1.3 – Superposition (*Interference*) of Waves and Standing Waves

Two waves are said to interfere if they act at the same location at the same time when two or more waves act at the same position (*same particle*) at the same time, the net effect is obtained by adding the wave algebraically. If the waves $y_1, y_2, y_3, \ldots$ are acting at a certain position at the same time, then the net waves $\left(y_{net}\right)$ is obtained as

$$y_{net} = y_1 + y_2 + y_3 + \ldots$$

Superposition of Harmonic Waves with Different Phase Angles (*but same frequency and wavelength*)

At the point of interference (x = constant), the two waves become harmonic oscillation that depend on time only which can be represented as, $y_1 = A_1 \cos\left(\omega t - \beta_1\right)$ and $y_2 = A_2 \cos\left(\omega t - \beta_2\right)$. There are 3 factors that contribute to the phase angles (φ_1 and φ_2) of the harmonic oscillation at the point of interference. The first is the initial phase angle which is the argument of the wave at t = 0 and x = 0. The second is the difference between the distances travelled by the two waves by the time they reach the interference point. The third is the difference between the times the waves were initiated. Let the two waves be represented as

$$y_1 = A_1 \cos\left(\omega t - Kx_1 - \phi_1\right)$$

$$y_2 = A_2 \cos\left(\omega t - Kx_2 - \phi_2\right)$$

Let the difference between the distance travelled by the two waves (*commonly refereed as path difference*) be denoted by Δx; that is $\Delta x = x_2 - x_1 \quad or \quad x_2 = \Delta x + x_1$. Let the difference between the times the two waves were initiated (*commonly referred as time lag*) be represented by Δt; that is $\Delta t = t_2 - t_1 \quad or \quad t_2 = \Delta t + t_1$. Now the two waves can be written as

$$y_1 = A_1 \cos\left(\omega t - Kx_1 - \phi_1\right)$$

$$y_2 = A_2 \cos\left(\omega t - K\left(x_1 + \Delta x\right) - \phi_2\right)$$

Dropping the subscript 1, because it is not needed any more

$$y_1 = A_1 \cos\left(\omega t - Kx - \phi_1\right)$$

$$y_2 = A_2 \cos\left(\omega t - Kx - \left(K\Delta x - \omega\Delta t - \phi_2\right)\right)$$

Therefore the phase angles of the harmonic oscillations at the point of interference are

$$\beta_1' = Kx + \phi_1$$

$$\beta_2' = Kx + (K\Delta x - \omega\Delta t + \phi_2)$$

And the phase difference between the two harmonic oscillations at the point of interference is

$$\beta_2' - \beta_1' = Kx + \left(K\Delta x - \omega\Delta t + \phi_2\right) - \left(Kx + \phi_1\right)$$

$$= K\Delta x - \omega\Delta t + \left(\phi_2 - \phi_1\right)$$

Where Δx is the path difference, Δt is the time lag and $\phi_2 - \phi_1$ is the difference between the initial phase angles (*arguments at t = 0 and x = 0*) The net oscillation is obtained by adding the two oscillators algebraically.

$$\delta_{net} = A_1\cos(\omega t - \beta_1') + A_2\cos(\omega t - \beta_2')$$

And expanding the cosines

$$\delta_{net} = A_1(\cos\beta_1'\cos(\omega t) + \sin\beta_1'\sin(\omega t)) + A_2(\cos\beta_2'\cos(\omega t) + \sin\beta_2'\sin(\omega t))$$

$$= (A_1\cos\beta_1' + A_2\cos\beta_2')\cos\omega t + (A_1\sin\beta_1' + A_2\sin\beta_2')\sin\omega t$$

These two terms can be combined into one term by introducing two values A and δ by

$A\cos\delta = A_1\cos\beta_1' + A_2\cos\beta_2'$ eq.1

$A\sin\delta = A_1\sin\beta_1' + \sin\beta_2'$ equ.2

$$\therefore y_{net} = A\cos\delta\cos\omega t + A\sin\delta\sin\omega t = A\cos(\omega t - \delta)$$

Squaring and adding equations 1 and 2.

