Solution Section 3.2 – Recursive Definitions and Structural Induction

Exercise

Find $f(1)$, $f(2)$, $f(3)$, and $f(4)$ if $f(n)$ is defined recursively by $f(0)=1$ and for $n=0,\ 1,\ 2,\dots$

a) $f(n+1)=f(n)+2$

b) $f(n+1)=3f(n)$

c) $f(n+1)=2^{f(n)}$

d) $f(n+1)=f(n)^2+f(n)+1$

Solution

a)
$$f(1)=f(0)+2=1+2\ \underline{=3}$$
$$f(2)=f(1)+2=3+2\ \underline{=5}$$
$$f(3)=f(2)+2=5+2\ \underline{=7}$$
$$f(4)=f(3)+2=7+2\ \underline{=9}$$

b) $f(n+1)=3f(n)$
$$f(1)=3\cdot f(0)=3(1)\ \underline{=3}$$
$$f(2)=3\cdot f(1)=3(3)\ \underline{=9}$$
$$f(3)=3\cdot f(2)=3(9)\ \underline{=27}$$
$$f(4)=3\cdot f(3)=3(27)\ \underline{=81}$$

c) $f(n+1)=2^{f(n)}$
$$f(1)=2^{f(0)}=2^1\ \underline{=2}$$
$$f(2)=2^{f(1)}=2^2\ \underline{=4}$$
$$f(3)=2^{f(2)}=2^4\ \underline{=16}$$
$$f(4)=2^{f(3)}=2^{16}\ \underline{=65536}$$

d) $f(n+1)=f(n)^2+f(n)+1$
$$f(1)=f(0)^2+f(0)+1=1^2+1+1\ \underline{=3}$$
$$f(2)=f(1)^2+f(1)+1=3^2+3+1\ \underline{=13}$$
$$f(3)=f(2)^2+f(2)+1=13^2+13+1\ \underline{=183}$$
$$f(4)=f(3)^2+f(3)+1=183^2+183+1\ \underline{=33673}$$

Exercise

Find $f(1)$, $f(2)$, $f(3)$, $f(4)$ and $f(5)$ if $f(n)$ is defined recursively by $f(0)=3$ and for $n=0,\ 1,\ 2,\dots$

a) $f(n+1)=-2f(n)$

b) $f(n+1)=3f(n)+7$

c) $f(n+1)=3^{f(n)/3}$

d) $f(n+1)=f(n)^2-2f(n)-2$

Solution

a) $f(n+1)=-2f(n)$

$$f(1)=-2f(0)=-2(3)\ \underline{=-6}$$

$$f(2)=-2f(1)=-2(-6)\ \underline{=12}$$

$$f(3)=-2f(2)=-2(12)\ \underline{=-24}$$

$$f(4)=-2f(3)=-2(-24)\ \underline{=48}$$

$$f(5)=-2f(4)=-2(48)\ \underline{=-96}$$

b)

$$f(1)=3\cdot f(0)+7=3(3)+7\ \underline{=16}$$

$$f(2)=3\cdot f(1)+7=3(16)+7\ \underline{=55}$$

$$f(3)=3\cdot f(2)+7=3(55)+7\ \underline{=172}$$

$$f(4)=3\cdot f(3)+7=3(172)+7\ \underline{=523}$$

$$f(5)=3\cdot f(4)+7=3(523)+7\ \underline{=1576}$$

c)

$$f(1)=3^{f(0)/3}=3^{3/3}=3^1\ \underline{=3}$$

$$f(2)=3^{f(1)/3}=3^{3/3}=3^1\ \underline{=3}$$

$$f(3)=3^{f(2)/3}=3^{3/3}=3^1\ \underline{=3}$$

$$f(4)=3^{f(3)/3}=3^{3/3}=3^1\ \underline{=3}$$

$$f(5)=3^{f(3)/3}=3^{3/3}=3^1\ \underline{=1}$$

d)

