

Solutions **Section 3.4 – Solving Trigonometric Equations**

Exercise

Find all solutions of the equation: $\sin x = \frac{\sqrt{2}}{2}$

Solution

$$\sin x = \frac{\sqrt{2}}{2}$$

$$\hat{x} = \sin^{-1} \frac{\sqrt{2}}{2}$$

$$= 45^\circ \quad x \in QI, QII$$

$$x = 45^\circ \rightarrow \underline{x = 45^\circ + 360^\circ k}$$

$$x = 180^\circ - 45^\circ = 135^\circ \rightarrow \underline{x = 135^\circ + 360^\circ k}$$

Exercise

Find all solutions of the equation: $\cos x = -\frac{\pi}{3}$

Solution

$$\cos x = -\frac{\pi}{3} < -1 \text{ has no solution } ([-1, 1])$$

Exercise

Find all solutions of the equation: $2\cos\theta - \sqrt{3} = 0$

Solution

$$2\cos\theta = \sqrt{3}$$

$$\cos\theta = \frac{\sqrt{3}}{2} \quad \theta \in QI, QIV$$

$$\theta = \frac{\pi}{6} \rightarrow \underline{\theta = \frac{\pi}{6} + 2\pi k}$$

$$\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6} \rightarrow \underline{\theta = \frac{11\pi}{6} + 2\pi k}$$

Exercise

Find all solutions of the equation: $2 \cos 2\theta - \sqrt{3} = 0$

Solution

$$2 \cos 2\theta = \sqrt{3}$$

$$\cos 2\theta = \frac{\sqrt{3}}{2} \quad \theta \in \text{QI, QIV}$$

$$2\theta = \frac{\pi}{6} \rightarrow \underline{\theta = \frac{\pi}{12} + \pi n}$$

$$2\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6} \rightarrow \underline{\theta = \frac{11\pi}{12} + \pi n}$$

Exercise

Find all solutions of the equation: $\sqrt{3} \tan \frac{1}{3}x = 1$

Solution

$$\tan \frac{1}{3}x = \frac{1}{\sqrt{3}}$$

$$\frac{1}{3}x = \frac{\pi}{6} \text{ or } \frac{7\pi}{6}$$

$$\frac{1}{3}x = \frac{\pi}{6} + \pi n$$

$$\underline{x = \frac{\pi}{2} + 3\pi n}$$

Exercise

Find all solutions of the equation: $\cos\left(4x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

Solution

$$\cos\left(4x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \cos \frac{\pi}{4}$$

$$4x - \frac{\pi}{4} = \frac{\pi}{4} + 2\pi k$$

$$4x = \frac{\pi}{2} + 2\pi k$$

$$\underline{x = \frac{\pi}{4} + \frac{\pi}{2}k}$$

$$\cos\left(4x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \cos \frac{7\pi}{4}$$

$$4x - \frac{\pi}{4} = \frac{7\pi}{4} + 2\pi k$$

$$4x = 2\pi + 2\pi k$$

$$\underline{x = \frac{\pi}{2}k}$$

Exercise

Find all solutions of the equation: $(\cos \theta - 1)(\sin \theta + 1) = 0$

Solution

$$\cos \theta - 1 = 0$$

$$\cos \theta = 1$$

$$\underline{\theta = 0^\circ + 360^\circ k}$$

$$\sin \theta + 1 = 0$$

$$\sin \theta = -1$$

$$\underline{\theta = 270^\circ + 360^\circ k}$$

Exercise

Find all solutions of the equation: $\cot^2 x - 3 = 0$

Solution

$$\cot^2 x = 3$$

$$\cot x = \pm\sqrt{3}$$

$$\underline{x = \frac{\pi}{6} + 2\pi k, \frac{5\pi}{6} + 2\pi k, \frac{7\pi}{6} + 2\pi k, \frac{11\pi}{6} + 2\pi k}$$

Or

$$\underline{x = \frac{\pi}{6} + \pi n, \frac{5\pi}{6} + \pi n}$$

Exercise

Find all solutions of the equation: $\cos x + 1 = 2\sin^2 x$

Solution

$$\cos x + 1 = 2(1 - \cos^2 x)$$

$$\cos x + 1 = 2 - 2\cos^2 x$$

$$\cos x + 1 - 2 + 2\cos^2 x = 0$$

$$2\cos^2 x + \cos x - 1 = 0$$

| | |
|-------------------------------------|---|
| $\cos x = -1$ $x = \pi + 2\pi n$ | $\cos x = \frac{1}{2}$ $x = \frac{\pi}{3} + 2\pi n; \quad x = \frac{5\pi}{3} + 2\pi n$ |
|-------------------------------------|---|

