Lecture One

Section 1.1 – Propositional Logic

Introduction

The rules of logic give precise meaning to mathematical statements. These rules are used to distinguish between valid and invalid mathematical arguments.
The logic rules are used in the design of computer circuits, the construction of computer programs, and the verification of the correctness of programs.

Propositions

A proposition is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

All the following declarative sentences are propositions

- ✓ Washington, D.C. is the capital of the United States of America. ***True***
- ✓ $1 + 1 = 2$ ***True***
- ✓ $2 + 2 = 3$ ***False***

Example

Consider the following sentences

1. What time is it?
2. Read this carefully
3. $x+1=2$
4. $x+y=z$

Solution

Sentences 1 and 2 are not propositions because they are not declarative sentences.
Sentences 3 and 4 are not propositions because they are not true (***T***) or false (***F***).

Definition

Let p be a proposition. The negation of p, denoted by $\neg p$ (also denoted by $\overline{p}$), is the statement

"It is not the case that p."

The proposition $\neg p$ is read "not p". The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p.

Example

Find the negation of the proposition: "Michael's PC runs Linux" and express this in simple English.

Solution

The negation: Michael's PC does not run Linux

Example

Find the negation of the proposition: "Vandana's smartphone has at least 32GB of memory" and express this in simple English.

Solution

The negation: Vandana's smartphone has less than 32GB of memory
Vandana's smartphone does not have at least 32GB of memory

***Table*: Truth table of the Negation of a Proposition**

p	$\neg p$
T	***F***
F	***T***

Definition

Let p and q be propositions. The ***conjunction*** of p and q, denoted by $p \wedge q$, is the proposition "p and q."
The *conjunction* $p \wedge q$ p and q is true for both are true and it's false otherwise.

Example

Find the conjunction of the propositions p and q where p is the proposition "your PC has more than 16GB free hard disk space" and q is the proposition "your PC processor runs faster than 1 GHz."

Solution

The conjunction is $p \wedge q$ and can be expressed as:

- ✓ Your PC has more than 16GB free hard disk space and its processor runs faster than 1 GHz.

For this conjunction to be true, both conditions given must be true.
It is false when one or both of these conditions are false.

Definition

Let p and q be propositions. The ***disjunction*** of p and q, denoted by $p \vee q$, is the proposition "p or q."
The disjunction $p \vee q$ is false when both p and q are false and it's true otherwise.

Truth Table for the Conjunction of Two Propositions.		
p	q	$p \wedge q$
T	***T***	***T***
T	***F***	***F***
F	***T***	***F***
F	***F***	***F***

Truth Table for the Disjunction of Two Propositions.		
p	q	$p \vee q$
T	***T***	***T***
T	***F***	***T***
F	***T***	***T***
F	***F***	***F***

Example

Find the disjunction of the propositions p and q where p is the proposition "your PC has more than 16GB free hard disk space" and q is the proposition "your PC processor runs faster than 1 GHz."

Solution

The disjunction is $p \vee q$ and can be expressed as:

- ✓ Your PC has more than 16GB free hard disk space, or the processor in your PC runs faster than 1 GHz.

Definition

Let p and q be propositions. The ***exclusive or*** of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

Truth Table for the Exclusive Or of Two Propositions.		
p	q	$p \oplus q$
T	***T***	***F***
T	***F***	***T***
F	***T***	***T***
F	***F***	***F***

Truth Table for the Conditional Statement $p \to q$.		
p	q	$p \to q$
T	***T***	***T***
T	***F***	***F***
F	***T***	***T***
F	***F***	***T***

Definition

Let p and q be propositions. The ***conditional statement*** $p \to q$, is the proposition "if p, then q." The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \to q$, p called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).

If p, then q	***p implies q***
If p, q	***p only if q***
p is sufficient for q	***a sufficient condition for q is p***
q if p	***q whenever p***
q when p	***q necessary for p***
a necessary condition for p is q	***q follows from p***
q unless $\neg p$	

Example

Let p be the statement "Maria learns discrete mathematics" and q the statement "Maria will find a good job". Express the statement $p \to q$ as a statement in English.

Solution

$p \to q$ represents the statement:

- ✓ If Maria learns discrete mathematics, then she will find a good job.

There are many other way to express this conditional statement.

