# *Lecture Three – Infinite Sequences and Series*

## *Section* 3.1 – Sequences

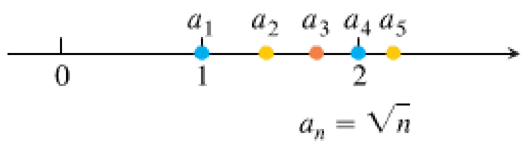
A sequence is a list of numbers

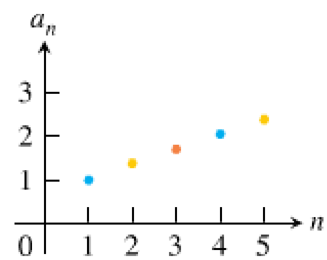
$$a_1,\ a_2,\ a_3,\ \dots,\ a_n,\ \dots$$

An ***infinite sequence*** of numbers is a function whose domain is the set of positive integers. These are the ***terms*** of the sequence. The integer ***n*** is called the ***index*** of $a_n$.

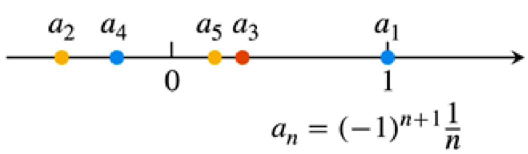
Sequences can be described by writing rules that specify their terms such as

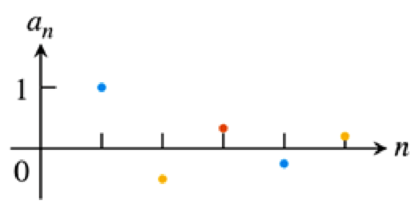
$$a_n = \sqrt{n} \quad \Rightarrow \quad \{a_n\} = \{\sqrt{1},\ \sqrt{2},\ \sqrt{3},\ \dots,\ \sqrt{n},\ \dots\}$$



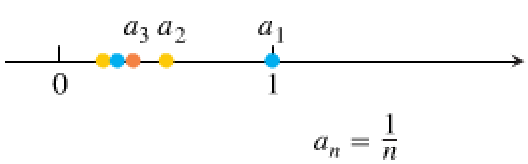



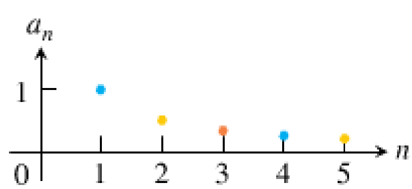

$$a_n = (-1)^{n+1}\frac{1}{n} \quad \Rightarrow \quad \{a_n\} = \left\{1,\ -\frac{1}{2},\ \frac{1}{3},\ -\frac{1}{4},\ \dots,\ (-1)^{n+1}\frac{1}{n},\ \dots\right\}$$



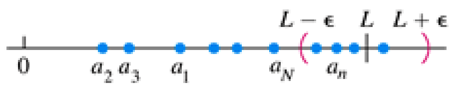

$$a_n = \frac{1}{n} \quad \Rightarrow \quad \{a_n\} = \left\{1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \dots,\ \frac{1}{n},\ \dots\right\}$$

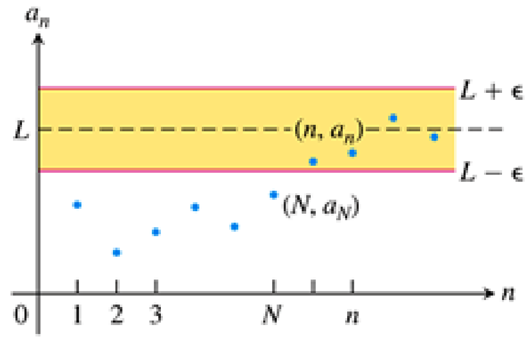



Also, we can write: $\{a_n\} = \{\sqrt{n}\}_{n=1}^{\infty}$

## Convergence and Divergence




$\left\{0,\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \ldots,\ 1-\frac{1}{n},\ \ldots\right\}$ Terms approach 1.

$\left\{1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots,\ \frac{1}{n},\ \ldots\right\}$ Terms approach 0.

### *Definition*

The sequence $\{a_n\}$ ***converges*** to the number $L$ if for every positive number $\varepsilon$ there corresponds an integer $N$ such that for all $n$,

$$n > N \quad \Rightarrow \quad |a_n - L| < \varepsilon$$

If no such number $L$ exists, we say $\{a_n\}$ ***diverges***.

The $\{a_n\}$ ***converges*** to $L$, we write $\lim\limits_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call $L$ the ***limit*** of the sequence.

### *Example*

Show that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

***Solution***

Let ε > 0 be given. We must show that there exists an integer $N$ such that for all $n$,

$$n > N \quad \Rightarrow \quad \left|\frac{1}{n} - 0\right| < \varepsilon$$

This implication will hold if $\frac{1}{n} < \varepsilon$ or $n > \frac{1}{\varepsilon}$. If $N$ is any integer greater than $\frac{1}{\varepsilon}$, the implication will hold for all $n > N$. This proves that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

### *Example*

Show that $\lim\limits_{n\to\infty} k = k \quad (\textit{any constant } k)$

***Solution***

Let ε > 0 be given. We must show that there exists an integer $N$ such that for all $n$,

$$n > N \quad \Rightarrow \quad |k - k| < \varepsilon$$

Since $k - k = 0$, we can use any positive integer for $N$ and the implication will hold for all $n > N$. This proves that $\lim\limits_{n\to\infty} k = k$

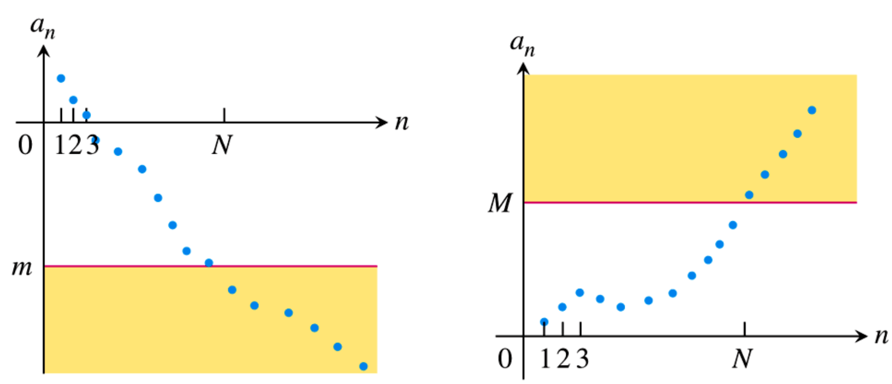
### *Definition*

The sequence $\{a_n\}$ ***diverges*** to infinity if for every number $M$ there is an integer $N$ such that for all $n$ larger than $N$, $a_n > M$. If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \quad or \quad a_n \to \infty$$

Similarly, if for every number $m$ there is an integer $N$ such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ ***diverges to negative infinity*** and write

$$\lim_{n\to\infty} a_n = -\infty \quad or \quad a_n \to -\infty$$



### *Theorem*

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let $A$ and $B$ real numbers. The following rules hold if $\lim\limits_{n\to\infty} a_n = A$ and $\lim\limits_{n\to\infty} b_n = B$

***Sum Rule***: $\lim\limits_{n\to\infty}\left(a_n + b_n\right) = A + B$

***Difference Rule***: $\lim\limits_{n\to\infty}\left(a_n - b_n\right) = A - B$

***Constant Multiple Rule***: $\lim\limits_{n\to\infty}\left(ka_n\right) = kA$

***Product Rule***: $\lim\limits_{n\to\infty}\left(a_n \bullet b_n\right) = A \bullet B$

***Quotient Rule***: $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{A}{B}$ *if* $B \ne 0$

### *Example*

***a)*** $\lim\limits_{n\to\infty}\left(-\dfrac{1}{n}\right) = -1 \bullet \lim\limits_{n\to\infty}\left(\dfrac{1}{n}\right) = -1(0) = 0$

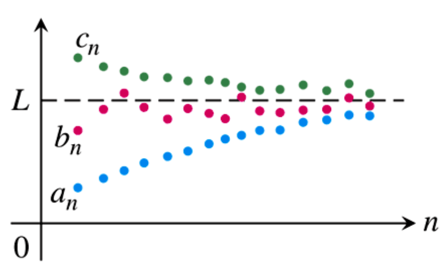
***b)*** $\lim\limits_{n\to\infty}\left(\dfrac{n-1}{n}\right) = \lim\limits_{n\to\infty}\left(1 - \dfrac{1}{n}\right) = \lim\limits_{n\to\infty} 1 - \lim\limits_{n\to\infty}\dfrac{1}{n} = 1 - 0 = 1$

***c)*** $\lim\limits_{n\to\infty}\left(\dfrac{5}{n^2}\right) = 5 \bullet \lim\limits_{n\to\infty}\left(\dfrac{1}{n^2}\right) = 5 \bullet \lim\limits_{n\to\infty}\left(\dfrac{1}{n}\right) \bullet \lim\limits_{n\to\infty}\left(\dfrac{1}{n}\right) = -1 \bullet 0 \bullet 0 = 0$

***d)*** $\lim\limits_{n\to\infty}\left(\dfrac{4 - 7n^6}{n^6 + 3}\right) = \lim\limits_{n\to\infty}\left(\dfrac{\frac{4}{n^6} - 7}{1 + \frac{3}{n^6}}\right) = \dfrac{0 - 7}{1 + 0} = -7$

### *Theorem* – *The Sandwich Theorem for Sequences*

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all $n$ beyond some index $N$, and if $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$ , then $\lim\limits_{n\to\infty} b_n = L$ also.



### *Example*

Since $\frac{1}{n} \to 0$, we know that

***a)*** $\frac{\cos n}{n} \to 0$ *because* $-\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$

***b)*** $\frac{1}{2^n} \to 0$ *because* $0 \le \frac{1}{2^n} \le \frac{1}{n}$

***c)*** $(-1)^n \frac{1}{n} \to 0$ *because* $-\frac{1}{n} \le (-1)^n \le \frac{1}{n}$

### *Theorem – The Continuous Function Theorem for Sequences*

Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if $f$ is a function that is continuous at L and defined at all $a_n$, then $f(a_n) \to f(L)$.

### *Example*

Show that $\sqrt{\frac{n+1}{n}} \to 1$

***Solution***

We know that $\frac{n+1}{n} \to 1$. Taking $f(x) = \sqrt{x}$ and $L = 1$ that gives $\sqrt{\frac{n+1}{n}} \to \sqrt{1} = 1$

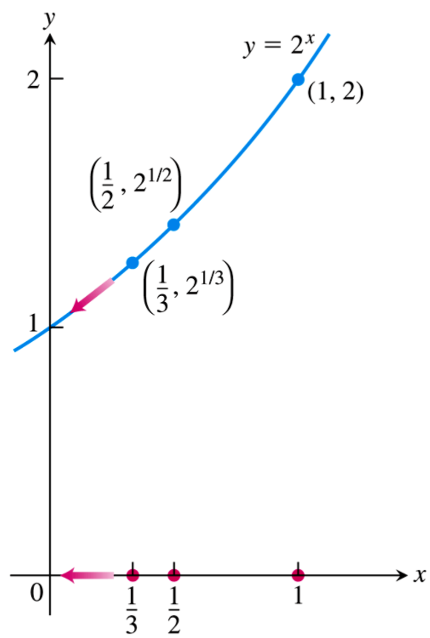
### *Example*

The sequence $\left\{\frac{1}{n}\right\}$ converges to 0.

By taking $a_n = \frac{1}{n}$, $f(x) = 2^x$, and $L = 0$.

We see that $2^{1/n} = f\left(\frac{1}{n}\right) \to f(L) = 2^0 = 1$.

The sequence $\left\{2^{1/n}\right\}$ converges to 1.



## *Using L'Hôpital's Rule*

### *Theorem*

Suppose that $f(x)$ is a function for all $x \ge n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \ge n_0$. Then

$$\lim_{x\to\infty} f(x) = L \quad \Rightarrow \quad \lim_{n\to\infty} a_n = L$$

### *Example*

Show that $\lim\limits_{n\to\infty} \frac{\ln n}{n} = 0$

***Solution***

The function $\frac{\ln x}{x}$ is defined for all $x \ge 1$ and agrees with the given sequence at positive integers.

Therefore;

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$

$$= 0$$

### *Example*

Does the sequence whose *n*th term is $a_n = \left(\frac{n+1}{n-1}\right)^n$ converge? If so, find $\lim\limits_{n\to\infty} a_n$

***Solution***

The limit leads to the indeterminate form $1^\infty$.

$$\ln a_n = \ln \left(\frac{n+1}{n-1}\right)^n$$

$$= n \ln \left(\frac{n+1}{n-1}\right) \qquad \infty.0 \textit{ form}$$

$$= \frac{\ln\left(\frac{n+1}{n-1}\right)}{\frac{1}{n}} \qquad 0.0 \textit{ form}$$

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{\frac{1}{n}} \qquad \left(\ln \frac{n+1}{n-1}\right)' = \frac{\frac{n-1-(n+1)}{(n-1)^2}}{\frac{n+1}{n-1}} = \frac{-2}{(n+1)(n-1)}$$

$$= \lim_{n\to\infty} \frac{\frac{-2}{n^2-1}}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{2n^2}{n^2-1}$$

$$\underline{= 2}$$

$$\boxed{\lim_{n\to\infty} a_n = e^2}$$

## *Theorem*

The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim_{n\to\infty} x^{1/n} = 1 \quad x > 0$

4. $\lim_{n\to\infty} x^n = 1 \quad |x| < 1$

5. $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x \quad (any\ x)$

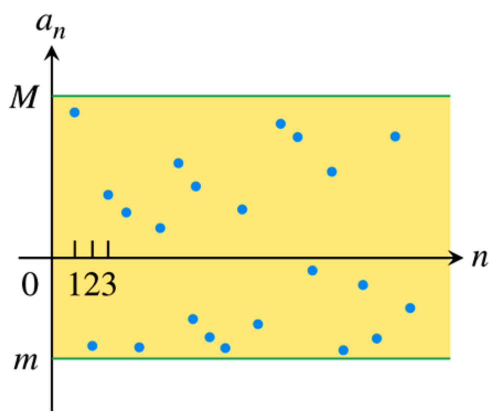
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad (any\ x)$

## Bounded *Monotonic* Sequences

### *Definitions*

A sequence $\{a_n\}$ is ***bounded from above*** if there exists a number $M$ such that $a_n \le M$ for all $n$. The number $M$ is an ***upper bound*** for $\{a_n\}$ but no number less than $M$ is an upper bound for $\{a_n\}$, then $M$ is the ***least upper bound*** for $\{a_n\}$.

A sequence $\{a_n\}$ is ***bounded from below*** if there exists a number $m$ such that $a_n \ge m$ for all $n$. The number $m$ is an ***lower bound*** for $\{a_n\}$. If $m$ is a lower bound for $\{a_n\}$ but no number greater than $m$ is a lower bound for $\{a_n\}$, then $m$ is the ***greatest lower bound*** for $\{a_n\}$.




If $\{a_n\}$ is bounded from above and below, the $\{a_n\}$ is ***bounded***.

If $\{a_n\}$ is not bounded, then $\{a_n\}$ is an ***unbounded*** sequence.

## *Definition*

A sequence $\{a_n\}$ is ***nondecreasing*** if $a_n \le a_{n+1}$ for all $n$. That is $a_1 \le a_2 \le a_3 \le \dots$

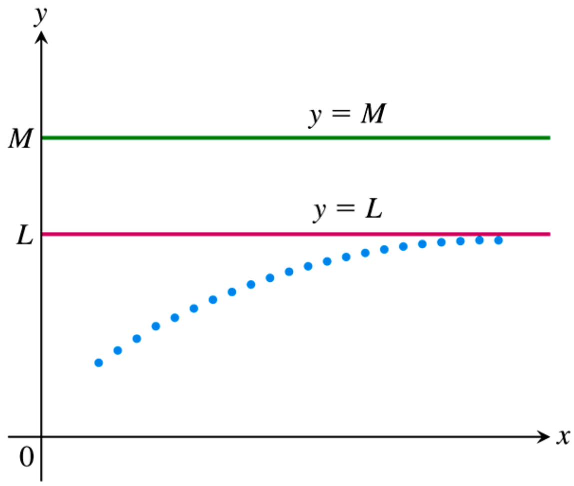
Which each term is greater than or equal to its predecessor $\left(a_{n+1} \ge a_n\right)$

***Example***: $\left\{1 - \frac{1}{n}\right\}_{n=1}^{\infty} = \left\{0,\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \dots\right\}$

A sequence $\{a_n\}$ is ***nonincreasing*** if $a_n \ge a_{n+1}$ for all $n$, which each term is less than or equal to its predecessor $\left(a_{n+1} \le a_n\right)$

***Example***: $\left\{1 + \frac{1}{n}\right\}_{n=1}^{\infty} = \left\{2,\ \frac{3}{2},\ \frac{4}{3},\ \frac{5}{4},\ \dots\right\}$

The sequence $\{a_n\}$ is ***monotonic*** if it is either nondecreasing or nonincreasing.




## *Theorem*

If a sequence $\{a_n\}$ is both *bounded* and *monotonic*, then the sequence converges.