Exercise

Find all solutions of the equation: $\cos(\ln x) = 0$

Solution

$$\cos(\ln x) = 0 \rightarrow \begin{cases} \ln x = \frac{\pi}{2} + 2\pi k \\ \ln x = \frac{3\pi}{2} + 2\pi k \end{cases}$$

$$\ln x = \frac{\pi}{2} + \pi n$$

$$x = e^{\pi/2 + \pi n}$$

Exercise

Find the solutions of the equation that are in the interval $[0, 2\pi)$: $2\sin^2 x = 1 - \sin x$

Solution

$$2\sin^2 x + \sin x - 1 = 0$$

| | |
|---------------------------------------|---|
| $\sin x = -1$ $x = \frac{3\pi}{2}$ | $\sin x = \frac{1}{2}$ $x = \frac{\pi}{6}; \quad x = \frac{5\pi}{6}$ |
|---------------------------------------|---|

Exercise

Find the solutions of the equation that are in the interval $[0, 2\pi)$: $\tan^2 x \sin x = \sin x$

Solution

$$\tan^2 x \sin x - \sin x = 0$$

$$\sin x (\tan^2 x - 1) = 0$$

| | |
|--|--|
| $\sin x = 0$ $x = 0; \quad x = \pi$ | $\tan^2 x - 1 = 0 \Rightarrow \tan^2 x = 1$ $\tan x = \pm 1$ $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ |
|--|--|

Exercise

Find the solutions of the equation that are in the interval $[0, 2\pi)$: $1 - \sin x = \sqrt{3} \cos x$

Solution

$$(1 - \sin x)^2 = (\sqrt{3} \cos x)^2$$

$$1 - 2 \sin x + \sin^2 x = 3 \cos^2 x$$

$$1 - 2 \sin x + \sin^2 x = 3(1 - \sin^2 x)$$

$$1 - 2 \sin x + \sin^2 x = 3 - 3 \sin^2 x$$

$$1 - 2 \sin x + \sin^2 x - 3 + 3 \sin^2 x = 0$$

$$4 \sin^2 x - 2 \sin x - 2 = 0$$

| | | |
|---|--|---|
| $\sin x = 1$ $x = \frac{\pi}{2} \rightarrow (\text{check})$ $1 - \sin \frac{\pi}{2} = \sqrt{3} \cos \frac{\pi}{2}$ $1 - (1) = \sqrt{3}(0)$ $0 = 0 \quad \checkmark$ | $\sin x = -\frac{1}{2}$ $x = \frac{7\pi}{6}$ $1 - \sin \frac{7\pi}{6} = \sqrt{3} \cos \frac{7\pi}{6}$ $1 - \left(-\frac{1}{2}\right) = \sqrt{3} \left(-\frac{\sqrt{3}}{2}\right)$ $\frac{3}{2} = -\frac{3}{2}$ | $x = \frac{11\pi}{6}$ $1 - \sin \frac{11\pi}{6} = \sqrt{3} \cos \frac{11\pi}{6}$ $1 - \left(-\frac{1}{2}\right) = \sqrt{3} \left(\frac{\sqrt{3}}{2}\right)$ $\frac{3}{2} = \frac{3}{2} \quad \checkmark$ |
|---|--|---|

The solutions are: $x = \frac{\pi}{2}, \frac{11\pi}{6}$

Exercise

Find the solutions of the equation that are in the interval $[0, 2\pi)$: $\sin x + \cos x \cot x = \csc x$

Solution

$$\sin x + \cos x \frac{\cos x}{\sin x} = \frac{1}{\sin x}$$

Multiply by $\sin x$ both sides ($\sin x \neq 0$)

$$\sin^2 x + \cos^2 x = 1$$

$$1 = 1 \quad (\text{True})$$

The solutions are: $x \in [0, 2\pi)$ except 0 and π .

