Lecture Two – Differentiation

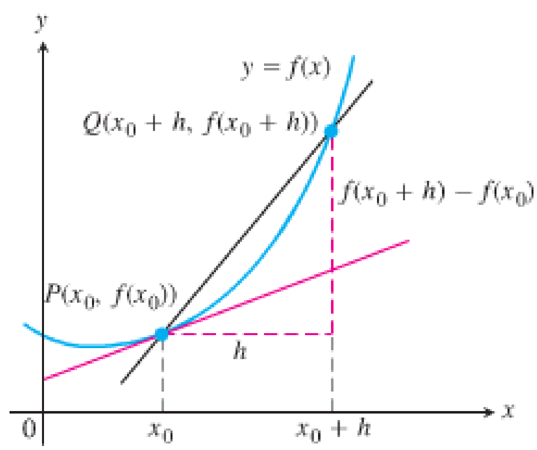
Section 2.1 –Introducing the Derivative

Definition

The slope of the curve $y = f(x)$ at the point $P\left(x_0, f\left(x_0\right)\right)$ is the number

$$m = \lim_{h \to 0} \frac{f\left(x_0 + h\right) - f\left(x_0\right)}{h}, \qquad (\lim \exists)$$

The tangent line to the curve at *P* is the line through *P* with this slope.



Example

a) Find the slope of the curve $y = \frac{1}{x}$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?

b) Where does the slope equal $-\frac{1}{4}$?

c) What happens to the tangent to the curve at the point $\left(a, \frac{1}{a}\right)$ as a changes?

Solution

a) The slope of $f(x) = \frac{1}{x}$ at $\left(a, \frac{1}{a}\right)$ is

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\frac{a-(a+h)}{a(a+h)}$$

$$= \lim_{h\to 0} \frac{1}{h}\frac{a-a-h}{a(a+h)}$$

$$= \lim_{h\to 0} \frac{1}{h}\frac{-h}{a(a+h)}$$

$$= \lim_{h\to 0} \frac{-1}{a(a+h)}$$

$$\boxed{= -\frac{1}{a^2}}$$

The slope at $x=-1$ is: $= -\frac{1}{(-1)^2}$ $\boxed{= -1}$

b) The slope equals to $x = -\frac{1}{4}$

$\Rightarrow -\frac{1}{a^2} = -\frac{1}{4}$

$a^2 = 4 \quad \rightarrow \quad \boxed{a = \pm 2}$

$$\left.\begin{array}{r} x=-2 \Rightarrow y=-\frac{1}{2} \\ x=2 \Rightarrow y=\frac{1}{2} \end{array}\right\} \Rightarrow \left(-2,\ -\frac{1}{2}\right) \text{ and } \left(2,\ \frac{1}{2}\right)$$

c) The slope $\left(-\frac{1}{a^2}\right)$ is always negative if $a \neq 0$

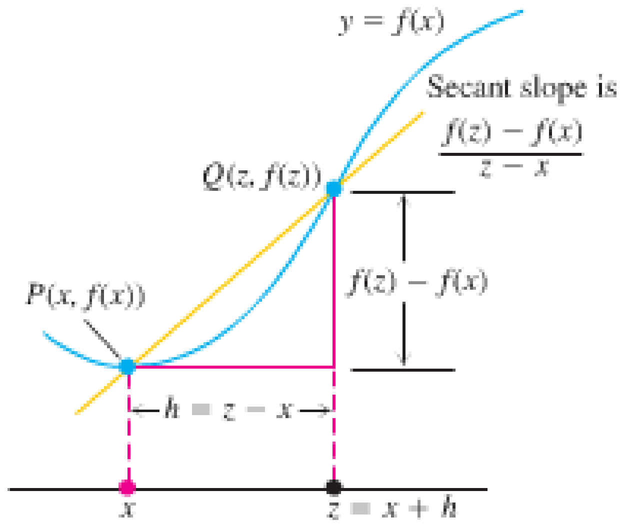
$\lim_{x\to\pm\infty}\left(-\frac{1}{a^2}\right) = 0$ The slope approaches 0 and the tangent becomes horizontal.

$\lim_{x\to 0^-}\left(-\frac{1}{a^2}\right) = -\infty$ The slope approaches $-\infty$ and the tangent increasingly steep.

Definition of the Derivative

The derivative of a function f at a point x_0, denoted $f'(x_0)$ is

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}, \qquad (\lim \exists)$$



If f' exists at a particular x, we say that f is ***differentiable*** (has a ***derivative***) at x.

If f' exists at every point in the domain of f, we call f ***differentiable***

The process of finding derivatives is called ***differentiation***.

Notations

Some common alternative notations for the derivative are

$$f'(x), \quad f', \quad \frac{d}{dx}[f(x)], \quad \frac{d}{dx}f, \quad \frac{dy}{dx}, \quad y', \quad \dot{y}, \textit{ and } D_x[y]$$

Example

Differentiate $f(x) = \frac{x}{x-1}$

Solution

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f(x+h) = \frac{(x+h)}{(x+h)-1}$$

$$= \lim_{h\to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)}$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{x^2 - x + hx - h - x^2 - hx + x}{(x+h-1)(x-1)}$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{-h}{(x+h-1)(x-1)}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h-1)(x-1)}$$

$$= \frac{-1}{(x-1)(x-1)}$$

$$= \frac{-1}{(x-1)^2}$$

Example

Find the derivative of $f(x) = x^2$

Solution

$$f(x+h) = (x+h)^2$$

$$= x^2 + 2hx + h^2$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2hx + h^2}{h}$$

$$= \lim_{h \to 0} (2x + h)$$

$$= 2x$$

Example

a) Find the derivative of $f(x)=\sqrt{x}$ for $x>0$

b) Find the tangent line to the curve $y=\sqrt{x}$ *at* $x=4$

Solution

a) $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$=\lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$=\frac{1}{\sqrt{x}+\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x}}$$

b) The slope of the curve at $x=4$ is: $f'(4)=\frac{1}{2\sqrt{4}}=\frac{1}{4}$

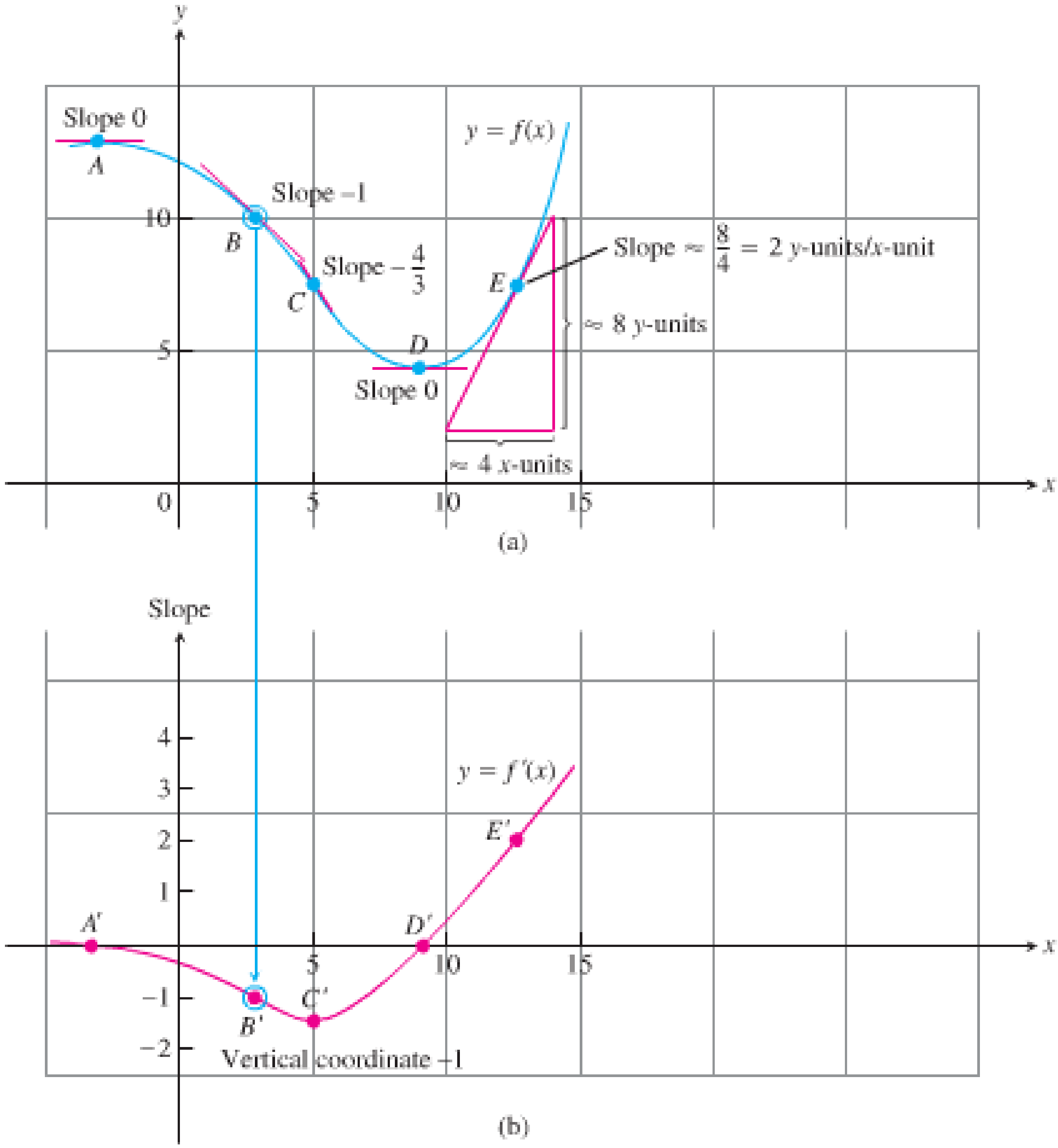
The tangent is the line through the point (4, 2) with slope $\frac{1}{4}$:

$$y-2=\frac{1}{4}(x-4)$$

$$y=\frac{1}{4}x-1+2$$

$$y=\frac{1}{4}x+1$$

Graphing



✓ The rate of change of *f* is positive, negative, or zero

✓ The rough size of the growth rate at any *x* and its size in relation to the size of $f(x)$

✓ Where the rate of change itself is increasing or decreasing.

Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

Theorem – Differentiability Implies Continuity

If f has a derivative at $x = c$, then f is continuous at $x = c$

Proof

Given that $f'(c)$ exists, we must show that $\lim\limits_{x\to c} f(x) = f(c)$, or equivalently, that $\lim\limits_{h\to 0} f(c+h) = f(c)$. If $h \neq 0$, then

$$f(c+h) = f(c) + \big(f(c+h) - f(c)\big)$$

$$= f(c) + \frac{f(c+h) - f(c)}{h} \cdot h$$

Take the limits as $h \to 0$.

$$\lim_{h\to 0} f(c+h) = \lim_{h\to 0} f(c) + \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \cdot \lim_{h\to 0} h$$

$$= f(c) + f'(c) \cdot 0$$

$$= f(c)$$

Summary

The following are all interpretations for the limit of the difference quotient, $\lim\limits_{h\to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$

1. The slope of the graph of $y = f(x)$ @ $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ @ $x = x_0$
3. The rate of change of $f(x)$ with respect to x @ $x = x_0$
4. The derivative $f'(x_0)$ at a point

Exercises *Section* 2.1 – Introducing the Derivative

(1 – 8) Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at *P*; then graph the curve and the tangent line.

1. $y = 4 - x^2; \quad P(-1,\ 3)$

2. $y = \frac{1}{x^2}; \quad P(-1,\ 1)$

3. $f(x) = 2\sqrt{x}; \quad P(1,\ 2)$

4. $f(x) = x^3 + 3x; \quad P(1,\ 4)$

5. $f(x) = 4x^2 - 7x + 5; \quad P(2,\ 7)$

6. $f(x) = 5x^3 + x; \quad P(1,\ 6)$

7. $f(x) = \frac{x+3}{2x+1}; \quad P(0,\ 3)$

8. $f(x) = \frac{1}{2\sqrt{3x+1}}; \quad P\left(0,\ \frac{1}{2}\right)$

9. Find the slope of the curve $y = 1 - x^2$ at the point $x = 2$

10. Find the slope of the curve $y = \frac{1}{x-1}$ at the point $x = 3$

11. Find the slope of the curve $y = \frac{x-1}{x+1}$ at the point $x = 0$

12. Find equations of all lines having slope -1 that are tangent to the curve $y = \frac{1}{x-1}$

13. What is the rate of change of the area of a circle $\left(A = \pi r^2\right)$ with respect to the radius when the radius is $r = 3$?

14. Find the slope of the tangent to the curve $y = \frac{1}{\sqrt{x}}$ at the point where $x = 4$

15. Find the values of the derivatives of the function $f(x) = 4 - x^2$. Then find the values of $f'(-3), f'(0), f'(1)$

16. Find the values of the derivatives of the function $r(s) = \sqrt{2s+1}$. Then find the values of $r'(0),\ r'\left(\frac{1}{2}\right),\ r'(1)$

17. Find the derivative of $f(x) = 3x^2 - 2x$

18. Find the derivative of *y* with the respect to *t* for the function $y = \frac{4}{t}$

19. Find the derivative of $\frac{dy}{dx}$ *if* $y = 2x^3$

20. Find the equation of the tangent line to $f(x) = x^2 + 1$ that is parallel to $2x + y = 0$

21. Differentiate the function $y = \frac{x+3}{1-x}$ and find the slope of the tangent line at the given value of the independent variable.

22. Use the definition of limits to find the derivative: $f(x) = \frac{3}{\sqrt{x}}$

23. Use the definition of limits to find the derivative: $f(x) = \sqrt{x+2}$

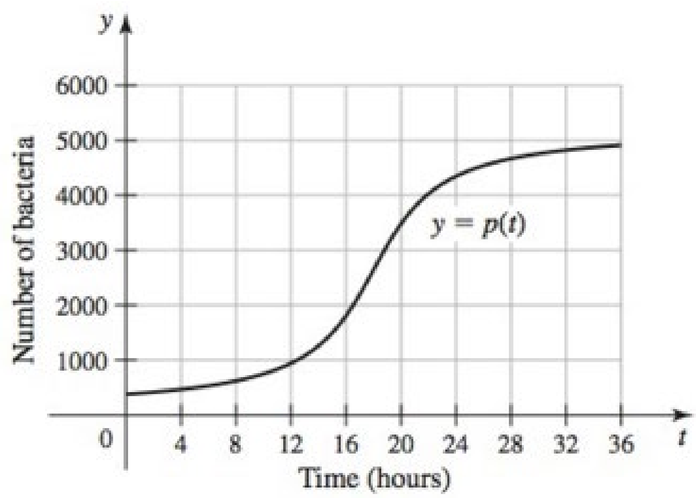
24. Suppose the height *s* of an object (in *m*) above the ground after *t* seconds is approximated by the function $s = -4.9t^2 + 25t + 1$

a) Make a table showing the average velocities of the object from time $t = 1$ to $t = 1 + h$, for $h = 0.01,\ 0.001,\ 0.0001,\ \textit{and}\ 0.00001$.

b) Use the table in part (*a*) to estimate the instantaneous velocity of the object at $t = 1$.

c) Use limits to verify your estimate in part (*b*).

25. Suppose the following graph represents the number of bacteria in a culture *t* hours after the start of an experiment.



a) At approximately what time is the instantaneous growth rate the greatest, for $0 \le t \le 36$? Estimate the growth rate at this time.

b) At approximately what time is the instantaneous growth rate the least, for $0 \le t \le 36$? Estimate the growth rate at this time.

c) What is the average growth rate over the interval $0 \le t \le 36$?

Section 2.2 – Rules of Differentiation

Notations for the Derivative

The derivative of $y = f(x)$ may be written in any of the following ways:

1st *derivative*	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}[f(x)]$	$D_x[y]$

Derivative of a *constant Function*

If f has the constant value $f(x) = c$

$$\frac{d}{dx}[c] = f'(c) = 0 \qquad c \text{ is constant}$$

Proof

Let $f(x) = c$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{c-c}{h}$$

$$= 0$$

So, $\frac{d}{dx}[c] = 0$

Example

Find the derivative

a) $f(x) = 9$

$f' = 0$

b) $h(t) = \pi$

$D_t[h(t)] = 0$

Power Rule

$$f(x) = x^n \quad \Rightarrow f'(x) = nx^{n-1} \qquad \textbf{\textit{n}} \text{ is any real number}$$

Proof

Let $f(x) = x^n$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + h^n - x^n}{h}$$

$$= \lim_{h \to 0} \left(nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h + \ldots + h^{n-1} \right)$$

$$= nx^{n-1}$$

Example

Find the derivative of: $a)\ x^3 \quad b)\ x^{2/3} \quad c)\ \frac{1}{x^4} \quad d)\ x^{\sqrt{2}} \quad e)\ \sqrt{x^{2+\pi}}$

Solution

a) $y = x^3$

$$\frac{dy}{dx} = 3x^{3-1}$$

$$= 3x^2$$

b) $y = x^{2/3}$

$$y' = \frac{2}{3}x^{2/3-1}$$

$$= \frac{2}{3}x^{-1/3}$$

c) $y = \frac{1}{x^4} = x^{-4}$

$$y' = -4x^{-4-1}$$

$$= -4x^{-5}$$

$$= -\frac{4}{x^5}$$

d) $D_x\left(x^{\sqrt{2}}\right)=\underline{\sqrt{2}x^{\sqrt{2}-1}}$

e) $y=\left(x^{2+\pi}\right)^{1/2}=x^{(2+\pi)/2}$

$$y'=\left(\frac{2+\pi}{2}\right)x^{1+\pi/2-1}$$

$$=\frac{1}{2}(2+\pi)\sqrt{x^{\pi}}$$

Derivative Constant Multiple Rule

If f is a differentiable function of x, and c is a real number (constant), then $\frac{d}{dx}(cf)=c\frac{df}{dx}$

In particular, if n is any real number, then $\frac{d}{dx}\left(cx^n\right)=cnx^{n-1}$

Proof

$$\frac{d}{dx}(cf)=\lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}$$

$$=c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$ ***Factor c***

$$=c\frac{df}{dx}$$

Example

If $y=8x^4$, find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx}=8\left(4x^3\right)\ \underline{=32x^3}$$

Example

If $y=-\frac{3}{4}x^{12}$, find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx}=-\frac{3}{4}\left(12x^{11}\right)$$

$$\underline{=-9x^{11}}$$

Sum or Difference Rule

The derivative of the sum or difference of two differentiable functions is the sum or difference of their derivatives.

$$\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx} \qquad\qquad \frac{d}{dx}(u-v)=\frac{du}{dx}-\frac{dv}{dx}$$

$$=u'+v' \qquad\qquad =u'-v'$$

Proof

$$f(x)=u(x)+v(x)$$

$$\begin{aligned}\frac{d}{dx}\left[u(x)+v(x)\right] &= \lim_{h\to 0}\frac{\left[u(x+h)+v(x+h)\right]-\left[u(x)+v(x)\right]}{h}\\ &= \lim_{h\to 0}\frac{u(x+h)+v(x+h)-u(x)-v(x)}{h}\\ &= \lim_{h\to 0}\left[\frac{u(x+h)-u(x)}{h}+\frac{v(x+h)-v(x)}{h}\right]\\ &= \lim_{h\to 0}\left[\frac{u(x+h)-u(x)}{h}\right]+\lim_{h\to 0}\left[\frac{v(x+h)-v(x)}{h}\right]\\ &= \frac{du}{dx}+\frac{dv}{dx}\end{aligned}$$

Example

Find the derivative of the polynomial $\quad y=x^3+\frac{4}{3}x^2-5x+1$

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}x^3+\frac{d}{dx}\left(\frac{4}{3}x^2\right)-\frac{d}{dx}(5x)+\frac{d}{dx}(1)\\ &= 3x^2+\frac{8}{3}x-5+0\\ &= 3x^2+\frac{8}{3}x-5\end{aligned}$$

Example

Find the derivative of $\quad y=x^{5/2}+x^3+\frac{1}{2}x^2+4$

Solution

$$y'=\frac{5}{2}x^{3/2}+3x^2+x$$

Example

Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

Solution

$$y' = 4x^3 - 4x$$

$$y' = 0 \quad \Rightarrow \quad 4x^3 - 4x = 0$$

$$4x\left(x^2 - 1\right) = 0$$

$$\underline{x = 0, \ \pm 1}\,\big|$$

The curve has horizontal tangents at $x = 0,\ 1,\ and\ \ -1$.

The corresponding points on the curve are; (0, 2), (1, 1) and (−1, 1)

***Second*– and *Higher*–Order Derivatives**

Notation for Higher-Order Derivatives							
1.	***1st derivative***	y'	***y*** prime	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}[f(x)]$	$D_x[y]$
2.	***2nd derivative***	y''	***y*** double prime	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}[f(x)]$	$D_x^2[y]$
3.	***3rd derivative***	y'''	***y*** triple prime	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}[f(x)]$	$D_x^3[y]$
4.	***4th derivative***	$y^{(4)}$		$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}[f(x)]$	$D_x^4[y]$
5.	***nth derivative***	$y^{(n)}$		$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	$\frac{d^n}{dx^n}[f(x)]$	$D_x^n[y]$

Example

Find the first four derivatives of $y = x^3 - 3x^2 + 2$

Solution

$$y' = 3x^2 - 6x$$
$$y'' = 6x - 6$$
$$y''' = 6$$
$$y^{(4)} = 0$$

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \quad \Rightarrow \quad \boxed{f^{(n)}(x) = n!a_n}$$

Exercises *Section* 2.2 – Rules of Differentiation

(**1 – 26**) Find the derivative of each function

1. $y = \frac{1}{x^3}$

2. $D_x\left(x^{4/3}\right)$

3. $y = \sqrt{z}$

4. $D_t\left(-8t\right)$

5. $y = \frac{9}{4x^2}$

6. $y = 6x^3 + 15x^2$

7. $y = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$

8. $p(t) = 12t^4 - 6\sqrt{t} + \frac{5}{t}$

9. $f(x) = \frac{x^3 + 3\sqrt{x}}{x}$

10. $y = \frac{x^3 - 4x}{\sqrt{x}}$

11. $f(x) = \left(4x^2 - 3x\right)^2$

12. $y = 3x\left(2x^2 + 5x\right)$

13. $y = 3\left(2x^2 + 5x\right)$

14. $y = (3x - 2)(2x + 3)$

15. $y = \frac{x^2 + 4x}{5}$

16. $y = \frac{3x^4}{5}$

17. $g(s) = \frac{s^2 - 2s + 5}{\sqrt{s}}$

18. $f(x) = \frac{x + 1}{\sqrt{x}}$

19. $f(x) = 4x^{5/3} + 6x^{-3/2} - 11x$

20. $f(x) = \frac{2}{3}x^3 + \pi x^2 + 7x + 1$

21. $f(x) = \frac{x^5 - x^3}{15}$

22. $f(x) = x^{1/3} + 2x^{1/4} - 3x^{1/5}$

23. $f(t) = 3\sqrt[3]{t^2} - \frac{2}{\sqrt{t^3}}$

24. $f(t) = \sqrt{t}\left(5 - t - \frac{1}{3}t^2\right)$

25. $f(x) = \frac{3}{5}x^{5/3} + \frac{5}{3}x^{-3/5}$

26. $f(x) = x^{23} - x^{-23}$

(**27 – 32**) Find the ***first*** and ***second*** derivatives

27. $y = -x^3 + 3$

28. $y = 3x^7 - 7x^3 + 21x^2$

29. $y = 6x^2 - 10x - \frac{1}{x}$

30. $f(x) = \frac{1}{2}x^4 + \pi x^3 - 7x + 1$

31. $y = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$

32. $y = (2x - 3)(1 - 5x)$

(**33 – 38**) Find the derivatives

33. $f(x)=3x^4-6x^3+\frac{x^2}{8}+5,\quad f^{(4)}(x)$

34. $f(x)=3x^4-6x^3+\frac{x^2}{8}+5,\quad f^{(5)}(x)$

35. $f(x)=2x^6+4x^4-x+2,\quad f^{(6)}(x)$

36. $f(x)=4x^5+4x^4+x^2-2,\quad f^{(5)}(x)$

37. $f(x)=4x^5+4x^4+x^2-2,\quad f^{(6)}(x)$

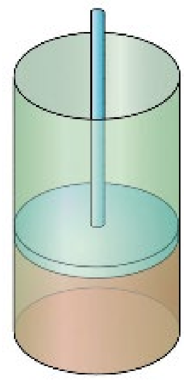
38. $f(x)=4x^4-2x^3+x+2,\quad f^{(4)}(x)$

39. Find an equation for the line perpendicular to the tangent to the curve $y=x^3-4x+1$ at the point $(2,\ 1)$.

40. If gas in a cylinder is maintained at a constant temperature *T*, the pressure *P* is related to the volume *V* by a formula of the form

$$P=\frac{nRT}{V-nb}-\frac{an^2}{V^2}$$

In which *a*, *b*, *n*, and *R* are constants. Find $\frac{dP}{dV}$



41. Show that if $(a,\ f(a))$ is any point on the graph of $f(x)=x^2$, then the slope of the tangent line at that point is $m=2a$

42. Show that if $(a,\ f(a))$ is any point on the graph of $f(x)=bx^2+cx+d$, then the slope of the tangent line at that point is $m=2ab+c$

43. Let $f(x)=x^2$

a) Show that $\frac{f(x)-f(y)}{x-y}=f'\left(\frac{x+y}{2}\right)$, for all $x\neq y$

b) Is this property true for $f(x)=ax^2$, where *a* is a nonzero real number?

c) Give a geometrical interpretation of this property.

d) Is this property true for $f(x)=ax^3$?

