1\right)}=-8,\quad f_{yy}\Big|_{\left(\frac{1}{2},1\right)}=\left(-\frac{1}{y^2}\right)\Big|_{\left(\frac{1}{2},1\right)}=-1,\quad f_{xy}\Big|_{\left(\frac{1}{2},1\right)}=0$$

$$f_{xx}f_{yy}-f_{xy}^2=(-8)(-1)-0^2=8>0 \quad \textit{and} \quad f_{xx}<0$$

The function f has a local maximum at $\left(\frac{1}{2},1\right)$ and the value is $f\left(\frac{1}{2},1\right)\ \underline{=-3-2\ln 2}$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x,y)=\ln(x+y)+x^2-y$

Solution

$$f_x=\frac{1}{x+y}+2x=0 \qquad f_y=\frac{1}{x+y}-1=0$$

$$\rightarrow \begin{cases} \frac{1}{x+y}=-2x & \rightarrow -2x(x+y)=1 \\ \frac{1}{x+y}=1 & \rightarrow 1=x+y \end{cases} \Rightarrow -2x(1)=1 \rightarrow x=-\frac{1}{2} \quad y=\frac{3}{2}$$

$\therefore$ The critical point is $\left(-\frac{1}{2},\frac{3}{2}\right)$

$$f_{xx}=-\frac{1}{(x+y)^2}+2, \quad f_{yy}=-\frac{1}{(x+y)^2}, \quad f_{xy}=-\frac{1}{(x+y)^2}$$

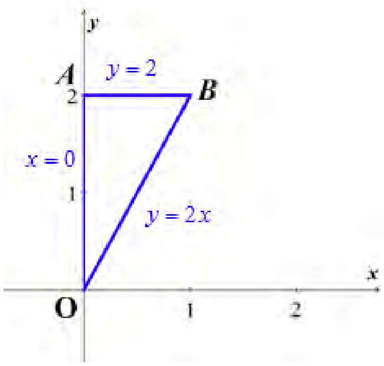
$$f_{xx}\Big|_{\left(-\frac{1}{2},\frac{3}{2}\right)}=1, \quad f_{yy}\Big|_{\left(-\frac{1}{2},\frac{3}{2}\right)}=-1, \quad f_{xy}\Big|_{\left(-\frac{1}{2},\frac{3}{2}\right)}=-1$$

$$f_{xx}f_{yy}-f_{xy}^2=(1)(-1)-(-1)^2=-2<0 \quad \textit{and} \quad \textit{Saddle Point}$$

Exercise

Find the absolute maxima and minima of the function $f(x,y)=2x^2-4x+y^2-4y+1$ on the closed triangular plate bounded by the lines $x=0, \quad y=2, \quad y=2x$ in the first quadrant.

Solution



$f_x=4x-4=0 \quad f_y=2y-4=0$

$x=1 \quad y=2$

The critical point is $(1,\ 2)$ and the value is $f(1,\ 2)\ \underline{=-5|}$

i. On the segment *OA*. The function $f(0,\ y)=y^2-4y+1$

This function is defined on the closed interval $0\le y\le 2$.

$$f'(0,\ y)=2y-4=0 \quad \rightarrow \quad y=2$$

$$\begin{cases} y=0 & \rightarrow f(0,\ 0)=\underline{1|} \\ y=2 & \rightarrow f(0,\ 2)=\underline{-3|} \end{cases}$$

ii. On the segment *OB*

$$f(x,\ 2x)=2x^2-4x+(2x)^2-4(2x)+1=6x^2-12x+1 \qquad 0\le x\le 1$$

$$f'(x,2x)=12x-12=0 \quad \rightarrow \quad x=1$$

$$\begin{cases} x=0 & \to f(0,\ 0)=\underline{1|} \\ x=1 & \to f(1,\ 2)=\underline{-5|} \end{cases} \qquad \therefore\ (1,\ 2) \text{ is not interior point of } OB$$

iii. On the segment AB

$$f(x,\ 2)=2x^2-4x+(2)^2-4(2)+1=2x^2-4x-3 \qquad 0\le x\le 1$$

$$f'(x,2)=4x-4=0 \quad \to \quad x=1$$

$$\begin{cases} x=0 & \to f(0,\ 2)=\underline{-3|} \\ x=1 & \to f(1,\ 2)=\underline{-5|} \end{cases}$$

$\Rightarrow (1,\ 2)$ is not interior point of triangular region.

Therefore; the absolute maximum is 1 at (0, 0) and the absolute minimum is –5 at $(1,\ 2)$

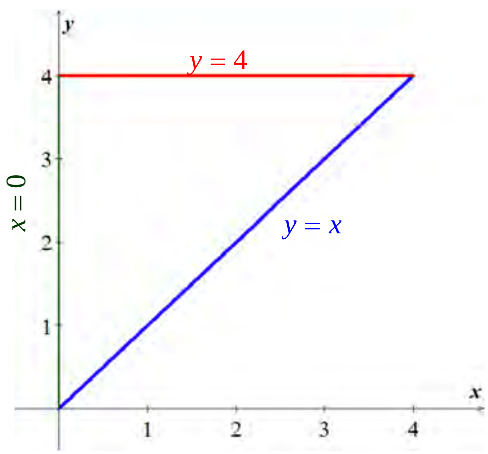
Exercise

Find the absolute maxima and minima of the function $D(x,y)=x^2-xy+y^2+1$ on the closed triangular plate bounded by the lines $x=0,\quad y=4,\quad y=x$ in the first quadrant.

Solution

$D_x=2x-y=0,\quad D_y=-x+2y=0,\quad \Rightarrow x=y=0$

The critical point is $(0,\ 0)$ and the value is $D(0,\ 0)\ \underline{=1|}$



i. On the segment OA.

$$D(0,\ y)=y^2+1,\quad 0\le y\le 4$$

$$D'(0,\ y)=2y=0 \quad \to \quad y=0$$

$$\begin{cases} y=0 & \to D(0,\ 0)=\underline{1|} \\ y=4 & \to D(0,\ 4)=\underline{17|} \end{cases}$$

ii. On the segment OB

$$D(x,\ x)=x^2+1 \qquad 0\le x\le 4$$

$$D'(x,x)=2x=0 \quad \to \quad x=0$$

$$x=0 \quad \to D(0,\ 0)=\underline{1|}$$

iii. On the segment AB

$$D(x,4)=x^2-4x+17 \qquad 0\le x\le 4$$

$$D'(x,2)=2x-4=0 \quad \to \quad x=2$$

$$\begin{cases} x=2 & \to D(2,\ 4)=13 \\ x=4 & \to D(4,\ 4)=\underline{17}| \end{cases}$$

$\Rightarrow$ $(0,0)$ is not interior point of triangular region.

