

## ***Solution***      **Section 2.7 – Euler's & Runge-Kutta Methods**

### ***Exercise***

Calculate the first five iterations of Euler's method with step  $h = 0.1$  of

$$y' = ty \quad y(0) = 1$$

### **Solution**

$t$	$y$
0.1	1.00000000
0.2	1.01000000
0.3	1.03020000
0.4	1.06110600
0.5	1.10355024

$$\frac{dy}{dt} = ty$$

$$\int \frac{dy}{y} = \int t dt$$

$$\ln y = \frac{1}{2}t^2 + C$$

$$y(t) = e^{t^2/2+C}$$

$$y(0) = e^C = 1 \quad \rightarrow \quad \boxed{C = 0}$$

$$\boxed{y(t) = e^{t^2/2}}$$

### ***Exercise***

Calculate the first five iterations of Euler's method with step  $h = 0.1$  of

$$z' = x - 2z \quad z(0) = 1$$

### **Solution**

$x$	$z$
0.0	1.00000000
0.1	0.80000000
0.2	0.65000000
0.3	0.54000000
0.4	0.46200000
0.5	0.40960000

### Exercise

Calculate the first five iterations of Euler's method with step  $h = 0.1$  of:  $z' = 5 - z$   $z(0) = 0$

### Solution

$$x_0 = 0; \quad z_0 = 0$$

The *first* step:

$$z_1 = z_0 + h(5 - z_0) = 0 + 0.1(5 - 0) = 0.5$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

The *second* step:

$$z_2 = z_1 + h(5 - z_1) = 0.5 + 0.1(5 - 0.5) = 0.95$$

$$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$$

#### *Euler Method*

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	0.00000000	0.00000000	0.00000000
0.10	0.50000000	0.47581291	-0.02418709
0.20	0.95000000	0.90634623	-0.04365377
0.30	1.35500000	1.29590890	-0.05909110
0.40	1.71950000	1.64839977	-0.07110023
0.50	2.04755000	1.96734670	-0.08020330

$$y(t) = 5 - 5e^{-t}$$

### Exercise

Given:  $y' + 2xy = x$   $y(0) = 8$

- Use a computer and Euler's method to calculate three separate approximate solutions on the interval  $[0, 1]$ , one with step size  $h = 0.2$ , a second with step size  $h = 0.1$ , a second with step size  $h = 0.05$ .
- Use the appropriate analytic to compute the exact solution
- Plot the exact solution and approximate solutions as discrete points.

### Solution

<i>x</i>	<i>y</i>
0.0	8.00000000
0.2	8.00000000
0.4	7.40000000
0.6	6.29600000
0.8	4.90496000
1.0	3.49537280
<i>x</i>	<i>y</i>
0.0	8.00000000
0.1	8.00000000

0.2	7.85000000
0.3	7.55600000
0.4	7.13264000
0.5	6.60202880
0.6	5.99182592
0.7	5.33280681
0.8	4.65621386
0.9	3.99121964
1.0	3.36280010

$x$	$y$	$x$	$y$
0.0	8.00000000		
0.05	8.00000000	0.55	6.16870319
0.10	7.96250000	0.60	5.85692451
0.15	7.88787500	0.65	5.53550904
0.20	7.77705688	0.70	5.20820096
0.25	7.63151574	0.75	4.87862689
0.30	7.45322784	0.80	4.55022987
0.35	7.24463101	0.85	4.22621148
0.40	7.00856892	0.90	3.90948351
0.45	6.74822617	0.95	3.60262999
0.50	6.46705599	1.00	3.30788014

$$y(t) = \frac{15}{2}e^{-t^2} + \frac{1}{2}$$

$t$	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
-----			
0.00	8.00000000	8.00000000	0.00000000
0.20	8.00000000	7.70592079	-0.29407921
0.40	7.40000000	6.89107842	-0.50892158
0.60	6.29600000	5.73257245	-0.56342755
0.80	4.90496000	4.45469318	-0.45026682
1.00	3.49537280	3.25909581	-0.23627699

$t$	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
-----			
0.00	8.00000000	8.00000000	0.00000000
0.10	8.00000000	7.92537375	-0.07462625
0.20	7.85000000	7.70592079	-0.14407921
0.30	7.55600000	7.35448389	-0.20151611
0.40	7.13264000	6.89107842	-0.24156158
0.50	6.60202880	6.34100587	-0.26102293
0.60	5.99182592	5.73257245	-0.25925347
0.70	5.33280681	5.09469796	-0.23810885
0.80	4.65621386	4.45469318	-0.20152068
0.90	3.99121964	3.83643550	-0.15478414
1.00	3.36280010	3.25909581	-0.10370430

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	8.00000000	8.00000000	0.00000000
0.05	8.00000000	7.98127342	-0.01872658
0.10	7.96250000	7.92537375	-0.03712625
0.15	7.88787500	7.83313428	-0.05474072
0.20	7.77705688	7.70592079	-0.07113608
0.25	7.63151574	7.54559797	-0.08591777
0.30	7.45322784	7.35448389	-0.09874395
0.35	7.24463101	7.13529429	-0.10933672
0.40	7.00856892	6.89107842	-0.11749051
0.45	6.74822617	6.62514862	-0.12307755
0.50	6.46705599	6.34100587	-0.12605012
0.55	6.16870319	6.04226366	-0.12643953
0.60	5.85692451	5.73257245	-0.12435207
0.65	5.53550904	5.41554691	-0.11996214
0.70	5.20820096	5.09469796	-0.11350300
0.75	4.87862689	4.77337119	-0.10525570
0.80	4.55022987	4.45469318	-0.09553669
0.85	4.22621148	4.14152671	-0.08468477
0.90	3.90948351	3.83643550	-0.07304801
0.95	3.60262999	3.54165879	-0.06097120
1.00	3.30788014	3.25909581	-0.04878433

### Exercise

Given:  $y' + 2y = 2 - e^{-4t}$   $y(0) = 1$

- Solve the differential equation
- Use Euler's method and Runge-Kutta methods to calculate three separate approximate solutions on the interval  $[0, 1]$ , one with step size  $h = 0.2$ , a second with step size  $h = 0.1$ , a second with step size  $h = 0.05$  Plot the exact solution and approximate solutions as discrete points.

### Solution

$$a) \int 2dt = e^{2t}$$

$$\int (2 - e^{-4t}) e^{2t} dt = \int (2e^{2t} - e^{-2t}) dt = e^{2t} + \frac{1}{2} e^{-2t}$$

$$y(t) = \frac{1}{e^{2t}} \left( e^{2t} + \frac{1}{2} e^{-2t} + C \right)$$

$$= 1 + \frac{1}{2} e^{-4t} + C e^{-2t}$$

$$y(0) = 1 \rightarrow 1 = 1 + \frac{1}{2} + C \Rightarrow C = -\frac{1}{2}$$

$$y(t) = 1 + \frac{1}{2} e^{-4t} - \frac{1}{2} e^{-2t}$$

b)  $h = 0.2$

Euler Method

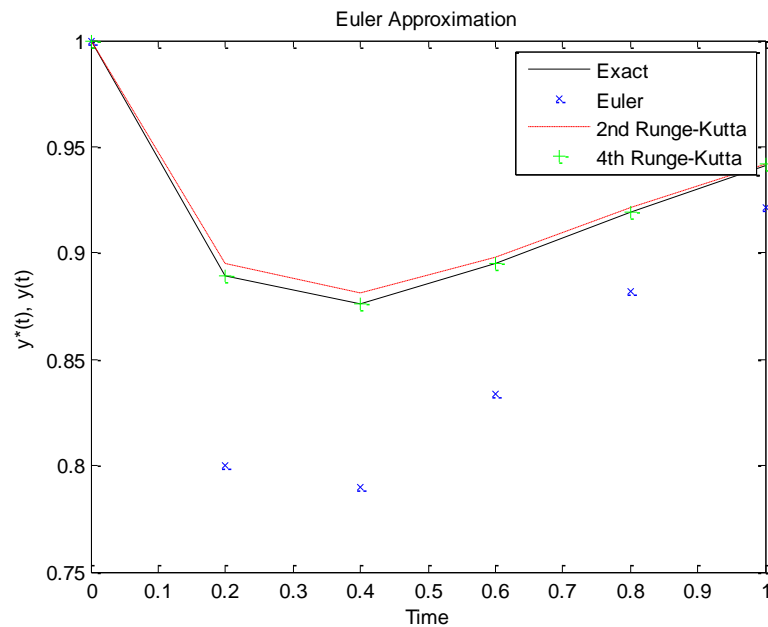
$t$	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	1.00000000	1.00000000	0.00000000
0.20	0.80000000	0.88950446	0.08950446
0.40	0.79013421	0.87628378	0.08614957
0.60	0.83370122	0.89476187	0.06106065
0.80	0.88207714	0.91943284	0.03735570
1.00	0.92109384	0.94149018	0.02039633