Section 2.3 – Product and Quotient Rules

Product Rule

The derivative of the product of two differentiable functions is equal to the first function times the derivative of the second plus the second function times the derivative of the first,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$(f.g)' = f'.g + g'.f$$

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'gh + fg'h + fgh'$$

Example

Find the derivative of $f(x) = (2x+3)\left(3x^2\right)$

Solution

$$f' = (2x+3)\left(3x^2\right)' + (2x+3)'\left(3x^2\right) \qquad f(x) = 6x^3 + 9x^2$$

$$= (2x+3)(6x) + (2)\left(3x^2\right)$$

$$= 12x^2 + 18x + 6x^2$$

$$= 18x^2 + 18x$$

$$= 18x(x+1)$$

Proof of the Derivative Product Rule

$$\frac{d}{dx}(uv) = \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

$$= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h}$$

$$= \lim_{h\to 0} \left[\frac{u(x+h)v(x+h) - u(x+h)v(x)}{h} + \frac{u(x+h)v(x) - u(x)v(x)}{h}\right]$$

$$= \lim_{h\to 0} \left[u(x+h)\frac{v(x+h) - v(x)}{h} + v(x)\frac{u(x+h) - u(x)}{h}\right]$$

$$= \lim_{h\to 0} u(x+h) \cdot \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} + v(x) \cdot \lim_{h\to 0} \frac{u(x+h) - u(x)}{h}$$

$$= u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

Example

Find the derivative of $y=\left(3x^2+1\right)\left(x^3+3\right)$

Solution

$u=3x^2+1 \quad v=x^3+3$

$u'=6x \qquad v'=3x^2$

$y=3x^5+9x^2+x^3+3$

$y'=15x^4+18x+3x^2$

$y'=(6x)\left(x^3+3\right)+\left(3x^2\right)\left(3x^2+1\right)$

$=6x^4+18x+9x^4+3x^2$

$=15x^4+3x^2+18x$

Example

Find the derivative of $y=\left(3x^3+2x+5\right)\left(x^2-2x+4\right)$

Solution

$y'=\underbrace{\left(9x^2+2\right)}_{u'}\underbrace{\left(x^2-2x+4\right)}_{v}+\underbrace{(2x-2)}_{v'}\underbrace{\left(3x^3+2x+5\right)}_{u}$

$=9x^4-18x^3+36x^2+2x^2-4x+8+6x^4+4x^2+10x-6x^3-4x-10$

$=15x^4-24x^3+42x^2+2x-2$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{f'(x)g(x)-g'(x)f(x)}{[g(x)]^2} \quad =\frac{f'g-g'f}{g^2}$$

$$\left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$$

$$\left(\frac{ax^n+b}{cx^n+d}\right)'=\frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}$$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)=\frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix}x^2+2\begin{vmatrix} a & c \\ d & f \end{vmatrix}x+\begin{vmatrix} b & c \\ e & f \end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$

Example

Find $f'(x)$ if $f(x)=\frac{2x-1}{4x+3}$

Solution

$$f'=\frac{(2x-1)'(4x+3)-(2x-1)(4x+3)'}{(4x+3)^2} \qquad \begin{array}{ll} u=2x-1 & v=4x+3 \\ u'=2 & v'=4 \end{array}$$

$$=\frac{(2)(4x+3)-(2x-1)(4)}{(4x+3)^2}$$

$$=\frac{8x+6-8x+4}{(4x+3)^2}$$

$$=\boxed{\frac{10}{(4x+3)^2}}$$

$$f'(x)=\frac{2(3)-(-1)(4)}{4x+3} \qquad \left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$$

$$=\boxed{\frac{10}{(4x+3)^2}}$$

Example

Find the derivative of $y=\dfrac{(x-1)\left(x^2-2x\right)}{x^4}$

Solution

$$y=\frac{x^3-2x^2-x^2+2x}{x^4}$$

$$=\frac{x^3-3x^2+2x}{x^4}$$

$$=\frac{x^3}{x^4}-\frac{3x^2}{x^4}+\frac{2x}{x^4}$$

$$=x^{-1}-3x^{-2}+2x^{-3}$$

$$y'=-x^{-2}+6x^{-3}-6x^{-4}$$

$$=-\frac{1}{x^2}+\frac{6}{x^3}-\frac{6}{x^4}$$

Combining the product and Quotient Rules

Example

Find the derivative of $y=\dfrac{(1+x)(2x-1)}{x-1}$

Solution

$$y=\frac{(1+x)(2x-1)}{x-1}$$

$$=\frac{2x^2+x-1}{x-1}$$

$$y'=\frac{\begin{vmatrix}2&1\\0&1\end{vmatrix}x^2+2\begin{vmatrix}2&-1\\0&-1\end{vmatrix}x+\begin{vmatrix}1&-1\\1&-1\end{vmatrix}}{x-1}$$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)=\frac{\begin{vmatrix}a&b\\d&e\end{vmatrix}x^2+2\begin{vmatrix}a&c\\d&f\end{vmatrix}x+\begin{vmatrix}b&c\\e&f\end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$

$$=\frac{2x^2-4x}{(x-1)^2}$$

Or

$$y'=\frac{(x-1)\frac{d}{dx}\left[(1+x)(2x-1)\right]-(1+x)(2x-1)\frac{d}{dx}[x-1]}{(x-1)^2}$$

$$= \frac{(x-1)\left[(1)(2x-1)+2(1+x)\right]-(1+x)(2x-1)(1)}{(x-1)^2}$$

$$= \frac{(x-1)(2x-1+2+2x)-(2x-1+2x^2-x)}{(x-1)^2}$$

$$= \frac{(x-1)(4x+1)-2x+1-2x^2+x}{(x-1)^2}$$

$$= \frac{4x^2+x-4x-1-2x+1-2x^2+x}{(x-1)^2}$$

$$= \frac{2x^2-4x}{(x-1)^2}$$

Exercises *Section* 2.3 – Product and Quotient Rules

(**1 – 59**) Find the derivative of each function

1. $y=(x+1)\left(\sqrt{x}+2\right)$

2. $y=\left(4x+3x^2\right)(6-3x)$

3. $y=\left(\frac{1}{x}+1\right)(2x+1)$

4. $y=\dfrac{3-\frac{2}{x}}{x+4}$

5. $g(x)=\dfrac{x^2-4x+2}{x^2+3}$

6. $f(x)=\dfrac{(3-4x)(5x+1)}{7x-9}$

7. $f(x)=x\left(1-\dfrac{2}{x+1}\right)$

8. $f(x)=\left(\sqrt{x}+3\right)\left(x^2-5x\right)$

9. $y=(2x+3)\left(5x^2-4x\right)$

10. $y=\left(x^2+1\right)\left(x+5+\frac{1}{x}\right)$

11. $y=\dfrac{x+4}{5x-2}$

12. $z=\dfrac{4-3x}{3x^2+x}$

13. $y=(2x-7)^{-1}(x+5)$

14. $f(x)=\dfrac{\sqrt{x}-1}{\sqrt{x}+1}$

15. $y=\dfrac{1}{\left(x^2-1\right)\left(x^2+x+1\right)}$

16. $f(x)=\dfrac{x^{3/2}\left(x^2+1\right)}{x+1}$

17. $f(x)=\dfrac{x^3-4x^2+x}{x-2}$

18. $g(x)=\dfrac{x(3-x)}{2x^2}$

19. $y=\dfrac{2x^2}{3x+1}$

20. $f(x)=\dfrac{x^9+x^8+4x^5-7x}{x^4-3x^2+2x+1}$

21. $f(x)=\dfrac{x}{1+x^2}$

22. $y=\dfrac{x^2-2ax+a^2}{x-a}$

23. $f(x)=\dfrac{x^2+4x^{1/2}}{x^2}$

24. $f(x)=(2x+1)\left(3x^2+2\right)$

25. $f(x)=\dfrac{x^2-1}{x^2+1}$

26. $y=\dfrac{4x^3+3x+1}{2x^5}$

27. $y=\dfrac{4}{3-x}$

28. $y=\dfrac{2}{1-x^2}$

29. $f(x)=\dfrac{\pi}{2-\pi x}$

30. $y=\dfrac{x-4}{5x-2}$

31. $y=\dfrac{3x-4}{2x-1}$

32. $y=\dfrac{3x+4}{2x+1}$

33. $y=\dfrac{-3x+4}{2x+1}$

34. $y = \dfrac{-3x-4}{2x-1}$

35. $y = \dfrac{2x-3}{x+1}$

36. $y = \dfrac{3x}{3x-2}$

37. $y = \dfrac{x-3}{2x+5}$

38. $y = \dfrac{5x-3}{2x+5}$

39. $y = \dfrac{6x-8}{2x-3}$

40. $y = \dfrac{x^2-4}{5x^2-2}$

41. $y = \dfrac{3x^2-4}{2x^2-1}$

42. $y = \dfrac{3x^2+4}{2x^2+1}$

43. $y = \dfrac{2x^2-3}{x^2+1}$

44. $y = \dfrac{3x^2}{3x^2-2}$

45. $y = \dfrac{5x^2-3}{2x^2+5}$

46. $y = \dfrac{6x^2-8}{2x^2+1}$

47. $y = \dfrac{6x^3+8}{2x^3+1}$

48. $y = \dfrac{5x^3-3}{2x^3+5}$

49. $y = \dfrac{x^3}{3x^3-2}$

50. $y = \dfrac{2x^3-3}{2x^3+1}$

51. $y = \dfrac{2x^4-3}{2x^4+1}$

52. $y = \dfrac{x^2-4x+1}{5x^2-2x-1}$

53. $y = \dfrac{3x^2-4x+2}{2x^2+x-1}$

54. $y = \dfrac{3x^2+x-4}{2x^2+1}$

55. $y = \dfrac{2x^2-3}{x^2+5x+1}$

56. $y = \dfrac{3x^2}{3x^2+6x-8}$

57. $y = \dfrac{x^2+2x}{2x^2+x-5}$

58. $y = \dfrac{x^2+5x+1}{x^2}$

59. $y = \dfrac{x^2-3x+1}{x^2-8x+5}$

60. Find the first and second derivative $y = \dfrac{x^2+5x-1}{x^2}$

61. Find an equation of the tangent line to the graph of $y = \dfrac{x^2-4}{2x+5}$ when $x = 0$

62. For what value(s) of x is the line tangent to the curve $y = x\sqrt{6-x}$ horizontal? Vertical?

63. Find y', y'', y''': $\quad y = (x-3)\sqrt{x+2}$

Section 2.4 –Derivatives of Trigonometric Functions

Derivative of the *Sine* Function

If $f(x)=\sin x$, then

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{(\sin x\cos h+\cos x\sin h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h} \\
&= \lim_{h\to 0}\frac{\sin x(\cos h-1)}{h}+\lim_{h\to 0}\frac{\cos x\sin h}{h} \\
&= \sin x\,\underbrace{\lim_{h\to 0}\frac{\cos h-1}{h}}_{0}+\cos x\,\underbrace{\lim_{h\to 0}\frac{\sin h}{h}}_{1} \\
&= \sin x\bullet(0)+\cos x\bullet(1) \\
&= \cos x
\end{aligned}
$$

$$\cos h = 1-2\sin^2\left(\frac{h}{2}\right)$$

$$
\begin{aligned}
\lim_{h\to 0}\frac{\cosh-1}{h} &= \lim_{h\to 0}\frac{1-2\sin^2\left(\frac{h}{2}\right)-1}{h} \\
&= \lim_{h\to 0}\frac{-2\sin^2\left(\frac{h}{2}\right)}{h} \qquad \textit{Let } \theta=\frac{h}{2}
\end{aligned}
$$

Let $\theta=\frac{h}{2}$

$$
\begin{aligned}
&= -\lim_{\theta\to 0}\frac{2\sin^2(\theta)}{2\theta} \\
&= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\sin\theta \\
&= -(1)(0) \\
&= 0
\end{aligned}
$$

$$\boxed{\frac{d}{dx}(\sin x)=\cos x}$$

Example

Find the derivative of $y=x^2-\sin x$

Solution

$$
\begin{aligned}
y' &= 2x-(\sin x)' \\
&= 2x-\cos x
\end{aligned}
$$

Example

Find the derivative of $y=x^2\sin x$

Solution

$$y' = 2x\sin x+x^2\cos x$$

Example

Find the derivative of $y = \dfrac{\sin x}{x}$

Solution

$$y' = \frac{x\cos x - \sin x \cdot (1)}{x^2}$$

$$= \boxed{\frac{x\cos x - \sin x}{x^2}}$$

Derivative of the *Cosine* Function

If $f(x) = \cos x$, then

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0}\frac{(\cos x\cos h - \sin x\sin h) - \cos x}{h}$$

$$= \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$

$$= \lim_{h\to 0}\frac{\cos x(\cos h - 1) - \sin x\sin h}{h}$$

$$= \lim_{h\to 0}\frac{\cos x(\cos h - 1)}{h} - \lim_{h\to 0}\frac{\sin x\sin h}{h}$$

$$= \cos x \underbrace{\lim_{h\to 0}\frac{\cos h - 1}{h}}_{0} - \sin x \underbrace{\lim_{h\to 0}\frac{\sin h}{h}}_{1}$$

$$\lim_{h\to 0}\frac{\cosh - 1}{h} = 0$$

$$= \cos x \bullet (0) - \sin x \bullet (1)$$

$$= \boxed{-\sin x}$$

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

Example

Find the derivative of $y = 5x + \cos x$

Solution

$$\boxed{y' = 5 - \sin x}$$

Example

Find the derivative of $y = \sin x \cos x$

Solution

$$
\begin{aligned}
y' &= (\sin x)' \cos x + (\cos x)' \sin x \\
&= (\cos x)\cos x + (-\sin x)\sin x \\
&= \cos^2 x - \sin^2 x
\end{aligned}
$$

Example

Find the derivative of $y = \dfrac{\cos x}{1-\sin x}$

Solution

$$
\begin{aligned}
y' &= \frac{(1-\sin x)(\cos x)' - \cos x(1-\sin x)'}{(1-\sin x)^2} \\
&= \frac{(1-\sin x)(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2} \\
&= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} \qquad \sin^2 x + \cos^2 x = 1 \\
&= \frac{1-\sin x}{(1-\sin x)^2} \\
&= \frac{1}{1-\sin x}
\end{aligned}
$$

Derivatives of the Other Trigonometric Functions

$$\begin{cases} (\tan x)' = \sec^2 x & (\cot x)' = -\csc^2 x \\ (\sec x)' = \sec x \tan x & (\csc x)' = -\csc x \cot x \end{cases}$$

Prove

Example

Find $\frac{d}{dx}(\tan x)$

Solution

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{(\sin x)' \cos x - \sin x(\cos x)'}{\cos^2 x}$$

Quotient Rule

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$\frac{1}{\cos x} = \sec x$$

$$\underline{= \sec^2 x}\,|$$

Example

Find y'' *if* $y = \sec x$

Solution

$$y' = \sec x \tan x$$

$$y'' = (\sec x)' \tan x + \sec x(\tan x)'$$

$$= (\sec x \tan x)\tan x + \sec x\left(\sec^2 x\right)$$

$$\underline{= \sec x \tan^2 x + \sec^3 x}\,|$$

Exercises Section 2.4 – Derivatives of Trigonometric Functions

(1 – 29) Find the derivative of

1. $y=-10x+3\cos x$

2. $y=\csc x-4\sqrt{x}+7$

3. $y=x^2\cos x$

4. $y=\csc x\cot x$

5. $y=(\sin x+\cos x)\sec x$

6. $y=(\sec x+\tan x)(\sec x-\tan x)$

7. $y=\dfrac{\cos x}{x}+\dfrac{x}{\cos x}$

8. $y=x^2\cos x-2x\sin x-2\cos x$

9. $y=(2-x)\tan^2 x$

10. $y=t^2-\sec t+1$

11. $y=\dfrac{1+\csc t}{1-\csc t}$

12. $r=\theta\sin\theta+\cos\theta$

13. $y=\dfrac{3x+\tan x}{x\sec x}$

14. $p=\dfrac{\sin q+\cos q}{\cos q}$

15. $p=\dfrac{3q+\tan q}{q\sec q}$

16. $f(x)=\dfrac{\sin x+2x}{x}$

17. $f(x)=\dfrac{\sin x}{x^2}$

18. $f(x)=x^3\cos x$

19. $f(x)=\dfrac{1}{x}-12\sec x$

20. $f(\theta)=5\theta\sec\theta+\theta\tan\theta$

21. $y=\sec\pi x$

22. $y=\cos 5x$

23. $y=\cos(4-3x)$

24. $f(x)=\sin(4-3x)$

25. $f(\theta)=\dfrac{\sin a\theta}{\cos b\theta}$

26. $f(\theta)=\sin 2\theta-\cos 2\theta$

27. $f(\theta)=\tan\theta-\cot\theta$

28. $\dfrac{d}{dx}\left(5x^2\sin x\right)$

29. $\dfrac{d}{dx}\left(2x(\sin x)\sqrt{3x-1}\right)$

30. Find $y^{(4)}$ *if* $y=9\cos x$

31. Find $y',\ y'',\ y'''$: $\quad y=(x-3)\sqrt{x+2}$

32. Find $\dfrac{d^{999}}{dx^{999}}(\cos x)$

33. Find $\displaystyle\lim_{x\to-\frac{\pi}{6}}\sqrt{1+\cos(\pi\csc x)}$

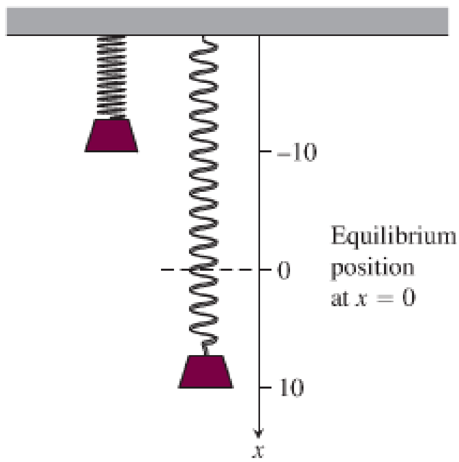
34. Assume that a particle's position on the x-axis is given by

$$x = 3\cos t + 4\sin t$$

a) Find the particle's position when $t = 0, \quad t = \frac{\pi}{2}, \quad \textit{and} \quad t = \pi$

b) Find the particle's velocity when $t = 0, \quad t = \frac{\pi}{2}, \quad \textit{and} \quad t = \pi$

35. A weight is attached to a spring and reaches its equilibrium position $(x = 0)$. It is then set in motion resulting in a displacement of $x = 10\cos t$



Where x is measured in centimeters and t is measured in seconds.

a) Find the spring's displacement when $t = 0, \quad t = \frac{\pi}{3}, \quad \textit{and} \quad t = \frac{3\pi}{4}$

b) Find the spring's velocity when $t = 0, \quad t = \frac{\pi}{3}, \quad \textit{and} \quad t = \frac{3\pi}{4}$

Section 2.5 – Derivatives as Rates of Change

Definition

The ***instantaneous rate of change*** of f with respect to x at x_0 is the derivative

$$f'\left(x_0\right) = \lim_{x \to \infty} \frac{f\left(x_0 + h\right) - f(x)}{h}$$

Provided the limit exists.

Example

The area **A** of the circle is related to its diameter by the equation $A = \frac{\pi}{4}D^2$

How fast does the area change with respect to the diameter when the diameter is 10 *m*?

