

Solution **Section 4.2 – General Linear Transformations**

Exercise

The matrix $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ gives a shearing transformation $T(x, y) = (x, 3x + y)$.

What happens to $(1, 0)$ and $(2, 0)$ on the x -axis.

What happens to the points on the vertical lines $x = 0$ and $x = a$?

Solution

The points $(1, 0)$ and $(2, 0)$ on the x -axis transform by T to $(1, 3)$ and $(2, 6)$. The horizontal x -axis transforms to the straight line with slope 3 (going through $(0, 0)$ of course). The points on the y -axis are not moved because $T(0, y) = (0, y)$. The y -axis is the line of eigenvectors of T with $\lambda = 1$.

The vertical line $x = a$ is moved up by $3a$, since $3a$ is added to the y component. This is *shearing*.

Vertical lines slide higher as you go from left to right.

Exercise

A nonlinear transformation T is invertible if every \vec{b} in the output space comes from exactly one x in the input space. $T(\vec{x}) = \vec{b}$ always has exactly one solution. Which of these transformation (on real numbers \vec{x} is invertible and what is T^{-1} ? None are linear, not even T_3 . When you solve $T(\vec{x}) = \vec{b}$, you are inverting T :

$$T_1(\vec{x}) = x^2 \quad T_2(\vec{x}) = x^3 \quad T_3(\vec{x}) = x + 9 \quad T_4(\vec{x}) = e^x \quad T_5(\vec{x}) = \frac{1}{x} \quad \text{for nonzero } x's$$

Solution

T_1 is not invertible because

$$x^2 = 1 \rightarrow x = \pm 1 \text{ and } x^2 = -1 \text{ has no solution.}$$

T_4 is not invertible because

$$e^x = -1 \text{ has no solution.}$$

T_2 is invertible.

$$\text{The solutions to } x^3 = b \rightarrow x = b^{1/3} = T_2^{-1}(b)$$

T_3 is invertible.

$$\text{The solutions to } x + 9 = b \rightarrow x = b - 9 = T_3^{-1}(b)$$

T_5 is invertible.

$$\text{The solutions to } \frac{1}{x} = b \rightarrow x = \frac{1}{b} = T_5^{-1}(b)$$

Exercise

If S and T are linear transformations, is $S(T(\vec{v}))$ linear or quadratic?

a) If $S(\vec{v}) = \vec{v}$ and $T(\vec{v}) = \vec{v}$, then $S(T(\vec{v})) = \vec{v}$ or \vec{v}^2 ?

b) $S(\vec{w}_1 + \vec{w}_2) = S(\vec{w}_1) + S(\vec{w}_2)$ and $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$ combine into

$$S(T(\vec{v}_1 + \vec{v}_2)) = S(\text{_____}) = \text{_____} + \text{_____}$$

Solution

a) $S(T(\vec{v})) = S(\vec{v})$ $T(\vec{v}) = \vec{v}$
 $\quad \quad \quad = \vec{v}$ $S(\vec{v}) = \vec{v}$

b) $S(T(\vec{v}_1 + \vec{v}_2)) = S(T(\vec{v}_1) + T(\vec{v}_2))$
 $\quad \quad \quad = S(T(\vec{v}_1)) + S(T(\vec{v}_2))$

It is quadratic.

Exercise

Find the range and kernel (like the column space and nullspace) of T :

a) $T(v_1, v_2) = (v_2, v_1)$

b) $T(v_1, v_2, v_3) = (v_1, v_2)$

c) $T(v_1, v_2) = (0, 0)$

d) $T(v_1, v_2) = (v_1, v_1)$

Solution

a) Range is the line $y = 0$, Kernel is the line $x = y$ in the xy plane.

b) Range is the xy plane, Kernel is the complementary line in \mathbb{R}^3 .

c) Range is the point $(0, 0)$, Kernel is plane

d) Range is the line $x = y$ in the xy plane, Kernel is the line $x = 0$.

Exercise

M is any 2 by 2 matrix and $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. The transformation T is defined by $T(M) = AM$. What rules of matrix multiplication show that T is linear?

Solution

The distribution law and the association law for multiplication give the linearity

$$\begin{aligned} A(cM + dN) &= A(cM) + A(dN) \\ &= (Ac)M + (Ad)N \\ &= cA(M) + dA(N) \end{aligned}$$

Exercise

Which of these transformations satisfy $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ and which satisfy $T(c\vec{v}) = cT(\vec{v})$?

- a) $T(\vec{v}) = \frac{\vec{v}}{\|\vec{v}\|}$
- b) $T(\vec{v}) = v_1 + v_2 + v_3$
- c) $T(\vec{v}) = \begin{pmatrix} v_1 \\ 2v_2 \\ 3v_3 \end{pmatrix}$
- d) $T(\vec{v}) = \text{largest component of } \vec{v}.$

Solution

- a) This is scaling the vector into a normal vector. This it is impossible that we get additivity, because the sums of normal vectors don't have to be normal. For example $T(0, 1)$ and $T(1, 0)$ for instance. However, true to its name this does have the scaling property. For c value, this value will be canceled from \vec{v} and $\|\vec{v}\|$.
- b) This satisfies both. One immediate way to see that it is matrix multiplication by $[1, 1, 1]$, which is a linear operation and thus satisfies both properties.
- c) This satisfies both. This a matrix multiplication by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
- d) Doesn't satisfy additivity $[(0, 1) \text{ and } (1, 0) \text{ still work}]$. Scaling doesn't work either, if we scale by -1 we now pick out the negative of the smallest component, which doesn't have to be related in any way to the largest component.