Therefore; the absolute maximum is 11 at $(0,4)$ *and* $(4,4)$ and the absolute minimum is 1 at $(0,0)$

Exercise

Find the absolute maxima and minima of the function $T(x,y)=x^2+xy+y^2-6x+2$ on the triangular plate $0\le x\le 5,\quad -3\le y\le 0$.

Solution

$$T_x=2x+y-6=0,\quad T_y=x+2y=0$$

$$\begin{cases} 2x+y=6 \\ x+2y=0 \end{cases} \to \boxed{x=4,\ y=-2}$$

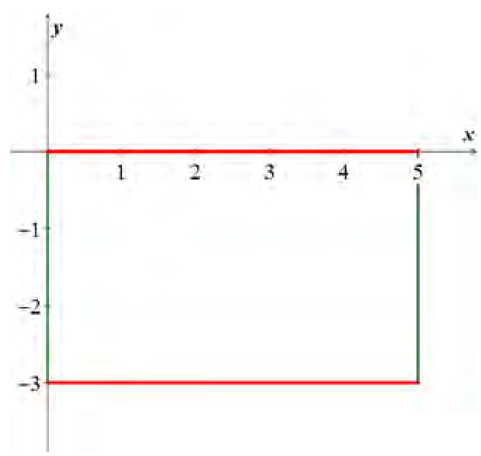
The critical point is $(4,-2)$ and the value is $T(4,-2)\ \underline{=-10}|$

i. On the segment *OA*.

$$T(0,\ y)=y^2+2,\quad -3\le y\le 0$$

$$T'(0,\ y)=2y=0 \quad\to\quad y=0$$

$$\begin{cases} y=0 & \to T(0,\ 0)=\underline{2}| \\ y=-3 & \to T(0,-3)=\underline{11}| \end{cases}$$



ii. On the segment *AB*

$$T(x,-3)=x^2-9x+11 \qquad 0\le x\le 5$$

$$T'(x,-3)=2x-9=0 \quad\to\quad x=\frac{9}{2}$$

$$\begin{cases} x=\frac{9}{2} & \to T\left(\frac{9}{2},-3\right)=\underline{-\frac{37}{4}}| \\ x=0 & \to T(0,-3)=\underline{11}| \end{cases}$$

iii. On the segment *BC*

$$T(5,y)=y^2+5y-3 \qquad -3\le y\le 0$$

$$T'(5,y)=2y+5=0 \quad\to\quad y=-\frac{5}{2}$$

$$\begin{cases} y=0 & \to T(5,0)=\underline{-3}| \\ y=-\frac{5}{2} & \to T\left(5,-\frac{5}{2}\right)=\underline{-\frac{37}{4}}| \\ y=-3 & \to T(5,-3)=\underline{-9}| \end{cases}$$

iv. On the segment CO

$$T(x,0) = x^2 - 6x + 2 \qquad 0 \le x \le 5$$

$$T'(x,0) = 2x - 6 = 0 \quad \to \quad x = 3$$

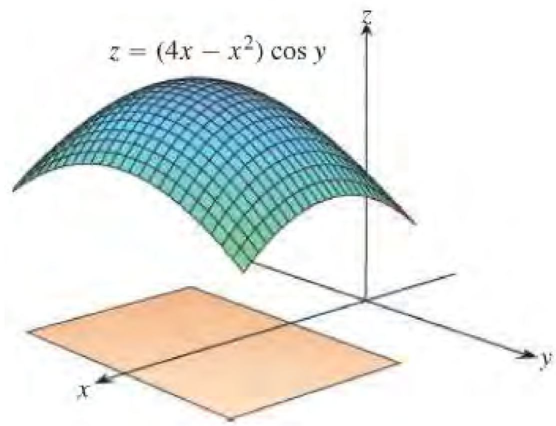
$$(3,0) \quad \to T(3,0) = \underline{-7}$$

Therefore; the absolute maximum is 11 at $(0,-3)$ and the absolute minimum is -10 at $(4,-2)$

Exercise

Find the absolute maxima and minima of the function $f(x,y) = \left(4x - x^2\right)\cos y$ on the triangular plate $1 \le x \le 3, \quad -\frac{\pi}{4} \le y \le \frac{\pi}{4}$.

Solution



$$f_x = (4-2x)\cos y = 0, \quad f_y = \left(x^2 - 4x\right)\sin y = 0$$

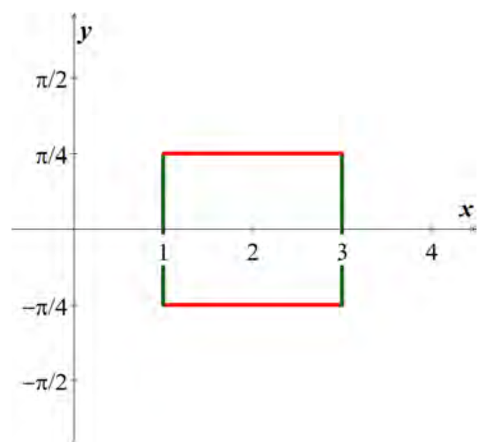
$$\begin{cases} (4-2x)\cos y = 0 & \to x = 2, \ y = \dfrac{(n+1)\pi}{2} \\ x(x-4)\sin y = 0 & \to x = 0, 4, \ y = n\pi \end{cases}$$

$\boxed{x = 2, \ y = 0}$ *because* $1 \le x \le 3, \quad -\frac{\pi}{4} \le y \le \frac{\pi}{4}$

The critical point is $(2,0)$ and the value is

$f(2,0) = \underline{4}$

Values of all 4 corner points:



$$A\left(1,-\frac{\pi}{4}\right) \quad \to f\left(1,-\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$$

$$B\left(1,\frac{\pi}{4}\right) \quad \to f\left(1,\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$$

$$C\left(3,\frac{\pi}{4}\right) \quad \to f\left(3,\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$$

$$A\left(3,-\frac{\pi}{4}\right) \quad \to f\left(3,-\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$$

i. On the segment AB

$$f(1,y) = 3\cos y \qquad -\frac{\pi}{4} \le y \le \frac{\pi}{4}$$

$$f'(1,y) = -3\sin y = 0 \quad \to \quad y = 0$$

$$x = 1 \quad \to f(1,0) = 3$$

ii. On the segment BC

$$f\left(x,\tfrac{\pi}{4}\right)=\frac{\sqrt{2}}{2}\left(4x-x^2\right) \qquad 1\le x\le 3$$

$$f'\left(x,\tfrac{\pi}{4}\right)=\frac{\sqrt{2}}{2}(4-2x)=0 \quad \Rightarrow \quad x=2$$

$$x=2 \quad \to f\left(2,\tfrac{\pi}{4}\right)=2\sqrt{2}$$

iii. On the segment CD

$$f(3,y)=3\cos y \qquad -\frac{\pi}{4}\le y\le\frac{\pi}{4}$$

$$f'(3,y)=-3\sin y=0 \quad \to \quad y=0$$

iv. On the segment DA

$$f\left(x,-\tfrac{\pi}{4}\right)=\frac{\sqrt{2}}{2}\left(4x-x^2\right) \qquad 1\le x\le 3$$

$$f'\left(x,-\tfrac{\pi}{4}\right)=\frac{\sqrt{2}}{2}(4-2x)=0 \quad \Rightarrow \quad x=2$$

Therefore; the absolute maximum is 4 at $(2,0)$ and the absolute minimum is $\frac{3\sqrt{2}}{2}$ at $\left(1,-\frac{\pi}{4}\right)$, $\left(1,\frac{\pi}{4}\right)$, $\left(3,-\frac{\pi}{4}\right)$, *and* $\left(3,\frac{\pi}{4}\right)$

Exercise

Find the point on the graph of $z=x^2+y^2+10$ nearest the plane $x+2y-z=0$

Solution

The point on $z=x^2+y^2+10$ where the tangent plane is parallel to the plane $x+2y-z=0$.

