Section 3.2 – Graphing Functions

Increasing and Decreasing Functions

Corollary

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b)
2. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on (a, b)
3. If $f'(x) = 0$ for all x in (a, b), then f is constant on (a, b)

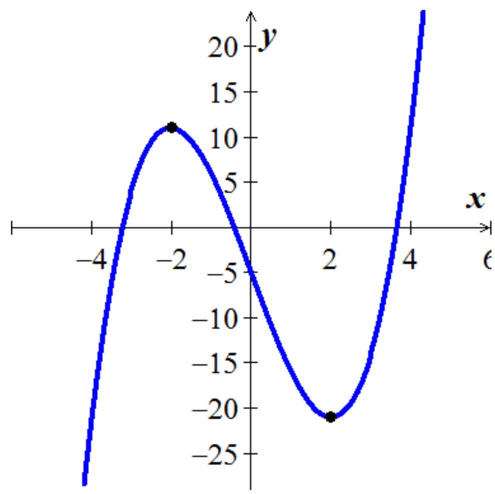
Example

Find the open intervals on which the function $f(x) = x^3 - 12x - 5$ is increasing or decreasing

Solution

$$f'(x) = 3x^2 - 12$$

$$3\left(x^2 - 4\right) = 0 \Rightarrow \boxed{x = \pm 2} \qquad (\textbf{\textit{CN}})$$

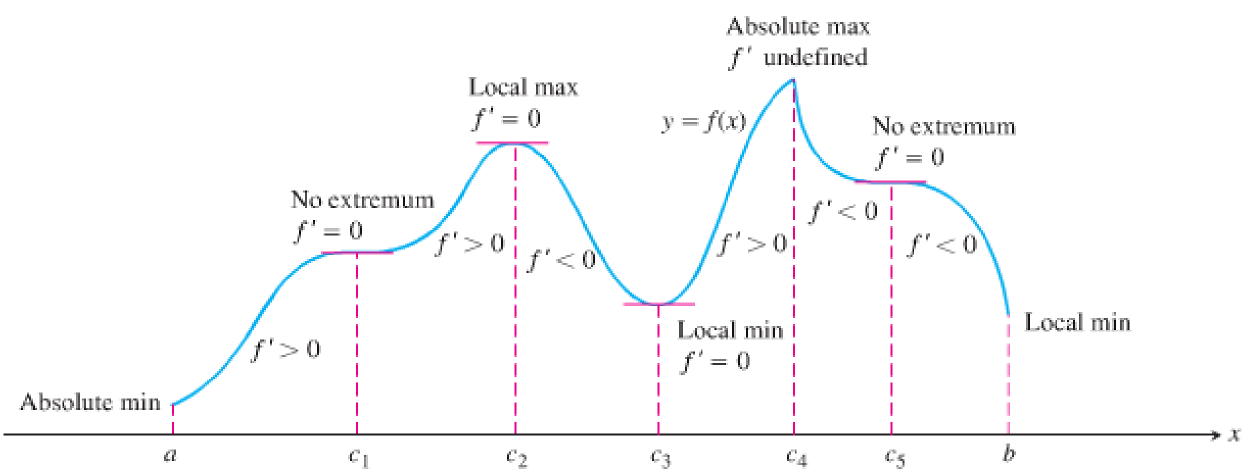


$-\infty$ $\quad -2 \quad$ 2 $\quad \infty$

$f'(-3) > 0$	$f'(0) < 0$	$f'(3) > 0$
Increasing	*Decreasing*	*Increasing*

Increasing: $(-\infty, -2)$ and $(2, \infty)$

Decreasing: $(-2, 2)$



First Derivative Test for Local Extrema

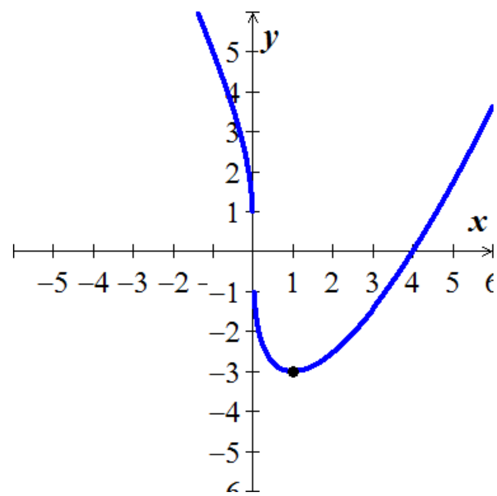
Suppose that c is a critical point of a continuous function f.

