Solution Section 2.1 – Introducing the Derivative

Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at *P*; then graph the curve and the tangent line.

$$y = 4 - x^2; \quad P(-1,\ 3)$$

Solution

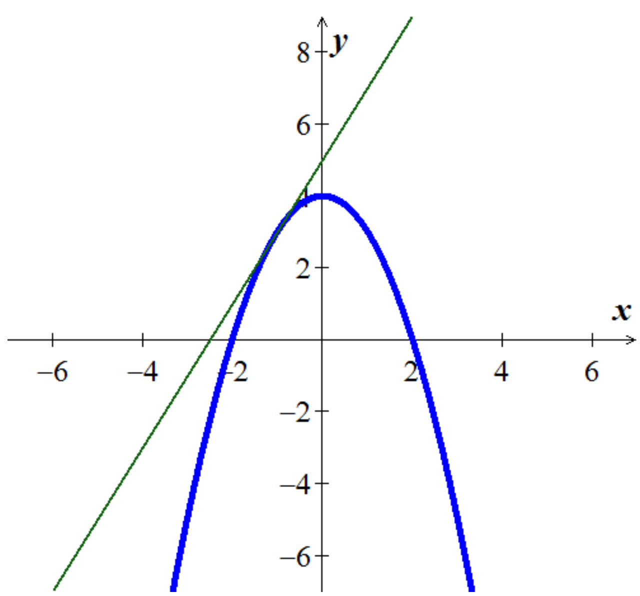
$$\begin{aligned}
m &= \lim_{h\to 0} \frac{4-(x+h)^2-\left(4-x^2\right)}{h} \\
&= \lim_{h\to 0} \frac{4-(-1+h)^2-\left(4-(-1)^2\right)}{h} \\
&= \lim_{h\to 0} \frac{4-\left(1-2h+h^2\right)-(4-1)}{h} \\
&= \lim_{h\to 0} \frac{4-1+2h-h^2-3}{h} \\
&= \lim_{h\to 0} \frac{2h-h^2}{h} \\
&= \lim_{h\to 0} (2-h) \\
&= 2
\end{aligned}$$

$$y - y_1 = m\left(x - x_1\right)$$

At (–1, 3) $\Rightarrow y - 3 = 2\left(x-(-1)\right)$

$$y - 3 = 2x + 2$$

$$y = 2x + 5$$



Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at *P*; then graph the curve and the tangent line.

$$y = \frac{1}{x^2}; \quad P(-1,\ 1)$$

Solution

$$m = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{(-1+h)^2} - \frac{1}{1}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{1-\left(1-2h+h^2\right)}{(-1+h)^2}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{1-1+2h-h^2}{(-1+h)^2}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{2h-h^2}{(-1+h)^2}\right)$$

$$= \lim_{h\to 0} \frac{h}{h}\left(\frac{2-h}{(-1+h)^2}\right)$$

$$= \lim_{h\to 0} \left(\frac{2-h}{(-1+h)^2}\right)$$

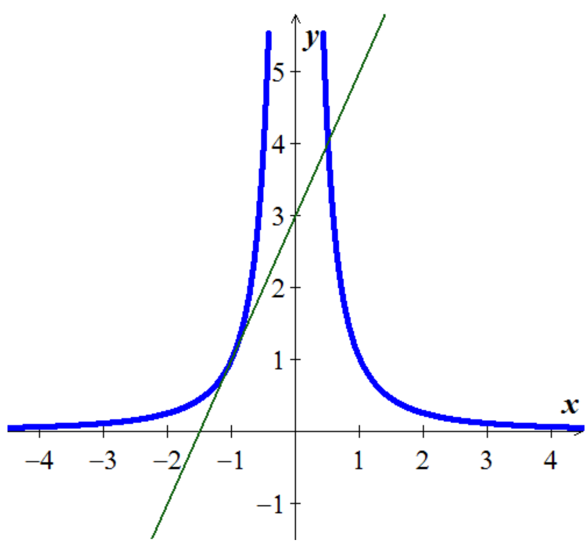
$$= \frac{2+0}{(-1+0)^2}$$

$$\underline{= 2}$$

At (–1, 3) $\Rightarrow y-1 = 2\left(x-(-1)\right)$

$$y - 1 = 2x + 2$$

$$\underline{y = 2x + 3}$$



Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at *P*; then graph the curve and the tangent line.

