

Solution **Section 1.2 – Definitions / Techniques of Limits**

Exercise

Find the limit: $\lim_{x \rightarrow 3} (-1)$

Solution

$$\lim_{x \rightarrow 3} (-1) = \underline{-1}$$

Exercise

Find the limit: $\lim_{x \rightarrow -1} (3)$

Solution

$$\lim_{x \rightarrow -1} (3) = \underline{3}$$

Exercise

Find the limit: $\lim_{x \rightarrow 1000} 18\pi^2$

Solution

$$\lim_{x \rightarrow 1000} 18\pi^2 = \underline{18\pi^2}$$

Exercise

Find the limit: $\lim_{x \rightarrow 1} \sqrt{5x+6}$

Solution

$$\lim_{x \rightarrow 1} \sqrt{5x+6} = \underline{\sqrt{11}}$$

Exercise

Find the limit: $\lim_{x \rightarrow 9} \sqrt{x}$

Solution

$$\begin{aligned} \lim_{x \rightarrow 9} \sqrt{x} &= \sqrt{9} \\ &= \underline{3} \end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow -3} (x^2 + 3x)$

Solution

$$\begin{aligned}\lim_{x \rightarrow -3} (x^2 + 3x) &= (-3)^2 + 3(-3) \\ &= 9 - 9 \\ &= \underline{0}\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow -4} |x - 4|$

Solution

$$\begin{aligned}\lim_{x \rightarrow -4} |x - 4| &= |-4 - 4| \\ &= |-8| \\ &= \underline{8}\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 4} (x + 2)$

Solution

$$\begin{aligned}\lim_{x \rightarrow 4} (x + 2) &= 4 + 2 \\ &= \underline{6}\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 4} (x - 4)$

Solution

$$\begin{aligned}\lim_{x \rightarrow 4} (x - 4) &= 4 - 4 \\ &= \underline{0}\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 2} (5x - 6)^{3/2}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 2} (5x - 6)^{3/2} &= (10 - 6)^{3/2} \\ &= \sqrt{4^3} \\ &= 8\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 9} \frac{x - 9}{\sqrt{x} - 3}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 9} \frac{x - 9}{\sqrt{x} - 3} &= \frac{9 - 9}{3 - 3} = \frac{0}{0} \\ &= \lim_{x \rightarrow 9} \frac{(\sqrt{x} - 3)(\sqrt{x} + 3)}{\sqrt{x} - 3} \\ &= \lim_{x \rightarrow 9} (\sqrt{x} + 3) \\ &= 6\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 1} (2x + 4)$

Solution

$$\begin{aligned}\lim_{x \rightarrow 1} (2x + 4) &= 2(1) + 4 \\ &= 6\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 1} \frac{x^2 - 4}{x - 2}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^2 - 4}{x - 2} &= \frac{1^2 - 4}{1 - 2} \\ &= \frac{-3}{-1}\end{aligned}$$

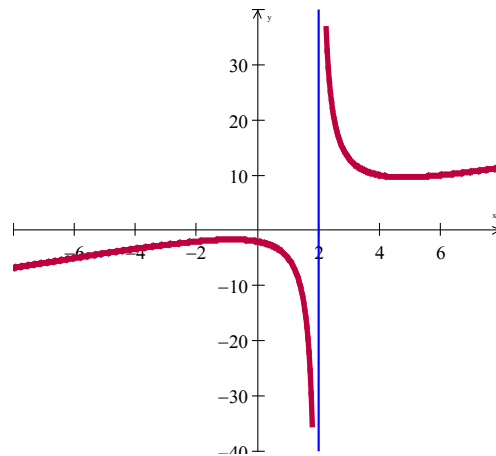
$$= 3$$

Exercise

Find the limit: $\lim_{x \rightarrow 2} \frac{x^2 + 4}{x - 2}$

Solution

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^2 + 4}{x - 2} &= \frac{2^2 + 4}{2 - 2} \\ &= \frac{8}{0} \\ &= \infty \end{aligned} \quad (\text{Doesn't exist})$$



Exercise

Find the limit: $\lim_{x \rightarrow 0} \frac{|x|}{x}$

Solution

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{|x|}{x} &= \frac{0}{0} \\ \lim_{x \rightarrow 0^-} \frac{|x|}{x} &= \frac{-x}{-x} = -1 \\ \lim_{x \rightarrow 0^+} \frac{|x|}{x} &= \frac{x}{x} = 1 \end{aligned}$$

Doesn't exist

Exercise

Find: $\lim_{x \rightarrow 3} \frac{x^2 - x - 1}{\sqrt{x} + 1}$

Solution

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{x^2 - x - 1}{\sqrt{x} + 1} &= \frac{3^2 - 3 - 1}{\sqrt{3} + 1} \\ &= \frac{5}{2} \end{aligned}$$

Exercise

Find: $\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2}$

Solution

$$\begin{aligned}
 \lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2} &= \frac{2^2 + 2 - 6}{2 - 2} = \frac{0}{0} \\
 &= \lim_{x \rightarrow 2} \frac{(x + 3)(x - 2)}{x - 2} \\
 &= \lim_{x \rightarrow 2} (x + 3) \\
 &= 5
 \end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 0} (3x - 2)$

Solution

$$\begin{aligned}
 \lim_{x \rightarrow 0} (3x - 2) &= 3(0) - 2 \\
 &= -2
 \end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 1} (2x^2 - x + 4)$

Solution

$$\begin{aligned}
 \lim_{x \rightarrow 1} (2x^2 - x + 4) &= 2(1)^2 - (1) + 4 \\
 &= 5
 \end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow -2} (x^3 - 2x^2 + 4x + 8)$

Solution

$$\begin{aligned}
 \lim_{x \rightarrow -2} (x^3 - 2x^2 + 4x + 8) &= (-2)^3 - 2(-2)^2 + 4(-2) + 8 \\
 &= -16
 \end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 2} \frac{x^2-4}{x-2} &= \frac{2^2-4}{2-2} = \frac{0}{0} \\
&= \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{x-2} \\
&= \lim_{x \rightarrow 2} (x+2) \\
&= 4
\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 2} \frac{x^3-8}{x-2}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 2} \frac{x^3-8}{x-2} &= \frac{0}{0} \\
&= \lim_{x \rightarrow 2} \frac{(x-2)(x^2+2x+4)}{x-2} \\
&= \lim_{x \rightarrow 2} x^2 + 2x + 4 \\
&= 2^2 + 2(2) + 4 \\
&= 12
\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 3} \frac{x^2+x-12}{x-3}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 3} \frac{x^2+x-12}{x-3} &= \frac{0}{0} \\
&= \lim_{x \rightarrow 3} \frac{(x-3)(x+4)}{x-3} \\
&= \lim_{x \rightarrow 3} (x+4) \\
&= 7
\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 0} \frac{\sqrt{x+4}-2}{x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{x+4}-2}{x} &= \frac{\sqrt{4}-2}{0} = \frac{0}{0} \\&= \lim_{x \rightarrow 0} \frac{\sqrt{x+4}-2}{x} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} \\&= \lim_{x \rightarrow 0} \frac{x+4-4}{x(\sqrt{x+4}+2)} \\&= \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{x+4}+2)} \\&= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+4}+2} \\&= \frac{1}{\sqrt{4}+2} \\&= \frac{1}{4} \quad \Big| \end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 0} \frac{3}{\sqrt{3x+1}+1}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{3}{\sqrt{3x+1}+1} &= \frac{3}{\sqrt{3(0)+1}+1} \\&= \frac{3}{1+1} \\&= \frac{3}{2} \quad \Big| \end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 0} f(x)$ $f(x) = \begin{cases} x^2+1 & x < 0 \\ 2x+1 & x > 0 \end{cases}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0^-} x^2+1 &= 1 \\ \lim_{x \rightarrow 0^+} 2x+1 &= 1\end{aligned}$$

$$\lim_{x \rightarrow 0} f(x) = 1$$

Exercise

Find the limit: $\lim_{x \rightarrow -2} \frac{5}{x+2}$

Solution

$$\lim_{x \rightarrow -2} \frac{5}{x+2} = \frac{5}{0}$$

$$= \infty$$

Exercise

Find the limit: $\lim_{x \rightarrow 3} \frac{\sqrt{x+1}-1}{x}$

Solution

$$\lim_{x \rightarrow 3} \frac{\sqrt{x+1}-1}{x} = \frac{\sqrt{3+1}-1}{3} = \frac{2-1}{3}$$

$$= \frac{1}{3}$$

Exercise

Find the limit: $\lim_{x \rightarrow 1} \frac{x^2-1}{x-1}$

Solution

$$\lim_{x \rightarrow 1} \frac{x^2-1}{x-1} = \frac{0}{0}$$

$$= \lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{x-1}$$

$$= \lim_{x \rightarrow 1} (x+1)$$

$$= 2$$

Exercise

Find the limit: $\lim_{x \rightarrow -2} \frac{|x+2|}{x+2}$

Solution

$$\lim_{x \rightarrow -2} \frac{|x+2|}{x+2} = \frac{|-2+2|}{-2+2} = \frac{0}{0}$$

$$\lim_{x \rightarrow -2^+} \frac{|x+2|}{x+2} = \frac{(x+2)}{(x+2)} = 1$$

$$\lim_{x \rightarrow -2^-} \frac{|x+2|}{x+2} = \frac{(x+2)}{-(x+2)} = -1$$

Doesn't exist

Exercise

Find the limit: $\lim_{x \rightarrow 0} (2x-8)^{1/3}$

Solution

$$\begin{aligned} \lim_{x \rightarrow 0} (2x-8)^{1/3} &= (2(0)-8)^{1/3} \\ &= (-8)^{1/3} \\ &= \underline{-2} \end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 2} \frac{x^2-7x+10}{x-2}$

