Solution Section 1.7 - Modeling Population Growth

Exercise

The rate of growth of bacteria in a petri dish is proportional to the number of bacteria in the dish.

Solution

$$\boxed{y' = ky(t)}$$

Exercise

The rate of growth of a population of field mice is inversely proportional to the square root of the population.

Solution

$$\boxed{y' = \frac{k}{\sqrt{y(t)}}}$$

Exercise

A biologist starts with 100 *cells* in a culture. After 24 *hrs*, he counts 300. Assuming a Malthusian model, what the reproductive rate? What will be the number of cells of the end of 5 *days*?

Solution

$$P = 100e^{rt}$$

$$300 = 100e^{r(1)} \qquad \textbf{24 } \textbf{\textit{hrs}} \textbf{ = 1 } \textbf{\textit{day P}} \textbf{ = 300}$$

$$3 = e^{r}$$

$$r = \ln 3$$

$$\approx 1.0986$$

$$P(t) = 100e^{1.0986t}$$

$$P(5) = 100e^{1.0986(5)}$$

$$\boxed{\approx 24300}$$

Exercise

A biologist prepares a culture. After 1 *day* of growth, the biologist counts 1000 cells. After 2*days,* he counts 3000. Assuming a Malthusian model, what the reproductive rate and how many cells were present initially?

Solution

Given: $P(1) = 1000, \quad P(2) = 3000$

The equation of the Malthusian model is $P(t) = Ce^{rt}$

$$P(1) = Ce^{r(1)}$$

$$1000 = Ce^{r}$$

$$e^{r} = \frac{1000}{C} \rightarrow \boxed{r = \ln\left(\frac{1000}{C}\right)}$$

$$P(2) = Ce^{r(2)}$$

$$3000 = Ce^{2r}$$

$$e^{2r} = \frac{3000}{C} \rightarrow \boxed{r = \frac{1}{2}\ln\left(\frac{3000}{C}\right)}$$

$$r = \frac{1}{2}\ln\left(\frac{3000}{C}\right) = \ln\left(\frac{1000}{C}\right)$$

$$\ln\left(\frac{3000}{C}\right) = 2\ln\left(\frac{1000}{C}\right)$$

$$\ln\left(\frac{3000}{C}\right) = \ln\left(\frac{1000}{C}\right)^2$$

$$\frac{3000}{C} = \frac{10^6}{C^2}$$

$$3000C^2 = 10^6 C$$

$$3000C^2 - 10^6 C = 0$$

$$C\left(3000C - 10^6\right) = 0 \quad \Rightarrow \quad \boxed{C = \frac{10^6}{3000} = \frac{1000}{3}}$$

$$r = \ln\left(\frac{1000}{\frac{1000}{3}}\right) = \ln 3$$

$$P(t) = \frac{1000}{3}e^{(\ln 3)t}$$

$$\boxed{P(t=0) = \frac{1000}{3}}$$

Exercise

A population of bacteria is growing according to the Malthusian model. If the population is triples in 10 *hrs*, what is the reproduction rate? How often does the population double itself?

Solution

$$P(t) = P_0 e^{t}$$

$$rT = \ln(X) \quad \rightarrow \quad \boxed{r = \frac{\ln 3}{10} \approx 0.1099}$$

$$P(t) = P_0 e^{(t\ln 3)/10}$$

$$t = \frac{\ln 2}{r}$$

$$= \frac{\ln 2}{\frac{\ln 3}{10}} = \frac{10\ln 2}{\ln 3}$$

$$\approx 6.3093 \ \ hrs$$

Exercise

Consider a lake that is stocked with walleye pike and that the population of pike is governed by the logistic equation

$$P' = 0.1P\left(1 - \frac{P}{10}\right)$$

where time is measured in days and P in thousands of fish. Suppose that fishing is started in this lake and that 100 fish are removed each day.

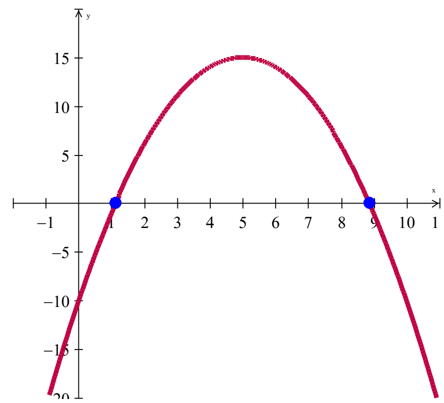
Solution

a) Modify the logistic model to account for the fishing.

The modified logistic model

$$P' = P' - \frac{100}{1000} \ (\text{in } \textit{thousand})$$

$$= 0.1P\left(1 - \frac{P}{10}\right) - 0.1$$



b) Find and classify the equilibrium points for your model.

$$P' = 0.1P - \frac{0.1P^2}{10} - 0.1 = 0 \qquad \textit{Multiply 100 each term}$$

$$10P - P^2 - 10 = 0$$

$$P^2 - 10P + 10 = 0 \qquad \textit{Solve for P}$$

$$P = 5 \pm \sqrt{15}$$

$P = 5 - \sqrt{15}$ Asymptotically stable

$P = 5 + \sqrt{15}$ Unstable

c) Use qualitative analysis to completely discuss the fate of the fish population with this model. In particular, if the initial fish population is 1000, what happens to the fish as time passes? what will happen to an initial population having 2000 fish?

$$P' > 0 \Rightarrow \ 5 - \sqrt{15} < P < 5 + \sqrt{15}$$

For the 1000 (= 1) population, the population decreases until it dies out (doomed);

For the 2000 (= 2) population, the population tend towards the equilibrium $P_2 = 5 + \sqrt{15}$

$$P' = 0.1(1)\left(1 - \frac{1}{10}\right) - .1 = -.01$$

$$P' = 0.1(2)\left(1 - \frac{2}{10}\right) - .1 = .06$$

Exercise

Suppose that in 1885 the population of a certain country was 50 *million* and was growing at the rate of 750,000 people per year at that time. Suppose also that in 1940 its population was 100 *million* and was then growing at the rate of 1 *million* per year. Assume that this population satisfies the logistic equation. Determine both the limiting population *M* and the predicted population for the year 2000.

Solution

$$P' = kP(M-P)$$

$$.75 = 50k(M-50) \quad (1) \quad (\textit{in million})$$

$$1 = 100k(M-100) \quad (2) \quad (\textit{in million})$$

$$\begin{cases} (1) \quad k = \dfrac{.75}{50(M-50)} \\ (2) \quad k = \dfrac{1}{100(M-100)} \end{cases}$$

$$\frac{.75}{50(M-50)} = \frac{1}{100(M-100)}$$

$$2(0.75)(M-100) = M-50$$

$$1.5M - 150 = M - 50$$

$$.5M = 100$$

$$\underline{M = 200|}$$

$$k = \frac{1}{100(200-100)} \ \underline{= 0.0001|}$$

$$P(60) = \frac{100 \bullet 200}{100 + (200-100)e^{-0.0001(200)(60)}} \qquad P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-kMt}}$$

$$\underline{\approx 153.7 \ \textit{million people}\ |}$$

Exercise

The time rate of change of a rabbit population *P* is proportional to the square root of *P*. At time $t = 0$ (months) the population numbers 100 rabbits and is increasing at the rate of 20 rabbits per month. How many rabbits will there be one year later?

Solution

Given: $P(0) = 100, \quad P'(0) = 20$

$$P' = k\sqrt{P}$$

$$at\ t = 0 \ \Rightarrow \ 20 = k\sqrt{100} \ \rightarrow \boxed{k = 2}$$

$$\frac{dP}{dt} = 2\sqrt{P}$$

$$\int \frac{1}{2\sqrt{P}} dP = \int dt$$

$\sqrt{P} = t + C \quad P(0) = 100 \quad \Rightarrow C = 10$

$\boxed{P(t) = (t+10)^2}$

$P(t=12) = (12+10)^2$

$= 484 \ \ rabbits$

Exercise

Suppose that the fish population $P(t)$ in a lake is attacked by a disease at time $t = 0$, with the result that the fish cease to reproduce (so that the birth rate is $\beta = 0$) and the death rate δ (deaths per week per fish) is thereafter proportional to $\frac{1}{\sqrt{P}}$. If there were initially 900 fish in the lake and 441 were left after 6 weeks, how long did it take all the fish in the lake to die?

Solution

Given: $P(0) = 900, \quad P(6) = 441$

$P' = -\delta P = -\frac{k}{\sqrt{P}} P = -k\sqrt{P}$

$\frac{dP}{dt} = -k\sqrt{P}$

$\int \frac{dP}{\sqrt{P}} = -\int k dt$

$2\sqrt{P} = -kt + C$

$2\sqrt{900} = -k(0) + C \quad \Rightarrow \quad \boxed{C = 60}$

$2\sqrt{441} = -k(6) + 60 \quad \Rightarrow \quad \boxed{k = 3}$

$2\sqrt{P} = -3t + 60$

$0 = -3t + 60 \quad \Rightarrow \quad t = 20$

It will take 20 weeks for the fish to die in the lake.

Exercise

Suppose that when a certain lake is stocked with fish, the birth and death rates β and δ are both inversely proportional to $\sqrt{P}$

a) Show that $P(t) = \left(\frac{1}{2}kt + \sqrt{P_0}\right)^2$, where k is a constant.

b) If $P_0 = 100$ and after 6 months there are 169 fish in the lake, how many will there be after 1 year?

Solution

a) $\frac{dP}{dt} = k\sqrt{P}$

$$\int \frac{dP}{\sqrt{P}} = \int k\,dt$$

$$2\sqrt{P} = kt + C_1$$

$$\sqrt{P} = \frac{1}{2}kt + C$$

$$P(t) = \left(\frac{1}{2}kt + C\right)^2$$

$$P(t=0) = \left(\frac{1}{2}k(0) + C\right)^2 = P_0 \quad \Rightarrow \quad C = \sqrt{P_0}$$

$$P(t) = \left(\frac{1}{2}kt + \sqrt{P_0}\right)^2$$

b) ***Given***: $P_0 = 100, \quad P(6) = 169$

$$169 = \left(\frac{1}{2}k(6) + \sqrt{100}\right)^2$$

$$13 = 3k + 10 \quad \Rightarrow \quad \boxed{k = 1}$$

$$P(t) = \left(\frac{1}{2}t + 10\right)^2$$

$$P(t = 1yr = 12mths) = (6 + 10)^2 = 256$$

There are 256 fish after 1 year.

Exercise

The time rate of change of an alligator population P in a swamp is proportional to the square of P. The swamp contained a dozen alligators in 1988, two dozen in 1998.

a) When will there be four dozen alligators in the swamp?

b) What happens thereafter?

Solution

Given: $P_0 = 12, \quad P(10) = 24$

a) $\frac{dP}{dt} = kP^2$

$$\int \frac{dP}{P^2} = \int k\,dt$$

$$-\frac{1}{P} = kt + C$$

$$P(t) = -\frac{1}{kt + C}$$

$$P(0) = -\frac{1}{C} = 12 \quad \Rightarrow \quad C = -\frac{1}{12}$$

$$P(t) = -\frac{1}{kt - \frac{1}{12}}$$

$$P(t) = \frac{12}{1-12kt}$$

$$P(10) = \frac{12}{1-120k} = 24 \quad \Rightarrow \quad 1-120k = \frac{1}{2}$$

$$\boxed{k = \frac{1}{240}}$$

$$P(t) = \frac{12}{1-\frac{1}{20}t}$$

$$= \frac{240}{20-t}$$

$$48 = \frac{240}{20-t}$$

$$20 - t = \frac{240}{48} = 5$$

$t = 15$, that is, in the year 2003

b) $P = \frac{240}{20-t} \xrightarrow[t \to 20]{} \infty$

The population approaches infinity as t approaches 20 years.

Exercise

Consider a prolific breed of rabbits whose birth and death rates, β and δ, are each proportional to the rabbit population $P = P(t)$, with $\beta > \delta$

a) Show that $P(t) = \frac{P_0}{1-kP_0 t}$, k *constant*

Note that $P(t) \to +\infty$ *as* $t \to \frac{1}{kP_0}$. This is doomsday

b) Suppose that $P_0 = 6$ and that there are nine rabbits after ten months. When does doomsday occur?

c) With $\beta < \delta$, repeat part (a)

d) What now happens to the rabbit population in the long run?

Solution

a) If the birth & death both are proportional to P^2 with $\beta > \delta$

$$\frac{dP}{dt} = kP^2$$

$$\int \frac{dP}{P^2} = \int k\,dt$$

$$-\frac{1}{P} = kt + C$$

$$P(t) = -\frac{1}{kt + C}$$

$$P(0) = -\frac{1}{C} = P_0 \quad \Rightarrow \quad \boxed{C = -\frac{1}{P_0}}$$

$$P(t) = -\frac{1}{kt - \frac{1}{P_0}}$$

$$\boxed{= \frac{P_0}{1 - P_0 kt}}$$

b) $P_0 = 6 \quad \Rightarrow \quad P(t) = \frac{6}{1 - 6kt}$

Given: $P(10) = 9$

$$\frac{6}{1 - 60k} = 9$$

$$1 - 60k = \frac{2}{3}$$

$$k = \frac{1}{180}$$

$$P(t) = \frac{6}{1 - \frac{1}{30}t}$$

$$\boxed{= \frac{180}{30 - t}}$$

$$P = \frac{180}{30 - t} \xrightarrow[t \to 30]{} \infty \text{ (doomsday)}$$

c) If the birth & death both are proportional to P^2 with $\beta < \delta$

$$\frac{dP}{dt} = -kP^2$$

$$\int \frac{dP}{P^2} = -\int kdt$$

$$-\frac{1}{P} = -kt - C$$

$$P(t) = \frac{1}{kt + C}$$

$$P(0) = \frac{1}{C} = P_0 \quad \Rightarrow \quad \boxed{C = \frac{1}{P_0}}$$

$$\boxed{P(t) = \frac{P_0}{1 + P_0 kt}}$$

d) $\frac{P_0}{1 + P_0 kt} \xrightarrow[t \to \infty]{} 0$

Therefore $P(t) \to 0$ *as* $t \to \infty$, so the population die out in the long run.

Exercise

Consider a population $P(t)$ satisfying the logistic equation $\frac{dP}{dt} = aP - bP^2$, where $B = aP$ is the time rate at which births occur and $D = bP^2$ is the rate at which deaths occur.

a) If the initial population is $P(0) = P_0$, and B_0 births per month and D_0 deaths per month are occurring at time $t = 0$, show that the limiting population is $M = \frac{B_0 P_0}{D_0}$.

b) If the initial population is 120 rabbits and there are 8 births per month and 6 deaths per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 95% of the limiting population *M*?

c) If the initial population is 240 rabbits and there are 9 births per month and 12 deaths per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 105% of the limiting population *M*?

Solution

a) $P' = aP - bP^2 = bP\left(\frac{a}{b} - P\right)$ $\qquad\qquad P' = kP(M - P)$

$\Rightarrow M = \frac{a}{b}$

$\frac{B_0 P_0}{D_0} = \frac{aP_0 P_0}{bP_0^2} = \frac{a}{b} = M$ ✓

b) ***Given***: $P_0 = 120, \quad B_0 = 8, \quad D_0 = 6$

$$a = \frac{B_0}{P_0} = \frac{8}{120} = \frac{1}{15}, \quad b = \frac{D_0}{P_0^2} = \frac{6}{120^2} = \frac{1}{2400}$$

$$M = \frac{B_0 P_0}{D_0} = \frac{(8)(120)}{6} = 160, \quad k = b = \frac{1}{2400}$$

$$P(t) = \frac{(160)(120)}{120 + (160 - 120)e^{-\frac{160}{2400}t}} \qquad\qquad P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}$$

$$= \frac{19200}{120 + 40e^{-\frac{1}{15}t}}$$

$$= \frac{480}{3 + e^{-\frac{1}{15}t}}$$

For $P = .95M$

$$.95(160) = \frac{480}{3 + e^{-\frac{1}{15}t}}$$

$$3+e^{-\frac{1}{15}t}=\frac{3}{.95}$$

$$e^{-\frac{1}{15}t}=\frac{3}{.95}-3=\frac{3}{19}$$

$$-\frac{t}{15}=\ln\frac{3}{19}$$

$$t=-15\ln\frac{3}{19}$$

$$\underline{\approx 27.69 \ \ months}\big|$$

c) ***Given***: $P_0=240,\quad B_0=9,\quad D_0=12$

$$M=\frac{B_0P_0}{D_0}=\frac{(9)(240)}{12}=180,\quad k=b=\frac{D_0}{P_0^2}=\frac{12}{240^2}=\frac{1}{4800}$$

$$P(t)=\frac{(180)(240)}{240+(180-240)e^{-\frac{180}{4800}t}} \qquad\qquad P(t)=\frac{MP_0}{P_0+\left(M-P_0\right)e^{-kMt}}$$

$$=\frac{43200}{240-60e^{-\frac{3}{80}t}}$$

$$=\frac{720}{4-e^{-\frac{3}{80}t}}$$

For $P=1.05M$

$$1.05(180)=\frac{720}{4-e^{-\frac{3}{80}t}}$$

$$4-e^{-\frac{3}{80}t}=\frac{720}{189}$$

$$e^{-\frac{3}{80}t}=4-\frac{720}{189}=\frac{36}{189}=\frac{4}{21}$$

$$-\frac{3t}{80}=\ln\frac{4}{21}$$

$$t=-\left(\frac{80}{3}\right)\ln\frac{4}{21}$$

$$\underline{\approx 44.22 \ \ months}\big|$$

Exercise

The amount of drug in the blood of a patient (in *mg*) due to an intravenous line is governed by the initial value problem

$$y'(t)=-0.02y+3,\quad y(0)=0 \ \ for\ t\ge 0$$

Where t is measured in hours

a) Find and graph the solution of the initial value problem.

b) What is the steady-state level of the drug?

c) When does the drug level reach 90% of the steady-state value?

Solution

a) $y' + 0.02y = 3$

$$e^{\int 0.02dt} = e^{0.02t}$$

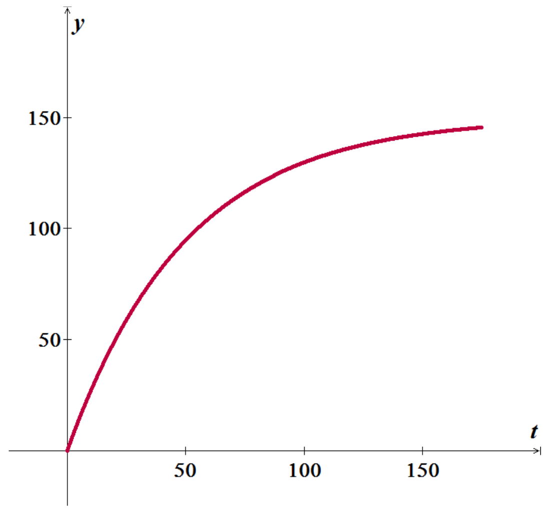
$$\int 3e^{0.02t}dt = 150e^{0.02t}$$

$$y = \frac{1}{e^{0.02t}}\left(150e^{0.02t} + C\right)$$

$$= 150 + Ce^{-0.02t}$$

$$y(0) = 0 \qquad 0 = 150 + C \to C = -150$$

$$y(t) = 150\left(1 - e^{-0.02t}\right)$$



b) The steady-state level is

$$\lim_{t\to\infty} 150\left(1 - e^{-0.02t}\right) = 150 \ \ mg$$

c) $150\left(1 - e^{-0.02t}\right) = 0.9(150)$

$$1 - e^{-0.02t} = 0.9$$

$$e^{-0.02t} = 0.1$$

$$-0.02t = \ln 0.1$$

$$t = \frac{\ln 0.1}{-0.02}$$

$$\approx 115 \ \ hrs$$

Exercise

A fish hatchery has 500 *fish* at time $t = 0$, when harvesting begins at a rate of b *fish/yr*. where $b > 0$. The fish population is modeled by the initial value problem.

$$y'(t) = 0.1y - b, \quad y(0) = 500 \qquad for \ \ t \geq 0$$

Where t is measured in years.

a) Find the fish population for $t \geq 0$ in terms of the harvesting rate b.

b) Graph the solution in the case that $b = 40$ $fish / yr$. Describe the solution.

c) Graph the solution in the case that $b = 60$ $fish / yr$. Describe the solution.

Solution

a) $y' - 0.1y = -b$

$$e^{\int -0.1dt} = e^{-0.1t}$$

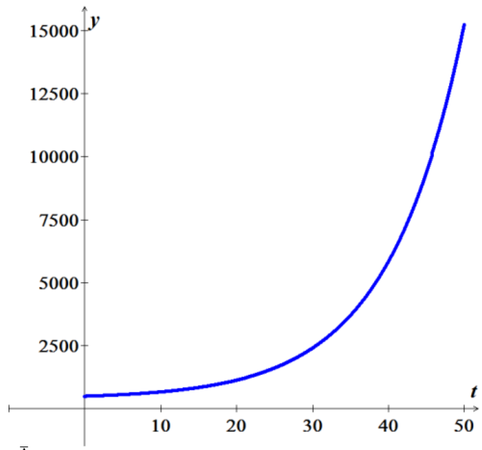
$$\int -be^{-0.1t}dt = 10be^{-0.1t}$$

$$y(t) = e^{0.1t}\left(10be^{-0.1t} + C\right)$$

$$\underline{= 10b + Ce^{0.1t}}$$

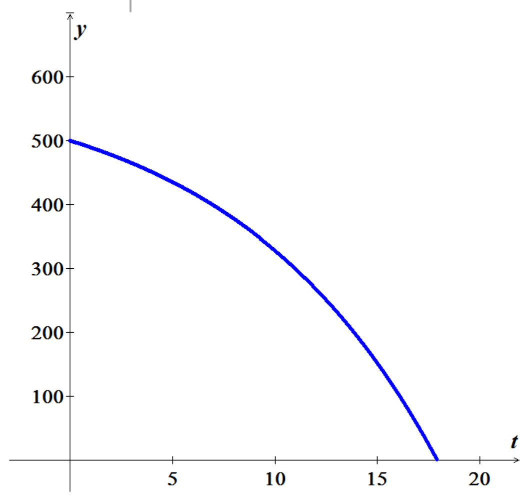
$$\color{red}{y(0) = 500} \quad \to \quad 500 = 10b + C \;\Rightarrow\; \underline{C = 500 - 10b}$$

$$\color{blue}{y(t) = 10b + (500 - 10b)e^{0.1t}}$$



b) For $b = 40$

$$y(t) = 400 + 100e^{0.1t}$$



c) For $b = 60$

$$y(t) = 600 - 100e^{0.1t}$$

Exercise

A community of hares on an island has a population of 50 when observations begin at $t = 0$. The population for $t \ge 0$ is modeled by the initial value problem.

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{200}\right), \quad P(0) = 50$$

d) Find the solution of the initial value problem.

e) What is the steady-state population?

