Solution Section 4.3 – Mathematical Induction

Exercise

Find all positive integers n for which the given statement is not true

$a)\quad 3^n > 6n \qquad b)\quad 3^n > 2n+1 \qquad c)\quad 2^n > n^2 \qquad d)\quad n! > 2n$

Solution

a) $n=1 \quad 3<6$

$n=2 \quad 3^2<18$

$n=3, \quad 27>18$

The statement is true for all $n \ge 3 \quad 3^n > 6n$

The statement is not true for $\boxed{n=1,\ 2}$

b) $n=1; \quad 3=3$

$n=2; \quad 9>5$

The statement is true for all $n \ge 2 \quad 3^n > 2n+1$

The statement is not true for $\boxed{n=1}$

c) $n=1; \quad 2<4$

$n=2; \quad 4=4$

$n=3; \quad 8<9$

$n=4; \quad 16=16$

$n=5; \quad 32>25$

The statement is true for all $n \ge 5; \quad 2^n > n^2$

The statement is not true for $\boxed{n=1,\ 2,\ 3,\ 4}$

d) $n=1; \quad 1<2$

$n=2; \quad 2<4$

$n=3; \quad 6=6$

$n=4; \quad 12>8$

The statement is true for all $n \ge 4; \quad n! > 2n$

The statement is not true for $\boxed{n=1,\ 2,\ 3}$

Exercise

Prove that the statement is true for every positive integer n. $2+4+6+\ldots+2n=n(n+1)$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 2=1(1+1)=2$; hence P_1 is true.

(2) Assume $2+4+6+\ldots+2k=k(k+1)$ is true

$$\Rightarrow 2+4+6+\ldots+2k+2(k+1)=(k+1)(k+1+1)?$$

$$\begin{aligned}2+4+6+\ldots+2k+2(k+1) &= 2+4+6+\ldots+2k+2(k+1)\\ &= k(k+1)+2(k+1) \qquad \textit{Factor } (k+1)\\ &= (k+1)(k+2)\\ &= (k+1)(k+1+1) \ \checkmark\end{aligned}$$

Hence P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $1+3+5+\ldots+(2n-1)=n^2$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 1\overset{?}{=}1^2=1$; hence P_1 is true.

(2) Assume $1+3+5+\ldots+(2k-1)=k^2$ is true

$$\Rightarrow 1+3+5+\ldots+(2(k+1)-1)=(k+1)^2 ?$$

$$\begin{aligned}1+3+5+\ldots+(2k-1)+(2(k+1)-1) &= 1+3+5+\ldots+(2k-1)+(2k+2-1)\\ &= k^2+(2k+1)\\ &= k^2+2k+1\\ &= (k+1)^2 \ \checkmark\end{aligned}$$

Hence P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $2+7+12+\ldots+(5n-3)=\frac{1}{2}n(5n-1)$

Solution

(1) For $\boldsymbol{n = 1} \Rightarrow 2\overset{?}{=}\frac{1}{2}(1)(5(1)-1)=\frac{1}{2}(4)=2$; hence P_1 is true.

(2) Assume $2+7+12+\ldots+(5k-3)=\frac{1}{2}k(5k-1)$ is true

$$2+7+12+\ldots+(5(k+1)-3)=\frac{1}{2}(k+1)(5(k+1)-1)?$$

$$\begin{aligned}
2+7+12+\ldots+(5k-3)+(5(k+1)-3) &= 2+7+12+\ldots+(5k-3)+(5k+5-3) \\
&= \frac{1}{2}k(5k-1)+(5k+2)\frac{2}{2} \\
&= \frac{1}{2}\left[5k^2-k+10k+4\right] \\
&= \frac{1}{2}\left[5k^2-k+5k+5k+5-1\right] \\
&= \frac{1}{2}\left[k(5k-1+5)+5k+5-1\right] \\
&= \frac{1}{2}\left[(k+1)(5k+5-1)\right] \\
&= \frac{1}{2}\left[(k+1)(5(k+1)-1)\right] \quad \checkmark
\end{aligned}$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $1+2.2+3.2^2+\ldots+n.2^{n-1}=1+(n-1).2^n$

Solution

(1) For $\boldsymbol{n = 1} \Rightarrow 1\overset{?}{=}1+(1-1)2^1=1-0=1$; hence P_1 is true.

