

Lecture One – Vectors and Vector-Values Functions

Solution ***Section 1.1 – Vectors***

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2 + z^2 = 4, \quad y = 0$

Solution

The circle $x^2 + z^2 = 4$ in the xz -plane

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2 + y^2 = 4, \quad z = -2$

Solution

The circle $x^2 + y^2 = 4$ in the plane $z = -2$

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2 + y^2 + z^2 = 1, \quad x = 0$

Solution

The circle $y^2 + z^2 = 1$ in the yz -plane

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2 + (y - 1)^2 + z^2 = 4, \quad y = 0$

Solution

$$x^2 + (0 - 1)^2 + z^2 = 4 \Rightarrow x^2 + z^2 = 3$$

The circle $x^2 + z^2 = 3$ in the xz -plane

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2 + y^2 + z^2 = 4$, $y = x$

Solution

The circle formed by the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $y = x$

Exercise

Find the distance between points $P_1(1, 1, 1)$, $P_2(3, 3, 0)$

Solution

$$\begin{aligned} |P_1 P_2| &= \sqrt{(3-1)^2 + (3-1)^2 + (0-1)^2} \\ &= \sqrt{4+4+1} \\ &= \sqrt{9} \\ &= 3 \end{aligned}$$

Exercise

Find the distance between points $P_1(-1, 1, 5)$, $P_2(2, 5, 0)$

Solution

$$\begin{aligned} |P_1 P_2| &= \sqrt{(2+1)^2 + (5-1)^2 + (0-5)^2} \\ &= \sqrt{9+16+25} \\ &= \sqrt{50} \\ &= 5\sqrt{2} \end{aligned}$$

Exercise

Find the distance between points $P_1(1, 4, 5)$, $P_2(4, -2, 7)$

Solution

$$\begin{aligned} |P_1 P_2| &= \sqrt{(4-1)^2 + (-2-4)^2 + (7-5)^2} \\ &= \sqrt{9+36+4} \\ &= 7 \end{aligned}$$

Exercise

Find the distance between points $P_1(3, 4, 5)$, $P_2(2, 3, 4)$

Solution

$$\begin{aligned} |P_1P_2| &= \sqrt{(2-3)^2 + (3-4)^2 + (4-5)^2} \\ &= \sqrt{1+1+1} \\ &= \sqrt{3} \end{aligned}$$

Exercise

Find the center and radii of the spheres $x^2 + y^2 + z^2 + 4x - 4z = 0$

Solution

$$\begin{aligned} (x^2 + 4x) + y^2 + (z^2 - 4z) &= 0 \\ (x^2 + 4x + 4) + y^2 + (z^2 - 4z + 4) &= 4 + 4 \\ (x+2)^2 + y^2 + (z-2)^2 &= 8 \end{aligned}$$

The center is at $(-2, 0, 2)$ and the radius is $\sqrt{8} = 2\sqrt{2}$

Exercise

Find the center and radii of the spheres $x^2 + y^2 + z^2 - 6y + 8z = 0$

Solution

$$\begin{aligned} x^2 + (y^2 - 6y) + (z^2 + 8z) &= 0 \\ x^2 + \left(y^2 - 6y + \left(-\frac{6}{2}\right)^2\right) + \left(z^2 + 8z + \left(\frac{8}{2}\right)^2\right) &= 9 + 16 \\ x^2 + (y-3)^2 + (z+4)^2 &= 25 \end{aligned}$$

The center is at $(0, 3, -4)$ and the radius is 5

Exercise

Find the center and radii of the spheres $2x^2 + 2y^2 + 2z^2 + x + y + z = 9$

Solution

$$x^2 + y^2 + z^2 + \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z = \frac{9}{2}$$

$$\left(x^2 + \frac{1}{2}x + \left(\frac{1}{2}\right)^2\right) + \left(y^2 + \frac{1}{2}y + \left(\frac{1}{2}\right)^2\right) + \left(z^2 + \frac{1}{2}z + \left(\frac{1}{2}\right)^2\right) = \frac{9}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$$

$$\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 + \left(z + \frac{1}{2}\right)^2 = \frac{9}{2} + \frac{3}{4}$$

$$\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 + \left(z + \frac{1}{2}\right)^2 = \frac{19}{4}$$

The center is at $\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$ and the radius is $\frac{\sqrt{19}}{2}$

Exercise

Find a formula for the distance from the point $P(x, y, z)$ to x -axis

Solution

The distance between (x, y, z) and $(x, 0, 0)$ is:

$$d = \sqrt{(x-x)^2 + (y-0)^2 + (z-0)^2}$$

$$= \sqrt{y^2 + z^2}$$

Exercise

Find a formula for the distance from the point $P(x, y, z)$ to xy -plane

Solution

The distance between (x, y, z) and $(x, 0, 0)$ is:

$$d = \sqrt{(x-x)^2 + (y-0)^2 + (z-0)^2}$$

$$= |z|$$

Exercise

Let $\mathbf{u} = \langle -3, 4 \rangle$ and $\mathbf{v} = \langle 2, -5 \rangle$. Find the component form and the magnitude of the vector

a) $3\mathbf{u} - 4\mathbf{v}$

b) $-2\mathbf{u}$

c) $\mathbf{u} + \mathbf{v}$

Solution

a) $3\vec{u} - 4\vec{v} = 3\langle -3, 4 \rangle - 4\langle 2, -5 \rangle$

$$= \langle -17, 32 \rangle \mid$$

$$b) \quad -2\vec{u} = -2\langle -3, 4 \rangle$$

$$= \langle 6, -8 \rangle \mid$$

$$c) \quad \vec{u} + \vec{v} = \langle -3, 4 \rangle + \langle 2, -5 \rangle$$

$$= \langle -1, -1 \rangle \mid$$

Exercise

Let $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle -2, 5 \rangle$. Find the component form and the magnitude if the vector

$$a) \quad 3\mathbf{u}$$

$$b) \quad \mathbf{u} - \mathbf{v}$$

$$c) \quad 2\mathbf{u} - 3\mathbf{v}$$

$$d) \quad -2\mathbf{u} + 5\mathbf{v}$$

$$e) \quad -\frac{5}{13}\mathbf{u} + \frac{12}{13}\mathbf{v}$$

Solution

$$a) \quad 3\mathbf{u} = 3\langle 3, -2 \rangle$$

$$= \langle 9, -6 \rangle \mid$$

$$b) \quad \mathbf{u} - \mathbf{v} = \langle 3, -2 \rangle - \langle -2, 5 \rangle$$

