Lecture Three

Section 3.1 – Inner Products

3.1–1 *Definition*

An ***inner product*** on a real vector space V is a function that associates a real number $\langle \vec{u}, \vec{v} \rangle$ with each pair of vectors in V in such a way that the following axioms are satisfies for all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in V and all scalars k.

1. $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ ***Symmetry axiom***
2. $\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ ***Additivity axiom***
3. $\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$ ***Homogeneity axiom***
4. $\langle \vec{v}, \vec{v} \rangle \ge 0$ and $\langle \vec{v}, \vec{v} \rangle = 0$ iff $\vec{v} = 0$ ***Positivity axiom***

A real vector space with an inner product is called a ***real inner product space***.

$$\begin{aligned}\langle \vec{u}, \vec{v} \rangle &= \vec{u} \bullet \vec{v} \\ &= u_1 v_1 + u_2 v_2 + \cdots u_n v_n\end{aligned}$$

This is called the ***Euclidean inner product*** (or the ***standard inner product***)

3.1–2 *Definition*

If V is a real inner product space, then the norm (or length) of a vector $\vec{v}$ in V is denoted by $\|\vec{v}\|$ and is defined by

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$$

And the ***distance*** between two vectors is denoted by $d(\vec{u}, \vec{v})$ and is defined by

$$\begin{aligned}d(\vec{u}, \vec{v}) &= \|\vec{u} - \vec{v}\| \\ &= \sqrt{\langle \vec{u}-\vec{v}, \ \vec{u}-\vec{v} \rangle}\end{aligned}$$

A vector of norm 1 is called a ***unit vector***.

3.1−3 *Theorem*

If ***u*** and ***v*** are vectors in a real inner product space *V*, and if *k* is a scalar, then:

a) $\|\vec{v}\| \geq 0$ with equality *iff* $\vec{v} = 0$

b) $\|k\vec{v}\| = |k|\|\vec{v}\|$

c) $d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$

d) $d(\vec{u}, \vec{v}) \geq 0$ with equality *iff* $\vec{u} = \vec{v}$

Although the Euclidean inner product is the most important inner product on $\mathbb{R}^n$, there are various applications in which is desirable to modify it by weighing each term differently.

More precisely, if $w_1, w_2, \ldots, w_n$ are positive real numbers, which we will call weighs, and if $\vec{u} = (u_1, u_2, \ldots, u_n)$ and are vectors in $\mathbb{R}^n$, then it can be shown that the formula

$$\langle \vec{u}, \vec{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \ldots + w_n u_n v_n$$

Defines an inner product on $\mathbb{R}^n$ that we call the ***weighted Euclidean inner product*** with weights $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n$

Example 3.1-1

Let $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ be vectors in $\mathbb{R}^2$, verify that the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 3u_1 v_1 + 2u_2 v_2$ satisfies the four inner product axioms.

Solution

Axiom 1: $\langle \vec{u}, \vec{v} \rangle = 3u_1 v_1 + 2u_2 v_2$

$$= 3v_1 u_1 + 2v_2 u_2$$

$$= \langle \vec{v}, \vec{u} \rangle$$

Axiom 2: $\langle \vec{u} + \vec{v}, \vec{w} \rangle = 3(u_1 + v_1) w_1 + 2(u_2 + v_2) w_2$

$$= 3(u_1 w_1 + v_1 w_1) + 2(u_2 w_2 + v_2 w_2)$$

$$= 3u_1 w_1 + 3v_1 w_1 + 2u_2 w_2 + 2v_2 w_2$$

$$= (3u_1 w_1 + 2u_2 w_2) + (3v_1 w_1 + 2v_2 w_2)$$

$$= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$$

Axiom 3: $\langle k\vec{u},\ \vec{v}\rangle = 3(ku_1)v_1 + 2(ku_2)v_2$

$$= k(3u_1v_1 + 2u_2v_2)$$

$$= k\langle \vec{u},\ \vec{v}\rangle$$

Axiom 4: $\langle \vec{v},\ \vec{v}\rangle = 3v_1v_1 + 2v_2v_2$

$$= 3v_1^2 + 2v_2^2 \ge 0$$

$$v_1 = v_2 = 0 \quad \textit{iff} \quad \vec{v} = \vec{0}$$

Exercises *Section* 3.1 – Inner Products

1. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (1, 1)$, $\vec{v} = (3, 2)$, $\vec{w} = (0, -1)$, and $k = 3$. Compute the following.

a) $\langle \vec{u}, \vec{v} \rangle$
b) $\langle k\vec{v}, \vec{w} \rangle$
c) $\langle \vec{u} + \vec{v}, \vec{w} \rangle$
d) $\|\vec{v}\|$
e) $d(\vec{u}, \vec{v})$
f) $\|\vec{u} - k\vec{v}\|$

2. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (3, -2)$, $\vec{v} = (4, 5)$, $\vec{w} = (-1, 6)$, and $k = -4$. Verify the following.

a) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$
b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$
c) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$
d) $\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$
e) $\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$

3. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^3$, and let $\vec{u} = (1, 0, -2)$, $\vec{v} = (5, 1, 2)$, $\vec{w} = (5, 2, -1)$, and $k = 3$. Compute the following.

a) $\langle \vec{u}, \vec{v} \rangle$
b) $\langle k\vec{v}, \vec{w} \rangle$
c) $\langle \vec{u} + \vec{v}, \vec{w} \rangle$
d) $\|\vec{v}\|$
e) $d(\vec{u}, \vec{v})$
f) $\|\vec{u} - k\vec{v}\|$

4. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^3$, and let $\vec{u} = (2, 4, 3)$, $\vec{v} = (0, -3, -4)$, $\vec{w} = (6, 3, 1)$, and $k = 2$. Verify the following.

a) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$
b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$
c) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$
d) $\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$
e) $\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$

5. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (1, 1)$, $\vec{v} = (3, 2)$, $\vec{w} = (0, -1)$ and $k = 3$. Compute the following for the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 2u_1v_1 + 3u_2v_2$.

a) $\langle \vec{u}, \vec{v} \rangle$
b) $\langle k\vec{v}, \vec{w} \rangle$
c) $\langle \vec{u} + \vec{v}, \vec{w} \rangle$
d) $\|\vec{v}\|$
e) $d(\vec{u}, \vec{v})$
f) $\|\vec{u} - k\vec{v}\|$

6. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (3, -2)$, $\vec{v} = (4, 5)$, $\vec{w} = (-1, 6)$, and $k = -4$. Verify the following for the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 4u_1v_1 + 5u_2v_2$

a) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

c) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

d) $\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$

e) $\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$

7. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (-3, 2)$, $\vec{v} = (5, 4)$, $\vec{w} = (1, -6)$, and $k = 2$. Verify the following for the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 4u_1v_1 + 5u_2v_2$

a) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

c) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

d) $\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$

e) $\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$

8. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^3$, and let $\vec{u} = (2, 1, -2)$, $\vec{v} = (-1, 3, 2)$, $\vec{w} = (2, 1, 0)$ and $k = 2$.

Compute the following for the weighted Euclidean inner product

$\langle \vec{u}, \vec{v} \rangle = 2u_1v_1 + 3u_2v_2 + u_3v_3$.

a) $\langle \vec{u}, \vec{v} \rangle$

b) $\langle k\vec{v}, \vec{w} \rangle$

c) $\langle \vec{u} + \vec{v}, \vec{w} \rangle$

d) $\|\vec{v}\|$

e) $d(\vec{u}, \vec{v})$

f) $\|\vec{u} - k\vec{v}\|$

9. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^3$, and let $\vec{u} = (3, 2, 2)$, $\vec{v} = (0, 2, 4)$, $\vec{w} = (1, -6, 3)$, and $k = -3$. Verify the following for the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 4u_1v_1 + 5u_2v_2 + 2u_3v_3$

a) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

c) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

d) $\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$

e) $\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$

10. Let $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$. Show that the following are inner product on $\mathbb{R}^2$ by verifying that the inner product axioms hold. $\langle \vec{u}, \vec{v} \rangle = 3u_1v_1 + 5u_2v_2$

11. Let $\vec{u}=(u_1,\ u_2,\ u_3)$ and $\vec{v}=(v_1,\ v_2,\ v_3)$. Show that the following are inner product on $\mathbb{R}^3$ by verifying that the inner product axioms hold. $\langle \vec{u},\ \vec{v}\rangle = 2u_1v_1 + 3u_2v_2 + 5u_3v_3$

12. Show that the following identity holds for the vectors in any inner product space

$$\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$$

13. Show that the following identity holds for the vectors in any inner product space

$$\langle \vec{u},\ \vec{v}\rangle = \frac{1}{4}\left(\|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2\right)$$

14. Prove that $\|k\vec{v}\| = |k|\ \|\vec{v}\|$

Section 3.2 – Angle and Orthogonality in Inner Product Spaces

3.2−1 Cosine Formula

If $\vec{u}$ and $\vec{v}$ are nonzero vectors that implies

$$\cos\theta = \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$$

$$\theta = \cos^{-1}\left(\frac{\langle \vec{u},\ \vec{v} \rangle}{\|\vec{u}\|\ \|\vec{v}\|}\right)$$

$$-1 \le \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\ \|\vec{v}\|} \le 1$$

Example 3.2-1

Let $\mathbb{R}^4$ have the Euclidean inner product. Find the cosine angle θ between the vectors $\vec{u} = (4,\ 3,\ 1,\ -2)$ and $\vec{v} = (-2,\ 1,\ 2,\ 3)$.

Solution

$$\|\vec{u}\| = \sqrt{4^2 + 3^2 + 1^2 + (-2)^2}$$

$$= \sqrt{30}$$

$$\|\vec{v}\| = \sqrt{4+1+4+9}$$

$$= \sqrt{18}$$

$$= 3\sqrt{2}$$

$$\langle \vec{u},\ \vec{v} \rangle = 4(-2) + 3(1) + 1(2) - 2(3)$$

$$= -9$$

$$\cos\theta = -\frac{9}{3\sqrt{30}\sqrt{2}} \qquad\qquad \cos\theta = \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$$

$$= -\frac{3}{\sqrt{60}}$$

$$= -\frac{3}{2\sqrt{15}}$$

3.2–2 *Theorem* – Cauchy-Schwarz Inequality

If $\vec{v}$ and $\vec{w}$ are vectors in a real inner product space V, then

$$\left\|\langle \vec{u},\ \vec{v}\rangle\right\| \le \|\vec{u}\|\ \|\vec{v}\|$$

Proof

If either $\vec{u}$ or $\vec{v}$ is equal to zero, then both sides equal to zero
Inequality holds.

