Section 4.2 – Calculus with Parametric Curves

Tangents and Areas

A parametrized curve $x = f(t)$ and $y = g(t)$ is differentiable at *t* if *f* and *g* are differentiable at *t*.

Parametric Formula for dy/dx

If all three derivatives exist and $\frac{dx}{dt} \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

The derivatives $\frac{dy}{dt}$, $\frac{dx}{dt}$, and $\frac{dy}{dx}$ are related by the Chain Rule $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

Parametric Formula for d^2y/dx^2

If the equations $x = f(t)$, $y = g(t)$ define *y* as a twice–differentiable function of *x*, then at any point where $\frac{dx}{dt} \neq 0$ and $y' = \frac{dy}{dx}$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

Example

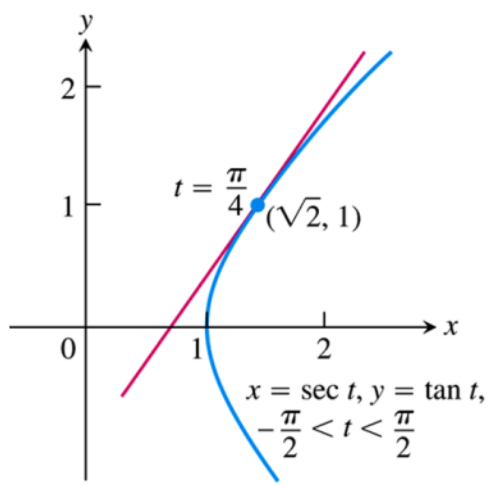
Find the tangent to the curve $x = \sec t$, $y = \tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$, at the point $(\sqrt{2},\ 1)$, where $t = \frac{\pi}{4}$

Solution

The slope of the curve at *t* is:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t}$$

$$t = \frac{\pi}{4} \quad \Rightarrow \quad \boxed{m = \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}}} = \frac{\sec\frac{\pi}{4}}{\tan\frac{\pi}{4}} = \frac{\sqrt{2}}{1} = \boxed{\sqrt{2}}$$



The tangent line is

$$y = m\left(x - x_1\right) + y_1$$

$$y = \sqrt{2}\left(x - \sqrt{2}\right) + 1 = \sqrt{2}x - 2 + 1 = \sqrt{2}\ x - 1$$

$$\boxed{= \sqrt{2}\ x - 1}$$

Example

Find $\dfrac{d^2y}{dx^2}$ as a function of t if $x = t - t^2, \quad y = t - t^3$

Solution

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t}$$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1-3t^2}{1-2t}\right)$$

$$u = 1-3t^2 \qquad v = 1-2t$$
$$u' = -6t \qquad v' = -2$$

$$= \frac{-6t(1-2t) - (-2)\left(1-3t^2\right)}{(1-2t)^2}$$

$$= \frac{-6t + 12t^2 + 2 - 6t^2}{(1-2t)^2}$$

$$= \frac{6t^2 - 6t + 2}{(1-2t)^2}$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$= \frac{6t^2 - 6t + 2}{(1-2t)^2} \div (1-2t)$$

$$\boxed{= \frac{6t^2 - 6t + 2}{(1-2t)^3}}$$

Example

Find the area enclosed by the asteroid: $x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le 2\pi$

Solution

By symmetry, the enclosed area is 4 times the area beneath the curve in the first quadrant where $0 \le t \le \frac{\pi}{2}$.