Let $w=z-x^2-y^2-10 \quad \to \quad \nabla w=-2x\boldsymbol{i}-2y\boldsymbol{j}+\boldsymbol{k}$ is normal to $z=x^2+y^2+10$ at (x,y).

The vector ∇w is parallel to $\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$ which is normal to the plane if $x=\frac{1}{2}$ *and* $y=1$

$(-2x=-1 \ \textit{and} \ -2y=-2)$, $z=\left(\frac{1}{2}\right)^2+1^2+10=\frac{45}{4}$

Thus the point $\left(\frac{1}{2},\ 1,\ \frac{45}{4}\right)$ is the point on the surface $z=x^2+y^2+10$ nearest the plane $x+2y-z=0$

Exercise

Find the minimum distance from the point $(2,\ -1,\ 1)$ to the plane $x+y-z=2$

Solution

$$d(x,y,z)=\sqrt{(x-2)^2+(y+1)^2+(z-1)^2}$$

$$x+y-z=2 \quad \Rightarrow \quad z=x+y-2$$

Let: $D(x,y,z)=(x-2)^2+(y+1)^2+(z-1)^2$

$$D(x,y)=(x-2)^2+(y+1)^2+(x+y-2-1)^2$$

$$=(x-2)^2+(y+1)^2+(x+y-3)^2$$

$$D_x=2(x-2)+2(x+y-3)$$
$$=4x+2y-10=0$$

$$D_y=2(y+1)+2(x+y-3)$$
$$=2x+4y-4=0$$

$$\begin{cases} 4x+2y=10 \\ 2x+4y=4 \end{cases} \quad \Rightarrow \quad \boxed{x=\frac{8}{3},\ y=-\frac{1}{3}}$$

$\therefore$ The critical point is $\left(\frac{8}{3},\ -\frac{1}{3}\right)$.

$$z=\frac{8}{3}-\frac{1}{3}-2=\frac{1}{3}$$

$$D_{xx}\Big|_{\left(\frac{8}{3},-\frac{1}{3}\right)}=4, \quad D_{yy}\Big|_{\left(\frac{8}{3},-\frac{1}{3}\right)}=4, \quad D_{xy}\Big|_{\left(\frac{8}{3},-\frac{1}{3}\right)}=2$$

$$D_{xx}D_{yy}-D_{xy}^2=(4)(4)-2^2=12>0 \quad \text{and} \quad D_{xx}>0$$

Therefore, the local minimum of the distance is

$$d\left(\frac{8}{3},-\frac{1}{3},\frac{1}{3}\right)=\sqrt{\left(\frac{8}{3}-2\right)^2+\left(-\frac{1}{3}+1\right)^2+\left(\frac{1}{3}-1\right)^2}=\frac{2}{\sqrt{3}}$$

Exercise

Find the maximum value of $s = xy + yz + xz$ where $x + y + z = 6$

Solution

$$x + y + z = 6 \quad \Rightarrow \quad z = 6 - x - y$$

$$s(x,y,z) = xy + yz + xz$$

$$s(x,y) = xy + y(6 - x - y) + x(6 - x - y)$$

$$= xy + 6y - xy - y^2 + 6x - x^2 - xy$$

$$= -x^2 - y^2 + 6y + 6x - xy$$

$$s_x = -2x + 6 - y = 0 \qquad s_y = -2y + 6 - x = 0$$

$$\begin{cases} 2x + y = 6 \\ x + 2y = 6 \end{cases} \Rightarrow \boxed{x = 2, \ \ y = 2}$$

$\therefore$ The critical point is $(2,\ 2)$.

$$\underline{z = 6 - 2 - 2 = 2}$$

$$s_{xx}\Big|_{(2,2)} = -2, \quad s_{yy}\Big|_{(2,2)} = -2, \quad s_{xy}\Big|_{(2,2)} = -1$$

$$s_{xx}s_{yy} - s_{xy}^2 = (-2)(-2) - (-1)^2 = 3 > 0 \quad \textit{and} \quad s_{xx} < 0$$

Therefore, the local maximum of the distance is

$$s(2,2,2) = (2)(2) + (2)(2) + (2)(2) \ \underline{= 12}$$

Solution   *Section* 2.8 – Lagrange Multipliers

Exercise

Find the points on the ellipse $x^2 + 2y^2 = 1$ where $f(x,y) = xy$ has its extreme values.

Solution

$$g(x,y) = x^2 + 2y^2 - 1$$

$$\nabla f = y\mathbf{i} + x\mathbf{j}, \quad \nabla g = 2x\mathbf{i} + 4y\mathbf{j}$$

$$\nabla f = \lambda \nabla g$$

$$y\mathbf{i} + x\mathbf{j} = 2\lambda x\mathbf{i} + 4\lambda y\mathbf{j}$$

$$y = 2\lambda x \qquad x = 4\lambda y$$

$$x = 8x\lambda^2$$

$$8x\lambda^2 - x = 0$$

$$x\left(8\lambda^2 - 1\right) = 0 \quad \Rightarrow \quad \begin{cases} \lambda^2 = \frac{1}{8} \to \lambda = \pm\frac{1}{2\sqrt{2}} \\ x = 0 \end{cases}$$

Case 1: If $x = 0 \Rightarrow y = 2\lambda x = 0$. But (0, 0) is not on the ellipse so $x \neq 0$

Case 2: If $x \neq 0$ *and* $\lambda = \pm\frac{\sqrt{2}}{4}$

$$\Rightarrow x = 4\lambda y = \pm\sqrt{2}y$$

$$\left(\pm\sqrt{2}y\right)^2 + 2y^2 = 1$$

$$2y^2 + 2y^2 = 1$$

$$y^2 = \frac{1}{4} \quad \Rightarrow \quad \boxed{y = \pm\frac{1}{2}}$$

$$x \pm \sqrt{2}y \quad \Rightarrow \quad \boxed{x = \pm\frac{\sqrt{2}}{2}}$$

$$f(x,y) = xy = \pm\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \pm\frac{\sqrt{2}}{4}$$

Therefore, f has extreme values at $\left(\pm\frac{\sqrt{2}}{2}, \ \pm\frac{1}{2}\right)$

The extreme values of f on the ellipse are $\pm\frac{\sqrt{2}}{4}$

Exercise

Find the extreme values of $f(x,y)=xy$ subject to the constraint $g(x,y)=x^2+y^2-10=0$.

