Solution *Section* 4.1 – System of linear Equations

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 3x+2y=-4 \\ 2x-y=-5 \end{cases}$$

Solution

$$\begin{cases} \quad 3x+2y=-4 \\ 2\times \quad 2x-y=-5 \end{cases}$$

$$\begin{array}{r} 3x+2y=-4 \\ \underline{4x-2y=-10} \\ 7x=-14 \end{array}$$

$$\underline{x=-2}|$$

$$\begin{aligned} y &= 2x+5 \\ &= -4+5 \\ &= 1 \end{aligned}$$

Solution: $(-2,\ 1)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 2x+5y=7 \\ 5x-2y=-3 \end{cases}$$

Solution

$$\begin{cases} -5\times \quad 2x+5y=7 \\ 2\times \quad 5x-2y=-3 \end{cases}$$

$$\begin{array}{r} -10x-25y=-35 \\ \underline{10x-4y=-6} \\ -29y=-41 \end{array}$$

$$y=\frac{41}{29}$$

$$x=\frac{1}{2}\left(7-5\left(\frac{41}{29}\right)\right)$$

$$x=\frac{1}{2}\left(-\frac{2}{29}\right)$$

$$= -\frac{1}{29}$$

$\therefore$ ***Solution***: $\left(-\frac{1}{29},\ \frac{41}{29}\right)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$$

Solution

$$\begin{cases} \quad 4x - 7y = -16 \\ -2\times \quad 2x + 5y = 9 \end{cases}$$

$$\begin{array}{r} 4x - 7y = -16 \\ -4x - 10y = -18 \\ \hline -17y = -34 \end{array}$$

$$y = 2$$

$$x = \frac{9 - 5y}{2}$$

$$= \frac{9 - 10}{2}$$

$$= -\frac{1}{2}$$

$\therefore$ ***Solution***: $\left(-\frac{1}{2},\ 2\right)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$$

Solution

$$\begin{cases} 3x + 2y = 4 & (1) \\ 2x + y = 1 & (2) \end{cases}$$

$(2) \rightarrow \quad y = 1 - 2x \quad (3)$

$(1) \rightarrow \quad 3x + 2 - 4x = 4$

$\underline{x = -2}|$

$(3) \rightarrow \quad y = 1 + 4$

$\quad\quad\quad\; \underline{= 5}|$

$\therefore$ ***Solution***: $\quad \underline{(-2,\ 5)}|$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 3x + 4y = 2 \\ 2x + 5y = -1 \end{cases}$$

Solution

$$\begin{cases} -2\times \quad 3x + 4y = 2 \\ \;\;3\times \quad 2x + 5y = -1 \end{cases}$$

$$\begin{array}{r} -6x - 8y = -4 \\ \underline{6x + 15y = -3} \\ 7y = -7 \end{array}$$

$\underline{y = -1}|$

$2x = -1 + 5$

$x = \frac{4}{2}$

$\quad \underline{= 2}|$

$\therefore$ ***Solution***: $\quad \underline{(2,\ -1)}|$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method) $\begin{cases} 5x - 2y = 4 \\ -10x + 4y = 7 \end{cases}$

Solution

$$\begin{cases} 2\times \quad 5x - 2y = 4 \\ \quad\quad -10x + 4y = 7 \end{cases}$$

$$\begin{array}{r} 10x - 4y = 8 \\ \underline{-10x + 4y = 7} \\ 0 = 15 \end{array} \quad (impossible)$$

$\therefore$ **No Solution**

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} x-4y=-8 \\ 5x-20y=-40 \end{cases}$$

Solution

$$\begin{cases} x-4y=-8 & (1) \\ 5x-20y=-40 & (2) \end{cases}$$

$(1) \to \quad x=4y-8$

$(2) \to \quad 5(4y-8)-20y=-40$

$20y-40-20y=-40$

$-40=-40 \quad (\mathit{True})$

$\therefore$ ***Solution***: $\quad \underline{x-4y=-8}|$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$$

Solution

$$\begin{cases} 2x+y=3 & (1) \\ x-y=3 & (2) \end{cases}$$

$(2) \to \quad x=3+y \quad (3)$

$(1) \to \quad 6+2y+y=3$

$3y=-3$

$\underline{y=-1}|$

$(3) \to \quad \underline{x=2}|$

$\therefore$ ***Solution***: $\quad \underline{(2,\ -1)}|$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 2x+10y=-14 \\ 7x-2y=-16 \end{cases}$$

Solution

$$\begin{cases} & 2x+10y=-14 \\ 5\times & 7x-2y=-16 \end{cases}$$

$$\begin{array}{r} 2x+10y=-14 \\ \underline{35x-10y=-80} \\ 37x=-94 \end{array}$$

$$\boxed{x=-\frac{94}{37}}$$

$$2y=7\left(-\frac{94}{37}\right)+16$$

$$y=-\frac{329}{37}+8$$

$$\boxed{=-\frac{33}{37}}$$

$\therefore$ ***Solution***: $\boxed{\left(-\frac{94}{37},\ -\frac{33}{37}\right)}$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 4x-3y=24 \\ -3x+9y=-1 \end{cases}$$

Solution

$$\begin{cases} 3\times & 4x-3y=24 \\ & -3x+9y=-1 \end{cases}$$

$$\begin{array}{r} 12x-9y=72 \\ \underline{-3x+9y=-1} \\ -9x=-71 \end{array}$$

$$\boxed{x=\frac{71}{9}}$$

$$3y=4\left(\frac{71}{9}\right)-24$$

$$y=\frac{284}{27}-8$$

$$= \boxed{\frac{68}{27}}$$

$\therefore$ ***Solution***: $\boxed{\left(\frac{71}{9},\ \frac{68}{27}\right)}$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 4x+2y=12 \\ 3x-2y=16 \end{cases}$$

Solution

$$\begin{array}{r} 4x+2y=12 \\ \underline{3x-2y=16} \\ 7x=28 \end{array}$$

$$\boxed{x=4}$$

$$2y=12-4(4)$$

$$y=-\frac{4}{2}$$

$$=\boxed{-2}$$

$\therefore$ ***Solution***: $\boxed{(4,\ -2)}$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} x+2y=-1 \\ 4x-2y=6 \end{cases}$$

Solution

$$\begin{array}{r} x+2y=-1 \\ \underline{4x-2y=6} \\ 5x=5 \end{array}$$

$$\boxed{x=1}$$

$$2y=-x-1$$

$$y=-\frac{2}{2}$$

$$=\boxed{-1}$$

$\therefore$ ***Solution***: $\boxed{(1,\ -1)}$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} x-2y=5 \\ -10x+2y=4 \end{cases}$$

Solution

$$\begin{array}{r} x-2y=5 \\ \underline{-10x+2y=4} \\ -9x=9 \end{array}$$

$\underline{x=-1}|$

$2y=x-5$

$y=-\frac{6}{2}$

$\quad \underline{=-3}|$

$\therefore$ ***Solution***: $\quad \underline{(-1,\ -3)}|$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 12x+15y=-27 \\ 30x-15y=-15 \end{cases}$$

Solution

$$\begin{array}{r} 12x+15y=-27 \\ \underline{30x-15y=-15} \\ 42x=-42 \end{array}$$

$\underline{x=-1}|$

$15y=-27-12(-1)$

$y=-\frac{15}{15}$

$\quad \underline{=-1}|$

$\therefore$ ***Solution***: $\quad \underline{(-1,\ -1)}|$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$$

Solution

$$\begin{array}{r} 4x - 4y = -12 \\ \underline{4x + 4y = -20} \\ 8x = -32 \end{array}$$

$$\underline{x = -4|}$$

$$4y = 4(-4) + 12$$

$$y = -\frac{4}{4}$$

$$\underline{= -1|}$$

$\therefore$ ***Solution***: $\quad \underline{(-4,\ -1)\ |}$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 1 & 4 & 7 \\ 3 & 5 & 0 \end{array}\right] \quad R_2 - 3R_1$$

Solution

$$\begin{array}{ccc} 3 & 5 & 0 \\ -3 & -12 & -21 \\ \hline 0 & -7 & -21 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 4 & 7 \\ 0 & -7 & -21 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -5 \end{array}\right] \quad R_2 - 2R_1$$

Solution

$$\begin{array}{ccc} 2 & 1 & -5 \\ -2 & 6 & -2 \\ \hline 0 & 7 & -7 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 7 & -7 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 5 & 2 & 19 \end{array}\right] \quad R_2 - 5R_1$$

Solution

$$\begin{array}{ccc} 5 & 2 & 19 \\ -5 & 15 & -15 \\ \hline 0 & 17 & -4 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 17 & -4 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 2 & -3 & 8 \\ -6 & 9 & 4 \end{array}\right] \quad R_2 + 3R_1$$

Solution

$$\begin{array}{ccc} -6 & 9 & 4 \\ 6 & -9 & 24 \\ \hline 0 & 0 & 28 \end{array}$$

$$\left[\begin{array}{cc|c} 2 & -3 & 8 \\ 0 & 0 & 28 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 2 & 3 & 11 \\ 1 & 2 & 8 \end{array}\right] \quad 2R_2 - R_1$$

Solution

$$\begin{array}{ccc} 2 & 4 & 16 \\ -2 & -3 & -11 \\ \hline 0 & 1 & 5 \end{array}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 11 \\ 0 & 1 & 5 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 3 & 5 & -13 \\ 2 & 3 & -9 \end{array}\right] \quad 3R_2 - 2R_1$$

Solution

$$\begin{array}{ccc} 6 & 9 & -27 \\ -6 & -10 & 26 \\ \hline 0 & -1 & -1 \end{array}$$

$$\left[\begin{array}{cc|c} 3 & 5 & -13 \\ 0 & -1 & -1 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 5 & 4 & 1 \end{array}\right] \quad R_3 - 5R_2$$

Solution

$$\begin{array}{cccc} 0 & 5 & 4 & 1 \\ 0 & -5 & 5 & -10 \\ \hline 0 & 0 & 9 & -9 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 9 & -9 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array}\right] \quad \begin{array}{c} R_2 - 3R_1 \\ R_3 - R_1 \end{array}$$

Solution

$$\begin{array}{cccc} 3 & 3 & -1 & 10 \\ -3 & 3 & -15 & 18 \\ \hline 0 & 6 & -16 & 28 \end{array} \qquad\qquad \begin{array}{cccc} 1 & 3 & 2 & 5 \\ -1 & 1 & -5 & 6 \\ \hline 0 & 4 & -3 & 11 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 0 & 4 & -3 & 11 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 2 & 4 & 4 & 22 \\ -1 & -2 & 3 & 15 \end{array}\right] \quad \begin{array}{c} 3R_2 - 2R_1 \\ 3R_3 + R_1 \end{array}$$

Solution

$$\begin{array}{cccc} 6 & 12 & 12 & 66 \\ -6 & -4 & -2 & -2 \\ \hline 0 & 8 & 10 & 64 \end{array} \qquad\qquad \begin{array}{cccc} -3 & -6 & 9 & 45 \\ 3 & 2 & 1 & 1 \\ \hline 0 & -4 & 10 & 46 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 0 & 8 & 10 & 64 \\ 0 & -4 & 10 & 46 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & -4 & 2 & -7 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

Solution

$$\begin{array}{cccc} 2 & 1 & 1 & 3 \\ -2 & -2 & -2 & -4 \\ \hline 0 & -1 & -1 & -1 \end{array} \qquad \begin{array}{cccc} 3 & -4 & 2 & -7 \\ -3 & -3 & -3 & -6 \\ \hline 0 & -7 & -1 & -13 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & -1 \\ 0 & -7 & -1 & -13 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & -2 \\ 2 & -3 & 5 & -1 & 0 \\ 1 & 0 & 3 & 1 & -4 \\ -4 & 3 & 2 & -1 & 3 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 + 4R_1 \end{array}$$

Solution

$$\begin{array}{ccccc} 2 & -3 & 5 & -1 & 0 \\ -2 & 4 & -2 & -6 & 4 \\ \hline 0 & 1 & 3 & -7 & 4 \end{array} \qquad \begin{array}{ccccc} 1 & 0 & 3 & 1 & -4 \\ -1 & 2 & -1 & -3 & 2 \\ \hline 0 & 2 & 2 & -2 & -2 \end{array} \qquad \begin{array}{ccccc} -4 & 3 & 2 & -1 & 3 \\ 4 & -8 & 4 & 12 & -8 \\ \hline 0 & -5 & 6 & 11 & -5 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & -2 \\ 0 & 1 & 3 & -7 & 4 \\ 0 & 2 & 2 & -2 & -2 \\ 0 & -5 & 6 & 11 & -5 \end{array}\right]$$

Exercise

Use the Gauss-Jordan method to solve the system
$$\begin{aligned} x-y+5z&=-6\\ 3x+3y-z&=10\\ x+3y+2z&=5 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6\\ 3 & 3 & -1 & 10\\ 1 & 3 & 2 & 5 \end{array}\right] \begin{array}{l} \\ R_2-3R_1\\ R_3-R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 3 & -1 & 10\\ -3 & 3 & -15 & 18\\ \hline 0 & 6 & -16 & 28 \end{array} \qquad \begin{array}{cccc} 1 & 3 & 2 & 5\\ -1 & 1 & -5 & 6\\ \hline 0 & 4 & -3 & 11 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6\\ 0 & 6 & -16 & 28\\ 0 & 4 & -3 & 11 \end{array}\right] \begin{array}{l} \\ \frac{1}{6}R_2\\ \\ \end{array}$$

$$\begin{array}{cccc} 0 & 1 & -\frac{8}{3} & \frac{14}{3} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6\\ 0 & 1 & -\frac{8}{3} & \frac{14}{3}\\ 0 & 4 & -3 & 11 \end{array}\right] \begin{array}{l} R_1+R_2\\ \\ R_3-4R_2 \end{array}$$

$$\begin{array}{cccc} 0 & 4 & -3 & 11\\ 0 & -4 & \frac{32}{3} & -\frac{56}{3}\\ \hline 0 & 0 & \frac{23}{3} & -\frac{23}{3} \end{array} \qquad \begin{array}{cccc} 1 & -1 & 5 & -6\\ 0 & 1 & -\frac{8}{3} & \frac{14}{3}\\ \hline 1 & 0 & \frac{7}{3} & -\frac{4}{3} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{3} & -\frac{4}{3}\\ 0 & 1 & -\frac{8}{3} & \frac{14}{3}\\ 0 & 0 & \frac{23}{3} & -\frac{23}{3} \end{array}\right] \begin{array}{l} \\ \\ \frac{3}{23}R_3 \end{array}$$

$$\begin{array}{cccc} 0 & 0 & 1 & -1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{3} & -\frac{4}{3}\\ 0 & 1 & -\frac{8}{3} & \frac{14}{3}\\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} R_1-\frac{7}{3}R_3\\ R_2+\frac{8}{3}R_3\\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & \frac{7}{3} & -\frac{4}{3}\\ 0 & 0 & -\frac{7}{3} & \frac{7}{3}\\ \hline 1 & 0 & 0 & 1 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -\frac{8}{3} & \frac{14}{3}\\ 0 & 0 & \frac{8}{3} & -\frac{8}{3}\\ \hline 0 & 1 & 0 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & -1 \end{array}\right]$$

S***olution***: $(1,\ 2,\ -1)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} 2x - y + 4z = -3 \\ x - 2y - 10z = -6 \\ 3x \qquad + 4z = 7 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & -1 & 4 & -3 \\ 1 & -2 & -10 & -6 \\ 3 & 0 & 4 & 7 \end{array}\right] \begin{array}{l} \frac{1}{2}R_1 \\ \\ \\ \end{array} \qquad \begin{array}{cccc} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 1 & -2 & -10 & -6 \\ 3 & 0 & 4 & 7 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 3R_1 \end{array} \qquad \begin{array}{cccc} 1 & -2 & -10 & -6 \\ -1 & \frac{1}{2} & -2 & \frac{3}{2} \\ \hline 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \end{array} \qquad \begin{array}{cccc} 3 & 0 & 4 & 7 \\ -3 & \frac{3}{2} & -6 & \frac{9}{2} \\ \hline 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \\ 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}\right] \begin{array}{l} \\ -\frac{2}{3}R_2 \\ \\ \end{array} \qquad \begin{array}{cccc} 0 & 1 & 8 & 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & 1 & 8 & 3 \\ 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}\right] \begin{array}{l} R_1 + \frac{1}{2}R_2 \\ \\ R_3 - \frac{3}{2}R_2 \end{array} \qquad \begin{array}{cccc} 0 & \frac{3}{2} & -2 & \frac{23}{2} \\ 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \\ \hline 0 & 0 & -14 & 7 \end{array} \qquad \begin{array}{cccc} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & \frac{1}{2} & 4 & \frac{3}{2} \\ \hline 1 & 0 & 6 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & -14 & 7 \end{array}\right] \begin{array}{l} \\ \\ -\frac{1}{14}R_3 \end{array} \qquad \begin{array}{cccc} 0 & 0 & 1 & -\frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right] \begin{array}{l} R_1 - 6R_3 \\ R_2 - 8R_3 \\ \\ \end{array} \qquad \begin{array}{cccc} 1 & 0 & 6 & 0 \\ 0 & 0 & -6 & 3 \\ \hline 1 & 0 & 0 & 3 \end{array} \qquad \begin{array}{cccc} 0 & 1 & 8 & 3 \\ 0 & 0 & -8 & 4 \\ \hline 0 & 1 & 0 & 7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right]$$

Solution: $\left(3,\ 7,\ -\frac{1}{2}\right)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} 4x+3y-5z=-29 \\ 3x-7y-z=-19 \\ 2x+5y+2z=-10 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 4 & 3 & -5 & -29 \\ 3 & -7 & -1 & -19 \\ 2 & 5 & 2 & -10 \end{array}\right] \begin{array}{l} \frac{1}{4}R_1 \\ \\ \\ \end{array} \qquad \begin{array}{cccc} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 3 & -7 & -1 & -19 \\ 2 & 5 & 2 & -10 \end{array}\right] \begin{array}{l} \\ R_2-3R_1 \\ R_3-2R_1 \end{array} \qquad \begin{array}{cccc} 3 & -7 & -1 & -19 \\ -3 & -\frac{9}{4} & \frac{15}{4} & \frac{87}{4} \\ \hline 0 & -\frac{37}{4} & \frac{11}{4} & \frac{11}{4} \end{array} \qquad \begin{array}{cccc} 2 & 5 & 2 & -10 \\ -2 & -\frac{3}{2} & \frac{5}{2} & \frac{29}{2} \\ \hline 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & -\frac{37}{4} & \frac{11}{4} & \frac{11}{4} \\ 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}\right] \begin{array}{l} \\ -\frac{4}{37}R_2 \\ \\ \end{array} \qquad \begin{array}{cccc} 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}\right] \begin{array}{l} R_1-\frac{3}{4}R_2 \\ \\ R_3-\frac{7}{2}R_2 \end{array} \qquad \begin{array}{cccc} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & -\frac{3}{4} & \frac{33}{148} & \frac{33}{148} \\ \hline 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \end{array} \qquad \begin{array}{cccc} 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \\ 0 & -\frac{7}{2} & \frac{77}{72} & \frac{77}{72} \\ \hline 0 & 0 & \frac{401}{72} & \frac{401}{72} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & \frac{401}{72} & \frac{401}{72} \end{array}\right] \begin{array}{l} \\ \\ \frac{72}{401}R_3 \end{array} \qquad \begin{array}{cccc} 0 & 0 & 1 & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1+\frac{38}{37}R_3 \\ R_2+\frac{11}{37}R_3 \\ \\ \end{array} \qquad \begin{array}{cccc} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 0 & \frac{38}{37} & \frac{38}{37} \\ \hline 1 & 0 & 0 & -6 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & \frac{11}{37} & \frac{11}{37} \\ \hline 0 & 1 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Solution: $\boxed{(-6,\ 0,\ 1)}$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} x+2y-3z=-15 \\ 2x-3y+4z=18 \\ -3x+y+z=1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 2 & -3 & 4 & 18 \\ -3 & 1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 2 & -3 & 4 & 18 \\ -3 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} R_2-2R_1 \\ R_3+3R_1 \end{array}$$

$$\begin{array}{cccc} -2 & -4 & 6 & 30 \\ 2 & -3 & 4 & 18 \\ \hline 0 & -7 & 10 & 48 \end{array} \qquad \begin{array}{cccc} 3 & 6 & -9 & -45 \\ -3 & 1 & 1 & 1 \\ \hline 0 & 7 & -8 & -44 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 0 & -7 & 10 & 48 \\ 0 & 7 & -8 & -44 \end{array}\right] \quad -\frac{1}{7}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 7 & -8 & -44 \end{array}\right] \begin{array}{l} R_1-2R_2 \\ \\ R_3-7R_2 \end{array}$$

$$\begin{array}{cccc} 1 & 2 & -3 & -15 \\ 0 & -2 & \frac{20}{7} & \frac{96}{7} \\ \hline 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \end{array} \qquad \begin{array}{cccc} 0 & -7 & 10 & 48 \\ 0 & 7 & -8 & -44 \\ \hline 0 & 0 & 2 & 4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & 2 & 4 \end{array}\right] \quad \frac{1}{2}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1+\frac{1}{7}R_3 \\ R_2+\frac{10}{7}R_3 \end{array}$$

$$\begin{array}{cccc} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 0 & \frac{1}{7} & \frac{2}{7} \\ \hline 1 & 0 & 0 & -1 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & \frac{10}{7} & \frac{20}{7} \\ \hline 0 & 1 & 0 & -4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solution: $(-1,\ -4,\ 2)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} x+2y+3z=10 \\ 4x+5y+6z=11 \\ 7x+8y+9z=12 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 4 & 5 & 6 & 11 \\ 7 & 8 & 9 & 12 \end{array}\right] \begin{array}{l} \\ R_2 - 4R_1 \\ R_3 - 7R_1 \end{array}$$

$$\begin{array}{cccc} -4 & -8 & -12 & -40 \\ 4 & 5 & 6 & 11 \\ \hline 0 & -3 & -6 & -29 \end{array} \qquad \begin{array}{cccc} -7 & -14 & -21 & -70 \\ 7 & 8 & 9 & 12 \\ \hline 0 & -6 & -12 & -58 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & -3 & -6 & -29 \\ 0 & -6 & -12 & -58 \end{array}\right] \ \tfrac{1}{3}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & \frac{29}{3} \\ 0 & -6 & -12 & -58 \end{array}\right] \ R_3 + 6R_2$$

$$\begin{array}{cccc} 0 & -6 & -12 & -58 \\ 0 & 6 & 12 & 58 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & \frac{29}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

let z be the variable

From Row 1 $\Rightarrow y + 2z = \frac{29}{3}$

$$y = \frac{29}{3} - 2z$$

From Row 1 $\Rightarrow x + 2y + 3z = 10$

$$x = 10 - 2y - 3z$$

$$x = 10 - 2\left(\frac{29}{3} - 2z\right) - 3z$$

$$x = 10 - \frac{58}{3} + 4z - 3z$$

$$x = z - \frac{28}{3}$$

Solution: $\left(z - \frac{28}{3},\ \frac{29}{3} - 2z,\ z\right)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} 2x+y+2z=4 \\ 2x+2y \qquad =5 \\ 2x-y+6z=2 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 4 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array}\right] \begin{array}{l} \frac{1}{2}R_1 \\ \\ \\ \end{array} \qquad \begin{array}{cccc} 1 & \frac{1}{2} & 1 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-2R_1 \end{array} \qquad \begin{array}{cccc} 2 & 2 & 0 & 5 \\ -2 & -1 & -2 & -4 \\ \hline 0 & 1 & -2 & 1 \end{array} \qquad \begin{array}{cccc} 2 & -1 & 6 & 2 \\ -2 & -1 & -2 & -4 \\ \hline 0 & -2 & 4 & -2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 4 & -2 \end{array}\right] \begin{array}{l} R_1-\frac{1}{2}R_2 \\ \\ R_3+2R_2 \end{array} \qquad \begin{array}{cccc} 1 & \frac{1}{2} & 1 & 2 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ \hline 1 & 0 & 2 & \frac{3}{2} \end{array} \qquad \begin{array}{cccc} 0 & -2 & 4 & -2 \\ 0 & 2 & -4 & 2 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & \frac{3}{2} \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From Row 3: 0 = 0 is a true statement. Let ***z*** be the variable.

From Row 2: $y - 2z = 1$

$$\underline{y = 1 + 2z}$$

From Row 1: $x + 2z = \frac{3}{2}$

$$\underline{x = -2z + \frac{3}{2}}$$

$\therefore$ ***Solution***: $\underline{\left(-2z+\frac{3}{2},\ 2z+1,\ z\right)}$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array} \qquad \begin{array}{cccc} -1 & -2 & 3 & 1 \\ 1 & 1 & 2 & 8 \\ \hline 0 & -1 & 5 & 9 \end{array} \qquad \begin{array}{cccc} 3 & -7 & 4 & 10 \\ -3 & -3 & -6 & -24 \\ \hline 0 & -10 & -2 & -14 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right] -R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 10R_2 \end{array} \qquad \begin{array}{cccc} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ \hline 1 & 0 & 7 & 17 \end{array} \qquad \begin{array}{cccc} 0 & -10 & -2 & -14 \\ 0 & 10 & -50 & -90 \\ \hline 0 & 0 & -52 & -104 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right] -\frac{1}{52}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 - 7R_3 \\ R_2 + 5R_3 \\ \\ \end{array} \qquad \begin{array}{cccc} 1 & 0 & 7 & 17 \\ 0 & 0 & -7 & -14 \\ \hline 1 & 0 & 0 & 3 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -5 & -9 \\ 0 & 0 & 5 & 10 \\ \hline 0 & 1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

∴ ***Solution***: $(3,\ 1,\ 2)$

Exercise

Use augmented elimination to solve linear system
$$\begin{matrix} 2x-5y+3z=1 \\ x-2y-2z=8 \end{matrix}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 2 & -5 & 3 & 1 \end{array}\right] \quad R_2 - 2R_1$$

$$\begin{array}{cccc} 2 & -5 & 3 & 1 \\ -2 & 4 & 4 & -16 \\ \hline 0 & -1 & 7 & -15 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 0 & -1 & 7 & -15 \end{array}\right] \quad -R_2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 0 & 1 & -7 & 15 \end{array}\right] \quad R_1 + 2R_2$$

$$\begin{array}{cccc} 1 & -2 & -2 & 8 \\ 0 & 2 & -14 & 30 \\ \hline 1 & 0 & -16 & 38 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -16 & 38 \\ 0 & 1 & -7 & 15 \end{array}\right] \begin{array}{l} \rightarrow \ x-16z=38 \\ \rightarrow \ y-7z=15 \end{array}$$

$$\begin{cases} x=16z+38 \\ y=7z+15 \end{cases}$$

$\therefore$ ***Solution***: $\underline{(16z+38,\ 7z+15,\ z)}$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} x+y+z=2 \\ 2x+y-z=5 \\ x-y+z=-2 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 5 \\ 1 & -1 & 1 & -2 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{cccc} 2 & 1 & -1 & 5 \\ -2 & -2 & -2 & -4 \\ \hline 0 & -1 & -3 & 1 \end{array} \qquad \begin{array}{cccc} 1 & -1 & 1 & -2 \\ -1 & -1 & -1 & -2 \\ \hline 0 & -2 & 0 & -4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & 1 \\ 0 & -2 & 0 & -4 \end{array}\right] \quad \begin{array}{c} (2) \\ (1) \\ -2y=-4 \end{array}$$

$\underline{y=2}$

$(1) \rightarrow \quad -y-3z=1$

$3z=-1-2$

$\underline{z=-1}$

$(2) \rightarrow \quad x+y+z=2$

$x = 2-2+1$

$\underline{=1|}$

$\therefore$ ***Solution***: $\underline{(1,\ 2,\ -1)}\ |$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x+y+z=9 \\ -x-y+z=1 \\ 3x-y+z=9 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ -1 & -1 & 1 & 1 \\ 3 & -1 & 1 & 9 \end{array}\right] \begin{array}{l} \\ 2R_2+R_1 \\ 2R_3-3R_1 \end{array}$$

$$\begin{array}{cccc} -2 & -2 & 2 & 2 \\ 2 & 1 & 1 & 9 \\ \hline 0 & -1 & 3 & 11 \end{array} \qquad \begin{array}{cccc} 6 & -2 & 2 & 18 \\ -6 & -3 & -3 & -27 \\ \hline 0 & -5 & -1 & -9 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ 0 & -1 & 3 & 11 \\ 0 & -5 & -1 & -9 \end{array}\right] \begin{array}{l} \\ \\ R_3-5R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -5 & -1 & -9 \\ 0 & 5 & -15 & -55 \\ \hline 0 & 0 & -16 & -64 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ 0 & -1 & 3 & 11 \\ 0 & 0 & -16 & -64 \end{array}\right] \begin{array}{l} (2) \\ (1) \\ -16z=-64 \end{array}$$

$\underline{z=4|}$

$(1) \rightarrow \quad -y+3z=11$

$y = 12-11$

$\underline{=1|}$

$(2) \rightarrow \quad 2x+y+z=9$

$2x = 9-1-4$

$\underline{x=2|}$

$\therefore$ ***Solution***: $\underline{(2,\ 1,\ 4)}\ |$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3y-z=-1 \\ x+5y-z=-4 \\ -3x+6y+2z=11 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ -3 & 6 & 2 & 11 \end{array}\right] \quad R_3 + 3R_1$$

$$\begin{array}{cccc} -3 & 6 & 2 & 11 \\ 3 & 15 & -3 & -12 \\ \hline 0 & 21 & -1 & -1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ 0 & 21 & -1 & -1 \end{array}\right] \quad R_3 - 7R_2$$

$$\begin{array}{cccc} 0 & 21 & -1 & -1 \\ 0 & -21 & 7 & 7 \\ \hline 0 & 0 & 6 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ 0 & 0 & 6 & 6 \end{array}\right] \quad \begin{array}{ll} \to x + 5y - z = -4 & (2) \\ \to 3y - z = -1 & (1) \\ \to 6z = 6 & \end{array}$$

$\underline{z = 1}|$

$(1) \to \quad 3y = -1 + 1$

$\qquad \underline{y = 0}|$

$(2) \to \quad x = -4 + 1$

$\qquad \underline{x = -3}|$

$\therefore$ ***Solution***: $\underline{(-3,\ 0,\ 1)}|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 2 & -3 & 2 & 10 \\ 3 & -1 & 1 & 9 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -3 & 2 & 10 \\ -2 & -6 & -8 & -28 \\ \hline 0 & -9 & -6 & -18 \end{array} \qquad \begin{array}{cccc} 3 & -1 & 1 & 9 \\ -3 & -9 & -12 & -42 \\ \hline 0 & -10 & -11 & -33 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & -9 & -6 & -18 \\ 0 & -10 & -11 & -33 \end{array}\right] \quad 9R_3 - 10R_2$$

$$\begin{array}{cccc} 0 & -90 & -99 & -297 \\ 0 & 90 & 60 & 180 \\ \hline 0 & 0 & -39 & -117 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & -9 & -6 & -18 \\ 0 & 0 & -39 & -117 \end{array}\right] \quad \begin{array}{ll} x + 3y + 4z = 14 & (3) \\ -9y - 6z = -18 & (2) \\ -39z = -117 & (1) \end{array}$$

$(1) \to \quad z = \dfrac{117}{39}$

$\qquad \underline{= 3}|$

$(2) \to \quad 9y = 18 - 6(3)$

$9y = 0$

$\underline{y = 0}|$

$(3) \to \quad x = 14 - 12$

$\underline{x = 2}|$

$\therefore$ ***Solution***: $\underline{(2,\ 0,\ 3)}\ |$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 3 & 2 & 1 & 8 \\ 2 & -3 & 2 & -16 \end{array}\right] \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix} \qquad \begin{array}{cccc} 3 & 2 & 1 & 8 \\ -3 & -12 & 3 & -60 \\ \hline 0 & -10 & 4 & -52 \end{array} \qquad \begin{array}{cccc} 2 & -3 & 2 & -16 \\ -2 & -8 & 2 & -40 \\ \hline 0 & -11 & 4 & -56 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 0 & -10 & 4 & -52 \\ 0 & -11 & 4 & -56 \end{array}\right] \begin{matrix} \\ \\ 10R_3 - 11R_2 \end{matrix} \qquad \begin{array}{cccc} 0 & -110 & 40 & -560 \\ 0 & 110 & -44 & 572 \\ \hline 0 & 0 & -4 & 12 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 0 & -10 & 4 & -52 \\ 0 & 0 & -4 & 12 \end{array}\right] \quad \begin{array}{rl} x + 4y - z = 20 & (3) \\ -10y + 4z = -52 & (2) \\ -4z = 12 & (1) \end{array}$$

$(1) \to \quad \underline{z = -3}|$

$(2) \to \quad -10y = -52 + 12$

$-10y = -40$

$\underline{y = 4}|$

$(3) \to \quad x = 20 - 16 - 3$

$\underline{x = 1}|$

$\therefore$ ***Solution***: $\underline{(1,\ 4,\ -3)}\ |$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 17 \\ 0 & 2 & -1 & 7 \\ 2 & -3 & 2 & -1 \end{array}\right] \quad R_3 - 2R_1 \qquad \begin{array}{cccc} 2 & -3 & 2 & -1 \\ -2 & -4 & -2 & -34 \\ \hline 0 & -7 & 0 & -35 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 17 \\ 0 & 2 & -1 & 7 \\ 0 & -7 & 0 & -35 \end{array}\right] \quad \begin{array}{rl} x + 2y + z = 17 & (3) \\ 2y - z = 7 & (2) \\ -7y = -35 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{y = 5}|$

