

Section 2.4 – Cross Product

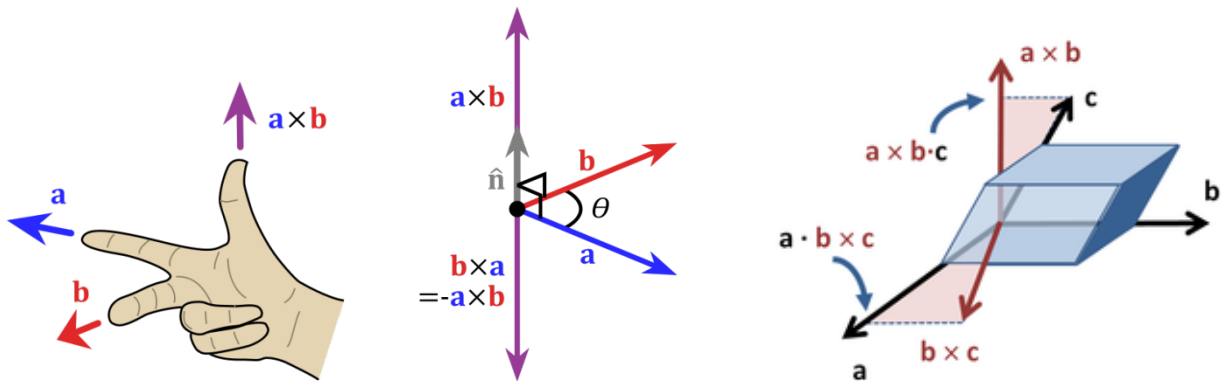
The Cross Product

To find a vector in 3-space that is perpendicular to two vectors; the type of vector multiplication that facilitates this construction is the cross product.

Definition

The cross product of $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ is the vector

$$\begin{aligned}\vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \hat{i} - \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} \hat{j} + \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \hat{k} \\ &= (u_2 v_3 - u_3 v_2) \hat{i} - (u_1 v_3 - u_3 v_1) \hat{j} + (u_1 v_2 - u_2 v_1) \hat{k} \\ &= \underline{(u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1)}\end{aligned}$$



In 1773, **Joseph Louis Lagrange** introduced the component form of both the dot and cross products in order to study the tetrahedron in three dimensions. In 1843 the Irish mathematical physicist Sir **William Rowan Hamilton** introduced the quaternion product, and with it the terms "vector" and "scalar". Given two quaternions $[0, \vec{u}]$ and $[0, \vec{v}]$, where \vec{u} and \vec{v} are vectors in \mathbb{R}^3 , their quaternion product can be summarized as $[-\vec{u} \cdot \vec{v}, \vec{u} \times \vec{v}]$. **James Clerk Maxwell** used Hamilton's quaternion tools to develop his famous **electromagnetism** equations, and for this and other reasons quaternions for a time were an essential part of physics education.

Example

Find $\vec{u} \times \vec{v}$, where $\vec{u} = (1, 2, -2)$ and $\vec{v} = (3, 0, 1)$

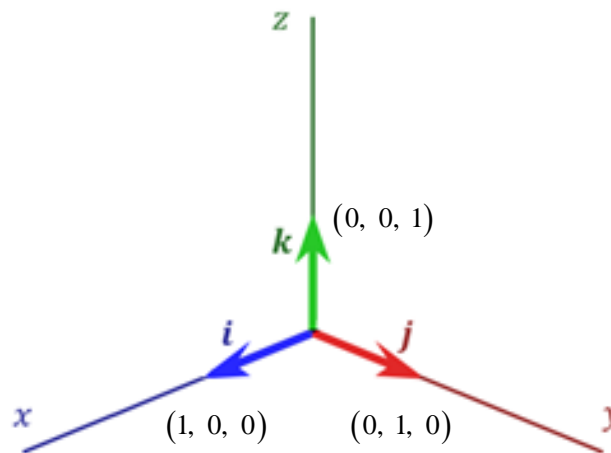
Solution

$$\begin{bmatrix} 1 & 2 & -2 \\ 3 & 0 & 1 \end{bmatrix}$$
$$\vec{u} \times \vec{v} = \left(\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \right)$$
$$= \underline{(2, -7, -6)}$$