$$= \langle 5, -7 \rangle \mid$$

$$c) \quad 2\mathbf{u} - 3\mathbf{v} = 2\langle 3, -2 \rangle - 3\langle -2, 5 \rangle$$

$$= \langle 6, -4 \rangle - \langle -6, 15 \rangle$$

$$= \langle 12, -19 \rangle \mid$$

$$d) \quad -2\mathbf{u} + 5\mathbf{v} = -2\langle 3, -2 \rangle + 5\langle -2, 5 \rangle$$

$$= \langle -6, 4 \rangle + \langle -10, 25 \rangle$$

$$= \langle -14, 29 \rangle \mid$$

$$e) \quad -\frac{5}{13}\mathbf{u} + \frac{12}{13}\mathbf{v} = -\frac{5}{13}\langle 3, -2 \rangle + \frac{12}{13}\langle -2, 5 \rangle$$

$$= \langle -6, 4 \rangle - \langle -10, 25 \rangle$$

$$= \langle 4, -21 \rangle \mid$$

Exercise

Find scalars a , b , and c such that $\langle 2, 2, 2 \rangle = a\langle 1, 1, 0 \rangle + b\langle 0, 1, 1 \rangle + c\langle 1, 0, 1 \rangle$

Solution

$$\langle 2, 2, 2 \rangle = a\langle 1, 1, 0 \rangle + b\langle 0, 1, 1 \rangle + c\langle 1, 0, 1 \rangle$$

$$= \langle a + c, a + b, b + c \rangle$$

$$\begin{cases} a + c = 2 \\ a + b = 2 \\ b + c = 2 \end{cases} \quad \begin{cases} c = 2 - a \\ b = 2 - a \end{cases} \quad \begin{cases} 2a - 4 = 2 \end{cases}$$

$$\underline{a = b = c = 1}$$

Exercise

Find the component form of the vector: The sum of \overrightarrow{AB} and \overrightarrow{CD} where

$$A = (1, -1), \quad B = (2, 0), \quad C = (-1, 3), \quad \text{and} \quad D = (-2, 2)$$

Solution

$$\overrightarrow{AB} = \langle 2 - 1, 0 - (-1) \rangle = \langle 1, 1 \rangle$$

$$\overrightarrow{CD} = \langle -2 - (-1), 2 - 3 \rangle = \langle -1, -1 \rangle$$

$$\overrightarrow{AB} + \overrightarrow{CD} = \langle 1, 1 \rangle + \langle -1, -1 \rangle$$

$$\underline{= \langle 0, 0 \rangle}$$

Exercise

Find the component form of the vector: The unit vector that makes an angle $\theta = \frac{2\pi}{3}$ with the positive x -axis

Solution

$$\left\langle \cos \frac{2\pi}{3}, \sin \frac{2\pi}{3} \right\rangle = \underline{\left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle}$$

Exercise

Find the component form of the vector: The unit vector obtained by rotating the vector $\langle 0, 1 \rangle$ 120° counterclockwise about the origin

Solution

The angle of unit vector $\langle 0, 1 \rangle$ is 90° , this unit vector rotates 120° which makes an angle of $90^\circ + 120^\circ = 210^\circ$ with the positive x -axis

$$\left\langle \cos 210^\circ, \sin 210^\circ \right\rangle = \underline{\left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle}$$

Exercise

Find the component form of the vector: The unit vector obtained by rotating the vector $\langle 1, 0 \rangle$ 135° counterclockwise about the origin

Solution

The angle of unit vector $\langle 1, 0 \rangle$ is 0° , this unit vector rotates 135° which makes an angle of $0^\circ + 135^\circ = 135^\circ$ with the positive x -axis

$$\langle \cos 135^\circ, \sin 135^\circ \rangle = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

Exercise

Find the component form of the vector: The unit vector that makes an angle $\theta = \frac{\pi}{6}$ with the positive x -axis

Solution

$$\left\langle \cos \frac{\pi}{6}, \sin \frac{\pi}{6} \right\rangle = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$$

Exercise

Find the component form of the vector: The vector 5 units long in the direction opposite to the direction of $\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$

Solution

$$\begin{aligned} -5 \left(\frac{1}{\sqrt{\frac{9}{25} + \frac{16}{25}}} \right) \left(\frac{3}{5}\hat{i} + \frac{4}{5}\hat{j} \right) &= -5(1) \left(\frac{3}{5}\hat{i} + \frac{4}{5}\hat{j} \right) \\ &= -3\hat{i} - 4\hat{j} \end{aligned}$$

Exercise

Express the velocity vector $\mathbf{v} = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \cos t + e^t \sin t)\mathbf{j}$ when $t = \ln 2$ in terms of its length and direction.

Solution

$$\vec{v}(t = \ln 2) = (e^{\ln 2} \cos(\ln 2) - e^{\ln 2} \sin(\ln 2))\hat{i} + (e^{\ln 2} \cos(\ln 2) + e^{\ln 2} \sin(\ln 2))\hat{j}$$

$$= (2 \cos(\ln 2) - 2 \sin(\ln 2)) \hat{i} + (2 \cos(\ln 2) + 2 \sin(\ln 2)) \hat{j}$$

$$\begin{aligned} \text{Length} = |v| &= \sqrt{(2 \cos(\ln 2) - 2 \sin(\ln 2))^2 + (2 \cos(\ln 2) + 2 \sin(\ln 2))^2} \\ &= 2 \sqrt{\cos^2(\ln 2) - 2 \cos(\ln 2) \sin(\ln 2) + \sin^2(\ln 2) \\ &\quad + \cos^2(\ln 2) + 2 \cos(\ln 2) \sin(\ln 2) + \sin^2(\ln 2)} \\ &= 2\sqrt{2} \end{aligned}$$

$$\begin{aligned} \text{Direction} &= \frac{v}{|v|} = \frac{2((\cos(\ln 2) - \sin(\ln 2)) \hat{i} + (\cos(\ln 2) + \sin(\ln 2)) \hat{j})}{2\sqrt{2}} \\ &= \frac{(\cos(\ln 2) - \sin(\ln 2))}{\sqrt{2}} \hat{i} + \frac{(\cos(\ln 2) + \sin(\ln 2))}{\sqrt{2}} \hat{j} \end{aligned}$$

