Solution Section 4.3 – LU−Decompositions

Exercise

What matrix E puts A into triangular form $EA = U$? Multiply by $E^{-1} = L$ to factor A into LU:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}$$

Solution

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{pmatrix} \quad R_3 - 3R_1 \quad : \ell_{31}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{pmatrix} = U$$

$$E_{31}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{pmatrix} = U$$

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

$$L = E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

$$A = LU$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{pmatrix}$$

Exercise

Solve $L\vec{c} = \vec{b}$ to find $\vec{c}$. Then solve $U\vec{x} = \vec{c}$ to find $\vec{x}$. What was A?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Solution

$$L\vec{c} = \vec{b}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

$$\begin{cases} \boxed{c_1 = 4} \\ c_1 + c_2 = 5 \Rightarrow c_2 = 5 - 4 = 1 \\ c_1 + c_2 + c_3 = 6 \Rightarrow c_3 = 6 - 4 - 1 = 1 \end{cases}$$

$$\vec{c} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

$$U\vec{x} = \vec{c}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x + y + z = 4 \\ y + z = 1 \\ z = 1 \end{cases} \rightarrow \begin{cases} x = 3 \\ y = 0 \end{cases}$$

$$\vec{x} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

$$L\vec{c} = \vec{b}$$

$$LU\vec{x} = \vec{b}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}}_{\vec{b}}$$

Exercise

Find L and U for the symmetric matrix

$$A=\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on $a,\ b,\ c,\ d$ to get $A=LU$ with four pivots

Solution

$$A=\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

Exercise

For which c is $A=LU$ impossible – with three pivots?

$$A=\begin{pmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad R_2-3R_1$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & c-6 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad R_3-R_1 \qquad \to c-6\neq 0 \Rightarrow \boxed{c\neq 6}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & c-6 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \frac{1}{c-6}R_1$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{1}{c-6} \\ 0 & 1 & 1 \end{pmatrix} \quad R_3 - R_2$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{1}{c-6} \\ 0 & 0 & \frac{c-7}{c-6} \end{pmatrix} \quad \rightarrow c-7 \neq 0$$

$\Rightarrow \underline{c \neq 7}$

LU will be impossible for $c = 6$ *and* $c = 7$

Exercise

Find an LU-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix} \quad \frac{1}{2}R_1 \quad \boxed{\frac{1}{2} : \ell_{11}}$$

$$\begin{bmatrix} 1 & 4 \\ -1 & -1 \end{bmatrix} \quad R_2 + R_1 \quad \boxed{1 : \ell_{21}}$$

$$\begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} \quad \frac{1}{3}R_2 \quad \boxed{\frac{1}{3} : \ell_{22}}$$

$$\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \quad \boldsymbol{U}$$

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{E_1^{-1}} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \xrightarrow{E_2^{-1}} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ -1 & \frac{1}{3} \end{bmatrix} \xrightarrow{E_3^{-1}} \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} \quad \boldsymbol{L}$$

$$\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$\begin{cases} 2y_1 = -2 \\ -y_1 + 3y_2 = -2 \end{cases}$$

$$\begin{cases} y_1 = -1 \\ y_2 = -1 \end{cases}$$

$$\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\begin{cases} x_1 + 4x_2 = -1 \\ \underline{x_2 = -1} \end{cases} \rightarrow \underline{x_1 = 3}$$

The solution: $x_1 = 3$ *and* $x_2 = -1$

Exercise

Find an LU-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} -5 & -10 \\ 6 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 19 \end{bmatrix}$$

Solution

$$\begin{bmatrix} -5 & -10 \\ 6 & 5 \end{bmatrix} \quad -\tfrac{1}{5}R_1 \quad \boxed{-\tfrac{1}{5} : \ell_1}$$

$$\begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix} \quad R_2 - 6R_1 \quad \boxed{-6 : \ell_{21}}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & -7 \end{bmatrix} \quad -\tfrac{1}{7}R_2 \quad \boxed{-\tfrac{1}{7} : \ell_2}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \boldsymbol{U}$$

$$\begin{bmatrix} -\tfrac{1}{5} & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{E_1^{-1}} \begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 0 \\ -6 & 1 \end{bmatrix} \xrightarrow{E_2^{-1}} \begin{bmatrix} -5 & 0 \\ 6 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 0 \\ 6 & -\frac{1}{7} \end{bmatrix} \xrightarrow{E_3^{-1}} \begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix} \quad \boldsymbol{L}$$

$$\begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 19 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 19 \end{bmatrix}$$

$$\begin{cases} -5y_1 = -10 & \rightarrow \underline{y_1 = 2} \\ 6y_1 - 7y_2 = 19 & \Rightarrow \underline{y_2 = -1} \end{cases}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{cases} x_1 + 2x_2 = 2 \\ \underline{x_2 = -1} \end{cases} \rightarrow \underline{x_1 = 4}$$

The solution: $\underline{x_1 = 4 \quad and \quad x_2 = -1}$

Exercise

Find an LU-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix} \quad \frac{1}{2}R_1 \quad \boxed{\frac{1}{2} : \ell_1}$$

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_1^{-1}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix}1 & -1 & -1\\ 0 & -2 & 2\\ -1 & 5 & 2\end{bmatrix} \quad R_3 + R_1 \quad \boxed{1 : \ell_{31}}$$

$$\begin{bmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 1 & 0 & 1\end{bmatrix} \xrightarrow{E_2^{-1}} \begin{bmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & -1 & -1\\ 0 & -2 & 2\\ 0 & 4 & 1\end{bmatrix} \quad -\tfrac{1}{2}R_2 \quad \boxed{-\tfrac{1}{2} : \ell_{22}}$$

$$\begin{bmatrix}2 & 0 & 0\\ 0 & -\tfrac{1}{2} & 0\\ -1 & 0 & 1\end{bmatrix} \xrightarrow{E_3^{-1}} \begin{bmatrix}2 & 0 & 0\\ 0 & -2 & 0\\ -1 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & -1 & -1\\ 0 & 1 & -1\\ 0 & 4 & 1\end{bmatrix} \quad R_3 - 4R_2 \quad \boxed{-4 : \ell_{32}}$$

