SOLUTION *Section* 3.6 – Alternating Series, Absolute and Conditional Convergence

Exercise

Determine if the alternating series converges or diverges $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$

Solution

$n \ge 1 \Rightarrow n+1 \ge n$

$\sqrt{n+1} \ge \sqrt{n}$

$\frac{1}{\sqrt{n+1}} \le \frac{1}{\sqrt{n}} \Rightarrow u_{n+1} \le u_n$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$

The series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$ ***converges*** by Alternating Convergence Test.

Exercise

Determine if the alternating series converges or diverges $\sum_{n=2}^{\infty}(-1)^{n}\frac{4}{(\ln n)^2}$

Solution

$n \ge 1 \Rightarrow n+1 \ge n$

$\ln(n+1) \ge \ln$

$(\ln(n+1))^2 \ge (\ln)^2$

$\frac{1}{(\ln(n+1))^2} \le \frac{1}{(\ln n)^2}$

$\frac{4}{(\ln(n+1))^2} \le \frac{4}{(\ln n)^2} \quad \Rightarrow \quad u_{n+1} \le u_n$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{(\ln n)^2} = 0$

The series $\sum_{n=2}^{\infty}(-1)^{n}\frac{4}{(\ln n)^2}$ ***converges*** by Alternating Series Test.

Exercise

Determine if the alternating series converges or diverges $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1}$

Solution

$$n \geq 1 \Rightarrow n^2+n \geq n^2+n+1$$

$$2n^2+2n \geq n^2+n+1$$

$$n^3+2n^2+2n \geq n^3+n^2+n+1$$

$$n\left(n^2+2n+2\right) \geq \left(n^2+1\right)(n+1)$$

$$n\left((n+1)^2+1\right) \geq \left(n^2+1\right)(n+1)$$

$$\frac{n}{n^2+1} \geq \frac{n+1}{(n+1)^2+1} \quad \Rightarrow \quad u_n \geq u_{n+1}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n^2+1} = 0$$

The series $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1}$ ***converges*** by Alternating Series Test.

Exercise

Determine if the alternating series converges or diverges $\sum_{n=1}^{\infty}(-1)^n \frac{n^2+5}{n^2+4}$

Solution

$$\lim_{n\to\infty} \frac{n^2+5}{n^2+4} = 1 \quad \Rightarrow \quad \lim_{n\to\infty} (-1)^n \frac{n^2+5}{n^2+4} = \textit{doesn't exist}$$

The given series ***diverges*** by nth Term Test for Divergence.

Exercise

Determine if the alternating series converges or diverges $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{10}\right)^n$

Solution

$$\lim_{n\to\infty} \left(\frac{n}{10}\right)^n \neq 0 \quad \Rightarrow \quad \lim_{n\to\infty} (-1)^{n+1}\left(\frac{n}{10}\right)^n \quad \textit{diverges}$$

The given series ***diverges*** by nth Term Test for Divergence.

Exercise

Determine if the alternating series converges or diverges $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}$

Solution

$$f(x)=\frac{\sqrt{x}+1}{x+1} \quad\Rightarrow\quad f'(x)=\frac{\frac{1}{2}x^{-1/2}(x+1)-(1)\left(\sqrt{x}+1\right)}{(x+1)^2}$$

$$=\frac{x+1-2\sqrt{x}\left(\sqrt{x}+1\right)}{2\sqrt{x}(x+1)^2}$$

$$=\frac{x+1-2x-2\sqrt{x}}{2\sqrt{x}(x+1)^2}$$

$$=\frac{1-x-2\sqrt{x}}{2\sqrt{x}(x+1)^2}<0 \quad\to\quad f(x) \ \textit{is decreasing}$$

$u_n \geq u_{n+1} \quad \Rightarrow$ The given series ***converges*** by Alternating Series Test.

Exercise

Determine if the alternating series converges or diverges $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+1}$

Solution

$$u_{n+1}=\frac{1}{n+2}<\frac{1}{n+1}=u_n$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{n+2}$$

$$=0$$

Therefore, the given series ***converges*** by Alternating Series Test.

Exercise

Determine if the alternating series converges or diverges $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{3n+2}$

Solution

$$\lim_{n\to\infty}\frac{n}{3n+2}=\frac{1}{3}<1$$

Therefore, the given series ***converges*** by n*th*-Term Test.

Exercise

Determine if the alternating series converges or diverges $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n}$

Solution

$$u_{n+1} = \frac{1}{3^{n+1}} < \frac{1}{3^n} = u_n$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{3^n}$$

$$= 0$$

Therefore, the given series ***converges*** by Alternating Series Test. (*Geometric series too*)

Exercise

Determine if the alternating series converges or diverges $\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n}$

Solution

$$u_{n+1} = \frac{1}{e^{n+1}} < \frac{1}{e^n} = u_n$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{e^n}$$

$$= 0$$

Therefore, the given series ***converges*** by Alternating Series Test. (*Geometric series too*)

Exercise

Determine if the alternating series converges or diverges $\sum_{n=1}^{\infty} (-1)^n \frac{5n-1}{4n+1}$

Solution

$$\lim_{n\to\infty} \frac{5n-1}{4n+1} = \frac{5}{4} > 1$$

Therefore, the given series ***diverges*** by n*th*-Term Test.

Exercise

Determine if the alternating series converges or diverges $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+5}$

Solution

$(n+1)^2+5>n^2+5$ ***or*** $f(x)=\dfrac{x}{x^2+5}$

$\dfrac{1}{(n+1)^2+5}<\dfrac{1}{n^2+5}$ $f'(x)=-\dfrac{x^2-5}{\left(x^2+5\right)^2}<0$ *for* $x\ge 3$

$u_{n+1}=\dfrac{n+1}{(n+1)^2+5}<\dfrac{n}{n^2+5}=u_n$

$\lim\limits_{n\to\infty}\dfrac{n}{n^2+5}\ \underline{=0|}$

Therefore, the given series ***converges*** by Alternating Series Test.

Exercise

Determine if the alternating series converges or diverges $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n}{\ln(n+1)}$

Solution

$\lim\limits_{n\to\infty}\dfrac{n}{\ln(n+1)}\ \underline{=\infty|}$

Therefore, the given series ***diverges*** by n*th*-Term Test.

