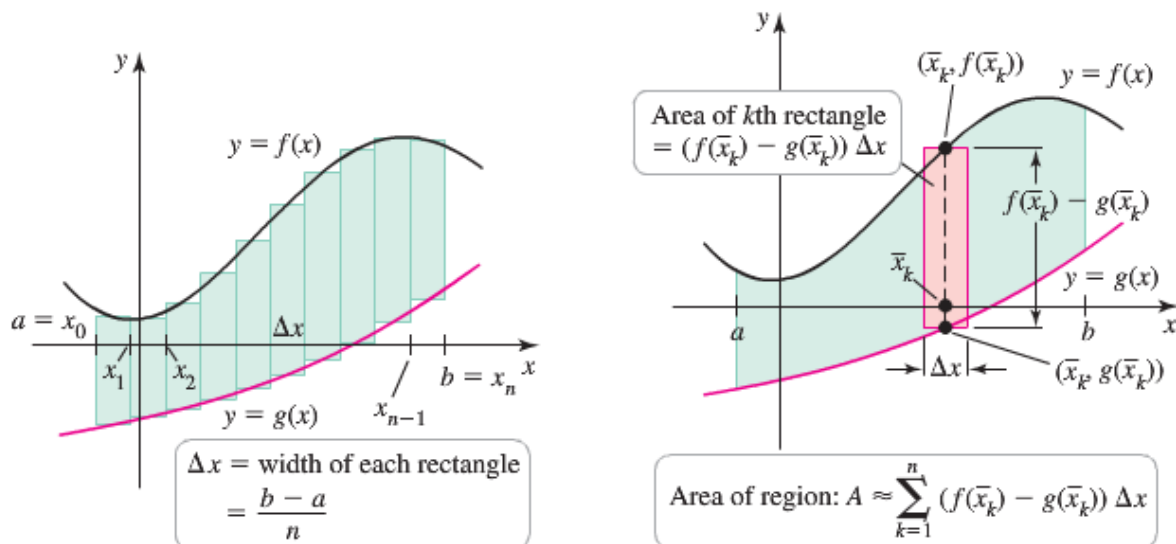


## Section 1.2 – Region between Curves

### Areas between Curves



### Definition

If  $f$  and  $g$  are continuous with  $f(x) \geq g(x)$  throughout  $[a, b]$ , then the **area of the region between the curves**  $y = f(x)$  and  $y = g(x)$  **from  $a$  to  $b$**  is:

$$A = \int_a^b [f(x) - g(x)] dx$$

### Example

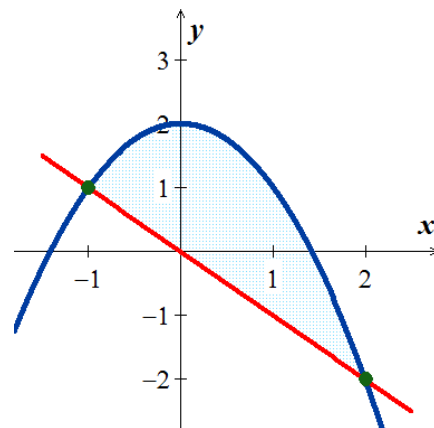
Find the area of the region enclosed by the parabola  $y = 2 - x^2$  and the line  $y = -x$ .

### Solution

The limits of integrations are found by letting:

$$2 - x^2 = -x \quad \Rightarrow \quad x^2 - x - 2 = 0 \quad \rightarrow \quad \underline{x = -1, 2}$$

$$\begin{aligned} A &= \int_{-1}^2 [f(x) - g(x)] dx \\ &= \int_{-1}^2 [2 - x^2 - (-x)] dx \\ &= \int_{-1}^2 (2 - x^2 + x) dx \end{aligned}$$



$$\begin{aligned}
&= \left[ 2x - \frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^2 \\
&= 4 - \frac{8}{3} + 2 - \left( -2 + \frac{1}{3} + \frac{1}{2} \right) \\
&= 8 - \frac{8}{3} - \frac{1}{3} - \frac{1}{2} \\
&= 5 - \frac{1}{2} \\
&= \frac{9}{2} \text{ unit}^2
\end{aligned}$$

### Example

Find the area of the region in the first quadrant that is bounded above by  $y = \sqrt{x}$  and below the  $x$ -axis and the line  $y = x - 2$