Exercise

Consider the basis $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ for \mathbb{R}^3 , where $\vec{v}_1 = (1, 1, 1)$ $\vec{v}_2 = (1, 1, 0)$ $\vec{v}_3 = (1, 0, 0)$ and let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation for which

$$T(\vec{v}_1) = (2, -1, 4), \quad T(\vec{v}_2) = (3, 0, 1), \quad T(\vec{v}_3) = (-1, 5, 1)$$

Find a formula for $T(\vec{x}_1, \vec{x}_2, \vec{x}_3)$, and then use that formula to compute $T(2, 4, -1)$

Solution

$$\text{Assume: } \vec{u} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$$

$$\begin{aligned} (\vec{x}_1, \vec{x}_2, \vec{x}_3) &= c_1 (1, 1, 1) + c_2 (1, 1, 0) + c_3 (1, 0, 0) \\ &= (c_1 + c_2 + c_3, c_1 + c_2, c_1) \end{aligned}$$

$$\begin{cases} c_1 + c_2 + c_3 = x_1 \\ c_1 + c_2 = x_2 \\ c_1 = x_3 \end{cases}$$

$$\begin{cases} c_3 = x_1 - x_2 \\ c_2 = x_2 - x_3 \\ c_1 = x_3 \end{cases}$$

$$\begin{aligned} T(\vec{x}_1, \vec{x}_2, \vec{x}_3) &= x_3 T(\vec{v}_1) + (x_2 - x_3) T(\vec{v}_2) + (x_1 - x_2) T(\vec{v}_3) \\ &= x_3 (2, -1, 4) + (x_2 - x_3) (3, 0, 1) + (x_1 - x_2) (-1, 5, 1) \\ &= (2x_3 + 3x_2 - 3x_3 - x_1 + x_2, -x_3 + 5x_1 - 5x_2, 4x_3 + x_2 - x_3 + x_1 - x_2) \\ &= (-x_1 + 4x_2 - x_3, 5x_1 - 5x_2 - x_3, x_1 + 3x_3) \end{aligned}$$

$$\begin{aligned} T(2, 4, -1) &= (-2 + 16 + 1, 10 - 20 + 1, 2 - 3) \\ &= \underline{(15, -9, -1)} \end{aligned}$$

Exercise

Consider the basis $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ for \mathbb{R}^3 , where $\vec{v}_1 = (1, 2, 1)$ $\vec{v}_2 = (2, 9, 0)$ $\vec{v}_3 = (3, 3, 4)$ and let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation for which

$$T(\vec{v}_1) = (1, 0), \quad T(\vec{v}_2) = (-1, 1), \quad T(\vec{v}_3) = (0, 1)$$

Find a formula for $T(\vec{x}_1, \vec{x}_2, \vec{x}_3)$, and then use that formula to compute $T(7, 13, 7)$

Solution

$$\text{Assume: } \vec{u} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$$

$$\begin{aligned} (\vec{x}_1, \vec{x}_2, \vec{x}_3) &= c_1 (1, 2, 1) + c_2 (2, 9, 0) + c_3 (3, 3, 4) \\ &= (c_1 + 2c_2 + 3c_3, 2c_1 + 9c_2 + 3c_3, c_1 + 4c_3) \end{aligned}$$

$$\begin{cases} c_1 + 2c_2 + 3c_3 = x_1 \\ 2c_1 + 9c_2 + 3c_3 = x_2 \\ c_1 + 4c_3 = x_3 \end{cases}$$

$$\begin{cases} c_1 + 7c_2 = x_2 - x_1 \\ c_1 + 4c_3 = x_3 \end{cases}$$

$$2c_1 + \frac{9}{7}x_2 - \frac{9}{7}x_1 - \frac{9}{7}c_1 + \frac{3}{4}x_3 - \frac{3}{4}c_1 = x_2$$

$$2c_1 - \frac{9}{7}c_1 - \frac{3}{4}c_1 = x_2 - \frac{9}{7}x_2 + \frac{9}{7}x_1 - \frac{3}{4}x_3$$

$$-\frac{1}{28}c_1 = \frac{9}{7}x_1 - \frac{2}{7}x_2 - \frac{3}{4}x_3$$

$$\underline{c_1 = -36x_1 + 8x_2 + 21x_3}$$

$$c_3 = \frac{1}{4}x_3 - \frac{1}{4}c_1$$

$$\underline{c_3 = 9x_1 - 2x_2 - 5x_3}$$

$$c_2 = \frac{1}{7}x_2 - \frac{1}{7}x_1 - \frac{1}{7}c_1$$

$$c_2 = \frac{1}{7}x_2 - \frac{1}{7}x_1 + \frac{36}{7}x_1 - \frac{8}{7}x_2 - 3x_3$$

$$\underline{c_2 = 5x_1 - x_2 - 3x_3}$$

$$\begin{aligned}
T(\vec{x}_1, \vec{x}_2, \vec{x}_3) &= (-36x_1 + 8x_2 + 21x_3)T(\vec{v}_1) + (5x_1 - x_2 - 3x_3)T(\vec{v}_2) \\
&\quad + (9x_1 - 2x_2 - 5x_3)T(\vec{v}_3) \\
&= (-36x_1 + 8x_2 + 21x_3)(1, 0) + (5x_1 - x_2 - 3x_3)(-1, 1) + (9x_1 - 2x_2 - 5x_3)(0, 1) \\
&= (36x_1 - 8x_2 + 21x_3 - 5x_1 + x_2 + 3x_3, 5x_1 - x_2 - 3x_3 + 9x_1 - 2x_2 - 5x_3) \\
&= (41x_1 + 9x_2 + 24x_3, 14x_1 - 3x_2 - 8x_3) \\
T(7, 13, 7) &= (37(7) - 13(13) + 24(7), 8(7) + 3(13) - 8(7)) \\
&= \underline{(-2, 3)}
\end{aligned}$$

