Solution   *Section* 1.1 – Velocity and Net Change

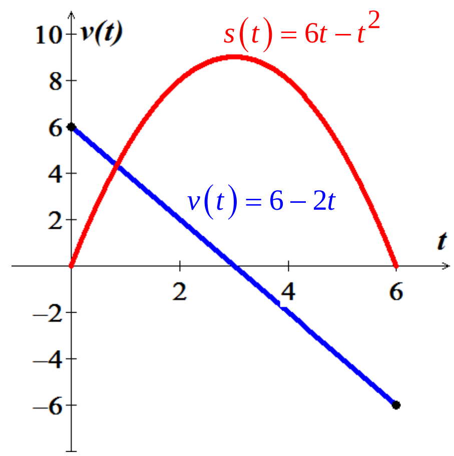
Exercise

Assume t is time measured in seconds and velocities have units of *m/s*. $v(t)=6-2t; \quad 0\le t\le 6$

a) Graph the velocity function over the given interval. Then determine when the motion is in the positive direction.
b) Find the displacement over the given interval.
c) Find the distance traveled over the given interval.

Solution

a) The motion is positive for $0\le t<3$ and negative for $3<t\le 6$



b) Displacement $=\int_0^6 (6-2t)\,dt$

$$=\left[6t-t^2\right]_0^6$$

$$\underline{=0\ \ m}$$

c) Distance traveled is $=\int_0^3 (6-2t)\,dt+\int_3^6 (2t-6)\,dt$

$$=\left[6t-t^2\right]_0^3+\left[t^2-6t\right]_3^6$$

$$=9+9$$

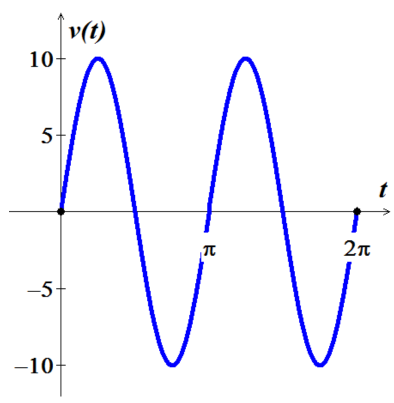
$$\underline{=18\ \ m}$$

Exercise

Assume t is time measured in seconds and velocities have units of *m/s*. $v(t)=10\sin 2t; \quad 0\le t\le 2\pi$

a) Graph the velocity function over the given interval. Then determine when the motion is in the positive direction.
b) Find the displacement over the given interval.
c) Find the distance traveled over the given interval.

Solution



a) The motion is positive for $0<t<\frac{\pi}{2}$ & $\pi<t<\frac{3\pi}{2}$

and negative for $\frac{\pi}{2}<t<\pi$ & $\frac{3\pi}{2}<t<2\pi$

b) Displacement $=\int_0^{2\pi} (10\sin 2t)\,dt$

$$=-5\cos 2t\Big|_0^{2\pi}$$

$$\underline{=0}$$

c) Distance traveled is $= \int_0^{2\pi} (10\sin 2t)\,dt = 4\cdot \int_0^{\pi/2} (10\sin 2t)\,dt$

$$= -20\cos 2t \Big|_0^{\pi/2}$$

$$= -20(-1-1)$$

$$= 40\ \ m$$

Exercise

Assume *t* is time measured in seconds and velocities have units of *m/s*. $v(t) = 50e^{-2t};\quad 0 \le t \le 4$

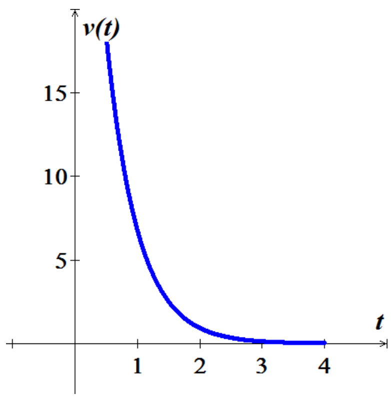
a) Graph the velocity function over the given interval. Then determine when the motion is in the positive direction.

b) Find the displacement over the given interval.

c) Find the distance traveled over the given interval.

Solution

a) The motion is positive for $0 \le t \le 4$



b) Displacement $= \int_0^4 50e^{-2t}\,dt$

$$= -25e^{-2t}\Big|_0^4$$

$$= -25\left(e^{-8} - 1\right)$$

$$= 25\left(1 - e^{-8}\right)\ \ m \qquad \approx 24.992\ \ m$$

c) Distance traveled is the same displacement si $\approx 24.992\ \ m$

Exercise

Consider an object moving along a line with the following velocities and initial positions

$v(t) = 6 - 2t \quad on\ [0,\ 5] \quad s(0) = 0$

a) Graph the velocity function on the given interval. Then determine when the object is moving in the positive direction and when it is moving in the negative direction.

b) Determine the position function for $t \ge 0$ using both the antiderivative method and the Fundamental Theorem of Calculus. Check for agreement between the two methods.

c) Graph the position function on the given interval.

Solution

a) The motion is positive for $0 \le t < 3$ and negative for $3 < t \le 5$

b) $s(t)=\int(6-2t)\,dt=6t-t^2+C$

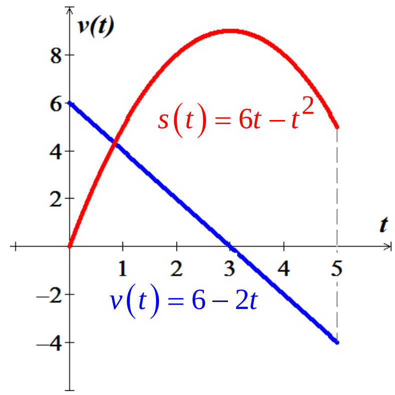
Given: $s(0)=0 \quad\rightarrow\quad \boxed{0=C}$

$\boxed{s(t)=6t-t^2}$

Also, $s(t)=s(0)+\int_0^t(6-2x)\,dx$

$=0+\left[6x-x^2\right]_0^t$

$\boxed{=6t-t^2}$



Exercise

Consider an object moving along a line with the following velocities and initial positions

$v(t)=9-t^2 \quad on\ [0,\ 4] \quad s(0)=-2$

a) Graph the velocity function on the given interval. Then determine when the object is moving in the positive direction and when it is moving in the negative direction.

b) Determine the position function for $t\ge 0$ using both the antiderivative method and the Fundamental Theorem of Calculus. Check for agreement between the two methods.

c) Graph the position function on the given interval.

Solution

a) The motion is positive for $0\le t<3$ and negative for $3<t\le 4$

b) $s(t)=\int\left(9-t^2\right)dt=9t-\frac{1}{3}t^3+C$

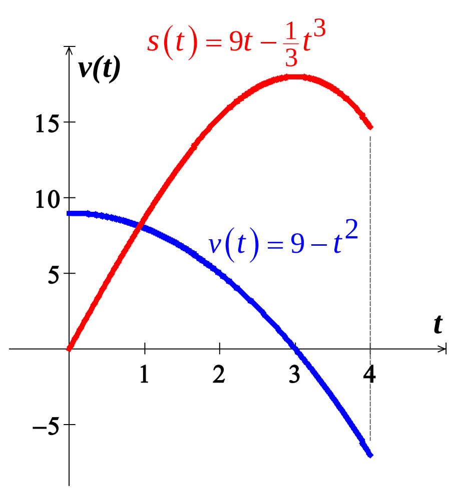
Given: $s(0)=-2 \quad\rightarrow\quad \boxed{-2=C}$

$\boxed{s(t)=9t-\frac{1}{3}t^3-2}$

Also, $s(t)=s(0)+\int_0^t\left(9-x^2\right)dx$

$=-2+\left[9x-\frac{1}{3}x^3\right]_0^t$

$\boxed{=9t-\frac{1}{3}t^3-2}$



Exercise

Find the position and velocity of an object moving along a straight line with the given acceleration, initial velocity, ansd initial position. Assume units of meters and seconds.

$a(t) = -9.8, \quad v(0) = 20, \quad s(0) = 0$

Solution

$$v(t) = \int a(t)\,dt = -\int 9.8\,dt \underline{= -9.8t + C_1|}$$

Since $v(0) = 20 \quad \to \quad \underline{20 = C_1|}$

$v(t) \underline{= -9.8t + 20|}$

$$s(t) = \int v(t)\,dt = \int (20 - 9.8t)\,dt$$

$$\underline{= 20t - 4.9t^2 + C_2|}$$

Since $s(0) = 0 \quad \to \quad \underline{0 = C_2|}$

$$\underline{s(t) = 20t - 4.9t^2|}$$

Exercise

Find the position and velocity of an object moving along a straight line with the given acceleration, initial velocity, ansd initial position. Assume units of meters and seconds.

$a(t) = e^{-t}, \quad v(0) = 60, \quad s(0) = 40$

Solution

$$v(t) = \int a(t)\,dt = \int e^{-t}\,dt \underline{= -e^{-t} + C_1|}$$

Since $v(0) = 60 \quad \to \quad 20 = -1 + C_1 \Rightarrow \underline{C_1 = 61|}$

$v(t) \underline{= -e^{-t} + 61|}$

$$s(t) = \int v(t)\,dt = \int \left(61 - e^{-t}\right)dt$$

$$\underline{= 61t + e^{-t} + C_2|}$$

Since $s(0) = 40 \quad \to \quad 40 = 1 + C_2 \Rightarrow \underline{C_2 = 39|}$

$$\underline{s(t) = 61t + e^{-t} + 39|}$$

Exercise

A mass hanging from a spring is set in motion and its ensuing velocity is given by $v(t) = 2\pi\cos\pi t$ for $t \ge 0$. Assume that the position direction is upward and $s(0) = 0$.

a) Determine the position function for $t \ge 0$.

b) Graph the position function on the interval [0, 3].

c) At what times does the mass reach its lowest point the first three times?

d) At what times does the mass reach its highest point the first three times?

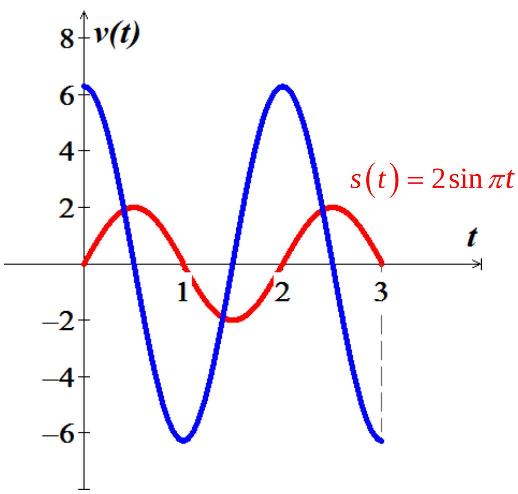
Solution

a) $s(t) = s(0) + \int_0^t (2\pi\cos\pi x)\,dx$

$$= 2\sin\pi x\Big|_0^t$$

$$= 2\sin\pi t$$

b)



c) The smallest value of Sine is -1, therefore the angle are

$$\frac{4k+3}{2}\pi = \pi t \quad \Rightarrow \quad t = \frac{4k+3}{2} \quad (k = 0,1,2)$$

The mass reaches its lowest point at $t = 1.5,\ t = 3.5,\ \textit{and}\ t = 5.5$

d) The Largest value of Sine is 1, therefore the angle are

$$\frac{4k+1}{2}\pi = \pi t \quad \Rightarrow \quad t = \frac{4k+1}{2} \quad (k = 0,1,2)$$

The mass reaches its highest point at $t = 0.5,\ t = 2.5,\ \textit{and}\ t = 4.5$

Exercise

The velocity of an airplane flying into a headwind is given by $v(t)=30\left(16-t^2\right)$ *mi/hr* for $0\le t\le 3$ *hr*.

Assume that $s(0)=0$

a) Determine and graph the position function for $0\le t\le 3$.

b) How far does the airplane travel in the first 2 *hr*.?

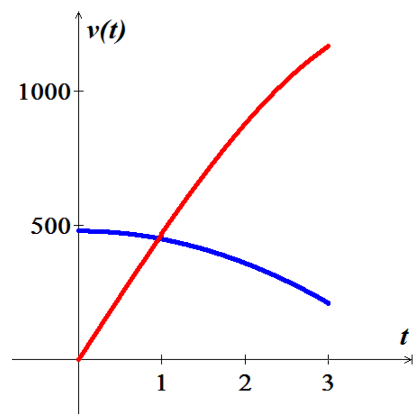
c) How far has the airplane traveled at the instant its velocity reaches 400 *mi/hr*.?

Solution

a) $s(t)=s(0)+30\int_0^t\left(16-x^2\right)dx$

$=30\left(16x-\frac{1}{3}x^3\right)\Big|_0^t$

$=480t-10t^3$



b) $s(2)=480(2)-10\left(2^3\right)$ $=880\ \ miles$

c) *Given*: $v=400;\ \rightarrow 480-30t^2=400$

$t^2=\frac{8}{3}\quad\Rightarrow\quad t=\sqrt{\frac{8}{3}}$

$s\left(\sqrt{\frac{8}{3}}\right)=480\sqrt{\frac{8}{3}}-10\left(\sqrt{\frac{8}{3}}\right)^3$ $\approx 740.290\ \ miles$

Exercise

A car slows down with an acceleration of $a(t)=-15\ ft/s^2$. Assume that $v(0)=60\ ft/s$ and $s(0)=0$

a) Determine and graph the position function for $t\ge 0$.

b) How far does the car travel in the time it takes to come to rest?

Solution

a) $v(t)=\int a(t)dt=-\int 15\ dt=-15t+C_1$

Given: $v(0)=60\quad\Rightarrow\quad C_1=60$

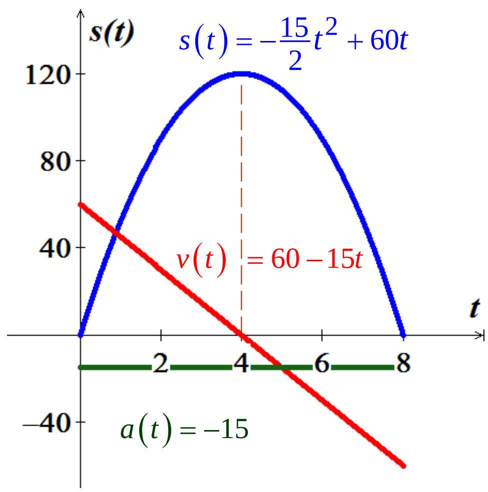
$v(t)\ =60-15t$

$s(t)=\int v(t)dt=\int(60-15t)\ dt$

$=60t-\frac{15}{2}t^2+C_2$

Given: $s(0)=0\quad\Rightarrow\quad C_2=0$

$s(t)=-\frac{15}{2}t^2+60t$



b) The car comes to rest when $v(t)=0=60-15t \quad \to \quad \underline{t=4}$

$$s(4)=-\frac{15}{2}(4)^2+60(4) \ \underline{=120\ ft}$$

Exercise

The owners of an oil reserve begin extracting oil at $t=0$. Based on estimates of the reserves, suppose the projected extraction rate is given by $Q'(t)=3t^2(40-t)^2$, where $0\le t\le 40$, Q is measured in millions of barrels, and t is measured in years.

a) When does the peak extraction rate occur?
b) How much oil is extracted in the first 10, 20, and 30 years?
c) What is the total amount of oil extracted in 40 year?
d) Is one-fourth of the total oil extracted in the first one-fourth of the extraction period? Explain.

Solution

a)
$$\begin{aligned} Q''(t) &= 6t(40-t)^2-6t^2(40-t) \\ &= 6t(40-t)(40-2t) \\ &= 12t(40-t)(20-t) = 0 \end{aligned}$$

0	**20**	**40**
+	–	

From the table, Q' is maximized at $t=20$; therefore the peak extraction rate is at $t=20\ yrs$

b) In the first **10 years**:

$$\begin{aligned} Q(t) &= \int Q'(t)\,dt = \int_0^{10}\left(3t^4-240t^3+4800t^2\right)dt \\ &= \left[\frac{3}{5}t^5-60t^4+1600t^3\right]_0^{10} \\ &= \frac{3}{5}10^5-6\left(10^5\right)+16\left(10^5\right) \\ &= \frac{53}{5}\cdot 10^5 \\ &\underline{=106\cdot 10^4} \end{aligned}$$

1,060,000 millions of barrels

In the first **20** years

$$\begin{aligned} Q(t) &= \int Q'(t)\,dt = \int_0^{20}\left(3t^4-240t^3+4800t^2\right)dt \\ &= \left[\frac{3}{5}t^5-60t^4+1600t^3\right]_0^{20} \\ &= \frac{3}{5}\cdot 2^5\cdot 10^5-6\cdot 2^4\cdot 10^5+16\cdot 2^3\cdot 10^5 \\ &= 2^5\cdot 10^5\left(\frac{3}{5}-3+4\right) \\ &= 8\cdot 2^6\cdot 10^4 \\ &\underline{=2^9\cdot 10^4} \end{aligned}$$

$5{,}120{,}000$ *millions* of barrels

In the first **30** years

$$Q(t)=\int Q'(t)\,dt=\int_0^{30}\left(3t^4-240t^3+4800t^2\right)dt$$

$$=\left[\frac{3}{5}t^5-60t^4+1600t^3\right]_0^{30}$$

$$=\frac{3}{5}\cdot 3^5\cdot 10^5-6\cdot 3^4\cdot 10^5+16\cdot 3^3\cdot 10^5$$

$$=3^3\cdot 10^5\left(\frac{27}{5}-18+16\right)$$

$$=3^3\cdot 10^5\cdot\frac{17}{5}$$

$=9{,}180{,}000$ *millions* of barrels

c) Total amount of oil extracted in 40 year

$$Q(t)=\int Q'(t)\,dt=\int_0^{40}\left(3t^4-240t^3+4800t^2\right)dt$$

$$=\left[\frac{3}{5}t^5-60t^4+1600t^3\right]_0^{40}$$

$$=\frac{3}{5}\cdot 4^5\cdot 10^5-6\cdot 4^4\cdot 10^5+16\cdot 4^3\cdot 10^5$$

$$=4^4\cdot 10^5\left(\frac{12}{5}-6+4\right)$$

$$=4^4\cdot 10^5\cdot\frac{2}{5}$$

$=10{,}240{,}000$ *millions* of barrels

d) $\frac{1}{4}(10{,}240{,}000)=2{,}560{,}000 \neq 5{,}120{,}000$

No, the amount extracted in the first 10 years is not $\frac{1}{4}$ of the total amount extracted.

Exercise

Starting with an initial value of $P(0)=55$, the population of a prairie dog community grows at a rate of $P'(t)=20-\frac{t}{5}$ (in units of prairie *dogs/month*), for $0\le t\le 200$.

a) What is the population 6 months later?

b) Find the population $P(t)$ for $0\le t\le 200$.

Solution

a) $P(t)=P(0)+\int_0^{6}\left(20-\frac{t}{5}\right)dt$

$$=55+\left[20t-\frac{1}{10}t^2\right]_0^{6}$$

$$= 55 + 120 - \frac{18}{5}$$

$$\underline{= \frac{857}{5} \approx 171.4}$$

b) $P(t) = 55 + \left[20t - \frac{1}{10}t^2 \right]_0^{200}$

$$= 55 + 4{,}000 - 4{,}000$$

$$\underline{= 55}$$

Exercise

The population of a community of foxes is observed to fluctuate on a 10-year cycle due to variations in the availability of prey. When population measurements began (t = 0 years), the population was 35 foxes. The growth rate in units of *foxes/yr*. was observed to be

$$P'(t) = 5 + 10\sin\left(\frac{\pi t}{5}\right)$$

a) What is the population 15 years later? 35 years later?

b) Find the population $P(t)$ at any time $t \ge 0$.

Solution

a) $P(t) = P(0) + \int_0^{15} \left(5 + 10\sin\frac{\pi t}{5}\right)dt$

$$= 35 + \left[5t - \frac{50}{\pi}\cos\frac{\pi t}{5} \right]_0^{15}$$

$$= 35 + 75 + \frac{50}{\pi} + \frac{50}{\pi}$$

$$\underline{= 110 + \frac{100}{\pi} \approx 142 \ \textit{foxes}}$$

$$P(t) = 35 + \left[5t - \frac{50}{\pi}\cos\frac{\pi t}{5} \right]_0^{35}$$

$$= 35 + 175 + \frac{50}{\pi} + \frac{50}{\pi}$$

$$\underline{= 210 + \frac{100}{\pi} \approx 242 \ \textit{foxes}}$$

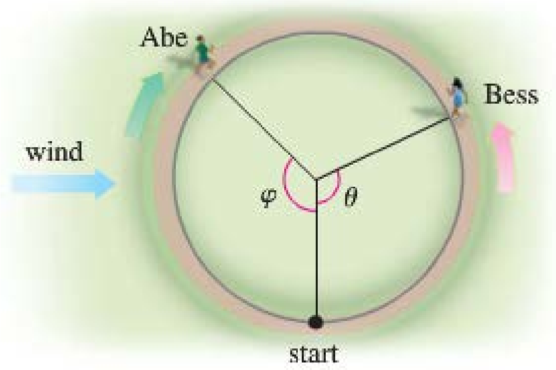
b) $P(t) = 35 + \int_0^{t} \left(5 + 10\sin\left(\frac{\pi}{5}x\right)\right)dx$

$$= 35 + \left[5x - \frac{50}{\pi}\cos\frac{\pi x}{5} \right]_0^{t}$$

$$\underline{= 35 + 5t - \frac{50}{\pi}\cos\frac{\pi t}{5} + \frac{50}{\pi} \ \textit{foxes}}$$

Exercise

A strong west wind blows across a circular running track. Abe and Bess start at the south end of the track and at the same time, Abe starts running clockwise and Bess starts running counterclockwise. Abe runs with a speed (in units of *mi/hr*.) given by $u(\varphi)=3-2\cos\varphi$ and Bess runs with a speed given by $v(\theta)=3+2\cos\theta$, where φ and θ are the central angles of the runners

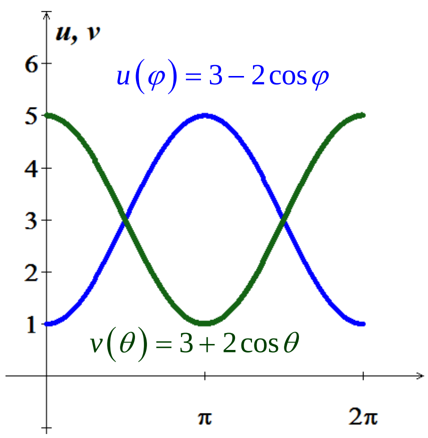


a) Graph the speed functions *u* and *v*, and explain why they describe the runners' speed (in light of the wind).

b) Which runner has the greater average speed for one lap?

c) If the track has a radius of $\frac{1}{10}$ *mi* , how long does it take each runner to complete one lap and who wins the race?

Solution



a) Abe starts out running into a headwind
Bess starts out running with a tailwind.

b) Abe's average speed $=\frac{1}{2\pi}\int_0^{2\pi}(3-2\cos\varphi)\,d\varphi$

$$=\frac{1}{2\pi}\left[3\varphi-2\sin\varphi\right]_0^{2\pi}$$

$$\underline{=3\ \ mph}$$

Bess' average speed $=\frac{1}{2\pi}\int_0^{2\pi}(3+2\cos\theta)\,d\theta$

$$=\frac{1}{2\pi}\left[3\theta+2\sin\theta\right]_0^{2\pi}$$

$$\underline{=3\ \ mph}$$

They have the same average speed.

c) The track is $\frac{1}{10}$ *mi* in radius $\Rightarrow s=\frac{1}{10}\varphi$ $\boxed{s=r\theta}$

We have $u=\frac{ds}{dt}=\frac{1}{10}\frac{d\varphi}{dt}=3-2\cos\varphi$

$$dt=\frac{1}{10(3-2\cos\varphi)}d\varphi$$

Abe's time:

$$T=\int_0^T dt=\int_0^{2\pi}\frac{1}{10(3-2\cos\varphi)}d\varphi$$

$$=\frac{2}{10\sqrt{5}}\tan^{-1}\left(\sqrt{5}\ \tan\frac{\varphi}{2}\right)\Bigg|_0^{2\pi}$$

$$=\frac{1}{5\sqrt{5}}[\pi-0]$$

$$\boxed{=\frac{\pi\sqrt{5}}{25}}$$

Let $\varphi=2\tan^{-1}u \quad\Rightarrow\quad u=\tan\frac{\varphi}{2}$

$$d\varphi=\frac{2}{1+u^2}du$$

$$\cos\varphi=\frac{1-\tan^2\frac{\varphi}{2}}{1+\tan^2\frac{\varphi}{2}}=\frac{1-u^2}{1+u^2}$$

$$\int\frac{d\varphi}{3-2\cos\varphi}=\int\frac{2}{1+u^2}\frac{1}{3-2\frac{1-u^2}{1+u^2}}du$$

$$=\int\frac{2}{1+5u^2}du$$

$$=\frac{2}{\sqrt{5}}\int\frac{1}{1+\left(\sqrt{5}u\right)^2}d\left(\sqrt{5}u\right)$$

$$=\frac{2}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}u\right)$$

$$=\frac{2}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\ \tan\frac{\varphi}{2}\right)$$

Bess' time:

$$T=\int_0^T dt=\int_0^{2\pi}\frac{1}{10(3+2\cos\theta)}d\theta$$

$$=\frac{1}{10}\int_0^{2\pi}\frac{2}{1+u^2}\frac{1}{3+2\frac{1-u^2}{1+u^2}}du$$

$$=\frac{1}{5}\int_0^{2\pi}\frac{du}{5+u^2}$$

$$=\frac{1}{5\sqrt{5}}\tan^{-1}\left(\frac{1}{\sqrt{5}}\ \tan u\right)\Bigg|_0^{2\pi}$$

$$=\frac{1}{5\sqrt{5}}\tan^{-1}\left(\frac{1}{\sqrt{5}}\ \tan\frac{\theta}{2}\right)\Bigg|_0^{2\pi}$$

$$\boxed{=\frac{\pi\sqrt{5}}{25}}$$

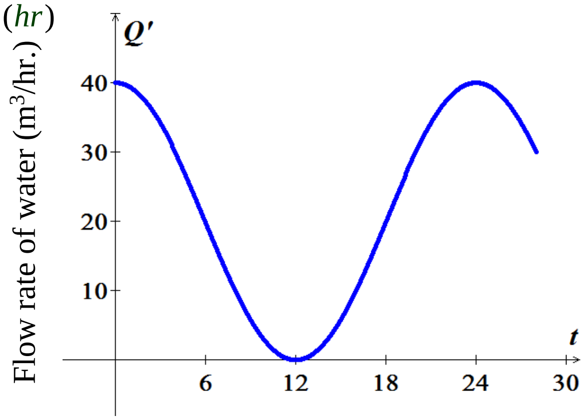
They tie the race, both have average speed $\frac{2\pi}{10T}=\frac{2\pi}{10\frac{\pi\sqrt{5}}{25}}=\frac{1}{\frac{\sqrt{5}}{5}}\ \boxed{=\sqrt{5}}$

Exercise

A reservoir with a capacity of 2500 m^3 is filled with a single inflow pipe. The reservoir is empty and the inflow pipe is opened at $t = 0$. Letting $Q(t)$ be the amount of water in the reservoir at time t, the flow rate of water into reservoir (in m^3 / hr) oscillates on 24-*hr* cycle and is given by

$$Q'(t) = 20\left[1 + \cos\frac{\pi t}{12}\right]$$

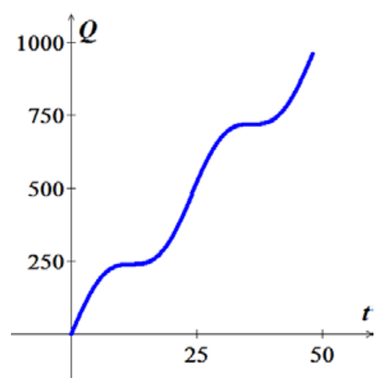
Time (hr)



a) How much water flows into the reservoir in the first 2 hr.?

b) Find and graph the function that gives the amount of water in the reservoir over the interval $[0,\ t]$ where $t \ge 0$.

c) When is the reservoir full?



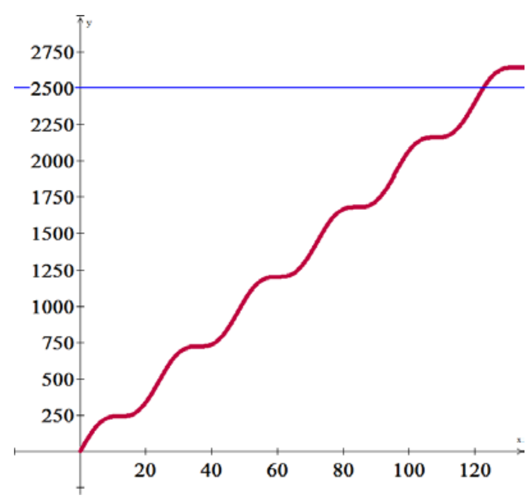
Solution

a) $Q(t) = \int_0^2 20\left(1 + \cos\frac{\pi t}{12}\right)dt$

$= \left(20t + \frac{240}{\pi}\sin\frac{\pi t}{12}\right)\Big|_0^2$

$= 400 + \frac{240}{\pi}\cdot\frac{1}{2}$

$= 400 + \frac{120}{\pi} \quad \approx 78.197\ m^3$



b) $Q(t) = \int_0^t 20\left(1 + \cos\frac{\pi x}{12}\right)dx$

$= \left(20x + \frac{240}{\pi}\sin\frac{\pi x}{12}\right)\Big|_0^t$

$= 20t + \frac{240}{\pi}\sin\frac{\pi t}{12}\quad m^3$

c) The reservoir is full when $20t + \frac{240}{\pi}\sin\frac{\pi t}{12} = 2500$

Using program: $T \approx 122.6\ \ hrs$

Solution   *Section* 1.2 – Region between Curves

Exercise

Find the area of the region bounded by the graphs of $y = 2x - x^2$ and $y = -3$

Solution

$$y = -3 \quad \to \quad 2x - x^2 = -3 \Rightarrow x^2 - 2x - 3 = 0 \quad \boxed{x = -1,\ 3}$$

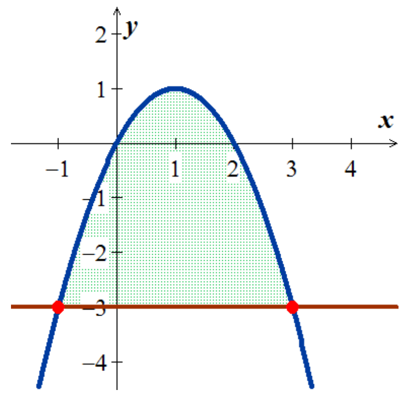
$$A = \int_{-1}^{3} \left[2x - x^2 - (-3)\right] dx$$

$$= \left[x^2 - \frac{x^3}{3} + 3x\right]_{-1}^{3}$$

$$= \left((3)^2 - \frac{(3)^3}{3} + 3(3)\right) - \left((-1)^2 - \frac{(-1)^3}{3} + 3(-1)\right)$$

$$= (9 - 9 + 9) - \left(1 + \frac{1}{3} - 3\right)$$

$$= \frac{32}{3}\ \ unit^2$$



Exercise

Find the area of the region bounded by the graphs of $y = 7 - 2x^2$ and $y = x^2 + 4$

Solution

$$7 - 2x^2 = x^2 + 4$$

$$-3x^2 = -3 \quad \to \quad x^2 = 1 \Rightarrow \boxed{x = \pm 1}$$

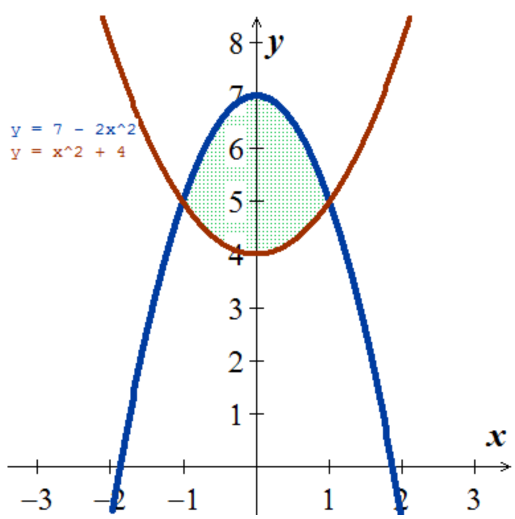
$$A = \int_{-1}^{1} \left[\left(7 - 2x^2\right) - \left(x^2 + 4\right)\right] dx$$

$$= \int_{-1}^{1} \left(3 - 3x^2\right) dx$$

$$= \left[3x - 3\frac{x^3}{3}\right]_{-1}^{1}$$

$$= \left(3(1) - (1)^3\right) - \left(3(-1) - (-1)^3\right)$$

$$= 4\ \ unit^2$$



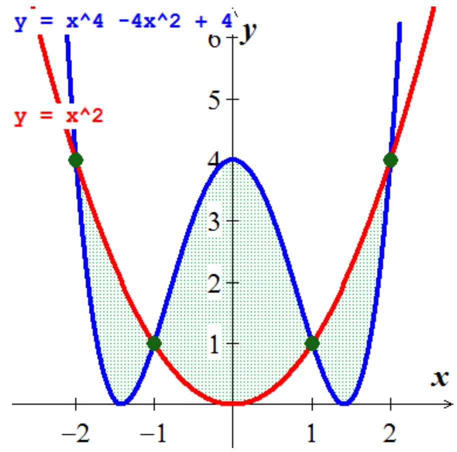
Exercise

Find the area of the region bounded by the graphs of $y = x^4 - 4x^2 + 4$ and $y = x^2$

Solution

$$x^4 - 4x^2 + 4 = x^2$$

$$x^4 - 5x^2 + 4 = 0 \quad \rightarrow \quad \boxed{x = \pm 1, \ \pm 2}$$



$$A = \int_{-2}^{-1} \left(x^2 - \left(x^4 - 4x^2 + 4\right)\right) dx + \int_{-1}^{1} \left(x^4 - 4x^2 + 4 - \left(x^2\right)\right) dx + \int_{1}^{2} \left(x^2 - \left(x^4 - 4x^2 + 4\right)\right) dx$$

$$= \int_{-2}^{-1} \left(-x^4 + 5x^2 - 4\right) dx + \int_{-1}^{1} \left(x^4 - 5x^2 + 4\right) dx + \int_{1}^{2} \left(-x^4 + 5x^2 - 4\right) dx$$

$$= \left[-\frac{x^5}{5} + \frac{5}{3}x^3 - 4x\right]_{-2}^{-1} + \left[\frac{x^5}{5} - \frac{5}{3}x^3 + 4x\right]_{-1}^{1} + \left[-\frac{x^5}{5} + \frac{5}{3}x^3 - 4x\right]_{1}^{2}$$

$$= \left[\left(-\frac{(-1)^5}{5} + \frac{5}{3}(-1)^3 - 4(-1)\right) - \left(-\frac{(-2)^5}{5} + \frac{5}{3}(-2)^3 - 4(-2)\right)\right]$$

$$+ \left[\left(\frac{(1)^5}{5} - \frac{5}{3}(1)^3 + 4(1)\right) - \left(\frac{(-1)^5}{5} - \frac{5}{3}(-1)^3 + 4(-1)\right)\right]$$

$$+ \left[\left(-\frac{(2)^5}{5} + \frac{5}{3}(2)^3 - 4(2)\right) - \left(-\frac{(1)^5}{5} + \frac{5}{3}(1)^3 - 4(1)\right)\right]$$

$$= \left(\tfrac{1}{5} - \tfrac{5}{3} + 4\right) - \left(\tfrac{32}{5} - \tfrac{40}{3} + 8\right) + \left(\tfrac{1}{5} - \tfrac{5}{3} + 4\right) - \left(-\tfrac{1}{5} + \tfrac{5}{3} - 4\right) + \left(-\tfrac{32}{5} + \tfrac{40}{3} - 8\right) - \left(-\tfrac{1}{5} + \tfrac{5}{3} - 4\right)$$

$$= 8 \ \ unit^2$$

Exercise

Find the area of the region bounded by the graphs of $x = 2y^2, \quad x = 0, \quad and \quad y = 3$

Solution

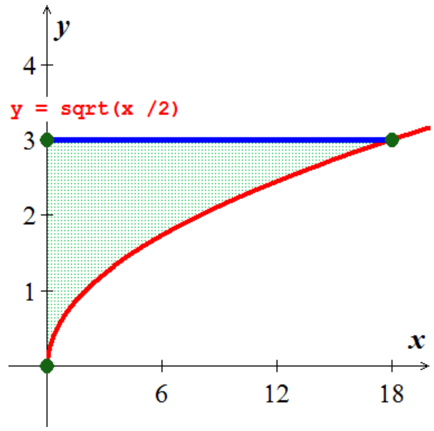
$$y = 3 \quad \to \quad \lfloor x = 2y^2 = 18 \rfloor$$

$$A = \int_0^3 2y^2 dy$$

$$= \frac{2}{3}\left[y^3 \right]_0^3$$

$$= \frac{2}{3}\left(3^3 - 0 \right)$$

$$= 18 \ \ unit^2$$



Exercise

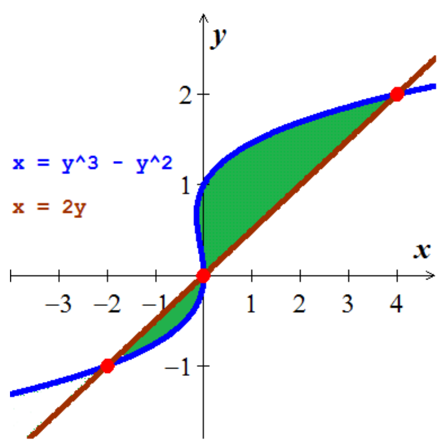
Find the area of the region bounded by the graphs of $x = y^3 - y^2 \quad and \quad x = 2y$

Solution

$$y^3 - y^2 = 2y$$

$$y^3 - y^2 - 2y = 0$$

$$y\left(y^2 - y - 2 \right) = 0 \quad \to \quad \boxed{y = 0, \ -1, \ 2}$$



$$A = \int_{-1}^{0} \left[y^3 - y^2 - (2y) \right] dy + \int_0^2 \left[2y - \left(y^3 - y^2 \right) \right] dy$$

$$= \int_{-1}^{0} \left(y^3 - y^2 - 2y \right) dy + \int_0^2 \left(2y - y^3 + y^2 \right) dy$$

$$= \left[\frac{y^4}{4} - \frac{y^3}{3} - y^2 \right]_{-1}^{0} + \left[y^2 - \frac{y^4}{4} + \frac{y^3}{3} \right]_0^2$$

$$= \left[0 - \left(\frac{1}{4} + \frac{1}{3} - 1 \right) \right] + \left[\left(4 - 4 + \frac{8}{3} \right) - 0 \right]$$

$$= \frac{5}{12} + \frac{8}{3}$$

$$= \frac{37}{12} \ \ unit^2$$

Exercise

Find the area of the region bounded by the graphs of $4x^2 + y = 4$ *and* $x^4 - y = 1$

Solution

$$4x^2 + y = 4 \quad \to \quad y = 4 - 4x^2$$

$$x^4 - y = 1 \quad and \quad y = x^4 - 1$$

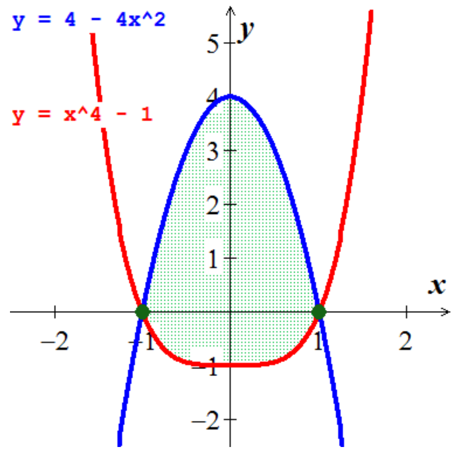
$$A = \int_{-1}^{1} \left[4 - 4x^2 - \left(x^4 - 1\right)\right] dx$$

$$= \int_{-1}^{1} \left(x^4 - 4x^2 + 5\right) dx$$

$$= \left[\frac{x^5}{5} - 4\frac{x^3}{3} + 5x\right]_{-1}^{1}$$

$$= \left(\frac{1}{5} - \frac{4}{3} + 5\right) - \left(-\frac{1}{5} + \frac{4}{3} - 5\right)$$

$$= \frac{105}{15} \ \ unit^2$$



Exercise

Find the area of the region bounded by the graphs of $x + 4y^2 = 4$ *and* $x + y^4 = 1,$ *for* $x \ge 0$

Solution

$$x = 4 - 4y^2 \quad x = 1 - y^4 \quad \to 4 - 4y^2 = 1 - y^4$$

$$y^4 - 4y^2 + 3 = 0 \quad \to \quad y^2 = 1,\ 3 \Rightarrow y = \pm 1,\ \pm\sqrt{3}$$

$$\begin{cases} y = \pm 1 & \to \boxed{x = 1 - (\pm 1)^4 = 0} \\ y = \pm\sqrt{3} & \to x = 1 - \left(\pm\sqrt{3}\right)^4 = -8 < 0 \times \end{cases}$$

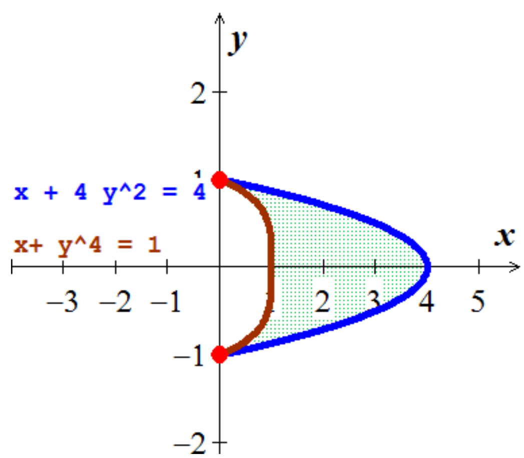
$$A = \int_{-1}^{1} \left[4 - 4y^2 - \left(1 - y^4\right)\right] dy$$

$$= \int_{-1}^{1} \left(3 - 4y^2 + y^4\right) dy$$

$$= \left[3y - 4\frac{y^3}{3} + \frac{y^5}{5}\right]_{-1}^{1}$$

$$= \left(3 - \frac{4}{3} + \frac{1}{5}\right) - \left(-3 + \frac{4}{3} - \frac{1}{5}\right)$$

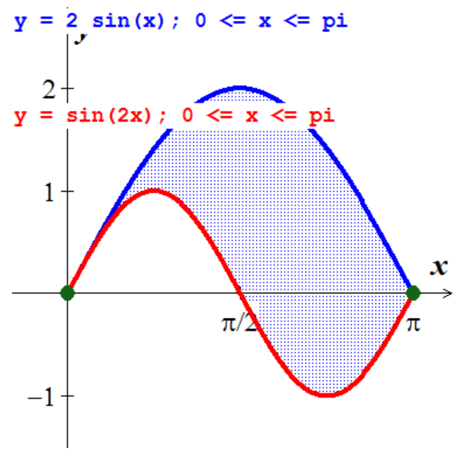
$$= \frac{56}{15} \ \ unit^2$$



Exercise

Find the area of the region bounded by the graphs of $y = 2\sin x$, *and* $y = \sin 2x$, $0 \le x \le \pi$

Solution



$$y = 2\sin x = \sin 2x$$

$$2\sin x = 2\sin x \cos x$$

$$2\sin x - 2\sin x \cos x = 0$$

$$2\sin x(1-\cos x) = 0 \quad \rightarrow \quad \begin{cases} \sin x = 0 & x = 0, \pi \\ \cos x = 1 & x = 0 \end{cases}$$

$$A = \int_0^{\pi} (2\sin x - \sin 2x)\,dx$$

$$= \left[-2\cos x + \frac{1}{2}\cos 2x\right]_0^{\pi}$$

$$= \left(-2(-1) + \frac{1}{2}(1)\right) - \left(-2 + \frac{1}{2}\right)$$

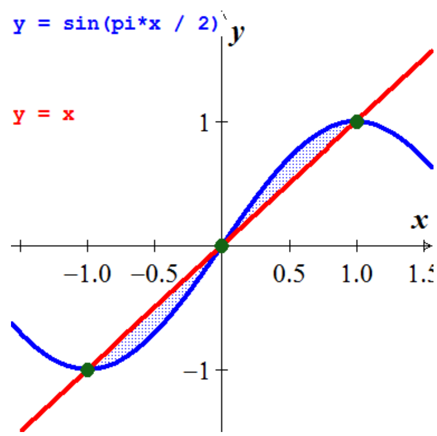
$$= 4 \ \ unit^2$$

Exercise

Find the area of the region bounded by the graphs of $y = \sin\frac{\pi x}{2}$ *and* $y = x$

Solution

$$y = \sin\frac{\pi x}{2} = x \quad \rightarrow \quad \boxed{x = -1,\ 1}$$



$$A = \int_{-1}^{0} \left(\sin\frac{\pi x}{2} - x\right)dx + \int_0^1 \left(\sin\frac{\pi x}{2} - x\right)dx$$

$$= 2\int_0^1 \left(\sin\frac{\pi x}{2} - x\right)dx$$

$$= 2\left[-\frac{2}{\pi}\cos\frac{\pi x}{2} - \frac{x^2}{2}\right]_0^1$$

$$= 2\left[\left(0 - \frac{1}{2}\right) - \left(-\frac{2}{\pi} - 0\right)\right]$$

$$= 2\left(-\frac{1}{2} + \frac{2}{\pi}\right)$$

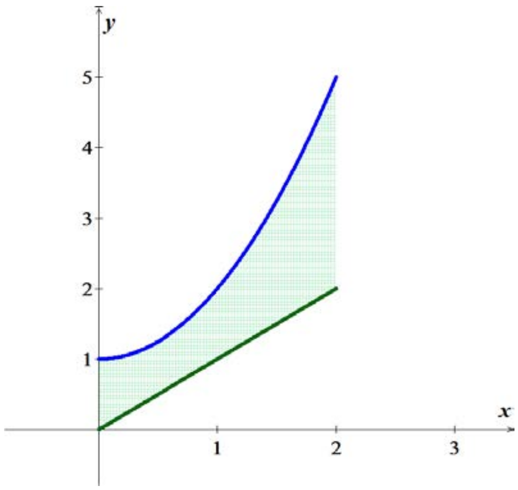
$$= 2\left(\frac{-\pi + 4}{2\pi}\right)$$

$$= \frac{4-\pi}{\pi} \ \ unit^2$$

Exercise

Find the area of the region bounded by the graphs of $y = x^2 + 1$ and $y = x$ for $0 \le x \le 2$

Solution



$$A = \int_0^2 [(x^2+1) - x]dx$$

$$= \int_0^2 \left(x^2 - x + 1\right)dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + 1x \Big|_0^2$$

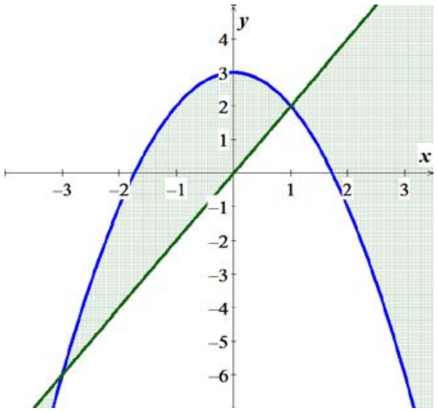
$$= \frac{2^3}{3} - \frac{2^2}{2} + 1(2) - 0$$

$$= \frac{8}{3} \ unit^2$$

Exercise

Find the area of the region bounded by the graphs of $y = 3 - x^2$ and $y = 2x$

Solution



$$x^2 + 2x - 3 = 0 \quad \rightarrow \boxed{x = 1, -3}$$

$$A = \int_{-3}^{1} \left(\left(3 - x^2\right) - 2x\right)dx$$

$$= \int_{-3}^{1} \left(-x^2 - 2x + 3\right)dx$$

$$= -\frac{x^3}{3} - 2\frac{x^2}{2} + 3x \Big|_{-3}^{1}$$

$$= -\frac{1^3}{3} - 1^2 + 3(1) - \left[-\frac{(-3)^3}{3} - (-3)^2 + 3(-3)\right] \quad = -\frac{1}{3} - 1 + 3 - [9 - 9 - 9]$$

$$= 11 - \frac{1}{3}$$

$$= \frac{32}{3} \ unit^2$$

Exercise

Find the area of the region bounded by the graphs of $y = x^2 - x - 2$ and x-axis

Solution

The intersection points: $x^2 - x - 2 = 0 \Rightarrow \boxed{x = -1, 2}$

$$A = \int_{-1}^{2} [0 - (x^2 - x - 2)]dx$$

$$= -\frac{x^3}{3} + \frac{x^2}{2} + 2x \Big|_{-1}^{2}$$

$$= -\frac{2^3}{3} + \frac{2^2}{2} + 2(2) - \left[-\frac{(-1)^3}{3} + \frac{(-1)^2}{2} + 2(-1) \right]$$

$$= -\frac{8}{3} + 2 + 4 - \left[\frac{1}{3} + \frac{1}{2} - 2 \right]$$

$$= \frac{10}{3} + \frac{7}{6}$$

$$= \frac{9}{2} \ \ unit^2$$

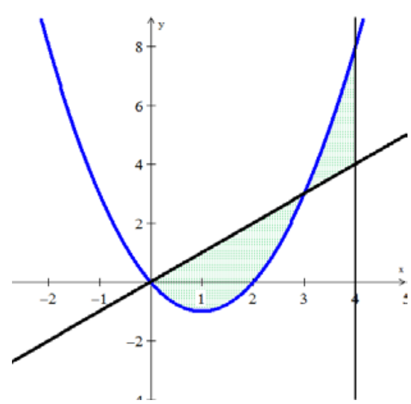
Exercise

Find the area between the curves $y = x^{1/2}$ *and* $y = x^3$

Solution

$x^3 = x^{1/2}$ *Square both sides* $\rightarrow x^6 = x$

$x\left(x^5 - 1\right) = 0 \quad \rightarrow x = 0 \quad x^5 - 1 = 0 \Rightarrow x = 1$



$$A = \int_{0}^{1} \left(x^{1/2} - x^3\right) dx$$

$$= \frac{2}{3}x^{3/2} - \frac{1}{4}x^4 \Big|_{0}^{1}$$

$$= \frac{2}{3}1^{3/2} - \frac{1}{4}1^4 - 0$$

$$= \frac{2}{3} - \frac{1}{4}$$

$$= \frac{8-3}{12}$$

$$= \frac{5}{12} \ \ unit^2$$

Exercise

Find the area of the region bounded by the graphs of $y = x^2 - 2x$ and $y = x$ on [0, 4].

Solution

$$x^2 - 2x = x \quad x^2 - 3x = 0$$

$$x(x-3) = 0 \Rightarrow \boxed{x = 0, 3}$$

$$A = \int_0^3 \left(x - \left(x^2 - 2x\right)\right) dx + \int_3^4 \left(x^2 - 2x - x\right) dx$$

$$= \int_0^3 \left(-x^2 + 3x\right) dx + \int_3^4 \left(x^2 - 3x\right) dx$$

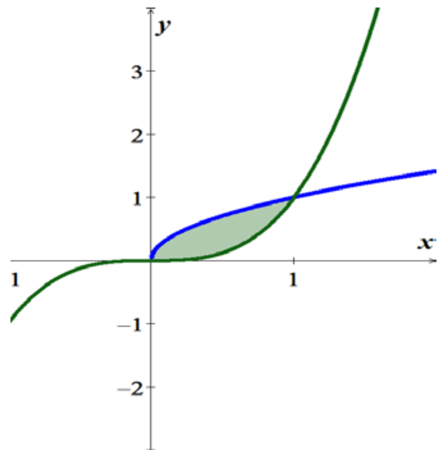
$$= \left(-\tfrac{1}{3}x^3 + \tfrac{3}{2}x^2\right)\Big|_0^3 + \left(\tfrac{1}{3}x^3 - \tfrac{3}{2}x^2\right)\Big|_3^4$$

$$= \left(-\tfrac{1}{3}3^3 + \tfrac{3}{2}3^2\right) + \left[\left(\tfrac{1}{3}4^3 - \tfrac{3}{2}4^2\right) - \left(\tfrac{1}{3}3^3 - \tfrac{3}{2}3^2\right)\right]$$

$$= \left(\tfrac{9}{2}\right) + \left[\left(-\tfrac{8}{3}\right) - \left(-\tfrac{9}{2}\right)\right]$$

$$= \tfrac{9}{2} - \tfrac{8}{3} + \tfrac{9}{2}$$

$$= \frac{19}{3} \ \ unit^2$$



Exercise

Find the area between the curves $x = 1, \quad x = 2, \quad y = x^3 + 2, \quad y = 0$

Solution

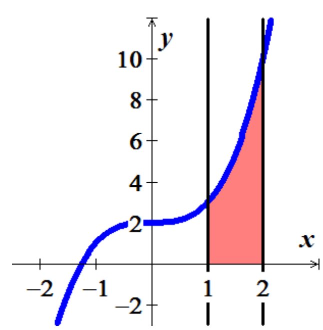
$$A = \int_1^2 \left(x^3 + 2 - 0\right) dx$$

$$= \tfrac{1}{4}x^4 + 2x\Big|_1^2$$

$$= \left(\tfrac{1}{4}2^4 + 2(2)\right) - \left(\tfrac{1}{4}1^4 + 2(1)\right)$$

$$= (8) - \left(\tfrac{9}{4}\right)$$

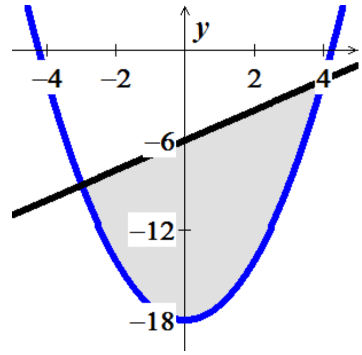
$$= \frac{23}{4} \ \ unit^2$$



Exercise

Find the area between the curves $y = x^2 - 18, \quad y = x - 6$

Solution



$$x^2 - 18 = x - 6$$

$$x^2 - x - 12 = 0 \rightarrow \boxed{x = -3,\ 4}$$

$$A = \int_{-3}^{4} \left(x^2 - 18 - (x - 6) \right) dx$$

$$= \int_{-3}^{4} \left(x^2 - x - 12 \right) dx$$

$$= \frac{1}{3}x^3 - \frac{1}{2}x^2 - 12x \Big|_{-3}^{4}$$

$$= \left(\frac{1}{3}4^3 - \frac{1}{2}4^2 - 12(4) \right) - \left(\frac{1}{3}(-3)^3 - \frac{1}{2}(-3)^2 - 12(-3) \right)$$

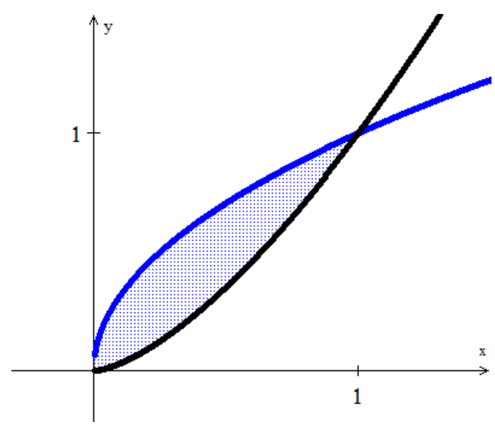
$$= \left(-\frac{104}{3} \right) - \left(\frac{45}{2} \right)$$

$$= \boxed{\frac{343}{6}\ unit^2}$$

Exercise

Find the area between the curves $y = \sqrt{x}, \quad y = x\sqrt{x}$

Solution



$$x\sqrt{x} = \sqrt{x} \quad \Rightarrow \quad \left(x\sqrt{x} \right)^2 = \left(\sqrt{x} \right)^2$$

$$x^2 x = x \quad \rightarrow \quad x\left(x^2 - 1 \right) = 0$$

$$\boxed{x = 0} \quad x^2 - 1 = 0 \Rightarrow x = \pm 1 (no\ negative) \quad \boxed{x = 1}$$

$$A = \int_0^1 \left(\sqrt{x} - x\sqrt{x} \right) dx$$

$$= \int_0^1 \left(x^{1/2} - x^{3/2} \right) dx$$

$$= \frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} \Big|_0^1$$

$$= \left(\frac{2}{3}1^{3/2} - \frac{2}{5}1^{5/2} \right) - \left(\frac{2}{3}0^{3/2} - \frac{2}{5}0^{5/2} \right)$$

$$= \left(\frac{2}{3} - \frac{2}{5} \right) - 0$$

$$= \boxed{\frac{4}{15}\ unit^2}$$

Exercise

Find the area of the region bounded by the graphs of $f(x)=x^3+2x^2-3x$ and $g(x)=x^2+3x$

Solution

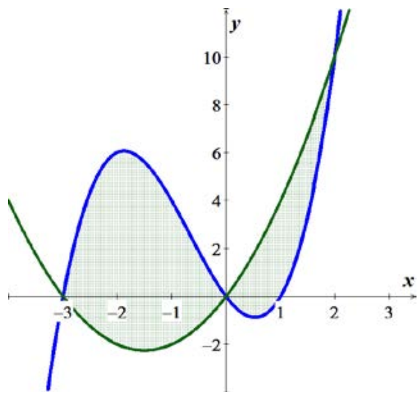
$$x^3+2x^2-3x=x^2+3x \quad\to\quad x^3+x^2-6x=0$$

$$x(x^2+x-6)=0 \Rightarrow \begin{cases} x=0 \\ x^2+x-6=0 \end{cases} \to \boxed{x=-3,\ 0,\ 2}$$

$$A=\int_{-3}^{0}(f-g)dx+\int_{0}^{2}(g-f)dx$$

$$=\int_{-3}^{0}\left(x^3+2x^2-3x-(x^2+3x)\right)dx+\int_{0}^{2}\left(x^2+3x-(x^3+2x^2-3x)\right)dx$$

$$=\int_{-3}^{0}\left(x^3+x^2-6x\right)dx+\int_{0}^{2}\left(-x^3-x^2+6x\right)dx$$



$$=\frac{x^4}{4}+\frac{x^3}{3}-3x^2\Big|_{-3}^{0}+\left[-\frac{x^4}{4}-\frac{x^3}{3}+3x^2\right]_0^2$$

$$=0-\left(\frac{(-3)^4}{4}+\frac{(-3)^3}{3}-3(-3)^2\right)+\left[\left(-\frac{2^4}{4}-\frac{2^3}{3}+32^2\right)-0\right]$$

$$=\frac{253}{12}\ unit^2 \qquad \approx 21.083$$

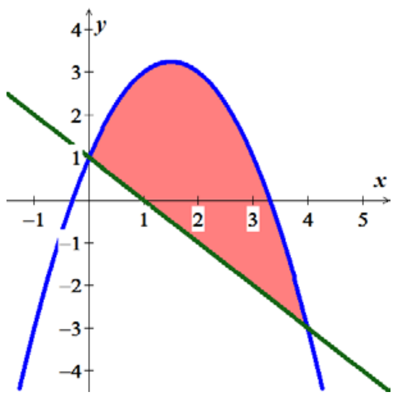
Exercise

Find the area of the region bounded by the graphs of $y=-x^2+3x+1,\quad y=-x+1$

Solution

$$y=-x^2+3x+1=-x+1 \;\to\; x^2-4x=0 \;\Rightarrow\; x=0,\ 4$$

$$A=\int_0^4\left[-x^2+3x+1-(-x+1)\right]dx$$



$$=\int_0^4\left(-x^2+4x\right)dx$$

$$=-\frac{1}{3}x^3+2x^2\Big|_0^4$$

$$=-\frac{64}{3}+32$$

$$=\frac{32}{3}\ unit^2$$

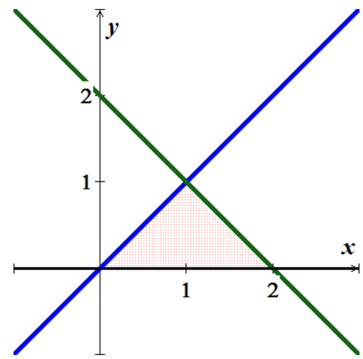
Exercise

Find the area of the region bounded by the graphs of $\quad y = x, \quad y = 2 - x, \quad y = 0$

Solution

$y = x = 2 - x \;\rightarrow\; \underline{x = 1}$

$y = 2 - x = 0 \;\rightarrow\; \underline{x = 2}$



$$A = \int_0^1 (x - 0)\,dx + \int_1^2 (2 - x - 0)\,dx$$

$$= \frac{1}{2}x^2 \Big|_0^1 + \left(2x - \frac{1}{2}x^2\right)\Big|_1^2$$

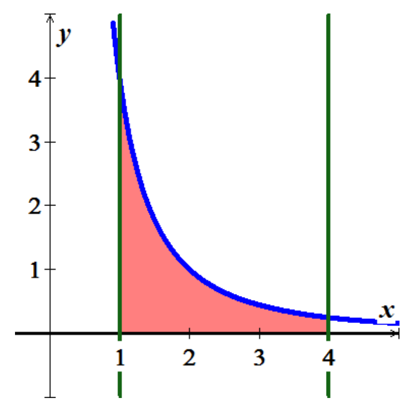
$$= \frac{1}{2} + 4 - 2 - 2 + \frac{1}{2}$$

$$\underline{= 1 \;\; unit^2}$$

Exercise

Find the area of the region bounded by the graphs of $\quad y = \frac{4}{x^2}, \quad y = 0, \quad x = 1, \quad x = 4$

Solution



$$A = \int_1^4 \frac{4}{x^2}\,dx$$

$$= -\frac{4}{x}\Big|_1^4$$

$$= 4\left(-\frac{1}{4} + 1\right)$$

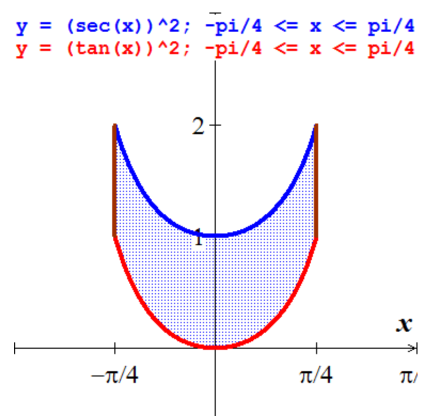
$$\underline{= 3 \;\; unit^2}$$

Exercise

Find the area of the region bounded by the graphs of

$$y = \sec^2 x, \quad y = \tan^2 x, \quad x = -\frac{\pi}{4}, \quad and \quad x = \frac{\pi}{4}$$

