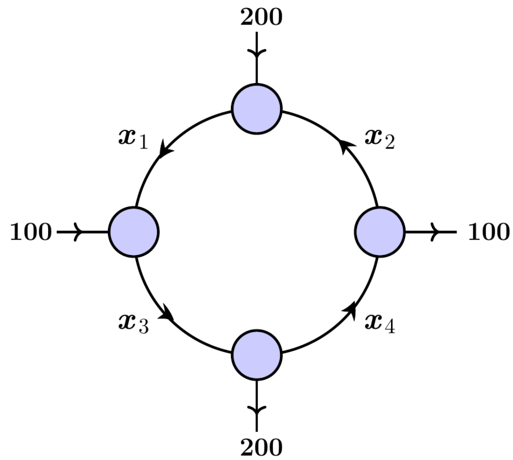
Solution Section 1.8 – Applications

Exercise

The flow of traffic, in vehicles per hour, through a network of streets as is shown below



a) Solve this system for $x_i, \quad i = 1, 2, 3, 4.$

b) Find the traffic flow when $x_4 = 0$.

c) Find the traffic flow when $x_4 = 100$.

d) Find the traffic flow when $x_1 = 2x_2$.

Solution

a) $$\begin{cases} x_1 + 100 = x_3 \\ x_2 + 200 = x_1 \\ x_2 + 100 = x_4 \\ x_4 + 200 = x_3 \end{cases}$$

$$\begin{cases} -x_1 + x_3 = 100 \\ x_1 - x_2 = 200 \\ -x_2 + x_4 = 100 \\ x_3 - x_4 = 200 \end{cases}$$

$$\left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 100 \\ 1 & -1 & 0 & 0 & 200 \\ 0 & -1 & 0 & 1 & 100 \\ 0 & 0 & 1 & -1 & 200 \end{array}\right) \quad R_2 + R_1$$

$$\begin{vmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} = -1\begin{vmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$\left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 100 \\ 0 & -1 & 1 & 0 & 300 \\ 0 & -1 & 0 & 1 & 100 \\ 0 & 0 & 1 & -1 & 200 \end{array}\right) \quad R_3 - R_2$$

$$\left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 100 \\ 0 & -1 & 1 & 0 & 300 \\ 0 & 0 & -1 & 1 & -200 \\ 0 & 0 & 1 & -1 & 200 \end{array}\right) \quad R_4 + R_3$$

$$\left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 100 \\ 0 & -1 & 1 & 0 & 300 \\ 0 & 0 & -1 & 1 & -200 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \to\ -x_1 + x_3 = 100 \\ \to\ -x_2 + x_3 = 100 \\ \to\ -x_3 + x_4 = 100 \\ \\ \end{array}$$

Let x_4 be the free variable

$$\begin{cases} \quad\quad\ \ \boxed{x_3 = x_4 + 200} \\ \quad\quad\ \ \boxed{x_2 = x_4 - 100} \\ x_1 = 200 + x_2 \ \boxed{= x_4 + 100} \end{cases}$$

Solution: $\left(x_4 + 100, \quad x_4 - 100, \quad x_4 + 200, \quad x_4\right)$

OR

$$\begin{vmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} = -1\begin{vmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= -1(1) - 1(-1)$$

$$= -1 + 1$$

$$\boxed{= 0}$$

$$\begin{cases} -x_1 + x_3 = 100 & \to\ x_1 = x_3 - 100 \ \boxed{= x_4 + 100} \\ x_1 - x_2 = 200 & \\ -x_2 + x_4 = 100 & \to\ \boxed{x_2 = x_4 - 100} \\ x_3 - x_4 = 200 & \to\ \boxed{x_3 = x_4 + 200} \end{cases}$$

b) The traffic flow when $x_4 = 0$ is:

$\therefore\ (100, \quad -100, \quad 200, \quad 0)$

c) The traffic flow when $x_4 = 100$ is:

$\therefore\ (200,\quad 0,\quad 300,\quad 100)$

d) The traffic flow when $x_1 = 2x_2$:

$$x_4 + 100 = 2(x_4 - 100)$$

$$x_4 + 100 = 2x_4 - 200$$

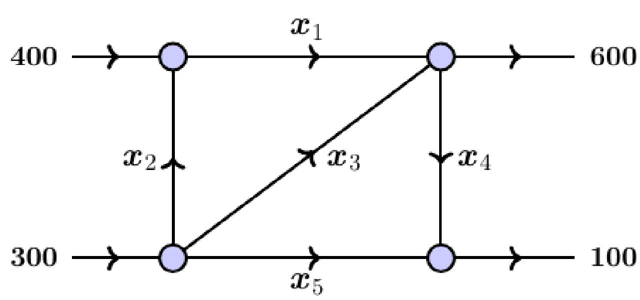
$$x_4 = 300$$

$\therefore\ (400,\quad 200,\quad 500,\quad 300)$

Exercise

Through a network, Express x_n's in terms of the parameters ***s*** and ***t***.