Exercise

let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be vectors in a vector space V , and let $T: V \rightarrow \mathbb{R}^3$ be the linear transformation for which

$$T(\vec{v}_1) = (1, -1, 2), \quad T(\vec{v}_2) = (0, 3, 2), \quad T(\vec{v}_3) = (-3, 1, 2).$$

Find $T(2\vec{v}_1 - 3\vec{v}_2 + 4\vec{v}_3)$

Solution

$$\begin{aligned}
T(2\vec{v}_1 - 3\vec{v}_2 + 4\vec{v}_3) &= 2T(\vec{v}_1) - 3T(\vec{v}_2) + 4T(\vec{v}_3) \\
&= 2(1, -1, 2) - 3(0, 3, 2) + 4(-3, 1, 2) \\
&= (2, -2, 4) - (0, 9, 6) + (-12, 4, 8) \\
&= \underline{(-10, -7, 6)}
\end{aligned}$$

Exercise

Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear operation given by the formula $T(x, y) = (2x - y, -8x + 4y)$

Which of the following vectors are in $R(T)$

a) $(1, -4)$ b) $(5, 0)$ c) $(-3, 12)$

Solution

$$\begin{aligned}
a) \quad T(x, y) &= (2x - y, -8x + 4y) = (1, -4) \\
&\begin{cases} 2x - y = 1 \\ -8x + 4y = -4 \end{cases}
\end{aligned}$$

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ -8 & 4 & -4 \end{array} \right] \quad R_2 + 4R_1$$

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 0 & 0 & 0 \end{array} \right] \quad \frac{1}{2}R_1$$

$$\left[\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{array} \right]$$

This is a consistent system, therefore $(1, -4)$ is in $R(T)$

b) $T(x, y) = (2x - y, -8x + 4y) = (5, 0)$

$$\begin{cases} 2x - y = 5 \\ -8x + 4y = 0 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & -1 & 5 \\ -8 & 4 & 0 \end{array} \right] \quad R_2 + 4R_1$$

$$\left[\begin{array}{cc|c} 2 & -1 & 5 \\ 0 & 0 & 20 \end{array} \right] \rightarrow 0 \neq 20$$

This is an inconsistent system, therefore $(5, 0)$ is not in $R(T)$

c) $T(x, y) = (2x - y, -8x + 4y) = (-3, 12)$

$$\begin{cases} 2x - y = -3 \\ -8x + 4y = 12 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & -1 & -3 \\ -8 & 4 & 12 \end{array} \right] \quad R_2 + 4R_1$$

$$\left[\begin{array}{cc|c} 2 & -1 & -3 \\ 0 & 0 & 0 \end{array} \right] \quad \frac{1}{2}R_1$$

$$\left[\begin{array}{cc|c} 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \end{array} \right]$$

This is a consistent system, therefore $(-3, 12)$ is in $R(T)$

Exercise

Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear operation given by the formula $T(x, y) = (2x - y, -8x + 4y)$

Which of the following vectors are in $\ker(T)$

a) $(5, 10)$ b) $(3, 2)$ c) $(1, 1)$

Solution

$$\begin{aligned} \text{a) } T(5, 10) &= (10 - 10, -40 + 40) \\ &= \underline{(0, 0)} \end{aligned}$$

Therefore $(5, 10)$ is in $\ker(T)$

$$\begin{aligned} \text{b) } T(3, 2) &= (6 - 2, -24 + 8) \\ &= \underline{(4, -16)} \end{aligned}$$

Therefore $(3, 2)$ is not in $\ker(T)$

$$\begin{aligned} \text{c) } T(1, 1) &= (2 - 1, -8 + 4) \\ &= \underline{(1, -4)} \end{aligned}$$

Therefore $(1, 1)$ is not in $\ker(T)$

Exercise

Let $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be the linear operation given by the formula

$$T(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = (4x_1 + x_2 - 2x_3 - 3x_4, 2x_1 + x_2 + x_3 - 4x_4, 6x_1 - 9x_3 + 9x_4)$$

Which of the following vectors are in $R(T)$

a) $(0, 0, 6)$ b) $(1, 3, 0)$ c) $(2, 4, 1)$

Solution

$$\text{a) } T(x_1, x_2, x_3, x_4) = (4x_1 + x_2 - 2x_3 - 3x_4, 2x_1 + x_2 + x_3 - 4x_4, 6x_1 - 9x_3 + 9x_4) = (0, 0, 6)$$

$$\begin{cases} 4x_1 + x_2 - 2x_3 - 3x_4 = 0 \\ 2x_1 + x_2 + x_3 - 4x_4 = 0 \\ 6x_1 - 9x_3 + 9x_4 = 6 \end{cases}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 0 \\ 2 & 1 & 1 & -4 & 0 \\ 6 & 0 & -9 & 9 & 6 \end{array} \right] \quad \begin{array}{l} \\ 2R_2 - R_1 \\ 2R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 0 \\ 0 & 1 & 4 & -5 & 0 \\ 0 & -3 & -12 & 27 & 12 \end{array} \right] \quad \begin{array}{l} R_1 - R_2 \\ R_3 + 3R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 2 & 0 \\ 0 & 1 & 4 & -5 & 0 \\ 0 & 0 & 0 & 12 & 12 \end{array} \right] \quad \frac{1}{12}R_3$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 2 & 0 \\ 0 & 1 & 4 & -5 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \quad \begin{array}{l} R_1 - 2R_3 \\ R_2 + 5R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 0 & -2 \\ 0 & 1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \quad \frac{1}{4}R_1$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -\frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right]$$