$$f(x) = 2\sqrt{x}; \quad P(1,\ 2)$$

Solution

$$m = \lim_{h\to 0} \frac{2\sqrt{x+h} - 2\sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{2\sqrt{1+h} - 2\sqrt{x}}{h} \cdot \frac{2\sqrt{1+h}+2}{2\sqrt{1+h}+2}$$

$$= \lim_{h\to 0} \frac{4(1+h)-4}{h\left(2\sqrt{1+h}+2\right)}$$

$$= \lim_{h\to 0} \frac{4+4h-4}{h\left(2\sqrt{1+h}+2\right)}$$

$$= \lim_{h\to 0} \frac{4h}{h\left(2\sqrt{1+h}+2\right)}$$

$$= \lim_{h\to 0} \frac{4}{2\sqrt{1+h}+2}$$

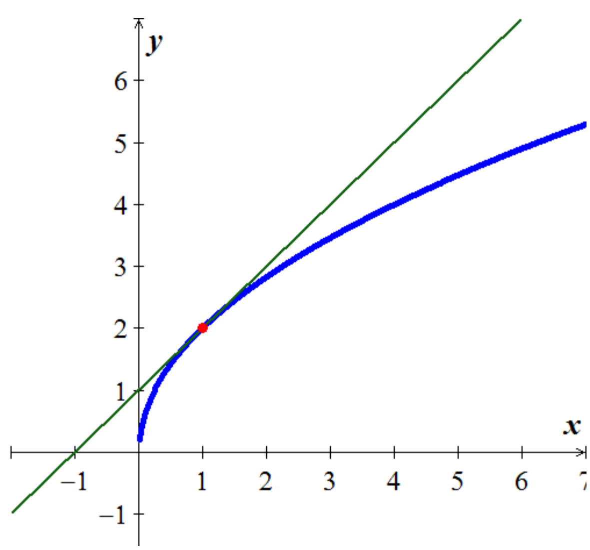
$$= \frac{4}{2+2}$$

$$= 1$$

At (1, 2) $\Rightarrow y - 2 = (x-1)$ $\qquad\qquad y - y_1 = m\left(x - x_1\right)$

$$y - 2 = x - 1$$

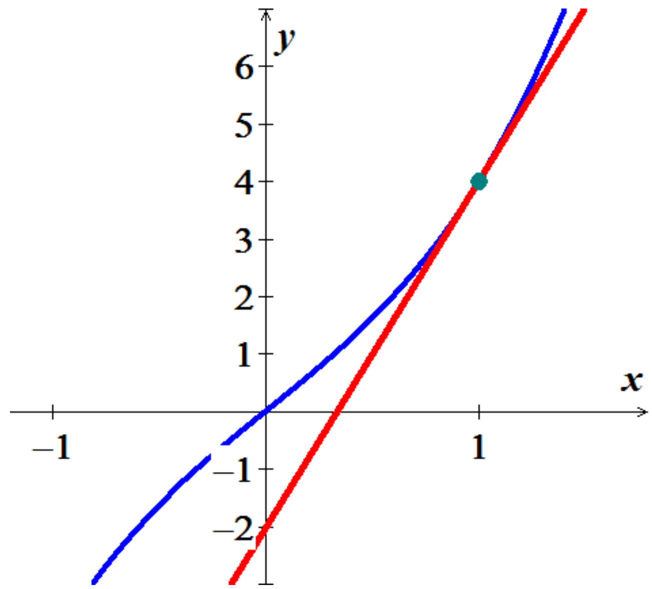
$$y = x + 1$$



Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P; then graph the curve and the tangent line.

$$f(x) = x^3 + 3x; \quad P(1,\ 4)$$



Solution

$$m = \lim_{h\to 0} \frac{(x+h)^3 + 3(x+h) - \left(x^3 + 3x\right)}{h}$$

$$= \lim_{h\to 0} \frac{1 + 3h + 3h^2 + h^3 + 3 + 3h - (4)}{h}$$

$$= \lim_{h\to 0} \frac{3h + 3h^2 + h^3 + 3h}{h}$$

$$= \lim_{h\to 0} \frac{3h^2 + h^3 + 6h}{h}$$

$$= 6$$

At $(1,\ 4)$

$$y - 4 = 6(x - 1) \qquad\qquad y - y_1 = m\left(x - x_1\right)$$

$$y - 4 = 6x - 6$$

$$y = 6x - 2$$

Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P; then graph the curve and the tangent line.