Runge-Kutta 2nd Order

$t$	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	1.00000000	1.00000000	0.00000000
0.20	0.89506710	0.88950446	-0.00556264
0.40	0.88149624	0.87628378	-0.00521246
0.60	0.89823186	0.89476187	-0.00346999
0.80	0.92127837	0.91943284	-0.00184552
1.00	0.94219199	0.94149018	-0.00070181

Runge-Kutta 4th Order

$t$	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	1.00000000	1.00000000	0.00000000
0.20	0.88960071	0.88950446	-0.00009625
0.40	0.87638272	0.87628378	-0.00009894
0.60	0.89483774	0.89476187	-0.00007587
0.80	0.91948402	0.91943284	-0.00005118
1.00	0.94152197	0.94149018	-0.00003179



$h = 0.1$

Euler Method

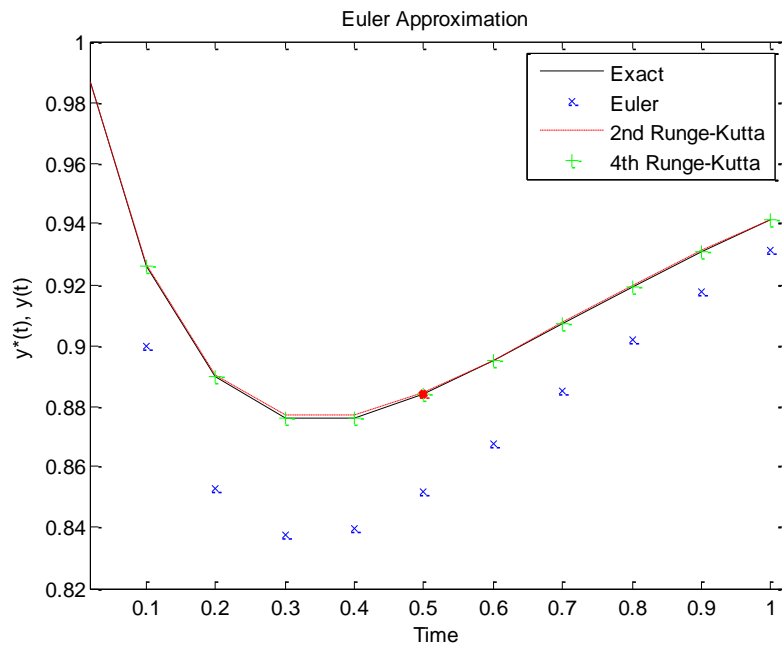
$t$	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	1.00000000	1.00000000	0.00000000
0.10	0.90000000	0.92579465	0.02579465
0.20	0.85296800	0.88950446	0.03653646
0.30	0.83744150	0.87619129	0.03874979
0.40	0.83983378	0.87628378	0.03645000
0.50	0.85167737	0.88372792	0.03205055
0.60	0.86780837	0.89476187	0.02695350
0.70	0.88517490	0.90710655	0.02193165
0.80	0.90205891	0.91943284	0.01737393
0.90	0.91757091	0.93101242	0.01344151
1.00	0.93132436	0.94149018	0.01016582

Runge-Kutta 2nd Order

$t$	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	1.00000000	1.00000000	0.00000000
0.10	0.92648400	0.92579465	-0.00068935
0.20	0.89043763	0.88950446	-0.00093317
0.30	0.87712599	0.87619129	-0.00093470
0.40	0.87710071	0.87628378	-0.00081694
0.50	0.88437996	0.88372792	-0.00065204
0.60	0.89524226	0.89476187	-0.00048039
0.70	0.90742943	0.90710655	-0.00032288
0.80	0.91962162	0.91943284	-0.00018878
0.90	0.93109305	0.93101242	-0.00008064
1.00	0.94148757	0.94149018	0.00000260

Runge-Kutta 4th Order

$t$	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	1.00000000	1.00000000	0.00000000
0.10	0.92579791	0.92579465	-0.00000326
0.20	0.88950913	0.88950446	-0.00000467
0.30	0.87619629	0.87619129	-0.00000500
0.40	0.87628854	0.87628378	-0.00000476
0.50	0.88373216	0.88372792	-0.00000424
0.60	0.89476548	0.89476187	-0.00000361
0.70	0.90710953	0.90710655	-0.00000298
0.80	0.91943524	0.91943284	-0.00000240
0.90	0.93101431	0.93101242	-0.00000189
1.00	0.94149164	0.94149018	-0.00000146



$h = 0.05$

Euler Method

$t$	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	1.00000000	1.00000000	0.00000000
0.05	0.95000000	0.95694667	0.00694667
0.10	0.91406346	0.92579465	0.01173118
0.15	0.88914111	0.90399671	0.01485559
0.20	0.87278642	0.88950446	0.01671804
0.25	0.86304133	0.88067439	0.01763306
0.30	0.85834323	0.87619129	0.01784806
0.35	0.85744919	0.87500583	0.01755664
0.40	0.85937442	0.87628378	0.01690935
0.45	0.86334216	0.87936461	0.01602246
0.50	0.86874300	0.88372792	0.01498492
0.55	0.87510193	0.88896604	0.01386410
0.60	0.88205158	0.89476187	0.01271029
0.65	0.88931053	0.90087089	0.01156037
0.70	0.89666579	0.90710655	0.01044076
0.75	0.90395871	0.91332845	0.00936974
0.80	0.91107349	0.91943284	0.00835936
0.85	0.91792803	0.92534487	0.00741684
0.90	0.92446656	0.93101242	0.00654586
0.95	0.93065372	0.93640108	0.00574736
1.00	0.93646981	0.94149018	0.00502037

Runge-Kutta 2nd Order

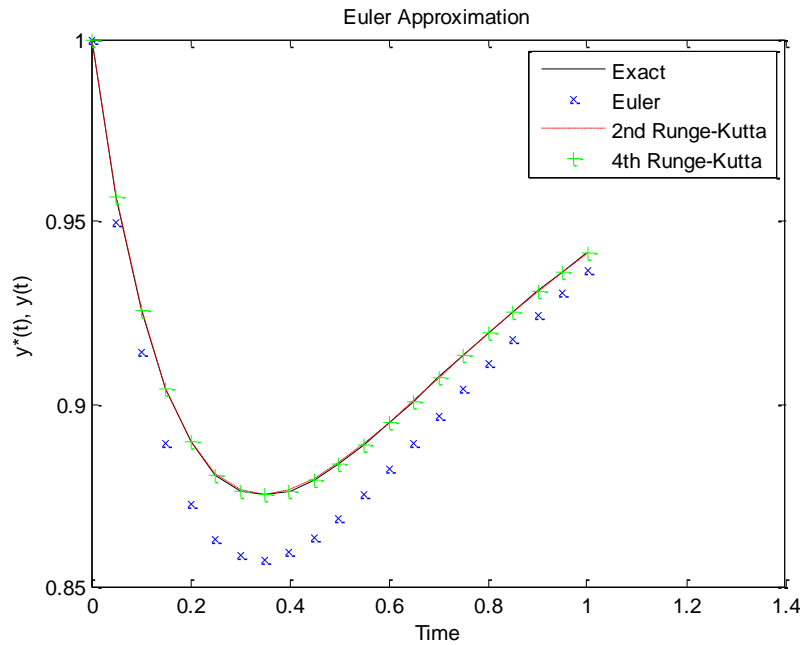
$t$	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	1.00000000	1.00000000	0.00000000
0.05	0.95703173	0.95694667	-0.00008506
0.10	0.92593427	0.92579465	-0.00013963

