

Lecture Six – Identities and Solving Trigonometric

Section 6.1 - Proving Identities

Reciprocal Identities

$$\begin{array}{lll} \csc \theta = \frac{1}{\sin \theta} & \sec \theta = \frac{1}{\cos \theta} & \cot \theta = \frac{1}{\tan \theta} \\ \sin \theta = \frac{1}{\csc \theta} & \cos \theta = \frac{1}{\sec \theta} & \tan \theta = \frac{1}{\cot \theta} \end{array}$$

Ratio Identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \qquad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

Pythagorean Identities

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$\cos \theta = \pm \sqrt{1 - \sin^2 \theta}$$

$$\sin \theta = \pm \sqrt{1 - \cos^2 \theta}$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

Example

Write $\sec \theta \tan \theta$ in terms of $\sin \theta$ and $\cos \theta$, and then simplify.

Solution

$$\sec \theta \tan \theta = \frac{1}{\cos \theta} \cdot \frac{\sin \theta}{\cos \theta} = \frac{\sin \theta}{\cos^2 \theta}$$

Example

Add $\frac{1}{\sin \theta} + \frac{1}{\cos \theta}$

Solution

$$\frac{1}{\sin \theta} + \frac{1}{\cos \theta} = \frac{\cos \theta + \sin \theta}{\sin \theta \cos \theta} \qquad \frac{1}{\sin \theta} \frac{\cos \theta}{\cos \theta} + \frac{1}{\cos \theta} \frac{\sin \theta}{\sin \theta}$$

Example

Write: $\tan \alpha + \cot \alpha$ in terms of $\sin \alpha$ and $\cos \alpha$

Solution

$$\begin{aligned}\tan \alpha + \cot \alpha &= \frac{\sin \alpha}{\cos \alpha} + \frac{\cos \alpha}{\sin \alpha} \\&= \frac{\sin \alpha}{\cos \alpha} \frac{\sin \alpha}{\sin \alpha} + \frac{\cos \alpha}{\sin \alpha} \frac{\cos \alpha}{\cos \alpha} \\&= \frac{\sin^2 \alpha + \cos^2 \alpha}{\cos \alpha \sin \alpha} \\&= \frac{1}{\cos \alpha \sin \alpha} \quad \left| \right.\end{aligned}$$

Example

Prove: $\tan x + \cos x = \sin x(\sec x + \cot x)$

Solution

$$\begin{aligned}\tan x + \cos x &= \frac{\sin x}{\cos x} + \cos x \\&= \sin x \frac{1}{\cos x} + \cos x \frac{\sin x}{\sin x} \\&= \sin x \sec x + \sin x \frac{\cos x}{\sin x} \\&= \sin x(\sec x + \cot x) \quad \checkmark\end{aligned}$$

or

$$\begin{aligned}\sin x(\sec x + \cot x) &= \sin x \left(\frac{1}{\cos x} + \frac{\cos x}{\sin x} \right) \\&= \frac{\sin x}{\cos x} + \sin x \frac{\cos x}{\sin x} \\&= \tan x + \cos x \quad \left| \right.\end{aligned}$$

Example

Prove: $\cot \alpha + 1 = \csc \alpha (\cos \alpha + \sin \alpha)$

Solution

$$\begin{aligned}\csc \alpha (\cos \alpha + \sin \alpha) &= \frac{1}{\sin \alpha} (\cos \alpha + \sin \alpha) \\ &= \frac{1}{\sin \alpha} \cos \alpha + \frac{1}{\sin \alpha} \sin \alpha \\ &= \cot \alpha + 1 \quad \checkmark\end{aligned}$$

Guidelines for Proving Identities

1. Work on the complicated side first (more trigonometry functions)
2. Look for trigonometry substitutions.
3. Look for algebraic operations
4. If not always change everything to sines and cosines
5. Keep an eye on the side you are not working.

