

Solution **Section 2.4 - Nonhomogeneous Equations; the Method of Undetermined Coefficients**

Exercise

Show that the 3 solutions $y_1 = x$, $y_2 = x \ln x$, $y_3 = x^2$ of the 3rd order equation

$x^3 y''' - x^2 y'' + 2xy' - 2y = 0$ are linearly independent on an open interval $x > 0$. Then find a particular solution that satisfies the initial conditions $y(1) = 3$, $y'(1) = 2$, $y''(1) = 1$

Solution

$$\begin{aligned}
 W &= \begin{vmatrix} x & x \ln x & x^2 \\ 1 & 1 + \ln x & 2x \\ 0 & \frac{1}{x} & 2 \end{vmatrix} \\
 &= 2x + 2x \ln x + x - 2x - 2x \ln x \\
 &= \underline{x \neq 0} \quad \text{since } x > 0
 \end{aligned}$$

$\therefore y_1, y_2, y_3$ are linearly independent.

$$\begin{aligned}
 y(x) &= C_1 x + C_2 x \ln x + C_3 x^2 & y(1) &= C_1 + C_3 = \mathbf{3} \\
 y'(x) &= C_1 + C_2 (1 + \ln x) + 2C_3 x & y'(1) &= C_1 + C_2 + 2C_3 = \mathbf{2} \\
 y''(x) &= C_2 \frac{1}{x} + 2C_3 & y''(1) &= C_2 + 2C_3 = \mathbf{1} \\
 \Rightarrow C_1 &= 1, \quad C_2 = -3, \quad \text{and} \quad C_3 = 2 \\
 y(x) &= \underline{x - 3x \ln x + 2x^2}
 \end{aligned}$$

Exercise

Find the particular solution for $y'' + 3y' + 2y = 4e^{-3t}$

Solution

$$\begin{aligned}
 y(t) &= Ae^{-3t} & \Rightarrow y' &= -3Ae^{-3t} \\
 & & y'' &= 9Ae^{-3t} \\
 y'' + 3y' + 2y &= 4e^{-3t} \\
 9Ae^{-3t} + 3(-3Ae^{-3t}) + 2Ae^{-3t} &= 4e^{-3t} \\
 2Ae^{-3t} &= 4e^{-3t} \\
 2A &= 4 \\
 A &= 2 \\
 \text{The particular solution: } y(t) &= \underline{2e^{-3t}}
 \end{aligned}$$

Exercise

Find the particular solution for $y'' + 6y' + 8y = -3e^{-t}$

Solution

$$y(t) = Ae^{-t}$$

$$y' = -Ae^{-t}$$

$$y'' = Ae^{-t}$$

$$y'' + 6y' + 8y = -3e^{-t}$$

$$Ae^{-t} - 6Ae^{-t} + 8Ae^{-t} = -3e^{-t}$$

$$A - 6A + 8A = -3$$

$$3A = -3 \Rightarrow \boxed{A = -1}$$

Therefore, the particular solution is: $\boxed{y(t) = -e^{-t}}$

Exercise

Find the particular solution for $y'' + 2y' + 5y = 12e^{-t}$

Solution

$$y(t) = Ae^{-t}$$

$$y' = -Ae^{-t}$$

$$y'' = Ae^{-t}$$

$$y'' + 2y' + 5y = 12e^{-t}$$

$$Ae^{-t} + 2(-Ae^{-t}) + 5Ae^{-t} = 12e^{-t}$$

$$4Ae^{-t} = 12e^{-t}$$

$$4A = 12 \rightarrow \boxed{A = 3}$$

The particular solution: $\boxed{y(t) = 3e^{-t}}$

Exercise

Find the particular solution for the given differential equation $y'' + 3y' - 18y = 18e^{2t}$

Solution

$$y(t) = Ae^{2t}$$

$$y' = 2Ae^{2t}$$

$$y'' = 4Ae^{2t}$$

$$y'' + 3y' - 18y = 18e^{2t}$$

$$4Ae^{2t} + 3(2Ae^{2t}) - 18Ae^{2t} = 18e^{2t}$$

$$4Ae^{2t} + 6Ae^{2t} - 18Ae^{2t} = 18e^{2t}$$

$$-8Ae^{2t} = 18e^{2t}$$

$$-8A = 18 \rightarrow A = -\frac{18}{8} = -\frac{9}{4}$$

The particular solution: $y(t) = -\frac{9}{4}e^{2t}$

Exercise

Use $y(t) = a \cos \omega t + b \sin \omega t$ to find the particular solution for $y'' + 4y = \cos 3t$

Solution

The particular solution: $y(t) = a \cos 3t + b \sin 3t$

$$y' = -3a \sin 3t + 3b \cos 3t$$

$$y'' = -9a \cos 3t - 9b \sin 3t$$

$$y'' + 4y = \cos 3t$$

$$-9a \cos 3t - 9b \sin 3t + 4(a \cos 3t + b \sin 3t) = \cos 3t$$

$$-9a \cos 3t - 9b \sin 3t + 4a \cos 3t + 4b \sin 3t = \cos 3t$$

$$-5a \cos 3t - 5b \sin 3t = \cos 3t$$

$$a = -\frac{1}{5} \quad b = 0$$

The particular solution: $y(t) = -\frac{1}{5} \cos 3t$

Exercise

Use $y(t) = a \cos \omega t + b \sin \omega t$ to find the particular solution for $y'' + 7y' + 6y = 3 \sin 2t$

Solution

The particular solution: $y(t) = a \cos 2t + b \sin 2t$

$$y' = -2a \sin 2t + 2b \cos 2t$$

$$y'' = -4a \cos 2t - 4b \sin 2t$$

$$y'' + 7y' + 6y = 3 \sin 2t$$

$$-4a \cos 2t - 4b \sin 2t + 7(-2a \sin 2t + 2b \cos 2t) + 6(a \cos 2t + b \sin 2t) = 3 \sin 2t$$

$$-4a \cos 2t - 4b \sin 2t - 14a \sin 2t + 14b \cos 2t + 6a \cos 2t + 6b \sin 2t = 3 \sin 2t$$

$$(14b + 2a)\cos 2t + (2b - 14a)\sin 2t = 3\sin 2t$$

$$\begin{cases} 14b + 2a = 0 \\ 2b - 14a = 3 \end{cases} \Rightarrow a = -\frac{21}{100} \quad b = \frac{3}{100}$$

The particular solution: $y_p(t) = -\frac{21}{100}\cos 2t + \frac{3}{100}\sin 2t$

Exercise

Find the particular solution for $y'' + 5y' + 4y = 2 + 3t$

Solution

The particular solution: $y(t) = at + b$

$$y' = a$$

$$y'' = 0$$

$$y'' + 5y' + 4y = 2 + 3t$$

$$0 + 5a + 4(at + b) = 2 + 3t$$

$$5a + 4b + 4at = 2 + 3t$$

$$\begin{cases} 5a + 4b = 2 \\ 4a = 3 \end{cases} \Rightarrow \begin{cases} b = -\frac{7}{16} \\ a = \frac{3}{4} \end{cases}$$

The particular solution: $y_p(t) = \frac{3}{4}t - \frac{7}{16}$

Exercise

Find the particular solution for $y'' + 6y' + 8y = 2t - 3$

Solution

The particular solution: $y(t) = at + b$

$$y'' + 6y' + 8y = 2t - 3$$

$$0 + 6a + 8(at + b) = 2t - 3$$

$$6a + 8at + 8b = 2t - 3$$

$$8at + 6a + 8b = 2t - 3$$

$$\begin{cases} 6a + 8b = -3 \\ 8a = 2 \end{cases} \Rightarrow \begin{cases} a = \frac{1}{4} \\ b = -\frac{9}{16} \end{cases}$$

The particular solution: $y_p(t) = \frac{1}{4}t - \frac{9}{16}$

Exercise

Find the particular solution for $y'' + 3y' + 4y = t^3$

Solution

The particular solution: $y(t) = at^3 + bt^2 + ct + d$

$$y' = 3at^2 + 2bt + c$$

$$y'' = 6at + 2b$$

$$y'' + 3y' + 4y = t^3$$

$$6at + 2b + 3(3at^2 + 2bt + c) + 4(at^3 + bt^2 + ct + d) = t^3$$

$$6at + 2b + 9at^2 + 6bt + 3c + 4at^3 + 4bt^2 + 4ct + 4d = t^3$$

$$4at^3 + (9a + 4b)t^2 + (6a + 6b + 4c)t + 2b + 3c + 4d = t^3$$

$$\begin{cases} 4a = 1 \\ 9a + 4b = 0 \\ 6a + 6b + 4c = 0 \\ 2b + 3c + 4d = 0 \end{cases} \Rightarrow \begin{cases} a = \frac{1}{4} \\ b = -\frac{9}{4}a = -\frac{9}{16} \\ c = -\frac{6}{4}a - \frac{6}{4}b = -\frac{3}{2}\frac{1}{4} + \frac{3}{2}\frac{9}{16} = \frac{15}{32} \\ d = -\frac{1}{2}b - \frac{1}{2}c = -\frac{9}{128} \end{cases}$$

The particular solution: $y_p(t) = \frac{1}{4}t^3 - \frac{9}{16}t^2 + \frac{15}{32}t - \frac{9}{128}$

Exercise

Find the particular solution for $y'' + 2y' + 2y = 2 + \cos 2t$

Solution

$$y'' + 2y' + 2y = 2 \text{ when } y = 1$$

The particular solution: $z = Ae^{i2t}$

$$z' = (2i)Ae^{i2t}$$

$$z'' = (2i)^2 Ae^{i2t}$$

$$z'' + 2z' + 2z = e^{i2t}$$

$$(2i)^2 Ae^{i2t} + 2(2i)Ae^{i2t} + 2Ae^{i2t} = e^{i2t}$$

$$(2i)^2 A + 2(2i)A + 2A = 1$$

$$(-4 + 4i + 2)A = 1$$

$$A = \frac{1}{-2 + 4i} \cdot \frac{-2 - 4i}{-2 - 4i}$$

$$= -\frac{2}{10} - \frac{4}{10}i$$

$$= -\frac{1}{10} - \frac{1}{5}i$$

$$z_p = \left(-\frac{1}{10} - \frac{1}{5}i\right)e^{i2t}$$

$$= \left(-\frac{1}{10} - \frac{1}{5}i\right)(\cos 2t + i \sin 2t)$$

$$= -\frac{1}{10}\cos 2t + \frac{1}{5}\sin 2t + i\left(-\frac{1}{5}\cos 2t - \frac{1}{10}\sin 2t\right)$$

The general solution: $\underline{y(t) = 1 - \frac{1}{10}\cos 2t + \frac{1}{5}\sin 2t}$

Exercise

Find the particular solution for $y'' - y = t - e^{-t}$

Solution

The characteristic eq.: $\lambda^2 - 1 = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = 1$

The particular solution: $y = at + b$

$$y' = a$$

$$y'' = 0$$

$$y'' - y = t$$

$$-at - b = t \Rightarrow \left\{ \boxed{a = -1}, \boxed{b = 0} \right\}$$

$$y(t) = -t$$

The homogenous solution: $y_h = C_1 e^{-t} + C_2 e^t$

Because the inhomogeneous part of $y'' - y = e^{-t}$ is also the solution.

Therefore: $y_p = Ate^{-t}$

$$y'_p = -Ate^{-t} + Ae^{-t} = Ae^{-t}(1-t)$$

$$y''_p = -Ae^{-t} + Ate^{-t} - Ae^{-t} = Ae^{-t}(t-2)$$

$$y'' - y = e^{-t}$$

$$Ate^{-t} - 2Ae^{-t} - Ate^{-t} = e^{-t}$$

$$-2Ae^{-t} = e^{-t}$$

$$-2A = 1 \Rightarrow \underline{A = -\frac{1}{2}}$$

$$y_p = -\frac{1}{2}te^{-t}$$

$$\underline{y(t) = -t + \frac{1}{2}te^{-t}}$$

Exercise

Find the particular solution for $y'' - 2y' + y = 10e^{-2t} \cos t$

Solution

The particular solution: $y_p = e^{-2t}(A \cos t + B \sin t)$

$$\begin{aligned} y' &= -2e^{-2t}(A \cos t + B \sin t) + e^{-2t}(-A \sin t + B \cos t) \\ &= e^{-2t}((B - 2A) \cos t - (A + 2B) \sin t) \end{aligned}$$

$$\begin{aligned} y'' &= -2e^{-2t}((B - 2A) \cos t - (A + 2B) \sin t) + e^{-2t}((2A - B) \sin t - (A + 2B) \cos t) \\ &= e^{-2t}((3A - 4B) \cos t + (4A + 3B) \sin t) \end{aligned}$$

$$y'' - 2y' + y = 10e^{-2t} \cos t$$

$$\begin{aligned} e^{-2t}((3A - 4B) \cos t + (4A + 3B) \sin t) - 2e^{-2t}((B - 2A) \cos t - (A + 2B) \sin t) \\ + e^{-2t}(A \cos t + B \sin t) = 10e^{-2t} \cos t \end{aligned}$$

$$((3A - 4B - 2B + 4A + A) \cos t + (4A + 3B + 2A + 4B + B) \sin t) = 10 \cos t$$

$$\begin{cases} 8A - 6B = 10 \\ 6A + 8B = 0 \end{cases} \rightarrow A = \frac{80}{100} = \frac{4}{5} \quad B = -\frac{3}{5}$$

$$\underline{y_p = e^{-2t} \left(\frac{4}{5} \cos t - \frac{3}{5} \sin t \right)}$$

Exercise

Find the particular solution for $y''' - 4y'' + 4y' = 5t^2 - 6t + 4t^2e^t + 3e^{5t}$

Solution

$$\text{Characteristic equation: } \lambda^3 - 4\lambda^2 + 4\lambda = \lambda(\lambda - 2)^2 = 0 \Rightarrow \lambda_1 = 0, \lambda_{2,3} = 2$$

$$\text{Homogeneous equation: } y_h = C_1 + (C_2 + C_3 t)e^{2t}$$

$$\text{The particular solution: } y_p = t(A t^2 + B t + C) + (E t^2 + F t + G)e^t + H e^{5t}$$

$$\begin{aligned} y'_p &= 3A t^2 + 2B t + C + (2E t + F)e^t + (E t^2 + F t + G)e^t + 5H e^{5t} \\ &= 3A t^2 + 2B t + C + (E t^2 + (2E + F)t + F + G)e^t + 5H e^{5t} \end{aligned}$$

$$\begin{aligned} y''_p &= 6A t + 2B + (2E t + 2E + F)e^t + (E t^2 + (2E + F)t + F + G)e^t + 25H e^{5t} \\ &= 6A t + 2B + (E t^2 + (4E + F)t + 2E + 2F + G)e^t + 25H e^{5t} \end{aligned}$$

$$y'''_p = 6A + (2E t + 4E + F)e^t + (E t^2 + (4E + F)t + 2E + 2F + G)e^t + 125H e^{5t}$$

$$\begin{aligned}
&= 6A + \left(Et^2 + (6E + F)t + 6E + 3F + G \right) e^t + 125He^{5t} \\
y''' - 4y'' + 4y' &= 6A + \left(Et^2 + (6E + F)t + 6E + 3F + G \right) e^t + 125He^{5t} \\
&\quad - 24At - 8B - 4 \left(Et^2 + (4E + F)t + 2E + 2F + G \right) e^t - 100He^{5t} \\
&\quad + 12At^2 + 8Bt + 4C + 4 \left(Et^2 + (2E + F)t + F + G \right) e^t + 20He^{5t} \\
&= 12At^2 + (8B - 24A)t + 6A - 8B + 4C + \left(Et^2 + (-2E + F)t - 2E - F + G \right) e^t + 45He^{5t} \\
&= 5t^2 - 6t + 4t^2 e^t + 3e^{5t}
\end{aligned}$$

$$\begin{cases} 12A = 5 \\ 8B - 24A = -6 \\ 6A - 8B + 4C = 0 \end{cases} \rightarrow \begin{cases} A = \frac{5}{12} \\ B = \frac{1}{2} \\ C = \frac{3}{8} \end{cases} \quad \begin{cases} E = 4 \\ F - 2E = 0 \\ -2E - F + G = 0 \end{cases} \Rightarrow \begin{cases} F = 8 \\ G = 16 \end{cases} \quad H = \frac{3}{45} = \frac{1}{15}$$

$$\underline{y_P = \frac{5}{12}t^3 + \frac{1}{2}t^2 + \frac{3}{8}t + (4t^2 + 8t + 16)e^t + \frac{1}{15}e^{5t}}$$

Exercise

Use the complex method to find the particular solution for $y'' + 4y' + 3y = \cos 2t + 3\sin 2t$

Solution

The characteristic eq.: $\lambda^2 + 4\lambda + 3 = 0 \Rightarrow \lambda_1 = -3, \lambda_2 = -1$

The homogenous solution: $y_h = e^{-t} (C_1 \cos t + C_2 \sin t)$

The particular solution: $z = Ae^{i2t}$

$$z' = (2i)Ae^{i2t}$$

$$z'' = (2i)^2 Ae^{i2t}$$

$$z'' + 4z' + 3z = e^{i2t}$$

$$(2i)^2 Ae^{i2t} + 4(2i)Ae^{i2t} + 3Ae^{i2t} = e^{i2t}$$

$$(-4 + 8i + 3)A = 1$$

$$(-1 + 8i)A = 1$$

$$A = \frac{1}{-1 + 8i} \cdot \frac{-1 - 8i}{-1 - 8i}$$

$$= \frac{-1 - 8i}{65}$$

$$= -\frac{1}{65} - i \frac{8}{65}$$

This gives the particular solution:

$$\begin{aligned}
 z &= \left(-\frac{1}{65} - i \frac{8}{65}\right) e^{i2t} \\
 &= \left(-\frac{1}{65} - i \frac{8}{65}\right) (\cos 2t + i \sin 2t) \\
 &= \left(-\frac{1}{65} \cos 2t + \frac{8}{65} \sin 2t\right) + i \left(-\frac{8}{65} \cos 2t - \frac{1}{65} \sin 2t\right)
 \end{aligned}$$

$$y = -\frac{1}{65} \cos 2t + \frac{8}{65} \sin 2t \text{ is a solution of } y'' + 4y' + 3y = \cos 2t$$

$$y = -\frac{8}{65} \cos 2t - \frac{1}{65} \sin 2t \text{ is a solution of } y'' + 4y' + 3y = \sin 2t$$

Therefore;

$$\begin{aligned}
 y(t) &= -\frac{1}{65} \cos 2t + \frac{8}{65} \sin 2t + 3 \left(-\frac{8}{65} \cos 2t - \frac{1}{65} \sin 2t\right) \\
 &= -\frac{1}{65} \cos 2t + \frac{8}{65} \sin 2t - \frac{24}{65} \cos 2t - \frac{3}{65} \sin 2t \\
 &= -\frac{25}{65} \cos 2t + \frac{5}{65} \sin 2t \\
 &= \underline{-\frac{5}{13} \cos 2t + \frac{1}{13} \sin 2t}
 \end{aligned}$$

Exercise

Use the complex method to find the particular solution for $y'' + 4y = \cos 3t$

Solution

The particular solution: $z = Ae^{i3t}$

$$z' = (3i) Ae^{i3t}$$

$$z'' = (3i)^2 Ae^{i3t}$$

$$z'' + 4z = \cos 3t = e^{i3t}$$

$$(3i)^2 Ae^{i3t} + 4Ae^{i3t} = e^{i3t}$$

$$(-9 + 4)A = 1 \rightarrow \underline{A = -\frac{1}{5}}$$

$$z = -\frac{1}{5} e^{i3t}$$

$$\underline{y(t) = -\frac{1}{5} \cos 3t}$$

Exercise

Find the general solution: $y'' + y = 2 \cos x$

Solution

$$\text{The characteristic equation: } \lambda^2 + 1 = 0 \rightarrow \underline{\lambda_{1,2} = \pm i}$$

$$\underline{y_h = C_1 \cos x + C_2 \sin x}$$

$$y_p = Ax \cos x + Bx \sin x \quad \text{Since } y_h \text{ in functions of cosine and sine}$$

$$y'_p = A \cos x - Ax \sin x + B \sin x + Bx \cos x$$

$$= (A + Bx) \cos x + (-Ax + B) \sin x$$

$$y''_p = B \cos x - (A + Bx) \sin x - A \sin x + (-Ax + B) \cos x$$

$$= (2B - Ax) \cos x + (-2A - Bx) \sin x$$

$$y'' + y = 2 \cos x$$

$$(2B - Ax) \cos x + (-2A - Bx) \sin x + Ax \cos x + Bx \sin x = 2 \cos x$$

$$2B \cos x - 2A \sin x = 2 \cos x$$

$$\begin{cases} -2A = 0 \\ 2B = 2 \end{cases} \rightarrow \underline{A = 0, B = 1}$$

$$\underline{y_p = x \sin x}$$

$$\underline{y(x) = C_1 \cos x + C_2 \sin x + x \sin x}$$

Exercise

Find the general solution for the given DE: $y'' + y = \cos 3x$

Solution

$$\text{The characteristic equation: } \lambda^2 + 1 = 0 \rightarrow \underline{\lambda_{1,2} = \pm i}$$

$$\underline{y_h = C_1 \cos x + C_2 \sin x}$$

$$y_p = A \cos 3x + B \sin 3x$$

$$y'_p = -3A \sin 3x + 3B \cos 3x$$

$$y''_p = -9A \cos 3x - 3B \sin 3x$$

$$y'' + y = \cos 3x$$

$$-9A \cos 3x - 3B \sin 3x + A \cos 3x + B \sin 3x = \cos 3x$$

$$\begin{cases} -8A = 1 \\ -2B = 0 \end{cases} \rightarrow \underline{A = -\frac{1}{8}, B = 0}$$

$$\underline{y(x) = C_1 \cos x + C_2 \sin x - \frac{1}{8} \cos 3x}$$

Exercise

Find the general solution for the given DE: $y'' + y = 2x \sin x$

Solution

The characteristic equation: $\lambda^2 + 1 = 0 \rightarrow \lambda_{1,2} = \pm i$

$$y_h = C_1 \cos x + C_2 \sin x$$

$$y_p = (Ax^2 + Bx) \cos x + (Cx^2 + Dx) \sin x$$

$$\begin{aligned} y'_p &= (2Ax + B) \cos x - (Ax^2 + Bx) \sin x + (2Cx + D) \sin x + (Cx^2 + Dx) \cos x \\ &= (Cx^2 + 2Ax + Dx + B) \cos x - (Ax^2 + Bx - 2Cx - D) \sin x \end{aligned}$$

$$\begin{aligned} y''_p &= (2Cx + 2A + D) \cos x - (Cx^2 + 2Ax + Dx + B) \sin x \\ &\quad - (2Ax + B - 2C) \sin x - (Ax^2 + Bx - 2Cx - D) \cos x \\ &= (-Ax^2 - Bx + 4Cx + 2A + 2D) \cos x - (Cx^2 + 4Ax + Dx + 2B - 2C) \sin x \end{aligned}$$

$$y'' + y = 2x \sin x$$

$$\cos x \quad x^2 \quad -A + A = 0$$

$$x \quad -B + 4C + B = 0 \quad \underline{C = 0}$$

$$x^0 \quad 2A + 2D = 0 \quad A = -D$$

$$\rightarrow \underline{D = \frac{1}{2}}$$

$$\sin x \quad x^2 \quad -C + C = 0$$

$$x \quad -4A = 2 \quad \underline{A = -\frac{1}{2}}$$

$$x^0 \quad 2B - 2C = 0 \quad \underline{B = C = 0}$$

$$y_p = -\frac{1}{2}x^2 \cos x + \frac{1}{2}x \sin x$$

$$y(x) = C_1 \cos x + C_2 \sin x - \frac{1}{2}x^2 \cos x + \frac{1}{2}x \sin x$$

Exercise

Find the general solution: $y'' - y = x^2 e^x + 5$

Solution

The characteristic equation: $\lambda^2 - 1 = 0 \rightarrow \lambda_{1,2} = \pm 1$

$$y_h = C_1 e^{-x} + C_2 e^x$$

$$y_p = (Ax^3 + Bx^2 + Cx) e^x + D$$

$$y'_p = (Ax^3 + Bx^2 + Cx + 3Ax^2 + 2Bx + C) e^x$$

$$y''_p = (Ax^3 + Bx^2 + 6Ax^2 + Cx + 4Bx + 6Ax + 2C + 2B)e^x$$

$$y'' - y = x^2 e^x + 5$$

$$(Ax^3 + Bx^2 + 6Ax^2 + Cx + 4Bx + 6Ax + 2C + 2B - Ax^3 - Bx^2 - Cx)e^x - D = x^2 e^x + 5$$

$$\underline{D = -5}$$

$$\textcolor{red}{x^2} \quad 6A = 1 \quad \underline{A = \frac{1}{6}}$$

$$\textcolor{red}{x} \quad 4B + 6A = 0 \quad \underline{B = -\frac{1}{4}}$$

$$\textcolor{red}{x^0} \quad 2C + 2B = 0 \quad \underline{C = \frac{1}{4}}$$

$$y_p = \left(\frac{1}{6}x^3 - \frac{1}{4}Bx^2 + \frac{1}{4}x \right) e^x - 5$$

$$\underline{y(x) = C_1 e^{-x} + C_2 e^x + \left(\frac{1}{6}x^3 - \frac{1}{4}Bx^2 + \frac{1}{4}x \right) e^x - 5}$$

Exercise

Find the general solution for the given DE: $y'' - y' = -3$

Solution

$$\text{The characteristic equation: } \lambda^2 - \lambda = 0 \rightarrow \underline{\lambda_{1,2} = 0, 1}$$

$$\underline{y_h = C_1 + C_2 e^x}$$

$$y_p = Ax$$

$$y'_p = A$$

$$y''_p = 0$$

$$y'' - y' = -3 \rightarrow \underline{A = 3}$$

$$\underline{y_p = 3x}$$

$$\underline{y(x) = C_1 + C_2 e^x - 3x}$$

Exercise

Find the general solution for the given DE: $y'' - y' = 2 \sin x$

Solution

$$\text{The characteristic equation: } \lambda^2 - \lambda = 0 \rightarrow \underline{\lambda_{1,2} = 0, 1}$$

$$\underline{y_h = C_1 + C_2 e^x}$$

$$y_p = A \cos x + B \sin x$$

$$y'_p = -A \sin x + B \cos x$$

$$y''_p = -A \cos x - B \sin x$$

$$y'' - y' = 2 \sin x$$

$$-A \cos x - B \sin x - (-A \sin x + B \cos x) = 2 \sin x$$

$$\begin{cases} -A + B = 0 \\ -A - B = 2 \end{cases} \rightarrow \underline{A = -1, B = -1}$$

$$\underline{y_p = -\cos x - \sin x}$$

$$\underline{y(x) = C_1 + C_2 e^x - \cos x - \sin x}$$

Exercise

Find the general solution for the given DE: $y'' - y' = \sin x$

Solution

$$\text{The characteristic equation: } \lambda^2 - \lambda = 0 \rightarrow \underline{\lambda_{1,2} = 0, 1}$$

$$\underline{y_h = C_1 + C_2 e^x}$$

$$y_p = A \cos x + B \sin x$$

$$y'_p = -A \sin x + B \cos x$$

$$y''_p = -A \cos x - B \sin x$$

$$y'' - y' = \sin x$$

$$-A \cos x - B \sin x - (-A \sin x + B \cos x) = \sin x$$

$$\begin{cases} -A + B = 0 \\ -A - B = 1 \end{cases} \rightarrow \underline{A = -\frac{1}{2}, B = -\frac{1}{2}}$$

$$\underline{y_p = -\frac{1}{2} \cos x - \frac{1}{2} \sin x}$$

$$\underline{y(x) = C_1 + C_2 e^x - \frac{1}{2} \cos x - \frac{1}{2} \sin x}$$

Exercise

Find the general solution for the given DE: $y'' - y' = -8x + 3$

Solution

$$\text{The characteristic equation: } \lambda^2 - \lambda = 0 \rightarrow \underline{\lambda_{1,2} = 0, 1}$$

$$\underline{y_h = C_1 + C_2 e^x}$$

$$y_p = Ax^2 + Bx$$

$$y'_p = 2Ax + B$$

$$y''_p = 2A$$

$$y'' - y' = -8x + 3$$

$$2A - 2Ax + B = -8x + 3$$

$$\begin{cases} -2A = -8 \\ 2A + B = 3 \end{cases} \rightarrow \underline{A = 4, B = -5}$$

$$\underline{y_p = 4x^2 - 5x}$$

$$\underline{y(x) = C_1 + C_2 e^x + 4x^2 - 5x}$$

Exercise

Find the general solution: $y'' + y = 2x + 3e^x$

Solution

The characteristic equation: $\lambda^2 + 1 = 0 \rightarrow \underline{\lambda_{1,2} = \pm i}$

$$\underline{y_h = C_1 \cos x + C_2 \sin x}$$

$$y_p = Ax + Be^x$$

$$y'_p = A + Be^x$$

$$y''_p = Be^x$$

$$y'' + y = 2x + 3e^x$$

$$2Be^x + Ax = 2x + 3e^x \rightarrow \underline{A = 2, B = \frac{3}{2}}$$

$$\underline{y_p = 2x + \frac{3}{2}e^x}$$

$$\underline{y(x) = C_1 \cos x + C_2 \sin x + 2x + \frac{3}{2}e^x}$$

Exercise

Find the general solution: $y'' - y = x^2 + e^x$

Solution

The characteristic equation: $\lambda^2 - 1 = 0 \rightarrow \underline{\lambda_{1,2} = \pm 1}$

$$\underline{y_h = C_1 e^{-x} + C_2 e^x}$$

$$y_p = Ax^2 + Bx + C + Dxe^x$$

$$y'_p = 2Ax + B + (D + Dx)e^x$$

$$y''_p = 2A + (2D + Dx)e^x$$

$$y'' - y = x^2 + e^x$$

$$2A + 2De^x - Ax^2 - Bx - C = x^2 + e^x$$

$$\begin{cases} -A = 1 \\ -B = 0 \\ 2A - C = 0 \\ 2D = 1 \end{cases} \rightarrow \underline{A = -1, B = 0, C = -2, D = \frac{1}{2}}$$

$$\underline{y_p = -x^2 - 2 + \frac{1}{2}xe^x}$$

$$\underline{y(x) = C_1 e^{-x} + \left(C_2 + \frac{1}{2}x\right)e^x - x^2 - 2}$$

Exercise

Find the general solution for the given DE: $y'' + y' = 10x^4 + 2$

Solution

The characteristic equation: $\lambda^2 + \lambda = 0 \rightarrow \underline{\lambda_{1,2} = 0, -1}$

$$\underline{y_h = C_1 + C_2 e^{-x}}$$

$$y_p = Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex$$

$$y'_p = 5Ax^4 + 4Bx^3 + 3Cx^2 + 2Dx + E$$

$$y''_p = 20Ax^3 + 12Bx^2 + 6Cx + 2D$$

$$y'' + y' = 10x^4 + 2$$

$$20Ax^3 + 12Bx^2 + 6Cx + 2D + 5Ax^4 + 4Bx^3 + 3Cx^2 + 2Dx + E = 10x^4 + 2$$

$$x^4 \quad 5A = 10$$

$$x^3 \quad 20A + 4B = 0$$

$$x^2 \quad 12B + 3C = 0 \rightarrow A = 2, B = -10, C = 40, D = -120, E = 242$$

$$x^1 \quad 6C + 2D = 0$$

$$x^0 \quad 1D + E = 2$$

$$\underline{y_p = 2x^5 - 10x^4 + 10x^3 - 120x^2 + 242x}$$

$$\underline{y(x) = C_1 + C_2 e^{-x} + 2x^5 - 10x^4 + 10x^3 - 120x^2 + 242x}$$

Exercise

Find the general solution for the given DE: $y'' - y' = 5e^x - \sin 2x$

Solution

The characteristic equation: $\lambda^2 - \lambda = 0 \rightarrow \lambda_{1,2} = 0, 1$

$$y_h = C_1 + C_2 e^x$$

$$y_p = Ax e^x + B \cos 2x + D \sin 2x$$

$$y'_p = (A + Ax)e^x - 2B \sin 2x + 2D \cos 2x$$

$$y''_p = (2A + Ax)e^x - 4B \cos 2x - 4D \sin 2x$$

$$y'' - y' = 5e^x - \sin 2x$$

$$(2A + Ax)e^x - 4B \cos 2x - 4D \sin 2x - (A + Ax)e^x + 2B \sin 2x - 2D \cos 2x = 5e^x - \sin 2x$$

$$Ae^x + (-4B - 2D)\cos 2x + (2B - 4D)\sin 2x = 5e^x - \sin 2x$$

$$\begin{cases} A = 5 \\ -4B - 2D = 0 \\ 2B - 4D = -1 \end{cases} \rightarrow \underline{A = 5, B = -\frac{1}{10}, D = \frac{1}{5}}$$

$$y_p = 5xe^x - \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x$$

$$y(x) = C_1 + C_2 e^x + 5xe^x - \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x$$