Solution

The rate of change of the area with respect to the diameter is

$$\frac{dA}{dD} = \frac{\pi}{4} \cdot 2D = \frac{\pi D}{2}$$

When $D = 10$ *m*, the area is changing with respect to the diameter at the rate of

$$\frac{dA}{dD} = \frac{\pi(10)}{2} \approx 15.71\ m^2 / m$$

Motion along a Line: Displacement, Velocity, Speed, Acceleration, and *Jerk*

Suppose that an object is moving along a coordinate line (an *s*-axis), usually horizontal or vertical, so that we know its position *s* on that line as a function of time *t*:

$$s = f(t)$$

The ***displacement*** of the object over the time interval from t to $t + \Delta t$ is

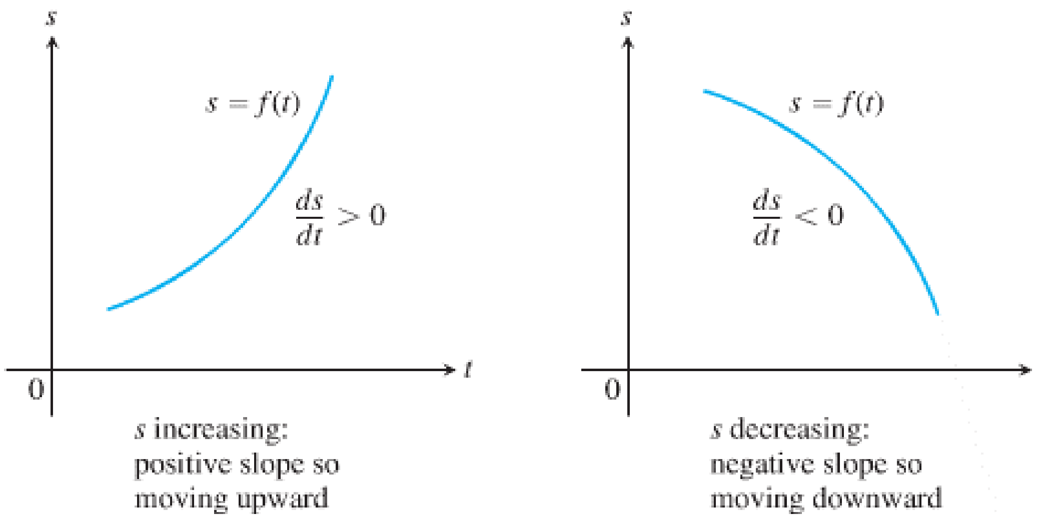
$$\Delta s = f(t + \Delta t) - f(t)$$

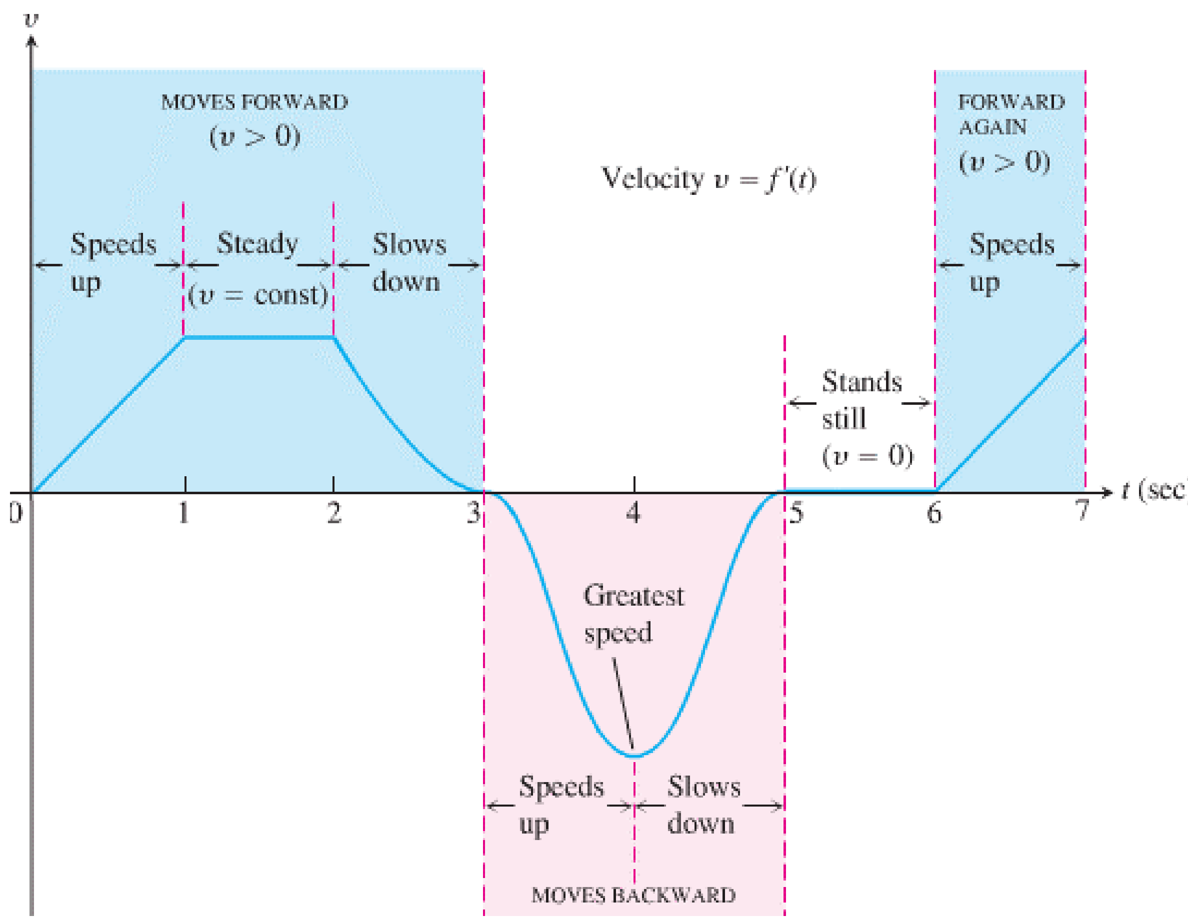
And the ***average velocity*** of the object over that time interval is

$$v_{avg} = \frac{displacement}{travel\ time} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

Definition

Speed is the absolute value of velocity $\quad speed = \left|v(t)\right| = \left|\frac{ds}{dt}\right|$


s
$s = f(t)$
$\frac{ds}{dt} > 0$
t
0
s increasing:
positive slope so
moving upward
s
$s = f(t)$
$\frac{ds}{dt} < 0$
t
0
s decreasing:
negative slope so
moving downward


v
MOVES FORWARD
$(v > 0)$
Velocity $v = f'(t)$
FORWARD
AGAIN
$(v > 0)$
Speeds
up
Steady
$(v = \text{const})$
Slows
down
Speeds
up
Stands
still
$(v = 0)$
0
1
2
3
4
5
6
7
t (sec
Greatest
speed
Speeds
up
Slows
down
MOVES BACKWARD

Definition

Acceleration is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

Jerk is the derivative of acceleration with the respect to time

$$j(t) = \frac{da}{dt} = \frac{d^3 s}{dt^3}$$

When a ride in a car is jerky, it is not that the accelerations involved are necessarily large but that the changes in acceleration are abrupt.

Example

The free fall of a heavy ball bearing released from rest at time $t = 0$ *sec*.

a) How many meters does the ball fall in the first 2 *sec*?

b) What is its velocity, speed, and acceleration when $t = 2$?

Solution

a) The metric free-fall equation is $s = 4.9t^2$.

During the first 2 *sec*: $s(2) = 4.9(2)^2 \ \underline{= 19.6\ m}$

b) At any time, the velocity is:

$$\begin{aligned} v &= \frac{ds}{dt} \\ &= \frac{d}{dt}\left(4.9t^2\right) \\ &\underline{= 9.8t} \end{aligned}$$

At $t = 2$,

Velocity: $v = 9.8(2) \ \underline{= 19.6\ m/\sec}$

$Speed = |v| \ \underline{= 19.6\ \ m/\sec}$

Acceleration:

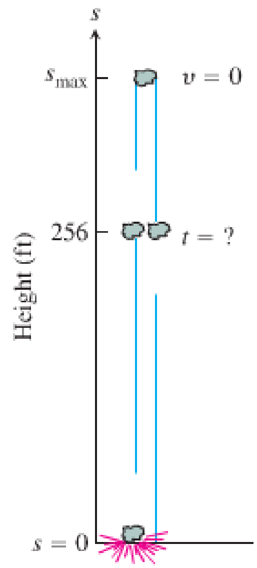
$a(t) = v'(t) \ \underline{= 9.8\ m/\sec^2}$

Example

A dynamic blast blows a heavy rock straight up with a launch velocity of 160 *ft/sec* (about 109 *mph*).

It reaches a height of $s = 160t - 16t^2$ after *t sec*.

a) How high does the rock go?
b) What are the velocity and speed of the rock when it is 256 *ft* above the ground on the way up? On the way down?
c) What is the acceleration of the rock at any time *t* during its flight (after the blast)?
d) When does the rock hit the ground again?



Solution

a) At any time *t* during the rock's motion, its velocity is

$$v = s' = 160 - 32t$$

The velocity is zero when it reaches maximum height:

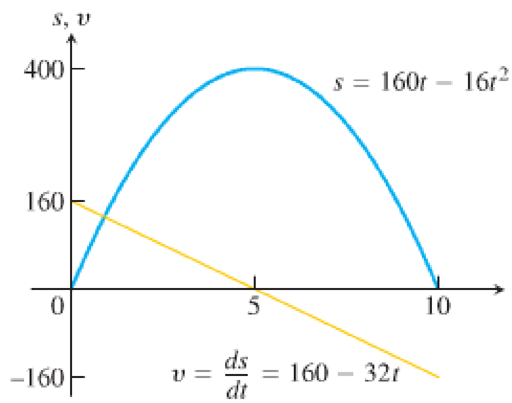
$$v = 160 - 32t = 0$$

$$160 = 32t$$

$$t = \frac{160}{32} = 5 \text{ sec}$$

The rock's height at $t = 5$ sec is

$$s(t=5) = 160(5) - 16(5)^2 = 400 \text{ ft}$$



b) $s = 160t - 16t^2 = 256$

$$-16t^2 + 160t - 256 = 0 \quad \Rightarrow \quad t = 2\text{sec}, \quad t = 8\text{sec}$$

$$\begin{cases} t = 2\text{sec} & \rightarrow \quad v = 160 - 32(2) = 96 \text{ ft / sec} \\ t = 8\text{sec} & \rightarrow \quad v = 160 - 32(8) = -96 \text{ ft / sec} \end{cases}$$

The rock's speed is 96 *ft/sec*.

Since $v(t=2) > 0$, the rock is moving upward and *s* is increasing.

$v(t=8) < 0$, the rock is moving downward and *s* is decreasing.

c) Acceleration at any time is: $a = v' = -32 \text{ ft / sec}^2$

d) $s = 160t - 16t^2 = 0$

$$t(160 - 16t) = 0 \quad \Rightarrow \quad t = 0, \ t = 10.$$

At $t = 0$, the blast occurred and the rock was thrown upward, it took 10 *sec* to return to ground.

Derivatives in Economics

Example

Suppose that it costs $C(x) = x^3 - 6x^2 + 15x$ dollars to produce x radiators when 8 to 30 radiators are produced and that $R(x) = x^3 - 3x^2 + 12x$ gives the dollar revenue from selling x radiators.

Your shop currently produces 10 radiators a day. About how much extra will it cost to produce one more radiator a day, and what is your estimated increase in revenue for selling 11 radiators a day?

Solution

The cost of producing one more radiator a day when 10 are produced is about $C'(10)$:

$$C'(x) = 3x^2 - 12x + 15$$

$$C'(x=10) = 3(10)^2 - 12(10) + 15$$

$$\underline{= 195}$$

The additional cost will be about \$195.00.

The marginal revenue is:

$$R'(x) = 3x^2 - 6x + 12$$

$$R'(x=10) = 3(10)^2 - 6(10) + 12$$

$$\underline{= \$252.00}$$

If you increase sales to 11 radiators a day, the revenue is an additional of \$252.00.

Exercises *Section* 2.5 – Derivatives as Rates of Change

1. The position $s(t)=t^2-3t+2, \quad 0\le t\le 2$ of a body moving on a coordinate line, with s in meters and t in seconds.

a) Find the body's displacement and average velocity for the given time interval.
b) Find the body's speed and acceleration at the endpoints of the interval.
c) When, if ever, during the interval does the body change direction?

2. The position $s(t)=\frac{25}{t+5}, \quad -4\le t\le 0$ of a body moving on a coordinate line, with s in meters and t in seconds.

a) Find the body's displacement and average velocity for the given time interval.
b) Find the body's speed and acceleration at the endpoints of the interval.
c) When, if ever, during the interval does the body change direction?

3. At time t, the position of a body moving along the s-axis is $s=t^3-6t^2+9t$ ***m***.

a) Find the body's acceleration each time the velocity is zero.
b) Find the body's speed each time the acceleration is zero.
c) Find the total distance traveled by the body from $t = 0$ to $t = 2$.

4. A rock thrown vertically upward from the surface of the moon at a velocity of 24 *m/sec* (about 86 *km/h*) reaches a height of $s(t)=24t-0.8t^2$ *m* in *t sec*.

a) Find the rock's velocity and acceleration at time t. (The acceleration in this case is the acceleration of gravity on the moon.)
b) How long does it take the rock to reach its highest point?
c) How high does the rock go?
d) How long does it take the rock to reach half its maximum height?
e) How long is the rock aloft?

5. Had Galileo dropped a cannonball from the Tower of Pisa, 179 *feet* above the ground, the ball's height above the ground *t sec* into the fall would have been $s=179-16t^2$.

a) What would have been the ball's velocity, speed, and acceleration at time t?
b) About how long would it have taken the ball to hit the ground?
c) What would have been the ball's velocity at the moment of impact?

6. A toy rocket fired straight up into the air has height $s(t)=160t-16t^2$ *feet* after t seconds.

a) What is the rocket's initial velocity (when $t=0$)?
b) What is the acceleration when $t=3$?
c) At what time will the rocket hit the ground?
d) At what velocity will the rocket be traveling just as it smashes into the ground?

7. A helicopter is rising straight up in the air. Its distance from the ground t seconds after takeoff is $s(t) = t^2 + t$ *feet*

a) How long will it take for the helicopter to rise 20 *feet* ?

b) Find the velocity and the acceleration of the helicopter when it is 20 *feet* above the ground.

8. The position of a particle moving on a line is given by $s(t) = 2t^3 - 21t^2 + 60t, \quad t \ge 0$, where t is measured in *seconds* and s in *feet*.

a) What is the velocity after 3 *seconds* and after 6 *seconds*?

b) When the particle moving in the positive direction?

c) Find the total distance traveled by the particle during the first 7 *seconds*.

9. A small probe is launched vertically from the ground. After it reaches its high point, a parachute deploys and the probe descends to Earth. The height of the probe the ground is

$$s(t) = \frac{300t - 50t^2}{t^3 + 2} \quad \text{for} \quad 0 \le t \le 6$$

a) Graph the height function and describe the motion of the probe.

b) Find the velocity of the probe.

c) Graph the velocity function and determine the approximate time at which the velocity is a maximum.

10. Suppose the cost of producing x lawn mowers is $C(x) = -0.02x^2 + 400x + 5000$

a) Determine the average and marginal costs for $x = 3000$ lawn mowers.

b) Interpret the meaning of your results in part (*a*)

11. Suppose a company produces fly rods. Assume $C(x) = -0.0001x^3 + 0.05x^2 + 60x + 800$ represents the cost of making x fly rods.

a) Determine the average and marginal costs for $x = 400$ fly rods.

b) Interpret the meaning of your results in part (*a*)

12. Suppose $p(t) = -1.7t^3 + 72t^2 + 7200t + 80{,}000$ is the population of a city t years after 1950.

a) Determine the average rate of growth of the city from 1950 to 2000.

b) What was the rate of growth of the city in 1990?

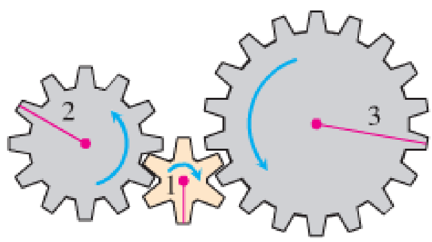
Section 2.6 – Chain Rule

Derivative of a Composite Function

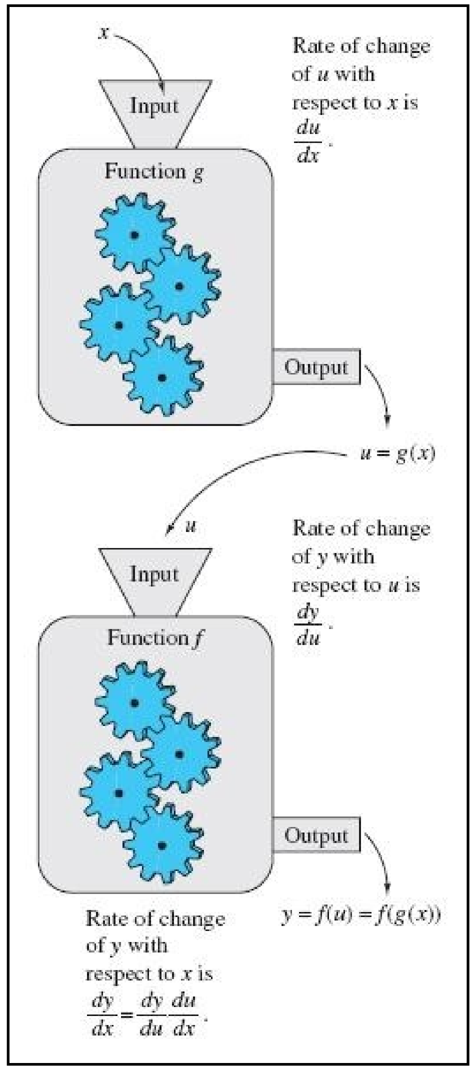
$$y = f(g(x)) = f(u)$$

$$\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$$

$$\frac{dy}{dx} = f'[g(x)] \bullet g'(x)$$



C: y turns B: u turns A: x turns



Example

Find the derivative of $y = \left(3x^2 + 1\right)^2$

Solution

$$u = 3x^2 + 1 \quad \Rightarrow \quad (u)' = 6x$$

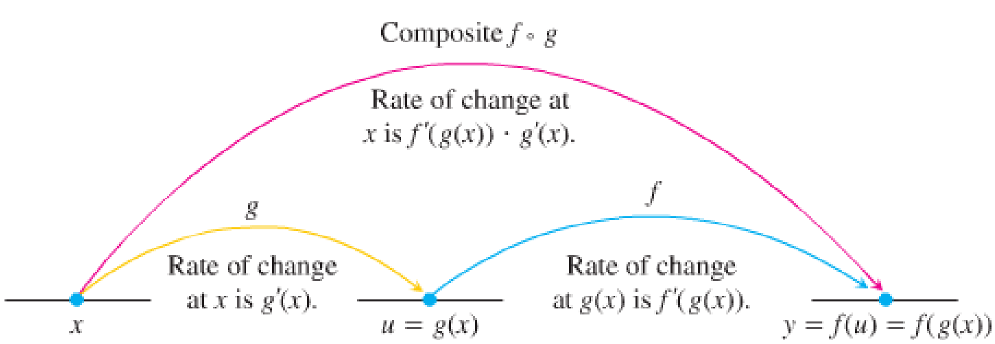
$$\frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot 6x$$

$$= 2\left(3x^2 + 1\right) \cdot 6x$$

$$\underline{= 36x^3 + 12x}$$

Calculating from the expand formula: $y = \left(3x^2 + 1\right)^2 = 9x^4 + 6x^2 + 1$

$$\underline{y' = 36x^3 + 12x}$$



Intuitive "*Proof*" of the Chain Rule

Let Δu be the change in u when x changes by Δx, so that

$$\Delta u = g(x+\Delta x) - g(x)$$

Let Δy be the change in y when u changes by Δu, so that

$$\Delta y = f(u+\Delta u) - f(u)$$

If $\Delta u \neq 0 \Rightarrow \quad \dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{\Delta u} \cdot \dfrac{\Delta u}{\Delta x}$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

$$= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

$$= \frac{dy}{du} \cdot \frac{du}{dx}$$

Example

An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos\left(t^2+1\right)$. Find the velocity of the object as a function of t.

Solution

Let: $u = t^2 + 1 \quad \Rightarrow \quad u' = 2t$

$x = \cos(u) \quad \Rightarrow \quad x' = -\sin(u)$

By the Chain Rule:

$$\frac{dx}{dt} = \frac{dx}{du} \cdot \frac{du}{dt}$$

$$= -\sin(u) \cdot 2t$$

$$\boxed{= -2t\sin\left(t^2+1\right)}$$

The General Power Rule

$$\frac{dy}{dx} = \frac{d}{dx}\left[u(x)^n\right]$$

$$= n\, u^{n\text{-}1}\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[u^n\right]$$

$$\boxed{= n\, u^{n\text{-}1}u'}$$

Example

Find the derivative of $\frac{d}{dx}\left(5x^3 - x^4\right)^7$

Solution

$$\frac{d}{dx}\overbrace{\left(5x^3 - x^4\right)^7}^{nu^n} = 7\overbrace{\left(5x^3 - x^4\right)^6}^{nu^{n-1}}\overbrace{\left(15x^2 - 4x^3\right)}^{u'}$$

Example

Find the derivative of $\frac{d}{dx}\left(\frac{1}{3x-2}\right)$

Solution

$$\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1}$$

$$= -3(3x-2)^{-2}$$

$$\boxed{= -\frac{3}{(3x-2)^2}}$$

Example

Find the derivative of $\frac{d}{dx}\left(\sin^5 x\right)$

Solution

$$\frac{d}{dx}\left(\sin^5 x\right) = 5\sin^4 x(\sin x)'$$

$$\boxed{= 5\sin^4 x\cos x}$$

Example

Find the derivative of $g(t) = \tan(5 - \sin 2t)$

Solution

$$g'(t) = \sec^2(5 - \sin 2t)\cdot(5 - \sin 2t)' \qquad u = 5 - \sin 2t \quad (\tan u)' = \sec^2 u \cdot (u')$$

$$= \sec^2(5 - \sin 2t)\cdot\left(0 - (\cos 2t)(2t)'\right)$$

$$= \sec^2(5 - \sin 2t)\cdot(-2\cos 2t)$$

$$= -2(\cos 2t)\sec^2(5 - \sin 2t)$$

Example

Show that the slope of every line tangent to the curve $y = \dfrac{1}{(1-2x)^3}$ is positive.

Solution

$$y = (1-2x)^{-3}$$

$$y' = -3(1-2x)^{-4}(-2)$$

$$= \frac{6}{(1-2x)^4}$$

At any point except $\left(x \neq \frac{1}{2}\right)$, the slope is $\dfrac{6}{(1-2x)^4}$ which is positive.