Solution



$$\begin{cases} x_1 = x_2 + 400 \\ x_1 + x_3 = x_4 + 600 \\ x_4 + x_5 = 100 \\ x_2 + x_3 + x_5 = 300 \end{cases}$$

$$\begin{cases} x_1 - x_2 = 400 \\ x_2 + x_3 - x_4 = 600 \\ x_4 + x_5 = 100 \\ x_2 + x_3 + x_5 = 300 \end{cases}$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 1 & 0 & 1 & -1 & 0 & 600 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 1 & 1 & 0 & 1 & 300 \end{array}\right) \quad \begin{array}{l} \\ R_2 - R_1 \\ \\ R_3 \leftrightarrow R_4 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 0 & 1 & 1 & -1 & 0 & 200 \\ 0 & 1 & 1 & 0 & 1 & 300 \\ 0 & 0 & 0 & 1 & 1 & 100 \end{array}\right) \quad \begin{array}{l} \\ \\ R_3 - R_2 \\ \\ \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 0 & 1 & 1 & -1 & 0 & 200 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 100 \end{array}\right) \quad \begin{array}{l} \\ \\ \\ R_4 - R_3 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 0 & 1 & 1 & -1 & 0 & 200 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{ll} x_1 - x_2 = 400 & \to x_1 = 400 + x_2 \\ x_2 + x_3 - x_4 = 200 & \to x_2 = 200 - x_3 + x_4 \\ x_4 + x_5 = 100 & \to \underline{x_4 = 100 - t} \end{array}$$

Let $x_5 = t$ & $x_3 = s$

$x_2 = 200 - s + 100 - t \ \underline{= 300 - s - t}$

$x_1 = 400 + 300 - s - t \ \underline{= 700 - s - t}$

Exercise

Water is flowing through a network of pipes. Express x_n's in terms of the parameters ***s*** and ***t***.

Solution

$x_1 + x_3 = 900$

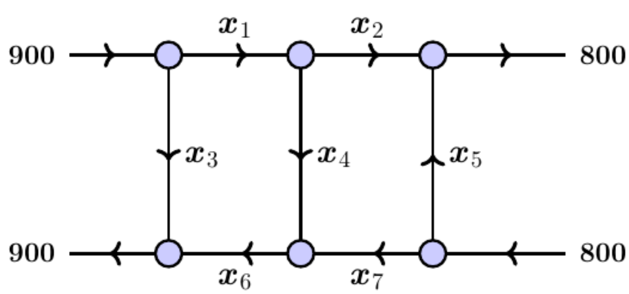
$x_1 = x_2 + x_4 \quad \to \quad x_1 - x_2 - x_4 = 0$

$x_2 + x_5 = 800$

$x_5 + x_7 = 800$

$x_6 = x_4 + x_7 \quad \to \quad x_4 - x_6 + x_7 = 0$

$x_3 + x_6 = 900$



$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 900 \end{array}\right] \quad \begin{array}{l} \\ R_2 - R_1 \\ \\ \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & -900 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 900 \end{array}\right] \quad \begin{array}{l} \\ \\ R_3 + R_2 \\ R_6 \\ \\ R_4 \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & -900 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 900 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \end{array}\right] \begin{array}{l} \\ -R_2 \\ \\ R_4 + R_3 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 900 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 800 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ R_5 + R_4 \\ \\ \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 900 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ \\ R_6 - R_5 \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 900 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{rl} x_2 = 900 - x_3 & (5) \\ x_2 = 900 - x_3 - x_4 & (4) \\ x_3 = 100 - x_4 + x_5 & (3) \\ -x_4 = 800 - x_5 - x_6 & (2) \\ x_5 = 800 - x_7 & (1) \end{array}$$