$$A = 4\int_0^1 y\,dx$$

$$dx = d\left(\cos^3 t\right) = 3\cos^2 t \sin t\,dt$$

$$= 4\int_0^{\pi/2} \sin^3 t \cdot 3\cos^2 t \sin t \; dt$$

$$= 12\int_0^{\pi/2} \sin^4 t \cdot \cos^2 t \; dt$$

$$= 12\int_0^{\pi/2} \left(\frac{1-\cos 2t}{2}\right)^2 \left(\frac{1+\cos 2t}{2}\right) \; dt$$

$$= \frac{3}{2}\int_0^{\pi/2} \left(1 - 2\cos 2t + \cos^2 2t\right)\left(1+\cos 2t\right) \; dt$$

$$= \frac{3}{2}\int_0^{\pi/2} \left(1 - 2\cos 2t + \cos^2 2t + \cos 2t - 2\cos^2 2t + \cos^3 2t\right) \; dt$$

$$= \frac{3}{2}\int_0^{\pi/2} \left(1 - \cos 2t - \cos^2 2t + \cos^3 2t\right) \; dt$$

$$= \frac{3}{2}\left[t - \frac{1}{2}\sin 2t\right]_0^{\pi/2} - \frac{3}{2}\int_0^{\pi/2} \cos^2 2t \; dt + \frac{3}{2}\int_0^{\pi/2} \cos^3 2t \; dt$$

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

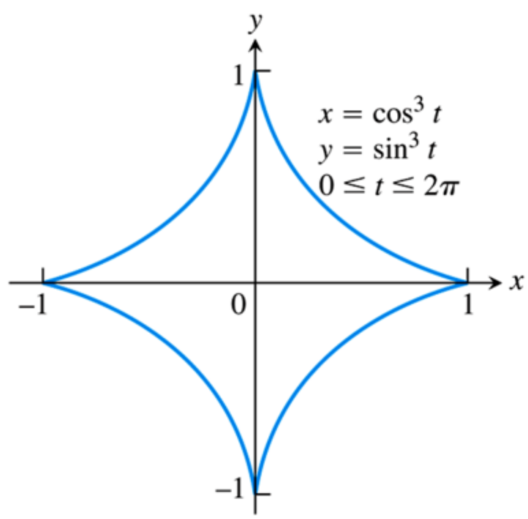
$$= \frac{3}{2}\left(\frac{\pi}{2} - 0\right) - \frac{3}{2}\int_0^{\pi/2} \frac{1+\cos 2t}{2} \; dt + \frac{3}{2}\int_0^{\pi/2} \left(1-\sin^2 2t\right)\cos 2t \; dt$$

$$= \frac{3\pi}{4} - \frac{3}{4}\left[t + \frac{1}{2}\sin 2t\right]_0^{\pi/2} + \frac{3}{4}\int_0^{\pi/2} \left(1-\sin^2 2t\right) d\left(\sin 2t\right)$$

$$= \frac{3\pi}{4} - \frac{3}{4}\left(\frac{\pi}{2} - 0\right) + \frac{3}{4}\left[\sin 2t - \frac{1}{3}\sin^3 2t\right]_0^{\pi/2}$$

$$= \frac{3\pi}{4} - \frac{3\pi}{8} + \frac{3}{4}(0-0)$$

$$\boxed{= \frac{3\pi}{8}}$$

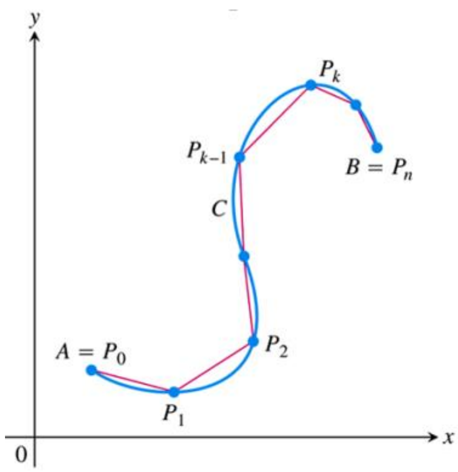


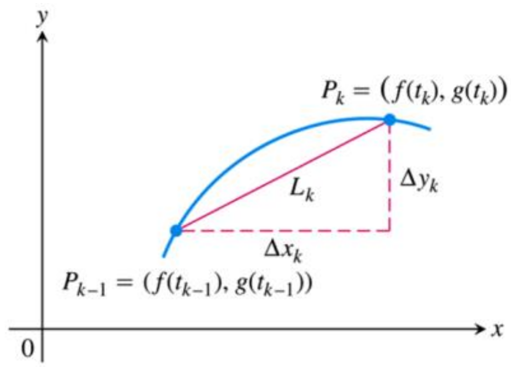
Length of a Parametrically Defined Curve

