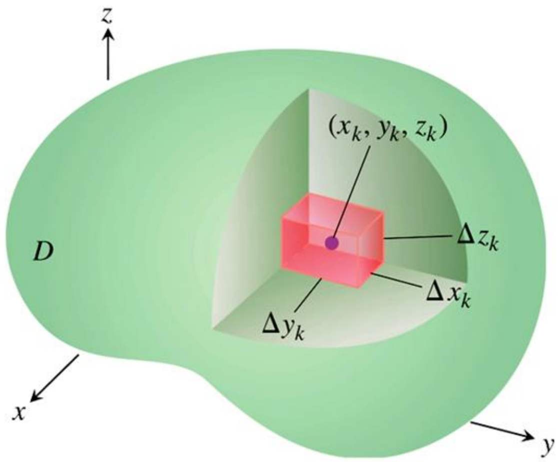
Section 3.4 – Triple Integrals

Triple Integrals

If $F(x,y,z)$ is a function define-d on a closed, bounded region *D* in space, such a solid ball or a lump of clay, then the integral of *F* over *D* may be defined in the following way.



$$\Delta V_k = \Delta x_k \Delta y_k \Delta z_k \quad \to \quad S_n = \sum_{k=1}^{n} F\left(x_k, y_k, z_k\right)\Delta V_k$$

The limit of this summation is the triple integral of *F* over *D*

$$\lim_{n\to\infty} S_n = \iiint_D F(x,y,z)\,dV \quad or \quad \lim_{\|P\|\to} S_n = \iiint_D F(x,y,z)\,dxdydz$$

Volume of a region in Space

Definition

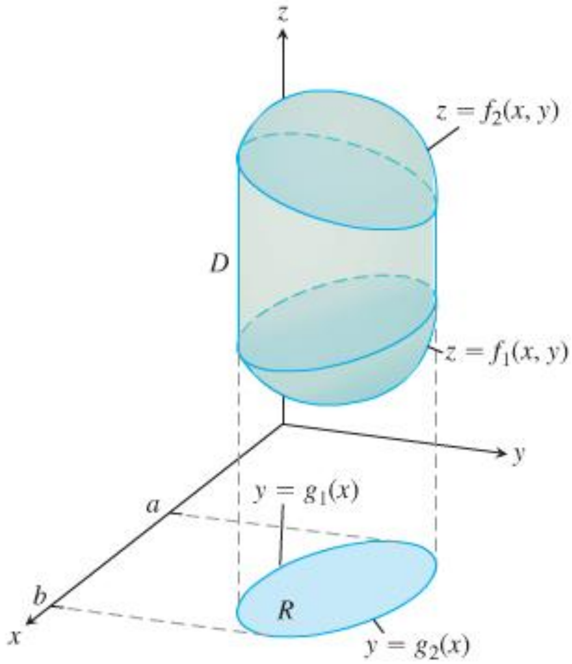
The volume of a closed, bounded region *D* in space is

$$V = \iiint_D dV$$

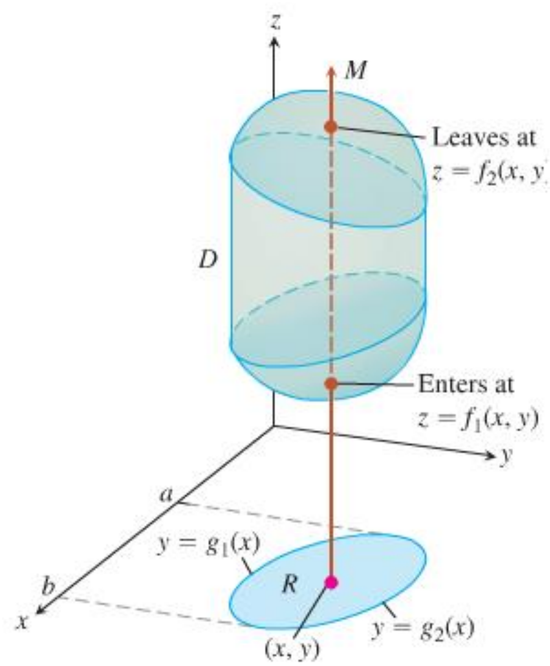
Find Limits of Integration in the Order *dz dy dx*

To evaluate $\iiint\limits_D F(x,y,z)\,dV$

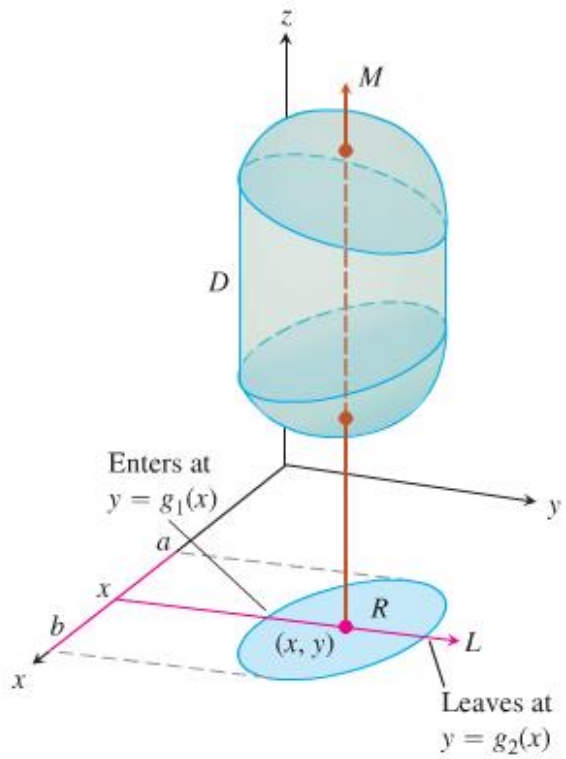
1. ***Sketch***: Sketch the region D along with its "shadow" R (vertical projection) in the xy-plane. Label the upper and lower bounding surfaces of D and R.



