4.3 LU decompositions

L → Lower, U → Upper

$$A\vec{x}=\vec{b}$$

$$LU\vec{x}=\vec{b}$$

$n\times 1 \rightarrow \quad U\vec{x}=\vec{b}$

$$\begin{tikzcd}
\vec{x} \arrow[rr, bend left, "A\vec{x}=\vec{b}"] \arrow[r, "U\vec{x}=\vec{y}"'] & \vec{y} \arrow[r, "L\vec{y}=\vec{b}"'] & \vec{b}
\end{tikzcd}$$

Ex $\quad A=\begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix} \qquad A=L.U$

$$\begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix} \quad R_2 - 3R_1$$

$$L=\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

$$l_{21} = -3 \rightarrow e_{21}$$

The Lower matrix has 1 in the main diagonal (opposite sign)

$$E_{21} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

$$U = EA$$

$$= \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix}} = \underset{L}{\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}}$$

Ex

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{or} \; (\quad)$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad R_2 - \frac{1}{2} R_1$$

ℓ_{21}

$$\boxed{\ell_{21} = -\frac{1}{2}} \qquad \boxed{\ell_{31} = 0}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad R_3 - \frac{2}{3} R_2 \qquad \ell_{32} = -\frac{2}{3}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$$

$$A = LU$$

4.4 $\underbrace{\text{Eigen}}$values & Eigenvectors

$(n \times n)$ square matrix

propa

proper value
characteristic value
latent roots

Defn eigenvalue : λ's of matrix A
the eigenvectors are corresponding to λ's

$$\underset{n\times 1}{\lambda \vec{x}} = \underset{n\times n}{A}\ \underset{n\times 1}{\vec{x}}$$

$A\vec{x} = \lambda\vec{x}$ $\qquad\qquad \lambda \in \mathbb{R}$

$A\vec{x} - \lambda I\vec{x} = 0$ $\qquad\qquad \in \mathbb{C}$

$(A - \lambda I)\vec{x} = 0$

$A - \lambda I \qquad \lambda \quad \vec{x} = \vec{0}$ #

Defn λ is an eigenvalue of A iff

$$\underbrace{\det(A - \lambda I) = 0}_{\text{characteric eqn}}$$

solve $|A - \lambda I| = 0$ for λ's.

Ex

$$A = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$$

$$|A - \lambda I| = \left| \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right|$$

$$= \begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix}$$

$$= -\lambda(3-\lambda) + 2$$

$$= \lambda^2 - 3\lambda + 2$$

characteristic eq: $\lambda^2 - 3\lambda + 2 = 0$

eigenvalues: $\lambda_{1,2} = 1, 2$

Theorem: Upper or lower or diagonal matrix, The Eigenvalues are the entries of main diagonal

Ex

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 3/2 & 0 \\ 5 & -8 & -\frac{1}{4} \end{bmatrix}$$ Lower. T..

eigenvalues: $\lambda_{1,2,3} = \frac{1}{2}, \frac{3}{2}, -\frac{1}{4}$

Theorem:

- ✓ λ is eigenvalue of A
- ✓ $(A - \lambda I) = \vec{0}$
- ✓ nonzero $\vec{x} \ni A\vec{x} = \lambda \vec{x}$
- ✓ $\det(A - \lambda I) = 0$

Eigenvectors

To find eigenvectors

$$(A - \lambda_i I) V_i = 0$$

Ex

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix}$$

$$\begin{matrix} 1 & 2 \\ 2 & 4 \end{matrix}$$

$$= \lambda^2 - 5\lambda = 0$$

eigenvalues : $\boxed{\lambda_{1,2} = 0, 5}$

For $\lambda_1 = 0 \Rightarrow (A - \lambda_1 I) V_1 = 0$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{matrix} x_1 + 2y_1 = 0 \\ \cancel{2x_1 + 4y_1 = 0} \end{matrix}$$