Definition

If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = a$, then the length of C is the definite integral

$$L = \int_a^b \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}\; dt$$





Example

Find the length of the circle of radius r defined parametrically by $x = r\cos t, \quad y = r\sin t, \quad 0 \le t \le 2\pi$

Solution

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

$$= \int_0^{2\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\; dt$$

$$= \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\; dt$$

$$= \int_0^{2\pi} \sqrt{r^2\left(\sin^2 t + \cos^2 t\right)}\; dt$$

$$= \int_0^{2\pi} r\; dt$$

$$= rt\Big|_0^{2\pi}$$

$$= 2\pi r \;\; unit$$

Example

Find the length of the asteroid: $x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le 2\pi$

Solution

Because of the curve's symmetry with respect to the coordinate axes, its length is 4 times the length of the first quadrant.

$$\left(\frac{dx}{dt}\right)^2 = \left[3\cos^2 t(-\sin t)\right]^2 = 9\cos^4 t \sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = \left[3\sin^2 t(\cos t)\right]^2 = 9\sin^4 t \cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}$$

$$= 3\left|\cos t \sin t\right| \sqrt{\cos^2 t + \sin^2 t} \qquad \cos^2 t + \sin^2 t = 1$$

$$= 3\cos t \sin t \qquad \cos t \sin t \ge 0, \quad 0 \le t \le \tfrac{\pi}{2}$$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 4\int_0^{\pi/2} 3\cos t \sin t\, dt \qquad \sin 2t = 2\cos t \sin t$$

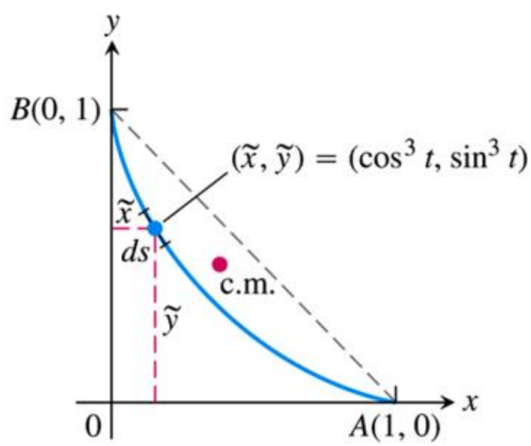
$$= \frac{12}{2}\int_0^{\pi/2} \sin 2t\, dt$$

$$= -\frac{6}{2}\cos 2t \Big|_0^{\pi/2}$$

$$= -3(-1-1)$$

$$= -3(-2)$$

$$\underline{= 6 \ \ unit|}$$



Area of Surface of Revolution for Parametrized Curves

If a smooth curve $x = f(t)$ and $y = g(t)$, $a \le t \le b$, is traversed exactly once as *t* increases from *a* to *b*, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. **Revolution about the *x*–axis ($y \ge 0$):**

$$S = 2\pi \int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

2. **Revolution about the *y*–axis ($x \ge 0$):**

$$S = 2\pi \int_a^b x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Example

The standard parametrization of the circle of radius 1 centered at the point (0, 1) in the *xy*–plane is

$$x = \cos t, \quad y = 1 + \sin t, \quad 0 \le t \le 2\pi$$

Use the parametrization to find the area of the surface swept out by revolving the circle about the x–axis.

