

## ***Solution***      **Section 2.7 – Implicit Differentiation**

### ***Exercise***

Find  $\frac{dy}{dx}$ :       $y^2 + x^2 - 2y - 4x = 4$

### **Solution**

$$\frac{d}{dx}(y^2 + x^2 - 2y - 4x) = \frac{d}{dx}(4)$$

$$\frac{d}{dx}(y^2) + \frac{d}{dx}(x^2) - \frac{d}{dx}(2y) - \frac{d}{dx}(4x) = \frac{d}{dx}(4)$$

$$2y \frac{dy}{dx} + 2x - 2 \frac{dy}{dx} - 4 = 0$$

$$2(y-1) \frac{dy}{dx} = 4 - 2x$$

$$(y-1) \frac{dy}{dx} = 2 - x$$

$$\boxed{\frac{dy}{dx} = \frac{2-x}{y-1}}$$

### ***Exercise***

Find  $\frac{dy}{dx}$ :       $x^2 y^2 - 2x = 3$

### **Solution**

$$2xy^2 + 2x^2 yy' - 2 = 0$$

$$2x^2 yy' = 2 - 2xy^2$$

$$y' = \frac{2(1 - xy^2)}{2x^2 y}$$

$$\boxed{\frac{dy}{dx} = \frac{1 - xy^2}{x^2 y}}$$

### Exercise

Find  $\frac{dy}{dx}$ :  $x + \sqrt{x}\sqrt{y} = y^2$

### Solution

$$\frac{d}{dx}(x + x^{1/2}y^{1/2}) = \frac{d}{dx}y^2$$

$$1 + \frac{d}{dx}(x^{1/2})y^{1/2} + x^{1/2}\frac{d}{dx}(y^{1/2}) = 2y\frac{dy}{dx}$$

$$1 + \frac{1}{2}x^{-1/2}y^{1/2} + x^{1/2}\frac{1}{2}y^{-1/2}\frac{dy}{dx} = 2y\frac{dy}{dx}$$

$$1 + \frac{y^{1/2}}{2x^{1/2}} + \frac{x^{1/2}}{2y^{1/2}}\frac{dy}{dx} = 2y\frac{dy}{dx}$$

$$1 + \frac{y^{1/2}}{2x^{1/2}} = 2y\frac{dy}{dx} - \frac{x^{1/2}}{2y^{1/2}}\frac{dy}{dx}$$

$$\left(\frac{4y^{3/2} - x^{1/2}}{2y^{1/2}}\right)\frac{dy}{dx} = \frac{2x^{1/2} + y^{1/2}}{2x^{1/2}}$$

$$\frac{dy}{dx} = \frac{2x^{1/2} + y^{1/2}}{2x^{1/2}} \cdot \frac{2y^{1/2}}{4y^{3/2} - x^{1/2}}$$

$$= \frac{4x^{1/2}y^{1/2} + 2y}{8x^{1/2}y^{3/2} - 2x}$$

*Divide every term by 2*

$$= \frac{2x^{1/2}y^{1/2} + y}{4x^{1/2}y^{3/2} - x}$$

### Exercise

Find  $\frac{dy}{dx}$ :  $x^2y + xy^2 = 6$

### Solution

$$\left(2xy + x^2\frac{dy}{dx}\right) + \left(y^2 + 2xy\frac{dy}{dx}\right) = 0$$