$$A^2sin^2\delta + A^2cos^2\delta = (A_1\cos\beta_1' + A_2\cos\beta_2')^2 + (A_1\sin\beta_1' + \sin\beta_2')^2$$

$$A^2(cos^2\delta + sin^2\delta) = A^2$$

$$= A_1^2cos^2\beta_1^2 + A_2^2cos^2\beta_2^1 + 2A_1A_2cos\beta_1'cos\beta_2' + A_1^2sin^2\beta_1' + A_2^2sin^2\beta_2' + 2A_1A_2sin\beta_1'sin\beta_2'$$

$$=A_1^2(cos^2\beta_1' + sin^2\beta_1') + A_2^2(cos^2\beta_2' + sin^2\beta_1') + 2A_1A_2(cos\beta_1'cos\beta_2' + sin\beta_1'sin\beta_2')$$

$$\Rightarrow A^2(cos^2\delta + sin^2\delta) = A^2$$

$$= A_1^2cos^2\beta_1'^2 + A_2^2cos^2\beta_2' + 2A_1A_2cos\beta_1'cos\beta_2' + A_1^2sin^2\beta_2' + 2A_1A_2sin\beta_1'sin\beta_2'$$

$$= A_1^2(cos^2\beta_1' + sin^2\beta_1') + A_2^2(cos^2\beta_2' + sin^2\beta_1') + 2A_1A_2(cos\beta_1'cos\beta_2' + sin\beta_1'sin\beta_2')$$

$$\Longrightarrow A^2 = A_1^2 + A_2^2 + 2A_1A_2cos(\beta_2' - \beta_1')$$

$$\Rightarrow A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2cos(\beta_2' - \beta_1')}$$

and with $\beta_2' = \beta_2$ and $\beta_1' = \beta_1$

$$\boxed{A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2cos\left(\beta_2 - \beta_1\right)}}$$

Where A is the amplitude of the net oscillation and $\beta_2 - \beta_1$ is the phase difference between the two waves at the point of interference similarly dividing equitation 2 by1 to get an expression for tanδ gives the following expression for δ

$$\boxed{\delta = tan^{-1}\left(\frac{A_1sin\beta_1 + A_2sin\beta_2}{A_1cos\beta_1 + A_2cos\beta_2}\right)}$$

(*+π if the denominator is <0*)

Where δ is their phase angle of the net harmonic oscillator at the point of inference. The net oscillation varies with time at the point of inference according to the equation

$$\boxed{y_{net} = A cos(\omega t - \delta)}$$

Where A and δ are given by the above expressions.

The following are some special cases of the general formula.

1. *Special Cases*

Wave with the same amplitude ($A_1 = A_2$) with $A_1 = A_2$

$$A = \sqrt{A_1^2 + A_1^2 + 2A_1^2 cos(\beta_2 - \beta_1)}$$

$$= \sqrt{2A_1^2 + 2A_1^2 cos(\beta_2 - \beta_1)}$$

$$= \sqrt{2}|A_1|\sqrt{1 + cos(\beta_2 - \beta_1)}$$

And using the trigonometric formula $\cos^2 x = \frac{1+\cos 2x}{2}$

$$A = \sqrt{2}|A_1|\sqrt{2cos^2\left(\frac{\beta_2 - \beta_1}{2}\right)}$$

$$A = 2|A_1|cos\left(\frac{\beta_2 - \beta_1}{2}\right)$$

Amplitude of the net wave due to the superposition of two harmonic waves with same amplitude.

2. *Constructive Interference*

Constructive interference is superposition of two waves resulting in the maximum possible amplitude.

The maximum valve of $A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2 cos(\beta_2 - \beta_1)}$ occurs when $cos(\beta_2 - \beta_1) = 1$ because the maximum value of cosine is one. Therefore for constructive interference.

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2} = \sqrt{(A_1 + A_2)^2} = |A_1 + A_2| = A_1 + A_2$$

Because $A_1 > 0$ and $A_2 > 0$

$$A = A_1 + A_2$$

(*maximum possible amplitude*)

