Math 2415 – Calculus III ***Exam* 4** ***Review***

Professor: Fred Khoury

1. Let C be the circle of radius 2 centered at the origin with counterclockwise orientation

a) Give the unit outward vector at any point $(x,\ y)$ on C.

b) Find the normal component of the vector field $\boldsymbol{F} = 2\langle y,\ -x\rangle$ at any point on C.

c) Find the normal component of the vector field $\boldsymbol{F} = \dfrac{\langle x,\ y\rangle}{x^2+y^2}$ at any point on C.

2. Evaluate the line integral $\displaystyle\int_C \left(x^2-2xy+y^2\right)ds$; C is the upper half of a circle $\boldsymbol{r}(t)=\langle 5\cos t,\ 5\sin t\rangle,\quad 0\le t\le \pi \quad (ccw)$

3. Evaluate the line integral $\displaystyle\int_C ye^{-xz}ds$; C is the path $\boldsymbol{r}(t)=\langle t,\ 3t,\ -6t\rangle,\quad 0\le t\le \ln 8$

4. Integrate $f(x,y,z)=\sqrt{x^2+z^2}$ over the circle $\boldsymbol{r}(t)=(a\cos t)\boldsymbol{j}+(a\sin t)\boldsymbol{k},\quad 0\le t\le 2\pi$

5. Integrate $f(x,y,z)=\sqrt{x^2+y^2}$ over the involute curve $\boldsymbol{r}(t)=(\cos t+t\sin t)\boldsymbol{i}+(\sin t-\cos t)\boldsymbol{j},\quad 0\le t\le\sqrt{3}$

6. Find the work required to move an object on the given curve $\boldsymbol{F} = \dfrac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^{3/2}}$ on the path $\boldsymbol{r}(t)=\left\langle t^2,\ 3t^2,\ -t^2\right\rangle,\quad 1\le t\le 2$

7. Find the work done by $\boldsymbol{F} = \dfrac{x\boldsymbol{i}+y\boldsymbol{j}}{\left(x^2+y^2\right)^{3/2}}$ over the plane curve $\boldsymbol{r}(t)=\left(e^t\cos t\right)\boldsymbol{i}+\left(e^t\sin t\right)\boldsymbol{j}$ from the point $(1,\ 0)$ to the point $\left(e^{2\pi},\ 0\right)$ by using the parametrization of the curve to evaluate the work integral

8. Find the circulation and the outward flux of the vector field $\boldsymbol{F} = \langle y-x,\ y\rangle$ for the curve $\boldsymbol{r}(t)=\langle 2\cos t,\ 2\sin t\rangle,\quad 0\le t\le 2\pi$

9. Find the flow of the field $\boldsymbol{F} = \nabla\left(x^2ze^y\right)$

a) Once around the ellipse C in which the plane $x+y+z=1$ intersects the cylinder $x^2+z^2=25$, clockwise as viewed from the positive y-axis.

b) Along the curved boundary of the helicoid $\boldsymbol{r}(r,\ \theta)=(r\cos\theta)\boldsymbol{i}+(r\sin\theta)\boldsymbol{j}+\theta\boldsymbol{k}$ from $(1,\ 0,\ 0)$ to $(1,\ 0,\ 2\pi)$

10. Evaluate the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ for the following vector field $\boldsymbol{F}$ and curves C in two ways

- ***i.*** By parameterizing C.
- ***ii.*** By using the Fundamental Theorem for the integrals, if possible.

a) $\boldsymbol{F} = \nabla(x, y, z)$; C: $\boldsymbol{r}(t) = \left\langle \cos t,\ \sin t,\ \frac{t}{\pi} \right\rangle, \quad 0 \le t \le \pi$

b) $\boldsymbol{F} = \langle x,\ -y \rangle$; C: is the square with vertices $(\pm 1,\ \pm 1)$ with *counterclockwise* orientation.

11. Prove that the radial field $\boldsymbol{F} = \frac{\boldsymbol{r}}{|\boldsymbol{r}|^p}$ where $\boldsymbol{r} = \langle x,\ y \rangle$ and p is a real number, is conservative on $\mathbb{R}^2$ with the origin removed. For what value of p is $\boldsymbol{F}$ conservative on $\mathbb{R}^2$ (including the origin)?

