3. Review of Vector Calculus

a) Vector functions of a single variable

Suppose that we have a particle whose position vector is a function of time:

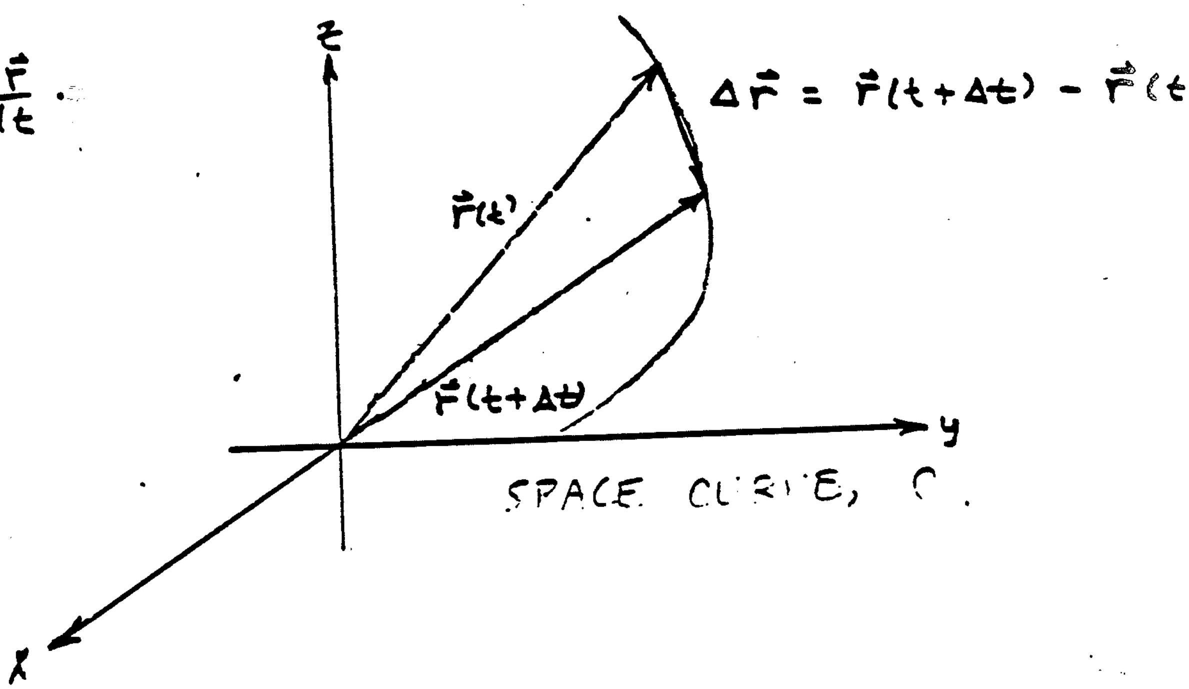
$$\vec{r} = \vec{r}(t) = x(t)\,\hat{x} + y(t)\,\hat{y} + z(t)\,\hat{z}.$$

We define the DERIVATIVE of $\vec{r}$ with respect to time, t, as

$$\lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}$$

which is denoted, as usual, by

$$\frac{d\vec{r}}{dt}.$$



Resolving r into components, we arive at the result that

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \left\{ \hat{x}\,\frac{x(t+\Delta t) - x(t)}{\Delta t} + \hat{y}\,\frac{y(t+\Delta t) - y(t)}{\Delta t} + \hat{z}\,\frac{z(t+\Delta t) - z(t)}{\Delta t} \right\}$$

$$= \hat{x}\frac{dx}{dt} + \hat{y}\frac{dy}{dt} + \hat{z}\frac{dz}{dt}.$$

Notice that $\Delta\vec{r}$ is a SECANT of the arc in the trajectory. Clearly, as $t \longrightarrow 0$, $\Delta\vec{r}$ has a direction which approaches the TANGENT to the arc.

Thus,

$$\frac{d\vec{r}}{dt} \Big/ \left|\frac{d\vec{r}}{dt}\right|$$

is the <u>UNIT</u> <u>TANGENT</u> to the curve C.

An important example:

If $\vec{r}(t)$ is the position of a particle as a function of time, then the particles velocity and acceleration are

$$\vec{v} = \frac{d\vec{r}}{dt} = \textit{velocity}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \text{acceleration}$$

EXAMPLE:

Suppose a particle moves in the helix, with $z = tm/s$ and

$$x^2 + y^2 = (1m)^2$$

with an angular speed of

$\omega = 3$ rad/s

with $x = 1m$ and $y = 0m$, at $t = 0$. Find the velocity and acceleration.

Then

$$\left.\begin{array}{l} x(t) = 1m\cos(3t/s) \\ y(t) = 1m\sin(3t/s) \\ z(t) = 1m/s \cdot t \end{array}\right\} \Longrightarrow$$

$$\vec{r}(t) = 1m \cdot \left\{ \hat{x}\cos(3t/s) + \hat{y}\sin(3t/s) + \hat{z} \cdot t/s \right\}$$

Differentiating,

$$\vec{v} = \frac{d\vec{r}}{dt} = -3\frac{m}{s}\hat{x}\sin(3t/s) + 3\frac{m}{s}\hat{y}\cos(3t/s) + \hat{z}\frac{1m}{s}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = -9\frac{m}{s^2}\hat{x}\cos(3t/s) - 9\frac{m}{s^2}\hat{y}\sin(3t/s).$$

The SPEED is

$$|\vec{v}| = \left[9\,\frac{m^2}{s^2}\sin^2(3t/s) + 9\,\frac{m^2}{s^2}\cos^2(3t/s) + 1\,\frac{m^2}{s^2}\right]^{1/2}$$

$$= \sqrt{10}\ \frac{m}{s}.$$

RULES FOR DIFFERENTIATION OF VECTORS:

$$\frac{d(\vec{u}+\vec{v})}{dt} = \frac{d\vec{u}}{dt} + \frac{d\vec{v}}{dt}$$

$$\frac{d(\varphi\vec{u})}{dt} = \frac{d\varphi}{dt}\vec{u} + \varphi\frac{d\vec{u}}{dt}$$

$$\frac{d(\vec{u}\cdot\vec{v})}{dt} = \vec{v}\cdot\frac{d\vec{u}}{dt} + \vec{u}\cdot\frac{d\vec{v}}{dt}$$

$$\frac{d(\vec{u}\times\vec{v})}{dt} = \frac{d\vec{u}}{dt}\times\vec{v} + \vec{u}\times\frac{d\vec{v}}{dt}$$

$$\frac{d[\vec{u}\,\vec{v}\,\vec{w}]}{dt} = \left[\frac{d\vec{u}}{dt}\,\vec{v}\,\vec{w}\right] + \left[\vec{u}\,\frac{d\vec{v}}{dt}\,\vec{w}\right] + \left[\vec{u}\,\vec{v}\,\frac{d\vec{w}}{dt}\right]$$

(where $[\vec{u}\,\vec{v}\,\vec{w}] = \vec{u}\cdot\vec{v}\times\vec{w} = \vec{w}\cdot\vec{u}\times\vec{v}$
$= \vec{v}\cdot\vec{w}\times\vec{u}$ = "triple scalar product.")

b) Vector functions of more than one variable

(1) Total Differential

Suppose we have a scalar field,

$$\varphi(x, y, z).$$

Then

$$\Delta\varphi = \varphi(x+\Delta x, y+\Delta y, z+\Delta z) - \varphi(x, y, z)$$

$$= \left[\frac{\varphi(x+\Delta x, y+\Delta y, z+\Delta z) - \varphi(x, y+\Delta y, z+\Delta z)}{\Delta x}\right]$$

$\sim \frac{\partial\varphi}{\partial x}\Delta x$

$$+ \left[\frac{\varphi(x, y+\Delta y, z+\Delta z) - \varphi(x, y, z+\Delta z)}{\Delta y}\right]$$

$\sim \frac{\partial\varphi}{\partial y}\Delta y$

$$+ \left[\frac{\varphi(x, y, z+\Delta z) - \varphi(x, y, z)}{\Delta z}\right]\Delta z.$$

$\sim \frac{\partial\varphi}{\partial z}\Delta z$

(Remember, practically all progress in mathematics is made by adding zero and multiplying by 1!)

Let Δx, Δy, and Δz approach 0. Then the TOTAL DIFFERENTIAL is

$$d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz.$$

(2) Directional derivative

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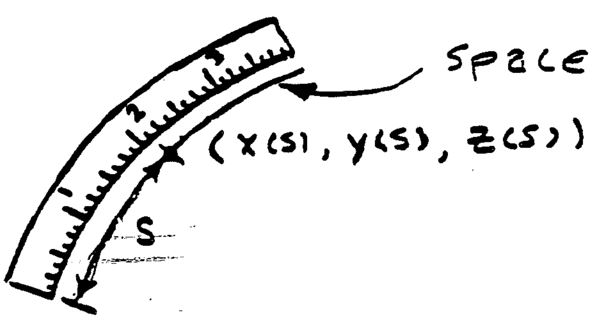
Suppose that we look at the variation of $\varphi(x,y,z)$ in a PARTICULAR DIRECTION.

To this end, let

$$(x(s),\ y(s),\ z(s))\ ;\quad \vec{r}(s) = x(s)\,\hat{x} + y(s)\,\hat{y} + z(s)\,\hat{z}$$

be a curve <u>parametrically</u> defined.

The parameter, **s**, can be taken to be the ARC LENGTH along the curve.



Since ds is an infintessimal section of <u>arc length</u>

$$ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

$$\therefore\quad \left|\frac{d\vec{r}}{ds}\right| = 1 .$$

Then

$$\frac{d\varphi}{ds} = \frac{\partial\varphi}{\partial x}\frac{dx}{ds} + \frac{\partial\varphi}{\partial y}\frac{dy}{ds} + \frac{\partial\varphi}{\partial z}\frac{dz}{ds}$$

$$= \left(\hat{x}\frac{\partial\varphi}{\partial x} + \hat{y}\frac{\partial\varphi}{\partial y} + \hat{z}\frac{\partial\varphi}{\partial z}\right)\cdot\left(\hat{x}\frac{dx}{ds} + \hat{y}\frac{dy}{ds} + \hat{z}\frac{dz}{ds}\right).$$

This derivative, which is the RATE OF CHANGE OF φ ALONG THE TANGENT OF THE CURVE, $\vec{r}(s)$, is called the DIRECTIONAL DERIVATIVE OF φ IN THE

$$\frac{d\vec{r}}{ds}$$

DIRECTION.

Question: In which <u>direction</u> is the directional derivative of φ the <u>greatest</u>?

The answer to this question leads us to the concept of the GRADIENT.

(3) Gradient

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To answer this question, we must remember that the magnitude of

$$\frac{d\vec{r}}{ds}$$

is one.

The dot product of a <u>unit vector</u> with another vector is maximum when the unit vector is PARALLEL to that vector.

Thus, the MAXIMUM rate of change of φ is in the direction of

$$\hat{x}\frac{\partial\varphi}{\partial x} + \hat{y}\frac{\partial\varphi}{\partial y} + \hat{z}\frac{\partial\varphi}{\partial z}.$$

The GRADIENT, denoted by

$$\text{grad } \varphi,$$

is defined as THE VECTOR WHICH POINTS IN THE DIRECTION OF THE GREATEST RATE OF CHANGE OF THE FUNCTION ON WHICH IT OPERATES AND HAS A MAGNITUDE EQUAL TO THE MAXIMUM DIRECTIONAL DERIVATIVE OF THIS FUNCTION.

In RECTANGULAR coordinates, the gradient, we have seen, turns out to be

$$\text{grad } \varphi = \hat{x}\frac{\partial\varphi}{\partial x} + \hat{y}\frac{\partial\varphi}{\partial y} + \hat{z}\frac{\partial\varphi}{\partial z}$$

(4) The "del" operator.
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This operator is defined, in rectangular coordinates, to be

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}.$$

Thus, we have

$$\text{grad}\,\varphi = \nabla\varphi.$$

Remember, however, that this expression is what we deduced from the mathematics. It is NOT the definition of grad.

Remember the **DEFINITION** given above as this will be important **later** when we consider other coordinate systems!

One of the important **uses** of the gradient is in the **construction** of

(5) Normals to level curves.
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Setting the function $\varphi(x,y,z) = c$, a constant, defines a surface.

This surface is called a LEVEL SURFACE with level c.

Let's see how this works. Suppose that (x,y,z) is a point which is constrained to lie on the level surface $\varphi(x,y,z) = c$.

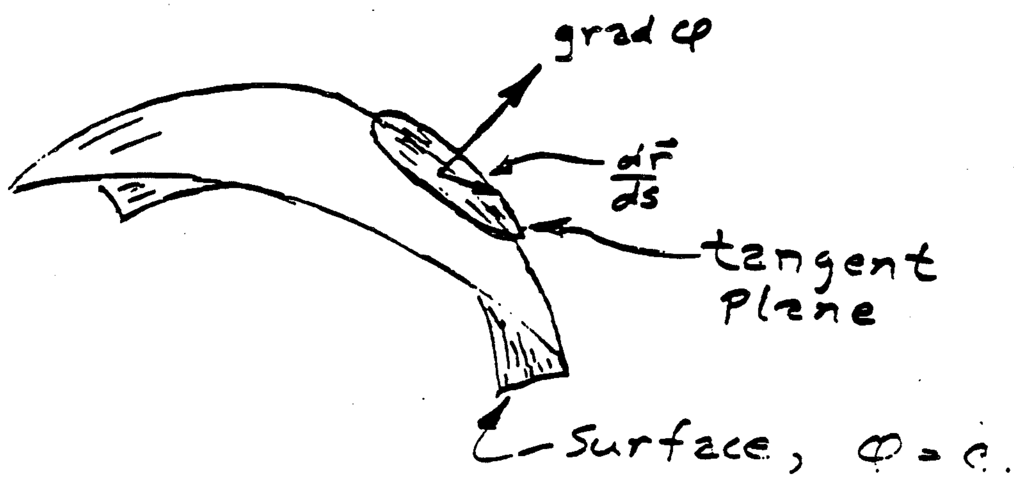
Then the directional derivative of $\varphi(x,y,z)$ in <u>any</u> direction TANGENT to the level surface is

remember $\varphi = c =$ constant.

$$\frac{d\varphi}{ds} = \frac{dc}{ds} = 0$$

$$= \operatorname{grad}\varphi \cdot \frac{d\vec{r}}{ds} \qquad \therefore \quad \operatorname{grad}\varphi \perp \frac{d\vec{r}}{ds}.$$

this vector lies in the tangent plane



Thus, the grad φ is NORMAL to the level surfaces of φ.

(6) An example of the use of grad.
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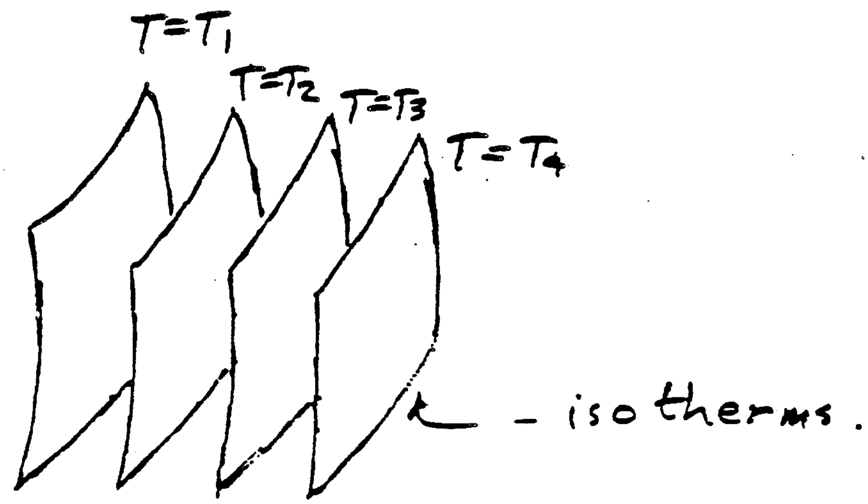
Suppose we have the following level curves for the temperature distribution some homogeneous material.



Which way will heat energy flow in this material?

Will it flow parallel to the isotherms?

How about perpendicular to the isotherms?

Of course, heat will flow in a direction NORMAL to the isotherms ie, in the direction of the GREATEST DECREASE in temperature. $T(x, y, z)$.

The faster $T(x,y,z)$ varies in a certain direction, the faster heat will flow in that direction.

Thus, if Q is the amount of heat which passed through the surface, $T = T_1$, then the rate of heat transfer is

$$\frac{dQ}{dt}$$

and is found, not surprisingly, to be

Thermal conductivity

$$\frac{dQ}{dt} = -(K\vec{\nabla}T)A$$

Area of $T = T_1$ surface through which Q passes

We will find other important uses for grad in the section on the equations of mathematical physics.

