1. $\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8-3 = \boxed{5}$

2. $\begin{vmatrix} 5 & 2 \\ -6 & 3 \end{vmatrix} = 15+12 = \boxed{27}$

3. $\begin{vmatrix} -7 & 6 \\ \frac{1}{2} & 3 \end{vmatrix} = -21-3 = \boxed{-24}$

4. $\begin{vmatrix} 0 & 8 \\ 0 & 4 \end{vmatrix} = 0$

5. $\begin{vmatrix} \lambda-3 & 2 \\ 4 & \lambda-1 \end{vmatrix} = (\lambda-3)(\lambda-1)-8$
$= \lambda^2-4\lambda+3-8$
$= \boxed{\lambda^2-4\lambda-5}$

10. $\begin{vmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{vmatrix} = 1\begin{vmatrix} 2 & 0 \\ 4 & 3 \end{vmatrix} - 4\begin{vmatrix} 3 & 0 \\ -1 & 3 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ -1 & 4 \end{vmatrix}$
$= 6-4(9)-2(12+2)$
$= 6-36-28$
$= \boxed{-58}$

13. $\begin{vmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{vmatrix} = 2(3)(-5)$
$= \boxed{-30}$

14. $\begin{vmatrix} x & y & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix}\begin{matrix} x & y \\ 3 & 2 \\ 1 & 1 \end{matrix} = 2x+0-3+2-3y$
$= \boxed{2x-3y-1}$

#17
$$\begin{vmatrix} 5 & 3 & 0 & 6 \\ 4 & 6 & 4 & 12 \\ 0 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \end{vmatrix} = 5\begin{vmatrix} 6 & 4 & 12 \\ 2 & -3 & 4 \\ 1 & -2 & 2 \end{vmatrix} - 4\begin{vmatrix} 3 & 0 & 6 \\ 2 & -3 & 4 \\ 1 & -2 & 2 \end{vmatrix}$$

$$= 5(-36+16-48+36+48-16)$$
$$-4(-18-24+18+24)$$
$$= 0.$$

#

$$\begin{vmatrix} x+3 & 2 \\ 1 & x+2 \end{vmatrix} = 0$$

$$(x+3)(x+2) - 2 = 0$$
$$x^2 + 5x + 4 = 0$$
$$\Rightarrow \boxed{x = -1, -4}$$

#18

$$\begin{vmatrix} \lambda+2 & 2 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 2\lambda - 2 = 0$$
$$\boxed{\lambda_{1,2} = -1 \pm \sqrt{3}}$$

#19

$$\begin{vmatrix} \lambda & 2 & 0 \\ 0 & \lambda+1 & 2 \\ 0 & 1 & \lambda \end{vmatrix} = \lambda(\lambda(\lambda+1) - 2)$$
$$= \lambda(\lambda^2 + \lambda - 2) = 0$$

$$\boxed{\lambda_{1,2,3} = 0, 1, -2}$$

6. $\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = 3e^{5x} - 2e^{5x}$

$= \underline{e^{5x}}$

7. $\begin{vmatrix} x & \ln x \\ 1 & \frac{1}{x} \end{vmatrix} = 1 - \ln x$

15 $\begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$

$= r(\cos^2\theta + \sin^2\theta)$

$= \underline{r}$

$$\begin{vmatrix} 1 & 7 & -3 \\ 1 & 3 & 1 \\ 4 & 8 & 1 \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - 4R_1 \end{matrix}$$

$$\begin{vmatrix} 1 & 7 & -3 \\ 0 & -4 & 4 \\ 0 & -20 & 13 \end{vmatrix} = 1 \begin{vmatrix} -4 & 4 \\ -20 & 13 \end{vmatrix} \qquad 4 \begin{vmatrix} -1 & 1 \\ -20 & 13 \end{vmatrix}$$

$$= 4(-13+20)$$

$$= \boxed{28}$$

8/ $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta$

$= \boxed{1}$

9/ $\begin{vmatrix} \sin\theta & 1 \\ 1 & \sin\theta \end{vmatrix} = \sin^2\theta - 1$

$= -\cos^2\theta$

11/ $\begin{vmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{vmatrix} = k$ diagonal

12/ $\begin{vmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$ lower triangular