Suppose that $\vec{u},\ \vec{v} \neq 0$ and if $\vec{w}$ any vector

$\|\vec{w}\| = \vec{w}\ \vec{w} \ge 0$

Let $\vec{w} = \vec{u} - t\vec{v}$, then:

$$\begin{aligned}
0 &\le \vec{w}\vec{w} \\
&= (\vec{u} - t\vec{v})(\vec{u} - t\vec{v}) \\
&= \vec{u} \bullet \vec{u} - t(\vec{u} \bullet \vec{v}) - t(\vec{v} \bullet \vec{u}) + t^2(\vec{v} \bullet \vec{v}) \\
&= \vec{u} \bullet \vec{u} - 2t\,(\vec{u} \bullet \vec{v}) + t^2(\vec{v} \bullet \vec{v}) \qquad \text{Let } t = \frac{\vec{u} \bullet \vec{v}}{\vec{v} \bullet \vec{v}} \\
&= \vec{u} \bullet \vec{u} - 2\left(\frac{\vec{u} \bullet \vec{v}}{\vec{v} \bullet \vec{v}}\right)(\vec{u} \bullet \vec{v}) + \left(\frac{\vec{u} \bullet \vec{v}}{\vec{v} \bullet \vec{v}}\right)^2(\vec{v} \bullet \vec{v}) \\
&= \vec{u} \bullet \vec{u} - 2\frac{(\vec{u} \bullet \vec{v})^2}{\vec{v} \bullet \vec{v}} + \frac{(\vec{u} \bullet \vec{v})^2}{\vec{v} \bullet \vec{v}} \\
&= \vec{u} \bullet \vec{u} - \frac{(\vec{u} \bullet \vec{v})^2}{\vec{v} \bullet \vec{v}} \\
&= \frac{(\vec{u} \bullet \vec{u})(\vec{v} \bullet \vec{v}) - (\vec{u} \bullet \vec{v})^2}{\vec{v} \bullet \vec{v}} \qquad \text{Since } \vec{v} \bullet \vec{v} > 0 \\
&\le (\vec{u} \bullet \vec{u})(\vec{v} \bullet \vec{v}) - (\vec{u} \bullet \vec{v})^2
\end{aligned}$$

$$(\vec{u} \bullet \vec{v})^2 \le (\vec{u} \bullet \vec{u})(\vec{v} \bullet \vec{v})$$

$$\left\|\langle \vec{u},\ \vec{v}\rangle\right\| \le \|\vec{u}\|\ \|\vec{v}\|$$

The following two alternative forms of the Cauchy-Schwarz inequality are useful to know:

$$\langle \vec{u},\ \vec{v}\rangle^2 \le \langle \vec{u},\ \vec{u}\rangle\langle \vec{v},\ \vec{v}\rangle$$

$$\langle \vec{u},\ \vec{v}\rangle^2 \le \|\vec{u}\|^2\ \|\vec{v}\|^2$$

3.2–3 *Theorem*

If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors in a real inner product space V, and if k is any scalar, then

a) $\|\vec{u}+\vec{v}\| \le \|\vec{u}\|+\|\vec{v}\|$ (***Triangle inequality for vectors***)

b) $d(\vec{u},\ \vec{v}) \le d(\vec{u},\ \vec{w})+d(\vec{w},\ \vec{v})$ (***Triangle inequality for distances***)

Proof **(*a*)**

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 &= \langle \vec{u}+\vec{v},\ \vec{u}+\vec{v}\rangle \\
&= \langle \vec{u},\ \vec{u}\rangle + 2\langle \vec{u},\ \vec{v}\rangle + \langle \vec{v},\ \vec{v}\rangle \\
&\le \langle \vec{u},\ \vec{u}\rangle + 2\left|\langle \vec{u},\ \vec{v}\rangle\right| + \langle \vec{v},\ \vec{v}\rangle \\
&\le \langle \vec{u},\ \vec{u}\rangle + 2\|\vec{u}\|\ \|\vec{v}\| + \langle \vec{v},\ \vec{v}\rangle \\
&= \|\vec{u}\|^2 + 2\|\vec{u}\|\ \|\vec{v}\| + \|\vec{v}\|^2 \\
&= \left(\|\vec{u}\|+\|\vec{v}\|\right)^2
\end{aligned}$$

$$\|\vec{u}+\vec{v}\|^2 \le \left(\|\vec{u}\|+\|\vec{v}\|\right)^2$$

$$\|\vec{u}+\vec{v}\| \le \|\vec{u}\|+\|\vec{v}\|$$

3.2–4 *Definition*

Two vectors $\vec{u}$ and $\vec{v}$ in an inner product space are called orthogonal if $\langle \vec{u},\ \vec{v}\rangle = 0$

Example 3.2-2

The vectors $\vec{u}=(1,\ 1)$ and $\vec{v}=(1,\ -1)$ are orthogonal with respect to the Euclidean inner product on $\mathbb{R}^2$, since

$$\begin{aligned}
\vec{u}\bullet\vec{v} &= 1(1)+1(-1) \\
&= 0
\end{aligned}$$

They are not orthogonal with the respect to the weighted Euclidean inner product $\langle \vec{u},\ \vec{v}\rangle = 3u_1v_1 + 2u_2v_2$, since

$$\begin{aligned}
\langle \vec{u},\ \vec{v}\rangle &= 3(1)(1) + 2(1)(-1) \\
&= 1 \ne 0
\end{aligned}$$

Example 3.2-3

$U=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ *and* $V=\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}$ are orthogonal, since

$$\begin{aligned} U\bullet V &= 1(0)+0(2)+1(0)+1(0) \\ &= 0 \end{aligned}$$

3.2−5 *Definition*

If W is a subspace of an inner product space V, then the set of all vectors are orthogonal to every vector in W is called the ***orthogonal complement*** of W and is denoted by the symbol $W^{\perp}$

3.2−6 *Theorem*

If W is a subspace of an inner product space V, then:

a) $W^{\perp}$ is a subspace of V.

b) $W\cap W^{\perp}=\{0\}$

Proof

a) Let set $W^{\perp}$ contains at least the zero vector, since $\langle\vec{0},\ \vec{w}\rangle=0$ for every vector $\vec{w}$ in W. We need to show that $W^{\perp}$ is closed under addition and scalar multiplication.

Suppose that $\vec{u}$ and $\vec{v}$ are vectors in $W^{\perp}$, so every vector $\vec{w}$ in W we have $\langle\vec{u},\ \vec{w}\rangle=0$ and $\langle\vec{v},\ \vec{w}\rangle=0$

$$\begin{aligned} \langle\vec{u}+\vec{v},\ \vec{w}\rangle &= \langle\vec{u},\ \vec{w}\rangle+\langle\vec{v},\ \vec{w}\rangle \\ &= 0+0 \\ &= 0 \end{aligned}$$

Closed under addition

$$\begin{aligned} \langle k\vec{u},\ \vec{w}\rangle &= k\langle\vec{u},\ \vec{w}\rangle \\ &= k(0) \\ &= 0 \end{aligned}$$

Closed under scalar multiplication

Which proves that $\vec{u}+\vec{w}$ and $k\vec{u}$ are in $W^{\perp}$

b) If $\vec{v}$ is any vector in both W and $W^{\perp}$, then $\vec{v}$ is orthogonal to itself; that is, $\langle\vec{v},\ \vec{v}\rangle=0$. It follows from the positivity axiom for inner products that $\vec{v}=\vec{0}$

3.2–7 *Theorem*

If W is a subspace of a finite-dimensional inner product space V, then the orthogonal complement of $W^{\perp}$ is W; that is

$$\left(W^{\perp}\right)^{\perp} = W$$

Example 3.2-4

Let W be the subspace of $\mathbb{R}^6$ spanned by the vectors

$$\vec{w}_1 = (1,\ 3,\ -2,\ 0,\ 2,\ 0), \qquad \vec{w}_2 = (2,\ 6,\ -5,\ -2,\ 4,\ -3)$$
$$\vec{w}_3 = (0,\ 0,\ 5,\ 10,\ 0,\ 15), \qquad \vec{w}_4 = (2,\ 6,\ 0,\ 8,\ 4,\ 18)$$

Find a basis for the orthogonal complement of W.

Solution

The Space W is the same as the row space of the matrix

$$A = \begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{pmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ \\ R_4 - 2R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 0 & 0 & 4 & 8 & 0 & 18 \end{pmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ R_3 + 5R_2 \\ R_4 + 4R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 6 \\ 0 & 0 & -1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix} \begin{matrix} \\ -R_2 \\ \\ \frac{1}{6}R_3 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 6 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} R_1 - 6R_3 \\ R_2 - 3R_3 \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The solution

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5, x_6) &= (-3x_2 - 4x_4 - 2x_5, x_2, -2x_4, x_4, x_5, 0) \\ &= x_2(-3, 1, 0, 0, 0, 0) + x_4(-4, 0, -2, 1, 0, 0) + x_5(-2, 0, 0, 0, 1, 0)\end{aligned}$$

$$\vec{v}_1 = (-3, 1, 0, 0, 0, 0), \quad \vec{v}_2 = (-4, 0, -2, 1, 0, 0), \quad \vec{v}_3 = (-2, 0, 0, 0, 1, 0)$$

3.2−8 *Definition*

A collection of vectors in $\mathbb{R}^n$ (or inner space) is called orthogonal if any 2 are perpendicular.

$$\vec{v}_i . \vec{v}_j = \vec{v}^T \vec{v} = \begin{cases} 0 & for\ i \neq j \quad (orthogonal\ vectors) \\ 1 & for\ i = j \quad (unit\ vectors) \end{cases}$$

3.2−9 *Theorem*

If $\vec{v}_1, \ldots, \vec{v}_m$ are nonzero orthogonal vectors, then they are linearly independent.

3.2−10 *Definition*

A vector $\vec{v}$ is called normal if $\|\vec{v}\| = 1$

A collection of vectors $\vec{v}_1, \ldots, \vec{v}_m$ is called orthonormal if they are orthogonal and each $\|\vec{v}_i\| = 1$.

An orthonormal basis is a basis made up of orthonormal vectors.

Example 3.2-5

Q rotates every vector in the plane through the angle θ.

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$Q^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad \cos^2\theta + \sin^2\theta = 1$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= Q^T$$

The dot product $\left(\cos\theta\sin\theta - \sin\theta\cos\theta = 0\right)$, the columns are orthogonal.

They are unit vectors because $\cos^2\theta + \sin^2\theta = 1$. Those columns give an orthonormal basis for the plane $\mathbb{R}^2$.

We have: $QQ^T = I = Q^T Q$ (This type is called ***rotation***)

Exercises Section 3.2 – Angle and Orthogonality in Inner Product Spaces

(1 – 10) Which of the following form ***orthonormal*** sets?

1. $\{(1,\ 0),\ (0,\ 2)\}\ \ in\ \mathbb{R}^2$

2. $\{(2,\ -4),\ (2,\ 1)\}\ \ in\ \mathbb{R}^2$

3. $\left\{\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)\right\}\ \ in\ \mathbb{R}^2$

4. $\left\{\left(-\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)\right\}\ \ in\ \mathbb{R}^2$

5. $\left\{\left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right),\ \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)\right\}\ \ in\ \mathbb{R}^3$

6. $\{(4,\ -1,\ 1),\ (-1,\ 0,\ 4),\ (-4,\ -17,\ -1)\}\ \ in\ \mathbb{R}^3$

7. $\left\{\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right),\ \left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right),\ \left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right)\right\}\ \ in\ \mathbb{R}^3$

8. $\left\{\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right),\ \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\right\}\ \ in\ \mathbb{R}^3$

9. $\left\{\left(\frac{1}{\sqrt{2}},\ 0,\ 0,\ \frac{1}{\sqrt{2}}\right),\ \left(0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right),\ \left(-\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2}\right)\right\}\ \ in\ \mathbb{R}^4$

10. $\left\{\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right),\ \left(0,\ 0,\ -\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right),\ \left(-\frac{1}{3},\ \frac{1}{3},\ -\frac{1}{3},\ \frac{1}{3}\right)\right\}\ \ in\ \mathbb{R}^4$

(11 – 18) Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.

11. $\vec{u}=(1,\ -3),\quad \vec{v}=(2,\ 4)$

12. $\vec{u}=(-1,\ 0),\quad \vec{v}=(3,\ 8)$

13. $\vec{u}=(-1,\ 5,\ 2),\quad \vec{v}=(2,\ 4,\ -9)$

14. $\vec{u}=(4,\ 1,\ 8),\quad \vec{v}=(1,\ 0,\ -3)$

15. $\vec{u}=(1,\ 0,\ 1,\ 0),\quad \vec{v}=(-3,\ -3,\ -3,\ -3)$

16. $\vec{u}=(2,\ 1,\ 7,\ -1),\quad \vec{v}=(4,\ 0,\ 0,\ 0)$

17. $\vec{u}=(1,\ 3,\ -5,\ 4),\quad \vec{v}=(2,\ -4,\ 4,\ 1)$

18. $\vec{u}=(1,\ 2,\ 3,\ 4),\quad \vec{v}=(-1,\ -2,\ -3,\ -4)$

(19 – 22) Find the cosine of the angle between ***A*** and ***B***.

