Solution Section 4.1 – Introduction and Review of Power Series

Exercise

Determine the centre, radius, and interval of convergencae of the power series $\sum_{n=0}^{\infty} \frac{x^{2n}}{\sqrt{n+1}}$

Solution

$$R = \lim_{n\to\infty} \left| \frac{1}{\sqrt{n+2}} \cdot \sqrt{n+1} \right| = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}} = 1$$

The radius of convergence is 1.
The centre of convergence is 0.
The interval of convergence is (–1, 1).
The series does not converge at $x = -1$ or $x = 1$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=0}^{\infty} 3n(x+1)^n$

Solution

$$R = \lim_{n\to\infty} \left| \frac{3n}{3(n+1)} \right| \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{3n}{3n}$$

$$\underline{= 1}$$

The radius of convergence is 1, and the centre of convergence is –1. $(x+1=0)$

$a - R < x < a + R \quad \Rightarrow \quad -1-1 < x < -1+1$

Therefore, the given series convergences absolutely on $(-2,\ 0)$

At $x = -2$, the series is $\sum_{n=0}^{\infty} 3n(-1)^n$ which diverges.

At $x = 0$, the series is $\sum_{n=0}^{\infty} 3n(1)^n = \sum_{n=0}^{\infty} 3n$ which diverges.

Hence, the interval of convergence is $(-2,\ 0)$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} x^n$

Solution

$$R = \lim_{n\to\infty} \left| \frac{(n+1)^4 2^{2n+2}}{n^4 2^{2n}} \right| \qquad\qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= 4 \lim_{n\to\infty} \left| \left(\frac{n+1}{n} \right)^4 \right|$$

$$\underline{= 4}$$

The radius of convergence is 4, and the centre of convergence is 0.

$a - R < x < a + R \quad \Rightarrow \quad -4 < x < 4$, the given series convergences absolutely on $(-4,\ 4)$

At $x = -4$, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} (-4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} (-1)^n 2^{2n} = \sum_{n=1}^{\infty} \frac{1}{n^4}$ which converges (p-series).

At $x = 4$, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} (4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} 2^{2n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ which also converges.

Hence, the interval of convergence is $[-4,\ 4]$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^3} (4 - x)^n$

Solution

$$R = \lim_{n\to\infty} \left| \frac{e^n}{n^3} \cdot \frac{(n+1)^3}{e^{n+1}} \right| \qquad\qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \frac{1}{e} \lim_{n\to\infty} \left| \left(\frac{n+1}{n} \right)^3 \right|$$

$$\underline{= \frac{1}{e}}$$

The radius of convergence is $\frac{1}{e}$.

The centre of convergence is 4. $(4 - x = 0 \ \Rightarrow \ x = 4)$

$a - R < x < a + R \quad \Rightarrow \quad 4 - \frac{1}{e} < x < 4 + \frac{1}{e}$, which the given series convergences absolutely

At $x=4-\frac{1}{e}$, the series is $\sum_{n=1}^{\infty}\frac{e^n}{n^3}\left(\frac{1}{e}\right)^n=\sum_{n=1}^{\infty}\frac{1}{n^3}$ which converges (p-series).

At $x=4+\frac{1}{e}$, the series is $\sum_{n=1}^{\infty}\frac{e^n}{n^3}\left(-\frac{1}{e}\right)^n=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$ which also converges (p-series).

Hence, the interval of convergence is $\left[4-\frac{1}{e},\ 4+\frac{1}{e}\right]$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=1}^{\infty}\frac{(4x-1)^n}{n^n}$

Solution

$$\sum_{n=1}^{\infty}\frac{(4x-1)^n}{n^n}=\sum_{n=1}^{\infty}\frac{4^n\left(x-\frac{1}{4}\right)^n}{n^n}$$

$$R=\lim_{n\to\infty}\left|\frac{4^n}{n^n}\cdot\frac{(n+1)^{n+1}}{4^{n+1}}\right| \qquad R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$=\frac{1}{4}\lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^n(n+1)\right|$$

$$=\infty$$

The radius of convergence is ∞ .

The centre of convergence is $x=\frac{1}{4}$.

The interval of convergence is the real line $(-\infty,\ \infty)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=0}^{\infty}\frac{1+5^n}{n!}x^n$

Solution

$$R=\lim_{n\to\infty}\left|\frac{1+5^n}{n!}\cdot\frac{(n+1)!}{1+5^{n+1}}\right| \qquad R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{1+5^n}{1+5^{n+1}}(n+1)\right|$$

$$=\infty$$

The radius of convergence is ∞ .
The centre of convergence is 0.
The interval of convergence is the real line $(-\infty, \infty)$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = e^{2x}, \quad a = 0$

Solution

$$f(x) = e^{2x} \quad \rightarrow \quad f(0) = e^{2(0)} = 1$$

$$f'(x) = 2e^{2x} \quad \rightarrow \quad f'(0) = 2$$

$$f''(x) = 4e^{2x} \quad \rightarrow \quad f''(0) = 4$$

$$f'''(x) = 8e^{2x} \quad \rightarrow \quad f'''(0) = 8$$

$$P_0(x) = f(0)$$
$$= 1$$

$$P_1(x) = f(0) + f'(0)(x-0)$$
$$= 1 + 2x$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$$
$$= 1 + 2x + 2x^2$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$
$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = \sin x, \quad a = 0$

Solution

$$f(x) = \sin x \quad \rightarrow \quad f(0) = 0$$

$$f'(x) = \cos x \quad \rightarrow \quad f'(0) = 1$$

$$f''(x) = -\sin x \quad \rightarrow \quad f''(0) = 0$$

$$f'''(x) = -\cos x \quad \rightarrow \quad f'''(0) = -1$$

$$P_0(x) = f(0)$$
$$= 0$$

$$P_1(x) = f(0) + f'(0)(x-0)$$

$$\boxed{= x}$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$\boxed{= x}$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$$\boxed{= x - \frac{1}{6}x^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = \ln(1+x), \quad a = 0$

Solution

$$f(x) = \ln(1+x) \quad \to \quad f(0) = 0$$

$$f'(x) = \frac{1}{1+x} \quad \to \quad f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} \quad \to \quad f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \quad \to \quad f'''(0) = 2$$

$$P_0(x) = f(0)$$

$$\boxed{= 0}$$

$$P_1(x) = f(0) + f'(0)(x-0)$$

$$\boxed{= x}$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$\boxed{= x - \frac{1}{2}x^2}$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$$\boxed{= x - \frac{1}{2}x^2 + \frac{1}{3}x^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\frac{1}{x+2}, \quad a=0$

Solution

$$f(x)=(x+2)^{-1} \quad \to \quad f(0)=\frac{1}{2}$$

$$f'(x)=-(x+2)^{-2} \quad \to \quad f'(0)=-\frac{1}{4}$$

$$f''(x)=2(x+2)^{-3} \quad \to \quad f''(0)=\frac{1}{4}$$

$$f'''(x)=-6(x+2)^{-4} \quad \to \quad f'''(0)=-\frac{3}{8}$$

$$P_0(x)=f(0)$$

$$=\frac{1}{2}$$

$$P_1(x)=f(0)+f'(0)(x-0)$$

$$=\frac{1}{2}-\frac{1}{4}x$$

$$P_2(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2$$

$$=\frac{1}{2}-\frac{1}{4}x+\frac{1}{8}x^2$$

$$P_3(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3$$

$$=\frac{1}{2}-\frac{1}{4}x+\frac{1}{8}x^2-\frac{1}{16}x^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\sqrt{1-x}, \quad a=0$

Solution

$$f(x)=(1-x)^{1/2} \quad \to \quad f(0)=1$$

$$f'(x)=-\frac{1}{2}(1-x)^{-1/2} \quad \to \quad f(0)=-\frac{1}{2}$$

$$f''(x)=-\frac{1}{4}(1-x)^{-3/2} \quad \to \quad f(0)=-\frac{1}{4}$$

$$f'''(x)=-\frac{3}{8}(1-x)^{-5/2} \quad \to \quad f(0)=-\frac{3}{8}$$

$$P_0(x)=f(0)$$

$$=1$$

$$P_1(x) = f(0) + f'(0)(x-0)$$

$$\boxed{= 1 - \frac{1}{2}x}$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$\boxed{= 1 - \frac{1}{2}x - \frac{1}{8}x^2}$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$$\boxed{= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3}$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = x^3, \quad a = 1$

Solution

$f(x) = x^3 \quad \rightarrow \quad f(1) = 1$

$f'(x) = 3x^2 \quad \rightarrow \quad f'(1) = 3$

$f''(x) = 6x \quad \rightarrow \quad f''(1) = 6$

$f'''(x) = 6 \quad \rightarrow \quad f'''(1) = 6$

$\boxed{P_0(x) = 1}$ $\qquad P_0(x) = f(a)$

$\boxed{P_1(x) = 1 + 3(x-1)}$ $\qquad P_1(x) = f(a) + f'(a)(x-a)$

$\boxed{P_2(x) = 1 + 3(x-1) + 3(x-1)^2}$ $\qquad P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$

$\boxed{P_3(x) = 1 + 3(x-1) + 3(x-1)^2 + (x-1)^3}$ $\qquad P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = 8\sqrt{x}, \quad a = 1$

Solution

$f(x) = 8x^{1/2} \quad \rightarrow \quad f(1) = 8$

$f'(x) = 4x^{-1/2} \quad \rightarrow \quad f'(1) = 4$

$f''(x) = -2x^{-3/2} \quad \rightarrow \quad f''(1) = -2$

$f'''(x) = 3x^{-5/2} \quad \rightarrow \quad f'''(1) = 3$

$$\boxed{P_0(x)=8}$$ $$P_0(x)=f(a)$$

$$\boxed{P_1(x)=8+4(x-1)}$$ $$P_1(x)=f(a)+f'(a)(x-a)$$

$$\boxed{P_2(x)=8+4(x-1)-(x-1)^2}$$ $$P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{P_3(x)=8+4(x-1)-(x-1)^2+3(x-1)^3}$$ $$P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\sin x, \quad a=\frac{\pi}{4}$

Solution

$$f(x)=\sin x \quad \to \quad f\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

$$f'(x)=\cos x \quad \to \quad f'\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

$$f''(x)=-\sin x \quad \to \quad f''\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$$

$$f'''(x)=-\cos x \quad \to \quad f'''\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$$

$$\boxed{P_0(x)=\frac{\sqrt{2}}{2}}$$ $$P_0(x)=f(a)$$

$$\boxed{P_1(x)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)}$$ $$P_1(x)=f(a)+f'(a)(x-a)$$

$$\boxed{P_2(x)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2}$$ $$P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{P_3(x)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2-\frac{\sqrt{2}}{12}\left(x-\frac{\pi}{4}\right)^3}$$ $$P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\cos x, \quad a=\frac{\pi}{6}$

Solution

$$f(x)=\cos x \quad \to \quad f\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$$

$$f'(x)=-\sin x \quad \to \quad f'\left(\frac{\pi}{6}\right)=-\frac{1}{2}$$

$$f''(x)=-\cos x \quad \to \quad f''\left(\frac{\pi}{6}\right)=-\frac{\sqrt{3}}{2}$$

$$f'''(x) = \sin x \quad \to \quad f'''\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$\boxed{P_0(x) = \frac{\sqrt{3}}{2}} \qquad\qquad P_0(x) = f(a)$$

$$\boxed{P_1(x) = \frac{\sqrt{2}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right)} \qquad\qquad P_1(x) = f(a) + f'(a)(x-a)$$

$$\boxed{P_2(x) = \frac{\sqrt{2}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{6}\right)^2} \qquad\qquad P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{P_3(x) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{6}\right)^2 + \frac{1}{12}\left(x - \frac{\pi}{6}\right)^3} \qquad P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = \sqrt{x}, \quad a = 9$

Solution

$$f(x) = x^{1/2} \quad \to \quad f(9) = 3$$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \quad \to \quad f'(9) = \frac{1}{6}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} = -\frac{1}{4x\sqrt{x}} \quad \to \quad f''(9) = -\frac{1}{4\times 3^3}$$

$$f'''(x) = \frac{3}{8}x^{-5/2} = \frac{3}{8x^2\sqrt{x}} \quad \to \quad f'''(9) = \frac{1}{2^3\times 3^4}$$

$$\boxed{P_0(x) = 3} \qquad\qquad P_0(x) = f(a)$$

$$\boxed{P_1(x) = 3 + \frac{1}{6}(x-9)} \qquad\qquad P_1(x) = f(a) + f'(a)(x-a)$$

$$\boxed{P_2(x) = 3 + \frac{1}{2\cdot 3}(x-9) - \frac{1}{2^3\cdot 3^3}(x-9)^2} \qquad\qquad P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{P_3(x) = 3 + \frac{1}{2\cdot 3}(x-9) - \frac{1}{2^2\cdot 3^3}(x-9)^2 + \frac{1}{2^4\cdot 3^5}(x-9)^3} \qquad P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = \sqrt[3]{x}, \quad a = 8$

Solution

$$f(x) = x^{1/3} \quad \to \quad f(8) = 2$$

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} \quad \to \quad f'(8) = \frac{1}{2^2\times 3}$$

$$f''(x) = -\frac{2}{9}x^{-5/3} = -\frac{2}{3^2 x^{5/3}} \quad \to \quad f''(8) = -\frac{1}{3^2 \times 2^4}$$

$$f'''(x) = \frac{10}{3^3}x^{-8/3} = \frac{2 \cdot 5}{3^3 x^{8/3}} \quad \to \quad f'''(8) = \frac{5}{2^7 \times 3^3}$$

$$\boxed{P_0(x) = 2} \qquad P_0(x) = f(a)$$

$$\boxed{P_1(x) = 2 + \frac{1}{2^2 \cdot 3}(x-8)} \qquad P_1(x) = f(a) + f'(a)(x-a)$$

$$\boxed{P_2(x) = 2 + \frac{1}{2^2 \cdot 3}(x-8) - \frac{1}{2^5 \cdot 3^2}(x-8)^2} \qquad P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{P_3(x) = 2 + \frac{1}{2^2 \cdot 3}(x-8) - \frac{1}{2^5 \cdot 3^2}(x-8)^2 + \frac{1}{2^8 \cdot 3^4}(x-8)^3} \qquad P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = \ln x, \quad a = e$

Solution

$$f(x) = \ln x \quad \to \quad f(e) = 1$$

$$f'(x) = \frac{1}{x} \quad \to \quad f'(e) = \frac{1}{e}$$

$$f''(x) = -\frac{1}{x^2} \quad \to \quad f''(e) = -\frac{1}{e^2}$$

$$f'''(x) = \frac{2}{x^3} \quad \to \quad f'''(e) = \frac{2}{e^3}$$

$$\boxed{P_0(x) = 1} \qquad P_0(x) = f(a)$$

$$\boxed{P_1(x) = 1 + \frac{1}{e}(x-e)} \qquad P_1(x) = f(a) + f'(a)(x-a)$$

$$\boxed{P_2(x) = 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2} \qquad P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{P_3(x) = 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3} \qquad P_3(x) = P_2(x) + \frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x) = \sqrt[4]{x}, \quad a = 8$

Solution

$$f(x) = x^{1/4} \quad \to \quad f(8) = \sqrt[4]{8}$$

$$f'(x)=\frac{1}{4}x^{-3/4}=\frac{1}{4x^{3/4}} \quad\to\quad f'\left(8=2^3\right)=\frac{1}{2^2\times 2^{9/4}}=\frac{1}{2^4\sqrt[4]{2}}$$

$$f''(x)=-\frac{3}{16}x^{-7/4}=-\frac{3}{2^4x^{7/4}} \quad\to\quad f''(8)=-\frac{3}{2^4\,2^{21/4}}=-\frac{3}{2^9\sqrt[4]{2}}$$

$$f'''(x)=\frac{21}{2^6}x^{-11/4}=\frac{21}{2^6x^{11/4}} \quad\to\quad f'''(8)=\frac{21}{2^6\times 2^{33/4}}=\frac{21}{2^{14}\sqrt[4]{2}}$$

$$\boxed{P_0(x)=\sqrt[4]{8}} \qquad P_0(x)=f(a)$$

$$\boxed{P_1(x)=\sqrt[4]{8}+\frac{1}{2^4\cdot\sqrt[4]{2}}(x-8)} \qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$\boxed{P_2(x)=\sqrt[4]{8}+\frac{1}{2^4\cdot\sqrt[4]{2}}(x-8)-\frac{3}{2^{10}\cdot\sqrt[4]{2}}(x-8)^2} \qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{P_3(x)=\sqrt[4]{8}+\frac{1}{2^4\cdot\sqrt[4]{2}}(x-8)-\frac{3}{2^{10}\cdot\sqrt[4]{2}}(x-8)^2+\frac{7}{2^{15}\cdot\sqrt[4]{2}}(x-8)^3} \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=\tan^{-1}x+x^2+1, \quad a=1$

Solution

$$f(x)=\tan^{-1}x+x^2+1 \quad\to\quad f(1)=\frac{\pi}{4}+2$$

$$f'(x)=\frac{1}{x^2+1}+2x \quad\to\quad f'(1)=\frac{5}{2}$$

$$f''(x)=-\frac{2x}{\left(x^2+1\right)^2}+2 \quad\to\quad f''(1)=-\frac{1}{2}+2=\frac{3}{2}$$

$$f'''(x)=-\frac{2x^2+2-8x^2}{\left(x^2+1\right)^3}=-\frac{2-2x^2}{\left(x^2+1\right)^3} \quad\to\quad f'''(1)=0 \qquad \left(U^nV^m\right)'=U^{n-1}V^{m-1}\left(nU'V+mUV'\right)$$

$$\boxed{P_0(x)=\frac{\pi}{4}+2} \qquad P_0(x)=f(a)$$

$$\boxed{P_1(x)=\frac{\pi}{4}+2+\frac{5}{2}(x-1)} \qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$\boxed{P_2(x)=\frac{\pi}{4}+2+\frac{5}{2}(x-1)-\frac{3}{4}(x-1)^2} \qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{P_3(x)=\frac{\pi}{4}+2+\frac{5}{2}(x-1)-\frac{3}{4}(x-1)^2} \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a: $f(x)=e^{x}, \quad a=\ln 2$

Solution

$$f(x)=e^{x} \quad \to \quad f(\ln 2)=2$$

$$f'(x)=e^{x} \quad \to \quad f'(\ln 2)=2$$

$$f''(x)=e^{x} \quad \to \quad f''(\ln 2)=2$$

$$f'''(x)=e^{x} \quad \to \quad f'''(\ln 2)=2$$

$$\boxed{P_0(x)=2} \qquad P_0(x)=f(a)$$

$$\boxed{P_1(x)=2+2(x-\ln 2)} \qquad P_1(x)=f(a)+f'(a)(x-a)$$

$$\boxed{P_2(x)=2+2(x-\ln 2)+(x-\ln 2)^2} \qquad P_2(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$\boxed{P_3(x)=2+2(x-\ln 2)+(x-\ln 2)^2+\frac{1}{3}(x-\ln 2)^3} \qquad P_3(x)=P_2(x)+\frac{f'''(a)}{3!}(x-a)^3$$

Exercise

Find the *n*th Maclaurin polynomial for the function $\quad f(x)=e^{4x}, \quad n=4$

Solution

$$f(x)=e^{4x} \quad \to \quad f(0)=1$$

$$f'(x)=4e^{4x} \quad \to \quad f'(0)=4$$

$$f''(x)=16e^{4x} \quad \to \quad f''(0)=16$$

$$f'''(x)=64e^{4x} \quad \to \quad f'''(0)=64$$

$$f^{(4)}(x)=256e^{4x} \quad \to \quad f^{(4)}(0)=256$$

$$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$

$$\boxed{P_4(x)=1+4x+8x^2+\frac{32}{3}x^3+\frac{32}{3}x^4}$$

Exercise

Find the *n*th Maclaurin polynomial for the function $\quad f(x)=e^{-x}, \quad n=5$

Solution

$$f(x)=e^{-x} \quad \to \quad f(0)=1$$

$$f'(x)=-e^{-x} \quad \to \quad f'(0)=-1$$

$$f''(x)=e^{-x} \quad \to \quad f''(0)=1$$

$$f'''(x)=-e^{-x} \quad \to \quad f'''(0)=-1$$

$$f^{(4)}(x)=e^{-x} \quad \to \quad f^{(4)}(0)=1$$

$$f^{(5)}(x)=-e^{-x} \quad \to \quad f^{(5)}(0)=-1$$

$$P_5(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\frac{f^{(5)}(0)}{5!}x^5$$

$$P_5(x)=1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{120}x^5$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x)=e^{-x/2}, \quad n=4$

Solution

$$f(x)=e^{-x/2} \quad \to \quad f(0)=1$$

$$f'(x)=-\frac{1}{2}e^{-x/2} \quad \to \quad f'(0)=-\frac{1}{2}$$

$$f''(x)=\frac{1}{4}e^{-x/2} \quad \to \quad f''(0)=\frac{1}{4}$$

$$f'''(x)=-\frac{1}{8}e^{-x/2} \quad \to \quad f'''(0)=-\frac{1}{8}$$

$$f^{(4)}(x)=\frac{1}{16}e^{-x/2} \quad \to \quad f^{(4)}(0)=\frac{1}{16}$$

$$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$

$$P_4(x)=1-\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{48}x^3+\frac{1}{384}x^4$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x)=e^{x/3}, \quad n=4$

Solution

$$f(x)=e^{x/3} \quad \to \quad f(0)=1$$

$$f'(x)=\frac{1}{3}e^{x/3} \quad \to \quad f'(0)=\frac{1}{3}$$

$$f''(x)=\frac{1}{9}e^{x/3} \quad \to \quad f''(0)=\frac{1}{9}$$

$$f'''(x)=\frac{1}{27}e^{x/3} \quad \to \quad f'''(0)=\frac{1}{27}$$

$$f^{(4)}(x)=\frac{1}{81}e^{x/3} \quad \to \quad f^{(4)}(0)=\frac{1}{81}$$

$$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$

$$\boxed{P_4(x)=1+\frac{1}{3}x+\frac{1}{18}x^2+\frac{1}{162}x^3+\frac{1}{1944}x^4}$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x)=\sin x, \quad n=5$

Solution

$$f(x)=\sin x \quad \to \quad f(0)=0$$

$$f'(x)=\cos x \quad \to \quad f'(0)=1$$

$$f''(x)=-\sin x \quad \to \quad f''(0)=0$$

$$f'''(x)=-\cos x \quad \to \quad f'''(0)=-1$$

$$f^{(4)}(x)=\sin x \quad \to \quad f^{(4)}(0)=0$$

$$f^{(5)}(x)=\cos x \quad \to \quad f^{(5)}(0)=1$$

$$P_5(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\frac{f^{(5)}(0)}{5!}x^5$$

$$\boxed{P_5(x)=x-\frac{1}{6}x^3+\frac{1}{120}x^5}$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x)=\cos \pi x, \quad n=4$

Solution

$$f(x)=\cos \pi x \quad \to \quad f(0)=1$$

$$f'(x)=-\pi \sin \pi x \quad \to \quad f'(0)=0$$

$$f''(x)=-\pi^2 \cos \pi x \quad \to \quad f''(0)=-\pi^2$$

$$f'''(x)=\pi^3 \sin \pi x \quad \to \quad f'''(0)=0$$

$$f^{(4)}(x)=\pi^4 \cos \pi x \quad \to \quad f^{(4)}(0)=\pi^4$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$P_4(x) = 1 - \frac{\pi^2}{2}x^2 + \frac{\pi^4}{24}x^4$$

Exercise

Find the nth Maclaurin polynomial for the function $f(x) = xe^x, \quad n = 4$

Solution

$$f(x) = xe^x \quad \to \quad f(0) = 0$$

$$f'(x) = e^x + xe^x \quad \to \quad f'(0) = 1$$

$$f''(x) = 2e^x + xe^x \quad \to \quad f''(0) = 2$$

$$f'''(x) = 3e^x + xe^x \quad \to \quad f'''(0) = 3$$

$$f^{(4)}(x) = 4e^x + xe^x \quad \to \quad f^{(4)}(0) = 4$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$P_4(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$$

Exercise

Find the nth Maclaurin polynomial for the function $f(x) = x^2e^{-x}, \quad n = 4$

Solution

$$f(x) = x^2e^{-x} \quad \to \quad f(0) = 0$$

$$f'(x) = 2xe^{-x} - x^2e^{-x} \quad \to \quad f'(0) = 0$$

$$f''(x) = 2e^{-x} - 4xe^{x} + x^2e^{-x} \quad \to \quad f''(0) = 2$$

$$f'''(x) = -6e^{-x} + 6xe^{-x} - x^2e^{-x} \quad \to \quad f'''(0) = -6$$

$$f^{(4)}(x) = 12e^{-x} - 8xe^{-x} + x^2e^{-x} \quad \to \quad f^{(4)}(0) = 12$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$P_4(x) = x^2 - x^3 + \frac{1}{2}x^4$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x)=\dfrac{1}{x+1}, \quad n=5$

Solution

$$f(x)=\frac{1}{x+1} \quad \to \quad f(0)=1$$

$$f'(x)=-(x+1)^{-2} \quad \to \quad f'(0)=-1$$

$$f''(x)=2(x+1)^{-3} \quad \to \quad f''(0)=2$$

$$f'''(x)=-6(x+1)^{-4} \quad \to \quad f'''(0)=-6$$

$$f^{(4)}(x)=24(x+1)^{-5} \quad \to \quad f^{(4)}(0)=24$$

$$f^{(5)}(x)=-120(x+1)^{-6} \quad \to \quad f^{(5)}(0)=-120$$

$$P_5(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\frac{f^{(5)}(0)}{5!}x^5$$

$$\boxed{P_5(x)=1-x+x^2-x^3+x^4-x^5}$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x)=\dfrac{x}{x+1}, \quad n=4$