(6) Line integrals

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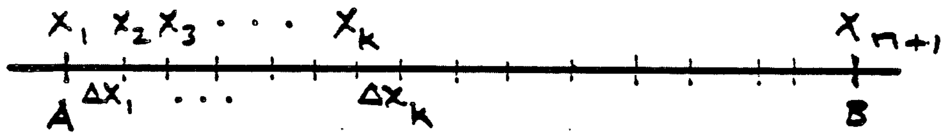
(a) 1-dimension

The 1-dimensional line integral is just the ordinary integral you learned in calculus:



If we let $n \to \infty$ letting each Δx_1, Δx_2, . . . , Δx_n go to zero at the same time such that

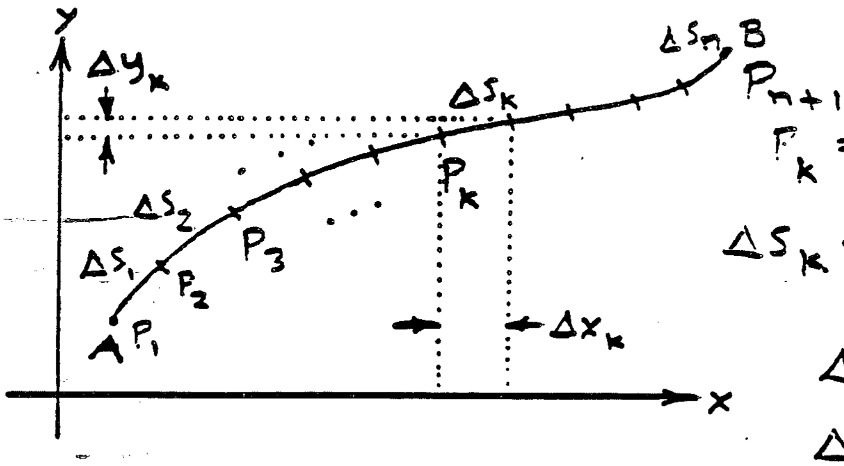
$$\Delta x_1 + \Delta x_2 + \cdots + \Delta x_n = B - A,$$

Then

$$\int_A^B F(x)\,dx = \lim_{n \to \infty} \sum_{k=1}^{n} F(x_k)\,\Delta x_k$$

(b) 2-dimensions

Just as in the 1-dimensional case, the line is broken up into small segments.



$$P_k = (x_k, y_k)$$

$$\Delta S_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

$$\Delta x_k = x_{k+1} - x_k$$

$$\Delta y_k = y_{k+1} - y_k$$

The line integral from point A to point B is defined as the limit of

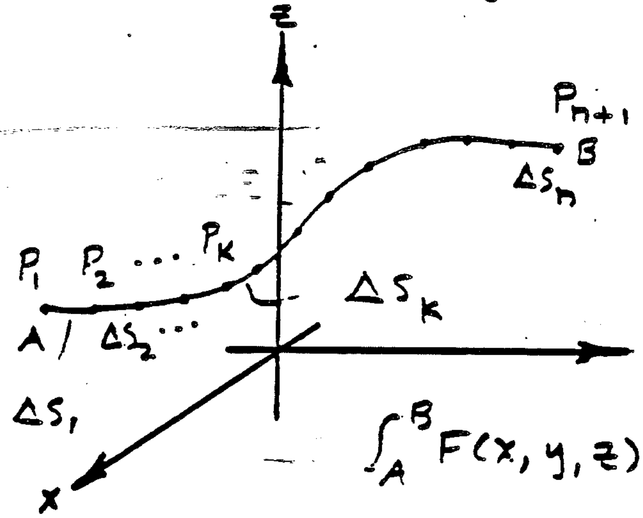
$$\lim_{n \to \infty} \sum_{k=1}^{n} F(x_k, y_k) \Delta S_k$$

and is denoted by

$$\int_A^B F(x,y)\, ds$$

(c) 3-dimensions

Of course, a line integral can be defined along a space curve as

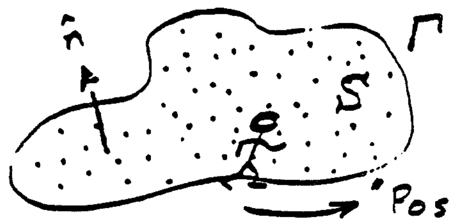


$$\Delta S_K = \sqrt{(\Delta X_K)^2 + (\Delta y_K)^2 + (\Delta z_K)^2}$$

$$P_K = (x_K, y_K, z_K)$$

$$\int_A^B F(x,y,z)\,ds = \lim_{n \to \infty} \sum_{K=1}^{n} F(x_K, y_K, z_K)\Delta S$$

One important class of line integrals is the integral over a closed curve.



Positive direction with respect to $\hat{n}$.

In most applications it is necessary to specify the SENSE in which the closed loop is traversed.

The curve is said to be traversed in the POSITIVE SENSE with respect to normal vector, $\hat{n}$, if, when you "walk the curve" on the side of the surface out of which $\hat{n}$ points, the enclosed surface is to your left.

This is often denoted by

$$\int_{\Gamma} F(x,y,z)\,ds = \oint_{\Gamma} F(x,y,z)\,ds$$

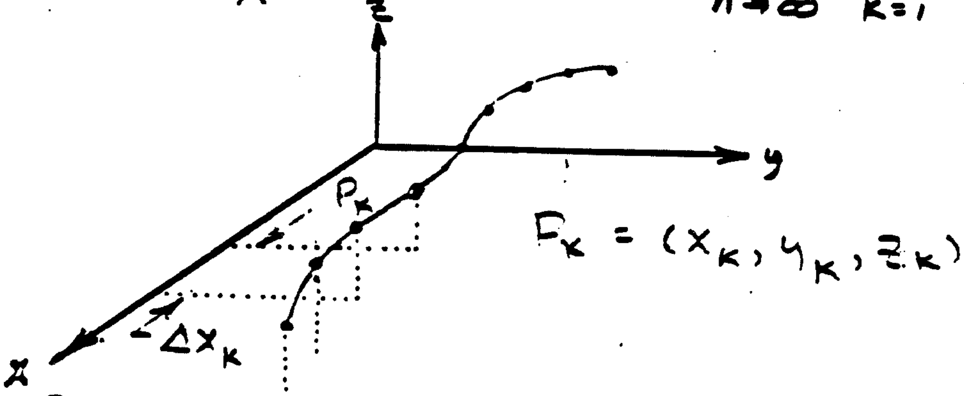
The types of line integrals we just discussed are independent of the direction along the path since the element of path length,

$$ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

is always positive.

Another, similar, type of line integral is defined by

$$\int_A^B F(x,y,z)\,dx = \lim_{n\to\infty} \sum_{K=1}^{n} F(x_K, y_K, z_K)\,\Delta x_K$$



Of course,

$$\int_A^B F(x,y,z)\,dy \quad \text{and} \quad \int_A^B F(x,y,z)\,dz$$

are similarly defined.

Notice that these DO depend on the direction in which the integral is taken since Δx_k can be positive or negative.

(d) Physical Interpretation

A very important line integral is

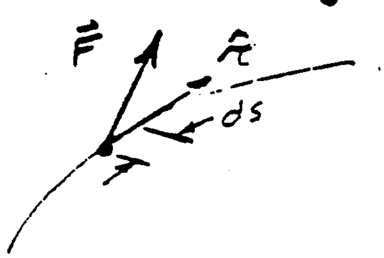
$$\int_A^B F_x\,dx + \int_A^B F_y\,dy + \int_A^B F_z\,dz$$

which can be written succinctly as

$$\int_A^B \vec{F}(\vec{r})\cdot d\vec{r} \quad \text{where } \vec{F} = F_x\hat{x} + F_y\hat{y} + F_z\hat{z}.$$

If $\vec{F}(\vec{r})$ is the force on a particle at position $\vec{r}$, then this line integral is the WORK done in moving a particle from point P_1 to point P_2.

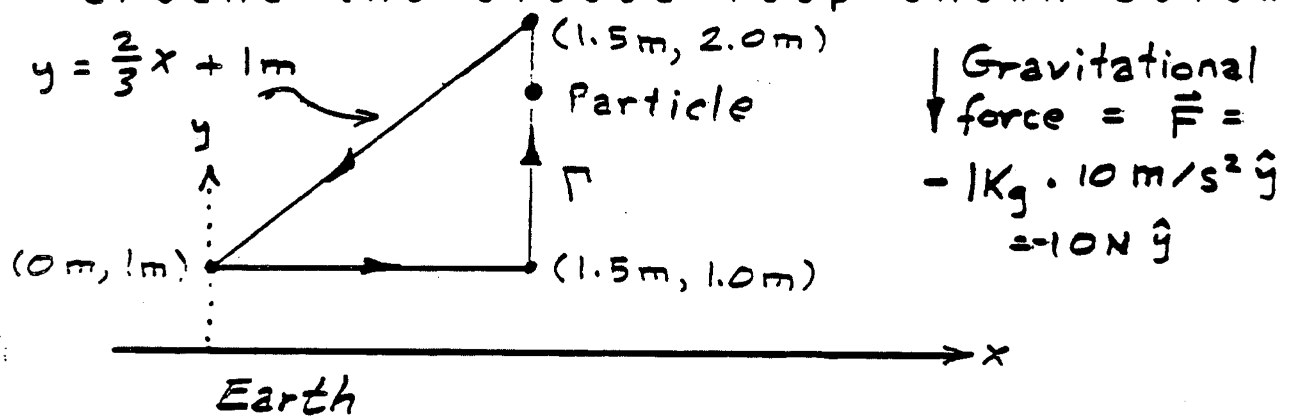
Remember that $d\vec{r}$ is tangent to the curve. If $\hat{\tau}$ is the unit tangent, then $d\vec{r} = \hat{\tau}\,ds$ and $\vec{F}\cdot\hat{\tau}$ is the component of force along the infintesimal arc segment of length ds.



Then the amount of energy released by allowing the force to move the particle over this length is $\vec{F}\cdot\hat{\tau}\,ds$.

For example:

Find the net amount of work required to move a 1Kg particle around the closed loop shown below.



The work required is $-\oint_\Gamma \vec{F}\cdot d\vec{r}$ in the sense indicated.

$$d\vec{r} = \hat{x}\,dx + \hat{y}\,dy.$$

$$-\oint_\Gamma \vec{F}\cdot d\vec{r} = -\int_{(0,1)m}^{(1.5,1)m} \vec{F}\cdot\hat{x}\,dx - \int_{(1.5,1)m}^{(1.5,2)m} \vec{F}\cdot\hat{y}\,dy$$

$$- \int_{(1.5,2)m}^{(0,1)m} \vec{F}\cdot(\hat{x}\,dx + \hat{y}\,dy)$$

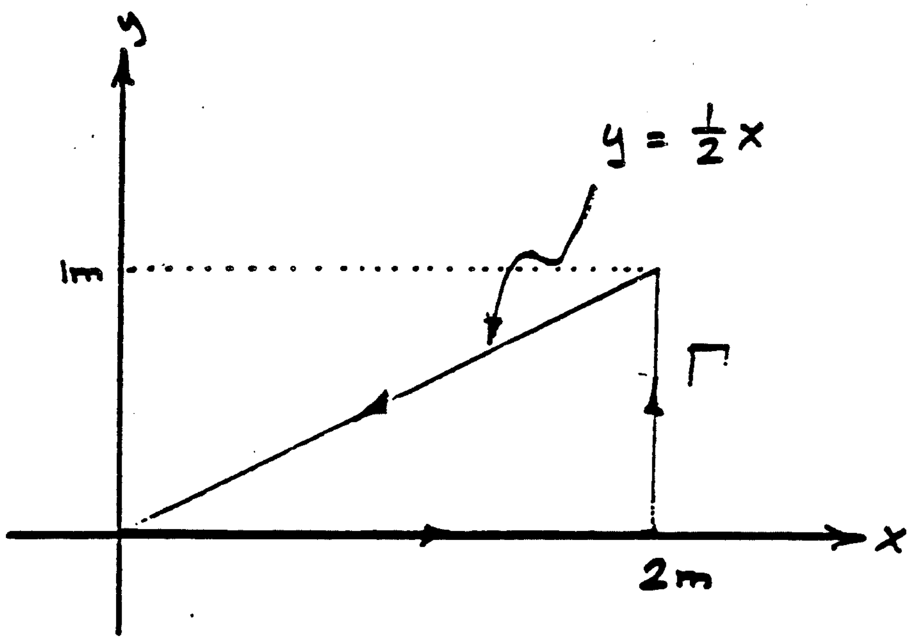
$$= 0 + \int_{1m}^{2m} 10N\,dy + \int_{2m}^{1m} 10N\,\hat{y}\cdot(\hat{x}\,dx + \hat{y}\,dy) = 0$$

(Vector fields which have the property that $\oint \vec{E}\cdot d\vec{r} = 0$ for any closed loop Γ are called "conservative.")

Find the amount of work required to move a point charge, q, around the loop shown in the presense of the non-conservative electric field,

$\vec{E} = y\hat{x} + 2x\hat{y}$ V/m^2

Force, $\vec{F} = q \cdot \vec{E}$



Work required $= -\oint_\Gamma \vec{F} \cdot d\vec{r}$

$$-\left\{ \int_{(0,0)}^{(2,0)\,m} \overset{0}{y}\,dx + \int_{(2,0)\,m}^{(2,1)\,m} 2\overset{2m}{x}\,dy + \int_{(2,1)\,m}^{(0,0)\,m} \left(\underbrace{y}_{\frac{1}{2}x}\,dx + 2x\,\underbrace{dy}_{\frac{1}{2}dx} \right) \right\} \mathrm{V/m^2} \cdot q$$

$$= -\left\{ 4m^2 + \frac{1}{2}\int_{2m}^{0m} x\,dx + \frac{1}{2}\int_{2m}^{0m} 2x\,dx \right\} q \cdot \frac{V}{m^2} = -1V \cdot q$$

(7) Surface Integrals

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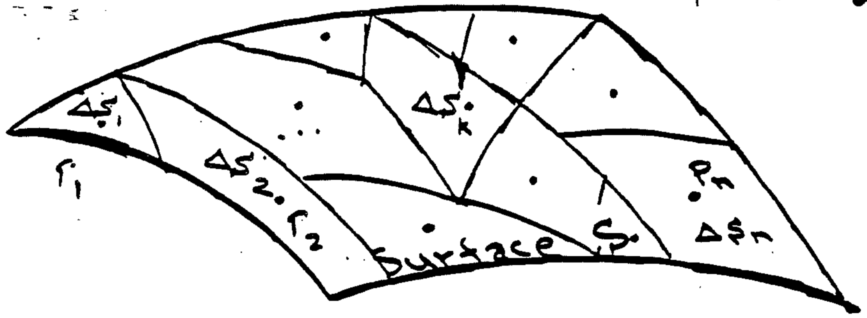
(a) Definition

$r_k = (x_k, y_k, z_k) =$ a point within ΔS_k



Divide the surface, S, into a set of sub-surfaces, ΔS_k.

Then form the sum

$$\sum_{k=1}^{n} F(x_k, y_k, z_k)\, \Delta S_k .$$

— i.e., for $k = 1, 2, \ldots, n$ $\Delta S_k \to 0$ as $n \to \infty$ $\sum_{k=1} \Delta S_k = S$

Continue to divide the surface into finer and finer sub-surfaces computing this sum every time.

Then, if this sequence of sums converges, the surface integral of F exists and is denoted by

$$\iint_S F(x,y,z)\, dS = \lim_{n \to \infty} \sum_{k=1}^{n} F(x_k, y_k, z_k)\, \Delta S_k .$$

If the surface, S, is closed, then the surface integral over S is denoted by

$$\oiint_S F\,ds$$

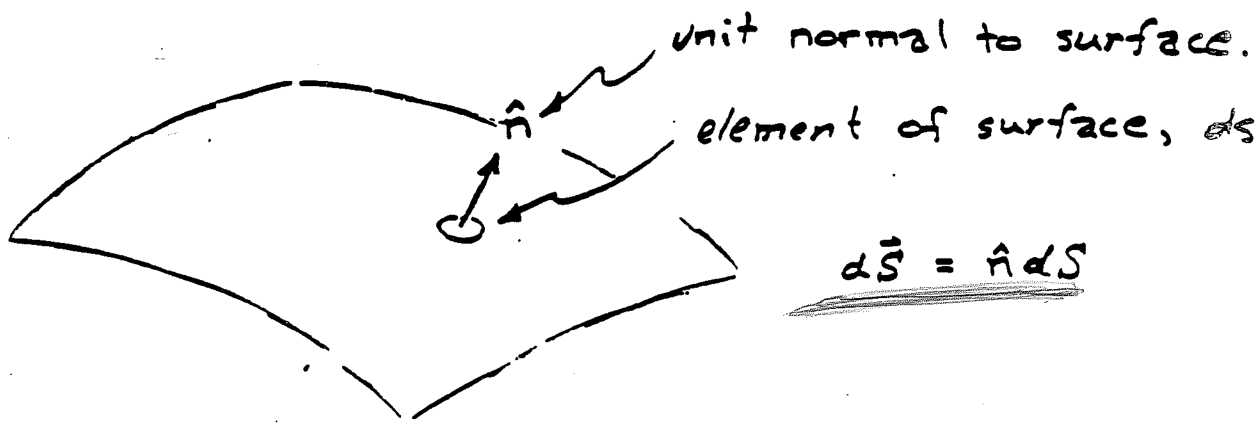
(d) Physical interpretation

A very important type of surface integral in physics is

$\hat{n}\,ds = d\vec{S}$

$$\iint_S \vec{F}(\vec{r}) \cdot \hat{n}\,dS = \iint_S \vec{F}(\vec{r}) \cdot d\vec{S}$$

where $\hat{n}(\vec{r})$ is the unit normal to the surface at the point, $\vec{r}$ on S.



$$\underline{d\vec{S} = \hat{n}\,dS}$$

If $\vec{F}$ is a FLUX DENSITY, then

$$\iint_S \vec{F} \cdot d\vec{S}$$

is called the flux through S.

For example, if $\vec{F} = \vec{J}(\vec{r})$ is a CURRENT DENSITY, then the TOTAL CURRENT flowing through surface S is

$$\iint_S \vec{J}(\vec{r}) \cdot \hat{n}\, ds.$$

Components of $\vec{J}$ perpendicular to $\hat{n}$ DO NOT PENETRATE S.

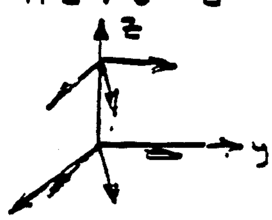
What are the units of $\vec{J}$?

Only component of $\vec{J}$ parallel to $\hat{n}$ will penetrate S.

