SOLUTION ***Section* 2.1 – Definitions of 2*nd* and Higher Order Equations**

Exercise

Decide whether the equation is linear or nonlinear. For the linear equation, state whether the equation is homogenous or inhomogeneous. $t^2 y'' = 4y' - \sin t$

Solution

$$y'' - \frac{4}{t^2} y' = -\frac{\sin t}{t^2}$$

$y'' + p(t)y' + q(t)y = g(t)$ It is linear and inhomogeneous

Exercise

Decide whether the equation is linear or nonlinear. For the linear equation, state whether the equation is homogenous or inhomogeneous. $ty'' + (\sin t)y' = 4y - \cos 5t$

Solution

$$y'' + \left(\frac{\sin t}{t}\right) y' - \frac{4}{t} y = -\frac{\cos 5t}{t}$$

$y'' + p(t)y' + q(t)y = g(t)$ It is linear and inhomogeneous $(g(t) \neq 0)$

Exercise

Decide whether the equation is linear or nonlinear. For the linear equation, state whether the equation is homogenous or inhomogeneous. $t^2 y'' + 4yy' = 0$

Solution

It is nonlinear $(4yy')$ $(g(t) \neq 0)$

Exercise

Decide whether the equation is linear or nonlinear. For the linear equation, state whether the equation is homogenous or inhomogeneous. $y'' + 4y' + 7y = 3e^{-t} \sin t$

Solution

Compare to $y'' + p(t)y' + q(t)y = g(t)$

$\Rightarrow p(t) = 4, \quad q(t) = 7, \quad g(t) = 3e^{-t} \sin t \qquad (g(t) \neq 0)$

Hence, the equation is linear and inhomogeneous.

Exercise

Show by direct substitution that the given functions $y_1(t)$ and $y_2(t)$ are solutions of the given differential equation. Then verify by direct substitution, that any linear combination $C_1y_1(t)+C_2y_2(t)$ of the 2 given solutions is also a solution.

$$y''+4y=0 \qquad y_1(t)=\cos 2t \qquad y_2(t)=\sin 2t$$

Solution

$$\begin{aligned} y &= C_1y_1+C_2y_2 \\ &= C_1\cos 2t+C_2\sin 2t \end{aligned}$$

$$y'=-2C_1\sin 2t+2C_2\cos 2t$$

$$y''=-4C_1\cos 2t-4C_2\sin 2t$$

If $C_1y_1(t)+C_2y_2(t)$, then

$$\begin{aligned} y''+4y &= -4C_1\cos 2t-4C_2\sin 2t+4\left(C_1\cos 2t+C_2\sin 2t\right) \\ &= -4C_1\cos 2t-4C_2\sin 2t+4C_1\cos 2t+4C_2\sin 2t \\ &= 0 \end{aligned}$$

Exercise

Show by direct substitution that the given functions $y_1(t)$ and $y_2(t)$ are solutions of the given differential equation. Then verify by direct substitution, that any linear combination $C_1y_1(t)+C_2y_2(t)$ of the 2 given solutions is also a solution.

$$y''-2y'+2y=0; \quad y_1(t)=e^t\cos t \quad y_2(t)=e^t\sin t$$

Solution

$$y_1(t)=e^t\cos t \quad \Rightarrow \quad y_1'(t)=e^t\cos t-e^t\sin t=e^t\left(\cos t-\sin t\right)$$

$$\begin{aligned} \Rightarrow \quad y_1''(t) &= e^t\left(\cos t-\sin t\right)+e^t\left(-\sin t-\cos t\right) \\ &= e^t\cos t-e^t\sin t-e^t\sin t-e^t\cos t \\ &= -2e^t\sin t \end{aligned}$$

$$\begin{aligned} y_1''-2y_1'+2y_1 &= -2e^t\sin t-2\left(e^t\cos t-e^t\sin t\right)+2e^t\cos t \\ &= -2e^t\sin t-2e^t\cos t+2e^t\sin t+2e^t\cos t \\ &= 0 \end{aligned}$$

$$y_2(t)=e^t\sin t$$

$$y_2'(t) = e^t \sin t + e^t \cos t$$

$$y_2''(t) = e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t$$

$$= 2e^t \cos t$$

$$y_1'' - 2y_1' + 2y_1 = 2e^t \cos t - 2\left(e^t \cos t + e^t \sin t\right) + 2e^t \sin t$$

$$= 2e^t \cos t - 2e^t \cos t - 2e^t \sin t + 2e^t \sin t$$

$$\underline{= 0}$$

If $y(t) = C_1 e^t \cos t + C_2 e^t \sin t$

$$y'(t) = C_1 e^t \cos t - C_1 e^t \sin t + C_2 e^t \sin t + C_2 e^t \cos t$$

$$= \left(C_1 + C_2\right) e^t \cos t + \left(C_2 - C_1\right) e^t \sin t$$

$$y''(t) = \left(C_1 + C_2\right) e^t \cos t - \left(C_1 + C_2\right) e^t \sin t + \left(C_2 - C_1\right) e^t \sin t + \left(C_2 - C_1\right) e^t \cos t$$

$$= \left(C_1 + C_2 + C_2 - C_1\right) e^t \cos t + \left(C_2 - C_1 - C_1 - C_2\right) e^t \sin t$$

$$= 2C_2 e^t \cos t - 2C_1 e^t \sin t$$

$$\begin{aligned} y'' - 2y' + 2y = 2C_2 e^t \cos t - 2C_1 e^t \sin t - 2\left(\left(C_1 + C_2\right) e^t \cos t + \left(C_2 - C_1\right) e^t \sin t\right) \\ + 2\left(C_1 e^t \cos t + C_2 e^t \sin t\right) \end{aligned}$$

$$\begin{aligned} = 2C_2 e^t \cos t - 2C_1 e^t \sin t - 2C_1 e^t \cos t - 2C_2 e^t \cos t + 2C_2 e^t \sin t - 2C_1 e^t \sin t \\ + 2C_1 e^t \cos t + 2C_2 e^t \sin t \end{aligned}$$

$$\underline{= 0}$$

Exercise

Explain why $y_1(t)$ and $y_2(t)$ are linearly independent solutions. Calculate Wronskian and use it to explain the independence of the given solutions.

$$y'' + 9y = 0 \qquad y_1(t) = \cos 3t \qquad y_2(t) = \sin 3t$$

Solution

$$w(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix}$$

$$= 3\cos^2 3t + 3\sin^2 3t$$
$$= 3\left(\cos^2 3t + \sin^2 3t\right)$$
$$= 3 \neq 0$$

The solutions $y_1(t)$ & $y_2(t)$ are linearly independent.

Exercise

Show that $y_1(t) = e^t$ and $y_2(t) = e^{-3t}$ form a fundamental set of solutions for $y'' + 2y' - 3y = 0,$ then find a solution satisfying $y(0) = 1$ and $y'(0) = -2$.

Solution

$$y_1(t) = e^t \ \Rightarrow y'' + 2y' - 3y = e^t + 2e^t - 3e^t = 0$$
$$y_2(t) = e^{-3t} \Rightarrow y'' + 2y' - 3y = 9e^{-3t} - 6e^{-3t} - 3e^{-3t} = 0$$
$$y(t) = C_1 e^t + C_2 e^{-3t} \qquad y(0) = C_1 + C_2 = 1$$
$$y'(t) = C_1 e^t - 3C_2 e^{-3t} \qquad y'(0) = C_1 - 3C_2 = -2$$
$$\Rightarrow C_1 = \frac{1}{4} \quad C_2 = \frac{3}{4}$$
$$\boxed{y(t) = \frac{1}{4}e^t + \frac{3}{4}e^{-3t}}$$

Exercise

Use the Wronskian to show that are linearly independence $\quad y_1(x) = e^{-3x}, \quad y_2(x) = e^{3x}$

Solution

$$W(x) = \begin{vmatrix} e^{-3x} & e^{3x} \\ -3e^{-3x} & 3e^{3x} \end{vmatrix}$$
$$= 3 + 3$$
$$\boxed{= 6 \ \neq 0}$$

Thus, the functions are linearly independent.

Exercise

Use the Wronskian to show that are linearly independence $\quad \mathbf{f}_1 = 1, \quad \mathbf{f}_2 = e^x, \quad \mathbf{f}_3 = e^{2x}$

Solution

The Wronskian is

$$W(x)=\begin{vmatrix} 1 & e^{x} & e^{2x} \\ 0 & e^{x} & 2e^{2x} \\ 0 & e^{x} & 4e^{2x} \end{vmatrix}$$

$$=e^{x}4e^{2x}-2e^{2x}e^{x}=2e^{3x}\neq 0$$

Thus, the functions are linearly independent.

Exercise

Use the Wronskian to show that are linearly independence $\left\{e^{x},\ xe^{x},\ (x+1)e^{x}\right\}$

Solution

$$W=\begin{vmatrix} e^{x} & xe^{x} & (x+1)e^{x} \\ e^{x} & (x+1)e^{x} & (x+2)e^{x} \\ e^{x} & (x+2)e^{x} & (x+3)e^{x} \end{vmatrix}$$

$$=(x+1)(x+3)e^{3x}+x(x+2)e^{3x}+(x+1)(x+2)e^{3x}-(x+1)^{2}e^{3x}-(x+2)^{2}e^{3x}-x(x+3)e^{3x}$$

$$=\left(x^{2}+4x+3+x^{2}+2x+x^{2}+3x+2-x^{2}-2x-1-x^{2}-4x-4-x^{2}-3x\right)e^{3x}$$

$$\boxed{=0}$$

Thus, the set $\left\{e^{x},\ xe^{x},\ (x+1)e^{x}\right\}$ is linearly dependent.

Exercise

Use the Wronskian to show that are linearly independence

$$y_{1}(x)=e^{-3x},\quad y_{2}(x)=\cos 2x,\quad y_{3}(x)=\sin 2x$$

Solution

$$W=\begin{vmatrix} e^{-3x} & \cos 2x & \sin 2x \\ -3e^{-3x} & -2\sin 2x & 2\cos 2x \\ 9e^{-3x} & -4\cos 2x & -4\sin 2x \end{vmatrix}$$

$$=8e^{-3x}\sin^{2}2x+18e^{-3x}\cos^{2}2x+12e^{-3x}\sin 2x\cos 2x$$
$$+18e^{-3x}\sin^{2}2x+8e^{-3x}\cos^{2}2x-12e^{-3x}\sin 2x\cos 2x$$

$$\boxed{=26e^{-3x}\neq 0}$$

$\therefore\ y_{1},\ y_{2},\ and\ y_{3}$ are linearly independent.

Exercise

Use the Wronskian to show that are linearly independence $y_1(x) = e^x, \quad y_2(x) = e^{2x}, \quad y_3(x) = e^{3x}$

Solution

$$W = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$= 18e^{6x} + 3e^{6x} + 4e^{6x} - 2e^{6x} - 12e^{6x} - 9e^{6x}$$

$$\boxed{= 2e^{6x} \neq 0}$$

$\therefore y_1, y_2, \textit{and } y_3$ are linearly independent.

Exercise

Use the Wronskian to show that are linearly independence

$$y_1(x) = \cos^2 x, \quad y_2(x) = \sin^2 x, \quad y_3(x) = \sec^2 x, \quad y_4(x) = \tan^2 x$$

Solution

Since $\cos^2 x + \sin^2 x = 1$ & $\sec^2 x = 1 + \tan^2 x$

$c_1 \cos^2 x + c_2 \sin^2 x + c_3 \sec^2 x + c_4 \tan^2 x = 0$

Let: $c_1 = c_2 = 0 \quad c_3 = -1 \quad c_4 = 1$

$\cos^2 x + \sin^2 x - \sec^2 x + \tan^2 x = 0$

The set of functions are linearly dependent.

Exercise

Determine whether the functions $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $(0, 1)$

$$y_1(t) = \cos t \sin t, \quad y_2(t) = \sin 2t$$

Solution

$y_1(t) = cy_2(t)$

$\cos t \sin t = c \sin 2t \quad \rightarrow \quad \boxed{c = \frac{1}{2}}$

The given functions are linearly dependent.

Exercise

Determine whether the functions $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $(0,\ 1)$

$$y_1(t) = e^{3t}, \quad y_2(t) = e^{-4t}$$

Solution

$y_1(t) = cy_2(t)$

$e^{3t} = ce^{-4t} \quad \rightarrow \quad e^{7t} = c$

Since an exponential function is strictly monotone, this is a contradiction.

Hence, given functions are linearly independent on $(0,\ 1)$

Exercise

Determine whether the functions $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $(0,\ 1)$

$$y_1(t) = te^{2t}, \quad y_2(t) = e^{2t}$$

Solution

$y_1(t) = cy_2(t)$

$te^{2t} = ce^{2t} \quad \rightarrow \quad \underline{c = t}|$

Hence, given functions are linearly independent on $(0,\ 1)$

Exercise

Determine whether the functions $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $(0,\ 1)$

$$y_1(t) = t^2\cos(\ln t), \quad y_2(t) = t^2\sin(\ln t)$$

Solution

$y_1(t) = cy_2(t)$

$t^2\cos(\ln t) = ct^2\sin(\ln t)$

$\cos(\ln t) = c\sin(\ln t)$

$\Rightarrow \ \underline{c = \cot(\ln t)}|$

Hence, given functions are linearly independent on $(0,\ 1)$

Exercise

Determine whether the functions $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $(0,\ 1)$

$$y_1(t) = \tan^2 t - \sec^2 t, \quad y_2(t) = 3$$

Solution

$$y_1(t) = cy_2(t)$$

$$\tan^2 t - \sec^2 t = 3c$$

$$-1 = 3c \ \Rightarrow \ \boxed{c = -\frac{1}{3}}$$

The given functions are linearly dependent.

Exercise

Determine whether the functions $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $(0,\ 1)$

$$y_1(t) \equiv 0, \quad y_2(t) = e^t$$

Solution

$$y_1(t) = cy_2(t)$$

$$0 \equiv ce^t \quad \rightarrow \quad \boxed{c \equiv 0}$$

The given functions are linearly dependent.

Exercise

Use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equation.

$$y'' + 2y' - 3y = 0$$

Solution

Let $v = y' \quad \Rightarrow \quad v' = y''$

$$y'' = -2y' + 3y$$

$$v' = -2v + 3y$$

The following system of the first-order equations:

$$\begin{cases} y' = v \\ v' = -2v + 3y \end{cases}$$

Exercise

Use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equation.

$$y'' + 3y' + 4y = 2\cos 2t$$

Solution

Let $v = y' \Rightarrow v' = y''$

$$y'' = -3y' - 4y + 2\cos 2t$$
$$v' = -3v - 4y + 2\cos 2t$$

The following system of the first-order equations: $\begin{cases} y' = v \\ v' = -3v - 4y + 2\cos 2t \end{cases}$

Exercise

Use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equation. $y'' + 2y' + 2y = 2\sin 2\pi t$

Solution

Let $v = y' \Rightarrow v' = y''$

$$y'' = -2y' - 2y + 2\sin 2\pi t$$

$$v' = -2v - 2y + 2\sin 2\pi t$$

The following system of the first-order equations: $\begin{cases} y' = v \\ v' = -2v - 2y + 2\sin 2\pi t \end{cases}$

Exercise

Use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equation. $y'' + \mu\left(t^2 - 1\right)y' + y = 0$

Solution

Let $v = y' \Rightarrow v' = y''$

$$y'' = -\mu\left(t^2 - 1\right)y' - y$$

$$v' = -\mu\left(t^2 - 1\right)v - y$$

The following system of the first-order equations:

$$\begin{cases} y' = v \\ v' = -\mu\left(t^2 - 1\right)v - y \end{cases}$$

Exercise

Use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equation. $4y'' + 4y' + y = 0$

Solution

Let $v = y' \quad \Rightarrow \quad v' = y''$

$$4y'' = -4y' - y$$

$$y'' = -y' - \frac{1}{4}y$$

$$v' = -v - \frac{1}{4}y$$

The following system of the first-order equations:

$$\begin{cases} y' = v \\ v' = -v - \frac{1}{4}y \end{cases}$$

Exercise

Find a particular solution satisfying the given initial conditions

$$y'' - 4y = 0\ ; \quad y_1(t) = e^{2t},\ \ y_2(t) = 2e^{-2t}; \quad y(0) = 1,\ \ y'(0) = -2$$

Solution

$$W = \begin{vmatrix} e^{2t} & 2e^{-2t} \\ 2e^{2t} & -4e^{-2t} \end{vmatrix}$$

$$= -8 \ \neq 0$$

$\therefore\ y_1$ *and* y_2 are linearly independent.

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$$y(t) = C_1 e^{2t} + 2C_2 e^{-2t} \qquad y(0) = 1 \ \rightarrow \ C_1 + 2C_2 = 1$$

$$y'(t) = 2C_1 e^{2t} - 4C_2 e^{-2t} \qquad y'(0) = -2 \ \rightarrow \ 2C_1 - 4C_2 = -2$$

$$\begin{cases} C_1 + 2C_2 = 1 \\ C_1 - 2C_2 = -1 \end{cases} \qquad \Delta = \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = -4 \quad \Delta_1 = \begin{vmatrix} 1 & 2 \\ -1 & -2 \end{vmatrix} = 0 \quad \Delta_2 = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

$$C_1 = 0, \quad C_2 = \frac{1}{2}$$

$$y(t) = e^{-2t}$$

Exercise

Find a particular solution satisfying the given initial conditions

$$y'' - y = 0\ ; \quad y_1(t) = 2e^t,\ y_2(t) = e^{-t+3}; \quad y(-1) = 1,\ y'(-1) = 0$$

Solution

$$W = \begin{vmatrix} 2e^t & e^{-t+3} \\ 2e^t & -e^{-t+3} \end{vmatrix}$$

$$= -4e^3 \neq 0$$

$\therefore\ y_1$ *and* y_2 are linearly independent.

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$$y(t) = 2C_1 e^t + C_2 e^{-t+3} \qquad y(-1) = 1 \ \rightarrow\ 2C_1 e^{-1} + C_2 e^4 = 1$$

$$y'(t) = 2C_1 e^t - C_2 e^{-t+3} \qquad y'(-1) = 0 \ \rightarrow\ 2C_1 e^{-1} - C_2 e^4 = 0$$

$$\begin{cases} 2C_1 + e^5 C_2 = e \\ 2C_1 - e^5 C_2 = 0 \end{cases} \quad \rightarrow\ 4C_1 = e$$

$$C_1 = \frac{e}{4},\quad C_2 = \frac{1}{2e^4}$$

$$y(t) = \frac{e}{4}e^t + \frac{1}{2e^4}e^{-t+3}$$

Exercise

Find a particular solution satisfying the given initial conditions

$$y'' + y = 0\ ; \quad y_1(t) = 0,\ y_2(t) = \sin t; \quad y\left(\frac{\pi}{2}\right) = 1,\ y'\left(\frac{\pi}{2}\right) = 1$$

Solution

$$W = \begin{vmatrix} 0 & \sin t \\ 0 & \cos t \end{vmatrix}$$

$$= 0$$

$\therefore\ y_1$ *and* y_2 are linearly dependent.

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$$y(t) = C_2 \sin t \qquad y\left(\frac{\pi}{2}\right) = 1 \ \rightarrow\ C_2 = 1$$

$y'(t) = C_2 \cos t \qquad y'\left(\frac{\pi}{2}\right) = 1 \quad \to \quad \cancel{0 = 1}$

$\boxed{y(t) = C_2 \sin t}$

Exercise

Find a particular solution satisfying the given initial conditions

$$y'' + y = 0\ ; \quad y_1(t) = \cos t, \ \ y_2(t) = \sin t; \quad y\left(\frac{\pi}{2}\right) = 1, \ \ y'\left(\frac{\pi}{2}\right) = 1$$

Solution

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}$$

$$= \cos^2 t + \sin^2 t$$

$$\boxed{= 1 \ \neq 0}$$

$\therefore\ y_1$ *and* y_2 are linearly independent.

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$$y(t) = C_1 \cos t + C_2 \sin t \qquad y\left(\frac{\pi}{2}\right) = 1 \quad \to \quad \boxed{C_2 = 1}$$

$$y'(t) = -C_1 \sin t + C_2 \cos t \qquad y'\left(\frac{\pi}{2}\right) = 1 \quad \to \quad \boxed{C_1 = -1}$$

$$\boxed{y(t) = -\cos t + \sin t}$$

Exercise

Find a particular solution satisfying the given initial conditions

$$y'' - 4y' + 4y = 0\ ; \quad y_1(t) = e^{2t}, \ \ y_2(t) = te^{2t}; \quad y(0) = 2, \ \ y'(0) = 0$$

Solution

$$W = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & (1+2t)e^{2t} \end{vmatrix}$$

$$= (1 + 2t - 2t)e^{4t}$$

$$\boxed{= e^{4t} \ \ \neq 0}$$

$\therefore\ y_1$ *and* y_2 are linearly independent.

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$$y(t)=C_1e^{2t}+C_2te^{2t} \qquad y(0)=2 \ \to \ \boxed{C_1=2}$$

$$y'(t)=2C_1e^{2t}+C_2(1+2t)e^{2t} \qquad y'(0)=0 \ \to \ 2C_1+C_2=0 \quad \to \boxed{C_2=-4}$$

$$\boxed{y(t)=2e^{2t}-4te^{2t}}$$

Exercise

Find a particular solution satisfying the given initial conditions

$$2y''-y'=0\ ; \quad y_1(t)=1,\ \ y_2(t)=e^{t/2}; \quad y(2)=0,\ \ y'(2)=2$$

Solution

$$W=\begin{vmatrix}1 & e^{t/2}\\ 0 & \frac{1}{2}e^{t/2}\end{vmatrix}$$

$$\boxed{=\frac{1}{2}e^{t/2} \ \neq 0}$$

$\therefore\ y_1$ *and* y_2 are linearly independent.

$$y(t)=C_1y_1(t)+C_2y_2(t)$$

$$y(t)=C_1+C_2e^{t/2} \qquad y(2)=0 \ \to \ C_1+eC_2=0$$

$$y'(t)=\frac{1}{2}C_2e^{t/2} \qquad y'(2)=2 \ \to \ \frac{1}{2}eC_2=2$$

$$\boxed{C_1=-4, \quad C_2=\frac{4}{e}}$$

$$\boxed{y(t)=-4+\frac{4}{e}e^{t/2}}$$

Exercise

Find a particular solution satisfying the given initial conditions

$$y''-3y'+2y=0\ ; \quad y_1(t)=2e^{t},\ \ y_2(t)=e^{2t}; \quad y(-1)=1,\ \ y'(-1)=0$$

Solution

$$W=\begin{vmatrix}2e^{t} & e^{2t}\\ 2e^{t} & 2e^{2t}\end{vmatrix}$$

$$\boxed{=2e^{3t} \ \neq 0}$$

$\therefore\ y_1$ *and* y_2 are linearly independent.

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$$y(t) = 2C_1 e^t + C_2 e^{2t} \qquad y(-1) = 1 \;\rightarrow\; 2e^{-1}C_1 + e^{-2}C_2 = 1$$

$$y'(t) = 2C_1 e^t + 2C_2 e^{2t} \qquad y'(-1) = 0 \;\rightarrow\; 2e^{-1}C_1 + 2e^{-2}C_2 = 2$$

$$\begin{cases} 2eC_1 + C_2 = e^2 \\ eC_1 + C_2 = e^2 \end{cases} \qquad \Delta = \begin{vmatrix} 2e & 1 \\ e & 1 \end{vmatrix} = e \quad \Delta_1 = \begin{vmatrix} e^2 & 1 \\ e^2 & 1 \end{vmatrix} = 0 \quad \Delta_2 = \begin{vmatrix} 2e & e^2 \\ e & e^2 \end{vmatrix} = e^3$$

$$\underline{C_1 = 0, \quad C_2 = e^2}$$

$$\underline{y(t) = e^{2t+2}}$$

Exercise

Find a particular solution satisfying the given initial conditions

$$ty'' + y' = 0\ ; \quad y_1(t) = \ln t, \;\; y_2(t) = \ln 3t; \quad y(3) = 0, \;\; y'(3) = 3$$

Solution

$$W = \begin{vmatrix} \ln t & \ln 3t \\ \frac{1}{t} & \frac{1}{t} \end{vmatrix}$$

$$= \frac{1}{t}(\ln t - \ln 3t) \;\; \neq 0$$

$\therefore\ y_1$ *and* y_2 are linearly independent.