### Solution

$$(y = \sqrt{x}) \cap (y = 0) \rightarrow (0, 0)$$

$$(y = \sqrt{x}) \cap (y = x - 2) \rightarrow \sqrt{x} = x - 2$$

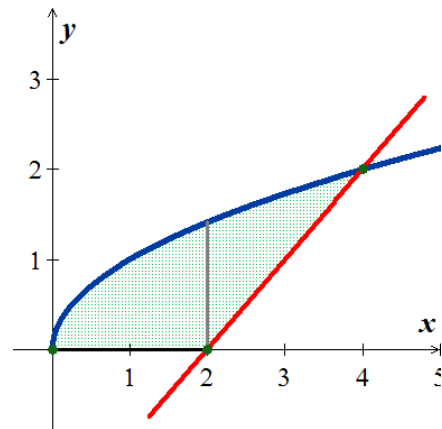
$$(\sqrt{x})^2 = (x - 2)^2$$

$$x = x^2 - 4x + 4$$

$$x^2 - 5x + 4 = 0$$

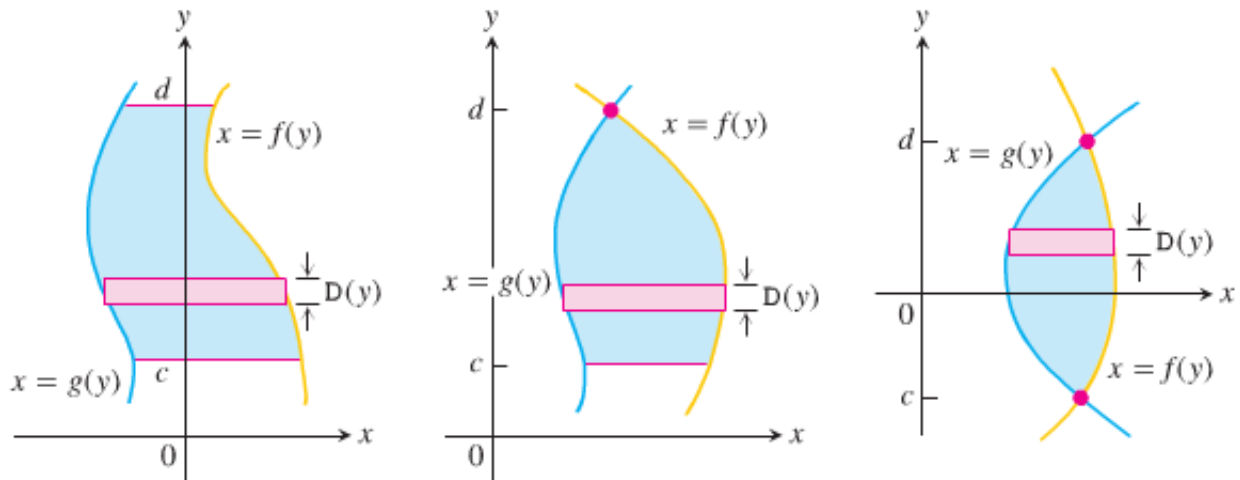
$$\rightarrow x = \cancel{X}, 4$$

$$(y = 0) \cap (y = x - 2) \rightarrow x = 2$$



$$\begin{aligned}
\text{Total Area} &= \int_0^2 [\sqrt{x} - 0] dx + \int_2^4 [\sqrt{x} - (-x + 2)] dx \\
&= \left[ \frac{2}{3} x^{3/2} \right]_0^2 + \left[ \frac{2}{3} x^{3/2} - \frac{x^2}{2} + 2x \right]_2^4 \\
&= \left[ \frac{2}{3} (2^{3/2}) - 0 \right] + \left( \frac{2}{3} 4^{3/2} - \frac{4^2}{2} + 2(4) \right) - \left( \frac{2}{3} 2^{3/2} - \frac{2^2}{2} + 2(2) \right) \\
&= \frac{2}{3} (2^{3/2}) + \frac{2}{3} 4^{3/2} - \frac{16}{2} + 8 - \frac{2}{3} 2^{3/2} + \frac{4}{2} - 4 \\
&= \frac{2}{3} (8) - 2 \\
&= \frac{10}{3} \text{ unit}^2
\end{aligned}$$

## Integration with Respect to $y$



$$A = \int_c^d [f(y) - g(y)] dy \quad (\text{From right hand to left hand})$$

### Example

Find the area of the region by integrating with respect to  $y$ , in the first quadrant that is bounded above by  $y = \sqrt{x}$  and below the  $x$ -axis and the line  $y = x - 2$ .

### Solution

$$y = \sqrt{x} \rightarrow x = y^2$$

$$y = x - 2 \rightarrow x = y + 2$$

$$(x = y^2) \cap (y = 0) \rightarrow (0, 0)$$

$$(x = y^2) \cap (x = y + 2) \rightarrow y^2 = y + 2$$

$$y^2 - y - 2 = 0 \rightarrow y = -1, 2$$

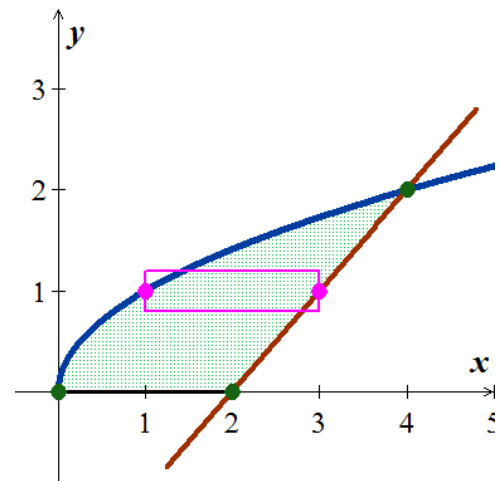
$$(y = 0) \cap (x = y + 2) \rightarrow y = 0$$