$$f(x) = 4x^2 - 7x + 5; \quad P(2,\ 7)$$

Solution

$$m = \lim_{h\to 0} \frac{4(x+h)^2 - 7(x+h) + 5 - 4x^2 + 7x - 5}{h}$$

$$= \lim_{h\to 0} \frac{4\left(x^2 + 2xh + h^2\right) - 7x - 7h - 4x^2 + 7x}{h}$$

$$= \lim_{h\to 0} \frac{4x^2 + 8xh + 4h^2 - 7h - 4x^2}{h}$$

$$= \lim_{h\to 0} \frac{8xh + 4h^2 - 7h}{h}$$

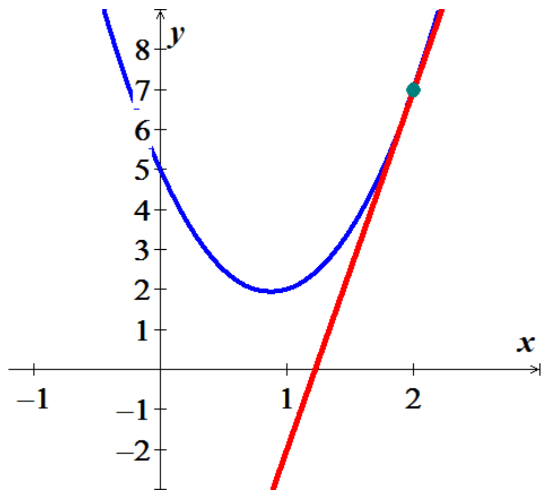
$$= \lim_{h\to 0}(8xh+4h-7)$$

$$\underline{=8x-7|}$$

At $(2,\ 7) \ \rightarrow\ \underline{m=9|}$

$y=9(x-2)+7$ $\qquad$ $y=m(x-x_1)+y_1$

$$\underline{=9x-11\ |}$$



Exercise

Use the definition of the derivative to determine the slope of the curve $y=f(x)$. Find an equation of the line tangent to the curve $y=f(x)$ at P; then graph the curve and the tangent line.

$$f(x)=5x^3+x; \quad P(1,\ 6)$$

Solution

$$m=\lim_{h\to 0}\frac{5(x+h)^3+(x+h)-5x^3-x}{h}$$

$$=\lim_{h\to 0}\frac{5\left(x^3+3x^2h+3xh^2+h^3\right)+h-5x^3}{h}$$

$$=\lim_{h\to 0}\frac{15x^2h+15xh^2+5h^3+h}{h}$$

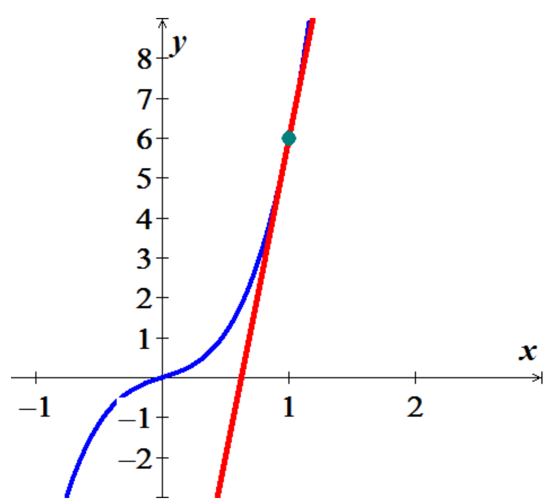
$$=\lim_{h\to 0}\left(15x^2+15xh+5h^2+1\right)$$

$$=15x^2+1\Big|_{(1,\ 6)}$$

$$\underline{=16|}$$

$y=16(x-1)+6$ $\qquad$ $y=m(x-x_1)+y_1$

$$\underline{=16x-10\ |}$$



Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at *P*; then graph the curve and the tangent line.

$$f(x) = \frac{x+3}{2x+1}; \quad P(0,\ 3)$$

Solution

$$m = \lim_{h \to 0} \frac{1}{h}\left[\frac{x+h+3}{2x+2h+1} - \frac{x+3}{2x+1}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{2x^2+2hx+6x+x+h+3-2x^2-2hx-x-6x-6h-3}{(2x+2h+1)(2x+1)}\right)$$