Formula $\qquad \left(U^m V^n W^p\right)' = U^{m-1}V^{n-1}W^{p-1}\left(mU'VW + nUV'W + pUVW'\right)$

Proof

$$\left(U^m V^n W^p\right)' = \left(U^m\right)' V^n W^p + U^m \left(V^n\right)' W^p + U^m V^n \left(W^p\right)'$$

$$= mU^{m-1}U'V^nW^p + nU^mV^{n-1}V'W^p + pU^mV^nW^{p-1}W' \qquad \text{factor } U^{m-1}V^{n-1}W^{p-1}$$

$$= U^{m-1}V^{n-1}W^{p-1}\left(mU'VW + nUV'W + pUVW'\right)$$

$$\left(U^m V^n\right)' = U^{m-1}V^{n-1}\left(mU'V + nUV'\right)$$

Exercises *Section* 2.6 – Chain Rule

(1 – 67) Find the derivative of

1. $y=\left(3x^4+1\right)^4\left(x^3+4\right)$
2. $p(t)=\dfrac{(2t+3)^3}{4t^2-1}$
3. $y=\left(x^3+1\right)^2$
4. $y=\left(x^2+3x\right)^4$
5. $y=\dfrac{4}{2x+1}$
6. $y=\dfrac{2}{(x-1)^3}$
7. $y=x^2\sqrt{x^2+1}$
8. $y=\left(\dfrac{x+1}{x-5}\right)^2$
9. $s(t)=\sqrt{2t^2+5t+2}$
10. $f(x)=\dfrac{1}{\left(x^2-3x\right)^2}$
11. $y=t^2\sqrt{t-2}$
12. $y=\left(\dfrac{6-5x}{x^2-1}\right)^2$
13. $y=4x(3x+5)^5$
14. $y=\left(3x^2-5x\right)^{1/2}$
15. $D_x\left(x^2+5x\right)^8$
16. $y=\dfrac{(3x+2)^7}{x-1}$
17. $y=\left(\dfrac{x^2}{8}+x-\dfrac{1}{x}\right)^4$
18. $y=\sqrt{3x^2-4x+6}$
19. $y=\cot\left(\pi-\dfrac{1}{x}\right)$
20. $y=5\cos^{-4}x$
21. $y=\sin\left(\dfrac{3\pi t}{2}\right)+\cos\left(\dfrac{3\pi t}{2}\right)$
22. $r=6(\sec\theta-\tan\theta)^{3/2}$
23. $g(x)=\dfrac{\tan 3x}{(x+7)^4}$
24. $f(\theta)=\left(\dfrac{\sin\theta}{1+\cos\theta}\right)^2$
25. $y=\sin^2(\pi t-2)$
26. $y=(t\tan t)^{10}$
27. $y=\cos\left(5\sin\left(\dfrac{t}{3}\right)\right)$
28. $y=4\sin\left(\sqrt{1+\sqrt{t}}\right)$
29. $y=\tan^2\left(\sin^3 x\right)$
30. $f(x)=\left(\left(x^2+3\right)^5+x\right)^2$
31. $y=\left(\dfrac{3x-1}{x^2+3}\right)^2$
32. $y=\cos\sqrt{\sin(\tan\pi x)}$
33. $f(x)=\dfrac{x}{\sqrt{x^2+1}}$

34. $y=\cos(1-2x)^2$

35. $f(x)=(4x-3)^2$

36. $f(x)=\dfrac{x}{\sqrt[3]{x^2+4}}$

37. $f(x)=\left(\dfrac{x^2}{x^3+2}\right)^2$

38. $y=\sin\sqrt[3]{x}+\sqrt[3]{\sin x}$

39. $f(\theta)=4\tan\left(\theta^2+3\theta+2\right)$

40. $f(\theta)=\tan(\sin\theta)$

41. $y=5x+\sin^3 x+\sin x^3$

42. $y=\csc^5 3x$

43. $y=2x\sqrt{x^2-2x+2}$

44. $\dfrac{d}{du}\left(\dfrac{4u^2+u}{8u+1}\right)^3$

45. $y=\frac{1}{2}x^2\sqrt{16-x^2}$

46. $y=\left(\dfrac{x-3}{2x+5}\right)^4$

47. $y=\left(\dfrac{5x-3}{2x+5}\right)^5$

48. $y=\left(\dfrac{6x-8}{2x-3}\right)^6$

49. $y=\left(\dfrac{3x^2-4}{2x^2-1}\right)^3$

50. $y=\left(\dfrac{3x^2+4}{2x^2+1}\right)^{-3}$

51. $y=\left(\dfrac{2x^2-3}{x^2+1}\right)^{1/3}$

52. $y=\sqrt{\dfrac{2x^3-3}{2x^3+1}}$

53. $y=\left(\dfrac{2x^4-3}{2x^4+1}\right)^5$

54. $y=\left(\dfrac{x^2-4x+1}{5x^2-2x-1}\right)^3$

55. $y=\left(\dfrac{3x^2-4x+2}{2x^2+x-1}\right)^{2/3}$

56. $f(x)=\left(\dfrac{3t^2-1}{3t^2+1}\right)^{-3}$

57. $f(x)=\left(\dfrac{x}{3x^2+2x+1}\right)^{1/3}$

58. $f(x)=\left(x^2+2x-3\right)^5(2x+3)^6$

59. $f(x)=\left(2x^2-4x+3\right)^4(3x-5)^5$

60. $f(x)=\left(x^2+2x-3\right)^4\left(x^2+3x+5\right)^6$

61. $f(x)=\left(2x^3-5x\right)^3\left(x^2+2x+1\right)^4(2x-3)^5$

62. $f(x)=\left(x^4+3x\right)^4\left(x^3+2x\right)^5(2x-3)^6$

63. $f(x)=\dfrac{\left(x^2-6x\right)^5}{\left(3x^2+5x-2\right)^4}$

64. $f(x)=\dfrac{\left(2x^2+3x1\right)^4}{\left(x^2+5x-6\right)^5}$

65. $f(x)=\dfrac{\left(x^3-3x\right)^3\left(x^2+4x\right)^4}{\left(x^2+4x+1\right)^2}$

66. $f(x)=\dfrac{x^2+3}{(2x-1)^3(3x+1)^4}$

67. $f(x)=\dfrac{\left(x^2-3x\right)^3\left(x^2+3x-3\right)^4}{\left(x^2-3x+2\right)^2}$

68. Find the ***second*** derivative $y = \dfrac{x^2+3}{(x-1)^3+(x+1)^3}$

69. Find the ***second*** derivative of $y = \left(1+\frac{1}{x}\right)^3$

70. Find the ***second*** derivative of $y = 9\tan\left(\frac{x}{3}\right)$

71. Find the tangent line to the graph of $y = \sqrt[3]{(x+4)^2}$ when $x = 4$

(72 – 73) Evaluate the limit

72. $\displaystyle\lim_{h\to 0} \frac{\sin^2\left(\frac{\pi}{4}+h\right)-\frac{1}{2}}{h}$

73. $\displaystyle\lim_{x\to 5} \frac{\tan\left(\pi\sqrt{3x-11}\right)}{x-5}$

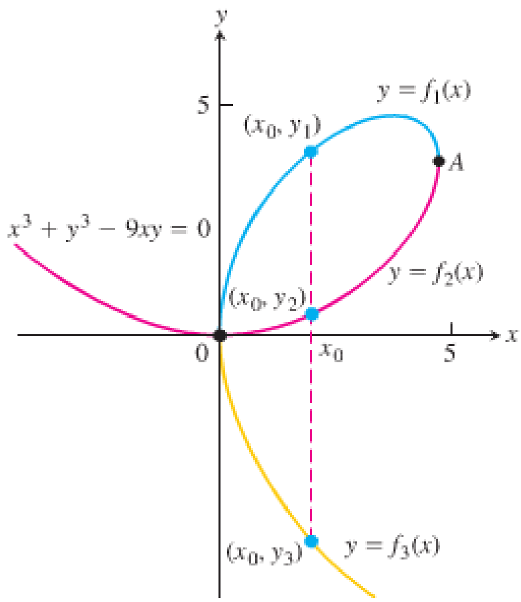
Section 2.7 – Implicit Differentiation

Definition

A relation $F(x,y)=0$ is said to define the function $y=f(x)$ implicitly if, for x in the domain of f

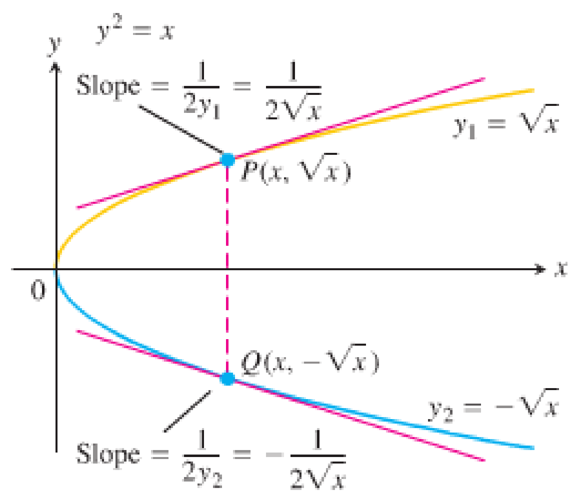
$\to F(x, f(x))=0$

Example: $x^3+y^3-9xy=0,\quad x^2+y^2=25$



Implicitly Defined Functions

It is always assumed that the given equation determines y implicitly as a differentiable function of x so that $\frac{dy}{dx}$ exists.



Example

Find $\frac{dy}{dx}$ if $y^2=x$

Solution

$$y^2=x$$

$$\frac{d}{dx}\left(y^2\right)=\frac{d}{dx}(x)$$

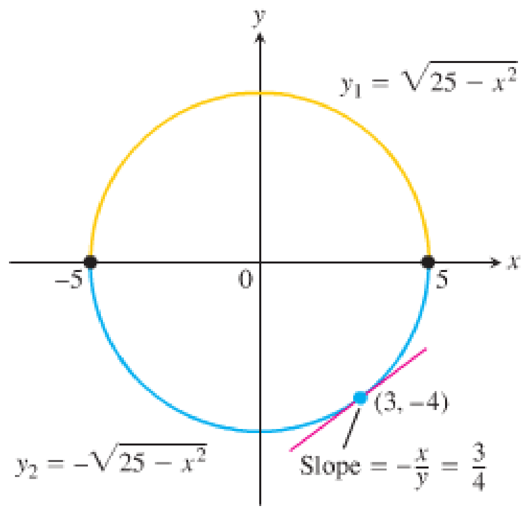
$$2y\frac{dy}{dx}=1$$

$$\boxed{\frac{dy}{dx}=\frac{1}{2y}}$$

Example

Find the slope of the circle $x^2 + y^2 = 25$ at the point (3, –4).

Solution



$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

The slope at (3, –4) is $\left.\frac{dy}{dx}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}$

Implicit Differentiation

- ✓ Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
- ✓ Collect the terms with $\frac{dy}{dx}$ on one side of the equation and solve for $\frac{dy}{dx}$.

Example

Find $\frac{dy}{dx}$ if $y^2 = x^2 + \sin xy$

Solution

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(\sin xy)$$

$$2y\frac{dy}{dx} = 2x + \cos(xy)\frac{d}{dx}(xy)$$

$$2y\frac{dy}{dx} = 2x + \cos(xy)\left(y + x\frac{dy}{dx}\right)$$

$$2y\frac{dy}{dx} = 2x + y\cos(xy) + x\cos(xy)\frac{dy}{dx}$$

$$2y\frac{dy}{dx} - x\cos(xy)\frac{dy}{dx} = 2x + y\cos(xy)$$

$$(2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy$$

$$\boxed{\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}}$$

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(\sin xy)$$

Let $y' = \frac{dy}{dx}$

$$2yy' = 2x + \cos(xy)(xy)'$$

$$2yy' = 2x + \cos(xy)(y + xy')$$

$$2yy' = 2x + y\cos(xy) + x\cos(xy)y'$$

$$2yy' - x\cos(xy)y' = 2x + y\cos(xy)$$

$$(2y - x\cos xy)y' = 2x + y\cos xy$$

$$\boxed{y' = \frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}}$$

Example

Find $\dfrac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 8$

Solution

Let $y' = \dfrac{dy}{dx}$

$$\frac{d}{dy}\left(2x^3 - 3y^2\right) = \frac{d}{dy}(8)$$

$$6x^2 - 6yy' = 0$$

$$6x^2 = 6yy'$$

$$\boxed{y' = \frac{6x^2}{6y} = \frac{x^2}{y}}$$

$$y'' = \left(\frac{x^2}{y}\right)'$$

$u = x^2 \quad v = y$

$u' = 2x \quad v' = y'$

$$y'' = \frac{2xy - x^2 y'}{y^2}$$

$$= \frac{2xy - x^2 \frac{x^2}{y}}{y^2}$$

$$= \frac{\frac{2xy^2 - x^4}{y}}{y^2}$$

$$\boxed{= \frac{2xy^2 - x^4}{y^3}}$$

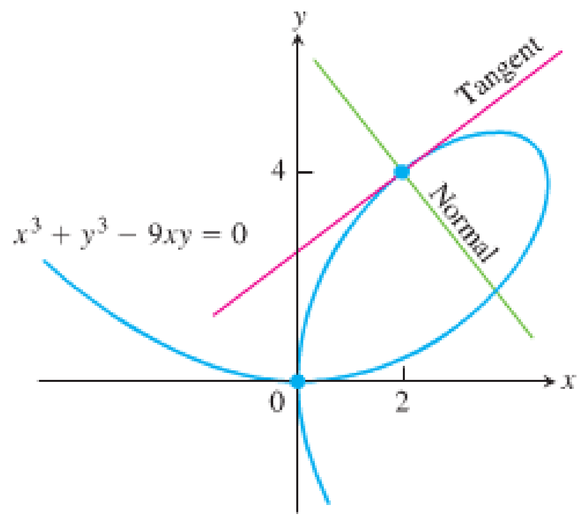
Normal Lines

The ***normal*** is the line ***perpendicular*** to the ***tangent*** of the profile curve at the point of entry.

Example

Show that the point (2, 4) lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve there.

Solution



$$\frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(y^3\right) - \frac{d}{dx}(9xy) = 0$$

$$3x^2 + 3y^2y' - 9\left(y + xy'\right) = 0$$

$$3x^2 + 3y^2y' - 9y - 9xy' = 0$$

$$3\left(y^2 - 3x\right)y' = 9y - 3x^2$$

$$y' = \frac{3\left(3y - x^2\right)}{3\left(y^2 - 3x\right)}$$

$$= \frac{3y - x^2}{y^2 - 3x}$$

The slope: $y'\Big|_{(2,4)} = \frac{3(4) - (2)^2}{(4)^2 - 3(2)}$

$$= \frac{8}{10}$$

$$= \frac{4}{5}$$

The tangent at (2, 4) is the line passes thru (2, 4) with slope $\frac{4}{5}$

$$y = \frac{4}{5}(x - 2) + 4 \qquad y = m\left(x - x_1\right) + y_1$$

$$y = \frac{4}{5}x - \frac{8}{5} + 4$$

$$y = \frac{4}{5}x + \frac{12}{5}$$

The Normal to the curve at (2, 4) is the line perpendicular to the tangent thru (2, 4) with slope $-\frac{5}{4}$

$$y = -\frac{5}{4}(x - 2) + 4 \qquad y = m\left(x - x_1\right) + y_1$$

$$y = -\frac{5}{4}x + \frac{5}{2} + 4$$

$$\boxed{y = -\frac{5}{4}x + \frac{13}{2}}$$

Exercises Section 2.7 – Implicit Differentiation

(1 – 13) Find $\frac{dy}{dx}$

1. $y^2 + x^2 - 2y - 4x = 4$

2. $x^2y^2 - 2x = 3$

3. $x + \sqrt{x}\sqrt{y} = y^2$

4. $x^2y + xy^2 = 6$

5. $x^3 - xy + y^3 = 1$

6. $y^2 = \frac{x-1}{x+1}$

7. $(3xy+7)^2 = 6y$

8. $xy = \cot(xy)$

9. $x + \tan(xy) = 0$

10. $x\cos(2x+3y) = y\sin x$

11. $y = \frac{e^y}{1+\sin x}$

12. $\sin x\cos(y-1) = \frac{1}{2}$

13. $y\sqrt{x^2+y^2} = 15$

(14 – 15) Find $\frac{dr}{d\theta}$

14. $r - 2\sqrt{\theta} = \frac{3}{2}\theta^{2/3} + \frac{4}{3}\theta^{3/4}$

15. $\sin(r\theta) = \frac{1}{2}$

(16 – 17) Find $\frac{d^2y}{dx^2}$

16. $x^{2/3} + y^{2/3} = 1$

17. $2\sqrt{y} = x - y$

18. If $x^3 + y^3 = 16$, find the value of $\frac{d^2y}{dx^2}$ at the point (2, 2).

19. Find dy/dx: $x^2 - xy + y^2 = 4$ and evaluate the derivative at the given point $(0,-2)$

20. Find the slope of the curve $\left(x^2+y^2\right)^2 = (x-y)^2$ at the point (–2, 1) and (–2, –1)

21. Find the slope of the tangent line to the circle $x^2 - 9y^2 = 16$ at the point (5, 1)

(22 – 26) Find an equation of the line tangent to the following curves at the given point

22. $y = 3x^3 + \sin x;\quad (0,\ 0)$

23. $y = \frac{4x}{x^2+3};\quad (3,\ 1)$

24. $y + \sqrt{xy} = 6;\quad (1,\ 4)$

25. $x^2y + y^3 = 75;\quad (4,\ 3)$

26. $x^3 + y^3 = 9xy;\quad (2,\ 4)$

27. Find the lines that are (***a***) tangent and (***b***) normal to the curve $x^2 + xy - y^2 = 1$ at the point (2, 3).

28. Find the lines that are (***a***) tangent and (***b***) normal to the curve $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$ at the point (−1, 0).

29. Find the lines that are (***a***) tangent and (***b***) normal to the curve $x^2 \cos^2 y - \sin y = 0$ at the point $(0,\ \pi)$.

30. Suppose that *x* and *y* are both functions of *t*, which can be considered to represent time, and that *x* and *y* are related by the equation

$$xy^2 + y = x^2 + 17$$

Suppose further that when $x = 2$ and $y = 3$, then $\frac{dx}{dt} = 13$. Find the value of the $\frac{dy}{dt}$ at that moment.

31. A cone-shaped icicle is dripping from the roof. The radius of the icicle is decreasing at a rate of 0.2 *cm* per hour, while the length is increasing at a rate of 0.8 *cm* per *hour*. If the icicle is currently 4 *cm* in radius and 20 *cm* long, is the volume of the icicle increasing or decreasing and at what rate?

Section 2.8 – Derivatives of Logarithmic & Exponential Functions

Derivative of $y = \ln x$

$$\boxed{\frac{d}{dx}\ln|x| = \frac{1}{x}} \qquad x \neq 0$$

The chain rule extends: $\boxed{\frac{d}{dx}\ln u = \frac{1}{u}\cdot\frac{du}{dx} = \frac{u'}{u}}$ $\quad u > 0$

Example

Find $\quad \frac{d}{dx}\ln 2x$

Solution

$$\frac{d}{dx}\ln 2x = \frac{(2x)'}{2x}$$

$$= \frac{2}{2x}$$

$$= \frac{1}{x}$$

Example

Find the derivative of $\ln\left(x^2+3\right)$

Solution

$$\frac{d}{dx}\ln\left(x^2+3\right) = \frac{2x}{x^2+3}$$

Properties of the Natural logarithm

Product Rule $\qquad \ln bx = \ln b + \ln x$

Quotient Rule $\qquad \ln\frac{b}{x} = \ln b - \ln x$

Reciprocal Rule $\qquad \ln\frac{1}{x} = -\ln x$

Power Rule $\qquad \ln x^r = r\ln x$

Example

a) $\ln(4\sin x) = \ln 4 + \ln \sin x$

b) $\ln \dfrac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$

c) $\ln \dfrac{1}{8} = -\ln 8 = -\ln 2^3 = -3\ln 2$

Example

Find $\dfrac{dy}{dx}$ if $y = \dfrac{(x^2+1)(x+3)^{1/2}}{x-1}, \quad x > 1$

Solution

$$\ln y = \ln \frac{(x^2+1)(x+3)^{1/2}}{x-1}$$

$$= \ln(x^2+1)(x+3)^{1/2} - \ln(x-1) \qquad \textit{Quotient Rule}$$

$$= \ln(x^2+1) + \ln(x+3)^{1/2} - \ln(x-1) \qquad \textit{Product Rule}$$

$$= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1) \qquad \textit{Power Rule}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2+1} + \frac{1}{2}\frac{1}{x+3} - \frac{1}{x-1}$$

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right)$$

$$\frac{dy}{dx} = \frac{(x^2+1)(x+3)^{1/2}}{x-1}\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right)$$

The Derivative and Integral of e^x

The natural exponential function is differentiable because it is the inverse of a differentiable function whose derivative is never zero.

$$\ln\left(e^x\right) = x \qquad \textit{Inverse relationship}$$

$$\frac{d}{dx}\ln\left(e^x\right) = 1 \qquad \textit{Differentiate both sides.}$$

$$\frac{1}{e^x}\frac{d}{dx}\left(e^x\right) = 1 \qquad \frac{d}{dx}\ln u = \frac{1}{u}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}e^x = e^x$$

If u is any differentiable function of x, then $\boxed{\frac{d}{dx}e^u = e^u\frac{du}{dx}}$ $\boxed{\left(e^u\right)' = u'e^u}$

Example

Find the derivative of $\frac{d}{dx}\left(5e^x\right)$

Solution

$$\frac{d}{dx}\left(5e^x\right) = 5\frac{d}{dx}e^x$$

$$= 5e^x$$

Example

Find the derivative of $\frac{d}{dx}\left(e^{\sin x}\right)$

Solution

$$\frac{d}{dx}\left(e^{\sin x}\right) = e^{\sin x}\frac{d}{dx}(\sin x)$$

$$= e^{\sin x}\cdot\cos x$$

Example

Find the derivative of $\frac{d}{dx}\left(e^{\sqrt{3x+1}}\right)$

Solution

$$\frac{d}{dx}\left(e^{\sqrt{3x+1}}\right) = e^{\sqrt{3x+1}} \frac{d}{dx}\left(\sqrt{3x+1}\right)$$

$$= e^{\sqrt{3x+1}} \cdot \tfrac{1}{2}(3x+1)^{-1/2} \cdot 3$$

$$= \boxed{\frac{3}{2\sqrt{3x+1}} e^{\sqrt{3x+1}}}$$

Definition

For any numbers $a > 0$ and x, the ***exponential function with base a*** is

$$a^x = e^{x \ln a}$$

When $a = e$, the function gives $a^x = e^{x \ln a} = e^{x \ln e} = e^x$

Power Rule – *Definition*

For any $x > 0$ and for any real number n, $\quad x^n = e^{n \ln x}$

General Power Rule for Derivatives

For any $x > 0$ and for any real number n, $\quad \frac{d}{dx} x^n = nx^{n-1}$

Proof

$$\frac{d}{dx} x^n = \frac{d}{dx} e^{n \ln x}$$

$$= e^{n \ln x} \frac{d}{dx}(n \ln x)$$

$$= x^n \cdot \frac{n}{x}$$

$$= nx^{n-1}$$

Example

Differentiate $f(x) = x^x, \quad x > 0$

Solution

$$f'(x) = \frac{d}{dx}\left(e^{x \ln x}\right)$$

$$= e^{x \ln x} \frac{d}{dx}(x \ln x)$$

$$= e^{x\ln x}\frac{d}{dx}(x\ln x)$$

$$= e^{x\ln x}\left(\ln x + x\cdot\frac{1}{x}\right)$$

$$= x^x(\ln x + 1) \quad x > 0$$

Theorem – The Number *e* as a Limit

The number e can be calculated as the limit $\quad e = \lim\limits_{x\to 0}(1+x)^{1/x}$

Proof

If $f(x) = \ln x \quad \to f'(x) = \frac{1}{x} \quad so \;\; f'(1) = 1$

$$f'(1) = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h}$$

$$= \lim_{x\to 0}\frac{f(1+x) - f(1)}{x}$$

$$= \lim_{x\to 0}\frac{\ln(1+x) - \ln(1)}{x}$$

$$= \lim_{x\to 0}\left[\frac{1}{x}\ln(1+x)\right]$$

$$= \lim_{x\to 0}\ln(1+x)^{1/x}$$

$$= \ln\left[\lim_{x\to 0}(1+x)^{1/x}\right] \qquad f'(1) = 1$$

$$= 1$$

$$\lim_{x\to 0}(1+x)^{1/x} = e$$

Definition

If $a > 0$ and u is a differentiable of x, then a^u is a differentiable function of x and

$$\boxed{\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}}$$

Example

- $\frac{d}{dx}3^x = 3^x \ln 3$
- $\frac{d}{dx}3^{-x} = 3^{-x} \ln 3 \frac{d}{dx}(-x)$

  $= -3^{-x} \ln 3$
- $\frac{d}{dx}3^{\sin x} = 3^{\sin x} \ln 3 \frac{d}{dx}(\sin x)$

  $= 3^{\sin x} \ln 3(\cos x)$

Logarithms with base *a*

For any positive number $a \neq 1$, $\log_a x$ is the inverse function of a^x

Inverse Equations for $\log_a x$ and a^x

$$a^{\log_a x} = x \quad (\forall x > 0)$$

$$\log_a \left(a^x\right) = x \quad (all\ x)$$

$$\log_a x = \frac{\ln x}{\ln a}$$

Derivative $\boxed{\frac{d}{dx}\left(\log_a u\right) = \frac{1}{u}\cdot\frac{1}{\ln a}\frac{du}{dx}}$

Example

- $\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{(3x+1)\cdot \ln 10}\frac{d}{dx}(3x+1)$