$$A = \int_0^2 [y + 2 - y^2] dy$$

$$= \left[ \frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_0^2$$

$$= \frac{2^2}{2} + 2(2) - \frac{2^3}{3} - 0$$

$$= \frac{10}{3} \text{ unit}^2$$



## Exercises      Section 1.2 – Region between Curves

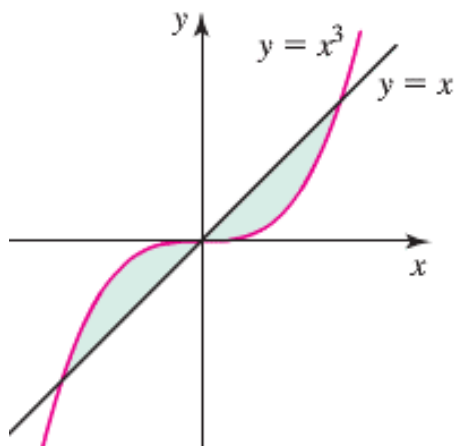
Find the area of the region bounded by the graphs of

1.  $y = 2x - x^2$  and  $y = -3$
2.  $y = 7 - 2x^2$  and  $y = x^2 + 4$
3.  $y = x^4 - 4x^2 + 4$  and  $y = x^2$
4.  $x = 2y^2$ ,  $x = 0$ , and  $y = 3$
5.  $x = y^3 - y^2$  and  $x = 2y$
6.  $4x^2 + y = 4$  and  $x^4 - y = 1$
7.  $y = \sin \frac{\pi x}{2}$  and  $y = x$
8.  $y = 3 - x^2$  and  $y = 2x$
9.  $y = x^2 - x - 2$  and  $x$ -axis
10.  $y = \sqrt{x}$ ,  $y = x\sqrt{x}$
11.  $y = x^{1/2}$  and  $y = x^3$
12.  $x + 4y^2 = 4$ ,  $x + y^4 = 1$ ,  $x \geq 0$
13.  $y = 2\sin x$ ,  $y = \sin 2x$ ,  $0 \leq x \leq \pi$
14.  $y = x^2 + 1$  and  $y = x$  for  $0 \leq x \leq 2$
15.  $y = x^2 - 2x$  and  $y = x$  on  $[0, 4]$
16.  $x = 1$ ,  $x = 2$ ,  $y = x^3 + 2$ ,  $y = 0$
17.  $y = x^2 - 18$ ,  $y = x - 6$
18.  $y = -x^2 + 3x + 1$ ,  $y = -x + 1$
19.  $y = x$ ,  $y = 2 - x$ ,  $y = 0$
20.  $y = \frac{4}{x^2}$ ,  $y = 0$ ,  $x = 1$ ,  $x = 4$
21.  $f(y) = y^2$ ,  $g(y) = y + 2$
22.  $f(x) = 2^x$ ,  $g(x) = \frac{3}{2}x + 1$
23.  $x = \sqrt[3]{y}$  and  $x = \sqrt[5]{y}$
24.  $f(x) = x^3 + 2x^2 - 3x$ ,  $g(x) = x^2 + 3x$
25.  $y = \sec^2 x$ ,  $y = \tan^2 x$ ,  $x = -\frac{\pi}{4}$ ,  $x = \frac{\pi}{4}$
26.  $f(x) = -x^2 + 1$ ,  $g(x) = 2x + 4$ ,  $x = -1$ ,  $x = 2$
27.  $f(x) = \sqrt{x} + 3$ ,  $g(x) = \frac{1}{2}x + 3$
28.  $f(x) = \sqrt[3]{x-1}$ ,  $g(x) = x - 1$
29.  $f(y) = y(2 - y)$ ,  $g(y) = -y$
30.  $f(y) = \frac{y}{\sqrt{16 - y^2}}$ ,  $g(y) = 0$ ,  $y = 3$
31.  $f(y) = y^2 + 1$ ,  $g(y) = 0$ ,  $y = -1$ ,  $y = 2$
32.  $f(x) = \frac{10}{x}$ ,  $x = 0$ ,  $y = 2$ ,  $y = 10$
33.  $g(x) = \frac{4}{2 - x}$ ,  $y = 4$ ,  $x = 0$
34.  $f(x) = \cos x$ ,  $g(x) = 2 - \cos x$ ,  $0 \leq x \leq 2\pi$
35.  $f(x) = \sin x$ ,  $g(x) = \cos 2x$ ,  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{6}$
36.  $f(x) = 2\sin x$ ,  $g(x) = \tan x$ ,  $-\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$
37.  $f(x) = \sec \frac{\pi x}{4} \tan \frac{\pi x}{4}$ ,  $g(x) = (\sqrt{2} - 4)x + 4$ ,  $x = 0$
38.  $f(x) = xe^{-x^2}$ ,  $y = 0$ ,  $0 \leq x \leq 1$
39.  $y = \sin x$  and  $y = x$   $0 \leq x \leq 2\pi$
40.  $y = x^2$ ,  $y = 2x^2 - 4x$  and  $y = 0$
41.  $y = 8\cos x$ ,  $y = \sec^2 x$ ,  $-\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$
42.  $y^2 = 4x + 4$ ,  $y = 4x - 16$
43.  $x = 2y^2$ ,  $x = 0$ ,  $y = 3$
44.  $x = y^3$  and  $y = x$