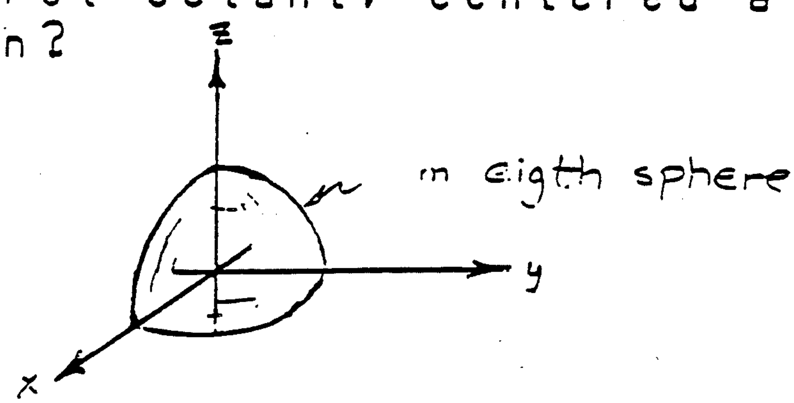
Let's do a specific example.

Suppose that we have a current density of

$$\vec{J} = (x\hat{x} + y\hat{y})\,\frac{A}{m^3}$$

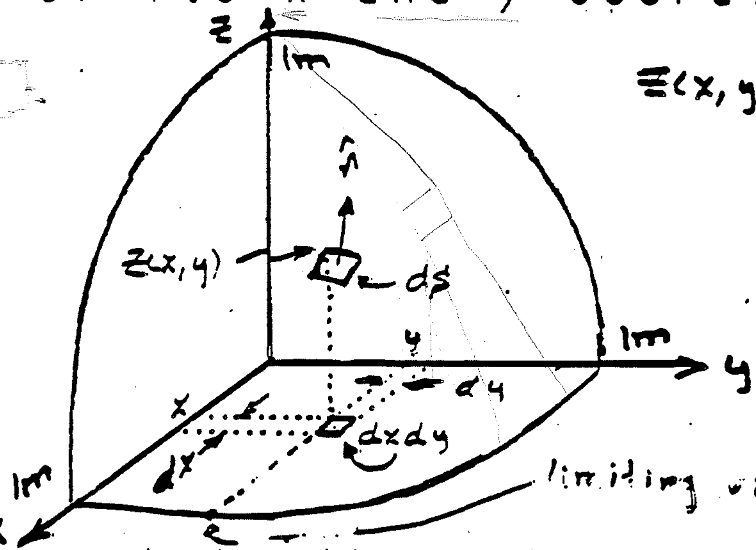


What is the current that flows through that section of a 1m sphere in the first octant, centered at the origin?



One way to define this surface is by specifying its z coordinate in terms of its x and y coordinates:

$$z(x,y) = \sqrt{1m^2 - x^2 - y^2}$$



The normal to the sphere is

$$\frac{\nabla(x^2+y^2+z^2)}{|\nabla(x^2+y^2+z^2)|} = \frac{x\hat{x}+y\hat{y}+z\hat{z}}{\underbrace{\sqrt{x^2+y^2+z^2}}_{1m^2}} = \left\{x\hat{x}+y\hat{y}+\sqrt{1m^2-x^2-y^2}\,\hat{z}\right\}\frac{1}{m}$$

Then the surface integral that must be computed is

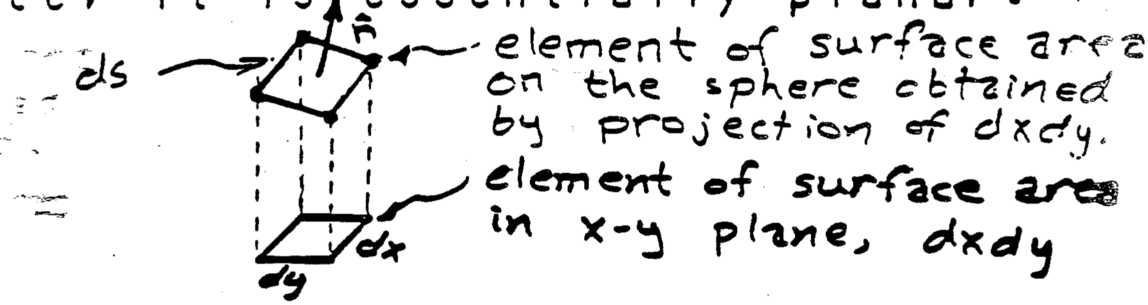
$$\frac{1A}{m^3}\iint_S (x\hat{x}+y\hat{y})\cdot\left(x\hat{x}+y\hat{y}+\hat{z}\sqrt{1m^2-x^2+y^2}\right)\frac{1}{m}\,dS$$

$$= \text{current}$$

Now we have everything in the integral in terms of x and y but the element of surface area, dS.

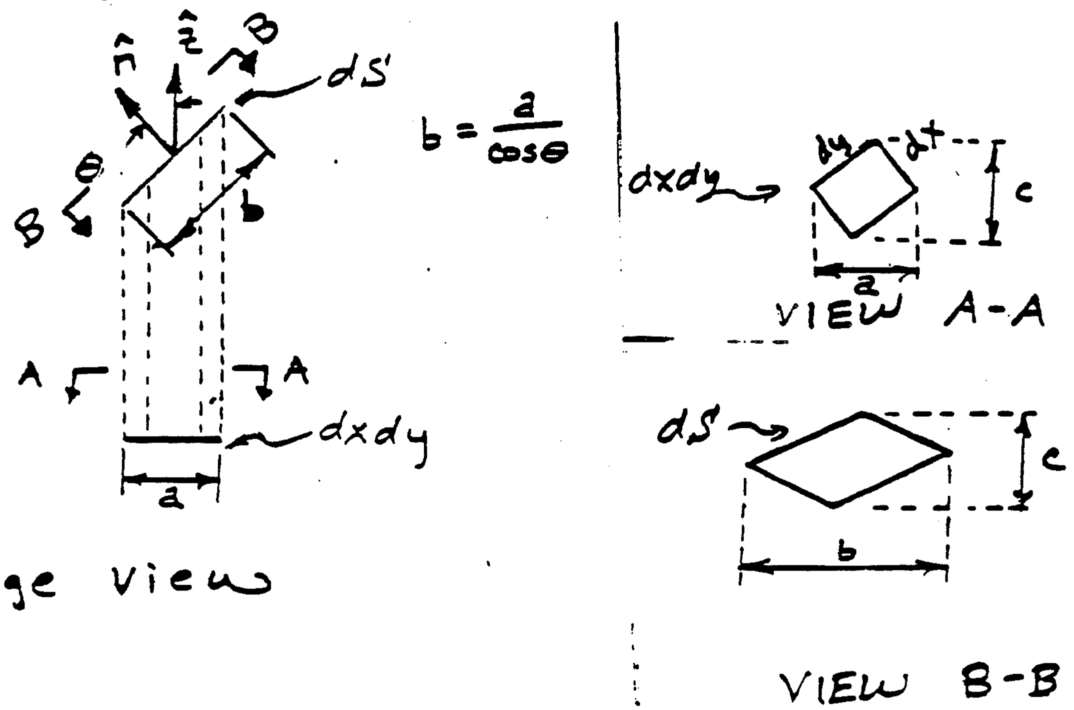
Can we write dS in terms of dx and dy?

YES. Since dS is (we can imagine) an exceedingly small piece of the surface, it is essentially planar.



If we were to PROJECT the element of surface area, dxdy, ONTO our surface, then we would create an element of surface area dS that we could use in the integration.

Let's look at this figure edge-on such that the normal to the surface is lying in the plane of the paper.



The dimension of dS in the direction perpendicular to the paper is the same as that of the element, dxdy.

But the remaining dimension of dS is streched by this projection by a factor of

$$\frac{1}{\cos\theta} = \frac{1}{\hat{z}\cdot\hat{n}}$$

Therefore,

$$dS = \frac{dx\,dy}{\hat{z}\cdot\hat{n}}$$

Substituting this expression for dS back into the integral,

Current =

$$1\frac{A}{m^3}\int_0^{1m}\int_0^{\sqrt{1m^2-y^2}} \frac{(x\hat{x}+y\hat{y})\cdot(x\hat{x}+y\hat{y}+\hat{z}\sqrt{1m^2-x^2-y^2})}{\hat{z}\cdot(x\hat{x}+y\hat{y}+\hat{z}\sqrt{1m^2-x^2-y^2})}\,dx\,dy$$

$$= \int_0^{1m} \int_0^{\sqrt{1m^2-y^2}} \frac{x^2+y^2}{\sqrt{1m^2-x^2-y^2}}\, dx\, dy \cdot 1A/m^3$$

Let $x = \sqrt{1m^2-y^2}\cos\varphi$. Then

$$dx = -\sqrt{1m^2-y^2}\sin\varphi\, d\varphi$$

at $x = 0$, $\varphi = \frac{\pi}{2}$

at $x = \sqrt{1m^2-y^2}$, $\varphi = 0$.

$$x^2+y^2 = 1m^2\cos^2\varphi - y^2\cos^2\varphi + y^2$$

$$= 1m^2\cos^2\varphi + y^2\sin^2\varphi$$

$$\sqrt{1m^2-x^2-y^2} = \sqrt{1m^2-y^2}\sin\varphi.$$

$$\therefore \int_0^{1m} \int_0^{\sqrt{1m^2-y^2}} \frac{x^2+y^2}{\sqrt{1m^2-x^2-y^2}}\, dx\, dy$$

$$= \int_0^{1m} \int_0^{\frac{\pi}{2}} \frac{1m^2\cos^2\varphi + y^2\sin^2\varphi}{\cancel{\sqrt{1m^2-y^2}\sin\varphi}} \cancel{\sqrt{1m^2-y^2}\sin\varphi}\, d\varphi\, dy$$

$$= \int_0^{1m}\left(\int_0^{\frac{\pi}{2}} 1m^2\cos^2\varphi\, d\varphi\right) dy + \int_0^1 y^2\left(\int_0^{\frac{\pi}{2}} \sin^2\varphi\, d\varphi\right) dy$$

But $\int_0^{\frac{\pi}{2}} \cos^2\varphi\, d\varphi = \int_0^{\frac{\pi}{2}} \sin^2\varphi\, d\varphi = \frac{\pi}{4}$.

$$\therefore \text{current} = \frac{\pi}{4}\frac{A}{m}\int_0^{1m} dy + \frac{\pi}{4}\frac{A}{m^3}\int_0^{1m} y^2\, dy$$

$$= \frac{\pi}{4}A + \frac{\pi}{12}A = \boxed{\frac{\pi}{3}A}$$

As we have seen in this example, if we hope to compute a surface integral analytically, we must

(1) Find an expression which describes the surface in terms of a pair of variables. (In this example, x and y).

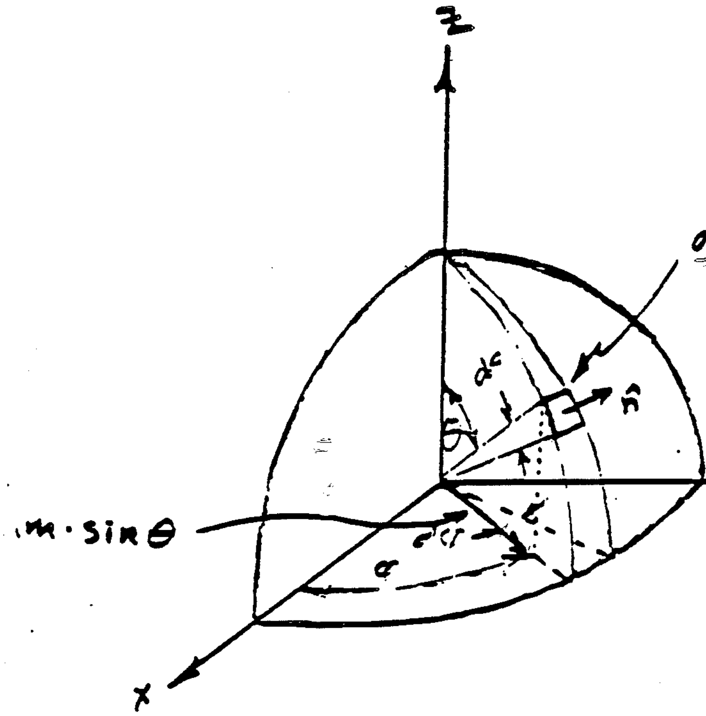
(2) Find the unit normal and our function, $\vec{J}$, in terms of that same pair of variables.

(3) And finally, write the element of surface area, dS, in terms of those variables.

However, when the surface, S, happens to be a part of a CONSTANT COORDINATE SURFACE in some coordinate system, it is usually more convenient to carry out the integration in that coordinate system.

In this example, the surface is part of a sphere.

Let's try to do the problem in spherical coordinates.



$$dS = 1\text{m}\, d\theta \cdot 1\text{m} \sin\theta\, d\varphi$$
$$= 1\text{m}^2 \sin\theta\, d\theta\, d\varphi$$

$$\hat{n} = \hat{r} = \hat{x}\cos\varphi\sin\theta + \hat{y}\sin\varphi\sin\theta + \hat{z}\cos\theta$$

$$\vec{J} = (x\hat{x} + y\hat{y})\ 1\text{A}/\text{m}^3$$
$$= \sin\theta\,(\hat{x}\cos\varphi + \hat{y}\sin\varphi)\,\frac{1}{m}$$

$$\vec{J}\cdot\hat{n} = \sin^2\theta\,(\cos^2\varphi + \sin^2\varphi)\,\frac{1\text{A}}{\text{m}^2}$$
$$= \sin^2\theta\ 1\text{A}/\text{m}^2$$

$$\therefore \iint_S \vec{J}\cdot\hat{n}\,dS = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} 1\text{A}/\text{m}^2 \sin^2\theta \cdot 1\text{m}^2 \sin\theta\, d\theta\, d\varphi$$

$$= 1\text{A}\cdot\frac{\pi}{2}\int_0^{\frac{\pi}{2}} \sin^3\theta\, d\theta$$

$$= 1\text{A}\,\frac{\pi}{2}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin\theta\, d\theta - 1\text{A}\,\frac{\pi}{2}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin\theta\cos(2\theta)\, d\theta$$

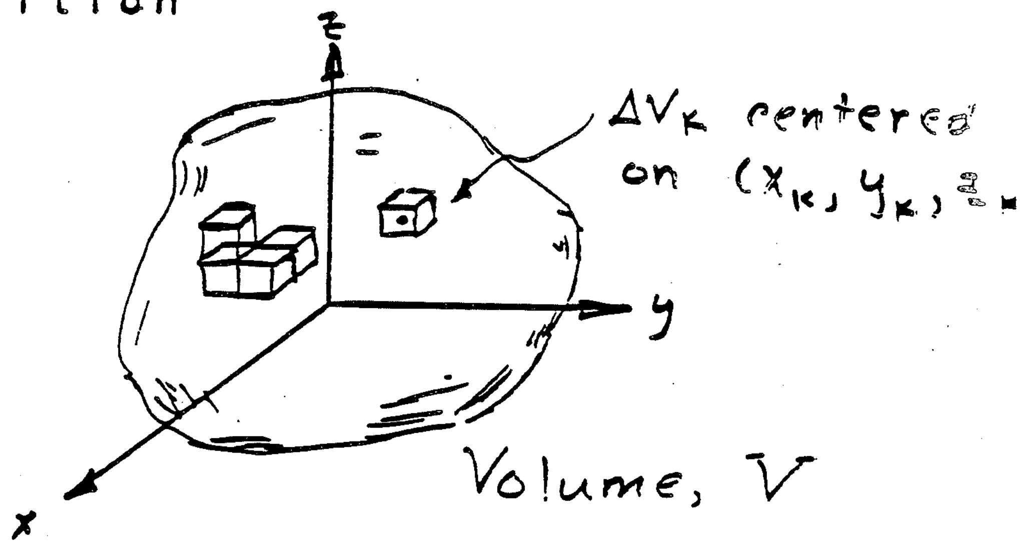
$$= \frac{\pi}{4}\text{A} - \frac{\pi}{8}\text{A}\int_0^{\frac{\pi}{2}} \sin(3\theta)\, d\theta + \frac{\pi}{8}\text{A}\int_0^{\frac{\pi}{2}} \sin\theta\, d\theta$$

$$= \frac{\pi}{4}\text{A} - \frac{\pi}{24}\text{A} + \frac{\pi}{8}\text{A} = \boxed{\frac{\pi}{3}\text{A}}$$

(8) Volume Integrals

<<<<<<<<<<<<<<<<<<>>>>>>>>>>>>>>>>>>>>

(a) Definition



Of course, we divide the volume up into small sub-volumes with centers (x_K, y_K, z_K) and with volumes ΔV_K.

Then forming a sequence of sums of the form

$$\sum_{K=1}^{n} F(x_K, y_K, z_K)\,\Delta V_K$$

by taking smaller and smaller sub-volumes, we end up with the volume integral

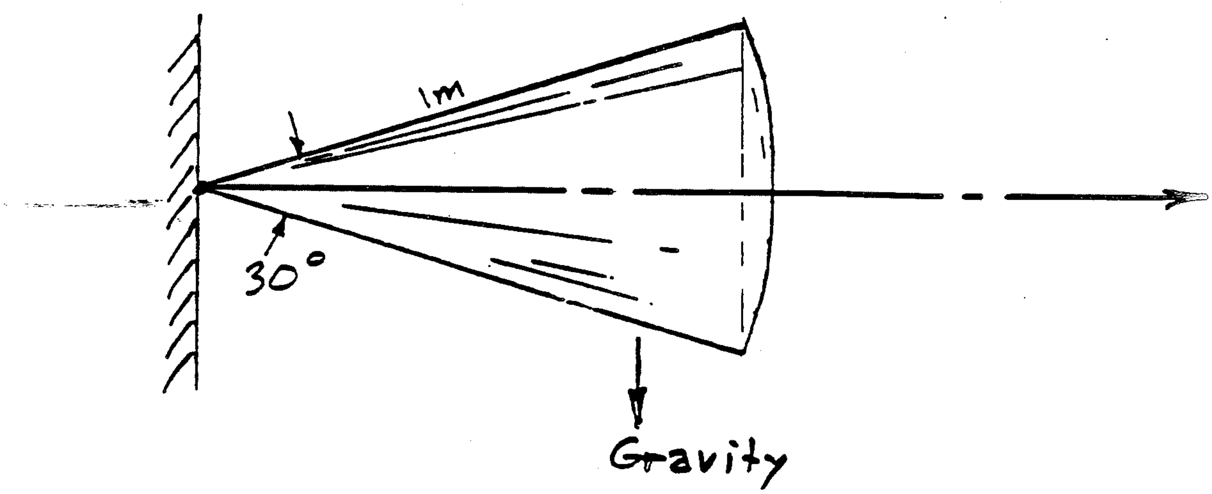
$$\iiint_V F(x,y,z)\,dV$$

in the limit.