19. $A=\begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix} \quad B=\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$

20. $A=\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} \quad B=\begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$

21. $A=\begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix} \quad B=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

22. $A=\begin{pmatrix} 1 & -2 & 7 \\ 6 & -3 & 4 \end{pmatrix} \quad B=\begin{pmatrix} 1 & -2 & 3 \\ 6 & 5 & -4 \end{pmatrix}$

(23 – 27) Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

23. $\vec{u}=(-1, 3, 2), \quad \vec{v}=(4, 2, -1)$

24. $\vec{u}=(a, b), \quad \vec{v}=(-b, a)$

25. $\vec{u}=(-2, -2, -2), \quad \vec{v}=(1, 1, 1)$

26. $\vec{u}=(-4, 6, -10, 1), \quad \vec{v}=(2, 1, -2, 9)$

27. $\vec{u}=(-4, 6, -10, 1), \quad \vec{v}=(2, 1, -2, 9)$

28. Do there exist scalars k and l such that the vectors $\vec{u}=(2, k, 6), \quad \vec{v}=(l, 5, 3), \quad and \quad \vec{w}=(1, 2, 3)$ are mutually orthogonal with respect to the Euclidean inner product?

29. Let $\mathbb{R}^3$ have the Euclidean inner product. For which values of k are $\vec{u}$ and $\vec{v}$ orthogonal?

a) $\vec{u}=(2, 1, 3), \quad \vec{v}=(1, 7, k)$ *b)* $\vec{u}=(k, k, 1), \quad \vec{v}=(k, 5, 6)$

30. Let V be an inner product space. Show that if $\vec{u}$ and $\vec{v}$ are orthogonal unit vectors in V, then $\|\vec{u}-\vec{v}\|=\sqrt{2}$

31. Let $\boldsymbol{S}$ be a subspace of $\mathbb{R}^n$. Explain what $\left(\boldsymbol{S}^{\perp}\right)^{\perp}=\boldsymbol{S}$ means and why it is true.

32. The methane molecule CH_4 is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$ – (***note*** that all six edges have length $\sqrt{2}$, so the tetrahedron is regular). What is the cosine of the angle between the rays going from the center $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ to the vertices?

33. Determine if the given vectors are orthogonal.

$$\vec{x}_1 = (1,\ 0,\ 1,\ 0),\quad \vec{x}_2 = (0,\ 1,\ 0,\ 1),\quad \vec{x}_3 = (1,\ 0,\ -1,\ 0),\quad \vec{x}_4 = (1,\ 1,\ -1,\ -1)$$

(34 – 35) Which of the following sets of vectors are orthogonal with respect to the Euclidean inner?

34. $\left\{\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right),\ \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\right\}$

35. $\left\{\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right),\ \left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right),\ \left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right)\right\}$

36. $\left\{\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right),\ \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\right\}$

37. $\left\{\left(\frac{1}{\sqrt{2}},\ 0,\ 0,\ \frac{1}{\sqrt{2}}\right),\ \left(0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right),\ \left(-\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2}\right)\right\}$

38. Consider vectors $\vec{u} = (2,\ 3,\ 5)\quad \vec{v} = (1,\ -4,\ 3)\quad in\quad \mathbb{R}^3$

a) $\langle \vec{u},\ \vec{v}\rangle$ *b)* $\|\vec{u}\|$ *c)* $\|\vec{v}\|$ *d)* Cosine between $\vec{u}$ *and* $\vec{v}$

39. Consider vectors $\vec{u} = (1,\ 1,\ 1)\quad \vec{v} = (1,\ 2,\ -3)\quad in\quad \mathbb{R}^3$

a) $\langle \vec{u},\ \vec{v}\rangle$ *b)* $\|\vec{u}\|$ *c)* $\|\vec{v}\|$ *d)* Cosine θ between $\vec{u}$ *and* $\vec{v}$

40. Consider vectors $\vec{u} = (1,\ 2,\ 5)\quad \vec{v} = (2,\ -3,\ 5)\quad \vec{w} = (4,\ 2,\ -3)\quad in\quad \mathbb{R}^3$

a) $\langle \vec{u},\ \vec{v}\rangle$
b) $\langle \vec{u},\ \vec{w}\rangle$
c) $\langle \vec{v},\ \vec{w}\rangle$
d) $\|\vec{u}\|$
e) $\|\vec{v}\|$
f) $\|\vec{w}\|$
g) Cosine α between $\vec{u}$ *and* $\vec{v}$
h) Cosine β between $\vec{u}$ *and* $\vec{w}$
i) Cosine θ between $\vec{v}$ *and* $\vec{w}$
j) $(\vec{u}+\vec{v})\bullet\vec{w}$

41. Consider vectors $\vec{u} = (-1,\ -1,\ -1)\quad \vec{v} = (2,\ 2,\ 2)\quad \vec{w} = (4,\ 2,\ -3)\quad in\quad \mathbb{R}^3$

a) $\langle \vec{u},\ \vec{v}\rangle$
b) $\langle \vec{u},\ \vec{w}\rangle$
c) $\langle \vec{v},\ \vec{w}\rangle$
d) $\|\vec{u}\|$
e) $\|\vec{v}\|$
f) $\|\vec{w}\|$
g) Cosine α between $\vec{u}$ *and* $\vec{v}$
h) Cosine β between $\vec{u}$ *and* $\vec{w}$
i) Cosine θ between $\vec{v}$ *and* $\vec{w}$
j) $(\vec{u}+\vec{v})\bullet\vec{w}$

42. Consider vectors $\vec{u}=(-2,\ 0,\ 1,\ 3)$ $\vec{v}=(1,\ 1,\ 1,\ 1)$ $\vec{w}=(3,\ -1,\ 5,\ 2)$ *in* $\mathbb{R}^4$

a) $\langle \vec{u},\ \vec{v}\rangle$ *d)* $\|\vec{u}\|$ *g)* Cosine α between $\vec{u}$ *and* $\vec{v}$

b) $\langle \vec{u},\ \vec{w}\rangle$ *e)* $\|\vec{v}\|$ *h)* Cosine β between $\vec{u}$ *and* $\vec{w}$

c) $\langle \vec{v},\ \vec{w}\rangle$ *f)* $\|\vec{w}\|$ *i)* Cosine θ between $\vec{v}$ *and* $\vec{w}$

j) $(\vec{u}+\vec{v})\bullet\vec{w}$

43. Consider polynomial $f(t)=3t-5;$ $g(t)=t^2$ *in* $\mathbb{P}(t)$

a) $\langle f,\ g\rangle$ *b)* $\|f\|$ *c)* $\|g\|$ *d)* Cosine between f *and* g

44. Consider polynomial $f(t)=t+2$; $g(t)=3t-2$; $h(t)=t^2-2t-3$ *in* $\mathbb{P}(t)$

a) $\langle f,\ g\rangle$ *d)* $\|f\|$ *g)* Cosine α between f *and* g

b) $\langle f,\ h\rangle$ *e)* $\|g\|$ *h)* Cosine β between f *and* h

c) $\langle g,\ h\rangle$ *f)* $\|h\|$ *i)* Cosine θ between g *and* h

45. Consider polynomial $f(x)=x^2-2x+3$; $g(x)=2x+3$; $h(x)=x-2$ *in* $\mathbb{P}(x)$

a) $\langle f,\ g\rangle$ *d)* $\|f\|$ *g)* Cosine α between f *and* g

b) $\langle f,\ h\rangle$ *e)* $\|g\|$ *h)* Cosine β between f *and* h

c) $\langle g,\ h\rangle$ *f)* $\|h\|$ *i)* Cosine θ between g *and* h

46. Suppose $\langle \vec{u},\ \vec{v}\rangle=3+2\,i$ in a complex inner product space V. Find:

a) $\langle (2-4\,i)\vec{u},\ \vec{v}\rangle$ *b)* $\langle \vec{u},\ (4+3\,i)\vec{v}\rangle$ *c)* $\langle (3-6\,i)\vec{u},\ (5-2\,i)\vec{v}\rangle$ *d)* $\|\vec{u},\ \vec{v}\|$

47. Suppose $\langle \vec{u},\ \vec{v}\rangle=2-3\,i$ in a complex inner product space V. Find:

a) $\langle (2+2\,i)\vec{u},\ \vec{v}\rangle$ *b)* $\langle \vec{u},\ (3-4\,i)\vec{v}\rangle$ *c)* $\langle (1+3\,i)\vec{u},\ (5-2\,i)\vec{v}\rangle$ *d)* $\|\vec{u},\ \vec{v}\|$

48. Find the Fourier coefficient c and the projection $c\vec{v}$ of $\vec{u}=(3+4\,i,\ 2-3\,i)$ along $\vec{v}=(5+i,\ 2i)$ in $\mathbb{C}^2$

49. Suppose $\vec{v}=(1,\ 3,\ 5,\ 7)$. Find the projection of $\vec{v}$ onto W or find $\vec{w}\in W$ that minimizes $\|\vec{v}-\vec{w}\|$, where W is the subspace of $\mathbb{R}^4$ spanned by:

a) $\vec{u}_1=(1,\ 1,\ 1,\ 1)$ *and* $\vec{u}_2=(1,\ -3,\ 4,\ -2)$

b) $\vec{v}_1=(1,\ 1,\ 1,\ 1)$ *and* $\vec{v}_2=(1,\ 2,\ 3,\ 2)$

50. Suppose $\left\{\vec{u}_1,\ \vec{u}_2,\ \ldots,\ \vec{u}_n\right\}$ is an orthogonal set of vectors. Prove that (*Pythagoras*)

$$\left\|\vec{u}_1+\vec{u}_2+\ \ldots+\vec{u}_n\right\|^2=\left\|\vec{u}_1\right\|^2+\left\|\vec{u}_2\right\|^2+\ \ldots+\left\|\vec{u}_n\right\|^2$$

51. Suppose A is an orthogonal matrix. Show that $\langle\vec{u}A,\ \vec{v}A\rangle=\langle\vec{u},\ \vec{v}\rangle$ for any $\vec{u},\ \vec{v}\ \in V$

52. Suppose A is an orthogonal matrix. Show that $\|\vec{u}A\|=\|\vec{u}\|$ for every $\vec{u}\ \in V$

53. Let V be an inner product space over $\mathbb{R}$ or $\mathbb{C}$. Show that

$$\|\vec{u}+\vec{v}\|=\|\vec{u}\|+\|\vec{v}\|$$

If and only if

$$\|s\vec{u}+t\vec{v}\|=s\|\vec{u}\|+t\|\vec{v}\|\quad \textit{for all}\quad s,\ t\geq 0$$

54. Let V be an inner product vector space over $\mathbb{R}$.

a) If $e_1,\ e_2,\ e_3$ are three vectors in V with pairwise product negative, that is,

$$\left\langle e_1,\ e_2\right\rangle<0,\quad i,\ j=1,\ 2,\ 3,\quad i\neq j$$

Show that $e_1,\ e_2,\ e_3$ are linearly independent.

b) Is it possible for three vectors on the xy–plane to have pairwise negative products?

c) Does part (a) remain valid when the word "negative: is replaced with positive?

d) Suppose $\vec{u},\ \vec{v},\ \textit{and}\ \vec{w}$ are three unit vectors in the xy–plane. What are the maximum and minimum values that

$$\langle\vec{u},\ \vec{v}\rangle+\langle\vec{v},\ \vec{w}\rangle+\langle\vec{w},\ \vec{u}\rangle$$

Can attain? And when?

Section 3.3 – Gram-Schmidt Process

3.3−1 *Definition*

A set of two or more vectors in a real inner product space is said to be ***orthogonal*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be ***orthonormal***.