0.15	0.90416803	0.90399671	-0.00017132
0.20	0.88969058	0.88950446	-0.00018612
0.25	0.88086308	0.88067439	-0.00018869
0.30	0.87637395	0.87619129	-0.00018266
0.35	0.87517663	0.87500583	-0.00017080
0.40	0.87643901	0.87628378	-0.00015523
0.45	0.87950216	0.87936461	-0.00013754
0.50	0.88384684	0.88372792	-0.00011892
0.55	0.88906627	0.88896604	-0.00010023
0.60	0.89484396	0.89476187	-0.00008208
0.65	0.90093579	0.90087089	-0.00006489
0.70	0.90715548	0.90710655	-0.00004893
0.75	0.91336281	0.91332845	-0.00003435
0.80	0.91945408	0.91943284	-0.00002123
0.85	0.92535446	0.92534487	-0.00000958
0.90	0.93101179	0.93101242	0.00000063
0.95	0.93639162	0.93640108	0.00000946
1.00	0.94147318	0.94149018	0.00001700

Runge-Kutta 4th Order

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
-----			
0.00	1.00000000	1.00000000	0.00000000
0.05	0.95694677	0.95694667	-0.00000011
0.10	0.92579483	0.92579465	-0.00000018
0.15	0.90399694	0.90399671	-0.00000023
0.20	0.88950472	0.88950446	-0.00000026
0.25	0.88067466	0.88067439	-0.00000027
0.30	0.87619156	0.87619129	-0.00000027
0.35	0.87500610	0.87500583	-0.00000027
0.40	0.87628404	0.87628378	-0.00000026
0.45	0.87936486	0.87936461	-0.00000025
0.50	0.88372815	0.88372792	-0.00000023
0.55	0.88896625	0.88896604	-0.00000021
0.60	0.89476207	0.89476187	-0.00000020
0.65	0.90087107	0.90087089	-0.00000018
0.70	0.90710671	0.90710655	-0.00000016
0.75	0.91332860	0.91332845	-0.00000015
0.80	0.91943297	0.91943284	-0.00000013
0.85	0.92534499	0.92534487	-0.00000011
0.90	0.93101252	0.93101242	-0.00000010
0.95	0.93640117	0.93640108	-0.00000009
1.00	0.94149026	0.94149018	-0.00000008





### Exercise

Given:  $z' - 2z = xe^{2x}$   $z(0) = 1$

- Use a computer and Euler's method to calculate three separate approximate solutions on the interval  $[0, 1]$ , one with step size  $h = 0.2$ , a second with step size  $h = 0.1$ , a third with step size  $h = 0.05$ .
- Use the appropriate analytic to compute the exact solution
- Plot the exact solution and approximate solutions as discrete points.

### Solution

a)

#### Euler Method

$t$	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.20	1.40000000	1.52166119	0.12166119
0.40	2.01967299	2.40358420	0.38391121
0.60	3.00558546	3.91773797	0.91215251
0.80	4.60623367	6.53800280	1.93176913
1.00	7.24121233	11.08358415	3.84237182

#### Euler Method

$t$	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.10	1.20000000	1.22750977	0.02750977
0.20	1.45221403	1.52166119	0.06944716
0.30	1.77249333	1.90411415	0.13162082
0.40	2.18165556	2.40358420	0.22192865
0.50	2.70700830	3.05806706	0.35105875
0.60	3.38432406	3.91773797	0.53341391

0.70		4.26039588		5.04872396		0.78832807
0.80		5.39633906		6.53800280		1.14166374
0.90		6.87184946		8.49975469		1.62790522
1.00		8.79068763		11.08358415		2.29289652

### *Euler Method*

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00			
0.05			
0.10			
0.15			
0.20			
0.25			
0.30			
0.35			
0.40			
0.45			
0.50			
0.55			
0.60			
0.65			
0.70			
0.75			
0.80			
0.85			
0.90			
0.95			
1.00			

b)  $z' - 2z = xe^{2x}$

$$P(x) = -2, \quad Q(x) = xe^{2x}$$

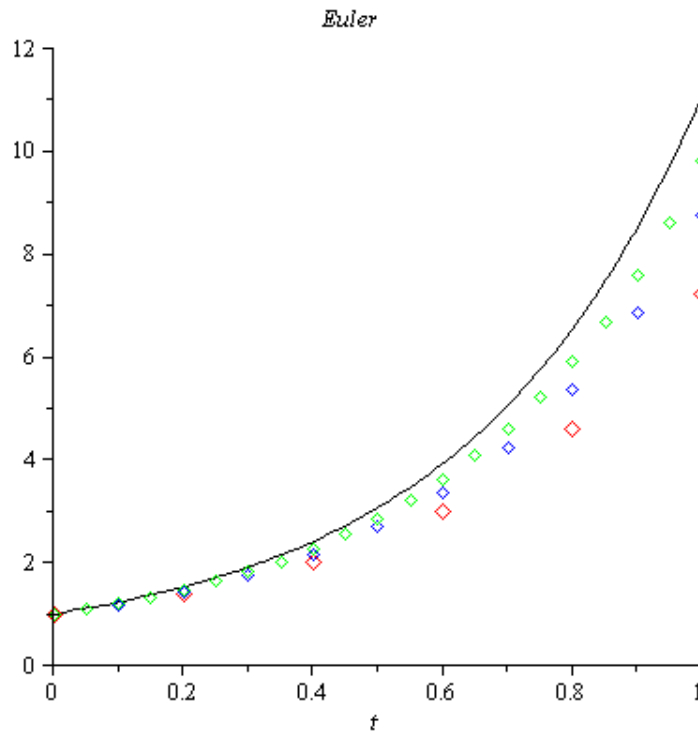
$$e^{\int -2dx} = e^{-2x}$$

$$\int xe^{2x} e^{-2x} dx = \int x dx = \frac{1}{2} x^2$$

$$\begin{aligned} z(x) &= \frac{1}{e^{-2x}} \left( \frac{1}{2} x^2 + C \right) \\ &= \frac{1}{2} x^2 e^{2x} + C e^{2x} \end{aligned}$$

$$z(0) = C = 1$$

$$z(x) = \frac{1}{2} x^2 e^{2x} + e^{2x}$$



### ***Exercise***

Consider the initial value problem  $y' = 12y(4 - y)$   $y(0) = 1$

Use Euler's method with step size  $h = 0.04$  to sketch solution on the interval  $[0, 2]$

### **Solution**

$$y(t) = \frac{4}{e^{\log 3 - 48t} + 1}$$

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	1.00000000	1.00000000	0.00000000
0.04	2.44000000	2.77812333	0.33812333
0.08	4.26707200	3.75770045	-0.50937155
0.12	3.72005658	3.96254078	0.24248419
0.16	4.21993115	3.99446397	-0.22546718
0.20	3.77444588	3.99918742	0.22474154
0.24	4.18308995	3.99988085	-0.18320910
0.28	3.81546672	3.99998253	0.18451582
0.32	4.15342541	3.99999744	-0.15342797
0.36	3.84754974	3.99999962	0.15244989
0.40	4.12909852	3.99999994	-0.12909858
0.44	3.87322947	3.99999999	0.12677052
0.48	4.10891492	4.00000000	-0.10891492
0.52	3.89410430	4.00000000	0.10589570
0.56	4.09204138	4.00000000	-0.09204138
0.60	3.91125556	4.00000000	0.08874444
0.64	4.07786461	4.00000000	-0.07786461
0.68	3.92545437	4.00000000	0.07454563
0.72	4.06591460	4.00000000	-0.06591460
0.76	3.93727310	4.00000000	0.06272690
0.80	4.05582011	4.00000000	-0.05582011
0.84	3.94714987	4.00000000	0.05285013
0.88	4.04728141	4.00000000	-0.04728141
0.92	3.95542805	4.00000000	0.04457195
0.96	4.04005260	4.00000000	-0.04005260

1.00	3.96238159	4.00000000	0.03761841
1.04	4.03392967	4.00000000	-0.03392967
1.08	3.96823212	4.00000000	0.03176788
1.12	4.02874204	4.00000000	-0.02874204
1.16	3.97316079	4.00000000	0.02683921
1.20	4.02434630	4.00000000	-0.02434630
1.24	3.97731688	4.00000000	0.02268312
1.28	4.02062150	4.00000000	-0.02062150
1.32	3.98082411	4.00000000	0.01917589
1.36	4.01746532	4.00000000	-0.01746532
1.40	3.98378549	4.00000000	0.01621451
1.44	4.01479115	4.00000000	-0.01479115
1.48	3.98628712	4.00000000	0.01371288
1.52	4.01252558	4.00000000	-0.01252558
1.56	3.98840115	4.00000000	0.01159885
1.60	4.01060636	4.00000000	-0.01060636
1.64	3.99018815	4.00000000	0.00981185
1.68	4.00898069	4.00000000	-0.00898069
1.72	3.99169905	4.00000000	0.00830095
1.76	4.00760380	4.00000000	-0.00760380
1.80	3.99297675	4.00000000	0.00702325
1.84	4.00643771	4.00000000	-0.00643771
1.88	3.99405741	4.00000000	0.00594259
1.92	4.00545023	4.00000000	-0.00545023
1.96	3.99497153	4.00000000	0.00502847
2.00	4.00461405	4.00000000	-0.00461405

### Exercise

You've seen that the error in Euler's method varies directly as the first power of the step size (i.e.  $E_h \approx \lambda h$ ). This makes Euler's method an order to halve the error? How does this affect the number of required iterations?

