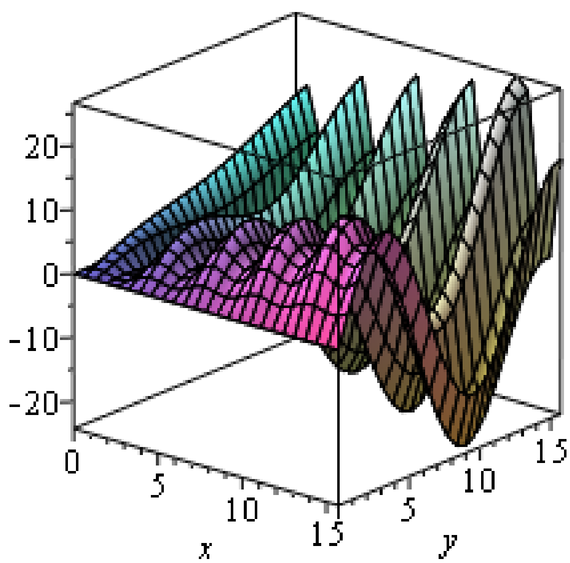
Notebook 15: Functions of Several Variables and Partial Derivatives

Functions of Several Variables

The *plot3d* command will display the graph of a function $z = f(x, y)$, and will also graph parameterized surfaces. The *implicitplot3d*, *contourplot*, and *contourplot3d* commands in the *plots* package are also useful for plotting and examining functions of multiple variables.

Consider the function

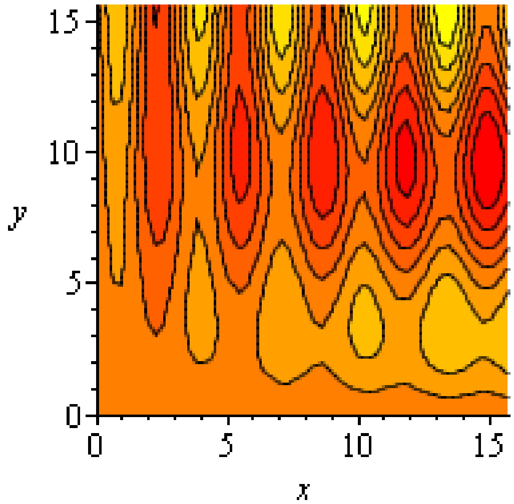
> $f(x, y) := x \cdot \sin\left(\frac{y}{2}\right) + y \cdot \sin(2x)$:
> $plot3d\big(f(x, y), x = 0..5\,\pi, y = 0..5\,\pi, axes = boxed, orientation = [-50, 70], lightmodel = light3\big)$



Notice the use of a special light model above to highlight the surface.

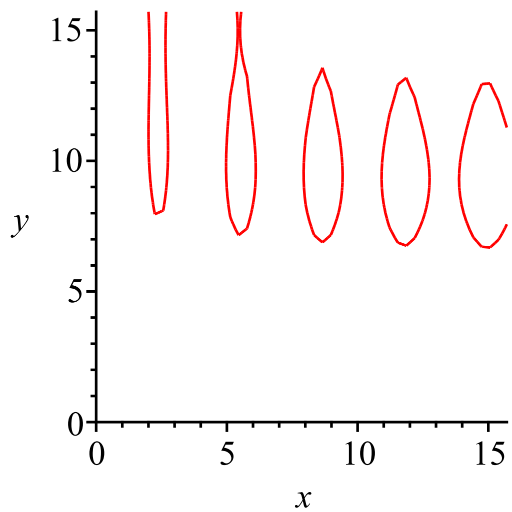
The *contourplot* command will give a plot of some level curves of the surface. The argument $grid = [m, n]$ specifies that m points are to be plotted in the x direction and n in the y direction; this smooths out the curves displayed. The argument $contours = 8$ specifies that 8 contour curves are to be plotted.