Example

Consider the vectors $\hat{i} = (1, 0, 0)$ $\hat{j} = (0, 1, 0)$ $\hat{k} = (0, 0, 1)$

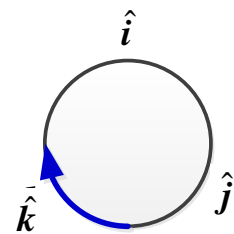
These vectors each have length of 1 and lie along the coordinate axes. They are called the **standard unit vectors** in 3-space.



For example: $(2, 3, -4) = 2\hat{i} + 3\hat{j} - 4\hat{k}$

Note:

- ✓ $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$
- ✓ $\hat{j} \times \hat{i} = -\hat{k}, \quad \hat{k} \times \hat{j} = -\hat{i}, \quad \hat{i} \times \hat{k} = -\hat{j}$
- ✓ $\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}$



Properties

1. $\vec{u} \times \vec{v}$ reverses rows 2 and 3 in the determinant so it is equals $-(\vec{u} \times \vec{v})$
2. The cross product $\vec{u} \times \vec{v}$ is perpendicular to \vec{u} , then $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$
3. The cross product $\vec{u} \times \vec{v}$ is perpendicular to \vec{v} , then $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$
4. The cross product of any vector with itself (two equal rows) is $\vec{u} \times \vec{u} = 0$.
5. Lagrange's identity: $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$
 $= \|\vec{u}\| \|\vec{v}\| |\sin \theta|$

$$|\vec{u} \cdot \vec{v}| = \|\vec{u}\| \|\vec{v}\| |\cos \theta|$$

Theorem

- a) $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
- b) $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
- c) $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$
- d) $k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{v})$
- e) $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$
- f) $\vec{u} \times \vec{u} = 0$

Definition

If \vec{u} , \vec{v} , and \vec{w} are vectors in 3-space, then $\boxed{\vec{u} \cdot (\vec{v} \times \vec{w})}$ is called the **scalar triple product** of \vec{u} , \vec{v} , and \vec{w} .

Example

Calculate the scalar triple product $\vec{u} \cdot (\vec{u} \times \vec{v})$ of the vectors:

$$\vec{u} = -2\hat{i} + 6\hat{k} \quad \vec{v} = \hat{i} - 3\hat{j} + \hat{k} \quad \vec{w} = -5\hat{i} - \hat{j} + \hat{k}$$

Solution

$$\begin{aligned} \vec{u} \cdot (\vec{u} \times \vec{v}) &= \begin{vmatrix} -2 & 0 & 6 \\ 1 & -3 & 1 \\ -5 & -1 & 1 \end{vmatrix} \\ &= \underline{-92} \end{aligned}$$

Area of a Parallelogram

Theorem

If \vec{u} and \vec{v} are vectors in 3-space, then $\|\vec{u} \times \vec{v}\|$ is equal to the area of the parallelogram determined by \vec{u} and \vec{v} .

Example

Find the area of the triangle determined by the points $P_1(2, 2, 0)$, $P_2(-1, 0, 2)$, and $P_3(0, 4, 3)$.

Solution

The area of the triangle is $\frac{1}{2}$ the area of the parallelogram determined by the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$

$$\begin{aligned}\overrightarrow{P_1P_2} &= (-1, 0, 2) - (2, 2, 0) \\ &= \underline{(-3, -2, 2)}\end{aligned}$$

$$\begin{aligned}\overrightarrow{P_1P_3} &= (0, 4, 3) - (2, 2, 0) \\ &= \underline{(-2, 2, 3)}\end{aligned}$$

$$\begin{aligned}\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} &= \left(\begin{vmatrix} -2 & 2 \\ 2 & 3 \end{vmatrix}, -\begin{vmatrix} -3 & 2 \\ -2 & 3 \end{vmatrix}, \begin{vmatrix} -3 & -2 \\ -2 & 2 \end{vmatrix} \right) \\ &= \underline{(-10, 5, -10)}\end{aligned}$$

