SOLUTION Section 3.7 – Power Series

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of *x* does the series converge (***b***) absolutely, (***c***) conditionally?

$$\sum_{n=0}^{\infty} x^n$$

Solution

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{x^n}\right| = |x| < 1 \quad \Rightarrow \quad -1 < x < 1$$

When $x = 1 \quad \Rightarrow \quad \sum_{n=0}^{\infty} 1$ and $x = -1 \quad \Rightarrow \quad \sum_{n=0}^{\infty} (-1)^n$ the series diverges.

a) The radius is 1; the interval of converges $-1 < x < 1$

b) The interval of absolute convergence is $-1 < x < 1$

c) There are no values for which the series converges conditionally

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of *x* does the series converge (***b***) absolutely, (***c***) conditionally?

$$\sum_{n=0}^{\infty} (x+5)^n$$

Solution

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(x+5)^{n+1}}{(x+5)^n}\right| = |x+5| < 1 \quad \Rightarrow \quad -6 < x < -4$$

When $x = -6 \quad \Rightarrow \quad \sum_{n=0}^{\infty} (-1)^n$ and $x = -4 \quad \Rightarrow \quad \sum_{n=0}^{\infty} 1$ the series diverges.

a) The radius is 1; the interval of converges $-6 < x < -4$

b) The interval of absolute convergence is $-6 < x < -4$

c) There are no values for which the series converges conditionally.

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of *x* does the series converge (***b***) absolutely, (***c***) conditionally?

$$\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$$

Solution

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(3x-2)^{n+1}}{n+1} \cdot \frac{n}{(3x-2)^n}\right| = \frac{n}{n+1}|3x-2| < 1$$

$$\lim_{n\to\infty} \frac{n}{n+1}|3x-2| < 1 \;\Rightarrow\; |3x-2| < 1$$

$$-1 < 3x-2 < 1 \;\Rightarrow\; 1 < 3x < 3 \;\Rightarrow\; \boxed{\frac{1}{3} < x < 1}$$

When $x = \frac{1}{3} \;\Rightarrow\; \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ which is the alternating harmonic series and is conditionally convergent.

$x = 1 \;\Rightarrow\; \sum_{n=0}^{\infty} \frac{1}{n}$ the series diverges harmonic.

a) The radius is $\frac{1}{3}$; the interval of converges $\frac{1}{3} \le x < 1$

b) The interval of absolute convergence is $\frac{1}{3} < x < 1$

c) The series converges conditionally at $x = \frac{1}{3}$

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of *x* does the series converge (***b***) absolutely, (***c***) conditionally?

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$$

Solution

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(x-2)^{n+1}}{10^{n+1}} \cdot \frac{10^n}{(x-2)^n}\right|$$

$$= \frac{|x-2|}{10} < 1$$

$$-1 < \frac{x-2}{10} < 1$$

$$-10 < x-2 < 10$$

$-8 < x < 12$

When $x = -8 \quad \Rightarrow \quad \sum_{n=0}^{\infty} (-1)^n$ which is a divergent series

$x = 12 \quad \Rightarrow \quad \sum_{n=0}^{\infty} 1$ the series diverges

a) The radius is 10; the interval of converges $-8 < x < 12$

b) The interval of absolute convergence is $-8 < x < 12$

c) There are no values for which the series converges conditionally

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of *x* does the series converge (***b***) absolutely, (***c***) conditionally?

$$\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$$

Solution

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{3^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n x^n}\right|$$

$$= \frac{3}{n+1}\left|x^n\right| < 1$$

$$3|x| \lim_{n\to\infty} \frac{1}{n+1} < 1 \quad \Rightarrow \quad \forall x$$

a) The radius is ∞; the series converges for all *x*.

b) The series convergence absolutely for all *x*.

c) There are no values for which the series converges conditionally

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of *x* does the series converge (***b***) absolutely, (***c***) conditionally?

$$\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2+3}}$$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{\sqrt{(n+1)^2+3}}\cdot\frac{\sqrt{n^2+3}}{x^n}\right|$$

$$= |x| \lim_{n\to\infty} \frac{n^2+3}{n^2+2n+4} < 1$$

$$|x| < 1 \quad \Rightarrow \quad -1 < x < 1$$

When $x = -1 \quad \Rightarrow \quad \sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n^2+3}}$ which is a convergent conditionally series

$x = 1 \quad \Rightarrow \quad \sum_{n=0}^{\infty}\frac{1}{\sqrt{n^2+3}}$ the series diverges

a) The radius is 1; the series converges for $-1 \le x < 1$.

b) The series convergence absolutely for $-1 < x < 1$.

c) The series convergence conditionally for $x = -1$

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of x does the series converge (***b***) absolutely, (***c***) conditionally?

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{\sqrt{n+3}}$$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+2}}{\sqrt{n+1+3}}\cdot\frac{\sqrt{n+3}}{x^{n+1}}\right|$$

$$= |x| \lim_{n\to\infty} \frac{\sqrt{n+3}}{\sqrt{n+1+3}} < 1$$

$$|x| < 1 \quad \Rightarrow \quad -1 < x < 1$$

When $x = -1 \quad \Rightarrow \quad \sum_{n=0}^{\infty}\frac{1}{\sqrt{n+3}}$ which is a divergent series

$x = 1 \quad \Rightarrow \quad \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n+3}}$ the series converges conditionally

a) The radius is 1; the series converges for $-1 < x \le 1$.

b) The series convergence absolutely for $-1 < x < 1$.

c) The series convergence conditionally for $x = 1$

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of *x* does the series converge (***b***) absolutely, (***c***) conditionally?

$$\sum_{n=1}^{\infty} \sqrt[n]{n}\,(2x+5)^n$$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt[n+1]{n+1}\,(2x+5)^{n+1}}{\sqrt[n]{n}\,(2x+5)^n}\right|$$

$$= |2x+5| \lim_{n\to\infty} \frac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}} < 1$$

$$= |2x+5| \frac{\lim\limits_{m\to\infty} \sqrt[m]{m}}{\lim\limits_{n\to\infty} \sqrt[n]{n}} < 1$$

$$= |2x+5| < 1$$

$$|2x+5| < 1 \quad \Rightarrow \quad -1 < 2x+5 < 1$$

$$-3 < x < -2$$

When $x = -3 \quad \Rightarrow \quad \sum_{n=0}^{\infty} (-1)^n \sqrt[n]{n}$ which is a divergent series

$x = -0 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \sqrt[n]{n}$ which is a divergent series

a) The radius is 1; the series converges for $-3 < x < -2$.

b) The series convergence absolutely for $-3 < x < -2$.

c) There are no values for which the series convergence conditionally

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of *x* does the series converge (***b***) absolutely, (***c***) conditionally?