1. If f' changes from negative to positive at c, then f has a local minimum (***LMIN***).
2. If f' changes from positive to negative at c, then f has a local maximum (***LMAX***).
3. If f' doesn't change sign at c, then f has no local extremum at c.

Example

Find the open intervals on which the function $f(x) = x^{1/3}(x-4)$ is increasing or decreasing

Solution



$$f(x) = x^{4/3} - 4x^{1/3}$$

$$f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3}\left(x^{1/3} - x^{-2/3}\right)\frac{x^{2/3}}{x^{2/3}}$$

$$= \frac{4}{3}\frac{x-1}{x^{2/3}} = 0$$

$$\Rightarrow \begin{cases} x = 1 \\ x \neq 0 \end{cases} \quad (\boldsymbol{CN})$$

-∞	0	1 ∞
$f'(-1) < 0$	$f'(0.5) < 0$	$f'(2) > 0$
Decreasing	***Decreasing***	***Increasing***

Increasing: $(1, \infty)$

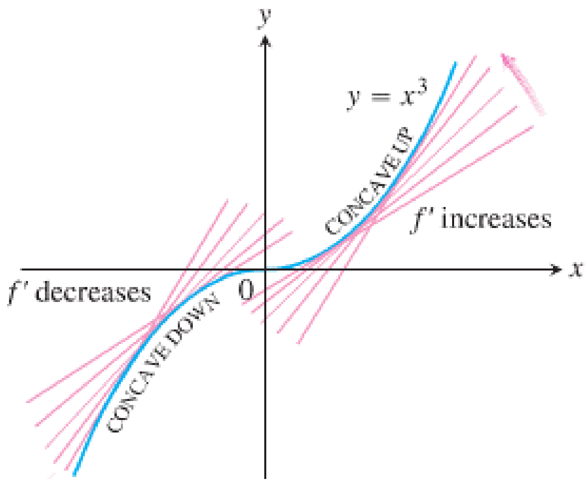
Decreasing: $(-\infty, 1)$

Concavity

Definition

Let f be differentiable on an open interval I . The graph of f is

1. **Concave *upward*** on I if f' is increasing on the interval.
2. **Concave *downward*** on I if f' is decreasing on the interval.



Test for Concavity

Let f be function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then f is ***concave up*** on I.
2. If $f''(x) < 0$ for all x in I, then f is ***concave down*** on I.

   i. Locate the x values @ which $f''(x) = 0$ or undefined
   ii. Use these test x-value to determine the test intervals
   iii. Test the sign of $f''(x)$ in each interval

Example

Determine the intervals on which the graph of the function is concave upward or concave downward.

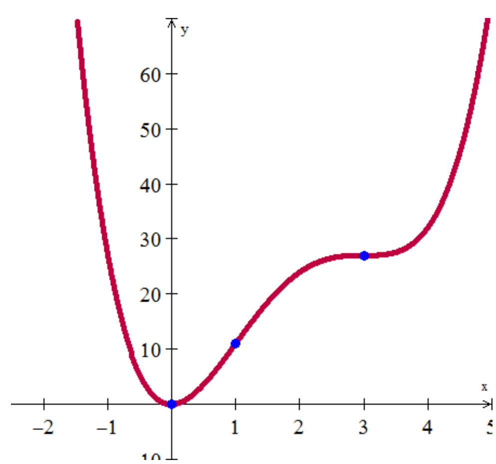
$$f(x) = x^4 - 8x^3 + 18x^2$$

Solution

$$f'(x) = 4x^3 - 24x^2 + 36x$$

$$f''(x) = 12x^2 - 48x + 36 = 0 \quad \to x = 1 \quad x = 3$$

$-\infty$ 1	3	∞
$f''(0) > 0$	$f''(2) < 0$	$f''(4) > 0$
upward	***downward***	***upward***