$$x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} = -2xy - y^2$$

$$\left(x^2 + 2xy\right)\frac{dy}{dx} = -2xy - y^2$$

$$\frac{dy}{dx} = \frac{-2xy - y^2}{x^2 + 2xy}$$

### Exercise

Find  $\frac{dy}{dx}$ :  $x^3 - xy + y^3 = 1$

### Solution

$$3x^2 - \left(y + x \frac{dy}{dx}\right) + 3y^2 \frac{dy}{dx} = 0$$

$$3x^2 - y - x \frac{dy}{dx} + 3y^2 \frac{dy}{dx} = 0$$

$$(3y^2 - x) \frac{dy}{dx} = y - 3x^2$$

$$\frac{dy}{dx} = \frac{y - 3x^2}{3y^2 - x}$$

### Exercise

Find  $\frac{dy}{dx}$ :  $y^2 = \frac{x-1}{x+1}$

### Solution

$$2yy' = \frac{1(x+1) - (1)(x-1)}{(x+1)^2}$$

$$2yy' = \frac{x+1-x+1}{(x+1)^2}$$

$$y' = \frac{2}{2y(x+1)^2}$$

$$\frac{dy}{dx} = \frac{1}{y(x+1)^2}$$

### Exercise

Find  $\frac{dy}{dx}$ :  $(3xy + 7)^2 = 6y$

### Solution

$$2(3xy + 7)(3y + 3xy') = 6y'$$

$$6(3xy + 7)(y + xy') = 6y'$$

$$(3xy + 7)(y + xy') = y'$$

$$3xy^2 + 3x^2yy' + 7y + 7xy' = y'$$

*Divide by 6 both sides*

$$\begin{aligned}
 3x^2 yy' + 7xy' - y' &= -3xy^2 - 7y \\
 (3x^2 y + 7x - 1)y' &= -(3xy^2 + 7y) \\
 \frac{dy}{dx} &= -\frac{3xy^2 + 7y}{3x^2 y + 7x - 1}
 \end{aligned}$$

### Exercise

Find  $\frac{dy}{dx}$ :  $xy = \cot(xy)$

### Solution

$$\begin{aligned}
 y + xy' &= -\csc^2(xy) (y + xy') \\
 y + xy' &= -y \csc^2(xy) - x \csc^2(xy) y' \\
 x \csc^2(xy) y' + xy' &= -y \csc^2(xy) - y \\
 x(\csc^2(xy) + 1)y' &= -y(\csc^2(xy) + 1) \\
 y' &= -\frac{y(\csc^2(xy) + 1)}{x(\csc^2(xy) + 1)} \\
 \frac{dy}{dx} &= -\frac{y}{x}
 \end{aligned}$$

### Exercise

Find  $\frac{dy}{dx}$ :  $x + \tan(xy) = 0$

### Solution

$$\begin{aligned}
 1 + \sec^2(xy)(y + xy') &= 0 \\
 1 + y \sec^2(xy) + x \sec^2(xy) y' &= 0 \\
 x \sec^2(xy) y' &= -y \sec^2(xy) - 1 \\
 y' &= -\frac{y \sec^2(xy)}{x \sec^2(xy)} - \frac{1}{x \sec^2(xy)} \\
 \frac{dy}{dx} &= -\frac{y}{x} - \frac{\cos^2 x}{x} \\
 &= \frac{-y - \cos^2 x}{x}
 \end{aligned}$$

### Exercise

Find  $\frac{dy}{dx}$ :  $x \cos(2x + 3y) = y \sin x$

### Solution

$$\cos(2x + 3y) - \sin(2x + 3y)(2x + 3y') = y' \sin x + y \cos x$$

$$\cos(2x + 3y) - 2x \sin(2x + 3y) - 3 \sin(2x + 3y) y' = y' \sin x + y \cos x$$

$$\cos(2x + 3y) - 2x \sin(2x + 3y) - y \cos x = y' \sin x + 3 \sin(2x + 3y) y'$$

$$\cos(2x + 3y) - 2x \sin(2x + 3y) - y \cos x = y'(\sin x + 3 \sin(2x + 3y))$$

$$y' = \frac{\cos(2x + 3y) - 2x \sin(2x + 3y) - y \cos x}{\sin x + 3 \sin(2x + 3y)}$$

### Exercise

Find  $\frac{dy}{dx}$ :  $y = \frac{e^y}{1 + \sin x}$

### Solution

$$y(1 + \sin x) = e^y$$

$$y'(1 + \sin x) + y \cos x = y' e^y$$

$$y'(e^y - 1 - \sin x) = y \cos x$$

$$\frac{dy}{dx} = \frac{y \cos x}{e^y - 1 - \sin x}$$

### Exercise

Find  $\frac{dy}{dx}$ :  $\sin x \cos(y - 1) = \frac{1}{2}$

### Solution

$$\cos x \cos(y - 1) - y' \sin x \sin(y - 1) = 0$$

$$y' \sin x \sin(y - 1) = \cos x \cos(y - 1)$$

$$y' = \frac{\cos x \cos(y - 1)}{\sin x \sin(y - 1)}$$

$$\frac{dy}{dx} = \cot x \cot(y - 1)$$

### Exercise

Find  $\frac{dy}{dx}$ :  $y\sqrt{x^2 + y^2} = 15$

### Solution

$$y'\sqrt{x^2 + y^2} + \frac{1}{2}y(2x + 2yy')\left(x^2 + y^2\right)^{-1/2} = 0 \quad \times \sqrt{x^2 + y^2}$$