(2) $1+2.2+3.2^2+\ldots+k.2^{k-1}=1+(k-1).2^k$ is true

$$1+2.2+3.2^2+\ldots+k.2^{k-1}+(k+1).2^{(k+1)-1}=1+((k+1)-1).2^{k+1}?$$

$$\begin{aligned}
1+2.2+3.2^2+\ldots+k.2^{k-1}+(k+1).2^{(k+1)-1} &= 1+(k-1).2^k+(k+1).2^{k+1-1} \\
&= 1+k.2^k-1.2^k+(k+1).2^k \\
&= 1+k.2^k-1.2^k+k.2^k+1.2^k \\
&= 1+2^1k.2^k \\
&= 1+(k+0).2^{k+1} \\
&= 1+((k+1)-1).2^{k+1} \quad \checkmark
\end{aligned}$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $1^2+2^2+3^2+\ldots+n^2=\dfrac{n(n+1)(2n+1)}{6}$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow 1^2 \overset{?}{=} \dfrac{1(1+1)(2(1)+1)}{6}=\dfrac{1(2)(3)}{6}=\dfrac{6}{6}=1\ \surd$; hence P_1 is true.

(2) $1^2+2^2+3^2+\ldots+k^2=\dfrac{k(k+1)(2k+1)}{6}$ is true

$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}?$$

$$\begin{aligned}
1^2+2^2+3^2+\ldots+k^2+(k+1)^2 &= \frac{k(k+1)(2k+1)}{6}+(k+1)^2 \\
&= \frac{k(k+1)(2k+1)+6(k+1)^2}{6} \\
&= \frac{(k+1)\left[k(2k+1)+6(k+1)\right]}{6} \\
&= \frac{(k+1)\left[2k^2+k+6k+6\right]}{6} \\
&= \frac{(k+1)\left[2k^2+7k+6\right]}{6} \\
&= \frac{(k+1)((k+2)(2k+3))}{6} \\
&= \frac{(k+1)((k+1+1)(2k+2+1))}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}\ \surd \qquad P_{k+1} \text{ is also true.}
\end{aligned}$$

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $\dfrac{1}{1\bullet 2}+\dfrac{1}{2\bullet 3}+\dfrac{1}{3\bullet 4}+\ldots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow \dfrac{1}{1.2}\overset{?}{=}\dfrac{1}{1+1}=\dfrac{1}{2}=\dfrac{1}{1.2}\ \surd$; hence P_1 is true.

(2) $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\ldots+\dfrac{1}{k(k+1)}=\dfrac{k}{k+1}$ is true

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{(k+1)+1}?$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$=\frac{(k+1)(k+1)}{(k+1)(k+2)}$$

$$=\frac{k+1}{(k+1+1)}$$

$$=\frac{k+1}{(k+1)+1}\ \sqrt{}\quad P_{k+1}\text{ is also true.}$$

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$

Solution

(1) For $\boldsymbol{n}$ **= 1** ⇒ $\frac{1}{2}\overset{?}{=}1-\frac{1}{2}=\frac{1}{2}\ \sqrt{}$; P_1 is true.

(2) $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}=1-\frac{1}{2^k}$ is true

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^{k+1}}\ ?$$

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^k}+\frac{1}{2^{k+1}}$$

$$=1-\frac{1}{2^k}+\frac{1}{2^k\cdot 2}$$

$$=\frac{2^{k+1}-2+1}{2^{k+1}}$$

$$=\frac{2^{k+1}-1}{2^{k+1}}$$

$$=\frac{2^{k+1}}{2^{k+1}}-\frac{1}{2^{k+1}}$$

$$=1-\frac{1}{2^{k+1}}\qquad \sqrt{}\quad P_{k+1}\text{ is also true.}$$

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots+\frac{1}{(3n-2)\cdot(3n+1)}=\frac{n}{3n+1}$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow \frac{1}{1\cdot 4}\overset{?}{=}\frac{1}{3(1)+1}=\frac{1}{4}$ $\checkmark$; P_1 is true.