Exercise

Find the general solution for the given DE $y'' + y = x \cos x - \cos x$

Solution

The characteristic equation: $\lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$

$$y_h = C_1 \cos x + C_2 \sin x$$

The particular equation: $y_p = (Ax + B)\cos x + (Cx + E)\sin x$

$$y'_p = A \cos x - (Ax + B)\sin x + C \sin x + (Cx + E)\cos x$$

$$= (Cx + A + E)\cos x - (Ax + B + C)\sin x$$

$$y''_p = C \cos x - (Cx + A + E)\sin x - A \sin x - (Ax + B + C)\cos x$$

$$-(Ax + B)\cos x - (Cx + 2A + E)\sin x + (Ax + B)\cos x + (Cx + E)\sin x = x \cos x - \cos x$$

$$\cancel{-2A \sin x = x \cos x - \cos x}$$

The particular equation: $y_p = (Ax^2 + Bx + C)\cos x + (Dx^2 + Ex + F)\sin x$

$$\begin{aligned} y'_p &= (2Ax + B)\cos x - (Ax^2 + Bx + C)\sin x + (2Dx + E)\sin x + (Dx^2 + Ex + F)\cos x \\ &= (Dx^2 + (2A + E)x + B + F)\cos x - (Ax^2 + (B - 2D)x + C - E)\sin x \end{aligned}$$

$$\begin{aligned} y''_p &= (2Dx + 2A + E)\cos x - (Dx^2 + (2A + E)x + B + F)\sin x \\ &\quad - (2Ax + B - 2D)\sin x - (Ax^2 + (B - 2D)x + C - E)\cos x \end{aligned}$$

$$\begin{aligned} &- (Ax^2 + (B - 4D)x - 2A + C - 2E)\cos x - (Dx^2 + (4A + E)x + 2B - 2D + F)\sin x \\ &+ (Ax^2 + Bx + C)\cos x + (Dx^2 + Ex + F)\sin x = x\cos x - \cos x \end{aligned}$$

$$4Dx\cos x + (2A + 2E)\cos x - 4Ax\sin x + (2D - 2B)\sin x = x\cos x - \cos x$$

$$\begin{cases} 4D = 1 & \rightarrow D = \frac{1}{4} \\ 2A + 2E = -1 & E = -\frac{1}{2} \\ -4A = 0 & \rightarrow A = 0 \\ 2D - 2B = 0 & B = \frac{1}{4} \end{cases}$$

$$y(x) = C_1 \cos x + C_2 \sin x + \frac{1}{4}x\cos x + \frac{1}{4}x^2\sin x - \frac{1}{2}x\sin x + C\cos x + F\sin x$$

$$y(x) = C_1 \cos x + C_2 \sin x + \frac{1}{4}x\cos x + \frac{1}{4}x^2\sin x - \frac{1}{2}x\sin x$$

Exercise

Find the general solution for the given DE: $y'' + y = e^x \sin x$

Solution

The characteristic equation: $\lambda^2 + \lambda = \lambda(\lambda + 1) = 0 \rightarrow \lambda_{1,2} = 0, -1$

$$y_h = C_1 + C_2 e^{-x}$$

$$y_p = e^x (A \cos x + B \sin x)$$

$$\begin{aligned} y'_p &= e^x (A \cos x + B \sin x - A \sin x + B \cos x) \\ &= e^x ((A + B) \cos x + (B - A) \sin x) \end{aligned}$$

$$y''_p = e^x ((A + B) \cos x + (B - A) \sin x - (A + B) \sin x + (B - A) \cos x)$$

$$y''_p = e^x (2B \cos x - 2A \sin x)$$

$$y'' + y = e^x \sin x$$

$$e^x (2B \cos x - 2A \sin x + A \cos x + B \sin x) = e^x \sin x$$

$$(A + 2B) \cos x + (B - 2A) \sin x = \sin x$$

$$\begin{cases} A + 2B = 0 \\ -2A + B = 1 \end{cases} \quad \Delta = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 5 \quad \Delta_A = \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} = -2 \quad \Delta_B = \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} = 1$$

$$\underline{A = -\frac{2}{5}, \quad B = \frac{1}{5}}$$

$$\underline{y_P = e^x \left(-\frac{2}{5} \cos x + \frac{1}{5} \sin x \right)}$$

$$\underline{y(x) = C_1 + C_2 e^{-x} e^x \left(-\frac{2}{5} \cos x + \frac{1}{5} \sin x \right)}$$

Exercise

Find the general solution: $y'' - 4y = 4x^2$

Solution

The characteristic equation: $\lambda^2 - 4 = 0 \rightarrow \underline{\lambda_{1,2} = \pm 2}$

$$\underline{y_h = C_1 e^{-2x} + C_2 e^{2x}}$$

$$y_P = Ax^2 + Bx + C$$

$$y'_P = 2Ax + B$$

$$y''_P = 2A$$

$$y'' - 4y = 4x^2$$

$$2A - 4Ax^2 - 4Bx - 4C = x^2$$

$$\textcolor{red}{x^2} \quad -4A = 1 \quad \rightarrow \underline{A = -\frac{1}{4}}$$

$$\textcolor{red}{x} \quad -4B = 0 \quad \rightarrow \underline{B = 0}$$

$$\textcolor{red}{x^0} \quad 2A - 4C = 0 \quad \rightarrow \underline{C = -\frac{1}{8}}$$

$$\underline{y_P = -\frac{1}{4}x^2 - \frac{1}{8}}$$

$$\underline{y(x) = C_1 e^{-2x} + C_2 e^{2x} - \frac{1}{4}x^2 - \frac{1}{8}}$$

Exercise

Find the general solution: $y'' - y' - 2y = 20\cos x$

Solution

The characteristic equation: $\lambda^2 - \lambda - 2 = 0 \rightarrow \lambda_{1,2} = -1, 2$

$$y_h = C_1 e^{-x} + C_2 e^{2x}$$

$$y_p = A \cos x + B \sin x$$

$$y'_p = -A \sin x + B \cos x$$

$$y''_p = -A \cos x - B \sin x$$

$$y'' - y' - 2y = 20\cos x$$

$$-A \cos x - B \sin x + A \sin x - B \cos x - 2A \cos x - 2B \sin x = 20\cos x$$

$$\begin{cases} \cos x & -3A - B = 20 \\ \sin x & A - 3B = 0 \end{cases} \rightarrow A = -6, B = -2$$

$$y_p = -6\cos x - 2\sin x$$

$$y(x) = C_1 e^{-x} + C_2 e^{2x} - 6\cos x - 2\sin x$$

Exercise

Find the general solution: $y'' - y' + \frac{1}{4}y = 3 + e^{x/2}$

Solution

The characteristic equation: $\lambda^2 - \lambda + \frac{1}{4} = 0 \rightarrow \lambda_{1,2} = \frac{1}{2}$

$$y_h = (C_1 + C_2 x) e^{x/2}$$

$$y_p = A + Bx^2 e^{x/2}$$

$$y'_p = \left(2Bx + \frac{1}{2} Bx^2 \right) e^{x/2}$$

$$y''_p = \left(2Bx + \frac{1}{4} Bx^2 + 2B \right) e^{x/2}$$

$$y'' - y' + \frac{1}{4}y = 3 + e^{x/2}$$

$$\left(2Bx + \frac{1}{4} Bx^2 + 2B - 2Bx - \frac{1}{2} Bx^2 + \frac{1}{4} Bx^2 \right) e^{x/2} + \frac{1}{4} A = 3 + e^{x/2}$$

$$\frac{1}{4} A = 3 \quad A = 12$$

$$x^0 \quad 2B = 1 \quad B = \frac{1}{2}$$

$$\underline{y_p = 12 + \frac{1}{2}x^2 e^{x/2}}$$

$$\underline{y(x) = (C_1 + C_2 x)e^{x/2} + 12 + \frac{1}{2}x^2 e^{x/2}}$$

Exercise

Find the general solution: $y'' + y' + \frac{1}{4}y = e^x (\sin 3x - \cos 3x)$

Solution

The characteristic equation: $\lambda^2 + \lambda + \frac{1}{4} = 0 \rightarrow \underline{\lambda_{1,2} = \pm \frac{1}{2}i}$

$$\underline{y_h = C_1 \cos x + C_2 \sin x}$$

$$y_p = e^x (A \cos 3x + B \sin 3x)$$

$$\begin{aligned} y'_p &= e^x (A \cos 3x + B \sin 3x - 3A \sin 3x + 3B \cos 3x) \\ &= e^x ((A + 3B) \cos 3x + (B - 3A) \sin 3x) \end{aligned}$$

$$\begin{aligned} y''_p &= ((A + 3B) \cos 3x + (B - 3A) \sin 3x - 3(A + 3B) \sin 3x + 3(B - 3A) \cos 3x) e^x \\ &= ((-8A + 6B) \cos 3x + (-8B - 6A) \sin 3x) e^x \end{aligned}$$

$$y'' + y' + \frac{1}{4}y = e^x (\sin 3x - \cos 3x)$$

$$e^x \begin{cases} \cos 3x & -8A + 6B + A + 3B + \frac{1}{4}A \\ \sin 3x & -8B - 6A + B - 3A + \frac{1}{4}B \end{cases} = e^x (\sin 3x - \cos 3x)$$

$$\begin{cases} -\frac{27}{4}A + 9B = -1 \\ -9A - \frac{27}{4}B = 1 \end{cases} \rightarrow \begin{cases} -27A + 36B = -4 \\ -36A - 27B = 4 \end{cases} \quad \Delta = \begin{vmatrix} -27 & 36 \\ -36 & -27 \end{vmatrix} = 2,025 \quad \Delta_A = \begin{vmatrix} -4 & 36 \\ 4 & -27 \end{vmatrix} = -36$$

$$\underline{A = -\frac{4}{225}, B = -\frac{28}{225}}$$

$$\underline{y_p = e^x \left(-\frac{4}{225} \cos 3x - \frac{28}{225} \sin 3x \right)}$$

$$\underline{y(x) = C_1 \cos x + C_2 \sin x + \left(-\frac{4}{225} \cos 3x - \frac{28}{225} \sin 3x \right) e^x}$$

Exercise

Find the general solution: $y'' - y' - 2y = e^{3x}$

Solution

The characteristic equation: $\lambda^2 - \lambda - 2 = 0 \rightarrow \underline{\lambda_{1,2} = -1, 2}$

$$\underline{y_h = C_1 e^{-x} + C_2 e^{2x}}$$

$$y_p = A e^{3x}$$

$$y'_p = 3A e^{3x}$$

$$y''_p = 9A e^{3x}$$

$$y'' - y' - 2y = e^{3x}$$

$$(9A - 3A - 2A)e^{3x} = e^{3x}$$

$$4A = 1 \rightarrow \underline{A = \frac{1}{4}}$$

$$\underline{y_p = \frac{1}{4} e^{3x}}$$

$$\underline{y(x) = C_1 e^{-x} + C_2 e^{2x} + \frac{1}{4} e^{3x}}$$

Exercise

Find the general solution: $y'' - y' - 6y = 20e^{-2x}$

Solution

The characteristic equation: $\lambda^2 - \lambda - 6 = 0 \rightarrow \underline{\lambda_{1,2} = -2, 3}$

$$\underline{y_h = C_1 e^{-2x} + C_2 e^{3x}}$$

$$y_p = A x e^{-2x}$$

$$y'_p = (A - 2Ax) e^{-2x}$$

$$y''_p = (-4A + 4Ax) e^{-2x}$$

$$y'' - y' - 6y = 20e^{-2x}$$

$$(-4A + 4Ax - A + 2Ax - 6Ax) e^{-2x} = 20e^{-2x}$$

$$-5A e^{-2x} = 20e^{-2x} \rightarrow A = -4$$

$$\underline{y_p = -4x e^{-2x}}$$

$$\underline{y(x) = C_1 e^{-2x} + C_2 e^{3x} - 4x e^{-2x}}$$

Exercise

Find the general solution: $y'' + y' - 6y = 2x$

Solution

The characteristic equation: $\lambda^2 + \lambda - 6 = 0 \rightarrow \lambda_{1,2} = -3, 2$

$$y_h = C_1 e^{-3x} + C_2 e^{2x}$$

$$y_p = Ax + B$$

$$y'_p = A$$

$$y''_p = 0$$

$$y'' + y' - 6y = 2x$$

$$A - 6Ax - 6B = 2x$$

$$\begin{cases} x & -6A = 2 \\ x^0 & A - 6B = 0 \end{cases} \rightarrow A = -\frac{1}{3}, B = -\frac{1}{18}$$

$$y(x) = C_1 e^{-3x} + C_2 e^{2x} - \frac{1}{3}x - \frac{1}{18}$$

Exercise

Find the general solution: $y'' - y' - 6y = e^{-x} - 7\cos x$

Solution

The characteristic equation: $\lambda^2 - \lambda - 6 = 0 \rightarrow \lambda_{1,2} = -2, 3$

$$y_h = C_1 e^{-2x} + C_2 e^{3x}$$

$$y_p = Ae^{-x} + B\cos x + C\sin x$$

$$y'_p = -Ae^{-x} - B\sin x + C\cos x$$

$$y''_p = Ae^{-x} - B\cos x - C\sin x$$

$$y'' - y' - 6y = e^{-x} - 7\cos x$$

$$Ae^{-x} - B\cos x - C\sin x + Ae^{-x} + B\sin x - C\cos x - 6Ae^{-x} - 6B\cos x - 6C\sin x = e^{-x} - 7\cos x$$

$$\begin{cases} e^{-x} & -4A = 1 \\ \cos x & -7B - C = -7 \\ \sin x & B - 7C = 0 \end{cases} \rightarrow A = -\frac{1}{4}, B = \frac{49}{50}, C = \frac{7}{50}$$

$$y_p = -\frac{1}{4}e^{-x} + \frac{49}{50}\cos x + \frac{7}{50}\sin x$$

$$y(x) = C_1 e^{-2x} + C_2 e^{3x} - \frac{1}{4}e^{-x} + \frac{49}{50}\cos x + \frac{7}{50}\sin x$$

Exercise

Find the general solution: $y'' + y' + 8y = x \cos 3x + (10x^2 + 21x + 9) \sin 3x$

Solution

The characteristic equation: $\lambda^2 + \lambda + 8 = 0 \rightarrow \lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{31}}{2}i$

$$y_h = e^{-x/2} \left(C_1 \cos \frac{\sqrt{31}}{2} x + C_2 \sin \frac{\sqrt{31}}{2} x \right)$$

$$y_p = (Ax^2 + Bx + C) \cos 3x + (Dx^2 + Ex + F) \sin 3x$$

$$\begin{aligned} y_p' &= (2Ax + B) \cos 3x - 3(Ax^2 + Bx + C) \sin 3x + (2Dx + E) \sin 3x + 3(Dx^2 + Ex + F) \cos 3x \\ &= (3Dx^2 + 2Ax + 3Ex + 3F + B) \cos 3x + (-3Ax^2 - 3Bx + 2Dx - 3C + E) \sin 3x \end{aligned}$$

$$\begin{aligned} y_p'' &= (6Dx + 2A + 3E) \cos 3x - 3(3Dx^2 + 2Ax + 3Ex + 3F + B) \sin 3x \\ &\quad + (-6Ax - 3B + 2D) \sin 3x + 3(-3Ax^2 - 3Bx + 2Dx - 3C + E) \cos 3x \\ &= (-9Ax^2 - 9Bx + 12Dx + 2A - 9C + 6E) \cos 3x \\ &\quad + (-9Dx^2 - 12Ax - 9Ex - 9F - 6B + 2D) \sin 3x \end{aligned}$$

$$y'' + y' + 8y = x \cos 3x + (10x^2 + 21x + 9) \sin 3x$$

$$\begin{cases} \cos 3x & x^2 & -A + 3D = 0 & (1) \\ & x & -B + 12D + 2A + 3E = 1 & (2) \\ & x^0 & 2A + B - C + 6E + 3F = 0 & (3) \end{cases}$$

$$\begin{cases} \sin 3x & x^2 & -3A - D = 10 & (4) \\ & x & -12A - 3B - E + 2D = 21 & (5) \\ & x^0 & -6B + 2D - 3C + E - F = 9 & (6) \end{cases}$$

$$\begin{cases} (1) & -A + 3D = 0 \\ (4) & 3A + D = -10 \end{cases} \rightarrow A = -\frac{30}{10} = -3, D = -\frac{10}{10} = -1$$

$$\begin{cases} (2) & -B + 3E = 19 \\ (5) & 3B + E = 13 \end{cases} \rightarrow B = \frac{20}{10} = 2, E = 7$$

$$\begin{cases} (3) & -C + 3F = -38 \\ (6) & -3C - F = 16 \end{cases} \rightarrow C = -\frac{10}{10} = -1, F = -13$$

$$y(x) = e^{-x/2} \left(C_1 \cos \frac{\sqrt{31}}{2} x + C_2 \sin \frac{\sqrt{31}}{2} x \right) + (-3x^2 + 2x - 1) \cos 3x + (-x^2 + 7x - 13) \sin 3x$$

Exercise

Find the general solution: $y'' - y' - 12y = e^{4x}$

Solution

The characteristic equation: $\lambda^2 - \lambda - 12 = 0 \rightarrow \lambda_{1,2} = -3, 4$

$$y_h = C_1 e^{-3x} + C_2 e^{4x}$$

$$y_p = A x e^{4x}$$

$$y'_p = (4Ax + A)e^{4x}$$

$$y''_p = (16Ax + 8A)e^{4x}$$

$$y'' - y' - 12y = e^{4x}$$

$$(16Ax + 8A - 4Ax - A - 12Ax)e^{4x} = e^{4x}$$

$$7Ae^{4x} = e^{4x} \rightarrow A = \frac{1}{7}$$

$$y_p = \frac{1}{7} x e^{4x}$$

$$y(x) = C_1 e^{-3x} + \left(C_2 + \frac{1}{7}x\right)e^{4x}$$

Exercise

Find the general solution: $y'' + 2y' = 2x + 5 - e^{-2x}$

Solution

The characteristic equation: $\lambda^2 + 2\lambda = 0 \rightarrow \lambda_{1,2} = 0, -2$

$$y_h = C_1 + C_2 e^{-2x}$$

$$y_p = Ax^2 + Bx + Cx e^{-2x}$$

$$y'_p = 2Ax + B + (-2Cx + C)e^{-2x}$$

$$y''_p = 2A + (4Cx - 4C)e^{-2x}$$

$$y'' + 2y' = 2x + 5 - e^{-2x}$$

$$2A + 4Ax + 2B - 2Ce^{-2x} = 2x + 5 - e^{-2x}$$

$$4A = 2 \quad A = \frac{1}{2}$$

$$2A + 2B = 5 \quad B = \frac{3}{2}$$

$$-2C = -1 \quad C = \frac{1}{2}$$

$$y_p = \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}$$

$$\underline{y(x) = C_1 + C_2 e^{-2x} + \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}}$$

Exercise

Find the general solution: $y'' - 2y' = 12x - 10$

Solution

The characteristic equation: $\lambda^2 - 2\lambda = 0 \rightarrow \underline{\lambda_{1,2} = 0, 2}$

$$\underline{y_h = C_1 + C_2 e^{2x}}$$

$$y_p = Ax^2 + Bx$$

$$y'_p = 2Ax + B$$

$$y''_p = 2A$$

$$y'' - 2y' = 12x - 10$$

$$2A - 4Ax - 2B = 12x - 10$$

$$\begin{cases} -4A = 12 \\ 2A - 2B = -10 \end{cases} \rightarrow \underline{A = -3, B = 2}$$

$$\underline{y_p = -3x^2 + 2x}$$

$$\underline{y(x) = C_1 + C_2 e^{2x} - 3x^2 + 2x}$$

Exercise

Find the general solution: $y'' + 2y' + y = \sin x + 3\cos 2x$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 1 = 0 \rightarrow \underline{\lambda_{1,2} = -1}$

$$\underline{y_h = (C_1 + C_2 x)e^{-x}}$$

$$y_p = A \cos x + B \sin x + C \cos 2x + D \sin 2x$$

$$y'_p = -A \sin x + B \cos x - 2C \sin 2x + 2D \cos 2x$$

$$y''_p = -A \cos x - B \sin x - 4C \cos 2x - 4D \sin 2x$$

$$y'' + 2y' + y = \sin x + 3\cos 2x$$

$$\begin{cases} \cos x & -A + 2B + A = 0 \\ \sin x & -B - 2A + B = 1 \\ \cos 2x & -4C + 4D + C = 3 \\ \sin 2x & -4D - 4C + D = 0 \end{cases} \rightarrow \begin{cases} B = 0, A = -\frac{1}{2} \\ -3C + 4D = 3 \\ -4C - 3D = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} -3 & 4 \\ -4 & -3 \end{vmatrix} = 25 \quad \Delta_C = \begin{vmatrix} 3 & 4 \\ 0 & -3 \end{vmatrix} = -9 \quad \Delta_D = \begin{vmatrix} -3 & 3 \\ -4 & 0 \end{vmatrix} = 12$$

$$\underline{C = -\frac{9}{25} \quad D = \frac{12}{25}}$$

$$\underline{y_P = -\frac{1}{2}\cos x - \frac{9}{25}\cos 2x + \frac{12}{25}\sin 2x}$$

$$\underline{y(x) = (C_1 + C_2 x)e^{-x} - \frac{1}{2}\cos x - \frac{9}{25}\cos 2x + \frac{12}{25}\sin 2x}$$

Exercise

Find the general solution: $y'' - 2y' + y = 6e^x$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 1 = 0 \rightarrow \lambda_{1,2} = 1$

$$\underline{y_h = (C_1 + C_2 x)e^x}$$

$$y_P = Ax^2 e^x$$

$$y'_P = (2Ax + Ax^2)e^x$$

$$y''_P = (2A + 4Ax + Ax^2)e^x$$

$$y'' - 2y' + y = 6e^x$$

$$(2A + 4Ax + Ax^2 - 4Ax - 2Ax^2 + Ax^2)e^x = 6e^x$$

$$2Ae^x = 6e^x \rightarrow A = 3$$

$$\underline{y_P = 3x^2 e^x}$$

$$\underline{y(x) = (C_1 + C_2 x)e^x + 3x^2 e^x}$$

Exercise

Find the general solution: $y'' + 2y' + y = x^2$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 1 = 0 \rightarrow \underline{\lambda_{1,2} = -1}$

$$\underline{y_h = (C_1 + C_2 x)e^{-x}}$$

$$y_p = Ax^2 + Bx + C$$

$$y'_p = 2Ax + B$$

$$y''_p = 2A$$

$$y'' + 2y' + y = x^2$$

$$2A + 4Ax + 2B + Ax^2 + Bx + C = x^2$$

$$\textcolor{red}{x^2} \quad A = 1$$

$$\textcolor{red}{x^1} \quad 4A + B = 0 \rightarrow \underline{A = 1, B = -4, C = -2}$$

$$\textcolor{red}{x^0} \quad 2A + C = 0$$

$$\underline{y_p = x^2 - 4x - 2}$$

$$\underline{y(x) = (C_1 + C_2 x)e^{-x} + x^2 - 4x - 2}$$

Exercise

Find the general solution: $y'' + 2y' + y = x^2 e^{-x}$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 1 = 0 \rightarrow \underline{\lambda_{1,2} = -1}$

$$\underline{y_h = (C_1 + C_2 x)e^{-x}}$$

$$y_p = (Ax^4 + Bx^3 + Cx^2)e^{-x}$$

$$y'_p = (4Ax^3 + 3Bx^2 + 2Cx - Ax^4 - Bx^3 - Cx^2)e^{-x}$$

$$y''_p = (12Ax^2 + 6Bx + 2C - 8Ax^3 - 6Bx^2 - 4Cx + Ax^4 + Bx^3 + Cx^2)e^{-x}$$

$$y'' + 2y' + y = x^2 e^{-x}$$

$$\textcolor{red}{x^4} \quad A - 2A + A = 0$$

$$\textcolor{red}{x^3} \quad -8A + B + 8A - 2B + B = 0$$

$$\textcolor{red}{x^2} \quad 12A - 6B + C + 6B - 2C + C = 1 \quad A = \frac{1}{12}$$

$$\textcolor{red}{x} \quad 6B - 4C + 4C = 0 \quad B = 0$$

$$\textcolor{red}{x^0} \quad 2C = 0 \quad C = 0$$

$$\underline{y_p = \frac{1}{12}x^4 e^{-x}}$$

$$\underline{y(x) = (C_1 + C_2 x)e^{-x} + \frac{1}{12}x^4 e^{-x}}$$

Exercise

Find the general solution: $y'' - 2y' + y = x^3 + 4x$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 1 = 0 \rightarrow \underline{\lambda_{1,2} = 1}$

$$\underline{y_h = (C_1 + C_2 x)e^x}$$

$$y_p = Ax^3 + Bx^2 + Cx + D$$

$$y'_p = 3Ax^2 + 2Bx + C$$

$$y''_p = 6Ax + 2B$$

$$y'' - 2y' + y = x^3 + 4x$$

$$\textcolor{red}{x^3} \quad \underline{A = 1}$$

$$\textcolor{red}{x^2} \quad -6A + B = 0 \quad B = 6$$

$$\textcolor{red}{x} \quad 6A - 4B + C = 4 \quad C = 22$$

$$\textcolor{red}{x^0} \quad 2B - 2C + D = 0 \quad D = 32$$

$$\underline{y_p = x^3 + 6x^2 + 22x + 32}$$

$$\underline{y(x) = (C_1 + C_2 x)e^x + x^3 + 6x^2 + 22x + 32}$$

Exercise

Find the general solution: $y'' + 2y' + y = 6\sin 2x$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 1 = 0 \rightarrow \underline{\lambda_{1,2} = -1}$

$$\underline{y_h = (C_1 + C_2 x)e^{-x}}$$

$$y_p = A \cos 2x + B \sin 2x$$

$$y'_p = -2A \sin 2x + 2B \cos 2x$$

$$y''_p = -4A \cos 2x - 4B \sin 2x$$

$$y'' + 2y' + y = 6\sin 2x$$

$$-4A\cos 2x - 4B\sin 2x - 4A\sin 2x + 4B\cos 2x + A\cos 2x + B\sin 2x = 6\sin 2x$$

$$\begin{cases} \cos 2x & -3A + 4B = 0 \\ \sin 2x & -4A - 3B = 6 \end{cases}$$

$$\rightarrow \Delta = \begin{vmatrix} -3 & 4 \\ -4 & -3 \end{vmatrix} = 25 \quad \Delta_A = \begin{vmatrix} 0 & 4 \\ 6 & -3 \end{vmatrix} = -24 \quad \Delta_B = \begin{vmatrix} -3 & 0 \\ -4 & 6 \end{vmatrix} = -18$$

$$\underline{A = -\frac{24}{25}, B = -\frac{18}{25}}$$

$$\underline{y_p = -\frac{24}{25}\cos 2x - \frac{18}{25}\sin 2x}$$

$$\underline{y(x) = (C_1 + C_2 x)e^{-x} - \frac{24}{25}\cos 2x - \frac{18}{25}\sin 2x}$$

Exercise

Find the general solution: $y'' - 2y' + y = (x^2 - 1)e^{2x} + (3x + 4)e^x$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 1 = 0 \rightarrow \underline{\lambda_{1,2} = 1}$

$$\underline{y_h = (C_1 + C_2 x)e^x}$$

$$y_p = (Ax^2 + Bx + C)e^{2x} + (Dx^3 + Ex^2)e^x$$

$$y'_p = (2Ax + B + 2Ax^2 + 2Bx + 2C)e^{2x} + (3Dx^2 + 2Ex + Dx^3 + Ex^2)e^x$$

$$y''_p = (2A + 8Ax + 4B + 4Ax^2 + 4Bx + 4C)e^{2x} + (6Dx + 2E + 6Dx^2 + 4Ex + Dx^3 + Ex^2)e^x$$

$$y'' - 2y' + y = (x^2 - 1)e^{2x} + (3x + 4)e^x$$

$$\begin{cases} e^{2x} & x^2 & \underline{A=1} \\ & x & 4A + B = 0 \rightarrow \underline{B=-4} \\ & x^0 & 2A + 2B + C = -1 \rightarrow \underline{C=5} \end{cases}$$

$$\begin{cases} e^x & x^3 & D - 2D + D = 0 \\ & x^2 & 6D + E - 6D - 2E + E = 0 \\ & x & 6D + 4E - 4E = 3 \rightarrow \underline{D = \frac{1}{2}} \\ & x^0 & 2E = 4 \rightarrow \underline{E = 2} \end{cases}$$

$$\underline{y_p = \left(x^2 - 4x + 5\right)e^{2x} + \left(\frac{1}{2}x^3 + 2x^2\right)e^x}$$

$$\underline{y(x) = \left(C_1 + C_2 x\right)e^x + \left(x^2 - 4x + 5\right)e^{2x} + \left(\frac{1}{2}x^3 + 2x^2\right)e^x}$$

Exercise

Find the general solution: $y'' + 2y' + 2y = 5e^{6x}$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 2 = 0 \rightarrow \lambda_{1,2} = \frac{-2 \pm 2i}{2} = -1 \pm i$

$$\underline{y_h = \left(C_1 \cos x + C_2 \sin x\right)e^{-x}}$$

$$y_p = Ae^{6x}$$

$$y'_p = 6Ae^{6x}$$

$$y''_p = 36Ae^{6x}$$

$$y'' + 2y' + 2y = 5e^{6x}$$

$$3(6A + 12A + 2A)e^{6x} = 5e^{6x} \rightarrow 50A = 5 \Rightarrow A = \frac{1}{10}$$

$$\underline{y_p = \frac{1}{10}e^{6x}}$$

$$\underline{y(x) = \left(C_1 \cos x + C_2 \sin x\right)e^{-x} + \frac{1}{10}e^{6x}}$$

Exercise

Find the general solution: $y'' + 2y' + 2y = x^3$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 2 = 0 \rightarrow \lambda_{1,2} = \frac{-2 \pm 2i}{2} = -1 \pm i$

$$\underline{y_h = \left(C_1 \cos x + C_2 \sin x\right)e^{-x}}$$

$$y_p = Ax^3 + Bx^2 + Cx + D$$

$$y'_p = 3Ax^2 + 2Bx + C$$

$$y''_p = 6Ax + 2B$$

$$y'' + 2y' + 2y = x^3$$

$$6Ax + 2B + 6Ax^2 + 4Bx + 2C + 2Ax^3 + 2Bx^2 + 2Cx + 2D = x^3$$

$$x^3 \quad 2A = 1 \quad \rightarrow \underline{A = \frac{1}{2}}$$

$$x^2 \quad 6A + 2B = 0 \quad \rightarrow \underline{B = -\frac{3}{2}}$$

$$x \quad 6A + 4B + 2C = 0 \quad \rightarrow \underline{C = \frac{3}{2}}$$

$$x^0 \quad 2B + 2C + 2D = 0 \quad \rightarrow \underline{D = 0}$$

$$y_p = \frac{1}{2}x^3 - \frac{3}{2}x^2 + \frac{3}{2}x$$

$$y(x) = \left(C_1 \cos x + C_2 \sin x \right) e^{-x} + \frac{1}{2}x^3 - \frac{3}{2}x^2 + \frac{3}{2}x$$

Exercise

Find the general solution: $y'' + 2y' + 2y = \cos x + e^{-x}$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 2 = 0 \rightarrow \lambda_{1,2} = \frac{-2 \pm 2i}{2} = -1 \pm i$

$$y_h = \left(C_1 \cos x + C_2 \sin x \right) e^{-x}$$

$$y_p = A \cos x + B \sin x + C e^{-x}$$

$$y'_p = -A \sin x + B \cos x - C e^{-x}$$

$$y''_p = -A \cos x - B \sin x + C e^{-x}$$

$$y'' + 2y' + 2y = \cos x + e^{-x}$$

$$-A \cos x - B \sin x + C e^{-x} - 2A \sin x + 2B \cos x - 2C e^{-x} + 2A \cos x + 2B \sin x + 2C e^{-x} = \cos x + e^{-x}$$

$$\cos x \quad A + 2B = 0$$

$$\sin x \quad -2A + B = 1$$

$$e^{-x} \quad \underline{C = 1}$$

$$\Delta = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 5 \quad \Delta_A = \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} = -2 \quad \Delta_B = \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} = 1$$

$$y_p = -\frac{2}{5} \cos x + \frac{1}{5} \sin x + e^{-x}$$

$$y(x) = \left(C_1 \cos x + C_2 \sin x + 1 \right) e^{-x} - \frac{2}{5} \cos x + \frac{1}{5} \sin x$$