$$\sum_{n=1}^{\infty}\left(2+(-1)^n\right)\cdot(x+1)^{n-1}$$

Solution

$$\sum_{n=1}^{\infty}\left(2+(-1)^n\right)\cdot(x+1)^{n-1}=\sum_{n=1}^{\infty}2(x+1)^{n-1}+\sum_{n=1}^{\infty}(-1)^n(x+1)^{n-1}$$

For the series $\sum_{n=1}^{\infty}2(x+1)^{n-1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{2(x+1)^n}{2(x+1)^{n-1}}\right|$$

$$=|x+1|<1$$

$$-1<x+1<1 \quad\rightarrow\quad -2<x<0$$

For the series $\sum_{n=1}^{\infty}(-1)^n(x+1)^{n-1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{(x+1)^n}{(x+1)^{n-1}}\right|$$

$$=|x+1|<1$$

$$-1<x+1<1 \quad\rightarrow\quad -2<x<0$$

When $x=-2 \quad\Rightarrow\quad \sum_{n=0}^{\infty}\left(2+(-1)^n\right)\cdot(-1)^{n-1}$ which is a divergent series

$x=0 \quad\Rightarrow\quad \sum_{n=0}^{\infty}\left(2+(-1)^n\right)$ which is a divergent series

a) The radius is $\frac{1}{2}$; the series converges for $-2<x<0$.

b) The series convergence absolutely for $-2<x<0$.

c) There are no values for which the series convergence conditionally

Exercise

Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} n!x^n$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right|$$

$$= |x|\lim_{n\to\infty}(n+1)$$

$$\underline{= \infty}$$

$$\to \quad R = \frac{1}{\infty} = 0$$

By the Ratio Test, the series diverges for $|x| > 0$ and converges only at its center, 0.

Therefore; the radius of convergence is $R = 0$.

Exercise

Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} 3(x-2)^n$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{3(x-2)^{n+1}}{3(x-2)^n}\right|$$

$$\underline{= |x-2|}$$

By the Ratio Test, the series converges for $|x-2| < 1$ and diverges for $|x-2| > 1$.

Therefore; the radius of convergence is $R = 1$.

Exercise

Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right|$$

$$= x^2\lim_{n\to\infty}\frac{1}{(2n+2)(2n+3)}$$

$$\underline{= 0}$$

$$\rightarrow \quad R = \frac{1}{0} = \infty$$

By the Ratio Test, the series converges for all x. Therefore; the radius of convergence is $R = \infty$.

Exercise

Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$

Solution

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{n+2} \cdot \frac{n+1}{x^n} \right|$$

$$= |x| \lim_{n\to\infty} \frac{n+1}{n+2}$$

$$\underline{= |x|}$$

$$\rightarrow \quad R = 1$$

By the Ratio Test, the series converges for $|x| < 1$ and diverges for $|x| > 1$.

Therefore; the radius of convergence is $R = 1$.

Exercise

Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} (3x)^n$

Solution

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{(3x)^{n+1}}{(3x)^n} \right|$$

$$\underline{= 3|x|}$$

$$\rightarrow \quad 3|x| < 1 \Rightarrow R = \frac{1}{3}$$

By the Ratio Test, the series converges for $|x| < \frac{1}{3}$ and diverges for $|x| > \frac{1}{3}$.

Therefore; the radius of convergence is $R = \frac{1}{3}$.

Exercise

Find the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}\frac{(4x)^n}{n^2}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(4x)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(4x)^n}\right|$$

$$= |4x| \lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\right|$$

$$\underline{= 4|x|}$$

$$\to \quad 4|x| < 1 \;\Rightarrow\; R = \frac{1}{4}$$

By the Ratio Test, the series converges for $|x| < \frac{1}{4}$ and diverges for $|x| > \frac{1}{4}$.

Therefore; the radius of convergence is $R = \frac{1}{4}$.

Exercise

Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{5^n}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{5^{n+1}}\cdot\frac{5^n}{x^n}\right|$$

$$\underline{= \frac{|x|}{5}}$$

$$\to \quad \frac{|x|}{5} < 1 \;\Rightarrow\; R = 5$$

By the Ratio Test, the series converges for $|x| < 5$ and diverges for $|x| > 5$.

Therefore; the radius of convergence is $R = 5$.

Exercise

Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right|$$

$$= x^2 \lim_{n\to\infty} \frac{1}{(2n+1)(2n+1)}$$

$$\underline{= 0}$$

$$R = \frac{1}{0} = \infty$$

By the Ratio Test, the series converges for all x. Therefore; the radius of convergence is $R = \infty$.

Exercise

Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!x^{2n}}{n!}$

Solution

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(2n+2)!\, x^{2n+2}}{(n+1)!} \cdot \frac{n!}{(2n)!x^{2n}}\right|$$

$$= x^2 \lim_{n\to\infty} \left(\frac{(2n+1)(2n+2)}{n+1}\right)$$

$$\underline{= \infty}$$

$$\rightarrow \quad R = \frac{1}{\infty} = 0$$

By the Ratio Test, the series diverges for $|x| > 0$ and converges only at its center, 0.

Therefore; the radius of convergence is $R = 0$.

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

Solution

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}\right|$$

$$= |x| \lim_{n\to\infty} \left(\frac{n}{n+1}\right)$$

$$\underline{= |x|}$$

$$\rightarrow \quad R = 1$$

So, by the Ratio Test, the radius of convergence is $R = 1$.

The series centered at 0, it converges in the interval $(-1, 1)$

When $x=1$ $\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\cdots$ *diverges*

When $x=-1$ $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}=-1+\frac{1}{2}-\frac{1}{3}+\cdots$ *converges*

Therefore; the interval of convergence $[-1,\ 1)$

Exercise

Find the interval of convergence of the power series $\sum_{n=0}^{\infty}(-1)^n\frac{(x+1)^n}{2^n}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{2^{n+1}}\cdot\frac{2^n}{(x+1)^n}\right|$$

$$= \frac{1}{2}|x+1|$$

$|x+1|<2 \to \quad R=2$

$$\begin{cases} x+1=-2 & x=-3 \\ x+1=2 & x=1 \end{cases}$$

So, by the Ratio Test, the radius of convergence is $R=2$.

The series centered at -1, it converges in the interval $(-3,\ 1)$

When $x=-3$ $\sum_{n=0}^{\infty}\frac{(-1)^n(-2)^n}{2^n}=\sum_{n=0}^{\infty}1$ *diverges*

When $x=1$ $\sum_{n=0}^{\infty}\frac{(-1)^n(2)^n}{2^n}=\sum_{n=0}^{\infty}(-1)^n$ *diverges*

Therefore; the interval of convergence $(-3,\ 1)$

Exercise

Find the interval of convergence of the power series $\sum_{n=1}^{\infty}\frac{x^n}{n^2}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2}\cdot\frac{n^2}{x^n}\right|$$

$$= |x| \lim_{n\to\infty} \left| \frac{n^2}{(n+1)^2} \right|$$

$$\underline{= |x|}$$

$\rightarrow \quad R = 1$

So, by the Ratio Test, the radius of convergence is $R = 1$.

