Section 4.3 – Legendre's Equation

Solving a homogeneous linear differential equation with constant coefficients can be reduced to the algebraic problem of finding the roots of its characteristic equation.

The Legendre's equation of order n is important in many applications. It has the form

$$\left(1-x^2\right)y''-2xy'+n(n+1)y=0$$

$$y''-\frac{2x}{1-x^2}y'+\frac{n(n+1)}{1-x^2}y=0$$

$$y''+P(x)y'+Q(x)y=0$$

Any solution of that equation is called a Legendre function.

Note that: $P(x)=\frac{2x}{1-x^2}$ and $Q(x)=\frac{n(n+1)}{1-x^2}$ are analytic at $x=0$. P are $x=\pm 1$.

Hence Legendre's equation has power series solutions of the form $y=\sum_{m=0}^{\infty}a_m x^m$

$$y'=\sum_{m=1}^{\infty}ma_m x^{m-1}$$

$$y''=\sum_{m=2}^{\infty}m(m-1)a_m x^{m-2}$$

$$\left(1-x^2\right)y''-2xy'+n(n+1)y=0$$

$$\left(1-x^2\right)\sum_{m=2}^{\infty}m(m-1)a_m x^{m-2}-2x\sum_{m=1}^{\infty}ma_m x^{m-1}+n(n+1)\sum_{m=0}^{\infty}a_m x^m=0$$

$$\sum_{m=2}^{\infty}m(m-1)a_m x^{m-2}-\sum_{m=2}^{\infty}m(m-1)a_m x^m-\sum_{m=1}^{\infty}2ma_m x^m+\sum_{m=0}^{\infty}n(n+1)a_m x^m=0$$

To obtain the same general power x^k, then we must set $m-2=k \quad\Rightarrow\quad m=k+2$

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k-\sum_{k=2}^{\infty}k(k-1)a_k x^k-\sum_{k=1}^{\infty}2ka_k x^k+\sum_{k=0}^{\infty}n(n+1)a_k x^k=0$$

$k=0$	$2\cdot 1\cdot a_2+n(n+1)a_0$

$k=1$	$3\cdot 2\cdot a_3+\left[-2+n(n+1)\right]a_1$
$k=2$	$4\cdot 3\cdot a_4+\left[-2-4+n(n+1)\right]a_2$
k	$(k+2)(k+1)a_{k+2}+\left[-k(k-1)-2k+n(n+1)\right]a_k$

$$(k+2)(k+1)a_{k+2}+\left[-k^2-k+n(n+1)\right]a_k=0$$

$$\begin{aligned} a_{k+2} &= -\frac{-k^2-k+n^2+n}{(k+2)(k+1)}a_k \\ &= -\frac{(n-k)(n+k+1)}{(k+2)(k+1)}a_k \end{aligned}$$

This is called a ***recurrence relation*** or ***recursion formula***.

$a_2=-\frac{n(n+1)}{2!}a_0$	$a_3=-\frac{(n-1)(n+2)}{3!}a_1$
$a_4=-\frac{(n-2)(n+3)}{4\cdot 3}a_2$	$a_5=-\frac{(n-3)(n+4)}{5\cdot 4}a_3$
$=\frac{(n-2)n(n+1)(n+3)}{4!}a_0$	$=\frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_1$
$\vdots \quad \vdots$	$\vdots \quad \vdots$

The general Legendre equation solution is: $y(x)=a_0y_1(x)+a_1y_2(x)$

Where

$$\begin{cases} y_1(x)=1-\frac{n(n+1)}{2!}x^2+\frac{(n-2)n(n+1)(n+3)}{4!}x^4-\cdots \\ y_2(x)=x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^4-\cdots \end{cases}$$

Legendre Polynomials $P_n(x)$

For Legendre's equation $\left(1-x^2\right)y''-2xy'+n(n+1)y=0$ will happen when the parameter $\boldsymbol{n}$ is nonnegative integer. Otherwise, when *n* is even, $y_1(x)$ reduces to a polynomial of degree *n*. If *n* is odd, $y_2(x)$ reduces (the same) to a polynomial of degree *n*.

$$a_n=\frac{(2n)!}{2^n(n!)^2}=\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}\qquad n\in\mathbb{Z}^+$$

If $n=0 \quad\Rightarrow\quad a_n=1$

$$a_k=-\frac{(k+2)(k+1)}{(n-k)(n+k+1)}a_{k+2}\qquad (k\le n-2)$$

If $k=n-2$

$$\begin{aligned}
a_{n-2}&=-\frac{n(n-1)}{2(2n-1)}a_n\\
&=-\frac{n(n-1)(2n)!}{2(2n-1)2^n(n!)^2}\\
&=-\frac{n(n-1)(2n)(2n-1)(2n-2)!}{2(2n-1)2^n\left[n(n-1)!\right]\left[n(n-1)(n-2)!\right]}\\
&=-\frac{(2n-2)!}{2^n(n-1)!(n-2)!}
\end{aligned}$$

$$\begin{aligned}
a_{n-4}&=-\frac{(n-2)(n-3)}{4(2n-3)}a_{n-2}\\
&=\frac{(n-2)(n-3)}{4(2n-3)}\cdot\frac{(2n-2)!}{2^n(n-1)!(n-2)!}\\
&=\frac{(n-2)(n-3)}{4(2n-3)}\cdot\frac{(2n-2)(2n-3)(2n-4)!}{2^n(n-1)(n-2)(n-3)(n-4)!(n-2)!}\\
&=\frac{2(n-1)(2n-4)!}{4\cdot 2^n(n-1)(n-4)!(n-2)!}\\
&=\frac{(2n-4)!}{2^n 2!(n-2)!(n-4)!}
\end{aligned}$$