Solution

$$x = \cos t \quad \Rightarrow \quad \left(\frac{dx}{dt}\right)^2 = (-\sin t)^2$$

$$y = 1 + \sin t \quad \Rightarrow \quad \left(\frac{dy}{dt}\right)^2 = (\cos t)^2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\sin^2 t + \cos^2 t} = 1$$

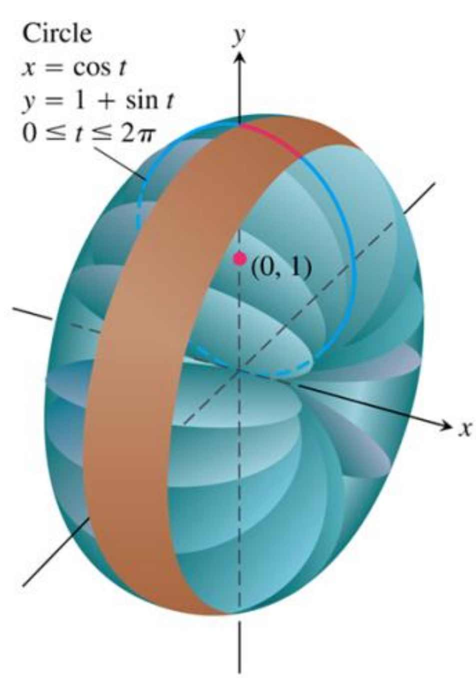
$$S = \int_a^b 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= \int_0^{2\pi} 2\pi(1 + \sin t)\ dt$$

$$= 2\pi\left[t - \cos t\right]_0^{2\pi}$$

$$= 2\pi\left(2\pi - 1 - (0 - 1)\right)$$

$$= 4\pi^2 \ \ unit^2$$



Exercises Section 4.2 – Calculus with Parametric Curves

Find all the points at which the curve has the given slope.

1. $x = 4\cos t, \quad y = 4\sin t; \quad slope = \frac{1}{2}$

2. $x = 2\cos t, \quad y = 8\sin t; \quad slope = -1$

3. $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}; \quad slope = 1$

4. $x = 2 + \sqrt{t}, \quad y = 2 - 4t; \quad slope = -8$

Find an equation of the line tangent to the curve at the point corresponding to the given value of t.

5. $x = \sin t, \quad y = \cos t, \quad t = \frac{\pi}{4}$

6. $x = t^2 - 1, \quad y = t^3 + t, \quad t = 2$

7. $x = e^t, \quad y = \ln(t+1), \quad t = 0$

8. $x = \cos t + t\sin t, \quad y = \sin t - t\cos t, \quad t = \frac{\pi}{4}$

9. $x = 6t, \quad y = t^2 + 4, \quad t = 1$

10. $x = t - 2, \quad y = \frac{1}{t} + 3, \quad t = 1$

11. $x = t^2 - t + 2, \quad y = t^3 - 3t, \quad t = -1$

12. $x = -t^2 + 3t, \quad y = 2t^{3/2}, \quad t = \frac{1}{4}$

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point

13. $x = \sin 2\pi t, \quad y = \cos 2\pi t, \quad t = -\frac{1}{6}$

14. $x = \cos t, \quad y = \sqrt{3}\cos t, \quad t = \frac{2\pi}{3}$

15. $x = t, \quad y = \sqrt{t}, \quad t = \frac{1}{4}$

16. $x = \sec^2 t - 1, \quad y = \tan t, \quad t = -\frac{\pi}{4}$

17. $x = \frac{1}{t+1}, \quad y = \frac{t}{t-1}, \quad t = 2$

18. $x = t + e^t, \quad y = 1 - e^t, \quad t = 0$

19. $x = 4t, \quad y = 3t - 2, \quad t = 3$

20. $x = \sqrt{t}, \quad y = 3t - 1, \quad t = 1$

21. $x = t + 1, \quad y = t^2 + 3t, \quad t = -1$

22. $x = t^2 + 5t + 4, \quad y = 4t, \quad t = 0$

23. $x = 4\cos\theta, \quad y = 4\sin\theta, \quad \theta = \frac{\pi}{4}$

24. $x = \cos\theta, \quad y = 3\sin\theta, \quad \theta = 0$

25. $x = 2 + \sec\theta, \quad y = 1 + 2\tan\theta, \quad \theta = \frac{\pi}{6}$

