Formulas

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Derivative

Formula

$$\left(U^m V^n W^p\right)' = U^{m-1} V^{n-1} W^{p-1} \left(mU'VW + nUV'W + pUVW'\right)$$

Proof

$$\left(U^m V^n W^p\right)' = \left(U^m\right)' V^n W^p + U^m \left(V^n\right)' W^p + U^m V^n \left(W^p\right)'$$

$$= mU^{m-1} U' V^n W^p + nU^m V^{n-1} V' W^p + pU^m V^n W^{p-1} W'$$

factor $U^{m-1} V^{n-1} W^{p-1}$

$$= U^{m-1} V^{n-1} W^{p-1} \left(mU'VW + nUV'W + pUVW'\right)$$

Derivative: Rational Function to Power 'n' in the form $\frac{ax^n+b}{cx^n+d}$

$$\left(\frac{ax^n+b}{cx^n+d}\right)' = \boxed{\frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}}$$

$$= \frac{n\begin{vmatrix} a & b \\ c & d \end{vmatrix} x^{n-1}}{\left(cx^n+d\right)^2}$$

Proof

$$u = ax^n + b \qquad v = cx^n + d$$

$$u' = nax^{n-1} \qquad v' = ncx^{n-1}$$

$$\left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{nax^{n-1}\left(cx^n+d\right) - ncx^{n-1}\left(ax^n+b\right)}{\left(cx^n+d\right)^2}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$= \frac{nacx^{2n-1} + nadx^{n-1} - nacx^{2n-1} - nbcx^{n-1}}{\left(cx^n+d\right)^2}$$

$$= \frac{nadx^{n-1} - nbcx^{n-1}}{\left(cx^n+d\right)^2}$$

$$= \boxed{\frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}}$$

Example

Find $\left(\frac{x+2}{3x-2}\right)'$

Solution

$$\left(\frac{x+2}{3x-2}\right)' = \frac{-2-6}{(3x-2)^2}$$

$$= \boxed{\frac{-8}{(3x-2)^2}}$$

Derivative: Rational Function in the form $\frac{\alpha+b}{\beta+d}$

$$\left(\frac{\alpha+b}{\beta+d}\right)' = \frac{\alpha'\beta-\alpha\beta'+\left(\alpha'd-\beta'b\right)}{\left(\beta+d\right)^2} \qquad (\alpha \ne \beta)$$

$$\left(\frac{\alpha+b}{\beta+d}\right)' = \frac{\alpha'd-\beta'b}{\left(\beta+d\right)^2} \qquad \left(\alpha \text{ same form } \beta \ \left(x^n,\ e^{*x}\right)\right)$$

Proof

$u=\alpha+b \quad v=\beta x+d$

$u'=\alpha' \qquad v'=\beta'$

$$\left(\frac{u}{v}\right)' = \frac{u'v-v'u}{v^2}$$

$$\begin{aligned}\left(\frac{\alpha+b}{\beta+d}\right)' &= \frac{\alpha'\left(\beta+d\right)-\beta'\left(\alpha+b\right)}{\left(\beta+d\right)^2}\\ &= \frac{\alpha'\beta+\alpha'd-\alpha\beta'-\beta'b}{\left(\beta+d\right)^2}\\ &= \frac{\left(\alpha'\beta-\alpha\beta'\right)+\left(\alpha'd-\beta'b\right)}{\left(\beta x+d\right)^2}\end{aligned}$$

Example

Find $\left(\frac{5x^2-3}{2x^2-4}\right)'$

Solution

$$\left(\frac{5x^2-3}{2x^2-4}\right)' = \frac{-40x+12x}{\left(2x^2-4\right)^2}$$

$$= \frac{-28x}{\left(2x^2-4\right)^2}$$

Example

Find $\left(\frac{4e^{2x}+1}{2e^{3x}+3}\right)'$

Solution

$$\left(\frac{4e^{2x}+1}{2e^{3x}+3}\right)' = \frac{(16-24)e^{5x}+24e^{2x}-6e^{3x}}{\left(2e^{3x}+3\right)^2}$$

$$= \frac{-8e^{5x}+24e^{2x}-6e^{3x}}{\left(2e^{3x}+3\right)^2}$$

Derivative: Rational Function to Power 'n' in the form $\left(\frac{ax^n+b}{cx^n+d}\right)^m$

$$\frac{d}{dx}\left(\frac{ax^n+b}{cx^n+d}\right)^m = mn(ad-bc)x^{n-1}\frac{\left(ax^n+b\right)^{m-1}}{\left(cx^n+d\right)^{m+1}}$$

Proof

$u = ax^n + b \quad v = cx^n + d$

$u' = nax^{n-1} \quad v' = ncx^{n-1}$

$$\frac{d}{dx}\left(\frac{ax^n+b}{cx^n+d}\right)^m = m\frac{nax^{n-1}\left(cx^n+d\right)-ncx^{n-1}\left(ax^n+b\right)}{\left(cx^n+d\right)^2}\left(\frac{ax^n+b}{cx^n+d}\right)^{m-1} \qquad \left(\frac{u}{v}\right)' = \frac{u'v-v'u}{v^2}$$

$$= \frac{m\left(nacx^{2n-1}+nadx^{n-1}-nacx^{2n-1}-nbcx^{n-1}\right)\left(ax^n+b\right)^{m-1}}{\left(cx^n+d\right)^2\left(cx^n+d\right)^{m-1}}$$

$$= \frac{m\left(nadx^{n-1}-nbcx^{n-1}\right)\left(ax^n+b\right)^{m-1}}{\left(cx^n+d\right)^{m+1}}$$

$$= \frac{mn(ad-bc)x^{n-1}\left(ax^n+b\right)^{m-1}}{\left(cx^n+d\right)^{m+1}}$$

Example

Find $\quad \frac{d}{dx}\left(\frac{5x^2-3}{2x^2-4}\right)^5$

Solution

$$\frac{d}{dx}\left(\frac{5x^2-3}{2x^2-4}\right)^5 = \frac{-140x\left(5x^2-3\right)^4}{\left(2x^2-4\right)^6}$$

