Solution   Section 3.1 – Inner Products

Exercise

Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (1, 1)$, $\vec{v} = (3, 2)$, $\vec{w} = (0, -1)$, and $k = 3$. Compute the following.

a) $\langle \vec{u}, \vec{v} \rangle$ *c)* $\langle \vec{u} + \vec{v}, \vec{w} \rangle$ *e)* $d(\vec{u}, \vec{v})$

b) $\langle k\vec{v}, \vec{w} \rangle$ *d)* $\|\vec{v}\|$ *f)* $\|\vec{u} - k\vec{v}\|$

Solution

a) $\langle \vec{u}, \vec{v} \rangle = 1(3) + 1(2)$

$$= 5$$

b) $\langle k\vec{v}, \vec{w} \rangle = \langle 3v, w \rangle$

$$= 9 \cdot 0 + 6 \cdot (-1)$$

$$= -6$$

c) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

$$= 1 \cdot 0 + 1 \cdot (-1) + 3 \cdot 0 + 2 \cdot (-1)$$

$$= -3$$

d) $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$

$$= \sqrt{3^2 + 2^2}$$

$$= \sqrt{13}$$

e) $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$

$$= \|(-2, -1)\|$$

$$= \sqrt{(-2)^2 + (-1)^2}$$

$$= \sqrt{5}$$

f) $\|\vec{u} - k\vec{v}\| = \|(1, 1) - 3(3, 2)\|$

$$= \|(-8, -5)\|$$

$$= \sqrt{(-8)^2 + (-5)^2}$$

$$= \sqrt{89}$$

Exercise

Let $\langle \vec{u},\ \vec{v}\rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u}=(1,\ 1)$, $\vec{v}=(3,\ 2)$, $\vec{w}=(0,\ -1)$ and $k=3$.

Compute the following for the weighted Euclidean inner product $\langle \vec{u},\ \vec{v}\rangle = 2u_1v_1 + 3u_2v_2$.

a) $\langle \vec{u},\ \vec{v}\rangle$ *c)* $\langle \vec{u}+\vec{v},\ \vec{w}\rangle$ *e)* $d(\vec{u},\ \vec{v})$

b) $\langle k\vec{v},\ \vec{w}\rangle$ *d)* $\|\vec{v}\|$ *f)* $\|\vec{u}-k\vec{v}\|$

Solution

a)
$$\langle \vec{u},\ \vec{v}\rangle = 2(1)(3)+3(1)(2)$$
$$=12$$

b)
$$\langle k\vec{v},\ \vec{w}\rangle = 2(3\cdot 3)(0)+3(3\cdot 2)(-1)$$
$$=-18$$

c)
$$\langle \vec{u}+\vec{v},\ \vec{w}\rangle = \langle \vec{u},\ \vec{w}\rangle + \langle \vec{v},\ \vec{w}\rangle$$
$$=1\cdot 0+1\cdot(-1)+3\cdot 0+2\cdot(-1)$$
$$=-3$$

d)
$$\|\vec{v}\| = \sqrt{\langle \vec{v},\ \vec{v}\rangle}$$
$$=\sqrt{2(3)(3)+3(2)(2)}$$
$$=\sqrt{30}$$

e)
$$d(\vec{u},\ \vec{v}) = \|\vec{u}-\vec{v}\|$$
$$=\|\langle(-2,\ -1)\rangle\|$$
$$=\sqrt{2(-2)(-2)+3(-1)(-1)}$$
$$=\sqrt{11}$$

f)
$$\|\vec{u}-k\vec{v}\| = \|(1,\ 1)-3(3,\ 2)\|$$
$$=\|\langle(-8,\ -5)\rangle\|$$
$$=\sqrt{2(-8)^2+3(-5)^2}$$
$$=\sqrt{203}$$

Exercise

Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (3, -2)$, $\vec{v} = (4, 5)$, $\vec{w} = (-1, 6)$, and $k = -4$. Verify the following.

a) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

c) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

d) $\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$

e) $\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$

Solution

a) $\langle \vec{u}, \vec{v} \rangle = 3 \cdot 4 + (-2) \cdot (5)$

$= 2$

$\langle \vec{v}, \vec{u} \rangle = 4 \cdot 3 + (5) \cdot (-2)$

$= 2$

$\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ ✓

b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle (7,3), (-1,6) \rangle$

$= 7(-1) + 3(6)$

$= 11$

$\langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle = (3)(-1) + (-2)(6) + (4)(-1) + (5)(6)$

$= 11$

$\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ ✓

c) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle (3, -2), (3, 11) \rangle$

$= 3(3) + (-2)(11)$

$= -13$

$\langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle = (3)(4) + (-2)(5) + (3)(-1) + (-2)(6)$

$= -13$

$\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$ ✓

d) $\langle k\vec{u}, \vec{v} \rangle = (-4 \cdot 3) \cdot 4 + ((-4)(-2)) \cdot (5)$

$= -8$

$k\langle \vec{u}, \vec{v} \rangle = (-4)(3 \cdot 4 + (-2) \cdot (5))$

$= -8$

$\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$ ✓

e) $\langle \vec{0},\ \vec{v}\rangle = 0\cdot 4 + 0\cdot(5)$

$\underline{=0}\rfloor$

$\langle \vec{v},\ \vec{0}\rangle = 4\cdot 0 + (5)\cdot(0)$

$\underline{=0}\rfloor$

$\langle \vec{0},\ \vec{v}\rangle = \langle \vec{v},\ \vec{0}\rangle = 0$ ✓

Exercise

Let $\langle \vec{u},\ \vec{v}\rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u}=(3,\ -2)$, $\vec{v}=(4,\ 5)$, $\vec{w}=(-1,\ 6)$, and $k=-4$. Verify the following for the weighted Euclidean inner product $\langle \vec{u},\ \vec{v}\rangle = 4u_1v_1 + 5u_2v_2$.

a) $\langle \vec{u},\ \vec{v}\rangle = \langle \vec{v},\ \vec{u}\rangle$

b) $\langle \vec{u}+\vec{v},\ \vec{w}\rangle = \langle \vec{u},\ \vec{w}\rangle + \langle \vec{v},\ \vec{w}\rangle$

c) $\langle \vec{u},\ \vec{v}+\vec{w}\rangle = \langle \vec{u},\ \vec{v}\rangle + \langle \vec{u},\ \vec{w}\rangle$

d) $\langle k\vec{u},\ \vec{v}\rangle = k\langle \vec{u},\ \vec{v}\rangle$

e) $\langle \vec{0},\ \vec{v}\rangle = \langle \vec{v},\ \vec{0}\rangle = 0$

Solution

a) $\langle \vec{u},\ \vec{v}\rangle = 4\cdot 3\cdot 4 + 5\cdot(-2)\cdot(5)$

$\underline{=-2}\rfloor$

$\langle \vec{v},\ \vec{u}\rangle = 4\cdot 4\cdot 3 + 5\cdot(5)\cdot(-2)$

$\underline{=-2}\rfloor$

$\langle \vec{u},\ \vec{v}\rangle = \langle \vec{v},\ \vec{u}\rangle$ ✓

b) $\langle \vec{u}+\vec{v},\ \vec{w}\rangle = \langle (7,3),\ (-1,6)\rangle$

$= 4\cdot 7(-1) + 5\cdot 3(6)$

$\underline{=62}\rfloor$

$\langle \vec{u},\ \vec{w}\rangle + \langle \vec{v},\ \vec{w}\rangle = 4\cdot(3)(-1) + 5\cdot(-2)(6) + 4\cdot(4)(-1) + 5\cdot(5)(6)$

$\underline{=62}\rfloor$

$\langle \vec{u}+\vec{v},\ \vec{w}\rangle = \langle \vec{u},\ \vec{w}\rangle + \langle \vec{v},\ \vec{w}\rangle$ ✓

c) $\langle \vec{u},\ \vec{v}+\vec{w}\rangle = \langle (3,\ -2),\ (3,\ 11)\rangle$

$= 4\cdot 3(3) + 5\cdot(-2)(11)$

$\underline{=-74}\rfloor$

$\langle \vec{u},\ \vec{v}\rangle + \langle \vec{u},\ \vec{w}\rangle = 4\cdot(3)(4) + 5\cdot(-2)(5) + 4\cdot(3)(-1) + 5\cdot(-2)(6)$

$$\underline{= -74}$$

$$\langle \vec{u},\ \vec{v}+\vec{w}\rangle = \langle \vec{u},\ \vec{v}\rangle + \langle \vec{u},\ \vec{w}\rangle \quad \checkmark$$

d) $\langle k\vec{u},\ \vec{v}\rangle = 4\cdot(-4\cdot 3)\cdot 4 + 5\cdot\big((-4)(-2)\big)\cdot(5)$

$$\underline{= 8}$$

$$k\langle \vec{u},\ \vec{v}\rangle = (-4)\big(4\cdot 3\cdot 4 + 5\cdot(-2)\cdot(5)\big)$$

$$\underline{= 8}$$

$$\langle k\vec{u},\ \vec{v}\rangle = k\langle \vec{u},\ \vec{v}\rangle \quad \checkmark$$

e) $\langle \vec{0},\ \vec{v}\rangle = 4\cdot 0\cdot 4 + 5\cdot 0\cdot(5)$

$$\underline{= 0}$$

$$\langle \vec{v},\ \vec{0}\rangle = 4\cdot 4\cdot 0 + 5\cdot(5)\cdot(0)$$

$$\underline{= 0}$$

$$\langle \vec{0},\ \vec{v}\rangle = \langle \vec{v},\ \vec{0}\rangle = 0 \quad \checkmark$$

Exercise

Let $\vec{u} = (u_1,\ u_2)$ and $\vec{v} = (v_1,\ v_2)$. Show that the following are inner product on $\mathbb{R}^2$ by verifying that the inner product axioms hold. $\langle \vec{u},\ \vec{v}\rangle = 3u_1v_1 + 5u_2v_2$

Solution

Axiom 1: $\langle \vec{u},\ \vec{v}\rangle = 3u_1v_1 + 5u_2v_2$

$$= 3v_1u_1 + 5v_2u_2$$

$$= \langle \vec{v},\ \vec{u}\rangle \quad \checkmark$$

Axiom 2: $\langle \vec{u}+\vec{v},\ \vec{w}\rangle = 3(u_1+v_1)w_1 + 5(u_2+v_2)w_2$

$$= 3(u_1w_1 + v_1w_1) + 5(u_2w_2 + v_2w_2)$$

$$= 3u_1w_1 + 3v_1w_1 + 5u_2w_2 + 5v_2w_2$$

$$= (3u_1w_1 + 5u_2w_2) + (3v_1w_1 + 5v_2w_2)$$

$$= \langle \vec{u},\ \vec{w}\rangle + \langle \vec{v},\ \vec{w}\rangle \quad \checkmark$$

Axiom 3: $\langle k\vec{u},\ \vec{v}\rangle = 3(ku_1)v_1 + 5(ku_2)v_2$

$$= k(3u_1v_1 + 5u_2v_2)$$

$$= k\langle \vec{u},\ \vec{v}\rangle \quad \checkmark$$

Axiom 4: $\langle \vec{v},\ \vec{v}\rangle = 3v_1 v_1 + 5v_2 v_2$

$$= 3v_1^2 + 5v_2^2 \geq 0$$

$$v_1 = v_2 = 0 \quad iff \quad \vec{v} = \vec{0} \quad \checkmark$$

Exercise

Show that the following identity holds for the vectors in any inner product space

$$\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$$

Solution

$$\begin{aligned}
\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 &= \langle \vec{u} + \vec{v},\ \vec{u} + \vec{v}\rangle + \langle \vec{u} - \vec{v},\ \vec{u} - \vec{v}\rangle \\
&= \langle \vec{u},\ \vec{u} + \vec{v}\rangle + \langle \vec{v},\ \vec{u} + \vec{v}\rangle + \langle \vec{u},\ \vec{u} - \vec{v}\rangle - \langle \vec{v},\ \vec{u} - \vec{v}\rangle \\
&= \langle \vec{u},\ \vec{u}\rangle + \langle \vec{u},\ \vec{v}\rangle + \langle \vec{v},\ \vec{u}\rangle + \langle \vec{v},\ \vec{v}\rangle + \langle \vec{u},\ \vec{u}\rangle - \langle \vec{u},\ \vec{v}\rangle - \langle \vec{v},\ \vec{u}\rangle + \langle \vec{v},\ \vec{v}\rangle \\
&= 2\langle \vec{u},\ \vec{u}\rangle + 2\langle \vec{v},\ \vec{v}\rangle \\
&= 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2 \quad \checkmark
\end{aligned}$$

Exercise

Show that the following identity holds for the vectors in any inner product space

$$\langle \vec{u},\ \vec{v}\rangle = \frac{1}{4}\left(\|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2\right)$$

Solution

$$\|\vec{u} + \vec{v}\|^2 = \langle \vec{u} + \vec{v},\ \vec{u} + \vec{v}\rangle = \|\vec{u}\|^2 + 2\langle \vec{u},\ \vec{v}\rangle + \|\vec{v}\|^2$$

$$\|\vec{u} - \vec{v}\|^2 = \langle \vec{u} - \vec{v},\ \vec{u} - \vec{v}\rangle = \|\vec{u}\|^2 - 2\langle \vec{u},\ \vec{v}\rangle + \|\vec{v}\|^2$$

$$\begin{array}{rl}
 & \|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + 2\langle \vec{u},\ \vec{v}\rangle + \|\vec{v}\|^2 \\
- & \underline{\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 - 2\langle \vec{u},\ \vec{v}\rangle + \|\vec{v}\|^2} \\
 & \|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 = 4\langle \vec{u},\ \vec{v}\rangle
\end{array}$$

$$\langle \vec{u},\ \vec{v}\rangle = \frac{1}{4}\left(\|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2\right) \quad \checkmark$$

Exercise

Prove that $\|k\vec{v}\| = |k|\ \|\vec{v}\|$

Solution

$$\|k\vec{v}\|^2 = \langle k\vec{v},\ \vec{v}\rangle$$
$$= k^2\langle \vec{v},\ \vec{v}\rangle$$
$$= k^2\|\vec{v}\|^2$$

$$\|k\vec{v}\| = k\ \|\vec{v}\| \quad \checkmark$$

Solution   *Section* 3.2 – Angle and Orthogonality in Inner Product Spaces

Exercise

Which of the following form orthonormal sets?

a) $(1, 0), (0, 2)$ in $\mathbb{R}^2$

b) $\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$ in $\mathbb{R}^2$

c) $\left(-\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$ in $\mathbb{R}^2$

d) $\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right),\ \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$ in $\mathbb{R}^3$

e) $\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right),\ \left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right),\ \left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right)$ in $\mathbb{R}^3$

f) $\left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right),\ \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)$ in $\mathbb{R}^3$

Solution

a) $$(1,\ 0)\cdot(0,\ 2)=1(0)+0(2)$$
$$\underline{=0}$$

They are ***orthonormal*** sets

b) $$\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)=\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}+\left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}$$
$$=\frac{1}{2}-\frac{1}{2}$$
$$\underline{=0}$$

They are ***orthonormal*** sets

c) $$\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)=\left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}+\left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}$$
$$=-\frac{1}{2}-\frac{1}{2}$$
$$\underline{=-1\ \neq 0}$$

They are ***not orthonormal*** sets

d) $$\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)\cdot\left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{2}}\right)+0+\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{3}}\right)\frac{1}{\sqrt{2}}$$

$$= -\frac{1}{2}\frac{1}{\sqrt{3}}-\frac{1}{2}\frac{1}{\sqrt{3}}$$

$$= -\frac{1}{\sqrt{3}} \neq 0$$

They are ***not orthonormal*** sets

e) $\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right)\cdot\left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right)\cdot\left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right)$

$$= \frac{2}{3}\frac{2}{3}\frac{1}{3}+\left(-\frac{2}{3}\right)\frac{1}{3}\frac{2}{3}+\frac{1}{3}\left(-\frac{2}{3}\right)\frac{2}{3}$$

$$= \frac{4}{27}-\frac{4}{27}-\frac{4}{27}$$

$$= -\frac{4}{27} \neq 0$$

They are ***not orthonormal*** sets

f) $\left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right)\cdot\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)=\frac{1}{\sqrt{6}}\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{6}}\left(-\frac{1}{\sqrt{2}}\right)+0$

$$= 0$$

They are ***orthonormal*** sets

Exercise

Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.

a) $\vec{u}=(1,\ -3),\quad \vec{v}=(2,\ 4)$

b) $\vec{u}=(-1,\ 0),\quad \vec{v}=(3,\ 8)$

c) $\vec{u}=(-1,\ 5,\ 2),\quad \vec{v}=(2,\ 4,\ -9)$

d) $\vec{u}=(4,\ 1,\ 8),\quad \vec{v}=(1,\ 0,\ -3)$

e) $\vec{u}=(1,\ 0,\ 1,\ 0),\quad \vec{v}=(-3,\ -3,\ -3,\ -3)$

f) $\vec{u}=(2,\ 1,\ 7,\ -1),\quad \vec{v}=(4,\ 0,\ 0,\ 0)$

g) $\vec{u}=(1,\ 3,\ -5,\ 4),\quad \vec{v}=(2,\ -4,\ 4,\ 1)$

h) $\vec{u}=(1,\ 2,\ 3,\ 4),\quad \vec{v}=(-1,\ -2,\ -3,\ -4)$

Solution

a) $\vec{u}=(1,\ -3),\quad \vec{v}=(2,\ 4)$

$$\|\vec{u}\|=\sqrt{1^2+(-3)^2}$$

$$=\sqrt{10}$$

$$\|\vec{v}\|=\sqrt{2^2+4^2}$$

$$=\sqrt{20}$$

$$\langle \vec{u}, \vec{v} \rangle = 1(2) + (-3)(4)$$
$$= -10$$

$$\cos\theta = \frac{-10}{\sqrt{10}\ \sqrt{20}} \qquad \cos\theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$
$$= -\frac{10}{\sqrt{200}}$$
$$= -\frac{1}{\sqrt{2}}$$

b) $\vec{u} = (-1,\ 0); \quad \vec{v} = (3,\ 8)$

$$\|\vec{u}\| = \sqrt{(-1)^2 + 0^2}$$
$$= 1$$

$$\|\vec{v}\| = \sqrt{3^2 + 8^2}$$
$$= \sqrt{73}$$

$$\langle \vec{u}, \vec{v} \rangle = (-1)(3) + (0)(8)$$
$$= -3$$

$$\cos\theta = \frac{-3}{1\sqrt{73}} \qquad \cos\theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$
$$= -\frac{3}{\sqrt{73}}$$

c) $\vec{u} = (-1,\ 5,\ 2); \quad \vec{v} = (2,\ 4,\ -9)$

$$\|\vec{u}\| = \sqrt{(-1)^2 + 5^2 + 2^2}$$
$$= \sqrt{30}$$

$$\|\vec{v}\| = \sqrt{2^2 + 4^2 + (-9)^2}$$
$$= \sqrt{101}$$

$$\langle \vec{u}, \vec{v} \rangle = (-1)(2) + (5)(4) + (2)(-9)$$
$$= 0$$

$$\cos\theta = 0 \qquad \cos\theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$

d) $\vec{u} = (4,\ 1,\ 8); \quad \vec{v} = (1,\ 0,\ -3)$

$$\|\vec{u}\| = \sqrt{4^2 + 1^2 + 8^2}$$

$$= 9$$

$$\|\vec{v}\| = \sqrt{1+0+9}$$

$$= \sqrt{10}$$

$$\langle \vec{u}, \vec{v} \rangle = (4)(1)+(1)(0)+(8)(-3)$$

$$= -20$$

$$\cos\theta = -\frac{20}{9\sqrt{10}} \qquad \cos\theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \, \|\vec{v}\|}$$

e) $\vec{u} = (1,\ 0,\ 1,\ 0);\quad \vec{v} = (-3,\ -3,\ -3,\ -3)$

$$\|\vec{u}\| = \sqrt{2}$$

$$\|\vec{v}\| = \sqrt{9+9+9+9}$$

$$= 12$$

$$\langle \vec{u}, \vec{v} \rangle = -3+0-3+0$$

$$= -6$$

$$\cos\theta = \frac{-6}{12\sqrt{2}} \qquad \cos\theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \, \|\vec{v}\|}$$

$$= -\frac{1}{2\sqrt{2}}$$

f) $\vec{u} = (2,\ 1,\ 7,\ -1);\quad \vec{v} = (4,\ 0,\ 0,\ 0)$

$$\|\vec{u}\| = \sqrt{2^2+1^2+7^2+(-1)^2}$$

$$= \sqrt{55}$$

$$\|\vec{v}\| = \sqrt{4^2+0}$$

$$= 4$$

$$\langle \vec{u}, \vec{v} \rangle = (2)(4)+(1)(0)+(7)(0)+(-1)(0)$$

$$= 8$$

$$\cos\theta = \frac{8}{4\sqrt{55}} \qquad \cos\theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \, \|\vec{v}\|}$$

$$= \frac{2}{\sqrt{55}}$$

g) $\vec{u} = (1,\ 3,\ -5,\ 4),\quad \vec{v} = (2,\ -4,\ 4,\ 1)$

$$\|\vec{u}\| = \sqrt{1+9+25+16}$$

$$= \sqrt{51}$$

$$\|\vec{v}\| = \sqrt{4+16+16+1}$$

$$= \sqrt{37}$$

$$\langle \vec{u}, \vec{v} \rangle = 2-12-20+4$$

$$= -26$$

$$\cos\theta = \frac{-26}{\sqrt{51}\sqrt{37}} \qquad \cos\theta = \frac{\langle \vec{u}, \vec{v}\rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$

h) $\vec{u} = (1,\ 2,\ 3,\ 4), \quad \vec{v} = (-1,\ -2,\ -3,\ -4)$

$$\|\vec{u}\| = \sqrt{1+4+9+16}$$

$$= \sqrt{30}$$

$$\|\vec{v}\| = \sqrt{1+4+9+16}$$

$$= \sqrt{30}$$

$$\langle \vec{u}, \vec{v} \rangle = -1-4-9-16$$

$$= -30$$

$$\cos\theta = \frac{-30}{\sqrt{30}\sqrt{30}} \qquad \cos\theta = \frac{\langle \vec{u}, \vec{v}\rangle}{\|\vec{u}\|\ \|\vec{v}\|}$$

$$= -1$$

Exercise

Find the cosine of the angle between ***A*** and ***B***.

a) $A = \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$

b) $A = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$

c) $A = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

d) $A = \begin{pmatrix} 1 & -2 & 7 \\ 6 & -3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 3 \\ 6 & 5 & -4 \end{pmatrix}$

Solution

a) $A = \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$

$$\|A\| = \sqrt{\langle A, A \rangle}$$

$$= \sqrt{2^2+6^2+1^2+(-3)^2}$$

$$= \sqrt{50}$$

$= 5\sqrt{2}$

$\|B\| = \sqrt{\langle B,\ B\rangle}$

$= \sqrt{9+4+1+0}$

$= \sqrt{14}$

$\langle A,\ B\rangle = 2(3) + 6(2) + 1(1) + (-3)(0)$

$= 19$

$\cos\theta = \dfrac{19}{5\sqrt{2}\sqrt{14}}$ $\qquad \cos\theta = \dfrac{\langle A,\ B\rangle}{\|A\|\ \|B\|}$

$= \dfrac{19}{10\sqrt{7}}$

b) $A = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$

$\|A\| = \sqrt{\langle A,\ A\rangle}$

$= \sqrt{2^2 + 4^2 + (-1)^2 + 3^2}$

$= \sqrt{30}$

$\|B\| = \sqrt{\langle B,\ B\rangle}$

$= \sqrt{(-3)^2 + 1^2 + 4^2 + 2^2}$

$= \sqrt{30}$

$\langle A,\ B\rangle = 2(-3) + 4(1) + (-1)(4) + 3(2)$

$= 0$

$\cos\theta = \dfrac{0}{30}$ $\qquad \cos\theta = \dfrac{\langle A,\ B\rangle}{\|A\|\ \|B\|}$

$= 0$

c) $A = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

$\|A\| = \sqrt{81+64+49+36+25+16}$ $\qquad \|A\| = \sqrt{\langle A,\ A\rangle}$

$= \sqrt{271}$

$\|B\| = \sqrt{1+4+9+16+25+36}$ $\qquad \|B\| = \sqrt{\langle B,\ B\rangle}$

$= \sqrt{91}$

$$\langle A,\ B\rangle = 9+16+21+24+25+24$$
$$= 119$$

$$\cos\theta = \frac{119}{\sqrt{271}\sqrt{91}} \qquad \cos\theta = \frac{\langle A,\ B\rangle}{\|A\|\ \|B\|}$$

d) $A = \begin{pmatrix} 1 & -2 & 7 \\ 6 & -3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 3 \\ 6 & 5 & -4 \end{pmatrix}$

$$\|A\| = \sqrt{1^2+(-2)^2+7^2+6^2+(-3)^2+4^2} \qquad \|A\| = \sqrt{\langle A,\ A\rangle}$$
$$= \sqrt{115}$$

$$\|B\| = \sqrt{1^2+(-2)^2+3^2+6^2+5^2+(-4)^2} \qquad \|B\| = \sqrt{\langle B,\ B\rangle}$$
$$= \sqrt{91}$$

$$\langle A,\ B\rangle = 1+4+21+36-15-16$$
$$= 31$$

$$\cos\theta = \frac{31}{\sqrt{115}\sqrt{91}} \qquad \cos\theta = \frac{\langle A,\ B\rangle}{\|A\|\ \|B\|}$$

Exercise

Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

a) $\vec{u} = (-1,\ 3,\ 2),\quad \vec{v} = (4,\ 2,\ -1)$

b) $\vec{u} = (a,\ b),\quad \vec{v} = (-b,\ a)$

c) $\vec{u} = (-2,\ -2,\ -2),\quad \vec{v} = (1,\ 1,\ 1)$

d) $\vec{u} = (-4,\ 6,\ -10,\ 1),\quad \vec{v} = (2,\ 1,\ -2,\ 9)$

e) $\vec{u} = (-4,\ 6,\ -10,\ 1),\quad \vec{v} = (2,\ 1,\ -2,\ 9)$

Solution

a) $\langle \vec{u},\ \vec{v}\rangle = (-1)(4)+3(2)+2(-1)$

$$= 0$$

Therefore, the given vectors are orthogonal.

b) $\langle \vec{u},\ \vec{v}\rangle = a(-b)+b(a)$

$$= 0$$

Therefore, the given vectors are orthogonal.

c) $\langle \vec{u},\ \vec{v}\rangle = (-2)(1)+(-2)(1)+(-2)(1)$

$$= -6$$

Therefore, the given vectors are ***not*** orthogonal.

d) $\langle\langle \vec{u}, \vec{v}\rangle\rangle = (-4)(2)+(6)(1)+(-10)(-2)+(1)(9)$

$= 27 \quad \neq 0$

Therefore, the given vectors are ***not*** orthogonal.

e) $\|\vec{u}\| = \sqrt{(-4)^2+6^2+(-10)^2+1^2}$

$= \sqrt{153}$

$= 3\sqrt{17}$

$\|\vec{v}\| = \sqrt{2^2+1^2+(-2)^2+9^2}$

$= \sqrt{90}$

$= 3\sqrt{10}$

$\langle \vec{u}, \vec{v}\rangle = (-4)(2)+6(1)-10(-2)+1(9)$

$= 27$

$\cos\theta = \dfrac{27}{3\sqrt{17}\left(3\sqrt{10}\right)} \qquad \cos\theta = \dfrac{\langle \vec{u}, \vec{v}\rangle}{\|\vec{u}\|\ \|\vec{v}\|}$

$= \dfrac{3}{\sqrt{170}}$

The vectors $\vec{u}$ *and* $\vec{v}$ are ***not*** orthogonal with respect to the Euclidean

Exercise

Do there exist scalars k and l such that the vectors $\vec{u}=(2, k, 6)$, $\vec{v}=(l, 5, 3)$, *and* $\vec{w}=(1, 2, 3)$ are mutually orthogonal with respect to the Euclidean inner product?

Solution

$\langle \vec{u}, \vec{w}\rangle = (2)(1)+(k)(2)+(6)(3)$

$= 20+2k = 0$

$\Rightarrow k = -10$

$\langle \vec{v}, \vec{w}\rangle = (l)(1)+(5)(2)+(3)(3)$

$= l+19 = 0$

$\Rightarrow l = -19$

$\langle \vec{u}, \vec{v}\rangle = (2)(l)+(k)(5)+(6)(3)$

$= 2l+5k+18 = 0$

$$2(-19)+5(-10)+18=-70\neq 0$$

Thus, there are no scalars such that the vectors are mutually orthogonal.

Exercise

Let $\mathbb{R}^3$ have the Euclidean inner product. For which values of k are $\vec{u}$ and $\vec{v}$ orthogonal?

a) $\vec{u}=(2,1,3), \quad \vec{v}=(1,7,k)$

b) $\vec{u}=(k,k,1), \quad \vec{v}=(k,5,6)$

Solution

a) $\langle\vec{u},\vec{v}\rangle=(2)(1)+(1)(7)+(3)(k)$

$$=9+3k=0$$

$\vec{u}$ and $\vec{v}$ are orthogonal for $\boxed{k=-3}$

b) $\langle\vec{u},\vec{v}\rangle=(k)(k)+(k)(5)+(1)(6)$

$$=k^2+5k+6=0$$

$\vec{u}$ and $\vec{v}$ are orthogonal for $\boxed{k=-2,\ -3}$

Exercise

Let V be an inner product space. Show that if $\vec{u}$ and $\vec{v}$ are orthogonal unit vectors in V, then $\|\vec{u}-\vec{v}\|=\sqrt{2}$

Solution

$$\begin{aligned}\|\vec{u}-\vec{v}\|^2 &=\langle\vec{u}-\vec{v},\ \vec{u}-\vec{v}\rangle\\ &=\langle\vec{u},\ \vec{u}-\vec{v}\rangle-\langle\vec{v},\ \vec{u}-\vec{v}\rangle\\ &=\langle\vec{u},\vec{u}\rangle-\langle\vec{u},\vec{v}\rangle-\langle\vec{u},\vec{v}\rangle+\langle\vec{v},\vec{v}\rangle\\ &=\|\vec{u}\|^2-0-0+\|\vec{v}\|^2 &&\text{since } \vec{u} \text{ and } \vec{v} \text{ are orthogonal unit vectors}\\ &=1+1\\ &=2\end{aligned}$$

Thus $\boxed{\|\vec{u}-\vec{v}\|=\sqrt{2}}$

Exercise

Let $\boldsymbol{S}$ be a subspace of $\mathbb{R}^n$. Explain what $\left(\boldsymbol{S}^{\perp}\right)^{\perp} = \boldsymbol{S}$ means and why it is true.