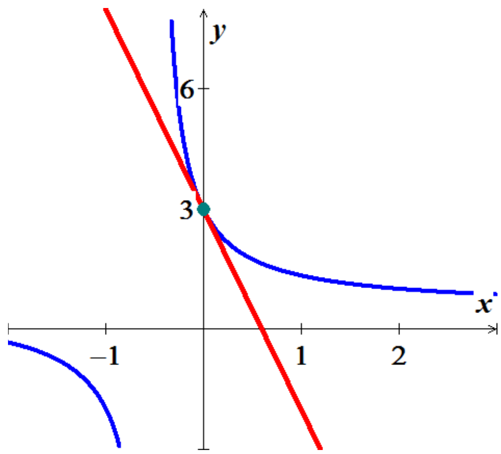
$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{-5h}{(2x+2h+1)(2x+1)}\right)$$

$$= \lim_{h \to 0} \left(\frac{-5}{(2x+2h+1)(2x+1)}\right)$$

$$= \frac{-5}{(2x+1)^2}\Bigg|_{(0,\ 3)}$$

$$= -5$$

$$y = -5x+3 \qquad\qquad y = m(x - x_1) + y_1$$



Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at *P*; then graph the curve and the tangent line.

$$f(x) = \frac{1}{2\sqrt{3x+1}}; \quad P\left(0,\ \frac{1}{2}\right)$$

Solution

$$m = \lim_{h \to 0} \frac{1}{h}\left[\frac{1}{2\sqrt{3x+3h+1}} - \frac{1}{2\sqrt{3x+1}}\right]$$

$$= \frac{1}{2}\lim_{h \to 0} \frac{1}{h}\left(\frac{\sqrt{3x+1}-\sqrt{3x+3h+1}}{\sqrt{3x+3h+1}\,\sqrt{3x+1}}\right) \quad = \frac{0}{0}$$

$$= \frac{1}{2}\lim_{h \to 0} \frac{1}{h}\left(\frac{\sqrt{3x+1}-\sqrt{3x+3h+1}}{\sqrt{3x+3h+1}\,\sqrt{3x+1}}\right)\frac{\sqrt{3x+1}+\sqrt{3x+3h+1}}{\sqrt{3x+1}+\sqrt{3x+3h+1}}$$

$$= \frac{1}{2}\lim_{h \to 0} \frac{1}{h}\left(\frac{3x+1-3x-3h-1}{\sqrt{3x+3h+1}\,\sqrt{3x+1}\left(\sqrt{3x+1}+\sqrt{3x+3h+1}\right)}\right)$$

$$= \frac{1}{2}\lim_{h\to 0}\frac{1}{h}\left(\frac{-3h}{\sqrt{3x+3h+1}\,\sqrt{3x+1}\left(\sqrt{3x+1}+\sqrt{3x+3h+1}\right)}\right)$$

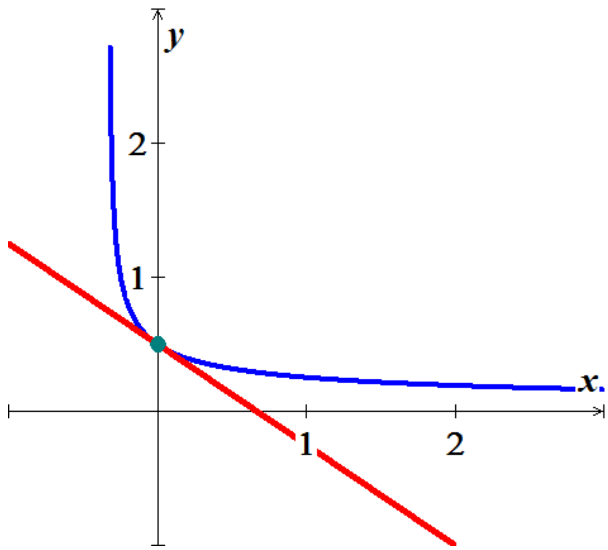
$$= -\frac{3}{2}\lim_{h\to 0}\left(\frac{1}{\sqrt{3x+3h+1}\,\sqrt{3x+1}\left(\sqrt{3x+1}+\sqrt{3x+3h+1}\right)}\right)$$

$$= -\frac{3}{2}\frac{1}{(3x+1)\left(2\sqrt{3x+1}\right)}$$

$$= -\frac{3}{4}\frac{1}{(3x+1)^{3/2}}\Bigg|_{\left(0,\ \frac{1}{2}\right)}$$

$$\boxed{= -\frac{3}{4}}$$

$$\boxed{y = -\frac{3}{4}x + \frac{1}{2}} \qquad y = m\left(x - x_1\right) + y_1$$