Exercise

Determine if the alternating series converges or diverges $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\ln(n+1)}$

Solution

$u_{n+1}=\dfrac{1}{\ln(n+2)}<\dfrac{1}{\ln(n+1)}=u_n$

$\lim\limits_{n\to\infty}\dfrac{1}{\ln(n+1)}\ \underline{=0|}$

Therefore, the given series ***converges*** by Alternating Series Test.

Exercise

Determine if the alternating series converges or diverges $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$

Solution

$u_{n+1}=\dfrac{1}{\sqrt{n+1}}<\dfrac{1}{\sqrt{n}}=u_n$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \underline{=0|}$$

Therefore, the given series ***converges*** by Alternating Series Test.

Exercise

Determine if the alternating series converges or diverges $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^2+4}$

Solution

$$\lim_{n\to\infty} \frac{n^2}{n^2+4} \underline{=1|}$$

Therefore, the given series ***converges*** by n*th*-Term Test.

Exercise

Determine if the alternating series converges or diverges $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{\ln(n+1)}$

Solution

$$\begin{aligned}\lim_{n\to\infty} \frac{n+1}{\ln(n+1)} &= \lim_{n\to\infty} \frac{1}{\frac{1}{n+1}} \\ &= \lim_{n\to\infty} (n+1) \\ &\underline{=\infty|}\end{aligned}$$

Therefore, the given series ***diverges*** by n*th*-Term Test.

Exercise

Determine if the alternating series converges or diverges $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln(n+1)}{n+1}$

Solution

$$u_{n+1} = \frac{\ln(n+2)}{n+2} < \frac{\ln(n+1)}{n+1} = u_n$$

$$\lim_{n\to\infty} \frac{\ln(n+1)}{n+1} \underline{=0|}$$

Therefore, the given series ***converges*** by Alternating Series Test.

Exercise

Determine if the alternating series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \sin\frac{(2n-1)\pi}{2}$

Solution

$$\sum_{n=1}^{\infty} \sin\frac{(2n-1)\pi}{2} = \sum_{n=1}^{\infty} (-1)^{n+1}$$

Therefore, the given series ***diverges*** by n*th*-Term Test.

Exercise

Determine if the alternating series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\cos n\pi$

Solution

$$\sum_{n=1}^{\infty} \frac{1}{n}\cos n\pi = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$u_{n+1} = \frac{1}{n+1} < \frac{1}{n} = u_n$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Therefore, the given series ***converges*** by Alternating Series Test.

Exercise

Determine if the alternating series converges or diverges $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

Solution

$$u_{n+1} = \frac{1}{(n+1)!} < \frac{1}{n!} = u_n$$

$$\lim_{n\to\infty} \frac{1}{n!} = 0$$

Therefore, the given series ***converges*** by Alternating Series Test.

Exercise

Determine if the alternating series converges or diverges $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$

Solution

$$u_{n+1} = \frac{1}{\left(2(n+1)+1\right)!} < \frac{1}{(2n+1)!} = u_n$$

$$\lim_{n\to\infty} \frac{1}{2n+1} \underline{= 0|}$$

Therefore, the given series ***converges*** by Alternating Series Test.

Exercise

Determine if the alternating series converges or diverges $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{n+2}$

Solution

$$u_{n+1} = \frac{\sqrt{n+1}}{(n+1)+2} < \frac{\sqrt{n}}{n+2} = u_n$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n+2} \underline{= 0|}$$

Therefore, the given series ***converges*** by Alternating Series Test.

Exercise

Determine if the alternating series converges or diverges $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{\sqrt[3]{n}}$

Solution

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt[3]{n}} = \lim_{n\to\infty} n^{2/3}$$

$$\underline{= \infty|}$$

Therefore, the given series ***diverges*** by n*th*-Term Test.

Exercise

Determine if the series converge absolutely and if it converges or diverges $\sum_{n=1}^{\infty} (-1)^{n+1} (0.1)^n$

Solution

$(0.1)^n$ converges geometric since $r = 0.1 < 1$

The given series ***converges*** absolutely.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.1)^n}{n}$

Solution

$$\left|(-1)^{n+1}\frac{(0.1)^n}{n}\right| = \frac{1}{n(10)^n} < \frac{1}{(10)^n} = \left(\frac{1}{10}\right)^n \text{ converges geometric}$$

The given series ***converges*** absolutely by Direct Comparison Test.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\sqrt{n}}$

Solution

$$\frac{1}{\sqrt{n}} > \frac{1}{1+\sqrt{n+1}} > 0 \qquad \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0 \quad \Rightarrow \ \textit{converges}$$

The given series ***converges*** conditionally, but $\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ is a divergent *p*-series.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum_{n=1}^{\infty}(-1)^n\frac{\sin n}{n^2}$

Solution

By Direct Comparison Test $\left|\frac{\sin n}{n^2}\right| \le \frac{1}{n^2}$

The given series ***converges*** absolutely.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3+n}{5+n}$

Solution

$$\lim_{n\to\infty}\frac{3+n}{5+n} = 1 \ne 0$$

The given series ***diverges*** by the n^{th}-Term Test.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n\ln n}$

Solution

$$f(x)=\frac{1}{x\ln x}$$

$$f'(x)=-\frac{\ln x+1}{(x\ln x)^2}<0 \quad \to f(x) \ \textit{is decreasing}$$

$$u_n>u_{n+1}>0 \quad \textit{for} \quad n\geq 2$$

$$\lim_{n\to\infty}\frac{1}{n\ln n}=0 \Rightarrow \textit{converges}$$

But by the Integral Test:

$$\int_2^{\infty}\frac{dx}{x\ln x}=\int_2^{\infty}\frac{d(\ln x)}{\ln x}$$

$$=\ln(\ln x)\Big|_2^{\infty}$$

$$=\infty$$

$$\Rightarrow \sum_{n=1}^{\infty}\left|a_n\right|=\sum_{n=1}^{\infty}\frac{1}{n\ln n} \ \textbf{\textit{diverges}}.$$

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2+2n+1}$

Solution

$\frac{1}{n^2+2n+1}<\frac{1}{n^2}$, which is a convergent p-series, since $p=2>1$.