Solution



$$A = \int_{-\pi/4}^{\pi/4} \left(\sec^2 x - \tan^2 x\right) dx$$

$$= \int_{-\pi/4}^{\pi/4} \left(\sec^2 x - \left(\sec^2 x - 1\right)\right) dx$$

$$= \int_{-\pi/4}^{\pi/4} dx$$

$$= x\Big|_{-\pi/4}^{\pi/4}$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right)$$

$$= \frac{\pi}{2}\ \ unit^2$$

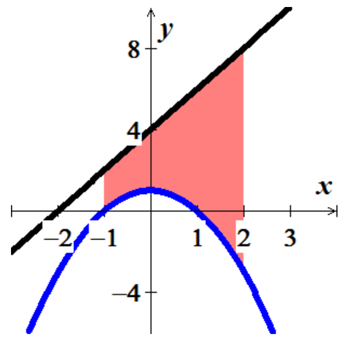
Exercise

Find the area bounded by $f(x) = -x^2 + 1,\ g(x) = 2x + 4,\ x = -1,\ x = 2$

Solution

$$f \cap g \Rightarrow \quad -x^2 + 1 = 2x + 4$$

$$x^2 + 2x + 3 = 0 \quad \Rightarrow x = -1 \pm i\sqrt{2}$$



$$A = \int_{-1}^{2} (g - f)dx$$

$$= \int_{-1}^{2} \left(2x + 4 - \left(-x^2 + 1\right)\right)dx$$

$$= \int_{-1}^{2} \left(x^2 + 2x + 3\right)dx$$

$$= \frac{1}{3}x^3 + x^2 + 3x\Big|_{-1}^{2}$$

$$= \left(\frac{1}{3}(2)^3 + (2)^2 + 3(2)\right) - \left(\frac{1}{3}(-1)^3 + (-1)^2 + 3(-1)\right)$$

$$= \left(\frac{8}{3} + 4 + 6\right) - \left(-\frac{1}{3} + 1 - 3\right)$$

$$= \frac{8}{3} + 10 + \frac{1}{3} + 2$$

$$= 15\ \ unit^2$$

Exercise

Find the area of the region bounded by the graphs of $\quad f(x) = \sqrt{x} + 3, \quad g(x) = \frac{1}{2}x + 3$

Solution

$$\sqrt{x} + 3 = \frac{1}{2}x + 3 \ \Rightarrow \ \left(\sqrt{x}\right)^2 = \left(\frac{1}{2}x\right)^2$$

$$x = \frac{1}{4}x^2 \to \underline{x = 0,\ 4}$$

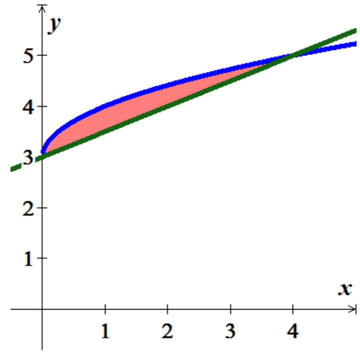
$$A=\int_0^4\left(\sqrt{x}+3-\frac{1}{2}x-3\right)dx$$

$$=\int_0^4\left(x^{1/2}-\frac{1}{2}x\right)dx$$

$$=\frac{2}{3}x^{3/2}-\frac{1}{4}x^2\Big|_0^4$$

$$=\frac{16}{3}-4$$

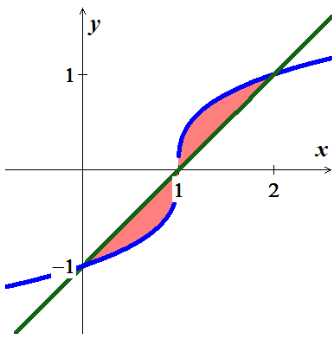
$$=\frac{4}{3}\ unit^2$$



Exercise

Find the area of the region bounded by the graphs of $f(x)=\sqrt[3]{x-1}, \quad g(x)=x-1$

Solution



$$\left(\sqrt[3]{x-1}\right)^3=(x-1)^3$$

$$x-1=x^3-3x^2+3x-1$$

$$x\left(x^2-3x+2\right)=0 \ \rightarrow \ \underline{x=0,\ 1,\ 2}$$

$$A=\int_0^1\left(x-1-\sqrt[3]{x-1}\right)dx+\int_1^2\left(\sqrt[3]{x-1}-x+1\right)dx$$

$$=\left[\frac{1}{2}x^2-x-\frac{3}{4}(x-1)^{4/3}\right]_0^1+\left[\frac{3}{4}(x-1)^{4/3}-\frac{1}{2}x^2+x\right]_1^2$$

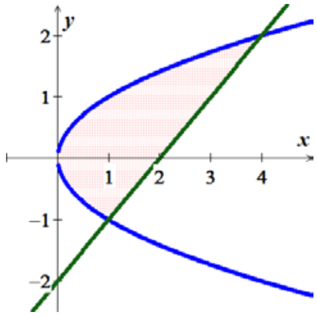
$$=\frac{1}{2}-1+\frac{3}{4}+\frac{3}{4}-2+2+\frac{1}{2}-1$$

$$=\frac{1}{2}\ unit^2$$

Exercise

Find the area of the region bounded by the graphs of $f(y)=y^2, \quad g(y)=y+2$

Solution



$$y^2=y+2 \ \Rightarrow \ y^2-y-2=0 \ \rightarrow \ \underline{y=-1,\ 2}$$

$$A=\int_{-1}^2\left(y+2-y^2\right)dy$$

$$= \frac{1}{2}y^2 + 2y - \frac{1}{3}y^3 \Big|_{-1}^{2}$$

$$= 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}$$

$$= \frac{9}{2} \ \ unit^2$$

Exercise

Find the area of the region bounded by the graphs of $f(y) = y(2-y), \quad g(y) = -y$

Solution

$$2y - y^2 = -y \quad \Rightarrow y^2 - 3y = 0 \ \rightarrow \ y = 0, \ 3$$

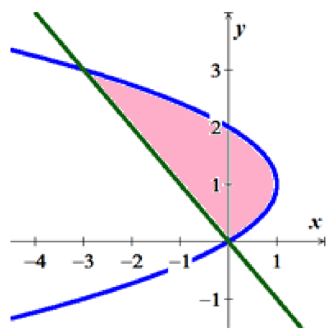
$$A = \int_0^3 \left(2y - y^2 + y\right) dy$$

$$= \int_0^3 \left(3y - y^2\right) dy$$

$$= \frac{3}{2}y^2 - \frac{1}{3}y^3 \Big|_0^3$$

$$= \frac{27}{2} - 9$$

$$= \frac{9}{2} \ \ unit^2$$



Exercise

Find the area of the region bounded by the graphs of $f(y) = y^2 + 1, \quad g(y) = 0, \quad y = -1, \quad y = 2$

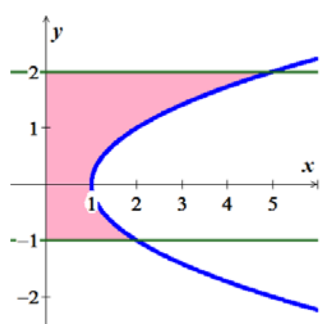
Solution

$$A = \int_{-1}^{2} \left(y^2 + 1 - 0\right) \ dy$$

$$= \frac{1}{3}y^3 + y \Big|_{-1}^{2}$$

$$= \frac{8}{3} + 2 + \frac{1}{3} + 1$$

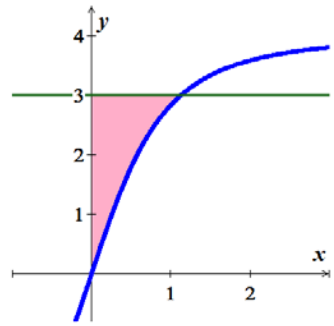
$$= 6 \ \ unit^2$$



Exercise

Find the area of the region bounded by the graphs of $f(y)=\dfrac{y}{\sqrt{16-y^2}}, \quad g(y)=0, \quad y=3$

Solution



$$A=\int_0^3\left(\frac{y}{\sqrt{16-y^2}}-0\right)dy$$

$$=-\frac{1}{2}\int_0^3\left(16-y^2\right)^{-1/2}\ d\left(16-y^2\right)$$

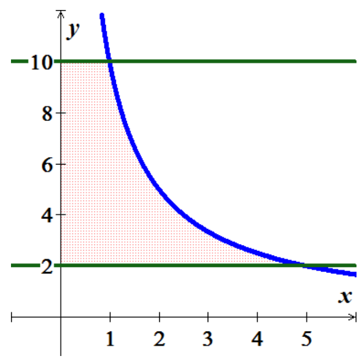
$$=-\sqrt{16-y^2}\Big|_0^3$$

$$\underline{=-\sqrt{7}+4\ \ unit^2}$$

Exercise

Find the area of the region bounded by the graphs of $f(x)=\dfrac{10}{x}, \quad x=0, \quad y=2, \quad y=10$

Solution



$$y=\frac{10}{x}\ \Rightarrow\ x=\frac{0}{y}$$

$$A=\int_2^{10}\frac{10}{y}\,dy$$

$$=10\ln y\Big|_2^{10}$$

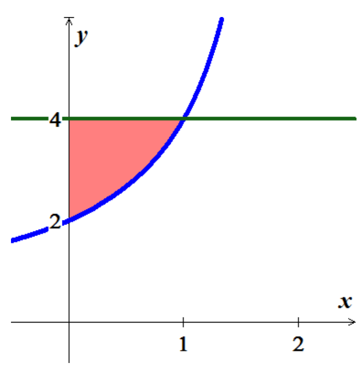
$$=10(\ln 10-\ln 2)$$

$$\underline{=10\ln 5\ \ unit^2}$$

Exercise

Find the area of the region bounded by the graphs of $g(x)=\dfrac{4}{2-x}, \quad y=4, \quad x=0$

Solution



$$\frac{4}{2-x}=4\ \Rightarrow\ 2-x=1\ \rightarrow\ \underline{x=1}$$

$$A=\int_0^1\left(4-\frac{4}{2-x}\right)dx \qquad\qquad \int\frac{dx}{ax+b}=\frac{1}{a}\ln|ax+b|$$

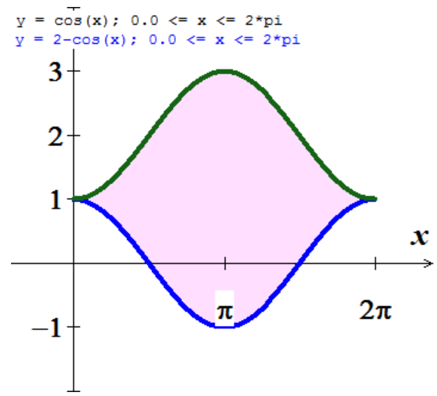
$$=4x+4\ln|2-x|\Big|_0^1$$

$$\underline{=4+4\ln 2\ \ unit^2}$$

Exercise

Find the area of the region bounded by the graphs of $f(x)=\cos x, \quad g(x)=2-\cos x, \quad 0\le x\le 2\pi$

Solution



$$A=\int_0^{2\pi}(2-\cos x-\cos x)\,dx$$

$$=2\int_0^{2\pi}(1-\cos x)\,dx$$

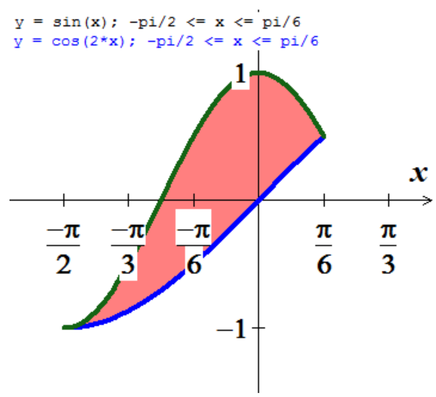
$$=2(x-\sin x)\Big|_0^{2\pi}$$

$$=4\pi$$

Exercise

Find the area of the region bounded by the graphs of $f(x)=\sin x, \quad g(x)=\cos 2x, \quad -\frac{\pi}{2}\le x\le \frac{\pi}{6}$

Solution



$$A=\int_{-\pi/2}^{\pi/6}(\cos 2x-\sin x)\,dx$$

$$=\frac{1}{2}\sin 2x+\cos x\Big|_{-\pi/2}^{\pi/6}$$

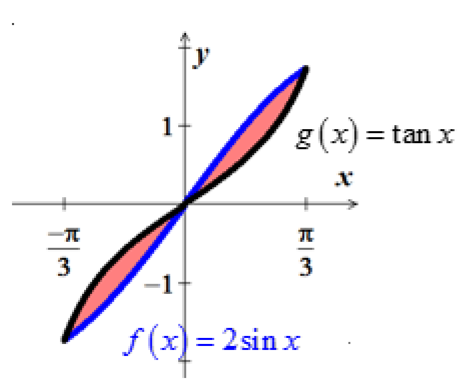
$$=\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{2}$$

$$=\frac{3\sqrt{3}}{4}$$

Exercise

Find the area of the region bounded by the graphs of $f(x)=2\sin x, \quad g(x)=\tan x, \quad -\frac{\pi}{3}\le x\le \frac{\pi}{3}$

Solution



$$A=2\int_0^{\pi/3}(2\sin x-\tan x)\,dx$$

$$=2\left(-2\cos x+\ln|\cos x|\right)\Big|_0^{\pi/3}$$

$$=2\left(-1+\ln\frac{1}{2}+2\right)$$

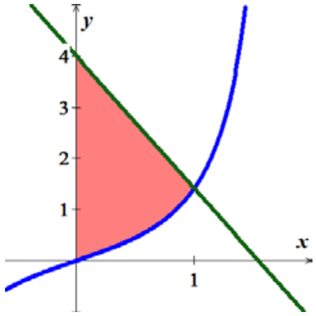
$$=2(1-\ln 2)$$

Exercise

Find the area of the region bounded by the graphs of

$$f(x)=\sec\frac{\pi x}{4}\tan\frac{\pi x}{4}, \quad g(x)=\left(\sqrt{2}-4\right)x+4, \quad x=0$$

Solution



$$A=\int_0^1\left(\left(\sqrt{2}-4\right)x+4-\sec\frac{\pi x}{4}\tan\frac{\pi x}{4}\right)dx$$

$$=\frac{1}{2}\left(\sqrt{2}-4\right)x^2+4x-\frac{4}{\pi}\sec\frac{\pi x}{4}\Big|_0^1$$

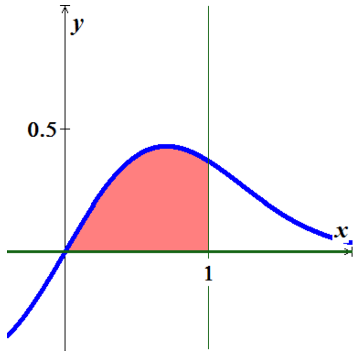
$$=\frac{1}{2}\sqrt{2}-2+4-\frac{4}{\pi}\sqrt{2}+\frac{4}{\pi}$$

$$=\frac{\sqrt{2}}{2}+2+\frac{4}{\pi}\left(1-\sqrt{2}\right)$$

Exercise

Find the area of the region bounded by the graphs of $f(x)=xe^{-x^2}, \quad y=0, \quad 0\le x\le 1$

Solution



$$A=\int_0^1 xe^{-x^2}dx$$

$$=-\frac{1}{2}\int_0^1 e^{-x^2}d\left(-x^2\right)$$

$$=-\frac{1}{2}e^{-x^2}\Big|_0^1$$

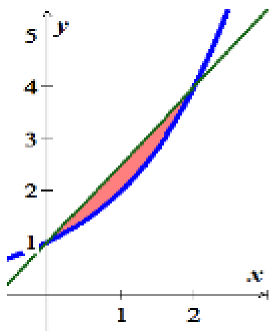
$$=-\frac{1}{2}\left(e^{-1}-1\right)$$

$$=\frac{1}{2}\left(1-\frac{1}{e}\right)$$

Exercise

Find the area of the region bounded by the graphs of $f(x)=2^x, \quad g(x)=\frac{3}{2}x+1$

Solution



$$A=\int_0^2\left(\frac{3}{2}x+1-2^x\right)dx$$

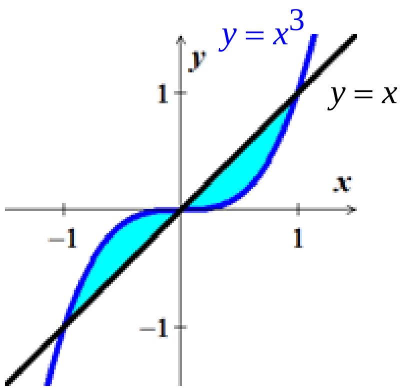
$$=\frac{3}{4}x^2+x-\frac{2^x}{\ln 2}\Big|_0^2$$

$$= 3 + 2 - \frac{4}{\ln 2} + \frac{1}{\ln 2}$$

$$= 5 - \frac{3}{\ln 2}$$

Exercise

Determine the area of the shaded region in



Solution

$$y = x^3 = x \quad \rightarrow \quad x\left(x^2 - 1\right) = 0 \quad \therefore x = 0, \ \pm 1$$

$$Area = \int_{-1}^{0} \left(x^3 - x\right) dx + \int_{0}^{1} \left(x - x^3\right) dx$$

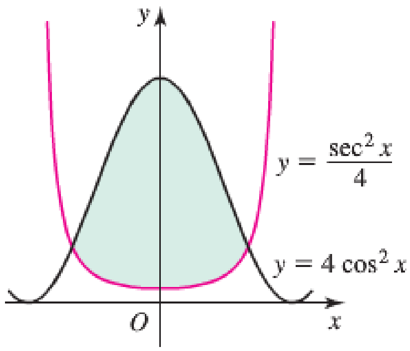
$$= \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_{-1}^{0} + \left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_{0}^{1}$$

$$= -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4}$$

$$= \frac{1}{2} \ unit^2$$

Exercise

Determine the area of the shaded region in



Solution

$$y = \frac{\sec^2 x}{4} = 4\cos^2 x \quad \rightarrow \quad \cos^4 x = \frac{1}{16}$$

$$\cos x = \pm\frac{1}{2} \quad \rightarrow \quad x = \pm\frac{\pi}{3}$$

By the symmetry;

$$Area = 2\int_{0}^{\pi/3} \left(4\cos^2 x - \frac{1}{4}\sec^2 x\right) dx$$

$$= 2\int_{0}^{\pi/3} \left(2 + 2\cos 2x - \frac{1}{4}\sec^2 x\right) dx$$

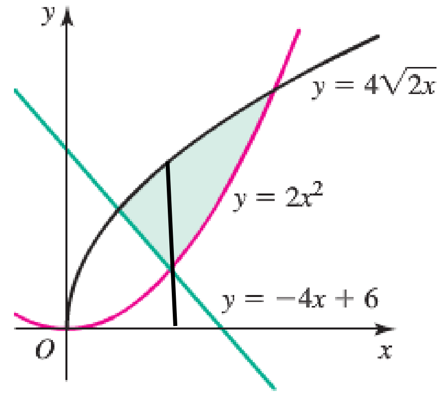
$$= 2\left[2x + \sin 2x - \frac{1}{4}\tan x\right]_{0}^{\pi/3}$$

$$= 2\left(\frac{2\pi}{3} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4}\right)$$

$$= \frac{4\pi}{3} + \frac{\sqrt{3}}{2} \ unit^2$$

Exercise

Determine the area of the shaded region in



Solution

$$y = 4\sqrt{2x} = -4x + 6 \quad \to \quad \left(4\sqrt{2x}\right)^2 = (-4x+6)^2$$

$$32x = 16x^2 - 48x + 36$$

$$16x^2 - 80x + 36 = 0 \quad \to \quad x = \frac{1}{2},\ \cancel{\frac{9}{2}}$$

$$y = 4\sqrt{2x} = 2x^2 \quad \to \quad \left(4\sqrt{2x}\right)^2 = \left(2x^2\right)^2$$

$$32x = 4x^4 \ \to \ 4x\left(x^3 - 8\right) = 0 \ \to \ x = 2,\ \cancel{0}$$

$$y = 2x^2 = -4x + 6 \quad \to \quad x^2 + 2x - 3 = 0 \quad \to \quad x = 1,\ \cancel{-3}$$

$$Area = \int_{1/2}^{1}\left(4\sqrt{2x} - (-4x+6)\right)dx + \int_1^2 \left(4\sqrt{2x} - 2x^2\right)dx$$

$$= \left(\frac{8\sqrt{2}}{3}x^{3/2} + 2x^2 - 6x\right)\Big|_{1/2}^{1} + \left(\frac{8\sqrt{2}}{3}x^{3/2} - \frac{2}{3}x^3\right)\Big|_1^2$$

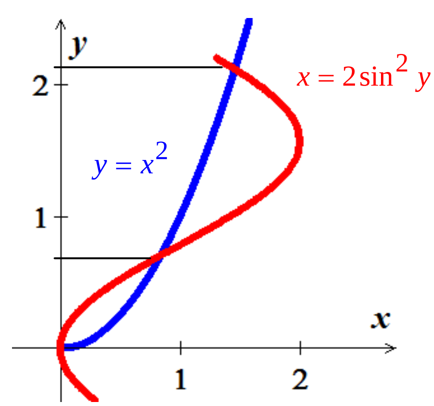
$$= \left(\frac{8\sqrt{2}}{3} + 2 - 6 - \frac{8\sqrt{2}}{3}\frac{1}{2\sqrt{2}} - \frac{1}{2} + 3\right) + \left(\frac{32}{3} - \frac{16}{3} - \frac{8\sqrt{2}}{3} + \frac{2}{3}\right)$$

$$= -1 - \frac{4}{3} - \frac{1}{2} + 6$$

$$= \frac{19}{6}\ \ unit^2$$

Exercise

Determine the area of the shaded region in



Solution

From the graph the intersection are: $y = 0,\ \ y \approx .705,\ \ y \approx 2.12$

$$A = \int_0^{.705}\left(\sqrt{y} - 2\sin^2 y\right)dy + \int_{.705}^{2.12}\left(2\sin^2 y - \sqrt{y}\right)dy$$

$$= \int_0^{.705}\left(y^{1/2} - 1 + \cos 2y\right)dy + \int_{.705}^{2.12}\left(1 - \cos 2y - y^{1/2}\right)dy$$

$$= \left(\frac{2}{3}y^{3/2} - y + \frac{1}{2}\sin 2y\right)\Big|_0^{.705} + \left(y - \frac{1}{2}\sin 2y - \frac{2}{3}y^{3/2}\right)\Big|_{.705}^{2.12}$$

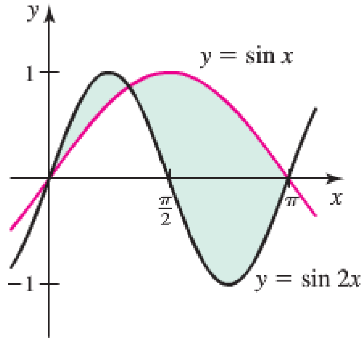
$$= \frac{2}{3}(.705)^{3/2} - 0.705 + \frac{1}{2}\sin(1.41) + 2.12 - \frac{1}{2}\sin(4.24) - \frac{2}{3}(2.12)^{3/2} - .705 + \frac{1}{2}\sin(1.41) + \frac{2}{3}(.705)^{3/2}$$

$$\approx .8738\ \ unit^2$$

Exercise

Determine the area of the shaded regions between $y = \sin x$ *and* $y = \sin 2x$, *for* $0 \le x \le \pi$

Solution



$$y = \sin x = \sin 2x$$

$$\sin x = 2\sin x \cos x \quad \rightarrow \quad \sin x(2\cos x - 1) = 0$$

$$\sin x = 0 \quad \rightarrow \underline{x = 0,\ \pi}|$$

$$\cos x = \frac{1}{2} \quad \rightarrow \underline{x = \frac{\pi}{3}}|$$

$$A = \int_0^{\pi/3} (\sin 2x - \sin x)\,dx + \int_{\pi/3}^{\pi} (\sin x - \sin 2x)\,dx$$

$$= \left[-\frac{1}{2}\cos 2x + \cos x\right]_0^{\pi/3} + \left[-\cos x + \frac{1}{2}\cos 2x\right]_{\pi/3}^{\pi}$$

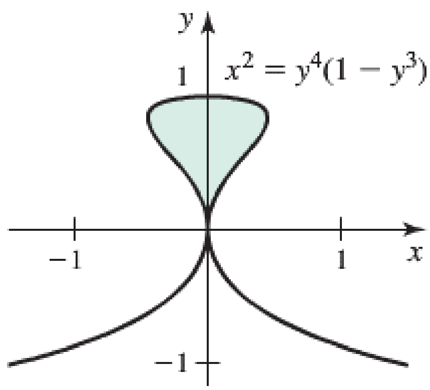
$$= \left(\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - 1\right) + \left(1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4}\right)$$

$$= \frac{5}{2}\ unit^2$$

Exercise

Determine the area of the shaded region bounded by the curve $x^2 = y^4\left(1 - y^3\right)$

Solution



$$x^2 = y^4\left(1 - y^3\right) \ \rightarrow \ x = y^2\sqrt{1 - y^3}$$

Since it is symmetric about y-axis, then

$$A = 2\int_0^1 y^2\sqrt{1 - y^3}\ dy$$

$$= -\frac{2}{3}\int_0^1 \left(1 - y^3\right)^{1/2}\ d\left(1 - y^3\right)$$

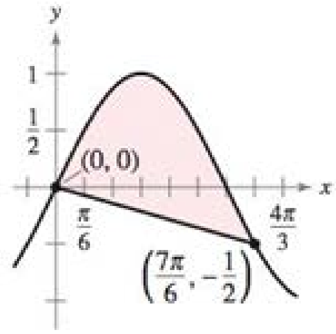
$$= -\frac{4}{9}\left(1 - y^3\right)^{3/2}\Big|_0^1$$

$$= \frac{4}{9}\ unit^2$$

Exercise

Find the area between the graph of $y = \sin x$ and the line segment joining the points $(0,\ 0)$ and $\left(\frac{7\pi}{6},\ -\frac{1}{2}\right)$.

Solution



Line: $y = \frac{-\frac{1}{2}}{\frac{7\pi}{6}}\left(x - \frac{7\pi}{6}\right) - \frac{1}{2}$

$= -\frac{3}{7\pi}\left(x - \frac{7\pi}{6}\right) - \frac{1}{2}$

$= -\frac{3}{7\pi}x$

$$A = \int_0^{7\pi/6} \left(\sin x + \frac{3}{7\pi}x\right)dx$$

$$= -\cos x + \frac{3}{14\pi}x^2 \Big|_0^{7\pi/6}$$

$$= \frac{\sqrt{3}}{2} + \frac{7\pi}{24} + 1$$

Exercise

The surface of a machine part is the region between the graphs of $y_1 = |x|$ and $y_2 = 0.08x^2 + k$

a) Find k where the parabola is tangent to the graph of y_1

b) Find the area of the surface of the machine part.

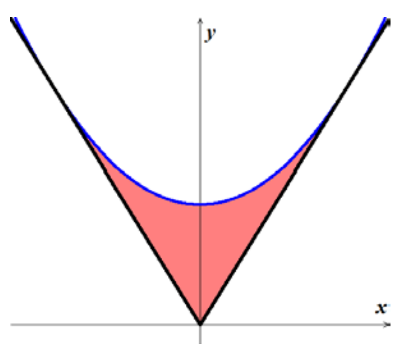
Solution

a) $y_1' = 1 \quad y_2' = 0.16x \quad \Rightarrow 0.16x = 1 \ \to \ x = \frac{1}{0.16} = 6.25$

$y_1 = y_2$

$6.25 = 0.08(6.25)^2 + k$

$k = 6.25 - 0.08(6.25)^2 \ = 3.125$



b) $A = 2\int_0^{6.25} \left(y_2 - y_1\right)dx$

$= 2\int_0^{6.25} \left(0.08x^2 + 3.125 - x\right)dx$

$= 2\left(\frac{.08}{3}x^3 + 3.125x - \frac{1}{2}x^2\right)\Big|_0^{6.25}$

$\approx 13.02083 \ \ unit^2$

Exercise

Concrete sections for a new building have the dimensions (in meters) and shape shown in figure

a) Find the area of the face of the section superimposed on the rectangularcoordinate system.

b) Find the volume of concrete in one of the sections by multiplying the area in part (*a*) by 2 *meters*.

c) One cubic meter of concrete weighs 5,000 *pounds*. Find the weight of the section.

Solution

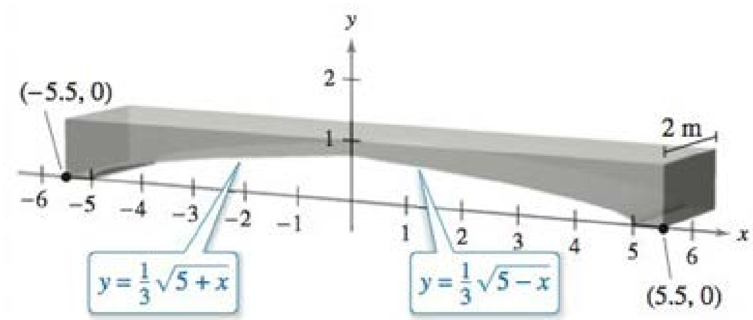
a) $$A = 2\int_0^5 \left(1 - \tfrac{1}{3}\sqrt{5+x}\right)dx + 2\int_5^{5.5} (1-0)\,dx$$

$$= 2\left[x + \tfrac{2}{9}(5-x)^{3/2}\right]_0^5 + 2x\Big|_5^{5.5}$$

$$= 2\left(5 - \tfrac{2}{9}5^{3/2}\right) + 2(5.5-5)$$

$$= 10 - \frac{20\sqrt{5}}{9} + 1$$

$$= 11 - \frac{20\sqrt{5}}{9}\ \ m^2$$



b) $V = 2A = 22 - \dfrac{40\sqrt{5}}{9}\ \ m^3$

c) $W = 5{,}000V = \left(11 - \dfrac{20\sqrt{5}}{9}\right) \times 10^4\ \ lb$

Exercise

A Lorenz curve is given by $y = L(x)$, where $0 \le x \le 1$ represents the lowest fraction of the population of a society in terms of wealth and $0 \le y \le 1$ represents the fraction of the total wealth that is owned by that fraction of the society. For example, the Lorenz curve in the figure shows that $L(0.5) = 0.2$, which means that the lowest 0.5 (50%) of the society owns 0.2 (20%) of the wealth.

a) A Lorenz curve $y = L(x)$ is accompanied by the line $y = x$, called the ***line of perfect equality***. Explain why this line is given the name.

b) Explain why a Lorenz curve satisfies the conditions $L(0) = 0,\ L(1) = 1,\ and\ L'(x) \ge 0\quad on\ [0,\ 1]$

c) Graph the Lorenz curves $L(x) = x^p$ corresponding to *p* = 1.1, 1.5, 2, 3, 4. Which value of *p* corresponds to the ***most*** equitable distribution of wealth (closest to the line of perfect equality)? Which value of *p* corresponds to the ***least*** equitable distribution of wealth? Explain.

d) The information in the Lorenz curve is often summarized in a single measure called the ***Gini index***, which is defined as follows. Let *A* be the area of the region between $y = x$ and $y = L(x)$ and Let *B* be the area of the region between $y = L(x)$ and the *x*-axis. Then the Gini index is $G = \dfrac{A}{A+B}$.

Show that $G = 2A = 1 - 2\int_0^1 L(x)\,dx$.

e) Compute the Gini index for the cases $L(x) = x^p$ and *p* = 1.1, 1.5, 2, 3, 4.

f) What is the smallest interval [*a*, *b*] on which values of the Gini index lie, for $L(x) = x^p$ with $p \ge 1$? Which endpoints of [*a*, *b*] correspond to the least and most equitable distribution of wealth?

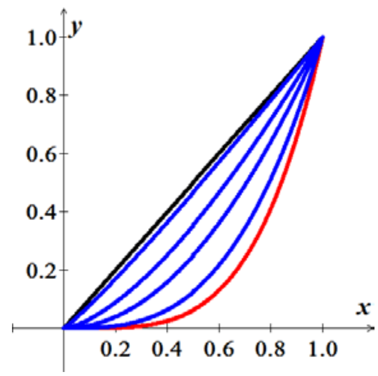
g) Consider the Lorenz curve described by $L(x) = \frac{5x^2}{6} + \frac{x}{6}$. Show that it satisfies the conditions $L(0) = 0,\ L(1) = 1,\ and\ L'(x) \ge 0 \quad on\ [0,\ 1]$. Find the Gini index for this function.

Solution

a) Let the point $N = (a,\ a)$ on the curve $y = x$ would represent the notion that the lowest *p*% of the society owns *p*% of the wealth, which would represent a form of equality.

b) The function must be increasing and concave up because the poorest *p*% cannot own more than *p*% of the wealth.

c) $y = x^{1.1}$ is closet to $y = x$, and $y = x^4$ is furthest from $y = x$



d) Since, $B = \int_0^1 L(x)\,dx$ and $A + B = \frac{1}{2}$

Then $A = \frac{1}{2} - B = \frac{1}{2} - \int_0^1 L(x)\,dx$

$$G = \frac{A}{A+B} = \frac{A}{\frac{1}{2}} = 2A = 1 - 2\int_0^1 L(x)\,dx$$

e) For $L(x) = x^p$

$$G = 1 - 2\int_0^1 x^p\,dx$$

$$= 1 - \frac{2}{p+1}\left(x^{p+1}\right)\Big|_0^1$$

$$= 1 - \frac{2}{p+1}$$

$$\boxed{= \frac{p-1}{p+1}}$$

P	1.1	1.5	2	3	4
G	$\frac{1}{21}$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{5}$

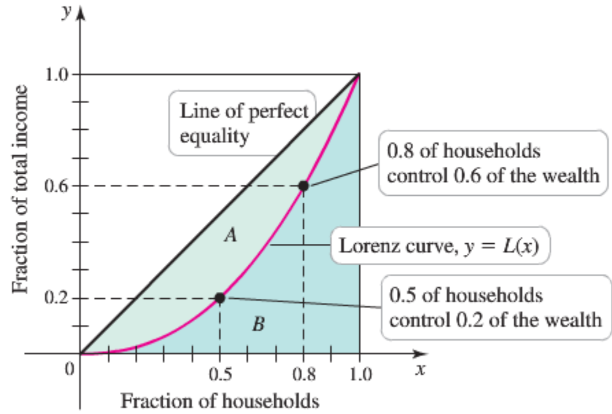
f) For $p = 1 \quad \rightarrow \quad \boxed{G = \frac{p-1}{p+1} = 0}$

$\lim\limits_{p \to \infty} \frac{p-1}{p+1} \boxed{= 1}$, the largest value of *G* approaches 1.

g) $L(x) = \frac{5x^2}{6} + \frac{x}{6} \qquad \rightarrow L(0) = 0, \quad L(0) = 1$

$$L'(x) = \frac{5}{3}x + \frac{1}{6} > 0 \quad x \in [0,\ 1]$$

$$L''(x) = \frac{5}{3} > 0$$



The Gini index is:

$$G = 1 - 2\int_0^1 \left(\frac{5x^2}{6} + \frac{x}{6}\right)dx$$

$$= 1 - 2\left(\frac{5x^3}{18} + \frac{x^2}{12}\right)\Bigg|_0^1$$

$$= 1 - 2\left(\frac{5}{18} + \frac{1}{12}\right)$$

$$= 1 - \frac{5}{9} - \frac{1}{6}$$

$$= \frac{5}{18}$$

Solution   Section 1.3 – Volumes by Slicing

Exercise

Find the volume of the solid formed by revolving the region about the *x-axis* : $y = -x + 1$

Solution

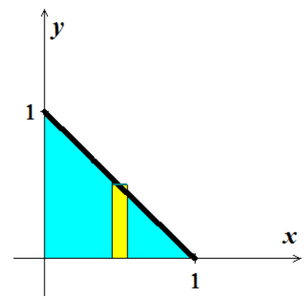
$$V = \pi\int_0^1 (-x+1)^2\ dx \qquad\qquad V = \pi\int_a^b \left(f(x)^2 - g(x)^2\right)dx$$

$$= \pi\int_0^1 \left(x^2 - 2x + 1\right)\ dx$$

$$= \pi\left(\frac{1}{3}x^3 - x^2 + x\right)\Big|_0^1$$

$$= \pi\left(\frac{1}{3} - 1 + 1\right)$$

$$= \frac{\pi}{3}$$



Exercise

Find the volume of the solid formed by revolving the region about the *x-axis* : $y = 4 - x^2$

Solution

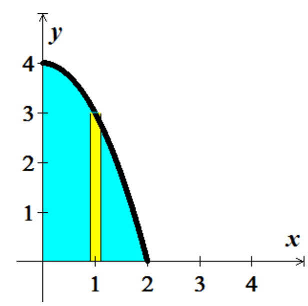
$$V = \pi\int_0^2 \left(4 - x^2\right)^2\ dx \qquad\qquad V = \pi\int_a^b \left(f(x)^2 - g(x)^2\right)dx$$

$$= \pi\int_0^2 \left(16 - 8x^2 + x^4\right)\ dx$$

$$= \pi\left(16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^2$$

$$= \pi\left(32 - \frac{64}{3} + \frac{32}{5}\right)$$

$$= \frac{256\pi}{15}$$



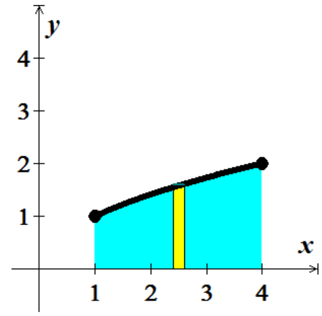
Exercise

Find the volume of the solid formed by revolving the region about the *x-axis* : $y = \sqrt{x}$

Solution

$$V = \pi\int_1^4 \left(\sqrt{x}\right)^2\ dx \qquad\qquad V = \pi\int_a^b \left(f(x)^2 - g(x)^2\right)dx$$

$$= \pi \int_1^4 x\, dx$$



$$= \pi \left(\frac{1}{2} x^2 \right) \Big|_1^4$$

$$= \pi \left(8 - \frac{1}{2} \right)$$

$$= \frac{15\pi}{2}$$

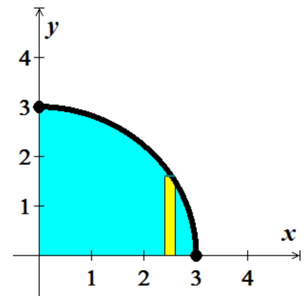
Exercise

Find the volume of the solid formed by revolving the region about the *x-axis* : $\quad y = \sqrt{9 - x^2}$

Solution

$$V = \pi \int_0^3 \left(\sqrt{9 - x^2} \right)^2 \, dx \qquad V = \pi \int_a^b \left(f(x)^2 - g(x)^2 \right) dx$$

$$= \pi \int_0^3 \left(9 - x^2 \right) \, dx$$



$$= \pi \left(9x - \frac{1}{3} x^3 \right) \Big|_0^3$$

$$= \pi (27 - 9)$$

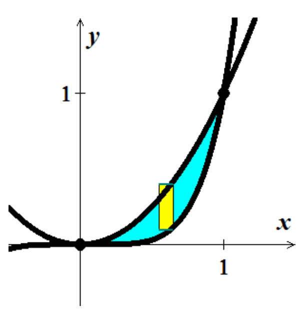
$$= 18\pi$$

Exercise

Find the volume of the solid formed by revolving the region about the *x-axis* : $\quad y = x^2, \quad y = x^5$

Solution

$$V = \pi \int_0^1 \left(x^4 - x^{10} \right) \, dx \qquad V = \pi \int_a^b \left(f(x)^2 - g(x)^2 \right) dx$$



$$= \pi \left(\frac{1}{5} x^5 - \frac{1}{11} x^{11} \right) \Big|_0^1$$

$$= \pi \left(\frac{1}{5} - \frac{1}{11} \right)$$

$$= \frac{6\pi}{55}$$

Exercise

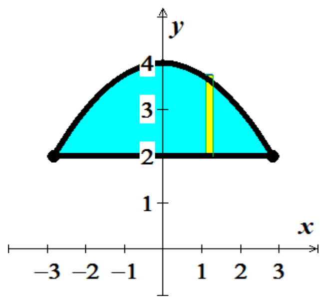
Find the volume of the solid formed by revolving the region about the *x-axis* : $y = 2, \quad y = 4 - \frac{x^2}{4}$

Solution

$$y = 4 - \frac{x^2}{4} = 2 \Rightarrow \frac{x^2}{4} = 2$$

$$x^2 = 8 \rightarrow \underline{x = \pm 2\sqrt{2}|}$$

$$V = \pi \int_{-2\sqrt{2}}^{2\sqrt{2}} \left(\left(4 - \frac{x^2}{4} \right)^2 - 4 \right) dx \qquad V = \pi \int_a^b \left(f(x)^2 - g(x)^2 \right) dx$$



$$= 2\pi \int_0^{2\sqrt{2}} \left(16 - 2x^2 + \frac{1}{16}x^4 - 4 \right) dx$$

$$= 2\pi \int_0^{2\sqrt{2}} \left(12 - 2x^2 + \frac{1}{16}x^4 \right) dx$$

$$= 2\pi \left(12x - \frac{2}{3}x^3 + \frac{1}{80}x^5 \right) \Big|_0^{2\sqrt{2}}$$

$$= 2\pi \left(24\sqrt{2} - \frac{32}{3}\sqrt{2} + \frac{128}{80}\sqrt{2} \right)$$

$$= 16\pi\sqrt{2} \left(3 - \frac{4}{3} + \frac{16}{80} \right)$$

$$= 16\pi\sqrt{2} \left(\frac{448}{240} \right)$$

$$\underline{= \frac{448\sqrt{2}}{15}\pi|}$$

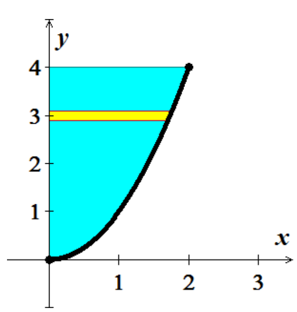
Exercise

Find the volume of the solid formed by revolving the region about the *y-axis* : $y = x^2$

Solution

$$y = x^2 \rightarrow x = \sqrt{y}$$

$$V = \pi \int_0^4 \left(\sqrt{y} \right)^2 dy \qquad V = \pi \int_c^d \left(p(y)^2 - q(y)^2 \right) dy$$



$$= \pi \int_0^4 y\, dy$$

$$= \pi \left(\frac{1}{2}y^2 \right) \Big|_0^4$$

$$\underline{= 8\pi|}$$

Exercise

Find the volume of the solid formed by revolving the region about the *y-axis*: $y=\sqrt{16-x^2}$

Solution

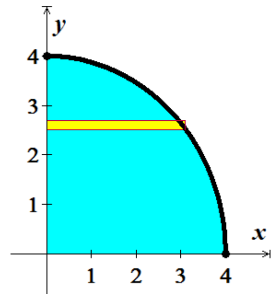
$$y=\sqrt{16-x^2} \rightarrow x=\sqrt{16-y^2}$$

$$V=\pi\int_0^4\left(16-y^2\right)\,dy \qquad V=\pi\int_c^d\left(p(y)^2-q(y)^2\right)dy$$

$$=\pi\left(16y-\frac{1}{3}y^3\right)\Big|_0^4$$

$$=\pi\left(64-\frac{64}{3}\right)$$

$$=\frac{128\pi}{3}$$



Exercise

Find the volume of the solid formed by revolving the region about the *y-axis*: $y=x^{2/3}$

Solution

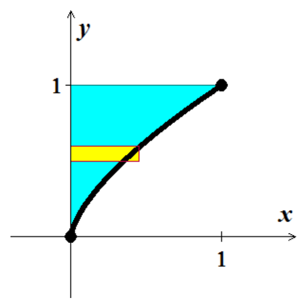
$$y=x^{2/3} \rightarrow x=y^{3/2}$$

$$V=\pi\int_0^1\left(y^{3/2}\right)^2\,dy \qquad V=\pi\int_c^d\left(p(y)^2-q(y)^2\right)dy$$

$$=\pi\int_0^1 y^3\,dy$$

$$=\pi\left(\frac{1}{4}y^4\right)\Big|_0^1$$

$$=\frac{\pi}{4}$$



Exercise

Find the volume of the solid formed by revolving the region about the *y-axis*: $x=-y^2+4y$

Solution

$$V=\pi\int_1^4\left(-y^2+4y\right)^2\,dy \qquad V=\pi\int_c^d\left(p(y)^2-q(y)^2\right)dy$$

$$=\pi\int_1^4\left(y^4-8y^3+16y^2\right)\,dy$$

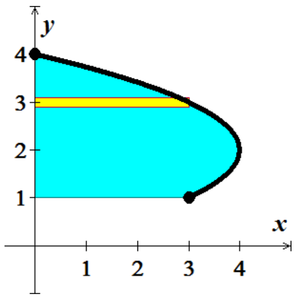
$$= \pi\left(\frac{1}{5}y^5 - 2y^4 + \frac{16}{3}y^3\right)\Big|_1^4$$

$$= \pi\left(\frac{1024}{5} - 512 + \frac{1024}{3} - \frac{1}{5} + 2 - \frac{16}{3}\right)$$

$$= \pi\left(\frac{1023}{5} + \frac{1008}{3} - 510\right)$$

$$= \pi\left(\frac{459}{15}\right)$$

$$= \frac{153\pi}{5}$$



Exercise

Find the volumes of the solids generated by revolving the region by the graphs of the equations about the given lines. $y = \sqrt{x}, \quad y = 0, \quad x = 3$

a) the x-*axis* *b)* the y-*axis* *c)* the line $x = 3$ *d)* the line $x = 6$

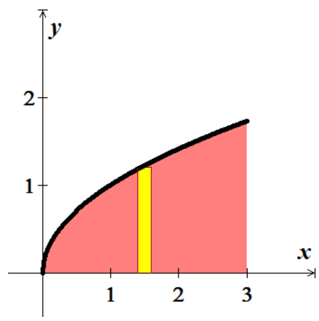
Solution

a) $\sqrt{x} = 0 \to \underline{x = 0}$

$$V = \pi\int_0^3 x\, dx \qquad V = \pi\int_a^b f(x)^2\, dx$$

$$= \pi\frac{1}{2}x^2\Big|_0^3$$

$$= \frac{9\pi}{2}$$



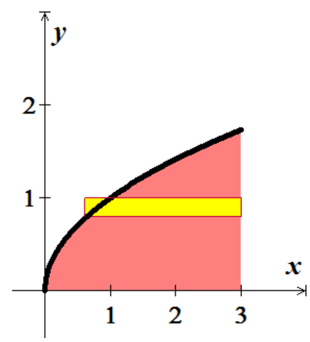
b) $y = \sqrt{x} \; \underline{= \sqrt{3}} \quad x = y^2$

$$V = \pi\int_0^{\sqrt{3}} \left(9 - y^4\right) dy \qquad V = \pi\int_c^d \left(p(y)^2 - q(y)^2\right) dy$$

$$= \pi\left(9x - \frac{1}{5}x^5\right)\Big|_0^{\sqrt{3}}$$

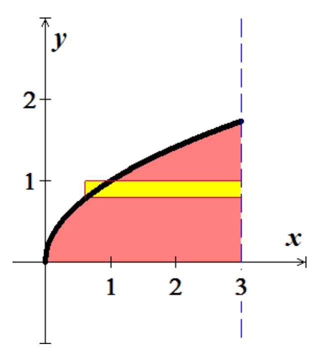
$$= \pi\left(9\sqrt{3} - \frac{9}{5}\sqrt{3}\right)$$

$$= \frac{36\pi\sqrt{3}}{5}$$



c) $V = \pi\int_0^{\sqrt{3}} \left(3 - y^2\right)^2\, dy$

$$= \pi\int_0^{\sqrt{3}} \left(9 - 6y^2 + y^4\right) dy$$

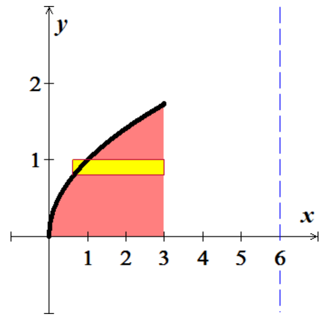


$$= \pi\left(9y - 2y^3 + \frac{1}{5}y^5\right)\Big|_0^{\sqrt{3}}$$

$$= \pi\left(9\sqrt{3} - 6\sqrt{3} + \frac{9}{5}\sqrt{3}\right)$$

$$= \boxed{\frac{24\sqrt{3}}{5}\pi}$$

d) $R(y) = 3 + \left(3 - y^2\right) = 6 - y^2 \qquad r(y) = 3$



$$V = \pi\int_0^{\sqrt{3}}\left(\left(6 - y^2\right)^2 - 9\right)\,dy \qquad V = \pi\int_c^d \left(R(y)^2 - r(y)^2\right)dy$$

$$= \pi\int_0^{\sqrt{3}}\left(36 - 12y^2 + y^4 - 9\right)\,dy$$

$$= \pi\left(27y - 4y^3 + \frac{1}{5}y^5\right)\Big|_0^{\sqrt{3}}$$

$$= \pi\left(27\sqrt{3} - 12\sqrt{3} + \frac{9}{5}\sqrt{3}\right)$$

$$= \boxed{\frac{84\sqrt{3}}{5}\pi}$$

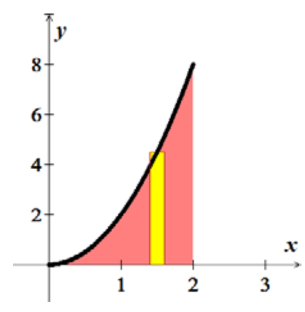
Exercise

Find the volumes of the solids generated by revolving the region by the graphs of the equations about the given lines. $\quad y = 2x^2, \quad y = 0, \quad x = 2$

a) the *x-axis* *b)* the *y-axis* *c)* the line $y = 8$ *d)* the line $x = 2$

Solution

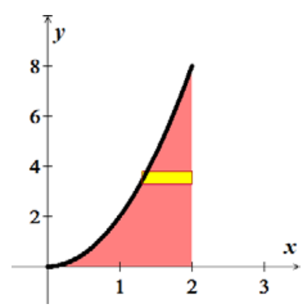
a) $V = \pi\int_0^2 4x^4\,dx \qquad V = \pi\int_a^b f(x)^2\,dx$



$$= \frac{4\pi}{5}x^5\Big|_0^2$$

$$= \boxed{\frac{128\pi}{5}}$$

b) $R(y) = 2 \quad x^2 = \frac{y}{2} \;\rightarrow\; r(y) = \sqrt{\frac{y}{2}}$



$x = 2 \;\rightarrow\; \boxed{y = 2(2)^2 = 8}$

$$V = \pi\int_0^8\left(4 - \frac{1}{2}y\right)\,dy \qquad V = \pi\int_c^d \left(R(y)^2 - r(y)^2\right)dy$$

$$= \pi\left(4y - \frac{1}{4}y^2\right)\Big|_0^8$$

$$= \pi(32 - 16)$$

$$\underline{= 16\pi}$$

c) $R(x) = 8 \quad r(x) = 8 - 2x^2$

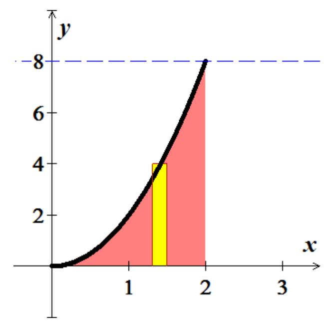
$$V = \pi\int_0^2 \left(64 - \left(64 - 32x^2 + 4x^4\right)\right)\, dx \qquad V = \pi\int_a^b \left(R(x)^2 - r(x)^2\right)dx$$

$$= 4\pi\int_0^2 \left(8x^2 - x^4\right)\, dx$$

$$= 4\pi\left(\frac{8}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^2$$

$$= 4\pi\left(\frac{64}{3} - \frac{32}{5}\right)$$

$$\underline{= \frac{896\pi}{15}}$$



d) $R(y) = 2 - \sqrt{\frac{1}{2}y} \quad r(y) = 0$

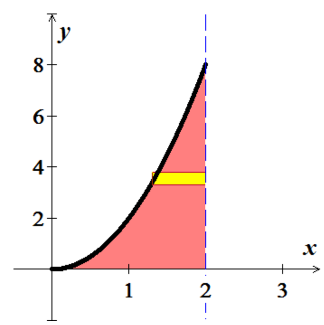
$$V = \pi\int_0^8 \left(4 - 4\sqrt{\frac{1}{2}}y^{1/2} + \frac{1}{2}y\right)\, dy \qquad V = \pi\int_c^d \left(R(y)^2 - r(y)^2\right)dy$$

$$= \pi\left(4y - \frac{4\sqrt{2}}{3}y^{3/2} + \frac{1}{4}y^2\right)\Big|_0^8$$

$$= \pi\left(32 - \frac{4\sqrt{2}}{3}\left(2^{9/2}\right) + 16\right)$$

$$= \pi\left(32 - \frac{128}{3} + 16\right)$$

$$\underline{= \frac{16\pi}{3}}$$



Exercise

Find the volumes of the solids generated by revolving the region by the graphs of the equations about the given lines. $y = x^2, \quad y = 4x - x^2$

a) the *x-axis* *b)* the line $y = 6$

Solution

$$x^2 = 4x - x^2 \quad \rightarrow \; 2x^2 - 4x = 0 \quad \underline{x = 0,\; 2|}$$

a) $R(x) = 4x - x^2 \quad r(x) = x^2$

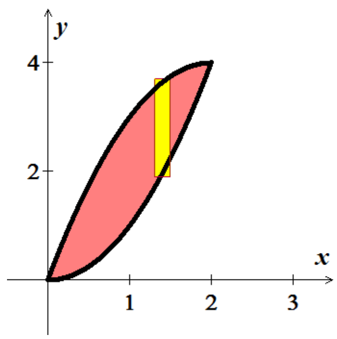
$$V = \pi \int_0^2 \left(16x^2 - 8x^3 + x^4 - x^4\right) dx \qquad V = \pi \int_a^b \left(R(x)^2 - r(x)^2\right) dx$$

$$= \pi \int_0^2 \left(16x^2 - 8x^3\right) dx$$

$$= \pi \left(\frac{16}{3}x^3 - 2x^4\right)\Big|_0^2$$

$$= \pi \left(\frac{128}{3} - 32\right)$$

$$= \frac{32\pi}{3}$$



b) $R(x) = 6 - x^2 \quad r(x) = 6 - 4x + x^2$

$$V = \pi \int_0^2 \left(36 - 12x^2 + x^4 - \left(36 - 48x + 28x^2 - 8x^3 + x^4\right)\right) dx$$

$$= \pi \int_0^2 \left(48x - 40x^2 + 8x^3\right) dx$$

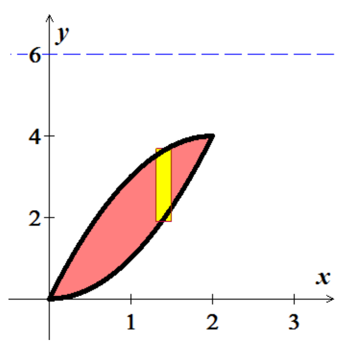
$$= 8\pi \int_0^2 \left(6x - 5x^2 + x^3\right) dx$$

$$= 8\pi \left(3x^2 - \frac{5}{3}x^3 + \frac{1}{4}x^4\right)\Big|_0^2$$

$$= 8\pi \left(12 - \frac{40}{3} + 4\right)$$

$$= 8\pi \left(\frac{8}{3}\right)$$

$$= \frac{64\pi}{3}$$



Exercise

Find the volumes of the solids generated by revolving the region by the graphs of the equations about the given lines. $y=-x^2+2x+4, \quad y=4-x$

a) the *x-axis* *b)* the line $y=1$

Solution

$$y=-x^2+2x+4=4-x \Rightarrow -x^2+3x=0 \quad \rightarrow \underline{x=0,\ 3}$$

a) $R(x)=-x^2+2x+4 \quad r(x)=4-x$

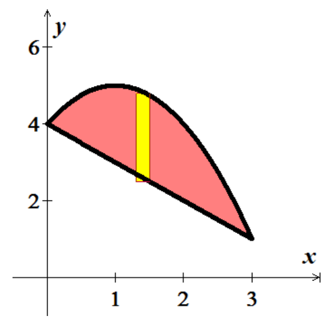
$$V=\pi\int_0^3\left(x^4-4x^3-4x^2+16x+16-\left(16-8x+x^2\right)\right)dx$$

$$=\pi\int_0^3\left(x^4-4x^3-5x^2+24x\right)dx$$

$$=\pi\left(\frac{1}{5}x^5-x^4-\frac{5}{3}x^3+12x^2\right)\Big|_0^3$$

$$=\pi\left(\frac{243}{5}-81-45+108\right)$$

$$=\frac{153\pi}{5}$$



b) $R(x)=\left(-x^2+2x+4\right)-1 \quad r(x)=(4-x)-1$

$$V=\pi\int_0^3\left(\left(-x^2+2x+3\right)^2-(3-x)^2\right)dx$$

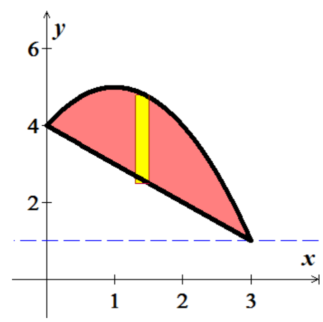
$$=\pi\int_0^3\left(x^4-4x^3-2x^2+12x+9-9+6x-x^2\right)dx$$

$$=\pi\int_0^3\left(x^4-4x^3-3x^2+18x\right)dx$$

$$=\pi\left(\frac{1}{5}x^5-x^4-x^3+9x^2\right)\Big|_0^3$$

$$=\pi\left(\frac{243}{5}-81-27+81\right)$$

$$=\frac{108\pi}{5}$$

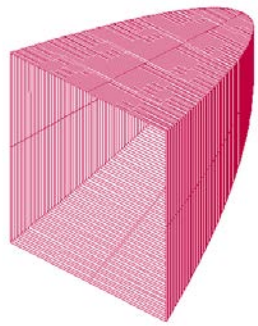


Exercise

The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross-sections perpendicular to the axis on the interval $0 \le x \le 4$ are squares whose diagonals run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$. Find the volume of the solid.

Solution

$$A(x) = \frac{1}{2}(diagonal)^2$$
$$= \frac{1}{2}\left(\sqrt{x} - \left(-\sqrt{x}\right)\right)^2$$
$$= \frac{1}{2}\left(2\sqrt{x}\right)^2$$
$$= \frac{1}{2}(4x)$$
$$= 2x \ \ unit^2 \qquad a = 0, \quad b = 4;$$



$$V = \int_a^b A(x)dx$$
$$= \int_0^4 2xdx$$
$$= \left[x^2\right]_0^4$$
$$= 4^2 - 0$$
$$= 16 \ \ unit^3$$

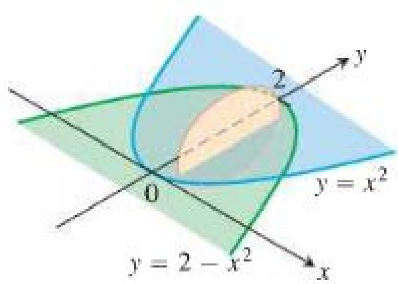
Exercise

The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis are circular whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$. Find the volume of the solid.

Solution

$$y = 2 - x^2 = x^2 \quad \Rightarrow \quad 2x^2 = 2 \to x^2 = 1 \ \Rightarrow \boxed{x = \pm 1}$$

$$A(x) = \frac{\pi}{4}(diameter)^2 = \frac{\pi}{4}\left(2 - x^2 - x^2\right)^2$$
$$= \frac{\pi}{4}\left(2\left(1 - x^2\right)\right)^2$$
$$= \frac{\pi}{4}4\left(1 - 2x^2 + x^4\right)$$
$$= \pi\left(1 - 2x^2 + x^4\right) \ \ unit^2 \qquad a = -1, \quad b = 1;$$



$$V = \int_a^b A(x)\,dx$$

$$= \int_{-1}^{1} \pi\left(1-2x^2+x^4\right)dx$$

$$= \pi\left[x-\frac{2}{3}x^3+\frac{1}{5}x^5\right]_{-1}^{1} \qquad = \pi\left[\left(1-\frac{2}{3}(1)^3+\frac{1}{5}(1)^5\right)-\left(-1-\frac{2}{3}(-1)^3+\frac{1}{5}(-1)^5\right)\right]$$

$$= \pi\left[\left(1-\frac{2}{3}+\frac{1}{5}\right)-\left(-1+\frac{2}{3}-\frac{1}{5}\right)\right]$$

$$= 2\pi\left(1-\frac{2}{3}+\frac{1}{5}\right)$$

$$= \frac{16\pi}{15}\ \ unit^3$$

Exercise

The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the axis between these planes are squares whose bases run from the semicircle $y=-\sqrt{1-x^2}$ to the semicircle $y=\sqrt{1-x^2}$. Find the volume of the solid.

Solution

$$A(x) = \frac{1}{2}(diagonal)^2$$

$$= \frac{1}{2}\left(\sqrt{1-x^2}-\left(-\sqrt{1-x^2}\right)\right)^2$$

$$= \frac{1}{2}\left(2\sqrt{1-x^2}\right)^2$$

$$= 2\left(1-x^2\right)\ \ unit^2 \qquad a=-1, \quad b=1;$$

$$V = \int_a^b A(x)\,dx$$

$$= \int_{-1}^{1} 2\left(1-x^2\right)dx$$

$$= 2\left[x-\frac{1}{3}x^3\right]_{-1}^{1}$$

$$= 2\left[\left(1-\frac{1}{3}1^3\right)-\left((-1)-\frac{1}{3}(-1)^3\right)\right] \qquad = 4\left(1-\frac{1}{3}\right)$$

$$= \frac{8}{3}\ \ unit^3$$

Exercise

The base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval [0, π] on the x-axis. The cross-sections perpendicular to the x-axis are
Find the volume of the solid.

a) Equilateral triangles with bases running from the x-axis to the curve as shown

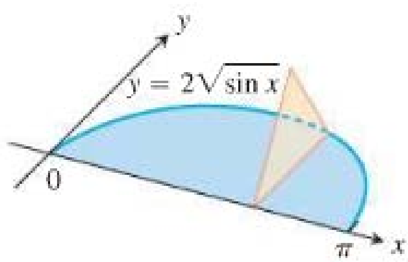
b) Squares with bases running from the x-axis to the curve.

Solution

a) $A(x) = \frac{1}{2}(side)(side)\cdot\sin\frac{\pi}{3}$ *Equilateral triangle* $\theta = \frac{\pi}{3}$

$$= \frac{1}{2}\left(2\sqrt{\sin x}\right)\left(2\sqrt{\sin x}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \sqrt{3}\sin x \ \ unit^2 \qquad a = 0, \quad b = \pi;$$



$$V = \sqrt{3}\int_0^{\pi} \sin x dx$$

$$= \sqrt{3}\left[-\cos x\right]_0^{\pi}$$

$$= -\sqrt{3}\left[\cos\pi - \cos 0\right]$$

$$= 2\sqrt{3} \ \ unit^3$$

b) $A(x) = (side)^2 = \left(2\sqrt{\sin x}\right)^2 = 4\sin x \ \ unit^2$ $\quad a = 0, \quad b = \pi;$

$$V = 4\int_0^{\pi} \sin x dx$$

$$= 4\left[-\cos x\right]_0^{\pi}$$

$$= -4\left[\cos\pi - \cos 0\right]$$

$$= 8 \ \ unit^3$$

Exercise

The base of the solid is the disk $x^2 + y^2 \le 1$. The cross-sections by planes perpendicular to the y-axis between $y = -1$ and $y = 1$ are isosceles right triangles with one leg in the disk.