2. ***Find the z-limits of integration***: Draw a line M passing through (x,y) in R parallel to the z-axis. As z increases, M enters D at $z = f_1(x,y)$ and leaves at $z = f_2(x,y)$.



3. ***Find the y-limits of integration***: Draw a line L passing through (x, y) parallel to the y-axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$.



4. ***Find the x-limits of integration***: Choose x-limits that include all lines through R parallel to the y-axis $(x = a \;\; and \;\; x = b)$.

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\, dzdydx$$

Example

Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

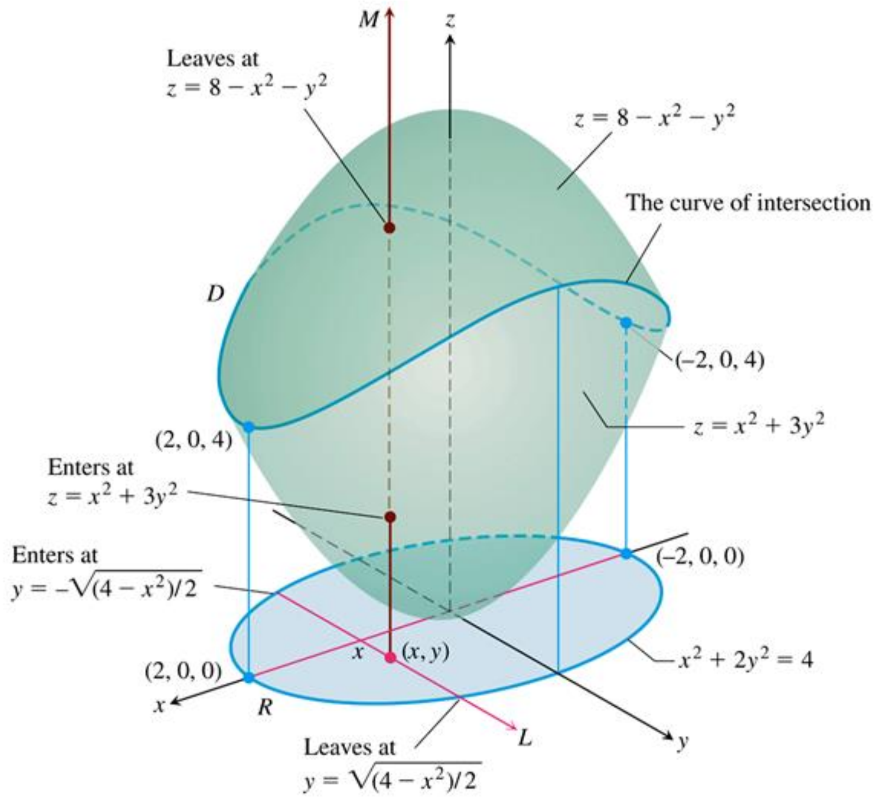
Solution

***z*-limits**: $x^2 + 3y^2 \le z \le 8 - x^2 - y^2$

***y*-limits**: $z = x^2 + 3y^2 = 8 - x^2 - y^2 \quad \rightarrow \quad 2x^2 + 4y^2 = 8 \;\Rightarrow\; x^2 + 2y^2 = 4$

$$y^2 = \frac{4 - x^2}{2} \quad \Rightarrow y = \pm\sqrt{\frac{4 - x^2}{2}} \quad \rightarrow -\sqrt{\frac{4 - x^2}{2}} \le y \le \sqrt{\frac{4 - x^2}{2}}$$

***x*-limits**: $x^2 + 2y^2 = 4 \quad (y = 0) \quad \rightarrow x = \pm 2$



$$V=\int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\int_{x^2+3y^2}^{8-x^2-y^2} dzdydx$$

$$=\int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \left[z\right]_{x^2+3y^2}^{8-x^2-y^2} dydx$$

$$=\int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \left(8-x^2-y^2-x^2-3y^2\right)dydx$$

$$=\int_{-2}^{2}\left[\left(8-2x^2\right)y-\frac{4}{3}y^3\right]_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} dx$$

$$=\int_{-2}^{2}\left[\left(8-2x^2\right)\sqrt{\frac{4-x^2}{2}}-\frac{4}{3}\left(\frac{4-x^2}{2}\right)^{3/2}+\left(8-2x^2\right)\sqrt{\frac{4-x^2}{2}}-\frac{4}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right]dx$$

$$=\int_{-2}^{2}\left[2\left(8-2x^2\right)\sqrt{\frac{4-x^2}{2}}-\frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right]dx$$

$$=\int_{-2}^{2}\left[2\left(\frac{2}{2}\right)(2)\left(4-x^2\right)\sqrt{\frac{4-x^2}{2}}-\frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right]dx$$

$$= \int_{-2}^{2} \left[8\left(\frac{4-x^2}{2}\right)\left(\frac{4-x^2}{2}\right)^{1/2} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2} \right] dx$$

$$= \int_{-2}^{2} \left[8\left(\frac{4-x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2} \right] dx$$

$$= \int_{-2}^{2} \left[\frac{16}{3}\left(\frac{4-x^2}{2}\right)^{3/2} \right] dx$$

$$= \frac{16}{3(2)^{3/2}} \int_{-2}^{2} \left(4-x^2\right)^{3/2} dx \qquad \frac{16}{3(2)^{3/2}} \frac{2^{1/2}}{2^{1/2}} = \frac{16\sqrt{2}}{3\cdot 4} = \frac{4\sqrt{2}}{3}$$

$$x = 2\sin u \quad dx = 2\cos u du \quad \left(4 - x^2 = 4 - 4\sin^2 u = 4\cos^2 u\right)$$

$$\begin{cases} x = 2 & \rightarrow u = \sin^{-1}\frac{x}{2} = \sin^{-1} 1 = \frac{\pi}{2} \\ x = -2 & \rightarrow u = \sin^{-1}(-1) = -\frac{\pi}{2} \end{cases}$$