Example

Prove $\frac{\cos^4 t - \sin^4 t}{\cos^2 t} = 1 - \tan^2 t$

Solution

$$\begin{aligned}\frac{\cos^4 t - \sin^4 t}{\cos^2 t} &= \frac{(\cos^2 t - \sin^2 t)(\cos^2 t + \sin^2 t)}{\cos^2 t} \\ &= \frac{(\cos^2 t - \sin^2 t)(1)}{\cos^2 t} \\ &= \frac{\cos^2 t - \sin^2 t}{\cos^2 t} \\ &= \frac{\cos^2 t}{\cos^2 t} - \frac{\sin^2 t}{\cos^2 t} \\ &= 1 - \tan^2 t \quad \checkmark\end{aligned}$$

$$a^2 - b^2 = (a - b)(a + b)$$

$$\cos^2 t + \sin^2 t = 1$$

Example

Prove: $1 + \cos \theta = \frac{\sin^2 \theta}{1 - \cos \theta}$

Solution

$$\frac{\sin^2 \theta}{1 - \cos \theta} = \frac{1 - \cos^2 \theta}{1 - \cos \theta}$$

$$= \frac{(1 - \cos \theta)(1 + \cos \theta)}{1 - \cos \theta}$$

$$= \underline{1 + \cos \theta} \quad \checkmark$$

$$\sin^2 \theta = 1 - \cos^2 \theta$$

$$a^2 - b^2 = (a - b)(a + b)$$

Example

Prove: $\tan^2 \alpha (1 + \cot^2 \alpha) = \frac{1}{1 - \sin^2 \alpha}$

Solution

$$\tan^2 \alpha (1 + \cot^2 \alpha) = \tan^2 \alpha + \tan^2 \alpha \cot^2 \alpha$$

$$= \tan^2 \alpha + \tan^2 \alpha \frac{1}{\tan^2 \alpha}$$

$$= \tan^2 \alpha + 1$$

$$= \sec^2 \alpha$$

$$= \frac{1}{\cos^2 \alpha}$$

$$= \underline{\frac{1}{1 - \sin^2 \alpha}} \quad \checkmark$$

$$\tan^2 \alpha + 1 = \sec^2 \alpha$$

$$\cos^2 \alpha = 1 - \sin^2 \alpha$$

Example

Prove: $\frac{\sin \alpha}{1 + \cos \alpha} + \frac{1 + \cos \alpha}{\sin \alpha} = 2 \csc \alpha$

Solution

$$\frac{\sin \alpha}{1 + \cos \alpha} + \frac{1 + \cos \alpha}{\sin \alpha} = \frac{\sin \alpha}{\sin \alpha} \cdot \frac{\sin \alpha}{1 + \cos \alpha} + \frac{1 + \cos \alpha}{\sin \alpha} \cdot \frac{1 + \cos \alpha}{1 + \cos \alpha}$$

$$= \frac{\sin^2 \alpha + (1 + \cos \alpha)^2}{\sin \alpha (1 + \cos \alpha)}$$

$$= \underline{\frac{\sin^2 \alpha + 1 + \cos^2 \alpha + 2 \cos \alpha}{\sin \alpha (1 + \cos \alpha)}}$$

$$\begin{aligned}
&= \frac{2 + 2\cos \alpha}{\sin \alpha(1 + \cos \alpha)} \\
&= \frac{2(1 + \cos \alpha)}{\sin \alpha(1 + \cos \alpha)} \\
&= \frac{2}{\sin \alpha} \\
&= 2 \csc \alpha \quad \checkmark
\end{aligned}$$

Example

Prove $\frac{1 + \sin t}{\cos t} = \frac{\cos t}{1 - \sin t}$

Solution

$$\begin{aligned}
\frac{1 + \sin t}{\cos t} &= \frac{1 + \sin t}{\cos t} \cdot \frac{1 - \sin t}{1 - \sin t} \\
&= \frac{1 - \sin^2 t}{\cos t(1 - \sin t)} \\
&= \frac{\cos^2 t}{\cos t(1 - \sin t)} \\
&= \frac{\cos t}{1 - \sin t} \quad \checkmark
\end{aligned}$$

Example

Show that $\cot^2 \theta + \cos^2 \theta = \cot^2 \theta \cos^2 \theta$ is not an identity by finding a counterexample