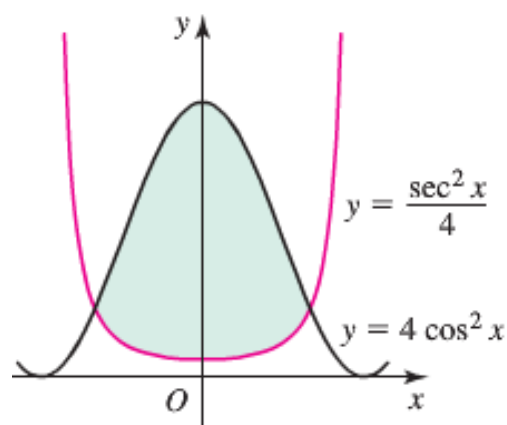
45. Find the area of the region in the first quadrant bounded by  $y = 4x$  and  $y = x\sqrt{25 - x^2}$
46. Find the area of the region in the first quadrant bounded by the curve  $\sqrt{x} + \sqrt{y} = 1$
47. Find the area of the region in the first quadrant bounded by  $y = \frac{x}{6}$  and  $y = 1 - \left|\frac{x}{2} - 1\right|$
48. Find the area of the region in the first quadrant bounded by  $y = x^p$  and  $y = \sqrt[p]{x}$  where  $p = 100$  and  $p = 1000$
49. Consider the functions  $y = \frac{x^2}{a}$  and  $y = \sqrt{\frac{x}{a}}$ , where  $a > 0$ . Find  $A(a)$ , the area of the region between the curves.
50. Find the area between the curves  $y = \ln x$  and  $y = \ln 2x$  from  $x = 1$  to  $x = 5$ .
51. Find the total area of the region enclosed by the curve  $x = y^{2/3}$  and lines  $x = y$  and  $y = -1$ .
52. Find the area of the “triangular region in the first quadrant bounded on the left by the  $y$ -axis and on the right by the curves  $\sin x$  and  $\cos x$ .
53. Find the area of the “triangular region in the first quadrant bounded above by the curve  $y = e^{2x}$ , below by the curve  $y = e^x$ , and on the right by the line  $x = \ln 3$ .
54. Find the area of the triangular region bounded on the left by  $x + y = 2$ , on the right by  $y = x^2$ , and above by  $y = 2$
55. Find the extreme values of  $f(x) = x^3 - 3x^2$  and find the area of the region enclosed by the graph of  $f$  and the  $x$ -axis.

(56 – 59) Determine the area of the shaded region in the following

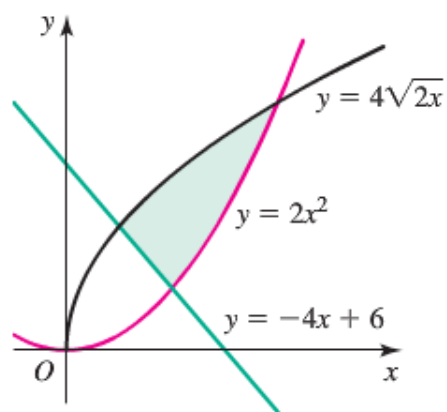
56.



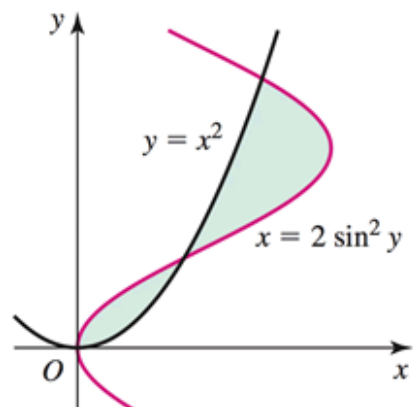
57.



58.

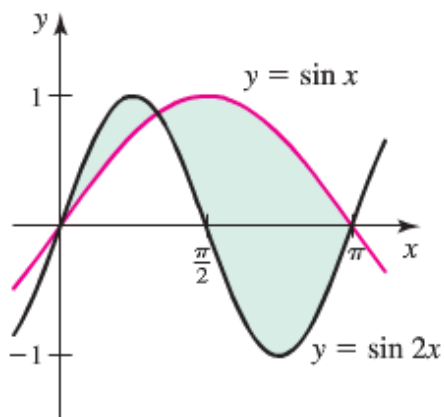


59.