Derivative: in the form $\qquad y = \dfrac{ax^2+bx+c}{dx^2+ex+f}$

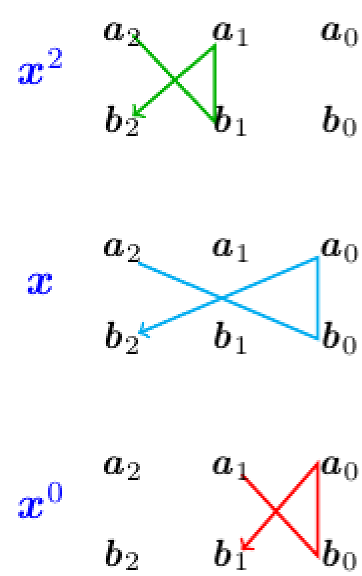
The numerator power of x is $2\boldsymbol{n}-2$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)=\frac{(2ax+b)\left(dx^2+ex+f\right)-(2dx+e)\left(ax^2+bx+c\right)}{\left(dx^2+ex+f\right)^2}$$

$$=\frac{2adx^3+2aex^2+2afx+bdx^2+bex+bf-2adx^3-2bdx^2-2cdx-aex^2-bex-ce}{\left(dx^2+ex+f\right)^2}$$

$$=\frac{(ae-bd)x^2+2(af-cd)x+bf-ce}{\left(dx^2+ex+f\right)^2}$$

$$=\frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix}x^2+2\begin{vmatrix} a & c \\ d & f \end{vmatrix}x+\begin{vmatrix} b & c \\ e & f \end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$



Example

$$f(x)=\frac{x^2-6x+8}{x^2-2x+1}$$

$$f'(x)=\frac{\begin{vmatrix} 1 & -6 \\ 1 & -2 \end{vmatrix}x^2+2\begin{vmatrix} 1 & 8 \\ 1 & 1 \end{vmatrix}x+\begin{vmatrix} -6 & 8 \\ -2 & 1 \end{vmatrix}}{\left(x^2-2x+1\right)^2}$$

$$=\frac{4x^2-14x+10}{\left(x^2-2x+1\right)^2}$$

Derivative: in the form $\quad f(x)=\dfrac{a_3x^3+a_2x^2+a_1x+a_0}{b_3x^3+b_2x^2+b_1x+b_0}$

$$u=a_3x^3+a_2x^2+a_1x+a_0 \quad\to\quad u'=3a_3x^2+2a_2x+a_1$$
$$v=b_3x^3+b_2x^2+b_1x+b_0 \quad\to\quad v'=3b_3x^2+2b_2x+b_1$$

$$\begin{aligned} u'v-v'u &= \left(3a_3x^2+2a_2x+a_1\right)\left(b_3x^3+b_2x^2+b_1x+b_0\right) \\ &\quad -\left(3b_3x^2+2b_2x+b_1\right)\left(a_3x^3+a_2x^2+a_1x+a_0\right) \end{aligned}$$

x^5	x^4	x^3	x^2	x^1	x^0
$3a_3b_3$	$3a_3b_2$	$3a_3b_1$	$3a_3b_0$		
$-3a_3b_3$	$2a_2b_3$	$2a_2b_2$	$2a_2b_1$	$2a_2b_0$	
	$-3a_2b_3$	a_1b_3	a_1b_2	a_1b_1	a_1b_0
	$-2a_3b_2$	$-3a_1b_3$	$-3a_0b_3$		
		$-2a_2b_2$	$-2a_1b_2$	$-2a_0b_2$	
		$-a_3b_1$	$-a_2b_1$	$-a_1b_1$	$-a_0b_1$

$$f'(x)=\frac{\left(a_3b_2-a_2b_3\right)x^4+2\left(a_3b_1-a_1b_3\right)x^3+\left(\left(a_2b_1-a_1b_2\right)+3\left(a_3b_0-a_0b_3\right)\right)x^2 + 2\left(a_2b_0-a_0b_2\right)x+a_1b_0-a_0b_1}{\left(b_3x^3+b_2x^2+b_1x+b_0\right)^2}$$

$$=\frac{\begin{vmatrix}a_3 & a_2\\ b_3 & b_2\end{vmatrix}x^4+2\begin{vmatrix}a_3 & a_1\\ b_3 & b_1\end{vmatrix}x^3+\left(\begin{vmatrix}a_2 & a_1\\ b_2 & b_1\end{vmatrix}+3\begin{vmatrix}a_3 & a_0\\ b_3 & b_0\end{vmatrix}\right)x^2+2\begin{vmatrix}a_2 & a_0\\ b_2 & b_0\end{vmatrix}x+\begin{vmatrix}a_1 & a_0\\ b_1 & b_0\end{vmatrix}}{\left(b_3x^3+b_2x^2+b_1x+b_0\right)^2}$$

Example

$$f(x)=\frac{x^3+2x^2-6x+2}{2x^3+x^2-2x+1}$$

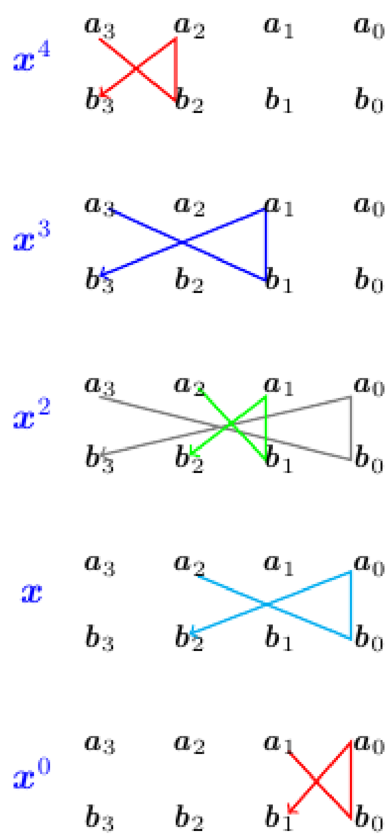
$1 \quad 2 \quad -6 \quad 2$

$2 \quad 1 \quad -2 \quad 1$

Solution

$$f'(x)=\frac{(1-4)x^4+\color{red}{2}(10)x^3+\left((-4+6)+\color{red}{3}(1-4)\right)x^2+\color{red}{2}(2-2)x+(-6+4)}{\left(2x^3+x^2-2x+1\right)^2}$$

$$=\frac{-3x^4+20x^3-7x^2-2}{\left(2x^3+x^2-2x+1\right)^2}$$



Derivative: in the form $$f(x)=\frac{a_4x^4+a_3x^3+a_2x^2+a_1x+a_0}{b_4x^4+b_3x^3+b_2x^2+b_1x+b_0}$$