Let $\quad x_6 = s \quad \& \quad x_7 = t$

$(1) \rightarrow \quad x_5 = 800 - t$

$(2) \rightarrow \quad x_4 = s - t$

$(3) \rightarrow \quad x_3 = 900 - s$

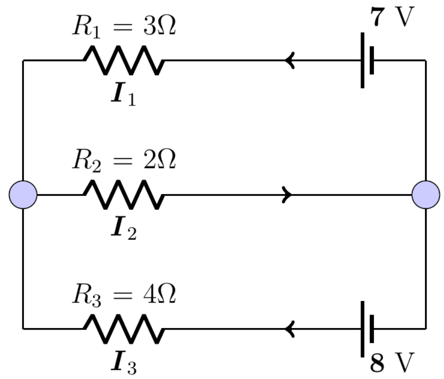
$(2) \rightarrow \quad x_2 = t$

$(1) \rightarrow \quad x_2 = s$

Solution: $\underline{(s,\ t,\ 900-s,\ s-t,\ 800-t,\ s,\ t)}$

Exercise

Determine the currents $I_1, I_2,$ *and* I_3 for the electrical network shown below



Solution

$$I_2 = I_1 + I_3$$
$$3I_1 + 2I_2 = 7$$
$$2I_2 + 4I_3 = 8$$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 3I_1 + 2I_2 \qquad = 7 \\ \qquad I_2 + 2I_3 = 4 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 3 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 13 \qquad D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 7 & 2 & 0 \\ 4 & 1 & 2 \end{vmatrix} = 13 \qquad D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 3 & 7 & 0 \\ 0 & 4 & 2 \end{vmatrix} = 26 \qquad D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 3 & 2 & 7 \\ 0 & 1 & 4 \end{vmatrix} = 13$$

$\underline{I_1 = 1\ A}\,|$ $\underline{I_2 = 2\ A}\,|$ $\underline{I_3 = 1\ A}\,|$

OR

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 3 & 2 & 0 & 7 \\ 0 & 1 & 2 & 4 \end{array}\right) \quad R_2 - 3R_1$$

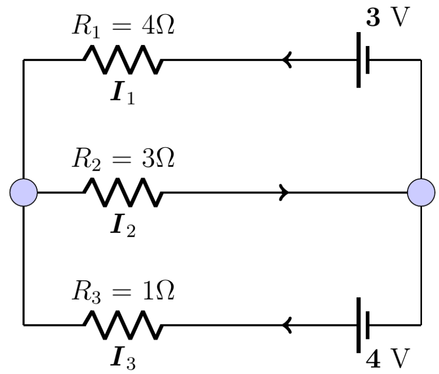
$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 5 & -3 & 7 \\ 0 & 1 & 2 & 4 \end{array}\right) \quad -5R_3 + R_2$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 5 & -3 & 7 \\ 0 & 0 & -13 & -13 \end{array}\right) \quad \begin{array}{l} I_1 = I_2 - I_3 \\ 5I_2 = 3I_3 + 7 \\ \underline{I_3 = 1}\,| \end{array}$$

$\underline{I_2 = 2}\,|$ $\underline{I_1 = 1}\,|$

Exercise

Determine the currents $I_1, I_2,$ *and* I_3 for the electrical network shown below



Solution

$$I_2 = I_1 + I_3$$

$$4I_1 + 3I_2 = 3$$

$$3I_2 + I_3 = 4$$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 4I_1 + 3I_2 \qquad = 3 \\ \qquad 3I_2 + I_3 = 4 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 4 & 3 & 0 \\ 0 & 3 & 1 \end{vmatrix} = 19 \qquad D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 3 & 3 & 0 \\ 4 & 3 & 1 \end{vmatrix} = 0 \qquad D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 4 & 3 & 0 \\ 0 & 4 & 1 \end{vmatrix} = 19 \qquad D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 4 & 3 & 3 \\ 0 & 3 & 4 \end{vmatrix} = 19$$

$\underline{I_1 = 0\ A}|$ $\underline{I_2 = 1\ A}|$ $\underline{I_3 = 1\ A}|$

OR

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 4 & 3 & 0 & 3 \\ 0 & 3 & 1 & 4 \end{array}\right) \quad R_2 - 4R_1$$