(b) Physical interpretation

Often physical quantities such as mass and charge are modeled as being continuously distributed in space with some density, ρ.

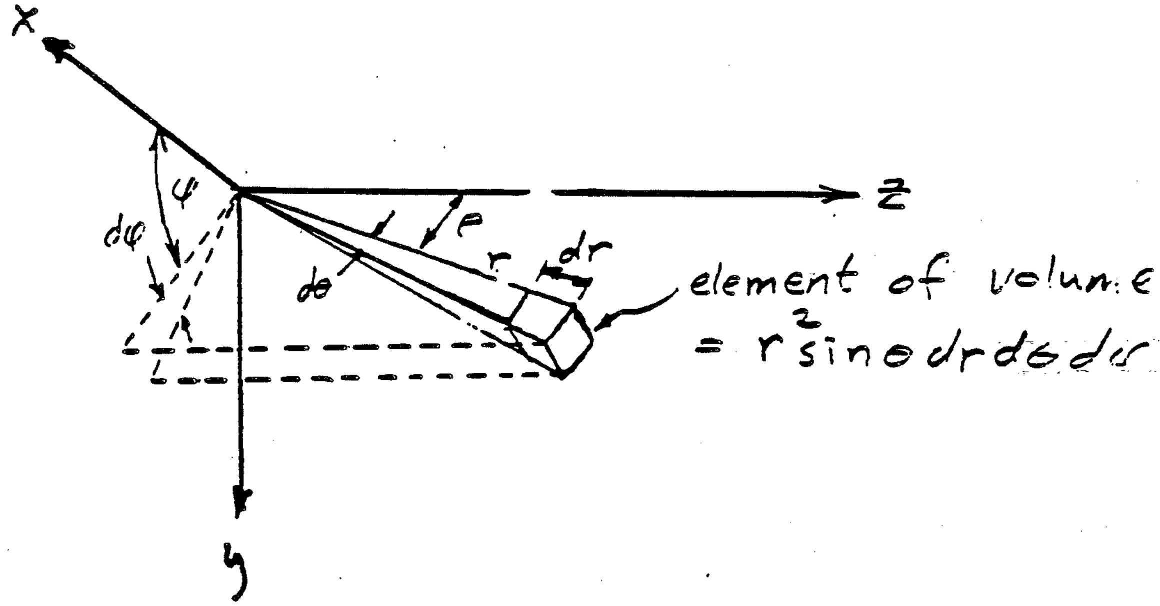
Suppose for example that we wish to find the bending moment felt by a pin holding the cone with uniform mass density, $1 Kg/m^3$.



Then the bending moment is the volume integral,

$$\iiint_V z \cdot 10 \frac{m}{s^2} \cdot 1 Kg/m^3 \, dV$$

Given the geometry of the volume, this integral is most easily performed in spherical coordinates.



$$\text{Moment} = \int_0^{1m}\int_0^{15^\circ}\int_0^{2\pi} r^2 \cdot z \sin\theta \, d\varphi \, d\theta \, dr \cdot 10\tfrac{N}{m^3}$$

$$= 10\frac{N}{m^3}\int_0^{1m}\int_0^{15^\circ}\int_0^{2\pi} r^3 \sin\theta \cos\theta \, d\varphi \, d\theta \, dr$$

$$= 10\frac{N}{m^3} \cdot 2\pi \int_0^{1m}\int_0^{15^\circ} r^3 \cdot \frac{1}{2}\sin(2\theta) \, d\theta \, dr$$

$$= 10\frac{N}{m^3} \, \frac{\pi}{2}\left[1-\cos(30^\circ)\right] \int_0^{1m} r^3 dr$$

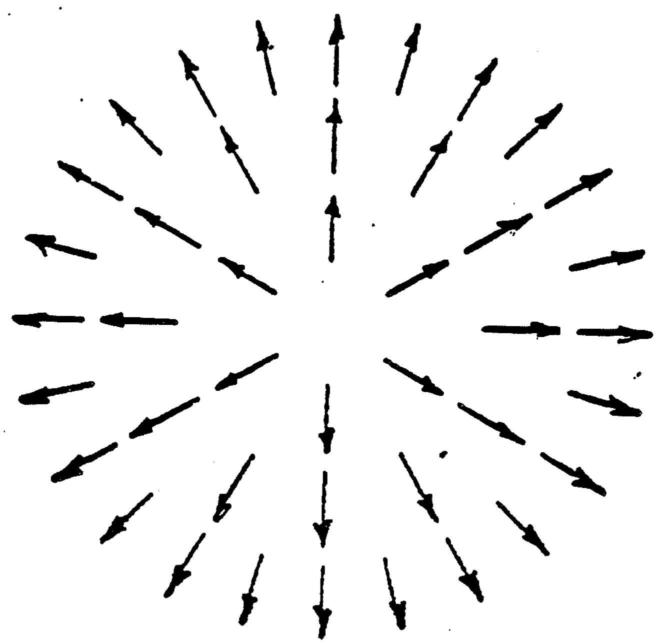
$$= \frac{5\pi}{4}\left[1-\cos(30^\circ)\right] \, N\cdot m$$

(9) Measuring the variation of a vector field

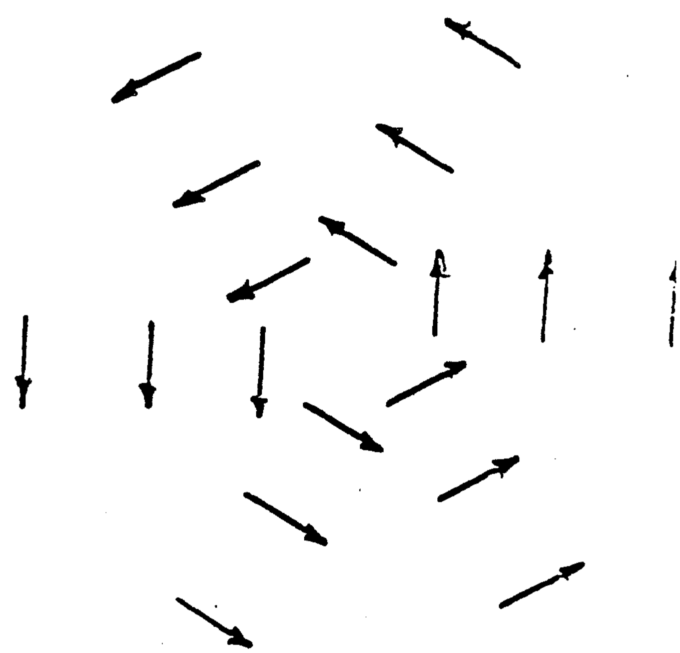
We have seen that the gradient of a scalar field is a measure of the variation of that field with position in space.

It is reasonable to ask if there is any systematic way of measuring the variation of a vector field with position.

Let's plot a two-dimensional vector field and see what types of variation the field can have.



(a)



(b)

In these illustrations, there seems to be two distinctly DIFFERENT types of variation in the vector field.

For field (a), the vectors seem to be diverging as we move along the "flow" of the field.

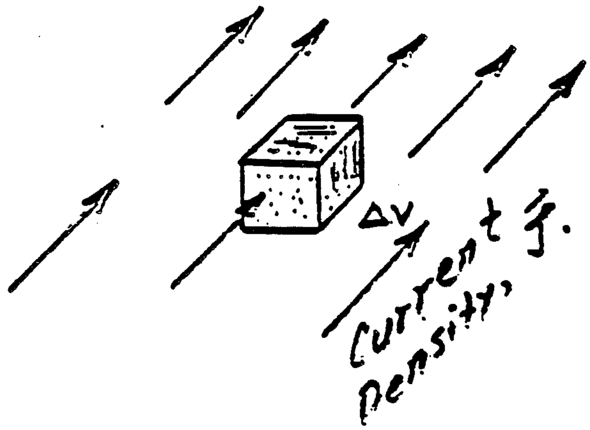
For field (b), the vectors do not seem to be diverging, but they seem to be "circulating" in a loop.

The fields in (a) do not exhibit this circulating property.

It appears that in order to fully characterize the variation of a vector field, TWO different types of measure will be required in contrast to the single type of measure needed for the gradient.

Indeed this is the case. The two types of measures are the DIVERGENCE and the CURL.

(10) Divergence



Suppose the vector field represents current density.

Then one way to measure how much the field diverges is to ask if any <u>charge</u> is <u>accumulated</u> <u>or</u> <u>depleted</u> <u>in</u> some small volume, <u>ΔV</u>.

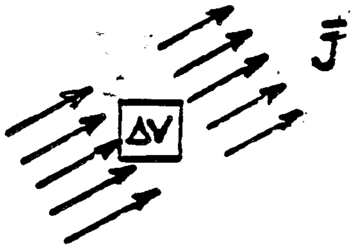
Since current density is the amount of charge per unit area flowing past a point in space in a particular direction, this question can be answered by finding the TOTAL FLUX leaving ΔV:

Rate that charge is leaving ΔV

$$= \oiint_{\partial \Delta V} \vec{J} \cdot \hat{n}\, dS .$$

Note, $\partial \Delta V$ is used to denote ~~the~~ the boundary of ΔV.

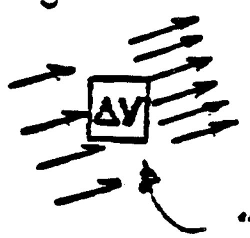
If this total flux is zero, then no net charge is being accumulated or depleted and the divergence of the field is zero.



The current in = current o
∴ No net accumulation of charge.

If the total flux is positive, then charge is being depleted since more current is leaving ΔV than entering and the divergence is positive.

Some current enters



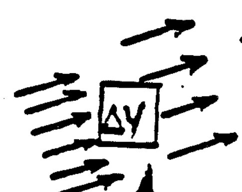
... But more current leaves
∴ There is a net depletion of cha

"Charge source"

If the total flux is negative, then charge is being accumulated since more current is entereing than leaving and the divergence is negative.

... But less current leaves

Current enters



∴ There is a net accumulation of charge.

"Charge sink"

Since the divergence properties of the field may vary from place to place, we would like to make ΔV as small as possible.

Indeed, we would like to allow ΔV to approach zero.

In order to obtain a non-vanishing measure, we must normalize the total flux by the volume ΔV.

This leads us to the DEFINITION of the divergence of a vector field:

$$\operatorname{div} \vec{J} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint_{\partial \Delta V} \vec{J} \cdot \hat{n}\, ds.$$

THE DIVERGENCE IS DEFINED AS THE NET FLUX LEAVING PER UNIT VOLUME.

For example, suppose we have a current density of

$$\vec{J} = \left(\frac{x^2}{m^2}\hat{x} - \frac{3y}{m}\hat{y} + \frac{xy^4}{m^5}\hat{z}\right)\frac{A}{m^2}$$

Then what is the time rate of change of the charge density?

If ρ is the charge density, then the charge stored in volume dV is $\rho\, dV$.

In small time interval, dt, div $\vec{J} \cdot dV \cdot dt$ amount of charge leaves dV.

The charge and therefore also the charge density inside dV must decrease.

If $d\rho$ is the change in charge density, then the amount of charge that has left dV in dt time is also,

$$dQ = -d\rho \, dV$$

Therefore,

$$-d\rho \cdot dV = \operatorname{div} \vec{J} \cdot dt \cdot dV \rightarrow \boxed{\frac{\partial \rho}{\partial t} = -\operatorname{div} \vec{J}}$$

This relation is called the <u>continuity equation</u>.

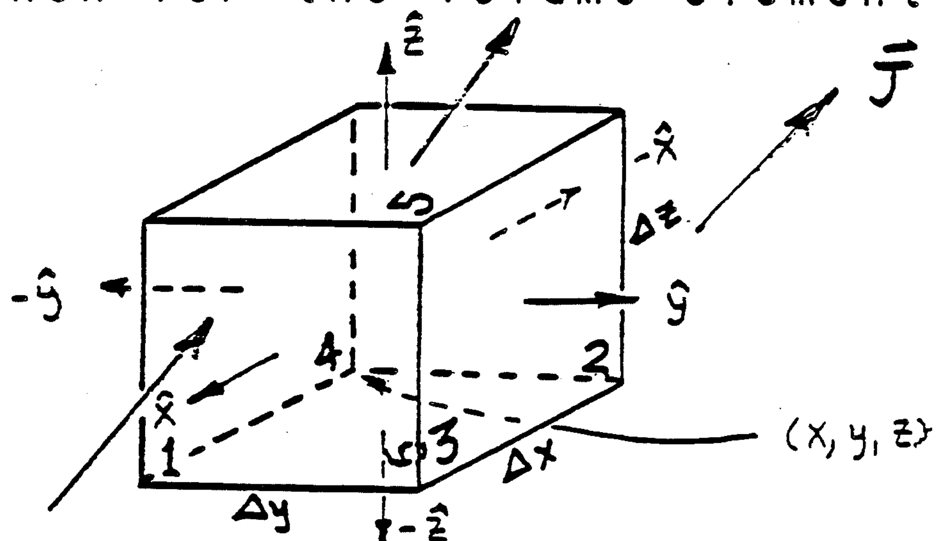
But what we need to do to come up with a number for this problem is to actually carry out the limiting process that defines div.

It would be a bit cumbersome to have to explicitly carry out this limiting process for every problem.

Fortunately, we can do it once and for all in general for all <u>continuously differentiable</u> vector fields to obtain a simple formula for finding div.

Suppose that $\vec{J}$ is such a vector field.

Then for the volume element, ΔV,



the net flux leaving, divided by ΔV is

$$\frac{1}{\Delta y \Delta z} \int_y^{y+\Delta y} \int_z^{z+\Delta z} \left[\frac{\overset{①}{J_x(x+\Delta x, \eta, \zeta)} - \overset{②}{J_x(x, \eta, \zeta)}}{\Delta x} \right] d\eta\, d$$

$$+ \frac{1}{\Delta x \Delta z} \int_x^{x+\Delta x} \int_z^{z+\Delta z} \left[\frac{\overset{③}{J_y(\xi, y+\Delta y, \zeta)} - \overset{④}{J_y(\xi, y, \zeta)}}{\Delta y} \right] d\xi\, d$$

$$+ \frac{1}{\Delta x \Delta y} \int_x^{x+\Delta x} \int_y^{y+\Delta y} \left[\frac{\overset{⑤}{J_z(\xi, \eta, z+\Delta z)} - \overset{⑥}{J_z(\xi, \eta, z)}}{\Delta z} \right] d\xi\, d\eta$$

= flux leaving faces 1, 2, 3, 4, 5, and 6.

Since $\vec{J}$ is <u>continuously differentiable</u>, the mean value theorem states that

$$\frac{J_x(x+\Delta x, \eta, \zeta) - J_x(x, \eta, \zeta)}{\Delta x} = \frac{\partial J_x}{\partial x}(\xi, \eta, \zeta)$$

for some ξ in the interval $x \leq \xi \leq x + \Delta x$.

Thus, the flux leaving faces 1 and 2 is

$$\frac{1}{\Delta y \Delta z} \int_y^{y+\Delta y} \int_z^{z+\Delta z} \frac{\partial J_x}{\partial x}(\xi, \eta, \zeta)\, d\eta\, d\zeta .$$

But this represents an <u>average</u> of a <u>continuous function over a small area</u>.

As the area becomes smaller and smaller, (that is, as Δx, Δy, and Δz go to zero) this average must approach

$$\frac{\partial J_x}{\partial x}(x, y, z)$$

Of course, we can do the same thing for the rest of the flux terms of div $\vec{J}$ and we end up with the simple formula that

$$\operatorname{div} \vec{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} .$$

This is what we get in <u>rectangular coordinates</u>. When we get to other coordinate systems, we will do the same thing.

The only difference will be that the volume, ΔV, will be constructed <u>in terms of changes in our new coordinates</u> instead of changes in x, y, and z.

Now we can complete the example we started.

Using the continuity equation, the rate at which charge density is changing is

$$\frac{\partial \rho}{\partial t} = -\operatorname{div} \hat{J} = -\left[\frac{\partial x^2}{\partial x}\frac{1}{m^2} - \frac{\partial (3y)}{\partial y}\frac{1}{m} + \frac{\partial (xy^4)}{\partial z}\frac{1}{m^5}\right]\frac{A}{m}$$
$$= (-2x/m^4 - 3/m^3)A$$

If we take a closer look at the simple formula we derived for the divergence, it has the form of our previously defined "del" operator dotted with $\vec{J}$.

$$\operatorname{div} \vec{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}$$
$$= \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial y}\right) \cdot \vec{J}$$

Thus,

$$div\ \vec{J} = \nabla \cdot \vec{J}$$

for continuously differentiable vector fields. This, however, is a CONCLUSION, NOT a definition.

As we saw earlier, the divergence of a vector only tells part of the story about its variation in space.

The next concept we must consider is the curl.