Solution

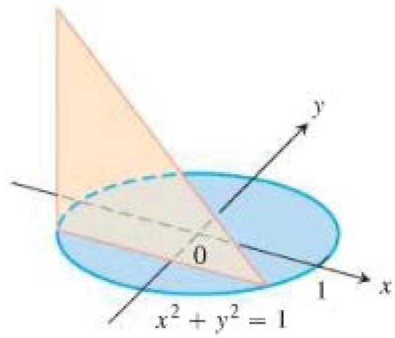
$$x^2 + y^2 = 1 \ \rightarrow \ x^2 = 1 - y^2 \Rightarrow \boxed{x = \pm\sqrt{1 - y^2}}$$

$$A(y) = \frac{1}{2}(leg)(leg) = \frac{1}{2}\left[\sqrt{1 - y^2} - \left(-\sqrt{1 - y^2}\right)\right]^2$$

$$= \frac{1}{2}\left[2\sqrt{1-y^2}\right]^2$$

$$\boxed{= 2\left(1-y^2\right)\ unit^2} \qquad c=-1, \quad d=1;$$

$$V = \int_{-1}^{1} 2\left(1-y^2\right)dy \qquad V = \int_c^d A(y)\,dy$$



$$= 2\left[y - \frac{1}{3}y^3\right]_{-1}^{1}$$

$$= 2\left[\left(1-\frac{1}{3}1^3\right) - \left((-1) - \frac{1}{3}(-1)^3\right)\right]$$

$$= 4\left(1-\frac{1}{3}\right)$$

$$\boxed{= \frac{8}{3}\ unit^3}$$

Exercise

Find the volume of the given tetrahedron. (***Hint***: Consider slices perpendicular to one of the labeled edges)

Solution

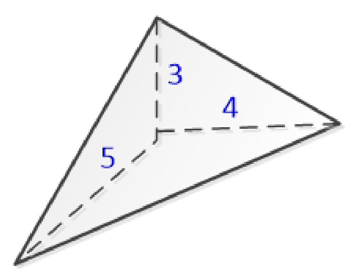
Let consider the slices perpendicular to edge labeled 5 are triangles.

By similar triangles, we have: $\frac{height}{base} = \frac{h}{b} = \frac{3}{4} \quad \Rightarrow \quad h = \frac{3}{4}b$

The equation of the line through (5, 0) and (0, 4) is: $y = \frac{4-0}{0-5}(x-0)+4 \quad \rightarrow \quad y = -\frac{4}{5}x+4$

Therefore, the length of the base: $b = -\frac{4}{5}x+4$

$$h = \frac{3}{4}b = \frac{3}{4}\left(-\frac{4}{5}x+4\right) = -\frac{3}{5}x+3$$



$$A(x) = \frac{1}{2}(base)\cdot(height) = \frac{1}{2}\left(-\frac{4}{5}x+4\right)\left(-\frac{3}{5}x+3\right)$$

$$= \frac{1}{2}\left(\frac{12}{25}x^2 - \frac{24}{5}x + 12\right)$$

$$= \frac{6}{25}x^2 - \frac{12}{5}x + 5 \quad unit^2$$

$$V = \int_0^5 \left(\frac{6}{25}x^2 - \frac{12}{5}x + 5\right)dx \qquad V = \int_a^b A(x)\,dx$$

$$= \left[\frac{2}{25}x^3 - \frac{6}{5}x^2 + 5x\right]_0^5$$

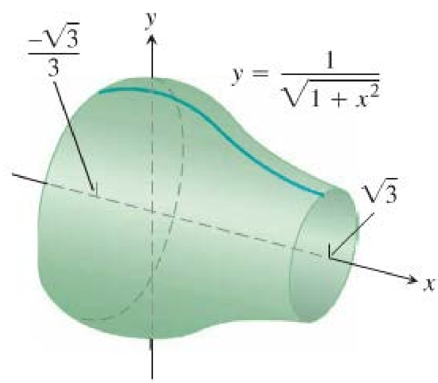
$$= \left[\frac{2}{25}(5)^3 - \frac{6}{5}(5)^2 + 5(5)\right]$$

$$\boxed{= 10\ unit^3}$$

Exercise

Find the volume of the solid of revolution

Solution



$$V = \pi \int_{-\sqrt{3}/3}^{\sqrt{3}} \left(\frac{1}{\sqrt{1+x^2}}\right)^2 dx$$

$$= \pi \int_{-\sqrt{3}/3}^{\sqrt{3}} \frac{1}{1+x^2} dx$$

$$= \pi \left[\tan^{-1} x\right]_{-\sqrt{3}/3}^{\sqrt{3}}$$

$$= \pi \left(\tan^{-1}\sqrt{3} - \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)\right)$$

$$= \pi \left(\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right)$$

$$= \pi \left(\frac{\pi}{3} + \frac{\pi}{6}\right)$$

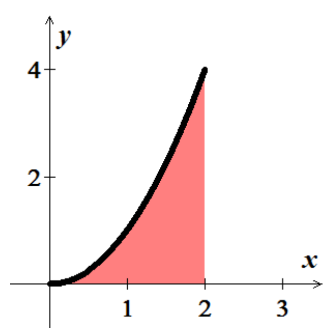
$$= \pi \left(\frac{\pi}{2}\right)$$

$$\boxed{= \frac{\pi^2}{2} \ unit^3}$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $y = x^2$ and the lines $y = 0$, $x = 2$ about the x-axis.

Solution



$$R(x) = x^2$$

$$V = \int_0^2 \pi \left[R(x)\right]^2 dx$$

$$= \pi \int_0^2 \left(x^2\right)^2 dx$$

$$= \pi \int_0^2 x^4 dx$$

$$= \pi \left[\frac{1}{5} x^5 \right]_0^2$$

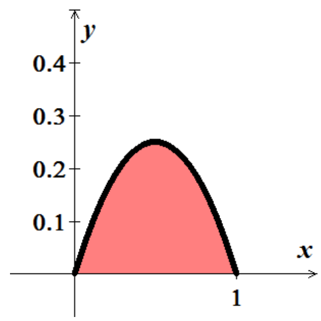
$$= \pi \left(\frac{1}{5} (2)^5 - 0 \right)$$

$$= \frac{32\pi}{5} \ \ unit^3$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $y = x - x^2$ and the line $y = 0$ about the *x*-axis.

Solution



$R(x) = x - x^2 \qquad\qquad x - x^2 = 0 \ \to \ x = 0, \ 1$

$$V = \int_0^1 \pi \left[x - x^2 \right]^2 dx$$

$$= \pi \int_0^1 \left(x^2 - 2x^3 + x^4 \right) dx$$

$$= \pi \left[\frac{1}{3} x^3 - \frac{1}{2} x^4 + \frac{1}{5} x^5 \right]_0^1$$

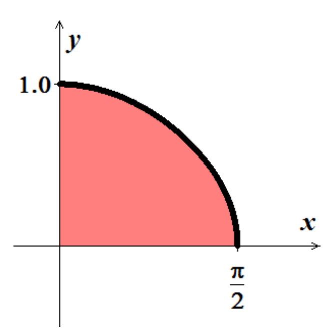
$$= \pi \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right)$$

$$= \frac{\pi}{30} \ \ unit^3$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{\cos x}$ and the lines $0 \le x \le \frac{\pi}{2}, \quad y = 0, \quad x = 0$ about the *x*-axis.

Solution



$R(x) = \sqrt{\cos x}$

$$V = \int_0^{\pi/2} \pi \left[\sqrt{\cos x} \right]^2 dx$$

$$= \pi \int_0^{\pi/2} \cos x dx$$

$$= \pi [\sin x]_0^{\pi/2}$$

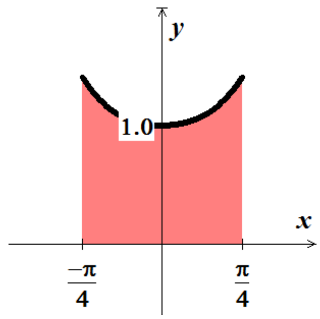
$$= \pi (1 - 0)$$

$$\underline{= \pi \ \ unit^3}$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $y = \sec x$ and the lines $y = 0, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$ about the x-axis.

Solution



$$R(x) = \sec x$$

$$V = \pi \int_{-\pi/4}^{\pi/4} \sec^2 x dx$$

$$= \pi [\tan x]_{-\pi/4}^{\pi/4}$$

$$= \pi \left(\tan\left(\frac{\pi}{4}\right) - \tan\left(-\frac{\pi}{4}\right) \right)$$

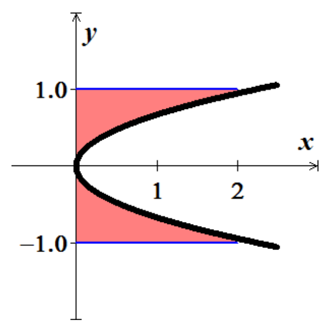
$$= \pi (1 - (-1))$$

$$\underline{= 2\pi \ \ unit^3}$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $x = \sqrt{5}\, y^2$ and the lines $x = 0, \quad y = -1, \quad y = 1$ about the y-axis.

Solution



$$R(y) = \sqrt{5} y^2$$

$$V = \pi \int_{-1}^{1} \left[\sqrt{5} y^2 \right]^2 dy$$

$$= \pi \int_{-1}^{1} 5y^4 dy$$

$$= \pi \left(y^5 \right)_{-1}^{1}$$

$$= \pi (1 - (-1))$$

$$\underline{= 2\pi \ \ unit^3}$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $y = 2\sqrt{x}$ and the lines $y = 2, \quad x = 0$ about the *x*-axis.

Solution

$$r(x) = 2\sqrt{x} \quad and \quad R(x) = 2$$

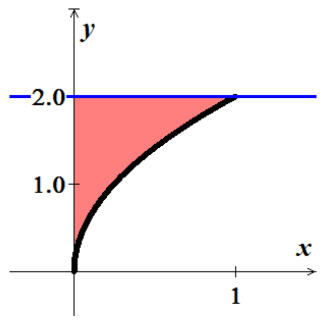
$$V = \pi \int_0^1 \left((2)^2 - \left(2\sqrt{x}\right)^2 \right) dx \qquad\qquad V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2 \right) dx$$

$$= \pi \int_0^1 (4 - 4x)\, dx$$

$$= 4\pi \left(x - \tfrac{1}{2}x^2 \right) \Big|_0^1$$

$$= 4\pi \left[\left(1 - \tfrac{1}{2}\right) - 0 \right]$$

$$= 2\pi \ \ unit^3$$



Exercise

Find the volume of the solid generated by revolving the region bounded by $y = \sec x, \quad y = \tan x$ and the lines $x = 0, \quad x = 1$ about the *x*-axis.

Solution

$$r(x) = \tan x \quad and \quad R(x) = \sec x$$

$$V = \pi \int_0^1 \left(\sec^2 x - \tan^2 x \right) dx \qquad\qquad V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2 \right) dx$$

$$= \pi \int_0^1 (1)\, dx$$

$$= \pi (x) \Big|_0^1$$

$$= \pi \ \ unit^3$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $x = \sqrt{2\sin 2y}$ and the lines $0 \le y \le \frac{\pi}{2}, \quad x = 0$ about the *y*-axis.

Solution

$$V = \pi \int_0^{\pi/2} \left(\sqrt{2\sin 2y}\right)^2 dy$$

$$= 2\pi \int_0^{\pi/2} (\sin 2y)\, dy$$

$$= -\pi \cos 2y \Big|_0^{\pi/2}$$

$$= -\pi(-1-1)$$

$$= 2\pi \ \ unit^3$$

Exercise

Find the volume of a solid ball having radius *a*.

Solution

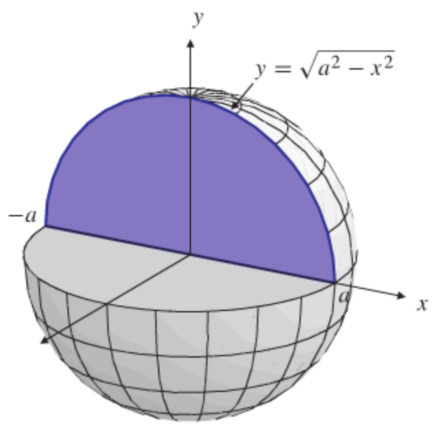
$$V = \pi \int_{-a}^{a} \left(\sqrt{a^2 - x^2}\right)^2 dx$$

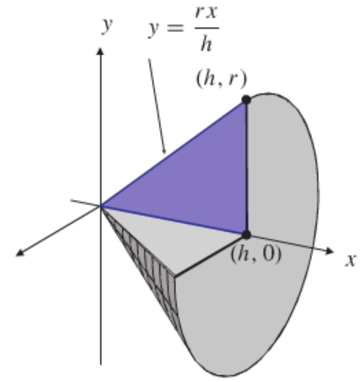
$$= 2\pi \int_0^{a} \left(a^2 - x^2\right) dx$$

$$= 2\pi \left[a^2 x - \frac{x^3}{3} \right]_0^a$$

$$= 2\pi \left(a^3 - \frac{a^3}{3} \right)$$

$$= \frac{4}{3}\pi a^3 \quad unit^3$$





Exercise

You are designing a wok frying pan that will be shaped like a spherical bowl with handles. A bit of experimentation at home persuades you that you can get one that hold about 3 *L* if you make it 9 *cm* deep and give the sphere a radius of 16 *cm*. To be sure, you picture the wok as a solid of revolution, and calculate its volume with an integral. To the nearest cubic centimeter, what volume do you really get? $\left(1\ L = 1{,}000\ cm^3\right)$

Solution

$$x^2 + y^2 = 256 \quad \Rightarrow \quad x^2 = 256 - y^2$$

$$R(y) = \sqrt{256 - y^2}$$

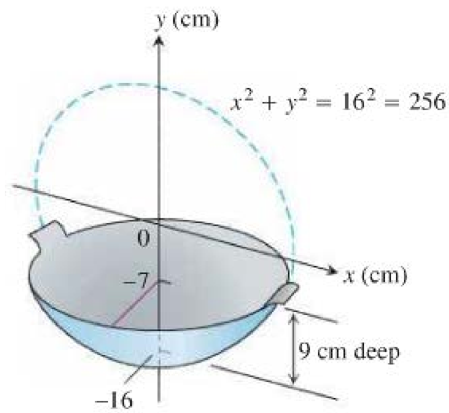
$$V = \pi \int_{-16}^{-7} \left[\sqrt{256 - y^2}\right]^2 dy$$

$$= \pi \int_{-16}^{-7} \left(256 - y^2\right) dy$$

$$= \pi \left[256y - \frac{1}{3}y^3\right]_{-16}^{-7}$$

$$= \pi \left[\left(256(-7) - \frac{1}{3}(-7)^3\right) - \left(256(-16) - \frac{1}{3}(-16)^3\right)\right]$$

$$= 1053\pi \ \ cm^3 \approx 3308 \ \ cm^3$$



Exercise

A curved wedge is cut from a circular cylinder of radius 3 by two planes. On plane is perpendicular to the axis of the cylinder. The second place crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

Solution

The base of the cylinder is a circle $x^2 + y^2 = 9$.

Since the second plane cut the base at the cylinder at the center, therefore, the base of the wedge is semi-circle. $y = \pm\sqrt{9 - x^2} = radius$.

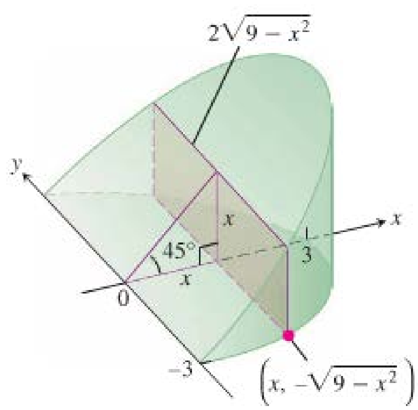
When we slice the wedge by a plane perpendicular to the axis of the cylinder, we obtained a cross-section at x which is a rectangle of height x.

The area of this cross-section is: $A(x) = height \times width$

$$= x\left(2\sqrt{9 - x^2}\right)$$

$$= 2x\sqrt{9 - x^2}$$

The rectangles run from $x = 0$ to $x = 3$, so



$$V = \int_0^3 2x\sqrt{9 - x^2}\, dx \qquad or \ \ u = 9 - x^2 \ \ \rightarrow \ \ du = -2xdx$$

$$= -\int_0^3 \left(9 - x^2\right)^{1/2} d\left(9 - x^2\right)$$

$$= -\frac{2}{3}\left[\left(9 - x^2\right)^{3/2}\right]_0^3$$

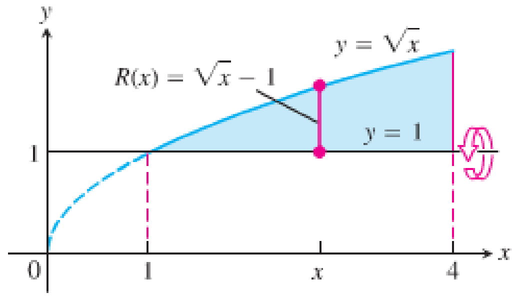
$$= -\frac{2}{3}\left[\left(9 - 3^2\right)^{3/2} - \left(9 - 0^2\right)^{3/2}\right]$$

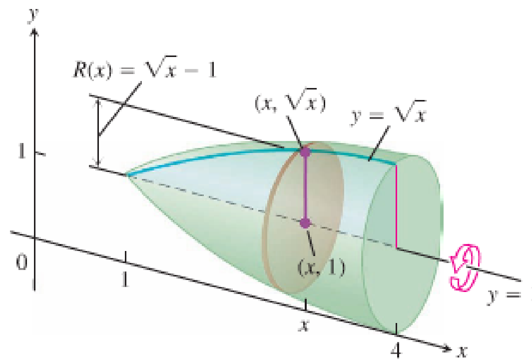
$$= 18 \ \ unit^3$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $y=\sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.

Solution





$$V=\int_1^4 \pi\left[R(x)\right]^2 dx$$

$$=\pi\int_1^4 \left[\sqrt{x}-1\right]^2 dx$$

$$=\pi\int_1^4 \left(x-2\sqrt{x}+1\right)dx$$

$$=\pi\left[\frac{x^2}{2}-2\frac{2}{3}x^{3/2}+x\right]_1^4$$

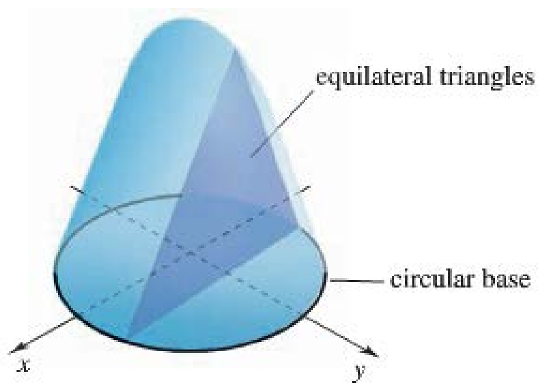
$$=\pi\left[\left(\frac{4^2}{2}-\frac{4}{3}4^{3/2}+4\right)-\left(\frac{1^2}{2}-\frac{4}{3}1^{3/2}+1\right)\right]$$

$$=\frac{7\pi}{6}\ \ unit^3$$

Exercise

The solid with a circular base of radius 5 whose cross sections perpendicular to the base and parallel to the x-axis are equilateral triangles. Use the general slicing method to find the volume of the solid.

Solution



$$x^2+y^2=5^2 \quad\rightarrow\quad y=\pm\sqrt{25-x^2}$$

$$A(x)=\left(2\sqrt{25-x^2}\right)^2 \ \ =100-4x^2$$

$$V=\int_0^5 \left(100-4x^2\right)dx$$

$$=\left(100x-\frac{4}{3}x^3\right)\Big|_0^5$$

$$=500-\frac{500}{3}$$

$$=\frac{1000}{3}\ \ unit^3$$

Exercise

The solid whose base is the region bounded by $y = x^2$ and the line $y = 1$ and whose cross sections perpendicular to the base and parallel to the *x*-axis squares. Use the general slicing method to find the volume of the solid.

Solution

$$y = x^2 \quad \Rightarrow \quad x = \sqrt{y}$$

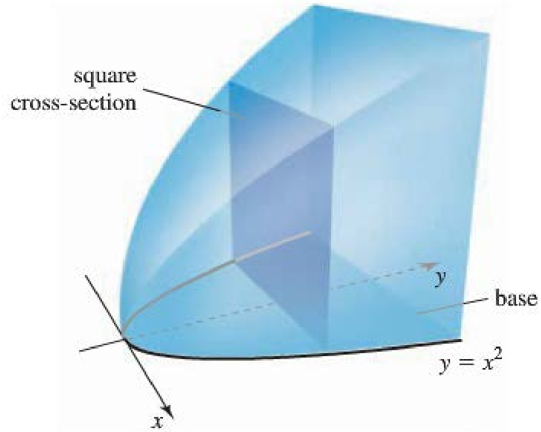
$$V = \int_0^1 A(y)\,dy$$

$$= \int_0^1 \left(2\sqrt{y}\right)^2 dy$$

$$= \int_0^1 (4y)\,dy$$

$$= 2y^2 \Big|_0^1$$

$$= 2 \ \ unit^3$$



Exercise

A circular cylinder of radius *r* and height *h* whose curved surface is at an angle of $\frac{\pi}{4}$ *rad* . Use the general slicing method to find the volume of the solid

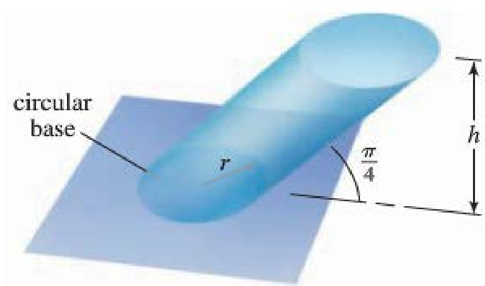
Solution

The cross sections are all circles with area πr^2

$$V = \int_0^h \left(\pi r^2\right) dz$$

$$= \pi r^2 z \Big|_0^h$$

$$= \pi r^2 h \ \ unit^3$$



∴ The 45° angle does not affect the volume.

Exercise

Let R be the region bounded by $y = e^{-x}$, $y = 0$, $x = 0$, $x = \ln 4$. Use the disk method to find the volume of the solid generated when R is revolved about x-axis.

Solution

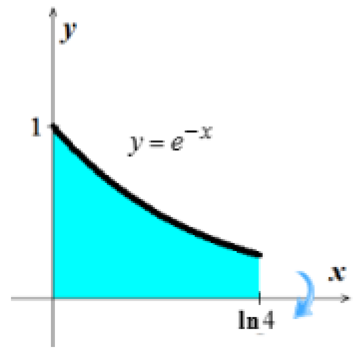
$$V = \pi \int_0^{\ln 4} e^{-2x} dx$$

$$= -\frac{\pi}{2} e^{-2x} \Big|_0^{\ln 4}$$

$$= -\frac{\pi}{2}\left(e^{-2\ln 4} - 1\right)$$

$$= -\frac{\pi}{2}\left(\frac{1}{16} - 1\right)$$

$$= \frac{15\pi}{32} \ unit^3$$



Exercise

Let R be the region bounded by $y = \cos x$, $y = 0$, $x = 0$. Use the disk method to find the volume of the solid generated when R is revolved about x-axis,

Solution

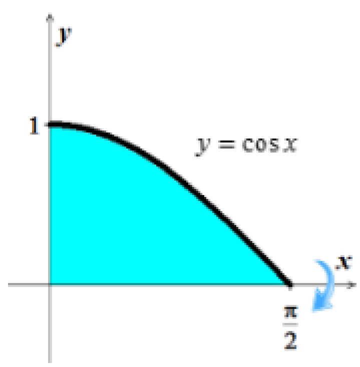
$$V = \pi \int_0^{\pi/2} \cos^2 x \ dx$$

$$= \frac{\pi}{2} \int_0^{\pi/2} (1 + \cos 2x) \ dx$$

$$= \frac{\pi}{2}\left(x + \frac{1}{2}\sin 2x\right)\Big|_0^{\pi/2}$$

$$= \frac{\pi}{2}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi^2}{4} \ unit^3$$



Exercise

Let R be the region bounded by $y = e^{x/2}$, $y = e^{-x}$, $x = \ln 2$, $x = \ln 8$. Use the disk method to find the volume of the solid generated when R is revolved about x-axis.

Solution

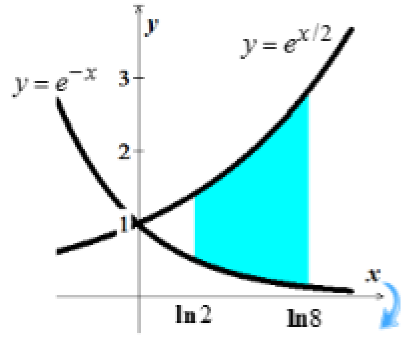
$$V = \pi \int_{\ln 2}^{\ln 8} \left(\left(e^{x/2}\right)^2 - \left(e^{-x}\right)^2 \right) dx$$

$$= \pi \int_{\ln 2}^{\ln 8} \left(e^{x} - e^{-2x}\right) dx$$

$$= \pi \left(e^{x} + \frac{1}{2}e^{-2x}\right) \Big|_{\ln 2}^{\ln 8}$$

$$= \pi \left(8 + \frac{1}{2}\frac{1}{64} - 2 - \frac{1}{8}\right)$$

$$= \frac{753\pi}{128} \ \ unit^3$$

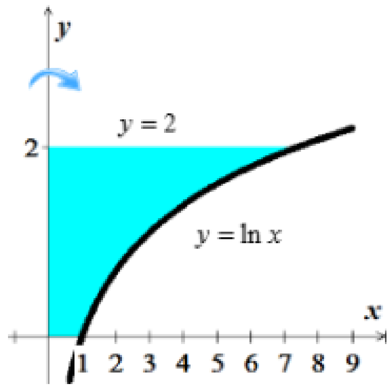


Exercise

Let R be the region bounded by $y = 0, \quad y = \ln x, \quad y = 2, \quad x = 0$. Use the disk method to find the volume of the solid generated when R is revolved about y-axis.

Solution

$$y = \ln x \quad \to \quad x = e^{y}$$



$$V = \pi \int_{0}^{2} e^{2y} dy$$

$$= \frac{\pi}{2} e^{2y} \Big|_{0}^{2}$$

$$= \frac{\pi}{2}\left(e^{4} - 1\right) \ \ unit^3$$

Exercise

Let R be the region bounded by $y = \sin^{-1} x, \quad x = 0, \quad y = \frac{\pi}{4}$. Use the disk method to find the volume of the solid generated when R is revolved about y-axis.

Solution

$$y = \sin^{-1} x \quad \to \quad x = \sin y$$

$$V = \pi \int_{0}^{\pi/4} \sin^2 y \ dy$$

$$= \frac{\pi}{2} \int_{0}^{\pi/4} (1 - \cos 2y) \ dy$$

$$= \frac{\pi}{2}\left(y - \frac{1}{2}\sin 2y\right) \Big|_{0}^{\pi/4}$$

$$= \frac{\pi}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right)$$

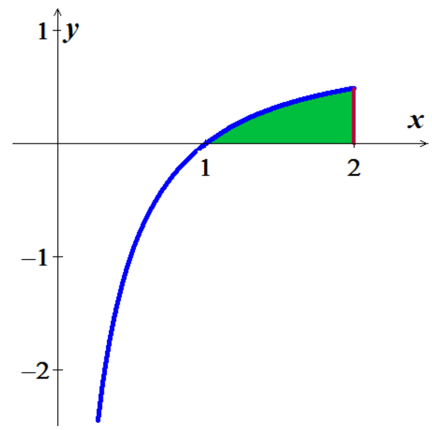
$$= \frac{\pi(\pi - 2)}{8} \ \ unit^3$$

Exercise

Find the volume of the solid of revolution bounded by $y = \frac{\ln x}{\sqrt{x}}$, $y = 0$, *and* $x = 2$ revolved about the x-axis. Sketch the region

Solution

$$y = \frac{\ln x}{\sqrt{x}} = 0 \rightarrow \underline{x = 1}$$



$$V = \pi \int_1^2 \left(\frac{\ln x}{\sqrt{x}}\right)^2 dx$$

$$= \pi \int_1^2 \frac{\ln^2 x}{x} dx$$

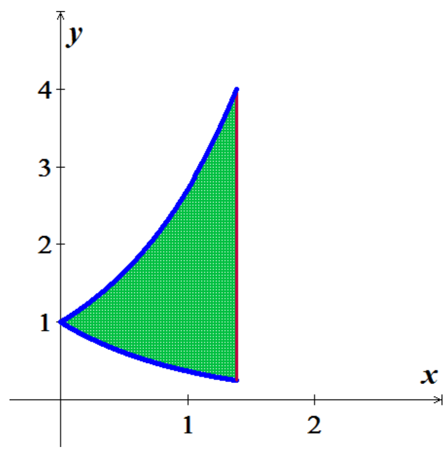
$$= \pi \int_1^2 \ln^2 x \, d(\ln x)$$

$$= \frac{\pi}{3}\left[\ln^3 x\right]_1^2$$

$$= \frac{\pi \ln^3 2}{3} \quad unit^3$$

Exercise

Find the volume of the solid of revolution bounded by $y = e^{-x}$, $y = e^x$, $x = 0$, $x = \ln 4$ revolved about the x-axis. Sketch the region



Solution

$$V = \pi \int_0^{\ln 4} \left(e^{2x} - e^{-2x}\right) dx$$

$$= \frac{\pi}{2}\left(e^{2x} + e^{-2x}\right)\Big|_0^{\ln 4}$$

$$= \frac{\pi}{2}\left(16 + \frac{1}{16}\right) \qquad e^{2\ln 4} = e^{\ln 4^2} = 16$$

$$= \frac{225\pi}{32} \quad unit^3$$

Exercise

Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $y=4$

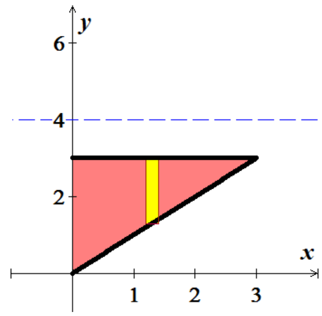
$$y=x, \quad y=3, \quad x=0$$

Solution

$y=\underline{x=3}|$

$R(x)=4-x \quad r(x)=4-3=1$

$$V=\pi\int_0^3\left((4-x)^2-1\right)dx \qquad V=\pi\int_a^b\left(R(x)^2-r(x)^2\right)dx$$



$$=\pi\int_0^3\left(16-8x+x^2-1\right)dx$$

$$=\pi\left(15x-4x^2+\frac{1}{3}x^3\right)\Big|_0^3$$

$$=\pi(45-36+9)$$

$$\underline{=18\pi}|$$

Exercise

Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $y=4$

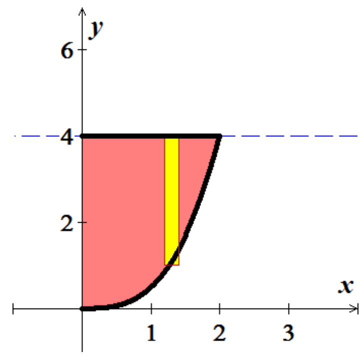
$$y=\frac{1}{2}x^3, \quad y=4, \quad x=0$$

Solution

$y=\frac{1}{2}x^3=4 \Rightarrow x^3=8 \quad \rightarrow \quad \underline{x=2}|$

$R(x)=4-\frac{1}{2}x^3 \quad r(x)=0$

$$V=\pi\int_0^2\left(4-\frac{1}{2}x^3\right)^2 dx \qquad V=\pi\int_a^b\left(R(x)^2-r(x)^2\right)dx$$



$$=\pi\int_0^2\left(16-4x^3+\frac{1}{4}x^6\right)dx$$

$$=\pi\left(16x-x^4+\frac{1}{28}x^7\right)\Big|_0^2$$

$$=\pi\left(32-16+\frac{128}{28}\right)$$

$$\underline{=\frac{144\pi}{7}}|$$

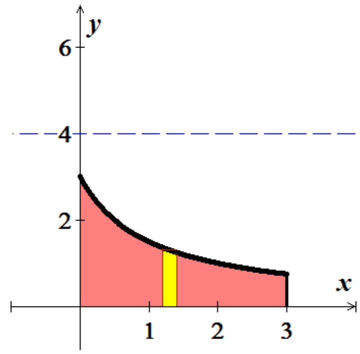
Exercise

Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $y = 4$

$$y = \frac{3}{1+x}, \quad y = 0, \quad x = 0, \quad x = 3$$

Solution

$$R(x) = 4 \quad r(x) = 4 - \frac{3}{1+x}$$



$$V = \pi \int_0^3 \left(16 - \left(4 - \frac{3}{1+x}\right)^2\right) dx \qquad V = \pi \int_a^b \left(R(x)^2 - r(x)^2\right) dx$$

$$= \pi \int_0^3 \left(\frac{24}{1+x} - \frac{9}{(1+x)^2}\right) dx \qquad \int \frac{d(x+1)}{(x+1)^2} = \frac{-1}{x+1}$$

$$= \pi \left(24 \ln(1+x) + \frac{9}{1+x}\right) \Big|_0^3$$

$$= \pi \left(24 \ln 4 + \frac{9}{4} - 9\right)$$

$$= \left(48 \ln 2 - \frac{27}{4}\right)\pi$$

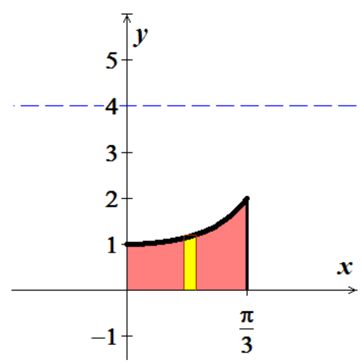
Exercise

Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $y = 4$

$$y = \sec x, \quad y = 0, \quad 0 \le x \le \frac{\pi}{3}$$

Solution

$$R(x) = 4 \quad r(x) = 4 - \sec x$$



$$V = \pi \int_0^{\pi/3} \left(16 - (4 - \sec x)^2\right) dx$$

$$= \pi \int_0^{\pi/3} \left(16 - 16 + 8 \sec x - \sec^2 x\right) dx$$

$$= \pi \left(8 \ln|\sec x + \tan x| - \tan x\right) \Big|_0^{\pi/3}$$

$$= \pi \left(8 \ln\left(2 + \sqrt{3}\right) - \sqrt{3}\right)$$

Exercise

Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $x=5$

$$y=x, \quad y=0, \quad y=4, \quad x=5$$

Solution

$R(y)=5-y \quad r(y)=0$

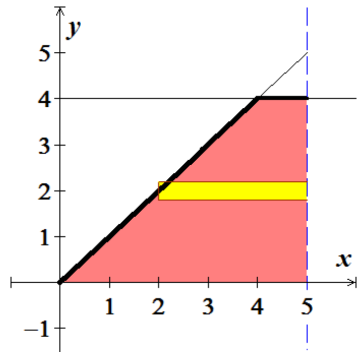
$$V=\pi\int_c^d \left(R(y)^2-r(y)^2\right)dy$$

$$V=\pi\int_0^4 (5-y)^2\,dy$$

$$=-\pi\frac{1}{3}(5-y)^3\Big|_0^4$$

$$=-\pi\frac{1}{3}\left(1-5^3\right)$$

$$=\frac{124\pi}{3}$$



Exercise

Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $x=5$

$$y=3-x, \quad y=0, \quad y=2, \quad x=0$$

Solution

$R(y)=5 \quad r(y)=5-(3-y)=2+y$

$$V=\pi\int_c^d \left(R(y)^2-r(y)^2\right)dy$$

$$V=\pi\int_0^2 \left(25-(2+y)^2\right)dy$$

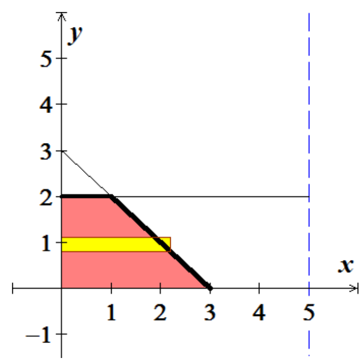
$$=\pi\int_0^2 \left(25-4-4y-y^2\right)dy$$

$$=\pi\int_0^2 \left(21-4y-y^2\right)dy$$

$$=\pi\left(21y-2y^2-\frac{1}{3}y^3\right)\Big|_0^2$$

$$=\pi\left(42-8-\frac{8}{3}\right)$$

$$=\frac{94\pi}{3}$$



Exercise

Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $x = 5$

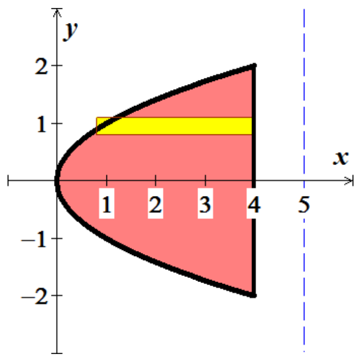
$$x = y^2, \quad x = 4$$

Solution

$x = y^2 = 4 \rightarrow \underline{y = \pm 2}|$

$R(y) = 5 - y^2 \quad r(y) = 5 - 4 = 1$

$$V = \pi \int_{-2}^{2} \left(\left(5 - y^2\right)^2 - 1 \right) dy \qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2 \right) dy$$



$$= 2\pi \int_0^2 \left(24 - 10y^2 + y^4 \right) dy$$

$$= 2\pi \left(24y - \frac{10}{3}y^3 + \frac{1}{5}y^5 \right) \Big|_0^2$$

$$= 2\pi \left(48 - \frac{80}{3} + \frac{32}{5} \right)$$

$$= \frac{832\pi}{15}$$

Exercise

Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $x = 5$

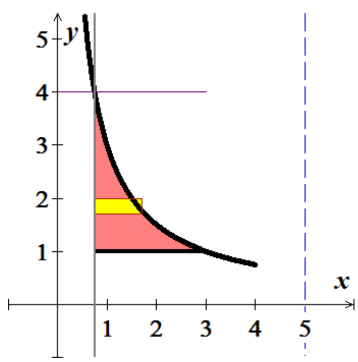
$$xy = 3, \quad y = 1, \quad y = 4, \quad x = 5$$

Solution

$xy = 3 \quad \rightarrow \ x = \frac{3}{y}$

$R(y) = 5 - \frac{3}{y} \quad r(y) = 0$

$$V = \pi \int_1^4 \left(5 - \frac{3}{y} \right)^2 dy \qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2 \right) dy$$



$$= \pi \int_1^4 \left(25 - \frac{30}{y} + \frac{9}{y^2} \right) dy$$

$$= \pi \left(25y - 30\ln y - \frac{9}{y} \right) \Big|_1^4$$

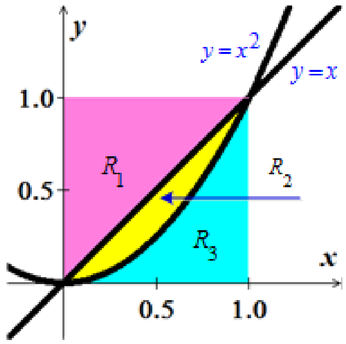
$$= \pi \left(100 - 30\ln 4 - \frac{9}{4} - 25 + 9 \right)$$

$$= \pi \left(84 - \frac{9}{4} - 30\ln 2^2 \right)$$

$$= \pi \left(\frac{327}{4} - 60\ln 2 \right)$$

Exercise

Find the volume generated by rotating the given region $y = x^2$ *and* $y = x$ about the specified line.



a) R_1 *about* $x=0$	*d)* R_2 *about* $y=1$	*g)* R_2 *about* $x=0$
b) R_1 *about* $x=1$	*e)* R_3 *about* $x=0$	*h)* R_2 *about* $x=1$
c) R_2 *about* $y=0$	*f)* R_3 *about* $x=1$	

Solution

a) $V = \pi \int_0^1 y^2 dy$ $\qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2 \right) dy$

$$= \frac{\pi}{3} y^3 \Big|_0^1$$

$$= \frac{\pi}{3}$$

b) $R(y) = 1 \quad r(y) = 1 - y$

$V = \pi \int_0^1 \left(1 - (1-y)^2 \right) dy$ $\qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2 \right) dy$

$$= \pi \int_0^1 \left(2y - y^2 \right) dy$$

$$= \pi \left(y^2 - \frac{1}{3} y^3 \right) \Big|_0^1$$

$$= \pi \left(1 - \frac{1}{3} \right)$$

$$= \frac{2\pi}{3}$$

c) R_2 *about* $y = 0$

$R(x) = x \quad r(x) = x^2$

$V = \pi \int_0^1 \left(x^2 - x^4 \right) dx$ $\qquad V = \pi \int_a^b \left(R(x)^2 - r(x)^2 \right) dx$

$$= \pi \left(\frac{1}{3}x^3 - \frac{1}{5}x^5 \right) \Big|_0^1$$

$$= \pi \left(\frac{1}{3} - \frac{1}{5} \right)$$

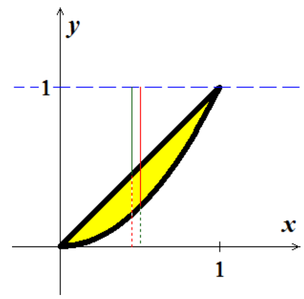
$$\underline{= \frac{2\pi}{15}}$$

d) R_2 *about* $y = 1$

$R(x) = 1 - x^2 \quad r(x) = 1 - x$

$$V = \pi \int_0^1 \left(\left(1 - x^2\right)^2 - (1-x)^2 \right) dx \qquad V = \pi \int_a^b \left(R(x)^2 - r(x)^2 \right) dx$$

$$= \pi \int_0^1 \left(1 - 2x^2 + x^4 - 1 + 2x - x^2 \right) dx$$



$$= \pi \int_0^1 \left(x^4 - 3x^2 + 2x \right) dx$$

$$= \pi \left(\frac{1}{5}x^4 - x^3 + x^2 \right) \Big|_0^1$$

$$= \pi \left(\frac{1}{5} - 1 + 1 \right)$$

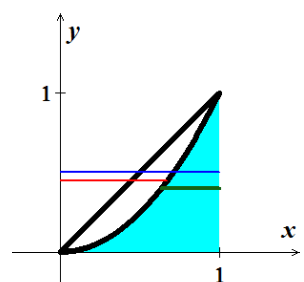
$$\underline{= \frac{\pi}{5}}$$

e) R_3 *about* $x = 0$

$y = x^2 \quad \to \quad x = \sqrt{y}$

$R(y) = 1 \quad r(y) = \sqrt{y}$

$$V = \pi \int_0^1 (1 - y) dy \qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2 \right) dy$$



$$= \pi \left(y - \frac{1}{2}y^2 \right) \Big|_0^1$$

$$= \pi \left(1 - \frac{1}{2} \right)$$

$$\underline{= \frac{\pi}{2}}$$

f) R_3 *about* $x = 1$

$R(y) = 1 - \sqrt{y} \quad r(y) = 1 - 1 = 0$

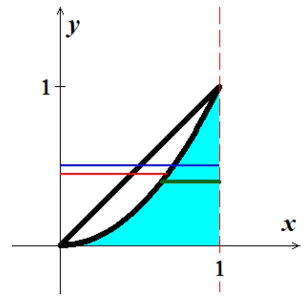
$$V = \pi \int_0^1 \left(1 - \sqrt{y} \right)^2 dy \qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2 \right) dy$$

$$= \pi \int_0^1 \left(1 - 2\sqrt{y} + y\right) dy$$

$$= \pi \left(y - \frac{4}{3} y^{3/2} + \frac{1}{2} y^2 \right) \Big|_0^1$$

$$= \pi \left(1 - \frac{4}{3} + \frac{1}{2}\right)$$

$$\boxed{= \frac{\pi}{6}}$$



g) R_2 *about* $x = 0$

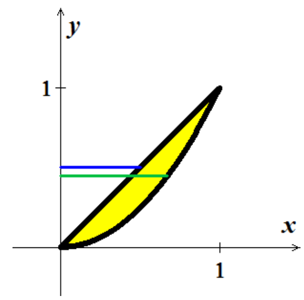
$R(y) = \sqrt{y} \quad r(y) = y$

$$V = \pi \int_0^1 \left(y - y^2\right) dy \qquad\qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2\right) dy$$

$$= \pi \left(\frac{1}{2} y^2 - \frac{1}{3} y^3 \right) \Big|_0^1$$

$$= \pi \left(\frac{1}{2} - \frac{1}{3} \right)$$

$$\boxed{= \frac{\pi}{6}}$$



h) R_2 *about* $x = 1$

$R(y) = 1 - y \quad r(y) = 1 - \sqrt{y}$

$$V = \pi \int_0^1 \left((1 - y)^2 - \left(1 - \sqrt{y}\right)^2 \right) dy \qquad\qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2\right) dy$$

$$= \pi \int_0^1 \left(1 - 2y + y^2 - 1 + 2\sqrt{y} - y\right) dy$$

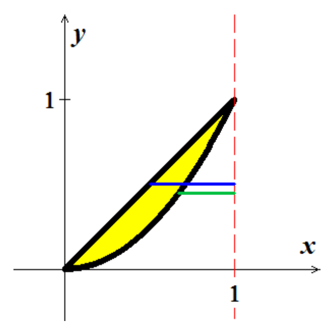
$$= \pi \int_0^1 \left(-3y + y^2 + 2y^{1/2}\right) dy$$

$$= \pi \left(-\frac{3}{2} y^2 + \frac{1}{3} y^3 + \frac{4}{3} y^{3/2} \right) \Big|_0^1$$

$$= \pi \left(-\frac{3}{2} + \frac{1}{3} + \frac{4}{3} \right)$$

$$= \pi \left(\frac{5}{3} - \frac{3}{2} \right)$$

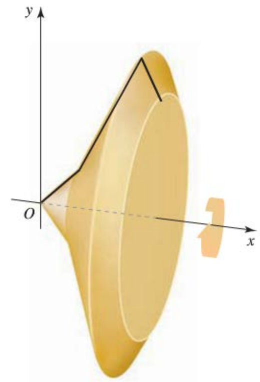
$$\boxed{= \frac{\pi}{6}}$$



Exercise

Let $f(x)=\begin{cases} x & if \quad 0\le x\le 2 \\ 2x-2 & if \quad 2<x\le 5 \\ -2x+18 & if \quad 5<x\le 6 \end{cases}$

Find the volume of the solid formed when the region bounded by the graph of *f*, the *x*-axis, and the line $x=6$ is revolved about the *x*-axis



Solution

$$V=\pi\int_0^2 x^2dx+\pi\int_2^5 (2x-2)^2\,dx+\pi\int_5^6 (18-2x)^2\,dx$$

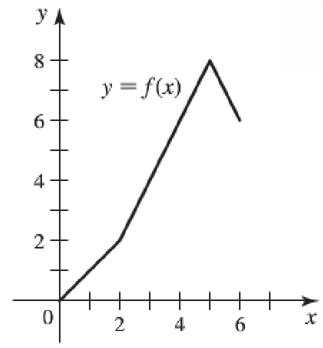
$$=\pi\int_0^2 x^2dx+\frac{\pi}{2}\int_2^5 (2x-2)^2\,d(2x-2)-\frac{\pi}{2}\int_5^6 (18-2x)^2\,d(18-2x)$$

$$=\frac{\pi}{3}x^3\Big|_0^2+\frac{\pi}{6}(2x-2)^3\Big|_2^5-\frac{\pi}{6}(18-2x)^3\Big|_5^6$$

$$=\pi\left[\frac{8}{3}+\frac{1}{6}(512-8)-\frac{1}{6}(216-512)\right]$$

$$=\pi\left(\frac{8}{3}+\frac{252}{3}+\frac{148}{3}\right)$$

$$=136\pi \ \ unit^3$$



Exercise

Consider the solid formed by revolving the region bounded by $y=\sqrt{x}$, $y=0$, and $x=4$ about the *x-axis*

a) Find the value of *x* in the interval $[0,\ 4]$ that divides the solids into two parts of equal volume.

b) Find the values of *x* in the interval $[0,\ 4]$ that divide the solids into three parts of equal volume.

Solution

$$V=\pi\int_0^4 \left(\sqrt{x}\right)^2 dx$$

$$=\pi\int_0^4 x\ dx$$

$$=\frac{\pi}{2}x^2\Big|_0^4$$

$$=8\pi$$

a) Let $0<c<4$

$$V=\pi\int_0^c x\ dx$$

$$= \frac{\pi}{2} x^2 \Big|_0^c$$

$$= \frac{\pi}{2} c^2 = \frac{1}{2}(8\pi)$$

$$\rightarrow c^2 = 8 \;\Rightarrow\; \underline{c = 2\sqrt{2}|}$$

$\therefore\ x = 2\sqrt{2}$, the solid is divided into 2 equal volume parts.

b) The first one third part:

$$\frac{\pi}{2} c^2 = \frac{1}{3}(8\pi) \quad\rightarrow\quad c^2 = \frac{16}{3} \quad\Rightarrow\quad \underline{c = \frac{4\sqrt{3}}{3}|}$$

The second part $\left(\frac{2}{3}\right)$:

$$\frac{\pi}{2} d^2 = \frac{2}{3}(8\pi) \quad\rightarrow\quad d^2 = \frac{32}{3} \quad\Rightarrow\quad \underline{d = \frac{4\sqrt{2}}{\sqrt{3}} = \frac{4\sqrt{6}}{3}|}$$

$\therefore\ x = \frac{4\sqrt{3}}{3},\ \frac{4\sqrt{6}}{3}$, the solid is divided into 3 equal volume parts.

Exercise

The arc of $y = 4 - \frac{1}{4}x^2$ on the interval $[0,\ 4]$ is revolved about the line $y = b$

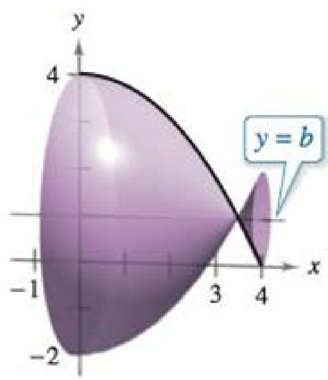
a) Find the volume of the resulting solid as a function of b.

b) Graph the function in part (a), and approximate the value of b that minimizes the volume of the solid.

c) Find the value of b that minimizes the volume of the solid, and compare the result with the answer in part (b).

Solution

a) $y = 4 - \frac{1}{4}x^2 = b \;\Rightarrow\; x^2 = 4(4-b) \;\rightarrow\; \underline{x = 2\sqrt{4-b}|}$



$$V = \pi\int_0^{2\sqrt{4-b}} \left(4 - \frac{1}{4}x^2 - b\right)^2 dx + \pi\int_{2\sqrt{4-b}}^{4} \left(b - 4 + \frac{1}{4}x^2\right)^2 dx$$

$$= \pi\int_0^4 \left(4 - \frac{1}{4}x^2 - b\right)^2 dx$$

$$= \pi\int_0^4 \left(16 - 2x^2 + \frac{1}{2}bx^2 + \frac{1}{16}x^4 - 8b + b^2\right)dx$$

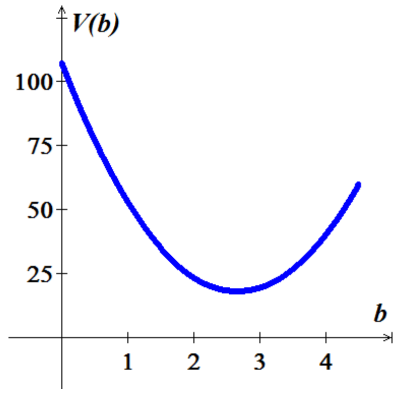
$$= \pi\left(16x - \frac{2}{3}x^3 + \frac{1}{6}bx^3 + \frac{1}{80}x^5 - 8bx + b^2x\right)\Big|_0^4$$

$$= \pi\left(64 - \frac{16}{3} + \frac{64}{6}b + \frac{64}{5} - 32b + 4b^2\right)$$

$$= \pi\left(4b^2 - \frac{64}{3}b + \frac{512}{15}\right)$$

$$V(b) = 4\pi\left(b^2 - \frac{16}{3}b + \frac{128}{15}\right)$$

b) From the graph (*using software*)



2.660	17.873
2.670	17.872
2.680	17.874

The minumium volume is 17.872 for $b = 2.67$

c) $V'(b) = 4\pi\left(2b - \frac{16}{3}\right) = 0 \quad \rightarrow \quad b = \frac{16}{6} = \frac{8}{3}$

$$V\left(b = \frac{8}{3}\right) = 4\pi\left(\left(\frac{8}{3}\right)^2 - \frac{16}{3}\frac{8}{3} + \frac{128}{15}\right)$$

$$= 4\pi\left(\frac{64}{9} - \frac{128}{9} + \frac{128}{15}\right)$$

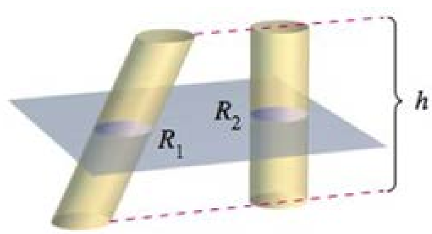
$$= 4\pi\left(\frac{128}{15} - \frac{64}{9}\right)$$

$$= 4\pi\left(\frac{64}{45}\right)$$

$$= \frac{256\pi}{45} \quad \approx 17.872$$

Exercise

Prove that if two solids have equal altitudes and all plane sections parallel to their bases and at equal distances from their bases have equal areas, then the solids hace the same volume.



Area of R_1 = *Area of* R_2

Solution

Since $A_1(x) = A_2(x)$ when $a \le x \le b$, then

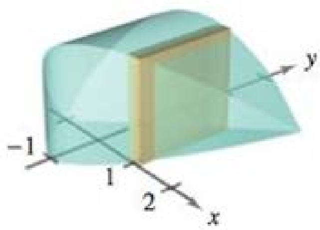
$$V_1 = \int_a^b A_1(x)\,dx = \int_a^b A_2(x)\,dx = V_2$$

∴ The volume are the same.

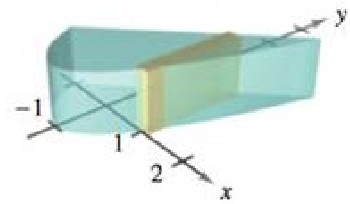
Exercise

Find the volumes of the solids whose bases are bounded by the graph of $y = x + 1$ and $y = x^2 - 1$, with the indicated cross sections taken perpendicular to the *x-axis*

a) Squares

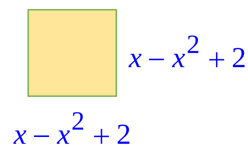


b) Rectangles of height 1



Solution

a) Base of cross section $= (x+1) - (x^2 - 1) = x - x^2 + 2$



$$A = b^2 = \left(2 + x - x^2\right)^2$$

$$= 4 + 4x - 3x^2 - 2x^3 + x^4$$

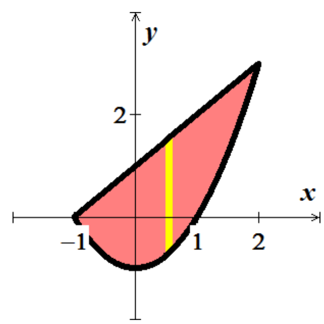
$$V = \int_{-1}^{2} \left(4 + 4x - 3x^2 - 2x^3 + x^4\right) dx$$

$$= \left(4x + 2x^2 - x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right) \Big|_{-1}^{2}$$

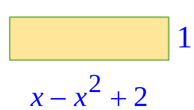
$$= 8 + 8 - 8 - 8 + \frac{32}{5} + 4 - 2 - 1 + \frac{1}{2} + \frac{1}{5}$$

$$= \frac{32}{5} + 1 + \frac{7}{10}$$

$$\boxed{= \frac{81}{10}}$$



b) $A = bh = \left(2 + x - x^2\right)(1)$



$$V = \int_{-1}^{2} \left(2 + x - x^2\right) dx$$

$$= 2x + \frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_{-1}^{2}$$

$$= 4 + 2 - \frac{8}{3} + 2 - \frac{1}{2} - \frac{1}{3}$$

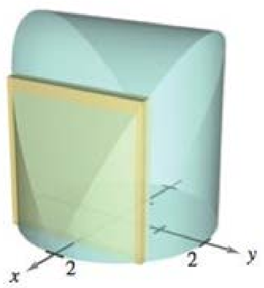
$$= 8 - 3 - \frac{1}{2}$$

$$\boxed{= \frac{9}{2}}$$

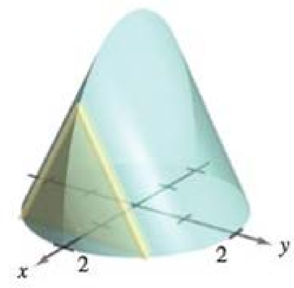
Exercise

Find the volumes of the solids whose bases are bounded by the circle $x^2 + y^2 = 4$, with the indicated cross sections taken perpendicular to the *x-axis*

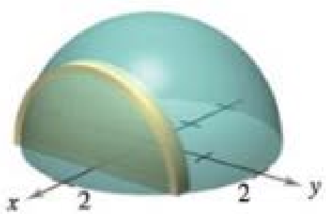
a) Squares



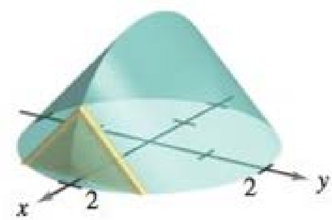
b) Equilateral triangles



c) Semicircles



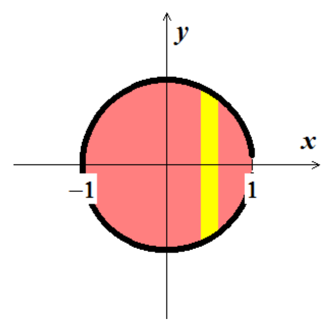
d) Iscosceles right triangles



Solution

$$y = \pm\sqrt{4-x^2}$$

Base cross section $y = 2\sqrt{4-x^2}$



a) $A = b^2 = 4\left(4-x^2\right)$

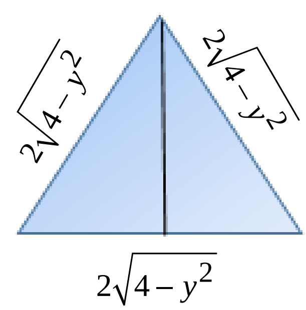
$$V = 4\int_{-2}^{2}\left(4-x^2\right)dx$$

$$= 8\left(4x - \frac{1}{3}x^3\right)\Big|_0^2$$

$$= 8\left(8 - \frac{8}{3}\right)$$

$$\underline{= \frac{128}{3}}$$

b) $\sin 60° = \dfrac{h}{2\sqrt{4-x^2}} \quad \rightarrow \quad \underline{h = \sqrt{3}\sqrt{4-x^2}}$



$$A = \frac{1}{2}bh = \frac{1}{2}\left(2\sqrt{4-x^2}\right)\sqrt{3}\sqrt{4-x^2}$$

$$= \sqrt{3}\left(4-x^2\right)$$

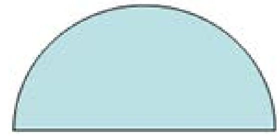
$$V = \sqrt{3}\int_{-2}^{2}\left(4-x^2\right)dx$$

$$= 2\sqrt{3}\left(4x - \frac{1}{3}x^3\right)\Big|_0^2$$

$$= 2\sqrt{3}\left(8 - \frac{8}{3}\right)$$

$$= \frac{32\sqrt{3}}{3}$$

c) $A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\sqrt{4 - x^2}\right)^2 = \frac{\pi}{2}\left(4 - x^2\right)$



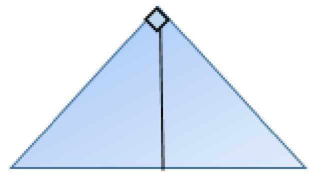
$$V = \frac{\pi}{2}\int_{-2}^{2}\left(4 - x^2\right)dx$$

$$= \pi\left(4x - \frac{1}{3}x^3\right)\Big|_0^2$$

$$= \pi\left(8 - \frac{8}{3}\right)$$

$$= \frac{16\pi}{3}$$

d) $\tan 45° = \frac{h}{\sqrt{4 - x^2}} \quad \to \quad h = \sqrt{4 - x^2}$



$$A = \frac{1}{2}bh = \frac{1}{2}\left(2\sqrt{4 - x^2}\right)\sqrt{4 - x^2} = 4 - x^2$$

$$V = \int_{-2}^{2}\left(4 - x^2\right)dx$$

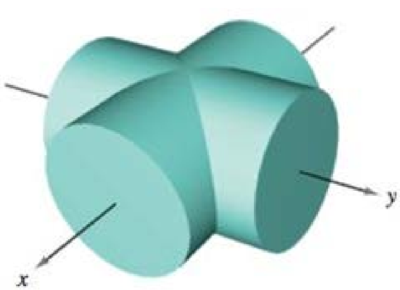
$$= 2\left(4x - \frac{1}{3}x^3\right)\Big|_0^2$$

$$= 2\left(8 - \frac{8}{3}\right)$$

$$= \frac{32}{3}$$

Exercise

Find the volume of the solid of intersection (the solid common to both) of the two right circular cylinders of radius *r* whose axes meet at right angles.



Two intersecting cylinders

Solid of intersection

Solution

The cross sections are squares.

By symmetry, we can divide the volume to 8 equal sections.

$$x^2 + y^2 = r^2 \;\rightarrow\; x = \sqrt{r^2 - y^2}$$

$$A = b^2 = \left(\sqrt{r^2 - y^2}\right)^2 = r^2 - y^2$$

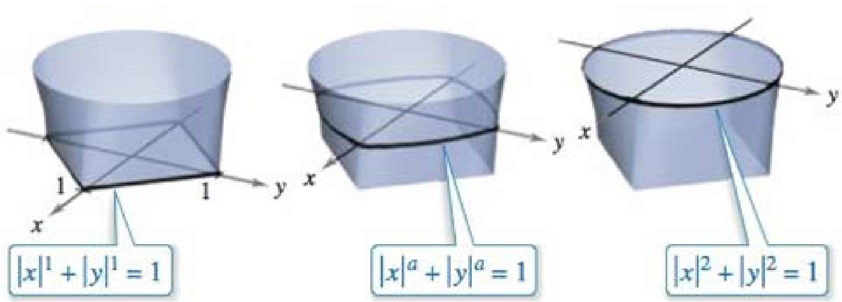
$$\begin{aligned} V &= 8\int_0^r \left(r^2 - y^2\right) dy \\ &= 8\left(r^2 y - \frac{1}{3}y^3\right)\Big|_0^r \\ &= 8\left(r^3 - \frac{1}{3}r^3\right) \\ &= \boxed{\frac{16}{3}r^3} \end{aligned}$$

Exercise

The solid shown in the figure has cross sections bounded by the graph $|x|^a + |y|^a = 1$ where $1 \le a \le 2$.

a) Describe the cross section when $a = 1$ and $a = 2$.