$$\begin{bmatrix}2 & 0 & 0\\ 0 & -2 & 0\\ -1 & -4 & 1\end{bmatrix} \xrightarrow{E_4^{-1}} \begin{bmatrix}2 & 0 & 0\\ 0 & -2 & 0\\ -1 & 4 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & -1 & -1\\ 0 & 1 & -1\\ 0 & 0 & 5\end{bmatrix} \quad \tfrac{1}{5}R_3 \quad \boxed{\tfrac{1}{5} : \ell_{32}}$$

$$\begin{bmatrix}2 & 0 & 0\\ 0 & -2 & 0\\ -1 & -4 & \tfrac{1}{5}\end{bmatrix} \xrightarrow{E_5^{-1}} \begin{bmatrix}2 & 0 & 0\\ 0 & -2 & 0\\ -1 & 4 & 5\end{bmatrix}$$

$$\begin{bmatrix}1 & -1 & -1\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix} \quad U$$

$$\begin{bmatrix}2 & 0 & 0\\ 0 & -2 & 0\\ -1 & 4 & 5\end{bmatrix} \quad L$$

$$\begin{bmatrix}2 & 0 & 0\\ 0 & -2 & 0\\ -1 & 4 & 5\end{bmatrix}\begin{bmatrix}1 & -1 & -1\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}-4\\ -2\\ 6\end{bmatrix}$$

$$\begin{bmatrix}2 & 0 & 0\\ 0 & -2 & 0\\ -1 & 4 & 5\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix} = \begin{bmatrix}-4\\ -2\\ 6\end{bmatrix}$$

$$\begin{cases} 2y_1 = -4 \\ -2y_2 = -2 \\ y_1 + 4y_2 + 5y_3 = 6 \end{cases} \Rightarrow \begin{array}{l} y_1 = -2 \\ y_2 = 1 \\ y_3 = 0 \end{array}$$

$$\begin{bmatrix}1 & -1 & -1\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}$$

$$\begin{cases} x_1 - x_2 - x_3 = -2 \rightarrow \boxed{x_1 = -1} \\ x_2 - x_3 = 1 \rightarrow \boxed{x_2 = 1} \\ \boxed{x_3 = 0} \end{cases}$$

Solution: $\boxed{x_1 = -1, \quad x_2 = 1, \quad x_3 = 0}$

Exercise

Find an *LU*-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$$

Solution

$$\begin{bmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \quad -\frac{1}{3}R_1 \quad \boxed{-\frac{1}{3} : \ell_1} \qquad \begin{bmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_1^{-1}} \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \quad R_2 - R_1 \quad \boxed{-1 : \ell_{21}} \qquad \begin{bmatrix} -3 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_2^{-1}} \begin{bmatrix} -3 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \frac{1}{2}R_2 \quad \boxed{\frac{1}{2} : \ell_{22}} \qquad \begin{bmatrix} -3 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_3^{-1}} \begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad R_3 - R_2 \quad \boxed{-1 : \ell_{32}} \qquad \begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{E_4^{-1}} \begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad U \qquad \begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad L$$

$$\begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$$

$$\begin{cases} -3y_1 = -33 \Rightarrow \underline{y_1 = 11} \\ y_1 + 2y_2 = 7 \Rightarrow \underline{y_2 = -2} \\ y_2 + y_3 = -1 \Rightarrow \underline{y_3 = 1} \end{cases}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{cases} x_1 - 4x_2 + 2x_3 = 11 \Rightarrow \underline{x_1 = 1} \\ \underline{x_2 = -2} \\ \underline{x_3 = 1} \end{cases}$$

Solution: $\underline{x_1 = 1, \quad x_2 = -2, \quad x_3 = 1}$

Exercise

Find an LU-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix} \quad \begin{matrix} R_2 + R_1 & \ell_{21} = 1 \\ R_3 - 2R_1 & \ell_{31} = -2 \end{matrix}$$

$$\begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{bmatrix} \quad R_3 + 5R_2 \quad \ell_{32} = 5$$

$$\begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}$$

$L\vec{y} = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$$

$$\begin{cases} \underline{y_1 = -7} \\ -y_1 + y_2 = 5 & \rightarrow \underline{y_2 = -2} \\ 2y_1 - 5y_2 + y_3 = 2 & \rightarrow \underline{y_3 = 6} \end{cases}$$

$U\vec{x} = \vec{y}$

$$\begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \\ 6 \end{bmatrix}$$

$$\begin{cases} 3x_1 - 7x_2 - 2x_3 = -7 & \rightarrow \underline{x_1 = 3} \\ -2x_2 - x_3 = -2 & \rightarrow \underline{x_2 = 4} \\ \underline{x_3 = -6} \end{cases}$$

Solution: $\underline{x_1 = 3, \quad x_2 = 4, \quad x_3 = -6}$

Exercise

Find an LU-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} 2 & -6 & 4 \\ -4 & 8 & 0 \\ 0 & -4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 2 & -6 & 4 \\ -4 & 8 & 0 \\ 0 & -4 & 6 \end{bmatrix} \quad R_2 + 2R_1 \quad \ell_{21} = 2$$

$$\begin{bmatrix} 2 & -6 & 4 \\ 0 & -4 & 8 \\ 0 & -4 & 6 \end{bmatrix} \quad R_3 - R_2 \qquad \ell_{32} = -1$$

$$\begin{bmatrix} 2 & -6 & 4 \\ 0 & -4 & 8 \\ 0 & 0 & -2 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$L\vec{y} = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$$

$$\begin{cases} \boxed{y_1 = 2} \\ -2y_1 + y_2 = -4 & \rightarrow \boxed{y_2 = 0} \\ y_2 + y_3 = 6 & \rightarrow \boxed{y_3 = 6} \end{cases}$$