Exercise

Find the general solution: $y'' - 2y' + 2y = e^x \sin x$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 2 = 0 \rightarrow \lambda_{1,2} = \frac{2 \pm 2i}{2} = 1 \pm i$

$$\underline{y_h = e^x (C_1 \cos x + C_2 \sin x)}$$

$$y_p = xe^x (A \cos x + B \sin x)$$

$$\begin{aligned} y'_p &= (1+x)e^x (A \cos x + B \sin x) + xe^x (-A \sin x + B \cos x) \\ &= \left((A + (A+B)x) \cos x + (B + (B-A)x) \sin x \right) e^x \end{aligned}$$

$$\begin{aligned} y''_p &= \left(\begin{aligned} &(A + (A+B)x) \cos x + (B + (B-A)x) \sin x + (A+B) \cos x - (A + (A+B)x) \sin x \\ &+ (B-A) \sin x + (B + (B-A)x) \cos x \end{aligned} \right) e^x \\ &= \left((2A + 2B + 2Bx) \cos x + (2B - 2A - 2Ax) \sin x \right) e^x \end{aligned}$$

$$y'' - 2y' + 2y = e^x \sin x$$

$$\begin{aligned} e^x \begin{cases} \cos x & 2A + 2B - 2A + (2B - 2A - 2B + 2A)x \\ \sin x & 2B - 2A - 2B + (-2A - 2B + 2A + 2B)x \end{cases} &= e^x \sin x \\ \begin{cases} 2B = 0 \\ -2A = 1 \end{cases} &\rightarrow \underline{A = -\frac{1}{2}, B = 0} \end{aligned}$$

$$\underline{y(x) = e^x (C_1 \cos x + C_2 \sin x) - \frac{1}{2} x e^x \cos x}$$

Exercise

Find the general solution: $y'' - 2y' + 2y = e^{2x} (\cos x - 3 \sin x)$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 2 = 0 \rightarrow \lambda_{1,2} = \frac{2 \pm 2i}{2} = 1 \pm i$

$$\underline{y_h = e^x (C_1 \cos x + C_2 \sin x)}$$

$$y_p = e^{2x} (A \cos x + B \sin x)$$

$$\begin{aligned} y'_p &= e^{2x} (2A \cos x + 2B \sin x - A \sin x + B \cos x) \\ &= e^{2x} ((2A + B) \cos x + (2B - A) \sin x) \end{aligned}$$

$$\begin{aligned} y''_p &= e^{2x} ((4A + 2B) \cos x + (4B - 2A) \sin x - (2A + B) \sin x + (2B - A) \cos x) \\ &= e^{2x} ((3A + 4B) \cos x + (3B - 4A) \sin x) \end{aligned}$$

$$y'' - 2y' + 2y = e^{2x} (\cos x - 3 \sin x)$$

$$e^{2x} \begin{cases} \cos x & 3A + 4B - 4A - 2B + 2A = 1 \\ \sin x & 3B - 4A - 4B + 2A + 2B = -3 \end{cases}$$

$$\rightarrow \left\{ \begin{array}{l} A + 2B = 1 \\ -2A + B = -3 \end{array} \Rightarrow A = \frac{7}{5}, B = -\frac{1}{5} \right|$$

$$\underline{y(x) = e^x \left(C_1 \cos x + C_2 \sin x \right) + e^{2x} \left(\frac{7}{5} \cos x + \frac{1}{5} \sin x \right)}$$

Exercise

Find the general solution: $y'' - 2y' - 3y = 1 - x^2$

Solution

The characteristic equation: $\lambda^2 - 2\lambda - 3 = 0 \rightarrow \underline{\lambda_{1,2} = -1, 3}$

$$\underline{y_h = C_1 e^{-x} + C_2 e^{3x}}$$

$$y_p = Ax^2 + Bx + C$$

$$y'_p = 2Ax + B$$

$$y''_p = 2A$$

$$y'' - 2y' - 3y = 1 - x^2$$

$$2A - 4Ax - 2B - 3Ax^2 - 3Bx - 3C = 1 - x^2$$

$$\textcolor{red}{x^2} \quad -3A = -1$$

$$\textcolor{red}{x^1} \quad -4A - 3B = 0 \rightarrow \underline{A = \frac{1}{3}, B = -\frac{4}{9}, C = \frac{5}{9}}$$

$$\textcolor{red}{x^0} \quad 2A - 2B - 3C = 1$$

$$\underline{y(x) = C_1 e^{-x} + C_2 e^{3x} + \frac{1}{3}x^2 - \frac{4}{9}x + \frac{5}{9}}$$

Exercise

Find the general solution: $y'' - 2y' - 3y = 4e^x - 9$

Solution

The characteristic equation: $\lambda^2 - 2\lambda - 3 = 0 \rightarrow \underline{\lambda_{1,2} = -1, 3}$

$$\underline{y_h = C_1 e^{-x} + C_2 e^{3x}}$$

$$y_p = Ae^x + B$$

$$y'_p = Ae^x$$

$$y''_p = Ae^x$$

$$\textcolor{red}{y'' - 2y' - 3y = 4e^x - 9}$$

$$(A - 2A - 3A)e^x - 3B = 4e^x - 9$$

$$\begin{cases} -4A = 4 \\ -3B = 9 \end{cases} \rightarrow \underline{A = -1, B = 3}$$

$$\underline{y(x) = C_1 e^{-x} + C_2 e^{3x} - e^x + 3}$$

Exercise

Find the general solution: $y'' - 2y' - 3y = 2e^{-x} \cos x + x^2 + xe^{3x}$

Solution

The characteristic equation: $\lambda^2 - 2\lambda - 3 = 0 \rightarrow \underline{\lambda_{1,2} = -1, 3}$

$$\underline{y_h = C_1 e^{-x} + C_2 e^{3x}}$$

$$y_p = e^{-x}(A \cos x + B \sin x) + Cx^2 + Dx + E + (Fx^3 + Gx^2 + Hx)e^{3x}$$

$$\triangleright y_{p1} = e^{-x}(A \cos x + B \sin x)$$

$$\begin{aligned} y'_{p1} &= e^{-x}(-A \cos x - B \sin x - A \sin x + B \cos x) \\ &= e^{-x}((B - A) \cos x - (A + B) \sin x) \end{aligned}$$

$$\begin{aligned} y''_{p2} &= e^{-x}(-(B - A) \cos x + (A + B) \sin x - (B - A) \sin x - (A + B) \cos x) \\ &= e^{-x}(-2B \cos x + 2A \sin x) \end{aligned}$$

$$y'' - 2y' - 3y = 2e^{-x} \cos x + x^2 + xe^{3x}$$

$$e^{-x}(-2B \cos x + 2A \sin x - 2(B - A) \cos x + 2(A + B) \sin x - 3A \cos x - 3B \sin x) = 2e^{-x} \cos x$$

$$e^{-x}((-A - 4B) \cos x + (4A - B) \sin x) = 2e^{-x} \cos x$$

$$\begin{cases} -A - 4B = 2 \\ 4A - B = 0 \end{cases} \quad \Delta = \begin{vmatrix} -1 & -4 \\ 4 & -1 \end{vmatrix} = 17 \quad \Delta_A = \begin{vmatrix} 2 & -4 \\ 0 & -1 \end{vmatrix} = -2 \quad \Delta_B = \begin{vmatrix} -1 & 2 \\ 4 & 0 \end{vmatrix} = -8$$

$$\underline{A = -\frac{2}{17}, \quad B = -\frac{8}{17}}$$

$$\triangleright y_{p2} = Cx^2 + Dx + E$$

$$y'_{p2} = 2Cx + D$$

$$y''_{p2} = 2C$$

$$y'' - 2y' - 3y = 2e^{-x} \cos x + x^2 + xe^{3x}$$

$$2C - 4Cx - 2D - 3Cx^2 - 3Dx - 3E = x^2$$

$$x^2 \quad -3C = 1 \quad \rightarrow C = -\frac{1}{3}$$

$$x \quad \frac{4}{3} - 3D = 0 \quad \rightarrow D = \frac{4}{9}$$

$$x^0 \quad -\frac{2}{3} - \frac{8}{9} - 3E = 0 \quad \rightarrow E = -\frac{14}{27}$$

$$\triangleright y_{p3} = (Fx^3 + Gx^2 + Hx)e^{3x}$$

$$\begin{aligned} y'_{p3} &= (3Fx^2 + 2Gx + H + 3Fx^3 + 3Gx^2 + 3Hx)e^{3x} \\ &= (3Fx^3 + (3F + 3G)x^2 + (2G + 3H)x + H)e^{3x} \end{aligned}$$

$$\begin{aligned} y''_{p3} &= (9Fx^2 + (6F + 6G)x + 2G + 3H + 9Fx^3 + (9F + 9G)x^2 + (6G + 9H)x + 3H)e^{3x} \\ &= (9Fx^3 + (18F + 9G)x^2 + (6F + 12G + 9H)x + 2G + 6H)e^{3x} \end{aligned}$$

$$9Fx^3 + (18F + 9G)x^2 + (6F + 12G + 9H)x + 2G + 6H - 6Fx^3 - (6F + 6G)x^2 - (4G + 6H)x - 2H - 3Fx^3 - 3Gx^2 - 3Hx$$

$$(12Fx^2 + (6F + 8G)x + 2G + 4H)e^{3x}$$

$$y'' - 2y' - 3y = 2e^{-x} \cos x + x^2 + xe^{3x}$$

$$x^2 \quad 12F = 0 \quad \rightarrow F = 0$$

$$x \quad 6F + 8G = 1 \quad \rightarrow G = \frac{1}{8}$$

$$x^0 \quad 2G + 4H = 0 \quad \rightarrow H = -\frac{1}{16}$$

$$y_p = -\left(\frac{2}{17} \cos x + \frac{8}{17} \sin x\right)e^{-x} - \frac{1}{3}x^2 + \frac{4}{9}x - \frac{14}{27} + \left(\frac{1}{8}x^2 - \frac{1}{16}x\right)e^{3x}$$

$$y(x) = C_1 e^{-x} + C_2 e^{3x} - \left(\frac{2}{17} \cos x + \frac{8}{17} \sin x\right)e^{-x} - \frac{1}{3}x^2 + \frac{4}{9}x - \frac{14}{27} + \left(\frac{1}{8}x^2 - \frac{1}{16}x\right)e^{3x}$$

Exercise

Find the general solution: $y'' - 2y' + 5y = 25x^2 + 12$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 5 = 0 \rightarrow \lambda_{1,2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$

$$y_h = e^x (C_1 \cos 2x + C_2 \sin 2x)$$

$$y_p = Ax^2 + Bx + C$$

$$y'_p = 2Ax + B$$

$$y''_p = 2A$$

$$y'' - 2y' + 5y = 25x^2 + 12$$

$$x^2 \quad 5A = 25$$

$$x^1 \quad -4A + 5B = 0 \quad \rightarrow \quad \underline{A = 5, B = 4, C = -10}$$

$$x^0 \quad 2A - 2B - 3C = 12$$

$$\underline{y(x) = e^x \left(C_1 \cos 2x + C_2 \sin 2x \right) + 5x^2 + 4x - 10}$$

Exercise

Find the general solution: $y'' - 2y' + 5y = e^x \cos 2x$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 5 = 0 \rightarrow \lambda_{1,2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$

$$\underline{y_h = e^x \left(C_1 \cos 2x + C_2 \sin 2x \right)}$$

$$y_p = e^x (Ax \cos 2x + Bx \sin 2x)$$

$$\begin{aligned} y_p' &= e^x (Ax \cos 2x + Bx \sin 2x + A \cos 2x - 2Ax \sin 2x + B \sin 2x + 2Bx \cos 2x) \\ &= e^x ((Ax + 2Bx + A) \cos 2x + (Bx - 2Ax + B) \sin 2x) \end{aligned}$$

$$\begin{aligned} y_p'' &= e^x \left((Ax + 2Bx + A) \cos 2x + (Bx - 2Ax + B) \sin 2x + (A + 2B) \cos 2x \right. \\ &\quad \left. - (2Ax + 4Bx + 2A) \sin 2x + (B - 2A) \sin 2x + (2Bx - 4Ax + 2B) \cos 2x \right) \\ &= e^x ((-3Ax + 4Bx + 2A + 4B) \cos 2x + (-4Ax - 3Bx - 4A + 2B) \sin 2x) \end{aligned}$$

$$y'' - 2y' + 5y = e^x \cos 2x$$

$$\cos 2x \quad -3Ax + 4Bx + 2A + 4B - 2Ax - 4Bx - 2A + 5Ax$$

$$\sin 2x \quad -4Ax - 3Bx - 4A + 2B - 2Bx + 4Ax - 2B + 5Bx$$

$$\rightarrow \begin{cases} 4B = 1 \\ -4A = 0 \end{cases} \Rightarrow \underline{\underline{B = \frac{1}{4}}} \\ \underline{\underline{A = 0}}$$

$$\underline{y(x) = e^x \left(C_1 \cos 2x + \left(C_2 + \frac{1}{4}x \right) \sin 2x \right)}$$

Exercise

Find the general solution: $y'' - 2y' + 5y = e^x \sin x$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 5 = 0 \rightarrow \lambda_{1,2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$

$$\underline{y_h = e^x \left(C_1 \cos 2x + C_2 \sin 2x \right)}$$

$$y_p = e^x (A \cos x + B \sin x)$$

$$\begin{aligned} y'_p &= e^x (A \cos x + B \sin x - A \sin x + B \cos x) \\ &= e^x ((A+B) \cos x + (B-A) \sin x) \end{aligned}$$

$$\begin{aligned} y''_p &= e^x ((A+B) \cos x + (B-A) \sin x - (A+B) \sin x + (B-A) \cos x) \\ &= e^x (2B \cos x - 2A \sin x) \end{aligned}$$

$$y'' - 2y' + 5y = e^x \sin x$$

$$\begin{array}{l} \cos x \quad 2B - 2A - 2B + 5A = 0 \\ \sin x \quad -2A - 2B + 2A + 5B = 1 \end{array} \rightarrow \begin{cases} A = 0 \\ B = \frac{1}{3} \end{cases}$$

$$\underline{y(x) = e^x \left(C_1 \cos 2x + C_2 \sin 2x + \frac{1}{3} \sin x \right)}$$

Exercise

Find the general solution: $y'' + 2y' - 24y = 16 - (x+2)e^{4x}$

Solution

The characteristic equation: $\lambda^2 + 2\lambda - 24 = 0 \rightarrow \lambda_{1,2} = \frac{-2 \pm 10}{2} = -6, 4$

$$\underline{y_h = C_1 e^{-6x} + C_2 e^{4x}}$$

$$y_p = A + (Bx^2 + Cx)e^{4x}$$

$$y'_p = (2Bx + C + 4Bx^2 + 4Cx)e^{4x}$$

$$y''_p = (2B + 8C + 16Bx + 16Bx^2 + 16Cx)e^{4x}$$

$$y'' + 2y' - 24y = 16 - (x+2)e^{4x}$$

$$\begin{array}{l} -24A = 16 \\ A = -\frac{2}{3} \end{array}$$

$$\begin{array}{l} x^2 \quad 16B + 8B - 24B = 0 \end{array}$$

$$\begin{array}{l} x \quad 16B + 16C + 8C + 4B - 24C = -1 \\ B = -\frac{1}{20} \end{array}$$

$$\begin{array}{l} x^0 \quad 2B + 8C + 2C = -2 \\ C = -\frac{19}{100} \end{array}$$

$$y_p = -\frac{2}{3} + \left(-\frac{1}{20}x^2 - \frac{19}{100}x \right) e^{4x}$$

$$\underline{y(x) = C_1 e^{-6x} + C_2 e^{4x} - \frac{2}{3} + \left(-\frac{1}{20}x^2 - \frac{19}{100}x \right) e^{4x}}$$

Exercise

Find the general solution: $y'' + 3y = -48x^2 e^{3x}$

Solution

The characteristic equation: $\lambda^2 + 3 = 0 \rightarrow \lambda_{1,2} = \pm i\sqrt{3}$

$$y_h = C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x$$

$$y_p = (Ax^2 + Bx + C)e^{3x}$$

$$y'_p = (2Ax + B + 3Ax^2 + 3Bx + 3C)e^{3x}$$

$$y''_p = (2A + 6Ax + 3B + 6Ax + 3B + 9Ax^2 + 9Bx + 9C)e^{3x}$$

$$y'' + 3y = -48x^2 e^{3x}$$

$$x^2 \quad 9A + 3A = -48 \quad A = -4$$

$$x \quad 12A + 9B + 3B = 0 \quad B = 4$$

$$x^0 \quad 2A + 6B + 9C + 3C = 0 \quad C = -\frac{4}{3}$$

$$y_p = \left(-4x^2 + 4x - \frac{4}{3}\right)e^{3x}$$

$$y(x) = C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x + \left(-4x^2 + 4x - \frac{4}{3}\right)e^{3x}$$

Exercise

Find the general solution: $y'' - 3y' = e^{3x} - 12x$

Solution

The characteristic equation: $\lambda^2 - 3\lambda = 0 \rightarrow \lambda_{1,2} = 0, 3$

$$y_h = C_1 + C_2 e^{3x}$$

$$y_p = Axe^{3x} + Bx^2 + Cx$$

$$y'_p = (A + 3Ax)e^{3x} + 2Bx + C$$

$$y''_p = (6A + 9Ax)e^{3x} + 2B$$

$$y'' - 3y' = e^{3x} - 12x$$

$$e^{3x} \quad 6A + 9Ax - 3A - 9Ax = 1 \quad A = \frac{1}{3}$$

$$x \quad -6B = -12 \quad B = 2$$

$$x^0 \quad 2B - 3C = 0 \quad C = \frac{4}{3}$$

$$\underline{y_p = \frac{1}{3}xe^{3x} + 2x^2 + \frac{4}{3}x}$$

$$\underline{y(x) = C_1 + C_2e^{3x} + \frac{1}{3}xe^{3x} + 2x^2 + \frac{4}{3}x}$$

Exercise

Find the general solution: $y'' + 3y' = 4x - 5$

Solution

The characteristic equation: $\lambda^2 + 3\lambda = 0 \rightarrow \underline{\lambda_{1,2} = 0, -3}$

$$\underline{y_h = C_1 + C_2e^{-3x}}$$

$$y_p = Ax^2 + Bx$$

$$y'_p = 2Ax + B$$

$$y''_p = 2A$$

$$y'' + 3y' = 4x - 5$$

$$\textcolor{red}{x} \quad 6A = 4 \quad \underline{A = \frac{2}{3}}$$

$$\textcolor{red}{x^0} \quad 2A + 3B = -5 \quad \underline{B = -\frac{19}{9}}$$

$$\underline{y_p = \frac{2}{3}x^2 - \frac{19}{9}x}$$

$$\underline{y(x) = C_1 + C_2e^{-3x} + \frac{2}{3}x^2 - \frac{19}{9}x}$$

Exercise

Find the general solution: $y'' - 3y' = 8e^{3x} + 4\sin x$

Solution

The characteristic equation: $\lambda^2 - 3\lambda = 0 \Rightarrow \underline{\lambda_{1,2} = 0, 3}$

$$\underline{y_h = C_1 + C_2e^{3x}}$$

The particular equation: $y_p = Ae^{3x} + B\cos x + C\sin x$

$$y'_p = 3Ae^{3x} - B\sin x + C\cos x$$

$$y''_p = 9Ae^{3x} - B\cos x - C\sin x$$

$$y'' - 3y' = 8e^{3x} + 4\sin x$$

$$\textcolor{red}{9Ae^{3x}} - B\cos x - C\sin x - \textcolor{red}{9Ae^{3x}} + 3B\sin x - 3C\cos x = 8e^{3x} + 4\sin x$$

The particular equation: $y_p = Axe^{3x} + B\cos x + C\sin x$

$$y'_p = 3Axe^{3x} + Ae^{3x} - B\sin x + C\cos x$$

$$y''_p = 3Ae^{3x} + 9Axe^{3x} + 3Ae^{3x} - B\cos x - C\sin x$$

$$6Ae^{3x} + 9Axe^{3x} - B\cos x - C\sin x - 9Axe^{3x} - 3Ae^{3x} + 3B\sin x - 3C\cos x = 8e^{3x} + 4\sin x$$

$$3Ae^{3x} - (B + 3C)\cos x + (3B - C)\sin x = 8e^{3x} + 4\sin x$$

$$\begin{cases} 3A = 8 & \rightarrow A = \frac{8}{3} \\ -B - 3C = 0 & B = \frac{6}{5} \\ 3B - C = 4 & C = -\frac{2}{5} \end{cases}$$

$$\underline{y(x) = C_1 + C_2e^{3x} + \frac{8}{3}xe^{3x} + \frac{6}{5}\cos x - \frac{2}{5}\sin x}$$

Exercise

Find the general solution: $y'' + 3y' + 2y = 6$

Solution

The characteristic equation: $\lambda^2 + 3\lambda + 2 = 0 \Rightarrow \underline{\lambda_{1,2} = -1, -2}$

$$\underline{y_h = C_1e^{-x} + C_2e^{-2x}}$$

$$y_p = A$$

$$y'_p = 0 = y''_p$$

$$y'' + 3y' + 2y = 6$$

$$2A = 6 \rightarrow \underline{A = 3}$$

$$\underline{y(x) = C_1e^{-x} + C_2e^{-2x} + 3}$$

Exercise

Find the general solution: $y'' + 3y' + 2y = 4x^2$

Solution

The characteristic equation: $\lambda^2 + 3\lambda + 2 = 0 \Rightarrow \underline{\lambda_{1,2} = -1, -2}$

$$\underline{y_h = C_1e^{-x} + C_2e^{-2x}}$$

The particular equation: $y_p = ax^2 + bx + c$

$$y'_p = 2ax + b$$

$$y''_p = 2a$$

$$y''_p + 3y'_p + 2y_p = 4x^2$$

$$2a + 6ax + 3b + 2ax^2 + 2bx + 2c = 4x^2$$

$$\begin{cases} 2a = 4 & \rightarrow a = 2 \\ 6a + 2b = 0 & \rightarrow b = -6 \\ 2a + 3b + 2c = 0 & \rightarrow c = 7 \end{cases}$$

$$\underline{y(x) = C_1 e^{-x} + C_2 e^{-2x} + 2x^2 - 6x + 7}$$

Exercise

Find the general solution: $y'' - 3y' + 2y = 5e^x$

Solution

The characteristic equation: $\lambda^2 - 3\lambda + 2 = 0 \rightarrow \lambda_{1,2} = 1, 2$

$$\underline{y_h = C_1 e^x + C_2 e^{2x}}$$

$$y_p = Axe^x$$

$$y'_p = (A + Ax)e^x$$

$$y''_p = (2A + Ax)e^x$$

$$y'' - 3y' + 2y = 5e^x$$

$$(2A + Ax - 3A - 3Ax + 2Ax)e^x = 5e^x$$

$$-Ae^x = 5e^x \rightarrow \underline{A = -5}$$

$$\underline{y_p = -5xe^x}$$

$$\underline{y(x) = C_1 e^x + C_2 e^{2x} - 5xe^x}$$

Exercise

Find the general solution: $y'' - 3y' + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$

Solution

The characteristic equation: $\lambda^2 - 3\lambda + 2 = 0 \rightarrow \lambda_{1,2} = 1, 2$

$$\underline{y_h = C_1 e^x + C_2 e^{2x}}$$

$$y_p = A_1 x^2 + A_2 x + A_3 + (A_4 x + A_5 x^2) e^x + A_6 e^{3x}$$

$$y'_p = 2A_1 x + A_2 + (A_4 + (2A_5 + A_4)x + A_5 x^2) e^x + 3A_6 e^{3x}$$

$$y''_p = 2A_1 + (2A_5 + 2A_4 + (4A_5 + A_4)x + A_5 x^2) e^x + 9A_6 e^{3x}$$

$$y'' - 3y' + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$$

$$\begin{cases} x^2 & 2A_1 = 2 & \rightarrow A_1 = 1 \\ x & -6A_1 + 2A_2 = 0 & \rightarrow A_2 = 3 \\ x^0 & 2A_1 - 3A_2 + 2A_3 = 0 & \rightarrow A_3 = \frac{7}{2} \end{cases}$$

$$x^2 e^x \quad A_5 - 3A_5 + 2A_5$$

$$x e^x \quad 4A_5 + A_4 - 6A_5 - 3A_4 + 2A_4 = 2 \rightarrow A_5 = -1$$

$$e^x \quad 2A_5 + 2A_4 - 3A_4 = 1 \rightarrow A_4 = -3$$

$$e^{3x} \quad 9A_6 - 9A_6 + 2A_6 = 4 \rightarrow A_6 = 2$$

$$y_p = x^2 + 3x + \frac{7}{2} - (x^2 + 3x) e^x + 2e^{3x}$$

$$y(x) = C_2 e^{2x} + x^2 + 3x + \frac{7}{2} - (x^2 + 3x + C_1) e^x + 2e^{3x}$$

Exercise

Find the general solution: $y'' - 3y' + 2y = 14\sin 2x - 18\cos 2x$

Solution

The characteristic equation: $\lambda^2 - 3\lambda + 2 = 0 \rightarrow \lambda_{1,2} = 1, 2$

$$y_h = C_1 e^x + C_2 e^{2x}$$

$$y_p = A \cos 2x + B \sin 2x$$

$$y'_p = -2A \sin 2x + 2B \cos 2x$$

$$y''_p = -4A \cos 2x - 4B \sin 2x$$

$$y'' - 3y' + 2y = 14\sin 2x - 18\cos 2x$$

$$-4A \cos 2x - 4B \sin 2x + 6A \sin 2x - 6B \cos 2x + 2A \cos 2x + 2B \sin 2x = 14\sin 2x - 18\cos 2x$$

$$(-2A - 6B) \cos 2x + (-2B + 6A) \sin 2x = 14\sin 2x - 18\cos 2x$$

$$\begin{cases} -2A - 6B = -18 \\ 6A - 2B = 14 \end{cases} \Rightarrow \begin{cases} A + 3B = 9 \\ 3A - B = 7 \end{cases} \rightarrow \underline{A = 3, B = 2}$$

$$\underline{y(x) = C_1 e^x + C_2 e^{2x} + 3\cos 2x + 2\sin 2x}$$

Exercise

Find the general solution: $y'' + 3y' + 2y = e^{-x} + e^{-2x} - x$

Solution

The characteristic equation: $\lambda^2 + 3\lambda + 2 = 0 \rightarrow \underline{\lambda_{1,2} = -2, -1}$

$$\underline{y_h = C_1 e^{-x} + C_2 e^{-2x}}$$

$$y_p = Ax e^{-x} + Bx e^{-2x} + Cx + D$$

$$y'_p = (A - Ax)e^{-x} + (B - 2Bx)e^{-2x} + C$$

$$y''_p = (-2A + Ax)e^{-x} + (-4B + 4Bx)e^{-2x}$$

$$y'' + 3y' + 2y = e^{-x} + e^{-2x} - x$$

$$Ae^{-x} - Be^{-2x} + 3C + 2Cx + 2D = e^{-x} + e^{-2x} - x$$

$$\begin{cases} e^{-x} & A = 1 \\ e^{-2x} & -B = 1 \\ x & 2C = -1 \\ x^0 & 3C + 2D = 0 \end{cases} \rightarrow \underline{B = -1, C = -\frac{1}{2}, D = -\frac{3}{4}}$$

$$\underline{y_p = x e^{-x} - x e^{-2x} - \frac{1}{2}x - \frac{3}{4}}$$

$$\underline{y(x) = (C_1 + x)e^{-x} + (C_2 - x)e^{-2x} - \frac{1}{2}x - \frac{3}{4}}$$

Exercise

Find the general solution: $y'' - 3y' - 10y = -3$

Solution

The characteristic equation: $\lambda^2 - 3\lambda - 10 = 0 \rightarrow \underline{\lambda_{1,2} = -2, 5}$

$$\underline{y_h = C_1 e^x + C_2 e^{2x}}$$

$$y_p = A$$

$$y'' - 3y' - 10y = -3$$

$$-10A = -3 \rightarrow \underline{A = \frac{3}{10}}$$

$$\underline{y(x) = C_1 e^x + C_2 e^{2x} + \frac{3}{10}}$$

Exercise

Find the general solution: $y'' - 3y' - 10y = 2x - 3$

Solution

The characteristic equation: $\lambda^2 - 3\lambda - 10 = 0 \rightarrow \underline{\lambda_{1,2} = -2, 5}$

$$\underline{y_h = C_1 e^x + C_2 e^{2x}}$$

$$y_p = Ax + B$$

$$y'_p = A$$

$$y''_p = 0$$

$$y'' - 3y' - 10y = 2x - 3$$

$$-3A - 10Ax - 10B = 2x - 3$$

$$\begin{cases} -10A = 2 \\ -3A - 10B = -3 \end{cases} \rightarrow \underline{A = -\frac{1}{5}, B = \frac{4}{25}}$$

$$\underline{y(x) = C_1 e^x + C_2 e^{2x} - \frac{1}{5}x + \frac{4}{25}}$$

Exercise

Find the general solution: $y'' + 3y' - 10y = 6e^{4x}$

Solution

The characteristic equation: $\lambda^2 + 3\lambda - 10 = 0 \rightarrow \underline{\lambda_{1,2} = -5, 2}$

$$\underline{y_h = C_1 e^{-5x} + C_2 e^{2x}}$$

$$y_p = Ae^{4x}$$

$$y'_p = 4Ae^{4x}$$

$$y''_p = 16Ae^{4x}$$

$$y'' + 3y' - 10y = 6e^{4x}$$

$$(16A + 12A - 10A)e^{4x} = 6e^{4x}$$

$$8Ae^{4x} = 6e^{4x} \rightarrow \underline{A = \frac{3}{4}}$$

$$\underline{y(x) = C_1 e^x + C_2 e^{2x} + \frac{3}{4} e^{4x}}$$

Exercise

Find the general solution: $y'' + 3y' - 10y = x(e^x + 1)$

Solution

The characteristic equation: $\lambda^2 + 3\lambda - 10 = 0 \rightarrow \underline{\lambda_{1,2} = -5, 2}$

$$\underline{y_h = C_1 e^{-5x} + C_2 e^{2x}}$$

$$y_p = (Ax + B)e^x + Cx + D$$

$$y'_p = (Ax + A + B)e^x + C$$

$$y''_p = (Ax + 2A + B)e^x$$

$$y'' + 3y' - 10y = xe^x + x$$

$$(Ax + 2A + B + 3Ax + 3A + 3B - 10Ax - 10B)e^x + 3B - 10Bx - 10C = xe^x + x$$

$$e^x \begin{cases} -6A = 1 & \underline{A = -\frac{1}{6}} \\ 5A - 6B = 0 & \underline{B = -\frac{5}{36}} \end{cases}$$

$$\begin{matrix} x & -10C = 1 \\ x^0 & 3C + D = 0 \end{matrix} \rightarrow \underline{C = -\frac{1}{10}, D = \frac{3}{10}}$$

$$\underline{y(x) = C_1 e^x + C_2 e^{2x} - \left(\frac{1}{6}x + \frac{5}{36}\right)e^x - \frac{1}{10}x + \frac{3}{10}}$$