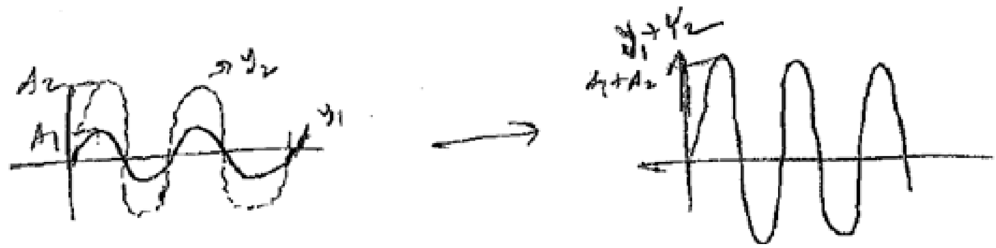
The condition for constructive interference is that $\cos(\beta_2 - \beta_1) = 1$ or

$$\beta_2 - \beta_1 = \cdots - 4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$$

or

$$\beta_2 - \beta_1 = 2n\pi \quad \text{where } n\text{: any integer}$$
(*condition for constructive interference*)

$\beta_2 - \beta_1$ is called the phase shift of wave 2 with respect to wave 1. And when $\beta_2 - \beta_1 = 2n\pi$ (*n integer*) we say the two waves are in phase shift.



Constructive interference

3. *Destructive Interference*

Destructive interference is superposition resulting in the minimum possible amplitude of the net wave. The minimum value of $A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2 cos\left(\beta_2 - \beta_1\right)}$ occurs when cos ($\beta_2 - \beta_1$) = –1 because the minimum value of cosine is –1. Therefore a destructive interference

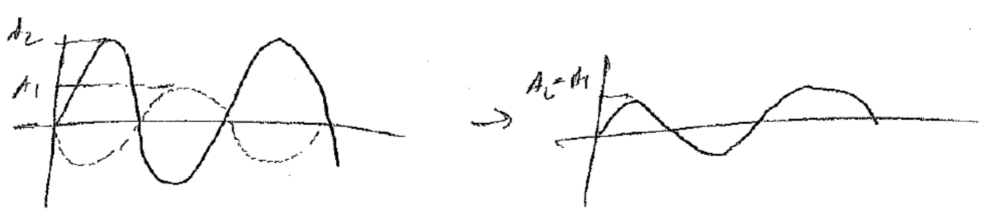
$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2(-1)} = \sqrt{(A_1 - A_2)^2}$$
$$A = |A_1 - A_2|$$

(*minimum possible amplitude*)

The condition for destructive interference is that $\cos\left(\beta_2 - \beta_1\right) = -1$ or $\beta_2 - \beta_1 = \ldots -3\pi, -\pi, \pi, 5\pi,$ … or

$$\beta_2 - \beta_1 = (2n+1)\pi \quad \text{where } n\text{: any integer}$$
(*Condition for destructive interference*)

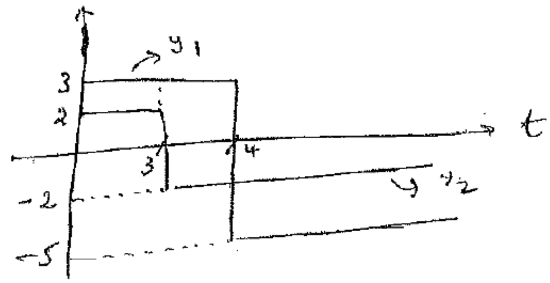
When $\beta_2 - \beta_1 = (2n+1)\pi$ (*n integer*) we say the two waves are out of phase.



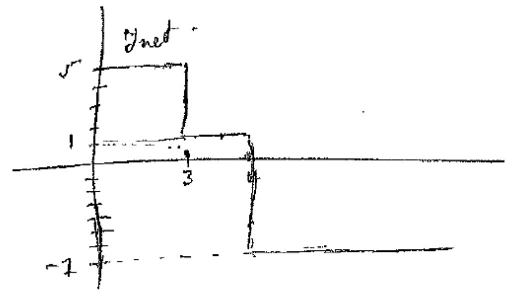
Destructive Interference

Example

Consider the two waves shown graphically find the net wave graphically.