$(2) \rightarrow \quad z = 10 - 7$

$\qquad\qquad \underline{= 3}|$

$(3) \rightarrow \quad x = 17 - 10 - 3$

$\qquad\qquad \underline{= 4}|$

$\therefore$ ***Solution***: $\underline{(4,\ 5,\ 3)}|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ -4 & 5 & 3 & 7 \\ -6 & 3 & 5 & -4 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \qquad \begin{array}{cccc} -4 & 5 & 3 & 7 \\ 4 & -12 & -14 & -6 \\ \hline 0 & -7 & -11 & 1 \end{array} \qquad \begin{array}{cccc} -6 & 3 & 5 & -4 \\ 6 & -18 & -21 & -9 \\ \hline 0 & -15 & -16 & -13 \end{array}$$

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ 0 & -7 & -11 & 1 \\ 0 & -15 & -16 & -13 \end{array}\right] \quad 7R_3 - 15R_1 \qquad \begin{array}{cccc} 0 & -105 & -112 & -91 \\ 0 & 105 & 165 & -15 \\ \hline 0 & 0 & 53 & -106 \end{array}$$

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ 0 & -7 & -11 & 1 \\ 0 & 0 & 53 & -106 \end{array}\right] \quad \begin{array}{rl} -2x + 6y + 7z = 3 & (3) \\ -7y - 11z = 1 & (2) \\ 53z = -106 & (1) \end{array}$$

$(1) \to \quad \underline{z = -2}$

$(2) \to \quad -7y = 1 - 22$

$-7y = -21$

$\underline{y = 3}$

$(3) \to \quad -2x = 3 - 18 + 14$

$-2x = -1$

$\underline{x = \frac{1}{2}}$

$\therefore$ ***Solution***: $\underline{\left(\frac{1}{2},\ 3,\ -2\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 3 & -3 & 4 & 5 \\ 4 & -2 & 3 & 4 \end{array}\right] \begin{array}{l} \\ 2R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\begin{array}{cccc} 6 & -6 & 8 & 10 \\ -6 & 3 & -3 & -3 \\ \hline 0 & -3 & 5 & 7 \end{array} \qquad \begin{array}{cccc} 4 & -2 & 3 & 4 \\ -4 & 2 & -2 & -2 \\ \hline 0 & 0 & 1 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 0 & -3 & 5 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \begin{array}{ll} 2x - y + z = 1 & (2) \\ -3y + 5z = 7 & (1) \\ \underline{z = 2} & \end{array}$$

$(1) \to \quad -3y = 7 - 10$

$-3y = -3$

$\underline{y = 1}$

$(3) \to \quad 2x = 1 + 1 - 2$

$\underline{x = 0}$

$\therefore$ ***Solution***: $\underline{(0,\ 1,\ 2)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x-4y+4z=7 \\ x-y-2z=2 \\ 2x-3y+6z=5 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 3 & -4 & 4 & 7 \\ 2 & -3 & 6 & 5 \end{array}\right] \begin{array}{l} \\ R_2-3R_1 \\ R_3-2R_1 \end{array}$$

$$\begin{array}{cccc} 3 & -4 & 4 & 7 \\ -3 & 3 & 6 & -6 \\ \hline 0 & -1 & 10 & 1 \end{array} \qquad \begin{array}{cccc} 2 & -3 & 6 & 5 \\ -2 & 2 & 4 & -4 \\ \hline 0 & -1 & 10 & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & -1 & 10 & 1 \\ 0 & -1 & 10 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3=R_2 \end{array} \qquad \begin{array}{ll} \to x-y-2z=2 & (2) \\ \to -y+10z=1 & (1) \end{array}$$

$(1) \to \quad \underline{y=10z-1}$

$(2) \to \quad x=2+10z-1+2z$

$\qquad\qquad \underline{=12z+1}$

$\therefore$ ***Solution***: $\underline{(12z+1,\ \ 10z-1,\ \ z)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x-2y-z=2 \\ 2x-y+z=4 \\ -x+y+z=4 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 2 & -1 & 1 & 4 \\ -1 & 1 & 1 & 4 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3+R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -1 & 1 & 4 \\ -2 & 4 & 2 & -4 \\ \hline 0 & 3 & 3 & 0 \end{array} \qquad \begin{array}{cccc} -1 & 1 & 1 & 4 \\ 1 & -2 & -1 & 2 \\ \hline 0 & -1 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 3 & 3 & 0 \\ 0 & -1 & 0 & 6 \end{array}\right] \quad \begin{array}{ll} x-2y-z=2 & (3) \\ 3y+3z=0 & (2) \\ -y=6 & (1) \end{array}$$

$(1) \to \quad \underline{y=-6}$

$(2) \to \quad z=-y$

$\qquad\qquad \underline{=6}$

$(3) \to \quad x=2-12+6$

$\qquad\qquad \underline{=-4}$

$\therefore$ ***Solution***: $\underline{(-4,\ \ -6,\ \ 6)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x+y+z=3 \\ -y+2z=1 \\ -x+z=0 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ -1 & 0 & 1 & 0 \end{array}\right] \quad R_3+R_1 \qquad \begin{array}{cccc} -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 3 \\ \hline 0 & 1 & 2 & 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 1 & 2 & 3 \end{array}\right] \quad R_3+R_2 \qquad \begin{array}{cccc} 0 & 1 & 2 & 3 \\ 0 & -1 & 2 & 1 \\ \hline 0 & 0 & 4 & 4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 4 & 4 \end{array}\right] \quad \begin{array}{rl} x+y+z=3 & (3) \\ -y+2z=1 & (2) \\ 4z=4 & (1) \end{array}$$

$(1) \to \quad \underline{z=1}$

$(2) \to \quad -y=1-2$

$\qquad\qquad \underline{y=1}$

$(3) \to \quad x=3-1-1$

$\qquad\qquad \underline{=1}$

$\therefore$ ***Solution***: $\underline{(1,\ 1,\ 1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x+y+3z=14 \\ 7x+5y+8z=37 \\ x+3y+2z=9 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 3 & 1 & 3 & 14 \\ 7 & 5 & 8 & 37 \end{array}\right] \quad \begin{array}{l} R_2-3R_1 \\ R_3-7R_1 \end{array} \qquad \begin{array}{cccc} 3 & 1 & 3 & 14 \\ -3 & -9 & -6 & -27 \\ \hline 0 & -8 & -3 & -13 \end{array} \qquad \begin{array}{cccc} 7 & 5 & 8 & 37 \\ -7 & -21 & -14 & -63 \\ \hline 0 & -16 & -6 & -26 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 0 & -8 & -3 & -13 \\ 0 & -16 & -6 & -26 \end{array}\right] \quad R_3-2R_2 \qquad \begin{array}{cccc} 0 & -16 & -6 & -26 \\ 0 & 16 & 6 & 26 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 0 & -8 & -3 & -13 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{ll} x+3y+2z=9 & (2) \\ -8y-3z=-13 & (1) \end{array}$$

$(1) \to \quad -8y = 3z - 13$

$$\boxed{y = -\frac{3}{8}z + \frac{13}{8}}$$

$(3) \to \quad x = 9 - 3\left(\frac{13}{8} - \frac{3}{8}z\right) - 2z$

$$= 9 - \frac{39}{8} + \frac{9}{8}z - 2z$$

$$\boxed{= \frac{33}{8} - \frac{7}{8}z}$$

$\therefore$ ***Solution***: $\boxed{\left(\frac{33}{8} - \frac{7}{8}z,\ \ \frac{13}{8} - \frac{3}{8}z,\ \ z\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 4x - 2y + z = 7 \\ x + y + z = -2 \\ 4x + 2y + z = 3 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 4 & -2 & 1 & 7 \\ 4 & 2 & 1 & 3 \end{array}\right] \begin{array}{l} \\ R_2 - 4R_1 \\ R_3 - 4R_1 \end{array}$$

4	−2	1	7
−4	−4	−4	8
0	−6	−3	15

4	2	1	3
−4	−4	−4	8
0	−2	−3	11

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -6 & -3 & 15 \\ 0 & -2 & -3 & 11 \end{array}\right] \begin{array}{l} \\ \\ -3R_3 + R_2 \end{array}$$

0	6	9	−33
0	−6	−3	15
0	0	6	−18

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -6 & -3 & 15 \\ 0 & 0 & 6 & -18 \end{array}\right] \quad \begin{array}{ll} x+y+z=-2 & (3) \\ -6y-3z=15 & (2) \\ 6z=-18 & (1) \end{array}$$

$(1) \to \quad \boxed{z = -3}$

$(2) \to \quad -6y = 15 - 9$

$$\boxed{y = -1}$$

$(3) \to \quad x = -2 + 1 + 3$

$$\boxed{= 2}$$

$\therefore$ ***Solution***: $\boxed{(2,\ \ -1,\ \ -3)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x-2y+z=-4 \\ 6x+4y-3z=-24 \\ x-2y+2z=1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 2 & -2 & 1 & -4 \\ 6 & 4 & -3 & -24 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-6R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -2 & 1 & -4 \\ -2 & 4 & -4 & -2 \\ \hline 0 & 2 & -3 & -6 \end{array} \qquad \begin{array}{cccc} 6 & 4 & -3 & -24 \\ -6 & 12 & -12 & -6 \\ \hline 0 & 16 & -15 & -30 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 2 & -3 & -6 \\ 0 & 16 & -15 & -30 \end{array}\right] \begin{array}{l} \\ \\ R_3-8R_2 \end{array}$$

$$\begin{array}{cccc} 0 & 16 & -15 & -30 \\ 0 & -16 & 24 & 48 \\ \hline 0 & 0 & 9 & 18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 2 & -3 & -6 \\ 0 & 0 & 9 & 18 \end{array}\right] \quad \begin{array}{rl} x-2y+2z=1 & (3) \\ 2y-3z=-6 & (2) \\ 9z=18 & (1) \end{array}$$

$(1)\rightarrow \quad \underline{z=2}$

$(2)\rightarrow \quad 2y=-6+6$

$\underline{y=0}$

$(3)\rightarrow \quad x=1-4$

$\underline{=-3}$

$\therefore$ ***Solution***: $\underline{(-3,\ 0,\ 2)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 9x+3y+z=4 \\ 16x+4y+z=2 \\ 25x+5y+z=2 \end{cases}$

Solution

$$\begin{cases} z+9x+3y=4 \\ z+16x+4y=2 \\ z+25x+5y=2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 1 & 16 & 4 & 2 \\ 1 & 25 & 5 & 2 \end{array}\right] \begin{array}{l} \\ R_2-R_1 \\ R_3-R_1 \end{array}$$

$$\begin{array}{cccc} 1 & 16 & 4 & 2 \\ -1 & -9 & -3 & -4 \\ \hline 0 & 7 & 1 & -2 \end{array} \qquad \begin{array}{cccc} 1 & 25 & 5 & 2 \\ -1 & -9 & -3 & -4 \\ \hline 0 & 16 & 2 & -2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 0 & 7 & 1 & -2 \\ 0 & 16 & 2 & -2 \end{array}\right] \quad 7R_3 - 16R_3 \qquad \begin{array}{cccc} 0 & 112 & 14 & -14 \\ 0 & -112 & -16 & 32 \\ \hline 0 & 0 & -2 & 18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 0 & 7 & 1 & -2 \\ 0 & 0 & -2 & 18 \end{array}\right] \quad \begin{array}{rl} z + 9x + 3y = 4 & (3) \\ 7x + y = -2 & (2) \\ -2y = 18 & (1) \end{array}$$

$(1) \rightarrow \quad y = -9$

$(2) \rightarrow \quad 7x = -2 + 9$

$\qquad\qquad = 1$

$(3) \rightarrow \quad z = 4 - 9 + 27$

$\qquad\qquad = 22$

∴ ***Solution***: $(1, \ -9, \ 22)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x - y + 2z = -8 \\ x + 2y - 3z = 9 \\ 3x - y - 4z = 3 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 2 & -8 \\ 3 & -1 & -4 & 3 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \qquad \begin{array}{cccc} 2 & -1 & 2 & -8 \\ -2 & -4 & 6 & -18 \\ \hline 0 & -5 & 8 & -26 \end{array} \qquad \begin{array}{cccc} 3 & -1 & -4 & 3 \\ -3 & -6 & 9 & -27 \\ \hline 0 & -7 & 5 & -24 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 0 & -7 & 5 & -24 \end{array}\right] \quad 5R_3 - 7R_2 \qquad \begin{array}{cccc} 0 & -35 & 25 & -120 \\ 0 & 35 & -56 & 182 \\ \hline 0 & 0 & -31 & 62 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 0 & 0 & -31 & 62 \end{array}\right] \quad \begin{array}{rl} x + 2y - 3z = 9 & (3) \\ -5y + 8z = -26 & (2) \\ -31z = 62 & (1) \end{array}$$

$(1) \rightarrow \quad z = -2$

$(2) \rightarrow \quad -5y = -26 + 16$

$\qquad\qquad -5y = 10$

$\qquad\qquad y = 2$

$(3) \rightarrow \quad x = 9 - 4 - 6$

$$= -1$$

$\therefore$ ***Solution***: $(-1,\ 2,\ -2)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x \quad\ \ - 3z = -5 \\ 2x - y + 2z = 16 \\ 7x - 3y - 5z = 19 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 2 & -1 & 2 & 16 \\ 7 & -3 & -5 & 19 \end{array}\right] \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 7R_1 \end{array} \qquad \begin{array}{cccc} 2 & -1 & 2 & 16 \\ -2 & 0 & 6 & 10 \\ \hline 0 & -1 & 8 & 26 \end{array} \qquad \begin{array}{cccc} 7 & -3 & -5 & 19 \\ -7 & 0 & 21 & 35 \\ \hline 0 & -3 & 16 & 54 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 0 & -1 & 8 & 26 \\ 0 & -3 & 16 & 54 \end{array}\right] \quad \begin{array}{l} \\ \\ R_3 - 3R_2 \end{array} \qquad \begin{array}{cccc} 0 & -3 & 16 & 54 \\ 0 & 3 & -24 & -78 \\ \hline 0 & 0 & -8 & -24 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 0 & -1 & 8 & 26 \\ 0 & 0 & -8 & -24 \end{array}\right] \quad \begin{array}{rl} x - 3z = -5 & (3) \\ -y + 8z = 26 & (2) \\ -8z = -24 & (1) \end{array}$$

$(1) \rightarrow \quad z = 3$

$(2) \rightarrow \quad -y = 26 - 24$

$\qquad y = -2$

$(3) \rightarrow \quad x = -5 + 9$

$\qquad = 4$

$\therefore$ ***Solution***: $(4,\ -2,\ 3)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + 2y - z = 5 \\ 2x - y + 3z = 0 \\ \quad\ \ 2y + z = 1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 2 & -1 & 3 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right] \quad R_2 - 2R_1 \qquad \begin{array}{cccc} 2 & -1 & 3 & 0 \\ -2 & -4 & 2 & -10 \\ \hline 0 & -5 & 5 & -10 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 5 & -10 \\ 0 & 2 & 1 & 1 \end{array}\right] \quad 5R_3 + 2R_2$$

$$\begin{array}{cccc} 0 & 10 & 5 & 5 \\ 0 & -10 & 10 & -20 \\ \hline 0 & 0 & 15 & -15 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 5 & -10 \\ 0 & 0 & 15 & -15 \end{array}\right] \quad \begin{array}{rr} x + 2y - z = 5 & (3) \\ -5y + 5z = -10 & (2) \\ 15z = -15 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z = -1}$

$(2) \rightarrow \quad -5y = -10 + 5$

$\qquad \underline{y = 1}$

$(3) \rightarrow \quad x = 5 - 2 - 1$

$\qquad \underline{= 2}$

$\therefore$ ***Solution***: $\underline{(2,\ 1,\ -1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + y + z = 6 \\ 3x + 4y - 7z = 1 \\ 2x - y + 3z = 5 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & 4 & -7 & 1 \\ 2 & -1 & 3 & 5 \end{array}\right] \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 4 & -7 & 1 \\ -3 & -3 & -3 & -18 \\ \hline 0 & 1 & -10 & -17 \end{array} \qquad \begin{array}{cccc} 2 & -1 & 3 & 5 \\ -2 & -2 & -2 & -12 \\ \hline 0 & -3 & 1 & -7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -10 & -17 \\ 0 & -3 & 1 & -7 \end{array}\right] \quad R_3 + 3R_2$$

$$\begin{array}{cccc} 0 & -3 & 1 & -7 \\ 0 & 3 & -30 & -51 \\ \hline 0 & 0 & -29 & -58 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -10 & -17 \\ 0 & 0 & -29 & -58 \end{array}\right] \quad \begin{array}{rr} x + y + z = 6 & (3) \\ y - 10z = -17 & (2) \\ -29z = -58 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z = 2}$

$(2) \rightarrow \quad y = -17 + 20$

$\qquad \underline{= 3}$

$(3) \rightarrow \quad x = 6 - 3 - 2$

$\qquad \underline{= 1}$

$\therefore$ ***Solution***: $\underline{(1,\ 3,\ 2)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x+2y+3z=3 \\ 4x-5y+7z=1 \\ 2x+3y-2z=6 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 4 & -5 & 7 & 1 \\ 2 & 3 & -2 & 6 \end{array}\right] \begin{array}{l} \\ 3R_2-4R_1 \\ 3R_3-2R_1 \end{array}$$

$$\begin{array}{cccc} 12 & -15 & 21 & 3 \\ -12 & -8 & -12 & -12 \\ \hline 0 & -23 & 9 & -9 \end{array} \qquad \begin{array}{cccc} 6 & 9 & -6 & 18 \\ -6 & -4 & -6 & -6 \\ \hline 0 & 5 & -12 & 12 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 0 & -23 & 9 & -9 \\ 0 & 5 & -12 & 12 \end{array}\right] \begin{array}{l} \\ \\ 23R_3+5R_2 \end{array}$$

$$\begin{array}{cccc} 0 & 115 & -276 & 276 \\ 0 & -115 & 45 & -45 \\ \hline 0 & 0 & -231 & 231 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 0 & -23 & 9 & -9 \\ 0 & 0 & -231 & 231 \end{array}\right] \begin{array}{rl} 3x+2y+3z=3 & (3) \\ -23y+9z=-9 & (2) \\ -231z=231 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z=-1}$

$(2) \rightarrow \quad -23y=-9+9$

$\qquad \underline{y=0}$

$(3) \rightarrow \quad 3x=3+3$

$\qquad \underline{x=2}$

$\therefore$ ***Solution***: $\underline{(2,\ 0,\ -1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x-3y+z=2 \\ 4x-12y+4z=8 \\ -2x+6y-2z=-4 \end{cases}$

Solution

$$\begin{cases} & x-3y+z=2 \\ \frac{1}{4}\times & 4x-12y+4z=8 \\ -\frac{1}{2}\times & -2x+6y-2z=-4 \end{cases}$$

$$\begin{cases} x-3y+z=2 \\ x-3y+z=2 \\ x-3y+z=2 \end{cases}$$

Since all three equations are the same.

∴ ***Solution***: is the plane $\underline{x-3y+z=2}|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x-2y+z=-1 \\ x+2y-z=2 \\ 6x+4y+3z=5 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & -2 & 1 & -1 \\ 6 & 4 & 3 & 5 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-6R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -2 & 1 & -1 \\ -2 & -4 & 2 & -4 \\ \hline 0 & -6 & 3 & -5 \end{array} \qquad \begin{array}{cccc} 6 & 4 & 3 & 5 \\ -6 & -12 & 6 & -12 \\ \hline 0 & -8 & 9 & -7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -6 & 3 & -5 \\ 0 & -8 & 9 & -7 \end{array}\right] \begin{array}{l} \\ \\ 3R_3-4R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -24 & 27 & -21 \\ 0 & 24 & -12 & 20 \\ \hline 0 & 0 & 15 & -1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -6 & 3 & -5 \\ 0 & 0 & 15 & -1 \end{array}\right] \begin{array}{rl} x+2y-z=2 & (3) \\ -6y+3z=-5 & (2) \\ 15z=-1 & (1) \end{array}$$

$(1) \to \quad \underline{z=-\frac{1}{15}}|$

$(2) \to \quad -6y=-5+\frac{1}{5}$

$-6y=-\frac{24}{5}$

$\underline{y=\frac{4}{5}}|$

$(3) \to \quad x=2-\frac{8}{5}-\frac{1}{15}$

$\underline{=\frac{1}{3}}|$

∴ ***Solution***: $\underline{\left(\frac{1}{3},\ \frac{4}{5},\ -\frac{1}{15}\right)}|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 \\ x_2 - 3x_3 - x_4 = 0 \\ 3x_1 + 2x_3 - x_4 = 6 \\ -4x_1 + x_2 + 4x_3 + 2x_4 = -3 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 3 & 0 & 2 & -1 & 6 \\ -4 & 1 & 4 & 2 & -3 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 3R_1 \\ R_4 + 4R_1 \end{array}$$

$$\begin{array}{ccccc} 3 & 0 & 2 & -1 & 6 \\ -3 & 15 & -6 & 6 & -12 \\ \hline 0 & 15 & -4 & 5 & -6 \end{array} \qquad \begin{array}{ccccc} -4 & 1 & 4 & 2 & -3 \\ 4 & -20 & 8 & -8 & 16 \\ \hline 0 & -19 & 12 & -6 & 13 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 15 & -4 & 5 & -6 \\ 0 & -19 & 12 & -6 & 13 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 15R_2 \\ R_4 + 19R_2 \end{array}$$

$$\begin{array}{ccccc} 0 & 15 & -4 & 5 & -6 \\ 0 & -15 & 45 & 15 & 0 \\ \hline 0 & 0 & 41 & 20 & -6 \end{array} \qquad \begin{array}{ccccc} 0 & -19 & 12 & -6 & 13 \\ 0 & 19 & -57 & -19 & 0 \\ \hline 0 & 0 & -45 & -25 & 13 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 41 & 20 & -6 \\ 0 & 0 & -45 & -25 & 13 \end{array}\right] \begin{array}{l} \\ \\ \\ 41R_4 + 45R_2 \end{array}$$

$$\begin{array}{ccccc} 0 & 0 & -1845 & -1025 & 533 \\ 0 & 0 & 1845 & 900 & -270 \\ \hline 0 & 0 & 0 & -125 & 263 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 41 & 20 & -6 \\ 0 & 0 & 0 & -125 & 263 \end{array}\right] \qquad \begin{array}{rl} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 & (4) \\ x_2 - 3x_3 - x_4 = 0 & (3) \\ 41x_3 + 20x_4 = -6 & (2) \\ -125x_4 = 263 & (1) \end{array}$$

$(1) \rightarrow \quad \boxed{x_4 = -\frac{263}{125}}$

$(2) \rightarrow \quad 41x_3 = -6 + \frac{1,052}{25}$

$\qquad = \frac{902}{25}$

$\qquad \boxed{x_3 = \frac{22}{25}}$

$(3) \rightarrow \quad x_2 = \frac{66}{25} - \frac{263}{125}$

$\qquad \boxed{= \frac{67}{125}}$

$(4) \rightarrow \quad x_1 = 4 + \frac{67}{25} - \frac{44}{25} - \frac{526}{125}$

$$= 4 + \frac{23}{25} - \frac{526}{125}$$

$$= \frac{500 + 115 - 526}{125}$$

$$= \boxed{\frac{89}{125}}$$

$\therefore$ ***Solution***: $\boxed{\left(\frac{89}{125}, \frac{67}{125}, \frac{22}{25}, -\frac{263}{125}\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x_1 + x_2 + x_3 + x_4 = 5 \\ x_1 + 2x_2 - x_3 - 2x_4 = -1 \\ x_1 - 3x_2 - 3x_3 - x_4 = -1 \\ 2x_1 - x_2 + 2x_3 - x_4 = -2 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 1 & 2 & -1 & -2 & -1 \\ 1 & -3 & -3 & -1 & -1 \\ 2 & -1 & 2 & -1 & -2 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - 2R_1 \end{array}$$

$$\begin{array}{ccccc} 1 & 2 & -1 & -2 & -1 \\ -1 & -1 & -1 & -1 & -5 \\ \hline 0 & 1 & -2 & -3 & -6 \end{array} \qquad \begin{array}{ccccc} 1 & -3 & -3 & -1 & -1 \\ -1 & -1 & -1 & -1 & -5 \\ \hline 0 & -4 & -4 & -2 & -6 \end{array} \qquad \begin{array}{ccccc} 2 & -1 & 2 & -1 & -2 \\ -2 & -2 & -2 & -2 & -10 \\ \hline 0 & -3 & 0 & -3 & -12 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & -4 & -4 & -2 & -6 \\ 0 & -3 & 0 & -3 & -12 \end{array}\right] \begin{array}{l} \\ \\ R_3 + 4R_2 \\ R_4 + 3R_2 \end{array}$$

$$\begin{array}{ccccc} 0 & -4 & -4 & -2 & -6 \\ 0 & 4 & -8 & -12 & -24 \\ \hline 0 & 0 & -12 & -14 & -30 \end{array} \qquad \begin{array}{ccccc} 0 & -3 & 0 & -3 & -12 \\ 0 & 3 & -6 & -9 & -18 \\ \hline 0 & 0 & -6 & -12 & -30 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & 0 & -12 & -14 & -30 \\ 0 & 0 & -6 & -12 & -30 \end{array}\right] \begin{array}{l} \\ \\ \\ -2R_4 + R_3 \end{array}$$

$$\begin{array}{ccccc} 0 & 0 & 12 & 24 & 60 \\ 0 & 0 & -12 & -14 & -30 \\ \hline 0 & 0 & 0 & 10 & 30 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & 0 & -12 & -14 & -30 \\ 0 & 0 & 0 & 10 & 30 \end{array}\right] \begin{array}{rl} x_1 + x_2 + x_3 + x_4 = 5 & (4) \\ x_2 - 2x_3 - 3x_4 = -6 & (3) \\ -12x_3 - 14x_4 = -30 & (2) \\ 10x_4 = 30 & (1) \end{array}$$

$(1) \to \quad \boxed{x_4 = 3}$

$(2) \to \quad -12x_3 = -30 + 42$

$= 12$

$\boxed{x_3 = -1}$

$(3) \to \quad x_2 = -6 - 2 + 9$

$\boxed{= 1}$

$(4) \to \quad x_1 = 5 - 1 + 1 - 3$

$\boxed{= 2}$

$\therefore$ ***Solution***: $\boxed{(2,\ 1,\ -1,\ 3)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x + 8y - z + w = 0 \\ 4x + 16y - 3z - w = -10 \\ -2x + 4y - z + 3w = -6 \\ -6x + 2y + 5z + w = 3 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 4 & 16 & -3 & -1 & -10 \\ -2 & 4 & -1 & 3 & -6 \\ -6 & 2 & 5 & 1 & 3 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 + 3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - \frac{13}{6}R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \quad \textit{Interchange } R_2 \textit{ and } R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & \frac{19}{3} & -\frac{14}{3} & 16 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 + \frac{19}{3}R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & 0 & -\frac{71}{3} & -\frac{142}{3} \end{array}\right] \quad \begin{array}{ll} 2x+8y-z+w=0 & (3) \\ 12y-2z+4w=-6 & (2) \\ -z-3w=-10 & (1) \\ -\frac{71}{3}w=-\frac{142}{3} & \to \underline{w=2} \end{array}$$

$(1) \to \quad z = 10 - 3w = 4$

$(2) \to \quad 12y = 2z - 4w - 6$

$$y = -\frac{1}{2}$$

$(3) \to \quad 2x = -8y + z - w$

$$2x = 4 + 4 - 2$$

$$2x = 6$$

$$x = 3$$

∴ ***Solution***: $\left(3,\ -\frac{1}{2},\ 4,\ 2\right)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 + 2x_2 \qquad = 0 \\ \qquad x_2 + x_3 = 0 \end{cases}$

Solution

$$\begin{cases} x_1 = -2x_2 \\ x_3 = -x_2 \end{cases}$$

$$2x_1 + x_2 + 3x_3 = 0$$

$$-4x_2 + x_2 - 3x_2 = 0 \quad \to \quad x_2 = 0$$

∴ ***Solution***: $(0,\ 0,\ 0)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x + 2y + 4z \qquad = 0 \\ \qquad -y - 3z + w = 0 \\ 3x + y + z + 2w = 0 \\ x + 3y - 2z - 2w = 0 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & -1 & -3 & 1 & 0 \\ 3 & 1 & 1 & 2 & 0 \\ 1 & 3 & -2 & -2 & 0 \end{array}\right] \begin{array}{l} \\ -R_2 \\ 2R_3 - 3R_1 \\ 2R_4 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & -4 & -10 & 4 & 0 \\ 0 & 4 & -8 & -4 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 + 4R_2 \\ R_4 - 4R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & -20 & 0 & 0 \end{array}\right] \begin{array}{ll} 2x + 2y - 4z = 0 & (1) \\ y + 3z - w = 0 & (2) \\ \rightarrow \boxed{z = 0} & \\ & \end{array}$$

$(2) \quad \rightarrow \quad \boxed{y = w}$

$(1) \quad \rightarrow 2x = -2y \quad \boxed{x = -w}$

∴ ***Solution***: $\boxed{(-w,\ w,\ 0,\ w)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x + z + w = 5 \\ y - w = -1 \\ 3x - z - w = 0 \\ 4x + y + 2z + w = 9 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 3 & 0 & -1 & -1 & 0 \\ 4 & 1 & 2 & 1 & 9 \end{array}\right] \begin{array}{l} \\ \\ 2R_3 - 3R_1 \\ 2R_4 - 4R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -5 & -5 & -15 \\ 0 & 2 & 0 & -2 & -2 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - 2R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -5 & -5 & -15 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{ll} 2x + z + w = 5 & (1) \\ y - w = -1 & (2) \\ -5z - 5w = -15 & (3) \\ & \end{array}$$

$(2) \rightarrow \qquad \underline{y = 1 + w}|$

$(3) \rightarrow \qquad \underline{z = 3 - w}|$

$(1) \rightarrow \quad 2x = 5 - (3 - w) - w \quad \Rightarrow \quad \underline{x = 1}|$

$\therefore$ ***Solution***: $\underline{(1,\ 1+w,\ 3-w,\ w)}|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 4y + z = 20 \\ 2x - 2y + z = 0 \\ x \quad + z = 5 \\ x + y - z = 10 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 10 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 1 & 5 \\ 0 & 4 & 1 & 20 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - R_1 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 10 \\ 0 & -4 & 3 & -20 \\ 0 & -1 & 2 & -5 \\ 0 & 4 & 1 & 20 \end{array}\right] \begin{array}{l} \\ \\ 4R_3 - R_2 \\ R_4 + R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 10 \\ 0 & -4 & 3 & -20 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \begin{array}{l} x + y = 10 \\ \rightarrow -4y = -20 \\ \rightarrow z = 0 \\ \\ \end{array}$$

$\therefore$ ***Solution***: $\underline{(5,\ 5,\ 0)}|$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x \qquad\qquad - 3w = -3 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ -1 & 3 & -2 & 1 \\ 3 & 4 & -7 & 10 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 5 & -1 & 9 \\ 0 & -2 & -10 & -14 \end{array}\right] \begin{array}{l} \\ \\ 5R_3 + 2R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 5 & -1 & 9 \\ 0 & 0 & -52 & -52 \end{array}\right] \quad \begin{array}{rl} x + 2y + z = 8 & (3) \\ 5y - z = 9 & (2) \\ -52z = -52 & (1) \end{array}$$

(1) $\Rightarrow \quad z = 1$

(2) $\Rightarrow \quad 5y = 9 + 1 = 10 \rightarrow y = 2$

(3) $\Rightarrow \quad x = 8 - 4 - 1 = 3$

$\therefore$ ***Solution***: $\boxed{(3,\ 2,\ 1)}$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} 2u - 3v + w - x + y = 0 \\ 4u - 6v + 2w - 3x - y = -5 \\ -2u + 3v - 2w + 2x - y = 3 \end{cases}$$

Solution

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 4 & -6 & 2 & -3 & -1 & -5 \\ -2 & 3 & -2 & 2 & -1 & 3 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -3 & -5 \\ 0 & 0 & -1 & 1 & 0 & 3 \end{array}\right] \quad \begin{array}{rl} 2u - 3v + w - x + y = 0 & (3) \\ -x - 3y = -5 & (2) \\ -w + x = 3 & (1) \end{array}$$

(2) $\Rightarrow \quad x = 5 - 3y$

(1) $\Rightarrow \quad w = x - 3 = 2 - 3y$

(3) $\Rightarrow \quad 2u = 3v - 2 + 3y + 5 - 3y - y = 3v - y + 3$

$$u = \frac{3}{2}v - \frac{1}{2}y + \frac{3}{2}$$

$\therefore$ ***Solution***: $\boxed{\left(\frac{3}{2}v - \frac{1}{2}y + \frac{3}{2}, v, 2 - 3y, 5 - 3y, y\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 6x_3 + 2x_4 - 4x_5 - 8x_6 = 8 \\ 3x_3 + x_4 - 2x_5 - 4x_6 = 4 \\ 2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 + x_6 = 2 \\ 6x_1 - 9x_2 + 11x_4 - 19x_5 + 3x_6 = 1 \end{cases}$

Solution

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 6 & -9 & 0 & 11 & -19 & 3 & 1 \end{array}\right] \quad \begin{array}{l} \\ \\ \\ R_4 - 3R_1 \end{array}$$

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & -3 & -1 & 2 & 0 & -5 \end{array}\right] \quad \begin{array}{l} \\ \\ R_3 - 2R_2 \\ R_4 + R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & -1 \end{array}\right] \quad \begin{array}{r} 2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 = 2 - x_6 \\ 3x_3 + x_4 - 2x_5 = 4 + 4x_6 \\ \\ \to \ \underline{x_6 = \frac{1}{4}} \end{array}$$

$$\begin{cases} 3x_3 = 5 - x_4 + 2x_5 \\ 2x_1 = \frac{7}{4} + 3x_2 - \frac{1}{3}\left(5 - x_4 + 2x_5\right) - 4x_4 + 7x_5 \end{cases}$$

$$\begin{cases} \underline{x_3 = \frac{5}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5} \\ 2x_1 = \frac{1}{12} + 3x_2 - \frac{11}{3}x_4 + \frac{19}{3}x_5 \end{cases}$$

∴ ***Solution***: $\underline{\left(\frac{1}{24} + \frac{3}{2}x_2 - \frac{11}{6}x_4 + \frac{19}{6}x_5,\ x_2,\ \frac{5}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5,\ x_4,\ x_5,\ \frac{1}{4}\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x_1 + 2x_2 - x_3 = -15 \\ 5x_1 + 3x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 3x_3 = 11 \\ -6x_1 - 4x_2 + 2x_3 = 30 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ -6 & -4 & 2 & 30 \end{array}\right] \begin{array}{c} \\ 3R_2 - 5R_1 \\ R_3 - R_1 \\ R_4 + 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & -1 & 4 & 26 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \\ \\ R_3 - R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & 0 & -7 & -49 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{rl} 3x_1 + 2x_2 - x_3 = -15 & (3) \\ -x_2 + 11x_3 = 75 & (2) \\ -7x_3 = -49 & (1) \end{array}$$

$(1) \rightarrow x_3 = 7$

$(2) \rightarrow x_2 = 77 - 75 = 2$

$(1) \rightarrow 3x_1 = -15 - 4 + 7 = 12 \Rightarrow x_1 = -4$

$\therefore$ ***Solution***: $\boxed{(-4, 2, 7)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1 \\ 5x_3 + 10x_4 + 15x_6 = 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6 \end{cases}$

Solution

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right] \begin{array}{c} \\ R_2 - 2R_1 \\ \\ R_4 - 2R_1 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \begin{array}{c} \\ -R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 5R_2 \\ R_4 - 4R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right] \begin{array}{l} \\ \\ \\ \frac{1}{6}R_4 \quad \textit{then interchanging row3 and row4} \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_3 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \rightarrow \begin{cases} x_1 + 3x_2 \quad + 4x_4 + 2x_5 = 0 \\ x_3 + 2x_4 = 0 \\ + x_6 = \frac{1}{3} \end{cases}$$

The general solution of the system: $\underline{x_6 = \frac{1}{3}, \quad x_3 = -2x_4, \quad x_1 = -3x_2 - 4x_4 - 2x_5}$

$\therefore$ ***Solution***: $\underline{\left(-3x_2 - 4x_4 - 2x_5, \ x_2, \ -2x_4, \ x_4, \ x_5, \ \frac{1}{3}\right)}$

Exercise

At SnackMix, caramel corn worth \$2.50 per *pound* is mixed with honey roasted missed nuts worth \$7.50 per *pound* in order to get 20 *lbs*. of a mixture worth \$4.50 per *pound*. How much of each snack is used?