  $= \frac{3}{(3x+1)\cdot \ln 10}$

Exercises *Section* 2.8 – Derivatives of Logarithmic & Exponential Functions

(1 – 72) Find the derivative

1. $y = \ln\sqrt{x+5}$
2. $y = (3x+7)\ln(2x-1)$
3. $f(x) = \ln\sqrt[3]{x+1}$
4. $f(x) = \ln\left[x^2\sqrt{x^2+1}\right]$
5. $y = \ln\dfrac{x^2}{x^2+1}$
6. $y = \ln\left[\dfrac{x^2(x+1)^3}{(x+3)^{1/2}}\right]$
7. $y = \ln(x^2+1)$
8. $f(x) = \ln\left(x^2-4\right)$
9. $f(x) = 2\ln\left(x^2-3x+4\right)$
10. $f(x) = 3\ln\left(1+x^2\right)$
11. $f(x) = (1+\ln x)^3$
12. $f(x) = (x-2\ln x)^4$
13. $f(x) = x^2\ln x$
14. $f(x) = -\dfrac{\ln x}{x^2}$
15. $y = \ln\left(t^2\right)$
16. $y = \ln(2\theta+2)$
17. $y = (\ln x)^3$
18. $y = x(\ln x)^2$
19. $y = \dfrac{x^4}{4}\ln x - \dfrac{x^4}{16}$
20. $y = \dfrac{1+\ln t}{t}$
21. $f(x) = \dfrac{\ln x}{1+x}$
22. $f(x) = \dfrac{2x}{1+\ln x}$
23. $f(x) = x^3\ln x$
24. $f(x) = 6x^4\ln x$
25. $f(x) = \ln x^8$
26. $f(x) = 5x - \ln x^5$
27. $f(x) = \ln x^{10} + 2\ln x$
28. $f(x) = \dfrac{\ln x}{2x+5}$
29. $f(x) = -2\ln x + x^2 - 4$
30. $y = \ln\left(\dfrac{1}{x\sqrt{x+1}}\right)$
31. $y = \ln(\ln(\ln x))$
32. $y = \ln(\sec(\ln x))$
33. $y = \ln\left(\dfrac{\left(x^2+1\right)^5}{\sqrt{1-x}}\right)$
34. $y = \ln\sqrt{\dfrac{(x+1)^5}{(x+2)^{20}}}$
35. $f(x) = e^{3x}$
36. $f(x) = e^{-2x^3}$
37. $f(x) = 4e^{x^2}$
38. $f(x) = 2x^3e^x$
39. $f(x) = \dfrac{3e^x}{1+e^x}$
40. $f(x) = 5e^x + 3x + 1$
41. $f(x) = x^2e^x$
42. $f(x) = \dfrac{e^x+e^{-x}}{2}$
43. $f(x) = \dfrac{e^x}{x^2}$
44. $f(x) = x^2e^x - e^x$
45. $f(x) = (1+2x)e^{4x}$
46. $y = x^2e^{5x}$
47. $y = x^2e^{-2x}$
48. $f(x) = \dfrac{e^x}{x^2+1}$
49. $f(x) = \dfrac{1-e^x}{1+e^x}$
50. $y = \dfrac{e^x+e^{-x}}{x}$
51. $y = \sqrt{e^{2x^2}+e^{-2x^2}}$
52. $y = \dfrac{x}{e^{2x}}$
53. $y = 3\,e^{5x^3+1}$

54. $f(x)=\left(x^2-2x+2\right)e^x$

55. $f(\theta)=e^{\theta}(\sin\theta+\cos\theta)$

56. $f(\theta)=\ln\left(3\theta e^{-\theta}\right)$

57. $f(\theta)=\theta^3 e^{-2\theta}\cos 5\theta$

58. $f(\theta)=\ln\left(\dfrac{\sqrt{\theta}}{1+\sqrt{\theta}}\right)$

59. $f(t)=e^{(\cos t+\ln t)}$

60. $y=e^{\sin t}\left(\ln t^2+1\right)$

61. $y=e^{x^2}\ln x$

62. $f(x)=e^x+x-\ln x$

63. $f(x)=\ln x+2e^x-3x^2$

64. $f(x)=\ln x^2+4e^x$

65. $y=\ln\dfrac{1+e^x}{1-e^x}$

66. $y=\dfrac{\ln x}{e^{2x}}$

67. $f(x)=e^{2x}\ln\left(xe^x+1\right)$

68. $f(x)=\dfrac{xe^x}{\ln\left(x^2+1\right)}$

69. $f(x)=xe^{-10x}$

70. $f(x)=x\ln^2 x$

71. $f(x)=e^{-x}\ln x$

72. $f(x)=2^{x^2-x}$

(73 – 77) Use logarithmic differentiation to find the derivative of

73. $y=\sqrt{x(x+1)}$

74. $y=\sqrt{\left(x^2+1\right)(x-1)^2}$

75. $y=\sqrt{\dfrac{1}{t(t+1)}}$

76. $y=\dfrac{\theta+5}{\theta\cos\theta}$

77. $y=\sqrt[3]{\dfrac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}}$

(78 – 87) Find the derivative of

78. $y=t^{1-e}$

79. $y=2^{\sin 3t}$

80. $y=\log_3(1+\theta\ln 3)$

81. $y=\log_{25}e^x-\log_5\sqrt{x}$

82. $y=\log_3 r\cdot\log_9 r$

83. $y=\log_7\left(\dfrac{\sin\theta\cos\theta}{e^{\theta}2^{\theta}}\right)$

84. $y=3\log_8\left(\log_2 t\right)$

85. $y=t\log_3\left(e^{(\sin t)(\ln 3)}\right)$

86. $f(x)=2^{x^2-x}$

87. $f(x)=\log_3(x+8)$

(88 – 92) Use logarithmic differentiation to find the derivative of

88. $y=(x+1)^x$

89. $y=x^2+x^{2x}$

90. $y=(\sin x)^x$

91. $y=x^{\sin x}$

92. $y=(\ln x)^{\ln x}$

93. Find the second derivative of $y = 3\,e^{5x^3+1}$

94. Find the equation of the tangent line to $f(x) = e^x$ at the point (0, 1)

95. Find the equation of the tangent line to $f(x) = e^x$ at the point (1, *e*)

96. Find the equation of the tangent lines to $f(x) = 4e^{-8x}$ at the points (0, 4)

97. Find the equation of the tangent line to $y = 4xe^{-x} + 5$ at $x = 1$

98. The following formula accurately models the relationship between the size of a certain type of tumor and the amount of time that it has been growing:

$$V(t) = 450\left(1 - e - 0.0022t\right)^3$$

where *t* is in months and $V(t)$ is measured in cubic centimeters. Calculate the rate of change of tumor volume at 80 *months*.

99. A yeast culture at room temperature (68° F) is placed in a refrigerator set at a constant temperature of 38° F. After *t* hours, the temperature *T* of the culture is given approximately by

$$T = 30e^{-0.58t} + 38 \quad t \geq 0$$

What is the rate of change of temperature of the culture at the end of 1 *hour*? At the end of 4 hours?

100. A mathematical model for the average age of a group of people learning to type is given by

$$N(t) = 10 + 6\ln t \quad t \geq 1$$

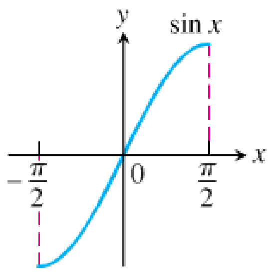
Where $N(t)$ is the number of words per minute typed after *t hours* of instruction and practice (2 hours per day, 5 days per week). What is the rate of learning after 10 *hours* of instruction and practice? After 100 *hours*?

101. The population of coyotes in the northwestern portion of Alabama is given by the formula $P(t) = \left(t^2 + 100\right)\ln\left(t + 2\right)$, where *t* represents the time in years since 2000 (the year 2000 corresponds to $(t = 0)$ Find the rate of change of the coyote population in 2013 $(t = 13)$.

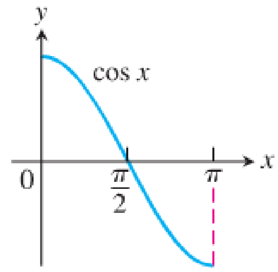
Section 2.9 – Derivatives of Inverse Trigonometric Functions

Defining the inverses

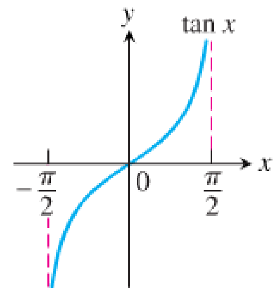
The six basic trigonometric functions are not one-to-one.



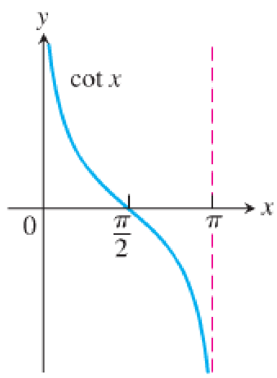
$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



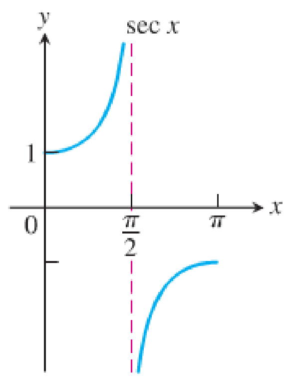
$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$



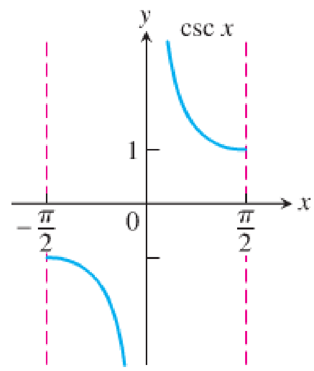
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$



$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$



$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

Since these restricted functions are now one-to-one, they have inverses, which we denoted by

$$y = \sin^{-1} x \quad or \quad y = \arcsin x$$

$$y = \cos^{-1} x \quad or \quad y = \arccos x$$

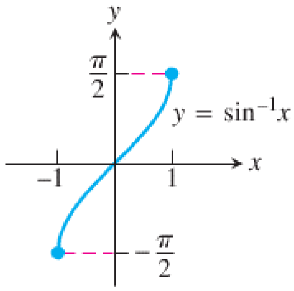
$$y = \tan^{-1} x \quad or \quad y = \arctan x$$

$$y = \cot^{-1} x \quad or \quad y = \operatorname{arc}\cot x$$

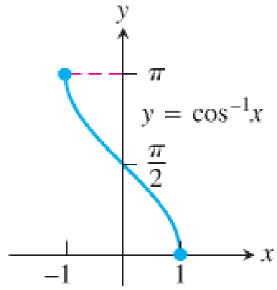
$$y = \sec^{-1} x \quad or \quad y = \operatorname{arc}\sec x$$

$$y = \csc^{-1} x \quad or \quad y = \operatorname{arc}\csc x$$

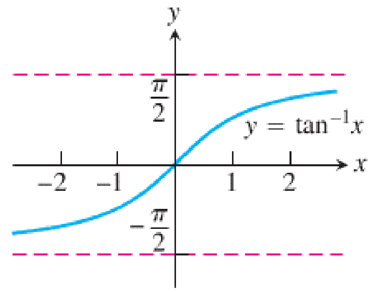
Domain: $-1 \le x \le 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



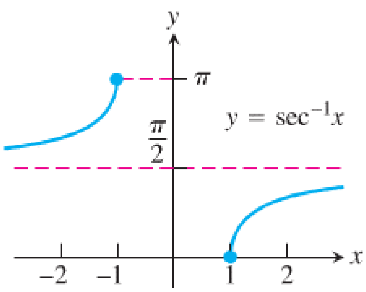
Domain: $-1 \le x \le 1$
Range: $0 \le y \le \pi$



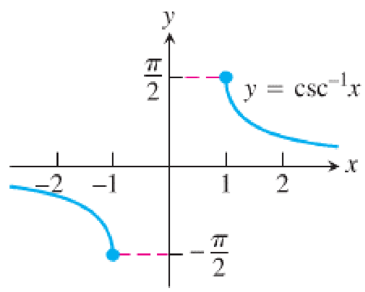
Domain: $-\infty < x < \infty$
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



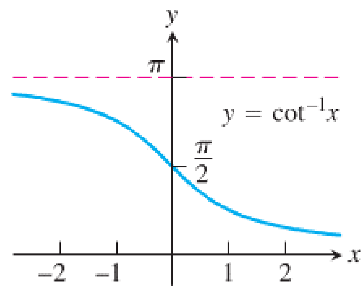
Domain: $x \le -1$ or $x \ge 1$
Range: $0 \le y \le \pi, y \ne \frac{\pi}{2}$



Domain: $x \le -1$ or $x \ge 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$



Domain: $-\infty < x < \infty$
Range: $0 < y < \pi$



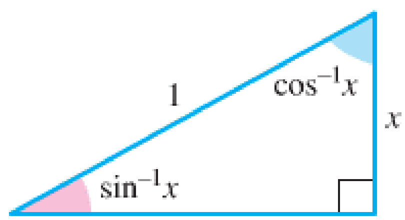
Definitions

✓ $y = \sin^{-1} x$ is the number in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for which $\sin y = x$

✓ $y = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$

✓ $y = \tan^{-1} x$ is the number in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which $\tan y = x$

✓ $y = \cot^{-1} x$ is the number in $(0, \pi)$ for which $\cot y = x$



Example $\quad \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

Inverse Function – Inverse Cofunction Identities

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$

$$\csc^{-1} x = \frac{\pi}{2} - \sec^{-1} x$$

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

$$\cot^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{x}\right)$$

$$\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{x}\right)$$

$$\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1}(x)$$

Derivative of $y=\sin^{-1}u$

$$\left(f^{-1}\right)'(x)=\frac{1}{f'\left(f^{-1}(x)\right)}$$

$$=\frac{1}{\cos\left(\sin^{-1}x\right)}$$

$$=\frac{1}{\sqrt{1-\sin^2\left(\sin^{-1}x\right)}}$$

$$=\frac{1}{\sqrt{1-x^2}}$$

$$\boxed{\frac{d}{dx}\sin^{-1}u=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}},\quad |u|<1 \qquad \boxed{\frac{d}{dx}\sin^{-1}u=\frac{u'}{\sqrt{1-u^2}}}$$

$\frac{d}{dx}\sin^{-1}u=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx},\quad \lvert u\rvert<1$	$\left(\sin^{-1}u\right)'=\frac{u'}{\sqrt{1-u^2}}$
$\frac{d}{dx}\cos^{-1}u=-\frac{1}{\sqrt{1-u^2}}\frac{du}{dx},\quad \lvert u\rvert<1$	$\left(\cos^{-1}u\right)'=-\frac{u'}{\sqrt{1-u^2}}$
$\frac{d}{dx}\tan^{-1}u=\frac{1}{1+u^2}\frac{du}{dx}$	$\left(\tan^{-1}u\right)'=\frac{u'}{1+u^2}$
$\frac{d}{dx}\cot^{-1}u=-\frac{1}{1+u^2}\frac{du}{dx}$	$\left(\cot^{-1}u\right)'=-\frac{u'}{1+u^2}$
$\frac{d}{dx}\sec^{-1}u=\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx},\quad \lvert u\rvert>1$	$\left(\sec^{-1}u\right)'=\frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$
$\frac{d}{dx}\csc^{-1}u=-\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx},\quad \lvert u\rvert>1$	$\left(\csc^{-1}u\right)'=-\frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$

Example

Find the derivative of $\frac{d}{dx}\left(\sin^{-1} x^2\right)$

Solution

$$\frac{d}{dx}\left(\sin^{-1} x^2\right) = \frac{1}{\sqrt{1-\left(x^2\right)^2}} \cdot \frac{d}{dx}\left(x^2\right)$$

$$= \frac{2x}{\sqrt{1-x^4}}$$

Example

Find the derivative of $\frac{d}{dx}\left(\sec^{-1} 5x^4\right)$

Solution

$$\frac{d}{dx}\left(\sec^{-1} 5x^4\right) = \frac{\left(5x^4\right)'}{5x^4\sqrt{\left(5x^4\right)^2-1}} \qquad 5x^4-1>0$$

$$= \frac{20x^3}{5x^4\sqrt{25x^8-1}}$$

$$= \frac{4}{x\sqrt{25x^8-1}}$$

Exercises *Section* 2.9 – Derivatives of Inverse Trigonometric Functions

(1 – 2) Find the value of

1. $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

2. $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

(3 – 5) Find the limit

3. $\lim\limits_{x\to -1^+} \cos^{-1} x$

4. $\lim\limits_{x\to -\infty} \tan^{-1} x$

5. $\lim\limits_{x\to \infty} \csc^{-1} x$

(6 – 17) Find the derivative

6. $y = \cos^{-1}\left(\frac{1}{x}\right)$

7. $y = \sin^{-1}\sqrt{2}t$

8. $y = \sec^{-1}(5s)$

9. $y = \cot^{-1}\sqrt{t-1}$

10. $y = \ln\left(\tan^{-1} x\right)$

11. $y = \tan^{-1}(\ln x)$

12. $y = \csc^{-1}\left(e^t\right)$

13. $y = x\sqrt{1-x^2} + \cos^{-1} x$

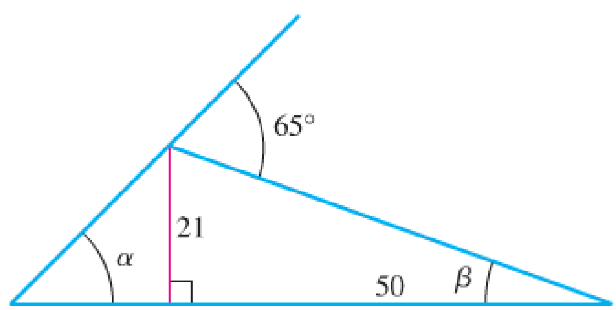
14. $y = \ln\left(x^2+4\right) - x\tan^{-1}\left(\frac{x}{2}\right)$

15. $f(x) = \sin^{-1}\frac{1}{x}$

16. $\left.\frac{d}{dx}\left(x\sec^{-1} x\right)\right|_{x=\frac{2}{\sqrt{3}}}$

17. $\left.\frac{d}{dx}\left(\tan^{-1} e^{-x}\right)\right|_{x=0}$

18. Find the angle α



Section 2.10 – Related Rates

The problem of finding a rate of change from other known rates of change is called a ***related rates problem***.

$$V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Example

Water runs into a conical tank at the rate of 9 $ft^3 / \min$. The tank stands point down and has a height of 10 *feet* and a base radius of 5 *feet*. How fast is the water level rising when the water is 6 *feet* deep?

Solution

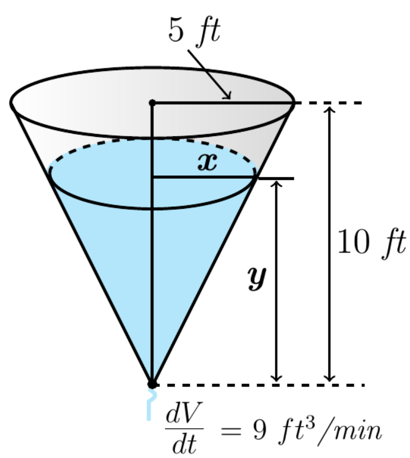
V = volume $\left(ft^3\right)$ of the water in the tank at time t (min)

x = radius (ft) of the surface of the water at time t

y = depth (ft) of the water in the tank at time t.

Given: $\frac{dV}{dt} = 9\frac{ft^3}{min}$ $\quad y = 6\,ft$

$h = 10\,ft \quad r = 5\,ft$



The water forms a cone with volume:

$$V = \frac{1}{3}\pi x^2 y$$

From the triangles:

$$\frac{x}{y} = \frac{5}{10} \quad \Rightarrow \quad x = \frac{y}{2}$$

$$V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y$$

$$= \frac{1}{3}\pi\left(\frac{y^2}{4}\right)y$$

$$= \frac{1}{12}\pi y^3$$

The derivative in function of time:

$$\frac{dV}{dt} = \frac{1}{12}\pi\left(3y^2\frac{dy}{dt}\right)$$

$$9 = \frac{\pi}{4}(6)^2\frac{dy}{dt}$$

$$\frac{(9)(4)}{36\pi} = \frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{1}{\pi} \approx 0.3183 \;\; ft / \min$$

The water level is rising at about 0.32 ***ft/min***.

Related Rates Problem Strategy

1. Draw a picture and name the variables and constants. (use *t* for time).
2. Write down the given information (numerical).
3. Write down what you are asked to find.
4. Write an equation that relates the variables.
5. Differentiate with respect to *t*.
6. Evaluate.

Example

A hot air balloon rising straight up from a level filed is tracked by a range finder 500 *feet* from the liftoff point. At the moment the range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of 0.14 *rad/min*. How fast is the balloon rising at that moment?

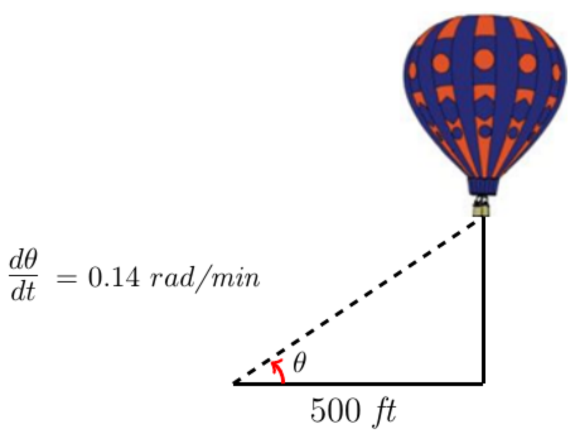
Solution

θ = the angle in *rad*.

y = the height in *feet* of the balloon

Given: $\frac{d\theta}{dt} = 0.14\ rad/\min \quad when \quad \theta = \frac{\pi}{4}$

$distance = 500\ ft$



$$\tan\theta = \frac{y}{500}$$

$$y = 500\tan\theta$$

$$\frac{dy}{dt} = 500\left(\sec^2\theta\right)\frac{d\theta}{dt}$$

$$= 500\left(\sec^2\frac{\pi}{4}\right)(0.14)$$

$$= 140$$

The balloon is rising at the rate of 140 *ft/min*.

Example

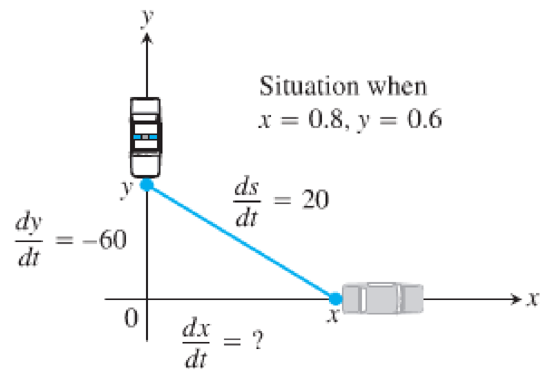
A police cruiser, approaching a right-angle intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 *mi* north of the intersection and the car is 0.8 *mi* to the east, the police determined with radar that the distance between them and the car is increasing at 20 *mph*. If the cruiser is moving at 60 *mph* at the instant of measurement, what is the speed of the car?

Solution

x = position of car at time t.

y = position of cruiser at time t.

s = distance between car and cruiser at time t.



Given: $x = 0.8\ mi \qquad y = 0.6\ mi$

$\frac{ds}{dt} = 20\ mph \qquad \frac{dy}{dt} = -60\ mph$

Using the Pythagorean Theorem to get the distance:

$$s^2 = x^2 + y^2$$

$$s = \sqrt{x^2 + y^2} = \sqrt{0.8^2 + 0.6^2} = 1$$

$$\frac{d}{dt}\left(s^2\right) = \frac{d}{dt}\left(x^2\right) + \frac{d}{dt}\left(y^2\right)$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$s\frac{ds}{dt} - y\frac{dy}{dt} = x\frac{dx}{dt}$$

$$x\frac{dx}{dt} = s\frac{ds}{dt} - y\frac{dy}{dt}$$

$$0.8\frac{dx}{dt} = (1)(20) - (0.6)(-60)$$

$$\frac{dx}{dt} = \frac{20 + 36}{0.8}$$

$$= 70\ \ \mathbf{mph}$$

The car’s speed is 70 mph.

Example

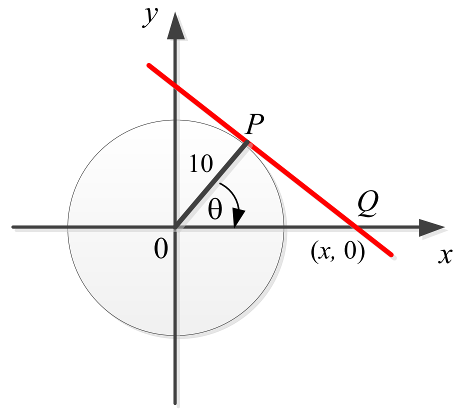
A particle moves clockwise at a constant rate along a circle of radius 10 *feet* centered at the origin. The particles initial position is (0, 10) on the *y*-axis and its final destination is the point (10, 0) on the *x*-axis. Once the particle is in motion, the tangent line at *P* intersects the *x*-axis at a point *Q* (which moves over time). If it takes the particle 30 *sec* to travel from start to finish, how fast is the point *Q* moving along the *x*-axis when it is 20 *feet* from the center of the circle?