Exercise

Find the general solution $y'' + 4y = 3x^3$

Solution

The characteristic equation: $\lambda^2 + 4 = 0 \rightarrow \lambda = \pm 2i$

$$\underline{y_h = A \cos 2t + B \sin 2t}$$

$$y_p = Cx^3 + Dx^2 + Ex + F$$

$$y'_p = 3Cx^2 + 2Dx + E$$

$$y''_p = 6Cx + 2D$$

$$6Cx + 2D + 4Cx^3 + 4Dx^2 + 4Ex + 4F = 3x^3$$

$$x^3 \quad 4C = 3 \quad \rightarrow C = \frac{3}{4}$$

$$x^2 \quad 4D = 0 \quad \rightarrow D = 0$$

$$x \quad 6C + 4E = 0 \quad \rightarrow E = -\frac{9}{8}$$

$$x^0 \quad 2D + 4F = 0 \quad \rightarrow F = 0$$

$$\rightarrow y_p = \frac{3}{4}x^3 - \frac{9}{8}x$$

$$y(x) = A \cos 2t + B \sin 2t + \frac{3}{4}x^3 - \frac{9}{8}x$$

Exercise

Find the general solution: $y'' + 4y = 3 \sin x$

Solution

The characteristic equation: $\lambda^2 + 4 = 0 \rightarrow \lambda_{1,2} = \pm 2i$

$$y_h = C_1 \cos 2x + C_2 \sin 2x$$

$$y_p = A \cos x + B \sin x$$

$$y'_p = -A \sin x + B \cos x$$

$$y''_p = -A \cos x - B \sin x$$

$$y'' + 4y = 3 \sin x$$

$$-A \cos x - B \sin x + 4A \cos x + 4B \sin x = 3 \sin x$$

$$\begin{cases} 3A = 0 & \rightarrow A = 0 \\ 3B = 3 & \rightarrow B = 1 \end{cases}$$

$$y(x) = C_1 \cos 2x + C_2 \sin 2x + \sin x$$

Exercise

Find the general solution: $y'' + 4y = 3 \sin 2x$

Solution

The characteristic equation: $\lambda^2 + 4 = 0 \rightarrow \lambda_{1,2} = \pm 2i$

$$y_h = C_1 \cos 2x + C_2 \sin 2x$$

$$y_p = Ax \cos 2x + Bx \sin 2x$$

$$y'_p = A \cos 2x - 2Ax \sin 2x + B \sin 2x + 2Bx \cos 2x$$

$$\begin{aligned}
&= (A + 2Bx)\cos 2x + (B - 2Ax)\sin 2x \\
y''_p &= 2B\cos 2x - 2A\sin 2x - 2(A + 2Bx)\sin 2x + 2(B - 2Ax)\cos 2x \\
&= (4B - 4Ax)\cos 2x + (-4A - 4Bx)\sin 2x \\
y'' + 4y &= 3\sin 2x \\
\begin{array}{rcl}
\cos 2x & x & -4A + 4A \\
x^0 & & 4B = 0
\end{array}
\quad
\begin{array}{rcl}
\sin 2x & x & -4B + 4B \\
x^0 & & -4A = 3
\end{array}
\rightarrow A = -\frac{3}{4}
\end{aligned}$$

$$y_p = -\frac{3}{4}x\cos 2x$$

$$y(x) = C_1 \cos 2x + C_2 \sin 2x - \frac{3}{4}x\cos 2x$$

Exercise

Find the general solution: $y'' + 4y = 4\cos x + 3\sin x - 8$

Solution

The characteristic equation: $\lambda^2 + 4 = 0 \rightarrow \lambda_{1,2} = \pm 2i$

$$y_h = C_1 \cos 2x + C_2 \sin 2x$$

$$y_p = A\cos x + B\sin x + C$$

$$y'_p = -A\sin x + B\cos x$$

$$y''_p = -A\cos x - B\sin x$$

$$y'' + 4y = 4\cos x + 3\sin x - 8$$

$$\cos x \quad -A + 4A = 4 \quad A = \frac{4}{3}$$

$$\sin x \quad -B + 4B = 3 \quad B = 1$$

$$4C = -8 \quad C = -2$$

$$y_p = \frac{4}{3}\cos x + \sin x - 2$$

$$y(x) = C_1 \cos 2x + C_2 \sin 2x + \frac{4}{3}\cos x + \sin x - 2$$

Exercise

Find the general solution: $y'' - 4y = (x^2 - 3)\sin 2x$

Solution

The characteristic equation: $\lambda^2 - 4 = 0 \rightarrow \lambda_{1,2} = \pm 2$

$$\underline{y_h = C_1 e^{-2x} + C_2 e^{2x}}$$

$$y_p = (Ax^2 + Bx + C) \cos 2x + (Dx^2 + Ex + F) \sin 2x$$

$$y'_p = (2Ax + B + 2Dx^2 + 2Ex + 2F) \cos 2x + (2Dx + E - 2Ax^2 - 2Bx - 2C) \sin 2x$$

$$\begin{aligned} y''_p &= (2A + 4Dx + 2E + 4Dx + 2E - 4Ax^2 - 4Bx - 4C) \cos 2x \\ &\quad + (2D - 4Ax - 2B - 4Ax - 2B - 4Dx^2 - 4Ex - 4F) \sin 2x \\ &= (-4Ax^2 + 8Dx - 4Bx + 2A - 4C + 4E) \cos 2x + (-4Dx^2 - 8Ax - 4Ex + 2D - 4B - 4F) \sin 2x \end{aligned}$$

$$y'' - 4y = (x^2 - 3) \sin 2x$$

$$\begin{array}{lll} \text{cos } 2x & x^2 & -4A - 4A = 0 \quad \underline{A = 0} \\ & x & 8D - 4B - 4B = 0 \quad D - B = 0 \\ & x^0 & 2A - 4C + 4E - 4C = 0 \quad E - 2C = 0 \end{array}$$

$$\begin{array}{lll} \text{sin } 2x & x^2 & -4D - 4D = 1 \quad \underline{D = -\frac{1}{8}} \\ & x & -8A - 4E - 4E = 0 \quad \underline{E = 0} \\ & x^0 & 2D - 4B - 8F = -3 \quad B + 2F = \frac{11}{16} \end{array}$$

$$\underline{B = D = -\frac{1}{8}} \quad \underline{C = \frac{1}{2}E = 0} \quad \underline{F = \frac{13}{32}}$$

$$\underline{y_p = -\frac{1}{8}x \cos 2x + \left(-\frac{1}{8}x^2 + \frac{13}{32}\right) \sin 2x}$$

$$\underline{y(x) = C_1 e^{-2x} + C_2 e^{2x} - \frac{1}{8}x \cos 2x + \left(-\frac{1}{8}x^2 + \frac{13}{32}\right) \sin 2x}$$

Exercise

Find the general solution: $y'' + 4y' + 4y = 2x + 6$

Solution

$$\text{The characteristic equation: } \lambda^2 + 4\lambda + 4 = 0 \rightarrow \underline{\lambda_{1,2} = -2}$$

$$\underline{y_h = (C_1 + C_2 x) e^{-2x}}$$

$$y_p = Ax + B$$

$$y'_p = A$$

$$y''_p = 0$$

$$y'' + 4y' + 4y = 2x + 6$$

$$4A + 4Ax + 4B = 2x + 6$$

$$x \quad 4A = 2 \quad \rightarrow \underline{A = \frac{1}{2}}$$

$$x^0 \quad 4A + 4B = 6 \quad \rightarrow \underline{B = 1}$$

$$y_p = \frac{1}{2}x + 1$$

$$\underline{y(x) = (C_1 + C_2 x)e^{-2x} + \frac{1}{2}x + 1}$$

Exercise

Find the general solution: $y'' + 4y' + 5y = 5x + e^{-x}$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 5 = 0 \rightarrow \lambda_{1,2} = \frac{-4 \pm 2i}{2} = \underline{-2 \pm i}$

$$\underline{y_h = e^{-2x}(C_1 \cos x + C_2 \sin x)}$$

$$y_p = Ax + B + Ce^{-x}$$

$$y'_p = A - Ce^{-x}$$

$$y''_p = Ce^{-x}$$

$$y'' + 4y' + 5y = 5x + e^{-x}$$

$$Ce^{-x} + 4A - 4Ce^{-x} + 5Ax + 5B + 5Ce^{-x} = \underline{5x + e^{-x}}$$

$$x \quad 5A = 5 \quad \rightarrow \underline{A = 1}$$

$$x^0 \quad 4A + 5B = 0 \quad \rightarrow \underline{B = -\frac{4}{5}}$$

$$e^{-x} \quad 2C = 1 \quad \rightarrow \underline{C = \frac{1}{2}}$$

$$\underline{y_p = x - \frac{4}{5} + \frac{1}{2}e^{-x}}$$

$$\underline{y(x) = e^{-2x}(C_1 \cos x + C_2 \sin x) + x - \frac{4}{5} + \frac{1}{2}e^{-x}}$$

Exercise

Find the general solution: $y'' + 4y' + 5y = 2e^{-2x} + \cos x$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 5 = 0 \rightarrow \lambda_{1,2} = \frac{-4 \pm 2i}{2} = \underline{-2 \pm i}$

$$\underline{y_h = e^{-2x} (C_1 \cos x + C_2 \sin x)}$$

$$y_p = Ae^{-2x} + B \cos x + C \sin x$$

$$y'_p = -2Ae^{-x} - B \sin x + C \cos x$$

$$y''_p = 4Ae^{-x} - B \cos x - C \sin x$$

$$y'' + 4y' + 5y = 2e^{-2x} + \cos x$$

$$4Ae^{-x} - B \cos x - C \sin x - 8Ae^{-x} - 4B \sin x + 4C \cos x + 5Ae^{-x} + 5B \cos x + 5C \sin x = 2e^{-2x} + \cos x$$

$$\cos x \quad 4B + 4C = 1 \quad \underline{B = \frac{1}{8}}$$

$$\sin x \quad -4B + 4C = 0 \quad \underline{C = \frac{1}{8}}$$

$$e^{-2x} \quad A = 2 \quad \rightarrow \underline{A = 2}$$

$$\underline{y_p = 2e^{-2x} + \frac{1}{8} \cos x + \frac{1}{8} \sin x}$$

$$\underline{y(x) = e^{-2x} (C_1 \cos x + C_2 \sin x) + 2e^{-2x} + \frac{1}{8} \cos x + \frac{1}{8} \sin x}$$

Exercise

Find the general solution: $y'' + 5y' = 15x^2$

Solution

The characteristic equation: $\lambda^2 + 5\lambda = 0 \rightarrow \underline{\lambda_{1,2} = 0, -5}$

$$\underline{y_h = C_1 + C_2 e^{-5x}}$$

$$y_p = Ax^3 + Bx^2 + Cx$$

$$y'_p = 3Ax^2 + 2Bx + C$$

$$y''_p = 6Ax + 2B$$

$$y'' + 5y' = 15x^2$$

$$6Ax + 2B + 15Ax^2 + 10Bx + 5C = 15x^2$$

$$15A = 15 \rightarrow \underline{A = 1}$$

$$6A + 10B = 0 \rightarrow \underline{B = -\frac{3}{5}}$$

$$2B + 5C = 0 \rightarrow \underline{C = \frac{6}{25}}$$

$$\Rightarrow \underline{y_p = x^3 - \frac{3}{5}x^2 + \frac{6}{25}x}$$

$$\underline{y(x) = C_1 + C_2 e^{-5x} + x^3 - \frac{3}{5}x^2 + \frac{6}{25}x}$$

Exercise

Find the general solution: $y'' - 5y' = 2x^3 - 4x^2 - x + 6$

Solution

The characteristic equation: $\lambda^2 - 5\lambda = 0 \rightarrow \underline{\lambda_{1,2} = 0, 5}$

$$\underline{y_h = C_1 + C_2 e^{5x}}$$

$$y_p = Ax^4 + Bx^3 + Cx^2 + Dx$$

$$y'_p = 4Ax^3 + 3Bx^2 + 2Cx + D$$

$$y''_p = 12Ax^2 + 6Bx + 2C$$

$$y'' - 5y' = 2x^3 - 4x^2 - x + 6$$

$$12Ax^2 + 6Bx + 2C - 20Ax^3 - 15Bx^2 - 10Cx - 5D = 2x^3 - 4x^2 - x + 6$$

$$\begin{matrix} x^3 & -20A = 2 & A = -\frac{1}{10} \end{matrix}$$

$$\begin{matrix} x^2 & 12A - 15B = -4 & B = \frac{14}{75} \end{matrix}$$

$$\begin{matrix} x & 6B - 10C = -1 & C = \frac{53}{250} \end{matrix}$$

$$\begin{matrix} x^0 & 2C - 5D = 6 & D = -\frac{697}{625} \end{matrix}$$

$$\underline{y_p = -\frac{1}{10}x^4 + \frac{14}{75}x^3 + \frac{53}{250}x^2 - \frac{697}{625}x}$$

$$\underline{y(x) = C_1 + C_2 e^{5x} - \frac{1}{10}x^4 + \frac{14}{75}x^3 + \frac{53}{250}x^2 - \frac{697}{625}x}$$

Exercise

Find the general solution: $y'' + 6y' + 8y = 3e^{-2x} + 2x$

Solution

The characteristic equation: $\lambda^2 + 6\lambda + 8 = 0 \rightarrow \underline{\lambda_{1,2} = -2, -4}$

$$\underline{y_h = C_1 e^{-2x} + C_2 e^{-4x}}$$

$$y_p = Axe^{-2x} + Bx + C$$

$$y'_p = (A - 2Ax)e^{-2x} + B$$

$$y_p'' = (-4A + 4Ax)e^{-2x}$$

$$y'' + 6y' + 8y = 3e^{-2x} + 2x$$

$$(-4A + 4Ax + 6A - 12Ax + 8Ax)e^{-2x} + 6B + 8Bx + 8C = 3e^{-2x} + 2x$$

$$e^{-2x} \quad 2A = 3 \quad \rightarrow \quad A = \frac{3}{2}$$

$$x \quad 8B = 2 \quad B = \frac{1}{4}$$

$$x^0 \quad 6B + 8C = 0 \quad C = -\frac{3}{16}$$

$$y_p = \frac{3}{2}xe^{-2x} + \frac{1}{4}x - \frac{3}{16}$$

$$y(x) = C_1e^{-2x} + C_2e^{-4x} + \frac{3}{2}xe^{-2x} + \frac{1}{4}x - \frac{3}{16}$$

Exercise

Find the general solution: $y'' - 6y' + 9y = e^{3x}$

Solution

The characteristic equation: $\lambda^2 - 6\lambda + 9 = 0 \rightarrow \lambda_{1,2} = 3$

$$y_h = (C_1 + C_2x)e^{3x}$$

$$y_p = Ax^2e^{3x}$$

$$y_p' = (2Ax + 3Ax^2)e^{3x}$$

$$y_p'' = (2A + 12Ax + 9Ax^2)e^{3x}$$

$$y'' - 6y' + 9y = e^{3x}$$

$$(2A + 12Ax + 9Ax^2 - 12Ax - 18Ax^2 + 9Ax^2)e^{3x} = e^{3x}$$

$$2Ae^{3x} = e^{3x} \rightarrow A = \frac{1}{2}$$

$$y(x) = \left(C_1 + C_2x + \frac{1}{2}\right)e^{3x}$$

Exercise

Find the general solution: $y'' + 6y' + 9y = -xe^{4x}$

Solution

The characteristic equation: $\lambda^2 + 6\lambda + 9 = 0 \rightarrow \lambda_{1,2} = -3$

$$y_h = (C_1 + C_2 x)e^{-3x}$$

$$y_p = (Ax + B)e^{4x}$$

$$y'_p = (A + 4Ax + 4B)e^{4x}$$

$$y''_p = (16Ax + 8A + 16B)e^{4x}$$

$$y'' + 6y' + 9y = -xe^{4x}$$

$$(16Ax + 8A + 16B + 6A + 24Ax + 24B + 9Ax + 9B)e^{4x} = -xe^{4x}$$

$$x \quad 49A = -1 \quad A = -\frac{1}{49}$$

$$x^0 \quad 14A + 25B = 0 \quad B = \frac{2}{175}$$

$$y_p = \left(-\frac{1}{49}x + \frac{2}{175}\right)e^{4x}$$

$$y(x) = (C_1 + C_2 x)e^{-3x} + \left(-\frac{1}{49}x + \frac{2}{175}\right)e^{4x}$$

Exercise

Find the general solution $y'' + 6y' + 13y = e^{-3x} \cos 2x$

Solution

$$\lambda^2 + 6\lambda + 13 = 0 \rightarrow \lambda_{1,2} = \frac{-6 \pm \sqrt{-16}}{2} = -3 \pm 2i$$

$$y_h = e^{-3x} (C_1 \cos 2x + C_2 \sin 2x)$$

$$y_p = e^{-3x} (Ax \cos 2x + Bx \sin 2x)$$

$$y'_p = e^{-3x} (A \cos 2x - 2Ax \sin 2x + B \sin 2x + 2Bx \cos 2x - 3Ax \cos 2x - 3Bx \sin 2x)$$

$$= e^{-3x} ((A - 3Ax + 2Bx) \cos 2x + (B - 2Ax - 3Bx) \sin 2x)$$

$$y''_p = e^{-3x} \left((-3A + 2B) \cos 2x + (-2A - 3B) \sin 2x - (2A - 6Ax + 4Bx) \sin 2x + (2B - 4Ax - 6Bx) \cos 2x \right. \\ \left. - 3(A - 3Ax + 2Bx) \cos 2x - 3(B - 2Ax - 3Bx) \sin 2x \right)$$

$$= e^{-3x} ((-6A + 4B + 5Ax - 12Bx) \cos 2x + (-4A - 6B + 12Ax + 5Bx) \sin 2x)$$

$$(-6A + 4B + 2Ax - 12Bx) \cos 2x + (-4A - 6B + 12Ax + 5Bx) \sin 2x + 6(A - 3Ax + 2Bx) \cos 2x$$

$$+ 6(B - 2Ax - 3Bx) \sin 2x + 13Ax \cos 2x + 13Bx \sin 2x = e^{-3x} \cos 2x$$

$$\begin{cases} \sin 2x & 13Bx + 6B - 12Ax - 18Bx - 4A - 6B + 12Ax + 5Bx = 0 \\ \cos 2x & 13Ax + 6A - 18Ax + 12Bx - 6A + 4B + 5Ax - 12Bx = 1 \end{cases}$$

$$\begin{cases} -4A = 0 & \rightarrow \underline{A = 0} \\ 4B = 1 & \rightarrow \underline{B = \frac{1}{4}} \end{cases} \quad y_p = \frac{1}{4} x e^{-3x} \sin 2x$$

$$\underline{y(x) = e^{-3x} \left(C_1 \cos 2x + C_2 \sin 2x + \frac{1}{4} x \sin 2x \right)}$$

Exercise

Find the general solution: $y'' - 7y' = -3$

Solution

The characteristic equation: $\lambda^2 - 7\lambda = 0 \rightarrow \underline{\lambda_{1,2} = 0, 7}$

$$\underline{y_h = C_1 + C_2 e^{7x}}$$

$$y_p = Ax^2 + Bx$$

$$y'_p = 2Ax + B$$

$$y''_p = 2A$$

$$y'' - 7y' = -3$$

$$2A - 14Ax - 7B = -3$$

$$\rightarrow \begin{cases} -14A = 0 & \underline{A = 0} \\ 2A - 7B = -3 & \underline{B = \frac{3}{7}} \end{cases}$$

$$\underline{y_p = \frac{3}{7}x}$$

$$\underline{y(x) = C_1 + C_2 e^{7x} + \frac{3}{7}x}$$

Exercise

Find the general solution: $y'' + 7y' = 42x^2 + 5x + 1$

Solution

The characteristic equation: $\lambda^2 + 7\lambda = 0 \rightarrow \underline{\lambda_{1,2} = 0, -7}$

$$\underline{y_h = C_1 + C_2 e^{-7x}}$$

$$y_p = Ax^3 + Bx^2 + Cx$$

$$y'_p = 3Ax^2 + 2Bx + C$$

$$y''_p = 6Ax + 2B$$

$$y'' + 7y' = 42x^2 + 5x + 1$$

$$6Ax + 2B + 21Ax^2 + 14Bx + 7C = 42x^2 + 5x + 1$$

$$21A = 42 \rightarrow A = 2$$

$$6A + 14B = 0 \rightarrow B = -\frac{6}{7}$$

$$2B + 7C = 0 \rightarrow C = \frac{12}{49}$$

$$y_p = 2x^3 - \frac{6}{7}x^2 + \frac{12}{49}x$$

$$y(x) = C_1 + C_2 e^{-7x} + 2x^3 - \frac{6}{7}x^2 + \frac{12}{49}x$$

Exercise

Find the general solution $y'' + 8y = 5x + 2e^{-x}$

Solution

The characteristic equation: $\lambda^2 + 8 = 0 \Rightarrow \lambda_{1,2} = \pm 2i\sqrt{2}$

$$y_h = C_1 \cos(2\sqrt{2}x) + C_2 \sin(2\sqrt{2}x)$$

$$y_p = A + Bx + Ce^{-x}$$

$$y'_p = B - Ce^{-x}$$

$$y''_p = Ce^{-x}$$

$$y'' + 8y = 5x + 2e^{-x}$$

$$Ce^{-x} + 8A + 8Bx + 8Ce^{-x} = 5x + 2e^{-x}$$

$$8A = 0 \rightarrow A = 0$$

$$8B = 5 \rightarrow B = \frac{5}{8}$$

$$9C = 2 \rightarrow C = \frac{2}{9}$$

$$y(x) = C_1 \cos(2\sqrt{2}x) + C_2 \sin(2\sqrt{2}x) + \frac{5}{8}x + \frac{2}{9}e^{-x}$$

Exercise

Find the general solution: $y'' - 8y' + 20y = 100x^2 - 26xe^x$

Solution

The characteristic equation: $\lambda^2 - 8\lambda + 20 = 0 \rightarrow \lambda_{1,2} = \frac{8 \pm 4i}{2} = 4 \pm 2i$

$$y_h = (C_1 \cos 2x + C_2 \sin 2x)e^{4x}$$

$$y_P = (Ax + B)e^x + Cx^2 + Dx + E$$

$$y'_P = (Ax + A + B)e^x + 2Cx + D$$

$$y''_P = (Ax + 2A + B)e^x + 2C$$

$$y'' - 8y' + 20y = 100x^2 - 26xe^x$$

$$\begin{array}{lcl} e^x & x & A - 8A + 20A = -26 \quad \underline{A = -2} \\ & x^0 & 2A + B - 8A - 8B + 20B = 0 \quad \underline{B = -\frac{12}{13}} \end{array}$$

$$x^2 \quad 20C = 100 \quad \underline{C = 5}$$

$$x \quad -16C + 20D = 0 \quad \underline{D = 4}$$

$$x^0 \quad 2C - 8D + 20E = 0 \quad \underline{E = \frac{11}{10}}$$

$$y_P = \left(-2x - \frac{12}{13}\right)e^x + 5x^2 + 4x + \frac{11}{10}$$

$$y(x) = \left(C_1 \cos 2x + C_2 \sin 2x\right)e^{4x} + \left(-2x - \frac{12}{13}\right)e^x + 5x^2 + 4x + \frac{11}{10}$$

Exercise

Find the general solution: $y'' - 9y = 54$

Solution

The characteristic equation: $\lambda^2 - 9 = 0 \rightarrow \underline{\lambda_{1,2} = \pm 3}$

$$y_h = C_1 e^{-3x} + C_2 e^{3x}$$

$$y_P = A$$

$$y'_P = y''_P = 0$$

$$y'' - 9y = 54$$

$$-9A = 54 \rightarrow \underline{A = -6}$$

$$y(x) = C_1 e^{-3x} + C_2 e^{3x} - 6$$

Exercise

Find the general solution: $y'' + 9y = x^2 \cos 3x + 4 \sin x$

Solution

The characteristic equation: $\lambda^2 + 9 = 0 \rightarrow \lambda_{1,2} = \pm 3i$

$$y_h = C_1 \cos 3x + C_2 \sin 3x$$

$$y_p = (Ax^3 + Bx^2 + Cx) \cos 3x + (Dx^3 + Ex^2 + Fx) \sin 3x + G \cos x + H \sin x$$

$$\triangleright y_{p1} = (Ax^3 + Bx^2 + Cx) \cos 3x$$

$$y'_{p1} = (3Ax^2 + 2Bx + C) \cos 3x - (3Ax^3 + 3Bx^2 + 3Cx) \sin 3x$$

$$y''_{p1} = (-9Ax^3 - 9Bx^2 - 9Cx + 6Ax + 2B) \cos 3x - (18Ax^2 + 12Bx + 6C) \sin 3x$$

$$y'' + 9y$$

$$\begin{aligned} & (-9Ax^3 - 9Bx^2 - 9Cx + 6Ax + 2B + 9Ax^3 + 9Bx^2 + 9Cx) \cos 3x - (18Ax^2 + 12Bx + 6C) \sin 3x \\ & (6Ax + 2B) \cos 3x - (18Ax^2 + 12Bx + 6C) \sin 3x \end{aligned}$$

$$\triangleright y_{p2} = (Dx^3 + Ex^2 + Fx) \sin 3x$$

$$y'_{p2} = (3Dx^2 + 2Ex + F) \sin 3x + (3Dx^3 + 3Ex^2 + 3Fx) \cos 3x$$

$$y''_{p2} = (6Dx + 2E - 9Dx^3 - 9Ex^2 - 9Fx) \sin 3x + 2(9Dx^2 + 6Ex + 3F) \cos 3x$$

$$y'' + 9y$$

$$\begin{aligned} & (6Dx + 2E - 9Dx^3 - 9Ex^2 - 9Fx + 9Dx^3 + 9Ex^2 + 9Fx) \sin 3x + 2(9Dx^2 + 6Ex + 3F) \cos 3x \\ & (6Dx + 2E) \sin 3x + (18Dx^2 + 12Ex + 6F) \cos 3x \end{aligned}$$

$$y'' + 9y = x^2 \cos 3x + 4 \sin x$$

$$(6Ax + 2B + 18Dx^2 + 12Ex + 6F) \cos 3x - (18Ax^2 + 12Bx + 6C - 6Dx - 2E) \sin 3x = x^2 \cos 3x$$

$$\cos 3x \quad x^2 \quad 18D = 1 \quad \rightarrow D = \frac{1}{18}$$

$$x \quad 6A + 12E = 0 \quad A + 2E = 0$$

$$x^0 \quad 2B + 6F = 0 \quad B + 3F = 0$$

$$\sin 3x \quad x^2 \quad -18A = 0 \quad \rightarrow A = 0$$

$$x \quad 12B - 6D = 0 \quad \rightarrow B = \frac{1}{36}$$

$$x^0 \quad 6C - 2E = 0$$

$$A + 2E = 0 \quad \rightarrow E = 0$$

$$B + 3F = 0 \quad \rightarrow F = -\frac{1}{108}$$

$$6C - 2E = 0 \rightarrow \underline{C = 0}$$

$$\underline{y_P = \frac{1}{36}x^2 \cos 3x + \left(\frac{1}{18}x^3 - \frac{1}{108}x\right) \sin 3x}$$

$$\triangleright y_{P3} = G \cos x + H \sin x$$

$$y'_{P3} = -G \sin x + H \cos x$$

$$y''_{P3} = -G \cos x - H \sin x$$

$$y'' + 9y = x^2 \cos 3x + 4 \sin x$$

$$-G \cos x - H \sin x + 9G \cos x + 9H \sin x = 4 \sin x$$

$$\cos x \quad 8G = 0 \rightarrow \underline{G = 0}$$

$$\sin x \quad 8H = 4 \rightarrow \underline{H = \frac{1}{2}}$$

$$\underline{y_{P3} = \frac{1}{2} \sin x}$$

$$\underline{y(x) = C_1 \cos 3x + C_2 \sin 3x + \frac{1}{36}x^2 \cos 3x + \left(\frac{1}{18}x^3 - \frac{1}{108}x\right) \sin 3x + \frac{1}{2} \sin x}$$

Exercise

Find the general solution: $y'' + 10y' + 25y = 14e^{-5x}$

Solution

$$\text{The characteristic equation: } \lambda^2 + 10\lambda + 25 = 0 \rightarrow \underline{\lambda_{1,2} = -5}$$

$$\underline{y_h = (C_1 + C_2 x)e^{-5x}}$$

$$y_P = Ax^2 e^{-5x}$$

$$y'_P = (2Ax - 5Ax^2)e^{-5x}$$

$$y''_P = (2A - 20Ax + 25Ax^2)e^{-5x}$$

$$y'' + 10y' + 25y = 14e^{-5x}$$

$$(2A - 20Ax + 25Ax^2 + 20Ax - 50Ax^2 + 25Ax^2)e^{-5x} = 14e^{-5x}$$

$$2A = 14 \rightarrow \underline{A = 7}$$

$$\underline{y(x) = (C_1 + C_2 x + 7x^2)e^{-5x}}$$

Exercise

Find the general solution: $y'' - 10y' + 25y = 30x + 3$

Solution

The characteristic equation: $\lambda^2 - 10\lambda + 25 = 0 \rightarrow \lambda_{1,2} = 5$

$$y_h = (C_1 + C_2 x)e^{5x}$$

$$y_p = Ax + B$$

$$y'_p = A$$

$$y''_p = 0$$

$$y'' - 10y' + 25y = 30x + 3$$

$$x \quad 25A = 30 \quad A = \frac{6}{5}$$

$$x^0 \quad -10A + 20B = 0 \quad B = \frac{3}{5}$$

$$y_p = \frac{6}{5}x + \frac{3}{5}$$

$$y(x) = (C_1 + C_2 x)e^{5x} + \frac{6}{5}x + \frac{3}{5}$$

Exercise

Find the general solution: $y'' - 16y = 2e^{4x}$

Solution

The characteristic equation: $\lambda^2 - 16 = 0 \rightarrow \lambda_{1,2} = \pm 4$

$$y_h = C_1 e^{-4x} + C_2 e^{4x}$$

$$y_p = A x e^{4x}$$

$$y'_p = (A + 4Ax)e^{4x}$$

$$y''_p = (8A + 16Ax)e^{4x}$$

$$y'' - 16y = 2e^{4x}$$

$$x \quad 16A - 16A$$

$$x^0 \quad 8A - 16A = 2 \quad A = -\frac{1}{4}$$

$$y_p = -\frac{1}{4} x e^{4x}$$

$$y(x) = C_1 e^{-4x} + C_2 e^{4x} - \frac{1}{4} x e^{4x}$$

Exercise

Find the general solution: $y'' + 25y = 6\sin x$

Solution

The characteristic equation: $\lambda^2 + 25 = 0 \rightarrow \lambda_{1,2} = \pm 5i$

$$y_h = C_1 \cos 5x + C_2 \sin 5x$$

$$y_p = A \cos x + B \sin x$$

$$y'_p = -A \sin x + B \cos x$$

$$y''_p = -A \cos x - B \sin x$$

$$y'' + 25y = 6\sin x$$

$$\cos x \quad -A + 25A = 0 \quad \underline{A = 0}$$

$$\sin x \quad -B + 25B = 6 \quad \underline{B = \frac{1}{4}}$$

$$y_p = \frac{1}{4} \sin x$$

$$y(x) = C_1 \cos 5x + C_2 \sin 5x + \frac{1}{4} \sin x$$

Exercise

Find the general solution $y'' + 25y = 20\sin 5x$

Solution

The characteristic equation: $\lambda^2 + 25 = 0 \Rightarrow \lambda_{1,2} = \pm 5i$

$$y_h = C_1 \cos 5x + C_2 \sin 5x$$

The particular equation: $y_p = A \cos 5x + B \sin 5x$

$$y'_p = -5A \sin 5x + 5B \cos 5x$$

$$y''_p = -25A \cos 5x - 25B \sin 5x$$

$$-25A \cos 5x - 25B \sin 5x + 25A \cos 5x + 25B \sin 5x = 20\sin 5x$$

$$\cancel{0 = 20\sin 5x}$$

The particular equation: $y_p = Ax \cos 5x + Bx \sin 5x$

$$y'_p = A \cos 5x - 5Ax \sin 5x + B \sin 5x + 5Bx \cos 5x = (A + 5Bx) \cos 5x + (B - 5Ax) \sin 5x$$

$$y''_p = 5B \cos 5x - 5(A + 5Bx) \sin 5x - 5A \sin 5x + 5(B - 5Ax) \cos 5x$$

$$= 10B \cos 5x - 25Ax \cos 5x - 10A \sin 5x - 25Bx \sin 5x$$

$$10B \cos 5x - 25Ax \cos 5x - 10A \sin 5x - 25Bx \sin 5x + 25Ax \cos 5x + 25Bx \cos 5x = 20\sin 5x$$

$$10B \cos 5x - 10A \sin 5x = 20\sin 5x$$

$$\begin{cases} 10B = 0 & \rightarrow B = 0 \\ -10A = 20 & \rightarrow A = -2 \end{cases}$$

$$\underline{y(x) = C_1 \cos 5x + C_2 \sin 5x - 2x \cos 5x}$$

Exercise

Find the general solution: $\frac{1}{4}y'' + y' + y = x^2 - 2x$

Solution

The characteristic equation: $\frac{1}{4}\lambda^2 + \lambda + 1 = \left(\frac{1}{2}\lambda + 1\right)^2 = 0 \rightarrow \underline{\lambda_{1,2} = -2}$

$$\underline{y_h = (C_1 + C_2 x)e^{-2x}}$$

$$y_p = Ax^2 + Bx + C$$

$$y'_p = 2Ax + B$$

$$y''_p = 2A$$

$$\frac{1}{4}y'' + y' + y = x^2 - 2x$$

$$\textcolor{red}{x^2} \quad \underline{\textcolor{green}{A=1}}$$

$$\textcolor{red}{x} \quad 2A + B = -2 \quad \underline{\textcolor{green}{B=-4}}$$

$$\textcolor{red}{x^0} \quad \frac{1}{2}A + B + C = 0 \quad \underline{\textcolor{green}{C=\frac{7}{2}}}$$

$$\underline{y_p = x^2 - 4x + \frac{7}{2}}$$

$$\underline{y(x) = (C_1 + C_2 x)e^{-2x} + x^2 - 4x + \frac{7}{2}}$$

Exercise

Find the general solution: $2y'' - 5y' + 2y = -6e^{x/2}$

Solution

The characteristic equation: $2\lambda^2 - 5\lambda + 2 = 0 \rightarrow \underline{\lambda_{1,2} = \frac{1}{2}, 2}$

$$\underline{y_h = C_1 e^{x/2} + C_2 e^{2x}}$$

$$y_p = (Ax^2 + Bx)e^{x/2}$$

$$y'_p = \left(2Ax + B + \frac{1}{2}Ax^2 + \frac{1}{2}Bx\right)e^{x/2}$$

$$y''_p = \left(2A + Ax + \frac{1}{2}B + Ax + \frac{1}{2}B + \frac{1}{4}Ax^2 + \frac{1}{4}Bx\right)e^{x/2}$$

$$= \left(\frac{1}{4}Ax^2 + 2Ax + \frac{1}{4}Bx + 2A + B \right) e^{x/2}$$

$$2y'' - 5y' + 2y = -6e^{x/2}$$

$$\left(\frac{1}{2}Ax^2 + 4Ax + \frac{1}{2}Bx + 4A + 2B - 10Ax - 5B - \frac{5}{2}Ax^2 - \frac{5}{2}Bx + 2Ax^2 + 2Bx \right) e^{x/2} = -6e^{x/2}$$

$$(-6Ax + 4A - 3B)e^{x/2} = -6e^{x/2}$$

$$\begin{cases} -6A = 0 & \rightarrow \underline{A = 0} \\ 4A - 3B = -6 & \rightarrow \underline{B = 2} \end{cases}$$

$$\underline{y_p = 2xe^{x/2}}$$

$$\underline{y(x) = C_1 e^{x/2} + C_2 e^{2x} + 2xe^{x/2}}$$

Exercise

Find the general solution: $2y'' - 7y' + 5y = -29$

Solution

The characteristic equation: $2\lambda^2 - 7\lambda + 5 = 0 \rightarrow \underline{\lambda_{1,2} = 1, \frac{5}{2}}$

$$\underline{y_h = C_1 e^x + C_2 e^{5x/2}}$$

$$y_p = A$$

$$y'_p = y''_p = 0$$

$$2y'' - 7y' + 5y = -29$$

$$5A = -29 \rightarrow \underline{A = -\frac{29}{5}}$$

$$\underline{y_p = -\frac{29}{5}}$$

$$\underline{y(x) = C_1 e^x + C_2 e^{5x/2} - \frac{29}{5}}$$

Exercise

Find the general solution: $4y'' + 9y = 15$

Solution

The characteristic equation: $4\lambda^2 + 9 = 0 \rightarrow \underline{\lambda_{1,2} = \pm \frac{3}{2}}$

$$\underline{y_h = C_1 e^{-3x/2} + C_2 e^{3x/2}}$$

$$y_p = A$$

$$y'_p = y''_p = 0$$

$$4y'' + 9y = 15$$

$$9A = 15 \rightarrow \underline{A = \frac{15}{9}}$$

$$\underline{y_P = \frac{15}{9}}$$

$$\underline{y(x) = C_1 e^{-3x/2} + C_2 e^{3x/2} + \frac{15}{9}}$$

Exercise

Find the general solution: $4y'' - 4y' - 3y = \cos 2x$

Solution

The characteristic equation: $4\lambda^2 - 4\lambda - 3 = 0 \rightarrow \lambda_{1,2} = \frac{4 \pm 8}{8} = -\frac{1}{2}, \frac{3}{2}$

$$\underline{y_h = C_1 e^{-x/2} + C_2 e^{3x/2}}$$

$$y_P = A \cos 2x + B \sin 2x$$

$$y'_P = -2A \sin 2x + 2B \cos 2x$$

$$y''_P = -4A \cos 2x - 4B \sin 2x$$

$$4y'' - 4y' - 3y = \cos 2x$$

$$\begin{array}{l} \cos 2x \quad -16A - 8B - 3A = 1 \\ \sin 2x \quad -16B + 8A - 3B = 0 \end{array} \rightarrow \begin{cases} -19A - 8B = 1 \\ 8A - 19B = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} -19 & -8 \\ 8 & -19 \end{vmatrix} = 425 \quad \Delta_A = \begin{vmatrix} 1 & -8 \\ 0 & -19 \end{vmatrix} = -19 \quad \Delta_B = \begin{vmatrix} -19 & 1 \\ 8 & 0 \end{vmatrix} = -8$$

$$\underline{A = -\frac{19}{425}, \quad B = -\frac{8}{425}}$$

$$\underline{y_P = -\frac{19}{425} \cos 2x - \frac{8}{425} \sin 2x}$$

$$\underline{y(x) = C_1 e^{-x/2} + C_2 e^{3x/2} - \frac{19}{425} \cos 2x - \frac{8}{425} \sin 2x}$$