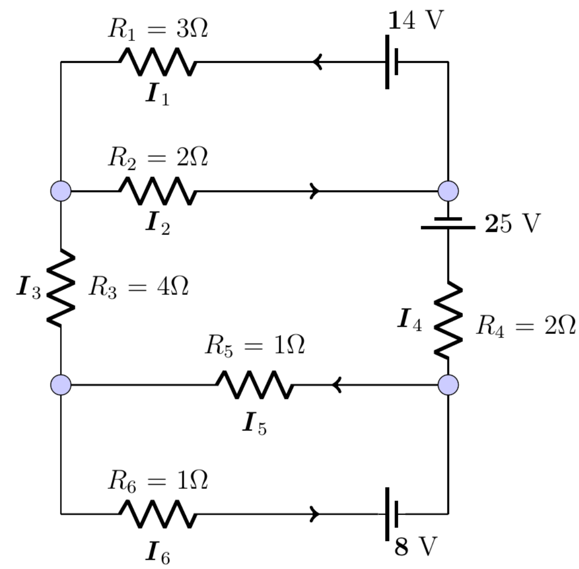
$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 7 & -4 & 3 \\ 0 & 3 & 1 & 4 \end{array}\right) \quad 7R_3 - 3R_2$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 7 & -4 & 3 \\ 0 & 0 & 19 & 19 \end{array}\right) \begin{array}{l} \to\ I_1 = I_2 - I_3 \quad (2) \\ \to\ 7I_2 = 4I_3 + 3 \quad (1) \\ \quad \underline{I_3 = 1}| \end{array}$$

$\underline{I_2 = 1}|$ $\underline{I_1 = 0}|$

Exercise

Determine the currents $I_1, I_2, I_3, I_4, I_5,$ *and* I_6 for the electrical network shown below



Solution

$$I_1 + I_3 = I_2 \quad \rightarrow \quad I_1 - I_2 + I_3 = 0$$

$$I_1 + I_4 = I_2 \quad \rightarrow \quad I_1 - I_2 + I_4 = 0$$

$$I_3 + I_6 = I_5 \quad \rightarrow \quad I_3 - I_5 + I_6 = 0$$

$$\begin{cases} 3I_1 + 2I_2 = 14 \\ 2I_2 + 4I_3 + I_5 + 2I_4 = 25 \\ I_5 + I_6 = 8 \end{cases}$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 & 14 \\ 0 & 2 & 4 & 2 & 1 & 0 & 25 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ \\ R_4 - 3R_1 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 2 & 4 & 2 & 1 & 0 & 25 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] \begin{array}{l} \\ R_4 \\ \\ R_2 \\ R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 2 & 4 & 2 & 1 & 0 & 25 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] \begin{array}{l} \\ \\ 5R_3 - 2R_2 \\ \\ R_5 + R_4 \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] \begin{array}{l} \\ \\ \\ 26R_4 + R_3 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & 0 & 36 & 5 & 0 & 97 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ 36R_5 - R_4 \\ \\ \end{array}$$

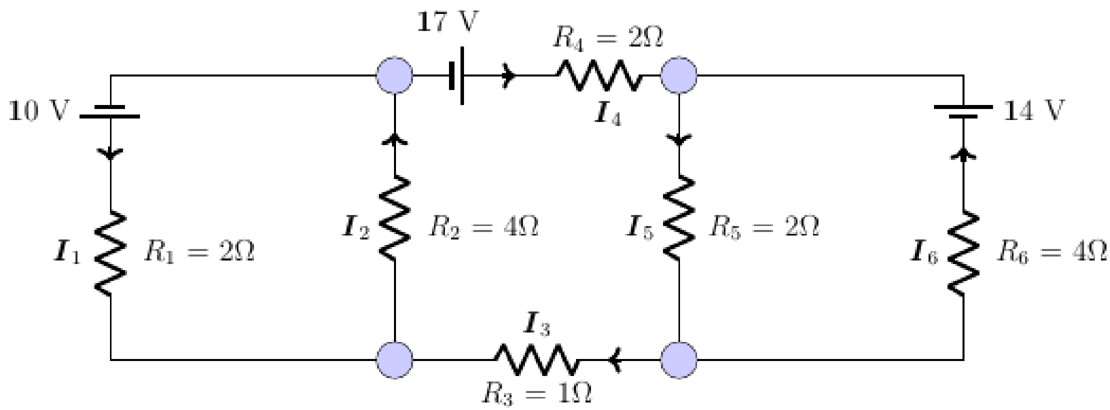
$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & 0 & 36 & 5 & 0 & 97 \\ 0 & 0 & 0 & 0 & -41 & 36 & -97 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ \\ 41R_6 + R_5 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & 0 & 36 & 5 & 0 & 97 \\ 0 & 0 & 0 & 0 & -41 & 36 & -97 \\ 0 & 0 & 0 & 0 & 0 & 77 & 231 \end{array}\right]$$

