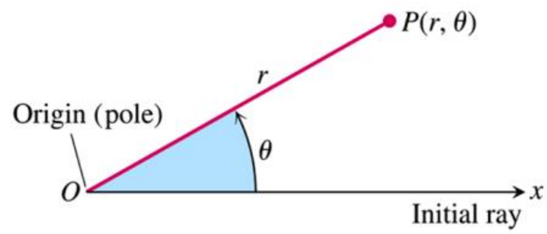
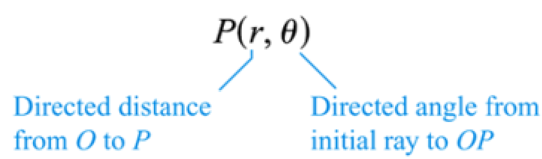
Section 4.3 – Polar Coordinates and Graphs

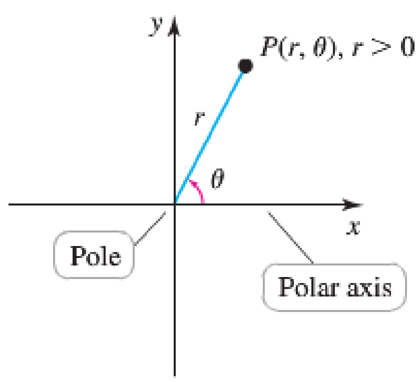
Definition of Polar Coordinates

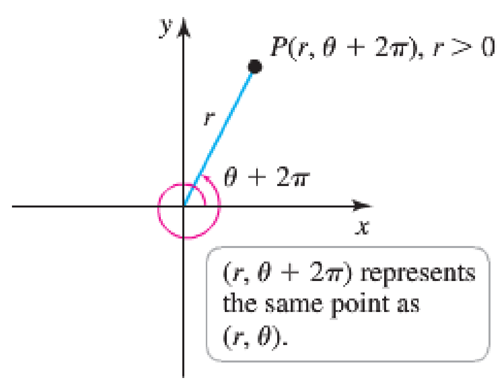
To define polar coordinates, let an ***origin*** O (called the ***pole***) and an ***initial ray*** from O. Then each point P can be located by assigning to it a ***polar coordinate pair*** $(r,\ \theta)$ in which r gives the directed from O to P and θ gives the directed angle from the initial ray to yay OP.

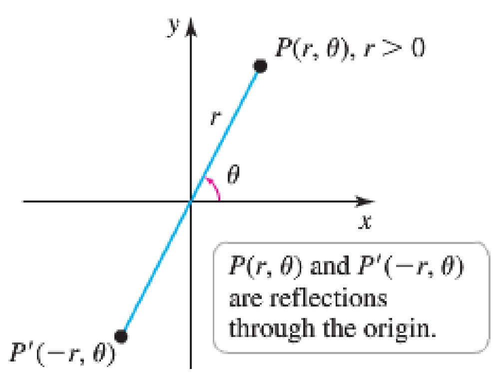


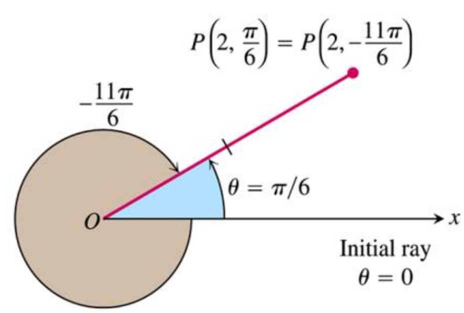
Polar Coordinates





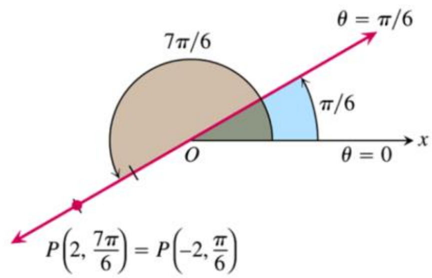






Example

Find all the polar coordinates of the point $P\left(2,\ \frac{\pi}{6}\right)$



Solution

For $r=2 \quad \Rightarrow \quad \theta=\frac{\pi}{6},\ \frac{\pi}{6}\pm 2\pi,\ \frac{\pi}{6}\pm 4\pi,\ \ldots$