Solution

$$g(x,y)=x^2+y^2-10$$

$$\nabla f=y\boldsymbol{i}+x\boldsymbol{j}, \quad \nabla g=2x\boldsymbol{i}+2y\boldsymbol{j}$$

$$\nabla f=\lambda\nabla g$$

$$y\boldsymbol{i}+x\boldsymbol{j}=2\lambda x\boldsymbol{i}+2\lambda y\boldsymbol{j}$$

$$y=2\lambda x \qquad x=2\lambda y=4x\lambda^2$$

$$x\left(4\lambda^2-1\right)=0 \quad \Rightarrow \quad \begin{cases}\lambda^2=\frac{1}{4}\to\lambda=\pm\frac{1}{2}\\ x=0\end{cases}$$

Case 1: If $x=0 \Rightarrow y=2\lambda x=0$. But (0, 0) is not on the circle so $x\neq 0$

Case 2: If $x\neq 0$ *and* $\lambda=\pm\frac{1}{2}$

$$\Rightarrow x=2\lambda y=\pm y$$

$$g(x,y)=x^2+y^2-10=0$$

$$(\pm y)^2+y^2=10$$

$$2y^2=10$$

$$y^2=5 \quad \Rightarrow \quad \boxed{y=\pm\sqrt{5}=\pm x}$$

$$f(x,y)=xy=\pm\sqrt{5}\left(\sqrt{5}\right)=\pm 5$$

Therefore, f has extreme values at $\left(\pm\sqrt{5},\ \pm\sqrt{5}\right)$

The extreme values of f on the circle are ± 5

Exercise

Find the maximum value of $f(x,y)=49-x^2-y^2$ on the line $x+3y=10$.

Solution

$$\nabla f=-2x\boldsymbol{i}-2y\boldsymbol{j}, \quad \nabla g=\boldsymbol{i}+3\boldsymbol{j}$$

$$\nabla f=\lambda\nabla g \quad \to \quad -2x\boldsymbol{i}-2y\boldsymbol{j}=\lambda\boldsymbol{i}+3\lambda\boldsymbol{j}$$

$$-2x=\lambda \qquad -2y=3\lambda$$

$$x=-\frac{\lambda}{2} \qquad y=-\frac{3\lambda}{2}$$

$$x + 3y = 10$$

$$-\frac{\lambda}{2} + 3\left(-\frac{3\lambda}{2}\right) = 10$$

$$-5\lambda = 10 \quad \Rightarrow \quad \boxed{\lambda = -2}$$

$$\underline{\left| x = -\frac{\lambda}{2} = 1 \right|} \quad and \quad \underline{\left| y = -\frac{3\lambda}{2} = 3 \right|}$$

$$f(x, y) = 49 - 1^2 - 3^2 = 39$$

Therefore, f has extreme values at $(1,\ 3)$.

The extreme values of f is 39

Exercise

Find the points on the curve $x^2 y = 2$ nearest the origin.

Solution

Let $f(x, y) = x^2 + y^2$, the square of the distance to the origin subject to the constraint

$$g(x, y) = x^2 y - 2 = 0$$

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j}, \quad \nabla g = 2xy\mathbf{i} + x^2\mathbf{j}$$

$$\nabla f = \lambda \nabla g \quad \rightarrow \quad 2x\mathbf{i} + 2y\mathbf{j} = 2xy\lambda\mathbf{i} + x^2\lambda\mathbf{j}$$

$$2x = 2xy\lambda \qquad 2y = x^2\lambda$$

$$y = \frac{1}{\lambda} \qquad x^2 = \frac{2y}{\lambda} = \frac{2}{\lambda^2}$$

$$x^2 y - 2 = 0$$

$$\left(\frac{2}{\lambda^2}\right)\left(\frac{1}{\lambda}\right) - 2 = 0$$

$$\frac{2}{\lambda^3} = 2 \quad \Rightarrow \quad \lambda^3 = 1 \rightarrow \boxed{\lambda = 1}$$

$$\boxed{y = 1} \quad x^2 = 2 \Rightarrow \boxed{x = \pm\sqrt{2}}$$

$\therefore\ \left(\pm\sqrt{2},\ 1\right)$ are the points on the curve $x^2 y = 2$ nearest the origin.

Exercise

Use the method of Lagrange multipliers to find

a) The minimum value of $x+y$, subject to the constraints $xy=16, \quad x>0, \quad y>0$

b) The maximum value of xy, subject to the constraints $x+y=16$

Solution

a) $\nabla f = \mathbf{i}+\mathbf{j}, \quad \nabla g = y\mathbf{i}+x\mathbf{j}$

$$\nabla f = \lambda \nabla g \quad \rightarrow \quad \mathbf{i}+\mathbf{j} = y\lambda \mathbf{i} + x\lambda \mathbf{j}$$

$$1 = y\lambda, \quad 1 = x\lambda$$

$$y = \frac{1}{\lambda}, \quad x = \frac{1}{\lambda}$$

$$g(x,y) = xy - 16 = 0$$

$$\frac{1}{\lambda^2} - 16 = 0 \quad \Rightarrow \quad \lambda^2 = \frac{1}{16} \rightarrow \boxed{\lambda = \pm\frac{1}{4}}$$

For $\lambda = -\frac{1}{4} \quad \rightarrow \quad \cancel{x = y = -4}$ *since* $x>0, \quad y>0$

For $\lambda = \frac{1}{4} \quad \rightarrow \quad \boxed{x = y = 4}$

The minimum value is $\underline{|f} = x+y = 4+4 \ \underline{=8|}$.

$xy=16, \quad x>0, \quad y>0$ is a branch of a hyperbola in the first quadrant with *x*- and *y*-axes as asymptotes.

The equations $x+y=c$ give a family of parallel lines with $m=-1$. Thus the minimum value of ***c*** occurs where $x+y=c$ is tangent to the hyperbola's branch.

b) $\nabla f = y\mathbf{i}+x\mathbf{j}, \quad \nabla g = \mathbf{i}+\mathbf{j}$

$$\nabla f = \lambda \nabla g \quad \rightarrow \quad y\mathbf{i}+x\mathbf{j} = \lambda \mathbf{i} + \lambda \mathbf{j}$$

$$y = \lambda, \quad x = \lambda$$

$$g(x,y) = x+y-16 = 0 \quad \rightarrow \quad 2\lambda = 16 \Rightarrow \boxed{\lambda = 8}$$

For $\lambda = 8 \quad \rightarrow \quad \boxed{x = y = 8}$

The maximum value is $\underline{|f} = xy = 8\times 8 \ \underline{=64|}$.