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$$y(t) = C_1 \ln t + C_2 \ln 3t \qquad y(3) = 0 \;\rightarrow\; (\ln 3)C_1 + (\ln 9)C_2 = 0$$

$$y'(t) = \frac{C_1}{t} + \frac{C_2}{t} \qquad y'(3) = 3 \;\rightarrow\; \frac{1}{3}(C_1 + C_2) = 3$$

$$\begin{cases} (\ln 3)C_1 + (2\ln 3)C_2 = 0 \\ C_1 + C_2 = 9 \end{cases}$$

$$\Delta = \begin{vmatrix} \ln 3 & 2\ln 3 \\ 1 & 1 \end{vmatrix} = -\ln 3 \quad \Delta_1 = \begin{vmatrix} 0 & 2\ln 3 \\ 9 & 1 \end{vmatrix} = -18\ln 3 \quad \Delta_2 = \begin{vmatrix} \ln 3 & 0 \\ 1 & 9 \end{vmatrix} = 9\ln 3$$

$$\underline{C_1 = 18, \quad C_2 = -9}$$

$$\underline{y(t) = 18\ln t - 9\ln 3t}$$

Exercise

Find a particular solution satisfying the given initial conditions

$$t^2y''-ty'-3y=0\ ;\quad y_1(t)=t^3,\ \ y_2(t)=-\frac{1}{t};\quad y(-1)=0,\ \ y'(-1)=-2\quad (t<0)$$

Solution

$$W=\begin{vmatrix} t^3 & -\frac{1}{t} \\ 3t^2 & \frac{1}{t^2} \end{vmatrix}$$

$$\underline{=4t\ \ \neq 0}$$

$\therefore\ y_1$ *and* y_2 are linearly independent.

$$y(t)=C_1y_1(t)+C_2y_2(t)$$

$$y(t)=C_1t^3-\frac{C_2}{t} \qquad\qquad y(-1)=0\ \ \rightarrow\ \ -C_1+C_2=0$$

$$y'(t)=3C_1t^2+\frac{C_2}{t^2} \qquad\qquad y'(-1)=-2\ \ \rightarrow\ \ 3C_1+C_2=-2$$

$$\begin{cases} -C_1+C_2=0 \\ 3C_1+C_2=-2 \end{cases} \qquad \Delta=\begin{vmatrix} -1 & 1 \\ 3 & 1 \end{vmatrix}=-4\quad \Delta_1=\begin{vmatrix} 0 & 1 \\ -2 & 1 \end{vmatrix}=2\quad \Delta_2=\begin{vmatrix} -1 & 0 \\ 3 & -2 \end{vmatrix}=2$$

$$\underline{C_1=-\frac{1}{2},\quad C_2=-\frac{1}{2}}$$

$$\underline{y(t)=-\frac{1}{2}t^3+\frac{1}{2t}}$$

Exercise

Find a particular solution satisfying the given initial conditions

$$y''+\pi^2y=0\ ;\quad y_1(t)=\sin\pi t+\cos\pi t,\ \ y_2(t)=\sin\pi t-\cos\pi t;\quad y\left(\frac{1}{2}\right)=1,\ \ y'\left(\frac{1}{2}\right)=0$$

Solution

$$W=\begin{vmatrix} \sin\pi t+\cos\pi t & \sin\pi t-\cos\pi t \\ \pi\cos\pi t-\pi\sin\pi t & \pi\cos\pi t+\pi\sin\pi t \end{vmatrix}$$

$$=\pi\sin^2\pi t+\pi\cos^2\pi t+2\pi\sin\pi t\cos\pi t-2\pi\sin\pi t\cos\pi t+\pi\sin^2\pi t+\pi\cos^2\pi t$$

$$=2\pi\left(\sin^2\pi t+\cos^2\pi t\right)$$

$$\underline{=2\pi\ \ \neq 0}$$

$\therefore\ y_1$ *and* y_2 are linearly independent.

$y(t) = C_1 y_1(t) + C_2 y_2(t)$

$y(t) = C_1(\sin \pi t + \cos \pi t) + C_2(\sin \pi t - \cos \pi t)$ $\qquad y\left(\frac{1}{2}\right) = 1 \quad \to \quad C_1 + C_2 = 1$

$y'(t) = C_1(\pi \cos \pi t - \pi \sin \pi t) + C_2(\pi \cos \pi t + \pi \sin \pi t)$ $\qquad y'\left(\frac{1}{2}\right) = 0 \quad \to \quad -\pi C_1 + \pi C_2 = 0$

$$\begin{cases} C_1 + C_2 = 1 \\ -C_1 + C_2 = 0 \end{cases} \quad \to \quad \underline{C_1 = \frac{1}{2}, \quad C_2 = \frac{1}{2}}$$

$$y(t) = \frac{1}{2}(\sin \pi t + \cos \pi t) + \frac{1}{2}(\sin \pi t - \cos \pi t)$$

$$\underline{= \sin \pi t}$$

Exercise

Find a particular solution satisfying the given initial conditions $\quad x^3 y^{(3)} - x^2 y'' + 2xy' - 2y = 0$

$y(1) = 3, \quad y'(1) = 2, \quad y''(1) = 1 \qquad y_1(x) = x, \quad y_2(x) = x \ln x, \quad y_3(x) = x^2$

Solution

$$W = \begin{vmatrix} x & x \ln x & x^2 \\ 1 & \ln x + 1 & 2x \\ 0 & \frac{1}{x} & 2 \end{vmatrix}$$

$$= 2x \ln x + 2x + x - 2x - 2x \ln x$$

$$\underline{= x \neq 0}$$

$\therefore\ y_1, y_2, \textit{ and } y_3$ are linearly independent.

$y(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x)$

$y(x) = C_1 x + C_2 x \ln x + C_3 x^2$ $\qquad y(1) = 3 \quad \to \quad C_1 + C_3 = 3$

$y'(x) = C_1 + C_2(1 + \ln x) + 2C_3 x$ $\qquad y'(1) = 2 \quad \to \quad C_1 + C_2 + 2C_3 = 2$

$y''(x) = \frac{C_2}{x} + 2C_3$ $\qquad y''(1) = 1 \quad \to \quad C_2 + 2C_3 = 1$

$$\begin{cases} C_1 + C_3 = 3 \\ C_1 + C_2 + 2C_3 = 2 \\ C_2 + 2C_3 = 1 \end{cases} \qquad \Delta = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{vmatrix} = 1 \quad \Delta_1 = \begin{vmatrix} 3 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{vmatrix} = 1$$

$$\underline{C_1 = 1, \quad C_2 = -3, \quad C_3 = 2}$$

$$\underline{y(x) = x - 3x \ln x + 2x^2}$$

Exercise

Find a particular solution satisfying the given initial conditions $y^{(3)} + 2y'' - y' - 2y = 0$

$y(0)=1, \quad y'(0)=2, \quad y''(0)=0 \qquad y_1(x)=e^{x}, \quad y_2(x)=e^{-x}, \quad y_3(x)=e^{-2x}$

Solution

$$W = \begin{vmatrix} e^{x} & e^{-x} & e^{-2x} \\ e^{x} & -e^{-x} & -2e^{-2x} \\ e^{x} & e^{-x} & 4e^{-2x} \end{vmatrix}$$

$$= -4e^{-2x} - 2e^{-2x} + e^{-2x} + e^{-2x} + 2e^{-2x} - 4e^{-2x}$$

$$= -6e^{-2x} \neq 0$$

$\therefore\ y_1, y_2, \textit{ and } y_3$ are linearly independent. $\quad y(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x)$

$$y(x) = C_1 e^{x} + C_2 e^{-x} + C_3 e^{-2x} \qquad y(0)=1 \ \rightarrow\ C_1 + C_2 + C_3 = 1$$

$$y'(x) = C_1 e^{x} - C_2 e^{-x} - 2C_3 e^{-2x} \qquad y'(0)=2 \ \rightarrow\ C_1 - C_2 - 2C_3 = 2$$

$$y''(x) = C_1 e^{x} + C_2 e^{-x} + 4C_3 e^{-2x} \qquad y''(0)=0 \ \rightarrow\ C_1 + C_2 + 4C_3 = 0$$

$$\begin{cases} C_1 + C_2 + C_3 = 1 \\ C_1 - C_2 - 2C_3 = 2 \\ C_1 + C_2 + 4C_3 = 0 \end{cases} \qquad \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{vmatrix} = -9 \quad \Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 0 & 1 & 4 \end{vmatrix} = -12 \quad \Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -2 \\ 1 & 0 & 4 \end{vmatrix} = 0$$

$$C_1 = \frac{4}{3}, \quad C_2 = 0, \quad C_3 = -\frac{1}{3}$$

$$y(x) = \frac{4}{3}e^{x} - \frac{1}{3}e^{-2x}$$

Exercise

Find a particular solution satisfying the given initial conditions $y^{(3)} - 6y'' + 11y' - 6y = 0$

$y(0)=0, \quad y'(0)=0, \quad y''(0)=3 \qquad y_1(x)=e^{x}, \quad y_2(x)=e^{2x}, \quad y_3(x)=e^{3x}$

Solution

$$W = \begin{vmatrix} e^{x} & e^{2x} & e^{3x} \\ e^{x} & 2e^{2x} & 3e^{3x} \\ e^{x} & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$= 18e^{6x} + 3e^{6x} + 4e^{6x} - 2e^{6x} - 12e^{6x} - 9e^{6x}$$

$$\boxed{= 2e^{6x} \neq 0}$$

$\therefore\ y_1, y_2, \text{ and } y_3$ are linearly independent.

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x)$$

$$y(x) = C_1 e^{x} + C_2 e^{2x} + C_3 e^{3x} \qquad y(0) = 0 \;\rightarrow\; C_1 + C_2 + C_3 = 0$$

$$y'(x) = C_1 e^{x} + 2C_2 e^{2x} + 3C_3 e^{3x} \qquad y'(0) = 0 \;\rightarrow\; C_1 + 2C_2 + 3C_3 = 0$$

$$y''(x) = C_1 e^{x} + 4C_2 e^{2x} + 9C_3 e^{3x} \qquad y''(0) = 3 \;\rightarrow\; C_1 + 4C_2 + 9C_3 = 3$$

$$\begin{cases} C_1 + C_2 + C_3 = 0 \\ C_1 + 2C_2 + 3C_3 = 0 \\ C_1 + 4C_2 + 9C_3 = 3 \end{cases} \qquad \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 2 \quad \Delta_1 = \begin{vmatrix} 0 & 1 & 1 \\ 0 & 2 & 3 \\ 3 & 4 & 9 \end{vmatrix} = 3 \quad \Delta_2 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 9 \end{vmatrix} = -6$$

$$\boxed{C_1 = \frac{3}{2}, \quad C_2 = -3, \quad C_3 = \frac{3}{2}}$$

$$\boxed{y(x) = \frac{3}{2} e^{x} - 3e^{2x} + \frac{3}{2} e^{3x}}$$

Exercise

Find a particular solution satisfying the given initial conditions $\quad y^{(3)} - 3y'' + 3y' - y = 0$

$y(0) = 2, \quad y'(0) = 0, \quad y''(0) = 0 \qquad y_1(x) = e^{x}, \quad y_2(x) = xe^{x}, \quad y_3(x) = x^2 e^{x}$

Solution

$$W = \begin{vmatrix} e^{x} & xe^{x} & x^2 e^{x} \\ e^{x} & (1+x)e^{x} & \left(2x + x^2\right)e^{x} \\ e^{x} & (2+x)e^{x} & \left(2 + 4x + x^2\right)e^{x} \end{vmatrix}$$

$$= \left(2 + 6x + 5x^2 + x^3 + 2x^2 + x^3 + 2x^2 + x^3 - x^2 - x^3 - 2x - 4x^2 - x^3 - 2x - 4x^2 - x^3\right)e^{3x}$$

$$\boxed{= (2 + 2x)e^{3x} \neq 0}$$

$\therefore\ y_1, y_2, \text{ and } y_3$ are linearly independent.

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x)$$

$$y(x) = C_1 e^{x} + C_2 xe^{x} + C_3 x^2 e^{x} \qquad y(0) = 2 \;\rightarrow\; C_1 = 2$$

$$y'(x) = C_1 e^{x} + C_2 (1+x)e^{x} + C_3 \left(2x + x^2\right)e^{x} \qquad y'(0) = 0 \;\rightarrow\; C_1 + C_2 = 0$$

$$y''(x)=C_1e^x+C_2(2+x)e^x+C_3\left(2+4x+x^2\right)e^x \qquad y''(0)=0 \quad\to\quad C_1+2C_2+2C_3=0$$

$$\underline{C_1=2,\quad C_2=-2,\quad C_3=1}$$

$$\underline{y(x)=2e^x-2xe^x+x^2e^x}$$

Exercise

Find a particular solution satisfying the given initial conditions $\quad y^{(3)}-5y''+8y'-4y=0$

$y(0)=1, \quad y'(0)=4, \quad y''(0)=0 \qquad y_1(x)=e^x, \quad y_2(x)=e^{2x}, \quad y_3(x)=xe^{2x}$

Solution

$$W=\begin{vmatrix} e^x & e^{2x} & xe^{2x} \\ e^x & 2e^{2x} & (1+2x)e^{2x} \\ e^x & 4e^{2x} & (4+4x)e^{2x} \end{vmatrix}$$

$$=(8+8x+4+8x+4x-2x-4-8x-4-4x)e^{5x}$$

$$\underline{=(6x+4)e^{5x} \neq 0}$$

$\therefore\ y_1, y_2, \text{ and } y_3$ are linearly independent.

$$y(x)=C_1y_1(x)+C_2y_2(x)+C_3y_3(x)$$

$$y(x)=C_1e^x+C_2e^{2x}+C_3xe^{2x} \qquad y(0)=1 \quad\to\quad C_1+C_2=1$$

$$y'(x)=C_1e^x+2C_2e^{2x}+C_3(1+2x)e^{2x} \qquad y'(0)=4 \quad\to\quad C_1+2C_2+C_3=4$$

$$y''(x)=C_1e^x+4C_2e^{2x}+(4+4x)C_3e^{2x} \qquad y''(0)=0 \quad\to\quad C_1+4C_2+4C_3=0$$

$$\begin{cases} C_1+C_2=1 \\ C_1+2C_2+C_3=4 \\ C_1+4C_2+4C_3=0 \end{cases}$$

$$\Delta=\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 4 & 4 \end{vmatrix}=1 \quad \Delta_1=\begin{vmatrix} 1 & 1 & 0 \\ 4 & 2 & 1 \\ 0 & 4 & 4 \end{vmatrix}=-12 \quad \Delta_2=\begin{vmatrix} 1 & 1 & 0 \\ 1 & 4 & 1 \\ 1 & 0 & 4 \end{vmatrix}=13$$

$$\underline{C_1=-12,\quad C_2=13,\quad C_3=-10}$$

$$\underline{y(x)=-12e^x+13e^{2x}-10xe^{2x}}$$

Exercise

Find a particular solution satisfying the given initial conditions $y^{(3)}+9y''=0$

$y(0)=3,\quad y'(0)=-1,\quad y''(0)=2 \qquad y_1(x)=1,\quad y_2(x)=\cos 3x,\quad y_3(x)=\sin 3x$

Solution

$$W=\begin{vmatrix}1 & \cos 3x & \sin 3x\\ 0 & -3\sin 3x & 3\cos 3x\\ 0 & -9\cos 3x & -9\sin 3x\end{vmatrix}$$

$$=27\sin^2 3x+27\cos^2 3x$$

$$=27 \neq 0$$

$\therefore\ y_1,\ y_2,\ and\ y_3$ are linearly independent.

$$y(x)=C_1y_1(x)+C_2y_2(x)+C_3y_3(x)$$

$$y(x)=C_1+C_2\cos 3x+C_3\sin 3x \qquad y(0)=3 \ \rightarrow\ C_1+C_2=3$$

$$y'(x)=-3C_2\sin 3x+3C_3\cos 3x \qquad y'(0)=-1 \ \rightarrow\ 3C_3=-1$$

$$y''(x)=-9C_2\cos 3x-9C_3\sin 3x \qquad y''(0)=0 \ \rightarrow\ -9C_2=0$$

$$C_1=3,\quad C_2=0,\quad C_3=-\frac{1}{3}$$

$$y(x)=3-\frac{1}{3}\sin 3x$$

Exercise

Find a particular solution satisfying the given initial conditions $y^{(3)}-3y''+4y'-2y=0$

$y(0)=1,\quad y'(0)=0,\quad y''(0)=0 \qquad y_1(x)=e^x,\quad y_2(x)=e^x\cos x,\quad y_3(x)=e^x\sin x$

Solution

$$W=\begin{vmatrix}e^x & e^x\cos x & e^x\sin x\\ e^x & (\cos x-\sin x)e^x & (\sin x+\cos x)e^x\\ e^x & -2e^x\sin x & 2e^x\cos x\end{vmatrix}$$

$$=\left(2\cos^2 x-\sin x\cos x+\cos^2 x-2\sin^2 x-\sin x\cos x+\sin^2 x+2\sin^2 x+2\sin x\cos x-2\cos^2 x\right)e^{3x}$$

$$=e^{3x} \neq 0$$

$\therefore\ y_1,\ y_2,\ and\ y_3$ are linearly independent.

$$y(x)=C_1y_1(x)+C_2y_2(x)+C_3y_3(x)$$

$$y(x)=C_1e^x+C_2e^x\cos x+C_3e^x\sin x \qquad y(0)=1 \ \rightarrow\ C_1+C_2=1$$

$$y'(x)=C_1e^x+C_2(\cos x-\sin x)e^x+C_3(\sin x+\cos x)e^x \qquad y'(0)=0 \ \rightarrow\ C_1+C_2+C_3=0$$

$$y''(x)=C_1e^x-2C_2e^x\sin x+2C_3e^x\cos x \qquad y''(0)=0 \ \rightarrow\ C_1+2C_3=0$$

$$\begin{cases} C_1+C_2=1 \\ C_1+C_2+C_3=0 \\ C_1+2C_3=0 \end{cases} \qquad \Delta=\begin{vmatrix}1&1&0\\1&1&1\\1&0&2\end{vmatrix}=1 \quad \Delta_1=\begin{vmatrix}1&1&0\\0&1&1\\0&0&2\end{vmatrix}=2$$

$$\underline{C_1=2, \quad C_2=-1, \quad C_3=-1}$$

$$\boxed{y(x)=2e^x-e^x\cos x-e^x\sin x}$$

Exercise

Find a particular solution satisfying the given initial conditions $x^3y^{(3)}-3x^2y''+6xy'-6y=0$

$y(1)=6, \quad y'(1)=14, \quad y''(1)=1 \qquad y_1(x)=x, \quad y_2(x)=x^2, \quad y_3(x)=x^3$

Solution

$$W=\begin{vmatrix}x&x^2&x^3\\1&2x&3x^2\\0&2&6x\end{vmatrix}$$

$$=12x^3+2x^3-6x^3-6x^3$$

$$\boxed{=2x^3 \neq 0}$$

$\therefore\ y_1, y_2,$ *and* y_3 are linearly independent.

$$y(x)=C_1y_1(x)+C_2y_2(x)+C_3y_3(x)$$

$$y(x)=C_1x+C_2x^2+C_3x^3 \qquad y(1)=1 \ \rightarrow\ C_1+C_2+C_3=6$$

$$y'(x)=C_1+2C_2x+3C_3x^2 \qquad y'(1)=14 \ \rightarrow\ C_1+2C_2+3C_3=14$$

$$y''(x)=2C_2+6C_3x \qquad y''(1)=1 \ \rightarrow\ 2C_2+6C_3=1$$

$$\begin{cases} C_1+C_2+C_3=6 \\ C_1+2C_2+3C_3=14 \\ 2C_2+6C_3=1 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix} = 2 \quad \Delta_1 = \begin{vmatrix} 6 & 1 & 1 \\ 14 & 2 & 3 \\ 1 & 2 & 6 \end{vmatrix} = -19 \quad \Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 14 & 3 \\ 0 & 1 & 6 \end{vmatrix} = 46$$

$$\underline{C_1 = -\frac{19}{2}, \quad C_2 = 23, \quad C_3 = -\frac{15}{2}}$$

$$\underline{y(x) = -\frac{19}{2}x + 23x^2 - \frac{15}{2}x^3}$$

Exercise

Find a particular solution satisfying the given initial conditions $x^3y^{(3)} + 6x^2y'' + 4xy' - 4y = 0$

$y(1) = 1, \quad y'(1) = 5, \quad y''(1) = -11 \qquad y_1(x) = x, \quad y_2(x) = x^{-2}, \quad y_3(x) = x^{-2}\ln x$

Solution

$$W = \begin{vmatrix} x & x^{-2} & x^{-2}\ln x \\ 1 & -2x^{-3} & (1-2\ln x)x^{-3} \\ 0 & 6x^{-4} & (-5+6\ln x)x^{-4} \end{vmatrix}$$

$$= (10 - 12\ln x + 6 - 6 + 12\ln x + 5 - 6\ln x)x^{-6}$$

$$\underline{= (15 - 6\ln x)x^{-6} \neq 0}$$

$\therefore\ y_1,\ y_2,\ and\ y_3$ are linearly independent.

$$y(x) = C_1y_1(x) + C_2y_2(x) + C_3y_3(x)$$

$$y(x) = C_1x + C_2x^{-2} + C_3x^{-2}\ln x \qquad y(1) = 1 \ \rightarrow\ C_1 + C_2 = 1$$

$$y'(x) = C_1 - 2C_2x^{-3} + C_3x^{-3}(1 - 2\ln x) \qquad y'(1) = 5 \ \rightarrow\ C_1 - 2C_2 + C_3 = 5$$

$$y''(x) = 6C_2x^{-4} + C_3x^{-4}(-5 + 6\ln x) \qquad y''(1) = -11 \ \rightarrow\ 6C_2 - 5C_3 = -11$$

$$\begin{cases} C_1 + C_2 = 1 \\ C_1 - 2C_2 + C_3 = 5 \\ 6C_2 - 5C_3 = -11 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 6 & -5 \end{vmatrix} = 9 \quad \Delta_1 = \begin{vmatrix} 1 & 1 & 0 \\ 5 & -2 & 1 \\ -11 & 6 & -5 \end{vmatrix} = 18 \quad \Delta_2 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 5 & 1 \\ 0 & -11 & -5 \end{vmatrix} = -9$$

$$\underline{C_1 = 2, \quad C_2 = -1, \quad C_3 = 1}$$

$$\underline{y(x) = 2x - x^{-2} + x^{-2}\ln x}$$

Exercise

Given the mass, damping, and spring constants of an undriven spring-mass system $my'' + \mu y' + ky = 0$

$$m = 1\ kg, \quad \mu = 0\ kg\,/\,s, \quad k = 4kg\,/\,s^2, \quad y(0) = -2\ m, \quad y'(0) = -2\ m\,/\,s$$

a) Provide separate plots of the position versus time (y vs. t) and the velocity versus time (v vs. t)

b) Provide a combined plot of both position and velocity versus time

c) Provide a plot of the velocity versus position (v vs. y) in the yv phase plane.