$$= \frac{4\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(4\cos^2 u\right)^{3/2} \left(2\cos u \; du\right)$$

$$= \frac{4\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} 16(\cos u)^3 (\cos u) du$$

$$= \frac{64\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \cos^4 u \; du$$

$$= \frac{64\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(\frac{1+\cos 2u}{2}\right)^2 du$$

$$= \frac{16\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(1 + 2\cos 2u + \cos^2 2u\right) du$$

$$= \frac{16\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(1 + 2\cos 2u + \frac{1}{2} + \frac{1}{2}\cos 4u\right) du$$

$$= \frac{16\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(\frac{3}{2} + 2\cos 2u + \frac{1}{2}\cos 4u\right) du$$

$$= \frac{16\sqrt{2}}{3} \left[\frac{3}{2}u + \sin 2u + \frac{1}{8}\sin 4u \right]_{-\pi/2}^{\pi/2}$$

$$= \frac{16\sqrt{2}}{3} \left[\frac{3\pi}{4} + \sin\pi + \frac{1}{8}\sin 2\pi - \left(-\frac{3\pi}{4} - \sin\pi - \frac{1}{8}\sin 2\pi \right) \right]$$

$$= \frac{16\sqrt{2}}{3}\left(\frac{3\pi}{2}\right)$$

$$\boxed{= 8\pi\sqrt{2} \quad unit^3}$$

Example

Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron *D* with vertices (0, 0, 0), (1, 1, 0), (0, 1, 0), and (0, 1, 1). Use the order of integration $dydzdx$.

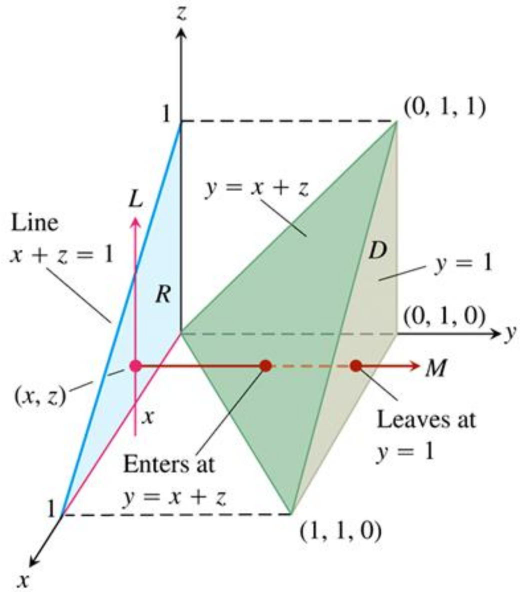
Solution

From the sketch, the upper (right-hand) bounding surface of *D* lies in the plane $y = 1$.

The lower (left-hand) bounding surface lies in the plane $y = x + z$.

The upper boundary of *R* is the line $z = 1 - x$.

The lower boundary is the line $z = 0$.



***y*-limits**: The line through $(x,\ z)$ in *R* parallel to the *y*-axis enters *D* at $y = x + z$ and leaves at $y = 1$.

***z*-limits**: The line through $(x,\ z)$ in *R* parallel to the *z*-axis enters *R* at $z = 0$ and leaves at $z = 1 - x$.

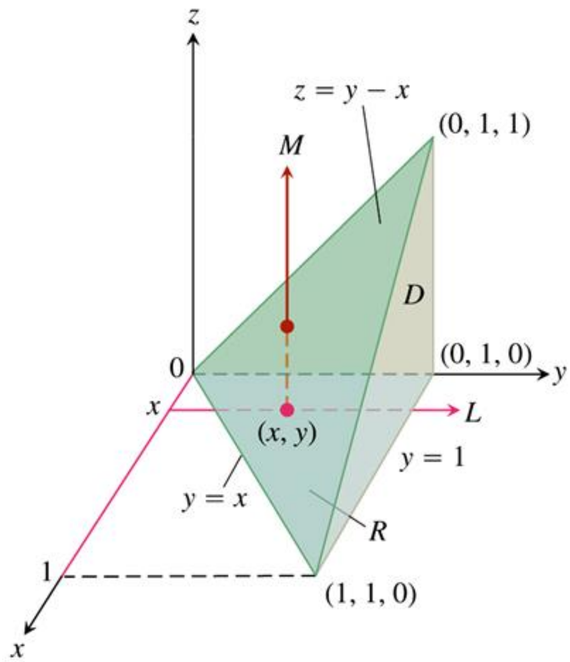
***x*-limits**: $0 \le x \le 1$

$$\int_0^1 \int_0^{1-x} \int_{x+z}^{1} F(x, y, z)\, dydzdx$$

Example

Integrate $F(x,y,z)=1$ over the tetrahedron *D* in the previous example in the order $dz\ dy\ dx$, and then integrate in the order $dy\ dz\ dx$.

Solution



z-limits of integration: A line *M* parallel to the *z*-axis through a typical point (*x*, *y*) in the *xy*-plane "shadow" enters the tetrahedron at *z* = 0 and exists through the upper plane where $z=y-x$. $0\le z\le y-x$

Line is given by: $ax+by+cz=0$ passes through the 2 points:

$(1,1,0)\to a+b=0 \quad\Rightarrow\quad a=-b$

and $(0,1,1)\to b+c=0 \quad\Rightarrow\quad c=-b$

$\to -bx+by-bz=0$

$-x+y-z=0 \quad\Rightarrow\quad z=y-x$

y-limits of integration: On the *xy*-plane, where *z* = 0, the sloped side of the tetrahedron crosses the plane along the line $y=x$. A line *L* through (*x*, *y*) parallel to the *y*-axis enters the shadow in the *xy*-plane at $y=x$ and exists at *y* = 1. $x\le y\le 1$

x-limits of integration: A line *L* parallel to the *y*-axis through a typical point (*x*, *y*) in the *xy*-plane sweeps out the shadow, where $0\le x\le 1$ at the point $(1,1,0)$