$$f(1)=f(0)^2-2f(0)-2=3^2-2(3)-2\ \underline{=1}$$

$$f(2)=f(1)^2-2f(1)-2=1^2-2(1)-2\ \underline{=-3}$$

$$f(3)=f(2)^2-2f(2)-2=(-3)^2-2(-3)-2\ \underline{=13}$$

$$f(4)=f(3)^2-2f(3)-2=(13)^2-2(13)-2\ \underline{=141}$$

$$f(5)=f(4)^2-2f(4)-2=(141)^2-2(141)-2\ \underline{=19,597}$$

Exercise

Find $f(2)$, $f(3)$, $f(4)$ and $f(5)$ if $f(n)$ is defined recursively by $f(0)=f(1)=1$ and for $n=1,\ 2,\ldots$

a) $f(n+1)=f(n)-f(n-1)$

b) $f(n+1)=f(n)f(n-1)$

c) $f(n+1)=f(n)^2+f(n-1)^3$

d) $f(n+1)=f(n)/f(n-1)$

Solution

a) $f(2)=f(1)-f(0)=1-1\ \underline{=0}|$

$f(3)=f(2)-f(1)=0-1\ \underline{=-1}|$

$f(4)=f(3)-f(2)=-1-0\ \underline{=-1}|$

$f(5)=f(4)-f(3)=-1-(-1)\ \underline{=0}|$

b) $f(2)=f(1)f(0)=(1)(1)\ \underline{=1}|$

$f(3)=f(2)f(1)=(1)(1)\ \underline{=1}|$

$f(4)=f(3)f(2)=(1)(1)\ \underline{=1}|$

$f(5)=f(4)f(3)=(1)(1)\ \underline{=1}|$

c) $f(2)=f(1)^2+f(0)^3=1^2+1^3\ \underline{=2}|$

$f(3)=f(2)^2+f(1)^3=2^2+1^3\ \underline{=5}|$

$f(4)=f(3)^2+f(2)^3=5^2+2^3\ \underline{=33}|$

$f(5)=f(4)^2+f(3)^3=33^2+5^3\ \underline{=1214}|$

d) $f(2)=\dfrac{f(1)}{f(0)}=\dfrac{1}{1}\ \underline{=1}|$

$f(3)=\dfrac{f(2)}{f(1)}=\dfrac{1}{1}\ \underline{=1}|$

$f(4)=\dfrac{f(3)}{f(2)}=\dfrac{1}{1}\ \underline{=1}|$

$f(5)=\dfrac{f(4)}{f(3)}=\dfrac{1}{1}\ \underline{=1}|$

Exercise

Determine whether each of these proposed definitions is a valid recursive definition of a function f from the set of nonnegative integers to the set of integers. If f is well defined, fund a formula for $f(n)$ when n is nonnegative integer and prove that your formula is valid.

a) $f(0)=0, \quad f(n)=2f(n-2) \quad for \quad n\ge 1$

b) $f(0)=1, \quad f(n)=-f(n-1) \quad for \quad n\ge 1$

c) $f(0)=1, \quad f(n)=f(n-1)-1 \quad for \quad n\ge 1$

d) $f(0)=2, \quad f(1)=3, \quad f(n)=f(n-1)-1 \quad for \quad n\ge 2$

e) $f(0)=1, \quad f(1)=2, \quad f(n)=2f(n-2) \quad for \quad n\ge 2$

f) $f(0)=1, \quad f(1)=0, \quad f(2)=2, \quad f(n)=2f(n-3) \quad for \quad n\ge 3$

g) $f(0)=0, \quad f(1)=1, \quad f(n)=2f(n+1) \quad for \quad n\ge 2$

h) $f(0)=0, \quad f(1)=1, \quad f(n)=2f(n-1) \quad for \quad n\ge 2$

i) $f(0)=2, \quad f(n)=f(n-1)$ if n is odd and $n\ge 1$ and $f(n)=2f(n-2)$ if n is even and $n\ge 2$

j) $f(0)=1, \quad f(n)=3f(n-1)$ if n is odd and $n\ge 1$ and $f(n)=9f(n-2)$ if n is even and $n\ge 2$

Solution

a) This is invalid, since $f(1)=2f(1-2)=\cancel{2f(-1)} \quad for \quad n\ge 1$, $f(-1)$ is not defined.

b) $f(1)=-f(0)=-1$, this is a valid, since $n=0$ is provided and each subsequent value is determined by the previous one. $f(n)=(-1)^n$, this is true for $n=0 \quad since \quad (-1)^0=1$.