The equations $xy=c, \quad x>0, \ y>0 \quad \textit{or} \quad x<0, \ y<0$ give a family of hyperbolas in the first and third quadrants with *x*- and *y*-axes as asymptotes. Thus the maximum value of ***c*** occurs where $xy=c$ is tangent to the line $x+y=16$.

Exercise

Find the radius and height of the open right circular cylinder of largest surface area that can be inscribed in a sphere of radius a. What is the largest surface area?

Solution

For a cylinder of radius r and height h, to maximize the surface area $S = 2\pi rh$ subject to the constraint $g(r,h) = r^2 + \left(\frac{h}{2}\right)^2 - a^2 = 0$

$$\nabla S = 2\pi h\boldsymbol{i} + 2\pi r\boldsymbol{j} \quad and \quad \nabla g = 2r\boldsymbol{i} + \frac{h}{2}\boldsymbol{j}$$

$$\nabla S = \lambda \nabla g \quad \rightarrow \quad 2\pi h\boldsymbol{i} + 2\pi r\boldsymbol{j} = 2r\lambda\boldsymbol{i} + \frac{h}{2}\lambda\boldsymbol{j}$$

$$2\pi h = 2r\lambda, \quad 2\pi r = \frac{h}{2}\lambda$$

$$\lambda = \frac{\pi h}{r} \quad \rightarrow 2\pi r = \frac{h}{2}\frac{\pi h}{r}$$

$$4r^2 = h^2 \quad \Rightarrow \quad h = 2r$$

$$r^2 + \left(\frac{h}{2}\right)^2 = a^2$$

$$r^2 + r^2 = a^2$$

$$2r^2 = a^2 \quad \rightarrow \quad \boxed{r = \frac{a}{\sqrt{2}}} \quad h = \frac{2a}{\sqrt{2}} = a\sqrt{2}$$

$$S = 2\pi rh = 2\pi\frac{a}{\sqrt{2}}a\sqrt{2} = 2\pi a^2$$

Exercise

Use the method of Lagrange multipliers to find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ with sides parallel to the coordinate axes.

Solution

The area of a rectangle is $A(x,y) = (2x)(2y) = 4xy$ subject to the constraint

$$g(x,y) = \frac{x^2}{16} + \frac{y^2}{9} - 1 = 0.$$

$$\nabla A = 4y\boldsymbol{i} + 4x\boldsymbol{j} \quad and \quad \nabla g = \frac{1}{8}x\boldsymbol{i} + \frac{2}{9}y\boldsymbol{j}$$

$$\nabla A = \lambda \nabla g \quad \Rightarrow \quad 4y\boldsymbol{i} + 4x\boldsymbol{j} = \frac{1}{8}x\lambda\boldsymbol{i} + \frac{2}{9}y\lambda\boldsymbol{j}$$

$$4y = \frac{1}{8}x\lambda \quad and \quad 4x = \frac{2}{9}y\lambda$$

$$\lambda = \frac{32y}{x} \quad \Rightarrow \quad 4x = \frac{2y}{9}\frac{32y}{x} \rightarrow x^2 = \frac{64y^2}{36} \quad x = \pm\frac{4}{3}y$$

$$\frac{1}{16}\frac{16y^2}{9}+\frac{1}{9}y^2=1$$

$$\frac{2}{9}y^2=1 \quad \to \quad y^2=\frac{9}{2} \Rightarrow y=\pm\frac{3\sqrt{2}}{2}$$

Since x and y represents distance, then $y=\frac{3\sqrt{2}}{2} \quad \to \quad x=\frac{4}{3}\frac{3\sqrt{2}}{2}=2\sqrt{2}$

$\therefore$ The length is $2x=4\sqrt{2}$ and the width is $2y=3\sqrt{2}$

Exercise

Find the maximum and minimum values of x^2+y^2 subject to the constraint $x^2-2x+y^2-4y=0$

Solution

$$\nabla f=2x\boldsymbol{i}+2y\boldsymbol{j} \quad \textit{and} \quad \nabla g=(2x-2)\boldsymbol{i}+(2y-4)\boldsymbol{j}$$

$$\nabla f=\lambda\nabla g \quad \Rightarrow \quad 2x\boldsymbol{i}+2y\boldsymbol{j}=(2x-2)\lambda\boldsymbol{i}+(2y-4)\lambda\boldsymbol{j}$$

$$\begin{array}{ll} 2x=2(x-1)\lambda \quad \textit{and} & 2y=2(y-2)\lambda \\ x=x\lambda-\lambda & y=y\lambda-2\lambda \\ x(\lambda-1)=\lambda & y(\lambda-1)=2\lambda \\ x=\dfrac{\lambda}{\lambda-1} & y=\dfrac{2\lambda}{\lambda-1}=2x \quad (\lambda\neq 1) \end{array}$$

$$x^2-2x+y^2-4y=0$$
$$x^2-2x+4x^2-8x=0$$
$$5x^2-10x=0$$
$$5x(x-2)=0 \quad \Rightarrow \quad x=0, \ 2$$

$$x=0 \quad y=2x=0 \to (0,\ 0)$$
$$x=2 \quad y=2x=4 \to (2,\ 4)$$

$\therefore$ $f(0,0)=0$ is the minimum value, and $f(2,4)=20$ is the maximum value.

Exercise

The temperature at a point (x, y) on a metal plate is $T(x, y) = 4x^2 - 4xy + y^2$. An ant on the plate walks around the circle of radius 5 centered at the origin. What are the highest and lowest temperatures encountered by the ant?

Solution

$$g(x,y) = x^2 + y^2 - 25 = 0$$

$$\nabla T = (8x - 4y)\boldsymbol{i} + (-4x + 2y)\boldsymbol{j} \quad \textit{and} \quad \nabla g = 2x\boldsymbol{i} + 2y\boldsymbol{j}$$

$$\nabla T = \lambda \nabla g \quad \to \quad (8x - 4y)\boldsymbol{i} + (-4x + 2y)\boldsymbol{j} = 2x\lambda\boldsymbol{i} + 2y\lambda\boldsymbol{j}$$

$$8x - 4y = 2x\lambda, \quad -4x + 2y = 2y\lambda$$

$$4x - 2y = x\lambda, \quad y - y\lambda = 2x \to y = \frac{2x}{1-\lambda} \quad (\lambda \neq 1)$$

$$4x - 2\frac{2x}{1-\lambda} = x\lambda$$

$$4x - \frac{4x}{1-\lambda} - x\lambda = 0$$

$$x\left(4 - 4\lambda - 4 - \lambda + \lambda^2\right) = 0$$

$$x\left(\lambda^2 - 5\lambda\right) = 0 \quad \Rightarrow \quad \boxed{x = 0}, \quad \boxed{\lambda = 0,\ 5}$$