## *Example*

The sequence $\{1, 2, 3, \dots, n, \dots\}$ is nondecreasing

The sequence $\left\{\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \dots,\ \frac{n}{n+1},\ \dots\right\}$ is nondecreasing

The sequence $\left\{1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \dots,\ \frac{1}{2^n},\ \dots\right\}$ is nonincreasing

# *Exercises* Section 3.1 – Sequences

**1.** Find the values of $a_1$, $a_2$, $a_3$, *and* $a_4$ for $a_n = \frac{1-n}{n^2}$

**2.** Find the values of $a_1$, $a_2$, $a_3$, *and* $a_4$ for $a_n = \frac{1}{n!}$

**3.** Find the values of $a_1$, $a_2$, $a_3$, *and* $a_4$ for $a_n = \frac{(-1)^{n+1}}{2n-1}$

**4.** Find the values of $a_1$, $a_2$, $a_3$, *and* $a_4$ for $a_n = 2+(-1)^n$

**5.** Find the values of $a_1$, $a_2$, $a_3$, *and* $a_4$ for $a_n = \frac{2^n-1}{2^n}$

**6.** Write the first ten terms of the sequence $a_1 = 1, \quad a_{n+1} = a_n + \frac{1}{2^n}$

**7.** Write the first ten terms of the sequence $a_1 = 1, \quad a_{n+1} = \frac{a_n}{n+1}$

**8.** Write the first ten terms of the sequence $a_1 = 2, \quad a_2 = -1, \quad a_{n+2} = \frac{a_{n+1}}{a_n}$

**9.** Find a formula for the *n*th term of the sequence $-1,\ 1,\ -1,\ 1,\ -1, \cdots$

**10.** Find a formula for the *n*th term of the sequence $1,\ -\frac{1}{4},\ \frac{1}{9},\ -\frac{1}{16},\ \frac{1}{25}, \cdots$

**11.** Find a formula for the *n*th term of the sequence $\frac{1}{9},\ \frac{2}{12},\ \frac{2^2}{15},\ \frac{2^3}{18},\ \frac{2^4}{21}, \cdots$

**12.** Find a formula for the *n*th term of the sequence $-3,\ -2,\ -1,\ 0,\ 1, \cdots$

**13.** Find a formula for the *n*th term of the sequence $\frac{1}{25},\ \frac{8}{125},\ \frac{27}{625},\ \frac{64}{3125},\ \frac{125}{15{,}625}, \cdots$

**14.** Find a formula for the *n*th term of the sequence $0,\ 1,\ 1,\ 2,\ 2,\ 3,\ 3,\ 4,\ \cdots$

**(15 – 43)** Determine if the sequence converge or diverge? Then find the limit of each convergent sequence.

**15.** $a_n = \frac{n+(-1)^n}{n}$

**16.** $a_n = \frac{1-2n}{1+2n}$

**17.** $a_n = \frac{1-n^3}{70-4n^2}$

**18.** $a_n = \left(2-\frac{1}{2^n}\right)\left(3+\frac{1}{2^n}\right)$

**19.** $a_n = n\pi\cos(n\pi)$

**20.** $a_n = n-\sqrt{n^2-n}$

**21.** $a_n = \sqrt{\frac{2n}{n+1}}$

**22.** $a_n = \frac{\sin^2 n}{2^n}$

**23.** $a_n = \frac{\ln n}{\ln 2n}$

**24.** $a_n = \frac{3^n \cdot 6^n}{2^{-n} \cdot n!}$

**25.** $a_n = \sqrt{n}\ \sin\frac{1}{\sqrt{n}}$

**26.** $a_n = \dfrac{n^2}{2^n - 1}$

**27.** $\{c_n\} = \left\{(-1)^n \dfrac{1}{n!}\right\}$

**28.** $a_n = \dfrac{5}{n+2}$

**29.** $a_n = 8 + \dfrac{5}{n}$

**30.** $a_n = (-1)^n \left(\dfrac{n}{n+1}\right)$

**31.** $a_n = \dfrac{1+(-1)^n}{n^2}$

**32.** $a_n = \dfrac{10n^2 + 3n + 7}{2n^2 - 6}$

**33.** $a_n = \dfrac{\sqrt[3]{n}}{\sqrt[3]{n}+1}$

**34.** $a_n = \dfrac{\ln\left(n^3\right)}{2n}$

**35.** $a_n = \dfrac{5^n}{3^n}$

**36.** $a_n = \dfrac{(n+1)!}{n!}$

**37.** $a_n = \dfrac{(n-2)!}{n!}$

**38.** $a_n = \dfrac{n^p}{e^n}, \quad p > 0$

**39.** $a_n = n \sin \dfrac{1}{n}$

**40.** $a_n = 2^{1/n}$

**41.** $a_n = -3^{-n}$

**42.** $a_n = \dfrac{\sin n}{n}$

**43.** $a_n = \dfrac{\cos \pi n}{n^2}$

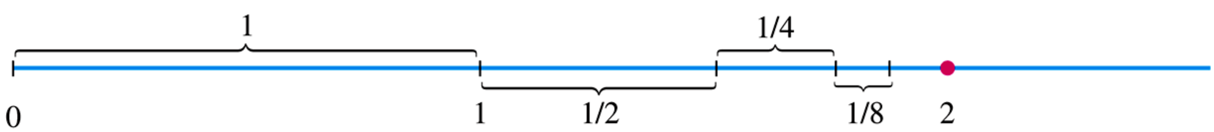
# *Section* 3.2 – Infinite Series

An ***infinite series*** is the sum of an infinite sequence of numbers

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

The sum of the first *n*th terms $\quad s_n = a_1 + a_2 + a_3 + \ldots + a_n$

| ***Partial Sum*** | | ***Value*** | ***Suggestive Expression For Partial Sum*** |
|---|---|---|---|
| *First* | $s_1 = 1$ | $1$ | $2 - 1$ |
| *Second* | $s_2 = 1 + \frac{1}{2}$ | $\frac{3}{2}$ | $2 - \frac{1}{2}$ |
| *Third* | $s_3 = 1 + \frac{1}{2} + \frac{1}{4}$ | $\frac{7}{4}$ | $2 - \frac{1}{4}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $n^{th}$ | $s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$ | $\frac{2^n - 1}{2^{n-1}}$ | $2 - \frac{1}{2^{n-1}}$ |




### *Definition*

Given a sequence of numbers $\{a_n\}$, an expression of the form $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ is an ***infinite series***. The number $a_n$ is the ***nth* term** of the series. The sequence $\{s_n\}$ is defined by

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$\vdots$$
$$s_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k$$
$$\vdots$$

Is the ***sequence of partial sums*** of the series, the number $\{s_n\}$ being the ***nth* partial sum**. If the sequence of partial sums converges to a limit *L*, we say that the series **converges** and that its **sum** is *L*.

In this case, we also write

$$a_1 + a_2 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n = L$$

If the sequence of partial sums of the series does not converge, we say that the series diverges.

## ***Geometric*** **Series**

Geometric series are series of the form

$$a + ar + ar^2 + \ldots + ar^{n-1} + \ldots = \sum_{n=1}^{\infty} ar^{n-1}$$

In which $a$ and $r$ are fixed real numbers, and $a \neq 0$.

The series can also be written as $\sum_{n=0}^{\infty} ar^n$

## Definition of *Geometric* Sequence

A sequence $a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots$ is a geometric sequence if $a_1 \neq 0$ and if there is a real number $r \neq 0$ such that for every positive integer $k$.

$$a_{k+1} = a_k r$$

The number $r = \dfrac{a_{k+1}}{a_k}$ is called the ***common ratio*** of the sequence.

***The formula for the n^th^ Term of a Geometric Sequence***:

$$\boxed{a_n = a_1 r^{n-1}}$$

## ***Theorem***: **Formula for** $S_n$

The $n$th partial sum $S_n$ of a geometric sequence with first term $a_1$ and common ratio $r \neq 1$ is

$$\boxed{S_n = a\frac{1-r^n}{1-r}}$$

### *Proof*

By definition, the $n^{th}$ partial sum $S_n$ of a geometric sequence is:

$$
\begin{aligned}
S_n &= a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} \\
-\quad rS_n &= ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n \\
\hline
S_n - rS_n &= a - ar^n
\end{aligned}
$$

$$(1-r)S_n = a\left(1-r^n\right)$$

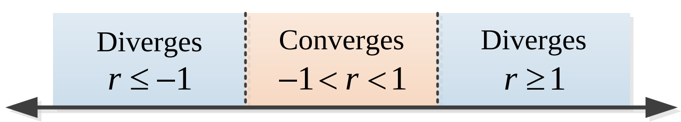
$$S_n = a\frac{1-r^n}{1-r}$$

### *Definition*

If $|r|<1$, the geometric series $a+ar+ar^2+\ldots+ar^{n-1}+\ldots$ converges to $\dfrac{a}{1-r}$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r|<1$$

If $|r|\geq 1$, the series diverges.




- The sequence may converge to a single value, which is the limit of the sequence.
- The sequence terms may increase in magnitude without bound (either with one sign or with mixed signs), in which case the sequence diverges.
- The sequence terms may remain bounded but settle into an oscillating pattern in which the terms approach two or more values, then the sequence diverges.
- The terms of a sequence may remain bounded, but wander chaotic forever without pattern, then the sequence diverges in this case.

### *Example*

Find the geometric series with $a = \frac{1}{9}$ *and* $r = \frac{1}{3}$

***Solution***

$$\frac{1}{9} + \frac{1}{9}\frac{1}{3} + \frac{1}{9}\left(\frac{1}{3}\right)^2 + \cdots = \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1}$$

$$= \frac{1/9}{1-1/3}$$

$$\boxed{= \frac{1}{6}}$$

### *Example*

The series $\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$ is geometric series with $a = 5$ *and* $r = -\frac{1}{4}$

***Solution***

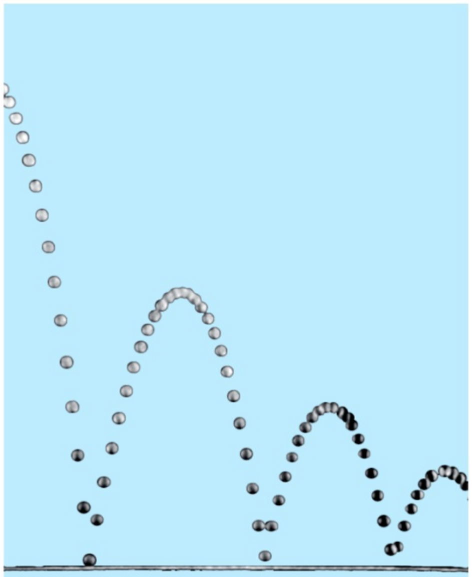
It converges to $\frac{a}{1-r} = \frac{5}{1-\left(-\frac{1}{4}\right)} = \frac{5}{\frac{5}{4}}$

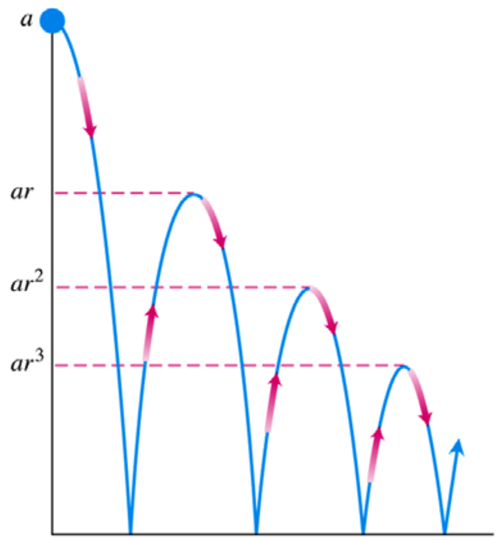
$$= 5 \cdot \frac{4}{5}$$

$$\boxed{= 4}$$

### *Example*

Your drop a ball from $a$ meters above a flat surface. Each time the ball hits the surface after falling a distance $h$, it rebounds a distance $rh$, where $r$ is positive but less than 1. Find the total distance the ball travels up and down. $\left(a = 6\ m \quad and \quad r = \frac{2}{3}\right)$






***Solution***

The total distance is

$$\begin{aligned} s &= a + 2ar + 2ar^2 + 2ar^3 + \dots \\ &= a + \frac{2ar}{1-r} \\ &= a\left(1 + \frac{2r}{1-r}\right) \\ &= a\frac{1+r}{1-r} \end{aligned}$$

If $a = 6\ m \quad and \quad r = \frac{2}{3}$, the distance is:

$$\begin{aligned} s &= 6\frac{1+\frac{2}{3}}{1-\frac{2}{3}} \\ &= 30\ \ m \end{aligned}$$

### *Example*

Express repeating decimal 5.232323… as the ratio of two integers.

***Solution***

$$5.232323\cdots = 5 + \frac{23}{100} + \frac{23}{(100)^2} + \frac{23}{(100)^3} + \cdots$$

$$= 5 + \frac{23}{100}\left(1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \cdots\right)$$

$$1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \cdots = \frac{1}{1 - \frac{1}{100}} = \frac{100}{99}$$

$$= 5 + \frac{23}{100}\left(\frac{100}{99}\right)$$

$$= 5 + \frac{23}{99}$$

$$= \frac{518}{99}$$

### *Example*

Find the sum of the "telescoping" series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

***Solution***

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$s_k = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \dots + \left(\cancel{\frac{1}{k}} - \frac{1}{k+1}\right)$$

$$= 1 - \frac{1}{k+1}$$

$$\lim_{k \to \infty} \frac{1}{k+1} = 0$$

$$s_k = 1 - 0 = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

## The *n*th-Term Test for a Divergent Series

### *Theorem*

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$

***The nth-Term Test for Divergence***

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

### *Example*

***a)*** The series $\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \cdots + \frac{n+1}{n} + \cdots$ diverges because each term is greater than 1, so the sum of the $n$ terms is greater than $n$. $\frac{n+1}{n} \to 1$

***b)*** $\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$

***c)*** $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty} (-1)^{n+1}$ doesn't exist.

***d)*** $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$

### *Theorem*

If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

***Sum Rule:*** $\sum (a_n + b_n) = \sum a_n + \sum b_n = A + B$

***Difference Rule:*** $\sum (a_n - b_n) = \sum a_n - \sum b_n = A - B$

***Constant Multiple Rule:*** $\sum ka_n = k\sum a_n = kA$

- Every nonzero constant multiple of a divergent series diverges.
- If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ both diverge.

### *Example*

Find the sums of the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}}$

***Solution***

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} &= \sum_{n=1}^{\infty}\left(\frac{3^{n-1}}{6^{n-1}} - \frac{1}{6^{n-1}}\right)\\ &= \sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}}\right)\\ &= \sum_{n=1}^{\infty}\frac{1}{2^{n-1}} - \sum_{n=1}^{\infty}\frac{1}{6^{n-1}}\\ &= \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{6}}\\ &= 2 - \frac{6}{5}\\ &= \frac{4}{5}\end{aligned}$$

### *Example*

Find the sums of the series $\displaystyle\sum_{n=0}^{\infty} \frac{4}{2^n}$

***Solution***

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{4}{2^n} = 4\sum_{n=0}^{\infty} \frac{1}{2^n} &= 4\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n\\ &= 4\frac{1}{1-\frac{1}{2}}\\ &= 4(2)\\ &= 8\end{aligned}$$

## Adding or Deleting Terms

We can add finite number of terms to a series or delete a finite number of terms without altering the series' convergence or divergence.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_{k-1} + \sum_{n=k}^{\infty} a_n$$

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \sum_{n=4}^{\infty} \frac{1}{5^n}$$

$$\sum_{n=4}^{\infty} \frac{1}{5^n} = \left( \sum_{n=1}^{\infty} \frac{1}{5^n} \right) - \frac{1}{5} - \frac{1}{25} - \frac{1}{125}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=5}^{\infty} \frac{1}{2^{n-5}} = \sum_{n=-4}^{\infty} \frac{1}{2^{n+4}}$$

# *Exercises* Section 3.2 – Infinite Series

**(1 – 19)** Find the limit of the following sequences or determine the limit does not exist

**1.** $a_n = \dfrac{n^3}{n^4+1}$

**2.** $a_n = n^{1/n}$

**3.** $\left\{\dfrac{n^{12}}{3n^{12}+4}\right\}$

**4.** $\left\{\dfrac{2e^{n+1}}{e^n}\right\}$

**5.** $\left\{\dfrac{\tan^{-1} n}{n}\right\}$

**6.** $\left\{\left(1+\frac{2}{n}\right)^n\right\}$

**7.** $\left\{\left(\dfrac{n}{n+5}\right)^n\right\}$

**8.** $\left\{\dfrac{\ln\left(\frac{1}{n}\right)}{n}\right\}$

**9.** $\{\ln\sin(1/n)+\ln n\}$

**10.** $a_n = \dfrac{n!}{n^n}$

**11.** $a_n = \dfrac{n^2+4}{\sqrt{4n^4+1}}$

**12.** $a_n = \dfrac{8^n}{n!}$

**13.** $a_n = \left(1+\frac{3}{n}\right)^{2n}$

**14.** $a_n = \sqrt[n]{n}$

**15.** $a_n = n-\sqrt{n^2-1}$

**16.** $a_n = \left(\frac{1}{n}\right)^{1/\ln n}$

**17.** $a_n = \sin\frac{\pi n}{6}$

**18.** $a_n = \dfrac{(-1)^n}{0.9^n}$

**19.** $a_n = \tan^{-1} n$

**(20 – 24)** Find a formula for the *n*th term partial sum of the series and use it to find the series' sum if the series converges

**20.** $2+\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\cdots+\frac{2}{3^{n-1}}+\cdots$

**21.** $\frac{9}{100}+\frac{9}{100^2}+\frac{9}{100^3}+\cdots+\frac{9}{100^n}+\cdots$

**22.** $1-2+4-8+\cdots+(-1)^{n-1}2^{n-1}+\cdots$

**23.** $\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{(n+1)(n+2)}+\cdots$

**24.** $\frac{5}{1\cdot 2}+\frac{5}{2\cdot 3}+\frac{5}{3\cdot 4}+\cdots+\frac{5}{n(n+1)}+\cdots$

**(25 – 28)** Write out the first few terms of each series to show how the series starts. Then find the sum of the series

**25.** $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}$

**26.** $\displaystyle\sum_{n=2}^{\infty}\frac{1}{4^n}$

**27.** $\displaystyle\sum_{n=0}^{\infty}\left(\frac{5}{2^n}-\frac{1}{3^n}\right)$

**28.** $\displaystyle\sum_{n=0}^{\infty}\frac{2^{n+1}}{5^n}$

(**29 – 31**) Determine if the geometric series converges or diverges. If a series converges, find its sum

**29.** $1+\left(\frac{2}{5}\right)+\left(\frac{2}{5}\right)^2+\left(\frac{2}{5}\right)^3+\left(\frac{2}{5}\right)^4+\cdots$

**30.** $1+(-3)+(-3)^2+(-3)^3+(-3)^4+\cdots$

**31.** $\left(-\frac{2}{3}\right)^2+\left(-\frac{2}{3}\right)^3+\left(-\frac{2}{3}\right)^4+\left(-\frac{2}{3}\right)^5+\cdots$

(**32 – 35**) Express each of the numbers as the ratio of two integers (fraction)

**32.** $0.\overline{23}=0.23\ 23\ 23\cdots$

**33.** $0.\overline{234}=0.234\ 234\ 234\cdots$

**34.** $1.\overline{414}=1.414\ 414\ 414\cdots$

**35.** $1.24\overline{123}=1.24\ 123\ 123\ 123\cdots$

(**36 – 43**) Use Divergence to determine if the series converges or diverges.