$U\vec{x} = \vec{y}$

$$\begin{bmatrix} 2 & -6 & 4 \\ 0 & -4 & 8 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$$

$$\begin{cases} 3x_1 - 7x_2 - 2x_3 = 2 & \rightarrow \boxed{x_1 = -11} \\ -4x_2 + 8x_3 = 0 & \rightarrow \boxed{x_2 = -6} \\ -2x_3 = 6 & \rightarrow \boxed{x_3 = -3} \end{cases}$$

Solution: $\boxed{x_1 = 3, \quad x_2 = 4, \quad x_3 = -6}$

Exercise

Find an LU-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} 2 & -4 & 2 \\ -4 & 5 & 2 \\ 6 & -9 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 2 & -4 & 2 \\ -4 & 5 & 2 \\ 6 & -9 & 1 \end{bmatrix} \quad \begin{matrix} R_2 + 2R_1 & \ell_{21} = 2 \\ R_3 - 3R_1 & \ell_{31} = -3 \end{matrix}$$

$$\begin{bmatrix} 2 & -4 & 2 \\ 0 & -3 & 6 \\ 0 & 3 & -5 \end{bmatrix} \quad R_3 + R_2 \quad \ell_{32} = 1$$

$$\begin{bmatrix} 2 & -4 & 2 \\ 0 & -3 & 6 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$$

$$L\vec{y} = \vec{b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}$$

$$\begin{cases} \underline{y_1 = 6} \\ -2y_1 + y_2 = 0 & \to \underline{y_2 = 12} \\ 3y_1 - y_2 + y_3 = 6 & \to \underline{y_3 = 0} \end{cases}$$

$$U\vec{x} = \vec{y}$$

$$\begin{bmatrix} 2 & -4 & 2 \\ 0 & -3 & 6 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 0 \end{bmatrix}$$

$$\begin{cases} 2x_1 - 4x_2 + 2x_3 = 6 & \rightarrow \quad \boxed{x_1 = -5} \\ -3x_2 + 6x_3 = 12 & \rightarrow \quad \boxed{x_2 = -4} \\ & \rightarrow \quad \boxed{x_3 = 0} \end{cases}$$

Solution: $\boxed{x_1 = -5, \quad x_2 = -4, \quad x_3 = 0}$

Exercise

Find an LU-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix} \quad \begin{matrix} \\ R_2 - R_1 \\ R_3 - 3R_1 \end{matrix} \quad \begin{matrix} \\ \ell_{21} = -1 \\ \ell_{31} = -3 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & -1 \\ 0 & 10 & -1 \end{bmatrix} \quad \begin{matrix} \\ \\ R_3 + 5R_2 \end{matrix} \quad \begin{matrix} \\ \\ \ell_{32} = 5 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}$$

$L\vec{y} = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$$

$$\begin{cases} \boxed{y_1 = 0} & \\ y_1 + y_2 = -5 & \rightarrow \quad \boxed{y_2 = -5} \\ 3y_1 - 5y_2 + y_3 = 7 & \rightarrow \quad \boxed{y_3 = -18} \end{cases}$$

$U\vec{x} = \vec{y}$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -18 \end{bmatrix}$$

$$\begin{cases} x_1 - x_2 + 2x_3 = 0 & \rightarrow \quad \underline{x_1 = -5} \\ -2x_2 - x_3 = -5 & \rightarrow \quad \underline{x_2 = 1} \\ -6x_3 = -18 & \rightarrow \quad \underline{x_3 = 3} \end{cases}$$

Solution: $\underline{x_1 = -5, \quad x_2 = 1, \quad x_3 = 3}$

Exercise

Find an LU-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 3 \\ 7 \end{bmatrix}$$

Solution

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad -R_1 \quad \boxed{-1 : \ell_{11}} \qquad \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_1^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad R_2 - 2R_1 \quad \boxed{-2 : \ell_{21}} \qquad \begin{bmatrix} -1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_2^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 3 & 0 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad \tfrac{1}{3}R_2 \quad \boxed{\tfrac{1}{3} : \ell_{22}} \qquad \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_3^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad R_3 + R_2 \quad \boxed{1 : \ell_{23}} \qquad \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_4^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix}1&0&-1&0\\0&1&0&2\\0&0&2&2\\0&0&1&5\end{bmatrix} \quad \tfrac{1}{2}R_3 \quad \boxed{\tfrac{1}{2}:\ell_{23}} \qquad \begin{bmatrix}-1&0&0&0\\2&3&0&0\\0&-1&\tfrac{1}{2}&0\\0&0&0&1\end{bmatrix} \xrightarrow{E_4^{-1}} \begin{bmatrix}-1&0&0&0\\2&3&0&0\\0&-1&2&0\\0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&-1&0\\0&1&0&2\\0&0&1&1\\0&0&1&5\end{bmatrix} \quad R_4-R_3 \quad \boxed{-1:\ell_{43}} \qquad \begin{bmatrix}-1&0&0&0\\2&3&0&0\\0&-1&2&0\\0&0&-1&1\end{bmatrix} \xrightarrow{E_5^{-1}} \begin{bmatrix}-1&0&0&0\\2&3&0&0\\0&-1&2&0\\0&0&1&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&-1&0\\0&1&0&2\\0&0&1&1\\0&0&0&4\end{bmatrix} \quad \tfrac{1}{4}R_4 \quad \boxed{\tfrac{1}{4}:\ell_{43}} \qquad \begin{bmatrix}-1&0&0&0\\2&3&0&0\\0&-1&2&0\\0&0&1&\tfrac{1}{4}\end{bmatrix} \xrightarrow{E_6^{-1}} \begin{bmatrix}-1&0&0&0\\2&3&0&0\\0&-1&2&0\\0&0&1&4\end{bmatrix}$$

$$\begin{bmatrix}1&0&-1&0\\0&1&0&2\\0&0&1&1\\0&0&0&1\end{bmatrix} \quad \boldsymbol{U} \qquad \begin{bmatrix}-1&0&0&0\\2&3&0&0\\0&-1&2&0\\0&0&1&4\end{bmatrix} \quad \boldsymbol{L}$$