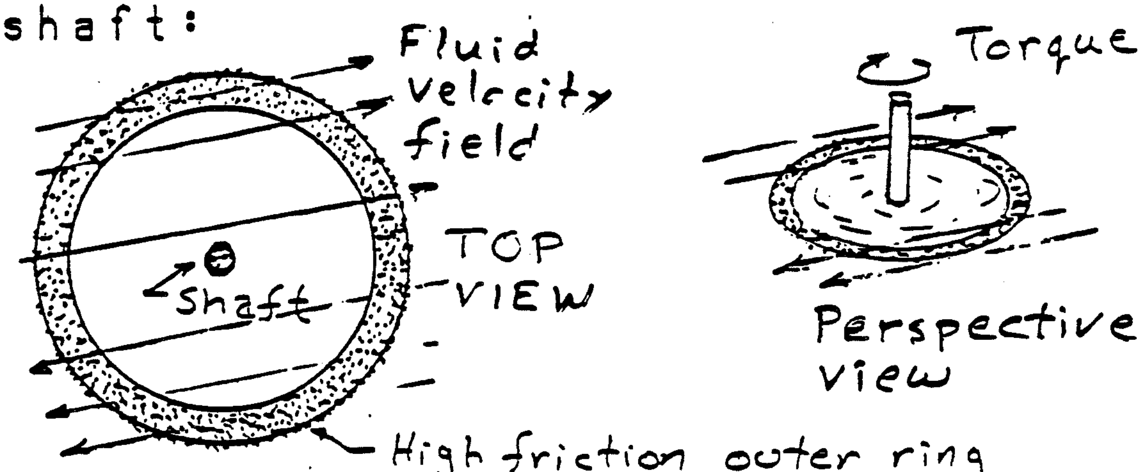
(11) Curl

<<<<<<<<<<<<<<<<<>>>>>>>>>>>>>>>>>>

The curl of a vector is intended to measure the <u>rotational</u> <u>tendency</u> of the vector field.

Suppose we have a fluid with a velocity field, $\vec{v}$.

To measure the rotational tendency of this velocity field, suppose we fashion a small disk mounted on a shaft:



The inner section will be made of a slick material while the outermost ring is coated with a gritty surface.

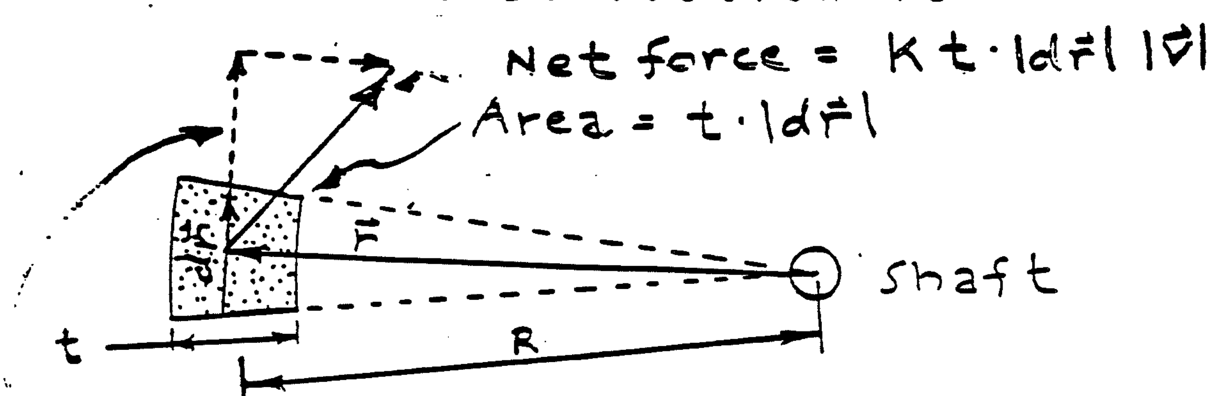
The fluid will flow over the slick portion of the disk with little friction.

Over the rough outer ring, however, the moving fluid will cause frictional forces to exist on the disk.

Any IMBALANCE in the symmetry of the velocity field should cause more frictional force on one side of the shaft than the other and therefore produce a torque on the shaft.

Thus, by measuring the torque, we should be able to learn something about the rotational tendency of the vector field.

Making the reasonable supposition that the force on a small section of the disk is proportional to the area of that section and the velocity of the fluid flowing across it, the torque produced because of that section is



Component of force $\perp$ to $\vec{r}$ is $Kt\,\vec{v} \cdot d\vec{r}$

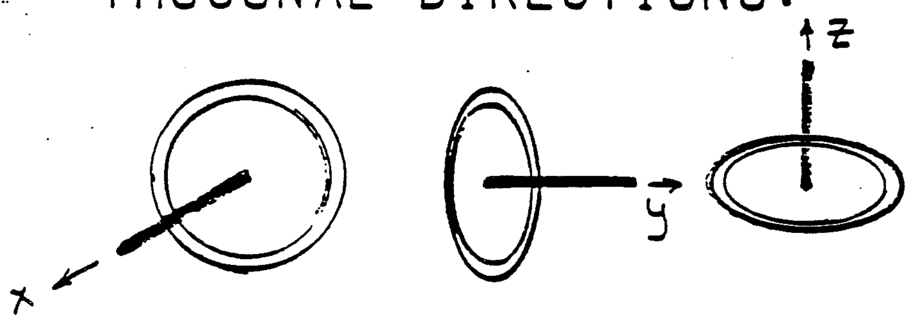
$\therefore$ Torque = $ktR\,\vec{v} \cdot d\vec{r}$

Thus, the total torque on the shaft is

$$\text{Torque} = kRt \oint_C \vec{V} \cdot d\vec{r}. \quad C \text{ is path around entire ring.}$$

This integral is given a special name. It is called the <u>net circulation</u> about the closed curve, C.

Since the torque on the shaft will depend in general on the ORIENTATION of the shaft, in order to fully characterize the rotational tendency of the field, we should do the measurement in THREE ORTHOGONAL DIRECTIONS.



Finally, since we would like to characterize the change in $\vec{V}$ at a point rather than an average change over a large region, we should make the radius of our disk as small as possible.

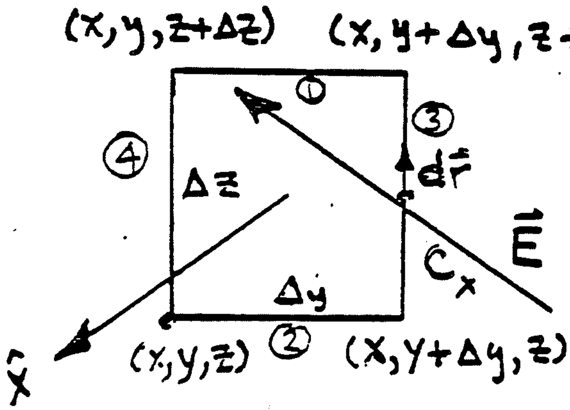
Indeed, we should let it approach zero.

In order to obtain a non-zero result, we need to normalize the circulation by the area of the disk.

By so doing, we arrive at the definition of the curl of a vector field:

THE CURL IS A VECTOR WHOSE COMPONENTS ARE THE NET CIRCULATION PER UNIT AREA IN THE POSITIVE SENSE IN A PLANE PERPENDICULAR TO THAT COMPONENT.

In rectangular components, for example, the curl of vector field $\vec{E}$ is



$$\hat{x}\cdot \operatorname{curl}\vec{E} = \lim_{\substack{\Delta y \to 0 \\ \Delta z \to 0}} \frac{1}{\Delta y \Delta z} \oint_{C_x} \vec{E}\cdot d\vec{r}$$

Similarly $$\hat{y}\cdot \operatorname{curl}\vec{E} = \lim_{\substack{\Delta x \to 0 \\ \Delta z \to 0}} \frac{1}{\Delta x \Delta z} \oint_{C_y} \vec{E}\cdot d\vec{r}$$

$$\hat{z}\cdot \operatorname{curl}\vec{E} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{1}{\Delta x \Delta y} \oint_{C_z} \vec{E}\cdot d\vec{r}$$

Again, it is cumbersome to have to actually perform this limiting process for every vector field.

But we are lucky again. If the vector field, $\vec{E}$, is continuously differentiable, then

$$\hat{x}\cdot \text{curl}\,\vec{E} = -\frac{1}{\Delta y}\int_{y}^{y+\Delta y}\left[\frac{\overset{①}{E_y(x,\eta,z+\Delta z)} - \overset{②}{E_y(x,\eta,z)}}{\Delta z}\right] + \frac{1}{\Delta z}\int_{z}^{z+\Delta z}\left[\frac{\overset{③}{E_z(x,y+\Delta y,\zeta)} - \overset{④}{E_z(x,y,\zeta)}}{\Delta y}\right]$$

$$\xrightarrow{\text{as } \Delta y,\, \Delta z \to 0} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}.$$ Similarly,

$$\hat{y}\cdot \text{curl}\,\vec{E} = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x};\quad \hat{z}\cdot \text{curl}\,\vec{E} = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}$$

We notice that this simple expression for the curl has the form of the cross product of the "del" operator with the vector, $\vec{E}$.

Therefore,

$$\text{curl}\,\vec{E} = \nabla \times \vec{E}$$

for continuously differentiable vector fields.

One way to remember how to expand this is to find the deterinant.

$$\text{curl } \hat{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

(12) Divergence Theorem

<<<<<<<<<<<<<<<<<<<<>>>>>>>>>>>>>>>>>>>>>>>

Suppose that we have a volume, V, containing charge of density, ρ.

Suppose that what we really want to know is the rate at which the TOTAL charge within V is varying with time.

We have already seen that the continuity equation relates the change in charge DENSITY to the divergence of current density:

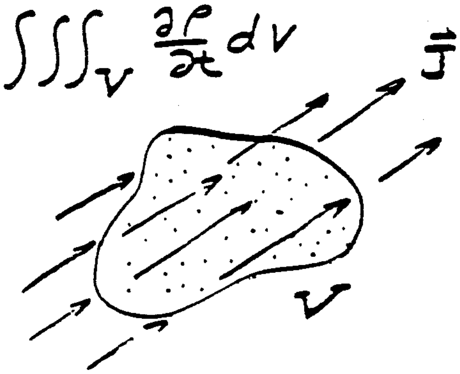
$$-\frac{\partial \rho}{\partial t} = \operatorname{div} \vec{J}.$$

To find out about the change in TOTAL charge, we must perform a volume integral over V:

$$Q = \iiint_V \rho \, dv$$

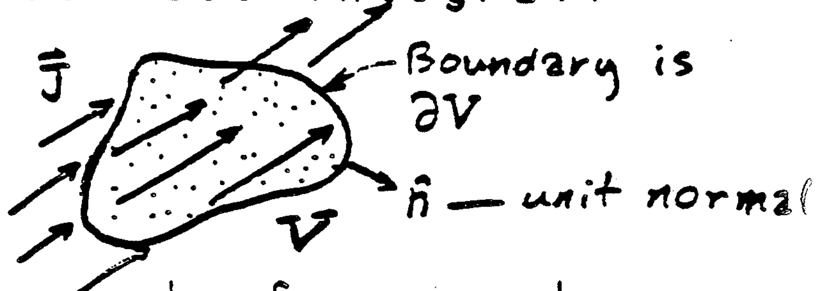
$$\therefore \frac{dQ}{dt} = \frac{d}{dt} \iiint_V \rho \, dv = \iiint_V \frac{\partial \rho}{\partial t} dv$$

$$= - \iiint_V \operatorname{div} \vec{J} \, dv$$



Now since the surface integral,

$$\oiint_{\partial V} \vec{J} \cdot \hat{n}\, ds$$



is the total amount of current leaving V, we know that the rate of change of total charge in V is

$$-\oiint_{\partial V} \vec{J} \cdot \hat{n}\, ds.$$

But this means that the volume and surface integrals should be the same:

$$\iiint_V \operatorname{div} \vec{J}\, dv = \oiint_{\partial V} \vec{J} \cdot \hat{n}\, ds.$$

This equivalence is the DIVERGENCE THEOREM.

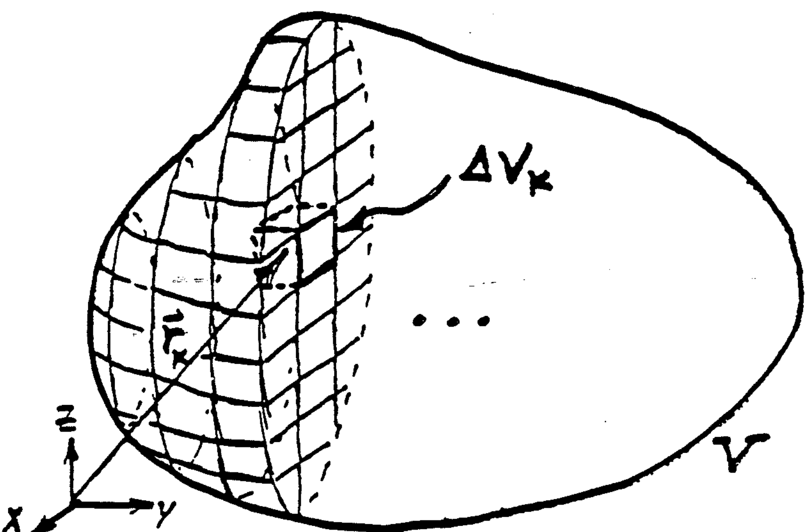
Let's prove this theorem.

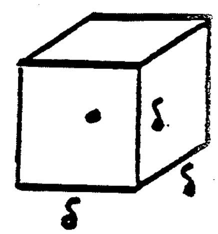
We know that from the definition that the volume integral,

$$\iiint_V \operatorname{div} \vec{J}\, dv$$

is formed by taking the limit of a sequence of sums:

$$\lim_{n \to \infty} \sum_{k=1}^{N} \operatorname{div} \vec{J}(\vec{r}_k)\, \Delta V_k$$





Volume element ΔV_k. Point at center is $\vec{r}_k$.

Since it really doesn't matter how we divide the volume up in our sequence as long as each sub-volume approaches zero, let's assume that each sub-volume is a cube of width, δ.

We will let the points at which we sample the integrand be at the centers of each of these cubes.

Then, by the definition of divergence,

$$\operatorname{div} \vec{J} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint_{\partial V} \vec{J} \cdot \hat{n}\, ds,$$

and the concept of a limit, we have that

$$\operatorname{div} \vec{J}(\vec{r}_k) = \frac{\oiint_{\partial \Delta V_k} \vec{J} \cdot \hat{n}\, ds}{\Delta V_k} + \epsilon(\vec{r}_k, \delta)$$

where $\epsilon(\vec{r}_k, \delta)$ is a function which approaches zero as δ, and hence, ΔV_k go to zero.

Then the sum,

$$\sum_{K=1}^{N} \operatorname{div} \vec{J}(\vec{r}_K)\, \Delta V_K$$

is equivalent to

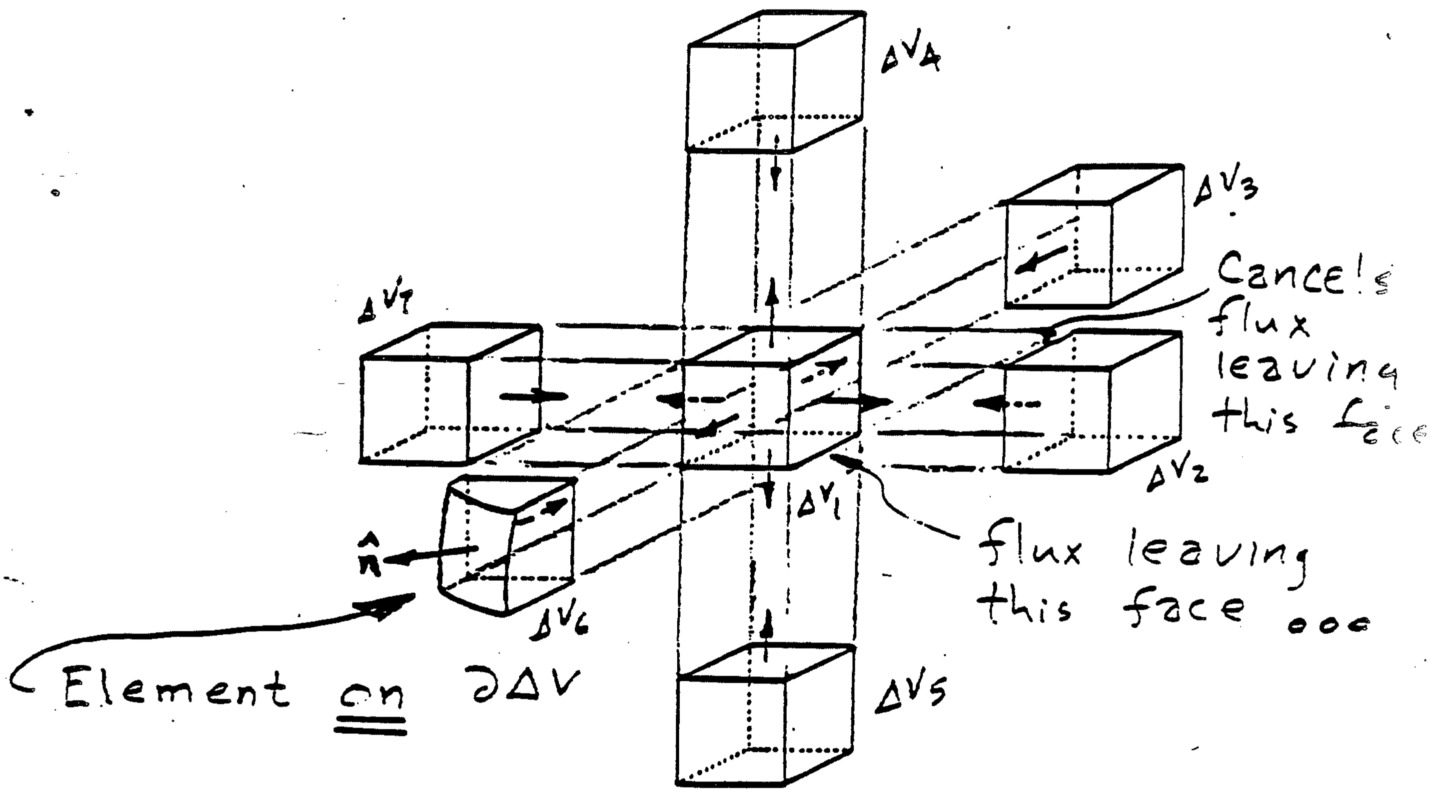
$$\sum_{K=1}^{N} \oiint_{\partial \Delta V_K} \vec{J} \cdot \hat{n}\, ds + \underbrace{\sum_{K=1}^{N} \epsilon(\vec{r}_K, \delta)\, \Delta V_K}$$

But $\left| \sum_{K=1}^{N} \epsilon(\vec{r}_K, \delta)\, \Delta V_K \right| \leq \epsilon_{max} \sum_{K=1}^{N} \Delta V_K = \epsilon_{max} V \to 0$ as $N \to \infty$.