Solution

$$f(x)=\frac{x}{x+1}=1-\frac{1}{x+1} \quad \to \quad f(0)=0$$

$$f'(x)=(x+1)^{-2} \quad \to \quad f'(0)=1$$

$$f''(x)=-2(x+1)^{-3} \quad \to \quad f''(0)=-2$$

$$f'''(x)=6(x+1)^{-4} \quad \to \quad f'''(0)=6$$

$$f^{(4)}(x)=-24(x+1)^{-5} \quad \to \quad f^{(4)}(0)=-24$$

$$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$

$$\boxed{P_4(x)=x-x^2+x^3-x^4}$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x)=\sec x, \quad n=2$

Solution

$$f(x)=\sec x \quad \to \quad f(0)=1$$

$$f'(x) = \sec x \tan x \quad \to \quad f'(0) = 0$$

$$f''(x) = \sec x \tan^2 x + \sec^3 x \quad \to \quad f''(0) = 1$$

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$

$$\boxed{P_2(x) = 1 + \frac{1}{2}x^2}$$

Exercise

Find the nth Maclaurin polynomial for the function $\quad f(x) = \tan x, \quad n = 3$

Solution

$$f(x) = \tan x \quad \to \quad f(0) = 0$$

$$f'(x) = \sec^2 x \quad \to \quad f'(0) = 1$$

$$f''(x) = 2\sec^2 x \tan x \quad \to \quad f''(0) = 0$$

$$f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x \quad \to \quad f'''(0) = 2$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3$$

$$\boxed{P_4(x) = x + \frac{1}{3}x^3}$$

Exercise

Find the Maclaurin series for: $\quad xe^x$

Solution

$$f(x) = xe^x \quad \to \quad f(0) = 0$$

$$f'(x) = e^x + xe^x \quad \to \quad f'(0) = 1$$

$$f''(x) = 2e^x + xe^x \quad \to \quad f''(0) = 2$$

$$f'''(x) = 3e^x + xe^x \quad \to \quad f'''(0) = 3$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$f^{(n)}(x) = ne^x + xe^x \quad \to \quad f^{(n)}(0) = n$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$xe^x = x + x^2 + \frac{1}{2}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{(n-1)!}x^n$$

Exercise

Find the Maclaurin series for: $5\cos \pi x$

Solution

$$f(x) = 5\cos \pi x \quad \rightarrow \quad f(0) = 5$$

$$f'(x) = -5\pi \sin \pi x \quad \rightarrow \quad f'(0) = 0$$

$$f''(x) = -5\pi^2 \cos \pi x \quad \rightarrow \quad f''(0) = -5\pi^2$$

$$f'''(x) = 5\pi^3 \sin \pi x \quad \rightarrow \quad f'''(0) = 0$$

$$5\cos \pi x = 5 - \frac{5\pi^2 x^2}{2!} + \frac{5\pi^4 x^4}{4!} - \frac{5\pi^6 x^6}{6!} + \cdots$$

$$= 5\sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!}$$

Exercise

Find the Maclaurin series for: $\dfrac{x^2}{x+1}$

Solution

$$f(x) = \frac{x^2}{x+1} \quad \rightarrow \quad f(0) = 0$$

$$f'(x) = \frac{2x(x+1) - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2} \quad \rightarrow \quad f'(0) = 0$$

$$f''(x) = \frac{(2x+2)(x+1)^2 - 2(x+1)\left(x^2+2x\right)}{(x+1)^4}$$

$$= \frac{2x^2 + 4x + 2 - 2x^2 - 4x}{(x+1)^3}$$

$$= \frac{2}{(x+1)^3} \quad \rightarrow \quad f''(0) = 2$$

$$f'''(x) = \frac{-6}{(x+1)^4} \quad \rightarrow \quad f'''(0) = -6$$

$$\frac{x^2}{x+1} = \frac{2}{2!}x^2 - \frac{6x^3}{3!} + \frac{24x^4}{4!} - \cdots = x^2 - x^3 + x^4 - \cdots$$

$$= \sum_{n=2}^{\infty} (-1)^n x^n$$

Exercise

Find the Maclaurin series for: e^{3x+1}

Solution

$$e^{3x+1} = e \cdot e^{3x}$$

$$= e\left(\sum_{n=0}^{\infty} \frac{(3x)^n}{n!}\right)$$

$$\boxed{= \sum_{n=0}^{\infty} \frac{e3^n x^n}{n!}} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\cos\left(2x^3\right)$

Solution

$$\cos\left(2x^3\right) = 1 - \frac{\left(2x^3\right)^2}{2!} + \frac{\left(2x^3\right)^4}{4!} - \frac{\left(2x^3\right)^6}{6!} + \cdots$$

$$= 1 - \frac{2^2 x^3}{2!} + \frac{2^4 x^{12}}{4!} - \frac{2^6 x^{18}}{6!} + \cdots$$

$$\boxed{= \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n)!} x^{6n}} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\cos(2x - \pi)$

Solution

$$\cos(2x - \pi) = \cos(2x)\cos\pi + \sin(2x)\sin\pi$$

$$= -\cos(2x)$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x)^{2n}$$

$$\boxed{= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 4^n}{(2n)!} x^{2n}} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $x^2 \sin\left(\frac{x}{3}\right)$

Solution

$$x^2 \sin\left(\frac{x}{3}\right) = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{x}{3}\right)^{2n+1}}{(2n+1)!}$$

$$= x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{3^{2n+1}(2n+1)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{3^{2n+1}(2n+1)!} \quad (\textit{for all } x)$$

Exercise

Find the Maclaurin series for: $\cos^2\left(\frac{x}{2}\right)$

Solution

$$\cos^2\left(\frac{x}{2}\right) = \frac{1}{2}(1 + \cos x)$$

$$= \frac{1}{2}\left(1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}\right)$$

$$= \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$= 1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad (\textit{for all } x)$$

Exercise

Find the Maclaurin series for: $\sin x \cos x$

Solution

$$\sin x \cos x = \frac{1}{2}\sin(2x)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(2x)^{2n+1}$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}2^{2n+1}x^{2n+1}$$

$$\boxed{=\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}}{(2n+1)!}x^{2n+1}} \quad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\tan^{-1}\left(5x^2\right)$

Solution

$$\tan^{-1}\left(5x^2\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(5x^2\right)^{2n+1} \qquad 5x^2\le 1 \rightarrow x^2\le\frac{1}{5} \;\Rightarrow\; -\frac{1}{\sqrt{5}}\le x\le\frac{1}{\sqrt{5}}$$

$$\boxed{=\sum_{n=0}^{\infty}\frac{(-1)^n 5^{2n+1}}{2n+1}x^{4n+2}} \quad \left(\text{for } -\frac{1}{\sqrt{5}}\le x\le\frac{1}{\sqrt{5}}\right)$$

Exercise

Find the Maclaurin series for: $\ln\left(2+x^2\right)$

Solution

$$\ln\left(2+x^2\right)=\ln 2\left(1+\frac{x^2}{2}\right)$$

$$=\ln 2+\ln\left(1+\frac{x^2}{2}\right)$$

$$=\ln 2+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{x^2}{2}\right)^n \qquad \frac{x^2}{2}\le 1 \rightarrow x^2\le 2 \;\Rightarrow\; -\sqrt{2}\le x\le\sqrt{2}$$

$$\boxed{=\ln 2+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}\frac{x^{2n}}{2^n}} \quad \left(\text{for } -\sqrt{2}\le x\le\sqrt{2}\right)$$

Exercise

Find the Maclaurin series for: $\dfrac{1+x^3}{1+x^2}$

Solution

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n = 1-x+x^2-x^3+\cdots$$

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+\cdots$$

$$\frac{1+x^3}{1+x^2}=\left(1+x^3\right)\left(1-x^2+x^4-x^6+\cdots\right)$$

$$=1-x^2+x^4-x^6+\cdots+x^3-x^5+x^7-x^9+\cdots$$

$$=1-x^2+x^3+x^4-x^5-x^6+x^7+x^8-x^9-\cdots$$

$$=1-x^2+\sum_{n=2}^{\infty}(-1)^n\left(x^{2n-1}+x^{2n}\right) \qquad \left(\text{for } |x|<1\right)$$

Exercise

Find the Maclaurin series for: $\ln\frac{1+x}{1-x}$

Solution

$$\ln\frac{1+x}{1-x}=\ln(1+x)-\ln(1-x) \qquad = x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots-\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots\right)=2x+2\frac{x^3}{3}+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}+\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$$

$$=\sum_{n=0}^{\infty}\left((-1)^n+1\right)\frac{x^{n+1}}{n+1}$$

$$=2\sum_{n=1}^{\infty}\frac{x^{2n-1}}{2n-1} \qquad (-1<x<1)$$

Exercise

Find the Maclaurin series for: $\frac{e^{2x^2}-1}{x^2}$

Solution

$$\frac{e^{2x^2}-1}{x^2}=\frac{1}{x^2}\left(e^{2x^2}-1\right)$$

$$=\frac{1}{x^2}\left(1+2x^2+\frac{\left(2x^2\right)^2}{2!}+\frac{\left(2x^2\right)^3}{3!}+\cdots-1\right)$$

$$=\frac{1}{x^2}\left(2x^2+\frac{2^2x^4}{2!}+\frac{2^3x^6}{3!}+\cdots\right)$$

$$=2+\frac{2^2x^2}{2!}+\frac{2^3x^4}{3!}+\frac{2^4x^6}{4!}+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{2^{n+1}}{(n+1)!}x^{2n} \qquad (\text{for all } \ x\neq 0)$$

Exercise

Find the Maclaurin series for: $\cosh x-\cos x$

Solution

$$\cosh x-\cos x=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} \qquad 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)=2\frac{x^2}{2!}+2\frac{x^6}{6!}+\cdots$$

$$=\sum_{n=0}^{\infty}\left(1-(-1)^n\right)\frac{x^{2n}}{(2n)!}$$

$$=2\sum_{n=0}^{\infty}\frac{x^{4n+2}}{(4n+2)!} \qquad (\text{for all } x)$$

Exercise

Find the Maclaurin series for: $\sinh x-\sin x$

Solution

$$\sinh x-\sin x=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

$$=\sum_{n=0}^{\infty}\left(1-(-1)^n\right)\frac{x^{2n+1}}{(2n+1)!}$$

$$=2\sum_{n=0}^{\infty}\frac{x^{4n+3}}{(4n+3)!} \qquad (\text{for all } x)$$

Exercise

Finding Taylor and Maclaurin Series generated by f at $x = a$: $f(x) = x^3 - 2x + 4, \quad a = 2$

Solution

$$f(x) = x^3 - 2x + 4 \quad \to \quad f(2) = 8$$

$$f'(x) = 3x^2 - 2 \quad \to \quad f'(2) = 10$$

$$f''(x) = 6x \quad \to \quad f''(2) = 12$$

$$f'''(x) = 6 \quad \to \quad f'''(2) = 6$$

$$f^{(n)}(x) = 0 \quad \to \quad f^{(n)}(2) = 0 \quad (n > 3)$$

$$P_n(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \cdots$$

$$x^3 - 2x + 4 = 8 + 10(x-2) + 6(x-2)^2 + (x-2)^3$$

Exercise

Finding Taylor and Maclaurin Series generated by f at $x = a$: $f(x) = 2x^3 + x^2 + 3x - 8, \quad a = 1$

Solution

$$f(x) = 2x^3 + x^2 + 3x - 8 \quad \to \quad f(1) = -2$$

$$f'(x) = 6x^2 + 2x + 3 \quad \to \quad f'(1) = 11$$

$$f''(x) = 12x + 2 \quad \to \quad f''(1) = 14$$

$$f'''(x) = 12 \quad \to \quad f'''(1) = 12$$

$$f^{(n)}(x) = 0 \quad \to \quad f^{(n)}(1) = 0 \quad (n \ge 4)$$

$$P_n(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots$$

$$2x^3 + x^2 + 3x - 8 = -2 + 11(x-1) + 7(x-1)^2 + 2(x-1)^3$$

Exercise

Finding Taylor and Maclaurin Series generated by f at $x = a$:

$$f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2, \quad a = -1$$

Solution

$$f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2 \quad \to \quad f(-1) = -7$$

$$f'(x) = 15x^4 - 4x^3 + 6x^2 + 2x \quad \to \quad f'(-1) = 23$$

$$f''(x) = 60x^3 - 12x^2 + 12x + 2 \quad \to \quad f''(-1) = -82$$

$$f'''(x) = 180x^2 - 24x + 12 \quad \to \quad f'''(-1) = 216$$

$$f^{(4)}(x) = 360x - 24 \quad \to \quad f^{(4)}(-1) = -384$$

$$f^{(5)}(x) = 360 \quad \to \quad f^{(5)}(-1) = 360$$

$$f^{(n)}(x) = 0 \quad \to \quad f^{(n)}(-1) = 0 \quad (n \ge 6)$$

$$P_n(x) = f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \frac{f'''(-1)}{3!}(x+1)^3 + \frac{f^{(4)}(-1)}{4!}(x+1)^2 + \frac{f^{(4)}(-1)}{5!}(x+1)^3$$

$$3x^5 - x^4 + 2x^3 + x^2 - 2 = -7 + 23(x+1) - \frac{82}{2!}(x+1)^2 + \frac{216}{3!}(x+1)^3 - \frac{384}{4!}(x+1)^4 + \frac{360}{5!}(x+1)^3$$

$$= -7 + 23(x+1) - 41(x+1)^2 + 36(x+1)^3 - 16(x+1)^4 + 3(x+1)^3$$

Exercise

Finding Taylor and Maclaurin Series generated by f *at* $x = a$: $\quad f(x) = \cos\left(2x + \frac{\pi}{2}\right), \quad a = \frac{\pi}{4}$

Solution

$$f(x) = \cos\left(2x + \frac{\pi}{2}\right) \quad \to \quad f\left(\frac{\pi}{4}\right) = -1$$

$$f'(x) = -2\sin\left(2x + \frac{\pi}{2}\right) \quad \to \quad f'\left(\frac{\pi}{4}\right) = 0$$

$$f''(x) = -4\cos\left(2x + \frac{\pi}{2}\right) \quad \to \quad f''\left(\frac{\pi}{4}\right) = 4$$

$$f'''(x) = 8\sin\left(2x + \frac{\pi}{2}\right) \quad \to \quad f'''\left(\frac{\pi}{4}\right) = 0$$

$$f^{(4)}(x) = 16\cos\left(2x + \frac{\pi}{2}\right) \quad \to \quad f^{(4)}\left(\frac{\pi}{4}\right) = -16$$

$$f^{(5)}(x) = -32\sin\left(2x + \frac{\pi}{2}\right) \quad \to \quad f^{(5)}\left(\frac{\pi}{4}\right) = 0$$

$$\to \quad f^{(2n)}\left(\frac{\pi}{4}\right) = (-1)^n 2^{2n}$$

$$\cos\left(2x + \frac{\pi}{2}\right) = -1 + \frac{4}{2!}\left(x - \frac{\pi}{4}\right)^2 - \frac{16}{4!}\left(x - \frac{\pi}{4}\right)^4 + \cdots$$

$$= -1 + 2\left(x - \frac{\pi}{4}\right)^2 - \frac{2}{3}\left(x - \frac{\pi}{4}\right)^4 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!}\left(x - \frac{\pi}{4}\right)^{2n}$$

Solution   Section 4.2 – Series Solutions near Ordinary Points

Exercise

Find a power series solution. $y' = 3y$

Solution

The equation $y' = 3y$ is separable with solution

$$\frac{dy}{dx} = 3y$$

$$\int \frac{dy}{y} = \int 3dx$$

$$y = Ce^{3x}$$

$$\ln(y) = 3x + C$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y' - 3y = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 3\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1) a_{n+1} - 3a_n \right] x^n = 0$$

$$(n+1) a_{n+1} - 3a_n = 0 \quad \Rightarrow \quad a_{n+1} = \frac{3a_n}{n+1}; \quad n \ge 0$$

With $y(0) = 3a_0$

$$a_1 = 3a_0$$

$$a_2 = \frac{3}{2} a_1 = \frac{3 \cdot 3}{2} a_0$$

$$a_3 = \frac{3}{3} a_2 = \frac{3 \cdot 3 \cdot 3}{2 \cdot 3} a_0$$

$$a_4 = \frac{3}{4} a_3 = \frac{3 \cdot 3 \cdot 3 \cdot 3}{2 \cdot 3 \cdot 4} a_0$$

$$\vdots \qquad \vdots$$

$$\boxed{a_n = \frac{3^n}{n!}a_0}$$

$$y(x) = \sum_{n=0}^{\infty} \frac{3^n}{n!}a_0 x^n \qquad\qquad y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$$

$$\boxed{= a_0 e^{3x}} \quad \checkmark$$

Exercise

Find a power series solution. $y' = 4y$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y' = 4y$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = 4\sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} 4a_n x^n$$

$$(n+1)a_{n+1}x^n = 4a_n x^n$$

$$(n+1)a_{n+1} = 4a_n$$

$$\boxed{a_{n+1} = \frac{4}{n+1}a_n}$$

$$n = 0 \quad \rightarrow \quad a_1 = 4a_0$$

$$n = 1 \quad \rightarrow \quad a_2 = \frac{4}{2}a_1 = \frac{4^2}{2!}a_0$$

$$n = 2 \quad \rightarrow \quad a_3 = \frac{4}{3}a_2 = \frac{4^3}{3!}a_0$$

$$n = 3 \quad \rightarrow \quad a_4 = \frac{4}{4}a_3 = \frac{4^4}{4!}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_n = \frac{4^n}{n!} a_0}$$

$$y(x) = \sum_{n=0}^{\infty} \frac{4^n}{n!} a_0 x$$

$$= a_0 \left(1 + 4x + \frac{4^2}{2!} x^2 + \frac{4^3}{3!} x^3 + \ \ldots \right)$$

$$\boxed{= a_0 e^{4x}}$$

Exercise

Find a power series solution. $y' = x^2 y$

Solution

$$\frac{dy}{dx} = x^2 y$$

$$\int \frac{dy}{y} = \int x^2 dx$$

$$\ln y = \frac{1}{3} x^3 + C_1$$

$$y = e^{\frac{1}{3} x^3 + C_1}$$

$$y = Ce^{\frac{1}{3} x^3}$$

$$y(0) = C(1) = a_0 \quad \rightarrow \quad C = a_0$$

$$\boxed{y = a_0 e^{x^3/3}}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y' - x^2 y = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{k=-2}^{\infty} (k+3) a_{k+3} x^{k+2}$$

$$= \sum_{n=-2}^{\infty} (n+3) a_{n+3} x^{n+2}$$

$$\sum_{n=-2}^{\infty} (n+3) a_{n+3} x^{n+2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$a_1 + 2a_2 x + \sum_{n=-2}^{\infty} (n+3) a_{n+3} x^{n+2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$a_1 + 2a_2 x + \sum_{n=-2}^{\infty} \left[(n+3) a_{n+3} - a_n \right] x^{n+2} = 0$$

If we set $a_1 = a_2 = 0$, then

$$(n+3) a_{n+3} - a_n = 0 \ \Rightarrow \quad a_{n+3} = \frac{a_n}{n+3}, \quad n \geq 0$$

$$a_3 = \frac{1}{3} a_0$$

$$a_4 = \frac{1}{4} a_1 = 0$$

$$a_5 = \frac{1}{5} a_2 = 0$$

$$a_6 = \frac{1}{6} a_3 = \frac{1}{3 \cdot 6} a_0$$

$$a_7 = \frac{1}{7} a_4 = 0$$

$$a_9 = \frac{1}{9} a_6 = \frac{1}{3 \cdot 6 \cdot 9} a_0 = \frac{1}{3^3 (1 \cdot 2 \cdot 3)} a_0$$

$$a_{12} = \frac{1}{12} a_9 = \frac{1}{3 \cdot 6 \cdot 9 \cdot 12} a_0 = \frac{1}{3^4 (1 \cdot 2 \cdot 3 \cdot 4)} a_0$$

$$\boxed{a_{3n} = \frac{1}{3^n \cdot n!} a_0}$$

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{3^n \cdot n!} a_0 x^{3n}$$

$$= a_0 \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{x^3}{3}\right)^n$$

$$\boxed{= a_0 e^{x^3/3}} \quad \checkmark \qquad\qquad P = \lim_{n\to\infty}\left|\frac{3^k k!}{1}\right| \boxed{= \infty}$$

Exercise

Find a power series solution. $y' + 2xy = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y' + 2xy = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0$$

$$a_1 + \sum_{n=0}^{\infty} (n+2) a_{n+2} x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0$$

$$a_1 + \sum_{n=0}^{\infty} \left[(n+2) a_{n+2} + 2a_n\right] x^{n+1} = 0$$

$$\begin{cases} \boxed{a_1 = 0} \\ (n+2) a_{n+2} + 2a_n = 0 \quad \to \quad \boxed{a_{n+2} = -\frac{2}{n+2} a_0} \end{cases}$$

$$n = 0 \quad \to \quad a_2 = -a_1 \qquad\qquad n = 1 \quad \to \quad a_3 = -\frac{2}{3} a_1 = 0$$

$$n = 2 \quad \to \quad a_4 = -\frac{1}{2} a_2 = \frac{1}{2} a_0 \qquad\qquad n = 3 \quad \to \quad a_5 = -\frac{2}{7} a_3 = 0$$

$$n = 4 \quad \to \quad a_6 = -\frac{1}{3} a_4 = -\frac{1}{2\cdot 3} a_0 \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n = 6 \quad \to \quad a_8 = -\frac{1}{4} a_6 = \frac{1}{4!} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_{2k} = \frac{(-1)^k}{k!} a_0}$$

$$y(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}a_0x^{2k}$$

$$=a_0\left(1-x^2+\frac{1}{2!}x^4-\frac{1}{3!}x^6+\cdots\right)$$

$$\boxed{=a_0e^{-x^2}}\qquad P=\lim_{n\to\infty}\left|\frac{n+2}{-2}\right|\ \boxed{=\infty}$$

Exercise

Find a power series solution. $(x-2)y'+y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n\quad\Rightarrow\quad y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}$$

$$(x-2)\sum_{n=1}^{\infty}na_nx^{n-1}+\sum_{n=0}^{\infty}a_nx^n=0$$

$$x\sum_{n=1}^{\infty}na_nx^{n-1}-2\sum_{n=1}^{\infty}na_nx^{n-1}+\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=1}^{\infty}na_nx^n-2\sum_{n=1}^{\infty}na_nx^{n-1}+\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}na_nx^n-2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}\left[na_n-2(n+1)a_{n+1}+a_n\right]x^n=0$$

$$na_n-2(n+1)a_{n+1}+a_n=0$$

$$2(n+1)a_{n+1}=(n+1)a_n$$

$$\boxed{a_{n+1}=\frac{1}{2}a_n}$$

$$n=0\quad\to\quad a_1=\frac{1}{2}a_0$$

$$n=1\quad\to\quad a_2=\frac{1}{2}a_1=\frac{1}{2^2}a_0$$

$$n=2\quad\to\quad a_3=\frac{1}{2}a_2=\frac{1}{2^3}a_0$$

$$n = 3 \quad \to \quad a_4 = \frac{1}{2}a_3 = \frac{1}{2^4}a_0$$

$$\vdots \quad \vdots \qquad \vdots \qquad \vdots$$

$$\boxed{a_n = \frac{1}{2^n}a_0}$$

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{2^n}a_0 x^n$$

$$= a_0\left(1 + \frac{1}{2}x + \frac{1}{2^2}x^2 + \frac{1}{2^3}x^3 + \cdots\right)$$

$$= a_0\left(1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + \cdots\right)$$

$$= a_0 \frac{1}{1 - \frac{x}{2}}$$

$$\boxed{= \frac{2a_0}{2 - x}}$$

Exercise

Find a power series solution. $(2x-1)y' + 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$$

$$(2x-1)\sum_{n=1}^{\infty} na_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$0 + \sum_{n=1}^{\infty} 2na_n x^n - \sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2na_n x^n - \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[2na_n - (n+1)a_{n+1} + 2a_n\right]x^n = 0$$

$$2na_n - (n+1)a_{n+1} + 2a_n = 0$$

$$(n+1)a_{n+1} = 2(n+1)a_n$$

$$\boxed{a_{n+1} = 2a_n}$$

$$n=0 \quad \to \quad a_1 = 2a_0$$

$$n=1 \quad \to \quad a_2 = 2a_1 = 2^2 a_0$$

$$n=2 \quad \to \quad a_3 = 2a_2 = 2^3 a_0$$

$$n=3 \quad \to \quad a_4 = 2a_3 = 2^4 a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_n = 2^n a_0}$$

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} 2^n a_0 x^n \\ &= a_0\left(1 + 2x + 2^2 x^2 + 2^3 x^3 + \cdots\right) \\ &= a_0\left(1 + 2x + (2x)^2 + (2x)^3 + \cdots\right) \\ &= \boxed{\frac{a_0}{1-2x}} \end{aligned}$$