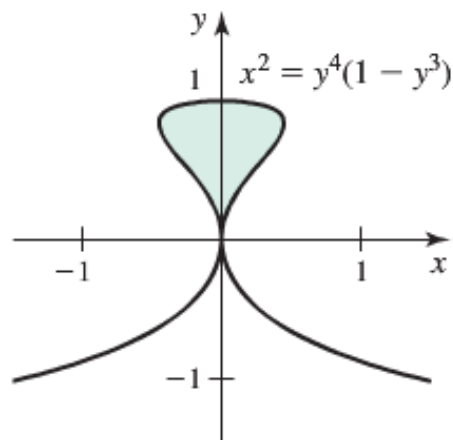


(60 – 71) Determine the area of the shaded regions

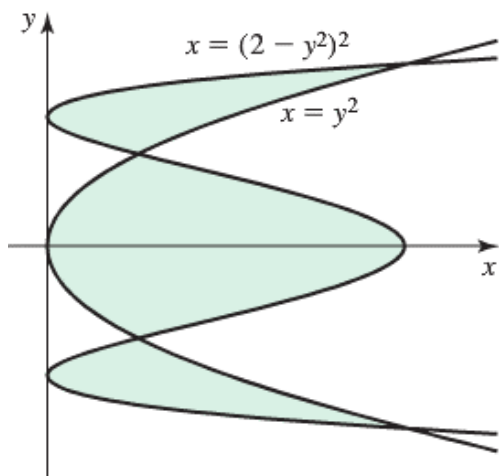
60.  $y = \sin x$  and  $y = \sin 2x$ , for  $0 \leq x \leq \pi$



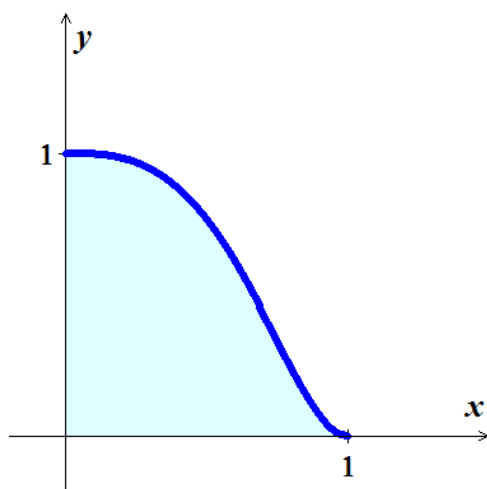
61. Bounded by  $x^2 = y^4(1 - y^3)$



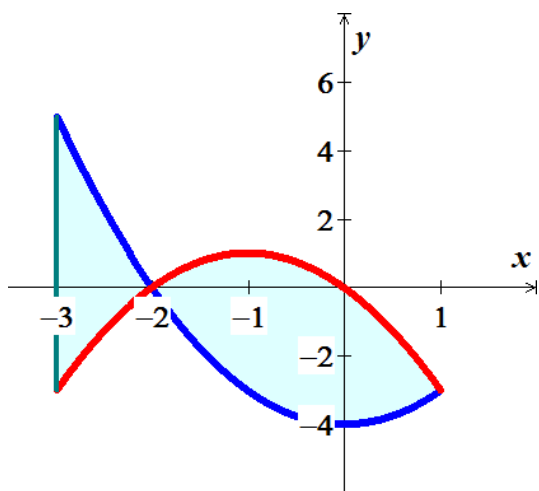
62. bounded by  $x = y^2$  and  $x = (2 - y^2)^2$



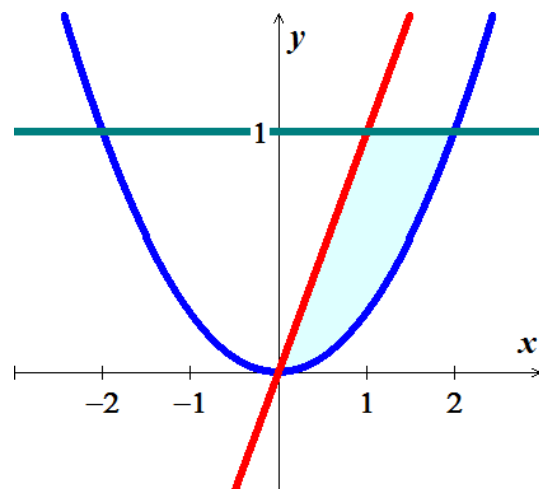
63.  $x^3 + \sqrt{y} = 1$ ,  $x = 0$ ,  $y = 0$ ,  $0 \leq x \leq 1$