This is a consistent system, therefore $(0, 0, 6)$ is in $R(T)$

b) $T(x_1, x_2, x_3, x_4) = (4x_1 + x_2 - 2x_3 - 3x_4, 2x_1 + x_2 + x_3 - 4x_4, 6x_1 - 9x_3 + 9x_4) = (1, 3, 0)$

$$\begin{cases} 4x_1 + x_2 - 2x_3 - 3x_4 = 1 \\ 2x_1 + x_2 + x_3 - 4x_4 = 3 \\ 6x_1 - 9x_3 + 9x_4 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 1 \\ 2 & 1 & 1 & -4 & 3 \\ 6 & 0 & -9 & 9 & 0 \end{array} \right] \quad \begin{array}{l} 2R_2 - R_1 \\ 2R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 1 \\ 0 & 1 & 4 & -5 & 5 \\ 0 & -3 & -12 & 27 & -3 \end{array} \right] \quad \begin{array}{l} R_1 - R_2 \\ R_3 + 3R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 2 & -4 \\ 0 & 1 & 4 & -5 & 5 \\ 0 & 0 & 0 & 12 & 12 \end{array} \right] \quad \frac{1}{12}R_3$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 2 & -4 \\ 0 & 1 & 4 & -5 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \quad \begin{array}{l} R_1 - 2R_3 \\ R_2 + 5R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 0 & -6 \\ 0 & 1 & 4 & 0 & 10 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \quad \frac{1}{4}R_1$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -\frac{3}{2} & 0 & -\frac{3}{2} \\ 0 & 1 & 4 & 0 & 10 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right]$$

This is a consistent system, therefore $(1, 3, 0)$ is in $R(T)$

c) $T(x_1, x_2, x_3, x_4) = (4x_1 + x_2 - 2x_3 - 3x_4, 2x_1 + x_2 + x_3 - 4x_4, 6x_1 - 9x_3 + 9x_4) = (2, 4, 1)$

$$\begin{cases} 4x_1 + x_2 - 2x_3 - 3x_4 = 2 \\ 2x_1 + x_2 + x_3 - 4x_4 = 4 \\ 6x_1 - 9x_3 + 9x_4 = 1 \end{cases}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 2 \\ 2 & 1 & 1 & -4 & 4 \\ 6 & 0 & -9 & 9 & 1 \end{array} \right] \quad \begin{array}{l} 2R_2 - R_1 \\ 2R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 2 \\ 0 & 1 & 4 & -5 & 6 \\ 0 & -3 & -12 & 27 & -4 \end{array} \right] \quad \begin{array}{l} R_1 - R_2 \\ R_3 + 3R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 2 & -4 \\ 0 & 1 & 4 & -5 & 6 \\ 0 & 0 & 0 & 12 & 14 \end{array} \right] \quad \frac{1}{12}R_3$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 2 & -4 \\ 0 & 1 & 4 & -5 & 6 \\ 0 & 0 & 0 & 1 & \frac{7}{6} \end{array} \right] \quad \begin{array}{l} R_1 - 2R_3 \\ R_2 + 5R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 0 & -\frac{19}{3} \\ 0 & 1 & 4 & 0 & \frac{71}{6} \\ 0 & 0 & 0 & 1 & \frac{7}{6} \end{array} \right] \quad \frac{1}{4}R_1$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -\frac{3}{2} & 0 & -\frac{19}{12} \\ 0 & 1 & 4 & 0 & \frac{71}{6} \\ 0 & 0 & 0 & 1 & \frac{7}{6} \end{array} \right]$$

This is a consistent system, therefore $(2, 4, 1)$ is in $R(T)$

Exercise

Let $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be the linear operation given by the formula

$$T(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = (4x_1 + x_2 - 2x_3 - 3x_4, 2x_1 + x_2 + x_3 - 4x_4, 6x_1 - 9x_3 + 9x_4)$$

Which of the following vectors are in $\ker(T)$

$$a) (3, -8, 2, 0) \quad b) (0, 0, 0, 1) \quad c) (0, -4, 1, 0)$$

Solution

$$a) \quad T(3, -8, 2, 0) = (12 - 8 - 4, 6 - 8 + 2, 18 - 18) \\ = (0, 0, 0)$$

Therefore, $(3, -8, 2, 0)$ is in $\ker(T)$

$$b) \quad T(0, 0, 0, 1) = (-3, -4, 9)$$

Therefore, $(0, 0, 0, 1)$ is **not** in $\ker(T)$

$$c) \quad T(0, -4, 1, 0) = (-4 - 2, -4 + 1, -9) \\ = (-6, -3, -9)$$

Therefore, $(0, -4, 1, 0)$ is **not** in $\ker(T)$

Exercise

Determine if the given function T is a linear transformation

$$T : M_{22} \rightarrow M_{22} \text{ by } T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2ab & 3cd \\ 0 & 0 \end{bmatrix}$$