Exercise

Find the solutions of the equation that are in the interval $[0, 2\pi)$: $2\sin^3 x + \sin^2 x - 2\sin x - 1 = 0$

Solution

$$\sin^2 x (2\sin x + 1) - (2\sin x + 1) = 0$$

Factor by grouping

$$(2\sin x + 1)(\sin^2 x - 1) = 0$$

| | |
|---|---|
| $2\sin x + 1 = 0$ $\sin x = -\frac{1}{2}$ $x = \frac{7\pi}{6}, \frac{11\pi}{6}$ | $\sin^2 x = 1$ $\sin x = \pm 1$ $x = \frac{\pi}{2}, \frac{3\pi}{2}$ |
|---|---|

Exercise

Find the solutions of the equation that are in the interval $[0, 2\pi)$: $2\tan x \csc x + 2\csc x + \tan x + 1 = 0$

Solution

$$2\tan x \csc x + \tan x + 2\csc x + 1 = 0$$

$$\tan x (2\csc x + 1) + (2\csc x + 1) = 0$$

$$(2\csc x + 1)(\tan x + 1) = 0$$

| | |
|---|---|
| $2\csc x + 1 = 0$ $\csc x = -\frac{1}{2} = \frac{1}{\sin x}$ $\sin x = -2$ (impossible) | $\tan x = -1$ $x = \frac{3\pi}{4}, \frac{7\pi}{4}$ |
|---|---|

Exercise

Solve $2\cos \theta + \sqrt{3} = 0$ if $0^\circ \leq \theta < 360^\circ$

Solution

$$2\cos \theta = -\sqrt{3}$$

$$\cos \theta = -\frac{\sqrt{3}}{2}$$

$$\hat{\theta} = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$\theta = 150^\circ, 210^\circ$$

Exercise

Solve $5\cos t + \sqrt{12} = \cos t$ if $0 \leq t < 2\pi$

Solution

$$5\cos t - \cos t = -\sqrt{12}$$

$$\sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3}$$

$$4\cos t = -2\sqrt{3}$$

$$4\cos t = -2\sqrt{3}$$

$$\cos t = -\frac{2\sqrt{3}}{4}$$

$$\cos t = -\frac{\sqrt{3}}{2} \Rightarrow t = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$\theta = \frac{5\pi}{6}, \frac{7\pi}{6}$$

Exercise

Solve $\tan \theta - 2\cos \theta \tan \theta = 0$ if $0^\circ \leq \theta < 360^\circ$

Solution

$$\tan \theta (1 - 2\cos \theta) = 0$$

$$\tan \theta = 0$$

$$1 - 2\cos \theta = 0$$

$$\theta = 0^\circ, 180^\circ$$

$$1 = 2\cos \theta$$

$$\cos \theta = \frac{1}{2} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = 60^\circ, 300^\circ$$

$$\theta = 0^\circ, 60^\circ, 180^\circ, 300^\circ$$

Exercise

Solve $2\sin^2 \theta - 2\sin \theta - 1 = 0$ if $0^\circ \leq \theta < 360^\circ$

Solution

$$\sin \theta = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{12}}{4}$$

$$= \frac{2 \pm 2\sqrt{3}}{4}$$

$$= \frac{1 \pm \sqrt{3}}{2}$$

$$\hat{\theta} = \sin^{-1}\left(\frac{1 - \sqrt{3}}{2}\right)$$

$$= -21.47^\circ$$

$$\sin \theta = \frac{1 + \sqrt{3}}{2} = 1.366 > 1$$

$$\theta = 360^\circ - 21.47^\circ = 338.53^\circ$$

$$\theta = 180^\circ + 21.47^\circ = 201.47^\circ$$

The solutions are: $\theta = 338.53^\circ, 201.47^\circ$

Exercise

Solve $\cos\left(A - \frac{\pi}{9}\right) = -\frac{1}{2}$

Solution

$-\frac{1}{2}$ is negative \rightarrow cosine is in QII or QIII.