Solution

$\left(\boldsymbol{S}^{\perp}\right)^{\perp}$ is the orthogonal complement of , $\boldsymbol{S}^{\perp}$, which is itself the orthogonal complement of $\boldsymbol{S}$, so $\left(\boldsymbol{S}^{\perp}\right)^{\perp} = \boldsymbol{S}$ means that $\boldsymbol{S}$ is the orthogonal of its orthogonal complement.

We need to show that $\boldsymbol{S}$ is contained in $\left(\boldsymbol{S}^{\perp}\right)^{\perp}$ and, conversely, that $\left(\boldsymbol{S}^{\perp}\right)^{\perp}$ is contained in $\boldsymbol{S}$ to be true.

i. Suppose $\vec{v} \in \boldsymbol{S}^{\perp}$ and $\vec{w} \in \boldsymbol{S}^{\perp}$. Then $\langle \vec{v},\ \vec{w} \rangle = 0$ by definition of $\boldsymbol{S}^{\perp}$.

Thus, $\boldsymbol{S}$ is certainly contained is $\left(\boldsymbol{S}^{\perp}\right)^{\perp}$ (which consists of all vectors in $\mathbb{R}^n$ which are orthogonal to $\boldsymbol{S}^{\perp}$).

ii. Suppose $\vec{v} \in \left(\boldsymbol{S}^{\perp}\right)^{\perp}$ (means $\vec{v}$ is orthogonal to all vectors in $\boldsymbol{S}^{\perp}$); then we need to show that $\vec{v} \in \boldsymbol{S}$.

Let assume $\left\{\vec{u}_1,\ \vec{u}_2,\ \ldots,\ \vec{u}_p\right\}$ be a basis for $\boldsymbol{S}$ and let $\left\{\vec{w}_1,\ \vec{w}_2,\ \ldots,\ \vec{w}_q\right\}$ be a basis for $\boldsymbol{S}^{\perp}$. If $\vec{v} \notin \boldsymbol{S}$, then $\left\{\vec{u}_1,\ \vec{u}_2,\ \ldots,\ \vec{u}_p,\ \vec{v}\right\}$ is linearly independent set. Since each vector ifs that set is orthogonal to all of $\boldsymbol{S}^{\perp}$, the set $\left\{\vec{u}_1,\ \vec{u}_2,\ \ldots,\ \vec{u}_p,\ \vec{v},\ \vec{w}_1,\ \vec{w}_2,\ \ldots,\ \vec{w}_q\right\}$ is linearly independent.

Since there are $p+q+1$ vectors in this set, this means that $p+q+1 \le n \quad \Leftrightarrow \quad p+q \le n-1$.

On the other hand, If A is the matrix whose i^{th} row is $\vec{u}_i^T$, then the row space of A is $\boldsymbol{S}$ and the nullspace of A is $\boldsymbol{S}^{\perp}$.

Since $\boldsymbol{S}$ is *p*-dimensional, the rank of A is *p*, meaning that the dimension of $\operatorname{nul}(A) = \boldsymbol{S}^{\perp}$ is $q = n - p$. Therefore,

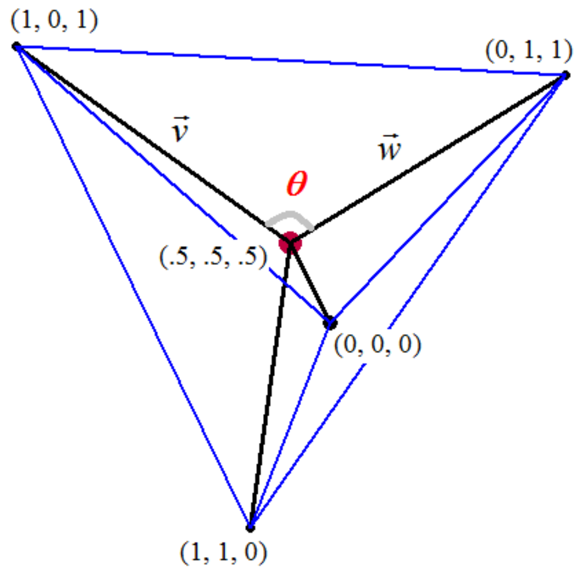
$$p+q = p+(n-p) = n$$

Which contradict the fact that $p+q \le n-1$. From this, we see that, if $\vec{v} \in \left(\boldsymbol{S}^{\perp}\right)^{\perp}$, it must be the case that $\vec{v} \in \boldsymbol{S}$.

Exercise

The methane molecule CH_4 is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at $(0,\ 0,\ 0)$, $(1,\ 1,\ 0)$, $(1,\ 0,\ 1)$ and $(0,\ 1,\ 1)$ – (***note*** that all six edges have length $\sqrt{2}$, so the tetrahedron is regular). What is the cosine of the angle between the rays going from the center $\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$ to the vertices?

Solution



Let $\vec{v}$ be the vector of the segment (1, 0, 1) and $\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

Let be the vector of the segment (0, 1, 1) and $\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$

$$\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$=\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

We have:

$$\cos\theta = \frac{\langle \vec{v},\ \vec{w}\rangle}{\|\vec{v}\|\ \|\vec{w}\|}$$

$$= \frac{\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)\cdot\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}{\sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}\ \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}}$$

$$= \frac{-\frac{1}{4}}{\frac{3}{4}}$$

$$= -\frac{1}{3}$$

$$\theta \approx 109.47°$$

Exercise

Determine if the given vectors are orthogonal.

$$\vec{x}_1 = (1,\ 0,\ 1,\ 0), \quad \vec{x}_2 = (0,\ 1,\ 0,\ 1), \quad \vec{x}_3 = (1,\ 0,\ -1,\ 0), \quad \vec{x}_4 = (1,\ 1,\ -1,\ -1)$$

Solution

$$\vec{x}_1 \bullet \vec{x}_2 = (1,\ 0,\ 1,\ 0) \bullet (0,\ 1,\ 0,\ 1)$$

$$= 0$$

$$\vec{x}_1 \bullet \vec{x}_3 = (1,\ 0,\ 1,\ 0) \bullet (1,\ 0,\ -1,\ 0)$$

$$= 1 - 1$$

$$= 0$$

$$\vec{x}_1 \bullet \vec{x}_4 = (1,\ 0,\ 1,\ 0) \bullet (1,\ 1,\ -1,\ -1)$$

$$= 1 - 1$$

$$= 0$$

$$\vec{x}_2 \bullet \vec{x}_3 = (0,\ 1,\ 0,\ 1) \bullet (1,\ 0,\ -1,\ 0)$$

$$= 0$$

$$\vec{x}_2 \bullet \vec{x}_4 = (0,\ 1,\ 0,\ 1) \bullet (1,\ 1,\ -1,\ -1)$$

$$= 1 - 1$$

$$\underline{=0}$$

$$\vec{x}_3 \bullet \vec{x}_4 = (1,\ 0,\ -1,\ 0) \bullet (1,\ 1,\ -1,\ -1)$$
$$= 1 - 1$$
$$\underline{=0}$$

The given vectors are ***orthogonal.***

Exercise

Which of the following sets of vectors are orthogonal with respect to the Euclidean inner

a) $\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\ \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)\ \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$

b) $\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right)\ \left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right)\ \left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right)$

Solution

a)
$$\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) \bullet \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}}$$
$$\underline{=0}$$

$$\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) \bullet \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) = -\frac{1}{2} + 0 + \frac{1}{2}$$
$$\underline{=0}$$

$$\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right) \bullet \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}}$$
$$\underline{= -\frac{2}{\sqrt{6}} \neq 0}$$

Therefore, the given vectors are ***not*** orthogonal.

b)
$$\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right) \bullet \left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right) = \frac{4}{9} - \frac{2}{9} - \frac{2}{9}$$
$$\underline{=0}$$

$$\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right) \bullet \left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right) = \frac{2}{9} - \frac{4}{9} + \frac{2}{9}$$
$$\underline{=0}$$

$$\left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right) \bullet \left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right) = \frac{2}{9} + \frac{2}{9} - \frac{4}{9}$$
$$\underline{=0}$$

Therefore, the given vectors are orthogonal.

Exercise

Consider vectors $\vec{u}=(2,\ 3,\ 5)\quad \vec{v}=(1,\ -4,\ 3)\quad in\quad \mathbb{R}^3$

a) $\langle \vec{u},\ \vec{v}\rangle$ *b)* $\|\vec{u}\|$ *c)* $\|\vec{v}\|$ *d)* Cosine between $\vec{u}$ *and* $\vec{v}$

Solution

a) $\langle \vec{u},\ \vec{v}\rangle=(2,\ 3,\ 5)\bullet(1,\ -4,\ 3)$

$$=2-12+15$$

$$\underline{=5}$$

b) $\|\vec{u}\|=\sqrt{4+9+25}$

$$\underline{=\sqrt{38}}$$

c) $\vec{v}=\sqrt{1+16+9}$

$$\underline{=\sqrt{26}}$$

d) $\cos\theta=\dfrac{5}{\sqrt{38}\sqrt{26}}$ $\qquad \cos\theta=\dfrac{\vec{u}\bullet\vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$

Exercise

Consider vectors $\vec{u}=(1,\ 1,\ 1)\quad \vec{v}=(1,\ 2,\ -3)\quad in\quad \mathbb{R}^3$

a) $\langle \vec{u},\ \vec{v}\rangle$ *b)* $\|\vec{u}\|$ *c)* $\|\vec{v}\|$ *d)* Cosine θ between $\vec{u}$ *and* $\vec{v}$

Solution

a) $\langle \vec{u},\ \vec{v}\rangle=(1,\ 1,\ 1)\bullet(1,\ 2,\ -3)$

$$=1+2-3$$

$$\underline{=0}$$

b) $\|\vec{u}\|=\sqrt{1+1+1}$

$$\underline{=\sqrt{3}}$$

c) $\|\vec{v}\|=\sqrt{1+4+9}$

$$\underline{=\sqrt{14}}$$

d) $\underline{\cos\theta=0}$ $\qquad \cos\theta=\dfrac{\vec{u}\bullet\vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$

$\vec{u}$ *and* $\vec{v}$ are orthogonal vectors.

Exercise

Consider vectors $\vec{u}=(1,\ 2,\ 5)$ $\vec{v}=(2,\ -3,\ 5)$ $\vec{w}=(4,\ 2,\ -3)$ *in* $\mathbb{R}^3$

a) $\langle\vec{u},\ \vec{v}\rangle$

b) $\langle\vec{u},\ \vec{w}\rangle$

c) $\langle\vec{v},\ \vec{w}\rangle$

d) $\|\vec{u}\|$

e) $\|\vec{v}\|$

f) $\|\vec{w}\|$

g) Cosine α between $\vec{u}$ *and* $\vec{v}$

h) Cosine β between $\vec{u}$ *and* $\vec{w}$

i) Cosine θ between $\vec{v}$ *and* $\vec{w}$

j) $(\vec{u}+\vec{v})\bullet\vec{w}$

Solution

a) $\langle\vec{u},\ \vec{v}\rangle=(1,\ 2,\ 5)\bullet(2,\ -3,\ 5)$

$$=2-6+25$$

$$\underline{=21}$$

b) $\langle\vec{u},\ \vec{w}\rangle=(1,\ 2,\ 5)\bullet(4,\ 2,\ -3)$

$$=4+4-15$$

$$\underline{=-7}$$

c) $\langle\vec{v},\ \vec{w}\rangle=(2,\ -3,\ 5)\bullet(4,\ 2,\ -3)$

$$=8-6-15$$

$$\underline{=-13}$$

d) $\|\vec{u}\|=\sqrt{1+4+25}$

$$\underline{=\sqrt{30}}$$

e) $\|\vec{v}\|=\sqrt{4+9+25}$

$$\underline{=\sqrt{38}}$$

f) $\|\vec{w}\|=\sqrt{16+4+9}$

$$\underline{=\sqrt{29}}$$

g) $\cos\alpha=\dfrac{21}{\sqrt{30}\sqrt{38}}$ $\qquad\cos\theta=\dfrac{\vec{u}\bullet\vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$

h) $\cos\beta=\dfrac{-7}{\sqrt{30}\sqrt{29}}$ $\qquad\cos\theta=\dfrac{\vec{u}\bullet\vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$

i) $\cos\theta=\dfrac{-13}{\sqrt{38}\sqrt{29}}$ $\qquad\cos\theta=\dfrac{\vec{u}\bullet\vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$

j) $(\vec{u}+\vec{v})\bullet\vec{w}=\left[(1,\ 2,\ 5)+(2,\ -3,\ 5)\right]\bullet(4,\ 2,\ -3)$

$$=(3,\ -1,\ 10)\bullet(4,\ 2,\ -3)$$

$$=12-2-30$$

$$\underline{=-20}$$

Exercise

Consider polynomial $f(t)=3t-5;\quad g(t)=t^2\quad in\quad \mathbb{P}(t)$

a) $\langle f,\, g\rangle$ *b)* $\|f\|$ *c)* $\|g\|$ *d)* Cosine between f *and* g

Solution

a)
$$\begin{aligned}\langle f,\, g\rangle &= \int_0^1 f(t)g(t)\,dt \\ &= \int_0^1 (3t-5)t^2 dt \\ &= \int_0^1 \left(3t^3-5t^2\right)dt \\ &= \frac{3}{4}t^4-\frac{5}{3}t^3\Big|_0^1 \\ &= \frac{3}{4}-\frac{5}{3} \\ &= -\frac{11}{12}\end{aligned}$$

b)
$$\begin{aligned}\langle f,\, f\rangle &= \int_0^1 f(t)\,f(t)\,dt \\ &= \int_0^1 (3t-5)^2\,dt \\ &= \frac{1}{3}\int_0^1 (3t-5)^2\,d(3t-5) \\ &= \frac{1}{9}(3t-5)^3\Big|_0^1 \\ &= \frac{1}{9}(8-125) \\ &= 13\end{aligned}$$

$$\begin{aligned}\|f\| &= \sqrt{\left|\langle f,\, f\rangle\right|} \\ &= \sqrt{13}\end{aligned}$$

c)
$$\langle g,\, g\rangle = \int_0^1 g(t)g(t)\,dt$$

$$= \int_0^1 t^4 dt$$

$$= \frac{1}{5}t^5 \Big|_0^1$$

$$= \frac{1}{5}$$

$$\|g\| = \sqrt{|\langle g, g\rangle|}$$

$$= \frac{1}{\sqrt{5}}$$

d) $\cos\theta = \dfrac{-\frac{11}{12}}{\sqrt{13}\frac{\sqrt{5}}{5}}$ $\qquad \cos\theta = \dfrac{f \bullet g}{\|f\|\,\|g\|}$

$$= \frac{-55}{12\sqrt{65}}$$

Exercise

Consider polynomial $f(t) = t+2$; $g(t) = 3t-2$; $h(t) = t^2 - 2t - 3$ *in* $\mathbb{P}(t)$

a) $\langle f, g\rangle$ *d)* $\|f\|$ *g)* Cosine α between f *and* g

b) $\langle f, h\rangle$ *e)* $\|g\|$ *h)* Cosine β between f *and* h

c) $\langle g, h\rangle$ *f)* $\|h\|$ *i)* Cosine θ between g *and* h

Solution

a) $\langle f, g\rangle = \int_0^1 (t+2)(3t-2)\,dt$ $\qquad \langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$

$$= \int_0^1 \left(3t^2 + 4t - 4\right)dt$$

$$= t^3 + 2t^2 - 4t \Big|_0^1$$

$$= 1 + 2 - 4$$

$$= -1$$

b) $\langle f, h\rangle = \int_0^1 (t+2)\left(t^2 - 2t - 3\right)dt$ $\qquad \langle f, h\rangle = \int_0^1 f(t)h(t)\,dt$

$$= \int_0^1 \left(t^3 - 7t - 6\right) dt$$

$$= \frac{1}{4}t^4 - \frac{7}{2}t^2 - 6t \Big|_0^1$$

$$= \frac{1}{4} - \frac{7}{2} - 6$$

$$\boxed{= -\frac{37}{4}}$$

c) $\langle g,\, h\rangle = \int_0^1 (3t-2)\left(t^2 - 2t - 3\right) dt$ $\qquad$ $\langle f,\, h\rangle = \int_0^1 g(t)h(t)\,dt$

$$= \int_0^1 \left(3t^3 - 8t^2 - 5t + 6\right) dt$$

$$= \frac{3}{4}t^4 - \frac{8}{3}t^3 - \frac{5}{2}t^2 + 6t \Big|_0^1$$

$$= \frac{3}{4} - \frac{8}{3} - \frac{5}{2} + 6$$

$$\boxed{= \frac{9}{4}}$$

d) $\langle f,\, f\rangle = \int_0^1 (t+2)^2\, dt$ $\qquad$ $\langle f,\, f\rangle = \int_0^1 f(t)\, f(t)\,dt$

$$= \frac{1}{3}(t+2)^3 \Big|_0^1$$

$$= \frac{1}{3}(27-8)$$

$$\boxed{= \frac{19}{3}}$$

$$\|f\| = \sqrt{|\langle f,\, f\rangle|}$$

$$\boxed{= \sqrt{\frac{19}{3}}}$$

e) $\langle g,\, g\rangle = \int_0^1 (3t-2)^2\, dt$ $\qquad$ $\langle g,\, g\rangle = \int_0^1 g(t)g(t)\,dt$

$$= \frac{1}{3}\int_0^1 (3t-2)^2\, d(3t-2)$$

$$= \frac{1}{9}(3t-2)^3 \Big|_0^1$$

$$=\frac{1}{9}(1+8)$$

$$\underline{=1}$$

$$\|g\| = \sqrt{|\langle g,\ g\rangle|}$$

$$\underline{=1}$$

f) $\langle g,\ g\rangle = \int_0^1 \left(t^2-2t-3\right)^2 dt$ $\qquad$ $\langle h,\ h\rangle = \int_0^1 h(t)h(t)\,dt$

$$= \int_0^1 \left(t^4-4t^3-2t^2+12t+9\right) dt$$

$$= \left(\frac{1}{5}t^5 - t^4 - \frac{2}{3}t^3 + 6t^2 + 9t\right)\Big|_0^1$$

$$= \frac{1}{5} - 1 - \frac{2}{3} + 6 + 9$$

$$\underline{= \frac{203}{15}}$$

$$\|h\| = \sqrt{|\langle h,\ h\rangle|}$$

$$\underline{= \sqrt{\frac{203}{15}}}$$

g) $\cos\alpha = \dfrac{-1}{\sqrt{\frac{19}{3}}}$ $\qquad$ $\cos\alpha = \dfrac{f \bullet g}{\|f\|\ \|g\|}$

$$\underline{= -\sqrt{\frac{3}{19}}}$$

h) $\cos\beta = -\dfrac{37}{4}\dfrac{1}{\sqrt{\frac{19}{3}}\sqrt{\frac{203}{15}}}$ $\qquad$ $\cos\beta = \dfrac{f \bullet g}{\|f\|\ \|g\|}$

$$\underline{= -\frac{111}{4}\sqrt{\frac{5}{3{,}857}}}$$

i) $\cos\theta = \dfrac{9}{4}\dfrac{1}{\sqrt{\frac{203}{15}}}$ $\qquad$ $\cos\theta = \dfrac{f \bullet g}{\|f\|\ \|g\|}$

$$\underline{= \frac{9}{4}\sqrt{\frac{15}{203}}}$$

Exercise

Suppose $\langle \vec{u}, \vec{v} \rangle = 3 + 2i$ in a complex inner product space *V*. Find:

a) $\langle (2-4i)\vec{u}, \vec{v} \rangle$ *b)* $\langle \vec{u}, (4+3i)\vec{v} \rangle$ *c)* $\langle (3-6i)\vec{u}, (5-2i)\vec{v} \rangle$ *d)* $\|\vec{u}, \vec{v}\|$

Solution

a) $\langle (2-4i)\vec{u}, \vec{v} \rangle = (2-4i)\langle \vec{u}, \vec{v} \rangle$

$= (2-4i)(3+2i)$

$= 6 + 4i - 12i + 8 \qquad i^2 = -1$

$\underline{= 14 - 8i}$

b) $\langle \vec{u}, (4+3i)\vec{v} \rangle = (4+3i)\langle \vec{u}, \vec{v} \rangle$

$= (4+3i)(3+2i)$

$= 12 + 8i + 9i - 6 \qquad i^2 = -1$

$\underline{= 14 - 8i}$

c) $\langle (3-6i)\vec{u}, (5-2i)\vec{v} \rangle = (3-6i)(5-2i)\langle \vec{u}, \vec{v} \rangle$

$= (15 - 36i - 12)(3+2i)$

$= (3 - 36i)(3+2i)$

$= 9 - 102i + 72$

$\underline{= 81 - 102i}$

d) $\|\vec{u}, \vec{v}\| = \sqrt{\langle \vec{u}, \vec{v} \rangle}$

$= \sqrt{9+4}$

$\underline{= \sqrt{13}}$

Exercise

Find the Fourier coefficient *c* and the projection $c\vec{v}$ of $\vec{u} = (3+4i,\ 2-3i)$ along $\vec{v} = (5+i,\ 2i)$ in $\mathbb{C}^2$

Solution

$$c = \frac{(3+4i)\left(\overline{5+i}\right) + (2-3i)\left(\overline{2i}\right)}{5^2 + 1^2 + 2^2} \qquad c = \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle}$$

$$= \frac{(3+4i)(5-i) + (2-3i)(-2i)}{25+1+4}$$

$$= \frac{15 + 17i + 4 - 4i - 6}{30}$$

$$= \frac{13 + 13i}{30}$$

$$= \frac{13}{30} + \frac{13}{30}i$$

$$proj(\vec{u}, \vec{v}) = c\vec{v}$$

$$= \left(\frac{13}{30} + \frac{13}{30}i\right)(5+i,\ 2i)$$

$$= \left(\frac{13}{6} + \frac{13}{6}i + \frac{13}{30}i - \frac{13}{30},\quad \frac{13}{15}i - \frac{13}{15}\right)$$

$$= \left(\frac{52}{30} + \frac{78}{30}i,\ -\frac{13}{15} + \frac{13}{15}i\right)$$

$$= \left(\frac{26}{15} + \frac{39}{30}i,\ -\frac{13}{15} + \frac{13}{15}i\right)$$

Exercise

Suppose $\vec{v} = (1,\ 3,\ 5,\ 7)$. Find the projection of $\vec{v}$ onto W or find $\vec{w} \in W$ that minimizes $\|\vec{v} - \vec{w}\|$, where W is the subspace of $\mathbb{R}^4$ spanned by:

a) $\vec{u}_1 = (1,\ 1,\ 1,\ 1)$ *and* $\vec{u}_2 = (1,\ -3,\ 4,\ -2)$

b) $\vec{v}_1 = (1,\ 1,\ 1,\ 1)$ *and* $\vec{v}_2 = (1,\ 2,\ 3,\ 2)$

Solution

a) $\vec{u}_1 \bullet \vec{u}_2 = (1,\ 1,\ 1,\ 1) \bullet (1,\ -3,\ 4,\ -2)$

$$= 1 - 3 + 4 - 2$$

$$= 0$$

Therefore, $\vec{u}_1$ *and* $\vec{u}_2$ are orthogonal.

$$c_1 = \frac{\langle \vec{v},\ \vec{u}_1 \rangle}{\langle \vec{u}_1,\ \vec{u}_1 \rangle}$$

$$= \frac{(1,\ 3,\ 5,\ 7) \bullet (1,\ 1,\ 1,\ 1)}{\|(1,\ 1,\ 1,\ 1)\|^2}$$

$$= \frac{1+3+5+7}{1+1+1+1}$$

$$= \frac{16}{4}$$

$$= 4$$

$$c_2 = \frac{\langle \vec{v},\ \vec{u}_2 \rangle}{\langle \vec{u}_2,\ \vec{u}_2 \rangle}$$

$$= \frac{(1,\ 3,\ 5,\ 7) \bullet (1,\ -3,\ 4,\ -2)}{\|(1,\ -3,\ 4,\ -2)\|^2}$$

$$= \frac{1-9+20-14}{1+9+16+4}$$

$$= \frac{-2}{30}$$

$$= \frac{1}{15}$$

$$w = proj(\vec{v},\ W)$$

$$= c_1\vec{u}_1 + c_2\vec{u}_2$$

$$= 4(1,\ 1,\ 1,\ 1) + \frac{1}{15}(1,\ -3,\ 4,\ -2)$$

$$= \left(\frac{59}{15},\ \frac{63}{5},\ \frac{56}{15},\ \frac{62}{15}\right)$$

b) $\vec{v}_1 \bullet \vec{v}_2 = (1,\ 1,\ 1,\ 1) \bullet (1,\ 2,\ 3,\ 2)$

$$= 1+2+3+2$$

$$= 8 \neq 0$$

Therefore, $\vec{v}_1$ *and* $\vec{v}_2$ are *not* orthogonal.

Applying Gram-Schmidt algorithm

$$\vec{w}_1 = \vec{v}_1 = (1,\ 1,\ 1,\ 1)$$

$$\vec{w}_2 = (1,\ 2,\ 3,\ 2) - \frac{(1,\ 2,\ 3,\ 2) \bullet (1,\ 1,\ 1,\ 1)}{4}(1,\ 1,\ 1,\ 1)$$

$$\vec{w}_2 = \vec{v}_2 - \frac{\langle \vec{v},\ \vec{w}_1 \rangle}{\|\vec{w}_1\|^2}\vec{w}_1$$

$$= (1,\ 2,\ 3,\ 2) - 2(1,\ 1,\ 1,\ 1)$$

$$= (-1,\ 0,\ 1,\ 0)$$

$$c_1 = \frac{(1,\ 3,\ 5,\ 7) \bullet (1,\ 1,\ 1,\ 1)}{\|(1,\ 1,\ 1,\ 1)\|^2}$$

$$c_1 = \frac{\langle \vec{v},\ \vec{w}_1 \rangle}{\langle \vec{w}_1,\ \vec{w}_1 \rangle}$$

$$= \frac{1+3+5+7}{1+1+1+1}$$

$$= \frac{16}{4}$$

$$= 4$$

$$c_2 = \frac{(1,\ 3,\ 5,\ 7) \bullet (-1,\ 0,\ 1,\ 0)}{\|(-1,\ 0,\ 1,\ 0)\|^2}$$

$$c_2 = \frac{\langle \vec{v},\ \vec{w}_2 \rangle}{\langle \vec{w}_2,\ \vec{w}_2 \rangle}$$

$$= \frac{-1+0+5+0}{2}$$

$$\underline{= -3}$$

$$w = proj(\vec{v},\ W)$$

$$= c_1 \vec{w}_1 + c_2 \vec{w}_2$$

$$= 4(1,\ 1,\ 1,\ 1) - 3(-1,\ 0,\ 1,\ 0)$$

$$\underline{= (7,\ 4,\ 1,\ 4)}$$

Exercise

Suppose $\{\vec{u}_1,\ \vec{u}_2,\ \ldots,\ \vec{u}_n\}$ is an orthogonal set of vectors. Prove that (*Pythagoras*)

$$\left\|\vec{u}_1 + \vec{u}_2 + \ \ldots + \vec{u}_n\right\|^2 = \left\|\vec{u}_1\right\|^2 + \left\|\vec{u}_2\right\|^2 + \ \ldots + \left\|\vec{u}_n\right\|^2$$

Solution

$$\begin{aligned}\left\|\vec{u}_1 + \vec{u}_2 + \ \ldots + \vec{u}_n\right\|^2 &= \left\langle \vec{u}_1 + \vec{u}_2 + \ \ldots + \vec{u}_n,\ \vec{u}_1 + \vec{u}_2 + \ \ldots + \vec{u}_n \right\rangle \\ &= \left\langle \vec{u}_1,\ \vec{u}_1 \right\rangle + \left\langle \vec{u}_2,\ \vec{u}_2 \right\rangle + \ldots + \left\langle \vec{u}_n,\ \vec{u}_n \right\rangle \\ &= \left\|\vec{u}_1\right\|^2 + \left\|\vec{u}_2\right\|^2 + \ \ldots + \left\|\vec{u}_n\right\|^2\end{aligned}$$

Exercise

Suppose A is an orthogonal matrix. Show that $\langle \vec{u}A,\ \vec{v}A \rangle = \langle \vec{u},\ \vec{v} \rangle$ for any $\vec{u},\ \vec{v}\ \in V$

Solution

A is an orthogonal matrix $\Rightarrow AA^T = I$

And $\langle \vec{u},\ \vec{v} \rangle = \vec{u}^T \vec{v}$

$$\begin{aligned}\langle \vec{u}A,\ \vec{v}A \rangle &= (A\vec{u})^T (A\vec{v}) \\ &= \vec{u}^T \left(A^T A\right) \vec{v} \\ &= \vec{u}^T I\ \vec{v} \\ &= \vec{u}^T \vec{v} \\ &= \langle \vec{u},\ \vec{v} \rangle \quad \checkmark\end{aligned}$$

Exercise

Suppose A is an orthogonal matrix. Show that $\|\vec{u}A\| = \|\vec{u}\|$ for every $\vec{u} \in V$

Solution

A is an orthogonal matrix

$\Rightarrow AA^T = I$ and $\langle \vec{u},\ \vec{u} \rangle = \vec{u}^T \vec{u}$

$$\|\vec{u}A\|^2 = \langle \vec{u}A,\ \vec{u}A \rangle$$
$$= (A\vec{u})^T (A\vec{u})$$
$$= \vec{u}^T \left(A^T A\right) \vec{u}$$
$$= \vec{u}^T I\ \vec{u}$$
$$= \vec{u}^T \vec{u}$$
$$= \langle \vec{u},\ \vec{u} \rangle \quad \checkmark$$

Exercise

Let V be an inner product space over $\mathbb{R}$ or $\mathbb{C}$. Show that

$$\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$$

If and only if

$$\|s\vec{u} + t\vec{v}\| = s\|\vec{u}\| + t\|\vec{v}\| \quad \textit{for all} \quad s,\ t \ge 0$$

Solution

Suppose that $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$. For $s,\ t \ge 0$

$$\|s\vec{u} + t\vec{v}\|^2 = s^2\|\vec{u}\|^2 + t^2\|\vec{v}\|^2 + 2st\,\vec{u}\vec{v}$$
$$\le s^2\|\vec{u}\|^2 + t^2\|\vec{v}\|^2$$
$$\le s\|\vec{u}\| + t\|\vec{v}\|$$

$$\|s\vec{u} + t\vec{v}\| \le s\|\vec{u}\| + t\|\vec{v}\|$$

$$\|s\vec{u} + t\vec{v}\| = \|s\vec{u} + t\vec{u} - t\vec{u} + t\vec{v}\|$$
$$= \|t\vec{u} + t\vec{v} - t\vec{u} + s\vec{u}\|$$
$$= \|t(\vec{u} + \vec{v}) - (t - s)\vec{u}\|$$
$$\ge \left|\, t\|\vec{u} + \vec{v}\| - (t - s)\|\vec{u}\|\, \right| \qquad \|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$$
$$= t\|\vec{u}\| + \|\vec{v}\| - t\|\vec{u}\| + s\|\vec{u}\|$$

$$= t\|\vec{v}\| + s\|\vec{u}\|$$

$$\begin{cases} \|s\vec{u}+t\vec{v}\| \le s\|\vec{u}\|+t\|\vec{v}\| \\ \quad\quad \text{and} \\ \|s\vec{u}+t\vec{v}\| \ge s\|\vec{u}\|+t\|\vec{v}\| \end{cases} \Rightarrow \quad \|s\vec{u}+t\vec{v}\| = s\|\vec{u}\|+t\|\vec{v}\|$$

Exercise

Let V be an inner product vector space over $\mathbb{R}$.

a) If e_1, e_2, e_3 are three vectors in V with pairwise product negative, that is,

$$\langle e_1, e_2 \rangle < 0, \quad i, j = 1, 2, 3, \quad i \ne j$$

Show that e_1, e_2, e_3 are linearly independent.

b) Is it possible for three vectors on the *xy*–plane to have pairwise negative products?

c) Does part (a) remain valid when the word "negative: is replaced with positive?

d) Suppose $\vec{u},\ \vec{v},\ \textit{and}\ \vec{w}$ are three–unit vectors in the *xy*–plane. What are the maximum and minimum values that

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle$$

Can attain? And when?