**36.** $\sum_{k=0}^{\infty}\frac{k}{2k+1}$

**37.** $\sum_{k=1}^{\infty}\frac{k}{k^2+1}$

**38.** $\sum_{k=2}^{\infty}\frac{k}{\ln k}$

**39.** $\sum_{k=1}^{\infty}\frac{k^2}{2^k}$

**40.** $\sum_{k=0}^{\infty}\frac{1}{k+100}$

**41.** $\sum_{k=0}^{\infty}\frac{k^3}{k^3+1}$

**42.** $\sum_{k=2}^{\infty}\frac{\sqrt{k}}{\ln^{10}k}$

**43.** $\sum_{k=1}^{\infty}\frac{\sqrt{k^2+1}}{k}$

(**44 – 65**) Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum.

**44.** $\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n$

**45.** $\sum_{n=1}^{\infty}(-1)^{n+1}n$

**46.** $\sum_{n=0}^{\infty}\frac{\cos n\pi}{5^n}$

**47.** $\sum_{n=0}^{\infty}e^{-2n}$

**48.** $\sum_{n=1}^{\infty}\ln\frac{1}{3^n}$

**49.** $\sum_{n=1}^{\infty}\frac{n^n}{n!}$

**50.** $\sum_{n=1}^{\infty}\frac{2^n+3^n}{4^n}$

**51.** $\sum_{n=0}^{\infty}\frac{e^{\pi n}}{\pi^{ne}}$

**52.** $\sum_{n=0}^{\infty}(1.075)^n$

53. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{1000}$

54. $\displaystyle\sum_{n=1}^{\infty} \frac{n+10}{10n+1}$

55. $\displaystyle\sum_{n=1}^{\infty} \frac{4n+1}{3n-1}$

56. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$

57. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

58. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

59. $\displaystyle\sum_{n=0}^{\infty} \frac{3}{5^n}$

60. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\ln n}$

61. $\displaystyle\sum_{n=1}^{\infty} \ln \frac{1}{n}$

62. $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{k}{n}\right)^n$

63. $\displaystyle\sum_{n=1}^{\infty} e^{-n}$

64. $\displaystyle\sum_{n=1}^{\infty} \arctan n$

65. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

(**66 – 71**) Determine if the series converges or diverges and describe whether they do so monotonically or by oscillation. Give the limit when the sequence converges.

66. $a_n = 0.3^n$

67. $a_n = 1.3^n$

68. $a_n = (-0.6)^n$

69. $a_n = (-1.01)^n$

70. $a_n = 2^n 3^{-n}$

71. $a_n = (-0.003)^n$

(**72 – 84**) Find the limit of the following sequences or state that they diverge

72. $a_n = \dfrac{\sin n}{2^n}$

73. $a_n = \dfrac{\cos\left(\frac{n\pi}{2}\right)}{\sqrt{n}}$

74. $a_n = \dfrac{2\tan^{-1} n}{n^3+4}$

75. $a_n = \dfrac{n\sin^3 n}{n+1}$

76. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{9}{10}\right)^k$

77. $\displaystyle\sum_{k=1}^{\infty} 3(1.001)^k$

78. $\displaystyle\sum_{k=0}^{\infty} \left(-\frac{1}{5}\right)^k$

79. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

80. $\displaystyle\sum_{k=2}^{\infty} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k-1}}\right)$

81. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{3}{3k-2} - \frac{3}{3k+1}\right)$

82. $\displaystyle\sum_{k=1}^{\infty} 4^{-3k}$

83. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{3^{k+2}}$

84. $\displaystyle\sum_{k=0}^{\infty} \left(\left(\frac{1}{3}\right)^k - \left(\frac{2}{3}\right)^{k+1}\right)$

**85.** Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+2)} = \frac{1}{2}\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+2}\right)$

*a)* Write the first four terms of the sequence of partial sums $S_1$, $S_2$, $S_3$, $S_4$.

*b)* Write the nth term of the sequence of partial sums $S_n$.

*c)* Find $\lim\limits_{n\to\infty} S_n$ and evaluate the series.

**86.** Many people take aspirin on a regular basis as a preventive measure for heart disease. Suppose a person take 80 *mg* of aspirin every 24 *hr*. Assume also that aspirin has a half-life of 24 *hrs*.; that is, every 24 *hr*. half of the drug in the blood is eliminated.

*a)* Find a recurrence relation for the sequence $\{d_n\}$ that gives the amount of drug in the blood after the $n^{th}$ dose, where $d_1 = 80$

*b)* Find the limit of $\{d_n\}$

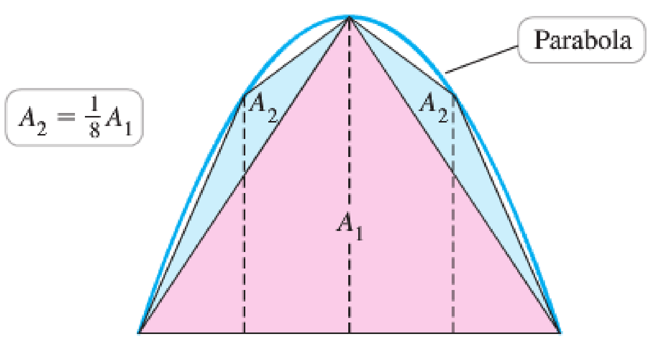
**87.** Suppose a tank is filled with 100 *L* of a 40% alcohol solution (by volume). You repeatedly perform the following operation: Remove 2 *L* of the solution from the tank and replace them with 2 *L* of 10% alcohol solution

*a)* Let $C_n$ be the concentration of the solution in the tank after the nth replacement, where $C_0 = 40\%$. Write the first five terms of the sequence $\{C_n\}$

*b)* After how many replacements does the alcohol concentration reach 15%?

*c)* Determine the limiting (steady-state) concentration of the solution that is approached after many replacements.

**88.** The Greeks solved several calculus problems almost 2000 *years* before the discovery of calculus. One example is Archimedes' calculation of the area of the region *R* bounded by a segment of a parabola, which he did using the "method of exhaustion".




The idea was to fill *R* with an infinite sequence of triangles. Archimedes began with an isosceles triangle inscribed in the parabola, with an area $A_1$, and proceeded in stages, with the number of new triangles doubling at each stage. He was able to show (the key to the solution) that at each stage, the area of a new triangle is $\frac{1}{8}$ of the area of a triangle at the previous stage; for example, $A_2 = \frac{1}{8} A_1$, and so forth. Show, as Archimedes did, that the area of $R$ is $\frac{4}{3}$ times the area of $A_1$.

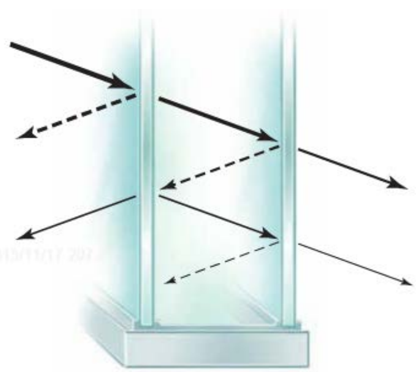
**89.**

*a)* Evalute the series $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{\left(3^{k+1}-1\right)\left(3^k-1\right)}$

*b)* For what values of a does the the series converge, and in those cases, what is its value?

$$\sum_{k=1}^{\infty} \frac{a^k}{\left(a^{k+1}-1\right)\left(a^k-1\right)}$$

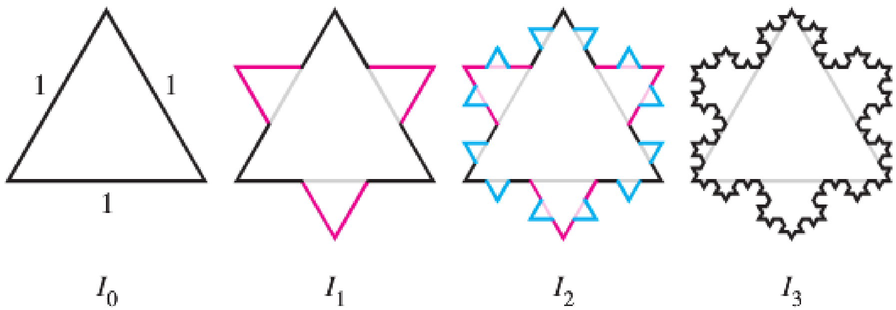
**90.** An insulated windows consists of two parallel panes of glass with a small spacing between them. Suppose that each pane reflects a fraction $p$ of the incoming light and transmits the remaining light. Considering all reflections of light between the panes, what fraction of the incoming light is ultimately transmitted by the windows? Assume the amount of incoming light is 1.



**91.** Suppose you borrow \$20,000 for a new car at a monthly interest rate of 0.75%. If you make payments of \$600/month, after how many months will the loan balance be zero? Estimate the answer by graphing the sequence of loan balances and then obtain an exact answer using infinite series.

**92.** Suppose a rubber ball, when dropped from a given height, returns to a fraction $p$ of that height. In the absence of air resistance, a ball dropped from a height $h$ requires $\sqrt{\frac{2h}{g}}$ seconds to fall to the ground, where $g \approx 9.8\ m/s^2$ is the acceleration due to gravity. The time taken to bounce up to a given to fall from that height to the ground. How long does it take a ball dropped from 10 $m$ to come to rest?

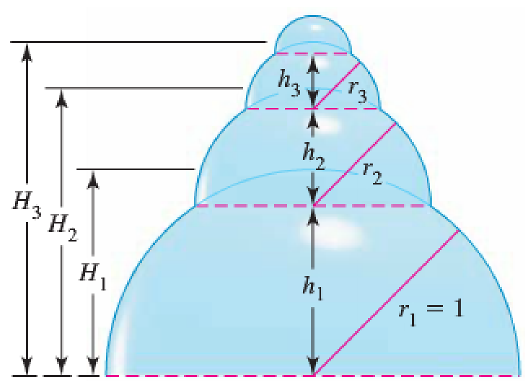
**93.** The fractal called the snowflake island (or Koch island) is constructed as flows: Let $I_0$ be an equilateral triangle with sides of length 1. The figure $I_1$ is obtained by replacing the middle third of each side of $I_0$ with a new outward equilateral triangle with sides of length $\frac{1}{3}$. The process is repeated where $I_{n+1}$ is obtained by replacing the middle third of each side of $I_n$ with a new outward equilateral triangle with sides of length of $\frac{1}{3^{n+1}}$. The limiting figure as $n \to \infty$ is called the snowflake island.




*a)* Let $L_n$ be the perimeter of $I_n$. Show that $\lim\limits_{n\to\infty} L_n = \infty$

*b)* Let $A_n$ be the area of $I_n$. Find $\lim\limits_{n\to\infty} A_n$. It exists!

**94.** Imagine a stack of hemispherical soap bubbles with decreasing radii $r_1 = 1,\ r_2,\ r_3, \ldots$ Let $h_n$ be the distance between the diameters of bubble *n* and bubble $n+1$, and let $H_n$ be the total height of the stack with *n* bubbles.




*a)* Use the Pythagorean theorem to show that in a stack with *n* bubbles $h_1^2 = r_1^2 - r_2^2,\ h_2^2 = r_2^2 - r_3^2,$ and so forth. Note that for the last bubble $h_n = r_n$.

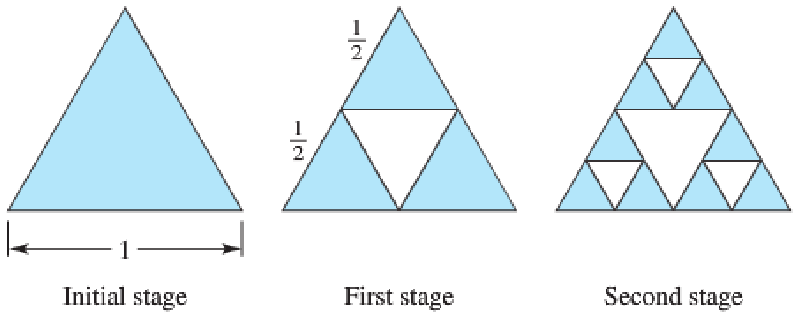
*b)* Use part (*a*) to show that the height of a stack with *n* bubbles is

$$H_n = \sqrt{r_1^2 - r_2^2} + \sqrt{r_2^2 - r_3^2} + \cdots + \sqrt{r_{n-1}^2 - r_n^2} + r_n$$

*c)* The height of a stack of bubbles depends on how the radii decrease. Suppose that $r_1 = 1,\ r_2 = a,\ r_3 = a^2,\ \ldots,\ r_n = a^{n-1}$ where $0 < a < 1$ is a fixed real number. In terms of *a*, find the height $H_n$ of a stack with *n* bubbles.

*d)* Suppose the stack in part (*c*) is extended indefinitely $(n \to \infty)$. In terms of *a*, how high would the stack be?

**95.** The fractal called the *Sierpinski triangle* is the limit of a sequence of figures. Starting with the equilateral triangle with sides of length 1, an inverted equilateral triangle with sides of length $\frac{1}{2}$ is removed. Then, the three inverted equilateral triangles with sides of length $\frac{1}{4}$ are removed from this figure.




The process continues in this way. Let $T_n$ be the total area of the removed triangles after stage *n* of the process. The area on equilateral triangle with side length *L* is $A = \frac{\sqrt{3}}{4} L^2$.

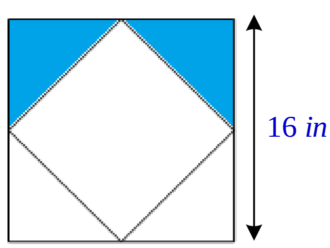
*a)* Find $T_1$ and $T_2$ the total area of the removed triangles after stages 1 and 2, respectively.

*b)* Find $T_n$ for $n = 1,\ 2,\ 3,\ \ldots$

*c)* Find $\lim\limits_{n \to \infty} T_n$

*d)* What is the area of the original triangle that remains as $n \to \infty$?

**96.** The sides of a ***square*** are 16 *inches* in length. A new square is formed by connecting the midpoints of the sides of the original square, and two of the triangles outside the second square are shaded.

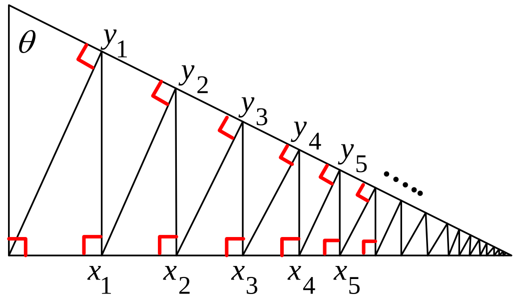



Determine the area of the shaded regions

*a)* When this process is continued five more times

*b)* When this pattern of shading is continued infinitely.

**97.** A right triangle $XYZ$ is shown below where $|XY| = z$ and $\angle X = \theta$. Line segments are continually drawn to be perpendicular to the triangle.

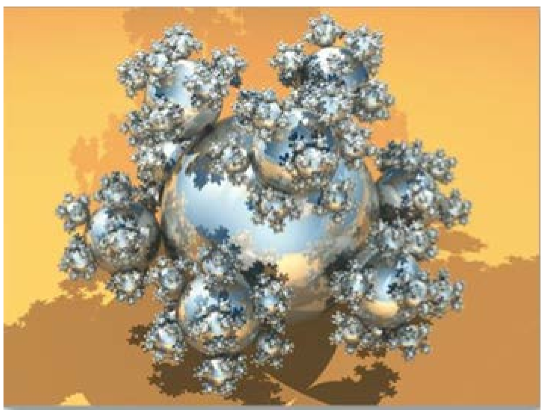



*a)* Find the total length of the perpendicular line segments $|Yy_1| + |x_1y_1| + |x_1y_2| + \cdots$ in terms of $z$ and $\theta$.

*b)* Find the total length of the perpendicular line segments when $z = 1$ and $\theta = \dfrac{\pi}{6}$

**98.** The sphereflake is a computer–generated fractal that was created by Eric Haines. The radius of the large sphere is 1. To the large sphere, nine spheres of radius $\dfrac{1}{3}$ are attached. To each of these, nine spheres of radius $\dfrac{1}{9}$ are attached. This process is continued infinitely.

Prove that the sphereflake has an infinite surface area.



# *Section* 3.3 – Integral Test

## Nondecreasing Partial Sums

Suppose that $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \ge 0$ for all $n$. Then each partial sum is greater than or equal to its predecessor because $s_{n+1} = s_n + a_n$:

$$s_1 \le s_2 \le s_3 \le \cdots \le s_n \le s_{n+1} \le \cdots$$

### *Corollary*

A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

### *Example*

The series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$

***Solution***

$$1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{>\frac{2}{4}=\frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{>\frac{4}{8}=\frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right)}_{>\frac{8}{16}=\frac{1}{2}} + \cdots$$

The sum of the first 2 terms is $\frac{3}{2}$.

The sum of the next 2 terms is $\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

The sum of the next 4 terms is $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$

$\vdots \qquad \vdots \qquad \vdots$

The sum of $2^n$ terms ending with $2^{n+1}$ is $> \frac{2^n}{2^{n+1}} = \frac{1}{2}$

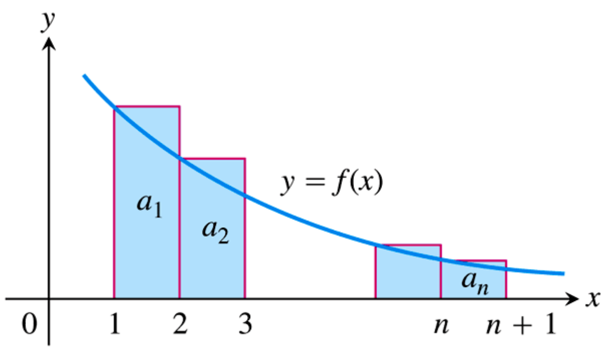
The sequence of partial sums is not bounded from above: If $2^k$, the partial sum $s_n > \frac{k}{2}$.