The series centered at 0, it converges in the interval $(-1,\ 1)$

When $x = -1$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -1 + \frac{1}{2^2} - \frac{1}{3^2} + \cdots$ *converges by alternating series*

$$u_{n+1} = \frac{1}{(n+1)^2} < \frac{1}{n^2} = u_n$$

$$\lim_{n\to\infty} \frac{1}{n^2} = 0$$

When $x = 1$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$ *converges by p-series*

Therefore; the interval of convergence $[-1,\ 1]$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$

Solution

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{4^{n+1}} \cdot \frac{4^n}{x^n} \right|$$

$$\underline{= \frac{|x|}{4}}$$

$\rightarrow \quad R = 4$

So, by the Ratio Test, the radius of convergence is $R = 4$.

The series centered at 0, it converges in the interval $(-4,\ 4)$

When $x = -4$ $\displaystyle\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + \cdots$ *diverges by alternating series*

When $x = 4$ $\displaystyle\sum_{n=1}^{\infty} 1$ *diverges*

Therefore; the interval of convergence $(-4,\ 4)$

Exercise

Find the interval of convergence of the power series $\sum_{n=0}^{\infty}(2x)^n$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2x)^{n+1}}{(2x)^n}\right|$$

$$= 2|x|$$

$2|x| = 1 \quad\rightarrow\quad R = \frac{1}{2}$

So, by the Ratio Test, the radius of convergence is $R = \frac{1}{2}$.

The series converges in the interval $\left(-\frac{1}{2},\ \frac{1}{2}\right)$

When $x = -\frac{1}{2}$ $\quad \sum_{n=0}^{\infty}(-1)^n = -1+1-1+\cdots$ *diverges by alternating series*

When $x = \frac{1}{2}$ $\quad \sum_{n=0}^{\infty}1$ *diverges*

Therefore; the interval of convergence $\left(-\frac{1}{2},\ \frac{1}{2}\right)$

Exercise

Find the interval of convergence of the power series $\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right|$$

$$= |x|\lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$

$$= |x|$$

$|x| = 1 \quad\rightarrow\quad R = 1$

So, by the Ratio Test, the radius of convergence is $R = 1$.

The series converges in the interval $(-1,\ 1)$

When $x = -1$ $\quad \sum_{n=1}^{\infty}\frac{1}{n}$ *diverges by p-series*

When $x=1$ $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ *converges* *by Alternating Series*

Therefore; the interval of convergence $(-1,\ 1]$

Exercise

Find the interval of convergence of the power series $\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right|$$

$$=|x|\lim_{n\to\infty}\left|\frac{n+2}{n+1}\right|$$

$$\underline{=|x|}$$

$|x|=1 \ \rightarrow \ R=1$

The series converges in the interval $(-1,\ 1)$

When $x=-1$ $\sum_{n=0}^{\infty}(n+1)$ *diverges*

When $x=1$ $\sum_{n=0}^{\infty}(-1)^n(n+1)$ *diverges*

Therefore; the interval of convergence $(-1,\ 1)$

Exercise

Find the interval of convergence of the power series $\sum_{n=0}^{\infty}\frac{x^{5n}}{n!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{5n+5}}{(n+1)!}\cdot\frac{n!}{x^{5n}}\right|$$

$$=\left|x^5\right|\lim_{n\to\infty}\left|\frac{1}{n+1}\right|$$

$$\underline{=0} \ \rightarrow \ R=\infty$$

The series converges for all x. Therefore; the interval of convergence $(-\infty,\ \infty)$

Exercise

Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(3x)^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{(3x)^n}\right|$$

$$=|3x|\lim_{n\to\infty}\left|\frac{1}{(2n+1)(2n+2)}\right|$$

$$=0$$

$\to \quad R=\infty$

The series converges for all x. Therefore; the interval of convergence $(-\infty,\ \infty)$

Exercise

Find the interval of convergence of the power series $\sum_{n=0}^{\infty}(2n)!\left(\frac{x}{3}\right)^n$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|(2n+2)!\left(\frac{x}{3}\right)^{n+1}\cdot(2n)!\left(\frac{x}{3}\right)^{-n}\right|$$

$$=\left|\frac{x}{3}\right|\lim_{n\to\infty}\left|(2n+1)(2n+2)\right|$$

$$=\infty$$

The series converges only for $x=0$

Exercise

Find the interval of convergence of the power series $\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{(n+1)(n+2)}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+2)(n+3)}\cdot\frac{(n+1)(n+2)}{x^n}\right|$$

$$=|x|\lim_{n\to\infty}\left|\frac{n+1}{n+3}\right|$$

$$=|x|$$

$|x|=1 \quad\to\quad R=1$

The series converges in the interval $(-1,\ 1)$

When $x=-1$ $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}$ *converges* *by Alternating Series*

$$u_{n+1}=\frac{1}{(n+3)(n+2)}<\frac{1}{(n+1)(n+2)}=u_n$$

$$\lim_{n\to\infty}\frac{1}{(n+1)(n+2)}=0$$

When $x=1$ $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)(n+2)}$ *converges* *by Limit Comparison Test to* $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$

Therefore; the interval of convergence $[-1,\ 1]$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{6^n}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{6^{n+1}}\cdot\frac{6^n}{x^n}\right|$$

$$=\frac{1}{6}|x|$$

$\frac{1}{6}|x|=1 \quad\to\quad R=6$

The series converges in the interval $(-6,\ 6)$

When $x=-6$ $\displaystyle\sum_{n=1}^{\infty}(-1)^n$ *diverges*

When $x=6$ $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}$ *diverges*

Therefore; the interval of convergence $(-6,\ 6)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{n!(x-5)^n}{3^n}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(x-5)^{n+1}}{3^{n+1}}\cdot\frac{3^n}{n!(x-5)^n}\right|$$

$$= \left|\frac{x-5}{3}\right|\lim_{n\to\infty}\ (n+1)$$

$$\underline{=\infty|}$$

The series converges only for $x = 5$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-4)^n}{n9^n}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-4)^{n+1}}{(n+1)9^{n+1}}\cdot\frac{n9^n}{(x-4)^n}\right|$$

$$= \frac{1}{9}|x-4|\lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$

$$\underline{= \frac{1}{9}|x-4|}$$

$\frac{1}{9}|x-4| = 1 \quad\rightarrow\quad R = 9$

$|x-4| = 9 \;\Rightarrow\; \begin{cases} x-4=-9 & x=-5 \\ x-4=9 & x=13 \end{cases}$

The series converges in the interval $(-5,\ 13)$ and center $x = 4$

When $x=-5$ $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-9)^n}{n9^n} = \sum_{n=1}^{\infty}\frac{-1}{n}$ *diverges*

When $x=13$ $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(9)^n}{n9^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ *converges by Alternating Series*

$$u_{n+1} = \frac{1}{n+1} < \frac{1}{n} = u_n \qquad \lim_{n\to\infty}\frac{1}{n} = 0$$