Therefore, we have the result that

$$\iiint_V \operatorname{div} \vec{J}\, dv = \lim_{N \to \infty} \sum_{K=1}^{N} \oiint_{\partial \Delta V_K} \vec{J} \cdot \hat{n}\, ds.$$

Now let's take an exploded view of a portion of our segmented volume.



Exploded view

Clearly, the flux leaving one face of one of our cubes <u>cancels</u> the flux leaving an adjacent face of another cube.

It cancels, that is, <u>except at</u> the <u>faces</u> of sub-volumes <u>which border</u> on the <u>boundary of</u> our original volume, <u>V</u>.

That is,

$$\sum_{K=1}^{N} \oiint_{\partial V_K} \vec{J}\cdot\hat{n}\,ds = \oiint_{\partial V} \vec{J}\cdot\hat{n}\,ds$$

and hence

$$\iiint_V \operatorname{div}\vec{J}\,dv = \lim_{N\to\infty}\sum_{K=1}^{N} \oiint_{\partial\Delta V_K} \vec{J}\cdot\hat{n}\,ds$$

$$= \oiint_{\partial V} \vec{J}\cdot\hat{n}\,ds$$

Q.E.D.

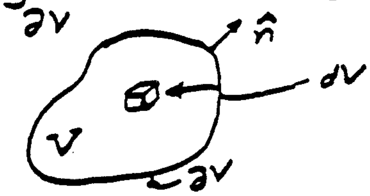
(13) Stoke's Theorem

<<<<<<<<<<<<<<<<<<>>>>>>>>>>>>>>>>>>

We have just seen that the TOTAL FLUX leaving the surface of a volume, V, is equal to the integral of the divergence of the vector field over V.

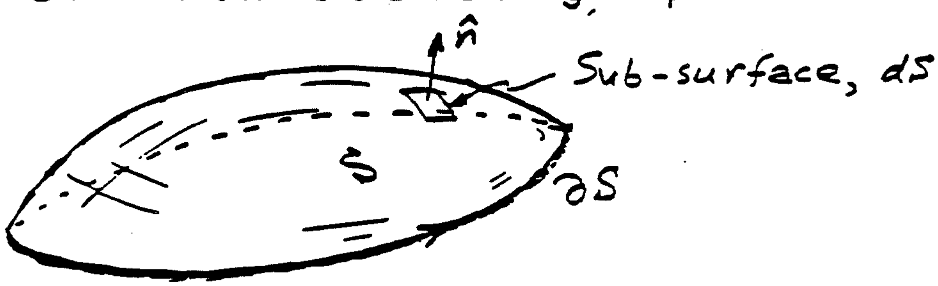
$$\text{Flux} = \oiint_{\partial V} \vec{F}\cdot\hat{n}\,ds = \iiint_V \underbrace{\text{div}\,\vec{F}\,dV}$$

flux leaving dV is $\text{div}\,\vec{F}\,dV = \oiint_{\partial dV} \vec{F}\cdot\hat{n}\ldots$



That is, the sum of the fluxes leaving each "local" sub-volume comprising V is equal to the net amount of flux leaving the surface of V.

Now suppose that we have a surface, S, with bounding space curve, ∂S.



We know that the net circulation about any sub-area, dS, comprising S is just

Circulation about ds $= \hat{n}\cdot \text{curl}\,\vec{F}\cdot dS$	Remember $\hat{n}\cdot \text{curl}\,\vec{F} = \lim_{\Delta s\to 0} \dfrac{\oint_{\partial \Delta s} \vec{F}\cdot d\vec{r}}{\Delta s}$

A reasonable question to ask is: Can the sum of the circulations over each sub-surface,

$$\iint_S \hat{n}\cdot \text{curl}\,\vec{F}\,ds,$$

be simply related to the NET CIRCULATION about the curve bounding S,

$$\oint_{\partial S} \vec{F}\cdot d\vec{r}\ ?$$

The answer, of course, is "<u>yes</u>."

<u>Stoke's</u> <u>theorem</u> states that

$$\oint_{\partial S} \vec{F}\cdot d\vec{r} = \iint_S \hat{n}\cdot \text{curl}\,\vec{F}\,dS.$$

in the positive sense with respect to $\hat{n}$.

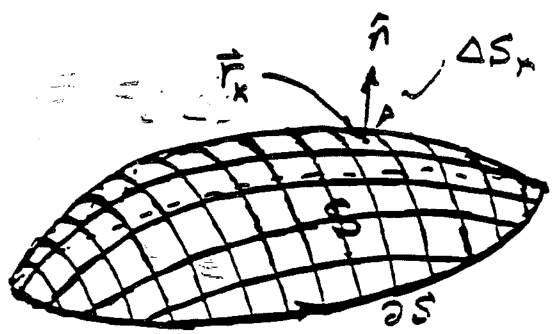
Let's prove it.

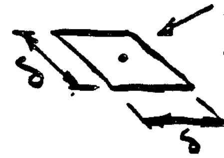
Just as we did in proving the divergence theorem, we will go back to the <u>definitions</u> of surface integrals and the curl.

The surface integral we need is defined as

$$\iint_S \hat{n}\cdot \text{curl}\, \vec{F}\, dS$$

$$= \lim_{N\to\infty} \sum_{K=1}^{N} \hat{n}\cdot \text{curl}\, \vec{F}(\vec{r}_K)\Delta S_K$$





Interior surface element, ΔS_K. $\vec{r}_K$ at center of ΔS_K.

Also, the definition of the curl requires that

$$\hat{n}\cdot \text{curl}\, \vec{F}(\vec{r}_K) = \lim_{\Delta S\to 0} \frac{\oint_{\partial\Delta S} \vec{F}\cdot d\vec{r}}{\Delta S} \longrightarrow$$

$$\hat{n}\cdot \text{curl}\, \vec{F}(\vec{r}_K) = \frac{\oint_{\partial\Delta S_K} \vec{F}\cdot d\vec{r}}{\Delta S_K} + \epsilon(\vec{r}_K, \delta)$$

where $\epsilon(\vec{r}_K, \delta)$ goes to zero as the parameter, δ, and hence also the sub-surfaces shrink to zero. Therefore,

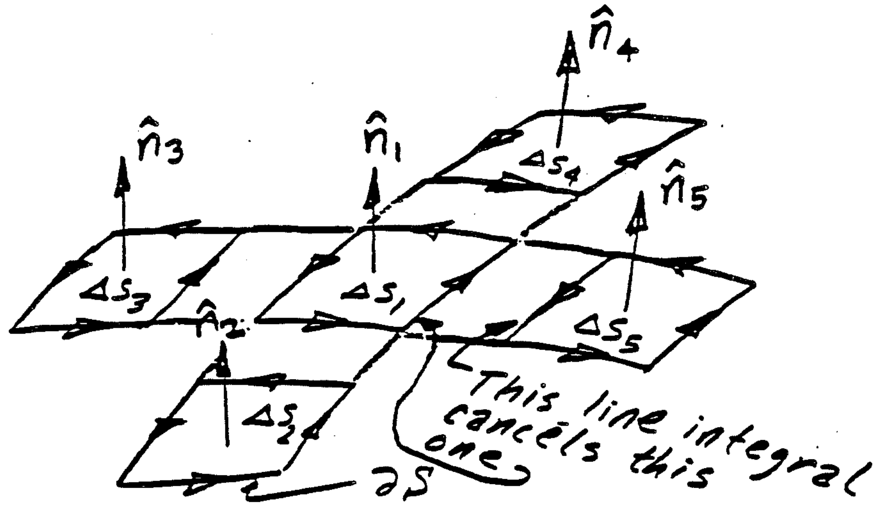
$$\sum_{K=1}^{N} \hat{n}\cdot \text{curl}\, \vec{F}(\vec{r}_K)\Delta S_K = \sum_{K=1}^{N} \oint_{\partial\Delta S_K} \vec{F}\cdot d\vec{r}$$

$$+ \boxed{\sum_{K=1}^{N} \epsilon(\vec{r}_K, \delta)\,\Delta S_K} \xrightarrow[\text{as } \delta\to 0]{} 0$$

$$\therefore \lim_{N\to\infty} \sum_{K=1}^{N} \hat{n}\cdot \text{curl}\, \vec{F}(\vec{r}_K)\Delta S_K = \lim_{N\to\infty} \sum_{K=1}^{N} \oint_{\Delta S_K} \vec{F}\cdot d\vec{r}$$

$\iint \hat{n}\cdot \text{curl}\, dS =$

Now if we take an exploded view of a portion of the surface, S.



We find that the line integral over each segment of one sub-surface exactly cancels that of an adjacent sub-surface EXCEPT along the boundary of S.

Therefore,

$$\sum_{k=1}^{N} \oint_{\partial \Delta S_k} \vec{F} \cdot d\vec{r} = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

which leads us to Stoke's theorem,

$$\iint_S \hat{n} \cdot \text{curl}\, \vec{F}\, dS = \lim_{N \to \infty} \sum_{k=1}^{N} \oint_{\partial \Delta S_k} \vec{F} \cdot d\vec{r} = \oint_{\partial S} \vec{F} \cdot d\vec{r}.$$

We will see a little later where Stoke's theorem has important applications in the Theory of Electromagnetics.

(14) The Laplacian ∇^2

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From the preceding two integral transformations, the divergence theorem and Stoke's theorem, a number of other important and useful integral transformations can be derived.

In order to derive one of the most important of these, Green's Theorem, we must introduce another scalar field and vector field operator, the Laplacian.

In many physical applications, it is necessary to know not only how fast a given quantity varies with location, but also the RATE OF THE RATE of variation —

that is, we often require the "second derivative" of our physical quantity.

Suppose we have a twice continuously differentiable scalar field, $u(\vec{r})$. Then the grad $u(\vec{r})$ is a vector quantity which measures how fast u varies with space.

Now the variation of this vector field, grad u($\vec{r}$), we have seen must be measured by <u>two operations</u> in order to fully characterize it, <u>the divergence</u> and the <u>curl</u>.

We will see later that <u>curl grad u is zero</u>. Thus, the "second derivative" variation of u is described by

Notation

$$\text{div grad } u = \nabla\cdot\nabla u \equiv \nabla^2 u \equiv \mathcal{L}ap\, u$$

$$\begin{tikzcd} & & \nabla\times\nabla u = 0 \\ u \arrow[r, "\text{grad}"] & \nabla u \arrow[ur, "\text{curl}"] \arrow[dr, "\text{div}"'] & \\ & & \nabla\cdot\nabla u \end{tikzcd}$$

scalar — vector — "1st derivative" — vector / "2nd derivative" — scalar

The Laplacian IN RECTANGULAR COORDINATES is

$$\mathcal{L}ap\, u = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)u$$

$$= \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\hat{x}\frac{\partial u}{\partial x} + \hat{y}\frac{\partial u}{\partial y} + \hat{z}\frac{\partial u}{\partial z}\right)$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

(In rectangular coordinates,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ — But the formula is not so simple in other coordinate systems as we will see later

For example, if $u(\vec{r}) = T(\vec{r})$ were a temperature distribution given by

$$T(\vec{r}) = x^2y^2z^2$$

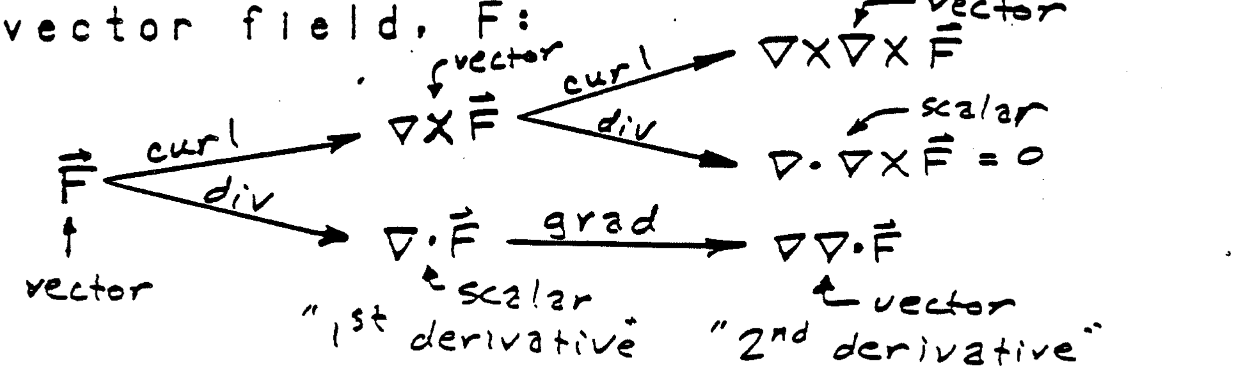
then Lap (T) is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) x^2y^2z^2 = 2y^2z^2 + 2x^2z^2 + 2x^2y^2$$

When measuring the "second derivative" variation of a scalar field, we saw that we basically have only <u>one</u> way to do it —— Lap.

On the other hand, we should expect that since vector fields have basically two types of "first derivative" variations, divergence and curl, there will also <u>be more than one type of "second derivative" variation</u>.

Let's consider the types of derivative operations we can perform on a twice continuously differentiable vector field, $\vec{F}$:



We will see later that the <u>divergence of the curl of any field is</u> automatically <u>zero</u>.

Thus, we have <u>only two types of second derivative</u> variational <u>measures</u> of $\vec{F}$,

$$\nabla\times\nabla\times\vec{F}, \qquad \nabla\nabla\cdot\vec{F}$$

One type of measure which combines both of these is the Laplacian APPLIED TO THE VECTOR $\vec{F}$,

$$\mathcal{L}_{ap}\vec{F} \equiv \nabla^2\vec{F} \equiv \nabla(\nabla\cdot\vec{F}) - \nabla\times\nabla\times\vec{F}.$$

In RECTANGULAR COORDINATES, this operator is

$$\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)$$

$$-\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\times\left[\hat{x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\right]$$

$$
\left.\begin{aligned}
&= \hat{x}\left(\frac{\partial^2 F_x}{\partial x^2} + \cancel{\frac{\partial^2 F_y}{\partial x \partial y}} + \cancel{\frac{\partial^2 F_z}{\partial x \partial z}}\right)\\
&+ \hat{y}\left(\cancel{\frac{\partial^2 F_x}{\partial y \partial x}} + \frac{\partial^2 F_y}{\partial y^2} + \cancel{\frac{\partial^2 F_z}{\partial y \partial z}}\right)\\
&+ \hat{z}\left(\cancel{\frac{\partial^2 F_x}{\partial z \partial x}} + \cancel{\frac{\partial^2 F_y}{\partial z \partial y}} + \frac{\partial^2 F_z}{\partial z^2}\right)
\end{aligned}\right\} \nabla\nabla\cdot\vec{F}
$$

$$
\left.\begin{aligned}
&- \hat{x}\left[\frac{\partial}{\partial y}\left(\cancel{\frac{\partial F_y}{\partial x}} - \frac{\partial F_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_x}{\partial z} - \cancel{\frac{\partial F_z}{\partial x}}\right)\right]\\
&- \hat{y}\left[\frac{\partial}{\partial z}\left(\cancel{\frac{\partial F_z}{\partial y}} - \frac{\partial F_y}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F_y}{\partial x} - \cancel{\frac{\partial F_x}{\partial y}}\right)\right]\\
&- \hat{z}\left[\frac{\partial}{\partial x}\left(\cancel{\frac{\partial F_x}{\partial z}} - \frac{\partial F_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_z}{\partial y} - \cancel{\frac{\partial F_y}{\partial z}}\right)\right]
\end{aligned}\right\} -\nabla\times\nabla\times\vec{F}
$$

$$
\begin{aligned}
&= \hat{x}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) F_x\\
&+ \hat{y}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) F_y\\
&+ \hat{z}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) F_z
\end{aligned}
$$

Or $\nabla^2\vec{F} = \nabla^2(\hat{x}F_x + \hat{y}F_y + \hat{z}F_z)$

$$= \hat{x}\,\nabla^2 F_x + \hat{y}\,\nabla^2 F_y + \hat{z}\,\nabla^2 F_z$$

can interchange only because $\hat{x}$, $\hat{y}$, and $\hat{z}$ are <u>constant</u>. Not so in other coordinate systems!

Although in RECTANGULAR COORDINATES it appears that Lap applied to a scalar and Lap applied to a vector are identical operators, DO NOT BE DECEIVED ...

They are quite <u>DIFFERENT</u> operators!

The differences become very apparent when writing them in OTHER coordinate systems.

(15) Green's Theorem

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Now that we have the divergence theorem, we can derive another integral theorem, GREEN'S THEOREM.

We will find important applications for this later when we represent the solution to partial differential equations in terms of the so-called Green's functions.

Rather than begin directly with Green's theorem, let's begin by reviewing the familiar concept of "integration by parts."

The formula for partial integration is

$$\int_a^b u\frac{dv}{dx}dx = u(b)v(b)-u(a)v(a)-\int_a^b v\frac{du}{dx}dx.$$

If we let

$$v=\frac{dw}{dx},$$

then

$$\int_a^b u\frac{d^2w}{dx^2}dx = u(b)\frac{dw(b)}{dx}-u(a)\frac{dw(a)}{dx}-\int_a^b \frac{dw}{dx}\frac{du}{dx}dx.$$

By interchaning u and w, we find also that

$$\int_a^b w\frac{d^2u}{dx^2}dx = w(b)\frac{du(b)}{dx}-w(a)\frac{du(a)}{dx}-\int_a^b \frac{dw}{dx}\frac{du}{dx}dx.$$

Subtracting these two equations, we arrive at

$$\int_a^b\left[u\frac{d^2w}{dx^2}-w\frac{d^2u}{dx^2}\right]dx =$$

$$\left[u(b)\frac{dw(b)}{dx}-w(b)\frac{du(b)}{dx}\right]$$

$$-\left[u(a)\frac{dw(a)}{dx}-w(a)\frac{du(a)}{dx}\right].$$

Thus, the integral of

$$\int_a^b \left[u\frac{d^2w}{dx^2} - w\frac{d^2u}{dx^2}\right]dx$$

over the interval from a to b can be written in terms of the values of u and w and their first derivatives at the boundary of the interval (the end points).