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{4\pi}{3}$$

$$\cos\left(A - \frac{\pi}{9}\right) = -\frac{1}{2} = \cos \frac{2\pi}{3}$$

$$\cos\left(A - \frac{\pi}{9}\right) = -\frac{1}{2} = \cos \frac{4\pi}{3}$$

$$A - \frac{\pi}{9} = \frac{2\pi}{3} + 2\pi k$$

$$A - \frac{\pi}{9} = \frac{4\pi}{3} + 2\pi k$$

$$A = \frac{2\pi}{3} + \frac{\pi}{9} + 2\pi k$$

$$A = \frac{4\pi}{3} + \frac{\pi}{9} + 2\pi k$$

$$A = \frac{7\pi}{9} + 2\pi k$$

$$A = \frac{13\pi}{9} + 2\pi k$$

Exercise

Solve: $4 \cos \theta - 3 \sec \theta = 0$ if $0^\circ \leq \theta < 360^\circ$

Solution

$$4 \cos \theta - 3 \frac{1}{\cos \theta} = 0$$

$$\cos \theta \neq 0 \rightarrow \theta \neq \frac{\pi}{2}, \frac{3\pi}{2}$$

$$4 \cos \theta \cos \theta - 3 \frac{1}{\cos \theta} \cos \theta = 0$$

$$4 \cos^2 \theta - 3 = 0$$

$$4 \cos^2 \theta = 3$$

$$\cos^2 \theta = \frac{3}{4}$$

$$\cos \theta = \pm \frac{\sqrt{3}}{2}$$

$$\theta = \cos^{-1}\left(\pm \frac{\sqrt{3}}{2}\right)$$

The solutions are: $\theta = 30^\circ, 150^\circ, 210^\circ, 330^\circ$

Exercise

Solve: $2 \sin^2 x - \cos x - 1 = 0$ if $0 \leq x < 2\pi$

Solution

$$2(1 - \cos^2 x) - \cos x - 1 = 0$$

$$2 - 2 \cos^2 x - \cos x - 1 = 0$$

$$-2 \cos^2 x - \cos x + 1 = 0$$

$$\cos x = -1 \qquad \cos x = \frac{1}{2}$$

$$x = \pi \qquad x = \frac{\pi}{3}, \frac{5\pi}{3}$$

The solutions are: $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

Exercise

Solve: $\sin \theta - \sqrt{3} \cos \theta = 1$ if $0^\circ \leq \theta < 360^\circ$

Solution

$$\sin \theta - 1 = -\sqrt{3} \cos \theta$$

$$(\sin \theta - 1)^2 = (-\sqrt{3} \cos \theta)^2$$

$$\sin^2 \theta - 2 \sin \theta + 1 = 3 \cos^2 \theta$$

$$\cos^2 \theta = 1 - \sin^2 \theta$$

$$\sin^2 \theta - 2 \sin \theta + 1 = 3(1 - \sin^2 \theta)$$

$$\sin^2 \theta - 2 \sin \theta + 1 = 3 - 3 \sin^2 \theta$$

$$\sin^2 \theta - 2 \sin \theta + 1 - 3 + 3 \sin^2 \theta = 0$$

$$4 \sin^2 \theta - 2 \sin \theta - 2 = 0$$

$$\sin \theta = 1 \Rightarrow \theta = 90^\circ$$

$$\sin \theta = -\frac{1}{2} \Rightarrow \theta = 210^\circ, 330^\circ$$

Check

| | | |
|--|---|---|
| $\theta = 90^\circ$ $\sin 90^\circ - \sqrt{3} \cos 90^\circ \stackrel{?}{=} 1$ $1 - \sqrt{3}(0) \stackrel{?}{=} 1$ $1 = 1 \quad \checkmark$ | $\theta = 210^\circ$ $\sin 210^\circ - \sqrt{3} \cos 210^\circ \stackrel{?}{=} 1$ $-\frac{1}{2} - \sqrt{3}\left(-\frac{\sqrt{3}}{2}\right) \stackrel{?}{=} 1$ $-\frac{1}{2} + \frac{3}{2} \stackrel{?}{=} 1$ $1 = 1 \quad \checkmark$ | $\theta = 330^\circ$ $\sin 330^\circ - \sqrt{3} \cos 330^\circ \stackrel{?}{=} 1$ $-\frac{1}{2} - \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) \stackrel{?}{=} 1$ $-\frac{1}{2} - \frac{3}{2} \stackrel{?}{=} 1$ $-2 \neq 1$ (False statement) |
|--|---|---|

The solutions are: $90^\circ, 210^\circ$

Exercise

Solve: $7 \sin^2 \theta - 9 \cos 2\theta = 0$ if $0^\circ \leq \theta < 360^\circ$

Solution

$$7 \sin^2 \theta - 9(1 - 2 \sin^2 \theta) = 0 \qquad \cos^2 \theta = 1 - 2 \sin^2 \theta$$