Solution

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^2-7x+10}{x-2} &= \frac{2^2-7(2)+10}{2-2} = \frac{0}{0} \\ &= \lim_{x \rightarrow 2} \frac{(x-2)(x-5)}{x-2} \\ &= \lim_{x \rightarrow 2} (x-5) \\ &= 2-5 \\ &= \underline{-3} \end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 0} \frac{5x^3 + 8x^2}{3x^4 - 16x^2}$

Solution

$$\lim_{x \rightarrow 0} \frac{5x^3 + 8x^2}{3x^4 - 16x^2} = \frac{0}{0}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{5x^3 + 8x^2}{3x^4 - 16x^2} &= \lim_{x \rightarrow 0} \frac{x^2(5x + 8)}{x^2(3x^2 - 16)} \\ &= \lim_{x \rightarrow 0} \frac{5x + 8}{3x^2 - 16} \\ &= \frac{8}{-16} \\ &= -\frac{1}{2} \end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 1} \frac{\frac{1}{x} - 1}{x - 1}$

Solution

$$\lim_{x \rightarrow 1} \frac{\frac{1}{x} - 1}{x - 1} = \frac{1 - 1}{1 - 1} = \frac{0}{0}$$

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\frac{1}{x} - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{\frac{1 - x}{x}}{x - 1} \\ &= \lim_{x \rightarrow 1} \left(\frac{1 - x}{x} \right) \left(\frac{1}{x - 1} \right) \\ &= \lim_{x \rightarrow 1} \left(\frac{-(x - 1)}{x} \right) \left(\frac{1}{x - 1} \right) \\ &= \lim_{x \rightarrow 1} \frac{-1}{x} \\ &= -1 \end{aligned}$$

Exercise

Find the limit: $\lim_{u \rightarrow 1} \frac{u^4 - 1}{u^3 - 1}$

Solution

$$\begin{aligned}
\lim_{u \rightarrow 1} \frac{u^4 - 1}{u^3 - 1} &= \lim_{u \rightarrow 1} \frac{(u^2 - 1)(u^2 + 1)}{(u - 1)(u^2 + u + 1)} \\
&= \lim_{u \rightarrow 1} \frac{(u - 1)(u + 1)(u^2 + 1)}{(u - 1)(u^2 + u + 1)} \\
&= \lim_{u \rightarrow 1} \frac{(u + 1)(u^2 + 1)}{u^2 + u + 1} \\
&= \frac{(1 + 1)(1^2 + 1)}{1^2 + 1 + 1} \\
&= \frac{4}{3}
\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 1} \frac{x - 1}{\sqrt{x + 3} - 2}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 1} \frac{x - 1}{\sqrt{x + 3} - 2} &= \frac{1 - 1}{\sqrt{1 + 3} - 2} = \frac{0}{\sqrt{4} - 2} = \frac{0}{0} \\
&= \lim_{x \rightarrow 1} \frac{x - 1}{\sqrt{x + 3} - 2} \cdot \frac{\sqrt{x + 3} + 2}{\sqrt{x + 3} + 2} \\
&= \lim_{x \rightarrow 1} \frac{(x - 1)(\sqrt{x + 3} + 2)}{x + 3 - 4} \\
&= \lim_{x \rightarrow 1} \frac{(x - 1)(\sqrt{x + 3} + 2)}{x - 1} \\
&= \lim_{x \rightarrow 1} (\sqrt{x + 3} + 2) \\
&= \sqrt{1 + 3} + 2 \\
&= 4
\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow -1} \frac{\sqrt{x^2+8}-3}{x+1} &= \frac{\sqrt{(-1)^2+8}-3}{-1+1} = \frac{\sqrt{9}-3}{0} = \frac{0}{0} \\
&= \lim_{x \rightarrow -1} \frac{\sqrt{x^2+8}-3}{x+1} \cdot \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3} \\
&= \lim_{x \rightarrow -1} \frac{x^2+8-9}{(x+1)(\sqrt{x^2+8}+3)} \\
&= \lim_{x \rightarrow -1} \frac{x^2-1}{(x+1)(\sqrt{x^2+8}+3)} \\
&= \lim_{x \rightarrow -1} \frac{(x-1)(x+1)}{(x+1)(\sqrt{x^2+8}+3)} \\
&= \lim_{x \rightarrow -1} \frac{(x-1)}{\sqrt{x^2+8}+3} \\
&= \frac{-2}{\sqrt{9}+3} = \frac{-2}{6} \\
&= -\frac{1}{3}
\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow -3} \frac{2-\sqrt{x^2-5}}{x+3}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow -3} \frac{2-\sqrt{x^2-5}}{x+3} &= \frac{2-\sqrt{(-3)^2-5}}{-3+3} = \frac{2-\sqrt{9-5}}{0} = \frac{2-\sqrt{4}}{0} = \frac{0}{0} \\
&= \lim_{x \rightarrow -3} \frac{2-\sqrt{x^2-5}}{x+3} \cdot \frac{2+\sqrt{x^2-5}}{2+\sqrt{x^2-5}} \\
&= \lim_{x \rightarrow -3} \frac{4-(x^2-5)}{(x+3)(2+\sqrt{x^2-5})} \\
&= \lim_{x \rightarrow -3} \frac{4-x^2+5}{(x+3)(2+\sqrt{x^2-5})} \\
&= \lim_{x \rightarrow -3} \frac{9-x^2}{(x+3)(2+\sqrt{x^2-5})}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow -3} \frac{(x-3)(x+3)}{(x+3)\left(2 + \sqrt{x^2 - 5}\right)} \\
&= \lim_{x \rightarrow -3} \frac{(x-3)}{2 + \sqrt{x^2 - 5}} \\
&= \frac{-6}{2 + \sqrt{9 - 5}} \\
&= \frac{-6}{2 + \sqrt{4}} \\
&= -\frac{3}{2}
\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 0} (2 \sin x - 1)$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 0} (2 \sin x - 1) &= 2 \sin(0) - 1 \\
&= 0 - 1 \\
&= -1
\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 0} \sin^2 x$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 0} \sin^2 x &= \sin^2(0) \\
&= 0
\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 0} \sec x$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 0} \sec x &= \sec(0) \\
&= \frac{1}{\cos(0)} \\
&= 1
\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 0} \frac{1+x+\sin x}{3\cos x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{1+x+\sin x}{3\cos x} &= \frac{1+\textcolor{red}{0}+\sin(\textcolor{red}{0})}{3\cos(\textcolor{red}{0})} \\ &= \frac{1}{3}\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow -\pi} \sqrt{x+4} \cos(x+\pi)$

Solution

$$\begin{aligned}\lim_{x \rightarrow -\pi} \sqrt{x+4} \cos(x+\pi) &= \sqrt{\textcolor{red}{-}\pi+4} \cos(\textcolor{red}{-}\pi+\pi) \\ &= \sqrt{\textcolor{red}{-}\pi+4} \cos(0) \\ &= \sqrt{4-\pi}\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow -0.5^-} \sqrt{\frac{x+2}{x+1}}$

Solution

$$\begin{aligned}\lim_{x \rightarrow -0.5^-} \sqrt{\frac{x+2}{x+1}} &= \sqrt{\frac{\textcolor{red}{-}0.5+2}{\textcolor{red}{-}0.5+1}} \\ &= \sqrt{\frac{1.5}{0.5}} \\ &= \sqrt{3}\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 1^+} \sqrt{\frac{x-1}{x+2}}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 1^+} \sqrt{\frac{x-1}{x+2}} &= \sqrt{\frac{1-1}{1+2}} \\ &= 0\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow -2^+} \left(\frac{x}{x+1} \right) \left(\frac{2x+5}{x^2+x} \right)$

Solution

$$\begin{aligned} \lim_{x \rightarrow -2^+} \left(\frac{x}{x+1} \right) \left(\frac{2x+5}{x^2+x} \right) &= \left(\frac{-2}{-2+1} \right) \left(\frac{2(-2)+5}{(-2)^2+(-2)} \right) \\ &= \left(\frac{-2}{-1} \right) \left(\frac{1}{2} \right) \\ &= \underline{1} \end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 0^+} \frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}$