Solution



$$0 \le t < 3, \quad y_1 = 3, \quad y_2 = 2 \quad \Rightarrow \quad y_{net} = y_1 + y_2 = 5$$

$$3 < t < 4, \quad y_1 = 3, \quad y_2 = -2 \Rightarrow \quad y_{net} = y_1 + y_2 = 1$$

$$t > 4, \quad y_1 = -5, \quad y_2 = 2 \qquad \Rightarrow \quad y_{net} = y_1 + y_2 = -7$$

Example

Determine the net amplitude when the following pair of waves meets at the same point at the same time.

a) $y_1 = 2\cos\left(20x - 30t + \frac{\pi}{2}\right) \quad y_2 = 3\cos(10x - 30t + \pi)$

b) $y_1 = -4\cos\left(20x - 50t + \frac{\pi}{3}\right) \quad y_2 = 2\cos(20x - 50t)$

Solution

a) $A_1 = 2$, $A_2 = 3$, $\beta_1 = \pi/2$, $\beta_2 = \pi$

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2 cos(\beta_2 - \beta_1)}$$

$$= \sqrt{2^2 + 3^2 + 2 \times 2 \times 3cos\left(\pi - \frac{\pi}{2}\right)}$$

$$= \sqrt{13}$$

b) First we should convert negative coefficients to positive (*remember A_1 and A_2 need to be positive and $-cos(x) = cos(x+\pi)$*)

$$y_1 = -4cos\left(20x - 50t + \frac{\pi}{3}\right) = 4cos\left(20x - 50t + \frac{\pi}{3} + \pi\right)$$

$A_1 = 4$, $\beta_1 = \pi/3 + \pi = 4/3\pi$, $A_2 = 2$, $\beta_2 = 0$

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2cos(\beta_2 - \beta_1)}$$

$$= \sqrt{4^2 + 2^2 + 2 \times 4 \times 2cos\left(\frac{4}{3}\pi - 0\right)}$$

$$= 2\sqrt{3}$$

Example

When two waves of the same amplitude at the same position and time, it is found that the amplitude of the net wave was found to be half of their amplitude. What are the possible values for the phase shift (*difference between their phase angles*) between the waves?

Solution

$$A = 2|A_1|cos\left(\frac{\beta_1 - \beta_2}{2}\right)$$

$$\left|cos\left(\frac{\beta_2 - \beta_1}{2}\right)\right| = 0.5$$

$$cos\left(\frac{\beta_2 - \beta_1}{2}\right) = 0.5\ or - 0.5$$

$$= \frac{\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, -\frac{2\pi}{3}$$

(*of course 2nπ can be added to give the general solution with n integers because periodicity of cosine is 2π*)

$$\beta_2 - \beta_1 = \frac{2\pi}{3}\ or - \frac{2\pi}{3}, \frac{4\pi}{3}\ or - \frac{4\pi}{3}$$

Example

When two waves of amplitude 4 and 5 meet at the same location and same time, what are the minimum and maximum amplitude of the net wave that can be obtained?

Solution

A_{max}= ? A_{min}= ? A_1= 4 A_2= 5

$$A_{max} = A_1 + A_2 = 4 + 5 = 9$$

$$A_{min} = |A_1 + A_2| = |4 - 5| = 1$$

Example

Determine whether the following pair of waves will interfere constructive, destructively or neither.

a) $y_1 = 2\cos\left(kx - \omega t + \frac{\pi}{2}\right) \quad y_2 = 4\cos\left(kx - \omega t + \frac{3\pi}{2}\right)$

b) $y_1 = -5\cos(20x - 30t); \quad y_2 = 3\cos(20x - 30t - \pi)$

c) $y_1 = 2\cos(30x - 50t) \quad y_2 = -3\cos\left(30x - 50t - \frac{\pi}{2}\right)$

Solution

a) $\beta_1 = \pi/2$; $\beta_2 = 3\pi/2$

$\beta_2 - \beta_1 = \frac{3\pi}{2} - \frac{\pi}{2} = \pi$ (*destructive interference*)

b) $\beta_1 = \pi$; $\beta_2 = -\pi$

$\beta_2 - \beta_1 = -\pi - \pi = -2\pi$ (*constructive interference*)

c) $\beta_1 = 0; \quad \beta_2 = -\frac{\pi}{2} + \pi = \frac{\pi}{2}$

$\beta_2 - \beta_1 = \frac{\pi}{2}$ (*neither constructive nor destructive*)

Example

Two waves of the form $y_1 = 5\cos\left(3000t_1 - 10x_1\right) \quad y_2 = 3\cos\left(3000t_2 - 10x_2\right)$ were initiated at the same time but at different points. By the time the two waves meet at a certain point, wave 2 has travelled a distance of 0.2m more than the distance travelled by wave 1.

a) Calculate the amplitude of the net oscillation at the point interference.
b) Calculate the phase angle of the net harmonic oscillation at the point of interference if the distance travelled by the first wave is 5m
c) Give a formula for the net harmonic oscillation at the point of interference as a function of time.