$(a, b, c \neq 0)$

16 $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = (1+a)(1+b)(1+c)+1+1-(1+b)-(1+c) - (1+a)$

$= (1+a+b+ab)(1+c) - a - b - c - 1$

$= 1+c+a+ac+b+bc+ab+abc - a - b - c - 1$

$= ac + ab + bc + abc$

$= abc\left(\frac{1}{b} + \frac{1}{c} + \frac{1}{a} + 1\right)$

20/ $A = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

a) $|A| = \begin{vmatrix} -2 & 1 \\ 4 & -2 \end{vmatrix} = 0$

b) $|B| = \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = -1$

c) $AB = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} -2 & -3 \\ 4 & 6 \end{pmatrix}$

d) $|AB| = \begin{vmatrix} -2 & -3 \\ 4 & 6 \end{vmatrix}$ $\quad -12+12$

$= \underline{0}$

$\Rightarrow |A||B| = |AB| = 0$ ✓

21/ $A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

a) $A = \begin{vmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 2$

b) $B = \begin{vmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = -6$

c) $AB = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 4 & 3 \\ -1 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix}$

d) $|AB| = \begin{vmatrix} 1 & 4 & 3 \\ -1 & 0 & 3 \\ 0 & 2 & 0 \end{vmatrix} = \underline{-12}$

$|A||B| = (2)(-6) = -12 = |AB|$ ✓

22) $A = \begin{pmatrix} 5 & 15 \\ 10 & -20 \end{pmatrix}$ $\underline{2 \times 2} \Rightarrow \underline{n = 2}$

$cA = \begin{pmatrix} 5c & 15c \\ 10c & -20c \end{pmatrix}$

$|cA| = \begin{vmatrix} 5c & 15c \\ 10c & -20c \end{vmatrix}$

$= -100c^2 - 150c^2$
$= -250c^2$

$|A| = \begin{vmatrix} 5 & 15 \\ 10 & -20 \end{vmatrix} = -250$

$\Rightarrow |cA| = (-250)c^2$
$= c^2 |A|$ ✓

$|A| = 5^2 \begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix}$
$= 25(-10)$
$= -250$

$c = 5$

23) $A = \begin{pmatrix} -3 & 6 & 9 \\ 6 & 9 & 12 \\ 9 & 12 & 15 \end{pmatrix}$ $3 \times 3 \Rightarrow n = 3$

$= 3 \begin{pmatrix} -1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$

$\underline{c = 3}$

$|A| = c^3 \begin{vmatrix} -1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$

$= 3^3(-15 + 24 + 24 - 27 + 16 - 20)$

$= 27(2)$

$= 54$

24/ $A = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$

a) $|A| = \begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} = \boxed{-2}$

b) $|B| = \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = \boxed{-2}$

c) $A+B = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

d) $|A+B| = 0$

$\Rightarrow |A|+|B| \stackrel{?}{=} |A+B|$

$-2-2 \neq 0$

$-4 \neq 0$ ✓

25/ $\begin{vmatrix} 5 & 4 \\ 10 & 8 \end{vmatrix} = 40 - 40 = \underline{0}$ Singular

26/ $\begin{vmatrix} \frac{1}{2} & \frac{3}{2} & 2 \\ \frac{2}{3} & -\frac{1}{3} & 0 \\ 1 & 1 & 1 \end{vmatrix} = -\frac{1}{6} + \frac{4}{3} + \frac{2}{3} - 1$

$= \frac{5}{6}$ Non singular

27/ $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \Rightarrow |A| = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 5$

$A^{-1} = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix} \rightarrow \underline{2\times 2}$ $\underline{c^n}$

$|A^{-1}| = \left(\frac{1}{5}\right)^2 \begin{vmatrix} 4 & -3 \\ -1 & 2 \end{vmatrix}$