The principle of "integration by parts" in higher dimensions is just Green's theorem which states that if u and w are twice continuously differentiable functions,

$$\iiint_V (u\nabla^2 w - w\nabla^2 u)\,dV = \oint\!\!\oint_{\partial V} [u\nabla w - w\nabla u]\cdot\hat{n}\,dS.$$

The proof of this depends first on the fact that

$$\nabla\cdot(\varphi\vec{F}) = \frac{\partial}{\partial x}(\varphi F_x) + \frac{\partial}{\partial y}(\varphi F_y) + \frac{\partial}{\partial z}(\varphi F_z)$$

$$= \left[\varphi\frac{\partial F_x}{\partial x} + F_x\frac{\partial\varphi}{\partial x}\right] + \left[\varphi\frac{\partial F_y}{\partial y} + F_y\frac{\partial\varphi}{\partial y}\right] + \left[\varphi\frac{\partial F_z}{\partial z} + F_z\frac{\partial\varphi}{\partial z}\right]$$

$$= \varphi\overbrace{\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)}^{div\,\vec{F}} + \underbrace{(\hat{x}F_x + \hat{y}F_y + \hat{z}F_z)}_{\vec{F}}\cdot\underbrace{\left(\hat{x}\frac{\partial\varphi}{\partial x} + \hat{y}\frac{\partial\varphi}{\partial y} + \hat{z}\frac{\partial\varphi}{\partial z}\right)}_{grad\,\varphi}$$

$$= \underline{\varphi\nabla\cdot\vec{F} + \vec{F}\cdot\nabla\varphi}$$

Then,

$$\nabla\cdot(u\nabla w) = u\nabla\cdot\nabla w + \nabla w\cdot\nabla u = u\nabla^2 w + \nabla w\cdot\nabla u$$
$$\nabla\cdot(w\nabla u) = w\nabla\cdot\nabla u + \nabla w\cdot\nabla u = w\nabla^2 u + \nabla w\cdot\nabla u$$
$$\therefore\ \nabla\cdot[u\nabla w - w\nabla u] = u\nabla^2 w - w\nabla^2 u$$

Thus,

$$\iiint_V [u\nabla^2 w - w\nabla^2 u]\,dV = \iiint_V \nabla\cdot[u\nabla w - w\nabla u]\,dV$$

But by the divergence theorem, we know that

$$\iiint_V \nabla\cdot[u\nabla w - w\nabla u]\,dV = \oiint_{\partial V}[u\nabla w - w\nabla u]\cdot\hat{n}\,dS$$

which completes our proof.

(16) Gauss's Theorem

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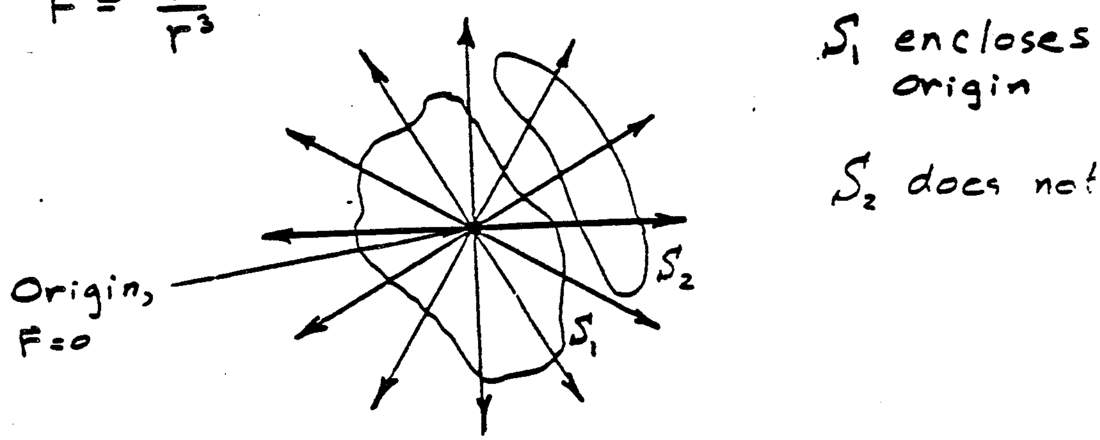
Another simple integral theorem which we will see has important applications in the theory of electrostatics and also for the concept of Green's function is Guass's theorem.

Gauss's theorem simply states that the net flux of the vector field $\vec{F} = \frac{\vec{r}}{r^3}$



that crosses any closed surface, S, is 4π if S encloses the origin and zero if S excludes the origin.

That is

$$\oiint_S \frac{\vec{r}}{r^3} \cdot \hat{n}\, dS = \begin{cases} 0 & \text{if } S \text{ excludes origin} \\ 4\pi & \text{if } S \text{ encloses origin} \end{cases}$$

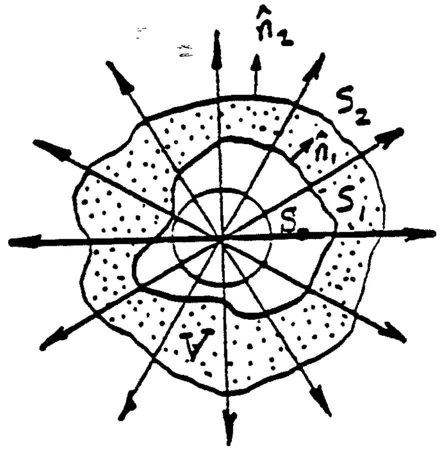
To see that this is true, it is only necessary to note that for any point other than the origin,

$$\mathrm{div}\,\frac{\vec{r}}{r^3} = \nabla\cdot\frac{\vec{r}}{r^3} = \nabla\cdot\left(\frac{x\hat{x}+y\hat{y}+z\hat{z}}{[x^2+y^2+z^2]^{3/2}}\right)$$

$$= \frac{1+1+1}{[x^2+y^2+z^2]^{3/2}} - \frac{3x^2+3y^2+3z^2}{[x^2+y^2+z^2]^{5/2}} = 0$$

If we think of this field as a current density, then the <u>ONLY</u> <u>source</u> for this current is a current injected <u>at</u> <u>the</u> <u>point</u> <u>at</u> <u>the</u> <u>origin</u>.

Then the current crossing surface, S_1, and surface S_2, must be the same.



$$\oint\!\!\oint_{S_1+S_2}\frac{\vec{r}}{r^3}\cdot\hat{n}\,dS = \iiint_V \cancelto{0}{\mathrm{div}\left(\frac{\vec{r}}{r^3}\right)}\,dv$$

$$= \oint\!\!\oint_{S_2}\frac{\vec{r}}{r^3}\cdot\hat{n}_2\,dS - \oint\!\!\oint_{S_1}\frac{\vec{r}}{r^3}\cdot\hat{n}_1\,dS$$

$$\longrightarrow$$

$$\oint\!\!\oint_{S_2}\frac{\vec{r}}{r^3}\cdot\hat{n}_2\,dS = \oint\!\!\oint_{S_1}\frac{\vec{r}}{r^3}\cdot\hat{n}_1\,dS$$

Therefore, if S encloses the origin, then the current crossing S is the same as the current crossing the sphere, S_0, of radius, r_0.

But it is easy to compute the current crossing this surface because of the symmetry of the field.

$$\oiint_{S_0} \frac{\vec{r}}{r^3}\cdot\hat{n}\,dS = \oiint_{S_0}\frac{\vec{r}}{r_0^3}\cdot\hat{r}\,dS = \frac{1}{r_0^2}\underbrace{\oiint_{S_0} dS}_{\text{area of sphere}} = \frac{4\pi}{r_0^2}\,r_0^2$$

For the case when S excludes the origin, the amount of flux entering the volume enclosed by S exactly equals the amount of flux leaving and hence the surface integral in this case is zero.

(17) Conservative vector fields and their representation in terms of scalar potentials.

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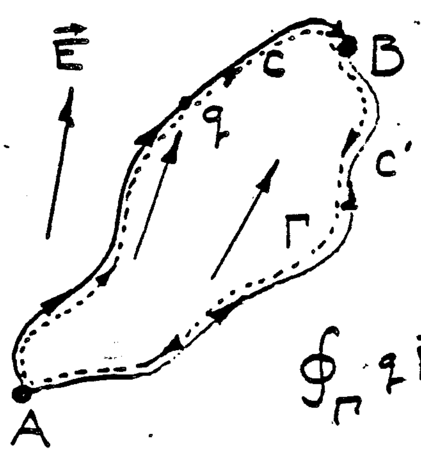
Suppose we move, say a point charge, q, along a line between two points, A, and B in an electric field, $\vec{E}$.



$$\text{Work} = \int_c q\vec{E}\cdot d\vec{r}$$

$$\neq \int_{c'} q\vec{E}\cdot d\vec{r},$$

That is, in general,

$$\oint_\Gamma q\vec{E}\cdot d\vec{r} = \int_c q\vec{E}\cdot d\vec{r} - \int_{c'} q\vec{E}\cdot d\vec{r} \neq 0$$

Then the work done, will, <u>in general</u>, <u>depend</u> <u>on</u> the precise <u>path</u> taken in the movement.

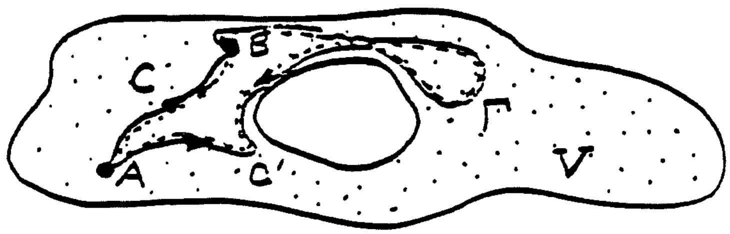
There is, <u>however</u>, a very <u>important class</u> of vector fields for which the <u>path of integration is NOT relevant</u>, ONLY the END POINTS.

That class of fields is called <u>CONSERVATIVE</u> vector fields.

They are "conservative" because the amount of enery expended in moving the charge in a closed loop is zero — hence all the energy is conserved within the system.

More precisely, a vector field, $\vec{F}$, is said to be conservative in a region V of space if the following condition holds:

Let A and B be ANY two points in V connected by a space curve, C lying entirely within V.



Then the line integral,

$$\int_C \vec{F} \cdot d\vec{r}$$

is UNCHANGED when C is CONTINUOUSLY (without any point ever leaving V) deformed into ANY OTHER path, C', having end points A and B.

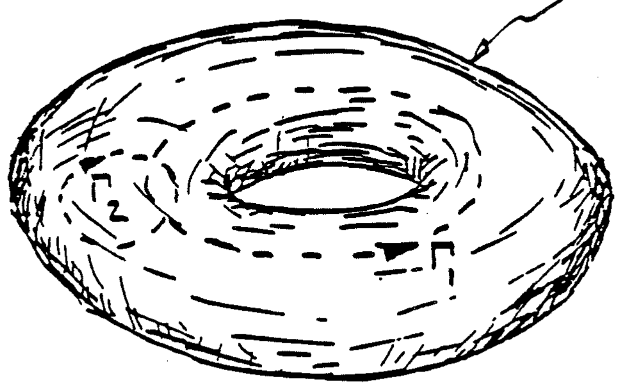
That is,

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C'} \vec{F} \cdot d\vec{r} \quad \underline{\underline{or}} \quad \oint_\Gamma \vec{F} \cdot d\vec{r} = 0.$$

Of course, this means that the line <u>integral</u> around a <u>CLOSED LOOP</u> formed by going from A to B on path C and returning to A on path C' <u>must be zero</u>.

This more precise definition is needed because the region <u>V may not be "simply connected</u>."

Roughly, a <u>multiply connected domain</u> is one which has "holes" or "tunnels" through it:



"donut" Region V

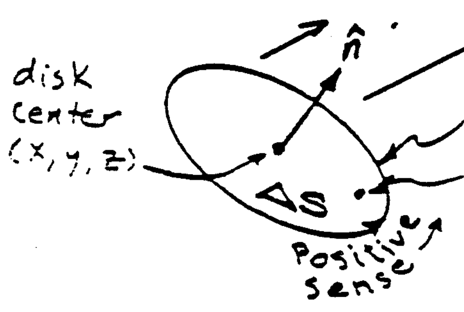
$\vec{F}$ is conservative in V

$$\int_{\Gamma_2} \vec{F}\cdot d\vec{r} = 0$$

But!

$\int_{\Gamma_1} \vec{F}\cdot d\vec{r}$ need <u>NOT</u> be zero.

Let's consider a small area, ΔS, which lies entirely within region V over which $\vec{F}$ is conservative.



Conservative field, $\vec{F}$.

boundary is $\partial \Delta S$

point on disk : (x',y',z')

Then the net circulation about the boundary of this small area is,

$$\oint_{\partial \Delta S} \vec{F} \cdot d\vec{r} = 0$$

Since $\vec{F}$ is conservative.

But by Stoke's theorem,

$$\oint_{\partial \Delta S} \vec{F} \cdot d\vec{r} = \iint_{\Delta S} \operatorname{curl} \vec{F} \cdot \hat{n}\, dS = 0$$

Let's assume that $\vec{F}$ is continuously differentiable in V. Then the value of

$$\operatorname{curl} \vec{F}(x', y', z') = \nabla \times \vec{F}(x', y', z')$$

anywhere on the disk is essentially the same as it is at the disk's center if the disk is small enough:

$$\nabla \times \vec{F}(x', y', z') \approx \nabla \times \vec{F}(x, y, z).$$

Thus, we have

$$0 = \iint_{\Delta S} \nabla \times \vec{F}(x', y', z') \cdot \hat{n}\, dS \approx \iint_{\Delta S} \nabla \times \vec{F}(x, y, z) \cdot \hat{n}\, dS$$

$$\nabla \times \vec{F}(x, y, z) \cdot \hat{n}\, \Delta S \longrightarrow$$

$$\nabla \times \vec{F}(x, y, z) \cdot \hat{n} \approx 0.$$

Of course, this becomes <u>exact in the limit</u> as the disk diameter shrinks to zero.

Furthermore, the orientation of the disk (and hence $\hat{n}$) can be taken in ANY direction and we therefore have arrived at the result that A CONTINUOUSLY DIFFERENTIABLE, CONSERVATIVE VECTOR FIELD HAS ZERO CURL.

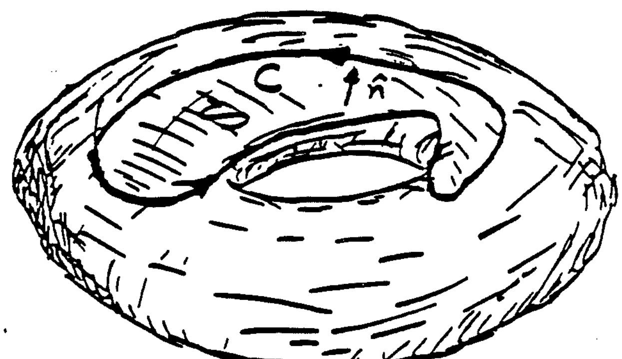
For this reason, a conservative vector field is also called IRROTATIONAL.

Of course the converse of this is also true.

If continuously differentiable vector field, $\vec{F}$, is irrotational over V,

$$\nabla \times \vec{F} = 0$$

$$C = \partial S$$



and if C is any closed path not encircling one of our holes or tunnels, then Stoke's theorem tells us that

$$\iint_S \nabla \times \vec{F} \cdot \hat{n}\, dS = 0 = \oint_C \vec{F} \cdot d\vec{r} = 0$$

and hence $\vec{F}$ is also conservative.

One extremely practical result of this is that any <u>conservative</u> vector <u>field can be REPRESENTED</u> as the <u>GRADIENT</u> of a <u>SCALAR POTENTIAL</u>.

That is, if $\vec{F}$ is conservative over V, there exists a scalar field, v, such that

$$\vec{F} = \text{grad}\, v = \nabla v.$$

Let's prove this result.

Let v be defined as

$$v = \int_{P_0}^{P} \vec{F} \cdot d\vec{r}.$$

where P_0 is an <u>ARBITRARY REFERENCE POINT</u> and P is the point with position vector, $\vec{r}$.

Then the grad v is

$$\nabla v = \frac{\partial v}{\partial x}\hat{x} + \frac{\partial v}{\partial y}\hat{y} + \frac{\partial v}{\partial z}\hat{z}.$$

Let's consider the computation of

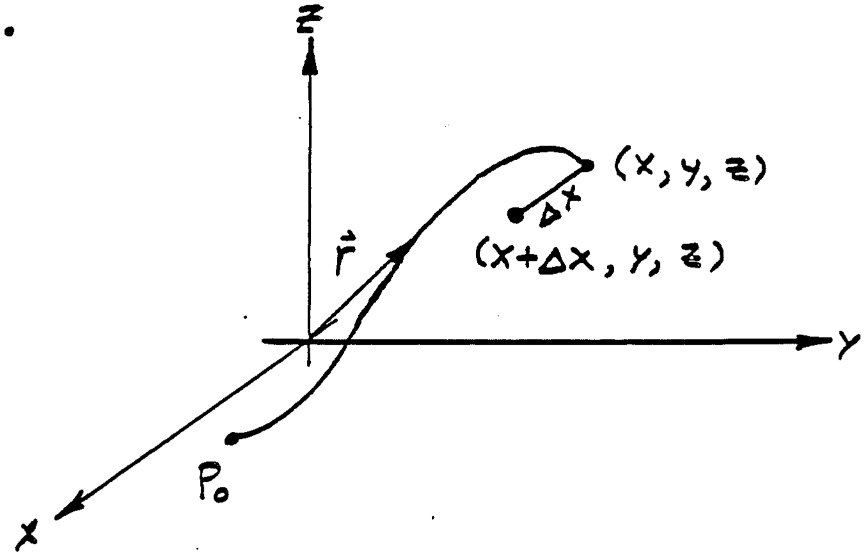
$$\frac{\partial v}{\partial x}$$

By definition,

$$\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \frac{V(x+\Delta x, y, z) - V(x, y, z)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\int_{P_0}^{(x+\Delta x, y, z)} \vec{F} \cdot d\vec{r} - \int_{P_0}^{(x, y, z)} \vec{F} \cdot d\vec{r}}{\Delta x}.$$

Since $\vec{F}$ is conservative, v is INDEPENDENT of the path of integration.