Solution

$$my'' = -\mu y' - ky$$

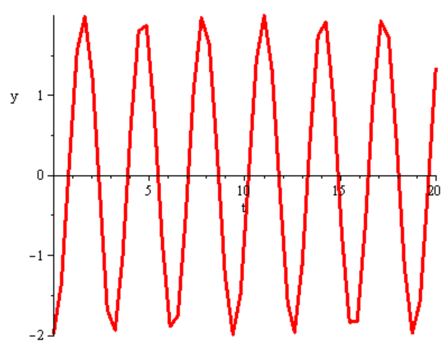
$$y'' = -\frac{\mu}{m}y' - \frac{k}{m}y$$

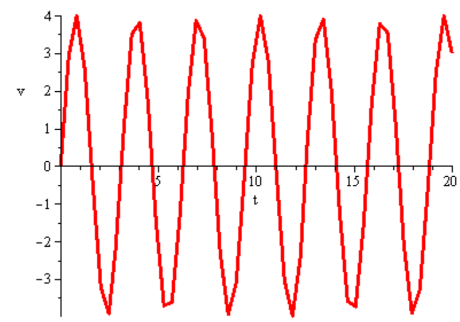
Let $\boxed{v = y'}$ $\Rightarrow v' = -\frac{\mu}{m}v - \frac{k}{m}y$

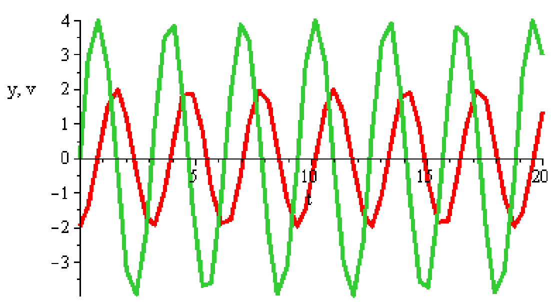
$$= -\frac{0}{1}v - \frac{4}{1}y$$

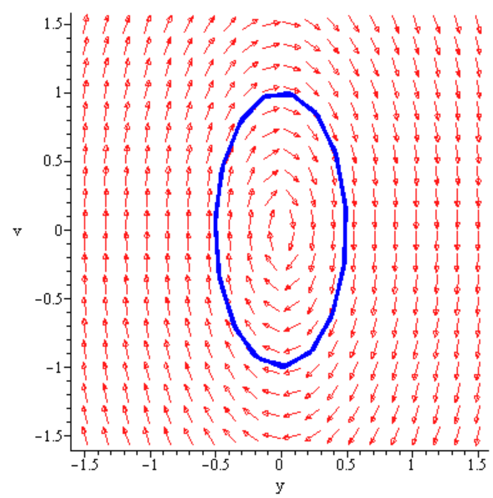
$$\boxed{v' = -4y}$$

$$y(0) = -2, \quad y'(0) = -2 = v(0)$$









Exercise

Given the mass, damping, and spring constants of an undriven spring-mass system $my'' + \mu y' + ky = 0$

$$m = 1\ kg, \quad \mu = 2\ kg/s, \quad k = 1 kg/s^2, \quad y(0) = -3\ m, \quad y'(0) = -2\ m/s$$

a) Provide separate plots of the position versus time (y vs. t) and the velocity versus time (v vs. t)

b) Provide a combined plot of both position and velocity versus time

c) Provide a plot of the velocity versus position (v vs. y) in the yv phase plane.

Solution

$$my'' = -\mu y' - ky \quad \Rightarrow \quad y'' = -\frac{\mu}{m}y' - \frac{k}{m}y$$

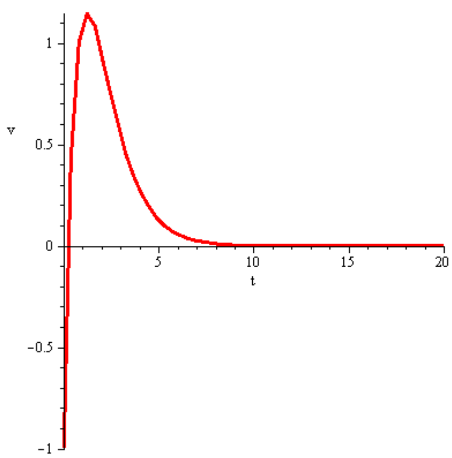
Let $v = y'$ $\Rightarrow v' = -\frac{\mu}{m}v - \frac{k}{m}y$

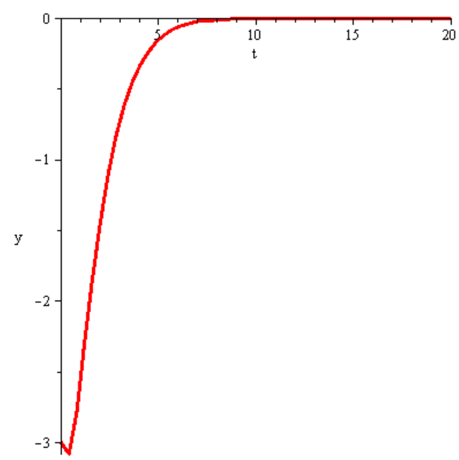
$$= -\frac{2}{1}v - \frac{1}{1}y$$

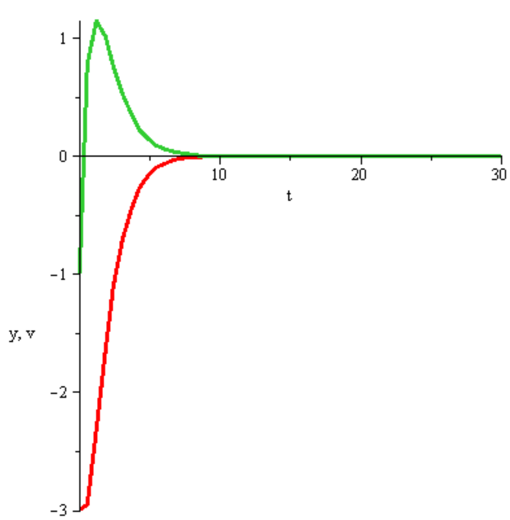
$$= -2v - y$$

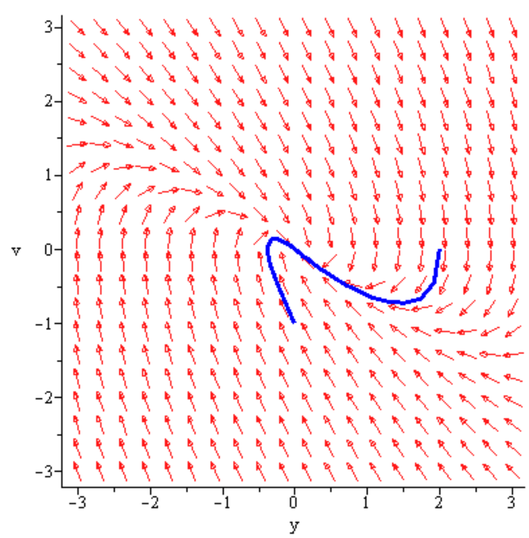
The following system of the first-order equations:

$$\begin{cases} y' = v \\ v' = -2v - y \end{cases} \quad with \quad \begin{aligned} y(0) &= -3 \\ y'(0) &= -2 = v(0) \end{aligned}$$









Exercise

When the values of a solution to a differential equation are specified at two different points, these conditions. (In contrast, initial conditions specify the values of a function and its derivative at the same point). The purpose of this is to show that for boundary value problems there is no existence-uniqueness theorem. Given that every solution to

$$y'' + y = 0 \qquad \text{is of the form} \qquad y(t) = c_1 \cos t + c_2 \sin t$$

Where c_1 and c_2 are arbitrary constants, show that

a) There is a unique solution to the given differential equation that satisfies the boundary conditions $y(0) = 2$ and $y\left(\frac{\pi}{2}\right) = 0$

b) There is no solution to given equation that satisfies $y(2) = 0$ and $y(\pi) = 0$

c) There are infinitely many solution to the given DE equation that satisfy $y(0) = 2$ and $y(\pi) = -2$

Solution

a) $\lambda^2 + 1 = 0 \quad \rightarrow \quad \underline{\lambda = \pm i}$

$y(t) = c_1 \cos t + c_2 \sin t$

$y(0) = 2 \quad \rightarrow \quad \underline{2 = c_1}$

$y\left(\frac{\pi}{2}\right) = 0 \quad \rightarrow \quad \underline{0 = c_2}$

$\underline{y(t) = 2\cos t}$

b) $y(0) = 2 \quad \rightarrow \quad \underline{2 = c_1}$

$y(\pi) = 0 \quad \rightarrow \quad \underline{0 = -c_1}$

This system is inconsistent, so there is no solution satisfying the given boundary.

c) $y(0) = 2 \quad \rightarrow \quad \underline{2 = c_1}$

$y(\pi) = -2 \quad \rightarrow \quad \underline{-2 = -c_1}$

$\underline{y(t) = 2\cos t + c_2 \sin t}$

Which has infinitely many solutions given c_2 is an arbitrary constant.

Solution Section 2.2 – Linear, Homogeneous Equations with Constant Coefficients

Exercise

Find the general solution: $y'' + y' = 0$

Solution

The characteristic equation: $\lambda^2 + \lambda = 0 \quad \rightarrow \quad \lambda_{1,2} = 0,\ -1$

$$y(x) = C_1 + C_2 e^{-x}$$

Exercise

Find the general solution: $y'' - 4y = 0$

Solution

The characteristic equation: $\lambda^2 - 4 = 0 \quad \rightarrow \quad \lambda_{1,2} = \pm 2$

$$y(x) = C_1 e^{-2x} + C_2 e^{2x}$$

Exercise

Find the general solution: $y'' + 8y = 0$

Solution

The characteristic equation: $\lambda^2 + 8\lambda = 0 \quad \rightarrow \quad \lambda_{1,2} = 0,\ 8$

$$y(x) = C_1 + C_2 e^{8x}$$

Exercise

Find the general solution: $y'' - 36y = 0$

Solution

The characteristic equation: $\lambda^2 - 36 = 0 \quad \rightarrow \quad \lambda_{1,2} = \pm 6$

$$y(x) = C_1 e^{-6x} + C_2 e^{6x}$$

Exercise

Find the general solution: $y'' + 9y = 0$

Solution

The characteristic equation: $\lambda^2 + 9 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 3i}$

$\boxed{y(x) = C_1 \cos 3x + C_2 \sin 3x}$

Exercise

Find the general solution: $y'' + 16y = 0$

Solution

The characteristic equation: $\lambda^2 + 16 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 4i}$

$\boxed{y(x) = C_1 \cos 4x + C_2 \sin 4x}$

Exercise

Find the general solution: $y'' + 25y = 0$

Solution

The characteristic equation: $\lambda^2 + 25 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 5i}$

$\boxed{y(x) = C_1 \cos 5x + C_2 \sin 5x}$

Exercise

Find the general solution: $y'' - 64y = 0$

Solution

The characteristic equation: $\lambda^2 - 64 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 8}$

$\boxed{y(x) = C_1 e^{-8x} + C_2 e^{8x}}$

Exercise

Find the general solution: $y'' + y' + y = 0$

Solution

The characteristic equation: $\lambda^2+\lambda+1=0$

$$\Rightarrow \quad \lambda_{1,2}=\frac{-1\pm\sqrt{1-4}}{2}=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$$

$$\boxed{y(x)=e^{-x/2}\left(C_1\cos\frac{\sqrt{3}}{2}x+C_2\sin\frac{\sqrt{3}}{2}x\right)}$$

Exercise

Find the general solution: $\quad y''+y'-y=0$

Solution

The characteristic equation: $\quad \lambda^2+\lambda-1=0$

$$\rightarrow \quad \boxed{\lambda_{1,2}=\frac{-1\pm\sqrt{5}}{2}}$$

$$\boxed{y(x)=C_1e^{\frac{-1-\sqrt{5}}{2}x}+C_2e^{\frac{-1+\sqrt{5}}{2}x}}$$

Exercise

Find the general solution: $\quad y''-y'-2y=0$

Solution

The characteristic equation: $\quad \lambda^2-\lambda-2=0 \quad \rightarrow \quad \boxed{\lambda_1=-1,\ \lambda_2=2}$

$$\boxed{y(x)=C_1e^{-x}+C_2e^{2x}}$$

Exercise

Find the general solution: $\quad y''-y'-6y=0$

Solution

The characteristic equation: $\quad \lambda^2-\lambda-6=0$

$$\rightarrow \quad \lambda_{1,2}=\frac{1\pm 5}{2}\ \boxed{=-2,\ 3}$$

$$\boxed{y(x)=C_1e^{-2x}+C_2e^{3x}}$$

Exercise

Find the general solution: $y'' + y' - 6y = 0$

Solution

The characteristic equation: $\lambda^2 + \lambda - 6 = 0$

$$\rightarrow \quad \lambda_{1,2} = \frac{-1 \pm 5}{2} = -3,\ 2$$

$$y(x) = C_1 e^{-3x} + C_2 e^{2x}$$

Exercise

Find the general solution: $y'' - y' - 11y = 0$

Solution

The characteristic equation: $\lambda^2 - \lambda - 11 = 0$

$$\rightarrow \quad \lambda_{1,2} = \frac{1 \pm \sqrt{45}}{2} = \frac{1 \pm 3\sqrt{5}}{2}$$

$$y(x) = C_1 e^{\frac{1-3\sqrt{5}}{2}x} + C_2 e^{\frac{1+3\sqrt{5}}{2}x}$$

Exercise

Find the general solution: $y'' - y' - 12y = 0$

Solution

The characteristic equation: $\lambda^2 - \lambda - 12 = 0$

$$\Rightarrow \lambda_{1,2} = -3,\ 4$$

$$y(t) = C_1 e^{-3t} + C_2 e^{4t}$$

Exercise

Find the general solution: $y'' + 2y' + y = 0$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 1 = 0$

$$\Rightarrow \lambda_{1,2} = -1$$

$$y(t) = \left(C_1 + C_2 t\right)e^{-t}$$

Exercise

Find the general solution: $y'' + 2y' + 3y = 0$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 3 = 0$

$$\Rightarrow \lambda_{1,2} = \frac{-2 \pm 2i\sqrt{2}}{2} = -1 \pm i\sqrt{2}$$

$$y(x) = e^{-x}\left(C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x\right)$$

Exercise

Find the general solution: $y'' + 2y' - 3y = 0$

Solution

The characteristic equation: $\lambda^2 + 2\lambda - 3 = 0$

$$\Rightarrow \lambda_{1,2} = 1, \ -3$$

$$y(x) = C_1 e^{-3x} + C_2 e^{x}$$

Exercise

Find the general solution: $y'' - 2y' - 3y = 0$

Solution

The characteristic equation: $\lambda^2 - 2\lambda - 3 = 0$

$$\Rightarrow \lambda_{1,2} = -1, \ 3$$

$$y(x) = C_1 e^{-x} + C_2 e^{3x}$$

Exercise

Find the general solution: $y'' - 2y' + 3y = 0$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 3 = 0$

$$\Rightarrow \lambda_{1,2} = \frac{2 \pm 2i\sqrt{2}}{2} = 1 \pm i\sqrt{2}$$

$$y(x) = e^{x}\left(C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x\right)$$

Exercise

Find the general solution: $y'' + 2y' + 4y = 0$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 4 = 0$

$$\Rightarrow \lambda_{1,2} = \frac{-2 \pm 2i\sqrt{3}}{2} = \underline{-1 \pm i\sqrt{3}}$$

$$\underline{y(x) = e^{-x}\left(C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x\right)}$$

Exercise

Find the general solution: $y'' + 2y' - 15y = 0$

Solution

The characteristic equation: $\lambda^2 + 2\lambda - 15 = 0$

$$\Rightarrow \lambda_{1,2} = \frac{-2 \pm 8}{2} = \underline{-5,\ 3}$$

$$\underline{y(x) = C_1 e^{-5x} + C_2 e^{3x}}$$

Exercise

Find the general solution: $y'' + 2y' + 17y = 0$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 17 = 0$

$$\Rightarrow \underline{\lambda_{1,2} = -1 \pm 4i}$$

$$\underline{y(t) = e^{-t}\left(C_1 \cos 4t + C_2 \sin 4t\right)}$$

Exercise

Find the general solution: $y'' - 2y' + 5y = 0$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 5 = 0$

$$\Rightarrow \lambda_{1,2} = \frac{-2 \pm 4i}{2} = \underline{-1 \pm 2i}$$

$$\underline{y(x) = e^{-x}\left(C_1 \cos 2x + C_2 \sin 2x\right)}$$

Exercise

Find the general solution: $y'' - 3y' + 2y = 0$

Solution

The characteristic equation: $\lambda^2 - 3\lambda + 2 = 0$

$\rightarrow \quad \boxed{\lambda_{1,2} = 1,\ 2}$

$\boxed{y(x) = C_1 e^{x} + C_2 e^{2x}}$

Exercise

Find the general solution: $y'' + 3y' - 4y = 0$

Solution

The characteristic equation: $\lambda^2 + 3\lambda - 4 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = 1,\ -4}$

$\boxed{y(x) = C_1 e^{x} + C_2 e^{-4x}}$

Exercise

Find the general solution: $y'' + 4y' - y = 0$

Solution

The characteristic equation: $\lambda^2 + 4\lambda - 1 = 0$

$\rightarrow \quad \lambda_{1,2} = \frac{-4 \pm 2\sqrt{5}}{2} \boxed{= -2 \pm \sqrt{5}}$

$\boxed{y(x) = C_1 e^{\left(-2-\sqrt{5}\right)x} + C_2 e^{\left(-2+\sqrt{5}\right)x}}$

Exercise

Find the general solution: $y'' - 4y' + 4y = 0$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 4 = 0 \quad \Rightarrow \lambda_{1,2} = 2$

$\boxed{y(t) = \left(C_1 + C_2 t\right)e^{2t}}$

Exercise

Find the general solution: $y'' + 4y' + 4y = 0$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 4 = 0 \quad \Rightarrow \lambda_{1,2} = -2$

$$\boxed{y(t) = \left(C_1 + C_2 t\right)e^{-2t}}$$

Exercise

Find the general solution: $y'' - 4y' + 5y = 0$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 5 = 0$

$\rightarrow \quad \lambda_{1,2} = \frac{4 \pm 2i}{2} \underline{= 2 \pm i}$

$$\boxed{y(x) = e^{2x}\left(C_1 \cos x + C_2 \sin x\right)}$$

Exercise

Find the general solution: $y'' + 4y' + 5y = 0$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 5 = 0$

$\rightarrow \quad \lambda_{1,2} = \frac{-4 \pm 2i}{2} \underline{= -2 \pm i}$

$$\boxed{y(x) = e^{-2x}\left(C_1 \cos x + C_2 \sin x\right)}$$

Exercise

Find the general solution: $y'' + 4y' - 5y = 0$

Solution

The characteristic equation: $\lambda^2 + 4\lambda - 5 = 0 \quad \rightarrow \quad \lambda_{1,2} \underline{= -5,\ 1}$

$$\boxed{y(x) = C_1 e^{-5x} + C_2 e^{x}}$$

Exercise

Find the general solution: $y'' + 4y' + 7y = 0$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 7 = 0$

$$\rightarrow \quad \lambda_{1,2} = \frac{-4 \pm 2i\sqrt{3}}{2} = -2 \pm i\sqrt{3}$$

$$y(x) = e^{-2x}\left(C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x\right)$$

Exercise

Find the general solution: $y'' + 4y' + 9y = 0$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 9 = 0$

$$\rightarrow \quad \lambda_{1,2} = \frac{-4 \pm 2i\sqrt{5}}{2} = -2 \pm i\sqrt{5}$$

$$y(x) = e^{-2x}\left(C_1 \cos\sqrt{5}x + C_2 \sin\sqrt{5}x\right)$$

Exercise

Find the general solution: $y'' + 5y' = 0$

Solution

The characteristic equation: $\lambda^2 + 5\lambda = 0 \quad \rightarrow \quad \lambda_1 = -5,\ \lambda_2 = 0$

$$y(x) = C_1 e^{-5x} + C_2$$

Exercise

Find the general solution: $y'' + 5y' + 6y = 0$

Solution

The characteristic equation: $\lambda^2 + 5\lambda + 6 = 0 \quad \rightarrow \quad \lambda_1 = -3,\ \lambda_2 = -2$

$$y(x) = C_1 e^{-3x} + C_2 e^{-2x}$$

Exercise

Find the general solution: $y'' + 6y' + 9y = 0$

Solution

The characteristic equation: $\lambda^2 + 6\lambda + 9 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -3}$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{-3x}}$$

Exercise

Find the general solution: $y'' - 6y' + 9y = 0$

Solution

The characteristic equation: $\lambda^2 - 6\lambda + 9 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = 3}$

$$\boxed{y(t) = \left(C_1 + C_2 t\right)e^{3t}}$$

Exercise

Find the general solution: $y'' - 6y' + 25y = 0$

Solution

The characteristic equation: $\lambda^2 - 6\lambda + 25 = 0$

$\Rightarrow \lambda_{1,2} = \frac{6 \pm 8i}{2} \boxed{= 3 \pm 4i}$

$$\boxed{y(x) = e^{3x}\left(C_1 \cos 4x + C_2 \sin 4x\right)}$$

Exercise

Find the general solution: $y'' + 8y' + 16y = 0$

Solution

The characteristic equation: $\lambda^2 + 8\lambda + 16 = (\lambda + 4)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -4}$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{-4x}}$$

Exercise

Find the general solution: $y'' + 8y' - 16y = 0$

Solution

The characteristic equation: $\lambda^2 + 8\lambda - 16 = 0$

$$\rightarrow \quad \lambda_{1,2} = \frac{-8 \pm 8\sqrt{2}}{2} = -4 \pm 4\sqrt{2}$$

$$y(x) = C_1 e^{\left(-4-4\sqrt{2}\right)x} + C_2 e^{\left(-4+4\sqrt{2}\right)x}$$

Exercise

Find the general solution: $y'' - 9y' + 20y = 0$

Solution

The characteristic equation: $\lambda^2 - 9\lambda + 20 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{9 \pm 1}{2} = 4,\ 5$

$$y(x) = C_1 e^{4x} + C_2 e^{5x}$$

Exercise

Find the general solution: $y'' - 10y' + 25y = 0$

Solution

The characteristic equation: $\lambda^2 - 10\lambda + 25 = (\lambda - 5)^2 = 0 \quad \rightarrow \quad \lambda_{1,2} = 5$

$$y(x) = \left(C_1 + C_2 x\right) e^{5x}$$

Exercise

Find the general solution: $y'' + 14y' + 49y = 0$

Solution

The characteristic equation: $\lambda^2 + 14\lambda + 49 = (\lambda + 7)^2 = 0 \quad \Rightarrow \quad \lambda_{1,2} = -7$

$$y(x) = \left(C_1 + C_2 x\right) e^{-7x}$$

Exercise

Find the general solution: $2y'' - y' - 3y = 0$

Solution

The characteristic equation: $2\lambda^2 - \lambda - 3 = 0 \quad \rightarrow \quad \lambda_{1,2} = -1, \ \frac{3}{2}$

$$y(x) = C_1 e^{-x} + C_2 e^{3x/2}$$

Exercise

Find the general solution: $2y'' + y' - y = 0$

Solution

The characteristic equation: $2\lambda^2 + \lambda - 1 = 0 \quad \rightarrow \quad \lambda_{1,2} = -1, \ \frac{1}{2}$

$$y(x) = C_1 e^{-x} + C_2 e^{x/2}$$

Exercise

Find the general solution: $2y'' + 2y' + y = 0$

Solution

The characteristic equation: $2\lambda^2 + 2\lambda + 1 = 0$

$$\rightarrow \quad \lambda_{1,2} = \frac{-2 \pm 2i}{4} = \frac{1}{2} \pm \frac{1}{2}i$$

$$y(x) = e^{x/2}\left(C_1 \cos\frac{1}{2}x + C_2 \sin\frac{1}{2}x\right)$$

Exercise

Find the general solution: $2y'' + 2y' + 3y = 0$

Solution

The characteristic equation: $2\lambda^2 + 2\lambda + 3 = 0$

$$\rightarrow \quad \lambda_{1,2} = \frac{-2 \pm 2i\sqrt{5}}{4} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}i$$

$$y(x) = e^{-x/2}\left(C_1 \cos\frac{\sqrt{5}}{2}x + C_2 \sin\frac{\sqrt{5}}{2}x\right)$$

