Section 2.3 – Orthogonality

Definition

Two nonzero vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are said to be ***orthogonal*** (or ***perpendicular***) if their dot product is zero $\vec{u} \bullet \vec{v} = 0$.

We will also agree that he zero vector in $\mathbb{R}^n$ is orthogonal to every vector in $\mathbb{R}^n$. A nonempty set of vectors $\mathbb{R}^n$ is called an ***orthogonal set*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set of unit vectors is called an ***orthonormal set***.

Example

The floor of your room (extended to infinity) is a subspace ***V***. The line where two walls meet is a subspace ***W*** (one-dimensional). Those subspaces are orthogonal. Every vector up the meeting line is perpendicular to every vector on the floor. The origin (0, 0, 0) is in the corner.

Example

Show that $\vec{u} = (-2, 3, 1, 4)$ and $\vec{v} = (1, 2, 0, -1)$ are orthogonal in $\mathbb{R}^4$

Solution

$$\begin{aligned}\vec{u} \bullet \vec{v} &= (-2)(1) + (3)(2) + (1)(0) + (4)(-1)\\ &= -2 + 6 + 0 - 4\\ &= 0\end{aligned}$$

These vectors are orthogonal in $\mathbb{R}^4$

Standard Unit Vectors

$$\hat{i} \bullet \hat{j} = \hat{i} \bullet \hat{k} = \hat{j} \bullet \hat{k} = \mathbf{0}$$

Proof

$$\begin{aligned}\hat{i} \bullet \hat{j} &= (1,\ 0,\ 0) \bullet (0,\ 1,\ 0)\\ &= 0\end{aligned}$$

Normal

To specify slope and inclination is to use a nonzero vector $\vec{n}$, called a ***normal***, that is orthogonal to the line or plane.

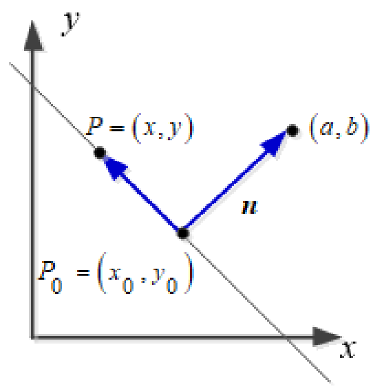
The line passes through a point $P_0\left(x_0, y_0\right)$ that has a normal $\vec{n} = (a, b)$

The plane through $P_0\left(x_0, y_0, z_0\right)$ that has a normal $\vec{n} = (a, b, c)$.

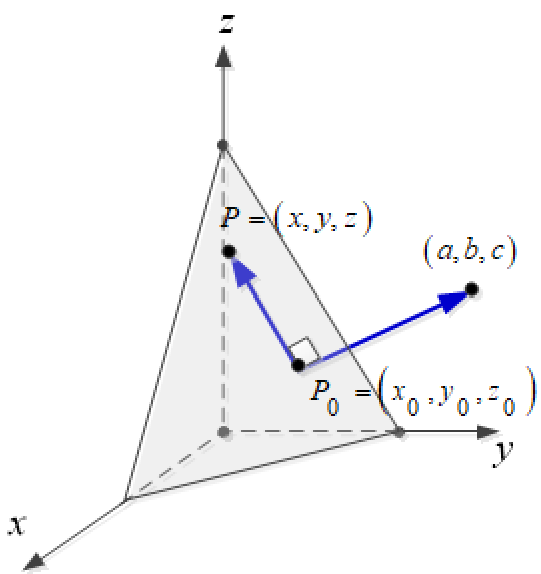
Both the line and the plane are represented by the vector equation

$$\vec{n} \bullet \overrightarrow{P_0 P} = 0$$

The line equation: $\quad a\left(x - x_0\right) + b\left(y - y_0\right) = 0$



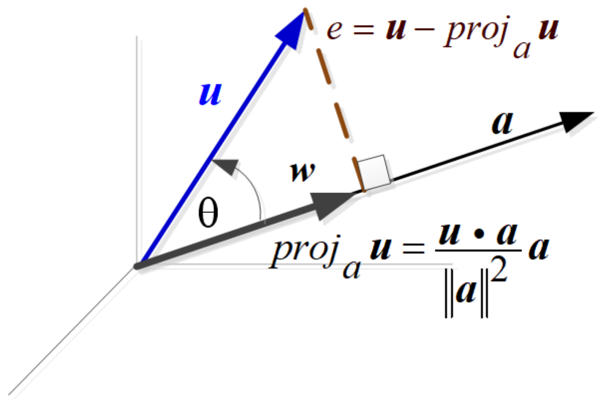
The plane equation: $\quad a\left(x - x_0\right) + b\left(y - y_0\right) + c\left(z - z_0\right) = 0$