Solution

Since the particle travels 30 *sec* from start to finish of angle $90° = \frac{\pi}{2}$, the particle is traveling along the circle at a constant rate of $\frac{\frac{\pi}{2}\ rad}{30\ \sec} \cdot \frac{60\ \sec}{1\ \min} = \pi\ rad / \min$

That implies $\frac{d\theta}{dt} = -\pi$

$$x = 20\ ft \quad and \quad \frac{d\theta}{dt} = -\pi rad / \min$$



$$\cos\theta = \frac{10}{x} \quad \Leftrightarrow \quad x = \frac{10}{\cos\theta} = 10\sec\theta$$

$$20 = 10\sec\theta \quad \Rightarrow \quad \sec\theta = 2$$

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt}(10\sec\theta) \\ &= 10\sec\theta\tan\theta\frac{d\theta}{dt} \\ &= 10\sec\theta\tan\theta(-\pi) \\ &= -10\pi\ \sec\theta\tan\theta \end{aligned}$$

$$20 = 10\sec\theta \quad \Rightarrow \quad \sec\theta = 2$$

$$\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{4-1} = \sqrt{3}$$

$$\begin{aligned} \frac{dx}{dt} &= -10\pi\ \sec\theta\tan\theta \\ &= -10\pi\ (2)\left(\sqrt{3}\right) \\ &= -20\pi\sqrt{3}\ \ ft / \min \end{aligned}$$

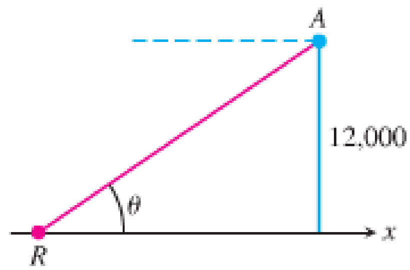
The point *Q* is moving towards the origin at the speed of $20\sqrt{3}\pi\ \approx 108.8\ ft / \min$

Example

A jet airliner is flying at a constant altitude of 12,000 *feet* above sea level as it approaches a Pacific island. The aircraft comes within the direct line of sight of a radar station located on the island, and the radar indicates the initial angle between sea level and its line of sight to the aircraft is 30°. How fast (in miles per hour) is the aircraft approaching the island when first detected by the radar instrument if it is turning upward (counterclockwise ***ccw***) at the rate of $\frac{2}{3}$ *deg / sec* in order to keep the aircraft within its direct line of sight?

Solution

From the triangle:



$$\tan\theta = \frac{12,000}{x}$$

$$x = \frac{12,000}{\tan\theta}$$

$$= \frac{12,000}{5,280}\cot\theta \quad mi$$

$$\frac{dx}{dt} = -\frac{12,000}{5,280}\csc^2\theta\frac{d\theta}{dt}$$

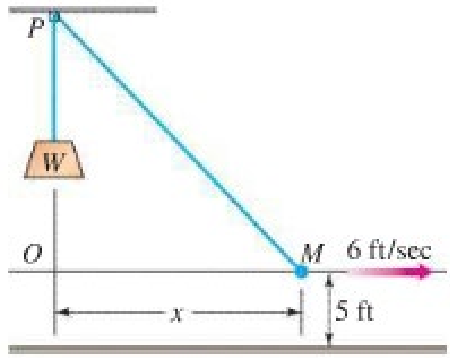
$$= -\frac{12,000}{5,280}\csc^2\left(\frac{\pi}{6}\right)\cdot\left(\frac{2}{3}\right)\frac{deg}{sec}\,\frac{\pi}{180^\circ}\,\frac{3600\ sec}{1\ hr}$$

$$\boxed{\approx -380\ \ mi/hr}$$

The negative appears because the distance x is decreasing, so the aircraft is approaching the island at a speed of approximately 380 ***mi/hr*** when first detected by the radar.

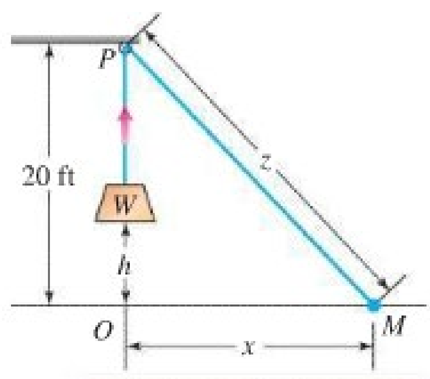
Example

A rope is running through a pulley at P and bearing a weight W at one end. The other end is held 5 *feet* above the ground in the hand M of a worker. Suppose the pulley is 25 *feet* above ground, the rope is 45 *feet* long, and the worker is walking rapidly away from the vertical line PW at the rate of 6 *ft/sec*. How fast is the weight being raised when the worker's hand is 21 *feet* away from PW?



Solution

Given: $when \;\; x = 21 \;\; \rightarrow \;\; \frac{dx}{dt} = 6$



$45 = z + 20 - h$

$z = 45 - 20 + h$

$\quad = 25 + h$

$z^2 = 20^2 + x^2$

$(25+h)^2 = 20^2 + x^2$

$\frac{d}{dt}\left((25+h)^2\right) = \frac{d}{dt}\left(20^2 + x^2\right)$

$2(25+h)\frac{dh}{dt} = 2x\frac{dx}{dt}$

$\frac{dh}{dt} = \frac{x}{25+h}\frac{dx}{dt}$

$when \;\; x = 21 \;\; \rightarrow \;\; (25+h)^2 = 20^2 + x^2 = 20^2 + 21^2 = 841$

$25 + h = \sqrt{841} = 29$

$\frac{dh}{dt} = \frac{21}{29}(6)$

$\quad \approx 4.3 \; ft\,/\,\text{sec}$

As the rate 4.3 ***ft/sec*** at which the weight is being raised when $x = 21$ *feet.*

Exercises *Section* 2.10 – Related Rates

1. If $y = x^2$ and $\frac{dx}{dt} = 3$, then what is $\frac{dy}{dt}$ *when* $x = -1$

2. If $x = y^3 - y$ and $\frac{dy}{dt} = 5$, then what is $\frac{dx}{dt}$ *when* $y = 2$

3. A cone-shaped icicle is dripping from the roof. The radius of the icicle is decreasing at a rate of 0.2 *cm* per *hour*, while the length is increasing at a rate of 0.8 *cm* per *hour*. If the icicle is currently 4 *cm* in radius and 20 *cm* long, is the volume of the icicle increasing or decreasing and at what rate?

4. A cube's surface area increases at the rate of $72\ in^2/\sec$. At what rate is the cube's volume changing when the edge length is $x = 3$ *in*?

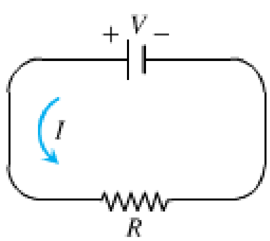
5. The radius r and height h of a right circular cone are related to the cone's volume V by the equation $V = \frac{1}{3}\pi r^2 h$.

a) How is $\frac{dV}{dt}$ related to $\frac{dh}{dt}$ if $\boldsymbol{r}$ is constant?

b) How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ if $\boldsymbol{h}$ is constant?

c) How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ and $\frac{dh}{dt}$ if neither $\boldsymbol{r}$ nor $\boldsymbol{h}$ is constant?

6. The voltage $\boldsymbol{V}$ (volts), current $\boldsymbol{I}$ (amperes), and resistance $\boldsymbol{R}$ (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 *volt/sec* while I is decreasing at the rate of $\frac{1}{3}\,amp/\sec$. Let t denote time in seconds.



a) What is the value of $\frac{dV}{dt}$?

b) What is the value of $\frac{dI}{dt}$?

c) What equation relates $\frac{dR}{dt}$ to $\frac{dV}{dt}$ and $\frac{dI}{dt}$?

d) Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amp. Is R increasing or decreasing?

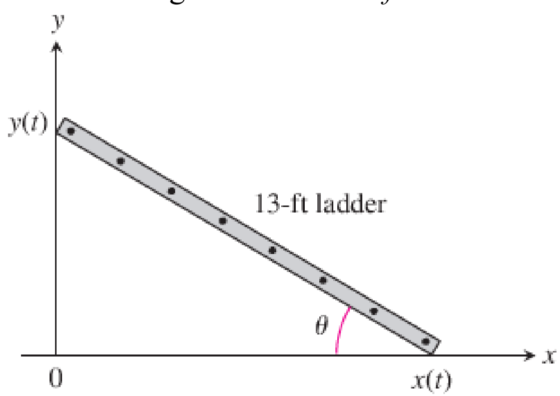
7. Let x and y be differentiable functions of t and let $s=\sqrt{x^2+y^2}$ be the distance between the points $(x, 0)$ and $(0, y)$ in the xy–plane.

a) How is $\frac{ds}{dt}$ related to $\frac{dx}{dt}$ if y is constant?

b) How is $\frac{ds}{dt}$ related to $\frac{dx}{dt}$ and $\frac{dy}{dt}$ if neither x nor y is constant?

c) How is $\frac{dx}{dt}$ related to $\frac{dy}{dt}$ if s is constant?

8. A 13-*ft* ladder is leaning against a house when its base starts to slide away. By the time the base is 12 *ft* from the house, the base is moving at the rate of 5 *ft/sec*.

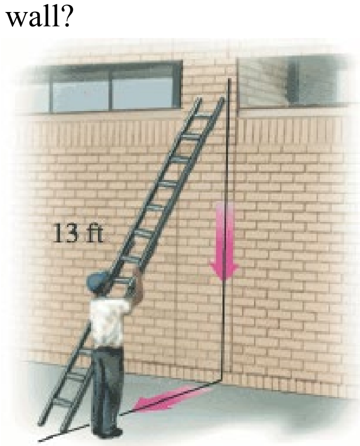


a) How fast is the top of the ladder sliding down the wall then?

b) At what rate is the area of the triangle formed by the ladder, wall, and the ground changing then?

c) At what rate is the angle θ between the ladder and the ground changing then?

9. A 13-*ft* ladder is leaning against a vertical wall when he begins pulling the foot of the ladder away from the wall at a rate of 0.5 *ft/s*. How fast is the top of the ladder sliding down the wall when the foot of the ladder is 5 *ft* from the wall?

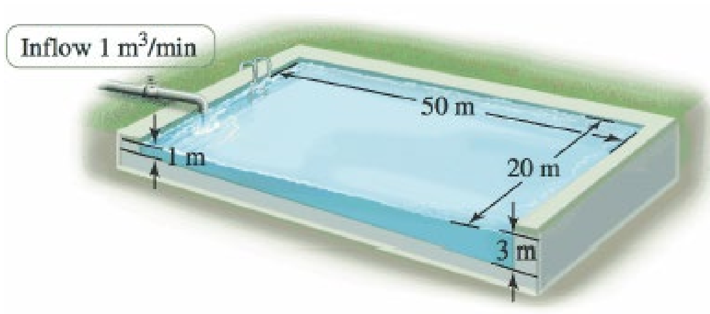


10. A 12-*ft* ladder is leaning against a vertical wall when he begins pulling the foot of the ladder away from the wall at a rate of 0.2 *ft/s*. What is the configuration of the ladder at the instant that the vertical speed of the top of the ladder equals the horizontal speed of the foot of the ladder?

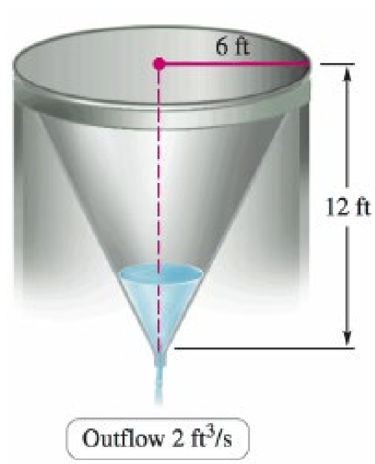
11. A swimming pool is 50 *m* long and 20 *m* wide. Its length decreases linearly along the length from 3 *m* to 1 *m*. It is initially empty and is filed at a rate of 1 $m^3/\min$.

a) How fast is the water level rising 250 *min* after the filling begins?

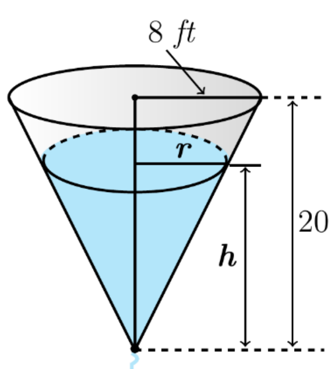
b) How long will it take to fill the pool?



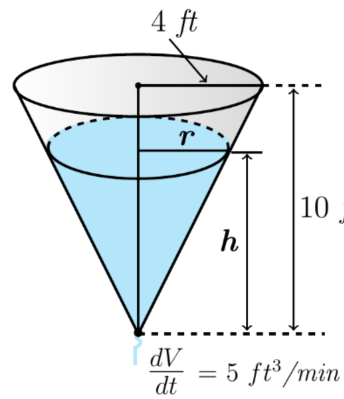
12. An inverted conical water tank with a height of 12 *feet* and a radius of 6 *feet* is drained through a hole in the vertex at a rate of 2 $ft^3/\sec$. What is the rate of change of the water depth when the water depth is 3 *feet*?



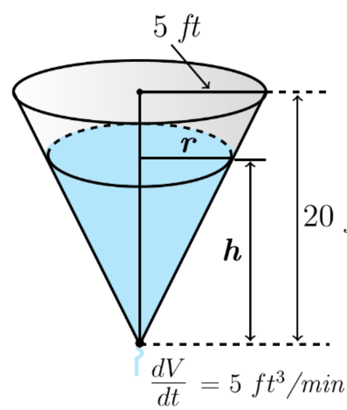
13. Water runs into a conical tank at the rate of $6\ ft^3/min$. The tank stands point down and has a height of 20 *feet* and a base radius of 8 *feet*. How fast is the water level rising when the water is 6 *feet* deep?



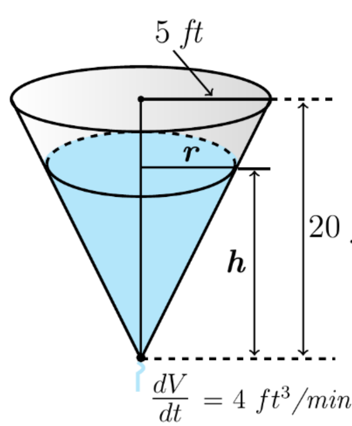
14. Water runs into a conical tank at the rate of $5\ ft^3/min$. The tank stands point down and has a height of 10 *feet* and a base radius of 4 *feet*. How fast is the water level rising when the water is 6 *feet* deep?



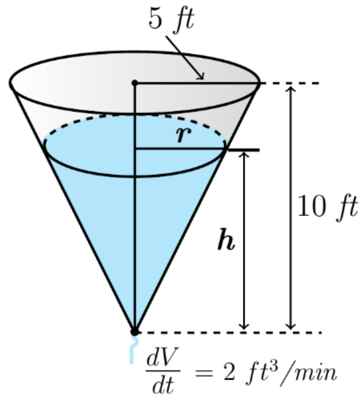
15. Water runs into a conical tank at the rate of $5\ ft^3/min$. The tank stands point down and has a height of 20 *feet* and a base radius of 5 *feet*. How fast is the water level rising when the water is 4 *feet* deep?



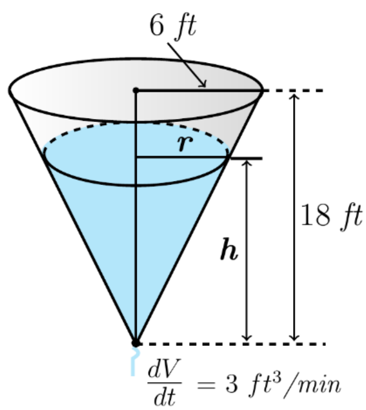
16. Water runs into a conical tank at the rate of $4\ ft^3/min$. The tank stands point down and has a height of 20 *feet* and a base radius of 5 *feet*. How fast is the water level rising when the water is 5 *feet* deep?



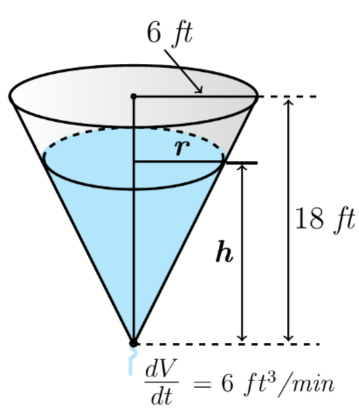
17. Water runs into a conical tank at the rate of $2\ ft^3/min$. The tank stands point down and has a height of 10 *feet* and a base radius of 5 *feet*. How fast is the water level rising when the water is 4 *feet* deep?



18. Water runs into a conical tank at the rate of $3\ ft^3/min$. The tank stands point down and has a height of 18 *feet* and a base radius of 6 *feet*. How fast is the water level rising when the water is 6 *feet* deep?



19. Water runs into a conical tank at the rate of $6\ ft^3/min$. The tank stands point down and has a height of 18 *feet* and a base radius of 6 *feet*. How fast is the water level rising when the water is 12 *feet* deep?

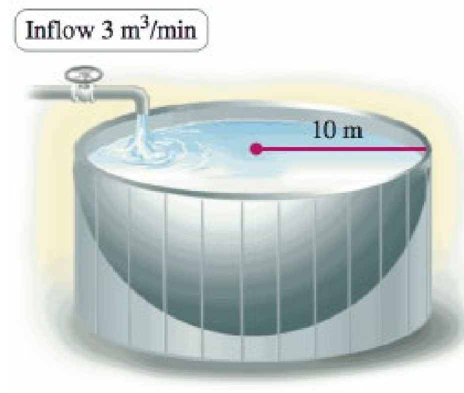


20. A hemispherical tank with a radius of 10 *m* is filled from an inflow pipe at a rate of 3 m^3 / min.

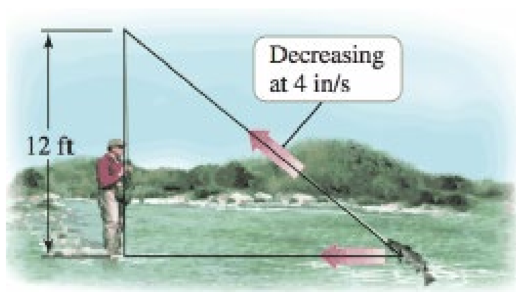
(Hint: The volume of a cap of thickness *h* sliced from a sphere of radius *r* is $\frac{\pi h^2(3r-h)}{3}$).

a) How fast is the water level rising when the water level is 5 *m* from the bottom of the tank?

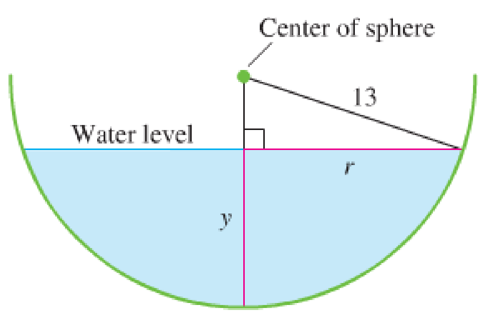
b) What is the rate of change of the surface area of the water when the water is 5 *m* deep?



21. A fisherman hooks a trout and reels in his line at 4 *in/sec*. Assume the trip of the fishing rod is 12 *ft* above the water directly above the fisherman and the fish is pulled horizontally directly towards the fisherman. Find the horizontal speed of the fish when it is 20 *ft* from the fisherman.



22. Water is flowing at the rate of 6 m^3 / min from a reservoir shaped like a hemispherical bowl of radius 13 *m*. Answer the following questions, given that the volume of water in a hemispherical bowl of radius *R* is $V = \frac{\pi}{3} y^2 (3R - y)$ when the water is y meters deep.



a) At what rate the water level changing when the water is 8 *m* deep?

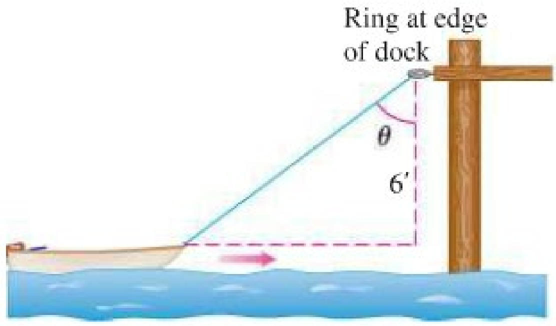
b) What is the radius *r* of the water's surface when the water is *y m* deep?

c) At what rate is the radius *r* changing when the water is 8 *m* deep?

23. A spherical balloon is inflated with helium at the rate of $100\pi\ ft^3 / \min$. How fast is the balloon's radius increasing at the instant the radius is 5 *feet*? How fast the surface area increasing?

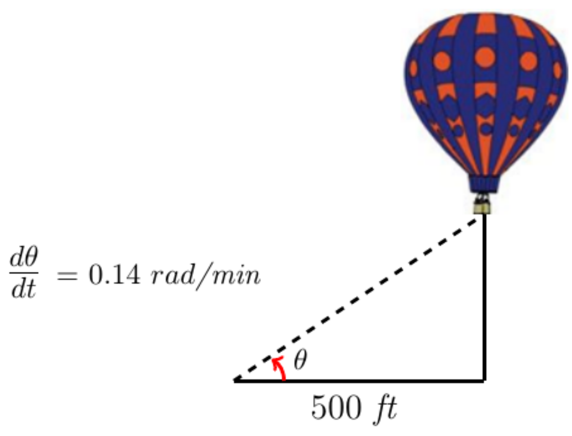
24. An observer stands 300 *feet* from the launch site of a hot-air balloon. The balloon is launched vertically and maintains a constant upward velocity of 20 *ft/sec*. what is the rate of change of the angle of elevation of the balloon when it is 400 *feet* from the ground? The angle of elevation is the angle θ between the observer's line of sight to the balloon and the ground.

25. A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 *feet* above the bow. The rope is hauled in at rate of 2 *ft/sec*.

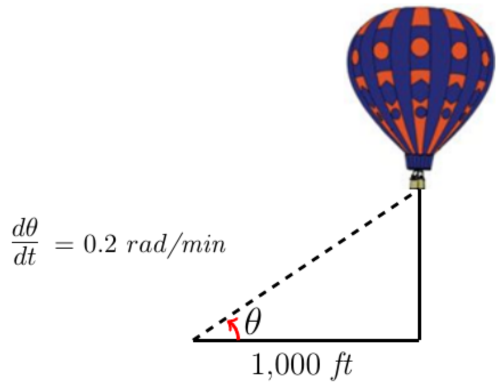


a) How fast is the boat approaching the dock when 10 *feet* of rope are out?
b) At what rate is the angle θ changing at this instant?

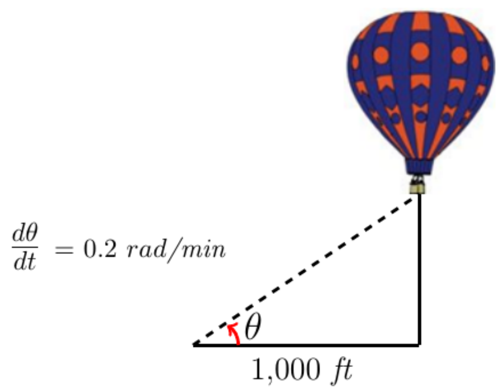
26. A hot air balloon rising straight up from a level filed is tracked by a range finder 500 *feet* from the liftoff point. At the moment the range finder's elevation angle is $\frac{\pi}{3}$, the angle is increasing at the rate of 0.14 *rad/min*. How fast is the balloon rising at that moment?