The integral is: $$\int_0^1\int_x^1\int_0^{y-x} F(x,y,z)\,dzdydx$$

$$V=\int_0^1\int_x^1\int_0^{y-x} dzdydx$$
$$=\int_0^1\int_x^1 [z]_0^{y-x} dydx$$
$$=\int_0^1\int_x^1 (y-x)dydx$$
$$=\int_0^1\left[\tfrac{1}{2}y^2-xy\right]_x^1 dx$$
$$=\int_0^1\left[\tfrac{1}{2}-x-\left(\tfrac{1}{2}x^2-x^2\right)\right]dx$$
$$=\int_0^1\left(\tfrac{1}{2}-x+\tfrac{1}{2}x^2\right)dx$$
$$=\left[\frac{1}{2}x-\frac{1}{2}x^2+\frac{1}{6}x^3\right]_0^1$$
$$=\frac{1}{2}-\frac{1}{2}+\frac{1}{6}$$
$$=\frac{1}{6}\quad unit^3$$

$$V=\int_0^1\int_0^{1-x}\int_{x+z}^1 dydzdx$$
$$=\int_0^1\int_0^{1-x}[y]_{x+z}^1 dzdx$$
$$=\int_0^1\int_0^{1-x}(1-x-z)dzdx$$
$$=\int_0^1\left[z-xz-\frac{1}{2}z^2\right]_0^{1-x} dx$$
$$=\int_0^1\left(1-x-x(1-x)-\tfrac{1}{2}(1-x)^2\right)dx$$
$$=\int_0^1\left((1-x)(1-x)-\tfrac{1}{2}(1-x)^2\right)dx$$

$$= \int_0^1 \left((1-x)^2 - \tfrac{1}{2}(1-x)^2 \right) dx$$

$$= \int_0^1 \tfrac{1}{2}(1-x)^2\, dx$$

$$= -\tfrac{1}{6}(1-x)^3 \Big|_0^1$$

$$= \tfrac{1}{6} \quad unit^3$$

Average Value of a Function in Space

The average value of a function F over a region D in space is defined by the formula

$$\textbf{\textit{Average value}} \text{ of } F \text{ over } D = \frac{1}{\text{volume of } D} \iiint_D F\,dV$$

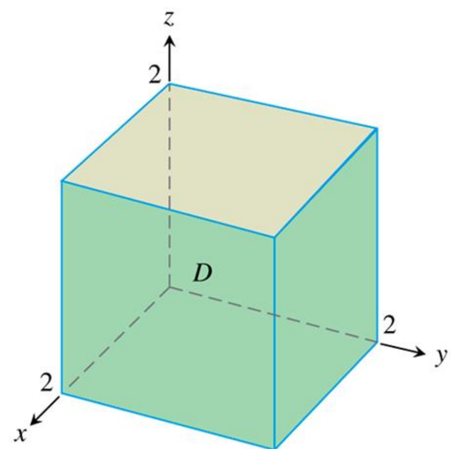
Example

Find the average of $F(x,y,z) = xyz$ throughout the cubical region D bounded by the coordinate planes and the planes $x=2$, $y=2$, and $z=2$ in the first octant.

Solution

$$Volume = 2 \cdot 2 \cdot 2 = 8$$

The value of the integral of F over the cube is



$$V = \int_0^2 \int_0^2 \int_0^2 xyz\,dx\,dy\,dz$$

$$= \int_0^2 z\,dz \int_0^2 y\,dy \int_0^2 x\,dx$$

$$= \left[\tfrac{1}{2}z^2\right]_0^2 \left[\tfrac{1}{2}y^2\right]_0^2 \left[\tfrac{1}{2}x^2\right]_0^2$$

$$= \tfrac{1}{8}(4)(4)(4)$$

$$= 8 \quad unit^3$$

$$\textbf{\textit{Average value}} \text{ of } xyz \text{ over } cube = \frac{1}{\text{volume of } D} \iiint_{cube} xyz\,dV$$

$$= \left(\tfrac{1}{8}\right)(8)$$

$$= 1$$

Exercises Section 3.4 – Triple Integrals

Evaluate the integral

1. $\displaystyle\int_0^1\int_0^1\int_0^1\left(x^2+y^2+z^2\right)dzdydx$

2. $\displaystyle\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dzdxdy$

3. $\displaystyle\int_0^{\pi/6}\int_0^1\int_{-2}^3 y\sin z\ dxdydz$

4. $\displaystyle\int_{-1}^1\int_0^1\int_0^2 (x+y+z)dydxdz$

5. $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2}} dzdydx$

6. $\displaystyle\int_0^1\int_0^{1-x^2}\int_0^{4-x^2-y} xdzdydx$

7. $\displaystyle\int_0^\pi\int_0^\pi\int_0^\pi \cos(u+v+w)dudvdw$

8. $\displaystyle\int_0^{\pi/4}\int_0^{\ln\sec v}\int_{-\infty}^{2t} e^x dxdtdv$

9. $\displaystyle\int_0^1\int_{-z}^z\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dydxdz$

10. $\displaystyle\int_0^\pi\int_0^y\int_0^{\sin x} dzdxdy$

11. $\displaystyle\int_0^9\int_0^1\int_{2y}^2 \frac{4\sin x^2}{\sqrt{z}}dxdydz$

12. $\displaystyle\int_0^\pi\int_0^\pi\int_0^\pi \cos(x+y+z)dxdydz$

13. $\displaystyle\int_1^e\int_1^x\int_0^z \frac{2y}{z^3}dydzdx$

14. $\displaystyle\int_{\ln 6}^{\ln 7}\int_0^{\ln 2}\int_{\ln 4}^{\ln 5} e^{(x+y+z)}dzdydx$