Solution

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x} &= \frac{\sqrt{5}-\sqrt{5}}{0} = \frac{0}{0} \\ &= \lim_{x \rightarrow 0^+} \frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x} \cdot \frac{\sqrt{x^2+4x+5}+\sqrt{5}}{\sqrt{x^2+4x+5}+\sqrt{5}} \\ &= \lim_{x \rightarrow 0^+} \frac{x^2+4x+5-5}{x(\sqrt{x^2+4x+5}+\sqrt{5})} \\ &= \lim_{x \rightarrow 0^+} \frac{x^2+4x}{x(\sqrt{x^2+4x+5}+\sqrt{5})} \\ &= \lim_{x \rightarrow 0^+} \frac{x(x+4)}{x(\sqrt{x^2+4x+5}+\sqrt{5})} \\ &= \lim_{x \rightarrow 0^+} \frac{x+4}{\sqrt{x^2+4x+5}+\sqrt{5}} \\ &= \frac{0+4}{\sqrt{0^2+4(0)+5}+\sqrt{5}} \\ &= \frac{4}{\sqrt{5}+\sqrt{5}} \\ &= \frac{4}{2\sqrt{5}} \\ &= \underline{\frac{2}{\sqrt{5}}} \end{aligned}$$

Exercise

Find $\lim_{x \rightarrow -2^+} (x+3) \frac{|x+2|}{x+2}$

Solution

$$\lim_{x \rightarrow -2^+} (x+3) \frac{|x+2|}{x+2} = (x+3) \frac{|-2+2|}{-2+2} = \frac{0}{0}$$

$$\text{Since } x \rightarrow -2^+ \Rightarrow x > -2$$

$$\Rightarrow |x+2| = (x+2)$$

$$\begin{aligned} \lim_{x \rightarrow -2^+} (x+3) \frac{|x+2|}{x+2} &= \lim_{x \rightarrow -2^+} (x+3) \frac{x+2}{x+2} \\ &= \lim_{x \rightarrow -2^+} (x+3) \\ &= -2+3 \\ &= 1 \end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|}$

Solution

$$\lim_{x \rightarrow 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|} = \frac{\sqrt{2(1)}(1-1)}{|1-1|} = \frac{0}{0}$$

$$\text{Since } x \rightarrow 1^+ \Rightarrow x > 1 \Rightarrow |x-1| = x-1$$

$$\begin{aligned} \lim_{x \rightarrow 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|} &= \lim_{x \rightarrow 1^+} \frac{\sqrt{2x}(x-1)}{x-1} \\ &= \lim_{x \rightarrow 1^+} \sqrt{2x} \\ &= \sqrt{2} \end{aligned}$$

Exercise

Find $\lim_{\theta \rightarrow 0} \frac{\sin \sqrt{2} \cdot \theta}{\sqrt{2} \cdot \theta}$

Solution

$$\text{Let: } \sqrt{2}\theta = x \rightarrow 0$$

$$\lim_{\theta \rightarrow 0} \frac{\sin \sqrt{2} \cdot \theta}{\sqrt{2} \cdot \theta} = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Exercise

Find $\lim_{x \rightarrow 0} \frac{\sin 3x}{4x}$

Solution

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin 3x}{4x} &= \lim_{x \rightarrow 0} \frac{\sin 3x}{4x} \cdot \frac{3}{3} \\ &= \frac{3}{4} \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \\ &= \frac{3}{4} \lim_{u \rightarrow 0} \frac{\sin u}{u} \\ &= \frac{3}{4} \end{aligned}$$

Let: $3x = u$

By definition: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

Exercise

Find $\lim_{x \rightarrow 0^-} \frac{x}{\sin 3x}$

Solution

$$\begin{aligned} \lim_{x \rightarrow 0^-} \frac{x}{\sin 3x} &= \lim_{x \rightarrow 0^-} \frac{x}{\sin 3x} \left(\frac{3}{3} \right) \\ &= \frac{1}{3} \lim_{x \rightarrow 0^-} \frac{3x}{\sin 3x} \\ &= \frac{1}{3} \lim_{x \rightarrow 0^-} \frac{1}{\frac{\sin 3x}{3x}} \\ &= \frac{1}{3} \end{aligned}$$

By definition: $\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1$

Exercise

Find $\lim_{x \rightarrow 0} \frac{\tan 2x}{x}$

Solution

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\tan 2x}{x} &= \lim_{x \rightarrow 0} \frac{\frac{\sin 2x}{\cos 2x}}{x} \\ &= \lim_{x \rightarrow 0} \left(\frac{\sin 2x}{x} \cdot \frac{1}{\cos 2x} \right) \end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 0} \left(2 \frac{\sin 2x}{2x} \right) \lim_{x \rightarrow 0} \left(\frac{1}{\cos 2x} \right) \\
&= 2 \cdot \frac{1}{\cos 0} \\
&= 2
\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 0} 6x^2 (\cot x) (\csc 2x)$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 0} 6x^2 (\cot x) (\csc 2x) &= \lim_{x \rightarrow 0} 6x^2 \left(\frac{\cos x}{\sin x} \right) \left(\frac{1}{\sin 2x} \right) \\
&= \lim_{x \rightarrow 0} 3 \cos x \left(\frac{x}{\sin x} \right) \left(\frac{2x}{\sin 2x} \right) \\
&= 3 \lim_{x \rightarrow 0} (\cos x) \lim_{x \rightarrow 0} \left(\frac{x}{\sin x} \right) \lim_{x \rightarrow 0} \left(\frac{2x}{\sin 2x} \right) = 3 \cdot 1 \cdot 1 \cdot 1 \\
&= 3
\end{aligned}$$

Exercise

Find $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\sin 2\theta}$

Solution

$$\begin{aligned}
\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\sin 2\theta} &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\sin 2\theta} \frac{2\theta}{2\theta} \\
&= \frac{1}{2} \lim_{\theta \rightarrow 0} \left(\frac{2\theta}{\sin 2\theta} \cdot \frac{\sin \theta}{\theta} \right) = \frac{1}{2} \cdot 1 \cdot 1 \\
&= \frac{1}{2}
\end{aligned}$$

Exercise

Find $\lim_{h \rightarrow 0} \frac{\sin(\sin h)}{\sin h}$

Solution

Let: $\sin h = \theta$ $\theta = \sin h \xrightarrow{h \rightarrow 0} 0$

$$\begin{aligned}
\lim_{h \rightarrow 0} \frac{\sin(\sin h)}{\sin h} &= \lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} \\
&= 1
\end{aligned}$$

Exercise

Find $\lim_{\theta \rightarrow 0} \frac{\theta \cot 4\theta}{\sin^2 \theta \cot^2 2\theta}$

Solution

$$\begin{aligned}\lim_{\theta \rightarrow 0} \frac{\theta \cot 4\theta}{\sin^2 \theta \cot^2 2\theta} &= \lim_{\theta \rightarrow 0} \frac{\theta \frac{\cos 4\theta}{\sin 4\theta}}{\sin^2 \theta \frac{\cos^2 2\theta}{\sin^2 2\theta}} \\&= \lim_{\theta \rightarrow 0} \theta \frac{\cos 4\theta}{2 \sin 2\theta \cos 2\theta} \frac{\sin^2 2\theta}{\sin^2 \theta \cos^2 2\theta} \\&= \lim_{\theta \rightarrow 0} \left(\frac{1}{2} \cdot \theta \cdot \cos 4\theta \cdot \frac{2 \sin \theta \cos \theta}{\sin^2 \theta} \cdot \frac{1}{\cos^3 2\theta} \right) \\&= \lim_{\theta \rightarrow 0} \left(\cos 4\theta \cdot \frac{\theta}{\sin \theta} \cdot \cos \theta \cdot \frac{1}{\cos^3 2\theta} \right) \\&= \lim_{\theta \rightarrow 0} (\cos 4\theta) \cdot \lim_{\theta \rightarrow 0} \left(\frac{\theta}{\sin \theta} \right) \cdot \lim_{\theta \rightarrow 0} \left(\frac{\cos \theta}{\cos^3 2\theta} \right) \\&= 1 \cdot 1 \cdot 1 \\&= \underline{1}\end{aligned}$$

Exercise

Find $\lim_{\theta \rightarrow \pi/4} \frac{\sin^2 \theta - \cos^2 \theta}{\sin \theta - \cos \theta}$

Solution

$$\begin{aligned}\lim_{\theta \rightarrow \pi/4} \frac{\sin^2 \theta - \cos^2 \theta}{\sin \theta - \cos \theta} &= \frac{\frac{1}{2} - \frac{1}{2}}{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}} = \frac{0}{0} \\&= \lim_{\theta \rightarrow \pi/4} \frac{(\sin \theta - \cos \theta)(\sin \theta + \cos \theta)}{\sin \theta - \cos \theta} \\&= \lim_{\theta \rightarrow \pi/4} (\sin \theta + \cos \theta) \\&= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \\&= \underline{\sqrt{2}}\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow \pi/2} \frac{\frac{1}{\sqrt{\sin x}} - 1}{x + \frac{\pi}{2}}$