$$u'v-v'u=\left(4a_4x^3+3a_3x^2+2a_2x+a_1\right)\left(b_4x^4+b_3x^3+b_2x^2+b_1x+b_0\right)$$
$$-\left(4b_4x^3+3b_3x^2+2b_2x+b_1\right)\left(a_4x^4+a_3x^3+a_2x^2+a_1x+a_0\right)$$

x^7 $\quad 4a_4b_4-4a_4b_4$

x^6 $\quad 4a_4b_3+3a_3b_4-4a_3b_4-3a_4b_3$

x^5 $\quad 4a_4b_2+3a_3b_3+2a_2b_4-4a_2b_4-3a_3b_3-2a_4b_2$

x^4 $\quad 4a_4b_1+3a_3b_2+2a_2b_3+a_1b_4-4a_1b_4-3a_2b_3-2a_3b_2-a_4b_1$

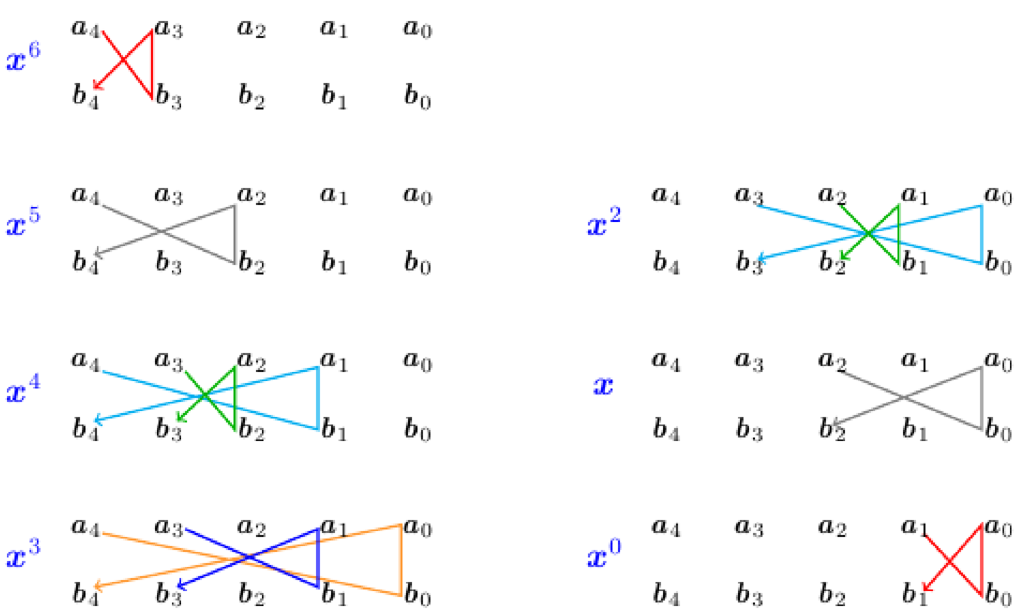
x^3 $\quad 4a_4b_0+3a_3b_1+2a_2b_2+a_1b_3-4a_0b_4-3a_1b_3-2a_2b_2-a_3b_1$

x^2 $\quad 3a_3b_0+2a_2b_1+a_1b_2-3a_0b_3-2a_1b_2-a_2b_1$

x^1 $\quad 2a_2b_0+a_1b_1-2a_0b_2-a_1b_1$

x^0 $\quad a_1b_0-a_0b_1$

$$f'(x)=\frac{\begin{array}{l}\left(a_4b_3-a_3b_4\right)x^6+2\left(a_4b_2-a_2b_4\right)x^5+\left(3\left(a_4b_1-a_1b_4\right)+\left(a_3b_2-a_2b_3\right)\right)x^4\\+\left(4\left(a_4b_0-a_0b_4\right)+2\left(a_3b_1-a_1b_3\right)\right)x^3\\+\left(\left(a_2b_1-a_1b_2\right)+3\left(a_3b_0-a_0b_3\right)\right)x^2+2\left(a_2b_0-a_0b_2\right)x+a_1b_0-a_0b_1\end{array}}{\left(b_4x^4+b_3x^3+b_2x^2+b_1x+b_0\right)^2}$$



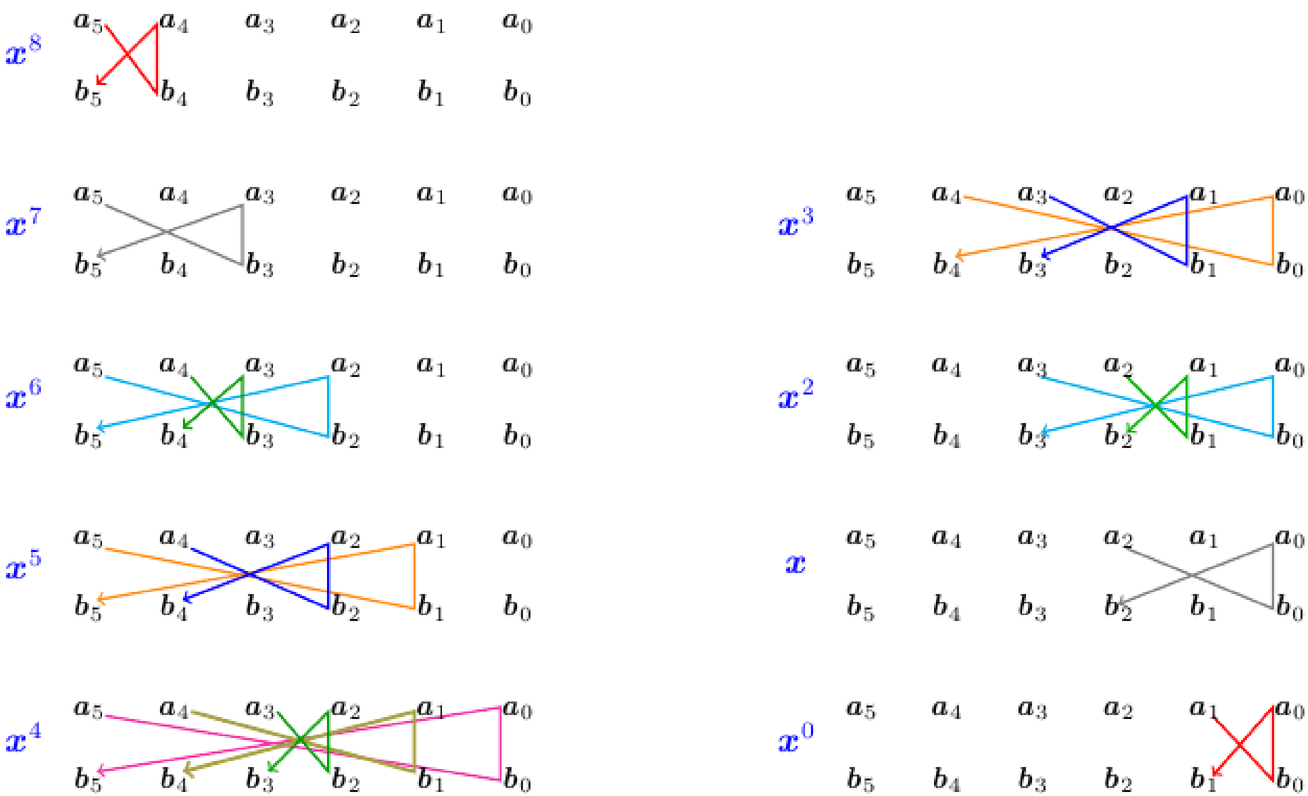
Derivative: in the form $\quad f(x)=\dfrac{a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0}{ba_5x^5+b_4x^4+b_3x^3+b_2x^2+b_1x+b_0}$