Solution

a) $\displaystyle\int \frac{200}{P(200 - P)}dP = \int 0.08dt$

$$\int \left(\frac{1}{P} + \frac{1}{200-P}\right) dP = \int 0.08\,dt$$

$$\ln P + \ln|200-P| = 0.08t + C$$

$$\ln\left|\frac{P}{200-P}\right| = 0.08t + C$$

$$P(0) = 50 \quad \to \ln\frac{50}{150} == C \ \Rightarrow \ \underline{C = -\ln 3|}$$

$$\ln\left|\frac{P}{200-P}\right| = 0.08t - \ln 3$$

$$\frac{P}{200-P} = e^{0.08t - \ln 3}$$

$$\frac{P}{200-P} = e^{0.08t} e^{\ln 3^{-1}}$$

$$\frac{P}{200-P} = \frac{1}{3} e^{0.08t}$$

$$3P = 200e^{0.08t} - Pe^{0.08t}$$

$$P(t) = \frac{200e^{0.08t}}{3 + e^{0.08t}}$$

$$= \frac{200}{3e^{-0.08t} + 1}$$

b) $$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{200}{3e^{-0.08t} + 1}$$

$$= 200$$

Exercise

When an infected person is introduced into a closed and otherwise healthy community, the number of people who become infected with the disease (in the absence of any intervention) may be modeled by the logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{A}\right), \quad P(0) = P_0$$

Where k is a positive infection rate, A is the number of people in the community, and P_0 is the number of infected people at $t = 0$. The model assumes no recovery or intervention.

a) Find the solution of the initial value problem in terms of k, A, and P_0.

b) Graph the solution in the case that $k = 0.025, \quad A = 300, \quad and \ P_0 = 1$.

c) For fixed values of k and A, describe the long-term behavior of the solutions for any P_0 with $0 < P_0 < A$

Solution

a) $$\frac{dP}{dt} = kP\left(\frac{A-P}{A}\right)$$

$$\int \frac{A}{P(A-P)}\,dP = \int k\,dt$$

$$\int \left(\frac{1}{P} + \frac{1}{A-P}\right) dP = \int k\,dt$$

$$\ln P - \ln|A-P| = kt + C_1$$

$$\ln\left|\frac{P}{A-P}\right| = kt + C_1$$

$$\frac{P}{A-P} = Ce^{kt}$$

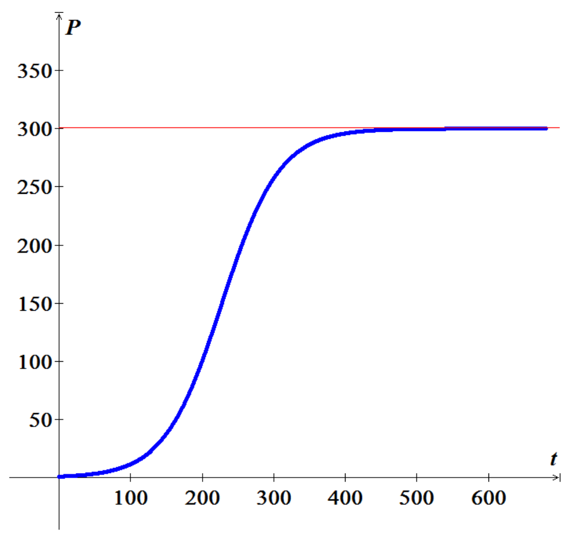
$$P(0) = P_0 \rightarrow \quad \frac{P_0}{A-P_0} = C$$

$$\frac{P}{A-P} = \frac{P_0}{A-P_0}e^{kt}$$

$$P = (A-P)\frac{P_0}{A-P_0}e^{kt}$$

$$\left(A - P_0 + P_0 e^{kt}\right)P = AP_0 e^{kt}$$

$$P(t) = \frac{AP_0 e^{kt}}{A - P_0 + P_0 e^{kt}} = \frac{AP_0}{P_0 + \left(A-P_0\right)e^{-kt}}$$



b) $k = 0.025, \quad A = 300, \quad and \quad P_0 = 1$

$$P(t) = \frac{300}{1 + 299e^{-0.025t}}$$

c) $\lim\limits_{t\to\infty} P(t) = \lim\limits_{t\to\infty} \dfrac{AP_0}{P_0 + \left(A-P_0\right)e^{-kt}}$

$$= \frac{AP_0}{P_0}$$

$$= A$$ Which is the ***steady-state*** solution

Exercise

The reaction of chemical compounds can often be modeled by differential equations. Let $y(t)$ be the concentration of a substance in reaction for $t \ge 0$ (typical units of *y* are *moles/L*). The change in the concentration of a substance, under appropriate conditions, is $\frac{dy}{dt} = -ky^n$, where $k > 0$ is a rate constant and the positive integer *n* is the order of the reaction.

a) Show that for a first-order reaction $(n = 1)$, the concentration obeys an exponential decay law.

b) Solve the initial value problem for a second-order reaction $(n = 2)$ assuming $y(0) = y_0$

c) Graph and compare the concentration for a first-order and second-order reaction with $k = 0.1$ and $y_0 = 1$

Solution

a) $\int \frac{dy}{y} = -\int k\,dt$

$\ln|y| = -kt + C_1$

$\boxed{y(t) = Ce^{-kt}}$

b) $n = 2 \rightarrow \frac{dy}{dt} = -ky^2$

$-\int \frac{dy}{y^2} = \int k\,dt$

$\frac{1}{y} = kt + C$

$y(0) = y_0 \rightarrow \boxed{\frac{1}{y_0} = C}$

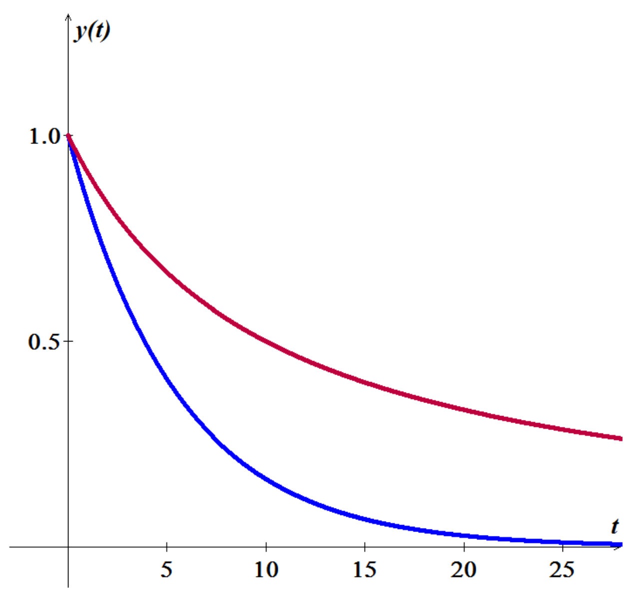
$\frac{1}{y} = kt + \frac{1}{y_0}$

$\boxed{y(t) = \frac{y_0}{1 + ky_0 t}}$

c) $y(t) = \frac{1}{1 + 0.1t}$

$y_0 = 1 \rightarrow C = 1$

$\boxed{y(t) = e^{-0.1t}}$



Exercise

The growth of cancer turmors may be modeled by the Gomperts growth equation. Let $M(t)$ be the mass of the tumor for $t \ge 0$. The relevant intial value problem is

$$\frac{dM}{dt} = -aM \ln\frac{M}{K}, \quad M(0) = M_0$$

Where a and K are positive constants and $0 < M_0 < K$

a) Graph the growth rate function $R(M) = -aM\ln\frac{M}{K}$ assuming $a=1$ and $K=4$. For what values of M is the growth rate positive? For what values of M is maximum?

b) Solve the initial cvalue problem and graph the solution for $a=1$, $K=4$, and $M_0=1$. Describe the groath pattern of the tumor. Is the growth unbounded? If not, what is the limiting size of the tumor?

c) In the general equation, what is the meaning of K?

Solution

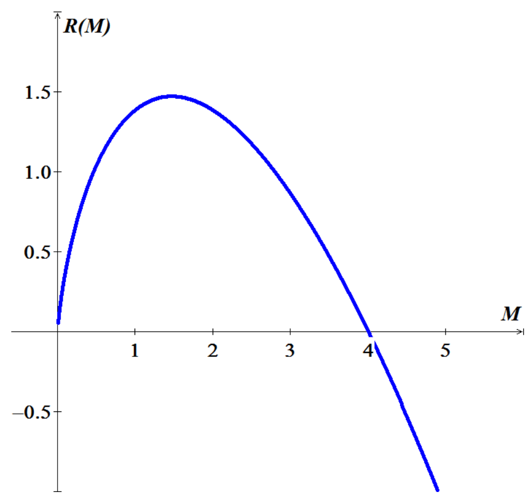
a) $R'(M) = -a\left(\ln\frac{M}{K} + M\frac{1}{K}\frac{K}{M}\right)$

$$= -a\left(\ln\frac{M}{K}+1\right) = 0$$

$$\Rightarrow \ln\frac{M}{K} = -1 \quad \to \quad \boxed{M = Ke^{-1} = \frac{K}{e}}$$

For $a=1$ and $K=4$

$$\boxed{R(M) = -M\ln\frac{M}{4}}$$



b) $\displaystyle\int\frac{dM}{M(\ln M - \ln K)} = -\int a\,dt$

$$d(\ln M - \ln K) = \frac{1}{M}dM$$

$$\int\frac{d(\ln M - \ln K)}{\ln M - \ln K} = -\int a\,dt$$

$$\ln|\ln M - \ln K| = -at + C_1$$

$$\ln\frac{M}{K} = Ce^{-at}$$

$$\boxed{M(t) = Ke^{Ce^{-at}}}$$

For $a=1$, $K=4$, and $M_0=1$

$$M(0) = 4e^C = 1 \quad\Rightarrow\quad C = \ln\frac{1}{4} = -\ln 4$$

$$\boxed{M(t) = 4e^{-(\ln 4)e^{-t}}}$$

$$\lim_{t\to\infty} M(t) = \lim_{t\to\infty} 4e^{-(\ln 4)e^{-t}} \;\boxed{= 4}$$

So the limiting size of the tumor is 4.

c) $\lim\limits_{t\to\infty} M(t) = \lim\limits_{t\to\infty} Ke^{Ce^{-at}}$ since $a > 0$

$\quad = K$

Exercise

The halibut fishery has been modeled by the differential equation

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right)$$

Where $y(t)$ is the biomass (the total mass of the members of the population) in kilograms at time t (measured in years), the carrying capacity is estimated to be $M = 8\times 10^7\ kg$ and $k = 0.71\ per\ year$.

a) If $y(0) = 2\times 10^7\ kg$, find the biomass a year later.

b) How long will it take for the biomass to reach $4\times 10^7\ kg$.

Solution

a) $\dfrac{M}{ky(M-y)}dy = dt \quad \to \quad \dfrac{M}{k}\dfrac{1}{y(M-y)}dy = dt$

$$\frac{1}{y(M-y)} = \frac{A}{y} + \frac{B}{M-y}$$

$$AM - Ay + By = 1 \quad \to \quad \begin{cases} AM = 1 \ \Rightarrow A = \dfrac{1}{M} \\ -A + B = 0 \ \Rightarrow B = A = \dfrac{1}{M} \end{cases}$$

$$\frac{M}{k}\frac{1}{M}\int\left(\frac{1}{y} + \frac{1}{M-y}\right)dy = \int dt$$

$$\frac{1}{k}\left(\ln y - \ln(M-y)\right) = t + C_1$$

$$\ln\frac{y}{M-y} = kt + C_2$$

$$\frac{y}{M-y} = e^{kt+C_2}$$

$$y = Me^{kt}e^{C_2} - ye^{kt}e^{C_2} \qquad C = e^{C_2}$$

$$y\left(1 + Ce^{kt}\right) = MCe^{kt}$$

$$y = \frac{MCe^{kt}}{1 + Ce^{kt}}$$

$$= \frac{M}{1 + Ce^{-kt}}$$

$$= \boxed{\frac{8\times 10^7}{1+Ce^{-0.71t}}}$$

$$y(0) = \frac{8\times 10^7}{1+C} = 2\times 10^7 \quad \Rightarrow \quad \boxed{C = \frac{8\times 10^7}{2\times 10^7} - 1 = 3}$$

$$\boxed{y(t) = \frac{8\times 10^7}{1+3e^{-0.71t}}}$$

$$y(1) = \frac{8\times 10^7}{1+3e^{-0.71}}$$

$$\boxed{\approx 3.23\times 10^7 \ kg}$$

b) $y(t) = \dfrac{8\times 10^7}{1+3e^{-0.71t}} = 4\times 10^7$

$$1+3e^{-0.71t} = \frac{8\times 10^7}{4\times 10^7} = 2$$

$$3e^{-0.71t} = 1$$

$$e^{-0.71t} = \frac{1}{3}$$

$$-.071t = \ln\frac{1}{3}$$

$$t = \frac{\ln 3}{0.71}$$

$$\boxed{\approx 1.55 \ \ years}$$

Exercise

Suppose a population $P(t)$ satisfies $\quad \dfrac{dP}{dt} = 0.4P - 0.001P^2, \quad P(0) = 50$

Where t is measured in years.

a) What is the carrying capacity?

b) What is $P'(0)$?

c) When will the population reach 50% of the carrying capacity?

Solution

a) $\dfrac{1}{0.4P(1-0.0025P)}dP = dt$

$$\frac{1}{P(1-0.0025P)} = \frac{A}{P} + \frac{B}{1-0.0025P}$$

$$A - .0025PA + PB = 1$$

$$\rightarrow \begin{cases} \underline{A=1} \\ -.0025A+B=0 \quad \underline{B=.0025} \end{cases}$$

$$\int\left(\frac{1}{P}+\frac{.0025}{1-.0025P}\right)dP = 0.4\int dt$$

$$\ln P - \ln(1-.0025P) = 0.4t + C_1$$

$$\ln\frac{P}{1-.0025P} = 0.4t + C_1$$

$$\frac{P}{1-.0025P} = e^{0.4t+C_1} = Ce^{0.4t} \qquad C = e^{C_1}$$

$$Ce^{-0.4t}P = 1-.0025P$$

$$Ce^{-0.4t}P + .0025P = 1$$

$$\left(Ce^{-0.4t} + .0025\right)P = 1$$

$$P(t) = \frac{1}{Ce^{-0.4t}+.0025}$$

$$P(0) = \frac{1}{C+.0025} = 50 \qquad \underline{C} = \frac{1}{50} - .0025 \; \underline{= .0175}$$

$$P(t) = \frac{1}{.0175e^{-0.4t}+.0025}$$

$$\boxed{P(t) = \frac{400}{7e^{-0.4t}+1}}$$

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{400}{1+7e^{-0.4t}}$$

$$\boxed{= 400}$$

The carrying capacity is 400.

b) $P'(0) = \frac{dP}{dt}\Big|_{t=0}$

$$= 0.4(50) - 0.001(50)^2$$

$$\boxed{= 17.5}$$

c) $P(t) = \dfrac{400}{7e^{-0.4t}+1} = 200$

$$7e^{-0.4t} + 1 = 2$$

$$e^{-0.4t} = \frac{1}{7}$$

$$-0.4t = \ln\left(\frac{1}{7}\right)$$

$$t = \frac{\ln\left(\frac{1}{7}\right)}{-0.4}$$

$$\boxed{\approx 4.86 \; \textit{years}}$$

Exercise

The board of directors of a corporation is calculating the price to pay for a business that is forecast to yield a continuous flow of profit of $500,000 per year. The money will earn a nominal rate of 5% per year compounded continuously. What is the present value of the business?

a) For 20 years?

b) Forever (in perpetuity)?

Solution

$$PV = \int_0^{t_0} 500,000e^{-0.05t}dt$$

$$= -10^7 e^{-0.05t}\Big|_0^{t_0}$$

$$= -10^7\left(e^{-0.05t_0} - 1\right)$$

a) $PV(20) = -10^7\left(e^{-0.05(20)} - 1\right)$

$$= \$6,321,205.59$$

b) $PV\left(t_0 \to \infty\right) = -10^7(0-1)$

$$= \$10,000,000$$

Exercise

The population of a community is known to increase at a rate proportional to the number of people present at a time t. If the population has doubled in 6 *years*, how long it will take to triple?

Solution

$$k = \frac{\ln 2}{6}$$

$$T = \frac{\ln 3}{k}$$

$$= 6\frac{\ln 3}{\ln 2}$$

$$\approx 9.5 \ yrs$$

Exercise

Let population of country be decreasing at the rate proportional to its population. If the population has decreased to 25% in 10 *years*, how long will it take to be half?

Solution

$$k = \frac{\ln .25}{10}$$

$$T = \frac{\ln \frac{1}{2}}{k}$$

$$= 10\frac{\ln 0.5}{\ln 0.25}$$

$$\boxed{\approx 5 \ \ yrs}$$

Exercise

Suppose that we have an artifact, say a piece of fossilized wood, and measurements show that the ratio of C–14 to carbon in the sample is 37% of the current ratio. Let us assume that the wood died at time 0, then compute the time T it would take for one gram of the radioactive carbon to decay this amount.

Solution

The half-life of carbon C-14 is about 5550.

$$k = \frac{\ln 0.5}{5550}$$

$$\boxed{\approx -0.000125}$$

$$T = \frac{\ln 0.37}{-0.000125}$$

$$\boxed{\approx 7955 \ \ yrs}$$

Exercise

A certain radioactive material is known to decay at a rate proportional to the amount present. If initially there is $50 \ \ mg$ of the material present and after 2 *hours* it is observed that the material has lost 10% of its original mass, find

a) An expression for the mass of the material remaining at any time t.

b) The mass of the material after 4 *hours*

c) The time at which the material has decayed to one half of its initial mass.

Solution

a) $$\frac{dN}{dt} = kN$$

$$\int \frac{dN}{N} = \int kdt$$

$$\ln N = kt + C$$

$$N = e^{kt+C} = Ae^{kt}$$

$$N(0) = 50 \quad \rightarrow \quad \boxed{A = 50}$$

$$\boxed{N = 50e^{kt}}$$

$t=2 \quad \to \boxed{N(2)=N(0)-0.1N(0)=0.9(50)=45}$

$45=50e^{2k}$

$2k=\ln\frac{45}{50}$

$k=\frac{1}{2}\ln\frac{9}{10}$

$\boxed{N(t)=50e^{\frac{1}{2}t\ln\frac{9}{10}}}$

$\boxed{N(t)=50e^{-0.053t}}$

b) $N(4)=50e^{-0.053(4)}$

$\boxed{\approx 40.5\ mg}$

c) $t=-\frac{1}{.053}\ln\frac{1}{2}$

$\boxed{\approx 13\ hrs}$

Exercise

The rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in given sample. Half of the original number of radioactive nuclei have undergone disintegration in a period of 1,500 *years*.

a) What percentage of the original radioactive nuclei will remain after 4,500 *years*?

b) In how many years will only one-tenth of the original number remain?

Solution

a) $\frac{dN}{dt}=-kN$

$\int\frac{dN}{N}=-\int kdt$

$\ln N=-kt+C$

$N(t)=e^{-kt+C}$

$\boxed{=-N_0e^{-kt}}$

$k=\frac{1}{1500}\ln\frac{1}{2}=-4.62\times 10^{-4}$ $\qquad kT=\ln\frac{N}{N_0}$

$N=-N_0e^{-4.62\times 10^{-4}t}$

$\frac{N}{N_0}=e^{-4.62\times 10^{-4}(4500)}=0.125$

The percentage of the original radioactive nuclei will remain after 4,500 *years*: $\boxed{12.5\%}$

b) $N = \frac{1}{10} N_0$

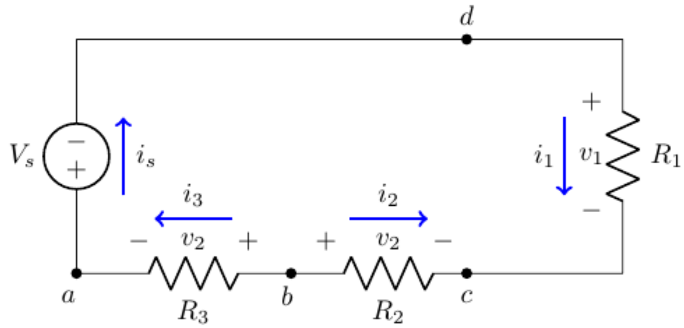
$$t = \frac{1}{-4.62\times10^{-4}} \ln\frac{1}{10} \qquad kT = \ln\frac{N}{N_0}$$

$$\approx 4985 \ \ years$$

Solution Section 1.8 - Basic Electrical Circuit

Exercise

Sum the currents at each node in he circuit



Solution

node ***a*** : $i_s - i_1 = 0$

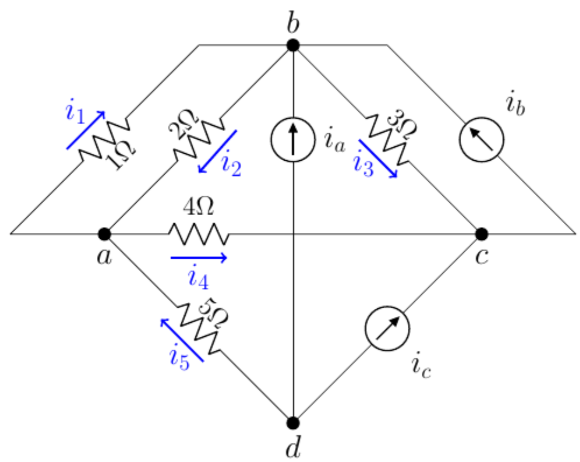
node ***b*** : $i_1 + i_c = 0$

node ***c*** : $-i_c - i_1 = 0$

node ***d*** : $i_1 - i_s = 0$

Exercise

Sum the currents at each node in he circuit



Solution

node ***a*** : $i_1 + i_4 - i_2 - i_5 = 0$

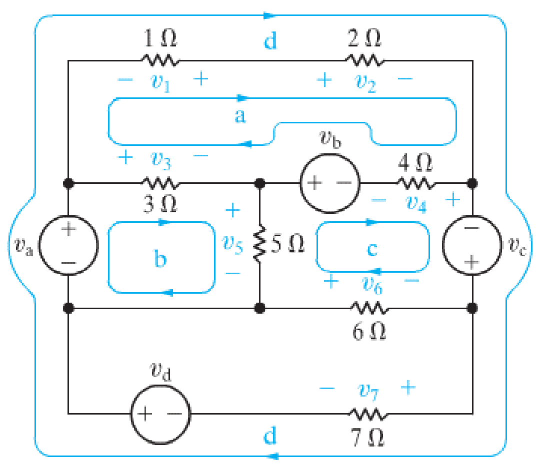
node ***b*** : $i_2 + i_3 - i_1 - i_b - i_a = 0$

node ***c*** : $i_b - i_3 - i_4 - i_c = 0$

node ***d*** : $i_5 + i_a + i_c = 0$

Exercise

Sum the voltges around rach designated path in the circuit



Solution

path ***a*** : $-v_1 + v_2 + v_4 - v_b - v_3 = 0$

path ***b*** : $-v_a + v_3 + v_5 = 0$

path ***c*** : $v_b - v_4 - v_c - v_6 - v_5 = 0$

path ***d*** : $-v_a - v_1 + v_2 - v_c + v_7 - v_d = 0$

Exercise

A resistor $R = 20\ \Omega$ and a capacitor of $C = 0.1\ F$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge on the capacitor at time t for the given $E(t) = 100\sin 2t$

Solution

$$20Q' + \frac{1}{0.1}Q = 100\sin 2t \qquad\qquad R\frac{dQ}{dt} + \frac{1}{C}Q = E$$

$$Q' + \frac{1}{2}Q = 5\sin 2t$$

$$e^{\int \frac{1}{2}dt} = e^{t/2}$$

$$\int (5\sin 2t)e^{t/2}dt = \left(-\frac{5}{2}\cos 2t + \frac{5}{8}\sin 2t\right)e^{t/2} - \frac{1}{16}\int (5\sin 2t)e^{t/2}dt$$

$$\frac{17}{16}\int (5\sin 2t)e^{t/2}dt = \frac{5}{8}(-4\cos 2t + \sin 2t)e^{t/2}$$

$$\int (5\sin 2t)e^{t/2}dt = \frac{10}{17}(-4\cos 2t + \sin 2t)e^{t/2}$$

		$\int 5\sin 2t$
+	$e^{t/2}$	$-\frac{5}{2}\cos 2t$
–	$\frac{1}{2}e^{t/2}$	$-\frac{5}{4}\sin 2t$
+	$\frac{1}{4}e^{t/2}$	

$$Q(t) = e^{-t/2}\left(\frac{10}{17}(-4\cos 2t + \sin 2t)e^{t/2} + K\right)$$

$$= \frac{10}{17}(-4\cos 2t + \sin 2t) + Ke^{-t/2}$$

$$Q(0) = 0 \quad \to \ 0 = \frac{10}{17}(-4) + K$$

$$\Rightarrow \ \boxed{K = \frac{40}{17}}$$

$$\boxed{Q(t) = \frac{10}{17}(\sin 2t - 4\cos 2t) + \frac{40}{17}e^{-t/2}}$$

Exercise

A resistor $R = 20\ \Omega$ and a capacitor of $C = 0.1\ F$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge on the capacitor at time t for the given $E(t) = 100e^{-0.1t}$