The given series ***converges absolutely*** by Direct Comparison Test.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}$

Solution

Let $a_n=\frac{(-1)^{n-1}}{2n-1} \quad \to \quad b_n=\frac{1}{2n-1}>\frac{1}{n}$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n-1}}{2n-1}\right|\cdot\frac{n}{1}=\lim_{n\to\infty}\frac{n}{2n-1}$$

$$\boxed{=\frac{1}{2}>0}$$

Since the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ ***diverges*** to infinity, then the given series doesn't converge absolutely.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum_{n=1}^{\infty}\frac{n\cos(n\pi)}{2^n}$

Solution

$$\rho=\lim_{n\to\infty}\left|\frac{(n+1)\cos((n+1)\pi)}{2^{n+1}}\cdot\frac{2^n}{n\cos(n\pi)}\right|$$

$$=\lim_{n\to\infty}\frac{n+1}{2n}$$

$$\boxed{=\frac{1}{2}<1}$$

Therefore, by the ratio test, the given series ***converges absolutely***.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$

Solution

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$ converges by the alternating series test, since the terms alternate in sign (decrease in size) and approach 0.

$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ ***diverges*** to infinity, then the series converge conditionally only.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + \ln n}$

Solution

$$n^2 + \ln n \ge n^2 \quad \Rightarrow \quad \frac{1}{n^2 + \ln n} \le \frac{1}{n^2}$$

$$\left| \frac{(-1)^n}{n^2 + \ln n} \right| \le \frac{1}{n^2} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges,}$$

Therefore, the given series ***converges absolutely***.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \left(n^2 - 1\right)}{n^2 + 1}$

Solution

$$\lim_{n \to \infty} \left| \frac{(-1)^n \left(n^2 - 1\right)}{n^2 + 1} \right| = \lim_{n \to \infty} \frac{n^2}{n^2} = 1$$

The given series ***diverges*** (since its terms do not approach 0.)

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{(n+1)\ln(n+1)}$

Solution

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{(n+1)\ln(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)\ln(n+1)} \text{ converges by alternating series test.}$$

let $x = n$, then $\displaystyle\int_1^{\infty} \frac{dx}{(x+1)\ln(x+1)} = \ln\left(\ln(x+1)\right)\Big|_1^{\infty}$

$$= \ln(\ln \infty) - \ln(\ln 2)$$

$$= \infty$$

The series ***converges conditionally*** since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)}$ diverges to infinity by the integral test.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$

Solution

$$\lim_{n\to\infty} \left| \frac{(-2)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-2)^n} \right| = 2 \lim_{n\to\infty} \frac{1}{n+1}$$

$$\underline{= 0|}$$

Therefore, the given series ***converges*** absolutely by the ratio test.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi^n}$

Solution

$\left| \frac{(-1)^n}{n\pi^n} \right| \le \frac{1}{\pi^n}$, and since $\sum \frac{1}{\pi^n}$ is convergent geometric series, then the given series ***converges*** absolutely.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum_{n=1}^{\infty} \frac{100\cos(n\pi)}{2n+3}$

Solution

$$\sum_{n=1}^{\infty} \frac{100\cos(n\pi)}{2n+3} = \sum_{n=1}^{\infty} \frac{100(-1)^n}{2n+3}$$

$$\lim_{n\to\infty} \left| \frac{100(-1)^n}{2n+3} \right| = \lim_{n\to\infty} \frac{1}{2n+3}$$

$$\underline{= 0|}$$

The series converges by alternating series test but only conditionally.

The given series ***diverges*** to infinity.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(-100)^n}$

Solution

$$\lim_{n\to\infty} \left| \frac{n!}{(-100)^n} \right| = \lim_{n\to\infty} \frac{n!}{100^n}$$

$$\lim_{n\to\infty} \frac{(n+1)!}{100^{n+1}} \cdot \frac{100^n}{n!} = \lim_{n\to\infty} \frac{n+1}{100}$$

$$= \infty$$

The given series ***diverges***.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=10}^{\infty} \frac{\sin\left(n+\frac{1}{2}\right)\pi}{\ln \ln n}$

Solution

$$\sum_{n=10}^{\infty} \frac{\sin\left(n+\frac{1}{2}\right)\pi}{\ln \ln n} = \sum_{n=10}^{\infty} \frac{(-1)^n}{\ln \ln n}$$

$$0 < \ln(\ln n) < n \quad \Rightarrow \quad \frac{1}{\ln(\ln n)} > \frac{1}{n} \quad \text{for } n \ge 10$$

Since $\displaystyle\sum_{n=10}^{\infty} \frac{1}{n}$ diverges to infinity (it is a harmonic series), so does $\displaystyle\sum_{n=10}^{\infty} \frac{1}{\ln(\ln n)}$ by comparison.

The series converges conditionally by the alternating series.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$

Solution

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges gemoemtric series

The given series ***converges*** *absolutely*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

Solution

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ***p***-series

The given series ***converges*** *absolutely*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n!}$

Solution

$\displaystyle\frac{1}{n!} < \frac{1}{n^2}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ***p***-series

The given series ***converges*** *absolutely*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+3}$

Solution

$\displaystyle\frac{1}{n+3} < \frac{1}{n}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by comparison

$$u_{n+1} = \frac{1}{(n+1)+3} < \frac{1}{n+3} = u_n$$

$\displaystyle\lim_{n\to\infty} \frac{1}{n+3} = 0$ converges by Alternating Series Test

The given series ***converges*** *absolutely*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

Solution

$$u_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = u_n$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0 \qquad \text{converges by Alternating Series Test}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \qquad \text{diverges by } \boldsymbol{p}\text{-series } \left(p = \frac{1}{2} < 1\right)$$

The given series ***converges*** *conditionally*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$

Solution

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} \qquad \text{converges by } \boldsymbol{p}\text{-series } \left(p = \frac{3}{2} > 1\right)$$

The given series ***converges*** *conditionally*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{(n+1)^2}$

Solution

$$\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$$

The given series ***diverges*** by the nth-Term Test.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+3}{n+10}$