Solution

$$\begin{aligned} \lim_{x \rightarrow \pi/2} \frac{\frac{1}{\sqrt{\sin x}} - 1}{x + \frac{\pi}{2}} &= \frac{1 - 1}{\frac{\pi}{2} + \frac{\pi}{2}} \\ &= 0 \end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 1} \frac{x^3 - 7x^2 + 12x}{4 - x}$

Solution

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^3 - 7x^2 + 12x}{4 - x} &= \frac{1 - 7 + 12}{4 - 1} \\ &= 2 \end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 4} \frac{x^3 - 7x^2 + 12x}{4 - x}$

Solution

$$\begin{aligned} \lim_{x \rightarrow 4} \frac{x^3 - 7x^2 + 12x}{4 - x} &= \frac{64 - 112 + 48}{4 - 4} = \frac{0}{0} \\ &= \lim_{x \rightarrow 4} \frac{x(x-3)(x-4)}{4-x} \\ &= \lim_{x \rightarrow 4} -x(x-3) \\ &= -4 \end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 1} \frac{1 - x^2}{x^2 - 8x + 7}$

Solution

$$\lim_{x \rightarrow 1} \frac{1 - x^2}{x^2 - 8x + 7} = \frac{0}{0}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 1} \frac{(1-x)(1+x)}{(x-1)(x-7)} \\
&= - \lim_{x \rightarrow 1} \frac{1+x}{x-7} \\
&= \frac{1}{3} \quad \Big|
\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 3} \frac{\sqrt{3x+16}-5}{x-3}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 3} \frac{\sqrt{3x+16}-5}{x-3} &= \frac{\sqrt{9+16}-5}{3-3} = \frac{5-5}{0} = \frac{0}{0} \\
&= \lim_{x \rightarrow 3} \frac{\sqrt{3x+16}-5}{x-3} \cdot \frac{\sqrt{3x+16}+5}{\sqrt{3x+16}+5} \\
&= \lim_{x \rightarrow 3} \frac{3x+16-25}{(x-3)(\sqrt{3x+16}+5)} \\
&= \lim_{x \rightarrow 3} \frac{3(x-3)}{(x-3)(\sqrt{3x+16}+5)} \\
&= \lim_{x \rightarrow 3} \frac{3}{\sqrt{3x+16}+5} \\
&= \frac{3}{5+5} \\
&= \frac{3}{10} \quad \Big|
\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 3} \frac{1}{x-3} \left(\frac{1}{\sqrt{x+1}} - \frac{1}{2} \right)$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 3} \frac{1}{x-3} \left(\frac{1}{\sqrt{x+1}} - \frac{1}{2} \right) &= \frac{1}{0} \left(\frac{1}{2} - \frac{1}{2} \right) = \frac{0}{0} \\
&= \lim_{x \rightarrow 3} \frac{1}{x-3} \left(\frac{2-\sqrt{x+1}}{\sqrt{x+1}} \right) \left(\frac{2+\sqrt{x+1}}{2+\sqrt{x+1}} \right) \\
&= \lim_{x \rightarrow 3} \frac{1}{x-3} \left(\frac{4-x-1}{2\sqrt{x+1}+x+1} \right) \\
&= \lim_{x \rightarrow 3} \frac{x-3}{x-3} \left(\frac{-1}{2\sqrt{x+1}+x+1} \right)
\end{aligned}$$

$$= \lim_{x \rightarrow 3} \frac{-1}{2\sqrt{x+1} + x+1}$$

$$= -\frac{1}{8}$$

Exercise

Find $\lim_{x \rightarrow 1/3} \frac{x - \frac{1}{3}}{(3x-1)^2}$

Solution

$$\lim_{x \rightarrow 1/3} \frac{x - \frac{1}{3}}{(3x-1)^2} = \frac{\frac{1}{3} - \frac{1}{3}}{\left(3\frac{1}{3} - 1\right)^2} = \frac{0}{0}$$

$$= \lim_{x \rightarrow 1/3} \frac{x - \frac{1}{3}}{9\left(x - \frac{1}{3}\right)^2}$$

$$= \lim_{x \rightarrow 1/3} \frac{1}{9\left(x - \frac{1}{3}\right)} = \frac{1}{0}$$

$$= \infty$$

Exercise

Find $\lim_{x \rightarrow 3} \frac{x^4 - 81}{x - 3}$

Solution

$$\lim_{x \rightarrow 3} \frac{x^4 - 81}{x - 3} = \frac{81 - 81}{3 - 3} = \frac{0}{0}$$

$$= \lim_{x \rightarrow 3} \frac{(x-3)(x+3)(x^2+9)}{x-3} \quad a^4 - b^4 = (a^2 - b^2)(a^2 + b^2) = (a-b)(a+b)(a^2 + b^2)$$

$$= \lim_{x \rightarrow 3} (x+3)(x^2+9) = 6(18)$$

$$= 108$$

Exercise

Find $\lim_{x \rightarrow 1} \frac{x^5 - 1}{x - 1}$

Solution

$$\begin{aligned}
 \lim_{x \rightarrow 1} \frac{x^5 - 1}{x - 1} &= \frac{1 - 1}{1 - 1} = \frac{0}{0} & (a^5 - b^5) &= (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4) \\
 &= \lim_{x \rightarrow 1} \frac{(x - 1)(x^4 + x^3 + x^2 + x + 1)}{x - 1} \\
 &= \lim_{x \rightarrow 1} (x^4 + x^3 + x^2 + x + 1) \\
 &= 5
 \end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 81} \frac{\sqrt[4]{x} - 3}{x - 81}$

Solution

$$\begin{aligned}
 \lim_{x \rightarrow 81} \frac{\sqrt[4]{x} - 3}{x - 81} &= \frac{3 - 3}{81 - 81} = \frac{0}{0} \\
 &= \lim_{x \rightarrow 81} \frac{\sqrt[4]{x} - 3}{(\sqrt{x} + 9)(\sqrt{x} - 9)} \\
 &= \lim_{x \rightarrow 81} \frac{\sqrt[4]{x} - 3}{(\sqrt{x} + 9)(\sqrt[4]{x} + 3)(\sqrt[4]{x} - 3)} \\
 &= \lim_{x \rightarrow 81} \frac{1}{(\sqrt{x} + 9)(\sqrt[4]{x} + 3)} \\
 &= \frac{1}{(18)(6)} \\
 &= \frac{1}{108}
 \end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{x - 1}$

Solution

$$\begin{aligned}
 \lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{x - 1} &= \frac{0}{0} \\
 &= \lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{(\sqrt[3]{x})^3 - 1^3}
 \end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{(\sqrt[3]{x} - 1)(x^{2/3} + \sqrt[3]{x} + 1)} \\
&= \lim_{x \rightarrow 1} \frac{1}{x^{2/3} + \sqrt[3]{x} + 1} \\
&= \frac{1}{3}
\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 2} \frac{x^5 - 32}{x - 2}$

Solution

$$\begin{array}{r|rrrrrr}
2 & 1 & 0 & 0 & 0 & 0 & -32 \\
& & 2 & 4 & 8 & 16 & 32 \\
\hline
& 1 & 2 & 4 & 8 & 16 & 0
\end{array}$$

$$\lim_{x \rightarrow 2} \frac{x^5 - 32}{x - 2} = \frac{2^5 - 32}{2 - 2} = \frac{0}{0}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 2} \frac{(x - 2)(x^4 + 2x^3 + 4x^2 + 8x + 16)}{x - 2} \\
&= \lim_{x \rightarrow 2} (x^4 + 2x^3 + 4x^2 + 8x + 16) \\
&= 16 + 16 + 16 + 16 + 16 \\
&= 80
\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 1} \frac{x^6 - 1}{x - 1}$