Exercise

Find the general solution: $9y'' - 6y' + y = 9xe^{x/3}$

Solution

The characteristic equation: $9\lambda^2 - 6\lambda + 1 = 0 \rightarrow \lambda_{1,2} = \frac{1}{3}$

$$\underline{y_h = (C_1 + C_2 x) e^{x/3}}$$

$$y_P = (Ax^3 + Bx^2)e^{x/3}$$

$$y'_P = \left(3Ax^2 + 2Bx + \frac{1}{3}Ax^3 + \frac{1}{3}Bx^2\right)e^{x/3}$$

$$y''_P = \left(6Ax + 2B + Ax^2 + \frac{2}{3}Bx + Ax^2 + \frac{2}{3}Bx + \frac{1}{9}Ax^3 + \frac{1}{9}Bx^2\right)e^{x/3}$$

$$= \left(\frac{1}{9}Ax^3 + 2Ax^2 + \frac{1}{9}Bx^2 + 6Ax + \frac{4}{3}Bx + 2B\right)e^{x/3}$$

$$9y'' - 6y' + y = 9xe^{x/3}$$

$$\left(Ax^3 + 18Ax^2 + Bx^2 + 54Ax + 12Bx + 18B - 18Ax^2 - 12Bx - 2Ax^3 - 2Bx^2 + Ax^3 + Bx^2\right)e^{x/3} = 9xe^{x/3}$$

$$(54Ax + 18B)e^{x/3} = 9xe^{x/3}$$

$$x \quad 54A = 9 \rightarrow A = \frac{1}{6}$$

$$x^0 \quad 18B = 0 \rightarrow B = 0$$

$$y_P = \frac{1}{6}x^3e^{x/3}$$

$$y(x) = \left(C_1 + C_2x + \frac{1}{6}x^3\right)e^{x/3}$$

Exercise

Find the general solution $y''' + y'' = 8x^2$

Solution

The characteristic equation: $\lambda^3 + \lambda^2 = \lambda^2(\lambda + 1) = 0 \rightarrow \lambda_{1,2} = 0, \lambda_3 = -1$

$$y_h = C_1 + C_2x + C_3e^{-x}$$

$$y_P = Ax^4 + Bx^3 + Cx^2$$

$$y'_P = 4Ax^3 + 3Bx^2 + 2Cx$$

$$y''_P = 12Ax^2 + 6Bx + 2C$$

$$y'''_P = 24Ax + 6B$$

$$y''' + y'' = 8x^2$$

$$x^2 \quad 12A = 8 \quad A = \frac{2}{3}$$

$$x \quad 24A + 6B = 0 \quad B = -\frac{8}{3}$$

$$x^0 \quad 6B + 2C = 0 \quad C = 8$$

$$y_p = \frac{2}{3}x^4 - \frac{8}{3}x^3 + 8x^2$$

$$\therefore y(x) = C_1 + C_2x + C_3e^{-x} + \frac{2}{3}x^4 - \frac{8}{3}x^3 + 8x^2$$

Exercise

Find the general solution $y''' - y'' - 4y' + 4y = 5 - e^x + e^{2x}$

Solution

The characteristic equation: $\lambda^3 - \lambda^2 - 4\lambda + 4 = 0 \rightarrow \lambda_1 = 1$

$$\begin{array}{c|cccc} 1 & 1 & -1 & -4 & 4 \\ & & 1 & 0 & 4 \\ \hline & 1 & 0 & -4 & 0 \end{array} \rightarrow \lambda^2 - 4 = 0 \Rightarrow \lambda_{2,3} = \pm 2$$

$$y_h = C_1e^{2x} + C_2e^x + C_3e^{2x}$$

$$y_p = A + Bxe^x + Cxe^{2x}$$

$$y'_p = (Bx + B)e^x + (2Cx + C)e^{2x}$$

$$y''_p = (Bx + 2B)e^x + (4Cx + 4C)e^{2x}$$

$$y'''_p = (Bx + 3B)e^x + (8Cx + 12C)e^{2x}$$

$$y''' - y'' - 4y' + 4y = 5 - e^x + e^{2x}$$

$$4A = 5 \quad A = \frac{5}{4}$$

$$e^x \begin{cases} x & B - B - 4B + 4B \\ x^0 & 3B - 2B - 4B = -1 \end{cases} \quad B = -\frac{1}{3}$$

$$e^{2x} \begin{cases} x & \\ x^0 & 12C - 4C - 4C = 1 \end{cases} \quad C = \frac{1}{4}$$

$$y_p = \frac{5}{4} - \frac{1}{3}xe^x + \frac{1}{4}xe^{2x}$$

$$\therefore y(x) = C_1e^{2x} + C_2e^x + C_3e^{2x} + \frac{5}{4} - \frac{1}{3}xe^x + \frac{1}{4}xe^{2x}$$

Exercise

Find the general solution $y^{(3)} + y'' = 3e^x + 4x^2$

Solution

$$\lambda^3 + \lambda^2 = 0 \quad \lambda^2(\lambda + 1) = 0 \rightarrow \lambda_{1,2} = 0, \lambda_3 = -1$$

$$\underline{y_h = C_1 + C_2x + C_3e^{-x}}$$

$$y_p = Ae^x + x^2(Bx^2 + Cx + D)$$

$$= Ae^x + Bx^4 + Cx^3 + Dx^2$$

$$y'_p = Ae^x + 4Bx^3 + 3Cx^2 + 2Dx$$

$$y''_p = Ae^x + 12Bx^2 + 6Cx + 2D$$

$$y'''_p = Ae^x + 24Bx + 6C$$

$$Ae^x + 24Bx + 6C + Ae^x + 12Bx^2 + 6Cx + 2D = 3e^x + 4x^2$$

$$2A=3 \quad A=\frac{3}{2}$$

$$12B=4 \quad B=\frac{1}{3}$$

$$24B+6C=0 \quad C=-\frac{4}{3}$$

$$6C+2D=0 \quad D=4$$

$$y_p = 3e^x + \frac{1}{3}x^4 - \frac{4}{3}x^3 + 4x^2$$

$$\therefore \underline{y(x) = C_1 + C_2x + C_3e^{-x} + 3e^x + \frac{1}{3}x^4 - \frac{4}{3}x^3 + 4x^2}$$

Exercise

Find the general solution $y''' + 2y'' + y' = 10$

Solution

The characteristic equation: $\lambda^3 + 2\lambda^2 + \lambda = 0 \rightarrow \underline{\lambda_{1,2,3} = 0, -1, -1}$

$$\underline{y_h = C_1 + (C_2 + C_3x)e^{-x}}$$

$$y_p = Ax$$

$$y'_p = A$$

$$y''_p = y'''_p = 0$$

$$y''' + 2y'' + y' = 10$$

$$\underline{A=10}$$

$$\underline{y_p = 10x}$$

$$\underline{y(x) = C_1 + (C_2 + C_3x)e^{-x} + 10x}$$

Exercise

Find the general solution $y''' - 2y'' - 4y' + 8y = 6xe^{2x}$

Solution

The characteristic equation: $\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$

$$\lambda^2(\lambda - 2) - 4(\lambda - 2) = 0$$

$$(\lambda^2 - 4)(\lambda - 2) = 0 \rightarrow \lambda_{1,2,3} = 2, 2, -2$$

$$y_h = C_1 e^{-2x} + (C_2 + C_3 x) e^{2x}$$

$$y_p = (Ax^3 + Bx^2) e^{2x}$$

$$y'_p = (2Ax^3 + 3Ax^2 + 2Bx^2 + 2Bx) e^{2x}$$

$$y''_p = (4Ax^3 + 12Ax^2 + 6Ax + 4Bx^2 + 8Bx + 2B) e^{2x}$$

$$y'''_p = (8Ax^3 + 36Ax^2 + 36Ax + 6A + 8Bx^2 + 24Bx + 12B) e^{2x}$$

$$y''' - 2y'' - 4y' + 8y = 6xe^{2x}$$

$$x^3 \quad 8A - 8A - 8A + 8A$$

$$x^2 \quad 36A - 24A - 12A + 8B - 8B - 8B + 8B$$

$$x \quad 36A - 12A + 24B - 16B - 8B = 6 \quad 24A = 6 \rightarrow A = \frac{1}{4}$$

$$x^0 \quad 6A + 12B - 4B = 0 \quad 6A + 8B = 0 \rightarrow B = -\frac{3}{16}$$

$$y_p = \left(\frac{1}{4}x^3 - \frac{3}{16}x^2 \right) e^{2x}$$

$$y(x) = C_1 e^{-2x} + (C_2 + C_3 x) e^{2x} + \left(\frac{1}{4}x^3 - \frac{3}{16}x^2 \right) e^{2x}$$

Exercise

Find the general solution: $y^{(3)} - 3y'' + 3y' - y = 3e^x$

Solution

The characteristic equation: $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3 = 0 \rightarrow \lambda_{1,2,3} = 1$

$$y_h = (C_1 + C_2 x + C_3 x^2) e^x$$

$$y_p = Ax^3 e^x$$

$$y'_p = (Ax^3 + 3Ax^2)e^x$$

$$y''_p = (Ax^3 + 6Ax^2 + 6Ax)e^x$$

$$y'''_p = (Ax^3 + 9Ax^2 + 18Ax + 6A)e^x$$

$$y^{(3)} - 3y'' + 3y' - y = 3e^x$$

$$6Ae^x = 3e^x \rightarrow \underline{A = \frac{1}{2}}$$

$$\underline{y(x) = (C_1 + C_2x + C_3x^2)e^x + \frac{1}{2}x^3e^x}$$

Exercise

Find the general solution $y''' - 3y'' + 3y' - y = x - 4e^x$

Solution

The characteristic equation: $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3 = 0 \rightarrow \underline{\lambda_{1,2,3} = 1}$

$$\underline{y_h = (C_1 + C_2x + C_3x^2)e^x}$$

$$y_p = Ax + B + Cx^3e^x$$

$$y'_p = A + (Cx^3 + 3Cx^2)e^x$$

$$y''_p = (Cx^3 + 6Cx^2 + 6Cx)e^x$$

$$y'''_p = (Cx^3 + 9Cx^2 + 18Cx + 6C)e^x$$

$$y''' - 3y'' + 3y' - y = x - 4e^x$$

$$\begin{cases} x & \underline{-A = 1} \\ x^0 & 3A - B = 0 \quad \underline{B = -3} \end{cases}$$

$$x^3 \quad C - 3C + 3C - C = 0$$

$$x^2 \quad 9C - 18C + 9C = 0$$

$$x \quad 18C - 18C = 0$$

$$x^0 \quad 6C = -4 \rightarrow \underline{C = -\frac{2}{3}}$$

$$\underline{y_p = -x - 3 - \frac{2}{3}x^3e^x}$$

$$\underline{y(x) = (C_1 + C_2x + C_3x^2)e^x - x - 3 - \frac{2}{3}x^3e^x}$$

Exercise

Find the general solution: $y^{(3)} - 4y'' + y' + 6y = 4\sin 2x$

Solution

The characteristic equation: $\lambda^3 - 4\lambda^2 + \lambda + 6 = 0 \rightarrow \lambda_1 = -1$

$$\begin{array}{c|cccc} -1 & 1 & -4 & 1 & 6 \\ & & -1 & 5 & -6 \\ \hline & 1 & -5 & 6 & 0 \end{array} \quad \lambda^2 - 5\lambda + 6 = 0 \rightarrow \lambda_{2,3} = 2, 3$$

$$y_h = C_1 e^{-x} + C_2 e^{2x} + C_3 e^{3x}$$

$$y_p = A \cos 2x + B \sin 2x$$

$$y'_p = -2A \sin 2x + 2B \cos 2x$$

$$y''_p = -4A \cos 2x - 4B \sin 2x$$

$$y'''_p = 8A \sin 2x - 8B \cos 2x$$

$$y^{(3)} - 4y'' + y' + 6y = 4\sin 2x$$

$$8A \sin 2x - 8B \cos 2x + 16A \cos 2x + 16B \sin 2x - 2A \sin 2x + 2B \cos 2x + 6A \cos 2x + 6B \sin 2x = 4\sin 2x$$

$$\begin{cases} \cos 2x & 22A - 6B = 0 \\ \sin 2x & 6A + 22B = 4 \end{cases}$$

$$A = \frac{24}{520} = \frac{3}{65} \quad B = \frac{88}{520} = \frac{11}{65}$$

$$y_p = \frac{3}{65} \cos 2x + \frac{11}{65} \sin 2x$$

$$y(x) = C_1 e^{-x} + C_2 e^{2x} + C_3 e^{3x} + \frac{3}{65} \cos 2x + \frac{11}{65} \sin 2x$$

Exercise

Find the general solution for the given differential equation $y''' - 3y'' + 3y' - y = e^x - x + 16$

Solution

The characteristic equation: $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3 = 0 \Rightarrow \lambda_{1,2,3} = 1$

$$y_h = (C_1 + C_2 x + C_3 x^2) e^x$$

The particular equation: $y_p = A + Bx + (Cx^3 + Ex^4) e^x$

$$y'_p = B + (3Cx^2 + 4Ex^3 + Cx^3 + Ex^4) e^x = B + (3Cx^2 + (4E + C)x^3 + Ex^4) e^x$$

$$y_p'' = \left(6Cx + 3(4E + C)x^2 + 4Ex^3\right)e^x + \left(3Cx^2 + (4E + C)x^3 + Ex^4\right)e^x$$

$$= \left(6Cx + (12E + 6C)x^2 + (8E + C)x^3 + Ex^4\right)e^x$$

$$y_p''' = \left(6C + (24E + 12C)x + (24E + 3C)x^2 + 4Ex^3\right)e^x$$

$$+ \left(6Cx + (12E + 6C)x^2 + (8E + C)x^3 + Ex^4\right)e^x$$

$$= \left(6C + (24E + 18C)x + (36E + 9C)x^2 + (12E + C)x^3 + Ex^4\right)e^x$$

$$\left(6C + (24E + 18C)x + (36E + 9C)x^2 + (12E + C)x^3 + Ex^4\right)e^x - 3\left(6Cx + (12E + 6C)x^2 + (8E + C)x^3 + Ex^4\right)e^x + 3B + 3\left(3Cx^2 + (4E + C)x^3 + Ex^4\right)e^x - A - Bx - \left(Cx^3 + Ex^4\right)e^x = \underline{e^x - x + 16}$$

$$(6C + 24Ex)e^x - A + 3B - Bx = \underline{e^x - x + 16}$$

$$\begin{cases} 6C = 1 & \rightarrow C = \frac{1}{6} \\ 24E = 0 & \rightarrow E = 0 \\ -B = -1 & \rightarrow B = 1 \\ 3B - A = 16 & \rightarrow A = -13 \end{cases}$$

$$\underline{y_p = x - 13 + \frac{1}{6}x^3e^x}$$

$$\underline{y(x) = \left(C_1 + C_2x + C_3x^2\right)e^x + \frac{1}{6}x^3e^x + x - 13}$$

Exercise

Find the general solution $y''' - 6y'' = 3 - \cos x$

Solution

The characteristic equation: $\lambda^3 - 6\lambda^2 = 0 \rightarrow \underline{\lambda_{1,2,3} = 0, 0, 6}$

$$\underline{y_h = C_1 + C_2x + C_3e^{6x}}$$

$$y_p = Ax^2 + B\cos x + C\sin x$$

$$y_p' = 2Ax - B\sin x + C\cos x$$

$$y_p'' = 2A - B\cos x - C\sin x$$

$$y_p''' = B\sin x - C\cos x$$

$$y''' - 6y'' = 3 - \cos x$$

$$\begin{array}{rcl}
 & -12A = 3 & \underline{A = -\frac{1}{4}} \\
 \cos x & -C + 6B = -1 & \underline{B = -\frac{6}{37}} \\
 \sin x & B + 6C = 0 & \underline{C = \frac{1}{37}}
 \end{array}$$

$$y_p = -\frac{1}{4}x^2 - \frac{6}{37}\cos x + \frac{1}{37}\sin x$$

$$y(x) = C_1 + C_2x + C_3e^{6x} - \frac{1}{4}x^2 - \frac{6}{37}\cos x + \frac{1}{37}\sin x$$

Exercise

Find the general solution $y^{(3)} - 6y'' + 11y' - 6y = 2xe^{-x}$

Solution

The characteristic equation: $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \rightarrow \lambda_1 = 1$

$$\begin{array}{c|cccc}
 1 & 1 & -6 & 11 & -6 \\
 & & 1 & -5 & 6 \\
 \hline
 & 1 & -5 & 6 & 0
 \end{array}
 \quad \lambda^2 - 5\lambda + 6 = 0 \rightarrow \lambda_{2,3} = 2, 3$$

$$y_h = C_1e^x + C_2e^{2x} + C_3e^{3x}$$

$$y_p = (Ax + B)e^{-x}$$

$$y'_p = (-Ax + A - B)e^{-x}$$

$$y''_p = (Ax - 2A + B)e^{-x}$$

$$y'''_p = (-Ax + 3A - B)e^{-x}$$

$$y^{(3)} - 6y'' + 11y' - 6y = 2xe^{-x}$$

$$(-24Ax + 26A - 24B)e^{-x} = 2xe^{-x}$$

$$\begin{cases}
 -24A = 2 & \rightarrow A = -\frac{1}{12} \\
 26A - 24B = 0 & \rightarrow B = -\frac{13}{144}
 \end{cases}$$

$$y_p = \left(-\frac{1}{12}x - \frac{13}{144}\right)e^{-x}$$

$$y(x) = C_1e^x + C_2e^{2x} + C_3e^{3x} - \frac{1}{12}xe^{-x} - \frac{13}{144}e^{-x}$$

Exercise

Find the general solution: $y''' + 8y'' = -6x^2 + 9x + 2$

Solution

The characteristic equation: $\lambda^3 + 8\lambda^2 = \lambda^2(\lambda + 8) = 0 \Rightarrow \lambda_{1,2} = 0, \lambda_3 = -8$

$$y_h = (C_1 + C_2 x)e^0 + e^{-8x} = C_1 + C_2 x + e^{-8x}$$

The particular equation: $y_p = Ax^4 + Bx^3 + Cx^2$

$$y'_p = 4Ax^3 + 3Bx^2 + 2Cx$$

$$y''_p = 12Ax^2 + 6Bx + 2C$$

$$y'''_p = 24Ax + 6B$$

$$y''' + 8y'' = -6x^2 + 9x + 2$$

$$24Ax + 6B + 96Ax^2 + 48Bx + 16C = -6x^2 + 9x + 2$$

$$\begin{cases} 96A = -6 & \rightarrow A = -\frac{1}{16} \\ 24A + 48B = 9 & \rightarrow B = \frac{1}{96}\left(9 + \frac{3}{2}\right) = \frac{21}{108} = \frac{7}{32} \\ 6B + 16C = 2 & \rightarrow C = \frac{1}{16}\left(2 - \frac{21}{16}\right) = \frac{11}{256} \end{cases}$$

$$y(x) = C_1 + C_2 x + e^{-8x} - \frac{1}{16}x^4 + \frac{7}{32}x^3 + \frac{11}{256}x^2$$

Exercise

Find the general solution: $y^{(4)} + y'' = 3x^2 + 4\sin x - 2\cos x$

Solution

The characteristic equation: $\lambda^4 + 2\lambda^2 = 0 \rightarrow \lambda_{1,2} = 0 \mid \lambda_{3,4} = \pm 2i$

$$y_h = C_1 + C_2 x + C_3 \cos x + C_4 \sin x$$

$$y_p = Ax^4 + Bx^3 + Cx^2 + Ex \cos x + Fx \sin x$$

$$y'_p = 4Ax^3 + 3Bx^2 + 2Cx + E \cos x - Ex \sin x + F \sin x + Fx \cos x$$

$$= 4Ax^3 + 3Bx^2 + 2Cx + (E + Fx) \cos x + (F - Ex) \sin x$$

$$y''_p = 12Ax^2 + 6Bx + 2C + F \cos x - (E + Fx) \sin x - E \sin x + (F - Ex) \cos x$$

$$= 12Ax^2 + 6Bx + 2C + (2F - Ex) \cos x - (2E + Fx) \sin x$$

$$y'''_p = 24Ax + 6B - E \cos x - (2F - Ex) \sin x - F \sin x - (2E + Fx) \cos x$$

$$= 24Ax + 6B - (3E + Fx)\cos x - (3F - Ex)\sin x$$

$$\begin{aligned} y^{(4)} &= 24A - F\cos x + (3E + Fx)\sin x + E\sin x - (3F - Ex)\cos x \\ &= 24A - (4F - Ex)\cos x + (4E + Fx)\sin x \end{aligned}$$

$$y^{(4)} + y'' = 3x^2 + 4\sin x - 2\cos x$$

$$\left\{ \begin{array}{lll} x^2 & 12A = 3 & \rightarrow A = \frac{1}{4} \\ x & 6B = 0 & \rightarrow B = 0 \\ x^0 & 24A + 2C = 0 & \rightarrow C = -3 \end{array} \right|$$

$$\left\{ \begin{array}{lll} \cos x & x & E - E \\ & x^0 & -4F + 2F = -2 \rightarrow F = 1 \\ \sin x & x & F - F \\ & x^0 & 4E - 2E = 4 \rightarrow E = 2 \end{array} \right|$$

$$y_p = \frac{1}{4}x^4 - 3x^2 + 2x\cos x + x\sin x$$

$$y(x) = C_1 + C_2x + C_3\cos x + C_4\sin x + \frac{1}{4}x^4 - 3x^2 + 2x\cos x + x\sin x$$

Exercise

Find the general solution $y^{(4)} + 2y'' + y = (x-2)^2$

Solution

The characteristic equation: $\lambda^4 + 2\lambda^2 + 1 = (\lambda^2 + 1)^2 = 0 \rightarrow \lambda_{1,2} = i \quad \lambda_{3,4} = -i$

$$y_h = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x$$

$$y_p = Ax^2 + Bx + C$$

$$y'_p = 2Ax + B$$

$$y''_p = 2A$$

$$y'''_p = y^{(4)}_p = 0$$

$$y^{(4)} + 2y'' + y = x^2 - 4x + 4$$

$$x^2 \quad A = 1$$

$$x \quad B = -4$$

$$x^0 \quad 4A + C = 4 \quad C = 0$$

$$\underline{y_p = x^2 - 4x}$$

$$\underline{y(x) = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x + x^2 - 4x}$$

Exercise

Find the general solution $y^{(4)} - y'' = 4x + 2xe^{-x}$

Solution

The characteristic equation: $\lambda^4 - \lambda^2 = \lambda^2(\lambda^2 - 1) = 0 \rightarrow \underline{\lambda_{1,2,3,4} = 0, 0, \pm 1}$

$$\underline{y_h = C_1 + C_2 x + C_3 e^{-x} + C_4 e^x}$$

$$y_p = Ax^3 + (Bx^2 + Cx)e^{-x}$$

$$y'_p = 3Ax^2 + (2Bx + C - Bx^2 - Cx)e^{-x}$$

$$y''_p = 6Ax + (Bx^2 - 4Bx + Cx + 2B - 2C)e^{-x}$$

$$\begin{aligned} y'''_p &= 6A + (2Bx - 4B + C - Bx^2 + 4Bx - Cx - 2B + 2C)e^{-x} \\ &= 6A + (-Bx^2 + 6Bx - Cx - 6B + 3C)e^{-x} \end{aligned}$$

$$\begin{aligned} y^{(4)}_p &= (Bx^2 - 6Bx + Cx + 6B - 3C - 2Bx + 6B - C)e^{-x} \\ &= (Bx^2 - 8Bx + Cx + 12B - 4C)e^{-x} \end{aligned}$$

$$y^{(4)} - y'' = 4x + 2xe^{-x}$$

$$-6A = 4 \rightarrow \underline{A = -\frac{2}{3}}$$

$$\begin{cases} e^{-x} & x^2 & -B - B = 0 \\ & x & -8B + C + 4B - C = 2 \\ & x^0 & 12B - 4C - 2B + 2C = 0 \end{cases} \quad \begin{aligned} B &= -\frac{1}{2} \\ C &= -\frac{5}{2} \end{aligned}$$

$$\underline{y_p = -\frac{2}{3}x^3 + \left(-\frac{1}{2}x^2 - \frac{5}{2}x\right)e^{-x}}$$

$$\underline{y(x) = C_1 + C_2 x + C_3 e^{-x} + C_4 e^x - \frac{2}{3}x^3 - \left(\frac{1}{2}x^2 + \frac{5}{2}x\right)e^{-x}}$$

Exercise

Find the general solution $(D^2 + D - 2)y = 2x - 40\cos 2x$

Solution

The characteristic equation: $\lambda^2 + \lambda - 2 = 0 \rightarrow \lambda_{1,2} = 1, -2$

$$y_h = C_1 e^x + C_2 e^{-2x}$$

$$y_p = Ax + B + C \cos 2x + D \sin 2x$$

$$y'_p = A - 2C \sin 2x + 2D \cos 2x$$

$$y''_p = -4C \cos 2x - 4D \sin 2x$$

$$y'' + y' - 2y = 2x - 40\cos 2x$$

$$-4C \cos 2x - 4D \sin 2x + A - 2C \sin 2x + 2D \cos 2x - 2Ax - 2B - 2C \cos 2x - 2D \sin 2x = 2x - 40\cos 2x$$

$$\begin{cases} x & -2A = 2 & A = -1 \\ x^0 & A - 2B = 0 & B = -\frac{1}{2} \\ \cos 2x & -6C + 2D = -40 & C = \frac{240}{40} = 6 \\ \sin 2x & -2C - 6D = 0 & D = -\frac{80}{40} = -2 \end{cases}$$

$$y_p = -x - \frac{1}{2} + 6\cos 2x - 2\sin 2x$$

$$y(x) = C_1 e^x + C_2 e^{-2x} - x - \frac{1}{2} + 6\cos 2x - 2\sin 2x$$

Exercise

Find the general solution $(D^2 - 3D + 2)y = 2\sin x$

Solution

The characteristic equation: $\lambda^2 + \lambda - 2 = 0 \rightarrow \lambda_{1,2} = 1, 2$

$$y_h = C_1 e^x + C_2 e^{2x}$$

$$y_p = A \cos x + B \sin x$$

$$y'_p = -A \sin x + B \cos x$$

$$y''_p = -A \cos x - B \sin x$$

$$y'' - 3y' + 2y = 2\sin x$$

$$-A \cos x - B \sin x + 3A \sin x - 3B \cos x + 2A \cos x + 2B \sin x = 2\sin x$$

$$\begin{cases} \cos x & A - 3B = 0 & \underline{A = \frac{6}{10} = \frac{3}{5}} \\ \sin x & 3A + B = 2 & \underline{B = \frac{2}{10} = \frac{1}{5}} \end{cases}$$

$$y_p = \frac{3}{5} \cos x + \frac{1}{5} \sin x$$

$$y(x) = C_1 e^x + C_2 e^{2x} + \frac{3}{5} \cos x + \frac{1}{5} \sin x$$

Exercise

Find the general solution $(D-2)^3(D^2+9)y = x^2 e^{2x} + x \sin 3x$

Solution

$$(\lambda-2)^3(\lambda^2+9)=0 \rightarrow \lambda_{1,2,3}=2; \lambda_{4,5}=\pm 3i$$

$$y_h = (C_1 + C_2 x + C_3 x^2) e^{2x} + C_4 \cos 3x + C_5 \sin 3x$$

$$y_p = (Ax^2 + Bx + C) e^{2x} + (Dx^2 + Ex) \cos 3x + (Fx^2 + Gx) \sin 3x$$

$$\begin{aligned} (D^3 - 6D^2 + 12D - 8)(D^2 + 9)y &= (D^5 - 6D^4 + 21D^3 - 62D^2 + 108D - 72)y \\ &= y^{(5)} - 6y^{(4)} + 21y^{(3)} - 62y'' + 108y' - 72y \end{aligned}$$

$$y_{p1} = (Ax^5 + Bx^4 + Cx^3) e^{2x} \quad x^3 (Ax^2 + Bx + C)$$

$$y'_{p1} = (2Ax^5 + 5Ax^4 + 2Bx^4 + 2Cx^3 + 4Bx^3 + 3Cx^2) e^{2x}$$

$$\begin{aligned} y''_{p1} &= (10Ax^4 + 20Ax^3 + 8Bx^3 + 6Cx^2 + 12Bx^2 + 6Cx + 4Ax^5 + 10Ax^4 + 4Bx^4 + 4Cx^3 + 8Bx^3 + 6Cx^2) e^{2x} \\ &= (4Ax^5 + 20Ax^4 + 4Bx^4 + 20Ax^3 + 16Bx^3 + 4Cx^3 + 12Cx^2 + 12Bx^2 + 6Cx) e^{2x} \end{aligned}$$

$$\begin{aligned} y^{(3)}_{p1} &= \left(\begin{aligned} &20Ax^4 + 80Ax^3 + 16Bx^3 + 60Ax^2 + 48Bx^2 + 12Cx^2 + 24Cx + 24Bx + 6C \\ &+ 8Ax^5 + 40Ax^4 + 8Bx^4 + 40Ax^3 + 32Bx^3 + 8Cx^3 + 24Cx^2 + 24Bx^2 + 12Cx \end{aligned} \right) e^{2x} \\ &= (8Ax^5 + 60Ax^4 + 8Bx^4 + 120Ax^3 + 48Bx^3 + 8Cx^3 + 60Ax^2 + 72Bx^2 + 36Cx^2 + 24Bx + 36Cx + 6C) e^{2x} \end{aligned}$$

$$\begin{aligned} y^{(4)}_{p1} &= \left(\begin{aligned} &16Ax^5 + 120Ax^4 + 16Bx^4 + 240Ax^3 + 96Bx^3 + 16Cx^3 + 120Ax^2 + 144Bx^2 + 72Cx^2 + 48Bx \\ &+ 72Cx + 12C + 40Ax^4 + 240Ax^3 + 32Bx^3 + 360Ax^2 + 144Bx^2 + 24Cx^2 + 120Ax + 144Bx + 36Cx \\ &+ 24B + 12C \end{aligned} \right) e^{2x} \\ &= \left(\begin{aligned} &16Ax^5 + 160Ax^4 + 16Bx^4 + 480Ax^3 + 128Bx^3 + 16Cx^3 + 480Ax^2 + 288Bx^2 + 96Cx^2 \\ &+ 120Ax + 192Bx + 144Cx + 24B + 48C \end{aligned} \right) e^{2x} \end{aligned}$$

$$y_{p1}^{(5)} = \left(\begin{array}{l} 32Ax^5 + 320Ax^4 + 32Bx^4 + 960Ax^3 + 256Bx^3 + 32Cx^3 + 960Ax^2 + 576Bx^2 + 192Cx^2 \\ + 240Ax + 384Bx + 288Cx + 48B + 96C + 80Ax^4 + 640Ax^3 + 64Bx^3 + 1440Ax^2 + 384Bx^2 \\ + 48Cx^2 + 960Ax + 576Bx + 192Cx + 120A + 192B + 144C \end{array} \right) e^{2x}$$

$$= \left(\begin{array}{l} 32Ax^5 + 400Ax^4 + 32Bx^4 + 1600Ax^3 + 320Bx^3 + 32Cx^3 + 2400Ax^2 + 960Bx^2 + 240Cx^2 \\ + 1200Ax + 960Bx + 480Cx + 120A + 240B + 240C \end{array} \right) e^{2x}$$

$$e^{2x} y^{(5)} - 6y^{(4)} + 21y^{(3)} - 62y'' + 108y' - 72y = x^2 e^{2x}$$

	x^5	x^4	x^3	x^2	x	x^0	
	32	400	1600	2400	1200	120	
	-96	-960	-2880	-2880	-720		
	168	1260	2520	1260			
A	-248	-1240	-1240				
	216	540					
	-72						
							$x^5 \quad 0=0$
							$x^4 \quad 0=0$
							$x^3 \quad 0=0$
		32	320	960	960	240	$x^2 \quad 780A=1 \quad A=\frac{1}{780}$
		-96	-768	-1728	-1152	-144	
B		168	1008	1512	504		$x \quad 480A+312B=0 \quad B=-\frac{1}{507}$
		-248	-992	-744			
		216	432				$x^0 \quad 120A+96B+78C=0 \quad C=\frac{1}{2197}$
		-72					
			32	240	480	240	
			-96	-576	-864	-288	
C			168	756	756	126	
			-248	-744	372		
			216	324			
			-72				

$$\Rightarrow A = \frac{1}{780}; \quad B = -\frac{1}{507}; \quad C = \frac{1}{2197} \rightarrow y_{p1} = \left(\frac{1}{780}x^5 - \frac{1}{507}x^4 + \frac{1}{2197}x^3 \right) e^{2x}$$

$$y_{p2} = (Dx^2 + Ex) \cos 3x + (Fx^2 + Gx) \sin 3x$$

$$y'_{p2} = (3Fx^2 + 3Gx + 2Dx + E) \cos 3x + (-3Dx^2 - 3Ex + 2Fx + G) \sin 3x$$

$$y''_{p2} = (-9Dx^2 - 9Ex + 12Fx + 6G + 2D) \cos 3x + (-9Fx^2 - 9Gx - 12Dx - 9E + 2F) \sin 3x$$

$$y^{(3)}_{p2} = (-27Fx^2 - 27Gx - 54Dx - 36E + 18F) \cos 3x + (27Dx^2 + 27Ex - 54Fx - 27G - 18D) \sin 3x$$

$$y^{(4)}_{p2} = (81Dx^2 + 81Ex - 216Fx - 108G - 108D) \cos 3x + (81Fx^2 + 81Gx + 216Dx + 135E - 108F) \sin 3x$$

$$y^{(5)}_{p2} = (243Fx^2 + 243Gx + 810Dx - 108E - 540F) \cos 3x + (-243Dx^2 - 243Ex + 810Fx + 405G + 540D) \sin 3x$$

$$y^{(5)} - 6y^{(4)} + 21y^{(3)} - 62y'' + 108y' - 72y = x \sin 3x$$

$\cos 3x$	x^2	x	x^0	
D	-486	810	648	
	558	-1134	-124	
	-72	216		
E		-486	-162	x^2
		558	-756	$0=0$
		-72	108	x
F	243	1296	-540	x^0
	-567	-744	378	$524D-810E-162F+276G=0$
	324			
G		243	648	
		-567	-372	
		324		