Exercise

Find the slope of the curve $y = 1 - x^2$ at the point $x = 2$

Solution

$$m = \lim_{h\to 0}\frac{1-(x+h)^2-\left(1-x^2\right)}{h}$$

$$= \lim_{h\to 0}\frac{1-(2+h)^2-\left(1-2^2\right)}{h}$$

$$= \lim_{h\to 0}\frac{1-\left(4+4h+h^2\right)-(-3)}{h}$$

$$= \lim_{h\to 0} \frac{1-4-4h-h^2+3}{h}$$

$$= \lim_{h\to 0} \frac{-4h-h^2}{h}$$

$$= \lim_{h\to 0} (-4-h)$$

$$\underline{= -4}$$

Exercise

Find the slope of the curve $y = \frac{1}{x-1}$ at the point $x = 3$

Solution

$$m = \lim_{h\to 0} \frac{\frac{1}{x+h-1} - \frac{1}{x-1}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{3+h-1} - \frac{1}{3-1}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{2-2-h}{2+h}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{2+h}\right)$$

$$= \lim_{h\to 0} \left(\frac{-1}{2+h}\right)$$

$$\underline{= -\frac{1}{2}}$$

Exercise

Find the slope of the curve $y = \frac{x-1}{x+1}$ at the point $x = 0$

Solution

$$m = \lim_{h\to 0} \frac{\frac{x+h-1}{x+h+1} - \frac{x-1}{x+1}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{0+h-1}{0+h+1} - \frac{0-1}{0+1}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{h-1}{h+1} + 1\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{h-1+h+1}{h+1}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{2h}{h+1}\right)$$

$$= \lim_{h\to 0} \left(\frac{2}{h+1}\right)$$

$$\underline{= 2}\,|$$

Exercise

Find equations of all lines having slope -1 that are tangent to the curve $y=\dfrac{1}{x-1}$

Solution

$$m = \lim_{h\to 0} \frac{\frac{1}{x+h-1}-\frac{1}{x-1}}{h}$$

$$-1 = \lim_{h\to 0} \frac{\frac{1}{x+h-1}-\frac{1}{x-1}}{h}$$

$$-1 = \lim_{h\to 0} \frac{1}{h}\left(\frac{x-1-(x+h-1)}{x+h-1}\right)$$ ***Cross multiplication***

$$-1 = \lim_{h\to 0} \frac{1}{h}\left(\frac{x-1-x-h+1}{x+h-1}\right)$$

$$-1 = \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{x+h-1}\right)$$

$$-1 = \lim_{h\to 0} \left(\frac{-1}{x+h-1}\right)$$

$$-1 = \frac{-1}{x-1}$$

$$-x+1=-1$$

$$\underline{x=2}\,|$$

$$y=\frac{1}{x-1}$$

$$=\frac{1}{2-1}$$

$$\underline{=1}\,|$$

At $(2,\ 1)$ $\Rightarrow y-1=-1(x-2)$

$$y-1=-x+2$$

$$\underline{y=-x+3}\,|$$

Exercise

What is the rate of change of the area of a circle $\left(A = \pi r^2\right)$ with respect to the radius when the radius is $r = 3$?

Solution

$$m = \lim_{h\to 0} \frac{\pi(3+h)^2 - \pi(3)^2}{h}$$

$$= \lim_{h\to 0} \frac{\pi\left(9+6h+h^2\right)-9\pi}{h}$$

$$= \lim_{h\to 0} \frac{9\pi + 6\pi h + \pi h^2 - 9\pi}{h}$$

$$= \lim_{h\to 0} \frac{6\pi h + \pi h^2}{h}$$

$$= \lim_{h\to 0} \frac{\pi h(6+h)}{h}$$

$$= \lim_{h\to 0} \pi(6+h)$$

$$= 6\pi$$

Exercise

Find the slope of the tangent to the curve $y = \frac{1}{\sqrt{x}}$ at the point where $x = 4$