For lower triangular: $\begin{bmatrix}-1&0&0&0\\-2&\tfrac{1}{3}&0&0\\0&1&\tfrac{1}{2}&0\\0&0&-1&\tfrac{1}{4}\end{bmatrix} \xrightarrow{E^{-1}} \begin{bmatrix}-1&0&0&0\\2&3&0&0\\0&-1&2&0\\0&0&1&4\end{bmatrix}$

$$\begin{bmatrix}-1&0&0&0\\2&3&0&0\\0&-1&2&0\\0&0&1&4\end{bmatrix}\begin{bmatrix}1&0&-1&0\\0&1&0&2\\0&0&1&1\\0&0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}5\\-1\\3\\7\end{bmatrix}$$

$$\begin{bmatrix}-1&0&0&0\\2&3&0&0\\0&-1&2&0\\0&0&1&4\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}=\begin{bmatrix}5\\-1\\3\\7\end{bmatrix} \to \begin{cases}-y_1=5\to \underline{y_1=-5}|\\ 2y_1+3y_2=-1\to \underline{y_2=3}|\\ -y_2+2y_3=3\to \underline{y_3=3}|\\ y_3+4y_4=7\to \underline{y_4=1}|\end{cases}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 3 \\ 1 \end{bmatrix} \rightarrow \begin{cases} x_1 - x_3 = -5 \rightarrow \underline{x_1 = -3|} \\ x_2 + 2x_4 = 3 \rightarrow \underline{x_2 = 1|} \\ x_3 + x_4 = 3 \rightarrow \underline{x_3 = 2|} \\ \underline{x_4 = 1|} \end{cases}$$

Solution: $\underline{x_1 = -3, \quad x_2 = 1, \quad x_3 = 2, \quad x_4 = 3 \,|}$

Exercise

Find an LU-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} 1 & -2 & -2 & -3 \\ 3 & -9 & 0 & -9 \\ -1 & 2 & 4 & 7 \\ -3 & -6 & 26 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \\ 3 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & -2 & -2 & -3 \\ 3 & -9 & 0 & -9 \\ -1 & 2 & 4 & 7 \\ -3 & -6 & 26 & 2 \end{bmatrix} \begin{array}{ll} & \\ R_2 - 3R_1 & \ell_{21} = -3 \\ R_3 + R_1 & \ell_{31} = 1 \\ R_4 + 3R_1 & \ell_{41} = 3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & -2 & -3 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & -12 & 20 & -7 \end{bmatrix} \begin{array}{ll} & \\ & \\ & \\ R_4 - 4R_2 & \ell_{42} = -4 \end{array}$$

$$\begin{bmatrix} 1 & -2 & -2 & -3 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & -7 \end{bmatrix} \begin{array}{ll} & \\ & \\ & \\ R_4 + 2R_3 & \ell_{43} = 2 \end{array}$$

$$\begin{bmatrix} 1 & -2 & -2 & -3 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -3 & 4 & -2 & 1 \end{bmatrix}$$

$L\vec{y} = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -3 & 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \\ 3 \end{bmatrix}$$

$$\begin{cases} \boxed{y_1 = 1} & \\ 3y_1 + y_2 = 6 & \rightarrow \ \boxed{y_2 = 3} \\ -y_1 + y_3 = 0 & \rightarrow \ \boxed{y_3 = 1} \\ -3y_1 + 4y_2 - 2y_3 + y_4 = 3 & \rightarrow \ \boxed{y_4 = -4} \end{cases}$$

$U\vec{x} = \vec{y}$

$$\begin{bmatrix} 1 & -2 & -2 & -3 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ -4 \end{bmatrix}$$

$$\begin{cases} x_1 - x_2 + 2x_3 - 3x_4 = 1 & \rightarrow \ \boxed{x_1 = 38} \\ -3x_2 + 6x_3 = 3 & \rightarrow \ \boxed{x_2 = 16} \\ 2x_3 + 4x_4 = 1 & \rightarrow \ \boxed{x_3 = \frac{17}{2}} \\ & \rightarrow \ \boxed{x_4 = -4} \end{cases}$$

Solution: $\boxed{x_1 = 38, \quad x_2 = 16, \quad x_3 = \frac{17}{2}, \quad x_4 = -4}$

Exercise

Find an LU factorization matrix $\begin{pmatrix} 2 & 5 \\ -3 & -4 \end{pmatrix}$

Solution

$$\begin{pmatrix} 2 & 5 \\ -3 & -4 \end{pmatrix} \quad R_2 + \frac{3}{2}R_1 \quad \ell_{21} = \frac{3}{2}$$

$$\begin{pmatrix} 2 & 5 \\ 0 & \frac{7}{2} \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 \\ -\frac{3}{2} & 1 \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} 2 & 5 \\ -3 & -4 \end{pmatrix}} \underset{=}{=} \underset{L}{\begin{pmatrix} 1 & 0 \\ -\frac{3}{2} & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} 2 & 5 \\ 0 & \frac{7}{2} \end{pmatrix}}$$

Exercise

Find an LU factorization matrix $\begin{pmatrix} 6 & 4 \\ 12 & 5 \end{pmatrix}$

Solution

$$\begin{pmatrix} 6 & 4 \\ 12 & 5 \end{pmatrix} \quad R_2 - 2R_1 \quad \ell_{21} = -2$$

$$\begin{pmatrix} 6 & 4 \\ 0 & -3 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} 6 & 4 \\ 12 & 5 \end{pmatrix}} \underset{=}{=} \underset{L}{\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} 6 & 4 \\ 0 & -3 \end{pmatrix}}$$

Exercise

Find an LU factorization matrix $\begin{pmatrix} 3 & 1 & 2 \\ -9 & 0 & -4 \\ 9 & 9 & 14 \end{pmatrix}$