Case 1: $x = 0 \quad \boxed{y = \frac{2x}{1-\lambda} = 0}$, but (0, 0) is not on the circle $x^2 + y^2 = 25$

Case 2: $\lambda = 0 \quad y = 2x$

$$\Rightarrow x^2 + (2x)^2 = 25 \quad \to 5x^2 = 25 \quad \boxed{x = \pm\sqrt{5}} \quad \left(\sqrt{5}, 2\sqrt{5}\right) \quad \left(-\sqrt{5}, -2\sqrt{5}\right)$$

Case 3: $\lambda = 5 \quad y = -\frac{x}{2}$

$$\Rightarrow x^2 + \frac{x^2}{4} = 25 \quad \to 5x^2 = 100 \quad \boxed{x = \pm 2\sqrt{5}} \quad \left(2\sqrt{5}, -\sqrt{5}\right) \quad \left(-2\sqrt{5}, \sqrt{5}\right)$$

$$T\left(\sqrt{5}, 2\sqrt{5}\right) = 4\left(\sqrt{5}\right)^2 - 4\left(\sqrt{5}\right)\left(2\sqrt{5}\right) + \left(2\sqrt{5}\right)^2 = 0°$$

$$T\left(-\sqrt{5}, -2\sqrt{5}\right) = 4\left(-\sqrt{5}\right)^2 - 4\left(-\sqrt{5}\right)\left(-2\sqrt{5}\right) + \left(-2\sqrt{5}\right)^2 = 0°$$

$$T\left(2\sqrt{5}, -\sqrt{5}\right) = 4\left(2\sqrt{5}\right)^2 - 4\left(2\sqrt{5}\right)\left(-\sqrt{5}\right) + \left(-\sqrt{5}\right)^2 = 125°$$

$$T\left(-2\sqrt{5}, \sqrt{5}\right) = 4\left(-2\sqrt{5}\right)^2 - 4\left(-2\sqrt{5}\right)\left(\sqrt{5}\right) + \left(\sqrt{5}\right)^2 = 125°$$

$\therefore$ The minimum temperature is 0° at $\left(\sqrt{5}, 2\sqrt{5}\right) \quad \left(-\sqrt{5}, -2\sqrt{5}\right)$

The maximum temperature is 125° at $\left(2\sqrt{5}, -\sqrt{5}\right) \quad \left(-2\sqrt{5}, \sqrt{5}\right)$

Exercise

Your firm has been asked to design a storage tank for liquid petroleum gas. The customer's specifications call for a cylindrical tank with hemispherical ends, and the tank is to hold $8000\ m^3$ of gas. He customer also wants to use the smallest amount of material possible in building the tank. What radius and height do you recommend for the cylindrical portion of the tank?

Solution

The surface area is: $S = 4\pi r^2 + 2\pi rh$ subject to the constraint $V = \frac{4}{3}\pi r^3 + \pi r^2 h = 8000$.

$$\nabla S = (8\pi r + 2\pi h)\boldsymbol{i} + 2\pi r\boldsymbol{j} \quad \textit{and} \quad \nabla V = \left(4\pi r^2 + 2\pi rh\right)\boldsymbol{i} + \pi r^2 \boldsymbol{j}$$

$$\nabla S = \lambda \nabla V \quad \to \quad (8\pi r + 2\pi h)\boldsymbol{i} + 2\pi r\boldsymbol{j} = \left(4\pi r^2 + 2\pi rh\right)\lambda\boldsymbol{i} + \pi r^2 \lambda \boldsymbol{j}$$

$$8\pi r + 2\pi h = 2\pi r(2r + h)\lambda \quad \textit{and} \quad 2\pi r = \pi r^2 \lambda$$

$$4r + h = r(2r + h)\lambda \qquad r^2\lambda - 2r = 0 \to r(\lambda r - 2) = 0$$

$$r = 0 \quad \textit{and} \quad \lambda = \frac{2}{r} \ (r \ne 0)$$

$$4r + h = r(2r + h)\frac{2}{r} \quad \to \quad 4r + h = 4r + 2h \ \Rightarrow \ \boxed{h = 0}$$

The tank is a sphere, there is no cylindrical part, and

$$\frac{4}{3}\pi r^3 + \pi r^2 (0) = 8000$$

$$r^3 = \frac{6000}{\pi} \quad \to \quad \boxed{r = 10\left(\frac{6}{\pi}\right)^{1/3} \approx 12.4}$$

Exercise

Find the point on the plane $x + 2y + 3z = 13$ closest to the point $(1,\ 1,\ 1)$

Solution

Let $f(x,y,z) = (x-1)^2 + (y-1)^2 + (z-1)^2$ (be the square of the distance from $(1,1,1)$)

$$\nabla f = 2(x-1)\boldsymbol{i} + 2(y-1)\boldsymbol{j} + 2(z-1)\boldsymbol{k} \quad \textit{and} \quad \nabla g = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$$

$$\nabla f = \lambda \nabla g \quad \Rightarrow \quad 2(x-1)\boldsymbol{i} + 2(y-1)\boldsymbol{j} + 2(z-1)\boldsymbol{k} = \lambda\boldsymbol{i} + 2\lambda\boldsymbol{j} + 3\lambda\boldsymbol{k}$$

$$\to \begin{cases} 2(x-1) = \lambda \to x = \frac{\lambda}{2} + 1 \\ 2(y-1) = 2\lambda \to y = \lambda + 1 \\ 2(z-1) = 3\lambda \to z = \frac{3\lambda}{2} + 1 \end{cases} \quad \to \quad \frac{\lambda}{2} + 1 + 2(\lambda + 1) + 3\left(\frac{3\lambda}{2} + 1\right) = 13$$

$$\frac{\lambda}{2} + 1 + 2\lambda + 2 + \frac{9\lambda}{2} + 3 = 13$$

$$7\lambda = 7 \quad \Rightarrow \quad \boxed{\lambda = 1}$$

$$x = \frac{\lambda}{2} + 1 = \frac{3}{2}, \quad y = \lambda + 1 = 2, \quad z = \frac{3\lambda}{2} + 1 = \frac{5}{2}$$

$\therefore$ The point $\left(\frac{3}{2},\ 2,\ \frac{5}{2}\right)$ is closet.