$$u=a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0 \quad\to\quad u'=5a_5x^4+4a_4x^3+3a_3x^2+2a_2x+a_1$$
$$v=b_5x^5+b_4x^4+b_3x^3+b_2x^2+b_1x+b_0 \quad\to\quad v'=5b_5x^4+4b_4x^3+3b_3x^2+2b_2x+b_1$$

$$\begin{aligned}u'v-v'u&=\left(5a_5x^4+4a_4x^3+3a_3x^2+2a_2x+a_1\right)\left(b_5x^5+b_4x^4+b_3x^3+b_2x^2+b_1x+b_0\right)\\&\quad-\left(5b_5x^4+4b_4x^3+3b_3x^2+2b_2x+b_1\right)\left(a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0\right)\end{aligned}$$

x^9	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x^1	x^0
$5a_5b_5$	$5a_5b_4$	$5a_5b_3$	$5a_5b_2$	$5a_5b_1$	$5a_5b_0$				
$-5a_5b_5$	$4a_4b_5$	$4a_4b_4$	$4a_4b_3$	$4a_4b_2$	$4a_4b_1$	$4a_4b_0$			
	$-5a_4b_5$	$3a_3b_5$	$3a_3b_4$	$3a_3b_3$	$3a_3b_2$	$3a_3b_1$	$3a_3b_0$		
	$-4a_5b_4$	$-5a_3b_5$	$2a_2b_5$	$2a_2b_4$	$2a_2b_3$	$2a_2b_2$	$2a_2b_1$	$2a_2b_0$	
		$-4a_4b_4$	$-5a_2b_5$	a_1b_5	a_1b_4	a_1b_3	a_1b_2	a_1b_1	a_1b_0
		$-3a_5b_3$	$-4a_3b_4$	$-5a_1b_5$	$-5a_0b_5$	$-4a_0b_4$	$-3a_0b_3$	$-2a_0b_2$	$-a_0b_1$
			$-3a_4b_3$	$-4a_2b_4$	$-4a_1b_4$	$-3a_1b_3$	$-2a_1b_2$	$-a_1b_1$	
			$-2a_5b_2$	$-3a_3b_3$	$-3a_2b_3$	$-2a_2b_2$	$-a_2b_1$		
				$-2a_4b_2$	$-2a_3b_2$	$-a_3b_1$			
				$-a_5b_1$	$-a_4b_1$				

$$f'(x)=\frac{\begin{aligned}&\left(a_5b_4-a_4b_5\right)x^8\quad+2\left(a_5b_3-a_3b_5\right)x^7\\&+\left(3\left(a_5b_2-a_2b_5\right)+\left(a_4b_3-a_3b_4\right)\right)x^6\\&+\left(4\left(a_5b_1-a_1b_5\right)+2\left(a_4b_2-a_2b_4\right)\right)x^5\\&+\left(5\left(a_5b_0-a_0b_5\right)+3\left(a_4b_1-a_1b_4\right)+\left(a_3b_2-a_2b_3\right)\right)x^4\\&+\left(4\left(a_4b_0-a_0b_4\right)+2\left(a_3b_1-a_1b_3\right)\right)x^3\\&+\left(3\left(a_3b_0-a_0b_3\right)+\left(a_2b_1-a_1b_2\right)\right)x^2\\&+2\left(a_2b_0-a_0b_2\right)x\quad+\left(a_1b_0-a_0b_1\right)\end{aligned}}{\left(b_5x^5+b_4x^4+b_3x^3+b_2x^2+b_1x+b_0\right)^2}$$


x^8 a_5 a_4 a_3 a_2 a_1 a_0 / b_5 b_4 b_3 b_2 b_1 b_0
x^7 a_5 a_4 a_3 a_2 a_1 a_0 / b_5 b_4 b_3 b_2 b_1 b_0
x^6 a_5 a_4 a_3 a_2 a_1 a_0 / b_5 b_4 b_3 b_2 b_1 b_0
x^5 a_5 a_4 a_3 a_2 a_1 a_0 / b_5 b_4 b_3 b_2 b_1 b_0
x^4 a_5 a_4 a_3 a_2 a_1 a_0 / b_5 b_4 b_3 b_2 b_1 b_0
x^3 a_5 a_4 a_3 a_2 a_1 a_0 / b_5 b_4 b_3 b_2 b_1 b_0
x^2 a_5 a_4 a_3 a_2 a_1 a_0 / b_5 b_4 b_3 b_2 b_1 b_0
x a_5 a_4 a_3 a_2 a_1 a_0 / b_5 b_4 b_3 b_2 b_1 b_0
x^0 a_5 a_4 a_3 a_2 a_1 a_0 / b_5 b_4 b_3 b_2 b_1 b_0

Exponential **Function**

$$a^{mx+n} = b^{px+q} \quad \Rightarrow \quad x = \frac{q\ln b - n\ln a}{m\ln a - p\ln b} \qquad \text{coefficient}\,\frac{no\ x's}{x's}$$