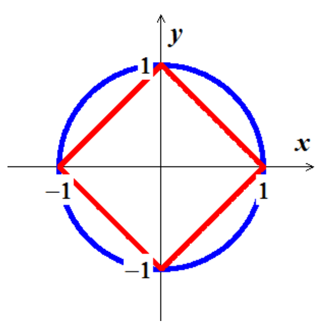
b) Describe a procedure for approximating the volume of the solid.



Solution

a) When $a = 1 \;\Rightarrow\; |x| + |y| = 1$ represents a square.

When $a = 2 \;\Rightarrow\; |x|^2 + |y|^2 = 1$ represents a circle.



b) $|x|^a + |y|^a = 1 \;\rightarrow\; |y| = \left(1 - |x|^a\right)^{1/a}$

$$\begin{aligned} A &= 2\int_{-1}^{1} \left(1 - |x|^a\right)^{1/a} dx \\ &= 4\int_0^1 \left(1 - |x|^a\right)^{1/a} dx \end{aligned}$$

To approximate the volume of the solid, from *n* slices, each of whose area is approximated by the integral above,

Then sum the volumes of these *n* slices.

Exercise

Two planes cut a right circular cylinder to form a wedge. One plane is perpendicular to the axis of the cylinder and the second makes an angle of θ degrees with the first.

a) Find the volume of the wedge if $\theta = 45°$.

b) Find the volume of the wedge for an arbitrary angle θ. Assuming that the cylinder has sufficient length, how does the volume of the wedge change as θ increases from 0° to 90°?

Solution

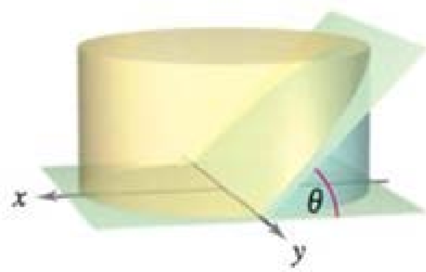
a) Since $\theta = 45°$, then the cross sections are isosceles right triangles.

$$x^2 + y^2 = r^2 \;\rightarrow\; x = \pm\sqrt{r^2 - y^2}$$

$$\tan 45° = \frac{h}{\sqrt{r^2 - y^2}} = 1 \;\rightarrow\; h = \sqrt{r^2 - y^2}$$

$$A(y) = \frac{1}{2}bh = \frac{1}{2}\sqrt{r^2 - y^2}\sqrt{r^2 - y^2}$$

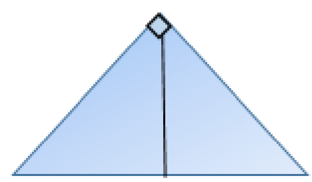
$$= \frac{1}{2}\left(r^2 - y^2\right)$$



$$V = \frac{1}{2}\int_{-r}^{r}\left(r^2 - y^2\right)dy$$

$$= \left(r^2 y - \frac{1}{3}y^3\right)\Big|_0^r$$

$$\boxed{= \frac{2}{3}r^3}$$



$2\sqrt{r^2 - y^2}$

b) $\tan\theta = \dfrac{h}{\sqrt{r^2 - y^2}} \;\rightarrow\; h = (\tan\theta)\sqrt{r^2 - y^2}$

$$A(y) = \frac{1}{2}bh = \frac{1}{2}\left(r^2 - y^2\right)\tan\theta$$

$$V = \frac{1}{2}\tan\theta\int_{-r}^{r}\left(r^2 - y^2\right)dy$$

$$= \tan\theta\left(r^2 y - \frac{1}{3}y^3\right)\Big|_0^r$$

$$\boxed{= \frac{2}{3}r^3\tan\theta}$$

As $\theta \rightarrow 90° \;\Rightarrow\; V \rightarrow \infty$

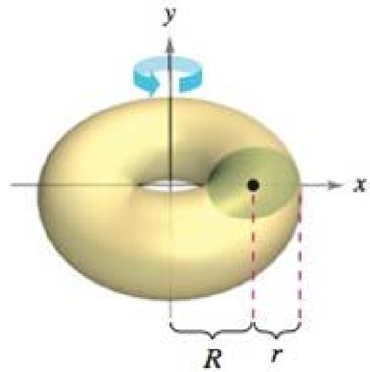
Exercise

For the given torus (donut).

a) Show that the volume of the torus is given by the integral

$$8\pi R\int_0^r \sqrt{r^2-y^2}\,dy \quad \text{where} \quad R>r>0$$

b) Find the volume of the torus



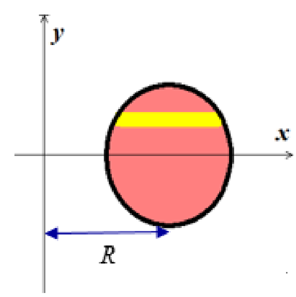
Solution

a) $(x-R)^2+y^2=r^2 \quad \to \quad x=R\pm\sqrt{r^2-y^2}$

$$V=\pi\int_{-r}^{r}\left[\left(R+\sqrt{r^2-y^2}\right)^2-\left(R-\sqrt{r^2-y^2}\right)^2\right]dy$$

$$=2\pi\int_0^r\left[R^2+2R\sqrt{r^2-y^2}+r^2-y^2-R^2+2R\sqrt{r^2-y^2}-r^2+y^2\right]dy$$

$$=8\pi R\int_0^r\sqrt{r^2-y^2}\,dy \quad \checkmark$$



b) $\int_0^r\sqrt{r^2-y^2}\,dy$ is $\frac{1}{4}$ of the area of a circle of radius r.

$$A=\frac{1}{4}\pi r^2$$

$$V=8\pi R\int_0^r\sqrt{r^2-y^2}\,dy$$

$$=8\pi R\left(\frac{1}{4}\pi r^2\right)$$

$$\underline{=2\pi^2r^2R}$$

Solution *Section* 1.4 – Volumes by Shells

Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the y-*axis*: $y = x$

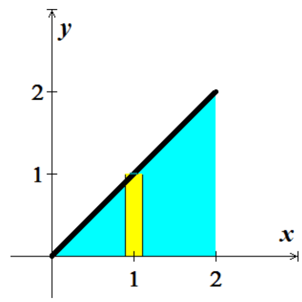
Solution

$$V = 2\pi \int_0^2 x(x)\,dx \qquad\qquad V = 2\pi \int_a^b x\,f(x)\,dx$$

$$= 2\pi \int_0^2 x^2\,dx$$

$$= \frac{2\pi}{3} x^3 \Big|_0^2$$

$$= \frac{16\pi}{3}$$



Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the y-*axis* $y = 1 - x$

Solution

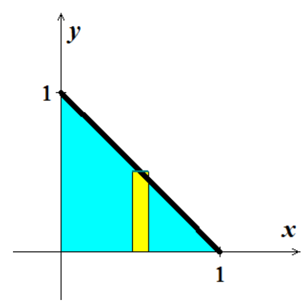
$$V = 2\pi \int_0^1 x(1-x)\,dx \qquad\qquad V = 2\pi \int_a^b x\,f(x)\,dx$$

$$= 2\pi \int_0^1 \left(x - x^2\right)\,dx$$

$$= 2\pi \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1$$

$$= 2\pi \left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{\pi}{3}$$



Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the y-*axis* $y = \sqrt{x}$

Solution

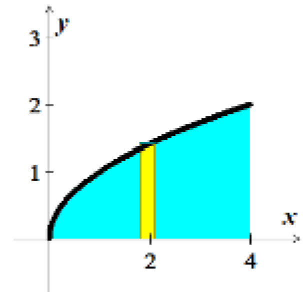
$$V = 2\pi \int_0^4 x\sqrt{x}\,dx \qquad\qquad V = 2\pi \int_a^b x\,f(x)\,dx$$

$$= 2\pi\int_0^4 x^{3/2}\, dx$$

$$= 2\pi\left(\frac{2}{5}x^{5/2}\right)\Big|_0^4$$

$$= \frac{4\pi}{5}\left(2^2\right)^{5/2}$$

$$= \frac{128\pi}{5}$$



Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis* $\quad y = \frac{1}{2}x^2 + 1$

Solution

$$f(x) = 3 - \left(\frac{1}{2}x^2 + 1\right) = 2 - \frac{1}{2}x^2$$

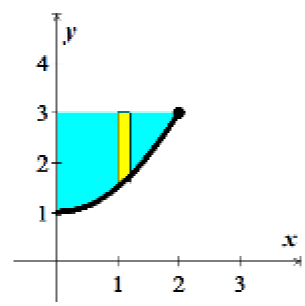
$$V = 2\pi\int_0^2 x\left(2 - \frac{1}{2}x^2\right)\, dx \qquad V = 2\pi\int_a^b x\, f(x)\, dx$$

$$= 2\pi\int_0^2 \left(2x - \frac{1}{2}x^3\right)\, dx$$

$$= 2\pi\left(x^2 - \frac{1}{8}x^4\right)\Big|_0^2$$

$$= 2\pi(4 - 2)$$

$$= 4\pi$$



Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

$$y = \frac{1}{4}x^2, \quad y = 0, \quad x = 4$$

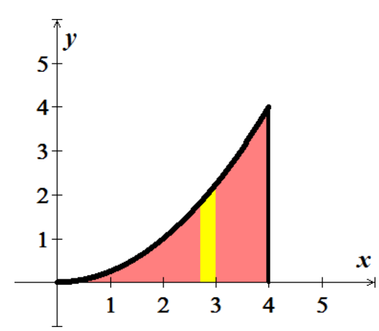
Solution

$$V = 2\pi\int_0^4 x\left(\frac{1}{4}x^2\right)dx \qquad V = 2\pi\int_a^b x\big(f(x) - g(x)\big)\, dx$$

$$= \frac{\pi}{2}\int_0^4 x^3 dx$$

$$= \frac{\pi}{8}x^4\Big|_0^4$$

$$= 32\pi$$



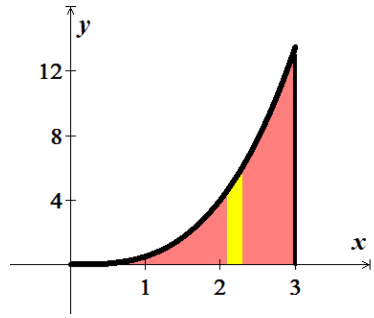
Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

$$y=\frac{1}{2}x^3, \quad y=0, \quad x=3$$

Solution

$$V=2\pi\int_0^3 x\left(\frac{1}{2}x^3\right)dx \qquad V=2\pi\int_a^b x\left(f(x)-g(x)\right)dx$$



$$=\pi\int_0^3 x^4dx$$

$$=\frac{\pi}{5}x^5\Big|_0^3$$

$$=\frac{243\pi}{5}$$

Exercise

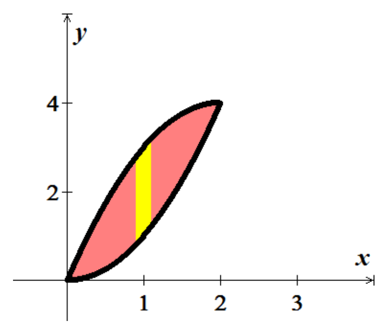
Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

$$y=x^2, \quad y=4x-x^2$$

Solution

$$y=4x-x^2=x^2 \Rightarrow 2x^2-4x=0 \rightarrow \underline{x=0,\ 2}$$

$$f(x)=4x-x^2, \quad g(x)=x^2$$



$$V=2\pi\int_0^2 x\left(4x-x^2-x^2\right)dx \qquad V=2\pi\int_a^b x\left(f(x)-g(x)\right)dx$$

$$=4\pi\int_0^2\left(2x^2-x^3\right)dx$$

$$=4\pi\left(\frac{2}{3}x^3-\frac{1}{4}x^4\right)\Big|_0^2$$

$$=4\pi\left(\frac{16}{3}-4\right)$$

$$=\frac{16\pi}{3}$$

Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

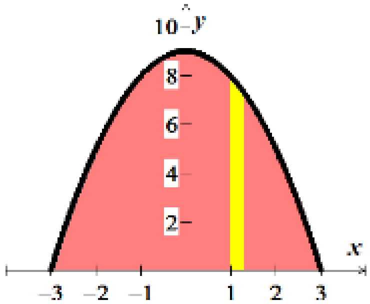
$$y = 9 - x^2, \quad y = 0$$

Solution

$y = 9 - x^2 = 0 \quad \to \quad \underline{x = \pm 3}$

$f(x) = 9 - x^2, \quad g(x) = 0$

$$V = 2\pi \int_0^3 x\left(9 - x^2\right) dx \qquad\qquad V = 2\pi \int_a^b x\left(f(x) - g(x)\right) dx$$



$$= 2\pi \int_0^3 \left(9x - x^3\right) dx$$

$$= 2\pi \left(\frac{9}{2}x^2 - \frac{1}{4}x^4\right)\Big|_0^3$$

$$= 2\pi \left(\frac{81}{2} - \frac{81}{4}\right)$$

$$= \boxed{\frac{81\pi}{2}}$$

Exercise

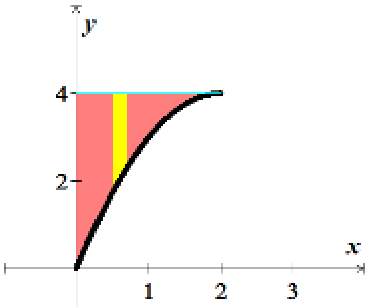
Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

$$y = 4x - x^2, \quad x = 0, \quad y = 4$$

Solution

$y = 4x - x^2 = 4 \Rightarrow x^2 - 4x + 4 \quad \to \quad \underline{x = 2}$

$f(x) = 4, \quad g(x) = 4x - x^2$



$$V = 2\pi \int_0^2 x\left(4 - 4x + x^2\right) dx \qquad\qquad V = 2\pi \int_a^b x\left(f(x) - g(x)\right) dx$$

$$= 2\pi \int_0^2 \left(4x - 4x^2 + x^3\right) dx$$

$$= 2\pi \left(2x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4\right)\Big|_0^2$$

$$= 2\pi \left(8 - \frac{32}{3} + 4\right)$$

$$= \boxed{\frac{8\pi}{3}}$$

Exercise

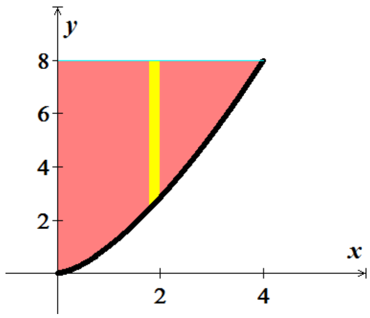
Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

$$y = x^{3/2}, \quad y = 8, \quad x = 0$$

Solution

$y = x^{3/2} = 8 \Rightarrow x = \left(2^3\right)^{2/3} \quad \rightarrow \underline{x = 4}$

$f(x) = 8, \quad g(x) = x^{3/2}$



$$V = 2\pi \int_0^4 x\left(8 - x^{3/2}\right) dx \qquad V = 2\pi \int_a^b x\left(f(x) - g(x)\right) dx$$

$$= 2\pi \int_0^4 \left(8x - x^{5/2}\right) dx$$

$$= 2\pi \left(4x^2 - \frac{2}{7}x^{7/2}\right)\Big|_0^4$$

$$= 2\pi \left(64 - \frac{256}{7}\right)$$

$$= \frac{384\pi}{7}$$

Exercise

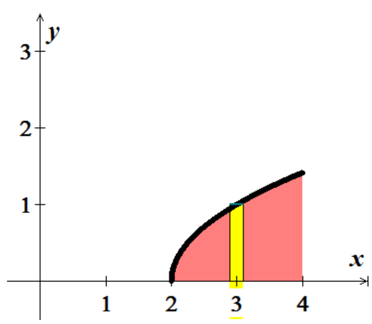
Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

$$y = \sqrt{x-2}, \quad y = 0, \quad x = 4$$

Solution

$y = \sqrt{x-2} = 0 \quad \rightarrow \underline{x = 2}$

$f(x) = \sqrt{x-2}, \quad g(x) = 0$



$$V = 2\pi \int_2^4 x\left(\sqrt{x-2}\right) dx \qquad V = 2\pi \int_a^b x\left(f(x) - g(x)\right) dx$$

$$= 2\pi \int_2^4 (u+2)u^{1/2}\, du \qquad u = x - 2 \quad x = u + 2$$

$$du = dx$$

$$= 2\pi \int_2^4 \left(u^{3/2} + 2u^{1/2}\right) du$$

$$= 2\pi \left(\frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2}\right)\Big|_2^4$$

$$= 2\pi \left(\frac{2}{5}(x-2)^{5/2} + \frac{4}{3}(x-2)^{3/2}\right)\Big|_2^4$$

$$= 2\pi\left(\frac{8\sqrt{2}}{5} + \frac{8\sqrt{2}}{3}\right)$$

$$= 16\pi\sqrt{2}\left(\frac{1}{5} + \frac{1}{3}\right)$$

$$= \frac{128\pi\sqrt{2}}{15}$$

Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

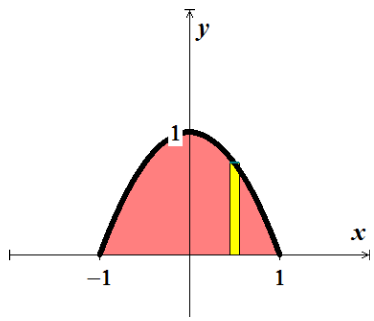
$$y = -x^2 + 1, \quad y = 0$$

Solution

$$y = -x^2 + 1 = 0 \quad \to \quad x = \pm 1$$

$$f(x) = -x^2 + 1, \quad g(x) = 0$$

$$V = 2\pi \int_0^1 x\left(-x^2 + 1\right) dx \qquad V = 2\pi \int_a^b x\big(f(x) - g(x)\big) dx$$



$$= 2\pi \int_0^1 \left(-x^3 + x\right) dx$$

$$= 2\pi\left(-\frac{1}{4}x^4 + \frac{1}{2}x^2\right)\Big|_0^1$$

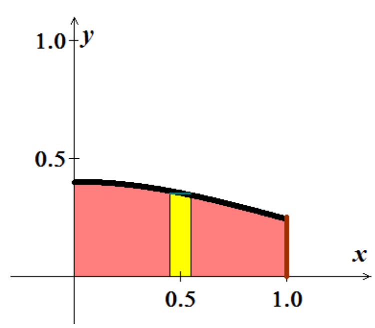
$$= 2\pi\left(-\frac{1}{4} + \frac{1}{2}\right)$$

$$= \frac{\pi}{2}$$

Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

$$y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad y = 0, \quad x = 0, \quad x = 1$$



Solution

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad g(x) = 0$$

$$V = 2\pi \int_0^1 x\left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\right) dx \qquad V = 2\pi \int_a^b x\big(f(x) - g(x)\big) dx$$

$$= -\sqrt{2\pi} \int_0^1 e^{-x^2/2}\, d\left(-\frac{x^2}{2}\right)$$

$$= -\sqrt{2\pi}\left(e^{-x^2/2}\right)\Big|_0^1$$

$$= -\sqrt{2\pi}\left(e^{-1/2} - 1\right)$$

$$= \sqrt{2\pi}\left(1 - \frac{1}{\sqrt{e}}\right)$$

Exercise

Use the shell method to find the volume of the solid generated by revolving the shaded region about the indicated axis

Solution

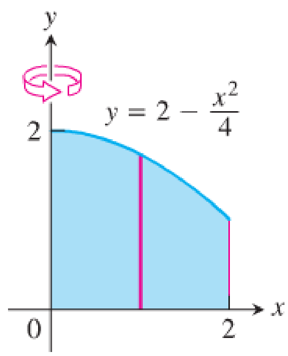
$$V = \int_0^2 2\pi(x)\left(2 - \frac{x^2}{4}\right)dx \qquad V = \int_a^b 2\pi\left(\begin{array}{c} shell \\ radius \end{array}\right)\left(\begin{array}{c} shell \\ height \end{array}\right)dx$$

$$= 2\pi\int_0^2\left(2x - \frac{x^3}{4}\right)dx$$

$$= 2\pi\left(x^2 - \frac{x^4}{16}\right)\Big|_0^2$$

$$= 2\pi\left[\left(2^2 - \frac{2^4}{16}\right) - 0\right]$$

$$= 6\pi \ \ unit^3$$



Exercise

Use the shell method to find the volume of the solid generated by revolving the shaded region about the indicated axis

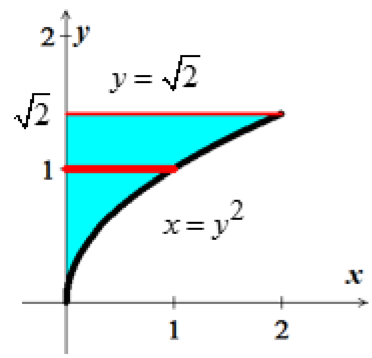
Solution

$$V = \int_0^{\sqrt{2}} 2\pi(y)\left(y^2\right)dy$$

$$= 2\pi\int_0^{\sqrt{2}} y^3 dy$$

$$= 2\pi\left(\frac{y^4}{4}\right)\Big|_0^{\sqrt{2}}$$

$$= 2\pi \ \ unit^3$$



Exercise

Use the shell method to find the volume of the solid generated by revolving the shaded region about the y-axis

Solution

$$V=\int_0^3 2\pi(x)\left(\frac{9x}{\sqrt{x^3+9}}\right)dx \qquad V=\int_a^b 2\pi\binom{shell}{radius}\binom{shell}{height}dx$$

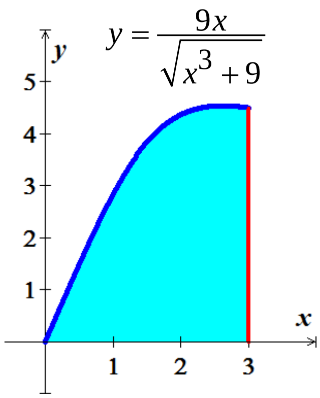
$$=2\pi\int_0^3\left(\frac{9x^2}{\sqrt{x^3+9}}\right)dx$$

$$=2\pi\int_0^3 3\left(x^3+9\right)^{1/2}d\left(x^3+9\right) \qquad d\left(x^3+9\right)=3x^2dx$$

$$=6\pi\left[2\left(x^3+9\right)^{1/2}\right]_0^3 \qquad =12\pi\left[\left(3^3+9\right)^{1/2}-(0+9)^{1/2}\right]$$

$$=12\pi[6-3]$$

$$=36\pi\ \ unit^3$$



Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y=x^2$, $y=2-x$, $x=0$, *for* $x\ge 0$ about the y-axis.

Solution

$$V=\int_0^1 2\pi(x)\left((2-x)-x^2\right)dx \qquad V=\int_a^b 2\pi\binom{shell}{radius}\binom{shell}{height}dx$$

$$=2\pi\int_0^1 x\left(2-x-x^2\right)dx$$

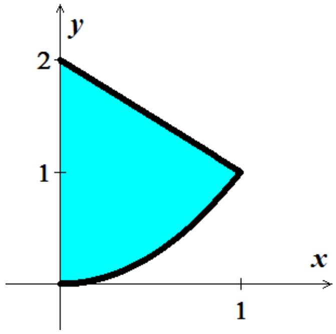
$$=2\pi\int_0^1\left(2x-x^2-x^3\right)dx$$

$$=2\pi\left[x^2-\tfrac{1}{3}x^3-\tfrac{1}{4}x^4\right]_0^1$$

$$=2\pi\left(1-\tfrac{1}{3}-\tfrac{1}{4}\right)$$

$$=12\pi\left(\tfrac{5}{12}\right)$$

$$=\frac{5\pi}{6}\ \ unit^3$$



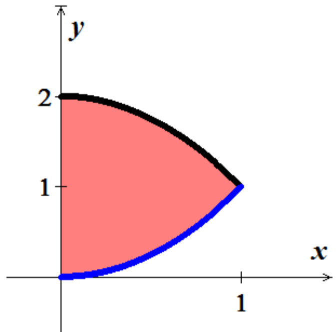
Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y=2-x^2, \quad y=x^2, \quad x=0$ about the y-axis.

Solution

$$y=2-x^2=x^2 \quad \rightarrow \quad 2x^2=2 \Rightarrow x^2=1 \rightarrow \boxed{x=\pm 1}$$

Since *about* $y-axis, \quad a=x=0 \quad b=1$

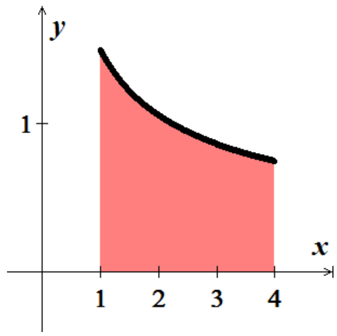


$$\begin{aligned}
V &= \int_0^1 2\pi(x)\left(\left(2-x^2\right)-x^2\right)dx \\
&= 2\pi\int_0^1 x\left(2-2x^2\right)dx \\
&= 4\pi\int_0^1 \left(x-x^3\right)dx \\
&= 4\pi\left[\frac{1}{2}x^2-\frac{1}{4}x^4\right]_0^1 \\
&= 4\pi\left[\frac{1}{2}-\frac{1}{4}\right]_0^1 \\
&= 4\pi\left(\frac{1}{4}\right) \\
&= \pi \ \ unit^3
\end{aligned}$$

Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y=\frac{3}{2\sqrt{x}}, \quad y=0, \quad x=1, \quad x=4$ about the y-axis.

Solution



$$\begin{aligned}
V &= \int_1^4 2\pi(x)\left(\frac{3}{2\sqrt{x}}-0\right)dx \\
&= \pi\int_1^4 x\left(3x^{-1/2}\right)dx \\
&= 3\pi\int_1^4 x^{1/2}dx \\
&= 3\pi\left[\frac{2}{3}x^{3/2}\right]_1^4
\end{aligned}$$

$$= 2\pi\left[4^{3/2} - 1^{3/2}\right]$$

$$= 2\pi(7)$$

$$\boxed{= 14\pi \quad unit^3}$$

Exercise

Let $g(x) = \begin{cases} \dfrac{(\tan x)^2}{x} & 0 < x \le \frac{\pi}{4} \\ 0 & x = 0 \end{cases}$

a) Show that $x \cdot g(x) = (\tan x)^2, \quad 0 \le x \le \frac{\pi}{4}$

b) Find the volume of the solid generated by revolving the shaded region about the *y*-axis.

Solution

a) $x \cdot g(x) = \begin{cases} x \cdot \dfrac{(\tan x)^2}{x} & 0 < x \le \frac{\pi}{4} \\ x \cdot 0 & x = 0 \end{cases} \quad \Rightarrow x \cdot g(x) = \begin{cases} \tan^2 x & 0 < x \le \frac{\pi}{4} \\ 0 & x = 0 \end{cases}$

Since $x = 0 \quad \to \quad \tan x = 0 \qquad \Rightarrow x \cdot g(x) = \begin{cases} \tan^2 x & 0 < x \le \frac{\pi}{4} \\ \tan^2 x & x = 0 \end{cases}$

$$\Rightarrow \boxed{x \cdot g(x) = \tan^2 x \qquad 0 \le x \le \frac{\pi}{4}}$$

b) $V = 2\pi \displaystyle\int_0^{\pi/4} x \cdot g(x)\,dx \qquad\qquad V = \int_a^b 2\pi \begin{pmatrix} shell \\ radius \end{pmatrix}\begin{pmatrix} shell \\ height \end{pmatrix} dx$

$$= 2\pi \int_0^{\pi/4} \tan^2 x \; dx$$

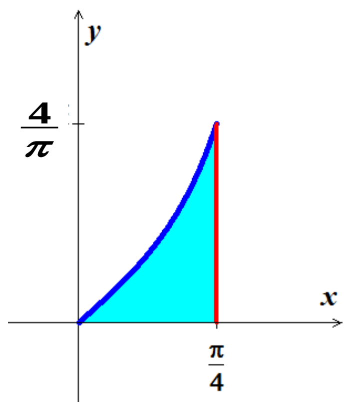
$$= 2\pi\left[\tan x - x\right]_0^{\pi/4}$$

$$= 2\pi\left[\left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) - (\tan 0 - 0)\right]$$

$$= 2\pi\left(1 - \frac{\pi}{4}\right)$$

$$= 2\pi\left(\frac{4 - \pi}{4}\right)$$

$$\boxed{= \frac{4\pi - \pi^2}{2} \quad unit^3}$$

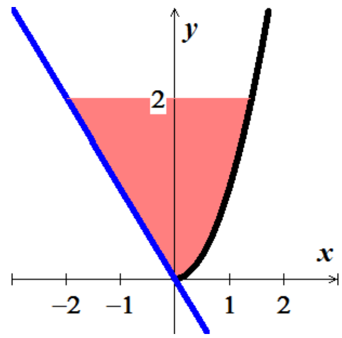


Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $x=\sqrt{y}, \quad x=-y, \quad y=2$ about the *x*-axis.

Solution

$$x=\sqrt{y}=-y \quad \rightarrow y=0=c$$



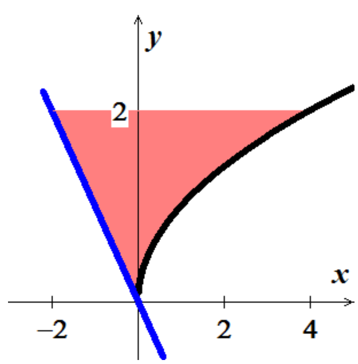
$$\begin{aligned}
V &= \int_0^2 2\pi(y)\left(\sqrt{y}-(-y)\right)dy \\
&= 2\pi \int_0^2 \left(y^{3/2}+y^2\right)dy \\
&= 2\pi\left[\tfrac{2}{5}y^{5/2}+\tfrac{1}{3}y^3\right]_0^2 \\
&= 2\pi\left[\tfrac{2}{5}(2)^{5/2}+\tfrac{1}{3}(2)^3\right] \\
&= 2\pi\left[\frac{8\sqrt{2}}{5}+\frac{8}{3}\right] \\
&= 16\pi\left(\frac{3\sqrt{2}+5}{15}\right) \\
&= \frac{16}{15}\pi\left(3\sqrt{2}+5\right) \ \ unit^3
\end{aligned}$$

Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $x=y^2, \quad x=-y, \quad y=2, \quad y\geq 0$ about the *x*-axis.

Solution

$$x=y^2=-y \quad \rightarrow y=0=c \quad d=2$$



$$\begin{aligned}
V &= \int_0^2 2\pi(y)\left(y^2-(-y)\right)dy \\
&= 2\pi \int_0^2 \left(y^3+y^2\right)dy \\
&= 2\pi\left[\tfrac{1}{4}y^4+\tfrac{1}{3}y^3\right]_0^2 \\
&= 2\pi\left(\tfrac{1}{4}(2)^4+\tfrac{1}{3}(2)^3\right) \\
&= 2\pi\left(4+\tfrac{8}{3}\right)
\end{aligned}$$

$$= 2\pi\left(\frac{20}{3}\right)$$

$$\boxed{= \frac{40\pi}{3}\ unit^3}$$

Exercise

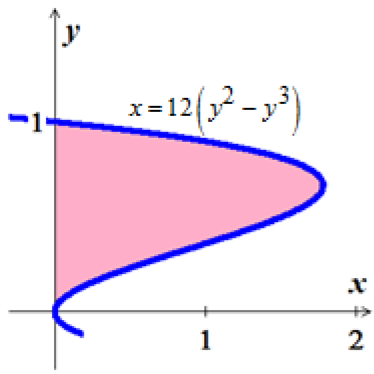
Use the shell method to find the volumes of the solids generated by revolving the shaded regions about the *indicated* axes.

a) The x-axis

b) The line $y = 1$

c) The line $y = \frac{8}{5}$

d) The line $y = -\frac{2}{5}$



Solution

a) $$V = \int_0^1 2\pi(y)\cdot\left[12\left(y^2 - y^3\right)\right]dy$$

$$= 24\pi\int_0^1 \left(y^3 - y^4\right)dy$$

$$= 24\pi\left[\frac{y^4}{4} - \frac{y^5}{5}\right]_0^1$$

$$= 24\pi\left(\frac{1}{4} - \frac{1}{5}\right)$$

$$= 24\pi\left(\frac{1}{20}\right)$$

$$\boxed{= \frac{6\pi}{5}\ unit^3}$$

b) $$V = \int_0^1 2\pi(1-y)\cdot\left[12\left(y^2 - y^3\right)\right]dy$$

$$= 24\pi\int_0^1 \left(y^2 - y^3 - y^3 + y^4\right)dy$$

$$= 24\pi\int_0^1 \left(y^2 - 2y^3 + y^4\right)dy$$

$$= 24\pi\left[\frac{y^3}{3} - \frac{y^4}{2} + \frac{y^5}{5}\right]_0^1$$

$$= 24\pi\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right)$$

$$= 24\pi\left(\frac{1}{30}\right)$$

$$\underline{= \frac{4\pi}{5} \quad unit^3}$$

c) $$V = \int_c^d 2\pi \begin{pmatrix} shell \\ radius \end{pmatrix} \begin{pmatrix} shell \\ height \end{pmatrix} dy$$

$$= 2\pi \int_0^1 \left(\frac{8}{5} - y\right) \cdot \left[12\left(y^2 - y^3\right)\right] dy$$

$$= 24\pi \int_0^1 \left(\frac{8}{5} y^2 - \frac{8}{5} y^3 - y^3 + y^4\right) dy$$

$$= 24\pi \int_0^1 \left(\frac{8}{5} y^2 - \frac{13}{5} y^3 + y^4\right) dy$$

$$= 24\pi \left[\frac{8}{15} y^3 - \frac{13}{20} y^4 + \frac{y^5}{5}\right]_0^1$$

$$= 24\pi\left(\frac{8}{15} - \frac{13}{20} + \frac{1}{5}\right)$$

$$= 24\pi\left(\frac{5}{60}\right)$$

$$\underline{= 2\pi \quad unit^3}$$

d) $$V = \int_0^1 2\pi\left(y + \frac{2}{5}\right) \cdot \left[12\left(y^2 - y^3\right)\right] dy$$

$$= 24\pi \int_0^1 \left(y^3 - y^4 + \frac{2}{5} y^2 - \frac{2}{5} y^3\right) dy$$

$$= 24\pi \int_0^1 \left(\frac{3}{5} y^3 - y^4 + \frac{2}{5} y^2\right) dy$$

$$= 24\pi \left[\frac{3}{20} y^4 - \frac{1}{5} y^4 + \frac{2}{15} y^3\right]_0^1$$

$$= 24\pi\left(\frac{3}{20} - \frac{1}{5} + \frac{2}{15}\right)$$

$$= 24\pi\left(\frac{5}{60}\right)$$

$$\underline{= 2\pi \quad unit^3}$$

Exercise

Compute the volume of the solid generated by revolving the region bounded by the lines $y = x \quad and \quad y = x^2$ about each coordinate axis using

a) The *shell* method

b) The *washer* method

Solution

$$y = x = x^2 \Rightarrow x^2 - x = 0 \to \boxed{x = 0,\ 1}$$

a) ***x-axis***

$$V = \int_0^1 2\pi(y)\cdot\left[\sqrt{y} - y\right]dy \qquad\qquad V = \int_c^d 2\pi\binom{shell}{radius}\binom{shell}{height}dy$$

$$= 2\pi\int_0^1 \left(y^{3/2} - y^2\right)dy$$

$$= 2\pi\left[\frac{2}{5}y^{5/2} - \frac{1}{3}y^3\right]_0^1$$

$$= 2\pi\left(\frac{2}{5} - \frac{1}{3}\right)$$

$$= \frac{2\pi}{15} \ unit^3$$

y-axis

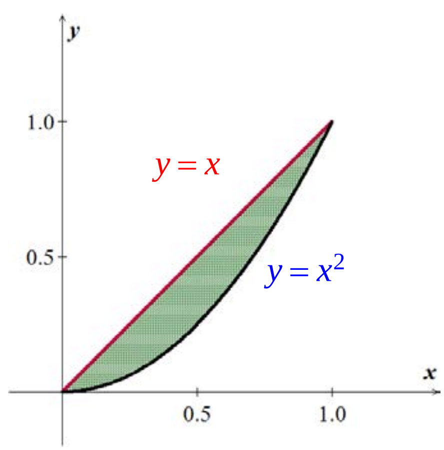
$$V = 2\pi\int_0^1 (x)\left(x - x^2\right)dx \qquad\qquad V = \int_a^b 2\pi\binom{shell}{radius}\binom{shell}{height}dx$$

$$= 2\pi\int_0^1 \left(x^2 - x^3\right)dx$$

$$= 2\pi\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1$$

$$= 2\pi\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= \frac{\pi}{6} \ unit^3$$



b) ***x-axis*** $\quad R(x) = x \quad and \quad r(x) = x^2$

$$V = \int_a^b \pi\left[R(x)^2 - r(x)^2\right]dx$$

$$= \pi\int_0^1 \left(x^2 - x^4\right)dx$$

$$= \pi\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1$$

$$= \pi\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$\boxed{= \frac{2\pi}{15}\ \ unit^3}$$

y-axis $\quad R(y) = \sqrt{y}$ *and* $r(y) = y$

$$V = \int_c^d \pi\left[R(y)^2 - r(y)^2\right]dy$$

$$= \pi\int_0^1 \left(y - y^2\right)dy$$

$$= \pi\left[\frac{1}{2}y^2 - \frac{1}{3}y^3\right]_0^1$$

$$= \pi\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$\boxed{= \frac{\pi}{6}\ \ unit^3}$$

Exercise

Use the *washer* method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = \sqrt{x}, \quad y = 2, \quad x = 0$ about

a) the x-axis

b) the y-axis

c) the line $x = 4$

d) the line $y = 1$

Solution

a) x-axis

$$V = \int_0^2 2\pi(y)\cdot\left(y^2 - 0\right)dy \qquad\qquad V = \int_c^d 2\pi\left(\begin{matrix} shell \\ radius \end{matrix}\right)\left(\begin{matrix} shell \\ height \end{matrix}\right)dy$$

$$= 2\pi\int_0^2 y^3 dy$$

$$= \frac{1}{2}\pi y^4 \Big|_0^2$$

$$= \frac{1}{2}\pi(2)^4$$

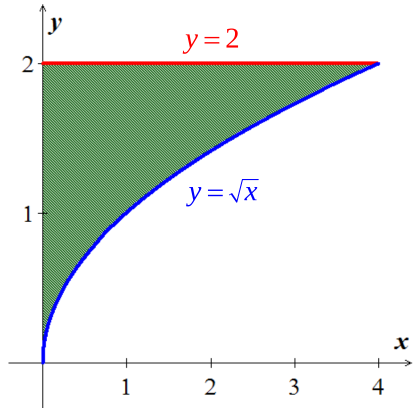
$$\boxed{= 8\pi\ \ unit^3}$$

b) y-axis

$$V = 2\pi \int_0^4 (x)\cdot\left(2-\sqrt{x}\right)dx$$

$$V = \int_a^b 2\pi \left(\begin{matrix} shell \\ radius \end{matrix}\right)\left(\begin{matrix} shell \\ height \end{matrix}\right)dx$$

$$= 2\pi \int_0^4 \left(2x - x^{3/2}\right)dx$$



$$= 2\pi \left[x^2 - \frac{2}{5}x^{5/2} \right]_0^4$$

$$= 2\pi \left(16 - \frac{64}{5}\right)$$

$$= \frac{32\pi}{5} \quad unit^3$$

c) the line $x = 4$

$$V = \int_0^4 2\pi(4-x)\left(2-\sqrt{x}\right)dx$$

$$V = \int_a^b 2\pi \left(\begin{matrix} shell \\ radius \end{matrix}\right)\left(\begin{matrix} shell \\ height \end{matrix}\right)dx$$

$$= 2\pi \int_0^4 \left(8 - 4x^{1/2} - 2x - x^{3/2}\right)dx$$

$$= 2\pi \left[8x - \frac{8}{3}x^{3/2} - x^2 - \frac{2}{5}x^{5/2} \right]_0^4$$

$$= 2\pi \left(32 - \frac{64}{3} - 16 + \frac{64}{5}\right)$$

$$= \frac{224\pi}{15} \quad unit^3$$

d) the line $y = 1$

$$V = 2\pi \int_0^2 (2-y)\left(y^2\right)dy$$

$$V = \int_c^d 2\pi \left(\begin{matrix} shell \\ radius \end{matrix}\right)\left(\begin{matrix} shell \\ height \end{matrix}\right)dy$$

$$= 2\pi \int_0^2 \left(2y^2 - y^3\right)dy$$

$$= 2\pi \left[\frac{2}{3}y^3 - \frac{1}{4}y^4 \right]_0^2$$

$$= 2\pi \left(\frac{16}{3} - \frac{16}{4}\right)$$

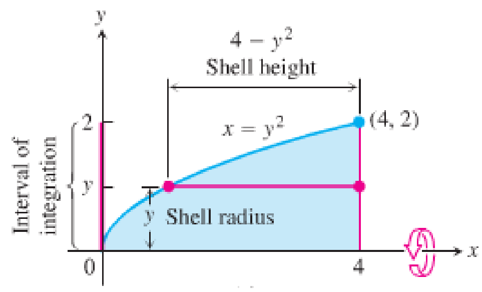
$$= \frac{32\pi}{12}$$

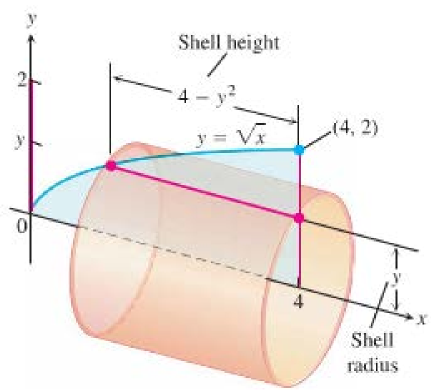
$$= \frac{8\pi}{3} \quad unit^3$$

Exercise

The region bounded by the curve $y=\sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution





$$V=2\pi\int_c^d\left(\begin{array}{c}shell\\ radius\end{array}\right)\left(\begin{array}{c}shell\\ height\end{array}\right)dy$$

$$=2\pi\int_0^2(y)\left(4-y^2\right)dy$$

$$=2\pi\int_0^2\left(4y-y^3\right)dy$$

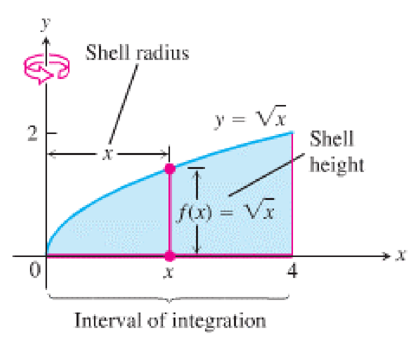
$$=2\pi\left[2y^2-\frac{y^4}{4}\right]_0^2$$

$$=2\pi\left(2(2)^2-\frac{(2)^4}{4}\right)$$

$$\underline{=8\pi\ \ unit^3}$$

Exercise

The region bounded by the curve $y=\sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.



Solution

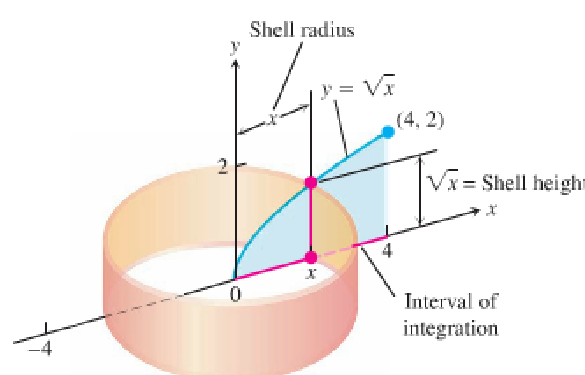
$$V=2\pi\int_0^4(x)\left(\sqrt{x}\right)dx \qquad V=\int_a^b 2\pi\left(\begin{array}{c}shell\\ radius\end{array}\right)\left(\begin{array}{c}shell\\ height\end{array}\right)dx$$

$$=2\pi\int_0^4 x^{3/2}dx$$

$$=2\pi\left[\frac{2}{5}x^{5/2}\right]_0^4$$

$$=\frac{4}{5}\pi\left[4^{5/2}\right]$$

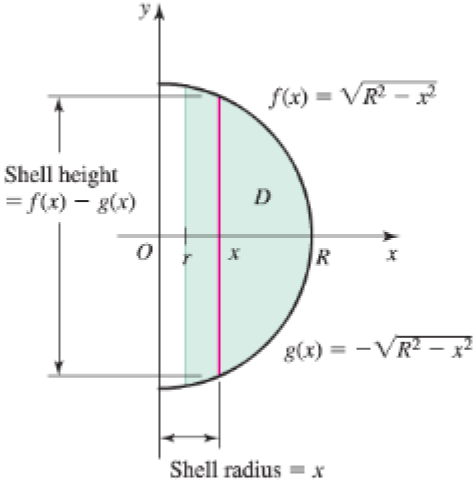
$$\underline{=\frac{128\pi}{5}\ \ unit^3}$$

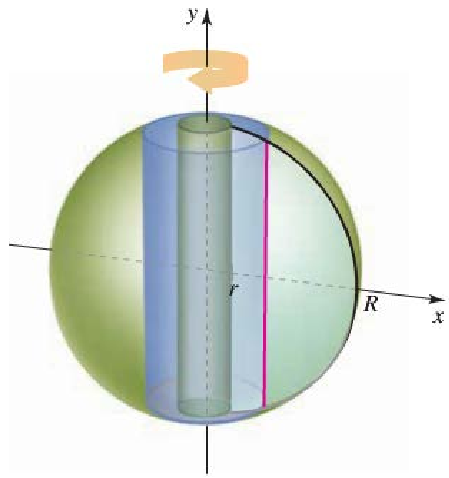


Exercise

A cylinder hole with radius r is drilled symmetrically through the center of a sphere with radius R, where $r \le R$. What is the volume of the remaining material?

Solution





Let D be the region in the xy-plane bounded above by $f(x) = \sqrt{R^2 - x^2}$, the upper half of the circle of radius R, and bounded below by $g(x) = -\sqrt{R^2 - x^2}$, the lower half of the circle of radius R, for $r \le x \le R$.

The radius of a typical shell is x. Height is $f(x) - g(x) = 2\sqrt{R^2 - x^2}$

$$V = 2\pi \int_r^R x\left(2\sqrt{R^2 - x^2}\right) dx$$

$$= -2\pi \int_r^R \left(R^2 - x^2\right)^{1/2} d\left(R^2 - x^2\right)$$

$$= -\frac{4}{3}\pi \left(R^2 - x^2\right)^{3/2} \Big|_r^R$$

$$= \frac{4}{3}\pi \left(R^2 - r^2\right)^{3/2} \quad unit^3$$

Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = \sqrt{x}, \quad y = 2 - x, \quad y = 0$ about the x-axis.

Solution

$x = y^2$

$y = 2 - x^2 = 2 - y^2 \;\Rightarrow\; y^2 + y - 2 = 0 \to y = \cancel{-2}, 1$

Given: $y = 0$

$$V = 2\pi \int_0^1 y\left(2 - y - y^2\right)dy$$

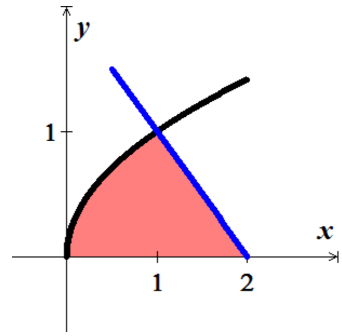
$$= 2\pi \int_0^1 \left(2y - y^2 - y^3\right)dy$$

$$= 2\pi \left(y^2 - \frac{1}{3}y^3 - \frac{1}{4}y^4\right)\Big|_0^1$$

$$= 2\pi \left(1 - \frac{1}{3} - \frac{1}{4}\right)$$

$$= 2\pi \left(\frac{5}{12}\right)$$

$$= \frac{5\pi}{6} \ \ unit^3$$



Exercise

Find the volume of the region bounded by $y = \frac{\ln x}{x^2}$, $y = 0$, $x = 1$, *and* $x = 3$ revolved about the *y-axis*

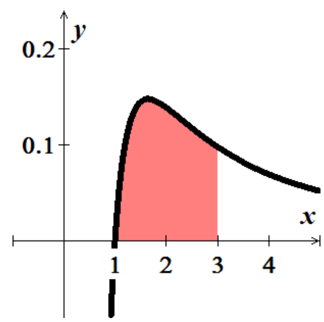
Solution

$$V = 2\pi \int_1^3 x \frac{\ln x}{x^2} dx$$

$$= 2\pi \int_1^3 \ln x \ d(\ln x)$$

$$= \pi (\ln x)^2 \Big|_1^3$$

$$= \pi (\ln 3)^2 \ \ unit^3$$



Exercise

Find the volume of the region bounded by $y = \frac{e^x}{x}$, $y = 0$, $x = 1$, *and* $x = 2$ revolved about the *y-axis*

Solution

$$V = 2\pi \int_1^2 x \frac{e^x}{x} dx$$

$$= 2\pi \int_1^2 e^x dx$$

$$= 2\pi e^x \Big|_1^2$$

$$\boxed{= 2\pi\left(e^2 - e\right) \ \ unit^3}$$

Exercise

Find the volume of the region bounded by $y^2 = \ln x, \quad y^2 = \ln x^3, \quad and \ y = 2$ revolved about the *x-axis*

Solution

$$\begin{cases} y^2 = \ln x & \rightarrow \quad x = e^{y^2} \\ y^2 = \ln x^3 & \rightarrow \quad x = e^{y^2/3} \end{cases}$$

$$V = 2\pi \int_0^2 y\left(e^{y^2} - e^{y^2/3}\right) dy$$

$$= \pi \int_0^2 \left(e^{y^2} - e^{y^2/3}\right) d\left(y^2\right)$$

$$= \pi \left(e^{y^2} - 3e^{y^2/3} \right) \Big|_0^2$$

$$= \pi\left(e^4 - 3e^{4/3} - 1 + 3\right)$$

$$\boxed{= \pi\left(2 + e^4 - 3e^{4/3}\right) \ \ unit^3}$$

Exercise

Find the volume using both the disk/washer and shell methods of

$$y = (x-2)^3 - 2, \quad x = 0, \quad y = 25; \text{ revolved about the } y\text{-}axis$$

Solution

Using ***washers***:

$$(x-2)^3 = y + 2 \quad \rightarrow \quad x = 2 + \sqrt[3]{y+2}$$

$$x = 0 \ \Rightarrow \ y = (-2)^3 - 2 = -10$$

$$V = \pi \int_{-10}^{25} \left(2 + \sqrt[3]{y+2}\right)^2 dy \qquad\qquad V = \pi \int_c^d f(y)^2 \, dy$$

$$= \pi \int_{-10}^{25} \left(4 + 4(y+2)^{1/3} + (y+2)^{2/3}\right) d(y+2)$$

$$= \pi\left(4(y+2)+3(y+2)^{4/3}+\frac{3}{5}(y+2)^{5/3}\right)\Big|_{-10}^{25}$$

$$= \pi\left(108+3(27)^{4/3}+\frac{3}{5}(27)^{5/3}-\left(-32+3(-8)^{4/3}+\frac{3}{5}(-8)^{5/3}\right)\right)$$

$$= \pi\left(108+243+\frac{729}{5}+32-48+\frac{96}{5}\right)$$

$$= \pi(335+165)$$

$$\underline{= 500\pi \ \ unit^3}$$

Using ***Shells***:

$$y = 25 \quad \to \ x = 2+\sqrt[3]{27} = 5$$

$$V = 2\pi\int_0^5 x\left(25-(x-2)^3+2\right)dx \qquad\qquad V = 2\pi\int_a^b x\left(f(x)-g(x)\right)dx$$

$$= 2\pi\int_0^5 x\left(27-x^3+6x^2-12x+8\right)dx$$

$$= 2\pi\int_0^5 \left(-x^4+6x^3-12x^2+35x\right)dx$$

$$= 2\pi\left(-\frac{1}{5}x^5+\frac{3}{2}x^4-4x^3+\frac{35}{2}x^2\right)\Big|_0^5$$

$$= 2\pi\left(-5^4+\frac{3}{2}5^4-4(5)^3+\frac{35}{2}(5)^2\right)$$

$$= 2\pi\left(-625+\frac{1875}{2}-500+\frac{875}{2}\right)$$

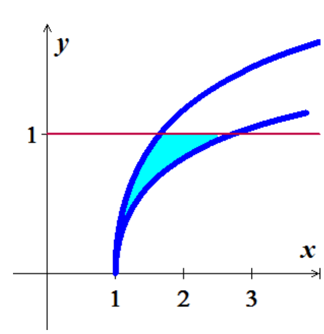
$$= 2\pi(250)$$

$$\underline{= 500\pi \ \ unit^3}$$

Exercise

Find the volume using both the disk/washer and shell methods of $y=\sqrt{\ln x}$, $y=\sqrt{\ln x^2}$, $y=1$; revolved about the x-*axis*

Solution



Using ***washers***:

$$y=\sqrt{\ln x}=\sqrt{\ln x^2} \quad \to \quad \ln x = \ln x^2$$

$$x = x^2 \ \Rightarrow \ \underline{x = 0,\ 1}$$

$$y = 1 = \sqrt{\ln x} \quad \Rightarrow \quad \underline{x = e}$$

$$y = 1 = \sqrt{\ln x^2} \quad \Rightarrow \quad x^2 = e \ \to \ \underline{x = \sqrt{e}}$$

$$V = \pi \int_1^{\sqrt{e}} \left(\ln x^2 - \ln x\right)dx + \pi \int_{\sqrt{e}}^{e} (1 - \ln x)\,dx \qquad V = \pi \int_a^b \left(f(x)^2 - g(x)^2\right)dx$$

$$= \pi \int_1^{\sqrt{e}} (2\ln x - \ln x)\,dx + \pi \int_{\sqrt{e}}^{e} (1 - \ln x)\,dx$$

$$= \pi \int_1^{\sqrt{e}} (\ln x)\,dx + \pi \int_{\sqrt{e}}^{e} (1 - \ln x)\,dx \qquad \int \ln x\,dx = x\ln x - x$$

$$= \pi (x\ln x - x)\Big|_1^{\sqrt{e}} + \pi(2x - x\ln x)\Big|_{\sqrt{e}}^{e}$$

$$= \pi\left(\tfrac{1}{2}\sqrt{e} - \sqrt{e} + 1\right) + \pi\left(2e - e - 2\sqrt{e} + \tfrac{1}{2}\sqrt{e}\right)$$

$$= \pi\left(-\tfrac{1}{2}\sqrt{e} + 1 + e - 2\sqrt{e} + \tfrac{1}{2}\sqrt{e}\right)$$

$$= \pi\left(e - 2\sqrt{e} + 1\right)$$

$$\underline{= \pi\left(\sqrt{e} - 1\right)^2 \quad unit^3}$$

Using ***Shells***:

$$y = \sqrt{\ln x} \quad \Rightarrow \quad \underline{x = e^{y^2}}$$

$$y = \sqrt{\ln x^2} \quad \Rightarrow \quad 2\ln x = y^2 \quad \to \quad \underline{x = e^{y^2/2}}$$

$$V = 2\pi \int_0^1 y\left(e^{y^2} - e^{y^2/2}\right)dy \qquad V = 2\pi \int_c^d y\left(p(y) - q(y)\right)dy$$

$$= \pi \int_0^1 e^{y^2} d\left(y^2\right) - 2\pi \int_0^1 e^{y^2/2} d\left(\tfrac{1}{2}y^2\right)$$

$$= \pi\left(e^{y^2} - 2e^{y^2/2}\right)_0^1$$

$$= \pi\left(e - 2e^{1/2} - 1 + 2\right)$$

$$= \pi\left(e - 2\sqrt{e} + 1\right)$$

$$\underline{= \pi\left(\sqrt{e} - 1\right)^2 \quad unit^3}$$

Exercise

Find the volume using both the disk/washer and shell methods of $y=\dfrac{6}{x+3}$, $\quad y=2-x$; revolved about the *x-axis*

Solution

Using ***washers***:

$$y=\frac{6}{x+3}=2-x$$

$$-x^2-x+6=6$$

$$x(x+1)=0 \;\Rightarrow\; \underline{x=-1,\ 0}$$

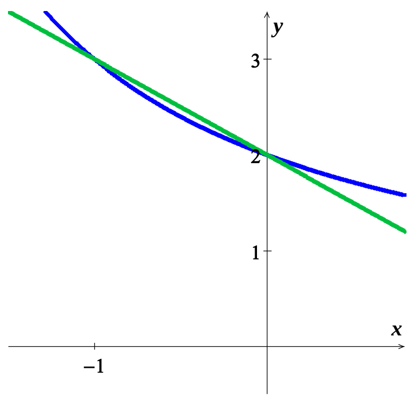
$$V=\pi\int_{-1}^{0}\left((2-x)^2-\frac{36}{(x+3)^2}\right)dx \qquad\qquad V=\pi\int_a^b\left(f(x)^2-g(x)^2\right)dx$$

$$=\pi\int_{-1}^{0}-(2-x)^2\,d(2-x)-\pi\int_{-1}^{0}\frac{36}{(x+3)^2}\,d(x+3)$$

$$=\pi\left(-\frac{1}{3}(2-x)^3+\frac{36}{x+3}\right)_{-1}^{0}$$

$$=\pi\left(-\frac{8}{3}+12+9-18\right)$$

$$=\frac{\pi}{3}\ unit^3$$



Using ***Shells***:

$$y=\frac{6}{x+3}\ \rightarrow\ x=\frac{6}{y}-3$$

$$y=2-x\ \rightarrow\ x=2-y$$

$$V=2\pi\int_2^3 y\left(2-y-\frac{6}{y}+3\right)dy \qquad\qquad V=2\pi\int_c^d y\left(p(y)-q(y)\right)dy$$

$$=2\pi\int_2^3\left(5y-y^2-6\right)dy$$

$$=2\pi\left(\frac{5}{2}y^2-\frac{1}{3}y^3-6y\right)_2^3$$

$$=2\pi\left(\frac{45}{2}-9-18-10+\frac{8}{3}+12\right)$$

$$=2\pi\left(\frac{151}{6}-25\right)$$

$$=\frac{\pi}{3}\ unit^3$$

Exercise

Use the shell method to find the volume of the solid generated by the revolving the plane region about the given line

$$y = 2x - x^2, \quad y = 0, \quad \textit{about the line} \quad x = 4$$

Solution

$y = 2x - x^2 = 0 \quad \underline{x = 0,\ 2}$

$p(x) = 4 - x, \quad f(x) = 2x - x^2$

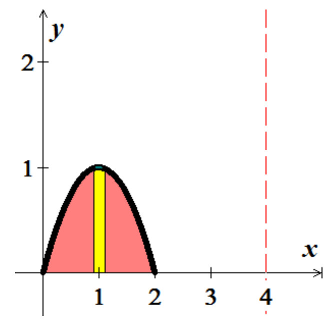
$$V = 2\pi \int_0^2 (4-x)\left(2x - x^2\right) dx \qquad\qquad V = 2\pi \int_a^b p(x)\, f(x)\, dx$$

$$= 2\pi \int_0^2 \left(8x - 6x^2 + x^3\right) dx$$

$$= 2\pi \left(4x^2 - 2x^3 + \frac{1}{4}x^4\right)\Big|_0^2$$

$$= 2\pi (16 - 16 + 4)$$

$$\underline{= 8\pi}$$



Exercise

Use the shell method to find the volume of the solid generated by the revolving the plane region about the given line

$$y = \sqrt{x}, \quad y = 0, \quad x = 4, \quad \textit{about the line} \quad x = 6$$

Solution

$y = \sqrt{x} = 0 \quad \underline{x = 0}$

$p(x) = 6 - x, \quad f(x) = \sqrt{x}$

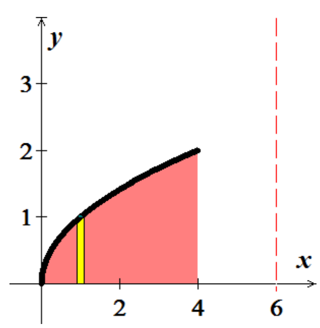
$$V = 2\pi \int_0^4 (6-x)\left(\sqrt{x}\right) dx \qquad\qquad V = 2\pi \int_a^b p(x)\, f(x)\, dx$$

$$= 2\pi \int_0^4 \left(6x^{1/2} - x^{3/2}\right) dx$$

$$= 2\pi \left(4x^{3/2} - \frac{2}{5}x^{5/2}\right)\Big|_0^4$$

$$= 2\pi \left(32 - \frac{64}{5}\right)$$

$$\underline{= \frac{192\pi}{5}}$$



Exercise

Use the shell method to find the volume of the solid generated by the revolving the plane region about the given line

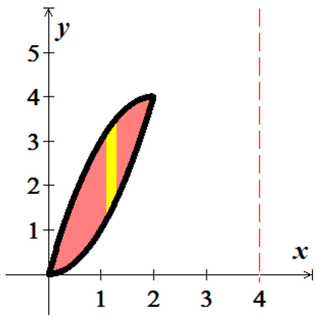
$$y = x^2, \quad y = 4x - x^2, \quad \text{about the line} \quad x = 4$$

Solution

$$y = x^2 = 4x - x^2 \Rightarrow 2x^2 - 4x = 0 \quad \underline{x = 0,\ 2|}$$

$$p(x) = 4 - x, \quad f(x) = 4x - x^2, \quad g(x) = x^2$$

$$V = 2\pi \int_0^2 (4 - x)\left(4x - x^2 - x^2\right) dx \qquad\qquad V = 2\pi \int_a^b p(x)\left(f(x) - g(x)\right) dx$$



$$= 2\pi \int_0^2 (4 - x)\left(4x - 2x^2\right) dx$$

$$= 2\pi \int_0^2 \left(16x - 12x^2 + 2x^3\right) dx$$

$$= 2\pi \left(8x^2 - 4x^3 + \frac{1}{2}x^4\right)\Big|_0^2$$

$$= 2\pi(32 - 32 + 8)$$

$$\underline{= 16\pi|}$$

Exercise

Use the shell method to find the volume of the solid generated by the revolving the plane region about the given line

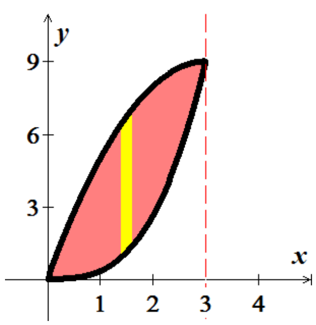
$$y = \frac{1}{3}x^3, \quad y = 6x - x^2, \quad \text{about the line} \quad x = 3$$

Solution

$$y = \frac{1}{3}x^3 = 6x - x^2 \Rightarrow x\left(x^2 - 3x + 18\right) = 0 \quad \underline{x = 0,\ 3,\ \cancel{-6}|}$$

$$p(x) = 3 - x, \quad f(x) = 6x - x^2, \quad g(x) = \frac{1}{3}x^3$$

$$V = 2\pi \int_0^3 (3 - x)\left(3x - x^2 - \frac{1}{3}x^3\right) dx \qquad V = 2\pi \int_a^b p(x)\left(f(x) - g(x)\right) dx$$



$$= 2\pi \int_0^3 \left(18x - 9x^2 + \frac{1}{3}x^4\right) dx$$

$$= 2\pi \left(9x^2 - 3x^3 + \frac{1}{15}x^5\right)\Big|_0^3$$

$$= 2\pi\left(81 - 81 + \frac{81}{5}\right)$$

$$\underline{= \frac{162\pi}{5}|}$$

Exercise

Use the disk method or shell method to find the volume of the solid generated vy revolving the region bounded by the graph of the equations about the given lines.