Exercise

Find a power series solution. $2(x-1)y' = 3y$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$(2x-2)\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} 3a_n x^n$$

$$2x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} 2n a_n x^{n-1} = \sum_{n=0}^{\infty} 3a_n x^n$$

$$0 + \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=1}^{\infty} 2n a_n x^{n-1} = \sum_{n=0}^{\infty} 3a_n x^n$$

$$\sum_{n=0}^{\infty} 2na_n x^n - \sum_{n=0}^{\infty} 2(n+1)a_{n+1} x^n = \sum_{n=0}^{\infty} 3a_n x^n$$

$$\sum_{n=0}^{\infty} \left[2na_n - 2(n+1)a_{n+1}\right] x^n = \sum_{n=0}^{\infty} 3a_n x^n$$

$$2na_n - 2(n+1)a_{n+1} = 3a_n$$

$$-2(n+1)a_{n+1} = (3-2n)a_n$$

$$\boxed{a_{n+1} = \frac{2n-3}{2n+2} a_n} \qquad \rho = \lim_{n\to\infty} \frac{2n-3}{2n+2} \boxed{=1}$$

$$n=0 \quad \to \quad a_1 = -\frac{3}{2} a_0$$

$$n=1 \quad \to \quad a_2 = -\frac{1}{4} a_1 = \frac{3}{8} a_0$$

$$n=2 \quad \to \quad a_3 = \frac{1}{6} a_2 = \frac{1}{16} a_0$$

$$n=3 \quad \to \quad a_4 = \frac{3}{8} a_3 = \frac{3}{128} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y(x) = a_0 \left(1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \frac{3}{128}x^4 + \cdots\right)}$$

$$\frac{y'}{y} = \frac{3}{2} \frac{1}{x-1}$$

$$\int \frac{dy}{y} = \frac{3}{2} \int \frac{1}{x-1} dx$$

$$\ln y = \frac{3}{2} \ln(x-1) + \ln C$$

$$\ln y = \ln C(x-1)^{3/2}$$

$$\boxed{y(x) = C(x-1)^{3/2}}$$

Exercise

Find a power series solution. $(1+x)y' - y = 0$

Solution

$$(1+x)\frac{dy}{dx} = y$$

$$\frac{dy}{y} = \frac{dx}{1+x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{1+x}$$

$$\ln(y) = \ln(x+1) + C \quad \Rightarrow \quad \boxed{y = C(x+1)}$$

With $y(0) = C(0+1) = a_0 \quad \rightarrow \quad C = a_0$

$$\Rightarrow \quad \boxed{y = a_0(x+1)}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$(1+x)y' - y = 0$$

$$(1+x)\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left((n+1)a_{n+1}x^n + n a_n x^n - a_n x^n\right) = 0$$

$$(n+1)a_{n+1} + (n-1)a_n = 0 \quad \rightarrow \quad a_{n+1} = \frac{1-n}{n+1} a_n; \quad n \ge 0$$

$$a_1 = a_0 \qquad a_2 = 0a_1 = 0 \qquad a_3 = \frac{-1}{3} a_2 = 0$$

$$a_n = 0 \quad \text{for} \quad n \ge 2$$

$$\begin{aligned} y(x) &= a_0 + a_1 x \\ &= a_0 + a_0 x \\ &= \boxed{a_0(1+x)} \quad \checkmark \end{aligned}$$

Exercise

Find a power series solution. $(2-x)y' + 2y = 0$

Solution

$$(2-x)\frac{dy}{dx}+2y=0$$

$$(2-x)\frac{dy}{dx}=-2y$$

$$\int\frac{dy}{y}=\int\frac{2d(2-x)}{2-x}$$

$$\ln y=2\ln(2-x)+C_1$$

$$\ln y=\ln(2-x)^2+C_1$$

$$\ln y=\ln C(2-x)^2$$

$$y=C(2-x)^2$$

$$y(0)=C(2-0)^2=a_0 \quad\rightarrow\quad \boxed{C=\tfrac{1}{4}a_0}$$

$$\boxed{y=\tfrac{1}{4}a_0(2-x)^2}$$

$$y(x)=\sum_{n=0}^{\infty}a_n x^n \qquad y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$(2-x)y'+2y=0$$

$$(2-x)\sum_{n=1}^{\infty}na_n x^{n-1}+2\sum_{n=0}^{\infty}a_n x^n=0$$

$$2\sum_{n=1}^{\infty}na_n x^{n-1}-\sum_{n=1}^{\infty}na_n x^n+2\sum_{n=0}^{\infty}a_n x^n=0 \qquad \sum_{n=1}^{\infty}na_n x^n=0+\sum_{n=1}^{\infty}na_n x^n=\sum_{n=0}^{\infty}na_n x^n$$

$$2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-\sum_{n=0}^{\infty}na_n x^n+2\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[2(n+1)a_{n+1}-na_n+2a_n\right]x^n=0$$

$$2(n+1)a_{n+1}-(n-2)a_n=0$$

$$2(n+1)a_{n+1}=(n-2)a_n$$

$$a_{n+1}=\frac{n-2}{2(n+1)}a_n, \qquad n\ge 0$$

$$a_1=\tfrac{-2}{2}a_0=-a_0$$

$$a_2=\tfrac{-1}{4}a_1=\tfrac{1}{4}a_0$$

$$a_3=\tfrac{0}{6}a_0=0$$

$$\vdots$$

$$a_n = 0$$

$$\begin{aligned} y(x) &= a_0 - a_0 x + \frac{1}{4}a_0 x^2 \\ &= a_0\left(1 - x + \frac{1}{4}x^2\right) \\ &= \frac{1}{4}a_0\left(4 - 4x + x^2\right) \\ &= \boxed{\frac{1}{4}a_0 (2-x)^2} \quad \checkmark \end{aligned}$$

Exercise

Find a power series solution. $(x-4)y' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$(x-4)y' + y = 0$$

$$(x-4)\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty}4(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty}\left[a_n - 4(n+1)a_{n+1}\right]x^n = 0$$

$$\sum_{n=0}^{\infty} na_n x^n + a_0 - 4a_1 + \sum_{n=1}^{\infty}\left[a_n - 4(n+1)a_{n+1}\right]x^n = 0$$

$$a_0 - 4a_1 + \sum_{n=1}^{\infty}\left[(n+1)a_n - 4(n+1)a_{n+1}\right]x^n = 0$$

$$a_0 - 4a_1 = 0 \quad \rightarrow \quad \boxed{a_1 = \frac{1}{4}a_0}$$

$$(n+1)a_n - 4(n+1)a_{n+1} = 0 \quad \rightarrow \quad a_{n+1} = \frac{1}{4}a_n$$

$$a_2 = \frac{1}{4}a_1 = \frac{1}{4^2}a_0$$

$$a_3 = \frac{1}{4}a_2 = \frac{1}{4^3}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_n = \frac{1}{4^n}a_0}$$

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{4^n} a_0 x^n$$

$$= a_0 \sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$$

$$= a_0 \left(\frac{1}{1-\frac{x}{4}}\right)$$

$$= a_0 \left(\frac{4}{4-x}\right)$$

$$\boxed{= \frac{-4a_0}{x-4}} \quad \checkmark$$

Exercise

Find a power series solution. $x^2 y' = y - x - 1$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$x^2 y' = y - x - 1$$

$$x^2 \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = -x - 1 + \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+2} = -x - 1 + \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=2}^{\infty} (n-1) a_{n-1} x^n = -x - 1 + \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=2}^{\infty} (n-1) a_{n-1} x^n = -x - 1 + a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n$$

$$-x - 1 + a_0 + a_1 x = 0$$

$$a_0 + a_1 x = 1 + x \quad \Rightarrow \quad a_0 = 1; \ \ a_1 = 1$$

$$(n-1)a_{n-1} = a_n$$

$$a_2 = a_1 = 1$$

$$a_3 = 2a_2 = 2$$

$$a_4 = 3a_3 = 1 \cdot 2 \cdot 3$$

$$\vdots \quad \vdots$$

$$a_n = (n-1)!$$

$$\boxed{y(x) = 1 + x + x^2 + 2!x^3 + 3!x^4 + \cdots}$$

Exercise

Find a power series solution. $(x-3)y' + 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$(x-3)y' + 2y = 0$$

$$(x-3)\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty}3(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}2a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty}3(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty}3(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}2a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[n a_n - 3(n+1)a_{n+1} + 2a_n\right]x^n = 0$$

$$-3(n+1)a_{n+1} + (n+2)a_n = 0 \quad \to \quad \boxed{a_{n+1} = \frac{n+2}{3(n+1)}a_n} \quad n = 0, 1, 2, \cdots$$

$$a_1 = \frac{2}{3}a_0 \qquad\qquad a_2 = \frac{3}{3\cdot 2}a_1 = \frac{3}{3^2}a_0$$

$$a_3 = \frac{4}{3\cdot 3}a_2 = \frac{4}{3^3}a_0 \qquad\qquad a_4 = \frac{5}{3\cdot 4}a_3 = \frac{5}{3^4}a_0$$

$$\vdots \quad \vdots$$

$$a_n = \frac{n+1}{3^n} a_0 \qquad n \ge 1$$

$$y(x) = \left(1 + \frac{2}{3}x + \frac{3}{3^2}x^2 + \frac{4}{3^3}x^3 + \dots\right) = a_0 \sum_{n=0}^{\infty} \frac{n+1}{3^n} x^n$$

$$\rho = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{3n+3}{n+2} = 3$$

$$\boxed{y(x) = \frac{1}{(3-x)^2}}$$

Exercise

Find a power series solution. $xy' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$xy' + y = 0$$

$$x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_n x^n = 0$$

$$(n+1) a_n = 0 \quad \to \quad \boxed{a_n = 0}$$

$$\boxed{y(x) \equiv 0}$$

$\therefore$ The equation has no non-trivial power series.

Exercise

Find a power series solution. $x^3y' - 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$x^3y' - 2y = 0$$

$$x^3 \sum_{n=1}^{\infty} n a_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n+2} - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=3}^{\infty} (n-2) a_{n-2} x^n - 2\left(a_0 + a_1 x + a_2 x^2\right) - \sum_{n=3}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=3}^{\infty} \left[(n-2) a_{n-2} - 2a_n\right] x^n - 2\left(a_0 + a_1 x + a_2 x^2\right) = 0$$

$$\begin{cases} a_0 = a_1 = a_2 = 0 \\ (n-2)a_{n-2} = 2a_n \quad \to ka_k = 2a_{k+2} \quad (k = n-2) \end{cases}$$

$$\boxed{a_{k+2} = \frac{k}{2} a_k}$$

$$a_3 = \frac{1}{2} a_1 = 0$$

$$a_4 = a_2 = 0$$

$$a_n \equiv 0$$

$$\boxed{y(x) \equiv 0}$$

$\therefore$ The equation has no non-trivial power series.

Exercise

Find a power series solution. $y'' = 4y$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y''=4y$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n=4\sum_{n=0}^{\infty}a_n x^n$$

$$(n+2)(n+1)a_{n+2}=4a_n$$

$$a_{n+2}=\frac{4}{(n+2)(n+1)}a_n$$

$n=0 \quad \to \quad a_2=2a_0=\frac{4}{2}a_0 \qquad\qquad n=1 \quad \to \quad a_3=\frac{4}{2\cdot 3}a_1$

$n=2 \quad \to \quad a_4=\frac{4}{3\cdot 4}a_2=\frac{4^2}{4!}a_0 \qquad\qquad n=3 \quad \to \quad a_5=\frac{4}{4\cdot 5}a_3=\frac{4^2}{5!}a_1$

$n=4 \quad \to \quad a_6=\frac{4}{5\cdot 6}a_4=\frac{4^3}{6!}a_0 \qquad\qquad n=5 \quad \to \quad a_7=\frac{4}{6\cdot 7}a_5=\frac{4^3}{7!}a_1$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{a_{2k}=\frac{2^{2k}}{(2k)!}a_0} \qquad\qquad \boxed{a_{2k+1}=\frac{2^{2k}}{(2k+1)!}a_1}$$

$$y(x)=a_0\left(1+\frac{2^2}{2!}x^2+\frac{2^4}{4!}x^4+\frac{2^6}{6!}x^6+\ \cdots\right)+a_1\left(x+\frac{2^3}{3!}x^3+\frac{2^5}{5!}x^5+\ \cdots\right)$$

$$=a_0\left(1+\frac{1}{2!}(2x)^2+\frac{1}{4!}(2x)^4+\frac{1}{6!}(2x)^6+\ \cdots\right)+a_1\left(x+\frac{1}{3!}(2x)^3+\frac{1}{5!}(2x)^5+\ \cdots\right)$$

$$\boxed{=a_0\cosh 2x+a_1\sinh 2x}$$

Exercise

Find a power series solution. $y''=9y$

Solution

The equation $y''=9y$ has a characteristic equation $\lambda^2-9=0 \quad \Rightarrow \quad \boxed{\lambda=\pm 3}$

$\therefore$ The general solution: $y(x)=C_1e^{3x}+C_2e^{-3x}$

With $y(0)=a_0$ *and* $y'(0)=a_1$

$$y(0)=C_1e^{3(0)}+C_2e^{-3(0)} \quad \rightarrow \quad C_1+C_2=a_0$$

$$y'(x)=3C_1e^{3x}-3C_2e^{-3x}$$

$$y(0)=3C_1e^{3(0)}-3C_2e^{-3(0)} \quad \rightarrow \quad 3C_1-3C_2=a_1$$

$$\begin{cases} C_1+C_2=a_0 \\ 3C_1-3C_2=a_1 \end{cases} \rightarrow \begin{cases} 3C_1+3C_2=3a_0 \\ 3C_1-3C_2=a_1 \end{cases}$$

$$6C_1=3a_0+a_1 \quad \rightarrow \quad \boxed{C_1=\frac{3a_0+a_1}{6}}$$

$$C_2=a_0-C_1 \quad \rightarrow \quad \boxed{C_2=a_0-\frac{3a_0+a_1}{6}=\frac{3a_0-a_1}{6}}$$

$$\boxed{y(x)=\frac{3a_0+a_1}{6}e^{3x}+\frac{3a_0-a_1}{6}e^{-3x}}$$

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''-9y=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-9\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-9\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-9a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}-9a_n=0$$

$$a_{n+2}=\frac{9}{(n+2)(n+1)}a_n, \quad n\ge 0$$

$$a_2=\frac{9}{(2)(1)}a_0=\frac{9}{2}a_0 \qquad\qquad a_3=\frac{9}{(3)(2)}a_1=\frac{9}{2\cdot 3}a_1$$

$$a_4 = \frac{3^2}{(4)(3)}a_2 = \frac{9\cdot 9}{2\cdot 3\cdot 4}a_0 = \frac{3^4}{2\cdot 3\cdot 4}a_0 \qquad a_5 = \frac{9}{(5)(4)}a_3 = \frac{3^4}{2\cdot 3\cdot 4\cdot 5}a_1$$

$$a_6 = \frac{3^2}{(6)(5)}a_4 = \frac{3^6}{6!}a_0 \qquad a_7 = \frac{9}{(7)(6)}a_5 = \frac{3^6}{7!}a_1$$

$$a_{2n} = \frac{3^{2n}}{(2n)!}a_0 \qquad a_{2n+1} = \frac{3^{2n}}{(2n+1)!}a_1$$

$$y(x) = a_0\left(1+\frac{3^2}{2!}x^2+\frac{3^4}{4!}x^4+\frac{3^6}{6!}x^6+\cdots\right)+a_1\left(x+\frac{3^2}{3!}x^3+\frac{3^4}{5!}x^5+\frac{3^6}{7!}x^7+\cdots\right)$$

$$y(x) = \frac{3a_0+a_1}{6}e^{3x}+\frac{3a_0-a_1}{6}e^{-3x}$$

$$= \frac{3a_0+a_1}{6}\left[1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\cdots\right]+\frac{3a_0-a_1}{6}\left[1-3x+\frac{(-3x)^2}{2!}+\frac{(-3x)^3}{3!}+\cdots\right]$$

$$= \frac{3a_0}{6}\left[1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\frac{(3x)^4}{4!}+\cdots\right]+\frac{a_1}{6}\left[1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\frac{(3x)^4}{4!}+\cdots\right]$$
$$+\frac{3a_0}{6}\left[1-3x+\frac{(3x)^2}{2!}-\frac{(3x)^3}{3!}+\cdots\right]-\frac{a_1}{6}\left[1-3x+\frac{(3x)^2}{2!}-\frac{(3x)^3}{3!}+\cdots\right]$$

$$= \frac{1}{2}a_0\left[1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\frac{(3x)^4}{4!}+\cdots+1-3x+\frac{(3x)^2}{2!}-\frac{(3x)^3}{3!}+\frac{(3x)^4}{4!}+\cdots\right]$$
$$+\frac{a_1}{6}\left[1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\frac{(3x)^4}{4!}+\cdots-1+3x-\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}-\frac{(3x)^4}{4!}+\cdots\right]$$

$$= \frac{1}{2}a_0\left[2+2\frac{(3x)^2}{2!}+2\frac{(3x)^4}{4!}+\cdots\right]+\frac{a_1}{6}\left[6x+2\frac{(3x)^3}{3!}+2\frac{(3x)^5}{5!}+\cdots\right]$$

$$= a_0\left(1+\frac{(3x)^2}{2!}+\frac{(3x)^4}{4!}+\cdots\right)+a_1\left(x+\frac{3^2x^3}{3!}+\frac{3^4x^5}{5!}+\cdots\right)$$

Which are identical.

Exercise

Find a power series solution. $y''+y=0$

Solution

The equation $y''+y=0$ has a characteristic equation $\lambda^2+1=0 \quad \Rightarrow \quad \boxed{\lambda=\pm i}$

$\therefore$ The general solution: $y(x)=C_1\sin x+C_2\cos x$

With $y(0)=a_0$ *and* $y'(0)=a_1$

$$y(0)=C_1\sin(0)+C_2\cos(0) \quad\rightarrow\quad C_2=a_0$$

$$y'(x)=C_1\cos x-C_2\sin x$$

$$y(0)=C_1\cos(0)-C_2\sin(0) \quad\rightarrow\quad C_1=a_1$$

$$y(x)=a_1\sin x+a_0\cos x$$

$$\boxed{=a_0\cos x+a_1\sin x}$$

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''+y=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}+a_n=0 \quad\rightarrow\quad a_{n+2}=\frac{-1}{(n+2)(n+1)}a_n, \quad n\ge 0$$

$$a_2=\frac{1}{(2)(1)}a_0=-\frac{1}{2}a_0 \qquad a_3=\frac{1}{(3)(2)}a_1=-\frac{1}{2\cdot 3}a_1$$

$$a_4=-\frac{1}{(4)(3)}a_2=\frac{1}{2\cdot 3\cdot 4}a_0 \qquad a_5=-\frac{1}{(5)(4)}a_3=\frac{1}{2\cdot 3\cdot 4\cdot 5}a_1$$

$$a_6=-\frac{1}{(6)(5)}a_4=-\frac{1}{6!}a_0 \qquad a_7=-\frac{1}{(7)(6)}a_5=-\frac{1}{7!}a_1$$

$$a_{2n}=\frac{(-1)^n}{(2n)!}a_0 \qquad a_{2n+1}=\frac{(-1)^n}{(2n+1)!}a_1$$

$$y(x)=a_0\left(1-\tfrac{1}{2!}x^2+\tfrac{1}{4!}x^4-\cdots+\frac{(-1)^n}{(2n)!}x^{2n}+\cdots\right)+a_1\left(x-\tfrac{1}{3!}x^3+\tfrac{1}{5!}x^5-\cdots+\frac{(-1)^n}{(2n+1)!}x^{2n+1}+\cdots\right)$$

$$\boxed{=a_0\cos x+a_1\sin x} \quad \checkmark$$

Exercise

Find a power series solution. $y'' - y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_n \right] x^n = 0$$

$$\boxed{a_{n+2} = \frac{1}{(n+2)(n+1)} a_n}$$

$$n = 0 \rightarrow a_2 = \frac{1}{1 \cdot 2} a_0 \qquad\qquad n = 1 \rightarrow a_3 = \frac{1}{2 \cdot 3} a_1$$

$$n = 2 \rightarrow a_4 = \frac{1}{3 \cdot 4} a_2 = \frac{1}{4!} a_0 \qquad\qquad n = 3 \rightarrow a_5 = \frac{1}{4 \cdot 5} a_3 = \frac{1}{5!} a_1$$

$$n = 4 \rightarrow a_6 = \frac{1}{5 \cdot 6} a_4 = \frac{1}{6!} a_0 \qquad\qquad n = 5 \rightarrow a_7 = \frac{1}{6 \cdot 7} a_5 = \frac{1}{7!} a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_{2k} = \frac{1}{(2k)!} a_0} \qquad\qquad \boxed{a_{2k+1} = \frac{1}{(2k+1)!} a_1}$$

$$y(x) = a_0 \left(1 + \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \frac{1}{6!} x^6 + \cdots \right) + a_1 \left(x + \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots \right)$$

$$\boxed{= a_0 \cosh x + a_1 \sinh x}$$

Exercise

Find a power series solution. $y'' + y = x$

Solution

$$y(x)=\sum_{n=0}^{\infty} a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y''+y=x$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}+\sum_{n=0}^{\infty} a_n x^n=x$$

$$2a_2+6a_3x+\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n+a_0+a_1x+\sum_{n=2}^{\infty}a_n x^n-x=0$$

$$\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_n\right]x^n+a_0+2a_2+\left(6a_3+a_1-1\right)x=0$$

$$\begin{cases} a_0+2a_2=0 \\ 6a_3+a_1-1=0 \\ (n+2)(n+1)a_{n+2}+a_n=0 \end{cases}$$

$$\begin{cases} a_2=-\frac{1}{2}a_0 \\ a_3=-\frac{1}{6}\left(a_1-1\right) \\ a_{n+2}=-\frac{1}{(n+2)(n+1)}a_n \end{cases}$$

$$n=2 \;\rightarrow\; a_4=-\frac{1}{3\cdot 4}a_2=\frac{1}{4!}a_0 \qquad n=3 \;\rightarrow\; a_5=-\frac{1}{4\cdot 5}a_3=\frac{1}{5!}\left(a_1-1\right)$$

$$n=4 \;\rightarrow\; a_6=-\frac{1}{5\cdot 6}a_4=-\frac{1}{6!}a_0 \qquad n=5 \;\rightarrow\; a_7=-\frac{1}{6\cdot 7}a_5=-\frac{1}{7!}\left(a_1-1\right)$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_{2k}=\frac{(-1)^k}{(2k)!}a_0} \qquad \boxed{a_{2k+1}=\frac{(-1)^k}{(2k+1)!}\left(a_1-1\right)}$$

$$y(x)=a_0+a_1x+a_0\sum_{k=2}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}+\left(a_1-1\right)\sum_{k=3}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}$$

$$= a_0 + a_0\left(-\frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right) + \left(a_1 - 1\right)\left(-\frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right) + a_1 x - x + x$$

$$= a_0\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right) + \left(a_1 - 1\right)\left(-\frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right) + \left(a_1 - 1\right)x + x$$

$$= x + a_0\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right) + \left(a_1 - 1\right)\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right)$$

$$= x + a_0 \cos x + \left(a_1 - 1\right)\sin x$$

Exercise

Find a power series solution. $y'' - xy = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - xy = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_{n-1}\right] x^n = 0$$

$$2a_2 = 0 \rightarrow a_2 = 0$$

$$(n+2)(n+1) a_{n+2} - a_{n-1} = 0 \quad \rightarrow \quad a_{n+2} = \frac{a_{n-1}}{(n+1)(n+2)}$$

$$a_3 = \frac{a_0}{2\cdot 3} = \frac{1}{6}a_0 \qquad a_4 = \frac{a_1}{3\cdot 4} = \frac{1}{12}a_1 \qquad a_5 = \frac{a_2}{4\cdot 5} = 0$$

$$a_6 = \frac{a_3}{5\cdot 6} = \frac{1}{180}a_0 \qquad a_7 = \frac{a_3}{6\cdot 7} = \frac{1}{504}a_1 \qquad a_8 = \frac{a_5}{7\cdot 8} = 0$$

$$\vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x) = \left(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots\right)a_0 \\ y_2(x) = \left(x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $y'' + xy = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + xy = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_{n-1}\right] x^n = 0$$

$$\begin{cases} 2a_2 = 0 \ \rightarrow \ \boxed{a_2 = 0} \\ (n+2)(n+1)a_{n+2} + a_{n-1} = 0 \end{cases}$$

$$\boxed{a_{n+2} = -\frac{a_{n-1}}{(n+1)(n+2)}}$$

$$a_0 \qquad\qquad a_1 \qquad\qquad a_2 = 0$$

$$n=1 \ \rightarrow \ a_3 = -\frac{a_0}{2\cdot 3} = -\frac{1}{6}a_0 \qquad n=2 \ \rightarrow \ a_4 = -\frac{1}{3\cdot 4}a_1 \qquad n=3 \ \rightarrow \ a_5 = -\frac{a_2}{20} = 0$$

$$n=4 \ \rightarrow \ a_6 = -\frac{a_3}{5\cdot 6} = \frac{1}{180}a_0 \qquad n=5 \ \rightarrow \ a_7 = -\frac{a_3}{6\cdot 7} = \frac{1}{504}a_1 \qquad n=6 \ \rightarrow \ a_8 = -\frac{a_5}{56} = 0$$

$$n=7 \ \rightarrow \ a_9 = -\frac{a_6}{8\cdot 9} = -\frac{1}{12{,}960}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x) = \left(1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 - \frac{1}{12{,}960}x^9 + \cdots\right)a_0 \\ y_2(x) = \left(x - \frac{1}{12}x^4 + \frac{1}{504}x^7 - \cdots\right)a_1 \end{cases}}$$

$$\boxed{y(x) = \left(1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 - \frac{1}{12{,}960}x^9 + \cdots\right)a_0 + \left(x - \frac{1}{12}x^4 + \frac{1}{504}x^7 - \cdots\right)a_1}$$