$\sin 3x$	x^2	x	x^0	
D	-243	-1296	540	
	567	744	-378	
	-324			
E		-243	486	x^2
		567	558	$0=0$
		-324		x
F	-486	810	648	x^0
	558	-1134	-124	$162D+1044E+524F-54G=0$
	-72	216		
G		-486	405	
		558	-567	
		-72	108	

$$\Rightarrow D = -\frac{23}{13,182}; \quad E = -\frac{251}{114,244}; \quad F = -\frac{3}{8788} \quad G = \frac{1379}{514,098}$$

$$y_{p2} = \left(-\frac{23}{13,182}x^2 - \frac{251}{114,244}x \right) \cos 3x + \left(-\frac{3}{8788}x^2 + \frac{1379}{514,098}x \right) \sin 3x$$

$$y(x) = \left(C_1 + C_2x + C_3x^2 + \frac{1}{780}x^5 - \frac{1}{507}x^4 + \frac{1}{2197}x^3 \right) e^{2x} - \left(+\frac{23}{13,182}x^2 + \frac{251}{114,244}x + C_4 \right) \cos 3x + \left(-\frac{3}{8788}x^2 + \frac{1379}{514,098}x + C_5 \right) \sin 3x$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + y = \cos x; \quad y(0)=1, \quad y'(0)=-1$$

Solution

The characteristic equation: $\lambda^2 + 1 = 0 \rightarrow \lambda_{1,2} = \pm i$

$$y_h = C_1 \cos x + C_2 \sin x$$

$$y_p = Ax \cos x + Bx \sin x$$

$$\begin{aligned} y'_p &= A \cos x - Ax \sin x + B \sin x + Bx \cos x \\ &= (A + Bx) \cos x + (B - Ax) \sin x \end{aligned}$$

$$y''_p = B \cos x - A \sin x - (A + Bx) \sin x + (B - Ax) \cos x$$

$$= (2B - Ax) \cos x - (2A + Bx) \sin x$$

$$y'' + y = \cos x$$

$$(2B - Ax) \cos x - (2A + Bx) \sin x + Ax \cos x + Bx \sin x = \cos x$$

$$2B \cos x - 2A \sin x = \cos x$$

$$\begin{cases} 2B = 1 \\ -2A = 0 \end{cases} \rightarrow \underline{A = 0, B = \frac{1}{2}}$$

$$y_p = \frac{1}{2} x \sin x$$

$$y(x) = C_1 \cos x + C_2 \sin x + \frac{1}{2} x \sin x$$

$$y(0) = 1 \rightarrow \underline{C_1 = 1}$$

$$y'(x) = -C_1 \sin x + C_2 \cos x + \frac{1}{2} (\sin x + x \cos x)$$

$$y'(0) = -1 \rightarrow \underline{C_2 = -1}$$

$$\underline{y(x) = \cos x - \sin x + \frac{1}{2} x \sin x}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + y' = x; \quad y(1) = 0, \quad y'(1) = 1$$

Solution

$$\text{The characteristic equation: } \lambda^2 + \lambda = 0 \rightarrow \underline{\lambda_{1,2} = 0, 1}$$

$$\underline{y_h = C_1 + C_2 e^x}$$

$$y_p = Ax^2 + Bx$$

$$y'_p = 2Ax + B$$

$$y''_p = 2A$$

$$y'' + y' = x$$

$$\begin{matrix} x & 2A = 1 \\ x^0 & 2A + B = 0 \end{matrix} \rightarrow \underline{A = \frac{1}{2}, B = -1}$$

$$\underline{y_p = \frac{1}{2} x^2 - x}$$

$$y(x) = C_1 + C_2 e^x + \frac{1}{2} x^2 - x$$

$$y(1) = 0 \rightarrow C_1 + C_2 e + \frac{1}{2} - 1 = 0 \quad \underline{C_1 + eC_2 = \frac{1}{2}}$$

$$y(x) = C_2 e^x + x - 1$$

$$y'(1) = 1 \rightarrow eC_2 = 1 \quad \underline{C_2 = \frac{1}{e}}$$

$$C_1 + e\frac{1}{e} = \frac{1}{2} \rightarrow \underline{C_1 = \frac{1}{2}}$$

$$y(x) = \frac{1}{2} + \frac{1}{e}e^x + \frac{1}{2}x^2 - x$$

$$\underline{= \frac{1}{2} + e^{x-1} + \frac{1}{2}x^2 - x}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + y' = -x ; \quad y(0) = 1, \quad y'(0) = 0$$

Solution

$$\text{The characteristic equation: } \lambda^2 + \lambda = 0 \rightarrow \underline{\lambda_{1,2} = 0, 1}$$

$$\underline{y_h = C_1 + C_2 e^x}$$

$$y_p = Ax^2 + Bx$$

$$y'_p = 2Ax + B$$

$$y''_p = 2A$$

$$y'' + y' = -x$$

$$\begin{array}{l} x \quad 2A = -1 \\ x^0 \quad 2A + B = 0 \end{array} \rightarrow \underline{A = -\frac{1}{2}, B = 1}$$

$$\underline{y_p = -\frac{1}{2}x^2 + x}$$

$$y(x) = C_1 + C_2 e^x - \frac{1}{2}x^2 + x$$

$$y(0) = 1 \rightarrow C_1 + C_2 = 1$$

$$y(x) = C_2 e^x - x + 1$$

$$y'(0) = 0 \rightarrow \underline{C_2 = -1} \Rightarrow \underline{C_1 = 2}$$

$$\underline{y(x) = 2 - e^x - \frac{1}{2}x^2 + x}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + y = 8\cos 2t - 4\sin t \quad y\left(\frac{\pi}{2}\right) = -1, \quad y'\left(\frac{\pi}{2}\right) = 0$$

Solution

The characteristic equation: $\lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$

$$y_h = C_1 \cos t + C_2 \sin t$$

The particular equation: $y_p = At \cos t + Bt \sin t + C \cos 2t + E \sin 2t$

$$y'_p = A \cos t - At \sin t + B \sin t + Bt \cos t - 2C \sin 2t + 2E \cos 2t$$

$$y''_p = -A \sin t - A \sin t - At \cos t + B \cos t + B \cos t - Bt \sin t - 4C \cos 2t - 4E \sin 2t$$

$$\begin{aligned} & -2A \sin t + 2B \cos t - At \cos t - Bt \sin t - 4C \cos 2t - 4E \sin 2t + At \cos t + Bt \sin t + C \cos 2t + E \sin 2t \\ & = 8\cos 2t - 4\sin t \end{aligned}$$

$$\begin{cases} -2A = -4 & \rightarrow A = 2 \\ 2B = 0 & \rightarrow B = 0 \\ -3C = 8 & \rightarrow C = -\frac{8}{3} \\ -3E = 0 & \rightarrow E = 0 \end{cases}$$

$$y_p = 2t \cos t - \frac{8}{3} \cos 2t$$

$$y(x) = C_1 \cos t + C_2 \sin t + 2t \cos t - \frac{8}{3} \cos 2t$$

$$y\left(\frac{\pi}{2}\right) = -1 \rightarrow C_2 + \frac{8}{3} = -1 \Rightarrow C_2 = -\frac{11}{3}$$

$$y'(x) = -C_1 \sin t + C_2 \cos t + 2 \cos t - 2t \sin t + \frac{16}{3} \sin 2t$$

$$y'\left(\frac{\pi}{2}\right) = 0 \rightarrow -C_1 - 2\frac{\pi}{2} = 0 \Rightarrow C_1 = -\pi$$

$$y(x) = -\pi \cos t - \frac{11}{3} \sin t + 2t \cos t - \frac{8}{3} \cos 2t$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - y' - 2y = 4x^2; \quad y(0) = 1, \quad y'(0) = 4$$

Solution

$$\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda_{1,2} = -1, 2$$

$$y_h = C_1 e^{-x} + C_2 e^{2x}$$

$$y_p = Ax^2 + Bx + C$$

$$y'_p = 2Ax + B$$

$$y''_p = 2A$$

$$y'' - y' - 2y = 4x^2$$

$$2A - 2Ax - B - 2Ax^2 - 2Bx - 2C = 4x^2$$

$$\begin{cases} x^2 & -2A = 4 & \rightarrow \underline{A = -2} \\ x & -2A - 2B = 0 & \rightarrow \underline{B = 2} \\ x^0 & 2A - B - 2C = 0 & \rightarrow \underline{C = -3} \end{cases}$$

$$y_p = -2x^2 + 2x - 3$$

$$y(x) = C_1 e^{-x} + C_2 e^{2x} - 2x^2 + 2x - 3$$

$$y(0) = 1 \rightarrow C_1 + C_2 - 3 = 1 \Rightarrow \underline{C_1 + C_2 = 4}$$

$$y' = -C_1 e^{-x} + 2C_2 e^{2x} - 4x + 2$$

$$y'(0) = 4 \rightarrow -C_1 + 2C_2 + 2 = 4 \Rightarrow -C_1 + 2C_2 = 2$$

$$\begin{cases} C_1 + C_2 = 4 \\ -C_1 + 2C_2 = 2 \end{cases} \rightarrow \underline{C_1 = 2, C_2 = 2}$$

$$y(x) = 2e^{-x} + 2e^{2x} - 2x^2 + 2x - 3$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - y' - 2y = e^{3x}; \quad y(1) = 2, \quad y'(1) = 1$$

Solution

$$\lambda^2 - \lambda - 2 = 0 \Rightarrow \underline{\lambda_{1,2} = -1, 2}$$

$$y_h = C_1 e^{-x} + C_2 e^{2x}$$

$$y_p = Ae^{3x}$$

$$y'_p = 3Ae^{3x}$$

$$y''_p = 9Ae^{3x}$$

$$y'' - y' - 2y = e^{3x}$$

$$9A - 3A - 2A = 1 \rightarrow \underline{A = \frac{1}{4}}$$

$$\underline{y_p = \frac{1}{4}e^{3x}}$$

$$y(x) = C_1 e^{-x} + C_2 e^{2x} + \frac{1}{4}e^{3x}$$

$$y(1) = 2 \rightarrow C_1 e^{-1} + C_2 e^2 + \frac{1}{4}e^3 = 2 \Rightarrow \frac{1}{e}C_1 + e^2 C_2 = 2 - \frac{1}{4}e^3$$

$$y'(x) = -C_1 e^{-x} + 2C_2 e^{2x} + \frac{3}{4}e^{3x}$$

$$y'(1) = 1 \rightarrow -\frac{1}{e}C_1 + 2e^2 C_2 + \frac{3}{4}e^3 = 1 \Rightarrow -C_1 + 2e^3 C_2 = e - \frac{3}{4}e^4$$

$$\begin{cases} C_1 + e^3 C_2 = 2e - \frac{1}{4}e^4 \\ -C_1 + 2e^3 C_2 = e - \frac{3}{4}e^4 \end{cases} \rightarrow 3e^3 C_2 = 3e - e^4$$

$$3e^3 C_2 = 3e - e^4 \rightarrow \underline{C_2 = e^{-2} - \frac{1}{3}e}$$

$$C_1 = 2e - \frac{1}{4}e^4 - e^3 \left(e^{-2} - \frac{1}{3}e \right) \rightarrow \underline{C_1 = e - \frac{1}{12}e^4}$$

$$\underline{y(x) = \left(e - \frac{1}{12}e^4 \right) e^{-x} + \left(e^{-2} - \frac{1}{3}e \right) e^{2x} + \frac{1}{4}e^{3x}}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - y' - 2y = e^{3x}; \quad y(0) = 1, \quad y'(0) = 2$$

Solution

$$\lambda^2 - \lambda - 2 = 0 \Rightarrow \underline{\lambda_{1,2} = -1, 2}$$

$$\underline{y_h = C_1 e^{-x} + C_2 e^{2x}}$$

$$y_p = Ae^{3x}$$

$$y'_p = 3Ae^{3x}$$

$$y''_p = 9Ae^{3x}$$

$$y'' - y' - 2y = e^{3x}$$

$$9A - 3A - 2A = 1 \rightarrow \underline{A = \frac{1}{4}}$$

$$\underline{y_p = \frac{1}{4}e^{3x}}$$

$$y(x) = C_1 e^{-x} + C_2 e^{2x} + \frac{1}{4} e^{3x}$$

$$y(0) = 1 \rightarrow C_1 + C_2 + \frac{1}{4} = 1 \Rightarrow C_1 + C_2 = \frac{3}{4}$$

$$y'(x) = -C_1 e^{-x} + 2C_2 e^{2x} + \frac{3}{4} e^{3x}$$

$$y'(0) = 2 \rightarrow -C_1 + 2C_2 + \frac{3}{4} = 2 \Rightarrow -C_1 + 2C_2 = \frac{5}{4}$$

$$\begin{cases} C_1 + C_2 = \frac{3}{4} \\ -C_1 + 2C_2 = \frac{5}{4} \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 3 \quad \Delta_{C_1} = \begin{vmatrix} \frac{3}{4} & 1 \\ \frac{5}{4} & 2 \end{vmatrix} = \frac{1}{4} \quad \Delta_{C_2} = \begin{vmatrix} 1 & \frac{3}{4} \\ -1 & \frac{5}{4} \end{vmatrix} = 2$$

$$C_1 = \frac{1}{12} \quad C_2 = \frac{2}{3}$$

$$y(x) = \frac{1}{12} e^{-x} + \frac{2}{3} e^{2x} + \frac{1}{4} e^{3x}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - y' - 2y = e^{3x}; \quad y(0) = 2, \quad y'(0) = 1$$

Solution

$$\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda_{1,2} = -1, 2$$

$$y_h = C_1 e^{-x} + C_2 e^{2x}$$

$$y_p = A e^{3x}$$

$$y'_p = 3A e^{3x}$$

$$y''_p = 9A e^{3x}$$

$$y'' - y' - 2y = e^{3x}$$

$$9A - 3A - 2A = 1 \rightarrow A = \frac{1}{4}$$

$$y_p = \frac{1}{4} e^{3x}$$

$$y(x) = C_1 e^{-x} + C_2 e^{2x} + \frac{1}{4} e^{3x}$$

$$y(0) = 2 \rightarrow C_1 + C_2 + \frac{1}{4} = 2 \Rightarrow C_1 + C_2 = \frac{7}{4}$$

$$y'(x) = -C_1 e^{-x} + 2C_2 e^{2x} + \frac{3}{4} e^{3x}$$

$$y'(0) = 1 \rightarrow -C_1 + 2C_2 + \frac{3}{4} = 1 \Rightarrow -C_1 + 2C_2 = \frac{1}{4}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 3 \quad \Delta_{C_1} = \begin{vmatrix} \frac{7}{4} & 1 \\ \frac{1}{4} & 2 \end{vmatrix} = \frac{13}{4} \quad \Delta_{C_2} = \begin{vmatrix} 1 & \frac{7}{4} \\ -1 & \frac{1}{4} \end{vmatrix} = 2$$

$$C_1 = \frac{13}{12} \quad C_2 = \frac{2}{3}$$

$$y(x) = \frac{13}{12} e^{-x} + \frac{2}{3} e^{2x} + \frac{1}{4} e^{3x}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + 2y' + y = 2 \cos t ; \quad y(0) = 3, \quad y'(0) = 0$$

Solution

$$\lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda_{1,2} = -1$$

$$y_h = (C_1 + C_2 t) e^{-t}$$

$$y_p = A \cos t + B \sin t$$

$$y'_p = -A \sin t + B \cos t$$

$$y''_p = -A \cos t - B \sin t$$

$$y'' + 2y' + y = 2 \cos t$$

$$\begin{cases} \cos t & -A + 2B + A = 2 \\ \sin t & -B - 2A + B = 0 \end{cases} \rightarrow A = 0, B = 1$$

$$y_p(t) = \sin t$$

$$y(t) = (C_1 + C_2 t) e^{-t} + \sin t$$

$$y(0) = 3 \rightarrow C_1 = 3$$

$$y' = (C_2 - C_1 - C_2 t) e^{-t} + \cos t$$

$$y'(0) = 0 \rightarrow C_2 - C_1 + 1 = 0 \Rightarrow C_2 = 2$$

$$y(t) = (3 + 2t) e^{-t} + \sin t$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - 2y' + y = t^3; \quad y(0) = 1, \quad y'(0) = 0$$

Solution

The homogeneous eq.: $y'' - 2y' + y = 0$

The characteristic eq.: $\lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda_{1,2} = 1$

$$y_h = (C_1 + C_2 t)e^t$$

$$y_p = at^3 + bt^2 + ct + d$$

$$y'_p = 3at^2 + 2bt + c$$

$$y''_p = 6at + 2b$$

$$\begin{aligned} y'' - 2y' + y &= 6at + 2b - 6at^2 - 4bt - 2c + at^3 + bt^2 + ct + d \\ &= at^3 + (b - 6a)t^2 + (6a - 4b + c)t + 2b - 2c + d \end{aligned}$$

$$\begin{cases} a = 1 \\ b - 6a = 0 \Rightarrow b = 6 \\ 6a - 4b + c = 0 \Rightarrow c = 18 \\ 2b - 2c + d = 0 \Rightarrow d = 24 \end{cases}$$

The particular solution is: $y_p = t^3 + 6t^2 + 18t + 24$

The general solution: $y = (C_1 + C_2 t)e^t + t^3 + 6t^2 + 18t + 24$

$$y(0) = (C_1 + C_2(0))e^{(0)} + (0)^3 + 6(0)^2 + 18(0) + 24$$

$$1 = C_1 + 24$$

$$y' = C_2 e^t (C_1 + C_2 t)e^t + 3t^2 + 12t + 18$$

$$y'(0) = C_2 e^{(0)} (C_1 + C_2(0))e^{(0)} + 3(0)^2 + 12(0) + 18$$

$$0 = C_2 + C_1 + 18$$

$$\underline{C_1 = -23 \quad C_2 = 5}$$

The general solution: $\underline{y(t) = (-23 + 5t)e^t + t^3 + 6t^2 + 18t + 24}$

Exercise

Find the solution of the given initial value problem

$$y'' - 2y' + y = -3 - x + x^2; \quad y(0) = -2, \quad y'(0) = 1$$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 1 = 0 \rightarrow \lambda_{1,2} = 1$

$$y_h = (C_1 + C_2 x)e^x$$

$$y_p = Ax^2 + Bx + C$$

$$y'_p = 2Ax + B$$

$$y''_p = 2A$$

$$y'' - 2y' + y = -3 - x + x^2$$

$$2A - 4Ax - 2B + Ax^2 + Bx + C = -3 - x + x^2$$

$$\begin{cases} x^2 & A = 1 \\ x & -4A + B = -1 \rightarrow B = 3 \\ x^0 & 2A - 2B + C = -3 \rightarrow C = 1 \end{cases}$$

$$y_p = x^2 + 3x + 1$$

$$y(x) = (C_1 + C_2 x)e^x + x^2 + 3x + 1$$

$$y(0) = -2 \rightarrow C_1 + 1 = -2 \quad C_1 = -3$$

$$y' = (C_2 + C_1 + C_2 x)e^x + 2x + 3$$

$$y'(0) = 1 \rightarrow C_2 - 3 + 3 = 1 \quad C_2 = 1$$

$$y(x) = (x - 3)e^x + x^2 + 3x + 1$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - 2y' + 2y = x + 1; \quad y(0) = 3, \quad y'(0) = 0$$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 2 = 0 \rightarrow \lambda_{1,2} = \frac{2 \pm 2i}{2} = 1 \pm i$

$$y_h = e^x (C_1 \cos x + C_2 \sin x)$$

$$y_p = Ax + B$$

$$y'_P = A$$

$$y''_P = 0$$

$$y'' - 2y' + 2y = x + 1$$

$$-2A + 2Ax + 2B = x + 1$$

$$\begin{cases} 2A = 1 \\ -2A + 2B = 1 \end{cases} \rightarrow \underline{A = \frac{1}{2}, B = 1}$$

$$y_P = \frac{1}{2}x + 1$$

$$y(x) = e^x \left(C_1 \cos x + C_2 \sin x \right) + \frac{1}{2}x + 1$$

$$\textcolor{red}{y(0) = 3} \rightarrow C_1 + 1 = 3 \quad \underline{C_1 = 2}$$

$$y'(x) = e^x \left(C_1 \cos x + C_2 \sin x - C_1 \sin x + C_2 \cos x \right) + \frac{1}{2}$$

$$\textcolor{red}{y'(0) = 0} \rightarrow C_1 + C_2 + \frac{1}{2} = 0 \Rightarrow \underline{C_2 = -\frac{5}{2}}$$

$$\underline{y(x) = e^x \left(2 \cos x - \frac{5}{2} \sin x \right) + \frac{1}{2}x + 1}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + 2y' + 2y = \sin 3x; \quad \textcolor{blue}{y(0) = 2}, \quad \textcolor{blue}{y'(0) = 0}$$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 2 = 0 \rightarrow \lambda_{1,2} = \frac{-2 \pm 2i}{2} = \underline{-1 \pm i}$

$$\underline{y_h = e^{-x} \left(C_1 \cos x + C_2 \sin x \right)}$$

$$y_P = A \cos 3x + B \sin 3x$$

$$y'_P = -3A \sin 3x + 3B \cos 3x$$

$$y''_P = -9A \cos 3x - 9B \sin 3x$$

$$y'' + 2y' + 2y = \sin 3x$$

$$-9A \cos 3x - 9B \sin 3x - 6A \sin 3x + 6B \cos 3x + 2A \cos 3x + 2B \sin 3x = \sin 3x$$

$$\begin{cases} \textcolor{red}{\cos 3x} & -7A + 6B = 0 \\ \textcolor{red}{\sin 3x} & -6A - 7B = 1 \end{cases}$$

$$\rightarrow \Delta = \begin{vmatrix} -7 & 6 \\ -6 & -7 \end{vmatrix} = 85 \quad \Delta_A = \begin{vmatrix} 0 & 6 \\ 1 & -7 \end{vmatrix} = -6 \quad \Delta_B = \begin{vmatrix} -7 & 0 \\ -6 & 1 \end{vmatrix} = -7$$

$$\underline{A = -\frac{6}{85}, B = -\frac{7}{85}}$$

$$y_p = -\frac{6}{85}\cos 3x - \frac{7}{85}\sin 3x$$

$$y(x) = e^{-x} \left(C_1 \cos x + C_2 \sin x \right) - \frac{6}{85}\cos 3x - \frac{7}{85}\sin 3x$$

$$y(0) = 2 \rightarrow C_1 - \frac{6}{85} = 2 \quad C_1 = \frac{176}{85}$$

$$y'(x) = e^{-x} \left(-C_1 \sin x + C_2 \cos x - C_1 \cos x - C_2 \sin x \right) + \frac{18}{85}\sin 3x - \frac{21}{85}\cos 3x$$

$$y'(0) = 0 \rightarrow C_2 - C_1 - \frac{21}{85} = 0 \quad C_2 = \frac{106}{85}$$

$$y(x) = e^{-x} \left(\frac{176}{85}\cos x + \frac{106}{85}\sin x \right) - \frac{6}{85}\cos 3x - \frac{7}{85}\sin 3x$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + 2y' + 2y = 2\cos 2t; \quad y(0) = -2, \quad y'(0) = 0$$

Solution

$$\text{The characteristic equation: } \lambda^2 + 2\lambda + 2 = 0 \rightarrow \lambda_{1,2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$\text{The homogenous solution: } y_h = e^{-t} (C_1 \cos t + C_2 \sin t)$$

$$\text{The particular solution: } z = Ae^{i2t}$$

$$z' = (2i)Ae^{i2t}$$

$$z'' = (2i)^2 Ae^{i2t}$$

$$z'' + 2z' + 2z = 2e^{i2t}$$

$$(2i)^2 Ae^{i2t} + 2(2i)Ae^{i2t} + 2Ae^{i2t} = 2e^{i2t}$$

$$(-4 + 4i + 2)A = 2$$

$$(-2 + 4i)A = 2$$

$$A = \frac{2}{-2 + 4i} \cdot \frac{-2 - 4i}{-2 - 4i}$$

$$= \frac{-4 - 8i}{20}$$

$$= \frac{-4}{20} - \frac{8i}{20}$$

$$= -\frac{1}{5} - \frac{2}{5}i$$

This gives the particular solution:

$$z = \left(-\frac{1}{5} - \frac{2}{5}i \right) e^{i2t}$$

$$= \left(-\frac{1}{5} - \frac{2}{5}i\right)(\cos 2t + i \sin 2t)$$

$$= -\frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t - i\left(\frac{1}{5}\sin 2t + \frac{2}{5}\cos 2t\right)$$

The real part of this solution $\left(-\frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t\right)$ is the particular solution of the system.