Exercise

Find the general solution: $2y'' - 3y' - 2y = 0$

Solution

The characteristic equation: $2\lambda^2 - 3\lambda - 2 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{3 \pm 5}{4} = -\frac{1}{2},\ 2$

$$y(x) = C_1 e^{-x/2} + C_2 e^{2x}$$

Exercise

Find the general solution: $2y'' - 3y' + 4y = 0$

Solution

The characteristic equation: $2\lambda^2 - 3\lambda + 4 = 0$

$$\rightarrow \quad \lambda_{1,2} = \frac{3 \pm i\sqrt{23}}{4} = \frac{3}{4} \pm \frac{\sqrt{23}}{4} i$$

$$y(x) = e^{3x/4}\left(C_1 \cos\frac{\sqrt{23}}{4}x + C_2 \sin\frac{\sqrt{23}}{4}x\right)$$

Exercise

Find the general solution: $2y'' - 4y' + 8y = 0$

Solution

The characteristic equation: $2\lambda^2 - 4\lambda + 8 = 0$

$$\rightarrow \quad \lambda_{1,2} = \frac{4 \pm 4i\sqrt{3}}{4} = 1 \pm i\sqrt{3}$$

$$y(x) = e^{x}\left(C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x\right)$$

Exercise

Find the general solution: $2y'' + 5y' = 0$

Solution

The characteristic equation: $2\lambda^2 + 5\lambda = \lambda(2\lambda + 5) = 0 \quad \Rightarrow \quad \lambda_1 = 0, \quad \lambda_2 = -\frac{5}{2}$

$$y(x) = C_1 + C_2 e^{-5x/2}$$

Exercise

Find the general solution: $2y'' - 5y' - 3y = 0$

Solution

The characteristic equation: $2\lambda^2 - 5\lambda - 3 = 0$

$\rightarrow \quad \lambda_{1,2} = \frac{-5 \pm 7}{4} \boxed{= -3, \ \frac{1}{2}}$

$\boxed{y(x) = C_1 e^{-3x} + C_2 e^{x/2}}$

Exercise

Find the general solution: $2y'' + 7y' - 4y = 0$

Solution

The characteristic equation: $2\lambda^2 + 7\lambda - 4 = 0$

$\rightarrow \quad \lambda_{1,2} = \frac{-7 \pm 9}{4} \boxed{= -4, \ \frac{1}{2}}$

$\boxed{y(x) = C_1 e^{-4x} + C_2 e^{x/2}}$

Exercise

Find the general solution: $3y'' + y = 0$

Solution

The characteristic equation: $3\lambda^2 + 1 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm \frac{1}{\sqrt{3}} i}$

$\boxed{y(x) = C_1 \cos \frac{\sqrt{3}}{3} x + C_2 \sin \frac{\sqrt{3}}{3} x}$

Exercise

Find the general solution: $3y'' - y' = 0$

Solution

The characteristic equation: $\lambda^2 + \lambda = \lambda(\lambda + 1) = 0 \quad \Rightarrow \quad \lambda_1 = 0, \quad \lambda_2 = -1$

$\boxed{y(x) = C_1 + C_2 e^{-x}}$

Exercise

Find the general solution: $3y'' + 2y' + y = 0$

Solution

The characteristic equation: $3\lambda^2 + 2\lambda + 1 = 0$

$$\Rightarrow \quad \lambda_{1,2} = \frac{-2 \pm \sqrt{4-12}}{6} = \boxed{-\frac{1}{3} \pm i\frac{\sqrt{2}}{3}}$$

$$\boxed{y(x) = e^{-x/3}\left(C_1 \cos\frac{\sqrt{2}}{3}x + C_2 \sin\frac{\sqrt{2}}{3}x\right)}$$

Exercise

Find the general solution: $3y'' + 11y' - 7y = 0$

Solution

The characteristic equation: $3\lambda^2 + 11\lambda - 7 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \frac{-11 \pm \sqrt{205}}{6}}$

$$\boxed{y(x) = C_1 e^{\frac{-11-\sqrt{205}}{6}x} + C_2 e^{\frac{-11+\sqrt{205}}{6}x}}$$

Exercise

Find the general solution: $3y'' - 20y' + 12y = 0$

Solution

The characteristic equation: $3\lambda^2 - 20\lambda + 13 = 0$

$$\rightarrow \quad \lambda_{1,2} = \frac{20 \pm \sqrt{244}}{6} = \boxed{\frac{10 \pm \sqrt{61}}{3}}$$

$$\boxed{y(x) = C_1 e^{\frac{10-\sqrt{61}}{3}x} + C_2 e^{\frac{10+\sqrt{61}}{3}x}}$$

Exercise

Find the general solution: $4y'' + y' = 0$

Solution

The characteristic equation: $4\lambda^2 + \lambda = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = 0, \ -\frac{1}{4}}$

$$\boxed{y(x) = C_1 + C_2 e^{-x/4}}$$

Exercise

Find the general solution: $4y'' + 4y' + y = 0$

Solution

The characteristic equation: $4\lambda^2 + 4\lambda + 1 = (2\lambda + 1)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -\frac{1}{2}}$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{-x/2}}$$

Exercise

Find the general solution: $4y'' - 4y' + y = 0$

Solution

The characteristic equation: $4\lambda^2 - 4\lambda + 1 = (2\lambda - 1)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \frac{1}{2}}$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{x/2}}$$

Exercise

Find the general solution: $4y'' + 4y' + 2y = 0$

Solution

The characteristic equation: $4\lambda^2 + 4\lambda + 2 = 0$

$\rightarrow \quad \lambda_{1,2} = \frac{-4 \pm 4i}{8} \boxed{= -\frac{1}{2} \pm \frac{1}{2}i}$

$$\boxed{y(x) = e^{-x/2}\left(C_1 \cos\frac{x}{2} + C_2 \sin\frac{x}{2}\right)}$$

Exercise

Find the general solution: $4y'' - 4y' + y = 0$

Solution

The characteristic equation: $4\lambda^2 - 4\lambda + 1 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \frac{1}{2}}$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{x/2}}$$

Exercise

Find the general solution: $4y'' - 4y' + 13y = 0$

Solution

The characteristic equation: $4\lambda^2 - 4\lambda + 13 = 0$

$\rightarrow \quad \lambda_{1,2} = \frac{4 \pm 8i\sqrt{3}}{8} \;\underline{= \frac{1}{2} \pm i\sqrt{3}}$

$$\underline{y(x) = \left(C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x\right)e^{x/2}}$$

Exercise

Find the general solution: $4y'' - 8y' + 7y = 0$

Solution

The characteristic equation: $4\lambda^2 - 8\lambda + 7 = 0$

$\rightarrow \quad \lambda_{1,2} = \frac{8 \pm 4i\sqrt{3}}{8} \;\underline{= 1 \pm \frac{1}{2} i\sqrt{3}}$

$$\underline{y(x) = e^{x}\left(C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right)}$$

Exercise

Find the general solution: $4y'' - 12y' + 9y = 0$

Solution

The characteristic equation: $4\lambda^2 - 12\lambda + 9 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{12 \pm 0}{8} \;\underline{= \frac{3}{2}}$

$$\underline{y(x) = \left(C_1 + C_2 x\right)e^{3x/2}}$$

Exercise

Find the general solution: $4y'' + 20y' + 25y = 0$

Solution

The characteristic equation: $4\lambda^2 + 20\lambda + 25 = (2\lambda + 5)^2 = 0 \quad \rightarrow \quad \underline{\lambda_{1,2} = -\frac{5}{2}}$

$$\underline{y(x) = \left(C_1 + C_2 x\right)e^{-5x/2}}$$

Exercise

Find the general solution: $6y'' + y' - 2y = 0$

Solution

The characteristic equation: $6\lambda^2 + \lambda - 2 = 0$

$\rightarrow \quad \lambda_{1,2} = \frac{-1 \pm 7}{2} = -4,\ 3$

$y(x) = C_1 e^{-4x} + C_2 e^{3x}$

Exercise

Find the general solution: $6y'' + 5y' - 6y = 0$

Solution

The characteristic equation: $6\lambda^2 + 5\lambda - 6 = 0 \quad \Rightarrow \quad \lambda_{1,2} = -3,\ \frac{4}{3}$

$y(t) = C_1 e^{-3t} + C_2 e^{4t/3}$

Exercise

Find the general solution: $6y'' - 7y' - 20y = 0$

Solution

The characteristic equation: $6\lambda^2 - 7\lambda - 20 = 0$

$\Rightarrow \quad \lambda_{1,2} = \frac{7 \pm \sqrt{529}}{12} = \frac{7 \pm 23}{12}$

$\lambda_1 = -\frac{4}{3}, \ \lambda_2 = \frac{5}{2}$

$y(t) = C_1 e^{-4t/3} + C_2 e^{5t/2}$

Exercise

Find the general solution: $6y'' + 13y' - 5y = 0$

Solution

The characteristic equation: $6\lambda^2 + 13\lambda - 5 = 0$

$\rightarrow \quad \lambda_{1,2} = \frac{-13 \pm 17}{12} = -\frac{5}{2},\ \frac{1}{3}$

$y(x) = C_1 e^{-5x/2} + C_2 e^{x/3}$

Exercise

Find the general solution: $6y''+13y'+7y=0$

Solution

The characteristic equation: $6\lambda^2+13\lambda+7=0 \rightarrow \underline{\lambda_{1,2}=-1,\ -\frac{7}{6}}$

$$\underline{y(x)=C_1e^{-x}+C_2e^{-7x/6}}$$

Exercise

Find the general solution: $6y''-13y'+7y=0$

Solution

The characteristic equation: $6\lambda^2-13\lambda+7=0 \rightarrow \underline{\lambda_{1,2}=1,\ \frac{7}{6}}$

$$\underline{y(x)=C_1e^{x}+C_2e^{7x/6}}$$

Exercise

Find the general solution: $8y''-10y'-3y=0$

Solution

The characteristic equation: $8\lambda^2-10\lambda-3=0$

$$\rightarrow \lambda_{1,2}=\frac{10\pm 14}{16}\underline{=-\frac{1}{4},\ \frac{3}{2}}$$

$$\underline{y(x)=C_1e^{-x/4}+C_2e^{3x/2}}$$

Exercise

Find the general solution: $9y''-y=0$

Solution

The characteristic equation: $9\lambda^2-1=0 \rightarrow \underline{\lambda_{1,2}=\pm\frac{1}{3}}$

$$\underline{y(x)=C_1e^{-x/3}+C_2e^{x/3}}$$

Exercise

Find the general solution: $9y''+6y'+y=0$

Solution

The characteristic equation: $9\lambda^2 + 6\lambda + 1 = (3\lambda + 1)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -\frac{1}{3}}$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{-x/3}}$$

Exercise

Find the general solution: $9y'' - 12y' + 4y = 0$

Solution

The characteristic equation: $9\lambda^2 - 12\lambda + 4 = (3\lambda - 2)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \frac{2}{3}}$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{2x/3}}$$

Exercise

Find the general solution: $9y'' + 24y' + 16y = 0$

Solution

The characteristic equation: $9\lambda^2 + 24\lambda + 16 = (3\lambda + 4)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -\frac{4}{3}}$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{-4x/3}}$$

Exercise

Find the general solution: $12y'' - 5y' - 2y = 0$

Solution

The characteristic equation: $12\lambda^2 - 5\lambda - 2 = 0$

$$\rightarrow \quad \lambda_{1,2} = \frac{5 \pm 11}{24} = \boxed{-\frac{1}{4}, \ \frac{2}{3}}$$

$$\boxed{y(x) = C_1 e^{-x/4} + C_2 e^{2x/3}}$$

Exercise

Find the general solution: $16y'' - 8y' + 7y = 0$

Solution

The characteristic equation: $16\lambda^2 - 8\lambda + 7 = 0$

$$\rightarrow \quad \lambda_{1,2} = \frac{8 \pm 8i\sqrt{6}}{32} = \boxed{\frac{1}{4} \pm i\frac{\sqrt{6}}{4}}$$

$$\boxed{y(x) = e^{x/4}\left(C_1 \cos\frac{\sqrt{6}}{4}x + C_2 \sin\frac{\sqrt{6}}{4}x\right)}$$

Exercise

Find the general solution: $16y'' - 12y' - 4y = 0$

Solution

The characteristic equation: $16\lambda^2 - 12\lambda - 4 = 0$

$$\rightarrow \quad \lambda_{1,2} = \frac{12 \pm 20}{32} \boxed{= -\frac{1}{4},\ 1}$$

$$\boxed{y(x) = C_1 e^{-x/4} + C_2 e^{x}}$$

Exercise

Find the general solution: $16y'' - 24y' + 9y = 0$

Solution

The characteristic equation: $16\lambda^2 - 24\lambda + 9 = (4\lambda - 3)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \frac{3}{4}}$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{3x/4}}$$

Exercise

Find the general solution: $25y'' + 10y' + y = 0$

Solution

The characteristic equation: $25\lambda^2 + 10\lambda + 1 = (5\lambda + 1)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -\frac{1}{5}}$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{-x/5}}$$

Exercise

Find the general solution: $25y'' - 10y' + y = 0$

Solution

The characteristic equation: $25\lambda^2 - 10\lambda + 1 = (5\lambda - 1)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \frac{1}{5}}$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{x/5}}$$

Exercise

Find the general solution: $35y'' - y' - 12y = 0$

Solution

The characteristic equation: $35\lambda^2 - \lambda - 12 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{1 \pm \sqrt{1681}}{70} = \frac{1 \pm 41}{70}$

$$\boxed{\lambda_1 = -\frac{4}{7}, \ \ \lambda_2 = \frac{3}{5}}$$

$$\boxed{y(x) = C_1 e^{-4x/5} + C_2 e^{3x/5}}$$

Exercise

Find the general solution of the given higher-order differential equation: $y''' + 3y'' + 3y' + y = 0$

Solution

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 3)^3 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2,3} = -3}$$

$$\boxed{y(x) = \left(C_1 + C_2 x + C_3 x^2\right) e^{-3x}}$$

Exercise

Find the general solution of the given higher-order differential equation: $y''' + 3y'' - y' - 3y = 0$

Solution

$$\lambda^3 + 3\lambda^2 - \lambda - 3 = 0$$

$$\lambda^2(\lambda + 3) - (\lambda + 3) = 0$$

$$(\lambda + 3)\left(\lambda^2 - 1\right) = 0$$

$$\boxed{\lambda_{1,2,3} = -3, \ \ \pm 1}$$

$$\boxed{y(x) = C_1 e^{-3x} + C_2 e^{-x} + C_3 e^{x}}$$

Exercise

Find the general solution of the given higher-order differential equation: $y^{(3)} + 3y'' - 4y = 0$

Solution

$$\lambda^3 + 3\lambda^2 - 4 = 0 \quad \rightarrow \quad \lambda_1 = 1$$

$$\begin{array}{r|rrrr} 1 & 1 & 3 & 0 & -4 \\ & & 1 & 4 & 4 \\ \hline & 1 & 4 & 4 & 0 \end{array} \qquad \lambda^2+4\lambda+4=0=(\lambda+2)^2$$

$$\boxed{\lambda_1=1,\ \lambda_{2,3}=-2}$$

$$\boxed{y(x)=C_1e^{x}+\left(C_2+C_3x\right)e^{-2x}}$$

Exercise

Find the general solution of the given higher-order differential equation: $3y'''-19y''+36y'-10y=0$

Solution

$$3\lambda^3-19\lambda^2+36\lambda-10=0$$

$$\boxed{\lambda_1=\frac{1}{3}},\quad \boxed{\lambda_{2,3}=3\pm i}$$

$$\boxed{y(x)=C_1e^{x/3}+e^{3x}\left(C_2\cos x+C_3\sin x\right)}$$

Exercise

Find the general solution of the given higher-order differential equation: $y'''-6y''+12y'-8y=0$

Solution

The characteristic equation: $\lambda^3-6\lambda^2+12\lambda-8=(\lambda-2)^3=0$

$$\Rightarrow\ \boxed{\lambda_{1,2,3}=2}$$

$$\boxed{y(x)=\left(C_1+C_2x+C_3x^2\right)e^{2x}}$$

Exercise

Find the general solution of the given higher–ODE: $y'''+5y''+7y'+3y=0$

Solution

The characteristic equation: $\lambda^3+5\lambda^2+7\lambda+3=0\quad\Rightarrow \boxed{\lambda_1=-3}$

$$\begin{array}{r|rrrr} -3 & 1 & 5 & 7 & 3 \\ & & -3 & -6 & -3 \\ \hline & 1 & 2 & 1 & 0 \end{array}$$

$$\lambda^2+2\lambda+1=0\quad\Rightarrow \boxed{\lambda_{2,3}=-1}$$

The general solution is: $\boxed{y(x) = C_1 e^{-3x} + C_2 e^{-x} + C_3 x e^{-x}}$

Exercise

Find the general solution of the given higher–ODE: $y^{(3)} + y' - 10y = 0$

Solution

The characteristic equation:

$\lambda^3 + \lambda - 10 = 0 \quad \Rightarrow \boxed{\lambda_1 = 2}$ (*Rational Zero Theorem*)

$$\begin{array}{r|rrrr} 2 & 1 & 0 & 1 & -10 \\ & & 2 & 4 & 10 \\ \hline & 1 & 2 & 5 & \boxed{0} \end{array}$$

$\lambda^2 + 2\lambda + 5 = 0 \quad \Rightarrow \boxed{\lambda_{2,3} = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i}$

The general solution is: $\boxed{y(x) = C_1 e^{2x} + e^{-x}\left(C_2 \cos 2x + C_3 \sin 2x\right)}$

Exercise

Find the general solution of the given higher ODE: $y''' + y'' - 6y' + 4y = 0$

Solution

The characteristic equation: $\lambda^3 + \lambda^2 - 6\lambda + 4 = 0$

$\lambda_1 = 1$

$$\begin{array}{r|rrrr} 1 & 1 & 1 & -6 & 4 \\ & & 1 & 2 & -4 \\ \hline & 1 & 2 & -4 & \boxed{0} \end{array}$$

$\lambda^2 + 2\lambda - 4 = 0 \quad \to \lambda_{2,3} = \frac{-2 \pm 2\sqrt{5}}{2} \boxed{= -1 \pm \sqrt{5}}$

$\boxed{y(x) = C_1 e^{x} + C_2 e^{\left(-1-\sqrt{5}\right)x} + C_3 e^{\left(-1+\sqrt{5}\right)x}}$

Exercise

Find the general solution of the given higher ODE: $y''' - 6y'' - y' + 6y = 0$

Solution

The characteristic equation: $\lambda^3 - 6\lambda^2 - \lambda + 6 = 0$

$$\lambda_1 = 1 \qquad \begin{array}{r|rrrr} 1 & 1 & -6 & -1 & 6 \\ & & 1 & -5 & -6 \\ \hline & 1 & -5 & -6 & \lfloor 0 \end{array}$$

$$\lambda^2 - 5\lambda - 6 = 0 \quad \to \quad \underline{\lambda_{2,3} = -1,\ 6}$$

$$\underline{y(x) = C_1 e^{-x} + C_2 e^{x} + C_3 e^{6x}}$$

Exercise

Find the general solution of the given higher ODE: $y''' + 2y'' - 4y' - 8y = 0$

Solution

The characteristic equation: $\lambda^3 - 2\lambda^2 - 4\lambda - 8 = 0$

$$\lambda_1 = -2 \qquad \begin{array}{r|rrrr} -2 & 1 & 2 & -4 & -8 \\ & & -2 & 0 & 8 \\ \hline & 1 & 0 & -4 & \lfloor 0 \end{array}$$

$$\lambda^2 - 4 = 0 \quad \to \quad \underline{\lambda_{2,3} = \pm 2}$$

$$\underline{y(x) = \left(C_1 + C_2 x\right) e^{-2x} + C_3 e^{2x}}$$

Exercise

Find the general solution of the given higher ODE: $y''' - 7y'' + 7y' + 15y = 0$

Solution

The characteristic equation: $\lambda^3 - 7\lambda^2 + 7\lambda + 15 = 0$

$$\lambda_1 = -1 \qquad \begin{array}{r|rrrr} -1 & 1 & -7 & 7 & 15 \\ & & -1 & 8 & -15 \\ \hline & 1 & -8 & 15 & \lfloor 0 \end{array}$$

$$\lambda^2 - 8\lambda + 15 = 0 \quad \to \quad \underline{\lambda_{2,3} = 3,\ 5}$$

$$\underline{y(x) = C_1 e^{-x} + C_2 e^{3x} + C_3 e^{5x}}$$

Exercise

Find the general solution of the given higher ODE: $y''' + 3y'' - 4y' - 12y = 0$

Solution

The characteristic equation: $\lambda^3 + 3\lambda^2 - 4\lambda - 12 = 0$

$$\lambda^2(\lambda+3)-4(\lambda+3)=0$$

$$(\lambda+3)\left(\lambda^2-4\right)=0 \quad \to \quad \lambda_1=-3,\ \lambda_2=-2,\ \lambda_3=2$$

$$y(x)=C_1e^{-3x}+C_2e^{-2x}+C_3e^{2x}$$

Exercise

Find the general solution of the given higher ODE: $\quad y'''-4y''-5y'=0$

Solution

The characteristic equation: $\quad \lambda^3-4\lambda^2-5\lambda=\lambda\left(\lambda^2-4\lambda-5\right)=0$

$$\lambda_1=0,\ \ \lambda_2=-1,\ \ \lambda_3=5$$

$$y(x)=C_1+C_2e^{-x}+C_3e^{5x}$$

Exercise

Find the general solution of the given higher ODE: $\quad y'''-y=0$

Solution

The characteristic equation: $\quad \lambda^3-1=(\lambda-1)\left(\lambda^2+\lambda+1\right)=0$

$$\lambda_1=2,\ \ \lambda_{2,3}=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$$

$$y(x)=C_1e^{2x}+e^{-x/2}\left(C_2\cos\frac{\sqrt{3}}{2}x+C_3\sin\frac{\sqrt{3}}{2}x\right)$$

Exercise

Find the general solution of the given higher ODE: $\quad y'''-5y''+3y'+9y=0$

Solution

The characteristic equation: $\quad \lambda^3-5\lambda^2+3\lambda+9=0$

$$\lambda_1=-1 \qquad \begin{array}{r|rrrr} -1 & 1 & -5 & 3 & 9 \\ & & -1 & 6 & -9 \\ \hline & 1 & -6 & 9 & \underline{|0} \end{array}$$

$$\lambda^2-6\lambda+9=0 \quad \to \quad \lambda_{2,3}=3,\ 3$$

$$y(x)=C_1e^{-x}+\left(C_2+C_3x\right)e^{3x}$$

Exercise

Find the general solution of the given higher ODE: $y''' + 3y'' - 4y' - 12y = 0$

Solution

The characteristic equation: $\lambda^3 + 3\lambda^2 - 4\lambda - 12 = 0$

$$\lambda^2(\lambda+3) - 4(\lambda+3) = 0$$

$$(\lambda+3)\left(\lambda^2 - 4\right) = 0 \quad \to \quad \underline{\lambda_{2,3} = -3, \ \pm 2}$$

$$\underline{y(x) = C_1 e^{-3x} + C_2 e^{-2x} + C_3 e^{2x}}$$

Exercise

Find the general solution of the given higher ODE: $y''' + y'' - 2y = 0$

Solution

The characteristic equation: $\lambda^3 + \lambda^2 - 2 = 0$

$$\lambda_1 = 1 \qquad \begin{array}{r|rrrr} 1 & 1 & 1 & 0 & -2 \\ & & 1 & 2 & 2 \\ \hline & 1 & 2 & 2 & \boxed{0} \end{array}$$

$$\lambda^2 + 2\lambda + 2 = 0 \quad \to \quad \lambda_{2,3} = \frac{-2 \pm 2i}{2} \ \underline{= -1 \pm i}$$

$$\underline{y(x) = C_1 e^{x} + e^{-x}\left(C_2 \cos x + C_3 \sin x\right)}$$

Exercise

Find the general solution of the given higher ODE: $y''' - y'' - 4y = 0$

Solution

The characteristic equation: $\lambda^3 - \lambda^2 - 4 = 0$

$$\lambda_1 = 2 \qquad \begin{array}{r|rrrr} 2 & 1 & -1 & 0 & -4 \\ & & 2 & 2 & 4 \\ \hline & 1 & 1 & 2 & \boxed{0} \end{array}$$

$$\lambda^2 + \lambda + 2 = 0 \quad \to \quad \underline{\lambda_{2,3} = \frac{-1 \pm i\sqrt{7}}{2}}$$

$$\underline{y(x) = C_1 e^{2x} + e^{-x/2}\left(C_2 \cos \frac{\sqrt{7}}{2} x + C_3 \sin \frac{\sqrt{7}}{2} x\right)}$$