### Solution

Because  $E_h \approx \lambda h$  halving the step size should halve the error.

$$E \approx \lambda \left( \frac{1}{2} h \right) \approx \frac{1}{2} \lambda h \approx \frac{1}{2} E_h$$

The number of iterations is given by:  $N = \frac{b-a}{h}$ , therefore halving the step size should double the number of iterations.

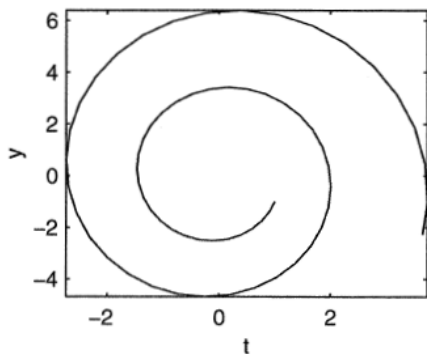
$$N = \frac{b-a}{\frac{1}{2}h} = 2 \frac{b-a}{h} \approx 2N_h$$

### Exercise

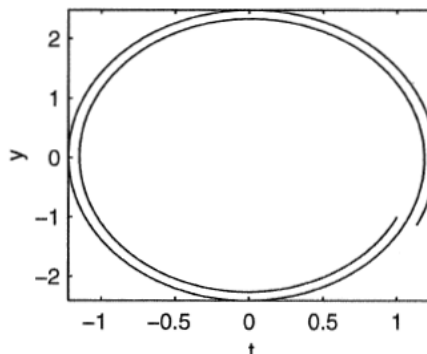
Use Euler's method to provide an approximate solution over the given time interval using the given steps sizes. Provide a plot of  $y$  versus  $x$  for each step size

$$y'' + 4y = 0, \quad y(0) = 4, \quad y'(0) = 0, \quad [0, 2\pi]; \quad h = 0.1, 0.01, 0.001$$

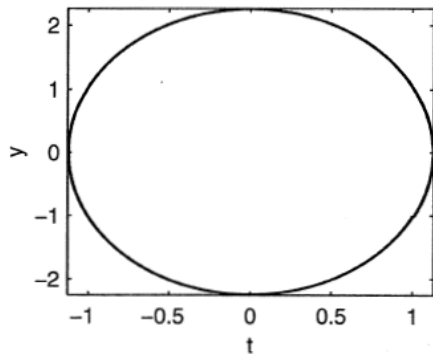
### Solution



$h = 0.1$



$h = 0.01$



$h = 0.001$

## Exercise

$$z' + z = \cos x \quad z(0) = 1$$

- Use a computer and Runge-Kutta method to calculate three separate approximate solutions on the interval  $[0, 1]$ , one with step size  $h = 0.2$ , a second with step size  $h = 0.1$ , a second with step size  $h = 0.05$ .
- Use the appropriate analytic to compute the exact solution
- Plot the exact solution and approximate solutions as discrete points.

## Solution

$$z(x) = \frac{1}{2}e^{-x} + \frac{1}{2}\cos x + \frac{1}{2}\sin x$$

### Runge-Kutta 2nd Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.20	0.99800666	0.99873333	0.00072667
0.40	0.98887689	0.99039969	0.00152281
0.60	0.96709749	0.96939486	0.00229738
0.80	0.92871746	0.93169588	0.00297842
1.00	0.87131508	0.87482637	0.00351128

### *Runge-Kutta 4th Order*

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.20	0.99873272	0.99873333	0.00000061
0.40	0.99039822	0.99039969	0.00000147
0.60	0.96939245	0.96939486	0.00000241
0.80	0.93169258	0.93169588	0.00000330
1.00	0.87482232	0.87482637	0.00000405

### *Runge-Kutta 2nd Order*

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.10	0.99975021	0.99983750	0.00008729
0.20	0.99855245	0.99873333	0.00018088
0.30	0.99555979	0.99583746	0.00027767
0.40	0.99002480	0.99039969	0.00037489
0.50	0.98129932	0.98176938	0.00047006
0.60	0.96883388	0.96939486	0.00056098
0.70	0.95217687	0.95282259	0.00064572
0.80	0.93097330	0.93169588	0.00072258
0.90	0.90496314	0.90575327	0.00079013
1.00	0.87397921	0.87482637	0.00084716

### *Runge-Kutta 4th Order*

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.10	0.99983748	0.99983750	0.00000002
0.20	0.99873329	0.99873333	0.00000004
0.30	0.99583739	0.99583746	0.00000007
0.40	0.99039960	0.99039969	0.00000009
0.50	0.98176926	0.98176938	0.00000012
0.60	0.96939471	0.96939486	0.00000015
0.70	0.95282241	0.95282259	0.00000018
0.80	0.93169568	0.93169588	0.00000020
0.90	0.90575304	0.90575327	0.00000023
1.00	0.87482612	0.87482637	0.00000025

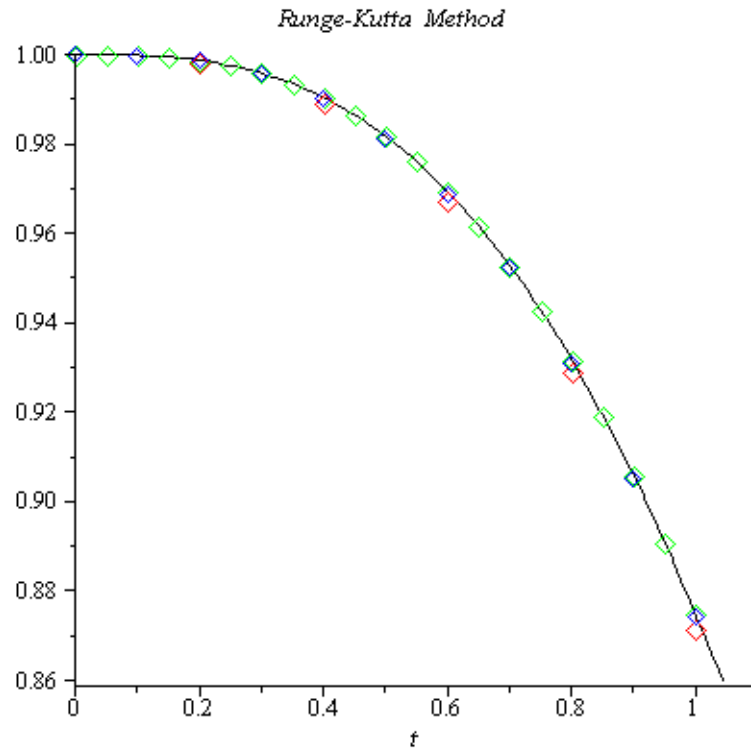
### *Runge-Kutta 2nd Order*

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.05	0.99996876	0.99997943	0.00001067
0.10	0.99981570	0.99983750	0.00002180
0.15	0.99942531	0.99945859	0.00003328