Exercise

Find a power series solution. $y'' + xy' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$y'' + xy' + y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}na_nx^n+\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}na_nx^n+\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+(n+1)a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}+(n+1)a_n=0$$

$$\boxed{a_{n+2}=-\frac{a_n}{n+2}}$$

a_0

$$n=0 \ \to\ a_2=-\tfrac{1}{2}a_0$$

$$n=2 \ \to\ a_4=-\tfrac{1}{4}a_2=\tfrac{1}{4\cdot 2}a_0$$

$$n=4 \ \to\ a_6=-\frac{a_4}{6}=-\frac{1}{6\cdot 4\cdot 2}a_0$$

$$n=6 \ \to\ a_8=-\frac{a_6}{8}=\frac{1}{8\cdot 6\cdot 4\cdot 2}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n}=\frac{(-1)^n a_0}{(2n)(2n-2)\cdots 4\cdot 2}=\frac{(-1)^n}{n!\ 2^n}a_0$$

a_1

$$n=1 \ \to\ a_3=-\tfrac{1}{3}a_1$$

$$n=3 \ \to\ a_5=-\frac{a_3}{5}=\frac{1}{3\cdot 5}a_1$$

$$n=5 \ \to\ a_7=-\frac{a_5}{7}=-\frac{1}{7\cdot 5\cdot 3}a_1$$

$$n=7 \ \to\ a_9=-\frac{a_7}{9}=\frac{1}{9\cdot 7\cdot 5\cdot 3}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n}=\frac{(-1)^n a_1}{(2n+1)(2n-1)\cdots 5\cdot 3}=\frac{(-1)^n}{(2n+1)!!}a_1$$

$$\boxed{y(x)=a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\ 2^n}x^{2n}+a_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!!}x^{2n+1}}$$

$$\boxed{y(x)=a_0\left(1-\tfrac{1}{2}x^2+\tfrac{1}{8}x^4-\tfrac{1}{48}x^6+\cdots\right)+a_1\left(x-\tfrac{1}{3}x^3+\tfrac{1}{15}x^5-\tfrac{1}{105}x^7+\cdots\right)}$$

Exercise

Find a power series solution. $y''-xy'-y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty} na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''-xy'-y=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x\sum_{n=1}^{\infty} na_n x^{n-1}-\sum_{n=0}^{\infty} a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} na_n x^{n}-\sum_{n=0}^{\infty} a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} na_n x^{n}-\sum_{n=0}^{\infty} a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-(n+1)a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}-(n+1)a_n=0$$

$$\boxed{a_{n+2}=\frac{a_n}{n+2}}$$

a_0	a_1
$n=0 \rightarrow a_2=\frac{1}{2}a_0$	$n=1 \rightarrow a_3=\frac{1}{3}a_1$
$n=2 \rightarrow a_4=\frac{1}{4}a_2=\frac{1}{4\cdot 2}a_0$	$n=3 \rightarrow a_5=\frac{a_3}{5}=\frac{1}{3\cdot 5}a_1$
$n=4 \rightarrow a_6=\frac{a_4}{6}=\frac{1}{6\cdot 4\cdot 2}a_0$	$n=5 \rightarrow a_7=\frac{a_5}{7}=\frac{1}{7\cdot 5\cdot 3}a_1$
$n=6 \rightarrow a_8=\frac{a_6}{8}=\frac{1}{8\cdot 6\cdot 4\cdot 2}a_0$	$n=7 \rightarrow a_9=\frac{a_7}{9}=\frac{1}{9\cdot 7\cdot 5\cdot 3}a_1$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$
$a_{2n}=\frac{a_0}{(2n)(2n-2)\cdots 4\cdot 2}=\frac{1}{n!\,2^n}a_0$	$a_{2n}=\frac{a_1}{(2n+1)(2n-1)\cdots 5\cdot 3}=\frac{1}{(2n+1)!!}a_1$

$$\boxed{y(x)=a_0\sum_{n=0}^{\infty}\frac{1}{n!\,2^n}x^{2n}+a_1\sum_{n=0}^{\infty}\frac{1}{(2n+1)!!}x^{2n+1}}$$

$$y(x)=a_0\left(1+\frac{x^2}{2}+\frac{x^4}{2^2\cdot 2}+\cdots+\frac{x^{2n}}{2^n\cdot n!}\right)+a_1\left(x+\frac{x^3}{3}+\frac{x^5}{5\cdot 3}+\cdots+\frac{2^n\cdot n!}{(2n+1)!!}x^{2n+1}\right)$$

Exercise

Find a power series solution. $y''+x^2y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''+x^2y=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+x^2\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}a_n x^{n+2}=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}a_{n-2}x^n=0$$

$$2a_2+6a_3x+\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}a_{n-2}x^n=0$$

$$2a_2+6a_3x+\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_{n-2}\right]x^n=0$$

$$2a_2=0\to \underline{a_2=0} \qquad 6a_3=0\to \underline{a_3=0}$$

$$(n+2)(n+1)a_{n+2}+a_{n-2}=0 \ \to\ a_{n+2}=\frac{-a_{n-2}}{(n+1)(n+2)}$$

$$a_4=\frac{-a_0}{3\cdot 4}=-\frac{1}{12}a_0 \qquad a_5=\frac{-a_1}{4\cdot 5}=-\frac{1}{20}a_1 \qquad a_6=\frac{-a_2}{5\cdot 6}=0 \qquad a_7=\frac{-a_3}{6\cdot 7}=0$$

$$a_8=\frac{-a_4}{7\cdot 8}=\frac{1}{672}a_0 \qquad a_9=\frac{-a_5}{8\cdot 9}=\frac{1}{1,440}a_1 \qquad a_{10}=\frac{-a_6}{9\cdot 10}=0 \qquad a_{11}=\frac{a_7}{10\cdot 11}=0$$

$\vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots$

$$\boxed{\begin{cases} y_1(x) = \left(1 - \frac{1}{12}x^4 + \frac{1}{672}x^8 - \cdots\right)a_0 \\ y_2(x) = \left(x - \frac{1}{20}x^5 + \frac{1}{1,440}x^9 + \cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $y'' + k^2x^2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$y'' + k^2x^2y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + k^2x^2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$2a_2 + 6a_3x + \sum_{n=2}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty} k^2a_{n-2}x^n = 0$$

$$2a_2 + 6a_3x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} + k^2a_{n-2}\right]x^n = 0$$

$$\begin{cases} a_2 = 0 \\ a_3 = 0 \\ (n+2)(n+1)a_{n+2} + k^2a_{n-2} = 0 \end{cases}$$

$$\boxed{a_{n+2} = -\frac{k^2}{(n+1)(n+2)}a_{n-2}} \quad (n \ge 2)$$

$n = 2 \rightarrow a_4 = -\frac{k^2}{3\cdot 4}a_0 \qquad\qquad n = 3 \rightarrow a_5 = -\frac{k^2}{4\cdot 5}a_1$

$$n=6 \ \rightarrow \ a_8 = -\frac{k^2}{7\cdot 8}a_4 = \frac{k^4}{3\cdot 4\cdot 7\cdot 8}a_0 \qquad\qquad n=7 \rightarrow \ a_9 = -\frac{k^2}{8\cdot 9}a_5 = \frac{k^4}{4\cdot 5\cdot 8\cdot 9}a_1$$

$$n=10 \ \rightarrow \ a_{12} = -\frac{k^2}{11\cdot 12}a_8 = \frac{k^6}{3\cdot 4\cdot 7\cdot 8\cdot 11\cdot 12}a_0 \qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{4m} = -\frac{k^2}{(4m)(4m-1)}a_{4m-4} \qquad\qquad a_{4m+1} = -\frac{k^2}{(4m)(4m+1)}a_{4m-3}$$

$$n=4 \rightarrow \ a_6 = -\frac{k^2}{5\cdot 6}a_2 = 0 \qquad\qquad n=5 \rightarrow \ a_7 = -\frac{k^2}{6\cdot 7}a_3 = 0$$

$$n=8 \ \rightarrow \ a_{10} = -\frac{k^2}{7\cdot 8}a_6 = 0 \qquad\qquad n=9 \rightarrow \ a_{11} = -\frac{k^2}{10\cdot 11}a_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y(x) = a_0\left(1 - \frac{k^2}{3\cdot 4}x^4 + -\frac{k^4}{3\cdot 4\cdot 7\cdot 8}x^8 - \frac{k^6}{3\cdot 4\cdot 7\cdot 8\cdot 11\cdot 12}x^{12} + \cdots\right)$$
$$+ a_1\left(x - \frac{k^2}{4\cdot 5}x^5 + -\frac{k^4}{4\cdot 5\cdot 8\cdot 9}x^9 - \frac{k^6}{4\cdot 5\cdot 8\cdot 9\cdot 12\cdot 13}x^{13} + \cdots\right)$$

Exercise

Find a power series solution. $y'' + 3xy' + 3y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + 3xy' + 3y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 3x\sum_{n=1}^{\infty} n a_n x^{n-1} + 3\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 3n a_n x^n + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+(3n+3)a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}+3(n+1)a_n=0$$

$$\boxed{a_{n+2}=-\frac{3}{n+2}a_n}$$

a_0

$n=2 \rightarrow a_2=-\frac{3}{2}a_0$

$n=4 \rightarrow a_4=-\frac{3}{4}a_2=\frac{3^2}{2^2\cdot 2}a_0$

$n=6 \rightarrow a_6=-\frac{3}{6}a_3=\frac{3^3}{2^3\cdot 2\cdot 3}a_0$

$n=8 \rightarrow a_8=-\frac{3}{8}a_6=\frac{3^4}{2^4\cdot 2\cdot 3\cdot 4}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$a_{2k}=\frac{(-3)^k}{2^k k!}a_0$$

a_1

$n=3 \rightarrow a_3=-\frac{3}{3}a_1$

$n=5 \rightarrow a_5=-\frac{3}{5}a_3=\frac{3^2}{3\cdot 5}a_1$

$n=7 \rightarrow a_7=-\frac{3}{7}a_5=\frac{3^3}{3\cdot 5\cdot 7}a_1$

$n=9 \rightarrow a_9=-\frac{3}{9}a_7=\frac{3^3}{3\cdot 5\cdot 7\cdot 9}a_1$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$a_{2k+1}=\frac{(-3)^k}{3\cdot 5\cdot 7\cdots(2k+1)}a_1$$

$$y(x)=a_0\left(1+\sum_{k=1}^{\infty}\frac{(-3)^k}{2^k k!}x^{2k}\right)+a_1\left(x+\sum_{k=1}^{\infty}\frac{(-3)^k}{3\cdot 5\cdot 7\cdots(2k+1)}x^{2k+1}\right)$$

$$\boxed{=a_0\left(1-\frac{3}{2}x^2+\frac{9}{8}x^4-\frac{27}{56}x^6+\cdots\right)+a_1\left(x-x^3+\frac{3^2}{3\cdot 5}x^5-\frac{27}{3\cdot 5\cdot 7}x^7+\cdots\right)}$$

Exercise

Find a power series solution. $y''-2xy'+y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y''-2xy'+y=0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - 2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}2na_n x^n + \sum_{n=0}^{\infty}a_n x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}2na_n x^n + a_0 + \sum_{n=1}^{\infty}a_n x^n = 0$$

$$2a_2 + a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - 2na_n + a_n\right]x^n = 0$$

$$2a_2 + a_0 = 0 \ \rightarrow\ \boxed{a_2 = -\frac{1}{2}a_0}$$

$$(n+2)(n+1)a_{n+2} - (2n-1)a_n = 0 \ \rightarrow\ a_{n+2} = \frac{(2n-1)a_n}{(n+1)(n+2)}$$

$$a_3 = \frac{a_1}{2\cdot 3} = \frac{1}{6}a_1 \qquad a_4 = \frac{3a_2}{3\cdot 4} = -\frac{1}{4}\frac{1}{4}a_0 = -\frac{1}{8}a_0$$

$$a_5 = \frac{5a_3}{4\cdot 5} = \frac{1}{2\cdot 3\cdot 4}a_1 = \frac{1}{4!}a_1 \qquad a_6 = \frac{7a_4}{5\cdot 6} = -\frac{7}{240}a_0$$

$$a_7 = \frac{9a_5}{6\cdot 7} = \frac{1}{112}a_1$$

$$\vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x) = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{240}x^6 - \cdots\right)a_0 \\ y_2(x) = \left(x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{1}{112}x^7 + \cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $y'' - xy' + 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty}a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty}na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y''-xy'+2y=0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 2\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}na_n x^n + \sum_{n=0}^{\infty}2a_n x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}na_n x^n + 2a_0 + \sum_{n=1}^{\infty}2a_n x^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - na_n + 2a_n\right]x^n = 0$$

$$2a_2 + 2a_0 = 0 \ \rightarrow\ \boxed{a_2 = -a_0}$$

$$(n+2)(n+1)a_{n+2} - (n-2)a_n = 0$$

$$\rightarrow\ a_{n+2} = \frac{(n-2)a_n}{(n+1)(n+2)}$$

$$a_3 = \frac{-a_1}{2\cdot 3} = -\frac{1}{6}a_1$$

$$a_5 = \frac{a_3}{4\cdot 5} = -\frac{1}{5!}a_1$$

$$a_7 = \frac{3a_5}{6\cdot 7} = \frac{3}{7!}a_1$$

$$a_9 = \frac{5a_7}{8\cdot 9} = \frac{3\cdot 5}{9!}a_1$$

$\vdots \quad \vdots$

$$a_4 = 0$$

$$a_6 = \frac{2a_4}{5\cdot 6} = 0$$

$$a_8 = \frac{4a_6}{5\cdot 6} = 0$$

$\vdots \quad \vdots$

$$\boxed{\begin{cases} y_1(x) = 1 - x^2 \\ y_2(x) = \left(x - \frac{1}{6}x^3 + \frac{1}{5!}x^5 + \frac{3}{7!}x^7 + \cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $y'' - xy' - x^2 y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - xy' - x^2 y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x \sum_{n=1}^{\infty} n a_n x^{n-1} - x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n - a_1 x - \sum_{n=2}^{\infty} n a_n x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$2a_2 + 6a_3 x - a_1 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} - n a_n - a_{n-2} \right] x^n = 0$$

$$\begin{cases} 2a_2 = 0 \ \rightarrow \ \boxed{a_2 = 0} \\ \left(6a_3 - a_1\right) x = 0 \ \rightarrow \ \boxed{a_3 = \frac{1}{6} a_1} \\ (n+1)(n+2) a_{n+2} - n a_n - a_{n-2} = 0 \end{cases}$$

$$\boxed{a_{n+2} = \frac{n a_n + a_{n-2}}{(n+1)(n+2)}}$$

$$a_0 \qquad\qquad a_1$$

$$a_2 = 0 \qquad\qquad a_3 = \frac{1}{6} a_1$$

$$n = 2 \ \rightarrow \ a_4 = \frac{2a_2 + a_0}{3 \cdot 4} = \frac{1}{12} a_0 \qquad\qquad n = 3 \ \rightarrow \ a_5 = \frac{3a_3 + a_1}{20} = \frac{1}{20}\left(\frac{3}{6} + 1\right) a_1 = \frac{1}{12} a_1$$

$$n=4 \;\rightarrow\; a_6 = \frac{4a_4 + a_2}{5\cdot 6} = \frac{1}{90}a_0 \qquad\qquad n=5 \;\rightarrow\; a_7 = \frac{5a_5 + a_3}{6\cdot 7} = \frac{1}{42}\left(\frac{5}{12}+\frac{1}{6}\right)a_1 = \frac{1}{72}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n=6 \;\rightarrow\; a_8 = \frac{6a_6 + a_4}{7\cdot 8} = \frac{1}{56}\left(\frac{1}{15}+\frac{1}{12}\right)a_0 = \frac{3}{1120}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y(x) = a_0\left(1+\frac{1}{12}x^2+\frac{1}{90}x^4+\frac{3}{1120}x^6+\cdots\right)+a_1\left(x+\frac{1}{12}x^3+\frac{1}{72}x^5+\cdots\right)}$$

Exercise

Find a power series solution. $y'' + x^2y' + xy = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$y'' + x^2y' + xy = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x^2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + x\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+2} + \sum_{n=0}^{\infty}a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty}(n-1)a_{n-1}x^n + \sum_{n=1}^{\infty}a_{n-1}x^n = 0$$

$$2a_2 + 6a_3x + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty}(n-1)a_{n-1}x^n + a_0x + \sum_{n=2}^{\infty}a_{n-1}x^n = 0$$

$$2a_2 + \left(6a_3 + a_0\right)x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + (n-1)a_{n-1} + a_{n-1}\right]x^n = 0$$

$$2a_2 + \left(6a_3 + a_0\right)x = 0 \;\Rightarrow\; \begin{cases} a_2 = 0 \\ a_3 = -\frac{1}{6}a_0 \end{cases}$$

$$(n+2)(n+1)a_{n+2} + na_{n-1} = 0 \;\Rightarrow a_{n+2} = -\frac{n}{(n+1)(n+2)}a_{n-1}$$

$$a_4 = -\frac{2}{3\cdot 4}a_1 = -\frac{1}{6}a_1 \qquad a_5 = -\frac{3}{4\cdot 5}a_2 = 0 \qquad a_6 = -\frac{4}{5\cdot 6}a_3 = \frac{1}{45}a_0$$

$$a_7 = -\frac{5}{6\cdot 7}a_4 = \frac{5}{252}a_1 \qquad a_8 = -\frac{6}{7\cdot 8}a_5 = 0 \qquad a_9 = -\frac{7}{8\cdot 9}a_3 = -\frac{7}{3,240}a_0$$

$$\vdots \;\; \vdots \qquad\qquad \vdots \;\; \vdots \qquad\qquad \vdots \;\; \vdots$$

$$\begin{cases} y_1(x) = \left(1 - \frac{1}{6}x^3 + \frac{1}{45}x^6 - \frac{7}{3,240}x^9 + \cdots\right)a_0 \\ y_2(x) = \left(x - \frac{1}{6}x^4 + \frac{5}{252}x^7 - \cdots\right)a_1 \end{cases}$$

Exercise

Find a power series solution. $y'' + x^2y' + 2xy = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$y'' + x^2y' + 2xy = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x^2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 2x\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+2} + \sum_{n=0}^{\infty}2a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty}(n-1)a_{n-1}x^n + \sum_{n=1}^{\infty}2a_{n-1}x^n = 0$$

$$2a_2+6a_3x+\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}(n-1)a_{n-1}x^n+2a_0x+\sum_{n=2}^{\infty}2a_{n-1}x^n=0$$

$$2a_2+\left(6a_3+2a_0\right)x+\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2}+(n-1)a_{n-1}+2a_{n-1}\right]x^n=0$$

$$2a_2+\left(6a_3+2a_0\right)x=0\ \Rightarrow\ \begin{cases}a_2=0\\ a_3=-\frac{1}{3}a_0\end{cases}$$

$$(n+2)(n+1)a_{n+2}+(n+1)a_{n-1}=0\ \Rightarrow a_{n+2}=-\frac{a_{n-1}}{n+2}$$

$$\boxed{a_{n+3}=-\frac{a_n}{n+3}}$$

a_0	a_1	$a_2=0$
$a_3=-\frac{1}{3}a_0$	$n=1\ \rightarrow\ a_4=-\frac{1}{4}a_1$	$n=2\rightarrow\ a_5=-\frac{a_2}{5}=0$
$n=3\ \rightarrow\ a_6=-\frac{a_3}{6}=\frac{1}{2\cdot 3^2}a_0$	$n=4\ \rightarrow\ a_7=-\frac{a_4}{7}=\frac{1}{7\cdot 4}a_1$	$n=5\rightarrow\ a_8=-\frac{a_5}{8}=0$
$n=6\ \rightarrow\ a_9=-\frac{a_6}{9}=-\frac{1}{3!\ 3^3}a_0$	$n=7\ \rightarrow\ a_{10}=-\frac{a_7}{10}=-\frac{1}{10\cdot 7\cdot 4\cdot 1}a_1$	$n=8\rightarrow\ a_{11}=0$
$\vdots\quad\vdots\quad\vdots\quad\vdots$	$\vdots\quad\vdots\quad\vdots\quad\vdots$	$\vdots\quad\vdots\quad\vdots\quad\vdots$
$a_{3n}=\frac{(-1)^n}{n!\ 3^n}$	$a_{3n+1}=\frac{(-1)^n}{1\cdot 4\cdot 7\cdots(3n+1)}$	$a_{3n+2}=0$

$$\boxed{y(x)=a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\ 3^n}x^{3n}+a_1\sum_{n=0}^{\infty}\frac{(-1)^n}{1\cdot 4\cdot 7\cdots(3n+1)}x^{3n+1}}$$

$$\boxed{y(x)=a_0\left(1-\frac{1}{3}x^3+\frac{1}{18}x^6-\frac{1}{162}x^9+\cdots\right)+a_1\left(x-\frac{1}{4}x^4+\frac{1}{28}x^7-\frac{1}{280}x^{10}+\cdots\right)}$$

Exercise

Find a power series solution. $y'' - x^2 y' - 3xy = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - x^2 y' - 3xy = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x^2 \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 3x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+2} - \sum_{n=0}^{\infty} 3 a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n - \sum_{n=1}^{\infty} 3 a_{n-1} x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n - 3a_0 x - \sum_{n=2}^{\infty} 3 a_{n-1} x^n = 0$$

$$2a_2 + 3(2a_3 - a_0)x + \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} - (n+2) a_{n-1} \right] x^n = 0$$

$$\begin{cases} 2a_2 = 0 & \rightarrow \boxed{a_2 = 0} \\ 2a_3 - a_0 = 0 & \rightarrow \boxed{a_3 = \frac{1}{2} a_0} \end{cases}$$

$$(n+2)(n+1) a_{n+2} = (n+2) a_{n-1}$$

$$a_{n+2} = \frac{a_{n-1}}{n+1} \quad \rightarrow \quad \boxed{a_{n+3} = \frac{a_n}{n+2}}$$

a_0	a_1	$a_2 = 0$
$a_3 = \frac{1}{2} a_0$	$n = 1 \rightarrow a_4 = \frac{1}{3} a_1$	$n = 2 \rightarrow a_5 = \frac{a_2}{5} = 0$

$n=3 \rightarrow a_6 = \frac{a_3}{5} = \frac{1}{2\cdot 5} a_0$ $\quad n=4 \rightarrow a_7 = \frac{a_4}{6} = \frac{1}{2\cdot 3^2} a_1$ $\quad n=5 \rightarrow a_8 = 0$

$n=6 \rightarrow a_9 = \frac{a_6}{8} = -\frac{1}{2\cdot 5\cdot 8} a_0$ $\quad n=7 \rightarrow a_{10} = \frac{a_7}{9} = \frac{1}{2\cdot 3\cdot 3^3} a_1$ $\quad n=8 \rightarrow a_{11} = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$a_{3n} = \frac{1}{2\cdot 5\cdot 8\cdots(3n-1)} \qquad a_{3n+1} = \frac{1}{n!\ 3^n} \qquad a_{3n+2} = 0$$

$$\underline{y(x) = a_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{2\cdot 5\cdot 8\cdots(3n-1)} + a_1 \sum_{n=0}^{\infty} \frac{x^{3n+1}}{n!\ 3^n}}$$

$$\underline{y(x) = a_0\left(1+\frac{1}{2}x^3+\frac{1}{10}x^6+\frac{1}{80}x^9+\cdots\right) + a_1\left(x+\frac{1}{3}x^4+\frac{1}{18}x^7+\frac{1}{162}x^{10}+\cdots\right)}$$

Exercise

Find a power series solution. $y'' + 2xy' + 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$y'' + 2xy' + 2y = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 2x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} 2n a_n x^n + 2a_0 + \sum_{n=1}^{\infty} 2 a_n x^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + 2na_n + 2a_n\right]x^n = 0$$

$$2a_2 + 2a_0 = 0 \quad \rightarrow \quad \boxed{a_2 = -a_0}$$

$$(n+2)(n+1)a_{n+2} + 2(n+1)a_n = 0$$

$$\rightarrow \quad \boxed{a_{n+2} = -\frac{2}{n+2}a_n} \quad n = 1,\ 2,\ \cdots$$

$$a_3 = -\frac{2}{3}a_1 \qquad\qquad a_4 = -\frac{2}{4}a_2 = \frac{1}{2}a_0$$

$$a_5 = -\frac{2}{5}a_3 = \frac{4}{15}a_1 \qquad\qquad a_6 = -\frac{2}{6}a_4 = -\frac{1}{6}a_0$$

$$a_7 = -\frac{2}{7}a_3 = -\frac{8}{105}a_1 \qquad\qquad a_8 = -\frac{2}{8}a_6 = \frac{1}{24}a_0$$

$$\vdots \quad \vdots \qquad\qquad \vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x) = \left(1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \frac{1}{4!}x^8 - \cdots\right)a_0 \\ y_2(x) = \left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7 + \cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $2y'' + xy' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$2y'' + xy' + y = 0$$

$$2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} + \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$4a_2 + \sum_{n=1}^{\infty} 2(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} na_n x^n + a_0 + \sum_{n=1}^{\infty} a_n x^n = 0$$

$$4a_2 + a_0 + \sum_{n=1}^{\infty} \left[2(n+2)(n+1)a_{n+2} + (n+1)a_n \right] x^n = 0$$

$$4a_2 + a_0 = 0 \quad \rightarrow \quad \boxed{a_2 = -\frac{1}{4}a_0}$$

$$2(n+2)(n+1)a_{n+2} + (n+1)a_n = 0$$

$$\boxed{a_{n+2} = -\frac{1}{2(n+2)}a_n}$$

a_0

$$n=0 \rightarrow a_2 = -\frac{1}{4}a_0$$

$$n=2 \rightarrow a_4 = -\frac{1}{8}a_2 = \frac{1}{2^4 \cdot 2}a_0$$

$$n=4 \rightarrow a_6 = -\frac{1}{2\cdot 6}a_4 = -\frac{1}{2^6 \cdot 2\cdot 3}a_0$$

$$n=6 \rightarrow a_8 = -\frac{1}{2\cdot 8}a_6 = \frac{1}{2^8 \cdot 4!}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n \ge 1 \quad a_{2n} = \frac{(-1)^n}{2^{2n}\ n!}a_0$$

a_1

$$n=1 \rightarrow a_3 = -\frac{1}{6}a_1$$

$$n=3 \rightarrow a_5 = -\frac{1}{2\cdot 5}a_3 = \frac{1}{2^2 \cdot 3\cdot 5}a_1$$

$$n=5 \rightarrow a_7 = -\frac{1}{2\cdot 7}a_5 = -\frac{1}{2^3 \cdot 3\cdot 5\cdot 7}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n \ge 1 \quad a_{2n+1} = \frac{(-1)^n}{2^n\ 1\cdot 3\cdot 5\cdots(2n+1)}a_1 = \frac{(-1)^n}{2^n\ (2n+1)!!}a_1$$

$$\boxed{y(x) = a_0\left(1 - \frac{1}{4}x^2 + \frac{1}{32}x^4 - \ldots\right) + a_1\left(x - \frac{1}{6}x^3 + \frac{1}{60}x^5 - \cdots\right)}$$

$$\boxed{y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}\ n!}x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n\ (2n+1)!!}x^{2n+1}}$$