(2) $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1}$ is true

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3(k+1)-2)(3(k+1)+1)}\overset{?}{=}\frac{k+1}{3(k+1)+1}$$

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3(k+1)-2)(3(k+1)+1)}=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$$

$$=\frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$=\frac{(3k+1)(k+1)}{(3k+1)(3k+3+1)}$$

$$=\frac{k+1}{3(k+1)+1}\quad \checkmark \quad P_{k+1} \text{ is also true}$$

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\frac{4}{5}+\frac{4}{5^2}+\frac{4}{5^3}+\cdots+\frac{4}{5^n}=1-\frac{1}{5^n}$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow \frac{4}{5}\overset{?}{=}1-\frac{1}{5}=\frac{4}{5}$ $\checkmark$; P_1 is true.

(2) $\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}=1-\frac{1}{5^k}$ is true

$$\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}+\frac{4}{5^{k+1}}\overset{?}{=}1-\frac{1}{5^{k+1}}$$

$$\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}+\frac{4}{5^{k+1}}=1-\frac{1}{5^k}+\frac{4}{5^{k+1}}$$

$$=1-\left(\frac{1}{5^k}-\frac{4}{5^{k+1}}\right)$$

$$=1-\frac{5-4}{5^{k+1}}$$

$$=1-\frac{1}{5^{k+1}}\quad \checkmark \quad P_{k+1} \text{ is also true.}$$

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $1^3+2^3+3^3+\cdots+n^3=\dfrac{n^2(n+1)^2}{4}$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow 1^3 \overset{?}{=} \dfrac{1^2(1+1)^2}{4}=\dfrac{4}{4}=1$ $\checkmark$; P_1 is true.

(2) $\dfrac{4}{5}+\dfrac{4}{5^2}+\cdots+\dfrac{4}{5^k}=1-\dfrac{1}{5^k}$ is true

$$1^3+2^3+\cdots+k^3+(k+1)^3 \overset{?}{=} \frac{(k+1)^2((k+1)+1)^2}{4}$$

$$\begin{aligned}
1^3+2^3+\cdots+k^3+(k+1)^3 &= \frac{k^2(k+1)^2}{4}+(k+1)^3 \\
&= \frac{k^2(k+1)^2+4(k+1)^3}{4} \\
&= \frac{(k+1)^2\left[k^2+4(k+1)\right]}{4} \\
&= \frac{(k+1)^2\left(k^2+4k+4\right)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \frac{(k+1)^2((k+1)+1)^2}{4} \quad \checkmark \quad P_{k+1} \text{ is also true.}
\end{aligned}$$

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $3+3^2+3^3+\ldots+3^n=\dfrac{3}{2}\left(3^n-1\right)$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow 3 \overset{?}{=} \dfrac{3}{2}\left(3^1-1\right)=\dfrac{3}{2}2=3$ $\checkmark$; P_1 is true.

(2) $3+3^2+\cdots+3^k=\dfrac{3}{2}\left(3^k-1\right)$ is true $\rightarrow$ Is $3+3^2+\cdots+3^k+3^{k+1} \overset{?}{=} \dfrac{3}{2}\left(3^{k+1}-1\right)$

$$\begin{aligned}
3+3^2+\cdots+3^k+3^{k+1} &= \frac{3}{2}\left(3^k-1\right)+3^{k+1} \\
&= \frac{1}{2}3^{k+1}-\frac{3}{2}+3^{k+1} \\
&= \frac{3}{2}3^{k+1}-\frac{3}{2} \\
&= \frac{3}{2}\left(3^{k+1}-1\right) \quad \checkmark \quad P_{k+1} \text{ is also true.}
\end{aligned}$$