Exercise

Find the point on the sphere $x^2 + y^2 + z^2 = 4$ farthest from the point $(1,\ -1,\ 1)$

Solution

Let $f(x,y,z) = (x-1)^2 + (y+1)^2 + (z-1)^2$ (be the square of the distance from $(1,-1,1)$)

$$\nabla f = 2(x-1)\boldsymbol{i} + 2(y+1)\boldsymbol{j} + 2(z-1)\boldsymbol{k} \quad \text{and} \quad \nabla g = 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k}$$

$$\nabla f = \lambda \nabla g \quad \Rightarrow \quad 2(x-1)\boldsymbol{i} + 2(y+1)\boldsymbol{j} + 2(z-1)\boldsymbol{k} = 2x\lambda\boldsymbol{i} + 2y\lambda\boldsymbol{j} + 2z\lambda\boldsymbol{k}$$

$$\rightarrow \begin{cases} x - 1 = x\lambda \rightarrow x = \frac{1}{1-\lambda} \\ y + 1 = y\lambda \rightarrow y = -\frac{1}{1-\lambda} \\ z - 1 = z\lambda \rightarrow z = \frac{1}{1-\lambda} \end{cases} \rightarrow \left(\frac{1}{1-\lambda}\right)^2 + \left(-\frac{1}{1-\lambda}\right)^2 + \left(\frac{1}{1-\lambda}\right)^2 = 4$$

$$3\left(\frac{1}{1-\lambda}\right)^2 = 4 \quad \rightarrow \left(\frac{1}{1-\lambda}\right)^2 = \frac{4}{3} \quad \rightarrow \frac{1}{1-\lambda} = \pm\frac{2}{\sqrt{3}}$$

$$x = \pm\frac{2}{\sqrt{3}}, \quad y = \mp\frac{2}{\sqrt{3}}, \quad z = \pm\frac{2}{\sqrt{3}}$$

$$\left(\frac{2}{\sqrt{3}},\ -\frac{2}{\sqrt{3}},\ \frac{2}{\sqrt{3}}\right) \quad \text{and} \quad \left(-\frac{2}{\sqrt{3}},\ \frac{2}{\sqrt{3}},\ -\frac{2}{\sqrt{3}}\right)$$

$$f\left(\frac{2}{\sqrt{3}},-\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}}\right) = \left(\frac{2}{\sqrt{3}}-1\right)^2 + \left(-\frac{2}{\sqrt{3}}+1\right)^2 + \left(\frac{2}{\sqrt{3}}-1\right)^2 \approx 0.72$$

$$f\left(-\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}},-\frac{2}{\sqrt{3}}\right) = \left(-\frac{2}{\sqrt{3}}-1\right)^2 + \left(\frac{2}{\sqrt{3}}+1\right)^2 + \left(-\frac{2}{\sqrt{3}}-1\right)^2 \approx 13.928$$

$\therefore$ The largest value of *f* occurs at $\left(-\frac{2}{\sqrt{3}},\ \frac{2}{\sqrt{3}},\ -\frac{2}{\sqrt{3}}\right)$ on the sphere.

Exercise

Find the minimum distance from the surface $x^2 - y^2 - z^2 = 1$ to the origin

Solution

Let $f(x,y,z) = x^2 + y^2 + z^2$ (be the square of the distance from origin)

$\nabla f = 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k}$ *and* $\nabla g = 2x\boldsymbol{i} - 2y\boldsymbol{j} - 2z\boldsymbol{k}$

$\nabla f = \lambda \nabla g \quad \Rightarrow \quad 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k} = 2x\lambda\boldsymbol{i} - 2y\lambda\boldsymbol{j} - 2z\lambda\boldsymbol{k}$

$$\rightarrow \begin{cases} 2x = 2x\lambda \\ 2y = -2y\lambda \\ 2z = -2z\lambda \end{cases} \rightarrow \quad \lambda = 1 \text{ or } x = 0$$

Case 1: $\lambda = 1 \quad \rightarrow \begin{cases} 2y = -2y\lambda & \boxed{y=0} \\ 2z = -2z\lambda & \boxed{z=0} \end{cases} \quad x^2 - y^2 - z^2 = 1 \Rightarrow \boxed{x = \pm 1}$

Case 2: $x = 0 \quad \rightarrow -y^2 - z^2 = 1$ *No solution*

$\therefore$ The points on the unit circle $y^2 + z^2 = 1$ are the points on the surface $x^2 - y^2 - z^2 = 1$ closest to the origin.

Exercise

Find the maximum and minimum values of $f(x,y,z) = x - 2y + 5z$ on the sphere $x^2 + y^2 + z^2 = 30$

Solution

$\nabla f = \boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}$ *and* $\nabla g = 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k}$

$\nabla f = \lambda \nabla g \quad \Rightarrow \quad \boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k} = 2x\lambda\boldsymbol{i} + 2y\lambda\boldsymbol{j} + 2z\lambda\boldsymbol{k}$

$$\rightarrow \begin{cases} 2x\lambda = 1 \\ 2y\lambda = -2 \\ 2z\lambda = 5 \end{cases} \rightarrow \quad \begin{aligned} x &= \frac{1}{2\lambda} \\ y &= -\frac{1}{\lambda} \\ z &= \frac{5}{2\lambda} \end{aligned}$$

$$\left(\frac{1}{2\lambda}\right)^2 + \left(-\frac{1}{\lambda}\right)^2 + \left(\frac{5}{2\lambda}\right)^2 = 30$$

$$\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{25}{4\lambda^2} = 30$$

$$\frac{30}{4\lambda^2} = 30 \quad \rightarrow \quad \lambda^2 = \frac{1}{4} \Rightarrow \boxed{\lambda = \pm\frac{1}{2}}$$

$\lambda = \frac{1}{2} \quad \Rightarrow \quad \boxed{x = 1,\ y = -2,\ z = 5}$

$\lambda = -\frac{1}{2} \quad \Rightarrow \quad \boxed{x = -1,\ y = 2,\ z = -5}$

$f(1,-2,5)=1+4+25=30$

$f(-1,2,-5)=-1-4-25=-30$

$\therefore$ The maximum value $f(1,-2,5)=30$ and the minimum is $f(-1,2,-5)=-30$

Exercise

Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.

Solution

$$f(x,y,z)=x^2+y^2+z^2 \quad \text{and} \quad g(x,y,z)=x+y+z-9=0$$

$$\nabla f=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k} \quad \text{and} \quad \nabla g=\mathbf{i}+\mathbf{j}+\mathbf{k}$$

$$\nabla f=\lambda\nabla g \quad \Rightarrow \quad 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}=\lambda\mathbf{i}+\lambda\mathbf{j}+\lambda\mathbf{k}$$

$$\to \begin{cases} 2x=\lambda \\ 2y=\lambda \\ 2z=\lambda \end{cases} \quad \to \quad x=y=z=\frac{1}{2\lambda}$$

$$\frac{1}{2\lambda}+\frac{1}{2\lambda}+\frac{1}{2\lambda}=9$$

$$\frac{3}{2\lambda}=9 \quad \Rightarrow \quad \lambda=\frac{1}{6}$$

$$x=y=z=\frac{1}{2\frac{1}{6}}\ \underline{=3}$$

Exercise

A space probe in the shape of the ellipsoid $4x^2+y^2+4z^2=16$ enters Earth's atmosphere and its surface begins to heat. After 1 hour, the temperature at the point (x,y,z) on the probe's surface is $T(x,y,z)=8x^2+4yz-16z+600$. Find the hottest point on the probe's surface.

