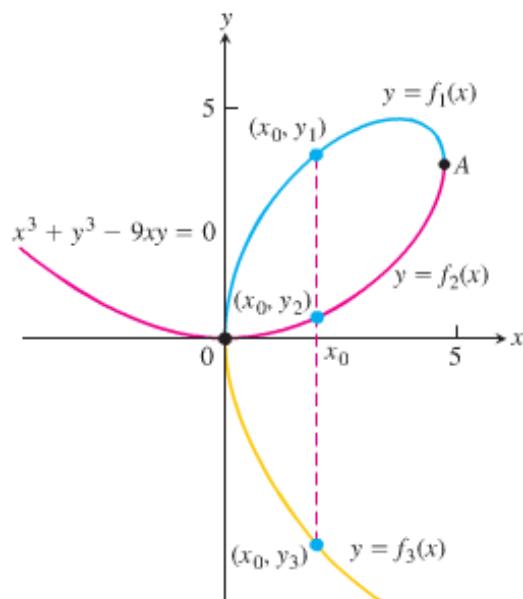


Section 2.7 – Implicit Differentiation

Definition

A relation $F(x, y) = 0$ is said to define the function $y = f(x)$ implicitly if, for x in the domain of f
 $\rightarrow F(x, f(x)) = 0$

Example: $x^3 + y^3 - 9xy = 0, \quad x^2 + y^2 = 25$



Implicitly Defined Functions

It is always assumed that the given equation determines y implicitly as a differentiable function of x so that $\frac{dy}{dx}$ exists.

Example

Find $\frac{dy}{dx}$ if $y^2 = x$

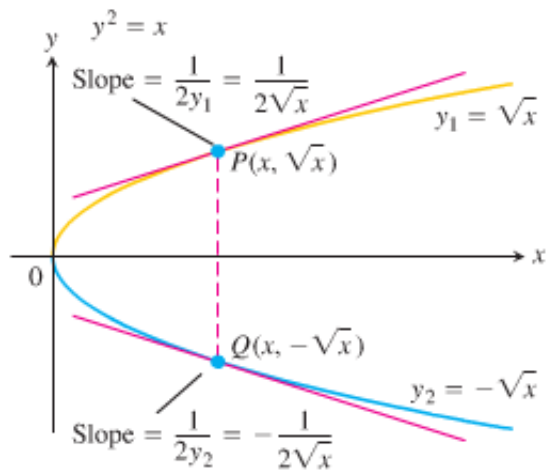
Solution

$$y^2 = x$$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x)$$

$$2y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2y}$$



Example

Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution

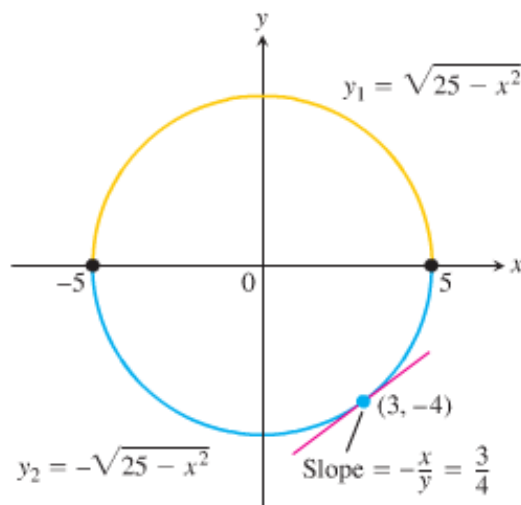
$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + 2y \frac{dy}{dx} = 0$$

$$2y \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

The slope at $(3, -4)$ is $\frac{dy}{dx}\bigg|_{(3, -4)} = -\frac{3}{-4} = \frac{3}{4}$



Implicit Differentiation

- ✓ Differentiate both sides of the equation with respect to x , treating y as a differentiable function of x .
- ✓ Collect the terms with $\frac{dy}{dx}$ on one side of the equation and solve for $\frac{dy}{dx}$.