$$x_1 = -2y_1$$

$x_1 = -2y_1$

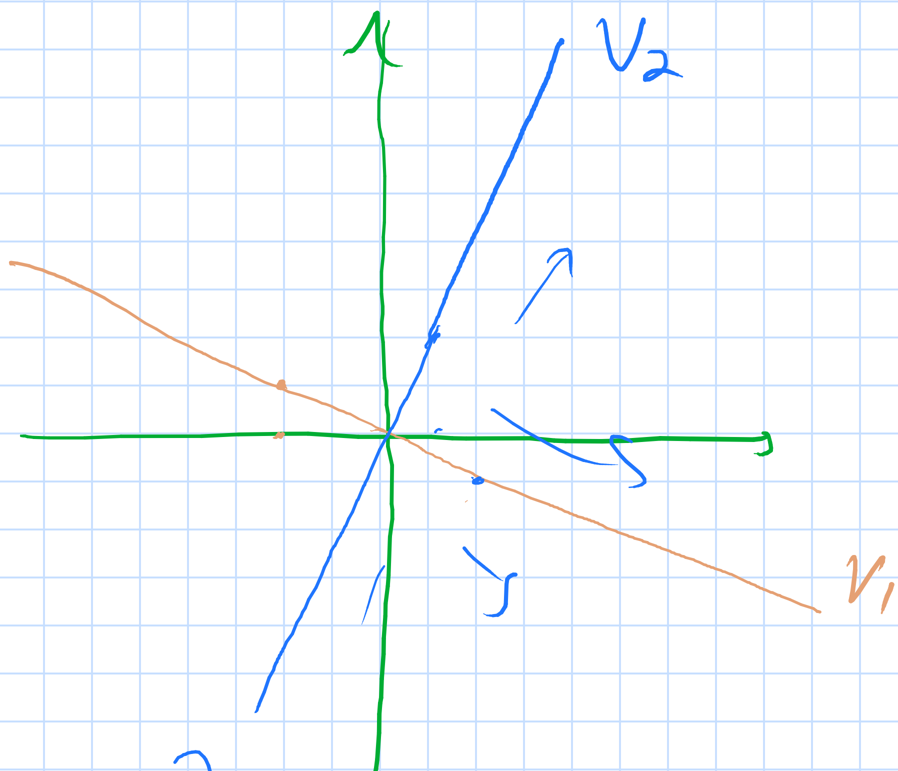
$V_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -2y_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} y_1$

$V_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ $\quad \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

For $\lambda_2 = 5 \Rightarrow (A - \lambda_2 I) V_2 = 0$

$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow 2x_2 = y_2$

$V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



λ_1	λ_2
+	+
−	−
−	+

Ex $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$ A^7

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix}$$

$$= -\lambda^3 + 5\lambda^2 - 6\lambda + 4 - 2\lambda$$

$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

$\lambda_1 = 1$

$\lambda_{2,3} = 2$

$$\begin{array}{c|cccc} 1 & -1 & 5 & -8 & 4 \\ & & -1 & 4 & -4 \\ \hline & -1 & 4 & -4 & 0 \end{array}$$

$-\lambda^2 + 4\lambda - 4 = 0$

$-(\lambda - 2)^2$

$\lambda_1 = 1^7 = 1$

$\lambda_{2,3} = 2^7 = 128$

Ex $A = \begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} -2-\lambda & -1 \\ 5 & 2-\lambda \end{vmatrix}$$

$$= \lambda^2 + 1 = 0$$

$\lambda_{1,2} = \pm i$ eigenvalues:

For $\lambda_1 = i \Rightarrow (A - \lambda_1 I) V_1 = 0$

$$\begin{pmatrix} -2-i & -1 \\ 5 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(2+i)x_1 = -y_1$

$V_1 = \begin{pmatrix} -1 \\ 2+i \end{pmatrix}$, $V_2 = \begin{pmatrix} -1 \\ 2-i \end{pmatrix}$

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