Solution   Section 3.7 – Trigonometric Form

Exercise

Write $-\sqrt{3}+i$ in trigonometric form. (Use radian measure)

Solution

$$-\sqrt{3}+i \quad \Rightarrow \quad \begin{cases} x=-\sqrt{3} \\ y=1 \end{cases}$$

$$r=\sqrt{\left(-\sqrt{3}\right)^2+1^2}$$

$$\underline{=2}$$

$$\hat{\theta}=\tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) \qquad\qquad \theta=\tan^{-1}\frac{y}{x}$$

$$\underline{=\frac{\pi}{6}}$$

The angle is in quadrant *II*, therefore;

$$\theta=\pi-\frac{\pi}{6}$$

$$\underline{=\frac{5\pi}{6}}$$

$$\underline{-\sqrt{3}+i=2\ cis\frac{5\pi}{6}}$$

Exercise

Write $3-4i$ in trigonometric form.

Solution

$$3-4i \quad \Rightarrow \quad \begin{cases} x=3 \\ y=-4 \end{cases}$$

$$r=\sqrt{3^2+(-4)^2}$$

$$\underline{=2}$$

$$\hat{\theta}=\tan^{-1}\left(\frac{4}{3}\right) \qquad\qquad \theta=\tan^{-1}\frac{y}{x}$$

$$\underline{\approx 53^\circ}$$

The angle is in quadrant *II*, therefore;

$$\theta=180^\circ-53^\circ$$

$$\underline{=127^\circ}$$

$$\underline{3-4i=5\ cis 127^\circ}$$

Exercise

Write $-21-20i$ in trigonometric form.

Solution

$$-21-20i \quad \Rightarrow \quad \begin{cases} x=-21 \\ y=-20 \end{cases}$$

$$r=\sqrt{(-21)^2+(-20)^2}$$
$$=29$$

$$\hat{\theta}=\tan^{-1}\left(\frac{20}{21}\right) \qquad\qquad \theta=\tan^{-1}\frac{y}{x}$$
$$\approx 43.6°$$

The angle is in quadrant *III*, therefore;

$$\theta=180°+43.6°$$
$$=223.6°$$

$$-21-20i=29\ cis223.6°$$

Exercise

Write $11+2i$ in trigonometric form.

Solution

$$11+2i \quad \Rightarrow \quad \begin{cases} x=11 \\ y=2 \end{cases}$$

$$r=\sqrt{11^2+2^2}$$
$$=\sqrt{125}$$
$$=5\sqrt{5}$$

$$\theta=\tan^{-1}\left(\frac{2}{11}\right) \qquad\qquad \theta=\tan^{-1}\frac{y}{x}$$
$$\approx 10.3°$$

$$11+2i=5\sqrt{5}\ cis10.3°$$

Exercise

Write $\sqrt{3}-i$ in trigonometric form.

Solution

$$\sqrt{3}-i \quad \Rightarrow \quad \begin{cases} x=\sqrt{3} \\ y=-1 \end{cases}$$

$$r=\sqrt{3+1}$$
$$\underline{=2}$$

$$\hat{\theta}=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \qquad\qquad \theta=\tan^{-1}\frac{y}{x}$$
$$\underline{\approx 30^\circ}$$

The angle is in quadrant *IV*, therefore;

$$\theta=360^\circ-30^\circ$$
$$\underline{=330^\circ}$$

$$\underline{\sqrt{3}-i=2\ cis330^\circ}$$

Exercise

Write $1-\sqrt{3}i$ in trigonometric form.

Solution

$$1-\sqrt{3}i \quad \Rightarrow \quad \begin{cases} x=1 \\ y=-\sqrt{3} \end{cases}$$

$$r=\sqrt{1+3}$$
$$\underline{=2}$$

$$\hat{\theta}=\tan^{-1}\left(\frac{\sqrt{3}}{1}\right) \qquad\qquad \theta=\tan^{-1}\frac{y}{x}$$
$$\underline{=60^\circ}$$

The angle is in quadrant *IV*, therefore;

$$\theta=360^\circ-60^\circ$$
$$\underline{=300^\circ}$$

$$\underline{1-\sqrt{3}i=2\ cis300^\circ}$$

Exercise

Write $9\sqrt{3}+9i$ in trigonometric form.