Solution

$x + y = 20 \qquad (1)$

$2.50x + 7.50y = 90 \qquad (2)$

(1) $y = 20 - x$

(2) $2.5x + 7.5\,(20 - x) = 90$

$2.5x + 150 - 7.5x = 90$

$-5x = 90 - 150$

$-5x = -60$

$x = \frac{-60}{-5} = 12$

$y = 20 - x$
$= 20 - 12$
$= 8$

The mixture consists of 12 *lbs*. of caramel and 8 *lbs*. of nuts

Solution Section 4.2 – Matrix operations and Their Applications

Exercise

Find values for the variables so that the matrices are equal. $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

Solution

$$\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$$

$$\Rightarrow \begin{cases} w = 9 & x = 17 \\ y = 8 & z = -12 \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal. $\begin{bmatrix} x & y+3 \\ 2z & 8 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 6 & 8 \end{bmatrix}$

Solution

$$\begin{cases} \underline{x = 12} \\ y + 3 = 5 \rightarrow \ \underline{y = 2} \\ 2z = 6 \rightarrow \ \underline{z = 3} \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal. $\begin{bmatrix} 5 & x-4 & 9 \\ 2 & -3 & 8 \\ 6 & 0 & 5 \end{bmatrix} = \begin{bmatrix} y+3 & 2 & 9 \\ z+4 & -3 & 8 \\ 6 & 0 & w \end{bmatrix}$

Solution

$$\begin{bmatrix} 5 = y+3 & x-4 = 2 & 9 = 9 \\ 2 = z+4 & -3 = -3 & 8 = 8 \\ 6 = 6 & 0 = 0 & 5 = w \end{bmatrix}$$

$$\rightarrow \begin{cases} y = 2 & z = -2 \\ x = 6 & w = 5 \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} a+2 & 3b & 4c \\ d & 7f & 8 \end{bmatrix} + \begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$$

Solution

$$\begin{bmatrix} a-5 & 5b & 4c+6 \\ -2d & 7f-6 & 6 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$$

$$\begin{cases} a-5=15 & \rightarrow & a=20 \\ 5b=25 & \rightarrow & b=5 \\ 4c+6=6 & \rightarrow & 4c=0 \rightarrow c=0 \\ -2d=-8 & \rightarrow & d=4 \\ 7f-6=1 & \rightarrow & 7f=7 \rightarrow f=1 \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} a+11 & 12z+1 & 5m \\ 11k & 3 & 1 \end{bmatrix} + \begin{bmatrix} 9a & 9z & 4m \\ 12k & 5 & 3 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

Solution

$$\begin{bmatrix} a+11+9a & 12z+1+9z & 5m+4m \\ 11k+12k & 3+5 & 1+3 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 10a+11 & 21z+1 & 9m \\ 23k & 8 & 4 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

$10a+11=41 \quad \rightarrow \quad 10a=30$

$\underline{a=3}$

$21z+1=-62 \quad \rightarrow \quad 21z=-63$

$\underline{z=-3}$

$9m=72 \quad \rightarrow \quad \underline{m=8}$

$23k=92 \quad \rightarrow \quad \underline{k=\frac{92}{23}=4}$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} x+2 & 3y+1 & 5z \\ 8w & 2 & 3 \end{bmatrix} + \begin{bmatrix} 3x & 2y & 5z \\ 2w & 5 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 7 & -2 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 4x+2 & 5y+1 & 10z \\ 10w & 7 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 7 & -2 \end{bmatrix}$$

$$\begin{cases} 4x+2=10 & \rightarrow \quad \underline{x=2} \\ 5y+1=-14 & \rightarrow \quad \underline{y=-3} \\ 10z=80 & \rightarrow \quad \underline{z=8} \\ 10w=10 & \rightarrow \quad \underline{w=1} \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} 2x-3 & y-2 & 2z+1 \\ 5 & 2w & 7 \end{bmatrix} + \begin{bmatrix} 3x-3 & y+2 & z-1 \\ -5 & 5w+1 & 3 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 9 \\ 0 & 8 & 10 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 5x-6 & 2y & 3z \\ 0 & 7w+1 & 10 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 9 \\ 0 & 8 & 10 \end{bmatrix}$$

$$\begin{cases} 5x-6=20 & \rightarrow \quad \underline{x=\frac{26}{5}} \\ 2y=8 & \rightarrow \quad \underline{y=4} \\ 3z=9 & \rightarrow \quad \underline{z=3} \\ 7w+1=8 & \rightarrow \quad \underline{w=1} \end{cases}$$

Exercise

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -3 & 6 \\ -3 & 1 & -4 \end{bmatrix}$$

Solution

$$\begin{aligned} A-B &= \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \end{bmatrix} - \begin{bmatrix} 2 & -3 & 6 \\ -3 & 1 & -4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 & -5 \\ 2 & 1 & 9 \end{bmatrix} \end{aligned}$$

$$3A+2B=3\begin{bmatrix}3 & 1 & 1\\ -1 & 2 & 5\end{bmatrix}+2\begin{bmatrix}2 & -3 & 6\\ -3 & 1 & -4\end{bmatrix}$$

$$=\begin{bmatrix}9 & 3 & 3\\ -3 & 6 & 15\end{bmatrix}+\begin{bmatrix}4 & -6 & 12\\ -6 & 2 & -8\end{bmatrix}$$

$$=\begin{bmatrix}13 & -3 & 15\\ -9 & 8 & 7\end{bmatrix}$$

Exercise

Given $A=\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}$ $\quad F=\begin{bmatrix}3 & 3\\ -1 & -1\end{bmatrix}$ $\quad$ Find $\quad 3F+2A$

Solution

$$3F+2A=3\begin{bmatrix}3 & 3\\ -1 & -1\end{bmatrix}+2\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}$$

$$=\begin{bmatrix}3(3) & 3(3)\\ 3(-1) & 3(-1)\end{bmatrix}+\begin{bmatrix}2(1) & 2(2)\\ 2(4) & 2(3)\end{bmatrix}$$

$$=\begin{bmatrix}9 & 9\\ -3 & -3\end{bmatrix}+\begin{bmatrix}2 & 4\\ 8 & 6\end{bmatrix}$$

$$=\begin{bmatrix}9+2 & 9+4\\ -3+8 & -3+6\end{bmatrix}$$

$$=\begin{bmatrix}11 & 13\\ 5 & 3\end{bmatrix}$$

Exercise

Evaluate $\quad \begin{bmatrix}2\\ 5\\ 8\end{bmatrix}+\begin{bmatrix}-6\\ 3\\ 12\end{bmatrix}$

Solution

$$\begin{bmatrix}2\\ 5\\ 8\end{bmatrix}+\begin{bmatrix}-6\\ 3\\ 12\end{bmatrix}=\begin{bmatrix}-4\\ 8\\ 20\end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 5 & 8 \\ 6 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 9 & 1 \\ 4 & 2 & 5 \end{bmatrix}$

Solution

$$\begin{bmatrix} 5 & 8 \\ 6 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 9 & 1 \\ 4 & 2 & 5 \end{bmatrix}$$

It is ***impossible****;* 2×2 and 2×3 are not the same size.

Exercise

Evaluate $\begin{bmatrix} -5 & 0 \\ 4 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & 3 \end{bmatrix}$

Solution

$$\begin{bmatrix} -5 & 0 \\ 4 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -5+6 & 0+(-3) \\ 4+2 & \frac{1}{2}+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 \\ 6 & \frac{7}{2} \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix}$

Solution

$$\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix} = \begin{bmatrix} 5-4 & -6+6 \\ 8+8 & 9-3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 16 & 6 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} -5 & 6 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -3 & 2 \\ 5 & -8 \end{bmatrix}$

Solution

$$\begin{bmatrix} -5 & 6 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -3 & 2 \\ 5 & -8 \end{bmatrix} = \begin{bmatrix} -5-(-3) & 6-2 \\ 2-5 & 4-(-8) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 4 \\ -3 & 12 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 8 & 6 & -4 \end{bmatrix} - \begin{bmatrix} 3 & 5 & -8 \end{bmatrix}$

Solution

$$\begin{bmatrix} 8 & 6 & -4 \end{bmatrix} - \begin{bmatrix} 3 & 5 & -8 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 4 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1(4)+3(1) & 1(6)+3(0) \\ 2(4)+5(1) & 2(6)+5(0) \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 6 \\ 13 & 12 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{bmatrix}\begin{bmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{bmatrix}$

Solution

$$\begin{bmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{bmatrix}\begin{bmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -3(-6)+4(2)+2(3) & -3(4)+4(3)+2(-2) \\ 5(-6)+0(2)+4(3) & 5(4)+0(3)+4(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 32 & -4 \\ -18 & 12 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix} . \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix} . \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1(1)-1(1)+4(1) & 1(1)-1(2)+4(-1) & 1(0)-1(4)+4(3) \\ 4(1)-1(1)+3(1) & 4(1)-1(2)+3(-1) & 4(0)-1(4)+3(3) \\ 2(1)+0(1)-2(1) & 2(1)+0(2)-2(-1) & 2(0)+0(4)-2(3) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -5 & 8 \\ 6 & -1 & 5 \\ 0 & 4 & -6 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix} . \begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix} . \begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1(1)+1(4)+0(2) & 1(-1)+1(-1)+0(0) & 1(4)+1(3)+0(-2) \\ 1(1)+2(4)+4(2) & 1(-1)+2(-1)+4(0) & 1(4)+2(3)+4(-2) \\ 1(1)-1(4)+3(2) & 1(-1)-1(-1)+3(0) & 1(4)-1(3)+3(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -2 & 7 \\ 17 & -3 & 2 \\ 3 & 0 & -5 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} -2 & -3 & -4 \\ 2 & -1 & 0 \\ 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix}$

Solution

$$\begin{bmatrix} -2 & -3 & -4 \\ 2 & -1 & 0 \\ 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix} = \begin{bmatrix} -3-12 & -2-6-8 & -8+3+8 \\ -1 & 2-2 & 8+1 \\ -2+9 & 4-4+6 & 16+2-6 \end{bmatrix}$$

$$=\begin{bmatrix} -15 & -16 & 3 \\ -1 & 0 & 9 \\ 7 & 6 & 12 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} \sqrt{2} & \sqrt{2} & -\sqrt{18} \\ \sqrt{3} & \sqrt{27} & 0 \end{bmatrix}\begin{bmatrix} 8 & -10 \\ 9 & 12 \\ 0 & 2 \end{bmatrix}$

Solution

$$\begin{bmatrix} \sqrt{2} & \sqrt{2} & -\sqrt{18} \\ \sqrt{3} & \sqrt{27} & 0 \end{bmatrix}\begin{bmatrix} 8 & -10 \\ 9 & 12 \\ 0 & 2 \end{bmatrix}=\begin{pmatrix} 17\sqrt{2} & -4\sqrt{2} \\ 35\sqrt{3} & 26\sqrt{3} \end{pmatrix}$$

Exercise

Evaluate $\begin{bmatrix} x & 2x+1 & 4 \\ 5 & x-1 & 8 \\ -2 & 3x & 2x+1 \end{bmatrix}+\begin{bmatrix} 2x-1 & -2x-1 & 4x \\ -5 & 6 & x+1 \\ -5 & 2 & -2x \end{bmatrix}$

Solution

$$\begin{bmatrix} x & 2x+1 & 4 \\ 5 & x-1 & 8 \\ -2 & 3x & 2x+1 \end{bmatrix}+\begin{bmatrix} 2x-1 & -2x-1 & 4x \\ -5 & 6 & x+1 \\ -5 & 2 & -2x \end{bmatrix}=\begin{bmatrix} 3x-1 & 0 & 4x+4 \\ 0 & x+5 & x+9 \\ -7 & 3x+2 & 1 \end{bmatrix}$$

Exercise

Given $A=\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ *and* $B=\begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}$. Find AB and BA.

Solution

$$AB=\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}=\begin{bmatrix} -2 & 13 \\ 4 & -4 \end{bmatrix}$$

$$BA=\begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}=\begin{bmatrix} -16 & 29 \\ -4 & 10 \end{bmatrix}$$

Note: $AB \neq BA$

Exercise

Given $A=\begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$ $B=\begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix}$$

$$=\begin{pmatrix} -6 & 17 \\ 6 & -8 \end{pmatrix}$$

$$BA=\begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 14 \\ 1 & -20 \end{pmatrix}$$

Exercise

Given $A=\begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$ $B=\begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} -3 & -11 \\ 4 & 0 \end{pmatrix}$$

$$BA=\begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} -7 & 1 \\ 16 & 4 \end{pmatrix}$$

Exercise

Given $A=\begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}$ $B=\begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix}$$

$$=\begin{pmatrix} 10 & 6 \\ 14 & -7 \end{pmatrix}$$

$$BA = \begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 14 & -1 \\ 0 & -11 \end{pmatrix}$$

Exercise

Given $A = \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}$$
$$= \begin{pmatrix} -4 & 2 \\ 4 & -4 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix}$$
$$= \begin{pmatrix} 4 & -4 \\ 14 & -12 \end{pmatrix}$$

Exercise

Given $A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -4 & 5 \\ 6 & 0 & 3 \\ -3 & -2 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}$$
$$= \begin{pmatrix} 5 & -13 \\ 3 & -4 \end{pmatrix}$$

Exercise

Given $A = \begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 5 & 4 \\ 2 & -3 & 8 \\ -3 & 8 & -5 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -14 & 7 \\ -4 & 5 \end{pmatrix}$$

Exercise

Given $A = \begin{pmatrix} 2 & 4 \\ 0 & -1 \\ -3 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 3 & 0 & -2 \\ -2 & 6 & 2 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} 2 & 4 \\ 0 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 & -2 \\ -2 & 6 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 24 & 4 \\ 2 & -6 & -2 \\ -13 & 12 & 10 \end{pmatrix}$$

$$BA = \begin{pmatrix} 3 & 0 & -2 \\ -2 & 6 & 2 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 0 & -1 \\ -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 12 & 8 \\ -10 & 10 \end{pmatrix}$$

Exercise

Given $A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$ $A = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & -2 & -6 \\ 0 & -1 & 2 \\ 5 & -3 & -4 \end{pmatrix}$$

$$BA = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -3 & 9 \\ 2 & -3 & 4 \\ 4 & -6 & 3 \end{pmatrix}$$

Exercise

Given $A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix}$ $B = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 8 & -2 \\ 3 & -8 & 4 \\ -2 & 13 & -5 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix}\begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 5 & -1 \\ 11 & -5 & 6 \\ -8 & 7 & -4 \end{pmatrix}$$

Exercise

Given $A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix}$ $B = \begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix}\begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & -7 & 2 \\ -6 & 2 & 2 \\ -8 & -6 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 6 & 1 \\ 7 & 0 & 5 \\ 4 & -4 & -2 \end{pmatrix}$$

Exercise

Given $A = \begin{bmatrix} -3 & 4 \\ 2 & -3 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 1 \\ 1 & -2 \\ 3 & -4 \end{bmatrix}$, Find

a) $A+B$
b) $A-B$
c) $3A$
d) $-2B$
e) $2A+3B$
f) A^2
g) AB
h) BA

Solution

a) $A+B = \begin{bmatrix} -3 & 4 \\ 2 & -3 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 1 & -2 \\ 3 & -4 \end{bmatrix}$

$$=\begin{bmatrix}1 & 5\\ 3 & -5\\ 2 & -4\end{bmatrix}$$

b) $$A-B=\begin{bmatrix}-3 & 4\\ 2 & -3\\ -1 & 0\end{bmatrix}-\begin{bmatrix}4 & 1\\ 1 & -2\\ 3 & -4\end{bmatrix}$$

$$=\begin{bmatrix}-7 & 3\\ 1 & -1\\ -4 & 4\end{bmatrix}$$

c) $$3A=3\begin{bmatrix}-3 & 4\\ 2 & -3\\ -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}-9 & 12\\ 6 & -9\\ -3 & 0\end{bmatrix}$$

d) $$-2B=-2\begin{bmatrix}4 & 1\\ 1 & -2\\ 3 & -4\end{bmatrix}$$

$$=\begin{bmatrix}-8 & -2\\ -2 & 4\\ -6 & 8\end{bmatrix}$$

e) $$2A+3B=2\begin{bmatrix}-3 & 4\\ 2 & -3\\ -1 & 0\end{bmatrix}+3\begin{bmatrix}4 & 1\\ 1 & -2\\ 3 & -4\end{bmatrix}$$

$$=\begin{bmatrix}-6 & 8\\ 4 & -6\\ -2 & 0\end{bmatrix}+\begin{bmatrix}12 & 3\\ 3 & -6\\ 9 & -12\end{bmatrix}$$

$$=\begin{bmatrix}6 & 11\\ 7 & -12\\ 7 & -12\end{bmatrix}$$

f) $A^2 = \textit{doesn't exist}$ (not a square matrix)

g) $AB = \nexists$ $(2\times 3 \quad 2\times 3)$ the inner not equal

h) $BA = \nexists$ $(2\times 3 \quad 2\times 3)$ the inner not equal

Exercise

Given $A=\begin{bmatrix} 2 & -2 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}$ $B=\begin{bmatrix} -1 & 8 \\ 2 & -2 \\ -4 & 3 \end{bmatrix}$, Find

a) $A+B$
b) $A-B$
c) $3A$
d) $-2B$
e) $2A+3B$
f) A^2
g) AB
h) BA

Solution

a) $A+B=\begin{bmatrix} 2 & -2 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}+\begin{bmatrix} -1 & 8 \\ 2 & -2 \\ -4 & 3 \end{bmatrix}$

$=\begin{bmatrix} 1 & 6 \\ 5 & 2 \\ -3 & 3 \end{bmatrix}$

b) $A-B=\begin{bmatrix} 2 & -2 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}-\begin{bmatrix} -1 & 8 \\ 2 & -2 \\ -4 & 3 \end{bmatrix}$

$=\begin{bmatrix} 3 & -10 \\ 1 & 6 \\ 5 & -3 \end{bmatrix}$

c) $3A=3\begin{bmatrix} 2 & -2 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}$

$=\begin{bmatrix} 6 & -6 \\ 9 & 12 \\ 3 & 0 \end{bmatrix}$

d) $-2B=-2\begin{bmatrix} -1 & 8 \\ 2 & -2 \\ -4 & 3 \end{bmatrix}$

$=\begin{bmatrix} 2 & -16 \\ -4 & 4 \\ 8 & -6 \end{bmatrix}$

e) $2A+3B=2\begin{bmatrix} 2 & -2 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}+3\begin{bmatrix} -1 & 8 \\ 2 & -2 \\ -4 & 3 \end{bmatrix}$

$$=\begin{bmatrix} 4 & -4 \\ 6 & 8 \\ 2 & 0 \end{bmatrix}+\begin{bmatrix} -3 & 24 \\ 6 & -6 \\ -12 & 9 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 20 \\ 12 & 2 \\ -10 & 9 \end{bmatrix}$$

f) $A^2 = doesn't\ exist$ (not a square matrix)

g) $AB = \nexists$ $(2\times 3 \quad 2\times 3)$ the inner not equal

h) $BA = \nexists$ $(2\times 3 \quad 2\times 3)$ the inner not equal

Exercise

Given $A=\begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}$ $B=\begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$, Find

a) $A+B$
b) $A-B$
c) $3A$
d) $-2B$
e) $2A+3B$
f) A^2
g) AB
h) BA

Solution

a) $A+B=\begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}+\begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$

$$=\begin{bmatrix} -1 & 1 & -1 \\ 2 & 2 & 1 \\ -1 & 2 & 5 \end{bmatrix}$$

b) $A-B=\begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}-\begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$

$$=\begin{bmatrix} -3 & 5 & -1 \\ -2 & -4 & 3 \\ -7 & 4 & 1 \end{bmatrix}$$

c) $3A=3\begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}$

$$=\begin{bmatrix} -6 & 9 & -3 \\ 0 & -3 & 6 \\ -12 & 9 & 9 \end{bmatrix}$$

d) $-2B=-2\begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$

$$=\begin{bmatrix} -2 & 4 & 0 \\ -4 & -6 & 2 \\ -6 & 2 & -4 \end{bmatrix}$$

e) $2A+3B=2\begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}+3\begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$

$$=\begin{bmatrix} -4 & 6 & -2 \\ 0 & -2 & 4 \\ -8 & 6 & 6 \end{bmatrix}+\begin{bmatrix} 3 & -6 & 0 \\ 6 & 9 & -3 \\ 9 & -3 & 6 \end{bmatrix}$$

$$=\begin{bmatrix} -1 & 0 & -2 \\ 6 & 7 & 1 \\ 1 & 3 & 12 \end{bmatrix}$$

f) $A^2=\begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}\begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}$

$$=\begin{bmatrix} 4+4 & -6-3-3 & 2+6-3 \\ -8 & 1+6 & -2+6 \\ 8-12 & -12-3+9 & 4+6+9 \end{bmatrix}$$

$$=\begin{bmatrix} 8 & -12 & 5 \\ -8 & 7 & 4 \\ -4 & -6 & 19 \end{bmatrix}$$

g) $AB=\begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$

$$=\begin{bmatrix} -2+6-3 & 4+9+1 & -3-2 \\ -2+6 & -3-2 & 1+4 \\ -4+6+9 & 8+9-3 & -3+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 14 & -5 \\ 4 & -5 & 5 \\ 11 & 14 & 3 \end{bmatrix}$$

h) $BA = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}\begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -2 & 3+2 & -1-4 \\ -4+4 & 6-3-3 & -2+6-3 \\ -6-8 & 9+1+6 & -3-2+6 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 5 & -5 \\ 0 & 0 & 1 \\ -14 & 16 & 1 \end{bmatrix}$$

Exercise

Given $A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix}$, Find

a) $A+B$

b) $A-B$

c) $3A$

d) $-2B$

e) $2A+3B$

f) A^2

g) AB

h) BA

Solution

a) $A+B = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 4 & 4 \\ 4 & 0 & 1 \\ 1 & 8 & 1 \end{bmatrix}$$

b) $A-B = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & -4 \\ -2 & -6 & 5 \\ 9 & 0 & -5 \end{bmatrix}$$

c) $3A = 3\begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 6 & 0 \\ 3 & -9 & 9 \\ 15 & 12 & -6 \end{bmatrix}$$

d) $-2B = -2\begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 2 & -4 & -8 \\ -6 & -6 & 4 \\ 8 & -8 & -6 \end{bmatrix}$$

e) $2A + 3B = 2\begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix} + 3\begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 4 & 0 \\ 2 & -6 & 6 \\ 10 & 8 & -4 \end{bmatrix} + \begin{bmatrix} -3 & 6 & 12 \\ 9 & 9 & -6 \\ -12 & 12 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 10 & 12 \\ 11 & 3 & 0 \\ -2 & 20 & 5 \end{bmatrix}$$

f) $A^2 = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 2 & -6 & 6 \\ -3+15 & 2+9+12 & -9-6 \\ 4-10 & 10-12-8 & 12+4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -6 & 6 \\ 12 & 23 & -15 \\ -6 & -10 & 16 \end{bmatrix}$$

g) $AB = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}\begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 6 & 6 & -4 \\ -1-9-12 & 2-9+12 & 4+6+9 \\ -5+12+8 & 10+12-8 & 20-8-6 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 6 & -4 \\ -22 & 5 & 19 \\ 15 & 14 & 6 \end{bmatrix}$$

h) $BA = \begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 2+10 & -2-6+16 & 6-8 \\ 3-10 & 6-9-8 & 9+4 \\ 4+15 & -8-12+12 & 12-6 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 8 & -2 \\ -7 & -11 & 13 \\ 19 & -8 & 6 \end{bmatrix}$$

Exercise

Given $A = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$ $C = \begin{pmatrix} 4 & 3 & 2 \\ -1 & 2 & 1 \end{pmatrix}$ $D = \begin{pmatrix} -2 & 3 \\ 2 & -1 \\ 3 & 2 \end{pmatrix}$, Find

a) $4A-2B$
b) $3A+C$
c) $3A+B$
d) $2A-3B$
e) AB
f) BA
g) A^2
h) B^3
i) AC
j) CA
k) CD
l) DC

Solution

a) $4A-2B = 4\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$

$$= \begin{pmatrix} -4 & 8 \\ -8 & 4 \end{pmatrix} - \begin{pmatrix} 2 & -4 \\ 4 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & 12 \\ -12 & 6 \end{pmatrix}$$

b) $3A+C = \nexists$

They are not the same order.

c) $3A+B = 3\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$

$$= \begin{pmatrix} -3 & 6 \\ -6 & 3 \end{pmatrix} + \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 4 \\ -4 & 2 \end{pmatrix}$$

d) $2A-3B=2\begin{pmatrix}-1 & 2\\ -2 & 1\end{pmatrix}-3\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}$

$=\begin{pmatrix}-2 & 4\\ -4 & 2\end{pmatrix}-\begin{pmatrix}3 & -6\\ 6 & -3\end{pmatrix}$

$=\begin{pmatrix}-5 & 10\\ -10 & 5\end{pmatrix}$

e) $AB=\begin{pmatrix}-1 & 2\\ -2 & 1\end{pmatrix}\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}$

$=\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}$

f) $BA=\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}\begin{pmatrix}-1 & 2\\ -2 & 1\end{pmatrix}$

$=\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}$

g) $A^2=\begin{pmatrix}-1 & 2\\ -2 & 1\end{pmatrix}\begin{pmatrix}-1 & 2\\ -2 & 1\end{pmatrix}$

$=\begin{pmatrix}-3 & 0\\ 0 & -3\end{pmatrix}$

h) $B^3=\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}$

$=\begin{pmatrix}-3 & 0\\ 0 & -3\end{pmatrix}\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}$

$=\begin{pmatrix}-3 & 6\\ -6 & 3\end{pmatrix}$

i) $AC=\begin{pmatrix}-1 & 2\\ -2 & 1\end{pmatrix}\begin{pmatrix}4 & 3 & 2\\ -1 & 2 & 1\end{pmatrix}$ $\qquad 2\times 2 \quad 2\times 3 \quad \to 2\times 3$

$=\begin{pmatrix}-6 & 1 & 0\\ -9 & -4 & -3\end{pmatrix}$

j) $CB=\nexists$ $\qquad 2\times 3 \quad 2\times 2$

C and B are not the same order.

k) $CD=\begin{pmatrix}4 & 3 & 2\\ -1 & 2 & 1\end{pmatrix}\begin{pmatrix}-2 & 3\\ 2 & -1\\ 3 & 2\end{pmatrix}$ $\qquad 2\times 3 \quad 3\times 2 \quad \to 2\times 2$

$=\begin{pmatrix}-8+6+6 & 12-3+4\\ 2+4+3 & -3+2+2\end{pmatrix}$

$$=\begin{pmatrix} 4 & 13 \\ 9 & 1 \end{pmatrix}$$

l) $DC=\begin{pmatrix} -2 & 3 \\ 2 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 4 & 3 & 2 \\ -1 & 2 & 1 \end{pmatrix}$ $\quad 3\times 2 \quad 2\times 3 \quad \to 3\times 3$

$$=\begin{pmatrix} -8-3 & -6+6 & -4+3 \\ 8+1 & 6-2 & 4-1 \\ 12-2 & 9+4 & 6+2 \end{pmatrix}$$

$$=\begin{pmatrix} -11 & 0 & -1 \\ 9 & 4 & 3 \\ 10 & 13 & 8 \end{pmatrix}$$

Exercise

Given $A=\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$ $\quad B=\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$ $\quad C=\begin{pmatrix} 1 & 4 & 5 \\ -2 & 3 & 4 \\ -1 & 0 & -2 \end{pmatrix}$ $\quad D=\begin{pmatrix} 2 & 4 & -2 \\ 0 & 3 & 5 \\ -3 & 1 & 1 \end{pmatrix}$, Find

a) $4A-2B$
b) $3A+C$
c) $3A+B$
d) $2A-3B$
e) AB
f) BA
g) A^2
h) B^3
i) AC
j) CB
k) CD
l) DC

Solution

a) $4A-2B=4\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}-2\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

$$=\begin{pmatrix} 8 & 16 \\ 12 & -4 \end{pmatrix}-\begin{pmatrix} -2 & 6 \\ 4 & -2 \end{pmatrix}$$

$$=\begin{pmatrix} 10 & 10 \\ 8 & -2 \end{pmatrix}$$

b) $3A+C=\nexists$

They are not the same order.

c) $3A+B=3\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}+\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

$$=\begin{pmatrix} 6 & 12 \\ 9 & -3 \end{pmatrix}+\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$$

$$=\begin{pmatrix} 5 & 15 \\ 11 & -4 \end{pmatrix}$$

d) $2A-3B=2\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}-3\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 4 & 8 \\ 6 & -2 \end{pmatrix} - \begin{pmatrix} -3 & 9 \\ 6 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -1 \\ 0 & 1 \end{pmatrix}$$

e) $AB = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 6 & 2 \\ -5 & 10 \end{pmatrix}$$

f) $BA = \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 7 & -7 \\ 1 & 9 \end{pmatrix}$$

g) $A^2 = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 14 & 4 \\ 3 & 13 \end{pmatrix}$$

h) $B^3 = \left(\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix} \right) \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 7 & -6 \\ -4 & 7 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -19 & 27 \\ 18 & -19 \end{pmatrix}$$

i) $AC = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 & 2 \\ -1 & 2 & 1 \end{pmatrix}$ $\qquad 2 \times 2 \quad 2 \times 3 \quad \to 2 \times 3$

$$= \begin{pmatrix} -6 & 1 & 0 \\ -9 & -4 & -3 \end{pmatrix}$$

j) $CB = \nexists$ $\qquad 2 \times 3 \quad 2 \times 2$

C and B are not the same order.

k) $CD = \begin{pmatrix} 1 & 4 & 5 \\ -2 & 3 & 4 \\ -1 & 0 & -2 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 0 & 3 & 5 \\ -3 & 1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} -12 & 21 & 13 \\ -16 & 5 & 23 \\ 4 & -6 & 0 \end{pmatrix}$$

l) $DC = \begin{pmatrix} 2 & 4 & -2 \\ 0 & 3 & 5 \\ -3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 & 5 \\ -2 & 3 & 4 \\ -1 & 0 & -2 \end{pmatrix}$

$= \begin{pmatrix} 2-8+2 & 8+12 & 10+16+4 \\ -6-5 & 9 & 12-10 \\ -3-2-1 & -12+3 & -15+4-2 \end{pmatrix}$

$= \begin{pmatrix} -4 & 20 & 30 \\ -11 & 9 & 2 \\ -6 & -9 & -12 \end{pmatrix}$

Exercise

A contractor builds three kinds of houses, models *A*, *B*, and *C*, with a choice of two styles, Spanish and contemporary. Matrix *P* shows the number of each kind of house planned for a new 100-home subdivision. The amounts for each of the exterior materials depend primarily on the style of the house. These amounts are shown in matrix *Q*. (concrete is in cubic yards, lumber in units of 1000 board feet, brick in 1000s, and shingles in units of 100 ft^2.) Matrix *R* gives the cost in dollars for each kind of material.

a) What is the total cost of these materials for each model?

b) How much of each of four kinds of material must be ordered

c) What is the total cost for exterior materials?

