Section 1.4 – Rational Functions

A function f is a ***rational function*** if $f(x)=\dfrac{g(x)}{h(x)}$,

Where $g(x)$ and $h(x)$ are polynomials. The domain of f consists of all real numbers ***except*** the zeros of the denominator $h(x)$.

Notation	***Terminology***
$x \to a^-$	x approaches a from the left (through values ***less*** than a)
$x \to a^+$	x approaches a from the right (through values ***greater*** than a)
$f(x) \to \infty$	$f(x)$ increases without bound (can be made as large positive as desired)
$f(x) \to -\infty$	$f(x)$ decreases without bound (can be made as large negative as desired)

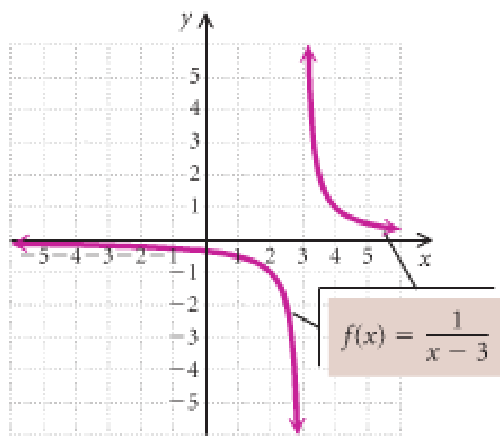
The Domain of a Rational Function

Example

Consider: $f(x)=\dfrac{1}{x-3}$

Find the domain and graph *f*.

Solution



$x-3=0 \quad \Rightarrow \quad \boxed{x=3}$

Thus the domain is: $\{x \mid x \neq 3\}$ *or* $(-\infty,\ 3)\cup(3,\ \infty)$

Function	***Domain***	
$f(x)=\dfrac{1}{x}$	$\{x \mid x \neq 0\}$	$(-\infty,\ 0)\cup(0,\ \infty)$
$f(x)=\dfrac{1}{x^2}$	$\{x \mid x \neq 0\}$	$(-\infty,\ 0)\cup(0,\ \infty)$
$f(x)=\dfrac{x-3}{x^2+x-2}$	$\{x \mid x \neq -2 \text{ and } x \neq 1\}$	$(-\infty,\ -2)\cup(-2,\ 1)\cup(1,\ \infty)$
$f(x)=\dfrac{2x+5}{2x-6}=\dfrac{2x+5}{2(x-3)}$	$\{x \mid x \neq 3\}$	$(-\infty,\ 3)\cup(3,\ \infty)$

Asymptotes

Vertical Asymptote (VA) - *Think Domain*

The line $x = a$ is a ***vertical asymptote*** for the graph of a function f if

$$f(x) \to \infty \quad or \quad f(x) \to -\infty$$

As ***x*** approaches ***a*** from either the left or the right

When the denominator and the numerator have both 0, then both the numerator and denominator can be factored by using $(x-a)$ and can be cancelled out. This means there is a ***hole*** in the function at this point.

Horizontal Asymptote (HA)

The line $y = c$ is a ***horizontal asymptote*** for the graph of a function f if

$$f(x) \to c \quad as \quad x \to -\infty \quad or \quad x \to -\infty$$

Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0} = \dfrac{a_n x^n}{b_m x^m}$ be a rational function.

1. If the degree of numerator is less than of denominator $(n < m) \Rightarrow y = 0$

$$y = \frac{2x+1}{4x^2+5} \quad \Rightarrow \quad \underline{y = 0}$$

2. If the degree of numerator is equal of denominator $(n = m) \Rightarrow y = \dfrac{a_n}{b_m}$

$$y = \frac{2x^2+1}{4x^2+5} \quad \Rightarrow \quad y = \frac{2}{4} = \frac{1}{2}$$

3. If the degree of numerator is greater than of denominator $(n > m) \Rightarrow$ No horizontal asymptote

$$y = \frac{2x^3+1}{4x^2+5} \quad \Rightarrow No \ \ HA$$

If ***no*** Horizontal Asymptote, then there is an ***Oblique Asymptote.***

Slant or *Oblique* Asymptotes

When the degree of the numerator is one greater than the degree of the numerator, the graph has a slant or oblique asymptote and it is a line $y = ax + b, \quad a \neq 0$. To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