Solution

$$\lim_{n\to\infty} \frac{2n+3}{n+10} = 2$$

The given series ***diverges*** by the nth-Term Test.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n\ln n}$

Solution

$$\int_2^{\infty} \frac{dx}{x\ln x} = \int_2^{\infty} \frac{d(\ln x)}{\ln x}$$

$$= \ln(\ln x)\Big|_2^{\infty}$$

$$= \infty \qquad \text{By the Integral Test, the series diverges}$$

$$u_{n+1} = \frac{1}{(n+1)\ln(n+1)} < \frac{1}{n\ln n} = u_n$$

$$\lim_{n\to\infty} \frac{1}{n\ln n} = 0 \qquad \text{converges by Alternating Series Test}$$

The given series ***converges*** *conditionally*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=0}^{\infty} (-1)^n e^{-n^2}$

Solution

$$b_n = \left(\frac{1}{e}\right)^n \qquad \text{Convverges by geometric series } \left(r = \frac{1}{e} < 1\right)$$

$$\left(\frac{1}{e}\right)^{n^2} < \left(\frac{1}{e}\right)^n \text{ converges by Compearison Test}$$

The given series ***converges*** *absolutely*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{n}{n^3-5}$

Solution

$$\frac{n}{n^3-5} = \frac{n}{n^3} = \frac{1}{n^2}$$

$b_n = \frac{1}{n^2}$ converges by p-series $\left(p = 2 > 1\right)$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^3 - 5} \frac{n^2}{1}$$

$\underline{= 1}$ converges by Limit Compearison Test

The given series ***converges*** *absolutely*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{4/3}}$

Solution

$\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ converges by p-series $\left(p = \frac{4}{3} > 1\right)$

$$u_{n+1} = \frac{1}{(n+1)^{4/3}} < \frac{1}{n^{4/3}} = u_n$$

$\lim_{n\to\infty} \frac{1}{n^{4/3}} \underline{= 0}$ converges by Alternating Series Test

The given series ***converges*** *absolutely*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+4}}$

Solution

$$u_{n+1} = \frac{1}{\sqrt{(n+1)+4}} < \frac{1}{\sqrt{n+4}} = u_n$$

$\lim_{n\to\infty} \frac{1}{\sqrt{n+4}} \underline{= 0}$ converges by Alternating Series Test

$b_n = \frac{1}{\sqrt{n}}$ diverges by ***p***-series $\left(p = \frac{1}{2} < 1\right)$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n+4}} \frac{\sqrt{n}}{1} \underline{= 1}$ diverges by Limit Compearison Test using ***p***-series

The given series ***converges*** *conditionally*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1}$

Solution

$$\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

$$u_{n+1} = \frac{1}{(n+1)+1} < \frac{1}{n+1} = u_n$$

$$\lim_{n\to\infty} \frac{1}{n+1} = 0$$ converges by Alternating Series Test

$$\sum_{n=0}^{\infty} \frac{|\cos n\pi|}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$$ diverges by a Limit Comparison to the divergent harmonic series

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{|\cos n\pi|}{n+1} \cdot \frac{n}{1}$$

$$= 1$$

The given series ***converges*** *conditionally*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \arctan n$

Solution

$$\lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$$

The given series ***diverges*** by the n*th*-Term Test.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}$

Solution

$$\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$u_{n+1} = \frac{1}{(n+1)^2} < \frac{1}{n^2} = u_n$$

$$\lim_{n\to\infty} \frac{1}{n^2} = 0 \qquad \text{converges by Alternating Series Test}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{converges by } p\text{-series } (p = 2 > 1)$$

The given series ***converges*** *absolutely*.

Exercise

Determine if the series converge absolutely or conditionally, or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\left[(n-1)\frac{\pi}{2}\right]}{n}$

Solution

$$\sum_{n=1}^{\infty} \frac{\sin\left[(n-1)\frac{\pi}{2}\right]}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

$$u_{n+1} = \frac{1}{n+1} < \frac{1}{n} = u_n$$

$$\lim_{n\to\infty} \frac{1}{n} = 0 \qquad \text{converges by Alternating Series Test}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges by } \boldsymbol{p}\text{-series } (p = 1)$$

The given series ***converges*** *conditionally*.

Exercise

For what values of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n\,2^n}$ converge absolutely? Converge conditionally?

Solution

$$\lim_{n\to\infty} \left| \frac{(x-5)^{n+1}}{(n+1)\,2^{n+1}} \frac{n\,2^n}{(x-5)^n} \right| = \lim_{n\to\infty} \frac{n}{n+1} \left| \frac{x-5}{2} \right|$$

$$= \left| \frac{x-5}{2} \right|$$

$$\left| \frac{x-5}{2} \right| < 1 \quad \to |x-5| < 2$$

$$\Rightarrow -2 < x-5 < 2$$

$\Rightarrow \quad 3 < x < 7$ Then the series converges absolutely

$$\left|\frac{x-5}{2}\right|>1 \quad \rightarrow |x-5|>2 \quad \Rightarrow x-5<-2 \quad and \quad x-5>2$$

$\Rightarrow \quad x<3 \quad and \quad x>7$ Then the series diverges (the term does approach zero)

$$\left|\frac{x-5}{2}\right|=1 \quad \rightarrow |x-5|=2 \quad \Rightarrow x-5=-2 \quad and \quad x-5=2$$

If $x=3$, the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{n\ 2^n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ which converges conditionally (it is an alternating harmonic series).

If $x=7$, the series $\displaystyle\sum_{n=1}^{\infty}\frac{(2)^n}{n\ 2^n}=\sum_{n=1}^{\infty}\frac{1}{n}$ the series is harmonic which diverges.

Hence, The series ***converges absolutely*** on the open interval (3, 7), **converges conditionally** at $x=3$, and ***diverges*** everywhere else.

Exercise

For what values of *x* does the series $\displaystyle\sum_{n=1}^{\infty}\frac{(x-2)^n}{n^2\ 2^{2n}}$ converge absolutely? Converge conditionally? Diverge?