$$7 \sin^2 \theta - 9 + 18 \sin^2 \theta = 0$$

$$25 \sin^2 \theta - 9 = 0$$

$$25 \sin^2 \theta = 9$$

$$\sin^2 \theta = \frac{9}{25} \Rightarrow \sin \theta = \pm \frac{3}{5}$$

$$\hat{\theta} = \sin^{-1}\left(\frac{3}{5}\right) \approx 36.87^\circ$$

$$\theta \approx 36.87^\circ$$

$$\theta \approx 180^\circ - 36.87^\circ \approx 143.13^\circ$$

$$\theta \approx 180^\circ + 36.87^\circ \approx 216.87^\circ$$

$$\theta \approx 360^\circ - 36.87^\circ \approx 323.13^\circ$$

The solutions are: $36.87^\circ, 143.13^\circ, 216.87^\circ, 323.13^\circ$

Exercise

Solve: $2\cos^2 t - 9\cos t = 5$ if $0 \leq t < 2\pi$

Solution

$$2\cos^2 t - 9\cos t - 5 = 0$$

$$(2\cos t + 1)(\cos t - 5) = 0$$

$$2\cos t + 1 = 0 \qquad \cos t - 5 = 0$$

$$\cos t = -\frac{1}{2} \qquad \cos t = 5$$

$$\cos t = -\frac{1}{2} \qquad \cos t = 5$$

$$\hat{t} = \cos^{-1}\left(-\frac{1}{2}\right) \qquad \text{No solution}$$

$$\hat{t} = \frac{\pi}{3}$$

Negative sign \rightarrow cosine is in QII or QIII

$$t = \pi - \frac{\pi}{3} \qquad t = \pi + \frac{\pi}{3}$$

$$t = \frac{2\pi}{3} \qquad t = \frac{4\pi}{3}$$

The solutions are: $\frac{2\pi}{3}, \frac{4\pi}{3}$

Exercise

Solve $\sin \theta \tan \theta = \sin \theta$ if $0^\circ \leq \theta < 360^\circ$

Solution

$$\sin \theta \tan \theta - \sin \theta = 0$$

$$\sin \theta (\tan \theta - 1) = 0$$

$$\sin \theta = 0 \qquad \tan \theta - 1 = 0$$

$$\theta = 0^\circ, 180^\circ \qquad \tan \theta = 1$$

$$\theta = 45^\circ, 225^\circ$$

The solutions are: $0^\circ, 45^\circ, 180^\circ, 225^\circ$

Exercise

Solve $\tan^2 x + \tan x - 2 = 0$ if $0 \leq x < 2\pi$

Solution

$$\tan^2 x + \tan x - 2 = 0$$

$$\tan x = 1$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$\tan x = -2$$

$$\hat{x} = \tan^{-1}(2) \approx 1.107$$

$$x = 2.034, 5.176$$

$$x \in QII, QIV$$

The solutions are: $\frac{\pi}{4}, \frac{5\pi}{4}, 2.034, 5.176$

Exercise

Solve $\tan x + \sqrt{3} = \sec x$ if $0 \leq x < 2\pi$

Solution

$$(\tan x + \sqrt{3})^2 = (\sec x)^2$$

$$\tan^2 x + 2\sqrt{3} \tan x + 3 = \sec^2 x$$

$$\tan^2 x + 2\sqrt{3} \tan x + 3 = 1 + \tan^2 x$$

$$\tan^2 x + 2\sqrt{3} \tan x + 3 - 1 - \tan^2 x = 0$$

$$2\sqrt{3} \tan x + 2 = 0$$

$$2\sqrt{3} \tan x = -2$$

$$\tan x = -\frac{2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

$$x = \frac{5\pi}{6} \quad \text{or} \quad x = \frac{11\pi}{6}$$

$$\tan \frac{5\pi}{6} + \sqrt{3} \stackrel{?}{=} \sec \frac{5\pi}{6}$$

$$-\frac{\sqrt{3}}{3} + \sqrt{3} \stackrel{?}{=} -\frac{2\sqrt{3}}{3}$$

$$\frac{2\sqrt{3}}{3} \neq -\frac{2\sqrt{3}}{3}$$

False

$$\tan \frac{11\pi}{6} + \sqrt{3} \stackrel{?}{=} \sec \frac{11\pi}{6}$$

$$-\frac{\sqrt{3}}{3} + \sqrt{3} \stackrel{?}{=} \frac{2\sqrt{3}}{3}$$

$$\frac{2\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$$

The solutions are: $\frac{11\pi}{6}$

Exercise

Solve $2\cos\theta + \sqrt{3} = 0$ if $0^\circ \leq \theta < 360^\circ$

Solution

$$2\cos\theta = -\sqrt{3}$$

$$\cos\theta = -\frac{\sqrt{3}}{2}$$

$$\hat{\theta} = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$\underline{\theta = 150^\circ, 210^\circ}$$