15. $\displaystyle\int_0^1\int_0^{x^2}\int_0^{x+y} (2x-y-z)dzdydx$

16. $\displaystyle\int_{-2}^2\int_3^6\int_0^2 dxdydz$

17. $\displaystyle\int_{-1}^1\int_{-1}^2\int_0^1 6xyz\ dydxdz$

18. $\displaystyle\int_{-2}^2\int_1^2\int_1^e \frac{xy^2}{z}dzdxdy$

19. $\displaystyle\int_0^{\ln 4}\int_0^{\ln 3}\int_0^{\ln 2} e^{-x+y+z}dxdydz$

20. $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^1\int_0^{\frac{\pi}{2}} \sin\pi x\cos y\sin 2z\ dydxdz$

21. $\displaystyle\int_0^2\int_1^2\int_0^1 yze^x dxdzdy$

22. $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dzdydx$

23. $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} 2xz\ dzdydx$

24. $\displaystyle\int_0^4\int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}}\int_0^{16-\frac{1}{4}x^2-y^2} dzdxdy$

25. $\displaystyle\int_1^6\int_0^{4-\frac{2}{3}y}\int_0^{12-2y-3z}\frac{1}{y}\,dxdzdy$

26. $\displaystyle\int_0^3\int_0^{\sqrt{9-z^2}}\int_0^{\sqrt{1+x^2+z^2}}dydxdz$

27. $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{\sin x}\sin y\,dzdxdy$

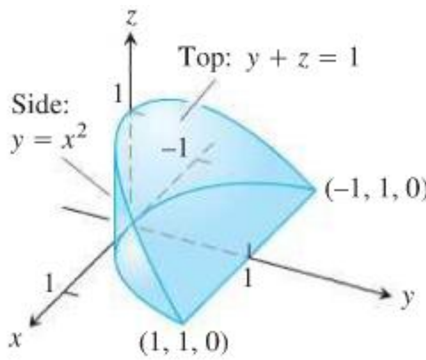
28. $\displaystyle\int_0^{\ln 8}\int_1^{\sqrt{z}}\int_{\ln y}^{\ln 2y}e^{x+y^2-z}dxdydz$

29. $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{2-x}4yz\,dzdydx$

30. $\displaystyle\int_0^2\int_0^4\int_{y^2}^4\sqrt{x}\,dzdxdy$

31. $\displaystyle\int_0^1\int_y^{2-y}\int_0^{2-x-y}xy\,dzdxdy$

32. Here is the region of integration of the integral $\displaystyle\int_{-1}^1\int_{x^2}^1\int_0^{1-y}dzdydx$



a) *dydzdx* ***b***) *dydxdz* ***c***) *dxdydz* ***d***) *dxdzdy* ***e***) *dzdxdy*

Use another order to evaluate

33. $\displaystyle\int_0^5\int_{-1}^0\int_0^{4x+4}dydxdz$

34. $\displaystyle\int_0^1\int_{-2}^2\int_0^{\sqrt{4-y^2}}dzdydx$

35. $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}}dydzdx$

36. $\displaystyle\int_0^4\int_0^{\sqrt{16-x^2}}\int_0^{\sqrt{16-x^2-z^2}}dydzdx$

37. $\displaystyle\int_1^4\int_z^{4z}\int_0^{\pi^2}\frac{\sin\sqrt{yz}}{x^{3/2}}dydxdz$

Evaluate

38. $\iiint\limits_D (xy + xz + yz)\,dV$; $\quad D = \{(x,\ y,\ z):\ -1 \le x \le 1,\quad -2 \le y \le 2,\quad -3 \le z \le 3\}$

39. $\iiint\limits_D xyze^{-x^2-y^2}\,dV$; $\quad D = \{(x,\ y,\ z):\ 0 \le x \le \sqrt{\ln 2},\quad 0 \le y \le \sqrt{\ln 4},\quad 0 \le z \le 1\}$

40. Let $D = \{(x, y, z):\ 0 \le x \le y^2,\ 0 \le y \le z^3,\ 0 \le z \le 2\}$

a) Use a triple integral to find the volume of *D*.

b) In theory, how many other possible orderings of the variables (besides the one used in part (*a*)) can be used to find the volume of *D*? Verify the result of part (*a*) using one of these other ordering.

c) What is the volume of the region $D = \{(x, y, z):\ 0 \le x \le y^p,\ 0 \le y \le z^q,\ 0 \le z \le 2\}$, where *p* and *q* are positive real numbers?

41. Find the volume the parallelepiped (slanted box) with vertices $(0,\ 0,\ 0)$, $(1,\ 0,\ 0)$, $(0,\ 1,\ 0)$, $(1,\ 1,\ 0)$, $(0,\ 1,\ 1)$, $(1,\ 1,\ 1)$, $(0,\ 2,\ 1)$, $(1,\ 2,\ 1)$

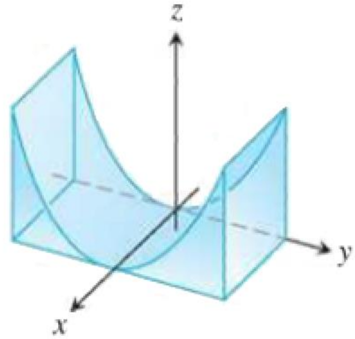
42. Find the volume the larger of two solids formed when the parallelepiped with vertices $(0,\ 0,\ 0)$, $(2,\ 0,\ 0)$, $(0,\ 2,\ 0)$, $(2,\ 2,\ 0)$, $(0,\ 1,\ 1)$, $(2,\ 1,\ 1)$, $(0,\ 3,\ 1)$, $(2,\ 3,\ 1)$ is sliced by the plane $y = 2$.