Numerator: multiply q with $\ln b$ ***minus*** multiply n with $\ln a$

Denominator: multiply m with $\ln a$ ***minus*** multiply p with $\ln b$

Proof

$$\ln a^{mx+n} = \ln b^{px+q}$$

$$(mx+n)\ln a = (px+q)\ln b$$

$$mx\ln a + n\ln a = px\ln b + q\ln b$$

$$mx\ln a - px\ln b = q\ln b - n\ln a$$

$$x(m\ln a - p\ln b) = q\ln b - n\ln a$$

$$\boxed{x = \frac{q\ln b - n\ln a}{m\ln a - p\ln b}}$$

$mx\ln a + n\ln a = px\ln b + q\ln b$

Example

Solve: $3^{2x-1} = 7^{x+1}$

Solution

$$\boxed{x = \frac{\ln 3 + \ln 7}{2\ln 3 - \ln 7}}$$

Example

Solve: $4^{x+3} = 3^{-x}$

Solution

$$\boxed{x = \frac{-3\ln 4}{\ln 4 + \ln 3}}$$

$$\ln 4^{\boxed{1}x+3} - \ln 3^{\boxed{-1}x}$$

$-\ln$

Growth & Decay Formula

$$A = A_0 e^{kt} \quad \Rightarrow \quad kT = \ln \frac{A}{A_0}$$

Proof

$$A = A_0 e^{kt}$$

$$\frac{A}{A_0} = e^{kt}$$

$$\ln \frac{A}{A_0} = \ln e^{kt}$$

$$\boxed{\ln \frac{A}{A_0} = kt} \quad \checkmark$$

Integration by Part

Evaluate $\int x^n e^{ax}\, dx$

		$\int e^{ax}$
+	x^n	$\frac{1}{a}e^{ax}$
−	nx^{n-1}	$\frac{1}{a^2}e^{ax}$
+	$n(n-1)x^{n-2}$	$\frac{1}{a^3}e^{ax}$
−	$n(n-1)(n-2)x^{n-3}$	$\frac{1}{a^4}e^{ax}$
	$\vdots\ \vdots$	$\vdots\ \vdots$

$$\int x^n e^{ax}\, dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a^2}x^{n-1}e^{ax} + \frac{n(n-1)}{a^3}x^{n-2}e^{ax} - \frac{n(n-1)(n-2)}{a^4}x^{n-3}e^{ax} + \dots$$

$$= e^{ax}\sum_{k=0}^{n}(-1)^k \cdot \frac{n!}{(n-k)!}\cdot\frac{1}{a^{k+1}}\cdot x^{n-k}$$

Inverse Functions

$$f(x) = \frac{ax+b}{cx+d} \quad \Rightarrow \quad f^{-1}(x) = \frac{-dx+b}{cx-a}$$

Proof

$$y = \frac{ax+b}{cx+d}$$

$$x = \frac{ay+b}{cy+d}$$

$$cxy + dx = ay + b$$

$$cxy - ay = -dx + b$$

$$(cx-a)y = -dx + b$$

$$y = \frac{-dx+b}{cx-a}$$

$$\boxed{f^{-1}(x) = \frac{-dx+b}{cx-a}} \quad \surd$$

Interchange ***a*** and ***d*** and change there signs.

Example

Find the inverse function of: $f(x) = \frac{1}{3x-2}$

Solution

$$f^{-1}(x) = \frac{2x+1}{3x} \qquad\qquad f(x) = \frac{0x+1}{3x-2}$$

Example

Find the inverse function of: $f(x) = \frac{3x+2}{2x-5}$

Solution

$$f^{-1}(x) = \frac{5x+2}{2x-3} \qquad\qquad f(x) = \frac{3x+2}{2x-5}$$

Example

Find the inverse function of: $f(x) = \frac{4x}{x+2}$

Solution

$$f^{-1}(x) = \frac{-2x}{x-4} \qquad\qquad f(x) = \frac{4x}{x+2}$$

Jose's Method

Evaluate $\int e^{ax}\cos bx\,dx$

$$\int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx$$

$$\left(1+\frac{a^2}{b^2}\right)\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{b^2}(b\sin bx + a\cos bx)$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(b\sin bx + a\cos bx) + C$$

		$\int \cos bx\,dx$
$+$	e^{ax}	$\frac{1}{b}\sin bx$
$-$	ae^{ax}	$-\frac{1}{b^2}\cos bx$
$+$	a^2e^{ax}	$-\frac{1}{b^2}\int \cos bx\,dx$

Proof

Find $\int e^{ax}\cos bx\,dx$

Solution

Let: $u = e^{ax}$ $\quad dv = \cos bx dx$

$du = ae^{ax}dx \quad v = \int \cos bx dx = \frac{1}{b}\sin bx$

$$\int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx dx \qquad \int udv = uv - \int vdu$$

Let: $u = e^{ax}$ $\quad dv = \sin bx dx$

$du = ae^{ax}dx \quad v = \int \sin bx dx = -\frac{1}{b}\cos bx$

$$\int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\left[-\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx\right]$$

$$= \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx$$

$$\int e^{ax}\cos bx\,dx + \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx + C_1$$

$$\frac{a^2+b^2}{b^2}\int e^{ax}\cos bx\,dx = \frac{1}{b^2}e^{ax}(b\sin bx + a\cos bx) + C_1$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(b\sin bx + a\cos bx) + C$$

Length

***Length* of a curve** $y = f(x)$ is given by the formula:

$$L = \int_c^d \sqrt{1 + [f'(x)]^2}\, dx = \int_c^d \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If $f(x) = ax^m + bx^n$, then

$$\boldsymbol{L} = \int_c^d \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx = \left(ax^m - bx^n\right)\Big|_c^d$$

Iff $f(x)$ satisfies these 2 conditions:

1. $m + n = 2$

2. $abmn = -\frac{1}{4}$

Proof

$$f'(x) = max^{m-1} + nbx^{n-1}$$

$$1 + (f')^2 = 1 + \left(max^{m-1} + nbx^{n-1}\right)^2$$

$$= 1 + m^2a^2x^{2m-2} + 2abmnx^{m+n-2} + n^2b^2x^{2n-2}$$

We need to combined to a perfect square $\quad a^2 - 2ab + b^2 = (a-b)^2$

➢ If $x^{m+n-2} = 1 = x^0 \quad \rightarrow \quad \boxed{m + n = 2}$

$$= m^2a^2x^{2m-2} + (1 + 2abmn) + n^2b^2x^{2n-2} \qquad a^2 - 2ab + b^2 = (a-b)^2$$