0.20	0.99868831	0.99873333	0.00004502
0.25	0.99750164	0.99755858	0.00005694
0.30	0.99576852	0.99583746	0.00006894
0.35	0.99339836	0.99347931	0.00008094
0.40	0.99030682	0.99039969	0.00009287
0.45	0.98641574	0.98652039	0.00010465
0.50	0.98165315	0.98176938	0.00011623
0.55	0.97595326	0.97608078	0.00012752
0.60	0.96925639	0.96939486	0.00013847
0.65	0.96150896	0.96165799	0.00014903
0.70	0.95266344	0.95282259	0.00015915
0.75	0.94267832	0.94284709	0.00016877
0.80	0.93151803	0.93169588	0.00017785
0.85	0.91915289	0.91933924	0.00018635
0.90	0.90555903	0.90575327	0.00019423
0.95	0.89071835	0.89091981	0.00020146
1.00	0.87461836	0.87482637	0.00020801

### *Runge-Kutta 4th Order*

t	y	y(t)	Difference
-----			
0.00	1.00000000	1.00000000	0.00000000
0.05	0.99997943	0.99997943	0.00000000
0.10	0.99983750	0.99983750	0.00000000
0.15	0.99945859	0.99945859	0.00000000
0.20	0.99873333	0.99873333	0.00000000
0.25	0.99755858	0.99755858	0.00000000
0.30	0.99583745	0.99583746	0.00000000
0.35	0.99347930	0.99347931	0.00000001
0.40	0.99039969	0.99039969	0.00000001
0.45	0.98652039	0.98652039	0.00000001
0.50	0.98176937	0.98176938	0.00000001
0.55	0.97608077	0.97608078	0.00000001
0.60	0.96939485	0.96939486	0.00000001
0.65	0.96165798	0.96165799	0.00000001
0.70	0.95282258	0.95282259	0.00000001
0.75	0.94284708	0.94284709	0.00000001
0.80	0.93169587	0.93169588	0.00000001
0.85	0.91933923	0.91933924	0.00000001
0.90	0.90575325	0.90575327	0.00000001
0.95	0.89091979	0.89091981	0.00000001
1.00	0.87482635	0.87482637	0.00000002



### ***Exercise***

Given  $x' = \frac{t}{x}$   $x(0) = 1$

- Use a computer and Runge-Kutta method to calculate three separate approximate solutions on the interval  $[0, 1]$ , one with step size  $h = 0.2$ , a second with step size  $h = 0.1$ , a second with step size  $h = 0.05$ .
- Use the appropriate analytic to compute the exact solution
- Plot the exact solution and approximate solutions as discrete points.

### **Solution**

*a)*

#### ***Runge-Kutta 2th Order***

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
-----			
0.00	1.00000000	1.00000000	0.00000000
0.20	1.01961161	1.01980390	0.00019229
0.40	1.07636229	1.07703296	0.00067067
0.60	1.16495094	1.16619038	0.00123944
0.80	1.27887002	1.28062485	0.00175483
1.00	1.41205020	1.41421356	0.00216336

#### ***Runge-Kutta 4th Order***

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
-----			
0.00	1.00000000	1.00000000	0.00000000
0.20	1.01980437	1.01980390	-0.00000046
0.40	1.07703431	1.07703296	-0.00000135



0.60	1.16619234	1.16619038	-0.00000196
0.80	1.28062701	1.28062485	-0.00000216
1.00	1.41421570	1.41421356	-0.00000214

### *Runge-Kutta 2th Order*

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	1.00000000	1.00000000	0.00000000
0.10	1.00497519	1.00498756	0.00001238
0.20	1.01975618	1.01980390	0.00004772
0.30	1.04392938	1.04403065	0.00010127
0.40	1.07686631	1.07703296	0.00016665
0.50	1.11779652	1.11803399	0.00023747
0.60	1.16588199	1.16619038	0.00030839
0.70	1.22027989	1.22065556	0.00037567
0.80	1.28018776	1.28062485	0.00043708
0.90	1.34487075	1.34536240	0.00049165
1.00	1.41367433	1.41421356	0.00053923

### *Runge-Kutta 4th Order*

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	1.00000000	1.00000000	0.00000000
0.10	1.00498757	1.00498756	-0.00000001
0.20	1.01980393	1.01980390	-0.00000003
0.30	1.04403071	1.04403065	-0.00000006
0.40	1.07703304	1.07703296	-0.00000008
0.50	1.11803409	1.11803399	-0.00000010
0.60	1.16619050	1.16619038	-0.00000012
0.70	1.22065569	1.22065556	-0.00000013
0.80	1.28062498	1.28062485	-0.00000013
0.90	1.34536254	1.34536240	-0.00000013
1.00	1.41421369	1.41421356	-0.00000013

### *Runge-Kutta 2th Order*

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	1.00000000	1.00000000	0.00000000
0.05	1.00124844	1.00124922	0.00000078
0.10	1.00498447	1.00498756	0.00000309
0.15	1.01118058	1.01118742	0.00000684
0.20	1.01979199	1.01980390	0.00001191
0.25	1.03075829	1.03077641	0.00001812
0.30	1.04400537	1.04403065	0.00002528
0.35	1.05944783	1.05948101	0.00003317
0.40	1.07699136	1.07703296	0.00004160
0.45	1.09653524	1.09658561	0.00005037
0.50	1.11797470	1.11803399	0.00005929
0.55	1.14120301	1.14127122	0.00006821
0.60	1.16611337	1.16619038	0.00007701
0.65	1.19260047	1.19268604	0.00008557
0.70	1.22056174	1.22065556	0.00009382
0.75	1.24989830	1.25000000	0.00010170
0.80	1.28051568	1.28062485	0.00010917
0.85	1.31232426	1.31244047	0.00011621
0.90	1.34523959	1.34536240	0.00012281
0.95	1.37918245	1.37931142	0.00012898

1.00 | 1.41407885 | 1.41421356 | 0.00013471

### Runge-Kutta 4th Order

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	1.00000000	1.00000000	0.00000000
0.05	1.00124922	1.00124922	-0.00000000
0.10	1.00498756	1.00498756	-0.00000000
0.15	1.01118742	1.01118742	-0.00000000
0.20	1.01980390	1.01980390	-0.00000000
0.25	1.03077641	1.03077641	-0.00000000
0.30	1.04403065	1.04403065	-0.00000000
0.35	1.05948101	1.05948101	-0.00000000
0.40	1.07703297	1.07703296	-0.00000001
0.45	1.09658562	1.09658561	-0.00000001
0.50	1.11803400	1.11803399	-0.00000001
0.55	1.14127123	1.14127122	-0.00000001
0.60	1.16619039	1.16619038	-0.00000001
0.65	1.19268605	1.19268604	-0.00000001
0.70	1.22065557	1.22065556	-0.00000001
0.75	1.25000001	1.25000000	-0.00000001
0.80	1.28062486	1.28062485	-0.00000001
0.85	1.31244048	1.31244047	-0.00000001
0.90	1.34536241	1.34536240	-0.00000001
0.95	1.37931143	1.37931142	-0.00000001
1.00	1.41421357	1.41421356	-0.00000001

b) The equation is separable:

$$x dx = t dt$$

$$\int x dx = \int t dt$$

$$\frac{1}{2} x^2 = \frac{1}{2} t^2 + C$$

$$x^2 = t^2 + 2C$$

$$x = \sqrt{t^2 + 2C}$$

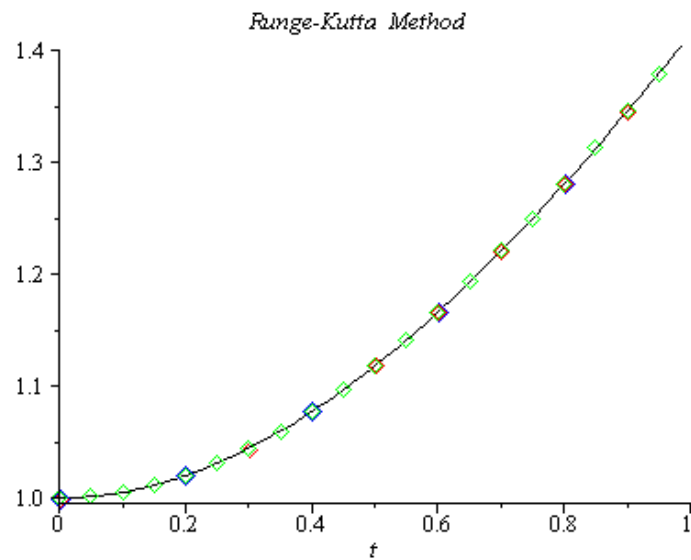
$$x(0) = \sqrt{2C}$$

$$1 = \sqrt{2C}$$

$$1 = 2C$$

$$C = \frac{1}{2}$$

$$x(t) = \sqrt{t^2 + 1}$$



### Exercise

Consider the initial value problem  $y' = \frac{t}{y^2}$   $y(0) = 1$

Use Runge-Kutta method with step size  $h = 0.04$  to sketch solution on the interval  $[0, 2]$