Solution

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \text{ and } B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

$$\begin{aligned} T(A+B) &= T \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} \\ &= \begin{bmatrix} 2(a_1 + a_2)(b_1 + b_2) & 3(c_1 + c_2)(d_1 + d_2) \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2a_1b_1 + 2a_1b_2 + 2a_2b_1 + 2a_2b_2 & 3c_1d_1 + 3c_1d_2 + 3c_2d_1 + 3c_2d_2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2a_1b_1 & 3c_1d_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2a_2b_2 & 3c_2d_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2a_1b_2 + 2a_2b_1 & 3c_1d_2 + 3c_2d_1 \\ 0 & 0 \end{bmatrix} \\ &= T(A) + T(B) + \begin{bmatrix} 2a_1b_2 + 2a_2b_1 & 3c_1d_2 + 3c_2d_1 \\ 0 & 0 \end{bmatrix} \\ &\quad \neq T(A) + T(B) \end{aligned}$$

Function T is NOT a linear transformation.

Exercise

Determine if the given function T is a linear transformation

$$T : M_{22} \rightarrow M_{22} \text{ by } T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+d & 0 \\ 0 & b+c \end{bmatrix}$$

Solution

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \text{ and } B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

$$T(A+B) = T \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

$$\begin{aligned}
&= \begin{bmatrix} a_1 + a_2 + d_1 + d_2 & 0 \\ 0 & b_1 + b_2 + c_1 + c_2 \end{bmatrix} \\
&= \begin{bmatrix} a_1 + d_1 & 0 \\ 0 & b_1 + c_1 \end{bmatrix} + \begin{bmatrix} a_2 + d_2 & 0 \\ 0 & b_2 + c_2 \end{bmatrix} \\
&= T(A) + T(B) \quad \checkmark
\end{aligned}$$

$$\begin{aligned}
T(kA) &= T\left(k \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} ka_1 & kb_1 \\ kc_1 & kd_1 \end{bmatrix}\right) \\
&= \begin{bmatrix} ka_1 + kd_1 & 0 \\ 0 & kb_1 + kc_1 \end{bmatrix} \\
&= \begin{bmatrix} k(a_1 + d_1) & 0 \\ 0 & k(b_1 + c_1) \end{bmatrix} \\
&= k \begin{bmatrix} a_1 + d_1 & 0 \\ 0 & b_1 + c_1 \end{bmatrix} \\
&= kT(A) \quad \checkmark
\end{aligned}$$

Since $T(A + B) = T(A) + T(B)$ and $T(kA) = kT(A)$, then function T is a linear transformation.

Exercise

Determine if the given function T is a linear transformation where A is fixed 2×3 matrix

$$T : M_{22} \rightarrow M_{23} \text{ by } T(B) = BA$$

Solution

$$\begin{aligned}
T(B + C) &= (B + C)A \\
&= BA + CA \\
&= T(B) + T(C)
\end{aligned}$$

$$\begin{aligned}
T(rB) &= (rB)A \\
&= r(BA) \\
&= rT(B)
\end{aligned}$$

Function T is a linear transformation

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T ; if T is linear, find the A such $T = f_A$

$$T(x, y) = (x^2, y)$$

Solution

$$\text{Let } \vec{u} = (x_1, y_1) \text{ and } \vec{v} = (x_2, y_2)$$

$$\begin{aligned} T(\vec{u} + \vec{v}) &= T(x_1 + x_2, y_1 + y_2) \\ &= \left((x_1 + x_2)^2, y_1 + y_2 \right) \\ &= \left(x_1^2 + x_2^2 + 2x_1x_2, y_1 + y_2 \right) \\ &= \left(x_1^2, y_1 \right) + \left(x_2^2, y_2 \right) + (2x_1x_2, 0) \\ &= T(\vec{u}) + T(\vec{v}) + (2x_1x_2, 0) \\ &\neq T(\vec{u}) + T(\vec{v}) \end{aligned}$$

The function T is **not** a linear transformation.

Domain: $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

Exercise

Determine if the given function T is a linear transformation. Also, give the domain and range of T ; if T is linear, find the A such $T = f_A$.

$$T(x, y, z) = (2x + y, x - y + z)$$

Solution

$$\text{Let } \vec{u} = (x_1, y_1, z_1) \text{ and } \vec{v} = (x_2, y_2, z_2)$$

$$\begin{aligned} T(\vec{u} + \vec{v}) &= T(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= \left(2(x_1 + x_2) + y_1 + y_2, x_1 + x_2 - (y_1 + y_2) + z_1 + z_2 \right) \\ &= \left(2x_1 + y_1 + 2x_2 + y_2, x_1 - y_1 + z_1 + x_2 - y_2 + z_2 \right) \\ &= \left(2x_1 + y_1, x_1 - y_1 + z_1 \right) + \left(2x_2 + y_2, x_2 - y_2 + z_2 \right) \\ &= T(x_1, y_1, z_1) + T(x_2, y_2, z_2) \end{aligned}$$

$$= T(\vec{u}) + T(\vec{v})$$

$$\begin{aligned} T(r\vec{u}) &= T(rx_1, ry_1, rz_1) \\ &= (2rx_1 + ry_1, rx_1 - ry_1 + rz_1) \\ &= r(2x_1 + y_1, x_1 - y_1 + z_1) \\ &= rT(\vec{u}) \end{aligned}$$

Since $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(r\vec{u}) = rT(\vec{u})$, then function T is a linear transformation.