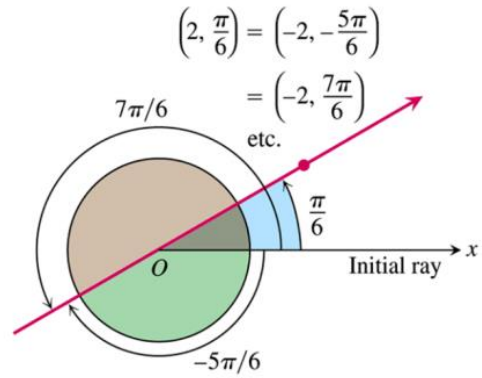
For $r=-2 \quad \Rightarrow \quad \theta=-\frac{5\pi}{6},\ -\frac{5\pi}{6}\pm 2\pi,\ -\frac{5\pi}{6}\pm 4\pi,\ \ldots$

The corresponding coordinate pairs of P are

$$\left(2,\ \frac{\pi}{6}+2n\pi\right), \quad n=0,\ \pm 1,\ \pm 2,\ \ldots$$

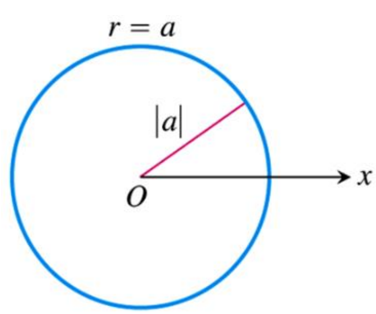
And

$$\left(-2,\ -\frac{5\pi}{6}+2n\pi\right), \quad n=0,\ \pm 1,\ \pm 2,\ \ldots$$



Polar Equations and Graphs

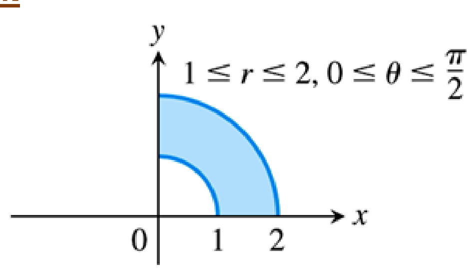
Equation	***Graph***
$r=a$	Circle of radius $\lvert a\rvert$ centered at O
$\theta=\theta_0$	Line through O making an angle θ_0 with the initial ray



Example

Graph the polar coordinate $1\le r\le 2 \quad \textit{and} \quad 0\le\theta\le\frac{\pi}{2}$

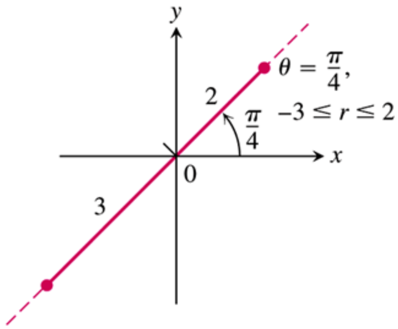
Solution



Example

Graph the polar coordinate $-3 \le r \le 2$ *and* $\theta = \frac{\pi}{4}$

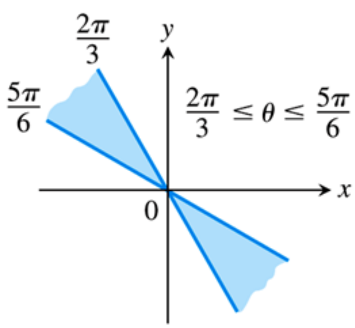
Solution



Example

Graph the polar coordinate $\frac{2\pi}{3} \le \theta \le \frac{5\pi}{6}$ $\quad$ (*no restriction on* r)