43. Find the volume of the pyramid with vertices $(0,\ 0,\ 0)$, $(2,\ 0,\ 0)$, $(2,\ 2,\ 0)$, $(0,\ 2,\ 0)$, $(0,\ 0,\ 4)$

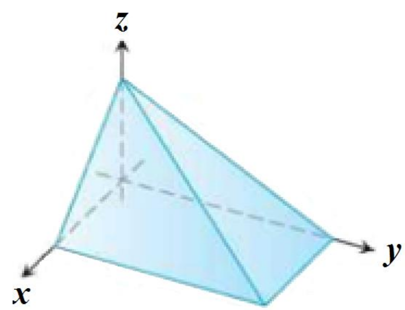
44. Two different tetrahedrons fill the region in the first octant bounded by the coordinate planes and the plane $x + y + z = 4$. Both solids have densities that vary in the *z*-direction between $\rho = 4$ and $\rho = 8$, according to the functions $\rho_1 = 8 - z$ and $\rho_2 = 4 + z$. Find the mass of each solid

45. Suppose a wedge of cheese fills the region in the first octant bounded by the planes $y = z$, $y = 4$ and $x = 4$. You could divide the wedge into two equal pieces (by volume) if you sliced the wedge with the plane $x = 2$. Instead find *a* with $0 < a < 1$ such that slicing the wedge with the plane $y = a$ divides the wedge into two pieces of equal volume

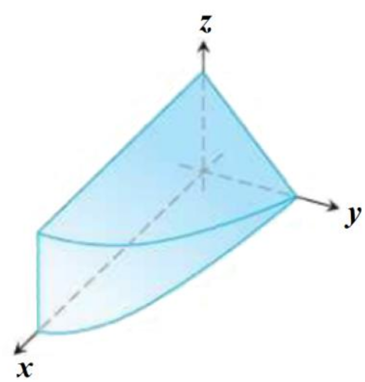
46. Find the volumes of the region between the cylinder $z = y^2$ and the *xy*-plane that is bounded by the planes $x = 0, \quad x = 1, \quad y = -1, \quad y = 1$



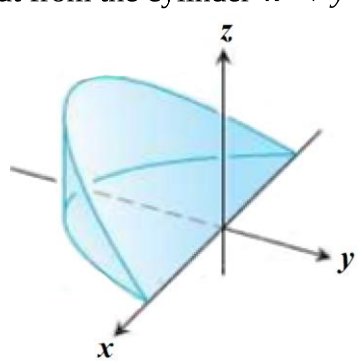
47. Find the volumes of the region in the first octant bounded by the coordinate planes and the planes $x + z = 1, \quad y + 2z = 2$



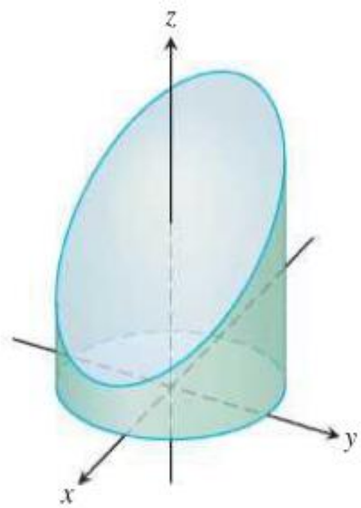
48. Find the volumes of the region in the first octant bounded by the coordinate planes and the plane $y + z = 2$, and the cylinder $x = 4 - y^2$



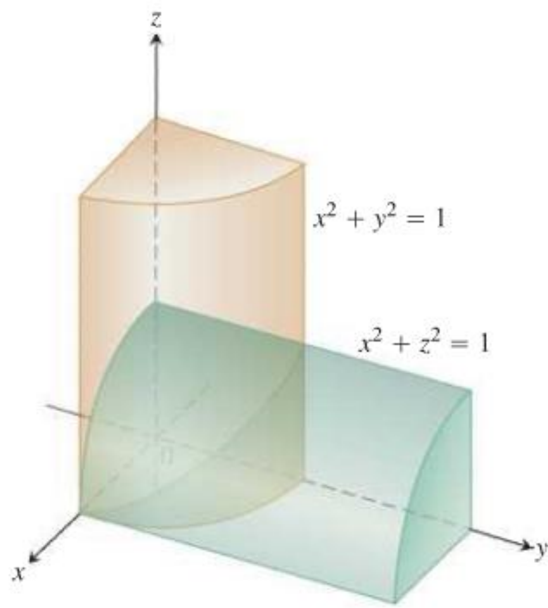
49. Find the volumes of the wedge cut from the cylinder $x^2 + y^2 = 1$ by the planes $z = -y, \quad z = 0$



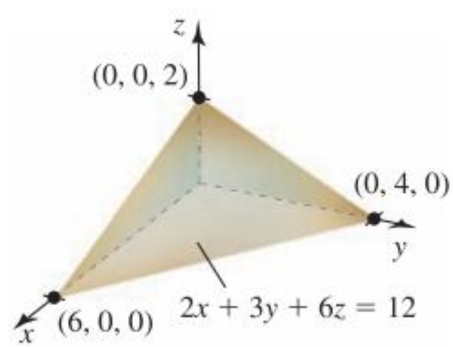
50. Find the volumes of the region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$



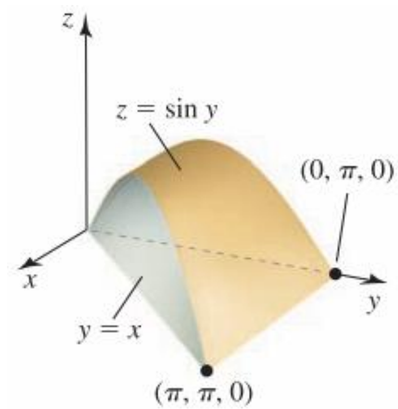
51. Find the volumes of the region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$, one-eighth of which is shown below



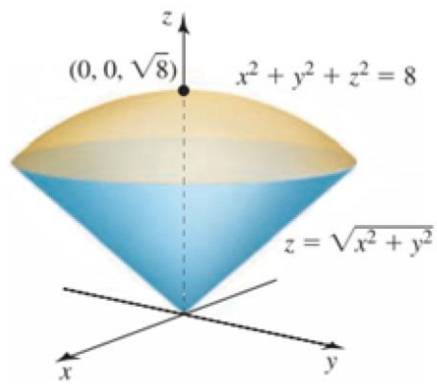
52. Find the volume of the solid in the first octant bounded by the plane $2x + 3y + 6z = 12$ and the coordinate planes