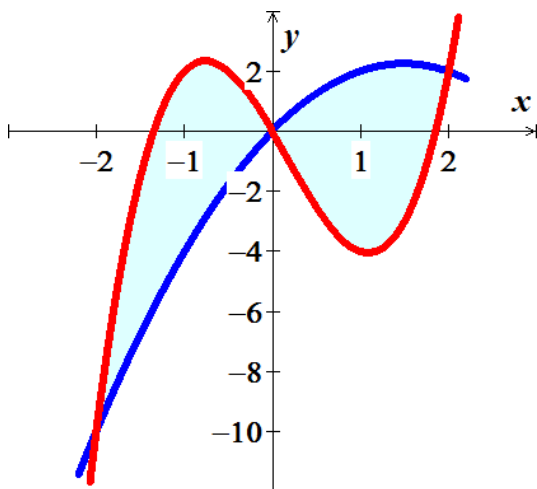
64.  $y = x^2 - 4$ ,  $y = -x^2 - 2x$ ,  $-3 \leq x \leq 1$



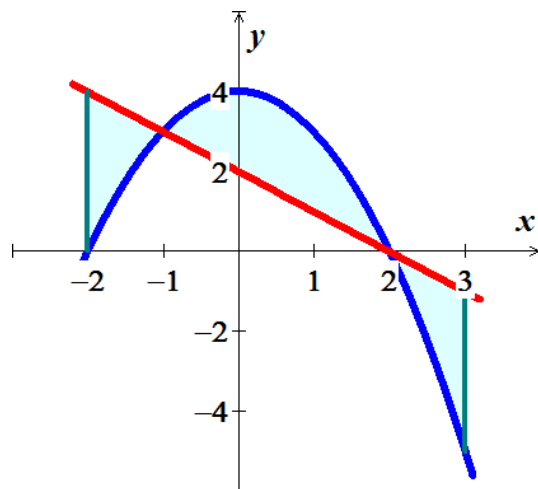
65.  $y = \frac{1}{4}x^2$ ,  $y = x$ ,  $y = 1$



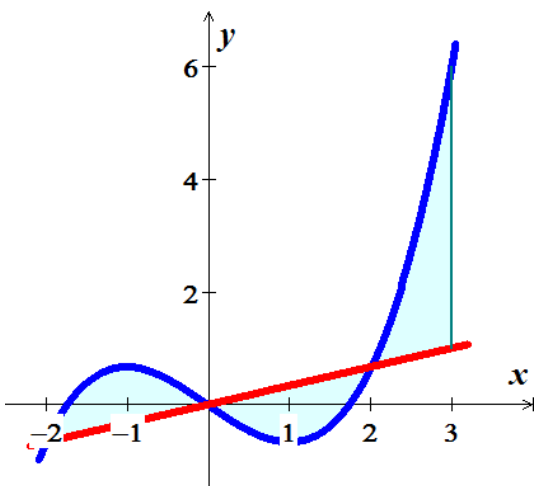
66.  $y = -x^2 + 3x$ ,  $y = 2x^3 - x^2 - 5x$ ,  $-2 \leq x \leq 2$



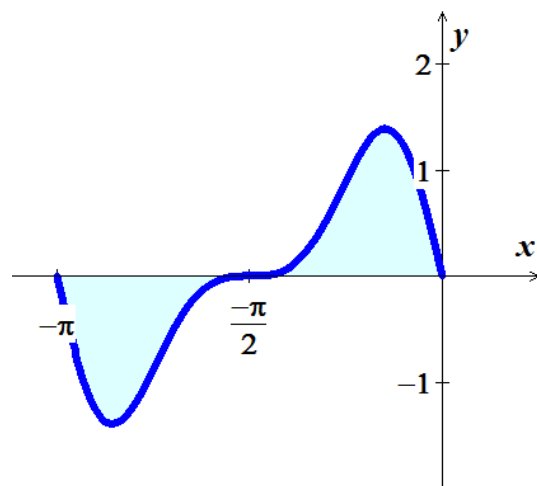
67.  $y = 4 - x^2$ ,  $y = -x + 2$ ,  $-2 \leq x \leq 3$



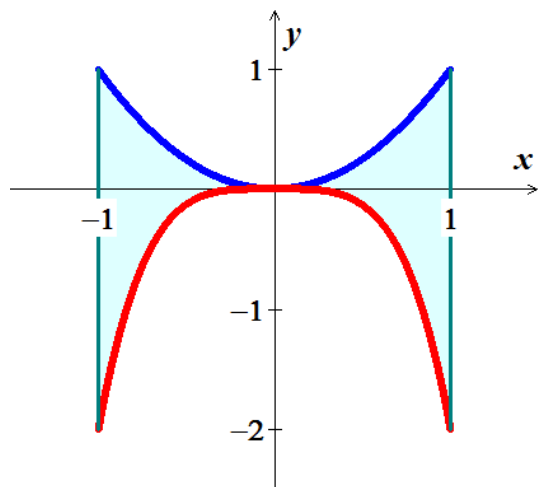
68.  $y = \frac{1}{3}x^3 - x$ ,  $y = \frac{1}{3}x$ ,  $-2 \leq x \leq 3$