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $x^{2n}+x^{2n-1}y+\cdots+xy^{2n-1}+y^{2n}=\dfrac{x^{2n+1}-y^{2n+1}}{x-y}$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow$ $x^2+xy+y^2\overset{?}{=}\dfrac{x^3-y^3}{x-y}$

$$=\frac{(x-y)\left(x^2+xy+y^2\right)}{x-y}$$

$$=x^2+xy+y^2 \quad \surd \quad ; \ P_1 \text{ is true.}$$

(2) $x^{2k}+x^{2k-1}y+\cdots+xy^{2k-1}+y^{2k}=\dfrac{x^{2k+1}-y^{2k+1}}{x-y}$ is true

$$x^{2(k+1)}+x^{2(k+1)-1}y+\cdots+xy^{2(k+1)-1}+y^{2(k+1)}\overset{?}{=}\frac{x^{2(k+1)+1}-y^{2(k+1)+1}}{x-y}$$

$$x^{2k+2}+x^{2k+1}y+\cdots+xy^{2k+1}+y^{2k+2}=x^2\left(x^{2k}+x^{2k-1}y+\cdots+y^{2k}\right)+xy^{2k+1}+y^{2k+2}$$

$$=x^2\left(\frac{x^{2k+1}-y^{2k+1}}{x-y}\right)+xy^{2k+1}+y^{2k+2}$$

$$=\frac{x^{2k+3}-x^2y^{2k+1}+x^2y^{2k+1}+xy^{2k+2}-xy^{2k+2}-y^{2(k+1)+1}}{x-y}$$

$$=\frac{x^{2(k+1)+1}-y^{2(k+1)+1}}{x-y} \quad \surd \qquad P_{k+1} \text{ is also true.}$$

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $5\cdot 6+5\cdot 6^2+5\cdot 6^3+\cdots+5\cdot 6^n=6\left(6^n-1\right)$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow$ $5\cdot 6\overset{?}{=}6\left(6^1-1\right)=6(5)$ $\surd$; P_1 is true.

(2) $5\cdot 6+5\cdot 6^2+\cdots+5\cdot 6^k=6\left(6^k-1\right)$ is true

$$5\cdot 6+5\cdot 6^2+\cdots+5\cdot 6^k+5\cdot 6^{k+1}\overset{?}{=}6\left(6^{k+1}-1\right)$$

$$5\cdot 6+5\cdot 6^2+\cdots+5\cdot 6^k+5\cdot 6^{k+1}=6\left(6^k-1\right)+5\cdot 6^{k+1}$$

$$=6^{k+1}-6+5\cdot 6^{k+1}$$

$$=6^{k+1}(1+5)-6$$

$$=6\cdot 6^{k+1}-6$$

$$=6\left(6^{k+1}-1\right) \quad \checkmark \qquad P_{k+1} \text{ is also true.}$$

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $7{\bullet}8+7{\bullet}8^2+7{\bullet}8^3+\cdots+7{\bullet}8^n=8\left(8^n-1\right)$

Solution

(1) For ***n*** **= 1** ⇒ $7\cdot 8\overset{?}{=}8\left(8^1-1\right)=8(7)$ ✓ ; P_1 is true.

(2) $7\cdot 8+7\cdot 8^2+\cdots+7\cdot 8^k=8\left(8^k-1\right)$ is true

$$7\cdot 8+7\cdot 8^2+\cdots+7\cdot 8^k+7\cdot 8^{k+1}\overset{?}{=}8\left(8^{k+1}-1\right)$$

$$\begin{aligned}7\cdot 8+7\cdot 8^2+\cdots+7\cdot 8^k+7\cdot 8^{k+1}&=8\left(8^k-1\right)+7\cdot 8^{k+1}\\&=8^{k+1}-8+7\cdot 8^{k+1}\\&=8^{k+1}(1+7)-8\\&=8\cdot 8^{k+1}-8\\&=8\left(8^{k+1}-1\right) \quad \checkmark \qquad P_{k+1} \text{ is also true.}\end{aligned}$$

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $3+6+9+\cdots+3n=\dfrac{3n(n+1)}{2}$

Solution

(1) For ***n*** **= 1** ⇒ $3\overset{?}{=}\dfrac{3(1)(1+1)}{2}=3$ ✓; P_1 is true.