Solution

$$\begin{array}{r|rrrrrrr}
1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
& & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
& 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{array}$$

$$\lim_{x \rightarrow 1} \frac{x^6 - 1}{x - 1} = \frac{0}{0}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 1} \frac{(x - 1)(x^5 + x^4 + x^3 + x^2 + x + 1)}{x - 1} \\
&= \lim_{x \rightarrow 1} (x^5 + x^4 + x^3 + x^2 + x + 1) \\
&= 1 + 1 + 1 + 1 + 1 + 1 \\
&= 6
\end{aligned}$$

$$= \lim_{x \rightarrow 1} (x^5 + x^4 + x^3 + x^2 + x + 1)$$

$$\underline{= 6}$$

Exercise

Find the limit: $\lim_{x \rightarrow -1} \frac{x^7 + 1}{x + 1}$

Solution

$$\begin{array}{c|cccccccc} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ & & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & -1 & 1 & -1 & 1 & \textcolor{red}{0} \end{array}$$

$$\lim_{x \rightarrow -1} \frac{x^7 + 1}{x + 1} = \frac{-1 + 1}{-1 + 1} = \frac{\textcolor{red}{0}}{\textcolor{red}{0}}$$

$$= \lim_{x \rightarrow -1} \frac{(x + 1)(x^6 - x^5 + x^4 - x^3 + x^2 - x + 1)}{x + 1}$$

$$= \lim_{x \rightarrow -1} (x^6 - x^5 + x^4 - x^3 + x^2 - x + 1)$$

$$\underline{= 1}$$

Exercise

Find the limit: $\lim_{x \rightarrow a} \frac{x^5 - a^5}{x - a}$

Solution

$$\begin{array}{c|cccccc} a & 1 & 0 & 0 & 0 & 0 & -a^5 \\ & & a & a^2 & a^3 & a^4 & a^5 \\ \hline & 1 & a & a^2 & a^3 & a^4 & \textcolor{red}{0} \end{array}$$

$$\lim_{x \rightarrow a} \frac{x^5 - a^5}{x - a} = \frac{a^5 - a^5}{a - a} = \frac{\textcolor{red}{0}}{\textcolor{red}{0}}$$

$$= \lim_{x \rightarrow a} \frac{(x - a)(x^4 + ax^3 + a^2x^2 + a^3x + a^4)}{x - a}$$

$$= \lim_{x \rightarrow a} (x^4 + ax^3 + a^2x^2 + a^3x + a^4)$$

$$= a^4 + a^4 + a^4 + a^4 + a^4$$

$$\underline{= 5a^4}$$

Exercise

Find the limit: $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} \quad n \in \mathbb{Z}^+$

Solution

$$\begin{array}{c|ccccccc} a & 1 & 0 & 0 & 0 & \cdots & 0 & -a^n \\ & & a & a^2 & a^3 & \cdots & a^{n-1} & a^n \\ \hline & 1 & a & a^2 & a^3 & \cdots & a^{n-1} & \textcolor{red}{0} \end{array}$$

$$\begin{aligned} \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} &= \frac{a^n - a^n}{a - a} = \frac{\textcolor{red}{0}}{\textcolor{red}{0}} \\ &= \lim_{x \rightarrow a} \frac{(x - a) \left(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-2}x + a^{n-1} \right)}{x - a} \\ &= \lim_{x \rightarrow a} \left(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-2}x + a^{n-1} \right) \\ &= a^{n-1} + a^{n-1} + a^{n-1} + \dots + a^{n-1} + a^{n-1} \\ &= \underline{\textcolor{blue}{na^{n-1}}} \end{aligned}$$

Exercise

Find the limit: $\lim_{h \rightarrow 0} \frac{100}{(10h - 1)^{11} + 2}$

Solution

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{100}{(10h - 1)^{11} + 2} &= \frac{100}{(-1)^{11} + 2} \\ &= \frac{100}{-1 + 2} \\ &= \underline{\textcolor{blue}{100}} \end{aligned}$$

Exercise

Find the limit: $\lim_{h \rightarrow 0} \frac{(5 + h)^2 - 25}{h}$

Solution

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{(5 + h)^2 - 25}{h} &= \frac{5^2 - 25}{0} = \frac{\textcolor{red}{0}}{\textcolor{red}{0}} \\ &= \lim_{h \rightarrow 0} \frac{((5 + h) - 5)((5 + h) + 5)}{h} \end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{h(h+10)}{h} \\
&= \lim_{h \rightarrow 0} (h+10) \\
&= \underline{10}
\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 3} \frac{\frac{1}{x^2+2x} - \frac{1}{15}}{x-3}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 3} \frac{\frac{1}{x^2+2x} - \frac{1}{15}}{x-3} &= \frac{\frac{1}{15} - \frac{1}{15}}{0} = \frac{0}{0} \\
&= \lim_{x \rightarrow 3} \frac{1}{x-3} \left(\frac{1}{x(x+2)} - \frac{1}{15} \right) \\
&= \lim_{x \rightarrow 3} \frac{1}{x-3} \left(\frac{15 - x^2 - 2x}{15x(x+2)} \right) \\
&= \lim_{x \rightarrow 3} \frac{-(x-3)(x+5)}{15x(x+2)(x-3)} \\
&= \lim_{x \rightarrow 3} \frac{-(x+5)}{15x(x+2)} \\
&= -\frac{8}{15(3)(5)} \\
&= \underline{-\frac{8}{225}}
\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 1} \frac{\sqrt{10x-9}-1}{x-1}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 1} \frac{\sqrt{10x-9}-1}{x-1} &= \frac{1-1}{0} = \frac{0}{0} \\
&= \lim_{x \rightarrow 1} \frac{\sqrt{10x-9}-1}{x-1} \cdot \frac{\sqrt{10x-9}+1}{\sqrt{10x-9}+1} \\
&= \lim_{x \rightarrow 1} \frac{10x-9-1}{(x-1)(\sqrt{10x-9}+1)}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 1} \frac{10(x-1)}{(x-1)(\sqrt{10x-9}+1)} \\
&= \lim_{x \rightarrow 1} \frac{10}{\sqrt{10x-9}+1} \\
&= \frac{10}{2} \\
&= 5
\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{2}{x^2-2x} \right)$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{2}{x^2-2x} \right) &= \frac{1}{0} - \frac{2}{0} = \infty - \infty \\
&= \lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{2}{x(x-2)} \right) \\
&= \lim_{x \rightarrow 2} \frac{x-2}{x(x-2)} \\
&= \lim_{x \rightarrow 2} \frac{1}{x} \\
&= \frac{1}{2}
\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow c} \frac{x^2-2cx+c^2}{x-c}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow c} \frac{x^2-2cx+c^2}{x-c} &= \frac{c^2-2c^2+c^2}{0} = \frac{0}{0} \\
&= \lim_{x \rightarrow c} \frac{(x-c)^2}{x-c} \\
&= \lim_{x \rightarrow c} (x-c) \\
&= 0
\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow -c} \frac{x^2 + 5cx + 4c^2}{x^2 + cx}$

Solution

$$\begin{aligned}\lim_{x \rightarrow -c} \frac{x^2 + 5cx + 4c^2}{x^2 + cx} &= \frac{c^2 - 5c^2 + 4c^2}{c^2 - c^2} = \frac{0}{0} \\&= \lim_{x \rightarrow -c} \frac{(x+c)(x+4c)}{x(x+c)} \\&= \lim_{x \rightarrow -c} \frac{x+4c}{x} \\&= \frac{-c+4c}{-c} \\&= \frac{3c}{-c} \\&= \underline{-3}\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 16} \frac{\sqrt[4]{x} - 2}{x - 16}$

Solution

$$\lim_{x \rightarrow 16} \frac{\sqrt[4]{x} - 2}{x - 16} = \frac{\sqrt[4]{16} - 2}{16 - 16} = \frac{2 - 2}{0} = \frac{0}{0}$$

$$\lim_{x \rightarrow 16} \frac{\sqrt[4]{x} - 2}{\left(\sqrt[4]{x}\right)^4 - 2^4}$$

$$= \lim_{x \rightarrow 16} \frac{\sqrt[4]{x} - 2}{\left(\sqrt{x} + 2^2\right)\left(\sqrt[4]{x} + 2\right)\left(\sqrt[4]{x} - 2\right)}$$

$$= \lim_{x \rightarrow 16} \frac{1}{\left(\sqrt{x} + 4\right)\left(\sqrt[4]{x} + 2\right)}$$

$$= \frac{1}{\left(\sqrt{16} + 4\right)\left(\sqrt[4]{16} + 2\right)}$$

$$= \frac{1}{(4+4)(2+2)}$$

$$= \frac{1}{(8)(4)}$$

$$= \underline{\frac{1}{32}}$$

$$a^4 - b^4 = (a^2 + b^2)(a - b)(a + b)$$

Exercise

Find the limit: $\lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x}-1}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x}-1} &= \frac{0}{0} \\&= \lim_{x \rightarrow 1} \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{\sqrt{x}-1} \\&= \lim_{x \rightarrow 1} (\sqrt{x}+1) \\&= 2\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 1} \frac{x-1}{\sqrt{4x+5}-3}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x-1}{\sqrt{4x+5}-3} &= \frac{0}{0} \\&= \lim_{x \rightarrow 1} \frac{x-1}{\sqrt{4x+5}-3} \cdot \frac{\sqrt{4x+5}+3}{\sqrt{4x+5}+3} \\&= \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{4x+5}+3)}{4x+5-9} \\&= \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{4x+5}+3)}{4(x-1)} \\&= \frac{1}{5} \lim_{x \rightarrow 1} (\sqrt{4x+5}+3) \\&= \frac{6}{5}\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}}$