3.3−2 *Theorem*

1. If $S=\left\{\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_n\right\}$ is an orthogonal basis for an inner product space V, and if $\vec{u}$ is any vector in V, then

$$\vec{u}=\frac{\left\langle \vec{u},\ \vec{v}_1\right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1+\frac{\left\langle \vec{u},\ \vec{v}_2\right\rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2+\ \cdots+\ \frac{\left\langle \vec{u},\ \vec{v}_n\right\rangle}{\left\|\vec{v}_n\right\|^2}\vec{v}_n$$

2. If $S=\left\{\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_n\right\}$ is an orthonormal basis for an inner product space V, and if $\vec{u}$ is any vector in V, then

$$\vec{u}=\left\langle \vec{u},\ \vec{v}_1\right\rangle\vec{v}_1+\left\langle \vec{u},\ \vec{v}_2\right\rangle\vec{v}_2+\ \cdots+\left\langle \vec{u},\ \vec{v}_n\right\rangle\vec{v}_n$$

Proof

1. Since $S=\left\{\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_n\right\}$ is a basis for V, every vector $\vec{u}$ in V can be expressed in the form

$$\vec{u}=c_1\vec{v}_1+c_2\vec{v}_2+\ \cdots+c_n\vec{v}_n$$

Let show that $c_i=\dfrac{\left\langle \vec{u},\ \vec{v}_i\right\rangle}{\left\|\vec{v}_i\right\|^2}\quad$ *for* $i=1,\ 2,\ \ldots,\ n$

$$\begin{aligned}\left\langle \vec{u},\ \vec{v}_i\right\rangle&=\left\langle c_1\vec{v}_1+c_2\vec{v}_2+\ \cdots+c_n\vec{v}_n,\ \vec{v}_i\right\rangle\\&=c_1\left\langle \vec{v}_1,\ \vec{v}_i\right\rangle+c_2\left\langle \vec{v}_2,\ \vec{v}_i\right\rangle+\ \cdots\ +c_n\left\langle \vec{v}_n,\ \vec{v}_i\right\rangle\end{aligned}$$

Since S is an orthogonal set, all of the inner products in the last equality are zero except the i^{th}, so we have

$$\begin{aligned}\left\langle \vec{u},\ \vec{v}_i\right\rangle&=c_i\left\langle \vec{v}_i,\ \vec{v}_i\right\rangle\\&=c_i\left\|\vec{v}_i\right\|^2\end{aligned}$$

3.3–3 The *Gram-Schmidt* Process

To convert a basis $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r\}$ into an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$, perform the following computations:

***Step* 1**: $\vec{v}_1 = \vec{u}_1$

***Step* 2**: $\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$

***Step* 3**: $\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$

***Step* 4**: $\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$

To convert the orthogonal basis into an orthonormal basis $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$, normalize the orthogonal basis vectors. $\boxed{\vec{q}_i = \frac{\vec{v}_i}{\|\vec{v}_i\|}}$

Example 3.3-1

Assume that the vector space $\mathbb{R}$ has the Euclidean inner product. Apply the Gram-Schmidt process to transform the basis vectors:

$$\vec{u}_1 = (1,\ 1,\ 1) \quad \vec{u}_2 = (0,\ 1,\ 1) \quad \vec{u}_3 = (0,\ 0,\ 1)$$

Into the orthogonal basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, and then normalize the ***orthogonal*** basis vectors to obtain an orthonormal basis $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$

Solution

$$\vec{v}_1 = \vec{u}_1$$
$$= (1,\ 1,\ 1)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (0,\ 1,\ 1) - \frac{0+1+1}{1^2+1^2+1^2}(1,\ 1,\ 1)$$

$$= (0,\ 1,\ 1) - \frac{2}{3}(1,\ 1,\ 1)$$

$$\underline{= \left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)\Big|}$$

$$\vec{v}_3 = \vec{u}_3 - proj_{\vec{v}_2}\vec{u}_3$$

$$= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (0,\ 0,\ 1) - \frac{0+0+1}{1^2+1^2+1^2}(1,\ 1,\ 1) - \frac{0+0+\frac{1}{3}}{\left(\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$= (0,\ 0,\ 1) - \frac{1}{3}(1,\ 1,\ 1) - \frac{\frac{1}{3}}{\frac{2}{3}}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$= (0,\ 0,\ 1) - \frac{1}{3}(1,\ 1,\ 1) - \frac{1}{2}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$\underline{= \left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)\Big|}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ 1,\ 1)}{\sqrt{3}}$$

$$\underline{= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\Big|}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}{\sqrt{\left(\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2}}$$

$$= \frac{\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}{\frac{\sqrt{6}}{3}}$$

$$= \boxed{\left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$= \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\sqrt{0^2+\left(-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}}$$

$$= \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\frac{\sqrt{2}}{2}}$$

$$= \boxed{\left(0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)}$$

3.3–4 *Gram-Schmidt* Process (*Orthonormal*)

Suppose $\vec{v}_1, \ldots, \vec{v}_n$ linearly independent in $\mathbb{R}^n$, construct n ***orthonormal*** $\vec{u}_1, \ldots, \vec{u}_n$ that span the same space: span $\{\vec{u}_1, \ldots, \vec{u}_k\}$ = span $\{\vec{v}_1, \ldots, \vec{v}_k\}$

***Step* 1**: Since $\vec{v}_i$ are linearly independent ($\neq 0$), so $\left\|\vec{v}_1\right\| \neq 0$ (to create a normal vector)

Let $\vec{u}_1 = \dfrac{\vec{v}_1}{\left\|\vec{v}_1\right\|} = \vec{q}_1$, then $\left\|\vec{u}_1\right\| = 1$ since $\vec{u}_1$ is orthonormal and span $\{\vec{u}_1\} = span\{\vec{v}_1\}$

$$\vec{w}_1 = \vec{v}_1 \Rightarrow \vec{v}_1 = \left\|\vec{w}_1\right\|\vec{u}_1$$

***Step* 2**: $\vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2 \bullet \vec{q}_1\right)\vec{q}_1$

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \bullet \vec{u}_1}{\left\|\vec{v}_1\right\|}\vec{v}_1 \qquad \left(\vec{w}_2 \perp \vec{u}_1\right)$$

$$\vec{v}_2 = \left\|\vec{w}_2\right\|\vec{u}_2 + \left(\vec{v}_2 \bullet \vec{u}_1\right)\vec{u}_1 \qquad \vec{w}_2 = \left\|\vec{w}_2\right\|\vec{u}_2$$

$$\boxed{\vec{q}_2 = \frac{\vec{w}_2}{\left\|\vec{w}_2\right\|}}$$

***Step* 3**: $\vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \bullet \vec{q}_1\right)\vec{q}_1 - \left(\vec{v}_3 \bullet \vec{q}_2\right)\vec{q}_2$

$$\boxed{\vec{q}_3 = \frac{\vec{w}_3}{\left\|\vec{w}_3\right\|}}$$

	$\vec{q}_1 = \dfrac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$
$\vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2 \bullet \vec{q}_1\right)\vec{q}_1$	$\vec{q}_2 = \dfrac{\vec{w}_2}{\left\|\vec{w}_2\right\|}$
$\vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \bullet \vec{q}_1\right)\vec{q}_1 - \left(\vec{v} \bullet \vec{q}_2\right)\vec{q}_2$	$\vec{q}_3 = \dfrac{\vec{w}_3}{\left\|\vec{w}_3\right\|}$
$\vec{w}_n = \vec{v}_n - \left(\vec{v}_n \bullet \vec{q}_1\right)\vec{q}_1 - \left(\vec{v}_n \bullet \vec{q}_2\right)\vec{q}_2 - \ldots - \left(\vec{v}_n \bullet \vec{q}_{n-1}\right)\vec{q}_{n-1}$	$\vec{q}_n = \dfrac{\vec{w}_n}{\left\|\vec{w}_n\right\|}$

Example 3.3-2

Use the Gram-Schmidt process to find an ***orthonormal*** basis for the subspaces of

$$\vec{v}_1 = (1,\ 1,\ 0,\ 0) \quad \vec{v}_2 = (0,\ 1,\ 1,\ 0) \quad \vec{v}_3 = (1,\ 0,\ 1,\ 1)$$

Solution

***Step* 1**: $\vec{q}_1 = \dfrac{(1,\ 1,\ 0,\ 0)}{\sqrt{1^2 + 1^2 + 0 + 0}}$ $\qquad\qquad \vec{q}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$

$$= \frac{(1,\ 1,\ 0,\ 0)}{\sqrt{2}}$$

$$= \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0 \right)$$

***Step* 2**: $\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \bullet \vec{q}_1)\vec{q}_1$

$$= (0,\ 1,\ 1,\ 0) - \left[(0,\ 1,\ 1,\ 0) \bullet \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0 \right) \right] \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0 \right)$$

$$= (0,\ 1,\ 1,\ 0) - \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0 \right)$$

$$= (0,\ 1,\ 1,\ 0) - \left(\frac{1}{2},\ \frac{1}{2},\ 0,\ 0 \right)$$

$$= \left(-\frac{1}{2},\ \frac{1}{2},\ 1,\ 0 \right)$$

$$\|\vec{w}_2\| = \sqrt{\left(-\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 + 1}$$

$$= \frac{\sqrt{3}}{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{6}}{2}$$

$$\vec{q}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{\left(-\frac{1}{2},\ \frac{1}{2},\ 1,\ 0 \right)}{\frac{\sqrt{6}}{2}}$$

$$= \frac{2}{\sqrt{6}} \left(-\frac{1}{2},\ \frac{1}{2},\ 1,\ 0 \right)$$

$$= \left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ 0 \right)$$

Step 3: $\vec{v}_3 \cdot \vec{q}_1 = (1,\ 0,\ 1,\ 1) \cdot \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)$

$$= \boxed{\frac{1}{\sqrt{2}}}$$

$$\vec{v}_3 \cdot \vec{q}_2 = (1,\ 0,\ 1,\ 1) \cdot \left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ 0\right)$$

$$= -\frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}}$$

$$= \boxed{\frac{1}{\sqrt{6}}}$$

$$\vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \cdot \vec{q}_1\right)\vec{q}_1 - \left(\vec{v} \cdot \vec{q}_2\right)\vec{q}_2$$

$$= (1,\ 0,\ 1,\ 1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right) - \frac{1}{\sqrt{6}}\left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ 0\right)$$

$$= (1,\ 0,\ 1,\ 1) - \left(\frac{1}{2},\ \frac{1}{2},\ 0,\ 0\right) - \left(-\frac{1}{6},\ \frac{1}{6},\ \frac{2}{6},\ 0\right)$$

$$= \boxed{\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)}$$

$$\vec{q}_3 = \frac{\vec{w}_3}{\left\|\vec{w}_3\right\|}$$

$$= \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + 1^2}}\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$= \frac{1}{\sqrt{\frac{21}{9}}}\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$= \frac{3}{\sqrt{21}}\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$= \boxed{\left(\frac{2}{\sqrt{21}},\ -\frac{2}{\sqrt{21}},\ \frac{2}{\sqrt{21}},\ \frac{3}{\sqrt{21}}\right)}$$

The ***orthonormal*** basis:

$$\left\{ \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right),\ \left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ 0\right),\ \left(\frac{2}{\sqrt{21}},\ -\frac{2}{\sqrt{21}},\ \frac{2}{\sqrt{21}},\ \frac{3}{\sqrt{21}}\right) \right\}$$

3.3−5 *QR*–Decomposition

3.3−6 Problem

If A is an $m\times n$ matrix with linearly independent column vectors, and if Q is the matrix that results by applying the Gram-Schmidt process to the column vectors of A, what relationship, if any, exists between A and Q?

To solve this problem, suppose that the column vectors of A are $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ and the orthonormal column vectors of Q are $\vec{q}_1, \vec{q}_2, \dots, \vec{q}_n$.

$$\begin{aligned}
\vec{u}_1 &= \langle \vec{u}_1, \vec{q}_1\rangle \vec{q}_1 + \langle \vec{u}_1, \vec{q}_2\rangle \vec{q}_2 + \cdots + \langle \vec{u}_1, \vec{q}_n\rangle \vec{q}_n \\
\vec{u}_2 &= \langle \vec{u}_2, \vec{q}_1\rangle \vec{q}_1 + \langle \vec{u}_2, \vec{q}_2\rangle \vec{q}_2 + \cdots + \langle \vec{u}_2, \vec{q}_n\rangle \vec{q}_n \\
\vdots \quad & \qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
\vec{u}_n &= \langle \vec{u}_n, \vec{q}_1\rangle \vec{q}_1 + \langle \vec{u}_n, \vec{q}_2\rangle \vec{q}_2 + \cdots + \langle \vec{u}_n, \vec{q}_n\rangle \vec{q}_n
\end{aligned}$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1\rangle & \langle \vec{u}_2, \vec{q}_1\rangle & \cdots & \langle \vec{u}_n, \vec{q}_1\rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2\rangle & \cdots & \langle \vec{u}_n, \vec{q}_2\rangle \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \langle \vec{u}_n, \vec{q}_n\rangle \end{bmatrix}$$

The equation $A = QR$ is a factorization of A into the product of a matrix Q with orthonormal column vectors and an invertible upper triangular matrix R. We call it the ***QR-decomposition of A***.