f is concave upward on $(-\infty,\ 1)$ *and* $(3,\ \infty)$

f is concave downward on $(1,\ 3)$

Example

Determine the concavity of $y = 3 + \sin x$ *on* $[0,\ 2\pi]$

Solution

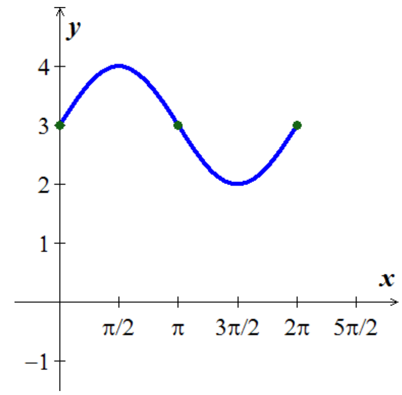
$y' = \cos x$

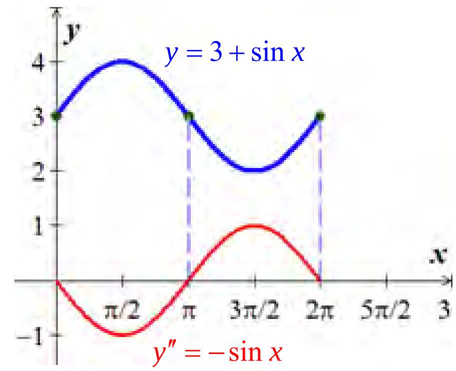
$y'' = -\sin x = 0 \quad \Rightarrow \quad \boxed{x = 0,\ \pi,\ 2\pi}$

0	π	2π
$f''\left(\frac{\pi}{2}\right) < 0$		$f''\left(\frac{3\pi}{2}\right) > 0$
downward		***upward***

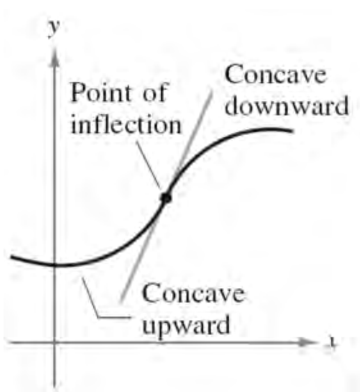
The graph y is ***concave down*** on $(0,\ \pi)$

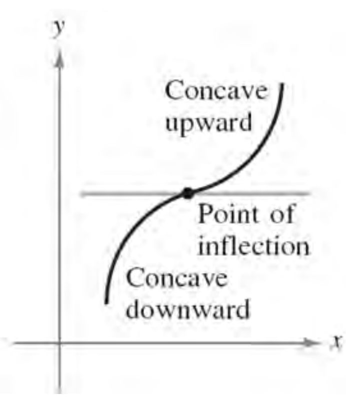
The graph y is ***concave up*** on $(\pi,\ 2\pi)$