Solution

$$9\sqrt{3}+9i \quad \Rightarrow \quad \begin{cases} x = 9\sqrt{3} \\ y = 9 \end{cases}$$

$$r = 9\sqrt{3+1}$$
$$= 18$$

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$= 30^\circ$$

$$9\sqrt{3}+9i \ = 18\ cis 30^\circ$$

Exercise

Write $-2+3i$ in trigonometric form.

Solution

$$-2+3i \quad \Rightarrow \quad \begin{cases} x = -2 \\ y = 3 \end{cases}$$

$$r = \sqrt{4+9}$$
$$= \sqrt{13}$$

$$\hat{\theta} = \tan^{-1}\left(\frac{3}{2}\right) \qquad\qquad \theta = \tan^{-1}\frac{y}{x}$$
$$\approx 56.31^\circ$$

The angle is in quadrant *II*, therefore;

$$\theta = 180^\circ - 56.31^\circ$$
$$\approx 123.69^\circ$$

$$-2+3i \ = \sqrt{13}\ cis 123.69^\circ$$

Exercise

Write $4\left(\cos 30^\circ + i\sin 30^\circ\right)$ in standard form.

Solution

$$4\left(\cos 30^\circ + i\sin 30^\circ\right) = 4\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$
$$= 2\sqrt{3}+2i$$

Exercise

Write $\sqrt{2}\ cis\frac{7\pi}{4}$ in standard form.

Solution

$$\sqrt{2}\ cis\frac{7\pi}{4} = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

$$= 1 - i$$

Exercise

Write $3cis210°$ in standard form.

Solution

$$3cis210° = 3\left(\cos 210° + i\sin 210°\right)$$

$$= -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$$

Exercise

Write $4\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$ in standard form.

Solution

$$4\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = 4\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$= 2\sqrt{2} - 2i\sqrt{2}$$

Exercise

Write $4cis\frac{\pi}{2}$ in standard form.

Solution

$$4cis\frac{\pi}{2} = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= 4i$$

Exercise

Find the quotient $\dfrac{20cis(75°)}{4cis(40°)}$. Write the result in rectangular form.

Solution

$$\frac{20cis(75°)}{4cis(40°)} = \frac{20}{4}cis(75° - 40°)$$
$$= 5cis(35°)$$
$$= 5(\cos 35° + i\sin 35°)$$
$$\underline{= 4.1 + 2.87i}$$

Exercise

Divide $z_1 = 1 + i\sqrt{3}$ *by* $z_2 = \sqrt{3} + i$. Write the result in rectangular form.

Solution

$$\frac{z_1}{z_2} = \frac{1+i\sqrt{3}}{\sqrt{3}+i}$$
$$= \frac{1+i\sqrt{3}}{\sqrt{3}+i}\,\frac{\sqrt{3}-i}{\sqrt{3}-i}$$
$$= \frac{\sqrt{3} - i + 3i - \sqrt{3}\,i^2}{3+1}$$
$$= \frac{2\sqrt{3}+2i}{4}$$
$$= \frac{2\sqrt{3}}{4} + \frac{2i}{4}$$
$$\underline{= \frac{\sqrt{3}}{2} + \frac{i}{2}}$$

$\boxed{or}$

$$1 + i\sqrt{3} \rightarrow \begin{cases} r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} \\ \theta = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3} \end{cases}$$

$$\sqrt{3} + i \rightarrow \begin{cases} r = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} \\ \theta = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6} \end{cases}$$

$$\frac{z_1}{z_2} = \frac{2cis\frac{\pi}{3}}{2cis\frac{\pi}{6}}$$
$$= \frac{2}{2}cis\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$
$$= \frac{2}{2}cis\left(\frac{\pi}{6}\right)$$

$$= cis\left(\frac{\pi}{6}\right)$$

$$\boxed{= \frac{\sqrt{3}}{2} + \frac{i}{2}}$$

Exercise

Find $(1+i)^8$ and express the result in rectangular form.