Therefore; the interval of convergence $(-5,\ 13]$

Exercise

Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(x-3)^{n+1}}{(n+1)4^{n+1}}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+2}}{(n+2)4^{n+2}}\cdot\frac{(n+1)4^{n+1}}{(x-3)^{n+1}}\right|$$

$$=\frac{1}{4}|x-3| \lim_{n\to\infty}\left|\frac{n+1}{n+2}\right|$$

$$\underline{=\frac{1}{4}|x-3|}$$

$\frac{1}{4}|x-3| = 1 \;\rightarrow\; R = 4$

$|x-3| = 4 \;\Rightarrow\; \begin{cases} x-3=-4 & x=-1 \\ x-3=4 & x=7 \end{cases}$

The series converges in the interval $(-1,\ 7)$ and center $x=3$

When $x=-1$ $\sum_{n=0}^{\infty}\frac{(-4)^{n+1}}{(n+1)4^{n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1}$ *converges by Alternating Series*

$$u_{n+1} = \frac{1}{n+2} < \frac{1}{n+1} = u_n$$

$$\lim_{n\to\infty}\frac{1}{n+1} = 0$$

When $x=4$ $\sum_{n=0}^{\infty}\frac{4^{n+1}}{(n+1)4^{n+1}} = \sum_{n=1}^{\infty}\frac{1}{n+1}$ *diverges by Integral Test*

$$\int_0^{\infty}\frac{dx}{x+1} = \ln(x+1)\Big|_0^{\infty} \;\underline{=\infty}$$

Therefore; the interval of convergence $[-1,\ 7)$

Exercise

Find the interval of convergence of the power series $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-1)^{n+1}}{n+1}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^{n+2}}{n+2}\cdot\frac{n+1}{(x-1)^{n+1}}\right|$$

$$=|x-1| \lim_{n\to\infty}\left|\frac{n+1}{n+2}\right|$$

$$\boxed{= |x-1|}$$

$|x-1| = 1 \quad \to \quad R = 1$

$$|x-1| = 1 \Rightarrow \begin{cases} x-1=-1 & x=0 \\ x-1=1 & x=2 \end{cases}$$

The series converges in the interval $(0,\ 2)$ and center $x=1$

When $x=0$ $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty}\frac{1}{n+1}$ *diverges by Integral Test*

$$\int_0^{\infty}\frac{dx}{x+1} = \ln(x+1)\Big|_0^{\infty} \ \boxed{=\infty}$$

When $x=1$ $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1}$ *converges by Alternative Test*

$$u_{n+1} = \frac{1}{n+2} < \frac{1}{n+1} = u_n$$

$$\lim_{n\to\infty}\frac{1}{n+1} = 0$$

Therefore; the interval of convergence $(0,\ 2]$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-2)^n}{n2^n}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n2^n}{(x-2)^n}\right|$$

$$= \frac{1}{2}|x-2|\lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$

$$\boxed{= \frac{1}{2}|x-2|}$$

$|x-2| = 2 \quad \to \quad R = 2$

$$|x-2| = 2 \Rightarrow \begin{cases} x-2=-2 & x=0 \\ x-2=2 & x=4 \end{cases}$$

The series converges in the interval $(0,\ 4)$ and center $x=2$

When $x=0$ $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-2)^n}{n2^n} = \sum_{n=0}^{\infty}\frac{-1}{n}$ *diverges by Integral Test*

$$\int_0^\infty \frac{-dx}{x} = -\ln(x)\Big|_0^\infty \underline{= \infty|}$$

When $x = 4$ $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n2^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ *converges by Alternative Test*

$$u_{n+1} = \frac{1}{n+1} < \frac{1}{n} = u_n$$

$$\lim_{n\to\infty}\frac{1}{n} = 0$$

Therefore; the interval of convergence $(0,\ 4]$

Exercise

Find the interval of convergence of the power series $\sum_{n=1}^{\infty}\frac{(x-3)^{n-1}}{3^{n-1}}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^n}{3^n}\cdot\frac{3^{n-1}}{(x-3)^{n-1}}\right|$$

$$\underline{=\frac{1}{3}|x-3||}$$

$\frac{1}{3}|x-3| = 1 \quad\rightarrow\quad R = 3$

$|x-3| = 3 \;\Rightarrow\; \begin{cases} x-3=-3 & x=0 \\ x-3=3 & x=6 \end{cases}$

The series converges in the interval $(0,\ 6)$

When $x = 0$ $\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty}(-1)$ *diverges*

When $x = 6$ $\sum_{n=1}^{\infty}\frac{3^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty}1$ *diverges*

Therefore; the interval of convergence $(0,\ 6)$

Exercise

Find the interval of convergence of the power series $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{2n+3}\cdot\frac{2n+1}{x^{2n+1}}\right|$$

$$= x^2 \lim_{n\to\infty}\left|\frac{2n+1}{2n+3}\right|$$

$$\underline{= x^2}$$

$\to \quad R = 1$

The series converges in the interval $(-1,\ 1)$

When $x=-1$ $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}$ *converges by Alternating Series*

When $x=1$ $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n}}{2n+1}$ *converges by Alternating Series*

Therefore; the interval of convergence $[-1,\ 1]$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n+1}(-2x)^{n-1}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n+2}(-2x)^n\cdot\frac{n+1}{n(-2x)^{n-1}}\right|$$

$$= |-2x| \lim_{n\to\infty}\left|\frac{(n+1)^2}{n(n+2)}\right|$$

$$\underline{= 2|x|}$$

$2|x| = 1 \quad\to\quad R = \frac{1}{2}$

The series converges in the interval $\left(-\frac{1}{2},\ \frac{1}{2}\right)$

When $x = -\frac{1}{2}$ $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n+1}$ *diverges by nth Term Test*

$$\lim_{n\to\infty}\frac{n}{n+1} = 1 \neq 0$$

When $x = \frac{1}{2}$ $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n+1}$ *diverges by Alternating Series*

$$\lim_{n\to\infty}\frac{n}{n+1} = 1 \neq 0$$

Therefore; the interval of convergence $\left(-\frac{1}{2}, \frac{1}{2}\right)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{(n+1)!}\cdot\frac{n!}{x^{2n}}\right|$$

$$= x^2 \lim_{n\to\infty}\left|\frac{1}{n+1}\right|$$

$$\underline{=0|}$$

$\to \quad R=\infty$

Therefore; the interval of convergence $(-\infty, \infty)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{3n+4}}{(3n+4)!}\cdot\frac{(3n+1)!}{x^{3n+1}}\right|$$

$$=\left|x^3\right| \lim_{n\to\infty}\left|\frac{1}{(3n+2)(3n+3)(3n+4)}\right|$$

$$\underline{=0|}$$

$\to \quad R=\infty$

Therefore; the interval of convergence $(-\infty, \infty)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{n!x^n}{(2n)!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{n!x^n}\right|$$