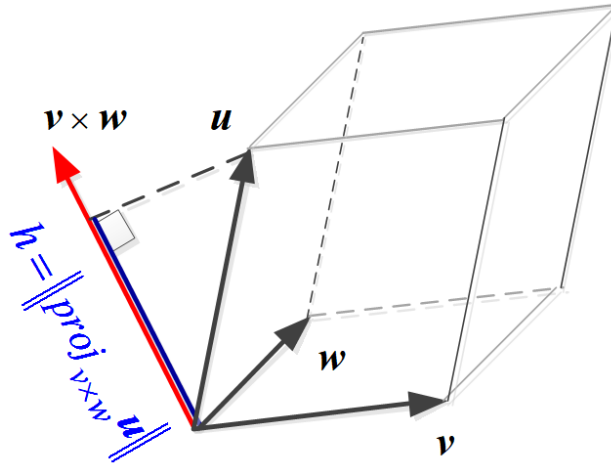
$$\begin{aligned}\text{Area} &= \frac{1}{2} \|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\| \\ &= \frac{1}{2} \sqrt{(-10)^2 + 5^2 + (-10)^2} \\ &= \underline{\frac{15}{2} \text{ unit}^2}\end{aligned}$$

Volume

The Volume of the Parallelepiped is

$$V = (\text{area of base}) \cdot (\text{height}) = \|\vec{v} \times \vec{w}\| \frac{|\vec{u} \cdot (\vec{v} \times \vec{w})|}{\|\vec{v} \times \vec{w}\|} = |\vec{u} \cdot (\vec{v} \times \vec{w})|$$

$$V = \left| \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \right|$$



Theorem

If the vectors $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, and $\vec{w} = (w_1, w_2, w_3)$ have the initial point, then they lie in the same plane if and only if

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0$$

Example

Find the volume of the parallelepiped with sides $\vec{u} = (2, -6, 2)$, $\vec{v} = (0, 4, -2)$, and $\vec{w} = (2, 2, -4)$

Solution

$$V = \left| \det \begin{bmatrix} 2 & -6 & 2 \\ 0 & 4 & -2 \\ 2 & 2 & -4 \end{bmatrix} \right|$$

=16

Exercises Section 2.4 – Cross Product

1. Prove when the cross product $\vec{u} \times \vec{v}$ is perpendicular to \vec{u} , then $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$
2. Find $\vec{u} \times \vec{v}$, where $\vec{u} = (1, 2, -2)$ and $\vec{v} = (3, 0, 1)$ and show that $\vec{u} \times \vec{v}$ is perpendicular to \vec{u} and to \vec{v} .
3. Given $\vec{u} = (3, 2, -1)$, $\vec{v} = (0, 2, -3)$, and $\vec{w} = (2, 6, 7)$ Compute the vectors
 - a) $\vec{u} \times \vec{v}$
 - b) $\vec{v} \times \vec{w}$
 - c) $\vec{u} \times (\vec{v} \times \vec{w})$
 - d) $(\vec{u} \times \vec{v}) \times \vec{w}$
 - e) $\vec{u} \times (\vec{v} - 2\vec{w})$
4. Use the cross product to find a vector that is orthogonal to both
 - a) $\vec{u} = (-6, 4, 2)$, $\vec{v} = (3, 1, 5)$
 - b) $\vec{u} = (1, 1, -2)$, $\vec{v} = (2, -1, 2)$
 - c) $\vec{u} = (-2, 1, 5)$, $\vec{v} = (3, 0, -3)$
5. Find the area of the parallelogram determined by the given vectors
 - a) $\vec{u} = (1, -1, 2)$ and $\vec{v} = (0, 3, 1)$
 - b) $\vec{u} = (3, -1, 4)$ and $\vec{v} = (6, -2, 8)$
 - c) $\vec{u} = (2, 3, 0)$ and $\vec{v} = (-1, 2, -2)$
6. Find the area of the parallelogram with the given vertices
$$P_1(3, 2), P_2(5, 4), P_3(9, 4), P_4(7, 2)$$
7. Find the area of the triangle with the given vertices:
 - a) $A(2, 0)$ $B(3, 4)$ $C(-1, 2)$
 - b) $A(1, 1)$ $B(2, 2)$ $C(3, -3)$
 - c) $P(2, 6, -1)$ $Q(1, 1, 1)$ $R(4, 6, 2)$
8.
 - a) Find the area of the parallelogram with edges $\vec{v} = (3, 2)$ and $\vec{w} = (1, 4)$
 - b) Find the area of the triangle with sides \vec{v} , \vec{w} , and $\vec{v} + \vec{w}$. Draw it.
 - c) Find the area of the triangle with sides \vec{v} , \vec{w} , and $\vec{v} - \vec{w}$. Draw it.
9. Find the volume of the parallelepiped with sides \vec{u} , \vec{v} , and \vec{w} .
 - a) $\vec{u} = (2, -6, 2)$, $\vec{v} = (0, 4, -2)$, $\vec{w} = (2, 2, -4)$
 - b) $\vec{u} = (3, 1, 2)$, $\vec{v} = (4, 5, 1)$, $\vec{w} = (1, 2, 4)$