Solution

$$1+i \quad \Rightarrow \quad \begin{cases} x = 1 \\ y = 1 \end{cases}$$

$$r = \sqrt{1^2 + 1^2}$$
$$\boxed{= \sqrt{2}}$$

$$\theta = \tan^{-1}(1)$$
$$\boxed{= \frac{\pi}{4}}$$

$$\boxed{1 + i = \sqrt{2}\,cis\,\frac{\pi}{4}}$$

$$\begin{aligned}
(1+i)^8 &= \left(\sqrt{2}\,cis\,\frac{\pi}{4}\right)^8 \\
&= \left(\sqrt{2}\right)^8 cis\left[8\left(\frac{\pi}{4}\right)\right] \\
&= 16\,cis\,2\pi \\
&= 16(\cos 2\pi + i\sin 2\pi) \\
&= 16(1 + i0) \\
&\boxed{= 16}
\end{aligned}$$

Exercise

Find $(1+i)^{10}$ and express the result in rectangular form.

Solution

$$1+i \quad \Rightarrow \quad \begin{cases} x = 1 \\ y = 1 \end{cases}$$

$$r = \sqrt{1^2 + 1^2}$$
$$\boxed{= \sqrt{2}}$$

$$\theta = \tan^{-1}(1)$$

$$=\frac{\pi}{4}$$

$$1+i=\sqrt{2}\,cis\frac{\pi}{4}$$

$$\begin{aligned}(1+i)^{10}&=\left(\sqrt{2}\,cis\frac{\pi}{4}\right)^{10}\\&=\left(\sqrt{2}\right)^{10}cis\left[10\left(\frac{\pi}{4}\right)\right]\\&=32cis\frac{5\pi}{2}\\&=32cis\frac{\pi}{2}\\&=32\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\\&=32(0+i)\\&=32i\end{aligned}$$

Exercise

Find and express the result in rectangular form $(1-i)^5$

Solution

$$1-i\ \Rightarrow\ \begin{cases}x=1\\y=-1\end{cases}$$

$$\begin{aligned}r&=\sqrt{1^2+1^2}\\&=\sqrt{2}\end{aligned}$$

$$\begin{aligned}\hat{\theta}&=\tan^{-1}(1)\\&=\frac{\pi}{4}\end{aligned}$$

The angle is in quadrant *IV*, therefore;

$$\begin{aligned}\theta&=2\pi-\frac{\pi}{4}\\&=\frac{7\pi}{4}\end{aligned}$$

$$1-i=\sqrt{2}\ cis\frac{7\pi}{4}$$

$$\begin{aligned}(1-i)^5&=\left(\sqrt{2}\ cis\frac{7\pi}{4}\right)^5\\&=4\sqrt{2}\left(\ cis\left(5\times\frac{7\pi}{4}\right)\right)\end{aligned}$$

$$= 4\sqrt{2}\left(cis\frac{35\pi}{4}\right) \qquad 35\pi - 8\times 2\pi = 3\pi$$

$$= 4\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$= 4\sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$\underline{= -4 + 4i}$$

Exercise

Find and express the result in rectangular form $\left(1-\sqrt{5}i\right)^8$

Solution

$$1-\sqrt{5}i \;\Rightarrow\; \begin{cases} x = 1 \\ y = -\sqrt{5} \end{cases}$$

$$r = \sqrt{1+5}$$

$$\underline{= \sqrt{6}}$$

$$\hat{\theta} = \tan^{-1}\left(\frac{\sqrt{5}}{1}\right)$$

$$\underline{\approx 66^\circ}$$

The angle is in quadrant *IV*, therefore;

$$\theta = 360^\circ - 66^\circ$$

$$\underline{= 294^\circ}$$

$$\underline{1-\sqrt{5}i = \sqrt{6}\ cis294^\circ}$$

$$\left(1-\sqrt{5}i\right)^8 = \left(\sqrt{6}cis294^\circ\right)^8$$

$$= \left(\sqrt{6}\right)^8\left(cis2352^\circ\right) \qquad 2352^\circ - 6\times 360^\circ = 192^\circ$$