Solution

$$20Q' + \frac{1}{0.1}Q = 100e^{-0.1t} \qquad\qquad R\frac{dQ}{dt} + \frac{1}{C}Q = E$$

$$Q' + \frac{1}{2}Q = 5e^{-0.1t}$$

$$e^{\int \frac{1}{2}dt} = e^{t/2}$$

$$\int \left(5e^{-0.1t}\right)e^{0.5t}dt = \int 5e^{0.4t}dt$$
$$= \frac{25}{2}e^{2t/5}$$

$$Q(t) = e^{-t/2}\left(\frac{25}{2}e^{2t/5} + K\right)$$
$$= \frac{25}{2}e^{-t/10} + Ke^{-t/2}$$

$$Q(0) = 0 \quad \rightarrow \quad 0 = \frac{25}{2} + K$$
$$\Rightarrow \boxed{K = -\frac{25}{2}}$$

$$\boxed{Q(t) = \frac{25}{2}\left(e^{-t/10} - e^{-t/2}\right)}$$

Exercise

A resistor $R = 20\ \Omega$ and a capacitor of $C = 0.1\ F$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge on the capacitor at time t for the given $E(t) = 100\left(1 - e^{-0.1t}\right)$

Solution

$$20Q' + \frac{1}{0.1}Q = 100\left(1 - e^{-0.1t}\right) \qquad\qquad R\frac{dQ}{dt} + \frac{1}{C}Q = E$$

$$Q' + \frac{1}{2}Q = 5\left(1 - e^{-0.1t}\right)$$

$$e^{\int \frac{1}{2}dt} = e^{t/2}$$

$$\int 5\left(1 - e^{-0.1t}\right)e^{0.5t}dt = 5\int \left(e^{t/2} - e^{0.4t}\right)dt$$
$$= 5\left(2e^{t/2} - \frac{5}{2}e^{2t/5}\right)$$

$$Q(t) = e^{-t/2}\left(10e^{t/2} - \frac{25}{2}e^{2t/5} + K\right)$$
$$= 10 - \frac{25}{2}e^{-t/10} + Ke^{-t/2}$$

$$Q(0) = 0 \quad \rightarrow \quad 0 = 10 - \frac{25}{2} + K$$
$$\Rightarrow \boxed{K = \frac{5}{2}}$$

$$\boxed{Q(t) = 10 - \frac{25}{2}e^{-t/10} + \frac{5}{2}e^{-t/2}}$$

Exercise

A resistor $R = 20\ \Omega$ and a capacitor of $C = 0.1\,F$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge on the capacitor at time *t* for the given $E(t) = 100\cos 3t$

Solution

$$20Q' + \frac{1}{0.1}Q = 100\cos 3t \qquad R\frac{dQ}{dt} + \frac{1}{C}Q = E$$

$$Q' + \frac{1}{2}Q = 5\cos 3t$$

$$e^{\int \frac{1}{2}dt} = e^{t/2}$$

$$\int (5\cos 3t)e^{t/2}dt = \left(\frac{5}{3}\sin 3t + \frac{5}{18}\cos 3t\right)e^{t/2} - \frac{1}{36}\int (5\cos 3t)e^{t/2}dt$$

$$\frac{37}{36}\int (5\cos 3t)e^{t/2}dt = \frac{5}{18}(6\sin 3t + \cos 3t)e^{t/2}$$

$$\int (5\cos 3t)e^{t/2}dt = \frac{10}{37}(6\sin 3t + \cos 3t)e^{t/2}$$

		$\int 5\cos 3t$
+	$e^{t/2}$	$\frac{5}{3}\sin 3t$
−	$\frac{1}{2}e^{t/2}$	$-\frac{5}{9}\cos 3t$
+	$\frac{1}{4}e^{t/2}$	

$$Q(t) = e^{-t/2}\left(\frac{10}{37}(6\sin 3t + \cos 3t)e^{t/2} + K\right)$$

$$= \frac{10}{37}(6\sin 3t + \cos 3t) + Ke^{-t/2}$$

$$Q(0) = 0 \quad \to \quad 0 = \frac{10}{37}(1) + K$$

$$\Rightarrow \ \boxed{K = -\frac{10}{37}}$$

$$\boxed{Q(t) = \frac{10}{37}\left(6\sin 3t + \cos 3t - e^{-t/2}\right)}$$

Exercise

An inductor $(L = 1\ H)$ and a resistor $(R = 0.1\ \Omega)$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge current in the current at time *t* for the given $E(t) = 10 - 2t$

Solution

$$\frac{dI}{dt} + 0.1I = 10 - 2t \qquad L\frac{dI}{dt} + RI = E(t)$$

$$e^{\int .1dt} = e^{t/10}$$

$$\int (10 - 2t)e^{t/10}dt = (100 - 20t + 200)e^{t/10}$$

$$= (300 - 20t)e^{t/10}$$

$$I(t)=e^{-10t}\left((300-20t)e^{t/10}+K\right)$$

$$=300-20t+Ke^{-10t}$$

$$I(0)=0 \quad \rightarrow \quad \underline{K=-300}$$

$$\boxed{I(t)=300-20t-300e^{-t/10}}$$

Exercise

An inductor $(L=1\ H)$ and a resistor $(R=0.1\ \Omega)$ are joined in series with an electronic force (*emf*) $E=E(t)$ and no charge on the capacitor at $t=0$. Find the ensuing current in the current at time t for the given $E(t)=4\cos 3t$

Solution

$$\frac{dI}{dt}+0.1I=4\cos 3t \qquad\qquad L\frac{dI}{dt}+RI=E(t)$$

$$e^{\int .1dt}=e^{t/10}$$

$$\int(4\cos 3t)e^{t/10}=\left(\frac{4}{3}\sin 3t+\frac{2}{45}\cos 3t\right)e^{t/10}-\frac{1}{900}\int(4\cos 3t)e^{t/10}$$

$$\frac{901}{900}\int(4\cos 3t)e^{t/10}=\frac{2}{45}(30\sin 3t+\cos 3t)e^{t/10}$$

$$\int(4\cos 3t)e^{t/10}=\frac{40}{901}(30\sin 3t+\cos 3t)e^{t/10}$$

		$\int 4\cos 3t$
+	$e^{t/10}$	$\frac{4}{3}\sin 3t$
−	$\frac{1}{10}e^{t/10}$	$-\frac{4}{9}\cos 3t$
+	$\frac{1}{100}e^{t/10}$	

$$I(t)=e^{-t/10}\left(\frac{40}{901}(30\sin 3t+\cos 3t)e^{t/10}+K\right)$$

$$=\frac{40}{901}(30\sin 3t+\cos 3t)+Ke^{-t/10}$$

$$I(0)=0 \quad \rightarrow \quad \underline{K=-\frac{40}{901}}$$

$$\boxed{I(t)=\frac{40}{901}\left(30\sin 3t+\cos 3t-e^{-t/10}\right)}$$

Exercise

An inductor $(L=1\ H)$ and a resistor $(R=0.1\ \Omega)$ are joined in series with an electronic force (*emf*) $E=E(t)$ and no charge on the capacitor at $t=0$. Find the ensuing current in the current at time t for the given $E(t)=4\sin 2\pi t$

Solution

$$\frac{dI}{dt}+0.1I=4\sin 2\pi t \qquad\qquad L\frac{dI}{dt}+RI=E(t)$$

$$e^{\int .1dt} = e^{t/10}$$

$$\int (4\sin 2\pi t) e^{t/10} = \left(-\frac{2}{\pi}\cos 2\pi t + \frac{1}{10\pi^2}\sin 2\pi t\right)e^{t/10} - \frac{1}{400\pi^2}\int (4\sin 2\pi t)e^{t/10}$$

$$\frac{1+400\pi^2}{400\pi^2}\int (4\sin 2\pi t)e^{t/10} = \frac{1}{10\pi^2}(-20\pi\cos 2\pi t + \sin 2\pi t)e^{t/10}$$

$$\int (4\sin 2\pi t)e^{t/10} = \frac{40}{1+400\pi^2}(-20\pi\cos 2\pi t + \sin 2\pi t)e^{t/10}$$

		$\int 4\sin 2\pi t$
+	$e^{t/10}$	$-\frac{2}{\pi}\cos 2\pi t$
−	$\frac{1}{10}e^{t/10}$	$-\frac{1}{\pi^2}\sin 2\pi t$
+	$\frac{1}{100}e^{t/10}$	

$$I(t) = e^{-t/10}\left(\frac{40}{1+400\pi^2}(-20\pi\cos 2\pi t + \sin 2\pi t)e^{t/10} + K\right)$$

$$= \frac{40}{1+400\pi^2}(-20\pi\cos 2\pi t + \sin 2\pi t) + Ke^{-t/10}$$

$$I(0) = 0 \quad \to 0 = \frac{40}{1+400\pi^2}(-20\pi) + K$$

$$\Rightarrow \ K = \frac{800\pi}{1+25\pi^2}$$

$$I(t) = \frac{40}{1+400\pi^2}(-20\pi\cos 2\pi t + \sin 2\pi t) + \frac{800}{1+400\pi^2}e^{-t/10}$$

$$= \frac{40}{1+400\pi^2}\left(\sin 2\pi t - 20\pi\cos 2\pi t + 20\pi e^{-t/10}\right)$$

Exercise

An *RL* circuit with a $1-\Omega$ resistor and a 0.1-H inductor is driven by a voltage $E(t) = \sin 100t\ V$. If the initial inductor current is zero, determine the subsequence resistor and inductor current and the voltages.

Solution

$$0.1\frac{dI}{dt} + I = \sin 100t \qquad\qquad L\frac{dI}{dt} + RI = E(t)$$

$$\frac{dI}{dt} + 10I = 10\sin 100t$$

$$e^{\int 10dt} = e^{10t}$$

$$\int (10\sin 100t)e^{10t}dt = \left(-\frac{1}{10}\cos 100t + \frac{1}{100}\sin 100t\right)e^{10t} - \int \left(\frac{1}{10}\sin 100t\right)e^{10t}dt$$

$$\left(10 + \frac{1}{10}\right)\int (\sin 100t)e^{10t}dt = \frac{1}{100}(-10\cos 100t + \sin 100t)e^{10t}$$

$$\frac{101}{10}\int (10\sin 100t)e^{10t}dt = \frac{1}{100}(-10\cos 100t + \sin 100t)e^{10t}$$

$$\int(10\sin 100t)e^{10t}dt=\frac{1}{1010}(-10\cos 100t+\sin 100t)e^{10t}$$

$$I(t)=e^{-10t}\left(\frac{1}{1010}(-10\cos 100t+\sin 100t)e^{10t}+K\right)$$

$$=\frac{1}{1010}(-10\cos 100t+\sin 100t)+Ke^{-10t}$$

		$\int 10\sin 100t$
+	e^{10t}	$-\frac{1}{10}\cos 100t$
−	$10e^{10t}$	$-\frac{1}{1000}\sin 100t$
+	$100e^{10t}$	

$$I(0)=0 \quad \rightarrow \ 0=\frac{1}{1010}(-10)+K$$

$$\Rightarrow \ K=\frac{1}{101}$$

$$I(t)=\frac{1}{1010}(-10\cos 100t+\sin 100t)+\frac{1}{101}e^{-10t}$$

The voltage at the resistor:

$$E_R(t)=RI=\frac{1}{1010}(-10\cos 100t+\sin 100t)+\frac{1}{101}e^{-10t}$$

The voltage at the inductor:

$$E_L(t)=L\frac{dI}{dt}=0.1\left(\frac{1}{1010}\left(10^3\sin 100t+100\cos 100t\right)-\frac{10}{101}e^{-10t}\right)$$

$$=\frac{10}{101}\sin 100t+\frac{1}{101}\sin 100t-\frac{1}{101}e^{-10t}$$

Exercise

An *RL* circuit with a $1-\Omega$ resistor and a $0.01\text{-}H$ inductor is driven by a voltage $E(t)=\sin 100t\ V$. If the initial inductor current is zero, determine the subsequence resistor and inductor current and the voltages.

Solution

$$0.01\frac{dI}{dt}+I=\sin 100t \qquad\qquad L\frac{dI}{dt}+RI=E(t)$$

$$\frac{dI}{dt}+100I=100\sin 100t$$

$$e^{\int 100dt}=e^{100t}$$

$$\int(100\sin 100t)e^{100t}dt=(-\cos 100t+\sin 100t)e^{100t}-\int(100\sin 100t)e^{100t}dt$$

$$2\int(100\sin 100t)e^{100t}dt=(-\cos 100t+\sin 100t)e^{100t}$$

$$\int(100\sin 100t)e^{100t}dt=\frac{1}{2}(-\cos 100t+\sin 100t)e^{100t}$$

		$\int 100\sin 100t$
+	e^{100t}	$-\cos 100t$
−	$100e^{100t}$	$-\frac{1}{100}\sin 100t$
+	$10000e^{100t}$	

$$I(t)=e^{-100t}\left(\frac{1}{2}(-\cos 100t+\sin 100t)e^{100t}+K\right)$$

$$=\frac{1}{2}(-\cos 100t+\sin 100t)+Ke^{-100t}$$

$$I(0)=0 \quad \to \quad 0=\frac{1}{2}(-1)+K$$

$$\Rightarrow \quad K=\frac{1}{2}$$

$$I(t)=\frac{1}{2}(-\cos 100t+\sin 100t)+\frac{1}{2}e^{-100t}$$

The voltage at the resistor:

$$E_R(t)=RI \;=\frac{1}{2}\left(\sin 100t-\cos 100t+e^{-100t}\right)$$

The voltage at the inductor:

$$E_L(t)=L\frac{dI}{dt}=(0.01)\frac{1}{2}\left(100\cos 100t+100\sin 100t-100e^{-100t}\right)$$

$$=\frac{1}{2}\left(\cos 100t+\sin 100t+e^{-10t}\right)$$

Exercise

An *RL* circuit with a $5-\Omega$ resistor and a $0.05\text{-}H$ inductor is driven by a voltage $E(t)=5\cos 120t\ V$. If the initial inductor current is 1 *A*, determine the subsequence resistor and inductor current and the voltages.

Solution

$$0.05\frac{dI}{dt}+5I=5\cos 120t \qquad\qquad L\frac{dI}{dt}+RI=E(t)$$

$$\frac{dI}{dt}+100I=100\cos 120t$$

$$e^{\int 100dt}=e^{100t}$$

$$\int(100\cos 120t)e^{100t}dt=\left(\frac{5}{6}\sin 120t+\frac{25}{36}\cos 120t\right)e^{100t}-\frac{25}{36}\int(100\cos 120t)e^{100t}dt$$

$$\left(1+\frac{25}{36}\right)\int(100\cos 120t)e^{100t}dt=\left(\frac{5}{6}\sin 120t+\frac{25}{36}\cos 120t\right)e^{100t}$$

$$\frac{61}{36}\int(100\cos 120t)e^{100t}dt=\frac{5}{36}(6\sin 120t+5\cos 120t)e^{100t}$$

$$\int(100\cos 120t)e^{100t}dt=\frac{5}{61}(6\sin 120t+5\cos 120t)e^{100t}$$

$$I(t)=e^{-100t}\left(\frac{5}{61}(6\sin 120t+5\cos 120t)e^{100t}+K\right)$$

$$=\frac{5}{61}(6\sin 120t+5\cos 120t)+Ke^{-100t}$$

$$I(0)=1 \quad \to \quad 1=\frac{5}{61}(5)+K$$

$$\Rightarrow\ \boxed{K=\frac{36}{61}}$$

		$\int 100\cos 120t$
+	e^{100t}	$\frac{5}{6}\sin 120t$
−	$100e^{100t}$	$-\frac{1}{144}\cos 120t$
+	$10^4 e^{100t}$	

$$\boxed{I(t)=\frac{5}{61}(6\sin 120t+5\cos 120t)+\frac{36}{61}e^{-100t}}$$

The voltage at the resistor:

$$\boxed{E_R(t)=RI}=\boxed{\frac{25}{61}(6\sin 120t+5\cos 120t)+\frac{180}{61}e^{-100t}}$$

The voltage at the inductor:

$$E_L(t)=L\frac{dI}{dt}=(0.05)\left(\frac{25}{61}(720\cos 120t-600\sin 120t)-\frac{18000}{61}e^{-100t}\right)$$

$$\boxed{=14.754\cos 120t-12.295\sin 120t-14.754e^{-100t}}$$

Exercise

An RC circuit with a $1-\Omega$ resistor and a 10^{-6}-F capacitor is driven by a voltage $E(t)=\sin 100t\ V$. If the initial capacitor current is zero, determine the subsequence resistor and capacitor current and the voltages.

Solution

$$Q'+10^6Q=\sin 100t \qquad\qquad R\frac{dQ}{dt}+\frac{1}{C}Q=E$$

$$e^{\int 10^6 dt}=e^{10^6 t}$$

$$\int(\sin 100t)e^{10^6 t}dt=\left(-\frac{1}{100}\cos 100t+100\sin 100t\right)e^{10^6 t}-10^8\int(\sin 100t)e^{10^6 t}dt$$

$$\left(10^8+1\right)\int(\sin 100t)e^{10^6 t}dt=\frac{1}{100}\left(-\cos 100t+10^4\sin 100t\right)e^{10^6 t}$$

$$\int(\sin 100t)e^{10^6 t}dt=\frac{1}{100\left(10^8+1\right)}\left(-\cos 100t+10^4\sin 100t\right)e^{10^6 t}$$

		$\int \sin 100t$
+	$e^{10^6 t}$	$-\frac{1}{100}\cos 100t$
−	$10^6 e^{10^6 t}$	$-10^{-4}\sin 100t$
+	$10^{12}e^{10^6 t}$	

$$Q(t)=e^{-10^6 t}\left(\frac{1}{10^{10}+100}\left(-\cos 100t+10^4\sin 100t\right)e^{10^6 t}+K\right)$$

$$=\frac{1}{10^{10}+100}\left(-\cos 100t+10^4\sin 100t\right)+Ke^{-10^6 t}$$

$$Q(0)=0 \quad\to\quad 0=\frac{1}{10^{10}+100}(-1)+K$$

$$\Rightarrow\ \boxed{K=\frac{1}{10^{10}+100}}$$

$$\boxed{Q(t)=\frac{1}{10^{10}+100}\left(-\cos 100t+10^4\sin 100t+e^{-10^6 t}\right)}$$

The voltage across the capacitor is:

$$V_C = \frac{Q}{C} = \frac{10^6}{10^{10}+100}\left(-\cos 100t + 10^4 \sin 100t + e^{-10^6 t}\right)$$

The current is:

$$I(t) = \frac{dQ}{dt} = \frac{1}{10^{10}+100}\left(100\sin 100t + 10^6 \cos 100t - 10^6 e^{-10^6 t}\right)$$

$$= \frac{1}{10^8+1}\left(\sin 100t + 10^4 \cos 100t - 10^4 e^{-10^6 t}\right)$$

The voltage across the resistor is:

$$V_R = RI = \frac{1}{10^8+1}\left(\sin 100t + 10^4 \cos 100t - 10^4 e^{-10^6 t}\right)$$

Exercise

Solve the general initial value problem modeling the *RC* circuit $R\frac{dQ}{dt} + \frac{1}{C}Q = E, \quad Q(0) = 0$

Where E is a constant source of *emf*

Solution

$$\frac{dQ}{dt} + \frac{1}{RC}Q = \frac{E}{R}$$

$$e^{\int \frac{1}{RC}dt} = e^{t/RC}$$

$$\int \frac{E}{R}e^{t/RC}dt = \frac{E}{R}RCe^{t/RC} = ECe^{t/RC}$$

$$Q(t) = \frac{1}{e^{t/RC}}\left(ECe^{t/RC} - K\right) = EC - Ke^{-t/RC}$$

$$Q(t=0) = EC - K$$

$$0 = EC - K \quad \Rightarrow \quad K = EC$$

$$Q(t) = EC\left(1 - e^{-t/RC}\right)$$

Exercise

Solve the general initial value problem modeling the *LR* circuit $L\frac{dI}{dt} + RI = E, \quad I(0) = I_0$

Where E is a constant source of emf

Solution

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L}$$

$$e^{\int \frac{R}{L}dt} = e^{(R/L)t}$$

$$\int \frac{E}{L}e^{(R/L)t}dt = \frac{E}{L}\frac{L}{R}e^{(R/L)t}$$

$$= \frac{E}{R}e^{(R/L)t}$$

$$I(t) = \frac{1}{e^{(R/L)t}}\left(\frac{E}{R}e^{(R/L)t} - K\right)$$

$$= \frac{E}{R} - Ke^{-(R/L)t}$$

$$I(t=0) = \frac{E}{R} - K$$

$$I_0 = \frac{E}{R} - K \quad \Rightarrow \quad K = \frac{E}{R} - I_0$$

$$I(t) = \frac{E}{R} - \left(\frac{E}{R} - I_0\right)e^{-Rt/L}$$

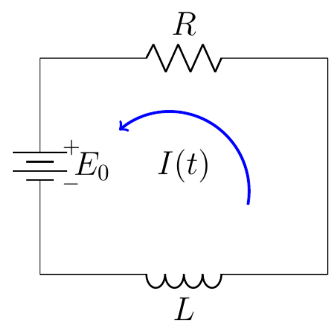
$$\boxed{I(t) = \frac{1}{R}\left(E + \left(RI_0 - E\right)e^{-Rt/L}\right)}$$

Exercise

For the given $RL-$circuit

Where E_0 is a constant source of *emf* at time $t = 0$.

Find the current $I(t)$ flowing in the circuit.



Solution

$$L\frac{dI}{dt} + RI = E_0$$

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E_0}{L}$$

$$e^{\int \frac{R}{L}dt} = e^{(R/L)t}$$

$$\int \frac{E_0}{L}e^{(R/L)t}dt = \frac{E_0}{L}\frac{L}{R}e^{(R/L)t}$$

$$= \frac{E_0}{R}e^{(R/L)t}$$

$$I(t) = \frac{1}{e^{(R/L)t}}\left(\frac{E_0}{R}e^{(R/L)t} - K\right)$$

$$= \frac{E_0}{R} - Ke^{-(R/L)t}$$

$$I(t=0) = \frac{E_0}{R} - K = I_0$$

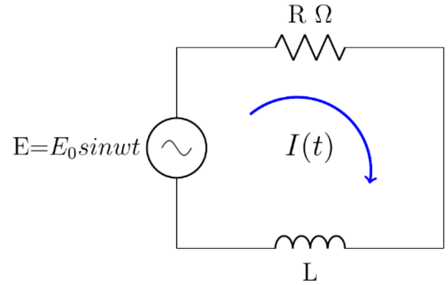
$$K = \frac{E_0}{R} - I_0$$

$$I(t) = \frac{E_0}{R} - \left(\frac{E_0}{R} - I_0\right)e^{-(R/L)t}$$

Exercise

For the given $RL-$circuit, wWhere $E = E_0 \sin \omega t$ is the impressed voltage.

Find the current $I(t)$ flowing in the circuit.



