3.1 Inner Products

$\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v}$

Euclidean
standard

Axiom

1- $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

2. $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

3- $\langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle$

4- $\langle \vec{v}, \vec{v} \rangle \geq 0$ and $\langle \vec{v}, \vec{v} \rangle = 0$ iff $\vec{v} = \vec{0}$

Defn V is a real inner product space,

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$$

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$$
$$= \sqrt{\langle \vec{u} - \vec{v}, \vec{u} - \vec{v} \rangle}$$

$\|\vec{v}\| \geq 0$

$\|k\vec{v}\| = |k| \, \|\vec{v}\|$

$d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u}) \geq 0$

$= 0$ iff $\vec{u} = \vec{v}$

Weighted Euclidean Inner Product

$$\langle \vec{u}, \vec{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n$$

$w_1, w_2, \ldots, w_n \quad \mathbb{R}^+$

$\vec{u} = (u_1, u_2, \ldots, u_n)$

$\vec{v} = (v_1, v_2, \ldots, v_n)$

Ex $\vec{u} = (u_1, u_2)$ — $\vec{v} = (v_1, v_2)$ $\quad \mathbb{R}^2$

$$\langle \vec{u}, \vec{v} \rangle = 3u_1 v_1 + 2u_2 v_2 \Leftarrow$$

a) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= 3u_1 v_1 + 2u_2 v_2 \\ &= 3v_1 u_1 + 2v_2 u_2 \\ &= \langle \vec{v}, \vec{u} \rangle \ \checkmark \end{aligned}$$

b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

let $\vec{w} = \langle w_1, w_2 \rangle$

$$\begin{aligned} \langle \vec{u} + \vec{v}, \vec{w} \rangle &= \langle (u_1, u_2) + (v_1, v_2), \vec{w} \rangle \\ &= \langle (u_1 + v_1, u_2 + v_2), \vec{w} \rangle \\ &= 3(u_1 + v_1) w_1 + 2(u_2 + v_2) w_2 \\ &= (3u_1 w_1 + 2u_2 w_2) + (3v_1 w_1 + 2v_2 w_2) \\ &= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \ \checkmark \end{aligned}$$

c) $\langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle$

$$\begin{aligned} \langle k\vec{u}, \vec{v} \rangle &= 3k u_1 v_1 + 2k u_2 v_2 \\ &= k(3u_1 v_1 + 2u_2 v_2) \\ &= k \langle \vec{u}, \vec{v} \rangle \ \checkmark \end{aligned}$$

d) $\langle \vec{v}, \vec{v} \rangle \geq 0$ & $\langle \vec{v}, \vec{v} \rangle = 0$ iff $\vec{v} = 0$

$$\langle \vec{v}, \vec{v} \rangle = 3 v_1 v_1 + 2 v_2 v_2$$
$$= 3 v_1^2 + 2 v_2^2 \geq 0$$

If $v_1 = v_2 = 0 \Rightarrow \vec{v} = \vec{0}$

dot $\bullet$ to Inner Product $\langle \ \rangle$

3.2 Angle & Orthogonality

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$

$$\theta = \cos^{-1} \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \, \|\vec{v}\|}$$

Ex cosine angle θ $\cos\theta$?

$\vec{u} = (4, 3, 1, -2)$ $\vec{v} = (-2, 1, 2, 3)$

$$\cos\theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \, \|\vec{v}\|}$$
$$= \frac{(4,3,1,-2) \cdot (-2,1,2,3)}{\sqrt{16+9+1+4}\,\sqrt{4+1+4+9}}$$
$$= \frac{-8+3+2-6}{\sqrt{30}\,\sqrt{18}}$$
$$= \frac{-9}{\sqrt{30}\;3\sqrt{2}}$$
$$= \frac{-3}{2\sqrt{15}}$$

Theorem: $\| \langle \vec{u}, \vec{v} \rangle \| \leq \|\vec{u}\| \, \|\vec{v}\|$