Exercise

Find a power series solution. $3y'' + xy' - 4y = 0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$3y''+xy'-4y=0$$

$$3\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-4\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}3(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}-\sum_{n=0}^{\infty}4a_n x^n=0$$

$$\sum_{n=0}^{\infty}3(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}na_n x^n-\sum_{n=0}^{\infty}4a_n x^n=0$$

$$6a_2+\sum_{n=1}^{\infty}3(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}na_n x^n-4a_0-\sum_{n=1}^{\infty}4a_n x^n=0$$

$$6a_2-4a_0+\sum_{n=1}^{\infty}\left[3(n+2)(n+1)a_{n+2}+(n-4)a_n\right]x^n=0$$

$$6a_2-4a_0 \quad\rightarrow\quad \boxed{a_2=\frac{2}{3}a_0}$$

$$3(n+2)(n+1)a_{n+2}+(n-4)a_n=0$$

$$\boxed{a_{n+2}=-\frac{(n-4)}{3(n+1)(n+2)}a_n}$$

a_0

$$a_2=\frac{2}{3}a_0$$

$$n=2\ \rightarrow\ a_4=\frac{2}{36}a_2=\frac{1}{27}a_0$$

$$n=4\ \rightarrow\ a_6=0$$

a_1

$$n=1\ \rightarrow\ a_3=\frac{3}{2\cdot3\cdot3}a_1=\frac{1}{2\cdot3}a_1$$

$$n=3\ \rightarrow\ a_5=\frac{1}{4\cdot5\cdot3}a_3=\frac{1}{5!\cdot3}a_1$$

$$n=5\ \rightarrow\ a_7=-\frac{1}{3\cdot6\cdot7}a_5=-\frac{1}{7!\cdot3^2}a_1$$

$$n=7\ \rightarrow\ a_9=-\frac{3}{3\cdot9\cdot8}a_7=\frac{3}{9!\cdot3^3}a_1$$

$$n=9\ \rightarrow\ a_{11}=-\frac{5}{3\cdot11\cdot10}a_9=-\frac{3\cdot5}{11!\cdot3^4}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n \ge 3 \quad a_{2n+1} = \frac{1\cdot 3\cdot 5\cdots(2n-5)(-1)^n}{(2n+1)!\ 3^n}a_1$$

$$= \frac{(2n-5)!!(-1)^n}{(2n+1)!\ 3^n}a_1$$

$$y(x) = a_0\left(1+\frac{2}{3}x^2+\frac{1}{27}x^4\right)+a_1\left(x+\frac{1}{6}x^3+\frac{1}{360}x^5+\sum_{n=3}^{\infty}\frac{(2n-5)!!(-1)^n}{(2n+1)!\ 3^n}\right)$$

$$= a_0\left(1+\frac{2}{3}x^2+\frac{1}{27}x^4\right)+a_1\left(x+\frac{1}{6}x^3+\frac{1}{360}x^5-\frac{1}{45,360}x^7+\cdots\right)$$

Exercise

Find a power series solution. $5y''-2xy'+10y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$5y''-2xy'+10y=0$$

$$5\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+10\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}5(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}2(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}10a_n x^n=0$$

$$\sum_{n=0}^{\infty}5(n+2)(n+1)a_{n+2}x^n-\sum_{n=1}^{\infty}2na_n x^n+\sum_{n=0}^{\infty}10a_n x^n=0$$

$$10a_2+\sum_{n=1}^{\infty}5(n+2)(n+1)a_{n+2}x^n-\sum_{n=1}^{\infty}2na_n x^n+10a_0+\sum_{n=1}^{\infty}10a_n x^n=0$$

$$10a_2 + 10a_0 + \sum_{n=1}^{\infty} \left[5(n+2)(n+1)a_{n+2} - 2(n-5)a_n \right] x^n = 0$$

$$10a_2 + 10a_0 \quad \rightarrow \quad \underline{a_2 = -a_0}$$

$$5(n+2)(n+1)a_{n+2} - 2(n-5)a_n = 0$$

$$\underline{a_{n+2} = \frac{2(n-5)}{5(n+1)(n+2)} a_n}$$

a_0

$$a_2 = -a_0$$

$$n=2 \rightarrow a_4 = -\frac{6}{60} a_2 = \frac{1}{10} a_0$$

$$n=4 \rightarrow a_6 = -\frac{2}{5 \cdot 5 \cdot 6} a_4 = -\frac{1}{750} a_0$$

$$n=6 \rightarrow a_8 = \frac{2}{5 \cdot 7 \cdot 8} a_6 = -\frac{2}{8! \cdot 5^2} a_0$$

$$n=8 \rightarrow a_{10} = \frac{2 \cdot 3}{5 \cdot 9 \cdot 10} a_8 = -\frac{2^2 \cdot 3}{10! \cdot 5^3} a_0$$

$$n=10 \rightarrow a_{12} = \frac{2 \cdot 5}{5 \cdot 11 \cdot 12} a_8 = -\frac{2^3 \cdot 3 \cdot 5}{12! \cdot 5^4} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n \geq 4 \quad a_{2n} = -15 \cdot \frac{2^n (2n-7)!!}{5^n \, (2n)!} a_0$$

a_1

$$n=1 \rightarrow a_3 = -\frac{8}{30} a_1 = -\frac{4}{15} a_1$$

$$n=3 \rightarrow a_5 = -\frac{4}{100} a_3 = \frac{4}{375} a_1$$

$$n=5 \rightarrow a_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underline{y(x) = a_0 \left(1 - x^2 + \frac{1}{10} x^4 - \frac{1}{750} x^6 - \frac{1}{105{,}000} x^8 - \cdots \right) + a_1 \left(x - \frac{4}{15} x^3 + \frac{4}{375} x^5 \right)}$$

Exercise

Find a power series solution. $(x-1)y'' + y' = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$(x-1)y''+y'=0$$

$$(x-1)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n=0$$

$$x\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+1}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n=0$$

$$\sum_{n=1}^{\infty}n(n+1)a_{n+1}x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n=0$$

$$\sum_{n=1}^{\infty}n(n+1)a_{n+1}x^n-2a_2-\sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n+a_1+\sum_{n=1}^{\infty}(n+1)a_{n+1}x^n=0$$

$$a_1-2a_2+\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1}-(n+2)(n+1)a_{n+2}+(n+1)a_{n+1}\right]x^n=0$$

$$a_1-2a_2+\sum_{n=1}^{\infty}\left[-(n+2)(n+1)a_{n+2}+(n+1)^2a_{n+1}\right]x^n=0$$

$$a_1-2a_2=0 \quad\rightarrow\quad \boxed{a_2=\frac{1}{2}a_1}$$

$$-(n+2)(n+1)a_{n+2}+(n+1)^2a_{n+1}$$

$$\rightarrow\quad \boxed{a_{n+2}=\frac{n+1}{n+2}a_{n+1}} \qquad n=1,\ 2,\ \ldots$$

$$a_3=\frac{2}{3}a_2=\frac{1}{3}a_1$$

$$a_4=\frac{3}{4}a_3=\frac{1}{4}a_1$$

$$a_5=\frac{4}{5}a_4=\frac{1}{5}a_1$$

$$\vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x)=a_0 \\ y_2(x)=\left(x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $(x+2)y''+xy'-y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$(x+2)y''+xy'-y=0$$

$$(x+2)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+1}+\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}-\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=1}^{\infty}n(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}na_n x^n-\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=1}^{\infty}n(n+1)a_{n+1}x^n+4a_2+\sum_{n=1}^{\infty}2(n+2)(n+1)a_{n+2}x^n+4a_2+\sum_{n=1}^{\infty}na_n x^n-a_0-\sum_{n=1}^{\infty}a_n x^n=0$$

$$4a_2-a_0+\sum_{n=1}^{\infty}\left(n(n+1)a_{n+1}+2(n+2)(n+1)a_{n+2}+(n-1)a_n\right)x^n=0$$

$$4a_2-a_0=0 \quad\rightarrow\quad \boxed{a_2=\frac{1}{4}a_0} \quad \boxed{a_1}$$

$$n(n+1)a_{n+1}+2(n+2)(n+1)a_{n+2}+(n-1)a_n=0$$

$$\boxed{a_{n+2}=-\frac{n-1}{2(n+2)(n+1)}a_n-\frac{n}{2(n+2)}a_{n+1}} \quad n=1,\ 2,\ \ldots$$

$$a_3=-\frac{1}{6}a_2=-\frac{1}{24}a_0$$

$$a_4=-\frac{1}{24}a_2-\frac{1}{4}a_3=-\frac{1}{96}a_0+\frac{1}{96}a_0=0$$

$$a_5=-\frac{1}{20}a_3-\frac{3}{10}a_4=\frac{1}{480}a_0$$

$$a_6 = -\frac{3}{60}a_4 - \frac{1}{3}a_5 = -\frac{1}{1,440}a_0$$

$$\vdots \ \vdots \qquad \vdots \ \vdots$$

$$\boxed{\begin{cases} y_1(x) = a_1 \\ y_2(x) = \left(1 + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{480}x^5 - \cdots\right)a_0 \end{cases}}$$

Exercise

Find a power series solution. $y'' - (x+1)y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - (x+1)y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_n\right] x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 - a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_n\right] x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 - a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_n - a_{n-1}\right] x^n = 0$$

$$2a_2 - a_0 = 0 \quad \rightarrow \quad \boxed{a_2 = \frac{1}{2}a_0}$$

$$(n+2)(n+1) a_{n+2} - a_n - a_{n-1} = 0$$

$$\boxed{a_{n+2}=\frac{a_n+a_{n-1}}{(n+1)(n+2)}}$$

$a_0 \qquad a_1$

$a_2=\frac{1}{2}a_0$

$n=1 \ \rightarrow \ a_3=\frac{1}{6}\left(a_1+a_0\right)$

$n=2 \ \rightarrow \ a_4=\frac{1}{12}\left(a_2+a_1\right)=\frac{1}{12}\left(\frac{1}{2}a_0+a_1\right)$

$n=3 \ \rightarrow \ a_5=\frac{1}{20}\left(a_3+a_2\right)=\frac{1}{20}\left(\frac{2}{3}a_0+\frac{1}{6}a_1\right)$

$n=4 \ \rightarrow \ a_6=\frac{1}{30}\left(a_4+a_3\right)=\frac{1}{30}\left(\frac{1}{2}a_0+a_1+\frac{1}{6}a_0+\frac{1}{6}a_1\right)=\frac{1}{30}\left(\frac{2}{3}a_0+\frac{7}{6}a_1\right)$

$a_0\neq 0 \quad a_1=0$	$a_0=0 \quad a_1\neq 0$
$a_2=\frac{1}{2}a_0$	$a_2=0$
$a_3=\frac{1}{6}a_0$	$a_3=\frac{1}{6}a_1$
$a_4=\frac{1}{24}a_0$	$a_4=\frac{1}{12}a_1$
$a_5=\frac{1}{30}a_0$	$a_5=\frac{1}{120}a_1$
$a_6=\frac{1}{45}a_0$	$a_6=\frac{7}{180}a_1$

$$\boxed{\begin{cases} y_1(x)=\left(1+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{30}x^5+\frac{1}{45}x^6+\cdots\right)a_0 \\ y_2(x)=\left(x+\frac{1}{6}x^3+\frac{1}{12}x^4+\frac{1}{120}x^5+\frac{7}{180}x^6+\cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $y''-(x+1)y'-y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y'' - (x+1)y' - y = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - (x+1)\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}na_n x^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}na_n x^n - a_1 - \sum_{n=1}^{\infty}(n+1)a_{n+1}x^n - a_0 - \sum_{n=1}^{\infty}a_n x^n = 0$$

$$2a_2 - a_1 - a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - na_n - (n+1)a_{n+1} - a_n\right]x^n = 0$$

$$2a_2 - a_1 - a_0 = 0 \quad \to \quad a_2 = \frac{1}{2}a_0 + \frac{1}{2}a_1$$

$$(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} - (n+1)a_n = 0$$

$$\boxed{a_{n+2} = \frac{1}{n+2}a_{n+1} + \frac{1}{n+2}a_n}$$

$$a_3 = \frac{1}{3}a_2 + \frac{1}{3}a_1 = \frac{1}{6}a_0 + \frac{1}{2}a_1$$

$$a_4 = \frac{1}{4}a_3 + \frac{1}{4}a_2 = \frac{1}{24}a_0 + \frac{1}{8}a_1 + \frac{1}{8}a_0 + \frac{1}{8}a_1 = \frac{1}{6}a_0 + \frac{1}{4}a_1$$

$$a_5 = \frac{1}{5}a_4 + \frac{1}{5}a_3 = \frac{1}{30}a_0 + \frac{1}{20}a_1 + \frac{1}{30}a_0 + \frac{1}{10}a_1 = \frac{1}{15}a_0 + \frac{3}{20}a_1$$

$$\vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x) = \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \cdots\right)a_0 \\ y_2(x) = \left(x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $\left(x^2+1\right)y'' - 6y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty} na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+1\right)y''-6y=0$$

$$\left(x^2+1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-6\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2}+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-6\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-6\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+2a_2+6a_3x+\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n-6a_0-6a_1x-6\sum_{n=2}^{\infty}a_n x^n=0$$

$$2a_2-6a_0+\left(6a_3-6a_1\right)x+\sum_{n=2}^{\infty}\left[\left(n^2-n-6\right)a_n+(n+2)(n+1)a_{n+2}\right]x^n=0$$

$$2a_2-6a_0+\left(6a_3-6a_1\right)x=0 \quad\rightarrow\quad \begin{cases} a_2=3a_0 \\ a_3=a_1 \end{cases}$$

$$(n+2)(n-3)a_n+(n+2)(n+1)a_{n+2}=0$$

$$\Rightarrow\quad a_{n+2}=-\frac{n-3}{n+1}a_n \qquad n=2,3,\ldots$$

$$a_4=\frac{1}{3}a_2=a_0 \qquad\qquad a_5=0$$

$$a_6=-\frac{1}{5}a_4=-\frac{1}{5}a_0 \qquad\qquad a_7=-\frac{1}{3}a_5=0$$

$$a_8=-\frac{3}{7}a_6=\frac{3}{35}a_0$$

$$\vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x)=\left(1+3x^2+x^4-\frac{1}{5}x^6+\cdots\right)a_0 \\ y_2(x)=\left(x+x^3\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $\left(x^2+2\right)y''+3xy'-y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+2\right)y''+3xy'-y=0$$

$$\left(x^2+2\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+3x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2}+\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}3(n+1)a_{n+1}x^{n+1}-\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+\sum_{n=0}^{\infty}\left[2(n+2)(n+1)a_{n+2}-a_n\right]x^n+\sum_{n=1}^{\infty}3na_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+4a_2-a_0+\left(12a_3-a_1\right)x+\sum_{n=2}^{\infty}\left[2(n+2)(n+1)a_{n+2}-a_n\right]x^n$$

$$+3a_1x+\sum_{n=2}^{\infty}3na_n x^n=0$$

$$4a_2-a_0+\left(12a_3+2a_1\right)x+\sum_{n=2}^{\infty}\left[n(n-1)a_n+2(n+2)(n+1)a_{n+2}+(3n-1)a_n\right]x^n=0$$

$$4a_2-a_0+\left(12a_3+2a_1\right)x \quad\Rightarrow\quad \boxed{a_2=\frac{1}{4}a_0} \quad \boxed{a_3=-\frac{1}{6}a_1}$$

$$2(n+2)(n+1)a_{n+2}+\left(n^2+2n-1\right)a_n=0$$

$$\rightarrow\quad a_{n+2}=-\frac{n^2+2n-1}{2(n+2)(n+1)}a_n \qquad n=2,3,\ldots$$

$$a_4=-\frac{7}{24}a_2=-\frac{7}{96}a_0 \qquad\qquad a_5=-\frac{7}{20}a_3=\frac{7}{120}a_1$$

$$a_6=-\frac{23}{60}a_4=\frac{161}{5760}a_0 \qquad\qquad a_7=-\frac{17}{42}a_5=-\frac{17}{720}a_1$$

$$\vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x)=\left(1+\frac{1}{4}x^2-\frac{7}{96}x^4+\frac{161}{5760}x^6-\cdots\right)a_0 \\ y_2(x)=\left(1-\frac{1}{6}x^3+\frac{7}{120}x^5-\frac{17}{720}x^7+\cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $\left(x^2-1\right)y''+xy'-y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2-1\right)y''+xy'-y=0$$

$$\left(x^2-1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}-\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_nx^n-\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_n\right]x^n+\sum_{n=1}^{\infty}na_nx^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_nx^n-\left(2a_2+a_0\right)-\left(6a_3+a_1\right)x-\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_n\right]x^n$$

$$+a_1x+\sum_{n=2}^{\infty}na_nx^n=0$$

$$-2a_2-a_0-6a_3x+\sum_{n=2}^{\infty}\left[n(n-1)a_n-(n+2)(n+1)a_{n+2}+(n-1)a_n\right]x^n=0$$

$$-2a_2 - a_0 - 6a_3 x = 0 \quad \boxed{\to a_2 = -\tfrac{1}{2}a_0} \quad \boxed{a_3 = 0}$$

$$-(n+2)(n+1)a_{n+2} + (n+1)(n-1)a_n = 0$$

$$\to \quad a_{n+2} = \frac{n-1}{n+2}a_n \qquad n = 2, 3, \ldots$$

$$a_4 = \tfrac{1}{4}a_2 = -\tfrac{1}{8}a_0 \qquad\qquad a_5 = -\tfrac{2}{5}a_3 = 0$$

$$a_6 = \tfrac{1}{2}a_4 = -\tfrac{1}{16}a_0 \qquad\qquad a_7 = \tfrac{4}{7}a_5 = 0$$

$$\vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x) = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \cdots\right)a_0 \\ y_2(x) = a_1 \end{cases}}$$

$$\begin{cases} y_1(x) = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \cdots \\ y_2(x) = x \end{cases}$$

Exercise

Find a power series solution. $\left(x^2+1\right)y'' + xy' - y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+1\right)y'' + xy' - y = 0$$

$$\left(x^2+1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$x^2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2}+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n}+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}-\sum_{n=0}^{\infty}a_{n}x^{n}=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_{n}x^{n}+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n}+\sum_{n=1}^{\infty}na_{n}x^{n}-\sum_{n=0}^{\infty}a_{n}x^{n}=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_{n}x^{n}+\sum_{n=1}^{\infty}na_{n}x^{n}+\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-a_{n}\right]x^{n}=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_{n}x^{n}+a_{1}x+\sum_{n=2}^{\infty}na_{n}x^{n}+(2a_{2}-a_{0})+(6a_{3}-a_{1})x+\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2}-a_{n}\right]x^{n}=0$$

$$2a_{2}-a_{0}+6a_{3}x+\sum_{n=2}^{\infty}\left[\left(n^{2}-n+n-1\right)a_{n}+(n+2)(n+1)a_{n+2}\right]x^{n}=0$$

$$\begin{cases}2a_{2}-a_{0}=0 & \rightarrow \boxed{a_{2}=\frac{1}{2}a_{0}}\\ 6a_{3}x=0 & \rightarrow \boxed{a_{3}=0}\end{cases}$$

$$(n-1)(n+1)a_{n}+(n+2)(n+1)a_{n+2}=0$$

$$\boxed{a_{n+2}=-\frac{n-1}{n+2}a_{n}}$$

a_0	a_1
$n=0\rightarrow a_{2}=\frac{1}{2}a_{0}$	$n=1\rightarrow a_{3}=0$
$n=2\rightarrow a_{4}=-\frac{1}{4}a_{2}=-\frac{1}{2^{3}}a_{0}$	$n=3\rightarrow a_{5}=-\frac{2}{5}a_{3}=0$
$n=4\rightarrow a_{6}=-\frac{3}{6}a_{4}=\frac{1\cdot 3}{2^{3}\cdot 3!}a_{0}$	$n=5\rightarrow a_{7}=0$
$n=6\rightarrow a_{8}=-\frac{5}{8}a_{6}=-\frac{1\cdot 3\cdot 5}{2^{4}\cdot 4!}a_{0}$	$\vdots\quad\vdots\quad\vdots\quad\vdots$
$\vdots\quad\vdots\quad\vdots\quad\vdots$	

$$a_{2n}=(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^{n}\cdot n!}a_{0}\qquad (n\geq 2)$$

$$\boxed{y(x)=a_{0}\left(1+\frac{1}{2}x^{2}-\frac{1}{2^{2}\cdot 2!}x^{4}+\frac{1\cdot 3}{2^{3}\cdot 3!}x^{6}-\frac{1\cdot 3\cdot 5}{2^{4}\cdot 4!}x^{8}+\cdots\right)+a_{1}x}$$

$$\boxed{=a_{0}\left(1+\frac{1}{2}x^{2}+\sum_{n=2}^{\infty}(-1)^{n-1}\frac{(2n-3)!!}{2^{n}\cdot n!}x^{2n}\right)+a_{1}x}$$

Exercise

Find a power series solution. $\left(x^2+1\right)y''-xy'+y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+1\right)y''-xy'+y=0$$

$$\left(x^2+1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$x^2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2}+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=1}^{\infty}na_n x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n-\sum_{n=1}^{\infty}na_n x^n+\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_n\right]x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n-a_1x-\sum_{n=2}^{\infty}na_n x^n+\left(2a_2+a_0\right)+\left(6a_3+a_1\right)x+\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_n\right]x^n=0$$

$$2a_2+a_0+6a_3x+\sum_{n=2}^{\infty}\left[\left(n^2-2n+1\right)a_n+(n+2)(n+1)a_{n+2}\right]x^n=0$$

$$\begin{cases}2a_2+a_0=0 & \rightarrow \boxed{a_2=-\frac{1}{2}a_0}\\ 6a_3x=0 & \rightarrow \boxed{a_3=0}\end{cases}$$

$$(n-1)^2a_n+(n+1)(n+2)a_{n+2}=0$$

$$\boxed{a_{n+2} = -\frac{(n-1)^2}{(n+1)(n+2)} a_n}$$

a_0

$n=0 \to a_2 = -\frac{1}{2} a_0$

$n=2 \to a_4 = -\frac{1}{12} a_2 = \frac{1}{4!} a_0$

$n=4 \to a_6 = -\frac{3^2}{5\cdot 6} a_4 = -\frac{3^2}{6!} a_0$

$n=6 \to a_8 = -\frac{5^2}{7\cdot 8} a_6 = \frac{1\cdot 3^2\cdot 5^2}{8!} a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

a_1

$n=1 \to a_3 = 0$

$n=3 \to a_5 = -\frac{2}{5} a_3 = 0$

$n=5 \to a_7 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$a_{2n} = (-1)^{n-1} \frac{1\cdot 3^2\cdot 5^2\cdots(2n-3)^2}{(2n)!} a_0 \quad (n \ge 3)$$

$$\boxed{y(x) = a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{9}{6!}x^6 - \frac{1\cdot 3^2\cdot 5^2}{8!}x^8 - \cdots\right) + a_1 x}$$

$$\boxed{= a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \sum_{n=3}^{\infty}(-1)^n \frac{(2n-3)^2!!}{(2n)!}x^{2n}\right) + a_1 x}$$

Exercise

Find a power series solution. $\left(1-x^2\right)y'' - 6xy' - 4y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(1-x^2\right)y'' - 6xy' - 4y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 6x \sum_{n=1}^{\infty} n a_n x^{n-1} - 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty}n(n-1)a_n x^n - \sum_{n=1}^{\infty}6na_n x^n - \sum_{n=0}^{\infty}4a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}n(n-1)a_n x^n - \sum_{n=0}^{\infty}6na_n x^n - \sum_{n=0}^{\infty}4a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - \big(n(n-1)+6n+4\big)a_n\right]x^n = 0$$

$$(n+2)(n+1)a_{n+2} - \left(n^2+5n+4\right)a_n = 0$$

$$(n+2)(n+1)a_{n+2} = (n+4)(n+1)a_n$$

$$\boxed{a_{n+2} = \frac{n+4}{n+2}a_n}$$

a_0	a_1
$n=2 \rightarrow a_2 = 2a_0$	$n=3 \rightarrow a_3 = \frac{5}{3}a_1$
$n=4 \rightarrow a_4 = \frac{6}{4}a_2 = 3a_0$	$n=5 \rightarrow a_5 = \frac{7}{5}a_3 = \frac{7}{3}a_1$
$n=6 \rightarrow a_6 = \frac{8}{6}a_3 = 4a_0$	$n=7 \rightarrow a_7 = \frac{11}{7}a_5 = \frac{11}{3}a_1$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$
$a_{2k} = (k+1)a_0$	$a_{2k+1} = \frac{2k+3}{3}a_1$

$$y(x) = a_0\left(1+2x^2+3x^4+4x^6+\cdots\right) + a_1\left(x+\frac{5}{3}x^3+\frac{7}{3}x^5+\frac{11}{3}x^7+\cdots\right)$$

$$\boxed{= \frac{a_0}{\left(1-x^2\right)^2} + \frac{3x-x^3}{3\left(1-x^2\right)^2}a_1}$$