$$y'(x^2 + y^2) + y(x + yy') = 0$$

$$y'(x^2 + y^2) + y^2y' = -xy$$

$$y'(x^2 + 2y^2) = -xy$$

$$\frac{dy}{dx} = -\frac{xy}{x^2 + 2y^2}$$

### Exercise

Find  $\frac{dr}{d\theta}$   $r - 2\sqrt{\theta} = \frac{3}{2}\theta^{2/3} + \frac{4}{3}\theta^{3/4}$

### Solution

$$r - 2\theta^{1/2} = \frac{3}{2}\theta^{2/3} + \frac{4}{3}\theta^{3/4}$$

$$\frac{dr}{d\theta} - 2\frac{1}{2}\theta^{-1/2} = \frac{3}{2}\frac{2}{3}\theta^{-1/3} + \frac{4}{3}\frac{3}{4}\theta^{-1/4}$$

$$\frac{dr}{d\theta} = \theta^{-1/3} + \theta^{-1/4} + \theta^{-1/2}$$

### Exercise

Find  $\frac{dr}{d\theta}$   $\sin(r\theta) = \frac{1}{2}$

### Solution

$$\cos(r\theta)\left(\theta\frac{dr}{d\theta} + r\right) = 0$$

$$\theta\frac{dr}{d\theta} + r = 0 \quad \cos(r\theta) \neq 0$$

$$\frac{dr}{d\theta} = -\frac{r}{\theta} \quad \cos(r\theta) \neq 0$$

### Exercise

Find  $\frac{d^2y}{dx^2}$   $x^{2/3} + y^{2/3} = 1$

### Solution

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

*Multiply all terms by  $\frac{3}{2}$*

$$x^{-1/3} + y^{-1/3}y' = 0$$

$$y^{-1/3}y' = -x^{-1/3}$$

$$y' = -\frac{x^{-1/3}}{y^{-1/3}}$$

$$= -\frac{y^{1/3}}{x^{1/3}} = -\left(\frac{y}{x}\right)^{1/3}$$

$$y'' = -\frac{1}{3}\left(\frac{y}{x}\right)^{-2/3}\left(\frac{xy' - y}{x^2}\right)$$

$$= -\frac{1}{3}\left(\frac{x}{y}\right)^{2/3}\left(\frac{-x\left(\frac{y}{x}\right)^{1/3} - y}{x^2}\right) = \frac{1}{3}\left(\frac{x^{4/3}y^{1/3}}{y^{2/3}x^2} + \frac{x^{2/3}y}{y^{2/3}x^2}\right)$$

$$= \frac{1}{3}\left(\frac{x}{y}\right)^{2/3}\left(\frac{x\frac{y^{1/3}}{x^{1/3}} + y}{x^2}\right)$$

$$= \frac{1}{3}\frac{x^{2/3}}{y^{2/3}}\frac{x^{2/3}y^{1/3} + y}{x^2}$$

$$= \frac{1}{3}\left(\frac{1}{y^{1/3}x^{2/3}} + \frac{y^{1/3}}{x^{4/3}}\right)$$

### Exercise

Find  $\frac{d^2y}{dx^2}$   $2\sqrt{y} = x - y$

### Solution

$$2\frac{1}{2}y^{-1/2}y' = 1 - y'$$

$$2\frac{1}{2}y^{-1/2}y' + y' = 1$$

$$\left(y^{-1/2} + 1\right)y' = 1$$

$$y' = \frac{1}{y^{-1/2} + 1} = \frac{1}{\frac{1}{\sqrt{y}} + 1} = \frac{\sqrt{y}}{1 + \sqrt{y}}$$