Exercise

Find the general solution of the given higher ODE: $y''' + 3y'' + 3y' + y = 0$

Solution

The characteristic equation: $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$

$\lambda_1 = -1$

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 3 & 1 \\ & & -1 & -2 & -1 \\ \hline & 1 & 2 & 1 & \boxed{0} \end{array}$$

$\lambda^2 + 2\lambda + 1 = 0 \rightarrow \boxed{\lambda_{2,3} = -1}$

$$\boxed{y(x) = \left(C_1 + C_2 x + C_3 x^2\right)e^{-x}}$$

Exercise

Find the general solution of the given higher ODE: $y''' - 6y'' + 12y' - 8y = 0$

Solution

The characteristic equation: $\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$

$\lambda_1 = 2$

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 12 & -8 \\ & & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

$\lambda^2 - 4\lambda + 4 = 0 \rightarrow \boxed{\lambda_{2,3} = 2}$

$$\boxed{y(x) = \left(C_1 + C_2 x + C_3 x^2\right)e^{2x}}$$

Exercise

Find the general solution of the given higher ODE: $y^{(4)} + y''' + y'' = 0$

Solution

The characteristic equation: $\lambda^4 + \lambda^3 + \lambda^2 = \lambda^2\left(\lambda^2 + \lambda + 1\right) = 0$

$\lambda_{1,2} = 0$

$\lambda^2 + \lambda + 1 = 0 \rightarrow \boxed{\lambda_{3,4} = \dfrac{-1 \pm i\sqrt{3}}{2}}$

$$\boxed{y(x) = C_1 + C_2 x + e^{-x/2}\left(C_3 \cos\frac{\sqrt{3}}{2}x + C_4 \sin\frac{\sqrt{3}}{2}x\right)}$$

Exercise

Find the general solution of the given higher ODE: $y^{(4)} - 2y'' + y = 0$

Solution

The characteristic equation: $\lambda^4 - 2\lambda^2 + 1 = \left(\lambda^2 - 1\right)^2 = 0$

$\lambda^2 = 1$ $\qquad$ $\boxed{\lambda_{1,2} = -1 \quad \lambda_{3,4} = 1}$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{-x} + \left(C_3 + C_4 x\right)e^{x}}$$

Exercise

Find the general solution of the given higher ODE: $16y^{(4)} + 24y'' + 9y = 0$

Solution

The characteristic equation: $16\lambda^4 + 24\lambda^2 + 9 = \left(4\lambda^2 + 3\right)^2 = 0$

$\lambda^2 = -\frac{3}{4}$

$\boxed{\lambda_{1,2} = \pm\frac{\sqrt{3}}{2}i \quad \lambda_{3,4} = \pm\frac{\sqrt{3}}{2}i}$

$$\boxed{y(x) = C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x + x\left(C_3 \cos\frac{\sqrt{3}}{2}x + C_4 \sin\frac{\sqrt{3}}{2}x\right)}$$

Exercise

Find the general solution of the given higher ODE: $y^{(4)} - 7y'' - 18y = 0$

Solution

The characteristic equation: $\lambda^4 - 7\lambda^2 - 18 = 0$

$\lambda^2 = \frac{7 \pm 11}{2}$

$\boxed{\lambda_{1,2} = \pm 2i \quad \lambda_{3,4} = \pm 3}$

$$\boxed{y(x) = C_1 \cos 2x + C_2 \sin 2x + C_3 e^{-3x} + C_4 e^{3x}}$$

Exercise

Find the general solution of the given higher-order differential equation: $y^{(4)}+2y''+y=0$

Solution

$$\lambda^4+2\lambda^2+1=\left(\lambda^2+1\right)^2=0$$

$$\lambda^2=-1 \quad\Rightarrow\quad \lambda=\pm i \quad\Rightarrow \lambda_{1,2}=-i,\ \lambda_{3,4}=i$$

$$\underline{y(x)=\left(C_1+C_2x\right)e^{-ix}+\left(C_3+C_4x\right)e^{ix}}$$

OR $\quad \underline{y(x)=C_1\cos x+C_2\sin x+C_3x\cos x+C_4x\sin x}$

Exercise

Find the general solution of the given higher-order differential equation: $y^{(4)}+y'''+y''=0$

Solution

$$\lambda^4+\lambda^3+\lambda^2=\lambda^2\left(\lambda^2+\lambda+1\right)=0$$

$$\lambda^2=0\to\lambda_{1,2}=0$$

$$\lambda=\frac{-1\pm\sqrt{1-4}}{2}\quad\Rightarrow\quad \lambda_{3,4}=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$$

$$\underline{y(x)=C_1+C_2x+e^{-x/2}\left(C_3\cos\left(\frac{\sqrt{3}}{2}x\right)+C_4\sin\left(\frac{\sqrt{3}}{2}x\right)\right)}$$

Exercise

Find the general solution of the given higher–ODE: $y^{(4)}+4y=0$

Solution

The characteristic equation: $\lambda^4+4=0 \quad\Rightarrow \lambda^2=\pm 2i \quad\to\lambda_{1,2,3,4}=\pm\sqrt{\pm 2i}$

Since $i=e^{\frac{\pi}{2}i} \quad -i=e^{\frac{3\pi}{2}i}$

$$\sqrt{2i}=\left(2e^{i\pi/2}\right)^{1/2}=\sqrt{2}e^{i\pi/4}=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=1+i$$

$$\sqrt{-2i}=\left(2e^{i3\pi/2}\right)^{1/2}=\sqrt{2}e^{i3\pi/4}=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=-1+i$$

$$\lambda=\pm(\pm1+i)=\begin{cases}1\pm i\\-1\pm i\end{cases}$$

$$\boxed{y(x) = e^{x}\left(C_1 \cos x + C_2 \sin x\right) + e^{-x}\left(C_3 \cos x + C_4 \sin x\right)}$$

Exercise

Find the general solution of the given higher–ODE: $y^{(4)} + 2y''' + 9y'' - 2y' - 10y = 0$

Solution

The characteristic equation:

$\lambda^4 + 2\lambda^3 + 9\lambda^2 - 2\lambda - 10 = 0 \quad \Rightarrow \boxed{\lambda_1 = 1}$ (*Rational Zero Theorem*)

$$\begin{array}{r|rrrrr} 1 & 1 & 2 & 9 & -2 & -10 \\ & & 1 & 3 & 12 & 10 \\ \hline & 1 & 3 & 12 & 10 & \boxed{0} \end{array} \quad \lambda^3 + 3\lambda^2 + 12\lambda + 10 = 0 \quad \Rightarrow \boxed{\lambda_2 = -1}$$

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 12 & 10 \\ & & -1 & -2 & -10 \\ \hline & 1 & 2 & 10 & \boxed{0} \end{array} \quad \lambda^2 + 2\lambda + 10 = 0 \quad \Rightarrow \boxed{\lambda_{3,4} = -1 \pm 3i}$$

The general solution is: $\boxed{y(x) = C_1 e^{x} + C_2 e^{-x} + e^{-x}\left(C_3 \cos 3x + C_4 \sin 3x\right)}$

Exercise

Find the solution of the given initial value problem $x^{(4)} - 4x^{(3)} + 7x'' - 4x' + 6x = 0$

Solution

The characteristic equation: $\lambda^4 - 4\lambda^3 + 7\lambda^2 - 4\lambda + 6 = 0 \quad \to \quad \boxed{\lambda_{1,2,3,4} = \pm i, \ 2 \pm i\sqrt{2}}$

$$\boxed{x(t) = C_1 e^{\left(2+i\sqrt{2}\right)t} + C_2 e^{\left(2-i\sqrt{2}\right)t} + C_3 \cos t + C_4 \sin t}$$

Exercise

Find the solution of the given initial value problem $x^{(4)} + 8x^{(3)} + 24x'' + 32x' + 16x = 0$

Solution

The characteristic equation: $\lambda^4 + 8\lambda^3 + 24\lambda^2 + 32\lambda + 16 = 0 \quad \to \quad \boxed{\lambda_1 = -2}$

$$\begin{array}{r|rrrrr} -2 & 1 & 8 & 24 & 32 & 16 \\ & & -2 & -12 & -24 & -16 \\ \hline & 1 & 6 & 12 & 8 & \boxed{0} \end{array} \quad \lambda^3 + 6\lambda^2 + 12\lambda + 8 = 0 \quad \Rightarrow \boxed{\lambda_2 = -2}$$

$$\begin{array}{r|rrrr} -2 & 1 & 6 & 12 & 8 \\ & & -2 & -8 & -8 \\ \hline & 1 & 4 & 4 & \lfloor 0 \end{array} \qquad \lambda^2 + 4\lambda + 4 = 0 \quad \Rightarrow \underline{\lambda_{3,4} = -2|}$$

The eigenvalues: $\underline{\lambda_{1,2,3,4} = -2|}$

$$\underline{x(t) = \left(C_1 + C_2 t + C_3 t^2 + C_4 t^3\right)e^{-2t}\ |}$$

Exercise

Find the solution of the given initial value problem $x^{(4)} - 4x'' + 16x' + 32x = 0$

Solution

The characteristic equation: $\lambda^4 - 4\lambda^2 + 16\lambda + 32 = 0 \quad \rightarrow \quad \underline{\lambda_1 = -2|}$

$$\begin{array}{r|rrrrr} -2 & 1 & 0 & -4 & 16 & 32 \\ & & -2 & 4 & 0 & -32 \\ \hline & 1 & -2 & 0 & 16 & \lfloor 0 \end{array} \qquad \lambda^3 - 2\lambda^2 + 16 = 0 \quad \Rightarrow \underline{\lambda_2 = -2|}$$

$$\begin{array}{r|rrrr} -2 & 1 & -2 & 0 & 16 \\ & & -2 & 8 & -16 \\ \hline & 1 & -4 & 8 & \lfloor 0 \end{array} \qquad \lambda^2 - 4\lambda + 8 = 0 \quad \Rightarrow \underline{\lambda_{3,4} = 2 \pm 2i|}$$

The eigenvalues: $\underline{\lambda = -2,\ -2,\ 2 \pm 2i|}$

$$\underline{x(t) = \left(C_1 + C_2 t\right)e^{-2t} + e^{2t}\left(C_3 \cos 2t + C_4 \sin 2t\right)\ |}$$

Exercise

Find the solution of the given initial value problem $x^{(4)} + 4x^{(3)} + 6x'' + 4x' + x = 0$

Solution

The characteristic equation: $\lambda^4 + 4\lambda^3 + 6\lambda^2 + 4\lambda + 1 = (\lambda + 1)^4 = 0$

$\rightarrow \quad \underline{\lambda_{1,2,3,4} = -1|}$

$$\underline{x(t) = \left(C_1 + C_2 t + C_3 t^2 + C_4 t^3\right)e^{-t}\ |}$$

Exercise

Find the solution of the given initial value problem $y^{(4)} - y^{(3)} + y'' - 3y' - 6y = 0$

Solution

The characteristic equation: $\lambda^4 - \lambda^3 + \lambda^2 - 3\lambda - 6 = 0 \quad \rightarrow \quad \lambda_1 = -1$

$$\begin{array}{r|rrrrr} -1 & 1 & -1 & 1 & -3 & -6 \\ & & -1 & 2 & -3 & 6 \\ \hline & 1 & -2 & 3 & -6 & \lfloor 0 \end{array} \qquad \lambda^3 - 2\lambda^2 + 3\lambda - 6 = 0 \quad \Rightarrow \underline{\lambda_2 = 2}$$

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 3 & -6 \\ & & 2 & 0 & 6 \\ \hline & 1 & 0 & 3 & \lfloor 0 \end{array} \qquad \lambda^2 + 3 = 0 \quad \Rightarrow \underline{\lambda_{3,4} = \pm i\sqrt{3}}$$

The eigenvalues: $\underline{\lambda = -1,\ 2,\ \pm i\sqrt{3}}$

$$\underline{y(t) = C_1 e^{-t} + C_2 e^{2t} + C_3 \cos\sqrt{3}\,t + C_4 \sin\sqrt{3}\,t}$$

Exercise

Find the solution of the given initial value problem $\quad y^{(4)} + y^{(3)} - 3y'' - 5y' - 2y = 0$

Solution

The characteristic equation: $\lambda^4 + \lambda^3 - 3\lambda^2 - 5\lambda - 2 = 0 \quad \to \quad \lambda_1 = -1$

$$\begin{array}{r|rrrrr} -1 & 1 & 1 & -3 & -5 & -2 \\ & & -1 & 0 & 3 & 2 \\ \hline & 1 & 0 & -3 & -2 & \lfloor 0 \end{array} \qquad \lambda^3 - 3\lambda - 2 = 0 \quad \Rightarrow \underline{\lambda_2 = 2}$$

$$\begin{array}{r|rrrr} 2 & 1 & 0 & -3 & -2 \\ & & 2 & 4 & 2 \\ \hline & 1 & 2 & 1 & \lfloor 0 \end{array} \qquad \lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \underline{\lambda_{3,4} = -1}$$

The eigenvalues: $\underline{\lambda_{1,2,3} = -1,\ \lambda_4 = 2}$

$$\underline{y(t) = \left(C_1 + C_2 t + C_3 t^2\right)e^{-t} + C_4 e^{2t}}$$

Exercise

Find the solution of the given initial value problem $\quad x^{(5)} - x^{(4)} - 2x^{(3)} + 2x'' + x' - x = 0$

Solution

The characteristic equation: $\lambda^5 - \lambda^4 - 2\lambda^3 + 2\lambda^2 + \lambda - 1 = 0 \quad \to \quad \underline{\lambda_1 = 1}$

$$\begin{array}{r|rrrrrr} 1 & 1 & -1 & -2 & 2 & 1 & -1 \\ & & 1 & 0 & -2 & 0 & 1 \\ \hline & 1 & 0 & -2 & 0 & 1 & \lfloor 0 \end{array}$$

$$\lambda^4 - 2\lambda^2 + 1 = \left(\lambda^2 - 1\right)^2 = 0 \quad \to \quad \lambda^2 = 1,\ 1$$

The eigenvalues: $\boxed{\lambda = 1,\ 1,\ 1,\ -1,\ -1}$

$$\boxed{x(t) = \left(C_1 + C_2 t + C_3 t^2\right)e^{t} + \left(C_4 + C_5 t\right)e^{-t}}$$

Exercise

Find the solution of the given initial value problem $x^{(5)} + 5x^{(4)} + 10x^{(3)} + 10x'' + 5x' + x = 0$

Solution

The characteristic equation: $\lambda^5 + 5\lambda^4 + 10\lambda^3 + 10\lambda^2 + 5\lambda + 1 = (\lambda + 1)^5 = 0$

The eigenvalues: $\boxed{\lambda = 1,\ 1,\ 1,\ -1,\ -1}$

$$\boxed{x(t) = \left(C_1 + C_2 t + C_3 t^2\right)e^{t} + \left(C_4 + C_5 t\right)e^{-t}}$$

Exercise

Find the general solution of the given higher ODE: $y^{(5)} + 5y^{(4)} - 2y''' - 10y'' + y' + 5y = 0$

Solution

The characteristic equation: $\lambda^5 + 5\lambda^4 - 2\lambda^3 - 10\lambda^2 + \lambda + 5 = 0$

$\boxed{\lambda_1 = 1}$

$$\begin{array}{r|rrrrrr} 1 & 1 & 5 & -2 & -10 & 1 & 5 \\ & & 1 & 6 & 4 & -6 & -5 \\ \hline & 1 & 6 & 4 & -6 & -5 & \boxed{0} \end{array}$$

$\to \lambda^4 + 6\lambda^3 + 4\lambda^2 - 6\lambda - 5 = 0 \quad \Rightarrow \boxed{\lambda_2 = 1}$

$$\begin{array}{r|rrrrr} 1 & 1 & 6 & 4 & -6 & -5 \\ & & 1 & 7 & 11 & 5 \\ \hline & 1 & 7 & 11 & 5 & \boxed{0} \end{array}$$

$\to \lambda^3 + 7\lambda^2 + 11\lambda + 5 = 0 \quad \Rightarrow \boxed{\lambda_3 = -1}$

$$\begin{array}{r|rrrr} -1 & 1 & 7 & 11 & 5 \\ & & -1 & -6 & -5 \\ \hline & 1 & 6 & 5 & \boxed{0} \end{array}$$

$\to \lambda^2 + 6\lambda + 5 = 0 \quad \Rightarrow \boxed{\lambda_{4,5} = -1,\ -5}$

$\boxed{\lambda = -5,\ -1,\ -1,\ 1,\ 1}$

$$\boxed{y(x) = C_1 e^{-x} + \left(C_2 + C_3 x\right)e^{-x} + \left(C_4 + C_5 x\right)e^{x}}$$

Exercise

Find the general solution of the given higher ODE: $2y^{(5)} - 7y^{(4)} + 12y''' + 8y'' = 0$

Solution

The characteristic equation: $2\lambda^5 - 7\lambda^4 + 12\lambda^3 + 8\lambda^2 = \lambda^2\left(2\lambda^3 - 7\lambda^2 + 12\lambda + 8\right) = 0$

$\underline{\lambda_{1,2} = 0,\ \lambda_3 = -\frac{1}{2}}$

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & -7 & 12 & 8 \\ & & -1 & 4 & -8 \\ \hline & 2 & -8 & 16 & 0 \end{array}$$

$\rightarrow\ 2\lambda^2 - 8\lambda + 16 = 0 \quad \Rightarrow \lambda_{4,5} = 2 \pm \frac{8i}{4} = 2 \pm 2i$

$\lambda = 0,\ 0,\ -\frac{1}{2},\ 2 \pm 2i$

$y(x) = C_1 + C_2 x + C_3 e^{-x/2} + e^{2x}\left(C_4 \cos 2x + C_5 \sin 2x\right)$

Exercise

Find the general solution of the given higher-order differential equation: $y^{(5)} - 2y^{(4)} + 17y''' = 0$

Solution

$\lambda^5 - 2\lambda^4 + 17\lambda^3 = \lambda^3\left(\lambda^2 - 2\lambda + 17\right) = 0$

$\lambda^3 = 0 \rightarrow \lambda_{1,2,3} = 0$

$\Rightarrow \quad \lambda = \frac{2 \pm \sqrt{4 - 68}}{2} \quad \Rightarrow \lambda_{4,5} = 1 \pm 4i$

$y(x) = C_1 + C_2 x + C_3 x^2 + e^x\left(C_4 \cos 4x + C_5 \sin 4x\right)$

Exercise

Find the solution of the given initial value problem $x^{(6)} - 5x^{(4)} + 16x^{(3)} + 36x'' - 16x' - 32x = 0$

Solution

The characteristic equation: $\lambda^6 - 5\lambda^4 + 16\lambda^3 + 36\lambda^2 - 16\lambda - 32 = 0 \quad \rightarrow \quad \lambda_1 = 1$

$$\begin{array}{r|rrrrrrr} 1 & 1 & 0 & -5 & 16 & 36 & -16 & -32 \\ & & 1 & 1 & -4 & 12 & 48 & 32 \\ \hline & 1 & 1 & -4 & 12 & 48 & 32 & 0 \end{array}$$

$$\lambda^5+\lambda^4-4\lambda^3+12\lambda^2+48\lambda+32=0 \quad \rightarrow \quad \underline{\lambda_2=-1}$$

$$\begin{array}{r|rrrrrr}
-1 & 1 & 1 & -4 & 12 & 48 & 32 \\
 & & -1 & 0 & 4 & -16 & -32 \\
\hline
 & 1 & 0 & -4 & 16 & 32 & \boxed{0}
\end{array}$$

$$\lambda^4-4\lambda^2+16\lambda+32=0 \quad \rightarrow \quad \lambda_3=-2$$

$$\begin{array}{r|rrrrr}
-2 & 1 & 0 & -4 & 16 & 32 \\
 & & -2 & 4 & 0 & -32 \\
\hline
 & 1 & -2 & 0 & 16 & \boxed{0}
\end{array}$$

$$\lambda^3-2\lambda^2+16=0 \quad \Rightarrow \underline{\lambda_4=-2}$$

$$\begin{array}{r|rrrr}
-2 & 1 & -2 & 0 & 16 \\
 & & -2 & 8 & -16 \\
\hline
 & 1 & -4 & 8 & \boxed{0}
\end{array}$$

$$\lambda^2-4\lambda+8=0 \quad \Rightarrow \underline{\lambda_{3,4}=2\pm 2i}$$

The eigenvalues: $\underline{\lambda=1,\ -1,\ -2,\ -2,\ 2\pm 2i}$

$$\underline{x(t)=C_1e^t+C_2e^{-t}+\left(C_3+C_4t\right)e^{-2t}+e^{2t}\left(C_5\cos 2t+C_6\sin 2t\right)}$$

Exercise

Find the general solution of the given higher-order differential equation: $\left(D^2+6D+13\right)^2 y=0$

Solution

The characteristic equation: $\left(\lambda^2+6\lambda+13\right)=0$

$$\Rightarrow \lambda=\frac{-6\pm\sqrt{-16}}{2}=-3\pm 2i \quad \textit{multiplicity } k=2$$

$$\underline{y(x)=e^{-3x}\left(C_1\cos 2x+C_2\sin 2x\right)+xe^{-3x}\left(C_3\cos 2x+C_4\sin 2x\right)}$$

Exercise

Find the general solution of the given higher-order differential equation $\lambda^3(\lambda-1)(\lambda-2)^3\left(\lambda^2+9\right)=0$

Solution

$$\lambda^2+9=0 \quad \Rightarrow \quad \lambda^2=-9 \ \rightarrow \ \lambda=\pm 3i$$

The solution: $\lambda = 0,\ 0,\ 0,\ 1,\ 2,\ 2,\ 2,\ \pm 3i$

$$\boxed{y(x) = C_1 + C_2 x + C_3 x^2 + C_4 e^x + \left(C_5 + C_6 x + C_7 x^2\right)e^{2x} + C_8 \cos 3x + C_9 \sin 3x}$$

Exercise

Find the solution of the given initial value problem $\quad y'' + y = 0, \quad y\left(\frac{\pi}{3}\right) = 0, \quad y'\left(\frac{\pi}{3}\right) = 2$

Solution

The characteristic equation: $\lambda^2 + 1 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm i}$

$$y(x) = C_1 \cos x + C_2 \sin x$$

$$y\left(\frac{\pi}{3}\right) = 0 \quad \rightarrow \quad \frac{1}{2}C_1 + \frac{\sqrt{3}}{2}C_2 = 0 \quad \Rightarrow \boxed{C_1 + \sqrt{3}C_2 = 0}$$

$$y'(x) = -C_1 \sin x + C_2 \cos x$$

$$y'\left(\frac{\pi}{3}\right) = 2 \quad \rightarrow \quad -\frac{\sqrt{3}}{2}C_1 + \frac{1}{2}C_2 = 2 \quad \Rightarrow \quad \boxed{-\sqrt{3}C_1 + C_2 = 4}$$

$$\begin{cases} C_1 + \sqrt{3}C_2 = 0 \\ -\sqrt{3}C_1 + C_2 = 4 \end{cases} \quad \rightarrow \quad \begin{cases} C_1 = -\sqrt{3}C_2 \\ 3C_2 + C_2 = 4 \end{cases}$$

$$\Rightarrow \quad \boxed{C_2 = 1,\ C_1 = -\sqrt{3}}$$

$$\boxed{y(x) = -\sqrt{3}\cos x + \sin x}$$

Exercise

Find the solution of the given initial value problem $\quad y'' + y = 0; \quad y(0) = 0, \quad y'\left(\frac{\pi}{2}\right) = 0$