Projections

Theorem Projection onto a line

If $\vec{u}$ and $\vec{a}$ are vectors in $\mathbb{R}^n$, and if $\vec{a} \neq 0$, then $\vec{u}$ can be expressed in exactly one way in the form $\vec{u} = \vec{w} + \vec{e}$, where $\vec{w}$ is a scalar multiple of $\vec{a}$ and $\vec{e}$ is orthogonal to $\vec{a}$.



The vector $\vec{w}$ is called the ***orthogonal projection*** of $\vec{u}$ on $\vec{a}$ or sometimes ***component*** of $\vec{u}$ along $\vec{a}$. The vector $\vec{e}$ is called the vector ***component*** of $\vec{u}$ ***orthogonal*** to $\vec{a}$ (error vector and should be perpendicular to $\vec{a}$)

$$proj_{\vec{a}} \vec{u} = \frac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|^2} \vec{a} = \vec{p} \qquad (\textit{vector component of } \vec{u} \textit{ along } \vec{a})$$

$$\vec{u} - proj_{\vec{a}} \vec{u} = \vec{u} - \frac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|^2} \vec{a} \qquad (\textit{vector component of } \vec{u} \textit{ orthogonal to } \vec{a})$$

The length is $\left\| proj_{\vec{a}} \vec{u} \right\| = \|\vec{u}\| |\cos\theta|$

$$\left\| proj_{\vec{a}} \vec{u} \right\| = \frac{|\vec{u} \bullet \vec{a}|}{\|\vec{a}\|}$$

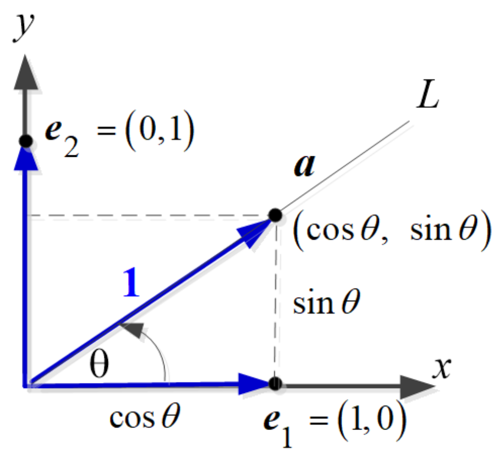
Special case: If $\vec{u} = \vec{a}$ then $\dfrac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|^2} = 1$. The projection of $\vec{a}$ onto $\vec{a}$ is itself.

Special case: If $\vec{u}$ is perpendicular to $\vec{a}$ then $\vec{u} \bullet \vec{a} = 0$. The projection is $\vec{p} = \vec{0}$.

Example

Find the orthogonal projections of the vectors $\hat{e}_1 = (1,\ 0)$ and $\hat{e}_2 = (0,\ 1)$ on the line L that makes an angle θ with the positive x-axis in $\mathbb{R}^2$

Solution



Let $\vec{a} = (\cos\theta,\ \sin\theta)$ be the unit vector along the line L.

$$\|\vec{a}\| = \sqrt{\cos^2\theta + \sin^2\theta}$$
$$= 1$$

$$\hat{e}_1 \bullet \vec{a} = (1,0)(\cos\theta, \sin\theta)$$
$$= (1)\cos\theta + (0)\sin\theta$$
$$= \cos\theta$$

$$proj_{\vec{a}}\hat{e}_1 = \frac{\hat{e}_1 \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a}$$
$$= \frac{\cos\theta}{1}(\cos\theta,\ \sin\theta)$$
$$= \left(\cos^2\theta,\ \cos\theta\sin\theta\right)$$

$$proj_{\vec{a}}\hat{e}_2 = \frac{\hat{e}_2 \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a}$$
$$= \frac{(0,\ 1)(\cos\theta,\ \sin\theta)}{1}(\cos\theta,\ \sin\theta)$$
$$= \sin\theta(\cos\theta,\ \sin\theta)$$
$$= \left(\sin\theta\cos\theta,\ \sin^2\theta\right)$$