> $with(plots):$
$contourplot\left(f(x, y), x = 0..5\pi, y = 0..5\pi, grid = [50, 50], contours = 8, filled = true\right)$



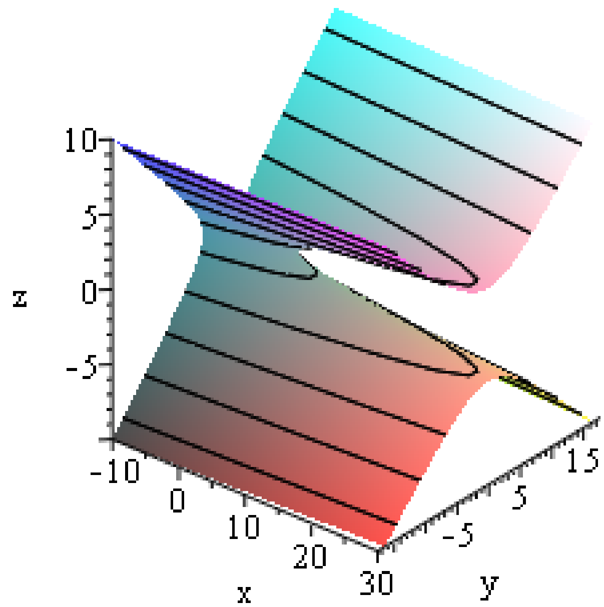
Specific contour curves may also be specified by the argument *contours* = [...].

> $contourplot\left(f(x, y), x = 0..5\pi, y = 0..5\pi, grid = [50, 50], contours = \left[f\left(3\pi, 3\pi\right)\right],\right.$
$\left. view = \left[0..5\pi, 0..5\pi\right]\right)$

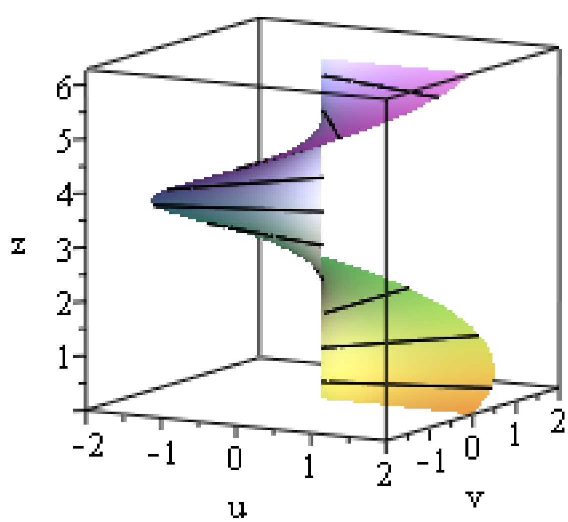


Below, an implicitly-defined function and a parameterized surface are plotted. Often experimentation is required, changing the *x*, *y*, and *z* ranges, the axes, the style of the surface, and other options in order to produce the best image.

> $implicitplot3d\left(x + y^2 - 3\,z^2 = 1, x = -10\,..30, y = -20\,..20, z = -10\,..10, axes = framed, style = patchcontour, grid = [30, 30, 30], orientation = [-50, 60]\right)$



> $plot3d\left([u \cdot \cos(v), u \cdot \sin(v), v], u = 0\,..2, v = 0\,..2\,\pi, axes = boxed, orientation = [-60, 80], style = patchcontour, lightmodel = light1, labels = ["u", "v", "z"]\right)$



Extreme Values and Saddle Points

The *diff* command makes no distinction between regular and partial derivatives.

> *diff*$(g(x), x)$

$$\frac{\mathrm{d}}{\mathrm{d}x}\, g(x)$$

> *diff*$(h(x, y), x)$, *diff*$(h(x, y), y)$

$$\frac{\partial}{\partial x}\, h(x, y),\ \frac{\partial}{\partial y}\, h(x, y)$$

The partial derivative template $\frac{\partial}{\partial x}$ can be found in the Expression palette or can be entered manually by typing diff, pressing **[esc]**, then selecting the second option from the contextual menu. An optional argument can be used with the D command to calculate the function that is the partial derivative of a function. For example, the partial derivative of a function $f(x, y)$ with respect to the first variable is shown below.