Exercise

Sketch the indicated vector

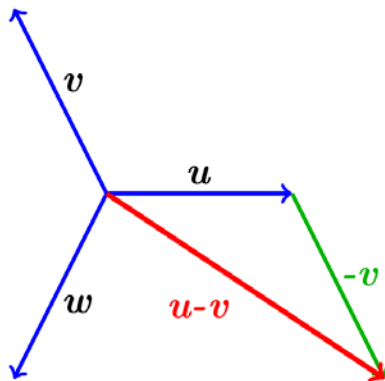
a) $u - v$

b) $2u - v$

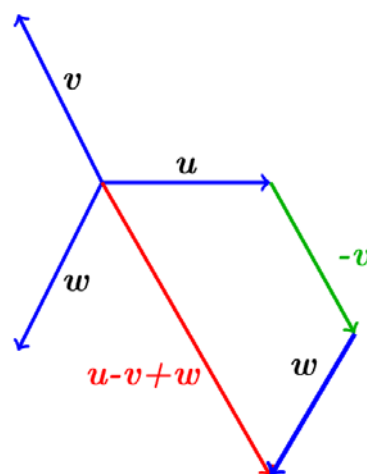
c) $u - v + w$

Solution

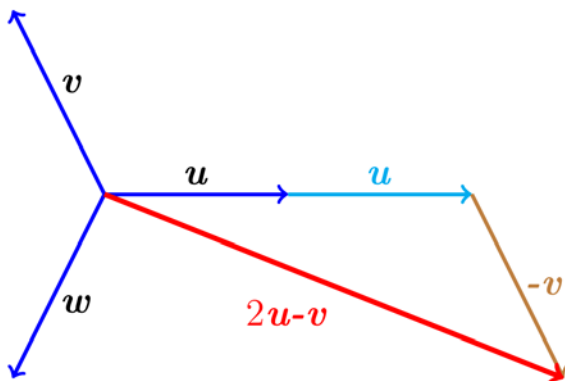
a)



b)



c)



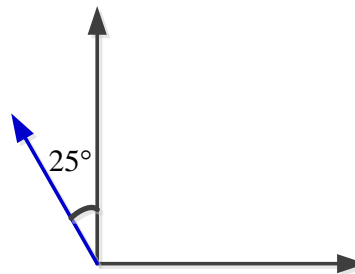
Exercise

An Airplane is flying in the direction 25° west of north at 800 km/h . Find the component form of the velocity of the airplane, assuming that the positive x -axis represents due east and the positive y -axis represents due north.

Solution

25° west of north is $25^\circ + 90^\circ = 115^\circ$ north of east

$$800 \langle \cos 115^\circ, \sin 115^\circ \rangle \approx \langle -338.095, 725.046 \rangle$$



Exercise

A jet airliner, flying due east at 500 mph in still air, encounters a 70-mph tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What speed and direction should the jetliner have in order for the resultant vector to be 500 mph due east?

Solution

$\mathbf{u} = \langle x, y \rangle$ = the velocity of the airplane;

\mathbf{v} = the velocity of the tailwind

$$\mathbf{v} = \langle 70 \cos 60^\circ, 70 \sin 60^\circ \rangle$$

$$= \langle 35, 35\sqrt{3} \rangle$$

$$\mathbf{u} + \mathbf{v} = \langle 500, 0 \rangle$$

$$\langle x, y \rangle + \langle 35, 35\sqrt{3} \rangle = \langle 500, 0 \rangle$$

$$\langle x, y \rangle = \langle 500, 0 \rangle - \langle 35, 35\sqrt{3} \rangle = \langle 465, -35\sqrt{3} \rangle$$

$$\mathbf{u} = \langle 465, -35\sqrt{3} \rangle$$

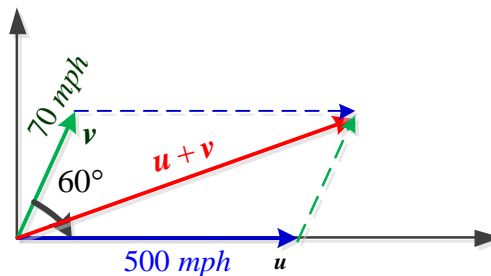
$$|\mathbf{u}| = \sqrt{465^2 + (-35\sqrt{3})^2}$$

$$\approx 468.9 \text{ mph}$$

$$|\theta| = \tan^{-1} \frac{-35\sqrt{3}}{465}$$

$$\approx -7.4^\circ$$

The direction is 7.4° south of east



Exercise

Consider a 100-N weight suspended by two wires. Find the magnitudes and components of the force vectors \vec{F}_1 and \vec{F}_2

Solution

$$\begin{aligned}\vec{F}_1 &= \left\langle -|\vec{F}_1| \cos 30^\circ, |\vec{F}_1| \sin 30^\circ \right\rangle \\ &= \left\langle -\frac{\sqrt{3}}{2}|\vec{F}_1|, \frac{1}{2}|\vec{F}_1| \right\rangle\end{aligned}$$

$$\begin{aligned}\vec{F}_2 &= \left\langle |\vec{F}_2| \cos 45^\circ, |\vec{F}_2| \sin 45^\circ \right\rangle \\ &= \left\langle \frac{\sqrt{2}}{2}|\vec{F}_2|, \frac{\sqrt{2}}{2}|\vec{F}_2| \right\rangle\end{aligned}$$

$$\vec{F}_1 + \vec{F}_2 = \langle 0, 100 \rangle$$

$$\left\langle -\frac{\sqrt{3}}{2}|\vec{F}_1|, \frac{1}{2}|\vec{F}_1| \right\rangle + \left\langle \frac{\sqrt{2}}{2}|\vec{F}_2|, \frac{\sqrt{2}}{2}|\vec{F}_2| \right\rangle = \langle 0, 100 \rangle$$

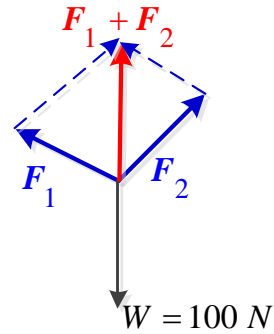
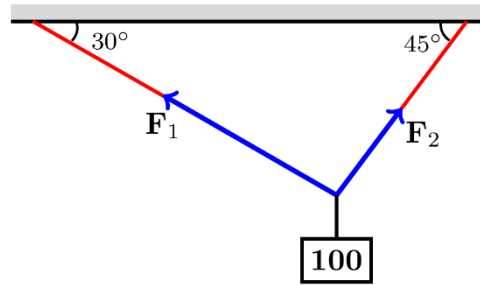
$$\left\langle -\frac{\sqrt{3}}{2}|\vec{F}_1| + \frac{\sqrt{2}}{2}|\vec{F}_2|, \frac{1}{2}|\vec{F}_1| + \frac{\sqrt{2}}{2}|\vec{F}_2| \right\rangle = \langle 0, 100 \rangle$$

$$\begin{cases} -\frac{\sqrt{3}}{2}|\vec{F}_1| + \frac{\sqrt{2}}{2}|\vec{F}_2| = 0 \\ \frac{1}{2}|\vec{F}_1| + \frac{\sqrt{2}}{2}|\vec{F}_2| = 100 \end{cases}$$

$$\Rightarrow \boxed{|\vec{F}_1| \approx 73.205 \text{ N}} \quad \boxed{|\vec{F}_2| \approx 89.658 \text{ N}}$$

$$\begin{aligned}\vec{F}_1 &= \left\langle -\frac{\sqrt{3}}{2}(73.205), \frac{1}{2}(73.205) \right\rangle \\ &\approx \langle -63.397, 36.603 \rangle\end{aligned}$$