Solution

$$L\frac{dI}{dt} + RI = E_0 \sin \omega t$$

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E_0}{L}\sin \omega t$$

$$e^{\int \frac{R}{L}dt} = e^{(R/L)t}$$

$$\int \frac{E_0}{L}(\sin \omega t)e^{(R/L)t}dt = \frac{E_0}{L}\left(\sin \omega t - \frac{L}{R}\omega \cos \omega t\right)e^{(R/L)t} - \omega^2 \frac{E_0}{L}\frac{L^2}{R^2}\int (\sin \omega t)e^{(R/L)t}dt$$

$$\frac{E_0}{L}\left(1 + \frac{LE_0\omega^2}{R^2}\right)\int (\sin \omega t)e^{(R/L)t}dt = \frac{E_0}{L}\left(\sin \omega t - \frac{L}{R}\omega \cos \omega t\right)e^{(R/L)t}$$

$$\int \frac{E_0}{L}(\sin \omega t)e^{(R/L)t}dt = \frac{R^2}{1 + LE_0\omega^2}\frac{E_0}{L}\left(\sin \omega t - \frac{L}{R}\omega \cos \omega t\right)e^{(R/L)t}$$

$$I(t) = \frac{1}{e^{(R/L)t}}\left(\frac{R^2E_0}{L + E_0L^2\omega^2}\left(\sin \omega t - \frac{L}{R}\omega \cos \omega t\right)e^{(R/L)t} + K\right)$$

$$= \frac{R^2E_0}{L + E_0L^2\omega^2}\left(\sin \omega t - \frac{L\omega}{R}\cos \omega t\right) + Ke^{-(R/L)t}$$

		$\int e^{(R/L)t}$
$+$	$\sin \omega t$	$\frac{L}{R}e^{(R/L)t}$
$-$	$\omega \cos \omega t$	$\left(\frac{L}{R}\right)^2 e^{(R/L)t}$
$+$	$-\omega^2 \sin \omega t$	

$$I(t = 0) = -\frac{L\omega}{R}\frac{R^2E_0}{L + E_0L^2\omega^2} + K = I_0$$

$$K = I_0 + \frac{R\omega E_0}{1 + E_0L\omega^2}$$

$$I(t) = \frac{R^2E_0}{L + E_0L^2\omega^2}\left(\sin \omega t - \frac{L\omega}{R}\cos \omega t\right) + \left(I_0 + \frac{R\omega E_0}{1 + E_0L\omega^2}\right)e^{-(R/L)t}$$

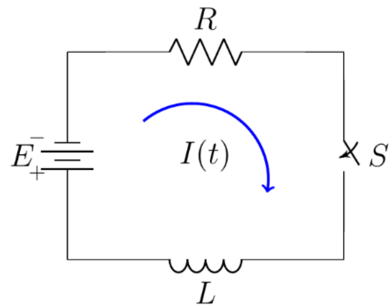
Exercise

For the given $RL-$circuit

Which has a constant impressed voltage E, a resistor of resistance R, and a coil of impedance L.

Find the current $I(t)$ flowing in the circuit.

Solution



$$L\frac{dI}{dt} + RI = E$$

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L}$$

$$e^{\int \frac{R}{L}dt} = e^{(R/L)t}$$

$$\int \frac{E}{L}e^{(R/L)t}dt = \frac{E}{L}\frac{L}{R}e^{(R/L)t}$$

$$= \frac{E}{R}e^{(R/L)t}$$

$$I(t) = \frac{1}{e^{(R/L)t}}\left(\frac{E}{R}e^{(R/L)t} - K\right)$$

$$= \frac{E}{R} - Ke^{-(R/L)t}$$

$$I(t=0) = \frac{E}{R} - K = 0$$

$$\boxed{K = \frac{E}{R}}$$

$$\boxed{I(t) = \frac{E}{R} - \frac{E}{R}e^{-(R/L)t}}$$

$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty}\left(\frac{E}{R} - \frac{E}{R}e^{-(R/L)t}\right)$$

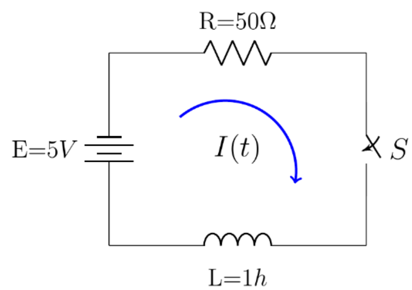
$$\boxed{= \frac{E}{R}}$$

Exercise

For the given $RL-$circuit

Which has a constant impressed voltage E, a resistor of resistance R, and a coil of impedance L.

Find the current $I(t)$ flowing in the circuit.



Solution

$$L\frac{dI}{dt} + RI = E$$

$$\frac{dI}{dt} + 50I = 5$$

$$e^{\int 50dt} = e^{50t}$$

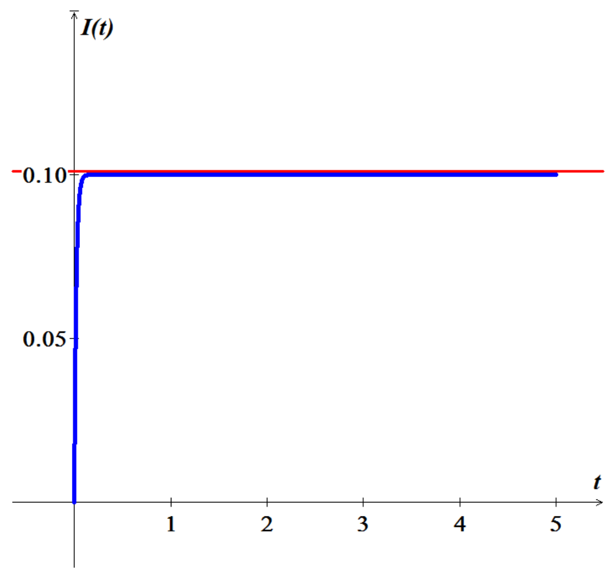
$$\int 5e^{50t}dt = \frac{1}{10}e^{50t}$$

$$I(t) = \frac{1}{e^{50t}}\left(\frac{1}{10}e^{50t} + K\right)$$

$$= \frac{1}{10} + Ke^{-50t}$$

$$I(t=0) = \frac{1}{10} + K = 0 \quad \rightarrow \quad \underline{K = -\frac{1}{10}}$$

$$\underline{I(t) = \frac{1}{10}\left(1 - e^{-50t}\right)}$$



Exercise

Consider the circuit shown and let I_1, I_2, and I_3 be the currents through the capacitor, resistor, and inductor, respectively. Let V_1, V_2, and V_3 be the corresponding voltage drops. The arrows denote the arbitrary chosen directions in which currents and voltage drops will be taken to be positive.

a) Applying Kirchhoff's second law to the upper loop in the circuit, show that

$$V_1 - V_2 = 0 \quad \textit{and} \quad V_2 - V_3 = 0$$

b) Applying Kirchhoff's first law to either node in the circuit, show that $I_1 + I_2 + I_3 = 0$

c) Use the current-voltage relation through each element in the circuit to obtain the equations

$$CV_1' = I_1, \quad V_2 = RI_2, \quad LI_3' = V_3$$

d) Eliminate V_2, V_3, I_1 and I_2 to obtain $\quad CV_1' = -I_3 - \frac{V_1}{R}, \quad LI_3' = V_1$

Solution

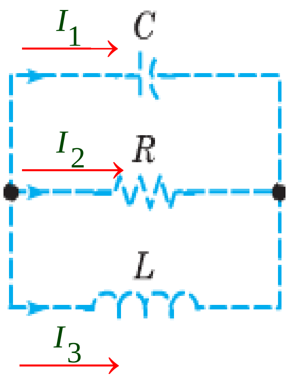
a) Taking the clockwise loop around each paths, it is easy to see that voltage drops are given by

$$V_1 - V_2 = 0 \quad and \quad V_2 - V_3 = 0$$

b) Consider the right node. The current is given by $I_1 + I_2$. The current leaving the node is $-I_3$.

Hence the cursing through the node is $I_1 + I_2 - (-I_3)$.

Based on Kirchhoff's first law, $I_1 + I_2 + I_3 = 0$



c) In the capacitor $CV_1' = I_1$

In the resistor $V_2 = RI_2$

In the inductor $LI_3' = V_3$

d) Based on part (a), $V_1 = V_2 = V_3$. Based on part (*b*),

$$CV_1' + \frac{1}{R}V_2 + I_3 = 0$$

It follows: $CV_1' = -I_3 - \frac{V_1}{R}, \quad LI_3' = V_1$

Exercise

Consider the circuit. Use the method outlined to show that the current *I* through the inductor and the voltage *V* across the capacitor satisfy the system of differential equations.

$$L\frac{dI}{dt} = -R_1 I - V, \quad C\frac{dV}{dt} = I - \frac{V}{R_2}$$

Solution

let I_1, I_2, I_3 and I_4 be the current through the resistors, inductor, and capacitor, respectively.

Assign V_1, V_2, V_3 and V_4 to be the corresponding voltage drops.

Based on Kirchhoff's second law, the net voltage drops, around each loop, satisfy

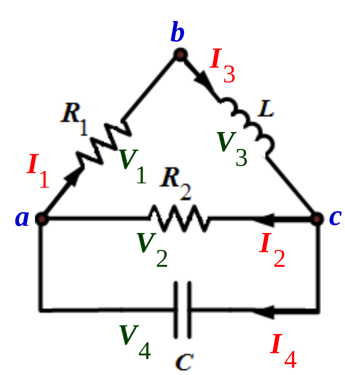
$$V_1 + V_3 + V_4 = 0, \quad V_1 + V_3 + V_2 = 0 \quad and \quad V_4 - V_2 = 0$$

Applying Kirchhoff's first law:

Node ***a***: $I_1 - (I_2 + I_4) = 0$

Node ***b***: $I_1 - I_3 = 0 \rightarrow I_1 = I_3$

Node ***c***: $I_2 + I_4 - I_1 = 0 \rightarrow I_2 + I_4 = I_1$



$$I_2 + I_4 = I_3 \Rightarrow \boxed{I_2 + I_4 - I_3 = 0}$$

Using the current-voltage relations:

$$V_1 = R_1 I_1 = R_1 I_3 \qquad V_2 = R_2 I_2$$

$$LI_3' = V_3 \qquad CV_4' = I_4$$

$$V_1+V_3+V_4=0 \quad \Rightarrow \quad \boxed{R_1I_3+LI'_3+V_4=0}$$

$$I_4=I_3-I_2 \quad \Rightarrow \quad \boxed{CV'_4=I_3-\frac{V_2}{R_2}}$$

Let $I_3=I$ *and* $V_4=V$

$$R_1I+LI'+V=0 \quad \Rightarrow \quad \boxed{LI'=-R_1I-V}$$

$$CV'_4=I_3-\frac{V_2}{R_2} \quad \Rightarrow \quad \boxed{CV'=I-\frac{V}{R_2}}$$

Exercise

Consider an electric circuit containing a capacitor, resistor, and battery.
The charge $Q(t)$ on the capacitor satisfies the equation

$$R\frac{dQ}{dt}+\frac{Q}{C}=V$$

Where *R* is the resistance, *C* is the capacitance, and *V* is the constant voltage supplied by the battery.

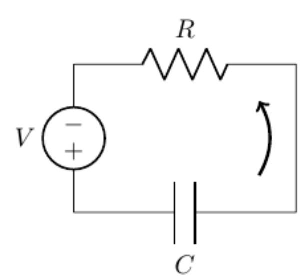
a) If $Q(0)=0$, find $Q(t)$ at time *t*.

b) Find the limiting value Q_L that $Q(t)$ approaches after a long time.

c) Suppose that $Q(t_1)=Q(t)$ and that at time $t=t_1$ the battery is removed and the circuit is closed again. Find $Q(t)$ for $t>t_1$.

Solution

a) $R\frac{dQ}{dt}+\frac{Q}{C}=V \quad \Rightarrow \quad R\frac{dQ}{dt}=V-\frac{Q}{C}$



$$R\frac{dQ}{dt}=\frac{CV-Q}{C}$$

$$\frac{dQ}{dt}=\frac{CV-Q}{RC}$$

$$\frac{dQ}{Q-CV}=-\frac{1}{RC}dt$$

$$\int\frac{dQ}{Q-CV}=-\int\frac{1}{RC}dt$$

$$\ln|Q-CV|=-\frac{t}{CR}+A$$

$$Q-CV=e^{-t/CR+A}$$

$$Q=De^{-t/CR}+CV$$

$$Q(0)=0$$

$$0 = De^{-0} + CV$$
$$D = -CV$$

$$Q = -CVe^{-t/CR} + CV$$
$$\boxed{Q(t) = CV\left(1 - e^{-t/CR}\right)}$$

b) $\lim\limits_{t\to\infty} Q(t) = CV \lim\limits_{t\to\infty}\left(1 - e^{-t/CR}\right)$

$$= CV(1-0)$$

$$\lim_{t\to\infty} e^{-t/CR} = \lim_{t\to\infty}\frac{1}{e^{t/CR}} \to \frac{1}{\infty} = 0$$

$$\boxed{= CV}$$

c) In this case, $R\dfrac{dQ}{dt} + \dfrac{Q}{C} = 0$, $Q(t_1) = CV$

$$\frac{dQ}{dt} = -\frac{Q}{CR}$$

$$\int \frac{dQ}{Q} = -\frac{1}{CR}\int dt$$

$$\ln|Q| = -\frac{t}{CR} + A$$

The solution is $Q = Ee^{-t/CR}$

So, $Q(t_1) = Ee^{-t_1/CR} = CV$

$$E = CVe^{t_1/CR}$$

$$Q(t) = CVe^{t_1/CR}e^{-t/CR}$$
$$= CVe^{(t_1-t)/CR}$$
$$\boxed{= CVe^{-(t-t_1)/CR}} \quad \text{for} \quad t \ge t_1$$

Exercise

A circuit containing an electromotive force, a capacitor with a capacitance of C farads (F), and a resistor with a resistance of R ohms (Ω). The voltage drop across the capacitor is $\dfrac{Q}{C}$, where Q is the charge (in coulombs), so in this case ***Kirchhoff's Law*** gives

$$RI + \frac{Q}{C} = E(t)$$

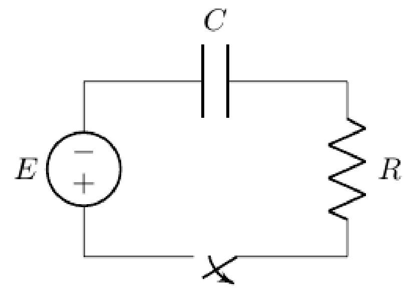
But $I = \dfrac{dQ}{dt}$, so we have $\qquad R\dfrac{dQ}{dt} + \dfrac{1}{C}Q = E(t)$

Find the charge and the current at time t

a) Suppose the resistance is $5\ \Omega$, the capacitance is 0.05 *F*, a battery gives voltage of 60 *V* and initial charge is $Q(0)=0\ C$

b) Suppose the resistance is $2\ \Omega$, the capacitance is 0.01 *F*, $E(t)=10\sin 60t$ and initial charge is $Q(0)=0\ C$

Solution



a) $5\frac{dQ}{dt}+\frac{1}{.05}Q=60 \quad\to\quad \frac{dQ}{dt}+4Q=12$

$$e^{\int 4dt}=e^{4t}$$

$$\int 12e^{4t}dt=3e^{4t}$$

$$Q(t)=\frac{1}{e^{4t}}\left(3e^{4t}+C\right)$$

$$=3+Ce^{-4t}$$

$$Q(0)=3+C=0 \ \Rightarrow\ C=-3$$

$$Q(t)=3\left(1-e^{-4t}\right)$$

$$I=\frac{dQ}{dt}=12e^{-4t}$$

b) $2\frac{dQ}{dt}+\frac{1}{.01}Q=10\sin 60t \quad\to\quad \frac{dQ}{dt}+50Q=5\sin 60t$

$$e^{\int 50dt}=e^{50t}$$

$$5\int e^{50t}(\sin 60t)\,dt=$$

$$\int e^{50t}(\sin 60t)\,dt=\left(-\frac{1}{60}\cos 60t+\frac{1}{72}\sin 60t\right)e^{50t}-\frac{25}{36}\int e^{50t}(\sin 60t)\,dt$$

$$\frac{61}{36}\int e^{50t}(\sin 60t)\,dt=\left(-\frac{1}{60}\cos 60t+\frac{1}{72}\sin 60t\right)e^{50t}$$

$$\int e^{50t}(\sin 60t)\,dt=\frac{36}{21,960}(-6\cos 60t+5\sin 60t)e^{50t}$$

$$5\int e^{50t}(\sin 60t)\,dt=\frac{1}{122}(-6\cos 60t+5\sin 60t)e^{50t}$$

		$\int \sin 60t$
+	e^{50t}	$-\frac{1}{60}\cos 60t$
−	$50e^{50t}$	$-\frac{1}{3600}\sin 60t$
+	$2500e^{50t}$	$-\frac{1}{3600}\int \sin 60t$

$$Q(t)=\frac{1}{e^{50t}}\left(\frac{1}{122}(-6\cos 60t+5\sin 60t)e^{50t}+C\right)$$

$$=\frac{1}{122}(-6\cos 60t+5\sin 60t)+Ce^{-50t}$$

$$Q(0)=-\frac{6}{122}+C=0 \ \Rightarrow\ C=\frac{3}{61}$$

$$\boxed{Q(t)=\frac{1}{122}\left(-5\cos 60t+6\sin 60t+6e^{-50t}\right)}$$

$$I=\frac{dQ}{dt}=\frac{1}{122}\left(300\sin 60t+360\cos 60t-300e^{-50t}\right)$$

$$=\boxed{\frac{30}{61}\left(5\sin 60t+6\cos 60t-5e^{-50t}\right)}$$

Exercise

A heart pacemaker consists of a switch, a battery voltage E_0, a capacitor with constant capacitance *C*, and the heart as a resistor with constant resistance *R*. When the switch is closed, the capacitor charges; when the switch is open, the capacitor discharges, sending an electrical stimulus to the heart. During the time the heart is being stimulated, the voltage *E* across the heart satisfies the linear differential equation

$$\frac{dE}{dt}=-\frac{1}{RC}E$$

Solve the *DE*, subject to $E(4)=E_0$

Solution

$$\int\frac{dE}{E}=-\frac{1}{RC}\int dt$$

$$\ln E=-\frac{1}{RC}t+C$$

$$E=e^{-\frac{1}{RC}t+C}=Ae^{-\frac{1}{RC}t}$$

$$E(4)=E_0 \quad\to\quad E_0=Ae^{-\frac{4}{RC}}$$

$$\Rightarrow \boxed{A=E_0e^{\frac{4}{RC}}}$$

$$E(t)=E_0e^{\frac{4}{RC}}e^{-\frac{1}{RC}t}$$

$$=\boxed{E_0e^{\frac{4-t}{RC}}}$$

Exercise

A 30–volt electromotive force is applied to an *LR*-series circuit in which the inductance is 0.1 *henry* and the resistance is 50 *ohms*.

a) Find the current $i(t)$ if $i(0)=0$

b) Determine the current as $t\to\infty$

c) Solve the equation when $E(t)=E_0\sin\omega t$ and $i(0)=i_0$

Solution

a) $0.1\frac{di}{dt}+50i=30$ $\qquad L\frac{di}{dt}+Ri=E(t)$

$$\frac{di}{dt}+500i=300$$

$$e^{\int 500dt}=e^{500t}$$

$$\int 300e^{500t}dt=\frac{3}{5}e^{500t}$$

$$i(t)=e^{-500t}\left(\frac{3}{5}e^{500t}+C\right) \qquad i(0)=0$$

$$0=\frac{3}{5}+C \;\to\; \boxed{C=-\frac{3}{5}}$$

$$\boxed{i(t)=\frac{3}{5}-\frac{3}{5}e^{-500t}}$$

b) $\lim\limits_{t\to\infty} i(t)=\lim\limits_{t\to\infty}\left(\frac{3}{5}-\frac{3}{5}e^{-500t}\right)$

$$\boxed{=\frac{3}{5}}$$

c) $\frac{di}{dt}+500i=10E_0\sin\omega t$

		$\int \sin\omega t$
+	e^{500t}	$-\frac{1}{\omega}\cos\omega t$
−	$500e^{500t}$	$-\frac{1}{\omega^2}\sin\omega t$
+	$25\times10^4e^{500t}$	$-\int\frac{1}{\omega^2}\sin\omega t$

$$\int 10E_0(\sin\omega t)e^{500t}dt=$$

$$\int(\sin\omega t)e^{500t}dt=\left(-\frac{1}{\omega}\cos\omega t+\frac{500}{\omega^2}\sin\omega t\right)e^{500t}-\frac{25\times10^4}{\omega^2}\int(\sin\omega t)e^{500t}dt$$

$$\left(\frac{\omega^2+25\times10^4}{\omega^2}\right)\int(\sin\omega t)e^{500t}dt=\frac{1}{\omega^2}(-\omega\cos\omega t+500\sin\omega t)e^{500t}$$

$$\int 10E_0(\sin\omega t)e^{500t}dt=\frac{10E_0}{\omega^2+25\times10^4}(-\omega\cos\omega t+500\sin\omega t)e^{500t}$$

$$i(t)=e^{-500t}\left(\frac{10E_0}{\omega^2+25\times10^4}(-\omega\cos\omega t+500\sin\omega t)e^{500t}+C\right) \qquad i(0)=i_0$$

$$0=-\frac{10\omega tE_0}{\omega^2+25\times10^4}+C \;\to\; \boxed{C=\frac{10\omega tE_0}{\omega^2+25\times10^4}}$$

$$\boxed{i(t)=\frac{10E_0}{\omega^2+25\times10^4}\left(-\omega\cos\omega t+500\sin\omega t-\omega e^{-500t}\right)}$$

Exercise

A 100–volt electromotive force is applied to an *RC*-series circuit in which the resistance is 200 *ohms* and the capacitance is 10^{-4} *farad*.

a) Find the charge $q(t)$ if $q(0)=0$

b) Find the current as $i(t)$

Solution

Given: $R=200\ \Omega,\quad C=10^{-4}\ F\quad E(t)=100\ V$

a) $200\frac{dq}{dt}+10^4 q=100 \qquad\qquad R\frac{dq}{dt}+\frac{1}{C}q=E(t)$

$$\frac{dq}{dt}+50q=\frac{1}{2}$$

$$e^{\int 50dt}=e^{50t}$$

$$\int \frac{1}{2}e^{50t}dt=\frac{1}{100}e^{50t}$$

$$q(t)=\frac{1}{e^{50t}}\left(\frac{1}{100}e^{50t}+C\right)$$

$$=\frac{1}{100}+Ce^{-50t}$$

$$q(0)=0\quad \to 0=\frac{1}{100}+C$$

$$\Rightarrow\ \boxed{C=-\frac{1}{100}}$$

$$\boxed{q(t)=\frac{1}{100}-\frac{1}{100}e^{-50t}}$$

b) $\boxed{i(t)=\frac{1}{2}e^{-50t}} \qquad\qquad i(t)=\frac{dq}{dt}$

Exercise

A 200–volt electromotive force is applied to an *RC*-series circuit in which the resistance is 1000 *ohms* and the capacitance is 5×10^{-6} *farad*.

