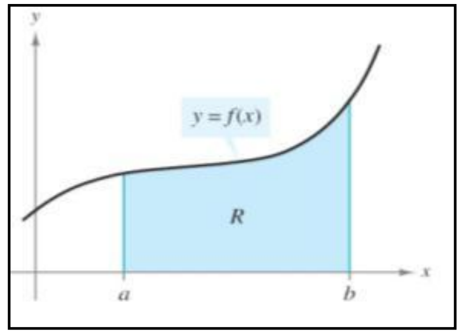
Section 4.4 – Area and the Fundamental Theorem of Calculus

Area and Definite Integrals

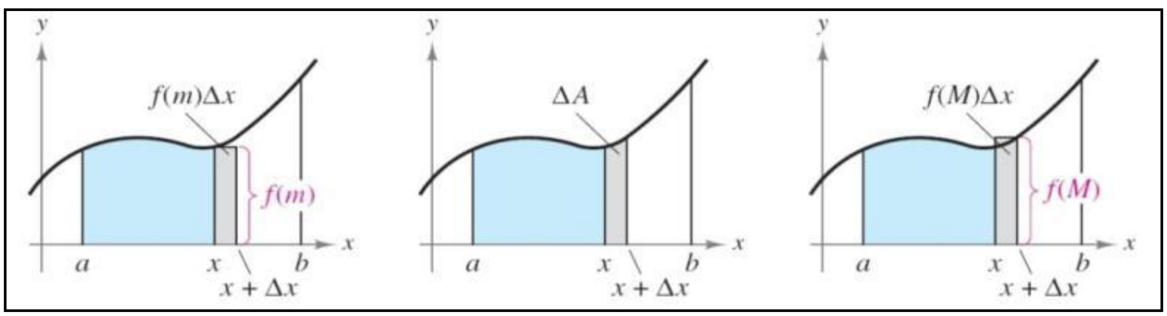
Definition of a Definite Integral

Let f be nonnegative and continuous on the closed interval $[a, b]$. The area of the region bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is denoted by

$$Area = \int_a^b f(x)dx$$



The expression $\int_a^b f(x)dx$ is called the definite integral from a to b, where a is the ***lower limit of integration*** and b is the ***upper limit of integration***.



The Fundamental Theorem of Calculus

If f is nonnegative and continuous on the closed interval $[a, b]$, then

$$\int_{\color{blue}a}^{\color{red}b} f(x)dx = F({\color{red}b}) - F({\color{blue}a})$$

Where F is any function such that $F'(x) = f(x)$ for all x in $[a, b]$.

Guidelines for Using the Fundamental Theorems of Calculus

1. The Fundamental Theorem of Calculus describes a way of evaluating a definite integral, not a procedure for finding anti-derivatives.

2. In applying the Fundamental Theorem, it is helpful to use the notation

$$\int_a^b f(x)dx = F(x)\Big]_a^b = F(b) - F(a)$$

3. The constant of integration C can be dropped because

$$\int_a^b f(x)dx = \left[F(x)+C\right]_a^b = F(b) - F(a)$$

$$= \left[F(b)+C\right] - \left[F(a)+C\right]$$

$$= F(b) + C - F(a) - C$$

$$= F(b) - F(a)$$

Properties of Definite Integrals

Let *f* and *g* be continuous on the closed interval [*a*, *b*].

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx\,, \qquad k \text{ is a constant}$$

$$\int_a^b \left[f(x) \pm g(x)\right]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \qquad a < c < b$$

$$\int_a^a f(x)dx = 0$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Example

Evaluate: $\int_0^3 4x\,dx$

<u>*Solution*</u>

$$\int_0^3 4x\,dx = 4\frac{1}{2}x^2\Big|_0^3$$

$$= 2x^2\Big|_0^3$$

$$= 2\left[3^2 - 0^2\right]$$

$$= 2(9)$$

$$\boxed{= 18}$$

Example

Find the area of the region bounded by the x-axis and the graph of $f(x) = x^2 + 1, \quad 2 \le x \le 3$