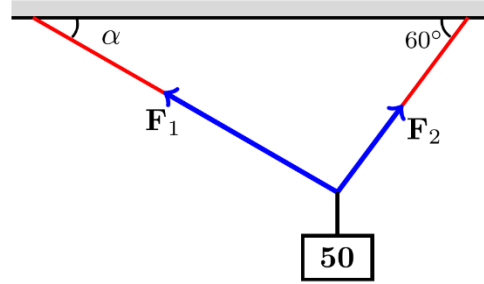
$$\begin{aligned}\vec{F}_2 &= \left\langle \frac{\sqrt{2}}{2}(89.658), \frac{\sqrt{2}}{2}(89.658) \right\rangle \\ &\approx \langle 63.397, 63.397 \rangle\end{aligned}$$



Exercise

Consider a 50-N weight suspended by two wires, If the magnitude of vector $\vec{F}_1 = 35 \text{ N}$, find the angle α and the magnitude of vector \vec{F}_2

Solution



$$\begin{aligned}\vec{F}_1 &= \langle -|\vec{F}_1| \cos \alpha, |\vec{F}_1| \sin \alpha \rangle \\ &= \langle -35 \cos \alpha, 35 \sin \alpha \rangle\end{aligned}$$

$$\begin{aligned}\vec{F}_2 &= \langle |\vec{F}_2| \cos 60^\circ, |\vec{F}_2| \sin 60^\circ \rangle \\ &= \left\langle \frac{1}{2} |\vec{F}_2|, \frac{\sqrt{3}}{2} |\vec{F}_2| \right\rangle\end{aligned}$$

$$w = \langle 0, -50 \rangle \Rightarrow \vec{F}_1 + \vec{F}_2 = \langle 0, 50 \rangle$$

$$\langle -35 \cos \alpha, 35 \sin \alpha \rangle + \left\langle \frac{1}{2} |\vec{F}_2|, \frac{\sqrt{3}}{2} |\vec{F}_2| \right\rangle = \langle 0, 50 \rangle$$

$$\left\langle -35 \cos \alpha + \frac{1}{2} |\vec{F}_2|, 35 \sin \alpha + \frac{\sqrt{3}}{2} |\vec{F}_2| \right\rangle = \langle 0, 50 \rangle$$

$$\rightarrow \begin{cases} -35 \cos \alpha + \frac{1}{2} |\vec{F}_2| = 0 \\ 35 \sin \alpha + \frac{\sqrt{3}}{2} |\vec{F}_2| = 50 \end{cases} \rightarrow \begin{cases} |\vec{F}_2| = 70 \cos \alpha \end{cases}$$

$$35 \sin \alpha + \frac{\sqrt{3}}{2} (70 \cos \alpha) = 50$$

$$35\sqrt{3} \cos \alpha = 50 - 35 \sin \alpha$$

$$\sqrt{3} \cos \alpha = \frac{10}{7} - \sin \alpha$$

$$(\sqrt{3} \cos \alpha)^2 = \left(\frac{10}{7} - \sin \alpha \right)^2$$

$$3 \cos^2 \alpha = \frac{100}{49} - \frac{20}{7} \sin \alpha + \sin^2 \alpha$$

$$3(1 - \sin^2 \alpha) = \frac{100}{49} - \frac{20}{7} \sin \alpha + \sin^2 \alpha$$

$$3 - 3 \sin^2 \alpha - \frac{100}{49} + \frac{20}{7} \sin \alpha - \sin^2 \alpha = 0$$

$$-4 \sin^2 \alpha + \frac{20}{7} \sin \alpha + \frac{47}{49} = 0$$

$$-196 \sin^2 \alpha + 140 \sin \alpha + 47 = 0 \Rightarrow \sin \alpha = \frac{5 \pm 6\sqrt{2}}{14}$$

Since $\alpha > 0 \Rightarrow \sin \alpha > 0$

$$\rightarrow \sin \alpha = \frac{5+6\sqrt{2}}{14} \approx 0.963$$

$$|\alpha \approx \sin^{-1}(0.963) = \underline{74.42^\circ}|$$

$$\begin{aligned} |\vec{F}_2| &= 70 \cos \alpha \\ &= 70 \cos 74.42^\circ \\ &\approx \underline{18.81 \text{ N}} \end{aligned}$$

Exercise

Consider a w -N weight suspended by two wires, If the magnitude of vector $\vec{F}_2 = 100 \text{ N}$, find w and the magnitude of vector \vec{F}_1

Solution

$$\vec{F}_1 = \langle -|\vec{F}_1| \cos 40^\circ, |\vec{F}_1| \sin 40^\circ \rangle$$

$$\begin{aligned} \vec{F}_2 &= \langle |\vec{F}_2| \cos 35^\circ, |\vec{F}_2| \sin 35^\circ \rangle \\ &= \langle 100(0.819), 100(0.5736) \rangle \\ &= \langle 81.915, 57.358 \rangle \end{aligned}$$

$$\vec{F}_1 + \vec{F}_2 = \langle 0, w \rangle$$

$$\langle -|\vec{F}_1| \cos 40^\circ, |\vec{F}_1| \sin 40^\circ \rangle + \langle 81.915, 57.358 \rangle = \langle 0, w \rangle$$

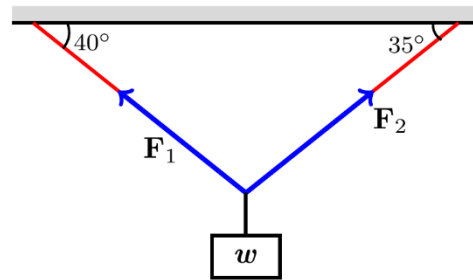
$$\langle -|\vec{F}_1| \cos 40^\circ + 81.915, |\vec{F}_1| \sin 40^\circ + 57.358 \rangle = \langle 0, w \rangle$$

$$-|\vec{F}_1| \cos 40^\circ + 81.915 = 0$$

$$|\vec{F}_1| \cos 40^\circ = 81.915$$

$$\begin{aligned} |\vec{F}_1| &= \frac{81.915}{\cos 40^\circ} \\ &\approx \underline{106.933 \text{ N}} \end{aligned}$$

$$\begin{aligned} w &= |\vec{F}_1| \sin 40^\circ + 57.358 \\ &= 106.933 \sin 40^\circ + 57.358 \\ &\approx \underline{126.093 \text{ N}} \end{aligned}$$



Exercise

Consider a 25-N weight suspended by two wires, If the magnitude of vector \vec{F}_1 and \vec{F}_2 are both 75 N, then angles α and β are equal. Find α .