Solution

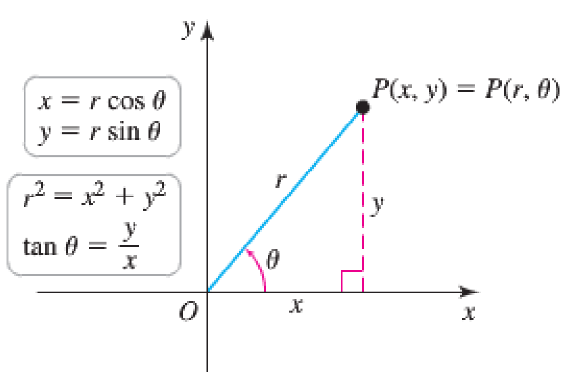


Relating Polar and Cartesian Coordinates

When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial polar ray as the positive *x*-axis. The ray $\theta = \frac{\pi}{2}$, $r > 0$ becomes the positive *y*-axis. The two coordinate systems are then related by the following equations

Equations Relating Polar and Cartesian Coordinates

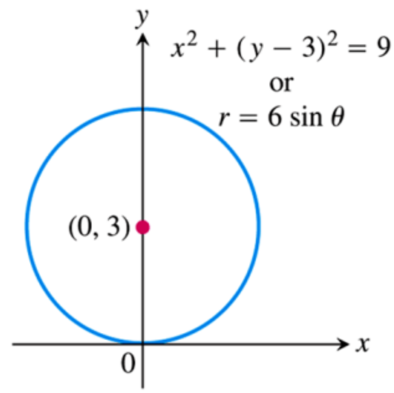
$$\begin{cases} x = r\cos\theta, & y = r\sin\theta \\ r^2 = x^2 + y^2, & \tan\theta = \dfrac{y}{x} \end{cases}$$



Polar equation	*Cartesian equation*
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

Example

Find a polar equation for the circle $x^2 + (y-3)^2 = 9$



Solution

$$x^2 + (y-3)^2 = 9$$

$$x^2 + y^2 - 6y + 9 = 9$$

$$x^2 + y^2 - 6y = 0 \qquad\qquad x^2 + y^2 = r^2$$

$$r^2 - 6r\sin\theta = 0$$

$$r(r - 6\sin\theta) = 0 \quad \Rightarrow \quad \boxed{r = 0} \quad \boxed{r = 6\sin\theta}$$

Example

Replace the polar equation by equivalent Cartesian equation and identify the graph: $r\cos\theta = -4$

Solution

$$r\cos\theta = -4 \quad \Rightarrow \quad x = -4$$

The graph: Vertical line through $x = -4$

Example

Replace the polar equation by equivalent Cartesian equation and identify the graph: $r^2 = 4r\cos\theta$

Solution

$$r^2 = 4r\cos\theta$$

$$x^2 + y^2 = 4x$$

$$x^2 - 4x + y^2 = 0$$

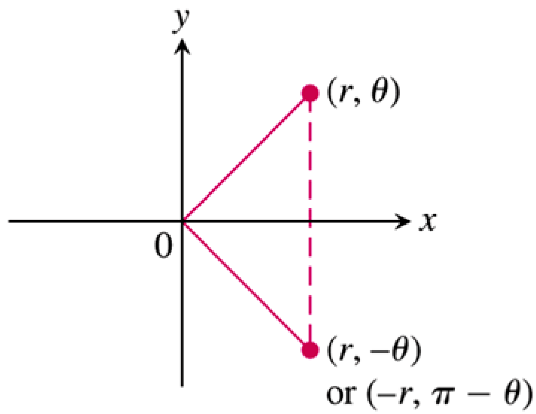
$$x^2 - 4x + 4 + y^2 = 4$$

$$(x-2)^2 + y^2 = 4$$

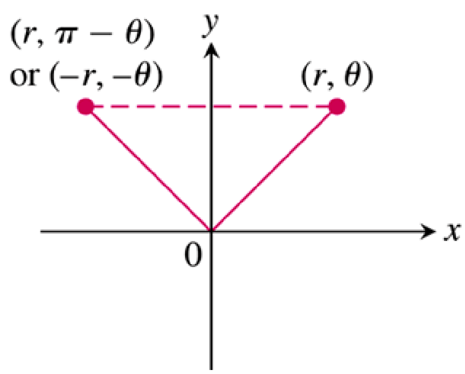
The ***graph***: Circle with center (2, 0) and radius 2.