Solution

$$\lim_{x \rightarrow 4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}} = \frac{0}{3-3} = \frac{0}{0}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}} \cdot \frac{3+\sqrt{x+5}}{3+\sqrt{x+5}} \\
&= 3 \lim_{x \rightarrow 4} \frac{(x-4)(3+\sqrt{x+5})\sqrt{x+5}}{9-(x+5)} \\
&= 3 \lim_{x \rightarrow 4} \frac{(x-4)(3+\sqrt{x+5})\sqrt{x+5}}{4-x} \\
&= -3 \lim_{x \rightarrow 4} (3+\sqrt{x+5})\sqrt{x+5} \\
&= -3 (6)(3) \\
&= \underline{\underline{-54}}
\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 0} \frac{x}{\sqrt{ax+1}-1} \quad (a \neq 0)$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{x}{\sqrt{ax+1}-1} &= \frac{0}{0} \\
&= \lim_{x \rightarrow 0} \frac{x}{\sqrt{ax+1}-1} \cdot \frac{\sqrt{ax+1}+1}{\sqrt{ax+1}+1} \\
&= \lim_{x \rightarrow 0} \frac{x(\sqrt{ax+1}+1)}{ax+1-1} \\
&= \lim_{x \rightarrow 0} \frac{x(\sqrt{ax+1}+1)}{ax} \\
&= \frac{1}{a} \lim_{x \rightarrow 0} (\sqrt{ax+1}+1) \\
&= \underline{\underline{\frac{2}{a}}}
\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow \pi} \frac{\cos^2 x + 3\cos x + 2}{\cos x + 1}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow \pi} \frac{\cos^2 x + 3\cos x + 2}{\cos x + 1} &= \frac{1-3+2}{-1+1} = \frac{0}{0} \\
&= \lim_{x \rightarrow \pi} \frac{(\cos x + 1)(\cos x + 2)}{\cos x + 1}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow \pi} (\cos x + 2) \\
&= -1 + 2 \\
&= 1
\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow \frac{3\pi}{2}} \frac{\sin^2 x + 6 \sin x + 5}{\sin^2 x - 1}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow \frac{3\pi}{2}} \frac{\sin^2 x + 6 \sin x + 5}{\sin^2 x - 1} &= \frac{1 - 6 + 5}{1 - 1} = \frac{0}{0} \\
&= \lim_{x \rightarrow \frac{3\pi}{2}} \frac{(\sin x + 1)(\sin x + 5)}{(\sin x - 1)(\sin x + 1)} \\
&= \lim_{x \rightarrow \frac{3\pi}{2}} \frac{\sin x + 5}{\sin x - 1} \\
&= \frac{-1 + 5}{-1 - 1} \\
&= -2
\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x - 1}{\sqrt{\sin x} - 1}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x - 1}{\sqrt{\sin x} - 1} &= \frac{1 - 1}{1 - 1} = \frac{0}{0} \\
&= \lim_{x \rightarrow \frac{\pi}{2}} \frac{(\sqrt{\sin x} - 1)(\sqrt{\sin x} + 1)}{\sqrt{\sin x} - 1} \\
&= \lim_{x \rightarrow \frac{\pi}{2}} (\sqrt{\sin x} + 1) \\
&= 2
\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 0} \frac{\frac{1}{2 + \sin x} - \frac{1}{2}}{\sin x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\frac{1}{2 + \sin x} - \frac{1}{2}}{\sin x} &= \frac{\frac{1}{2} - \frac{1}{2}}{0} = \frac{0}{0} \\&= \lim_{x \rightarrow 0} \frac{1}{\sin x} \cdot \frac{2 - \sin x - 2}{2(2 + \sin x)} \\&= \frac{1}{2} \lim_{x \rightarrow 0} \frac{1}{\sin x} \frac{-\sin x}{(2 + \sin x)} \\&= -\frac{1}{2} \lim_{x \rightarrow 0} \frac{1}{2 + \sin x} \\&= -\frac{1}{2} \left(\frac{1}{2} \right) \\&= -\frac{1}{4} \quad \boxed{}\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{e^x - 1}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{e^x - 1} &= \frac{1 - 1}{1 - 1} = \frac{0}{0} \\&= \lim_{x \rightarrow 0} \frac{(e^x - 1)(e^x + 1)}{e^x - 1} \\&= \lim_{x \rightarrow 0} (e^x + 1) \\&= 2 \quad \boxed{}\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow \frac{\pi}{4}} \csc x$

Solution

$$\begin{aligned}\lim_{x \rightarrow \frac{\pi}{4}} \csc x &= \csc \frac{\pi}{4} \\&= \frac{1}{\cos \frac{\pi}{4}} \\&= \sqrt{2} \quad \boxed{}\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 4} \frac{x-5}{(x^2-10x+24)^2}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 4} \frac{x-5}{(x^2-10x+24)^2} &= \frac{-1}{(16-41+24)^2} \\ &= -1\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 0} \frac{\cos x - 1}{\sin^2 x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\cos x - 1}{\sin^2 x} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \frac{\cos x - 1}{(1 - \cos x)(1 + \cos x)} \\ &= - \lim_{x \rightarrow 0} \frac{1}{1 + \cos x} \\ &= -\frac{1}{2}\end{aligned}$$

Exercise

Find the limit: $\lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{\sin x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{\sin x} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{\sin x} \\ &= \lim_{x \rightarrow 0} \sin x \\ &= 0\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 0} \frac{x^3 - 5x^2}{x^2}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{x^3 - 5x^2}{x^2} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 0} (x - 5) \\ &= -5\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 5} \frac{4x^2 - 100}{x - 5}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 5} \frac{4x^2 - 100}{x - 5} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 5} \frac{4(x - 5)(x + 5)}{x - 5} \\ &= \lim_{x \rightarrow 5} 4(x + 5) \\ &= 40\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 3} \frac{\sqrt{9 - 6x + x^2}}{x - 3}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{\sqrt{9 - 6x + x^2}}{x - 3} &= \frac{\sqrt{9 - 18 + 9}}{3 - 3} = \frac{0}{0} \\ &= \lim_{x \rightarrow 3} \frac{\sqrt{(x - 3)^2}}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{x - 3}{x - 3} \\ &= 1\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 3} \frac{\sqrt{9+6x+x^2}}{x-3}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{\sqrt{9+6x+x^2}}{x-3} &= \frac{\sqrt{9+18+9}}{3-3} \\ &= \frac{\sqrt{36}}{0} \\ &= \infty\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 3} \frac{\sqrt{x^2-9}}{x-3}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{\sqrt{x^2-9}}{x-3} &= \frac{\sqrt{9-9}}{3-3} = \frac{0}{0} \\ &= \lim_{x \rightarrow 3} \frac{\sqrt{(x-3)(x+3)}}{x-3} \\ &= \lim_{x \rightarrow 3} \sqrt{\frac{x+3}{x-3}} \\ &= \sqrt{\frac{6}{0}} \\ &= \infty\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow \frac{4\pi}{3}} \sin x$

Solution

$$\begin{aligned}\lim_{x \rightarrow \frac{4\pi}{3}} \sin x &= \sin \frac{4\pi}{3} \\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow \frac{2\pi}{3}} \cos x$

Solution

$$\begin{aligned} \lim_{x \rightarrow \frac{2\pi}{3}} \cos x &= \cos \frac{2\pi}{3} \\ &= -\frac{1}{2} \end{aligned}$$

Exercise

Find $\lim_{x \rightarrow \frac{7\pi}{4}} \sin x$

Solution

$$\begin{aligned} \lim_{x \rightarrow \frac{7\pi}{4}} \sin x &= \sin \frac{7\pi}{4} \\ &= -\frac{\sqrt{2}}{2} \end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 1} \frac{\sin \sqrt{1-x}}{\sqrt{1-x^2}}$