### Solution

$$y(t) = 3^{1/3} \left( \frac{1}{2}t^2 + \frac{1}{3} \right)^{1/3}$$

#### *Runge-Kutta 4th Order*

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	1.00000000	1.00000000	0.00000000
0.04	1.00079936	1.00079936	-0.00000000
0.08	1.00318981	1.00318981	-0.00000000
0.12	1.00714877	1.00714877	-0.00000000
0.16	1.01263957	1.01263957	-0.00000000
0.20	1.01961283	1.01961282	-0.00000000
0.24	1.02800822	1.02800822	-0.00000000
0.28	1.03775651	1.03775651	-0.00000000
0.32	1.04878166	1.04878166	-0.00000001
0.36	1.06100297	1.06100297	-0.00000001
0.40	1.07433708	1.07433707	-0.00000001
0.44	1.08869975	1.08869974	-0.00000001
0.48	1.10400743	1.10400742	-0.00000001
0.52	1.12017855	1.12017854	-0.00000001
0.56	1.13713450	1.13713449	-0.00000001
0.60	1.15480036	1.15480035	-0.00000001
0.64	1.17310545	1.17310544	-0.00000001
0.68	1.19198361	1.19198360	-0.00000001
0.72	1.21137336	1.21137335	-0.00000001
0.76	1.23121787	1.23121787	-0.00000001
0.80	1.25146496	1.25146495	-0.00000001
0.84	1.27206683	1.27206682	-0.00000001
0.88	1.29297992	1.29297991	-0.00000001
0.92	1.31416464	1.31416463	-0.00000001
0.96	1.33558509	1.33558508	-0.00000001
1.00	1.35720882	1.35720881	-0.00000001
1.04	1.37900650	1.37900650	-0.00000001
1.08	1.40095174	1.40095173	-0.00000001
1.12	1.42302075	1.42302075	-0.00000001
1.16	1.44519217	1.44519216	-0.00000001
1.20	1.46744679	1.46744678	-0.00000001
1.24	1.48976740	1.48976739	-0.00000001
1.28	1.51213855	1.51213854	-0.00000001
1.32	1.53454641	1.53454640	-0.00000001
1.36	1.55697860	1.55697859	-0.00000001
1.40	1.57942403	1.57942403	-0.00000001
1.44	1.60187281	1.60187281	-0.00000001
1.48	1.62431609	1.62431608	-0.00000001
1.52	1.64674596	1.64674596	-0.00000001

1.56		1.66915540		1.66915539		-0.00000001
1.60		1.69153812		1.69153811		-0.00000001
1.64		1.71388854		1.71388853		-0.00000001
1.68		1.73620170		1.73620169		-0.00000001
1.72		1.75847320		1.75847319		-0.00000001
1.76		1.78069914		1.78069913		-0.00000001
1.80		1.80287607		1.80287606		-0.00000000
1.84		1.82500094		1.82500094		-0.00000000
1.88		1.84707109		1.84707109		-0.00000000
1.92		1.86908417		1.86908417		-0.00000000
1.96		1.89103813		1.89103813		-0.00000000
2.00		1.91293119		1.91293118		-0.00000000

### Exercise

Consider the initial value problem  $y' - y = -\frac{1}{2}e^{t/2} \sin 5t + 5e^{t/2} \cos 5t$   $y(0) = 0$

Use Runge-Kutta method with step size  $h = 0.05$  to sketch solution on the interval  $[0, 5]$

### Solution

$$e^{\int -dt} = e^{-t}$$

$$\int \left( -\frac{1}{2}e^{t/2} \sin 5t + 5e^{t/2} \cos 5t \right) e^{-t} dt = -\frac{1}{2} \int \left( e^{-t/2} \sin 5t \right) dt + 5 \int \left( e^{-t/2} \cos 5t \right) dt$$

		$\int \sin 5t$
+	$e^{-t/2}$	$-\frac{1}{5} \cos 5t$
-	$-\frac{1}{2}e^{-t/2}$	$-\frac{1}{25} \sin 5t$
+	$\frac{1}{4}e^{-t/2}$	

		$\int \cos 5t$
+	$e^{-t/2}$	$\frac{1}{5} \sin 5t$
-	$-\frac{1}{2}e^{-t/2}$	$-\frac{1}{25} \cos 5t$
+	$\frac{1}{4}e^{-t/2}$	

$$\int \left( e^{-t/2} \sin 5t \right) dt = \left( -\frac{1}{5} \cos 5t - \frac{1}{50} \sin 5t \right) e^{-t/2} - \frac{1}{100} \int \left( e^{-t/2} \sin 5t \right) dt$$

$$\frac{101}{100} \int \left( e^{-t/2} \sin 5t \right) dt = -\frac{1}{50} (10 \cos 5t + \sin 5t) e^{-t/2}$$

$$\int \left( e^{-t/2} \sin 5t \right) dt = -\frac{2}{101} (10 \cos 5t + \sin 5t) e^{-t/2}$$

$$\int \left( e^{-t/2} \cos 5t \right) dt = e^{-t/2} \left( \frac{1}{5} \sin 5t - \frac{1}{50} \cos 5t \right) - \frac{1}{100} \int \left( e^{-t/2} \cos 5t \right) dt$$

$$\frac{101}{100} \int \left( e^{-t/2} \cos 5t \right) dt = \frac{1}{50} e^{-t/2} (10 \sin 5t - \cos 5t)$$

$$\int \left( e^{-t/2} \cos 5t \right) dt = \frac{2}{101} e^{-t/2} (10 \sin 5t - \cos 5t)$$

$$\int \left( -\frac{1}{2} e^{t/2} \sin 5t + 5 e^{t/2} \cos 5t \right) e^{-t} dt = \left( \frac{10}{101} \cos 5t + \frac{1}{101} \sin 5t + \frac{100}{101} \sin 5t - \frac{10}{101} \cos 5t \right) e^{-t/2}$$

$$= e^{-t/2} \sin 5t$$

$$y(t) = e^t \left( e^{-t/2} \sin 5t + C \right)$$

$$= e^{t/2} \sin 5t + C e^t$$

$$\textcolor{red}{y(0)=0} \rightarrow \underline{C=0}$$

$$\underline{y(t) = \textcolor{blue}{e^{t/2} \sin 5t}}$$

Euler Method

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	0.00000000	0.00000000	0.00000000
0.05	0.25000000	0.25366702	0.00366702
0.10	0.50451846	0.50400621	-0.00051225
0.15	0.74778853	0.73472762	-0.01306091
0.20	0.96397872	0.92996926	-0.03400946
0.25	1.13821003	1.07534045	-0.06286957
0.30	1.25756378	1.15892383	-0.09863994
0.35	1.31201510	1.17216954	-0.13984556
0.40	1.29522788	1.11061839	-0.18460949
0.45	1.20515309	0.97439874	-0.23075435
0.50	1.04438175	0.76845344	-0.27592831
0.55	0.82021731	0.50246840	-0.31774891
0.60	0.54444836	0.19049209	-0.35395627
0.65	0.23282095	-0.14974538	-0.38256633
0.70	-0.09577085	-0.49778510	-0.40201424
0.75	-0.42033859	-0.83161681	-0.41127822
0.80	-0.71904181	-1.12901665	-0.40997485
0.85	-0.97054890	-1.36896715	-0.39841824
0.90	-1.15543496	-1.53307240	-0.37763744
0.95	-1.25752830	-1.60687699	-0.34934869
1.00	-1.26511659	-1.58099885	-0.31588225
1.05	-1.17192748	-1.45199341	-0.28006593
1.10	-0.97780916	-1.22287990	-0.24507073
1.15	-0.68905162	-0.90327826	-0.21422665
1.20	-0.31830941	-0.50912823	-0.19081882
1.25	0.11588941	-0.06198694	-0.17787635
1.30	0.59003789	0.41207112	-0.17796677
1.35	1.07691137	0.88390140	-0.19300997
1.40	1.54713288	1.32300854	-0.22412434
1.45	1.97095749	1.69944069	-0.27151680
1.50	2.32017202	1.98574597	-0.33442605
1.55	2.56999372	2.15886828	-0.41112544
1.60	2.70084550	2.20185727	-0.49898823
1.65	2.69988727	2.10527280	-0.59461447
1.70	2.56219211	1.86817786	-0.69401425
1.75	2.29147346	1.49863486	-0.79283860