Assume it is true for $n=k$, then

$f(k+1)=-f(k+1-1)=-f(k)=-(-1)^k=(-1)^{k+1}$

c) $f(1)=f(0)-1=1-1=0$, this is a valid.

$f(2)=f(1)-1=0-1=-1$

The sequence: $1,\ 0,\ -1,\ -2,\ -3,\ \dots \Rightarrow f(n)=1-n$

By induction:

The basis step: $f(0)=1-0=1$

If $\quad f(k)=1-k$

Then $\quad f(k+1)=f(k)-1=1-k-1=1-(k+1)$

d) $f(2)=f(1)-1=3-1=2$

$f(3)=f(2)-1=2-1=1$

Given: $f(0)=2, \quad f(1)=3$

Then the sequence: $2, 3, 2, 1, 0, \dots \Rightarrow f(n)=4-n$

By induction: Basis step: $f(0)=2$ and $f(1)=4-1=3$

If $\quad f(k)=4-k$

Then $\quad f(k+1)=f(k)-1=4-k-1=4-(k+1)$

e) $f(2)=2f(0)=2 \qquad f(1)=2$

$f(3)=2f(1)=2(2)=4 \qquad f(4)=2f(2)=2(2)=4$

$f(5)=2f(3)=2(4)=8 \qquad f(6)=2f(4)=2(4)=8$

Then the sequence: 1, 2, 2, 4, 4, 8, 8, … $\Rightarrow f(n)=2^{(n+1)/2}$

By induction: Basis step: $f(0)=2^{(0+1)/2}=1$ *and* $f(1)=2^{(1+1)/2}=2$ and

If $\quad f(k)=2^{(k+1)/2}$

Then

$f(k+1)=2f(k-1)=2\cdot 2^{(k-1+1)/2}=2\cdot 2^{k/2}=2^{(k/2)+1}=2^{(k+2)/2}=2^{((k+1)+1)/2}$

f) $f(3)=2f(0)=2(1)=2 \qquad f(4)=2f(1)=2(0)=0 \qquad f(5)=2f(2)=2(2)=4$

$f(6)=2f(3)=2(2)=4 \qquad f(7)=2f(4)=2(0)=0 \qquad f(8)=2f(5)=2(4)=8$

This is valid, since the values n = 0, 1, 2 are given. The sequence: 1, 0, 2, 2, 0, 4, 4, 0, 8, …

We conjecture the formula	***Prove***
$f(n)=2^{n/3}$ when $n\equiv 0(\boldsymbol{mod}\,3)$	$f(0)=2^{0/3}=1$
$f(n)=0$ when $n\equiv 1(\boldsymbol{mod}\,3)$	$f(1)=0$
$f(n)=2^{(n+1)/3}$ when $n\equiv 2(\boldsymbol{mod}\,3)$	$f(2)=2^{(2+1)/3}=2^1=2$

Assume the inductive hypothesis that the formula is valid for smaller inputs. Then

For $n\equiv 0(\boldsymbol{mod}\,3)$ we have $f(n)=2f(n)=2\cdot 2^{(n-3)/3}=2\cdot 2^{n/3}\cdot 2^{-1}=2^{n/3}$ as desired

For $n\equiv 1(\boldsymbol{mod}\,3)$ we have $f(n)=2f(n-3)=2\cdot 0=0$ as desired