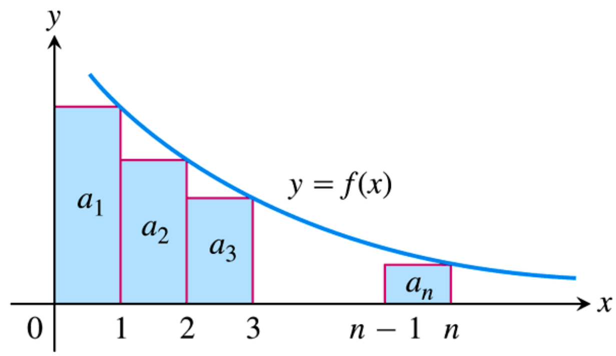
The harmonic series diverges.

## The Integral Test

### *Theorem*

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where $f$ is a continuous, positive, decreasing function of $x$ for all $x \ge N$ ($N$ a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)dx$ both converge or both diverge.




### *Example*

Does the following series converge? $\sum_{n=1}^{\infty} \frac{1}{n^2}$

***Solution***

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$$

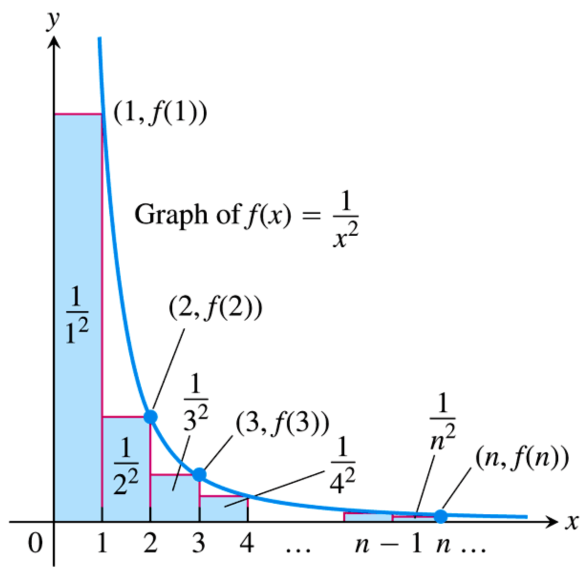
$$S_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2}$$

$$= f(1) + f(2) + f(3) + \cdots + f(n)$$

$$< 1 + \int_1^n \frac{1}{x^2} dx$$

$$< 1 + \lim_{b \to \infty} \left( -\frac{1}{x} \right) \Bigg|_1^b$$

$$= 1 - \left( \frac{1}{\infty} - 1 \right)$$



$$\underline{= 2}$$

Thus, the partial sums are bounded from above by 2 and the series converges.

$$\sum_{n-N}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\approx 1.64493$$

## *p*-series

### Example

Show that the ***p****-series* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$

($p$ a real constant) converges if $p > 1$, and diverges if $p \le 1$.

***Solution***

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \left( \frac{x^{-p+1}}{-p+1} \Bigg|_1^b \right.$$

$$= \frac{1}{1-p} \lim_{b\to\infty} \left[ b^{-p+1} - 1 \right]$$

$$= \frac{1}{1-p} \lim_{b\to\infty} \left[ \frac{1}{b^{p-1}} - 1 \right] \qquad \boldsymbol{p > 1}$$

$$= \frac{1}{1-p} (0 - 1)$$

$$= \frac{1}{p-1}$$

The series ***converges*** when $p > 1$.

if $p \le 1$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{1}{1-p} \lim_{b\to\infty} \left( b^{1-p} - 1 \right)$$

$$= \infty$$

The series ***diverges***.

### *Example*

Does the following series converge $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

**Solution**

$$f(x) = \frac{1}{x^2+1} \quad \rightarrow \quad \int_1^{\infty} \frac{1}{x^2+1} dx = \lim_{b \to \infty} \left[\arctan x\right]_1^b$$

$$= \lim_{b \to \infty} \left(\arctan b - \arctan 1\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

The series ***converges***, but we do not know the value of its sum

## Bounds for the Remainder in the Integral Test

Suppose $\{a_n\}$ is a sequence of positive terms with $a_k = f(k)$, where $f$ is a continuous positive decreasing function of $x$ for all $x \ge n$, and that $\sum a_n$ converges to $S$. Then the remainder $R_n = S - s_n$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$$

### *Example*

Estimate the sum of the series $\sum \frac{1}{n^2}$ with $n = 10$

***Solution***

$$\int_n^{\infty} \frac{1}{x^2}dx = \lim_{b\to\infty} \left(-\frac{1}{x}\right|_n^b$$

$$= -\lim_{b\to\infty}\left(\frac{1}{b} - \frac{1}{n}\right)$$

$$\boxed{= \frac{1}{n}}$$

$$s_{10} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{100} \approx 1.5497677$$

$$s_{10} + \frac{1}{11} \le S \le s_{10} + \frac{1}{10}$$

$$1.5497677 + \frac{1}{11} \le S \le 1.5497677 + \frac{1}{10}$$

$$1.64067679 \le S \le 1.6497677$$

If we approximate the sum $S$ by the midpoint of this interval, then

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \approx 1.645222$$

The error in this approximation is less than half the length of the interval, so the error is less than 0.005.

# ***Exercises*** *Section* 3.3 – Integral Test

(**1 – 22**) Use the ***Integral Test*** to determine if the series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{0.2}}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

3. $\displaystyle\sum_{n=1}^{\infty} e^{-2n}$

4. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n-4}{n^2-2n+1}$

7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

8. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{\sqrt{k^2+4}}$

9. $\displaystyle\sum_{k=1}^{\infty} ke^{-2k^2}$

10. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{\sqrt{k+8}}$

11. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)\ln(\ln k)}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/4}}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$

18. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{e^n}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$

20. $\displaystyle\sum_{k=1}^{\infty} \frac{|\sin k|}{k^2}$

21. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{\left(k^2+1\right)^3}$

22. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k+10}}$

(**23 – 28**) Use the ***p-series Test*** to determine if the series converge or diverge.

23. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^9}$

24. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^6}$

25. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{1/9}}$

26. $\displaystyle\sum_{k=1}^{\infty} k^{-2}$

27. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k}}$

28. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{16k^2}}$

(**29 – 54**) Determine if the series converge or diverge

**29.** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^8}$

**30.** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3^k}$

**31.** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{5/2}}$

**32.** $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

**33.** $\displaystyle\sum_{n=0}^{\infty} \frac{10}{n^2+9}$

**34.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n+1)(3n+4)}$

**35.** $\displaystyle\sum_{n=1}^{\infty} e^{-n}$

**36.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

**37.** $\displaystyle\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$

**38.** $\displaystyle\sum_{n=1}^{\infty} \frac{-8}{n}$

**39.** $\displaystyle\sum_{n=2}^{\infty} \frac{\ln}{n}$

**40.** $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{4^n+3}$

**41.** $\displaystyle\sum_{n=1}^{\infty} n\sin\frac{1}{n}$

**42.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{10}}$

**43.** $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(n-2)^4}$

**44.** $\displaystyle\sum_{n=2}^{\infty} \frac{n^e}{n^\pi}$

**45.** $\displaystyle\sum_{k=1}^{\infty} \frac{2^k+3^k}{4^k}$

**46.** $\displaystyle\sum_{n=1}^{\infty} 2n^{-3/2}$

**47.** $\displaystyle\sum_{k=0}^{\infty} \frac{k}{\sqrt{k^2+4}}$

**48.** $\displaystyle\sum_{k=2}^{\infty} \frac{4}{k\ln^2 k}$

**49.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$

**50.** $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$

**51.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$

**52.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^\pi}$

**53.** $1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\frac{1}{4\sqrt{4}}+\frac{1}{5\sqrt{5}}+\cdots$

**54.** $1+\frac{1}{\sqrt[3]{4}}+\frac{1}{\sqrt[3]{9}}+\frac{1}{\sqrt[3]{16}}+\frac{1}{\sqrt[3]{25}}+\cdots$

**55.** Consider the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$, where $p$ is a real number.

*a)* Use the Integral Test to determine the values of $p$ for which this series converges.

*b)* Does this series converge faster for $p=2$ or $p=3$? Explain.

**56.** Consider the series $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)\left(\ln(\ln k)\right)^{p}}$, where $p$ is a real number.

*a)* For what values of $p$ does this series converge?

*b)* Which he following series converge faster? Explain.

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2} \quad \text{or} \quad \sum_{k=2}^{\infty} \frac{1}{k(\ln k)\left(\ln(\ln k)\right)^2}$$

**57.** Consider a wedding cake of infinite height, each layer of which is a right circular cylinder of height 1. The bottom layer of the cake has a radius of 1, the second layer has a radius of $\frac{1}{2}$, the third layer has a radius of $\frac{1}{3}$, and the $n^{th}$ layer has a radius of $\frac{1}{n}$.

*a)* To determine how much frosting is needed to cover the cake, find the area of the lateral (vertical sides of the wedding cake. What is the area of the horizontal surfaces of the cake?

*b)* Determine the volume of the cake.

*c)* Comment on your answer to parts (*a*) and (*b*)

**58.** The Riemann zeta function is the subject of extensive research and is associated with several renowned unsolved problems. Is its defined by $\zeta(x) = \sum_{k=1}^{\neq} \frac{1}{k^x}$, when $x$ is a real number, the zeta function becomes a $p$-series. For even positive integers $\rho$, the value of $\zeta(\rho)$ is known exactly. For example,

$$\sum_{k=1}^{\neq} \frac{1}{k^2} = \frac{\pi^2}{6}, \quad \sum_{k=1}^{\neq} \frac{1}{k^4} = \frac{\pi^4}{90}, \quad \text{and} \quad \sum_{k=1}^{\neq} \frac{1}{k^6} = \frac{\pi^6}{945}, \quad \ldots$$

*a)* Use the estimation techniques to approximate $\zeta(3)$ and $\zeta(5)$ (whose values are not known exactly) with a remainder less than $10^{-3}$.

*b)* Determine the sum of the reciprocals of the squares of the odd positive integers by rearranging the terms of the series $(x=2)$ without changing the value of the series.

**59.** Consider a set of identical dominoes that are 2 *inches* long. The dominoes are stacked on top of each other with their long edges aligned so that each domino overhangs the one beneath is as far as possible

*a)* If there are $n$ dominoes in the stack, what is the greatest distance that the top domino can be made to overhang the bottom domino? (*Hint*: Put the $n^{th}$ domino beneath the previous $n-1$ dominoes.)

*b)* If we allow for infinitely many dominoes in the stack, what is the greatest distance that the top domino can be made to overhang the bottom domino?

**60.** A theorem states that the sequence of prime numbers $\{p_k\}$ satisfies $\lim\limits_{k\to\infty} \dfrac{p_k}{k\ln k}$.

Show that $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\ln k}$ diverges, which implies that the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{p_k}$

(A prime number is a positive integer number that is divisible only by 1 and itself).

# *Section* 3.4 – Comparison Tests

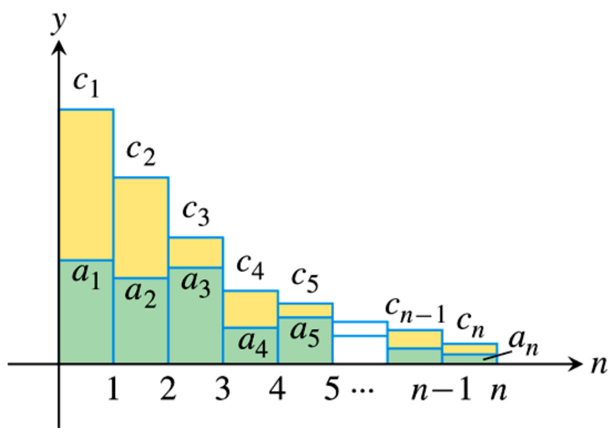
### *Theorem*

Let $\sum a_n$, $\sum c_n$, and $\sum d_n$ be series with nonnegative terms. Suppose that for some integer $N$.

$$d_n \le a_n \le c_n \quad \text{for all} \quad n > N$$

***a)*** If $\sum c_n$ converges, then $\sum a_n$ also converges.

***b)*** If $\sum d_n$ diverges, then $\sum a_n$ also diverges.




### *Example*

Use the comparison Test to determine if $\displaystyle\sum_{n=1}^{\infty} \frac{5}{5n-1}$ converges or diverges.

***Solution***

$$\frac{5}{5n-1} = \frac{1}{n-\frac{1}{5}}$$

$$> \frac{1}{n}$$

The series ***diverges*** because its $n$th term is greater than the $n$th term of the divergent harmonic series.

### *Example*

Use the comparison Test to determine if $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges or diverges.

***Solution***

$$\sum_{n=0}^{\infty} \frac{1}{n!} < 1 + \sum_{n=0}^{\infty} \frac{1}{2^n}$$

$$= 1 + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ is a geometric series $|r| = \frac{1}{2} < 1$

$$1 + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{1 - \frac{1}{2}}$$

$$\underline{= 3}$$

The series ***converges***.

## Limit Comparison Test

### *Theorem*

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \ge N$ ($N$ an integer)

**1.** If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge

**2.** If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges

**3.** If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges

### *Example*

Does the series $\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots$ converge or diverge?

***Solution***

$$\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots = \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$$

$$= \sum_{n=1}^{\infty} \frac{2n+1}{n^2 + 2n + 1}$$

Let $a_n = \frac{2n+1}{n^2+2n+1} \rightarrow \frac{2n}{n^2} = \frac{2}{n}$

$\frac{2}{n} > b_n = \frac{1}{n}$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ *diverges*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n+1}{n^2+2n+1} \cdot \frac{n}{1}$$

$$= 2$$

By the limit Comparison test $\sum a_n$ diverges

### *Example*

Does the series $\frac{1}{1}+\frac{1}{3}+\frac{1}{7}+\frac{1}{15}+\cdots=\sum_{n=1}^{\infty}\frac{1}{2^n-1}$ converge or diverge?

***Solution***

Let $a_n=\frac{1}{2^n-1} \quad\rightarrow\quad b_n=\frac{1}{2^n}$

Since $\sum_{n=1}^{\infty} b_n=\sum_{n=1}^{\infty}\frac{1}{2^n}$ *converges*

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{2^n}{2^n-1}$$

$$=\lim_{n\to\infty}\frac{1}{1-\frac{1}{2^n}}$$

$$=1$$

$\Rightarrow \sum a_n$ converges by the Limit Comparison Test.

### *Example*

Does the series $\frac{1+2\ln 2}{9}+\frac{1+3\ln 3}{14}+\frac{1+4\ln 4}{21}+\cdots=\sum_{n=2}^{\infty}\frac{1+n\ln n}{n^2+5}$ converge or diverge?

***Solution***

Let $a_n=\frac{1+n\ln n}{n^2+5} \quad\rightarrow\quad b_n=\frac{n\ln n}{n^2}=\frac{\ln n}{n}>\frac{1}{n}$

$\sum_{n=1}^{\infty} b_n=\sum_{n=1}^{\infty}\frac{1}{n}$ *diverges*

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{1+n\ln n}{n^2+5}\cdot\frac{n}{1}$$

$$=\lim_{n\to\infty}\frac{n+n^2\ln n}{n^2+5}$$

$$=\infty$$

$\Rightarrow \sum a_n$ diverges by the Limit Comparison Test.

## ***Example***

Does the series $\sum_{n=2}^{\infty} \frac{\ln n}{n^{3/2}}$ converge?

***Solution***

Let $a_n = \frac{\ln n}{n^{3/2}} < \frac{n^{1/4}}{n^{3/2}}$

$$= \frac{1}{n^{5/4}} = b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n^{3/2}} \cdot \frac{n^{5/4}}{1}$$

$$= \lim_{n\to\infty} \frac{\ln n}{n^{1/4}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{4} n^{-3/4}} \qquad \textit{L'hôpital Rule}$$

$$= \lim_{n\to\infty} \frac{4}{n^{1/4}}$$

$$\underline{= 0}$$

$\Rightarrow \sum a_n$ converges by the Limit Comparison Test.

# *Exercises* *Section* 3.4 – Comparison Tests

Use the Comparison Test to determine if the series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+30}$

2. $\displaystyle\sum_{n=1}^{\infty}\frac{n-1}{n^4+2}$

3. $\displaystyle\sum_{n=2}^{\infty}\frac{n+2}{n^2-n}$

4. $\displaystyle\sum_{n=1}^{\infty}\frac{\cos^2 n}{n^{3/2}}$

5. $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}+1}{\sqrt{n^2+3}}$

6. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n+1}$

7. $\displaystyle\sum_{n=1}^{\infty}\frac{3n+1}{n^3+1}$

8. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\ln n}$

9. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n-1}$

10. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3n^2+2}$

11. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}-1}$

12. $\displaystyle\sum_{n=0}^{\infty}\frac{4^n}{5^n+3}$

13. $\displaystyle\sum_{n=2}^{\infty}\frac{\ln n}{n+1}$

14. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3+1}}$

15. $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}$

16. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{4\sqrt[3]{n}-1}$

17. $\displaystyle\sum_{n=0}^{\infty}e^{-n^2}$

18. $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{2^n-1}$

Use the Limit Comparison Test to determine if the series converges or diverges.