Exercise

Find a power series solution. $y'' + (x-1)^2 y' - 4(x-1)y = 0$

Solution

Let $\quad z = x-1 \quad \rightarrow \quad dz = dx$

$$y(x) = \sum_{n=0}^{\infty}a_n z^n$$

$$y'(x) = \sum_{n=1}^{\infty}na_n z^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}z^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n$$

$$y''+z^2y'-4zy=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n+z^2\sum_{n=1}^{\infty}na_nz^{n-1}-4z\sum_{n=0}^{\infty}a_nz^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n+\sum_{n=1}^{\infty}na_nz^{n+1}-\sum_{n=0}^{\infty}4a_nz^{n+1}=0$$

$$2a_2+\sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}z^n+\sum_{n=1}^{\infty}(n-4)a_nz^{n+1}=0$$

$$2a_2+\sum_{n=0}^{\infty}(n+2)(n+3)a_{n+3}z^{n+1}+\sum_{n=1}^{\infty}(n-4)a_nz^{n+1}=0$$

$$2a_2+\sum_{n=0}^{\infty}\left[(n+2)(n+3)a_{n+3}+(n-4)a_n\right]z^{n+1}=0$$

$$\boxed{a_2=0}$$

$$(n+2)(n+3)a_{n+3}+(n-4)a_n=0$$

$$\boxed{a_{n+3}=-\frac{n-4}{(n+2)(n+3)}a_n}$$

a_0	a_1	$a_2=0$
$n=0\rightarrow\ a_3=\frac{4}{2\cdot 3}a_0$	$n=1\rightarrow\ a_4=\frac{3}{3\cdot 4}a_1$	$n=2\ \rightarrow\ a_5=\frac{2}{20}a_2=0$
$n=3\rightarrow\ a_6=\frac{1}{5\cdot 6}a_3=\frac{4}{2\cdot 3\cdot 5\cdot 6}a_0$	$n=4\rightarrow\ a_7=0$	$n=5\ \rightarrow\ a_8=0$
$n=6\rightarrow\ a_9=-\frac{2}{8\cdot 9}a_6=-\frac{8}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}a_0$	$n=7\rightarrow\ a_{10}=0$	$n=8\ \rightarrow\ a_5=0$
$\vdots\quad\vdots\quad\vdots\quad\vdots$		$\vdots\quad\vdots\quad\vdots\quad\vdots$

$$y(x)=a_0\left(1+\frac{4}{2\cdot 3}z^3+\frac{4}{2\cdot 3\cdot 5\cdot 6}z^6-\frac{8}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 8}z^9+\cdots\right)+a_1\left(z+\frac{1}{4}z^4\right)$$

$$\boxed{=a_0\left(1+\tfrac{2}{3}(x-1)^3+\tfrac{1}{45}(x-1)^6-\tfrac{1}{1,620}(x-1)^9+\cdots\right)+a_1\left(x-1+\tfrac{1}{4}(x-1)^4\right)}$$

Exercise

Find a power series solution. $\left(2-x^2\right)y''-xy'+16y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(2-x^2\right)y''-xy'+16y=0$$

$$2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-x^2\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}-x\sum_{n=1}^{\infty}na_n x^{n-1}+16\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n-\sum_{n=2}^{\infty}n(n-1)a_n x^n-\sum_{n=1}^{\infty}na_n x^n+\sum_{n=0}^{\infty}16a_n x^n=0$$

$$\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}n(n-1)a_n x^n-\sum_{n=0}^{\infty}na_n x^n+\sum_{n=0}^{\infty}16a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[2(n+2)(n+1)a_{n+2}-\left(n^2-n+n-16\right)a_n\right]x^n=0$$

$$2(n+1)(n+2)a_{n+2}-\left(n^2-16\right)a_n=0$$

$$2(n+1)(n+2)a_{n+2}=(n+4)(n-4)a_n$$

$$\boxed{a_{n+2}=\frac{(n+4)(n-4)}{2(n+1)(n+2)}a_n}$$

a_0	a_1
$n=0\to\ a_2=-4a_0$	$n=1\ \to\ a_3=-\frac{5}{4}a_1$
$n=2\ \to\ a_4=-\frac{1}{2}a_2=2a_0$	$n=3\ \to\ a_5=-\frac{7}{40}a_3=\frac{7}{32}a_1$
$n=4\ \to\ a_6=0$	$n=5\ \to\ a_7=\frac{9}{70}a_5=\frac{9}{320}a_1$
$\vdots\quad\vdots\quad\vdots\quad\vdots$	$n=7\ \to\ a_9=\frac{33}{144}a_7=\frac{33}{5,120}a_1$

$$\vdots \quad \vdots \qquad \vdots \qquad \vdots$$

$$a_{2n+1} = \frac{(2n-5)!!\ (2n+3)!!}{2^n\ (2n+1)!} a_1$$

$$\boxed{y(x) = a_0\left(1-4x^2+2x^4\right) + a_1\left(x - \frac{5}{4}x^3 + \frac{7}{32}x^5 + \frac{9}{320}x^7 + \cdots\right)}$$

$$\boxed{= a_0\left(1-4x^2+2x^4\right) + a_1\sum_{n=0}^{\infty}\frac{(2n-5)!!\ (2n+3)!!}{2^n\ (2n+1)!} x^{2n+1}}$$

Exercise

Find a power series solution. $\left(x^2+1\right)y'' - y' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+1\right)y'' - y' + y = 0$$

$$x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2} - (n+1)a_{n+1} + \left(n^2-n+1\right)a_n\right]x^n = 0$$

$$(n+1)(n+2)a_{n+2} - (n+1)a_{n+1} + \left(n^2-n+1\right)a_n = 0$$

$$\boxed{a_{n+2} = \frac{(n+1)a_{n+1} - \left(n^2 - n + 1\right)a_n}{(n+1)(n+2)}}$$

$$n = 0 \rightarrow\ a_2 = \frac{1}{2}\left(a_1 - a_0\right)$$

$$n = 1 \rightarrow\ a_3 = \frac{1}{6}\left(2a_2 - a_1\right) = \frac{1}{6}\left(a_1 - a_0 - a_1\right) = -\frac{1}{6}a_0$$

$$n = 2 \rightarrow\ a_4 = \frac{1}{12}\left(3a_3 - 3a_2\right) = \frac{1}{4}\left(-\frac{1}{6}a_0 - \frac{1}{2}a_1 + \frac{1}{2}a_0\right) = \frac{1}{12}a_0 - \frac{1}{8}a_1$$

$$n = 3 \rightarrow\ a_5 = \frac{1}{20}\left(4a_4 - 7a_3\right) = \frac{1}{20}\left(\frac{1}{3}a_0 - \frac{1}{2}a_1 + \frac{7}{6}a_0\right) = \frac{3}{40}a_0 - \frac{1}{40}a_1$$

$$n = 4 \rightarrow\ a_6 = \frac{1}{30}\left(5a_5 - 13a_4\right) = \frac{1}{30}\left(\frac{3}{8}a_0 - \frac{1}{8}a_1 - \frac{13}{12}a_0 + \frac{13}{8}a_1\right) = -\frac{17}{720}a_0 + \frac{1}{20}a_1$$

$$y(x) = a_0 + a_1 x + \left(\frac{1}{2}a_0 - \frac{1}{2}a_1\right)x^2 - \frac{1}{6}a_0 x^3 + \left(\frac{1}{12}a_0 - \frac{1}{8}a_1\right)x^4 + \left(\frac{3}{40}a_0 - \frac{1}{40}a_1\right)x^5$$
$$+\left(-\frac{17}{720}a_0 + \frac{1}{20}a_1\right)x^6 + \cdots$$

$$\boxed{\begin{aligned} y(x) = {} & a_0\left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 - \frac{17}{720}x^6 + \cdots\right) \\ & + a_1\left(x - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{40}x^5 + \frac{1}{20}x^6 + \cdots\right) \end{aligned}}$$

Exercise

Find a power series solution. $\left(x^2 + 1\right)y'' + 6xy' + 4y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

$$\left(x^2 + 1\right)y'' + 6xy' + 4y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + 6x \sum_{n=1}^{\infty} n a_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} 6na_n x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} 6na_n x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + \left(n^2+5n+4\right)a_n\right]x^n = 0$$

$$(n+1)(n+2)a_{n+2} + (n+1)(n+4)a_n = 0$$

$$\boxed{a_{n+2} = -\frac{n+4}{n+2}a_n}$$

a_0

$n=0 \to a_2 = -2a_0$

$n=2 \to a_4 = -\frac{3}{2}a_2 = 3a_0$

$n=4 \to a_6 = -\frac{8}{6}a_4 = -4a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

a_1

$n=1 \to a_3 = -\frac{5}{3}a_1$

$n=3 \to a_5 = -\frac{7}{5}a_3 = \frac{7}{3}a_1$

$n=5 \to a_7 = -\frac{9}{7}a_5 = -\frac{9}{3}a_1$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$a_{2n} = (-1)^n(n+1)a_0 \qquad\qquad a_{2n+1} = (-1)^n(2n+3)a_1$$

$$\boxed{y(x) = a_0\sum_{n=0}^{\infty}(-1)^n(n+1)x^{2n} + \frac{1}{3}a_1\sum_{n=0}^{\infty}(-1)^n(2n+3)x^{2n+1}}$$

$$\boxed{y(x) = a_0\left(1-2x^2+3x^4-4x^6+\cdots\right) + \frac{1}{3}a_1\left(x-\frac{5}{3}x^3+\frac{7}{3}x^5-\frac{9}{3}x^7+\cdots\right)}$$

Exercise

Find a power series solution. $\left(x^2-1\right)y'' - 6xy' + 12y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2-1\right)y''-6xy'+12y=0$$

$$x^2\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-6x\sum_{n=1}^{\infty}na_n x^{n-1}+12\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=1}^{\infty}6na_n x^n+\sum_{n=0}^{\infty}12a_n x^n=0$$

$$\sum_{n=0}^{\infty}n(n-1)a_n x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}6na_n x^n+\sum_{n=0}^{\infty}12a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[-(n+2)(n+1)a_{n+2}+\left(n^2-7n+12\right)a_n\right]x^n=0$$

$$-(n+1)(n+2)a_{n+2}+(n-3)(n-4)a_n=0$$

$$\boxed{a_{n+2}=\frac{(n-3)(n-4)}{(n+1)(n+2)}a_n}$$

a_0	a_1
$n=0\to\ a_2=6a_0$	$n=1\ \to\ a_3=a_1$
$n=2\ \to\ a_4=\frac{1}{6}a_2=a_0$	$n=3\ \to\ a_5=0$
$n=4\ \to\ a_6=0$	$\vdots\quad\vdots\quad\vdots\quad\vdots$
$\vdots\quad\vdots\quad\vdots\quad\vdots$	

$$\boxed{y(x)=a_0\left(1+6x^2+x^4\right)+a_1\left(x+x^3\right)}$$

Exercise

Find a power series solution. $\left(x^2-1\right)y''+8xy'+12y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2-1\right)y''+8xy'+12y=0$$

$$x^2\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+8x\sum_{n=1}^{\infty}na_n x^{n-1}+12\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}8na_n x^n+\sum_{n=0}^{\infty}12a_n x^n=0$$

$$\sum_{n=0}^{\infty}n(n-1)a_n x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}8na_n x^n+\sum_{n=0}^{\infty}12a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[-(n+2)(n+1)a_{n+2}+\left(n^2+7n+12\right)a_n\right]x^n=0$$

$$-(n+1)(n+2)a_{n+2}+(n+3)(n+4)a_n=0$$

$$\boxed{a_{n+2}=\frac{(n+3)(n+4)}{(n+1)(n+2)}a_n}$$

a_0	a_1
$n=0\rightarrow\ a_2=6a_0$	$n=1\ \rightarrow\ a_3=\frac{10}{3}a_1$
$n=2\ \rightarrow\ a_4=\frac{5}{2}a_2=15a_0$	$n=3\ \rightarrow\ a_5=\frac{42}{20}a_3=\frac{21}{3}a_1$
$n=4\ \rightarrow\ a_6=\frac{5}{2}a_4=28a_0$	$n=5\ \rightarrow\ a_7=\frac{12}{7}a_5=\frac{36}{3}a_1$
$\vdots\quad\vdots\quad\vdots\quad\vdots$	$\vdots\quad\vdots\quad\vdots\quad\vdots$
$a_{2n}=(n+1)(2n+1)a_0$	$a_{2n+1}=\frac{1}{3}(n+1)(2n+3)a_0$

$$\boxed{y(x)=a_0\left(1+6x^2+15x^4+28x^6+\cdots\right)+a_1\left(x+\frac{10}{3}x^3+7x^5+12x^7+\cdots\right)}$$

$$\boxed{=a_0\sum_{n=0}^{\infty}(n+1)(2n+1)x^{2n}+\frac{1}{3}a_1\sum_{n=0}^{\infty}(n+1)(2n+3)x^{2n+1}}$$

Exercise

Find a power series solution. $\left(x^2-1\right)y''+4xy'+2y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2-1\right)y''+4xy'+2y=0$$

$$x^2\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+4x\sum_{n=1}^{\infty}na_n x^{n-1}+2\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}4na_n x^n+\sum_{n=0}^{\infty}2a_n x^n=0$$

$$\sum_{n=0}^{\infty}n(n-1)a_n x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}4na_n x^n+\sum_{n=0}^{\infty}2a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[-(n+2)(n+1)a_{n+2}+\left(n^2+3n+2\right)a_n\right]x^n=0$$

$$-(n+1)(n+2)a_{n+2}+(n+1)(n+2)a_n=0$$

$$\underline{a_{n+2}=a_n}|$$

a_0	a_1
$n=0\to\ a_2=a_0$	$n=1\ \to\ a_3=a_1$
$n=2\ \to\ a_4=a_2=a_0$	$n=3\ \to\ a_5=a_1$
$\vdots\quad\vdots\quad\vdots\quad\vdots$	$\vdots\quad\vdots\quad\vdots\quad\vdots$

$$\underline{y(x)=a_0\left(1+x^2+x^4+x^6+\cdots\right)+a_1\left(x+x^3+x^5+x^7+\cdots\right)}|$$

$$=a_0\sum_{n=0}^{\infty}x^{2n}+a_1\sum_{n=0}^{\infty}x^{2n+1}$$

$$= \boxed{\frac{a_0 + a_1 x}{1 - x^2}}$$

Exercise

Find a power series solution. $\left(x^2+1\right)y'' - 4xy' + 6y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(x^2+1\right)y'' - 4xy' + 6y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 4x \sum_{n=1}^{\infty} n a_n x^{n-1} + 6 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=0}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} + \left(n^2 - 5n + 6\right) a_n \right] x^n = 0$$

$$(n+1)(n+2) a_{n+2} + (n-2)(n-3) a_n = 0$$

$$\boxed{a_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)} a_n}$$

a_0	a_1
$n = 0 \rightarrow a_2 = -3a_0$	$n = 1 \rightarrow a_3 = -\frac{1}{3} a_1$
$n = 2 \rightarrow a_4 = 0$	$n = 3 \rightarrow a_5 = 0$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x) = a_0\left(1-3x^2\right) + a_1\left(x - \frac{1}{3}x^3\right)}$$

Exercise

Find a power series solution. $\left(x^2+2\right)y'' + 4xy' + 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(x^2+2\right)y'' + 4xy' + 2y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 4x\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} 2(n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} 2(n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[2(n+2)(n+1)a_{n+2} + \left(n^2+3n+2\right)a_n\right]x^n = 0$$

$$2(n+1)(n+2)a_{n+2} + (n+1)(n+2)a_n = 0$$

$$\boxed{a_{n+2} = -\frac{1}{2}a_n}$$

a_0	a_1
$n=0 \rightarrow a_2 = -\frac{1}{2}a_0$	$n=1 \rightarrow a_3 = -\frac{1}{2}a_1$
$n=2 \rightarrow a_4 = -\frac{1}{2}a_2 = \frac{1}{2^2}a_0$	$n=3 \rightarrow a_5 = -\frac{1}{2}a_3 = \frac{1}{2^2}a_1$
$n=4 \rightarrow a_6 = -\frac{1}{2}a_4 = -\frac{1}{2^3}a_0$	$n=5 \rightarrow a_7 = -\frac{1}{2}a_5 = -\frac{1}{2^3}a_1$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad\qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n} = (-1)^n \frac{1}{2^n} a_0 \qquad\qquad a_{2n+1} = (-1)^n \frac{1}{2^n} a_1$$

$$\boxed{y(x) = a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{4}x^4 - \frac{1}{8}x^6 + \cdots\right) + a_1\left(x - \frac{1}{2}x^3 + \frac{1}{4}x^5 - \frac{1}{8}x^7 + \cdots\right)}$$

$$\boxed{= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^{2n+1}}$$

Exercise

Find a power series solution. $\left(x^2 - 3\right)y'' + 2xy' = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(x^2 - 3\right)y'' + 2xy' = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 3\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 2x \sum_{n=1}^{\infty} n a_n x^{n-1} = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} 3(n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} 2n a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} 3(n+1)(n+2) a_{n+2} x^n + \sum_{n=0}^{\infty} 2n a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[-3(n+1)(n+2) a_{n+2} + \left(n^2 + n\right) a_n\right] x^n = 0$$

$$-3(n+1)(n+2) a_{n+2} + n(n+1) a_n = 0$$

$$\boxed{a_{n+2} = \frac{1}{3} \frac{n}{n+2} a_n}$$

a_0

$n=0 \to a_2 = 0$

$n=2 \to a_4 = \frac{2}{12} a_2 = 0$

$n=4 \to a_6 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

a_1

$n=1 \to a_3 = \frac{1}{3^2} a_1$

$n=3 \to a_5 = \frac{1}{3}\frac{3}{5} a_3 = \frac{1}{3^2 \cdot 5} a_1$

$n=5 \to a_7 = \frac{1}{3}\frac{5}{7} a_5 = \frac{1}{3^3 \cdot 7} a_1$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$a_{2n+1} = (-1)^n \frac{1}{(2n+1)3^n} a_1$$

$$y(x) = a_0 + a_1 \left(x + \frac{1}{9}x^3 + \frac{1}{45}x^5 + \frac{1}{189}x^7 + \cdots \right)$$

$$= a_0 + a_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)3^n} x^{2n+1}$$

Exercise

Find a power series solution. $\left(x^2+3\right)y'' - 7xy' + 16y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(x^2+3\right)y'' - 7xy' + 16y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 3\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 7x \sum_{n=1}^{\infty} n a_n x^{n-1} + 16 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} 3(n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} 7n a_n x^n + \sum_{n=0}^{\infty} 16 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} 3(n+1)(n+2)a_{n+2} x^n - \sum_{n=0}^{\infty} 7na_n x^n + \sum_{n=0}^{\infty} 16a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[3(n+1)(n+2)a_{n+2} + \left(n^2 - 8n + 16\right)a_n\right] x^n = 0$$

$$3(n+1)(n+2)a_{n+2} + (n-4)^2 a_n = 0$$

$$\boxed{a_{n+2} = -\frac{(n-4)^2}{3(n+1)(n+2)} a_n}$$

a_0	a_1
$n=0 \rightarrow a_2 = -\frac{16}{6}a_0 = -\frac{8}{3}a_0$	$n=1 \rightarrow a_3 = -\frac{9}{18}a_1 = -\frac{1}{2}a_1$
$n=2 \rightarrow a_4 = -\frac{1}{9}a_2 = \frac{8}{27}a_0$	$n=3 \rightarrow a_5 = -\frac{1}{60}a_3 = \frac{1}{120}a_1$
$n=4 \rightarrow a_6 = 0$	$n=5 \rightarrow a_7 = -\frac{1}{126}a_5 = -\frac{1}{560\cdot 3^3}a_1$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x) = a_0\left(1 - \frac{8}{3}x^2 + \frac{8}{27}x^4\right) + a_1\left(x - \frac{1}{2}x^3 + \frac{1}{120}x^5 + \frac{1}{15,120}x^7 + \cdots\right)}$$

Exercise

Find the series solution to the initial value problem $y'' + 4y = 0$; $y(0) = 0,$ $y'(0) = 3$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

$y'' + 4y = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+4a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}+4a_n=0$$

$$\boxed{a_{n+2}=-\frac{4}{(n+1)(n+2)}a_n}$$

Given: $y(0)=0=a_0,\quad y'(0)=3=a_1$

$a_0=0$

$n=0 \;\rightarrow\; a_2=-2a_0=0$

$n=2 \;\rightarrow\; a_4=-\frac{4}{12}a_2=0$

$n=4 \;\rightarrow\; a_6=0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$a_1=3$

$n=1 \;\rightarrow\; a_3=-\frac{4}{6}a_1=-\frac{2^2}{3!}a_1=-2$

$n=3 \;\rightarrow\; a_5=-\frac{4}{20}a_3=-\frac{2^4}{5!}a_1=\frac{2}{5}$

$n=5 \;\rightarrow\; a_7=-\frac{4}{42}a_5=-\frac{2^6}{7!}a_1=-\frac{4}{105}$

$$a_{2k+1}=\frac{(-1)^k\,2^{2k}}{(2k+1)!}a_1$$

$$\begin{aligned}y(x)&=3x-2x^3+\frac{2}{5}x^5-\frac{4}{105}x^7+\cdots\\&=3\left(x-\frac{2^2}{3!}x^3+\frac{2^4}{5!}x^5-\frac{2^6}{7!}x^7+\cdots\right)\\&=\frac{3}{2}\left((2x)-\frac{1}{3!}(2x)^3+\frac{1}{5!}(2x)^5-\frac{1}{7!}(2x)^7+\cdots\right)\\&=\boxed{\frac{3}{2}\sin 2x}\end{aligned}$$

Exercise

Find the series solution to the initial value problem $y''+x^2y=0$; $\quad y(0)=1,\quad y'(0)=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''+x^2y=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x^2\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty}a_{n-2} x^{n} = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty}a_{n-2} x^{n} = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty}\left[(n+1)(n+2)a_{n+2} + a_{n-2}\right]x^n = 0$$

$$(n+1)(n+2)a_{n+2} + a_{n-2} = 0$$

$$\boxed{a_{n+2} = -\frac{1}{(n+1)(n+2)}a_{n-2}}$$

Given: $y(0) = 1 = a_0, \quad y'(0) = 0 = a_1$

$2a_2 + 6a_3 x = 0 \quad \rightarrow \quad \boxed{a_2 = a_3 = 0}$

$a_0 = 1$

$n=2 \quad \rightarrow \quad a_4 = -\frac{1}{12}a_0 = -\frac{1}{12}$

$a_1 = a_2 = a_3 = 0$

$n=3 \rightarrow a_5 = -\frac{1}{20}a_1 = 0$

$n=4 \rightarrow a_6 = {}^*a_2 = 0$

$n=5 \rightarrow a_7 = {}^*a_3 = 0$

$n=6 \rightarrow a_8 = -\frac{1}{56}a_4 = \frac{1}{672}$

$n=10 \rightarrow a_{12} = -\frac{1}{132}a_8 = \frac{1}{88,704}$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x) = 1 - \frac{1}{12}x^4 + \frac{1}{672}x^8 - \frac{1}{88,704}x^{12} + \cdots}$$

Exercise

Find the series solution to the initial value problem $y'' - 2xy' + 8y = 0; \quad y(0) = 3, \quad y'(0) = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty} na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y''-2xy'+8y=0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+8\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}2(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}8a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+8a_n\right]x^n-\sum_{n=1}^{\infty}2na_n x^n=0$$

$$2a_2+8a_0+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}+8a_n\right]x^n-\sum_{n=1}^{\infty}2na_n x^n=0$$

$$2a_2+8a_0+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}+(8-2n)a_n\right]x^n=0$$

Given: $y(0)=3=a_0$, $\quad y'(0)=0=a_1$

$2a_2+8a_0=0 \quad\rightarrow\quad \boxed{a_2=-4a_0=-12}$

$(n+2)(n+1)a_{n+2}+(8-2n)a_n=0$

$\rightarrow a_{n+2}=\dfrac{2n-8}{(n+1)(n+2)}a_n \qquad n=1,2,\ldots$

$a_3=-a_1=0 \qquad\qquad a_4=-\frac{1}{3}a_2=4$

$a_5=-\frac{1}{10}a_3=0 \qquad\qquad a_6=0a_4=0$

$a_7=\frac{1}{21}a_5=0 \qquad\qquad a_6=0a_4=0$

$\boxed{y(x)=3-12x^2+4x^4}$

Exercise

Find the series solution to the initial value problem $y''+y'-2y=0$; $\quad y(0)=1, \quad y'(0)=-2$