Solution

$$m = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{\sqrt{4}-\sqrt{4+h}}{2\sqrt{4+h}}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{2-\sqrt{4+h}}{2\sqrt{4+h}}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{2-\sqrt{4+h}}{2\sqrt{4+h}}\cdot\frac{2+\sqrt{4+h}}{2+\sqrt{4+h}}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{4-(4+h)}{2\sqrt{4+h}\left(2+\sqrt{4+h}\right)}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{2\sqrt{4+h}\left(2+\sqrt{4+h}\right)}\right)$$

$$= \lim_{h\to 0} \left(\frac{-1}{2\sqrt{4+h}\left(2+\sqrt{4+h}\right)} \right)$$

$$= \frac{-1}{2\sqrt{4}\left(2+\sqrt{4}\right)}$$

$$= \frac{-1}{2(2)(2+2)}$$

$$= \frac{-1}{16}$$

Exercise

Fin the values of the derivatives of the function $f(x) = 4 - x^2$. Then find the values of $f'(-3)$, $f'(0)$, $f'(1)$

Solution

$$\frac{f(x+h)-f(x)}{h} = \frac{4-(x+h)^2-\left(4-x^2\right)}{h}$$

$$= \frac{4-\left(x^2+2xh+h^2\right)-\left(4-x^2\right)}{h}$$

$$= \frac{4-x^2-2xh-h^2-4+x^2}{h}$$

$$= \frac{-2xh-h^2}{h}$$

$$= -2x-h$$

$$f'(x) = \lim_{h\to 0}(-2x-h)$$

$$= -2x$$

$$f'(-3) = 6$$

$$f'(0) = 0$$

$$f'(1) = -2$$

Exercise

Fin the values of the derivatives of the function $r(s) = \sqrt{2s+1}$. Then find the values of $r'(0)$, $r'\left(\frac{1}{2}\right)$, $r'(1)$

Solution

$$r'(s) = \lim_{h\to 0} \frac{r(s+h) - r(s)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{2(s+h)+1} - \sqrt{2s+1}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{2s+2h+1} - \sqrt{2s+1}}{h} \cdot \frac{\sqrt{2s+2h+1} + \sqrt{2s+1}}{\sqrt{2s+2h+1} + \sqrt{2s+1}}$$

$$= \lim_{h\to 0} \frac{2s+2h+1-(2s+1)}{h\left(\sqrt{2s+2h+1} + \sqrt{2s+1}\right)}$$

$$= \lim_{h\to 0} \frac{2s+2h+1-2s-1}{h\left(\sqrt{2s+2h+1} + \sqrt{2s+1}\right)}$$

$$= \lim_{h\to 0} \frac{2h}{h\left(\sqrt{2s+2h+1} + \sqrt{2s+1}\right)}$$

$$= \lim_{h\to 0} \frac{2}{\sqrt{2s+2h+1} + \sqrt{2s+1}}$$

$$= \frac{2}{\sqrt{2s+1} + \sqrt{2s+1}}$$

$$= \frac{2}{2\sqrt{2s+1}}$$

$$= \frac{1}{\sqrt{2s+1}}$$

$$r'(0) = \frac{1}{\sqrt{2(0)+1}} \quad \underline{= 1}$$

$$r'\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2\frac{1}{2}+1}} \quad \underline{= \frac{1}{\sqrt{2}}}$$

$$r'(1) = \frac{1}{\sqrt{2(1)+1}} \quad \underline{= \frac{1}{\sqrt{3}}}$$

Exercise

Find the derivative of $f(x) = 3x^2 - 2x$

Solution

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{3(x+\Delta x)^2 - 2(x+\Delta x) - (3x^2 - 2x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{3(x^2 + \Delta x^2 + 2x\Delta x) - 2x - 2\Delta x - 3x^2 + 2x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2 + 3\Delta x^2 + 6x\Delta x - 2x - 2\Delta x - 3x^2 + 2x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3\Delta x^2 + 6x\Delta x - 2\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 3\Delta x + 6x - 2$$

$$\underline{= 6x - 2 \,|}$$

Exercise

Find the derivative of y with the respect to t for the function $y = \frac{4}{t}$