19. $\displaystyle\sum_{n=1}^{\infty}\frac{n-2}{n^3-n^2+3}$

20. $\displaystyle\sum_{n=2}^{\infty}\frac{n(n+1)}{\left(n^2+1\right)(n-1)}$

21. $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{3+4^n}$

22. $\displaystyle\sum_{n=1}^{\infty}\frac{5^n}{\sqrt{n}\,4^n}$

23. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n+3}{5n+4}\right)^n$

24. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\ln n}$

25. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{1+\sqrt{n}}$

26. $\displaystyle\sum_{n=1}^{\infty}\frac{n+5}{n^3-2n+3}$

27. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^2+1}$

28. $\displaystyle\sum_{n=1}^{\infty}\frac{5}{4^n+1}$

29. $\displaystyle\sum_{n=0}^{\infty}\frac{1}{\sqrt{n^2+1}}$

30. $\displaystyle\sum_{n=1}^{\infty}\frac{2^n+1}{5^n+1}$

31. $\displaystyle\sum_{n=1}^{\infty}\frac{2n^2-1}{3n^5+2n+1}$

32. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2(n+3)}$

33. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n^2+1}}$

Use any method to determine if the series converges or diverges

**34.** $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+\sqrt[3]{n}}$

**35.** $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$

**36.** $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$

**37.** $\sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$

**38.** $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$

**39.** $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$

**40.** $\sum_{n=1}^{\infty} \frac{1+\sin n}{n^2}$

**41.** $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

**42.** $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$

**43.** $\sum_{n=1}^{\infty} \frac{1}{an+b}$

**44.** $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

**45.** $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$

**46.** $\sum_{n=0}^{\infty} 5\left(-\frac{4}{3}\right)^n$

**47.** $\sum_{n=1}^{\infty} \frac{1}{5^n+1}$

**48.** $\sum_{n=2}^{\infty} \frac{1}{n^3-8}$

**49.** $\sum_{n=1}^{\infty} \frac{2n}{3n-2}$

**50.** $\sum_{n=1}^{\infty} \left(\frac{1}{n+1}-\frac{1}{n+2}\right)$

**51.** $\sum_{n=1}^{\infty} \frac{n}{\left(n^2+1\right)^2}$

**52.** $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

**53.** $\sum_{n=1}^{\infty} \frac{n\,2^n}{4n^3+1}$

**54.** $\sum_{k=1}^{\infty} \frac{|\sin k|}{k^2}$

**55.** $\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2}$

**56.** $\sum_{k=1}^{\infty} \sin^2 \frac{1}{k}$

**57.** $\sum_{k=1}^{\infty} \sin \frac{1}{k}$

**58.** $\sum_{k=1}^{\infty} \frac{1}{k} \sin \frac{1}{k}$

**59.** $\sum_{k=1}^{\infty} \frac{1}{k^2} \sin \frac{1}{k}$

**60.** $\sum_{k=1}^{\infty} (-1)^k\, k \sin \frac{1}{k}$

**61.** $\sum_{k=1}^{\infty} \tan \frac{1}{k}$

**62.** $\sum_{k=1}^{\infty} (-1)^k \tan^{-1} k$

**63.** $\sum_{n=1}^{\infty} \frac{\cos n}{n^3}$

**64.** $\sum_{k=2}^{\infty} \frac{k}{\ln k}$

**65.** $\sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{\pi}{2n}$

**66.** $\frac{1}{1+\sqrt{1}}+\frac{1}{1+\sqrt{2}}+\frac{1}{1+\sqrt{3}}+\cdots$

# *Section* 3.5 – The Ratio and Root Tests

## *Theorem* – The Ratio Test

Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$$

Then

***a)*** the series ***converges*** if $\rho < 1$,

***b)*** the series ***diverges*** if $\rho > 1$, or $\rho$ is infinite

***c)*** the test is ***inconclusive*** if $\rho = 1$,

The value $\rho$ doesn't mean the sum of the series.

### *Example*

Investigate the convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$

***Solution***

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}+5}{3^{n+1}}}{\frac{2^n+5}{3^n}}$$

$$= \frac{1}{3}\frac{2^{n+1}+5}{2^n+5}$$

$$= \frac{1}{3}\cdot\frac{2+5\cdot 2^{-n}}{1+5\cdot 2^{-n}}$$

$$\rho = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}$$

$$= \lim_{n\to\infty}\frac{1}{3}\cdot\frac{2+5\cdot 2^{-n}}{1+5\cdot 2^{-n}}$$

$$= \frac{1}{3}\cdot\frac{2}{1}$$

$$= \frac{2}{3} < 1$$

The series ***converges*** since $\rho < 1$.

$$\sum_{n=0}^{\infty}\frac{2^n+5}{3^n}=\sum_{n=0}^{\infty}\frac{2^n}{3^n}+\sum_{n=0}^{\infty}\frac{5}{3^n}$$

$$=\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n+\sum_{n=0}^{\infty}\frac{5}{3^n}$$

$$=\frac{1}{1-\frac{2}{3}}+\frac{5}{1-\frac{1}{3}}$$

$$=\frac{21}{2}$$

### *Example*

Investigate the convergence of the series $\sum_{n=1}^{\infty}\frac{(2n)!}{n!n!}$

***Solution***

$$\frac{a_{n+1}}{a_n}=\frac{\frac{(2(n+1))!}{(n+1)!(n+1)!}}{\frac{(2n)!}{n!n!}}$$

$$=\frac{1}{(n+1)(n+1)}\frac{(2n+2)!}{(2n)!}$$

$$=\frac{(2n+2)(2n+1)}{(n+1)(n+1)}$$

$$=\frac{2(n+1)(2n+1)}{(n+1)(n+1)}$$

$$=\frac{4n+1}{n+1}$$

$$\rho=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$$

$$=\lim_{n\to\infty}\frac{4n+1}{n+1}$$

$$=4>1$$

The series ***diverges*** since $\rho > 1$.

### ***Example***

Investigate the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{4^n n!n!}{(2n)!}$

***Solution***

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}(n+1)!(n+1)!}{(2(n+1))!} \cdot \frac{(2n)!}{4^n n!n!}$$

$$= \frac{4(n+1)(n+1)}{(2n+2)(2n+1)}$$

$$= \frac{4(n+1)}{2(2n+1)}$$

$$= \frac{2(n+1)}{2n+1}$$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

$$= \lim_{n\to\infty} \frac{2n+2}{2n+1}$$

$$= 1$$

Because the limit is $\rho = 1$, we can't decide from the *Ratio Test* whether the series converges.

However, since $a_{n+1}$ *always* $>$ $a_n$, then the series ***diverges***.

### *Theorem* – The Root Test

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$$

Then

- ***a)*** the series ***converges*** if $\rho < 1$,
- ***b)*** the series ***diverges*** if $\rho > 1$, or $\rho$ is infinite
- ***c)*** the test is ***inconclusive*** if $\rho = 1$,

### *Example*

Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges or diverges using the Root Test

***Solution***

$$\sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}}$$

$$= \frac{\left(\sqrt[n]{n}\right)^2}{2}$$

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n}$$

$$= \lim_{n\to\infty} \frac{n^{2/n}}{2}$$

$$= \frac{\infty^0}{2}$$

$$= \frac{1}{2} < 1$$

The series ***converges*** by the *Root Test*.

### *Example*

Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ converges or diverges using the Root Test

***Solution***

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{2^n}{n^3}}$$

$$= \frac{\sqrt[n]{2^n}}{\sqrt[n]{n^3}}$$

$$= \frac{2}{\left(\sqrt[n]{n}\right)^3}$$

$$\rho = \lim_{n\to\infty} \frac{2}{\left(\sqrt[n]{n}\right)^3}$$

$$= \frac{2}{1}$$

$$\boxed{= 2 > 1}$$

The series ***diverges*** by the *Root Test*.

## *Example*

Determine if the series $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$ converges or diverges using the Root Test

***Solution***

$$\sqrt[n]{\left(\frac{1}{1+n}\right)^n} = \frac{1}{1+n}$$

$$\boxed{\lim_{n\to\infty} \frac{1}{1+n} = 0 < 1}$$

The series ***converges*** by the *Root Test*.

# *Exercises* *Section* 3.5 – The Ratio and Root Tests

Use the ***Ratio Test*** to determine if the series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}}{n3^{n-1}}$

3. $\displaystyle\sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2(n+2)!}{n!3^{2n}}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n5^n}{(2n+3)\ln(n+1)}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{99^n}{n!}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{2^n}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

12. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{3^n}$

13. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

14. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{6}{5}\right)^n$

15. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{7}{8}\right)^n$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4^n}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n^4}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$

20. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(\frac{3}{2}\right)^n}{n^2}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n3^n}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

Use the ***Root Test*** to determine if the series converges or diverges.

24. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$

25. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5}\right)^n$

26. $\displaystyle\sum_{n=1}^{\infty} \left(\ln\left(e^2+\frac{1}{n}\right)\right)^{n+1}$

27. $\displaystyle\sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$

28. $\displaystyle\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

32. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

33. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n}{n+1}\right)^{n}$

34. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3n+2}{n+3}\right)^{n}$

35. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n-2}{5n+1}\right)^{n}$

36. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n}}{(\ln n)^{n}}$

37. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{-3n}{2n+1}\right)^{3n}$

38. $\displaystyle\sum_{n=1}^{\infty}\left(2\sqrt[n]{n}+1\right)^{n}$

39. $\displaystyle\sum_{n=0}^{\infty}e^{-3n}$

40. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{3^{n}}$

41. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{500}\right)^{n}$

42. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n^{2}}\right)^{n}$

43. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\ln n}{n}\right)^{n}$

44. $\displaystyle\sum_{n=2}^{\infty}\frac{n}{(\ln n)^{n}}$

Use any method to determine if the series converges or diverges.

45. $\displaystyle\sum_{n=1}^{\infty}\frac{n^{\sqrt{2}}}{2^{n}}$

46. $\displaystyle\sum_{n=1}^{\infty}n^{2}e^{-n}$

47. $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{10^{n}}$

48. $\displaystyle\sum_{n=1}^{\infty}\frac{(\ln n)^{n}}{n^{n}}$

49. $\displaystyle\sum_{n=1}^{\infty}\frac{n2^{n}(n+1)!}{3^{n}n!}$

50. $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^{2}}{(2n)!}$

51. $\displaystyle\sum_{n=1}^{\infty}\frac{n^{2}+1}{n^{3}+1}$

52. $\displaystyle\sum_{n=1}^{\infty}\left|\sin\frac{1}{n^{2}}\right|$

53. $\displaystyle\sum_{n=8}^{\infty}\frac{1}{\pi^{n}+5}$

54. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{(\ln n)^{3}}$

55. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\pi^{n}-n^{\pi}}$

56. $\displaystyle\sum_{n=0}^{\infty}\frac{1+n}{2+n}$

57. $\displaystyle\sum_{n=1}^{\infty}\frac{1+n^{4/3}}{2+n^{5/3}}$

58. $\displaystyle\sum_{n=1}^{\infty}\frac{n^{2}}{1+n\sqrt{n}}$

59. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)(\ln\ln n)^{2}}$

60. $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n(\ln n)\sqrt{\ln\ln n}}$

61. $\displaystyle\sum_{n=1}^{\infty}\frac{1+(-1)^{n}}{\sqrt{n}}$

62. $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{n^{2}e^{n}}$

63. $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)!6^{n}}{(3n)!}$

64. $\displaystyle\sum_{n=2}^{\infty}\frac{\sqrt{n}}{3^{n}\ln n}$

65. $\displaystyle\sum_{n=0}^{\infty}\frac{n^{100}2^{n}}{\sqrt{n!}}$

66. $\displaystyle\sum_{n=1}^{\infty}\frac{1+n!}{(1+n)!}$

67. $\displaystyle\sum_{n=1}^{\infty}\frac{2^{n}}{3^{n}-n^{3}}$

68. $\displaystyle\sum_{n=1}^{\infty}\frac{n^{n}}{\pi^{n}n!}$

**69.** $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

**70.** $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$

**71.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

**72.** $\displaystyle\sum_{n=1}^{\infty} \frac{100}{n}$

**73.** $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$

**74.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2\pi}{3}\right)^n$

**75.** $\displaystyle\sum_{n=1}^{\infty} \frac{5n}{2n-1}$

**76.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$

**77.** $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3^{n-2}}{2^n}$

**78.** $\displaystyle\sum_{n=1}^{\infty} \frac{10}{3\sqrt{n^3}}$

**79.** $\displaystyle\sum_{n=1}^{\infty} \frac{10n+3}{n2^n}$

**80.** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{4n^2-1}$

**81.** $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{3^n}$

**82.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n\,7^n}$

**83.** $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

**84.** $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$

**85.** $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{\ln(k+1)}\right)^k$

**86.** $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2(\ln k)^2}$

**87.** $\displaystyle\sum_{k=3}^{\infty} \frac{1}{\ln k}$

**88.** $\displaystyle\sum_{k=2}^{\infty} \frac{5\ln k}{k}$

**89.** $\displaystyle\sum_{k=1}^{\infty} \ln\left(\frac{k+2}{k+1}\right)$

**90.** $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2 \ln k}$

**91.** $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^{\ln k}}$

**92.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

**93.** $\displaystyle\frac{1+\sqrt{2}}{2}+\frac{1+\sqrt{3}}{4}+\frac{1+\sqrt{4}}{8}+\cdots$

**94.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{3\cdot 5\cdot 7\cdots(2n+1)}$

**95.** $\displaystyle\sum_{n=1}^{\infty} \frac{3\cdot 5\cdot 7\cdots(2n+1)}{18^n(2n-1)n!}$

**96.** Use the integral test to show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges. Show that the sum $s$ of the series is less than $\frac{\pi}{2}$

**97.** Use the root test to show that $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}}{n^n}$ converges

**98.** Use the root test to test that $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ converges

**99.** Try to use the ratio test to determine whether $\sum_{n=1}^{\infty} \frac{2^{2n}(n!)^2}{(2n)!}$ converges. What happen?

Now observe that $\frac{2^{2n}(n!)^2}{(2n)!} = \frac{\left[2n(2n-2)(2n-4)\ \cdots\ 6\times 4\times 2\right]^2}{2n(2n-1)(2n-2)\ \cdots\ 3\times 2\times 1}$

$$= \frac{2n}{2n-1}\times\frac{2n-2}{2n-3}\times\frac{4}{3}\times\frac{2}{1}$$

Does the given series converge? Why or why not?

**100.** Suppose $a_n > 0$ and $\frac{a_{n+1}}{a_n} \geq \frac{n}{n+1}$ for all $n$. Show that $\sum_{n=1}^{\infty} a_n$ diverges.

$\left(a_n \geq \frac{K}{n}\ \textit{for some constant } K\right)$

**101.** Working in the early 1600s, the mathematicians Wallis, Pascal, and Fermat were calculating the area of the region under the curve $y = x^p$ between $x = 0$ and $x = 1$, where $p$ is the positive integer. Using arguments that predated the Fundamental Theorem of Calculus, they were able to prove that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{k}{n}\right)^p = \frac{1}{p+1}$$

Use Riemann sums and integrals to verify this limit.

**102.** Complete the following steps to find the values of $p > 0$ for which the series $\sum_{k=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2k-1)}{p^k k!}$ converges

*a)* Use the Ratio Test to show that $\sum_{k=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2k-1)}{p^k k!}$ converges for $p > 2$.

*b)* Use Stirling's formula, $k! = \sqrt{2\pi k}\, k^k e^{-k}$ for large $k$, to determine whether the series converges when $p = 2$.

$$\left(\textit{Hint}:\ 1\cdot 3\cdot 5\cdots(2k-1) = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2k-1)2k}{2\cdot 4\cdot 6\cdots 2k}\right)$$

# *Section* 3.6 – Alternating Series, Absolute and Conditional Convergence

A series in which the terms are alternately positive and negative is an alternating series.

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots+\frac{(-1)^{n+1}}{n}+\cdots$$

$$-2+1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots+\frac{(-1)^{n}4}{2^{n}}+\cdots$$

## *Theorem* – The Alternating Series Test (Leibniz's Test)

The series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n=u_1-u_2+u_3-u_4+\cdots$

Converges if all three of the following conditions are satisfied:

1. The $u_n$'s are all positive.
2. The positive $u_n$'s are (eventually) non-increasing: $u_n \ge u_{n+1}$ for all $n \ge N$, for some integer N.
3. $u_n \to 0$

### *Example*

The alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$

***Solution***

1. $\frac{1}{n}>0$
2. $n<n+1 \rightarrow \frac{1}{n}>\frac{1}{n+1}$
3. $\frac{1}{n}\to 0$

Therefore, the series converges.

## *Theorem* – The Alternating Series Estimation Theorem

IF the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the three conditions, then for $n \geq N$

$$s_n = u_1 - u_2 + u_3 - \cdots + (-1)^{n+1}u_n$$

Approximates the sum $L$ of the series with an error whose absolute values is less than $u_{n+1}$, the absolute value of the first unused term. Furthermore, the sum L lies between any two successive partial sums $s_n$ and $s_{n+1}$ and the remainder, $L - s_n$, has the same sign as the first unused term.

## Absolute and Conditional Convergence

### *Definition*

A series $\sum a_n$ ***converges absolutely*** (is ***absolutely convergent***) if the corresponding series of absolute values, $\sum |a_n|$, converges.

### *Definition*

A series converges but does not converge absolutely ***converges conditionally***.

### *Theorem*

If $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty}a_n$ converges.

## *Example*

For $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$ the corresponding series of absolute values is the convergent series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

The original series converges because it converges absolutely.

***Example***

For $\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$ , which contains both positive and negative terms, the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| = \frac{|\sin 1|}{1} + \frac{|\sin 2|}{4} + \frac{|\sin 3|}{9} + \cdots$$

Which converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ because $|\sin n| \le 1$ for every $n$.

The original series converges absolutely; therefore, it converges.