3.3−7 *Theorem*

If A is an $m\times n$ matrix with linearly independent column vectors, then A can be factored as

$$A = QR$$

Where Q is an $m\times n$ matrix with orthonormal column vectors, and R is an $n\times n$ invertible upper triangular matrix.

Example 3.3-3

Find the QR-decomposition of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Solution

The column vectors of are

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

From the previous example

$$\vec{q}_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \qquad \vec{q}_2 = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right) \qquad \vec{q}_3 = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle & \langle \vec{u}_3, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle & \langle \vec{u}_3, \vec{q}_2 \rangle \\ 0 & 0 & \langle \vec{u}_3, \vec{q}_3 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} (1)\frac{1}{\sqrt{3}} + (1)\frac{1}{\sqrt{3}} + (1)\frac{1}{\sqrt{3}} & 0 + (1)\frac{1}{\sqrt{3}} + (1)\frac{1}{\sqrt{3}} & 0 + 0 + (1)\frac{1}{\sqrt{3}} \\ 0 & 0\left(\frac{-2}{\sqrt{6}}\right) + (1)\frac{1}{\sqrt{6}} + (1)\frac{1}{\sqrt{6}} & 0\left(\frac{-2}{\sqrt{6}}\right) + 0\frac{1}{\sqrt{6}} + (1)\frac{1}{\sqrt{6}} \\ 0 & 0 & 0 + (0)\frac{-1}{\sqrt{2}} + (1)\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\underset{A}{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}} = \underset{Q}{\begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}} \cdot \underset{R}{\begin{pmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}}$$

3.3−8 *Calculus*: Applying the Gram-Schmidt Process

We can apply the Gram-Schmidt orthogonalization procedure to generate some polynomials that are orthonormal on the interval $x \in [-1, 1]$ with inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

Example 3.3-4

Apply the Gram-Schmidt orthonormalization process to the basis $B = \{1, x, x^2\}$ in $\mathbb{P}_2$ using the inner product.

Solution

$B = \{1, x, x^2\}$

Let $\vec{u}_1 = 1, \quad \vec{u}_2 = x, \quad \vec{u}_3 = x^2$

$\vec{v}_1 = \vec{u}_1 = 1$

$$\begin{aligned}\langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^{1} dx \\ &= x \Big|_{-1}^{1} \\ &= 2\end{aligned}$$

$$\begin{aligned}\langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^{1} x\,dx \\ &= \frac{1}{2}x^2 \Big|_{-1}^{1} \\ &= 0\end{aligned}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 \\ &= x - \frac{0}{2}(1) \\ &= x\end{aligned}$$

$$\left\langle \vec{v}_2, \vec{v}_2 \right\rangle = \int_{-1}^{1} x^2 \, dx$$

$$= \frac{1}{3}x^3 \Big|_{-1}^{1}$$

$$\boxed{= \frac{2}{3}}$$

$$\left\langle \vec{u}_3, \vec{v}_1 \right\rangle = \int_{-1}^{1} x^2 \, dx$$

$$= \frac{1}{3}x^3 \Big|_{-1}^{1}$$

$$\boxed{= \frac{2}{3}}$$

$$\left\langle \vec{u}_3, \vec{v}_2 \right\rangle = \int_{-1}^{1} x^3 \, dx$$

$$= \frac{1}{4}x^4 \Big|_{-1}^{1}$$

$$\boxed{= 0}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\left\langle \vec{u}_3, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2}\vec{v}_1 - \frac{\left\langle \vec{u}_3, \vec{v}_2 \right\rangle}{\left\| \vec{v}_2 \right\|^2}\vec{v}_2$$

$$= x^2 - \frac{3}{2}(x)(0) - \frac{1}{2}\frac{2}{3}$$

$$\boxed{= x^2 - \frac{1}{3}}$$

$$\left\langle \vec{v}_3, \vec{v}_3 \right\rangle = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx$$

$$= \int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) dx$$

$$= \left(\frac{1}{5}x^5 - \frac{2}{9}x^3 + \frac{1}{9}x\right) \Big|_{-1}^{1}$$

$$= \frac{1}{5} - \frac{2}{9} + \frac{1}{9} + \frac{1}{5} - \frac{2}{9} + \frac{1}{9}$$

$$\boxed{= \frac{8}{45}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\boxed{= \frac{1}{\sqrt{2}}}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{x}{\sqrt{2/3}}$$

$$\boxed{= \frac{\sqrt{3}}{\sqrt{2}} x}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3}\right)$$

$$\boxed{= \frac{3\sqrt{5}}{2\sqrt{2}} \left(x^2 - \frac{1}{3}\right)}$$

The ***orthonormal*** basis is $\left\{ \frac{1}{\sqrt{2}},\ \sqrt{\frac{3}{2}}\, x,\ \frac{1}{2}\sqrt{\frac{5}{2}} \left(3x^2 - 1\right) \right\}$

Exercises *Section* 3.3 – Gram-Schmidt Process

(1 – 40) Use the Gram-Schmidt process to find an ***orthogonal*** basis and an ***orthonormal*** basis for the subspaces of $\mathbb{R}^m$.

1. $\vec{u}_1 = (1,\ -3),\ \ \vec{u}_2 = (2,\ 2)$

2. $\vec{u}_1 = (1,\ 0),\ \ \vec{u}_2 = (3,\ -5)$

3. $\vec{u}_1 = (1,\ 0,\ 0),\ \ \vec{u}_2 = (3,\ 7,\ -2),\ \ \vec{u}_3 = (0,\ 4,\ 1)$

4. $\vec{u}_1 = (1,\ 1,\ 0,\ -1),\ \ \vec{u}_2 = (1,\ 3,\ 0,\ 1),\ \ \vec{u}_3 = (4,\ 2,\ 2,\ 0)$

5. $\vec{u}_1 = (1,\ 1,\ 1,\ 1),\ \ \vec{u}_2 = (1,\ 2,\ 4,\ 5),\ \ \vec{u}_3 = (1,\ -3,\ -4,\ -2)$

6. $\vec{u}_1 = (1,\ 1,\ 1,\ 1),\ \ \vec{u}_2 = (1,\ 1,\ 2,\ 4),\ \ \vec{u}_3 = (1,\ 2,\ -4,\ -3)$

7. $\vec{u}_1 = (0,\ 2,\ 1,\ 0),\ \ \vec{u}_2 = (1,\ -1,\ 0,\ 0),\ \ \vec{u}_3 = (1,\ 2,\ 0,\ -1),\ \ \vec{u}_4 = (1,\ 0,\ 0,\ 1)$

8. $\{(3,\ 4),\ (1,\ 0)\}$

9. $\{(3,\ 0,\ -1),\ (8,\ 5,\ -6)\}$

10. $\{(0,\ 4,\ 2),\ (5,\ 6,\ -7)\}$

11. $\{(1,\ 1,\ 1),\ (-1,\ 1,\ 0),\ (1,\ 2,\ 1)\}$

12. $\{(1,\ 1,\ 1),\ (0,\ 1,\ 1),\ (0,\ 0,\ 1)\}$

13. $\{(1,\ 1,\ 1),\ (0,\ 2,\ 1),\ (1,\ 0,\ 3)\}$

14. $\{(2,\ 2,\ 2),\ (1,\ 0,\ -1),\ (0,\ 3,\ 1)\}$

15. $\{(1,\ -1,\ 0),\ (0,\ 1,\ 0),\ (2,\ 3,\ 1)\}$

16. $\{(3,\ 0,\ 4),\ (-1,\ 0,\ 7),\ (2,\ 9,\ 11)\}$

17. $\{(2,\ 1,\ -1),\ (1,\ 2,\ 2),\ (2,\ -2,\ 1)\}$

18. $\{(1,\ 1,\ 0),\ (1,\ 2,\ 0),\ (0,\ 1,\ 2)\}$

19. $\{(1,\ -2,\ 2),\ (2,\ 2,\ 1),\ (2,\ -1,\ -2)\}$

20. $\{(1,\ 0,\ 0),\ (1,\ 1,\ 1),\ (1,\ 1,\ -1)\}$

21. $\{(4,\ -3,\ 0),\ (1,\ 2,\ 0),\ (0,\ 0,\ 4)\}$

22. $\{(0,\ 1,\ 2),\ (2,\ 0,\ 0),\ (1,\ 1,\ 1)\}$

23. $\{(0,\ 1,\ 1),\ (1,\ 1,\ 0),\ (1,\ 0,\ 1)\}$

24. $\{(2,\ -1,\ 3),\ (3,\ 4,\ 1),\ (2,\ -3,\ 4)\}$

25. $\{(-3,\ 0,\ 4),\ (5,\ -1,\ 2),\ (1,\ 1,\ 3)\}$

26. $\{(-2,\ 0,\ 1),\ (3,\ 2,\ 5),\ (6,-1,\ 1),\ (7,\ 0,-2)\}$

27. $\{(1,\ 2,\ -2),\ (1,\ 0,\ -4),\ (5,\ 2,\ 0),\ (1,\ 1,\ -1)\}$

28. $\{(-3,\ 1,\ 2),\ (1,\ 1,\ 1),\ (2,\ 0,\ -1),\ (1,\ -3,\ 2)\}$

29. $\{(2,\ 1,\ 1),\ (0,\ 3,\ -1),\ (3,\ -4,\ -2),\ (-1,\ -1,\ 3)\}$

30. $\{(1,\ 1,\ 1,\ 1),\ (0,\ 1,\ 1,\ 1),\ (0,\ 0,\ 1,\ 1)\}$

31. $\{(1,\ 2,\ -1,\ 0),\ (2,\ 2,\ 0,\ 1),\ (1,\ 1,\ -1,\ 0)\}$

32. $\{(1,\ 1,\ 1,\ 1),\ (1,\ 2,\ 1,\ 0),\ (1,\ 3,\ 0,\ 0)\}$

33. $\{(0,\ 2,\ -1,\ 1),\ (0,\ 0,\ 1,\ 1),\ (-2,\ 1,\ 1,\ -1)\}$

34. $\{(-1,\ 3,\ 1,\ 1),\ (6,\ -8,\ -2,\ -4),\ (6,\ 3,\ 6,\ -3)\}$

35. $\{(0,\ 0,\ 2,\ 2),\ (3,\ 3,\ 0,\ 0),\ (1,\ 1,\ 0,-1)\}$

36. $\{(1,\ 0,\ 0,\ 0),\ (1,\ 1,\ 0,\ 0),\ (1,\ 1,\ 1,\ 0),\ (1,\ 1,\ 1,\ 1)\}$

37. $\{(3,\ 8,\ 7,\ -3),\ (1,\ 5,\ 3,\ -1),\ (2,\ -1,\ 2,\ 6),\ (1,\ 4,\ 0,\ 3)\}$

38. $\{(0,\ 3,\ -3,\ -6),\ (-2,\ 0,\ 0,\ -6),\ (0,\ -4,\ -2,\ -2),\ (0,\ -8,\ 4,\ -4)\}$

39. $\{(3,\ 0,\ -3,\ 6),\ (0,\ 2,\ 3,\ 1),\ (0,\ -2,\ -2,\ 0),\ (-2,\ 1,\ 2,\ 1)\}$

40. $\{(3,\ 4,\ 0,\ 0),\ (-1,\ 1,\ 0,\ 0),\ (2,\ 1,\ 0,\ -1),\ (0,\ 1,\ 1,\ 0)\}$

(**41 – 55**) Find the ***QR***-decomposition of

41. $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

42. $\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix}$

43. $\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$

44. $\begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}$

45. $\begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix}$

46. $\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}$

47. $\begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix}$

48. $\begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}$

49. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$

50. $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{pmatrix}$

51. $\begin{pmatrix} 7 & 1 & 2 \\ 2 & -1 & 3 \\ -3 & 0 & 4 \end{pmatrix}$

52. $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$

53. $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

54. $\begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{pmatrix}$

55. $\begin{pmatrix} -1 & 2 & 2 \\ 2 & 5 & -1 \\ 2 & 8 & 0 \\ -3 & 3 & 5 \end{pmatrix}$