- ✓ Maria will find a good job when she learns discrete mathematics.
- ✓ For Maria to get a good job, it is sufficient for her to learn discrete mathematics.
- ✓ Maria will find a good job unless she does not learn discrete mathematics.

Example

What is the value of the variable x after the statement

if $2+2=4$ ***then*** $x := x+1$

If $x=0$ before the statement is encountered? (The symbol $:=$ stands for assignment. The statement $x := x+1$ means the assignment of the value of $x+1$ to x.)

Solution

Because 2 + 2 = 4 is true, the assignment statement $x := x+1$ is executed.

Hence, x has the value $0+1=1$ after this statement is encountered.

Converse, Contrapositive, and Inverse.

The ***converse*** of $p \to q$ is the proposition $q \to p$

The ***contrapositive*** of $p \to q$ is the proposition $\neg q \to \neg p$

The ***inverse*** of $p \to q$ is the proposition $\neg p \to \neg q$

Example

What are the contrapositive, the converse, and the inverse of the conditional statement

"The home team wins whenever it is raining?"

Solution

Because "q whenever p" is one of these ways to express the conditional statement $p \to q$, the original statement can be written as

- ✓ If it is raining, then the home team wins.

The contrapositive of this conditional statement is:

- ✓ If the home team does not win, then it is not raining.

The converse: If the home team wins, then it is raining.

The inverse: If it is not raining, then the home team does not win.

Only the contrapositive is equivalent to the original statement.

Definition

Let p and q be propositions. The ***biconditional statement*** $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called ***bi-implications***.

p is necessary and sufficient for q

If p then q, and conversely

p iff q

$p \leftrightarrow q$ has exactly the truth value as $(p \to q) \wedge (q \to p)$

Example

Let p be the statement "You can take the flight," and let q be the statement "You buy a ticket." Then $p \leftrightarrow q$ is the statement: "You can take the flight if and only if you buy a ticket."

Solution

This statement is true if p and q are either both true or false, that is, if you buy a ticket and can take the flight or if you do not buy a ticket and you cannot take the flight.

It is false when p and q have opposite truth values, that is, when you do not buy a ticket, but you can take the flight, and when you buy a ticket but you cannot take the flight.

Truth Tables of Compound Propositions

Example

Construct the truth table of the compound proposition $(p \vee \neg q) \rightarrow (p \wedge q)$

Solution

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	*T*	*F*	*T*	*T*	*T*
T	*F*	*T*	*T*	*F*	*F*
F	*T*	*F*	*F*	*F*	*T*
F	*F*	*T*	*T*	*F*	*F*

Precedence of Logical Operators

Precedence of Logical Operators	
Operator	*Precedence*
$\neg$	*1*
$\wedge$	*2*
$\vee$	*3*
$\rightarrow$	*4*
$\leftrightarrow$	*5*

$p \wedge q \vee r$ ***means*** $(p \wedge q) \vee r$

Logic and Bit Opertions

True Value	*Bit*
T	*1*
F	*0*

Bit Operators OR, AND, and XOR				
x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	*0*	*0*	*0*	*0*
0	*1*	*1*	*0*	*1*
1	*0*	*1*	*0*	*1*
1	*1*	*1*	*1*	*0*

Definition

A bit string is a sequence of zero or more bits. The length of this string is the number of bits in the string.

Exercises *Section* 1.1 – Propositional Logic

1. Which of these sentences are propositions? What are truth values of those that are propositions?

a) Boston is the capital of Massachusetts.
b) Miami is the capital of Florida
c) $2+3=5$
d) $5+7=10$
e) $x+2=11$
f) Answer this question
g) Do not pass go
h) What time is it?
i) The moon is made of green cheese
j) $2^n \geq 100$

2. What is the negation if each of these propositions?

a) Mei has an MP3 player
b) There is no pollution in Texas
c) $2+1=3$
d) There are 13 items in a baker's dozen,
e) 121 is a perfect square

3. Suppose the Smartphone *A* has 256 MB RAM and 32 GB ROM, and the resolution of its camera is 8 MP; Smartphone *B* has 288 MB RAM and 64 GB ROM, and the resolution of its camera is 4 MP; Smartphone *C* has 128 MB RAM and 32 GB ROM, and the resolution of its camera is 5 MP. Determine the truth value of each of these propositions.

a) Smartphone *B* has the most RAM of these three smartphones
b) Smartphone *C* has more ROM or higher resolution camera than Smartphone *B*.
c) Smartphone *B* has more RAM, more ROM, and a higher resolution camera than Smartphone *A*.
d) If Smartphone *B* has more RAM and more ROM than Smartphone *C*, then it also has a higher resolution camera.
e) Smartphone *A* has more RAM than Smartphone *B* if and only if Smartphone *B* has more RAM than Smartphone *A*.