Solution

$$\begin{array}{l} \\ Model\ A \\ Model\ B \\ Model\ C \end{array}\begin{array}{c} \begin{array}{cc} Spanish & Contemporary \end{array} \\ \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix} \end{array} = P$$

$$\begin{array}{l} \\ Spanish \\ Contemporary \end{array}\begin{array}{c} \begin{array}{cccc} Concrete & Lumber & Brick & Shingles \end{array} \\ \begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix} \end{array} = Q$$

$$\begin{array}{l} \\ Concrete \\ Lumber \\ Brick \\ Shingles \end{array}\begin{array}{c} Cost\ per\ unit \\ \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix} \end{array} = R$$

a) What is the total cost of these materials for each model?

$$PQ = \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix}\begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{cccc} \textit{Concrete} & \textit{Lumber} & \textit{Brick} & \textit{Shingles} \end{array} \\ \begin{bmatrix} 1500 & 30 & 600 & 60 \\ 100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix} \end{array} \begin{array}{l} \textit{Model } A \\ \textit{Model } \mathrm{B} \\ \textit{Model } \mathrm{C} \end{array}$$

$$(PQ)R = \begin{bmatrix} 1500 & 30 & 600 & 60 \\ 100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix} \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix}$$

$$= \begin{bmatrix} 72,900 \\ 54,700 \\ 60,800 \end{bmatrix} \begin{array}{l} \textit{Model } A \\ \textit{Model } B \\ \textit{Model } C \end{array}$$

The total cost of materials is \$72,900 for model *A*, \$54,700 for model *B*, \$60,800 for model *C*.

b) How much of each of four kinds of material must be ordered

$$\begin{array}{cccc} \left[1500 \right. & 30 & 600 & \left. 60 \right] \\ 100 & 40 & 400 & 60 \\ \left[1200 \right. & 60 & 400 & \left. 80 \right] \\ \hline 3800 & 130 & 1400 & 200 \end{array}$$

$$T = \begin{bmatrix} 3800 & 130 & 1400 & 200 \end{bmatrix}$$

3800 yd^3 of concrete, 130,000 board feet of lumber, 1,400,000 bricks, and 20,000 ft^2 of shingles are needed.

c) What is the total cost for exterior materials?

$$TR = \begin{bmatrix} 3800 & 130 & 1400 & 200 \end{bmatrix} \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix}$$

$$= \begin{bmatrix} 188,400 \end{bmatrix}$$

The total cost for exterior materials is \$188,400

Exercise

Mitchell Fabricators manufactures three styles of bicycle frames in its two plants. The following table shows the number of each style produced at each plant

	Mountain Bike	***Racing Bike***	***Touring Bike***
North Plant	150	120	100
South Plant	180	90	130

a) Write a 2 *x* 3 matrix ***A*** that represents the information in the table

b) The manufacturer increased production of each style by 20%. Find a Matrix ***M*** that represents the increased production figures.

c) Find the matrix ***A* + *M*** and tell what it represents

Solution

a) $A = \begin{bmatrix} 150 & 120 & 100 \\ 180 & 90 & 130 \end{bmatrix}$

b) The 20% production will represent

A + 20%(***A***)

$\rightarrow$ ***A*** + .2 ***A*** = 1.2***A***

$$M = (1.2)\begin{bmatrix} 150 & 120 & 100 \\ 180 & 90 & 130 \end{bmatrix}$$

$$= \begin{bmatrix} 180 & 144 & 120 \\ 216 & 108 & 156 \end{bmatrix}$$

c) $$A + M = \begin{bmatrix} 150 & 120 & 100 \\ 180 & 90 & 130 \end{bmatrix} + \begin{bmatrix} 180 & 144 & 120 \\ 216 & 108 & 156 \end{bmatrix}$$

$$= \begin{bmatrix} 330 & 264 & 220 \\ 396 & 198 & 286 \end{bmatrix}$$

The matrix *A* + *M* represents the total production of each style at each plant for the time period (2 months)

Exercise

Sal's Shoes and Fred's Footwear both have outlets in California and Arizona. Sal's sells shoes for $80, sandals for $40, and boots for $120. Fred's prices are $60, $30, and $150 for shoes, sandals and boots, respectively. Half of all sales in California stores are shoes, 1/4 are *sandals*, and 1/4 are *boots*. In Arizona the fractions are 1/5 *shoes*, 1/5 are *sandals*, and 3/5 are *boots*.

a) Write a 2 x 3 matrix called P representing prices for the two stores and three types of footwear.

b) Write a 2 x 3 matrix called F representing fraction of each type of footwear sold in each state.

c) Only one of the two products *PF* and *FP* is meaningful. Determine which one it is, calculate the product, and describe what the entries represent.

Solution

a) Write a 2 x 3 matrix called *P* representing prices for the two stores and three types of footwear.

$$P = \begin{array}{c} \begin{bmatrix} 80 & 40 & 120 \\ 60 & 30 & 150 \end{bmatrix} \end{array} \begin{array}{l} \textit{Sal's} \\ \textit{Fred's} \end{array}$$

b) Write a 2 *x* 3 matrix called F representing fraction of each type of footwear sold in each state.

$$F = \begin{array}{c} \begin{array}{cc} CA & AR \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{3}{5} \end{bmatrix} \end{array}$$

c)

$$PF = \begin{bmatrix} 80 & 40 & 120 \\ 60 & 30 & 150 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{3}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 80\frac{1}{2}+40\frac{1}{4}+120\frac{1}{4} & 80\frac{1}{5}+40\frac{1}{5}+120\frac{3}{5} \\ 60\frac{1}{2}+30\frac{1}{4}+150\frac{1}{4} & 60\frac{1}{5}+30\frac{1}{5}+150\frac{3}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 80 & 96 \\ 75 & 108 \end{bmatrix}$$

Solution   Section 4.3 – Multiplicative Inverses of Matrices

Exercise

Show that B is Multiplicative inverse of A

$$A = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

Solution

$$\begin{aligned} AB &= \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \\ &= \begin{pmatrix} -6 & \\ & \end{pmatrix} \\ &\neq I \end{aligned}$$

B is not multiplicative inverse of A

Exercise

Show that B is Multiplicative inverse of A

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \quad \& \quad B = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

Solution

$$\begin{aligned} AB &= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I \end{aligned}$$

$\therefore B$ is Multiplicative inverse of A

Exercise

Find the inverse, if exists, of $A=\begin{bmatrix}2 & -6\\ 1 & -2\end{bmatrix}$

Solution

$$A^{-1}=\frac{1}{-4+6}\begin{bmatrix}-2 & 6\\ -1 & 2\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}-2 & 6\\ -1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}-1 & 3\\ -\frac{1}{3} & 1\end{bmatrix}$$

Exercise

Find the inverse, if exists, of $A=\begin{bmatrix}10 & -2\\ -5 & 1\end{bmatrix}$

Solution

$$A^{-1}=\frac{1}{10-10}\begin{bmatrix}10 & -2\\ -5 & 1\end{bmatrix}$$

$$=\frac{1}{0}\begin{bmatrix}10 & -2\\ -5 & 1\end{bmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A=\begin{bmatrix}-2 & 3\\ -3 & 4\end{bmatrix}$

Solution

$$\left[\begin{array}{cc|cc}-2 & 3 & 1 & 0\\ -3 & 4 & 0 & 1\end{array}\right] \quad -\tfrac{1}{2}R_1$$

$$\begin{array}{cccc}1 & -\frac{3}{2} & -\frac{1}{2} & 0\end{array}$$

$$\left[\begin{array}{cc|cc}1 & -\frac{3}{2} & -\frac{1}{2} & 0\\ -3 & 4 & 0 & 1\end{array}\right] \quad R_2+3R_1$$

$$\begin{array}{cccc}-3 & 4 & 0 & 1\\ 3 & -\frac{9}{2} & -\frac{3}{2} & 0\\ \hline 0 & -\frac{1}{2} & -\frac{3}{2} & 1\end{array}$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right] \quad -2R_2$$

$$\begin{array}{cccc} 0 & 1 & 3 & -2 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 3 & -2 \end{array}\right] \quad R_1 + \frac{3}{2}R_2$$

$$\begin{array}{cccc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & \frac{9}{2} & -3 \\ \hline 1 & 0 & 4 & -3 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 4 & -3 \\ 0 & 1 & 3 & -2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{bmatrix} a & b \\ 3 & 3 \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{3a-3b}\begin{bmatrix} 3 & -b \\ -3 & a \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{3(a-b)} & \frac{-b}{3(a-b)} \\ \frac{-3}{3(a-b)} & \frac{a}{3(a-b)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{a-b} & \frac{-b}{3(a-b)} \\ \frac{-1}{a-b} & \frac{a}{3(a-b)} \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{bmatrix} -2 & a \\ 4 & a \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{-2a-4a}\begin{bmatrix} a & -a \\ -4 & -2 \end{bmatrix}$$

$$=\begin{bmatrix} \frac{a}{-6a} & \frac{-a}{-6a} \\ \frac{-4}{-6a} & \frac{-2}{-6a} \end{bmatrix}$$

$$=\begin{bmatrix} -\frac{1}{6} & \frac{1}{6} \\ \frac{2}{3a} & \frac{1}{3a} \end{bmatrix}$$

Exercise

Find the inverse of $\quad A=\begin{bmatrix} 4 & 4 \\ b & a \end{bmatrix}$

Solution

$$A^{-1}=\frac{1}{4a-4b}\begin{bmatrix} a & -4 \\ -b & 4 \end{bmatrix}$$

$$=\begin{bmatrix} \frac{a}{4(a-b)} & \frac{-4}{4(a-b)} \\ \frac{-b}{4(a-b)} & \frac{4}{4(a-b)} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{a}{4(a-b)} & \frac{-1}{a-b} \\ \frac{-b}{4(a-b)} & \frac{1}{a-b} \end{bmatrix}$$

Exercise

Find the inverse of $\quad A=\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$

Solution

$$A^{-1}=\frac{1}{-1+4}\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{3}\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$

Solution

$$A^{-1} = -\frac{1}{14}\begin{pmatrix} -1 & -4 \\ -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{14} & \frac{2}{7} \\ \frac{3}{14} & -\frac{1}{7} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

Solution

$$A^{-1} = -\frac{1}{5}\begin{pmatrix} -1 & -3 \\ -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{11}\begin{pmatrix} 5 & -3 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{11} & -\frac{3}{11} \\ \frac{2}{11} & \frac{1}{11} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{0}\begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{pmatrix} -6 & 9 \\ 2 & -3 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{18-18}\begin{pmatrix} & \\ & \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{pmatrix} -2 & 7 \\ 0 & 2 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -7 \\ 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{7}{4} \\ 0 & -\frac{1}{2} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$

Solution

$$A = \frac{1}{-16+16}\begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Solution

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 2 & 1 \\ a & a \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{a}\begin{pmatrix} a & -1 \\ -a & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\frac{1}{a} \\ -1 & \frac{2}{a} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} b & 3 \\ b & 2 \end{pmatrix}$

Solution

$$A^{-1} = -\frac{1}{b}\begin{pmatrix} 2 & -3 \\ -b & b \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{b} & \frac{3}{b} \\ 1 & -1 \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & a \\ 3 & a \end{pmatrix}$

Solution

$$A^{-1} = -\frac{1}{2a}\begin{pmatrix} a & -a \\ -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2a} & -\frac{1}{2a} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{a^2-4}\begin{pmatrix} a & -2 \\ -2 & a \end{pmatrix}$$

$$= \begin{pmatrix} \frac{a}{a^2-4} & \frac{-2}{a^2-4} \\ \frac{-2}{a^2-4} & \frac{a}{a^2-4} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{0}\begin{pmatrix} & \\ & \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{pmatrix} -3 & \frac{1}{2} \\ 6 & -1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{0}\begin{pmatrix} & \\ & \end{pmatrix}$$

∴ Inverse ***doesn't exist***

Exercise

Find A^{-1} *if* $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccccc} 2 & -2 & -1 & 0 & 1 & 0 \\ -2 & 0 & -2 & -2 & 0 & 0 \\ \hline 0 & -2 & -3 & -2 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 3 & 0 & 0 & 0 & 0 & 1 \\ -3 & 0 & -3 & -3 & 0 & 0 \\ \hline 0 & 0 & -3 & -3 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] -\tfrac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] -\tfrac{1}{3}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 - \frac{3}{2}R_3 \\ \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & \frac{1}{3} \\ \hline 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{array} \qquad \begin{array}{cccccc} 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & 0 & \frac{1}{2} \\ \hline 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

Exercise

Find A^{-1}, where $A=\begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2-3R_1 \\ R_3+2R_1 \end{array}$$

$$\begin{array}{rrrrrr} 3 & 5 & 3 & 0 & 1 & 0 \\ -3 & -6 & 3 & -3 & 0 & 0 \\ \hline 0 & -1 & 6 & -3 & 1 & 0 \end{array} \qquad \begin{array}{rrrrrr} 2 & 4 & 3 & 0 & 0 & 1 \\ -2 & -4 & 2 & -2 & 0 & 0 \\ \hline 0 & 0 & 5 & -2 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right] \quad -R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right] \quad R_1-2R_2$$

$$\begin{array}{rrrrrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 12 & -6 & 2 & 0 \\ \hline 1 & 0 & 11 & -5 & 2 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right] \quad \tfrac{1}{5}R_3$$

$$\begin{array}{rrrrrr} 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}\right] \quad \begin{array}{l} R_1-11R_3 \\ R_2+6R_3 \end{array}$$

$$\begin{array}{rrrrrr} 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 6 & -\frac{12}{5} & 0 & \frac{6}{5} \\ \hline 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \end{array} \qquad \begin{array}{rrrrrr} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 0 & -11 & \frac{22}{5} & 0 & -\frac{11}{5} \\ \hline 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}\right]$$

$$A^{-1}=\begin{bmatrix} -\frac{3}{5} & 2 & -\frac{11}{5} \\ \frac{3}{5} & -1 & \frac{6}{5} \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{bmatrix}$$

Exercise

Find A^{-1}, where $A=\begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

<u>***Solution***</u>

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2+2R_1 \\ R_3-R_1 \end{array}$$

$$\begin{array}{cccccc} -2 & 0 & 1 & 0 & 1 & 0 \\ 2 & 4 & -2 & 2 & 0 & 0 \\ \hline 0 & 4 & -1 & 2 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 1 & -1 & 0 & 0 & 0 & 1 \\ -1 & -2 & 1 & -1 & 0 & 0 \\ \hline 0 & -3 & 1 & -1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 4 & -1 & 2 & 1 & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \frac{1}{4}R_2 \\ \\ \end{array}$$

$$\begin{array}{cccccc} 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} R_1-2R_2 \\ \\ R_3+3R_2 \end{array}$$

$$\begin{array}{cccccc} 0 & -3 & 1 & -1 & 0 & 1 \\ 0 & 3 & -\frac{3}{4} & \frac{3}{2} & \frac{3}{4} & 0 \\ \hline 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{array} \qquad \begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & \frac{1}{2} & -1 & -\frac{1}{2} & 0 \\ \hline 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{array}\right] \begin{array}{l} \\ \\ 4R_3 \end{array}$$

$$\begin{array}{cccccc} 0 & 0 & 1 & 2 & 3 & 4 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array}\right] \begin{array}{l} R_1+\frac{1}{2}R_3 \\ R_2+\frac{1}{4}R_3 \\ \\ \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 1 & \frac{3}{2} & 2 \\ \hline 1 & 0 & 0 & 1 & 1 & 2 \end{array} \qquad \begin{array}{cccccc} 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \\ \hline 0 & 1 & 0 & 1 & 1 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array}\right]$$

$$A^{-1}=\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

Exercise

Find A^{-1}, where $A=\begin{bmatrix} -2 & 5 & 3 \\ 4 & -1 & 3 \\ 7 & -2 & 5 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} -2 & 5 & 3 & 1 & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \qquad \frac{1}{-2}R_1 \qquad 1 \quad -\frac{5}{2} \quad -\frac{3}{2} \quad -\frac{1}{2} \quad 0 \quad 0$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \qquad R_2-4R_1 \qquad \begin{array}{cccccc} 4 & -1 & 3 & 0 & 1 & 0 \\ -4 & 10 & 6 & 2 & 0 & 0 \\ \hline 0 & 9 & 9 & 2 & 1 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \qquad R_3-7R_1 \qquad \begin{array}{cccccc} 7 & -2 & 5 & 0 & 0 & 1 \\ -7 & \frac{35}{2} & \frac{21}{2} & \frac{7}{2} & 0 & 0 \\ \hline 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}\right] \qquad \frac{1}{9}R_2 \qquad 0 \quad 1 \quad 1 \quad \frac{2}{9} \quad \frac{1}{9} \quad 0$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}\right] \qquad R_3-\frac{31}{2}R_2 \qquad \begin{array}{cccccc} 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \\ 0 & -\frac{31}{2} & -\frac{31}{2} & -\frac{31}{9} & -\frac{31}{18} & 0 \\ \hline 0 & 0 & 0 & \frac{1}{18} & -\frac{31}{18} & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{18} & -\frac{31}{18} & 1 \end{array}\right]$$

$\therefore$ The inverse matrix ***doesn't exist***

Exercise

Find the inverse, if exists, of $A=\begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right) \quad R_2 + R_1 \qquad \begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ \hline 0 & 4 & 4 & 1 & 1 & 0 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 4 & 4 & 1 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right) \quad \tfrac{1}{4}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 4R_2 \end{array} \qquad \begin{array}{cccccc} 0 & 4 & 3 & 0 & 0 & 1 \\ 0 & -4 & -4 & -1 & -1 & 0 \\ \hline 0 & 0 & -1 & -1 & -1 & 1 \end{array} \qquad \begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ \hline 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right) \quad -R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right) \quad \begin{array}{l} R_1 + R_3 \\ R_2 - R_3 \end{array} \qquad \begin{array}{cccccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ \hline 1 & 0 & 0 & \frac{7}{4} & \frac{3}{4} & -1 \end{array} \qquad \begin{array}{cccccc} 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \\ \hline 0 & 1 & 0 & -\frac{3}{4} & -\frac{3}{4} & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{4} & \frac{3}{4} & -1 \\ 0 & 1 & 0 & -\frac{3}{4} & -\frac{3}{4} & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{7}{4} & \frac{3}{4} & -1 \\ -\frac{3}{4} & -\frac{3}{4} & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ -2 & -3 & 0 & 0 & 0 & 1 \end{array}\right) \quad R_3 + 2R_1 \qquad \begin{array}{cccccc} -2 & -3 & 0 & 0 & 0 & 1 \\ 2 & -2 & 2 & 2 & 0 & 0 \\ \hline 0 & -5 & 2 & 2 & 0 & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & -5 & 2 & 2 & 0 & 1 \end{array}\right) \quad -\frac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & -5 & 2 & 2 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 + R_2 \\ \\ R_3 + 5R_2 \end{array} \qquad \begin{array}{cccccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ \hline 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \end{array} \qquad \begin{array}{cccccc} 0 & -5 & 2 & 2 & 0 & 1 \\ 0 & 5 & -\frac{5}{2} & 0 & -\frac{5}{2} & 0 \\ \hline 0 & 0 & -\frac{1}{2} & 2 & -\frac{5}{2} & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 2 & -\frac{5}{2} & 1 \end{array}\right) \quad -2R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -4 & 5 & -2 \end{array}\right) \quad \begin{array}{l} R_1 - \frac{1}{2}R_3 \\ R_2 + \frac{1}{2}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & -4 & 5 & -2 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A=\begin{pmatrix}1 & 0 & 2\\ -1 & 2 & 3\\ 1 & -1 & 0\end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc}1 & 0 & 2 & 1 & 0 & 0\\ -1 & 2 & 3 & 0 & 1 & 0\\ 1 & -1 & 0 & 0 & 0 & 1\end{array}\right) \begin{matrix} R_2+R_1\\ R_3-R_1\end{matrix}$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & 2 & 1 & 0 & 0\\ 0 & 2 & 5 & 1 & 1 & 0\\ 0 & -1 & -2 & -1 & 0 & 1\end{array}\right) \tfrac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & 2 & 1 & 0 & 0\\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0\\ 0 & -1 & -2 & -1 & 0 & 1\end{array}\right) R_3+R_2$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & 2 & 1 & 0 & 0\\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1\end{array}\right) 2R_3$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & 2 & 1 & 0 & 0\\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 1 & -1 & 1 & 2\end{array}\right) \begin{matrix} R_1-2R_3\\ R_2-\frac{5}{2}R_3\end{matrix}$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & 0 & 3 & -2 & -4\\ 0 & 1 & 0 & -2 & -2 & -5\\ 0 & 0 & 1 & 3 & 1 & 2\end{array}\right)$$

$$A^{-1}=\begin{pmatrix}3 & -2 & -4\\ -2 & -2 & -5\\ 3 & 1 & 2\end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A=\begin{pmatrix}1&1&1\\3&2&-1\\3&1&2\end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\3&2&-1&0&1&0\\3&1&2&0&0&1\end{array}\right)\begin{array}{l}\\R_2-3R_1\\R_3-3R_1\end{array}$$

$$\left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&-1&-4&-3&1&0\\0&-2&-1&-3&0&1\end{array}\right)\begin{array}{l}\\-R_2\\ \end{array}$$

$$\left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&4&3&-1&0\\0&-2&-1&-3&0&1\end{array}\right)\begin{array}{l}R_1-R_2\\ \\R_3+2R_2\end{array}$$

$$\left(\begin{array}{ccc|ccc}1&0&-3&-2&1&0\\0&1&4&3&-1&0\\0&0&7&3&-2&1\end{array}\right)\begin{array}{l}\\ \\ \frac{1}{7}R_3\end{array}$$

$$\left(\begin{array}{ccc|ccc}1&0&-3&-2&1&0\\0&1&4&3&-1&0\\0&0&1&\frac{3}{7}&-\frac{2}{7}&\frac{1}{7}\end{array}\right)\begin{array}{l}R_1+3R_3\\R_2-4R_3\\ \end{array}$$

$$\left(\begin{array}{ccc|ccc}1&0&0&-\frac{5}{7}&\frac{1}{7}&\frac{3}{7}\\0&1&0&\frac{9}{7}&\frac{1}{7}&-\frac{4}{7}\\0&0&1&\frac{3}{7}&-\frac{2}{7}&\frac{1}{7}\end{array}\right)$$

$$A^{-1}=\begin{pmatrix}-\frac{5}{7}&\frac{1}{7}&\frac{3}{7}\\\frac{9}{7}&\frac{1}{7}&-\frac{4}{7}\\\frac{3}{7}&-\frac{2}{7}&\frac{1}{7}\end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A=\begin{pmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 3 & 3 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \quad \tfrac{1}{3}R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & -3 & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 3R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & \frac{7}{3} & -\frac{5}{3} & 3 & 1 \end{array}\right) \quad \begin{array}{l} \\ \\ \frac{3}{7}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 & -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{array}\right) \quad \begin{array}{l} R_1 + \frac{1}{3}R_3 \\ R_2 - \frac{2}{3}R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ 0 & 1 & 0 & \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ 0 & 0 & 1 & -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{array}\right)$$

$$A^{-1}=\begin{pmatrix} \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A=\begin{pmatrix}-3 & 1 & -1\\ 1 & -4 & -7\\ 1 & 2 & 5\end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc}-3 & 1 & -1 & 1 & 0 & 0\\ 1 & -4 & -7 & 0 & 1 & 0\\ 1 & 2 & 5 & 0 & 0 & 1\end{array}\right)\quad -\frac{1}{3}R_1$$

$$\left(\begin{array}{ccc|ccc}1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0\\ 1 & -4 & -7 & 0 & 1 & 0\\ 1 & 2 & 5 & 0 & 0 & 1\end{array}\right)\quad \begin{array}{l} \\ R_2-R_1\\ R_3-R_1\end{array}$$

$$\left(\begin{array}{ccc|ccc}1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0\\ 0 & -\frac{11}{3} & -\frac{22}{3} & \frac{1}{3} & 1 & 0\\ 0 & \frac{7}{3} & \frac{14}{3} & \frac{1}{3} & 0 & 1\end{array}\right)\quad -\frac{3}{11}R_2$$

$$\left(\begin{array}{ccc|ccc}1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0\\ 0 & 1 & 2 & -\frac{1}{11} & -\frac{3}{11} & 0\\ 0 & \frac{7}{3} & \frac{14}{3} & \frac{1}{3} & 0 & 1\end{array}\right)\quad R_3-\frac{7}{3}R_2$$

$$\left(\begin{array}{ccc|ccc}1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0\\ 0 & 1 & 2 & -\frac{1}{11} & -\frac{3}{11} & 0\\ 0 & 0 & 0 & & & \end{array}\right)$$

$\therefore$ Inverse ***does not exist***

Exercise

Find the inverse, if exists, of $A=\begin{pmatrix}1 & 1 & -3\\ 2 & -4 & 1\\ -5 & 7 & 1\end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 2 & -4 & 1 & 0 & 1 & 0 \\ -5 & 7 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + 5R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & -6 & 7 & -2 & 1 & 0 \\ 0 & 12 & -14 & 5 & 0 & 1 \end{array}\right) \quad -\tfrac{1}{6}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 12 & -14 & 5 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 12R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & & & \end{array}\right)$$

$\therefore$ Inverse ***does not exist***

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_3 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right) \quad -R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 - 2R_2 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ \\ \tfrac{1}{5}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}\right) \begin{array}{l} R_1 - 11R_3 \\ R_2 + 6R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{3}{5} & 2 & -\frac{11}{5} \\ \frac{3}{5} & -1 & \frac{6}{5} \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} -2 & -3 & 4 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & -6 & 1 \\ -2 & -2 & 5 & 1 \end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc} -2 & -3 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & 5 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} -\frac{1}{2}R_1 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & 5 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 + 2R_1 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - R_2 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$\therefore$ Inverse ***does not exist***

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 1 & -14 & 7 & 38 \\ -1 & 2 & 1 & -2 \\ 1 & 2 & -1 & -6 \\ 1 & -2 & 3 & 6 \end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 2 & -1 & -6 & 0 & 0 & 1 & 0 \\ 1 & -2 & 3 & 6 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ 0 & -12 & 8 & 36 & 1 & 1 & 0 & 0 \\ 0 & 16 & -8 & -44 & -1 & 0 & 1 & 0 \\ 0 & 12 & -4 & -32 & -1 & 0 & 0 & 1 \end{array}\right] \quad -\frac{1}{12}R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 16 & -8 & -44 & -1 & 0 & 1 & 0 \\ 0 & 12 & -4 & -32 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + 14R_2 \\ \\ R_3 - 16R_2 \\ R_4 - 12R_2 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -\frac{7}{3} & -4 & -\frac{1}{6} & -\frac{7}{6} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & \frac{8}{3} & 4 & \frac{1}{3} & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 & 1 & 0 & 1 \end{array}\right] \quad \frac{3}{8}R_3$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -\frac{7}{3} & -4 & -\frac{1}{6} & -\frac{7}{6} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 4 & 4 & 0 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + \frac{7}{3}R_3 \\ R_2 + \frac{2}{3}R_3 \\ \\ R_4 - 4R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ 0 & 1 & 0 & -2 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & -2 & -\frac{1}{2} & -1 & -\frac{3}{2} & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ -\frac{1}{2}R_4 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ 0 & 1 & 0 & -2 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{array}\right] \begin{array}{l} R_1 + \frac{1}{2}R_4 \\ R_2 + 2R_4 \\ R_3 - \frac{3}{2}R_4 \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{5}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{5}{4} & \frac{7}{4} & -1 \\ 0 & 0 & 1 & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{5}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{5}{4} & \frac{7}{4} & -1 \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{bmatrix}$$

Exercise

Find the inverse, if exists, of $A=\begin{bmatrix}10 & 20 & -30 & 15\\ 3 & -7 & 14 & -8\\ -7 & -2 & -1 & 2\\ 4 & 4 & -3 & 1\end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc}10 & 20 & -30 & 15 & 1 & 0 & 0 & 0\\ 3 & -7 & 14 & -8 & 0 & 1 & 0 & 0\\ -7 & -2 & -1 & 2 & 0 & 0 & 1 & 0\\ 4 & 4 & -3 & 1 & 0 & 0 & 0 & 1\end{array}\right] \quad \frac{1}{10}R_1$$

$$\left[\begin{array}{cccc|cccc}1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0\\ 3 & -7 & 14 & -8 & 0 & 1 & 0 & 0\\ -7 & -2 & -1 & 2 & 0 & 0 & 1 & 0\\ 4 & 4 & -3 & 1 & 0 & 0 & 0 & 1\end{array}\right] \quad \begin{array}{l}R_2-3R_1\\ R_3+7R_1\\ R_4-4R_1\end{array}$$

$$\left[\begin{array}{cccc|cccc}1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0\\ 0 & -13 & 23 & -\frac{25}{2} & -\frac{3}{10} & 1 & 0 & 0\\ 0 & 12 & -22 & \frac{25}{2} & \frac{7}{10} & 0 & 1 & 0\\ 0 & -4 & 9 & -5 & -\frac{2}{5} & 0 & 0 & 1\end{array}\right] \quad -\frac{1}{13}R_2$$

$$\left[\begin{array}{cccc|cccc}1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0\\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0\\ 0 & 12 & -22 & \frac{25}{2} & \frac{7}{10} & 0 & 1 & 0\\ 0 & -4 & 9 & -5 & -\frac{2}{5} & 0 & 0 & 1\end{array}\right] \quad \begin{array}{l}R_1-2R_2\\ \\ R_3-12R_2\\ R_4+4R_2\end{array}$$

$$\left[\begin{array}{cccc|cccc}1 & 0 & \frac{7}{13} & -\frac{11}{26} & \frac{7}{130} & \frac{2}{13} & 0 & 0\\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0\\ 0 & 0 & -\frac{10}{13} & \frac{25}{26} & \frac{11}{26} & \frac{12}{13} & 1 & 0\\ 0 & 0 & \frac{25}{13} & -\frac{15}{13} & -\frac{4}{13} & -\frac{4}{13} & 0 & 1\end{array}\right] \quad -\frac{13}{10}R_3$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \frac{7}{13} & -\frac{11}{26} & \frac{7}{130} & \frac{2}{13} & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 0 & 1 & -\frac{5}{4} & -\frac{11}{20} & -\frac{6}{5} & -\frac{13}{10} & 0 \\ 0 & 0 & \frac{25}{13} & -\frac{15}{13} & -\frac{4}{13} & -\frac{4}{13} & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 - \frac{7}{13}R_3 \\ R_2 + \frac{23}{13}R_3 \\ \\ R_4 - \frac{25}{13}R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & -\frac{5}{4} & -\frac{19}{20} & -\frac{11}{5} & -\frac{23}{10} & 0 \\ 0 & 0 & 1 & -\frac{5}{4} & -\frac{11}{20} & -\frac{6}{5} & -\frac{13}{10} & 0 \\ 0 & 0 & 0 & \frac{5}{4} & \frac{3}{4} & 2 & \frac{5}{2} & 1 \end{array}\right] \quad \begin{array}{l} R_2 + R_4 \\ R_3 + R_4 \end{array}.$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & \frac{5}{4} & \frac{3}{4} & 2 & \frac{5}{2} & 1 \end{array}\right] \quad \begin{array}{l} \\ \\ \\ \frac{4}{5}R_4 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{array}\right] \quad \begin{array}{l} R_1 - \frac{1}{4}R_4 \\ \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{bmatrix}$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1} $\quad A = [x]$

Solution

For A^{-1} exists, $x \neq 0$

$$AA^{-1} = I$$

$$[x][a] = [1]$$

$$xa = 1$$

$$\boxed{a = \frac{1}{x}}$$

$$A^{-1} = \left[\frac{1}{x}\right]$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1} $\quad A = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$

Solution

For A^{-1} exists, $\boxed{x, y \neq 0}$

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad AA^{-1} = I$$

$$\begin{cases} ax = 1 & bx = 0 \\ cy = 0 & dy = 1 \end{cases}$$

$$\begin{cases} a = \frac{1}{x} & b = 0 \\ c = 0 & d = \frac{1}{y} \end{cases}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{x} & 0 \\ 0 & \frac{1}{y} \end{bmatrix}$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1} $\quad A=\begin{bmatrix} 0 & 0 & x \\ 0 & y & 0 \\ z & 0 & 0 \end{bmatrix}$