Solution

$$\begin{aligned}\vec{F}_1 &= \langle -|\vec{F}_1| \cos \alpha, |\vec{F}_1| \sin \alpha \rangle \\ &= \langle -75 \cos \alpha, 75 \sin \alpha \rangle\end{aligned}$$

$$\begin{aligned}\vec{F}_2 &= \langle |\vec{F}_2| \cos \beta, |\vec{F}_2| \sin \beta \rangle \\ &= \langle 75 \cos \beta, 75 \sin \beta \rangle\end{aligned}$$

$$w = \langle 0, -25 \rangle \Rightarrow \vec{F}_1 + \vec{F}_2 = \langle 0, 25 \rangle$$

$$\langle -75 \cos \alpha, 75 \sin \alpha \rangle + \langle 75 \cos \beta, 75 \sin \beta \rangle = \langle 0, 25 \rangle$$

$$\langle -75 \cos \alpha + 75 \cos \beta, 75 \sin \alpha + 75 \sin \beta \rangle = \langle 0, 25 \rangle$$

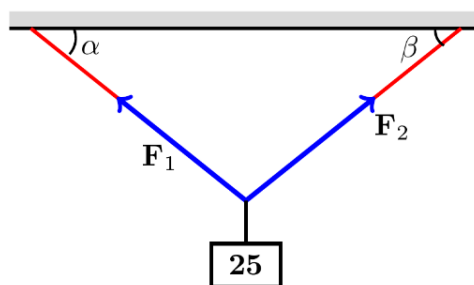
$$-75 \cos \alpha + 75 \cos \beta = 0 \Rightarrow \cos \alpha = \cos \beta$$

$$150 \sin \alpha = 25$$

$$\sin \alpha = \frac{25}{150}$$

$$|\alpha| = \sin^{-1} \frac{25}{150}$$

$$\approx 9.59^\circ$$



since $\alpha = \beta$

Exercise

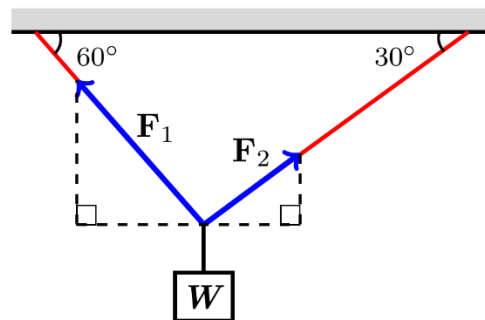
Consider a $W = 100$ N weight suspended by two wires. Find the magnitudes and components of the force vectors \vec{F}_1 and \vec{F}_2

Solution

$$\begin{aligned}\vec{F}_1 &= \langle -|\vec{F}_1| \cos 60^\circ, |\vec{F}_1| \sin 60^\circ \rangle \\ &= \left\langle -\frac{1}{2}|\vec{F}_1|, \frac{\sqrt{3}}{2}|\vec{F}_1| \right\rangle\end{aligned}$$

$$\begin{aligned}\vec{F}_2 &= \langle |\vec{F}_2| \cos 30^\circ, |\vec{F}_2| \sin 30^\circ \rangle \\ &= \left\langle \frac{\sqrt{3}}{2}|\vec{F}_2|, \frac{1}{2}|\vec{F}_2| \right\rangle\end{aligned}$$

$$\vec{F}_1 + \vec{F}_2 = \langle 0, 100 \rangle$$



$$\left\langle -\frac{1}{2}|\vec{F}_1|, \frac{\sqrt{3}}{2}|\vec{F}_1| \right\rangle + \left\langle \frac{\sqrt{3}}{2}|\vec{F}_2|, \frac{1}{2}|\vec{F}_2| \right\rangle = \langle 0, 100 \rangle$$

$$\left\langle -\frac{1}{2}|\vec{F}_1| + \frac{\sqrt{3}}{2}|\vec{F}_2|, \frac{\sqrt{3}}{2}|\vec{F}_1| + \frac{1}{2}|\vec{F}_2| \right\rangle = \langle 0, 100 \rangle$$

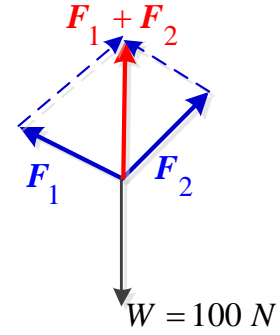
$$\begin{cases} -\frac{1}{2}|\vec{F}_1| + \frac{\sqrt{3}}{2}|\vec{F}_2| = 0 \\ \frac{\sqrt{3}}{2}|\vec{F}_1| + \frac{1}{2}|\vec{F}_2| = 100 \end{cases}$$

$$\Delta = \begin{vmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix} = -1 \quad \Delta_1 = \begin{vmatrix} 0 & \frac{\sqrt{3}}{2} \\ 100 & \frac{1}{2} \end{vmatrix} = -50\sqrt{3} \quad \Delta_2 = \begin{vmatrix} -\frac{1}{2} & 0 \\ \frac{\sqrt{3}}{2} & 100 \end{vmatrix} = -50$$

$$\Rightarrow \boxed{|\vec{F}_1| = 50\sqrt{3} \text{ N}} \quad \boxed{|\vec{F}_2| = 50 \text{ N}}$$

$$\begin{aligned} \vec{F}_1 &= \left\langle -\frac{1}{2}(50\sqrt{3}), \frac{\sqrt{3}}{2}(50\sqrt{3}) \right\rangle \\ &= \boxed{\langle -25\sqrt{3}, 75 \rangle} \end{aligned}$$

$$\begin{aligned} \vec{F}_2 &= \left\langle \frac{\sqrt{3}}{2}(50), \frac{1}{2}(50) \right\rangle \\ &= \boxed{\langle 25\sqrt{3}, 25 \rangle} \end{aligned}$$