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta t \to 0} \frac{\frac{4}{t+\Delta t} - \frac{4}{t}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\frac{4t-4(t+\Delta t)}{t(t+\Delta t)}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{4t-4(t+\Delta t)}{t(t+\Delta t)}$$

$$= \lim_{\Delta t \to 0} \frac{-4\Delta t}{t(t+\Delta t)\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{-4}{t(t+\Delta t)}$$

$$\underline{= -\frac{4}{t^2} \,|}$$

Exercise

Find the derivative of $\frac{dy}{dx}$ *if* $y = 2x^3$

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{2(x + \Delta x)^3 - 2x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2\left(x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3\right) - 2x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x^3 + 6x^2 \Delta x + 6x(\Delta x)^2 + 3(\Delta x)^3 - 2x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x\left(6x^2 + 6x(\Delta x) + 3(\Delta x)^2\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(6x^2 + 6x(\Delta x) + 3(\Delta x)^2\right)$$

$$\boxed{= 6x^2}$$

Exercise

Differentiate the function $y = \frac{x+3}{1-x}$ and find the slope of the tangent line at the given value of the independent variable.

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{\frac{x+\Delta x+3}{1-x-\Delta x} - \frac{x+3}{1-x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\frac{1}{\Delta x}\right)\left(\frac{(x+\Delta x+3)(1-x)-(x+3)(1-x-\Delta x)}{(1-x-\Delta x)(1-x)}\right)$$

$$= \lim_{\Delta x \to 0} \left(\frac{1}{\Delta x}\right)\left(\frac{x+\Delta x+3-x^2-x\Delta x-3x-\left(x-x^2-x\Delta x+3-3x-3\Delta x\right)}{(1-x-\Delta x)(1-x)}\right)$$

$$= \lim_{\Delta x \to 0} \left(\frac{1}{\Delta x}\right)\left(\frac{x+\Delta x+3-x^2-x\Delta x-3x-x+x^2+x\Delta x-3+3x+3\Delta x}{(1-x-\Delta x)(1-x)}\right)$$

$$= \lim_{\Delta x \to 0} \left(\frac{1}{\Delta x}\right)\left(\frac{4\Delta x}{(1-x-\Delta x)(1-x)}\right)$$

$$= \lim_{\Delta x \to 0} \frac{4}{(1-x-\Delta x)(1-x)}$$

$$= \frac{4}{(1-x)(1-x)}$$

$$\boxed{= \frac{4}{(1-x)^2}}$$

Exercise

Find the equation of the tangent line to $f(x) = x^2 + 1$ that is parallel to $2x + y = 0$

Solution

$2x + y = 0 \Rightarrow y = -2x \Rightarrow \text{slope} = -2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 + 1 - (x^2+1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + \Delta x^2 + 2x\Delta x + 1 - x^2 - 1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x^2 + 2x\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \Delta x + 2x = 2x$$

$f' = 2x = -2$

$\Rightarrow x = -1$

$f(-1) = (-1)^2 + 1 = 2$

$\rightarrow \ (-1,\ 2)$

The line equation is given by $\quad y = m\left(x - x_1\right) + y_1$

$y = -2\left(x+1\right) + 2$

$\boxed{y = -2x}$

Exercise

Use the definition of limits to find the derivative: $f(x) = \dfrac{3}{\sqrt{x}}$ $y - 2 = -2x - 2$

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left(\frac{3}{\sqrt{x+\Delta x}}\right) - \left(\frac{3}{\sqrt{x}}\right)}{\Delta x} \cdot \frac{\sqrt{x} \cdot \sqrt{x+\Delta x}}{\sqrt{x} \cdot \sqrt{x+\Delta x}}$$

$$= \lim_{\Delta x \to 0} \frac{3\sqrt{x} - 3\sqrt{x+\Delta x}}{\Delta x\left(\sqrt{x}\cdot\sqrt{x+\Delta x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{3\left(\sqrt{x} - \sqrt{x+\Delta x}\right)}{\Delta x\left(\sqrt{x}\cdot\sqrt{x+\Delta x}\right)} \cdot \frac{\sqrt{x}+\sqrt{x+\Delta x}}{\sqrt{x}+\sqrt{x+\Delta x}}$$

$$= \lim_{\Delta x \to 0} \frac{3\left(x - (x+\Delta x)\right)}{\Delta x\left(\sqrt{x}\cdot\sqrt{x+\Delta x}\right)\left(\sqrt{x}+\sqrt{x+\Delta x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{-3\Delta x}{\Delta x\left(\sqrt{x}\cdot\sqrt{x+\Delta x}\right)\left(\sqrt{x}+\sqrt{x+\Delta x}\right)}$$