Solution

a) Suppose that e_1, e_2, e_3 are linearly dependent.

Then, assume that e_1, e_2, e_3 are unit vectors and that

$$e_3 = c_1 e_1 + c_2 e_2$$

Then

$$\langle e_1, e_3 \rangle = c_1 \langle e_1, e_1 \rangle + c_2 \langle e_1, e_2 \rangle \qquad \langle e_1, e_1 \rangle = 1$$

$$= c_1 + c_2 \langle e_1, e_2 \rangle < 0$$

$$c_1 < -c_2 \langle e_1, e_2 \rangle$$

$$\langle e_2, e_3 \rangle = c_1 \langle e_2, e_1 \rangle + c_2 \langle e_2, e_2 \rangle \qquad \langle e_2, e_2 \rangle = 1$$

$$= c_1 \langle e_2, e_1 \rangle + c_2 < 0$$

$$c_2 < -c_1 \langle e_2, e_1 \rangle$$

$$c_2 < -c_1 \langle e_2, e_1 \rangle$$

$$< -\left(-c_2 \langle e_1, e_2 \rangle\right) \langle e_2, e_1 \rangle$$

$$= c_2 \langle e_1, e_2 \rangle^2 \qquad\qquad \langle e_1, e_2 \rangle^2 > 1$$

$c_2 < c_2$ *Contradiction*

Therefore, e_1, e_2, e_3 are linearly independent.

b) To have all three vectors on the *xy*–plane which is in 2 dimensional.
Therefore, it is ***impossible*** for three to have pairwise negative products.

c) No

d) Given: $\vec{u},\ \vec{v},\ and\ \vec{w}$ are three–unit vectors in the *xy*–plane and

$$|\langle \vec{u}, \vec{u} \rangle| = |\langle \vec{v}, \vec{v} \rangle| = |\langle \vec{w}, \vec{w} \rangle| = 1$$

$$\cos\alpha_1 = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\|\ \|\vec{v}\|} \rightarrow \cos\alpha_1 = \langle \vec{u}, \vec{v} \rangle$$

$$\cos\alpha_2 = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\|\ \|\vec{w}\|} \rightarrow \cos\alpha_2 = \langle \vec{v}, \vec{w} \rangle$$

$$\cos\alpha_3 = \frac{\langle \vec{u}, \vec{w} \rangle}{\|\vec{u}\|\ \|\vec{w}\|} \rightarrow \cos\alpha_3 = \langle \vec{u}, \vec{w} \rangle$$

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle = \cos\alpha_1 + \cos\alpha_2 + \cos\alpha_3$$

Since $-1 \le \cos\theta \le 1$

$$\begin{aligned}\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle &= 1+1+1\\ &= 3\end{aligned}$$

Since the 3 vectors are unit vectors in the *xy*–plane and which it will divide the plane into a three equal angles $\alpha = \frac{2\pi}{3}$

$$\begin{aligned}\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle &= \cos\frac{2\pi}{3} + \cos\frac{2\pi}{3} + \cos\frac{2\pi}{3}\\ &= 3\cos\frac{2\pi}{3}\\ &= 3\left(-\frac{1}{2}\right)\\ &= -\frac{3}{2}\end{aligned}$$

Therefore, the minimum $\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle = -\frac{3}{2}$

The maximum: $\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle = 3$

Solution Section 3.3 – Gram-Schmidt Process

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\mathbb{R}^m$.

$\vec{u}_1 = (1,\ -3),\quad \vec{u}_2 = (2,\ 2)$

Solution

$$\vec{v}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|}$$

$$= \frac{(1,\ -3)}{\sqrt{1^2 + (-3)^2}}$$

$$= \frac{(1,\ -3)}{\sqrt{10}}$$

$$= \left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right)$$

$$\vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2 \bullet \vec{u}_1\right)\vec{u}_1$$

$$= (2,\ 2) - \left[(2,\ 2).\left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right)\right]\left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right)$$

$$= (2,\ 2) - \left[\frac{2}{\sqrt{10}} - \frac{6}{\sqrt{10}}\right]\left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right)$$

$$= (2,\ 2) - \left[-\frac{4}{\sqrt{10}}\right]\left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right)$$

$$= (2,\ 2) - \left(-\frac{4}{10},\ \frac{12}{10}\right)$$

$$= (2,\ 2) - \left(-\frac{2}{5},\ \frac{6}{5}\right)$$

$$= \left(\frac{12}{5},\ \frac{4}{5}\right)$$

$$\|w_2\| = \sqrt{\left(\frac{12}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$$

$$= \sqrt{\frac{144}{25} + \frac{16}{25}}$$

$$= \sqrt{\frac{160}{25}}$$

$$= \frac{4\sqrt{10}}{5}$$

$$\vec{v}_2 = \frac{\vec{w}_2}{\left\|\vec{w}_2\right\|}$$

$$= \frac{4\sqrt{10}}{5}\left(\frac{12}{5},\ \frac{4}{5}\right)$$

$$= \left(\frac{48\sqrt{10}}{25},\ \frac{16\sqrt{10}}{25}\right)$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\mathbb{R}^m$.

$\vec{u}_1 = (1,\ 0), \quad \vec{u}_2 = (3,\ -5)$

Solution

$$\vec{v}_1 = \frac{\vec{u}_1}{\left\|\vec{u}_1\right\|}$$

$$= \frac{(1,\ 0)}{\sqrt{1^{2+}0^2}}$$

$$= (1,\ 0)$$

$$\vec{w}_2 = \vec{u}_2 - \left(\vec{u}_2 \cdot \vec{v}_1\right)\vec{v}_1$$

$$= (0,\ -5)$$

$$= (3,\ -5) - \left[(3,\ -5).(1,\ 0)\right](1,\ 0)$$

$$= (3,\ -5) - [3](1,\ 0)$$

$$= (3,\ -5) - (3,\ 0)$$

$$= (0,\ -5)$$

$$\left\|\vec{w}_2\right\| = \sqrt{0^2 + (-5)^2}$$

$$= 5$$

$$\vec{v}_2 = \frac{\vec{w}_2}{\left\|\vec{w}_2\right\|}$$

$$= \frac{1}{5}(0,\ -5)$$

$$= (0,\ -1)$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\mathbb{R}^m$.

$$\{(1,\ 1,\ 1),\ (-1,\ 1,\ 0),\ (1,\ 2,\ 1)\}$$

Solution

$$\vec{u}_1 = \frac{(1,\,1,\,1)}{\sqrt{1^2+1^2+1^2}} \qquad\qquad \vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ 1,\ 1)}{\sqrt{3}}$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2 \bullet \vec{u}_1\right)\vec{u}_1$$

$$= (-1,\ 1,\ 0) - \left[(-1,\ 1,\ 0) \bullet \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (-1,\ 1,\ 0) - \left[-\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + 0\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (-1,\ 1,\ 0) - 0\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (-1,\ 1,\ 0)$$

$$\|\vec{w}_2\| = \sqrt{(-1)^2 + 1^2}$$

$$= \sqrt{2}$$

$$\vec{u}_2 = \frac{(-1,\ 1,\ 0)}{\sqrt{2}} \qquad\qquad \vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$\vec{v}_3 \bullet \vec{u}_1 = (1,\ 2,\ 1) \bullet \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}$$

$$= \frac{3}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}}$$

$$= \sqrt{3}$$

$$\vec{v}_3 \bullet \vec{u}_2 = (1,\ 2,\ 1) \bullet \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= -\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} + 0$$

$$= \frac{1}{\sqrt{2}}$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \bullet \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \bullet \vec{u}_2)\vec{u}_2$$

$$= (1,\ 2,\ 1) - \sqrt{3}\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) - \sqrt{2}\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= (1,\ 2,\ 1) - (1,\ 1,\ 1) - (-1,\ 1,\ 0)$$

$$= (1,\ 0,\ 0)$$

$$\vec{u}_3 = \frac{(1,\ 0,\ 0)}{\sqrt{1}} \qquad\qquad \vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$= (1,\ 0,\ 0)$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\boldsymbol{R}^m$.

$$\{(1,\ 1,\ 1),\ (0,\ 1,\ 1),\ (0,\ 0,\ 1)\}$$

Solution

$$\vec{u}_1 = \frac{(1,\ 1,\ 1)}{\sqrt{1+1+1}} \qquad\qquad \vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ 1,\ 1)}{\sqrt{3}}$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\vec{w}_2 = (0,\ 1,\ 1) - \left[(0,\ 1,\ 1) \bullet \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) \qquad \vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \bullet \vec{u}_1)\vec{u}_1$$

$$= (0,\ 1,\ 1) - \left[\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (0,\ 1,\ 1) - \left[\frac{2}{\sqrt{3}}\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (0,\ 1,\ 1) - \left(\frac{2}{3},\ \frac{2}{3},\ \frac{2}{3}\right)$$

$$=\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$\|\vec{w}_2\| = \sqrt{\left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2}$$

$$= \sqrt{\frac{4}{9}+\frac{1}{9}+\frac{1}{9}}$$

$$= \sqrt{\frac{6}{9}}$$

$$= \frac{\sqrt{6}}{3}$$

$$\vec{u}_2 = \frac{\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}{\frac{\sqrt{6}}{3}} \qquad \vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{3}{\sqrt{6}}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$= \left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$\vec{v}_3 \bullet \vec{u}_1 = (0,\ 0,\ 1) \bullet \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}}$$

$$\vec{v}_3 \bullet \vec{u}_2 = (0,\ 0,\ 1) \bullet \left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$= \frac{1}{\sqrt{6}}$$

$$\vec{w}_3 = (0,\ 0,\ 1) - \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) - \frac{1}{\sqrt{6}}\left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right) \qquad \vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \bullet \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \bullet \vec{u}_2)\vec{u}_2$$

$$= (0,\ 0,\ 1) - \left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right) - \left(-\frac{1}{3},\ \frac{1}{6},\ \frac{1}{6}\right)$$

$$= \left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)$$

$$\vec{u}_3 = \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}} \qquad \vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$= \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\sqrt{\frac{1}{2}}}$$

$$= \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\frac{1}{\sqrt{2}}}$$

$$= \sqrt{2}\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)$$

$$= \left(0,\ -\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\mathbb{R}^m$.

$$\{(1,\ 1,\ 1),\ (0,\ 2,\ 1),\ (1,\ 0,\ 3)\}$$

Solution

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ 1,\ 1)}{\sqrt{1+1+1}}$$

$$= \frac{(1,\ 1,\ 1)}{\sqrt{3}}$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2 \bullet \vec{u}_1\right)\vec{u}_1$$

$$= (0,\ 2,\ 1) - \left[(0,\ 2,\ 1) \bullet \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (0,\ 2,\ 1) - \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (0,\ 2,\ 1) - \frac{3}{\sqrt{3}}\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (0,\ 2,\ 1) - \sqrt{3}\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (0,\ 2,\ 1) - (1,\ 1,\ 1)$$

$$= (-1,\ 1,\ 0)$$

$$\|w_2\| = \sqrt{(-1)^2 + (1)^2 + (0)^2}$$

$$= \sqrt{2}$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{(-1,\ 1,\ 0)}{\sqrt{2}}$$

$$\boxed{= \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)}$$

$$\vec{v}_3 \bullet \vec{u}_1 = (1,\ 0,\ 3) \bullet \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}} + \frac{3}{\sqrt{3}}$$

$$\boxed{= \frac{4}{\sqrt{3}}}$$

$$\vec{v}_3 \bullet \vec{u}_2 = (1,\ 0,\ 3) \bullet \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$\boxed{= -\frac{1}{\sqrt{2}}}$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \bullet \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \bullet \vec{u}_2)\vec{u}_2$$

$$= (1,\ 0,\ 3) - \frac{4}{\sqrt{3}}\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) + \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= (1,\ 0,\ 3) - \left(\frac{4}{3},\ \frac{4}{3},\ \frac{4}{3}\right) + \left(-\frac{1}{2},\ \frac{1}{2},\ 0\right)$$

$$\boxed{= \left(-\frac{5}{6},\ -\frac{5}{6},\ \frac{5}{3}\right)}$$

$$\vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$= \frac{1}{\sqrt{\left(-\frac{5}{6}\right)^2 + \left(-\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2}}\left(-\frac{5}{6},\ -\frac{5}{6},\ \frac{5}{3}\right)$$

$$= \frac{1}{\sqrt{\frac{25}{36} + \frac{25}{36} + \frac{25}{9}}}\left(-\frac{5}{6},\ -\frac{5}{6},\ \frac{5}{3}\right)$$

$$= \frac{1}{\frac{5}{6}\sqrt{6}}\left(-\frac{5}{6},\ -\frac{5}{6},\ \frac{5}{3}\right)$$

$$\boxed{= \left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\mathbb{R}^m$.

$$\{(2,\ 2,\ 2),\ (1,\ 0,\ -1),\ (0,\ 3,\ 1)\}$$

Solution

$$\vec{u}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \frac{(2,\ 2,\ 2)}{\sqrt{2^2+2^2+2^2}}$$

$$= \frac{(2,\ 2,\ 2)}{\sqrt{12}}$$

$$\boxed{= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)}$$

$$\vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \bullet \vec{u}_1\right)\vec{u}_1 - \left(\vec{v}_3 \bullet \vec{u}_2\right)\vec{u}_2$$

$$= (1,\ 0,\ -1) - \left[(1,\ 0,\ -1) \bullet \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (1,\ 0,\ -1) - \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (0,\ 2,\ 1) - (0)\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\boxed{= (1,\ 0,\ -1)}$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\left\|\vec{w}_2\right\|}$$

$$= \frac{(1,\ 0,\ -1)}{\sqrt{2}}$$

$$\boxed{= \left(\frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right)}$$

$$\vec{v}_3 \bullet \vec{u}_1 = (0,\ 3,\ 1) \bullet \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= \frac{3}{\sqrt{3}} + \frac{1}{\sqrt{3}}$$

$$\boxed{= \frac{4}{\sqrt{3}}}$$

$$\vec{v}_3 \bullet \vec{u}_2 = (0,\ 3,\ 1) \bullet \left(\frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right)$$

$$= -\frac{1}{\sqrt{2}}$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \bullet \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \bullet \vec{u}_2)\vec{u}_2$$

$$= (0,\ 3,\ 1) - \frac{4}{\sqrt{3}}\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right)$$

$$= (0,\ 3,\ 1) - \left(\frac{4}{3},\ \frac{4}{3},\ \frac{4}{3}\right) + \left(\frac{1}{2},\ 0,\ -\frac{1}{2}\right)$$

$$= \left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right)$$

$$\vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$= \frac{1}{\sqrt{\left(-\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(-\frac{5}{6}\right)^2}}\left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right)$$

$$= \frac{1}{\sqrt{\frac{25}{36} + \frac{25}{36} + \frac{25}{9}}}\left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right)$$

$$= \frac{1}{\frac{5}{6}\sqrt{6}}\left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right)$$

$$= \left(-\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}}\right)$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\mathbb{R}^m$.

$$\{(1,\ -1,\ 0),\ (0,\ 1,\ 0),\ (2,\ 3,\ 1)\}$$

Solution

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ -1,\ 0)}{\sqrt{2}}$$

$$= \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \bullet \vec{u}_1)\vec{u}_1$$

$$=(0,\ 1,\ 0)-\left[(0,\ 1,\ 0)\bullet\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)\right]\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)$$

$$=(0,\ 1,\ 0)+\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)$$

$$=(0,\ 1,\ 0)+\left(\frac{1}{2},\ -\frac{1}{2},\ 0\right)$$

$$\underline{=\left(\frac{1}{2},\ \frac{1}{2},\ 0\right)\Big|}$$

$$\vec{u}_2=\frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$=\frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}}}\left(\frac{1}{2},\ \frac{1}{2},\ 0\right)$$

$$\underline{=\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)\Big|}$$

$$\vec{v}_3\bullet\vec{u}_1=(2,\ 3,\ 1)\bullet\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)$$

$$=\frac{2}{\sqrt{2}}-\frac{3}{\sqrt{2}}$$

$$\underline{=-\frac{1}{\sqrt{2}}\Big|}$$

$$\vec{v}_3\bullet\vec{u}_2=(2,\ 3,\ 1)\bullet\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$=\frac{4}{\sqrt{2}}$$

$$\underline{=2\sqrt{2}\,\Big|}$$

$$\vec{w}_3=\vec{v}_3-\left(\vec{v}_3\bullet\vec{u}_1\right)\vec{u}_1-\left(\vec{v}_3\bullet\vec{u}_2\right)\vec{u}_2$$

$$=(2,\ 3,\ 1)+\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)-2\sqrt{2}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$=(2,\ 3,\ 1)+\left(\frac{1}{2},\ -\frac{1}{2},\ 0\right)-(2,\ 2,\ 0)$$

$$\underline{=\left(\frac{1}{2},\ \frac{1}{2},\ 1\right)\Big|}$$

$$\vec{u}_3=\frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$=\frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}}\left(\frac{1}{2},\ \frac{1}{2},\ 1\right)$$

$$=\frac{2}{\sqrt{6}}\left(\frac{1}{2},\ \frac{1}{2},\ 1\right)$$

$$\boxed{=\left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\mathbb{R}^m$.

$$\{(3,\ 0,\ 4),\ (-1,\ 0,\ 7),\ (2,\ 9,\ 11)\}$$

Solution

$$\vec{u}_1=\frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$=\frac{(3,\ 0,\ 4)}{\sqrt{9+16}}$$

$$\boxed{=\left(\frac{3}{5},\ 0,\ \frac{4}{5}\right)}$$

$$\vec{w}_2=\vec{v}_2-\left(\vec{v}_2\bullet\vec{u}_1\right)\vec{u}_1$$

$$=(-1,\ 0,\ 7)-\left[(-1,\ 0,\ 7)\bullet\left(\frac{3}{5},\ 0,\ \frac{4}{5}\right)\right]\left(\frac{3}{5},\ 0,\ \frac{4}{5}\right)$$

$$=(-1,\ 0,\ 7)-\left(-\frac{3}{5}+\frac{28}{5}\right)\left(\frac{3}{5},\ 0,\ \frac{4}{5}\right)$$

$$=(-1,\ 0,\ 7)-5\left(\frac{3}{5},\ 0,\ \frac{4}{5}\right)$$

$$=(-1,\ 0,\ 7)-(3,\ 0,\ 4)$$

$$\boxed{=(-4,\ 0,\ 3)}$$

$$\vec{u}_2=\frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$=\frac{1}{\sqrt{16+9}}(-4,\ 0,\ 3)$$

$$\boxed{=\left(-\frac{4}{5},\ 0,\ \frac{3}{5}\right)}$$

$$\vec{v}_3\bullet\vec{u}_1=(2,\ 9,\ 11)\bullet\left(\frac{3}{5},\ 0,\ \frac{4}{5}\right)$$

$$=\frac{6}{5}+\frac{44}{5}$$

$$\underline{=10}$$

$$\vec{v}_3 \cdot \vec{u}_2 = (2,\ 9,\ 11) \cdot \left(-\frac{4}{5},\ 0,\ \frac{3}{5}\right)$$

$$=-\frac{8}{5}+\frac{33}{5}$$

$$\underline{=5}$$

$$\vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \cdot \vec{u}_1\right)\vec{u}_1 - \left(\vec{v}_3 \cdot \vec{u}_2\right)\vec{u}_2$$

$$\vec{w}_3 = (2,\ 9,\ 11) - 10\left(\frac{3}{5},\ 0,\ \frac{4}{5}\right) - 5\left(-\frac{4}{5},\ 0,\ \frac{3}{5}\right)$$

$$\underline{=(0,\ 9,\ 0)}$$

$$\vec{u}_3 = \frac{\vec{w}_3}{\left\|\vec{w}_3\right\|}$$

$$=\frac{1}{9}(0,\ 9,\ 0)$$

$$\underline{=(0,\ 1,\ 0)}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\mathbb{R}^m$.

$$\{(1,\ 1,\ 1,\ 1),\ (1,\ 2,\ 1,\ 0),\ (1,\ 3,\ 0,\ 0)\}$$

Solution

$$\vec{v}_1 = \vec{u}_1$$

$$\underline{=(1,\ 1,\ 1,\ 1)}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$=\frac{(1,\ 1,\ 1,\ 1)}{\sqrt{4}}$$

$$\underline{=\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2,\ \vec{v}_1\right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$= (1,\ 2,\ 1,\ 0) - \frac{(1,\ 2,\ 1,\ 0) \bullet (1,\ 1,\ 1,\ 1)}{\|(1,\ 1,\ 1,\ 1)\|^2}(1,\ 1,\ 1,\ 1)$$

$$= (1,\ 2,\ 1,\ 0) - \frac{1+2+1}{4}(1,\ 1,\ 1,\ 1)$$

$$= (1,\ 2,\ 1,\ 0) - \frac{4}{4}(1,\ 1,\ 1,\ 1)$$

$$= (1,\ 2,\ 1,\ 0) - (1,\ 1,\ 1,\ 1)$$

$$= (0,\ 1,\ 0,\ -1)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{(0,\ 1,\ 0,\ -1)}{\sqrt{1+1}}$$

$$= \left(0,\ \frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (1,\ 3,\ 0,\ 0) - \frac{(1,\ 3,\ 0,\ 0) \bullet (1,\ 1,\ 1,\ 1)}{\|(1,\ 1,\ 1,\ 1)\|^2}(1,\ 1,\ 1,\ 1) - \frac{(1,\ 3,\ 0,\ 0) \bullet (0,\ 1,\ 0,\ -1)}{\|(0,\ 1,\ 0,\ -1)\|^2}(0,\ 1,\ 0,\ -1)$$

$$= (1,\ 3,\ 0,\ 0) - \frac{4}{4}(1,\ 1,\ 1,\ 1) - \frac{3}{2}(0,\ 1,\ 0,\ -1)$$

$$= (1,\ 3,\ 0,\ 0) - (1,\ 1,\ 1,\ 1) - \left(0,\ \frac{3}{2},\ 0,\ -\frac{3}{2}\right)$$

$$= \left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{\left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)}{\sqrt{\frac{1}{4}+1+\frac{1}{4}}}$$

$$= \frac{\left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)}{\frac{\sqrt{6}}{2}}$$

$$= \frac{2}{\sqrt{6}}\left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)$$

$$= \left(0,\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\mathbb{R}^m$.

$$\{(0,\ 2,\ -1,\ 1),\ (0,\ 0,\ 1,\ 1),\ (-2,\ 1,\ 1,\ -1)\}$$

Solution

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(0,\ 2,\ -1,\ 1)}{\sqrt{6}}$$

$$= \left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$\vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2 \cdot \vec{u}_1\right)\vec{u}_1$$

$$= (0,\ 0,\ 1,\ 1) - \left[(0,\ 0,\ 1,\ 1) \cdot \left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)\right]\left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$= (0,\ 0,\ 1,\ 1) - \left[-\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}}\right]\left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$= (0,\ 0,\ 1,\ 1) - [0]\left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$= (0,\ 0,\ 1,\ 1)$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{(0,\ 0,\ 1,\ 1)}{\sqrt{2}}$$

$$= \left(0,\ 0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

$$\vec{v}_3 \cdot \vec{u}_1 = (-2,\ 1,\ 1,\ -1) \cdot \left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$= \frac{2}{\sqrt{6}} - \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}}$$

$$= 0$$

$$\vec{v}_3 \cdot \vec{u}_2 = (-2,\ 1,\ 1,\ -1) \cdot \left(0,\ 0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$

$$= 0$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$$

$$= (-2,\ 1,\ 1,\ -1) - 0 - 0$$

$$= (-2,\ 1,\ 1,\ -1)$$

$$\vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$= \frac{(-2,\ 1,\ 1,\ -1)}{\sqrt{(-2)^2 + 1^2 + 1^2 + (-1)^2}}$$

$$= \frac{(-2,\ 1,\ 1,\ -1)}{\sqrt{7}}$$

$$= \left(-\frac{2}{\sqrt{7}},\ \frac{1}{\sqrt{7}},\ \frac{1}{\sqrt{7}},\ -\frac{1}{\sqrt{7}}\right)$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\mathbb{R}^m$.

$$\vec{u}_1 = (1,\ 0,\ 0),\quad \vec{u}_2 = (3,\ 7,\ -2),\quad \vec{u}_3 = (0,\ 4,\ 1)$$

Solution

$$\vec{v}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|}$$

$$= \frac{(1,\ 0,\ 0)}{\sqrt{1^2 + 0^2 + 0^2}}$$

$$= (1,\ 0,\ 0)$$

$$\vec{w}_2 = \vec{u}_2 - (\vec{u}_2 \cdot \vec{v}_1)\vec{v}_1$$

$$= (3,\ 7,\ -2) - \left[(3,\ 7,\ -2) \cdot (1,\ 0,\ 0)\right](1,\ 0,\ 0)$$

$$= (3,7,-2) - 3(1,0,0)$$

$$= (0,\ 7,\ -2)$$

$$\vec{v}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{1}{\sqrt{53}}(0,\ 7,\ -2)$$

$$= \left(0,\ \frac{7}{\sqrt{53}},\ -\frac{2}{\sqrt{53}}\right)$$

$$\vec{u}_3 \bullet \vec{v}_1 = (0,\ 4,\ 1) \bullet (1,\ 0,\ 0)$$

$$= 0$$

$$\vec{u}_3 \bullet \vec{v}_2 = (0,\ 4,\ 1) \bullet \left(0,\ \frac{7}{\sqrt{53}},\ -\frac{2}{\sqrt{53}}\right)$$

$$= \frac{26}{\sqrt{53}}$$

$$\vec{w}_3 = \vec{u}_3 - (\vec{u}_3 \bullet \vec{v}_1)\vec{v}_1 - (\vec{u}_3 \bullet \vec{v}_2)\vec{v}_2$$

$$= (0,\ 4,\ 1) - 0 - \frac{26}{\sqrt{53}}\left(0,\ \frac{7}{\sqrt{53}},\ -\frac{2}{\sqrt{6}}\right)$$

$$= (0,\ 4,\ 1) - 0 - \frac{26}{\sqrt{53}}\left(0,\ \frac{7}{\sqrt{53}},\ -\frac{2}{\sqrt{6}}\right)$$

$$= (0,\ 4,\ 1) - \left(0,\ \frac{182}{53},\ -\frac{52}{53}\right)$$

$$= \left(0,\ \frac{30}{53},\ \frac{105}{53}\right)$$

$$\vec{v}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$= \frac{\left(0,\ \frac{30}{53},\ \frac{105}{53}\right)}{\sqrt{\left(\frac{30}{53}\right)^2 + \left(\frac{105}{53}\right)^2}}$$

$$= \frac{53}{\sqrt{11925}}\left(0,\ \frac{30}{53},\ \frac{105}{53}\right)$$

$$= \frac{53}{15\sqrt{53}}\left(0,\ \frac{30}{53},\ \frac{105}{53}\right)$$

$$= \left(0,\ \frac{2}{\sqrt{15}},\ \frac{7}{\sqrt{15}}\right)$$

Exercise

Use the Gram-Schmidt process to find an *orthonormal* basis for the subspaces of $\mathbb{R}^m$.

$$\vec{u}_1 = (1, 1, 1, 1), \quad \vec{u}_2 = (1, 2, 4, 5), \quad \vec{u}_3 = (1, -3, -4, -2)$$

Solution

$$\vec{v}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|}$$

$$= \frac{(1, 1, 1, 1)}{\sqrt{4}}$$

$$= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$\vec{w}_2 = \vec{u}_2 - (\vec{u}_2 \cdot \vec{v}_1)\vec{v}_1$$

$$= (1, 2, 4, 5) - \left[(1, 2, 4, 5) \cdot \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right]\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$= (1, 2, 4, 5) - 6\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$= (1, 2, 4, 5) - (3, 3, 3, 3)$$

$$= (-2, -1, 1, 2)$$

$$\vec{v}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{1}{\sqrt{10}}(-2, -1, 1, 2)$$

$$= \left(-\frac{2}{\sqrt{10}}, -\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right)$$

$$\vec{u}_3 \cdot \vec{v}_1 = (1, -3, -4, -2) \cdot \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$= \frac{1-3-4-2}{2}$$

$$= -4$$

$$\vec{u}_3 \cdot \vec{v}_2 = (1, -3, -4, -2) \cdot \left(-\frac{2}{\sqrt{10}}, -\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right)$$

$$= \frac{-2+3-4-4}{\sqrt{10}}$$

$$= -\frac{7}{\sqrt{10}}$$

$$\vec{w}_3 = \vec{u}_3 - (\vec{u}_3 \cdot \vec{v}_1)\vec{v}_1 - (\vec{u}_3 \cdot \vec{v}_2)\vec{v}_2$$

$$= (1,\ -3,\ -4,\ -2) + 4\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right) + \frac{7}{\sqrt{10}}\left(-\frac{2}{\sqrt{10}},\ -\frac{1}{\sqrt{10}},\ \frac{1}{\sqrt{10}},\ \frac{2}{\sqrt{10}}\right)$$

$$= (1,\ -3,\ -4,\ -2) + (2,\ 2,\ 2,\ 2) + \left(-\frac{7}{5},\ -\frac{7}{10},\ \frac{7}{10},\ \frac{7}{5}\right)$$

$$= \left(\frac{8}{5},\ -\frac{17}{10},\ -\frac{27}{10},\ \frac{7}{5}\right)$$

$$\vec{v}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$= \frac{1}{\sqrt{\frac{64}{25} + \frac{289}{100} + \frac{289}{100} + \frac{49}{25}}}\left(\frac{8}{5},\ -\frac{17}{10},\ -\frac{27}{10},\ \frac{7}{5}\right)$$

$$= \frac{1}{\sqrt{\frac{1030}{100}}}\left(\frac{8}{5},\ -\frac{17}{10},\ -\frac{27}{10},\ \frac{7}{5}\right)$$

$$= \left(\frac{16}{\sqrt{1030}},\ -\frac{17}{\sqrt{1030}},\ -\frac{27}{\sqrt{1030}},\ \frac{14}{\sqrt{1030}}\right)$$