$$= 1296\left(\cos192^\circ + i\sin192^\circ\right)$$

$$= 1296\left(-.978 - 0.208i\right)$$

$$\underline{= -1267.488 - 269.568\ i}$$

Exercise

Find and express the result in rectangular form $\left(3cis80^\circ\right)^3$

Solution

$$\begin{aligned}\left(3cis80^\circ\right)^3 &= 3^3\left(cis240^\circ\right)\\ &= 27\left(\cos 240^\circ + i\sin 240^\circ\right)\\ &= 27\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\\ &= -\frac{27}{2} - i\frac{27\sqrt{3}}{2}\end{aligned}$$

Exercise

Find and express the result in rectangular form $\left(\sqrt{3}cis10^\circ\right)^6$

Solution

$$\begin{aligned}\left(\sqrt{3}cis10^\circ\right)^6 &= 27\left(cis60^\circ\right)\\ &= 27\left(\cos 60^\circ + i\sin 60^\circ\right)\\ &= \frac{27}{2} + i\frac{27\sqrt{3}}{2}\end{aligned}$$

Exercise

Find and express the result in rectangular form $\left(\sqrt{2} - i\right)^6$

Solution

$$\sqrt{2} - i \quad \Rightarrow \quad \begin{cases} x = \sqrt{2} \\ y = -1 \end{cases}$$

$$\begin{aligned} r &= \sqrt{2+1}\\ &= \sqrt{3}\end{aligned}$$

$$\begin{aligned}\hat{\theta} &= \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\\ &\approx 35.26^\circ\end{aligned}$$

The angle is in quadrant *IV*, therefore;

$$\begin{aligned}\theta &= 360^\circ - 35.26^\circ\\ &= 324.74^\circ\end{aligned}$$

$$\sqrt{2} - i = \sqrt{3}\ cis324.74^\circ$$

$$\left(\sqrt{2}-i\right)^6=\left(\sqrt{3}\ cis324.74^\circ\right)^6$$

$$=27\left(cis1948.44^\circ\right) \qquad 1948.44^\circ-5\times 360^\circ=148.44^\circ$$

$$=27\left(\cos 148.44^\circ+i\sin 148.44^\circ\right)$$

$$\underline{=-23+14.142i}$$

Exercise

Find and express the result in rectangular form $\left(4cis40^\circ\right)^6$

Solution

$$\left(4cis40^\circ\right)^6=4^6\left(cis\left(6\times 40^\circ\right)\right)$$

$$=4^6\left(\cos 240^\circ+i\sin 240^\circ\right)$$

$$=4096\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$

$$\underline{=-2048+2048\ i\sqrt{3}}$$

Exercise

Find and express the result in rectangular form $\left(2cis30^\circ\right)^5$

Solution

$$\left(2cis30^\circ\right)^5=2^5cis\left(5(30^\circ)\right)$$

$$=32\left(\cos 150^\circ+i\sin 150^\circ\right)$$

$$=32\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$$

$$\underline{=-16\sqrt{3}+16i}$$

Exercise

Find and express the result in rectangular form $\left(\frac{1}{2}cis72^\circ\right)^5$

Solution

$$\left(\frac{1}{2}cis72^\circ\right)^5=\frac{1}{2^5}cis\left(5\times 72^\circ\right)$$

$$=\frac{1}{32}cis\left(\cos 360^\circ+i\sin 360^\circ\right)$$

$$\underline{=\frac{1}{32}}$$

Exercise

Find ***fifth*** roots of $z = 1 + i\sqrt{3}$ and express the result in rectangular form.