a) Find the charge $q(t)$ if $i(0)=0.4$

b) Determine the charge as $t\to\infty$

Solution

Given: $R=1000\ \Omega,\quad C=5\times10^{-6}\ F\quad E(t)=200\ V$

a) $1000\frac{dq}{dt}+\frac{1}{5}10^6 q=200 \qquad\qquad R\frac{dq}{dt}+\frac{1}{C}q=E(t)$

$$\frac{dq}{dt}+200q=\frac{1}{5}$$

$$e^{\int 200dt} = e^{200t}$$

$$\int \frac{1}{5}e^{200t}dt = \frac{1}{1000}e^{200t}$$

$$q(t) = \frac{1}{e^{200t}}\left(\frac{1}{1000}e^{200t} + C\right)$$

$$= \frac{1}{1000} + Ce^{-200t}$$

$$i(t) = -200Ce^{-200t} \qquad\qquad i(t) = \frac{dq}{dt}$$

$$i(0) = 0.4 \quad \to 0.4 = -200C$$

$$\Rightarrow \boxed{C = -\frac{1}{500}}$$

$$\boxed{q(t) = \frac{1}{1000} - \frac{1}{500}e^{-200t}}$$

b) $\lim\limits_{t\to\infty} q(t) = \lim\limits_{t\to\infty}\left(\frac{1}{1000} - \frac{1}{500}e^{-200t}\right)$ $\qquad \lim\limits_{t\to\infty}\left(e^{-200t}\right) = \lim\limits_{t\to\infty}\left(\frac{1}{e^{200t}}\right) = 0$

$$\boxed{= \frac{1}{1000}}$$

Exercise

An electromotive force

$$E(t) = \begin{cases} 120, & 0 \le t \le 20 \\ 0, & t > 20 \end{cases}$$

Is applied to an *LR*-series circuit in which the inductance is 20 *henries* and resistance is 2 *ohms*. Find the current $i(t)$ if $i(0) = 0$

Solution

For $0 \le t \le 20$

$$20\frac{di}{dt} + 2i = 120 \qquad\qquad L\frac{di}{dt} + Ri = E(t)$$

$$\frac{di}{dt} + \frac{1}{10}i = 6$$

$$e^{\int \frac{1}{10}dt} = e^{t/10}$$

$$\int 6e^{t/10}dt = 60e^{t/10}$$

$$i(t) = e^{-t/10}\left(60e^{t/10} + C\right)$$

$$= 60 + Ce^{-t/10}$$

$$i(0)=0 \quad \to 0=60+C$$

$$\Rightarrow \underline{C=-60}$$

$$\boxed{i(t)=60-60e^{-t/10}}$$

For $t>20$

$$20\frac{di}{dt}+2i=0 \qquad\qquad L\frac{di}{dt}+Ri=E(t)$$

$$\frac{di}{dt}=-\frac{1}{10}i$$

$$\int\frac{di}{i}=-\int\frac{1}{10}dt$$

$$\ln i=-\frac{1}{10}t+C$$

$$i(t)=C_1e^{-t/10}$$

$$i(t=20)=60-60e^{-2}$$

$$i(20)=60-60e^{-2}=Ce^{-2}$$

$$\Rightarrow \boxed{C=60\left(e^2-1\right)}$$

$$\boxed{i(t)=60\left(e^2-1\right)e^{-t/10}}$$

$$i(t)=\begin{cases}60-60e^{-t/10} & 0\le t\le 20\\ 60\left(e^2-1\right)e^{-t/10} & t>20\end{cases}$$

Exercise

Suppose an *RC*-series circuit has a variable resistor. If the resistance at time t is given by $R=k_1+k_2t,$ where k_1 and k_2 are known positive constants, then

$$\left(k_1+k_2t\right)\frac{dq}{dt}+\frac{1}{C}q=E(t)$$

If $E(t)=E_0$ and $q(0)=q_0$, where E_0 and q_0 are constants, show that

$$q(t)=E_0C+\left(q_0-E_0C\right)\left(\frac{k_1}{k_1+k_2t}\right)^{1/Ck_2}$$

Solution

$$\left(k_1+k_2t\right)\frac{dq}{dt}=E_0-\frac{1}{C}q$$

$$\int \frac{dq}{E_0 - \frac{1}{C}q} = \int \frac{dt}{k_1 + k_2 t}$$

$$-C\ln\left(E_0 - \frac{1}{C}q\right) = \frac{1}{k_2}\ln\left(k_1 + k_2 t\right) + \ln A$$

$$\ln\left(E_0 - \frac{1}{C}q\right)^{-C} = \ln\left(k_1 + k_2 t\right)^{1/k_2} + \ln A$$

$$\left(\frac{CE_0 - q}{C}\right)^{-C} = A\left(k_1 + k_2 t\right)^{1/k_2}$$

$$q(0) = q_0 \quad \rightarrow \quad \left(\frac{C}{CE_0 - q_0}\right)^{C} = A\left(k_1\right)^{1/k_2}$$

$$A = \left(\frac{C}{CE_0 - q_0}\right)^{C} \left(k_1\right)^{-1/k_2}$$

$$\left(\frac{C}{CE_0 - q}\right)^{C} = \left(\frac{C}{CE_0 - q_0}\right)^{C} \left(\frac{1}{k_1}\right)^{1/k_2} \left(k_1 + k_2 t\right)^{1/k_2}$$

$$\left(\frac{1}{CE_0 - q}\right)^{C} = \left(\frac{1}{CE_0 - q_0}\right)^{C} \left(\frac{k_1 + k_2 t}{k_1}\right)^{1/k_2}$$

$$\frac{1}{CE_0 - q} = \frac{1}{CE_0 - q_0}\left(\frac{k_1 + k_2 t}{k_1}\right)^{1/Ck_2}$$

$$CE_0 - q = \left(CE_0 - q_0\right)\left(\frac{k_1}{k_1 + k_2 t}\right)^{1/Ck_2}$$

$$q(t) = E_0 C - \left(CE_0 - q_0\right)\left(\frac{k_1}{k_1 + k_2 t}\right)^{1/Ck_2}$$

$$\boxed{q(t) = E_0 C + \left(q_0 - E_0 C\right)\left(\frac{k_1}{k_1 + k_2 t}\right)^{1/Ck_2}}$$

Exercise

A heart pacemaker, consists of a switch, a battery, a capacitor, and the heart as a resistor.

When the switch S is at P, the capacitor charges; when S is at Q, the capacitor discharges, sending an electrical stimulus to the heart. The electrical stimulus is being applied to the heart, the voltage E across the heart satisfies the linear DE.

$$\frac{dE}{dt} = -\frac{1}{RC}E$$

a) Let assume that over the time interval of length t_1, $0 < t < t_1$, the switch *S* is at position *P* and the capacitor is being charges. When the switch is moved to position *Q* at time t_1 the capacitor discharges, sending an impulse to the heart over the time interval of length t_2 : $t_1 \le t < t_1 + t_2$. Thus over the initial charging/discharging interval $0 < t < t_1 + t_2$ the voltage to the heart is actually modeled by the piecewise-defined differential equation

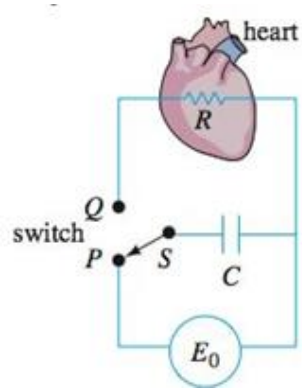
$$\frac{dE}{dt} = \begin{cases} 0, & 0 < t < t_1 \\ -\frac{1}{RC}E, & t_1 \le t < t_1 + t_2 \end{cases}$$

By moving *S* between *P* and *Q*, the charging and discharging over time intervals of lengths t_1 and t_2 is repeated indefinitely. Suppose $t_1 = 4\ s$, $t_2 = 2\ s$. $E_0 = 12\ V$, and $E(0) = 0$, $E(4) = 12$, $E(6) = 0$, $E(10) = 12$, $E(12) = 0$, and so on.

Solve for $E(t)$ for $0 \le t \le 24$

b) Suppose for the sake of illustration that $R = C = 1$. Graph the solution in part (*a*) for $0 \le t \le 24$

Solution



a) $\frac{dE}{dt} = -\frac{1}{RC}E \quad \to \quad \frac{dE}{E} = -\frac{1}{RC}dt$

$$\int \frac{dE}{E} = -\int \frac{1}{RC}dt$$

$$\ln E = -\frac{1}{RC}t + C$$

$$E(t) = e^{-\frac{1}{RC}t + A_1} = Ae^{-\frac{1}{RC}t}$$

For $0 \le t < 4$, $6 \le t < 10$, and $12 \le t < 16$

$\to \boxed{E(t) = 0}$

For $E(4) = E(10) = E(16) = 12$

$$Ae^{-\frac{1}{RC}t} = 12 \Rightarrow \boxed{A = 12e^{\frac{1}{RC}t}}$$

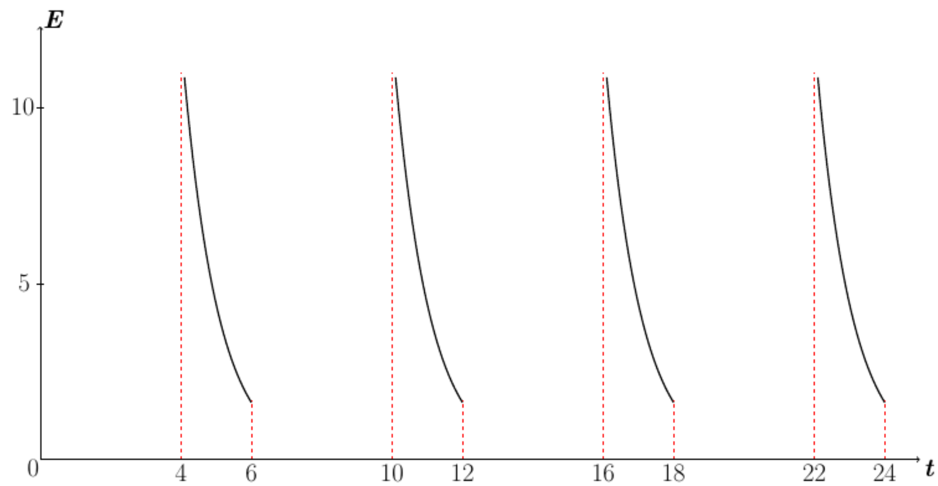
$E(4) \to A = 12e^{4/RC}$

$E(10) \to A = 12e^{10/RC}$

$E(16) \to A = 12e^{16/RC}$

$$E(t)=\begin{cases} 0 & 0 \le t < 4,\quad 6 \le t < 10,\quad 12 \le t < 16 \\ 12e^{\frac{4-t}{RC}} & 4 \le t < 6 \\ 12e^{\frac{10-t}{RC}} & 10 \le t < 12 \\ 12e^{\frac{16-t}{RC}} & 16 \le t < 18 \\ 12e^{\frac{22-t}{RC}} & 22 \le t < 24 \end{cases}$$

b)



Solution   Section 1.9 - Existence and Uniqueness of Solutions

Exercise

Which of the initial value problems are guaranteed a unique solution. $y' = 4 + y^2, \quad y(0) = 1$

Solution

$f(t, y) = 4 + y^2 \to f$ is continuous

$\dfrac{\partial f}{\partial y} = 2y$ is also continuous on the whole plane.

Hence the hypotheses are satisfied and guarantee a unique solution.

Exercise

Which of the initial value problems are guaranteed a unique solution? $y' = \sqrt{y}, \quad y(4) = 0$

Solution

$f(t, y) = \sqrt{y} \quad \Rightarrow y \ge 0$

$\dfrac{\partial f}{\partial y} = \dfrac{1}{2\sqrt{y}} \quad \to y > 0 \ \ (only)$

Initial condition: $y(4) = 0 \Rightarrow y_0 = 0$ *and* $t_0 = 4$

Both f *and* $\dfrac{\partial f}{\partial y}$ are not continuous in the rectangle containing (t_0, y_0)

Hence the hypotheses are not satisfied.

Exercise

Which of the initial value problems are guaranteed a unique solution? $y' = t\tan^{-1} y, \quad y(0) = 2$

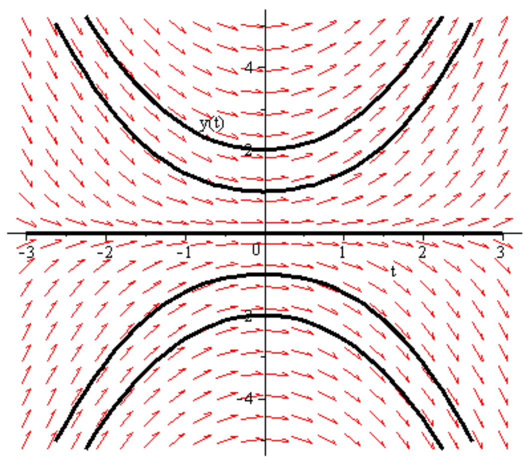
Solution

The right hand side of the equation is $f(t, y) = t\tan^{-1} y$,

which is continuous in the whole plane. $\dfrac{\partial f}{\partial y} = \dfrac{t}{t + y^2}$ is also

continuous in the whole plane.

Hence the hypotheses are satisfied and the theorem guarantees a unique solution.



Exercise

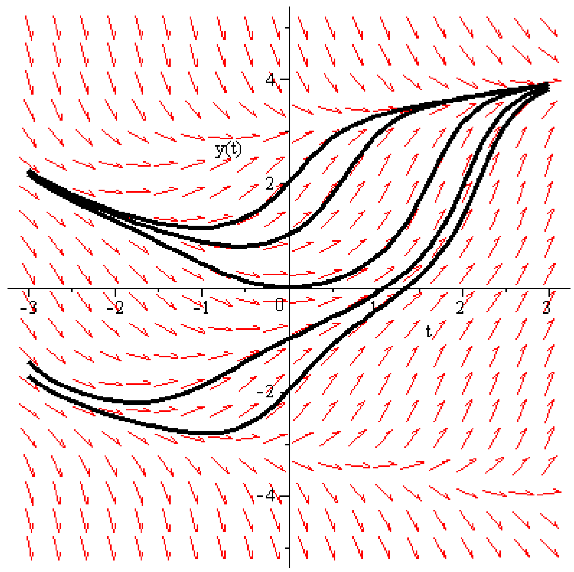
Which of the initial value problems are guaranteed a unique solution? $\omega' = \omega \sin \omega + s, \quad \omega(0) = -1$

Solution

The right hand side of the equation is $f(s, \omega) = \omega \sin \omega + s$, which is continuous in the whole plane.

$\dfrac{\partial f}{\partial \omega} = \sin \omega + \omega \cos \omega$ is also continuous in the whole plane.

Hence the hypotheses are satisfied and the theorem guarantees a unique solution.



Exercise

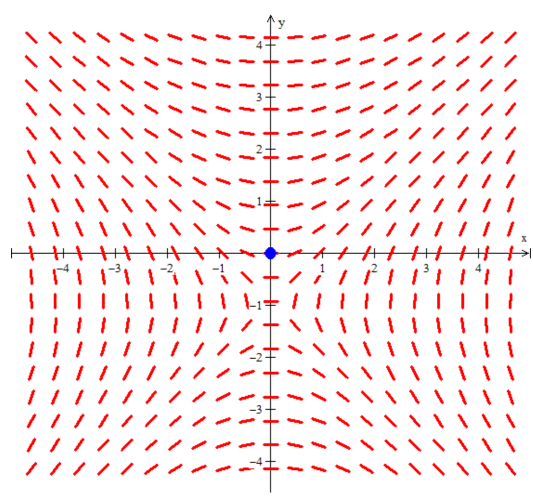
Which of the initial value problems are guaranteed a unique solution? $x' = \dfrac{t}{x+1}, \quad x(0) = 0$

Solution

The right hand side of the equation is $f(t, x) = \dfrac{t}{x+1}$, which is continuous in the whole plane, except where $x = -1$.

$\dfrac{\partial f}{\partial x} = -\dfrac{t}{(x+1)^2}$ is also continuous in the whole plane, except where $x = -1$.

Hence the hypotheses are satisfied in a rectangle containing the initial point (0, 0), so the theorem guarantees a unique solution.



Exercise

Which of the initial value problems are guaranteed a unique solution? $y' = \frac{1}{x}y + 2, \quad y(0) = 1$

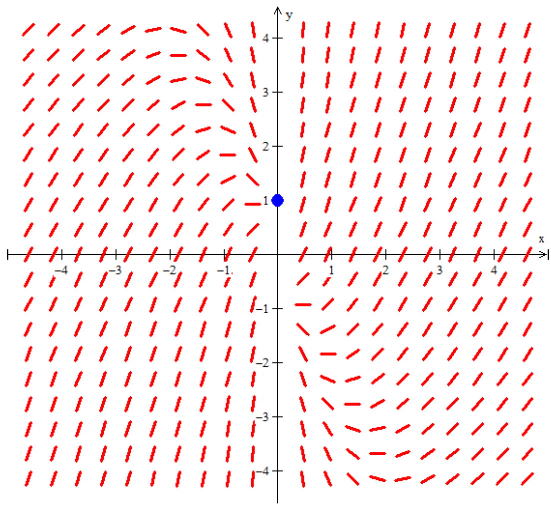
Solution

The right hand side of the equation is $f(x, y) = \frac{1}{x}y + 2$, which is continuous in the whole plane, except where $x = 0$.

Since the initial point is (0, 1), *f* is discontinuous there.

Consequently there is no rectangle containing this point in which *f* is continuous.

The hypotheses are not satisfied, so the theorem doesn't guarantee a unique solution.



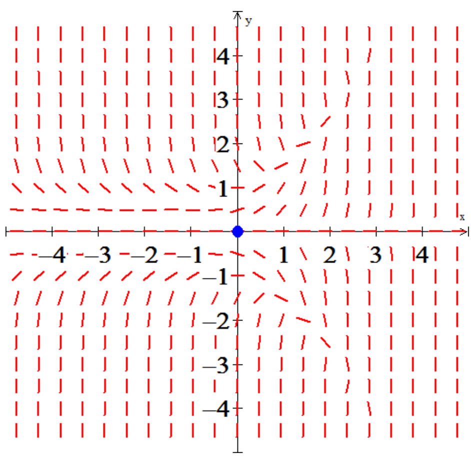
Exercise

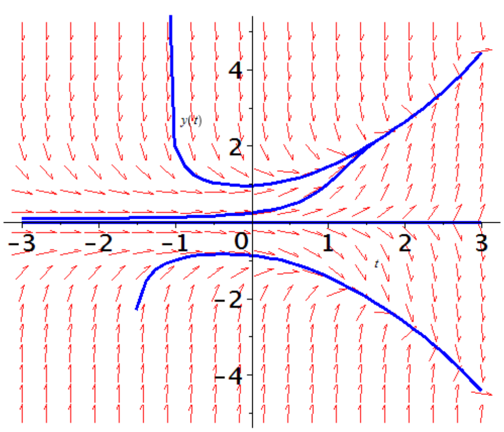
Which of the initial value problems are guaranteed a unique solution? $y' = e^t y - y^3, \quad y(0) = 0$

Solution

The right hand side of the equation is $f(t, y) = e^t y - y^3$, which is continuous in the whole plane.

$\frac{\partial f}{\partial y} = e^t - 3y^2$ is also continuous in the whole plane.





Hence the hypotheses are satisfied in a rectangle containing the initial point (0, 0), so the theorem guarantees a unique solution.

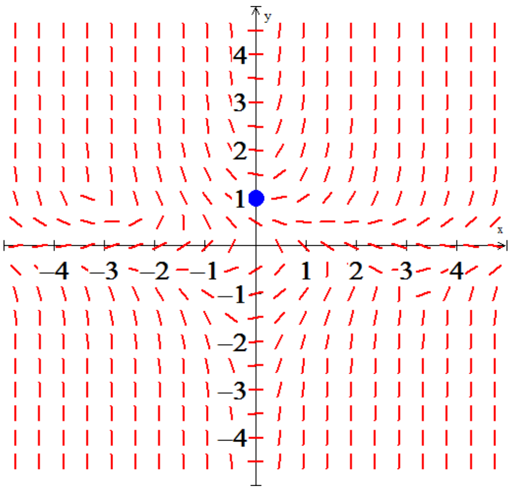
Exercise

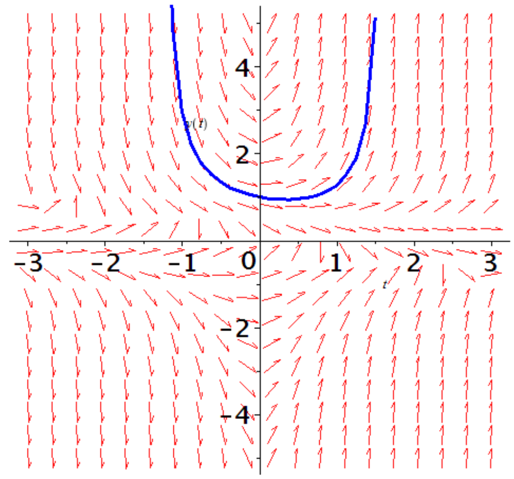
Which of the initial value problems are guaranteed a unique solution? $\quad y' = ty^2 - \dfrac{1}{3y+t}, \quad y(0)=1$

Solution

The right hand side of the equation is $f(t,y) = ty^2 - \dfrac{1}{3y+t}$, which is continuous in the whole plane.

$\dfrac{\partial f}{\partial y} = 2ty + \dfrac{3}{(3y+t)^2}$ is also continuous in the whole plane.





Hence the hypotheses are satisfied in a rectangle containing the initial point $(0,\ 1)$, so the theorem guarantees a unique solution.

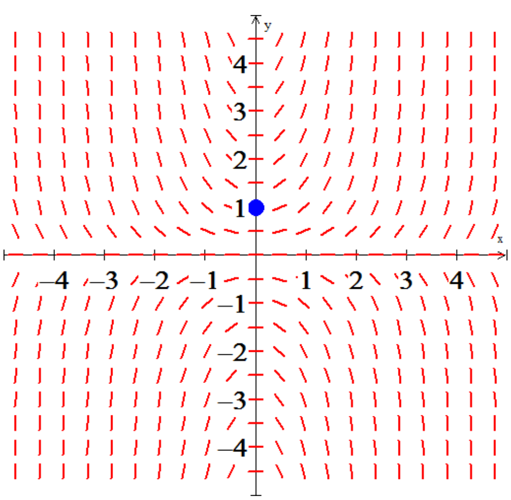
Exercise

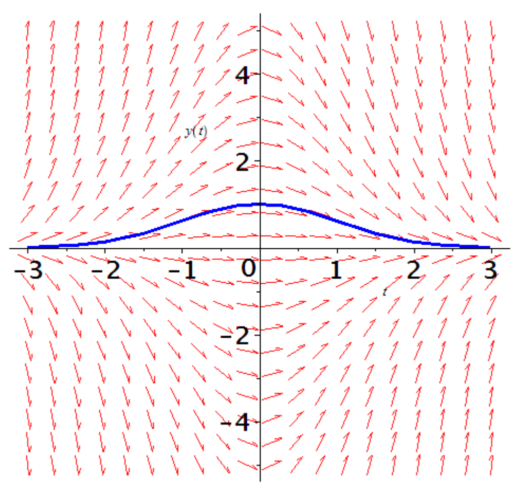
Which of the initial value problems are guaranteed a unique solution? $\quad y' = xy, \quad y(0)=1$

Solution

The right hand side of the equation is $f(x,y) = xy$, which is continuous in the whole plane.

$\dfrac{\partial f}{\partial y} = x$ is also continuous in the whole plane.





Hence the hypotheses are satisfied in a rectangle containing the initial point $(0,\ 1)$, so the theorem guarantees a unique solution.