$y = x^3, \quad y = 0, \quad x = 2$

a) *the x-axis* *b)* *the y-axis* *c)* *the line* $x = 4$

Solution

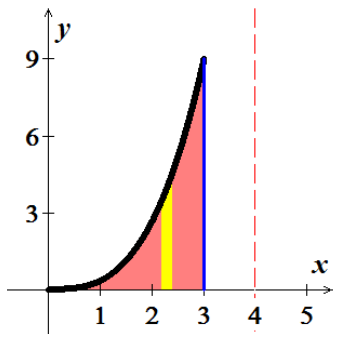
a) Using ***Disk method***:

$f(x) = x^3, \quad g(x) = 0$

$$V = \pi \int_0^2 x^6 dx \qquad V = \pi \int_a^b \left(\left(f(x) \right)^2 - \left(g(x) \right)^2 \right) dx$$

$$= \frac{\pi}{7} x^7 \Big|_0^2$$

$$= \frac{128\pi}{7}$$



b) Using ***Shell method***:

$p(x) = x, \quad f(x) = x^3, \quad g(x) = 0$

$$V = 2\pi \int_0^2 x\left(x^3\right) dx \qquad V = 2\pi \int_a^b p(x)\left(f(x) - g(x) \right) dx$$

$$= 2\pi \int_0^2 x^4 \, dx$$

$$= \frac{2\pi}{5} x^5 \Big|_0^2$$

$$= \frac{164\pi}{5}$$

c) Using ***Shell method***:

$p(x) = 4 - x, \quad f(x) = x^3, \quad g(x) = 0$

$$V = 2\pi \int_0^2 (4 - x)\left(x^3\right) dx \qquad V = 2\pi \int_a^b p(x)\left(f(x) - g(x) \right) dx$$

$$= 2\pi \int_0^2 \left(4x^3 - x^4\right) dx$$

$$= 2\pi \left(x^4 - \frac{1}{5} x^5 \right) \Big|_0^2$$

$$= 2\pi \left(16 - \frac{32}{5} \right)$$

$$= \frac{96\pi}{5}$$

Exercise

Use the disk method or shell method to find the volume of the solid generated vy revolving the region bounded by the graph of the equations about the given lines.

$y = \dfrac{10}{x^2}, \quad y = 0, \quad x = 1, \quad x = 5$

a) *the x-axis*   *b)* *the y-axis*   *c)* *the line* $y = 10$

Solution

a) Using ***Disk method***:

$R(x) = \dfrac{10}{x^2}, \quad r(x) = 0$

$$V = \pi \int_1^5 100x^{-4}\, dx \qquad V = \pi \int_a^b \left((R(x))^2 - (r(x))^2 \right) dx$$

$$= -\frac{100}{3} \pi x^{-3} \Big|_1^5$$

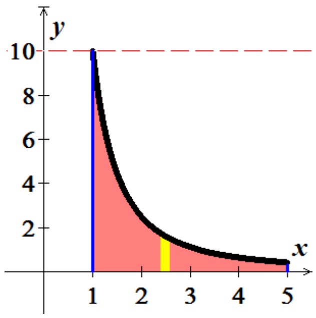
$$= -\frac{100}{3} \pi \left(\frac{1}{125} - 1 \right)$$

$$= \frac{496\pi}{15}$$

b) Using ***Shell method***:

$p(x) = x, \quad f(x) = \dfrac{10}{x^2}, \quad g(x) = 0$

$$V = 2\pi \int_1^5 x \left(\frac{10}{x^2} \right) dx \qquad V = 2\pi \int_a^b p(x)\left(f(x) - g(x) \right) dx$$



$$= 20\pi \int_1^5 \frac{1}{x}\, dx$$

$$= 20\pi \ln x \Big|_1^5$$

$$= 20\pi \ln 5$$

c) Using ***Disk method***:

$R(x) = 10, \quad r(x) = 10 - \dfrac{10}{x^2}$

$$V = \pi \int_1^5 \left(100 - \left(10 - 10x^{-2} \right)^2 \right) dx \qquad V = \pi \int_a^b \left((R(x))^2 - (r(x))^2 \right) dx$$

$$= \pi \int_1^5 \left(200x^{-2} - 100x^{-4} \right) dx$$

$$= 100\pi \left(-\frac{2}{x} + \frac{1}{3x^3} \right) \Big|_1^5$$

$$= 100\pi\left(-\frac{2}{5}+\frac{1}{375}+2-\frac{1}{3}\right)$$

$$= 100\pi\left(2-\frac{274}{375}\right)$$

$$= 100\pi\left(\frac{476}{375}\right)$$

$$\boxed{= \frac{1904\pi}{15}}$$

Exercise

Let V_1 and V_2 be the volumes of the solids that result when the plane region bounded by $y=\frac{1}{x}$, $y=0$, $x=\frac{1}{4}$, and $x=c$ $\left(where\ \ c>\frac{1}{4}\right)$ is revolved about the $x\text{-}axis$ and the $y\text{-}axis$, respectively. Find the value of c for which $V_1=V_2$

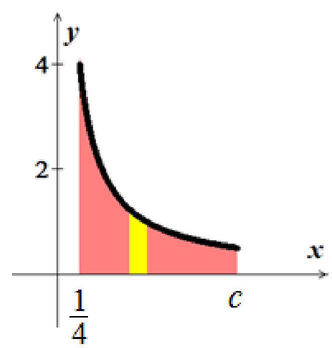
Solution

$$V_1=\pi\int_{1/4}^{c}\frac{1}{x^2}dx \qquad\qquad V=\pi\int_a^b\left(R(x)^2-r(x)^2\right)dx$$

$$= -\pi\frac{1}{x}\Big|_{1/4}^{c}$$

$$= -\pi\left(\frac{1}{c}-4\right)$$

$$\boxed{= \frac{4c-1}{c}\pi}$$



$$V_2=2\pi\int_{1/4}^{c}x\frac{1}{x}dx$$

$$= 2\pi x\Big|_{1/4}^{c}$$

$$\boxed{= 2\pi\left(c-\frac{1}{4}\right)}$$

Since $V_1=V_2$

$$\frac{4c-1}{c}\pi=2\pi\left(c-\frac{1}{4}\right)$$

$$4c-1=2c^2-\frac{1}{2}c$$

$$2c^2-\frac{9}{2}c+1=0$$

$$4c^2-9c+2=0 \quad\rightarrow\quad \boxed{c=2},\ \cancel{\frac{1}{4}} \qquad \left(\frac{1}{4}\text{ has no volume}\right)$$

Exercise

The region bounded by $y = r^2 - x^2$, $y = 0$, and $x = 0$ is revolved about the *y-axis* to form a paraboloid. A hole, centered along the axis of revolution, is drilled through this solid. The hole has a radius k, $0 < k < r$. Find the volume of the resulting ring

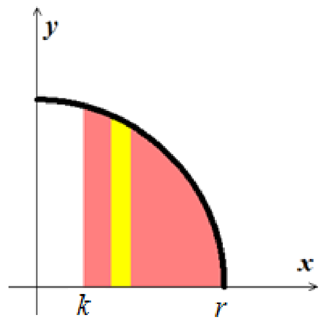
a) By integrating with respect to x

b) By integrating with respect to y.

Solution

a) $f(x) = r^2 - x^2, \quad g(x) = 0$

$$V = 2\pi \int_k^r x\left(r^2 - x^2\right)dx \qquad V = 2\pi \int_a^b x\left(f(x) - g(x)\right)dx \quad (\textbf{\textit{Shell Method}})$$



$$= 2\pi \int_k^r \left(r^2 x - x^3\right)dx$$

$$= 2\pi \left(\frac{1}{2}r^2x^2 - \frac{1}{4}x^4\right)\Big|_k^r$$

$$= \frac{1}{2}\pi\left(2r^4 - r^4 - 2r^2k^2 + k^4\right)$$

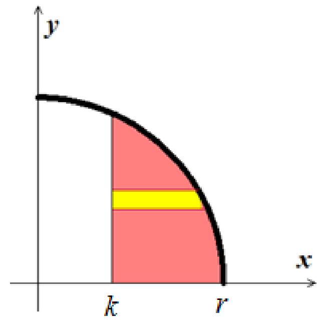
$$= \frac{1}{2}\pi\left(r^4 - 2r^2k^2 + k^4\right)$$

$$= \frac{1}{2}\pi\left(r^2 - k^2\right)^2$$

b) $y = r^2 - x^2 \quad \rightarrow \quad x = \sqrt{r^2 - y}$

$R(y) = \sqrt{r^2 - y}, \quad r(y) = k$

$$V = \pi \int_0^{r^2 - k} \left(r^2 - y - k^2\right)dy \qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2\right)dy$$



$$= \pi\left(\left(r^2 - k^2\right)y - \frac{1}{2}y^2\right)\Big|_0^{r^2 - k}$$

$$= \pi\left(\left(r^2 - k^2\right)^2 - \frac{1}{2}\left(r^2 - k^2\right)^2\right)$$

$$= \frac{1}{2}\pi\left(r^2 - k^2\right)^2$$

Solution Section 1.5 – Length of Curves

Exercise

Find the length of the curve $y = \frac{1}{3}\left(x^2+2\right)^{3/2}$ from $x = 0$ to $x = 3$.

Solution

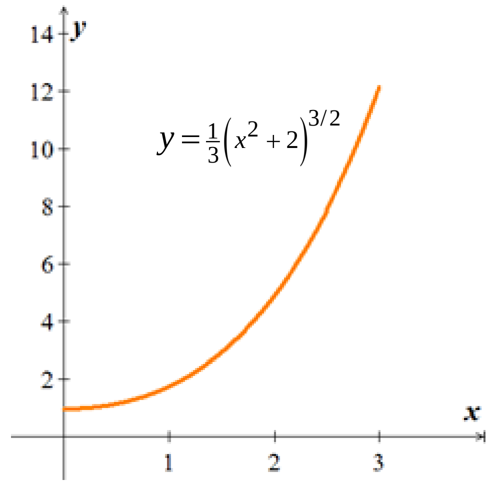
$$\frac{dy}{dx} = \frac{1}{3}\frac{3}{2}\left(x^2+2\right)^{1/2}(2x) = x\left(x^2+2\right)^{1/2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+x^2\left(x^2+2\right)}$$

$$= \sqrt{1+x^4+2x^2}$$

$$= \sqrt{\left(x^2+1\right)^2}$$

$$= x^2+1$$



$$L = \int_0^3 \left(x^2+1\right)\,dx$$

$$= \left[\frac{1}{3}x^3 + x\right]_0^3$$

$$= \frac{1}{3}(3)^3 + (3) - 0$$

$$= 12 \ \ unit$$

Exercise

Find the length of the curve $y = (x)^{3/2}$ from $x = 0$ to $x = 4$.

Solution

$$\frac{dy}{dx} = \frac{3}{2}x^{1/2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{9}{4}x}$$

$$= \sqrt{\frac{4+9x}{4}}$$

$$= \frac{1}{2}\sqrt{4+9x}$$

$$L = \int_0^4 \frac{1}{2}(4+9x)^{1/2}\,dx \qquad u = 4+9x \ \Rightarrow du = 9dx \ \rightarrow \frac{1}{9}du = dx \quad \begin{cases} x=4 & \rightarrow u = 40 \\ x = 0 & \rightarrow u = 4 \end{cases}$$

$$= \frac{1}{2}\int_4^{40} \frac{1}{9}u^{1/2}du$$

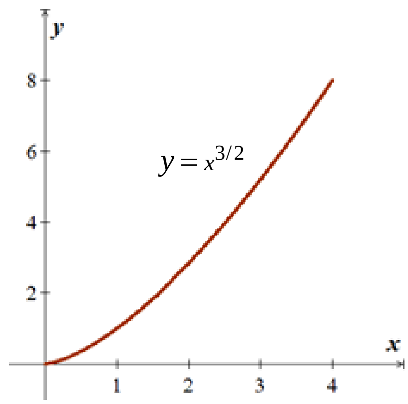
$$= \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_4^{40}$$

$$= \frac{1}{27}\left(40^{3/2} - 4^{3/2}\right)$$

$$= \frac{1}{27}\left(\sqrt{40^3} - \sqrt{4^3}\right)$$

$$= \frac{1}{27}\left(80\sqrt{10} - 8\right)$$

$$= \frac{8}{27}\left(10\sqrt{10} - 1\right) \ \ unit$$



Exercise

Find the length of the curve $x = \dfrac{y^{3/2}}{3} - y^{1/2}$ from $y = 1$ to $y = 9$.

Solution

$$\frac{dx}{dy} = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2} = \frac{1}{2}\left(y^{1/2} - \frac{1}{y^{1/2}}\right)$$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+\frac{1}{4}\left(y^{1/2} - \frac{1}{y^{1/2}}\right)^2}$$

$$= \sqrt{1+\frac{1}{4}\left(y - 2 + \frac{1}{y}\right)}$$

$$= \sqrt{1+\frac{1}{4}y - \frac{1}{2} + \frac{1}{4y}}$$

$$= \sqrt{\frac{1}{4}y + \frac{1}{2} + \frac{1}{4y}}$$

$$= \sqrt{\frac{1}{4}\left(y + 2 + \frac{1}{y}\right)}$$

$$= \frac{1}{2}\sqrt{\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)^2}$$

$$= \frac{1}{2}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)$$

$$L = \frac{1}{2}\int_1^9 \left(y^{1/2} + y^{-1/2}\right)dy$$

$a = \frac{1}{3}, \quad m = \frac{3}{2}, \quad b = -1, \quad n = \frac{1}{2}$

1. $m + n = \frac{3}{2} + \frac{1}{2} = 2$ ✓
2. $abmn = \frac{1}{3}(-1)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$ ✓

$$L = \left(\frac{1}{3}y^{3/2} + y^{1/2}\right)_1^9$$

$$= 9 + 3 - \frac{4}{3}$$

$$= \frac{32}{3} \ \ unit$$

$$=\frac{1}{2}\left[\frac{2}{3}y^{3/2}+2y^{1/2}\right]_1^9$$

$$=\left[\frac{1}{3}y^{3/2}+y^{1/2}\right]_1^9$$

$$=\left[\frac{1}{3}9^{3/2}+9^{1/2}-\left(\frac{1}{3}1^{3/2}+1^{1/2}\right)\right]$$

$$=\frac{1}{3}3^3+3-\left(\frac{1}{3}+1\right)$$

$$=9+3-\frac{4}{3}$$

$$=\frac{32}{3}\ \text{unit}$$

Exercise

Find the length of the curve $x=\frac{y^3}{6}+\frac{1}{2y}$ from $y = 2$ to $y = 3$.

Solution

$$\frac{dx}{dy}=\frac{1}{2}y^2-\frac{1}{2y^2}=\frac{1}{2}\left(y^2-y^{-2}\right)$$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2}=\sqrt{1+\frac{1}{4}\left(y^2-y^{-2}\right)^2}$$

$$=\frac{1}{2}\sqrt{4+\left(y^4-2+y^{-4}\right)}$$

$$=\frac{1}{2}\sqrt{y^4+2+y^{-4}}$$

$$=\frac{1}{2}\sqrt{\left(y^2+y^{-2}\right)^2}$$

$$=\frac{1}{2}\left(y^2+y^{-2}\right)$$

$$L=\frac{1}{2}\int_2^3\left(y^2+y^{-2}\right)dy$$

$$=\frac{1}{2}\left[\frac{1}{3}y^3-y^{-1}\right]_2^3$$

$$=\frac{1}{2}\left[\left(\frac{1}{3}3^3-3^{-1}\right)-\left(\frac{1}{3}2^3-2^{-1}\right)\right]$$

$$=\frac{1}{2}\left[9-\frac{1}{3}-\left(\frac{8}{3}-\frac{1}{2}\right)\right]$$

$$=\frac{1}{2}\left(\frac{26}{3}-\frac{13}{6}\right)$$

$$=\frac{13}{4}\ \text{unit}$$

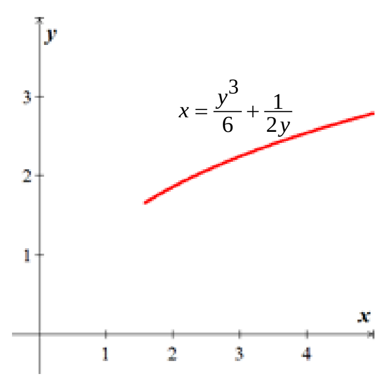
$a=\frac{1}{6},\quad m=3,\quad b=\frac{1}{2},\quad n=-1$

1. $m+n=3-1=2$ ✓
2. $abmn=\frac{1}{6}\left(\frac{1}{2}\right)(3)(-1)=-\frac{1}{4}$ ✓

$$L=\left(\frac{y^3}{6}-\frac{1}{2y}\right)_1^9$$

$$=\frac{1}{2}\left[9-\frac{1}{3}-\left(\frac{8}{3}-\frac{1}{2}\right)\right]$$

$$=\frac{13}{4}\ \text{unit}$$



Exercise

Find the length of the curve $f(x) = x^3 + \frac{1}{12x}$ *for* $\frac{1}{2} \le x \le 2$

Solution

$$a = 1, \quad m = 3, \quad b = \frac{1}{12}, \quad n = -1$$

1. $m + n = 2$ ✓ **2.** $abmn = -\frac{1}{4}$ ✓

$$L = \left(x^3 - \frac{1}{12x}\right)\Big|_{1/2}^{2}$$

$$= 8 - \frac{1}{24} - \frac{1}{8} + \frac{1}{6}$$

$$= 8 \ \ unit$$

Exercise

Find the length of the curve of $f(x) = \frac{1}{5}x^5 + \frac{1}{12x^3}$ $1 \le x \le 2$

Solution

$$a = \frac{1}{5}, \quad m = 5, \quad b = \frac{1}{12}, \quad n = -3$$

1. $m + n = 5 - 3 = 2$ ✓ **2.** $abmn = \frac{1}{5}\left(\frac{1}{12}\right)(5)(-3) = -\frac{1}{4}$ ✓

$$L = \frac{1}{5}x^5 - \frac{1}{12x^3}\Big|_{1}^{2}$$

$$= \frac{32}{5} - \frac{1}{96} - \frac{1}{5} + \frac{1}{12}$$

$$= \frac{31}{5} + \frac{7}{96}$$

$$= \frac{3011}{480}$$

Exercise

Find the length of the curve of $y = \frac{1}{3}x^{1/2} - x^{3/2}, \quad 0 \le x \le \frac{1}{3}$

Solution

$$a = \frac{1}{3}, \quad m = \frac{1}{2}, \quad b = -1, \quad n = \frac{3}{2}$$

3. $m + n = \frac{1}{2} + \frac{3}{2} = 2$ ✓ **4.** $abmn = \frac{1}{3}(-1)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right) = -\frac{1}{4}$ ✓

$$L = \frac{1}{3}x^{1/2} + x^{3/2}\Big|_{0}^{1/3}$$

$$= \frac{1}{3\sqrt{3}} + \frac{1}{3\sqrt{3}}$$

$$= \frac{2}{3\sqrt{3}} \qquad\qquad = \frac{2\sqrt{3}}{9}$$

Exercise

Find the length of the curve of $\quad y = \frac{1}{3}x^3 + \frac{1}{4x}, \quad 1 \le x \le 2$

Solution

$a = \frac{1}{3}, \quad m = 3, \quad b = \frac{1}{4}, \quad n = -1$

5. $m + n = 3 - 1 = 2$ ✓ **6.** $abmn = \frac{1}{3}\left(\frac{1}{4}\right)(3)(-1) = -\frac{1}{4}$ ✓

$$L = \frac{1}{3}x^3 - \frac{1}{4x}\Big|_1^2$$

$$= \frac{8}{3} - \frac{1}{8} - \frac{1}{3} + \frac{1}{4} \qquad = \frac{7}{3} + \frac{1}{8}$$

$$= \frac{59}{24}$$

Exercise

Find the length of the curve of $\quad y = 2e^x + \frac{1}{8}e^{-x} \quad 0 \le x \le \ln 2$

Solution

$a = 2, \quad m = 1, \quad b = \frac{1}{8}, \quad n = -1$

7. $m = -n = 1$ ✓ **8.** $abmn = 2\left(\frac{1}{8}\right)(1)(-1) = -\frac{1}{4}$ ✓

$$L = 2e^x - \frac{1}{8}e^{-x}\Big|_0^{\ln 2}$$

$$= 2e^{\ln 2} - \frac{1}{8}e^{-\ln 2} - 2 + \frac{1}{8}$$

$$= 4 - \frac{1}{16} - \frac{15}{8}$$

$$= \frac{33}{16}$$

Exercise

Find the length of the curve of $\quad y = e^{2x} + \frac{1}{16}e^{-2x}, \quad 0 \le x \le \ln 3$

Solution

$a = 1, \quad m = 2, \quad b = \frac{1}{16}, \quad n = -2$

9. $m=-n=2$ ✓ **10.** $abmn=1\left(\frac{1}{16}\right)(2)(-2)=-\frac{1}{4}$ ✓

$$L=e^{2x}-\frac{1}{16}e^{-2x}\Big|_0^{\ln 3}$$

$$=e^{2\ln 3}-\frac{1}{16}e^{-2\ln 3}-1+\frac{1}{16}$$

$$=9-\frac{1}{16}\left(\frac{1}{9}\right)-\frac{15}{16}$$

$$=\frac{1,160}{144}$$

$$=\frac{145}{18}$$

Exercise

Find the length of the curve $y=\ln(\cos x)\quad 0\le x\le\frac{\pi}{4}$

Solution

$$\frac{dy}{dx}=\frac{-\sin x}{\cos x}=-\tan x$$

$$L=\int_0^{\pi/4}\sqrt{1+\tan^2 x}\;dx$$

$$=\int_0^{\pi/4}\sqrt{\sec^2 x}\;dx$$

$$=\int_0^{\pi/4}\sec x\;dx$$

$$=\left[\ln\left|\sec x+\tan x\right|\right]_0^{\pi/4}$$

$$=\ln\left|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right|-\ln\left|\sec 0+\tan 0\right|$$

$$=\ln\left|\sqrt{2}+1\right|-\ln\left|1+0\right|$$

$$=\ln\left|\sqrt{2}+1\right|-0$$

$$=\ln\left(\sqrt{2}+1\right)\ \textit{unit}$$

Exercise

Find the length of the curve $f(y)=2e^{\sqrt{2}y}+\frac{1}{16}e^{-\sqrt{2}y}\quad \textit{for}\quad 0\le y\le\frac{\ln 2}{\sqrt{2}}$

Solution

$$a=2,\quad m=\sqrt{2},\quad b=\frac{1}{16},\quad n=-\sqrt{2}$$

1. $m = -n$ ✓ 2. $abmn = 2\left(\sqrt{2}\right)\left(\frac{1}{16}\right)\left(-\sqrt{2}\right) = -\frac{1}{4}$ ✓

$$L = \left(2e^{\sqrt{2}y} + \frac{1}{16}e^{-\sqrt{2}y}\right)\Bigg|_0^{\ln 2/\sqrt{2}}$$

$$= 2e^{\ln 2} + \frac{1}{16}e^{-\ln 2} - 2 - \frac{1}{16}$$

$$= 4 + \frac{1}{32} - \frac{33}{16}$$

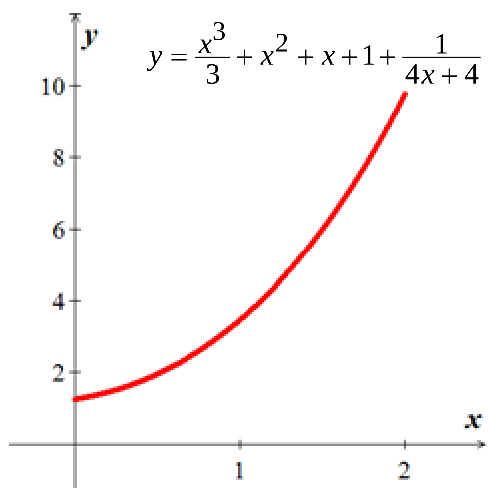
$$= \frac{63}{32} \ \ unit$$

Exercise

Find the length of the curve $y = \frac{x^3}{3} + x^2 + x + 1 + \frac{1}{4x+4} \quad 0 \le x \le 2$

Solution

$$\frac{dy}{dx} = x^2 + 2x + 1 - \frac{1}{4}\frac{1}{(x+1)^2} = (x+1)^2 - \frac{1}{4}\frac{1}{(x+1)^2}$$



$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left((x+1)^2 - \frac{1}{4}\frac{1}{(x+1)^2}\right)^2}$$

$$= \sqrt{1 + (x+1)^4 - \frac{1}{2} - \frac{1}{16}\frac{1}{(x+1)^4}}$$

$$= \sqrt{(x+1)^4 + \frac{1}{2} - \frac{1}{16}\frac{1}{(x+1)^4}}$$

$$= \sqrt{\left((x+1)^2 + \frac{1}{4}\frac{1}{(x+1)^2}\right)^2}$$

$$= (x+1)^2 + \frac{1}{4}(x+1)^{-2}$$

$$L = \int_0^2 \left((x+1)^2 + \frac{1}{4}(x+1)^{-2}\right)dx \qquad u = x+1 \ \Rightarrow du = dx \quad \begin{cases} x = 2 & \to u = 3 \\ x = 0 & \to u = 1 \end{cases}$$

$$= \int_1^3 \left(u^2 + \frac{1}{4}u^{-2}\right)du$$

$$= \left[\frac{1}{3}u^3 - \frac{1}{4}u^{-1}\right]_1^3$$

$$= 9 - \frac{1}{12} - \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= \frac{53}{6} \ \ unit$$

Exercise

Find the length of the curve $\quad y=\ln\left(e^x-1\right)-\ln\left(e^x+1\right) \quad \ln 2 \le x \le \ln 3$

Solution

$$y=\ln\left(e^x-1\right)-\ln\left(e^x+1\right) \quad \Rightarrow \frac{dy}{dx}=\frac{e^x}{e^x-1}-\frac{e^x}{e^x+1}$$

$$=\frac{e^{2x}+e^x-e^{2x}+e^x}{e^{2x}-1}$$

$$=\frac{2e^x}{e^{2x}-1}$$

$$L=\int_{\ln 2}^{\ln 3}\sqrt{1+\left(\frac{2e^x}{e^{2x}-1}\right)^2}\,dx$$

$$=\int_{\ln 2}^{\ln 3}\sqrt{1+\frac{4e^{2x}}{e^{4x}-2e^{2x}+1}}\,dx$$

$$=\int_{\ln 2}^{\ln 3}\sqrt{\frac{e^{4x}-2e^{2x}+1+4e^{2x}}{\left(e^{2x}-1\right)^2}}\,dx$$

$$=\int_{\ln 2}^{\ln 3}\sqrt{\frac{e^{4x}+2e^{2x}+1}{\left(e^{2x}-1\right)^2}}\,dx$$

$$=\int_{\ln 2}^{\ln 3}\sqrt{\frac{\left(e^{2x}+1\right)^2}{\left(e^{2x}-1\right)^2}}\,dx$$

$$=\int_{\ln 2}^{\ln 3}\frac{e^{2x}+1}{e^{2x}-1}\,dx$$

$$=\int_{\ln 2}^{\ln 3}\frac{\frac{e^{2x}+1}{e^x}}{\frac{e^{2x}-1}{e^x}}\,dx$$

$$=\int_{\ln 2}^{\ln 3}\frac{\frac{e^{2x}}{e^x}+\frac{1}{e^x}}{\frac{e^{2x}}{e^x}-\frac{1}{e^x}}\,dx$$

$$=\int_{\ln 2}^{\ln 3}\frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx \qquad \text{or Let } u=e^x-e^{-x} \quad \Rightarrow du=\left(e^x+e^{-x}\right)dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{1}{e^x - e^{-x}} \, d\left(e^x - e^{-x}\right) \qquad\qquad d\left(e^x - e^{-x}\right) = \left(e^x + e^{-x}\right)dx$$

$$= \left[\ln\left|e^x - e^{-x}\right|\right]_{\ln 2}^{\ln 3}$$

$$= \ln\left(3 - \frac{1}{3}\right) - \ln\left(2 - \frac{1}{2}\right)$$

$$= \ln\left(\frac{8}{3}\right) - \ln\left(\frac{3}{2}\right)$$

$$= \ln\left(\frac{16}{9}\right) \ \ unit$$

Exercise

Find the length of the curve $\quad f(x) = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2} \quad 1 \le x \le 4$

Solution

$$a = \frac{2}{3}, \quad m = \frac{3}{2}, \quad b = -\frac{1}{2}, \quad n = \frac{1}{2}$$

1. $m + n = \frac{3}{2} + \frac{1}{2} = 2$ ✓ **2.** $abmn = \frac{2}{3}\left(\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$ ✓

$$L = \left(\frac{2}{3}x^{3/2} + \frac{1}{2}x^{1/2}\right)\Big|_1^4$$

$$= \frac{2}{3}4^{3/2} + 1 - \frac{2}{3} - \frac{1}{2}$$

$$= \frac{16}{3} - \frac{2}{3} + \frac{1}{2}$$

$$= \frac{31}{6} \ \ unit$$

Exercise

Find the length of the curve $\quad f(x) = x^3 + \frac{1}{12x} \quad 1 \le x \le 4$

Solution

$$a = 1, \quad m = 3, \quad b = \frac{1}{12}, \quad n = -1$$

1. $m + n = 3 - 1 = 2$ ✓ **2.** $abmn = (1)\left(\frac{1}{12}\right)(3)(-1) = -\frac{1}{4}$ ✓

$$L = \left(x^3 - \frac{1}{12x}\right)\Big|_1^4$$

$$= 4^3 - \frac{1}{48} - 1 + \frac{1}{12}$$

$$= 63 + \frac{3}{48}$$

$$= \frac{3,027}{48} \text{ unit}$$

Exercise

Find the length of the curve $f(x) = \frac{1}{8}x^4 + \frac{1}{4x^2} \quad 1 \le x \le 10$

Solution

$$a = \frac{1}{8}, \quad m = 4, \quad b = \frac{1}{4}, \quad n = -2$$

1. $m + n = 4 - 2 = 2$ ✓ **2.** $abmn = \left(\frac{1}{8}\right)\left(\frac{1}{4}\right)(4)(-2) = -\frac{1}{4}$ ✓

$$L = \left(\frac{1}{8}x^4 - \frac{1}{4x^2}\right)\Big|_1^{10}$$

$$= \frac{10^4}{8} - \frac{1}{400} - \frac{1}{8} + \frac{1}{4}$$

$$= \frac{9,999}{8} + \frac{99}{400}$$

$$= \frac{9}{8}\left(1111 + \frac{11}{50}\right)$$

$$= \frac{9}{8}\left(\frac{55,561}{50}\right)$$

$$= \frac{500,049}{400} \text{ unit}$$

Exercise

Find the length of the curve $f(x) = \frac{1}{4}x^4 + \frac{1}{8x^2} \quad 3 \le x \le 8$

Solution

$$a = \frac{1}{4}, \quad m = 4, \quad b = \frac{1}{8}, \quad n = -2$$

1. $m + n = 4 - 2 = 2$ ✓ **2.** $abmn = \left(\frac{1}{4}\right)\left(\frac{1}{8}\right)(4)(-2) = -\frac{1}{4}$ ✓

$$L = \left(\frac{1}{4}x^4 - \frac{1}{8x^2}\right)\Big|_3^{8}$$

$$= \frac{8^4}{4} - \frac{1}{8^3} - \frac{81}{4} + \frac{1}{72}$$

$$= \frac{4,015}{4} - \frac{1}{512} + \frac{1}{72}$$

$$= \frac{1}{4}\left(4,015 - \frac{1}{128} + \frac{1}{18}\right)$$

$$= \frac{1}{4}\left(4,015 + \frac{55}{1152}\right)$$

$$\boxed{= \frac{4,625,335}{4,608} \ \ unit}$$

Exercise

Find the length of the curve $f(x) = \frac{1}{10}x^5 + \frac{1}{6x^3} \quad 1 \le x \le 7$

Solution

$$a = \frac{1}{10}, \quad m = 5, \quad b = \frac{1}{6}, \quad n = -3$$

1. $m + n = 5 - 3 = 2$ ✓ **2.** $abmn = \left(\frac{1}{10}\right)\left(\frac{1}{6}\right)(5)(-3) = -\frac{1}{4}$ ✓

$$L = \left(\frac{1}{10}x^5 - \frac{1}{6x^3}\right)\Big|_1^7$$

$$= \frac{7^5}{10} - \frac{1}{2058} - \frac{1}{10} + \frac{1}{6}$$

$$= \frac{8403}{5} + \frac{57}{343}$$

$$\boxed{= \frac{2,882,514}{1,715} \ \ unit}$$

Exercise

Find the length of the curve $f(x) = \frac{3}{10}x^{1/3} - \frac{3}{2}x^{5/3} \quad 0 \le x \le 12\,\text{b}$

Solution

$$a = \frac{1}{10}, \quad m = 5, \quad b = \frac{1}{6}, \quad n = -3$$

1. $m + n = 5 - 3 = 2$ ✓ **2.** $abmn = \left(\frac{1}{10}\right)\left(\frac{1}{6}\right)(5)(-3) = -\frac{1}{4}$ ✓

$$L = \left(\frac{3}{10}x^{1/3} + \frac{3}{2}x^{5/3}\right)\Big|_0^{12}$$

$$= \frac{3}{10}\sqrt[3]{12} + \frac{3}{2}12\sqrt[3]{144}$$

$$\boxed{= \frac{3}{10}\sqrt[3]{12} + 18\sqrt[3]{144} \ \ unit} \qquad = \frac{3}{10}\sqrt[3]{12}\left(1 + 600\sqrt[3]{12}\right)$$

Exercise

Find the length of the curve $f(x) = x^{1/2} - \frac{1}{3}x^{3/2} \quad 2 \le x \le 9$

Solution

$a = 1, \quad m = \frac{1}{2}, \quad b = -\frac{1}{3}, \quad n = \frac{3}{2}$

1. $m + n = \frac{1}{2} + \frac{3}{2} = 2$ ✓ **2.** $abmn = (1)\left(-\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right) = -\frac{1}{4}$ ✓

$$L = \left(x^{1/2} + \frac{1}{3}x^{3/2}\right)\Big|_2^9$$

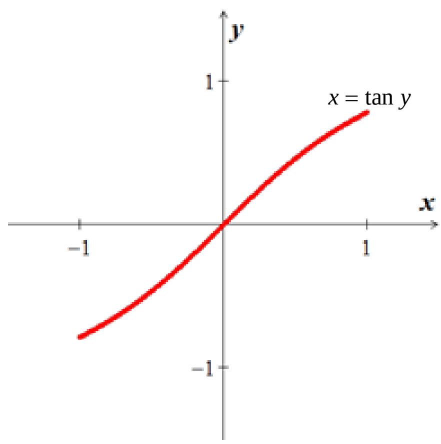
$$= 3 + 9 - \sqrt{2} - \frac{2\sqrt{2}}{3}$$

$$= \frac{1}{3}\left(36 - 5\sqrt{2}\right) \ \textit{unit} \qquad = \frac{3}{10}\sqrt[3]{12}\left(1 + 600\sqrt[3]{12}\right)$$

Exercise

Find the length of the curve $x = \int_0^y \sqrt{\sec^4 t - 1}\, dt \quad -\frac{\pi}{4} \le y \le \frac{\pi}{4}$

Solution



$$\frac{dx}{dy} = \sqrt{\sec^4 y - 1}$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \sec^4 y - 1}$$

$$= \sqrt{\sec^4 y}$$

$$= \sec^2 y$$

$$L = \int_{-\pi/4}^{\pi/4} \sec^2 y \ dy$$

$$= \tan y \Big|_{-\pi/4}^{\pi/4}$$

$$= 1 - (-1)$$

$$= 2 \ \textit{unit}$$

Exercise

Find the length of the curve $y = 3 - 2x \quad 0 \le x \le 2$. Check your answer by finding the length of the segment as the hypotenuse of a right triangle.

Solution

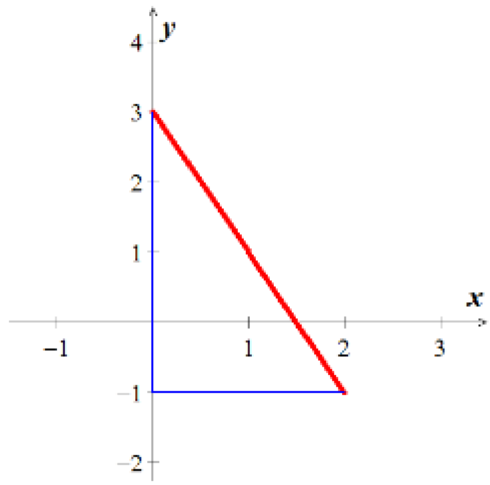
$$\frac{dy}{dx} = -2$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1+4} = \sqrt{5}$$

$$L = \int_0^2 \sqrt{5}\ dx$$

$$= \sqrt{5}x \Big|_0^2$$

$$= 2\sqrt{5}\ \ unit$$



$$\begin{cases} x = 0 & \to y = 3 \\ x = 2 & \to y = -1 \end{cases}$$

$$d = \sqrt{(2-0)^2 + (3+1)^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}\ \ unit$$

Exercise

Find a curve through the origin in the xy-plane whose length from $x = 0$ to $x = 1$ is $L = \int_0^1 \sqrt{1 + \frac{1}{4}e^x}\ dx$

Solution

$$L = \int_0^1 \sqrt{1 + \frac{1}{4}e^x}\ dx \quad \Rightarrow \frac{dy}{dx} = \frac{e^{x/2}}{2} \qquad \to dy = \frac{e^{x/2}}{2}dx$$

$$y = \int \frac{e^{x/2}}{2}dx = e^{x/2} + C$$

$$0 = e^{0/2} + C$$

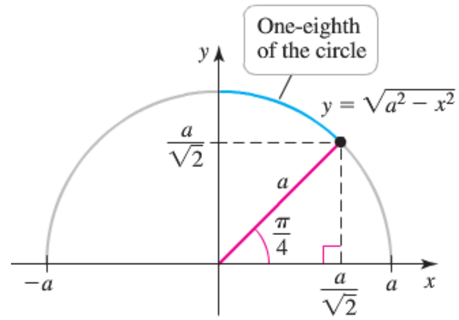
$$0 = 1 + C \qquad \Rightarrow \underline{C = -1}$$

$$y = e^{x/2} - 1$$

Exercise

Confirm that the circumference of a circle of radius a is $2\pi a$.

Solution



$$f(x) = \sqrt{a^2 - x^2} \quad \text{for} \quad -a \le x \le a$$

$$f'(x) = -\frac{x}{\sqrt{a^2 - x^2}} \qquad \text{but } \boxed{x \ne \pm a}$$

$$\sqrt{1 + f'(x)^2} = \sqrt{1 + \frac{x^2}{a^2 - x^2}}$$

$$= \frac{a}{\sqrt{a^2 - x^2}}$$

Let's compute the length of $\frac{1}{8}$ of the circle on $\left[0, \frac{a}{\sqrt{2}}\right]$

$$L = 8a \int_0^{a/\sqrt{2}} \frac{dx}{\sqrt{a^2 - x^2}}$$

$$= 8a \sin^{-1}\left(\frac{x}{a}\right)\Big|_0^{a/\sqrt{2}}$$

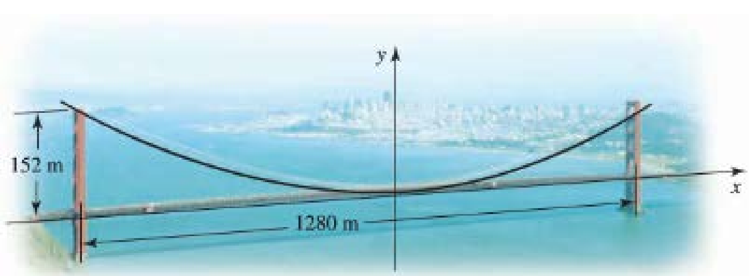
$$= 8a \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= 8a\left(\frac{\pi}{4}\right)$$

$$\boxed{= 2\pi a \ \ unit}$$

Exercise

The profile of the cables on a suspension bridge may be modeled by a parabola. The central span of the Golden Gate Bridge is 1280 *m* long and 152 *m* high. The parabola $y = 0.00037x^2$ gives a good fit to the shape of the cables, where $|x| \le 640$, and x and y are measured in *meters*. Approximate the length of the cables that stretch between the tops of the two towers.



Solution

$$y' = 0.00074x$$

$$L = \int_{-640}^{640} \sqrt{1+(.00074x)^2}\ dx \qquad \int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right|$$

$$= \left(\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\ln\left|x+\sqrt{1+x^2}\right|\right)\Bigg|_{-640}^{640}$$

$$= 320\sqrt{1+640^2} + \frac{1}{2}\ln\left|640+\sqrt{1+640^2}\right| + 320\sqrt{1+x^2} - \frac{1}{2}\ln\left|-640+\sqrt{1+640^2}\right|$$

$$\approx 1326.4\ m$$

Exercise

Electrical wires suspended between two towers form a caternary modeled by the equation

$$y = 20\cosh\frac{x}{20}, \quad -20 \le x \le 20$$

Where x and y are measured in *meters*. The towers are 40 *meters* apart. Find the length of the suspended cable.

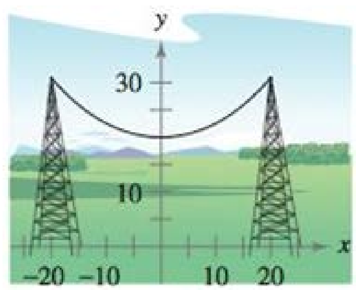
Solution

$$y = 20\cosh\frac{x}{20} \quad \to y' = \sinh\frac{x}{20}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\sinh^2\frac{x}{20}}$$

$$= \sqrt{\cosh^2\frac{x}{20}}$$

$$= \cosh\frac{x}{20}$$



$$L = \int_{-20}^{20} \cosh\frac{x}{20}\,dx$$

$$= 2(20)\sinh\frac{x}{20}\Bigg|_{0}^{20}$$

$$= 40(\sinh 1 - \sinh 0)$$

$$= 40\sinh 1$$

Exercise

A barn is 100 *feet* long and 40 *feet* wide. A cross section of the roof is the inverted caternary $y = 31 - 10\left(e^{x/20} + e^{-x/20}\right)$. Find the number of ***square feet*** of roofing on the barn.

Solution

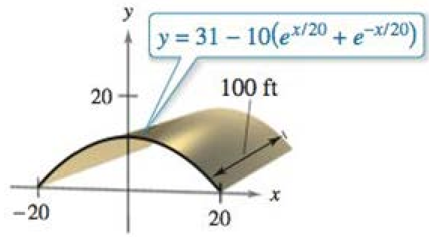
$$a = 10, \quad m = \frac{1}{20}, \quad b = 10, \quad n = -\frac{1}{20}$$

1. $m = -n$ ✓ **2.** $abmn = 10(10)\left(\frac{1}{20}\right)\left(-\frac{1}{20}\right) = -\frac{1}{4}$ ✓

$$L = 10\left(e^{x/20} - e^{-x/20}\right)\Big|_{-20}^{20}$$

$$= 10\left(e - \frac{1}{e} - \frac{1}{e} + e\right)$$

$$= 20\left(e - \frac{1}{e}\right) \approx 47\ ft$$



$\therefore$ There are $100(47) = 4,700\ ft^2$ of roofing on the barn

Exercise

A cable for a suspension bridge has the shape of a parabola with equation $y = kx^2$. Let h represent the height of the cable from it lowest point to its highest point and let $2w$ represent the total span of the bridge.

Show that the length C of the cable is given by $\quad C = 2\int_0^w \sqrt{1 + \frac{4h^2}{w^4}x^2}\ dx$

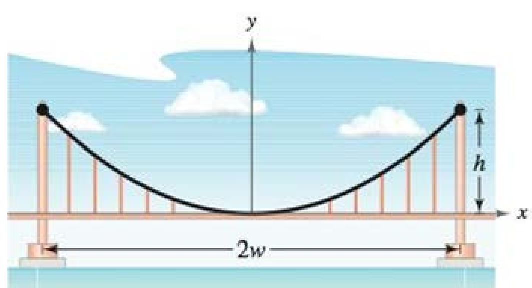
Solution

$y' = 2kx$

$\sqrt{1 + (y')^2} = \sqrt{1 + 4k^2x^2}$

At $(w,\ h) \to h = kw^2 \quad \Rightarrow \quad k = \frac{h}{w^2}$

$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{4h^2}{w^4}x^2}$



$\therefore$ By symmetry: $\quad C = 2\int_0^w \sqrt{1 + \frac{4h^2}{w^4}x^2}\ dx$

Exercise

Find the total length of the graph of the astroid $x^{2/3} + y^{2/3} = 4$

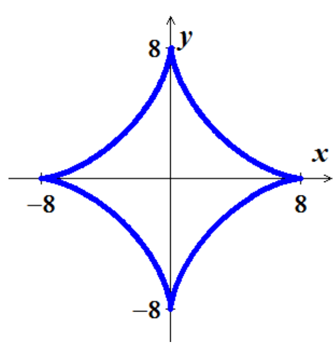
Solution

$x^{2/3} + y^{2/3} = 4 \quad \Rightarrow \quad y = \left(4 - x^{2/3}\right)^{3/2}$

$y' = \frac{3}{2}\left(-\frac{2}{3}x^{-1/3}\right)\left(4 - x^{2/3}\right)^{1/2}$

$= -\frac{1}{x^{1/3}}\left(4 - x^{2/3}\right)^{1/2}$

$1 + (y')^2 = 1 + \frac{1}{x^{2/3}}\left(4 - x^{2/3}\right)$



$$= \frac{4}{x^{2/3}}$$

$$y = 0 \quad \to x^{2/3} = 4 \;\Rightarrow\; \underline{x = 4^{3/2} = 8}$$

$$L = 4\int_0^8 \sqrt{\frac{4}{x^{2/3}}}\, dx$$

$$= 8\int_0^8 x^{-1/3}\, dx$$

$$= 12x^{2/3} \Big|_0^8$$

$$= 12(4 - 0)$$

$$\underline{= 48}$$

Exercise

Find the arc length from $(0,\ 3)$ clockwise to $(2,\ \sqrt{5})$ along the circle $x^2 + y^2 = 9$

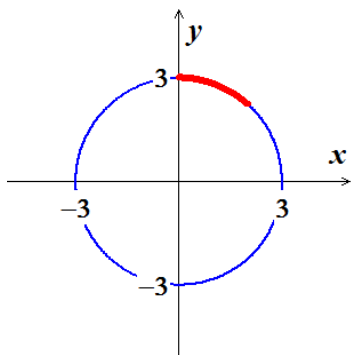
Solution

$$y = \sqrt{9 - x^2} \quad \Rightarrow \quad y' = -\frac{x}{\sqrt{9 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{9 - x^2}}$$

$$= \sqrt{\frac{9}{9 - x^2}}$$

$$= \frac{3}{\sqrt{9 - x^2}}$$



$$L = \int_0^2 \frac{3}{\sqrt{9 - x^2}}\, dx$$

$$= 3\arcsin\frac{x}{3} \Big|_0^2$$

$$\underline{= 3\arcsin\frac{2}{3}} \quad \underline{\approx 2.1892}$$

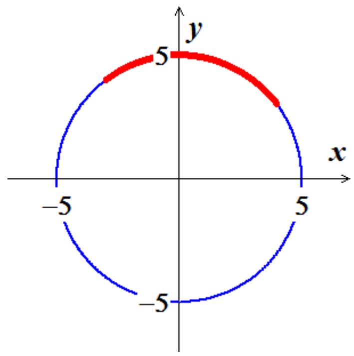
Exercise

Find the arc length from $(-3,\ 4)$ clockwise to $(4,\ 3)$ along the circle $x^2 + y^2 = 25$. Show that the result is one-fourth the circumference of the circle.

Solution

$$y = \sqrt{25 - x^2} \quad \Rightarrow \quad y' = -\frac{x}{\sqrt{25 - x^2}}$$

$$\begin{aligned}
\sqrt{1 + (y')^2} &= \sqrt{1 + \frac{x^2}{25 - x^2}} \\
&= \sqrt{\frac{25}{25 - x^2}} \\
&= \frac{5}{\sqrt{25 - x^2}}
\end{aligned}$$



$$L = \int_{-3}^{4} \frac{5}{\sqrt{25 - x^2}}\, dx$$

$$= 5 \arcsin \frac{x}{5} \Big|_{-3}^{4}$$

$$\underline{= 5\left(\arcsin \frac{4}{5} + \arcsin \frac{3}{5}\right)} \quad \underline{\approx 7.854}$$

Solution Section 1.6 – Surface Area

Exercise

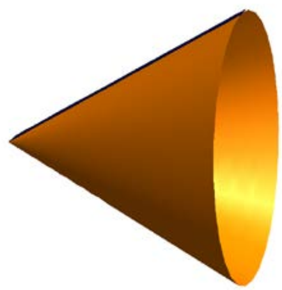
Find the lateral (side) surface area of the cone generated by revolving the line segment $y = \frac{x}{2}, \quad 0 \le x \le 4$, about the x-axis. Check your answer with the geometry formula

$$Lateral\ surface\ area = \frac{1}{2} \times base\ circumference \times slant\ height$$

Solution

$$\frac{dy}{dx} = \frac{1}{2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{4}} = \frac{\sqrt{5}}{2}$$



$$\begin{aligned} S &= 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= 2\pi \int_0^4 \left(\frac{x}{2}\right)\frac{\sqrt{5}}{2}\,dx \\ &= \frac{\pi\sqrt{5}}{2}\int_0^4 x\,dx \\ &= \frac{\pi\sqrt{5}}{2}\frac{1}{2}x^2\Big|_0^4 \\ &= \frac{\pi\sqrt{5}}{4}\left(4^2-0\right) \\ &= 4\pi\sqrt{5}\ \ unit^2 \end{aligned}$$

$$base\ circumference = 2\pi r = 2\pi(2) = 4\pi$$

$$slant\ height = \sqrt{4^2+2^2} = \sqrt{20} = 2\sqrt{5}$$

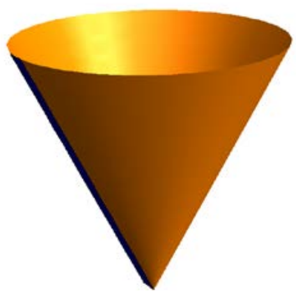
$$\begin{aligned} Lateral\ surface\ area &= \frac{1}{2} \times base\ circumference \times slant\ height \\ &= \frac{1}{2}\times(4\pi)\times\left(2\sqrt{5}\right) \\ &= 4\pi\sqrt{5} \end{aligned}$$

Exercise

Find the lateral surface area of the cone generated by revolving the line segment $y=\frac{x}{2}, \quad 0\le x\le 4$, about the y-axis. Check your answer with the geometry formula

$$\textit{Lateral surface area} = \frac{1}{2}\times \textit{base circumference}\times \textit{slant height}$$

Solution



$$y=\frac{x}{2} \quad \Rightarrow x=2y \to \begin{cases} x=0 & \to y=0 \\ x=4 & \to y=2 \end{cases}$$

$$\frac{dx}{dy}=2$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+4}=\sqrt{5}$$

$$\begin{aligned} S &= 2\pi\int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \\ &= 2\pi\int_0^2 2y\sqrt{5}\,dy \\ &= 4\pi\sqrt{5}\int_0^2 y\,dy \\ &= 4\pi\sqrt{5}\,\frac{1}{2}y^2\Big|_0^2 \\ &= 2\pi\sqrt{5}(4-0) \\ &= 8\pi\sqrt{5}\ \textit{unit}^2 \end{aligned}$$

$$\textit{base circumference} = 2\pi(4)=8\pi$$

$$\textit{slant height} = \sqrt{4^2+2^2}=\sqrt{20}=2\sqrt{5}$$

$$\begin{aligned} \textit{Lateral surface area} &= \frac{1}{2}\times \textit{base circumference}\times \textit{slant height} \\ &= \frac{1}{2}\times(8\pi)\times\left(2\sqrt{5}\right) \\ &= 8\pi\sqrt{5} \end{aligned}$$

Exercise

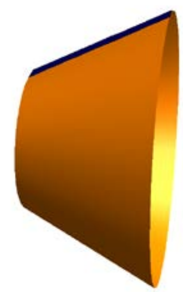
Find the lateral surface area of the cone frustum generated by revolving the line segment $y = \frac{x}{2} + \frac{1}{2}, \quad 1 \le x \le 3$, about the x-axis. Check your answer with the geometry formula

$$\text{Frustum surface area} = \pi\left(r_1 + r_2\right) \times \text{slant height}$$

Solution

$$\frac{dy}{dx} = \frac{1}{2}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$$



$$S = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_1^3 \left(\frac{x}{2} + \frac{1}{2}\right)\left(\frac{\sqrt{5}}{2}\right) dx$$

$$= \pi \frac{\sqrt{5}}{2} \int_1^3 (x + 1)\, dx$$

$$= \pi \frac{\sqrt{5}}{2} \left[\frac{1}{2}x^2 + x\right]_1^3$$

$$= \pi \frac{\sqrt{5}}{2} \left[\frac{1}{2}(3)^2 + (3) - \left(\frac{1}{2}(1)^2 + (1)\right)\right]$$

$$= \pi \frac{\sqrt{5}}{2} \left[\frac{9}{2} + 3 - \frac{3}{2}\right]$$

$$= \pi \frac{\sqrt{5}}{2}(6)$$

$$= 3\pi\sqrt{5}\ unit^2$$

$$r_1 = \frac{1}{2} + \frac{1}{2} = 1 \quad r_2 = \frac{3}{2} + \frac{1}{2} = 2$$

$$\text{slant height} = \sqrt{(2-1)^2 + (3-1)^2} = \sqrt{5}$$

$$\text{Frustum surface area} = \pi\left(r_1 + r_2\right) \times \text{slant height}$$

$$= \pi(1 + 2)\sqrt{5}$$

$$= 3\pi\sqrt{5}$$

Exercise

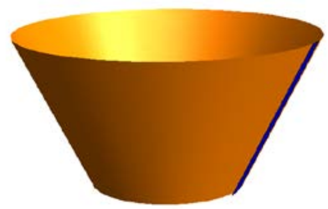
Find the lateral surface area of the cone frustum generated by revolving the line segment $y = \frac{x}{2} + \frac{1}{2}, \quad 1 \le x \le 3$, about the *y*-axis. Check your answer with the geometry formula

$$Frustum\ surface\ area = \pi\left(r_1 + r_2\right) \times slant\ height$$

Solution

$y = \frac{x}{2} + \frac{1}{2} \quad \to 2y = x + 1 \Rightarrow \boxed{x = 2y - 1}$

$\frac{dx}{dy} = 2$



$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1+4} = \sqrt{5}$

$S = 2\pi \int_1^2 (2y-1)\left(\sqrt{5}\right) dy$

$\quad = 2\pi\sqrt{5} \int_1^2 (2y-1) dy$

$\quad = 2\pi\sqrt{5} \left[y^2 - y \right]_1^2$

$\quad = 2\pi\sqrt{5} \left[(2)^2 - 2 - \left(1^2 - 1\right) \right]$

$\quad = 4\pi\sqrt{5}\ unit^2$

$r_1 = 1 \quad r_2 = 3$

$slant\ height = \sqrt{(2-1)^2 + (3-1)^2} = \sqrt{5}$

$Frustum\ surface\ area = \pi\left(r_1 + r_2\right) \times slant\ height$

$\qquad = \pi(1+3)\sqrt{5}$

$\qquad = 4\pi\sqrt{5}$

Exercise

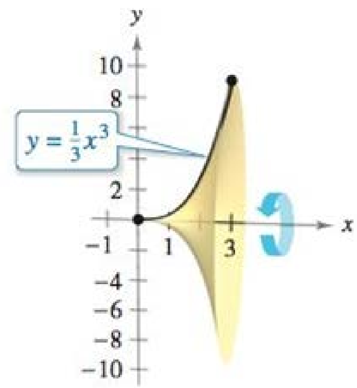
Set up an evaluate the definite integral for the area of the surface generated by revolving the curve $y = \frac{1}{3}x^3$ about the *x-axis*

Solution

$y = \frac{1}{3}x^3 \quad \to \quad y' = x^2$

$$\sqrt{1+(y')^2} = \sqrt{1+x^4}$$

$$S = 2\pi\int_0^3 \frac{1}{3}x^3\sqrt{1+x^4}\ dx \qquad\qquad S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$



$$= \frac{\pi}{6}\int_0^3 \left(1+x^4\right)^{1/2}\ d\left(1+x^4\right)$$

$$= \frac{\pi}{9}\left(1+x^4\right)^{3/2}\Big|_0^3$$

$$= \frac{\pi}{9}\left((82)^{3/2}-1\right)$$

$$= \frac{\pi}{9}\left(82\sqrt{82}-1\right)$$

Exercise

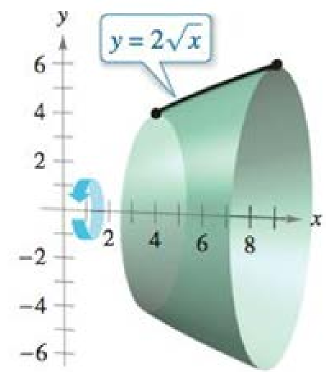
Set up an evaluate the definite integral for the area of the surface generated by revolving the curve $y = 2\sqrt{x}$ about the *x-axis*

Solution

$$y = 2\sqrt{x} \ \rightarrow\ y' = \frac{1}{\sqrt{x}}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{1}{x}}$$

$$S = 2\pi\int_4^9 2\sqrt{x}\sqrt{\frac{x+1}{x}}\ dx \qquad\qquad S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$



$$= 4\pi\int_4^9 (1+x)^{1/2}\ d(1+x)$$

$$= \frac{8}{3}\pi(1+x)^{3/2}\Big|_4^9$$

$$= \frac{8}{3}\pi\left(10^{3/2}-5^{3/2}\right) \qquad \approx 171.285$$

Exercise

Find the area of the surface generated by $y = \frac{x^3}{9}, \quad 0 \le x \le 2, \quad x - axis$

Solution

$$\frac{dy}{dx} = \frac{1}{3}x^2$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{1}{9}x^4} = \frac{1}{3}\sqrt{9 + x^4}$$

$$S = 2\pi \int_0^2 \frac{x^3}{9}\frac{1}{3}\sqrt{9 + x^4}\,dx \qquad\qquad S = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \frac{2\pi}{27}\int_0^2 x^3\sqrt{9 + x^4}\,dx \qquad\qquad u = 9 + x^4 \quad \to du = 4x^3dx \Rightarrow \frac{1}{4}du = x^3dx$$

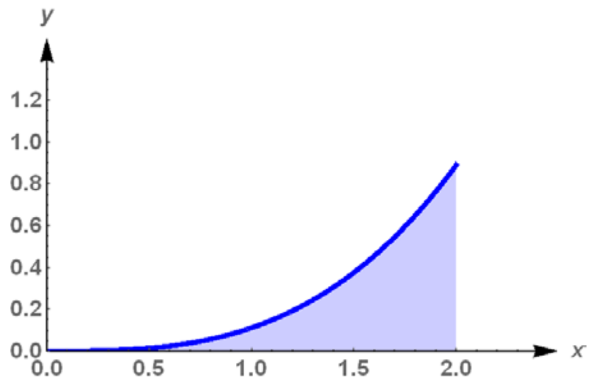
$$= \frac{2\pi}{27}\int_9^{25} u^{1/2}\left(\frac{1}{4}du\right) \qquad\qquad \to \begin{cases} x = 2 & \to u = 25 \\ x = 0 & \to u = 9 \end{cases}$$

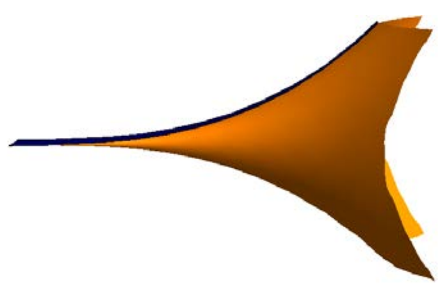
$$= \frac{\pi}{54}\int_9^{25} u^{1/2}du$$

$$= \frac{\pi}{54}\frac{2}{3}u^{3/2}\Big|_9^{25}$$

$$= \frac{\pi}{81}\left(25^{3/2} - 9^{3/2}\right)$$

$$= \frac{98\pi}{81}\ unit^2$$

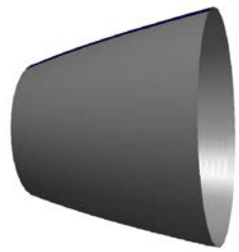




Exercise

Find the area of the surface generated by $y=\sqrt{x+1}, \quad 1\le x\le 5, \quad x-axis$

Solution



$$y=\sqrt{x+1}=(x+1)^{1/2}$$

$$\frac{dy}{dx}=\frac{1}{2}(x+1)^{-1/2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\frac{1}{4}(x+1)^{-1}}$$

$$=\sqrt{1+\frac{1}{4(x+1)}}$$

$$=\sqrt{\frac{4x+4+1}{4(x+1)}}$$

$$=\frac{1}{2}\sqrt{\frac{4x+5}{x+1}}$$

$$S=2\pi\int_1^5 \sqrt{x+1}\,\frac{1}{2}\frac{\sqrt{4x+5}}{\sqrt{x+1}}dx \qquad S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\pi\int_1^5 \sqrt{4x+5}\,dx$$

$$=\frac{\pi}{4}\int_1^5 (4x+5)^{1/2}\,d(4x+5)$$

$$=\frac{\pi}{6}(4x+5)^{3/2}\Big|_1^5$$

$$=\frac{\pi}{6}\left(25^{3/2}-9^{3/2}\right)$$

$$=\frac{\pi}{6}(98)$$

$$=\frac{49\pi}{3}\ unit^2$$

Exercise

Find the area of the surface generated by $y = \sqrt{2x - x^2}$, $0.5 \le x \le 1.5$, $x - axis$