Solution

For A^{-1} exists, $\underline{x, y, z \neq 0}$

$$\begin{pmatrix} 0 & 0 & x \\ 0 & y & 0 \\ z & 0 & 0 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad AA^{-1}=I$$

$$\begin{pmatrix} xg & xh & xi \\ yd & ye & yf \\ za & zb & zc \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{cases} xg=1 & xh=0 & xi=0 \\ yd=0 & ye=1 & yf=0 \\ za=0 & zb=0 & zc=1 \end{cases}$$

$$\begin{cases} g=\frac{1}{x} & h=0 & i=0 \\ d=0 & e=\frac{1}{y} & f=0 \\ a=0 & b=0 & c=\frac{1}{z} \end{cases}$$

$$A^{-1}=\begin{pmatrix} 0 & 0 & \frac{1}{z} \\ 0 & \frac{1}{y} & 0 \\ \frac{1}{z} & 0 & 0 \end{pmatrix}$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1} $\quad A=\begin{bmatrix} x & 1 & 1 & 1 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{bmatrix}$

Solution

For A^{-1} exists, $\underline{x, y, z, w \neq 0}$

$$\begin{pmatrix} x & 1 & 1 & 1 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad AA^{-1} = I$$

$$\begin{cases} xa_{11} + xa_{21} + xa_{31} + xa_{41} = 1 \\ xa_{12} + xa_{22} + xa_{32} + xa_{42} = 0 \\ xa_{13} + xa_{23} + xa_{33} + xa_{43} = 0 \\ xa_{14} + xa_{24} + xa_{34} + xa_{44} = 0 \end{cases}$$

$$\begin{cases} ya_{21} = 0 & a_{21} = 0 \\ ya_{22} = 1 & a_{22} = \dfrac{1}{y} \\ ya_{23} = 0 & a_{23} = 0 \\ ya_{24} = 0 & a_{24} = 0 \end{cases}$$

$$\begin{cases} za_{31} = 0 & a_{31} = 0 \\ za_{32} = 0 & a_{32} = 0 \\ za_{33} = 1 & a_{33} = \dfrac{1}{z} \\ za_{34} = 0 & a_{34} = 0 \end{cases}$$

$$\begin{cases} wa_{41} = 0 & a_{41} = 0 \\ wa_{42} = 0 & a_{42} = 0 \\ wa_{43} = 0 & a_{43} = 0 \\ wa_{44} = 1 & a_{44} = \dfrac{1}{w} \end{cases}$$

$$\rightarrow \begin{cases} xa_{11} = 1 & a_{11} = \dfrac{1}{x} \\ xa_{12} + \dfrac{x}{y} = 0 & a_{12} = -\dfrac{1}{y} \\ xa_{13} + \dfrac{x}{z} = 0 & a_{13} = -\dfrac{1}{z} \\ xa_{14} + \dfrac{x}{w} = 0 & a_{14} = -\dfrac{1}{w} \end{cases}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{x} & -\frac{1}{y} & -\frac{1}{z} & -\frac{1}{w} \\ 0 & \frac{1}{y} & 0 & 0 \\ 0 & 0 & \frac{1}{z} & 0 \\ 0 & 0 & 0 & \frac{1}{w} \end{pmatrix}$$

Exercise

Solve the system using A^{-1} $\begin{cases} x \qquad + 2z = 6 \\ -x + 2y + 3z = -5 \\ x - y \qquad = 6 \end{cases}$ Given $A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

Solution

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3(6) - 2(-5) - 4(6) \\ 3(6) - 2(-5) - 5(6) \\ -1(6) + 1(-5) + 2(6) \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

Solution: $(4, \ -2, \ 1)$

Exercise

Solve the system using A^{-1} $\begin{cases} x + 2y + 5z = 2 \\ 2x + 3y + 8z = 3 \\ -x + \ y + 2z = 3 \end{cases}$

a) Write the linear system as a matrix equation in the form $AX = B$

b) Solve the system using the inverse that is given for the coefficient matrix

the inverse of $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{bmatrix}$ *is* $\begin{bmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{bmatrix}$

Solution

a) $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$

b) $\begin{bmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$

Exercise

Solve the system using A^{-1} $\quad \begin{cases} x - y + z = 8 \\ \quad 2y - z = -7 \\ 2x + 3y \quad = 1 \end{cases}$

a) Write the linear system as a matrix equation in the form $AX = B$

b) Solve the system using the inverse that is given for the coefficient matrix

the inverse is $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

Solution

a) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -7 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\quad \begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$

Solution

$A = \begin{pmatrix} 3 & 2 \\ 2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$

$A^{-1} = -\frac{1}{7}\begin{pmatrix} -1 & -2 \\ -2 & 3 \end{pmatrix}$

$$=\begin{pmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{pmatrix}$$

$$X=\begin{pmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{pmatrix}\begin{pmatrix} -4 \\ 5 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} \frac{6}{7} \\ -\frac{23}{7} \end{pmatrix}$$

$\therefore$ ***Solution***: $\left(\frac{6}{7},\ -\frac{23}{7}\right)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x+5y=7 \\ 5x-2y=-3 \end{cases}$

Solution

$$A=\begin{pmatrix} 2 & 5 \\ 5 & -2 \end{pmatrix} \quad B=\begin{pmatrix} 7 \\ -3 \end{pmatrix}$$

$$A^{-1}=-\frac{1}{29}\begin{pmatrix} -2 & -5 \\ -5 & 2 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{2}{29} & \frac{5}{29} \\ \frac{5}{29} & -\frac{2}{29} \end{pmatrix}$$

$$X=\begin{pmatrix} \frac{2}{29} & \frac{5}{29} \\ \frac{5}{29} & -\frac{2}{29} \end{pmatrix}\begin{pmatrix} 7 \\ -3 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -\frac{1}{29} \\ -\frac{41}{7} \end{pmatrix}$$

$\therefore$ ***Solution***: $\left(-\frac{1}{29},\ -\frac{41}{29}\right)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$

Solution

$$A = \begin{pmatrix} 4 & -7 \\ 2 & 5 \end{pmatrix} \quad B = \begin{pmatrix} -16 \\ 9 \end{pmatrix}$$

$$A^{-1} = \frac{1}{34}\begin{pmatrix} 5 & 7 \\ -2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{34} & \frac{7}{34} \\ -\frac{1}{17} & \frac{2}{17} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{5}{34} & \frac{7}{34} \\ -\frac{1}{17} & \frac{2}{17} \end{pmatrix}\begin{pmatrix} -16 \\ 9 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 2 \end{pmatrix}$$

$\therefore$ ***Solution***: $\left(-\frac{1}{2},\ 2\right)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$

Solution

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$A^{-1} = -1\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$$

$$X = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

$\therefore$ ***Solution***: $(-2,\ 5)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 3x+4y=2 \\ 2x+5y=-1 \end{cases}$

Solution

$$A=\begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} \quad B=\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$A^{-1}=\frac{1}{7}\begin{pmatrix} 5 & -4 \\ -2 & 3 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{5}{7} & -\frac{4}{7} \\ -\frac{2}{7} & \frac{3}{7} \end{pmatrix}$$

$$X=\begin{pmatrix} \frac{5}{7} & -\frac{4}{7} \\ -\frac{2}{7} & \frac{3}{7} \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$\therefore$ ***Solution***: $\underline{(2,\ -1)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 5x-2y=4 \\ -10x+4y=7 \end{cases}$

Solution

$$A=\begin{pmatrix} 5 & -2 \\ -10 & 4 \end{pmatrix} \quad B=\begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

$$A^{-1}=\frac{1}{0}\begin{pmatrix} 4 & 2 \\ 10 & 5 \end{pmatrix}$$

Inverse matrix ***doesn't exist***.

$$-\frac{1}{2}\begin{cases} 5x-2y=4 \\ 5x-2y=-\frac{7}{2} \end{cases}$$

$$4\neq-\frac{7}{2}$$

$\therefore$ ***No Solution***

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x-4y=-8 \\ 5x-20y=-40 \end{cases}$

Solution

$$A=\begin{pmatrix} 1 & -4 \\ 5 & -20 \end{pmatrix} \quad B=\begin{pmatrix} -8 \\ -40 \end{pmatrix}$$

$$A^{-1}=\frac{1}{0}\begin{pmatrix} & \\ & \end{pmatrix}$$

Inverse matrix ***doesn't exist***.

$$\frac{1}{5}\begin{cases} x-4y=-8 \\ x-4y=-8 \end{cases}$$

$\therefore$ ***Solution***: $\quad \underline{(4y-8,\ y)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$

Solution

$$A=\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \quad B=\begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$A^{-1}=-\frac{1}{3}\begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$$

$$X=\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}\begin{pmatrix} 3 \\ 3 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$\therefore$ ***Solution***: $\quad \underline{(2,\ -1)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x+10y=-14 \\ 7x-2y=-16 \end{cases}$

Solution

$$A=\begin{pmatrix} 2 & 10 \\ 7 & -2 \end{pmatrix} \quad B=\begin{pmatrix} -14 \\ -16 \end{pmatrix}$$

$$A^{-1}=-\frac{1}{74}\begin{pmatrix} -2 & -10 \\ -7 & 2 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{37} & \frac{5}{37} \\ \frac{7}{74} & -\frac{1}{37} \end{pmatrix}$$

$$X=\begin{pmatrix} \frac{1}{37} & \frac{5}{37} \\ \frac{7}{74} & -\frac{1}{37} \end{pmatrix}\begin{pmatrix} -14 \\ -16 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -\frac{94}{37} \\ -\frac{33}{37} \end{pmatrix}$$

$\therefore$ ***Solution***: $\left(-\frac{94}{37},\ -\frac{33}{37}\right)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 4x-3y=24 \\ -3x+9y=-1 \end{cases}$

Solution

$$A=\begin{pmatrix} 4 & -3 \\ -3 & 9 \end{pmatrix} \quad B=\begin{pmatrix} 24 \\ -1 \end{pmatrix}$$

$$A^{-1}=\frac{1}{27}\begin{pmatrix} 9 & 3 \\ 3 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{3} & \frac{1}{9} \\ \frac{1}{9} & \frac{4}{27} \end{pmatrix}$$

$$X=\begin{pmatrix} \frac{1}{3} & \frac{1}{9} \\ \frac{1}{9} & \frac{4}{27} \end{pmatrix}\begin{pmatrix} 24 \\ -1 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{71}{9} \\ \frac{68}{27} \end{pmatrix}$$

$\therefore$ ***Solution***: $\underline{\left(\frac{71}{9}, \frac{68}{27}\right)\big|}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$

Solution

$$A = \begin{pmatrix} 4 & 2 \\ 3 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 12 \\ 16 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{14}\begin{pmatrix} -2 & -2 \\ -3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{3}{14} & -\frac{2}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{3}{14} & -\frac{2}{7} \end{pmatrix}\begin{pmatrix} 12 \\ 16 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$\therefore$ ***Solution***: $\underline{(4, \ -2)\big|}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & 2 \\ 4 & -2 \end{pmatrix} \quad B = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{10}\begin{pmatrix} -2 & -2 \\ -4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{10} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{10} \end{pmatrix} \begin{pmatrix} -1 \\ 6 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

∴ ***Solution***: $(1, \ -1)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x - 2y = 5 \\ -10x + 2y = 4 \end{cases}$

Solution

$$\frac{1}{2} \begin{cases} x - 2y = 5 \\ -5x + y = 2 \end{cases}$$

$$A = \begin{pmatrix} 1 & -2 \\ -5 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{9} \begin{pmatrix} 1 & 2 \\ 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{9} & -\frac{2}{9} \\ -\frac{5}{9} & -\frac{1}{9} \end{pmatrix}$$

$$X = \begin{pmatrix} -\frac{1}{9} & -\frac{2}{9} \\ -\frac{5}{9} & -\frac{1}{9} \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

∴ ***Solution***: $(-1, \ -3)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 12x+15y=-27 \\ 30x-15y=-15 \end{cases}$

Solution

$$\begin{matrix} \frac{1}{3} \\ \frac{1}{15} \end{matrix} \to \begin{cases} 4x+5y=-9 \\ 2x-y=-1 \end{cases}$$

$$A=\begin{pmatrix} 4 & 5 \\ 2 & -1 \end{pmatrix} \quad B=\begin{pmatrix} -9 \\ -1 \end{pmatrix}$$

$$A^{-1}=-\frac{1}{14}\begin{pmatrix} -1 & -5 \\ -2 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{14} & \frac{5}{14} \\ \frac{1}{7} & -\frac{2}{7} \end{pmatrix}$$

$$X=\begin{pmatrix} \frac{1}{14} & \frac{5}{14} \\ \frac{1}{7} & -\frac{2}{7} \end{pmatrix}\begin{pmatrix} -9 \\ -1 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$\therefore$ ***Solution***: $\quad \underline{(-1,\ -1)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 4x-4y=-12 \\ 4x+4y=-20 \end{cases}$

Solution

$$\frac{1}{4} \to \begin{cases} x-y=-3 \\ x+y=-5 \end{cases}$$

$$A=\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad B=\begin{pmatrix} -3 \\ -5 \end{pmatrix}$$

$$A^{-1}=\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -3 \\ -5 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$$

∴ ***Solution***: $(-4, \ -1)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} -2x + 3y = 4 \\ -3x + 4y = 5 \end{cases}$

Solution

$$A = \begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}$$

$$X = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

∴ ***Solution***: $(1, \ 2)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x - 2y = 6 \\ 4x + 3y = 2 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & -2 \\ 4 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{11} \begin{pmatrix} 3 & 2 \\ -4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{11} & \frac{2}{11} \\ -\frac{4}{11} & \frac{1}{11} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{3}{11} & \frac{2}{11} \\ -\frac{4}{11} & \frac{1}{11} \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$\therefore$ ***Solution***: $\quad \underline{(2, \ -2)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x - 3y = 7 \\ 4x + y = -7 \end{cases}$

Solution

$$A = \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 7 \\ -7 \end{pmatrix}$$

$$A^{-1} = \frac{1}{14}\begin{pmatrix} 1 & 3 \\ -4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{14} & \frac{3}{14} \\ -\frac{2}{7} & \frac{1}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{14} & \frac{3}{14} \\ -\frac{2}{7} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} 7 \\ -7 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

$\therefore$ ***Solution***: $\quad \underline{(-1, \ -3)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$

Solution

$$A=\begin{pmatrix}1&1&1\\2&1&-1\\1&-1&1\end{pmatrix}\quad B=\begin{pmatrix}2\\5\\-2\end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\2&1&-1&0&1&0\\1&-1&1&0&0&1\end{array}\right)\quad \begin{array}{l}R_2-2R_1\\R_3-R_1\end{array}$$

$$\left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&-1&-3&-2&1&0\\0&-2&0&-1&0&1\end{array}\right)\quad -R_2$$

$$\left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&3&2&-1&0\\0&-2&0&-1&0&1\end{array}\right)\quad \begin{array}{l}R_1-R_2\\ \\R_3+2R_2\end{array}$$

$$\left(\begin{array}{ccc|ccc}1&0&-2&-1&1&0\\0&1&3&2&-1&0\\0&0&6&3&-2&1\end{array}\right)\quad \frac{1}{6}R_3$$

$$\left(\begin{array}{ccc|ccc}1&0&-2&-1&1&0\\0&1&3&2&-1&0\\0&0&1&\frac{1}{2}&-\frac{1}{3}&\frac{1}{6}\end{array}\right)\quad \begin{array}{l}R_1+2R_3\\R_2-3R_3\end{array}$$

$$\left(\begin{array}{ccc|ccc}1&0&0&0&\frac{1}{3}&\frac{1}{3}\\0&1&0&\frac{1}{2}&0&-\frac{1}{2}\\0&0&1&\frac{1}{2}&-\frac{1}{3}&\frac{1}{6}\end{array}\right)$$

$$A^{-1}=\begin{pmatrix}0&\frac{1}{3}&\frac{1}{3}\\\frac{1}{2}&0&-\frac{1}{2}\\\frac{1}{2}&-\frac{1}{3}&\frac{1}{6}\end{pmatrix}$$

$$X=\begin{pmatrix}0&\frac{1}{3}&\frac{1}{3}\\\frac{1}{2}&0&-\frac{1}{2}\\\frac{1}{2}&-\frac{1}{3}&\frac{1}{6}\end{pmatrix}\begin{pmatrix}2\\5\\-2\end{pmatrix}\qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$\therefore$ ***Solution***: $\underline{(1,\ 2,\ \ -1)}\,|$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & -1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -1 \\ 9 \\ 9 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 - 2R_1 \\ R_2 - 3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 0 & -4 & 4 & -3 & 0 & 1 \end{array}\right) \quad -R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & -4 & 4 & -3 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 4R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & -8 & 5 & -4 & 1 \end{array}\right) \quad -\tfrac{1}{8}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{array}\right) \quad \begin{array}{l} R_1 - 2R_3 \\ R_2 + 3R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{8} & \frac{1}{2} & -\frac{3}{8} \\ 0 & 0 & 1 & -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & -\frac{3}{8} \\ -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & -\frac{3}{8} \\ -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{pmatrix}\begin{pmatrix} -1 \\ 9 \\ 9 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$$

$\therefore$ ***Solution***: $(2,\ 1,\ 4)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & 5 & -1 \\ 0 & 3 & -1 \\ -3 & 6 & 2 \end{pmatrix} \quad B = \begin{pmatrix} -4 \\ -1 \\ 11 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 5 & -1 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ -3 & 6 & 2 & 0 & 0 & 1 \end{array}\right) \quad R_3 + 3R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 5 & -1 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 0 & 21 & -1 & 3 & 0 & 1 \end{array}\right) \quad \frac{1}{3}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 5 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 21 & -1 & 3 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 - 5R_2 \\ \\ R_3 - 21R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & 1 & -\frac{5}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 6 & 3 & -7 & 1 \end{array}\right) \begin{array}{l} \\ \\ \frac{1}{6}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & 1 & -\frac{5}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{array}\right) \begin{array}{l} R_1 - \frac{2}{3}R_3 \\ R_2 + \frac{1}{3}R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & \frac{8}{9} & -\frac{1}{9} \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{1}{18} & \frac{1}{18} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{8}{9} & -\frac{1}{9} \\ \frac{1}{6} & -\frac{1}{18} & \frac{1}{18} \\ \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{2}{3} & \frac{8}{9} & -\frac{1}{9} \\ \frac{1}{6} & -\frac{1}{18} & \frac{1}{18} \\ \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} -4 \\ -1 \\ 11 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore$ ***Solution***: $\underline{(-3,\ 0,\ 1)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x+3y+4z=14 \\ 2x-3y+2z=10 \\ 3x-y+z=9 \end{cases}$

Solution

$$A=\begin{pmatrix} 1 & 3 & 4 \\ 2 & -3 & 2 \\ 3 & -1 & 1 \end{pmatrix} \quad B=\begin{pmatrix} 14 \\ 10 \\ 9 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 2 & -3 & 2 & 0 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_2-2R_1 \\ R_3-3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & -9 & -6 & -2 & 1 & 0 \\ 0 & -10 & -11 & -3 & 0 & 1 \end{array}\right) -\frac{1}{9}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{9} & -\frac{1}{9} & 0 \\ 0 & -10 & -11 & -3 & 0 & 1 \end{array}\right) \begin{array}{l} R_1-3R_2 \\ \\ R_3+10R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{9} & -\frac{1}{9} & 0 \\ 0 & 0 & -\frac{13}{3} & -\frac{7}{9} & -\frac{10}{9} & 1 \end{array}\right) -\frac{3}{13}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{9} & -\frac{1}{9} & 0 \\ 0 & 0 & 1 & \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{array}\right) \begin{array}{l} R_2-2R_3 \\ R_2-\frac{2}{3}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{39} & -\frac{7}{39} & \frac{6}{13} \\ 0 & 1 & 0 & \frac{4}{39} & -\frac{11}{39} & \frac{2}{13} \\ 0 & 0 & 1 & \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{1}{39} & -\frac{7}{39} & \frac{6}{13} \\ \frac{4}{39} & -\frac{11}{39} & \frac{2}{13} \\ \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{pmatrix}$$

$$X = \begin{pmatrix} -\frac{1}{39} & -\frac{7}{39} & \frac{6}{13} \\ \frac{4}{39} & -\frac{11}{39} & \frac{2}{13} \\ \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{pmatrix} \begin{pmatrix} 14 \\ 10 \\ 9 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

∴ ***Solution***: $\underline{(2,\ 0,\ 3)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \end{array}\right) \quad R_2 - R_1$$

∴ ***Solution***: $\underline{(1,\ 4,\ -3)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 2 & -3 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 17 \\ 7 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 2 & -3 & 2 & 0 & 0 & 1 \end{array}\right) \quad R_3 - 2R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & -7 & 0 & -2 & 0 & 1 \end{array}\right) \quad \frac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -7 & 0 & -2 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 7R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{7}{2} & -2 & \frac{7}{2} & 1 \end{array}\right) \quad -\frac{2}{7}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{4}{7} & -1 & -\frac{2}{7} \end{array}\right) \quad \begin{array}{l} R_1 - 2R_3 \\ R_2 + \frac{1}{2}R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{7} & 1 & \frac{4}{7} \\ 0 & 1 & 0 & \frac{2}{7} & 0 & -\frac{1}{7} \\ 0 & 0 & 1 & \frac{4}{7} & -1 & -\frac{2}{7} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{1}{7} & 1 & \frac{4}{7} \\ \frac{2}{7} & 0 & -\frac{1}{7} \\ \frac{4}{7} & -1 & -\frac{2}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} -\frac{1}{7} & 1 & \frac{4}{7} \\ \frac{2}{7} & 0 & -\frac{1}{7} \\ \frac{4}{7} & -1 & -\frac{2}{7} \end{pmatrix} \begin{pmatrix} 17 \\ 7 \\ -1 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$$

$\therefore$ ***Solution***: $\underline{(4,\ 5,\ 3)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$

Solution

$$A = \begin{pmatrix} -2 & 6 & 7 \\ -4 & 5 & 3 \\ -6 & 3 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 3 \\ 7 \\ -4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} -2 & 6 & 7 & 1 & 0 & 0 \\ -4 & 5 & 3 & 0 & 1 & 0 \\ -6 & 3 & 5 & 0 & 0 & 1 \end{array}\right) \quad -\tfrac{1}{2}R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & -3 & -\frac{7}{2} & -\frac{1}{2} & 0 & 0 \\ -4 & 5 & 3 & 0 & 1 & 0 \\ -6 & 3 & 5 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} \\ R_2 + 4R_1 \\ R_3 + 6R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -3 & -\frac{7}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & -7 & -11 & -2 & 1 & 0 \\ 0 & -15 & -16 & -3 & 0 & 1 \end{array}\right) \quad -\tfrac{1}{7}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & -3 & -\frac{7}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{11}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & -15 & -16 & -3 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 + 3R_2 \\ \\ R_3 + 15R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{17}{14} & \frac{5}{14} & -\frac{3}{7} & 0 \\ 0 & 1 & \frac{11}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & \frac{53}{7} & \frac{9}{7} & -\frac{15}{7} & 1 \end{array}\right) \quad \tfrac{7}{53}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{17}{14} & \frac{5}{14} & -\frac{3}{7} & 0 \\ 0 & 1 & \frac{11}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & 1 & \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{array}\right) \quad \begin{array}{l} R_1 - \frac{17}{14}R_3 \\ R_2 - \frac{11}{7}R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{8}{53} & -\frac{9}{106} & -\frac{17}{106} \\ 0 & 1 & 0 & \frac{1}{53} & \frac{16}{53} & -\frac{11}{53} \\ 0 & 0 & 1 & \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{8}{53} & -\frac{9}{106} & -\frac{7}{106} \\ \frac{1}{53} & \frac{16}{53} & -\frac{11}{53} \\ \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{8}{53} & -\frac{9}{106} & -\frac{7}{106} \\ \frac{1}{53} & \frac{16}{53} & -\frac{11}{53} \\ \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{pmatrix} \begin{pmatrix} 3 \\ 7 \\ -4 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 3 \\ -2 \end{pmatrix}$$

∴ ***Solution***: $\left(\frac{1}{2},\ 3,\ -2\right)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$

Solution

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -3 & 4 \\ 4 & -2 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 1 & 1 & 0 & 0 \\ 3 & -3 & 4 & 0 & 1 & 0 \\ 4 & -2 & 3 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} \\ 2R_2 - 3R_1 \\ 2R_3 - 4R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 1 & 1 & 0 & 0 \\ 0 & -3 & 5 & -3 & 2 & 0 \\ 0 & 0 & 2 & -4 & 0 & 2 \end{array}\right) \quad \begin{array}{l} 3R_1 - R_2 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 6 & 0 & -2 & 6 & -2 & 0 \\ 0 & -3 & 5 & -3 & 2 & 0 \\ 0 & 0 & 2 & -4 & 0 & 2 \end{array}\right) \quad \begin{array}{l} R_1 + R_3 \\ 2R_2 - 5R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 6 & 0 & 0 & 2 & -2 & 2 \\ 0 & -6 & 0 & 14 & 4 & -10 \\ 0 & 0 & 2 & -4 & 0 & 2 \end{array}\right) \quad \begin{array}{l} \frac{1}{6}R_1 \\ -\frac{1}{6}R_2 \\ \frac{1}{2}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{7}{6} & -\frac{2}{3} & \frac{5}{3} \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{7}{6} & -\frac{2}{3} & \frac{5}{3} \\ -2 & 0 & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{7}{6} & -\frac{2}{3} & \frac{5}{3} \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

∴ ***Solution***: $\underline{(0,\ 1,\ 2)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x-2y-z=2 \\ 2x-y+z=4 \\ -x+y+z=4 \end{cases}$

Solution

$$A=\begin{pmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \quad B=\begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{c} \\ R_2-2R_1 \\ R_3+R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 3 & -2 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{array}\right) \quad \tfrac{1}{3}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{array}\right) \quad \begin{array}{c} R_1+2R_2 \\ \\ R_3+R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 1 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & 1 \end{array}\right) \quad \begin{array}{c} R_1-R_3 \\ R_2-R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{3} & \frac{1}{3} & -1 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & 1 \end{array}\right)$$

$$A^{-1}=\begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & -1 \\ -1 & 0 & -1 \\ \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}$$

$$X=\begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & -1 \\ -1 & 0 & -1 \\ \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \\ 6 \end{pmatrix}$$

∴ ***Solution***: $(-4,\ -6,\ 6)$

Solution *Section* 4.4 – Determinants and Cramer's Rule

Exercise

Evaluate $\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix}$

Solution

$$\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix} = -9 - (-6)$$

$$\underline{= -3}$$

Exercise

Evaluate $\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix} = -6 - (0)$$

$$\underline{= -6}$$

Exercise

Evaluate $\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix} = x(8x) - 4x(2x)$$

$$= 8x^2 - 8x^2$$

$$\underline{= 0}$$

Exercise

Evaluate $\begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix} = 3x - 2x(4)$$

$$= 3x - 8x$$

$$\underline{= -5x}$$

Exercise

Evaluate $\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix}$

Solution

$$\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix} = \underline{-3x^4 - 2x}$$

Exercise

Evaluate $\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix}$

Solution

$$\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix} = \underline{-8a + 5b}$$

Exercise

Evaluate $\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix}$

Solution

$$\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix} = 15 - 14$$

$$\underline{= 1}$$

Exercise

Evaluate $\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix} = 5 - 20$$

$$\underline{= -16}$$

Exercise

Evaluate $\begin{vmatrix} 5 & 3 \\ -2 & 3 \end{vmatrix}$

Solution

$$\begin{vmatrix} 5 & 3 \\ -2 & 3 \end{vmatrix} = 15 + 6$$

$$\underline{= 21}$$

Exercise

Evaluate $\begin{vmatrix} -4 & -1 \\ 5 & 6 \end{vmatrix}$

Solution

$$\begin{vmatrix} -4 & -1 \\ 5 & 6 \end{vmatrix} = -24 + 5$$

$$\underline{= -19}$$

Exercise

Evaluate $\begin{vmatrix} \sqrt{3} & -2 \\ -3 & \sqrt{3} \end{vmatrix}$

Solution

$$\begin{vmatrix} \sqrt{3} & -2 \\ -3 & \sqrt{3} \end{vmatrix} = 3 - 6$$

$$\underline{= -3}$$

Exercise

Evaluate $\begin{vmatrix} \sqrt{7} & 6 \\ -3 & \sqrt{7} \end{vmatrix}$

Solution

$$\begin{vmatrix} \sqrt{7} & 6 \\ -3 & \sqrt{7} \end{vmatrix} = 7 + 18$$

$$= 25$$

Exercise

Evaluate $\begin{vmatrix} \sqrt{5} & 3 \\ -2 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} \sqrt{5} & 3 \\ -2 & 2 \end{vmatrix} = 2\sqrt{5} + 6$$

Exercise

Evaluate $\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{vmatrix}$

Solution

$$\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{vmatrix} = -\frac{3}{8} - \frac{1}{16}$$

$$= -\frac{7}{16}$$

Exercise

Evaluate $\begin{vmatrix} \frac{1}{5} & \frac{1}{6} \\ -6 & -5 \end{vmatrix}$

Solution

$$\begin{vmatrix} \frac{1}{5} & \frac{1}{6} \\ -6 & -5 \end{vmatrix} = -1 + 1$$

$$= 0$$

Exercise

Evaluate $\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{2} & \frac{3}{4} \end{vmatrix}$

Solution

$$\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{2} & \frac{3}{4} \end{vmatrix} = \frac{1}{2} + \frac{1}{6}$$

$$= \frac{2}{3}$$

Exercise

Evaluate $\begin{vmatrix} x & x^2 \\ 4 & x \end{vmatrix}$

Solution

$$\begin{vmatrix} x & x^2 \\ 4 & x \end{vmatrix} = x^2 - 4x^2$$

$$= -3x^2$$

Exercise

Evaluate $\begin{vmatrix} x & x^2 \\ x & 9 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & x^2 \\ x & 9 \end{vmatrix} = 9x - x^3$$

Exercise

Evaluate $\begin{vmatrix} x^2 & x \\ -3 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} x^2 & x \\ -3 & 2 \end{vmatrix} = 2x^2 + 3x$$

Exercise

Evaluate $\begin{vmatrix} x+2 & 6 \\ x-2 & 4 \end{vmatrix}$

Solution

$$\begin{vmatrix} x+2 & 6 \\ x-2 & 4 \end{vmatrix} = 4(x+2) - 6(x-2)$$

$$= 4x + 8 - 6x + 12$$

$$\underline{= -2x + 20}$$

Exercise

Evaluate $\begin{vmatrix} x+1 & -6 \\ x+3 & -3 \end{vmatrix}$

Solution

$$\begin{vmatrix} x+1 & -6 \\ x+3 & -3 \end{vmatrix} = -3x - 3 + 6x + 18$$

$$\underline{= -2x + 20}$$

Exercise

Evaluate $\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & -5 \\ 2 & 5 & -1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & -5 \\ 2 & 5 & -1 \end{vmatrix} \begin{matrix} 3 & 0 \\ 2 & 1 \\ 2 & 5 \end{matrix}$$

$$= -3 + 0 + 0 - 0 + 75 - 0$$

$$\underline{= 72}$$

Exercise

Evaluate $\begin{vmatrix} 4 & 0 & 0 \\ 3 & -1 & 4 \\ 2 & -3 & 6 \end{vmatrix}$

Solution

$$\begin{vmatrix} 4 & 0 & 0 \\ 3 & -1 & 4 \\ 2 & -3 & 6 \end{vmatrix} \begin{matrix} 4 & 0 \\ 3 & -1 \\ 2 & -3 \end{matrix} \qquad or \quad = 4\begin{vmatrix} -1 & 4 \\ -3 & 6 \end{vmatrix}$$

$$= -24 + 48$$

$$\underline{= 24}$$

Exercise

Evaluate $\begin{vmatrix} 3 & 1 & 0 \\ -3 & -4 & 0 \\ -1 & 3 & 5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 3 & 1 & 0 \\ -3 & -4 & 0 \\ -1 & 3 & 5 \end{vmatrix} \begin{matrix} 3 & 1 \\ -3 & -4 \\ -1 & 3 \end{matrix}$$

$$= -60 + 15$$

$$\underline{= -45}$$

Exercise

Evaluate $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & -4 & 5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & -4 & 5 \end{vmatrix} \begin{matrix} 1 & 1 \\ 2 & 2 \\ 3 & -4 \end{matrix}$$

$$= 10 + 6 - 8 - 6 + 8 - 10$$

$$\underline{= 0}$$

Exercise

Evaluate $\begin{vmatrix} x & 0 & -1 \\ 2 & 1 & x^2 \\ -3 & x & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 0 & -1 \\ 2 & 1 & x^2 \\ -3 & x & 1 \end{vmatrix} \begin{matrix} x & 0 \\ 2 & 1 \\ -3 & x \end{matrix}$$

$$= x - 2x - 3 - x^4$$

$$\underline{= -x^4 - x - 3}$$

Exercise

Evaluate $\begin{vmatrix} x & 1 & -1 \\ x^2 & x & x \\ 0 & x & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 1 & -1 \\ x^2 & x & x \\ 0 & x & 1 \end{vmatrix} \begin{matrix} x & 1 \\ x^2 & x \\ 0 & x \end{matrix}$$

$$= x^2 - x^3 - x^3 - x^2$$

$$\underline{= -2x^3}$$

Exercise

Evaluate $\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix}$

Solution

$$\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix} = 0 + 126 + 48 - (-48 + 36 + 0)$$

$$\underline{= 90}$$

Exercise

Evaluate $\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix} = 0-4-16-(-14-16+0)$$

$$\underline{= 10}$$

Exercise

Evaluate $\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix}$

Solution

$$\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix} \begin{matrix} 3 & 1 \\ -2 & 3 \\ 3 & 4 \end{matrix}$$

$$= -54+3-16-18-12-12$$

$$\underline{= -109}$$

Exercise

Evaluate $\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix} \begin{matrix} 2x & 1 \\ 0 & 4 \\ 3 & 0 \end{matrix}$$

$$= 16x+3x+12$$

$$\underline{= 19x+12}$$

Exercise

Evaluate $\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix} \begin{matrix} 0 & x \\ x & x^2 \\ x & 7 \end{matrix}$$

$$= 5x^2 + 7x^2 - x^4 + 5x^2$$

$$\underline{= 17x^2 - x^4}$$

Exercise

Evaluate $\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix} \begin{matrix} 2 & x \\ -3 & 1 \\ 2 & 1 \end{matrix}$$

$$= 8 - 3 - 2 + 12x$$

$$\underline{= 12x + 3}$$

Exercise

Evaluate $\begin{vmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix} \begin{matrix} 1 & x \\ 3 & 1 \\ 0 & -2 \end{matrix}$$

$$= 2 + 12 + 2 - 6x$$

$$\underline{= -6x + 16}$$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y=-4 \\ 2x-y=-5 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}=-3-4=-7$$

$$D_x=\begin{vmatrix} -4 & 2 \\ -5 & -1 \end{vmatrix}=4-(-10)=14$$

$$D_y=\begin{vmatrix} 3 & -4 \\ 2 & -5 \end{vmatrix}=-15-(-8)=-7$$

$$x=\frac{D_x}{D}=\frac{14}{-7}=-2$$

$$y=\frac{D_y}{D}=\frac{-7}{-7}=1$$

$\therefore$ ***Solution***: $(-2,\ 1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x+5y=7 \\ 5x-2y=-3 \end{cases}$