In general; $a_{n-2k}=(-1)^k\dfrac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!}$

The resulting solution of Legendre's differential equation is called the Legendre polynomial of degree ***n*** and is denoted by $P_n(x)$.

$$P_n(x) = \sum_{k=0}^{K} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}$$

$$= \frac{(2n)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n 1!(n-1)!(n-2)!} x^{n-2} + \cdots$$

$P_0(x) = 1$

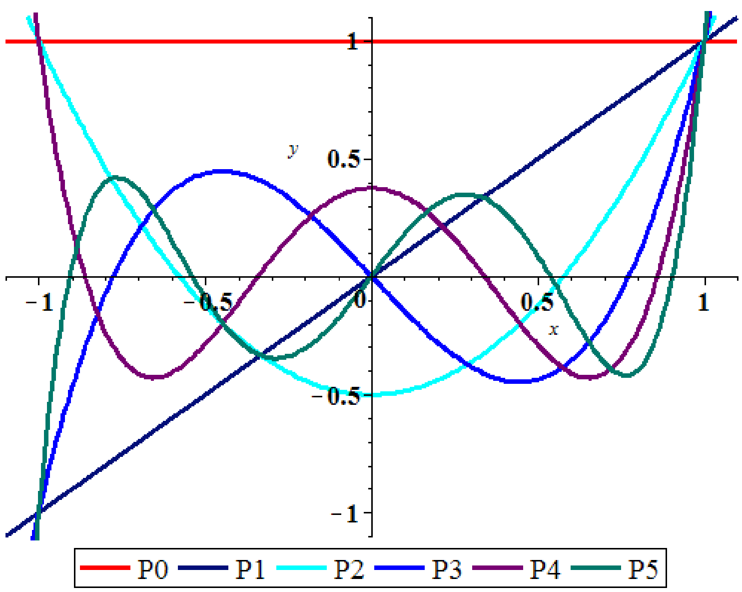
$P_1(x) = x$

$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$

$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$

$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$

$P_5(x) = \frac{1}{8}\left(65x^5 - 70x^3 + 15x\right)$



Exercise *Section* 4.3 – Legendre's Equation

1. Establish the recursion formula using the following two steps

a) Differentiate both sides of equation

$$g(t,\,x)=\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n \text{ with respect to } t \text{ to show that}$$

$$(x-t)\sum_{n=0}^{\infty}P_n(x)t^n=\left(1-2xt+t^2\right)\sum_{n=1}^{\infty}nP_n(x)t^{n-1}$$

b) Equate the coefficients of t^n in this equation to show that

$$P_1(x)=xP_0(x)$$

and

$$(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x) \quad \textit{for } \ n\ge 1$$

2. Show that $P_{2n+1}(0)=0$ *and* $P_{2n}(0)=(-1)^n\dfrac{2n!}{2^{2n}(n!)^2}$

3. Show that $P_n'(0)=(-1)^{n+1}P_n'(-1)=\dfrac{n(n+1)}{2}$

Hint: Use Legendre's equation $\left(1-x^2\right)y''-2xy'+n(n+1)y=0$

4. The differential equation $y''+xy=0$ is called ***Airy's equation***, and its solutions are called ***Airy functions***. Find the series for the solutions y_1 and y_2 where $y_1(0)=1$ and $y_1'(0)=0$, while $y_2(0)=0$ and $y_2'(0)=1$. What is the radius of convergence for these two series?

5. The Hermite equation of order α is $\quad y''-2xy'+2\alpha y=0$

a) Find the general solution is $y(x)=a_0y_1(x)+a_1y_2(x)$

Show that $y_1(x)$ is a polynomial if α is an even integer, whereas $y_2(x)$ is a polynomial if α is an odd integer.

b) When $\alpha=n$, use $y_1(x)$ to find polynomial solutions for $n=0$, $n=2$, and $n=4$, then use $y_2(x)$ to find polynomial solutions for $n=1$, $n=3$, and $n=5$.

c) The Hermite polynomial of degree n is denoted by $H_n(x)$. It is the nth-degree polynomial solution of Hermite's equation, multiplied by a suitable constant so that the coefficient of x^n is 2^n. Use part (b) to show the first six Hermite polynomials are

$H_0(x) = 1$

$H_1(x) = 2x$

$H_2(x) = 4x^2 - 2$

$H_3(x) = 8x^3 - 12x$

$H_4(x) = 16x^4 - 48x^2 + 12$

$H_5(x) = 32x^5 - 160x^3 + 120x$

d) A general formula for the Hermite polynomials is

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

Verify that this formula does in fact give an nth-degree polynomial.

6. Rodrigues's Formula is given by: $P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}\left(x^2-1\right)^n$

For the nth-degree Legendre polynomial.

a) Show that $v = \left(x^2-1\right)^n$ satisfies the differential equation $\left(1-x^2\right)v' + 2nxv = 0$

Differentiate each side of this equation to obtain

$$\left(1-x^2\right)v'' + 2(n-1)xv' + 2nv = 0$$

b) Differentiate each side of the last equation n times in succession to obtain

$$\left(1-x^2\right)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} = 0$$

which satisfies Legendre's equation of order n.

c) Show that the coefficient of x^n in u is $\frac{(2n)!}{n!}$; then state why this proves Rodrigues. Formula.

Note: that the coefficient of x^n in $P_n(x)$ is $\frac{(2n)!}{2^n (n!)^2}$