Exercise

Solve $5\cos t + \sqrt{12} = \cos t$ if $0 \leq t < 2\pi$

Solution

$$5\cos t - \cos t = -\sqrt{12}$$

$$4\cos t = -2\sqrt{3}$$

$$4\cos t = -2\sqrt{3}$$

$$\cos t = -\frac{2\sqrt{3}}{4}$$

$$\cos t = -\frac{\sqrt{3}}{2}$$

$$t = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$\underline{\theta = \frac{5\pi}{6}, \frac{7\pi}{6}}$$

Exercise

Solve $\tan\theta - 2\cos\theta \tan\theta = 0$ if $0^\circ \leq \theta < 360^\circ$

Solution

$$\tan\theta(1 - 2\cos\theta) = 0$$

$$\tan\theta = 0$$

$$\theta = 0^\circ, 180^\circ$$

$$1 - 2\cos\theta = 0$$

$$1 = 2\cos\theta$$

$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = 60^\circ, 300^\circ$$

$$\underline{\theta = 0^\circ, 60^\circ, 180^\circ, 300^\circ}$$

Exercise

Solve $2\sin^2 \theta - 2\sin \theta - 1 = 0$ if $0^\circ \leq \theta < 360^\circ$

Solution

$$\begin{aligned}\sin \theta &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)} \\ &= \frac{2 \pm \sqrt{12}}{4} \\ &= \frac{2 \pm 2\sqrt{3}}{4} \\ &= \frac{1 \pm \sqrt{3}}{2}\end{aligned}$$

$$\hat{\theta} = \sin^{-1}\left(\frac{1 - \sqrt{3}}{2}\right) = -21.47^\circ$$

$$\sin \theta = \frac{1 + \sqrt{3}}{2} = 1.366 > 1$$

$$\theta = 360^\circ - 21.47^\circ = 338.53^\circ$$

$$\theta = 180^\circ + 21.47^\circ = 201.47^\circ$$

Exercise

Solve $\cos\left(A - \frac{\pi}{9}\right) = -\frac{1}{2}$

Solution

$-\frac{1}{2}$ is negative \rightarrow cosine is in QII or QIII.

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{4\pi}{3}$$

$$\cos\left(A - \frac{\pi}{9}\right) = -\frac{1}{2} = \cos \frac{2\pi}{3}$$

$$\cos\left(A - \frac{\pi}{9}\right) = -\frac{1}{2} = \cos \frac{4\pi}{3}$$

$$A - \frac{\pi}{9} = \frac{2\pi}{3} + 2\pi k$$

$$A - \frac{\pi}{9} = \frac{4\pi}{3} + 2\pi k$$

$$A = \frac{2\pi}{3} + \frac{\pi}{9} + 2\pi k$$

$$A = \frac{4\pi}{3} + \frac{\pi}{9} + 2\pi k$$

$$A = \frac{7\pi}{9} + 2\pi k$$

$$A = \frac{13\pi}{9} + 2\pi k$$

Exercise

Solve: $4 \cos \theta - 3 \sec \theta = 0$ if $0^\circ \leq \theta < 360^\circ$

Solution

$$4 \cos \theta - 3 \frac{1}{\cos \theta} = 0 \quad \boxed{\cos \theta \neq 0}$$

$$4 \cos \theta \cos \theta - 3 \frac{1}{\cos \theta} \cos \theta = 0$$

$$4 \cos^2 \theta - 3 = 0$$

$$4 \cos^2 \theta = 3$$

$$\cos^2 \theta = \frac{3}{4}$$

$$\cos \theta = \pm \frac{\sqrt{3}}{2}$$

$$\theta = \cos^{-1} \left(\pm \frac{\sqrt{3}}{2} \right)$$

The solutions are: $\theta = 30^\circ, 150^\circ, 210^\circ, 330^\circ$

Exercise

Solve: $2 \sin^2 x - \cos x - 1 = 0$ if $0 \leq x < 2\pi$

Solution

$$2(1 - \cos^2 x) - \cos x - 1 = 0$$

$$2 - 2 \cos^2 x - \cos x - 1 = 0$$

$$-2 \cos^2 x - \cos x + 1 = 0$$

$$\cos x = -1 \quad \cos x = \frac{1}{2}$$

$$x = \pi \quad x = \frac{\pi}{3}, \frac{5\pi}{3}$$

The solutions are: $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

Exercise

Solve: $\sin \theta - \sqrt{3} \cos \theta = 1$ if $0^\circ \leq \theta < 360^\circ$