Exercise

Use the Gram-Schmidt process to find an *orthonormal* basis for the subspaces of $\mathbb{R}^m$.

$$\vec{u}_1 = (1,\ 1,\ 1,\ 1), \quad \vec{u}_2 = (1,\ 1,\ 2,\ 4), \quad \vec{u}_3 = (1,\ 2,\ -4,\ -3)$$

Solution

$$\vec{v}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|}$$

$$= \frac{(1,\ 1,\ 1,\ 1)}{\sqrt{4}}$$

$$= \left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$\vec{w}_2 = \vec{u}_2 - (\vec{u}_2 \cdot \vec{v}_1)\vec{v}_1$$

$$= (1,\ 1,\ 2,\ 4) - \left[(1,\ 1,\ 2,\ 4) \cdot \left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)\right]\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$= (1,\ 1,\ 2,\ 4) - 4\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$= (1,\ 1,\ 2,\ 4) - (2,\ 2,\ 2,\ 2)$$

$$= (-1,\ -1,\ 0,\ 2)$$

$$\vec{v}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{1}{\sqrt{1+1+4}}(-1,\ -1,\ 0,\ 2)$$

$$\boxed{= \left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0,\ \frac{2}{\sqrt{6}}\right)}$$

$$\vec{u}_3 \bullet \vec{v}_1 = (1,\ 2,\ -4,\ -3) \bullet \left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$= \frac{1+2-4-3}{2}$$

$$\boxed{= -2}$$

$$\vec{u}_3 \bullet \vec{v}_2 = (1,\ 2,\ -4,\ -3) \bullet \left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0,\ \frac{2}{\sqrt{6}}\right)$$

$$= \frac{-1-2-6}{\sqrt{6}}$$

$$\boxed{= -\frac{9}{\sqrt{6}}}$$

$$\vec{w}_3 = \vec{u}_3 - (\vec{u}_3 \bullet \vec{v}_1)\vec{v}_1 - (\vec{u}_3 \bullet \vec{v}_2)\vec{v}_2$$

$$= (1,\ 2,\ -4,\ -3) + 2\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right) + \frac{9}{\sqrt{6}}\left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0,\ \frac{2}{\sqrt{6}}\right)$$

$$= (1,\ 2,\ -4,\ -3) + (1,\ 1,\ 1,\ 1) + \left(-\frac{3}{2},\ -\frac{3}{2},\ 0,\ 3\right)$$

$$\boxed{= \left(\frac{1}{2},\ \frac{3}{2},\ -3,\ 1\right)}$$

$$\vec{v}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$= \frac{1}{\sqrt{\frac{1}{4}+\frac{9}{4}+9+1}}\left(\frac{1}{2},\ \frac{3}{2},\ -3,\ 1\right)$$

$$= \frac{2}{5\sqrt{2}}\left(\frac{1}{2},\ \frac{3}{2},\ -3,\ 1\right)$$

$$\boxed{= \left(\frac{1}{5\sqrt{2}},\ \frac{3}{5\sqrt{2}},\ -\frac{6}{5\sqrt{2}},\ \frac{2}{5\sqrt{2}}\right)}$$

Exercise

Use the Gram-Schmidt process to find an *orthonormal* basis for the subspaces of $\mathbb{R}^m$.

$$\vec{u}_1 = (0,\ 2,\ 1,\ 0);\quad \vec{u}_2 = (1,\ -1,\ 0,\ 0);\quad \vec{u}_3 = (1,\ 2,\ 0,\ -1);\quad \vec{u}_4 = (1,\ 0,\ 0,\ 1)$$

Solution

$$\vec{v}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|}$$

$$= \frac{(0,\ 2,\ 1,\ 0)}{\sqrt{5}}$$

$$= \left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)$$

$$\vec{w}_2 = \vec{u}_2 - (\vec{u}_2 \cdot \vec{v}_1)\vec{v}_1$$

$$= (1, -1,\ 0,\ 0) - \left[(1, -1,\ 0,\ 0) \cdot \left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)\right]\left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)$$

$$= (1, -1,\ 0,\ 0) - \left(-\frac{2}{\sqrt{5}}\right)\left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right).$$

$$= \left(1,\ -\frac{1}{5},\ \frac{2}{5},\ 0\right)$$

$$\vec{v}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{1}{\sqrt{1 + \frac{1}{25} + \frac{4}{25} + 0}}\left(1,\ -\frac{1}{5},\ \frac{2}{5},\ 0\right)$$

$$= \frac{5}{\sqrt{30}}\left(1, -\frac{1}{5},\ \frac{2}{5},\ 0\right)$$

$$= \left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right)$$

$$u_3 \cdot v_1 = (1,\ 2,\ 0,\ -1) \cdot \left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)$$

$$= \frac{4}{\sqrt{5}}$$

$$u_3 \cdot v_2 = (1,\ 2,\ 0,\ -1) \cdot \left(\frac{5}{\sqrt{30}},\ -\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right)$$

$$= \frac{5}{\sqrt{30}} - \frac{2}{\sqrt{30}}$$

$$= \boxed{\frac{3}{\sqrt{30}}}$$

$$\vec{w}_3 = \vec{u}_3 - (\vec{u}_3 \bullet \vec{v}_1)\vec{v}_1 - (\vec{u}_3 \bullet \vec{v}_2)\vec{v}_2$$

$$= (1,\ 2,\ 0,\ -1) - \left(\frac{4}{\sqrt{5}}\right)\left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right) - \left(\frac{3}{\sqrt{30}}\right)\left(\frac{5}{\sqrt{30}},\ -\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right)$$

$$= (1,\ 2,\ 0,\ -1) - \left(0,\ \frac{8}{5},\ \frac{4}{5},\ 0\right) - \left(\frac{1}{2},\ -\frac{1}{10},\ \frac{1}{5},\ 0\right)$$

$$= \boxed{\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)}$$

$$\vec{v}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$= \frac{\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + (-1)^2 + (-1)^2}}$$

$$= \frac{1}{\sqrt{\frac{5}{2}}}\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)$$

$$= \frac{\sqrt{2}}{\sqrt{5}}\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right) = \frac{2}{\sqrt{10}}\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)$$

$$= \boxed{\left(\frac{1}{\sqrt{10}},\ \frac{1}{\sqrt{10}},\ -\frac{2}{\sqrt{10}},\ -\frac{2}{\sqrt{10}}\right)}$$

$$u_4 \bullet v_1 = (1,\ 0,\ 0,\ 1) \bullet \left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)$$

$$= \boxed{0}$$

$$u_4 \bullet v_2 = (1,\ 0,\ 0,\ 1) \bullet \left(\frac{5}{\sqrt{30}},\ -\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right)$$

$$= \boxed{\frac{5}{\sqrt{30}}}$$

$$u_4 \bullet v_3 = (1,\ 0,\ 0,\ 1) \bullet \left(\frac{1}{\sqrt{10}},\ \frac{1}{\sqrt{10}},\ -\frac{2}{\sqrt{10}},\ -\frac{2}{\sqrt{10}}\right)$$

$$= \boxed{-\frac{1}{\sqrt{10}}}$$

$$\vec{w}_4 = \vec{u}_4 - (\vec{u}_4 \bullet \vec{v}_1)\vec{v}_1 - (\vec{u}_4 \bullet \vec{v}_2)\vec{v}_2 - (\vec{u}_4 \bullet \vec{v}_3)\vec{v}_3$$

$$= (1,\ 2,\ 0,\ -1) - (0) - \left(\frac{5}{\sqrt{30}}\right)\left(\frac{5}{\sqrt{30}},\ -\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right) + \left(\frac{1}{\sqrt{10}}\right)\left(\frac{1}{\sqrt{10}},\ \frac{1}{\sqrt{10}},\ -\frac{2}{\sqrt{10}},\ -\frac{2}{\sqrt{10}}\right)$$

$$= (1,\ 2,\ 0,\ -1) - \left(\frac{5}{6},\ -\frac{1}{6},\ \frac{1}{3},\ 0\right) + \left(\frac{1}{10},\ \frac{1}{10},\ -\frac{1}{5},\ -\frac{1}{5}\right)$$

$$= \left(\frac{4}{15},\ \frac{4}{15},\ -\frac{8}{15},\ \frac{4}{5}\right)$$

$$\vec{v}_4 = \frac{\vec{w}_4}{\left\|\vec{w}_4\right\|}$$

$$= \frac{\left(\frac{4}{15},\ \frac{4}{15},\ -\frac{8}{15},\ \frac{4}{5}\right)}{\sqrt{\left(\frac{4}{15}\right)^2 + \left(\frac{4}{15}\right)^2 + \left(-\frac{8}{15}\right)^2 + \left(\frac{4}{5}\right)^2}}$$

$$= \frac{1}{\sqrt{\frac{240}{225}}}\left(\frac{4}{15},\ \frac{4}{15},\ -\frac{8}{15},\ \frac{4}{5}\right)$$

$$= \frac{1}{\frac{4}{\sqrt{15}}}\left(\frac{4}{15},\ \frac{4}{15},\ -\frac{8}{15},\ \frac{4}{5}\right)$$

$$= \frac{15}{4\sqrt{15}}\left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right)$$

$$= \left(\frac{1}{\sqrt{15}},\ \frac{1}{\sqrt{15}},\ -\frac{2}{\sqrt{15}},\ \frac{3}{\sqrt{15}}\right)$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\{(1,\ 1,\ 0),\ (1,\ 2,\ 0),\ (0,\ 1,\ 2)\}$$

Solution

$$\vec{v}_1 = (1,\ 1,\ 0)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$= (0,\ 2,\ 1) - \frac{(1,\ 2,\ 0) \bullet (1,\ 1,\ 0)}{1+1+0}(1,\ 1,\ 0)$$

$$= (0,\ 2,\ 1) - \frac{3}{2}(1,\ 1,\ 0)$$

$$= \left(-\frac{3}{2},\ \frac{1}{2},\ 1\right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1 = \frac{(0,\ 1,\ 2) \bullet (1,\ 1,\ 0)}{2}(1,\ 1,\ 0)$$

$$= \left(\frac{1}{2},\ \frac{1}{2},\ 0\right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{1}{\frac{9}{4}+\frac{1}{4}+1}(0,\ 1,\ 2) \bullet \left(-\frac{3}{2},\ \frac{1}{2},\ 1\right)\left(-\frac{3}{2},\ \frac{1}{2},\ 1\right)$$

$$= \frac{2}{7}\frac{5}{2}\left(-\frac{3}{2},\ \frac{1}{2},\ 1\right)$$

$$= \left(-\frac{15}{14},\ \frac{5}{14},\ \frac{5}{7}\right)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (0,\ 1,\ 2) - \left(\frac{1}{2},\ \frac{1}{2},\ 0\right) - \left(-\frac{15}{14},\ \frac{5}{14},\ \frac{5}{7}\right)$$

$$= \left(\frac{4}{7},\ \frac{1}{7},\ \frac{9}{7}\right)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{2}}(1,\ 1,\ 0)$$

$$= \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{\sqrt{\frac{9}{4}+\frac{1}{4}+1}}\left(-\frac{3}{2},\ \frac{1}{2},\ 1\right)$$

$$= \frac{2}{\sqrt{14}}\left(-\frac{3}{2},\ \frac{1}{2},\ 1\right)$$

$$= \left(-\frac{2}{\sqrt{14}},\ \frac{1}{\sqrt{14}},\ \frac{2}{\sqrt{14}}\right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\sqrt{\frac{16}{49}+\frac{1}{49}+\frac{81}{49}}}\left(\frac{4}{7},\ \frac{1}{7},\ \frac{9}{7}\right)$$

$$= \frac{7}{\sqrt{98}}\left(\frac{4}{7},\ \frac{1}{7},\ \frac{9}{7}\right)$$

$$= \frac{7}{7\sqrt{2}}\left(\frac{4}{7},\ \frac{1}{7},\ \frac{9}{7}\right)$$

$$\boxed{= \left(\frac{4}{7\sqrt{2}},\ \frac{1}{7\sqrt{2}},\ \frac{9}{7\sqrt{2}}\right)}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\{(1,\ -2,\ 2),\ (2,\ 2,\ 1),\ (2,\ -1,\ -2)\}$$

Solution

$$\boxed{\vec{v}_1 = \vec{u}_1 = (1,\ -2,\ 2)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (2,\ 2,\ 1) - \frac{(2,\ 2,\ 1) \bullet (1,\ -2,\ 2)}{9}(1,\ -2,\ 2)$$

$$= (2,\ 2,\ 1) - \frac{0}{9}(1,\ -2,\ 2)$$

$$\boxed{= (2,\ 2,\ 1)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(2,\ -1,\ -2) \bullet (1,\ -2,\ 2)}{9}(1,\ -2,\ 2)$$

$$= \frac{0}{9}(1,\ -2,\ 2)$$

$$\boxed{= (0,\ 0,\ 0)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{1}{9}\left[(2,\ -1,\ -2) \bullet (2,\ 2,\ 1)\right](2,\ 2,\ 1)$$

$$\boxed{= (0,\ 0,\ 0)}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (2,\ -1,\ -2) - (0,\ 0,\ 0) - (0,\ 0,\ 0)$$

$$\boxed{= (2,\ -1,\ -2)}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \frac{1}{3}(1,\ -2,\ 2)$$

$$\underline{= \left(\frac{1}{3},\ -\frac{2}{3},\ \frac{2}{3}\right)}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \frac{1}{3}(2,\ 2,\ 1)$$

$$\underline{= \left(\frac{2}{3},\ \frac{2}{3},\ \frac{1}{3}\right)}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$= \frac{1}{3}(2,\ -1,\ -2)$$

$$\underline{= \left(\frac{2}{3},\ -\frac{1}{3},\ -\frac{2}{3}\right)}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\{(1,\ 0,\ 0),\ (1,\ 1,\ 1),\ (1,\ 1,\ -1)\}$$

Solution

$$\underline{\vec{v}_1 = \vec{u}_1 = (1,\ 0,\ 0)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2,\ \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$= (1,\ 1,\ 1) - \frac{(1,\ 1,\ 1) \bullet (1,\ 0,\ 0)}{1}(1,\ 0,\ 0)$$

$$= (1,\ 1,\ 1) - (1,\ 0,\ 0)$$

$$\underline{= (0,\ 1,\ 1)}$$

$$\frac{\left\langle \vec{u}_3,\ \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1 = \frac{(1,\ 1,\ -1) \bullet (1,\ 0,\ 0)}{1}(1,\ 0,\ 0)$$

$$\underline{= (1,\ 0,\ 0)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{1}{1}\left[(1,\ 1,\ -1)\cdot(0,\ 1,\ 1)\right](0,\ 1,\ 1)$$

$$= 0(0,\ 1,\ 1)$$

$$= (0,\ 0,\ 0)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (1,\ 1,\ -1) - (1,\ 0,\ 0) - (0,\ 0,\ 0)$$

$$= (0,\ 1,\ -1)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{1}(1,\ 0,\ 0)$$

$$= (1,\ 0,\ 0)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{\sqrt{2}}(0,\ 1,\ 1)$$

$$= \left(0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\sqrt{2}}(0,\ 1,\ -1)$$

$$= \left(0,\ \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right)$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\{(4,\ -3,\ 0),\ (1,\ 2,\ 0),\ (0,\ 0,\ 4)\}$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (4,\ -3,\ 0)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (1,\ 2,\ 0) - \frac{(1,\ 2,\ 0) \bullet (4,\ -3,\ 0)}{25}(4,\ -3,\ 0)$$

$$= (1,\ 2,\ 0) + \frac{2}{25}(4,\ -3,\ 0)$$

$$= \left(\frac{33}{25},\ \frac{44}{25},\ 0\right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(0,\ 0,\ 4) \bullet (4,\ -3,\ 0)}{25}(4,\ -3,\ 0)$$

$$= (0,\ 0,\ 0)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{225}{3,025}\left[(0,\ 0,\ 4) \bullet \left(\frac{33}{25},\ \frac{44}{25},\ 0\right)\right]\left(\frac{33}{25},\ \frac{44}{25},\ 0\right)$$

$$= (0,\ 0,\ 0)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (0,\ 0,\ 4) - (0,\ 0,\ 0) - (0,\ 0,\ 0)$$

$$= (0,\ 0,\ 4)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{16+9}}(4,\ -3,\ 0)$$

$$= \left(\frac{4}{5},\ -\frac{3}{5},\ 0\right)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{25}{\sqrt{3,025}}\left(\frac{33}{25},\ \frac{44}{25},\ 0\right)$$

$$= \frac{25}{55}\left(\frac{33}{25},\ \frac{44}{25},\ 0\right)$$

$$= \left(\frac{3}{5},\ \frac{4}{5},\ 0\right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{4}(0,\ 0,\ 4)$$

$$= (0,\ 0,\ 1)$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\{(0,\ 1,\ 2),\ (2,\ 0,\ 0),\ (1,\ 1,\ 1)\}$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (0,\ 1,\ 2)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (2,\ 0,\ 0) - \frac{(2,\ 0,\ 0) \bullet (0,\ 1,\ 2)}{5}(0,\ 1,\ 2)$$

$$= (2,\ 0,\ 0) + \frac{0}{5}(0,\ 1,\ 2)$$

$$= (2,\ 0,\ 0)$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(1,\ 1,\ 1) \bullet (0,\ 1,\ 2)}{5}(0,\ 1,\ 2)$$

$$= \frac{3}{5}(0,\ 1,\ 2)$$

$$= \left(0,\ \frac{3}{5},\ \frac{6}{5}\right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{1}{4}\left[(1,\ 1,\ 1) \bullet (2,\ 0,\ 0)\right](2,\ 0,\ 0)$$

$$= \frac{1}{2}(2,\ 0,\ 0)$$

$$= (1,\ 0,\ 0)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (1,\ 1,\ 1) - (1,\ 0,\ 0) - \left(0,\ \frac{3}{5},\ \frac{6}{5}\right)$$

$$= \boxed{\left(0,\ \frac{2}{5},\ -\frac{1}{5}\right)}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{5}}(0,\ 1,\ 2)$$

$$= \boxed{\left(0,\ \frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right)}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{2}(2,\ 0,\ 0)$$

$$= \boxed{(1,\ 0,\ 0)}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{5}{\sqrt{5}}\left(0,\ \frac{2}{5},\ -\frac{1}{5}\right)$$

$$= \boxed{\left(0,\ \frac{2}{\sqrt{5}},\ -\frac{1}{\sqrt{5}}\right)}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\{(0,\ 1,\ 1),\ (1,\ 1,\ 0),\ (1,\ 0,\ 1)\}$$

Solution

$$\boxed{\vec{v}_1 = \vec{u}_1 = (0,\ 1,\ 1)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (1,\ 1,\ 0) - \frac{(1,\ 1,\ 0) \bullet (0,\ 1,\ 1)}{2}(0,\ 1,\ 1)$$

$$= (1,\ 1,\ 0) - \frac{1}{2}(0,\ 1,\ 1)$$

$$= \boxed{\left(1,\ \frac{1}{2},\ -\frac{1}{2}\right)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(1,\ 0,\ 1)\bullet(0,\ 1,\ 1)}{2}(0,\ 1,\ 1)$$

$$= \left(0,\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{4}{6}\left[(1,\ 0,\ 1)\bullet\left(1,\ \frac{1}{2},\ -\frac{1}{2}\right)\right]\left(1,\ \frac{1}{2},\ -\frac{1}{2}\right)$$

$$= \frac{1}{3}\left(1,\ \frac{1}{2},\ -\frac{1}{2}\right)$$

$$= \left(\frac{1}{3},\ \frac{1}{6},\ -\frac{1}{6}\right)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (1,\ 0,\ 1) - \left(0,\ \frac{1}{2},\ \frac{1}{2}\right) - \left(\frac{1}{3},\ \frac{1}{6},\ -\frac{1}{6}\right)$$

$$= \left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3}\right)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{2}}(0,\ 1,\ 1)$$

$$= \left(0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{2}{\sqrt{6}}\left(1,\ \frac{1}{2},\ -\frac{1}{2}\right)$$

$$= \left(\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}}\right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{3}{\sqrt{12}}\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3}\right)$$

$$= \left(\frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\{(1,\ 2,\ -2),\ (1,\ 0,\ -4),\ (5,\ 2,\ 0),\ (1,\ 1,\ -1)\}$$

Solution

$$\vec{v}_1=\vec{u}_1=(1,\ 2,\ -2)$$

$$\vec{v}_2=\vec{u}_2-\frac{\langle\vec{u}_2,\vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$=(1,\ 0,\ -4)-\frac{(1,\ 0,\ -4)\bullet(1,\ 2,\ -2)}{9}(1,\ 2,\ -2)$$

$$=(1,\ 0,\ -4)-(1,\ 2,\ -2)$$

$$=(0,\ -2,\ -2)$$

$$\frac{\langle\vec{u}_3,\vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1=\frac{(5,\ 2,\ 0)\bullet(1,\ 2,\ -2)}{9}(1,\ 2,\ -2)$$

$$=(1,\ 2,\ -2)$$

$$\frac{\langle\vec{u}_3,\vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2=\frac{1}{8}\left[(5,\ 2,\ 0)\bullet(0,\ -2,\ -2)\right](0,\ -2,\ -2)$$

$$=-\frac{1}{2}(0,\ -2,\ -2)$$

$$=(0,\ 1,\ 1)$$

$$\vec{v}_3=\vec{u}_3-\frac{\langle\vec{u}_3,\vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1-\frac{\langle\vec{u}_3,\vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$=(5,\ 2,\ 0)-(1,\ 2,\ -2)-(0,\ 1,\ 1)$$

$$=(4,\ -1,\ 1)$$

$$\frac{\langle\vec{u}_4,\vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1=\frac{(1,\ 1,\ -1)\bullet(1,\ 2,\ -2)}{9}(1,\ 2,\ -2)$$

$$=\frac{5}{9}(1,\ 2,\ -2)$$

$$=\left(\frac{5}{9},\ \frac{10}{9},\ -\frac{10}{9}\right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{1}{8}\left[(1,\ 1,\ -1) \bullet (0,\ -2,\ -2)\right](0,\ -2,\ -2)$$

$$= \boxed{(0,\ 0,\ 0)}$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3 = \frac{1}{18}\left[(1,\ 1,\ -1) \bullet (4,\ -1,\ 1)\right](4,\ -1,\ 1)$$

$$= \frac{1}{9}(4,\ -1,\ 1)$$

$$= \boxed{\left(\frac{4}{9},\ -\frac{1}{9},\ \frac{1}{9}\right)}$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$

$$= (1,\ 1,\ -1) - \left(\frac{5}{9},\ \frac{10}{9},\ -\frac{10}{9}\right) - \left(\frac{4}{9},\ -\frac{1}{9},\ \frac{1}{9}\right)$$

$$= \boxed{(0,\ 0,\ 0)}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{3}(1,\ 2,\ -2)$$

$$= \boxed{\left(\frac{1}{3},\ \frac{2}{3},\ -\frac{2}{3}\right)}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{2\sqrt{2}}(0,\ -2,\ -2)$$

$$= \boxed{\left(0,\ -\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right)}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{3\sqrt{2}}(4,\ -1,\ 1)$$

$$= \boxed{\left(\frac{4}{3\sqrt{2}},\ -\frac{1}{3\sqrt{2}},\ \frac{1}{3\sqrt{2}}\right)}$$

$$\vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

$$= \boxed{(0,\ 0,\ 0)}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\{(-3,\ 1,\ 2),\ (1,\ 1,\ 1),\ (2,\ 0,\ -1),\ (1,\ -3,\ 2)\}$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (-3,\ 1,\ 2)$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 \\
&= (1,\ 1,\ 1) - \frac{(1,\ 1,\ 1)\bullet(-3,\ 1,\ 2)}{14}(-3,\ 1,\ 2) \\
&= (1,\ 1,\ 1) - \frac{0}{14}(1,\ 2,\ -2) \\
&= (1,\ 1,\ 1)
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 &= \frac{(2,\ 0,\ -1)\bullet(-3,\ 1,\ 2)}{14}(-3,\ 1,\ 2) \\
&= -\frac{4}{7}(-3,\ 1,\ 2) \\
&= \left(\frac{12}{7},\ -\frac{4}{7},\ -\frac{8}{7}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 &= \frac{1}{3}\left[(2,\ 0,\ -1)\bullet(1,\ 1,\ 1)\right](1,\ 1,\ 1) \\
&= \frac{1}{3}(1,\ 1,\ 1) \\
&= \left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right)
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 \\
&= (2,\ 0,\ -1) - \left(\frac{12}{7},\ -\frac{4}{7},\ -\frac{8}{7}\right) - \left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right) \\
&= \left(-\frac{1}{21},\ \frac{5}{21},\ -\frac{4}{21}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 &= \frac{(1,\ -3,\ 2)\bullet(-3,\ 1,\ 2)}{14}(-3,\ 1,\ 2) \\
&= -\frac{1}{7}(-3,\ 1,\ 2)
\end{aligned}$$

$$=\left(\frac{3}{7},\ -\frac{1}{7},\ -\frac{2}{7}\right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{(1,\ -3,\ 2)\bullet(1,\ 1,\ 1)}{3}(1,\ 1,\ 1)$$

$$=(0,\ 0,\ 0)$$

$$\frac{\langle \vec{u}_4, \vec{v}_3\rangle}{\|\vec{v}_3\|^2}\vec{v}_3 = \frac{441}{42}\left[(1,\ -3,\ 2)\bullet\left(-\frac{1}{21},\ \frac{5}{21},\ -\frac{4}{21}\right)\right]\left(-\frac{1}{21},\ \frac{5}{21},\ -\frac{4}{21}\right)$$

$$=\frac{441}{42}\left(-\frac{24}{21}\right)\left(-\frac{1}{21},\ \frac{5}{21},\ -\frac{4}{21}\right)$$

$$=\left(\frac{4}{7},\ -\frac{20}{7},\ \frac{16}{7}\right)$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3\rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$

$$=(1,\ -3,\ 2)-\left(\frac{3}{7},\ -\frac{1}{7},\ -\frac{2}{7}\right)-(0,\ 0,\ 0)-\left(\frac{4}{7},\ -\frac{20}{7},\ \frac{16}{7}\right)$$

$$=(0,\ 0,\ 0)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$=\frac{1}{\sqrt{14}}(-3,\ 1,\ 2)$$

$$=\left(-\frac{3}{\sqrt{14}},\ \frac{1}{\sqrt{14}},\ \frac{2}{\sqrt{14}}\right)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$=\frac{1}{\sqrt{3}}(1,\ 1,\ 1)$$

$$=\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$=\frac{21}{\sqrt{42}}\left(-\frac{1}{21},\ \frac{5}{21},\ -\frac{4}{21}\right)$$

$$=\left(-\frac{1}{\sqrt{42}},\ \frac{5}{\sqrt{42}},\ -\frac{4}{\sqrt{42}}\right)$$

$$\vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

$$= (0,\ 0,\ 0)$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\{(2,\ 1,\ 1),\ (0,\ 3,\ -1),\ (3,\ -4,\ -2),\ (-1,\ -1,\ 3)\}$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (2,\ 1,\ 1)$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 \\
&= (0,\ 3,\ -1) - \frac{(0,\ 3,\ -1)\bullet(2,\ 1,\ 1)}{6}(2,\ 1,\ 1) \\
&= (0,\ 3,\ -1) - \frac{1}{3}(2,\ 1,\ 1) \\
&= \left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 &= \frac{(3,\ -4,\ -2)\bullet(2,\ 1,\ 1)}{6}(2,\ 1,\ 1) \\
&= (0,\ 0,\ 0)
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 &= \frac{9}{84}\left[(3,\ -4,\ -2)\bullet\left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right)\right]\left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right) \\
&= \frac{3}{28}(-10)\left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right) \\
&= \left(\frac{5}{7},\ -\frac{20}{7},\ \frac{10}{7}\right)
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 \\
&= (3,\ -4,\ -2) - (0,\ 0,\ 0) - \left(\frac{5}{7},\ -\frac{20}{7},\ \frac{10}{7}\right) \\
&= \left(\frac{16}{7},\ -\frac{8}{7},\ -\frac{24}{7}\right)
\end{aligned}$$

$$\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(-1, -1, 3)\cdot(2, 1, 1)}{6}(2, 1, 1)$$

$$= \boxed{(0, 0, 0)}$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{9}{84}\left[(-1, -1, 3)\cdot\left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3}\right)\right]\left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3}\right)$$

$$= \frac{3}{28}\left(-\frac{18}{3}\right)\left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3}\right)$$

$$= \boxed{\left(\frac{3}{7}, -\frac{12}{7}, \frac{6}{7}\right)}$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3 = \frac{49}{896}\left[(-1, -1, 3)\cdot\left(\frac{16}{7}, -\frac{8}{7}, -\frac{24}{7}\right)\right]\left(\frac{16}{7}, -\frac{8}{7}, -\frac{24}{7}\right)$$

$$= \frac{7}{128}\left(-\frac{80}{7}\right)\left(\frac{16}{7}, -\frac{8}{7}, -\frac{24}{7}\right)$$

$$= \boxed{\left(-\frac{10}{7}, \frac{5}{7}, \frac{15}{7}\right)}$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$

$$= (-1, -1, 3) - (0, 0, 0) - \left(\frac{3}{7}, -\frac{12}{7}, \frac{6}{7}\right) - \left(-\frac{10}{7}, \frac{5}{7}, \frac{15}{7}\right)$$

$$= \boxed{(0, 0, 0)}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{6}}(2, 1, 1)$$

$$= \boxed{\left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{3}{2\sqrt{21}}\left(-\frac{2}{3}, \frac{8}{3}, -\frac{4}{3}\right)$$

$$= \boxed{\left(-\frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, -\frac{2}{\sqrt{21}}\right)}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{7}{8\sqrt{14}}\left(\frac{16}{7},\ -\frac{8}{7},\ -\frac{24}{7}\right)$$

$$= \left(\frac{2}{\sqrt{14}},\ -\frac{1}{\sqrt{14}},\ -\frac{3}{\sqrt{14}}\right)$$

$$\vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

$$= (0,\ 0,\ 0)$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\vec{u}_1 = (1, 1, 0, -1), \quad \vec{u}_2 = (1, 3, 0, 1), \quad \vec{u}_3 = (4, 2, 2, 0)$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (1, 1, 0, -1)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (1,\ 3,\ 0,\ 1) - \frac{(1,\ 3,\ 0,\ 1)\bullet(1,\ 1,\ 0,\ -1)}{3}(1,\ 1,\ 0,\ -1)$$

$$= (1,\ 3,\ 0,\ 1) - (1,\ 1,\ 0,\ -1)$$

$$= (0,\ 2,\ 0,\ 2)$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(4,\ 2,\ 2,\ 0)\bullet(1,\ 1,\ 0,\ -1)}{3}(1,\ 1,\ 0,\ -1)$$

$$= (2,\ 2,\ 0,\ -2)$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{(4,\ 2,\ 2,\ 0)\bullet(0,\ 2,\ 0,\ 2)}{8}(0,\ 2,\ 0,\ 2)$$

$$= \frac{1}{2}(0,\ 2,\ 0,\ 2)$$

$$= (0,\ 1,\ 0,\ 1)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (4,\ 2,\ 2,\ 0) - (2,\ 2,\ 0,\ -2) - (0,\ 1,\ 0,\ 1)$$

$$\underline{= (2,\ -1,\ 2,\ 1)}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{3}}(1,\ 1,\ 0,\ -1)$$

$$\underline{= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ 0,\ -\frac{1}{\sqrt{3}}\right)}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{2\sqrt{2}}(0,\ 2,\ 0,\ 2)$$

$$\underline{= \left(0,\ \frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\sqrt{10}}(2,\ -1,\ 2,\ 1)$$

$$\underline{= \left(\frac{2}{\sqrt{10}},\ -\frac{1}{\sqrt{10}},\ \frac{2}{\sqrt{10}},\ -\frac{1}{\sqrt{10}}\right)}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\vec{u}_1 = (1,\ 1,\ 1,\ 1), \quad \vec{u}_2 = (1,\ 1,\ 2,\ 4), \quad \vec{u}_3 = (1,\ 2,\ -4,\ -3)$$