Solution

$$\frac{dy}{dx} = \frac{1}{2}\left(2x - x^2\right)^{-1/2}(2 - 2x) = (1 - x)\left(2x - x^2\right)^{-1/2}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + (1 - x)^2\left(2x - x^2\right)^{-1}}$$

$$= \sqrt{1 + \frac{1 - 2x + x^2}{2x - x^2}}$$

$$= \sqrt{\frac{2x - x^2 + 1 - 2x + x^2}{2x - x^2}}$$

$$= \sqrt{\frac{1}{2x - x^2}}$$

$$= \frac{1}{\sqrt{2x - x^2}}$$

$$S = 2\pi \int_{.5}^{1.5} \sqrt{2x - x^2}\, \frac{1}{\sqrt{2x - x^2}}\, dx$$

$$S = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_{.5}^{1.5} dx$$

$$= 2\pi x \Big|_{.5}^{1.5} \qquad = 2\pi(1.5 - .5)$$

$$\underline{= 2\pi \ unit^2}$$

Exercise

Find the area of the surface generated by $y = 3x + 4$, $0 \le x \le 6$, *revolved about* $x - axis$

Solution

$y' = 3$

$$S = 2\pi \int_0^6 (3x + 4)\sqrt{1 + 9}\, dx$$

$$S = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi\sqrt{10}\left(\frac{3}{2}x^2 + 4x\right)_0^6$$

$$= 2\pi\sqrt{10}(54 + 24)$$

$$\underline{= 156\pi\sqrt{10} \ \ unit^2}$$

Exercise

Find the area of the surface generated by $y = 12 - 3x, \quad 1 \le x \le 3, \quad revolved\ \ about\ \ x - axis$

Solution

$$y' = -3$$

$$S = 2\pi \int_1^3 (12 - 3x)\sqrt{1+9}\ dx \qquad\qquad S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= 2\pi\sqrt{10}\left(12x - \frac{3}{2}x^2\right)\Big|_1^3$$

$$= 2\pi\sqrt{10}\left(36 - \frac{27}{2} - 12 + \frac{3}{2}\right)$$

$$= 24\pi\sqrt{10}\ \ unit^2$$

Exercise

Find the area of the surface generated by $y = 8\sqrt{x}, \quad 9 \le x \le 20, \quad revolved\ \ about\ \ x - axis$

Solution

$$y' = \frac{4}{\sqrt{x}}$$

$$S = 2\pi \int_9^{20} 8\sqrt{x}\sqrt{1+\frac{16}{x}}\ dx \qquad\qquad S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= 16\pi \int_9^{20} \sqrt{x}\frac{\sqrt{x+16}}{\sqrt{x}}\ dx$$

$$= 16\pi \int_9^{20} (x+16)^{1/2}\ d(x+16)$$

$$= \frac{32\pi}{3}(x+16)^{3/2}\Big|_9^{20}$$

$$= \frac{32\pi}{3}\left((36)^{3/2} - (25)^{3/2}\right)$$

$$= \frac{32\pi}{3}(216 - 125)$$

$$= \frac{2912\pi}{3}\ \ unit^2$$

Exercise

Find the area of the surface generated by $y = x^3, \quad 0 \le x \le 1, \quad revolved\ \ about\ \ x - axis$

Solution

$$y' = 3x^2$$

$$S = 2\pi\int_0^1 x^3\sqrt{1+9x^4}\ dx \qquad\qquad S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \frac{\pi}{18}\int_0^1 \left(1+9x^4\right)^{1/2}\ d\left(1+9x^4\right)$$

$$= \frac{\pi}{27}\left(1+9x^4\right)^{3/2}\Big|_0^1$$

$$= \frac{\pi}{27}\left((10)^{3/2}-1\right)$$

$$= \frac{\pi}{27}\left(10\sqrt{10}-1\right)\ unit^2$$

Exercise

Find the area of the surface generated by $y = x^{3/2} - \frac{1}{3}x^{1/2}, \quad 1 \le x \le 2, \quad revolved\ about\ x-axis$

Solution

$$y' = \frac{3}{2}x^{1/2} - \frac{1}{6}x^{-1/2} = \frac{1}{2}\left(3\sqrt{x} - \frac{1}{3\sqrt{x}}\right) = \frac{9x-1}{6\sqrt{x}}$$

$$S = 2\pi\int_1^2 \left(x^{3/2} - \frac{1}{3}x^{1/2}\right)\sqrt{1+\frac{(9x-1)^2}{36x}}\ dx \qquad\qquad S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \frac{2}{3}\pi\int_1^2 \left(3x^{3/2} - x^{1/2}\right)\frac{\sqrt{36x+81x^2-18x+1}}{6\sqrt{x}}\ dx$$

$$= \frac{\pi}{9}\int_1^2 (3x-1)\sqrt{81x^2+18x+1}\ dx$$

$$= \frac{\pi}{9}\int_1^2 (3x-1)\sqrt{(9x+1)^2}\ dx$$

$$= \frac{\pi}{9}\int_1^2 (3x-1)(9x+1)\ dx$$

$$= \frac{\pi}{9}\int_1^2 \left(27x^2-6x-1\right)\ dx$$

$$= \frac{\pi}{9}\left(9x^3-3x^2-x\right)\Big|_1^2 = \frac{\pi}{9}(72-12-2-9+3+1)$$

$$= \frac{53\pi}{9}\ unit^2$$

Exercise

Find the area of the surface generated by $y=\sqrt{4x+6}, \quad 0\le x\le 5, \quad revolved\ about\ x-axis$

Solution

$$y'=\frac{2}{\sqrt{4x+6}}$$

$$S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\begin{aligned}
S &= 2\pi\int_0^5 \sqrt{4x+6}\sqrt{1+\frac{4}{4x+6}}\,dx \\
&= 2\pi\int_0^5 \sqrt{4x+6}\sqrt{\frac{4x+6+4}{4x+6}}\,dx \\
&= 2\pi\int_0^5 (4x+10)^{1/2}\,dx \\
&= \frac{\pi}{2}\int_0^5 (4x+10)^{1/2}\,d(4x+10) \\
&= \frac{\pi}{3}(4x+10)^{3/2}\Big|_0^5 \\
&= \frac{\pi}{3}\left(30^{3/2}-10^{3/2}\right) \\
&= \frac{\pi}{3}\left(30\sqrt{30}-10\sqrt{10}\right) \\
&= \frac{10\pi\sqrt{10}}{3}\left(3\sqrt{3}-1\right)\ unit^2
\end{aligned}$$

Exercise

Find the area of the surface generated by $y=\frac{1}{4}\left(e^{2x}+e^{-2x}\right), \quad -2\le x\le 2, \quad revolved\ about\ x-axis$

Solution

$$y'=\frac{1}{2}\left(e^{2x}-e^{-2x}\right)$$

$$S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\begin{aligned}
S &= 2\pi\int_{-2}^{2}\frac{1}{4}\left(e^{2x}+e^{-2x}\right)\sqrt{1+\frac{1}{4}\left(e^{2x}-e^{-2x}\right)^2}\,dx \\
&= \frac{\pi}{2}\int_{-2}^{2}\left(e^{2x}+e^{-2x}\right)\frac{1}{2}\sqrt{4+e^{4x}-2+e^{-4x}}\,dx \\
&= \frac{\pi}{4}\int_{-2}^{2}\left(e^{2x}+e^{-2x}\right)\sqrt{e^{4x}+2+e^{-4x}}\,dx
\end{aligned}$$

$$= \frac{\pi}{4}\int_{-2}^{2}\left(e^{2x}+e^{-2x}\right)\sqrt{\left(e^{2x}+e^{-2x}\right)^2}\ dx$$

$$= \frac{\pi}{4}\int_{-2}^{2}\left(e^{2x}+e^{-2x}\right)^2\ dx$$

$$= \frac{\pi}{4}\int_{-2}^{2}\left(e^{4x}+2+e^{-4x}\right)\ dx$$

$$= \frac{\pi}{4}\left(\frac{1}{4}e^{4x}+2x-\frac{1}{4}e^{-4x}\right)\Big|_{-2}^{2}$$

$$= \frac{\pi}{4}\left(\frac{1}{4}e^{8}+4-\frac{1}{4}e^{-8}-\frac{1}{4}e^{-8}+4+\frac{1}{4}e^{8}\right)$$

$$= \frac{\pi}{4}\left(\frac{1}{2}e^{8}+8-\frac{1}{2}e^{-8}\right)$$

$$\boxed{= \frac{\pi}{8}\left(e^{8}+16-e^{-8}\right)\ unit^2}$$

Exercise

Find the area of the surface generated by $y=\frac{1}{8}x^4+\frac{1}{4x^2},\quad 1\le x\le 2,\quad revolved\ about\ x-axis$

Solution

$$y' = \frac{1}{2}x^3-\frac{1}{2x^3}=\frac{1}{2}\frac{x^6-1}{x^3}$$

$$S = 2\pi\int_{1}^{2}\left(\frac{1}{8}x^4+\frac{1}{4x^2}\right)\sqrt{1+\left(\frac{x^6-1}{2x^3}\right)^2}\ dx \qquad S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \frac{\pi}{4}\int_{1}^{2}\left(\frac{x^6+2}{x^2}\right)\sqrt{1+\frac{x^{12}-2x^6+1}{4x^6}}\ dx$$

$$= \frac{\pi}{4}\int_{1}^{2}\left(\frac{x^6+2}{x^2}\right)\sqrt{\frac{x^{12}+2x^6+1}{4x^6}}\ dx$$

$$= \frac{\pi}{4}\int_{1}^{2}\left(\frac{x^6+2}{x^2}\right)\frac{\sqrt{\left(x^6+1\right)^2}}{2x^3}\ dx$$

$$= \frac{\pi}{4}\int_{1}^{2}\frac{\left(x^6+2\right)\left(x^6+1\right)}{2x^5}\ dx$$

$$= \frac{\pi}{4}\int_{1}^{2}\frac{x^{12}+3x^6+2}{2x^5}\ dx$$

$$= \frac{\pi}{8}\int_1^2 \left(x^7 + 3x + 2x^{-5}\right)\, dx$$

$$= \frac{\pi}{8}\left(\frac{1}{8}x^8 + \frac{3}{2}x^2 - \frac{1}{2}x^{-4}\right)\Big|_1^2$$

$$= \frac{\pi}{8}\left(32 + 6 - \frac{1}{32} - \frac{1}{8} - \frac{3}{2} + \frac{1}{2}\right)$$

$$= \frac{\pi}{8}\left(37 - \frac{5}{32}\right)$$

$$= \frac{1179\pi}{256} \quad unit^2$$

Exercise

Find the area of the surface generated by $y = \frac{1}{3}x^3 + \frac{1}{4x}, \quad \frac{1}{2} \le x \le 2, \quad revolved\ about\ x - axis$

Solution

$$y' = x^2 - \frac{1}{4x^2} = \frac{4x^4 - 1}{4x^2}$$

$$S = 2\pi\int_{1/2}^2 \left(\frac{1}{3}x^3 + \frac{1}{4x}\right)\sqrt{1 + \left(\frac{4x^4 - 1}{4x^2}\right)^2}\, dx \qquad S = 2\pi\int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi\int_{1/2}^2 \left(\frac{4x^4 + 3}{12x}\right)\sqrt{1 + \frac{16x^8 - 8x^4 + 1}{16x^4}}\, dx$$

$$= \frac{\pi}{6}\int_{1/2}^2 \left(\frac{4x^4 + 3}{x}\right)\sqrt{\frac{16x^8 + 8x^4 + 1}{16x^4}}\, dx$$

$$= \frac{\pi}{6}\int_{1/2}^2 \left(\frac{4x^4 + 3}{x}\right)\frac{\sqrt{\left(4x^4 + 1\right)^2}}{4x^2}\, dx$$

$$= \frac{\pi}{24}\int_{1/2}^2 \left(\frac{4x^4 + 3}{x^3}\right)\left(4x^4 + 1\right)\, dx$$

$$= \frac{\pi}{24}\int_{1/2}^2 \left(4x + 3x^{-3}\right)\left(4x^4 + 1\right)\, dx$$

$$= \frac{\pi}{24}\int_{1/2}^2 \left(16x^5 + 16x + 3x^{-3}\right)\, dx$$

$$= \frac{\pi}{24}\left(\frac{8}{3}x^6 + 8x^2 - \frac{3}{2}x^{-2}\right)\Big|_{1/2}^2$$

$$= \frac{\pi}{24}\left(\frac{512}{3} + 32 - \frac{3}{8} - \frac{1}{24} - 2 + 6\right)$$

$$= \frac{\pi}{24}\left(\frac{4086}{24} + 36\right)$$

$$= \frac{\pi}{24}\left(\frac{681}{4} + 36\right)$$

$$= \frac{\pi}{24}\left(\frac{825}{4}\right)$$

$$\boxed{= \frac{275\pi}{32} \quad unit^2}$$

Exercise

Find the area of the surface generated by $y = \sqrt{5x - x^2}, \quad 1 \le x \le 4, \quad revolved\ \ about\ \ x - axis$

Solution

$$y' = \frac{5 - 2x}{2\sqrt{5x - x^2}}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{(5 - 2x)^2}{4\left(5x - x^2\right)}$$

$$= \frac{20x - 4x^2 + 25 - 20.x + 4x^2}{4\left(5x - x^2\right)}$$

$$= \frac{25}{4\left(5x - x^2\right)}$$

$$S = 2\pi \int_1^4 \sqrt{5x - x^2} \sqrt{\frac{25}{4\left(5x - x^2\right)}}\ dx \qquad\qquad S = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$= 5\pi \int_1^4 dx$$

$$= 5\pi x \Big|_1^4$$

$$\boxed{= 15\pi \ \ unit^2}$$

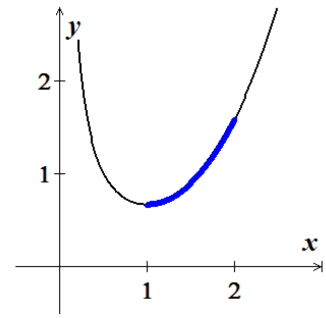
Exercise

Set up an evaluate the definite integral for the area of the surface generated by revolving the curve about the x-*axis* $\qquad y = \frac{1}{6}x^3 + \frac{1}{2x}, \quad 1 \le x \le 2$

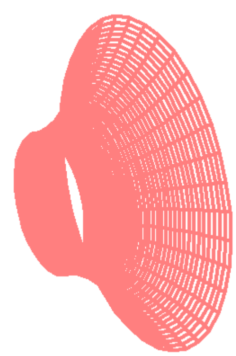
Solution

$$y' = \frac{1}{2}x^2 - \frac{1}{2x^2}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{1}{4}x^4 - \frac{1}{2} + \frac{1}{4x^4}}$$
$$= \sqrt{\frac{1}{4}x^4 + \frac{1}{2} + \frac{1}{4x^4}}$$
$$= \sqrt{\left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right)^2}$$
$$= \frac{1}{2}x^2 + \frac{1}{2x^2}$$



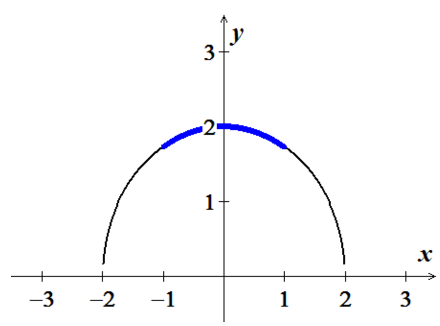
$$S = 2\pi\int_1^2 \left(\frac{1}{6}x^3 + \frac{1}{2x}\right)\left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right)dx \qquad S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$= 2\pi\int_1^2 \left(\frac{x^5}{12} + \frac{x}{3} + \frac{1}{4x^3}\right)dx$$
$$= 2\pi\left(\frac{1}{72}x^6 + \frac{1}{6}x^2 - \frac{1}{8x^2}\right)\Bigg|_1^2$$
$$= 2\pi\left(\frac{64}{72} + \frac{2}{3} - \frac{1}{32} - \frac{1}{72} - \frac{1}{6} + \frac{1}{8}\right)$$
$$= 2\pi\left(\frac{63}{72} + \frac{19}{32}\right)$$
$$= 2\pi\left(\frac{423}{288}\right)$$
$$= \frac{47\pi}{16}$$



Exercise

Set up an evaluate the definite integral for the area of the surface generated by revolving the curve about the *x-axis* $\qquad y = \sqrt{4-x^2}, \quad -1 \le x \le 1$

Solution



$$y' = \frac{-x}{\sqrt{4-x^2}}$$
$$\sqrt{1+(y')^2} = \sqrt{1+\frac{x^2}{4-x^2}}$$
$$= \sqrt{\frac{4}{4-x^2}}$$
$$S = 2\pi\int_{-1}^1 \sqrt{4-x^2}\,\frac{2}{\sqrt{4-x^2}}\,dx \qquad S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 4\pi \int_{-1}^{1} dx$$

$$= 4\pi x \Big|_{-1}^{1}$$

$$\underline{= 8\pi}$$

Exercise

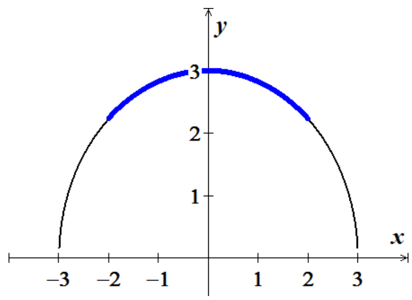
Set up an evaluate the definite integral for the area of the surface generated by revolving the curve about the *x-axis* $\quad y = \sqrt{9 - x^2}, \quad -2 \le x \le 2$

Solution

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{9 - x^2}}$$

$$= \sqrt{\frac{9}{9 - x^2}}$$

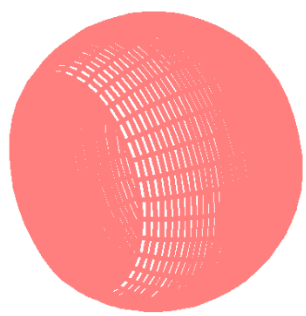


$$S = 2\pi \int_{-2}^{2} \sqrt{9 - x^2} \frac{3}{\sqrt{9 - x^2}} dx \qquad S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= 6\pi \int_{-2}^{2} dx$$

$$= 6\pi x \Big|_{-2}^{2}$$

$$\underline{= 24\pi}$$



Exercise

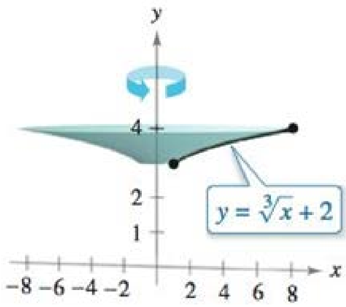
Set up an evaluate the definite integral for the area of the surface generated by revolving the curve about the *y-axis*

Solution

$$y' = \frac{1}{3} x^{-2/3} = \frac{1}{3x^{2/3}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{1}{9x^{4/3}}}$$

$$= \frac{\sqrt{9x^{4/3} + 1}}{3x^{2/3}}$$



$$S = 2\pi \int_1^8 x \frac{\sqrt{9x^{4/3}+1}}{3x^{2/3}} dx \qquad S = 2\pi \int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \frac{2}{3}\pi \int_1^8 x^{1/3}\sqrt{9x^{4/3}+1}\ dx$$

$$= \frac{\pi}{18}\int_1^8 \left(9x^{4/3}+1\right)^{1/2}\ d\left(9x^{4/3}+1\right)$$

$$= \frac{\pi}{27}\left(9x^{4/3}+1\right)^{3/2}\Big|_1^8$$

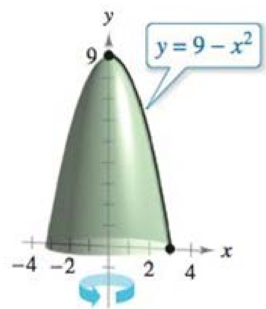
$$= \frac{\pi}{27}\left(\left(72(8)^{1/3}+1\right)^{3/2} - 10^{3/2}\right)$$

$$= \frac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right)$$

Exercise

Set up an evaluate the definite integral for the area of the surface generated by revolving the curve about the *y-axis*

Solution



$$y' = -2x$$

$$\sqrt{1+(y')^2} = \sqrt{1+4x^2}$$

$$S = 2\pi \int_0^3 x\sqrt{1+4x^2}\ dx \qquad S = 2\pi \int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \frac{\pi}{4}\int_0^3 \left(1+4x^2\right)^{1/2}\ d\left(1+4x^2\right)$$

$$= \frac{\pi}{6}\left(1+4x^2\right)^{3/2}\Big|_0^3$$

$$= \frac{\pi}{6}\left(37\sqrt{37} - 1\right)$$

Exercise

Find the area of the surface generated by $y = (3x)^{1/3}; \quad 0 \le x \le \frac{8}{3}$ *about y-axis*

Solution

$$3x = y^3 \to x = \frac{1}{3}y^3 \quad \Rightarrow \ x' = y^2 \qquad \begin{cases} x = 0 & \to y = 0 \\ x = \frac{8}{3} & \to y = \left(3\frac{8}{3}\right)^{1/3} = 2 \end{cases}$$

$$S = 2\pi \int_0^2 \frac{1}{3} y^3 \sqrt{1+y^4}\, dy \qquad\qquad S = 2\pi \int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \frac{\pi}{6} \int_0^2 \left(1+y^4\right)^{1/2} d\left(1+y^4\right)$$

$$= \frac{\pi}{9} \left(1+y^4\right)^{3/2} \Big|_0^2$$

$$= \frac{\pi}{9} \left((17)^{3/2} - 1\right)$$

$$= \frac{\pi}{9} \left(17\sqrt{17} - 1\right)$$

Exercise

Find the area of the surface generated of the curve $y = 4x - 1$ between the points $(1,\ 3)$ and $(4,\ 15)$ *about y-axis*

Solution

$$y = 4x - 1 \to x = \frac{1}{4}(y+1) \quad \Rightarrow\ x' = \frac{1}{4}$$

$$S = 2\pi \int_3^{15} \frac{1}{4}(y+1)\sqrt{1+\frac{1}{16}}\, dy \qquad\qquad S = 2\pi \int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \frac{\pi}{2} \int_3^{15} (y+1)\sqrt{\frac{17}{16}}\, dy$$

$$= \frac{\pi\sqrt{17}}{8} \left(\frac{1}{2} y^2 + y\right) \Big|_3^{15}$$

$$= \frac{\pi\sqrt{17}}{8} \left(\frac{225}{2} + 15 - \frac{9}{2} - 3\right)$$

$$= \frac{\pi\sqrt{17}}{8} (120)$$

$$= 15\pi\sqrt{17}$$

Exercise

Find the area of the surface generated of the curve $y = \frac{1}{2}\ln\left(2x + \sqrt{4x^2 - 1}\right)$ between the points $\left(\frac{1}{2},\ 0\right)$ and $\left(\frac{17}{16},\ \ln 2\right)$ *about y-axis*

Solution

$$2y = \ln\left(2x + \sqrt{4x^2 - 1}\right) \ \to\ \left(2x + \sqrt{4x^2 - 1}\right)^2 = \left(e^{2y}\right)^2$$

$$4x^2 + 4x\sqrt{4x^2 - 1} + 4x^2 - 1 = e^{4y}$$

$$4x\left(2x + \sqrt{4x^2 - 1}\right) = e^{4y} + 1 \qquad\qquad 2x + \sqrt{4x^2 - 1} = e^{2y}$$

$$4x\left(e^{2y}\right) = e^{4y} + 1$$

$$x = \frac{e^{4y} + 1}{4e^{2y}} = \frac{1}{4}\left(e^{2y} + e^{-2y}\right)$$

$$x' = \frac{1}{2}\left(e^{2y} - e^{-2y}\right)$$

$$S = 2\pi\int_0^{\ln 2} \frac{1}{4}\left(e^{2y} + e^{-2y}\right)\sqrt{1 + \frac{1}{4}\left(e^{2y} - e^{-2y}\right)^2}\, dy \qquad\qquad S = 2\pi\int_c^d x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \frac{\pi}{4}\int_0^{\ln 2} \left(e^{2y} + e^{-2y}\right)\sqrt{4 + e^{4y} - 2 + e^{-4y}}\, dy$$

$$= \frac{\pi}{4}\int_0^{\ln 2} \left(e^{2y} + e^{-2y}\right)\sqrt{\left(e^{2y} + e^{-2y}\right)^2}\, dy$$

$$= \frac{\pi}{4}\int_0^{\ln 2} \left(e^{2y} + e^{-2y}\right)^2\, dy$$

$$= \frac{\pi}{4}\int_0^{\ln 2} \left(e^{4y} + 2 + e^{-4y}\right)\, dy$$

$$= \frac{\pi}{4}\left(\frac{1}{4}e^{4y} + 2y - \frac{1}{4}e^{-4y}\right)\Big|_0^{\ln 2}$$

$$= \frac{\pi}{4}\left(\frac{1}{4}e^{4\ln 2} + 2\ln 2 - \frac{1}{4}e^{-4\ln 2} - \frac{1}{4} + \frac{1}{4}\right)$$

$$= \frac{\pi}{4}\left(\frac{1}{4}e^{\ln 2^4} + 2\ln 2 - \frac{1}{4}e^{\ln 2^{-4}}\right)$$

$$= \frac{\pi}{4}\left(\frac{1}{4}2^4 + 2\ln 2 - \frac{1}{4}2^{-4}\right)$$

$$= \frac{\pi}{4}\left(4 + 2\ln 2 - \frac{1}{64}\right)$$

$$\boxed{= \frac{\pi}{4}\left(\frac{255}{64} + 2\ln 2\right)\ unit^2}$$

Exercise

Find the area of the surface generated by $x=\sqrt{12y-y^2}$; $2\le y\le 10$ *about y-axis*

Solution

$$x'=\frac{6-y}{\sqrt{12y-y^2}}$$

$$S=2\pi\int_2^{10}\sqrt{12y-y^2}\sqrt{1+\frac{(6-y)^2}{12y-y^2}}\,dy \qquad S=2\pi\int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$=2\pi\int_2^{10}\sqrt{12y-y^2+36-12y+y^2}\,dy$$

$$=12\pi\int_2^{10}dy$$

$$=12\pi y\Big|_2^{10}$$

$$\underline{=96\pi\ \ unit^2}$$

Exercise

Find the area of the surface generated by $x=4y^{3/2}-\frac{1}{12}y^{1/2}$; $1\le y\le 4$ *about y-axis*

Solution

$$x'=6y^{1/2}-\frac{1}{24\sqrt{y}}=\frac{144y-1}{24\sqrt{y}}$$

$$S=2\pi\int_1^4\left(4y^{3/2}-\frac{1}{12}y^{1/2}\right)\sqrt{1+\frac{(144y-1)^2}{576y}}\,dy \qquad S=2\pi\int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$=2\pi\int_1^4\left(4y^{3/2}-\frac{1}{12}y^{1/2}\right)\sqrt{\frac{576y+(144y)^2-288y+1}{576y}}\,dy$$

$$=\frac{\pi}{12}\int_1^4\left(4y^{3/2}-\frac{1}{12}y^{1/2}\right)\frac{1}{\sqrt{y}}\sqrt{(144y+1)^2}\,dy$$

$$=\frac{\pi}{144}\int_1^4(48y-1)(144y+1)\,dy$$

$$=\frac{\pi}{144}\int_1^4\left(6{,}912y^2-96y-1\right)dy$$

$$=\frac{\pi}{144}\left(2304y^3-48y^2-y\right)\Big|_1^4$$

$$= \frac{\pi}{144}(147,456 - 768 - 4 - 2304 + 48 + 1)$$

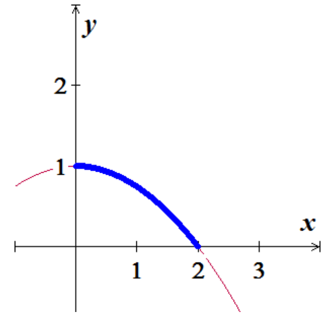
$$= \frac{144,429\pi}{144}$$

$$= \frac{48,143\,\pi}{48}\ \ unit^2$$

Exercise

Set up an evaluate the definite integral for the area of the surface generated by revolving the curve about the *y-axis* $\qquad y = 1 - \frac{1}{4}x^2, \quad 0 \le x \le 2$

Solution



$$y' = -\frac{1}{2}x$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{x^2}{4}}$$

$$= \frac{1}{2}\sqrt{4+x^2}$$

$$S = 2\pi\int_0^2 x\frac{\sqrt{4+x^2}}{2}\,dx \qquad S = 2\pi\int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \frac{\pi}{2}\int_0^2 \left(4+x^2\right)^{1/2}\,d\left(4+x^2\right)$$

$$= \frac{\pi}{3}\left(4+x^2\right)^{3/2}\Big|_0^2$$

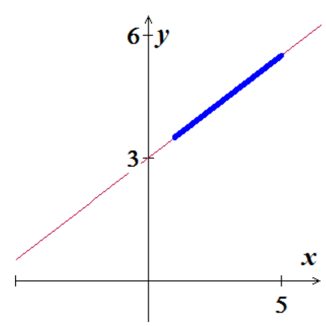
$$= \frac{\pi}{3}\left(8^{3/2} - 4^{3/2}\right)$$

$$= \frac{\pi}{3}\left(16\sqrt{2} - 8\right) \quad \approx 15.318\ \ unit^2$$

Exercise

Set up an evaluate the definite integral for the area of the surface generated by revolving the curve about the *y-axis* $\qquad y = \frac{1}{2}x + 3, \quad 1 \le x \le 5$

Solution



$$y' = \frac{1}{2}$$

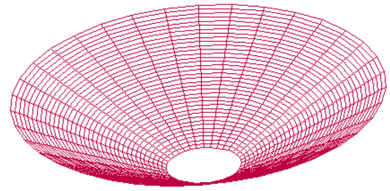
$$\sqrt{1+(y')^2} = \sqrt{1+\frac{1}{4}} = \frac{\sqrt{5}}{2}$$

$$S = \pi\sqrt{5}\int_1^5 x\,dx \qquad S = 2\pi\int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \pi\sqrt{5}\left(\frac{1}{2}x^2\right)\Big|_1^5$$

$$= \frac{\sqrt{5}}{2}\pi(25-1)$$

$$\underline{= 12\pi\sqrt{5}} \quad unit^2$$



Exercise

A right circular cone is generated by revolving the region bounded by $y = \frac{3}{4}x$, $y = 3$, and $x = 0$ about the y-*axis*. Find the lateral surface area of the cone.

Solution

$$y' = \frac{3}{4}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{9}{16}} = \frac{5}{4}$$

$$y = 3 = \frac{3}{4}x \;\Rightarrow\; \underline{x = 4}$$

$$S = \frac{5\pi}{2}\int_0^4 x\,dx \qquad\qquad S = 2\pi\int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \frac{5\pi}{4}x^2\Big|_0^4$$

$$\underline{= 20\pi \quad unit^2}$$

Exercise

A right circular cone is generated by revolving the region bounded by $y = \frac{h}{r}x$, $y = h$, and $x = 0$ about the y-*axis*. Verify that the lateral surface area of the cone is $S = \pi r\sqrt{r^2+h^2}$

Solution

$$y' = \frac{h}{r}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{h^2}{r^2}}$$

$$= \frac{\sqrt{r^2+h^2}}{r}$$

$$y = h = \frac{h}{r}x \;\Rightarrow\; \underline{x = r}$$

$$S = 2\pi \int_0^r x \frac{\sqrt{r^2+h^2}}{r}\, dx \qquad\qquad S = 2\pi \int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \frac{\pi\sqrt{r^2+h^2}}{r}\left(x^2\right)\Big|_0^r$$

$$\boxed{= \pi r\sqrt{r^2+h^2}}$$

Exercise

Find the area of the zone of a sphere formed by revolving the graph of $y = \sqrt{9-x^2}, \quad 0 \le x \le 2$, about the y-*axis*

Solution

$$y' = \frac{-x}{\sqrt{9-x^2}}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{x^2}{9-x^2}} = \frac{3}{\sqrt{9-x^2}}$$

$$S = 2\pi \int_0^2 x \frac{3}{\sqrt{9-x^2}}\, dx \qquad\qquad S = 2\pi \int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= -3\pi \int_0^2 \left(9-x^2\right)^{-1/2}\, d\left(9-x^2\right)$$

$$= -6\pi\left(9-x^2\right)^{1/2}\Big|_0^2$$

$$= -6\pi\left(\sqrt{5}-3\right)$$

$$\boxed{= 6\pi\left(3-\sqrt{5}\right)}$$

Exercise

Find the area of the zone of a sphere formed by revolving the graph of $y = \sqrt{r^2-x^2}, \quad 0 \le x \le a$, about the y-*axis*. Assume that $a < r$.

Solution

$$y' = \frac{-x}{\sqrt{r^2-x^2}}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{x^2}{r^2-x^2}} = \frac{r}{\sqrt{r^2-x^2}}$$

$$S = 2\pi\int_0^a x\frac{r}{\sqrt{r^2-x^2}}\,dx \qquad\qquad S = 2\pi\int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= -\pi r\int_0^a \left(r^2-x^2\right)^{-1/2}\,d\left(r^2-x^2\right)$$

$$= -2\pi r\sqrt{r^2-x^2}\,\Big|_0^a$$

$$= -2\pi r\left(\sqrt{r^2-a^2}-r\right)$$

$$= 2\pi r\left(r-\sqrt{r^2-a^2}\right)$$

Exercise

Find the area of the surface generated by the curve $y = 1+\sqrt{1-x^2}$ between the points $(1,\ 1)$ and $\left(\frac{\sqrt{3}}{1},\ \frac{3}{2}\right)$ *about y-axis*

Solution

$$\left(\sqrt{1-x^2}\right)^2 = (y-1)^2 \quad\Rightarrow\quad 1-x^2 = y^2-2y+1$$

$$x = \sqrt{2y-y^2} \;\rightarrow\; x' = \frac{1-y}{\sqrt{2y-y^2}}$$

$$S = 2\pi\int_1^{3/2} \sqrt{2y-y^2}\sqrt{1+\frac{(1-y)^2}{2y-y^2}}\,dy \qquad\qquad S = 2\pi\int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$= 2\pi\int_1^{3/2} \sqrt{2y-y^2+1-2y+y^2}\,dy$$

$$= 2\pi\int_1^{3/2} dy$$

$$= 2\pi y\,\Big|_1^{3/2}$$

$$= \pi\ \ unit^2$$

Exercise

Find the area of the surface generated by $x = 2\sqrt{4-y} \quad 0 \le y \le \frac{15}{4}, \quad y-axis$

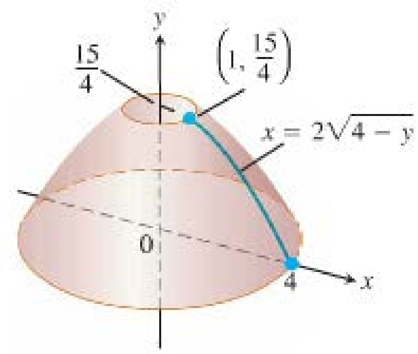
Solution

$$\frac{dy}{dx} = 2\frac{1}{2}(4-y)^{-1/2}(-1) = \frac{-1}{\sqrt{4-y}}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{4-y}}$$

$$= \sqrt{\frac{4-y+1}{4-y}}$$

$$= \sqrt{\frac{5-y}{4-y}}$$



$$S = 2\pi\int_0^{15/4} 2\sqrt{4-y}\frac{\sqrt{5-y}}{\sqrt{4-y}}dy$$

$$= 4\pi\int_0^{15/4} \sqrt{5-y}\ dy \qquad\qquad d(5-y) = -dy$$

$$= 4\pi\int_0^{15/4} (5-y)^{1/2}\ \left(-d(5-y)\right)$$

$$= -4\pi\frac{2}{3}(5-y)^{3/2}\Big|_0^{15/4}$$

$$= -\frac{8\pi}{3}\left[\left(5-\frac{15}{4}\right)^{3/2} - (5-0)^{3/2}\right]$$

$$= -\frac{8\pi}{3}\left[\left(\frac{5}{4}\right)^{3/2} - 5^{3/2}\right]$$

$$= -\frac{8\pi}{3}\left[\frac{5\sqrt{5}}{8} - 5\sqrt{5}\right]$$

$$= -\frac{8\pi}{3}5\sqrt{5}\left(\frac{1}{8}-1\right)$$

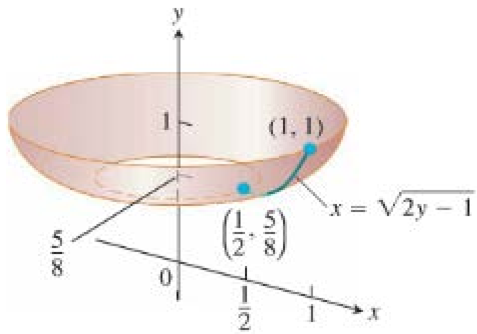
$$= -\frac{8\pi}{3}5\sqrt{5}\left(-\frac{7}{8}\right)$$

$$= \frac{35\pi\sqrt{5}}{3}\ unit^2$$

Exercise

Find the area of the surface generated by $x=\sqrt{2y-1} \quad \frac{5}{8}\le y\le 1, \quad y-axis$

Solution



$$\frac{dy}{dx}=\frac{1}{2}(2y-1)^{-1/2}(2)=\frac{1}{\sqrt{2y-1}}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\frac{1}{2y-1}}$$

$$=\sqrt{\frac{2y}{2y-1}}$$

$$S=2\pi\int_{5/8}^{1}\sqrt{2y-1}\frac{\sqrt{2y}}{\sqrt{2y-1}}dy$$

$$=2\pi\int_{5/8}^{1}\sqrt{2y}\ dy \qquad\qquad u=2y\to du=2dy$$

$$=2\pi\int_{5/8}^{1}u^{1/2}\ \left(\tfrac{1}{2}du\right)$$

$$=\pi\int_{5/8}^{1}u^{1/2}\ du$$

$$=\frac{2\pi}{3}(2y)^{3/2}\Big|_{5/8}^{1}$$

$$=\frac{2\pi}{3}\left((2)^{3/2}-\left(\tfrac{5}{4}\right)^{3/2}\right)$$

$$=\frac{2\pi}{3}\left(2\sqrt{2}-\tfrac{5\sqrt{5}}{8}\right)$$

$$=\frac{2\pi}{3}\left(\tfrac{16\sqrt{2}-5\sqrt{5}}{8}\right)$$

$$=\frac{\pi}{12}\left(16\sqrt{2}-5\sqrt{5}\right)\ \ unit^2$$

Exercise

$y=\frac{1}{3}\left(x^2+2\right)^{3/2}, \quad 0\le x\le\sqrt{2}; \quad y-axis$ (Hint: Express $ds=\sqrt{dx^2+dy^2}$ in terms of dx, and evaluate the integral $S=\int 2\pi x\ ds$ with appropriate limits.)

Solution

$$dy = \frac{1}{3}\frac{3}{2}\left(x^2+2\right)^{1/2}(2x)\,dx = x\sqrt{x^2+2}\ dx$$

$$ds = \sqrt{dx^2 + \left(x\sqrt{x^2+2}\ dx\right)^2}$$

$$= \sqrt{dx^2 + x^2\left(x^2+2\right)dx^2}$$

$$= \sqrt{1+x^4+2x^2}\,dx$$

$$= \sqrt{\left(1+x^2\right)^2}\ dx$$

$$= \left(1+x^2\right)dx$$

$$S = \int 2\pi x\ ds$$

$$= 2\pi\int_0^{\sqrt{2}} x\left(1+x^2\right)\ dx \qquad\qquad d\left(1+x^2\right) = 2xdx$$

$$= \pi\int_0^{\sqrt{2}} \left(1+x^2\right)\ d\left(1+x^2\right)$$

$$= \pi\frac{1}{2}u^2\Big|_1^3$$

$$= \frac{\pi}{2}\left(3^2-1^2\right)$$

$$= \frac{\pi}{2}(8)$$

$$= 4\pi\ unit^2$$

Exercise

Find the area of the surface generated by revolving the curve $x = \frac{1}{2}\left(e^y + e^{-y}\right), \quad 0 \le y \le \ln 2,$ about *y*-*axis*

Solution

$$S = 2\pi\int_0^{\ln 2} \frac{1}{2}\left(e^y+e^{-y}\right)\sqrt{1+\left(\frac{e^y-e^{-y}}{2}\right)^2}dy$$

$$= \pi\int_0^{\ln 2} \left(e^y+e^{-y}\right)\sqrt{1+\frac{e^{2y}+e^{-2y}-2}{4}}dy$$

$$= \pi \int_0^{\ln 2} \left(e^y + e^{-y}\right)\sqrt{\frac{4 + e^{2y} + e^{-2y} - 2}{4}}\, dy$$

$$= \frac{\pi}{2} \int_0^{\ln 2} \left(e^y + e^{-y}\right)\sqrt{e^{2y} + e^{-2y} + 2}\, dy$$

$$= \frac{\pi}{2} \int_0^{\ln 2} \left(e^y + e^{-y}\right)\sqrt{\left(e^y + e^{-y}\right)^2}\, dy$$

$$= \frac{\pi}{2} \int_0^{\ln 2} \left(e^y + e^{-y}\right)^2\, dy$$

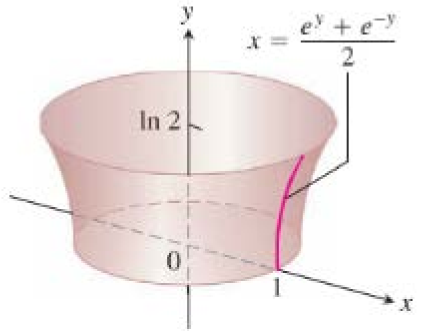
$$= \frac{\pi}{2} \int_0^{\ln 2} \left(e^{2y} + e^{-2y} + 2\right) dy$$

$$= \frac{\pi}{2}\left[\frac{1}{2}e^{2y} - \frac{1}{2}e^{-2y} + 2y\right]_0^{\ln 2}$$

$$= \frac{\pi}{2}\left[\left(\frac{1}{2}e^{2\ln 2} - \frac{1}{2}e^{-2\ln 2} + 2\ln 2\right) - \left(\frac{1}{2}e^0 - \frac{1}{2}e^0 + 0\right)\right]$$

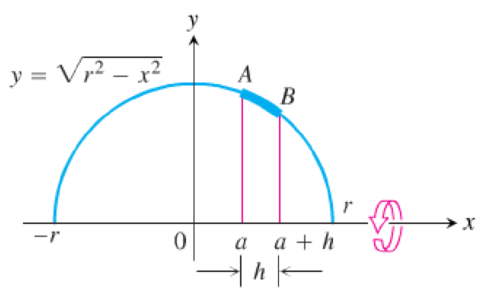
$$= \frac{\pi}{2}\left(\frac{1}{2}\cdot 4 - \frac{1}{2}\cdot\frac{1}{4} + 2\ln 2\right)$$

$$= \frac{\pi}{2}\left(\frac{15}{8} + 2\ln 2\right) \text{ unit}^2$$



Exercise

Did you know that if you can cut a spherical loaf of bread into slices of equal width, each slice will have the same amount of crust? To see why, suppose the semicircle $y = \sqrt{r^2 - x^2}$ shown here is revolved about the x-axis to generate a sphere. Let ***AB*** be an arc of the semicircle that lies above an interval of length ***h*** on the x-axis. Show that the area swept out by ***AB*** does not depend on the location of the interval. (It does depend on the length of the interval.)



Solution

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{r^2 - x^2}} = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^2}{r^2 - x^2}}$$

$$= \sqrt{\frac{r^2}{r^2 - x^2}}$$

$$S = 2\pi \int_a^{a+h} \sqrt{r^2 - x^2}\, \frac{r}{\sqrt{r^2 - x^2}}\, dx$$

$$= 2\pi r \int_a^{a+h} dx$$

$$= 2\pi r x \Big|_a^{a+h}$$

$$= 2\pi r (a + h - a)$$

$$\boxed{= 2\pi r h \ \ unit^2}$$

Example

The curved surface of a funnel is generated by revolving the graph of $y = f(x) = x^3 + \frac{1}{12x}$ on the interval $[1,\ 2]$ about the x-*axis* . Approximately what volume of paint is needed to cover the outside of the funnel with a layer of paint 0.05 *cm* thick? Assume that *x* and *y* measured in centimeters.

Solution

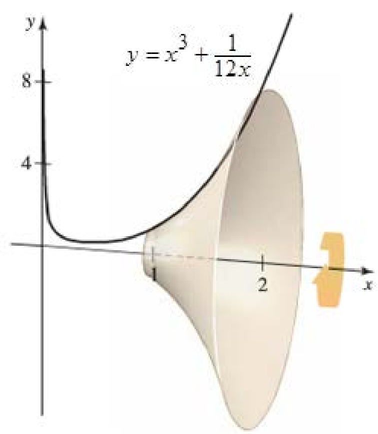
$$f'(x) = 3x^2 - \frac{1}{12x^2}$$

$$1 + f'(x)^2 = 1 + \left(3x^2 - \frac{1}{12x^2}\right)^2$$

5
$$= 1 + 9x^4 - \frac{1}{2} + \frac{1}{144x^4}$$

$$= 9x^4 + \frac{1}{2} + \frac{1}{144x^4}$$

$$= \left(3x^2 + \frac{1}{12x^2}\right)^2$$



$$S = 2\pi \int_1^2 \left(x^3 + \frac{1}{12x}\right) \sqrt{\left(3x^2 + \frac{1}{12x^2}\right)^2}\, dx$$

$$= 2\pi \int_1^2 \left(x^3 + \frac{1}{12x}\right)\left(3x^2 + \frac{1}{12x^2}\right) dx$$

$$= 2\pi \int_1^2 \left(3x^5 + \frac{x}{3} + \frac{1}{144} x^{-3}\right) dx$$

$$= 2\pi \left(\frac{1}{2}x^6 + \frac{1}{6}x^2 - \frac{1}{288}x^{-2}\right)\Big|_1^2$$

$$= 2\pi\left(32 + \frac{2}{3} - \frac{1}{1152} - \frac{1}{2} - \frac{1}{6} + \frac{1}{288}\right)$$

$$= 2\pi\left(\frac{36864 + 768 - 1 - 576 - 192 + 4}{1152}\right)$$

$$= \frac{12,289}{192}\pi \ \ cm^2$$

Because the paint layer is 0.05 *cm* thick, the approximate volume of paint needed is

$$= \left(\frac{12,289}{192}\pi \ cm^2\right)(0.05 \ \ cm) \approx 10.1 \ \ cm^3$$

Exercise

When the circle $x^2 + (y-a)^2 = r^2$ on the interval $[-r, \ r]$ is revolved about the x-*axis*, the result is the surface of a torus, where $0 < r < a$. Show that the surface area of the torus is $S = 4\pi^2 ar$.

Solution

$$x^2 + (y-a)^2 = r^2 \quad \Rightarrow \quad (y-a)^2 = r^2 - x^2$$

$$y = a \pm \sqrt{r^2 - x^2}$$

$$f(x) = a + \sqrt{r^2 - x^2}$$

$$1 + f'(x)^2 = 1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2$$

$$= 1 + \frac{x^2}{r^2 - x^2}$$

$$= \frac{r^2}{r^2 - x^2}$$

$$S_1 = 2\pi\int_{-r}^{r}\left(a + \sqrt{r^2 - x^2}\right)\frac{r}{\sqrt{r^2 - x^2}}\ dx$$

$$= 4\pi\int_0^r\left(\frac{ar}{\sqrt{r^2 - x^2}} + r\right) dx$$

$$= 4\pi\left[ar\sin^{-1}\left(\frac{x}{r}\right) + rx\right]_0^r$$

$$= 4\pi\left(ar\frac{\pi}{2} + r^2\right)$$

$$= 2\pi^2 ar + 4\pi r^2$$

$$S_2 = 2\pi \int_{-r}^{r} \left(a - \sqrt{r^2 - x^2}\right) \frac{r}{\sqrt{r^2 - x^2}}\, dx$$

$$= 4\pi \int_{0}^{r} \left(\frac{ar}{\sqrt{r^2 - x^2}} - r \right) dx$$

$$= 4\pi \left[ar \sin^{-1}\left(\frac{x}{r}\right) - rx \right]_{0}^{r}$$

$$= 4\pi \left(ar \frac{\pi}{2} - r^2 \right)$$

$$= 2\pi^2 ar - 4\pi r^2$$

$$S = 2\pi^2 ar + 4\pi r^2 + 2\pi^2 ar - 4\pi r^2 = 4\pi^2 ar \ \ unit^2$$

Exercise

A 1.5–*mm* layer of paint is applied to one side. Find the approximate volume of paint needed of the spherical zone generated when the curve $y = \sqrt{8x - x^2}$ on the interval $[1,\ 7]$ is revolved about the *x-axis* Assume *x* and *y* are in *meters*.

Solution

$$y' = \frac{4 - x}{\sqrt{8x - x^2}}$$

$$S = 2\pi \int_{1}^{7} \sqrt{8x - x^2} \sqrt{1 + \frac{(4 - x)^2}{8x - x^2}}\, dx \qquad S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_{1}^{7} \sqrt{8x - x^2} \frac{\sqrt{8x - x^2 + 16 - 8x + x^2}}{\sqrt{8x - x^2}}\, dx$$

$$= 2\pi \int_{1}^{7} \sqrt{16}\, dx$$

$$= 8\pi x \Big|_{1}^{7}$$

$$= 48\pi \ \ m^2$$

The volume of paint required to cover the surface to a thickness 0.0015 *m* is

$$V = 48\pi (0.0015) \approx 0.226195 \ \ m^3 \qquad 1\ m^3 = 264.172052\ gal$$

$$= 0.226195 \times 264.172052 \approx 59.75 \ \ gal$$

Exercise

A 1.5–*mm* layer of paint is applied to one side. Find the approximate volume of paint needed of the spherical zone generated when the upper portion of the circle $x^2 + y^2 = 100$ on the interval $[-8,\ 8]$ is revolved about the x-*axis*. Assume *x* and *y* are in *meters*.

Solution

$$y = \sqrt{100 - x^2} \quad \Rightarrow \quad y' = \frac{-x}{\sqrt{100 - x^2}}$$

$$S = 2\pi \int_{-8}^{8} \sqrt{100 - x^2}\sqrt{1 + \frac{x^2}{100 - x^2}}\, dx \qquad\qquad S = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_{-8}^{8} \sqrt{100 - x^2}\,\frac{\sqrt{100 - x^2 + x^2}}{\sqrt{100 - x^2}}\, dx$$

$$= 20\pi \int_{-8}^{8} dx$$

$$= 20\pi x \Big|_{-8}^{8}$$

$$\underline{= 320\pi \ \ m^2}$$

The volume of paint required to cover the surface to a thickness 0.0015 *m* is

$$V = 320\pi(0.0015) \ \underline{\approx 1.507965 \ \ m^3} \qquad\qquad 1\ m^3 = 264.172052\ gal$$

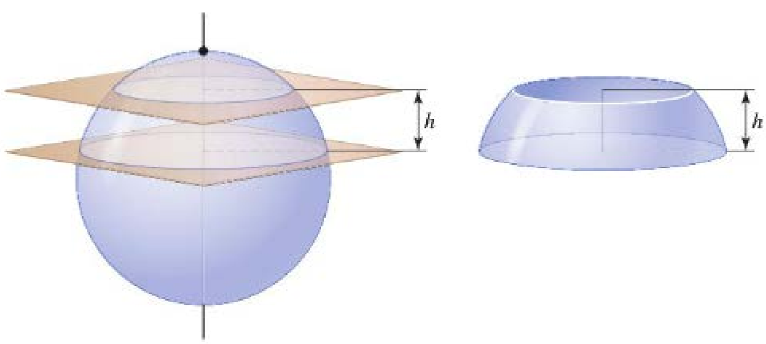
$$= 1.507965 \times 264.172052 \ \ \underline{\approx 398.36 \ \ gal}$$

Exercise

Suppose a sphere of radius *r* is sliced by two horizontal planes *h* units apart. Show that the surface area of the resulting zone on the sphere is $2\pi h$, independent of the location of the cutting planes.

Solution

$$f(x) = \sqrt{r^2 - x^2}$$



$$1 + f'(x)^2 = 1 + \left(\frac{x}{\sqrt{r^2 - x^2}}\right)^2$$

$$= 1 + \frac{x^2}{r^2 - x^2}$$

$$= \frac{r^2}{r^2 - x^2}$$

$$S = 2\pi \int_a^{a+h} \sqrt{r^2 - x^2} \frac{r}{\sqrt{r^2 - x^2}} \, dx \qquad\qquad S = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx$$

$$= 2\pi r x \Big|_a^{a+h}$$

$$= 2\pi r(a + h - a)$$

$$= 2\pi r h \ \ unit^2$$

Exercise

An ornamental light bulb is designed by revolving the graph of $y = \frac{1}{3}x^{1/2} - x^{3/2}, \quad 0 \le x \le \frac{1}{3}$ about the x-*axis* , where x and y are mesured in *feet*. Find the surface area of the bulb and use the result to approximate the amount of glass needed to make the bulb.
(Assume that the glass is 0.015 *inch* thick)

Solution

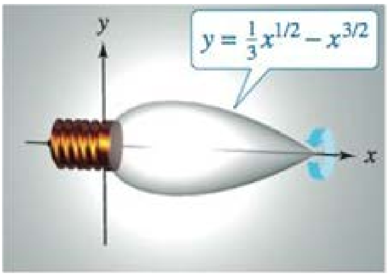
$$y' = \frac{1}{6}x^{-1/2} - \frac{3}{2}x^{1/2}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{1}{36}x^{-1} - \frac{1}{2} + \frac{9}{4}x}$$

$$= \frac{1}{6}\sqrt{x^{-1} + 18 + 81x}$$

$$= \frac{1}{6}\sqrt{\left(x^{-1/2} + 9x^{1/2}\right)^2}$$

$$= \frac{1}{6}\left(x^{-1/2} + 9x^{1/2}\right)$$



$$S = 2\pi \frac{1}{6} \int_0^{1/3} \left(\frac{1}{3}x^{1/2} - x^{3/2}\right)\left(x^{-1/2} + 9x^{1/2}\right) dx$$

$$= \frac{\pi}{3} \int_0^{1/3} \left(\frac{1}{3} + 2x - 9x^2\right) dx$$

$$= \frac{\pi}{3}\left(\frac{1}{3}x + x^2 - 3x^3\right)\Big|_0^{1/3}$$

$$= \frac{\pi}{3}\left(\frac{1}{9} + \frac{1}{9} - \frac{1}{9}\right)$$

$$= \frac{\pi}{27} \quad \approx 0.1164 \ ft^2 \quad \approx 16.8 \ in^2$$

Amount of glass needed: $V = \frac{\pi}{2}\left(\frac{0.015}{12}\right) \approx 0.00015 \ ft^3 \ \ \approx 0.25 \ in^3$

Solution *Section* 1.7 – Physical Applications

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = 1 + \sin x; \quad 0 \le x \le \pi$

Solution

$$m = \int_0^{\pi} (1 + \sin x)\,dx \qquad\qquad m = \int_a^b \rho(x)\,dx$$

$$= x - \cos x \Big|_0^{\pi}$$

$$= \pi + 1 + 1$$

$$\underline{= \pi + 2}$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = 1 + x^3; \quad 0 \le x \le 1$

Solution

$$m = \int_0^{1} \left(1 + x^3\right)dx \qquad\qquad m = \int_a^b \rho(x)\,dx$$

$$= x + \frac{1}{4}x^4 \Big|_0^{1}$$

$$= 1 + \frac{1}{4}$$

$$\underline{= \frac{5}{4}}$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = 2 - \frac{x}{2}; \quad 0 \le x \le 2$

Solution

$$m = \int_0^{2} \left(2 - \frac{x}{2}\right)dx \qquad\qquad m = \int_a^b \rho(x)\,dx$$

$$= 2x - \frac{1}{4}x^2 \Big|_0^{2}$$

$$= 4 - 1$$

$$\underline{= 3}$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = 5e^{-2x}; \quad 0 \le x \le 4$

Solution

$$m = \int_0^4 5e^{-2x} dx \qquad\qquad m = \int_a^b \rho(x) dx$$

$$= -\frac{5}{2} e^{-2x} \Big|_0^4$$

$$= -\frac{5}{2}\left(e^{-8} - 1\right)$$

$$= \frac{5}{2}\left(1 - e^{-8}\right)$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = x\sqrt{2 - x^2}; \quad 0 \le x \le 1$

Solution

$$m = \int_0^1 x\sqrt{2 - x^2}\ dx \qquad\qquad m = \int_a^b \rho(x) dx$$

$$= -\frac{1}{2}\int_0^1 \left(2 - x^2\right)^{1/2}\ d\left(2 - x^2\right)$$

$$= -\frac{1}{3}\left(2 - x^2\right)^{3/2} \Big|_0^1$$

$$= -\frac{1}{3}\left(1 - 2\sqrt{2}\right)$$

$$= \frac{1}{3}\left(2\sqrt{2} - 1\right)$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = \begin{cases} 1 & if\ \ 0 \le x \le 2 \\ 2 & if\ \ 2 < x \le 3 \end{cases}$

Solution

$$m = \int_0^2 1\ dx + \int_2^3 2\ dx \qquad\qquad m = \int_a^b \rho(x) dx$$

$$= x \Big|_0^2 + (2x) \Big|_2^3$$

$$= 2 + (6 - 4)$$

$$= 4$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = \begin{cases} 1 & if \;\; 0 \le x \le 2 \\ 1+x & if \;\; 2 < x \le 4 \end{cases}$

Solution

$$m = \int_0^2 1\, dx + \int_2^4 (1+x)\, dx \qquad\qquad m = \int_a^b \rho(x)dx$$

$$= x \Big|_0^2 + \left(x + \frac{1}{2}x^2\right)\Big|_2^4$$

$$= 2 + (4 + 8 - 2 - 2)$$

$$= 10$$

Exercise

Find the mass of a thin bar with the given density function $\rho(x) = \begin{cases} x^2 & if \;\; 0 \le x \le 1 \\ x(2-x) & if \;\; 1 < x \le 2 \end{cases}$

Solution

$$m = \int_0^1 x^2\, dx + \int_1^2 \left(2x - x^2\right) dx \qquad\qquad m = \int_a^b \rho(x)dx$$

$$= \frac{1}{3}x^3 \Big|_0^1 + \left(x^2 - \frac{1}{3}x^3\right)\Big|_1^2$$

$$= \frac{1}{3} + 4 - \frac{8}{3} - 1 + \frac{1}{3}$$

$$= 1$$

Exercise

A heavy-duty shock absorber is compressed 2 *cm* from its equilibrium position by a mass of 500 *kg*. How much work is required tocompress the shock absorber 4 *cm* from its equilibrium position? (A mass of 500 *kg* exerts a force (in newtons) of 500 *g*)

Solution

Given: $F(0.02) = 500 \quad g = 9.8\ m/s^2$

$$F(0.02) = 0.02k = 500 \times 9.8 \qquad\qquad F(x) = kx = mg$$

$$k = \frac{4900}{0.02} = 245{,}000$$

$$W = \int_0^{.04} 245{,}000x\, dx \qquad\qquad W = \int_a^b F(x)dx$$

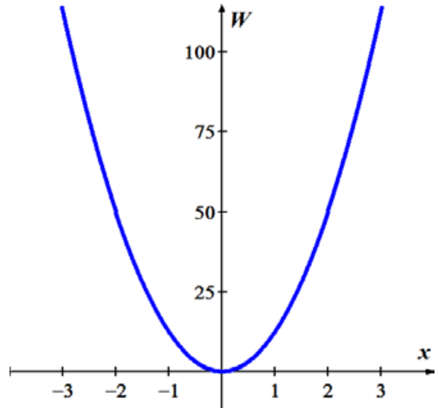
$$= 122{,}500x^2 \Big|_0^{0.04}$$

$$= 195\ J$$

Exercise

A spring has a restoring force given by $F(x)=25x$. Let $W(x)$ be the work required to stretch the spring from its equilibrium position $(x=0)$ to a variable distance x. Graph the work function. Compare the work required to stretch the spring x units from equilibrium to the work required to compress the spring x units from equilibrium.

Solution



$$W=\int_0^x 25t\ dt \qquad\qquad W=\int_a^b F(x)\,dx$$

$$=\frac{25}{2}t^2\Big|_0^x$$

$$\boxed{=\frac{25}{2}x^2}$$

Since $W(x)$ is an even function.

So that $W(-x)=W(x)$, and thus the work is the same to compress or stretch the spring a given distance from its equilibrium position,

Exercise

A swimming pool has the shape of a box with a base that measures 25 *m* by 15 *m* and a depth of 2.5 *m*. How much work is required to pump the water out of the pool when it is full?

Solution

$$W=\int_0^{2.5}\rho gA(y)(2.5-y)\,dy$$

$$=\int_0^{2.5}(1000)(9.8)(25\times15)(2.5-y)\,dy$$

$$=3,675,000\left(2.5y-\frac{1}{2}y^2\right)\Big|_0^{2.5}$$

$$=3,675,000\left(6.25-\frac{6.25}{2}\right)$$

$$\boxed{=11,484,375\ \ J}$$

Exercise

It took 1800 *J* of work to stretch a spring from its natural length of 2 *m* to a length of 5 *m*. Find the spring's force constant

Solution

$$W = \int_0^3 F(x)\,dx$$

$$1800 = \int_0^3 kx\,dx$$

$$1800 = \frac{1}{2}kx^2 \Big|_0^3$$

$$1800 = \frac{1}{2}k(9-0)$$

$$1800 = \frac{9}{2}k$$

$$1800\left(\frac{2}{9}\right) = k$$

$$\boxed{k = 400 \;\; N/m}$$

Exercise

How much work is required to move am object from $x=1$ to $x=5$ (measired in meters) in the presence of a constant force at 5 *N* acting along the *x-axis* .

Solution

$$W = \int_1^5 5\,dx$$

$$= 5(5-1)$$

$$\underline{= 20 \;\; J|}$$

Exercise

How much work is required to move am object from $x=0$ to $x=3$ (measired in meters) with a force (in *N*) is given by $F(x) = \frac{2}{x^2}$ acting along the *x-axis* .

Solution

$$W = \int_1^3 \frac{2}{x^2}\,dx$$

$$= -2\frac{1}{x}\Big|_1^3$$

$$= -2\left(\frac{1}{3}-1\right)$$

$$\boxed{= \frac{4}{3}\ J}$$

Exercise

A spring on a horizontal surface can be stretched and held 0.5 m from its equilibrium position with a force of 50 N.

a) How much work is done in stretching the spring 1.5 m from its equilibrium position?

b) How much work is done in compressing the spring 0.5 m from its equilibrium position?

Solution

a) $f(x) = kx \quad \rightarrow \quad f(0.5) = 50 = 0.5k \ \Rightarrow \ \underline{k = 100}$

$$W = \int_0^{1.5} 100x\,dx$$

$$= 50x^2 \Big|_0^{1.5}$$

$$\boxed{= 112.5\ J}$$

b) $$W = \int_0^{-.5} 100x\,dx$$

$$= 50x^2 \Big|_0^{-0.5}$$

$$\boxed{= 12.5\ J}$$

Exercise

Suppose a force of 10 N is required to stretch a spring 0.1 m from its equilibrium position and hold it in that position.

a) Assuming that the spring obeys Hooke's law, find the spring constant k.

b) How much work is needed to ***compress*** the spring 0.5 m from its equilibrium position?

c) How much work is needed to ***stretch*** the spring 0.25 m from its equilibrium position?

d) How much additional work is required to stretch the spring 0.25 m if it has already been stretched 0.1 m from its equilibrium position?