Example

Let $\vec{u} = (2, -1, 3)$ and $\vec{a} = (4, -1, 2)$. Find the vector component of $\vec{u}$ along $\vec{a}$ and the vector component of $\vec{u}$ orthogonal to $\vec{a}$.

Solution

$$\begin{aligned} proj_{\vec{a}}\vec{u} &= \frac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a} \\ &= \frac{(2,\ -1,\ 3)\bullet(4,\ -1,\ 2)}{\left(\sqrt{4^2+(-1)^2+2^2}\right)^2}(4,\ -1,\ 2) \\ &= \frac{8+1+6}{21}(4,\ -1,\ 2) \\ &= \frac{15}{21}(4,\ -1,\ 2) \\ &= \frac{5}{7}(4,\ -1,\ 2) \\ &= \left(\frac{20}{7},\ -\frac{5}{7},\ \frac{10}{7}\right) \end{aligned}$$

The vector component of $\vec{u}$ orthogonal to $\vec{a}$ is

$$\begin{aligned} \vec{u} - proj_{\vec{a}}\vec{u} &= (2,\ -1,\ 3) - \left(\frac{20}{7},\ -\frac{5}{7},\ \frac{10}{7}\right) \\ &= \left(-\frac{6}{7},\ -\frac{2}{7},\ \frac{11}{7}\right) \end{aligned}$$

***Theorem* of *Pythagoras* in $\mathbb{R}^n$**

If $\vec{u}$ and $\vec{v}$ are orthogonal vectors in $\mathbb{R}^n$ with the Euclidean inner product, then

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$

Proof

Since $\vec{u}$ and $\vec{v}$ are orthogonal, then $\vec{u} \bullet \vec{v} = 0$

$$\begin{aligned} \|\vec{u}+\vec{v}\|^2 &= (\vec{u}+\vec{v}) \bullet (\vec{u}+\vec{v}) \\ &= \|\vec{u}\|^2 + 2(\vec{u}\bullet\vec{v}) + \|\vec{v}\|^2 \\ &= \|\vec{u}\|^2 + \|\vec{v}\|^2 \end{aligned}$$

Distance

Theorem

In $\mathbb{R}^2$ the distance D between the point $P_0 = (x_0,\ y_0)$ and the line $ax + by + c = 0$ is

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

In $\mathbb{R}^3$ the distance D between the point $P_0 = (x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Exercises Section 2.3 – Orthogonality

1. Determine whether $\vec{u}$ and $\vec{v}$ are orthogonal

a) $\vec{u}=(-6,\ -2),\quad \vec{v}=(5,\ -7)$

b) $\vec{u}=(6,\ 1,\ 4),\quad \vec{v}=(2,\ 0,\ -3)$

c) $\vec{u}=(1,\ -5,\ 4),\quad \vec{v}=(3,\ 3,\ 3)$

d) $\vec{u}=(-2,\ 2,\ 3),\quad \vec{v}=(1,\ 7,\ -4)$

2. Determine whether the vectors form an orthogonal set

a) $\vec{v}_1=(2,\ 3),\quad \vec{v}_2=(3,\ 2)$

b) $\vec{v}_1=(1,\ -2),\quad \vec{v}_2=(-2,\ 1)$

c) $\vec{u}=(-4,\ 6,\ -10,\ 1)\quad \vec{v}=(2,\ 1,\ -2,\ 9)$

d) $\vec{u}=(a,\ b)\quad \vec{v}=(-b,\ a)$

e) $\vec{v}_1=(-2,\ 1,\ 1),\quad \vec{v}_2=(1,\ 0,\ 2),\quad \vec{v}_3=(-2,\ -5,\ 1)$

f) $\vec{v}_1=(1,\ 0,\ 1),\quad \vec{v}_2=(1,\ 1,\ 1),\quad \vec{v}_3=(-1,\ 0,\ 1)$

g) $\vec{v}_1=(2,\ -2,\ 1),\quad \vec{v}_2=(2,\ 1,\ -2),\quad \vec{v}_3=(1,\ 2,\ 2)$