1.80	1.90029341	1.01364791	-0.88664550
1.85	1.40971217	0.43852637	-0.97118580
1.90	0.84837261	-0.19431898	-1.04269159
1.95	0.25104980	-0.84709881	-1.09814861
2.00	-0.34326837	-1.47880270	-1.13553433
2.05	-0.89366994	-2.04767466	-1.15400472
2.10	-1.35984868	-2.51386357	-1.15401490
2.15	-1.70472418	-2.84208687	-1.13736269
2.20	-1.89698837	-3.00413660	-1.10714824
2.25	-1.91341049	-2.98105850	-1.06764801
2.30	-1.74073987	-2.76484685	-1.02410698
2.35	-1.37706328	-2.35952135	-0.98245807
2.40	-0.83250141	-1.78148483	-0.94898341
2.45	-0.12916591	-1.05910196	-0.92993604
2.50	0.69965839	-0.23148617	-0.93114456
2.55	1.61109302	0.65346778	-0.95762524
2.60	2.55494455	1.54171751	-1.01322704
2.65	3.47657170	2.37623761	-1.10033409
2.70	4.32018626	3.10053857	-1.21964769
2.75	5.03239764	3.66233161	-1.37006604
2.80	5.56578542	4.01711092	-1.54867451
2.85	5.88227111	4.13141845	-1.75085267
2.90	5.95606196	3.98556468	-1.97049728
2.95	5.77595524	3.57560239	-2.20035285
3.00	5.34682192	2.91438791	-2.43243401
3.05	4.69013206	2.03161484	-2.65851722
3.10	3.84343922	0.97276538	-2.87067384
3.15	2.85880347	-0.20300879	-3.06181226
3.20	1.80020034	-1.42599446	-3.22619481
3.25	0.74003061	-2.61986419	-3.35989480
3.30	-0.24509055	-3.70625192	-3.46116137
3.35	-1.07903060	-4.60969444	-3.53066383
3.40	-1.69104556	-5.26263929	-3.57159374
3.45	-2.02058926	-5.61020009	-3.58961083
3.50	-2.02171162	-5.61434002	-3.59262841
3.55	-1.66674125	-5.25718495	-3.59044370
3.60	-0.94897881	-4.54320809	-3.59422929
3.65	0.11582328	-3.50008777	-3.61591105
3.70	1.48935225	-2.17811492	-3.66746718
3.75	3.11207493	-0.64811430	-3.76018923
3.80	4.90601419	1.00206320	-3.90395099
3.85	6.77885700	2.67232428	-4.10653272
3.90	8.62919773	4.25615056	-4.37304717
3.95	10.35263755	5.64712789	-4.70550966
4.00	11.84839105	6.74580367	-5.10258738
4.05	13.02600082	7.46644758	-5.55955324
4.10	13.81173327	7.74327526	-6.06845801
4.15	14.15422757	7.53570683	-6.61852074
4.20	14.02899591	6.83227210	-7.19672381
4.25	13.44142635	5.65284162	-7.78858473
4.30	12.42801840	4.04895406	-8.37906435
4.35	11.05568119	2.10212134	-8.95355985

4.40	9.41904025	-0.07988319	-9.49892344
4.45	7.63582435	-2.36861780	-10.00444215
4.50	5.84053115	-4.62218308	-10.46271423
4.55	4.17669135	-6.69366883	-10.87036018
4.60	2.78815793	-8.44035671	-11.22851463
4.65	1.80993130	-9.73312476	-11.54305605
4.70	1.35908774	-10.46545870	-11.82454643
4.75	1.52640052	-10.56146949	-12.08787001
4.80	2.36922910	-9.98235001	-12.35157911
4.85	3.90619931	-8.73077509	-12.63697440
4.90	6.11410924	-6.85285609	-12.96696533
4.95	8.92737431	-4.43739883	-13.36477314
5.00	12.24017767	-1.61237440	-13.85255207

Runge-Kutta 2nd Order

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	0.00000000	0.00000000	0.00000000
0.05	0.25225923	0.25366702	0.00140779
0.10	0.50126928	0.50400621	0.00273693
0.15	0.73083289	0.73472762	0.00389473
0.20	0.92517466	0.92996926	0.00479460
0.25	1.06997848	1.07534045	0.00536197
0.30	1.15338415	1.15892383	0.00553968
0.35	1.16687699	1.17216954	0.00529254
0.40	1.10600766	1.11061839	0.00461072
0.45	0.97088706	0.97439874	0.00351168
0.50	0.76641304	0.76845344	0.00204040
0.55	0.50220045	0.50246840	0.00026796
0.60	0.19220390	0.19049209	-0.00171182
0.65	-0.14595823	-0.14974538	-0.00378715
0.70	-0.49195043	-0.49778510	-0.00583466
0.75	-0.82389011	-0.83161681	-0.00772670
0.80	-1.11967729	-1.12901665	-0.00933937
0.85	-1.35840643	-1.36896715	-0.01056072
0.90	-1.52177383	-1.53307240	-0.01129857
0.95	-1.59538951	-1.60687699	-0.01148748
1.00	-1.56990449	-1.58099885	-0.01109436
1.05	-1.44187115	-1.45199341	-0.01012226
1.10	-1.21426784	-1.22287990	-0.00861206
1.15	-0.89663646	-0.90327826	-0.00664180
1.20	-0.50480467	-0.50912823	-0.00432356
1.25	-0.06018891	-0.06198694	-0.00179803
1.30	0.41129806	0.41207112	0.00077306
1.35	0.88068514	0.88390140	0.00321626
1.40	1.31765160	1.32300854	0.00535694
1.45	1.69241026	1.69944069	0.00703043
1.50	1.97765283	1.98574597	0.00809314
1.55	2.15043546	2.15886828	0.00843282
1.60	2.19387987	2.20185727	0.00797740

1.65	2.09857119	2.10527280	0.00670161
1.70	1.86354684	1.86817786	0.00463102
1.75	1.49679192	1.49863486	0.00184294
1.80	1.01518384	1.01364791	-0.00153593
1.85	0.44386158	0.43852637	-0.00533521
1.90	-0.18496945	-0.19431898	-0.00934953
1.95	-0.83374872	-0.84709881	-0.01335009
2.00	-1.46170451	-1.47880270	-0.01709819
2.05	-2.02731452	-2.04767466	-0.02036014
2.10	-2.49094092	-2.51386357	-0.02292265
2.15	-2.81747925	-2.84208687	-0.02460762
2.20	-2.97885110	-3.00413660	-0.02528550
2.25	-2.95617232	-2.98105850	-0.02488618
2.30	-2.74144026	-2.76484685	-0.02340659
2.35	-2.33860692	-2.35952135	-0.02091443
2.40	-1.76393735	-1.78148483	-0.01754748
2.45	-1.04559354	-1.05910196	-0.01350842
2.50	-0.22243099	-0.23148617	-0.00905518
2.55	0.65795501	0.65346778	-0.00448723
2.60	1.54184603	1.54171751	-0.00012852
2.65	2.37254558	2.37623761	0.00369203
2.70	3.09387721	3.10053857	0.00666136
2.75	3.65382846	3.66233161	0.00850315
2.80	4.00811352	4.01711092	0.00899739
2.85	4.12342111	4.13141845	0.00799734
2.90	3.98012211	3.98556468	0.00544257
2.95	3.57423517	3.57560239	0.00136722
3.00	2.91848542	2.91438791	-0.00409752
3.05	2.04234183	2.03161484	-0.01072699
3.10	0.99097829	0.97276538	-0.01821290
3.15	-0.17682977	-0.20300879	-0.02617901
3.20	-1.39179241	-1.42599446	-0.03420205
3.25	-2.57802748	-2.61986419	-0.04183671
3.30	-3.65760857	-3.70625192	-0.04864335
3.35	-4.55547762	-4.60969444	-0.05421682
3.40	-5.20442486	-5.26263929	-0.05821444
3.45	-5.54981858	-5.61020009	-0.06038151
3.50	-5.55376757	-5.61434002	-0.06057245
3.55	-5.19841904	-5.25718495	-0.05876591
3.60	-4.48813526	-4.54320809	-0.05507283
3.65	-3.45035150	-3.50008777	-0.04973627
3.70	-2.13499213	-2.17811492	-0.04312279
3.75	-0.61240881	-0.64811430	-0.03570549
3.80	1.03010243	1.00206320	-0.02803923
3.85	2.69305372	2.67232428	-0.02072944
3.90	4.27054658	4.25615056	-0.01439602
3.95	5.65676237	5.64712789	-0.00963448
4.00	6.75278040	6.74580367	-0.00697673
4.05	7.47330147	7.46644758	-0.00685388
4.10	7.75283900	7.74327526	-0.00956374
4.15	7.55095199	7.53570683	-0.01524515
4.20	6.85613350	6.83227210	-0.02386140