<u>*Solution*</u>

$$\int_2^3 (x^2+1)\,dx = \left[\frac{1}{3}x^3 + x\right]_2^3$$

$$= \left(\frac{1}{3}3^3 + 3\right) - \left(\frac{1}{3}2^3 + 2\right)$$

$$= (9+3) - \left(\frac{8}{3} + 2\right)$$

$$= 12 - \left(\frac{14}{3}\right)$$

$$= \frac{22}{3}$$

$$\boxed{= 7.3}$$

Example

Evaluate: $\int_0^1 (2t+3)^3 dt$

<u>Solution</u>

$$u = 2t+3 \Rightarrow du = 2dt \to \frac{du}{2} = dt$$

$$\int_0^1 (2t+3)^3 dt = \int_0^1 u^3 \frac{1}{2} du$$

$$= \frac{1}{2}\int_0^1 u^3 du$$

$$= \frac{1}{2}\frac{u^4}{4}\Big|_0^1$$

$$= \frac{1}{8}(2t+3)^4\Big|_0^1$$

$$= \frac{1}{8}\left[(2(1)+3)^4 - (2(0)+3)^4\right]$$

$$= \frac{1}{8}\left[5^4 - 3^4\right]$$

$$= 68$$

Example

Evaluate:

a) $\int_0^1 e^{4x} dx$

<u>Solution</u>

$$\int_0^1 e^{4x} dx = \frac{1}{4}e^{4x}\Big|_0^1$$

$$= \frac{1}{4}\left[e^4 - e^0\right]$$

$$= \frac{1}{4}\left[e^4 - 1\right]$$

$$\underline{\approx 13.4}$$

b) $\displaystyle\int_2^5 -\frac{1}{x}\,dx$

<u>*Solution*</u>

$$\int_2^5 -\frac{1}{x}\,dx = -\ln x\Big|_2^5$$

$$= -(\ln 5 - \ln 2)$$

$$\boxed{\approx -0.916}$$

Example

Evaluate: $\displaystyle\int_0^5 |x-2|\,dx$

<u>*Solution*</u>

$$|x-2| = \begin{cases} x-2 & x>2 \\ -(x-2) & x<2 \end{cases}$$

$$\int_0^5 |x-2|\,dx = \int_0^2 -(x-2)dx + \int_2^5 (x-2)dx$$

$$= -\frac{x^2}{2} + 2x\Big|_0^2 + \left(\frac{x^2}{2} - 2x\right)\Big|_2^5$$

$$= -\frac{4}{2} + 4 - 0 + \left(\frac{25}{2} - 10 - (\frac{4}{2} - 4)\right)$$

$$= -2 + 4 + \frac{25}{2} - 10 - 2 + 4$$

$$= \frac{25}{2} - 6$$

$$\boxed{= \frac{13}{2}}$$

Marginal Analysis

Example

The marginal profit for a product is modeled by $\frac{dP}{dx} = -0.0002x + 14.2$

a. Find the change in profit when sales increase from 100 to 101 units.
b. Find the change in profit when sales increase from 100 to 110 units.

<u>*Solution*</u>

a. $$\int_{100}^{101} \frac{dP}{dx}dx = \int_{100}^{101} (-0.0002x + 14.2)dx$$

$$= -\frac{0.0002}{2}x^2 + 14.2x \Big|_{100}^{101}$$

$$= -\frac{0.0002}{2}101^2 + 14.2(101) - \left[-\frac{0.0002}{2}100^2 + 14.2(100)\right]$$

$= \$14.18$

b. $$\int_{100}^{110} \frac{dP}{dx}dx = \int_{100}^{110} (-0.0002x + 14.2)dx$$

$$= -\frac{0.0002}{2}x^2 + 14.2x \Big|_{100}^{110}$$

$$= -\frac{0.0002}{2}110^2 + 14.2(110) - \left[-\frac{0.0002}{2}100^2 + 14.2(100)\right]$$

$\approx \$141.79$

Average Value

Definition of the Average Value of a Definition

If f is continuous on $[a, b]$, then the average value of f on $[a, b]$.

$$\text{Average Value of } f \text{ on } [a, b] = \frac{1}{b-a}\int_a^b f(x)dx$$

Example

Find the average cost per unit over a two-year period if the cost per unit c of roller blades is given by $c = 0.005t^2 + 0.02t + 12.5$, for $0 \le t \le 24$, where t is the time in months.