➢ Let $1 + 2abmn = -2abmn \quad \rightarrow \quad \boxed{abmn = -\frac{1}{4}}$

$$= m^2a^2x^{2m-2} - 2abmn + n^2b^2x^{2n-2} \qquad x^{2(m+n-2)} = 1$$

$$= \left(max^{m-1} - nbx^{n-1}\right)^2$$

$$\boldsymbol{L} = \int_c^d \sqrt{\left(max^{m-1} - nbx^{n-1}\right)^2}\, dx$$

$$= \int_c^d \left(max^{m-1} - nbx^{n-1}\right) dx$$

$$= \left(ax^m - bx^n\right)\Big|_c^d \quad \checkmark$$

Example

Find the length of the graph of $f(x)=\frac{x^3}{12}+\frac{1}{x}, \quad 1\le x\le 4$

Solution

$$f'(x)=\frac{x^2}{4}-\frac{1}{x^2}$$

$$\begin{aligned}1+\left[f'(x)\right]^2 &=1+\left(\frac{x^2}{4}-\frac{1}{x^2}\right)^2\\ &=1+\frac{x^4}{16}-\frac{1}{2}+\frac{1}{x^4}\\ &=\frac{x^4}{16}+\frac{1}{2}+\frac{1}{x^4}\\ &=\left(\frac{x^2}{4}+\frac{1}{x^2}\right)^2\end{aligned}$$

$a=\frac{1}{12}, \quad m=3, \quad b=1, \quad n=-1$

1. $m+n=3-1=2$ ✓
2. $abmn=\frac{1}{12}(1)(3)(-1)=-\frac{1}{4}$ ✓

$$L=\left(\frac{x^3}{12}-\frac{1}{x}\right)\Big|_1^4 \underline{=6\ \ unit}$$

$$\begin{aligned}L&=\int_1^4\sqrt{1+\left(f'(x)\right)^2}\,dx\\ &=\int_1^4\sqrt{\left(\frac{x^2}{4}+\frac{1}{x^2}\right)^2}\,dx\\ &=\int_1^4\left(\frac{x^2}{4}+\frac{1}{x^2}\right)dx\\ &=\left(\frac{x^3}{12}-\frac{1}{x}\right)\Big|_1^4\\ &=\left(\frac{4^3}{12}-\frac{1}{4}\right)-\left(\frac{1}{12}-\frac{1}{1}\right)\\ &=\frac{72}{12}\\ &\underline{=6\ \ unit}\end{aligned}$$

Examples

$f(x)=\frac{1}{3}x^{3/2}-x^{1/2} \quad\to\quad L=\frac{1}{3}x^{3/2}+x^{1/2}+C$

$f(x)=\frac{2}{3}x^{3/2}-\frac{1}{2}x^{1/2} \quad\to\quad L=\frac{2}{3}x^{3/2}+\frac{1}{2}x^{1/2}$

$f(x)=\frac{1}{6}x^3+\frac{1}{2x} \quad\to\quad L=\frac{1}{6}x^3-\frac{1}{2x}+C$

$f(x)=x^3+\frac{1}{12x} \quad\to\quad L=x^3-\frac{1}{12x}+C$

$f(x)=\frac{1}{6}x^3+\frac{1}{2x} \quad\to\quad L=\frac{1}{6}x^3-\frac{1}{2x}+C$

$f(x)=\frac{1}{8}x^4+\frac{1}{4x^2} \quad\to\quad L=\frac{1}{8}x^4-\frac{1}{4x^2}+C$

$f(x)=\frac{1}{4}x^4+\frac{1}{8x^2} \quad\to\quad L=\frac{1}{4}x^4-\frac{1}{8x^2}+C$

$f(x)=x^{1/2}-\frac{1}{3}x^{3/2} \quad\to\quad L=x^{1/2}+\frac{1}{3}x^{3/2}$

If $f(x) = ae^{mx} + be^{nx}$, then

$$\boldsymbol{L} = \int_c^d \sqrt{1+\left(\frac{df}{dx}\right)^2}\, dx \;=\; \left(ae^{mx} - be^{nx}\right|_c^d$$

Iff $f(x)$ satisfies these 2 conditions:

1. $m = -n$

2. $abmn = -\frac{1}{4}$

Proof

$$f'(x) = ame^{mx} + bne^{nx}$$

$$1+(f')^2 = 1+\left(ame^{mx} + bne^{nx}\right)^2$$

$$= 1 + m^2a^2e^{2mx} + 2abmne^{(m+n)x} + n^2b^2e^{2nx}$$

➢ If $e^{(m+n)x} = 1 = e^{(x=0)} \quad \rightarrow \quad \boxed{m-n}$

$$= m^2a^2x^{2m-2} + (1+2abmn) + n^2b^2x^{2n-2}$$ $\qquad a^2 - 2ab + b^2 = (a-b)^2$

➢ Let $\quad 1+2abmn = -2abmn \quad \rightarrow \quad \boxed{abmn = -\frac{1}{4}}$

$$= m^2a^2e^{2mx} - 2abmne^{(m+n)x} + n^2b^2e^{2nx}$$ $\qquad x^{2(m+n-2)} = 1$

$$= \left(ame^{mx} - bne^{nx}\right)^2$$

$$\left(ame^{mx} - bne^{nx}\right)^2 = m^2a^2e^{2mx} - 2abmne^{(m+n)x} + n^2b^2e^{2nx}$$

Example

$$f(x) = 2e^x + \tfrac{1}{8}e^{-x} \quad \rightarrow \quad L = 2e^x - \tfrac{1}{8}e^{-x}$$

$$f(x) = 2e^{\sqrt{2}x} + \frac{1}{16}e^{-\sqrt{2}x} \quad \rightarrow \quad L = 2e^{\sqrt{2}x} - \frac{1}{16}e^{-\sqrt{2}x}$$

Matrix: *Upper triangular with 1' to the Power n*

m x m

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & \frac{1}{(j-1)!}\frac{(n+j-2)!}{(n-1)!} & & \\ 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 1 & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

<u>Note:</u> ***j*** is the column number.