Solution

$$D=\begin{vmatrix} 2 & 5 \\ 5 & -2 \end{vmatrix}=-29 \qquad D_x=\begin{vmatrix} 7 & 5 \\ -3 & -2 \end{vmatrix}=1 \qquad D_y=\begin{vmatrix} 2 & 7 \\ 5 & -3 \end{vmatrix}=-41$$

$$x=\frac{1}{-29}=-\frac{1}{29} \qquad y=\frac{41}{29}$$

$\therefore$ ***Solution***: $\left(-\frac{1}{29},\ \frac{41}{29}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y=-4 \\ 2x-y=-5 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}=-7 \qquad D_x=\begin{vmatrix} -4 & -5 \\ 2 & -1 \end{vmatrix}=14 \qquad D_y=\begin{vmatrix} 3 & -4 \\ 2 & -5 \end{vmatrix}=-7$$

$x = -\frac{14}{7} \underline{= -2}$ $\qquad$ $x = \frac{D_x}{D}$

$y = \frac{7}{7} \underline{= 1}$ $\qquad$ $y = \frac{D_y}{D}$

Solution: $\underline{(-2,\ 1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$

Solution

$D = \begin{vmatrix} 2 & 5 \\ 5 & -2 \end{vmatrix} = -29 \qquad D_x = \begin{vmatrix} 7 & 5 \\ -3 & -2 \end{vmatrix} = 1 \qquad D_y = \begin{vmatrix} 2 & 7 \\ 5 & -3 \end{vmatrix} = -41$

$\underline{x = -\frac{1}{29}}$ $\qquad$ $x = \frac{D_x}{D}$

$\underline{y = \frac{41}{29}}$ $\qquad$ $y = \frac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{\left(-\frac{1}{29},\ \frac{41}{29}\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$

Solution

$D = \begin{vmatrix} 4 & -7 \\ 2 & 5 \end{vmatrix} = 34 \qquad D_x = \begin{vmatrix} -16 & -7 \\ 9 & 5 \end{vmatrix} = -17 \qquad D_y = \begin{vmatrix} 4 & -16 \\ 2 & 9 \end{vmatrix} = 68$

$x = -\frac{17}{34} \underline{= -\frac{1}{2}}$ $\qquad$ $x = \frac{D_x}{D}$

$y = \frac{68}{34} \underline{= 2}$ $\qquad$ $y = \frac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{\left(-\frac{1}{2},\ 2\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y=4 \\ 2x+y=1 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}=-1 \qquad D_x=\begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix}=2 \qquad D_y=\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix}=-5$$

$\underline{x=-2}$ $\qquad x=\dfrac{D_x}{D}$

$\underline{y=5}$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(-2,\ 5)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+4y=2 \\ 2x+5y=-1 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix}=7 \qquad D_x=\begin{vmatrix} 2 & 4 \\ -1 & 5 \end{vmatrix}=14 \qquad D_y=\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}=-7$$

$x=\frac{14}{7}\underline{=2}$ $\qquad x=\dfrac{D_x}{D}$

$y=-\frac{7}{7}\underline{=-1}$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(2,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 5x-2y=4 \\ -10x+4y=7 \end{cases}$

Solution

$$D=\begin{vmatrix} 5 & -2 \\ -10 & 4 \end{vmatrix}=0 \qquad D_y=\begin{vmatrix} 5 & 4 \\ -10 & 7 \end{vmatrix}=75\neq 0$$

$\therefore$ ***No Solution***

Exercise

Use Cramer's rule to solve the system $\begin{cases} x-4y=-8 \\ 5x-20y=-40 \end{cases}$

Solution

$$D=\begin{vmatrix}1 & -4 \\ 5 & -20\end{vmatrix}=0 \qquad D_y=\begin{vmatrix}1 & -8 \\ 5 & -40\end{vmatrix}=0$$

$$\frac{1}{5}\times \begin{cases} x-4y=-8 \\ 5x-20y=-40 \end{cases}$$

$$\begin{cases} x-4y=-8 \\ x-4y=-8 \end{cases}$$

$\therefore$ ***Solution***: $(4y-8,\ y)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$

Solution

$$D=\begin{vmatrix}2 & 1 \\ 1 & -1\end{vmatrix}=-3 \qquad D_x=\begin{vmatrix}3 & 1 \\ 3 & -1\end{vmatrix}=-6 \qquad D_y=\begin{vmatrix}2 & 3 \\ 1 & 3\end{vmatrix}=3$$

$x=\frac{6}{3}=2$ $\qquad x=\frac{D_x}{D}$

$y=-\frac{3}{3}=-1$ $\qquad y=\frac{D_y}{D}$

$\therefore$ ***Solution***: $(2,\ -1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x+10y=-14 \\ 7x-2y=-16 \end{cases}$

Solution

$$D=\begin{vmatrix}2 & 10 \\ 7 & -2\end{vmatrix}=-74 \qquad D_x=\begin{vmatrix}-14 & 10 \\ -16 & -2\end{vmatrix}=188 \qquad D_y=\begin{vmatrix}2 & -14 \\ 7 & -16\end{vmatrix}=66$$

$x=-\frac{188}{74}=-\frac{94}{37}$ $\qquad x=\frac{D_x}{D}$

$y=-\dfrac{66}{74}=-\boxed{\dfrac{33}{37}}$ $\qquad$ $y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\boxed{\left(-\dfrac{94}{37},\ -\dfrac{33}{37}\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x-3y=24 \\ -3x+9y=-1 \end{cases}$

Solution

$D=\begin{vmatrix} 4 & -3 \\ -3 & 9 \end{vmatrix}=27$ $\qquad$ $D_x=\begin{vmatrix} 24 & -3 \\ -1 & 9 \end{vmatrix}=213$ $\qquad$ $D_y=\begin{vmatrix} 4 & 24 \\ -3 & -1 \end{vmatrix}=68$

$x=\dfrac{213}{27}=\boxed{\dfrac{71}{9}}$ $\qquad$ $x=\dfrac{D_x}{D}$

$y=\boxed{\dfrac{68}{27}}$ $\qquad$ $y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\boxed{\left(\dfrac{71}{9},\ \dfrac{68}{27}\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x+2y=12 \\ 3x-2y=16 \end{cases}$

Solution

$D=\begin{vmatrix} 4 & 2 \\ 3 & -2 \end{vmatrix}=-14$ $\qquad$ $D_x=\begin{vmatrix} 12 & 2 \\ 16 & -2 \end{vmatrix}=-56$ $\qquad$ $D_y=\begin{vmatrix} 4 & 12 \\ 3 & 16 \end{vmatrix}=28$

$x=\dfrac{56}{14}=\boxed{4}$ $\qquad$ $x=\dfrac{D_x}{D}$

$y=-\dfrac{28}{14}=\boxed{-2}$ $\qquad$ $y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\boxed{(4,\ -2)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+2y=-1 \\ 4x-2y=6 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix}=-10 \qquad D_x=\begin{vmatrix} -1 & 2 \\ 6 & -2 \end{vmatrix}=-10 \qquad D_y=\begin{vmatrix} 1 & -1 \\ 4 & 6 \end{vmatrix}=10$$

$\underline{x=1}$ $\qquad x=\dfrac{D_x}{D}$

$\underline{y=-1}$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(1,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x-2y=5 \\ -10x+2y=4 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & -2 \\ -10 & 2 \end{vmatrix}=-18 \qquad D_x=\begin{vmatrix} 5 & -2 \\ 4 & 2 \end{vmatrix}=18 \qquad D_y=\begin{vmatrix} 1 & 5 \\ -10 & 4 \end{vmatrix}=54$$

$\underline{x=-1}$ $\qquad x=\dfrac{D_x}{D}$

$\underline{y=-\frac{54}{18}=-3}$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(-1,\ -3)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 12x+15y=-27 \\ 30x-15y=-15 \end{cases}$

Solution

$$\begin{array}{l} \frac{1}{3}\times \\ \frac{1}{15}\times \end{array} \begin{cases} 12x+15y=-27 \\ 30x-15y=-15 \end{cases}$$

$$\begin{cases} 4x+5y=-9 \\ 2x-y=-1 \end{cases}$$

$$D = \begin{vmatrix} 4 & 5 \\ 2 & -1 \end{vmatrix} = -14 \qquad D_x = \begin{vmatrix} -9 & 5 \\ -1 & -1 \end{vmatrix} = 14 \qquad D_y = \begin{vmatrix} 4 & -9 \\ 2 & -1 \end{vmatrix} = 14$$

$\underline{x = -1}$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = -1}$ $\qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\quad \underline{(-1, \ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$

Solution

$$\begin{array}{l} \frac{1}{4}\times \\ \frac{1}{4}\times \end{array} \begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$$

$$\begin{cases} x - y = -3 \\ x + y = -5 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \qquad D_x = \begin{vmatrix} -3 & -1 \\ -5 & 1 \end{vmatrix} = -8 \qquad D_y = \begin{vmatrix} 1 & -3 \\ 1 & -5 \end{vmatrix} = -2$$

$\underline{x = -4}$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = -1}$ $\qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\quad \underline{(-4, \ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x + y = 7 \\ x - y = 3 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \qquad D_x = \begin{vmatrix} 7 & 1 \\ 3 & -1 \end{vmatrix} = -10 \qquad D_y = \begin{vmatrix} 1 & 7 \\ 1 & 3 \end{vmatrix} = -4$$

$\underline{x = 5}$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = 2}\,\rvert$ $\qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(5,\ 2)}\,\rvert$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$

Solution

$D = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3 \qquad D_x = \begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix} = -6 \qquad D_y = \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = 3$

$\underline{x = 2}\,\rvert$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = -1}\,\rvert$ $\qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(2,\ -1)}\,\rvert$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 12x + 3y = 15 \\ 2x - 3y = 13 \end{cases}$

Solution

$D = \begin{vmatrix} 12 & 3 \\ 2 & -3 \end{vmatrix} = -42 \qquad D_x = \begin{vmatrix} 15 & 3 \\ 13 & -3 \end{vmatrix} = -84 \qquad D_y = \begin{vmatrix} 12 & 15 \\ 2 & 13 \end{vmatrix} = 126$

$\underline{x = 2}\,\rvert$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = -3}\,\rvert$ $\qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(2,\ -3)}\,\rvert$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x - 2y = 5 \\ 5x - y = -2 \end{cases}$

Solution

$D = \begin{vmatrix} 1 & -2 \\ 5 & -1 \end{vmatrix} = 9 \qquad D_x = \begin{vmatrix} 5 & -2 \\ -2 & -1 \end{vmatrix} = -9 \qquad D_y = \begin{vmatrix} 1 & 5 \\ 5 & -2 \end{vmatrix} = -27$

$\boxed{x=-1}$ $\quad x=\dfrac{D_x}{D}$

$\boxed{y=-3}$ $\quad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\boxed{(-1,\ -3)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases}4x-5y=17\\ 2x+3y=3\end{cases}$

Solution

$D=\begin{vmatrix}4 & -5\\ 2 & 3\end{vmatrix}=22$ $\quad D_x=\begin{vmatrix}17 & -5\\ 3 & 3\end{vmatrix}=66$ $\quad D_y=\begin{vmatrix}4 & 17\\ 2 & 3\end{vmatrix}=-22$

$\boxed{x=3}$ $\quad x=\dfrac{D_x}{D}$

$\boxed{y=-1}$ $\quad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\boxed{(3,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases}3x+2y=2\\ 2x+2y=3\end{cases}$

Solution

$D=\begin{vmatrix}3 & 2\\ 2 & 2\end{vmatrix}=2$ $\quad D_x=\begin{vmatrix}2 & 2\\ 3 & 2\end{vmatrix}=-2$ $\quad D_y=\begin{vmatrix}3 & 2\\ 2 & 3\end{vmatrix}=5$

$\boxed{x=-1}$ $\quad x=\dfrac{D_x}{D}$

$\boxed{y=\frac{5}{2}}$ $\quad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\boxed{\left(-1,\ \frac{5}{2}\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x-3y=4 \\ 3x-4y=12 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & -3 \\ 3 & -4 \end{vmatrix}=5 \qquad D_x=\begin{vmatrix} 4 & -3 \\ 12 & -4 \end{vmatrix}=20 \qquad D_y=\begin{vmatrix} 1 & 4 \\ 3 & 12 \end{vmatrix}=0$$

$\underline{x=4}$ $\qquad x=\dfrac{D_x}{D}$

$\underline{y=0}$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(4,\ 0)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x-9y=5 \\ 3x-3y=11 \end{cases}$

Solution

$$D=\begin{vmatrix} 2 & -9 \\ 3 & -3 \end{vmatrix}=21 \qquad D_x=\begin{vmatrix} 5 & -9 \\ 11 & -3 \end{vmatrix}=84 \qquad D_y=\begin{vmatrix} 2 & 5 \\ 3 & 11 \end{vmatrix}=7$$

$\underline{x=4}$ $\qquad x=\dfrac{D_x}{D}$

$\underline{y=\frac{1}{3}}$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{\left(4,\ \frac{1}{3}\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x-4y=4 \\ x+y=6 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & -4 \\ 1 & 1 \end{vmatrix}=7 \qquad D_x=\begin{vmatrix} 4 & -4 \\ 6 & 1 \end{vmatrix}=28 \qquad D_y=\begin{vmatrix} 3 & 4 \\ 1 & 6 \end{vmatrix}=14$$

$\underline{x=4}$ $\qquad x=\dfrac{D_x}{D}$

$\underline{y=2}$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(4,\ 2)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x = 7y + 1 \\ 2x = 3y - 1 \end{cases}$

Solution

$$\begin{cases} 3x - 7y = 1 \\ 2x - 3y = -1 \end{cases}$$

$$D = \begin{vmatrix} 3 & -7 \\ 2 & -3 \end{vmatrix} = 5 \qquad D_x = \begin{vmatrix} 1 & -7 \\ -1 & -3 \end{vmatrix} = -10 \qquad D_y = \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} = -5$$

$\underline{x = -2}$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = -1}$ $\qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(-2,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x = 3y + 2 \\ 5x = 51 - 4y \end{cases}$

Solution

$$\begin{cases} 2x - 3y = 2 \\ 5x + 4y = 51 \end{cases}$$

$$D = \begin{vmatrix} 2 & -3 \\ 5 & 4 \end{vmatrix} = 23 \qquad D_x = \begin{vmatrix} 2 & -3 \\ 51 & 4 \end{vmatrix} = 161 \qquad D_y = \begin{vmatrix} 2 & 2 \\ 5 & 51 \end{vmatrix} = 92$$

$\underline{x = 7}$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = 4}$ $\qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(7,\ 4)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} y = -4x + 2 \\ 2x = 3y - 1 \end{cases}$

Solution

$$\begin{cases} 4x + y = 2 \\ 2x - 3y = -1 \end{cases}$$

$$D = \begin{vmatrix} 4 & 1 \\ 2 & -3 \end{vmatrix} = -14 \qquad D_x = \begin{vmatrix} 2 & 1 \\ -1 & -3 \end{vmatrix} = -5 \qquad D_y = \begin{vmatrix} 4 & 2 \\ 2 & -1 \end{vmatrix} = -8$$

$$x = \frac{5}{14} \qquad x = \frac{D_x}{D}$$

$$y = \frac{4}{7} \qquad y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $\left(\frac{15}{4},\ \frac{4}{7}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x = 2 - 3y \\ 2y = 3 - 2x \end{cases}$

Solution

$$\begin{cases} 3x + 3y = 2 \\ 2x + 2y = 3 \end{cases}$$

$$D = \begin{vmatrix} 3 & 3 \\ 2 & 2 \end{vmatrix} = 0 \qquad D_y = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 5 \neq 0$$

$\therefore$ ***No Solution***

Exercise

Use Cramer's rule to solve the system $\begin{cases} x + 2y - 3 = 0 \\ 12 = 8y + 4x \end{cases}$

Solution

$$\begin{cases} x + 2y = 3 \\ 4x + 8y = 12 \end{cases}$$

$$\begin{cases} x + 2y = 3 \\ x + 2y = 3 \end{cases}$$

$\therefore$ ***Solution:*** $(3 - 2y,\ y)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 7x - 2y = 3 \\ 3x + y = 5 \end{cases}$

Solution

$D = \begin{vmatrix} 7 & -2 \\ 3 & 1 \end{vmatrix} = 13 \qquad D_x = \begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} = 13 \qquad D_y = \begin{vmatrix} 7 & 3 \\ 3 & 5 \end{vmatrix} = 26$

$\underline{x = 1}$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = 2}$ $\qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(1,\ 2)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x + 2y - z = 4 \\ 3x - 2y + z = 5 \\ 4x - 5y - z = -1 \end{cases}$

Solution

$D = \begin{vmatrix} 3 & 2 & -1 \\ 3 & -2 & 1 \\ 4 & -5 & -1 \end{vmatrix} = 42 \qquad D_x = \begin{vmatrix} 4 & 2 & -1 \\ 5 & -2 & 1 \\ -1 & -5 & -1 \end{vmatrix} = 63$

$D_y = \begin{vmatrix} 3 & 4 & -1 \\ 3 & 5 & 1 \\ 4 & -1 & -1 \end{vmatrix} = 39 \qquad D_z = \begin{vmatrix} 3 & 2 & 4 \\ 3 & -2 & 5 \\ 4 & -5 & -1 \end{vmatrix} = 99$

$x = \dfrac{63}{42} = \underline{\dfrac{3}{2}}$ $\qquad x = \dfrac{D_x}{D}$

$y = \dfrac{39}{42} = \underline{\dfrac{13}{14}}$ $\qquad y = \dfrac{D_y}{D}$

$z = \dfrac{99}{42} = \underline{\dfrac{33}{14}}$ $\qquad z = \dfrac{D_z}{D}$

Solution: $\underline{\left(\dfrac{3}{2},\ \dfrac{13}{14},\ \dfrac{33}{14}\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+y+z=2 \\ 2x+y-z=5 \\ x-y+z=-2 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}=-6 \qquad D_x=\begin{vmatrix} 2 & 1 & 1 \\ 5 & 1 & -1 \\ -2 & -1 & 1 \end{vmatrix}=-6$$

$$D_y=\begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & -1 \\ 1 & -2 & 1 \end{vmatrix}=-12 \qquad D_z=\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 5 \\ 1 & -2 & 2 \end{vmatrix}=6$$

$x=1 \qquad x=\dfrac{D_x}{D}$

$y=2 \qquad y=\dfrac{D_y}{D}$

$z=-1 \qquad z=\dfrac{D_z}{D}$

$\therefore$ ***Solution***: $(1,\ 2,\ -1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x+y+z=9 \\ -x-y+z=1 \\ 3x-y+z=9 \end{cases}$

Solution

$$D=\begin{vmatrix} 2 & 1 & 1 \\ -1 & -1 & 1 \\ 3 & -1 & 1 \end{vmatrix}\begin{matrix} 2 & 1 \\ -1 & -1 \\ 3 & -1 \end{matrix} = -2+3+1+3+2+1$$

$$=8$$

$$D_x=\begin{vmatrix} 9 & 1 & 1 \\ 1 & -1 & 1 \\ 9 & -1 & 1 \end{vmatrix}\begin{matrix} 9 & 1 \\ 1 & -1 \\ 9 & -1 \end{matrix} = -9+9-1+9+9-1$$

$$=16$$

$$D_y=\begin{vmatrix} 2 & 9 & 1 \\ -1 & 1 & 1 \\ 3 & 9 & 1 \end{vmatrix}\begin{matrix} 2 & 9 \\ -1 & 1 \\ 3 & 9 \end{matrix} = 2+27-9-3-18+9$$

$$=8$$

$$D_z = \begin{vmatrix} 2 & 1 & 9 \\ -1 & -1 & 1 \\ 3 & -1 & 9 \end{vmatrix} \begin{matrix} 2 & 1 \\ -1 & -1 \\ 3 & -1 \end{matrix} = -18+3+9+27+2+9$$

$$= 32$$

$x = 2$ $\qquad x = \dfrac{D_x}{D}$

$y = 1$ $\qquad y = \dfrac{D_y}{D}$

$z = \dfrac{32}{8} = 4$ $\qquad z = \dfrac{D_z}{D}$

$\therefore$ ***Solution***: $(2,\ 1,\ 4)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$

Solution

$$D = \begin{vmatrix} 0 & 3 & -1 \\ 1 & 5 & -1 \\ -3 & 6 & 2 \end{vmatrix} \begin{matrix} 0 & 3 \\ 1 & 5 \\ -3 & 6 \end{matrix} = 9-6-15-6$$

$$= -18$$

$$D_x = \begin{vmatrix} -1 & 3 & -1 \\ -4 & 5 & -1 \\ 11 & 6 & 2 \end{vmatrix} \begin{matrix} -1 & 3 \\ -4 & 5 \\ 11 & 6 \end{matrix} = -10-33+24+55-6+24$$

$$= 54$$

$$D_y = \begin{vmatrix} 0 & -1 & -1 \\ 1 & -4 & -1 \\ -3 & 11 & 2 \end{vmatrix} \begin{matrix} 0 & -1 \\ 1 & -4 \\ -3 & 11 \end{matrix} = -3-11+12+2$$

$$= 0$$

$$D_z = \begin{vmatrix} 0 & 3 & -1 \\ 1 & 5 & -4 \\ -3 & 6 & 11 \end{vmatrix} \begin{matrix} 0 & 3 \\ 1 & 5 \\ -3 & 6 \end{matrix} = 36-6-15-33$$

$$= -18$$

$x = -3$ $\qquad x = \dfrac{D_x}{D}$

$y = 0$ $\qquad y = \dfrac{D_y}{D}$

$z = 1$ $\qquad z = \dfrac{D_z}{D}$

$\therefore$ ***Solution***: $(-3,\ 0,\ 1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & 3 & 4 \\ 2 & -3 & 2 \\ 3 & -1 & 1 \end{vmatrix} \begin{matrix} 1 & 3 \\ 2 & -3 \\ 3 & -1 \end{matrix} = -3 + 18 - 8 + 36 + 2 - 6$$

$$= 39$$

$$D_x = \begin{vmatrix} 14 & 3 & 4 \\ 10 & -3 & 2 \\ 9 & -1 & 1 \end{vmatrix} \begin{matrix} 14 & 3 \\ 10 & -3 \\ 9 & -1 \end{matrix} = -42 + 54 - 40 + 108 + 28 - 30$$

$$= 78$$

$$D_y = \begin{vmatrix} 1 & 14 & 4 \\ 2 & 10 & 2 \\ 3 & 9 & 1 \end{vmatrix} \begin{matrix} 1 & 14 \\ 2 & 10 \\ 3 & 9 \end{matrix} = 10 + 84 + 72 - 120 - 18 - 28$$

$$= 0$$

$$D_z = \begin{vmatrix} 1 & 3 & 14 \\ 2 & -3 & 10 \\ 3 & -1 & 9 \end{vmatrix} \begin{matrix} 1 & 3 \\ 2 & -3 \\ 3 & -1 \end{matrix} = -27 + 90 - 28 + 126 + 10 - 54$$

$$= 117$$

$x = \dfrac{78}{39} = 2$ $\qquad x = \dfrac{D_x}{D}$

$y = 0$ $\qquad y = \dfrac{D_y}{D}$

$z = \dfrac{117}{39} = 3$ $\qquad z = \dfrac{D_z}{D}$

$\therefore$ ***Solution***: $(2,\ 0,\ 3)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+4y-z=20 \\ 3x+2y+z=8 \\ 2x-3y+2z=-16 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & 4 & -1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix}\begin{matrix} 1 & 4 \\ 3 & 2 \\ 2 & -3 \end{matrix} = 4+8+9+4+3-24$$

$$\underline{=4}$$

$$D_x=\begin{vmatrix} 20 & 4 & -1 \\ 8 & 2 & 1 \\ -16 & -3 & 2 \end{vmatrix}\begin{matrix} 20 & 4 \\ 8 & 2 \\ -16 & -3 \end{matrix} = 80-64+24-32+60-64$$

$$\underline{=4}$$

$$D_y=\begin{vmatrix} 1 & 20 & -1 \\ 3 & 8 & 1 \\ 2 & -16 & 2 \end{vmatrix}\begin{matrix} 1 & 20 \\ 3 & 8 \\ 2 & -16 \end{matrix} = 16+40+48+16+16-120$$

$$\underline{=16}$$

$$D_z=\begin{vmatrix} 1 & 4 & 20 \\ 3 & 2 & 8 \\ 2 & -3 & -16 \end{vmatrix}\begin{matrix} 1 & 4 \\ 3 & 2 \\ 2 & -3 \end{matrix} = -32+64-180-80+24+192$$

$$\underline{=-12}$$

$x=\frac{4}{4}\ \underline{=1}$ $\qquad x=\frac{D_x}{D}$

$y=\frac{16}{4}\ \underline{=4}$ $\qquad y=\frac{D_y}{D}$

$z=-\frac{12}{4}\ \underline{=-3}$ $\qquad z=\frac{D_z}{D}$

$\therefore$ ***Solution***: $\underline{(1,\ 4,\ -3)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} -2x+6y+7z=3 \\ -4x+5y+3z=7 \\ -6x+3y+5z=-4 \end{cases}$

Solution

$$D=\begin{vmatrix} -2 & 6 & 7 \\ -4 & 5 & 3 \\ -6 & 3 & 5 \end{vmatrix}\begin{matrix} -2 & 6 \\ -4 & 5 \\ -6 & 3 \end{matrix} = -50-108-84+210+18+120$$

$$\underline{=106|}$$

$$D_x=\begin{vmatrix} 3 & 6 & 7 \\ 7 & 5 & 3 \\ -4 & 3 & 5 \end{vmatrix}\begin{matrix} 3 & 6 \\ 7 & 5 \\ -4 & 3 \end{matrix} = 75-72+147+140-27-210$$

$$\underline{=53|}$$

$$D_y=\begin{vmatrix} -2 & 3 & 7 \\ -4 & 7 & 3 \\ -6 & -4 & 5 \end{vmatrix}\begin{matrix} -2 & 3 \\ -4 & 7 \\ -6 & -4 \end{matrix} = -70-54+112+294-24+60$$

$$\underline{=318|}$$

$$D_z=\begin{vmatrix} -2 & 6 & 3 \\ -4 & 5 & 7 \\ -6 & 3 & -4 \end{vmatrix}\begin{matrix} -2 & 6 \\ -4 & 5 \\ -6 & 3 \end{matrix} = 40-252-36+90+42-96$$

$$\underline{=-212|}$$

$$x=\frac{53}{106}\ \underline{=\frac{1}{2}|} \qquad x=\frac{D_x}{D}$$

$$y=\frac{318}{106}\ \underline{=3|} \qquad y=\frac{D_y}{D}$$

$$z=-\frac{212}{106}\ \underline{=-2|} \qquad z=\frac{D_z}{D}$$

$\therefore$ ***Solution***: $\underline{\left(\frac{1}{2},\ 3,\ -2\right)\ |}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x-y+z=1 \\ 3x-3y+4z=5 \\ 4x-2y+3z=4 \end{cases}$

Solution

$$D=\begin{vmatrix} 2 & -1 & 1 \\ 3 & -3 & 4 \\ 4 & -2 & 3 \end{vmatrix}\begin{matrix} 2 & -1 \\ 3 & -3 \\ 4 & -2 \end{matrix} = -18-16-6+12+16+9$$

$$\underline{=-3|}$$

$$D_x = \begin{vmatrix} 1 & -1 & 1 \\ 5 & -3 & 4 \\ 4 & -2 & 3 \end{vmatrix} \begin{matrix} 1 & -1 \\ 5 & -3 \\ 4 & -2 \end{matrix} = -9 - 16 - 10 + 12 + 8 + 15$$

$$\underline{= 0}$$

$$D_y = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 5 & 4 \\ 4 & 4 & 3 \end{vmatrix} \begin{matrix} 2 & 1 \\ 3 & 5 \\ 4 & 4 \end{matrix} = 30 + 16 + 12 - 20 - 32 - 9$$

$$\underline{= -3}$$

$$D_z = \begin{vmatrix} 2 & -1 & 1 \\ 3 & -3 & 5 \\ 4 & -2 & 4 \end{vmatrix} \begin{matrix} 2 & -1 \\ 3 & -3 \\ 4 & -2 \end{matrix} = -24 - 20 - 6 + 12 + 20 + 12$$

$$\underline{= -6}$$

$$x = -\frac{0}{3} \ \underline{= 0} \qquad\qquad x = \frac{D_x}{D}$$

$$y = \frac{-3}{-3} \ \underline{= 1} \qquad\qquad y = \frac{D_y}{D}$$

$$z = \frac{-6}{-3} \ \underline{= 2} \qquad\qquad z = \frac{D_z}{D}$$

$\therefore$ ***Solution***: $\underline{(0,\ 1,\ 2)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x - 4y + 4z = 7 \\ x - y - 2z = 2 \\ 2x - 3y + 6z = 5 \end{cases}$

Solution

$$D = \begin{vmatrix} 3 & -4 & 4 \\ 1 & -1 & -2 \\ 2 & -3 & 6 \end{vmatrix} \begin{matrix} 3 & -4 \\ 1 & -1 \\ 2 & -3 \end{matrix} = -18 + 16 - 12 + 8 - 18 + 24$$

$$\underline{= 0}$$

$$D_z = \begin{vmatrix} 3 & -4 & 7 \\ 1 & -1 & 2 \\ 2 & -3 & 5 \end{vmatrix} \begin{matrix} 3 & -4 \\ 1 & -1 \\ 2 & -3 \end{matrix} = -15 - 16 - 21 + 14 + 18 + 20$$

$$\underline{= 0}$$

$$-3 \times (2) \quad \begin{cases} -3x + 3y + 6z = -6 \\ 2x - 3y + 6z = 5 \end{cases}$$

$$\overline{\quad -x + 12z = -1 \quad}$$

$\underline{x = 12z + 1}$

$(2) \rightarrow \quad y = 12z + 1 - 2z - 2$

$\qquad \underline{= 10z - 1}$

$\therefore$ ***Solution***: $\underline{(12z+1,\ \ 10z-1,\ \ z)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x - 2y - z = 2 \\ 2x - y + z = 4 \\ -x + y + z = 4 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} \begin{matrix} 1 & -2 \\ 2 & -1 \\ -1 & 1 \end{matrix} = -1 + 2 - 2 + 1 - 1 + 4$$

$$\underline{= 3}$$

$$D_x = \begin{vmatrix} 2 & -2 & -1 \\ 4 & -1 & 1 \\ 4 & 1 & 1 \end{vmatrix} \begin{matrix} 2 & -2 \\ 4 & -1 \\ 4 & 1 \end{matrix} = -2 - 8 - 4 - 4 - 2 + 8$$

$$\underline{= -12}$$

$$D_y = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ -1 & 4 & 1 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & 4 \\ -1 & 4 \end{matrix} = 4 - 2 - 8 - 4 - 4 - 4$$

$$\underline{= -18}$$

$$D_z = \begin{vmatrix} 1 & -2 & 2 \\ 2 & -1 & 4 \\ -1 & 1 & 4 \end{vmatrix} \begin{matrix} 1 & -2 \\ 2 & -1 \\ -1 & 1 \end{matrix} = -4 + 8 + 4 - 2 - 4 + 16$$

$$\underline{= 18}$$

$x = -\dfrac{12}{3} \ \underline{= -4} \qquad\qquad x = \dfrac{D_x}{D}$

$y = -\dfrac{18}{3} \ \underline{= -6} \qquad\qquad y = \dfrac{D_y}{D}$

$z = \dfrac{18}{3} \ \underline{= 6} \qquad\qquad z = \dfrac{D_z}{D}$

$\therefore$ ***Solution***: $\underline{(-4,\ \ -6,\ \ 6)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+y+z=3 \\ -y+2z=1 \\ -x+z=0 \end{cases}$