Symmetry Test for Polar Graphs

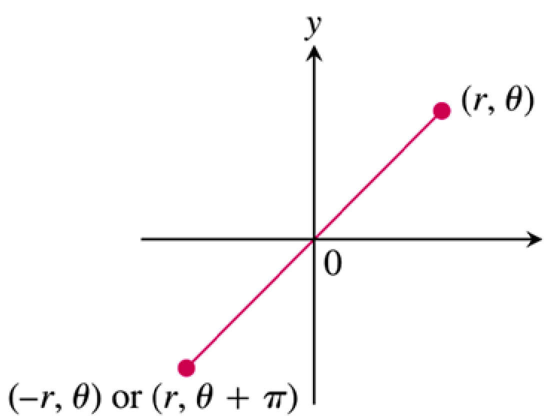
1. ***Symmetry about the x-axis***: If the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph.



2. ***Symmetry about the y-axis***: If the point (r, θ) lies on the graph, then the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph.



3. ***Symmetry about the origin***: If the point (r, θ) lies on the graph, then the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph.



Example

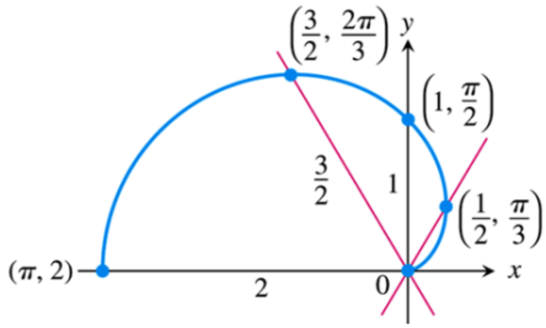
Graph the curve $r = 1 - \cos\theta$

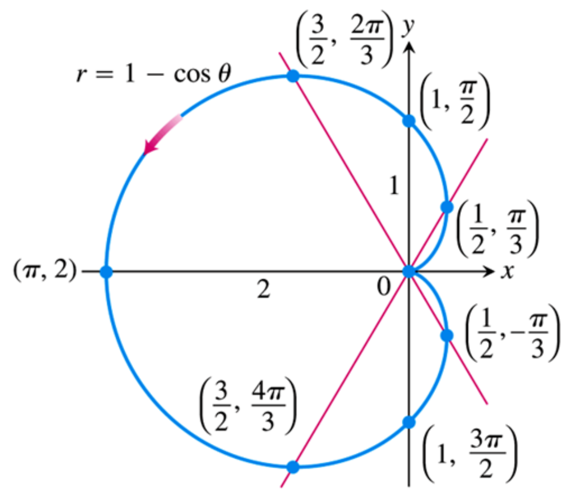
Solution

The curve is symmetric about the *x*-axis:

$1 - \cos(-\theta) = 1 - \cos\theta = r$

θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2





Example

Graph the curve $r^2 = 4\cos\theta$

Solution

The curve is symmetric about the *x*-axis:

$r^2 = 4\cos\theta$

$r^2 = 4\cos(-\theta)$

$(r, -\theta)$

The curve is symmetric about the *origin*:

$r^2 = 4\cos\theta$

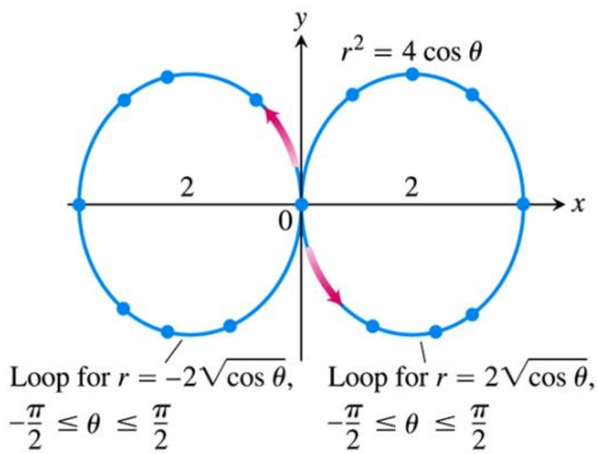
$(-r)^2 = 4\cos\theta$

$(-r, \theta)$

Therefore, the curve is also symmetric about the *y*-axis.

$r^2 = 4\cos\theta \quad \Rightarrow \quad r = \pm 2\sqrt{\cos\theta}$

θ	$r = \pm 2\sqrt{\cos\theta}$
0	± 2
$\pm\frac{\pi}{6}$	$\approx \pm 1.9$
$\pm\frac{\pi}{4}$	$\approx \pm 1.7$
$\pm\frac{\pi}{3}$	$\approx \pm 1.4$
$\pm\frac{\pi}{2}$	0



A Technique for Graphing

One way to graph a polar equation $r = f(\theta)$ is to make a table of (r, θ) values, plot the corresponding points, and connect them in order of increasing.

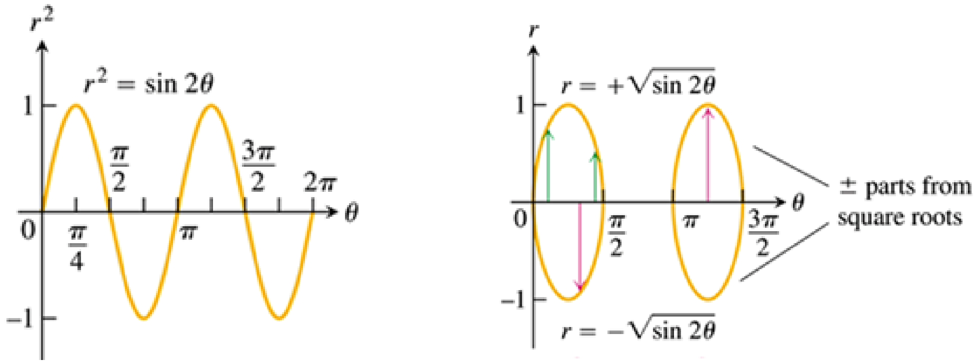
Another method of graphing more reliable is

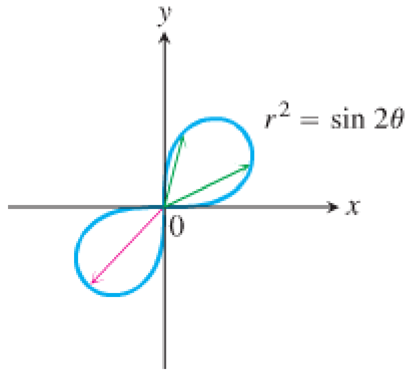
1. First graph $r = f(\theta)$ in the *Cartesian* $r\theta-$*plane*,
2. Then use the *Cartesian* graph as a table and guide to sketch the *polar coordinate* graph.