Solution

$$\text{Average cost} = \frac{1}{24-0}\int_0^{24}(0.005t^2 + 0.02t + 12.5)dt$$

$$= \frac{1}{24}\left[\frac{0.005}{3}t^3 + \frac{0.02}{2}t^2 + 12.5t\right]_0^{24}$$

$$= \frac{1}{24}\left[\left(\frac{0.005}{3}24^3 + \frac{0.02}{2}24^2 + 12.5(24)\right) - 0\right]$$

$$\approx 13.7$$

Even and Odd Function

Integration of Even and Odd Functions

1. If f is an *even* function, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

2. If f is an *odd* function, then $\int_{-a}^{a} f(x)dx = 0$

Example

Evaluate each definite integral

a) $\int_{-1}^{1} x^4 dx$

$$\int_{-1}^{1} x^4 dx = 2\int_{0}^{1} x^4 dx$$

$$= 2\left.\frac{x^5}{5}\right|_0^1$$

$$= 2\left[\frac{1^5}{5} - 0\right]$$

$$= \frac{2}{5}$$

b) $\int_{-1}^{1} x^5 dx = 0$

$$or \int_{-1}^{1} x^5 dx = \left.\frac{x^6}{6}\right|_{-1}^{1} = \frac{1^6}{6} - \frac{(-1)^6}{6} = \frac{1^6}{6} - \frac{1^6}{6} = 0$$

Annuity

Amount of an Annuity

If ***c*** represents a continuous income function in dollars per year (where ***t*** is the time in years), ***r*** represents the interest rate compounded continuously and **T** represents the term of the annuity in years, then the amount of an annuity is

$$\text{Amount of an annuity} = e^{rT}\int_0^T c(t)e^{-rt}\,dt$$

Example

If you deposit $1000 in a savings account every year, paying 4% interest, how much will be in the account after 10 years?

Solution

$$c(t) = 1000$$

$$\begin{aligned}\text{Amount of an annuity} &= e^{rT}\int_0^T c(t)e^{-rt}\,dt\\
&= e^{(0.04)(10)}\int_0^{10} 1000e^{-0.04t}\,dt\\
&= 1000e^{0.4}\left[-\frac{e^{-0.04t}}{0.04}\right]_0^{10}\\
&= 1000e^{0.4}\left[-\frac{e^{-0.4}}{0.04}-\left(-\frac{e^{0}}{0.04}\right)\right]\\
&\approx \$\ 12{,}295.62\end{aligned}$$

Exercise *Section* 4.4 – Area and the Fundamental Theorem of Calculus

Evaluate each integral

1. $\displaystyle\int_0^3 (2x+1)dx$

2. $\displaystyle\int_{-1}^4 |x-2|\,dx$

3. $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$

4. $\displaystyle\int_0^1 x^2e^x dx$

5. $\displaystyle\int_0^2 x(x-3)\,dx$

6. $\displaystyle\int_0^4 \left(3x-\frac{x^3}{4}\right)dx$

7. $\displaystyle\int_{-2}^2 \left(x^3-2x+3\right)dx$

8. $\displaystyle\int_0^1 \left(x^2+\sqrt{x}\right)dx$

9. $\displaystyle\int_{-3}^{-1} \frac{y^5-2y}{y^3}\,dy$

10. $\displaystyle\int_1^8 \frac{\left(x^{1/3}+1\right)\left(2-x^{2/3}\right)}{x^{1/3}}\,dx$

11. $\displaystyle\int_0^3 \sqrt{y+1}\,dy$

12. $\int_{-1}^{1} r\sqrt{1-r^2}\ dr$

13. $\int_{0}^{1} \mathrm{t}^3\left(1+t^4\right)^3\ dt$

14. A company manufactures *x* HDTVs per month. The monthly marginal profit (in dollars) is given by

$$P'(x) = 165 - 0.1x \qquad 0 \le x \le 4{,}000$$

The company is currently manufacturing 1,500 HDTVs per month, but is planning to increase production. Find the change in the monthly profit if monthly production is increased to 1,600 HDTVs.

15. An amusement company maintains records for each video game installed in an arcade. Suppose that $C(t)$ and $R(t)$ represent the total accumulated costs and revenues (in thousands of dollars), respectively, *t* years after a particular game has been installed. Suppose also that

$$C'(t) = 2 \quad R'(t) = 9e^{-0.5t}$$

The value of *t* for which $C'(t) = R'(t)$ is called the ***useful life*** of the game.

a) Find the useful life of the game, to the nearest year.
b) Find the total profit accumulated during the useful life of the game.

16. The total cost (in dollars) of printing x dictionaries is $C(x) = 20{,}000 + 10x$

a) Find the average cost per unit if 1,000 dictionaries are produced.
b) Find the average value of the cost function over the interval [0, 1,000}
c) Discuss the difference between parts (***a***) and (***b***)

17. If the rate of labor is $g(x) = 2{,}000\ x^{-1/3}$, then approximately how many labor–hours will be required to assemble the 9th through the 27th.