Solution

$$\nabla T=16x\mathbf{i}+4z\mathbf{j}+(4y-16)\mathbf{k} \quad \text{and} \quad \nabla g=8x\mathbf{i}+2y\mathbf{j}+8z\mathbf{k}$$

$$\nabla f=\lambda\nabla g \quad \Rightarrow \quad 16x\mathbf{i}+4z\mathbf{j}+(4y-16)\mathbf{k}=8x\lambda\mathbf{i}+2y\lambda\mathbf{j}+8z\lambda\mathbf{k}$$

$$\to \begin{cases} 16x=8x\lambda \\ 4z=2y\lambda \\ 4y-16=8z\lambda \end{cases} \quad \to \quad \lambda=2 \text{ or } x=0$$

$$\textbf{Case } 1\text{: } \lambda=2 \quad \to \begin{cases} 2z=y\lambda \\ y-4=2z\lambda \end{cases} \quad \begin{array}{l} \to 2z=2y \Rightarrow z=y \\ y-4=2y(2) \end{array} \quad 3y=-4 \to \boxed{y=-\frac{4}{3}=z}$$

$$4x^2+\frac{16}{9}+4\left(\frac{16}{9}\right)=16 \quad \to x^2=\frac{16}{9} \quad \boxed{x=\pm\frac{4}{3}}$$

Case 2: $x=0 \quad \to \lambda=\frac{2z}{y} \quad \Rightarrow y-4=2z\frac{2z}{y}$

$$y^2-4y=4z^2$$

$$4x^2+y^2+4z^2=16 \quad \to \quad y^2+y^2-4y=16$$

$$2y^2-4y-16=0 \quad \to \quad \boxed{y=4,\ -2}$$

$$\begin{cases} y=4 & \to 4z^2=4^2-16=0 \Rightarrow \boxed{z=0} \\ y=-2 & \to 4z^2=(-2)^2+8=13 \Rightarrow \boxed{z=\pm\sqrt{3}} \end{cases}$$

$$T\left(-\frac{4}{3},-\frac{4}{3},-\frac{4}{3}\right)=8\left(-\frac{4}{3}\right)^2+4\left(-\frac{4}{3}\right)\left(-\frac{4}{3}\right)-16\left(-\frac{4}{3}\right)+600 \approx 642.667^\circ$$

$$T\left(\frac{4}{3},-\frac{4}{3},-\frac{4}{3}\right)=8\left(\frac{4}{3}\right)^2+4\left(-\frac{4}{3}\right)\left(-\frac{4}{3}\right)-16\left(-\frac{4}{3}\right)+600 \approx 642.667^\circ$$

$$T(0,4,0)=0+0-0+600 \approx 600^\circ$$

$$T\left(0,-2,-\sqrt{3}\right)=0+4(-2)\left(-\sqrt{3}\right)-16\left(-\sqrt{3}\right)+600 \approx 641.6^\circ$$

$$T\left(0,-2,\sqrt{3}\right)=0+4(-2)\left(\sqrt{3}\right)-16\left(\sqrt{3}\right)+600 \approx 558.43^\circ$$

$\therefore \left(\pm\frac{4}{3},-\frac{4}{3},-\frac{4}{3}\right)$ are the hottest points on the space probe.

Exercise

Find the extreme values of $f(x,y,z)=xyz$

Subject to the constraint $\begin{cases} x+y+z=32 \\ x-y+z=0 \end{cases}$

Solution

$f(x,y,z)=xyz$

$\nabla f=yz\hat{i}+xz\hat{j}+xy\hat{k}$

$g_1(x,y,z)=x+y+z-32=0$

$\nabla g_1=\hat{i}+\hat{j}+\hat{k}$

$g_2(x,y,z)=x-y+z=0$

$\nabla g_2=\hat{i}-\hat{j}+\hat{k}$

$$\nabla f=\lambda\nabla g_1+\mu\nabla g_2$$

$$yz\hat{i}+xz\hat{j}+xy\hat{k}=\lambda\hat{i}+\lambda\hat{j}+\lambda\hat{k}+\mu\hat{i}+\mu\hat{j}+\mu\hat{k}$$

$$\begin{cases} yz=\lambda+\mu & (1) \\ xz=\lambda-\mu & (2) \\ xy=\lambda+\mu & (3) \end{cases} \Rightarrow \begin{cases} (1)+(2)\to & 2\lambda=yz+zx \\ (3)+(2)\to & 2\lambda=xy+zx \end{cases} \to 2\lambda=yz+zx=xy+zx$$

$$yz=xy \quad \Rightarrow \quad y=0 \quad or \quad x=z \quad (y\neq 0)$$

Case 1:

$$\text{If } y=0 \Rightarrow \begin{cases} g_1(x,y,z) = x+z-32=0 & \cancel{-32=0} \\ g_2(x,y,z) = x+z=0 & \to x=-z \end{cases}$$

Case 2:

$$\text{If } x=z \Rightarrow \begin{cases} g_1(x,y,z) = 2x+y-32=0 & 4x=32 \to \boxed{x=8=z} \\ g_2(x,y,z) = 2x-y=0 & \to y=2x \quad \boxed{y=16} \end{cases}$$

$$f(x,y,z) = xyz = (8)(16)(8) \ \underline{=1024}$$

The extreme point is $(8,\ 16,\ 8)$ with a value of 1024.

Exercise

Find the extreme values of $f(x,y,z) = x^2+y^2+z^2$

Subject to the constraint $\begin{cases} x+2z=6 \\ x+y=12 \end{cases}$

Solution

$f(x,y,z) = x^2+y^2+z^2$ | $g_1(x,y,z) = x+2z-6=0$ | $g_2(x,y,z) = x+y-12=0$

$\nabla f = 2x\hat{i}+2y\hat{j}+2z\hat{k}$ | $\nabla g_1 = \hat{i}+2\hat{k}$ | $\nabla g_2 = \hat{i}+\hat{j}$

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

$$2x\hat{i}+2y\hat{j}+2z\hat{k} = \lambda\hat{i}+2\lambda\hat{k}+\mu\hat{i}+\mu\hat{j}$$

$$\begin{cases} 2x=\lambda+\mu & (1) \\ 2y=\mu & (2) \\ 2z=2\lambda & (3) \end{cases} \Rightarrow 2x=z+2y \Rightarrow z=2x-2y$$

$$\begin{cases} x+2z=6 \\ x+y=12 \end{cases} \quad \begin{cases} x+4x-4y=6 \\ x+y=12 \end{cases} \Rightarrow \begin{cases} 5x-4y=6 \\ x+y=12 \end{cases} \quad x=6, \quad y=6 \Rightarrow z=0$$

$$f(x,y,z) = x^2+y^2+z^2 \ \underline{=72}$$

The extreme point is $(6,\ 6,\ 0)$ with a value of 72.