For $n\equiv 2(\boldsymbol{mod}\,3)$ we have $f(n)=2f(n-3)=2\cdot 2^{(n-3+1)/3}=2\cdot 2^{(n+1)/3}\cdot 2^{-1}=2^{(n+1)/3}$ as desired

g) $f(2)=2f(3)$ This is not valid, since $f(3)$ has not been defined

h) $f(2)=2\cdot f(1)=2(1)=2 \qquad f(3)=2f(2)=2(2)=4$

This is ***invalid***, because the value at $n=1$ is defined in 2 conflicting ways, first as $f(1)=1$ and then as $f(1)=2f(1-1)=2f(0)=2(0)=0$

i) $f(1)=f(0)=2 \qquad f(2)=2f(0)=2(2)=4$

$f(3)=f(2)=4 \qquad f(4)=2f(2)=2(4)=8$

This is ***invalid***, since we have a conflict for odd $n\ge 3$.

On one hand $f(3)=f(2)$, but the other hand $f(3)=2f(1)$.

However, $f(1)=f(0)=2$ and $f(2)=2f(0)=4$, so these apparently conflicting rules tell us that $f(3)=2\cdot 2=4$ on the other hand. We got the same answer either way.

The sequence: 2, 2, 4, 4, 8, 8, …

j) $f(1)=3f(0)=3(1)=3$ $\quad f(2)=9f(0)=9(1)=9$

$f(3)=3f(2)=3(9)=27$ $\quad f(4)=9f(2)=9(9)=81$

The sequence: 1, 3, 9, 27, 81, …

This is a valid, since we conjecture the formula $f(n)=3^n$

By induction: Basis step: $f(0)=3^0=1$

If $\quad f(k)=3^k$

Then $f(k+1)=3f(k)=3\cdot 3^k=\underline{3^{k+1}|}$

Exercise

Give a recursive definition of the sequence $\{a_n\}$, $n=1,\ 2,\ 3,\dots$ if

a) $a_n=6n$ *b)* $a_n=2n+1$ *c)* $a_n=10^n$ *d)* $a_n=5$

e) $a_n=4n-2$ *f)* $a_n=1+(-1)^n$ *g)* $a_n=n(n+1)$ *h)* $a_n=n^2$

Solution

a) $a_1=6,\quad a_2=12=6+6,\quad a_3=18=12+6,\ \dots$

$\rightarrow \underline{a_{n+1}=a_n+6|}$ *with* $a_1=6$ *for all* $n\ge 1$

b) $a_1=3,\quad a_2=5=3+2,\quad a_3=7=5+2,\ \dots \rightarrow \underline{a_{n+1}=a_n+2|}$ *with* $a_1=3$ *for all* $n\ge 1$

c) $a_1=10,\quad a_2=10^2=10\cdot 10,\quad a_3=10^3=10\cdot 10^2,\ \dots$

$\underline{a_{n+1}=10a_n|}$ *with* $a_1=10$ *for all* $n\ge 1$

d) $a_1=5,\quad a_2=5,\quad a_3=5,\ \dots \rightarrow \underline{a_{n+1}=a_1|}$ *with* $a_1=5,$ *for all* $n\ge 1$

e) $a_1=2,\quad a_2=6=2+4,\quad a_3=10=6+4,\ \dots \rightarrow \underline{a_{n+1}=a_n+4|}$ *with* $a_1=2,$ *for all* $n\ge 1$

f) $a_1=1-1=0,\quad a_2=1+1=2,\quad a_3=1-1=0,\ \dots$, the sequence alternate: 0,2, 0, 2, …

$\underline{a_n=a_{n-2}|}$ *with* $a_1=0,\ a_2=2,$ *for all* $n\ge 3$

g) $a_1=1(2)=2,\quad a_2=2(3)=6,\quad a_3=12,\ \dots$, the sequence alternate: 2,6, 12, 20, 30, …

The difference between successive terms are 4, 6, 8, 10, ….