Solution

$$\underline{\vec{v}_1 = \vec{u}_1 = (1,\ 1,\ 1,\ 1)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (1,\ 1,\ 2,\ 4) - \frac{(1,\ 1,\ 2,\ 4)\bullet(1,\ 1,\ 1,\ 1)}{4}(1,\ 1,\ 1,\ 1)$$

$$= (1,\ 1,\ 2,\ 4) - 2(1,\ 1,\ 1,\ 1)$$

$$\underline{= (-1,\ -1,\ 0,\ 2)\,\big|}$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(1,\ 2,\ -4,\ -3) \bullet (1,\ 1,\ 1,\ 1)}{4}(1,\ 1,\ 1,\ 1)$$

$$\underline{= (-1,\ -1,\ -1,\ -1)\,\big|}$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{(1,\ 2,\ -4,\ -3) \bullet (-1,\ -1,\ 0,\ 2)}{6}(-1,\ -1,\ 0,\ 2)$$

$$= -\frac{3}{2}(-1,\ -1,\ 0,\ 2)$$

$$\underline{= \left(\frac{3}{2},\ \frac{3}{2},\ 0,\ -3\right)\,\big|}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (1,\ 2,\ -4,\ -3) - (-1,\ -1,\ -1,\ -1) - \left(\frac{3}{2},\ \frac{3}{2},\ 0,\ -3\right)$$

$$\underline{= \left(\frac{1}{2},\ \frac{3}{2},\ -3,\ 1\right)\,\big|}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{2}(1,\ 1,\ 1,\ 1)$$

$$\underline{= \left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)\,\big|}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{1}{\sqrt{6}}(-1,\ -1,\ 0,\ 2)$$

$$\underline{= \left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0,\ \frac{2}{\sqrt{6}}\right)\,\big|}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{2}{\sqrt{50}}\left(\frac{1}{2},\ \frac{3}{2},\ -3,\ 1\right)$$

$$= \frac{2}{5\sqrt{2}}\left(\frac{1}{2},\ \frac{3}{2},\ -3,\ 1\right)$$

$$\boxed{= \left(\frac{1}{5\sqrt{2}},\ \frac{3}{5\sqrt{2}},\ -\frac{6}{5\sqrt{2}},\ \frac{2}{5\sqrt{2}}\right)}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

$$\{(3,\ 4,\ 0,\ 0),\ (-1,\ 1,\ 0,\ 0),\ (2,\ 1,\ 0,\ -1),\ (0,\ 1,\ 1,\ 0)\}$$

Solution

$$\boxed{\vec{v}_1 = \vec{u}_1 = (3,\ 4,\ 0,\ 0)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (-1,\ 1,\ 0,\ 0) - \frac{(-1,\ 1,\ 0,\ 0)\cdot(3,\ 4,\ 0,\ 0)}{25}(3,\ 4,\ 0,\ 0)$$

$$= (-1,\ 1,\ 0,\ 0) - \frac{1}{25}(3,\ 4,\ 0,\ 0)$$

$$\boxed{= \left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(2,\ 1,\ 0,\ -1)\cdot(3,\ 4,\ 0,\ 0)}{25}(3,\ 4,\ 0,\ 0)$$

$$= \frac{10}{25}(3,\ 4,\ 0,\ 0)$$

$$\boxed{= \left(\frac{6}{5},\ \frac{8}{5},\ 0,\ 0\right)}$$

$$\frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{625}{1,225}\left[(2,\ 1,\ 0,\ -1)\cdot\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)\right]\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)$$

$$= \frac{25}{49}\left(-\frac{35}{25}\right)\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)$$

$$\boxed{= \left(\frac{4}{5},\ -\frac{3}{5},\ 0,\ 0\right)}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (2,\ 1,\ 0,\ -1) - \left(\frac{6}{5},\ \frac{8}{5},\ 0,\ 0\right) - \left(\frac{4}{5},\ -\frac{3}{5},\ 0,\ 0\right)$$

$$= (0,\ 0,\ 0,\ -1)$$

$$\frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(0,\ 1,\ 1,\ 0) \cdot (3,\ 4,\ 0,\ 0)}{25}(3,\ 4,\ 0,\ 0)$$

$$= \frac{4}{25}(3,\ 4,\ 0,\ 0)$$

$$= \left(\frac{12}{25},\ \frac{16}{25},\ 0,\ 0\right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{25}{49}\left[(0,\ 1,\ 1,\ 0) \cdot \left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)\right]\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)$$

$$= \frac{25}{49}\left(\frac{21}{25}\right)\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)$$

$$= \left(-\frac{12}{25},\ \frac{9}{25},\ 0,\ 0\right)$$

$$\frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3 = \left[(0,\ 1,\ 1,\ 0) \cdot (0,\ 0,\ 0,\ -1)\right](0,\ 0,\ 0,\ -1)$$

$$= (0,\ 0,\ 0,\ 0)$$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$

$$= (0,\ 1,\ 1,\ 0) - \left(\frac{12}{25},\ \frac{16}{25},\ 0,\ 0\right) - \left(-\frac{12}{25},\ \frac{9}{25},\ 0,\ 0\right) - (0,\ 0,\ 0,\ 0)$$

$$= (0,\ 0,\ 1,\ 0)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{5}(3,\ 4,\ 0,\ 0)$$

$$= \left(\frac{3}{5},\ \frac{4}{5},\ 0,\ 0\right)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{25}{35}\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)$$

$$= \left(-\frac{4}{5},\ \frac{3}{5},\ 0,\ 0\right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= (0,\ 0,\ 0,\ -1)$$

$$\vec{q}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

$$= (0,\ 0,\ 1,\ 0)$$

Exercise

Find the QR–decomposition of

a) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

Solution

a) Since $\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5 \neq 0$, The matrix is invertible

$\vec{u}_1 (1,\ 2), \quad \vec{u}_2 = (-1,\ 3)$

$$\vec{v}_1 = \vec{u}_1 = (1,\ 2)$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ 2)}{\sqrt{1^2 + 2^2}}$$

$$= \frac{(1,\ 2)}{\sqrt{5}}$$

$$= \left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right)$$

$$\vec{v}_2 = \vec{u}_2 - \left(\vec{u}_2 \cdot \vec{v}_1\right)\vec{v}_1$$

$$= (-1,\ 3) - \left[(-1,\ 3) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)\right]\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$= (-1,\ 3) - \left(\frac{5}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$
$$= (-1,\ 3) - (1,\ 2)$$
$$\underline{= (-2,\ 1)}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$
$$= \frac{(-2,\ 1)}{\sqrt{(-2)^2 + 1^2}}$$
$$\underline{= \left(-\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right)}$$

$$\left\langle \vec{u}_1, \vec{q}_1 \right\rangle = (1,\ 2) \bullet \left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right)$$
$$= \frac{5}{\sqrt{5}}$$
$$\underline{= \sqrt{5}}$$

$$\left\langle \vec{u}_2, \vec{q}_1 \right\rangle = (-1,\ 3) \bullet \left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right)$$
$$= \frac{5}{\sqrt{5}}$$
$$\underline{= \sqrt{5}}$$

$$\left\langle \vec{u}_2, \vec{q}_2 \right\rangle = (-1,\ 3) \bullet \left(-\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right)$$
$$= \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{5}}$$
$$\underline{= \sqrt{5}}$$

$$R = \begin{bmatrix} \left\langle \vec{u}_1, \vec{q}_1 \right\rangle & \left\langle \vec{u}_2, \vec{q}_1 \right\rangle \\ 0 & \left\langle \vec{u}_2, \vec{q}_2 \right\rangle \end{bmatrix}$$
$$= \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}$$

The *QR*-decomposition of the matrix is

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}$$

b) The column vectors of are: $\vec{u}_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$$\boxed{\vec{v}_1 = \vec{u}_1 = (3,\ -4)}$$

$$\begin{aligned}
\vec{q}_1 &= \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|} \\
&= \frac{(3,\ -4)}{\sqrt{9+16}} \\
&= \boxed{\left(\frac{3}{5},\ -\frac{4}{5}\right)}
\end{aligned}$$

$$\begin{aligned}
\vec{v}_2 &= \vec{u}_2 - \frac{\left\langle \vec{u}_2,\ \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1 \\
&= (5,\ 0) - \frac{(5,\ 0)\bullet(3,\ -4)}{25}(3,\ -4) \\
&= (5,\ 0) - \frac{15}{25}(3,\ -4) \\
&= (5,\ 0) - \frac{3}{5}(3,\ -4) \\
&= (5,\ 0) - \left(\frac{9}{5},\ -\frac{12}{5}\right) \\
&= \boxed{\left(\frac{16}{5},\ \frac{12}{5}\right)}
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|} \\
&= \frac{\left(\frac{16}{5},\ \frac{12}{5}\right)}{\sqrt{\frac{256}{25}+\frac{144}{25}}} \\
&= \frac{1}{\sqrt{\frac{400}{25}}}\left(\frac{16}{5},\ \frac{12}{5}\right) \\
&= \frac{1}{\sqrt{16}}\left(\frac{16}{5},\ \frac{12}{5}\right) \\
&= \frac{1}{4}\left(\frac{16}{5},\ \frac{12}{5}\right) \\
&= \boxed{\left(\frac{4}{5},\ \frac{3}{5}\right)}
\end{aligned}$$

$$R = \begin{bmatrix} \left\langle \vec{u}_1,\ \vec{q}_1 \right\rangle & \left\langle \vec{u}_2,\ \vec{q}_1 \right\rangle \\ 0 & \left\langle \vec{u}_2,\ \vec{q}_2 \right\rangle \end{bmatrix}$$

$$= \begin{bmatrix} 3\left(\frac{3}{5}\right) - 4\left(-\frac{4}{5}\right) & 5\left(\frac{3}{5}\right) + 0\left(-\frac{4}{5}\right) \\ 0 & 5\left(\frac{4}{5}\right) - 0\left(\frac{3}{5}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 \\ 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \cdot \begin{bmatrix} 5 & 3 \\ 0 & 4 \end{bmatrix}$$

$$\boldsymbol{A} \quad = \quad \boldsymbol{Q} \qquad \boldsymbol{R}$$

c) Since the column vectors $\vec{u}_1\,(1,\ 0,\ 1),\quad \vec{u}_2 = (2,\ 1,\ 4)$ are linearly independent, so has a QR–decomposition.

$$\underline{\vec{v}_1 = \vec{u}_1 = (1,\ 0,\ 1)}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ 0,\ 1)}{\sqrt{1^2 + 0 + 1^2}}$$

$$= \frac{(1,\ 0,\ 1)}{\sqrt{2}}$$

$$\underline{= \left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)}$$

$$\vec{v}_2 = \vec{u}_2 - \langle \vec{u}_2,\ \vec{v}_1 \rangle \vec{v}_1$$

$$= (2,1,4) - \left[(2,1,4) \bullet \left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\right]\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$= (2,1,4) - \left(\frac{6}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$= (2,1,4) - (3,\ 0,\ 3)$$

$$\underline{= (-1,\ 1,\ 1)}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{(-1,\ 1,\ 1)}{\sqrt{(-1)^2 + 1^2 + 1^2}}$$

$$=\underline{\left(-\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)}$$

$$\left\langle \vec{u}_1,\ \vec{q}_1 \right\rangle = (1,\ 0,\ 1) \cdot \left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$= \frac{2}{\sqrt{2}}$$

$$= \underline{\sqrt{2}}$$

$$\left\langle \vec{u}_2,\ \vec{q}_1 \right\rangle = (2,\ 1,\ 4) \cdot \left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$= \frac{2}{\sqrt{2}} + \frac{4}{\sqrt{2}}$$

$$= \underline{3\sqrt{2}}$$

$$\left\langle \vec{u}_2,\ \vec{q}_2 \right\rangle = (2,\ 1,\ 4) \cdot \left(-\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= -\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{4}{\sqrt{3}}$$

$$= \frac{3}{\sqrt{3}}$$

$$= \underline{\sqrt{3}}$$

$$R = \begin{bmatrix} \left\langle \vec{u}_1,\ \vec{q}_1 \right\rangle & \left\langle \vec{u}_2,\ \vec{q}_1 \right\rangle \\ 0 & \left\langle \vec{u}_2,\ \vec{q}_2 \right\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$$

The QR–decomposition of the matrix is

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}}_{R}$$

d) Since $\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{vmatrix} = -4 \neq 0$,

The matrix is invertible, so it has a QR-decomposition.

$$\vec{u}_1\,(1,\ 1,\ 0),\quad \vec{u}_2=(2,\ 1,\ 3),\quad \vec{u}_3=(1,\ 1,\ 1)$$

$$\vec{v}_1=\vec{u}_1=(1,\ 1,\ 0)$$

$$\vec{q}_1=\frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$=\frac{(1,\ 1,\ 0)}{\sqrt{1^2+1^2+0}}$$

$$=\frac{(1,\ 1,\ 0)}{\sqrt{2}}$$

$$=\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$\vec{v}_2=\vec{u}_2-(\vec{u}_2\bullet\vec{v}_1)\vec{v}_1$$

$$=(2,\ 1,\ 3)-\left[(2,\ 1,\ 3)\bullet\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)\right]\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$=(2,\ 1,\ 3)-\frac{3}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$=(2,\ 1,\ 3)-\left(\frac{3}{2},\ \frac{3}{2},\ 0\right)$$

$$=\left(\frac{1}{2},\ -\frac{1}{2},\ 3\right)$$

$$\vec{q}_2=\frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$=\frac{\left(\frac{1}{2},\ -\frac{1}{2},\ 3\right)}{\sqrt{\left(\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^2+3^2}}$$

$$=\frac{\left(\frac{1}{2},\ -\frac{1}{2},\ 3\right)}{\sqrt{\frac{19}{2}}}$$

$$=\frac{\sqrt{2}}{\sqrt{19}}\left(\frac{1}{2},\ -\frac{1}{2},\ 3\right)$$

$$=\left(\frac{1}{\sqrt{38}},\ -\frac{1}{\sqrt{38}},\ \frac{6}{\sqrt{38}}\right)$$

$$\vec{v}_3=\vec{u}_3-(\vec{u}_3\bullet\vec{v}_1)\vec{v}_1-(\vec{u}_3\bullet\vec{v}_2)\vec{v}_2$$

$$= (1,1,1) - \left[(1,1,1).\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)\right]\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$$

$$- \left[(1,1,1).\left(\frac{1}{\sqrt{38}},-\frac{1}{\sqrt{38}},\frac{6}{\sqrt{38}}\right)\right]\left(\frac{1}{\sqrt{38}},-\frac{1}{\sqrt{38}},\frac{6}{\sqrt{38}}\right)$$

$$= (1,1,1) - \frac{2}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right) - \frac{6}{\sqrt{38}}\left(\frac{1}{\sqrt{38}},-\frac{1}{\sqrt{38}},\frac{6}{\sqrt{38}}\right)$$

$$= (1,1,1) - (1,1,0) - \left(\frac{3}{19},-\frac{3}{19},\frac{18}{19}\right)$$

$$= \left(-\frac{3}{19},\ \frac{3}{19},\ \frac{1}{19}\right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$= \frac{\left(-\frac{3}{19},\ \frac{3}{19},\ \frac{1}{19}\right)}{\sqrt{\left(-\frac{3}{19}\right)^2 + \left(\frac{3}{19}\right)^2 + \left(\frac{1}{19}\right)^2}}$$

$$= \frac{19}{\sqrt{19}}\left(-\frac{3}{19},\ \frac{3}{19},\ \frac{1}{19}\right)$$

$$= \left(-\frac{3}{\sqrt{19}},\ \frac{3}{\sqrt{19}},\ \frac{1}{\sqrt{19}}\right)$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix}$$

$$\left\langle \vec{u}_1,\ \vec{q}_1 \right\rangle = (1,\ 1,\ 0) \bullet \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= \frac{2}{\sqrt{2}}$$

$$= \sqrt{2}$$

$$\left\langle \vec{u}_2,\ \vec{q}_1 \right\rangle = (2,\ 1,\ 3) \bullet \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= \frac{3}{\sqrt{2}}$$

$$\left\langle \vec{u}_2,\ \vec{q}_2 \right\rangle = (2,\ 1,\ 3) \bullet \left(\frac{1}{\sqrt{38}},-\frac{1}{\sqrt{38}},\frac{6}{\sqrt{38}}\right)$$

$$
\begin{aligned}
&= \frac{2-1+18}{\sqrt{38}} \\
&= \frac{19}{\sqrt{38}} \\
&= \frac{19}{\sqrt{2}\sqrt{19}} \\
&= \boxed{\frac{\sqrt{19}}{\sqrt{2}}}
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \vec{u}_3,\ \vec{q}_1 \right\rangle &= (1,\ 1,\ 1) \bullet \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0 \right) \\
&= \frac{2}{\sqrt{2}} \\
&= \boxed{\sqrt{2}}
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \vec{u}_3,\ \vec{q}_2 \right\rangle &= (1,\ 1,\ 1) \bullet \left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}} \right) \\
&= \frac{1-1+6}{\sqrt{38}} \\
&= \frac{6}{\sqrt{2}\sqrt{19}} \frac{\sqrt{2}}{\sqrt{2}} \\
&= \boxed{\frac{3\sqrt{2}}{\sqrt{19}}}
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \vec{u}_3,\ \vec{q}_3 \right\rangle &= (1,\ 1,\ 1) \bullet \left(-\frac{3}{\sqrt{19}},\ \frac{3}{\sqrt{19}},\ \frac{1}{\sqrt{19}} \right) \\
&= \frac{-3+3+1}{\sqrt{19}} \\
&= \boxed{\frac{1}{\sqrt{19}}}
\end{aligned}
$$

$$
\begin{aligned}
R &= \begin{bmatrix} \left\langle \vec{u}_1,\ \vec{q}_1 \right\rangle & \left\langle \vec{u}_2,\ \vec{q}_1 \right\rangle & \left\langle \vec{u}_3,\ \vec{q}_1 \right\rangle \\ 0 & \left\langle \vec{u}_2,\ \vec{q}_2 \right\rangle & \left\langle \vec{u}_3,\ \vec{q}_2 \right\rangle \\ 0 & 0 & \left\langle \vec{u}_3,\ \vec{q}_3 \right\rangle \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix}
\end{aligned}
$$

The QR–decomposition of the matrix is

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix}$$

$$\boldsymbol{A} \qquad = \qquad \boldsymbol{Q} \qquad\qquad \boldsymbol{R}$$

e) $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \quad \begin{matrix} R_2 + R_1 \\ R_3 - R_1 \\ R_4 + R_1 \end{matrix}$

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \quad R_4 - R_2$

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The matrix is linearly dependent, so ***doesn't*** have a QR–decomposition.

Exercise

Verify that the Cauchy-Schwarz inequality holds for the given vectors using the Euclidean inner product

$$\vec{u} = (0,\ -2,\ 2,\ 1), \quad \vec{v} = (-1,\ -1,\ 1,\ 1)$$

Solution

$$\langle \vec{u},\ \vec{v} \rangle = 0 - 2(-1) + 2(1) + 1(1)$$
$$= 5$$

$$\|\langle \vec{u},\ \vec{v} \rangle\| = \sqrt{5}$$

$$\|\vec{u}\|\ \|\vec{v}\| = \sqrt{0+4+4+1}\ \sqrt{1+1+1+1}$$
$$= \sqrt{9}\sqrt{4}$$
$$= 6$$

$$\sqrt{5} < 6 \quad \Rightarrow \quad \|\langle \vec{u},\ \vec{v} \rangle\| \le \|\vec{u}\|\ \|\vec{v}\|$$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1,\ 1]$ spanned by the functions, using the inner product $f_1(x)=x+2, \quad f_2(x)=x^2-3x+4$

Solution

Let $\vec{u}_1 = f_1 = x+2, \quad \vec{u}_2 = f_2 = x^2-3x+4$

$$\boxed{\vec{v}_1 = \vec{u}_1 = x+2}$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^{1} (x+2)^2\, dx$$

$$= \frac{1}{3}(x+2)^3 \Big|_{-1}^{1}$$

$$= \frac{1}{3}(27-1)$$

$$= \boxed{\frac{26}{3}}$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_{-1}^{1} \left(x^2-3x+4\right)(x+2)\ dx$$

$$= \int_{-1}^{1} \left(x^3-x^2-2x+8\right)\ dx$$

$$= \left(\frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 + 8x\right)\Big|_{-1}^{1}$$

$$= \frac{1}{4} - \frac{1}{3} - 1 + 8 - \frac{1}{4} - \frac{1}{3} + 1 + 8$$

$$= \boxed{\frac{46}{3}}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$\vec{v}_2 = x^2 - 3x + 4 - \frac{46}{3}\left(\frac{3}{26}\right)(x+2)$$

$$= x^2 - 3x + 4 - \frac{23}{13}x - \frac{46}{13}$$

$$\boxed{= x^2 - \frac{62}{13}x + \frac{6}{13}}$$

The orthogonal basis is $\left\{ x+2, \quad x^2 - \frac{62}{13}x + \frac{6}{13} \right\}$

$$\left\langle \vec{v}_2, \vec{v}_2 \right\rangle = \int_{-1}^{1} \left(x^2 - \frac{62}{13}x + \frac{6}{13} \right)^2 \, dx$$

$$= \frac{1}{169} \int_{-1}^{1} \left(13x^2 - 62x + 6 \right)^2 \, dx$$

$$= \frac{1}{169} \int_{-1}^{1} \left(169x^4 + 3,844x^2 + 36 - 1,612x^3 + 156x^2 - 744x \right) dx$$

$$= \frac{1}{169} \left(\frac{169}{5}x^5 + \frac{4,000}{3}x^3 + 36x - 403x^4 - 372x^2 \right) \Big|_{-1}^{1}$$

$$= \frac{1}{169} \left(\frac{169}{5} + \frac{4,000}{3} + 36 - 403 - 372 + \frac{169}{5} + \frac{4,000}{3} + 36 + 403 + 372 \right)$$

$$= \frac{1}{169} \left(\frac{338}{5} + \frac{8,000}{3} + 72 \right)$$

$$= \boxed{\frac{3,238}{195}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\left\| \vec{v}_1 \right\|}$$

$$= \boxed{\frac{\sqrt{3}}{\sqrt{26}} \left(x + 2 \right)}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\left\| \vec{v}_2 \right\|}$$

$$= \boxed{\sqrt{\frac{195}{3238}} \left(x^2 - \frac{62}{13}x + \frac{6}{13} \right)}$$

The orthonormal basis is $\left\{ \frac{\sqrt{3}}{\sqrt{26}} \left(x + 2 \right), \ \sqrt{\frac{195}{3238}} \left(x^2 - \frac{62}{13}x + \frac{6}{13} \right) \right\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1,\ 1]$ spanned by the functions, using the inner product $f_1(x)=x, \quad f_2(x)=x^3, \quad f_3(x)=x^5$

Solution

Let $\vec{u}_1=f_1=x, \quad \vec{u}_2=f_2=x^3, \quad \vec{u}_3=f_3=x^5$

$$\boxed{\vec{v}_1=\vec{u}_1=x}$$

$$\begin{aligned}\langle\vec{v}_1,\ \vec{v}_1\rangle &= \int_{-1}^{1} x^2 dx \\ &= \frac{1}{3}x^3\Big|_{-1}^{1} \\ &= \frac{2}{3}\end{aligned}$$

$$\begin{aligned}\langle\vec{u}_2,\ \vec{v}_1\rangle &= \int_{-1}^{1} x^4\ dx \\ &= \frac{1}{5}x^5\Big|_{-1}^{1} \\ &= \frac{2}{5}\end{aligned}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2-\frac{\langle\vec{u}_2,\ \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 \\ &= x^3-\frac{2}{5}\left(\frac{3}{2}\right)(x) \\ &= x^3-\frac{3}{5}x\end{aligned}$$

$$\begin{aligned}\langle\vec{v}_2,\ \vec{v}_2\rangle &= \int_{-1}^{1}\left(x^3-\frac{3}{5}x\right)^2\ dx \\ &= \int_{-1}^{1}\left(x^6-\frac{6}{5}x^4+\frac{9}{25}x^2\right)\ dx \\ &= \left(\frac{1}{7}x^7-\frac{6}{25}x^5+\frac{3}{25}x^3\right)\Big|_{-1}^{1} \\ &= 2\left(\frac{1}{7}-\frac{6}{25}+\frac{3}{25}\right)\end{aligned}$$

$$= \boxed{\frac{8}{175}}$$

$$\left\langle \vec{u}_3, \vec{v}_1 \right\rangle = \int_{-1}^{1} x^6 \, dx$$

$$= \frac{1}{7} x^7 \Big|_{-1}^{1}$$

$$= \boxed{\frac{2}{7}}$$

$$\left\langle \vec{u}_3, \vec{v}_2 \right\rangle = \int_{-1}^{1} x^5 \left(x^3 - \frac{3}{5} x \right) dx$$

$$= \int_{-1}^{1} \left(x^8 - \frac{3}{5} x^6 \right) dx$$

$$= \left(\frac{1}{9} x^9 - \frac{3}{35} x^7 \right) \Big|_{-1}^{1}$$

$$= 2\left(\frac{1}{9} - \frac{3}{35} \right)$$

$$= \boxed{\frac{16}{315}}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\left\langle \vec{u}_3, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1 - \frac{\left\langle \vec{u}_3, \vec{v}_2 \right\rangle}{\left\| \vec{v}_2 \right\|^2} \vec{v}_2$$

$$= x^5 - \frac{16}{315}\left(\frac{175}{8} \right)\left(x^3 - \frac{3}{5} x \right) - \frac{2}{7}\left(\frac{3}{2} \right) x$$

$$= x^5 - \frac{70}{63}\left(x^3 - \frac{3}{5} x \right) - \frac{3}{7} x$$

$$= x^5 - \frac{70}{63} x^3 + \frac{14}{21} x - \frac{3}{7} x$$

$$= \boxed{x^5 - \frac{70}{63} x^3 + \frac{5}{21} x}$$

The orthogonal basis is $\left\{ x,\ x^3 - \frac{3}{5} x\ ,\ x^5 - \frac{70}{63} x^3 + \frac{5}{21} x \right\}$

$$\left\langle \vec{v}_3, \vec{v}_3 \right\rangle = \int_{-1}^{1} \left(x^5 - \frac{70}{63} x^3 + \frac{5}{21} x \right)^2 dx$$

$$= \int_{-1}^{1} \frac{1}{3{,}969} \left(63x^5 - 70x^3 + 15x \right)^2 dx$$

$$= \frac{1}{3,969}\int_{-1}^{1}\left(3,969x^{10} - 8,820x^{8} + 1,890x^{6} - 2,100x^{4} + 4,900x^{6} + 225x^{2}\right)dx$$

$$= \frac{1}{3,969}\left(\frac{3,969}{11}x^{11} - 980x^{9} + 970x^{7} - 420x^{5} + 75x^{3}\right)\Big|_{-1}^{1}$$

$$= \frac{2}{3,969}\left(\frac{3,969}{11} - 980 + 970 - 420 + 75\right)$$

$$= \frac{2}{3,969}\left(\frac{3,969}{11} - 355\right)$$

$$= \frac{2}{3,969}\left(\frac{64}{11}\right)$$

$$= \boxed{\frac{128}{43,659}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \frac{x}{\sqrt{2/3}}$$

$$= \boxed{\frac{\sqrt{3}}{\sqrt{2}}x}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \sqrt{\frac{175}{8}}\left(x^3 - \frac{3}{5}x\right)$$

$$= \boxed{\frac{5\sqrt{7}}{2\sqrt{2}}\left(x^3 - \frac{3}{5}x\right)}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$= \sqrt{\frac{43,659}{128}}\left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x\right)$$

$$= \boxed{\frac{63\sqrt{11}}{8\sqrt{2}}\left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x\right)}$$

The orthonormal basis is $\left\{\frac{\sqrt{3}}{\sqrt{2}}x,\ \ \frac{5}{2}\sqrt{\frac{7}{2}}\left(x^3 - \frac{3}{5}x\right),\ \ \frac{63}{8}\sqrt{\frac{11}{2}}\left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x\right)\right\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1,\ 1]$ spanned by the functions, using the inner product $\quad f_1(x)=1, \quad f_2(x)=x, \quad f_3(x)=\frac{1}{2}\left(3x^2-1\right)$