(2) $3+6+9+\cdots+3k=\dfrac{3k(k+1)}{2}$ is true

$$3+6+9+\cdots+3k+3(k+1)\overset{?}{=}\frac{3(k+1)(k+2)}{2}$$

$$\begin{aligned}3+6+9+\cdots+3k+3(k+1)&=\frac{3k(k+1)}{2}+3(k+1)\\&=\frac{3k(k+1)+6(k+1)}{2}\\&=\frac{(k+1)(3k+6)}{2}\\&=\frac{3(k+1)(k+2)}{2} \quad \checkmark \qquad P_{k+1} \text{ is also true.}\end{aligned}$$

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $5+10+15+\cdots+5n=\dfrac{5n(n+1)}{2}$

Solution

(1) For ***n* = 1** $\Rightarrow$ $5\overset{?}{=}\dfrac{5(1)(1+1)}{2}=5$ $\checkmark$; P_1 is true.

(2) $5+10+15+\cdots+5k=\dfrac{5k(k+1)}{2}$ is true

$$5+10+15+\cdots+5k+5(k+1)\overset{?}{=}\frac{5(k+1)(k+2)}{2}$$

$$\begin{aligned}5+10+15+\cdots+5k+5(k+1)&=\frac{5k(k+1)}{2}+5(k+1)\\&=\frac{5k(k+1)+10(k+1)}{2}\\&=\frac{(k+1)(5k+10)}{2}\\&=\frac{5(k+1)(k+2)}{2}\ \checkmark\end{aligned}$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $1+3+5+\cdots+(2n-1)=n^2$

Solution

(1) For ***n* = 1** $\Rightarrow$ $1\overset{?}{=}1^2=1$ $\checkmark$; P_1 is true.

(2) $1+3+5+\cdots+(2k-1)=k^2$ is true

$$1+3+5+\cdots+(2k-1)+(2(k+1)-1)\overset{?}{=}(k+1)^2$$

$$\begin{aligned}1+3+5+\cdots+(2k-1)+(2(k+1)-1)&=k^2+2k+2-1\\&=k^2+2k+1\\&=(k+1)^2\ \checkmark\end{aligned}$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $4+7+10+\cdots+(3n+1)=\frac{n(3n+5)}{2}$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow$ $4\overset{?}{=}\frac{1(3+5)}{2}=4$ $\surd$; P_1 is true.

(2) $4+7+10+\cdots+(3k+1)=\frac{k(3k+5)}{2}$ is true

$$4+7+10+\cdots+(3k+1)+(3(k+1)+1)\overset{?}{=}\frac{(k+1)(3(k+1)+5)}{2}=\frac{(k+1)(3k+8)}{2}$$

$$\begin{aligned}4+7+10+\cdots+(3k+1)+(3k+4)&=\frac{k(3k+5)}{2}+3k+4\\&=\frac{3k^2+5k+6k+8}{2}\\&=\frac{3k^2+5k+3k+3k+8}{2}\\&=\frac{k(3k+8)+(3k+8)}{2}\\&=\frac{(3k+8)(k+1)}{2}\quad\surd\end{aligned}$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $\left(a^m\right)^n=a^{mn}$ (a and m are constant)

Solution

- For $\boldsymbol{n}$ **= 1** $\Rightarrow$ $\left(a^m\right)^1\overset{?}{=}a^{m(1)}$ $\rightarrow$ $a^m=a^m$ $\surd$; P_1 is true.
- $\left(a^m\right)^k=a^{mk}$ is true

$$\left(a^m\right)^{(k+1)}\overset{?}{=}a^{m(k+1)}$$

$$\begin{aligned}\left(a^m\right)^{(k+1)}&=\left(a^m\right)^k a^m\\&=a^{km}a^m\\&=a^{km+m}\\&=a^{m(k+1)}\quad\surd\end{aligned}$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $n < 2^n$