Domain: $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$$\begin{aligned} T(x, y, z) &= (2x + y, x - y + z) \\ &= \begin{pmatrix} 2x + y \\ x - y + z \end{pmatrix} \end{aligned}$$

$$A = \begin{matrix} & \begin{matrix} x & y & z \end{matrix} \\ \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \end{matrix}$$

Exercise

Determine if the given function T is a linear transformation. Also, give the domain and range of T ; if T is linear, find the A such $T = f_A$.

$$T(x, y, z) = (z - x, z - y)$$

Solution

Let $\vec{u} = (x_1, y_1, z_1)$ and $\vec{v} = (x_2, y_2, z_2)$

$$\begin{aligned} T(\vec{u} + \vec{v}) &= T(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= (z_1 + z_2 - (x_1 + x_2), z_1 + z_2 - (y_1 + y_2)) \\ &= (z_1 + z_2 - x_1 - x_2, z_1 + z_2 - y_1 - y_2) \\ &= (z_1 - x_1, z_1 - y_1) + (z_2 - x_2, z_2 - y_2) \\ &= T(x_1, y_1, z_1) + T(x_2, y_2, z_2) \\ &= T(\vec{u}) + T(\vec{v}) \end{aligned}$$

$$T(r\vec{u}) = T(rx_1, ry_1, rz_1)$$

$$\begin{aligned}
&= (rz_1 - rx_1, rz_1 - ry_1) \\
&= r(z_1 - x_1, z_1 - y_1) \\
&= rT(\vec{u})
\end{aligned}$$

Since $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(r\vec{u}) = rT(\vec{u})$, then function T is a linear transformation.

Domain: $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$$T(x, y, z) = (z - x, z - y)$$

$$= \begin{pmatrix} -x + z \\ -y + z \end{pmatrix}$$

$$A = \begin{matrix} & \begin{matrix} x & y & z \end{matrix} \\ \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \end{matrix}$$

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T ; if T is linear, find the A such $T = f_A$

$$T(x_1, x_2, x_3) = (2x_1 - x_2 + x_3, x_2 - 4x_3)$$

Solution

Let $\vec{u} = (x_1, x_2, x_3)$ and $\vec{v} = (y_1, y_2, y_3)$

$$\begin{aligned}
T(\vec{u} + \vec{v}) &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\
&= (2(x_1 + y_1) - x_2 - y_2 + x_3 + y_3, x_2 + y_2 - 4(x_3 + y_3)) \\
&= (2x_1 + 2y_1 - x_2 - y_2 + x_3 + y_3, x_2 + y_2 - 4x_3 - 4y_3) \\
&= (2x_1 - x_2 + x_3, x_2 - 4x_3) + (2y_1 - y_2 + y_3, y_2 - 4y_3) \\
&= T(\vec{u}) + T(\vec{v})
\end{aligned}$$

$$\begin{aligned}
T(r\vec{u}) &= T(rx_1, rx_2, rx_3) \\
&= (2rx_1 - rx_2 + rx_3, rx_2 - 4rx_3) \\
&= r(2x_1 - x_2 + x_3, x_2 - 4x_3) \\
&= rT(\vec{u})
\end{aligned}$$

Since $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(r\vec{u}) = rT(\vec{u})$, then function T is a linear transformation.

Domain: $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$$\begin{aligned} T(x_1, x_2, x_3) &= (2x_1 - x_2 + x_3, x_2 - 4x_3) \\ &= \begin{pmatrix} 2x_1 - x_2 + x_3 \\ x_2 - 4x_3 \end{pmatrix} \end{aligned}$$

$$A = \begin{pmatrix} \overset{x_1}{2} & \overset{x_2}{-1} & \overset{x_3}{1} \\ 0 & -1 & -4 \end{pmatrix}$$

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T ; if T is linear, find the A such $T = f_A$

$$T(x_1, x_2) = (2x_1 - x_2, -3x_1 + x_2, 2x_1 - 3x_2)$$

Solution

$$\text{Let } \vec{u} = (u_1, u_2) \text{ and } \vec{v} = (v_1, v_2)$$

$$\begin{aligned} T(\vec{u} + \vec{v}) &= T(u_1 + v_1, u_2 + v_2) \\ &= (2(u_1 + v_1) - (u_2 + v_2), -3(u_1 + v_1) + (u_2 + v_2), 2(u_1 + v_1) - 3(u_2 + v_2)) \\ &= (2u_1 + 2v_1 - u_2 - v_2, -3u_1 - 3v_1 + u_2 + v_2, 2u_1 + 2v_1 - 3u_2 - 3v_2) \\ &= ((2u_1 - u_2) + (2v_1 - v_2), (-3u_1 + u_2) + (-3v_1 + v_2), (2u_1 - 3u_2) + (2v_1 - 3v_2)) \\ &= (2u_1 - u_2, -3u_1 + u_2, 2u_1 - 3u_2) + (2v_1 - v_2, -3v_1 + v_2, 2v_1 - 3v_2) \\ &= T(\vec{u}) + T(\vec{v}) \end{aligned}$$

$$\begin{aligned} T(r\vec{u}) &= T(r(u_1, u_2)) \\ &= T(ru_1, ru_2) \\ &= (2ru_1 - ru_2, -3ru_1 + ru_2, 2ru_1 - 3ru_2) \\ &= r(2u_1 - u_2, -3u_1 + u_2, 2u_1 - 3u_2) \\ &= rT(\vec{u}) \end{aligned}$$

Since $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(r\vec{u}) = rT(\vec{u})$, then function T is a linear transformation.

