2.4

149 $y'' + 4y' + 4y = 4 - t \quad y(0) = -1 \quad y'(0) = 0$

$\lambda^2 + 4\lambda + 4 = 0$

$\lambda_{1,2} = -2$

$y_h = (C_1 + C_2 t)e^{-2t}$

$y_p = a + bt$

$y' = b$

$y'' = 0$

$4b + 4a + 4bt = 4 - t$

$$\begin{cases} 4(a+b) = 4 \to a + b = 1 \Rightarrow a = \frac{5}{4} \\ 4b = -1 \to b = -\frac{1}{4} \end{cases}$$

$y_p = \frac{5}{4} - \frac{1}{4}t$

$y = (C_1 + C_2 t)e^{-2t} + \frac{5}{4} - \frac{1}{4}t$

$-1 = C_1 + \frac{5}{4} \Rightarrow C_1 = -\frac{9}{4}$

$y' = (C_2 - 2C_1 - 2C_2 t)e^{-2t} - \frac{1}{4}$

$0 = C_2 + \frac{9}{2} - \frac{1}{4} \Rightarrow C_2 = -\frac{17}{4}$

$y(t) = \left(-\frac{9}{4} - \frac{17}{4}t\right)e^{-2t} + \frac{5}{4} - \frac{1}{4}t$

~~Ex~~ #155 $y'' - 4y' - 12y = \sin 2t$

$y(0) = 0, \quad y'(0) = 0$

$\lambda^2 - 4\lambda - 12 = 0$

$\lambda_{1,2} = -2, 6$

$y_h = C_1 e^{-2t} + C_2 e^{6t}$

$-12 \quad y_p = a\cos 2t + b\sin 2t$

$-4) \quad y' = -2a\sin 2t + 2b\cos 2t$

$y'' = -4a\cos 2t - 4b\sin 2t$

$\cos 2t$	$\sin 2t$
$-12a - 8b$	$-12b + 8a$
$-4a$	$-4b$

$-16a - 8b = 0 \qquad -16b + 8a = 1$

$\begin{cases} 2a + b = 0 \\ 8a - 16b = 1 \end{cases} \rightarrow \quad -24a = 1$

$a = -\frac{1}{24}$

$b = \frac{1}{12}$

$y(t) = C_1 e^{-2t} + C_2 e^{6t} - \frac{1}{24}\cos 2t + \frac{1}{12}\sin 2t$

$0 = C_1 + C_2 - \frac{1}{24}$

$y' = -2C_1 e^{-2t} + 6C_2 e^{6t} + \frac{1}{12}\sin 2t + \frac{1}{6}\cos 2t$

$0 = -2C_1 + 6C_2 + \frac{1}{6}$

$\begin{cases} C_1 + C_2 = +\frac{1}{24} \\ -2C_1 + 6C_2 = \frac{1}{6} \end{cases} \quad 4C_2 = \frac{1}{12} \rightarrow C_2 = \frac{1}{48}$

$C_1 = \frac{1}{24} - \frac{1}{48} = \frac{1}{48}$

$y(t) = \frac{1}{48}e^{-2t} + \frac{1}{48}e^{6t} - \frac{1}{24}\cos 2t + \frac{1}{12}\sin 2t$

$$y^{(3)} - 3y'' + 3y' - y = 0$$

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

$$(\lambda - 1)^3 = 0$$

$$\lambda_{1,2,3} = 1$$

$$y(t) = (C_1 + C_2 t + C_3 t^2)e^t$$

$$y^{(3)} - 4y'' + y' + 6y = 0$$

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$$

$$\begin{array}{r|rrrr} & 1 & -4 & 1 & 6 \\ -1 & & -1 & 5 & -6 \\ \hline & 1 & -5 & 6 & \end{array}$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$\lambda = -1,\ 2,\ 3$$

$$y(t) = C_1 e^{-t} + C_2 e^{2t} + C_3 e^{3t}$$

2.5 $y'' + 4y = \sin 2t$

$$\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$$

$$y_h = C_1 \cos 2t + C_2 \sin 2t$$

$$y_1 = \cos 2t, \quad y_2 = \sin 2t$$

$$W = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix}$$

$$= \underline{2}$$

$$v_1 = -\int \frac{\sin^2 2t}{2}\,dt$$

$$= -\frac{1}{4}\int (1 - \cos 2t)\,dt$$

$$= -\frac{1}{4}\left(t - \frac{1}{2}\sin 2t\right)$$

$$v_2 = \frac{1}{2}\int \cos 2t \sin 2t\,dt$$

$$= \frac{1}{4}\int \sin 4t\,dt$$

$$= -\frac{1}{16}\cos 4t$$

$$y(t) = C_1 \cos 2t + C_2 \sin 2t + -\frac{1}{4}\left(t - \frac{1}{2}\sin 2t\right)\cos 2t$$
$$- \frac{1}{16}\cos 4t \sin 2t.$$

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$f(t) = e^{at}$$

$$F(s) = \int_0^\infty e^{at} e^{-st}\, dt$$

$$= \int_0^\infty e^{(a-s)t}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= -\frac{1}{s-a} e^{-(s-a)t} \Big|_0^\infty$$

$$= -\frac{1}{s-a} \left(\underset{0}{e^{-\infty}} - \underset{1}{e^{0}} \right)$$

$$= \boxed{\frac{1}{s-a}}$$

Ex $f(t) = t$.

		$\int e^{-st}$
$+$	t	$-\frac{1}{s}e^{-st}$
$-$	1	$\frac{1}{s^2}e^{-st}$

$$F(s) = \int_0^\infty t e^{-st}\,dt$$

$$= \left(-\frac{t}{s} - \frac{1}{s^2}\right) e^{-st} \Big|_0^\infty$$

$$= -\left(-\frac{1}{s^2}\right) e^{0}$$

$$= \frac{1}{s^2}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$f(t) = \sin at$

$$F(s) = \int_0^\infty \sin at \, e^{-st} \, dt$$

		$\int \sin at$
$+$	e^{-st}	$-\frac{1}{a}\cos at$
$-$	$-se^{-st}$	$\frac{1}{a^2}\sin at$
$+$	$s^2 e^{-st}$	

$$\int \sin at \, e^{-st} \, dt = \left(-\frac{1}{a}\cos at + \frac{s}{a^2}\sin at\right)e^{-st} + \frac{s^2}{a^2}\int e^{-st}\sin at \, dt$$

$$\frac{s^2+a^2}{a^2}\int \sin at \, e^{-st} \, dt = \frac{1}{a^2}\left(-a\cos at + s\sin at\right)e^{-st}$$

$$\int_0^\infty \sin at \, e^{-st} \, dt = \frac{1}{s^2+a^2}\left(-a\cos at + s\sin at\right)e^{-st}\Big|_0^\infty$$

$$= \frac{1}{s^2+a^2}\left(+a + 0\right)$$

$$= \frac{a}{s^2+a^2}$$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$