Solution

Let $\quad \vec{u}_1=f_1=1, \quad \vec{u}_2=f_2=x, \quad \vec{u}_3=f_3=\frac{3}{2}x^2-\frac{1}{2}$

$$\vec{v}_1=\vec{u}_1=1$$

$$\left\langle \vec{v}_1,\ \vec{v}_1\right\rangle=\int_{-1}^{1}1\ dx$$

$$=x\Big|_{-1}^{1}$$

$$=2$$

$$\left\langle \vec{u}_2,\ \vec{v}_1\right\rangle=\int_{-1}^{1}x\ dx$$

$$=\frac{1}{2}x^2\Big|_{-1}^{1}$$

$$=0$$

$$\vec{v}_2=\vec{u}_2-\frac{\left\langle \vec{u}_2,\ \vec{v}_1\right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$=x-\frac{0}{2}(1)$$

$$=x$$

$$\left\langle \vec{v}_2,\ \vec{v}_2\right\rangle=\int_{-1}^{1}x^2\ dx$$

$$=\frac{1}{3}x^3\Big|_{-1}^{1}$$

$$=\frac{2}{3}$$

$$\left\langle \vec{u}_3,\ \vec{v}_1\right\rangle=\frac{1}{2}\int_{-1}^{1}\left(3x^2-1\right)\ dx$$

$$=\frac{1}{2}\left(x^3-x\right)\Big|_{-1}^{1}$$

$$\underline{= 0}$$

$$\left\langle \vec{u}_3, \vec{v}_2 \right\rangle = \frac{1}{2}\int_{-1}^{1} x\left(3x^2-1\right)\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(3x^3-x\right)\, dx$$

$$= \frac{1}{2}\left(\frac{3}{4}x^4-\frac{1}{2}x^2\right)\Big|_{-1}^{1}$$

$$\underline{= 0}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\left\langle \vec{u}_3, \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1 - \frac{\left\langle \vec{u}_3, \vec{v}_2 \right\rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2$$

$$= \frac{3}{2}x^2 - \frac{1}{2} - \frac{0}{1}(1) - \frac{0}{\frac{2}{3}}(x)$$

$$\boxed{= \frac{1}{2}\left(3x^2-1\right)}$$

The orthogonal basis is $\left\{1,\ x\ ,\ \ \frac{1}{2}\left(3x^2-1\right)\ \right\}$

$$\left\langle \vec{v}_3, \vec{v}_3 \right\rangle = \frac{1}{4}\int_{-1}^{1} \left(3x^2-1\right)^2\, dx$$

$$= \frac{1}{4}\int_{-1}^{1} \left(9x^4-6x^2+1\right)\, dx$$

$$= \frac{1}{4}\left(\frac{9}{5}x^5-2x^3+x\right)\Big|_{-1}^{1}$$

$$= \frac{1}{2}\left(\frac{9}{5}-2+1\right)$$

$$\boxed{= \frac{2}{5}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$\boxed{= \frac{1}{\sqrt{2}}}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$\boxed{= \sqrt{\frac{3}{2}}\, x}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{1}{\sqrt{\frac{2}{5}}}\frac{1}{2}\left(3x^2 - 1\right)$$

$$\boxed{= \frac{1}{2}\sqrt{\frac{5}{2}}\left(3x^2 - 1\right)}$$

The orthonormal basis is $\left\{\frac{1}{\sqrt{2}},\ \sqrt{\frac{3}{2}}x\ ,\ \frac{1}{2}\sqrt{\frac{5}{2}}\left(3x^2 - 1\right)\right\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1,\ 1]$ spanned by the functions, using the inner product $\quad f_1(x) = 1,\quad f_2(x) = \sin \pi x,\quad f_3(x) = \cos \pi x$

Solution

Let $\quad \vec{u}_1 = f_1 = 1,\quad \vec{u}_2 = f_2 = \sin \pi x,\quad \vec{u}_3 = f_3 = \cos \pi x$

$$\boxed{\vec{v}_1 = \vec{u}_1 = 1}$$

$$\left\langle \vec{v}_1,\ \vec{v}_1 \right\rangle = \int_{-1}^{1} 1dx$$

$$= x\Big|_{-1}^{1}$$

$$\boxed{= 2}$$

$$\left\langle \vec{u}_2,\ \vec{v}_1 \right\rangle = \int_{-1}^{1} \sin \pi x\ dx$$

$$= -\frac{1}{\pi}\cos \pi x\Big|_{-1}^{1}$$

$$\boxed{= 0}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2,\ \vec{v}_1 \right\rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$\boxed{= \sin \pi x}$$

$$\left\langle \vec{v}_2,\ \vec{v}_2 \right\rangle = \int_{-1}^{1} \sin^2 \pi x\ dx$$

$$=\frac{1}{2}\int_{-1}^{1}(1-\cos 2\pi x)\,dx$$

$$=\frac{1}{2}\left(x-\frac{1}{2\pi}\sin 2\pi x\right)\Big|_{-1}^{1}$$

$$\underline{=1}$$

$$\langle \vec{u}_3, \vec{v}_1\rangle = \int_{-1}^{1}\cos \pi x\,dx$$

$$=\frac{1}{\pi}\sin \pi x\Big|_{-1}^{1}$$

$$\underline{=0}$$

$$\langle \vec{u}_3, \vec{v}_2\rangle = \int_{-1}^{1}\cos \pi x \sin \pi x\,dx$$

$$=\frac{1}{2}\int_{-1}^{1}\sin 2\pi x\,dx$$

$$=-\frac{1}{4\pi}\cos 2\pi x\Big|_{-1}^{1}$$

$$\underline{=0}$$

$$\vec{v}_3=\vec{u}_3-\frac{\langle \vec{u}_3, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1-\frac{\langle \vec{u}_3, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$=\cos \pi x-\frac{0}{\;}-\frac{0}{\;}$$

$$\underline{=\cos \pi x}$$

The orthogonal basis is $\left\{1,\ \sin \pi x-\frac{1}{\pi},\ \cos \pi x\right\}$

$$\langle \vec{v}_3, \vec{v}_3\rangle = \int_{-1}^{1}\cos^2 \pi x\,dx$$

$$=\frac{1}{2}\int_{-1}^{1}(1+\cos 2\pi x)\,dx$$

$$=\frac{1}{2}\left(x+\frac{1}{2\pi}\sin 2\pi x\right)\Big|_{-1}^{1}$$

$$\underline{=1}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{2}}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \sin \pi x$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \cos \pi x$$

The orthonormal basis is $\left\{ \frac{1}{\sqrt{2}}, \ \sin \pi x, \ \cos \pi x \right\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $\quad f_1(x) = \sin \pi x, \quad f_2(x) = \sin 2\pi x, \quad f_3(x) = \sin 3\pi x$

Solution

Let $\vec{u}_1 = f_1 = \sin \pi x, \quad \vec{u}_2 = f_2 = \sin 2\pi x, \quad \vec{u}_3 = f_3 = \sin 3\pi x$

$$\vec{v}_1 = \vec{u}_1 = \sin \pi x$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^{1} \sin^2 \pi x \, dx$$

$$= \frac{1}{2} \int_{-1}^{1} (1 - \cos 2\pi x) \, dx$$

$$= \frac{1}{2} \left(x - \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^{1}$$

$$= 1$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_{-1}^{1} \sin \pi x \sin 2\pi x \, dx \qquad \sin a \sin b = \frac{1}{2}\left[\cos(a+b) - \cos(a-b)\right]$$

$$= \frac{1}{2} \int_{-1}^{1} \left(\cos 3\pi x - \cos(-\pi x)\right) \, dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(\cos 3\pi x - \cos \pi x\right)\, dx$$

$$= \frac{1}{2}\left(\frac{1}{3\pi}\sin 3\pi x - \frac{1}{\pi}\sin \pi x\right) \Big|_{-1}^{1}$$

$$\underline{= 0}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$\underline{= \sin 2\pi x}$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^{1} \sin^2 2\pi x\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(1 - \cos 4\pi x\right)\, dx$$

$$= \frac{1}{2}\left(x - \frac{1}{4\pi}\sin 4\pi x\right) \Big|_{-1}^{1}$$

$$\underline{= 1}$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_{-1}^{1} \sin \pi x \sin 3\pi x\, dx \qquad \sin a \sin b = \frac{1}{2}\left[\cos(a+b) - \cos(a-b)\right]$$

$$= \frac{1}{2}\int_{-1}^{1} \left(\cos 4\pi x - \cos(-2\pi x)\right)\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(\cos 4\pi x - \cos 2\pi x\right)\, dx$$

$$= \frac{1}{2}\left(\frac{1}{4\pi}\sin 4\pi x - \frac{1}{2\pi}\sin 2\pi x\right) \Big|_{-1}^{1}$$

$$\underline{= 0}$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^{1} \sin 3\pi x \sin 2\pi x\, dx \qquad \sin a \sin b = \frac{1}{2}\left[\cos(a+b) - \cos(a-b)\right]$$

$$= \frac{1}{2}\int_{-1}^{1} \left(\cos 5\pi x - \cos \pi x\right)\, dx$$

$$= \frac{1}{2}\left(\frac{1}{5\pi}\sin 5\pi x - \frac{1}{\pi}\sin \pi x\right) \Big|_{-1}^{1}$$

$$\underline{= 0}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= \sin 3\pi x$$

The orthogonal basis is $\{\sin \pi x, \ \sin 2\pi x, \ \sin 3\pi x\}$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_{-1}^{1} \sin^2 3\pi x \, dx$$

$$= \frac{1}{2}\int_{-1}^{1} (1 - \cos 6\pi x) \, dx$$

$$= \frac{1}{2}\left(x - \frac{1}{6\pi}\sin 6\pi x\right) \Big|_{-1}^{1}$$

$$= 1$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \sin \pi x$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \sin 2\pi x$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \sin 3\pi x$$

The orthonormal basis is $\{\sin \pi x, \ \sin 2\pi x, \ \sin 3\pi x\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $\quad f_1(x) = \cos \pi x, \quad f_2(x) = \cos 2\pi x, \quad f_3(x) = \cos 3\pi x$

Solution

Let $\vec{u}_1 = f_1 = \cos \pi x, \quad \vec{u}_2 = f_2 = \cos 2\pi x, \quad \vec{u}_3 = f_3 = \cos 3\pi x$

$$\vec{v}_1 = \vec{u}_1 = \cos \pi x$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^{1} \cos^2 \pi x \, dx$$

$$= \frac{1}{2}\int_{-1}^{1} (1 + \cos 2\pi x) \, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^{1}$$

$$\underline{= 1}$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_{-1}^{1} \cos 2\pi x \cos \pi x \, dx \qquad \cos a \cos b = \frac{1}{2}\left[\cos(a+b) + \cos(a-b) \right]$$

$$= \frac{1}{2}\int_{-1}^{1} (\cos 3\pi x + \cos \pi x) \, dx$$

$$= \frac{1}{2}\left(\frac{1}{3\pi} \sin 3\pi x + \frac{1}{\pi} \sin \pi x \right) \Big|_{-1}^{1}$$

$$\underline{= 0}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1$$

$$\underline{= \cos 2\pi x}$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^{1} \cos^2 2\pi x \, dx$$

$$= \frac{1}{2}\int_{-1}^{1} (1 + \cos 4\pi x) \, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{4\pi} \sin 4\pi x \right) \Big|_{-1}^{1}$$

$$\underline{= 1}$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_{-1}^{1} \cos 3\pi x \cos \pi x \, dx \qquad \cos a \cos b = \frac{1}{2}\left[\cos(a+b) + \cos(a-b) \right]$$

$$= \frac{1}{2}\int_{-1}^{1} (\cos 4\pi x + \cos 2\pi x) \, dx$$

$$= \frac{1}{2}\left(\frac{1}{4\pi} \sin 4\pi x + \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^{1}$$

$$\underline{=0}$$

$$\left\langle \vec{u}_3,\ \vec{v}_2 \right\rangle = \int_{-1}^{1} \cos 3\pi x \cos 2\pi x\ dx \qquad \cos a \cos b = \frac{1}{2}\left[\cos(a+b)+\cos(a-b)\right]$$

$$= \frac{1}{2}\int_{-1}^{1} \left(\cos 5\pi x + \cos \pi x\right)\ dx$$

$$= \frac{1}{2}\left(\frac{1}{5\pi}\sin 5\pi x + \frac{1}{\pi}\sin \pi x\right)\Big|_{-1}^{1}$$

$$\underline{=0}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\left\langle \vec{u}_3,\ \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1 - \frac{\left\langle \vec{u}_3,\ \vec{v}_2 \right\rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2$$

$$\underline{= \cos 3\pi x}$$

The orthogonal basis is $\{\cos \pi x,\ \ \cos 2\pi x\ ,\ \ \cos 3\pi x\}$

$$\left\langle \vec{v}_3,\ \vec{v}_3 \right\rangle = \int_{-1}^{1} \cos^2 3\pi x\ dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(1 + \cos 6\pi x\right)\ dx$$

$$= \frac{1}{2}\left(x + \frac{1}{6\pi}\sin 6\pi x\right)\Big|_{-1}^{1}$$

$$\underline{=1}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$\underline{= \cos \pi x}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$\underline{= \cos 2\pi x}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$\underline{= \cos 3\pi x}$$

The orthonormal basis is $\{\cos \pi x,\ \ \cos 2\pi x\ ,\ \ \cos 3\pi x\}$

Exercise

For $\mathbb{P}_3[x]$, define the inner product over $\mathbb{R}$ as

$$\langle f,\ g\rangle = \int_{-1}^{1} f(x)g(x)\ dx$$

a) If $f(x)=1$ is a unit vector in $\mathbb{P}_3[x]$?

b) Find an orthonormal basis for the subspace spanned by $x\ and\ x^2$.

c) Complete the basis in part (*b*) to an orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product.

d) Is

$$[f,\ g] = \int_{0}^{1} f(x)g(x)\ dx$$

Also, an inner product for $\mathbb{P}_3[x]$

e) Find a pair of vectors $\vec{v}$ and $\vec{w}$ such that

$$\langle \vec{v},\ \vec{w}\rangle = 0 \quad but \quad [\vec{v},\ \vec{w}] \neq 0$$

f) Is the basis found in part (*c*) are orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product in part (*d*)?

Solution

a) $f(x)=1$

$$\begin{aligned}\langle f,\ f\rangle &= \int_{-1}^{1} f(x)\,f(x)\ dx\\ &= \int_{-1}^{1} dx\\ &= x\Big|_{-1}^{1}\\ &= 1+1\\ &= 2 \neq 1\end{aligned}$$

Therefore, when $f(x)=1$ is ***not*** a unit vector in $\mathbb{P}_3[x]$

b) Let $\vec{u}_1 = f = x,\quad \vec{u}_2 = g = x^2$

$$\vec{v}_1 = \vec{u}_1 = x$$

$$\left\langle \vec{v}_1, \vec{v}_1 \right\rangle = \int_{-1}^{1} x^2 \, dx$$

$$= \frac{1}{3} x^3 \Big|_{-1}^{1}$$

$$= \frac{1}{3}(1+1)$$

$$= \frac{2}{3}$$

$$\left\langle \vec{u}_2, \vec{v}_1 \right\rangle = \int_{-1}^{1} x^2 (x) \, dx$$

$$= \int_{-1}^{1} x^3 \, dx$$

$$= \frac{1}{4} x^4 \Big|_{-1}^{1}$$

$$= \frac{1}{4}(1-1)$$

$$= 0$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1$$

$$= x^2 - \frac{0}{*}(\quad)$$

$$= x^2$$

$$\left\langle \vec{v}_2, \vec{v}_2 \right\rangle = \int_{-1}^{1} x^4 \, dx$$

$$= \frac{1}{5} x^5 \Big|_{-1}^{1}$$

$$= \frac{1}{5}(1+1)$$

$$= \frac{2}{5}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\left\| \vec{v}_1 \right\|}$$

$$= \frac{x}{\sqrt{\frac{2}{3}}}$$

$$= \boxed{\sqrt{\frac{3}{2}}\, x}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{x^2}{\sqrt{\frac{2}{5}}}$$

$$= \boxed{\sqrt{\frac{5}{2}}\, x^2}$$

The orthonormal basis is $\left\{ \sqrt{\frac{3}{2}}\, x,\ \ \sqrt{\frac{5}{2}}\, x^2 \right\}$

c) Since $\vec{u}_1 = x, \quad \vec{u}_2 = x^2$ in $\mathbb{P}_3[x]$

Then, let $\vec{u}_3 = 1$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_{-1}^{1} (1)(x)\, dx$$

$$= \int_{-1}^{1} x\, dx$$

$$= \frac{1}{2}x^2 \Big|_{-1}^{1}$$

$$= \frac{1}{2}(1-1)$$

$$= \boxed{0}$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^{1} (1)\left(x^2\right) dx$$

$$= \int_{-1}^{1} x^2\, dx$$

$$= \frac{1}{3}x^3 \Big|_{-1}^{1}$$

$$= \frac{1}{3}(1+1)$$

$$= \boxed{\frac{2}{3}}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$=1-\frac{0}{\frac{2}{3}}(x)-\frac{\frac{2}{3}}{\frac{2}{5}}\left(x^2\right)$$

$$=1-\frac{5}{3}x^2$$

$$\left\langle \vec{v}_3,\ \vec{v}_3 \right\rangle = \int_{-1}^{1}\left(1-\frac{5}{3}x^2\right)^2\ dx$$

$$= \int_{-1}^{1}\left(1-\frac{10}{3}x^2+\frac{25}{9}x^4\right)\ dx$$

$$= \left(x-\frac{10}{9}x^3+\frac{5}{9}x^5\right)\Big|_{-1}^{1}$$

$$= 2\left(1-\frac{10}{9}+\frac{5}{9}\right)$$

$$= 2\left(\frac{9-5}{9}\right)$$

$$= \frac{8}{9}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$= \left(\sqrt{\frac{9}{8}}\right)\left(1-\frac{5}{3}x^2\right)$$

$$= \frac{3}{2\sqrt{2}}\left(1-\frac{5}{3}x^2\right)$$

$$= \frac{3}{2\sqrt{2}}-\frac{5}{2\sqrt{2}}\ x^2$$

The orienthormal basis is $\left\{ \sqrt{\frac{3}{2}}\ x,\ \ \sqrt{\frac{5}{2}}\ x^2,\ \ \frac{3}{2\sqrt{2}}-\frac{5}{2\sqrt{2}}\ x^2 \right\}$

d) $[f,\ g] = \int_{0}^{1} f(x)g(x)\ dx$

Let $\vec{u}_1 = 1,\quad \vec{u}_2 = x,\quad \vec{u}_3 = x^2$

$$\vec{v}_1 = \vec{u}_1 = 1$$

$$\left\langle \vec{v}_1,\ \vec{v}_1 \right\rangle = \int_{0}^{1} 1\ dx$$

$$= x \Big|_0^1$$

$$\boxed{= 1}$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_0^1 x(1)\, dx$$

$$= \int_0^1 x\, dx$$

$$= \frac{1}{2}x^2 \Big|_0^1$$

$$\boxed{= \frac{1}{2}}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= x - \frac{\frac{1}{2}}{1}(1)$$

$$\boxed{= x - \frac{1}{2}}$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_0^1 \left(x - \frac{1}{2}\right)^2 dx$$

$$= \int_0^1 \left(x - \frac{1}{2}\right)^2 d\left(x - \frac{1}{2}\right)$$

$$= \frac{1}{3}\left(x - \frac{1}{2}\right)^3 \Big|_0^1$$

$$= \frac{1}{3}\left(\left(\frac{1}{2}\right)^3 - \left(-\frac{1}{2}\right)^3\right)$$

$$= \frac{1}{3}\left(\frac{1}{8} + \frac{1}{8}\right)$$

$$\boxed{= \frac{1}{12}}$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_0^1 \left(x^2\right)(1)\, dx$$

$$= \int_0^1 x^2\, dx$$

$$= \frac{1}{3}x^3 \Big|_0^1$$

$$\boxed{= \frac{1}{3}}$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_0^1 \left(x^2\right)\left(x - \frac{1}{2}\right) dx$$

$$= \int_0^1 \left(x^3 - \frac{1}{2}x^2\right) dx$$

$$= \left(\frac{1}{4}x^4 - \frac{1}{6}x^3\right) \Big|_0^1$$

$$= \frac{1}{4} - \frac{1}{6}$$

$$\boxed{= \frac{1}{12}}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= x^2 - \frac{\frac{1}{3}}{1}(1) - \frac{\frac{1}{12}}{\frac{1}{12}}\left(x - \frac{1}{2}\right)$$

$$= x^2 - \frac{1}{3} - x + \frac{1}{2}$$

$$\boxed{= x^2 - x + \frac{1}{6}}$$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 dx$$

$$= \int_0^1 \left(x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36}\right) dx$$

$$= \left(\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{4}{9}x^3 - \frac{1}{6}x^2 + \frac{1}{36}x\right) \Big|_0^1$$

$$= \frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36}$$

$$= \frac{2-5}{10} + \frac{16-6+1}{36}$$

$$= -\frac{3}{10} + \frac{11}{36}$$

$$= \frac{-108+110}{360}$$

$$= \frac{1}{180}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= 1$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{x - \frac{1}{2}}{\sqrt{\frac{1}{12}}}$$

$$= 2\sqrt{3}\left(x - \frac{1}{2}\right)$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \left(\sqrt{180}\right)\left(x^2 - x + \frac{1}{6}\right)$$

$$= \left(6\sqrt{5}\right)\left(x^2 - x + \frac{1}{6}\right)$$

$$= 6\sqrt{5}\,x^2 - 6\sqrt{5}\,x + \sqrt{5}$$

The orthonormal basis is $\left\{ 1,\ 2\sqrt{3}\left(x - \frac{1}{2}\right),\ \sqrt{5}\left(6x^2 - 6x + 1\right) \right\}$

Therefore, $[f, g] = \int_0^1 f(x)\,g(x)\,dx$ is an inner product for $\mathbb{P}_3[x]$

e) Let assume: $\vec{v} = 1$ *and* $\vec{w} = x$

$$\langle \vec{v}, \vec{w} \rangle = \int_{-1}^{1} 1(x)\,dx$$

$$= \int_{-1}^{1} x\,dx$$

$$= \frac{1}{2}x^2 \Big|_{-1}^{1}$$

$$= \frac{1}{2}(1 - 1)$$

$$= 0 \quad \checkmark$$

$$[\vec{v},\ \vec{w}] = \int_0^1 1(x)\, dx$$

$$= \frac{1}{2}x^2 \Big|_0^1$$

$$\boxed{= \frac{1}{2} \neq 0} \quad \checkmark$$

f) The orthonormal basis in part (c) $\left\{ \sqrt{\frac{3}{2}}\, x,\ \sqrt{\frac{5}{2}}\, x^2,\ \frac{3}{2\sqrt{2}} - \frac{5}{2\sqrt{2}}\, x^2 \right\}$ are ***not*** the same as

the orthonormal basis in part (d) $\left\{ 1,\ 2\sqrt{3}\left(x - \frac{1}{2}\right),\ \sqrt{5}\left(6x^2 - 6x + 1\right) \right\}$

Solution Section 3.4 – Orthogonal Matrices

Exercise

Show that the matrix is orthogonal $A=\begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$

Solution

$$AA^T = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}\begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$A^T A = \begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix}\begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$AA^T = A^T A = I$

$\therefore$ ***A*** is an orthogonal

Exercise

Show that the matrix is orthogonal $A=\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

Solution

$$AA^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$A^T A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$AA^T = A^T A = I$$

$\therefore$ ***A*** is an orthogonal.

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is orthogonal with inverse $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Solution

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ is orthogonal with inverse $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ (It is a standard matrix for a rotation of 45°)

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Solution

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$\therefore \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is orthogonal with inverse $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Solution

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}^T = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$\therefore \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ is orthogonal with an inverse $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 7 \\ 1 & 3 & -5 \\ -1 & 4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & & \\ & & \\ & & \end{pmatrix} \neq I$$

$\therefore \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$ is not an orthogonal

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Solution

$$\begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}^T = \begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2} & & \\ & & \\ & & \end{pmatrix} \neq I$$

Or $\left\|r_1\right\| = \sqrt{0+1^2+\left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{3}{2}} \underline{\neq 1}$ $\quad \therefore$ ***A*** is ***not*** orthogonal

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Solution

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\therefore \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \text{is orthogonal with inverse} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

Solution

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\therefore \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \text{is orthogonal with inverse} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

Solution

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\therefore \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix} \text{ is orthogonal with inverse } \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{bmatrix}$$

Solution

$$\left\|\boldsymbol{r}_2\right\| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{1}{3} + \frac{1}{4}}$$

$$= \sqrt{\frac{7}{12}} \neq 1$$

Or

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{5}{6} & & \\ & & & \\ & & & \end{pmatrix} \neq I$$

∴ The matrix is ***not*** an orthogonal

Exercise

Find a last column so that the resulting matrix is orthogonal

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & \cdots \end{bmatrix}$$

Solution

$$\vec{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}^T$$

$$\left\|\vec{q}_1\right\| = \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}}$$

$$\underline{= 1}$$

$$\vec{q}_2 = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}^T$$

$$\left\|\vec{q}_2\right\| = \sqrt{\frac{1}{6}+\frac{1}{6}+\frac{4}{6}}$$

$$= 1$$

Let $\vec{q}_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$

$$\vec{q}_1 \bullet \vec{q}_3 = \frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y - \frac{1}{\sqrt{3}}z = 0 \quad \rightarrow \quad x + y - z = 0$$

$$\vec{q}_2 \bullet \vec{q}_3 = \frac{1}{\sqrt{6}}x + \frac{1}{\sqrt{6}}y - \frac{2}{\sqrt{6}}z = 0 \quad \rightarrow \quad x + y - 2z = 0$$

$$\begin{cases} x + y - z = 0 \\ x + y - 2z = 0 \end{cases} \rightarrow \quad z = 0 \quad and \quad x + y = 0 \Rightarrow x = -y$$

$$\vec{q}_3 = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}^T$$

Exercise

Determine if the given matrix is orthogonal. If it is, find its inverse

$$\begin{bmatrix} \frac{1}{9} & \frac{4}{5} & \frac{3}{7} \\ \frac{4}{9} & \frac{3}{5} & -\frac{2}{7} \\ \frac{8}{9} & -\frac{2}{5} & \frac{3}{7} \end{bmatrix}$$

Solution

$$\vec{q}_1 = \begin{bmatrix} \frac{1}{9} & \frac{4}{9} & \frac{8}{9} \end{bmatrix}^T \qquad \vec{q}_2 = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} & -\frac{2}{5} \end{bmatrix}^T \qquad \vec{q}_3 = \begin{bmatrix} \frac{3}{7} & -\frac{2}{7} & \frac{3}{7} \end{bmatrix}^T$$

$$\vec{q}_1 \bullet \vec{q}_2 = \frac{4}{45} + \frac{12}{45} - \frac{16}{45}$$

$$= 0$$

$$\vec{q}_1 \bullet \vec{q}_3 = \frac{3}{63} - \frac{8}{63} + \frac{24}{63}$$

$$= \frac{19}{63} \neq 0$$

$$\vec{q}_2 \bullet \vec{q}_3 = \frac{12}{35} - \frac{6}{35} + \frac{6}{35}$$

$$= \frac{12}{35} \neq 0$$

The given matrix is ***not*** orthogonal

Exercise

Prove that if A is orthogonal, then A^T is orthogonal.

Solution

Since A is orthogonal then $A^T = A^{-1}$ and $A = \left(A^T\right)^T$

Then $\left(A^T\right)^T A^T = AA^T = I \ \Rightarrow \ A^T$ is orthogonal

Another word, since A is orthogonal, then both column and row vectors of A form an orthonormal set.

A^T is just A with its row and column vectors are swapped.

The column vectors of A^T (which are the row vectors of A) and row vectors of A^T (which are the column vectors of A) form orthonormal sets, therefore A^T is orthogonal

Exercise

Prove that if A is orthogonal, then A^{-1} is orthogonal

Solution

Since A is orthogonal then $A^T = A^{-1}$ and $A = \left(A^{-1}\right)^{-1}$

$$\left(A^{-1}\right)^{-1} = \left(A^T\right)^{-1} \qquad A^T = A^{-1}$$

$$= \left(A^{-1}\right)^T$$

$\therefore \ A^{-1}$ is orthogonal

Exercise

Prove that if A and B are orthogonal, then AB is orthogonal

Solution

Since A is orthogonal then $A^T = A^{-1}$

and B is orthogonal then $B^T = B^{-1}$

$$(AB)^T = B^T A^T$$

$$= B^{-1}A^{-1}$$

$$= (AB)^{-1}$$

$\therefore \ AB$ is orthogonal

Exercise

Let Q be an $n \times n$ orthogonal matrix, and let A be an $n \times n$ matrix.

Show that $\det\left(QAQ^T\right) = \det(A)$

Solution

$$\det\left(QAQ^T\right) = \det(Q)\det(A)\det\left(Q^T\right)$$

$$= \det(A)\det\left(QQ^T\right) \qquad \text{Since } Q \text{ is an orthogonal matrix } \det\left(QQ^T\right) = \det(I)$$

$$= \det(A)\det(I)$$

$$= \det(A) \quad \checkmark$$

Exercise

Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$

a) Is matrix A an orthogonal matrix?

b) Let B be the matrix obtained by normalizing each row of A, find B.

c) Is B an orthogonal matrix?

d) Are the columns of B orthogonal?