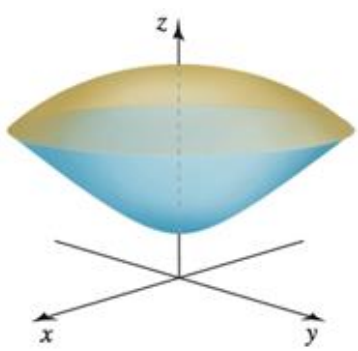
53. Find the volume of the solid in the first octant formed when the cylinder $z = \sin y$, for $0 \le y \le \pi$, is sliced by the planes $y = x$ and $x = 0$



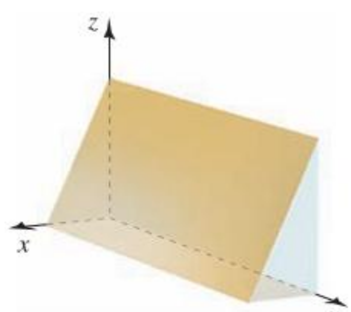
54. Find the volume of the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and bounded above the sphere $x^2 + y^2 + z^2 = 8$



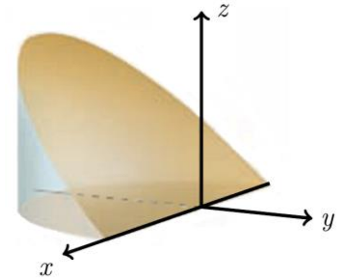
55. The solid between the sphere $x^2 + y^2 + z^2 = 19$ and the hyperboloid $z^2 - x^2 - y^2 = 1$, for $z > 0$



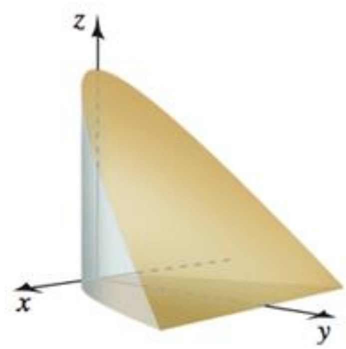
56. Find the volume of the prism in the first octant bounded below by $z = 2 - 4x$ and $y = 8$



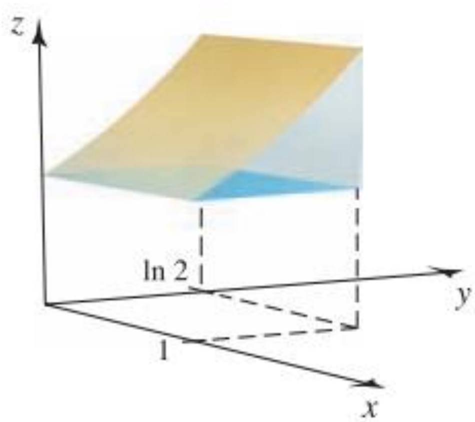
57. Find the volume of the wedge above the *xy*-plane formed when the cylinder $x^2 + y^2 = 4$ is cut by the planes $z = 0$ and $y = -z$



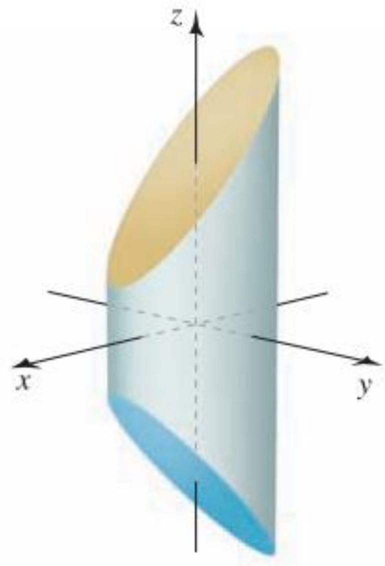
58. The wedge bounded by the parabolic cylinder $y = x^2$ and the planes $z = 3 - y$ and $z = 0$



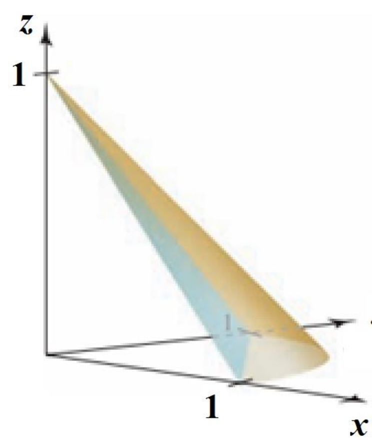
59. Find the volume of the solid bounded by the surfaces $z = e^y$ and $z = 1$ over the rectangle $\{(x, y):\ 0 \le x \le 1,\ 0 \le y \le \ln 2\}$



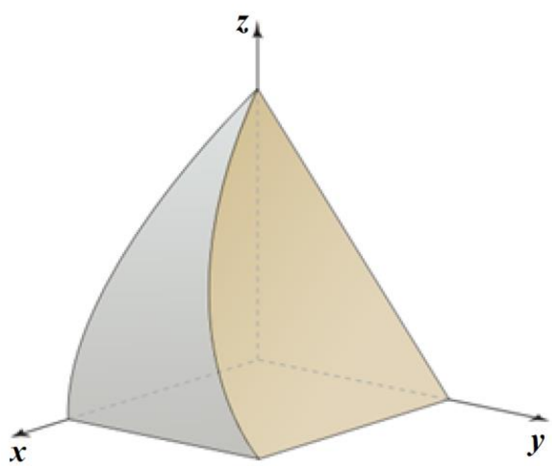
60. Find the volume of the wedge of the cylinder $x^2 + 4y^2 = 4$ created by the planes $z = 3 - x$ and $z = x - 3$