$= \frac{5}{25}$

$= \frac{1}{5}$

$= \frac{1}{|A|}$ ✓

28

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & -1 & 2 \\ 3 & 0 & 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & 2 \\ 3 & 0 & 3 \end{vmatrix} = -3$$

$$A^{-1} = \frac{1}{-3}\begin{pmatrix} -3 & 6 & -1 \\ 3 & -3 & -1 \\ 3 & -6 & 0 \end{pmatrix}$$

$$|A^{-1}| = \left(\frac{1}{-3}\right)^3 \begin{vmatrix} -3 & 6 & -1 \\ 3 & -3 & -1 \\ 3 & -6 & 0 \end{vmatrix}$$

$$= -\frac{1}{27}(-18+18-9+18)$$

$$= -\frac{9}{27}$$

$$= -\frac{1}{3}$$

$$= \frac{1}{|A|}$$

A is singular $\Leftrightarrow$ $|A| = 0$

$$A = \begin{bmatrix} k-1 & 3 \\ 2 & k-2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} k-1 & 3 \\ 2 & k-2 \end{vmatrix}$$

$$= k^2 - 3k - 4 = 0$$

$$\therefore \boxed{k = -1, 4}$$

29. $A, B\ (n\times n) \ni AB = I$

Prove $|A| \neq 0$ & $|B| \neq 0$.

$|AB| = |I|$

$|A||B| = 1$

$\therefore |A| \neq 0$ & $|B| = 0$

30. AB singular

Prove A or B is singular.

(AB) is singular $\Rightarrow |AB| = 0$

$|AB| = |A||B| = 0$

$\Rightarrow$ Either $|A|$ or $|B|$ is equal to zero

$\therefore$ Either A or B is singular.

31. $|A| + |B| = |A+B|$ Find A & $B \ni$.

Assume $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$

$|B| = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$

$A + B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

$|A+B| = \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0$

$\Rightarrow |A| + |B| = |A+B|$ ✓.

$0 + 0 = 0$

32/ $\begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix} \overset{?}{=} b^2(3a+b)$

$$\begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix} = (a+b)^3 + a^3 + a^3 - 3a^2(a+b)$$
$$= \underline{a^3} + \underline{3a^2b} + 3ab^2 + b^3 + \underline{2a^3} - \underline{3a^3} - \underline{3a^2b}$$
$$= 3ab^2 + b^3$$
$$= b^2(3a+b) \checkmark.$$

33/ $A(n \times n)$

Sum of the entries of each row $= 0$

$\Rightarrow$ One column will result $= 0$.

$|A| = 0$ $\qquad$ $A = 0$

$A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix} \Rightarrow |A| = -ab + ab$
$= 0.$

34/

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & -4 & -12 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & -4 & -12 \end{vmatrix} = 0$$

$$\Rightarrow A^{-1} \nexists$$

35/

$$A = \begin{bmatrix} -3 & -5 & -7 \\ 2 & 4 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} -3 & -5 & -7 \\ 2 & 4 & 3 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= \underline{-3}$$

$$A^{-1} = -\frac{1}{3}\begin{bmatrix} -7 & -12 & 13 \\ 2 & 3 & -5 \\ 2 & 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{3} & 4 & -\frac{13}{3} \\ -\frac{2}{3} & -1 & \frac{5}{3} \\ -\frac{2}{3} & -1 & \frac{2}{3} \end{bmatrix}$$

38/ $\begin{cases} 20x + 8y = 11 \\ 12x - 24y = 21 \end{cases}$

$$\Delta = \begin{vmatrix} 20 & 8 \\ 12 & -24 \end{vmatrix} = -573$$

$$\Delta_x = \begin{vmatrix} 11 & 8 \\ 21 & -24 \end{vmatrix} = -432$$

$$\Delta_y = \begin{vmatrix} 20 & 11 \\ 12 & 21 \end{vmatrix} = 288$$

$$x = \frac{-432}{-573} = \frac{3}{4}$$

$$y = \frac{288}{-573} = -\frac{1}{2}$$

$\therefore$ soln : $\left(\frac{3}{4}, -\frac{1}{2}\right)$

39/ $\begin{cases} 4x - y - z = 1 \\ 2x + 2y + 3z = 10 \\ 5x - 2y - 2z = -1 \end{cases}$

$$\Delta = \begin{vmatrix} 4 & -1 & -1 \\ 2 & 2 & 3 \\ 5 & -2 & -2 \end{vmatrix} = 3 \qquad \Delta_x = \begin{vmatrix} 1 & -1 & -1 \\ 10 & 2 & 3 \\ -1 & -2 & -2 \end{vmatrix} = 3$$