Solution

$$\begin{pmatrix} 3 & 1 & 2 \\ -9 & 0 & -4 \\ 9 & 9 & 14 \end{pmatrix} \quad \begin{matrix} R_2 + 3R_1 & \ell_{21} = 3 \\ R_3 - 3R_1 & \ell_{31} = -3 \end{matrix}$$

$$\begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 6 & 8 \end{pmatrix} \quad R_3 - 2R_2 \quad \ell_{32} = -2$$

$$\begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 3 & 1 & 2 \\ -9 & 0 & -4 \\ 9 & 9 & 14 \end{pmatrix}}_{A} \underbrace{=}_{=} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix}}_{U}$$

Exercise

Find an *LU* factorization matrix $\begin{pmatrix} -5 & 0 & 4 \\ 10 & 2 & -5 \\ 10 & 10 & 16 \end{pmatrix}$

Solution

$$\begin{pmatrix} -5 & 0 & 4 \\ 10 & 2 & -5 \\ 10 & 10 & 16 \end{pmatrix} \quad \begin{matrix} R_2 + 2R_1 & \ell_{21} = -2 \\ R_3 + 2R_1 & \ell_{31} = -2 \end{matrix}$$

$$\begin{pmatrix} -5 & 0 & 4 \\ 0 & 2 & 3 \\ 0 & 10 & 24 \end{pmatrix} \quad R_3 - 5R_2 \quad \ell_{31} = -5$$

$$\begin{pmatrix} -5 & 0 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 9 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 5 & 1 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} -5 & 0 & 4 \\ 10 & 2 & -5 \\ 10 & 10 & 16 \end{pmatrix}}_{A} \underbrace{=}_{=} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 5 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} -5 & 0 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 9 \end{pmatrix}}_{U}$$

Exercise

Find an LU factorization matrix $\begin{pmatrix} 3 & 7 & 2 \\ 6 & 19 & 4 \\ 9 & 9 & 14 \end{pmatrix}$

Solution

$$\begin{pmatrix} 3 & 7 & 2 \\ 6 & 19 & 4 \\ 9 & 9 & 14 \end{pmatrix} \quad \begin{matrix} R_2 - 2R_1 & \ell_{21} = -2 \\ R_3 - 3R_1 & \ell_{31} = -3 \end{matrix}$$

$$\begin{pmatrix} 3 & 7 & 2 \\ 0 & 5 & 0 \\ 0 & -12 & 8 \end{pmatrix} \quad R_3 + \frac{12}{5}R_2 \quad \ell_{32} = \frac{12}{5}$$

$$\begin{pmatrix} 3 & 7 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -\frac{12}{5} & 1 \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} 3 & 7 & 2 \\ 6 & 19 & 4 \\ 9 & 9 & 14 \end{pmatrix}} = \underset{L}{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -\frac{12}{5} & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} 3 & 7 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{pmatrix}}$$

Exercise

Find an LU factorization matrix $\begin{pmatrix} 2 & 3 & 2 \\ 4 & 13 & 9 \\ -6 & 5 & 4 \end{pmatrix}$

Solution

$$\begin{pmatrix} 2 & 3 & 2 \\ 4 & 13 & 9 \\ -6 & 5 & 4 \end{pmatrix} \quad \begin{matrix} R_2 - 2R_1 & \ell_{21} = -2 \\ R_3 + 3R_1 & \ell_{31} = 3 \end{matrix}$$

$$\begin{pmatrix} 2 & 3 & 2 \\ 0 & 7 & 5 \\ 0 & 14 & 10 \end{pmatrix} \quad R_3 - 2R_2 \quad \ell_{32} = -2$$

$$\begin{pmatrix} 2 & 3 & 2 \\ 0 & 7 & 5 \\ 0 & 0 & 0 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 2 & 1 \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} 2 & 3 & 2 \\ 4 & 13 & 9 \\ -6 & 5 & 4 \end{pmatrix}} \underset{=}{=} \underset{L}{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 2 & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} 2 & 3 & 2 \\ 0 & 7 & 5 \\ 0 & 0 & 0 \end{pmatrix}}$$

Exercise

Find an LU factorization matrix $\begin{pmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ -2 & -4 & 7 & 5 \end{pmatrix}$

Solution

$$\begin{pmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ -2 & -4 & 7 & 5 \end{pmatrix} \qquad \begin{matrix} R_2 + R_1 & \ell_{21} = 1 \\ R_3 - 4R_1 & \ell_{31} = -4 \\ R_4 + 2R_1 & \ell_{41} = 2 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & -10 & 15 & 5 \\ 0 & 2 & -3 & -1 \end{pmatrix} \qquad \begin{matrix} R_3 - 5R_2 & \ell_{32} = -5 \\ R_4 + R_2 & \ell_{42} = 1 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 4 & 5 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ -2 & -4 & 7 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 4 & 5 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A \quad = \quad L \quad\quad U$$

Exercise

Find an LU factorization matrix
$$\begin{pmatrix} 1 & 3 & 1 & 5 \\ 5 & 20 & 6 & 31 \\ -2 & -1 & -1 & -4 \\ -1 & 7 & 1 & 7 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 1 & 3 & 1 & 5 \\ 5 & 20 & 6 & 31 \\ -2 & -1 & -1 & -4 \\ -1 & 7 & 1 & 7 \end{pmatrix} \quad \begin{matrix} R_2 - 5R_1 & \ell_{21} = -5 \\ R_3 + 2R_1 & \ell_{31} = 2 \\ R_4 + R_1 & \ell_{41} = 1 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 1 & 5 \\ 0 & 5 & 1 & 5 \\ 0 & 5 & 1 & 6 \\ 0 & 10 & 2 & 12 \end{pmatrix} \quad \begin{matrix} R_3 - R_2 & \ell_{32} = -1 \\ R_4 - 2R_2 & \ell_{42} = -2 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 1 & 5 \\ 0 & 5 & 1 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 1 & 5 \\ 5 & 20 & 6 & 31 \\ -2 & -1 & -1 & -4 \\ -1 & 7 & 1 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 & 5 \\ 0 & 5 & 1 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$A \quad = \quad L \quad\quad U$$

Exercise

Find an LU factorization matrix $\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{pmatrix}$