Solution

The characteristic equation: $\lambda^2 + 1 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm i}$

$$y(x) = C_1 \cos x + C_2 \sin x$$

$$y(0) = 0 \quad \rightarrow \quad \boxed{C_1 = 0}$$

$$y'(x) = -C_1 \sin x + C_2 \cos x$$

$$y'\left(\frac{\pi}{2}\right) = 0 \quad \rightarrow \quad \boxed{C_1 = 0}$$

$$\boxed{y(x) = C_2 \sin x}$$

Exercise

Find the solution of the given initial value problem $y'' + y' = 0; \quad y(0) = 2, \quad y'(0) = 1$

Solution

The characteristic equation: $\lambda^2 + \lambda = 0 \quad \rightarrow \quad \underline{\lambda_{1,2} = 0, \ -1}$

$\underline{y(x) = C_1 + C_2 e^{-x}}$

$y(0) = 2 \quad \rightarrow C_1 + C_2 = 2$

$y'(x) = -C_2 e^{-x}$

$y'(0) = 1 \quad \rightarrow \underline{C_2 = -1}$

$C_1 + C_2 = 2 \quad \rightarrow \quad \underline{C_1 = 3}$

$\underline{y(x) = 3e^{-4x}}$

Exercise

Find the general solution: $y'' - y' - 2y = 0; \quad y(0) = -1, \quad y'(0) = 2$

Solution

The characteristic equation: $\lambda^2 - \lambda - 2 = 0 \qquad \Rightarrow \underline{\lambda_1 = 2; \ \lambda_2 = -1}$

$y(t) = C_1 e^{2t} + C_2 e^{-t}$

$y(0) = C_1 + C_2 = -1$

$y'(t) = 2C_1 e^{2t} - C_2 e^{-t}$

$y'(0) = 2C_1 - C_2 = 2$

$\underline{C_1 = \frac{1}{3} \quad C_2 = -\frac{4}{3}}$

$\underline{y(t) = \frac{1}{3}e^{2t} - \frac{4}{3}e^{-t}}$

Exercise

Find the solution of the given initial value problem $y'' + y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 0$

Solution

The characteristic equation: $\lambda^2 + \lambda + 2 = 0 \quad \rightarrow \quad \underline{\lambda_{1,2} = \frac{-1 \pm i\sqrt{7}}{2}}$

$$y(x) = e^{-x/2}\left(C_1 \cos\frac{\sqrt{7}}{2}x + C_2 \sin\frac{\sqrt{7}}{2}x\right)$$

$y(0) = 0 \quad \rightarrow \quad \boxed{C_1 = 0}$

$$y'(x) = e^{-x/2}\left(-\tfrac{1}{2}C_1 \cos\frac{\sqrt{7}}{2}x - \tfrac{1}{2}C_2 \sin\frac{\sqrt{7}}{2}x - \frac{\sqrt{7}}{2}C_1 \sin\frac{\sqrt{7}}{2}x + \frac{\sqrt{7}}{2}C_2 \cos\frac{\sqrt{7}}{2}x\right)$$

$y'(0) = 0 \quad \rightarrow \quad -\tfrac{1}{2}C_1 + \frac{\sqrt{7}}{2}C_2 = 0 \ \Rightarrow \ \boxed{C_2 = 0}$

$\boxed{y(x) = 0}$

Exercise

Find the solution of the given initial value problem $\quad y'' + 2y' + y = 0; \quad y(0) = 1, \quad y'(0) = -3$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -1}$

$\boxed{y(x) = (C_1 + C_2 x)e^{-x}}$

$y(0) = 1 \quad \rightarrow \quad \boxed{C_1 = 1}$

$y'(x) = (C_2 - C_1 - C_2 x)e^{-x}$

$y'(0) = -3 \quad \rightarrow \quad C_2 - C_1 = -3 \quad \Rightarrow \boxed{C_2 = -2}$

$\boxed{y(x) = (1 - 2x)e^{-x}}$

Exercise

Find the solution of the given initial value problem $\quad y'' - 2y' + y = 0\,; \quad y(0) = 5, \quad y'(0) = 10$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = 1}$

$y(x) = (C_1 + C_2 x)e^{x}$

$y(0) = 5 \quad \rightarrow \quad \boxed{C_1 = 5}$

$y'(x) = (C_2 + C_1 + C_2 x)e^{x}$

$y'(0) = 10 \quad \rightarrow \quad C_2 + C_1 = 10 \quad \Rightarrow \boxed{C_2 = 5}$

$\boxed{y(x) = 5(1 + x)e^{x}}$

Exercise

Find the solution of the given initial value problem $y'' - 2y' - 2y = 0$; $y(0) = 0$, $y'(0) = 3$

Solution

The characteristic equation: $\lambda^2 - 2\lambda - 2 = 0 \rightarrow \lambda_{1,2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$

$$y(x) = C_1 e^{(1-\sqrt{3})x} + C_2 e^{(1+\sqrt{3})x}$$

$$y(0) = 0 \rightarrow C_1 + C_2 = 0$$

$$y'(x) = (1-\sqrt{3})C_1 e^{(1-\sqrt{3})x} + (1+\sqrt{3})C_2 e^{(1+\sqrt{3})x}$$

$$y'(0) = 3 \rightarrow (1-\sqrt{3})C_1 + (1+\sqrt{3})C_2 = 3$$

$$\begin{cases} C_1 + C_2 = 0 \rightarrow C_1 = -C_2 \\ (1-\sqrt{3})C_1 + (1+\sqrt{3})C_2 = 3 \end{cases} \rightarrow C_1(1-\sqrt{3}-1-\sqrt{3}) = 3$$

$$C_1 = -\frac{\sqrt{3}}{2},\ C_2 = \frac{\sqrt{3}}{2}$$

$$y(x) = -\frac{\sqrt{3}}{2} e^{(1-\sqrt{3})x} + \frac{\sqrt{3}}{2} e^{(1+\sqrt{3})x}$$

Exercise

Find the solution of the given initial value problem $y'' - 2y' + 2y = 0$; $y(0) = 1$, $y(\pi) = 1$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 2 = 0 \rightarrow \lambda_{1,2} = \frac{2 \pm 2i}{2} = 1 \pm i$

$$y(x) = e^x (C_1 \cos x + C_2 \sin x)$$

$$y(0) = 1 \rightarrow C_1 = 1$$

$$y(\pi) = 1 \rightarrow -e^{\pi} C_1 = 1\ C_1 = -e^{-\pi}$$

$$C_1 = -e^{-\pi} \neq 1$$

There is No solution the *ODE* under the given conditions.

Exercise

Find the solution of the given initial value problem. $y'' - 2y' - 3y = 0;$ $y(0) = 2$, $y'(0) = -3$

Solution

The characteristic equation: $\lambda^2 - 2\lambda - 3 = 0 \Rightarrow \lambda_1 = -1;\ \lambda_2 = 3$

$y(t) = C_1 e^{-t} + C_2 e^{3t}$

$y(0) = C_1 + C_2 = 2$

$y'(t) = -C_1 e^{-t} + 3C_2 e^{3t}$

$y'(0) = -C_1 + 3C_2 = -3$

$$\begin{cases} C_1 + C_2 = 2 \\ -C_1 + 3C_2 = -3 \end{cases} \rightarrow \ C_1 = \frac{9}{4} \quad C_2 = -\frac{1}{4}$$

$$\boxed{y(t) = \frac{9}{4} e^{-t} - \frac{1}{4} e^{3t}}$$

Exercise

Find the solution of the given initial value problem $y'' + 2y' - 8y = 0$; $y(0) = 3,\quad y'(0) = -12$

Solution

The characteristic equation: $\lambda^2 + 2\lambda - 8 = 0 \ \rightarrow \ \boxed{\lambda_{1,2} = -4,\ 2}$

$$\boxed{y(x) = C_1 e^{-4x} + C_2 e^{2x}}$$

$y(0) = 3 \ \rightarrow \ C_1 + C_2 = 3$

$y'(x) = -4C_1 e^{-4x} + 2C_2 e^{2x}$

$y'(0) = -12 \ \rightarrow \ -4C_1 + 2C_2 = -12$

$$\begin{cases} C_1 + C_2 = 3 \\ -4C_1 + 2C_2 = -12 \end{cases}$$

$$\rightarrow \ \boxed{C_1 = 3,\ \ C_2 = 0}$$

$$\boxed{y(x) = 3e^{-4x}}$$

Exercise

Find the general solution: $y'' - 2y' + 17y = 0;\quad y(0) = -2,\quad y'(0) = 3$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 17 = 0$

$\Rightarrow \lambda_{1,2} = 1 \pm 4i$

$$y(t)=e^{t}\left(C_1\cos 4t+C_2\sin 4t\right)$$

$$y(t)=e^{t}\left(C_1\cos 4t+C_2\sin 4t\right)\ \rightarrow y(0)=\underline{C_1=-2}|$$

$$y'(t)=e^{t}\left(C_1\cos 4t+C_2\sin 4t\right)+e^{t}\left(-4C_1\sin 4t+4C_2\cos 4t\right)$$

$$\Rightarrow y'(0)=C_1+4C_2=3$$

$$\Rightarrow\ \underline{C_2=\tfrac{5}{4}}|$$

$$\underline{y(t)=e^{t}\left(-2\cos 4t+\tfrac{5}{4}\sin 4t\right)}|$$

Exercise

Find the general solution: $y''+2\sqrt{2}y'+2y=0$; $y(0)=1,\quad y'(0)=0$

Solution

The characteristic equation: $\lambda^2+2\sqrt{2}\,\lambda+2=\left(\lambda+\sqrt{2}\right)^2=0$

$$\Rightarrow \underline{\lambda_{1,2}=\pm\sqrt{2}\,i}|$$

$$\underline{y(t)=C_1\cos\sqrt{2}\,t+C_2\sin\sqrt{2}\,t}|$$

$$y(0)=1\quad\rightarrow\ \underline{C_1=1}|$$

$$y'(t)=-\sqrt{2}C_1\sin\sqrt{2}\,t+\sqrt{2}C_2\cos\sqrt{2}\,t$$

$$y'(0)=0\quad\rightarrow\sqrt{2}C_2=0\quad\Rightarrow\ \underline{C_2=0}|$$

$$\underline{y(t)=\cos\sqrt{2}\,t}\ |$$

Exercise

Find the general solution: $y''+3y'-10y=0;\quad y(0)=4,\quad y'(0)=-2$

Solution

The characteristic equation: $\lambda^2+3\lambda+10=0\quad\rightarrow\quad \underline{\lambda_{1,2}=-5,\ 2}|$

$$\underline{y(x)=C_1e^{-5x}+C_2e^{2x}}|$$

$$y(0)=4\quad\rightarrow\ C_1+C_2=4$$

$$y' = -5C_1 e^{-5x} + 2C_2 e^{2x}$$

$$y'(0) = -2 \quad \rightarrow \quad -5C_1 + 2C_2 = -2$$

$$\begin{cases} C_1 + C_2 = 4 \\ -5C_1 + 2C_2 = -2 \end{cases} \quad \rightarrow \quad \boxed{C_1 = \frac{10}{7}, \ C_2 = \frac{18}{7}}$$

$$\boxed{y(x) = \frac{10}{7} e^{-5x} + \frac{18}{7} e^{2x}}$$

Exercise

Find the solution of the given initial value problem $y'' + 4y = 0$; $y(0) = 0$, $y(\pi) = 0$

Solution

The characteristic equation: $\lambda^2 + 4 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 2i}$

$$y(x) = C_1 \cos 2x + C_2 \sin 2x$$

$$y(0) = 0 \quad \rightarrow \quad \boxed{C_1 = 0}$$

$$y(\pi) = 0 \quad \rightarrow \quad \boxed{C_1 = 0}$$

$$\boxed{y(x) = C_2 \sin 2x}$$

Exercise

Find the solution of the given initial value problem $y'' + 4y = 0$; $y\left(\frac{\pi}{4}\right) = -2$, $y'\left(\frac{\pi}{4}\right) = 1$

Solution

The characteristic equation: $\lambda^2 + 4 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 2i}$

$$\boxed{y(x) = C_1 \cos 2x + C_2 \sin 2x}$$

$$y\left(\frac{\pi}{4}\right) = -2 \quad \rightarrow \quad \boxed{C_2 = -2}$$

$$y' = -2C_1 \sin 2x + 2C_2 \cos 2x$$

$$y'\left(\frac{\pi}{4}\right) = 1 \quad \rightarrow \quad -2C_1 = 1 \quad \Rightarrow \quad \boxed{C_1 = -\frac{1}{2}}$$

$$\boxed{y(x) = -2\cos 2x - \frac{1}{2}\sin 2x}$$

Exercise

Find the solution of the given initial value problem $y''+4y'+2y=0; \quad y(0)=-1, \quad y'(0)=2$

Solution

The characteristic equation: $\lambda^2+4\lambda+2=0$

$\rightarrow \lambda_{1,2}=\frac{-4\pm 2\sqrt{2}}{2}=-2\pm\sqrt{2}$

$y(x)=C_1e^{-(2+\sqrt{2})x}+C_2e^{(-2+\sqrt{2})x}$

$y(0)=-1 \quad \rightarrow C_1+C_2=-1$

$y'(x)=-(2+\sqrt{2})C_1e^{-(2+\sqrt{2})x}+(-2+\sqrt{2})C_2e^{(-2+\sqrt{2})x}$

$y'(0)=2 \quad \rightarrow (-2-\sqrt{2})C_1+(-2+\sqrt{2})C_2=2$

$$\begin{cases} C_1+C_2=-1 \\ (-2-\sqrt{2})C_1+(-2+\sqrt{2})C_2=2 \end{cases}$$

$\rightarrow \Delta=\begin{vmatrix} 1 & 1 \\ -2-\sqrt{2} & -2+\sqrt{2} \end{vmatrix}=2\sqrt{2} \qquad \Delta_{C_c}=\begin{vmatrix} -1 & 1 \\ 2 & -2+\sqrt{2} \end{vmatrix}=-\sqrt{2}$

$C_1=-\frac{1}{2}, \quad C_2=-\frac{1}{2}$

$y(x)=-\frac{1}{2}e^{-(2+\sqrt{2})x}-\frac{1}{2}e^{(-2+\sqrt{2})x}$

Exercise

Find the solution of the given initial value problem $y''-4y'+3y=0, \quad y(0)=1, \quad y'(0)=\frac{1}{3}$

Solution

The characteristic equation: $\lambda^2-4\lambda+3=0 \quad \rightarrow \quad \lambda_{1,2}=1,\ 3$

$y(x)=C_1e^{x}+C_2e^{3x}$

$y(0)=1 \quad \rightarrow C_1+C_2=1$

$y'(x)=C_1e^{x}+3C_2e^{3x}$

$y'(0)=\frac{1}{3} \quad \rightarrow C_1+3C_2=\frac{1}{3}$

$$\begin{cases} C_1 + C_2 = 1 \\ C_1 + 3C_2 = \frac{1}{3} \end{cases} \rightarrow C_2 = -\frac{1}{3} \quad C_1 = \frac{4}{3}$$

$$\boxed{y(x) = \frac{4}{3}e^x - \frac{1}{3}e^{3x}}$$

Exercise

Find the solution of the given initial value problem $\quad y'' - 4y' + 4y = 0; \quad y(1) = 1, \quad y'(1) = 1$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = 2}$

$$\boxed{y(x) = (C_1 + C_2 x)e^{2x}}$$

$$y(1) = 1 \quad \rightarrow \quad (C_1 + C_2)e^2 = 1 \quad \Rightarrow \quad C_1 + C_2 = e^{-2}$$

$$y'(x) = (C_2 + 2C_1 + 2C_2 x)e^{2x}$$

$$y'(1) = 1 \quad \rightarrow \quad (2C_1 + 3C_2)e^2 = 1 \quad \Rightarrow \quad 2C_1 + 3C_2 = e^{-2}$$

$$\begin{cases} C_1 + C_2 = e^{-2} \\ 2C_1 + 3C_2 = e^{-2} \end{cases} \rightarrow \Delta = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1 \quad \Delta_{C_1} = \begin{vmatrix} e^{-2} & 1 \\ e^{-2} & 3 \end{vmatrix} = 2e^{-2}$$

$$\Rightarrow \boxed{C_1 = 2e^{-2}} \quad \boxed{C_2 = -e^{-2}}$$

$$y(x) = \left(2e^{-2} - e^{-2}x\right)e^{-x}$$

$$\boxed{= 2e^{-x-2} - xe^{-x-2}}$$

Exercise

Find the solution of the given initial value problem $\quad y'' + 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 3$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -2}$

$$\boxed{y(x) = (C_1 + C_2 x)e^{-2x}}$$

$$y(0) = 1 \quad \rightarrow \quad \boxed{C_1 = 1}$$

$$y'(x) = (C_2 - 2C_1 - 2C_2 x)e^{-2x}$$

$$y'(0)=3 \ \rightarrow \ C_2-2=3 \ \Rightarrow \ \boxed{C_2=5}$$

$$\boxed{y(x)=(1+5x)e^{-2x}}$$

Exercise

Find the solution of the given initial value problem: $y''-4y'+5y=0; \quad y(0)=1, \ \ y'(0)=5$

Solution

The characteristic equation: $\lambda^2-4\lambda+5=0 \ \Rightarrow \ \lambda_{1,2}=2\pm i$

$$y(x)=e^{2x}\left(C_1\cos x+C_2\sin x\right)$$

$$\Rightarrow y(0)=\boxed{C_1=1}$$

$$y'(x)=2e^{2x}\left(C_1\cos x+C_2\sin x\right)+e^{2x}\left(-C_1\sin x+C_2\cos x\right)$$

$$\Rightarrow y'(0)=2C_1+C_2=5 \quad \Rightarrow \ \boxed{C_2=3}$$

$$\boxed{y(x)=e^{2x}\left(\cos x+3\sin x\right)}$$

Exercise

Find the solution of the given initial value problem $y''+4y'+5y=0\ ; \ \ y(0)=1, \ \ y'(0)=0$

Solution

The characteristic equation: $\lambda^2+4\lambda+5=0$

$$\rightarrow \ \lambda_{1,2}=\frac{-4\pm 2i}{2}\ \boxed{=-2\pm i}$$

$$\boxed{y(x)=e^{-2x}\left(C_1\cos x+C_2\sin x\right)}$$

$$y(0)=1 \ \rightarrow \ \boxed{C_1=1}$$

$$y'(x)=e^{-2x}\left(-2C_1\cos x-2C_2\sin x-C_1\sin x+C_2\cos x\right)$$

$$y'(0)=0 \ \rightarrow \ -2C_1+C_2=0 \ \Rightarrow \ \boxed{C_2=2}$$

$$\boxed{y(x)=e^{-2x}\left(\cos x+2\sin x\right)}$$

Exercise

Find the solution of the given initial value problem $\quad y''+4y'+5y=0; \quad y\left(\frac{\pi}{2}\right)=\frac{1}{2}, \quad y'\left(\frac{\pi}{2}\right)=-2$

Solution

The characteristic equation: $\quad \lambda^2+4\lambda+5=0$

$$\to \lambda_{1,2}=\frac{-4\pm 2i}{2}=-2\pm i$$

$$y(x)=e^{-2x}\left(C_1\cos x+C_2\sin x\right)$$

$$y\left(\frac{\pi}{2}\right)=\frac{1}{2} \quad \to e^{-\pi}C_2=\frac{1}{2} \quad \Rightarrow C_2=\frac{e^{\pi}}{2}$$

$$y'(x)=e^{-2x}\left(-2C_1\cos x-2C_2\sin x-C_1\sin x+C_2\cos x\right)$$

$$y'\left(\frac{\pi}{2}\right)=-2 \quad \to e^{-\pi}\left(-2\frac{e^{\pi}}{2}-C_1\right)=-2$$

$$\Rightarrow 1+e^{-\pi}C_1=2 \to C_1=e^{\pi}$$

$$y(x)=e^{-2x}\left(e^{\pi}\cos x+\frac{1}{2}e^{\pi}\sin x\right)$$

$$=e^{\pi-2x}\left(\cos x+\frac{1}{2}\sin x\right)$$

Exercise

Find the solution of the given initial value problem $\quad y''-4y'-5y=0\ ; \quad y(1)=0, \quad y'(1)=2$

Solution

The characteristic equation: $\quad \lambda^2-4\lambda-5=0 \quad \to \quad \lambda_{1,2}=-1,\ 5$

$$y(x)=C_1e^{-x}+C_2e^{5x}$$

$$y(1)=0 \quad \to C_1e^{-1}+C_2e^{5}=0$$

$$y'(x)=-C_1e^{-x}+5C_2e^{5x}$$

$$y'(1)=2 \quad \to e^{-1}C_1+5e^{5}C_2=2$$

$$\begin{cases} e^{-1}C_1+e^5C_2=0 \\ -e^{-1}C_1+5e^5C_2=2 \end{cases} \to \Delta=\begin{vmatrix} e^{-1} & e^5 \\ -e^{-1} & 5e^5 \end{vmatrix}=6e^4 \quad \Delta_{C_1}=\begin{vmatrix} 0 & e^5 \\ 2 & 5e^5 \end{vmatrix}=-2e^5$$

$$\Rightarrow C_1=-\frac{1}{3}e, \ C_2=\frac{1}{3}e^{-5}$$

$$y(x)=-\frac{1}{3}e^{1-x}+\frac{1}{3}e^{-5x-5}$$

Exercise

Find the solution of the given initial value problem $y'' - 4y' - 5y = 0$, $y(-1) = 3$, $y'(-1) = 9$

Solution

The characteristic equation: $\lambda^2 - 4\lambda - 5 = 0 \rightarrow \lambda_{1,2} = -1,\ 5$

$$y(x) = C_1 e^{-x} + C_2 e^{5x}$$

$$y(-1) = 3 \rightarrow C_1 e + C_2 e^{-5} = 3$$

$$y'(x) = -C_1 e^{-x} + 5C_2 e^{5x}$$

$$y'(-1) = 9 \rightarrow -C_1 e + 5C_2 e^{-5} = 9$$

$$\begin{cases} eC_1 + e^{-5}C_2 = 3 \\ -eC_1 + 5e^{-5}C_2 = 9 \end{cases} \rightarrow \Delta = \begin{vmatrix} e & e^{-5} \\ -e & 5e^{-5} \end{vmatrix} = 6e^{-4} \quad \Delta_{C_1} = \begin{vmatrix} 3 & e^{-5} \\ 9 & 5e^{-5} \end{vmatrix} = 6e^{-5}$$

$$\Rightarrow C_1 = e^{-1},\ C_2 = 2e^5$$

$$y(x) = e^{-1-x} + 2e^{5x+5}$$

Exercise

Find the solution of the given initial value problem $y'' - 4y' + 9y = 0$, $y(0) = 0$, $y'(0) = -8$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 9 = 0$

$$\rightarrow \lambda_{1,2} = \frac{4 \pm 2i\sqrt{5}}{2} = 2 \pm i\sqrt{5}$$

$$y(t) = e^{2t}\left(C_1 \cos\sqrt{5}t + C_2 \sin\sqrt{5}t\right)$$

$$y(0) = 0 \rightarrow C_1 = 0$$

$$y' = e^{2t}\left(2C_1 \cos\sqrt{5}t + 2C_2 \sin\sqrt{5}t - \sqrt{5}C_1 \sin\sqrt{5}t + \sqrt{5}C_2 \cos\sqrt{5}t\right)$$

$$y'(0) = 8 \rightarrow -\sqrt{5}C_2 = 8 \Rightarrow C_2 = -\frac{8}{\sqrt{5}}$$

$$y(t) = -\frac{8}{\sqrt{5}} e^{2t} \sin\sqrt{5}t$$

Exercise

Find the solution of the given initial value problem. $y'' - 4y' + 13y = 0; \quad y(0) = -1, \quad y'(0) = 2$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 13 = 0$

$$\Rightarrow \lambda_{1,\,2} = \frac{4 \pm \sqrt{16-52}}{2} = 2 \pm 3i$$

$$\underline{y(x) = e^{2x}\left(C_1 \cos 3x + C_2 \sin 3x\right)}$$

$$y(0) = e^0\left(C_1 \cos(0) + C_2 \sin(0)\right) \quad \to \underline{C_1 = -1}$$

$$y'(x) = 2e^{2x}\left(C_1 \cos 3x + C_2 \sin 3x\right) + e^{2x}\left(-3C_1 \sin 3x + 3C_2 \cos 3x\right)$$

$$y'(0) = 2C_1 + 3C_2 = 2 \quad \Rightarrow \quad \underline{C_2 = \frac{2-2(-1)}{3} = \frac{4}{3}}$$

$$\underline{y(x) = e^{2x}\left(-\cos 3x + \tfrac{4}{3}\sin 3x\right)}$$

Exercise

Find the solution of the given initial value problem $y'' - 5y' + 6y = 0; \quad y(1) = e^2, \quad y'(1) = 3e^2$