$$\begin{array}{rcl} I_1 = 4 - 2 & \rightarrow & \boxed{I_1 = 2} \\ 5I_2 = 14 + 3(2) & \rightarrow & \boxed{I_2 = 4} \\ 26I_3 = 97 - 10(2) - 5(5) & \rightarrow & \boxed{I_3 = 2} \\ 36I_4 = 97 - 5(5) & \rightarrow & \boxed{I_4 = 2} \\ -41I_5 = -97 - 36(3) & \rightarrow & \boxed{I_5 = 5} \\ 77I_6 = 231 & \rightarrow & \boxed{I_6 = 3} \end{array}$$

Exercise

Determine the currents $I_1, I_2, I_3, I_4, I_5,$ *and* I_6 for the electrical network shown below



Solution

$1 \rightarrow \quad I_1 + I_3 = I_2$

$2 \rightarrow \quad I_1 + I_4 = I_2$

$3 \rightarrow \quad I_3 + I_6 = I_5$

$4 \rightarrow \quad I_4 + I_6 = I_5$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ I_1 - I_2 + I_4 = 0 \\ I_3 - I_5 + I_6 = 0 \\ I_4 - I_5 + I_6 = 0 \\ 2I_1 + 4I_2 = 10 \\ 4I_2 + I_3 + 2I_4 + 2I_5 = 17 \\ 2I_5 + 4I_6 = 14 \end{cases}$$

$$\begin{array}{cl} \left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 5 \\ 0 & 4 & 1 & 2 & 2 & 0 & 17 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \end{array}\right) & \begin{array}{l} \\ R_2 - R_1 \\ \\ \\ R_5 - R_1 \\ \\ \\ \end{array} \end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 4 & 1 & 2 & 2 & 0 & 17 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \end{array}\right) \begin{array}{l} \\ \\ R_3 + R_2 \\ \\ \\ 3R_6 - 4R_5 \\ \\ \end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 7 & 6 & 6 & 0 & 31 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \end{array}\right) \begin{array}{l} \\ \\ \\ R_4 - R_3 \\ \\ \\ \\ \end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 7 & 6 & 6 & 0 & 31 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \end{array}\right) \begin{array}{l} \\ \\ \\ \\ \\ R_6 + 7R_2 \\ \\ \end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 13 & 6 & 0 & 31 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \\ \\ \\ \\ R_5 - 13R_4 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 19 & -13 & 31 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \\ \\ \\ \\ \\ 19R_6 - R_5 \\ \\ \end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 19 & -13 & 31 \\ 0 & 0 & 0 & 0 & 0 & 51 & 102 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} (4) \\ (3) \\ (2) \\ (1) \\ I_5 = \frac{1}{19}\left(31 + 13I_6\right) \\ \boxed{I_6 = 2} \\ \end{array}$$

$\boxed{I_5 = 3}$

$(1) \to \boxed{I_4 = I_5 - I_6 = 1}$

$(2) \to \boxed{I_3 = I_4 = 1}$

$(3) \to \boxed{I_2 = \frac{1}{3}\left(I_3 + 5\right) = 2}$

$(4) \to \boxed{I_1 = I_2 - I_3 = 1}$

Exercise

Consider the invertible matrix: $A = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{pmatrix}$

The message: ***ICEBERG DEAD AHEAD***

a) Write the uncoded for the message.

b) Use the matrix A to encode the message.

c) Decode a message from part *b*) given the matrix A.

Solution

a)