56. Verify that the Cauchy-Schwarz inequality holds for the given vectors using the Euclidean inner product
$\vec{u} = (0,\ -2,\ 2,\ 1), \quad \vec{v} = (-1,\ -1,\ 1,\ 1)$

(**57 – 70**) Apply the Gram-Schmidt ***orthonormalization*** process in $\mathbb{C}^0\,[-1,\ 1]$ spanned by the functions, using the inner product

57. $f_1(x) = x+2, \quad f_2(x) = x^2 - 3x + 4$

58. $f_1(x) = 1+3x^2, \quad f_2(x) = x - x^2$

59. $f_1(x) = 5x-3, \quad f_2(x) = x^3 - x^2$

60. $f_1(x) = 1, \quad f_2(x) = 2x - 1$

61. $f_1(x) = e^x, \quad f_2(x) = x$

62. $f_1(x) = x, \quad f_2(x) = x^3, \quad f_3(x) = x^5$

63. $f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x) = \frac{1}{2}\left(3x^2 - 1\right)$

64. $f_1(x)=5, \quad f_2(x)=x^2-6x, \quad f_3(x)=(3-x)^2$

65. $f_1(x)=1, \quad f_2(x)=\sin \pi x, \quad f_3(x)=\cos \pi x$

66. $f_1(x)=1, \quad f_2(x)=\sin x, \quad f_3(x)=\sin 2x$

67. $f_1(x)=6, \quad f_2(x)=3\sin^2 x, \quad f_3(x)=2\cos^2 x$

68. $f_1(x)=\cos 2x, \quad f_2(x)=\sin^2 x, \quad f_3(x)=\cos^2 x$

69. $f_1(x)=\sin \pi x, \quad f_2(x)=\sin 2\pi x, \quad f_3(x)=\sin 3\pi x$

70. $f_1(x)=\cos \pi x, \quad f_2(x)=\cos 2\pi x, \quad f_3(x)=\cos 3\pi x$

71. For $\mathbb{P}_3[x]$, define the inner product over $\mathbb{R}$ as

$$\langle f,\, g\rangle = \int_{-1}^{1} f(x)g(x)\, dx$$

a) If $f(x)=1$ is a unit vector in $\mathbb{P}_3[x]$?

b) Find an orthonormal basis for the subspace spanned by $x \text{ and } x^2$.

c) Complete the basis in part (*b*) to an orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product.

d) Is

$$[f,\, g] = \int_{0}^{1} f(x)g(x)\, dx$$

Also, an inner product for $\mathbb{P}_3[x]$

e) Find a pair of vectors $\vec{v}$ and $\vec{w}$ such that

$$\langle \vec{v},\, \vec{w}\rangle = 0 \quad but \quad [\vec{v},\, \vec{w}] \neq 0$$

f) Is the basis found in part (*c*) are orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product in part (*d*)?

Section 3.4 – Orthogonal Matrices

3.4−1 *Definition*

A square matrix A is said to be orthogonal if its transpose is the same as its inverse, that is, if

$$A^{-1} = A^T$$

or, equivalently, if

$$AA^T = A^T A = I$$

Example 3.4-1

The matrix $A = \begin{pmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{pmatrix}$

Solution

$$\begin{aligned} A^T A &= \begin{pmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{pmatrix} \begin{pmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Example 3.4-2

The matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Solution

$$\begin{aligned} A^T A &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

3.4−2 *Theorem*

The following are equivalent for $n \times n$ matrix A.

a) A is orthogonal.

b) The row vectors of A form an orthonormal set in R^n with the Euclidean inner product.

c) The column vectors of A form an orthonormal set in R^n with the Euclidean inner product.

3.4−3 *Theorem*

a) The inverse of an orthogonal matrix is orthogonal

b) A product of orthogonal matrices is orthogonal

c) If A is orthogonal, then $\det(A) = 1$ or $\det(A) = -1$

3.4−4 *Theorem*

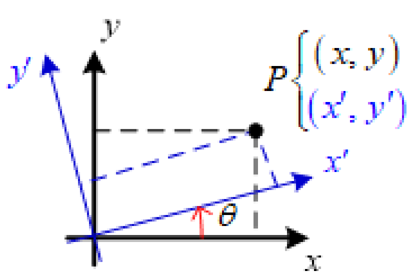
If A is an $n \times n$ matrix, then the following are equivalent

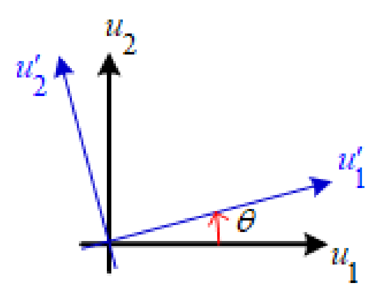
a) A is orthogonal.

b) $\|A\vec{x}\| = \|\vec{x}\|$ for all $\boldsymbol{x}$ in R^n.

c) $A\vec{x} \bullet A\vec{y} = \vec{x} \bullet \vec{y}$ for all $\vec{x}$ and $\vec{y}$ in R^n.

Let $\vec{u}_1$ and $\vec{u}_2$ be the unit vectors along the x- and y-axes and unit vectors $\vec{u}'_1$ and $\vec{u}'_2$ along the x'- and y'-axes.

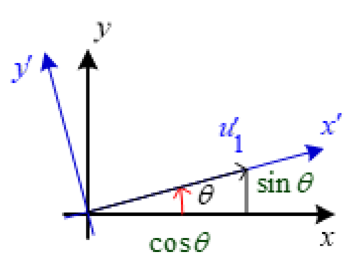




The new coordinates (x', y') and the old coordinates (x, y) of a point P will be related by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$$

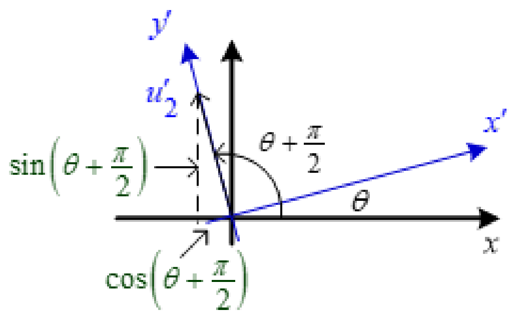
$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



$$P^{-1} = P^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \vec{x}' \\ \vec{y}' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix}$$

$$\to \begin{cases} x' = x\cos\theta + y\sin\theta \\ y' = -x\sin\theta + y\cos\theta \end{cases}$$



These are sometimes called the ***rotation equations***.

Example 3.4-3

Use the form $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ to find the new coordinates of the point $Q(2,\ 1)$ if the coordinate axes of a rectangular coordinate system are rotated through an angle of $\theta = \frac{\pi}{4}$.

Solution

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

The new coordinates of Q are $(x',\ y') = \left(\frac{3}{\sqrt{2}},\ -\frac{1}{\sqrt{2}} \right)$

Exercises *Section* 3.4 – Orthogonal Matrices

(1 – 7) Show that the matrix is orthogonal

1. $A=\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$

2. $A=\begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix}$

3. $A=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$

4. $A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{7}{\sqrt{53}} & \frac{2}{\sqrt{53}} \\ 0 & -\frac{2}{\sqrt{53}} & \frac{7}{\sqrt{53}} \end{pmatrix}$

5. $A=\begin{pmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{pmatrix}$

6. $A=\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$

7. $A=\begin{pmatrix} 0 & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{15}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{15}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & -\frac{2}{\sqrt{10}} & -\frac{2}{\sqrt{15}} \\ 0 & 0 & -\frac{2}{\sqrt{10}} & \frac{3}{\sqrt{15}} \end{pmatrix}$

(8 – 30) Determine if the matrix is orthogonal. For those that is orthogonal find the inverse.

8. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

9. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

10. $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

11. $\begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix}$

12. $\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$

13. $\begin{pmatrix} \frac{2}{\sqrt{6}} & 0 \\ 0 & \frac{3}{\sqrt{6}} \end{pmatrix}$

14. $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

15. $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

16. $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$

17. $\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$

18. $\begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

19. $\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

20. $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

21. $\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix}$

22. $\begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{5\sqrt{3}} & \frac{4}{5\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{7}{5\sqrt{3}} & \frac{3}{5\sqrt{2}} \end{pmatrix}$

23. $\begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$

24. $\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{pmatrix}$

25. $\begin{pmatrix} -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{42}} \\ \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{42}} \\ \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{3}} & -\frac{4}{\sqrt{42}} \end{pmatrix}$

26. $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$

27. $\begin{pmatrix} \frac{7}{\sqrt{62}} & \frac{9}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ \frac{2}{\sqrt{62}} & -\frac{24}{\sqrt{682}} & \frac{1}{\sqrt{11}} \\ -\frac{3}{\sqrt{62}} & \frac{5}{\sqrt{682}} & \frac{3}{\sqrt{11}} \end{pmatrix}$

28. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$

29. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{bmatrix}$

30. $\begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{\sqrt{174}} & \frac{9}{2\sqrt{29}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{2} & -\frac{10}{\sqrt{174}} & -\frac{1}{2\sqrt{29}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{2} & -\frac{8}{\sqrt{174}} & \frac{5}{2\sqrt{29}} \\ -\frac{2}{\sqrt{6}} & -\frac{1}{2} & -\frac{1}{\sqrt{174}} & -\frac{3}{2\sqrt{29}} \end{pmatrix}$

31. Determine if the given matrix is orthogonal. If it is, find its inverse

$$\begin{bmatrix} \frac{1}{9} & \frac{4}{5} & \frac{3}{7} \\ \frac{4}{9} & \frac{3}{5} & -\frac{2}{7} \\ \frac{8}{9} & -\frac{2}{5} & \frac{3}{7} \end{bmatrix}$$

(31 – 39) Find a last column so that the resulting matrix is orthogonal

32. $\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & \cdots \end{pmatrix}$

33. $\begin{pmatrix} \frac{3}{5} & 0 & \cdots \\ \frac{4}{5} & 0 & \cdots \\ 0 & 1 & \cdots \end{pmatrix}$

34. $\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & \cdots \\ 0 & 0 & \cdots \\ \frac{4}{5} & \frac{3}{5} & \cdots \end{pmatrix}$

35. $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots \\ 0 & 0 & \cdots \end{pmatrix}$

36. $\begin{pmatrix} \frac{2}{7} & \frac{3}{7} & \cdots \\ \frac{6}{7} & \frac{2}{7} & \cdots \\ -\frac{3}{7} & \frac{6}{7} & \cdots \end{pmatrix}$

37. $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \cdots \\ \frac{2}{3} & \frac{1}{3} & \cdots \\ -\frac{2}{3} & \frac{2}{3} & \cdots \end{pmatrix}$

38. $\begin{pmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & \cdots \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} & \cdots \\ -\frac{1}{3} & 0 & \cdots \end{pmatrix}$

39. $\begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{3}{\sqrt{70}} & \cdots \\ -\frac{1}{\sqrt{5}} & \frac{6}{\sqrt{70}} & \cdots \\ 0 & -\frac{5}{\sqrt{70}} & \cdots \end{pmatrix}$

40. Determine if the given matrix is orthogonal. If it is, find its inverse

$$\begin{bmatrix} \frac{1}{9} & \frac{4}{5} & \frac{3}{7} \\ \frac{4}{9} & \frac{3}{5} & -\frac{2}{7} \\ \frac{8}{9} & -\frac{2}{5} & \frac{3}{7} \end{bmatrix}$$

41. Prove that if A is orthogonal, then A^T is orthogonal.

42. Prove that if A is orthogonal, then A^{-1} is orthogonal.

43. Prove that if A and B are orthogonal, then AB is orthogonal.

44. Let Q be an $n \times n$ orthogonal matrix, and let A be an $n \times n$ matrix. Show that $\det\left(QAQ^T\right) = \det(A)$

45. Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$

a) Is matrix A an orthogonal matrix?
b) Let B be the matrix obtained by normalizing each row of A, find B.
c) Is B an orthogonal matrix?
d) Are the columns of B orthogonal?