$$= \frac{-3}{x\left(2\sqrt{x}\right)}$$

$$= \frac{-3}{2x^{3/2}}$$

Exercise

Use the definition of limits to find the derivative: $f(x) = \sqrt{x+2}$

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x+2} - \sqrt{x+2}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x+2} + \sqrt{x+2}}{\sqrt{x+\Delta x+2} + \sqrt{x+2}}$$

$$= \lim_{\Delta x \to 0} \frac{x+\Delta x+2-(x+2)}{\Delta x\left(\sqrt{x+\Delta x+2} + \sqrt{x+2}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\left(\sqrt{x+\Delta x+2} + \sqrt{x+2}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x+2} + \sqrt{x+2}}$$

$$= \frac{1}{2\sqrt{x+2}}$$

Exercise

Suppose the height s of an object (in m) above the ground after t seconds is approximated by the function $s(t) = -4.9t^2 + 25t + 1$

a) Make a table showing the average velocities of the object from time $t = 1$ to $t = 1 + h$, for $h = 0.01,\ 0.001,\ 0.0001,\ \textit{and}\ 0.00001$.

b) Use the table in part (a) to estimate the instantaneous velocity of the object at $t = 1$.

c) Use limits to verify your estimate in part (b).

Solution

a)

$$\begin{aligned}\frac{f(1+h)-f(1)}{h} &= \frac{1}{h}\left(-4.9(1+h)^2 + 25(1+h) + 1 + 4.9 - 25 - 1\right)\\ &= \frac{1}{h}\left(-4.9 - 9.8h - 4.9h^2 + 25h + 4.9\right)\\ &= \frac{1}{h}\left(-4.9h^2 + 15.2h\right)\\ &= 15.2 - 4.9h\end{aligned}$$

h	$\dfrac{f(1+h)-f(1)}{h}$
0.01	15.151
0.001	15.1951
0.0001	15.1995
0.00001	15.2
0.000001	15.2

b)

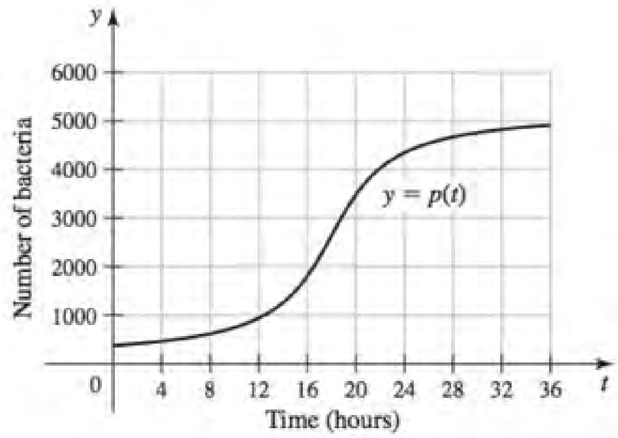
$$\begin{aligned}f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}\\ &\approx 15.2\ \ m/sec\end{aligned}$$

c)

$$\begin{aligned}f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}\\ &= 15.2\ \ m/sec\end{aligned}$$

Exercise

Suppose the following graph represents the number of bacteria in a culture t hours after the start of an experiment.



a) At approximately what time is the instantaneous growth rate the greatest, for $0 \le t \le 36$? Estimate the growth rate at this time.

b) At approximately what time is the instantaneous growth rate the least, for $0 \le t \le 36$? Estimate the growth rate at this time.

c) What is the average growth rate over the interval $0 \le t \le 36$?

Solution

a) $t = \frac{36}{2} = 18$

Point the rate $= \dfrac{N(20) - N(16)}{20 - 16}$

$= \dfrac{2500 - 1900}{4}$

$= 400 \ \ bacteria/hr$

b) It is smallest at $t = 0 \quad or \quad t = 36$

$\dfrac{N(36) - N(32)}{4} = \dfrac{4900 - 4800}{4}$

$= 25 \ \ bacteria/hr$

c) Growth rate $= \dfrac{N(36) - N(0)}{36}$

$\approx \dfrac{4900 - 400}{36}$

$= 125 \ \ bacteria/hr$