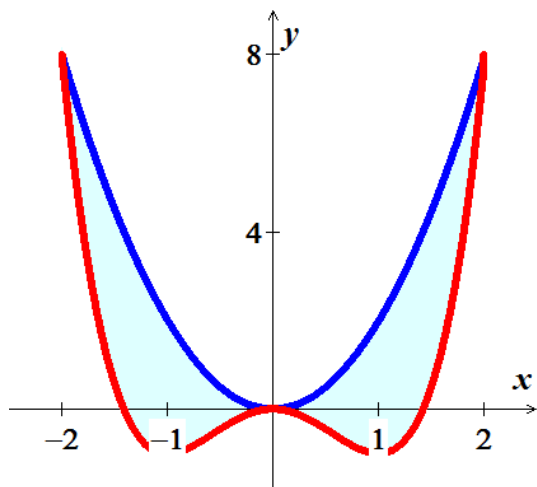
69.  $y = \frac{\pi}{2} \cos x \sin(\pi + \pi \sin x)$ ,  $-\pi \leq x \leq 0$



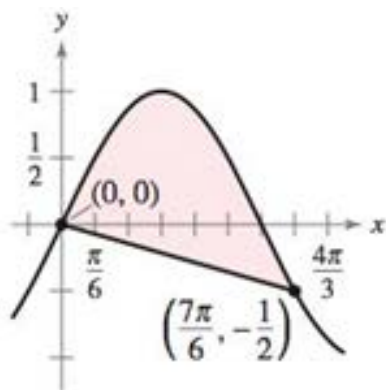
70.  $y = x^2$ ,  $y = -2x^4$ ,  $-1 \leq x \leq 1$



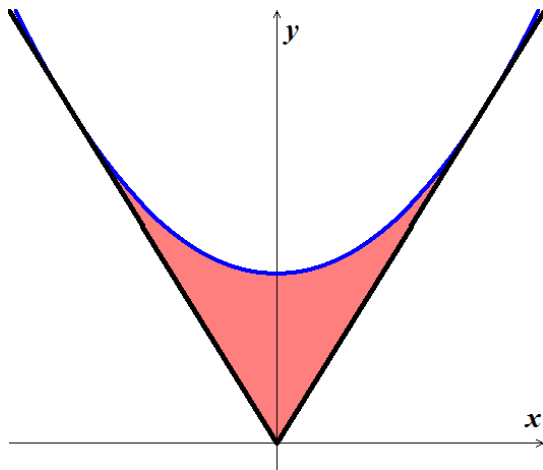
71.  $y = 2x^2$ ,  $y = x^4 - 2x^2$ ,  $-2 \leq x \leq 2$



72. Find the area between the graph of  $y = \sin x$  and the line segment joining the points  $(0, 0)$  and  $(\frac{7\pi}{6}, -\frac{1}{2})$ .



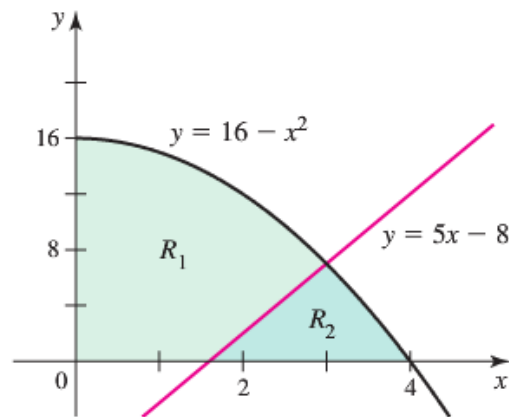
73. The surface of a machine part is the region between the graphs of  $y_1 = |x|$  and  $y_2 = 0.08x^2 + k$



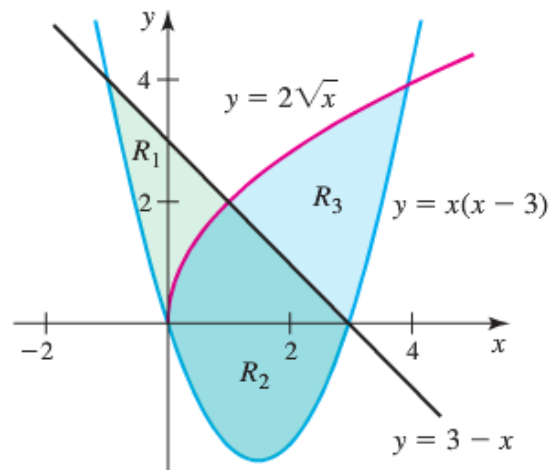
- Find  $k$  where the parabola is tangent to the graph of  $y_1$
- Find the area of the surface of the machine part.