<u>Solution</u>

a) $\varphi_1 = \varphi_2 = 0; \quad \Delta t = 0$ (*initiated at the same time*)

$\Delta x = x_2 - x_1 = 0.2m$ (*were 2 travelled 0.2m more than wave 1*)

$\Omega = 3{,}000$ rad/s $\quad K = 10$ 1/m $\quad A_1 = 5$m $\quad A_2 = 3$m

$$\beta_2 - \beta_1 = k\Delta x - \omega\Delta t + \left(\varphi_2 - \varphi_1\right)$$
$$= 10(0.2) + 3{,}000(0) + (0 - 0)$$
$$= 2\ rad$$

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\left(\beta_2 - \beta_1\right)}$$
$$= \sqrt{5^2 + 3^2 + 2(3)(5)\cos(2)}$$
$$= 4.638\ m$$

b) $x_1 = 5m \quad x_2 = 5 + 0.2 = 5.2m$

$\beta_1 = kx_1 + \varphi = 10(5) = 50; \quad \beta_2 = kx_2 = 10(5.2) = 52$

$$\delta = \tan^{-1}\left(\frac{A_1\sin\beta_1 + A_2\sin\beta_2}{A_1\cos\beta_1 + A_2\cos\beta_2}\right)$$
$$= \tan^{-1}\left(\frac{5\sin(50) + 3\sin(52)}{5\cos(50) + 3\cos(52)}\right)$$
$$= 0.363\ rad$$

c) $y_{net} = y_1 + y_2 = A\cos(\omega t - \delta)$

$$y_{net} = 4.628\cos\left(3{,}000t - 0.363\right)$$

Example

Two waves of the form $y_1 = 5\cos\left(3000t_1 - 10x_1\right)$ $y_2 = 3\cos\left(3000t_2 - 10x_2\right)m$ were initiated at the same point. Wave 1 was initiated 0.004s earlier than wave 2 calculate the amplitude of the net oscillation by the time they meet at a certain point (*actually any point*)

Solution

$\omega = 3{,}000$ $K = 10$ $A_1 = 5$ $A_2 = 3$

$\Delta x = x_2 - x_1 = 0m$; $\Delta t = t_2 - t_1 = 0.004$ $\phi_1 = \phi_2 = 0$

$$\beta_2 - \beta_1 = k\Delta x - \omega\Delta t + \left(\varphi_2 - \varphi_1\right)$$

$$= 10(0) + 3{,}000(0.004) + (0 - 0)$$

$$\underline{= -102\ rad|}$$

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\left(\beta_2 - \beta_1\right)}$$

$$= \sqrt{5^2 + 3^2 + 2(3)(5)\cos(-102)}$$

$$\underline{= 6.7\ m|}$$

Example

Two waves of the form $y_1 = 5\cos\left(3{,}000t_1 - 10x_1 + 0.5\right)$ $y_2 = 3\cos\left(3{,}000t_2 - 10x_2 - 0.2\right)$ are initiated at different times and different points wave 2 travelled 0.4m more than that of wave 1 by the time they meet at the point of interference wave 1 was initiated 0.002s earlier than wave 2. Calculate the amplitude of the net harmonic oscillation at the point of interference.

Solution

Given: $\omega = 3{,}000$; $K = 10$; $A_1 = 5$; $A_2 = 3$ $\Delta x = x_2 - x_1 = 0.4m$; $\Delta t = t_2 - t_1 = 0.002$

$\phi_1 = -0.5$; $\phi_2 = 0.2$

$$\beta_2 - \beta_1 = k\Delta x - \omega\Delta t + \left(\varphi_2 - \varphi_1\right)$$

$$= 10(0.4) + 3{,}000(0.002) + \left(0.2 - (-0.5)\right)$$

$$\underline{= -1.3\ rad|}$$

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\left(\beta_2 - \beta_1\right)}$$

$$= \sqrt{5^2 + 3^2 + 2(3)(5)\cos(-1.3)}$$

$$\underline{= 6.48\ m|}$$