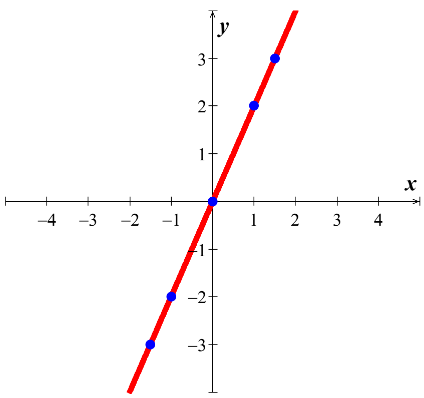
Section 3.5 – Least Squares Analysis

The use to ***best*** fit data, we will use results about orthogonal projections in inner product spaces to obtain a technique for fitting a line or other polynomial. The outcome will let us predict the next desire value that we are looking for.

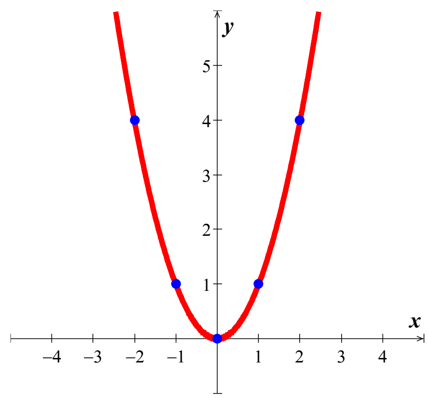
From the shape of the scatter output data will be the best fit to determine the function that you need to use to reduce margin error.

3.5–1 Fitting a Curve to Data

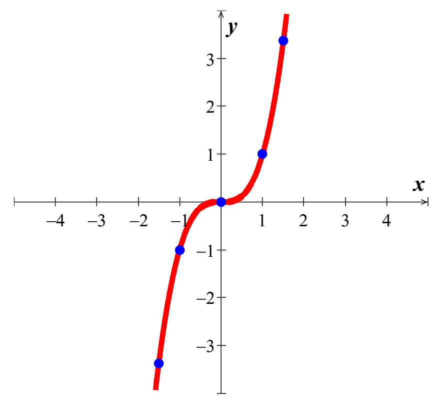
The common problem is to obtain a mathematical relationship between 2 variables x and y by ***fitting*** a curve to points in the xy-plane.
Some possibility of fitting the data



$y = ax + b$



$y = ax^2 + bx + c$



$y = ax^3 + bx^2 + cx + d$

3.5–2 Least Squares Fit of a Straight Line

Recall that a system of equations $A\vec{x} = \vec{y}$ is called inconsistent if it does not have a solution. Suppose we want to fit a straight line $y = mx + b$ to the determined points $\left(x_1, y_1\right), \ \ldots, \ \left(x_n, y_n\right)$

If the data points were collinear, the line would pass through all n points and the unknown coefficients m and b would satisfy the equations

$$\begin{matrix} y_1 = mx_1 + b \\ y_2 = mx_2 + b \\ \vdots \quad \vdots \quad \vdots \\ y_n = mx_n + b \end{matrix} \quad \Rightarrow \quad \underbrace{\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ b \end{bmatrix}}_{\vec{x}} \underset{=}{=} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\vec{y}}$$

The problem is to find m and b that minimize the errors is some sense.

3.5−3 Least Square Problem

Given a linear system $A\vec{x} = \vec{y}$ of m equations in n unknowns, find a vector $\vec{x}$ that minimizes $\|\vec{y} - A\vec{x}\|$ with respect to the Euclidean inner product on $\mathbb{R}^m$. We call such as $\vec{x}$ a least squares solution of the system, we call $\vec{y} - A\vec{x}$ the least squares error vectors, and we call $\|\vec{y} - A\vec{x}\|$ the least squares error.

$$A\boldsymbol{x} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{pmatrix}$$

The term "***least square solution***" results from the fact the minimizing $\|\vec{y} - A\vec{x}\| = e_1^2 + e_2^2 + \ldots + e_m^2$

Example 3.5-1

Find the sums of squares of the errors of (2, 4), (4, 8), (6, 6)

Solution

$$4 = 2m + b \quad \Rightarrow \quad 4 - 2m - b = e_1$$

$$8 = 4m + b \quad \Rightarrow \quad 8 - 4m - b = e_2$$

$$6 = 6m + b \quad \Rightarrow \quad 6 - 6m - b = e_3$$

$$e_1^2 + e_2^2 + \ldots + e_m^2 = (4 - 2m - b)^2 + (8 - 4m - b)^2 + (6 - 6m - b)^2$$

The least squares problem for this example to find the values m and b for which $e_1^2 + e_2^2 + \ldots + e_m^2$ is a minimum.

3.5–4 *Theorem*

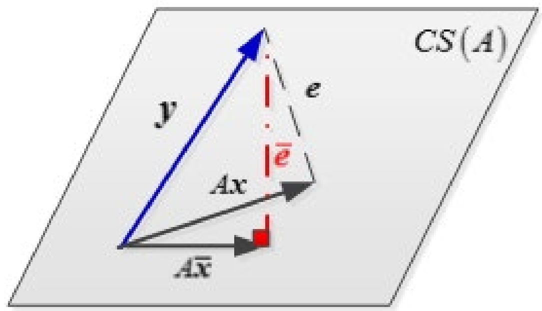
If A is an $m \times n$ matrix, the equation $A\vec{x} = \vec{y}$ has a solution if and only if $\vec{y}$ is in the column space of A.

$$\vec{y} - A\vec{x} = \vec{e}$$

$A\vec{x}$ is a vector that is in the column space of A. For this A the column space is a plane is $\mathbb{R}^m$

$\vec{y}$ is a vector, not in the column space of A (otherwise $A\vec{x} = \vec{y}$ has an exact solution)

$\vec{e}$ is the error vector, the difference between $\vec{y}$ and $A\vec{x}$



The length $\|\vec{e}\|$ is a *minimum* exactly when $\vec{e} \perp CS(A)$

3.5–5 Best Approximation *Theorem*

If $CS(A)$ is a finite dimensional subspace of an inner product space, and if $\vec{y}$ is a vector in ***V***, then $proj_{CS(A)}\vec{y}$ is the best approximation to ***y*** from $CS(A)$ is the sense that

$$\left\| \vec{y} - proj_{CS(A)}\vec{y} \right\| < \| \vec{y} - CS(A) \|$$

For every vector $\vec{w}$ in $CS(A)$ that is different from $proj_{CS(A)}\vec{y}$

3.5–6 *Theorem*

For every linear system $A\vec{x} = \vec{y}$, the associated normal system

$$A^T A\vec{x} = A^T \vec{y}$$

Is consistent, and all solutions are least squares solutions of $A\vec{x} = \vec{y}$

If the columns of A are linearly independent, then $A^T A$ is invertible so has a unique solution $\bar{x}$. This solution is often expressed theoretically as

$$\left(A^T A\right)^{-1} A^T A\overline{x} = \left(A^T A\right)^{-1} A^T \vec{y}$$

$$\overline{x} = \left(A^T A\right)^{-1} A^T \vec{y}$$

Proof

Let the vector $\overline{\boldsymbol{x}}$ is a least squares solution to $A\vec{x} = \vec{y} \quad \Leftrightarrow \quad \left(\vec{y} - A\overline{x}\right) \perp CS\left(A\right)$

$$\left(\vec{y} - A\overline{x}\right) \cdot \vec{z} = 0 \qquad \vec{z} \text{ in } CS\left(A\right) \quad \& \quad \vec{z} = A\vec{w}$$

$$\left(\vec{y} - A\overline{x}\right) \cdot A\vec{w} = 0 \qquad \vec{w} \text{ in } \mathbb{R}^n$$

$$A^T\left(\vec{y} - A\overline{x}\right) \cdot \vec{w} = 0$$

$$A^T\left(\vec{y} - A\overline{\boldsymbol{x}}\right) = 0$$

$$A^T\vec{y} - A^T A\overline{\boldsymbol{x}} = 0$$

$$A^T\vec{y} = A^T A\overline{x}$$

3.5–6 ***Theorem***

If A is an $m \times n$ matrix, then the following are equivalent

a) A has linearly independent column vectors.

b) $A^T A$ is invertible.

Example 3.5-2

Find the equation of the line that best fits the given points in the least-squares sense.

(40, 482), (45, 467), (50, 452), (55, 432), (60, 421)

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 40 & 1 \\ 45 & 1 \\ 50 & 1 \\ 55 & 1 \\ 60 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 482 \\ 467 \\ 452 \\ 432 \\ 421 \end{pmatrix}$$

Where $A = \begin{pmatrix} 40 & 1 \\ 45 & 1 \\ 50 & 1 \\ 55 & 1 \\ 60 & 1 \end{pmatrix} \quad \boldsymbol{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \boldsymbol{y} = \begin{pmatrix} 482 \\ 467 \\ 452 \\ 432 \\ 421 \end{pmatrix}$

Using the normal equation formula: $A^T Ax = A^T y$

$$\begin{pmatrix} 40 & 45 & 50 & 55 & 60 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 40 & 1 \\ 45 & 1 \\ 50 & 1 \\ 55 & 1 \\ 60 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 40 & 45 & 50 & 55 & 60 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 482 \\ 467 \\ 452 \\ 432 \\ 421 \end{pmatrix}$$

$$\begin{pmatrix} 12,750 & 250 \\ 250 & 5 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 111,970 \\ 2,255 \end{pmatrix}$$

$$X = A^{-1}B$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{1250} \begin{pmatrix} 5 & -250 \\ -250 & 12,750 \end{pmatrix} \begin{pmatrix} 111,970 \\ 2,255 \end{pmatrix}$$

$$= \begin{pmatrix} -3.12 \\ 607 \end{pmatrix}$$

Or

$$m = \frac{\begin{vmatrix} 111,970 & 250 \\ 2,255 & 5 \end{vmatrix}}{\begin{vmatrix} 12,750 & 250 \\ 250 & 5 \end{vmatrix}}$$

$$= \frac{-3,900}{1,250}$$

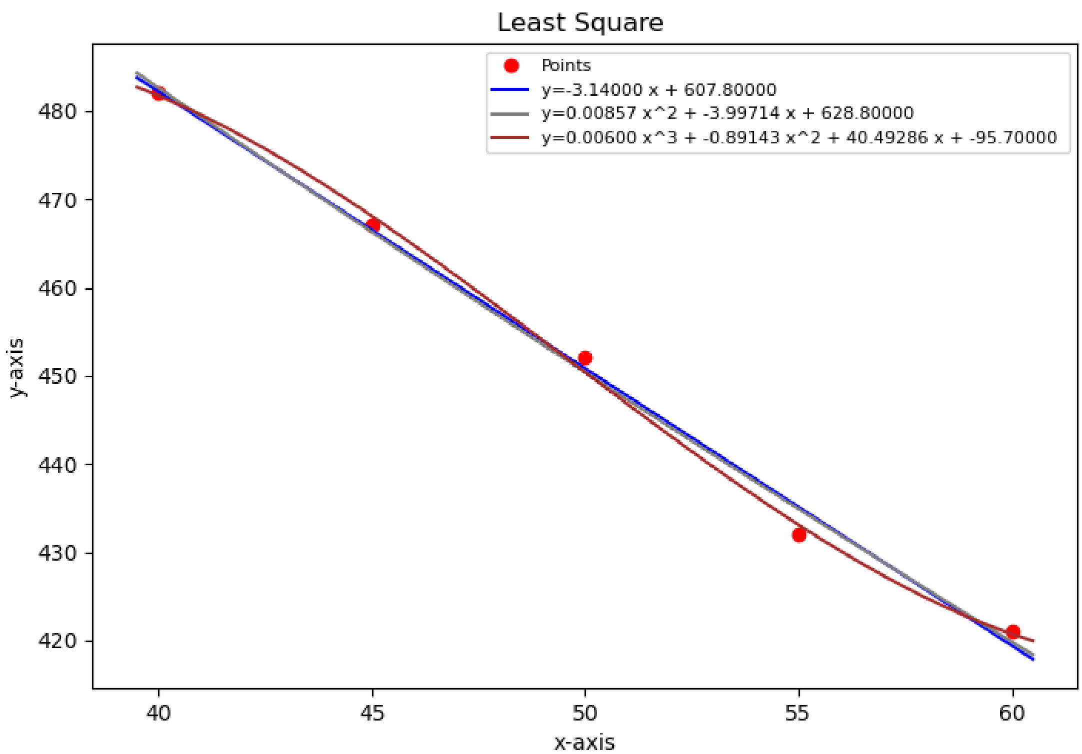
$$= -\frac{78}{25}$$

$$b = \frac{\begin{vmatrix} 12,750 & 111,970 \\ 250 & 2,255 \end{vmatrix}}{1,250}$$

$$= \frac{758,750}{1,250}$$

$$= 607$$

Thus, $\boxed{y=-\frac{78}{25}x+607}$ *or* $\boxed{y=-3.12x+607}$



Example 3.5-3

Given the system equation: $\begin{cases} x_1 - x_2 = 4 \\ 3x_1 + 2x_2 = 1 \\ -2x_1 + 4x_2 = 3 \end{cases}$

a) Find the least-squares solution of the linear system $A\vec{x}=\vec{y}$

b) Find the orthogonal projection of $\vec{y}$ on the column space of A

c) Find the ***error vector*** and the ***error***

Solution

a) $A=\begin{pmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix} \quad \vec{x}=\begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y}=\begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$

$$A^T A\vec{x}=A^T\vec{y}$$

$$\begin{pmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix}=\begin{pmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 14 & -3 \\ -3 & 21 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{285}\begin{pmatrix} 21 & 3 \\ 3 & 14 \end{pmatrix}\begin{pmatrix} 1 \\ 10 \end{pmatrix} \qquad X = A^{-1}B$$

$$= \begin{pmatrix} \frac{51}{285} \\ \frac{143}{285} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{17}{95} \\ \frac{143}{285} \end{pmatrix}$$