Solution

$$\cot^2 \frac{\pi}{4} + \cos^2 \frac{\pi}{4} = \cot^2 \frac{\pi}{4} \cos^2 \frac{\pi}{4}$$

$$1^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1^2 \left(\frac{1}{\sqrt{2}}\right)^2$$

$$1 + \frac{1}{2} = \frac{1}{2}$$

$$\frac{3}{2} \neq \frac{1}{2} \quad \checkmark$$

Exercises

Section 6.1 – Proving Identities

(1–80) Prove the identity

1. $\cos \theta \cot \theta + \sin \theta = \csc \theta$

2. $\sec \theta \cot \theta - \sin \theta = \frac{\cos^2 \theta}{\sin \theta}$

3. $\frac{\csc \theta \tan \theta}{\sec \theta} = 1$

4. $(\sin \theta + \cos \theta)^2 = 1 + 2 \sin \theta \cos \theta$

5. $\sin \theta (\sec \theta + \cot \theta) = \tan \theta + \cos \theta$

6. $\cos \theta (\csc \theta + \tan \theta) = \cot \theta + \sin \theta$

7. $\cot \theta + \tan \theta = \csc \theta \sec \theta$

8. $\tan x (\cos x + \cot x) = \sin x + 1$

9. $\frac{1 - \cos^4 \theta}{1 + \cos^2 \theta} = \sin^2 \theta$

10. $\frac{1 - \sec x}{1 + \sec x} = \frac{\cos x - 1}{\cos x + 1}$

11. $\frac{\cos x}{1 + \sin x} - \frac{1 - \sin x}{\cos x} = 0$

12. $\frac{1 + \cot^3 t}{1 + \cot t} = \csc^2 t - \cot t$

13. $\tan x + \cot x = \sec x \csc x$

14. $\frac{\tan x - \cot x}{\sin x \cos x} = \sec^2 x - \csc^2 x$

15. $\frac{\sec x + \tan x}{\sec x - \tan x} = \frac{1 + 2 \sin x + \sin^2 x}{\cos^2 x}$

16. $\sin^2 x - \cos^2 x = 2 \sin^2 x - 1$

17. $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$

18. $\frac{\cos \alpha}{1 + \sin \alpha} = \sec \alpha - \tan \alpha$

19. $\frac{\sin \alpha}{1 - \sin \alpha} - \frac{\cos \alpha}{1 - \sin \alpha} = \frac{1 - \cot \alpha}{\csc \alpha - 1}$

20. $\frac{\frac{1}{\tan x} + \cot x}{\frac{1}{\tan x} + \tan x} = \frac{2}{\sec^2 x}$

21. $\frac{\cot^2 \theta + 3 \cot \theta - 4}{\cot \theta + 4} = \cot \theta - 1$

22. $\frac{\sin \theta}{1 + \cos \theta} = \frac{1 - \cos \theta}{\sin \theta}$

23. $\tan x (\csc x - \sin x) = \cos x$

24. $\sin x (\tan x \cos x - \cot x \cos x) = 1 - 2 \cos^2 x$

25. $(1 + \tan x)^2 + (\tan x - 1)^2 = 2 \sec^2 x$

26. $\sec x + \tan x = \frac{\cos x}{1 - \sin x}$

27. $\frac{\tan x - 1}{\tan x + 1} = \frac{1 - \cot x}{1 + \cot x}$

28. $7 \csc^2 x - 5 \cot^2 x = 2 \csc^2 x + 5$

29. $1 - \frac{\cos^2 x}{1 - \sin x} = -\sin x$

30. $\frac{1 - \cos x}{1 + \cos x} = \frac{\sec x - 1}{\sec x + 1}$

31. $\frac{\sec x - 1}{\tan x} = \frac{\tan x}{\sec x + 1}$

32. $\frac{\cos x}{\cos x - \sin x} = \frac{1}{1 - \tan x}$

33. $(\sec x + \tan x)^2 = \frac{1 + \sin x}{1 - \sin x}$

34. $\frac{\cos x}{1 + \tan x} - \frac{\sin x}{1 + \cot x} = \cos x - \sin x$

35. $\frac{\cot x + \csc x - 1}{\cot x - \csc x + 1} = \csc x + \cot x$

36. $\frac{\tan x + \cot x}{\tan x - \cot x} = \frac{1}{\sin^2 x - \cos^2 x}$

37. $\frac{1 - \cot^2 x}{1 + \cot^2 x} + 1 = 2 \sin^2 x$

38. $\frac{1 + \cos x}{1 - \cos x} - \frac{1 - \cos x}{1 + \cos x} = 4 \cot x \csc x$