$\boxed{a_n = a_{n-1} + 2n}$ *with* $a_1 = 2,$ *for all* $n \ge 2$

h) $a_1 = 1^2 = 1,\quad a_2 = 2^2 = 4,\quad a_3 = 3^2 = 9,\ \ldots$, the sequence alternate: 1, 4, 9, 16, 25, …

The difference between successive terms are 3, 5, 7, 9, ….

$\boxed{a_n = a_{n-1} + 2n - 1}$ *with* $a_1 = 1,$ *for all* $n \ge 2$

Exercise

Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ when *n* is a positive integer and f_n is the *n*th Fibonacci number.

Solution

For $n = 1$: $f_1^2 = f_1 f_2 = 1 \cdot 1 = 1$ is true since both values are 1

Assume the inductive hypothesis. Then

$$\begin{aligned} f_1^2 + f_2^2 + \cdots + f_n^2 + f_{n+1}^2 &= f_n f_{n+1} + f_{n+1}^2 \\ &= f_{n+1}\left(f_n + f_{n+1}\right) \\ &= f_{n+1} f_{n+2} \end{aligned}$$

Exercise

Prove that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$ when *n* is a positive integer and f_n is the *n*th Fibonacci number.

Solution

Using the principle of mathematical induction

For $n = 1$: $f_1 = f_2$ is true since both values are 1

Let assume that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$

We need to prove that $f_1 + f_3 + \cdots + f_{2n-1} + f_{2n+1} = f_{2(n+1)}$

$$\begin{aligned} f_1 + f_3 + \cdots + f_{2n-1} + f_{2n+1} &= f_{2n} + f_{2n+1} \\ &= f_{2n+2} \end{aligned}$$ **(by the *definition* of the Fibonacci numbers)**

Exercise

Give a recursive definition of

a) The set of odd positive integers

b) The set of positive integers powers of 3

c) The set of polynomial with integer coefficients

d) The set of even integers

e) The set of positive integers congruent to 2 modulo 3.

f) The set of positive integers not divisible by 5

Solution

a) Off integers are obtained from other odd integers by adding 2. Thus we can define this set *S* as follows $1 \in S$; and if $n \in S,\ then\ \ n+2 \in S$.

b) Powers of 3 are obtained from other powers of 3 by multiplying by 3. Thus we can define this set *S* as follows $3 \in S$; and if $n \in S,\ then\ \ 3n \in S$.

c) There are several ways to do this. One that is suggested by Horner's method is as follows. We assume that the variable for these polynomials is the letter *x*. All integers are in *S*; if $p(x) \in S$ and *n* is any integer, then $xp(x)+n$ is in *S*.

Another method constructs the polynomials term by term. Its base case is to let 0 be in *S*; and its inductive step is to say that if $p(x) \in S$, *c* is an integer, and *n* is a nonnegative integer, then $p(x)+cx^n$ is in *S*.

d) Off integers are obtained from other even integers by adding 2. Thus we can define this set *S* as follows $2 \in S$; and if $n \in S,\ then\ \ n-2 \in S\ \ and\ \ n+2 \in S$.

e) The smallest positive integer congruent to 2 modulo 3 is 2, so $2 \in S$. All the others can be obtained by adding multiples of 3,so the inductive step is that $n \in S,\ then\ \ n+3 \in S$

f) The positive integers no divisible by 5 are the ones congruent to 1, 2, 3, or 4 *modulo* 5. Thus we can define this set *S* as follows $1 \in S,\ 2 \in S,\ 3 \in S,\ and\ 4 \in S$; and if $n \in S,\ then\ \ n+5 \in S$

Exercise

Let *S* be the subset of the set of ordered pairs of integers defined recursively by

Basis step: (0, 0) ∈ *S*.

Recursive step: If (*a*, *b*) ∈ *S*, then $(a+2,\ b+3) \in S\ \ and\ \ (a+3,\ b+2) \in S$

a) List the elements of *S* produced by the first five applications of the recursive definition.

b) Use strong induction on the number of applications of the recursive step of the definition to show that $5 \mid a+b$ when (*a*, *b*) ∈ *S*.

c) Use structural induction to show that $5 \mid a+b$ when (*a*, *b*) ∈ *S*.