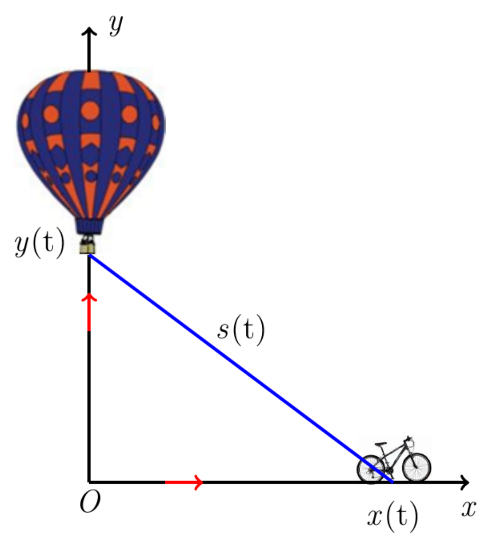
27. A hot air balloon rising straight up from a level filed is tracked by a range finder 1,000 *feet* from the liftoff point. At the moment the range finder's elevation angle is $\frac{\pi}{3}$, the angle is increasing at the rate of $0.2\ rad/min$. How fast is the balloon rising at that moment?



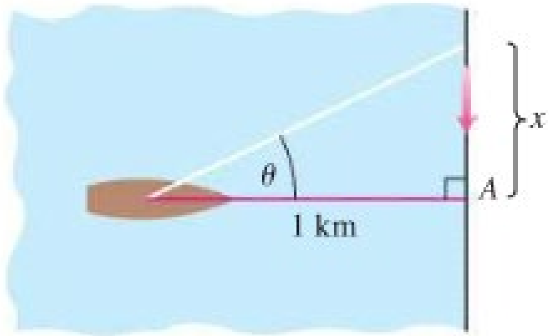
28. A hot air balloon rising straight up from a level filed is tracked by a range finder 1,000 *feet* from the liftoff point. At the moment the range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of $0.2\ \ rad/min$. How fast is the balloon rising at that moment?



29. A balloon rising vertically above a level, straight road at a constant rate of 1 *ft/sec*. Just when the balloon is 65 *feet* above the ground, a bicycle moving at a constant rate of 17 *ft/sec* passes under it. How fast is the distance $s(t)$ between the bicycle and the balloon increasing 3 *sec* later?



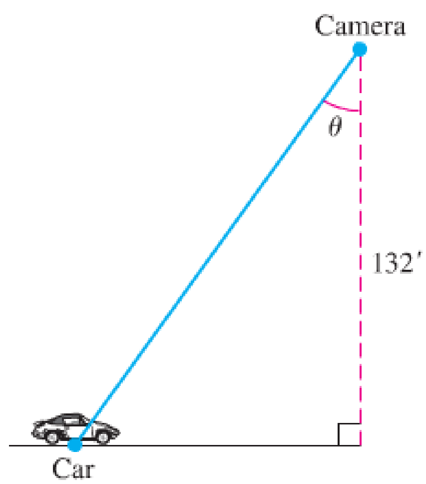
30. The figure shows a boat 1 *km* offshore, sweeping the shore with a searchlight. The light turns at a constant rate, $\frac{d\theta}{dt} = -0.6\ rad/\sec$.



a) How fast is the light moving along the shore when it reaches point *A*?

b) How many revolutions per minute is 0.3 rad/sec?

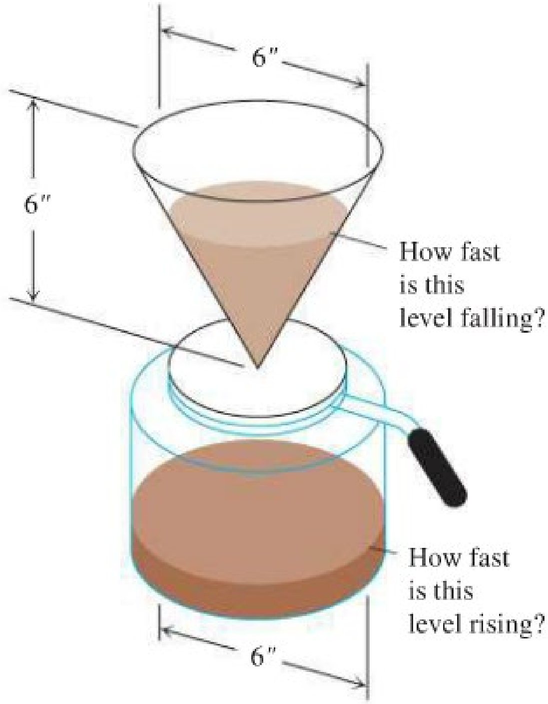
31. You are videotaping a race from a stand 132 *feet* from the track, following a car that is moving at 180 *mi/h* (264 *ft/sec*). How fast will your camera angle θ be changing when the car is right in front of you? A half second later?



32. The coordinates of a particle in the metric *xy*–plane are differentiable functions of time *t* with $\frac{dx}{dt} = -1\ m/\sec$ and $\frac{dy}{dt} = -5\ m/\sec$. How fast is the particle's distance from the origin changing as it passes through the point (5, 12)?

33. A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its *x*-coordinate (measure in meters) increases at a steady 10 *m/sec*. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = 3m$?

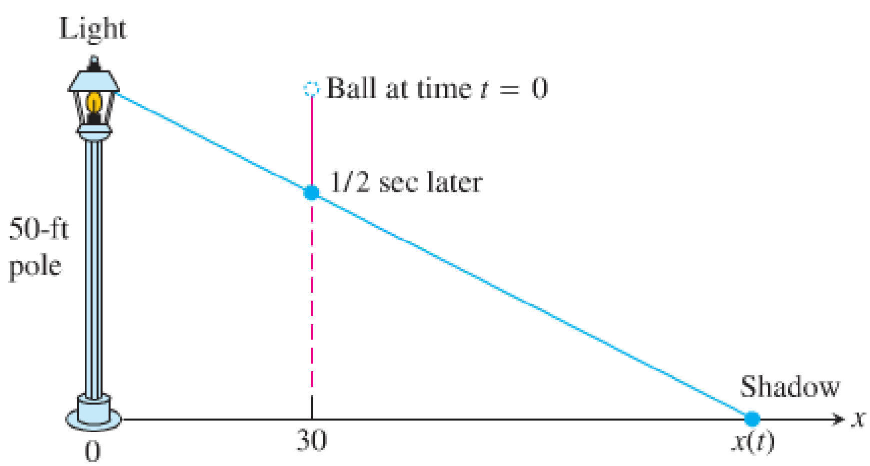
34. Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 in^3 / min .



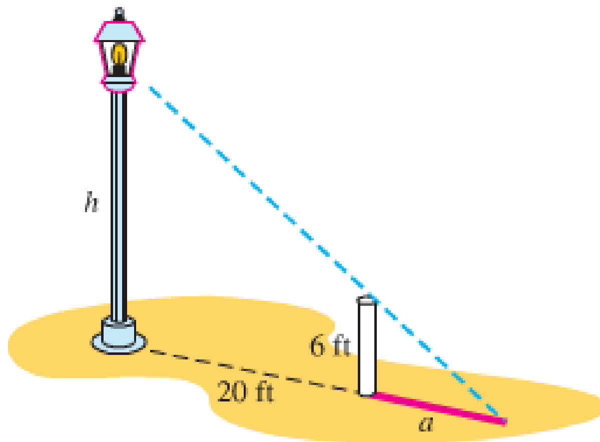
a) How fast is the level in the pot rising when the coffee in the cone is 5 *in*. deep?

b) How fast is the level in the cone falling then?

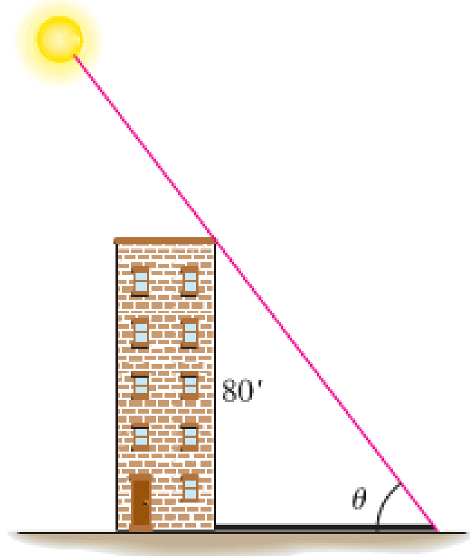
35. A light shines from the top of a pole 50 *feet* high. A ball is dropped from the same height from a point 30 *feet* away from the light. How fast is the shadow of the ball moving along the ground $\frac{1}{2}$ *sec* later? (Assume the ball falls a distance $s = 16t^2$ *foot* in *t* sec.)



36. To find the height of a lamppost, you stand a 6 *feet* pole 20 *feet* from the lamp and measure the length a of its shadow, finding it to be 15 *feet*, give or take an inch. Calculate the height of the lamppost using the value of $a = 15$ and estimate the possible error in the result.

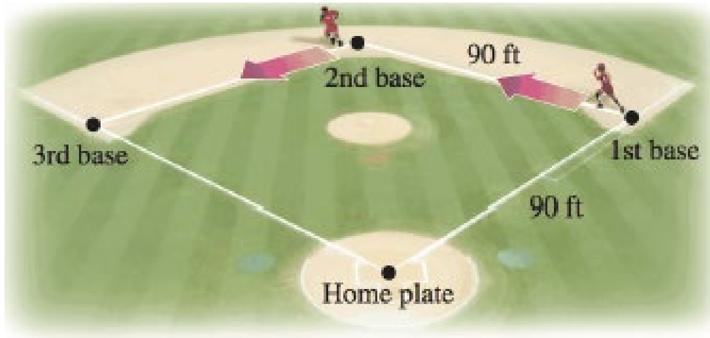


37. On a morning of a day when the sun will pass directly overhead, the shadow of an 80–*feet* building on level ground is 60 *feet* long. At the moment in question, the angle θ the sun makes with the ground is increasing at the rate of $0.27° / min$. At what rate is the shadow decreasing?



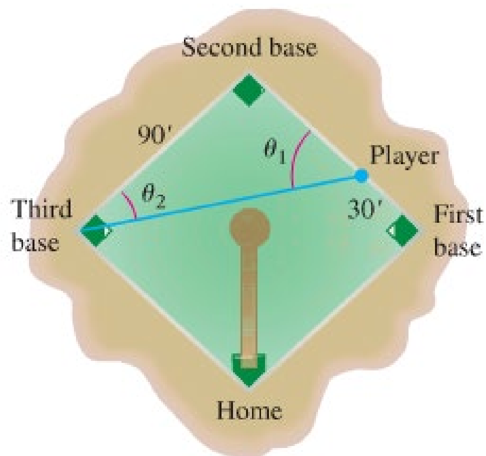
38. A spherical iron ball 8 *in*. in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the rate of $10\ in^3 / \text{min}$, how fast is the thickness of the ice decreasing when it is 2 *in*. thick? How fast is the outer surface area of ice decreasing?

39. Runners stand at first and second base in a baseball game. At the moment a ball is hit the runner at first base runs to second base at 18 *ft/s*; simultaneously the runner on second runs to third base at 20 *ft/s*. How fast is the distance between the runners changing 1 *sec* after the ball is hit?



(***Hint***: The distance between consecutive bases I 90 *feet* and the bases lie at the corners of a square.)

40. A baseball diamond is a square 90 *feet* on a side. A player runs from first base to second at a rate of 16 *ft/sec*.

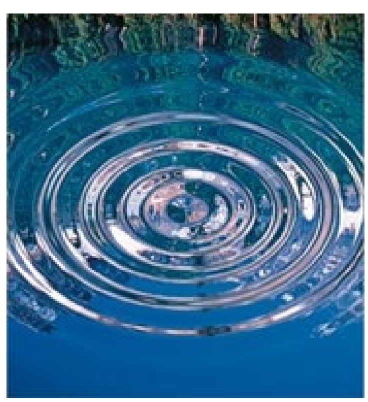


a) At what rate is the player's distance from third base changing when the player is 30 *ft* from first base?

b) At what rates are angles θ_1 *and* θ_2 changing at that time?

c) The player slides into second base at the rate of 15 *ft/sec*. At what rates are angles θ_1 *and* θ_2 changing as the player touches base?

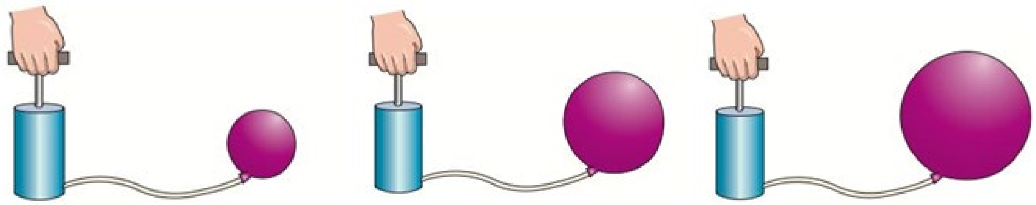
41. A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius *r* of the outer ripple is increasing at a constant rate of 1 *foot* per *second*.



When the radius is 4 *feet*, at what rate is the total area ***A*** of the disturbed water changing?

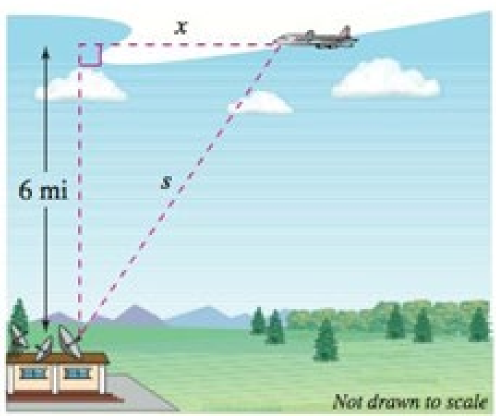
42. The variables x and y are both differentiable functions of t and are related by the equation $y = x^2 + 3$. Find $\frac{dy}{dt}$ when $x = 1$, given $\frac{dx}{dt} = 2$ when $x = 1$.

43. Air is being pumped into a spherical balloon at a rate of 4.5 ft^3 / min.

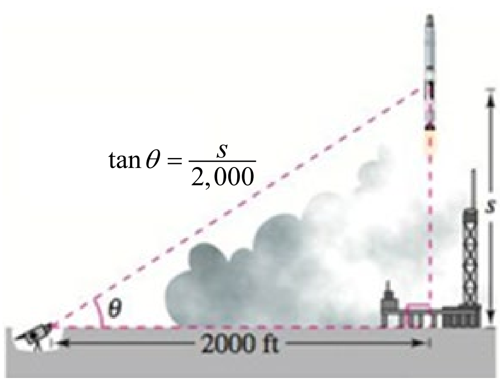


Find the rate of change of the radius when the radius is 2 *feet*.

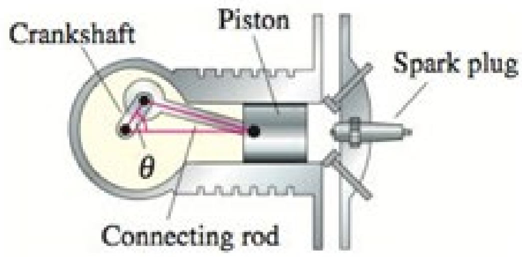
44. An Airplane is flying on a flight path that will take it directly over a radar tracking station. The distance s is decreasing at a rate of 400 *mph* when $s = 10$ mi. What is the speed of the plane?



45. Find the rate of change in the angle of elevation of the camera at 10 *seconds* after lift-off.

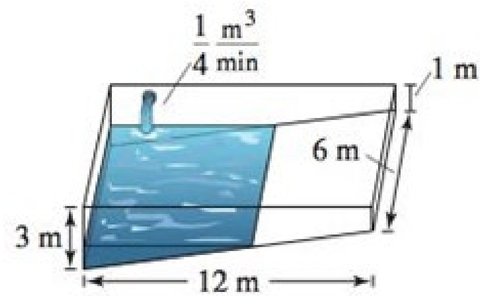


46. In the engine, a 7–inch connecting rod is fastened to a crank of radius 3 *inches*, the crankshaft rotates counterclockwise at a constant rate of 200 *revolutions per minute*.



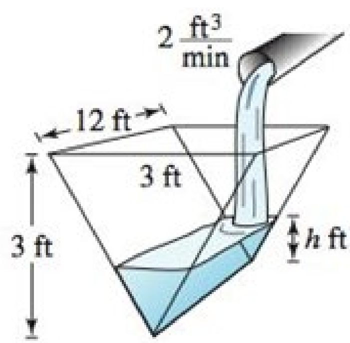
Find the velocity of the piston when $\theta = \frac{\pi}{3}$.

47. A swimming pool is 12 *meters* long, 6 *meters* wide, 1 *meter* deep at the shallow end, and 3 *meters* deep at the deep end. Water is being pumped into the pool at $\frac{1}{4}\ m^3/\text{min}$, and there is 1 *meter* of water at the deep end.



a) What percent of the pool is filled?
b) At what rate is the water level rising?

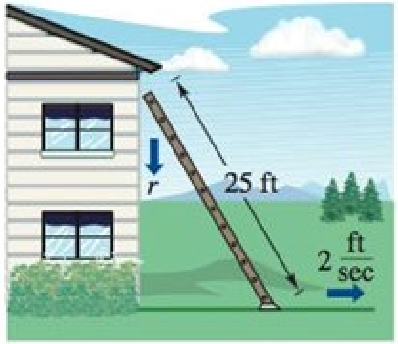
48. A trough is 12 *feet* long and 3 *feet* across the top. Its ends are isosceles triangles with altitudes of 3 *feet*.



a) Water is being pumped into the trough at $2\ ft^3/\min$. How fast is the water level rising when the depth h is 1 *foot*?

b) The water is rising at a rate of $\frac{3}{8}\ in/\min$ when $h = 2$. Determine the rate at which water is being pumped into the trough.

49. A ladder 25 *feet* long is leaning against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 2 *feet* per *second*.

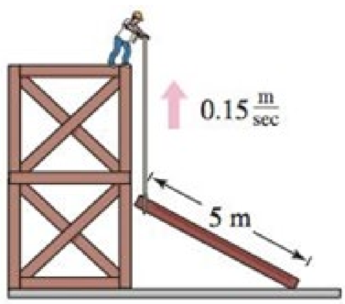


a) How fast is the top of the ladder moving down the wall when its base is 7 *feet*, 15 *feet*, and 24 *feet* from the wall?

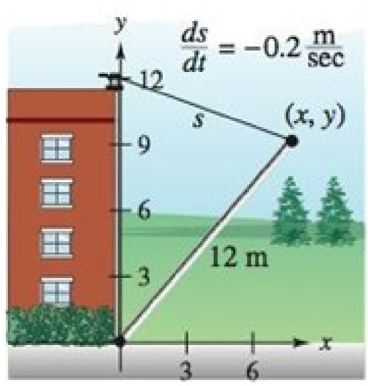
b) Consider the triangle formed by the side of the house, the ladder, and the ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 7 *feet* from the wall.

c) Find the rate at which the angle between the ladder and the wall of the house is changing when the base of the ladder is 7 *feet* from the wall.

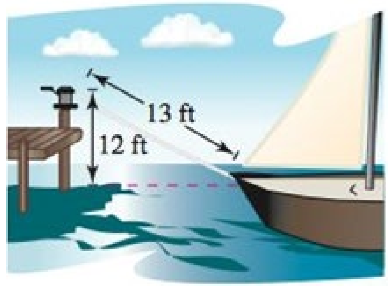
50. A construction worker pulls a five-*meter* plank up the side of a building under construction by means of a rope tied to one end of the plank. Assume the opposite end of the plank follows a path perpendicular to the wall of the building and the worker pulls the rope at a rate of $0.15\ m/\sec$. How fast is the end of the plank sliding along the ground when it is 2.5 *m* from the wall of the building?



51. A winch at the top of a 12-*meter* building pulls a pipe of the same length to a vertical position. The winch pulls in rope at a rate of $-0.2\ m/\sec$. Find the rate of vertical change and the rate of horizontal change at the end of the pipe when $y = 6$



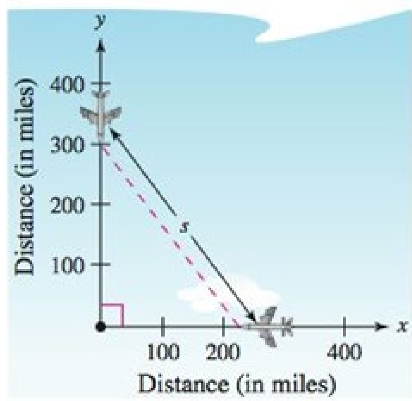
52. A boat is pulled into a dock by means of a winch 12 *feet* above the deck of the boat.



a) The winch pulls in rope at a rate of 4 *feet* per *second*. Determine the speed of the boat when there is 13 *feet* of rope out. What happens to the speed of the boat as it gets closer to the dock?

b) Suppose the boat is moving at a constant rate of 4 *feet* per *second*. Determine the speed at which the winch pulls in rope when there is a total of 13 *feet* of rope out. What happens to the speed at which the winch pulls in rope as the boat gets closer to the dock?

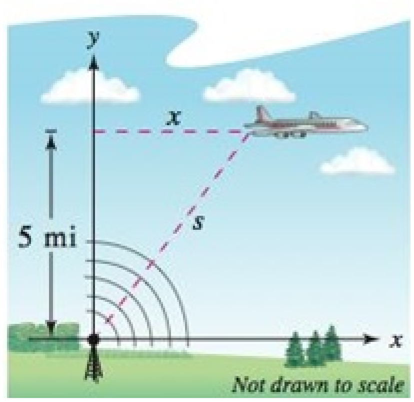
53. An air traffic controller spots two planes at the same altitude converging on a point as they fly at right angles to each other. One plane is 225 *miles* from the point moving at 450 *mph*. The other plane is 300 *miles* from the point moving at 600 *mph*.



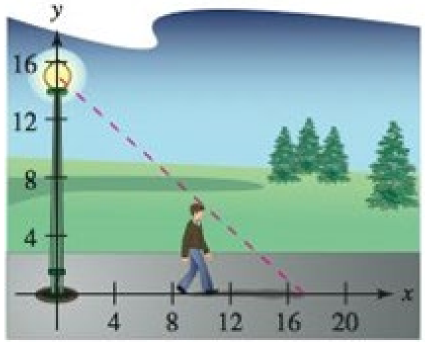
a) At what rate is the distance between the planes decreasing?

b) How much time does the air traffic controller have to get one of the planes on a different flight path?

54. An airplane is flying at an altitude of 5 *miles* and passes directly over a radar antenna. When the plane is 10 *miles* away $(s = 10)$, the radar detects that the distance s is changing at a rate of 240 *mph*. What is the speed of the plane?

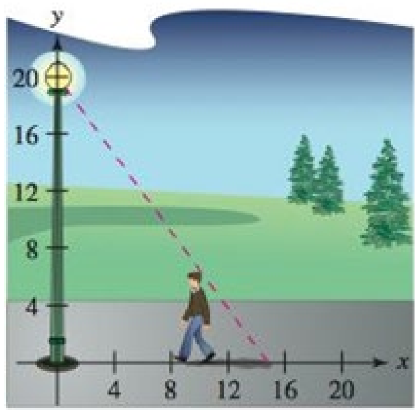


55. A man 6 *feet* tall walks at a rate of 5 *feet* per *second* away from a light that is 15 *feet* above the ground.



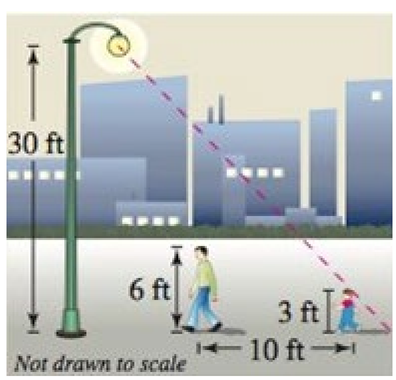
a) When he is 10 *feet* from the base of the light, at what rate is the tip of his shadow moving?
b) When he is 10 *feet* from the base of the light, at what rate is the length of his shadow changing?