Thus, the general solution is:

$$y = e^{-t}\left(C_1 \cos t + C_2 \sin t\right) - \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t$$

The initial condition: $y(0) = -2$

$$-2 = e^{-0}\left(C_1 \cos 0 + C_2 \sin 0\right) - \frac{1}{5}\cos 0 + \frac{2}{5}\sin 0$$

$$-2 = C_1 - \frac{1}{5} \Rightarrow \boxed{C_1 = -\frac{9}{5}}$$

$$y' = e^{-t}\left(-C_1 \sin t + C_2 \cos t\right) - e^{-t}\left(C_1 \cos t + C_2 \sin t\right) + \frac{2}{5}\sin 2t + \frac{4}{5}\cos 2t$$

$$= -C_1 e^{-t} \sin t + C_2 e^{-t} \cos t - C_1 e^{-t} \cos t - C_2 e^{-t} \sin t + \frac{2}{5}\sin 2t + \frac{4}{5}\cos 2t$$

$$= (C_2 - C_1)e^{-t} \cos t - (C_1 + C_2)e^{-t} \sin t + \frac{2}{5}\sin 2t + \frac{4}{5}\cos 2t$$

The initial condition: $y'(0) = 0$

$$0 = (C_2 - C_1)e^{-0} \cos 0 - (C_1 + C_2)e^{-0} \sin 0 + \frac{2}{5}\sin 0 + \frac{4}{5}\cos 0$$

$$0 = C_2 - C_1 + \frac{4}{5}$$

$$\boxed{C_2 = C_1 - \frac{4}{5} = -\frac{9}{5} - \frac{4}{5} = -\frac{13}{5}}$$

The general solution: $\underline{y(t) = e^{-t}\left(-\frac{9}{5}\cos t - \frac{13}{5}\sin t\right) - \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t}$

Exercise

Find the solution of the given initial value problem

$$y'' - 2y' - 3y = 2e^x - 10\sin x ; \quad y(0) = 2, \quad y'(0) = 4$$

Solution

The characteristic equation: $\lambda^2 - 2\lambda - 3 = 0 \rightarrow \underline{\lambda_{1,2} = -1, 3}$

$$\underline{y_h = C_1 e^{-x} + C_2 e^{3x}}$$

$$y_p = Ae^x + B\cos x + C\sin x$$

$$y'_p = Ae^x - B\sin x + C\cos x$$

$$y_p'' = Ae^x - B \cos x - C \sin x$$

$$y'' - 2y' - 3y = 2e^x - 10 \sin x$$

$$\begin{cases} e^x & A - 2A - 3A = 2 \rightarrow A = -\frac{1}{2} \\ \cos x & -B - 2C - 3B = 0 \\ \sin x & -C + 2B - 3C = -10 \end{cases}$$

$$\begin{cases} -4B - 2C = 0 \\ 2B - 4C = -10 \end{cases} \rightarrow \begin{cases} B = -\frac{20}{20} = -1 \\ C = 2 \end{cases}$$

$$y_p = -\frac{1}{2}e^x - \cos x + 2 \sin x$$

$$y(x) = C_1 e^{-x} + C_2 e^{3x} - \frac{1}{2}e^x - \cos x + 2 \sin x$$

$$y(0) = 2 \rightarrow C_1 + C_2 - \frac{1}{2} - 1 = 2 \Rightarrow C_1 + C_2 = \frac{7}{2}$$

$$y' = -C_1 e^{-x} + 3C_2 e^{3x} - \frac{1}{2}e^x + \sin x + 2 \cos x$$

$$y'(0) = 4 \rightarrow -C_1 + 3C_2 - \frac{1}{2} + 2 = 4 \Rightarrow -C_1 + 3C_2 = \frac{5}{2}$$

$$\begin{cases} 2C_1 + 2C_2 = 7 \\ -2C_1 + 6C_2 = 5 \end{cases} \rightarrow \begin{cases} C_1 = \frac{32}{16} = 2 \\ C_2 = \frac{24}{16} = \frac{3}{2} \end{cases}$$

$$y(x) = 2e^{-x} + \frac{3}{2}e^{3x} - \frac{1}{2}e^x - \cos x + 2 \sin x$$

Exercise

Find the solution of the given initial value problem

$$y'' + 2y' + 10y = 4 + 26x + 6x^2 + 10x^3; \quad y(0) = 2, \quad y'(0) = 9$$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 10 = 0 \rightarrow \lambda_{1,2} = -1 \pm 3i$

$$y_h(x) = e^{-x} (C_1 \cos 3x + C_2 \sin 3x)$$

$$y_p = Ax^3 + Bx^2 + Cx + D$$

$$y_p' = 3Ax^2 + 2Bx + C$$

$$y_p'' = 6Ax + 2B$$

$$y'' + 2y' + 10y = 4 + 26x + 6x^2 + 10x^3$$

$$\begin{cases} x^3 & 10A = 10 & \rightarrow \underline{A = 1} \\ x^2 & 6A + 10B = 6 & \rightarrow \underline{B = 0} \\ x & 6A + 4B + 10C = 26 & \rightarrow \underline{C = 2} \\ x^0 & 2B + 2C + 10D = 4 & \rightarrow \underline{D = 0} \end{cases}$$

$$\underline{y_p = x^3 + 2x}$$

$$y(x) = e^{-x} (C_1 \cos 3x + C_2 \sin 3x) + x^3 + 2x$$

$$y(0) = 2 \rightarrow \underline{C_1 = 2}$$

$$y' = e^{-x} (-C_1 \cos 3x - C_2 \sin 3x - 3C_1 \sin 3x + 3C_2 \cos 3x) + 3x^2 + 2$$

$$y'(0) = 9 \rightarrow -C_1 + 3C_2 + 2 = 9 \Rightarrow \underline{C_2 = 3}$$

$$\underline{y(x) = e^{-x} (2\cos 3x + 3\sin 3x) + x^3 + 2x}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - 2y' + 10y = 6\cos 3t - \sin 3t ; \quad y(0) = 2, \quad y'(0) = -8$$

Solution

$$\lambda^2 - 2\lambda + 10 = 0 \rightarrow \underline{\lambda_{1,2} = 1 \pm 3i}$$

$$\underline{y_h(t) = e^t (C_1 \cos 3t + C_2 \sin 3t)}$$

$$y_p = A \cos 3t + B \sin 3t$$

$$y'_p = -3A \sin 3t + 3B \cos 3t$$

$$y''_p = -9A \cos 3t - 9B \sin 3t$$

$$y'' - 2y' + 10y = 6\cos 3t - \sin 3t$$

$$\begin{cases} \cos 3t & -9A - 6B + 10A = 6 \\ \sin 3t & -9B + 6A + 10B = -1 \end{cases} \rightarrow \begin{cases} A - 6B = 6 \\ 6A + B = -1 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & -6 \\ 6 & 1 \end{vmatrix} = 37 \quad \Delta_A = \begin{vmatrix} 6 & -6 \\ -1 & 1 \end{vmatrix} = 0 \quad \Delta_B = \begin{vmatrix} 1 & 6 \\ 6 & -1 \end{vmatrix} = -37$$

$$\underline{A = 0 \quad B = -1}$$

$$\underline{y_p(t) = -\sin 3t}$$

$$y(t) = e^t (C_1 \cos 3t + C_2 \sin 3t) - \sin 3t$$

$$y(0) = 2 \rightarrow \underline{C_1 = 2}$$

$$y' = e^t (C_1 \cos 3t + C_2 \sin 3t - 3C_1 \sin 3t + 3C_2 \cos 3t) - 3 \cos 3t$$

$$y'(0) = -8 \rightarrow C_1 + 3C_2 - 3 = -8 \Rightarrow \underline{C_2 = -\frac{7}{3}}$$

$$\underline{y(t) = e^t \left(2 \cos 3t - \frac{7}{3} \sin 3t \right) - \sin 3t}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + 3y' + 2y = e^x; \quad y(0) = 0, \quad y'(0) = 3$$

Solution

$$\text{The characteristic equation: } \lambda^2 + 3\lambda + 2 = 0 \rightarrow \underline{\lambda_{1,2} = -2, -1}$$

$$\underline{y_h = C_1 e^{-x} + C_2 e^{-2x}}$$

$$y_p = Ae^x$$

$$y'_p = Ae^x$$

$$y''_p = Ae^x$$

$$y'' + 3y' + 2y = e^x$$

$$(A + 3A + 2A)e^x = e^x \rightarrow \underline{A = \frac{1}{6}}$$

$$\underline{y_p = \frac{1}{6}e^x}$$

$$y(x) = C_1 e^{-x} + C_2 e^{-2x} + \frac{1}{6}e^x$$

$$y(0) = 0 \rightarrow C_1 + C_2 = -\frac{1}{6}$$

$$y'(x) = -C_1 e^{-x} - 2C_2 e^{-2x} + \frac{1}{6}e^x$$

$$y'(0) = 3 \rightarrow -C_1 - 2C_2 + \frac{1}{6} = 3 \Rightarrow C_1 + 2C_2 = -\frac{17}{6}$$

$$\begin{cases} C_1 + C_2 = -\frac{1}{6} \\ C_1 + 2C_2 = \frac{17}{6} \end{cases} \rightarrow \Delta = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \quad \Delta_{C_1} = \begin{vmatrix} -\frac{1}{6} & 1 \\ -\frac{17}{6} & 2 \end{vmatrix} = \frac{5}{2} \quad \Delta_{C_2} = \begin{vmatrix} 1 & -\frac{1}{6} \\ 1 & -\frac{17}{6} \end{vmatrix} = -\frac{8}{3}$$

$$\underline{C_1 = \frac{5}{2}, C_2 = -\frac{8}{3}}$$

$$\underline{y(x) = \frac{5}{2}e^{-x} - \frac{8}{3}e^{-2x} + \frac{1}{6}e^x}$$

Exercise

Find the general solution $y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x$ $y(0) = 1$ $y'(0) = 2$

Solution

$$\lambda^2 - 3\lambda + 2 = 0 \rightarrow \lambda_{1,2} = 1, 2$$

$$y_h = C_1 e^x + C_2 e^{2x}$$

$$y_p = Ae^{-x} + B\cos 3x + C\sin 3x$$

$$y'_p = -Ae^{-x} - 3B\sin 3x + 3C\cos 3x$$

$$y''_p = Ae^{-x} - 9B\cos 3x - 9C\sin 3x$$

$$Ae^{-x} - 9B\cos 3x - 9C\sin 3x + 3Ae^{-x} + 9B\sin 3x - 9C\cos 3x + 2Ae^{-x} + 2B\cos 3x + 2C\sin 3x =$$

$$3e^{-x} - 10\cos 3x$$

$$6A = 3 \quad A = \frac{1}{2}$$

$$-7B - 9C = -10 \quad B = \frac{7}{13}$$

$$9B - 7C = 0 \quad C = \frac{9}{13}$$

$$\Delta = \begin{vmatrix} 7 & 9 \\ 9 & -7 \end{vmatrix} = -130 \quad \Delta_B = \begin{vmatrix} 10 & 9 \\ 0 & -7 \end{vmatrix} = -70 \quad \Delta_C = \begin{vmatrix} 7 & 10 \\ 9 & 0 \end{vmatrix} = -90$$

$$\therefore y(x) = C_1 e^x + C_2 e^{2x} + \frac{1}{2}e^{-x} + \frac{7}{13}\cos 3x + \frac{9}{13}\sin 3x$$

$$y(0) = C_1 + C_2 + \frac{1}{2} + \frac{7}{13} = 1$$

$$y'(x) = C_1 e^x + 2C_2 e^{2x} - \frac{1}{2}e^{-x} - \frac{21}{13}\sin 3x + \frac{27}{13}\cos 3x \rightarrow y'(0) = C_1 + 2C_2 - \frac{1}{2} + \frac{27}{13} = 2$$

$$\begin{cases} C_1 + C_2 = -\frac{1}{26} \\ C_1 + 2C_2 = \frac{11}{26} \end{cases} \quad C_2 = \frac{12}{26} = \frac{6}{13} \quad C_1 = -\frac{1}{26} - \frac{6}{13} = -\frac{1}{2}$$

$$\therefore y(x) = -\frac{1}{2}e^x + \frac{6}{13}e^{2x} + \frac{1}{2}e^{-x} + \frac{7}{13}\cos 3x + \frac{9}{13}\sin 3x$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + 4y = -2; \quad y\left(\frac{\pi}{8}\right) = \frac{1}{2}, \quad y'\left(\frac{\pi}{8}\right) = 2$$

Solution

$$\text{The characteristic equation: } \lambda^2 + 4 = 0 \rightarrow \lambda_{1,2} = \pm 2i$$

$$y_h = C_1 \cos 2x + C_2 \sin 2x$$

$$y_P = A$$

$$y'_P = y''_P = 0$$

$$y'' + 4y = -2$$

$$4A = -2 \rightarrow \underline{A = -\frac{1}{2}}$$

$$\underline{y_P = -\frac{1}{2}}$$

$$y(x) = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{2}$$

$$y\left(\frac{\pi}{8}\right) = \frac{1}{2} \rightarrow \frac{\sqrt{2}}{2}C_1 + \frac{\sqrt{2}}{2}C_2 - \frac{1}{2} = \frac{1}{2} \Rightarrow \sqrt{2}C_1 + \sqrt{2}C_2 = 2$$

$$y'(x) = -2C_1 \sin 2x + 2C_2 \cos 2x$$

$$y\left(\frac{\pi}{8}\right) = 2 \rightarrow -\sqrt{2}C_1 + \sqrt{2}C_2 = 2$$

$$\begin{cases} \sqrt{2}C_1 + \sqrt{2}C_2 = 2 \\ -\sqrt{2}C_1 + \sqrt{2}C_2 = 2 \end{cases} \rightarrow \underline{C_2 = \sqrt{2}, C_1 = 0}$$

$$\underline{y(x) = \sqrt{2} \sin 2x - \frac{1}{2}}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + 4y = 2x ; \quad y(0) = 1, \quad y'(0) = 2$$

Solution

$$\text{The characteristic equation: } \lambda^2 + 4 = 0 \rightarrow \underline{\lambda_{1,2} = \pm 2i}$$

$$\underline{y_h = C_1 \cos 2x + C_2 \sin 2x}$$

$$y_P = Ax + B$$

$$y'_P = A$$

$$y''_P = 0$$

$$y'' + 4y = 2x$$

$$4Ax + B = 2x$$

$$\begin{cases} 4A = 2 \\ B = 0 \end{cases} \rightarrow \underline{A = \frac{1}{2}, B = 0}$$

$$\underline{y_P = \frac{1}{2}x}$$

$$y(x) = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{2}x$$

$$y(0)=1 \rightarrow \underline{C_1=1}$$

$$y'(x) = -2C_1 \sin 2x + 2C_2 \cos 2x + \frac{1}{2}$$

$$y'(0)=2 \rightarrow 2C_2 + \frac{1}{2} = 2 \quad \underline{C_2 = \frac{3}{4}}$$

$$\underline{y(x) = \cos 2x + \frac{3}{4} \sin 2x + \frac{1}{2} x}$$

Exercise

Find the solution of the given initial value problem

$$y'' - 4y' + 4y = e^x; \quad y(0) = 2, \quad y'(0) = 0$$

Solution

$$\text{The characteristic equation: } \lambda^2 - 4\lambda + 4 = 0 \rightarrow \underline{\lambda_{1,2} = 2}$$

$$\underline{y_h = (C_1 + C_2 x)e^{2x}}$$

$$y_p = Ae^x$$

$$y'_p = Ae^x$$

$$y''_p = Ae^x$$

$$y'' - 4y' + 4y = e^x$$

$$\rightarrow \underline{A=1}$$

$$\underline{y_p = e^x}$$

$$y(x) = (C_1 + C_2 x)e^{2x} + e^x$$

$$y(0)=2 \rightarrow C_1 + 1 = 2 \Rightarrow \underline{C_1 = 1}$$

$$y' = (C_2 + 2C_1 + 2C_2 x)e^{2x} + e^x$$

$$y'(0)=0 \rightarrow C_2 + 2C_1 + 1 = 0 \Rightarrow \underline{C_2 = -3}$$

$$\underline{y(x) = (1 - 3x)e^{2x} + e^x}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - 4y' + 8y = x^3; \quad y(0) = 2, \quad y'(0) = 4$$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 8 = 0 \rightarrow \lambda_{1,2} = \frac{4 \pm 4i}{2} = 2 \pm 2i$

$$\underline{y_h = e^{2x} (C_1 \cos 2x + C_2 \sin 2x)}$$

$$y_p = Ax^3 + Bx^2 + Cx + D$$

$$y'_p = 3Ax^2 + 2Bx + C$$

$$y''_p = 6Ax + 2B$$

$$y'' - 4y' + 8y = x^3$$

$$\begin{cases} x^3 & 8A = 1 \\ x^2 & -12A + 8B = 0 \\ x & 6A - 8B + 8C = 0 \\ x^0 & 2B - 4C + 8D = 0 \end{cases} \rightarrow \underline{A = \frac{1}{8}, B = \frac{3}{16}, C = \frac{3}{32}, D = 0}$$

$$\underline{y_p = \frac{1}{8}x^3 + \frac{3}{16}x^2 + \frac{3}{32}x}$$

$$y(x) = e^{2x} (C_1 \cos 2x + C_2 \sin 2x) + \frac{1}{8}x^3 + \frac{3}{16}x^2 + \frac{3}{32}x$$

$$y(0) = 2 \rightarrow \underline{C_1 = 2}$$

$$y'(x) = e^{2x} (2C_1 \cos 2x + 2C_2 \sin 2x - 2C_1 \sin 2x + 2C_2 \cos 2x) + \frac{3}{8}x^2 + \frac{3}{8}x + \frac{3}{32}$$

$$y'(0) = 4 \rightarrow 2C_1 + 2C_2 + \frac{3}{32} = 4 \quad \underline{C_2 = -\frac{3}{64}}$$

$$\underline{y(x) = e^{2x} \left(2 \cos 2x - \frac{3}{64} \sin 2x \right) + \frac{1}{8}x^3 + \frac{3}{16}x^2 + \frac{3}{32}x}$$

Exercise

Find the general solution $y'' + 4y = \sin^2 2t; \quad x\left(\frac{\pi}{8}\right) = 0 \quad x'\left(\frac{\pi}{8}\right) = 0$

Solution

$$\text{Characteristic Eqn.: } \lambda^2 + 4 = 0 \Rightarrow \underline{\lambda_{1,2} = \pm 2i}$$

$$\underline{y_h = C_1 \cos 2t + C_2 \sin 2t}$$

$$y'' + 4y = \frac{1}{2} - \frac{1}{2} \cos 4t$$

$$y_p = A + B \cos 4t + D \sin 4t$$

$$y'_p = -4B \sin 4t + 4D \cos 4t$$

$$y''_p = -16B \cos 4t - 16D \sin 4t$$

$$-16B \cos 4t - 16D \sin 4t + 4A + 4B \cos 4t + 4D \sin 4t = \frac{1}{2} - \frac{1}{2} \cos 4t$$

$$\begin{cases} 4A = \frac{1}{2} & \underline{A = \frac{1}{8}} \\ \cos 4t & -12B = -\frac{1}{2} & \underline{B = \frac{1}{24}} \\ \sin 4t & -12D = 0 & \underline{D = 0} \end{cases}$$

$$\underline{y_p = \frac{1}{8} + \frac{1}{24} \cos 4t}$$

$$y(t) = C_1 \cos 2t + C_2 \sin 2t + \frac{1}{8} + \frac{1}{24} \cos 4t$$

$$x\left(\frac{\pi}{8}\right) = 0 \rightarrow \frac{1}{\sqrt{2}}C_1 + \frac{1}{\sqrt{2}}C_2 + \frac{1}{8} = 0 \Rightarrow \underline{C_1 + C_2 = -\frac{\sqrt{2}}{8}}$$

$$y' = -2C_1 \sin 2t + 2C_2 \cos 2t - \frac{1}{6} \sin 4t$$

$$x\left(\frac{\pi}{8}\right) = 0 \rightarrow -\sqrt{2}C_1 + \sqrt{2}C_2 - \frac{1}{6} = 0 \Rightarrow \underline{C_1 - C_2 = \frac{\sqrt{2}}{12}}$$

$$\begin{cases} C_1 + C_2 = -\frac{\sqrt{2}}{8} \\ C_1 - C_2 = \frac{\sqrt{2}}{12} \end{cases} \rightarrow C_1 = -\frac{\sqrt{2}}{48}, \quad C_2 = -\frac{5\sqrt{2}}{48}$$

The **general** solution:

$$\underline{y(t) = -\frac{5\sqrt{2}}{48} \cos 2t - \frac{\sqrt{2}}{48} \sin 2t + \frac{1}{8} + \frac{1}{24} \cos 4t}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + 4y' + 4y = (3+x)e^{-2x}; \quad y(0) = 2, \quad y'(0) = 5$$

Solution

$$\text{The characteristic equation: } \lambda^2 + 4\lambda + 4 = 0 \Rightarrow \underline{\lambda_{1,2} = -2}$$

$$\underline{y_h = (C_1 + C_2 x)e^{-2x}}$$

$$y_p = (Ax^3 + Bx^2)e^{-2x}$$

$$y'_p = (-2Ax^3 - 2Bx^2 + 3Ax^2 + 2Bx)e^{-2x}$$

$$\begin{aligned} y''_p &= (4Ax^3 + 4Bx^2 - 6Ax^2 - 4Bx - 6Ax^2 - 4Bx + 6Ax + 2B)e^{-2x} \\ &= (4Ax^3 + 4Bx^2 - 12Ax^2 - 8Bx + 6Ax + 2B)e^{-2x} \end{aligned}$$

$$y'' + 4y' + 4y = (3+x)e^{-2x}$$

$$\begin{array}{rcl}
 x^3 & 4A - 8A + 4A = 0 & \\
 x^2 & 4B - 12A - 8B + 12A + 4B = 0 & \rightarrow \underline{A = \frac{1}{6}, B = \frac{3}{2}} \\
 x & -8B + 6A + 8B = 1 & \\
 x^0 & 2B = 3 &
 \end{array}$$

$$y_P = \left(\frac{1}{6}x^3 + \frac{3}{2}x^2 \right) e^{-2x}$$

$$y(x) = \left(C_1 + C_2 x + \frac{3}{2}x^2 + \frac{1}{6}x^3 \right) e^{-2x}$$

$$y(0) = 2 \rightarrow \underline{C_1 = 2}$$

$$y'(x) = \left(C_2 + 3x + \frac{1}{2}x^2 - 2C_1 - 2C_2 x - 3x^2 - \frac{1}{3}x^3 \right) e^{-2x}$$

$$y'(0) = 5 \rightarrow C_2 - 2C_1 = 5 \quad \underline{C_2 = 9}$$

$$\therefore y(x) = \left(2 + 9x + \frac{3}{2}x^2 + \frac{1}{6}x^3 \right) e^{-2x}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + 4y' + 4y = 4 - t; \quad y(0) = -1, \quad y'(0) = 0$$

Solution

$$\text{The characteristic eqn. : } \lambda^2 + 4\lambda + 4 = 0 \Rightarrow \underline{\lambda_{1,2} = -2}$$

$$y_h = \left(C_1 + C_2 t \right) e^{-2t}$$

$$y_p = at + b$$

$$y'_p = a$$

$$y''_p = 0$$

$$y'' + 4y' + 4y = 4a + 4at + 4b = 4at + 4a + 4b$$

$$\begin{cases}
 4a = -1 & \Rightarrow a = -\frac{1}{4} \\
 4a + 4b = 4 & \Rightarrow b = \frac{1}{4}
 \end{cases}$$

$$\text{The particular solution is: } y_p = -\frac{1}{4}t + \frac{1}{4}$$

$$\text{The general solution is: } y(t) = \left(C_1 + C_2 t \right) e^{-2t} - \frac{1}{4}t + \frac{1}{4}$$

$$y(0) = \left(C_1 + C_2 \cdot 0 \right) e^{-2(0)} - \frac{1}{4} \cdot 0 + \frac{1}{4}$$

$$-1 = C_1 + \frac{1}{4} \Rightarrow \underline{C_1 = -\frac{5}{4}}$$

$$y'(t) = C_2 e^{-2t} - 2(C_1 + C_2 t)e^{-2t} - \frac{1}{4}$$

$$0 = C_2 - 2C_1 - \frac{1}{4}$$

$$C_2 = -\frac{10}{4} + \frac{1}{4} \Rightarrow \underline{C_2 = -\frac{9}{4}}$$

The general solution is: $\underline{y(t) = (C_1 + C_2 t)e^{-2t} - \frac{1}{4}t + \frac{1}{4}}$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - 4y' - 5y = 4e^{-2t}; \quad y(0) = 0, \quad y'(0) = -1$$

Solution

The characteristic eq.: $\lambda^2 - 4\lambda - 5 = 0 \Rightarrow \lambda_1 = 5, \lambda_2 = -1$

$$y_h = C_1 e^{5t} + C_2 e^{-t}$$

$$y_p = A e^{-2t}$$

$$y' = -2A e^{-2t}$$

$$y'' = 4A e^{-2t}$$

$$y'' - 4y' - 5y = 4e^{-2t}$$

$$4A e^{-2t} + 8A e^{-2t} - 5A e^{-2t} = 4e^{-2t}$$

$$7A = 4$$

$$A = \frac{4}{7}$$

$$y_p = \frac{4}{7} e^{-2t}$$

$$y = C_1 e^{5t} + C_2 e^{-t} + \frac{4}{7} e^{-2t}$$

$$y(0) = C_1 e^{5(0)} + C_2 e^{-(0)} + \frac{4}{7} e^{-2(0)}$$

$$0 = C_1 + C_2 + \frac{4}{7}$$

$$C_1 + C_2 = -\frac{4}{7} \quad (1)$$

$$y' = 5C_1 e^{5t} - C_2 e^{-t} - \frac{8}{7} e^{-2t}$$

$$y'(0) = 5C_1 e^{5(0)} - C_2 e^{-(0)} - \frac{8}{7} e^{-2(0)}$$

$$-1 = 5C_1 - C_2 - \frac{8}{7}$$

$$5C_1 - C_2 = \frac{1}{7} \quad (2)$$

$$\underline{C_1 = -\frac{1}{14} \text{ and } C_2 = -\frac{1}{2}} \\ \underline{y(t) = -\frac{1}{14}e^{5t} - \frac{1}{2}e^{-t} + \frac{4}{7}e^{-2t}}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + 4y' + 5y = 35e^{-4x} ; \quad y(0) = -3, \quad y'(0) = 1$$

Solution

$$\text{The characteristic equation: } \lambda^2 + 4\lambda + 5 = 0 \Rightarrow \lambda_{1,2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

$$\underline{y_h = (C_1 \cos x + C_2 \sin x)e^{-2x}}$$

$$y_p = Ae^{-4x}$$

$$y'_p = -4Ae^{-4x}$$

$$y''_p = 16Ae^{-4x}$$

$$y'' + 4y' + 5y = 35e^{-4x}$$

$$16A - 16A + 5A = 35 \rightarrow \underline{A = 7}$$

$$\underline{y_p = 7e^{-4x}}$$

$$y(x) = (C_1 \cos x + C_2 \sin x)e^{-2x} + 7e^{-4x}$$

$$\textcolor{red}{y(0) = -3} \rightarrow C_1 + 7 = -3 \quad \underline{C_1 = -10}$$

$$y'(x) = (-C_1 \sin x + C_2 \cos x - 2C_1 \cos x - 2C_2 \sin x)e^{-2x} - 28e^{-4x}$$

$$\textcolor{red}{y'(0) = 1} \rightarrow C_2 - 2C_1 - 28 = 1 \quad \underline{C_2 = 9}$$

$$\underline{y(x) = (-10\cos x + 9\sin x)e^{-2x} + 7e^{-4x}}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + 4y' + 8y = \sin t ; \quad y(0) = 1, \quad y'(0) = 0$$

Solution

$$\lambda^2 + 4\lambda + 8 = 0 \Rightarrow \lambda_{1,2} = -2 \pm 2i$$

$$\underline{y_h = (C_1 \cos 2t + C_2 \sin 2t)e^{-2t}}$$

$$y_p = A \cos t + B \sin t$$

$$y_p = -A \sin t + B \cos t$$

$$y_p = -A \cos t - B \sin t$$

$$y'' + 4y' + 8y = \sin t$$

$$\begin{cases} \cos t & -A + 4B + 8A = 0 \\ \sin t & -B - 4A + 8B = 1 \end{cases} \rightarrow \begin{cases} 7A + 4B = 0 \\ -4A + 7B = 1 \end{cases}$$

$$\Delta = \begin{vmatrix} 7 & 4 \\ -4 & 7 \end{vmatrix} = 65 \quad \Delta_A = \begin{vmatrix} 0 & 4 \\ 1 & 7 \end{vmatrix} = -4 \quad \Delta_B = \begin{vmatrix} 7 & 0 \\ -4 & 1 \end{vmatrix} = 7$$

$$\underline{A = -\frac{4}{65}, \quad B = \frac{7}{65}}$$

$$\underline{y_p = -\frac{4}{65} \cos t + \frac{7}{65} \sin t}$$

$$y(t) = (C_1 \cos 2t + C_2 \sin 2t)e^{-2t} - \frac{4}{65} \cos t + \frac{7}{65} \sin t$$

$$y(0) = 1 \rightarrow C_1 - \frac{4}{65} = 1 \Rightarrow \underline{C_1 = \frac{69}{65}}$$

$$y' = (-2C_1 \sin 2t + 2C_2 \cos 2t - 2C_1 \cos 2t - 2C_2 \sin 2t)e^{-2t} + \frac{4}{65} \sin t + \frac{7}{65} \cos t$$

$$y'(0) = 0 \rightarrow 2C_2 - \frac{138}{65} + \frac{7}{65} = 0 \Rightarrow \underline{C_2 = \frac{131}{130}}$$

$$\underline{y(t) = \left(\frac{69}{65} \cos 2t + \frac{131}{130} \sin 2t \right) e^{-2t} - \frac{4}{65} \cos t + \frac{7}{65} \sin t}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - 4y' - 12y = 3e^{5t} ; \quad y(0) = \frac{18}{7}, \quad y'(0) = -\frac{1}{7}$$

Solution

$$\lambda^2 - 4\lambda - 12 = 0 \Rightarrow \underline{\lambda_{1,2} = -2, 6}$$

$$\underline{y_h = C_1 e^{-2t} + C_2 e^{6t}}$$

$$y_p = Ae^{5t}$$

$$y'_p = 5Ae^{5t}$$

$$y''_p = 25Ae^{5t}$$

$$y'' - 4y' - 12y = 3e^{5t}$$

$$25A - 20A - 12A = 3 \rightarrow \underline{A = -\frac{3}{7}}$$

$$\underline{y_p = -\frac{3}{7}e^{5t}}$$

$$y(t) = C_1 e^{-2t} + C_2 e^{6t} - \frac{3}{7}e^{5t}$$

$$\textcolor{red}{y(0) = \frac{18}{7}} \rightarrow C_1 + C_2 - \frac{3}{7} = \frac{18}{7} \Rightarrow C_1 + C_2 = 3$$

$$y' = -2C_1 e^{-2t} + 6C_2 e^{6t} - \frac{15}{7}e^{5t}$$

$$\textcolor{red}{y'(0) = -\frac{1}{7}} \rightarrow -2C_1 + 6C_2 - \frac{15}{7} = -\frac{1}{7} \Rightarrow -C_1 + 3C_2 = 1$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 4 \quad \Delta_{C_1} = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 8 \quad \Delta_{C_2} = \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4$$

$$\underline{C_1 = 2 \quad C_2 = 1}$$

$$\underline{y(t) = 2e^{-2t} + e^{6t} - \frac{3}{7}e^{5t}}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - 4y' - 12y = \sin 2t ; \quad y(0) = 0, \quad y'(0) = 0$$

Solution

$$\lambda^2 - 4\lambda - 12 = 0 \Rightarrow \underline{\lambda_{1,2} = -2, 6}$$

$$\underline{y_h = C_1 e^{-2t} + C_2 e^{6t}}$$

$$y_p = A \cos 2t + B \sin 2t$$

$$y'_p = -2A \sin 2t + 2B \cos 2t$$

$$y''_p = -4A \cos 2t - 4B \sin 2t$$

$$y'' - 4y' - 12y = \sin 2t$$

$$\begin{cases} \textcolor{red}{\cos 2t} & -4A - 8B - 12A = 0 \\ \textcolor{red}{\sin 2t} & -4B + 8A - 12B = 1 \end{cases} \rightarrow \begin{cases} 2A + B = 0 \\ 8A - 16B = 1 \end{cases}$$

$$\Delta = \begin{vmatrix} 2 & 1 \\ 8 & -16 \end{vmatrix} = -40 \quad \Delta_A = \begin{vmatrix} 0 & 1 \\ 1 & -16 \end{vmatrix} = -1 \quad \Delta_B = \begin{vmatrix} 2 & 0 \\ 8 & 1 \end{vmatrix} = 2$$

$$\underline{A = \frac{1}{40} \quad B = -\frac{1}{20}}$$

$$\underline{y_p = \frac{1}{40} \cos 2t - \frac{1}{20} \sin 2t}$$

$$y(t) = C_1 e^{-2t} + C_2 e^{6t} + \frac{1}{40} \cos 2t - \frac{1}{20} \sin 2t$$

$$y(0) = 0 \rightarrow C_1 + C_2 = -\frac{1}{40}$$

$$y' = -2C_1 e^{-2t} + 6C_2 e^{6t} - \frac{1}{20} \sin 2t - \frac{1}{10} \cos 2t$$

$$y'(0) = 0 \rightarrow -2C_1 + 6C_2 - \frac{1}{10} = 0 \Rightarrow -C_1 + 3C_2 = \frac{1}{20}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 4 \quad \Delta_{C_1} = \begin{vmatrix} -\frac{1}{40} & 1 \\ \frac{1}{20} & 3 \end{vmatrix} = -\frac{1}{8} \quad \Delta_{C_2} = \begin{vmatrix} 1 & -\frac{1}{40} \\ -1 & \frac{1}{20} \end{vmatrix} = \frac{1}{40}$$

$$C_1 = -\frac{1}{32} \quad C_2 = \frac{1}{160}$$

$$y(t) = -\frac{1}{32} e^{-2t} + \frac{1}{160} e^{6t} + \frac{1}{40} \cos 2t - \frac{1}{20} \sin 2t$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - 5y' = t - 2 \quad y(0) = 0, \quad y'(0) = 2$$

Solution

The characteristic equation: $\lambda^2 - 5\lambda = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = 5$

$$y_h = C_1 + C_2 e^{5t}$$

The particular equation: $y_p = At + Bt^2 \Rightarrow y'_p = A + 2Bt \rightarrow y''_p = 2B$

$$2B - 5A - 10Bt = t - 2$$

$$\begin{cases} -10B = 1 & \rightarrow B = -\frac{1}{10} \\ 2B - 5A = -2 & \rightarrow A = \frac{1}{5} \left(-\frac{1}{5} + 2 \right) = \frac{9}{25} \end{cases}$$

$$\Rightarrow y_p = -\frac{1}{10} t^2 + \frac{9}{25} t$$

$$y(t) = C_1 + C_2 e^{5t} - \frac{1}{10} t^2 + \frac{9}{25} t$$

$$y(0) = 0 \Rightarrow C_1 + C_2 = 0 \Rightarrow C_1 = -C_2$$

$$y'(t) = 5C_2 e^{5t} - \frac{1}{5} t + \frac{9}{25}$$

$$y'(0) = 2 \Rightarrow 5C_2 + \frac{9}{25} = 2$$

$$C_2 = \frac{1}{5} \left(2 - \frac{9}{25} \right) = \frac{41}{125} \quad C_1 = -\frac{41}{125}$$

$$y(t) = -\frac{41}{125} + \frac{41}{125} e^{5t} - \frac{1}{10} t^2 + \frac{9}{25} t$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + 5y' - 6y = 10e^{2x} ; \quad y(0) = 1, \quad y'(0) = 1$$