Solution

a) $AA^T = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 7 \\ 1 & 3 & -5 \\ -1 & 4 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 3 & & \\ & & \\ & & \end{pmatrix} \neq I$$

$\therefore \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$ is ***not*** an orthogonal

b) $\|(1,\ 1,\ -1)\| = \sqrt{1+1+1}$

$$\underline{= \sqrt{3}}$$

$\|(1,\ 3,\ 4)\| = \sqrt{1+9+16}$

$$\underline{= \sqrt{26}}$$

$$\left\|(7,\ -5,\ 2)\right\| = \sqrt{49+25+4}$$
$$= \sqrt{78}$$

$$B = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{26}} & \frac{3}{\sqrt{26}} & \frac{4}{\sqrt{26}} \\ \frac{7}{\sqrt{78}} & -\frac{5}{\sqrt{78}} & \frac{2}{\sqrt{78}} \end{pmatrix}$$

c) *Yes*, since the rows are orthogonal with unit vectors.

$$BB^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{26}} & \frac{3}{\sqrt{26}} & \frac{4}{\sqrt{26}} \\ \frac{7}{\sqrt{78}} & -\frac{5}{\sqrt{78}} & \frac{2}{\sqrt{78}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{26}} & \frac{7}{\sqrt{78}} \\ \frac{1}{\sqrt{3}} & \frac{3}{\sqrt{26}} & -\frac{5}{\sqrt{78}} \\ -\frac{1}{\sqrt{3}} & \frac{4}{\sqrt{26}} & \frac{2}{\sqrt{78}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

d) *Yes*, since the rows of *B* form an orthonormal set of vectors.
Then, the column of *B* must form an orthonormal set.

$$\left\|\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{26}},\ \frac{7}{\sqrt{78}}\right)\right\| = \sqrt{\frac{1}{3}+\frac{1}{26}+\frac{49}{78}}$$
$$= \sqrt{\frac{26+3+49}{78}}$$
$$= \sqrt{\frac{78}{78}}$$
$$= 1$$

$$\left\|\left(\frac{1}{\sqrt{3}},\ \frac{3}{\sqrt{26}},\ -\frac{5}{\sqrt{78}}\right)\right\| = \sqrt{\frac{1}{3}+\frac{9}{26}+\frac{25}{78}}$$
$$= \sqrt{\frac{26+27+25}{78}}$$
$$= \sqrt{\frac{78}{78}}$$
$$= 1$$

$$\left\|\left(-\frac{1}{\sqrt{3}},\ \frac{4}{\sqrt{26}},\ \frac{2}{\sqrt{78}}\right)\right\| = \sqrt{\frac{1}{3}+\frac{16}{26}+\frac{4}{78}}$$

$$= \sqrt{\frac{78}{78}}$$

$$\underline{= 1}$$

Solution   *Section* 3.5 – Least Squares Analysis

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(0,\ 2),\ (1,\ 2),\ (2,\ 0)\}$$

Solution

$$\{(0,\ 2),\ (1,\ 2),\ (2,\ 0)\}$$

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

where $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} m \\ b \end{bmatrix}$ $\vec{y} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

The normal equation formula: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$m = \frac{\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 3 & 3 \end{vmatrix}}$$

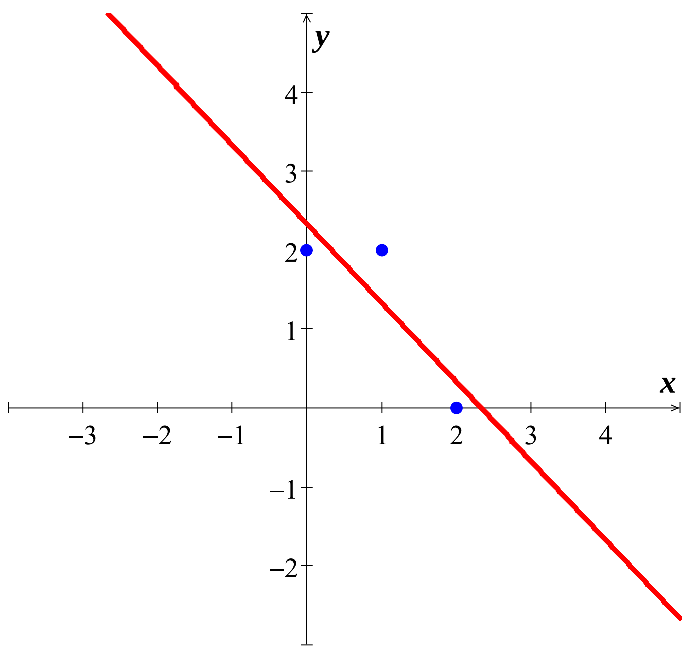
$$= \frac{-6}{6}$$

$$= -1$$

$$b = \frac{\begin{vmatrix} 5 & 2 \\ 3 & 4 \end{vmatrix}}{6}$$

$$= \frac{7}{3}$$

Thus, $y = -x + \frac{7}{3}$

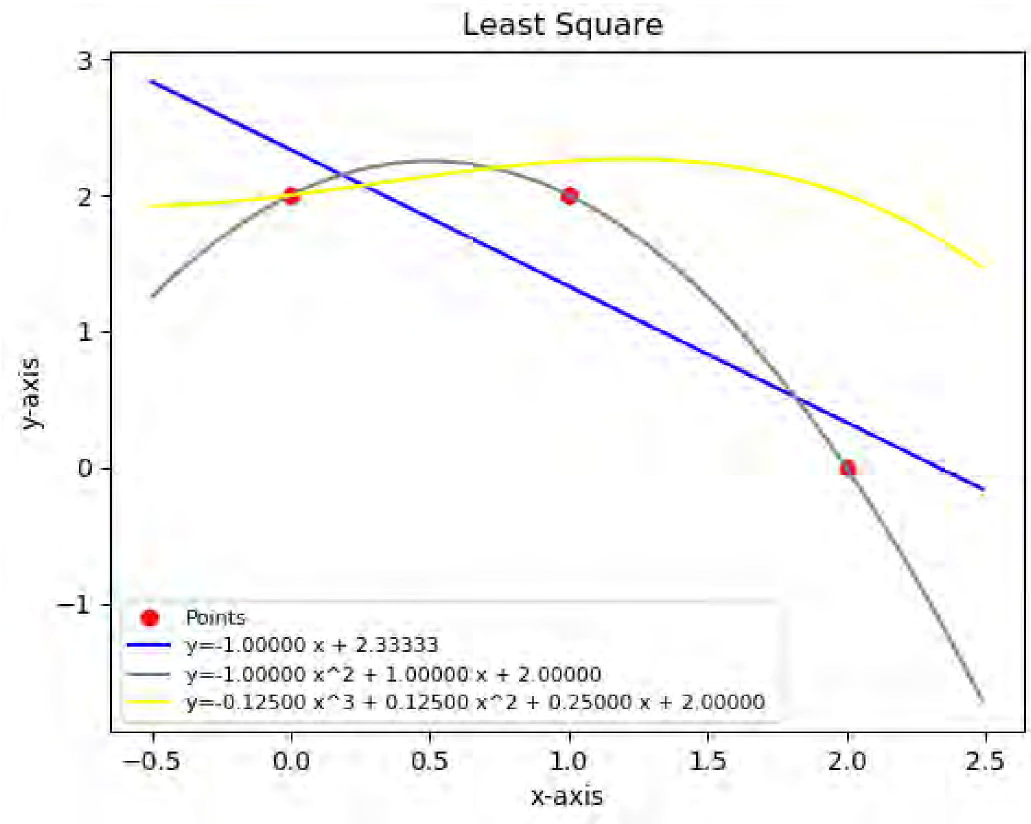


$$A\vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ \frac{7}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{7}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{7}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$$



Error:

$$\|\vec{y} - A\vec{x}\| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{1}{9}}$$

$$= \frac{\sqrt{6}}{3}$$

$$\approx 0.8164966$$

****

The ***second*** *order* equation:

$$y = -x^2 + x + 2$$

Error = 0.00000

****

The ***third order*** equation:

$$y = -.1250x^3 - 0.1250x^2 + 0.25x + 2$$

Error = 2.01556

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(1,\ 5),\ (2,\ 4),\ (3,\ 1),\ (4,\ 1),\ (5,-1)\}$$

Solution

$$\{(1,\ 5),\ (2,\ 4),\ (3,\ 1),\ (4,\ 1),\ (5,-1)\}$$

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix} \qquad \text{where} \quad A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} m \\ b \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

The normal equation: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 55 & 15 \\ 15 & 5 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \end{bmatrix}$$

$$m = \frac{\begin{vmatrix} 15 & 15 \\ 10 & 5 \end{vmatrix}}{\begin{vmatrix} 55 & 15 \\ 15 & 5 \end{vmatrix}}$$

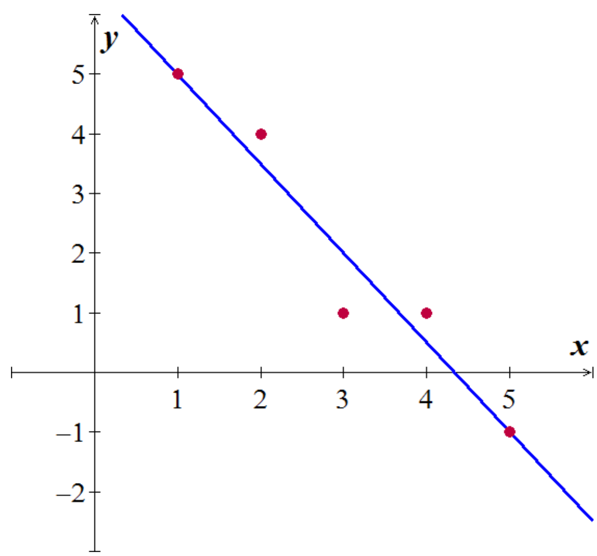
$$= \frac{-75}{50}$$

$$\boxed{= -\frac{3}{2}}$$

$$b = \frac{\begin{vmatrix} 55 & 15 \\ 15 & 10 \end{vmatrix}}{50}$$

$$= \frac{325}{50}$$

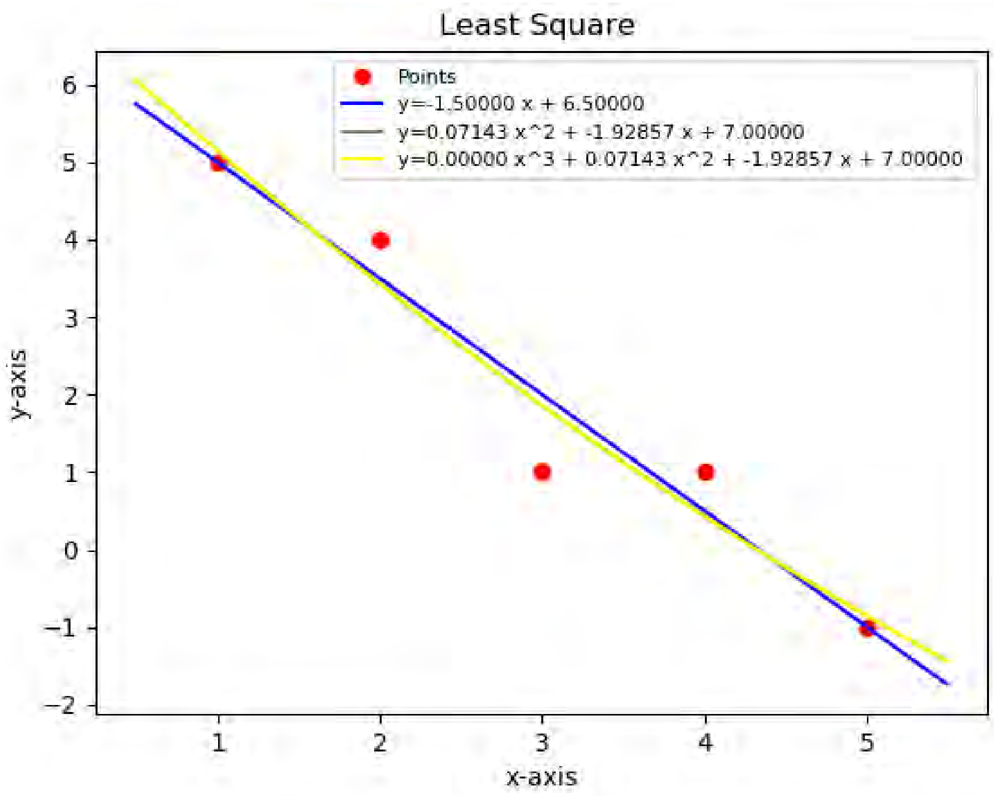
$$\boxed{= \frac{13}{2}}$$



Thus, $\boxed{y = -\frac{3}{2}x + \frac{13}{2}}$ *or* $\boxed{y = -1.5x + 6.5}$

$$A\vec{x} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ \frac{13}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ \frac{7}{2} \\ 2 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$



$$\vec{y} - A\vec{x} = \begin{pmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ \frac{7}{2} \\ 2 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \frac{1}{2} \\ -1 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

Error:

$$\|\vec{y} - A\vec{x}\| = \sqrt{\frac{1}{4} + 1 + \frac{1}{4}}$$

$$= \frac{\sqrt{6}}{2}$$

$$\approx 1.224745$$

The ***second order*** equation:

$$y = 0.07143x^2 - 1.92857x + 7$$

Error = 1.19523

****

The ***third order*** equation:

$$y = 0.0x^3 + 0.07143x^2 - 1.92857x + 7$$

Error = 1.19523

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

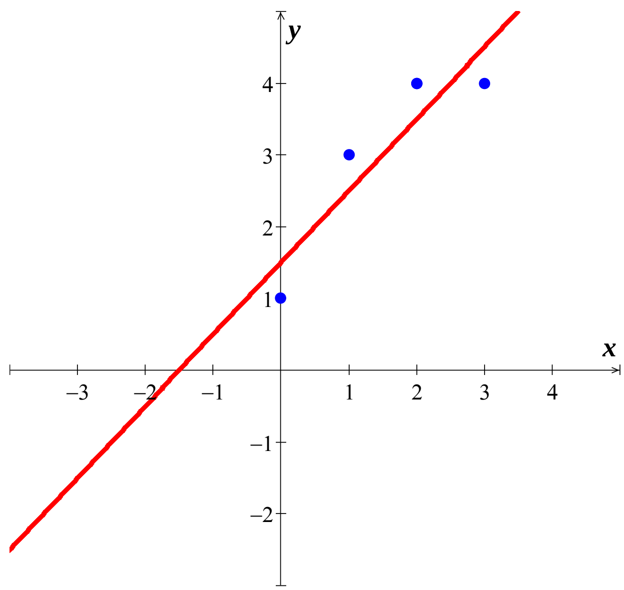
$$\{(0,\ 1),\ (1,\ 3),\ (2,\ 4),\ (3,\ 4)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

where $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$ $\quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$ $\quad \vec{y} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$



The normal equation: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 6 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 23 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{20}\begin{pmatrix} 4 & -6 \\ -6 & 14 \end{pmatrix}\begin{pmatrix} 23 \\ 12 \end{pmatrix} \qquad X = A^{-1}B$$

$$= \frac{1}{20}\begin{pmatrix} 20 \\ 30 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$$

We have: $m = 1 \quad and \quad b = \frac{3}{2}$.

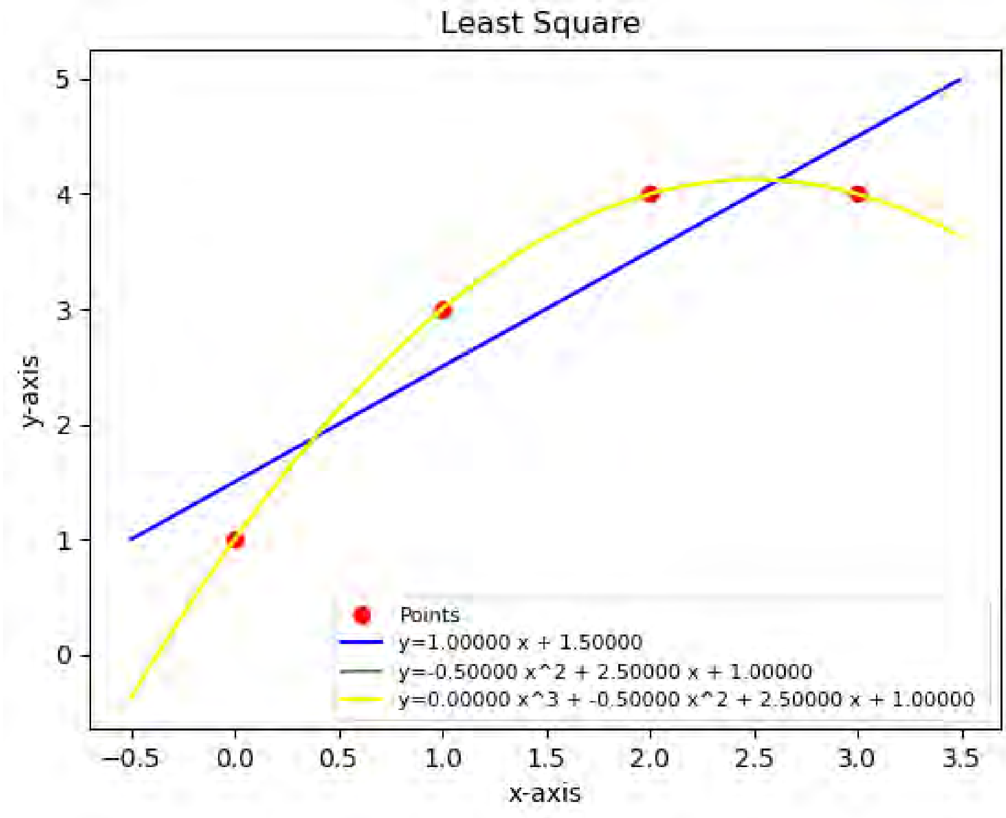
Thus, $\boxed{y = x + \frac{3}{2}}$

$$A\vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2} \\ \frac{5}{2} \\ \frac{7}{2} \\ \frac{9}{2} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} \frac{3}{2} \\ \frac{5}{2} \\ \frac{7}{2} \\ \frac{9}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$



Error:

$$\|\vec{y} - A\vec{x}\| = \sqrt{4\left(\frac{1}{4}\right)}$$

$$= 1$$

The ***second order*** equation:

$$y = -0.50x^2 + 2.5x + 1.0$$

Error = 0.0000

****

The ***third order*** equation:

$$y = 0.0x^3 - 0.5x^2 + 2.5x + 1$$

Error = 0.00000

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-2,\ 0),\ (-1,\ 0),\ (0,\ 1),\ (1,\ 3),\ (2,\ 5)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

where $A = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$

The normal equation: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

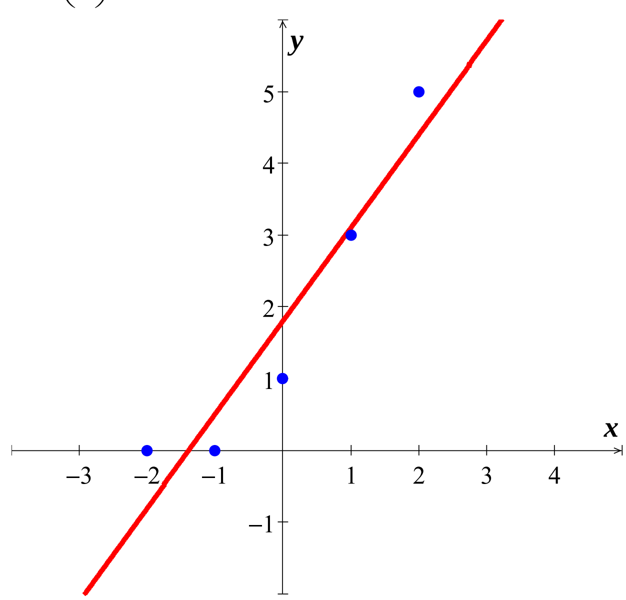
$$\begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 13 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{50} \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} 13 \\ 9 \end{pmatrix}$$

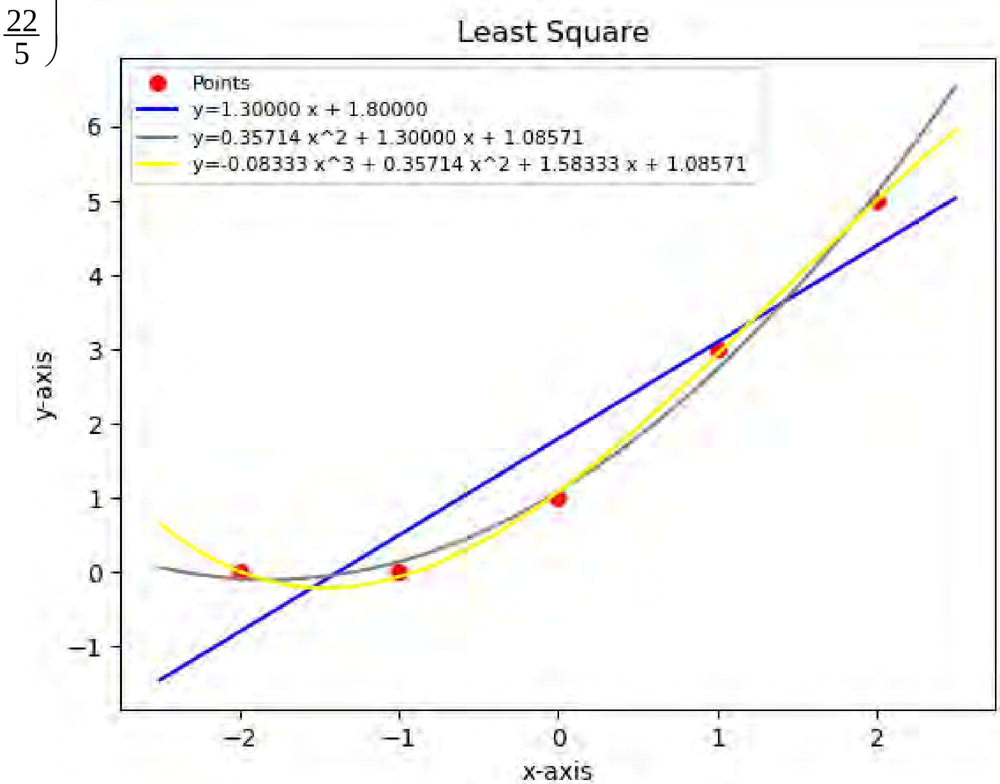
$$= \begin{pmatrix} \frac{13}{10} \\ \frac{9}{5} \end{pmatrix}$$

We have: $m = 1.3 \quad \textit{and} \quad b = 1.8$

Thus, $\boxed{y = \frac{13}{10}x + \frac{9}{5}}$



$$A\vec{x}=\begin{pmatrix}-2&1\\-1&1\\0&1\\1&1\\2&1\end{pmatrix}\begin{pmatrix}\frac{13}{10}\\\frac{9}{5}\end{pmatrix}=\begin{pmatrix}-\frac{4}{5}\\\frac{1}{2}\\\frac{9}{5}\\\frac{31}{10}\\\frac{22}{5}\end{pmatrix}$$



$$\vec{y}-A\vec{x}=\begin{pmatrix}0\\0\\1\\3\\5\end{pmatrix}-\begin{pmatrix}-\frac{4}{5}\\\frac{1}{2}\\\frac{9}{5}\\\frac{31}{10}\\\frac{22}{5}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{4}{5}\\-\frac{1}{2}\\-\frac{4}{5}\\-\frac{1}{10}\\\frac{3}{5}\end{pmatrix}$$

Error: $\|\vec{y}-A\vec{x}\|=\sqrt{\frac{16}{25}+\frac{1}{4}+\frac{16}{25}+\frac{1}{100}+\frac{9}{25}}$

$$=\sqrt{\frac{41}{25}+\frac{26}{100}}$$

$$=\frac{\sqrt{190}}{10}$$

$$\approx 1.37840$$

The *second order* equation:

$y=0.35714x^2+1.30x+1.08571$

Error = 0.33806

****

The *third order* equation:

$y=-0.08333x^3+0.35714x^2+1.58333x+1.08571$

Error = 0.11952

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(2,\ 3),\ (3,\ 2),\ (5,\ 1),\ (6,\ 0)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

where $A = \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix}$ $\quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$ $\quad \vec{y} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$

The normal equation: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 2 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 74 & 16 \\ 16 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 17 \\ 6 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 74 & 16 \\ 16 & 4 \end{vmatrix} = 40 \quad \Delta_m = \begin{vmatrix} 17 & 16 \\ 6 & 4 \end{vmatrix} = -28 \quad \Delta_b = \begin{vmatrix} 74 & 17 \\ 16 & 6 \end{vmatrix} = 172$$

$$\boxed{m = -\frac{28}{40} = -\frac{7}{10}}$$

$$\boxed{b = \frac{172}{40} = \frac{43}{10}}$$

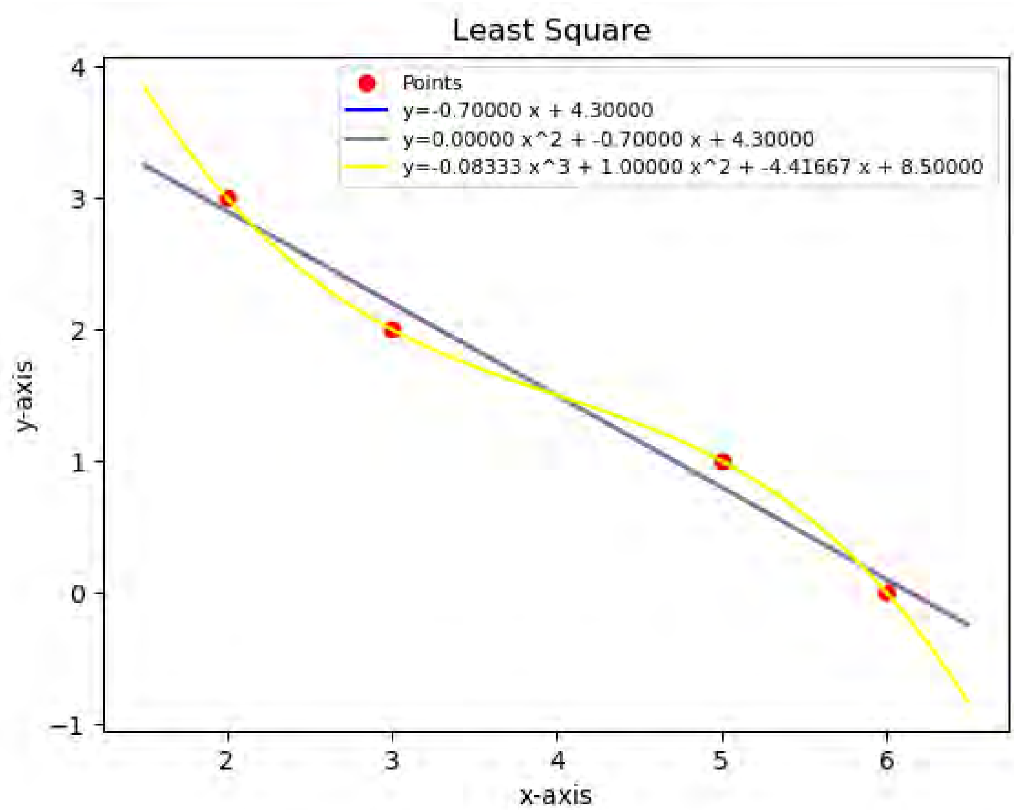
Thus, $\boxed{y = -\frac{7}{10}x + \frac{43}{10}}$

$$A\vec{x} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} -\frac{7}{10} \\ \frac{43}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{29}{10} \\ \frac{22}{10} \\ \frac{8}{10} \\ \frac{1}{10} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{29}{10} \\ \frac{22}{10} \\ \frac{8}{10} \\ \frac{1}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{10} \\ -\frac{1}{5} \\ \frac{1}{5} \\ -\frac{1}{10} \end{pmatrix}$$



Error:

$$\|\vec{y} - A\vec{x}\| = \sqrt{2\left(\frac{1}{100} + \frac{1}{25}\right)}$$

$$= \frac{\sqrt{10}}{10}$$

$$= 0.31623$$

****

The ***second order*** equation:

$$y = 0.0x^2 - 0.7x + 4.3$$

Error = 0.31623

****

The ***third order*** equation:

$$y = -0.08333x^3 + x^2 - 4.41667x + 8.5$$

Error = 0.00000

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-1,\ 0),\ (0,\ 1),\ (1,\ 2),\ (2,\ 4)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$$

where $A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$ $\quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$ $\quad \vec{y} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$

The normal equation: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 10 \\ 7 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 6 & 2 \\ 2 & 4 \end{vmatrix} = 20 \quad \Delta_m = \begin{vmatrix} 10 & 2 \\ 7 & 4 \end{vmatrix} = 26 \quad \Delta_b = \begin{vmatrix} 6 & 10 \\ 2 & 7 \end{vmatrix} = 22$$

$$m = \frac{26}{20} = \frac{13}{10}$$

$$b = \frac{22}{20} = \frac{11}{10}$$

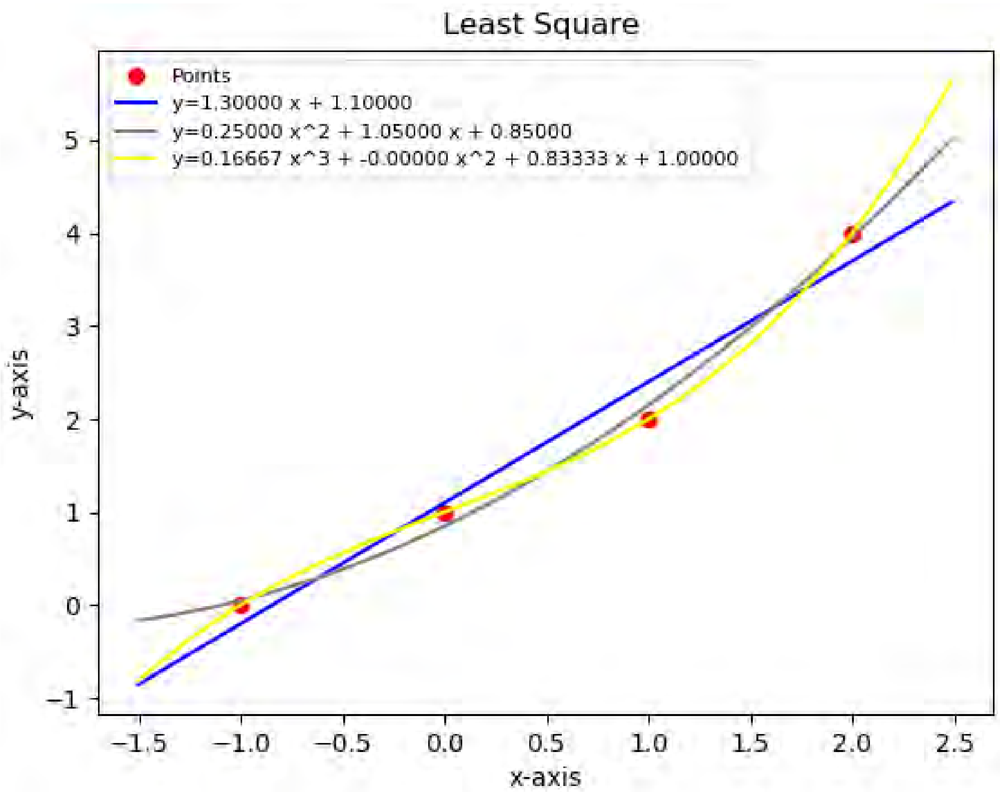
Thus, $y = \frac{13}{10}x + \frac{11}{10}$

$$A\vec{x} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \frac{13}{10} \\ \frac{11}{10} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{5} \\ \frac{11}{10} \\ \frac{12}{5} \\ \frac{37}{10} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} -\frac{1}{5} \\ \frac{11}{10} \\ \frac{12}{5} \\ \frac{37}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} \\ -\frac{1}{10} \\ -\frac{2}{5} \\ \frac{3}{10} \end{pmatrix}$$