Exercise

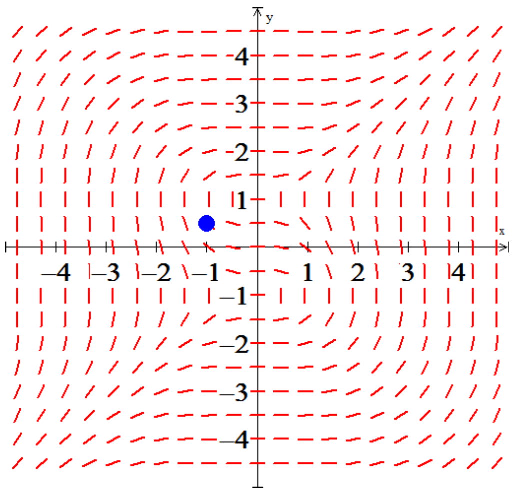
Which of the initial value problems are guaranteed a unique solution? $\quad y' = -\dfrac{t^2}{1-y^2}, \quad y(-1) = \dfrac{1}{2}$

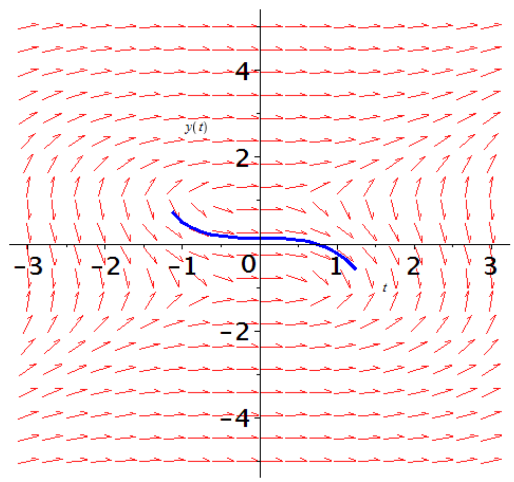
Solution

The right hand side of the equation is $f(t,y) = -\dfrac{t^2}{1-y^2}$, which is continuous in the whole plane, except where $y = \pm 1$.

$\dfrac{\partial f}{\partial y} = -\dfrac{2t^2 y}{\left(1-y^2\right)^2}$ is also continuous in the whole plane, except where $y = \pm 1$.

Since $t = -1 \rightarrow y_0 = \frac{1}{2} \quad (\neq \pm 1)$





Hence the hypotheses are satisfied in a rectangle containing the initial point $\left(-1,\ \frac{1}{2}\right)$, so the theorem guarantees a unique solution.

Exercise

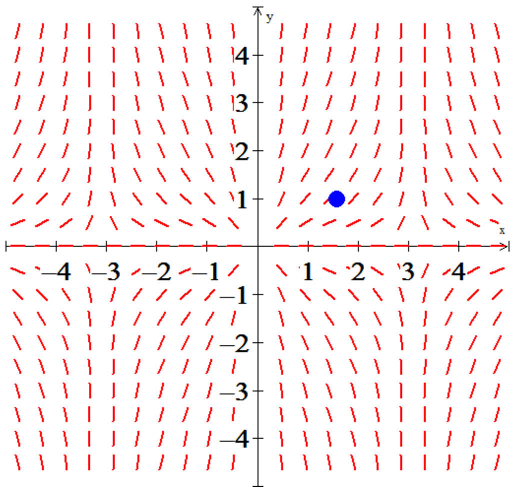
Which of the initial value problems are guaranteed a unique solution? $\quad y' = \dfrac{y}{\sin t}, \quad y\left(\frac{\pi}{2}\right) = 1$

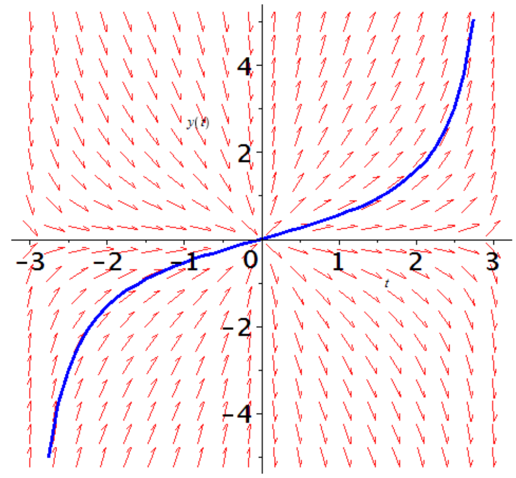
Solution

The right hand side of the equation is $f(t,y) = \dfrac{y}{\sin t}$, which is continuous in the whole plane, except where $t = n\pi$.

$\dfrac{\partial f}{\partial y} = \dfrac{1}{\sin t}$ is also continuous in the whole plane, except where $t = n\pi$

Since $t=\frac{\pi}{2} \rightarrow y_0=1 \ (\neq n\pi)$





Hence the hypotheses are satisfied in a rectangle containing the initial point $\left(\frac{\pi}{2},\ 1\right)$, so the theorem guarantees a unique solution.

Exercise

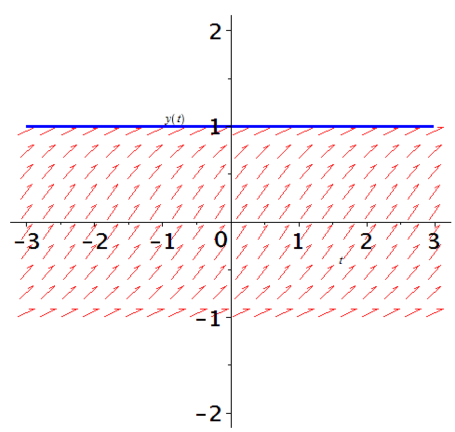
Which of the initial value problems are guaranteed a unique solution? $\quad y'=\sqrt{1-y^2}, \quad y(0)=1$

Solution

The right hand side of the equation is $f(t,y)=\sqrt{1-y^2}$, which is continuous in the whole plane except where $y<-1$ & $y>1$.

$$\frac{\partial f}{\partial y}=-\frac{y}{\sqrt{1-y^2}}\bigg|_{y=1}=\infty \text{ undefined.}$$

So the uniqueness theorem doesn't apply



$$\frac{dy}{dt}=\sqrt{1-y^2}$$

$$\int \frac{dy}{\sqrt{1-y^2}}=\int dt$$

$$\arccos y=t+C$$

$$y(t)=\cos(t+C)$$

$$y(0)=1 \quad \rightarrow 1=\cos C \quad \Rightarrow C=0(=2n\pi)$$

$$y(t)=\cos t \quad (2n-1)\pi\leq t\leq 2\pi$$

Exercise

Show that $y(t)=0$ and $y(t)=t^3$ are both solutions of the initial value problem $y'=3y^{2/3}$, where $y(0)=0$. Explain why this fact doesn't contradict Theorem

Solution

$$f(t,y)=3y^{2/3}$$

$f'=2y^{-1/3}$ which is not continuous at $y=0$

Exercise

Use a numerical solver to sketch the solution of the given initial value problem

$$\frac{dy}{dt}=\frac{t}{y+1}, \qquad y(2)=0$$

a) Where does your solver experience difficulty? why? Use the image of your solution to estimate the interval of existence.

b) Find an explicit solution; then use your formula to determine the interval of existence. How does it compare with the approximation found in part (a).

Solution

a) $(y+1)dy=tdt$

$$\int (y+1)dy=\int tdt$$

$$\tfrac{1}{2}y^2+y=\tfrac{1}{2}t^2+C$$

$$y^2+2y=t^2+C$$

$$(0)^2+2(0)=2^2+C$$

$$0=4+C$$

$$C=-4$$

$$y^2+2y=t^2-4$$

$$y^2+2y-t^2+4=0$$

Solve for y:

$$y=\frac{-2\pm\sqrt{4t^2-12}}{2}$$

$$=\frac{-2\pm2\sqrt{t^2-3}}{2}$$

$$\underline{=-1\pm\sqrt{t^2-3}}$$

b) The only solution is: $y=-1+\sqrt{t^2-3}$ and $t^2-3>0 \Rightarrow t>\sqrt{3}$

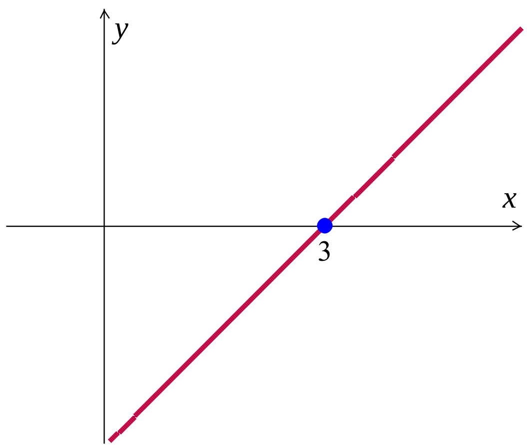
The interval of the solution $\left(\sqrt{3},\ \infty\right)$

Solution   Section 1.10 - Autonomous Equations and Stability

Exercise

The graph of the right-hand side $y' = f(y)$ is shown. Identify the equilibrium points and sketch the equilibrium solutions in the *ty*-plane. Classify each equilibrium point as either unstable or asymptotically stable.

Solution

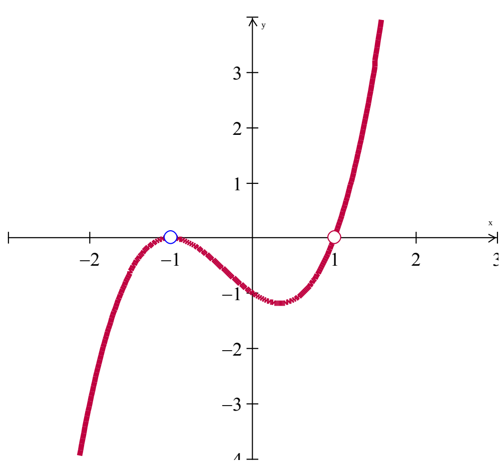


The equilibrium point is: 3 and is stable

Exercise

The graph of the right-hand side $y' = f(y)$ is shown. Identify the equilibrium points and sketch the equilibrium solutions in the *ty*-plane. Classify each equilibrium point as either unstable or asymptotically stable.

Solution

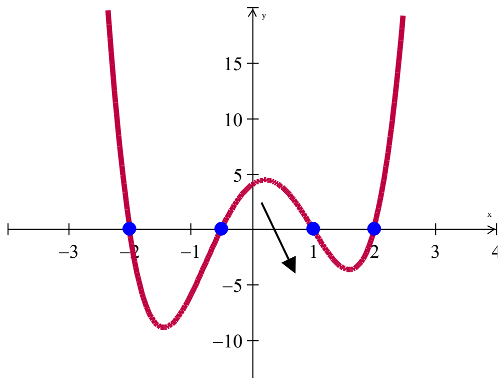


The equilibrium points are: $-1,\ 1$ and both are unstable

Exercise

The graph of the right-hand side $y' = f(y)$ is shown. Identify the equilibrium points and sketch the equilibrium solutions in the *ty*-plane. Classify each equilibrium point as either unstable or asymptotically stable.

Solution



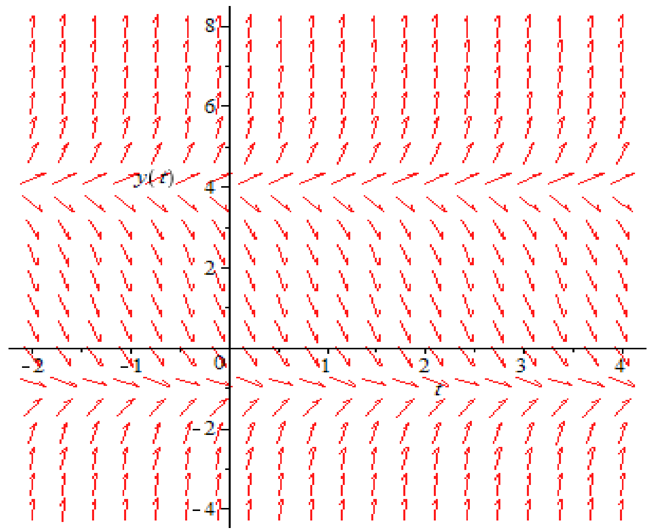
The equilibrium points are: $-2,\ -\frac{1}{2},\ 1,\ 2$

$-2,\ 1$ are asymptotically stable

$-\frac{1}{2},\ 2$ are unstable

Exercise

Impose the equilibrium solution(s), classifying each as either unstable or asymptotically stable



Solution

Because the $y' = f(y)$ is autonomous, the slope at any point $(t,\ x)$ in the direction field does not depend on *t*, only on *y*.

There are two equilibrium points. The smaller of them is unstable and the other is asymptotically stable.

Exercise

An autonomous differential equation is given by $y' = 2 - y$

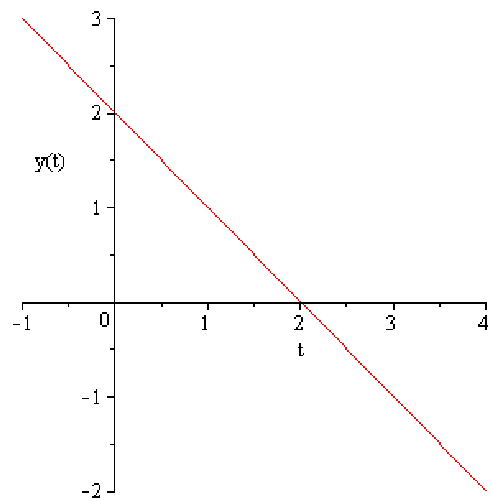
a) Sketch a graph of $f(y)$

b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

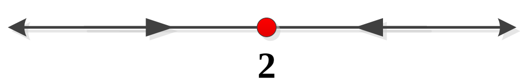
c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

Solution

a) $f(y) = 2 - y$



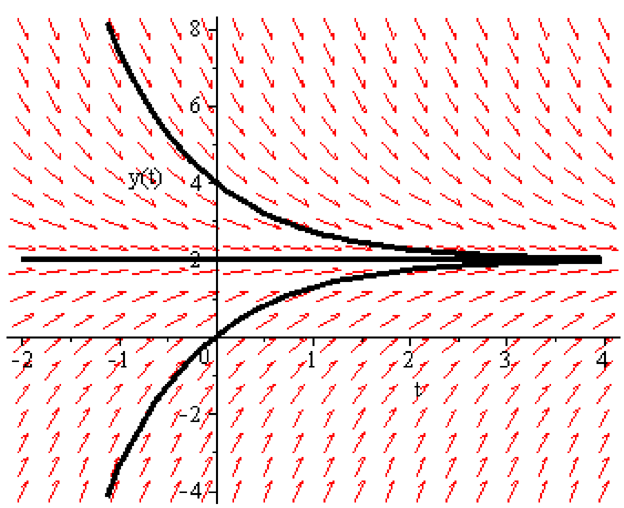
b) The phase line for the autonomous equation is



$y = 2$ is asymptotically stable

c) The phase line indicates that the solutions increase if $y < 2$ and decrease if $y > 2$.

The stable equilibrium solution is $y = 2$



Exercise

An autonomous differential equation is given by $y' = (y+1)(y-4)$

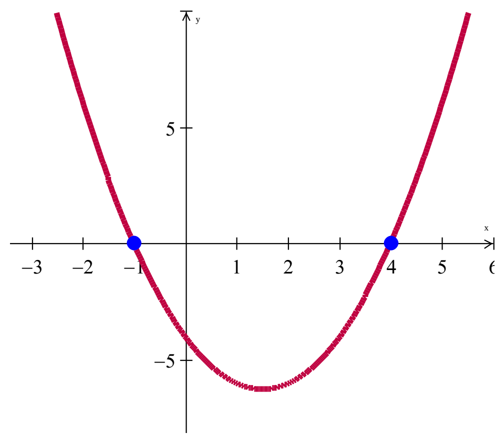
a) Sketch a graph of $f(y)$

b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

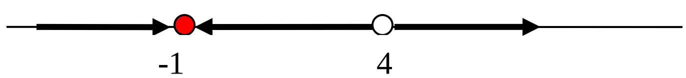
c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

Solution

a) $f(y) = (y+1)(y-4)$

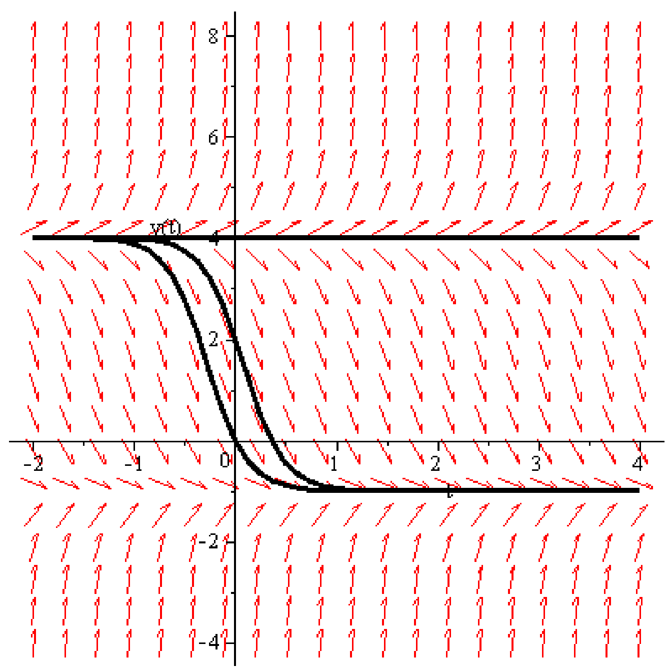


b) The phase line for the autonomous equation is



$y = -1$ is asymptotically stable and $y = 4$ is unstable

c)



Exercise

An autonomous differential equation is given by $y' = 9y - y^3$

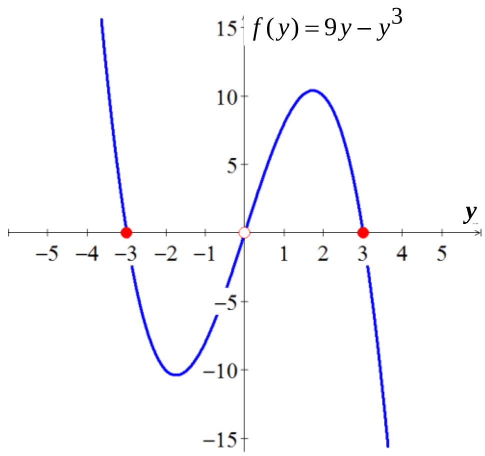
a) Sketch a graph of $f(y)$

b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

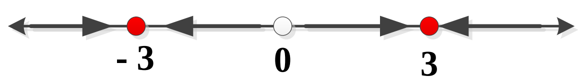
c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

Solution

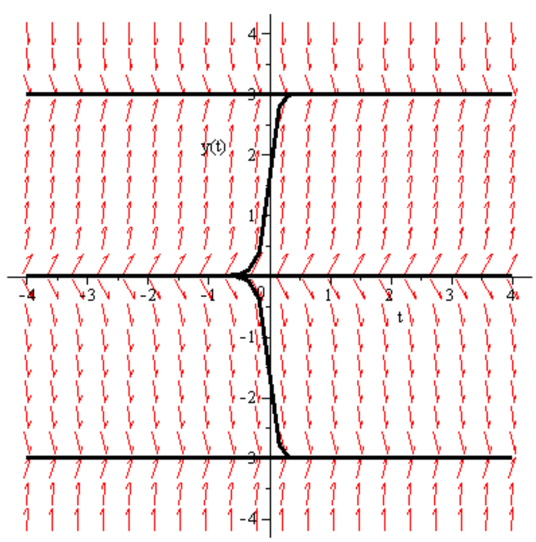
a) $f(y) = 9y - y^3 = y\left(9 - y^2\right)$



b) The phase line for the autonomous equation is



c) The solutions increase if $y < -3$, decrease for $-3 < y < 0$, increase if $0 < y < 3$, and decrease for $y >$ 3.



The stable equilibrium solutions at $y(t) = -3$, $y(t) = 3$ and unstable equilibrium solutions at $y(t) = 0$

Exercise

An autonomous differential equation is given by $y' = \sin y$

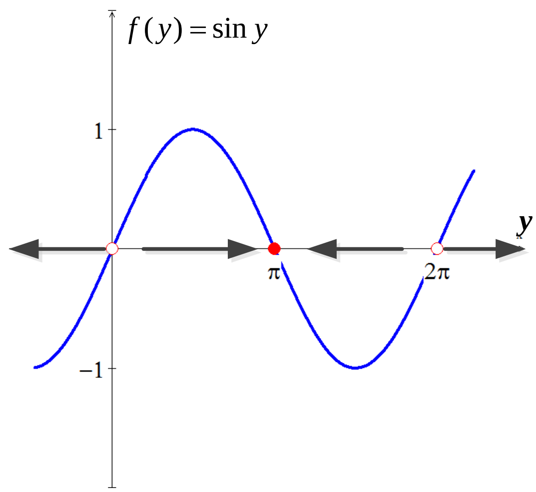
a) Sketch a graph of $f(y)$

b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

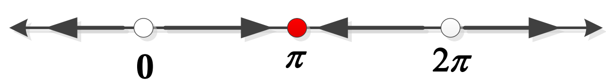
c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

Solution

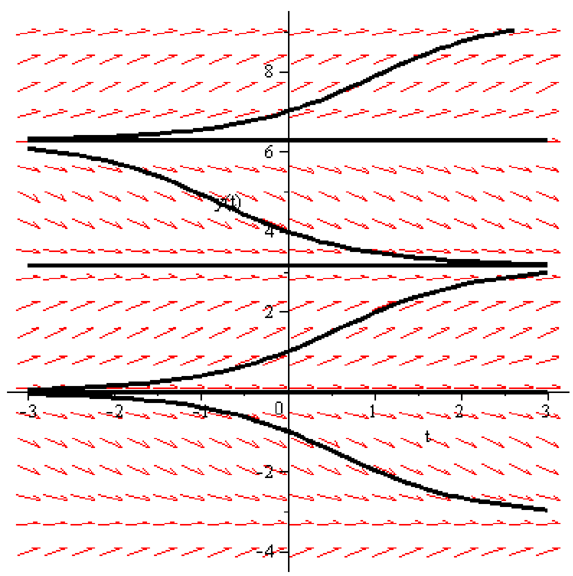
a) $f(y) = \sin y$



b) The phase line for the autonomous equation is



c) The solutions decrease if $-\pi < y < 0$, increase for $0 < y < \pi$, increase if $\pi < y < 2\pi$



The stable equilibrium solutions at $y(t) = \pi$ and unstable equilibrium solutions at $y(t) = 0,\ 2\pi$

Exercise

An autonomous differential equation is given by $y' = y^2 - 3y$

a) Sketch a graph of $f(y)$

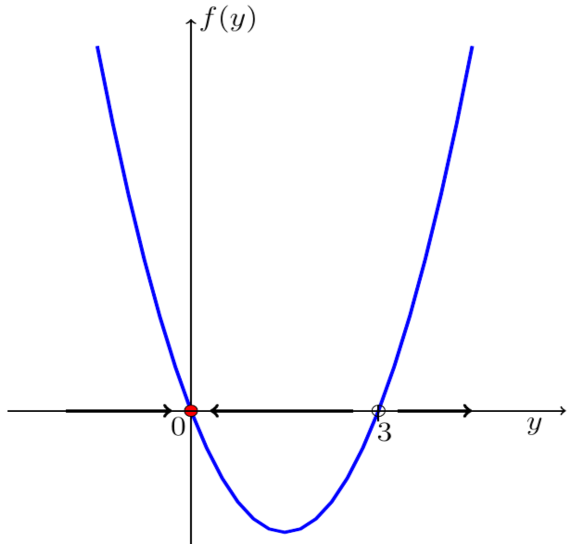
b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

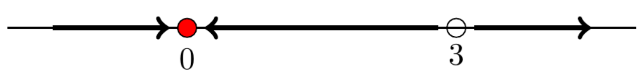
Solution

a) $f(y) = y^2 - 3y = 0 \quad \rightarrow \quad \underline{y = 0,\ 3|}$

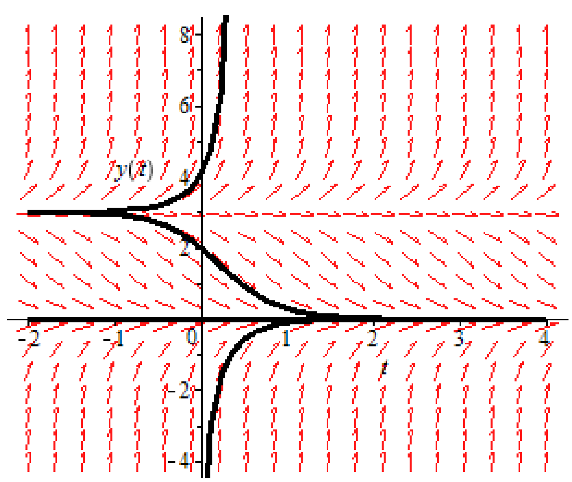
The critical points are 0 and 3.



b) The phase line for the autonomous equation is



c) The solutions increase if $-\infty < y < 0$ and $0 < y < \infty$, decrease $0 < y < 3$



The asymptotically stable equilibrium solution at $y = 0$ (attractor) and unstable equilibrium solutions at $y = 3$ (repeller).