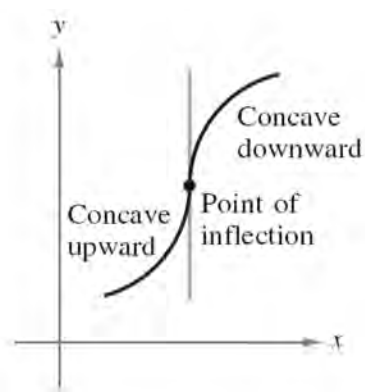




Points of Inflection



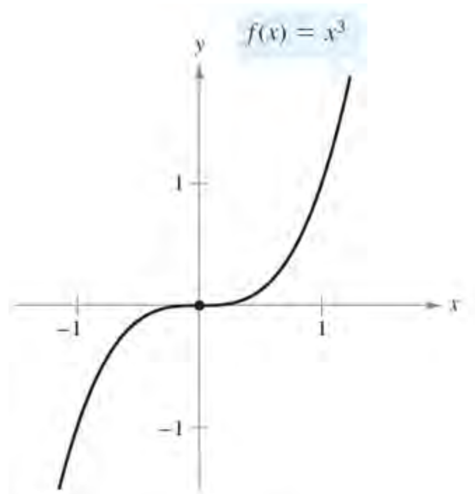




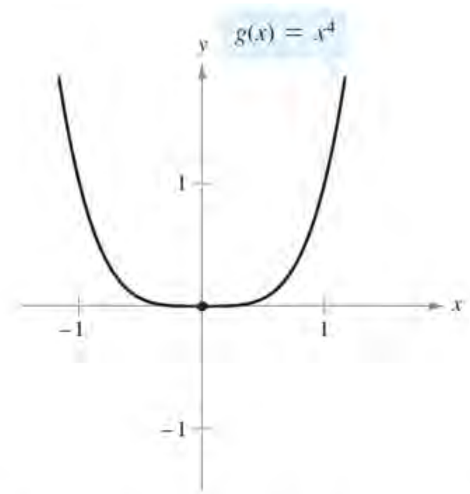
Definition

If the graph of a continuous function has a tangent line @ a point where its concavity changes from upward to downward (or down to upward) then the point is a ***point of inflection***.

At a point of inflection $(c, f(c))$, either $f''(c) = 0$ *or* $f''(c)$ fails to exist.



$f''(0) = 0$, and $(0, 0)$ is a point of inflection.



$g''(0) = 0$, but $(0, 0)$ is not a point of inflection.

Example

A particle is moving along a horizontal coordinate line (positive to the right) with position function

$$s(t) = 2t^3 - 14t^2 + 22t - 5, \quad t \ge 0$$

Find the velocity and acceleration, and describe the motion of the particle.

Solution

The velocity is: $v(t) = s'(t) = 6t^2 - 28t + 22 = 0 \quad \Rightarrow \boxed{t = 1, \ \frac{11}{3}}$

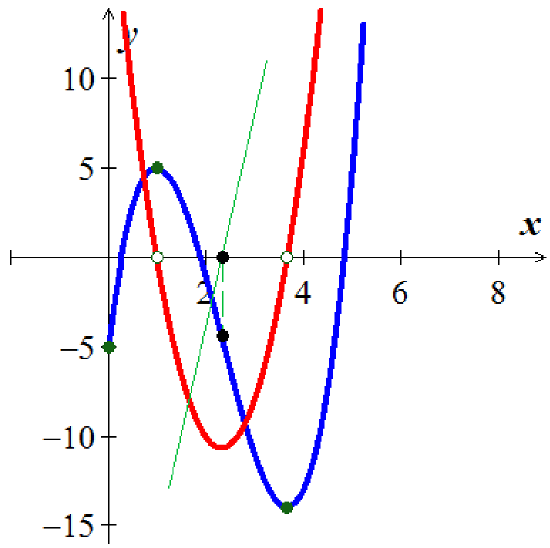
The acceleration is: $a(t) = v'(t) = 12t - 28 = 0 \quad \Rightarrow \boxed{t = \frac{7}{3}}$

0	1	$\frac{7}{3}$	$\frac{11}{3}$

$f'(.5) > 0$	$f'(2) < 0$	$f'(4) > 0$
Increasing	***Decreasing***	***Increasing***
right	***left***	***right***