4. Let *p* and *q* be the proposition

p: I bought a lottery ticket this week
q: I won the million dollar jackpot

a) $\neg p$ *b)* $p \lor q$ *c)* $p \to q$ *d)* $p \land q$
e) $p \leftrightarrow q$ *f)* $\neg p \to \neg q$ *g)* $\neg p \land \neg q$ *h)* $\neg p \lor (p \land q)$

5. Let p and q be the proposition

p: Swimming at the New Jersey shore is allowed

q: Sharks have been spotted new the shore

$a)\quad \neg q$ $\qquad b)\quad p \wedge q$ $\qquad c)\quad \neg p \vee q$ $\qquad d)\quad p \rightarrow \neg q$

$e)\quad \neg q \rightarrow p$ $\qquad f)\quad \neg p \rightarrow \neg q$ $\qquad g)\quad p \leftrightarrow \neg q$ $\qquad h)\quad \neg p \wedge (p \vee \neg q)$

6. Let p, q and r be the proposition

p: You have the flu

q: You miss the final examination

r: You pass the course

Express each of these proposition as an English sentence

$a)\quad p \rightarrow q$ $\qquad b)\quad \neg q \leftrightarrow r$ $\qquad c)\quad q \rightarrow \neg r$ $\qquad d)\quad p \vee q \vee r$

$e)\quad (p \rightarrow \neg r) \vee (q \rightarrow \neg r)$ $\qquad f)\quad (p \wedge q) \vee (\neg q \wedge r)$

7. Determine whether each of these conditional statements is true or false.

a) If $1+1=2$, then 2 + 2 = 5
b) If 1 + 1 = 3, then 2 + 2 = 4
c) If 1 + 1 = 3, then 2 + 2 = 5
d) If monkeys can fly, then 1 + 1 = 3
e) If 1 + 1 = 3, then unicorns exist
f) If 1 + 1 = 3, then dogs can fly
g) If 1 + 1 = 2, the dogs can fly
h) If 2 + 2 = 4, then 1 + 2 = 3

8. Write each of these propositions in the form "*p* if and only if *q*" in English

a) If it is hot outside you buy an ice cream cone, and if you buy an ice cream cone it is hot outside.
b) For you to win the contest it is necessary and sufficient that you have only winning ticket.
c) You get promoted only if you have connections, and you have connections only if you get promoted.
d) If you watch television your mind will decay, and conversely.
e) The trains run late on exactly those days when I take it.
f) For you to get an ***A*** in this course, it is necessary and sufficient that you learn how to solve discrete mathematics problems.
g) If you read the newspaper every day, you will be informed, and conversely.
h) It rains if it is a weekend day, and it is a weekend day if it rains.
i) You can see the wizard only if the wizard is not in, and the wizard is not in only if you can see him

9. Construct a truth table for each of these compound propositions.

a) $p \wedge \neg p$

b) $p \vee \neg p$

c) $p \rightarrow \neg p$

d) $p \leftrightarrow \neg p$

e) $p \rightarrow \neg q$

f) $\neg p \leftrightarrow q$

g) $(p \vee \neg q) \rightarrow q$

h) $(p \vee q) \rightarrow (p \wedge q)$

i) $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$

j) $(p \rightarrow q) \rightarrow (q \rightarrow p)$

k) $p \oplus (p \vee q)$

l) $(p \wedge q) \rightarrow (p \vee q)$

m) $(q \rightarrow \neg p) \leftrightarrow (p \leftrightarrow q)$

n) $(p \rightarrow q) \vee (\neg p \rightarrow q)$

o) $(p \rightarrow q) \wedge (\neg p \rightarrow q)$

p) $(p \vee q) \vee r$

q) $(p \vee q) \wedge r$

r) $(p \wedge q) \vee r$

s) $(p \wedge q) \wedge r$

t) $(p \vee q) \wedge \neg r$