Exercise

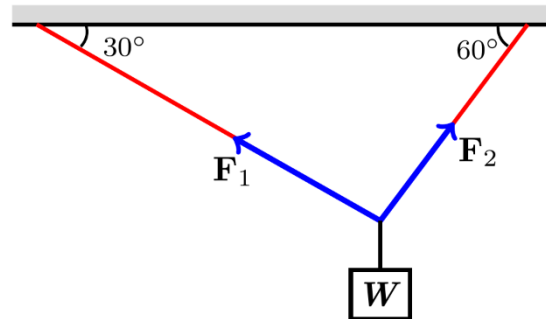
Consider a $W = 50 \text{ N}$ weight suspended by two wires. Find the magnitudes and components of the force vectors \vec{F}_1 and \vec{F}_2

Solution

$$\begin{aligned} \vec{F}_1 &= \left\langle -|\vec{F}_1|\cos 30^\circ, |\vec{F}_1|\sin 30^\circ \right\rangle \\ &= \left\langle -\frac{\sqrt{3}}{2}|\vec{F}_1|, \frac{1}{2}|\vec{F}_1| \right\rangle \end{aligned}$$

$$\begin{aligned} \vec{F}_2 &= \left\langle |\vec{F}_2|\cos 60^\circ, |\vec{F}_2|\sin 60^\circ \right\rangle \\ &= \left\langle \frac{1}{2}|\vec{F}_2|, \frac{\sqrt{3}}{2}|\vec{F}_2| \right\rangle \end{aligned}$$

$$\vec{F}_1 + \vec{F}_2 = \langle 0, 50 \rangle$$



$$\left\langle -\frac{\sqrt{3}}{2}|\vec{F}_1|, \frac{1}{2}|\vec{F}_1| \right\rangle + \left\langle \frac{1}{2}|\vec{F}_2|, \frac{\sqrt{3}}{2}|\vec{F}_2| \right\rangle = \langle 0, 50 \rangle$$

$$\left\langle -\frac{\sqrt{3}}{2}|\vec{F}_1| + \frac{1}{2}|\vec{F}_2|, \frac{1}{2}|\vec{F}_1| + \frac{\sqrt{3}}{2}|\vec{F}_2| \right\rangle = \langle 0, 50 \rangle$$

$$\begin{cases} -\frac{\sqrt{3}}{2}|\vec{F}_1| + \frac{1}{2}|\vec{F}_2| = 0 \\ \frac{1}{2}|\vec{F}_1| + \frac{\sqrt{3}}{2}|\vec{F}_2| = 50 \end{cases}$$

$$\Delta = \begin{vmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{vmatrix} = -1 \quad \Delta_1 = \begin{vmatrix} 0 & \frac{1}{2} \\ 50 & \frac{\sqrt{3}}{2} \end{vmatrix} = -25 \quad \Delta_2 = \begin{vmatrix} -\frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & 50 \end{vmatrix} = -25\sqrt{3}$$

$$\Rightarrow \boxed{|\vec{F}_1| = 25 \text{ N}} \quad \boxed{|\vec{F}_2| = 25\sqrt{3} \text{ N}}$$

$$\boxed{\vec{F}_1 = \left\langle -\frac{25\sqrt{3}}{2}, \frac{25}{2} \right\rangle}$$

$$\boxed{\vec{F}_2 = \left\langle \frac{25\sqrt{3}}{2}, \frac{75}{2} \right\rangle}$$

Exercise

Consider a $W = 100 \text{ N}$ weight suspended by two wires. Find the magnitudes and components of the force vectors \vec{F}_1 and \vec{F}_2

Solution

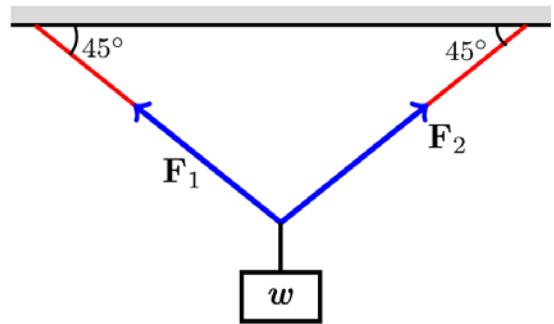
$$\begin{aligned} \vec{F}_1 &= \left\langle -|\vec{F}_1|\cos 45^\circ, |\vec{F}_1|\sin 45^\circ \right\rangle \\ &= \left\langle -\frac{\sqrt{2}}{2}|\vec{F}_1|, \frac{\sqrt{2}}{2}|\vec{F}_1| \right\rangle \end{aligned}$$

$$\begin{aligned} \vec{F}_2 &= \left\langle |\vec{F}_2|\cos 45^\circ, |\vec{F}_2|\sin 45^\circ \right\rangle \\ &= \left\langle \frac{\sqrt{2}}{2}|\vec{F}_2|, \frac{\sqrt{2}}{2}|\vec{F}_2| \right\rangle \end{aligned}$$

$$\vec{F}_1 + \vec{F}_2 = \langle 0, 100 \rangle$$

$$\left\langle -\frac{\sqrt{2}}{2}|\vec{F}_1|, \frac{\sqrt{2}}{2}|\vec{F}_1| \right\rangle + \left\langle \frac{\sqrt{2}}{2}|\vec{F}_2|, \frac{\sqrt{2}}{2}|\vec{F}_2| \right\rangle = \langle 0, 100 \rangle$$

$$\left\langle -\frac{\sqrt{2}}{2}|\vec{F}_1| + \frac{\sqrt{2}}{2}|\vec{F}_2|, \frac{\sqrt{2}}{2}|\vec{F}_1| + \frac{\sqrt{2}}{2}|\vec{F}_2| \right\rangle = \langle 0, 100 \rangle$$



$$\begin{cases} -\frac{\sqrt{2}}{2}|\vec{F}_1| + \frac{\sqrt{2}}{2}|\vec{F}_2| = 0 \\ \frac{\sqrt{2}}{2}|\vec{F}_1| + \frac{\sqrt{2}}{2}|\vec{F}_2| = 100 \end{cases}$$

$$\Delta = \begin{vmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{vmatrix} = -1 \quad \Delta_1 = \begin{vmatrix} 0 & \frac{\sqrt{2}}{2} \\ 100 & \frac{\sqrt{2}}{2} \end{vmatrix} = -50\sqrt{2} \quad \Delta_2 = \begin{vmatrix} -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 100 \end{vmatrix} = -50\sqrt{2}$$

$$\Rightarrow \boxed{|\vec{F}_1| = 50\sqrt{2} \text{ N}} \quad \boxed{|\vec{F}_2| = 50\sqrt{2} \text{ N}}$$

$$\begin{aligned} \vec{F}_1 &= \left\langle -\frac{\sqrt{2}}{2}(50\sqrt{2}), \frac{\sqrt{2}}{2}(50\sqrt{2}) \right\rangle \\ &= \langle -50, 50 \rangle \end{aligned}$$

$$\begin{aligned} \vec{F}_2 &= \left\langle \frac{\sqrt{2}}{2}(50\sqrt{2}), \frac{\sqrt{2}}{2}(50\sqrt{2}) \right\rangle \\ &= \langle 50, 50 \rangle \end{aligned}$$