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

$$\begin{array}{cccccc} I\ C\ E & B\ E\ R & G\ \_\ D & E\ A\ D & \_\ A\ H & E\ A\ D \\ [9\ \ 3\ \ 5] & [2\ \ 5\ \ 18] & [7\ \ 0\ \ 4] & [5\ \ 1\ \ 4] & [0\ \ 1\ \ 8] & [5\ \ 1\ \ 4] \end{array}$$

b) Let encode the message ***ICEBERG DEAD AHEAD***

$$\begin{bmatrix}9 & 3 & 5\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 3 & 7 & 9\\ -3 & -2 & 7\end{bmatrix}=\begin{bmatrix}3 & 29 & 80\end{bmatrix}$$

$$\begin{bmatrix}2 & 5 & 18\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 3 & 7 & 9\\ -3 & -2 & 7\end{bmatrix}=\begin{bmatrix}-37 & 3 & 175\end{bmatrix}$$

$$\begin{bmatrix}7 & 0 & 4\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 3 & 7 & 9\\ -3 & -2 & 7\end{bmatrix}=\begin{bmatrix}-5 & 6 & 42\end{bmatrix}$$

$$\begin{bmatrix}5 & 1 & 4\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 3 & 7 & 9\\ -3 & -2 & 7\end{bmatrix}=\begin{bmatrix}-4 & 9 & 47\end{bmatrix}$$

$$\begin{bmatrix}0 & 1 & 8\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 3 & 7 & 9\\ -3 & -2 & 7\end{bmatrix}=\begin{bmatrix}-21 & -5 & 65\end{bmatrix}$$

$$\begin{bmatrix}5 & 1 & 4\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 3 & 7 & 9\\ -3 & -2 & 7\end{bmatrix}=\begin{bmatrix}-4 & 9 & 47\end{bmatrix}$$

The sequence of coded row matrices is

$$\begin{bmatrix}3 & 29 & 80\end{bmatrix}\ \begin{bmatrix}-37 & 3 & 175\end{bmatrix}\ \begin{bmatrix}-5 & 6 & 42\end{bmatrix}\ \begin{bmatrix}-4 & 9 & 47\end{bmatrix}\begin{bmatrix}-21 & -9 & 65\end{bmatrix}\begin{bmatrix}-4 & 9 & 47\end{bmatrix}$$

The cryptogram:

$$3\ \ 29\ \ 80\ \ -37\ \ 3\ \ 175\ \ -5\ \ 6\ \ 42\ \ -4\ \ 9\ \ 47\ \ -21\ \ -9\ \ 65\ \ -4\ \ 9\ \ 47$$

c) To decode a message given the matrix A.

$$|A|=\begin{vmatrix}1 & 2 & 2\\ 3 & 7 & 9\\ -3 & -2 & 7\end{vmatrix}=1$$

$$A^{-1}=\begin{bmatrix}67 & -18 & 4\\ -48 & 13 & -3\\ 15 & -4 & 1\end{bmatrix}$$

With the cryptogram:

$$\begin{bmatrix}3 & 29 & 80\end{bmatrix}\ \begin{bmatrix}-37 & 3 & 175\end{bmatrix}\ \begin{bmatrix}-5 & 6 & 42\end{bmatrix}\ \begin{bmatrix}-4 & 9 & 47\end{bmatrix}\begin{bmatrix}-21 & -9 & 65\end{bmatrix}\begin{bmatrix}-4 & 9 & 47\end{bmatrix}$$

$$\begin{bmatrix}3 & 29 & 80\end{bmatrix}\begin{bmatrix}67 & -18 & 4\\ -48 & 13 & -3\\ 15 & -4 & 1\end{bmatrix}=\begin{bmatrix}9 & 3 & 5\end{bmatrix}$$

$$\begin{bmatrix}-37 & 3 & 175\end{bmatrix}\begin{bmatrix}67 & -18 & 4\\ -48 & 13 & -3\\ 15 & -4 & 1\end{bmatrix}=\begin{bmatrix}2 & 5 & 18\end{bmatrix}$$

$$\begin{bmatrix}-5 & 6 & 42\end{bmatrix}\begin{bmatrix}67 & -18 & 4\\ -48 & 13 & -3\\ 15 & -4 & 1\end{bmatrix}=\begin{bmatrix}7 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}-4 & 9 & 47\end{bmatrix}\begin{bmatrix}67 & -18 & 4\\ -48 & 13 & -3\\ 15 & -4 & 1\end{bmatrix}=\begin{bmatrix}5 & 1 & 4\end{bmatrix}$$

$$\begin{bmatrix}-21 & -9 & 65\end{bmatrix}\begin{bmatrix}67 & -18 & 4\\ -48 & 13 & -3\\ 15 & -4 & 1\end{bmatrix}=\begin{bmatrix}0 & 1 & 8\end{bmatrix}$$

$$\begin{bmatrix}-4 & 9 & 47\end{bmatrix}\begin{bmatrix}67 & -18 & 4\\ -48 & 13 & -3\\ 15 & -4 & 1\end{bmatrix}=\begin{bmatrix}5 & 1 & 4\end{bmatrix}$$

The message is:

9	3	5	2	5	18	7	0	1	5	1	4	0	1	8	5	1	4
I	*C*	*E*	*B*	*E*	*R*	*G*	_	*D*	*E*	*A*	*D*	_	*A*	*H*	*E*	*A*	*D*

Exercise

You want to send the message: ***LINEAR ALGEBRA*** with a key word ***MATH***

a) Write the matrix A.

b) Write the uncoded for the message.

c) Use the matrix A to encode the message.

d) Decode a message from part *b*) given the matrix A.