56. A man 6 *feet* tall walks at a rate of 5 *feet* per *second toward* a light that is 20 *feet* above the ground.



a) When he is 10 *feet* from the base of the light, at what rate is the tip of his shadow moving?
b) When he is 10 *feet* from the base of the light, at what rate is the length of his shadow changing?

57. A man 6 *feet* tall walks at a rate of 5 $ft/\sec$ toward a streetlight that is 30 *feet* high. The man's 3-*foot*-tall child follows at the same speed, but 10 *feet* behind the man. At times, the shadow behind the child is caused by the man, and at other times, by the child.

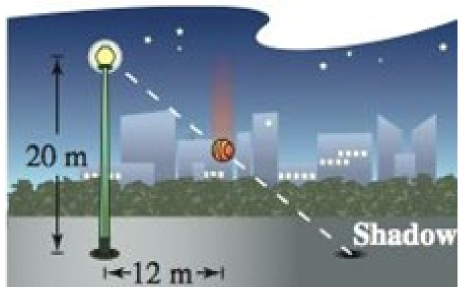


a) Suppose the man is 90 *feet* from the streetlight. Show that the man's shadow extends beyond the child's shadow.
b) Suppose the man is 60 *feet* from the streetlight. Show that the child's shadow extends beyond the man's shadow.

c) Determine the distance d from the man to the streetlight at which the tips of the two shadows are exactly the same distance from the streetlight.

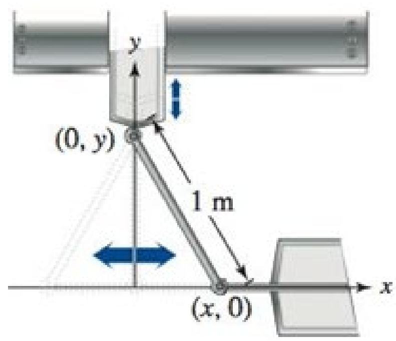
d) Determine how fast the tip of the man's shadow is moving as a function of x, the distance between the man and the streetlight. Discuss the continuity of this shadow speed function.

58. A ball is dropped from a height of 20 *m*, 12 *m* away from the top of a 20-meter lamppost. The ball's shadow, caused by the light at the top of the lamppost, is moving along the level ground. How fast is the shadow moving 1 *second* after the ball is released?



59. The endpoints of a movable rod of length 1 *meter* have coordinates $(x,\ 0)$ and $(0,\ y)$. The position of the end on the $x\text{-}axis$ is

$$x(t)=\frac{1}{2}\sin\frac{\pi t}{6} \qquad \text{where } t \text{ is the time in } seconds.$$

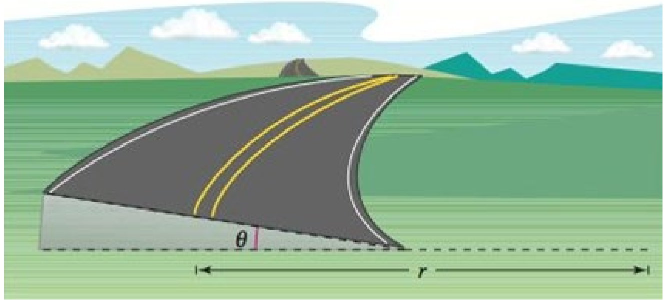


a) Find the time of one complete cycle of the rod.

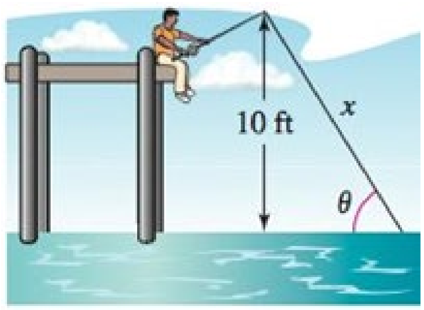
b) What is the lowest point reached by the end of the rod on the $y\text{-}axis$?

c) Find the speed of the $y\text{-}axis$ endpoint when the $x\text{-}axis$ endpoints is $\left(\frac{1}{4},\ 0\right)$

60. Cars on a certain roadway travel on a circular arc of radius r. in order not to rely on friction alone to overcome to centrifugal force, the road is banked at an angle of magnitude θ from the horizontal. The banking angle must satisfy the equation $rg\tan\theta = v^2$, where v is the velocity of the cars and $g=32\ ft/\sec^2$ is the acceleration due to gravity. Find the relationship between the related rates $\frac{dv}{dt}$ and $\frac{d\theta}{dt}$.

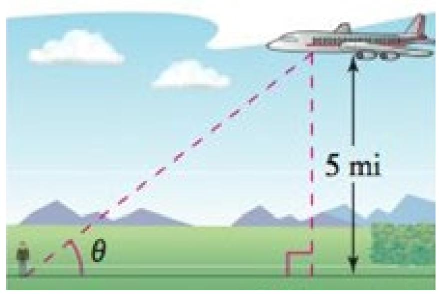


61. A fish is reeled in at a rate of 1 $ft/\sec$ from a point 10 *feet* above the water. At what rate is the angle θ between the line and the water changing when there is a total of 25 *feet* of line from the end of the rod to the water?



62. An airplane flies at an altitude of 5 *miles* toward a point directly over an observer. The speed of the plane is 600 mph. Find the rates at which the angle of elevation θ is changing when the angle is

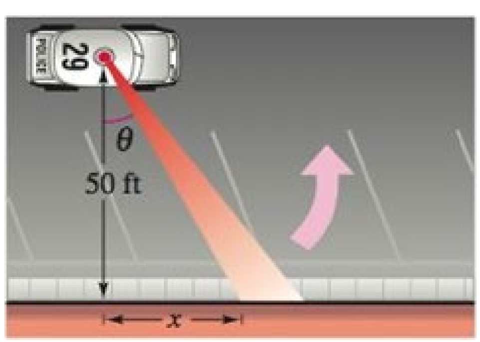
$a)\ \theta = 30° \quad b)\ \theta = 60° \quad c)\ \theta = 75°$



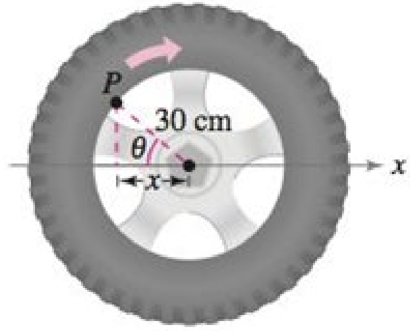
63. A patrol car is parked 50 feet from a long warehouse. The revolving light on top of the car turns at a rate of 30 revolutions per minute. How fast is the light beam moving along the wall when the beam makes angles of

$a)\ \theta = 30° \quad b)\ \theta = 60° \quad c)\ \theta = 70°$

With the perpendicular line from the light to the wall?



64. A wheel of radius 30 *cm* revolves at a rate of 10 *revolutions* per *second*. A dot is painted at a point *P* on the rim of the wheel.



a) Find $\frac{dx}{dt}$ as a function of θ.

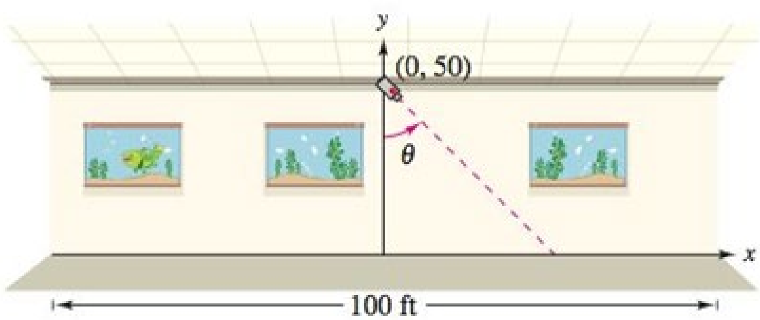
b) Graph the function.

c) When is the absolute value of the rate of change of *x* greatest?

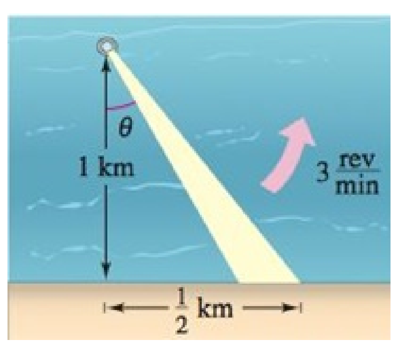
d) When is it least?

e) Find $\frac{dx}{dt}$ when $\theta = 30°$ and $\theta = 60°$

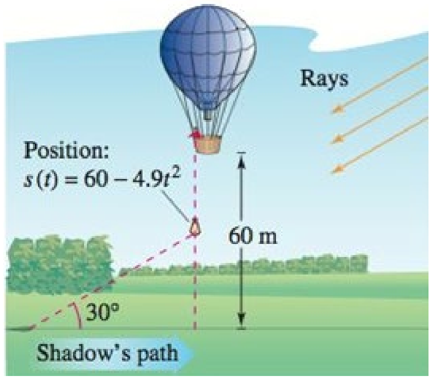
65. A security camera is centered 50 *feet* above a 100-*foot* hallway. It is easiest to design the camera with a constant angular rate of rotation, but this results in recording the images of the surveillance area at a variable rate. So, it is desirable to design a system with a variable rate of rotation and a constant rate of movement of the scanning beam along the hallway. Find a model for the variable rate of rotation when $\left|\frac{dx}{dt}\right| = 2$ $ft/\sec$



66. A rotating beacon is located 1 *km* off a straight shoreline. The beacon rotates at a rate of 3 $rev/\min$. How fast (in km/hr) does the beam of light appear to be moving to a viewer who is $\frac{1}{2}$ *km* down the shoreline?



67. A sandbag is dropped from a balloon at a height of 60 *m* when the angle of elevation to the sun is 30°. The position of the sandbag is $s(t) = 60 - 4.9t^2$



Find the rate at which the shadow of the sandbag is traveling along the ground when the sandbag is at height of 35 *m*.

68. The distance between the head of a piston and the end of a cylindrical chamber is given by $x(t) = \frac{8t}{t+1}$ *cm*, for $t \ge 0$ (measured in seconds). The radius of the cylinder is 4 *cm*.

a) Find the volume of the chamber, for $t \ge 0$.

b) Find the rate of change of the volume $V'(t)$ for $t \ge 0$.

c) Graph the derivative of the volume function. On what intervals is the volume increasing? Decreasing?

69. Two boats leave a dock at the same time. One boat travels south at 30 *mi/hr* and the other travels east at 40 *mi/hr*. after half an hour, how fast is the distance between the boats increasing?

70. A spherical balloon is inflated at a rate of 10 cm^3 / min. At what rate is the diameter of the balloon increasing when the balloon has a diameter of 5 *cm*.

71. A rope is attached to the bottom of a hot-air balloon that is floating above the flat field. If the angle of the rope to the ground remains 65° and the rope is pulled in at 5 *ft/s*, how quickly is the elevation of the balloon changing?

72. Water flows into a conical tank at a rate of 2 ft^3 / min. If the radius of the top of the tank is 4 *feet* and the height is 6 *feet*, determine how quickly the water level is rising when the water is 2 *feet* deep in the tank.

73. A jet flies horizontally 500 *feet* directly above a spectator at an air show at 450 *mi/hr*. How quickly is the angle of elevation (between the ground and the line from the spectator to the jet) changing 2 seconds later?

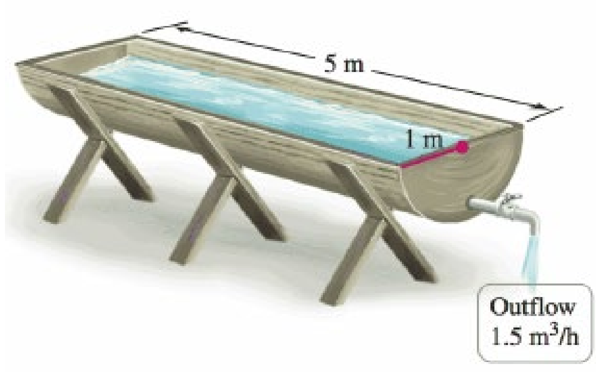
74. A man whose eyelevel is 6 *feet* above the ground walks toward a billboard at a rate of 2 *ft/s*. The bottom of the billboard is 10 *feet* above the ground, and it is 15 *feet* high. The man's viewing angle is the angle formed by the lines between the man's eyes and the top and bottom of the billboard. At what rate is the viewing angle changing when the man is 30 *feet* from the billboard?

75. A trough is shaped like a half cylinder with length 5 *m* and radius 1 *m*. The trough is full of water when a valve is opened and water flows out of the bottom of the trough at a rate of 1.5 m^3 / hr.

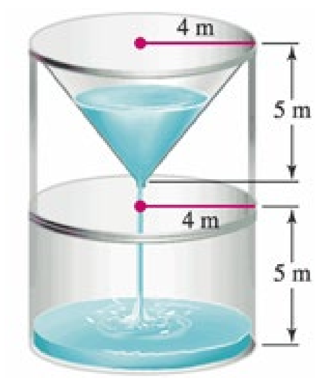
(***Hint***: Area of the sector = $\frac{1}{2}r^2\theta$, *r* is the radius of a sector of the circle subtended by an angle of θ)

a) How fast is the water level changing when the water level is 0.5 *m* from the bottom of the through?

b) What is the rate of change of the surface area of the water when the water is 0.5 *m* deep?

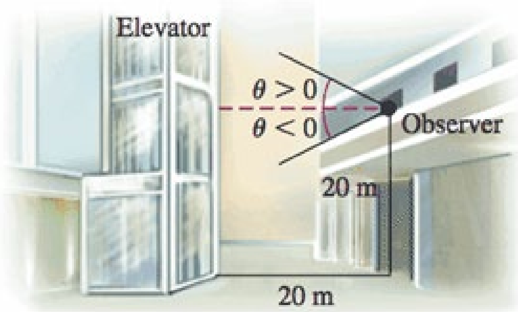


76. A conical tank with an upper radius of 4 *m* and a height of 5 *m* drains into a cylindrical tank with a radius of 4 *m* and a height of 5 *m*. If the water level in the conical tank drops at a rate of 0.5 *m/min*, at what rate does the water in the cylindrical tank rise when the water level in the conical tank is 3 *m*? 1 *m*?



77. Two cylindrical swimming pools are being filled simultaneously at the same rate (in m^3/min). The smaller pool has a radius of 5 *m*, and the water level rises at a rate of 0.5 *m/min*. The larger pool has a radius of 8 *m*. How fast is the water level rising in the larger pool?

78. An observer is 20 *m* above the ground floor of a large hotel atrium looking at a glass enclosed elevator shaft that is 20 *m* horizontally from the observer. The angle of elevation of the elevator is the angle that the observer's line of sight makes with the horizontal (it may be positive or negative).



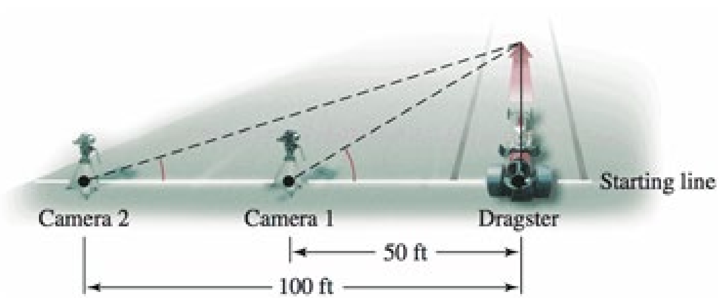
a) Assuming that the elevator rises at a rate of 5 *m/s*, what is the rate of change of the angle of the angle of elevation when the elevator is 10 *m* above the ground?

b) When the elevator is 40 *m* above the ground?

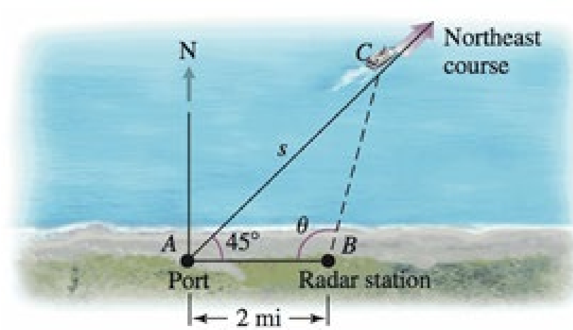
79. A camera is set up at the starting line of a drag race 50 *ft.* from a dragster at the starting line (camera 1). Two seconds after the start race, the dragster has traveled 100 *ft.* and the camera is turning at 0.75 *rad/s* while filming the dragster.

a) What is the speed of the dragster at this point?

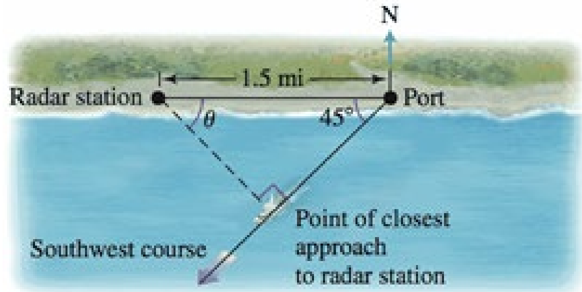
b) A second camera (camera 2) filming the dragster is located on the starting line 100 *ft.* away from the dragster at the start of the race. How fast is this camera turning 2 *sec* after the start of the race?



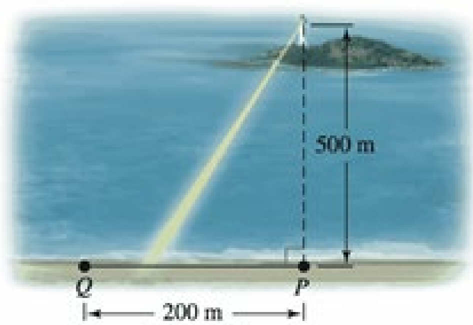
80. A port and a radar station are 2 *mi* apart on a straight shore running east and west. A ship leaves the port at noon traveling northeast at a rate of 15 *mi/hr*. If the ship maintains its speed and course, what is the rate of change of the tracking angle θ between the shore and the line between the radar station and the ship at 12:30 PM?



81. A ship leaves port traveling southwest at a rate of 12 *mi/hr*. At noon, the ship reaches its closest approach to a radar station, which is on the shore 1.5 *mi* from the port. If the ship maintains its speed and course, what is the rate of change of the tracking angle θ between the radar station and the ship at 1:30 PM?

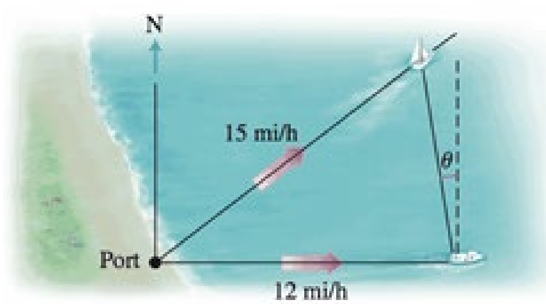


82. A lighthouse stands 500 *m* off of a straight shore, the focused beam of its light revolving four times each minute. *P* is the point on shore closest to the lighthouse and *Q* is a point on the shore 200 *m* from *P*.



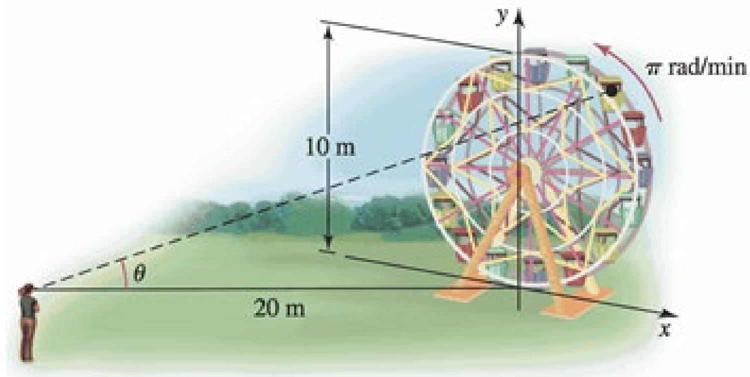
a) What is the speed of the beam along the shore when it strikes the point *Q*?
b) Describe how the speed of the beam along the shore varies with the distance between *P* and *Q*. (neglect the height of the lighthouse)

83. A boat leaves a port traveling due east at 12 *mi/hr*. At the same time, another boat leaves the same port traveling northeast at 15 *mi/hr*. The angle θ of the line between the boats is measured relative to due north. What is the rate of change of this angle 30 *min*. after the boats leave the port? 2 *hr*. after the boats leave the port?

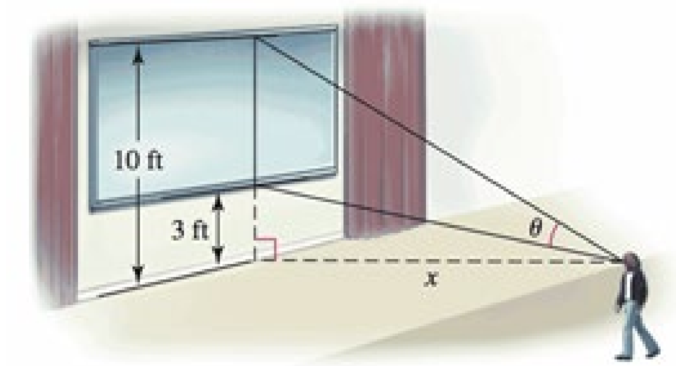


84. An observer stands 20 *m* from the bottom of a 10-*m* tall Ferris wheel on a line that is perpendicular to the face of the Ferris wheel. The wheel revolves at a rate of π *rad/min* and the observer's line of sight with a specific seat on the wheel makes an angle θ with the ground. 40 *seconds* after that seat leaves the lowest point on the wheel, what is the rate of change of θ?

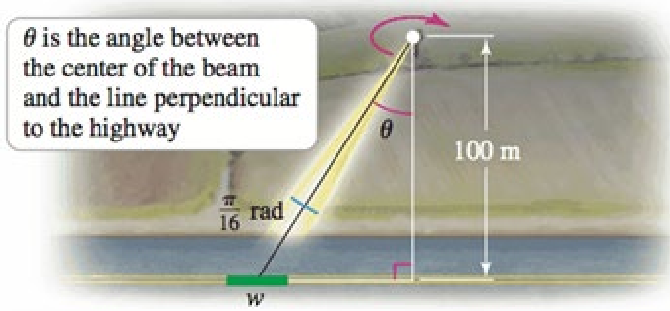
Assume the observer's eyes are level with the bottom of the wheel.



85. The bottom of a large theater screen is 3 *ft*. above your eye level and the top of the screen is 10 *ft*. above your eye level. Assume you walk away from the screen (perpendicular to the screen) at a rate of 3 *ft/s* while looking at the screen. What is the rate of change of the viewing angle $\boldsymbol{\theta}$ when you are 30 *ft*. from the wall on which the screen hangs, assuming the floor is flat?



86. A revolving searchlight, 100 *m* from the nearest point on the center line of a straight highway, casts a horizontal beam along a highway. The beam leaves the spotlight at an angle of $\frac{\pi}{16}$ *rad* and revolves at a rate $\frac{\pi}{16}$ *rad/s*. Let $\boldsymbol{w}$ be the width of the beam as it sweeps along the highway and $\boldsymbol{\theta}$ be the angle that the center of the beam makes with the perpendicular to the highway. What is the rate of change of $\boldsymbol{w}$ when $\theta = \frac{\pi}{3}$? *Neglect the height of the lighthouse.*



87. A piston is seated at the top of a cylindrical chamber with radius 5 *cm* when it starts moving into the chamber at a constant speed of 3 *cm/sec*. What is the rate of change of the volume of the cylinder when the piston is 2 *cm* from the base of the chamber?