Example

Find $\frac{dy}{dx}$ if $y^2 = x^2 + \sin xy$

Solution

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin xy)$$

$$2y \frac{dy}{dx} = 2x + \cos(xy) \frac{d}{dx}(xy)$$

$$2y \frac{dy}{dx} = 2x + \cos(xy) \left(y + x \frac{dy}{dx} \right)$$

$$2y \frac{dy}{dx} = 2x + y \cos(xy) + x \cos(xy) \frac{dy}{dx}$$

$$2y \frac{dy}{dx} - x \cos(xy) \frac{dy}{dx} = 2x + y \cos(xy)$$

$$(2y - x \cos xy) \frac{dy}{dx} = 2x + y \cos xy$$

$$\frac{dy}{dx} = \frac{2x + y \cos xy}{2y - x \cos xy}$$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin xy)$$

$$\text{Let } y' = \frac{dy}{dx}$$

$$2yy' = 2x + \cos(xy)(xy)'$$

$$2yy' = 2x + \cos(xy)(y + xy')$$

$$2yy' = 2x + y \cos(xy) + x \cos(xy) y'$$

$$2yy' - x \cos(xy) y' = 2x + y \cos(xy)$$

$$(2y - x \cos xy) y' = 2x + y \cos xy$$

$$y' = \frac{dy}{dx} = \frac{2x + y \cos xy}{2y - x \cos xy}$$

Example

Find $\frac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 8$

Solution

$$\text{Let } y' = \frac{dy}{dx}$$

$$\frac{d}{dy}(2x^3 - 3y^2) = \frac{d}{dy}(8)$$

$$6x^2 - 6yy' = 0$$

$$6x^2 = 6yy'$$

$$\boxed{y' = \frac{6x^2}{6y} = \frac{x^2}{y}}$$

$$y'' = \left(\frac{x^2}{y} \right)'$$

$$\begin{aligned} u &= x^2 & v &= y \\ u' &= 2x & v' &= y' \end{aligned}$$

$$y'' = \frac{2xy - x^2y'}{y^2}$$

$$= \frac{2xy - x^2 \frac{x^2}{y}}{y^2}$$

$$= \frac{2xy^2 - x^4}{y^2}$$

$$\boxed{= \frac{2xy^2 - x^4}{y^3}}$$

Normal Lines

The **normal** is the line **perpendicular** to the **tangent** of the profile curve at the point of entry.

Example

Show that the point (2, 4) lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve there.

Solution

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - \frac{d}{dx}(9xy) = 0$$

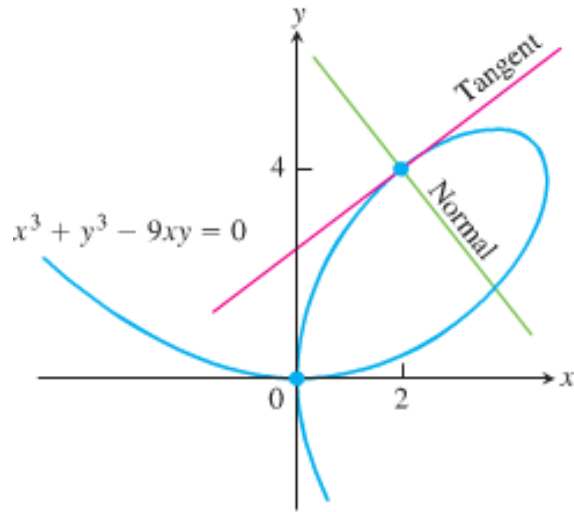
$$3x^2 + 3y^2y' - 9(y + xy') = 0$$

$$3x^2 + 3y^2y' - 9y - 9xy' = 0$$

$$3(y^2 - 3x)y' = 9y - 3x^2$$

$$y' = \frac{3(3y - x^2)}{3(y^2 - 3x)}$$

$$= \frac{3y - x^2}{y^2 - 3x}$$



$$\text{The slope: } y' \Big|_{(2,4)} = \frac{3(4) - (2)^2}{(4)^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}$$