Solution

a)

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

$$\begin{matrix} M & A & T & H \\ 13 & 1 & 20 & 8 \end{matrix}$$

$$A = \begin{pmatrix} 13 & 1 \\ 20 & 8 \end{pmatrix}$$

b)

$$\begin{matrix} L & I & N & E & A & R & \_ & A & L & G & E & B & R & A \\ 12 & 9 & 14 & 5 & 1 & 18 & 0 & 1 & 12 & 7 & 5 & 2 & 18 & 1 \end{matrix}$$

$$\begin{bmatrix} 12 & 9 \end{bmatrix} \quad \begin{bmatrix} 14 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 18 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 12 & 7 \end{bmatrix} \quad \begin{bmatrix} 5 & 2 \end{bmatrix} \quad \begin{bmatrix} 18 & 1 \end{bmatrix}$$

c) Encoding the message

$$\begin{bmatrix} 12 & 9 \end{bmatrix} \begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = \begin{bmatrix} 336 & 84 \end{bmatrix}$$

$$\begin{bmatrix} 14 & 5 \end{bmatrix} \begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = \begin{bmatrix} 282 & 54 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 18 \end{bmatrix} \begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = \begin{bmatrix} 373 & 145 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = \begin{bmatrix} 20 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 12 & 7 \end{bmatrix} \begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = \begin{bmatrix} 296 & 68 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 2 \end{bmatrix} \begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = \begin{bmatrix} 105 & 21 \end{bmatrix}$$

$$\begin{bmatrix} 18 & 1 \end{bmatrix} \begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = \begin{bmatrix} 254 & 26 \end{bmatrix}$$

The cryptogram:

336 84 282 54 373 145 20 8 296 68 105 21 254 26

d) To decode a message given the matrix *A*.

$$A^{-1} = \frac{1}{84} \begin{bmatrix} 8 & -1 \\ -20 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix}$$

With the cryptogram:

$$\begin{bmatrix} 336 & 84 \end{bmatrix} \quad \begin{bmatrix} 282 & 54 \end{bmatrix} \quad \begin{bmatrix} 373 & 145 \end{bmatrix} \quad \begin{bmatrix} 20 & 8 \end{bmatrix} \quad \begin{bmatrix} 296 & 68 \end{bmatrix} \quad \begin{bmatrix} 105 & 21 \end{bmatrix} \quad \begin{bmatrix} 254 & 26 \end{bmatrix}$$

$$\begin{bmatrix}336 & 84\end{bmatrix}\begin{bmatrix}\frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84}\end{bmatrix}=\begin{bmatrix}12 & 9\end{bmatrix}$$

$$\begin{bmatrix}282 & 54\end{bmatrix}\begin{bmatrix}\frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84}\end{bmatrix}=\begin{bmatrix}14 & 5\end{bmatrix}$$

$$\begin{bmatrix}373 & 145\end{bmatrix}\begin{bmatrix}\frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84}\end{bmatrix}=\begin{bmatrix}1 & 18\end{bmatrix}$$

$$\begin{bmatrix}20 & 8\end{bmatrix}\begin{bmatrix}\frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84}\end{bmatrix}=\begin{bmatrix}0 & 1\end{bmatrix}$$

$$\begin{bmatrix}296 & 68\end{bmatrix}\begin{bmatrix}\frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84}\end{bmatrix}=\begin{bmatrix}12 & 7\end{bmatrix}$$

$$\begin{bmatrix}105 & 21\end{bmatrix}\begin{bmatrix}\frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84}\end{bmatrix}=\begin{bmatrix}5 & 2\end{bmatrix}$$

$$\begin{bmatrix}254 & 26\end{bmatrix}\begin{bmatrix}\frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84}\end{bmatrix}=\begin{bmatrix}18 & 1\end{bmatrix}$$

$$\begin{array}{cccccccccccccc} 12 & 9 & 14 & 5 & 1 & 18 & 0 & 1 & 12 & 7 & 5 & 2 & 18 & 1 \\ L & I & N & E & A & R & \_ & A & L & G & E & B & R & A \end{array}$$