26. $x = \sqrt{t}, \quad y = \sqrt{t-1}, \quad t = 2$

27. $x = \cos^3\theta, \quad y = \sin^3\theta, \quad \theta = \frac{\pi}{4}$

28. $x = \theta - \sin\theta, \quad y = 1 - \cos\theta, \quad \theta = \pi$

Find the equations of the tangent lines at the point where the curve crosses itself

29. $x = 2\sin 2t, \quad y = 3\sin t$

30. $x = 2 - \pi\cos t, \quad y = 2t - \pi\sin t$

31. $x = t^2 - t, \quad y = t^3 - 3t - 1$

32. $x = t^3 - 6t, \quad y = t^2$

Find the slope of the curve $x = f(t),\ y = g(t)$ at the given value of t. Define x and y as differentiable functions.

33. $x^3 + 2t^2 = 9, \quad 2y^3 - 3t^2 = 4, \quad t = 2$

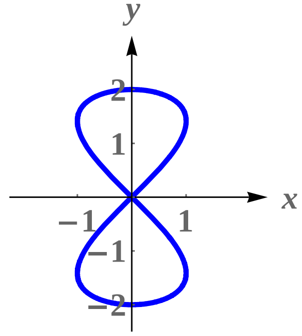
34. $x + 2x^{3/2} = t^2 + t, \quad y\sqrt{t+1} + 2t\sqrt{y} = 4, \quad t = 0$

35. $t = \ln(x - t), \quad y = te^t, \quad t = 0$

36. Consider Lissajous curve, estimate the coordinates of the points on the curve at which there is

$$x = \sin 2t, \quad y = 2\sin t; \quad 0 \le t \le 2\pi$$

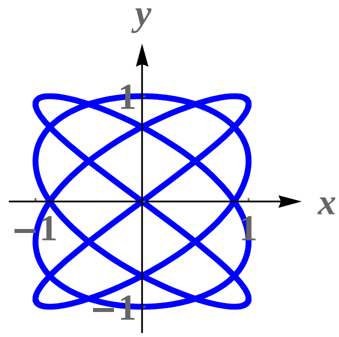
a) A horizontal tangent line
b) A vertical tangent line.



37. Consider Lissajous curve, estimate the coordinates of the points on the curve at which there is

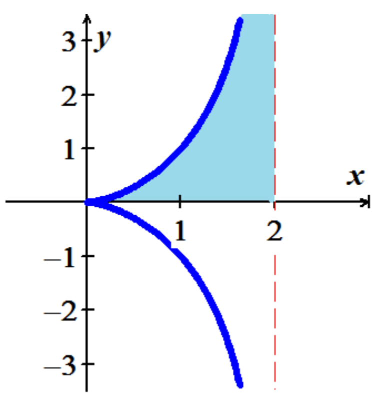
$$x = \sin 4t, \quad y = \sin 3t; \quad 0 \le t \le 2\pi$$

a) A horizontal tangent line
b) A vertical tangent line.

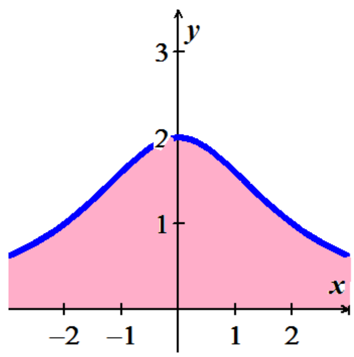


Find the area of the region

38. $x = 2\sin^2\theta, \quad y = 2\sin^2\theta\tan\theta, \quad 0 \le \theta < \frac{\pi}{2}$



39. $x = 2\cot\theta, \quad y = 2\sin^2\theta, \quad 0 \le \theta < \pi$



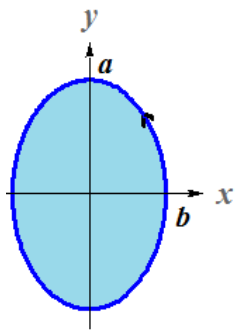
40. Find the area under one arch of the cycloid $x = a(t - \sin t), \quad y = a(1 - \cos t)$