> $f(x, y) := x^2 - \sin(x \cdot y)$;
  $\mathrm{D}[1](f)$

$$f := (x, y) \to x^2 - \sin(x\, y)$$
$$(x, y) \to 2\, x - \cos(x\, y)\, y$$

Evaluating this derivative function at (x, y) gives the partial derivative as an expression.

> $\mathrm{D}[1](f)\ (x, y)$;
  $\frac{\partial}{\partial x} f(x, y)$

$$2\, x - \cos(x\, y)\, y$$
$$2\, x - \cos(x\, y)\, y$$

Of course, the partial derivative of *f* with respect to its second variable is given by $\mathrm{D}[2](f)$. The second mixed partial derivative function is given by $\mathrm{D}[1, 2](f)$

> $\mathrm{D}[1, 2](f)$;
  $\mathrm{D}[1, 2](f)\ (x, y)$

$$(x, y) \to \sin(x\, y)\, y\, x - \cos(x\, y)$$
$$\sin(x\, y)\, y\, x - \cos(x\, y)$$

This can also be obtained using the *diff* command or the $\frac{\partial}{\partial x}$ template.

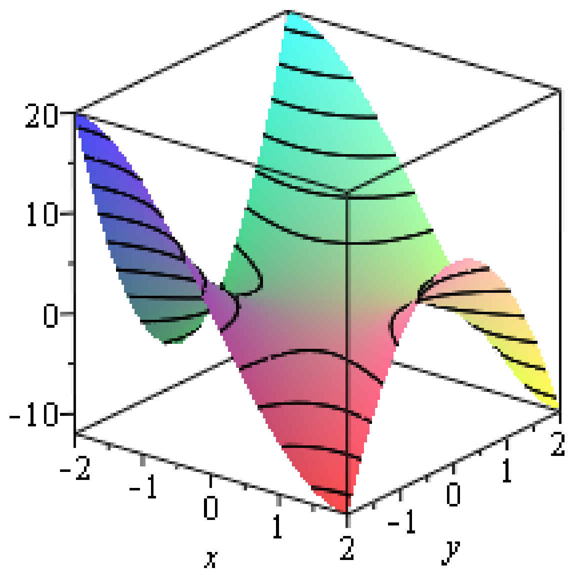
> *diff*$(f(x, y), x, y)$;
  $\frac{\partial^2}{\partial y\, \partial x} f(x, y)$

$$\sin(x\, y)\, y\, x - \cos(x\, y)$$
$$\sin(x\, y)\, y\, x - \cos(x\, y)$$

The $\frac{\partial}{\partial x}$ template must be manually edited to calculate the second partial derivative. The second partial derivative in the denominator by typing Partial, then **[esc]/[enter]**.

Now, consider the problem of finding and classifying critical points of a surface.

> $f(x,y) := x^3 - 3\,x \cdot y^2 + y^2$:
> $plot3d(f(x,y), x = -2..2, y = -2..2, axes = boxed, orientation = [-52, 68], style = patchcontour)$;
> $Surface := \%$:



There appears to be a saddle point. To find the critical points, the equations $\frac{\partial}{\partial x} f(x,y) = 0$ and $\frac{\partial}{\partial y} f(x,y) = 0$ must be solved simultaneously.