Solution

The characteristic equation: $\lambda^2 - 5\lambda + 6 = 0 \quad \to \quad \underline{\lambda_{1,2} = 2,\ 3}$

$$y(x) = C_1 e^{2x} + C_2 e^{3x}$$

$$y(1) = e^2 \quad \to C_1 e^2 + C_2 e^3 = e^2 \quad \Rightarrow \underline{C_1 + eC_2 = 1}$$

$$y'(x) = 2C_1 e^{2x} + 3C_2 e^{3x}$$

$$y'(1) = 3e^2 \quad \to 2C_1 e^2 + 3C_2 e^3 = 3e^2 \quad \Rightarrow \underline{2C_1 + 3eC_2 = 3}$$

$$-2\times\begin{cases} C_1 + eC_2 = 1 \\ 2C_1 + 3eC_2 = 3 \end{cases} \quad \to eC_2 = 1 \quad \underline{C_2 = e^{-1};\ C_1 = 0}$$

$$\underline{y(x) = e^{3x-1}}$$

Exercise

Find the solution of the given initial value problem $y'' + 6y' + 5y = 0, \quad y(1) = 0, \quad y'(0) = 3$

Solution

The characteristic equation: $\lambda^2+6\lambda+5=0 \quad \rightarrow \quad \boxed{\lambda_{1,2}=-1,\ -5}$

$y(x)=C_1e^{-x}+C_2e^{-5x}$

$y(1)=0 \quad \rightarrow \quad C_1e^{-1}+C_2e^{-5}=0 \quad \Rightarrow C_1e^4+C_2=e^5$

$y'(x)=-C_1e^{-x}-5C_2e^{-5x}$

$y'(0)=3 \quad \rightarrow \quad -C_1-5C_2=3$

$$\begin{cases} e^4C_1+C_2=e^5 \\ C_1+5C_2=-3 \end{cases}$$

$$\rightarrow \Delta=\begin{vmatrix} e^4 & 1 \\ 1 & 5 \end{vmatrix}=5e^4-1 \quad \Delta_{C_1}=\begin{vmatrix} e^5 & 1 \\ -3 & 5 \end{vmatrix}=5e^5+3 \quad \Delta_{C_2}=\begin{vmatrix} e^4 & e^5 \\ 1 & -3 \end{vmatrix}=-3e^4-e^5$$

$$\boxed{C_1=\frac{5e^5+3}{5e^4-1};\ C_2=-\frac{3e^4+e^5}{5e^4-1}}$$

$$\boxed{y(x)=\frac{5e^5+3}{5e^4-1}e^{-x}-\frac{3e^4+e^5}{5e^4-1}e^{-5x}}$$

Exercise

Find the solution of the given initial value problem $\quad y''-6y'+5y=0; \quad y(0)=3, \quad y'(0)=11$

Solution

The characteristic equation: $\lambda^2-6\lambda+5=0 \quad \rightarrow \quad \boxed{\lambda_{1,2}=1,\ 5}$

$y(x)=C_1e^{x}+C_2e^{5x}$

$y(0)=3 \quad \rightarrow \quad \boxed{C_1+C_2=3}$

$y'(x)=C_1e^{x}+5C_2e^{5x}$

$y'(0)=11 \quad \rightarrow \quad \boxed{C_1+5C_2=11}$

$$\begin{cases} C_1+C_2=3 \\ C_1+5C_2=11 \end{cases} \rightarrow \quad \boxed{C_2=2;\ C_1=1}$$

$$\boxed{y(x)=e^{x}+2e^{5x}}$$

Exercise

Find the solution of the given initial value problem $y'' - 6y' + 9y = 0, \quad y(0) = 2, \quad y'(0) = \frac{25}{3}$

Solution

The characteristic equation: $\lambda^2 - 6\lambda + 9 = 0 \quad \to \quad \underline{\lambda_{1,2} = 3}$

$$\underline{y(x) = (C_1 + C_2 x)e^{3x}}$$

$$y(0) = 2 \quad \to \quad \underline{C_1 = 2}$$

$$y'(x) = (C_2 + 3C_1 + 3C_2 x)e^{3x}$$

$$y'(0) = \frac{25}{3} \quad \to C_2 + 3C_1 = \frac{25}{3} \quad \underline{C_2 = \frac{7}{3}}$$

$$\underline{y(x) = \left(2 + \frac{7}{3}x\right)e^{3x}}$$

Exercise

Find the solution of the given initial value problem $y'' - 6y' + 9y = 0; \quad y(0) = 0, \quad y'(0) = 5$

Solution

The characteristic equation: $\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0 \quad \to \quad \underline{\lambda_{1,2} = 3}$

$$y(x) = (C_1 + C_2 x)e^{3x}$$

$$y(0) = 0 \quad \to \quad \underline{C_1 = 0}$$

$$y'(x) = (C_2 + 3C_1 + 3C_2 x)e^{3x}$$

$$y'(0) = 5 \quad \to 3C_1 + C_2 = 5 \Rightarrow \underline{C_2 = 5}$$

$$\underline{y(x) = 5xe^{3x}}$$

Exercise

Find the solution of the given initial value problem $y'' + 6y' + 9y = 0; \quad y(0) = 2, \quad y'(0) = -2$

Solution

The characteristic equation: $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0 \quad \to \quad \underline{\lambda_{1,2} = -3}$

$$y(x) = (C_1 + C_2 x)e^{-3x}$$

$$y(0) = 2 \quad \to \quad \underline{C_1 = 2}$$

$$y'(x) = (C_2 - 3C_1 - 3C_2 x)e^{-3x}$$

$$y'(0) = -2 \quad \rightarrow C_2 - 6 = -2 \;\Rightarrow\; \boxed{C_2 = 4}$$

$$\boxed{y(x) = (2 + 4x)e^{-3x}}$$

Exercise

Find the solution of the given initial value problem $\quad y'' + 8y' - 9y = 0; \quad y(1) = 2, \quad y'(1) = 0$

Solution

The characteristic equation: $\lambda^2 + 8\lambda - 9 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = 1, \ -9}$

$$y(x) = C_1 e^{-9x} + C_2 e^x$$

$$y(1) = 2 \quad \rightarrow C_1 e^{-9} + C_2 e = 2 \;\Rightarrow\; \boxed{C_1 + e^{10} C_2 = 2e^9}$$

$$y'(x) = -9C_1 e^{-9x} + C_2 e^x$$

$$y'(1) = 0 \quad \rightarrow \; -9C_1 e^{-9} + C_2 e = 0 \Rightarrow \boxed{C_2 = 9e^{-10} C_1}$$

$$C_1 + e^{10}\left(9e^{-10} C_1\right) = 2e^9$$

$$\Rightarrow \; \boxed{C_1 = \frac{e^9}{5}, \quad C_2 = \frac{9}{5e}}$$

$$\boxed{y(x) = \frac{1}{5} e^{9-9x} + \frac{9}{5} e^{x-1}}$$

Exercise

Find the solution of the given initial value problem. $\quad y'' - 8y' + 17y = 0; \quad y(0) = 4, \quad y'(0) = -1$

Solution

The characteristic equation: $\lambda^2 - 8\lambda + 17 = 0$

$$\Rightarrow \lambda_{1,\,2} = \frac{8 \pm \sqrt{64 - 68}}{2} = \frac{8 \pm i2}{2} = 4 \pm i$$

$$\boxed{y(x) = e^{4x}\left(C_1 \cos x + C_2 \sin x\right)}$$

$$y(0) = e^0\left(C_1 \cos(0) + C_2 \sin(0)\right) \;\rightarrow\; \boxed{4 = C_1}$$

$$y'(x) = 4e^{4x}\left(C_1 \cos x + C_2 \sin x\right) + e^{4x}\left(-C_1 \sin x + C_2 \cos x\right)$$

$$y'(0) = 4C_1 + C_2 = -1 \quad \Rightarrow \quad \boxed{C_2 = -1 - 16 = -17}$$

$$\boxed{y(x) = e^{4x}\left(4\cos x - 17\sin x\right)}$$

Exercise

Find the solution of the given initial value problem $\quad y'' - 9y = 0, \quad y(0) = 2, \quad y'(0) = -1$

Solution

The characteristic equation: $\lambda^2 - 9 = 0 \quad \rightarrow \quad \underline{\lambda_{1,2} = \pm 3|}$

$$y(x) = C_1 e^{-3x} + C_2 e^{3x}$$

$$y(0) = 2 \quad \rightarrow \quad \Rightarrow \underline{C_1 + C_2 = 2|}$$

$$y'(x) = -3C_1 e^{-3x} + 3C_2 e^{3x}$$

$$y'(0) = -1 \quad \rightarrow \quad \underline{-3C_1 + 3C_2 = -1|}$$

$$\begin{cases} C_1 + C_2 = 2 \\ -3C_1 + 3C_2 = -1 \end{cases} \quad \rightarrow \quad \underline{C_1 = \frac{7}{6}, \; C_2 = \frac{5}{6}|}$$

$$\underline{y(x) = \frac{7}{6} e^{-3x} + \frac{5}{6} e^{3x}|}$$

Exercise

Find the solution of the given initial value problem $\quad y'' - 10y' + 25y = 0, \quad y(0) = 1, \quad y'(1) = 0$

Solution

The characteristic equation: $\lambda^2 - 10\lambda + 25 = (\lambda - 5)^2 = 0 \quad \rightarrow \quad \underline{\lambda_{1,2} = 5|}$

$$y(x) = (C_1 + C_2 x) e^{5x}$$

$$y(0) = 1 \quad \rightarrow \underline{C_1 = 1|}$$

$$y'(x) = (C_2 + 5C_1 + 5C_2 x) e^{5x}$$

$$y'(1) = 0 \quad \rightarrow \quad (6C_2 + 5C_1) e^5 = 0$$

$$\Rightarrow 6C_2 + 5C_1 = 0 \rightarrow \underline{C_2 = -\frac{5}{6}|}$$

$$\underline{y(x) = \left(1 - \frac{5}{6} x\right) e^{5x}|}$$

Exercise

Find the general solution: $\quad y'' + 10y' + 25y = 0; \quad y(0) = 2, \quad y'(0) = -1$

Solution

The characteristic equation: $\lambda^2 + 10\lambda + 25 = 0$

$\Rightarrow \lambda_{1,2} = -5$

$y(t) = \left(C_1 + C_2 t\right)e^{-5t}$

$$y(t) = \left(C_1 + C_2 t\right)e^{-5t} \qquad y(0) = \boxed{C_1 = 2}$$

$$y' = C_2 e^{-5t} - 5\left(C_1 + C_2 t\right)e^{-5t} \quad y'(0) = C_2 - 5C_1 = -1 \quad \Rightarrow \boxed{C_2 = 9}$$

$$\underline{y(t) = (2 + 9t)e^{-5t}}$$

Exercise

Find the general solution: $y'' + 11y' + 24y = 0;\quad y(0) = 0,\quad y'(0) = -7$

Solution

The characteristic equation: $\lambda^2 + 11\lambda + 24 = 0 \quad \to \lambda_{1,2} = \frac{-11 \pm 5}{2} \quad \underline{\lambda_{1,2} = -8,\ -3}$

$y(x) = C_1 e^{-8x} + C_2 e^{-3x}$

$$y(0) = 0 \quad \to \quad \Rightarrow \underline{C_1 + C_2 = 0}$$

$y' = -8C_1 e^{-8x} - 3C_2 e^{-3x}$

$$y'(0) = -7 \quad \to \quad \underline{-8C_1 - 3C_2 = -7}$$

$$\begin{cases} C_1 + C_2 = 0 \\ -8C_1 - 3C_2 = -7 \end{cases} \quad \to \quad \underline{C_1 = \frac{7}{5},\ C_2 = -\frac{7}{5}}$$

$$\underline{y(x) = \frac{7}{5}e^{-8x} - \frac{7}{5}e^{-3x}}$$

Exercise

Find the solution of the given initial value problem $\quad y'' + 12y = 0,\quad y(0) = 0,\quad y'(0) = 1$

Solution

The characteristic equation: $\lambda^2 + 12 = 0 \quad \to \quad \underline{\lambda_{1,2} = \pm 2i\sqrt{3}}$

$y(x) = C_1 \cos 2\sqrt{3}x + C_2 \sin 2\sqrt{3}x$

$$y(0) = 0 \quad \to \quad \underline{C_1 = 0}$$

$y'(x) = -2\sqrt{3}C_1 \sin 2\sqrt{3}x + 2\sqrt{3}C_2 \cos 2\sqrt{3}x$

$y'(0)=2 \quad \to \ 2\sqrt{3}C_2 = 2 \quad \Rightarrow \ \boxed{C_2 = \frac{\sqrt{3}}{3}}$

$\boxed{y(x) = \frac{\sqrt{3}}{3}\sin 2\sqrt{3}x}$

Exercise

Find the solution of the given initial value problem $\quad y''+16y=0, \quad y(\pi)=2, \quad y'(0)=-2$

Solution

The characteristic equation: $\quad \lambda^2+16=0 \quad \to \quad \boxed{\lambda_{1,2} = \pm 4i}$

$y(x)=C_1\cos 4x + C_2\sin 4x$

$y(0)=2 \quad \to \ \boxed{C_1 = 2}$

$y'(x) = -4C_1\sin 4x + 4C_2\cos 4x$

$y'(0)=-2 \quad \to \ 4C_2 = -2 \quad \Rightarrow \ \boxed{C_2 = -\frac{1}{2}}$

$\boxed{y(x) = 2\cos 4x - \frac{1}{2}\sin 4x}$

Exercise

Find the solution of the given initial value problem $\quad y''+16y=0, \quad y\left(\frac{\pi}{2}\right)=-10, \quad y'\left(\frac{\pi}{2}\right)=3$

Solution

The characteristic equation: $\quad \lambda^2+16=0 \quad \to \quad \boxed{\lambda_{1,2} = \pm 4i}$

$y(x)=C_1\cos 4x + C_2\sin 4x$

$y\left(\frac{\pi}{2}\right)=-10 \quad \to \ \boxed{C_1 = -10}$

$y' = -4C_1\sin 4x + 4C_2\cos 4x$

$y'\left(\frac{\pi}{2}\right)=3 \quad \to \ 4C_2 = 3 \quad \Rightarrow \ \boxed{C_2 = \frac{3}{4}}$

$\boxed{y(x) = -10\cos 4x + \frac{3}{4}\sin 4x}$

Exercise

Find the solution of the given initial value problem $y'' + 16y = 0,\quad y(\pi) = 2,\quad y'(0) = -2$

Solution

The characteristic equation: $\lambda^2 + 16 = 0 \;\rightarrow\; \underline{\lambda_{1,2} = \pm 4i}|$

$y(x) = C_1 \cos 4x + C_2 \sin 4x$

$y(\pi) = 2 \;\rightarrow\; \underline{C_1 = 2}|$

$y'(x) = -4C_1 \sin 4x + 4C_2 \cos 4x$

$y'(0) = -2 \;\rightarrow\; 4C_2 = -2 \;\Rightarrow\; \underline{C_2 = -\frac{1}{2}}|$

$\underline{y(x) = 2\cos 4x - \frac{1}{2}\sin 4x}|$

Exercise

Find the general solution: $y'' + 25y = 0;\quad y(0) = 1,\quad y'(0) = -1$

Solution

The characteristic equation: $\lambda^2 + 25 = 0$

$\Rightarrow \lambda_{1,2} = \pm 5i$

$y(t) = e^{0t}\left(C_1 \cos 5t + C_2 \sin 5t\right)$

$y(t) = C_1 \cos 5t + C_2 \sin 5t$

$y(t) = C_1 \cos 5t + C_2 \sin 5t \;\rightarrow y(0) = \boxed{C_1 = 1}$

$y'(t) = -5C_1 \sin 5t + 5C_2 \cos 5t$

$\Rightarrow y'(0) = 5C_2 = -1 \rightarrow \boxed{C_2 = -\frac{1}{5}}$

$\underline{y(t) = 5\cos 5t - \frac{1}{5}\sin 5t}|$

Exercise

Find the solution of the given initial value problem $2y'' - 2y' + y = 0\;;\quad y(-\pi) = 1,\quad y'(-\pi) = -1$

Solution

The characteristic equation: $2\lambda^2 - 2\lambda + 1 = 0 \;\rightarrow\; \lambda_{1,2} = \frac{2 \pm 2i}{4} = \underline{\frac{1}{2} \pm \frac{1}{2}i}|$

$$\boxed{y(x)=e^{x/2}\left(C_1\cos\frac{x}{2}+C_2\sin\frac{x}{2}\right)}$$

$$y(-\pi)=1 \ \rightarrow\ e^{-\pi/2}\left(-C_2\right)=1 \ \Rightarrow\ \boxed{C_2=-e^{\pi/2}}$$

$$y'(x)=e^{x/2}\left(\frac{1}{2}C_1\cos\frac{x}{2}+\frac{1}{2}C_2\sin\frac{x}{2}-\frac{1}{2}C_1\sin\frac{x}{2}+\frac{1}{2}C_2\cos\frac{x}{2}\right)$$

$$y'(-\pi)=-1 \ \rightarrow\ e^{-\pi/2}\left(\frac{1}{2}e^{\pi/2}+\frac{1}{2}C_1\right)=-1$$

$$\frac{1}{2}+\frac{1}{2}e^{-\pi/2}C_1=-1 \ \rightarrow\ \boxed{C_1=-3e^{\pi/2}}$$

$$y(x)=e^{x/2}\left(-3e^{\pi/2}\cos\frac{x}{2}-e^{\pi/2}\sin\frac{x}{2}\right)$$

$$\boxed{=-e^{(x+\pi)/2}\left(3\cos\frac{x}{2}+\sin\frac{x}{2}\right)}$$

Exercise

Find the solution of the given initial value problem $3y''+y'-14y=0,\quad y(0)=2,\quad y'(0)=-1$

Solution

The characteristic equation: $3\lambda^2+\lambda-14=0 \ \rightarrow\ \lambda_{1,2}=\frac{-1\pm 13}{6}=\boxed{-\frac{7}{3},\ 2}$

$$y(x)=C_1e^{-7x/3}+C_2e^{2x}$$

$$y(0)=2 \ \rightarrow\ C_1+C_2=2$$

$$y'(x)=-\frac{7}{3}C_1e^{-7x/3}+2C_2e^{2x}$$

$$y'(0)=-1 \ \rightarrow\ -\frac{7}{3}C_1+2C_2=-1$$

$$\begin{cases} C_1+C_2=2 \\ -7C_1+6C_2=-3 \end{cases} \ \rightarrow\ \boxed{C_2=\frac{11}{13},\ C_1=\frac{15}{13}}$$

$$\boxed{y(x)=\frac{11}{13}e^{-7x/3}+\frac{15}{13}e^{2x}}$$

Exercise

Find the solution of the given initial value problem $3y''+2y'-8y=0,\quad y(0)=-6,\quad y'(0)=-18$

Solution

The characteristic equation: $3\lambda^2+2\lambda-8=0 \ \rightarrow\ \lambda_{1,2}=\frac{-2\pm 10}{6}=\boxed{-2,\ \frac{4}{3}}$

$$y(x)=C_1e^{-2x}+C_2e^{4x/3}$$

$$y(0)=-6 \quad \rightarrow \ C_1+C_2=-6$$

$$y'=-2C_1e^{-2x}+\frac{4}{3}C_2e^{4x/3}$$

$$y'(0)=-18 \quad \rightarrow \ -2C_1+\frac{4}{3}C_2=-18$$

$$\begin{cases} C_1+C_2=-6 \\ -3C_1+2C_2=-27 \end{cases} \quad \rightarrow \ \underline{C_1=\frac{15}{5}=3, \ \ C_2=-9}$$

$$\underline{y(x)=3e^{-2x}-9e^{4x/3}}$$

Exercise

Find the solution of the given initial value problem $4y''-4y'+y=0, \quad y(0)=4, \quad y'(0)=4$

Solution

The characteristic equation: $4\lambda^2-4\lambda+1=(2\lambda-1)^2=0 \quad \rightarrow \ \underline{\lambda_{1,2}=\frac{1}{2}}$

$$y(x)=\left(C_1+C_2x\right)e^{x/2}$$

$$y(0)=4 \quad \rightarrow \ \underline{C_1=4}$$

$$y'(x)=\left(C_2+\frac{1}{2}C_1+\frac{1}{2}C_2x\right)e^{x/2}$$

$$y'(0)=4 \quad \rightarrow \ C_2+\frac{1}{2}C_1=4 \quad \Rightarrow \ \underline{C_2=2}$$

$$\underline{y(x)=2(1+x)e^{x/2}}$$

Exercise

Find the solution of the given initial value problem $4y''-4y'+y=0; \quad y(1)=-4, \quad y'(1)=0$

Solution

The characteristic equation: $4\lambda^2-4\lambda+1=(2\lambda-1)^2=0 \quad \rightarrow \ \underline{\lambda_{1,2}=\frac{1}{2}}$

$$\underline{y(x)=\left(C_1+C_2x\right)e^{x/2}}$$

$$y(1)=-4 \quad \rightarrow \ \left(C_1+C_2\right)e^{1/2}=-4$$

$$y'(x)=\left(C_2+\frac{1}{2}C_1+\frac{1}{2}C_2x\right)e^{x/2}$$

$$y'(1)=0 \quad \rightarrow \ \left(C_2+\frac{1}{2}C_1+\frac{1}{2}C_2\right)e^{1/2}=0 \quad \Rightarrow \ C_1+3C_2=0$$

$$\begin{cases} C_1 + C_2 = -4e^{-1/2} \\ C_1 + 3C_2 = 0 \end{cases} \rightarrow \underline{C_2 = 2e^{-1/2} \quad C_1 = -6e^{-1/2}}$$

$$y(x) = \left(-6e^{-1/2} + 2e^{-1/2}x\right)e^{x/2}$$

$$\underline{= 2(x-3)e^{(x-1)/2}}$$

Exercise

Find the solution of the given initial value problem $4y'' - 4y' - 3y = 0, \quad y(0) = 1, \quad y'(0) = 5$

Solution

The characteristic equation: $4\lambda^2 - 4\lambda - 3 = 0 \rightarrow \lambda_{1,2} = \frac{4 \pm 8}{8} \underline{= -\frac{1}{2}, \ \frac{3}{2}}$

$$y(x) = C_1 e^{-x/2} + C_2 e^{3x/2}$$

$$y(0) = 1 \rightarrow C_1 + C_2 = 1$$

$$y'(x) = -\frac{1}{2}C_1 e^{-x/2} + \frac{3}{2}C_2 e^{3x/2}$$

$$y'(0) = 5 \rightarrow -\frac{1}{2}C_1 + \frac{3}{2}C_2 = 5$$

$$\begin{cases} C_1 + C_2 = 1 \\ -C_1 + 3C_2 = 10 \end{cases} \rightarrow C_2 = \frac{11}{4} \quad C_1 = -\frac{7}{4}$$

$$\underline{y(x) = -\frac{7}{4}e^{-x/2} + \frac{11}{4}e^{3x/2}}$$

Exercise

Find the solution of the given initial value problem $4y'' + 4y' + 5y = 0, \quad y(\pi) = 1, \quad y'(\pi) = 0$