$$=|x|\lim_{n\to\infty}\left|\frac{n+1}{(2n+1)(2n+2)}\right|$$

$$\underline{=0}$$

$\to \quad R=\infty$

Therefore; the interval of convergence $(-\infty,\ \infty)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}\frac{2\cdot 3\cdot 4\cdots(n+1)x^n}{n!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{2\cdot 3\cdot 4\cdots(n+1)(n+2)x^{n+1}}{(n+1)!}\cdot\frac{n!}{2\cdot 3\cdot 4\cdots(n+1)x^n}\right|$$

$$=|x|\lim_{n\to\infty}\left|\frac{n+2}{n+1}\right|$$

$$\underline{=|x|}$$

$|x|=1 \quad\to\quad R=1$

The series converges in the interval $(-1,\ 1)$

When $x=-1$ $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{2\cdot 3\cdot 4\cdots(n+1)}{n!}=\sum_{n=1}^{\infty}(-1)^n(n+1)$ *diverges*

When $x=1$ $\displaystyle\sum_{n=1}^{\infty}\frac{2\cdot 3\cdot 4\cdots(n+1)}{n!}=\sum_{n=1}^{\infty}(n+1)$ *diverges*

Therefore; the interval of convergence $(-1,\ 1)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n}}{\sqrt{n+1}}$

Solution

$$R=\lim_{n\to\infty}\left|\frac{1}{\sqrt{n+2}}\cdot\sqrt{n+1}\right| = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}}=1$$

The radius of convergence is 1.

The centre of convergence is 0.

The interval of convergence is (–1, 1).

The series does not converge at $x=-1$ or $x=1$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} 3n(x+1)^n$

Solution

$$R=\lim_{n\to\infty}\left|\frac{3n}{3(n+1)}\right| \qquad\qquad R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$=\lim_{n\to\infty}\frac{3n}{3n}$$

$$\underline{=1}$$

The radius of convergence is 1, and the centre of convergence is –1. $(x+1=0)$

$a-R<x<a+R \quad\Rightarrow\quad -1-1<x<-1+1$

Therefore; the given series convergences absolutely on $(-2,\ 0)$

At $x=-2$, the series is $\displaystyle\sum_{n=0}^{\infty} 3n(-1)^n$ which diverges.

At $x=0$, the series is $\displaystyle\sum_{n=0}^{\infty} 3n(1)^n=\sum_{n=0}^{\infty} 3n$ which diverges.

Hence, the interval of convergence is $(-2,\ 0)$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4 2^{2n}}x^n$

Solution

$$R=\lim_{n\to\infty}\left|\frac{(n+1)^4 2^{2n+2}}{n^4 2^{2n}}\right| \qquad\qquad R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$=4\lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^4\right|$$

$$\underline{=4}$$

The radius of convergence is 4, and the centre of convergence is 0.

$a - R < x < a + R \;\; \Rightarrow \;\; -4 < x < 4$, the given series convergences absolutely on $(-4,\ 4)$

At $x = -4$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}}(-4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}}(-1)^n 2^{2n}$

$= \sum_{n=1}^{\infty} \frac{1}{n^4}$ which converges (p-series).

At $x = 4$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}}(4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} 2^{2n}$

$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ which also converges.

Hence, the interval of convergence is $[-4,\ 4]$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{e^n}{n^3}(4-x)^n$

Solution

$$R = \lim_{n\to\infty} \left| \frac{e^n}{n^3} \cdot \frac{(n+1)^3}{e^{n+1}} \right| \qquad\qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \frac{1}{e} \lim_{n\to\infty} \left| \left(\frac{n+1}{n} \right)^3 \right|$$

$$= \frac{1}{e}$$

The radius of convergence is $\frac{1}{e}$.

The centre of convergence is 4. $(4 - x = 0 \;\; \Rightarrow \;\; x = 4)$

$a - R < x < a + R \;\; \Rightarrow \;\; 4 - \frac{1}{e} < x < 4 + \frac{1}{e}$, which the given series convergences absolutely

At $x = 4 - \frac{1}{e}$, the series is $\sum_{n=1}^{\infty} \frac{e^n}{n^3}\left(\frac{1}{e}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ which converges (p-series).

At $x = 4 + \frac{1}{e}$, the series is $\sum_{n=1}^{\infty} \frac{e^n}{n^3}\left(-\frac{1}{e}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ which also converges (p-series).

Hence, the interval of convergence is $\left[4 - \frac{1}{e},\ 4 + \frac{1}{e} \right]$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{1+5^n}{n!}x^n$

Solution

$$R = \lim_{n\to\infty}\left|\frac{1+5^n}{n!}\cdot\frac{(n+1)!}{1+5^{n+1}}\right| \qquad\qquad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1+5^n}{1+5^{n+1}}(n+1)\right|$$

$$\underline{= \infty|}$$

The radius of convergence is ∞ .

The centre of convergence is 0.

The interval of convergence is the real line $(-\infty, \infty)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n}$

Solution

$$\sum_{n=1}^{\infty}\frac{(4x-1)^n}{n^n} = \sum_{n=1}^{\infty}\frac{4^n\left(x-\frac{1}{4}\right)^n}{n^n}$$

$$R = \lim_{n\to\infty}\left|\frac{4^n}{n^n}\cdot\frac{(n+1)^{n+1}}{4^{n+1}}\right| \qquad\qquad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= \frac{1}{4}\lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^n(n+1)\right|$$

$$\underline{= \infty\ |}$$

The radius of convergence is ∞ .

$$4x-1=0 \ \Rightarrow\ x=\frac{1}{4}$$

The centre of convergence is $\underline{x=\frac{1}{4}|}$

The interval of convergence is the real line $(-\infty, \infty)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{1+5^n}{n!} x^n$

Solution

$$a_n = \frac{1+5^n}{n!}$$

$$R = \lim_{n\to\infty} \left| \frac{\left(1+5^n\right)}{n!} \cdot \frac{(n+1)!}{\left(1+5^{n+1}\right)} \right| \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| (n+1) \frac{1+5^n}{1+5^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| (n+1) \frac{1}{5} \right|$$

$$= \infty$$

The radius of convergence is ∞ .

The centre of convergence is $x = 0$.

The interval of convergence is the real line $(-\infty, \infty)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum \frac{n^2 x^n}{n!}$

Solution

$$a_n = \frac{n^2}{n!}$$

$$R = \lim_{n\to\infty} \left| \frac{n^2 x^n}{n!} \cdot \frac{(n+1)!}{(n+1)^2 x^{n+1}} \right| \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| (n+1) \left(\frac{n}{n+1} \right)^2 \right|$$

$$= \infty$$

The radius of convergence is ∞ .

The centre of convergence is $x = 0$.