Proof

$\vec{u} = \vec{v} = \vec{0} \Rightarrow 0 = 0$

if $\vec{u}$ & $\vec{v} \neq \vec{0}$

let $\vec{w}$ any vector $\ni \|\vec{w}\| \geq 0$

let $\vec{w} = \vec{u} - t\vec{v}$

$$0 \leq \vec{w} \cdot \vec{w}$$

$$= (\vec{u} - t\vec{v}) \cdot (\vec{u} - t\vec{v})$$

$$= \vec{u} \cdot \vec{u} - t(\vec{u} \cdot \vec{v}) - t(\vec{v} \cdot \vec{u}) + t^2 \vec{v}^2$$

$$= \vec{u} \cdot \vec{u} - 2t(\vec{u} \cdot \vec{v}) + t^2 (\vec{v} \cdot \vec{v})$$

let $t = \dfrac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$ $(\vec{u} \cdot \vec{v})$

$$= \vec{u} \cdot \vec{u} - 2 \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} + \left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \right)^2 (\vec{v} \cdot \vec{v})$$

$$= \vec{u} \cdot \vec{u} - 2 \frac{(\vec{u} \cdot \vec{v})^2}{\vec{v} \cdot \vec{v}} + \frac{(\vec{u} \cdot \vec{v})^2}{\vec{v} \cdot \vec{v}}$$

$$= \vec{u} \cdot \vec{u} - \frac{(\vec{u} \cdot \vec{v})^2}{\vec{v} \cdot \vec{v}}$$

$$= \frac{(\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v}) - (\vec{u} \cdot \vec{v})^2}{\vec{v} \cdot \vec{v}}$$

$\vec{v} \cdot \vec{v} > 0$

$$\leq (\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v}) - (\vec{u} \cdot \vec{v})^2$$

$$(\vec{u} \cdot \vec{v})^2 \leq (\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v})$$

$$\| \langle \vec{u}, \vec{v} \rangle \| \leq \|\vec{u}\| \, \|\vec{v}\| \checkmark$$

Proof $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

$$\|\vec{u}+\vec{v}\| = \sqrt{\langle \vec{u}+\vec{v},\ \vec{u}+\vec{v} \rangle} \qquad \vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$$

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 &= \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v} \\
&= \langle \vec{u}, \vec{u} \rangle + 2\langle \vec{u}, \vec{v} \rangle \quad \langle \vec{v}, \vec{v} \rangle \\
&\leq \|\vec{u}\|^2 + 2\|\vec{u}\|\,\|\vec{v}\| + \|\vec{v}\|^2 \\
&= (\|\vec{u}\| + \|\vec{v}\|)^2
\end{aligned}$$

$$\|\vec{u}+\vec{v}\|^2 \leq (\|\vec{u}\| + \|\vec{v}\|)^2$$

$$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

Defn 2 vectors $\vec{u}$ & $\vec{v}$ are orthogonal if $\langle \vec{u}, \vec{v} \rangle = 0$

Ex $U = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad V = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$

$$U \bullet V = 0 + 0 + 0 + 0$$
$$= \underline{0}$$

$\therefore$ U & V are orthogonal

Defn If W is a subspace of an inner product V (space), then the set of all vectors are orthogonal to every vector in W is an orthogonal complement of W: $W^\perp$

$W^\perp$ is a subspace of V

$W \cap W^\perp = \{0\}$

$(W^\perp)^\perp = W$

Ex W in $\mathbb{R}^6$

$\vec{w}_1 = (1, 3, -2, 0, 2, 0)$

$\vec{w}_2 = (2, 6, -5, -2, 4, -3)$

$\vec{w}_3 = (0, 0, 5, 10, 0, 15)$

$\vec{w}_4 = (2, 6, 0, 8, 4, 18)$

$$A = \begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{pmatrix} \xrightarrow{rref}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \boxed{\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= 0 \end{aligned}}$$

$(x_1, x_2, x_3, x_4, x_5, x_6) = (-3x_2 - 4x_4 - 2x_5, x_2, -2x_4, x_4, x_5, 0)$

$$\vec{v}_1 = (-3, 1, 0, 0, 0, 0)$$
$$\vec{v}_2 = (-4, 0, -2, 1, 0, 0)$$
$$\vec{v}_3 = (-2, 0, 0, 0, 1, 0)$$

Ex

$$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$Q^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= Q^T$$

3.3 Gram_Schmidt Process

$$\vec{u} = \frac{\langle \vec{u}, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 + \cdots + \frac{\langle \vec{u}, \vec{v}_n \rangle}{\|v_n\|^2} \vec{v}_n$$

$\{\vec{u}_1, \vec{u}_2, ---, \vec{u}_n\} \perp \to \{\vec{v}_1, ---, \vec{v}_n\}$

let $\vec{v}_1 = \vec{u}_1$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

orthogonal Basis

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$