Exercise

An autonomous differential equation is given by $y' = y^2 - y^3$

a) Sketch a graph of $f(y)$

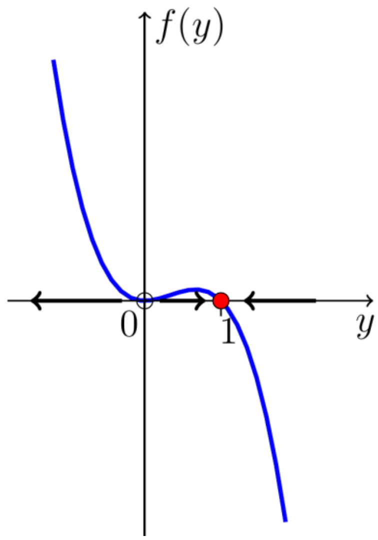
b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

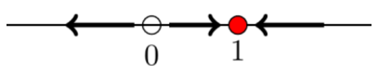
Solution

a) $f(y) = y^2 - y^3 = 0 \quad \to \quad \underline{y = 0,\ 1}$

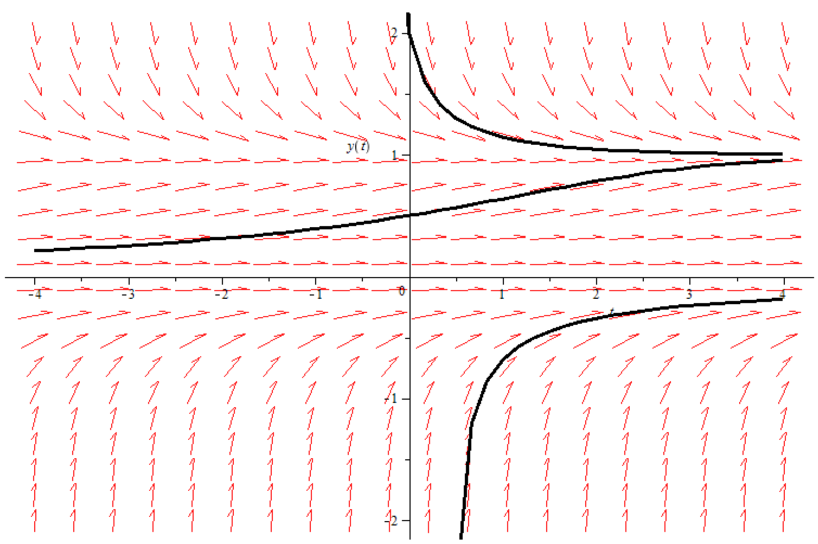
The critical points are 0 and 1.



b) The phase line for the autonomous equation is



c) The solutions increase if $0 < y < 1$ and $1 < y < \infty$, decrease $-\infty < y < 0$



The asymptotically stable at $y = 1$ (attractor) and semi-stable at $y = 0$.

Exercise

An autonomous differential equation is given by $y' = (y-2)^4$

a) Sketch a graph of $f(y)$

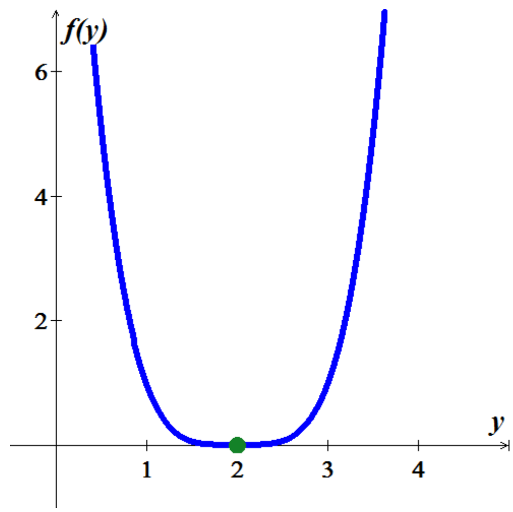
b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

Solution

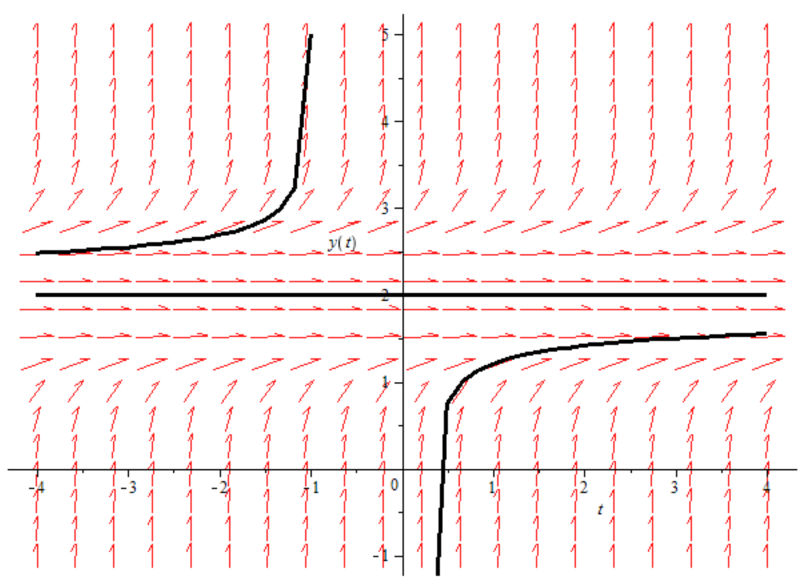
a) $f(y) = (y-2)^4 = 0 \quad \rightarrow \quad \underline{y = 2|}$

The critical point is 2.



b) The phase line for the autonomous equation is

c) The solutions increase if $-\infty < y < 2$ and $2 < y < \infty$



The semi-stable at $y = 2$.

Exercise

An autonomous differential equation is given by $y' = 10 + 3y - y^2$

a) Sketch a graph of $f(y)$

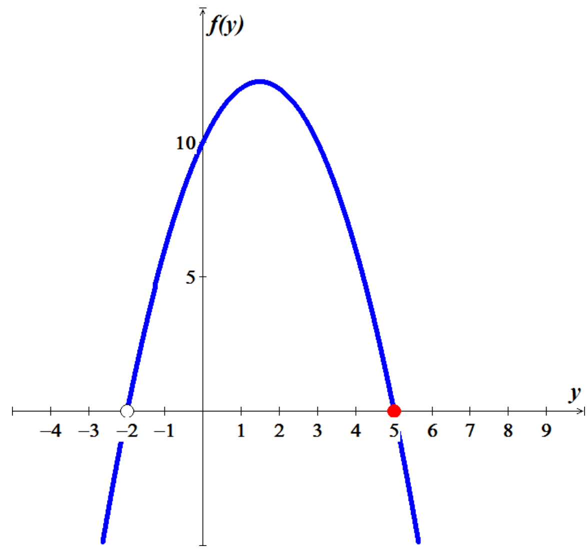
b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

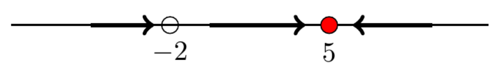
Solution

a) $f(y) = 10 + 3y - y^2 = 0 \quad \to \quad \underline{y = -2,\ 5}$

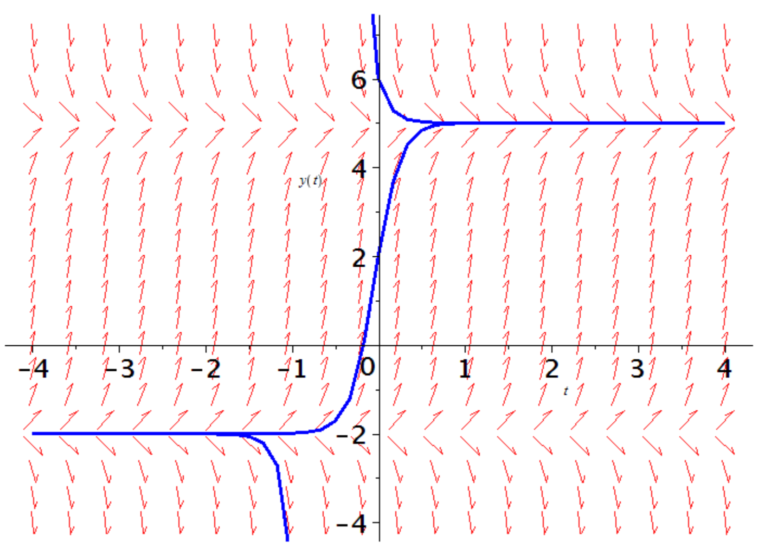
The critical points are -2 and 5.



b) The phase line for the autonomous equation is



c) The solutions increase if $-2 < y < 5$, decrease $-\infty < y < -2$ and $5 < y < \infty$



The asymptotically stable at $y = 5$ (attractor) and unstable at $y = -2$ (repeller).

Exercise

An autonomous differential equation is given by $\frac{dy}{dt} = y^2\left(4 - y^2\right)$

a) Sketch a graph of $f(y)$

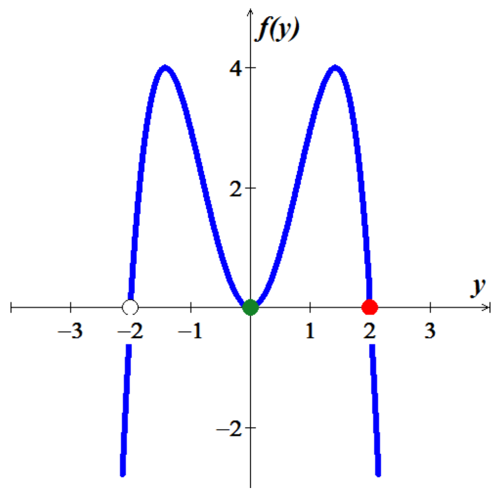
b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

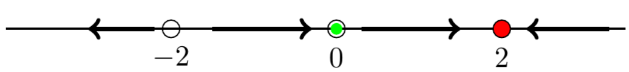
Solution

a) $f(y) = y^2\left(4 - y^2\right) = 0 \quad \to \quad \underline{y = \pm 2,\ 0}$

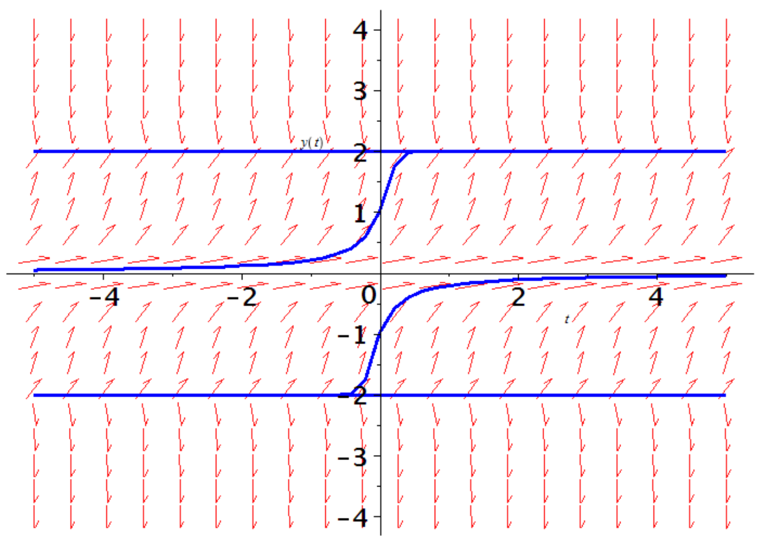
The critical points are ± 2 and 0.



b) The phase line for the autonomous equation is



c) The solutions increase if $-2 < y < 2$, decrease $-\infty < y < -2$ and $2 < y < \infty$



The asymptotically stable at $y = 2$ (attractor), semi-stable at $y = 0$, and unstable at $y = -2$ (repeller).

Exercise

An autonomous differential equation is given by $\dfrac{dy}{dt} = y(2-y)(4-y)$

a) Sketch a graph of $f(y)$

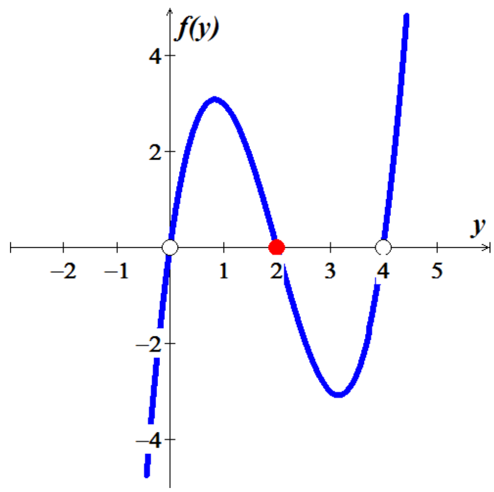
b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

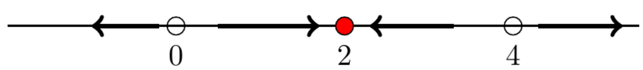
Solution

a) $f(y) = y(2-y)(4-y) = 0 \quad \to \quad \underline{y = 0,\ 2,\ 4}$

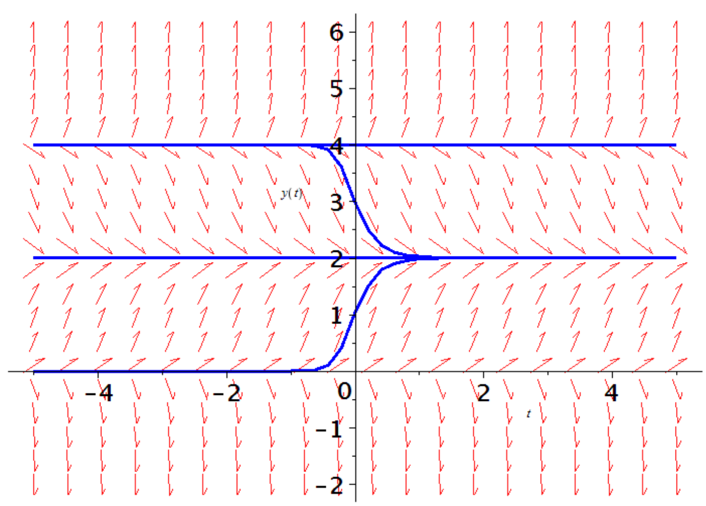
The critical points are $0,\ 2,\ 4$.



b) The phase line for the autonomous equation is



c) The solutions increase if $0 < y < 2$ and $4 < y < \infty$, decrease $-\infty < y < 0$ and $2 < y < 4$



The asymptotically stable at $y = 2$ (attractor) and unstable at $y = 0,\ 4$ (repellers).

Exercise

An autonomous differential equation is given by $\dfrac{dy}{dt} = y\ln(y+2)$

a) Sketch a graph of $f(y)$

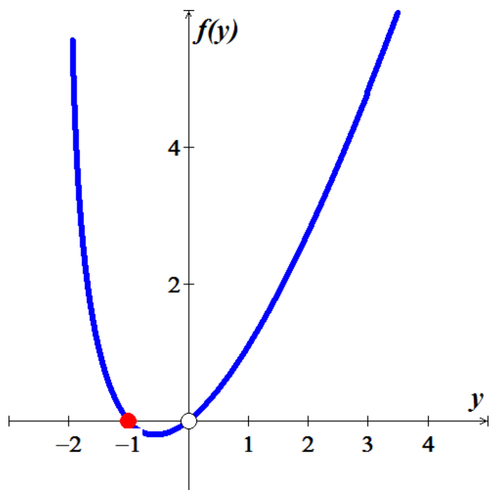
b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

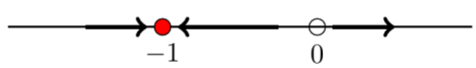
Solution

a) $f(y) = y\ln(y+2) = 0 \quad \to \quad \underline{y = 0, \ -1}$

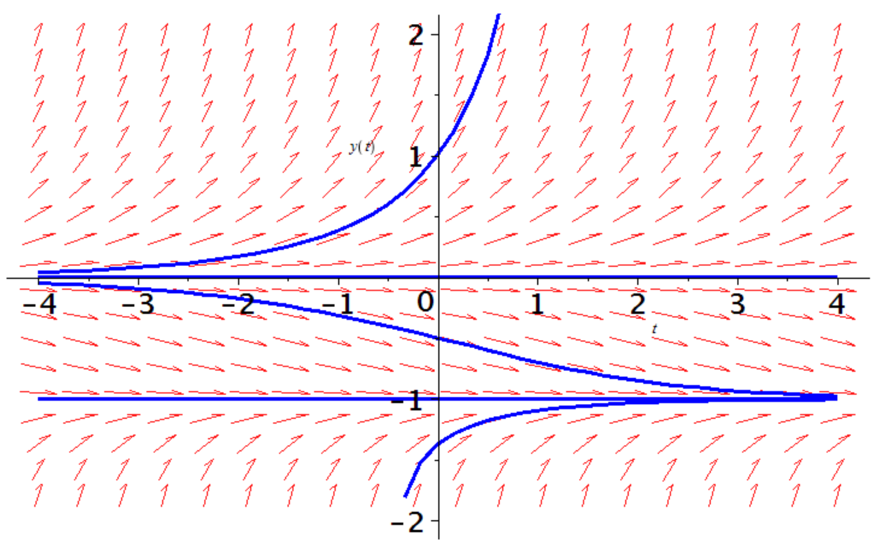
The critical points are $0, \ -1$.



b) The phase line for the autonomous equation is



c) The solutions increase if $-\infty < y < -1$ and $0 < y < \infty$, decrease $-1 < y < 0$



The asymptotically stable at $y = -1$ (attractor) and unstable at $y = 0$ (repeller).

Exercise

An autonomous differential equation is given by $\dfrac{dy}{dt} = \dfrac{ye^y - 9y}{e^y}$

a) Sketch a graph of $f(y)$

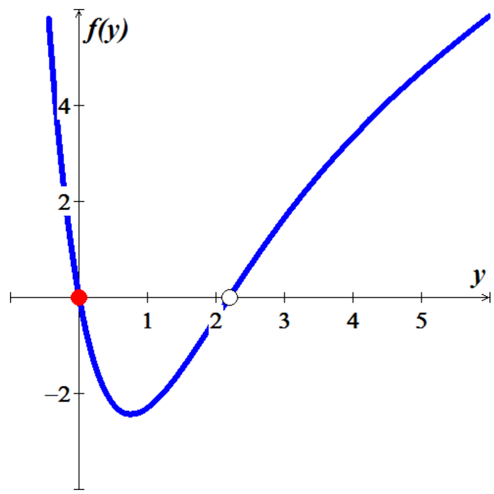
b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

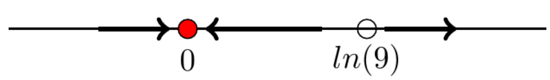
Solution

a) $f(y) = \dfrac{y\left(e^y - 9\right)}{e^y} = 0 \quad \to \quad \underline{y = 0,\ \ln 9}$

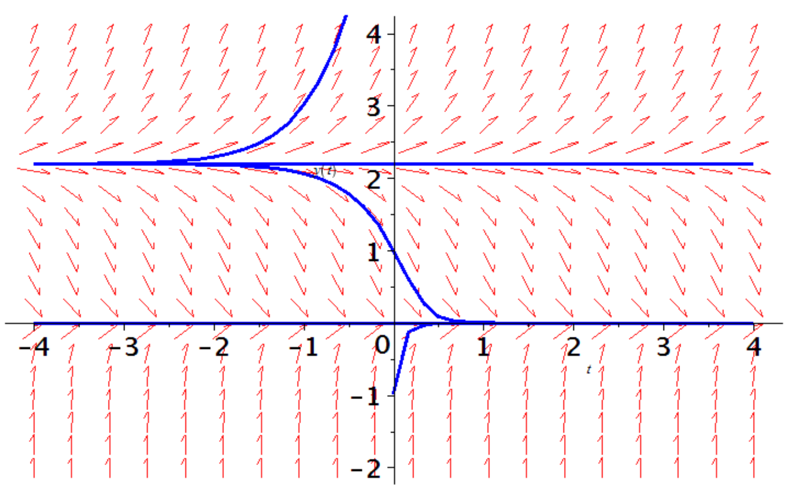
The critical points are $0,\ \ln 9$.



b) The phase line for the autonomous equation is



c) The solutions increase if $-\infty < y < -1$ and $0 < y < \infty$, decrease $-1 < y < 0$



The asymptotically stable at $y = 0$ (attractor) and unstable at $y = \ln 9$ (repeller).

Exercise

An autonomous differential equation is given by $y' = \dfrac{2}{\pi} y - \sin y$

a) Sketch a graph of $f(y)$

b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

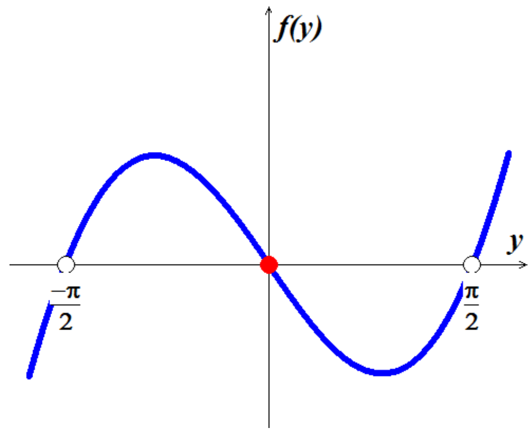
c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

Solution

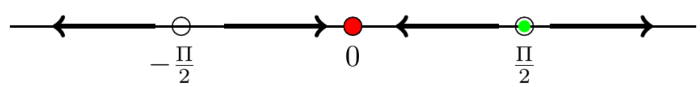
a) $f(y) = \dfrac{2}{\pi} y - \sin y = 0$

$$\frac{2}{\pi} y = \sin y \quad \to \quad y = 0, \ \pm\frac{\pi}{2}$$

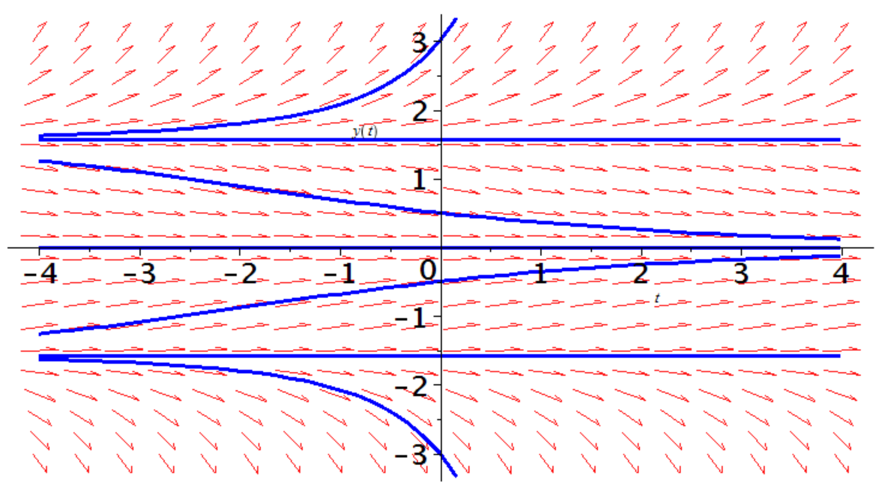
The critical points are $0, \ \pm\frac{\pi}{2}$.



b) The phase line for the autonomous equation is



c) The solutions increase if $-\frac{\pi}{2} < y < 0$ and $y > \frac{\pi}{2}$, decrease $\frac{\pi}{2} < y < 0$ and $y < -\frac{\pi}{2}$



The asymptotically stable at $y = 0$ (attractor) and unstable at $y = \pm\frac{\pi}{2}$ (repeller).