The interval of convergence is the real line $(-\infty, \infty)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum \frac{x^{4n}}{n^2}$

Solution

$$a_n = \frac{1}{n^2} x^{4n}$$

$$R = \lim_{n\to\infty} \left(\frac{1}{n^2} \cdot \frac{(n+1)^2}{1} \right) \left| \frac{x^{4n}}{x^{4n+4}} \right| \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^2 \left| \frac{1}{x^4} \right|$$

$$= 1$$

The radius of convergence is 1

The centre of convergence is $x = 0$

$-1 < x < 1$ $\qquad a - R < x < a + R$

which the given series convergences absolutely

At $x = -1$, the series is $\sum \frac{(-1)^{4n}}{n^2} = \sum \frac{1}{n^2}$ which converges (p-series).

At $x = 1$, the series is $\sum \frac{(1)^{4n}}{n^2} = \sum \frac{1}{n^2}$ which also converges (p-series).

The interval of convergence is the real line $[-1, 1]$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum (-1)^n \frac{(x+1)^{2n}}{n!}$

Solution

$$a_n = \frac{1}{n!}(x+1)^{2n}$$

$$R = \lim_{n\to\infty} \left(\frac{1}{n!} \cdot \frac{(n+1)!}{1} \right) \left| \frac{(x+1)^{2n}}{(x+1)^{2n+2}} \right| \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} (n+1) \left| \frac{1}{(x+1)^2} \right|$$

$$= \infty$$

The radius of convergence is ∞

$x+1=0 \quad \rightarrow \quad x=-1$

The centre of convergence is $\underline{x=-1}$

The interval of convergence is the real line $(-\infty, \infty)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\displaystyle\sum \frac{(x-1)^n}{n \cdot 5^n}$

Solution

$$a_n = \frac{1}{n \cdot 5^n}(x-1)^n$$

By Ratio Test:

$$R = \lim_{n\to\infty} \left(\frac{1}{n \cdot 5^n} \cdot \frac{(n+1)\cdot 5^{n+1}}{1} \right) \left| \frac{(x-1)^n}{(x-1)^{n+1}} \right| \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= 5 \lim_{n\to\infty} \frac{n+1}{n} \left| \frac{1}{x-1} \right|$$

$$\underline{= 5}$$

The radius of convergence is $\underline{5}$

$x-1=0 \quad \rightarrow \quad x=1$

The centre of convergence is $\underline{x=1}$

$$-5+1 < x < 5+1 \qquad\qquad a-R < x < a+R$$

$$-4 < x < 6$$

which the given series convergences absolutely

At $x=-4$, the series is $\displaystyle\sum \frac{(-5)^n}{n \cdot 5^n} = \sum \frac{(-1)^n}{n}$

$$\frac{1}{n} > \frac{1}{n+1}$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

which converges *Alternating Harmonic Series*.

At $x=6$, the series is $\displaystyle\sum \frac{(5)^n}{n \cdot 5^n} = \sum \frac{1}{n}$ which *diverges* (p-series $p=1 \le 1$)

The interval of convergence is the real line $[-4, 6)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum \left(\frac{x}{9}\right)^{3n}$

Solution

$$a_n = \left(\frac{x}{9}\right)^{3n}$$

By Root Test:

$$R = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{x}{9}\right)^{3n}\right|} \qquad\qquad R = \lim_{n\to\infty} \sqrt[n]{\left|a_n\right|}$$

$$= \lim_{n\to\infty} \left(\frac{|x|}{9}\right)^3$$

$$= \frac{1}{729}\left|x^3\right| < 1$$

$$\left|\frac{x}{9}\right|^3 < 1$$

$$\left|\frac{x}{9}\right| < 1$$

$$-9 < x < 9$$

The radius of convergence is 9

The centre of convergence is $x = 0$

At $x = -9$, the series is $\sum \left(\frac{-9}{9}\right)^{3n} = \sum (-1)$ which diverges by the *divergence Test*.

At $x = 9$, the series is $\sum \left(\frac{9}{9}\right)^{3n} = \sum (1)$ which diverges by the *divergence Test*.

The interval of convergence is the real line $(-9,\ 9)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum \frac{(x+2)^n}{\sqrt{n}}$

Solution

$$a_n = \frac{(x+2)^n}{\sqrt{n}}$$

By Ratio Test:

$$R=\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}\cdot\frac{\sqrt{n+1}}{1}\right)\left|\frac{(x+2)^n}{(x+2)^{n+1}}\right| \qquad\qquad R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$=\lim_{n\to\infty}\sqrt{\frac{n+1}{n}}\left|\frac{1}{x+2}\right|$$

$$\underline{=1}$$

The radius of convergence is $\underline{1}$

$x+2=0 \quad\to\quad x=-2$

The centre of convergence is $\underline{x=-2}$

$-2-1<x<-2+1 \qquad\qquad a-R<x<a+R$

$-3<x<-1$

which the given series convergences absolutely

At $x=-3$, the series is $\sum \frac{(-1)^n}{\sqrt{n}}$

$$\frac{1}{\sqrt{n}}>\frac{1}{\sqrt{n+1}}$$

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$$

which converges *Alternating Series*.

At $x=-1$, the series is $\sum \frac{(1)^n}{\sqrt{n}}=\sum \frac{1}{\sqrt{n}}$ which *diverges* (p-series $p=\frac{1}{2}\le 1$)

The interval of convergence is the real line $[-3,\ -1)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum \frac{(x+2)^k}{2^k \ln k}$

Solution

$$a_k=\frac{(x+2)^k}{2^k \ln k}$$

By Ratio Test:

$$R=\lim_{k\to\infty}\left(\frac{1}{2^k \ln k}\cdot\frac{2^{k+1}\ln(k+1)}{1}\right)\left|\frac{(x+2)^k}{(x+2)^{k+1}}\right| \qquad R=\lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right|$$

$$=2\lim_{n\to\infty}\frac{\ln(k+1)}{\ln k}\left|\frac{1}{x+2}\right|$$

$$= 2 \lim_{n\to\infty} \frac{\frac{1}{k+1}}{\frac{1}{k}}$$

$$= 2 \lim_{n\to\infty} \frac{k}{k+1}$$

$$\underline{= 2}$$

The radius of convergence is $\underline{2}$

$x + 2 = 0 \quad \to \quad x = -2$

The centre of convergence is $\underline{x = -2}$

$-2 - 2 < x < -2 + 2$ $\qquad\qquad a - R < x < a + R$

$-4 < x < 0$

which the given series convergences absolutely

At $x = -4$, the series is $\sum \frac{(-2)^k}{2^k \ln k} = \sum \frac{(-1)^k}{\ln k}$

$$\frac{1}{\ln k} > \frac{1}{\ln(k+1)}$$

$$\lim_{n\to\infty} \frac{1}{\ln k} = 0$$

which converges *Alternating Series*.