Solution

$$y(x)=\sum_{n=0}^{\infty} a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''+y'-2y=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-2\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+(n+1)a_{n+1}-2a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}+(n+1)a_{n+1}-2a_n=0$$

$$\boxed{a_{n+2}=\frac{2a_n-(n+1)a_{n+1}}{(n+1)(n+2)}}$$

Given: $y(0)=1=a_0$, $y'(0)=-2=a_1$

$$a_0=1 \qquad\qquad a_1=-2$$

$$n=0 \;\rightarrow\; a_2=\frac{2a_0-a_1}{2}=2 \qquad\qquad n=1 \;\rightarrow\; a_3=\frac{2a_1-2a_2}{6}=-\frac{8}{6}$$

$$n=2 \;\rightarrow\; a_4=\frac{2a_2-3a_3}{12}=\frac{2}{3} \qquad\qquad n=3 \;\rightarrow\; a_5=\frac{2a_3-4a_4}{20}=-\frac{4}{15}$$

$$n=4 \;\rightarrow\; a_6=\frac{2a_4-5a_5}{30}=\frac{4}{45} \qquad\qquad n=5 \;\rightarrow\; a_7=\frac{2a_5-6a_6}{42}=\frac{8}{315}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y(x)=1-2x+2x^2-\frac{4}{3}x^3+\frac{2}{3}x^4-\frac{4}{15}x^5+\frac{4}{45}x^6-\frac{8}{315}x^7+\cdots}$$

$$=1+(-2x)+\frac{1}{2!}(-2x)^2+\frac{1}{3!}(-2x)^3+\frac{1}{4!}(-2x)^4+\frac{1}{5!}(-2x)^5+\frac{1}{6!}(-2x)^6+\frac{1}{7!}(-2x)^7+\cdots$$

$$\boxed{=e^{-2x}}$$

Exercise

Find the series solution to the initial value problem $y'' - 2y' + y = 0$; $y(0) = 0,$ $y'(0) = 1$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$y'' - 2y' + y = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - 2(n+1) a_{n+1} + a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} - 2(n+1) a_{n+1} + a_n = 0$$

$$\boxed{a_{n+2} = \frac{2(n+1) a_{n+1} - a_n}{(n+1)(n+2)}}$$

Given: $y(0) = 0 = a_0,$ $y'(0) = 1 = a_1$

$a_0 = 0$ $\qquad$ $a_1 = 1$

$n = 0 \quad \to \quad a_2 = \frac{2a_1 - a_0}{2} = 1$ $\qquad$ $n = 1 \quad \to \quad a_3 = \frac{4a_2 - a_1}{6} = \frac{1}{2}$

$n = 2 \quad \to \quad a_4 = \frac{6a_3 - a_2}{12} = \frac{1}{6}$ $\qquad$ $n = 3 \quad \to \quad a_5 = \frac{8a_4 - a_3}{20} = \frac{1}{20}\left(\frac{4}{3} - \frac{1}{2}\right) = \frac{1}{24}$

$n = 4 \quad \to \quad a_6 = \frac{10a_5 - a_4}{30} = \frac{1}{30}\left(\frac{5}{12} - \frac{1}{6}\right) = \frac{1}{120}$ $\qquad$ $n = 5 \quad \to \quad a_7 = \frac{12a_6 - a_5}{42} = \frac{1}{42}\left(\frac{1}{10} - \frac{1}{24}\right) = \frac{1}{720}$

$\vdots \quad \vdots \quad \vdots \quad \vdots$ $\qquad$ $\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \frac{1}{120}x^6 + \frac{1}{720}x^7 + \cdots$$

$$= x\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots\right)$$

$$\boxed{= xe^x}$$

Exercise

Find the series solution to the initial value problem $y'' + xy' + y = 0 \quad y(0) = 1 \quad y'(0) = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$y'' + xy' + y = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + n a_n + a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} + (n+1) a_n = 0$$

$$(n+2)(n+1) a_{n+2} = -(n+1) a_n$$

$$a_{n+2} = -\frac{1}{n+2} a_n$$

$$a_0 = y(0) = 1 \qquad a_1 = y'(0) = 0$$

$$a_2 = -\frac{1}{2} a_0 = -\frac{1}{2} \qquad a_3 = -\frac{1}{3} a_1 = 0$$

$$a_4 = -\frac{1}{4} a_2 = \frac{1}{2 \cdot 4} = \frac{1}{2^2 \cdot 1 \cdot 2} \qquad a_5 = -\frac{1}{5} a_3 = 0$$

$$a_6 = -\frac{1}{6} a_4 = -\frac{1}{2^3 \cdot 1 \cdot 2 \cdot 3} \qquad a_7 = -\frac{1}{7} a_7 = 0$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + +a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$= 1 - \frac{1}{2} x^2 + \frac{1}{2^2 2!} x^4 - \frac{1}{2^3 3!} x^5 + \cdots$$

Exercise

Find the series solution to the initial value problem $y'' - xy' - y = 0 \quad y(0) = 2 \quad y'(0) = 1$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - xy' - y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - n a_n - a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} - (n+1) a_n = 0$$

$$(n+2)(n+1) a_{n+2} = -(n+1) a_n$$

$$\boxed{a_{n+2} = \frac{1}{n+2} a_n}$$

$$a_0 = y(0) = 2 \qquad\qquad a_1 = y'(0) = 1$$

$$a_2 = \frac{1}{2} a_0 = 1 \qquad\qquad a_3 = \frac{1}{3} a_1 = \frac{1}{3}$$

$$a_4 = \frac{1}{4} a_2 = \frac{1}{4} \qquad\qquad a_5 = \frac{1}{5} a_3 = \frac{1}{3 \cdot 5}$$

$$a_6 = \frac{1}{6} a_4 = \frac{1}{4}\frac{1}{6} = \frac{1}{24} \qquad\qquad a_7 = \frac{1}{7} a_7 = \frac{1}{3 \cdot 5 \cdot 7}$$

$$\vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y(x) = 2 + x + x^2 + \frac{1}{3} x^3 + \frac{1}{4} x^4 + \frac{1}{15} x^5 + \frac{1}{24} x^6 + \cdots}$$

Exercise

Find the series solution to the initial value problem $y'' - xy' - y = 0$; $y(0)=1$ $y'(0)=0$

Solution

$$y(x)=\sum_{n=0}^{\infty} a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y'' - xy' - y = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - n a_n - a_n\right]x^n = 0$$

$$(n+2)(n+1)a_{n+2} - (n+1)a_n = 0$$

$$(n+2)(n+1)a_{n+2} = -(n+1)a_n$$

$$\boxed{a_{n+2} = \frac{1}{n+2}a_n}$$

Given: $a_0 = y(0) = 1$ $a_1 = y'(0) = 0$

$a_2 = \frac{1}{2}a_0 = \frac{1}{2}$ $a_3 = \frac{1}{3}a_1 = 0$

$a_4 = \frac{1}{4}a_2 = \frac{1}{2\cdot 2^2}$ $a_5 = \frac{1}{5}a_3 = 0$

$a_6 = \frac{1}{6}a_4 = \frac{1}{2^3\cdot 3!}$ $\vdots \quad \vdots \quad \vdots$

$\vdots \quad \vdots \quad \vdots$

$$\boxed{y(x) = 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + \cdots}$$

Exercise

Find a power series solution. $y'' + xy' - 2y = 0$; $y(0) = 1$ $y'(0) = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$y'' + xy' - 2y = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} + (n-2) a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} + (n-2) a_n = 0$$

$$\boxed{a_{n+2} = -\frac{n-2}{(n+1)(n+2)} a_n}$$

$a_0 = y(0) = 1$ $a_1 = y'(0) = 0$

$n = 0 \rightarrow a_2 = \frac{2}{2} a_0 = 1$ $n = 1 \rightarrow a_3 = \frac{1}{6} a_1 = 0$

$n = 2 \rightarrow a_4 = 0$ $n = 3 \rightarrow a_5 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$ $\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x) = 1 + x^2}$$

Exercise

Find the series solution to the initial value problem $y'' + (x-1)y' + y = 0 \qquad y(1) = 2 \qquad y'(1) = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$y'' + (x-1)y' + y = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n + (x-1) \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n + \sum_{n=1}^{\infty} n a_n (x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + n a_n + a_n \right] x^{n-1} = 0$$

$$(n+2)(n+1) a_{n+2} + (n+1) a_n = 0$$

$$\boxed{a_{n+2} = -\frac{1}{n+2} a_n}$$

$a_0 = y(1) = 2 \qquad\qquad a_1 = y'(1) = 0$

$a_2 = -\frac{1}{2} a_0 = -1 \qquad\qquad a_3 = -\frac{1}{3} a_1 = 0$

$a_4 = -\frac{1}{4} a_2 = \frac{1}{2 \bullet 4} a_0 = \frac{1}{4} \qquad\qquad a_5 = -\frac{1}{5} a_3 = 0$

$a_6 = -\frac{1}{6} a_4 = -\frac{1}{24} \qquad\qquad a_7 = -\frac{1}{7} a_5 = 0$

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n = a_0 + a_1 (x-1) + a_2 (x-1) + a_3 (x-1)^3 + a_4 (x-1)^4 + \cdots$$

$$\boxed{= 2 - (x-1)^2 + \frac{1}{4}(x-1)^4 - \frac{1}{24}(x-1)^6 + \cdots}$$

$$\boxed{= \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{2n}}{n!\ 2^n}}$$

Exercise

Find the series solution to the initial value problem $(x-1)y''-xy'+y=0;\quad y(0)=-2,\quad y'(0)=6$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$(x-1)y''-xy'+y=0$$

$$(x-1)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+1}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=1}^{\infty}n(n+1)a_{n+1}x^n-\sum_{n=1}^{\infty}na_nx^n+\sum_{n=0}^{\infty}\left[a_n-(n+2)(n+1)a_{n+2}\right]x^n=0$$

$$\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1}-na_n\right]x^n+a_0-2a_2+\sum_{n=1}^{\infty}\left[a_n-(n+2)(n+1)a_{n+2}\right]x^n=0$$

$$a_0-2a_2+\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1}-(n-1)a_n-(n+2)(n+1)a_{n+2}\right]x^n=0$$

Given: $y(0)=-2=a_0,\quad y'(0)=6=a_1$

$$a_0-2a_2=0\quad\to\quad a_2=\frac{1}{2}a_0=-1$$

$$n(n+1)a_{n+1}-(n-1)a_n-(n+2)(n+1)a_{n+2}=0$$

$$\to\ a_{n+2}=\frac{n}{n+2}a_{n+1}-\frac{n-1}{(n+2)(n+1)}a_n$$

$$a_3=\frac{1}{3}a_2-0a_1=-\frac{1}{3}\qquad\qquad a_4=\frac{1}{2}a_3-\frac{1}{12}a_2=-\frac{1}{6}+\frac{1}{12}=-\frac{1}{12}$$

$$a_5=\frac{3}{5}a_4-\frac{1}{10}a_3=-\frac{3}{60}+\frac{1}{30}=-\frac{1}{60}\qquad\qquad a_6=\frac{2}{3}a_5-\frac{1}{10}a_4=-\frac{1}{90}+\frac{1}{120}=-\frac{1}{360}$$

$$y(x)=\left(-2-x^2-\frac{1}{3}x^3-\frac{1}{12}x^4-\ldots\right)+a_1x$$

$$= -2\left(1+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\ldots\right)+6x$$

$$= -2\left(1+x+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\ldots\right)+6x+2x$$

$$\boxed{= 8x-2e^x}$$

Exercise

Find the series solution to the initial value problem

$$(x+1)y''-(2-x)y'+y=0; \quad y(0)=2, \quad y'(0)=-1$$

Solution

$$y(x)=\sum_{n=0}^{\infty} a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty} na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$(x+1)y''-(2-x)y'+y=0$$

$$(x+1)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-(2-x)\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+1}+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}2(n+1)a_{n+1}x^n$$

$$+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+(n+1)a_{n+1}\right]x^{n+1}+\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-2(n+1)a_{n+1}+a_n\right]x^n=0$$

$$\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1}+na_n\right]x^n+\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-2(n+1)a_{n+1}+a_n\right]x^n=0$$

$$\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1}+na_n\right]x^n+2a_2-2a_1+a_0+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}-2(n+1)a_{n+1}+a_n\right]x^n=0$$

$$2a_2 - 2a_1 + a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + (n-2)(n+1)a_{n+1} + (n+1)a_n\right]x^n = 0$$

Given: $y(0) = 2 = a_0, \quad y'(0) = -1 = a_1$

$$2a_2 - 2a_1 + a_0 = 0 \quad \to \quad \boxed{a_2 = \frac{1}{2}\left(2a_1 - a_0\right) = -2}$$

$$(n+2)(n+1)a_{n+2} + (n-2)(n+1)a_{n+1} + (n+1)a_n = 0$$

$$\to \ a_{n+2} = -\frac{n-2}{n+2}a_{n+1} - \frac{1}{n+2}a_n$$

$$a_3 = \frac{1}{3}a_2 - \frac{1}{3}a_1 = \frac{2}{3} + \frac{1}{3} = 1 \qquad a_4 = 0a_3 - \frac{1}{4}a_2 = \frac{1}{2}$$

$$a_5 = -\frac{1}{5}a_4 - \frac{1}{5}a_3 = -\frac{1}{10} - \frac{1}{5} = -\frac{3}{10} \qquad a_6 = -\frac{1}{3}a_5 - \frac{1}{6}a_4 = \frac{1}{10} - \frac{1}{12} = \frac{1}{60}$$

$$\boxed{y(x) = 2 - x - 2x^2 + x^3 + \frac{1}{2}x^4 - \frac{3}{10}x^5 + \dots}$$

Exercise

Find the series solution to the initial value problem

$$(1-x)y'' + xy' - 2y = 0\ ; \quad y(0) = 0, \quad y'(0) = 1$$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$(1-x)y'' + xy' - 2y = 0$$

$$(1-x)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - 2\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+1} + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty}2a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - 2a_n\right]x^n - \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - (n+1)a_{n+1}\right]x^{n+1} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-2a_n\right]x^n-\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1}-na_n\right]x^n=0$$

$$2a_2-2a_0+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}-2a_n\right]x^n-\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1}-na_n\right]x^n=0$$

$$2a_2-2a_0+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}-n(n+1)a_{n+1}+(n-2)a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}-n(n+1)a_{n+1}+(n-2)a_n=0$$

$$\boxed{a_{n+2}=\frac{n(n+1)a_{n+1}-(n-2)a_n}{(n+1)(n+2)}}$$

Given: $y(0)=0=a_0,\quad y'(0)=1=a_1$

$$2a_2-2a_0=0\ \rightarrow\ \boxed{a_2=a_0=0}$$

$$n=1\rightarrow\ a_3=\frac{2a_2+a_1}{6}=\frac{1}{6}$$

$$n=2\ \rightarrow\ a_4=\frac{6a_3}{12}=\frac{1}{12}$$

$$n=3\rightarrow\ a_5=\frac{1}{20}\left(12a_4-a_3\right)=\frac{1}{20}\left(1-\frac{1}{6}\right)=\frac{1}{24}$$

$$n=4\rightarrow\ a_6=\frac{1}{30}\left(20a_5-2a_4\right)=\frac{1}{30}\left(\frac{5}{6}-\frac{1}{6}\right)=\frac{1}{45}$$

$$n=5\rightarrow\ a_7=\frac{1}{42}\left(30a_6-3a_5\right)=\frac{1}{30}\left(\frac{2}{3}-\frac{1}{8}\right)=\frac{13}{1008}$$

$$\vdots\quad\vdots\quad\vdots\quad\vdots$$

$$\boxed{y(x)=x+\frac{1}{6}x^3+\frac{1}{12}x^4+\frac{1}{24}x^5+\frac{1}{45}x^6+\frac{13}{1008}x^7+\cdots}$$

Exercise

Find the series solution to the initial value problem

$$\left(x^2+1\right)y''+2xy'=0;\quad y(0)=0,\quad y'(0)=1$$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+1\right)y''+2xy'=0$$

$$\left(x^2+1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2}+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}2(n+1)a_{n+1}x^{n+1}=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}2na_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+2a_2+6a_3x+\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n+2a_1x+\sum_{n=2}^{\infty}2na_n x^n=0$$

$$2a_2+\left(6a_3+2a_1\right)x+\sum_{n=2}^{\infty}\left[\left(n(n-1)+2n\right)a_n+(n+2)(n+1)a_{n+2}\right]x^n=0$$

Given: $y(0)=0=a_0, \quad y'(0)=1=a_1$

$$2a_2+\left(6a_3+2a_1\right)x=0 \quad \rightarrow \quad \begin{cases} a_2=0 \\ a_3=-\frac{1}{3}a_1=-\frac{1}{3} \end{cases}$$

$$n(n+1)a_n+(n+2)(n+1)a_{n+2}=0$$

$$\rightarrow\ a_{n+2}=-\frac{n}{n+2}a_n \qquad n=2,3,\ldots$$

$$a_4=-\frac{1}{2}a_2=0 \qquad\qquad a_5=-\frac{3}{5}a_3=\frac{1}{5}$$

$$a_6=-\frac{2}{3}a_4=0 \qquad\qquad a_7=-\frac{5}{7}a_5=-\frac{1}{7}$$

$$\boxed{y(x)=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\cdots}$$

Exercise

Find the series solution to the initial value problem

$$\left(x^2-1\right)y''+3xy'+xy=0\ ; \qquad y(0)=4, \qquad y'(0)=6$$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2-1\right)y''+3xy'+xy=0$$

$$\left(x^2-1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+3x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+x\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}3(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}a_n x^{n+1}=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^{n+2}+\sum_{n=0}^{\infty}\left[(3n+3)a_{n+1}+a_n\right]x^{n+1}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+\sum_{n=1}^{\infty}\left(3na_n+a_{n-1}\right)x^n-\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+\left(3a_1+a_0\right)x+\sum_{n=2}^{\infty}\left(3na_n+a_{n-1}\right)x^n-2a_2-6a_3x-\sum_{n=2}^{\infty}(n+1)(n+2)a_{n+2}x^n=0$$

$$-2a_2+\left(3a_1+a_0-6a_3\right)x+\sum_{n=2}^{\infty}\left[\left(n^2+2n\right)a_n+a_{n-1}-(n+2)(n+1)a_{n+2}\right]x^n=0$$

Given: $y(0)=4=a_0,\quad y'(0)=6=a_1$

$$\begin{cases}-2a_2=0 & \rightarrow \boxed{a_2=0}\\ 3a_1+a_0-6a_3=0 & \rightarrow \boxed{a_3=\frac{22}{6}=\frac{11}{3}}\end{cases}$$

$$\left(n^2+2n\right)a_n+a_{n-1}-(n+2)(n+1)a_{n+2}=0$$

$$\boxed{a_{n+2}=\frac{\left(n^2+2n\right)a_n+a_{n-1}}{(n+1)(n+2)}}$$

$$n=2\ \rightarrow\ a_4=\frac{8a_2+a_1}{12}=\frac{6}{12}=\frac{1}{2}$$

$$n=3 \to a_5 = \frac{1}{20}\left(15a_3 + a_2\right) = \frac{1}{20}(55) = \frac{11}{4}$$

$$n=4 \to a_6 = \frac{1}{30}\left(24a_4 + a_3\right) = \frac{1}{30}\left(12 + \frac{11}{3}\right) = \frac{47}{90}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y(x) = 4 + 6x + \frac{11}{3}x^3 + \frac{1}{2}x^4 + \frac{11}{4}x^5 + \frac{47}{90}x^6 + \cdots}$$

Exercise

Find the series solution to the initial value problem

$$\left(2+x^2\right)y'' - xy' + 4y = 0 \qquad y(0) = -1 \qquad y'(0) = 3$$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

$$\left(2+x^2\right)y'' - xy' + 4y = 0$$

$$\left(2+x^2\right)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} na_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} na_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$2(n+2)(n+1)a_{n+2} + n(n-1)a_n - na_n + 4a_n = 0$$

$$2(n+2)(n+1)a_{n+2} + \left(n^2 - 2n + 4\right)a_n = 0$$

$$\boxed{a_{n+2} = -\frac{n^2 - 2n + 4}{2(n+2)(n+1)}a_n}$$

$$a_0 = y(0) = -1 \qquad\qquad a_1 = y'(0) = 3$$

$n=0 \rightarrow \quad a_2 = -\frac{4}{4}a_0 = 1 \qquad\qquad n=1 \rightarrow \quad a_3 = -\frac{3}{12}a_1 = -\frac{1}{4}(3) = -\frac{3}{4}$

$n=2 \rightarrow \quad a_4 = -\frac{4}{24}a_2 = -\frac{1}{6} \qquad\qquad n=3 \rightarrow \quad a_5 = -\frac{7}{40}a_3 = -\frac{7}{40}\left(-\frac{3}{4}\right) = \frac{21}{160}$

$$\boxed{y(x) = -1 + 3x + x^2 - \frac{3}{4}x^3 - \frac{1}{6}x^4 + \frac{21}{160}x^5 + \cdots}$$

Exercise

Find the series solution to the initial value problem

$$\left(2-x^2\right)y'' - xy' + 4y = 0 \qquad y(0)=1 \qquad y'(0)=0$$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$\left(2-x^2\right)y'' - xy' + 4y = 0$$

$$\left(2-x^2\right)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[2(n+1)(n+2)a_{n+2} - \left(n^2 - n + n - 4\right)a_n\right]x^n = 0$$

$$2(n+1)(n+2)a_{n+2} - (n-2)(n+2)a_n = 0$$

$$\boxed{a_{n+2} = \frac{n-2}{2(n+1)}a_n}$$

$a_0 = y(0) = 1$ $\qquad$ $a_1 = y'(0) = 0$

$n = 0 \rightarrow \quad a_2 = \frac{-2}{2} a_0 = -1$ $\qquad$ $n = 1 \rightarrow \quad a_3 = -\frac{1}{4} a_1 = 0$

$n = 2 \rightarrow \quad a_4 = 0$ $\qquad$ $n = 3 \rightarrow \quad a_5 = {}^*a_3 = 0$

$\vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x) = 1 - x^2}$$

Exercise

Find the series solution to the initial value problem

$$\left(4 - x^2\right) y'' + 2y = 0 \qquad y(0) = 0 \qquad y'(0) = 1$$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(4 - x^2\right) y'' + 2y = 0$$

$$\left(4 - x^2\right) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} 4n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 4(n+1)(n+2) a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1) a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[4(n+1)(n+2) a_{n+2} - \left(n^2 - n - 2\right) a_n \right] x^n = 0$$

$$4(n+1)(n+2) a_{n+2} - (n+1)(n-2) a_n = 0$$

$$\boxed{a_{n+2} = \frac{n-2}{4(n+2)} a_n}$$

$a_0 = y(0) = 0$ $\qquad$ $a_1 = y'(0) = 1$

$n = 0 \rightarrow \quad a_2 = \frac{-2}{8} a_0 = 0$ $\qquad$ $n = 1 \rightarrow \quad a_3 = -\frac{1}{12} a_1 = -\frac{1}{12}$

$n = 2 \rightarrow \quad a_4 = 0$ $\qquad$ $n = 3 \rightarrow \quad a_5 = \frac{1}{20} a_3 = -\frac{1}{240}$

$\vdots \quad \vdots \quad \vdots$ $\qquad$ $n = 5 \rightarrow \quad a_7 = \frac{3}{28} a_5 = -\frac{1}{2,240}$

$\vdots \quad \vdots \quad \vdots$

$$\boxed{y(x) = x - \frac{1}{12} x^3 - \frac{1}{240} x^5 - \frac{1}{2240} x^7 - \frac{1}{16,128} x^9 - \cdots}$$

Exercise

Find a power series solution. $\left(x^2 - 4\right) y'' + 3xy' + y = 0$; $\quad y(0) = 4, \quad y'(0) = 1$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(x^2 - 4\right) y'' + 3xy' + y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 4 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 3x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=2}^{\infty} 4n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} 3n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} 4(n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 3n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[\left(n^2 - n + 3n + 1\right) a_n - 4(n+2)(n+1) a_{n+2} \right] x^n = 0$$

$$\left(n^2 + 2n + 1\right) a_n - 4(n+2)(n+1) a_{n+2} = 0$$

$$4(n+2)(n+1)a_{n+2} = (n+1)^2 a_n$$

$$\boxed{a_{n+2} = \frac{n+1}{4(n+2)} a_n}$$

$$a_0 = y(0) = 4 \qquad\qquad a_1 = y'(0) = 1$$

$$n=0 \quad \to \quad a_2 = \frac{1}{8}a_0 = \frac{1}{2} \qquad\qquad n=1 \quad \to \quad a_3 = \frac{2}{4\cdot 3}a_1 = \frac{1}{6}$$

$$n=2 \quad \to \quad a_4 = \frac{3}{16}a_2 = \frac{3}{32} \qquad\qquad n=3 \quad \to \quad a_5 = \frac{1}{5}a_3 = \frac{1}{30}$$

$$n=4 \quad \to \quad a_6 = \frac{5}{24}a_4 = \frac{5}{256} \qquad\qquad n=5 \quad \to \quad a_7 = \frac{6}{4\cdot 7}a_5 = \frac{1}{140}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y(x) = 4 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{3}{32}x^4 + \frac{1}{30}x^5 + \frac{5}{256}x^6 + \frac{1}{140}x^7 + \cdots}$$

Exercise

Find a power series solution. $\left(x^2+1\right)y'' + 2xy' - 2y = 0$; $y(0) = 0,\quad y'(0) = 1$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(x^2+1\right)y'' + 2xy' - 2y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2x \sum_{n=1}^{\infty} n a_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}+\left(n^2+n-2\right)a_n\right]x^n=0$$

$$(n+1)(n+2)a_{n+2}+(n-1)(n+2)a_n=0$$

$$\boxed{a_{n+2}=-\frac{n-1}{n+1}a_n}$$

$$a_0=y(0)=0 \qquad\qquad a_1=y'(0)=1$$

$$\begin{array}{llll} n=0 & \rightarrow & a_2=a_0 & \\ n=2 & \rightarrow & a_4=-\frac{1}{3}a_2=-\frac{1}{3}a_0 & \\ n=4 & \rightarrow & a_6=-\frac{3}{5}a_4=\frac{1}{5}a_0 & \\ n=6 & \rightarrow & a_8=-\frac{5}{7}a_6=-\frac{1}{7}a_0 & \\ \vdots & \vdots & \vdots & \vdots \end{array} \qquad \begin{array}{llll} n=1 & \rightarrow & a_3=0 & \\ n=3 & \rightarrow & a_5=0 & \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

$$a_{2n}=\frac{(-1)^n}{2n-1}a_0$$

$$\boxed{y(x)=a_1x+a_0\left(1+x^2-\frac{1}{3}x^4+\frac{1}{5}x^6-\frac{1}{7}x^8+\cdots\right)}$$

$$y(x)=a_1x+a_0\left(1+x\left(x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\cdots\right)\right)$$

$$\boxed{=a_1x+a_0\left(1+x\tan^{-1}x\right)}$$

Exercise

Find a power series solution. $\left(2x-x^2\right)y''-6(x-1)y'-4y=0$; $y(1)=0,\quad y'(1)=1$