Solution

The characteristic equation: $\lambda^2 + 5\lambda - 6 = 0 \Rightarrow \lambda_{1,2} = 1, -6$

$$y_h = C_1 e^x + C_2 e^{-6x}$$

$$y_p = Ae^{2x}$$

$$y'_p = 2Ae^{2x}$$

$$y''_p = 4Ae^{2x}$$

$$y'' + 5y' - 6y = 10e^{2x}$$

$$4A + 10A - 6A = 10 \rightarrow A = \frac{5}{4}$$

$$y_p = \frac{5}{4}e^{2x}$$

$$y(x) = C_1 e^x + C_2 e^{-6x} + \frac{5}{4}e^{2x}$$

$$y(0) = 1 \rightarrow C_1 + C_2 + \frac{5}{4} = 1 \Rightarrow C_1 + C_2 = -\frac{1}{4}$$

$$y'(x) = C_1 e^x - 6C_2 e^{-6x} + \frac{5}{2}e^{2x}$$

$$y'(0) = 1 \rightarrow C_1 - 6C_2 + \frac{5}{2} = 1 \Rightarrow C_1 - 6C_2 = -\frac{3}{2}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 1 & -6 \end{vmatrix} = -7 \quad \Delta_{C_1} = \begin{vmatrix} -\frac{1}{4} & 1 \\ -\frac{3}{2} & -6 \end{vmatrix} = 3 \quad \Delta_{C_2} = \begin{vmatrix} 1 & -\frac{1}{4} \\ 1 & -\frac{3}{2} \end{vmatrix} = -\frac{5}{4}$$

$$C_1 = -\frac{3}{7}, \quad C_2 = \frac{5}{28}$$

$$y(x) = -\frac{3}{7}e^x + \frac{5}{28}e^{-6x} + \frac{5}{4}e^{2x}$$

Exercise

Find the solution of the given initial value problem

$$y'' + 6y' + 10y = 22 + 20x ; \quad y(0) = 2, \quad y'(0) = -2$$

Solution

The characteristic equation: $\lambda^2 + 6\lambda + 10 = 0 \rightarrow \lambda_{1,2} = -3 \pm i$

$$\underline{y_h = e^{-3x}(C_1 \cos x + C_2 \sin x)}$$

$$y_p = Ax + B$$

$$y'_p = A$$

$$y_p = 0$$

$$y'' + 6y' + 10y = 22 + 20x$$

$$\begin{cases} x & 10A = 20 & \rightarrow \underline{A = 2} \\ x^0 & 6A + 10B = 22 & \rightarrow \underline{B = 1} \end{cases}$$

$$\underline{y_p = 2x + 1}$$

$$y(x) = e^{-3x}(C_1 \cos x + C_2 \sin x) + 2x + 1$$

$$y(0) = 2 \rightarrow C_1 + 1 = 2 \Rightarrow \underline{C_1 = 1}$$

$$y' = e^{-3x}(-3C_1 \cos x - 3C_2 \sin x - C_1 \sin x + C_2 \cos x) + 2$$

$$y'(0) = -2 \rightarrow -3 + C_2 + 2 = -2 \Rightarrow \underline{C_2 = -1}$$

$$\therefore \underline{y(x) = e^{-3x}(\cos x - \sin x) + 2x + 1}$$

Exercise

Find the solution of the given initial value problem

$$y'' + 7y' + 12y = -2\cos 2x + 36\sin 2x; \quad y(0) = -3, \quad y'(0) = 3$$

Solution

$$\text{The characteristic equation: } \lambda^2 + 7\lambda + 12 = 0 \rightarrow \underline{\lambda_{1,2} = -4, -3}$$

$$\underline{y_h = C_1 e^{-4x} + C_2 e^{-3x}}$$

$$y_p = A \cos 2x + B \sin 2x$$

$$y'_p = -2A \sin 2x + 2B \cos 2x$$

$$y_p = -4A \cos 2x - 4B \sin 2x$$

$$y'' + 7y' + 12y = -2\cos 2x + 36\sin 2x$$

$$\begin{cases} \cos 2x & -4A + 14B + 12A = -2 & \rightarrow 4A + 7B = -1 \\ \sin 2x & -4B - 14A + 12B = 36 & \rightarrow -7A + 4B = 18 \end{cases}$$

$$A = -\frac{130}{65} = -2 \quad B = \frac{65}{65} = 1$$

$$\underline{y_p = -2\cos 2x + \sin 2x}$$

$$y(x) = C_1 e^{-4x} + C_2 e^{-3x} - 2\cos 2x + \sin 2x$$

$$y(0) = -3 \rightarrow C_1 + C_2 - 2 = -3 \Rightarrow C_1 + C_2 = -1$$

$$y' = -4C_1 e^{-4x} - 3C_2 e^{-3x} + 4\sin 2x + 2\cos 2x$$

$$y'(0) = 3 \rightarrow -4C_1 - 3C_2 + 2 = 3 \Rightarrow -4C_1 - 3C_2 = 1$$

$$\begin{cases} C_1 + C_2 = -1 \\ -4C_1 - 3C_2 = 1 \end{cases} \rightarrow \underline{C_1 = 2} \quad \underline{C_2 = -3}$$

$$\underline{y(x) = 2e^{-4x} - 3e^{-3x} - 2\cos 2x + \sin 2x}$$

Exercise

Find the solution of the given initial value problem

$$y'' + 8y' + 7y = 10e^{-2x}; \quad y(0) = -2, \quad y'(0) = 10$$

Solution

$$\text{The characteristic equation: } \lambda^2 + 8\lambda + 7 = 0 \rightarrow \underline{\lambda_{1,2} = -1, -7}$$

$$\underline{y_h = C_1 e^{-x} + C_2 e^{-7x}}$$

$$y_p = Ae^{-2x}$$

$$y'_p = -2Ae^{-2x}$$

$$y''_p = 4Ae^{-2x}$$

$$y'' + 8y' + 7y = 10e^{-2x}$$

$$(4 - 16 + 7)Ae^{-2x} = 10e^{-2x} \rightarrow \underline{A = -2}$$

$$\underline{y_p = -2e^{-2x}}$$

$$y(x) = C_1 e^{-x} + C_2 e^{-7x} - 2e^{-2x}$$

$$y(0) = -2 \rightarrow C_1 + C_2 - 2 = -2 \Rightarrow \underline{C_1 + C_2 = 0}$$

$$y' = -C_1 e^{-x} - 7C_2 e^{-7x} + 4e^{-2x}$$

$$y'(0) = 10 \rightarrow -C_1 - 7C_2 + 4 = 10 \Rightarrow \underline{-C_1 - 7C_2 = 6}$$

$$\begin{cases} C_1 + C_2 = 0 \\ C_1 + 7C_2 = -6 \end{cases} \rightarrow C_1 = \frac{6}{6} = 1 \quad \underline{C_1 = -1}$$

$$\underline{y(x) = e^{-x} - e^{-7x} - 2e^{-2x}}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + 9y = \sin 2x; \quad y(0) = 1, \quad y'(0) = 0$$

Solution

$$\text{The characteristic equation: } \lambda^2 + 9 = 0 \rightarrow \underline{\lambda_{1,2} = \pm 3i}$$

$$\underline{y_h = C_1 \cos 3x + C_2 \sin 3x}$$

$$y_p = A \cos 2x + B \sin 2x$$

$$y'_p = -2A \sin 2x + 2B \cos 2x$$

$$y''_p = -4A \cos 2x - 4B \sin 2x$$

$$y'' + 9y = \sin 2x$$

$$-4A \cos 2x - 4B \sin 2x + 9A \cos 2x + 9B \sin 2x = \sin 2x$$

$$\begin{cases} \cos 2x & 5A = 0 \\ \sin 2x & 5B = 1 \end{cases} \rightarrow \underline{A = 0, B = \frac{1}{5}}$$

$$y_p = \frac{1}{5} \sin 2x$$

$$y(x) = C_1 \cos 3x + C_2 \sin 3x + \frac{1}{5} \sin 2x$$

$$\underline{y(0) = 1} \rightarrow \underline{C_1 = 1}$$

$$y'(x) = -3C_1 \sin 3x + 3C_2 \cos 3x + \frac{2}{5} \cos 2x$$

$$\underline{y'(0) = 0} \rightarrow 3C_2 + \frac{2}{5} = 0 \quad \underline{C_2 = -\frac{2}{15}}$$

$$\underline{y(x) = \cos 3x - \frac{2}{15} \sin 3x + \frac{1}{5} \sin 2x}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - 64y = 16; \quad y(0) = 1, \quad y'(0) = 0$$

Solution

$$\text{The characteristic equation: } \lambda^2 - 64 = 0 \Rightarrow \underline{\lambda_{1,2} = \pm 8}$$

$$\underline{y_h = C_1 e^{-8x} + C_2 e^{8x}}$$

The particular equation: $y_p = A \Rightarrow y'_p = y''_p = 0$

$$-64A = 16 \Rightarrow A = -\frac{1}{4} \Rightarrow \underline{y_p = -\frac{1}{4}}$$

$$y(x) = C_1 e^{-8x} + C_2 e^{8x} - \frac{1}{4}$$

$$y(0) = 1 \rightarrow C_1 + C_2 - \frac{1}{4} = 1$$

$$y' = -8C_1 e^{-8x} + 8C_2 e^{8x}$$

$$y'(0) = 0 \rightarrow -8C_1 + 8C_2 = 0$$

$$\begin{cases} C_1 + C_2 = \frac{5}{4} \\ -C_1 + C_2 = 0 \end{cases} \rightarrow C_2 = \frac{5}{8} = C_1$$

$$\underline{y(x) = \frac{5}{8}e^{-8x} + \frac{5}{8}e^{8x} - \frac{1}{4}}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$2y'' + 3y' - 2y = 14x^2 - 4x + 11; \quad y(0) = 0, \quad y'(0) = 0$$

Solution

The characteristic equation: $2\lambda^2 + 3\lambda - 2 = 0 \rightarrow \lambda_{1,2} = \frac{-3 \pm 5}{4} = -2, \frac{1}{2}$

$$\underline{y_h = C_1 e^{-2x} + C_2 e^{x/2}}$$

$$y_p = Ax^2 + Bx + C$$

$$y'_p = 2Ax + B$$

$$y''_p = 2A$$

$$2y'' + 3y' - 2y = 14x^2 - 4x + 11$$

$$x^2 \quad -2A = 14$$

$$x \quad 6A - 2B = -4 \rightarrow \underline{A = -7, B = -19, C = -37}$$

$$x^0 \quad 4A + 3B - 2C = 11$$

$$\underline{y_p = -7x^2 - 19x - 37}$$

$$y(x) = C_1 e^{-2x} + C_2 e^{x/2} - 7x^2 - 19x - 37$$

$$y(0) = 0 \rightarrow C_1 + C_2 = 37$$

$$y'(x) = -2C_1 e^{-2x} + \frac{1}{2}C_2 e^{x/2} - 14x - 19$$

$$y'(0) = 0 \rightarrow -2C_1 + \frac{1}{2}C_2 = 19$$

$$\begin{cases} C_1 + C_2 = 37 \\ -4C_1 + C_2 = 38 \end{cases} \rightarrow \underline{C_1 = -\frac{1}{5}, C_2 = \frac{186}{5}}$$

$$\underline{y(x) = -\frac{1}{5}e^{-2x} + \frac{186}{5}e^{x/2} - 7x^2 - 19x - 37}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$5y'' + y' = -6x; \quad y(0) = 0, \quad y'(0) = -10$$

Solution

$$\text{The characteristic equation: } 5\lambda^2 + \lambda = 0 \rightarrow \underline{\lambda_{1,2} = 0, -\frac{1}{5}}$$

$$\underline{y_h = C_1 + C_2 e^{-x/5}}$$

$$y_p = Ax^2 + Bx$$

$$y'_p = 2Ax + B$$

$$y''_p = 2A$$

$$5y'' + y' = -6x$$

$$\begin{cases} x & 2A = -6 \\ x^0 & 10A + B = 0 \end{cases} \rightarrow \underline{A = -3, B = 30}$$

$$\underline{y_p = -3x^2 + 30x}$$

$$y(x) = C_1 + C_2 e^{-x/5} - 3x^2 + 30x$$

$$y(0) = 0 \rightarrow C_1 + C_2 = 0$$

$$y(x) = -\frac{1}{5}C_2 e^{-x/5} - 6x + 30$$

$$y'(0) = -10 \rightarrow -\frac{1}{5}C_2 + 30 = -10 \quad \underline{C_2 = 200}$$

$$C_1 = -C_2 \rightarrow \underline{C_1 = -200}$$

$$\underline{y(x) = 200e^{-x/5} - 200 + 30x - 3x^2}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$x'' + 9x = 10\cos 2t; \quad x(0) = x'(0) = 0$$

Solution

The characteristic equation: $\lambda^2 + 9 = 0 \rightarrow \lambda_{1,2} = \pm 3i$

$$x_h = C_1 \cos 3t + C_2 \sin 3t$$

$$x_p = A \cos 2t + B \sin 2t$$

$$x'_p = -2A \sin 2t + 2B \cos 2t$$

$$x''_p = -4A \cos 2t - 4B \sin 2t$$

$$x'' + 9x = 10\cos 2t$$

$$-4A \cos 2t - 4B \sin 2t + 9A \cos 2t + 9B \sin 2t = 10\cos 2t$$

$$\begin{cases} \cos 2t & 5A = 10 \\ \sin 2t & 5B = 0 \end{cases} \rightarrow A = 2, B = 0$$

$$x_p = 2 \cos 2t$$

$$x(t) = C_1 \cos 3t + C_2 \sin 3t + 2 \cos 2t$$

$$x(0) = 0 \rightarrow C_1 = -2$$

$$x'(t) = -3C_1 \sin 3t + 3C_2 \cos 3t - 4 \sin 2t$$

$$x'(0) = 0 \rightarrow C_2 = 0$$

$$x(t) = -2 \cos 3t + 2 \cos 2t$$

Exercise

Find the general solution that satisfy the given initial conditions

$$x'' + 4x = 5\sin 3t; \quad x(0) = x'(0) = 0$$

Solution

The characteristic equation: $\lambda^2 + 4 = 0 \rightarrow \lambda_{1,2} = \pm 2i$

$$x_h = C_1 \cos 2t + C_2 \sin 2t$$

$$x_p = A \cos 3t + B \sin 3t$$

$$x'_p = -3A \sin 3t + 3B \cos 3t$$

$$x''_p = -9A \cos 3t - 9B \sin 3t$$

$$x'' + 4x = 5\sin 3t$$

$$-9A \cos 3t - 9B \sin 3t + 4A \cos 3t + 4B \sin 3t = 5 \sin 3t$$

$$\begin{cases} \cos 3t & -5A = 0 \\ \sin 3t & -5B = 5 \end{cases} \rightarrow \underline{A = 0, B = -1}$$

$$\underline{x_p = -\sin 3t}$$

$$x(t) = C_1 \cos 2t + C_2 \sin 2t - \sin 3t$$

$$\underline{x(0) = 0 \rightarrow C_1 = 0}$$

$$x'(t) = -2C_1 \sin 2t + 2C_2 \cos 2t - 3 \cos 3t$$

$$\underline{x'(0) = 0 \rightarrow C_2 = \frac{3}{2}}$$

$$\underline{x(t) = \frac{3}{2} \sin 2t - \sin 3t}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$x'' + 100x = 225 \cos 5t + 300 \sin 5t ; \quad \underline{x(0) = 375, \quad x'(0) = 0}$$

Solution

$$\text{The characteristic equation: } \lambda^2 + 100 = 0 \rightarrow \underline{\lambda_{1,2} = \pm 10i}$$

$$\underline{x_h = C_1 \cos 10t + C_2 \sin 10t}$$

$$x_p = A \cos 5t + B \sin 5t$$

$$x'_p = -5A \sin 5t + 5B \cos 5t$$

$$x''_p = -25A \cos 5t - 25B \sin 5t$$

$$x'' + 100x = 225 \cos 5t + 300 \sin 5t$$

$$-25A \cos 5t - 25B \sin 5t + 100 \cos 5t + 100 \sin 10t = 225 \cos 5t + 300 \sin 5t$$

$$\begin{cases} \cos 5t & 75A = 225 \\ \sin 5t & 75B = 300 \end{cases} \rightarrow \underline{A = 3, B = 4}$$

$$\underline{x_p = 3 \cos 5t + 4 \sin 5t}$$

$$x(t) = C_1 \cos 10t + C_2 \sin 10t + 3 \cos 5t + 4 \sin 5t$$

$$\underline{x(0) = 375 \rightarrow C_1 + 3 = 375 \quad C_1 = 372}$$

$$x'(t) = -10C_1 \sin 10t + 10C_2 \cos 10t - 15 \sin 5t + 20 \cos 5t$$

$$\underline{x'(0) = 0 \rightarrow 10C_2 + 20 = 0 \quad C_2 = -2}$$

$$\underline{x(t) = 372 \cos 10t - 2 \sin 10t + 3 \cos 5t + 4 \sin 5t}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$x'' + 25x = 90\cos 4t ; \quad x(0) = 0, \quad x'(0) = 90$$

Solution

The characteristic equation: $\lambda^2 + 25 = 0 \rightarrow \lambda_{1,2} = \pm 5i$

$$x_h = C_1 \cos 5t + C_2 \sin 5t$$

$$x_p = A \cos 4t + B \sin 4t$$

$$x'_p = -4A \sin 4t + 4B \cos 4t$$

$$x''_p = -16A \cos 4t - 16B \sin 4t$$

$$x'' + 25x = 90\cos 4t$$

$$-16A \cos 4t - 16B \sin 4t + 25A \cos 4t + 25B \sin 4t = 90\cos 4t$$

$$\begin{cases} \cos 4t & 9A = 90 \\ \sin 4t & 9B = 0 \end{cases} \rightarrow \underline{A = 10, B = 0}$$

$$x_p = 10\cos 4t$$

$$x(t) = C_1 \cos 5t + C_2 \sin 5t + 10\cos 4t$$

$$x(0) = 0 \rightarrow \underline{C_1 = -10}$$

$$x'(t) = -5C_1 \sin 5t + 5C_2 \cos 5t - 40\sin 4t$$

$$x'(0) = 90 \rightarrow 5C_2 = 90 \quad \underline{C_2 = 18}$$

$$x(t) = \underline{-10\cos 5t + 18\sin 5t + 10\cos 4t}$$

Exercise

Find the solution of the given initial value problem

$$y^{(3)} - y' = 4e^{-x} + 3e^{2x} ; \quad y(0) = 0, \quad y'(0) = -1, \quad y''(0) = 2$$

Solution

The characteristic equation: $\lambda^3 - \lambda = \lambda(\lambda^2 - 1) = 0 \rightarrow \lambda_{1,2,3} = 0, \pm 1$

$$y_h = C_1 + C_2 e^{-x} + C_3 e^x$$

$$y_p = A x e^{-x} + B e^{2x}$$

$$y'_p = (A - A x) e^{-x} + 2B e^{2x}$$

$$y_p'' = (-2A + Ax)e^{-x} + 4Be^{2x}$$

$$y_p''' = (3A - Ax)e^{-x} + 8Be^{2x}$$

$$y^{(3)} - y' = 4e^{-x} + 3e^{2x}$$

$$(2A)e^{-x} + 6Be^{2x} = 4e^{-x} + 3e^{2x}$$

$$\begin{cases} 2A = 4 & \rightarrow A = 2 \\ 6B = 3 & \rightarrow B = \frac{1}{2} \end{cases}$$

$$y_p = 2xe^{-x} + \frac{1}{2}e^{2x}$$

$$y(x) = C_1 + C_2e^{-x} + C_3e^x + 2xe^{-x} + \frac{1}{2}e^{2x}$$

$$y(0) = 0 \rightarrow C_1 + C_2 + C_3 + \frac{1}{2} = 0 \Rightarrow C_1 + C_2 + C_3 = -\frac{1}{2}$$

$$y' = -C_2e^{-x} + C_3e^x + 2e^{-x} - 2xe^{-x} + e^{2x}$$

$$y'(0) = -1 \rightarrow -C_2 + C_3 + 2 + 1 = -1 \Rightarrow -C_2 + C_3 = -4$$

$$y'' = C_2e^{-x} + C_3e^x - 4e^{-x} + 2xe^{-x} + 2e^{2x}$$

$$y''(0) = 2 \rightarrow C_2 + C_3 - 4 + 2 = 2 \Rightarrow C_2 + C_3 = 4$$

$$\begin{cases} C_1 + C_2 + C_3 = -\frac{1}{2} \\ -C_2 + C_3 = -4 \\ C_2 + C_3 = 4 \end{cases} \rightarrow \begin{cases} C_3 = 0 \\ C_2 = 4 \\ C_1 = -\frac{9}{2} \end{cases}$$

$$y(x) = -\frac{9}{2} + 4e^{-x} + 2xe^{-x} + \frac{1}{2}e^{2x}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y^{(3)} + y'' = x + e^{-x}; \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1$$

Solution

$$\text{The characteristic equation: } \lambda^3 + \lambda^2 = 0 \rightarrow \lambda_{1,2,3} = 0, 0, -1$$

$$y_h = C_1 + C_2x + C_3e^{-x}$$

$$y_p = Ax^3 + Bx^2 + Cxe^{-x}$$

$$y_p' = 3Ax^2 + 2Bx + (C - Cx)e^{-x}$$

$$y_p'' = 6Ax + 2B + (-2C + Cx)e^{-x}$$

$$y_p''' = 6A + (3C - Cx)e^{-x}$$

$$y^{(3)} + y'' = x + e^{-x}$$

$$6Ax + 2B + 6A + Ce^{-x} = x + e^{-x}$$

$$\begin{cases} e^{-x} & C=1 \\ x & 6A=1 \\ x^0 & 2B+6A=0 \end{cases} \rightarrow \underline{A=\frac{1}{6}, B=-\frac{1}{2}, C=1}$$

$$\underline{y_p = \frac{1}{6}x^3 - \frac{1}{2}x^2 + xe^{-x}}$$

$$y(x) = C_1 + C_2x + C_3e^{-x} + \frac{1}{6}x^3 - \frac{1}{2}x^2 + xe^{-x}$$

$$y(0)=1 \rightarrow C_1 + C_3 = 1$$

$$y'(x) = C_2 - C_3e^{-x} + \frac{1}{2}x^2 - x + (1-x)e^{-x}$$

$$y'(0)=0 \rightarrow C_2 - C_3 = -1$$

$$y''(x) = C_3e^{-x} + x - 1 + (-2+x)e^{-x}$$

$$y''(0)=1 \rightarrow C_3 - 1 - 2 = 1 \Rightarrow \underline{C_3 = 4}$$

$$\begin{cases} C_1 + C_3 = 1 \\ C_2 - C_3 = -1 \end{cases} \rightarrow \underline{C_1 = -3, C_2 = 3}$$

$$\underline{y(x) = -3 + 3x + 4e^{-x} + \frac{1}{6}x^3 - \frac{1}{2}x^2 + xe^{-x}}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y^{(3)} - 2y'' + y' = 1 + xe^x; \quad y(0) = y'(0) = 0, \quad y''(0) = 1$$

Solution

$$\text{The characteristic equation: } \lambda^3 - 2\lambda^2 + \lambda = \lambda(\lambda^2 - 2\lambda + 1) = 0 \rightarrow \underline{\lambda_{1,2,3} = 0, 1, 1}$$

$$\underline{y_h = C_1 + (C_2 + C_3x)e^x}$$

$$y_p = Ax^2 + Bx + (Cx^3 + Dx^2)e^x$$

$$y_p' = 2Ax + B + (2Dx + (3C + D)x^2 + Cx^3)e^x$$

$$y''_p = 2A + \left(2Dx + (3C + D)x^2 + Cx^3 + 2D + (6C + 2D)x + 3Cx^2 \right) e^x$$

$$= 2A + \left(2D + (6C + 4D)x + (6C + D)x^2 + Cx^3 \right) e^x$$

$$y'''_p = \left(2D + (6C + 4D)x + (6C + D)x^2 + Cx^3 + (6C + 4D) + (12C + 2D)x + 3Cx^2 \right) e^x$$

$$= \left(6C + 6D + (18C + 6D)x + (9C + D)x^2 + Cx^3 \right) e^x$$

$$y^{(3)} - 2y'' + y' = 1 + xe^x$$

$$(6C + 2D + (6C + 4D)x) e^x - 4A + 2Ax + B = 1 + xe^x$$

$$\begin{cases} e^x & 6C + 2D = 0 \\ & 6C = 1 \\ x & 2A = 0 \\ x^0 & B - 4A = 1 \end{cases} \rightarrow \underline{A = 0, B = 1, C = \frac{1}{6}, D = -\frac{1}{2}}$$

$$\underline{y_p = x + \left(\frac{1}{6}x^3 - \frac{1}{2}x^2 \right) e^x}$$

$$y(x) = C_1 + (C_2 + C_3x)e^x + x + \left(\frac{1}{6}x^3 - \frac{1}{2}x^2 \right) e^x$$

$$y(0) = 0 \rightarrow C_1 + C_2 = 0$$

$$y'(x) = (C_2 + C_3 + C_3x)e^x + 1 + \left(-x + \frac{1}{6}x^3 \right) e^x$$

$$y'(0) = 0 \rightarrow C_2 + C_3 = -1$$

$$y''(x) = (C_2 + 2C_3 + C_3x)e^x + \left(-x + \frac{1}{6}x^3 - 1 + \frac{1}{2}x^2 \right) e^x$$

$$y''(0) = 1 \rightarrow C_2 + 2C_3 - 1 = 1 \Rightarrow C_2 + 2C_3 = 2$$

$$C_1 + C_2 = 0$$

$$\begin{cases} C_2 + C_3 = -1 \\ C_2 + 2C_3 = 2 \end{cases} \rightarrow \underline{C_3 = 3, C_2 = -4, C_1 = 4}$$

$$\underline{y(x) = 4 + x + \left(-4 + 3x - \frac{1}{2}x^2 + \frac{1}{6}x^3 \right) e^x}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y^{(4)} - y = 5; \quad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0$$

Solution

The characteristic equation: $\lambda^4 - 1 = 0 \rightarrow \lambda^2 = \pm 1 \quad \lambda_{1,2,3,4} = \pm 1, \pm i$

$$\underline{y_h = C_1 e^{-x} + C_2 e^x + C_3 \cos x + C_4 \sin x}$$

$$y_p = A$$

$$y'_p = y''_p = y'''_p = y^{(4)}_p = 0$$

$$y^{(4)} - y = 5 \rightarrow \underline{-A = 5}$$

$$\underline{y_p = -5}$$

$$y(x) = C_1 e^{-x} + C_2 e^x + C_3 \cos x + C_4 \sin x + 5$$

$$y(0) = 0 \rightarrow C_1 + C_2 + C_3 = -5$$

$$y'(x) = -C_1 e^{-x} + C_2 e^x - C_3 \sin x + C_4 \cos x$$

$$y'(0) = 0 \rightarrow -C_1 + C_2 + C_4 = 0$$

$$y''(x) = C_1 e^{-x} + C_2 e^x - C_3 \cos x - C_4 \sin x$$

$$y''(0) = 0 \rightarrow C_1 + C_2 - C_3 = 0$$

$$y'''(x) = -C_1 e^{-x} + C_2 e^x + C_3 \sin x - C_4 \cos x$$

$$y^{(3)}(0) = 0 \rightarrow -C_1 + C_2 - C_4 = 0$$

$$C_1 + C_2 + C_3 = -5$$

$$C_1 + C_2 - C_3 = 0$$

$$-C_1 + C_2 + C_4 = 0$$

$$-C_1 + C_2 - C_4 = 0$$

$$\rightarrow \underline{C_1 = \frac{5}{4}, C_2 = \frac{5}{4}, C_3 = \frac{5}{2}, C_4 = 0}$$

$$\underline{y(x) = \frac{5}{4}e^{-x} + \frac{5}{4}e^x + \frac{5}{2}\cos x - 5}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y^{(4)} - 4y'' = x^2; \quad y(0) = y'(0) = 1, \quad y''(0) = y^{(3)}(0) = -1$$

Solution

The characteristic equation: $\lambda^4 - 4\lambda^2 = 0 \rightarrow \lambda_{1,2,3,4} = 0, 0, \pm 2$

$$\underline{y_h = C_1 + C_2 x + C_3 e^{-2x} + C_4 e^{2x}}$$

$$y_p = Ax^4 + Bx^3 + Cx^2$$

$$y'_p = 4Ax^3 + 3Bx^2 + 2Cx$$

$$y''_p = 12Ax^2 + 6Bx + 2C$$

$$y'''_p = 24Ax + 6B$$

$$y^{(4)}_p = 24A$$

$$y^{(4)} - 4y'' = x^2$$

$$24A - 48Ax^2 - 24Bx - 8C = x^2$$

$$\begin{cases} -48A = 1 \\ 24B = 0 \\ 24A - 8C = 0 \end{cases} \rightarrow \underline{A = -\frac{1}{48}, B = 0, C = -\frac{1}{16}}$$

$$\underline{y_p = \frac{1}{48}x^4 + \frac{1}{16}x^2}$$

$$y(x) = C_1 + C_2x + C_3e^{-2x} + C_4e^{2x} - \frac{1}{48}x^4 - \frac{1}{16}x^2$$

$$y(0) = 1 \rightarrow C_1 + C_3 + C_4 = 1$$

$$y'(x) = C_2 - 2C_3e^{-2x} + 2C_4e^{2x} - \frac{1}{12}x^3 - \frac{1}{8}x$$

$$y'(0) = 1 \rightarrow C_2 - 2C_3 + 2C_4 = 1$$

$$y''(x) = 4C_3e^{-2x} + 4C_4e^{2x} - \frac{1}{4}x^2 - \frac{1}{8}$$

$$y''(0) = -1 \rightarrow 4C_3 + 4C_4 - \frac{1}{8} = -1 \quad 4C_3 + 4C_4 = -\frac{7}{8}$$

$$y^{(3)}(x) = -8C_3e^{-2x} + 8C_4e^{2x} - \frac{1}{2}x$$

$$y^{(3)}(0) = -1 \rightarrow -8C_3 + 8C_4 = -1$$

$$C_1 + C_3 + C_4 = 1$$

$$C_2 - 2C_3 + 2C_4 = 1$$

$$4C_3 + 4C_4 = -\frac{7}{8}$$

$$-8C_3 + 8C_4 = -1$$

$$\rightarrow \underline{C_1 = \frac{117}{96}, C_2 = \frac{5}{4}, C_3 = -\frac{3}{64}, C_4 = -\frac{11}{64}}$$

$$\underline{y(x) = \frac{117}{96} + \frac{5}{4}x - \frac{3}{64}e^{-2x} - \frac{11}{64}e^{2x} - \frac{1}{48}x^4 - \frac{1}{16}x^2}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y^{(4)} - y''' = x + e^x; \quad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0$$

Solution

The characteristic equation: $\lambda^4 - \lambda^3 = 0 \rightarrow \lambda_{1,2,3,4} = 0, 0, 0, 1$

$$y_h = C_1 + C_2 x + C_3 x^2 + C_4 e^x$$

$$y_p = Ax^4 + Bx^3 + Cxe^x$$

$$y'_p = 4Ax^3 + 3Bx^2 + (Cx + C)e^x$$

$$y''_p = 12Ax^2 + 6Bx + (Cx + 2C)e^x$$

$$y'''_p = 24Ax + 6B + (Cx + 3C)e^x$$

$$y^{(4)}_p = 24A + (Cx + 4C)e^x$$

$$y^{(4)} - y''' = x + e^x$$

$$\begin{array}{rcl} x & -24A = 1 \\ x^0 & 24A - 6B = 0 \end{array} \rightarrow \underline{A = -\frac{1}{24}, B = -\frac{1}{6}}$$

$$\begin{array}{rcl} e^x & C - C = 0 \\ x^0 & 4C - 3C = 1 \end{array} \rightarrow \underline{C = 1}$$

$$y_p = -\frac{1}{24}x^4 - \frac{1}{6}x^3 + xe^x$$

$$y(x) = C_1 + C_2 x + C_3 x^2 + C_4 e^x - \frac{1}{24}x^4 - \frac{1}{6}x^3 + xe^x$$

$$y(0) = 0 \rightarrow C_1 + C_4 = 0$$

$$y'(x) = C_2 + 2C_3 x + C_4 e^x - \frac{1}{6}x^3 - \frac{1}{2}x^2 + (1+x)e^x$$

$$y'(0) = 0 \rightarrow C_2 + C_4 + 1 = 0$$

$$y''(x) = 2C_3 + C_4 e^x - \frac{1}{2}x^2 - x + (2+x)e^x$$

$$y''(0) = 0 \rightarrow 2C_3 + C_4 = -2$$

$$y'''(x) = C_4 e^x - x - 1 + (3+x)e^x$$

$$y'''(0) = 0 \rightarrow \underline{C_4 = -2}$$

$$2C_3 + C_4 = -2 \rightarrow \underline{C_3 = 0}$$

$$C_2 + C_4 + 1 = 0 \rightarrow \underline{C_2 = 1}$$

$$C_1 + C_4 = 0 \rightarrow \underline{C_1 = 2}$$

$$\underline{y(x) = 2 + x - 2e^x - \frac{1}{24}x^4 - \frac{1}{6}x^3 + xe^x}$$

Exercise

If k and b are positive constants, then find the general solution of $y'' + k^2y = \sin bx$

Solution

The characteristic equation: $\lambda^2 + k^2 = 0 \rightarrow (k > 0) \quad \underline{\lambda_{1,2} = \pm ki}$

$$\underline{y_h = C_1 \cos kx + C_2 \sin kx}$$

If $k \neq b$

$$y_p = A \cos bx + B \sin bx$$

$$y'_p = -bA \sin bx + bB \cos bx$$

$$y''_p = -b^2 A \cos bx - b^2 B \sin bx$$

$$y'' + k^2 y = \sin bx$$

$$-b^2 A \cos bx - b^2 B \sin bx + kA \cos bx + kB \sin bx = \sin bx$$

$$(k - b^2) A \cos bx + (k - b^2) B \sin bx = \sin bx$$

$$\begin{cases} (k - b^2) A = 0 \\ (k - b^2) B = 1 \end{cases} \rightarrow \underline{A = 0} \quad \underline{B = \frac{1}{k - b^2}}$$

$$\underline{y_p = \frac{1}{k - b^2} \sin bx}$$

$$\underline{y(x) = C_1 \cos kx + C_2 \sin kx + \frac{1}{k - b^2} \sin bx}$$