Solution

***Step* 1**. For $\boldsymbol{n} = \mathbf{1} \Rightarrow 1 \overset{?}{<} 2^1$ $\surd$ $\Rightarrow P_1$ is true.

***Step* 2**. Assume that P_k is true $k < 2^k$

We need to prove that P_{k+1} is true, that is $k+1 < 2^{k+1}$

$$\begin{aligned} k+1 &< k+k = 2k \\ &< 2 \cdot 2^k \\ &= 2^{k+1} \quad \surd \end{aligned}$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. 3 is a factor of $n^3 - n + 3$

Solution

- For $\boldsymbol{n} = \mathbf{1} \Rightarrow 1^3 - 1 + 3 = 3 = 3(1)$ $\surd$ $\Rightarrow P_1$ is true.
- Assume that P_k is true 3 is a factor of $k^3 - k + 3$

  We need to prove that P_{k+1} is true, that is $(k+1)^3 - (k+1) + 3$

$$\begin{aligned} (k+1)^3 - (k+1) + 3 &= k^3 + 3k^2 + 3k + 1 - k - 1 + 3 \\ &= \left(k^3 - k + 3\right) + 3k^2 + 3k \\ &= 3K + 3k^2 + 3k \\ &= 3\left(K + k^2 + k\right) \quad \surd \end{aligned}$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. 4 is a factor of $5^n - 1$

Solution

- For $\boldsymbol{n} = \mathbf{1} \Rightarrow 5^1 - 1 = 4 = 4(1)$ $\surd$ $\Rightarrow P_1$ is true.
- Assume that P_k is true 4 is a factor of $5^k - 1$

  We need to prove that P_{k+1} is true, that is $5^{k+1} - 1$

$$5^{k+1} - 1 = 5^k 5^1 - 5 + 4$$

$= 5\left(5^{k} - 1\right) + 4$

$= 5\left(5^{k} - 1\right) + 4$

By the induction hypothesis, 4 is a factor of $5^{k} - 1$ and 4 is a factor of 4, so 4 is a factor of the $(k+1)$ term. √

Thus, P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $2^{n} > 2n \quad if \quad n \ge 3$

Solution

- For ***n* = 3** $\Rightarrow 2^{3} \ge 2(3) \Rightarrow 8 \ge 6$ √ $\Rightarrow P_1$ is true.
- Assume that P_k is true: $2^{k} > 2k$; we need to prove that P_{k+1}: $2^{k+1} > 2(k+1)$ is true

$$
\begin{aligned}
&2^{k} > 2k \\
&2^{k} \cdot 2 > 2k \cdot 2 \\
&2^{k+1} > 4k = 2k + 2k \qquad k \ge 3 \\
&\qquad > 2k + 2 \\
&\qquad = 2(k+1) \quad \surd \qquad P_{k+1} \text{ is also true.}
\end{aligned}
$$

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: If $0 < a < 1$, then $a^{n} < a^{n-1}$

Solution

- For ***n* = 1** $\Rightarrow a^{1} < a^{1-1} \Rightarrow a < 1$ √ since $0 < a < 1 \Rightarrow P_1$ is true.
  - Assume that P_k is true: $a^{k} < a^{k-1}$; we need to prove that P_{k+1}: $a^{k+1} < a^{k}$ is true

$$
\begin{aligned}
a^{k} < a^{k-1} \quad \rightarrow \quad & a^{k} \cdot a < a^{k-1} \cdot a \\
& a^{k+1} < a^{k} \quad \surd \qquad P_{k+1} \text{ is also true.}
\end{aligned}
$$

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: If $n \ge 4$, then $n! > 2^n$

Solution

- For $n = 4 \Rightarrow 4! > 2^4 \Rightarrow 24 > 16$ ✓ $\Rightarrow P_1$ is true.
- Assume that P_k is true: $k! > 2^k$; we need to prove that $P_{k+1}: (k+1)! > 2^{k+1}$ is true

$$\begin{aligned}(k+1)! &= k!\,(k+1)\\ &> 2^k(k+1) && k \ge 4 \Rightarrow k+1 > 2\\ &> 2^k \cdot 2\\ &= 2^{k+1}\ ✓\end{aligned}$$