Solution

$$\sin \theta - 1 = -\sqrt{3} \cos \theta$$

$$(\sin \theta - 1)^2 = (-\sqrt{3} \cos \theta)^2$$

$$\sin^2 \theta - 2 \sin \theta + 1 = 3 \cos^2 \theta$$

$$\cos^2 \theta = 1 - \sin^2 \theta$$

$$\sin^2 \theta - 2 \sin \theta + 1 = 3(1 - \sin^2 \theta)$$

$$\sin^2 \theta - 2 \sin \theta + 1 - 3 + 3 \sin^2 \theta = 0$$

$$4 \sin^2 \theta - 2 \sin \theta - 2 = 0$$

$$\sin \theta = 1$$

$$\sin \theta = -\frac{1}{2}$$

$$\theta = 90^\circ$$

$$\theta = 210^\circ, 330^\circ$$

Check

| $\theta = 90^\circ$ | $\theta = 210^\circ$ | $\theta = 330^\circ$ |
|--|---|--|
| $\sin 90^\circ - \sqrt{3} \cos 90^\circ \stackrel{?}{=} 1$ | $\sin 210^\circ - \sqrt{3} \cos 210^\circ \stackrel{?}{=} 1$ | $\sin 330^\circ - \sqrt{3} \cos 330^\circ \stackrel{?}{=} 1$ |
| $1 - \sqrt{3}(0) \stackrel{?}{=} 1$ | $-\frac{1}{2} - \sqrt{3}\left(-\frac{\sqrt{3}}{2}\right) \stackrel{?}{=} 1$ | $-\frac{1}{2} - \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) \stackrel{?}{=} 1$ |
| $1 = 1$ | $-\frac{1}{2} + \frac{3}{2} \stackrel{?}{=} 1$ | $-\frac{1}{2} - \frac{3}{2} \stackrel{?}{=} 1$ |
| | $1 = 1$ | $-2 \neq 1$ (<i>False statement</i>) |

The solutions are: $90^\circ, 210^\circ$

Exercise

Solve: $7 \sin^2 \theta - 9 \cos 2\theta = 0$ if $0^\circ \leq \theta < 360^\circ$

Solution

$$7 \sin^2 \theta - 9(1 - 2 \sin^2 \theta) = 0$$

$$\cos^2 \theta = 1 - 2 \sin^2 \theta$$

$$7 \sin^2 \theta - 9 + 18 \sin^2 \theta = 0$$

$$25 \sin^2 \theta - 9 = 0$$

$$25 \sin^2 \theta = 9$$

$$\sin^2 \theta = \frac{9}{25}$$

$$\sin \theta = \pm \frac{3}{5}$$

$$\hat{\theta} = \sin^{-1}\left(\frac{3}{5}\right)$$

$$\approx 36.87^\circ$$

$$\theta \approx 36.87^\circ$$

$$\theta \approx 180^\circ - 36.87^\circ \approx 143.13^\circ$$

$$\theta \approx 180^\circ + 36.87^\circ \approx 216.87^\circ$$

$$\theta \approx 360^\circ - 36.87^\circ \approx 323.13^\circ$$

The solutions are: 36.87°, 143.13°, 216.87°, 323.13°

Exercise

Solve: $2\cos^2 t - 9\cos t = 5$ if $0 \leq t < 2\pi$

Solution

$$2\cos^2 t - 9\cos t - 5 = 0$$

$$(2\cos t + 1)(\cos t - 5) = 0$$

$$2\cos t + 1 = 0$$

$$\cos t - 5 = 0$$

$$\cos t = -\frac{1}{2}$$

$$\cos t = 5$$

$$\cos t = -\frac{1}{2}$$

$$\cos t = 5$$

$$\hat{t} = \cos^{-1}\left(-\frac{1}{2}\right) \quad \text{No solution}$$

$$\hat{t} = \frac{\pi}{3}$$

Negative sign \rightarrow cosine is in QII or QIII

$$t = \pi - \frac{\pi}{3}$$

$$t = \pi + \frac{\pi}{3}$$

$$t = \frac{2\pi}{3}$$

$$t = \frac{4\pi}{3}$$

The solutions are: $\frac{2\pi}{3}, \frac{4\pi}{3}$

Exercise

Solve $\sin \theta \tan \theta = \sin \theta$ if $0^\circ \leq \theta < 360^\circ$