Solution

a) Apply each recursive step rules to the only element given in the basis step, we see that (2, 3) and (3, 2) are in *S*. If we apply the recursive step to these we add (4, 6), (5, 5) and (6, 4). The next round gives us (6, 9), (7, 8), and (9, 6). Add (8, 12), (9, 11), (10, 10), (11, 9), and (12, 8); and a fifth set of applications adds (10, 15), (11, 4), (12, 13), (13, 12), (14, 1), and (15, 10).

b) Let $P(n)$ be the statement that $5 | a+b$ when $(a, b) \in S$ is obtained by n applications to the recursive step.

For $n=0$, $P(0)$ is true, since the only element of S obtained with no applications of the recursive step is (0, 0), and $5 | 0+0$ $\checkmark$

Assume the inductive hypothesis that $5 | a+b$ whenever $(a, b) \in S$ is obtained by k or fewer applications of the recursive step, and consider an element obtained with $k+1$ applications of the recursive step. Since the final application of the recursive step to an element (a, b) must applied to an element, that $5 | a+b$.

We need to check that this inequality implies $5 | a+2+b+3$ and $5 | a+3+b+2$.

This is clear, since each is equivalent to $5 | a+b+5$ and 5 divides both $a+b$ and 5.

c) This holds for the basis step, since $5 | 0+0$

If this holds for (a, b), then it also holds for the elements obtained from (a, b) in the recursive step by the same argument as in part (*b*).

Exercise

Let *S* be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(0, 0) \in S$.

Recursive step: If $(a, b) \in S$, then $(a,\ b+1) \in S$, $(a+1,\ b+1) \in S$ *and* $(a+2,\ b+1) \in S$

a) List the elements of *S* produced by the first five applications of the recursive definition.

b) Use strong induction on the number of applications of the recursive step of the definition to show that $a \le 2b$ whenever $(a, b) \in S$.

c) Use structural induction to show that $a \le 2b$ whenever $(a, b) \in S$.

Solution

a) Apply each recursive step rules to the only element given in the basis step, we see that (0, 1), (1, 1) and (2, 1) are in *S*.

2nd step: (0, 2), (1, 2), (2, 2), (3, 2) and (4, 2).

3rd step: (0,3), (1, 3), (2, 3), (3, 3), (4, 3), (5, 3) and (6, 3).

4th step: (0, 4), (1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4), (7, 4) and (8,4)

5th step: (0, 5), (1, 5), (2, 5), (3, 5), (4, 5), (5, 5), (6, 5), (7, 5), (8, 5), (9, 5), and (10, 5)

b) Let $P(n)$ be the statement that $a \le 2b$ whenever $(a, b) \in S$ is obtained with no applications of the recursive step.

For the basis step, the only element of *S* obtained with no applications of the recursive step is $(0,\ 0)$, then $0 \le 2 \cdot 0$ is true. Therefore $P(0)$ is true.

Assume that $a \le 2b$ whenever $(a, b) \in S$ is obtained by k or fewer applications of the recursive step. Consider an element obtained with $k + 1$ applications of the recursive step.

We know that $a \le 2b$, we need to check this inequality implies $a \le 2(b+1)$, $a+1 \le 2(b+1)$, and $a+2 \le 2(b+1)$.

Thus is clear that $0 \le 2,\ 1 \le 2\ \ and\ \ 2 \le 2$, respectively, to $a \le 2b$ to obtain these inequalities.

c) This holds for the basis step, since $0 \le 0$.

If this holds for $(a,\ b)$, then it also holds for the elements obtained from $(a,\ b)$ in the recursive step, since adding $0 \le 2,\ 1 \le 2\ \ and\ \ 2 \le 2$, respectively, to $a \le 2b$ yiekds $a \le 2(b+1)$, $a+1 \le 2(b+1)$, and $a+2 \le 2(b+1)$.