$f''(1) < 0$	$f''(4) > 0$
Concave down	***Concave up***

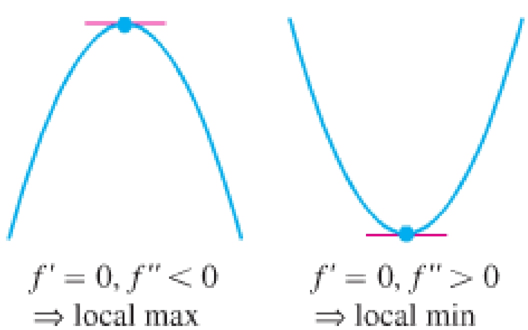
The particle starts moving to the right while slowing down, and then reverses by moving to the left at $t=1$ under the influence of the leftward acceleration over the time interval $\left[0,\ \frac{7}{3}\right)$. The acceleration then changes direction changes direction at $t=\frac{7}{3}$ but the particles continues moving leftward, while slowing down under the rightward acceleration. At $t=\frac{11}{3}$ the particle reverses direction again; moving to the right in the same direction as the acceleration.



Second Derivative Test for local Extrema

Let $f'(c)=0$ and let f'' exist ($\exists$)

1. If $f'(c)=0$ *and* $f''(c)>0 \Rightarrow f$ is a local Minimum at $x=c$
2. If $f'(c)=0$ *and* $f''(c)<0 \Rightarrow f$ is a local Maximum at $x=c$
3. If $f'(c)=0$ *and* $f''(c)=0 \Rightarrow$ Test fails $\rightarrow$ use f' to determine Max, Min.



Example

Sketch a graph of the function $f(x) = x^4 - 4x^3 + 10$ using the following steps

a) Identify where the extrema of f occur

b) Find the intervals on which f is increasing and decreasing

c) Find where the graph of f is concave up and down

d) Sketch the general shape of the graph for f

e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

Solution

$$f'(x) = 4x^3 - 12x^2$$

$$= 4x^2(x-3) = 0 \quad \Rightarrow \quad \boxed{x = 0,\ 0} \quad \boxed{x = 3} \quad (CN)$$

$-\infty$ 0	0 3	3 ∞
$f'(-1) < 0$ ↘	$f'(1) < 0$ ↘	↗ $f'(4) > 0$
decreasing	***decreasing***	***increasing***

a) $x = 3 \quad \Rightarrow \quad \lfloor y = 3^4 - 4(3)^3 + 10 = \underline{-17} \rfloor$

A local minimum at (3, −17)

b) f is ***decreasing***: $(-\infty,\ 0] \cup [0,\ 3)$

f is ***increasing***: $(3,\ \infty)$

c) $f''(x) = 12x^2 - 24x = 12x(x-2) = 0$

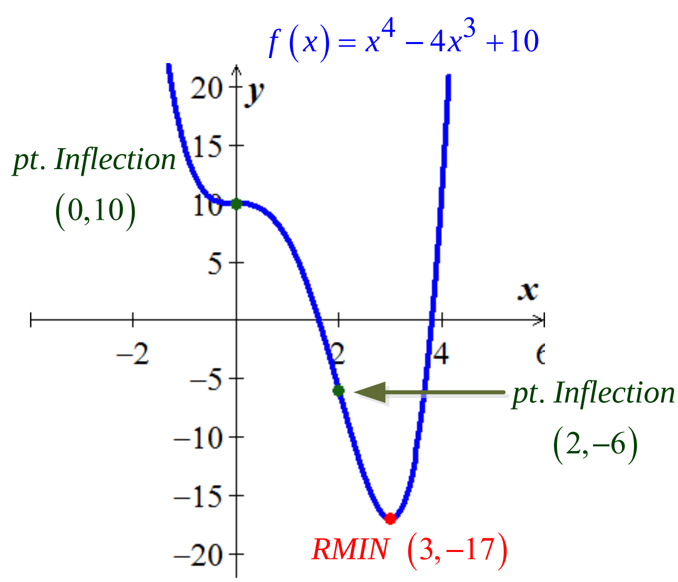
$$\Rightarrow \begin{cases} \boxed{x = 0} & \rightarrow f(0) = 10 \\ \boxed{x = 2} & \rightarrow f(2) = -6 \end{cases}$$