Solution

Using the ratio test

$$\rho=\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^2\ 2^{2n+2}}\frac{n^2\ 2^{2n}}{(x-2)^n}\right|=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2\frac{|x-2|}{4}$$

$$=\frac{|x-2|}{4}<1$$

$$\frac{|x-2|}{4}<1 \quad \rightarrow |x-2|<4 \quad \Rightarrow -4<x-2<4$$

$\Rightarrow \quad -2<x<6$ Then the series converges absolutely

$$\frac{|x-2|}{4}>1 \quad \rightarrow |x-2|>4 \quad \Rightarrow x-2<-4 \quad and \quad x-2>4$$

$\Rightarrow \quad x<-2 \quad and \quad x>6$ Then the series diverges (the term does approach zero)

$$\frac{|x-2|}{4}=1 \quad \rightarrow |x-2|=4 \quad \Rightarrow x-2=-4 \quad and \quad x-2=4$$

If $x=-2$, the series:

$$\sum_{n=1}^{\infty}\frac{(x-2)^n}{n^2\ 2^{2n}}=\sum_{n=1}^{\infty}\frac{(-4)^n}{n^2\ 2^{2n}}$$

$$=\sum_{n=1}^{\infty}\frac{\left(-2^2\right)^n}{n^2\ 2^{2n}}$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

which converges absolutely (it is an alternating harmonic series)..

If $x=6$, the series

$$\sum_{n=1}^{\infty}\frac{(x-2)^n}{n^2\ 2^{2n}}=\sum_{n=1}^{\infty}\frac{(4)^n}{n^2\ 2^{2n}}$$

$$=\sum_{n=1}^{\infty}\frac{2^{2n}}{n^2\ 2^{2n}}$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

the series converges absolutely.

Hence, the series converges absolutely if $-2\le x\le 6$ and diverges everywhere else.

Exercise

For what values of x does the series $\sum_{n=1}^{\infty}(n+1)^2\left(\frac{x}{x+2}\right)^n$ converge absolutely? Converge conditionally? Diverge?

Solution

Using the ratio test

$$\lim_{n\to\infty}\left|\frac{(n+2)^2\left(\frac{x}{x+2}\right)^{n+1}}{(n+1)^2\left(\frac{x}{x+2}\right)^n}\right|=\lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)^2\left|\frac{x}{x+2}\right|$$

$$=\left|\frac{x}{x+2}\right|$$

$$\left|\frac{x}{x+2}\right|=1\quad\Rightarrow\quad\frac{x}{x+2}=1 \qquad\Big|\qquad \frac{x}{x+2}=-1$$

$$x=x+2\quad(\text{impossible}) \qquad\Big|\qquad x=-x-2$$

$$\underline{x=-1}$$

If $\left|\frac{x}{x+2}\right| < 1 \quad \Rightarrow \quad -2 < x < 0$.

Hence $x > -1$ the series converges absolutely.

$-\infty$	-2	-1	$0 \quad \infty$
+	–	–	–
+	–	+	+
		–	–

If $\frac{|x|}{|x+2|} > 1 \quad \Rightarrow \quad x < -1$, the series diverges.

If $x = -1$, the series is $\sum_{n=1}^{\infty} (-1)^n (n+1)^2$ which diverges

The series converges absolutely for $x > -1$, converges conditionally nowhere, and diverges for $x \le -1$

Exercise

For what values of x does the series $\sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{2n+3}$ converge absolutely? Converge conditionally? Diverge?

Solution

Using the ratio test

$$\rho = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{2(n+1)+3} \cdot \frac{2n+3}{(x-1)^n} \right|$$

$$= \lim_{n\to\infty} \frac{2n+3}{2n+5} |x-1| \qquad\qquad \lim_{n\to\infty} \frac{2n+3}{2n+5} = \lim_{n\to\infty} \frac{2n}{2n} = 1$$

$$= |x-1|$$

If $|x-1| < 1 \quad -1 < x-1 < 1 \quad \Rightarrow \quad 0 < x < 2$ Then the series converges absolutely

If $|x-1| > 1 \quad \Rightarrow \quad x < 0 \quad$ *and* $\quad x > 2$ Then the series diverges

If $|x-1| = 1 \quad \Rightarrow \quad x = 0 \quad$ *and* $\quad x = 2$

If $x = 0$, the series $\sum_{n=0}^{\infty} (-1)^n \frac{(0-1)^n}{2n+3} = \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{2n+3} = \sum_{n=0}^{\infty} \frac{1}{2n+3}$ is harmonic which diverges.

If $x = 2$, the series $\sum_{n=0}^{\infty} (-1)^n \frac{(2-1)^n}{2n+3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+3}$ which converges absolutely (it is an alternating harmonic series).

Therefore, the series converges absolutely if and converges conditionally if $x = 2$ and diverges everywhere else.

Exercise

For what values of x does the series $\sum_{n=1}^{\infty} \frac{1}{2n-1}\left(\frac{3x+2}{-5}\right)^n$ converge absolutely? Converge conditionally? Diverge?

Solution

Using the ratio test

$$\rho = \lim_{n\to\infty} \left| \frac{1}{2(n+1)-1}\left(\frac{3x+2}{-5}\right)^{n+1} \cdot (2n-1)\left(\frac{3x+2}{-5}\right)^{-n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2n-1}{2n+1} \cdot \frac{3x+2}{-5} \right| \qquad \lim_{n\to\infty} \frac{2n-1}{2n+1} = \lim_{n\to\infty} \frac{2n}{2n} = 1$$

$$= \frac{1}{5}|3x+2|$$

If $\frac{1}{5}|3x+2| < 1 \quad -5 < 3x+2 < 5 \quad \Rightarrow \quad -\frac{7}{3} < x < 1$ Then the series converges absolutely

If $|3x+2| > 5 \quad \Rightarrow \quad 3x+2 < -5 \quad \textit{and} \quad 3x+2 > 5$

$x < -\frac{7}{3} \quad \textit{and} \quad x > 1$. Then the series diverges

If $x = -\frac{7}{3}$, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2n-1}(1)^n$

$$= \sum_{n=1}^{\infty} \frac{1}{2n-1} \qquad \text{is harmonic which diverges.}$$

If $x = 1$, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ converges absolutely (it is an alternating harmonic series).