4.25	5.68803555	5.65284162	-0.03519394
4.30	4.09780169	4.04895406	-0.04884764
4.35	2.16638901	2.10212134	-0.06426767
4.40	0.00088437	-0.07988319	-0.08076755
4.45	-2.27105067	-2.36861780	-0.09756712
4.50	-4.50834452	-4.62218308	-0.11383856
4.55	-6.56491101	-6.69366883	-0.12875783
4.60	-8.29879815	-8.44035671	-0.14155855
4.65	-9.58153975	-9.73312476	-0.15158501
4.70	-10.30711803	-10.46545870	-0.15834067
4.75	-10.39994039	-10.56146949	-0.16152910
4.80	-9.82126597	-9.98235001	-0.16108404
4.85	-8.57358877	-8.73077509	-0.15718632
4.90	-6.70259041	-6.85285609	-0.15026568
4.95	-4.29641205	-4.43739883	-0.14098678
5.00	-1.48215502	-1.61237440	-0.13021938

Runge-Kutta 4th Order

<i>t</i>	<i>Approx.</i>	<i>Exact</i>	<i>Difference</i>
0.00	0.00000000	0.00000000	0.00000000
0.05	0.25366730	0.25366702	-0.00000028
0.10	0.50400675	0.50400621	-0.00000054
0.15	0.73472836	0.73472762	-0.00000075
0.20	0.92997015	0.92996926	-0.00000089
0.25	1.07534142	1.07534045	-0.00000096
0.30	1.15892478	1.15892383	-0.00000095
0.35	1.17217037	1.17216954	-0.00000084
0.40	1.11061902	1.11061839	-0.00000064
0.45	0.97439909	0.97439874	-0.00000035
0.50	0.76845344	0.76845344	0.00000001
0.55	0.50246799	0.50246840	0.00000042
0.60	0.19049122	0.19049209	0.00000086
0.65	-0.14974669	-0.14974538	0.00000131
0.70	-0.49778684	-0.49778510	0.00000175
0.75	-0.83161894	-0.83161681	0.00000213
0.80	-1.12901910	-1.12901665	0.00000245
0.85	-1.36896982	-1.36896715	0.00000267
0.90	-1.53307519	-1.53307240	0.00000279
0.95	-1.60687978	-1.60687699	0.00000279
1.00	-1.58100151	-1.58099885	0.00000266
1.05	-1.45199583	-1.45199341	0.00000242
1.10	-1.22288197	-1.22287990	0.00000207
1.15	-0.90327990	-0.90327826	0.00000164
1.20	-0.50912938	-0.50912823	0.00000115
1.25	-0.06198757	-0.06198694	0.00000063
1.30	0.41207099	0.41207112	0.00000013
1.35	0.88390173	0.88390140	-0.00000033
1.40	1.32300925	1.32300854	-0.00000070
1.45	1.69944165	1.69944069	-0.00000096
1.50	1.98574704	1.98574597	-0.00000108

1.55	2.15886931	2.15886828	-0.00000103
1.60	2.20185808	2.20185727	-0.00000081
1.65	2.10527321	2.10527280	-0.00000041
1.70	1.86817772	1.86817786	0.00000014
1.75	1.49863402	1.49863486	0.00000084
1.80	1.01364626	1.01364791	0.00000165
1.85	0.43852385	0.43852637	0.00000252
1.90	-0.19432240	-0.19431898	0.00000342
1.95	-0.84710311	-0.84709881	0.00000430
2.00	-1.47880780	-1.47880270	0.00000510
2.05	-2.04768043	-2.04767466	0.00000577
2.10	-2.51386985	-2.51386357	0.00000628
2.15	-2.84209346	-2.84208687	0.00000659
2.20	-3.00414329	-3.00413660	0.00000669
2.25	-2.98106506	-2.98105850	0.00000656
2.30	-2.76485306	-2.76484685	0.00000621
2.35	-2.35952701	-2.35952135	0.00000566
2.40	-1.78148978	-1.78148483	0.00000496
2.45	-1.05910610	-1.05910196	0.00000414
2.50	-0.23148944	-0.23148617	0.00000327
2.55	0.65346536	0.65346778	0.00000242
2.60	1.54171585	1.54171751	0.00000165
2.65	2.37623657	2.37623761	0.00000104
2.70	3.10053793	3.10053857	0.00000064
2.75	3.66233110	3.66233161	0.00000051
2.80	4.01711023	4.01711092	0.00000068
2.85	4.13141725	4.13141845	0.00000119
2.90	3.98556265	3.98556468	0.00000203
2.95	3.57559921	3.57560239	0.00000319
3.00	2.91438329	2.91438791	0.00000462
3.05	2.03160857	2.03161484	0.00000627
3.10	0.97275731	0.97276538	0.00000808
3.15	-0.20301874	-0.20300879	0.00000995
3.20	-1.42600625	-1.42599446	0.00001179
3.25	-2.61987770	-2.61986419	0.00001351
3.30	-3.70626694	-3.70625192	0.00001502
3.35	-4.60971069	-4.60969444	0.00001625
3.40	-5.26265642	-5.26263929	0.00001712
3.45	-5.61021769	-5.61020009	0.00001760
3.50	-5.61435770	-5.61434002	0.00001767
3.55	-5.25720229	-5.25718495	0.00001735
3.60	-4.54322475	-4.54320809	0.00001666
3.65	-3.50010344	-3.50008777	0.00001567
3.70	-2.17812940	-2.17811492	0.00001448
3.75	-0.64812749	-0.64811430	0.00001319
3.80	1.00205127	1.00206320	0.00001193
3.85	2.67231345	2.67232428	0.00001083
3.90	4.25614054	4.25615056	0.00001002
3.95	5.64711827	5.64712789	0.00000962
4.00	6.74579394	6.74580367	0.00000973
4.05	7.46643715	7.46644758	0.00001043
4.10	7.74326349	7.74327526	0.00001177

4.15	7.53569307	7.53570683	0.00001376
4.20	6.83225573	6.83227210	0.00001637
4.25	5.65282207	5.65284162	0.00001955
4.30	4.04893088	4.04895406	0.00002318
4.35	2.10209420	2.10212134	0.00002714
4.40	-0.07991446	-0.07988319	0.00003128
4.45	-2.36865321	-2.36861780	0.00003542
4.50	-4.62222247	-4.62218308	0.00003939
4.55	-6.69371186	-6.69366883	0.00004303
4.60	-8.44040289	-8.44035671	0.00004618
4.65	-9.73317349	-9.73312476	0.00004873
4.70	-10.46550929	-10.46545870	0.00005060
4.75	-10.56152123	-10.56146949	0.00005174
4.80	-9.98240218	-9.98235001	0.00005217
4.85	-8.73082706	-8.73077509	0.00005196
4.90	-6.85290732	-6.85285609	0.00005123
4.95	-4.43744898	-4.43739883	0.00005015
5.00	-1.61242331	-1.61237440	0.00004891