> $CPeqns := \{ \mathrm{D}[1](f)(x,y) = 0, \mathrm{D}[2](f)(x,y) = 0 \}$

$$CPeqns := \{3\,x^2 - 3\,y^2 = 0,\ -6\,x\,y + 2\,y = 0\}$$

> $CP := solve(\,CPeqns, \{x, y\})$

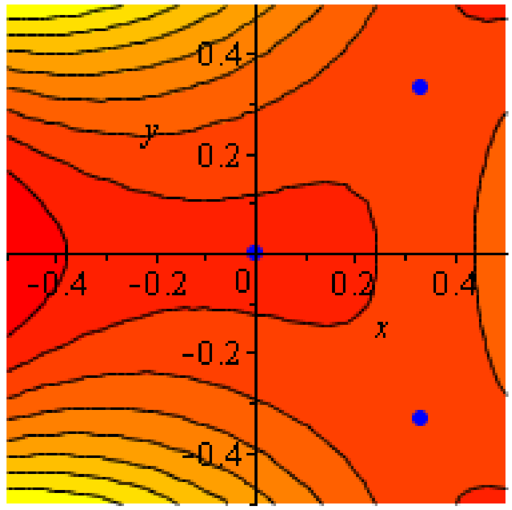
$$CP := \{x = 0, y = 0\},\ \{x = 0, y = 0\},\ \left\{x = \frac{1}{3}, y = \frac{1}{3}\right\},\ \left\{x = \frac{1}{3}, y = -\frac{1}{3}\right\}$$

There are three critical points. The repeated solution can be removed by placing the solutions *CP* in a set. Let's then add these points to a contour plot and investigate

> $CPset := \{CP\}$

$$CPset := \left\{ \{x = 0, y = 0\},\ \left\{x = \frac{1}{3}, y = -\frac{1}{3}\right\},\ \left\{x = \frac{1}{3}, y = \frac{1}{3}\right\} \right\}$$

> *points* := *pointplot*(['*subs*'(*CPset*[*k*], [*x*, *y*]) $*k* = 1 ..3], *symbol* = *solidcircle*, *symbolsize* = 30, *color* = *blue*) :
display(*contourplot*($f(x, y)$, $x = -0.5 .. 0.5$, $y = -0.5 .. 0.5$, *filled* = *true*), *points*)



The f_{xx} partial derivative and the Hessian determinant ("The Second Derivative Test") calculation can be repeated for each critical point using a **for** .. **do** loop.

> **for** *k* **from** 1 **to** 3 **do**
Point := *subs*(*CPset*[*k*], [*x*, *y*]);
$f_{xx} = \mathrm{D}[1, 1](f)(op(Point)), Hessian = \mathrm{D}[1, 1](f)(op(Point)) \cdot \mathrm{D}[2, 2](f)(op(Point)) - \mathrm{D}[1, 2](f)(op(Point))^2$;
end do; *unassign*('*k*') :

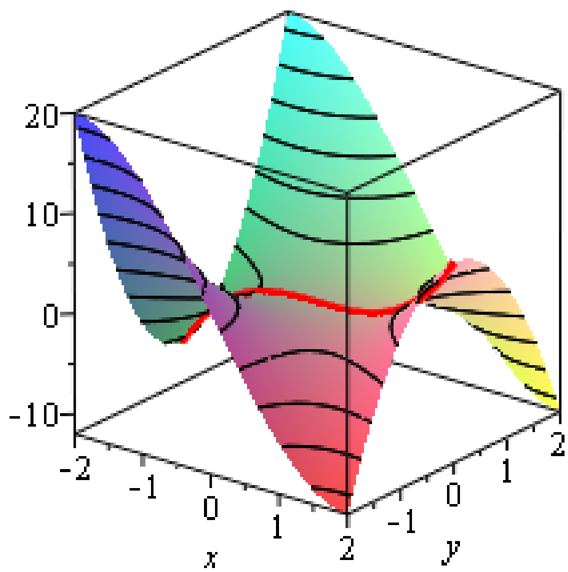
$$Point := [0, 0]$$
$$f_{xx} = 0, Hessian = 0$$
$$Point := \left[\frac{1}{3}, -\frac{1}{3}\right]$$
$$f_{xx} = 2, Hessian = -4$$
$$Point := \left[\frac{1}{3}, \frac{1}{3}\right]$$
$$f_{xx} = 2, Hessian = -4$$

The points $\left(\frac{1}{3}, -\frac{1}{3}\right)$ and $\left(\frac{1}{3}, \frac{1}{3}\right)$ are clearly saddle points, but no information is given about the point $(0, 0)$. Notice that when $y = 0$, the curve on the surface is a cubic.