41. Find the area enclosed by the y–axis and the curve $x = t - t^2, \quad y = 1 + e^{-t}$

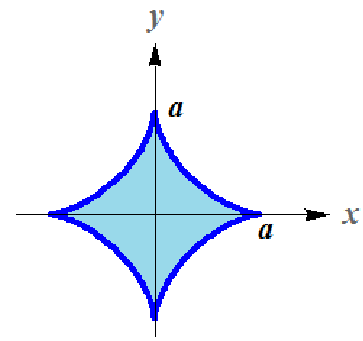
42. Find the area enclosed by the ellipse $x = a\cos t, \quad y = b\sin t, \quad 0 \le t \le 2\pi$

Find the area of the closed curve

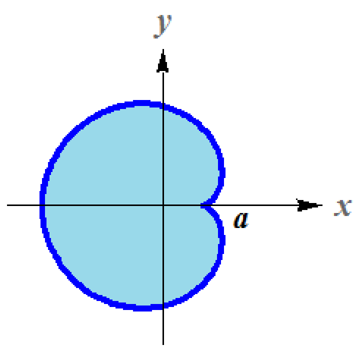
43. *Ellipse* $\begin{cases} x = b\cos t \\ y = a\sin t \end{cases}$ $0 \le t \le 2\pi$



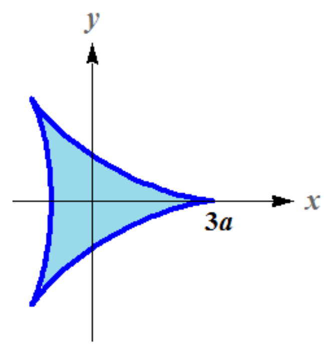
44. *Astroid* $\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases}$ $0 \le t \le 2\pi$



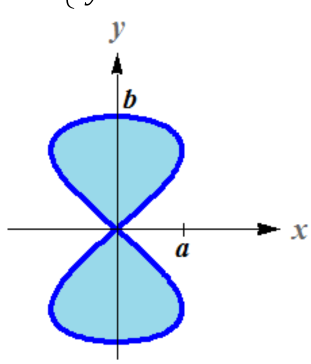
45. *Cardioid* $\begin{cases} x = 2a\cos t - a\cos 2t \\ y = 2a\sin t - a\sin 2t \end{cases}$ $0 \le t \le 2\pi$



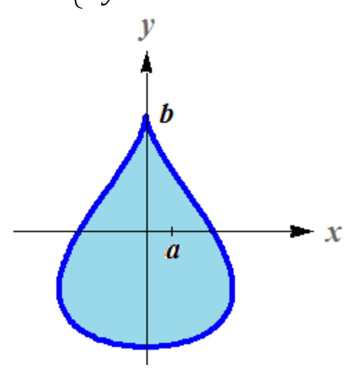
46. *Deltoid* $\begin{cases} x = 2a\cos t + a\cos 2t \\ y = 2a\sin t - a\sin 2t \end{cases}$ $0 \le t \le 2\pi$



47. *Hourglass* $\begin{cases} x = a\sin 2t \\ y = b\sin t \end{cases}$ $0 \le t \le 2\pi$



48. *Teardrop* $\begin{cases} x = 2a\cos t - a\sin 2t \\ y = b\sin t \end{cases}$ $0 \le t \le 2\pi$



Find the lengths of the curves

49. $x = \cos t, \quad y = t + \sin t, \quad 0 \le t \le \pi$

50. $x = t^3, \quad y = \frac{3}{2}t^2, \quad 0 \le t \le \sqrt{3}$

51. $x = 8\cos t + 8t\sin t, \quad y = 8\sin t - 8t\cos t, \quad 0 \le t \le \frac{\pi}{2}$

52. $x = \ln(\sec t + \tan t) - \sin t, \quad y = \cos t, \quad 0 \le t \le \frac{\pi}{3}$