Solution

$$D=\left|\begin{array}{ccc}1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 0 & 1\end{array}\right|\begin{array}{cc}1 & 1 \\ 0 & -1 \\ -1 & 0\end{array} \underline{=-4}$$

$$D_x=\left|\begin{array}{ccc}3 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & 0 & 1\end{array}\right|\begin{array}{cc}3 & 1 \\ 1 & -1 \\ 0 & 0\end{array} \underline{=-4}$$

$$D_y=\left|\begin{array}{ccc}1 & 3 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 1\end{array}\right|\begin{array}{cc}1 & 1 \\ 0 & -1 \\ -1 & 0\end{array} \underline{=-4}$$

$$D_z=\left|\begin{array}{ccc}1 & 1 & 3 \\ 0 & -1 & 1 \\ -1 & 0 & 0\end{array}\right|\begin{array}{cc}1 & 1 \\ 0 & -1 \\ -1 & 0\end{array} \underline{=-4}$$

$$x=\frac{4}{4} \underline{=1} \qquad x=\frac{D_x}{D}$$

$$y=\frac{4}{4} \underline{=1} \qquad y=\frac{D_y}{D}$$

$$z=\frac{4}{4} \underline{=1} \qquad z=\frac{D_z}{D}$$

$\therefore$ ***Solution***: $\underline{(1,\ 1,\ 1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+y+3z=14 \\ 7x+5y+8z=37 \\ x+3y+2z=9 \end{cases}$

Solution

$$D=\left|\begin{array}{ccc}3 & 1 & 3 \\ 7 & 5 & 8 \\ 1 & 3 & 2\end{array}\right|\begin{array}{cc}3 & 1 \\ 7 & 5 \\ 1 & 3\end{array} =30+8+62-15-72-14 \underline{=0}$$

$$D_z=\left|\begin{array}{ccc}3 & 1 & 14 \\ 7 & 5 & 37 \\ 1 & 3 & 9\end{array}\right|\begin{array}{cc}3 & 1 \\ 7 & 5 \\ 1 & 3\end{array} =135+37+294-70-333-63 \underline{=0}$$

$$\begin{array}{r} -3\times(1) \\ (3) \end{array}\begin{cases} -9x-3y-9z=-42 \\ \quad x+3y+2z=9 \end{cases}$$

$$-8x-7z=-33$$

$$x=-\frac{7}{8}z+\frac{33}{8}$$

$$(1)\rightarrow \quad y=14-3z-3\left(-\frac{7}{8}z+\frac{33}{8}\right)$$

$$=\frac{13}{8}-\frac{3}{8}z$$

$\therefore$ ***Solution***: $\left(\frac{33}{8}-\frac{7}{8}z,\ \ \frac{13}{8}-\frac{3}{8}z,\ \ z\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x-2y+z=7 \\ x+y+z=-2 \\ 4x+2y+z=3 \end{cases}$

Solution

$$D=\begin{vmatrix} 4 & -2 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix}\begin{matrix} 4 & -2 \\ 1 & 1 \\ 4 & 2 \end{matrix} = -12$$

$$D_x=\begin{vmatrix} 7 & -2 & 1 \\ -2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix}\begin{matrix} 7 & -2 \\ -2 & 1 \\ 3 & 2 \end{matrix} = -24$$

$$D_y=\begin{vmatrix} 4 & 7 & 1 \\ 1 & -2 & 1 \\ 4 & 3 & 1 \end{vmatrix}\begin{matrix} 4 & 7 \\ 1 & -2 \\ 4 & 3 \end{matrix} = 12$$

$$D_z=\begin{vmatrix} 4 & -2 & 7 \\ 1 & 1 & -2 \\ 4 & 2 & 3 \end{vmatrix}\begin{matrix} 4 & -2 \\ 1 & 1 \\ 4 & 2 \end{matrix} = 36$$

$$x=\frac{24}{12}=2 \qquad\qquad x=\frac{D_x}{D}$$

$$y=-\frac{12}{12}=-1 \qquad\qquad y=\frac{D_y}{D}$$

$$z=-\frac{36}{12}=-3 \qquad\qquad z=\frac{D_z}{D}$$

$\therefore$ ***Solution***: $(2,\ -1,\ -3)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$

Solution

$$D = \begin{vmatrix} 0 & 2 & -1 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix} \begin{matrix} 0 & 2 \\ 1 & 2 \\ 2 & 3 \end{matrix} \underline{=1}$$

$$D_x = \begin{vmatrix} 7 & 2 & -1 \\ 17 & 2 & 1 \\ -1 & 3 & 2 \end{vmatrix} \begin{matrix} 7 & 2 \\ 17 & 2 \\ -1 & 3 \end{matrix} \underline{=-116}$$

$$D_y = \begin{vmatrix} 0 & 7 & -1 \\ 1 & 17 & 1 \\ 2 & -1 & 2 \end{vmatrix} \begin{matrix} 0 & 7 \\ 1 & 17 \\ 2 & -1 \end{matrix} \underline{=35}$$

$$D_z = \begin{vmatrix} 0 & 2 & 7 \\ 1 & 2 & 17 \\ 2 & 3 & -1 \end{vmatrix} \begin{matrix} 0 & 2 \\ 1 & 2 \\ 2 & 3 \end{matrix} \underline{=63}$$

$x \underline{=-116}$ $\qquad x = \dfrac{D_x}{D}$

$y \underline{=35}$ $\qquad y = \dfrac{D_y}{D}$

$z \underline{=63}$ $\qquad z = \dfrac{D_z}{D}$

$\therefore$ ***Solution***: $\underline{(-116,\ 35,\ 63)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x - 2y + z = -4 \\ 6x + 4y - 3z = -24 \\ x - 2y + 2z = 1 \end{cases}$

Solution

$$D = \begin{vmatrix} 2 & -2 & 1 \\ 6 & 4 & -3 \\ 1 & -2 & 2 \end{vmatrix} \begin{matrix} 2 & -2 \\ 6 & 4 \\ 1 & -2 \end{matrix} \underline{=18}$$

$$D_x = \begin{vmatrix} -4 & -2 & 1 \\ -24 & 4 & -3 \\ 1 & -2 & 2 \end{vmatrix} \begin{matrix} -4 & -2 \\ -24 & 4 \\ 1 & -2 \end{matrix} \underline{=-54}$$

$$D_y = \begin{vmatrix} 2 & -4 & 1 \\ 6 & -24 & -3 \\ 1 & 1 & 2 \end{vmatrix} \begin{matrix} 2 & -4 \\ 6 & -24 \\ 1 & 1 \end{matrix} \underline{=0|}$$

$$D_z = \begin{vmatrix} 2 & -2 & -4 \\ 6 & 4 & -24 \\ 1 & -2 & 1 \end{vmatrix} \begin{matrix} 2 & -2 \\ 6 & 4 \\ 1 & -2 \end{matrix} \underline{=36|}$$

$x = -\frac{54}{18} \underline{=-3|}$ $\qquad x = \frac{D_x}{D}$

$y \underline{=0|}$ $\qquad y = \frac{D_y}{D}$

$z \underline{=2|}$ $\qquad z = \frac{D_z}{D}$

$\therefore$ ***Solution***: $\underline{(-3,\ 0,\ 2)\ |}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 9x+3y+z=4 \\ 16x+4y+z=2 \\ 25x+5y+z=2 \end{cases}$

Solution

$$D = \begin{vmatrix} 9 & 3 & 1 \\ 16 & 4 & 1 \\ 25 & 5 & 1 \end{vmatrix} \begin{matrix} 9 & 3 \\ 16 & 4 \\ 25 & 5 \end{matrix} \underline{=-2|}$$

$$D_x = \begin{vmatrix} 4 & 3 & 1 \\ 2 & 4 & 1 \\ 2 & 5 & 1 \end{vmatrix} \begin{matrix} 4 & 3 \\ 2 & 4 \\ 2 & 5 \end{matrix} \underline{=-2|}$$

$$D_y = \begin{vmatrix} 9 & 4 & 1 \\ 16 & 2 & 1 \\ 25 & 2 & 1 \end{vmatrix} \begin{matrix} 9 & 4 \\ 16 & 2 \\ 25 & 2 \end{matrix} \underline{=18|}$$

$$D_z = \begin{vmatrix} 9 & 3 & 4 \\ 16 & 4 & 2 \\ 25 & 5 & 2 \end{vmatrix} \begin{matrix} 9 & 3 \\ 16 & 4 \\ 25 & 5 \end{matrix} \underline{=-44|}$$

$x \underline{=1|}$ $\qquad x = \frac{D_x}{D}$

$y \underline{=-9|}$ $\qquad y = \frac{D_y}{D}$

$z = 22$ $\qquad z = \frac{D_z}{D}$

$\therefore$ ***Solution***: $(1,\ -9,\ 22)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x - y + 2z = -8 \\ x + 2y - 3z = 9 \\ 3x - y - 4z = 3 \end{cases}$

Solution

$$D = \begin{vmatrix} 2 & -1 & 2 \\ 1 & 2 & -3 \\ 3 & -1 & -4 \end{vmatrix} \begin{matrix} 2 & -1 \\ 1 & 2 \\ 3 & -1 \end{matrix} = -31$$

$$D_x = \begin{vmatrix} -8 & -1 & 2 \\ 9 & 2 & -3 \\ 3 & -1 & -4 \end{vmatrix} \begin{matrix} -8 & -1 \\ 9 & 2 \\ 3 & -1 \end{matrix} = 31$$

$$D_y = \begin{vmatrix} 2 & -8 & 2 \\ 1 & 9 & -3 \\ 3 & 3 & -4 \end{vmatrix} \begin{matrix} 2 & -8 \\ 1 & 9 \\ 3 & 3 \end{matrix} = -62$$

$$D_z = \begin{vmatrix} 2 & -1 & -8 \\ 1 & 2 & 9 \\ 3 & -1 & 3 \end{vmatrix} \begin{matrix} 2 & -1 \\ 1 & 2 \\ 3 & -1 \end{matrix} = 62$$

$x = -\frac{31}{31} = -1$ $\qquad x = \frac{D_x}{D}$

$y = \frac{62}{31} = 2$ $\qquad y = \frac{D_y}{D}$

$z = -\frac{62}{31} = -2$ $\qquad z = \frac{D_z}{D}$

$\therefore$ ***Solution***: $(-1,\ 2,\ -2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x \qquad - 3z = -5 \\ 2x - y + 2z = 16 \\ 7x - 3y - 5z = 19 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & 0 & -3 \\ 2 & -1 & 2 \\ 7 & -3 & -5 \end{vmatrix} \begin{matrix} 1 & 0 \\ 2 & -1 \\ 7 & -3 \end{matrix} \underline{=8|}$$

$$D_x = \begin{vmatrix} -5 & 0 & -3 \\ 16 & -1 & 2 \\ 19 & -3 & -5 \end{vmatrix} \begin{matrix} -5 & 0 \\ 16 & -1 \\ 19 & -3 \end{matrix} \underline{=32|}$$

$$D_y = \begin{vmatrix} 1 & -5 & -3 \\ 2 & 16 & 2 \\ 7 & 19 & -5 \end{vmatrix} \begin{matrix} 1 & -5 \\ 2 & 16 \\ 7 & 19 \end{matrix} \underline{=-16|}$$

$$D_z = \begin{vmatrix} 1 & 0 & -5 \\ 2 & -1 & 16 \\ 7 & -3 & 19 \end{vmatrix} \begin{matrix} 1 & 0 \\ 2 & -1 \\ 7 & -3 \end{matrix} \underline{=24|}$$

$x = \frac{32}{8} \underline{=4|}$ $\qquad x = \frac{D_x}{D}$

$y = -\frac{16}{8} \underline{=-2|}$ $\qquad y = \frac{D_y}{D}$

$z = \frac{24}{8} \underline{=3|}$ $\qquad z = \frac{D_z}{D}$

∴ ***Solution***: $\underline{(4,\ -2,\ 3)\ |}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x + 2y - z = 5 \\ 2x - y + 3z = 0 \\ \qquad 2y + z = 1 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 0 & 2 & 1 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & -1 \\ 0 & 2 \end{matrix} \underline{=-15|}$$

$$D_x = \begin{vmatrix} 5 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 2 & 1 \end{vmatrix} \begin{matrix} 5 & 2 \\ 0 & -1 \\ 1 & 2 \end{matrix} \underline{=-30|}$$

$$D_y = \begin{vmatrix} 1 & 5 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 1 \end{vmatrix} \begin{matrix} 1 & 5 \\ 2 & 0 \\ 0 & 1 \end{matrix} \underline{=-15|}$$

$$D_z = \begin{vmatrix} 1 & 2 & 5 \\ 2 & -1 & 0 \\ 0 & 2 & 1 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & -1 \\ 0 & 2 \end{matrix} \underline{=15|}$$

$x = \frac{30}{15} \underline{=2|}$ $\qquad x = \frac{D_x}{D}$

$y = \frac{15}{15} \underline{=1|}$ $\qquad y = \frac{D_y}{D}$

$z = -\frac{15}{15} \underline{=-1|}$ $\qquad z = \frac{D_z}{D}$

$\therefore$ ***Solution***: $\underline{(2,\ 1,\ -1)\ |}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x + y + z = 6 \\ 3x + 4y - 7z = 1 \\ 2x - y + 3z = 5 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 4 & -7 \\ 2 & -1 & 3 \end{vmatrix} \begin{matrix} 1 & 1 \\ 3 & 4 \\ 2 & -1 \end{matrix} \underline{=-29|}$$

$$D_x = \begin{vmatrix} 6 & 1 & 1 \\ 1 & 4 & -7 \\ 5 & -1 & 3 \end{vmatrix} \begin{matrix} 6 & 1 \\ 1 & 4 \\ 5 & -1 \end{matrix} \underline{=-29|}$$

$$D_y = \begin{vmatrix} 1 & 6 & 1 \\ 3 & 1 & -7 \\ 2 & 5 & 3 \end{vmatrix} \begin{matrix} 1 & 6 \\ 3 & 1 \\ 2 & 5 \end{matrix} \underline{=-87|}$$

$$D_z = \begin{vmatrix} 1 & 2 & 6 \\ 2 & -1 & 1 \\ 0 & 2 & 5 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & -1 \\ 0 & 2 \end{matrix} \underline{=-58|}$$

$x = \frac{29}{29} \underline{=1|}$ $\qquad x = \frac{D_x}{D}$

$y = \frac{87}{29} \underline{=3|}$ $\qquad y = \frac{D_y}{D}$

$z = \frac{58}{29} \underline{=2|}$ $\qquad z = \frac{D_z}{D}$

$\therefore$ ***Solution***: $\underline{(1,\ 3,\ 2)\ |}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y+3z=3 \\ 4x-5y+7z=1 \\ 2x+3y-2z=6 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & 2 & 3 \\ 4 & -5 & 7 \\ 2 & 3 & -2 \end{vmatrix}\begin{matrix} 3 & 2 \\ 4 & -5 \\ 2 & 3 \end{matrix} = 77$$

$$D_x=\begin{vmatrix} 3 & 2 & 3 \\ 1 & -5 & 7 \\ 6 & 3 & -2 \end{vmatrix}\begin{matrix} 3 & 2 \\ 1 & -5 \\ 6 & 3 \end{matrix} = 154$$

$$D_y=\begin{vmatrix} 3 & 3 & 3 \\ 4 & 1 & 7 \\ 2 & 6 & -2 \end{vmatrix}\begin{matrix} 3 & 3 \\ 4 & 1 \\ 2 & 6 \end{matrix} = 0$$

$$D_z=\begin{vmatrix} 3 & 2 & 3 \\ 4 & -5 & 1 \\ 2 & 3 & 6 \end{vmatrix}\begin{matrix} 3 & 2 \\ 4 & -5 \\ 2 & 3 \end{matrix} = -77$$

$x=\frac{154}{77}=2$ $\qquad x=\frac{D_x}{D}$

$y=0$ $\qquad y=\frac{D_y}{D}$

$z=-\frac{77}{77}=-1$ $\qquad z=\frac{D_z}{D}$

$\therefore$ ***Solution***: $(2,\ 0,\ -1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x+5y \quad\ \ =2 \\ 11x+y+2z=3 \\ x+5y+2z=1 \end{cases}$

Solution

$$D=\begin{vmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix}\begin{matrix} 4 & 5 \\ 11 & 1 \\ 1 & 5 \end{matrix} = -132$$

$$D_x=\begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix}\begin{matrix} 2 & 5 \\ 3 & 1 \\ 1 & 5 \end{matrix} = -36$$

$$D_y = \left|\begin{array}{ccc}4 & 2 & 0\\ 11 & 3 & 2\\ 1 & 1 & 2\end{array}\right| \begin{array}{cc}4 & 2\\ 11 & 3\\ 1 & 1\end{array} \boxed{=-24}$$

$$D_z = \left|\begin{array}{ccc}4 & 5 & 2\\ 11 & 1 & 3\\ 1 & 5 & 1\end{array}\right| \begin{array}{cc}4 & 5\\ 11 & 1\\ 1 & 5\end{array} \boxed{=12}$$

$$x = \frac{36}{132} \boxed{=\frac{3}{11}} \qquad\qquad x = \frac{D_x}{D}$$

$$y = \frac{24}{132} \boxed{=\frac{2}{11}} \qquad\qquad y = \frac{D_y}{D}$$

$$z = -\frac{12}{132} \boxed{=-\frac{1}{11}} \qquad\qquad z = \frac{D_z}{D}$$

$\therefore$ ***Solution***: $\boxed{\left(\frac{3}{11},\ \frac{2}{11},\ -\frac{1}{11}\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x - 4y + z = 6 \\ 4x - y + 2z = -1 \\ 2x + 2y - 3z = -20 \end{cases}$

<u>Solution</u>

$$D = \left|\begin{array}{ccc}1 & -4 & 1\\ 4 & -1 & 2\\ 2 & 2 & -3\end{array}\right| \begin{array}{cc}1 & -4\\ 4 & -1\\ 2 & 2\end{array} \boxed{=-55}$$

$$D_x = \left|\begin{array}{ccc}6 & -4 & 1\\ -1 & -1 & 2\\ -20 & 2 & -3\end{array}\right| \begin{array}{cc}6 & -4\\ -1 & -1\\ -20 & 2\end{array} \boxed{=144}$$

$$D_y = \left|\begin{array}{ccc}1 & 6 & 1\\ 4 & -1 & 2\\ 2 & -20 & -3\end{array}\right| \begin{array}{cc}1 & 6\\ 4 & -1\\ 2 & -20\end{array} \boxed{=61}$$

$$D_z = \left|\begin{array}{ccc}1 & -4 & 6\\ 4 & -1 & -1\\ 2 & 2 & -20\end{array}\right| \begin{array}{cc}1 & -4\\ 4 & -1\\ 2 & 2\end{array} \boxed{=-230}$$

$$\boxed{x = -\frac{144}{55}} \qquad\qquad x = \frac{D_x}{D}$$

$$y = -\frac{61}{55} \qquad\qquad y = \frac{D_y}{D}$$

$$z = \frac{230}{55} = \frac{46}{11} \qquad\qquad z = \frac{D_z}{D}$$

$\therefore$ ***Solution***: $\left(-\frac{144}{55},\ -\frac{61}{55},\ \frac{46}{11}\right)$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 2x - y + z = -1 \\ 3x + 4y - z = -1 \\ 4x - y + 2z = -1 \end{cases}$$

Solution

$$D = \begin{vmatrix} 2 & -1 & 1 \\ 3 & 4 & -1 \\ 4 & -1 & 2 \end{vmatrix} \begin{matrix} 2 & -1 \\ 3 & 4 \\ 4 & -1 \end{matrix} = 5$$

$$D_x = \begin{vmatrix} -1 & -1 & 1 \\ -1 & 4 & -1 \\ -1 & -1 & 2 \end{vmatrix} \begin{matrix} -1 & -1 \\ -1 & 4 \\ -1 & -1 \end{matrix} = -5$$

$$D_y = \begin{vmatrix} 2 & -1 & 1 \\ 3 & -1 & -1 \\ 4 & -1 & 2 \end{vmatrix} \begin{matrix} 2 & -1 \\ 3 & -1 \\ 4 & -1 \end{matrix} = 5$$

$$D_z = \begin{vmatrix} 2 & -1 & -1 \\ 3 & 4 & -1 \\ 4 & -1 & -1 \end{vmatrix} \begin{matrix} 2 & -1 \\ 3 & 4 \\ 4 & -1 \end{matrix} = 10$$

$$x = -1 \qquad\qquad x = \frac{D_x}{D}$$

$$y = 1 \qquad\qquad y = \frac{D_y}{D}$$

$$z = 2 \qquad\qquad z = \frac{D_z}{D}$$

$\therefore$ ***Solution***: $(-1,\ 1,\ 2)$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} -x_1 - 4x_2 + 2x_3 + x_4 = -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 = 14 \\ -x_1 + x_2 + 3x_3 + x_4 = 11 \\ -x_1 - 2x_2 + x_3 - 4x_4 = -4 \end{cases}$$

Solution

$$D = \begin{vmatrix} -1 & -4 & 2 & 1 \\ 2 & -1 & 7 & 9 \\ -1 & 1 & 3 & 1 \\ -1 & -2 & 1 & -4 \end{vmatrix} \underline{= -243}$$

$$D_1 = \begin{vmatrix} -32 & -4 & 2 & 1 \\ 14 & -1 & 7 & 9 \\ 11 & 1 & 3 & 1 \\ -4 & -2 & 1 & -4 \end{vmatrix} \underline{= -2115}$$

$$D_2 = \begin{vmatrix} -1 & -32 & 2 & 1 \\ 2 & 14 & 7 & 9 \\ -1 & 11 & 3 & 1 \\ -1 & -4 & 1 & -4 \end{vmatrix} \underline{= -1834}$$

$$D_3 = \begin{vmatrix} -1 & -4 & -32 & 1 \\ 2 & -1 & 14 & 9 \\ -1 & 1 & 11 & 1 \\ -1 & -2 & -4 & -4 \end{vmatrix} \underline{= -1279}$$

$$D_4 = \begin{vmatrix} -1 & -4 & 2 & -32 \\ 2 & -1 & 7 & 14 \\ -1 & 1 & 3 & 11 \\ -1 & -2 & 1 & -4 \end{vmatrix} \underline{= 883}$$

$\therefore$ ***Solution***: $\underline{\left(\frac{235}{27}, \ \frac{1834}{243}, \ \frac{1279}{243}, \ -\frac{883}{243} \right)}$

Exercise

Solve for x. $\begin{vmatrix} x & 3 \\ 2 & 1 \end{vmatrix} = 12$

Solution

$$\begin{vmatrix} x & 3 \\ 2 & 1 \end{vmatrix} = x - 6 = 12$$

$\therefore$ ***Solution***: $\underline{x = 18}$

Exercise

Solve for x. $\begin{vmatrix} x & 1 \\ 2 & x \end{vmatrix} = -1$

Solution

$$\begin{vmatrix} x & 1 \\ 2 & x \end{vmatrix} = x^2 - 2 = -1$$

$$x^2 = 1$$

$\therefore$ ***Solution***: $\underline{x = \pm 1}$

Exercise

Solve for x. $\begin{vmatrix} 3 & x \\ x & 4 \end{vmatrix} = -13$

Solution

$$\begin{vmatrix} 3 & x \\ x & 4 \end{vmatrix} = 12 - x^2 = -13$$

$$x^2 = 25$$

$\therefore$ ***Solution***: $\underline{x = \pm 5}$

Exercise

Solve for x. $\begin{vmatrix} x & 2 \\ 3 & x \end{vmatrix} = x$

Solution

$$\begin{vmatrix} x & 2 \\ 3 & x \end{vmatrix} = x^2 - 6 = x$$

$$x^2 - x - 6 = 0$$

$\therefore$ ***Solution***: $\underline{x = -2,\ 3}$

Exercise

Solve for x. $\begin{vmatrix} 4 & 6 \\ -2 & x \end{vmatrix} = 32$

Solution

$$\begin{vmatrix} 4 & 6 \\ -2 & x \end{vmatrix} = 4x + 12 = 32$$

$$4x = 20$$

$\therefore$ ***Solution***: $\underline{x = 5}$

Exercise

Solve for x. $\begin{vmatrix} x+2 & -3 \\ x+5 & -4 \end{vmatrix} = 3x - 5$

Solution

$$\begin{vmatrix} x+2 & -3 \\ x+5 & -4 \end{vmatrix} = -4x - 8 + 3x + 15 = 3x - 5$$

$$-4x = -12$$

$\therefore$ ***Solution***: $\underline{x = 3}$

Exercise

Solve for x. $\begin{vmatrix} x+3 & -6 \\ x-2 & -4 \end{vmatrix} = 28$

Solution

$$\begin{vmatrix} x+3 & -6 \\ x-2 & -4 \end{vmatrix} = -4x - 12 + 6x - 12 = 28$$

$$2x = 52$$

$\therefore$ ***Solution***: $\underline{x = 26}$

Exercise

Solve for x. $\begin{vmatrix} x & -3 \\ -1 & x \end{vmatrix} \ge 0$

Solution

$$\begin{vmatrix} x & -3 \\ -1 & x \end{vmatrix} = x^2 - 3 \geq 0$$

$x^2 \geq 3$

$\therefore$ ***Solution***: $\underline{x \leq -\sqrt{3} \quad x \geq \sqrt{3}}$

Exercise

Solve for x. $\begin{vmatrix} 2 & x & 1 \\ 1 & 2 & -1 \\ 3 & 4 & -2 \end{vmatrix} = -6$

Solution

$$\begin{vmatrix} 2 & x & 1 \\ 1 & 2 & -1 \\ 3 & 4 & -2 \end{vmatrix} = -8 - 3x + 4 - 6 + 8 + 2x = -6$$

$-x = -4$

$\therefore$ ***Solution***: $\underline{x = 4}$

Exercise

Solve for x. $\begin{vmatrix} 1 & x & -3 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix} = 8$

Solution

$$\begin{vmatrix} 1 & x & -3 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix} = 2 + 18 + 2 - 6x = 8$$

$-6x = -14$

$\therefore$ ***Solution***: $\underline{x = \frac{7}{3}}$

Exercise

Solve for x. $\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix} = 39$

Solution

$$\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix} = 8 - 3 - 2 + 12x = 39$$

$$12x = 36$$

$\therefore$ ***Solution***: $\underline{x = 3}|$

Exercise

Solve for x. $\begin{vmatrix} x & 0 & 0 \\ 7 & x & 1 \\ 7 & 2 & 1 \end{vmatrix} = -1$

Solution

$$\begin{vmatrix} x & 0 & 0 \\ 7 & x & 1 \\ 7 & 2 & 1 \end{vmatrix} = x^2 - 2x = -1$$

$$x^2 - 2x + 1 = 0$$

$\therefore$ ***Solution***: $\underline{x = 1}|$

Exercise

Find the quadratic function $f(x) = ax^2 + bx + c$ for which $f(1) = -10,\quad f(-2) = -31,\quad f(2) = -19$. What is the function?

Solution

$$f(1) = a(1)^2 + b(1) + c \quad \Rightarrow \quad -10 = a + b + c$$

$$f(-2) = a(-2)^2 + b(-2) + c \quad \Rightarrow \quad -31 = 4a - 2b + c$$

$$f(2) = a(2)^2 + b(2) + c \quad \Rightarrow \quad -19 = 4a + 2b + c$$

$$\begin{cases} a + b + c = -10 \\ 4a - 2b + c = -31 \\ 4a + 2b + c = -19 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 4 & -2 & 1 \\ 4 & 2 & 1 \end{vmatrix} = 12 \qquad D_a = \begin{vmatrix} -10 & 1 & 1 \\ -31 & -2 & 1 \\ -19 & 2 & 1 \end{vmatrix} = -48$$

$$D_b = \begin{vmatrix} 1 & -10 & 1 \\ 4 & -31 & 1 \\ 4 & -19 & 1 \end{vmatrix} = 36 \qquad D_c = \begin{vmatrix} 1 & 1 & -10 \\ 4 & -2 & -31 \\ 4 & 2 & -19 \end{vmatrix} = -108$$

$$a = \frac{D_a}{D} = \frac{-48}{12} \underline{= -4}$$

$$b = \frac{D_b}{D} = \frac{36}{12} \underline{= 3}$$

$$c = \frac{D_c}{D} = \frac{-108}{12} \underline{= -9}$$

∴ ***Solution***: $\underline{f(x) = -x^2 + 3x - 9}$

Exercise

you wish to mix candy worth \$3.44 per pound with candy worth \$9.96 per pound to form 24 pounds of a mixture worth \$8.33 per pound.

a) Write the system equations?

b) How many pounds of each candy should you use?

Solution

Let x: total pounds of \$3.44 candy

y: total pounds of \$9.96 candy

a) $\begin{cases} x + y = 24 \\ 3.44x + 9.96y = 8.33(24) \end{cases}$

$\begin{cases} x + y = 24 \\ 344x + 996y = 19,992 \end{cases}$

$\begin{cases} x + y = 24 \\ 86x + 249y = 4,998 \end{cases}$

b) $D = \begin{vmatrix} 1 & 1 \\ 86 & 249 \end{vmatrix} = 163$

$D_x = \begin{vmatrix} 24 & 1 \\ 4998 & 249 \end{vmatrix} = 978 \qquad D_y = \begin{vmatrix} 1 & 24 \\ 86 & 4998 \end{vmatrix} = 2,934$

Total pounds of \$3.44 candy: $\frac{978}{163} \underline{= 6 \ \ lbs}$

Total pounds of \$9.96 candy: $\frac{2,934}{163} \underline{= 18 \ \ lbs}$

Exercise

Anne and Nancy use a metal alloy that is 17.76% copper to make jewelry. How many ounces of a 15% alloy must be mixed with a 19% alloy to form 100 ounces of the desired alloy?

Solution

Let x: total ounces 15%

y: total ounces of 19%

$$\begin{cases} x+y=100 \\ 15x+19y=17.76(100) \end{cases}$$

$$\begin{cases} x+y=100 \\ 15x+19y=1776 \end{cases}$$

$$D=\begin{vmatrix} 1 & 1 \\ 15 & 19 \end{vmatrix}=4 \qquad D_x=\begin{vmatrix} 100 & 1 \\ 1776 & 19 \end{vmatrix}=124$$

$\therefore$ Total ounces 15%: $\frac{124}{4}\ \underline{=31\ \ ounces}$

Exercise

A company makes 3 types of cable. Cable ***A*** requires 3 black, 3 white, and 2 red wires. ***B*** requires 1 black, 2 white, and 1 red. ***C*** requires 2 black, 1 white, and 2 red. They used 95 black, 100 white and 80 red wires.

a) Write the system equations?

b) How many of each cable were made?

Solution

Let x: Cable ***A***

y: Cable ***B***

z: Cable ***C***

a) $$\begin{cases} 3x+y+2z=95 \\ 3x+2y+z=100 \\ 2x+y+2z=80 \end{cases}$$

b) $$D=\begin{vmatrix} 3 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix}\begin{matrix} 3 & 1 \\ 3 & 2 \\ 2 & 1 \end{matrix}\ \underline{=3}$$

$$D_x=\begin{vmatrix} 95 & 1 & 2 \\ 100 & 2 & 1 \\ 80 & 1 & 2 \end{vmatrix}\begin{matrix} 95 & 1 \\ 100 & 2 \\ 80 & 1 \end{matrix}\ \underline{=45}$$

$$D_y = \begin{vmatrix} 3 & 95 & 2 \\ 3 & 100 & 1 \\ 2 & 80 & 2 \end{vmatrix} \begin{matrix} 3 & 95 \\ 3 & 100 \\ 2 & 80 \end{matrix} \underline{=60|}$$

$$D_z = \begin{vmatrix} 3 & 1 & 95 \\ 3 & 2 & 100 \\ 2 & 1 & 80 \end{vmatrix} \begin{matrix} 3 & 1 \\ 3 & 2 \\ 2 & 1 \end{matrix} \underline{=45|}$$

$x = \frac{45}{3} \underline{=15|}$ $\qquad x = \frac{D_x}{D}$

$y = \frac{60}{3} \underline{=20|}$ $\qquad y = \frac{D_y}{D}$

$z = \frac{45}{3} \underline{=15|}$ $\qquad z = \frac{D_z}{D}$

∴ ***Solution***: 15 cable ***A*** 20 cable ***B*** 15 cable ***C***

Exercise

A basketball fieldhouse seats 15,000. Courtside seats sell for \$8.00, end zone for \$6.00, and balcony for \$5.00. Total for a sell-out is \$86,000. If half the courtside and balcony and all end zone seats are sold, ticket sales total \$49,000.

a) Write the system equations?

b) How many of each type of seat are there?