Solution

a) $F(0.1) = k(0.1) = 10$

$k = \frac{10}{0.1}\ \underline{= 100\ N/m}$ Therfore, Hooke's law for this spring: $F(x) = 100x$

b) Work is needed to ***compress*** the spring

$$W = \int_0^{-0.5} 100x\ dx$$

$$= 50x^2 \Big|_0^{-0.5}$$

$$= 50(-0.5)^2$$

$$\underline{= 12.5\ \ J}$$

c) Work is needed to ***stretch*** the spring

$$W = \int_0^{0.25} 100x\ dx$$

$$= 50x^2 \Big|_0^{0.25}$$

$$= 50(.25)^2$$

$$\underline{= 3.125\ \ J}$$

d) Work is required to ***stretch*** the spring

$$W = \int_{0.1}^{0.35} 100x\ dx$$

$$= 50x^2 \Big|_{0.1}^{0.35}$$

$$= 50\left(0.35^2 - 0.1^2\right)$$

$$\underline{= 5.625\ \ J}$$

Exercise

A force of 200 N will stretch a garage door spring 0.8-*m* beyond its unstressed length.

a) How far will a 300-N-force stretch the spring?

b) How much work does it take to stretch the spring this far?

Solution

$$k = \frac{F}{x} = \frac{200}{0.8} \underline{= 250\ \ N/m}$$

a) $300 = 250x \quad \to \quad \underline{x = 1.2\ m}$

b) $$W = \int_0^{1.2} 250x\ dx$$

$$= 125x^2 \Big|_0^{1.2}$$

$$\underline{= 180\ \ J\ \ (N-m)}$$

Exercise

A spring has a natural length of 10 *in*. An 800-*lb* force stretches the spring to 14 *in*.

a) Find the force constant.

b) How much work is done in stretching the spring from 10 *in* to 12 *in*?

c) How far beyond its natural length will a 1600-*lb* force stretch the spring?

Solution

a) $k=\frac{F}{x}=\frac{800}{14-10}=\frac{800}{4}$

$k\ \underline{=200\ \text{lb}/in}$

b) $\Delta x=12-10=2\ in$

$$W=k\int_0^2 x\,dx$$

$$=200\tfrac{1}{2}x^2\Big|_0^2$$

$$=100(4-0)$$

$$=400\ \ in.lb$$

$$=400\frac{1\ ft}{12\ in}\ \ in.lb$$

$$\underline{=33.3\ \ ft\cdot lb}$$

c) $F=200x$

$1600=200x$

$\frac{1600}{200}=x$

$x\ \underline{=8\ \ in}$

Exercise

It takes a force of 21,714 *lb* to compress a coil spring assembly on a Transit Authority subway car from its free height of 8 *in*. to its fully compressed height of 5 *in*.

a) What is the assembly's force constant?

b) How much work does it take to compress the assembly the first half inch? The second half inch? Answer to the nearest *in*-*lb*.

Solution

a) $F=kx$

$21714=k(8-5)$

$21714=3k$

$k\ \underline{=7238\ \ \text{lb}/in}$

b) $W = k\int_0^{0.5} x\,dx$

$$= 7238\left[\frac{1}{2}x^2\right]_0^{0.5}$$

$$= 7238\left[\frac{1}{2}(0.5)^2 - 0\right]$$

$$\underline{= 905\ \ in \cdot lb}$$

$$W = 7238\int_{0.5}^{1} x\,dx$$

$$= 7238\left[\frac{1}{2}x^2\right]_{0.5}^{1}$$

$$= 3619\left[1^2 - 0.5^2\right]_{0.5}^{1}$$

$$\underline{= 2714\ \ in \cdot lb}$$

Exercise

A mountain climber is about to haul up a 50 *m* length of hanging rope. How much work will it take if the rope weighs 0.624 *N/m*?

Solution

$$W = 0.624\int_0^{50} x\,dx$$

$$= 0.624\left[\frac{1}{2}x^2\right]_0^{50}$$

$$= \frac{0.624}{2}\left[(50)^2 - 0\right]$$

$$\underline{= 780\ \ J}$$

Exercise

A bag of sand originally weighing 144 *lb* was lifted a constant rate. As it rose, sand also leaked out at a constant rate. The sand was half gone by the time the bag had been lifted to 18 *ft*. How much work was done lifting the sand this far? (Neglect the weight of the bag and lifting equipment.)

Solution

The weight of sands decreases by $\frac{1}{2}144 = 72\ lb$ over the 18 ft. at rate $\frac{72}{18} = 4\ lb\ /\ ft$

$$F(x) = 144 - 4x$$

$$W = \int_0^{18} (144 - 4x)\,dx$$

$$= \left[144x - 2x^2\right]_0^{18}$$

$$= 144(18) - 2(18)^2 - (0)$$

$$= 1944\ ft \cdot lb$$

Exercise

An electric elevator with a motor at the top has a multistrand cable weighing 4.5 *lb/ft*. When the car is at the first floor, 180 *feet* of cable are paid out, and effectively 0 *foot* are out when the car is at the top floor. How much work does the motor do just lifting the cable when it takes the car from the first floor to the top?

Solution

$$F(x) = k\Delta x = 4.5(180 - x)$$

$$W = \int_0^{180} 4.5(180 - x)\,dx$$

$$= 4.5\left[180x - \tfrac{1}{2}x^2\right]_0^{180}$$

$$= 4.5\left[180(180) - \tfrac{1}{2}(180)^2 - 0\right]$$

$$= 72{,}900\ ft \cdot lb$$

Exercise

The rectangular cistern (storage tank for rainwater) shown has its top 10 *ft* below ground level. The cistern, currently full, is to be emptied for inspection by pumping its contents to ground level. Assume that the water weighs $62.4\ lb/ft^3$

a) How much work will it take to empty the cistern?

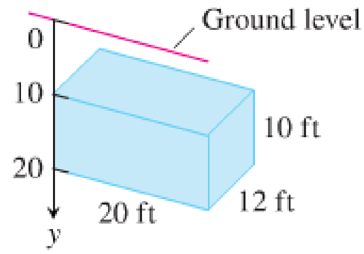
b) How long will it take a 1/-*hp* pump, rated at 275 *ft-lb/sec*, to pump the tank dry?

c) How long will it take the pump in part (*b*) to empty the tank hallway? (It will be less than half the time required to empty the tank completely)

d) What are the answers to parts (*a*) through (*c*) in a location where water weighs 62.6 lb/ft^3? 62.59 lb/ft^3?

Solution

a) $\Delta V = (20)(12)\Delta y = 240\Delta y$



$$\begin{aligned} F &= 62.4(\Delta V) \\ &= (62.4)240\Delta y \\ &= 14976\Delta y \end{aligned}$$

$$\begin{aligned} \Delta W &= \text{ force} \times \text{distance} \\ &= 14976\ \Delta y \times y \end{aligned}$$

$$\begin{aligned} W &= 14976\int_{10}^{20} y dy \\ &= 14976\left[\frac{1}{2}y^2\right]_{10}^{20} \\ &= \frac{14976}{2}\left(20^2 - 10^2\right) \\ &= 2,246,400\ \ ft \cdot lb \end{aligned}$$

b) $t = \dfrac{W}{275\frac{ft.lb}{\sec}}$

$$\begin{aligned} &= \frac{2,246,400\ ft.lb}{275}\frac{\sec}{ft.lb} \\ &\approx 8,168.73\ sec \frac{1\ \ hr}{3600\ sec} \\ &\approx 2.27\ \ hrs \end{aligned}$$

$2\ hrs\ \ \&\ \ 16.1\text{min}$

c)
$$\begin{aligned} W &= 14976\int_{10}^{15} y dy \\ &= 14976\left[\frac{1}{2}y^2\right]_{10}^{15} \\ &= \frac{14976}{2}\left(15^2 - 10^2\right) \\ &= 936,000\ \ ft \cdot lb \end{aligned}$$

$$\begin{aligned} t &= \frac{W}{275\frac{ft.lb}{\sec}} \\ &= \frac{936,000\ ft.lb}{275}\frac{\sec}{ft.lb} \\ &\approx 3403.64\ sec \frac{1\ \ \min}{60\ sec} \\ &\approx 56.7\ \ \min \end{aligned}$$

d) ***Water weighs*** $62.26\ lb / ft^3$

$W = (62.26)(240)(150)$

$$\boxed{= 2,214,360 \ \ ft \cdot lb}$$

$$t = \frac{W}{275 \frac{ft.lb}{\sec}}$$

$$= \frac{2,241,360 \ ft.lb}{275} \frac{\sec}{ft.lb}$$

$$\approx 8,150.4 \ sec \frac{1 \ \ hr}{3600 \ sec}$$

$$\boxed{\approx 2.264 \ \ hrs} \quad 2 \ hrs \ \ \& \ \ 15.8 \min$$

$$W = (62.26)(240)\left(\frac{150}{2}\right)$$

$$\boxed{= 933,900 \ \ ft \cdot lb}$$

$$t = \frac{W}{275 \frac{ft.lb}{\sec}}$$

$$= \frac{933,900 \ ft.lb}{275} \frac{\sec}{ft.lb}$$

$$\approx 3396 \ sec \frac{1 \ \ \min}{60 \ sec}$$

$$\boxed{\approx 56.6 \ \ \min}$$

Water weighs $62.59 \ lb / \ ft^3$

$$W = (62.59)(240)(150)$$

$$\boxed{= 2,253,240 \ \ ft \cdot lb}$$

$$t = \frac{W}{275 \frac{ft.lb}{\sec}}$$

$$= \frac{2,253,240 \ ft.lb}{275} \frac{\sec}{ft.lb}$$

$$\approx 8,193.60 \ sec \frac{1 \ \ hr}{3600 \ sec}$$

$$\boxed{\approx 2.276 \ \ hrs} \quad 2 \ hrs \ \ \& \ \ 16.56 \min$$

$$W = (62.59)(240)\left(\frac{150}{2}\right) \quad \boxed{= 938,850 \ \ ft \cdot lb}$$

$$t = \frac{W}{275 \frac{ft.lb}{\sec}}$$

$$= \frac{938,850 \ ft.lb}{275} \frac{\sec}{ft.lb}$$

$$\approx 3414 \ sec \frac{1 \ \ \min}{60 \ sec}$$

$$\boxed{\approx 56.9 \ \ \min}$$

Exercise

When a particle of mass *m* is at (*x*, 0), it is attracted toward the origin with a force whose magnitude is $\frac{k}{x^2}$. If the particle starts from rest at $x = b$ and is acted on by no other forces, find the work done on it by the time reaches $x = a$, $0 < a < b$.

Solution

$$F(x) = -\frac{k}{x^2}$$

$$W = \int_a^b -\frac{k}{x^2}\,dx$$

$$= k\int_a^b -\frac{1}{x^2}\,dx \qquad\qquad \int \frac{1}{x^2}\,dx = -\frac{1}{x}$$

$$= k\left[\frac{1}{x}\right]_a^b$$

$$= k\left(\frac{1}{b} - \frac{1}{a}\right)$$

$$= \frac{k(a-b)}{ab}$$

Exercise

The strength of Earth's gravitation field varies with the distance *r* from Earth's center, and the magnitude of the gravitational force experienced by a satellite of mass *m* during and after launch is

$$F(r) = \frac{mMG}{r^2}$$

Here, $M = 5.975\times 10^{24}\ kg$ is Earth's mass, $G = 6.6720\times 10^{-11}\ N\cdot m^2 kg^{-2}$ is the universal gravitational constant, and *r* is measured in meters. The work it takes to lift a 1000-*kg* satellite from Earth's surface to a circular orbit 35,780 *km* above Earth's center is therefore given by the integral

$$W = \int_{6,370,000}^{35,780,000} \frac{1000MG}{r^2}\,dr \ \ joules$$

Evaluate the integral. The lower limit of integration is Earth's radius in meters at the launch site. (This calculation does not take into account energy spend lifting the launch vehicle or energy spent bringing the satellite to orbit velocity.)

Solution

$$W = 1000MG\int_{6,370,000}^{35,780,000} \frac{dr}{r^2}$$

$$= 1000MG\left[-\frac{1}{r}\right]_{6,370,000}^{35,780,000}$$

$$= 1000\left(5.975\times10^{24}\right)\left(6.6720\times10^{-11}\right)\left(\frac{1}{6,370,000}-\frac{1}{35,780,000}\right)$$

$$\approx 5.144\times10^{10}\ J$$

Exercise

You drove an 800-*gal* truck from the base of Mt. Washington to the summit and discovered on arrival that the tank was only half full. You started with a full tank, climbed at a steady rate, and accomplished the 4750-*ft* elevation change in 50 *minutes*.

Assuming that the water leaked out at a steady rate, how much work was spent in carrying the water to the top? Do not count the work done in getting yourself and the truck there. Water weighs 8- *lb./gal*.

Solution

The force required to lift the water is equal to the water's weight which varies $8(800)$ *lbs*. to $8(400)$ *lbs*. over the 4750 *ft* change in elevation. Since it looses half of the water when the truck reaches its destination, it would loose all of the water if it went twice the distance. When the truck is *x* feet of the base of Mt. Washington, the water's weight is the following proportion.

$$F(x) = 8(800)\left(\frac{2(4750)-x}{2(4750)}\right) = 6400\left(1-\frac{x}{9500}\right)$$

$$W = 6400\int_0^{4750}\left(1-\frac{x}{9500}\right)dx$$

$$= 6400\left(x-\frac{x^2}{19000}\right)\Big|_0^{4750}$$

$$= 22,800,000\ \ ft-lbs$$

Exercise

A cylindrical water tank has height 8 *m* and radius 2 *m*

a) If the tank is full of water, how much work is required to pump the water to the level of the top of the tank and out of the tank?

b) Is it true it takes half as much work to pump the water out of the tank when it is half full as when it is full? Explain.

Solution

a) $W = \rho g\int_0^a (a-y)w(y)dy$

$$= 1{,}000(9.8)\int_0^8 (8-y)\left(2^2\pi\right)dy$$

$$= 39{,}200\pi\left(8y - \frac{1}{2}y^2\right)\Big|_0^8$$

$$= 39{,}200\pi(64-32)$$

$$= 125{,}400\pi$$

$$\approx 3.941\times 10^6\ J$$

b) The work done pumping the water from a half-full tank

$$W = 9{,}800(4\pi)\int_4^8 (8-y)dy$$

$$= 39{,}200\pi\left(8y - \frac{1}{2}y^2\right)\Big|_4^8$$

$$= 39{,}200\pi(32-32+8)$$

$$\approx 985203\ J$$

To empty a half-full tank, the work is

$$W = 39{,}200\pi\left(8y - \frac{1}{2}y^2\right)\Big|_0^4$$

$$= 39{,}200\pi(32-16)$$

$$\approx 2.9556\times 10^6\ J$$

NO, it is not true.

Exercise

A water tank is shaped like an inverted cone with height 6 *m* and base radius 1.5 *m*.

a) If the tank is full of water, how much work is required to pump the water to the level of the top of the tank and out of the tank?

b) Is it true it takes half as much work to pump the water out of the tank when it is half full as when it is full? Explain.

Solution

$$\frac{x}{1.5} = \frac{y}{6} \quad \rightarrow \quad x = \frac{y}{4}$$

$$Area = \pi x^2 = \frac{\pi}{16}y^2$$

a) $W = \rho g\int_0^a A(y)(a-y)dy$

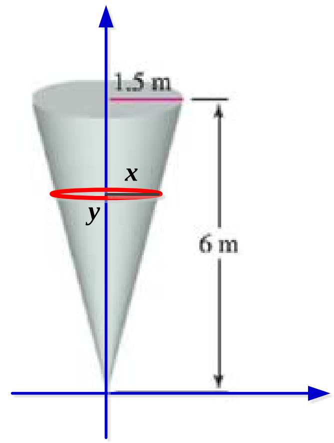
$$= 9{,}800 \int_0^6 \frac{\pi}{16} y^2 (6 - y)\, dy$$

$$= 612.5\pi \int_0^6 \left(6y^2 - y^3\right) dy$$

$$= 612.5\pi \left(2y^3 - \frac{1}{4} y^4\right) \Big|_0^6$$

$$= 612.5\pi (432 - 324)$$

$$= 66{,}150\pi \ J$$



b) $W = 9{,}800 \int_0^3 \frac{\pi}{16} y^2 (6 - y)\, dy$

$$= 612.5\pi \left(2y^3 - \frac{1}{4} y^4\right) \Big|_0^3$$

$$= 612.5\pi (54 - 20.25)$$

$$\approx 20{,}672\pi \ J$$

The work done is less than half the half amount from part (*a*).

It is not true, while the water must be raised further than water in the top half, due to the shape of the tank, there is far less water in the bottom half than in the top.

Exercise

A spherical water tank with an inner radius of 8 *m* has its lowest point 2 *m* above the ground. It is filled by a pipe that feed the tank at its lowest point.

a) Neglecting the volume of the inflow pipe, how much work is required to fill the tank if it is initially empty?

b) Now assume that the inflow pipe feeds the tank at the top of the tank. Neglecting the volume of the inflow pipe, how much work is required to fill the tank if it is initially empty?

Solution

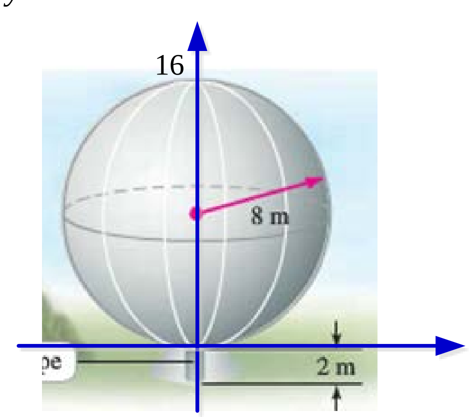
a) Equation of the tank: $x^2 + (y-8)^2 = 64 \quad \to \quad x^2 = 16y - y^2$

$$A(y) = \pi x^2 = \pi\left(16y - y^2\right)$$

$$W = \rho g \pi \int_0^{16} \left(16y - y^2\right) dy$$

$$= 9800\pi \left(8y^2 - \frac{1}{3} y^3\right) \Big|_0^{16}$$

$$= 9800\pi \left(16^2\right)\left(8 - \frac{16}{3}\right)$$



$$\approx 2.102 \times 10^8 \ J$$

b) The toal weight of the water lifted up for 18 *m* is

$$W = \frac{4\pi}{3} R^3 \rho g h$$

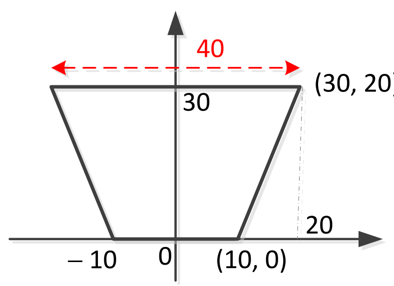
$$= \frac{4\pi}{3} 8^3 (9800)(18)$$

$$\approx 3.783 \times 10^8 \ J$$

Exercise

A large vertical dam in the shape of a symmetric trapezoid has a height of 30 *m*. a width of 20 *m*. at its base, and a width of 40 *m*. at the top. What is the total force on the face of the dam when the reservoir is full? $\left(\rho = 1000 \frac{kg}{m^3}, \ g = 9.8 \frac{m}{s^2} \right)$

Solution

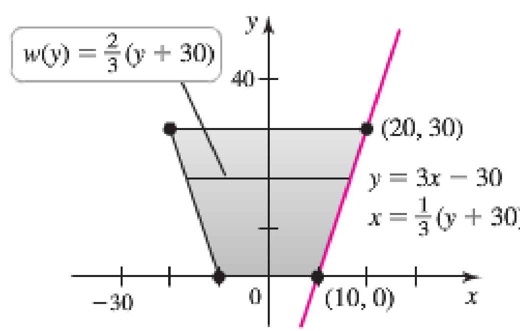


$$y - 0 = \frac{30}{10}(x - 10)$$

$$y = 3x - 30 \quad \rightarrow \quad x = \frac{1}{3}(y + 30)$$

Depth: $0 \le y \le 30$

Width: $w(y) = 2x = \frac{2}{3}(y + 30)$



$$F = \int_0^a \rho g (a - y) w(y) dy$$

$$= \int_0^{30} (10^3)(9.8)(30 - y)\frac{2}{3}(y + 30) dy$$

$$= \frac{19600}{3} \int_0^{30} (900 - y^2) dy$$

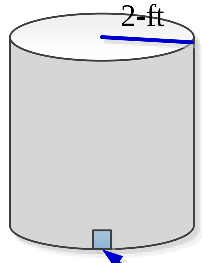
$$= \frac{19600}{3} \left(900y - \frac{1}{3}y^3\right) \Big|_0^{30}$$

$$= \frac{19600}{3}(27000 - 9000)$$

$$= 1.176 \times 10^8 \ kg$$

Exercise

Pumping water from a lake 15-*feet* below the bottom of the tank can fill the cylindrical tank shown here.



There are two ways to go about it. One is to pump the water through a hose attached to a valve in the bottom of the tank. The other is to attach the hose to the rim of the tank and let the water pour in. Which way will be faster? Give reason for answer

Solution

The water is being pumped from a lake that is 15-*feet* below the tank. That is the distance it takes to get the water from the lake to the valve, but this is not the total distance that the water is moved. We are forcing the water up into the tank, so the water travels a distance of y in the tank.

The total distance the water travels is $15 + y$.

The cross-section is a circle: $A(y) = \pi r^2 = 4\pi$

$$
\begin{aligned}
W &= 62.4\int_0^6 (4\pi)(15+y)\,dy \\
&= 249.6\pi\left(15y+\frac{1}{2}y^2\right)\Big|_0^6 \\
&= 249.6\pi(90+18) \\
&\approx 84,687.3\ \ ft-lbs
\end{aligned}
$$

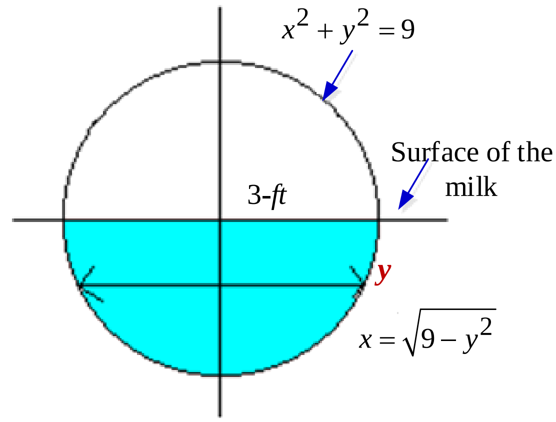
Now we are pumping the water to the top of the tank and letting it pour in.
Therefore, the distance that the water is pumped is $15+6=21\ ft$

$$
\begin{aligned}
W &= 62.4\int_0^6 21(4\pi)\,dy \\
&= 16,466.97(y)\Big|_0^6 \\
&\approx 98,801.83\ \ ft-lbs
\end{aligned}
$$

Exercise

A tank truck hauls milk in a 6-*feet* diameter horizontal right circular cylindrical tank. How must force does the milk exert on each end of the tank when the tank is half full?

Solution



Diameter = 6 $\rightarrow r=3$

Circular cylinder: $x^2+y^2=9$

$$L(y)=2x=2\sqrt{9-y^2}$$

Weight density of milk is $64.5\ \ lbs/\ ft^3$

$$F=64.5\int_{-3}^{0}2\sqrt{9-y^2}\,(0-y)\,dy$$

$$=64.5\int_{-3}^{0}\left(9-y^2\right)^{1/2}\ d\left(9-y^2\right)$$

$$=64.5\left(\tfrac{2}{3}\right)\left(9-y^2\right)^{3/2}\Big|_{-3}^{0}$$

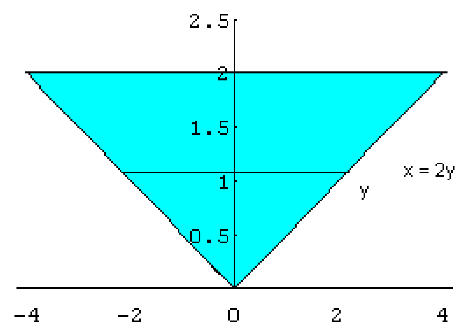
$$=43(27)$$

$$\underline{=1{,}161\ \ lbs|}$$

Exercise

The vertical triangular plate shown here is the end plate of a trough full of water. What is the fluid force against the plate?

Solution



$$L(y)=2x=4y$$

$$F=62.4\int_{0}^{2}(2-y)\cdot(4y)\,dy$$

$$=249.6\int_{0}^{2}\left(2y-y^2\right)dy$$

$$=249.6\left(y^2-\tfrac{1}{3}y^3\right)\Big|_{0}^{2}$$

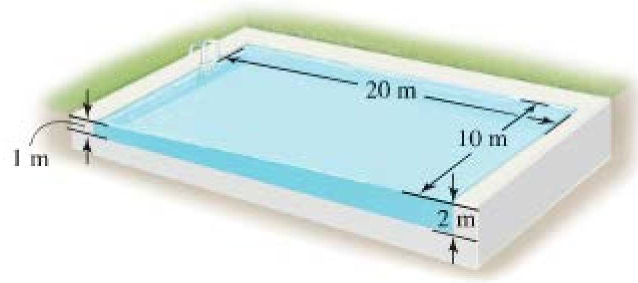
$$=249.6\left(4-\tfrac{8}{3}\right)$$

$$\underline{=332.8\ \ lbs|}$$

Exercise

A swimming pool is 20 *m* leg long and 10 *m* wide, with a bottom that slopes uniformly from a depth of 1 *m* at one end to a depth of 2 *m* at the other end.
Assuming the pool is full, how much work is required to pump the water to a level 0.2 *m* above the top of the pool?

Solution



Depth: $(2+.2)-y=2.2-y$

From 0–1 *m*:

$y=\frac{1}{20}(10-x) \quad \rightarrow \quad 10-x=2y$

$$A(y)=10(20y)=200y$$

From 1–2 *m*: $A(y)=10(20)=200$

$$W=\rho g\int_0^1 200y(2.2-y)dy+\rho g\int_1^2 200(2.2-y)dy$$

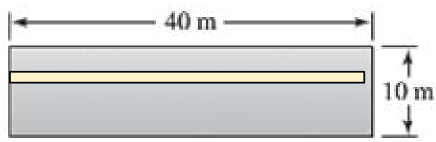
$$=(200\rho g)\left\{\left(1.1y^2-\frac{1}{3}y^3\right)\Big|_0^1+\left(2.2y-\frac{1}{2}y^2\right)\Big|_1^2\right\}$$

$$=\left(1.96\times10^6\right)\left(1.1-\frac{1}{3}+4.4-2-2.2+\frac{1}{2}\right)$$

$$\approx 2.875\times10^6\ J$$

Exercise

Find the total force on the face of the given dam



Solution

Freshwater Weight density: $\rho=62.4\ lb/ft^3=10^3\ kg/m^3$

$$F=\int_0^{10}\left(10^3\right)(9.8)(10-y)(40)dy \qquad F=\int_0^a \rho g(a-y)w(y)dy$$

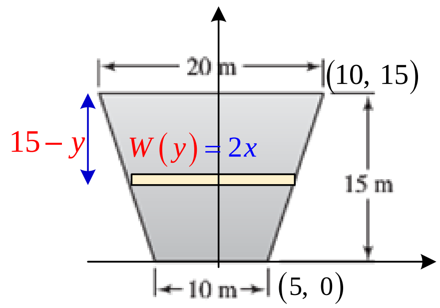
$$=392\times10^3\left(10y-\frac{1}{2}y^2\right)\Big|_0^{10}$$

$$=392\times10^3(100-50)$$

$$=196\times10^5\ N$$

Exercise

Find the total force on the face of the given dam



Solution

Freshwater Weight density: $\rho = 10^3\ kg/m^3$

$$y = \frac{15-0}{10-5}(x-5) = 3x - 15$$

$$x = \frac{1}{3}(y+15) \ \Rightarrow\ 2x = \frac{2}{3}(y+15)$$

$$F = \int_0^{15} \left(10^3\right)(9.8)(15-y)\frac{2}{3}(y+15)\,dy \qquad F = \int_0^a \rho g(a-y)w(y)\,dy$$

$$= \frac{19.6}{3}\times 10^3 \int_0^{15}\left(225 - y^2\right)dy$$

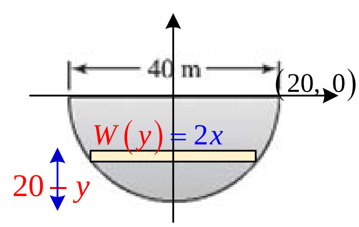
$$= \frac{19.6}{3}\times 10^3 \left(225y - \frac{1}{3}y^3\right)\Big|_0^{15}$$

$$= \frac{19.6}{3}\times 10^3 \left(15^5 - \frac{1}{3}15^3\right)$$

$$= 1.47\times 10^7\ \ N$$

Exercise

Find the total force on the face of the given dam



Solution

Freshwater Weight density: $\rho = 10^3\ kg/m^3$

$$x^2 + (y-20)^2 = 20^2$$

$$x = \sqrt{400 - \left(y^2 - 40y + 400\right)} \ \rightarrow\ 2x = 2\sqrt{40y - y^2}$$

$$F = \int_0^{20} \left(10^3\right)(9.8)(20-y)(2)\sqrt{40y-y^2}\,dy \qquad F = \int_0^a \rho g(a-y)w(y)\,dy$$

$$= 9.8\times 10^3 \int_0^{20}\left(40y - y^2\right)^{1/2} d\left(40y - y^2\right).$$

$$= \frac{19.6}{3}\times 10^3 \left(40y - y^2\right)^{3/2}\Big|_0^{20}$$

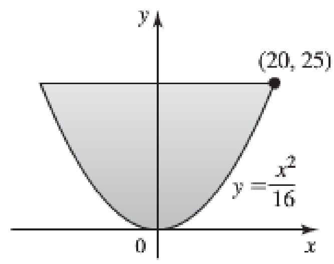
$$= \frac{19.6}{3}\times 10^3 (800-400)^{3/2}$$

$$= \frac{19.6}{3}\times 10^3 (20)^3$$

$$= 5.227\times 10^7\ \ N$$

Exercise

Find the total force on the face of the given dam



Solution

Freshwater Weight density: $\rho = 10^3\ kg/m^3$

$x^2 = 16y \to x = 4\sqrt{y} \quad \Rightarrow \underline{2x = 8\sqrt{y}}$

$$F = \int_0^{25} \left(10^3\right)(9.8)(25-y)(8)\sqrt{y}\ dy \qquad F = \int_0^a \rho g(a-y)w(y)dy$$

$$= 78.4\times 10^3 \int_0^{25}\left(25y^{1/2} - y^{3/2}\right)dy$$

$$= 78.4\times 10^3\left(\frac{50}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right)\Big|_0^{25}$$

$$= 78.4\times 10^3\left(\frac{2}{3}5^5 - \frac{2}{5}5^5\right)$$

$$= 78.4\times 5^5\times 10^3\left(\frac{4}{15}\right)$$

$$\underline{= 6.533\times 10^7\ \ N}$$

Exercise

A large building shaped like a box is 50 *m* high with a face that is 80 *m* wide. A strong wind blows directly at the face of the building, exerting a pressure of 150 N/m^2 at the ground and increasing with height according to $P(y) = 150 + 2y$, where *y* is the height above the ground. Calculate the total force on the building, which is a meaure of the resistance that must be included in the design of the building.

Solution

$$F = \int_0^{50} (150+2y)(80)\ dy \qquad F = \int_0^a P(y)w(y)dy$$

$$= 80\left(150y + y^2\right)\Big|_0^{50}$$

$$= 80(7500 + 2500)$$

$$\underline{= 8\times 10^5\ N}$$

Exercise

Adiving pool that is 4 *m* deep full of water has a viewing window on one of its vertical walls. Find the force of the a square window, 0.5 *m* on side, with the lower edge of the window on the bottom of the pool.

Solution

Freshwater Weight density: $\rho = 10^3\ kg/m^3$

$$F = \int_0^{0.5} 10^3 (9.8)(4-y)(0.5)\ dy \qquad\qquad F = \int_0^a \rho g(a-y)w(y)dy$$

$$= 4.9\times 10^3 \left(4y - \frac{1}{2}y^2\right)\Big|_0^{0.5}$$

$$= 4.9\times 10^3 \left(2 - \frac{1}{8}\right)$$

$$= 4.9\times 10^3 \left(\frac{15}{8}\right)$$

$$\underline{= 9187.5\ N}$$

Exercise

Adiving pool that is 4 *m* deep full of water has a viewing window on one of its vertical walls. Find the force of the a square window, 0.5 *m* on side, with the lower edge of the window 1 *m* from the bottom of the pool.

Solution

Freshwater Weight density: $\rho = 10^3\ kg/m^3$

$$F = \int_1^{1.5} 10^3 (9.8)(4-y)(0.5)\ dy \qquad\qquad F = \int_0^a \rho g(a-y)w(y)dy$$

$$= 4.9\times 10^3 \left(4y - \frac{1}{2}y^2\right)\Big|_1^{1.5}$$

$$= 4.9\times 10^3 \left(6 - \frac{9}{8} - 4 + \frac{1}{2}\right)$$

$$= 4.9\times 10^3 \left(2 - \frac{5}{8}\right)$$

$$= 4.9\times 10^3 \left(\frac{11}{8}\right)$$

$$\underline{= 6737.5\ N}$$

Exercise

Adiving pool that is 4 *m* deep full of water has a viewing window on one of its vertical walls. Find the force of the a circle window, with a radius of 0.5 *m*, tangent to the bottom of the pool.

Solution

Equation of the circle: $x^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4}$

$$x^2=\frac{1}{4}-y^2+y-\frac{1}{4}=y-y^2$$

$$x=\sqrt{y-y^2} \quad \Rightarrow \quad 2x=2\sqrt{y-y^2}$$

$$F=\int_0^1 10^3(9.8)(4-y)\left(2\sqrt{y-y^2}\right)\,dy \qquad F=\int_0^a \rho g(a-y)w(y)\,dy$$

$$=19.6\times 10^3\int_0^1\left(\frac{7}{2}+\frac{1}{2}-y\right)\sqrt{y-y^2}\,dy \qquad d\left(y-y^2\right)=1-2y=2\left(\frac{1}{2}-y\right)$$

$$=19.6\times 10^3\int_0^1 \frac{7}{2}\sqrt{y-y^2}\,dy+19.6\times 10^3\int_0^1\left(\frac{1}{2}-y\right)\sqrt{y-y^2}\,dy$$

$$=7\times 9.8\times 10^3\int_0^1\sqrt{y-y^2}\,dy+39.2\times 10^3\int_0^1\sqrt{y-y^2}\,d\left(y-y^2\right) \qquad \textit{Area of semicircle}: \frac{1}{2}\pi r^2$$

$$=7\times 9.8\times 10^3\left(\frac{1}{2}\frac{\pi}{4}\right)+39.2\times 10^3\left(\frac{1}{2}y^2-\frac{1}{3}y^3\right)\Big|_0^1$$

$$=\frac{7\pi}{8}\times 9.8\times 10^3+39.2\times 10^3\left(\frac{1}{2}-\frac{1}{3}\right)$$

$$=\frac{7\pi}{8}\times 9.8\times 10^3+39.2\times 10^3\left(\frac{1}{6}\right)$$

$$\approx 2.694\times 10^4\ N$$

Exercise

A rigid body with a mass of 2 *kg* moves along a line due to a force that produces a position function $x(t)=4t^2$, where *x* is measured in *meters* and *t* is measured in *seconds*. Find the work done during the first 5 *sec*. in two ways.

a) Note that $x''(t)=8$; then use Newton's second law, $\left(F=ma=mx''(t)\right)$ to evaluate the work integral $W=\int_{x_0}^{x_f}F(x)\,dx$, where x_0 and x_f are the initial and final positions, repectively.

b) Change variables in the work integral and integrate with respect to *t*.

Solution

a) $W = \int_{x_0}^{x_f} mx'' dx$

$= \int_0^{x(5)} 2\times 8 dx \qquad\qquad x(5) = 4(5)^2 = 100$

$= 16x \Big|_0^{100}$

$\underline{= 1600 \ \ J}$

b) $W = \int_{x_0}^{x_f} mx'' dx$

$= \int_0^{5} 2\times 8 \frac{dx}{dt} dt$

$= 16\int_0^{5} (8t)\, dt$

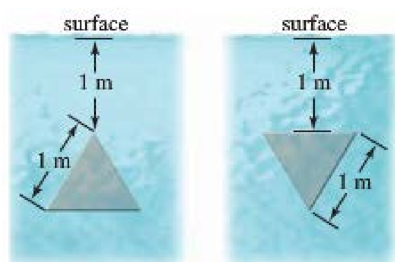
$= 64t^2 \Big|_0^{5}$

$\underline{= 1600 \ \ J}$

Exercise

A plate shaped like an equilateral triangle 1 *m* on a side is placed on a vetical wall 1 *m* below the surface of a pool filled with water. On which plate in the figure is the force is greater

Solution



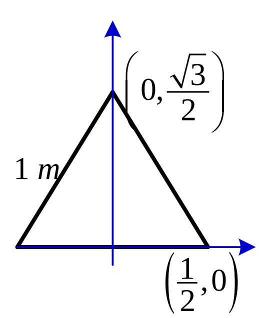
The left picture has more force than the right, because of the bottom part is wider side in the pool.

For the left side plate:

$y = h = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}$

Line segement: $y = \dfrac{0-\frac{\sqrt{3}}{2}}{\frac{1}{2}-0}\left(x-\frac{1}{2}\right) = -\sqrt{3}\left(x-\frac{1}{2}\right)$

$x = \frac{1}{2} - \frac{1}{\sqrt{3}}y \ \rightarrow \ \underline{2x = 1-\frac{2}{\sqrt{3}}y}$



$F = \int_0^{\sqrt{3}/2} \rho g\left(1+\frac{\sqrt{3}}{2}-y\right)\left(1-\frac{2}{\sqrt{3}}y\right)dy$

$$= \rho g \int_0^{\sqrt{3}/2} \left(1 + \frac{\sqrt{3}}{2} - y - \frac{2\sqrt{3}}{3} y - y + \frac{2\sqrt{3}}{3} y^2\right) dy$$

$$= \rho g \int_0^{\sqrt{3}/2} \left(1 + \frac{\sqrt{3}}{2} - \frac{2}{3}\left(3 + \sqrt{3}\right) y + \frac{2\sqrt{3}}{3} y^2\right) dy$$

$$= \rho g \left(\left(1 + \frac{\sqrt{3}}{2}\right) y - \frac{1}{3}\left(3 + \sqrt{3}\right) y^2 + \frac{2\sqrt{3}}{9} y^3\right) \Big|_0^{\sqrt{3}/2}$$

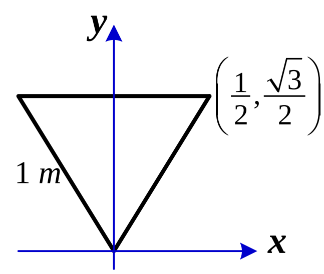
$$= 9{,}800\left(\frac{\sqrt{3}}{2} + \frac{3}{4} - \frac{3}{4} - \frac{\sqrt{3}}{4} + \frac{1}{4}\right) \qquad \rho = 1000 \ \frac{kg}{m^3} \quad g = 9.8 \ m/s^2$$

$$\boxed{= 2{,}450\left(1 + \sqrt{3}\right) \ N}$$

For the right side plate:

Line segement: $y = \dfrac{\frac{\sqrt{3}}{2} - 0}{\frac{1}{2} - 0}(x - 0) = \sqrt{3} x$

$$x = \frac{1}{\sqrt{3}} y \ \rightarrow \ \boxed{2x = \frac{2}{\sqrt{3}} y}$$



$$F = \int_0^{\sqrt{3}/2} \rho g \left(1 + \frac{\sqrt{3}}{2} - y\right)\left(\frac{2}{\sqrt{3}} y\right) dy$$

$$= \rho g \int_0^{\sqrt{3}/2} \left(\frac{2\sqrt{3}}{3} y + y - \frac{2\sqrt{3}}{3} y^2\right) dy$$

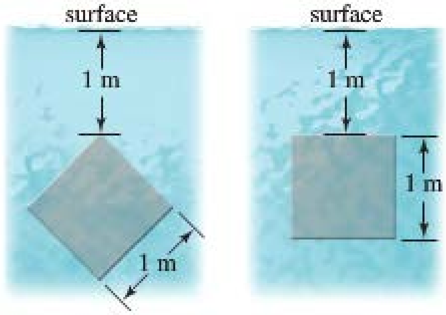
$$= \rho g \left(\frac{\sqrt{3}}{3} y^2 + \frac{1}{2} y^2 - \frac{2\sqrt{3}}{9} y^3\right) \Big|_0^{\sqrt{3}/2}$$

$$= 9{,}800\left(\frac{\sqrt{3}}{4} + \frac{3}{8} - \frac{1}{4}\right)$$

$$\boxed{= 2{,}450\left(1 + 2\sqrt{3}\right) \ N} \quad \boxed{\approx 10{,}937 \ N}$$

Exercise

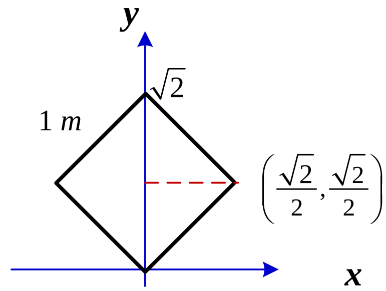
A square plate 1 *m* on a side is placed on a vetical wall 1 *m* below the surface of a pool filled with water.



On which plate in the figure is the force is greater

Solution

For the plate on left side :



$$2x^2 = 1 \;\rightarrow\; x = \frac{\sqrt{2}}{2}$$

Line segment OA: $\underline{y = x}|$ $\rightarrow 2x = 2y$

Line segment AB: $\lfloor y = \frac{\sqrt{2} - \frac{\sqrt{2}}{2}}{0 - \frac{\sqrt{2}}{2}} x + \sqrt{2} \;\underline{= -x + \sqrt{2}}|$ $\rightarrow\; x = \sqrt{2} - y$

$$F = \rho g \int_0^{\sqrt{2}/2} \left(1 + \sqrt{2} - y\right)(2y)\,dy + \rho g \int_{\sqrt{2}/2}^{\sqrt{2}} \left(1 + \sqrt{2} - y\right)(2)\left(\sqrt{2} - y\right)dy$$

$$= 2\rho g \int_0^{\sqrt{2}/2} \left(y + \sqrt{2}y - y^2\right)dy + 2\rho g \int_{\sqrt{2}/2}^{\sqrt{2}} \left(\sqrt{2} + 2 - y - 2\sqrt{2}y + y^2\right)dy$$

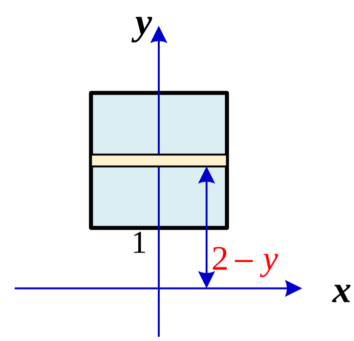
$$= 2\rho g \left(\frac{1}{2}y^2 + \frac{\sqrt{2}}{2}y^2 - \frac{1}{3}y^3\right)_0^{\sqrt{2}/2} + 2\rho g \left(\sqrt{2}y + 2y - \frac{1}{2}y^2 - \sqrt{2}y^2 + \frac{1}{3}y^3\right)_{\sqrt{2}/2}^{\sqrt{2}}$$

$$= 2\rho g \left(\frac{1}{4} + \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{12} + 2 + 2\sqrt{2} - 2 - 2\sqrt{2} + \frac{2\sqrt{2}}{3} - 1 - \sqrt{2} + \frac{1}{4} + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{12}\right)$$

$$= 2\rho g \left(\frac{1}{2} + \frac{\sqrt{2}}{4}\right) \qquad \rho = 1000 \;\frac{kg}{m^3} \quad g = 9.8 \;m/s^2$$

$$\underline{= 9{,}800\left(1 + \frac{\sqrt{2}}{2}\right) \;N}| \quad \underline{\approx 16{,}730 \;N}|$$

For the plate on right side:



$$F = \rho g \int_0^1 (2 - y)(1)\,dy$$

$$= \rho g \left(2y - \frac{1}{2}y^2\right)_0^1$$

$$= 9{,}800\left(\frac{3}{2}\right)$$

$$\underline{= 14{,}700 \;N}|$$

Solution   *Section* 1.8 – Exponential Models

Exercise

Find the derivative of $y = \ln\left(\dfrac{\sqrt{\sin\theta\cos\theta}}{1+2\ln\theta}\right)$

Solution

$$y = \ln(\sin\theta\cos\theta)^{1/2} - \ln(1+2\ln\theta)$$

$$= \tfrac{1}{2}\left(\ln(\sin\theta) + \ln(\cos\theta)\right) - \ln(1+2\ln\theta)$$

$$y' = \frac{1}{2}\left(\frac{(\sin\theta)'}{\sin\theta} + \frac{(\cos\theta)'}{\cos\theta}\right) - \frac{(1+2\ln\theta)'}{1+2\ln\theta} = \frac{1}{2}\left(\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}\right) - \frac{\frac{2}{\theta}}{1+2\ln\theta}$$

$$= \frac{1}{2}(\cot\theta - \tan\theta) - \frac{2}{\theta(1+2\ln\theta)}$$

Exercise

Find the derivative of $f(x) = e^{\left(4\sqrt{x}+x^2\right)}$

Solution

$$\frac{d}{dx}e^{\left(4\sqrt{x}+x^2\right)} = e^{\left(4\sqrt{x}+x^2\right)}\frac{d}{dx}\left(4\sqrt{x}+x^2\right) = \left(\frac{2}{\sqrt{x}} + 2x\right)e^{\left(4\sqrt{x}+x^2\right)}$$

Exercise

Find the derivative of $f(t) = \ln\left(3te^{-t}\right)$

Solution

$$\frac{d}{dt}\ln\left(3te^{-t}\right) = \frac{\left(3te^{-t}\right)'}{3te^{-t}}$$

$$= 3\frac{e^{-t} - te^{-t}}{3te^{-t}}$$

$$\frac{e^{-t}(1-t)}{te^{-t}}$$

$$= \frac{1-t}{t}$$

$$\ln\left(3te^{-t}\right) = \ln 3 + \ln t + \ln e^{-t}$$

$$= \ln 3 + \ln t - t$$

$$\left(\ln\left(3te^{-t}\right)\right)' = \frac{1}{t} - 1$$

$$= \frac{1-t}{t}$$

Exercise

Find the Derivative of $f(x)=\dfrac{e^{\sqrt{x}}}{\ln\left(\sqrt{x}+1\right)}$

Solution

$$f=e^{\sqrt{x}} \qquad U=\sqrt{x}=x^{1/2}\Rightarrow U'=\tfrac{1}{2}x^{-1/2} \qquad f'=\tfrac{1}{2}x^{-1/2}e^{\sqrt{x}}=\frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

$$g=\ln\left(\sqrt{x}+1\right) \qquad U=x^{1/2}+1\Rightarrow U'=\tfrac{1}{2}x^{-1/2} \qquad g'=\frac{\frac{1}{2}x^{-1/2}}{\sqrt{x}+1}=\frac{1}{2x^{1/2}\left(\sqrt{x}+1\right)}$$

$$f'(x)=\frac{\frac{e^{\sqrt{x}}}{2\sqrt{x}}\ln\left(\sqrt{x}+1\right)-\frac{1}{2\sqrt{x}\left(\sqrt{x}+1\right)}e^{\sqrt{x}}}{\left(\ln\left(\sqrt{x}+1\right)\right)^2}$$

$$=\frac{\frac{\left(\sqrt{x}+1\right)e^{\sqrt{x}}\ln\left(\sqrt{x}+1\right)-e^{\sqrt{x}}}{2\sqrt{x}\left(\sqrt{x}+1\right)}}{\left(\ln\left(\sqrt{x}+1\right)\right)^2}$$

$$\boxed{=\frac{e^{\sqrt{x}}\left[\left(\sqrt{x}+1\right)\ln\left(\sqrt{x}+1\right)-1\right]}{2\sqrt{x}\left(\sqrt{x}+1\right)\left(\ln\left(\sqrt{x}+1\right)\right)^2}}$$

Exercise

Find the Derivative of $y=\sqrt{\dfrac{(x+1)^{10}}{(2x+1)^5}}$

Solution

$$y=\left(\frac{(x+1)^{10}}{(2x+1)^5}\right)^{1/2}$$

$$\ln y=\ln\left(\frac{(x+1)^{10}}{(2x+1)^5}\right)^{1/2}$$

$$\ln y=\frac{1}{2}\ln\left(\frac{(x+1)^{10}}{(2x+1)^5}\right)$$

$$=\frac{1}{2}\left(\ln(x+1)^{10}-\ln(2x+1)^5\right)$$

$$=\frac{1}{2}\left(10\ln(x+1)-5\ln(2x+1)\right)$$

$$= 5\ln(x+1) - \frac{5}{2}\ln(2x+1)$$

$$\frac{y'}{y} = 5\frac{1}{x+1} - \frac{5}{2}\frac{2}{2x+1}$$

$$\frac{y'}{y} = \frac{5}{x+1} - \frac{5}{2x+1}$$

$$y' = y\left(\frac{5}{x+1} - \frac{5}{2x+1}\right)$$

$$y' = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}}\left(\frac{5}{x+1} - \frac{5}{2x+1}\right)$$

Exercise

Find the derivative of $f(x) = (2x)^{4x}$

Solution

$$\ln f(x) = 4x\ln(2x)$$

$$\frac{f'}{f} = 4\left(\ln 2x + x\frac{2}{2x}\right)$$

$$f'(x) = 4(\ln 2x + 1)(2x)^{4x}$$

Exercise

Find the derivative of $f(x) = 2^{x^2}$

Solution

$$f'(x) = 2x \cdot 2^{x^2} \ln 2$$

Exercise

Find the derivative of $h(y) = y^{\sin y}$

Solution

$$\ln h = \ln y^{\sin y} = \sin y \ln y$$

$$\frac{h'}{h} = \cos y \ln y + \frac{\sin y}{y}$$

$$h'(y) = y^{\sin y}\left(\cos y \ln y + \frac{\sin y}{y}\right)$$

Exercise

Find the derivative of $f(x) = x^{\pi}$

Solution

$$\ln f = \pi \ln x$$

$$\frac{f'}{f} = \frac{\pi}{x}$$

$$\boxed{f'(x) = \pi x^{\pi - 1}}$$

Exercise

Find the derivative of $h(t) = (\sin t)^{\sqrt{t}}$

Solution

$$\ln h = \ln(\sin t)^{\sqrt{t}} = \sqrt{t}\ln(\sin t)$$

$$\frac{h'}{h} = \frac{1}{2\sqrt{t}}\ln \sin t + \sqrt{t}\frac{\cos t}{\sin t}$$

$$\boxed{h'(t) = \frac{1}{2\sqrt{t}}(\ln \sin t + 2t\cot t)(\sin t)^{\sqrt{t}}}$$

Exercise

Find the derivative of $p(x) = x^{-\ln x}$

Solution

$$\ln p(x) = \ln x^{-\ln x} = -(\ln x)^2$$

$$\frac{p'}{p} = -\frac{2\ln x}{x}$$

$$p'(x) = -\frac{2\ln x}{x}x^{-\ln x} \boxed{= -\frac{2\ln x}{x^{1+\ln x}}}$$

Exercise

Find the derivative of $f(x) = x^{2x}$

Solution

$$\ln f = \ln x^{2x} = 2x\ln x$$

$$\frac{f'}{f} = 2\ln x + 2\frac{x}{x}$$

$$\boxed{f'(x) = 2(1 + \ln x)x^{2x}}$$

Exercise

Find the derivative of $f(x) = x^{\tan x}$

Solution

$$\ln f(x) = \ln x^{\tan x} = \tan x \ln x$$

$$\frac{f'}{f} = \sec^2 x \ln x + \frac{\tan x}{x}$$

$$\boxed{f'(x) = \left(\sec^2 x \ln x + \frac{\tan x}{x}\right) x^{\tan x}}$$

Exercise

Find the derivative of $f(x) = x^e + e^x$

Solution

$$\boxed{f'(x) = ex^{e-1} + e^x}$$

Exercise

Find the derivative of $f(x) = x^{x^{10}}$

Solution

$$\ln f = x^{10} \ln x$$

$$\frac{f'}{f} = 10x^9 \ln x + \frac{x^{10}}{x}$$

$$f'(x) = x^{x^{10}} \left(10x^9 \ln x + x^9\right)$$

$$\boxed{= x^{9+x^{10}} (10 \ln x + 1)}$$

Exercise

Find the derivative of $f(x) = \left(1 + \frac{4}{x}\right)^x$

Solution

$$\ln f = x \ln\left(1 + \frac{4}{x}\right)$$

$$\frac{f'}{f} = \ln\left(1 + \frac{4}{x}\right) + x \frac{-\frac{4}{x^2}}{1 + \frac{4}{x}}$$

$$\boxed{f'(x) = \left(1 + \frac{4}{x}\right)^x \left(\ln\left(1 + \frac{4}{x}\right) - \frac{4}{x+4}\right)}$$

Exercise

Find the derivative of $f(x)=\cos\left(x^{2\sin x}\right)$

Solution

$$f'=-\left(x^{2\sin x}\right)'\sin\left(x^{2\sin x}\right)$$

$$Let\ y=x^{2\sin x} \quad \to \quad \ln y=(2\sin x)\ln x$$

$$\frac{y'}{y}=2\cos x\ln x+\frac{2\sin x}{x}$$

$$\underline{f'=-x^{2\sin x}\left(2\cos x\ln x+\frac{2\sin x}{x}\right)\sin\left(x^{2\sin x}\right)}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{2ydy}{y^2-25}$

Solution

$$\int \frac{2ydy}{y^2-25}=\int \frac{d\left(y^2-25\right)}{y^2-25} \qquad d\left(y^2-25\right)=2ydy$$

$$\underline{=\ln\left|y^2-25\right|+C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sec y\tan y}{2+\sec y}dy$

Solution

$$\int \frac{\sec y\tan y}{2+\sec y}dy=\int \frac{d(2+\sec y)}{2+\sec y} \qquad d(2+\sec y)=\sec y\tan ydy$$

$$\underline{=\ln\left|2+\sec y\right|+C}$$

Exercise

Find the integral $\displaystyle\int \frac{5}{e^{-5x}+7}dx$

Solution

$$\int \frac{5}{e^{-5x}+7}\frac{e^{5x}}{e^{5x}}dx=\int \frac{5e^{5x}}{1+7e^{5x}}dx \qquad d\left(1+7e^{5x}\right)=35e^{5x}dx$$

$$= \frac{1}{7}\int \frac{d\left(1+7e^{5x}\right)}{1+7e^{5x}}$$

$$\underline{= \frac{1}{7}\ln\left|1+7e^{5x}\right| + C}$$

Exercise

Find the integral $\displaystyle\int \frac{e^{2x}}{4+e^{2x}}dx$

Solution

$$\int \frac{e^{2x}}{4+e^{2x}}dx = \frac{1}{2}\int \frac{d\left(4+e^{2x}\right)}{4+e^{2x}} \qquad d\left(4+e^{2x}\right) = 2e^{2x}dx$$

$$\underline{= \frac{1}{2}\ln\left(4+e^{2x}\right) + C}$$

Exercise

Find the integral $\displaystyle\int \frac{dx}{x\ln x\ln(\ln x)}$

Solution

$$\int \frac{dx}{x\ln x\ln(\ln x)} = \int \frac{d(\ln(\ln x))}{\ln(\ln x)} \qquad d(\ln(\ln x)) = \frac{1/x}{\ln x} = \frac{1}{x\ln x}$$

$$\underline{= \ln\ln(\ln x) + C}$$

Exercise

Find the integral $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

Solution

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = 2\int e^{\sqrt{x}}d\left(\sqrt{x}\right) \qquad d\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}dx$$

$$\underline{= 2e^{\sqrt{x}} + C}$$

Exercise

Find the integral $\displaystyle\int \frac{e^{\sin x}}{\sec x}dx$

Solution

$$\int \frac{e^{\sin x}}{\sec x}dx = \int e^{\sin x} d(\sin x) \qquad d(\sin x) = \cos x dx = \frac{dx}{\sec x}$$

$$\underline{= e^{\sin x} + C}$$

Exercise

Find the integral $\int \frac{e^x + e^{-x}}{e^x - e^{-x}}dx$

Solution

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}}dx = \int \frac{1}{e^x - e^{-x}} d\left(e^x - e^{-x}\right) \qquad d\left(e^x - e^{-x}\right) = \left(e^x + e^{-x}\right)dx$$

$$\underline{= \ln\left(e^x - e^{-x}\right) + C}$$

Exercise

Find the integral $\int \frac{4^{\cot x}}{\sin^2 x}dx$

Solution

$$\int \frac{4^{\cot x}}{\sin^2 x}dx = -\int 4^{\cot x} d(\cot x) \qquad d(\cot x) = -\csc^2 x dx = -\frac{dx}{\sin^2 x}$$

$$\underline{= \frac{4^{\cot x}}{\ln 4} + C} \qquad \int a^x dx = \frac{a^x}{\ln a} + C$$

Exercise

Find the integral $\int \frac{4x^2 + 2x + 4}{x+1}dx$

Solution

$$\int \frac{4x^2 + 2x + 4}{x+1}dx = \int \left(4x + 2 + \frac{6}{x+1}\right)dx$$

$$= \int (4x - 2)dx + \int \frac{6}{x+1}dx$$

$$= \int (4x - 2)dx + 6\int \frac{d(x+1)}{x+1} \qquad \int \frac{d(U)}{U} = \ln|U|$$

$$\underline{= 2x^2 - 2x + 6\ln|x+1| + C}$$

Exercise

Evaluate the integral $\displaystyle\int_{\ln 4}^{\ln 9} e^{x/2}dx$

Solution

$$\int_{\ln 4}^{\ln 9} e^{x/2}dx = 2e^{x/2}\Big|_{\ln 2^2}^{\ln 3^2}$$

$$= 2\left(e^{(2\ln 3)/2} - e^{(2\ln 2)/2}\right)$$

$$= 2\left(e^{\ln 3} - e^{\ln 2}\right)$$

$$= 2(3-2)$$

$$= 2$$

Exercise

Evaluate the integral $\displaystyle\int_0^3 \frac{2x-1}{x+1}dx$

Solution

$$\int_0^3 \frac{2x-1}{x+1}dx = \int_0^3 \left(2 - \frac{3}{x+1}\right)dx$$

$$= \left(2x - 3\ln|x+1|\right)\Big|_0^3$$

$$= 6 - 3\ln 4$$

$$\begin{array}{r} 2 \\ x+1\overline{)2x-1} \\ \underline{-2x-2} \\ -3 \end{array}$$

Exercise

Evaluate the integral $\displaystyle\int_e^{e^2} \frac{dx}{x\ln^3 x}$

Solution

$$\int_e^{e^2} \frac{dx}{x\ln^3 x} = \int_e^{e^2} \ln^{-3} x \, d(\ln x)$$

$$= -\frac{1}{2}\ln^{-2} x\Big|_e^{e^2}$$

$$= -\frac{1}{2}(2-1)$$

$$= -\frac{1}{2}$$

Exercise

Evaluate the integral $\displaystyle\int_{e^2}^{e^3} \frac{dx}{x\ln x\ln^2(\ln x)}$

Solution

$$\int_{e^2}^{e^3} \frac{dx}{x\ln x\ln^2(\ln x)} = \int_{e^2}^{e^3} (\ln(\ln x))^{-2}\, d(\ln(\ln x)) \qquad d(\ln(\ln x)) = \frac{1/x}{\ln x} = \frac{1}{x\ln x}$$

$$= -\frac{1}{\ln(\ln x)}\Big|_{e^2}^{e^3}$$

$$= -\frac{1}{\ln\left(\ln e^3\right)} + \frac{1}{\ln\left(\ln e^2\right)}$$

$$= -\frac{1}{\ln 3} + \frac{1}{\ln 2}$$

Exercise

Evaluate the integral $\displaystyle\int_0^1 \frac{y\ln^4\left(y^2+1\right)}{y^2+1}dy$

Solution

$$\int_0^1 \frac{y\ln^4\left(y^2+1\right)}{y^2+1}dy = \frac{1}{2}\int_0^1 \ln^4\left(y^2+1\right) d\left(\ln\left(y^2+1\right)\right) \qquad d\left(\ln\left(y^2+1\right)\right) = \frac{2y}{y^2+1}dy$$

$$= \frac{1}{10}\ln^5\left(y^2+1\right)\Big|_0^1$$

$$= \frac{1}{10}(\ln 2)^5$$

Exercise

Evaluate the integral $\displaystyle\int_{\ln 2}^{\ln 3} \frac{e^x+e^{-x}}{e^{2x}-2+e^{-2x}}dx$

Solution

$$\int_{\ln 2}^{\ln 3} \frac{e^x+e^{-x}}{e^{2x}-2+e^{-2x}}dx = \int_{\ln 2}^{\ln 3} \frac{e^x+e^{-x}}{\left(e^x-e^{-x}\right)^2}dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{1}{\left(e^x-e^{-x}\right)^2} d\left(e^x-e^{-x}\right)$$

$$= -\frac{1}{e^x - e^{-x}}\Bigg|_{\ln 2}^{\ln 3}$$

$$= -\frac{1}{e^{\ln 3} - e^{-\ln 3}} + \frac{1}{e^{\ln 2} - e^{-\ln 2}}$$

$$= \frac{1}{2 - \frac{1}{2}} - \frac{1}{3 - \frac{1}{3}}$$

$$= \frac{2}{3} - \frac{3}{8}$$

$$= \frac{7}{24}$$

Exercise

Evaluate the integral $\int_{-2}^{2} \frac{e^{z/2}}{e^{z/2}+1} dz$

Solution

$$\int_{-2}^{2} \frac{e^{z/2}}{e^{z/2}+1} dz = 2\int_{-2}^{2} \frac{1}{e^{z/2}+1} d\left(e^{z/2}+1\right)$$

$$d\left(e^{z/2}+1\right) = \frac{1}{2}e^{z/2}dz$$

$$= 2\ln\left(e^{z/2}+1\right)\Big|_{-2}^{2}$$

$$= 2\left(\ln(e+1) - \ln\left(e^{-1}+1\right)\right)$$

Exercise

Evaluate the integral $\int_{0}^{\pi/2} 4^{\sin x} \cos x \, dx$

Solution

$$\int_{0}^{\pi/2} 4^{\sin x} \cos x \, dx = \int_{0}^{\pi/2} 4^{\sin x} \, d(\sin x)$$

$$\int a^x dx = \frac{a^x}{\ln a}$$

$$= \frac{1}{\ln 4} 4^{\sin x}\Big|_{0}^{\pi/2}$$

$$= \frac{1}{\ln 4}(4-1)$$

$$= \frac{3}{\ln 4}$$

Exercise

Evaluate the integral $\int_{1/3}^{1/2} \frac{10^{1/p}}{p^2} dp$

Solution

$$\int_{1/3}^{1/2} \frac{10^{1/p}}{p^2} dp = -\int_{1/3}^{1/2} 10^{1/p} d\left(\frac{1}{p}\right)$$

$$= -\frac{1}{\ln 10} 10^{1/p} \Big|_{1/3}^{1/2}$$

$$= -\frac{1}{\ln 10}\left(10^2 - 10^3\right)$$

$$= \frac{900}{\ln 10}$$

Exercise

Evaluate the integral $\int_1^2 (1 + \ln x) x^x dx$

Solution

$$y = x^x \quad \to \quad \ln y = x \ln x$$

$$\frac{y'}{y} = 1 + \ln x \quad \Rightarrow \quad \left(x^x\right)' = x^x (1 + \ln x)$$

$$\int_1^2 (1 + \ln x) x^x dx = \int_1^2 d\left(x^x\right)$$

$$= x^x \Big|_1^2$$

$$= 2^2 - 1$$

$$= 3$$

Exercise

Find a curve through the origin in the xy-plane whose length from $x = 0$ to $x = 1$ is $L = \int_0^1 \sqrt{1 + \frac{1}{4} e^x}\, dx$

Solution

$$L = \int_0^1 \sqrt{1 + \frac{1}{4} e^x}\, dx \quad \Rightarrow \frac{dy}{dx} = \frac{e^{x/2}}{2}$$

$$dy = \frac{e^{x/2}}{2} dx$$

$$y = \int \frac{e^{x/2}}{2} dx = e^{x/2} + C$$

$$0 = e^{0/2} + C$$

$$0 = 1 + C \quad \to \quad C = -1$$

$$\boxed{y = e^{x/2} - 1}$$

Exercise

Find the length of the curve $y = \ln\left(e^x - 1\right) - \ln\left(e^x + 1\right)$ *from* $x = \ln 2$ *to* $x = \ln 3$

Solution

$$y = \ln\left(e^x - 1\right) - \ln\left(e^x + 1\right) \quad \Rightarrow \frac{dy}{dx} = \frac{e^x}{e^x - 1} - \frac{e^x}{e^x + 1}$$

$$= \frac{e^{2x} + e^x - e^{2x} + e^x}{e^{2x} - 1}$$

$$= \frac{2e^x}{e^{2x} - 1}$$

$$L = \int_{\ln 2}^{\ln 3} \sqrt{1 + \left(\frac{2e^x}{e^{2x} - 1}\right)^2}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{1 + \frac{4e^{2x}}{e^{4x} - 2e^{2x} + 1}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{e^{4x} - 2e^{2x} + 1 + 4e^{2x}}{\left(e^{2x} - 1\right)^2}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{e^{4x} + 2e^{2x} + 1}{\left(e^{2x} - 1\right)^2}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{\left(e^{2x} + 1\right)^2}{\left(e^{2x} - 1\right)^2}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{e^{2x} + 1}{e^{2x} - 1}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{\frac{e^{2x}+1}{e^x}}{\frac{e^{2x}-1}{e^x}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{\frac{e^{2x}}{e^x} + \frac{1}{e^x}}{\frac{e^{2x}}{e^x} - \frac{1}{e^x}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{e^x + e^{-x}}{e^x - e^{-x}}\, dx$$

Let $u = e^x - e^{-x} \quad \Rightarrow du = \left(e^x + e^{-x}\right)dx \quad \to \begin{cases} x = \ln 2 & u = e^{\ln 2} - e^{-\ln 2} = 2 - \frac{1}{2} = \frac{3}{2} \\ x = \ln 3 & u = e^{\ln 3} - e^{-\ln 3} = 3 - \frac{1}{3} = \frac{8}{3} \end{cases}$

$$L = \int_{3/2}^{8/3} \frac{du}{u}$$

$$= \left[\ln|u|\ \right]_{3/2}^{8/3}$$

$$= \ln\frac{8}{3} - \ln\frac{3}{2}$$

$$= \ln\frac{8/3}{3/2}$$

$$= \ln\left(\frac{16}{9}\right)$$

Exercise

Find the length of the curve $y = \ln(\cos x)$ *from* $x = 0$ *to* $x = \frac{\pi}{4}$

Solution

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$$

$$L = \int_0^{\pi/4} \sqrt{1+\tan^2 x}\ dx$$

$$= \int_0^{\pi/4} \sqrt{\sec^2 x}\ dx$$

$$= \int_0^{\pi/4} \sec x\ dx$$

$$= \left[\ln|\sec x + \tan x|\right]_0^{\pi/4}$$

$$= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln\left|\sec 0 + \tan 0\right|$$

$$= \ln\left|\sqrt{2} + 1\right| - \ln\left|1 + 0\right|$$

$$= \ln\left|\sqrt{2} + 1\right| - 0$$

$$= \ln\left(\sqrt{2} + 1\right)$$

Exercise

Find the area of the surface generated by revolving the curve $x = \frac{1}{2}\left(e^{y} + e^{-y}\right), \quad 0 \le y \le \ln 2,$ about y-axis

Solution

$$S = 2\pi \int_0^{\ln 2} \frac{1}{2}\left(e^{y} + e^{-y}\right)\sqrt{1 + \left(\frac{e^{y} - e^{-y}}{2}\right)^2}\, dy$$

$$= \pi \int_0^{\ln 2} \left(e^{y} + e^{-y}\right)\sqrt{1 + \frac{e^{2y} + e^{-2y} - 2}{4}}\, dy$$

$$= \pi \int_0^{\ln 2} \left(e^{y} + e^{-y}\right)\sqrt{\frac{4 + e^{2y} + e^{-2y} - 2}{4}}\, dy$$

$$= \frac{\pi}{2} \int_0^{\ln 2} \left(e^{y} + e^{-y}\right)\sqrt{e^{2y} + e^{-2y} + 2}\, dy$$

$$= \frac{\pi}{2} \int_0^{\ln 2} \left(e^{y} + e^{-y}\right)\sqrt{\left(e^{y} + e^{-y}\right)^2}\, dy$$

$$= \frac{\pi}{2} \int_0^{\ln 2} \left(e^{y} + e^{-y}\right)^2\, dy$$

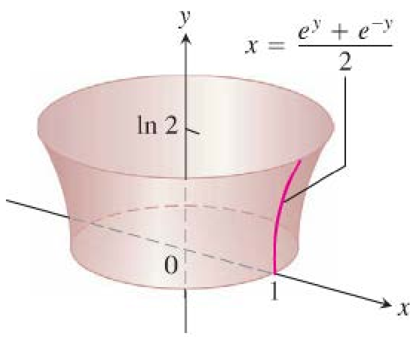
$$= \frac{\pi}{2} \int_0^{\ln 2} \left(e^{2y} + e^{-2y} + 2\right)\, dy$$

$$= \frac{\pi}{2}\left[\frac{1}{2}e^{2y} - \frac{1}{2}e^{-2y} + 2y\right]_0^{\ln 2}$$

$$= \frac{\pi}{2}\left[\left(\frac{1}{2}e^{2\ln 2} - \frac{1}{2}e^{-2\ln 2} + 2\ln 2\right) - \left(\frac{1}{2}e^{0} - \frac{1}{2}e^{0} + 0\right)\right]$$

$$= \frac{\pi}{2}\left(\frac{1}{2}\cdot 4 - \frac{1}{2}\cdot\frac{1}{4} + 2\ln 2\right)$$

$$= \frac{\pi}{2}\left(\frac{15}{8} + 2\ln 2\right)$$



Exercise

The population of a town with a 2010 population of 90,000 grows at a rate of 2.4% /yr. In what year will the population coudle its initial value (to 180,000)?