$$\Delta_y = \begin{vmatrix} 4 & 1 & -1 \\ 2 & 10 & 3 \\ 5 & -1 & -2 \end{vmatrix} = 3 \qquad \Delta_z = \begin{vmatrix} 4 & -1 & 1 \\ 2 & 2 & 10 \\ 5 & -2 & -1 \end{vmatrix} = 6$$

$$x = \frac{3}{3} = 1 \qquad y = \frac{3}{3} = 1 \qquad z = \frac{6}{3} = 2$$

$\therefore$ soln $(1, 1, 2)$

36 $\begin{cases} x_1 + 2x_2 = 5 \\ -x_1 + x_2 = 1 \end{cases}$

$$\Delta = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3$$

$$\Delta_{x_1} = \begin{vmatrix} 5 & 2 \\ 1 & 1 \end{vmatrix} = 3$$

$$\Delta_{x_2} = \begin{vmatrix} 1 & 5 \\ -1 & 1 \end{vmatrix} = 6$$

$x_1 = \frac{3}{3} = 1$ $\qquad$ $x_2 = \frac{6}{3} = 2$

$\therefore$ soln: $(1, 2)$

37 $\begin{cases} 3x + 4y = -2 \\ 5x + 3y = 4 \end{cases}$

$$\Delta = \begin{vmatrix} 3 & 4 \\ 5 & 3 \end{vmatrix} = -11$$

$$\Delta x = \begin{vmatrix} -2 & 4 \\ 4 & 3 \end{vmatrix} = -22$$

$$\Delta y = \begin{vmatrix} 3 & -2 \\ 5 & 4 \end{vmatrix} = 22$$

$x = \frac{-22}{-11} = 2$ $\qquad$ $y = \frac{22}{-11} = -2$

$\therefore$ soln: $(2, -2)$

40/ $\begin{cases} 3x + 4y + 4z = 11 \\ 4x - 4y + 6z = 11 \\ 6x - 6y \qquad = 3 \end{cases}$

$$\Delta = \begin{vmatrix} 3 & 4 & 4 \\ 4 & -4 & 6 \\ 6 & -6 & 0 \end{vmatrix} = 252$$

$$\Delta_x = \begin{vmatrix} 11 & 4 & 4 \\ 11 & -4 & 6 \\ 3 & -6 & 0 \end{vmatrix} = 252$$

$$\Delta_y = \begin{vmatrix} 3 & 11 & 4 \\ 4 & 11 & 6 \\ 6 & 3 & 0 \end{vmatrix} = 126$$

$$\Delta_z = \begin{vmatrix} 3 & 4 & 11 \\ 4 & -4 & 11 \\ 6 & -6 & 3 \end{vmatrix} = 378$$

$x = \frac{252}{252} = 1 \qquad y = \frac{126}{252} = \frac{1}{2} \qquad z = \frac{378}{252} = \frac{3}{2}$

<u>Soln</u>: $\left(1, \frac{1}{2}, \frac{3}{2}\right)$.

$$\begin{vmatrix} \cos x & 0 & \sin x \\ \sin x & 0 & \cos x \\ \sin x - \cos x & 1 & \sin x + \cos x \end{vmatrix} = 0$$

$$-\cos^2 x + \sin^2 x = 0$$

$$-\cos 2x = 0 \Rightarrow 2x = (2n+1)\frac{\pi}{2}$$

$$x = (2n+1)\frac{\pi}{4} \; ; \; n \in \mathbb{Z}$$

$$\begin{vmatrix} 1-n & 1 & 1 & \cdots & 1 \\ 1 & 1-n & 1 & \cdots & 1 \\ \vdots & & & & \vdots \\ 1 & 1 & 1 & \cdots & 1-n \end{vmatrix}$$

2×2: $\begin{vmatrix} 1-n & 1 \\ 1 & 1-n \end{vmatrix} = n^2 - 2n = n(n-2)$
$= 0$ since $n=2$

3×3: $\begin{vmatrix} 1-n & 1 & 1 \\ 1 & 1-n & 1 \\ 1 & 1 & 1-n \end{vmatrix} = (1-n)^3 + 1 + 1 - 3(1-n)$
$= n^3 - 3n^2$
$= n^2(n-3) = 0$ $(n=3)$.

$n \times n \Rightarrow \left| \quad \right| = n^{n-1}(n-n)$
$= \underline{0}$