Thus $\boxed{y = \frac{17}{95}x + \frac{143}{285}}$ *or* $\boxed{y = 0.1789x + 0.5018}$

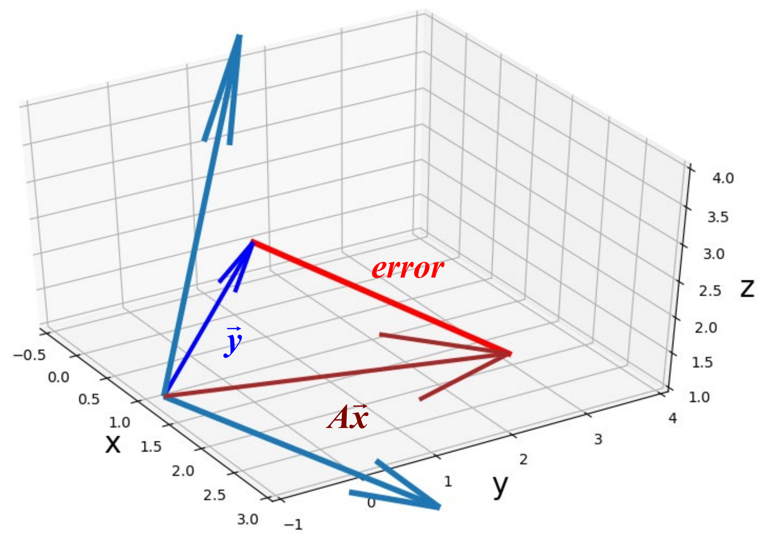
b) The orthogonal projection of $\vec{y}$ on the column space of A

$$A\vec{x} = \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} \frac{17}{95} \\ \frac{143}{285} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{pmatrix}$$

c) $$\vec{y} - A\vec{x} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1232}{285} \\ -\frac{154}{285} \\ \frac{4}{3} \end{pmatrix}$$



The ***error***: $\|\vec{y} - A\vec{x}\| = \sqrt{\left(\frac{1232}{285}\right)^2 + \left(-\frac{154}{285}\right)^2 + \left(\frac{4}{3}\right)^2}$

$$\boxed{\approx 4.556}$$

The ***red line*** is the distance error between the vectors (endpoint) $\vec{y}$ and $A\vec{x}$. However, do not let the distance length trick you. This picture had to be rotated to for our best view.

Exercises *Section* 3.5 – Least Squares Analysis

(1 – 15) Find the equation of the line that best fits the given points in the least-squares sense and find the error.

1. $\{(0,\ 2),\ (1,\ 2),\ (2,\ 0)\}$

2. $\{(0,\ 0),\ (1,\ 1),\ (2,\ 4)\}$

3. $\{(0,\ 0),\ (2,\ 1),\ (4,\ 1)\}$

4. $\{(-1,\ -1),\ (1,\ 0),\ (2,\ 4)\}$

5. $\{(0,\ 1),\ (1,\ 1),\ (2,\ 2),\ (3,\ 2)\}$

6. $\{(2,\ 1),\ (5,\ 2),\ (7,\ 3),\ (8,\ 3)\}$

7. $\{(-2,\ -2),\ (-1,\ 0),\ (0,\ -2),\ (1,\ 0)\}$

8. $\{(-2,\ 0),\ (-1,\ 1),\ (0,\ 1),\ (1,\ 2)\}$

9. $\{(0,\ 1),\ (1,\ 3),\ (2,\ 4),\ (3,\ 4)\}$

10. $\{(2,\ 3),\ (3,\ 2),\ (5,\ 1),\ (6,\ 0)\}$

11. $\{(-1,\ 0),\ (0,\ 1),\ (1,\ 2),\ (2,\ 4)\}$

12. $\{(1,\ 0),\ (2,\ 1),\ (4,\ 2),\ (5,\ 3)\}$

13. $\{(1,\ 5),\ (2,\ 4),\ (3,\ 1),\ (4,\ 1),\ (5,\ -1)\}$

14. $\{(-2,\ 0),\ (-1,\ 0),\ (0,\ 1),\ (1,\ 3),\ (2,\ 5)\}$

15. $\{(-5,\ 10),\ (-1,\ 8),\ (3,\ 6),\ (7,\ 4),\ (5,\ 5)\}$

(16 – 18) Find the orthogonal projection of the vector $\vec{u}$ on the subspace of $\mathbb{R}^4$ spanned by the vectors

16. $\vec{u}=(-3,\ -3,\ 8,\ 9);\ \ \vec{v}_1=(3,\ 1,\ 0,\ 1),\ \ \vec{v}_2=(1,\ 2,\ 1,\ 1),\ \ \vec{v}_3=(-1,\ 0,\ 2,\ -1)$

17. $\vec{u}=(6,\ 3,\ 9,\ 6);\ \ \vec{v}_1=(2,\ 1,\ 1,\ 1),\ \ \vec{v}_2=(1,\ 0,\ 1,\ 1),\ \ \vec{v}_3=(-2,\ -1,\ 0,\ -1)$

18. $\vec{u}=(-2,\ 0,\ 2,\ 4);\ \ v_1=(1,\ 1,\ 3,\ 0),\ \ \vec{v}_2=(-2,\ -1,\ -2,\ 1),\ \ \vec{v}_3=(-3,\ -1,\ 1,\ 3)$

(19 – 26) For the given matrix A and $\vec{y}$

a) Find the least-squares solution of the linear system $A\vec{x} = \vec{y}$

b) Find the orthogonal projection of $\vec{y}$ on the column space of A

c) Find the ***error vector*** and the ***error***

19. $A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 2 \\ 0 \\ 11 \end{pmatrix}$

20. $A = \begin{pmatrix} 1 & 2 \\ 2 & -3 \\ -1 & 3 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$

21. $A = \begin{pmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} -5 \\ 8 \\ 1 \end{pmatrix}$

22. $A = \begin{pmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 6 \end{pmatrix}$

23. $A = \begin{pmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix}$

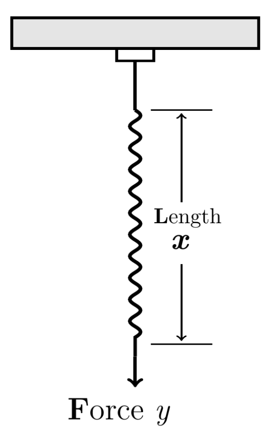
24. $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix}$

25. $A = \begin{pmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 3 \\ 5 \\ 7 \\ -3 \end{pmatrix}$

26. $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{pmatrix}$

27. Find the standard matrix for the orthogonal projection P of $\mathbb{R}^2$ on the line passes through the origin and makes an angle θ with the positive x-axis.

28. Hooke's law in physics states that the length x of a uniform spring is a linear function of the force y applied to it. If we express the relationship as $y = mx + b$, then the coefficient m is called the spring constant.



Suppose a particular unstretched spring has a measured length of 6.1 *inches*.(i.e., $x = 6.1$ when $y = 0$). Forces of 2 pounds, 4 pounds, and 6 pounds are then applied to the spring, and the corresponding lengths are found to be 7.6 inches, 8.7 inches, and 10.4 inches. Find the spring constant.

29. Prove: If A has a linearly independent column vectors, and if $\boldsymbol{b}$ is orthogonal to the column space of A, then the least squares solution of $A\vec{x} = \vec{b}$ is $\vec{x} = \vec{0}$.

30. Let A be an $m \times n$ matrix with linearly independent row vectors. Find a standard matrix for the orthogonal projection of $\mathbb{R}^n$ onto the row space of A.

31. Let W be the line with parametric equations $x = 2t, \quad y = -t, \quad z = 4t$

a) Find a basis for W.
b) Find the standard matrix for the orthogonal projection on W.
c) Use the matrix in part (b) to find the orthogonal projection of a point $P_0\left(x_0, y_0, z_0\right)$ on W.
d) Find the distance between the point $P_0\,(2,\ 1,\ -3)$ and the line W.

32. In $\mathbb{R}^3$, consider the line l given by the equations $x = t, \quad y = t, \quad z = t$

And the line m given by the equations $x = s, \quad y = 2s - 1, \quad z = 1$

Let P be the point on l, and let Q be a point on m.

Find the values of t and s that minimize the distance between the lines by minimizing the squared distance $\|P - Q\|^2$

33. Determine whether the statement is true or false,

a) If A is an $m \times n$ matrix, then $A^T A$ is a square matrix.
b) If $A^T A$ is invertible, then A is invertible.
c) If A is invertible, then $A^T A$ is invertible.
d) If $A\vec{x} = \vec{b}$ is a consistent linear system, then $A^T A\vec{x} = A^T\vec{b}$ is also consistent.
e) If $A\vec{x} = \vec{b}$ is an inconsistent linear system, then $A^T A\vec{x} = A^T\vec{b}$ is also inconsistent.
f) Every linear system has a least squares solution.
g) Every linear system has a unique least squares solution.
h) If A is an $m \times n$ matrix with linearly independent columns and $\vec{b}$ is in R^m, then $A\vec{x} = \vec{b}$ has a unique least squares solution.

34. A certain experiment produces the data $\{(1,\ 1.8),\ (2,\ 2.7),\ (3,\ 3.4),\ (4,\ 3.8),\ (5,\ 3.9)\}$.

Find the function that it will fit these data in the form of $y=\beta_1 x+\beta_2 x^2$

35. According to Kepler's first law, a comet should have an ellipse, parabolic, or hyperbolic orbit (with gravitational attractions from the planets ignored). In suitable polar coordinates, the position $(r,\ \upsilon)$ of a comet satisfies an equation of the form

$$r=\beta+e(r\cdot\cos\upsilon)$$

Where β is a constant and e is the eccentricity of the orbit, with $0\le e<1$ for an ellipse, $e=1$ for a parabolic, and $e>1$ for a hyperbola.

Suppose observations of a newly discovered comet provide the data below.

υ	.88	1.10	1.42	1.77	2.14
r	3.00	2.30	1.65	1.25	1.01

Determine the type of orbit, and predict where the orbit will be when $\upsilon=4.6$ (*radians*)?

36. To measure the takeoff performance of an airplane, the horizontal position of the plane was measured every second, from $t=0$ to $t=12$

The position (in *feet*) were:

0, 8.8, 29.9, 62.0, 104.7, 159.1, 222.0, 294.5, 380.4, 471.1, 571.7, 686.8, *and* 809.2

a) Find the least square cubic curve $y=\beta_0+\beta_1 t+\beta_2 t^2+\beta_3 t^3$ for these data.

b) Estimate the velocity of the plane when $t=4.5$ *sec*, using the result from part (*a*).