Solution

$$1 + i\sqrt{3} \;\Rightarrow\; \begin{cases} x = 1 \\ y = \sqrt{3} \end{cases}$$

$$r = \sqrt{1+3}$$

$$= 2$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$$

$$= 60^\circ$$

$$1 + i\sqrt{3} = 2\ cis 60^\circ$$

$$\left(1+i\sqrt{3}\right)^{1/5} = \left(2\ cis 60^\circ\right)^{1/5}$$

$$= \sqrt[5]{2}\left(cis \frac{60^\circ}{5} + \frac{360^\circ k}{5}\right)$$

$$= \sqrt[5]{2}\ cis\left(12^\circ + 72^\circ k\right) \qquad k = 0,\ 1,\ 2,\ 3,\ 4$$

For $k = 0$

$$\sqrt[5]{2}\ cis\left(12^\circ + 72^\circ 0\right) = \sqrt[5]{2}\ cis 12^\circ$$

For $k = 1$

$$\sqrt[5]{2}\ cis\left(12^\circ + 72^\circ.(1)\right) = \sqrt[5]{2}\ cis 84^\circ$$

For $k = 2$

$$\sqrt[5]{2}\ cis\left(12^\circ + 72^\circ.(2)\right) = \sqrt[5]{2}\ cis 156^\circ$$

For $k = 3$

$$\sqrt[5]{2}\ cis\left(12^\circ + 72^\circ.(3)\right) = \sqrt[5]{2}\ cis 228^\circ$$

For $k = 4$

$$\sqrt[5]{2}\ cis\left(12^\circ + 72^\circ.(4)\right) = \sqrt[5]{2}\ cis 300^\circ$$

Exercise

Find the ***fourth*** roots of $z = 16cis60°$

Solution

$$\sqrt[4]{z} = \sqrt[4]{16}\ cis\left(\frac{60°}{4} + \frac{360°}{4}k\right)$$

$$= 2cis\left(15° + 90°k\right) \qquad k = 0,\ 1,\ 2,\ 3$$

$For\ \ k = 0$

$$2\ cis\left(15° + 90°(0)\right) = 2cis15°$$

$For\ \ k = 1$

$$2\ cis\left(15° + 90°(1)\right) = 2cis105°$$

$For\ \ k = 2$

$$2\ cis\left(15° + 90°(2)\right) = 2cis195°$$

$For\ \ k = 3$

$$2\ cis\left(15° + 90°(3)\right) = 2cis285°$$

Exercise

Find the ***fourth*** roots of $\sqrt{3} - i$

Solution

$$\sqrt{3} - i \Rightarrow \begin{cases} x = \sqrt{3} \\ y = -1 \end{cases}$$

$$r = \sqrt{3+1}$$

$$= 2$$

$$\hat{\theta} = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{6}$$

The angle is in quadrant *IV*, therefore;

$$\theta = 2\pi - \frac{\pi}{6}$$

$$= \frac{11\pi}{6}$$

$$\sqrt{3} - i = 2\ cis\frac{11\pi}{6}$$

$$\sqrt[4]{\sqrt{3} - i} = \sqrt[4]{2\ cis\frac{11\pi}{6}}$$

$$= \sqrt[4]{2}\ cis\left(\frac{1}{4}\frac{11\pi}{6} + \frac{2\pi k}{4}\right)$$

$$\boxed{= \sqrt[4]{2}\ cis\left(\frac{11\pi}{24} + \frac{\pi k}{2}\right)} \qquad k = 0,\ 1,\ 2,\ 3$$

For $k = 0$

$$\sqrt[4]{2}\ cis\left(\frac{11\pi}{24} + 0\right) \boxed{= \sqrt[4]{2}\ cis\frac{11\pi}{24}}$$

For $k = 1$

$$\sqrt[4]{2}\ cis\left(\frac{11\pi}{24} + \frac{\pi}{2}\right) \boxed{= \sqrt[4]{2}\ cis\frac{23\pi}{24}}$$

For $k = 2$

$$\sqrt[4]{2}\ cis\left(\frac{11\pi}{24} + \pi\right) \boxed{= \sqrt[4]{2}\ cis\frac{35\pi}{24}}$$

For $k = 3$

$$k = 3 \Rightarrow \sqrt[4]{2}\ cis\left(\frac{11\pi}{24} + \frac{3\pi}{2}\right) \boxed{= \sqrt[4]{2}\ cis\frac{47\pi}{24}}$$