> $f(x, 0)$

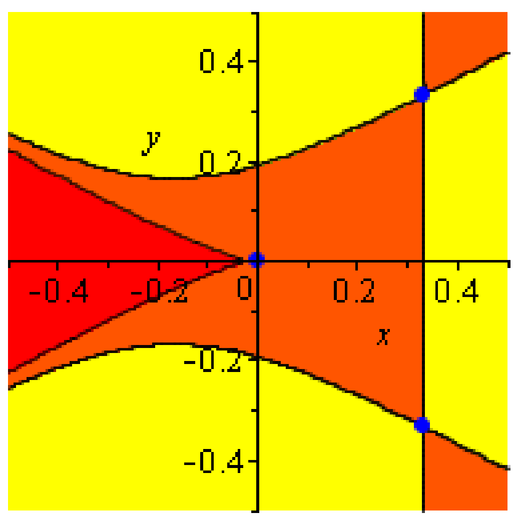
$$x^3$$

> $display(\,Surface, spacecurve(\,[\,x, 0, f(x, 0)\,], x=-2\,..2, color=red, thickness=3\,)\,)$



Therefore every neighborhood of $(0, 0)$ contains points where f is larger than $f(0, 0) = 0$ and points where f is smaller than 0, so $(0, 0)$ is a saddle point. The following graph supports these conclusions.

> $display(\,contourplot(f(x, y), x=-0.5\,..0.5, y=-0.5\,..0.5, contours = [\,'subs'(\,CPset[\,k\,], f(x, y)\,)\,\$k=1\,..3\,], grid=[\,80, 80\,], filled=true\,), points\,)$



Lagrange Multipliers

When the *solve* command is used to solve an equation or a system of equations, the output sometimes contains expressions that represent the roots of a polynomial or some other expression.

> $solve(x^4 + x + 1 = 0, x)$

$RootOf(\_Z^4 + \_Z + 1, index = 1), RootOf(\_Z^4 + \_Z + 1, index = 2), RootOf(\_Z^4 + \_Z + 1, index = 3),$
$RootOf(\_Z^4 + \_Z + 1, index = 4)$

When this happens, the *evalf* or the *allvalues* commands can be applied to see the values of the roots.
e

> $evalf(\%[1])$

$$0.7271360845 + 0.9340992895\,\mathrm{I}$$

The first root above evaluates numerically to a complex number.

Consider the following problem: Minimize $f(x, y, z) = x\cdot y + y\cdot z$ subject to the constraints $x^2 + y^2 = 2$ and $x^2 + z^2 = 2$ using Lagrange multipliers. The λ can be found in the Greek pallete or can be entered manually by typing lambda, then **[esc]/[enter]**.

First, define *f* and the constraint functions, then form the function *h*.

> $f(x, y, z) := x\cdot y + y\cdot z : g1(x, y, z) := x^2 + y^2 - 2 : g2(x, y, z) := x^2 + z^2 - 2 :$
> $h := unapply\left(f(x, y, z) - \lambda_1\cdot g1(x, y, z) - \lambda_2\cdot g2(x, y, z), x, y, z, \lambda_1, \lambda_2\right)$

$h := (x, y, z, lambda\_1, lambda\_2) \rightarrow x\,y + y\,z - lambda\_1\,(x^2 + y^2 - 2) - lambda\_2\,(x^2 + z^2 - 2)$

Next, calculate all partial derivatives of *h* and set them equal to 0.