Solution

$$\sin \theta \tan \theta - \sin \theta = 0$$

$$\sin \theta (\tan \theta - 1) = 0$$

$$\sin \theta = 0$$

$$\theta = 0^\circ, 180^\circ$$

$$\tan \theta - 1 = 0$$

$$\tan \theta = 1$$

$$\theta = 45^\circ, 225^\circ$$

The solutions are: $0^\circ, 45^\circ, 180^\circ, 225^\circ$

Exercise

Solve $\tan^2 x + \tan x - 2 = 0$ if $0 \leq x < 2\pi$

Solution

$$\tan^2 x + \tan x - 2 = 0$$

$$\tan x = 1$$

$$\tan x = -2$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$\hat{x} = \tan^{-1}(2) \approx 1.107$$

$$x \in QII, QIV$$

$$x = 2.034, 5.176$$

The solutions are: $\frac{\pi}{4}, \frac{5\pi}{4}, 2.034, 5.176$

Exercise

Solve $\tan x + \sqrt{3} = \sec x$ if $0 \leq x < 2\pi$

Solution

$$(\tan x + \sqrt{3})^2 = (\sec x)^2$$

$$\tan^2 x + 2\sqrt{3} \tan x + 3 = \sec^2 x$$

$$\tan^2 x + 2\sqrt{3} \tan x + 3 = 1 + \tan^2 x$$

$$\tan^2 x + 2\sqrt{3} \tan x + 3 - 1 - \tan^2 x = 0$$

$$2\sqrt{3} \tan x + 2 = 0$$

$$2\sqrt{3} \tan x = -2$$

$$\tan x = -\frac{2}{2\sqrt{3}}$$

$$= -\frac{1}{\sqrt{3}}$$

$$x = \frac{5\pi}{6} \quad \text{or} \quad x = \frac{11\pi}{6}$$

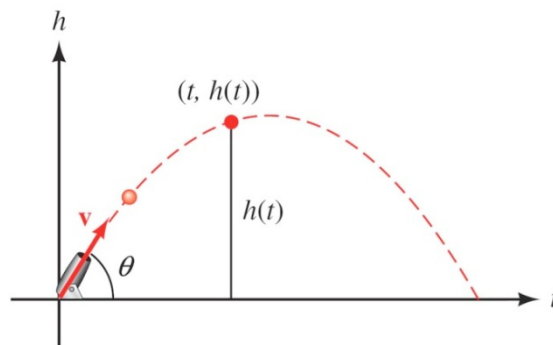
| | |
|--|---|
| $\tan \frac{5\pi}{6} + \sqrt{3} \stackrel{?}{=} \sec \frac{5\pi}{6}$ $-\frac{\sqrt{3}}{3} + \sqrt{3} \stackrel{?}{=} -\frac{2\sqrt{3}}{3}$ $\frac{2\sqrt{3}}{3} \neq -\frac{2\sqrt{3}}{3}$ <p>False</p> | $\tan \frac{11\pi}{6} + \sqrt{3} \stackrel{?}{=} \sec \frac{11\pi}{6}$ $-\frac{\sqrt{3}}{3} + \sqrt{3} \stackrel{?}{=} \frac{2\sqrt{3}}{3}$ $\frac{2\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$ |
|--|---|

The solutions are: $\frac{11\pi}{6}$

Exercise

If a projectile (such as a bullet) is fired into the air with an initial velocity v at an angle of elevation θ , then the height h of the projectile at time t is given by:

$$h(t) = -16t^2 + vt \sin \theta$$



- Give the equation for the height, if v is 600 ft./sec and $\theta = 45^\circ$.
- Use the equation in part (a) to find the height of the object after $\sqrt{3}$ seconds.
- Find the angle of elevation of θ of a rifle barrel, if a bullet fired at 1,500 ft./sec takes 3 seconds to reach a height of 750 feet. Give your answer in the nearest of a degree.

Solution

$$\begin{aligned}
 a) \quad h(t) &= -16t^2 + 600t \sin 45^\circ \\
 &= -16t^2 + 600t \frac{\sqrt{2}}{2} \\
 &= -16t^2 + 300\sqrt{2} t
 \end{aligned}$$

$$\begin{aligned}
 b) \quad h(t = \sqrt{3}) &= -16(\sqrt{3})^2 + 300\sqrt{2} \sqrt{3} \\
 &\approx 686.8 \text{ ft}
 \end{aligned}$$

$$c) \quad h(t) = -16t^2 + vt \sin \theta$$

$$750 = -16(3)^2 + 1500(3) \sin \theta$$

$$750 = -144 + 4500 \sin \theta$$

$$750 + 144 = 4500 \sin \theta$$

$$\frac{894}{4500} = \sin \theta$$

$$\theta = \sin^{-1} \left(\frac{894}{4500} \right)$$

$$\approx \underline{11.5^\circ}$$