Exercise

Find the ***fourth*** roots of $4 - 4\sqrt{3}i$

Solution

$$4 - 4\sqrt{3}i \ \Rightarrow \begin{cases} x = 4 \\ y = -4\sqrt{3} \end{cases}$$

$$r = 4\sqrt{3+1}$$

$$\boxed{= 8}$$

$$\hat{\theta} = \tan^{-1}\left(\frac{4\sqrt{3}}{4}\right)$$

$$= \tan^{-1}\left(\sqrt{3}\right)$$

$$\boxed{= \frac{\pi}{3}}$$

The angle is in quadrant *IV*, therefore;

$$\theta = 2\pi - \frac{\pi}{3}$$

$$\boxed{= \frac{5\pi}{3}}$$

$$\boxed{4 - 4\sqrt{3}i = 8\ cis\frac{5\pi}{3}}$$

$$\sqrt[4]{4 - 4\sqrt{3}i} = \sqrt[4]{8\ cis\frac{5\pi}{3}}$$

$$= \boxed{\sqrt[4]{8}\ cis\left(\frac{5\pi}{12} + \frac{\pi k}{2}\right)} \qquad k = 0,\ 1,\ 2,\ 3$$

For $k = 0$

$$\sqrt[4]{8}\ cis\left(\frac{5\pi}{12} + 0\right) = \boxed{\sqrt[4]{8}\ cis\frac{5\pi}{12}}$$

For $k = 1$

$$\sqrt[4]{8}\ cis\left(\frac{5\pi}{12} + \frac{\pi}{2}\right) = \boxed{\sqrt[4]{8}\ cis\frac{11\pi}{12}}$$

For $k = 2$

$$\sqrt[4]{8}\ cis\left(\frac{5\pi}{12} + \pi\right) = \boxed{\sqrt[4]{8}\ cis\frac{17\pi}{12}}$$

For $k = 3$

$$\sqrt[4]{8}\ cis\left(\frac{5\pi}{12} + \frac{3\pi}{2}\right) = \boxed{\sqrt[4]{8}\ cis\frac{23\pi}{12}}$$

Exercise

Find the ***fourth*** roots of $-16i$

Solution

$$-16i \ \Rightarrow \ \begin{cases} x = 0 \\ y = -16 \end{cases}$$

$$\boxed{r = 16}$$

$$\boxed{\theta = \frac{3\pi}{2}}$$

$$\boxed{-16i = 16\ cis\frac{3\pi}{2}}$$

$$\sqrt[4]{-16i} = \sqrt[4]{16\ cis\frac{3\pi}{2}}$$

$$= \boxed{2\ cis\left(\frac{3\pi}{8} + \frac{\pi k}{2}\right)} \qquad k = 0,\ 1,\ 2,\ 3$$

For $k = 0$

$$2\ cis\left(\frac{3\pi}{8} + 0\right) = \boxed{2\ cis\frac{3\pi}{8}}$$

For $k = 1$

$$2\ cis\left(\frac{3\pi}{8} + \frac{\pi}{2}\right) = \boxed{2\ cis\frac{7\pi}{8}}$$

For $k = 2$

$$2\ cis\left(\frac{3\pi}{8} + \pi\right) = \boxed{2\ cis\frac{11\pi}{8}}$$

For $k = 3$

$$2\ cis\left(\frac{3\pi}{8} + \frac{3\pi}{2}\right) = \boxed{2\ cis\frac{15\pi}{8}}$$

Exercise

Find the ***cube*** roots of 27.

Solution

$$\sqrt[3]{27} = (27cis0°)^{1/3}$$

$$= \sqrt[3]{27}\ cis\left(\frac{0°}{3} + \frac{360°}{3}k\right)$$

$$\boxed{= 3\ cis(0° + 120°k)} \qquad k = 0,\ 1,\ 2$$

For $k = 0$

$$z = 3\ cis(0° + 120°(0))\ \boxed{= 2\ cis0°}$$

For $k = 1$

$$z = 3\ cis(0° + 120°(1))\ \boxed{= 2\ cis120°}$$

For $k = 2$

$$z = 3\ cis(0° + 120°(2))\ \boxed{= 2\ cis240°}$$

Exercise

Find the ***cube*** roots of $8 - 8i$

Solution

$$8 - 8i \Rightarrow \begin{cases} x = 8 \\ y = -8 \end{cases}$$

$$r = 8\sqrt{1+1}$$

$$\boxed{= 8\sqrt{2}}$$

$$\hat{\theta} = \tan^{-1}\left(\frac{8}{8}\right)$$

$$= \tan^{-1}(1)$$

$$\boxed{= \frac{\pi}{4}}$$

The angle is in quadrant *IV*, therefore;