> $eqns := \{\,'\mathrm{D}'[k](h)\left(x, y, z, \lambda_1, \lambda_2\right) = 0\$k = 1\,..5\}$

$eqns := \{y - 2\,\lambda_2\,z = 0, 2 - x^2 - y^2 = 0, 2 - x^2 - z^2 = 0, x + z - 2\,\lambda_1\,y = 0, y - 2\,\lambda_1\,x - 2\,\lambda_2\,x = 0\}$

Now, solve the system of equations. By putting the unknowns in a list in the *solve* command, the output will be a list of lists.

> $solns := solve\left(eqns, [x, y, z, \lambda_1, \lambda_2]\right)$

$solns := \Big[\Big[x = -RootOf(\_Z^2 - 1 - RootOf(2\,\_Z^2 - 1)) + 2\,RootOf(2\,\_Z^2 - 1)\,RootOf(\_Z^2 - 1 - RootOf(2\,\_Z^2 - 1)), y = RootOf(\_Z^2 - 1 - RootOf(2\,\_Z^2 - 1)), z = RootOf(\_Z^2 - 1 - RootOf(2\,\_Z^2 - 1)), \lambda_1 = RootOf(2\,\_Z^2 - 1), \lambda_2 = \frac{1}{2}\Big], \Big[x = -RootOf(\_Z^2 - 1 + RootOf(2\,\_Z^2 - 1)) - 2\,RootOf(2\,\_Z^2 - 1)\,RootOf(\_Z^2 - 1 + RootOf(2\,\_Z^2 - 1)), y = -RootOf(\_Z^2 - 1 + RootOf(2\,\_Z^2 - 1)), z = RootOf(\_Z^2 - 1 + RootOf(2\,\_Z^2 - 1)), \lambda_1 = RootOf(2\,\_Z^2 - 1), \lambda_2 = -\frac{1}{2}\Big]\Big]$

There appear to be two families of solutions. This can be checked with the *nops* command.

> $nops(solns)$

$$2$$

These families will be analyzed separately. The first family of solutions will be called *S*. The *allvalues* command will show what the solutions look like.

> $S := allvalues(solns[1])$

$$S := \left[x = -\frac{1}{2}\sqrt{4+2\sqrt{2}} + \frac{1}{2}\sqrt{2}\sqrt{4+2\sqrt{2}}, y = \frac{1}{2}\sqrt{4+2\sqrt{2}}, z = \frac{1}{2}\sqrt{4+2\sqrt{2}}, \lambda_1 = \frac{1}{2}\sqrt{2}, \lambda_2 = \frac{1}{2}\right], \left[x = \frac{1}{2}\sqrt{4+2\sqrt{2}} - \frac{1}{2}\sqrt{2}\sqrt{4+2\sqrt{2}}, y = -\frac{1}{2}\sqrt{4+2\sqrt{2}}, z = -\frac{1}{2}\sqrt{4+2\sqrt{2}}, \lambda_1 = \frac{1}{2}\sqrt{2}, \lambda_2 = \frac{1}{2}\right], \left[x = -\frac{1}{2}\sqrt{4-2\sqrt{2}} - \frac{1}{2}\sqrt{2}\sqrt{4-2\sqrt{2}}, y = \frac{1}{2}\sqrt{4-2\sqrt{2}}, z = \frac{1}{2}\sqrt{4-2\sqrt{2}}, \lambda_1 = -\frac{1}{2}\sqrt{2}, \lambda_2 = \frac{1}{2}\right], \left[x = \frac{1}{2}\sqrt{4-2\sqrt{2}} + \frac{1}{2}\sqrt{2}\sqrt{4-2\sqrt{2}}, y = -\frac{1}{2}\sqrt{4-2\sqrt{2}}, z = -\frac{1}{2}\sqrt{4-2\sqrt{2}}, \lambda_1 = -\frac{1}{2}\sqrt{2}, \lambda_2 = \frac{1}{2}\right]$$

Approximations of each solution and the function value at each point are calculated using a **for** .. **do** loop.