$$\left(y^{-1/2} + 1\right)y'' + \left(-\frac{1}{2}y^{-3/2}y'\right)y' = 0$$

$$\left(y^{-1/2} + 1\right)y'' - \frac{1}{2}y^{-3/2}(y')^2 = 0$$

$$\left(y^{-1/2} + 1\right)y'' = \frac{1}{2}y^{-3/2}\left(\frac{1}{y^{-1/2} + 1}\right)^2$$

$$y'' = \frac{1}{2}y^{-3/2} \frac{1}{\left(y^{-1/2} + 1\right)^2} \frac{1}{y^{-1/2} + 1}$$

$$= \frac{1}{2}y^{-3/2} \frac{1}{\left(\frac{1}{\sqrt{y}} + 1\right)^3}$$

$$= \frac{1}{2}y^{-3/2} \frac{1}{\left(\frac{1 + \sqrt{y}}{\sqrt{y}}\right)^3}$$

$$= \frac{1}{2}y^{-3/2} \frac{1}{\frac{(1 + \sqrt{y})^3}{(y^{1/2})^3}}$$

$$= \frac{1}{2}y^{-3/2} \frac{y^{3/2}}{(1 + \sqrt{y})^3}$$

$$= \frac{1}{2(1 + \sqrt{y})^3}$$

### ***Exercise***

If  $x^3 + y^3 = 16$ , find the value of  $\frac{d^2y}{dx^2}$  at the point (2, 2).

### ***Solution***

$$3x^2 + 3y^2y' = 0$$

$$3y^2y' = -3x^2$$



$$y^2 y' = -x^2$$

$$2yy'y' + y^2 y'' = -2x$$

$$y^2 y'' = -2x - 2y(y')^2$$

$$y^2 y'' = -2x - 2y \left( \frac{-x^2}{y^2} \right)^2$$

$$y^2 y'' = -2x - 2 \frac{x^4}{y^3}$$

$$y'' = -2 \frac{x}{y^2} - 2 \frac{x^4}{y^5}$$

$$= \frac{-2xy^3 - 2x^4}{y^5}$$

$$y'' \Big|_{(2,2)} = \frac{-2(2)2^3 - 2(2)^4}{2^5}$$

$$= \frac{-2^5 - 2^5}{2^5}$$

$$\underline{\underline{= -2}}$$

### Exercise

Find  $dy/dx$ :  $x^2 - xy + y^2 = 4$  and evaluate the derivative at the given point  $(0, -2)$

### Solution

$$2x - (y + xy') + 2yy' = 0$$

$$-y - xy' + 2yy' = -2x$$

$$(2y - x)y' = y - 2x$$

$$\underline{\underline{\frac{dy}{dx} = \frac{y - 2x}{2y - x}}}$$

$$@ (0, -2) \rightarrow \frac{dy}{dx} = \frac{-2 - 2(0)}{2(-2) - (0)}$$

$$= \frac{-2}{-4}$$

$$\underline{\underline{= \frac{1}{2}}}$$

**Exercise**

Find the slope of the curve  $(x^2 + y^2)^2 = (x - y)^2$  at the point  $(-2, 1)$  and  $(-2, -1)$

**Solution**

1 and  $-1$

**Exercise**

Find the slope of the tangent line to the circle  $x^2 - 9y^2 = 16$  at the point  $(5, 1)$

**Solution**

$$2x - 18y \frac{dy}{dx} = 0$$

$$-18y \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{-18y} = \frac{x}{9y}$$

$$\begin{aligned} @ (5, 1) \rightarrow \frac{dy}{dx} &= \frac{5}{9(1)} \\ &= \frac{5}{9} \end{aligned}$$

**Exercise**

Find the slope of the tangent line to the circle  $x^2 + y^2 = 25$  at the point  $(3, -4)$

**Solution**

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$2x + 2y \frac{dy}{dx} = 0$$

$$2y \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\begin{aligned} \text{Slope: } \frac{dy}{dx} &= -\frac{3}{-4} \\ &= \frac{3}{4} \end{aligned}$$