3. Find a unit vector that is orthogonal to both $\vec{u}=(1, 0, 1)$ and $\vec{v}=(0, 1, 1)$

4. *a*) Show that $\vec{v}=(a, b)$ and $\vec{w}=(-b, a)$ are orthogonal vectors.

b) Use the result to find two vectors that are orthogonal to $\vec{v}=(2, -3)$.

c) Find two unit vectors that are orthogonal to $(-3, 4)$

5. Find the vector component of $\vec{u}$ along $\vec{a}$ and the vector component of $\vec{u}$ orthogonal to $\vec{a}$.

a) $\vec{u}=(6,\ 2),\quad \vec{a}=(3,\ -9)$

b) $\vec{u}=(3,\ 1,\ -7),\quad \vec{a}=(1,\ 0,\ 5)$

c) $\vec{u}=(1,\ 0,\ 0),\quad \vec{a}=(4,\ 3,\ 8)$

d) $\vec{u}=(1,\ 1,\ 1),\quad \vec{a}=(0,\ 2,\ -1)$

e) $\vec{u}=(2,\ 1,\ 1,\ 2),\quad \vec{a}=(4,\ -4,\ 2,\ -2)$

f) $\vec{u}=(5,\ 0,\ -3,\ 7),\quad \vec{a}=(2,\ 1,\ -1,-1)$

6. Project the vector $\vec{v}$ onto the line through $\vec{a}$, check that $\vec{e}=\vec{u}-proj_{\vec{a}}\vec{u}$ is perpendicular to $\vec{a}$:

a) $\vec{v}=\begin{pmatrix}1\\2\\2\end{pmatrix}$ *and* $\vec{a}=\begin{pmatrix}1\\1\\1\end{pmatrix}$

b) $\vec{v}=\begin{pmatrix}1\\3\\1\end{pmatrix}$ *and* $\vec{a}=\begin{pmatrix}-1\\-3\\-1\end{pmatrix}$

c) $\vec{v}=\begin{pmatrix}1\\1\\1\end{pmatrix}$ *and* $\vec{a}=\begin{pmatrix}1\\2\\2\end{pmatrix}$

7. Find the projection matrix $proj_{\vec{a}}\vec{u}=\dfrac{\vec{u}\bullet\vec{a}}{\|\vec{a}\|^2}\vec{a}$ onto the line through $\vec{a}=\begin{bmatrix}1\\2\\2\end{bmatrix}$

(8 – 9) Draw the projection of $\vec{b}$ onto $\vec{a}$ and also compute it from $proj_{\vec{a}}\vec{u} = \dfrac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a}$

8. $\vec{b} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ *and* $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

9. $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and* $\vec{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

10. Show that if $\vec{v}$ is orthogonal to both $\vec{w}_1$ and $\vec{w}_2$, then $\boldsymbol{v}$ is orthogonal to $k_1\vec{w}_1 + k_2\vec{w}_2$ for all scalars k_1 and k_2.

11. *a*) Project the vector $\vec{v} = (3, 4, 4)$ onto the line through $\vec{a} = (2, 2, 1)$ and then onto the plane that also contains $\vec{a}^* = (1, 0, 0)$.

b) Check that the first error vector $\vec{v} - \vec{p}$ is perpendicular to $\vec{a}$, and the second error vector $\vec{v} - \vec{p}^*$ is also perpendicular to $\vec{a}^*$.

12. Compute the projection matrices $\vec{a}\vec{a}^T / \vec{a}^T\vec{a}$ onto the lines through $\vec{a}_1 = (-1, 2, 2)$ and $\vec{a}_2 = (2, 2, -1)$. Multiply those projection matrices and explain why their product P_1P_2 is what it is. Project $\vec{v} = (1, 0, 0)$ onto the lines $\vec{a}_1$, $\vec{a}_2$, and also onto $\vec{a}_3 = (2, -1, 2)$.Add up the three projections $p_1 + p_2 + p_3$.