## Rearranging Series

***Theorem***

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1,\ b_2,\ \ldots,\ b_n,\ \ldots$ is any arrangement of the sequence $\{a_n\}$, then

$\sum b_n$ converges absolutely and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$

# *Exercises* *Section* 3.6 – Alternating Series, Absolute and Conditional Convergence

Determine if the alternating series converges or diverges

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$
2. $\displaystyle\sum_{n=2}^{\infty}(-1)^{n}\frac{4}{(\ln n)^2}$
3. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n^2+1}$
4. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{n^2+5}{n^2+4}$
5. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{10}\right)^n$
6. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}$
7. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+1}$
8. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{3n+2}$
9. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n}}{3^n}$
10. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n}}{e^n}$
11. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{5n-1}{4n+1}$
12. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+5}$
13. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{\ln(n+1)}$
14. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n}}{\ln(n+1)}$
15. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n}}{\sqrt{n}}$
16. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^2+4}$
17. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+1}{\ln(n+1)}$
18. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln(n+1)}{n+1}$
19. $\displaystyle\sum_{n=1}^{\infty}\sin\frac{(2n-1)\pi}{2}$
20. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\cos n\pi$
21. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!}$
22. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n}}{(2n+1)!}$
23. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}}{n+2}$
24. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}}{\sqrt[3]{n}}$

Determine if the series converge absolutely or conditionally, or diverges

25. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}(0.1)^n$
26. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.1)^n}{n}$
27. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n}}{1+\sqrt{n}}$
28. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{\sin n}{n^2}$
29. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3+n}{5+n}$
30. $\displaystyle\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n\ln n}$
31. $\displaystyle\sum_{n=1}^{\infty}\frac{n\cos(n\pi)}{2^n}$
32. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}$
33. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2+2n+1}$

34. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

35. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + \ln n}$

36. $\sum_{n=0}^{\infty} \frac{(-1)^n (n^2 - 1)}{n^2 + 1}$

37. $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{(n+1)\ln(n+1)}$

38. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$

39. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi^n}$

40. $\sum_{n=1}^{\infty} \frac{100\cos(n\pi)}{2n+3}$

41. $\sum_{n=1}^{\infty} \frac{n!}{(-100)^n}$

42. $\sum_{n=10}^{\infty} \frac{\sin\left(n+\frac{1}{2}\right)\pi}{\ln \ln n}$

43. $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$

44. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

45. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

46. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+3}$

47. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

48. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$

49. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{(n+1)^2}$

50. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+3}{n+10}$

51. $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$

52. $\sum_{n=0}^{\infty} (-1)^n e^{-n^2}$

53. $\sum_{n=2}^{\infty} (-1)^n \frac{n}{n^3 - 5}$

54. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{4/3}}$

55. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+4}}$

56. $\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1}$

57. $\sum_{n=1}^{\infty} (-1)^{n+1} \arctan n$

58. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}$

59. $\sum_{n=1}^{\infty} \frac{\sin\left[(n-1)\frac{\pi}{2}\right]}{n}$

For what values of $x$ does the series converge absolutely? Converge conditionally? Diverge?

60. $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n\, 2^n}$

61. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2\, 2^{2n}}$

62. $\sum_{n=1}^{\infty} (n+1)^2 \left(\frac{x}{x+2}\right)^n$

63. $\sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{2n+3}$

64. $\sum_{n=1}^{\infty} \frac{1}{2n-1} \left(\frac{3x+2}{-5}\right)^n$

65. $\sum_{n=2}^{\infty} \frac{x^n}{2^n \ln n}$

66. $\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^3}$

67. $\sum_{n=1}^{\infty} \frac{(2x+3)^n}{n^{1/3} 4^n}$

68. $\sum_{n=1}^{\infty} \frac{1}{n} \left(1 + \frac{1}{x}\right)^n$

Use any method to determine if the series converges or diverges.

**69.** $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$

**70.** $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

**71.** $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{5}{n}$

**72.** $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{n-1}}{n!}$

**73.** $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^n}{n 2^n}$

**74.** $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

**75.** $\sum_{n=1}^{\infty} \frac{n}{(-2)^{n-1}}$

**76.** $\sum_{n=1}^{\infty} \frac{10}{n^{3/2}}$

**77.** $\sum_{n=1}^{\infty} \frac{2}{n^2+5}$

**78.** $\sum_{n=1}^{\infty} \frac{3^n}{n^2}$

**79.** $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

**80.** $\sum_{n=1}^{\infty} 5\left(\frac{7}{8}\right)^n$

**81.** $\sum_{n=1}^{\infty} \frac{3n^2}{2n^2+1}$

**82.** $\sum_{n=1}^{\infty} 100e^{-n/2}$

**83.** $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+4}$

**84.** $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{4}{3n^2-1}$

**85.** $\sum_{n=2}^{\infty} \frac{\ln n}{n}$

**86.** $\sum_{k=1}^{\infty} \frac{2}{k^{3/2}}$

**87.** $\sum_{k=1}^{\infty} k^{-2/3}$

**88.** $\sum_{k=1}^{\infty} \frac{2k^2+1}{\sqrt{k^3+2}}$

**89.** $\sum_{k=1}^{\infty} \frac{2^k}{e^k}$

**90.** $\sum_{k=1}^{\infty} \left(\frac{k}{k+3}\right)^{2k}$

**91.** $\sum_{k=1}^{\infty} \frac{2^k k!}{k^k}$

**92.** $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}\sqrt{k+1}}$

**93.** $\sum_{k=1}^{\infty} \frac{3}{2+e^k}$

**94.** $\sum_{k=1}^{\infty} k \sin \frac{1}{k}$

**95.** $\sum_{k=1}^{\infty} \frac{\sqrt[k]{k}}{k^3}$

**96.** $\sum_{k=1}^{\infty} \frac{1}{1+\ln k}$

**97.** $\sum_{k=1}^{\infty} k^5 e^{-k}$

**98.** $\sum_{k=4}^{\infty} \frac{1}{k^2-10}$

**99.** $\sum_{k=1}^{\infty} \frac{\ln k^2}{k^2}$

**100.** $\sum_{k=1}^{\infty} k e^{-k}$

**101.** $\sum_{k=0}^{\infty} \frac{2 \cdot 4^k}{(2k+1)!}$

**102.** $\sum_{k=0}^{\infty} \frac{9^k}{(2k)!}$

**103.** $\sum_{k=1}^{\infty} \frac{\coth k}{k}$

**104.** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sinh k}$

**105.** $\displaystyle\sum_{k=1}^{\infty} \tanh k$

**106.** $\displaystyle\sum_{k=0}^{\infty} \operatorname{sech} k$

**107.** $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{k^2-1}$

**108.** $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2+4}{2k^2+1}$

**109.** $\displaystyle\sum_{k=1}^{\infty} (-1)^k \, ke^{-k}$

**110.** $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k^2+1}}$

**111.** $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{10^k}{k!}$

**112.** $\displaystyle\sum_{k=1}^{\infty} \frac{(-2)^{k+1}}{k^2}$

**113.** $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{e^k+e^{-k}}$

**114.** $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{k\ln k}$

**115.** $\displaystyle\sum_{k=2}^{\infty} (-1)^k \frac{\ln k}{k^2}$

**116.** $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{k\ln^2 k}$

**117.** $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}$

**118.** $\displaystyle\sum_{k=2}^{\infty} 3e^{-k}$

**119.** $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{e^k-1}$

**120.** $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{e^k}{(k+1)!}$

**121.** $\displaystyle\sum_{k=1}^{\infty} \frac{k}{\left(k^2+1\right)^3}$

**122.** $\displaystyle\sum_{k=2}^{\infty} \frac{k^e}{k^\pi}$

**123.** $\displaystyle\sum_{k=3}^{\infty} \frac{1}{(k-2)^4}$

**124.** $\displaystyle\sum_{k=1}^{\infty} \left(\frac{2k}{k+1}\right)^k$

**125.** $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{2^k}$

**126.** $\displaystyle\sum_{k=1}^{\infty} \frac{k^2-1}{k^3+4}$

**127.** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}\sin\frac{\pi}{2k}$

**128.** Use a Riemann sum argument to show that $ln\, n! \geq \int_1^n ln\, t \; dt = n ln\, n - n + 1$

Then for what values of $x$ does the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!x^n}{n^n}$ converge absolutely? Converge conditionally? Diverge? (*Use the ratio test first*)

**129.** Let $S_n$ be the $n$th partial sum of $\displaystyle\sum_{k=1}^{\infty} a_k = 8$. Find the $\lim\limits_{k\to\infty} a_k$ and $\lim\limits_{n\to\infty} S_n$

**130.** It can be proved that if a series converges absolutely, then its terms may be summed in any order without changing the value of the series. However, if a series converges conditionally, then the value of the series depends on the order of summation. For example, the (conditionally convergent) alternating harmonic series has the value

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots=\ln 2$$

Show that by rearranging the terms (so the sign pattern is ++–),

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\ldots=\frac{3}{2}\ln 2$$

**131.** A crew of workers is constructing a tunnel through a mountain. Understandably, the rate of construction decreases because rocks and earth must be removed a greater distance as the tunnel gets longer. Suppose that each week the crew digs 0.95 of the distance it dug the previous week. In the first week, the crew constructed 100 *m* of tunnel.

*a)* How far does the crew dig in 10 *weeks*? 20 *weeks*? *N* weeks?

*b)* What is the longest tunnel the crew can build at this rate?

*c)* The time required to dig 100 *m* increases by 10% each *week*, starting with 1 *week* to dig the first 100 *m*. Can the crew complete a 1.5 *km* tunnel in 10 *weeks*? Explain.

**132.** Consider the alternating series

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k \quad \text{where} \quad a_k=\begin{cases}\dfrac{4}{k+1} & \text{if } k \text{ is odd}\\[2mm] \dfrac{2}{k} & \text{if } k \text{ is even}\end{cases}$$

*a)* Write out the first ten terms of the series, group them in pairs, and show that the even partial sums of the series form the (divergent) harmonic series.

*b)* Show that $\lim\limits_{k\to\infty} a_k=0$

*c)* Explain why the series diverges even though the terms of the series approach zero.

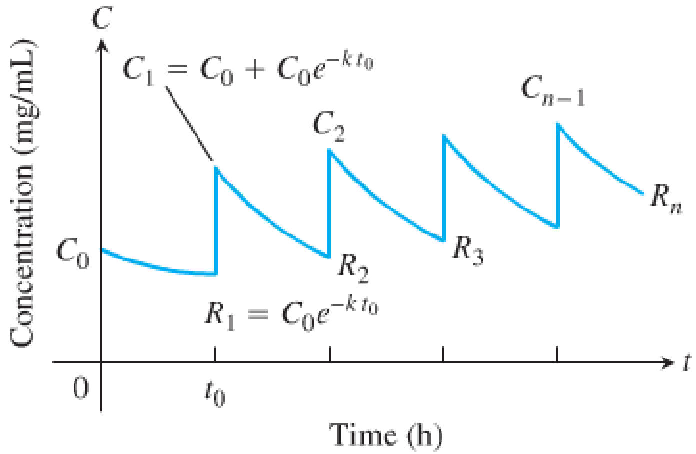
**133.** The concentration in the blood resulting from a single dose of a drug normally decreases with time as the drug is eliminated from the body. Doses may therefore need to be repeated periodically to keep the concentration from dropping below some particular level. One model for the effect of repeated doses gives the residual concentration just before the $(n+1)st$ does as

$$R_n=C_0e^{-kt_0}+C_0e^{-2kt_0}+\cdots+C_0e^{-nkt_0}$$

Where $C_0=$ the change in concentration achievable by a single dose $(mg/mL)$,

$k=$ the elimination constant $\left(h^{-1}\right)$, and

$t_0=$ time between doses ($h$).

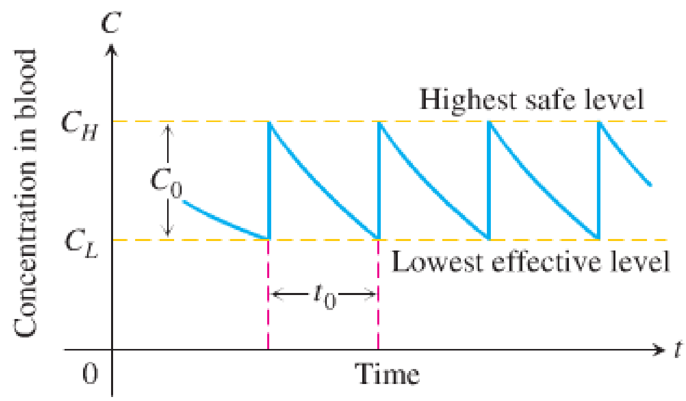



*a)* Write $R_n$ in closed form as a single fraction, and find $R = \lim\limits_{n\to\infty} R_n$

*b)* Calculate $R_1$ and $R_{10}$ for $C_0 = 1\ mg / \text{mL}$, $k = 0.1\ h^{-1}$, and $t_0 = 10\ h$. How good as estimate of $R$ is $R_{10}$

*c)* If $k = 0.01\ h^{-1}$ and $t_0 = 10\ h$, find the smallest $n$ such that $R_n > \frac{1}{2}R$.

**134.** If a drug is known to be ineffective below a concentration $C_L$ and harmful above some higher concentration $C_H$, one needs to find values of $C_0$ and $t_0$ that will produce a concentration that is safe (not above $C_H$) but effective (not below $C_L$).




We want to find values for $C_0$ and $t_0$ for which

$$R = C_L \quad \textit{and} \quad C_0 + R = C_H$$

Thus, $C_0 = C_H - C_L$. The resulting equation simplifies to

$$t_0 = \frac{1}{k}\ln\frac{C_H}{C_L}$$

To reach an effective level rapidly, one might administer a “loading” dose that would produce a concentration of $C_H\ mg / mL$. This could be followed every $t_0$ hours by a dose that raises the concentration by $C_0 = C_H - C_L\ mg / mL$.

*a)* Verify the preceding equation for $t_0$.

*b)* If $k = 0.05\ h^{-1}$ and the highest safe concentration is $e$ times the lowest effective concentration, find the length of time between doses that will assure safe and effective concentrations.

*c)* Given $C_H = 2\ mg\ /\ mL,\ C_L = 0.5\ mg\ /\ mL,$ and $k = 0.02\ h^{-1},$ determine a scheme for administering the drug.

*d)* Suppose that $k = 0.2\ h^{-1}$ and the smallest effective concentration is 0.03 *mg/mL*. A single dose that produces a concentration of 0.1 *mg/mL* is administered. About how long will the drug remain effective?

# *Section* 3.7 – Power Series

## Power Series and Converge

### *Definitions*

A **power series about $x = 0$** is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$

A **power series about $x = a$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots$$

In which the ***center*** $a$ and the ***coefficients*** $c_0, c_1, c_2, \cdots, c_n, \cdots$ are constants.

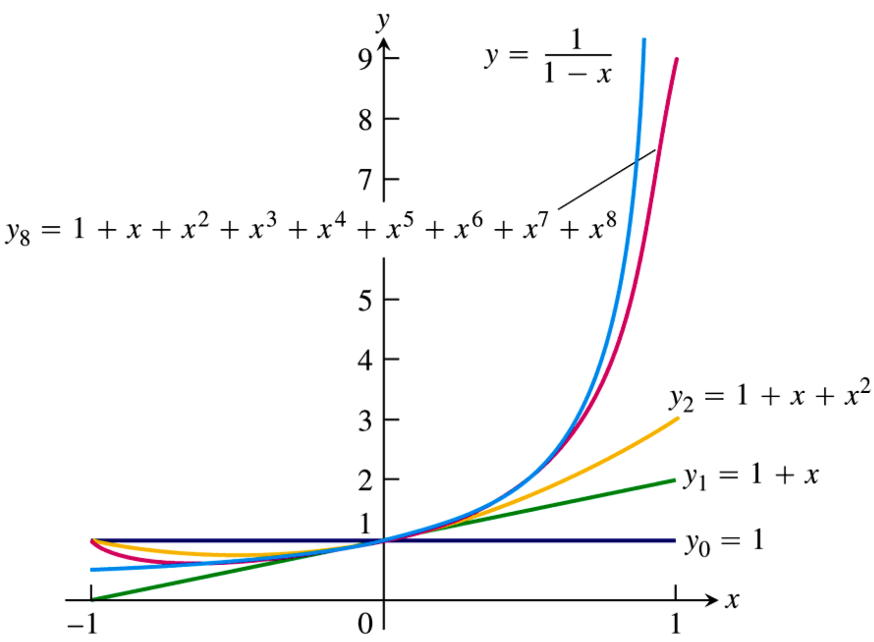
### *Example*

Find the convergence of $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$

***Solution***

This is the geometric series with first term 1 and ratio $x$. it converges to $\frac{1}{1-x}$ *for* $|x| < 1$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \quad -1 < x < 1$$



***Example***

The power series $1-\frac{1}{2}(x-2)+\frac{1}{4}(x-2)^2-\cdots+\left(-\frac{1}{2}\right)^n(x-2)^n+\cdots$

This is the geometric series with first term 1 and ratio $r=-\frac{x-2}{2}$. it converges to

$\left|\frac{x-2}{2}\right|<1 \quad \textit{for} \quad 0<x<4$. The sum

$$\frac{1}{1-r}=\frac{1}{1+\frac{x-2}{2}}$$

$$=\frac{2}{x}$$

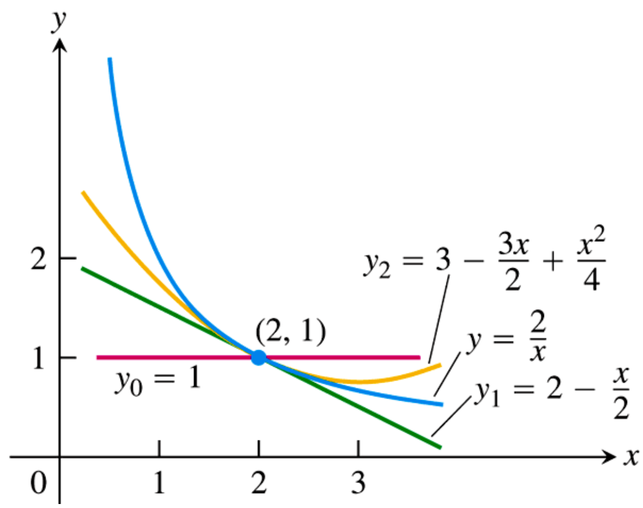
$$\frac{2}{x}=1-\frac{x-2}{2}+\frac{(x-2)^2}{4}-\cdots+\left(-\frac{1}{2}\right)^n(x-2)^n+\cdots$$

The series generates polynomial approximations of $f(x)=\frac{2}{x}$ for values of $x$ near 2:

$P_0(x)=1$

$P_1(x)=1-\frac{1}{2}(x-2)=2-\frac{x}{2}$

$P_2(x)=1-\frac{1}{2}(x-2)+\frac{1}{4}(x-2)^2=3-\frac{3x}{2}+\frac{x^2}{4}$



### *Example*

For what values of $x$ do the power series converges? $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

***Solution***

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x}\right|$$

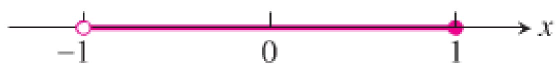
$$= \frac{n}{n+1}|x| \to |x|$$

The series converges absolutely for $|x| < 1$. It diverges if $|x| > 1$.