f is ***concave up***: $(-\infty,\ 0) \cup (2,\ \infty)$

f is ***concave down***: $(0,\ 2)$

$-\infty$ 0	0 2	2 ∞
$f''(-1) > 0$	$f''(1) < 0$	$f''(3) > 0$
Concave up	***Concave down***	***Concave up***

d) $f(x = 0) = 0^4 - 4(0)^3 + 10 = 10$



Example

Sketch the graph of $f(x)=\dfrac{(x+1)^2}{1+x^2}$

Solution

Domain of f is $(-\infty,\ \infty)$ **Horizontal Asymptotes** $y=1$

$$f'(x)=\frac{2(x+1)\left(1+x^2\right)-2x(x+1)^2}{\left(1+x^2\right)^2}$$

$u=(x+1)^2 \quad v=1+x^2$

$u'=2(x+1) \quad v'=2x$

$$=\frac{2(x+1)\left[\left(1+x^2\right)-x(x+1)\right]}{\left(1+x^2\right)^2}$$

$$=\frac{2(x+1)\left[1+x^2-x^2-x\right]}{\left(1+x^2\right)^2}$$

$$=\frac{2(x+1)(1-x)}{\left(1+x^2\right)^2}$$

$$=2\frac{1-x^2}{\left(1+x^2\right)^2}$$

$\rightarrow (x+1)(1-x)=0 \quad \Rightarrow \quad \boxed{x=\pm 1}$ (CN)

$$f'(x)=2\left(1-x^2\right)\left(1+x^2\right)^{-2}$$

$$f''(x)=2\left(1+x^2\right)^{-3}\left[-2x\left(1+x^2\right)-2(2x)\left(1-x^2\right)\right]$$

$$\left(U^mV^n\right)'=U^{m-1}V^{n-1}\left(mU'V+nUV'\right)$$

$$=2\frac{-2x-2x^3-4x+4x^3}{\left(1+x^2\right)^3}$$

$$=2\frac{2x^3-6x}{\left(1+x^2\right)^3}$$

$$=\frac{4x\left(x^2-3\right)}{\left(1+x^2\right)^3}=0$$

$\rightarrow \boxed{x=0}$ $\boxed{x=\pm\sqrt{3}}$ ***Point of inflections***

$-\infty$	$-\sqrt{3}$	-1	**0**	1	$\sqrt{3}$	∞
$-$ *Decreasing*		$+$ *Increasing*			$-$ *Decreasing*	
$-$ *Concave down*	$+$ *Concave up*		$-$ *Concave down*		$+$ *Concave up*	

RMAX: $(1,\ 2)$

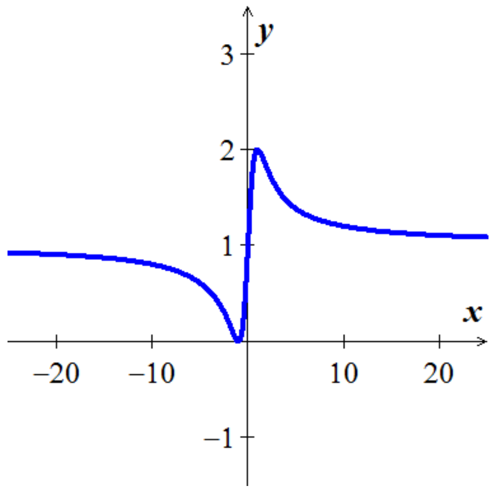
RMIN: $(-1,\ 0)$

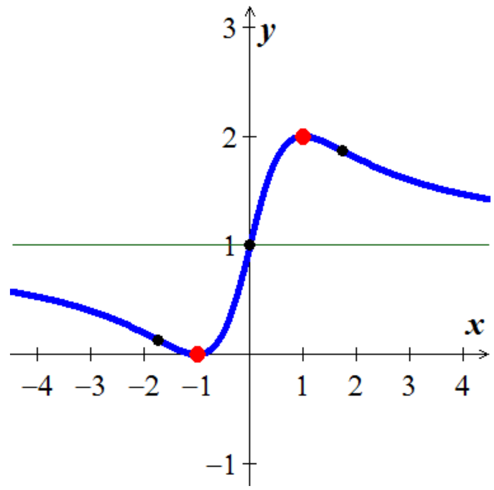
Decreasing: $(-\infty,\ -1)\cup(1,\ \infty)$