### Exercise

Find an equation of the line tangent to the following curves at the given point

$$y = 3x^3 + \sin x; \quad (0, 0)$$

### Solution

$$m = y' = 9x^2 + \cos x \Big|_{(0, 0)}$$

$$= 1 \Big|$$

$$\underline{y = x}$$

$$y = m(x - x_1) + y_1$$

### Exercise

Find an equation of the line tangent to the following curves at the given point

$$y = \frac{4x}{x^2 + 3}; \quad (3, 1)$$

### Solution

$$m = y' = \frac{4x^2 + 12 - 8x^2}{(x^2 + 3)^2}$$

$$= \frac{12 - 4x^2}{(x^2 + 3)^2} \Big|_{(3, 1)}$$

$$= \frac{-24}{144}$$

$$= -\frac{1}{6} \Big|$$

$$y = -\frac{1}{6}(x - 3) + 1$$

$$\underline{= -\frac{1}{6}x + \frac{3}{2}}$$

$$y = m(x - x_1) + y_1$$

### Exercise

Find an equation of the line tangent to the following curves at the given point

$$y + \sqrt{xy} = 6; \quad (1, 4)$$

### Solution

$$y' + \frac{1}{2}(y + xy') \frac{1}{\sqrt{xy}} = 0 \Big|_{(1, 4)}$$

$$y' + \frac{1}{2}(4 + y') \frac{1}{2} = 0$$

$$y' + \frac{1}{4}y' = -1$$

$$\frac{5}{4}y' = -1$$

$$m = y' = -\frac{4}{5} \quad \Bigg|$$

$$y = -\frac{4}{5}(x-1) + 4$$

$$= -\frac{4}{5}x + \frac{24}{5} \quad \Bigg|$$

$$y = m(x - x_1) + y_1$$

### Exercise

Find an equation of the line tangent to the following curves at the given point

$$x^2y + y^3 = 75; \quad (4, 3)$$

### Solution

$$2xy + x^2y' + 3y^2y' = 0 \quad \Bigg| \quad (4, 3)$$

$$(16 + 27)y' = -24$$

$$y' = -\frac{24}{43} = m \quad \Bigg|$$

$$y = -\frac{24}{43}(x-4) + 3$$

$$= -\frac{24}{43}x + \frac{225}{43} \quad \Bigg|$$

$$y = m(x - x_1) + y_1$$

### Exercise

Find the equation of the tangent line to the circle  $x^3 + y^3 = 9xy$  at the point (2, 4)

### Solution

$$3x^2 + 3y^2y' = 9y + 9xy'$$

$$3y^2y' - 9xy' = 9y - 3x^2$$

$$(3y^2 - 9x)y' = 9y - 3x^2$$

$$y' = \frac{3(3y - x^2)}{3(y^2 - 3x)}$$

$$= \frac{3y - x^2}{y^2 - 3x}$$

$$\begin{aligned}
 m \Big|_{(2,4)} &= \frac{3(4)-2^2}{4^2-3(2)} \\
 &= \frac{8}{10} \\
 &= \frac{4}{5} \Big|
 \end{aligned}$$

$$\begin{aligned}
 y &= \frac{4}{5}(x-2) + 4 \\
 &= \frac{4}{5}x - \frac{8}{5} + 4 \\
 y &= \frac{4}{5}x + \frac{12}{5} \Big|
 \end{aligned}$$

$$y = m(x - x_1) + y_1$$

### Exercise

Find the lines that are **(a)** tangent and **(b)** normal to the curve  $x^2 + xy - y^2 = 1$  at the point  $(2, 3)$ .

### Solution

$$2x + y + xy' - 2yy' = 0$$

$$(x - 2y)y' = -2x - y$$

$$y' = \frac{-2x-y}{x-2y} = \frac{2x+y}{2y-x}$$

$$\begin{aligned}
 \text{a) tangent slope} &= y' \Big|_{(2,3)} \\
 &= \frac{2(2)+3}{2(3)-2} \\
 &= \frac{7}{4} \Big|
 \end{aligned}$$

$$\begin{aligned}
 y &= \frac{7}{4}(x-2) + 3 \\
 &= \frac{7}{4}x - \frac{7}{2} + 3 \\
 &= \frac{7}{4}x - \frac{1}{2} \Big|
 \end{aligned}$$

$$y = m(x - x_1) + y_1$$

$$\text{b) normal slope} = -\frac{4}{7}$$

$$y = -\frac{4}{7}(x-2) + 3$$

$$y = \frac{4}{7}x - \frac{8}{7} + 3$$

$$y = -\frac{4}{7}x + \frac{29}{7} \Big|$$

$$y = m(x - x_1) + y_1$$

### Exercise

Find the lines that are **(a)** tangent and **(b)** normal to the curve  $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$  at the point  $(-1, 0)$ .