At $x = 1$, we get the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, which converges.

At $x = -1$, we get the alternating harmonic series $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots$, the negative of the harmonic series; it diverges.

The series ***converges*** for $-1 < x \le 1$ and ***diverges*** elsewhere.




### *Example*

For what values of $x$ do the power series converges? $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

***Solution***

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{2n+1}}{2n+1}\cdot\frac{2n-1}{x^{2n-1}}\right|$$

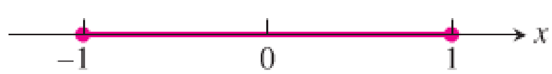
$$= \frac{2n-1}{2n+1}x^2 \to x^2$$

The series converges absolutely for $x^2 < 1$. It diverges if $x^2 > 1$.

At $x = 1$, we get the alternating harmonic series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$, which converges.

At $x = -1$, we get the alternating harmonic series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$, it converges.

The series ***converges*** for $-1 \le x \le 1$ and ***diverges*** elsewhere.



### *Example*

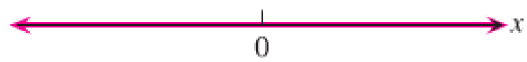
For what values of $x$ do the power series converges? $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

***Solution***

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right|$$

$$= \frac{|x|}{n+1} \to 0 \quad (\forall x)$$

The series ***converges absolutely*** for all $x$.




### *Example*

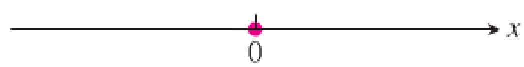
For what values of $x$ do the power series converges? $\displaystyle\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$

***Solution***

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right|$$

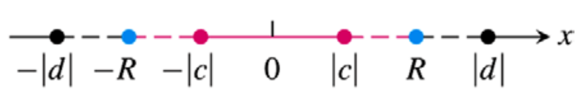
$$= (n+1)|x| \to \infty$$

The series ***diverges absolutely*** for all $x$ except $x = 0$.




### *Theorem*

If the power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ converges at $x = c \ne 0$, then it converges absolutely for all $x$ with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all $x$ with $|x| > |d|$.



## Radius of Convergence of a Power Series

### *Corollary to Theorem*

The convergence of the series $\sum c_n (x-a)^n$ is described by one of the following three cases:

1. There is a positive number $R$ such the series diverges for $x$ with $|x-a|>R$ but converges absolutely for $x$ with $|x-a|<R$. The series may or may not converge at either of the endpoints $x=a-R$ and $x=a+R$.
2. The series converges absolutely for every $x$ $(R=\infty)$.
3. The series converges at $x = a$ and diverges elsewhere ($R = 0$)

$R$ is called the ***radius of convergence*** of the power series, and the interval of radius $R$ centered at $x = a$ is called the ***interval of convergence***.

### *Definition*

Suppose that $L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists or is ∞. Then the power series $\sum c_n (x-a)^n$ has radius of convergence $R=\frac{1}{L}$. (If $L = 0$, then $R = \infty$; if $L = \infty$, then $R = 0$) and $R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$

### *How to Test a Power Series for Convergence*

1. Use the Ratio Test (or Root Test) to find the interval where the series converges. Ordinarily, this is an open interval
$$|x-a|<R \quad \text{or} \quad a-R<x<a+R$$
2. If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint. Use the Comparison Test, the Integral Test, or the Alternating Series Test.
3. If the interval of absolute convergence is $a-R<x<a+R$, the series diverges for $|x-a|>R$ (it does not even converge conditionally) because the $n$th term does not approach zero for those values of $x$.

### ***Example***

Determine the centre, radius, and interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(2x+5)^n}{(n^2+1)3^n}$

***Solution***

$$\sum_{n=0}^{\infty} \frac{(2x+5)^n}{(n^2+1)3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \frac{1}{n^2+1}\left(x+\frac{5}{2}\right)^n$$

The centre of convergence is

$$x+\frac{5}{2}=0 \quad \Rightarrow \quad x=-\frac{5}{2}$$

$$L = \lim_{n\to\infty} \left| \frac{\left(\frac{2}{3}\right)^{n+1} \frac{1}{(n+1)^2+1}}{\left(\frac{2}{3}\right)^n \frac{1}{n^2+1}} \right|$$

$$= \lim_{n\to\infty} \frac{2}{3} \frac{n^2+1}{(n+1)^2+1}$$

$$= \frac{2}{3}$$

$$R = \frac{1}{L} = \frac{3}{2}$$

The series converges absolutely on ***interval***

$$\left(-\frac{5}{2}-\frac{3}{2},\ -\frac{5}{2}+\frac{3}{2}\right) = (-4,\ -1) \qquad a-R<x<a+R$$

It diverges on $(-\infty,\ -4)\cup(-1,\ \infty)$

$$\text{At } x=-4 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$$

$$\text{At } x=-1 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \frac{3^n}{(n^2+1)3^n} = \sum_{n=0}^{\infty} \frac{1}{n^2+1}$$

Both series ***converge*** (*absolutely*).

Therefore; the interval of convergence of the given power is $[-4,\ -1]$

## *Example*

Determine the radius of convergence of $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

***Solution***

$$L = \lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \right|$$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{1}{n+1}$$

$$= 0$$

Thus $R = \infty$

***Or*** $R = \lim_{n\to\infty} \left| \frac{1}{n!} \cdot (n+1)! \right|$

$$= \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \right|$$

$$= \lim_{n\to\infty} (n+1)$$

$$= \infty$$

This series ***converges*** (*absolutely*) for all *x*.

## *Example*

Determine the radius of convergence of $\sum_{n=0}^{\infty} n!x^n$

***Solution***

$$L = \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \right|$$

$$= \lim_{n\to\infty} (n+1)$$

$$= \infty$$

Thus $R = 0$

This series ***converges*** only at its centre of convergence, $x = 0$.

### *Theorem* – The Series Multiplication Theorem for Power Series

If $A(x)=\sum_{n=0}^{\infty}a_n x^n$ and $B(x)=\sum_{n=0}^{\infty}b_n x^n$ converge absolutely for $|x|<R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}$$

Then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x|<R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\cdot\left(\sum_{n=0}^{\infty} b_n x^n\right)=\sum_{n=0}^{\infty} c_n x^n$$

Finding the coefficients $c_n$

$$\begin{aligned}\left(\sum_{n=0}^{\infty} x^n\right)\cdot\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}\right) &= \left(1+x+x^2+\cdots\right)\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right)\\ &= \underbrace{\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right)}_{by\ 1}+\underbrace{\left(x^2-\frac{x^3}{2}+\frac{x^4}{3}-\cdots\right)}_{by\ x}+\underbrace{\left(x^3-\frac{x^4}{2}+\frac{x^5}{3}-\cdots\right)}_{by\ x^2}+\cdots\\ &= x+\frac{x^2}{2}+\frac{5x^3}{6}-\frac{x^4}{6}\cdots\end{aligned}$$

### *Theorem*

If $\sum_{n=0}^{\infty}a_n x^n$ converges absolutely for $|x|<R$, then $\sum_{n=0}^{\infty}a_n\left(f(x)\right)^n$ converges absolutely for any continuous function $f$ on $|f(x)|<R$

***Theorem*** **– The term-by-Term Differentiation Theorem**

If $\sum_{n=0}^{\infty} c_n (x-a)^n$ has a radius of convergence $R > 0$, it defines a function.

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \textit{on the interval} \quad a - R < x < a + R$$

This function $f$ has derivatives of all order inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n (x-a)^{n-2}$$

And so on. Each of these derived series converges at every point of the interval $a - R < x < a + R$

### *Example*

Find the series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$

$$= \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1$$

***Solution***

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots$$

$$= \sum_{n=1}^{\infty} nx^{n-1}$$

$$f''(x) = \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + 20x^3 + \cdots + n(n-1)x^{n-1} + \cdots$$

$$= \sum_{n=2}^{\infty} n(n-1)x^{n-2} \qquad \boxed{-1 < x < 1}$$

### *Theorem* – The term-by-Term Integration Theorem

Suppose that $f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $a-R<x<a+R \quad (R>0)$. Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

Converges $a-R<x<a+R$ and

$$\int f(x)\,dx=\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}+C \quad \mathit{for} \quad a-R<x<a+R$$

### *Example*

Identify the function $f(x)=\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots, \quad -1\le x\le 1$

***Solution***

$$f'(x)=1-x^2+x^4-x^6+\cdots, \quad -1\le x\le 1$$

This is a geometric series with first term 1 and ratio $-x^2$, so

$$f'(x)=\frac{1}{1-\left(-x^2\right)}=\frac{1}{1+x^2}$$

$$\int f'(x)\,dx=\int \frac{dx}{1+x^2}=\tan^{-1} x+C$$

The series for $f(x=0)=0$

$$\tan^{-1} 0+C=0 \to \boxed{C=0}$$

$$f(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots$$

$$=\tan^{-1} x, \quad -1<x<1$$

# *Exercises* Section 3.7 – Power Series

**(1 – 9)**

***(a)*** Find the series' radius and interval of convergence. For what values of $x$ does the series converge (***b***) absolutely, (***c***) conditionally?

**1.** $\displaystyle\sum_{n=0}^{\infty} x^n$

**2.** $\displaystyle\sum_{n=0}^{\infty} (x+5)^n$

**3.** $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$

**4.** $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$

**5.** $\displaystyle\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$

**6.** $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2+3}}$

**7.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{\sqrt{n+3}}$

**8.** $\displaystyle\sum_{n=1}^{\infty} \sqrt[n]{n}\,(2x+5)^n$

**9.** $\displaystyle\sum_{n=1}^{\infty} \left(2+(-1)^n\right)\cdot(x+1)^{n-1}$

**(10 – 18)** Find the radius of convergence of the power series

**10.** $\displaystyle\sum_{n=0}^{\infty} n!x^n$

**11.** $\displaystyle\sum_{n=0}^{\infty} 3(x-2)^n$

**12.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

**13.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$

**14.** $\displaystyle\sum_{n=0}^{\infty} (3x)^n$

**15.** $\displaystyle\sum_{n=1}^{\infty} \frac{(4x)^n}{n^2}$

**16.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{5^n}$

**17.** $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

**18.** $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!x^{2n}}{n!}$

**(19 – 42)** Find the interval of convergence of the power series

**19.** $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

**20.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(x+1)^n}{2^n}$

**21.** $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

**22.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$

**23.** $\displaystyle\sum_{n=0}^{\infty} (2x)^n$

**24.** $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$

**25.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n (n+1) x^n$

**26.** $\displaystyle\sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$

**27.** $\displaystyle\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$

**28.** $\displaystyle\sum_{n=0}^{\infty} (2n)!\left(\frac{x}{3}\right)^n$

**29.** $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{(n+1)(n+2)}$

**30.** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{6^n}$

**31.** $\sum_{n=0}^{\infty}(-1)^n \frac{n!(x-5)^n}{3^n}$

**32.** $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{(x-4)^n}{n9^n}$

**33.** $\sum_{n=0}^{\infty} \frac{(x-3)^{n+1}}{(n+1)4^{n+1}}$

**34.** $\sum_{n=0}^{\infty}(-1)^{n+1} \frac{(x-1)^{n+1}}{n+1}$

**35.** $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{(x-2)^{n+1}}{n2^n}$

**36.** $\sum_{n=1}^{\infty} \frac{(x-3)^{n-1}}{3^{n-1}}$

**37.** $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

**38.** $\sum_{n=1}^{\infty} \frac{n}{n+1}(-2x)^{n-1}$

**39.** $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$

**40.** $\sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!}$

**41.** $\sum_{n=1}^{\infty} \frac{n!x^n}{(2n)!}$

**42.** $\sum_{n=1}^{\infty} \frac{2\cdot 3\cdot 4\cdots(n+1)x^n}{n!}$

(**43 – 56**) Determine the centre, radius, and interval of convergence of each of the power series

**43.** $\sum_{n=0}^{\infty} \frac{x^{2n}}{\sqrt{n+1}}$

**44.** $\sum_{n=0}^{\infty} 3n(x+1)^n$

**45.** $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} x^n$

**46.** $\sum_{n=1}^{\infty} \frac{e^n}{n^3}(4-x)^n$

**47.** $\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n}$

**48.** $\sum_{n=0}^{\infty} \frac{1+5^n}{n!} x^n$

**49.** $\sum \frac{n^2 x^n}{n!}$

**50.** $\sum \frac{x^{4n}}{n^2}$

**51.** $\sum (-1)^n \frac{(x+1)^{2n}}{n!}$

**52.** $\sum \frac{(x-1)^n}{n\cdot 5^n}$

**53.** $\sum \left(\frac{x}{9}\right)^{3n}$

**54.** $\sum \frac{(x+2)^n}{\sqrt{n}}$

**55.** $\sum \frac{(x+2)^k}{2^k \ln k}$

**56.** $x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\cdots$

**57.** For what values of $x$ does the series $1-\frac{1}{2}(x-3)+\frac{1}{4}(x-3)^2+\cdots+\left(-\frac{1}{2}\right)^n(x-3)^n+\cdots$ converges? What is its sum? What series do you get if you differentiate the given series term by term? For what values of $x$ does the new series converge? What is its sum?

**58.** The series $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\frac{x^{11}}{11!}+\cdots$ converges to sin$x$ for all $x$.

***a)*** Find the first six terms of a series for cos$x$. For what values of $x$ should the series converge?

***b)*** By replacing $x$ by $2x$ in the series for sin$x$, find a series that converges to sin$2x$ for all $x$.

***c)*** Using the result in part (a) and series multiplication, calculate the first six term of a series for $2\sin x\cos x$. Compare your answer with the answer in part (b).

**59.** Find the sum of the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ by the first finding the sum of the power series

$$\sum_{n=1}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + 16x^4 + \cdots$$

**60.** Find a series representation of $f(x) = \frac{1}{2+x}$ in powers of $x-1$. What is the interval of convergence of this series?

**61.** Determine the Cauchy product of the series $1 + x + x^2 + x^3 + \cdots$ *and* $-x + x^2 - x^3 + \cdots$. On what interval and to what function does the product series converge?

**62.** Determine the power series expansion of $\frac{1}{(1-x)^2}$ by formally dividing $1 - 2x + x^2$ into 1.

Use the power series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad -1 < x < 1$

**(63 – 65)** Determine the interval of convergence and the sum of each of the series

**63.** $1 - 4x + 16x^2 - 64x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n (4x)^n$

**64.** $3 + 4x + 5x^2 + 6x^3 + \cdots = \sum_{n=0}^{\infty} (n+3) x^n$

**65.** $\frac{1}{3} + \frac{x}{4} + \frac{x^2}{5} + \frac{x^3}{6} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n+3}$

# *Section* 3.8 – Taylor and Maclaurin Series

The sum of a power series:

$$f(x)=\sum_{n=0}^{\infty} a_n (x-a)^n$$

$$=a_0+a_1(x-a)+a_2(x-a)^2+\cdots+a_n(x-a)^n+\cdots$$

$$f'(x)=a_1+2a_2(x-a)+3a_3(x-a)^2+\cdots+na_n(x-a)^{n-1}+\cdots$$

$$f''(x)=2a_2+2\bullet 3a_3(x-a)+3\bullet 4a_4(x-a)^2+\cdots+(n-1)\bullet na_n(x-a)^{n-2}+\cdots$$

$$f'''(x)=1\bullet 2\bullet 3a_3+2\bullet 3\bullet 4a_4(x-a)++3\bullet 4\bullet 5a_5(x-a)^2\cdots+(n-2)\bullet(n-1)\bullet na_n(x-a)^{n-3}+\cdots$$

$$f^{(n)}(x)=n!a_n+\textit{a sum of terms with } (x-a) \textit{ as a factor}$$

In general: $\boxed{f^{(n)}(x)=n!a_n} \quad \Rightarrow \quad a_n=\dfrac{f^{(n)}(a)}{n!}$

If $f$ has a series representation, then the series must be

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots$$

## Taylor and Maclaurin Series

### *Definitions*

Let $f$ be a function with derivatives of all orders throughout some interval containing $\boldsymbol{a}$ as an interior point. Then the ***Taylor series generated by*** $f$ at $x=a$ is

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots$$

The ***Maclaurin series generated by*** $f$ is

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots,$$

The Taylor series generated by $f$ at $x=0$.

### *Example*

Find the Taylor series generated by $f(x)=\frac{1}{x}$ *at* $a=2$. Where, if anywhere, does the series converges to $\frac{1}{x}$.