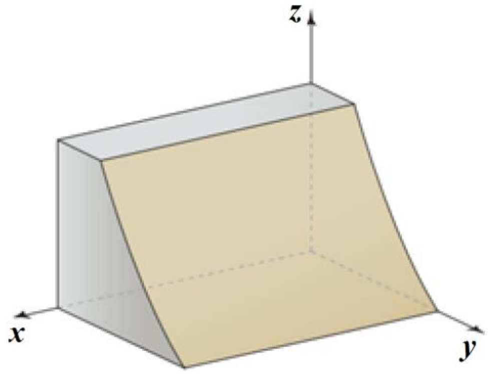
61. Find the volume of the solid in the first octant bounded by the cone $z = 1 - \sqrt{x^2 + y^2}$ and the plane $x + y + z = 1$



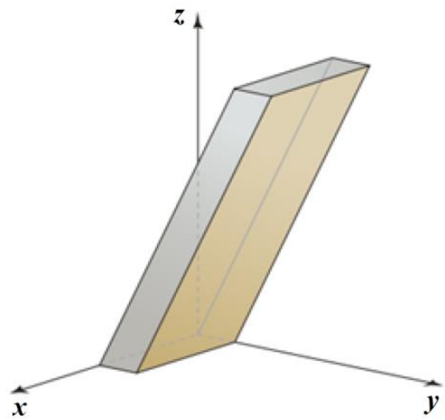
62. Find the volume of the solid bounded by $x = 0$, $x = 1 - z^2$, $y = 0$, $z = 0$, and $z = 1 - y$



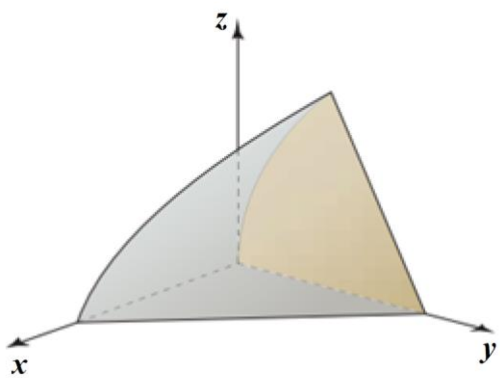
63. Find the volume of the solid bounded by $x = 0$, $x = 2$, $y = 0$, $y = e^{-z}$, $z = 0$, and $z = 1$



64. Find the volume of the solid bounded by $x = 0$, $x = 2$, $y = z$, $y = z + 1$, $z = 0$, and $z = 4$

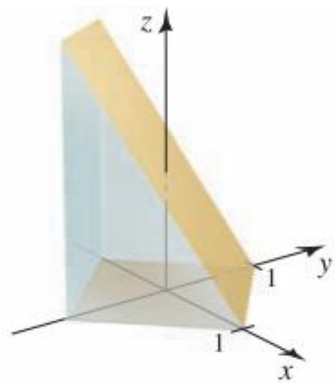


65. Find the volume of the solid bounded by $x = 0$, $y = z^2$, $z = 0$, and $z = 2 - x - y$



66. Find the volume of the solid common to the cylinders $z = \sin x$ and $z = \sin y$ over the square $R = \{(x, y): \ 0 \le x \le \pi, \ 0 \le y \le \pi\}$

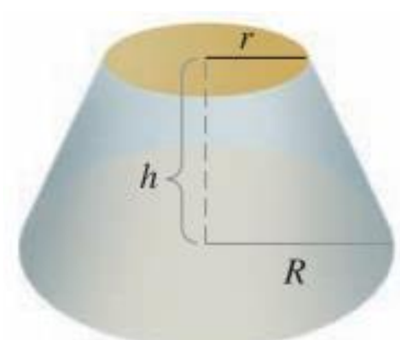
67. Find the volume of the wedge of the square column $|x|+|y|=1$ created by the planes $z=0$ and $x+y+z=1$



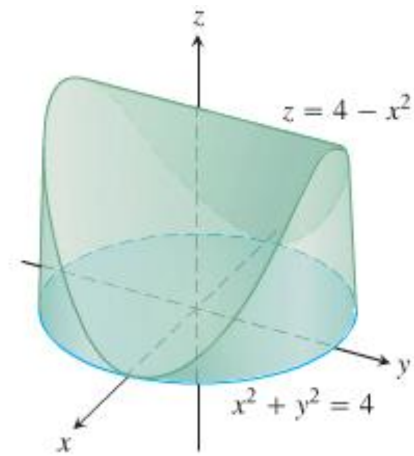
68. Find the volume of a right circular cone with height *h* and base radius *r*.

69. Find the volume of a tetrahedron whose vertices are located at $(0,\ 0,\ 0)$, $(a,\ 0,\ 0)$, $(0,\ b,\ 0)$, and $(0,\ 0,\ c)$

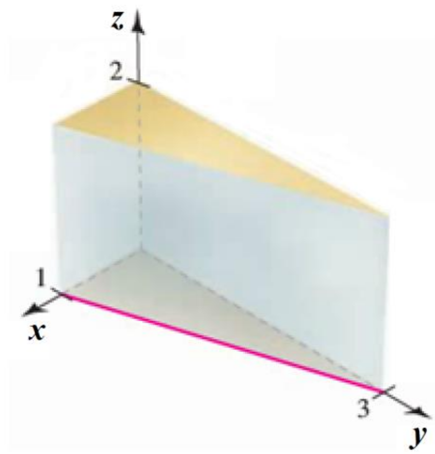
70. Find the volume of a truncated cone of height *h* whose ends have radii *r* and *R*.



71. Find the volume of the solid that is bounded above by the cylinder $z=4-x^2$, on the sides by the cylinder $x^2+y^2=4$, and below by the *xy*-plane.



72. Find the volume of the prism in the first octant bounded by the planes $y = 3 - 3x$ *and* $z = 2$



73. Find the volume of the prism in the first octant bounded by the planes
$x^2 + y^2 = 4$ *and* $x^2 + z^2 = 4$