12. Evaluate $\int_C y^2 dx + x^2 dy \quad C$ is the circle $x^2 + y^2 = 4$

13. Find the area of the elliptical region cut from the plane $x + y + z = 1$ by the cylinder $x^2 + y^2 = 1$

14. Find the area of the cap cut from the paraboloid $x^2 + y^2 + z^2 = 1$ by the plane $z = \frac{\sqrt{2}}{2}$

15. Use Green's Theorem to find the counterclockwise circulation and outward flux for the field and curves *Square*: $\boldsymbol{F} = (2xy + x)\boldsymbol{i} + (xy - y)\boldsymbol{j}$ C: *The square bounded by* $x = 0,\ x = 1,\ y = 0,\ y = 1$

16. Use Green's Theorem to find the counterclockwise circulation and outward flux for the field and curves *Triangle*: $\boldsymbol{F} = \left(y - 6x^2\right)\boldsymbol{i} + \left(x + y^2\right)\boldsymbol{j}$

C: *The triangle made by the lines* $y = 0,\ y = x,$ *and* $x = 1$

17. Show that $\oint_C \ln x \sin y\,dy - \frac{\cos y}{x} dx = 0$ for any closed curve C to which Green's Theorem applies.

18. Use either form of Green's Theorem to evaluate the line integral $\oint_C \left(x^3 + xy\right)dy + \left(2y^2 - 2x^2 y\right)dx$; C is the square with vertices $(\pm 1,\ \pm 1)$ with *counterclockwise* orientation

19. Use either form of Green's Theorem to evaluate the line integral $\oint_C 3x^3 dy - 3y^3 dx$; C is the circle of radius 4 centered at the origin with *clockwise* orientation.

20. Find the area of the region bounded by the hypocycloid $\boldsymbol{r}(t) = \left\langle \cos^3 t,\ \sin^3 t \right\rangle$ for $0 \le t \le 2\pi$, using a line integral

21. Compute the divergence and curl of the following vector fields. State whether the field is *source-free* or *irrotational*.

a) $\boldsymbol{F} = \langle yz,\ xz,\ xy \rangle$

b) $\boldsymbol{F} = \boldsymbol{r}|\boldsymbol{r}| = \langle x,\ y,\ z \rangle \sqrt{x^2 + y^2 + z^2}$

c) $\boldsymbol{F} = \langle \sin xy,\ \cos yz,\ \sin xz \rangle$

22. Prove that $\nabla\left(\dfrac{1}{|\boldsymbol{r}|^4}\right) = -\dfrac{4\boldsymbol{r}}{|\boldsymbol{r}|^6}$ and use the result to prove that $\nabla \cdot \nabla\left(\dfrac{1}{|\boldsymbol{r}|^4}\right) = \dfrac{12}{|\boldsymbol{r}|^6}$

23. Find the surface area of the helicoid $\boldsymbol{r}(r,\ \theta) = (r\cos\theta)\boldsymbol{i} + (r\sin\theta)\boldsymbol{j} + \theta\boldsymbol{k}, \quad 0 \le \theta \le 2\pi, \quad 0 \le r \le 1$

24. Use a surface integral to find the area of the hemisphere $x^2 + y^2 + z^2 = 9$ for $z \ge 0$ (excluding the base)

25. Use a surface integral to find the area of the surface $f(x,\ y) = \sqrt{2}\, xy$ above the origin $\{(r,\ \theta):\ 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

26. Find the flux of $\boldsymbol{F} = \dfrac{\boldsymbol{r}}{|\boldsymbol{r}|}$ across the sphere of radius a centered at the origin, where $\boldsymbol{r} = \langle x, y, z \rangle$. Assume the normal vectors to the surface point outward.

27. Evaluate the surface integrals $\iint_S (x - y + z)\, dS$; S is the entire surface including the base of the hemisphere $x^2 + y^2 + z^2 = 4$, for $z \ge 0$

28. Evaluate the line integral $\oint_C \boldsymbol{F}.d\boldsymbol{r}$ using the Stoke's Theorem $\boldsymbol{F} = \langle x^2 - y^2,\ x,\ 2yz \rangle$; C is the boundary of the plane $z = 6 - 2x - y$ in the first octant and has counterclockwise orientation.

29. Use Stoke's Theorem to evaluate the surface integral $\iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, dS$; $\boldsymbol{F} = \langle -z,\ x,\ y \rangle$, where S is the hyperboloid $z = 10 - \sqrt{1 + x^2 + y^2}$ *for* $z \ge 0$. Assume that $\boldsymbol{n}$ is the *outward normal*.

30. Use the Divergence Theorem to compute the outward flux of the vector field $\boldsymbol{F} = \langle -x,\ x - y,\ x - z \rangle$; S is the surface of the cube cut from the first octant by the planes $x = 1,\ y = 1,$ *and* $z = 1$.