$$\theta = 2\pi - \frac{\pi}{4}$$

$$\boxed{= \frac{7\pi}{4}}$$

$$\sqrt[3]{8-8i} = \sqrt[3]{8\sqrt{2}\ cis\frac{7\pi}{4}}$$

$$\boxed{= 2\sqrt[3]{2}\ cis\left(\frac{7\pi}{12} + \frac{2\pi k}{3}\right)} \qquad k = 0,\ 1,\ 2$$

For $k=0$

$$z=2\sqrt[3]{2}\ cis\left(\frac{7\pi}{12}+0\right)=\boxed{2\sqrt[3]{2}\ cis\frac{7\pi}{12}}$$

For $k=1$

$$z=2\sqrt[3]{2}\ cis\left(\frac{7\pi}{12}+\frac{2\pi}{3}\right)=\boxed{2\sqrt[3]{2}\ cis\frac{15\pi}{12}}$$

For $k=2$

$$z=2\sqrt[3]{2}\ cis\left(\frac{7\pi}{12}+\frac{4\pi}{3}\right)=\boxed{2\sqrt[3]{2}\ cis\frac{23\pi}{12}}$$

Exercise

Find the ***cube*** roots of -8

Solution

$\boxed{r=8}$

$\boxed{\theta=\frac{3\pi}{2}}$

$$\sqrt[3]{-8}=\sqrt[3]{8\ cis\frac{3\pi}{2}}$$

$$=\boxed{2\ cis\left(\frac{\pi}{2}+\frac{2\pi k}{3}\right)}\qquad k=0,\ 1,\ 2$$

For $k=0$

$$z=2\ cis\left(\frac{\pi}{2}+0\right)=\boxed{2\ cis\frac{\pi}{2}}$$

For $k=1$

$$z=2\ cis\left(\frac{\pi}{2}+\frac{2\pi}{3}\right)=\boxed{2\ cis\frac{7\pi}{6}}$$

For $k=2$

$$z=2\ cis\left(\frac{\pi}{2}+\frac{4\pi}{3}\right)=\boxed{2\ cis\frac{11\pi}{6}}$$

Exercise

Find all complex number solutions of $x^3+1=0$.

Solution

$x^3+1=0$

$x^3=-1$

$\boxed{r=1}$

$\theta=\tan^{-1}\left(\frac{0}{-1}\right)$

$$\boxed{=\pi}$$

$$x^3 = 1\, cis(\pi)$$

$$x = (1\, cis\pi)^{1/3}$$

$$= (1)^{1/3}\ cis\left(\frac{\pi}{3} + \frac{2\pi}{3}k\right)$$

$$\boxed{= cis\left(\frac{\pi}{3} + \frac{2\pi}{3}k\right)} \qquad k = 0,\ 1,\ 2$$

For $k = 0$

$$x = cis\left(\frac{\pi}{3} + \frac{2\pi}{3}(0)\right)$$

$$= cis\frac{\pi}{3}$$

$$= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

$$\boxed{= \frac{1}{2} + i\frac{\sqrt{3}}{2}}$$

For $k = 1$

$$x = cis\left(\frac{\pi}{3} + \frac{2\pi}{3}(1)\right)$$

$$= cis\left(\frac{3\pi}{3}\right)$$

$$= cis\pi$$

$$= \cos\pi + i\sin\pi$$

$$\boxed{= -1}$$

For $k = 2$

$$x = cis\left(\frac{\pi}{3} + \frac{2\pi}{3}(2)\right)$$

$$= cis\frac{5\pi}{3}$$

$$= \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$$

$$\boxed{= \frac{1}{2} - i\frac{\sqrt{3}}{2}}$$