At $x = 0$, the series is $\sum \frac{(2)^k}{2^k \ln k} = \sum \frac{1}{\ln k}$

$\ln k < k$

$\frac{1}{\ln k} > \frac{1}{k}$

$\frac{1}{k}$ *diverges* (p-series $p = 1 \le 1$)

$\therefore$ Which *diverges by Comprison Test.*

The interval of convergence is the real line $[-4, \ 0)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots$

Solution

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$$

$$a_k = \frac{x^{2k+1}}{2k+1}$$

By Ratio Test:

$$R = \lim_{k\to\infty} \left(\frac{1}{2k+1}\cdot\frac{2k+3}{1}\right)\left|\frac{x^{2k+1}}{x^{2k+3}}\right| \qquad R = \lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right|$$

$$= \lim_{n\to\infty} \frac{2k+3}{2k+1}\left(\frac{1}{x^2}\right)$$

$$\underline{=1}$$

The radius of convergence is $\underline{1}$

The centre of convergence is $\underline{x=0}$

$-1 < x < 1$

which the given series convergences absolutely

At $x=-1$, the series is $\sum \frac{(-1)^{2k+1}}{2k+1} = \sum \frac{-1}{2k+1}$

$$\int_0^\infty \frac{-1}{2x+1}dx = -\frac{1}{2}\int_0^\infty \frac{1}{2x+1}d(2x+1)$$

$$= -\frac{1}{2}\ln(2x+1)\Big|_0^\infty$$

$$= -\frac{1}{2}(\ln\infty - \ln 1)$$

$$\underline{= -\infty}$$

which diverges *Integral Test*.

At $x=1$, the series is $\sum \frac{(1)^{2k+1}}{2k+1} = \sum \frac{1}{2k+1}$

$$\int_0^\infty \frac{1}{2x+1}dx = \frac{1}{2}\int_0^\infty \frac{1}{2x+1}d(2x+1)$$

$$= \frac{1}{2}\ln(2x+1)\Big|_0^\infty$$

$$= \frac{1}{2}(\ln\infty - \ln 1)$$

$$\underline{= \infty}$$

which diverges *Integral Test*.

The interval of convergence is the real line $(-1,\ 1)$

Exercise

For what value of x does the series $1-\frac{1}{2}(x-3)+\frac{1}{4}(x-3)^2+\cdots+\left(-\frac{1}{2}\right)^n(x-3)^n+\cdots$ converges? What is its sum? What series do you get if you differentiate the given series term by term? For what value of x does the new series converge? What is its sum?

Solution

$$1-\frac{1}{2}(x-3)+\frac{1}{4}(x-3)^2+\cdots+\left(-\frac{1}{2}\right)^n(x-3)^n+\cdots=\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n(x-3)^n$$

$$\lim_{b\to\infty}\left|\frac{(x-3)^{n+1}}{2^{n+1}}\cdot\frac{2^n}{(x-3)^n}\right|=\left|\frac{x-3}{2}\right|<1$$

$$\Rightarrow\quad -1<\frac{x-3}{2}<1$$

$$-2<x-3<2\ \Rightarrow\ \boxed{1<x<5}$$

When $x=1\quad\Rightarrow\quad \sum_{n=1}^{\infty}(1)^n$ which is a divergent series

$x=5\quad\Rightarrow\quad \sum_{n=1}^{\infty}(-1)^n$ the series diverges

The series is a geometric series, the sum is

$$\frac{1}{1+\frac{x-3}{2}}=\frac{2}{x-1}$$

If $f(x)=1-\frac{1}{2}(x-3)+\frac{1}{4}(x-3)^2+\cdots+\left(-\frac{1}{2}\right)^n(x-3)^n+\cdots$

$$=\frac{2}{x-1}$$

Then $f'(x)=-\frac{1}{2}+\frac{1}{2}(x-3)+\cdots+\left(-\frac{1}{2}\right)^n n(x-3)^{n-1}+\cdots$

$f'(x)$ is convergent when $1<x<5$ and divergent when $x=1\ or\ 5$

The sum for $f'(x)$ is $\frac{-2}{(x-1)^2}$

Exercise

The series $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$ converges to sin$x$ for all x.

a) Find the first six terms of a series for cos$x$. For what values of x should the series converge?

b) By replacing x by $2x$ in the series for sin$x$, find a series that converges to sin$2x$ for all x.

c) Using the result in part (a) and series multiplication, calculate the first six term of a series for $2\sin x \cos x$. Compare your answer with the answer in part (b).

Solution

a) $(\sin x)' = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots$

$$\cos x = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n}}{(2n)!}$$

$$\lim_{n\to\infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right| = x^2 \lim_{n\to\infty} \left| \frac{1}{(2n+1)(2n+2)} \right| = 0 < 1 \quad (\forall x)$$

The series converges for all values of x.

b) $\sin 2x = 2x - \frac{2^3x^3}{3!} + \frac{2^5x^5}{5!} - \frac{2^7x^7}{7!} + \frac{2^9x^9}{9!} - \frac{2^{11}x^{11}}{11!} + \cdots$

$$= 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \frac{512x^9}{9!} - \frac{2048x^{11}}{11!} + \cdots$$

c) $2\sin x \cos x = 2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right)$

$$= 2x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right) - 2\frac{x^3}{3!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right)$$

$$+ 2\frac{x^5}{5!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right) - 2\frac{x^7}{7!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right)$$

$$+ 2\frac{x^9}{9!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right) - 2\frac{x^{11}}{11!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right)$$

$$= 2x - \frac{2x^3}{2!} + \frac{2x^5}{4!} - \frac{2x^7}{6!} + \frac{2x^9}{8!} - \frac{2x^{11}}{10!} - \frac{2x^3}{3!} + \frac{2x^5}{2!3!} - \frac{2x^7}{4!3!} + \frac{2x^9}{6!3!} - \frac{2x^{11}}{8!3!}$$

$$+ \frac{2x^5}{5!} - \frac{2x^7}{5!2!} + \frac{2x^9}{5!4!} - \frac{2x^{11}}{5!6!} - \frac{2x^7}{7!} + \frac{2x^9}{7!2!} - \frac{2x^{11}}{7!4!} + \frac{2x^9}{9!} - \frac{2x^{11}}{9!2!} - \frac{2x^{11}}{11!} + \cdots$$

$$= 2x - \frac{2^3x^3}{3!} + \frac{2^5x^5}{5!} - \frac{2^7x^7}{7!} + \frac{2^9x^9}{9!} - \frac{2^{11}x^{11}}{11!} + \cdots$$

Exercise

Find the sum of the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ by the first finding the sum of the power series

$$\sum_{n=1}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + 16x^4 + \cdots$$