Thus, P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $3^n > 2n+1 \quad if \quad n \ge 2$

Solution

- For $n = 2 \Rightarrow 3^2 > 2(2)+1 \Rightarrow 9 > 5$ ✓ $\Rightarrow P_1$ is true.
- Assume that P_k is true: $3^k > 2k+1$; we need to prove that $P_{k+1}: 3^{k+1} > 2(k+1)+1$ is true

$$3^k > 2k+1 \Rightarrow 3^k \cdot 3 > (2k+1)\cdot 3$$

$$\begin{aligned}3^{k+1} &> 6k+3\\ &> 2k+2+1\\ &= 2(k+1)+1\ ✓\end{aligned}$$

Thus, P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $2^n > n^2 \quad for \quad n > 4$

Solution

- For $n = 5 \Rightarrow 2^5 > 5^2 \Rightarrow 32 > 25$ ✓ $\Rightarrow P_1$ is true.
- Assume that P_k is true: $2^k > k^2$; we need to prove that $P_{k+1}: 2^{k+1} > (k+1)^2$ is true

$$2^k > k^2 \Rightarrow 2^k \cdot 2 > k^2 \cdot 2$$

$$\begin{aligned}2^{k+1} &> 2k^2 && k < k+1 \Rightarrow k^2 > 2k+1\\ &> (k+1)^2\ ✓\end{aligned}$$

Thus, P_{k+1} is also true

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $4^n > n^4 \quad for \quad n \ge 5$

Solution

- For $\boldsymbol{n = 5} \Rightarrow 4^5 > 5^4 \Rightarrow 1024 > 625$ √ $\Rightarrow P_1$ is true.
- Assume that P_k is true: $4^k > k^4$; we need to prove that $P_{k+1}: 4^{k+1} > (k+1)^4$ is true

$$4^k > k^4 \Rightarrow 4^k \cdot 4 > k^4 \cdot 4$$

$$4^{k+1} > 4k^4 \qquad k < k+1 \Rightarrow k^2 > 2k+1$$

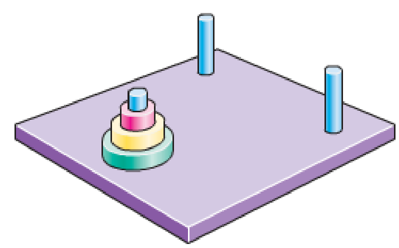
$$> (k+1)^4 \ \surd \qquad \text{Thus, } P_{k+1} \text{ is also true}$$

∴ By the mathematical induction, the proof is completed.

Exercise

A pile of *n* rings, each smaller than the one below it, is on a peg on board. Two other pegs are attached to the board. In the game called the Tower of Hanoi puzzle, all the rings must moved, one at a time, to a different peg with no ring ever placed on top of a smaller ring. Find the least number of moves that would be required. Prove your result by mathematical induction.

Solution



With 1 ring, 1 move is required.

With 2 rings, 3 moves are required $\Rightarrow 3 = 2 + 1$

With 3 rings, 7 moves are required $\Rightarrow 7 = 2^2 + 2 + 1$

With *n* rings, $2^{n-1} + \cdots + 2^2 + 2^1 + 2^0 = 2^n - 1$ moves are required

- For $\boldsymbol{n = 1} \Rightarrow 2^0 = 2^1 - 1 = 1$ √ $\Rightarrow P_1$ is true.
- Assume that P_k is true: $2^{k-1} + \cdots + 2^2 + 2^1 + 2^0 = 2^k - 1$

$$2^k + 2^{k-1} + \cdots + 2^2 + 2^1 + 1 \overset{?}{=} 2^{k+1} - 1$$

$$\begin{aligned} 2^k + 2^{k-1} + \cdots + 2^2 + 2^1 + 1 &= 2^k + 2^k - 1 \\ &= 2 \cdot 2^k - 1 \\ &= 2^{k+1} - 1 \quad \surd \end{aligned}$$

∴ By the mathematical induction, the proof is completed.