39. $\frac{\sin^3 x - \cos^3 x}{\sin x - \cos x} = 1 + \sin x \cos x$

40. $1 + \sec^2 x \sin^2 x = \sec^2 x$
41. $\frac{1 + \csc x}{\sec x} = \cos x + \cot x$
42. $\tan^2 x = \sec^2 x - \sin^2 x - \cos^2 x$
43. $\frac{\sin x}{1 - \cos x} + \frac{\sin x}{1 + \cos x} = 2 \csc x$
44. $\frac{\tan x + \sec x}{\sec x} - \frac{\tan x + \sec x}{\tan x} = -\cos x \cot x$
45. $\cos^2(\alpha - \beta) - \cos^2(\alpha + \beta) = \sin^2(\alpha + \beta) - \sin^2(\alpha - \beta)$
46. $\tan x \csc x - \sec^2 x \cos x = 0$
47. $(1 + \tan x)^2 - 2 \tan x = \frac{1}{(1 - \sin x)(1 + \sin x)}$
48. $\frac{3 \csc^2 x - 5 \csc x - 28}{\csc x - 4} = \frac{3}{\sin x} + 7$
49. $(\sec^2 x - 1)(\sec^2 x + 1) = \tan^4 x + 2 \tan^2 x$
50. $\frac{\csc x}{\cot x} - \frac{\cot x}{\csc x} = \frac{\sin x}{\cot x}$
51. $\frac{1 - \cos^2 x}{1 + \cos x} = \frac{\sec x - 1}{\sec x}$
52. $\frac{\cos x}{1 + \cos x} = \frac{\sec x - 1}{\tan^2 x}$
53. $\frac{1 - 2 \sin^2 x}{1 + 2 \sin x \cos x} = \frac{\cos x - \sin x}{\cos x + \sin x}$
54. $(\cos x - \sin x)^2 + (\cos x + \sin x)^2 = 2$
55. $\frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x} = 2 \csc x$
56. $\frac{\sin x + \tan x}{\cot x + \csc x} = \sin x \tan x$
57. $\csc^2 x \sec^2 x = \sec^2 x + \csc^2 x$
58. $\cos^2 x + 1 = 2 \cos^2 x + \sin^2 x$
59. $1 - \frac{\cos^2 x}{1 + \sin x} = \sin x$
60. $\cot^2 x = (\csc x - 1)(\csc x + 1)$
61. $\frac{\csc x + \cot x}{\tan x + \sin x} = \csc x \cot x$
62. $\frac{1 - \sec x}{\tan x} + \frac{\tan x}{1 - \sec x} = -2 \csc x$
63. $\csc x - \sin x = \cos x \cot x$
64. $\cot^3 x = \cot x (\csc^2 x - 1)$
65. $\frac{\cot^2 x}{\csc x - 1} = \frac{1 + \sin x}{\sin x}$
66. $\cot^2 x + \csc^2 x = 2 \csc^2 x - 1$
67. $\frac{\cot^2 x}{1 + \csc x} = \csc x - 1$
68. $\sec^4 x - \tan^4 x = \sec^2 x + \tan^2 x$
69. $\frac{\cos x}{1 + \sin x} + \frac{1 + \sin x}{\cos x} = 2 \sec x$
70. $\frac{\sin x + \cos x}{\sin x - \cos x} = \frac{1 + 2 \sin x \cos x}{2 \sin^2 x - 1}$
71. $\frac{\csc x - 1}{\csc x + 1} = \frac{\cot^2 x}{\csc^2 x + 2 \csc x + 1}$
72. $\csc^4 x - \cot^4 x = \csc^2 x + \cot^2 x$
73. $\tan\left(\frac{\pi}{4} + x\right) = \cot\left(\frac{\pi}{4} - x\right)$
74. $\frac{\sin \theta}{1 + \sin \theta} - \frac{\sin \theta}{1 - \sin \theta} = -2 \tan^2 \theta$
75. $\csc^2 x - \cos^2 x \csc^2 x = 1$
76. $1 - 2 \sin^2 x = 2 \cos^2 x - 1$
77. $\csc^2 x - \cos x \sec x = \cot^2 x$
78. $(\sec x - \tan x)(\sec x + \tan x) = 1$
79. $(1 + \tan^2 x)(1 - \sin^2 x) = 1$
80. $10 \csc^2 x - 6 \cot^2 x = 4 \csc^2 x + 6$