Exercise

A bird flies from its nest 5 km in the direction 60° north east, where it stops to rest on a tree. It then flies 10 km in the direction due southeast and lands atop a telephone pole. Place an xy -coordinate system so that the origin is the bird's nest, the x -axis points east, and the y -axis points north.

- At what point is the tree located?
- At what point is the telephone pole?

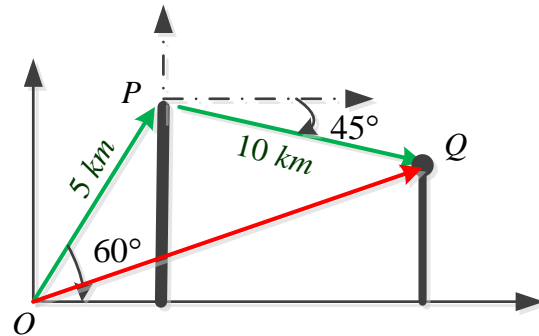
Solution

$$\begin{aligned} \text{a) } \vec{OP} &= (5\cos 60^\circ)\hat{i} + (5\sin 60^\circ)\hat{j} \\ &= \frac{5}{2}\hat{i} + \frac{5\sqrt{3}}{2}\hat{j} \end{aligned}$$

The tree is located at the point

$$\boxed{P = \left(\frac{5}{2}, \frac{5\sqrt{3}}{2} \right)}$$

$$\begin{aligned} \text{b) } \vec{OQ} &= \vec{OP} + \vec{PQ} \\ &= \frac{5}{2}\hat{i} + \frac{5\sqrt{3}}{2}\hat{j} + (10\cos 315^\circ)\hat{i} + (10\sin 315^\circ)\hat{j} \\ &= \frac{5}{2}\hat{i} + \frac{5\sqrt{3}}{2}\hat{j} + \left(10\frac{\sqrt{2}}{2}\right)\hat{i} + \left(10\left(-\frac{\sqrt{2}}{2}\right)\right)\hat{j} \end{aligned}$$



$$= \left(\frac{5}{2} + 5\sqrt{2} \right) \hat{i} + \left(\frac{5\sqrt{3}}{2} - \frac{10\sqrt{2}}{2} \right) \hat{j}$$

$$= \left(\frac{5+10\sqrt{2}}{2} \right) \hat{i} + \left(\frac{5\sqrt{3}-10\sqrt{2}}{2} \right) \hat{j}$$

The pole is located at the point $\underline{Q = \left(\frac{5+10\sqrt{2}}{2}, \frac{5\sqrt{3}-10\sqrt{2}}{2} \right)}$

Exercise

Suppose that A , B , and C are the corner points of the thin triangular plate of constant density.

- Find the vector from C to the midpoint M of side AB .
- Find the vector from C to the point that lies two-thirds of the way from C to M on the median CM .
- Find the coordinates of the point in which the medians of $\triangle ABC$ intersect (this point is the plate's center of mass).

Solution

- a) The midpoint of AB is:

$$M = \left(\frac{4+1}{2}, \frac{2+3}{2}, 0 \right)$$

$$= \left(\frac{5}{2}, \frac{5}{2}, 0 \right)$$

$$\overrightarrow{CM} = \left(\frac{5}{2} - 1 \right) \hat{i} + \left(\frac{5}{2} - 1 \right) \hat{j} + (0 - 3) \hat{k}$$

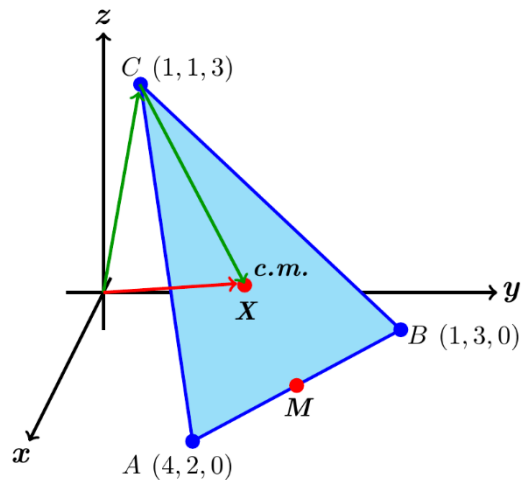
$$= \frac{3}{2} \hat{i} + \frac{3}{2} \hat{j} - 3 \hat{k}$$

- b) The desired vector is

$$\overrightarrow{CX} = \frac{2}{3} \overrightarrow{CM}$$

$$= \frac{2}{3} \left(\frac{3}{2} \hat{i} + \frac{3}{2} \hat{j} - 3 \hat{k} \right)$$

$$= \hat{i} + \hat{j} - 2 \hat{k}$$



- c) The vector whose sum is the vector from the origin to C and the result of part (b) will terminate at the center of mass.

$$\overrightarrow{OX} = \overrightarrow{OC} + \overrightarrow{CX}$$

$$= \hat{i} + \hat{j} + 3\hat{k} + \hat{i} + \hat{j} - 2\hat{k}$$

$$= 2\hat{i} + 2\hat{j} + \hat{k}$$

Therefore; the center of mass point is $\underline{(2, 2, 1)}$

Exercise

Show that a unit vector in the plane can be expressed as $\mathbf{u} = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j}$, obtained by rotating \mathbf{i} through an angle θ in the counterclockwise direction. Explain why this form gives *every* unit vector in the plane.

Solution

Let \mathbf{u} be any unit vector in the plane.

If \mathbf{u} is positioned so that its initial point and terminal point is at (x, y) , then \mathbf{u} makes an angle θ with \mathbf{i} , measured in the *ccw* direction.

Since $|\mathbf{u}| = 1 \Rightarrow x = \cos \theta$ and $y = \sin \theta$

That implies to: $\mathbf{u} = (\cos \theta)\hat{\mathbf{i}} + (\sin \theta)\hat{\mathbf{j}}$

Since \mathbf{u} is any unit vector in the plane; this holds for every unit vector in the plane.