53. Hypocycloid perimeter curve: $x = a\cos\theta, \quad y = a\sin\theta$

54. Circle circumference: $x = a\cos^3\theta, \quad y = a\sin^3\theta$

55. Cycloid arch: $x = a(\theta - \sin\theta), \quad y = a(1-\cos\theta)$

56. Involute of a circle: $x = \cos\theta + \theta\sin\theta, \quad y = \sin\theta - \theta\cos\theta$

Find the areas of the surfaces generated by revolving the curves

57. $x = \frac{1}{3}t^3, \quad y = t+1, \quad 1 \le t \le 2, \quad$ *y-axis*

58. $x = \frac{2}{3}t^{3/2}, \quad y = 2\sqrt{t}, \quad 0 \le t \le \sqrt{3}; \quad x-axis$

59. $x = t + \sqrt{2}, \quad y = \frac{t^2}{2} + \sqrt{2}t, \quad -\sqrt{2} \le t \le \sqrt{2}; \quad y-axis$

60. $x = 2t, \quad y = 3t; \quad 0 \le t \le 3 \quad$ *x-axis*

61. $x = 2t, \quad y = 3t; \quad 0 \le t \le 3 \quad$ *y-axis*

62. $x = t, \quad y = 4-2t; \quad 0 \le t \le 2 \quad$ *x-axis*

63. $x = t, \quad y = 4-2t; \quad 0 \le t \le 2 \quad$ *y - axis*

64. $x = 5\cos\theta, \quad y = 5\sin\theta, \quad 0 \le \theta \le \frac{\pi}{2}, \quad$ *y - axis*

65. $x = a\cos^3\theta, \quad y = a\sin^3\theta, \quad 0 \le \theta \le \pi, \quad$ *x-axis*

66. $x = a\cos\theta, \quad y = b\sin\theta, \quad 0 \le \theta \le 2\pi$

a) *x-axis* *b)* *y-axis*

67. $x = 2t, \quad y = 3t, \quad 0 \le t \le 3$

a) *x-axis* *b)* *y-axis*

68. $x = t, \quad y = 4-2t, \quad 0 \le t \le 2$

a) *x-axis* *b)* *y-axis*

69. Use the parametric equations $x = t^2\sqrt{3}$ *and* $y = 3t - \frac{1}{3}t^3$ to

a) Graph the curve on the interval $-3 \le t \le 3$.

b) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

c) Find the equation of the tangent line at the point $\left(\sqrt{3}, \frac{8}{3}\right)$

d) Find the length of the curve

e) Find the surface area generated by revolving the curve about the *x-axis*

70. Use the parametric equations $x = a(\theta - \sin\theta)$ *and* $y = a(1-\cos\theta) \quad a > 0$

a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

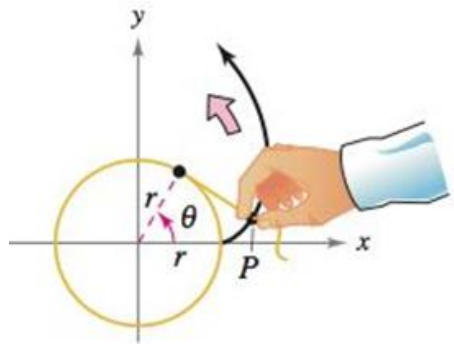
b) Find the equation of the tangent line at the point where $\theta = \frac{\pi}{6}$

c) Find all points (if any) of horizontal tangency.

d) Determine where the curve is concave upward or concave downward.

e) Find the length of one arc of the curve

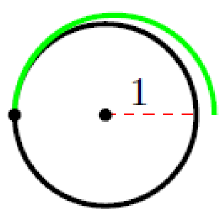
71. The involute of a circle is described by the endpoint *P* of a string that is held taut as it is unwound from a spool that does not turn.



Show that a parametric representation of the involute is

$$x = r(\cos\theta + \theta\sin\theta) \quad \textit{and} \quad y = r(\sin\theta - \theta\cos\theta)$$

72. The figure shows a piece of string tied to a circle with a radius of one unit. The string is just long enough to reach the opposite side if the circle.



Find the area that is covered when the string is unwounded counterclockwise.