Error:

$$\|\vec{y} - A\vec{x}\| = \sqrt{\frac{1}{25} + \frac{1}{100} + \frac{4}{25} + \frac{9}{100}}$$

$$= \sqrt{\frac{4+1+16+9}{100}}$$

$$= \frac{\sqrt{30}}{10}$$

$$= 0.54772$$

****

The ***second order*** equation:

$$y = 0.25x^2 + 1.05x + 0.85$$

Error = 0.22361

****

The ***third order*** equation:

$$y = 0.16667x^3 + 0.82222x + 1$$

Error = 0.00000

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(1,\ 0),\ (2,\ 1),\ (4,\ 2),\ (5,\ 3)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

where $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix}$ $\vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$ $\vec{y} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$

The normal equation: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 1 & 2 & 4 & 5 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 46 & 12 \\ 12 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 25 \\ 6 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 46 & 12 \\ 12 & 4 \end{vmatrix} = 40 \quad \Delta_m = \begin{vmatrix} 25 & 12 \\ 6 & 4 \end{vmatrix} = 28 \quad \Delta_b = \begin{vmatrix} 46 & 25 \\ 12 & 6 \end{vmatrix} = -24$$

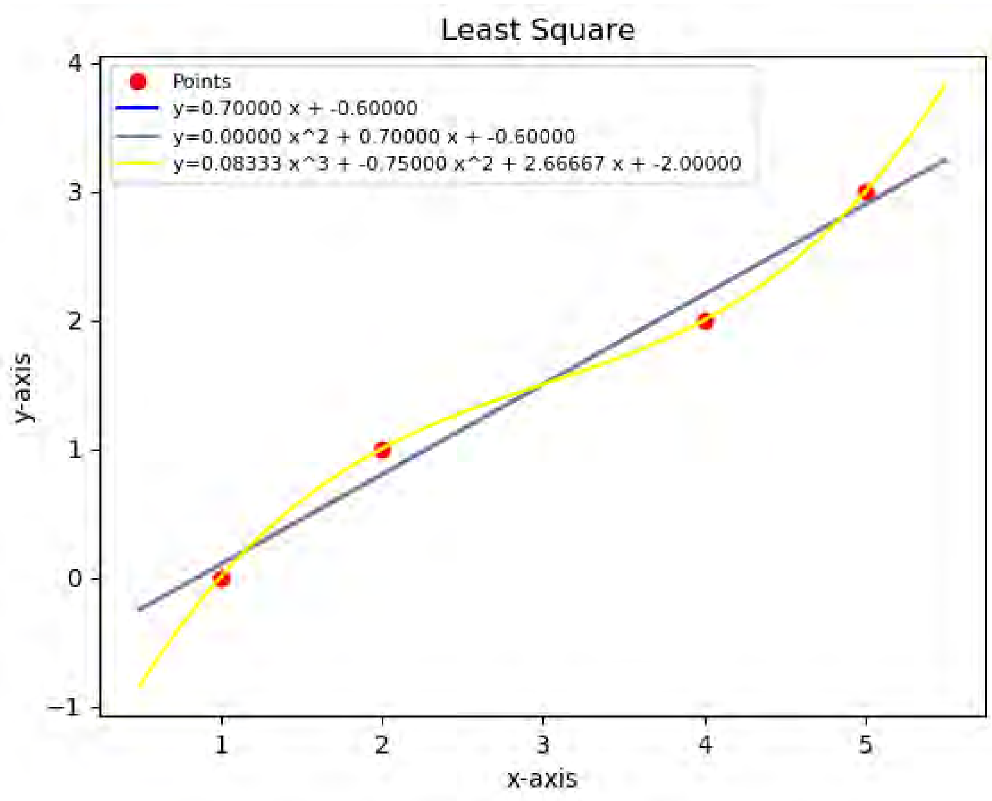
$$m = \frac{28}{40} = \frac{7}{10}$$

$$b = -\frac{24}{40} = -\frac{3}{5}$$

Thus, $y = \frac{7}{10}x - \frac{3}{5}$

$$A\vec{x} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} \frac{7}{10} \\ -\frac{3}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{10} \\ \frac{4}{5} \\ \frac{11}{5} \\ \frac{29}{10} \end{pmatrix}$$



$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} \frac{1}{10} \\ \frac{4}{5} \\ \frac{11}{5} \\ \frac{29}{10} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{10} \\ \frac{1}{5} \\ -\frac{1}{5} \\ \frac{1}{10} \end{pmatrix}$$

Error:

$$\|\vec{y} - A\vec{x}\| = \sqrt{2\left(\frac{1}{100} + \frac{1}{25}\right)}$$

$$= \frac{\sqrt{10}}{10}$$

$$= 0.31623$$

****

The ***second order*** equation:

$$y = 0.0x^2 + 0.7x - .6$$

Error = 0.31623

****

The ***third order*** equation:

$$y = 0.08333x^3 - 0.75x^2 + 2.66667x - 2$$

Error = 0.00000

Exercise

Find the orthogonal projection of the vector $\vec{u}$ on the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{u} = (-3,\ -3,\ 8,\ 9);\quad \vec{v}_1 = (3,\ 1,\ 0,\ 1),\quad \vec{v}_2 = (1,\ 2,\ 1,\ 1),\quad \vec{v}_3 = (-1,\ 0,\ 2,\ -1)$$

Solution

Let $A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{pmatrix}$$

$$A^T \vec{u} = \begin{pmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ -3 \\ 8 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 8 \\ 10 \end{pmatrix}$$

The normal solution is $A^T A\vec{x} = A^T \vec{u}$

$$\begin{pmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \\ 10 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{vmatrix} = 134 \qquad \Delta_1 = \begin{vmatrix} -3 & 6 & -4 \\ 8 & 7 & 0 \\ 10 & 0 & 6 \end{vmatrix} = -134 \qquad \Delta_2 = \begin{vmatrix} 11 & -3 & -4 \\ 6 & 8 & 0 \\ -4 & 10 & 6 \end{vmatrix} = 268$$

$$x_1 = \frac{-134}{134} = -1 \qquad x_2 = \frac{268}{134} = 2 \qquad x_3 = \frac{134}{134} = 1$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

So $proj_W \vec{u} = A\vec{x}$

$$= \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 3 \\ 4 \\ 0 \end{pmatrix}$$

$proj_W \vec{u} = (-2,\ 3,\ 4,\ 0)$

Exercise

Find the orthogonal projection of the vector $\vec{u}$ on the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{u} = (6,\ 3,\ 9,\ 6); \quad \vec{v}_1 = (2,\ 1,\ 1,\ 1), \quad \vec{v}_2 = (1,\ 0,\ 1,\ 1), \quad \vec{v}_3 = (-2,\ -1,\ 0,\ -1)$$

Solution

Let $A = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{pmatrix}$$

$$A^T \vec{u} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 9 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 30 \\ 21 \\ -21 \end{pmatrix}$$

The normal solution is $A^T A\vec{x} = A^T \vec{u}$

$$\begin{pmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 30 \\ 21 \\ -21 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{vmatrix} = 3 \qquad \Delta_1 = \begin{vmatrix} 30 & 4 & -6 \\ 21 & 3 & -3 \\ -21 & -3 & 6 \end{vmatrix} = 18 \qquad \Delta_2 = \begin{vmatrix} 7 & 30 & -6 \\ 4 & 21 & -3 \\ -6 & -21 & 6 \end{vmatrix} = 9$$

$$\underline{x_1 = \frac{18}{3} = 6} \quad \underline{x_2 = \frac{9}{3} = 3} \quad \underline{x_3 = 4}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$$

So $proj_W \vec{u} = A\vec{x}$

$$= \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 2 \\ 9 \\ 5 \end{pmatrix}$$

$$\underline{proj_W \vec{v} = (7,\ 2,\ 9,\ 5)}$$

Exercise

Find the orthogonal projection of the vector $\vec{u}$ on the subspace of $\mathbb{R}^4$ spanned by the vectors

$\vec{u} = (-2,\ 0,\ 2,\ 4);\quad v_1 = (1,\ 1,\ 3,\ 0),\quad \vec{v}_2 = (-2,\ -1,\ -2,\ 1),\quad \vec{v}_3 = (-3,\ -1,\ 1,\ 3)$

Solution

Let $A = \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 1 & 1 & 3 & 0 \\ -2 & -1 & -2 & 1 \\ -3 & -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{pmatrix}$$

$$A^T \vec{u} = \begin{pmatrix} 1 & 1 & 3 & 0 \\ -2 & -1 & -2 & 1 \\ -3 & -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 4 \\ 20 \end{pmatrix}$$

The normal solution is $A^T A\vec{x} = A^T \vec{u}$

$$\begin{pmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 20 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{vmatrix} = 10 \qquad \Delta_1 = \begin{vmatrix} 4 & -9 & -1 \\ 4 & 10 & 8 \\ 20 & 8 & 20 \end{vmatrix} = -8 \qquad \Delta_2 = \begin{vmatrix} 11 & 4 & -1 \\ -9 & 4 & 8 \\ -1 & 20 & 20 \end{vmatrix} = -16$$

$$\underline{x_1 = \frac{-8}{10} = -\frac{4}{5}} \quad \underline{x_2 = \frac{-16}{10} = -\frac{8}{5}} \quad \underline{x_3 = \frac{8}{5}}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} \\ -\frac{8}{5} \\ \frac{8}{5} \end{pmatrix}$$

$$proj_W \vec{u} = A\vec{x}$$

$$= \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} -\frac{4}{5} \\ -\frac{8}{5} \\ \frac{8}{5} \end{pmatrix}$$

$$=\begin{pmatrix} -\frac{12}{5} \\ -\frac{4}{5} \\ \frac{12}{5} \\ \frac{16}{5} \end{pmatrix}$$

$$\underline{proj_W \vec{u} = \left(-\frac{12}{5},\ -\frac{4}{5},\ \frac{12}{5},\ \frac{16}{5}\right)}$$

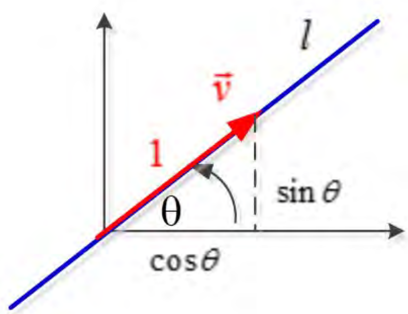
Exercise

Find the standard matrix for the orthogonal projection P of $\mathbb{R}^2$ on the line passes through the origin and makes an angle θ with the positive *x*-axis.

Solution

Since the line l in 2-dimensional, than we can take $\vec{v} = (\cos\theta,\ \sin\theta)$ as a basis for this subspace

$$A = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$



$$[P] = A^T A$$

$$= [\cos\theta \quad \sin\theta]\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

Exercise

Hooke's law in physics states that the length *x* of a uniform spring is a linear function of the force *y* applied to it. If we express the relationship as $y = mx + b$, then the coefficient *m* is called the spring constant.

Suppose a particular unstretched spring has a measured length of 6.1 *inches*.(i.e., *x* = 6.1 when *y* = 0). Forces of 2 pounds, 4 pounds, and 6 pounds are then applied to the spring, and the corresponding lengths are found to be 7.6 inches, 8.7 inches, and 10.4 inches. Find the spring constant.

Solution

$$M = \begin{pmatrix} 6.1 & 1 \\ 7.6 & 1 \\ 8.7 & 1 \\ 10.4 & 1 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \end{pmatrix}$$

The normal equation: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 6.1 & 7.6 & 8.7 & 10.4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6.1 & 1 \\ 7.6 & 1 \\ 8.7 & 1 \\ 10.4 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 6.1 & 7.6 & 8.7 & 10.4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 278.82 & 32.8 \\ 32.8 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 112.4 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{39.44} \begin{pmatrix} 4 & -32.8 \\ -32.8 & 278.2 \end{pmatrix} \begin{pmatrix} 112.4 \\ 12 \end{pmatrix}$$

$$= \frac{1}{39.44} \begin{pmatrix} 56 \\ -348.32 \end{pmatrix}$$

$$= \begin{pmatrix} 1.4 \\ -8.8 \end{pmatrix}$$

Thus, the estimated value of the spring constant is ≈ 1.4 *pounds*

Exercise

Prove:

If *A* has a linearly independent column vectors, and if $\vec{b}$ is orthogonal to the column space of *A*, then the least squares solution of $A\vec{x} = \vec{b}$ is $\vec{x} = \vec{0}$.

Solution

If *A* has linearly independent column vectors, then $A^T A$ is invertible and the least squares solution of $A\vec{x} = \vec{b}$ is the solution of $A^T A\vec{x} = A^T \vec{b}$, but since $\vec{b}$ is orthogonal to the column space of *A*.

$A^T \vec{b} = 0$, so $\vec{x}$ is a solution of $A^T A\vec{x} = 0$.

Thus $\vec{x} = \vec{0}$ since $A^T A$ is invertible.

Exercise

Let *A* be an $m \times n$ matrix with linearly independent row vectors. Find a standard matrix for the orthogonal projection of $\mathbb{R}^n$ onto the row space of *A*.

Solution

A^T will have linearly independent column vectors, and the column space A^T is the row space of *A*.

Thus, the standard matrix for the orthogonal projection of $\mathbb{R}^n$ onto the row space of *A* is

$$[P] = A^T\left[\left(A^T\right)^T A^T\right]^{-1}\left(A^T\right)^T$$

$$= A^T\left(AA^T\right)^{-1}A$$

Exercise

Let W be the line with parametric equations $x = 2t, \quad y = -t, \quad z = 4t$

a) Find a basis for W.

b) Find the standard matrix for the orthogonal projection on W.

c) Use the matrix in part (b) to find the orthogonal projection of a point $P_0\left(x_0, y_0, z_0\right)$ on W.

d) Find the distance between the point $P_0(2, 1, -3)$ and the line W.

Solution

a) $W = span\{(2, \ -1, \ 4)\}$

So that the vector $(2, \ -1, \ 4)$ forms a basis for W (linear independence)

b) Let $A = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$

$$[P] = A\left(A^T A\right)^{-1}A^T$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}\left(\begin{bmatrix} 2 & -1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}\right)^{-1}\begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}[21]^{-1}\begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \frac{1}{21}\begin{bmatrix} 4 & -2 & 8 \\ -2 & 1 & -4 \\ 8 & -4 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix}$$

c) $\begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} \frac{4}{21}x_0 - \frac{2}{21}y_0 + \frac{8}{21}z_0 \\ -\frac{2}{21}x_0 + y_0 - \frac{4}{21}z_0 \\ \frac{8}{21}x_0 - \frac{4}{21}y_0 + \frac{16}{21}z_0 \end{bmatrix}$

d) $\begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -\frac{6}{7} \\ \frac{3}{7} \\ -\frac{12}{7} \end{bmatrix}$

The distance between P_0 and W equals to the distance between P_0 and its projection on W.

The distance between $(2,\ 1,\ -3)$ and $\left(-\frac{6}{7},\ \frac{3}{7},\ -\frac{12}{7}\right)$ is

$$d = \sqrt{\left(2+\frac{6}{7}\right)^2 + \left(1-\frac{3}{7}\right)^2 + \left(-3+\frac{12}{7}\right)^2}$$

$$= \sqrt{\frac{400}{49} + \frac{16}{49} + \frac{81}{49}}$$

$$= \frac{\sqrt{497}}{7}$$

Exercise

In R^3, consider the line l given by the equations $\quad x = t, \quad y = t, \quad z = t$

And the line m given by the equations $\quad x = s, \quad y = 2s - 1, \quad z = 1$

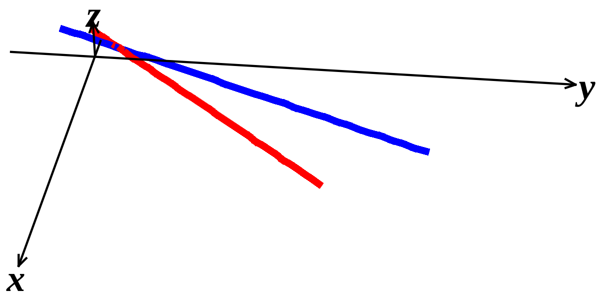
Let P be the point on l, and let Q be a point on m. Find the values of t and s that minimize the distance between the lines by minimizing the squared distance $\|P - Q\|^2$

Solution

When $t = 1 \quad \Rightarrow \quad Let\ P = (1,\ 1,\ 1)$ is on line l

When $s = 1 \quad \Rightarrow \quad Let\ Q = (1,\ 1,\ 1)$ is on line m

$$\|P - Q\| = \sqrt{(1-1)^2 + (1-1)^2 + (1-1)^2} = 0 \ge 0$$



Thus, these are the values $P=(1,\ 1,\ 1)$ and $Q=(1,\ 1,\ 1)$ are the values for $s=t=1$ that minimize the distance between the lines.

Exercise

Determine whether the statement is true or false,

a) If A is an $m\times n$ matrix, then $A^T A$ is a square matrix.

b) If $A^T A$ is invertible, then A is invertible.

c) If A is invertible, then $A^T A$ is invertible.

d) If $A\vec{x}=\vec{b}$ is a consistent linear system, then $A^T A\vec{x}=A^T\vec{b}$ is also consistent.

e) If $A\vec{x}=\vec{b}$ is an inconsistent linear system, then $A^T A\vec{x}=A^T\vec{b}$ is also inconsistent.

f) Every linear system has a least squares solution.

g) Every linear system has a unique least squares solution.

h) If A is an $m\times n$ matrix with linearly independent columns and $\vec{b}$ is in R^m, then $A\vec{x}=\vec{b}$ has a unique least squares solution.

Solution

a) True; $A^T A$ is an $n\times n$ matrix

b) False; only square matrix has inverses, but $A^T A$ can be invertible when A is not square matrix.

c) True; if A is invertible, so is A^T, so the product $A^T A$ is also invertible

d) True

e) False; the system $A^T A\vec{x}=A^T\vec{b}$ may be consistent

f) True

g) False; the least squares solution may involve a parameter

h) True; if A has linearly independent column vectors; then $A^T A$ is invertible, so $A^T A\vec{x}=A^T\vec{b}$ has a unique solution

Exercise

A certain experiment produces the data $\{(1,\ 1.8),\ (2,\ 2.7),\ (3,\ 3.4),\ (4,\ 3.8),\ (5,\ 3.9)\}$.

Find the function that it will fit these data in the form of $y = \beta_1 x + \beta_2 x^2$

Solution

Given: the equation $y = \beta_1 x + \beta_2 x^2$ that best fits the given points. Then

$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{pmatrix}$$

where $A = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{pmatrix}$

The normal equation formula: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \end{pmatrix}\begin{pmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{pmatrix}$$

$$\begin{pmatrix} 55 & 225 \\ 225 & 979 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 52.1 \\ 201.5 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 55 & 225 \\ 225 & 979 \end{vmatrix} = 3,220 \qquad \Delta_{\beta_1} = \begin{vmatrix} 52.1 & 225 \\ 201.5 & 979 \end{vmatrix} = 5,668.4 \qquad \Delta_{\beta_2} = \begin{vmatrix} 55 & 52.1 \\ 225 & 201.5 \end{vmatrix} = -640$$

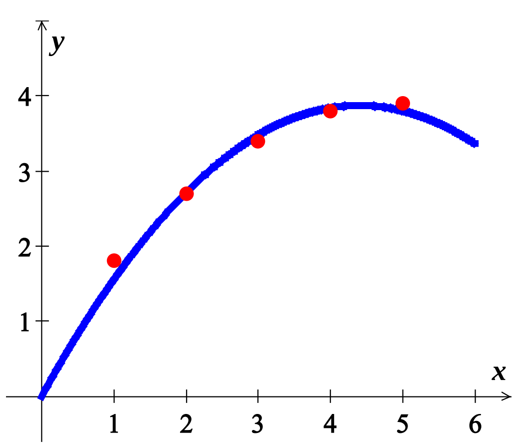
$$\beta_1 = \frac{5,668.4}{3,220}$$

$$\approx 1.76$$

$$\beta_2 = -\frac{640}{3,220}$$

$$\approx -0.199$$

$$y = 1.76x - .2x^2$$



Exercise

According to Kepler's first law, a comet should have an ellipse, parabolic, or hyperbolic orbit (with gravitational attractions from the planets ignored). In suitable polar coordinates, the position $(r,\ \upsilon)$ of a comet satisfies an equation of the form

$$r = \beta + e(r \cdot \cos\upsilon)$$

Where β is a constant and e is the eccentricity of the orbit, with $0 \le e < 1$ for an ellipse, $e = 1$ for a parabolic, and $e > 1$ for a hyperbola.

Suppose observations of a newly discovered comet provide the data below.

υ	.88	1.10	1.42	1.77	2.14
r	3.00	2.30	1.65	1.25	1.01

Determine the type of orbit, and predict where the orbit will be when $\upsilon = 4.6$ $(radians)$?

Solution

Given: the equation in the form $r = \beta + e(r \cdot \cos\upsilon)$

$3 = \beta + e(3 \cdot \cos(.88)) = \beta + 1.911e$

$2.3 = \beta + e(2.3\cos(1.1)) = \beta + 1.043e$

$1.65 = \beta + e(1.65\cos(1.42)) = \beta + .248e$

$1.25 = \beta + e(1.25\cos(1.77)) = \beta - .247e$

$1.01 = \beta + e(1.01\cos(2.14)) = \beta - .544e$

$$\begin{pmatrix} 1 & 1.911 \\ 1 & 1.043 \\ 1 & .248 \\ 1 & -.247 \\ 1 & -.544 \end{pmatrix} \begin{pmatrix} \beta \\ e \end{pmatrix} = \begin{pmatrix} 3 \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{pmatrix}$$

where $A = \begin{pmatrix} 1 & 1.911 \\ 1 & 1.043 \\ 1 & .248 \\ 1 & -.247 \\ 1 & -.544 \end{pmatrix}$ $\quad \vec{\upsilon} = \begin{pmatrix} \beta \\ e \end{pmatrix}$ $\quad \vec{r} = \begin{pmatrix} 3 \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{pmatrix}$

The normal equation formula: $A^T A\,\vec{\upsilon} = A^T \vec{r}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1.911 & 1.043 & .248 & -.247 & -.544 \end{pmatrix}\begin{pmatrix} 1 & 1.911 \\ 1 & 1.043 \\ 1 & .248 \\ 1 & -.247 \\ 1 & -.544 \end{pmatrix}\begin{pmatrix} \beta \\ e \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1.911 & 1.043 & .248 & -.247 & -.544 \end{pmatrix}\begin{pmatrix} 3 \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 2.411 \\ 2.411 & 5.158 \end{pmatrix}\begin{pmatrix} \beta \\ e \end{pmatrix} = \begin{pmatrix} 9.21 \\ 7.683 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 5 & 2.411 \\ 2.411 & 5.158 \end{vmatrix} = 19.98 \qquad \Delta_\beta = \begin{vmatrix} 9.21 & 2.411 \\ 7.683 & 5.158 \end{vmatrix} = 28.98 \qquad \Delta_e = \begin{vmatrix} 5 & 9.21 \\ 2.411 & 7.683 \end{vmatrix} = 16.21$$

$$\beta = \frac{28.98}{19.98}$$

$$\underline{\approx 1.45}$$

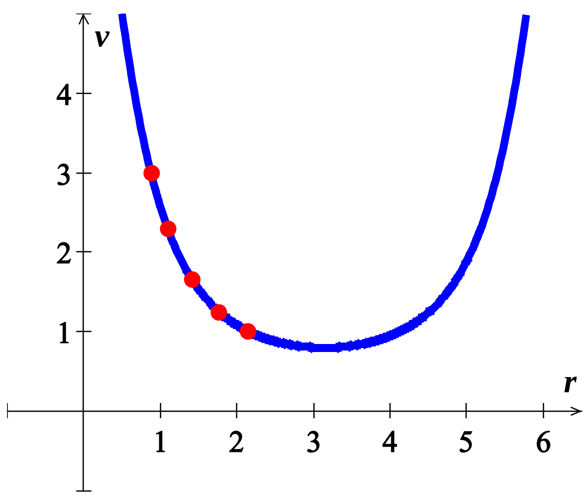
$$e = \frac{16.21}{19.98}$$

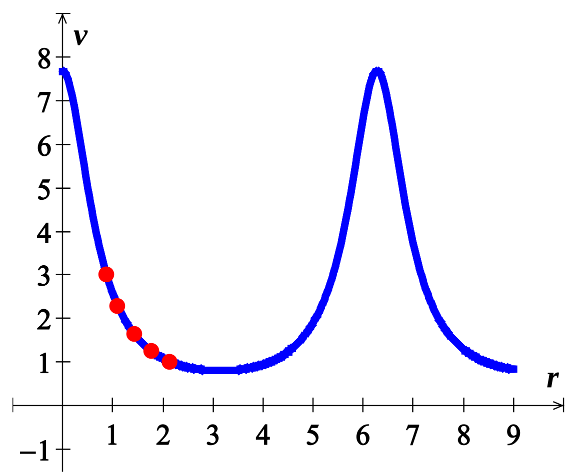
$$\underline{\approx 0.811 \quad < 1}$$

Therefore, the orbit is an ***ellipse*** type since $e \approx 0.811 \quad < 1$

Since $r = \beta + e(r \cdot \cos\upsilon)$

Then, $r(\upsilon) = \dfrac{1.45}{1 - 0.811 \cdot \cos\upsilon}$





$$r(4.6) = \frac{1.45}{1 - 0.811 \cdot \cos 4.6}$$

$$\underline{\approx 1.329}$$

Exercise

To measure the takeoff performance of an airplane, the horizontal position of the plane was measured every second, from $t = 0$ to $t = 12$

The position (in *feet*) were:

$0,\ 8.8,\ 29.9,\ 62.0,\ 104.7,\ 159.1,\ 222.0,\ 294.5,\ 380.4,\ 471.1,\ 571.7,\ 686.8,\ and\ 809.2$

a) Find the least square cubic curve $y = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$ for these data.

b) Estimate the velocity of the plane when $t = 4.5$ *sec*, using the result from part (*a*).

Solution

Given: the equation is in form $y = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \\ 1 & 6 & 36 & 216 \\ 1 & 7 & 49 & 343 \\ 1 & 8 & 64 & 512 \\ 1 & 9 & 81 & 729 \\ 1 & 10 & 100 & 1000 \\ 1 & 11 & 121 & 1331 \\ 1 & 12 & 144 & 1728 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}}_{\vec{t}} \underset{=}{=} \underbrace{\begin{pmatrix} 0 \\ 8.8 \\ 29.9 \\ 62 \\ 104.7 \\ 159.1 \\ 222.0 \\ 294.5 \\ 380.4 \\ 471.1 \\ 571.7 \\ 686.8 \\ 809.2 \end{pmatrix}}_{\vec{y}}$$

The normal equation formula: $A^T A\,\vec{t} = A^T\,\vec{y}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 & 81 & 100 & 121 & 144 \\ 0 & 1 & 8 & 27 & 64 & 125 & 216 & 343 & 512 & 729 & 1000 & 1331 & 1728 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \\ 1 & 6 & 36 & 216 \\ 1 & 7 & 49 & 343 \\ 1 & 8 & 64 & 512 \\ 1 & 9 & 81 & 729 \\ 1 & 10 & 100 & 1000 \\ 1 & 11 & 121 & 1331 \\ 1 & 12 & 144 & 1728 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 & 81 & 100 & 121 & 144 \\ 0 & 1 & 8 & 27 & 64 & 125 & 216 & 343 & 512 & 729 & 1000 & 1331 & 1728 \end{pmatrix} \begin{pmatrix} 0 \\ 8.8 \\ 29.9 \\ 62 \\ 104.7 \\ 159.1 \\ 222.0 \\ 294.5 \\ 380.4 \\ 471.1 \\ 571.7 \\ 686.8 \\ 809.2 \end{pmatrix}$$

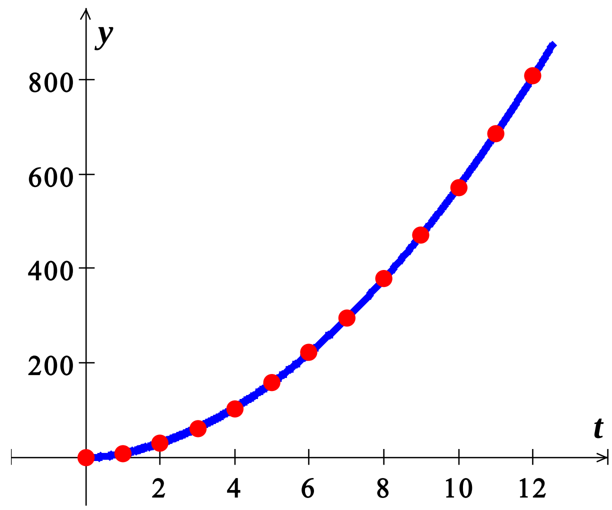
$$\begin{pmatrix} 13 & 78 & 650 & 6,084 \\ 78 & 650 & 6,084 & 60,710 \\ 650 & 6,084 & 60,710 & 630,708 \\ 6,084 & 60,710 & 630,708 & 6,735,950 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 3,800.2 \\ 35,127.7 \\ 348,063.9 \\ 3,599,800.9 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 13 & 78 & 650 & 6,084 \\ 78 & 650 & 6,084 & 60,710 \\ 650 & 6,084 & 60,710 & 630,708 \\ 6,084 & 60,710 & 630,708 & 6,735,950 \end{vmatrix} = 97,538,785,344$$

$$\Delta_0 = \begin{vmatrix} 3800.2 & 78 & 650 & 6,084 \\ 35,127.7 & 650 & 6,084 & 60,710 \\ 348,063.9 & 6,084 & 60,710 & 630,708 \\ 3,599,800.9 & 60,710 & 630,708 & 6,735,950 \end{vmatrix} = -83,470,691,303.8916$$

Or I use my program to find the values

rref = (Matrix([
[1, 0, 0, 0, -0.855769230765803],
[0, 1, 0, 0, 4.70248501498163],
[0, 0, 1, 0, 5.55536963037029],
[0, 0, 0, 1, -0.0273601398601744]]))



$\boxed{\beta_0 \approx -0.855769}$

$\boxed{\beta_1 \approx 4.702485}$

$\boxed{\beta_2 \approx 5.55537}$

$\boxed{\beta_3 \approx -0.02736}$

$\boxed{y(t) = -0.855769 + 4.702485\,t + 5.55537\,t^2 - 0.02736\,t^3}$

Error = 3.9734