The message is: ***Linear Algebra***

Exercise

Consider the invertible matrix: $A=\begin{pmatrix}2 & 1 \\ 1 & 1\end{pmatrix}$

Decode the cryptogram 27 14 48 28 5 5 21 20 50 25

Solution

$$A^{-1}=\begin{pmatrix}1 & -1 \\ -1 & 2\end{pmatrix}$$

$$\begin{bmatrix}27 & 14\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}=\begin{bmatrix}13 & 1\end{bmatrix}$$

$$\begin{bmatrix}48 & 28\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}=\begin{bmatrix}20 & 8\end{bmatrix}$$

$$\begin{bmatrix}5 & 5\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}=\begin{bmatrix}0 & 5\end{bmatrix}$$

$$\begin{bmatrix}21 & 20\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}=\begin{bmatrix}1 & 19\end{bmatrix}$$

$$\begin{bmatrix}50 & 25\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}=\begin{bmatrix}25 & 0\end{bmatrix}$$

$$\begin{array}{cccccccccc}13 & 1 & 20 & 8 & 0 & 5 & 1 & 19 & 25 & 0\\ M & A & T & H & \_ & E & A & S & Y & \_\end{array}$$

The message is: ***Math Easy***

Exercise

Consider the invertible matrix: $A=\begin{pmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{pmatrix}$

Decode the cryptogram

$$1 \quad -5 \quad 11 \quad 19 \quad -25 \quad -45 \quad 11 \quad -16 \quad -28 \quad 20 \quad -29 \quad -27$$
$$12 \quad -12 \quad -53 \quad 40 \quad -61 \quad -35 \quad 8 \quad -17 \quad 7$$

Solution

$|A|=1$

$$A^{-1}=\begin{pmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{pmatrix}$$

With the cryptogram:

$$\begin{bmatrix}1 & -5 & 11\end{bmatrix}\ \begin{bmatrix}19 & -25 & -45\end{bmatrix}\ \begin{bmatrix}11 & -16 & -28\end{bmatrix}\ \begin{bmatrix}20 & -29 & -27\end{bmatrix}$$
$$\begin{bmatrix}12 & -12 & -53\end{bmatrix}\ \begin{bmatrix}40 & -61 & -35\end{bmatrix}\ \begin{bmatrix}8 & -17 & 7\end{bmatrix}$$

$$\begin{bmatrix}1 & -5 & 11\end{bmatrix}\begin{pmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{pmatrix}=\begin{bmatrix}4 & 9 & 6\end{bmatrix}$$

$$\begin{bmatrix}19 & -25 & -45\end{bmatrix}\begin{pmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{pmatrix}=\begin{bmatrix}6 & 5 & 18\end{bmatrix}$$

$$\begin{bmatrix}11 & -16 & -28\end{bmatrix}\begin{pmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{pmatrix}=\begin{bmatrix}5 & 14 & 20\end{bmatrix}$$

$$\begin{bmatrix}20 & -29 & -27\end{bmatrix}\begin{pmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{pmatrix}=\begin{bmatrix}9 & 1 & 12\end{bmatrix}$$

$$\begin{bmatrix}12 & -12 & -53\end{bmatrix}\begin{pmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{pmatrix}=\begin{bmatrix}0 & 5 & 17\end{bmatrix}$$

$$\begin{bmatrix}40 & -61 & -35\end{bmatrix}\begin{pmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{pmatrix}=\begin{bmatrix}21 & 1 & 20\end{bmatrix}$$

$$\begin{bmatrix}8 & -17 & 7\end{bmatrix}\begin{pmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{pmatrix}=\begin{bmatrix}9 & 15 & 14\end{bmatrix}$$

4	9	6	6	5	18	5	14	20	9	1	12	0	5	17	21	1	20	9	15	14
D	*I*	*F*	*F*	*E*	*R*	*E*	*N*	*T*	*I*	*A*	*L*	_	*E*	*Q*	*U*	*A*	*T*	*I*	*O*	*N*

The message is: ***Differential Equation***.

Exercise

Solution