Solution

Let x: Courtside seats

y: end zone

z: balcony

***a)** $$\begin{cases} x+y+z=15,000 \\ 8x+6y+5z=86,000 \\ \frac{1}{2}(8x)+6y+\frac{1}{2}(5z)=49,000 \end{cases}$$

$$\begin{cases} x+y+z=15,000 \\ 8x+6y+5z=86,000 \\ 8x+12y+5z=98,000 \end{cases}$$

b) $$D = \begin{vmatrix} 1 & 1 & 1 \\ 8 & 6 & 5 \\ 8 & 12 & 5 \end{vmatrix} \begin{matrix} 1 & 1 \\ 8 & 6 \\ 8 & 12 \end{matrix} \underline{=18|}$$

$$D_x = \begin{vmatrix} 15,000 & 1 & 1 \\ 86,000 & 6 & 5 \\ 98,000 & 12 & 5 \end{vmatrix} \begin{matrix} 15,000 & 1 \\ 86,000 & 6 \\ 98,000 & 12 \end{matrix} \underline{= 54,000|}$$

$$D_y = \begin{vmatrix} 1 & 15,000 & 1 \\ 8 & 86,000 & 5 \\ 8 & 98,000 & 5 \end{vmatrix} \begin{matrix} 1 & 15,000 \\ 8 & 86,000 \\ 8 & 98,000 \end{matrix} \underline{= 36,000|}$$

$$D_z = \begin{vmatrix} 1 & 1 & 15,000 \\ 8 & 6 & 86,000 \\ 8 & 12 & 98,000 \end{vmatrix} \begin{matrix} 1 & 1 \\ 8 & 6 \\ 8 & 12 \end{matrix} \underline{= 180,000|}$$

$$x = \frac{54,000}{18} \underline{= 3,000|} \qquad x = \frac{D_x}{D}$$

$$y = \frac{36,000}{18} \underline{= 2,000|} \qquad y = \frac{D_y}{D}$$

$$z = \frac{180,000}{18} \underline{= 10,000|} \qquad z = \frac{D_z}{D}$$

∴ ***Solution***: **3,000** Courtside **2,000** End zone **10,000** Balcony

Exercise

A movie theater charges $9.00 for adults and $7.00 for senior citizens. On a day when 325 people paid admission, the total receipts were $2,495.

a) Write the system equations?

b) How many who paid were adults? How many were seniors?

Solution

Let x: Adults

y: Senior citizens

a) $\begin{cases} x + y = 325 \\ 9x + 7y = 2,495 \end{cases}$

b) $D = \begin{vmatrix} 1 & 1 \\ 9 & 7 \end{vmatrix} \underline{= -2|}$

$$D_x = \begin{vmatrix} 325 & 1 \\ 2,495 & 7 \end{vmatrix} \underline{= -220|} \qquad D_y = \begin{vmatrix} 1 & 325 \\ 9 & 2,495 \end{vmatrix} \underline{= 430|}$$

$$x = \frac{220}{2} \underline{= 110|} \qquad x = \frac{D_x}{D}$$

$$y = \frac{430}{2} \underline{= 215|} \qquad y = \frac{D_y}{D}$$

∴ ***Solution***: **110** Adults **215** Senior citizens

Exercise

A Broadway theater has 500 seats, divided into orchestra, main, and balcony seating. Orchestra seats sell for \$150, main seats for \$135, and balcony seats for \$110. If all the seats are sold, the gross revenue to the theater is \$64,250. If all the main and balcony seats are sold, but only half the orchestra seats are sold, the gross revenue is \$56,750.

a) Write the system equations?

b) How many of each kind of seat are there?

Solution

Let x: Numbers of orchestra seats

y: Numbers of main seats

z: Numbers of balcony seats

a) $$\begin{cases} x+y+z=500 \\ 150x+135y+110z=64,250 \\ \frac{1}{2}(150)x+135y+110z=56,750 \end{cases}$$

$$\begin{cases} x+y+z=500 \\ 30x+27y+22z=12,850 \\ 15x+27y+22z=11,350 \end{cases}$$

b) $D=\begin{vmatrix} 1 & 1 & 1 \\ 30 & 27 & 22 \\ 15 & 27 & 22 \end{vmatrix} \underline{=75|}$ $\qquad D_x=\begin{vmatrix} 500 & 1 & 1 \\ 12850 & 27 & 22 \\ 11350 & 27 & 22 \end{vmatrix} \underline{=7,500|}$

$D_y=\begin{vmatrix} 1 & 500 & 1 \\ 30 & 12,850 & 22 \\ 15 & 11,350 & 22 \end{vmatrix} \underline{=15,750|}$ $\qquad D_z=\begin{vmatrix} 1 & 1 & 500 \\ 30 & 27 & 12,850 \\ 15 & 27 & 11,350 \end{vmatrix} \underline{=14,250|}$

$x=\dfrac{7,500}{75} \underline{=100|}$ $\qquad x=\dfrac{D_x}{D}$

$y=\dfrac{15,750}{75} \underline{=210|}$ $\qquad y=\dfrac{D_y}{D}$

$z=\dfrac{14,250}{75} \underline{=190|}$ $\qquad z=\dfrac{D_z}{D}$

∴ ***Solution***: There are **100** orchestra seats, **210** main seats, and **190** balcony seats.

Exercise

A movie theater charges \$11 for adults, \$6.50 for children, and \$9 for senior citizens. One day the theater sold 405 tickets and collected \$3,315 in receipts. Twice as many children's tickets were sold as adult tickets.

a) Write the system equations?

b) How many adults, children, and senior citizens went to the theater that day?

Solution

Let x: Numbers of adults

y: Numbers of children

z: Numbers of senior citizens

a) $$\begin{cases} x+y+z=405 \\ 11x+6.5y+9z=3,315 \\ y=2x \end{cases}$$

b) $$\begin{cases} 3x+z=405 \\ 24x+9z=3,315 \end{cases}$$

$$D=\begin{vmatrix} 3 & 1 \\ 24 & 9 \end{vmatrix} \underline{=3}$$

$$D_x=\begin{vmatrix} 405 & 1 \\ 3,315 & 9 \end{vmatrix} \underline{=330} \qquad D_y=\begin{vmatrix} 3 & 405 \\ 24 & 3,315 \end{vmatrix} \underline{=225}$$

$$x=\frac{330}{3} \underline{=110} \qquad x=\frac{D_x}{D}$$

$$z=\frac{225}{3} \underline{=75} \qquad z=\frac{D_z}{D}$$

$$y=2(110) \underline{=220}$$

∴ ***Solution***: There are **110** adults, **220** children, and **75** senior citizens.

Exercise

Emma has \$20,000 to invest. As her financial planner, you recommend that she diversify into three investements: Treasure bills that yield 5% simple interest. Treasury bonds tht yield 7% simple interest, and corporate bonds that yield 10% simple interest. Emma wishes to earn \$1,390 per year in income. Also, Emma wants her investment in Treasury bills to be \$3,000 more than her investment in corporate bonds. How much money should Emma place in each investement?

Solution

Let x: Amount in Treasure bills.

y: Amount in Treasury bonds.

z: Amount in corporate bonds.

$$\begin{cases} x+y+z=20,000 \\ .05x+.07y+.1z=1,390 \\ x=3,000+z \end{cases}$$

$$\begin{cases} x+y+z=20,000 \\ 5x+7y+10z=139,000 \\ x-z=3,000 \end{cases}$$

$$D=\begin{vmatrix} 1 & 1 & 1 \\ 5 & 7 & 10 \\ 1 & 0 & -1 \end{vmatrix} \underline{=1|}$$

$$D_x=\begin{vmatrix} 20,000 & 1 & 1 \\ 139,000 & 7 & 10 \\ 3,000 & 0 & -1 \end{vmatrix} \underline{=8,000|}$$

$$D_y=\begin{vmatrix} 1 & 20,000 & 1 \\ 5 & 139,000 & 10 \\ 1 & 3,000 & -1 \end{vmatrix} \underline{=7,000|}$$

$$D_z=\begin{vmatrix} 1 & 1 & 20,000 \\ 5 & 7 & 139,000 \\ 1 & 0 & 3,000 \end{vmatrix} \underline{=5,000|}$$

∴ ***Solution***: Emma should invest **$8,000** in Treasure bills

$7,000 in Treasury bonds

$5,000 in corporate bonds.

Exercise

A person invested $17,000 for one year, part at 10%, part at 12%, and the remainder at 15%. The total annual income from these investements was $2,110. The amount of money invested at 12% was $1,000 less than the amounts invested at 10% and 15% combined. Find the amount invested at each rate.

Solution

Let x = Amount invested at 10%
Let y = Amount invested at 12%
Let z = Amount invested at 15%

$$\begin{cases} x+y+z=17,000 \\ .1x+.12y+.15z=2,110 \\ y=x+z-1,000 \end{cases}$$

$$\begin{cases} x+y+z=17{,}000 \\ 10x+12y+15z=211{,}000 \\ x-y+z=1{,}000 \end{cases}$$

$$D=\begin{vmatrix} 1 & 1 & 1 \\ 10 & 12 & 15 \\ 1 & -1 & 1 \end{vmatrix} \underline{=10|}$$

$$D_x=\begin{vmatrix} 17{,}000 & 1 & 1 \\ 211{,}000 & 12 & 15 \\ 1{,}000 & -1 & 1 \end{vmatrix} \underline{=40{,}000|}$$

$$D_y=\begin{vmatrix} 1 & 17{,}000 & 1 \\ 10 & 211{,}000 & 15 \\ 1 & 1{,}000 & 1 \end{vmatrix} \underline{=80{,}000|}$$

$$D_z=\begin{vmatrix} 1 & 1 & 17{,}000 \\ 10 & 12 & 211{,}000 \\ 1 & -1 & 1{,}000 \end{vmatrix} \underline{=50{,}000|}$$

$$x=\frac{40{,}000}{10} \underline{=4{,}000|} \qquad x=\frac{D_x}{D}$$

$$y=\frac{80{,}000}{10} \underline{=8{,}000|} \qquad y=\frac{D_y}{D}$$

$$z=\frac{50{,}000}{10} \underline{=5{,}000|} \qquad z=\frac{D_z}{D}$$

∴ ***Solution***: should invest **\$4,000** invested at 10%

\$8,000 invested at 12%

\$5,000 invested at 15%.

Exercise

At a production, 400 tickets were sold. The ticket prices were \$8, \$10, and \$12, and the total income from ticket sales was \$3,700. How many tickets of each type were sold if the combined number of \$8 and \$10 tickets sold was 7 times the number of \$12 tickets sold?

Solution

Let x = Numbers of tickets sold at \$8

Let y = Numbers of tickets sold at \$10

Let z = Numbers of tickets sold at \$12

$$\begin{cases} x + y + z = 400 \\ 8x + 10y + 12z = 3{,}700 \\ x + y = 7z \end{cases}$$

$$\begin{cases} x + y + z = 400 \\ 4x + 5y + 6z = 1{,}850 \\ x + y - 7z = 0 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 5 & 6 \\ 1 & 1 & -7 \end{vmatrix} \underline{= -8}$$

$$D_x = \begin{vmatrix} 400 & 1 & 1 \\ 1{,}850 & 5 & 6 \\ 0 & 1 & -7 \end{vmatrix} \underline{= -1{,}600}$$

$$D_y = \begin{vmatrix} 1 & 400 & 1 \\ 4 & 1850 & 6 \\ 1 & 0 & -7 \end{vmatrix} \underline{= -1{,}200}$$

$$D_z = \begin{vmatrix} 1 & 1 & 400 \\ 4 & 5 & 1{,}850 \\ 1 & 1 & 0 \end{vmatrix} \underline{= -400}$$

$$x = \frac{1600}{8} \underline{= 200} \qquad\qquad x = \frac{D_x}{D}$$

$$y = \frac{1200}{8} \underline{= 150} \qquad\qquad y = \frac{D_y}{D}$$

$$z = \frac{400}{8} \underline{= 50} \qquad\qquad z = \frac{D_z}{D}$$

∴ ***Solution***: **200** tickets sold at \$8

150 tickets sold at \$10

50 tickets sold at \$12

Exercise

A certain brand of razor blades comes in packages if 6, 12, and 24 blades, costing \$2, \$3, and \$4 per package, respectively. A store sold 12 packages containing a total of 162 razor blades and took in \$35. How many packages of each type were sold?

Solution

Let x = Numbers of packages sold at \$2
Let y = Numbers of packages sold at \$3
Let z = Numbers of packages sold at \$4

$$\begin{cases} x+y+z=12 \\ 2x+3y+4z=35 \\ 6x+12y+24z=162 \end{cases}$$

$$\begin{cases} x+y+z=12 \\ 2x+3y+4z=35 \\ x+2y+4z=27 \end{cases}$$

$$D=\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & 2 & 4 \end{vmatrix} \underline{=1|}$$

$$D_x=\begin{vmatrix} 12 & 1 & 1 \\ 35 & 3 & 4 \\ 27 & 2 & 4 \end{vmatrix} \underline{=5|}$$

$$D_y=\begin{vmatrix} 1 & 12 & 1 \\ 2 & 35 & 4 \\ 1 & 27 & 4 \end{vmatrix} \underline{=3|}$$

$$D_z=\begin{vmatrix} 1 & 1 & 12 \\ 2 & 3 & 35 \\ 1 & 2 & 27 \end{vmatrix} \underline{=4|}$$

$$x=\frac{5}{1}\underline{=5|} \qquad x=\frac{D_x}{D}$$

$$y=\frac{3}{1}\underline{=3|} \qquad y=\frac{D_y}{D}$$

$$z=\frac{4}{1}\underline{=4|} \qquad z=\frac{D_z}{D}$$

∴ ***Solution***: **5** packages sold at \$2
3 packages sold at \$3
4 packages sold at \$4

Exercise

A store sells cashews for \$5.00 per pound and peanuts for \$1.50 per pound. The manager decides to mix 30 pounds of peanuts with some cashews and sell the mixture for \$3.00 per pound.

a) Write the system equations?

b) How many pounds of cashews should be mixed with peanuts so that the mixture will produce the same revenue as selling the nuts separately?

Solution

Let x: pounds of cashews

y: pounds of in the mixture

a) $\begin{cases} x+30=y \\ 5x+\frac{3}{2}(30)=3y \end{cases}$

$\begin{cases} x-y=-30 \\ 5x-3y=-45 \end{cases}$

b) $D=\begin{vmatrix} 1 & -1 \\ 5 & -3 \end{vmatrix} \underline{=2|}$

$D_x=\begin{vmatrix} -30 & -1 \\ -45 & -3 \end{vmatrix} \underline{=45|}$

$x=\frac{90}{7}$ $\qquad x=\frac{D_x}{D}$

$y=\frac{120}{7}$ $\qquad y=\frac{D_y}{D}$

∴ ***Solution***: $\frac{45}{2}=22.5$ pounds of cashews

Exercise

A wireless store takes presale orders for a new smartphone and tablet. He gets 340 preorders for the smartphone and 250 preorders for the tablet. The combined value of the preorders is \$270,500.00. If the price of a smartphone and tablet together is \$965, how much does each device cost?

Solution

Let x: Cost of a smartphone

y: Cost of a tablet

$\begin{cases} 340x+250y=270,500 \\ x+y=965 \end{cases}$

$\begin{cases} 34x+25y=27,050 \\ x+y=965 \end{cases}$

$$D = \begin{vmatrix} 34 & 25 \\ 1 & 1 \end{vmatrix} = 9$$

$$D_x = \begin{vmatrix} 27,050 & 25 \\ 965 & 1 \end{vmatrix} = 2,925 \qquad D_y = \begin{vmatrix} 34 & 27,050 \\ 1 & 965 \end{vmatrix} = 5,760$$

$$x = \frac{2,925}{9} = \$325 \qquad x = \frac{D_x}{D}$$

$$y = \frac{5,760}{9} = \$640 \qquad y = \frac{D_y}{D}$$

∴ ***Solution***: Cost of a smartphone is $325

Cost of a tablet is $640

Exercise

A restaurant manager wants to purchase 200 sets of dishes. One design costs \$25 per set, and another costs \$45 per set. If she has only \$7400 to spend, how many sets of each design should be order?

Solution

Let x: Number of sets for \$25 set.

y: Number of sets for \$45 set.

$$\begin{cases} 25x + 45y = 7,400 \\ x + y = 200 \end{cases}$$

$$\begin{cases} 5x + 9y = 1,480 \\ x + y = 200 \end{cases}$$

$$D = \begin{vmatrix} 5 & 9 \\ 1 & 1 \end{vmatrix} = -4$$

$$D_x = \begin{vmatrix} 1480 & 9 \\ 200 & 1 \end{vmatrix} = -320 \qquad D_y = \begin{vmatrix} 5 & 1480 \\ 1 & 200 \end{vmatrix} = -480$$

$$x = \frac{320}{4} = 80 \qquad x = \frac{D_x}{D}$$

$$y = \frac{480}{4} = 120 \qquad y = \frac{D_y}{D}$$

∴ ***Solution***: 80 sets for \$25 set.

120 sets for \$45 set.

Exercise

One group of people purchased 10 hot dogs and 5 soft drinks at a cost of \$35.00. A second bought 7 hot dogs and 4 soft drinks at a cost of \$25.25. What is the cost of a single hot dog and a single soft drink?

Solution

Let x: Cost of a hot dog.

y: Cost of a drink

$$\begin{cases} 10x + 5y = 35 \\ 7x + 4y = 25.25 \end{cases}$$

$$\begin{cases} 2x + y = 7 \\ 700x + 400y = 2,525 \end{cases}$$

$$D = \begin{vmatrix} 2 & 1 \\ 700 & 400 \end{vmatrix} = 100$$

$$D_x = \begin{vmatrix} 7 & 1 \\ 2,525 & 400 \end{vmatrix} = 275 \qquad D_y = \begin{vmatrix} 2 & 7 \\ 700 & 2,525 \end{vmatrix} = 150$$

$$x = \frac{275}{100} = 2.75 \qquad x = \frac{D_x}{D}$$

$$y = \frac{150}{100} = 1.5 \qquad y = \frac{D_y}{D}$$

∴ ***Solution***: Cost of a hot dog is \$2.75

Cost of a soft drink is \$1.50

Exercise

The sum of three times the first number, plus the second number, and twice the third number is 5. If 3 times the second number is subtracted from the sum of the first number and 3 times the third number, the result is 2. If the third number is subtracted from the sum of 2 times the first number and 3 times the second number, the result is 1. Find the three numbers.

Solution

Let x: be the first number.

y: be the second number.

z: be the third number.

$$\begin{cases} 3x + y + 2z = 5 \\ (x + 3z) - 3y = 2 \\ 2x + 3y - z = 1 \end{cases}$$

$$\begin{cases} 3x + y + 2z = 5 \\ x - 3y + 3z = 2 \\ 2x + 3y - z = 1 \end{cases}$$

$$D = \begin{vmatrix} 3 & 1 & 2 \\ 1 & -3 & 3 \\ 2 & 3 & -1 \end{vmatrix} \underline{= 7|} \qquad D_x = \begin{vmatrix} 5 & 1 & 2 \\ 2 & -3 & 3 \\ 1 & 3 & -1 \end{vmatrix} \underline{= -7|}$$

$$D_y = \begin{vmatrix} 3 & 5 & 2 \\ 1 & 2 & 3 \\ 2 & 1 & -1 \end{vmatrix} \underline{= 14|} \qquad D_z = \begin{vmatrix} 3 & 1 & 5 \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix} \underline{= 21|}$$

$$x = -\frac{7}{7} \underline{= -1|} \qquad x = \frac{D_x}{D}$$

$$y = \frac{14}{7} \underline{= 2|} \qquad y = \frac{D_y}{D}$$

$$z = \frac{21}{7} \underline{= 3|} \qquad z = \frac{D_z}{D}$$

∴ ***Solution***: The three numbers are: $-1,\ 2,\ and\ \ 3$

Exercise

The sum of three numbers is 16. The sum of twice the first number, 3 times the second number, and 4 times the third number is 46. The difference between 5 times the first number and the second number is 31. Find the three numbers.

Solution

Let x: be the first number.
y: be the second number.
z: be the third number.

$$\begin{cases} x + y + z = 16 \\ 2x + 3y + 4z = 46 \\ 5x - y = 31 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 5 & -1 & 0 \end{vmatrix} \underline{= 7|} \qquad D_x = \begin{vmatrix} 16 & 1 & 1 \\ 46 & 3 & 4 \\ 31 & -1 & 0 \end{vmatrix} \underline{= 49|}$$

$$D_y = \begin{vmatrix} 1 & 16 & 1 \\ 2 & 46 & 4 \\ 5 & 31 & 0 \end{vmatrix} \underline{= 28|} \qquad D_z = \begin{vmatrix} 1 & 1 & 16 \\ 2 & 3 & 46 \\ 5 & -1 & 31 \end{vmatrix} \underline{= 35|}$$

$$x = \frac{49}{7} \ \underline{= 7|} \qquad\qquad x = \frac{D_x}{D}$$

$$y = \frac{28}{7} \ \underline{= 4|} \qquad\qquad y = \frac{D_y}{D}$$

$$z = \frac{35}{7} \ \underline{= 5|} \qquad\qquad z = \frac{D_z}{D}$$

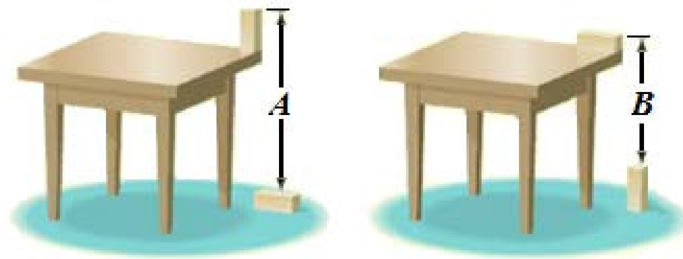
∴ ***Solution***: The three numbers are: 7, 4, *and* 5

Exercise

Two blocks of wood having the same length and width are placed on the top and bottom of a table. Length *A* measure 32 *cm*. The blocks are rearranged. Length *B* measures 28 *cm*. Determine the height of the table.

Solution

Let *h*: height of the table.
l: length of the block
w: width of the block



$$\begin{cases} (A) & h - w + l = 32 \\ (B) & h - l + w = 28 \end{cases}$$

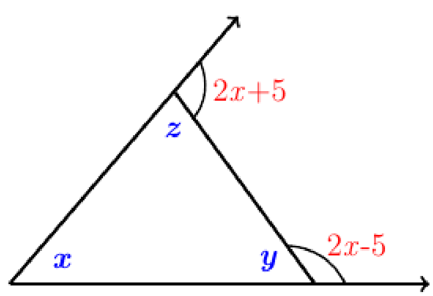
$$\overline{\quad 2h = 60 \quad}$$

∴ ***Solution***: The height of the table is **30** *cm*

Exercise

In the following triangle, the degree measures of the three interior angles and two of the exterior angles are represented with variables. Find the measure of each interior angle.

Solution



$$\begin{cases} x + y + z = 180 \\ z + 2x + 5 = 180 \\ y + 2x - 5 = 180 \end{cases}$$

$$\begin{cases} x + y + z = 180 \\ 2x \quad + z = 175 \\ 2x + y \quad = 185 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} \ \underline{= 3|}$$

$$D_x = \begin{vmatrix} 180 & 1 & 1 \\ 175 & 0 & 1 \\ 185 & 1 & 0 \end{vmatrix} = 180 \qquad D_y = \begin{vmatrix} 1 & 180 & 1 \\ 2 & 175 & 1 \\ 2 & 185 & 0 \end{vmatrix} = 195 \qquad D = \begin{vmatrix} 1 & 1 & 180 \\ 2 & 0 & 175 \\ 2 & 1 & 185 \end{vmatrix} = 165$$

$$x = \frac{180}{3} = 60^\circ \qquad x = \frac{D_x}{D}$$

$$y = \frac{195}{3} = 65^\circ \qquad y = \frac{D_y}{D}$$

$$z = \frac{165}{3} = 55^\circ \qquad z = \frac{D_z}{D}$$

Exercise

Three painters (Beth, Bill, and Edie), working together, can paint the exterior of a home in 10 *hours*. Bill and Edie together have painted similar house in 15 *hours*. One day, all three worked on this same kind of house for 4 *hours*, after which Edie left. Beth and Bill required 8 more *hours* to finish. Assuming no gain or loss in efficiency, how long should it take each person to complete such a job alone?

Solution

Let x: Beth's time

y: Bill's time

z: Edie's time

Let $\frac{1}{x} = a$: Beth's part of the job done in 1 *hour*.

$\frac{1}{y} = b$: Bill's part of the job done in 1 *hour*.

$\frac{1}{z} = c$: Edie's part of the job done in 1 *hour*.

All completed 1 job in 10 *hours*: $10\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 1$

Bill and Edie 1 job in 15 *hours*: $15\left(\frac{1}{y} + \frac{1}{z}\right) = 1$

All worked 1 job in 4 *hours* Beth and Bill required 8 *hours*: $4\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) + 8\left(\frac{1}{x} + \frac{1}{y}\right) = 1$

$$\begin{cases} 10a + 10b + 10c = 1 \\ 15b + 15c = 1 \\ 4a + 4b + 4c + 8a + 8b = 1 \end{cases}$$

$$\begin{cases} 10a + 10b + 10c = 1 \\ 15b + 15c = 1 \\ 12a + 12b + 4c = 1 \end{cases}$$

$$D = \begin{vmatrix} 10 & 10 & 10 \\ 0 & 15 & 15 \\ 12 & 12 & 4 \end{vmatrix} = -1200 \qquad D_a = \begin{vmatrix} 1 & 10 & 10 \\ 1 & 15 & 15 \\ 1 & 12 & 4 \end{vmatrix} = -40$$

$$D_b = \begin{vmatrix} 10 & 1 & 10 \\ 0 & 1 & 15 \\ 12 & 1 & 4 \end{vmatrix} = -50 \qquad D_c = \begin{vmatrix} 10 & 10 & 1 \\ 0 & 15 & 1 \\ 12 & 12 & 1 \end{vmatrix} = -30$$

$$a = \frac{40}{1200} = \frac{1}{30} \quad \to \quad x = \frac{1}{a} = 30$$

$$b = \frac{50}{1200} = \frac{1}{24} \quad \to \quad y = \frac{1}{b} = 24$$

$$c = \frac{30}{1200} = \frac{1}{40} \quad \to \quad z = \frac{1}{c} = 40$$

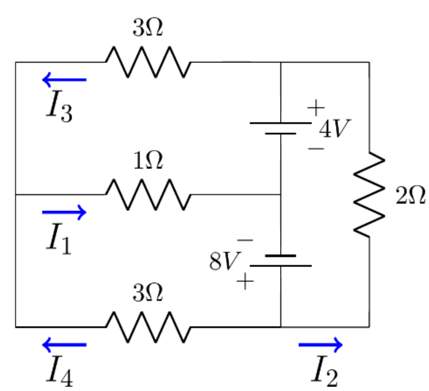
∴ ***Solution***: Took alone to complete a job: Beth **30** *hours*, Bill **24** *hours*, and Eddie **40** *hours*

Exercise

An application of Kirchhoff's Rules to the circuit shown results in the following system of equations:

$$\begin{cases} I_1 = I_3 + I_4 \\ I_1 + 5I_4 = 8 \\ I_1 + 3I_3 = 4 \\ 8 - 4 - 2I_2 = 0 \end{cases}$$

Find the currents I_1, I_2, I_3, and I_4



Solution

$$\begin{cases} I_1 - I_3 - I_4 = 0 \\ I_1 + 5I_4 = 8 \\ I_1 + 3I_3 = 4 \\ I_2 = 2 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & -1 \\ 1 & 0 & 5 \\ 1 & 3 & 0 \end{vmatrix} = -23$$

$$D_1 = \begin{vmatrix} 0 & -1 & -1 \\ 8 & 0 & 5 \\ 4 & 3 & 0 \end{vmatrix} = -44 \qquad D_3 = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 8 & 5 \\ 1 & 4 & 0 \end{vmatrix} = -16 \qquad D_4 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & 8 \\ 1 & 3 & 4 \end{vmatrix} = -28$$

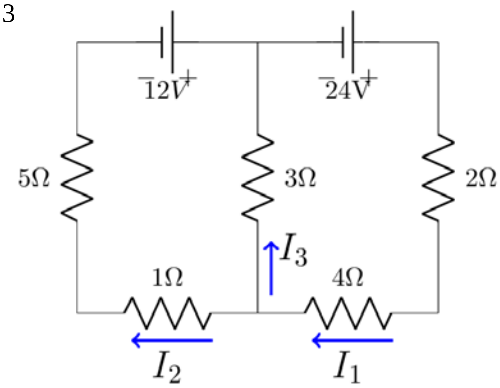
∴ ***Solution***: $I_1 = \frac{44}{23}$ $\quad I_2 = 2$ $\quad I_3 = \frac{16}{23}$ $\quad I_4 = \frac{28}{23}$

Exercise

An application of Kirchhoff's Rules to the circuit shown results in the following system of equations:

$$\begin{cases} I_1 = I_2 + I_3 \\ 24 - 6I_1 - 3I_3 = 0 \\ 12 + 24 - 6I_1 - 6I_2 = 0 \end{cases}$$

Find the currents I_1, I_2, and I_3



Solution

$$\begin{cases} I_1 - I_2 - I_3 = 0 \\ 2I_1 + I_3 = 8 \\ I_1 + I_2 = 6 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -4$$

$$D_1 = \begin{vmatrix} 0 & -1 & -1 \\ 8 & 0 & 1 \\ 6 & 1 & 0 \end{vmatrix} = -14 \qquad D_2 = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 8 & 1 \\ 1 & 6 & 0 \end{vmatrix} = -10 \qquad D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 0 & 8 \\ 1 & 1 & 6 \end{vmatrix} = -4$$

∴ ***Solution***: $I_1 = \frac{7}{2}$ $\quad I_2 = \frac{5}{2}$ $\quad I_3 = 1$

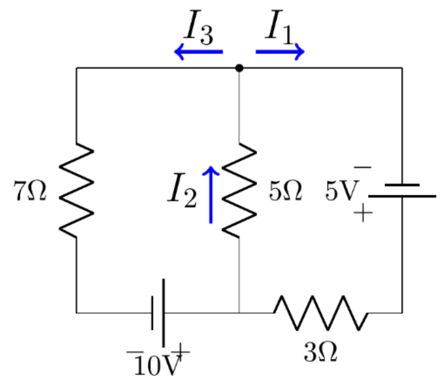
Exercise

An application of Kirchhoff's Rules to the circuit shown results in the following system of equations:

$$\begin{cases} I_2 = I_1 + I_3 \\ 5 - 3I_1 - 5I_2 = 0 \\ 10 - 5I_2 - 7I_3 = 0 \end{cases}$$

Find the currents I_1, I_2, and I_3

Solution



$$\begin{cases} -I_1 + I_2 - I_3 = 0 \\ 3I_1 + 5I_2 = 5 \\ 5I_2 + 7I_3 = 10 \end{cases}$$

$$D = \begin{vmatrix} -1 & 1 & -1 \\ 3 & 5 & 0 \\ 0 & 5 & 7 \end{vmatrix} = -71$$

$$D_1 = \begin{vmatrix} 0 & 1 & -1 \\ 5 & 5 & 0 \\ 10 & 5 & 7 \end{vmatrix} = -10 \qquad D_2 = \begin{vmatrix} -1 & 0 & -1 \\ 3 & 5 & 0 \\ 0 & 10 & 7 \end{vmatrix} = -65 \qquad D_3 = \begin{vmatrix} -1 & 1 & 0 \\ 3 & 5 & 5 \\ 0 & 5 & 10 \end{vmatrix} = -55$$

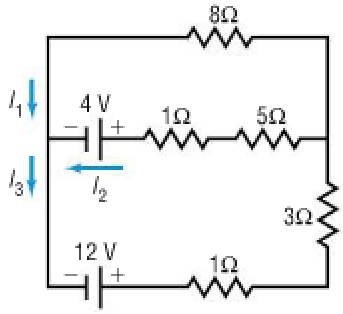
∴ ***Solution***: $I_1 = \frac{10}{71}$ $\quad I_2 = \frac{65}{71}$ $\quad I_3 = \frac{55}{71}$

Exercise

An application of Kirchhoff's Rules to the circuit shown results in the following system of equations:

$$\begin{cases} I_3 = I_1 + I_2 \\ 6I_2 + 4I_3 = 8 \\ 8I_1 = 4 + 6I_2 \end{cases}$$

Find the currents I_1, I_2, and I_3



Solution

$$\begin{cases} I_1 + I_2 - I_3 = 0 \\ 3I_2 + 2I_3 = 4 \\ 4I_1 - 3I_2 = 2 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 3 & 2 \\ 4 & -3 & 0 \end{vmatrix} = 26$$

$$D_1 = \begin{vmatrix} 0 & 1 & -1 \\ 4 & 3 & 2 \\ 2 & -3 & 0 \end{vmatrix} = 22 \qquad D_2 = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 4 & 2 \\ 4 & 2 & 0 \end{vmatrix} = 12 \qquad D_3 = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 3 & 4 \\ 4 & -3 & 2 \end{vmatrix} = 34$$

∴ ***Solution***: $I_1 = \frac{22}{26} = \frac{11}{13}$

$I_2 = \frac{12}{26} = \frac{6}{13}$

$I_3 = \frac{34}{26} = \frac{17}{13}$