Therefore, the series converges absolutely if and $-\frac{7}{3} < x < 1$, converges conditionally if $x = 1$ and diverges everywhere else.

Exercise

For what values of x does the series $\sum_{n=2}^{\infty} \frac{x^n}{2^n \ln n}$ converge absolutely? Converge conditionally? Diverge?

Solution

Using the ratio test

$$\rho = \lim_{n\to\infty} \left| \frac{x^{n+1}}{2^{n+1}\ln(n+1)} \cdot \frac{2^n \ln n}{x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x}{2} \cdot \frac{\ln n}{\ln(n+1)} \right| \qquad \lim_{n\to\infty} \left| \frac{\ln n}{\ln(n+1)} \right| = \lim_{n\to\infty} \frac{1/n}{1/n+1} = \lim_{n\to\infty} \frac{n+1}{n} = 1 \quad (L'Hôpital\ rule)$$

$$= \frac{|x|}{2}$$

If $\frac{|x|}{2} < 1 \quad \Rightarrow |x| < 2 \quad \rightarrow \quad -2 < x < 2$, the given series converges absolutely.

If $x = -2$, the series

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{(-2)^n}{2^n \ln n}$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$$ converges absolutely (it is an alternating harmonic series).

If $x = 2$, the series

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{(2)^n}{2^n \ln n}$$

$$= \sum_{n=2}^{\infty} \frac{1}{\ln n}$$ is harmonic which diverges

Therefore, the series converges absolutely if and $-2 < x < 2$, converges conditionally if $x = -2$ and diverges everywhere else.

Exercise

For what values of x does the series $\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^3}$ converge absolutely? Converge conditionally?

Diverge?

Solution

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^3}$ by using the ratio test

$$\rho = \lim_{n\to\infty} \left| \frac{(4x+1)^{n+1}}{(n+1)^3} \cdot \frac{n^3}{(4x+1)^n} \right|$$

$$= |4x+1| \lim_{n\to\infty} \left| \frac{n^3}{(n+1)^3} \right| \qquad \lim_{n\to\infty} \left| \frac{n^3}{(n+1)^3} \right| = \lim_{n\to\infty} \left| \left(\frac{n}{n+1}\right)^3 \right| = 1$$

$= |4x+1|$

If $|4x+1| < 1 \quad \Rightarrow -1 < 4x+1 < 1 \quad \to \quad -\frac{1}{2} < x < 0$, the given series converges absolutely.

If $x = -\frac{1}{2}$, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ converges absolutely.

If $x = 0$, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges absolutely (p-series)

Therefore, the series converges absolutely if $-\frac{1}{2} \le x \le 0$ and diverges everywhere else.

Exercise

For what values of x does the series $\sum_{n=1}^{\infty} \frac{(2x+3)^n}{n^{1/3}4^n}$ converge absolutely? Converge conditionally? Diverge?

Solution

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(2x+3)^n}{n^{1/3}4^n}$ by using the ratio test

$$\rho = \lim_{n\to\infty} \left| \frac{(2x+3)^{n+1}}{(n+1)^{1/3}4^{n+1}} \cdot \frac{n^{1/3}4^n}{(2x+3)^n} \right|$$

$$= \frac{|2x+3|}{4} \lim_{n\to\infty} \left| \frac{n^{1/3}}{(n+1)^{1/3}} \right|$$

$$\lim_{n\to\infty} \left| \frac{n^{1/3}}{(n+1)^{1/3}} \right| = \lim_{n\to\infty} \left| \left(\frac{n}{n+1} \right)^{1/3} \right| = 1$$

$$= \frac{|2x+3|}{4}$$

If $|2x+3| < 4 \quad \Rightarrow -4 < 2x+3 < 4 \quad \to \quad -\frac{7}{2} < x < \frac{1}{2}$, the given series converges absolutely.

If $x = -\frac{7}{2}$, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}4^n}$ converges conditionally (Alternating test).

If $x = \frac{1}{2}$, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}4^n}$ diverges.

Therefore, the series converges absolutely if $-\frac{7}{2} \le x \le \frac{1}{2}$, $x = -\frac{7}{2}$ ***converges conditionally*** and ***diverges*** everywhere else.

Exercise

For what values of x does the series $\sum_{n=1}^{\infty} \frac{1}{n}\left(1+\frac{1}{x}\right)^n$ converge absolutely? Converge conditionally? Diverge?

Solution

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}\left(1+\frac{1}{x}\right)^n$ by using the ratio test

$$\rho = \lim_{n\to\infty} \left| \frac{1}{n+1}\left(1+\frac{1}{x}\right)^{n+1} \cdot n\left(1+\frac{1}{x}\right)^{-n} \right|$$

$$= \left|1+\frac{1}{x}\right| \lim_{n\to\infty} \left|\frac{n}{n+1}\right| \qquad \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = 1$$

$$= \left|1+\frac{1}{x}\right| < 1$$

If $-1 < 1+\frac{1}{x} < 1 \quad \to \quad -2 < \frac{1}{x} < 0$

$\Rightarrow$ $\boxed{-\frac{1}{2} < x < 0}$, the given series converges absolutely.

If $x = -\frac{1}{2}$, the series

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally (Alternating test).

Therefore, the series converges absolutely if $-\frac{1}{2} < x < 0$, $x = -\frac{1}{2}$ converges conditionally, diverges everywhere else, and undefined at $x = 0$.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$

Solution

$$\lim_{n\to\infty} \frac{\sqrt{n+1}}{n+2} \frac{n+1}{\sqrt{n}} = \lim_{n\to\infty} \sqrt{\frac{n+1}{n}}\left(\frac{n+1}{n+2}\right)$$

$$= 1$$

$$\lim_{n\to\infty} \frac{\sqrt{n+1}}{n+2} = \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n+1}}}{1}$$

$$= 0$$

Therefore, the given series ***converges*** by the Alternating Series Test.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

Solution

$$a_{n+1} = \frac{1}{(n+1)\ln(n+1)} \le \frac{1}{n \ln n} = a_n$$

$$\lim_{n \to \infty} \frac{1}{n \ln n} = 0$$

Therefore, the given series ***converges*** by Alternating Series Test.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{5}{n}$