Domain: $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$

$$\begin{aligned} T(x_1, x_2) &= (2x_1 - x_2, -3x_1 + x_2, 2x_1 - 3x_2) \\ &= \begin{pmatrix} 2x_1 - x_2 \\ -3x_1 + x_2 \\ 2x_1 - 3x_2 \end{pmatrix} \end{aligned}$$

$$A = \begin{pmatrix} \overset{x_1}{2} & \overset{x_2}{-1} \\ -3 & 1 \\ 2 & -3 \end{pmatrix}$$

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T ; if T is linear, find the A such $T = f_A$

$$T(x_1, x_2) = (x_1 + 4x_2, 0, x_1 - 3x_2, x_1)$$

Solution

$$\text{Let } \vec{u} = (u_1, u_2) \text{ and } \vec{v} = (v_1, v_2)$$

$$\begin{aligned} T(\vec{u} + \vec{v}) &= T(u_1 + v_1, u_2 + v_2) \\ &= ((u_1 + v_1) + 4(u_2 + v_2), 0, (u_1 + v_1) - 3(u_2 + v_2), (u_1 + v_1)) \\ &= (u_1 + v_1 + 4u_2 + 4v_2, 0, u_1 + v_1 - 3u_2 - 3v_2, u_1 + v_1) \\ &= ((u_1 + 4u_2) + (v_1 + 4v_2), 0, (u_1 - 3u_2) + (v_1 - 3v_2), u_1 + v_1) \\ &= (u_1 + 4u_2, 0, u_1 - 3u_2, u_1) + (v_1 + 4v_2, 0, v_1 - 3v_2, v_1) \\ &= T(\vec{u}) + T(\vec{v}) \end{aligned}$$

$$\begin{aligned} T(r\vec{u}) &= T(r(u_1, u_2)) \\ &= T(ru_1, ru_2) \\ &= (ru_1 + 4ru_2, 0, ru_1 - 3ru_2, ru_1) \\ &= r(u_1 + 4u_2, 0, u_1 - 3u_2, u_1) \\ &= rT(\vec{u}) \end{aligned}$$

Since $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(r\vec{u}) = rT(\vec{u})$, then function T is a linear transformation.

Domain: $T: \mathbb{R}^2 \rightarrow \mathbb{R}^4$

$$\begin{aligned} T(x_1, x_2) &= (x_1 + 4x_2, 0, x_1 - 3x_2, x_1) \\ &= \begin{pmatrix} x_1 + 4x_2 \\ 0 \\ x_1 - 3x_2 \\ x_1 \end{pmatrix} \end{aligned}$$

$$A = \begin{pmatrix} \overset{x_1}{1} & \overset{x_2}{4} \\ 0 & 0 \\ 1 & -3 \\ 1 & 0 \end{pmatrix}$$

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T ; if T is linear, find the A such $T = f_A$

$$T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$$

Solution

$$\text{Let } \vec{u} = (u_1, u_2, u_3) \text{ and } \vec{v} = (v_1, v_2, v_3)$$

$$\begin{aligned} T(\vec{u} + \vec{v}) &= T(u_1 + v_1, u_2 + v_2, u_3 + v_3) \\ &= ((u_1 + v_1) - 5(u_2 + v_2) + 4(u_3 + v_3), (u_2 + v_2) - 6(u_3 + v_3)) \\ &= (u_1 + v_1 - 5u_2 - 5v_2 + 4u_3 + 4v_3, u_2 + v_2 - 6u_3 - 6v_3) \\ &= ((u_1 - 5u_2 + 4u_3) + (v_1 - 5v_2 + 4v_3), (u_2 - 6u_3) + (v_2 - 6v_3)) \\ &= (u_1 - 5u_2 + 4u_3, u_2 - 6u_3) + (v_1 - 5v_2 + 4v_3, v_2 - 6v_3) \\ &= T(\vec{u}) + T(\vec{v}) \end{aligned}$$

$$\begin{aligned} T(r\vec{u}) &= T(r(u_1, u_2, u_3)) \\ &= T(ru_1, ru_2, ru_3) \\ &= (ru_1 - 5ru_2 + 4ru_3, ru_2 - 6ru_3) \end{aligned}$$

$$\begin{aligned}
&= r(u_1 - 5u_2 + 4u_3, u_2 - 6u_3) \\
&= rT(\vec{u})
\end{aligned}$$

Since $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(r\vec{u}) = rT(\vec{u})$, then function T is a linear transformation.

Domain: $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$$\begin{aligned}
T(x_1, x_2, x_3) &= (x_1 - 5x_2 + 4x_3, x_2 - 6x_3) \\
&= \begin{pmatrix} x_1 - 5x_2 + 4x_3 \\ x_2 - 6x_3 \end{pmatrix}
\end{aligned}$$

$$A = \begin{pmatrix} \overset{x_1}{1} & \overset{x_2}{-5} & \overset{x_3}{4} \\ 0 & 1 & -6 \end{pmatrix}$$