31. Use the Divergence Theorem to compute the outward flux of the vector field $\boldsymbol{F} = \langle x^2,\ y^2,\ z^2 \rangle$; S is the cylinder $\{(x, y, z):\ x^2 + y^2 = 4,\ 0 \le z \le 8\}$

32. Compute the outward flux of the field $\boldsymbol{F} = \left\langle x^2 + x\sin y,\ y^2 + 2\cos y,\ z^2 + z\sin y \right\rangle$ across the surface S that is the boundary of the prism bounded by the planes $y = 1 - x,\ x = 0,\ y = 0,\ z = 0,\ z = 4$

33. Consider the surface S consisting of the quarter-sphere $x^2 + y^2 + z^2 = a^2$, for $z \geq 0$ *and* $x \geq 0$, and the half disk in the yz-plane $y^2 + z^2 \leq a^2$, for $z \geq 0$. The boundary of S in the xy-plane is C, which consists of the semicircle $x^2 + y^2 = a^2$, for $x \geq 0$, and the line segment $[-a,\ a]$ on the y-axis, with a counterclockwise orientation. Let $\boldsymbol{F} = \langle 2z - y,\ x - z,\ y - 2x \rangle$

a) Describe the direction in which the normal vectors point on S.

b) Evaluate $\oint_C \boldsymbol{F}.d\boldsymbol{r}$

c) Evaluate $\iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, dS$ and check for segment with part (*b*).

34. Let S be the hemisphere $x^2 + y^2 + z^2 = a^2$, for $z \geq 0$, and let T be the paraboloid $z = a - \frac{1}{a}\left(x^2 + y^2\right)$, for $z \geq 0$, where $a > 0$. Assume the surfaces have outward normal vectors.

a) Verify that S and T have the same base $\left(x^2 + y^2 \leq a^2\right)$ and the same high point $(0,\ 0,\ a)$.

b) Which surface has the greater area?

c) Show that the flux of the radial field $\boldsymbol{F} = \langle x,\ y,\ z \rangle$ across S is $2\pi a^3$.

d) Show that the flux of the radial field $\boldsymbol{F} = \langle x,\ y,\ z \rangle$ across T is $\frac{3\pi a^3}{2}$.

Solution

1. $a)\ \frac{1}{2}\langle x,y\rangle \quad b)\ 0 \quad c)\ \frac{1}{2}$

2. 125π

3. $\frac{\sqrt{46}}{4}\left(e^{54\ln 2}-1\right)$

4. $4a^2$

5. $\frac{7}{3}$

6. $\frac{3\sqrt{11}}{44}$

7. $1-e^{-2\pi}$

8. $Cir:-4\pi \quad flux:0$

9. $\boldsymbol{a})\ \ 0 \qquad \boldsymbol{b})\ \ 2\pi$

10. $\boldsymbol{a})$ $i)$ 0 $ii)$ 0 $\boldsymbol{b})$ $i)$ 0 $ii)$ 0

11. $\varphi=\dfrac{-1}{(p-2)|\mathbf{r}|^{p-2}}\ \left(for\ p\neq 2\right) \qquad \varphi=\frac{1}{2}\ln\left(|\mathbf{r}|^2\right)\ \left(for\ p=2\right)$

12. 0

13. $\pi\sqrt{3}$

14. $2\pi\left(1-\frac{1}{\sqrt{2}}\right)$

15. $\boldsymbol{a})\ \ FLux=\frac{3}{2} \qquad \boldsymbol{b})\ \ Cir=-\frac{1}{2}$

16. $\boldsymbol{a})\ \ FLux=-\frac{11}{3} \qquad \boldsymbol{b})\ \ Cir=0$

17. 0

18. $\frac{20}{3}$

19. -1152π

20. $\frac{3\pi}{8}$

21. $\boldsymbol{a})\ \ 0,\ \langle 0,\ 0,\ 0\rangle \qquad \boldsymbol{b})\ \ 4|\mathbf{r}|,\ \langle 0,\ 0,\ 0\rangle$

$\boldsymbol{c})\ \ y\cos xy - z\sin yz + x\cos xz,\ \ \langle y\sin yz,\ -z\cos xz,\ -x\cos xy\rangle$

22.

23. $\pi\left[\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right]$

24. 18π

25. $\frac{26}{3}\pi$

26. $4\pi a^2$

27. 8π

28.

29. 99π

30. -3

31. 256π

32. $\frac{32}{3}$

33. ***a***) ***b***) πa^2 ***c***) πa^2

34. ***a***) ***b***) $\frac{5\sqrt{5}-1}{6} \cdot \pi a^2$