Example

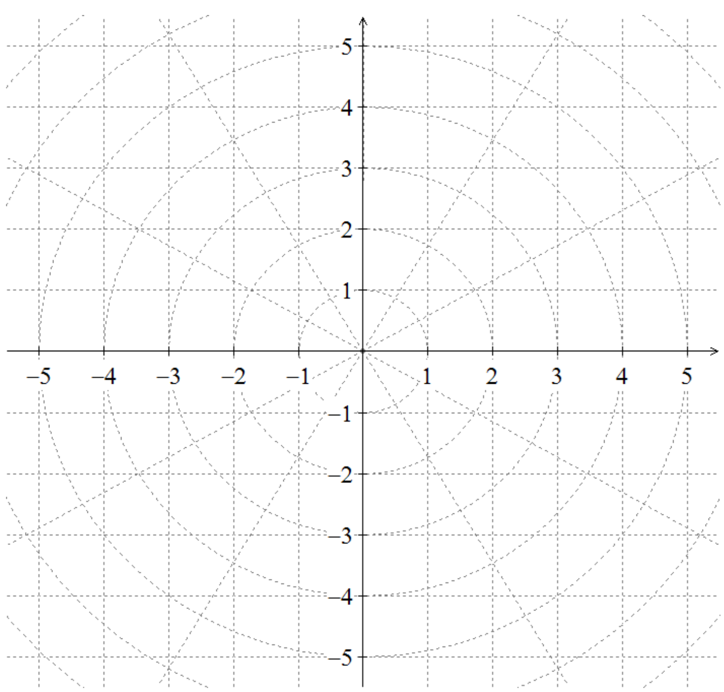
Graph the *lemniscate* curve $r^2 = \sin 2\theta$

Solution





θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{7\pi}{6}$
$r = \pm\sqrt{\sin 2\theta}$	0	$\pm\sqrt{\frac{\sqrt{3}}{2}} \approx \pm.93$	± 1	$\pm\sqrt{\frac{\sqrt{3}}{2}} \approx \pm.93$	0	0	$\pm\sqrt{\frac{\sqrt{3}}{2}} \approx \pm.93$



Exercises Section 4.3 – Polar Coordinates

1. Find the Cartesian coordinates of the following points (given in polar coordinates)

$a)\ \left(\sqrt{2},\ \frac{\pi}{4}\right)\quad b)\ (1,\ 0)\quad c)\ \left(0,\ \frac{\pi}{2}\right)\quad d)\ \left(-\sqrt{2},\ \frac{\pi}{4}\right)$

2. Find the polar coordinates, $0 \le \theta < 2\pi$ *and* $r \ge 0$, of the following points given in Cartesian coordinates

$a)\ (1,\ 1)\quad b)\ (-3,\ 0)\quad c)\ \left(\sqrt{3},\ -1\right)\quad d)\ (-3,\ 4)$

3. Find the polar coordinates, $-\pi \le \theta < \pi$ *and* $r \ge 0$, of the following points given in Cartesian coordinates

$a)\ (-2,\ -2)\quad b)\ (0,\ 3)\quad c)\ \left(-\sqrt{3},\ 1\right)\quad d)\ (5,\ -12)$

Graph

4. $1 \le r \le 2$

5. $0 \le \theta \le \frac{\pi}{6},\quad r \ge 0$

6. $\theta = \frac{\pi}{2},\quad r \le 0$

7. $-\frac{\pi}{4} \le \theta \le \frac{3\pi}{4},\quad 0 \le r \le 1$

8. $0 \le \theta \le \frac{\pi}{2},\quad 1 \le |r| \le 2$

Replace the polar equation with equivalent Cartesian equation and identify the graph

9. $r\cos\theta = 2$

10. $r\sin\theta = -1$

11. $r = -3\sec\theta$

12. $r\cos\theta + r\sin\theta = 1$

13. $r^2 = 4r\sin\theta$

14. $r = \dfrac{5}{\sin\theta - 2\cos\theta}$

15. $r = 4\tan\theta\sec\theta$

16. $r\sin\theta = \ln r + \ln\cos\theta$

17. $\cos^2\theta = \sin^2\theta$

18. $r = 2\cos\theta + 2\sin\theta$

19. $r\sin\left(\frac{2\pi}{3} - \theta\right) = 5$

20. $r = \dfrac{4}{2\cos\theta - \sin\theta}$

Replace the Cartesian equation with equivalent polar equation

21. $x = y$

22. $x^2 - y^2 = 1$

23. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

24. $xy = 1$

25. $x^2 + xy + y^2 = 1$

26. $x^2 + (y-2)^2 = 4$

27. $(x+2)^2 + (y-5)^2 = 16$

28. ***a)*** Show that every vertical line in the *xy*–plane has a polar equation of the form $r = a\sec\theta$
b) Find the analogous polar equation for horizontal lines in the *xy*–plane.