Solution

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}$$

$$\begin{aligned}\sum_{n=1}^{\infty} nx^n &= x + x^2 + 3x^3 + 4x^4 + \cdots \\ &= x\left(1 + x + 3x^2 + 4x^3 + \cdots\right) \\ &= x\frac{d}{dx}\left(1 + x + x^2 + \cdots + x^n + \cdots\right) \\ &= x\left(\frac{1}{1-x}\right)' \\ &= \frac{x}{(1-x)^2}\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\sum_{n=1}^{\infty} nx^n &= \left(\frac{x}{(1-x)^2}\right)' \\ &= \frac{(1-x)^2 + 2x(1-x)}{(1-x)^4} \\ &= \frac{1+x}{(1-x)^3}\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\sum_{n=1}^{\infty} nx^n &= \sum_{n=1}^{\infty} n^2 x^{n-1} \\ &= \frac{1+x}{(1-x)^3}\end{aligned}$$

Multiply by x both sides

$$x\sum_{n=1}^{\infty} n^2 x^{n-1} = x\frac{1+x}{(1-x)^3}$$

$$\sum_{n=1}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + 16x^4 + \cdots$$

$$= \frac{x(1+x)}{(1-x)^3}$$

Let $x = \frac{1}{2}$

$$\sum_{n=1}^{\infty} n^2 \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}\frac{3}{2}}{\left(\frac{1}{2}\right)^3}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = 6$$

Exercise

Find a series representation of $f(x) = \frac{1}{2+x}$ in powers of $x-1$. What is the interval of convergence of this series?

Solution

Let $t = x-1 \quad \Rightarrow \quad x = t+1$, we have

$$\frac{1}{2+x} = \frac{1}{3+t}$$

$$= \frac{1}{3}\frac{1}{1+\frac{t}{3}}$$

$$= \frac{1}{3}\left(1 - \frac{t}{3} + \frac{t^2}{3^2} - \frac{t^3}{3^3} + \cdots\right) \qquad \left(-1 < \frac{t}{3} < 1\right)$$

$$= \sum_{n-0}^{\infty} (-1)^n \frac{t^n}{3^{n+1}} \qquad (-3 < t < 3)$$

$$= \sum_{n-0}^{\infty} (-1)^n \frac{(x-1)^n}{3^{n+1}} \qquad (-2 < x < 4)$$

$$R = \lim_{n\to\infty} \frac{3^{n+2}}{3^{n+1}}$$

$$= 3$$

The radius of convergence of this series is 3.

The distance from the centre of convergence $x-1=0 \Rightarrow x=1$, to the point -2 where the denominator is 0.

Exercise

Determine the Cauchy product of the series $1+x+x^2+x^3+\cdots$ *and* $-x+x^2-x^3+\cdots$. On what interval and to what function does the product series converge?

Solution

$$1+x+x^2+x^3+\cdots=\frac{1}{1-x}$$

$$=\sum_{n=0}^{\infty}x^n$$

$$-x+x^2-x^3+\cdots=\frac{1}{1+x}$$

$$=\sum_{n=0}^{\infty}(-1)^n x^n$$

Let $a_n=1$ *and* $b_n=(-1)^n$, then the series holds for $-1<x<1$

We have

$$c_n=\sum_{j=0}^{n}(-1)^{n-j}=\begin{cases}0, & \text{if } n \text{ is odd}\\ 1 & \text{if } n \text{ is even}\end{cases}$$

Then the Cauchy product is

$$1+x^2+x^4+\cdots=\sum_{n=0}^{\infty}x^{2n}$$

$$=\frac{1}{1-x}\cdot\frac{1}{1+x}$$

$$\boxed{=\frac{1}{1-x^2}} \quad \text{for} \quad -1<x<1$$

Exercise

Determine the power series expansion of $\frac{1}{(1-x)^2}$ by formally dividing $1-2x+x^2$ into 1.

Use the power series $\frac{1}{1-x} = 1+x+x^2+x^3+\cdots \quad -1<x<1$

Solution

$$\begin{array}{r|l} 1-2x+x^2 & \underline{\;1+2x+3x^2+4x^3+\cdots\;} \\ & 1 \\ & \underline{1-2x+x^2} \\ & \quad 2x-x^2 \\ & \quad \underline{2x-4x^2+2x^3} \\ & \qquad 3x^2-2x^3 \\ & \qquad \underline{3x^2-6x^3+3x^4} \\ & \qquad\qquad 4x^3+3x^4 \\ & \qquad\qquad \underline{4x^3-8x^4+4x^5} \\ & \qquad\qquad\quad 11x^4-\cdots \end{array}$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n \quad \textit{for} \quad -1<x<1$$

Exercise

Determine the interval of convergence and the sum of the series

$$1-4x+16x^2-64x^3+\cdots = \sum_{n=0}^{\infty}(-1)^n(4x)^n$$

Solution

$$1-4x+16x^2-64x^3+\cdots = 1+(-4x)+(-4x)^2+(-4x)^3+\cdots \qquad \frac{1}{1-u} = 1+u+u^2+u^3+\cdots$$

$$= \frac{1}{1-(-4x)}$$

$$= \frac{1}{1+4x}$$

Therefore; the interval of convergence is $-\frac{1}{4}<x<\frac{1}{4}$

Exercise

Determine the interval of convergence and the sum of the series

$$3+4x+5x^2+6x^3+\cdots = \sum_{n=0}^{\infty}(n+3)x^n$$

Solution

$$\sum_{n=0}^{\infty}(n+3)x^n = \sum_{n=0}^{\infty} nx^n + \sum_{n=0}^{\infty} 3x^n$$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

$$\left(\sum_{n=0}^{\infty} x^n\right)' = 1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2} \qquad \textit{Differentiate}$$

$$x\left(1 + 2x + 3x^2 + 4x^3 + \cdots\right) = \frac{x}{(1-x)^2} \qquad \textit{Multiply by } x$$

$$\sum_{n=0}^{\infty} nx^n = x + 2x^2 + 3x^3 + \cdots = \frac{x}{(1-x)^2}$$

Then,

$$\sum_{n=0}^{\infty}(n+3)x^n = \sum_{n=0}^{\infty} nx^n + \sum_{n=0}^{\infty} 3x^n$$

$$= \frac{x}{(1-x)^2} + 3\frac{1}{1-x}$$

$$= \frac{3-2x}{(1-x)^2} \qquad (-1 < x < 1)$$

Exercise

Determine the interval of convergence and the sum of the series

$$\frac{1}{3} + \frac{x}{4} + \frac{x^2}{5} + \frac{x^3}{6} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n+3}$$

Solution

$$\frac{1}{3} + \frac{x}{4} + \frac{x^2}{5} + \frac{x^3}{6} + \cdots = \frac{1}{x^3}\left(\frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \cdots\right)$$

$$= \frac{1}{x^3}\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \cdots - x - \frac{x^2}{2}\right) \qquad \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$

$$= \frac{1}{x^3}\left(-\ln(1-x) - x - \frac{x^2}{2}\right)$$

$$= -\frac{1}{x^3}\ln(1-x) - \frac{1}{x^2} - \frac{1}{2x} \qquad (-1 \le x < 1, \ \ x \ne 0)$$