Exercise

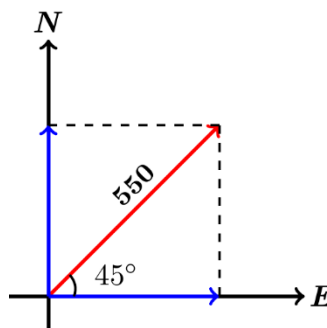
Assume the positive *x-axis* points east and the positive *y-axis* points north.

- An airliner flies northeast at a constant altitude at 550 *mi/hr* in calm air. Find a and b such that its velocity may be expressed in the form $\mathbf{v} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$
- An airliner flies northeast at a constant altitude at 550 *mi/hr* relative to the air in a southerly crosswind $\mathbf{w} = \langle 0, 40 \rangle$. Find the velocity of the airliner relative to the ground.

Solution

$$\begin{aligned} a) \quad \vec{v} &= 550 \langle -\cos 45^\circ, \sin 45^\circ \rangle \\ &= 550 \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle \\ &= \langle -275\sqrt{2}, 275\sqrt{2} \rangle \end{aligned}$$

$$\begin{aligned} b) \quad \vec{v} &= 550 \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle + \langle 0, 40 \rangle \\ &= \langle -275\sqrt{2}, 275\sqrt{2} \rangle \end{aligned}$$



Exercise

Let \overline{PQ} extended from $P(2, 0, 6)$ to $Q(2, -8, 5)$

- Find the position vector equal to \overline{PQ} .
- Find the midpoint M of the line segment PQ . Then find the magnitude of \overline{PM} .

c) Find a vector of length 8 with direction opposite that of \overrightarrow{PQ} .

Solution

$$a) \quad \overrightarrow{PQ} = \langle 2-2, -8-0, 5-6 \rangle \\ = \langle 0, -8, -1 \rangle$$

$$b) \quad M = \left(\frac{2+2}{2}, \frac{0-8}{2}, \frac{6+5}{2} \right) \\ = \left(2, -4, \frac{11}{2} \right)$$

$$\overrightarrow{PM} = \left\langle 0, -4, -\frac{1}{2} \right\rangle$$

$$|\overrightarrow{PM}| = \sqrt{16 + \frac{1}{4}} \\ = \frac{1}{2}\sqrt{65}$$

$$c) \quad |\overrightarrow{PQ}| = \sqrt{64+1} \\ = \sqrt{65}$$

$$\text{vector} = \frac{-8}{\sqrt{65}} \langle 0, -8, -1 \rangle \\ = \frac{8}{\sqrt{65}} \langle 0, 8, 1 \rangle$$

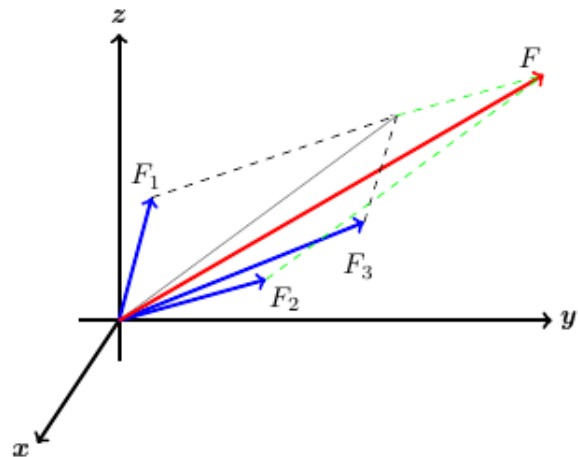
Exercise

An object at the origin is acted on by the forces $F_1 = -10\hat{i} + 20\hat{k}$, $F_2 = 40\hat{j} + 10\hat{k}$, and $F_3 = -50\hat{i} + 20\hat{j}$. Find the magnitude of the combined force and use a sketch to illustrate the direction of the combined force.

Solution

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \\ = -10\hat{i} + 20\hat{k} + 40\hat{j} + 10\hat{k} - 50\hat{i} + 20\hat{j} \\ = -60\hat{i} + 60\hat{j} + 30\hat{k}$$

$$|\vec{F}| = \sqrt{3600 + 3600 + 900} \\ = \sqrt{8100} \\ = 90$$



Exercise

A remote sensing probe falls vertically with a terminal of 60 m/s when it encounters a horizontal crosswind blowing north at 4 m/s and an updraft blowing vertically at 10 m/s. find the magnitude and direction of the resulting velocity relative to the ground.

Solution

The velocity relative to the ground is:

$$\langle 0, 4, 10 - 60 \rangle = \langle 0, 4, -50 \rangle$$

$$\begin{aligned} \text{Magnitude: } \sqrt{16 + 2500} &= \sqrt{2516} \\ &= 2\sqrt{629} \quad \approx 50.16 \text{ m/s} \end{aligned}$$

$$\begin{aligned} \text{Direction} &= \cos^{-1} \frac{4}{\sqrt{2516}} \\ &\approx 85.4^\circ \end{aligned}$$

Below the horizontal in the northerly horizontal direction.

Exercise

A small plane is flying north in calm air at 250 mi/hr when it is hit by a horizontal crosswind blowing northeast at 40 mi/hr and a 25 mi/hr downdraft. Find the resulting velocity and speed of the plane.

Solution

$$\text{Velocity vector} = \langle 250, 0, 0 \rangle$$

$$\begin{aligned} \text{Crosswind} &= \langle 40\cos 45^\circ, 40\sin 45^\circ, 0 \rangle \\ &= \langle 20\sqrt{2}, 20\sqrt{2}, 0 \rangle \end{aligned}$$

$$\text{Downdraft} = \langle 0, 0, -25 \rangle$$

$$\begin{aligned} \text{Resulting velocity} &= \langle 250, 0, 0 \rangle + \langle 20\sqrt{2}, 20\sqrt{2}, 0 \rangle + \langle 0, 0, -25 \rangle \\ &= \langle 250 + 20\sqrt{2}, 20\sqrt{2}, -25 \rangle \end{aligned}$$

$$\begin{aligned} \text{Speed} &= \sqrt{(250 + 20\sqrt{2})^2 + 800 + 625} \\ &= \sqrt{62500 + 10^4\sqrt{2} + 800 + 1,425} \\ &= \sqrt{64,725 + 10^4\sqrt{2}} \\ &= 5\sqrt{2,589 + 400\sqrt{2}} \\ &= 280.83 \text{ mph} \end{aligned}$$