Solution

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\sin \sqrt{1-x}}{\sqrt{1-x^2}} &= \frac{\sin 0}{0} = \frac{0}{0} \\ &= \lim_{x \rightarrow 1} \frac{\sin \sqrt{1-x}}{\sqrt{1-x}} \cdot \frac{1}{\sqrt{1+x}} \\ &= \lim_{(1-x) \rightarrow 0} \frac{\sin \sqrt{1-x}}{\sqrt{1-x}} \lim_{x \rightarrow 1} \frac{1}{\sqrt{1+x}} \\ &= 1 \left(\frac{1}{\sqrt{2}} \right) \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 2} \frac{\sin \sqrt{2-x}}{\sqrt{4-x^2}}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{\sin \sqrt{2-x}}{\sqrt{4-x^2}} &= \frac{\sin 0}{0} = \frac{0}{0} \\&= \lim_{x \rightarrow 2} \frac{\sin \sqrt{2-x}}{\sqrt{2-x}} \cdot \frac{1}{\sqrt{2+x}} \\&= \lim_{\sqrt{2-x} \rightarrow 0} \frac{\sin \sqrt{2-x}}{\sqrt{2-x}} \quad \lim_{x \rightarrow 2} \frac{1}{\sqrt{2+x}} \\&= 1 \left(\frac{1}{2} \right) \\&= \frac{1}{2} \quad \Bigg| \end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 0} \frac{\sin(\sqrt{5} x)}{\sin(\sqrt{3} x)}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin(\sqrt{5} x)}{\sin(\sqrt{3} x)} &= \frac{\sin 0}{\sin 0} = \frac{0}{0} \\&= \lim_{x \rightarrow 0} \frac{\sqrt{5} x}{\sqrt{3} x} \cdot \frac{\sin(\sqrt{5} x)}{\sqrt{5} x} \cdot \frac{1}{\frac{\sin(\sqrt{3} x)}{\sqrt{3} x}} \\&= \frac{\sqrt{5}}{\sqrt{3}} \lim_{\sqrt{5} x \rightarrow 0} \frac{\sin(\sqrt{5} x)}{\sqrt{5} x} \cdot \frac{1}{\lim_{\sqrt{3} x \rightarrow 0} \frac{\sin(\sqrt{3} x)}{\sqrt{3} x}} \\&= \frac{\sqrt{5}}{\sqrt{3}} \quad \Bigg| \end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 0} \frac{\sin(\sqrt{15} x)}{\sin(\sqrt{3} x)}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin(\sqrt{15} x)}{\sin(\sqrt{3} x)} &= \frac{\sin 0}{\sin 0} = \frac{0}{0} \\&= \lim_{x \rightarrow 0} \frac{\sqrt{15} x}{\sqrt{3} x} \cdot \frac{\sin(\sqrt{5} x)}{\sqrt{15} x} \cdot \frac{1}{\frac{\sin(\sqrt{3} x)}{\sqrt{3} x}} \\&= \sqrt{\frac{15}{3}} \lim_{x \rightarrow 0} \frac{\sin(\sqrt{15} x)}{\sqrt{15} x} \cdot \frac{1}{\lim_{x \rightarrow 0} \frac{\sin(\sqrt{3} x)}{\sqrt{3} x}} \\&= \sqrt{3} \quad | \end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 0^+} \frac{x - \sqrt{x}}{\sqrt{\sin x}}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{x - \sqrt{x}}{\sqrt{\sin x}} &= \frac{0}{0} \\&= \lim_{x \rightarrow 0^+} \frac{x - \sqrt{x}}{\sqrt{\sin x}} \cdot \frac{1}{\sqrt{x}} \\&= \lim_{x \rightarrow 0^+} \frac{1}{\sqrt{\frac{\sin x}{x}}} \cdot \lim_{x \rightarrow 0^+} \frac{x - \sqrt{x}}{\sqrt{x}} \\&= (1) \lim_{x \rightarrow 0^+} \left(\frac{x}{\sqrt{x}} - \frac{\sqrt{x}}{\sqrt{x}} \right) \\&= \lim_{x \rightarrow 0^+} (\sqrt{x} - 1) \\&= -1 \quad | \end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 1} \frac{x - \sqrt{x}}{\sqrt{\sin x}}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x - \sqrt{x}}{\sqrt{\sin x}} &= \frac{0}{\sqrt{\sin 1}} \\ &= 0\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow \pi} \frac{x - \sqrt{x}}{\sqrt{\sin x}}$

Solution

$$\begin{aligned}\lim_{x \rightarrow \pi} \frac{x - \sqrt{x}}{\sqrt{\sin x}} &= \frac{\pi - \sqrt{\pi}}{\sqrt{\sin \pi}} \\ &= \frac{\pi - \sqrt{\pi}}{0} \\ &= \infty\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 0} e^{x^3}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} e^{x^3} &= e^0 \\ &= 1\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 1} e^{x^2}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 1} e^{x^2} &= e^1 \\ &= e\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 1} e^{x^3-1}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 1} e^{x^3-1} &= e^{1-1} \\ &= e^0 \\ &= \underline{1} \end{aligned}$$

Exercise

Find $\lim_{x \rightarrow -1} e^{x^3-1}$

Solution

$$\begin{aligned}\lim_{x \rightarrow -1} e^{x^3-1} &= e^{-1-1} \\ &= e^{-2} \\ &= \underline{\frac{1}{e^2}} \end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 2} \left(e^{x^2} - \ln x \right)$

Solution

$$\lim_{x \rightarrow 2} \left(e^{x^2} - \ln x \right) = \underline{e^4 - \ln 2}$$

Exercise

Find $\lim_{x \rightarrow 1} \left(e^{x^2} - \ln x \right)$

Solution

$$\begin{aligned}\lim_{x \rightarrow 1} \left(e^{x^2} - \ln x \right) &= e - \ln 1 \\ &= \underline{e} \end{aligned}$$

Exercise

Find $\lim_{x \rightarrow e} \ln x$

Solution

$$\begin{aligned}\lim_{x \rightarrow e} \ln x &= \ln e \\ &= 1\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow e} \ln x^2$

Solution

$$\begin{aligned}\lim_{x \rightarrow e} \ln x^2 &= \ln e^2 \\ &= 2 \ln e \\ &= 2\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 0^+} \ln x$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0^+} \ln x &= \ln 0^+ \\ &= -\infty\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 1} \frac{1}{\ln x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{1}{\ln x} &= \frac{1}{\ln 1} \\ &= \frac{1}{0} \\ &= \infty\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow e} \ln e^{2x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow e} \ln e^{2x} &= \ln e^{2e} \\ &= 2e \ln e \\ &= \underline{2e}\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow 1} \ln e^{x^2}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 1} \ln e^{x^2} &= \ln e \\ &= \underline{1}\end{aligned}$$

Exercise

For the function $f(t)$ graphed, find the following limits or explain why they do not exist.

$$a) \lim_{t \rightarrow -2} f(t) \quad b) \lim_{t \rightarrow -1} f(t) \quad c) \lim_{t \rightarrow 0} f(t) \quad d) \lim_{t \rightarrow -0.5} f(t)$$

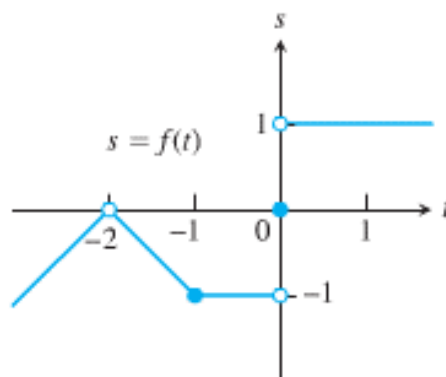
Solution

$$a) \lim_{t \rightarrow -2} f(t) = \underline{0}$$

$$b) \lim_{t \rightarrow -1} f(t) = \underline{-1}$$

$$c) \lim_{t \rightarrow 0} f(t) = \text{doesn't exist}$$

$$d) \lim_{t \rightarrow -0.5} f(t) = \underline{-1}$$



Exercise

Suppose $\lim_{x \rightarrow c} f(x) = 5$ and $\lim_{x \rightarrow c} g(x) = -2$. Find

a) $\lim_{x \rightarrow c} f(x)g(x)$

b) $\lim_{x \rightarrow c} 2f(x)g(x)$

c) $\lim_{x \rightarrow c} (f(x) + 3g(x))$

d) $\lim_{x \rightarrow c} \frac{f(x)}{f(x) - g(x)}$

Solution

$$\begin{aligned} \text{a) } \lim_{x \rightarrow c} f(x)g(x) &= \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x) \\ &= (5)(-2) \\ &= \underline{-10} \end{aligned}$$

$$\begin{aligned} \text{b) } \lim_{x \rightarrow c} 2f(x)g(x) &= 2 \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x) \\ &= 2(-10) \\ &= \underline{-20} \end{aligned}$$

$$\begin{aligned} \text{c) } \lim_{x \rightarrow c} (f(x) + 3g(x)) &= \lim_{x \rightarrow c} f(x) + 3 \lim_{x \rightarrow c} g(x) \\ &= 5 + 3(-2) \\ &= \underline{-1} \end{aligned}$$

$$\begin{aligned} \text{d) } \lim_{x \rightarrow c} \frac{f(x)}{f(x) - g(x)} &= \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} f(x) - \lim_{x \rightarrow c} g(x)} \\ &= \frac{5}{5 - (-2)} \\ &= \underline{\frac{5}{7}} \end{aligned}$$

Exercise

Explain why the limits do not exist for $\lim_{x \rightarrow 0} \frac{x}{|x|}$

Solution

$$\lim_{x \rightarrow 0} \frac{x}{|x|} = \frac{0}{0}$$

$$\lim_{x \rightarrow 0^-} \frac{x}{|x|} = \frac{-x}{x} = -1$$

$$\lim_{x \rightarrow 0^+} \frac{x}{|x|} = \frac{x}{x} = 1$$

Doesn't exist

Exercise

Evaluate the limit using the form $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ for $f(x) = x^2$, $x = 1$

Solution

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\&= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\&= \lim_{h \rightarrow 0} \left(\frac{2xh}{h} + \frac{h^2}{h} \right) \\&= \lim_{h \rightarrow 0} (2x + h) \\&= 2x \quad | \end{aligned}$$

Exercise

Evaluate the limit using the form $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ for $f(x) = \sqrt{3x+1}$, $x = 0$