***Solution***

$$f(x)=x^{-1}$$

$$f'(x)=-x^{-2}$$

$$f''(x)=2!x^{-3}$$

$$f'''(x)=-3!x^{-4}$$

$$f^{(n)}(x)=(-1)^n n!x^{-(n+1)}$$

$$f(2)=2^{-1}=\frac{1}{2}$$

$$f'(2)=-\frac{1}{2^2}$$

$$\frac{f''(2)}{2!}=2^{-3}=\frac{(-1)^3}{2^3}$$

$$\vdots \quad \vdots$$

$$\frac{f^{(n)}(2)}{n!}=\frac{(-1)^n}{2^{n+1}}$$

The Taylor series is:

$$f(2)+f'(2)(x-2)+\frac{f''(2)}{2!}(x-2)^2+\cdots+\frac{f^{(n)}(2)}{n!}(x-2)^n$$

$$=\frac{1}{2}-\frac{(x-2)}{2^2}+\frac{(x-2)^2}{2^3}+\cdots+(-1)^n\frac{(x-2)^n}{2^{n+1}}+\cdots$$

# *Taylor* Polynomials

## *Definition*

Let $f$ be a function with derivatives of order *k* for $k=1,\ 2,\ \ldots,\ N$ in some interval containing *a* as an interior point. Then for any integer n from 0 through *N*, the Taylor polynomial of order *n* generated by $f$ at $x=a$ is the polynomial

$$P_n(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n$$

### *Example*

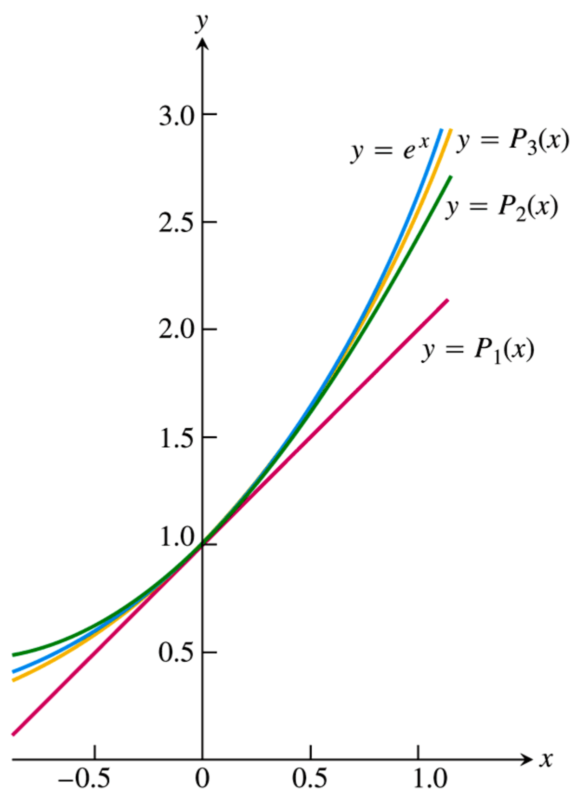
Find the Taylor series and the Taylor polynomials generated by $f(x) = e^x$ at $x = 0$

**Solution**

$$f^{(n)}(x) = e^x \quad \rightarrow \quad f^{(n)}(0) = 1$$

$$\begin{aligned} P_n(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}x^k \end{aligned}$$

This is also the Maclaurin series of $e^x$




The Taylor polynomial of order $n$ at $x = 0$ is

$$P_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$$

### *Example*

Find the Taylor series and the Taylor polynomials generated by $f(x)=\cos x$ at $x=0$

***Solution***

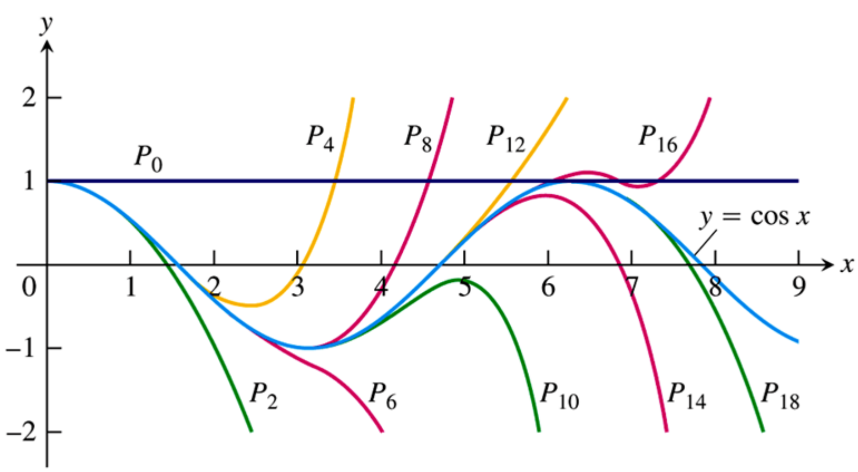
$$f(x)=\cos x, \qquad f'(x)=-\sin x,$$
$$f''(x)=-\cos x, \qquad f'''(x)=\sin x,$$
$$\vdots \qquad \vdots$$
$$f^{(2n)}(x)=(-1)^n\cos x, \qquad f^{(2n+1)}(x)=(-1)^{2n+1}\sin x$$

$$f^{(2n)}(0)=(-1)^n, \qquad f^{(2n+1)}(0)=0$$

The Taylor series generated by $f$ at $x=0$ is

$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots$$
$$=1+0\cdot x-\frac{x^2}{2!}+0\cdot x^3+\frac{x^4}{4!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots$$
$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots$$
$$=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}$$

$$P_{2n}(x)=P_{2n+1}(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}$$



### *Example*

Find the Taylor series for $\cos x$ about $\frac{\pi}{3}$. Where is the series valid?

***Solution***

$$\cos x = \cos\left(x - \frac{\pi}{3} + \frac{\pi}{3}\right)$$

$$= \cos\left(x - \frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right) - \sin\left(x - \frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)$$

$$= \frac{1}{2}\left[1 - \frac{1}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{3}\right)^4 - \cdots\right] - \frac{\sqrt{3}}{2}\left[\left(x - \frac{\pi}{3}\right) - \frac{1}{3!}\left(x - \frac{\pi}{3}\right)^3 + \frac{1}{5!}\left(x - \frac{\pi}{3}\right)^5 - \cdots\right]$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{2}\frac{1}{2!}\left(x - \frac{\pi}{3}\right)^2 - \frac{\sqrt{3}}{2}\frac{1}{3!}\left(x - \frac{\pi}{3}\right)^3 + \frac{1}{2}\frac{1}{4!}\left(x - \frac{\pi}{3}\right)^4 + \frac{1}{5!}\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)^5 - \cdots$$

This series representation is valid for all *x*.

### *Example*

Find the Taylor series for $\ln x$ in powers of $x-2$. Where does the series converge to $\ln x$?

***Solution***

Let $t = \frac{x-2}{2}$, then

$$\ln x = \ln\left(2+(x-2)\right)$$
$$= \ln\left[2\left(1+\frac{x-2}{2}\right)\right]$$
$$= \ln 2 + \ln(1+t)$$

| | |
|---|---|
| $f(t) = \ln(1+t)$ | $f(0) = \ln(1) = 0$ |
| $f'(t) = \frac{1}{1+t}$ | $f'(0) = 1$ |
| $f''(t) = \frac{-1}{(1+t)^2}$ | $f''(0) = -1$ |
| $f'''(t) = \frac{2}{(1+t)^3}$ | $f'''(0) = 2$ |
| $f^{(4)}(t) = \frac{-6}{(1+t)^4}$ | $f'''(0) = -6$ |

$$f^{(n)}(t) = (-1)^{n+1}\frac{(n-1)!}{(1+t)^n}$$

$$\ln(1+t) = f(0) + f'(0)t + \frac{f''(0)}{2!}t^2 + \frac{f'''(0)}{3!}t^3 + \frac{f^{(IV)}(0)}{4!}t^4 + \cdots$$
$$= t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots$$

$$\ln x = \ln 2 + \ln(1+t)$$
$$= \ln 2 + t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots$$
$$= \ln 2 + \frac{x-2}{2} - \frac{(x-2)^2}{2\times 2^2} + \frac{(x-2)^3}{3\times 2^3} - \frac{(x-2)^4}{4\times 2^4} + \cdots$$
$$= \ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\times 2^n}(x-2)^n$$

Since the series for $\ln(1+t)$ is valid for $-1 < t \le 1$, this series for $\ln x$ is valid for $-1 < \frac{x-2}{2} \le 1$

$-2 < x-2 \le 2 \quad \to \quad \boxed{0 < x \le 4}$

# *Exercises* Section 3.8 – Taylor and Maclaurin Series

**(1 – 23)** Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by $f$ *at a*

**1.** $f(x)=e^{2x}, \quad a=0$

**2.** $f(x)=\sin x, \quad a=0$

**3.** $f(x)=\ln(1+x), \quad a=0$

**4.** $f(x)=\frac{1}{x+2}, \quad a=0$

**5.** $f(x)=\sqrt{1-x}, \quad a=0$

**6.** $f(x)=x^3, \quad a=1$

**7.** $f(x)=8\sqrt{x}, \quad a=1$

**8.** $f(x)=\sin x, \quad a=\frac{\pi}{4}$

**9.** $f(x)=\cos x, \quad a=\frac{\pi}{6}$

**10.** $f(x)=\sqrt{x}, \quad a=9$

**11.** $f(x)=\sqrt[3]{x}, \quad a=8$

**12.** $f(x)=\ln x, \quad a=e$

**13.** $f(x)=\sqrt[4]{x}, \quad a=8$

**14.** $f(x)=\tan^{-1}x+x^2+1, \quad a=1$

**15.** $f(x)=e^{x}, \quad a=\ln 2$

**16.** $f(x)=e^{3x}; \quad a=0$

**17.** $f(x)=\frac{1}{x}; \quad a=1$

**18.** $f(x)=\cos x; \quad a=\frac{\pi}{2}$

**19.** $f(x)=\frac{1}{x+1}; \quad a=0$

**20.** $f(x)=\tan^{-1}4x; \quad a=0$

**21.** $f(x)=\sin 2x; \quad a=-\frac{\pi}{2}$

**22.** $f(x)=\cosh 3x; \quad a=0$

**23.** $f(x)=\frac{1}{4+x^2}; \quad a=0$

**(25 – 35)** Find the $n$th Maclaurin polynomial for the function

**24.** $f(x)=e^{4x}, \quad n=4$

**25.** $f(x)=e^{-x}, \quad n=5$

**26.** $f(x)=e^{-x/2}, \quad n=4$

**27.** $f(x)=e^{x/3}, \quad n=4$

**28.** $f(x)=\sin x, \quad n=5$

**29.** $f(x)=\cos \pi x, \quad n=4$

**30.** $f(x)=xe^{x}, \quad n=4$

**31.** $f(x)=x^2e^{-x}, \quad n=4$

**32.** $f(x)=\frac{1}{x+1}, \quad n=5$

**33.** $f(x)=\frac{x}{x+1}, \quad n=4$

**34.** $f(x)=\sec x, \quad n=2$

**35.** $f(x)=\tan x, \quad n=3$

**(36 – 39)** Find out the ***third*** term of the Maclaurin series for the following function.

**36.** $f(x)=(1+x)^{1/3}$

**37.** $f(x)=(1+x)^{-1/2}$

**38.** $f(x)=\left(1+\frac{x}{2}\right)^{-3}$

**39.** $f(x)=(1+2x)^{-5}$

**(40 – 55)** Find the Maclaurin series for

**40.** $xe^x$

**41.** $5\cos \pi x$

**42.** $\dfrac{x^2}{x+1}$

**43.** $e^{3x+1}$

**44.** $\cos\left(2x^3\right)$

**45.** $\cos(2x-\pi)$

**46.** $x^2 \sin\left(\frac{x}{3}\right)$

**47.** $\cos^2\left(\frac{x}{2}\right)$

**48.** $\sin x \cos x$

**49.** $\tan^{-1}\left(5x^2\right)$

**50.** $\ln\left(2+x^2\right)$

**51.** $\dfrac{1+x^3}{1+x^2}$

**52.** $\ln\dfrac{1+x}{1-x}$

**53.** $\dfrac{e^{2x^2}-1}{x^2}$

**54.** $\cosh x - \cos x$

**55.** $\sinh x - \sin x$

**(56 – 59)** Finding Taylor and Maclaurin Series generated by $f$ *at* $x = a$

**56.** $f(x) = x^3 - 2x + 4, \quad a = 2$

**57.** $f(x) = 2x^3 + x^2 + 3x - 8, \quad a = 1$

**58.** $f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2, \quad a = -1$

**59.** $f(x) = \cos\left(2x + \frac{\pi}{2}\right), \quad a = \frac{\pi}{4}$

**(60 – 68)** Find the Taylor series of the functions, where is each series representation valid?

**60.** $f(x) = e^{-2x}$ *about* $-1$

**61.** $f(x) = \sin x$ *about* $\frac{\pi}{2}$

**62.** $f(x) = \ln x$ *in powers of* $x-3$

**63.** $f(x) = \ln(2+x)$ *in powers of* $x-2$

**64.** $f(x) = e^{2x+3}$ *in powers of* $x+1$

**65.** $f(x) = \sin x - \cos x$ *about* $\frac{\pi}{4}$

**66.** $f(x) = \cos^2 x$ *about* $\frac{\pi}{8}$

**67.** $f(x) = \dfrac{x}{1+x}$ *in powers of* $x-1$

**68.** $f(x) = xe^x$ *in powers of* $x+2$

**(69 – 81)** Find the $n$th–order Taylor polynomial centered at $c$ for the function

**69.** $f(x) = \frac{2}{x}, \quad n = 3, \quad c = 1$

**70.** $f(x) = \frac{1}{x^2}, \quad n = 4, \quad c = 2$

**71.** $f(x) = \sqrt{x}, \quad n = 3, \quad c = 4$

**72.** $f(x) = \sqrt[3]{x}, \quad n = 3, \quad c = 8$

**73.** $f(x) = \ln x, \quad n = 4, \quad c = 2$

**74.** $f(x) = x^2 \cos x, \quad n = 2, \quad c = \pi$

**75.** $f(x) = \sin 2x; \quad n = 3, \quad c = 0$

**76.** $f(x) = \cos x^2; \quad n = 2, \quad c = 0$

**77.** $f(x) = e^{-x}; \quad n = 2, \quad c = 0$

**78.** $f(x) = \cos x; \quad n = 2, \quad c = \frac{\pi}{4}$

**79.** $f(x) = \ln x; \quad n = 2, \quad c = 1$

**80.** $f(x) = \sinh 2x; \quad n = 4, \quad c = 0$

**81.** $f(x) = \cosh x; \quad n = 3, \quad c = \ln 2$

**(82 – 84)** Find the sums of the series

**82.** $1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\frac{x^8}{4!}+\cdots$

**83.** $1+\frac{x^2}{3!}+\frac{x^4}{5!}+\frac{x^6}{7!}+\frac{x^8}{9!}+\cdots$

**84.** $x^3-\frac{x^9}{3!\times 4}+\frac{x^{15}}{5!\times 16}-\frac{x^{21}}{7!\times 64}+\frac{x^{27}}{9!\times 256}-\cdots$

**(85 – 90)** Use the geometric series $\sum_{k=0}^{\infty} x^k=\frac{1}{1-x}$, for $|x|<1$, to determine the Maclaurin series and the interval of convergence for the following functions.

**85.** $f(x)=\frac{1}{1-x^2}$

**86.** $f(x)=\frac{1}{1+x^3}$

**87.** $f(x)=\frac{1}{1+5x}$

**88.** $f(x)=\frac{10}{1+x}$

**89.** $f(x)=\frac{1}{(1-10x)^2}$

**90.** $f(x)=\ln(1-4x)$

**91.** The limit $\lim_{n\to\infty}\frac{n!}{\sqrt{2\pi}\; n^{n+1/2}e^{-n}}=1$ that is the relative error in the approximation

$n!\approx\sqrt{2\pi}\; n^{n+1/2}e^{-n}$

Approaches zero as $n$ increases. That is $n!$ grows at a rate comparable to $\sqrt{2\pi}\; n^{n+1/2}e^{-n}$. This result, known as Stirling's Formula, is often very useful in applied mathemmatics and statistics. Prove it by carrying out the following steps.

*a)* Use the identity $\ln(n!)=\sum_{j=1}^{n}\ln j$ and the increasing nature of ln to show that if $n\ge 1$,

$$\int_0^n \ln x\,dx<\ln(n!)<\int_1^{n+1}\ln x\,dx$$

And hence that $n\ln n-n<\ln(n!)<(n+1)\ln(n+1)-n$

*b)* If $c_n=\ln(n!)-\left(n+\frac{1}{2}\right)\ln n+n$, show that

$$c_n-c_{n+1}=\left(n+\frac{1}{2}\right)\ln\left(\frac{n+1}{n}\right)-1$$

$$=\left(n+\frac{1}{2}\right)\ln\left(\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}}\right)-1$$

*c)* Use the Maclaurin series for $\ln\frac{1+t}{1-t}$ to show that

$$0 < c_n - c_{n+1} < \frac{1}{3}\left(\frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^3} + \cdots\right)$$
$$= \frac{1}{12}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

and therefore that $\{c_n\}$ is decreasing and $\left\{c_n - \frac{1}{12n}\right\}$ is increasing. Hence conclude that $\lim_{n\to\infty} c_n = c$ exists, and that

$$\lim_{n\to\infty} \frac{n!}{n^{n+1/2}e^{-n}} = \lim_{n\to\infty} e^{c_n} = e^c$$

**92.** Suppose you want to approximate $\sqrt[3]{128}$ to within $10^{-4}$ of the exact value.

*a)* Use a Taylor polynomial for $f(x) = (125+x)^{1/3}$ centered at 0.

*b)* Use a Taylor polynomial for $f(x) = x^{1/3}$ centered at 125.

*c)* Compare the two approaches. Are they equivalent?

**93.** Consider the function

$$f(x) = \begin{cases} e^{-1/x^2} & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$

*a)* Use the definition of the derivative to show that $f'(0) = 0$

*b)* Assume the fact that $f^k(0) = 0$ for $k$ = 1, 2, 3, …. (prove using the definition of the derivative.) Write the Taylor series for $f$ centered at 0.

*c)* Explain why the Taylor series for $f$ does not converge to $f$ for $x \neq 0$

**94.** Teams *A* and *B* go into sudden death overtime after playing to a tie. The teams alternate possession of the ball and the first team to score wins. Each team has a $\frac{1}{6}$ chance of scoring when it has the ball, with Team *A* having the ball first.

*a)* The probability that Team *A* ultimately wins is $\sum_{k=0}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^{2k}$. Evaluate this series.

*b)* The expected number of rounds (possessions by either team) required for the overtime to end is $\frac{1}{6}\sum_{k=1}^{\infty} k\left(\frac{5}{6}\right)^{k-1}$. Evaluate this series.