Solution

$$\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{pmatrix} \quad \begin{array}{ll} R_2 + 2R_1 & \ell_{21} = 2 \\ R_3 - R_1 & \ell_{31} = -1 \\ R_4 + 3R_1 & \ell_{41} = 3 \end{array}$$

$$\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{pmatrix} \quad \begin{array}{ll} R_3 + 3R_2 & \ell_{32} = 3 \\ R_4 - 4R_2 & \ell_{42} = -4 \end{array}$$

$$\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{pmatrix} \quad R_4 - 2R_4 \quad \ell_{44} = -2$$

$$\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{pmatrix}} = \underset{L}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}}$$

Exercise

Find an LU factorization matrix $\begin{pmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{pmatrix}$

Solution

$$\begin{pmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{pmatrix} \quad \begin{array}{ll} R_2 - 3R_1 & \ell_{21} = -3 \\ R_3 - R_1 & \ell_{31} = -1 \\ R_4 - 2R_1 & \ell_{41} = -2 \\ R_5 + 3R_1 & \ell_{51} = 3 \end{array}$$

$$\begin{pmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & -3 & -1 & 6 \\ 0 & 6 & 2 & -7 \\ 0 & -9 & -3 & 13 \end{pmatrix} \quad \begin{array}{ll} R_3 + R_2 & \ell_{32} = 1 \\ R_4 - 2R_2 & \ell_{42} = -2 \\ R_5 + 3R_2 & \ell_{52} = 3 \end{array}$$

$$\begin{pmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 10 \end{pmatrix} \quad \begin{array}{ll} R_4 + R_3 & \ell_{43} = 1 \\ R_5 - 2R_3 & \ell_{53} = -2 \end{array}$$

$$\begin{pmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & -1 & 1 & 0 \\ -3 & -3 & 2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 1 & 0 \\ -3 & -3 & 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A \qquad = \qquad L \qquad\qquad U$$

Exercise

Let A be a lower triangular $n \times n$ matrix with nonzero entries on the diagonal. Show that A is invertible and A^{-1} is lower triangular.

Solution

Since A is a lower triangular $n \times n$ matrix with nonzero entries on the diagonal, then the determinant is equal to the products of the main diagonal entries.

Therefore, A^{-1} exists and A is invertible.

To find A^{-1}

$$\begin{bmatrix} A & I \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

$$\left[\begin{array}{cccc|cccc} a_{11} & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ * & a_{22} & \vdots & \vdots & 0 & 1 & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right]$$

To pivot the augmented matrix above and the upper triangular for A and I are zeros. The results from the pivot will not change the zero values in the upper triangular, since we are trying to get one in the main diagonal and zero elsewhere.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & b_{11} & 0 & \cdots & 0 \\ 0 & 1 & \vdots & \vdots & * & b_{22} & \vdots & \vdots \\ \vdots & \cdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & b_{n1} & b_{n2} & \cdots & b_{nn} \end{array}\right]$$

Therefore, A^{-1} is lower triangular

Exercise

Let $A = LU$ be an LU factorization. Explain why A can be row reduced to U using only replacement operations.

Solution

Let $A = LU$ be an LU factorization for A.

Since L is unit lower triangular, from previous problem, A is invertible. So, the matrix L can be row reduced to I by using the appropriate pivots to reduced to zero in the lower entries of the main diagonal which maintain one's.
The row operation done to L are row-replacement operations.

If elementary matrices $E_1, E_2, \cdots, E_n$, then

$$\begin{aligned} E_n \cdots E_2 E_1 A &= \left(E_n \cdots E_2 E_1\right) LU \\ &= IU \\ &= U \end{aligned}$$

That implies that A can be row reduced to U using only row-replacement operations.

Exercise

Suppose an $m \times n$ matrix A admits a factorization $A = CD$ where C is $m \times 4$ and D is $4 \times n$.

a) Show that A is the sum of four outer products.

b) Let $m = 400 \quad and \quad n = 100$. Explain why a computer programmer might prefer to store the data from A in the form of two matrices C and D.

Solution

a) C is $m \times 4$ that implies $C = \begin{bmatrix} c_{i1} & c_{i2} & c_{i3} & c_{i4} \end{bmatrix} \qquad 1 \le i \le m$

D is $4 \times n$ that implies $D = \begin{bmatrix} d_{1j} \\ d_{2j} \\ d_{3j} \\ d_{4j} \end{bmatrix} \qquad 1 \le j \le n$

$$\begin{aligned} A &= CD \\ &= \begin{bmatrix} c_{i1} & c_{i2} & c_{i3} & c_{i4} \end{bmatrix} \begin{bmatrix} d_{1j} \\ d_{2j} \\ d_{3j} \\ d_{4j} \end{bmatrix} \\ &= c_{i1} d_{1j} + c_{i2} d_{2j} + c_{i3} d_{3j} + c_{i4} d_{4j} \end{aligned}$$

= Sum of four outer products

b) ***Given***: $m = 400 \quad and \quad n = 100$

The size of matrix A is $400 \times 100 = 40,000$ *entries*

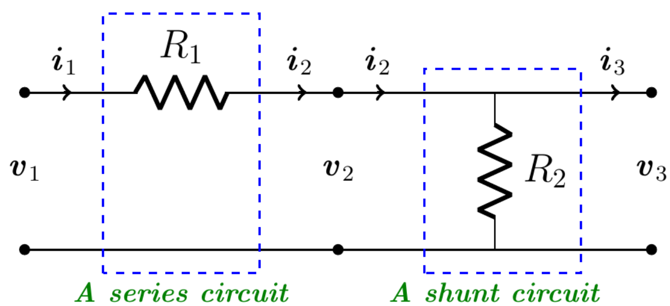
Matrix C has: $400 \times 4 = 1,600$ *entries*

Matrix D has: $100 \times 4 = 400$ *entries*

Both matrices C and D have: $1,600 + 400 = 2,000$ *entries* Only which is lot less than 40,000 entries.

Exercise

A ladder network, where two circuits are connected in series, so that the output of one circuit becomes the input of the next circuit.