Solution

The characteristic equation: $4\lambda^2 + 4\lambda + 5 = 0 \rightarrow \lambda_{1,2} = \frac{-4 \pm 8i}{8} \underline{= -\frac{1}{2} \pm i}$

$$y(x) = e^{-x/2}\left(C_1 \cos x + C_2 \sin x\right)$$

$$y(\pi) = 1 \rightarrow -C_1 e^{-\pi/2} = 1 \quad \underline{C_1 = -e^{\pi/2}}$$

$$y'(x) = e^{-x/2}\left(-C_1 \sin x + C_2 \cos x - \frac{1}{2}C_1 \cos x - \frac{1}{2}C_2 \sin x\right)$$

$$y'(\pi) = 0 \rightarrow \left(-C_2 + \frac{1}{2}C_1\right)e^{-\pi/2} = 0$$

$$\underline{C_2} = \frac{1}{2}C_1 \underline{= -\frac{1}{2}e^{\pi/2}}$$

$$y(x) = e^{-x/2}\left(-e^{\pi/2}\cos x - \frac{1}{2}e^{\pi/2}\sin x\right)$$

$$= -\frac{1}{2}e^{(\pi - x)/2}(2\cos x + \sin x)$$

Exercise

Find the solution of the given initial value problem $4y'' + 4y' + 17y = 0, \quad y(0) = -1, \quad y'(0) = 2$

Solution

The characteristic equation: $4\lambda^2 + 4\lambda + 17 = 0$

$$\rightarrow \quad \lambda_{1,2} = \frac{-4 \pm 16i}{8} = -\frac{1}{2} \pm 2i$$

$$y(x) = e^{-x/2}\left(C_1\cos 2x + C_2\sin 2x\right)$$

$$y(0) = -1 \quad \rightarrow \quad -1 = C_1$$

$$y'(x) = e^{-x/2}\left(-\frac{1}{2}C_1\cos 2x - \frac{1}{2}C_2\sin 2x - 2C_1\sin 2x + 2C_2\cos 2x\right)$$

$$y'(0) = 2 \quad \rightarrow \quad 2 = \frac{1}{2} + 2C_2 \quad \Rightarrow C_2 = \frac{3}{4}$$

$$y(x) = e^{-x/2}\left(-\cos 2x + \frac{3}{4}\sin 2x\right)$$

Exercise

Find the solution of the given initial value problem $4y'' - 5y' = 0, \quad y(-2) = 0, \quad y'(-2) = 7$

Solution

The characteristic equation: $4\lambda^2 - 5\lambda = 0 \quad \rightarrow \quad \lambda_{1,2} = 0, \ \frac{5}{4}$

$$y(t) = C_1 + C_2 e^{5t/4}$$

$$y(-2) = 0 \quad \rightarrow \quad C_1 + C_1 e^{-5/2} = 0$$

$$y' = \frac{5}{4}C_2 e^{5t/4}$$

$$y'(-2) = 7 \quad \rightarrow \frac{5}{4}C_2 e^{-5/2} = 7 \quad \Rightarrow \quad C_2 = \frac{28}{5}e^{5/2}$$

$$C_1 = -\frac{28}{5}$$

$$y(t) = -\frac{28}{5} + \frac{28}{5}e^{5t/4}$$

Exercise

Find the solution of the given initial value problem $4y'' + 12y' + 9y = 0, \quad y(0) = 2, \quad y'(0) = 1$

Solution

The characteristic equation: $4\lambda^2 + 12\lambda + 9 = (2\lambda + 3)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -\frac{3}{2}}$

$$y(x) = \left(C_1 + C_2 x\right)e^{-3x/2}$$

$$y(0) = 2 \quad \rightarrow \quad \boxed{C_1 = 2}$$

$$y'(x) = \left(C_2 - \frac{3}{2}C_1 - \frac{3}{2}C_2 x\right)e^{-3x/2}$$

$$y'(0) = 1 \quad \rightarrow \quad C_2 - \frac{3}{2}C_1 = 1 \quad \Rightarrow \boxed{C_2 = 4}$$

$$\boxed{y(x) = (2 + 4x)e^{-3x/2}}$$

Exercise

Find the solution of the given initial value problem $4y'' + 24y' + 37y = 0, \quad y(\pi) = 1, \quad y'(\pi) = 0$

Solution

The characteristic equation: $4\lambda^2 + 24\lambda + 37 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{-24 \pm 4i}{8} = \boxed{-3 \pm \frac{1}{2}i}$

$$y(t) = e^{-3t}\left(C_1 \cos\frac{1}{2}t + C_2 \sin\frac{1}{2}t\right)$$

$$y(\pi) = 1 \quad \rightarrow e^{-3\pi}C_2 = 1 \quad \Rightarrow \quad \boxed{C_2 = e^{3\pi}}$$

$$y' = e^{-3t}\left(-3C_1 \cos\frac{1}{2}t - 3C_2 \sin\frac{1}{2}t - \frac{1}{2}C_1 \sin\frac{1}{2}t + \frac{1}{2}C_2 \cos\frac{1}{2}t\right)$$

$$y'(\pi) = 0 \quad \rightarrow e^{-3\pi}\left(-3e^{3\pi} - \frac{1}{2}C_1\right) = 0 \quad \Rightarrow \quad \boxed{C_1 = -6e^{3\pi}}$$

$$y(t) = e^{-3t}\left(-6e^{3\pi} \cos\frac{1}{2}t + e^{3\pi} \sin\frac{1}{2}t\right)$$

$$\boxed{= -6e^{3(\pi - t)} \cos\frac{t}{2} + e^{3(\pi - t)} \sin\frac{t}{2}}$$

Exercise

Find the solution of the given initial value problem $9y'' + y = 0; \quad y\left(\frac{\pi}{2}\right) = 4, \quad y'\left(\frac{\pi}{2}\right) = 0$

Solution

The characteristic equation: $9\lambda^2 + 1 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm\frac{1}{3}i}$

$$\boxed{y(x) = C_1 \cos\frac{x}{3} + C_2 \sin\frac{x}{3}}$$

$$y\left(\frac{\pi}{2}\right) = 4 \quad \rightarrow \quad \frac{\sqrt{3}}{2}C_1 + \frac{1}{2}C_2 = 4$$

$$y'(x) = -\frac{1}{3}C_1 \sin\frac{x}{3} + \frac{1}{3}C_2 \cos\frac{x}{3}$$

$$y'\left(\frac{\pi}{2}\right) = 0 \quad \rightarrow \quad -\frac{1}{2}C_1 + \frac{\sqrt{3}}{2}C_2 = 0$$

$$\begin{cases} \sqrt{3}C_1 + C_2 = 8 \\ -C_1 + \sqrt{3}C_2 = 0 \end{cases} \qquad \Delta = \begin{vmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{vmatrix} = 4 \quad \Delta_1 = \begin{vmatrix} 8 & 1 \\ 0 & \sqrt{3} \end{vmatrix} = 8\sqrt{3} \quad \Delta_2 = \begin{vmatrix} \sqrt{3} & 8 \\ -1 & 0 \end{vmatrix} = 8$$

$$\boxed{C_1 = 2\sqrt{3} \quad C_2 = 2}$$

$$\boxed{y(x) = 2\sqrt{3}\cos\frac{x}{3} + 2\sin\frac{x}{3}}$$

Exercise

Find the solution of the given initial value problem $\quad 9y'' + \pi^2 y = 0; \quad y(3) = 2, \quad y'(3) = -\pi$

Solution

The characteristic equation: $\quad 9\lambda^2 + \pi^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm\frac{\pi}{3}i}$

$$\boxed{y(x) = C_1 \cos\frac{\pi}{3}x + C_2 \sin\frac{\pi}{3}x}$$

$$y(3) = 2 \quad \rightarrow \boxed{C_1 = -2}$$

$$y'(x) = -\frac{\pi}{3}C_1 \sin\frac{\pi}{3}x + \frac{\pi}{3}C_2 \cos\frac{\pi}{3}x$$

$$y'(3) = -\pi \quad \rightarrow \quad -\frac{\pi}{3}C_2 = -\pi \quad \Rightarrow \quad \boxed{C_2 = 3}$$

$$\boxed{y(x) = -2\cos\frac{\pi}{3}x + 3\sin\frac{\pi}{3}x}$$

Exercise

Find the solution of the given initial value problem $\quad 9y'' - 6y' + y = 0; \quad y(3) = -2, \quad y'(3) = -\frac{5}{3}$

Solution

The characteristic equation: $\quad 9\lambda^2 - 6\lambda + 1 = 0 = (3\lambda - 1)^2 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \frac{1}{3}}$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{x/3}}$$

$y(3) = -2 \quad \to \quad (C_1 + 3C_2)e = -2$

$y'(x) = \left(C_2 + \frac{1}{3}C_1 + \frac{1}{3}C_2 x\right)e^{x/3}$

$y'(3) = -\frac{5}{3} \quad \to \quad \left(2C_2 + \frac{1}{3}C_1\right)e = -\frac{5}{3}$

$$\begin{cases} C_1 + 3C_2 = -\frac{2}{e} \\ C_1 + 6C_2 = -\frac{5}{e} \end{cases} \qquad \Delta = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \quad \Delta_1 = \begin{vmatrix} -\frac{2}{e} & 3 \\ -\frac{5}{e} & 6 \end{vmatrix} = \frac{3}{e} \quad \Delta_2 = \begin{vmatrix} 1 & -\frac{2}{e} \\ 1 & -\frac{5}{e} \end{vmatrix} = -\frac{3}{e}$$

$\underline{C_1 = \frac{1}{e} \quad C_2 = -\frac{1}{e}}$

$\underline{y(x) = \frac{1}{e}(1-x)e^{x/3}}$

Exercise

Find the solution of the given initial value problem $\quad 9y'' + 6y' + 2y = 0$; $\quad y(3\pi) = 0, \quad y'(3\pi) = \frac{1}{3}$

Solution

The characteristic equation: $\quad 9\lambda^2 + 6\lambda + 2 = 0 \quad \to \quad \lambda_{1,2} = \frac{-6 \pm 6i}{18} = -\frac{1}{3} \pm \frac{1}{3}i$

$\underline{y(x) = \left(C_1 \cos\frac{x}{3} + C_2 \sin\frac{x}{3}\right)e^{-x/3}}$

$y(3\pi) = 0 \quad \to \quad -C_1 e^{-\pi} = 0 \quad \Rightarrow \quad \underline{C_1 = 0}$

$y'(x) = \left(-\frac{1}{3}C_1 \sin\frac{x}{3} + \frac{1}{3}C_2 \cos\frac{x}{3} - \frac{1}{3}C_1 \cos\frac{x}{3} - \frac{1}{3}C_2 \sin\frac{x}{3}\right)e^{-x/3}$

$y'(3\pi) = -\frac{5}{3} \quad \to \quad \left(-\frac{1}{3}C_2\right)e^{-\pi} = \frac{1}{3} \quad \Rightarrow \quad \underline{C_2 = -e^{\pi}}$

$\underline{y(x) = -e^{\pi} \sin\frac{x}{3} e^{-x/3}}$

Exercise

Find the solution of the given initial value problem $\quad 9y'' - 12y' + 4y = 0, \quad y(0) = -1, \quad y'(0) = 1$

Solution

The characteristic equation: $\quad 9\lambda^2 - 12\lambda + 4 = (3\lambda - 2)^2 = 0 \quad \to \quad \underline{\lambda_{1,2} = \frac{2}{3}}$

$y(x) = (C_1 + C_2 x)e^{2x/3}$

$y(0) = -1 \quad \to \quad \underline{C_1 = -1}$

$$y'(x) = \left(C_2 + \frac{2}{3}C_1 + \frac{2}{3}C_2 x\right)e^{2x/3}$$

$$y'(0) = 1 \quad \to C_2 + \frac{2}{3}C_1 = 1 \quad \Rightarrow \boxed{C_2 = \frac{5}{3}}$$

$$\boxed{y(x) = \left(-1 + \frac{5}{3}x\right)e^{2x/3}}$$

Exercise

Find the solution of the given initial value problem $12y'' + 5y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = -1$

Solution

The characteristic equation: $12\lambda^2 + 5\lambda - 2 = 0 \quad \to \quad \lambda_{1,2} = \frac{-5 \pm 11}{24} = \boxed{-\frac{2}{3}, \ \frac{1}{4}}$

$$y(x) = C_1 e^{2x/3} + C_2 e^{x/4}$$

$$y(0) = 1 \quad \to \quad C_1 + C_2 = 1$$

$$y'(x) = \frac{2}{3}C_1 e^{2x/3} + \frac{1}{4}C_2 e^{x/4}$$

$$y'(0) = -1 \quad \to \quad \frac{2}{3}C_1 + \frac{1}{4}C_2 = -1$$

$$\begin{cases} C_1 + C_2 = 1 \\ 8C_1 + 3C_2 = -12 \end{cases} \quad \to \quad \boxed{C_1 = -3, \ \ C_2 = 4}$$

$$\boxed{y(x) = -3e^{2x/3} + 4e^{x/4}}$$

Exercise

Find the solution of the given initial value problem $16y'' - 8y' + y = 0 \ ; \quad y(0) = -4, \quad y'(0) = 3$

Solution

The characteristic equation: $16\lambda^2 - 8\lambda + 1 = 0 \quad \to \quad \lambda_{1,2} = \frac{8 \pm 0}{32} = \boxed{\frac{1}{4}}$

$$y(x) = \left(C_1 + C_2 x\right)e^{x/4}$$

$$y(0) = -4 \quad \to \quad \boxed{C_1 = -4}$$

$$y'(x) = \left(C_2 + \frac{1}{4}C_1 + \frac{1}{4}C_2 x\right)e^{x/4}$$

$$y'(0) = 3 \quad \to C_2 - 1 = 3 \quad \Rightarrow \quad \boxed{C_2 = 4}$$

$$\boxed{y(x) = (-4 + 4x)e^{x/4}}$$

Exercise

Find the solution of the given initial value problem

$$25y''+20y'+4y=0\ ; \quad y(5)=4e^{-2}, \quad y'(5)=-\frac{3}{5}e^{-2}$$

Solution

The characteristic equation: $25\lambda^2+20\lambda+4=0 \ \rightarrow \ \lambda_{1,2}=\frac{-20\pm 0}{50}=-\frac{2}{5}$

$$y(x)=\left(C_1+C_2x\right)e^{-2x/5}$$

$$y(5)=4e^{-2} \ \rightarrow \ \left(C_1+5C_2\right)e^{-2}=4e^{-2} \ \Rightarrow C_1+5C_2=4$$

$$y'(x)=\left(C_2-\frac{2}{5}C_1-\frac{2}{5}C_2x\right)e^{-2x/5}$$

$$y'(5)=-\frac{3}{5}e^{-2} \ \rightarrow \ \left(C_2-\frac{2}{5}C_1-2C_2\right)e^{-2}=-\frac{3}{5}e^{-2} \ \Rightarrow \ 2C_1+5C_2=3$$

$$\begin{cases} C_1+5C_2=4 \\ 2C_1+5C_2=3 \end{cases} \ \rightarrow \ \Delta=\begin{vmatrix}1 & 5\\ 2 & 5\end{vmatrix}=-5 \quad \Delta_1=\begin{vmatrix}4 & 5\\ 3 & 5\end{vmatrix}=5$$

$$C_1=-1, \quad C_2=1$$

$$y(x)=(-1+x)e^{-2x/5}$$

Exercise

Find the solution of the given initial value problem

$$y'''+12y''+36y'=0, \quad y(0)=0, \quad y'(0)=1, \quad y''(0)=-7$$

Solution

$$\lambda^3+12\lambda^2+36\lambda=0$$

$$\lambda(\lambda+6)^2=0 \ \rightarrow \ \lambda_1=0,-6,-6$$

$$y(x)=C_1+\left(C_2+C_3x\right)e^{-6x}$$

$$y(0)=0 \ \rightarrow \ C_1+C_2=0$$

$$y'(x)=\left(C_3-6C_2-6C_3x\right)e^{-6x}$$

$$y'(0)=1 \ \rightarrow C_3-6C_2=1$$

$$y''=\left(-12C_3+36C_2+36C_3x\right)e^{-6x}$$

$$y''(0)=-7 \ \rightarrow \ -12C_3+36C_2=-7$$

$$\begin{cases} C_3 - 6C_2 = 1 \\ -12C_3 + 36C_2 = -7 \end{cases} \rightarrow \ C_3 = \frac{1}{6}, \ C_2 = -\frac{5}{36}, \ C_1 = \frac{5}{36}$$

$$\underline{y(x) = \frac{5}{36} - \frac{5}{36}e^{-6x} + \frac{1}{6}xe^{-6x}}$$

Exercise

Find the solution of the given initial value problem

$$y''' + 2y'' - 5y' - 6y = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1$$

Solution

The characteristic equation: $\lambda^3 + 2\lambda^2 - 5\lambda - 6 = 0 \ \rightarrow \ \underline{\lambda_1 = -1}$

$$\begin{array}{r|rrrr} -1 & 1 & 2 & -5 & -6 \\ & & -1 & -1 & 6 \\ \hline & 1 & 1 & -6 & \boxed{0} \end{array} \qquad \lambda^2 + \lambda - 6 = 0 \ \Rightarrow \underline{\lambda_3 = -3, \ \lambda_4 = 2}$$

$$y(x) = C_1 e^{-3x} + C_2 e^{-x} + C_3 e^{2x}$$

$$y(0) = 0 \ \rightarrow \ \underline{C_1 + C_2 + C_3 = 0}$$

$$y'(x) = -3C_1 e^{-3x} - C_2 e^{-x} + 2C_3 e^{2x}$$

$$y'(0) = 0 \ \rightarrow \ \underline{-3C_1 - C_2 + 2C_3 = 0}$$

$$y''(x) = 9C_1 e^{-3x} + C_2 e^{-x} + 4C_3 e^{2x}$$

$$y''(0) = 1 \ \rightarrow \ \underline{9C_1 + C_2 + 4C_3 = 1}$$

$$\begin{cases} C_1 + C_2 + C_3 = 0 \\ -3C_1 - C_2 + 2C_3 = 0 \\ 9C_1 + C_2 + 4C_3 = 1 \end{cases}$$

$$\rightarrow \ \Delta = \begin{vmatrix} 1 & 1 & 1 \\ -3 & -1 & 2 \\ 9 & 1 & 4 \end{vmatrix} = 30 \quad \Delta_{C_1} = \begin{vmatrix} 0 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix} = 3 \quad \Delta_{C_2} = \begin{vmatrix} 1 & 0 & 1 \\ -3 & 0 & 2 \\ 9 & 1 & 4 \end{vmatrix} = 5$$

$$\underline{C_1 = \frac{1}{10}, \ C_2 = \frac{1}{6}, \ C_3 = -\frac{4}{15}}$$

$$\underline{y(x) = \frac{1}{10}e^{-3x} + \frac{1}{6}e^{-x} - \frac{4}{15}e^{2x}}$$

Exercise

The roots of the characteristic equation of a certain differential equation are:

$$3, -5, 0, 0, 0, 0, -5, \ 2 \pm 3i \ \text{ and } \ 2 \pm 3i$$

Write a general solution of this homogeneous differential equation.

Solution

For $\lambda = 0, \ 0, \ 0, \ 0 \quad \Rightarrow \quad y_1 = C_1 + C_2 x + C_3 x^2 + C_4 x^3$

For $\lambda = 3 \quad \Rightarrow \quad y_2 = C_5 e^{3x}$

For $\lambda = -5, \ -5 \quad \Rightarrow \quad y_3 = C_6 e^{-5x} + C_7 x e^{-5x}$

For $\lambda = 2 \pm 3i, \ 2 \pm 3i$

$$\Rightarrow \quad y_4 = e^{2x}\left(C_8 \cos 3x + C_9 \sin 3x\right) + xe^{-5x}\left(C_{10} \cos 3x + C_{11} \sin 3x\right)$$

$$\boxed{\begin{aligned} y(x) = &C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 e^{3x} + C_6 e^{-5x} + C_7 x e^{-5x} \\ &+ e^{2x}\left(C_8 \cos 3x + C_9 \sin 3x\right) + xe^{-5x}\left(C_{10} \cos 3x + C_{11} \sin 3x\right) \end{aligned}}$$

Exercise

$y(x) = C_1 e^{2x} + C_2 e^{-2x} + C_3 \cos 2x + C_4 \sin 2x$ is the general solution of a homogeneous equation. What is the equation?

Solution

$$\lambda_1 = 2, \quad \lambda_2 = -2, \quad \lambda_{3,4} = \pm 2i$$

$$(\lambda - 2)(\lambda + 2)(\lambda - 2i)(\lambda + 2i) = 0$$

$$\left(\lambda^2 - 4\right)\left(\lambda^2 + 4\right) = 0$$

$$\lambda^4 - 16 = 0 \quad \Rightarrow \quad \boxed{y^{(4)} - 16y = 0}$$

Exercise

Show that the second differential equation $y'' + 4y = 0$

a) Has no solution to the boundary value $y(0) = 0, \quad y(\pi) = 1$

b) There are infinitely many solutions to the boundary value $y(0) = 0, \quad y(\pi) = 0$

Solution

The characteristic equation: $\lambda^2 + 4 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 2i}$

$$\boxed{y(x) = C_1 \cos 2x + C_2 \sin 2x}$$

a) $y(0) = 0 \quad \to \quad \boxed{C_1 = 0}$

$y(\pi) = 1 \quad \to \quad \boxed{C_1 = 1}$

Therefore, there is no solution since $\cancel{C_1 = 0 = 1}$

b) $y(0) = 0 \quad \to \quad \boxed{C_1 = 0}$

$y(\pi) = 0 \quad \to \quad \boxed{C_1 = 0}$

$$\boxed{y(x) = C_2 \sin 2x}$$

∴ There are infinite many solutions.

Exercise

Show that the general solution of the equation/ $\qquad y'' + Py' + Qy = 0$

(where P and Q are constant) approaches 0 as $x \to \infty$ if and only if P and Q are both positive.

Solution

$$\lambda^2 + P\lambda + Q = 0$$

The solutions: $\lambda_{1,2} = \dfrac{-P \pm \sqrt{P^2 - 4Q}}{2}$

If $P^2 - 4Q < 0 \quad \to \quad P < 2\sqrt{Q} \quad$ (P & Q are positives)

$$\lambda_{1,2} = -\frac{P}{2} \pm i\frac{\sqrt{4Q - P^2}}{2}$$

$$y(x) = e^{-Px/2}\left(C_1 \cos\tfrac{1}{2}\sqrt{4Q - P^2}\,x + C_2 \sin\tfrac{1}{2}\sqrt{4Q - P^2}\,x\right)$$

$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty}\left[e^{-Px/2}\left(C_1 \cos\tfrac{1}{2}\sqrt{4Q - P^2}\,x + C_2 \sin\tfrac{1}{2}\sqrt{4Q - P^2}\,x\right)\right]$$

$$\boxed{= 0} \qquad\qquad \lim_{x\to\infty}\left(e^{-Px/2}\right) = \lim_{x\to\infty}\left(\frac{1}{e^{Px/2}}\right) = \frac{1}{\infty} = 0 \quad (P > 0)$$

If $P^2 - 4Q = 0 \quad \to \quad \lambda_{1,2} = -\frac{1}{2}P$

$$y(x) = \left(C_1 + C_2 x\right)e^{-Px/2}$$

$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty}\left(C_1 + C_2 x\right)e^{-Px/2}$$

$$\boxed{= 0}$$

If $P^2 - 4Q > 0 \quad \rightarrow \quad \lambda_{1,2} = \dfrac{-P \pm \sqrt{P^2 - 4Q}}{2}$

$$y(x) = C_1 e^{\frac{-P-\sqrt{P^2-4Q}}{2}x} + C_2 e^{\frac{-P+\sqrt{P^2-4Q}}{2}x}$$

$$\sqrt{P^2 - 4Q} < \sqrt{P^2} = P \quad \rightarrow \quad \frac{-P + \sqrt{P^2 - 4Q}}{2} < 0$$

$$\lim_{x\to\infty} e^{\frac{-P+\sqrt{P^2-4Q}}{2}x} = 0$$

$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty}\left(C_1 e^{\frac{-P-\sqrt{P^2-4Q}}{2}x} + C_2 e^{\frac{-P+\sqrt{P^2-4Q}}{2}x} \right)$$

$$= 0$$