Solution

$$a_{n+1} = \frac{5}{n+1} \le \frac{5}{n} = a_n$$

$$\lim_{n \to \infty} \frac{5}{n} = 0$$

Therefore, the given series ***converges*** by Alternating Series Test.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{n-1}}{n!}$

Solution

$$\begin{aligned} \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \to \infty} \left| \frac{3^n}{(n+1)!} \cdot \frac{n!}{3^{n-1}} \right| \\ &= \lim_{n \to \infty} \frac{3}{n+1} \\ &= 0 \end{aligned}$$

Therefore, the given series ***converges absolutely*** by the Ratio Test.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3^n}{n2^n}$

Solution

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n2^n}{3^n}\right|$$

$$= \lim_{n\to\infty}\frac{3}{2}\frac{n}{n+1}$$

$$= \frac{3}{2} > 1$$

Therefore, the given series ***diverges*** by the Ratio Test.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$

Solution

$$a_{n+1} = \frac{1}{n+1} \le \frac{1}{n} = a_n$$

$$\lim_{n\to\infty}\frac{1}{n} = 0$$

Therefore, the given series ***converges*** by Alternating Series Test.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty}\frac{n}{(-2)^{n-1}}$

Solution

$$\frac{1}{2} \le \frac{n}{n+1}$$

$$\frac{2^{n-1}}{2^n} \le \frac{n}{n+1}$$

$$a_{n+1} = \frac{n+1}{2^n} \le \frac{n}{2^{n-1}} = a_n$$

$$\lim_{n\to\infty}\frac{n}{2^{n-1}} = \lim_{n\to\infty}\frac{1}{2^{n-1}(\ln 2)} = 0$$

Therefore, the given series ***converges*** by Alternating Series Test.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{10}{n^{3/2}}$

Solution

$$p = \frac{3}{2} > 1$$

Therefore, the given series ***converges*** by ***p****-series* Test.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{2}{n^2+5}$

Solution

$$b_n = \frac{1}{n^2}$$

$$\frac{2}{n^2+5} < \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{2}{n^2+5} \underline{=0|}$$

Therefore, the given series ***converges*** by the Limit Comparison Test with ***p****-series*.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{3^n}{n^2}$

Solution

$$\lim_{x\to\infty} \frac{3^{n+1}}{(n+1)^2} \cdot \frac{n^2}{3^n} = \lim_{x\to\infty} 3\left(\frac{n}{n+1}\right)^2$$

$$\underline{= 3 > 1|}$$

Therefore, the given series ***diverges*** by the Ratio Test.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

Solution

$$\frac{1}{2^n+1} < \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$$

Therefore, the given series ***converges*** by the Limit Comparison Test with Gemoetric series $r = \frac{1}{2} < 1$

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} 5\left(\frac{7}{8}\right)^n$

Solution

Therefore, the given series ***converges*** by Gemoetric series $|r| = \frac{7}{8} < 1$

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{3n^2}{2n^2+1}$

Solution

$$\lim_{n\to\infty} \frac{3n^2}{2n^2+1} = \frac{3}{2}$$

Therefore, the given series ***diverges***.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} 100e^{-\pi/2}$

Solution

$$\sum_{n=1}^{\infty} 100e^{-\pi/2} = 100\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{e}}\right)^n$$

Therefore, the given series ***converges*** by Gemoetric series $|r| = \frac{1}{\sqrt{e}} < 1$

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+4}$

Solution

$$a_{n+1} = \frac{1}{(n+1)+4} < \frac{1}{n+4} = a_n$$

$$\lim_{n\to\infty} \frac{1}{n+4} \underline{=0|}$$

Therfore, the given series ***converges*** conditionally by Alternating Sereis Test.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{4}{3n^2 - 1}$

Solution

$$3(n+1)^2 - 1 > 3n^2 - 1$$

$$a_{n+1} = \frac{4}{3(n+1)^2 - 1} < \frac{4}{3n^2 - 1} = a_n$$

$$\lim_{n\to\infty} \frac{4}{3n^2 - 1} \underline{=0|}$$

Therfore, the given series ***converges*** absolutely by Alternating Sereis Test.

Exercise

Use any method to determine if the series converges or diverges. $\sum_{n=2}^{\infty} \frac{\ln n}{n}$

Solution

$$\begin{aligned} \int_1^{\infty} \frac{\ln x}{x} dx &= \int_1^{\infty} \ln x \, d(\ln x) \\ &= \frac{1}{2}(\ln x)^2 \Big|_1^{\infty} \\ &\underline{=\infty|} \end{aligned}$$

Therfore, the given series ***diverges*** by Integral Test.

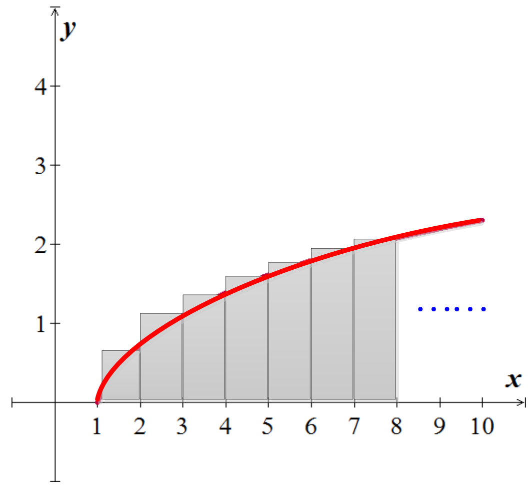
Exercise

Use a Riemann sum argument to show that $ln\, n! \ge \int_1^n ln\, t \; dt = n\, ln\, n - n + 1$

Then for what values of x does the series $\sum_{n=1}^{\infty} \frac{n!\, x^n}{n^n}$ converge absolutely? Converge conditionally?