Increasing: $(-1,\ 1)$

Concave down: $\boxed{\left(-\infty,\ -\sqrt{3}\right)\cup\left(0,\ \sqrt{3}\right)}$

Concave up: $\boxed{\left(-\sqrt{3},\ 0\right)\cup\left(\sqrt{3},\ \infty\right)}$





Exercises Section 3.2 – Graphing Functions

Find the critical numbers and the open intervals on which the function is increasing or decreasing.

1. $f(x) = x^3 + 3x^2 - 9x + 4$

2. $f(x) = (x-1)^{2/3}$

3. $f(x) = x\sqrt{x+1}$

4. $f(x) = \dfrac{x}{x^2+4}$

5. $f(x) = \dfrac{x}{x^2+1}$

6. $f(x) = x\sqrt{x+1}$

7. $f(x) = x^3 - 12x$

8. $f(x) = x^{2/3}$

Find the open intervals on which the function is increasing and decreasing. Then, identify the function's local and absolute extreme values, if any, saying where they occur.

9. $g(t) = -t^2 - 3t + 3$

10. $h(x) = 2x^3 - 18x$

11. $f(\theta) = 3\theta^2 - 4\theta^3$

12. $g(x) = x^4 - 4x^3 + 4x^2$

13. $f(x) = x - 6\sqrt{x-1}$

14. $f(x) = \dfrac{x^3}{3x^2+1}$

15. $f(x) = x^{1/3}(x+8)$

Find all relative Extrema as well as where the function is increasing and decreasing

16. $f(x) = 2x^3 - 6x + 1$

17. $f(x) = 6x^{2/3} - 4x$

18. $f(x) = x^4 - 4x^3$

19. $f(x) = 3x^{2/3} - 2x$

20. $y = \sqrt{4 - x^2}$

21. $f(x) = x\sqrt{x+1}$

22. $f(x) = \dfrac{x}{x^2+1}$

23. $f(x) = x^4 - 8x^2 + 9$

Find the local extrema of each function on the given interval, and say where they occur

24. $f(x) = \sin 2x \quad 0 \le x \le \pi$

25. $f(x) = \sqrt{3}\cos x + \sin x \quad 0 \le x \le 2\pi$

26. $f(x) = \frac{x}{2} - 2\sin\frac{x}{2} \quad 0 \le x \le 2\pi$

27. $f(x) = \sec^2 x - 2\tan x \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Determine the intervals on which the graph of the function is concave upward or concave downward.