13. If $P^2 = P$ show that $(I - P)^2 = I - P$. When P projects onto the column space of A, $I - P$ projects onto the ____.

14. What linear combination of $(1, 2, -1)$ and $(1, 0, 1)$ is closest to $\vec{v} = (2, 1, 1)$?

15. Show that $\vec{u} - \vec{v}$ is orthogonal to $\vec{u} + \vec{v}$ if and only if $\|\vec{u}\| = \|\vec{v}\|$

16. Given $\vec{u} = (3, -1, 2)$ $\vec{v} = (4, -1, 5)$ *and* $\vec{w} = (8, -7, -6)$

a) Find $3\vec{v} - 4(5\vec{u} - 6\vec{w})$

b) Find $\vec{u} \bullet \vec{v}$ and then the angle θ between $\vec{u}$ and $\vec{v}$.

17. Given: $\vec{u} = (3, 1, 3)$ $\vec{v} = (4, 1, -2)$

a) Compute the projection $\vec{w}$ of $\vec{u}$ on $\vec{v}$

b) Find $\vec{p} = \vec{u} - \vec{v}$ and show that $\vec{p}$ is perpendicular to $\vec{v}$.

18. *a*) Show that $\vec{v} = (a, b)$ and $\vec{w} = (-b, a)$ are orthogonal vectors

b) Use the result in part (*a*) to find two vectors that are orthogonal to $\vec{v} = (2, -3)$

c) Find two unit vectors that are orthogonal to $(-3, 4)$

19. Show that A (3, 0, 2), B (4, 3, 0), and C (8, 1, −1) are vertices of a right triangle. At which vertex is the right angle?

20. Establish the identity: $\vec{u} \bullet \vec{v} = \frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2$

21. Find the Euclidean inner product $\vec{u} \bullet \vec{v}$: $\vec{u} = (-1,\ 1,\ 0,\ 4,\ -3)$ $\vec{v} = (-2,\ -2,\ 0,\ 2,\ -1)$

22. Find the Euclidean distance between $\vec{u}$ and $\vec{v}$: $\vec{u} = (3,\ -3,\ -2,\ 0,\ -3)$ $\vec{v} = (-4,\ 1,\ -1,\ 5,\ 0)$

(*Exercises* **22 – 26**) Find

- *a)* $\vec{v} \bullet \vec{u},\ |\vec{v}|,\ |\vec{u}|$
- *b)* The cosine of the angle between $\vec{v}$ and $\vec{u}$
- *c)* The scalar component of $\vec{u}$ in the direction of $\vec{v}$
- *d)* The vector $proj_{\vec{v}}\vec{u}$

23. $\vec{v} = 2\hat{i} - 4\hat{j} + \sqrt{5}\,\hat{k}, \quad \vec{u} = -2\hat{i} + 4\hat{j} - \sqrt{5}\,\hat{k}$

24. $\vec{v} = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{k}, \quad \vec{u} = 5\hat{i} + 12\hat{j}$

25. $\vec{v} = 2\hat{i} + 10\hat{j} - 11\hat{k}, \quad \vec{u} = 2\hat{i} + 2\hat{j} + \hat{k}$

26. $\vec{v} = 5\hat{i} + \hat{j}, \quad \vec{u} = 2\hat{i} + \sqrt{17}\,\hat{j}$

27. $\vec{v} = \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}}\right), \quad \vec{u} = \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{3}}\right)$

28. Suppose Ted weighs 180 *lb.* and he is sitting on an inclined plane that drops 3 *units* for every 4 horizontal units. The gravitational force vector is $\vec{F}_g = \begin{pmatrix} 0 \\ -180 \end{pmatrix}$.

- *a)* Find the force pushing Ted down the slope.
- *b)* Find the force acting to hold Ted against the slope

29. Prove that is two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$ are orthogonal to nonzero vector $\vec{w}$ in $\mathbb{R}^2$, then $\vec{u}$ and $\vec{v}$ are scalar multiples of each other.