Solution

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\sqrt{3(x+h)+1} - \sqrt{3x+1}}{h} \\&= \lim_{h \rightarrow 0} \frac{\sqrt{3x+3h+1} - \sqrt{3x+1}}{h} \\&= \lim_{h \rightarrow 0} \frac{\sqrt{3x+3h+1} - \sqrt{3x+1}}{h} \cdot \frac{\sqrt{3x+3h+1} + \sqrt{3x+1}}{\sqrt{3x+3h+1} + \sqrt{3x+1}} \\&= \lim_{h \rightarrow 0} \frac{3x+3h+1 - (3x+1)}{h(\sqrt{3x+3h+1} + \sqrt{3x+1})} \\&= \lim_{h \rightarrow 0} \frac{3x+3h+1 - 3x-1}{h(\sqrt{3x+3h+1} + \sqrt{3x+1})} \\&= \lim_{h \rightarrow 0} \frac{3h}{h(\sqrt{3x+3h+1} + \sqrt{3x+1})} \\&= \lim_{h \rightarrow 0} \frac{3}{\sqrt{3x+3h+1} + \sqrt{3x+1}} \\&= \frac{3}{\sqrt{3(0)+1} + \sqrt{3(0)+1}} \quad \text{Given : } x = 0 \\&= \frac{3}{2} \quad | \end{aligned}$$

Exercise

If $\lim_{x \rightarrow 4} \frac{f(x) - 5}{x - 2} = 1$, find $\lim_{x \rightarrow 4} f(x)$

Solution

$$\lim_{x \rightarrow 4} \frac{f(x) - 5}{x - 2} = 1$$

$$\frac{\lim_{x \rightarrow 4} f(x) - 5}{4 - 2} = 1$$

$$\frac{\lim_{x \rightarrow 4} f(x) - 5}{2} = 1$$

Multiply both sides by 2

$$\lim_{x \rightarrow 4} f(x) - 5 = 2$$

Add 5 on both sides

$$\lim_{x \rightarrow 4} f(x) = 7$$

Exercise

If $\lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 1$, find $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 0} \frac{f(x)}{x}$

Solution

$$\lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 1$$

$$\frac{\lim_{x \rightarrow 0} f(x)}{\lim_{x \rightarrow 0} x^2} = 1$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x^2$$
$$\underline{= 0}$$

$$\lim_{x \rightarrow 0} \frac{f(x)}{x} = \lim_{x \rightarrow 0} \left(\frac{f(x)}{x^2} \cdot x \right)$$

$$= \lim_{x \rightarrow 0} \frac{f(x)}{x^2} \cdot \lim_{x \rightarrow 0} x$$

$$= 1 \cdot 0$$

$$\underline{= 0}$$

Exercise

If $x^4 \leq f(x) \leq x^2$; $-1 \leq x \leq 1$ and $x^2 \leq f(x) \leq x^4$; $x < -1$ and $x > 1$. At what points c do you automatically know $\lim_{x \rightarrow c} f(x)$? What can you say about the value of the limits at these points?

Solution

$$\lim_{x \rightarrow c} x^4 = \lim_{x \rightarrow c} x^2 \Rightarrow c^4 = c^2$$

$$c^4 - c^2 = 0$$

$$c^2(c^2 - 1) = 0$$

$$c^2 = 0$$

$$c^2 - 1 = 0$$

$$\boxed{c = 0}$$

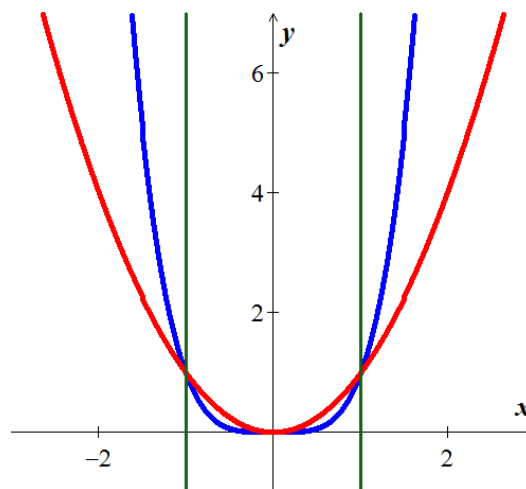
$$\boxed{c = \pm 1}$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x^2$$

$$\underline{= 0}$$

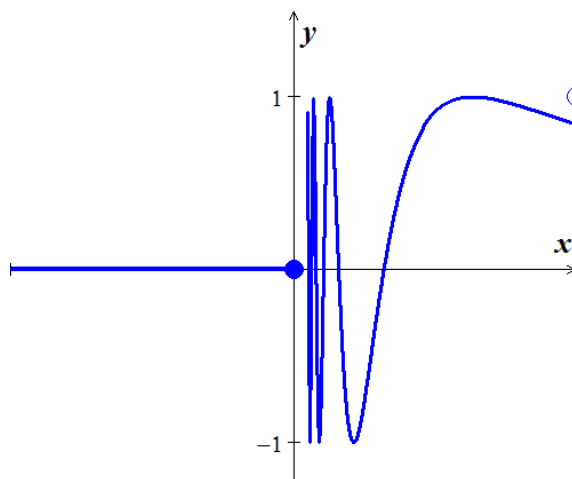
$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow 1} f(x)$$

$$\underline{= 1}$$



Exercise

$$\text{Let } f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \frac{1}{x}, & x > 0 \end{cases}$$



a) Does $\lim_{x \rightarrow 0^+} f(x)$ exist? If so, what is it? If not, why not?

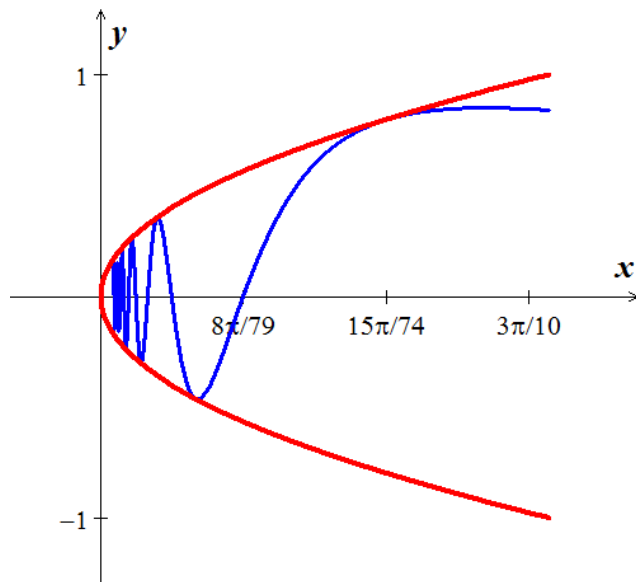
- b) Does $\lim_{x \rightarrow 0^-} f(x)$ exist? If so, what is it? If not, why not?
- c) Does $\lim_{x \rightarrow 0} f(x)$ exist? If so, what is it? If not, why not?

Solution

- a) $\lim_{x \rightarrow 0^+} f(x)$ doesn't exist, since $\sin\left(\frac{1}{x}\right)$ doesn't approach any single value as $x \rightarrow 0$
- b) $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} 0 = 0$
- c) $\lim_{x \rightarrow 0} f(x)$ doesn't exist, since $\lim_{x \rightarrow 0^+} f(x)$ doesn't exist

Exercise

Let $g(x) = \sqrt{x} \sin \frac{1}{x}$



- a) Does $\lim_{x \rightarrow 0^+} g(x)$ exist? If so, what is it? If not, why not?
- b) Does $\lim_{x \rightarrow 0^-} g(x)$ exist? If so, what is it? If not, why not?
- c) Does $\lim_{x \rightarrow 0} g(x)$ exist? If so, what is it? If not, why not?

Solution

- a) $\lim_{x \rightarrow 0^+} g(x)$ exists, by the sandwich theorem $-\sqrt{x} \leq g(x) \leq \sqrt{x}$. for $x > 0$
- b) $\lim_{x \rightarrow 0^-} g(x)$ doesn't exist, since \sqrt{x} is not defined for $x < 0$

- c) $\lim_{x \rightarrow 0} g(x)$ doesn't exist, since $\lim_{x \rightarrow 0^-} g(x)$ doesn't exist.

Exercise

Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

Solution

- a) $\lim_{x \rightarrow -1^+} f(x) = 1$ **True**
- b) $\lim_{x \rightarrow 0^-} f(x) = 0$ **True**
- c) $\lim_{x \rightarrow 0^-} f(x) = 1$ **False**
- d) $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x)$ **True**
- e) $\lim_{x \rightarrow 0} f(x)$ exists **True**
- f) $\lim_{x \rightarrow 0} f(x) = 0$ **True**
- g) $\lim_{x \rightarrow 0} f(x) = 1$ **False**
- h) $\lim_{x \rightarrow 1} f(x) = 1$ **False**
- i) $\lim_{x \rightarrow 1} f(x) = 0$ **False**
- j) $\lim_{x \rightarrow 2^-} f(x) = 2$ **False**
- k) $\lim_{x \rightarrow -1^-} f(x) = 0$ does not exist **True**
- l) $\lim_{x \rightarrow 2^+} f(x) = 0$ **False**