Diverge? (*Use the ratio test first*)

Solution



$$\ln n! = \ln 1 + \ln 2 + \ln 3 + \cdots + \ln n$$

$= \text{Sum of area of the shaded rectangles}$

$$> \int_1^n \ln t \, dt$$

$$= \left[t \ln t - t \right]_1^n$$

$$= n \ln n - n + 1$$

Using the ratio test

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n! x^n}{n^n}$

$$\rho = \lim_{n \to \infty} \left| \frac{(n+1)! x^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n! x^n} \right|$$

$$= |x| \lim_{n \to \infty} \left(\frac{n}{n+1} \right)^n \qquad \lim_{n \to \infty} \left(\frac{n}{n+1} \right)^n = e$$

$$= \frac{|x|}{e} < 1$$

If $|x| < e \quad \Rightarrow \quad -e < x < e \Rightarrow$ The given series converges absolutely.

If $x = \pm e$, then

$$\ln \left| \frac{n! x^n}{n^n} \right| = \ln n! + \ln \left| x^n \right| - \ln n^n$$

$$= \ln n! + \ln e^n - \ln n^n$$

$$> n \ln n - n + 1 + n - \ln n^n$$

$$= \ln n^n + 1 - \ln n^n$$

$$= 1$$

$$\Rightarrow \left| \frac{n! x^n}{n^n} \right| > e$$

Hence, the given series ***converges*** absolutely if $-e < x < e$ and ***diverges*** *elsewhere*.

Exercise

It can be proved that if a series converges absolutely, then its terms may be summed in any order without changing the value of the series. However, if a series converges conditionally, then the value of the series depends on the order of summation. For example, the (conditionally convergent) alternating harmonic series has the value

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \ln 2$$

Show that by rearranging the terms (so the sign pattern is ++–),

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\ldots=\frac{3}{2}\ln 2$$

Solution

$$S=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$$

$$=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\left(\frac{1}{7}-\frac{1}{8}\right)+\ldots$$

$$\frac{1}{2}S=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\right)$$

$$=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots$$

$$+\begin{cases} S=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\left(\frac{1}{7}-\frac{1}{8}\right)+\ldots=\ln 2 \\ \frac{1}{2}S=\quad\frac{1}{2}\quad-\quad\frac{1}{4}\quad+\quad\frac{1}{6}\quad-\quad\frac{1}{8}\quad+\cdots\qquad=\frac{1}{2}\ln 2 \end{cases}$$

$$\frac{3}{2}S=1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots\ \boxed{=\frac{3}{2}\ln 2}$$

Exercise

A crew of workers is constructing a tunnel through a mountain. Understandably, the rate of construction decreases because rocks and earth must be removed a greater distance as the tunnel gets longer. Suppose that each week the crew digs 0.95 of the distance it dug the previous week. In the first week, the crew constructed 100 *m* of tunnel.

a) How far does the crew dig in 10 *weeks*? 20 *weeks*? *N weeks*?

b) What is the longest tunnel the crew can build at this rate?

c) The time required to dig 100 *m* increases by 10% each week, starting with 1 *week* to dig the first 100 *m*. Can the crew complete a 1.5 *km* tunnel in 10 *weeks*? Explain.

Solution

a) Let T_n be the amount of additional tunnel dug during week *n*. Then

$$T_0=100$$

$$T_n=0.95\,T_{n-1}=(0.95)^n\ T_0\ \boxed{=100(0.95)^n}$$

So the total distance dug in *N* weeks is

$$S_N=100\sum_{k=0}^{N-1}(0.95)^k$$

$$=100\left(\frac{1-(0.95)^N}{1-0.95}\right)$$

$$= 2000\left(1-(0.95)^N\right)$$

For 10 weeks: $S_{10} = 2000\left(1-(0.95)^{10}\right) \approx 802.5\ m$

For 20 weeks: $S_{20} = 2000\left(1-(0.95)^{20}\right) \approx 1283.03\ m$

b) The longest possible tunnel is

$$S_\infty = 100\sum_{k=0}^{\infty}(0.95)^k$$
$$= \frac{100}{1-0.95}$$
$$= 2000\ \ m$$

c) The time required to dig $t_n = 100(n-1)$ through $n\cdot 100$

$$T_n = 1.1\,T_{n-1} = (1.1)^{n-1}\ T_1 = (1.1)^{n-1}\ weeks$$

The time required to dig 1500 *m* is:

$$\sum_{k=1}^{15} t_k = \sum_{k=1}^{15}(1.1)^{k-1} = \frac{1-1.1^{15}}{1-1.1} \approx 31.77\ weeks$$

So, it is not possible.

Exercise

Consider the alternating series

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k \quad where \quad a_k = \begin{cases} \frac{4}{k+1} & if\ \ k \text{ is odd} \\ \frac{2}{k} & if\ \ k \text{ is even} \end{cases}$$

a) Write out the first ten terms of the series, group them in pairs, and show that the even partial sums of the series form the (divergent) harmonic series.

b) Show that $\lim\limits_{k\to\infty} a_k = 0$

c) Explain why the series diverges even though the terms of the series approach zero.

Solution

a) The first ten terms of the series are:

$$(2-1)+\left(1-\tfrac{1}{2}\right)+\left(\tfrac{2}{3}-\tfrac{1}{3}\right)+\left(\tfrac{1}{2}-\tfrac{1}{4}\right)+\left(\tfrac{2}{5}-\tfrac{1}{5}\right)$$

Suppose that $\begin{cases} for\ even & k = 2i \\ for\ odd & k = 2i-1 \end{cases}$

Then the sum of the $(k-1)st$ term and the kth term is

$$\frac{4}{k}-\frac{2}{k}=\frac{2}{k}=\frac{2}{2i}=\frac{1}{i}$$

Then the sum of the even partial sums of the given series is $\displaystyle\sum_{i-1}^{n}\frac{1}{i}$

b) $\displaystyle\lim_{k\to\infty}\frac{4}{k+1}=\lim_{k\to\infty}\frac{4}{k}=0$

Given $\varepsilon>0$, $\exists\, N_1$ so that for $k>N_1$ we have $\frac{4}{k+1}<\varepsilon$.

Also $\exists\, N_2$ so that for $k>N_2$, $\frac{2}{k}<\varepsilon$.

Let N be the larger of N_1 or N_2. Then for $k>N$, we have $a_k<\varepsilon$ as desired.

c) The series can be seen to diverge because the even partial sums have limit ∞. This does not contradict the alternating series test because the terms a_k are not nonincreasing.