10. Compute the scalar triple product $\vec{u} \cdot (\vec{v} \times \vec{w})$

a) $\vec{u} = (-2, 0, 6), \quad \vec{v} = (1, -3, 1), \quad \vec{w} = (-5, -1, 1)$

b) $\vec{u} = (-1, 2, 4), \quad \vec{v} = (3, 4, -2), \quad \vec{w} = (-1, 2, 5)$

c) $\vec{u} = (a, 0, 0), \quad \vec{v} = (0, b, 0), \quad \vec{w} = (0, 0, c)$

d) $\vec{u} = 3\hat{i} - 2\hat{j} - 5\hat{k}, \quad \vec{v} = \hat{i} + 4\hat{j} - 4\hat{k}, \quad \vec{w} = 3\hat{j} + 2\hat{k}$

e) $\vec{u} = (3, -1, 6) \quad \vec{v} = (2, 4, 3) \quad \vec{w} = (5, -1, 2)$

11. Use the cross product to find the sine of the angle between the vectors

$$\vec{u} = (2, 3, -6), \quad \vec{v} = (2, 3, 6)$$

12. Simplify $(\vec{u} + \vec{v}) \times (\vec{u} - \vec{v})$

13. Prove Lagrange's identity: $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$

14. Polar coordinates satisfy $x = r \cos \theta$ and $y = r \sin \theta$. Polar area $J dr d\theta$ includes J :

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}$$

The two columns are orthogonal. Their lengths are _____. Thus $J =$ _____.

15. Prove that $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$ if and only if \vec{u} and \vec{v} are parallel vectors.

16. State the following statements as True or False

a) The cross product of two nonzero vectors \vec{u} and \vec{v} is a nonzero vector if and only if \vec{u} and \vec{v} are not parallel.

b) A normal vector to a plane can be obtained by taking the cross product of two nonzero and noncollinear vectors lying in the plane.

c) The scalar triple product of \vec{u} , \vec{v} , and \vec{w} determines a vector whose length is equal to the volume of the parallelepiped determined by \vec{u} , \vec{v} , and \vec{w} .

d) If \vec{u} and \vec{v} are vectors in 3-space, then $\|\vec{u} \times \vec{v}\|$ is equal to the area of the parallelogram determined by \vec{u} and \vec{v} .

e) For all vectors \vec{u} , \vec{v} , and \vec{w} in R^3 , the vectors $(\vec{u} \times \vec{v}) \times \vec{w}$ and $\vec{u} \times (\vec{v} \times \vec{w})$ are the same.

f) If \vec{u} , \vec{v} , and \vec{w} are vectors in R^3 , where \vec{u} is nonzero and $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$, then $\vec{v} = \vec{w}$