#### Solution

$$12x + 3y + 3xy' + 4yy' + 17y' = 0$$

$$(3x + 4y + 17)y' = -12x - 3y$$

$$y' = \frac{-12x - 3y}{3x + 4y + 17}$$

$$\begin{aligned} \text{a) tangent slope} &= y' \Big|_{(-1, 0)} \\ &= \frac{-12(-1) - 3(0)}{3(-1) + 4(0) + 17} \\ &= \frac{6}{7} \end{aligned}$$

$$y = \frac{6}{7}(x + 1)$$

$$= \frac{6}{7}x + \frac{6}{7}$$

$$y = m(x - x_1) + y_1$$

$$\text{b) normal slope} = -\frac{7}{6}$$

$$y = -\frac{7}{6}(x + 1) \Rightarrow y = -\frac{7}{6}x - \frac{7}{6}$$

$$y = m(x - x_1) + y_1$$

### Exercise

Find the lines that are **(a)** tangent and **(b)** normal to the curve  $x^2 \cos^2 y - \sin y = 0$  at the point  $(0, \pi)$ .

#### Solution

$$2x \cos^2 y + x^2 (2 \cos y (-\sin y) y') - (\cos y) y' = 0$$

$$(-2x^2 \cos y \sin y - \cos y) y' = -2x \cos^2 y$$

$$y' = \frac{-2x \cos^2 y}{-(2x^2 \sin y + 1) \cos y} = \frac{2x \cos y}{2x^2 \sin y + 1}$$

$$\begin{aligned} \text{a) tangent slope} &= y' \Big|_{(0, \pi)} \\ &= \frac{2(0) \cos(\pi)}{2(0)^2 \sin(\pi) + 1} \end{aligned}$$

$$\underline{=0}$$

$$y - \pi = 0(x - 0)$$

$$\underline{y = \pi}$$

**b)** normal slope = 0

$$\Rightarrow \underline{x = 0}$$

### Exercise

Suppose that  $x$  and  $y$  are both functions of  $t$ , which can be considered to represent time, and that  $x$  and  $y$  are related by the equation  $xy^2 + y = x^2 + 17$

Suppose further that when  $x = 2$  and  $y = 3$ , then  $\frac{dx}{dt} = 13$ . Find the value of the  $\frac{dy}{dt}$  at that moment.

### Solution

$$y^2 \frac{dx}{dt} + 2xy \frac{dy}{dt} + \frac{dy}{dt} = 2x \frac{dx}{dt}$$

$$3^2(13) + 2(2)(3) \frac{dy}{dt} + \frac{dy}{dt} = 2(2)(13)$$

$$117 + 12 \frac{dy}{dt} + \frac{dy}{dt} = 52$$

$$13 \frac{dy}{dt} = -65$$

$$\frac{dy}{dt} = \frac{-65}{13}$$

$$\underline{= -5}$$

### Exercise

A cone-shaped icicle is dripping from the roof. The radius of the icicle is decreasing at a rate of 0.2 cm per hour, while the length is increasing at a rate of 0.8 cm per hour. If the icicle is currently 4 cm in radius and 20 cm long, is the volume of the icicle increasing or decreasing and at what rate?

### Solution

The volume of the cone is given by the formula:  $V = \frac{1}{3}\pi r^2 h$ .

$$\frac{dV}{dt} = \frac{1}{3}\pi \left( 2rh \frac{dr}{dt} + r^2 \frac{dh}{dt} \right)$$

**Given** the values:

$$\frac{dr}{dt} = -0.2 \quad \frac{dh}{dt} = 0.8 \quad r = 4 \quad h = 20$$

$$\frac{dV}{dt} = \frac{1}{3}\pi \left( 2(4)(20)(-0.2) + 4^2(0.8) \right)$$

$$\underline{=-20}$$

The volume is decreasing at a rate of  $20 \text{ cm}^3$  per *hour*.