> **for** k **from** 1 **to** 3 **do**
> $evalf[4](S[k])$; $f(x, y, z) = evalf[4](eval(f(x, y, z), S[k]))$;
> **od**; $unassign('k')$

$$[x = 0.542, y = 1.306, z = 1.306, \lambda_1 = 0.7070, \lambda_2 = 0.5000]$$
$$f(x, y, z) = 2.415$$
$$[x = -0.542, y = -1.306, z = -1.306, \lambda_1 = 0.7070, \lambda_2 = 0.5000]$$
$$f(x, y, z) = 2.415$$
$$[x = -1.307, y = 0.5415, z = 0.5415, \lambda_1 = -0.7070, \lambda_2 = 0.5000]$$
$$f(x, y, z) = -0.4145$$

The last two steps are repeated for the second family of solutions, called *T*.

> $T := allvalues(solns[2])$

$$T := \left[x = -\frac{1}{2}\sqrt{4-2\sqrt{2}} - \frac{1}{2}\sqrt{2}\sqrt{4-2\sqrt{2}}, y = -\frac{1}{2}\sqrt{4-2\sqrt{2}}, z = \frac{1}{2}\sqrt{4-2\sqrt{2}}, \lambda_1 = \frac{1}{2}\sqrt{2}, \lambda_2 = -\frac{1}{2}\right], \left[x = \frac{1}{2}\sqrt{4-2\sqrt{2}} + \frac{1}{2}\sqrt{2}\sqrt{4-2\sqrt{2}}, y = \frac{1}{2}\sqrt{4-2\sqrt{2}}, z = -\frac{1}{2}\sqrt{4-2\sqrt{2}}, \lambda_1 = \frac{1}{2}\sqrt{2}, \lambda_2 = -\frac{1}{2}\right], \left[x = -\frac{1}{2}\sqrt{4+2\sqrt{2}} + \frac{1}{2}\sqrt{2}\sqrt{4+2\sqrt{2}}, y = -\frac{1}{2}\sqrt{4+2\sqrt{2}}, z = \frac{1}{2}\sqrt{4+2\sqrt{2}}, \lambda_1 = -\frac{1}{2}\sqrt{2}, \lambda_2 = -\frac{1}{2}\right], \left[x = \frac{1}{2}\sqrt{4+2\sqrt{2}} - \frac{1}{2}\sqrt{2}\sqrt{4+2\sqrt{2}}, y = \frac{1}{2}\sqrt{4+2\sqrt{2}}, z = -\frac{1}{2}\sqrt{4+2\sqrt{2}}, \lambda_1 = -\frac{1}{2}\sqrt{2}, \lambda_2 = -\frac{1}{2}\right]$$

> **for** k **from** 1 **to** 3 **do**
> $evalf[4](T[k])$; $f(x, y, z) = evalf[4](eval(f(x, y, z), T[k]))$;
> **od**; $unassign('k')$

$$[x = -1.307, y = -0.5415, z = 0.5415, \lambda_1 = 0.7070, \lambda_2 = -0.5000]$$
$$f(x, y, z) = 0.4145$$
$$[x = 1.307, y = 0.5415, z = -0.5415, \lambda_1 = 0.7070, \lambda_2 = -0.5000]$$
$$f(x, y, z) = 0.4145$$
$$[x = 0.542, y = -1.306, z = 1.306, \lambda_1 = -0.7070, \lambda_2 = -0.5000]$$
$$f(x, y, z) = -2.415$$

The minimum value of $f(x, y, z)$ is approximately -2.415 and is attained at approximately the point $(0.542, -1.306, 1.306)$. To find the exact minimum value, evaluate the function at the exact point where the minimum value is attained.

> $eval(f(x, y, z), T[3]); expand(\%)$

$$-\frac{1}{2}\left(-\frac{1}{2}\sqrt{4+2\sqrt{2}}+\frac{1}{2}\sqrt{2}\sqrt{4+2\sqrt{2}}\right)\sqrt{4+2\sqrt{2}}-1-\frac{1}{2}\sqrt{2}$$

$$-\sqrt{2}-1$$