The tangent at (2, 4) is the line passes thru (2, 4) with slope $\frac{4}{5}$

$$y = \frac{4}{5}(x - 2) + 4$$

$$y = m(x - x_1) + y_1$$

$$y = \frac{4}{5}x - \frac{8}{5} + 4$$

$$y = \frac{4}{5}x + \frac{12}{5}$$

The Normal to the curve at (2, 4) is the line perpendicular to the tangent there thru (2, 4) with slope $-\frac{5}{4}$

$$y = -\frac{5}{4}(x - 2) + 4$$

$$y = m(x - x_1) + y_1$$

$$y = -\frac{5}{4}x + \frac{5}{2} + 4$$

$$y = -\frac{5}{4}x + \frac{13}{2}$$

Exercises Section 2.7 – Implicit Differentiation

Find $\frac{dy}{dx}$

1. $y^2 + x^2 - 2y - 4x = 4$

6. $y^2 = \frac{x-1}{x+1}$

10. $x \cos(2x + 3y) = y \sin x$

2. $x^2 y^2 - 2x = 3$

7. $(3xy + 7)^2 = 6y$

11. $y = \frac{e^y}{1 + \sin x}$

3. $x + \sqrt{x}\sqrt{y} = y^2$

8. $xy = \cot(xy)$

12. $\sin x \cos(y - 1) = \frac{1}{2}$

4. $x^2 y + xy^2 = 6$

9. $x + \tan(xy) = 0$

13. $y\sqrt{x^2 + y^2} = 15$

5. $x^3 - xy + y^3 = 1$

Find $\frac{dr}{d\theta}$

14. $r - 2\sqrt{\theta} = \frac{3}{2}\theta^{2/3} + \frac{4}{3}\theta^{3/4}$

15. $\sin(r\theta) = \frac{1}{2}$

Find $\frac{d^2y}{dx^2}$

16. $x^{2/3} + y^{2/3} = 1$

17. $2\sqrt{y} = x - y$

18. If $x^3 + y^3 = 16$, find the value of $\frac{d^2y}{dx^2}$ at the point (2, 2).

19. Find dy/dx : $x^2 - xy + y^2 = 4$ and evaluate the derivative at the given point (0, -2)

20. Find the slope of the curve $(x^2 + y^2)^2 = (x - y)^2$ at the point (-2, 1) and (-2, -1)

21. Find the slope of the tangent line to the circle $x^2 - 9y^2 = 16$ at the point (5, 1)

Find an equation of the line tangent to the following curves at the given point

22. $y = 3x^3 + \sin x$; (0, 0)

25. $x^2 y + y^3 = 75$; (4, 3)

23. $y = \frac{4x}{x^2 + 3}$; (3, 1)

26. $x^3 + y^3 = 9xy$; (2, 4)

24. $y + \sqrt{xy} = 6$; (1, 4)

27. Find the lines that are (a) tangent and (b) normal to the curve $x^2 + xy - y^2 = 1$ at the point (2, 3).

28. Find the lines that are (a) tangent and (b) normal to the curve $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$ at the point (-1, 0).

29. Find the lines that are (a) tangent and (b) normal to the curve $x^2 \cos^2 y - \sin y = 0$ at the point $(0, \pi)$.
30. Suppose that x and y are both functions of t , which can be considered to represent time, and that x and y are related by the equation
- $$xy^2 + y = x^2 + 17$$
- Suppose further that when $x = 2$ and $y = 3$, then $\frac{dx}{dt} = 13$. Find the value of the $\frac{dy}{dt}$ at that moment.
31. A cone-shaped icicle is dripping from the roof. The radius of the icicle is decreasing at a rate of 0.2 *cm* per hour, while the length is increasing at a rate of 0.8 *cm* per *hour*. If the icicle is currently 4 *cm* in radius and 20 *cm* long, is the volume of the icicle increasing or decreasing and at what rate?