28. $f(x) = \dfrac{x^2-1}{2x+1}$

29. $f(x) = -4x^3 - 8x^2 + 32$

30. $f(x) = \dfrac{12}{x^2+4}$

31. Find the points of inflection. $f(x) = x^3 - 9x^2 + 24x - 18$

32. Does $f(x) = 2x^5 - 10x^4 + 20x^3 + x + 1$ have any inflection points? If so, identify them.

33. Find the second derivative of $f(x) = -2\sqrt{x}$ and discuss the concavity of the graph

34. Find the extrema using the second derivative test $f(x) = \frac{4}{x^2+1}$

35. Discuss the concavity of the graph of *f* and find its points of inflection. $f(x) = x^4 - 2x^3 + 1$

36. Find all relative extrema of $f(x) = x^4 - 4x^3 + 1$

Sketch the graph

37. $y = x^3 - 3x + 3$

38. $y = -x^4 + 6x^2 - 4$

39. $y = x\left(\frac{x}{2} - 5\right)^4$

40. $y = x + \sin x \quad 0 \le x \le 2\pi$

41. $y = \cos x + \sqrt{3}\sin x \quad 0 \le x \le 2\pi$

42. $y = \frac{x}{\sqrt{x^2+1}}$

43. $y = x^2 + \frac{2}{x}$

44. $y = \frac{x^2-3}{x-2}$

45. $y = \frac{5}{x^4+5}$

46. $y = \frac{x^2-49}{x^2+5x-14}$

47. $y = \frac{x^4+1}{x^2}$

48. $y = \frac{x^2-4}{x^2-1}$

49. $y = -\frac{x^2-x+1}{x-1}$

50. $y = \frac{x^3-3x^2+3x-1}{x^2+x-2}$

51. $y = \frac{4x}{x^2+4}$

52. $f(x) = \frac{x^2+4}{2x}$

53. $f(x) = \frac{1}{2}x^4 - 3x^2 + 4x + 1$

54. $f(x) = \frac{3x}{x^2+3}$

55. $f(x) = 4\cos(\pi(x-1)) \quad on\ [0,\ 2]$

56. $f(x) = \frac{x^2+x}{4-x^2}$

57. $f(x) = \sqrt[3]{x} - \sqrt{x} + 2$

58. $f(x) = \frac{\cos \pi x}{1+x^2} \quad on\ [-2,\ 2]$

59. $f(x) = x^{2/3} + (x+2)^{1/3}$

60. $f(x) = x(x-1)e^{-x}$

61. The revenue R generated from sales of a certain product is related to the amount x spent on advertising by

$$R(x) = \frac{1}{15{,}000}\left(600x^2 - x^3\right), \qquad 0 \le x \le 600$$

Where x and R are in thousands of dollars.
Is there a point of diminishing returns for this function?

62. Find the point of diminishing returns (x, y) for the function

$$R(x) = -x^3 + 45x^2 + 400x + 8000, \quad 0 \le x \le 20$$

where $R(x)$ represents revenue in thousands of dollars and x represents the amount spent on advertising in tens of thousands of dollars.

63. A county realty group estimates that the number of housing starts per year over the next three years will be

$$H(r) = \frac{300}{1 + 0.03r^2}$$

Where r is the mortgage rate (in percent).

a) Where is $H(r)$ increasing?

b) Where is $H(r)$ decreasing?

64. Suppose the total cost $C(x)$ to manufacture a quantity x of insecticide (in hundreds of liters) is given by $C(x) = x^3 - 27x^2 + 240x + 750$. Where is $C(x)$ decreasing?

65. A manufacturer sells telephones with cost function $C(x) = 6.14x - 0.0002x^2, \quad 0 \le x \le 950$ and revenue function $R(x) = 9.2x - 0.002x^2, \quad 0 \le x \le 950$. Determine the interval(s) on which the profit function is increasing.

66. The cost of a computer system increases with increased processor speeds. The cost C of a system as a function of processor speed is estimated as $C(x) = 14x^2 - 4x + 1200$, where x is the processor speed in *MHz*. Determine the intervals where the cost function $C(x)$ is decreasing.

67. The percent of concentration of a drug in the bloodstream t hours after the drug is administered is given by $K(t) = \dfrac{t}{t^2 + 36}$. On what time interval is the concentration of the drug increasing?

68. Coughing forces the trachea to contract, this in turn affects the velocity of the air through the trachea. The velocity of the air during coughing can be modeled by: $v = k\left(R - r\right)r^2, \quad 0 \le r < R$ where k is a constant, R is the normal radius of the trachea (also a constant) and r is the radius of the trachea during coughing. What radius r will produce the maximum air velocity?

69. $P(x) = -x^3 + 15x^2 - 48x + 450, \quad x \ge 3$ is an approximation to the total profit (in thousands of dollars) from the sale of x hundred thousand tires. Find the number of hundred thousands of tires that must be sold to maximize profit.

70. $P(x) = -x^3 + 3x^2 + 360x + 5000; \quad 6 \le x \le 20$ is an approximation to the number of salmon swimming upstream to spawn, where x represents the water temperature in degrees Celsius. Find the temperature that produces the maximum number of salmon.