The transformation $\begin{pmatrix} v_1 \\ i_1 \end{pmatrix} \longrightarrow \begin{pmatrix} v_2 \\ i_2 \end{pmatrix}$ is linear with a transfer matrix A of the ladder network.

Let the transfer matrix A_1 of the series circuit is given by $\begin{pmatrix} v_2 \\ i_2 \end{pmatrix} = A_1 \begin{pmatrix} v_1 \\ i_1 \end{pmatrix}$

Let the transfer matrix A_2 of the shunt circuit is given by $\begin{pmatrix} v_3 \\ i_3 \end{pmatrix} = A_2 \begin{pmatrix} v_2 \\ i_2 \end{pmatrix}$

a) Compute the transfer matrix of the ladder network.

b) Design a ladder network whose transfer matrix is $\begin{pmatrix} 1 & -8 \\ -\frac{1}{2} & 5 \end{pmatrix}$

Solution

a) For "series circuit":

The voltage across R_1 is: $v_1 = R_1 i_1$

The current: $i_2 = i_1$

The drop voltage: $v_2 = v_1 - R_1 i_1$

$$\begin{pmatrix} v_2 \\ i_2 \end{pmatrix} = \begin{pmatrix} v_1 - R_1 i_1 \\ i_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -R_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ i_1 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 1 & -R_1 \\ 0 & 1 \end{pmatrix} \qquad \textit{Upper Triangular}$$

For “shunt circuit”:

The voltage is: $v_3 = v_2$

Voltage R_2 : $current \times R_2$

The current: $i_2 = \dfrac{1}{R_2} v_2 + i_3$

$$i_3 = -\frac{1}{R_2} v_2 + i_2$$

$$\begin{pmatrix} v_3 \\ i_3 \end{pmatrix} = \begin{pmatrix} v_2 \\ -\dfrac{1}{R_2} v_2 + i_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{R_2} & 1 \end{pmatrix} \begin{pmatrix} v_2 \\ i_2 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{R_2} & 1 \end{pmatrix} \qquad \textit{Lower Triangular}$$

The transfer matrix of the ladder network A:

$$A = A_2 A_1$$

$$= \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{R_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & -R_1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -R_1 \\ -\dfrac{1}{R_2} & \dfrac{R_1}{R_2} + 1 \end{pmatrix}$$

b) $\begin{pmatrix} 1 & -8 \\ -\frac{1}{2} & 5 \end{pmatrix} = \begin{pmatrix} 1 & -R_1 \\ -\frac{1}{R_2} & \frac{R_1}{R_2}+1 \end{pmatrix}$

$\underline{R_1 = 8}|$

$-\frac{1}{R_2} = -\frac{1}{2} \quad \rightarrow \quad \underline{R_2 = 2}|$

$\frac{R_1}{R_2}+1 = \frac{8}{2}+1$

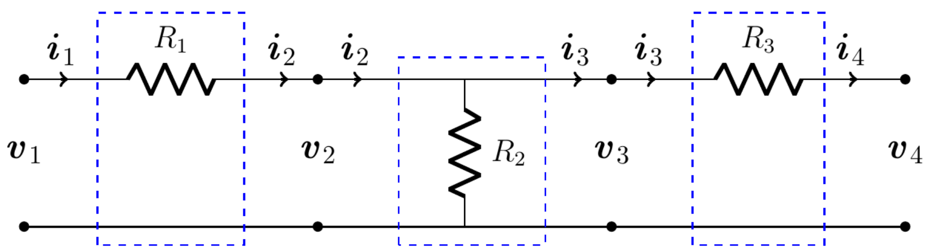
$= 4+1$

$\underline{=5}|$ ✓

The given ladder network whose transfer matrix is $\begin{pmatrix} 1 & -8 \\ -.5 & 5 \end{pmatrix}$ has the resistors $R_1 = 8\ \Omega$ and $R_2 = 2\ \Omega$

Exercise

A ladder network, where three circuits are connected in series, so that the output of one circuit becomes the input of the next circuit.



a) Compute the transfer matrix of the ladder network

b) Design a ladder network whose transfer matrix is $\begin{pmatrix} 3 & -12 \\ -\frac{1}{3} & \frac{5}{3} \end{pmatrix}$

Solution

a) Across R_1 is:

The current: $i_2 = i_1$

The drop voltage: $v_2 = v_1 - R_1 i_1$

$$\begin{pmatrix} v_2 \\ i_2 \end{pmatrix} = \begin{pmatrix} v_1 - R_1 i_1 \\ i_1 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & -R_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ i_1 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 1 & -R_1 \\ 0 & 1 \end{pmatrix} \qquad \textit{Upper Triangular}$$

For across R_2 :

The voltage is: $v_3 = v_2$

Voltage R_2 : *current* $\times R_2$

The current: $i_2 = \frac{1}{R_2} v_2 + i_3$

$$i_3 = -\frac{1}{R_2} v_2 + i_2$$

$$\begin{pmatrix} v_3 \\ i_3 \end{pmatrix} = \begin{pmatrix} v_2 \\ -\frac{1}{R_2} v_2 + i_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -\frac{1}{R_2} & 1 \end{pmatrix}\begin{pmatrix} v_2 \\ i_2 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 1 & 0 \\ -\frac{1}{R_2} & 1 \end{pmatrix} \qquad \textit{Lower Triangular}$$

Across R_3 is:

The current: $i_3 = i_4$

The drop voltage: $v_4 = v_3 - R_3 i_3$

$$\begin{pmatrix} v_4 \\ i_4 \end{pmatrix} = \begin{pmatrix} v_3 - R_3 i_3 \\ i_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -R_3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v_3 \\ i_3 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & -R_3 \\ 0 & 1 \end{pmatrix} \qquad \textit{Upper Triangular}$$

The transfer matrix of the ladder network A:

$A = A_3 A_2 A_1$

$$=\begin{pmatrix}1 & -R_3\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -\frac{1}{R_2} & 1\end{pmatrix}\begin{pmatrix}1 & -R_1\\ 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1+\frac{R_3}{R_2} & -R_3\\ -\frac{1}{R_2} & 1\end{pmatrix}\begin{pmatrix}1 & -R_1\\ 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1+\frac{R_3}{R_2} & \left(1+\frac{R_3}{R_2}\right)\left(-R_1\right)-R_3\\ -\frac{1}{R_2} & \frac{R_1}{R_2}+1\end{pmatrix}$$

$$=\begin{pmatrix}\frac{R_2+R_3}{R_2} & \frac{-R_1R_2-R_1R_3}{R_2}-R_3\\ -\frac{1}{R_2} & \frac{R_1+R_2}{R_2}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{R_2+R_3}{R_2} & \frac{-R_1R_2-R_1R_3-R_2R_3}{R_2}\\ -\frac{1}{R_2} & \frac{R_1+R_2}{R_2}\end{pmatrix}$$

b) $$\begin{pmatrix}3 & -12\\ -\frac{1}{3} & \frac{5}{3}\end{pmatrix}=\begin{pmatrix}\frac{R_2+R_3}{R_2} & \frac{-R_1R_2-R_1R_3-R_2R_3}{R_2}\\ -\frac{1}{R_2} & \frac{R_1+R_2}{R_2}\end{pmatrix}$$

$$\begin{cases}1+\frac{R_3}{R_2}=3 & \rightarrow \quad \frac{R_3}{3}=2 \quad (1)\\ \frac{-R_1R_2-R_1R_3-R_2R_3}{R_2}=-12 & \rightarrow \quad (2)\\ -\frac{1}{R_2}=-\frac{1}{3} & \rightarrow \quad \underline{R_2=3|}\\ \frac{R_1}{R_2}+1=\frac{5}{3} & \rightarrow \quad \frac{R_1}{3}=\frac{2}{3} \quad (3)\end{cases}$$

$$\begin{cases}(1) \rightarrow \underline{R_3=6|}\\ (3) \rightarrow \underline{R_1=2|}\end{cases}$$

$(2)\quad \frac{-(2)(3)-(2)(6)-(3)(6)}{3} \stackrel{?}{=} -12$

$\frac{-6-12-18}{3} \stackrel{?}{=} -12$

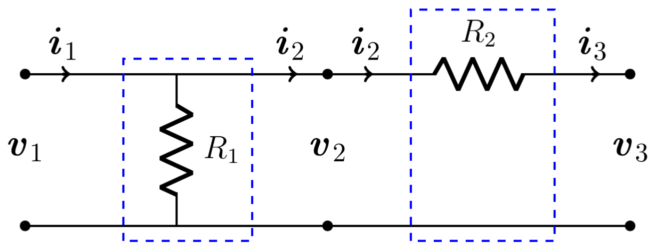
$-\frac{36}{3} \stackrel{?}{=} -12$

$-12 = -12 \quad \checkmark$

The given ladder network whose transfer matrix is $\begin{pmatrix} 3 & -12 \\ -\frac{1}{3} & \frac{5}{3} \end{pmatrix}$ has the resistors $R_1 = 2\ \Omega$, $R_2 = 3\ \Omega$, and $R_3 = 6\ \Omega$

Exercise

A ladder network, where two circuits are connected in series, so that the output of one circuit becomes the input of the next circuit.



a) Compute the transfer matrix of the ladder network

b) Find the values of the resistors when the input voltage is 12 volts and current is 6 amps if the output voltage is 9 volts and current is 4 amps

Solution

a) Across R_1 is:

The drop voltage: $current \times R_1$

The current: $i_1 = \frac{1}{R_1}v_1 + i_2$

$$i_2 = -\frac{1}{R_1}v_1 + i_1$$

$$\begin{pmatrix} v_2 \\ i_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ -\frac{1}{R_1}v_1 + i_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -\frac{1}{R_1} & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ i_1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 1 & 0 \\ -\frac{1}{R_1} & 1 \end{pmatrix} \qquad \textit{Lower Triangular}$$

For across R_2:

The current is: $i_3 = i_2$

The drop voltage: $v_3 = v_3 - R_2 i_2$

$$\begin{pmatrix} v_3 \\ i_3 \end{pmatrix} = \begin{pmatrix} v_2 - R_2 i_2 \\ i_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -R_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_2 \\ i_2 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 1 & -R_2 \\ 0 & 1 \end{pmatrix} \qquad \textit{Upper Triangular}$$

The transfer matrix of the ladder network A:

$$A = A_1 A_2$$

$$= \begin{pmatrix} 1 & -R_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{R_1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{R_2}{R_1} & -R_2 \\ -\frac{1}{R_1} & 1 \end{pmatrix}$$

b) $$\begin{pmatrix} v_2 \\ i_2 \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{12}{R_1} + 6 \end{pmatrix}$$

$$\begin{cases} v_2 = 12 \\ i_2 = -\frac{12}{R_1} + 6 \end{cases}$$

$$\begin{pmatrix} v_3 \\ i_3 \end{pmatrix} = \begin{pmatrix} v_2 - R_2 i_2 \\ i_2 \end{pmatrix}$$

$$\begin{pmatrix} 9 \\ 4 \end{pmatrix} = \begin{pmatrix} 12 - R_2\left(-\frac{12}{R_1} + 6\right) \\ -\frac{12}{R_1} + 6 \end{pmatrix}$$

$$\begin{cases} 12+12\dfrac{R_2}{R_1}-6R_2 = 9 \\ -\dfrac{12}{R_1}+6=4 \end{cases}$$

$$\begin{cases} 12\dfrac{R_2}{R_1}-6R_2 = -3 & (1) \\ -\dfrac{12}{R_1}=-2 & \rightarrow \boxed{R_1 = 6} \end{cases}$$

$$(1) \rightarrow \quad \left(4\frac{1}{6}-2\right)R_2 = -1$$

$$-\frac{8}{6}R_2 = -1$$

$$\boxed{R_2 = \frac{3}{4}}$$

The resistors $R_1 = 6\ \Omega$ and $R_2 = \frac{3}{4}\ \Omega$,