Solution

Let $z=x-1 \;\Rightarrow\; \begin{cases} x=z+1 \\ dz=dx \end{cases}$

$$\left(2x-x^2\right)y''-6(x-1)y'-4y=0$$

$$\left(2z+2-z^2-2z-1\right)y''-6zy'-4y=0$$

$$\left(1-z^2\right)y''-6zy'-4y=0$$

$$y(z)=\sum_{n=0}^{\infty}a_nz^n$$

$$y'(z)=\sum_{n=1}^{\infty} na_n z^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n$$

$$y''(z)=\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n$$

$$\left(1-z^2\right)y''-6zy'-4y=0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}-z^2\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}-6z\sum_{n=1}^{\infty} na_n z^{n-1}-4\sum_{n=0}^{\infty} a_n z^n=0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}-\sum_{n=2}^{\infty} n(n-1)a_n z^{n}-\sum_{n=1}^{\infty} 6na_n z^{n}-4\sum_{n=0}^{\infty} a_n z^n=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}z^n-\sum_{n=0}^{\infty} n(n-1)a_n z^{n}-\sum_{n=0}^{\infty} 6na_n z^{n}-4\sum_{n=0}^{\infty} a_n z^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}-\left(n^2+5n+4\right)a_n\right]z^n=0$$

$$(n+1)(n+2)a_{n+2}-(n+1)(n+4)a_n=0$$

$$a_{n+2}=\frac{n+4}{n+2}a_n$$

Given: $y(1)=0=a_0, \quad y'(1)=1=a_1$

$a_0=0$

$n=0 \quad \rightarrow \quad a_2=2a_0=0$

$n=2 \quad \rightarrow \quad a_4=\frac{3}{2}a_2=0$

$n=4 \quad \rightarrow \quad a_6=0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$a_1=1$

$n=1 \quad \rightarrow \quad a_3=\frac{5}{3}a_1=\frac{5}{3}$

$n=3 \quad \rightarrow \quad a_5=\frac{7}{5}a_3=\frac{7}{3}$

$n=5 \quad \rightarrow \quad a_7=\frac{9}{7}a_5=\frac{9}{3}$

$n=7 \quad \rightarrow \quad a_9=\frac{11}{9}a_7=\frac{11}{3}$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$a_{2n+1}=\frac{2n+3}{3}$

$$y(z)=z+\frac{5}{3}z^3+\frac{7}{3}z^5+3z^7+\frac{11}{3}z^9+\cdots+\frac{2n+3}{3}z^{2n+1}+\cdots$$

$$y(x)=(x-1)+\frac{5}{3}(x-1)^3+\frac{7}{3}(x-1)^5+3(x-1)^7+\frac{11}{3}(x-1)^9+\cdots$$

Exercise

Find a power series solution. $\left(x^2-6x+10\right)y''-4(x-3)y'+6y=0$; $y(3)=2,\quad y'(3)=0$

Solution

Let $z=x-3 \Rightarrow \begin{cases} x=z+3 \\ dz=dx \end{cases}$

$$\left(x^2-6x+10\right)y''-4(x-3)y'+6y=0$$

$$\left(z^2+6z+9-6z-18+10\right)y''-4zy'+6y=0$$

$$\boxed{\left(z^2+1\right)y''-4zy'+6y=0}$$

$$y(z)=\sum_{n=0}^{\infty}a_nz^n$$

$$y'(z)=\sum_{n=1}^{\infty}na_nz^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n$$

$$y''(z)=\sum_{n=2}^{\infty}n(n-1)a_nz^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n$$

$$\left(z^2+1\right)y''-4zy'+6y=0$$

$$z^2\sum_{n=2}^{\infty}n(n-1)a_nz^{n-2}+\sum_{n=2}^{\infty}n(n-1)a_nz^{n-2}-4z\sum_{n=1}^{\infty}na_nz^{n-1}+6\sum_{n=0}^{\infty}a_nz^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_nz^n+\sum_{n=2}^{\infty}n(n-1)a_nz^{n-2}-\sum_{n=1}^{\infty}4na_nz^n+\sum_{n=0}^{\infty}6a_nz^n=0$$

$$\sum_{n=0}^{\infty}n(n-1)a_nz^n+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n-\sum_{n=0}^{\infty}4na_nz^n+\sum_{n=0}^{\infty}6a_nz^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}+\left(n^2-5n+6\right)a_n\right]z^n=0$$

$$(n+1)(n+2)a_{n+2}+(n-2)(n-3)a_n=0$$

$$\boxed{a_{n+2}=-\frac{(n-2)(n-3)}{(n+1)(n+2)}a_n}$$

Given: $y(3)=2=a_0,\quad y'(3)=0=a_1$

$a_0 = 2$

$n = 0 \quad \to \quad a_2 = -3a_0 = -6$

$n = 2 \quad \to \quad a_4 = 0$

$n = 4 \quad \to \quad a_6 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$a_1 = 0$

$n = 1 \quad \to \quad a_3 = -\frac{2}{4}a_1 = 0$

$n = 3 \quad \to \quad a_5 = 0$

$n = 5 \quad \to \quad a_7 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$y(z) = 2 - 6z^2$

$\boxed{y(x) = 2 - 6(x-3)^2}$

Exercise

Find a power series solution. $\quad \left(4x^2 + 16x + 17\right)y'' - 8y = 0\ ; \quad y(-2) = 1, \quad y'(-2) = 0$

Solution

Let $\quad z = x + 2 \quad \Rightarrow \begin{cases} x = z - 2 \\ dz = dx \end{cases}$

$$\left(4z^2 - 16z + 16 + 16z - 32 + 17\right)y'' - 8y = 0$$

$$\boxed{\left(4z^2 + 1\right)y'' - 8y = 0}$$

$$y(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$y'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$$

$$y''(z) = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n$$

$$\left(4z^2 + 1\right)y'' - 8y = 0$$

$$4z^2 \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} - 8\sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=2}^{\infty} 4n(n-1) a_n z^n + \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} - 8\sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty} 4n(n-1) a_n z^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n - 8\sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}+\left(4n^2-4n-8\right)a_n\right]z^n=0$$

$$(n+1)(n+2)a_{n+2}+4(n+1)(n-2)a_n=0$$

$$\boxed{a_{n+2}=-\frac{4(n-2)}{n+2}a_n}$$

Given: $y(-2)=1=a_0,\quad y'(-2)=0=a_1$

$$a_0=1 \qquad\qquad a_1=0$$

$n=0 \;\rightarrow\; a_2=\frac{8}{2}a_0=4$ $\qquad$ $n=1 \;\rightarrow\; a_3=\frac{4}{3}a_1=0$

$n=2 \;\rightarrow\; a_4=-0a_2=0$ $\qquad$ $n=3 \;\rightarrow\; a_5=0$

$n=4 \;\rightarrow\; a_6=0$ $\qquad$ $n=5 \;\rightarrow\; a_7=0$

$\vdots \;\vdots\;\vdots\;\vdots\;\vdots \qquad \vdots\;\vdots\;\vdots\;\vdots\;\vdots$

$$y(z)=1+4z^2$$

$$\boxed{y(x)=1+4(x+2)^2}$$

Exercise

Find a power series solution. $\quad\left(x^2+6x\right)y''+(3x+9)y'-3y=0\ ;\quad y(-3)=0,\quad y'(-3)=2$

Solution

Let $\quad z=x+3 \;\Rightarrow\; \begin{cases} x=z-3 \\ dz=dx \end{cases}$

$$\left(x^2+6x\right)y''+(3x+9)y'-3y=0$$

$$\left(z^2-6z+9+6z-18\right)y''+3zy'-3y=0$$

$$\boxed{\left(z^2-9\right)y''+3zy'-3y=0}$$

$$y(z)=\sum_{n=0}^{\infty}a_nz^n$$

$$y'(z)=\sum_{n=1}^{\infty}na_nz^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n$$

$$y''(z)=\sum_{n=2}^{\infty}n(n-1)a_nz^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n$$

$$\left(z^2-9\right)y''+3zy'-3y=0$$

$$z^2\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}-9\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}+3z\sum_{n=1}^{\infty}na_n z^{n-1}-3\sum_{n=0}^{\infty}a_n z^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n z^n-\sum_{n=2}^{\infty}9n(n-1)a_n z^{n-2}+\sum_{n=1}^{\infty}3na_n z^n-\sum_{n=0}^{\infty}3a_n z^n=0$$

$$\sum_{n=0}^{\infty}n(n-1)a_n z^n-\sum_{n=0}^{\infty}9(n+2)(n+1)a_{n+2}z^n+\sum_{n=0}^{\infty}3na_n z^n-\sum_{n=0}^{\infty}3a_n z^n=0$$

$$\sum_{n=0}^{\infty}\left[-9(n+2)(n+1)a_{n+2}+\left(n^2-n+3n-3\right)a_n\right]z^n=0$$

$$-9(n+2)(n+1)a_{n+2}+(n+3)(n-1)a_n=0$$

$$\boxed{a_{n+2}=\frac{(n+3)(n-1)}{9(n+1)(n+2)}a_n}$$

Given: $y(-3)=0=a_0, \quad y'(-3)=2=a_1$

$a_0=0$	$a_1=2$
$n=0 \quad \to \quad a_2=\frac{-3}{18}a_0=0$	$n=1 \quad \to \quad a_3=0a_1=0$
$n=2 \quad \to \quad a_4=\frac{5}{108}a_2=0$	$n=3 \quad \to \quad a_5=0$
$n=4 \quad \to \quad a_6=0$	$n=5 \quad \to \quad a_7=0$
$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$y(z)=2z$$

$$\boxed{y(x)=2x+6}$$

Exercise

Find the series solution near the given value $\qquad y''-(x-2)y'+2y=0\ ; \quad near\ x=2$

Solution

$$y=\sum_{n=0}^{\infty}a_n(x-2)^n$$

$$y'=\sum_{n=1}^{\infty}na_n(x-2)^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-2)^n$$

$$y''=\sum_{n=2}^{\infty} n(n-1)a_n(x-2)^{n-2}=\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-2)^n$$

$$y''-(x-2)y'+2y=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-2)^n-(x-2)\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-2)^n+2\sum_{n=0}^{\infty}a_n(x-2)^n=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-2)^n-\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-2)^{n+1}+\sum_{n=0}^{\infty}2a_n(x-2)^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}+2a_n\right](x-2)^n-\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-2)^{n+1}=0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}+2a_n\right](x-2)^n-\sum_{n=1}^{\infty}na_n(x-2)^n=0$$

$$2a_2+2a_0+\sum_{n=1}^{\infty}\left[(n+1)(n+2)a_{n+2}+2a_n\right](x-2)^n-\sum_{n=1}^{\infty}na_n(x-2)^n=0$$

$$2a_2+2a_0+\sum_{n=1}^{\infty}\left[(n+1)(n+2)a_{n+2}-(n-2)a_n\right](x-2)^n=0$$

For $n=0 \;\rightarrow\; 2a_2+2a_0=0 \;\Rightarrow\; \boxed{a_2=-a_0}$

$$(n+1)(n+2)a_{n+2}-(n-2)a_n=0$$

$$\boxed{a_{n+2}=\frac{n-2}{(n+1)(n+2)}a_n}$$

a_0	a_1
$n=0 \rightarrow a_2=-a_0$	$n=1 \rightarrow a_3=-\frac{1}{6}a_1$
$n=2 \rightarrow a_4=0$	$n=3 \rightarrow a_5=\frac{1}{20}a_3=-\frac{1}{120}a_1$
$n=4 \rightarrow a_6=0$	$n=5 \rightarrow a_7=\frac{3}{42}a_5=-\frac{1}{1680}a_1$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x)=a_0\left(1-(x-2)^2\right)+a_1\left((x-2)-\tfrac{1}{6}(x-2)^3-\tfrac{1}{120}(x-2)^5-\tfrac{1}{1680}(x-2)^7-\cdots\right)}$$

Exercise

Find the series solution near the given value $y''+(x-1)^2 y'-4(x-1)y=0$; *near* $x=1$

Solution

$$y=\sum_{n=0}^{\infty} a_n (x-1)^n$$

$$y'=\sum_{n=1}^{\infty} na_n (x-1)^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n$$

$$y''=\sum_{n=2}^{\infty} n(n-1)a_n (x-1)^{n-2}=\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n$$

$$y''+(x-1)^2 y'-4(x-1)y=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n+(x-1)^2\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n-4(x-1)\sum_{n=0}^{\infty}a_n(x-1)^n=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^{n+2}-\sum_{n=0}^{\infty}4a_n(x-1)^{n+1}=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n+\sum_{n=2}^{\infty}(n-1)a_{n-1}(x-1)^n-\sum_{n=1}^{\infty}4a_{n-1}(x-1)^n=0$$

$$2a_2+6a_3(x-1)+\sum_{n=2}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n+\sum_{n=2}^{\infty}(n-1)a_{n-1}(x-1)^n$$

$$-4a_0(x-1)-\sum_{n=2}^{\infty}4a_{n-1}(x-1)^n=0$$

$$2a_2+(6a_3-4a_0)(x-1)+\sum_{n=2}^{\infty}\left[(n+1)(n+2)a_{n+2}+(n-5)a_{n-1}\right](x-1)^n=0$$

$$\begin{cases} 2a_2=0 & \rightarrow \boxed{a_2=0} \\ 6a_3-4a_0=0 & \rightarrow \boxed{a_3=\frac{2}{3}a_0} \end{cases}$$

$$(n+1)(n+2)a_{n+2}+(n-5)a_{n-1}=0$$

$$\boxed{a_{n+2}=-\frac{(n-5)}{(n+1)(n+2)}a_{n-1}}$$

$$a_0 \qquad\qquad a_1 \qquad\qquad a_2 = 0$$

$$n=1 \to a_3 = \frac{2}{3}a_0 \qquad n=2 \to a_4 = \frac{1}{4}a_1 \qquad n=3 \to a_5 = \frac{2}{20}a_2 = 0$$

$$n=4 \to a_6 = \frac{1}{30}a_3 = \frac{1}{45}a_0 \qquad n=5 \to a_7 = 0 \qquad n=6 \to a_8 = 0$$

$$n=7 \to a_9 = -\frac{2}{8\cdot 9}a_6 = -\frac{1}{1{,}620}a_0 \qquad n=8 \to a_{10} = 0 \qquad n=9 \to a_{11} = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad \vdots \quad \vdots \quad \vdots \quad \vdots \qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y(x) = a_0\left(1 + \frac{2}{3}(x-1)^3 + \frac{1}{45}(x-1)^6 - \frac{1}{1{,}620}(x-1)^9 + \cdots\right) + a_1\left((x-1) + \frac{1}{4}(x-1)^4\right)}$$

$$\boxed{y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n 4(x-1)^{3n}}{3^n(3n-1)(3n-4)n!} + a_1\left((x-1) + \frac{1}{4}(x-1)^4\right)}$$

Exercise

Find the series solution near the given value $y'' + (x-1)y = e^x$; *near* $x = 1$

Solution

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x-1)^n$$

$$y'' + (x-1)y = e^{x-1+1}$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x-1)^n + (x-1)\sum_{n=0}^{\infty} a_n (x-1)^n = e \cdot e^{x-1} \qquad\qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^{n+1} = e \cdot \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x-1)^n + \sum_{n=1}^{\infty} a_{n-1} (x-1)^n = e \cdot \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$$

$$2a_2+\sum_{n=1}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n+\sum_{n=1}^{\infty}a_{n-1}(x-1)^n=e+e\cdot\sum_{n=1}^{\infty}\frac{(x-1)^n}{n!}$$

$$2a_2+\sum_{n=1}^{\infty}\left[(n+1)(n+2)a_{n+2}+a_{n-1}\right](x-1)^n=e+e\cdot\sum_{n=1}^{\infty}\frac{(x-1)^n}{n!}$$

$$2a_2=e\quad\to\quad \boxed{a_2=\frac{e}{2}}$$

$$(n+1)(n+2)a_{n+2}+a_{n-1}=\frac{e}{n!}$$

$$\boxed{a_{n+2}=\frac{e}{(n+1)(n+2)\,n!}-\frac{1}{(n+1)(n+2)}a_{n-1}}$$

a_0

$$n=1\ \to\ a_3=\frac{e}{6}-\frac{1}{6}a_0$$

$$n=4\ \to\ a_6=\frac{e}{720}-\frac{1}{30}a_3=-\frac{11e}{720}+\frac{1}{180}a_0$$

$\vdots\quad\vdots\quad\vdots\quad\vdots$

a_1

$$n=2\to\ a_4=\frac{e}{24}-\frac{1}{12}a_1$$

$$n=5\to\ a_7=\frac{e}{5040}-\frac{1}{42}a_4=\frac{e}{1260}+\frac{1}{504}a_0$$

$\vdots\quad\vdots\quad\vdots\quad\vdots$

$$a_2=\frac{e}{2}$$

$$n=3\ \to\ a_5=\frac{e}{120}-\frac{1}{20}a_2=\frac{e}{120}-\frac{e}{40}=-\frac{e}{60}$$

$$n=6\ \to\ a_8=\frac{e}{40,320}-\frac{1}{56}a_5=\frac{e}{40,320}+\frac{e}{3360}=\frac{13e}{40,320}$$

$\vdots\quad\vdots\quad\vdots\quad\vdots$

$$y(x)=a_0+(x-1)a_1+\frac{e}{2}(x-1)^2+\left(\frac{e}{6}-\frac{1}{6}a_0\right)(x-1)^3+\left(\frac{e}{24}-\frac{1}{12}a_1\right)(x-1)^4-\frac{e}{60}(x-1)^5+\cdots$$

$$=a_0+(x-1)a_1+\frac{e}{2}(x-1)^2+\frac{e}{6}(x-1)^3+-\frac{1}{6}a_0(x-1)^3+\frac{e}{24}(x-1)^4$$

$$-\frac{1}{12}a_1(x-1)^4-\frac{e}{60}(x-1)^5+\cdots$$

$$\boxed{\begin{aligned}y(x)&=e\left(\frac{1}{2}(x-1)^2+\frac{1}{6}(x-1)^3+\frac{1}{24}(x-1)^4-\frac{1}{60}(x-1)^5+\cdots\right)\\&\quad+a_0\left(1-\frac{1}{6}(x-1)^3+\cdots\right)+a_1\left((x-1)-\frac{1}{12}(x-1)^4+\cdots\right)\end{aligned}}$$

Exercise

Find the series solution near the given value

$$y''+xy'+(2x-1)y=0\ ;\qquad \textit{near}\ x=-1\qquad y(-1)=2,\qquad y'(-1)=-2$$

Solution

$$t=x+1\quad\to\quad x=t-1$$

$$y = \sum_{n=0}^{\infty} a_n t^n$$

$$y' = \sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} t^n$$

$$y'' + xy' + (2x-1) y = 0$$

$$y'' + (t-1) y' + (2t-3) y = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} t^n + (t-1)\sum_{n=0}^{\infty} (n+1) a_{n+1} t^n + (2t-3)\sum_{n=0}^{\infty} a_n t^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} t^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} t^{n+1} - \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n + \sum_{n=0}^{\infty} 2a_n t^{n+1} - \sum_{n=0}^{\infty} 3a_n t^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} - (n+1) a_{n+1} - 3a_n \right] t^n + \sum_{n=0}^{\infty} \left[(n+1) a_{n+1} + 2a_n \right] t^{n+1} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} - (n+1) a_{n+1} - 3a_n \right] t^n + \sum_{n=1}^{\infty} \left[n a_n + 2a_{n-1} \right] t^n = 0$$

$$2a_2 - a_1 - 3a_0 + \sum_{n=1}^{\infty} \left[(n+1)(n+2) a_{n+2} - (n+1) a_{n+1} - 3a_n \right] t^n + \sum_{n=1}^{\infty} \left[n a_n + 2a_{n-1} \right] t^n = 0$$

$$2a_2 - a_1 - 3a_0 + \sum_{n=1}^{\infty} \left[(n+1)(n+2) a_{n+2} - (n+1) a_{n+1} + (n-3) a_n + 2a_{n-1} \right] t^n = 0$$

$$2a_2 - a_1 - 3a_0 = 0 \quad \rightarrow \quad \boxed{a_2 = \frac{1}{2}\left(a_1 + 3a_0\right)}$$

$$(n+1)(n+2) a_{n+2} - (n+1) a_{n+1} + (n-3) a_n + 2a_{n-1} = 0$$

$$\boxed{a_{n+2} = \frac{1}{n+2} a_{n+1} - \frac{n-3}{(n+1)(n+2)} a_n - \frac{2}{(n+1)(n+2)} a_{n-1}}$$

Given: $t = x + 1$

$$y(x=-1) = y(t=0) = 2 = a_0, \qquad y'(x=-1) = y(t=0) = -2 = a_1$$

$$\underline{a_2} = \frac{1}{2}\left(a_1 + 3a_0\right) = \frac{1}{2}(-2+6) \ \underline{= 2}$$

$$n=1 \rightarrow a_3=\frac{1}{3}a_2+\frac{1}{3}a_1-\frac{1}{3}a_0=\frac{2}{3}-\frac{2}{3}-\frac{2}{3}=\boxed{-\frac{2}{3}}$$

$$n=2 \rightarrow a_4=\frac{1}{4}a_3+\frac{1}{12}a_2-\frac{1}{6}a_1=-\frac{1}{6}+\frac{1}{6}+\frac{1}{3}=\boxed{\frac{1}{3}}$$

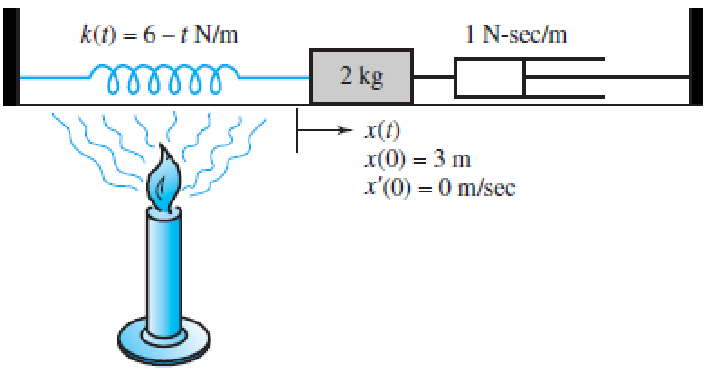
$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y(t)=2-2t+3t^2-\frac{1}{3}t^3+\frac{1}{3}t^4+\cdots$$

$$\boxed{y(x)=2-2(x+1)+3(x+1)^2-\frac{1}{3}(x+1)^3+\frac{1}{3}(x+1)^4+\cdots}$$

Exercise

As a spring is heated, its spring "constant" decreases. Suppose the spring is heated so that the spring "constant" at time *t* is $k(t)=6-t$ *N/m*.



If the unforced mass-spring system has mass $m=2$ *kg* and a damping constant $b=1$ *N-sec/m* with initial conditions $x(0)=3$ *m* and $x'(0)=0$ *m/sec*, then the displacement $x(t)$ is governed by the initial value problem

$$2x''(t)+x'(t)+(6-t)x(t)=0\ ; \quad x(0)=3, \quad x'(0)=0$$

Find at least the first four nonzero terms in a power series expansion about $t=0$ for the displacement.

Solution

$$x(t)=\sum_{n=0}^{\infty}a_nt^n$$

$$x'(t)=\sum_{n=1}^{\infty}na_nt^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n$$

$$x''(t)=\sum_{n=2}^{\infty}n(n-1)a_nt^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n$$

$$2x''+x'+(6-t)x=0$$

$$2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n+(6-t)\sum_{n=0}^{\infty}a_nt^n=0$$

$$\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}t^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n+\sum_{n=0}^{\infty}6a_nt^n-\sum_{n=0}^{\infty}a_nt^{n+1}=0$$

$$\sum_{n=0}^{\infty}\left[2(n+2)(n+1)a_{n+2}+(n+1)a_{n+1}+6a_n\right]t^n-\sum_{n=1}^{\infty}a_{n-1}t^n=0$$

$$4a_2+a_1+6a_0+\sum_{n=1}^{\infty}\left[2(n+2)(n+1)a_{n+2}+(n+1)a_{n+1}+6a_n\right]t^n-\sum_{n=1}^{\infty}a_{n-1}t^n=0$$

$$4a_2+a_1+6a_0+\sum_{n=1}^{\infty}\left[2(n+1)(n+2)a_{n+2}+(n+1)a_{n+1}+6a_n-a_{n-1}\right]t^n=0$$

Given: $x(0)=3=a_0,\quad x'(0)=0=a_1$

$$4a_2+a_1+6a_0=0\quad\rightarrow\quad \boxed{a_2=-\frac{9}{2}}$$

$$2(n+1)(n+2)a_{n+2}+(n+1)a_{n+1}+6a_n-a_{n-1}=0$$

$$\boxed{a_{n+2}=\frac{a_{n-1}-6a_n-(n+1)a_{n+1}}{2(n+1)(n+2)}}$$

$$n=1\ \rightarrow\ a_3=\frac{1}{12}\left(a_0-6a_1-2a_2\right)=\frac{1}{12}(3+9)=1$$

$$n=2\ \rightarrow\ a_4=\frac{1}{24}\left(a_1-6a_2-3a_3\right)=\frac{1}{24}(27-3)=1$$

$$\boxed{x(t)=3-\frac{9}{2}t^2+t^3+t^4+\cdots}$$