Matrix: *Upper triangular to the Power n*

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \dots & a_m \\ 0 & a_1 & a_2 & a_3 & \dots & a_{m-1} \\ 0 & 0 & a_1 & a_2 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & 0 & a_1 \end{bmatrix}^n = \begin{bmatrix} a_1^n & \Delta & \Delta & \Delta & \dots & \Delta \\ 0 & a_1^n & \Delta & \Delta & \dots & \Delta \\ 0 & 0 & a_1^n & & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & a_1^n & \Delta \\ 0 & 0 & 0 & 0 & 0 & a_1^n \end{bmatrix}$$

$$\Delta = \dots \sum_{u=0} \sum_{s=0} \sum_{q=0} \sum_{p=3} \sum_{r=p+1} \sum_{t=r+1} \dots \frac{1}{w!} \frac{1}{u!} \frac{1}{s!} \frac{1}{q!} \frac{\prod_{k=2+(pq-2q)+(rs-2s)+(ut-2u)+\dots}^{m} (n-m+k)}{(m-1+(q-pq)+(s-rs)+(u-ut)+\dots)!} a_1^{n-m+1+(pq-2q)+\dots} a_2^{m-1+(q-pq)+(s-rs)+\dots} a_p^q \, a_r^s \, a_t^u$$

$$\Delta = \sum_{i=3} \sum_{s_i=0} \sum_{r_i=i+1} \frac{1}{s_i!} \frac{\prod_{k=\alpha}^{m} (n-m+k)}{(m-\beta)!} a_1^{n-m-1+\alpha} a_2^{m-\beta} a_{r_i}^{s_i}$$

$$\alpha = 2 + \sum_{i=3} \left(r_i s_i - 2 s_i \right)$$

$$\beta = 1 + \sum_{i=3} \left(r_i s_i - s_i \right)$$

Quadratic equation

$$ax^2+bx+c=0$$

If $\boldsymbol{a+b+c=0} \Rightarrow x=1,\ \frac{c}{a}$

Proof

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \qquad a+b+c=0 \ \rightarrow\ b=-a-c$$

$$=\frac{-(-a-c)\pm\sqrt{(-a-c)^2-4ac}}{2a}$$

$$=\frac{a+c\pm\sqrt{a^2+2ac+c^2-4ac}}{2a}$$

$$=\frac{a+c\pm\sqrt{a^2-2ac+c^2}}{2a}$$

$$=\frac{a+c\pm\sqrt{(a-c)^2}}{2a}$$

$$=\frac{a+c\pm(a-c)}{2a}$$

$$x_1=\frac{a+c+(a-c)}{2a}=\frac{a+c+a-c}{2a}=\frac{2a}{2a}\ \boxed{=1}$$

$$x_2=\frac{a+c-(a-c)}{2a}=\frac{a+c-a+c}{2a}\quad\frac{2c}{2a}\ \boxed{=\frac{c}{a}}$$

Example

$$2x^2+x-3=0$$

$$2+1-3=0$$

$$\Rightarrow \boxed{x=1,\quad -\frac{3}{2}}$$

Quadratic equation

$$ax^2 + bx + c = 0$$

If $a - b + c = 0 \Rightarrow x = -1,\ -\frac{c}{a}$

Proof

$a - b + c = 0 \ \rightarrow\ b = a + c$

$$\begin{aligned}
x_{1,2} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(a+c) \pm \sqrt{(a+c)^2 - 4ac}}{2a} \\
&= \frac{-a - c \pm \sqrt{a^2 + 2ac + c^2 - 4ac}}{2a} \\
&= \frac{-a - c \pm \sqrt{a^2 - 2ac + c^2}}{2a} \\
&= \frac{-a - c \pm \sqrt{(a-c)^2}}{2a} \\
&= \frac{-a - c \pm (a-c)}{2a}
\end{aligned}$$

$$\begin{aligned}
x_1 &= \frac{-a - c + (a-c)}{2a} \\
&= \frac{-a - c + a - c}{2a} \\
&= \frac{2c}{2a} \\
&= \boxed{-\frac{c}{a}}
\end{aligned}$$

$$\begin{aligned}
x_2 &= \frac{-a - c - (a-c)}{2a} \\
&= \frac{-a - c - a + c}{2a} \\
&= \frac{-2a}{2a} \\
&= \boxed{-1}
\end{aligned}$$

Example

$$2x^2 - x - 3 = 0$$

$$2 - (-1) - 3 = 0$$

$$\Rightarrow \boxed{x = -1,\ \frac{3}{2}}$$

Square Root

$\sqrt{3} \approx 1.732050807568877293$

1 2 3 1.732050807568877293 3 4	$1 \times 1 = 1$	1 → $1 \times 2 = 2$
1 200	$27 \times 7 = 189$	2 → $17 \times 2 = 34$
189 1100	$343 \times 3 = 1029$	3 → $173 \times 2 = 346$
1029 7100	$3462 \times 2 = 6924$	4 → $1732 \times 2 = 3464$
6924 17600 1560000	3464_×_ = 34,64_>15,600 346405×5=1,732,025	17320×2=34640 173205×2=346410
1732025 279750 279750000	346410_×_=? 346410_>2797500 34641008×8=277128064	1732050×2=3464100 17320508×2=34641016
277128064 262193600 26219360000	34641016_×_=? 34641016_>262193600 346410167×7=2424871169	173205080×2=346410160 1732050807×2=3464101614
2424871169 197064875100	34641016145×5=173205080725	17320508075×2=34641016150
173205080725 2385979437500	346410161506×6=2078460969036	173205080756×2=346410161512
2078460969036 30751846846400	3464101615128×8=2771281291024	1732050807568×2=3464101615136
27712812921024 303903392537600	34641016151368×8=27712812910944	17320508075688×2=34641016151376
277128129210944 2677526332665600	346410161513767×7=2424871130596369	173205080756887×2=346410161513774
2424871130596369 25265520206923100	3464101615137747×7=24248711305964229	1732050807568877×2=3464101615137754
24248711305964229 101680890095887100	34641016151377542×2=69282032302755084	17320508075688772×2=34641016151377544
69282032302755084 3239885779313201600	346410161513775449×9=3117691453623979041	173205080756887729×2=346410161513775459
3117691453623979041 122219432568922255900	3464101615137754593×3=10392304845413263263779	
10392304845413263263779 1827127723508992121		

Surface

***Surface* of a curve** $y=f(x)$ is given by the formula:

$$S=2\pi\int_a^b f(x)\sqrt{1+\left(f'(x)\right)^2}\,dx$$

If $f(x)=ax^m+bx^n$, then

$$\sqrt{1+\left(f'(x)\right)^2}=\overline{f'(x)}$$

$\overline{f'(x)}$: is the conjugate of $f'(x)$

Iff $f(x)$ satisfies these 2 conditions:

3. $m+n=2$
4. $abmn=-\frac{1}{4}$

Proof

$$f'(x)=max^{m-1}+nbx^{n-1}$$

$$1+\left(f'\right)^2=1+\left(max^{m-1}+nbx^{n-1}\right)^2$$

$$=1+m^2a^2x^{2m-2}+2abmnx^{m+n-2}+n^2b^2x^{2n-2}$$

We need to combined to a perfect square $\quad a^2-2ab+b^2=(a-b)^2$

- If $x^{m+n-2}=1=x^0 \rightarrow \boxed{m+n=2}$

$$=m^2a^2x^{2m-2}+(1+2abmn)+n^2b^2x^{2n-2} \qquad a^2-2ab+b^2=(a-b)^2$$

- Let $1+2abmn=-2abmn \rightarrow \boxed{abmn=-\frac{1}{4}}$

$$=m^2a^2x^{2m-2}-2abmn+n^2b^2x^{2n-2} \qquad x^{2(m+n-2)}=1$$

$$=\left(max^{m-1}-nbx^{n-1}\right)^2$$

$$\sqrt{\left(max^{m-1}-nbx^{n-1}\right)^2}=max^{m-1}-nbx^{n-1} \quad \checkmark$$

$$f'(x)=max^{m-1}+nbx^{n-1} \Rightarrow \sqrt{1+\left(f'(x)\right)^2}=max^{m-1}-nbx^{n-1}=\overline{f'(x)}$$

Example

Find the surface of the graph of $f(x)=\frac{x^3}{12}+\frac{1}{x}, \quad 1\le x\le 4$

Solution

$$f'(x)=\frac{x^2}{4}-\frac{1}{x^2}$$

$$\begin{aligned}1+\left[f'(x)\right]^2 &= 1+\left(\frac{x^2}{4}-\frac{1}{x^2}\right)^2\\ &= 1+\frac{x^4}{16}-\frac{1}{2}+\frac{1}{x^4}\\ &= \frac{x^4}{16}+\frac{1}{2}+\frac{1}{x^4}\\ &= \left(\frac{x^2}{4}+\frac{1}{x^2}\right)^2\end{aligned}$$

$a=\frac{1}{12}, \quad m=3, \quad b=1, \quad n=-1$

3. $m+n=3-1=2$ ✓

4. $abmn=\frac{1}{12}(1)(3)(-1)=-\frac{1}{4}$ ✓

$$\begin{aligned}S &= 2\pi\int_1^4 f(x)\sqrt{1+\left(f'(x)\right)^2}\, dx\\ &= 2\pi\int_1^4 \left(\frac{x^3}{12}+\frac{1}{x}\right)\sqrt{\left(\frac{x^2}{4}+\frac{1}{x^2}\right)^2}\, dx\\ &= 2\pi\int_1^4 \left(\frac{x^3}{12}+\frac{1}{x}\right)\left(\frac{x^2}{4}+\frac{1}{x^2}\right) dx\\ &= 2\pi\int_1^4 \left(\frac{1}{48}x^5+\frac{1}{12}x+\frac{1}{4}x+x^{-3}\right) dx\\ &= 2\pi\left(\frac{1}{288}x^6+\frac{1}{6}x^2-\frac{1}{2x^2}\right)\Big|_1^4\\ &= \pi\left(\frac{256}{9}+\frac{16}{3}-\frac{1}{16}-\frac{1}{144}-\frac{1}{3}+1\right)\\ &= \boxed{\frac{275}{8}\pi \quad unit^2}\end{aligned}$$

$$S=2\pi\int_1^4 \left(\frac{x^3}{12}+\frac{1}{x}\right)\left(\frac{x^2}{4}+\frac{1}{x^2}\right) dx$$

If $f(x) = ae^{mx} + be^{nx}$, then

$$S = 2\pi \int_a^b f(x)\sqrt{1+\left(f'(x)\right)^2}\ dx\ =\ 2\pi \int_a^b f(x)\,\overline{f'(x)}\ dx$$

Iff $f(x)$ satisfies these 2 conditions:

3. $m = -n$

4. $abmn = -\frac{1}{4}$

Proof

$f'(x) = ame^{mx} + bne^{nx}$

$1+\left(f'\right)^2 = 1+\left(ame^{mx} + bne^{nx}\right)^2$

$= 1 + m^2a^2e^{2mx} + 2abmne^{(m+n)x} + n^2b^2e^{2nx}$

➢ If $e^{(m+n)x} = 1 = e^{(x=0)} \quad \rightarrow \quad \boxed{m-n}$

$= m^2a^2x^{2m-2} + (1+2abmn) + n^2b^2x^{2n-2}$ $\qquad a^2 - 2ab + b^2 = (a-b)^2$

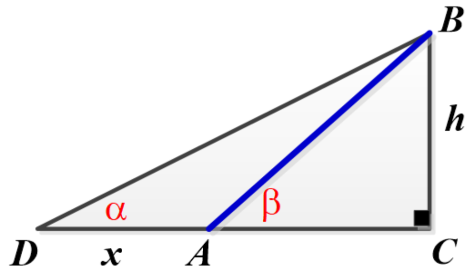
➢ Let $\quad 1+2abmn = -2abmn \quad \rightarrow \quad \boxed{abmn = -\frac{1}{4}}$

$= m^2a^2e^{2mx} - 2abmne^{(m+n)x} + n^2b^2e^{2nx}$ $\qquad x^{2(m+n-2)} = 1$

$= \left(ame^{mx} - bne^{nx}\right)^2$

$\sqrt{1+\left(f'\right)^2} = \sqrt{\left(ame^{mx} - bne^{nx}\right)^2}$

$\sqrt{1+f'(x)} = \overline{f'(x)}$ ✓



Proof

Triangle DCB: $\tan\alpha = \dfrac{h}{50+x} \quad \Rightarrow \quad h = (50+x)\tan\alpha$

Triangle ACB: $\tan\beta = \dfrac{h}{x} \quad \Rightarrow \quad h = x\tan\beta$

$x\tan\beta = (50+x)\tan\alpha$

$x\tan\beta = 50\tan\alpha + x\tan\alpha$

$x\tan\beta - x\tan\alpha = 50\tan\alpha$

$x(\tan\beta - \tan\alpha) = 50\tan\alpha$

$x = \dfrac{50\tan\alpha}{\tan\beta - \tan\alpha}$

$$\boxed{h = x\,\dfrac{50\tan\alpha\tan\beta}{\tan\beta - \tan\alpha}}$$

Height is equal to distance times (tan tan) divides by the (tan(larger angle) – tan) (difference between tangents)

Example

From a given point on the ground, the angle of elevation to the top of a tree is 36.7°. From a second point, 50 *feet* back, the angle of elevation to the top of the tree is 22.2°. Find the height of the tree to the nearest foot.

Solution

$h = 50\dfrac{\tan 22.2°\ \tan 36.7°}{\tan 36.7° - \tan 22.2°}$

$\approx 45\ ft$