Thus we can take the path shown in going from P_0 to $(x+\Delta x, y, z)$.

Therefore,

$$\frac{\int_{P_0}^{(x+\Delta x, y, z)} \vec{F}\cdot d\vec{r} - \int_{P_0}^{(x,y,z)} \vec{F}\cdot d\vec{r}}{\Delta x} = \frac{\int_{(x,y,z)}^{(x+\Delta x, y, z)} \vec{F}\cdot\hat{x}\, dx}{\Delta x}$$

$$\approx \frac{F_x \Delta x}{\Delta x} = F_x \longrightarrow \frac{\partial v}{\partial x} = F_x .$$

if $\vec{F}$ is continuous.

Of course, by the same argument we can show that

$$\frac{\partial v}{\partial y} = F_y , \quad \frac{\partial v}{\partial z} = F_z .$$

Therefore,

$$\nabla v = \hat{x}\frac{\partial v}{\partial x} + \hat{y}\frac{\partial v}{\partial y} + \hat{z}\frac{\partial v}{\partial z} = \hat{x}F_x + \hat{y}F_y + \hat{z}F_z = \vec{F}$$

QED.

We notice that the potential, v, is unique only up to an additive CONSTANT since

$$\vec{F} = \nabla(v + \underbrace{c}_{\text{constant}}) = \nabla v + \cancelto{0}{\nabla c} = \nabla v$$

For example, can the electric field defined by

$$\vec{E} = 3x^2yz^2\hat{x} + x^3z^2\hat{y} + 2x^3yz\,\hat{z}$$

be represented as the gradient of scalar potential, and if so what is that potential?

To answer the first question, we need only compute the curl $\vec{E}$ and see whether or not it is zero:

$$\nabla \times \vec{E} = \hat{x}\left[\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right] + \hat{y}\left[\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right] + \hat{z}\left[\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right]$$

$$= \hat{x}\left[2x^3z - 2x^3z\right] + \hat{y}\left[6x^2yz - 6x^2yz\right] + \hat{z}\left[3x^2z^2 - 3x^2z^2\right]$$

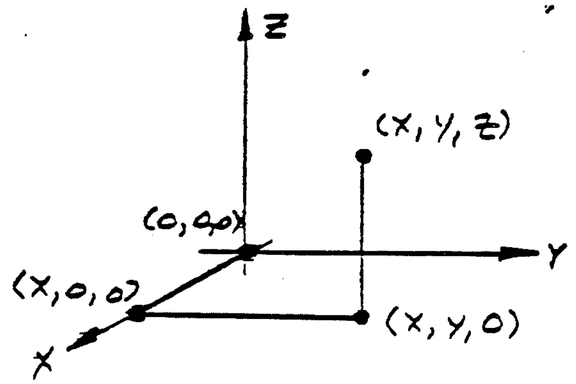
$$= 0$$

The curl vanishes everywhere and thus the electric field $\vec{E}$ is <u>irrotational</u>, or <u>conservative</u> and is representable by a scalar potential.

To explicity find this potential, we simply <u>choose</u> an <u>arbitrary reference point</u> <u>and</u> <u>integrate</u> along the most convenient path we can find,

$$V = \int_{(0,0,0)}^{(x,y,z)} \vec{E}\cdot d\vec{r} = \int_{(0,0,0)}^{(x,y,z)} \left[3x^2yz^2dx + x^3z^2dy + 2x^3yz\,dz\right]$$

Carrying out this integration,



$$\int_{(0,0,0)}^{(x,y,z)} \vec{E}\cdot d\vec{r} = \int_{(0,0,0)}^{(x,0,0)} \vec{E}\cdot d\vec{r} + \int_{(x,0,0)}^{(x,y,0)} \vec{E}\cdot d\vec{r} + \int_{(x,y,0)}^{(x,y,z)} \vec{E}\cdot d\vec{r}$$

$$\int_{(0,0,0)}^{(x,0,0)} \vec{E}\cdot d\vec{r} = \int_0^x [0\hat{x} + 0\hat{y} + 0\hat{z}]\cdot \hat{x}\, d\xi = 0$$

$$\int_{(x,0,0)}^{(x,y,0)} \vec{E}\cdot d\vec{r} = \int_0^y [0\hat{x} + 0\hat{y} + 0\hat{z}]\cdot \hat{y}\, d\eta = 0$$

$$\int_{(x,y,0)}^{(x,y,z)} \vec{E}\cdot d\vec{r} = \int_0^z [3x^2y\zeta^2\hat{x} + x^3\zeta^2\hat{y} + 2x^3y\zeta\hat{z}]\cdot \hat{z}\, d\zeta$$

$$= \int_0^z 2x^3y\zeta\, d\zeta = 2x^3y\int_0^z \zeta\, d\zeta$$

$$= 2x^3y\cdot\frac{1}{2}\zeta^2\Big|_0^z = \underline{\underline{x^3yz^2}}$$

Therefore, the potential, V, is

$$V = 0 + 0 + x^3yz^2 = x^3yz^2$$

We can check to see if this is indeed a valid scalar potential for E by taking its gradient:

$$\nabla(x^3yz^2) = \hat{x}\frac{\partial}{\partial x}(x^3yz^2) + \hat{y}\frac{\partial}{\partial y}(x^3yz^2) + \hat{z}\frac{\partial}{\partial z}(x^3y$$

$$= 3x^2yz^2\hat{x} + x^3z^2\hat{y} + 2x^3yz\,\hat{z},$$

which is identical to $\vec{E}$.

Since $\vec{E}$ times a small charge, q, is the force on that charge, the potential v is the WORK PER UNIT POSITIVE CHARGE EXPENDED IN MOVING THE CHARGE FROM THE REFERENCE POINT TO POINT P.

This potential is also refered to as the "voltage" between point P and the reference point (or "ground" node).

The statement,

$$\oint \vec{E}\cdot d\vec{r} = 0$$

is just Kirchoff's voltage law.

One reasonable question to ask about conservative fields is . . .

Is the gradient of ANY scalar field, v, an irrotational field?

What we can say is that if v is twice continously differentiable, then

$$\text{curl grad } \varphi =$$

$$\nabla \times (\nabla \varphi) = \nabla \times \left(\hat{x} \frac{\partial v}{\partial x} + \hat{y} \frac{\partial v}{\partial y} + \hat{z} \frac{\partial v}{\partial z} \right)$$

$$= \hat{x} \left[\frac{\partial^2 v}{\partial y \partial z} - \frac{\partial^2 v}{\partial z \partial y} \right] + \hat{y} \left[\frac{\partial^2 v}{\partial z \partial x} - \frac{\partial^2 v}{\partial x \partial z} \right]$$

$$+ \hat{z} \left[\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} \right] = 0$$

which implies that <u>every</u> <u>vector</u> <u>field</u>, $\vec{F}$, <u>defined</u> <u>by</u> <u>$\vec{E} = \nabla v$</u> (v twice continuously differentiable) <u>is</u> <u>conservative</u>.

Of course, not every field is irrotational and therefore conservative. At the other extreme are fields which are PURELY rotational which we will consider next.

(18) Solenoidal vector fields and their representation in terms of vector potentials.

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We have seen in the discussion of the diveregnece of a field that this type of variation of the field could be measured (say for a current density) by asking how much charge was being depleted or accumulated in a small volume, ΔV.

If a net amount of current is leaving this volume, we can think of the volume as being a "source."

If a net amount of current is entering this volume, we can think of the volume as being a "sink."

Suppose that no net amount of current enters or leaves ANY volume that we choose in a given region of space.

Then the only type of variation that such a current density can have is a ROTATIONAL variation.

We recall that the diverengence of a vector field is a "local" measure of the net flux per unit volume leaving a closed surface.

Therefore, since the net flux leaving any closed surface is zero for a solenodial field, the divergence of that field must also be zero.

$$\operatorname{div} \vec{F} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint_{\partial V} \vec{F} \cdot \hat{n}\, dS = 0$$

The converse of this is also true.

If the div $\vec{F}$ = 0 over all points inside V, then according to the divergence theorem,

$$\iiint_{V'} \operatorname{div} \vec{F}\, dV = 0 = \oiint_{\partial V'} \vec{F} \cdot \hat{n}\, dS$$ where V' is any volume in V.

Thus, a "divergenceless field" (try to say that three times in a row real fast!) is also a solenoidal field.

$$\operatorname{div} \vec{F} = 0 \longrightarrow \vec{F} \text{ is solenoidal}$$

$$\vec{F} \text{ solenoidal} \longrightarrow \operatorname{div} \vec{F} = 0$$

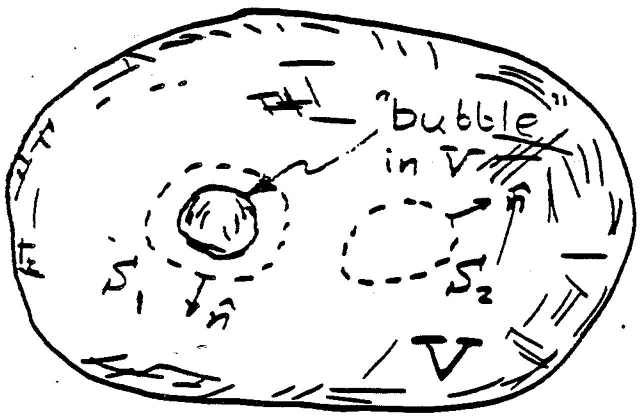
$\therefore$ $\vec{F}$ solenoidal is equivalent to $\operatorname{div} \vec{F} = 0$.

This leads us to the definition of a SOLENOIDAL field:

A vector field, $\vec{F}$, in a region V, is said to be SOLENOIDAL if for every closed surface, S (enclosing NO point NOT in V), the surface integral,

$$\oiint_S \vec{F} \cdot \hat{n}\, dS$$

is zero.



$\vec{F}$ solenoidal in V

Then

$$\oiint_{S_2} \vec{F} \cdot \hat{n}\, dS = 0$$

But

$\oiint_{S_1} \vec{F} \cdot \hat{n}\, dS$ may NOT be zero since S_1 encloses points not in V.

The reason for insisting that the surface enclose ONLY points in V is because this region may have "bubbles" in it.

Without this qualification, the theorems introduced below would be invalid.

One of the most important properties of a continuously differentiable solenoidal field, $\vec{F}$, is that it can be REPRESENTED as the curl of a VECTOR POTENTIAL, $\vec{A}$.

$$\vec{F} = \nabla \times \vec{A} .$$

We will prove this and at the same time, give a method for actually constructing such a vector potential.

To simplify matters, let's first observe that IF such a vector potential, $\vec{A}$, exists, then it can be MODIFIED by adding ANY CONSERVATIVE field, $\vec{E}$, since the curl $\vec{E}$ is zero.

That is,

$$\vec{F} = \nabla \times \vec{A} = \nabla \times (\underbrace{\vec{A} + \vec{E}}_{\vec{G}\text{ - just as valid as }\vec{A}}) = \nabla \times \vec{G}$$

which says that the vector potential is UNIQUE ONLY UP TO AN ADDITIVE CONSERVATIVE FIELD.

One intriguing thought is that maybe we can find a conservative $\vec{E}$ which has a z component which exactly cancels the z component of $\vec{A}$.

$$\vec{G} = A_x\hat{x} + A_y\hat{y} + \cancel{A_z\hat{z}} + E_x\hat{x} + E_y\hat{y} - \cancel{A_z\hat{z}}$$

$$= G_x\hat{x} + G_y\hat{y}.$$

If we can, then we need only find a two-component vector potential, $\vec{G}$, which satisfies:

$$\vec{F} = \nabla \times (G_x\hat{x} + G_y\hat{y}) \longrightarrow$$

$$F_x = -\frac{\partial G_y}{\partial z} \qquad (1)$$

$$F_y = \frac{\partial G_x}{\partial z} \qquad (2)$$

$$F_z = \left[\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}\right]. \qquad (3)$$

Let's construct a $\vec{G}$ which satisfies equations (1) and (2) and see if it can also be made to satisfy equation (3).

If we are successful, then we have found a valid vector potential for $\vec{F}$ and proved our assertion.

By integrating equations (1) and (2), we can take G_x and G_y to be

$$G_y = -\int_{z_0}^{z} F_x(x,y,\zeta)\,d\zeta \quad \text{and}$$

$$G_x = \int_{z_0}^{z} F_y(x,y,\zeta)\,d\zeta + w(x,y).$$

Here, the function $w(x,y)$ is some, as yet, unspecified function which in no way affects the validity of equations (1) and (2).

It is introduced, as we will see momentarily, to allow G_x and G_y to satisfy equation (3) by properly choosing w.

Now since,

$$\frac{\partial G_y}{\partial x} = -\int_{z_0}^{z} \frac{\partial F_x}{\partial x}(x,y,\zeta)\,d\zeta \quad \text{and}$$

$$\frac{\partial G_x}{\partial y} = \int_{z_0}^{z} \frac{\partial F_y}{\partial y}(x,y,\zeta)\,d\zeta + \frac{\partial w}{\partial y}(x,y)$$

equation (3) becomes

$$\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y} = -\int_{z_0}^{z}\left[\frac{\partial F_x}{\partial x}(x,y,\zeta) + \frac{\partial F_y}{\partial y}(x,y,\zeta)\right]d\zeta - \frac{\partial w}{\partial y}$$

At this point we will use the fact that $\bar{F}$ is solenoidal and hence has zero divergence:

$$\nabla \cdot \bar{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 0 \rightarrow \frac{\partial F_z}{\partial z} = -\left(\frac{\partial F_x}{\partial x} + \frac{\partial F}{\partial y}\right.$$

$$\longrightarrow$$

$$\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y} = \int_{z_0}^{z} \frac{\partial F_z(x,y,\zeta)}{\partial \zeta} d\zeta - \frac{\partial w}{\partial y}.$$

But we can integrate exactly this last integral:

$$\int_{z_0}^{z} \frac{\partial F_z(x,y,\zeta)}{\partial \zeta} d\zeta = F_z(x,y,z) - F_z(x,y,z_0)$$

Therefore,

$$\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y} = F_z(x,y,z) - F_z(x,y,z_0) - \frac{\partial w}{\partial y}(x,y)$$

Remember that w is still an arbitrary function. If we choose it so that

$$\frac{\partial w(x,y)}{\partial y} = -F_z(x,y,z_0) \rightarrow w(x,y) = -\int_{y_0}^{y} F_z(x,\eta,z_0) d\eta$$

then equation (3) is also satisfied and our proof is completed.

Let's do an example now. Suppose we have a solenoidal magnetic field (all PHYSICAL magnetic fields ARE solenoidal) given by

$$\bar{B} = \left(\frac{-2x^2yz}{m^4}\hat{x} + \frac{xy}{m^2}\hat{y} + \left(\frac{2xyz^2}{m^4} - \frac{xz}{m^2}\right)\hat{z}\right)\left(\frac{Wb}{m^2}\right)$$

Then what is its vector potential?

We need only implement the detailed construction considered above to this specific field:

$$\bar{G} = G_x\hat{x} + G_y\hat{y} \qquad , \text{ let } z_0 = y_0 = 0$$

$$G_y = -\int_{z_0}^{z} F_x(x,y,\zeta)\,d\zeta = \int_0^z 2x^2y\zeta\left(\frac{Wb}{m^6}\right)d\zeta = x^2yz^2\left(\frac{Wb}{m^6}\right)$$

$$\omega(x,y) = -\int_{y_0}^{y} F_z(x,\eta,z)\,d\eta = -\int_0^y \left(2x\eta z_0^2\left(\frac{Wb}{m^6}\right) - xz_0\left(\frac{Wb}{m^4}\right)\right)d\eta = -\int_0^y (0+0)\,d\eta$$

$$G_x = \int_{z_0}^{z} F_y(x,y,\zeta)\,d\zeta + \omega(x,y) = \int_0^z xy\left(\frac{Wb}{m^4}\right)d\zeta + 0 = xyz\left(\frac{Wb}{m^4}\right)$$

$$\therefore \; \bar{G} = \left(\frac{xyz}{m^3}\hat{x} + \frac{x^2yz^2}{m^5}\hat{y}\right)\left(\frac{Wb}{m}\right)$$

Lets check our result by deriving $\bar{B}$ from $\bar{G}$.

$$\bar{B} = \nabla\times\bar{G} = \left(-\frac{\partial}{\partial z}x^2yz^2\left(\frac{Wb}{m^6}\right)\right)\hat{x} + \left(\frac{\partial}{\partial z}xyz\left(\frac{Wb}{m^4}\right)\right)\hat{y}$$

$$+ \left(\frac{\partial}{\partial x}x^2yz^2\left(\frac{Wb}{m^6}\right) - \frac{\partial}{\partial y}xyz\left(\frac{Wb}{m^4}\right)\right)\hat{z}$$

$$= \left(\frac{-2x^2yz}{m^4}\hat{x} + \frac{xy}{m^2}\hat{y} + \left(\frac{2xyz^2}{m^4} - \frac{xz}{m^2}\right)\hat{z}\right)\left(\frac{Wb}{m^2}\right)$$