Exercise

An autonomous differential equation is given by $y' = 3y - ye^{y^2}$

a) Sketch a graph of $f(y)$

b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

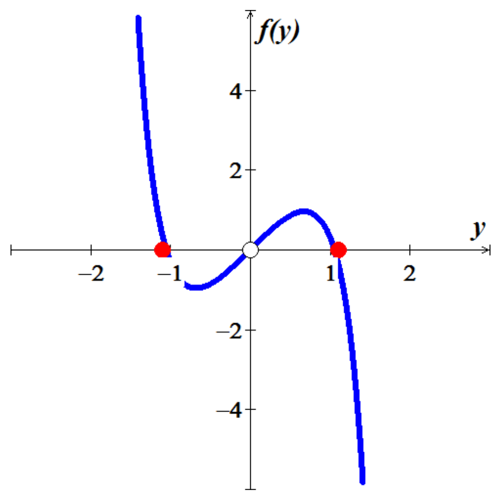
Solution

a) $f(y) = 3y - ye^{y^2} = 0$

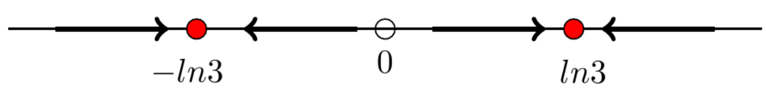
$y\left(3 - e^{y^2}\right) = 0 \quad \to \quad y = 0, \ e^{y^2} = 3$

$e^{y^2} = 3 \quad \to \quad \underline{y = \pm\sqrt{\ln 3}, \ 0|}$

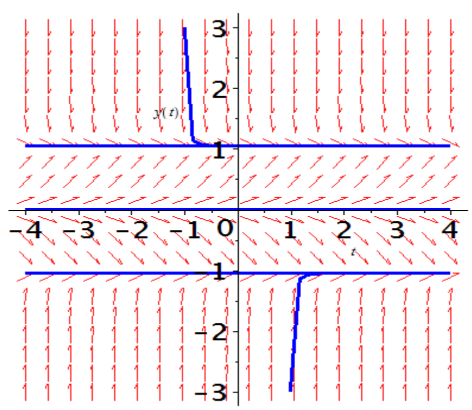
The critical points are $\pm\sqrt{\ln 3}, \ 0$.



b) The phase line for the autonomous equation is



c) The solutions increase if $y < -\ln 3$ and $0 < y < \ln 3$, decrease $-\ln 3 < y < 0$ and $y > \ln 3$



The asymptotically stable at $y = \pm\sqrt{\ln 3}$ (attractor) and unstable at $y = 0$ (repeller).

Exercise

An autonomous differential equation is given by $y' = (1-y)(y+1)^2$

a) Sketch a graph of $f(y)$

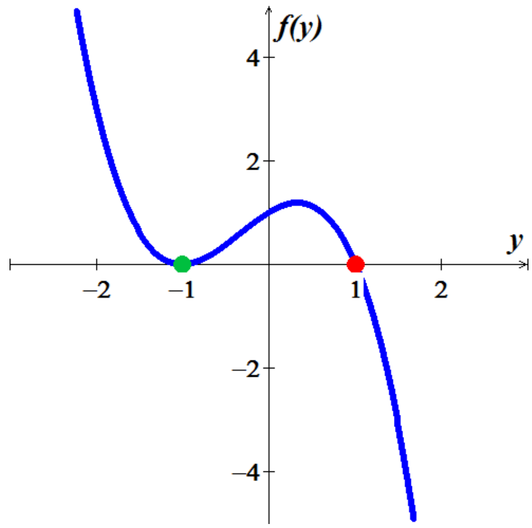
b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

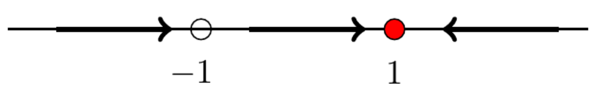
Solution

a) $f(y) = (1-y)(y+1)^2 = 0 \rightarrow \underline{y = -1,\ -1,\ 1}$

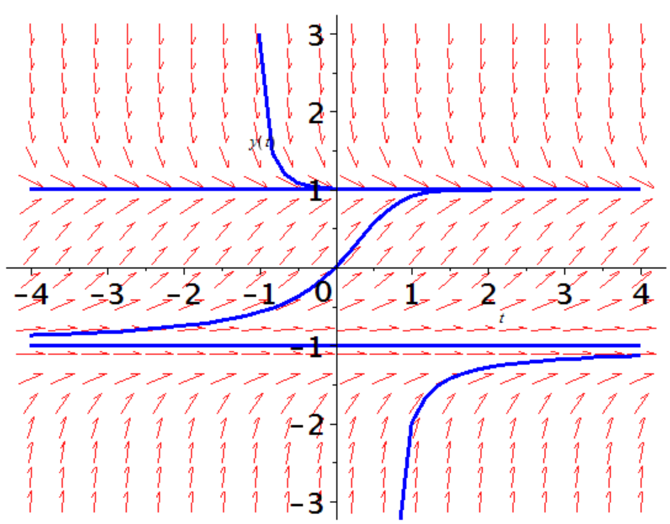
The critical points are ± 1.



b) The phase line for the autonomous equation is



c) The solutions increase if $y < -1$ and $-1 < y < 0$, decrease $y > 1$



The asymptotically stable at $y = 1$ (attractor) and semi-stable at $y = -1$.

Exercise

An autonomous differential equation is given by $y' = \sin\frac{y}{2}$

a) Sketch a graph of $f(y)$

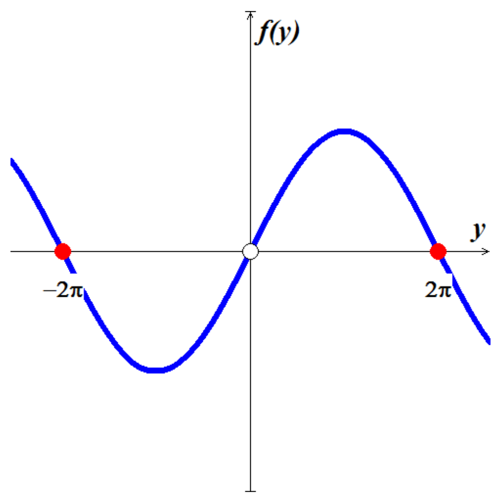
b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

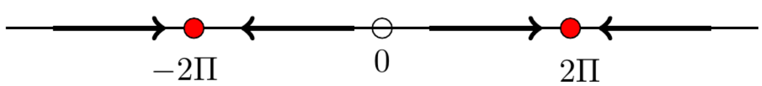
Solution

a) $f(y) = \sin\frac{y}{2} = 0 \rightarrow \underline{y = 2n\pi}$

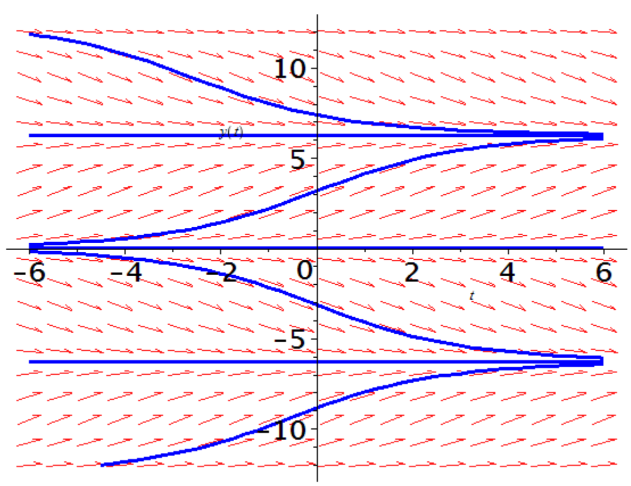
The critical points are $0, \ \pm 2\pi$.



b) The phase line for the autonomous equation is



c) The solutions increase if $0 < y < 2\pi$ and $-3\pi < y < -2\pi$, decrease $-2\pi < y < 0$ and $2\pi < y < 3\pi$



The asymptotically stable at $y = \pm 2\pi$ (attractor) and unstable at $y = 0$ (repeller).

Exercise

Determine the stability of the equilibrium solutions $x' = 4 - x^2$

Solution

$f(x) = x' = 4 - x^2 = 0$

$\Rightarrow x^2 = 4$

The equilibrium points $x = \pm 2$

$f'(x) = -2x$

$f'(-2) = -2(-2) > 0$ $\quad$ $x = -2$ is unstable

$f'(2) = -2(2) < 0$ $\quad$ $x = 2$ is asymptotically stable

Exercise

Determine the stability of the equilibrium solutions $x' = x(x-1)(x+2)$

Solution

The equation $f(x) = x(x-1)(x+2)$.

$f(x) = 0 \quad \Rightarrow$ The equilibrium points are $x = 0,\ 1,\ -2$.

$f(x) = x\left(x^2 + x - 2\right) = x^3 + x^2 - 2x$

$f'(x) = 3x^2 + 2x - 2$

$f'(0) = -2 < 0 \quad \Rightarrow \quad x = 0$ ***Asymptotically stable***

$f'(1) = 3 > 0 \quad \Rightarrow \quad x = 1$ ***Unstable***

$f'(-2) = 2 > 0 \quad \Rightarrow \quad x = -2$ ***Unstable***

Exercise

A tank contains 100 *gal* of pure water. A salt solution with concentration 3 *lb/gal* enters the tank at a rate of 2 *gal/min*. Solution drains from the tank at a rate of 2 *gal/min*. Use the qualitative analysis to find the eventual concentration of the salt in the tank.

Solution

Let $x(t)$ represents the amount of salt.

$Rate\ in = 2\frac{gal}{min} \times 3\frac{lb}{gal} = 6\frac{lb}{\min}$

$Rate\ out = 2\frac{gal}{min} \times \frac{x(t)}{100}\frac{lb}{gal} = \frac{x(t)}{50}\frac{lb}{gal}$

$\frac{dx}{dt} = 6 - \frac{1}{50}x$

Let $c(t)$ represents the concentration of salt. Thus, $c(t)=\frac{x(t)}{100} \quad \rightarrow \quad x'=100c'$

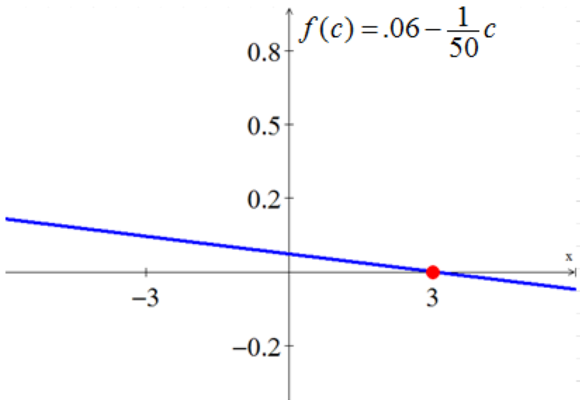
$100c'=6-\frac{1}{50}(100c)$

$100c'=.06-\frac{1}{50}c$

$\Rightarrow f(c)=.06-\frac{1}{50}c=0$

$\frac{1}{50}c=.06 \quad \Rightarrow \quad c=3$

$c=3$ is stable equilibrium point so a trajectory should approach the stable equilibrium solution $c(t)=3$



Exercise

A mathematical model for rate at which a drug disseminates into the bloodstream at time *t*.

$$\frac{dx}{dt}=r-kx$$

Where *r* and *k* are positive constants. The function $x(t)$ describes the concentration of the drug in the bloodstream at time *t*.

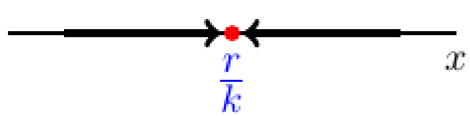
a) Since the DE is autonomous, use the phase portrait concept to find the limiting value of $x(t)$ as $t\rightarrow\infty$

b) Solve $x(t)$ subject to $x(0)=0$. Sketch the graph of $x(t)$ and verify your prediction in part (*a*). At what time is the concentration one-half this limiting value?

Solution

a) $\frac{dx}{dt}=r-kx=0 \quad \rightarrow \quad \boxed{x=\frac{r}{k}}$

The equilibrium solution $x=\frac{r}{k}$.



When $x<\frac{r}{k} \quad \Rightarrow \quad \frac{dx}{dt}>0$

When $x>\frac{r}{k} \quad \Rightarrow \quad \frac{dx}{dt}<0$

$\boxed{\lim\limits_{x\to\infty} x(t)=\frac{r}{k}}$

b) $\frac{dx}{dt}+kx=r$

$$e^{\int k\,dt} = e^{kt}$$

$$\int re^{kt}dt = \frac{r}{k}e^{kt}$$

$$x(t) = \frac{1}{e^{kt}}\left(\frac{r}{k}e^{kt} + C\right)$$

$$= \boxed{\frac{r}{k} + Ce^{-kt}}$$

$$x(0) = 0 \quad \rightarrow 0 = \frac{r}{k} + C \quad \Rightarrow \quad \boxed{C = -\frac{r}{k}}$$

$$\boxed{x(t) = \frac{r}{k} - \frac{r}{k}e^{-kt}}$$

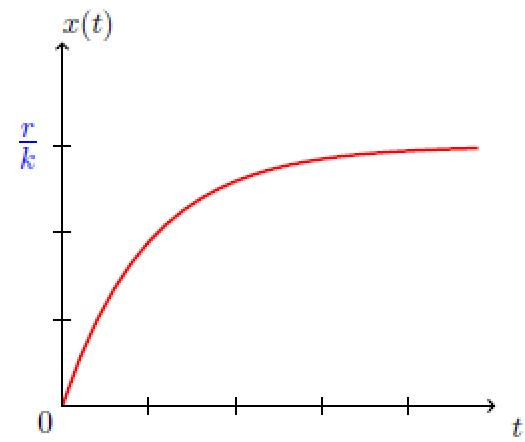
$x \rightarrow \frac{r}{k}$ as $t \rightarrow \infty$

If $x(T) = \frac{r}{2k}$

$$\frac{r}{2k} = \frac{r}{k} - \frac{r}{k}e^{-kT}$$

$$\frac{r}{2k} = \frac{r}{k}e^{-kT}$$

$$e^{-kT} = \frac{1}{2} \quad \rightarrow \quad \boxed{T = \frac{\ln 2}{k}}$$



Exercise

When forgetfulness is taken into account, the rate of memorization of a subject is given by

$$\frac{dA}{dt} = k_1(M - A) - k_2 A$$

Where $k_1 > 0$, $k_2 > 0$, $A(t)$ is the amount memorized in time *t*, *M* is the total amount to be memorized, and $M - A$ is the amount remaining to be memorized.

a) Since the DE is autonomous, use the phase portrait concept to find the limiting value of $A(t)$ as $t \rightarrow \infty$. Interpret the result

b) Solve $A(t)$ subject to $A(0) = 0$. Sketch the graph of $A(t)$ and verify your prediction in part (*a*).

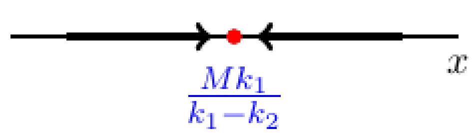
Solution

a) $\frac{dA}{dt} = k_1(M - A) - k_2 A = 0$

$k_1 M - k_1 A - k_2 A = 0$

$A = \frac{k_1 M}{k_1 + k_2}$; the equilibrium solution

$$\boxed{\lim_{t\to\infty} A(t) = \frac{k_1 M}{k_1 + k_2}}$$



Since $k_2 > 0$, the material will never be completely memorized and the larger k_2 is, the less the amount of material will be memorized over time.

b) $\frac{dA}{dt} = k_1 M - (k_1 + k_2)A$

$$\frac{dA}{dt} + (k_1 + k_2)A = k_1 M$$

$$e^{\int (k_1+k_2)dt} = e^{(k_1+k_2)t}$$

$$\int Mk_1 e^{(k_1+k_2)t}\, dt = \frac{Mk_1}{k_1+k_2} e^{(k_1+k_2)t}$$

$$A(t) = e^{-(k_1+k_2)t}\left(\frac{Mk_1}{k_1+k_2} e^{(k_1+k_2)t} + C\right)$$

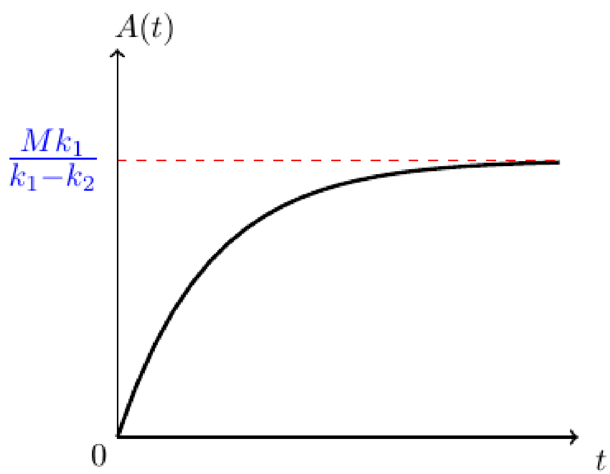
$$= \frac{Mk_1}{k_1+k_2} + Ce^{-(k_1+k_2)t}$$

$$A(0) = 0 \quad \to 0 = \frac{Mk_1}{k_1+k_2} + C \quad \Rightarrow C = -\frac{Mk_1}{k_1+k_2}$$

$$A(t) = \frac{Mk_1}{k_1+k_2} - \frac{Mk_1}{k_1+k_2} e^{-(k_1+k_2)t}$$

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(\frac{Mk_1}{k_1+k_2} - \frac{Mk_1}{k_1+k_2} e^{-(k_1+k_2)t}\right)$$

$$= \frac{Mk_1}{k_1+k_2}$$



Exercise

The number $N(t)$ of supermarkets throughout the country that are using a computerized checkout system is described by the initial-value problem

$$\frac{dN}{dt} = N(1-0.0005N), \quad N(0)=1$$

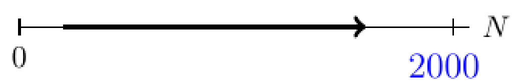
a) Use the phase portrait concept to predict how many supermarkets are expected to adopt the new procedure over a long period of time. Sketch a solution curve of the given initial-value problem.

b) Solve the initial-value problem and then graph it to verify the solution in part (*a*)

c) How many companies are expected to adopt the new technology when $t=10$?

Solution

a) $\frac{dN}{dt} = N(1-0.0005N) = 0$



$N=0 \quad N = \frac{1}{0.0005} = 2000$

When $0 < N < 2000 \quad \Rightarrow \quad \frac{dN}{dt} > 0$

From the phase portait:

$\lim\limits_{t\to\infty} N(t) \ \underline{= 2000}$

b) $\frac{dN}{N(1-0.0005N)} = dt$

$\frac{2000}{N(2000-N)} dN = dt$

$\int \left(\frac{1}{N} - \frac{1}{N-2000}\right) dN = \int dt$

$\ln N - \ln(N-2000) = t + C$

$\ln \frac{N}{N-2000} = t + C$

$\frac{N}{N-2000} = e^{t+C}$

$N = Ne^{t+C} - 2000e^{t+C}$

$N\left(e^C e^t + 1\right) = 2000e^C e^t$

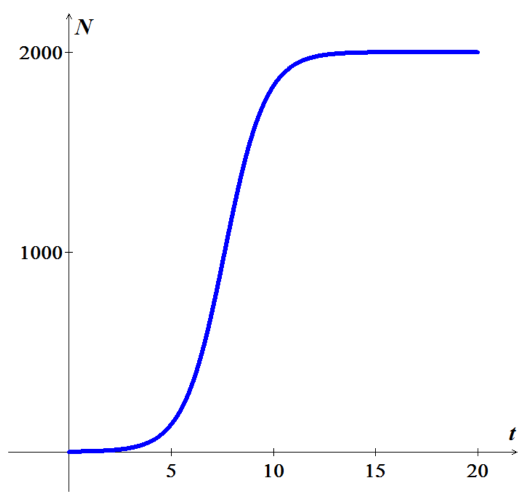
$N(t) = \frac{2000e^C e^t}{1+e^C e^t}$

$N(0)=1 \quad 1 = \frac{2000e^C}{1+e^C}$

$1+e^C = 2000e^C$

$e^C = \frac{1}{1999}$

$\boxed{N(t) = \frac{2000e^t}{1999+e^t}}$



$$N(10)=\frac{2000e^{10}}{1999+e^{10}}\ \underline{\approx 1833.59|}$$

About 1834 companies are expected to adopt the new technology when $t=10$

Exercise

For the linear *ODE* $\quad ty'+y=2t$

a) Find all solution of the given *DE* equation.

b) Show that the initial value $y(0)=0$, has exactly one solution.

c) But if $y(0)=y_0\neq 0$ there is no solution at all. Why doesn't this contradict the Existence and Uniqueness Theorem?

d) Plot several solutions of the *ODE* over the interval $-5\leq t\leq 5$

Solution

a) $\frac{1}{t}\times \quad ty'+y=2t$

$$y'+\frac{1}{t}y=2 \quad (t\neq 0)$$

$$e^{\int\frac{dt}{t}}=e^{\ln t}=t$$

$$\int 2t\,dt=t^2$$

$$y(t)=\frac{1}{t}\left(t^2+C\right)$$

$$\underline{=t+\frac{C}{t}|}$$

Each value of C gives 2 distinct solutions. On defined $-\infty<t<0$ and the other on $0<t<\infty$

b) ***Given***: $y(0)=0$

$0=0+\frac{C}{0}$, the only solution is when $C=0$, which gives us the general solution

$$\underline{y(t)=t|}$$

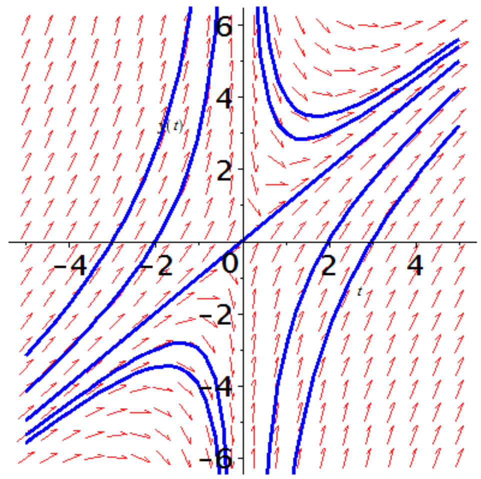
c) $y(0)=y_0\neq 0 \quad\rightarrow\quad y_0=0+\frac{C}{0}$

$\Rightarrow y_0=0$ which contradict the given information.

Which contradict the Existence and Uniqueness Theorem of the initial value existence.

d) The solution curve of $y=t$ goes through the origin.

All the other are curves in the shape of hyperbolas and are asymptotic to one end or the other of the *y*-axis



6
4
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