Solution

$$k = \frac{\ln \frac{1.024(90,000)}{90,000}}{1} = \ln(1.024)$$

$$T_2 = \frac{\ln 2}{\ln 1.024} \approx 29.226 \ \ yrs$$

It reaches 180,000 around the year 2039.

Exercise

How long will it take an initial deposit of \$1500 to increase in value to \$2500 in a saving account with an APY of 3.1%? Assume the interest rate reamins constant and no additional deposits or withdrawals are made.

Solution

$$y(t) = 1500\, e^{kt}$$

$$k = \frac{\ln 1.031}{1} = \ln(1.031)$$

$$T = \frac{\ln\left(\frac{2500}{1500}\right)}{\ln 1.031} \approx 16.7 \ \ yrs \qquad\qquad kT = \ln\left(\frac{y}{y_0}\right)$$

Exercise

The number of cells in a tumor doubles every 6 weeks starting with 8 cells. After how many weeks doses the tumor have 1500 cells?

Solution

$$k = \frac{\ln 2}{T_2} = \frac{\ln 2}{6} \quad \Rightarrow \quad y(t) = 8\, e^{(t \ln 2)/6}$$

$$t = 6\frac{\ln\left(\frac{1500}{8}\right)}{\ln 2} \approx 45.3 \ \ weeks \qquad\qquad kT = \ln\left(\frac{y}{y_0}\right)$$

Exercise

According to the 2010 census, the U.S. population was 309 million with an estimated growth rate of 0.8% */yr*.

a) Based on these figures, find the doubling time and project the population in 2050.

b) Suppose the actual growth rate is just 0.2 percentage point lower than 0.8% /*yr* (0.6%). What are the resulting doubling time and projected 2050 population? Repeat these calculstions assuming the growth rate is 0.2 percentage point higher than 0.8% /yr.

c) Comment on th sensitivity of these projections to the growth rate.

Solution

a) $T_2 = \dfrac{\ln 2}{\ln 1.008} \underline{\approx 87 \ \ yrs}$

The population in 2050: $P(50) = 309e^{40\ln 1.008} \underline{\approx 425 \ \ million}$

b) If the growth rate is 0.6%: $T_2 = \dfrac{\ln 2}{\ln 1.006} \underline{\approx 116 \ \ yrs}$

The population in 2050: $P(50) = 309e^{40\ln 1.006} \underline{\approx 392.5 \ \ million}$

If the growth rate is 1%: $T_2 = \dfrac{\ln 2}{\ln 1.01} \underline{\approx 69.7 \ \ yrs}$

The population in 2050: $P(50) = 309e^{40\ln 1.01} \underline{\approx 460.1 \ \ million}$

c) A growth rate of just 0.2% produces large differences in population growth.

Exercise

The homicide rate decreases at a rate of 3%/yr in a city that had 800 homicides /yr in 2010. At this rate, when will the homicide rate reach 600 homicides/yr?

Solution

The homicide rate is modeled by: $H(t) = 800e^{-kt}$

$k = \ln(1-.03) \underline{\approx -0.03}$

$H(t) = 800e^{-0.03t}$

$t = \dfrac{\ln(6/8)}{-0.03} \underline{\approx 9.6 \ \ yrs}$ $\qquad kT = \ln\left(\dfrac{y}{y_0}\right)$

So it should achieve this rate in 2019.

Exercise

A drug is eliminated from the body at a rate of 15% /hr. after how many hours does the amount of drug reach 10% of the initial dose?

Solution

$k = \ln(1-.15) \underline{\approx -\ln(.85)}$

$t = \dfrac{\ln(.1)}{\ln(.85)} \underline{\approx 14.17 \ \ hrs}$ $\qquad kT = \ln\left(\dfrac{y}{y_0}\right)$

Exercise

A large die-casting machine used to make automobile engine blocks is purchased for \$2.5 *million*. For tax purposes, the value of the machine can be depreciated by 6.8% of its current value each year.

a) What is the value of the machine after 10 *years*?

b) After how many years is the value of the machine 10% of its original value?

Solution

a) $V(t) = 2.5e^{-kt}$

$$k = \frac{\ln(1-.068)}{1} \approx \underline{\ln(.932)} \qquad\qquad kT = \ln\left(\frac{y}{y_0}\right)$$

$$V(t) = 2.5e^{-t\ln .932} \quad \rightarrow \quad V(10) = 2.5e^{-10\cdot\ln .932} \; \underline{\approx \$1.2 \; million}$$

b) $t = \dfrac{\ln(.1)}{\ln(.932)} \; \underline{\approx 32.7 \;\; yrs}$

Exercise

Roughly 12,000 Americans are diagmosed with thyroid cancer every year, which accounts for 1% of all cancer cases. It occurs in women three times as frequency as in men. Fortunately, thyroid cancer can be treated successfully in many cases with radioactive iodine, or I-131. This unstable form of iodine has a half-life of 8 days and is given in small doses meansured in millicuries.

a) Suppose a patient is given an initial dose of 100 millicuries. Find the function that gives the amount of I-131 in the body after $t \ge 0$ days.

b) How long does it take the amount of I-131 to reach 10% of the initial dose?

c) Finding the initial dose to give a particular patient is a critical calculation. How does the time reach 10% of the initial dose change if the initial dose is increased by 5%?

Solution

a) $\underline{k = \dfrac{\ln 2}{8}} \qquad\qquad kT = \ln(y/y_0)$

After *t* days would be: $y = 100e^{-(t\ln 2)/8}$ millicuries.

b) $t = \dfrac{-8\ln\left(\frac{10}{100}\right)}{\ln(2)} \; \underline{\approx 26.58 \;\; days}$

c) $t = \dfrac{-8\ln\left(\frac{10}{105}\right)}{\ln(2)} \; \underline{\approx 27.14 \;\; days}$

Exercise

City ***A*** has a current population of 500,000 people and grows at a rate of 3% /*yr*. City ***B*** has a cuurent population of 300,000 and grows at a rate of 5%/*yr*.

a) When will the cities have the same population?

b) Suppose City ***C*** has a current population of $y_0 < 500{,}000$ and a growth rate of $p > 3\%/yr$. What is the relationship between y_0 and p such that the Cities ***A*** and ***C*** have the same population in 10 years?

Solution

a) $500{,}000e^{\ln(1.03)t} = 300{,}000e^{\ln(1.05)t}$

$5e^{\ln(1.03)t} = 3e^{\ln(1.05)t}$

$\frac{5}{3} = e^{(\ln(1.05)-\ln 1.03)t}$

$\ln\frac{5}{3} = \left(\ln\frac{1.05}{1.03}\right)t \quad \to \quad t = \frac{\ln(5/3)}{\ln(1.05/1.03)} \approx 26.56\ yrs$

b) $500{,}000e^{\ln(1.03)(10)} = y_0 e^{\ln(1+p)(10)}$

$y_0 = 500{,}000e^{10(\ln(1.03)-\ln(1+p))}$

$= 500{,}000e^{\ln\left(\frac{1.03}{1+p}\right)^{10}}$

$= 500{,}000\left(\frac{1.03}{1+p}\right)^{10}$

Exercise

Suppose the acceleration of an object moving along a line is given by $a(t) = -kv(t)$, where k is a positive constant and v is the object's velocity. Assume that the initial velocity and position are given by $v(0) = 10$ and $s(0) = 0$, respectively.

a) Use $a(t) = v'(t)$ to find the velocity of the object as a function of time.

b) Use $v(t) = s'(t)$ to find the position of the object as a function of time.

c) Use the fact that $\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt}$ (by the *Chain Rule*) to find the velocity as a function of position.

Solution

a) If $a(t) = \frac{dv}{dt} = -kv \quad \to \quad \frac{dv}{v} = -kdt$

$\int \frac{dv}{v} = -k\int dt$

$\ln v = -kt + C \qquad$ Since $v(0) = 10$

$\ln 10 = C$

$\ln v = -kt + \ln 10$

$v = e^{-kt+\ln 10} = e^{-kt}e^{\ln 10}$

$v(t) = 10e^{-kt}$

b) $v(t)=\frac{ds}{dt}=10e^{-kt}$

$$\int ds=10\int e^{-kt}dt$$

$s(t)=-\frac{10}{k}e^{-kt}+C$ Since $s(0)=0$

$$0=-\frac{10}{k}+C \quad \rightarrow \quad \underline{C=\frac{10}{k}}$$

$$\underline{s(t)=-\frac{10}{k}e^{-kt}+\frac{10}{k}}$$

c) $\frac{dv}{dt}=\frac{dv}{ds}\frac{ds}{dt}$

$$-10ke^{-kt}=\frac{dv}{ds}\left(10e^{-kt}\right)$$

$$-k=\frac{dv}{ds}$$

$$\int dv=-k\int ds$$

$v=-ks+C$ Since $v(0)=10$

$$\underline{v=10-ks}$$

Exercise

On the first day of the year $(t=0)$, a city uses electricity at a rate of 2000 *MW*. That rate is projected to increase at a rate of 1.3% per *year*.

a) Based on these figures, find an exponential growth function for the power (rate of electricity use) for the city.

b) Find the total energy (in MW-*yr*) used by the city over four full years beginning at $t=0$

c) Find a function that gives the total energy used (in MW-*yr*) between $t=0$ and any future time $t>0$

Solution

a) $P(t)=2000e^{kt}$

At a rate of 1.3% per year: $k=\ln(1.013)$

$$\underline{P(t)=2000e^{t\ln 1.013}}$$

b) $\int_0^4 P(t)dt=2000\int_0^4 e^{t\ln 1.013}dt$

$$=\frac{2000}{\ln 1.013}e^{t\ln 1.013}\Big|_0^4$$

$$\underline{\approx 8210.3}$$

c) $\displaystyle\int_0^t P(s)\,ds = 2000\int_0^t e^{s\ln 1.013}ds$

$$= \frac{2000}{\ln 1.013} e^{s\ln 1.013}\Big|_0^t$$

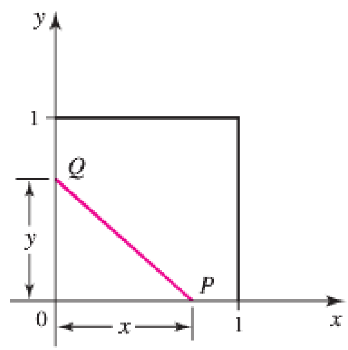
$$\underline{= -154{,}844\left(1 + e^{t\ln(1.013)}\right)}$$

Exercise

Two points P and Q are chosen randomly, one on each of two adjacent sides of a unit square.

What is the probability that the area of the triangle formed by the sides of the square and the line segment PQ is less than one-fourth the area of the square? Begin by showing that x and y must satisfy $xy < \frac{1}{2}$ in order for the area consition to be met. Then argue that the required probability is $\frac{1}{2} + \int_{1/2}^{1} \frac{dx}{2x}$ and evaluate the integral.

Solution



The area of the triangle is $\frac{1}{2}xy$

If $xy < \frac{1}{2}$, then if we let $0 < x < \frac{1}{2}$ we have $0 < y < 1$

Because there is a probability of $\frac{1}{2}$ of choosing $0 < x < \frac{1}{2}$, the probability we seek is at least $\frac{1}{2}$.

In addtion, for $\frac{1}{2} < x < 1$, if $y < \frac{1}{2x}$,

$$\int_{1/2}^{1} \frac{dx}{2x} = \frac{1}{2}\ln x\Big|_{1/2}^{1} = \frac{\ln 2}{2}$$

$$\frac{1}{2} + \int_{1/2}^{1} \frac{dx}{2x} = \frac{1}{2}(1 + \ln 2)$$

Exercise

Rewrite the expression $\cosh 3x - \sinh 3x$ in terms of exponentials and simplify the results as much as you can.

Solution

$$\begin{aligned}\cosh 3x - \sinh 3x &= \frac{e^{3x}+e^{-3x}}{2} - \frac{e^{3x}-e^{-3x}}{2} \\ &= \frac{e^{3x}+e^{-3x}-e^{3x}+e^{-3x}}{2} \\ &= \frac{2e^{-3x}}{2} \\ &= e^{-3x}\end{aligned}$$

Exercise

Rewrite the expression $\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x)$ in terms of exponentials and simplify the results as much as you can.

Solution

$$\begin{aligned}\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x) &= \ln\left[(\cosh x + \sinh x)(\cosh x - \sinh x)\right] \\ &= \ln\left(\cosh^2 x - \sinh^2 x\right) \\ &= \ln(1) \\ &= 0\end{aligned}$$

Exercise

Prove the identities

a) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

b) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

Solution

a)

$$\begin{aligned}\sinh x \cosh y + \cosh x \sinh y &= \frac{e^{x}-e^{-x}}{2}\frac{e^{y}+e^{-y}}{2} + \frac{e^{x}+e^{-x}}{2}\frac{e^{y}-e^{-y}}{2} \\ &= \frac{e^{x}e^{y}+e^{x}e^{-y}-e^{-x}e^{y}-e^{-x}e^{-y}+e^{x}e^{y}-e^{x}e^{-y}+e^{-x}e^{y}-e^{-x}e^{-y}}{4} \\ &= \frac{2e^{x}e^{y}-2e^{-x}e^{-y}}{4} \\ &= \frac{e^{x+y}-e^{-x-y}}{2}\end{aligned}$$

$$= \frac{e^{x+y} - e^{-(x+y)}}{2}$$

$$= \sinh(x+y)$$

b) $\cosh x \cosh y + \sinh x \sinh y = \frac{e^y + e^{-y}}{2} \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} \frac{e^y - e^{-y}}{2}$

$$= \frac{e^x e^y + e^x e^{-y} + e^{-x} e^y + e^{-x} e^{-y} + e^x e^y - e^x e^{-y} - e^{-x} e^y + e^{-x} e^{-y}}{4}$$

$$= \frac{2e^x e^y + 2e^{-x} e^{-y}}{4}$$

$$= \frac{e^{x+y} + e^{-x-y}}{2}$$

$$= \frac{e^{x+y} + e^{-(x+y)}}{2}$$

$$= \cosh(x+y)$$

Exercise

Find the derivative of $y = \frac{1}{2}\sinh(2x+1)$

Solution

$$y' = \frac{1}{2}\left[\cosh(2x+1)\right](2) \quad = \cosh(2x+1)$$

Exercise

Find the derivative of $y = 2\sqrt{t}\tanh\sqrt{t}$

Solution

$$y' = 2\left(\frac{1}{2}t^{-1/2}\tanh\sqrt{t} + t^{1/2}\left(\frac{1}{2}\right)\sec^2\sqrt{t}\right)$$

$$= \frac{\tanh\sqrt{t}}{\sqrt{t}} + \sqrt{t}\sec^2\sqrt{t}$$

Exercise

Find the derivative of $y = \ln(\cosh z)$

Solution

$$y' = \frac{\sinh z}{\cosh z} = \tanh z$$

Exercise

Find the derivative of $y = \operatorname{csch}\theta(1 - \ln \operatorname{csch}\theta)$

Solution

$$y' = (-\operatorname{csch}\theta \coth\theta)(1 - \ln \operatorname{csch}\theta) + \operatorname{csch}\theta\left(-\frac{-\operatorname{csch}\theta \coth\theta}{\operatorname{csch}\theta}\right)$$

$$= -\operatorname{csch}\theta \coth\theta + \operatorname{csch}\theta \coth\theta(\ln \operatorname{csch}\theta) + \operatorname{csch}\theta \coth\theta$$

$$= \operatorname{csch}\theta \coth\theta(\ln \operatorname{csch}\theta)$$

Exercise

Find the derivative of $y = \ln \sinh v - \frac{1}{2}\coth^2 v$

Solution

$$y' = \frac{\cosh v}{\sinh v} - \frac{1}{2}2\coth v\left(-\operatorname{csch}^2 v\right)$$

$$= \coth v + (\coth v)\left(\operatorname{csch}^2 v\right)$$

Exercise

Find the derivative of $y = \left(x^2 + 1\right)\operatorname{sech}(\ln x)$

Solution

$$y = \left(x^2 + 1\right)\left(\frac{2}{e^{\ln x} + e^{-\ln x}}\right)$$

$$= \left(x^2 + 1\right)\left(\frac{2}{x + x^{-1}}\right)$$

$$= \left(x^2 + 1\right)\left(\frac{2x}{x^2 + 1}\right)$$

$$= 2x$$

$$y' = 2$$

Exercise

Find the derivative of $y = \left(4x^2 - 1\right)\operatorname{csch}(\ln 2x)$

Solution

$$y = \left(4x^2 - 1\right)\left(\frac{2}{e^{\ln 2x} - e^{-\ln 2x}}\right)$$

$$= \left(4x^2 - 1\right)\left(\frac{2}{2x - (2x)^{-1}}\right)$$

$$= \left(4x^2 - 1\right)\left(\frac{4x}{4x^2 - 1}\right)$$

$$= 4x$$

$$\underline{y' = 4}$$

Exercise

Find the derivative of $y = \cosh^{-1} 2\sqrt{x+1}$

Solution

$$y = \cosh^{-1} 2\sqrt{x+1} = \cosh^{-1} 2(x+1)^{1/2}$$

$$y' = \frac{2\left(\frac{1}{2}\right)(x+1)^{-1/2}}{\sqrt{\left(2(x+1)^{1/2}\right)^2 - 1}}$$

$$= \frac{1}{(x+1)^{1/2}\sqrt{4(x+1)-1}}$$

$$= \frac{1}{\sqrt{x+1}\sqrt{4x+3}}$$

$$\underline{= \frac{1}{\sqrt{4x^2 + 7x + 3}}}$$

Exercise

Find the derivative of $y = \left(\theta^2 + 2\theta\right)\tanh^{-1}(\theta+1)$

Solution

$$y' = (2\theta + 2)\tanh^{-1}(\theta+1) + \left(\theta^2 + 2\theta\right)\left(\frac{1}{1 - (\theta+1)^2}\right)$$

$$= (2\theta + 2)\tanh^{-1}(\theta+1) + \frac{\theta^2 + 2\theta}{1 - \left(\theta^2 + 2\theta + 1\right)}$$

$$= (2\theta + 2)\tanh^{-1}(\theta+1) + \frac{\theta^2 + 2\theta}{-\theta^2 - 2\theta}$$

$$\underline{= (2\theta + 2)\tanh^{-1}(\theta+1) - 1}$$

Exercise

Find the derivative of $y = (1-t)\coth^{-1}\sqrt{t}$

Solution

$$y' = -\coth^{-1}\sqrt{t} + (1-t)\frac{\frac{1}{2}t^{-1/2}}{1-\left(t^{1/2}\right)^2}$$

$$= -\coth^{-1}\sqrt{t} + (1-t)\frac{1}{2\sqrt{t}(1-t)}$$

$$\boxed{= -\coth^{-1}\sqrt{t} + \frac{1}{2\sqrt{t}}}$$

Exercise

Find the derivative of $y = \ln x + \sqrt{1-x^2}\ \operatorname{sech}^{-1}x$

Solution

$$y' = \frac{1}{x} + \frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x)\ \operatorname{sech}^{-1}x + \sqrt{1-x^2}\left(\frac{-1}{x\sqrt{1-x^2}}\right)$$

$$= \frac{1}{x} - \frac{x}{\left(1-x^2\right)^{1/2}}\ \operatorname{sech}^{-1}x - \frac{1}{x}$$

$$\boxed{= -\frac{x}{\sqrt{1-x^2}}\ \operatorname{sech}^{-1}x}$$

Exercise

Find the derivative of $y = \operatorname{csch}^{-1}\left(\frac{1}{2}\right)^{\theta}$

Solution

$$y' = -\frac{\left[\ln\left(\frac{1}{2}\right)\right]\left(\frac{1}{2}\right)^{\theta}}{\left(\frac{1}{2}\right)^{\theta}\sqrt{1+\left[\left(\frac{1}{2}\right)^{\theta}\right]^2}}$$

$$= -\frac{-\ln 2}{\sqrt{1+\left(\frac{1}{2}\right)^{2\theta}}}$$

$$\boxed{= \frac{\ln 2}{\sqrt{1+\left(\frac{1}{2}\right)^{2\theta}}}}$$

Exercise

Find the derivative of $y = \cosh^{-1}(\sec x)$

Solution

$$y' = \frac{(\sec x)(\tan x)}{\sqrt{\sec^2 x - 1}}$$

$$= \frac{(\sec x)(\tan x)}{\sqrt{\tan^2 x}}$$

$$= \frac{(\sec x)(\tan x)}{|\tan x|}$$

$$\underline{= \sec x} \quad 0 < x < \frac{\pi}{2}$$

Exercise

Find the derivative of $y = -\sinh^3 4x$

Solution

$$\underline{y' = -12\left(\sinh^2 4x\right)(\cosh 4x)}$$

Exercise

Find the derivative of $y = \sqrt{\coth 3x}$

Solution

$$y' \underline{= \frac{-3\,\mathrm{csch}^2 3x}{2\sqrt{\coth 3x}}}$$

$$\left(\sqrt{u}\right)' = \frac{u'}{2\sqrt{u}}$$

Exercise

Find the derivative of $y = \dfrac{x}{\mathrm{csch}x}$

Solution

$$y' = \frac{\mathrm{csch}x + x\,\mathrm{csch}x\,\coth x}{\mathrm{csch}^2 x}$$

$$\frac{d}{dx}(\mathrm{csch}\,u) = -u'\,\mathrm{csch}\,u\ \coth u$$

$$= \frac{1 + x\coth x}{\mathrm{csch}x}$$

$$= \frac{1}{\mathrm{csch}x} + x\frac{\coth x}{\mathrm{csch}x}$$

$$\underline{= \sinh x + x\cosh x}$$

Exercise

Find the derivative of $y = \tanh^2 x$

Solution

$$y' = 2\tanh x \ \text{sech}^2 x \qquad \frac{d}{dx}(\tanh u) = u' \text{sech}^2 u$$

Exercise

Find the derivative of $y = \ln \ \text{sech} 2x$

Solution

$$y' = \frac{-2\ \text{sech} 2x \ \tanh 2x}{\text{sech} 2x} = -2\tanh 2x \qquad \frac{d}{dx}(\text{sech}\, u) = -u' \text{sech}\, u \ \tanh u$$

Exercise

Find the derivative of $y = x^2 \cosh^2 3x$

Solution

$$y' = 2x\cosh^2 3x + 6x^2 \cosh 3x \ \sinh 3x \qquad \frac{d}{dx}(\sinh u) = u' \cosh u$$

$$= 2x\cosh 3x(\cosh 3x + 3x \ \sinh 3x)$$

Exercise

Find the derivative of $f(t) = 2\tanh^{-1}\sqrt{t}$

Solution

$$f'(t) = 2\frac{\frac{1}{2\sqrt{t}}}{1 - \left(\sqrt{t}\right)^2} = \frac{1}{\sqrt{t}(1-t)} \qquad \frac{d}{dx}\left(\tanh^{-1} u\right) = \frac{u'}{1-u^2}$$

Exercise

Find the derivative of $f(x) = \sinh^{-1} x^2$

Solution

$$f'(x) = \frac{2x}{\sqrt{x^4+1}} \qquad \frac{d}{dx}\left(\sinh^{-1} u\right) = \frac{u'}{\sqrt{1+u^2}}$$

Exercise

Find the derivative of $f(x) = \operatorname{csch}^{-1}\left(\frac{2}{x}\right)$

Solution

$$f'(x) = \frac{-1}{\left|\frac{2}{x}\right|\sqrt{1+\frac{4}{x^2}}}\left(\frac{-2}{x^2}\right) \boxed{= \frac{1}{\sqrt{x^2+4}}}$$

$$\frac{d}{dx}\left(\operatorname{csch}^{-1} u\right) = -\frac{u'}{|u|\sqrt{1+u^2}}$$

Exercise

Find the derivative of $f(x) = x\sinh^{-1} x - \sqrt{x^2+1}$

Solution

$$f'(x) = \sinh^{-1} x + x\frac{1}{\sqrt{x^2+1}} - \frac{2x}{2\sqrt{x^2+1}}$$

$$\boxed{= \sinh^{-1} x}$$

$$\frac{d}{dx}\left(\sinh^{-1} u\right) = \frac{u'}{\sqrt{1+u^2}}$$

Exercise

Find the derivative of $f(x) = \sinh^{-1}(\tan x)$

Solution

$$f'(x) = \frac{\sec^2 x}{\sqrt{1+\tan^2 x}}$$

$$= \frac{\sec^2 x}{\sqrt{\sec^2 x}}$$

$$\boxed{= |\sec x|}$$

$$\frac{d}{dx}\left(\sinh^{-1} u\right) = \frac{u'}{\sqrt{1+u^2}}$$

Exercise

Verify the integration $\int x\operatorname{sech}^{-1} x\,dx = \frac{x^2}{2}\operatorname{sech}^{-1} x - \frac{1}{2}\sqrt{1-x^2} + C$

Solution

If $y = \frac{x^2}{2}\operatorname{sech}^{-1} x - \frac{1}{2}\sqrt{1-x^2} + C$

$$dy = \left[x\operatorname{sech}^{-1} x + \frac{x^2}{2}\left(\frac{-1}{x\sqrt{1-x^2}}\right) - \frac{1}{2}\left(\frac{1}{2}\right)\frac{-2x}{\sqrt{1-x^2}}\right]dx$$

$$dy = \left[x\operatorname{sech}^{-1} x - \frac{x}{2\sqrt{1-x^2}} + \frac{x}{2\sqrt{1-x^2}}\right]dx$$

$\boxed{dy = \left(x\operatorname{sech}^{-1} x\right)dx}$ ✓ Which verifies the formula

Exercise

Verify the integration $\int \tanh^{-1} x dx = x\tanh^{-1} x + \frac{1}{2}\ln\left(1-x^2\right) + C$

Solution

If $\quad y = x\tanh^{-1} x + \frac{1}{2}\ln\left(1-x^2\right) + C$

$$\frac{dy}{dx} = \tanh^{-1} x + x\left(\frac{1}{1-x^2}\right) + \frac{1}{2}\frac{-2x}{1-x^2}$$

$$= \tanh^{-1} x + \frac{x}{1-x^2} - \frac{x}{1-x^2}$$

$= \tanh^{-1} x$ ✓ which verifies the formula

Exercise

Evaluate the integral: $\int \sinh 2x dx$

Solution

$$\int \sinh 2x\ dx = \frac{1}{2}\int \sinh 2x\ d(2x)$$

$$= \frac{1}{2}\cosh 2x + C$$

Exercise

Evaluate the integral: $\int 4\cosh(3x - \ln 2)dx$

Solution

$$\int 4\cosh(3x-\ln 2)dx = \frac{4}{3}\int \cosh(3x-\ln 2)\ d(3x-\ln 2) \qquad d(3x-\ln 2) = 3dx$$

$$= \frac{4}{3}\sinh(3x-\ln 2) + C$$

Exercise

Evaluate the integral: $\int \tanh\frac{x}{7} dx$

Solution

$$\int \tanh\frac{x}{7}\ dx = \int \frac{\sinh\frac{x}{7}}{\cosh\frac{x}{7}}\ d(x) \qquad d\left(\cosh\frac{x}{7}\right) = \frac{1}{7}\left(\sinh\frac{x}{7}\right)dx$$

$$= 7\int \frac{1}{\cosh\frac{x}{7}}\ d\left(\cosh\frac{x}{7}\right)$$

$$= 7\ln\left|\cosh\frac{x}{7}\right| + C$$

$$= 7\ln\left[\frac{e^{x/7} + e^{-x/7}}{2}\right] + C$$

$$= 7\left[\ln\left(e^{x/7} + e^{-x/7}\right) - \ln 2\right] + C$$

$$= 7\ln\left(e^{x/7} + e^{-x/7}\right) - 7\ln 2 + C \qquad C_1 = -7\ln 2 + C$$

$$= 7\ln\left(e^{x/7} + e^{-x/7}\right) + C_1$$

Exercise

Evaluate the integral: $\int \coth\frac{\theta}{\sqrt{3}}\,d\theta$

Solution

$$\int \coth\frac{\theta}{\sqrt{3}}\,d\theta = \int \frac{\cosh\frac{\theta}{\sqrt{3}}}{\sinh\frac{\theta}{\sqrt{3}}}\,d\theta \qquad d\left(\sinh\frac{\theta}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}\left(\cosh\frac{\theta}{\sqrt{3}}\right)d\theta$$

$$= \sqrt{3}\int \frac{1}{\sinh\frac{\theta}{\sqrt{3}}}\,d\left(\sinh\frac{\theta}{\sqrt{3}}\right)$$

$$= \sqrt{3}\ln\left|\sinh\frac{\theta}{\sqrt{3}}\right| + C_1$$

$$= \sqrt{3}\ln\left|\frac{e^{\theta/\sqrt{3}} - e^{-\theta/\sqrt{3}}}{2}\right| + C_1$$

$$= \sqrt{3}\ln\left|e^{\theta/\sqrt{3}} - e^{-\theta/\sqrt{3}}\right| - \sqrt{3}\ln 2 + C_1 \qquad C = -\sqrt{3}\ln 2 + C_1$$

$$= \sqrt{3}\ln\left|e^{\theta/\sqrt{3}} - e^{-\theta/\sqrt{3}}\right| + C$$

Exercise

Evaluate the integral: $\int \operatorname{csch}^2(5-x)\,dx$

Solution

$$\int \operatorname{csch}^2(5-x)\,dx = -\int \operatorname{csch}^2(5-x)\,d(5-x) \qquad \int \operatorname{csch}^2 u\,du = -\coth u$$

$$= \coth(5-x) + C$$

Exercise

Evaluate the integral: $\int \frac{\operatorname{sech}\sqrt{t}\ \tanh\sqrt{t}}{\sqrt{t}}\,dt$

Solution

$$\int \frac{\operatorname{sech}\sqrt{t}\ \tanh\sqrt{t}}{\sqrt{t}}\,dt = 2\int \operatorname{sech}\sqrt{t}\ \tanh\sqrt{t}\ d\left(\sqrt{t}\right) \qquad d\left(\sqrt{t}\right) = \frac{1}{2\sqrt{t}}dt$$

$$\underline{= -2\operatorname{sec}h\sqrt{t} + C} \qquad \int \operatorname{sech} u \tanh u\ du = -\operatorname{sech} u$$

Exercise

Evaluate the integral: $\int \frac{\operatorname{csch}(\ln t)\ \coth(\ln t)}{t}\,dt$

Solution

$$\int \frac{\operatorname{csch}(\ln t)\ \coth(\ln t)}{t}\,dt = \int \operatorname{csch}(\ln t)\ \coth(\ln t)\ d(\ln t) \qquad d(\ln t) = \frac{dt}{t}$$

$$\underline{= -\operatorname{csc}h(\ln t) + C} \qquad \int \operatorname{csch} u \coth u\ du = -\operatorname{csch} u$$

Exercise

Evaluate the integral $\int \frac{\sinh x}{1+\cosh x}\,dx$

Solution

$$\int \frac{\sinh x}{1+\cosh x}\,dx = \int \frac{d(1+\cosh x)}{1+\cosh x} \qquad d(1+\cosh x) = \sinh x\ dx$$

$$\underline{= \ln\left|1+\cosh x\right| + C}$$

Exercise

Evaluate the integral $\int \operatorname{sech}^2 x\ \tanh x\ dx$

Solution

$$\int \operatorname{sech}^2 x\ \tanh x\ dx = \int \tanh x\ d(\tanh x) \qquad d(\tanh x) = \operatorname{sech}^2 x\ dx$$

$$\underline{= \frac{1}{2}\tanh^2 x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \coth^2 x \operatorname{csch}^2 x\, dx$

Solution

$$\int \coth^2 x \operatorname{csch}^2 x\, dx = -\int \coth^2 x\, d(\coth x) \qquad d(\coth x) = -\operatorname{csch}^2 x\, dx$$

$$\boxed{= -\frac{1}{3}\coth^3 x + C}$$

Exercise

Evaluate the integral $\displaystyle\int \tanh^2 x\, dx$

Solution

$$\int \tanh^2 x\, dx = \int \left(1 - \operatorname{sech}^2 x\right) dx \qquad \tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\boxed{= x - \tanh x + C} \qquad \int \operatorname{sech}^2 u\, du = \tanh u$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sinh(\ln x)}{x} dx$

Solution

$$\int \frac{\sinh(\ln x)}{x} dx = \int \sinh(\ln x)\, d(\ln x)$$

$$= \cosh(\ln x) + C$$

$$= \frac{e^{\ln x} + e^{-\ln x}}{2} + C$$

$$= \frac{1}{2}\left(x + \frac{1}{x}\right) + C$$

$$\boxed{= \frac{x^2 + 1}{2x} + C}$$

$$\sinh(\ln x) = \frac{e^{\ln x} - e^{-\ln x}}{2} = \frac{1}{2}\left(x - \frac{1}{x}\right)$$

$$\int \frac{\sinh(\ln x)}{x} dx = \frac{1}{2}\int \frac{1}{x}\left(x - \frac{1}{x}\right) d(x)$$

$$= \frac{1}{2}\int \left(1 - \frac{1}{x^2}\right) d(x)$$

$$= \frac{1}{2}\left(x + \frac{1}{x}\right) + C$$

$$\boxed{= \frac{x^2 + 1}{2x} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{8 - x^2} \quad x > 2\sqrt{2}$

Solution

$$\int \frac{dx}{8 - x^2} \boxed{= \frac{1}{2\sqrt{2}}\tanh^{-1}\left(\frac{x}{2\sqrt{2}}\right) + C} \qquad \int \frac{du}{a^2 - u^2} = \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right)$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{\sqrt{x^2-16}}$

Solution

$$\int \frac{dx}{\sqrt{x^2-16}} = \coth^{-1}\left(\frac{x}{4}\right) + C$$

$$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right)$$

Exercise

Evaluate the integral $\displaystyle\int_0^1 \cosh^3 3x \ \sinh 3x \ dx$

Solution

$$\int_0^1 \cosh^3 3x \ \sinh 3x \ dx = \frac{1}{3}\int_0^1 \cosh^3 3x \ d(\cosh 3x)$$

$$d(\cosh 3x) = 3\sinh x \ dx$$

$$= \frac{1}{12}\cosh^4 3x \Big|_0^1$$

$$= \frac{1}{12}\left(\cosh^4 3 - \cosh^4 0\right)$$

$$= \frac{1}{12}\left(\cosh^4 3 - 1\right) \quad \approx 856.034$$

Exercise

Evaluate the integral $\displaystyle\int_0^4 \frac{\operatorname{sech}^2\sqrt{x}}{\sqrt{x}} dx$

Solution

$$\int_0^4 \frac{\operatorname{sech}^2\sqrt{x}}{\sqrt{x}} dx = 2\int_0^4 \operatorname{sech}^2\sqrt{x} \ d\left(\sqrt{x}\right)$$

$$d\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}} dx$$

$$= 2\tanh\sqrt{x}\Big|_0^4$$

$$= 2\tanh 2 \quad \approx 1.93$$

Exercise

Evaluate the integral $\displaystyle\int_{\ln 2}^{\ln 3} \operatorname{csch} x \ dx$

Solution

$$\int_{\ln 2}^{\ln 3} \operatorname{csch} x\, dx = \ln\left|\tanh\frac{x}{2}\right|\Bigg|_{\ln 2}^{\ln 3}$$

$$\underline{= \ln\left|\tanh\frac{\ln 3}{2}\right| - \ln\left|\tanh\frac{\ln 2}{2}\right|} \quad \underline{\approx 0.405}$$

Exercise

Evaluate the integral: $\int_{\ln 2}^{\ln 4} \coth x\, dx$

Solution

$$\int_{\ln 2}^{\ln 4} \coth x\, dx = \int_{\ln 2}^{\ln 4} \frac{\cosh x}{\sinh x}\, dx$$

$$= \int_{\ln 2}^{\ln 4} \frac{1}{\sinh x}\, d(\sinh x) \qquad d(\sinh x) = \cosh x\, dx$$

$$= \ln|\sinh x|\Big|_{\ln 2}^{\ln 4}$$

$$= \ln|\sinh \ln 4| - \ln|\sinh \ln 2|$$

$$= \ln\left(\frac{e^{\ln 4} - e^{-\ln 4}}{2}\right) - \ln\left(\frac{e^{\ln 2} - e^{-\ln 2}}{2}\right)$$

$$= \ln\left(\frac{4 - \frac{1}{4}}{2}\right) - \ln\left(\frac{2 - \frac{1}{2}}{2}\right)$$

$$= \ln\left(\frac{15}{8}\right) - \ln\left(\frac{3}{4}\right)$$

$$= \ln\left(\frac{15}{8} \div \frac{3}{4}\right)$$

$$\underline{= \ln\left(\frac{5}{2}\right)}$$

Exercise

Evaluate the integral: $\int_0^{\pi/2} 2\sinh(\sin\theta)\cos\theta\, d\theta$

Solution

$$\int_0^{\pi/2} 2\sinh(\sin\theta)\cos\theta\, d\theta = 2\int_0^{\pi/2} \sinh(\sin\theta)\, d(\sin\theta) \qquad d(\sin\theta) = \cos\theta\, d\theta$$

$$= 2\cosh(\sin\theta)\Big|_0^{\pi/2}$$

$$= 2(\cosh 1 - \cosh 0)$$

$$= 2\left(\frac{e + e^{-1}}{2} - 1\right)$$

$$\underline{= e + e^{-1} - 2}$$

Exercise

Evaluate the integral: $\displaystyle\int_1^2 \frac{8\cosh\sqrt{x}}{\sqrt{x}}dx$

Solution

$$\int_1^2 \frac{8\cosh\sqrt{x}}{\sqrt{x}}dx = 16\int_1^2 \cosh\sqrt{x}\; d\left(\sqrt{x}\right)$$

$$= 16\sinh\sqrt{x}\Big|_1^2$$

$$= 16\left(\sinh\sqrt{2} - \sinh 1\right)$$

$$= 16\left(\frac{e^{\sqrt{2}} - e^{-\sqrt{2}}}{2} - \frac{e - e^{-1}}{2}\right)$$

$$\underline{= 8\left(e^{\sqrt{2}} - e^{-\sqrt{2}} - e + e^{-1}\right)}$$

Exercise

Evaluate the integral: $\displaystyle\int_{-\ln 2}^{0} \cosh^2\left(\frac{x}{2}\right)dx$

Solution

$$\int_{-\ln 2}^{0} \cosh^2\left(\frac{x}{2}\right)dx = \frac{1}{2}\int_{-\ln 2}^{0} (\cosh x + 1)dx$$

$$= \frac{1}{2}(\sinh x + x)\Big|_{-\ln 2}^{0}$$

$$= \frac{1}{2}(-\sinh(-\ln 2) + \ln 2)$$

$$= \frac{1}{2}\left(-\frac{e^{-\ln 2} - e^{\ln 2}}{2} + \ln 2\right)$$

$$= \frac{1}{2}\left(-\frac{\frac{1}{2} - 2}{2} + \ln 2\right)$$

$$=\frac{3}{8}+\frac{1}{2}\ln 2$$

$$\boxed{=\frac{3}{8}+\ln\sqrt{2}}$$

Exercise

Evaluate the integral: $\displaystyle\int_0^{\ln 2} 4e^{-\theta}\sinh\theta\, d\theta$

Solution

$$\int_0^{\ln 2} 4e^{-\theta}\sinh\theta d\theta=\int_0^{\ln 2} 4e^{-\theta}\,\frac{e^{\theta}-e^{-\theta}}{2}d\theta$$

$$=2\int_0^{\ln 2}\left(1-e^{-2\theta}\right)d\theta$$

$$=2\left[\theta+\frac{1}{2}e^{-2\theta}\right]_0^{\ln 2}$$

$$=2\left[\ln 2+\frac{1}{2}e^{-2\ln 2}-\left(0+\frac{1}{2}\right)\right]$$

$$=2\left[\ln 2+\frac{1}{2}e^{\ln 2^{-2}}-\frac{1}{2}\right]$$

$$=2\left(\ln 2+\frac{1}{2}2^{-2}-\frac{1}{2}\right)$$

$$=2\left(\ln 2+\frac{1}{8}-\frac{1}{2}\right)$$

$$=2\left(\ln 2-\frac{3}{8}\right)$$

$$=2\ln 2-\frac{3}{4}$$

$$\boxed{=\ln 4-\frac{3}{4}}$$

Exercise

Evaluate the integral: $\displaystyle\int_1^{e^2}\frac{dx}{x\sqrt{\ln^2 x+1}}$

Solution

$$\int_1^{e^2}\frac{dx}{x\sqrt{\ln^2 x+1}}=\int_1^{e^2}\frac{d(\ln x)}{\sqrt{\ln^2 x+1}}$$

$$\int\frac{du}{\sqrt{a^2+u^2}}=\sinh^{-1}\left(\frac{u}{a}\right)$$

$$= \sinh^{-1}(\ln x)\Big|_1^{e^2}$$

$$= \sinh^{-1} 2 - 0$$

$$\underline{= \sinh^{-1} 2}$$

Exercise

Evaluate the integral: $\displaystyle\int_{1/8}^{1} \frac{dx}{x\sqrt{1+x^{2/3}}}$

Solution

$$\int_{1/8}^{1} \frac{dx}{x\sqrt{1+x^{2/3}}} = 3\int_{1/8}^{1} \frac{u^2 du}{u^3\sqrt{1+u^2}}$$

$u = x^{1/3} \quad \to \quad du = \frac{1}{3}x^{-2/3}dx$

$u^3 = x \quad \& \quad dx = 3x^{2/3}du = 3u^2du$

$$= 3\int_{1/8}^{1} \frac{du}{u\sqrt{1+u^2}}$$

$\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right|$

$$= -3\operatorname{csch}^{-1}\left|x^{1/3}\right|\Big|_{1/8}^{1}$$

$$= -3\left(\operatorname{csch}^{-1}1 - \operatorname{csch}^{-1}\frac{1}{2}\right)$$

$$= 3\left(\sinh^{-1}2 - \sinh^{-1}1\right)$$

$x = \ln\left(y + \sqrt{y^2+1}\right)$

$$\underline{= 3\left(\ln\left(2+\sqrt{5}\right) - \ln\left(1+\sqrt{2}\right)\right)}$$

Exercise

Derive the formula $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$ for all real *x*. Explain in your derivation why the plus sign is used with the square root instead of the minus sign

Solution

$$y = \sinh^{-1} x \quad \Rightarrow \quad x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - e^{-y}$$

$$2xe^y = e^y e^y - e^{-y}e^y$$

$$e^{2y} - 2xe^y - 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

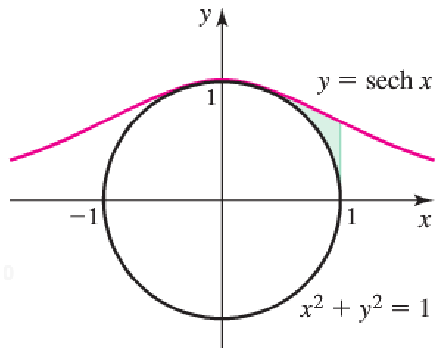
$$y = \ln\left(x \pm \sqrt{x^2+1}\right)$$

Since $x - \sqrt{x^2+1} < 0$ (***impossible***) $\Rightarrow \quad y = \ln\left(x - \sqrt{x^2+1}\right)$

$$\therefore \ y = \ln\left(x + \sqrt{x^2+1}\right)$$

Exercise

Find the area of the region bounded by $y = \text{sech}\, x$, $x = 1$, and the unit circle.



Solution

The area of a quarter circle $= \frac{1}{4}\left(\pi r^2\right) = \frac{\pi}{4}$

$$Area = \int_0^1 \text{sech}x\, dx - \frac{\pi}{4}$$

$$= \tan^{-1}|\sinh x|\Big|_0^1 - \frac{\pi}{4}$$

$$= \tan^{-1}(\sinh 1) - \frac{\pi}{4}$$

$$\approx 0.08$$

$$Area = \int_0^1 \left(\text{sech}x - \sqrt{1-x^2}\right) dx$$

$$= \left(\tan^{-1}|\sinh x| - \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\sin^{-1} x\right)\Big|_0^1$$

$$= \tan^{-1}(\sinh 1) - \frac{1}{2}\sin^{-1} 1$$

$$= \tan^{-1}(\sinh 1) - \frac{\pi}{4}$$

$$\approx 0.08$$

Exercise

A region in the first quadrant is bounded above the curve $y = \cosh x$, below by the curve $y = \sinh x$, and on the left and right by the y-axis and the line $x = 2$, respectively. Find the volume of the solid generated by revolving the region about the x-axis.

Solution

$$V = \pi \int_0^2 \left(\cosh^2 x - \sinh^2 x\right) dx$$

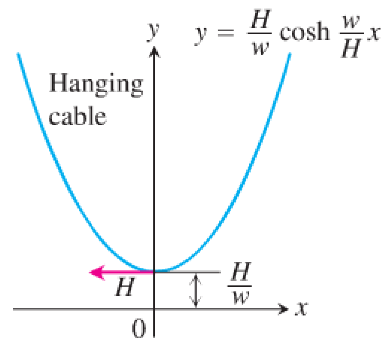
$$= \pi \int_0^2 dx$$

$$= \pi x\Big|_0^2$$

$$= 2\pi$$

Exercise

Imagine a cable, like a telephone line or TV cable, strung from one support to another and hanging freely. The cable's weight per unit length is a constant *w* and the horizontal tension at its lowest point is a vector of length *H*. If we choose a coordinate system for the plane of the cable in which the *x*-axis is horizontal, the force of gravity is straight down, the positive *y*-axis points straight up, and the lowest point of the cable lies at the point $y = \frac{H}{w}$ on the *y*-axis, then it can be shown that the cable lies along the graph of the hyperbolic cosine



$$y = \frac{H}{w} \cosh\left(\frac{w}{H} x\right)$$

Such a curve is sometimes called a ***chain curve*** or a ***catenary***, the latter deriving from the Latin *catena*, meaning "*chain*".

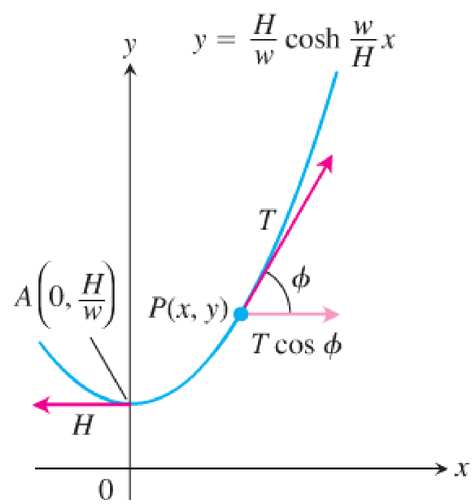
a) Let $P(x, y)$ denote an arbitrary point on the cable. The next accompanying displays the tension *H* at the lowest point *A*. Show that the cable's slope at *P* is

$$\tan\phi = \frac{dy}{dx} = \sinh\left(\frac{w}{H} x\right)$$

b) Using the result in part (*a*) and the fact that the horizontal tension at *P* must equal *H* (the cable is not moving), show that $T = wy$. Hence, the magnitude of the tension at $P(x, y)$ is exactly equal to the weight of *y* units of cable.

c) The length of arc *AP* is $s = \frac{1}{a}\sinh ax$, where $a = \frac{w}{H}$. Show that the coordinates of *P* may be expressed in terms of *s* as $x = \frac{1}{a}\sinh^{-1} as, \quad y = \sqrt{s^2 + \frac{1}{a^2}}$

Solution



a) $y = \frac{H}{w} \cosh\left(\frac{w}{H} x\right)$

$$\tan\phi = \frac{dy}{dx} = \frac{H}{w}\left[\frac{w}{H}\sinh\left(\frac{w}{H}x\right)\right]$$

$$\boxed{= \sinh\left(\frac{w}{H}x\right)}$$

b) The tension at *P* is given by $T\cos\phi = H$.

$$T = \frac{H}{\cos\phi}$$

$$= H\sec\phi$$

$$= H\sqrt{1 + \tan^2\phi}$$

$$= H\sqrt{1 + \sinh^2\left(\frac{w}{H}x\right)}$$

$\cosh^2 x - \sinh^2 x = 1 \rightarrow \cosh x = \sqrt{1 + \sinh^2 x}$

$$= H \cosh\left(\frac{w}{H}x\right) \qquad\qquad yw = H\cosh\left(\frac{w}{H}x\right)$$

$$\underline{= wy}$$

c) $s = \frac{1}{a}\sinh ax \quad \to \sinh ax = as$

$$ax = \sinh^{-1} as \ \to \ x = \frac{1}{a}\sinh^{-1} as$$

$$y = \frac{H}{w}\cosh\left(\frac{w}{H}x\right)$$

$$= \frac{1}{a}\cosh(ax) \qquad\qquad a = \frac{w}{H}$$

$$= \frac{1}{a}\sqrt{\cosh^2(ax)}$$

$$= \frac{1}{a}\sqrt{1+\sinh^2(ax)}$$

$$= \frac{1}{a}\sqrt{1+(as)^2}$$

$$\underline{= \sqrt{\frac{1}{a^2}+s^2}}$$

Exercise

The portion of the curve $y = \frac{17}{15} - \cosh x$ that lies above the x-axis forms a catenary arch. Find the average height of the arch above the x-axis.

Solution

By symmetry;

$$I = 2\int_0^{\cosh^{-1}(17/15)}\left(\frac{17}{15} - \cosh x\right)dx$$

$$= 2\left(\frac{17}{15}x - \sinh x\right)\Big|_0^{\cosh^{-1}(17/15)}$$

$$= 2\left[\frac{17}{15}\cosh^{-1}\left(\frac{17}{15}\right) - \sinh\left(\cosh^{-1}\left(\frac{17}{15}\right)\right)\right]$$

$$= \frac{34}{15}\cosh^{-1}\left(\frac{17}{15}\right) - \frac{16}{15}$$

$$\textit{Average height} = \frac{I}{2\cosh^{-1}\left(\frac{17}{15}\right)}$$

$$\underline{= \frac{17}{15} - \frac{8}{15\cosh^{-1}\left(\frac{17}{15}\right)}} \quad \underline{\approx 0.09}$$

Exercise

A power line is attached at the same height to two utility poles that are separated by a distance of 100 *ft*; the power line follows the curve $f(x) = a\cosh\left(\frac{x}{a}\right)$. Use the following steps to find the value of a that produces a sag of 10 *ft*. midway between the poles. Use the coordinate system that places the poles at $x = \pm 50$

a) Show that a satisfies the equation $\cosh\left(\frac{50}{a}\right) - 1 = \frac{10}{a}$

b) Let $t = \frac{10}{a}$, confirm that the equation in part (*a*) reduces to $\cosh 5t - 1 = t$, and solve for t using a graphing utility. (2 decimal places)

c) Use the answer in part (*b*) to find a and then compute the length of the power line.

Solution

a) Let $a = 10\ ft\ \ (sag)$.

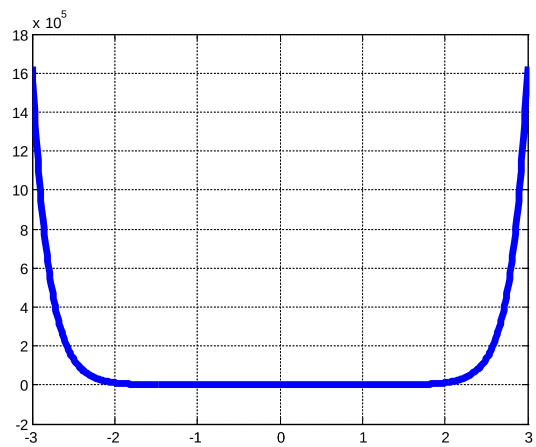
$$f(0) + sag = f(50)$$

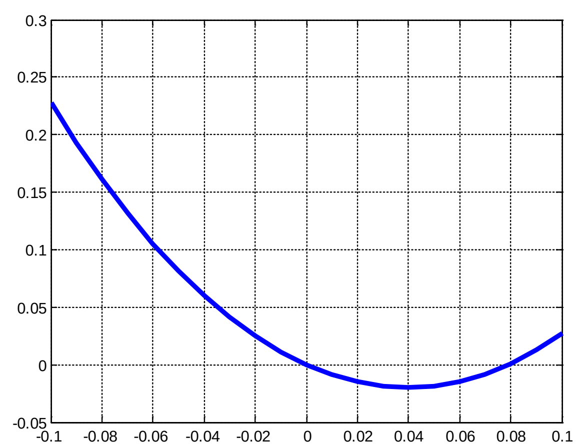
$$a + sag = a\cosh\left(\frac{50}{a}\right)$$

$$1 + \frac{sag}{a} = \cosh\left(\frac{50}{a}\right)$$

$$\cosh\left(\frac{50}{a}\right) - 1 = \frac{10}{a}$$

b) If $t = \frac{10}{a} \rightarrow \cosh(5t) - 1 = t$





$t \approx 0.08$

c) If $\frac{10}{a} = 0.08 \Rightarrow a = \frac{10}{0.08} = 125$

The length of the power line is:

$$L = 2\int_0^{50} \sqrt{1 - \sinh^2\left(\frac{x}{125}\right)}\, dx$$

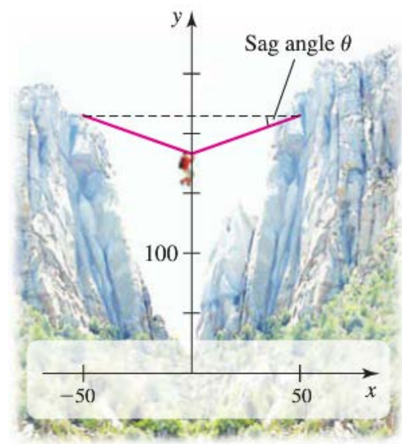
$$= 2\int_0^{50} \cosh\left(\frac{x}{125}\right)\, dx$$

$$= 250\sinh\left(\frac{x}{125}\right)\Big|_0^{50}$$

$$= 250\sinh\left(\frac{2}{5}\right) \approx 102.7\ ft$$

Exercise

Imagine a climber clipping onto the rope and pulling hinself to the rope's midpoint. Because the rope is supporting the weight of the climber, it no longer takes the shape of the catenary $y = 200\cosh\left(\frac{x}{200}\right)$. Instead, the rope (nearly) forms two sides of an isosceles triangle. Compute the sag angle illustrated in the figure, assuming that the rope does not stretch when weighted. Assume the length of the rope is 101 *feet*.

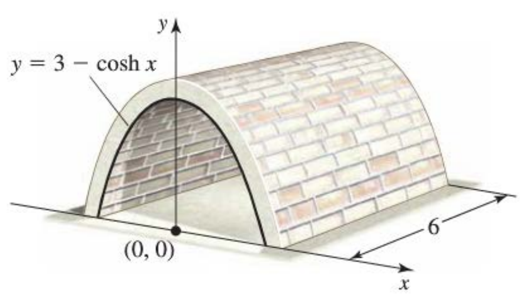


Solution

$$\theta = \cos^{-1}\left(\frac{50}{50.5}\right) \approx 0.14\ rad$$

Exercise

Find the volume interior to the inverted catenary kiln (an oven used to fire pottery).



Solution

$$y = 3 - \cosh x = 0 \quad \Rightarrow \quad \cosh x = 3 \to x = \cosh^{-1}(\pm 3)$$

Therfore; the area is between $\cosh^{-1}(-3)$ and $\cosh^{-1}(3)$.

$$A = 2\int_0^{\cosh^{-1}(3)} (3 - \cosh x)\, dx$$

$$= 2(3x - \sinh x)\Big|_0^{\cosh^{-1}(3)}$$

$$= 2\left(3\cosh^{-1}(3) - \sinh x\left(\cosh^{-1}(3)\right)\right)$$

$$\approx 4.92$$

$$Volume \approx 6(4.92) \approx 29.5$$