Identify the symmetries of the curves. Then sketch the curves.

29. $r = 2 - 2\cos\theta$

31. $r = 2 + \sin\theta$

33. $r^2 = -\sin\theta$

30. $r = 1 + \sin\theta$ **32.** $r^2 = \sin\theta$ **34.** $r^2 = -\cos\theta$

Graph the lemniscate. What symmetries do these curves have?

35. $r^2 = 4\cos 2\theta$ **36.** $r^2 = 4\sin 2\theta$ **37.** $r^2 = -\cos 2\theta$

Graph the limaçons is Old French for "snail". Equations for limaçons have the form $r = a \pm b\cos\theta$ *or* $r = a \pm b\sin\theta$

38. $r = \frac{1}{2} + \cos\theta$ **40.** $r = 1 - \cos\theta$ **42.** $r = 2 + \cos\theta$

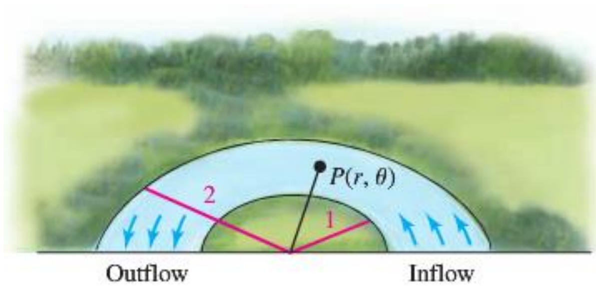
39. $r = \frac{1}{2} + \sin\theta$ **41.** $r = \frac{3}{2} - \sin\theta$

Graph the equation

43. $r = 1 - 2\sin 3\theta$ **44.** $r = \sin^2 \frac{\theta}{2}$ **45.** $r = 1 - \sin\theta$ **46.** $r^2 = 4\sin\theta$

47. Graph the ***nephroid*** of ***Freeth*** equation $r = 1 + 2\sin\frac{\theta}{2}$

48. Water flows in a shallow semicircular channel with inner and outer radii of 1 *m* and 2 *m*. At a point $P(r,\ \theta)$ in the channel, the flow is in the tangential direction (counterclockwise along circles), and it depends only on *r*, the distance from the center of the semicircles.



a) Express the region formed by the channel as a set in polar coordinates.
b) Express the inflow and outflow regions of the channel as sets in polar coordinates.
c) Suppose the tangential velocity of the water in *m/s* is given by $v(r) = 10r$, for $1 \le r \le 2$. Is the velocity greater at $\left(1.5,\ \frac{\pi}{4}\right)$ or $\left(1.2,\ \frac{3\pi}{4}\right)$? Explain.
d) Suppose the tangential velocity of the water is given by $v(r) = \frac{20}{r}$, for . Is the velocity greater $\left(1.8,\ \frac{\pi}{6}\right)$ or $\left(1.3,\ \frac{2\pi}{3}\right)$? Explain.
e) The total amount of water that flows through the channel (across a cross section of the channel $\theta = \theta_0$) is proportional to $\int_1^2 v(r)\,dr$. Is the total flow through the channel greater for the flow in part (*c*) or (*d*)?

49. A simplified model assumes that the orbits of Earth and Mars are circular with radii of 2 and 3, respectively, and that Earth completes one orbit in one year while Mars takes two years. When $t = 0$

. Earth is at $(2,\ 0)$ and Mars is at $(3,\ 0)$; both orbit the Sum (at $(0,\ 0)$) in the counterclockwise direction.

The position of Mars relative to Earth is given by the parametric equations

$$x = (3 - 4\cos \pi t)\cos \pi t + 2, \quad y = (3 - 4\cos \pi t)\sin \pi t$$

a) Graph the parametric equations, for $0 \le t \le 2$

b) Letting $r = 3 - 4\cos \pi t$, explain why the path of Mars relative to Earth is a limaçon.