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T ; if T is linear, find the A such $T = f_A$

$$T(x_1, x_2, x_3, x_4) = (x_1 + 2x_2, 0, 2x_2 + x_4, x_2 - x_4)$$

Solution

$$\text{Let } \vec{u} = (u_1, u_2, u_3, u_4) \text{ and } \vec{v} = (v_1, v_2, v_3, v_4)$$

$$\begin{aligned}
T(\vec{u} + \vec{v}) &= T(u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4) \\
&= ((u_1 + v_1) + 2(u_2 + v_2), 0, 2(u_2 + v_2) - 3(u_4 + v_4), (u_2 + v_2) - (u_4 + v_4)) \\
&= (u_1 + v_1 + 2u_2 + 2v_2, 0, 2u_2 + 2v_2 - 3u_4 - 3v_4, u_2 + v_2 - u_4 - v_4) \\
&= ((u_1 + 2u_2) + (v_1 + 2v_2), 0, (2u_2 - 3u_4) + (2v_2 - 3v_4), (u_2 - u_4) + (v_2 - v_4)) \\
&= (u_1 + 2u_2, 0, 2u_2 - 3u_4, u_2 - u_4) + (v_1 + 2v_2, 0, 2v_2 - 3v_4, v_2 - v_4) \\
&= T(\vec{u}) + T(\vec{v})
\end{aligned}$$

$$\begin{aligned}
T(r\vec{u}) &= T(r(u_1, u_2, u_3, u_4)) \\
&= T(ru_1, ru_2, ru_3, ru_4) \\
&= (ru_1 + 2ru_2, 0, 2ru_2 - 3ru_4, ru_2 - ru_4)
\end{aligned}$$

$$\begin{aligned}
&= r(u_1 + 2u_2, 0, 2u_2 - 3u_4, u_2 - u_4) \\
&= rT(\vec{u})
\end{aligned}$$

Since $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(r\vec{u}) = rT(\vec{u})$, then function T is a linear transformation.

Domain: $T: \mathbb{R}^4 \rightarrow \mathbb{R}^4$

$$\begin{aligned}
T(x_1, x_2, x_3, x_4) &= (x_1 + 2x_2, 0, 2x_2 + x_4, x_2 - x_4) \\
&= \begin{pmatrix} x_1 + 2x_2 \\ 0 \\ 2x_2 + x_4 \\ x_2 - x_4 \end{pmatrix}
\end{aligned}$$

$$A = \begin{matrix} & \begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix} \\ \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \end{matrix}$$

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T ; if T is linear, find the A such $T = f_A$

$$T(x_1, x_2, x_3, x_4) = 3x_1 + 4x_3 - 2x_4$$

Solution

$$\text{Let } \vec{u} = (u_1, u_2, u_3, u_4) \text{ and } \vec{v} = (v_1, v_2, v_3, v_4)$$

$$\begin{aligned}
T(\vec{u} + \vec{v}) &= T(u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4) \\
&= 3(u_1 + v_1) + 4(u_3 + v_3) - 2(u_4 + v_4) \\
&= 3u_1 + 3v_1 + 4u_3 + 4v_3 - 2u_4 - 2v_4 \\
&= (3u_1 + 4u_3 - 2u_4) + (3v_1 + 4v_3 - 2v_4) \\
&= T(\vec{u}) + T(\vec{v})
\end{aligned}$$

$$\begin{aligned}
T(r\vec{u}) &= T(r(u_1, u_2, u_3, u_4)) \\
&= T(ru_1, ru_2, ru_3, ru_4)
\end{aligned}$$

$$\begin{aligned}
&= 3ru_1 + 4ru_3 - 2ru_4 \\
&= r(3u_1 + 4u_3 - 2u_4) \\
&= rT(\vec{u})
\end{aligned}$$

Since $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(r\vec{u}) = rT(\vec{u})$, then function T is a linear transformation.

Domain: $T: \mathbb{R}^4 \rightarrow \mathbb{R}^1$

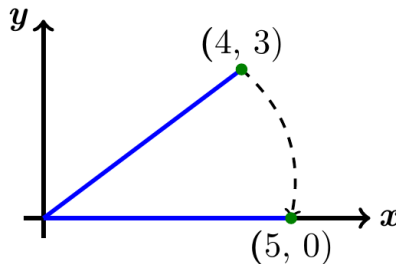
$$\begin{aligned}
T(x_1, x_2, x_3, x_4) &= 3x_1 + 4x_3 - 2x_4 \\
&= (3x_1 + 4x_3 - 2x_4)
\end{aligned}$$

$$A = \begin{pmatrix} \overset{x_1}{3} & \overset{x_2}{0} & \overset{x_3}{4} & \overset{x_4}{-2} \end{pmatrix}$$

Exercise

A Givens rotation is a linear transformation from \mathbb{R}^n to \mathbb{R}^n used in computer to create a zero entry in a vector (usually a column of a matrix). The standard matrix of a Givens rotation in \mathbb{R}^2 has the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}; \quad a^2 + b^2 = 1$$



A Givens rotation in \mathbb{R}^2

Find a and b that $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ is rotated into $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$.

Solution

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$$\begin{cases} 4a - 3b = 5 & (1) \\ 4b + 3a = 0 \end{cases} \rightarrow b = -\frac{3a}{4}$$

$$(1) \rightarrow 4a - 3\left(-\frac{3a}{4}\right) = 5$$

$$4a + \frac{9a}{4} = 5$$

$$\left(4 + \frac{9}{4}\right)a = 5$$

$$\frac{25}{4}a = 5$$

$$\underline{a = \frac{4}{5}}$$

$$b = -\frac{3}{4} \frac{4}{5}$$

$$\underline{b = -\frac{3}{5}}$$

$$A = \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix}$$