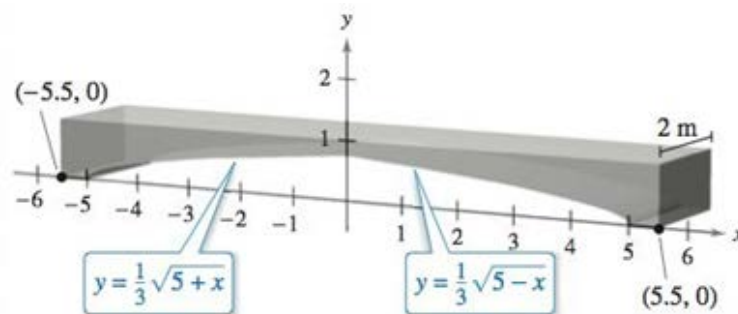
74. Find the area of the regions  $R_1$  and  $R_2$  (separately) shown in the figure, which are formed by the graphs of  $y = 16 - x^2$  and  $y = 5x - 8$



75. Find the area of the regions  $R_1$ ,  $R_2$  and  $R_3$  (separately) shown in the figure, which are formed by the graphs of  $y = 2\sqrt{x}$ ,  $y = 3 - x$ , and  $y = x(x - 3)$



76. Concrete sections for a new building have the dimensions (in meters) and shape shown in figure



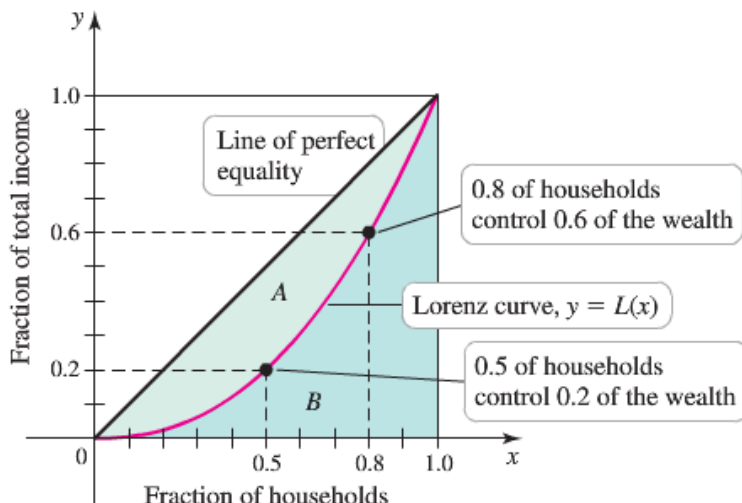
- Find the area of the face of the section superimposed on the rectangular coordinate system.
  - Find the volume of concrete in one of the sections by multiplying the area in part (a) by 2 meters.
  - One cubic meter of concrete weighs 5,000 pounds. Find the weight of the section.
77. A Lorenz curve is given by  $y = L(x)$ , where  $0 \leq x \leq 1$  represents the lowest fraction of the population of a society in terms of wealth and  $0 \leq y \leq 1$  represents the fraction of the total wealth that is owned by

that fraction of the society. For example, the Lorenz curve in the figure shows that  $L(0.5) = 0.2$ , which means that the lowest 0.5 (50%) of the society owns 0.2 (20%) of the wealth.

- a) A Lorenz curve  $y = L(x)$  is accompanied by the line  $y = x$ , called the **line of perfect equality**.

Explain why this line is given the name.

- b) Explain why a Lorenz curve satisfies the conditions  $L(0) = 0$ ,  $L(1) = 1$ , and  $L'(x) \geq 0$  on  $[0, 1]$



- c) Graph the Lorenz curves  $L(x) = x^p$  corresponding to  $p = 1.1, 1.5, 2, 3, 4$ . Which value of  $p$  corresponds to the **most** equitable distribution of wealth (closest to the line of perfect equality)? Which value of  $p$  corresponds to the **least** equitable distribution of wealth? Explain.
- d) The information in the Lorenz curve is often summarized in a single measure called the **Gini index**, which is defined as follows. Let  $A$  be the area of the region between  $y = x$  and  $y = L(x)$  and Let  $B$  be the area of the region between  $y = L(x)$  and the  $x$ -axis. Then the Gini index is  $G = \frac{A}{A+B}$ .

Show that  $G = 2A = 1 - 2 \int_0^1 L(x) dx$ .

- e) Compute the Gini index for the cases  $L(x) = x^p$  and  $p = 1.1, 1.5, 2, 3, 4$ .
- f) What is the smallest interval  $[a, b]$  on which values of the Gini index lie, for  $L(x) = x^p$  with  $p \geq 1$ ? Which endpoints of  $[a, b]$  correspond to the least and most equitable distribution of wealth?
- g) Consider the Lorenz curve described by  $L(x) = \frac{5x^2}{6} + \frac{x}{6}$ . Show that it satisfies the conditions  $L(0) = 0$ ,  $L(1) = 1$ , and  $L'(x) \geq 0$  on  $[0, 1]$ . Find the Gini index for this function.