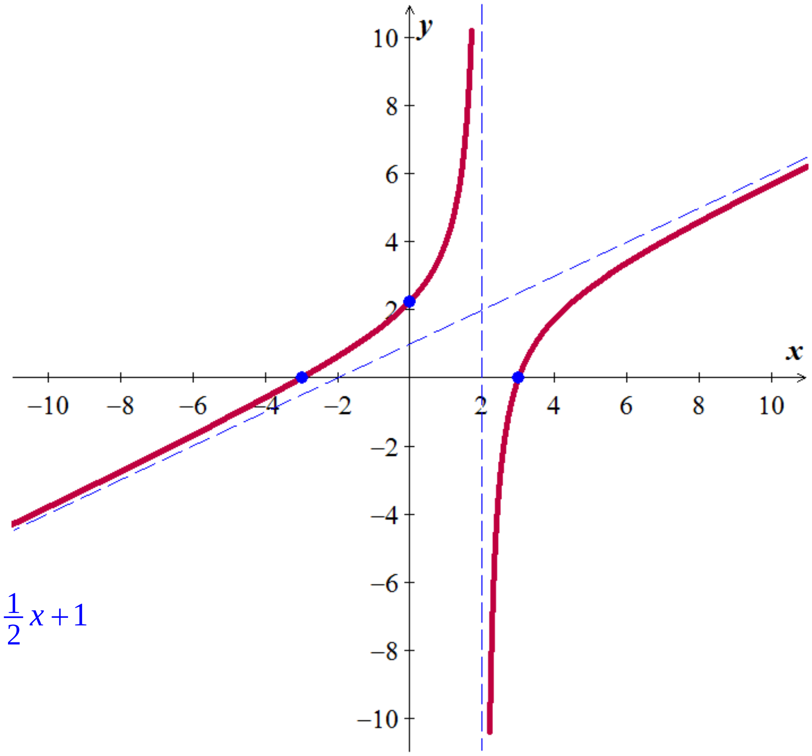
Example

Find all the asymptotes and sketch the graph of f if $f(x)=\dfrac{x^2-9}{2x-4}$

Solution

$$\begin{array}{r}
\frac{1}{2}x+1 \\
2x-4\overline{\big)\,x^2 \qquad -9} \\
\underline{x^2-2x} \qquad \\
2x-9 \\
\underline{2x-4} \\
\boxed{-5}
\end{array}$$

$$f(x)=\left(\frac{1}{2}x+1\right)-\frac{5}{2x-4}$$



VA: $x=2$ ***HA***: n/a

Hole: n/a ***Oblique asymptote***: $y=\frac{1}{2}x+1$

x	***y***
0	$\frac{9}{4}$
± 3	0

Example

Find the vertical asymptote of $f(x)=\dfrac{1}{x-2}$, and sketch the graph.

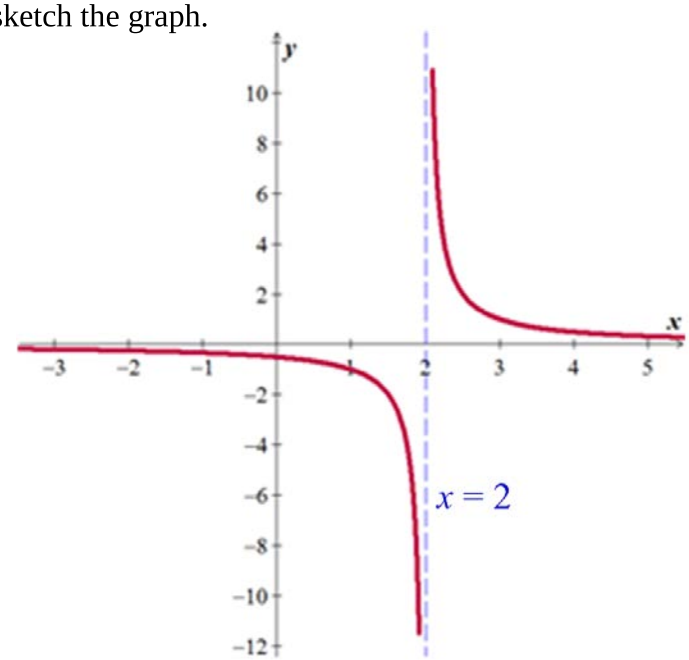
Solution

VA: $x=2$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

$f(x)\to\infty \quad as \quad x\to 2^+$

$f(x)\to -\infty \quad as \quad x\to 2^-$



Example

Sketch the graph of g if $g(x)=\dfrac{3x^2+x-4}{2x^2-7x+5}$

Solution

$$g(x)=\frac{(3x+4)(x-1)}{(2x-5)(x-1)}$$

$$=\frac{3x+4}{2x-5}$$

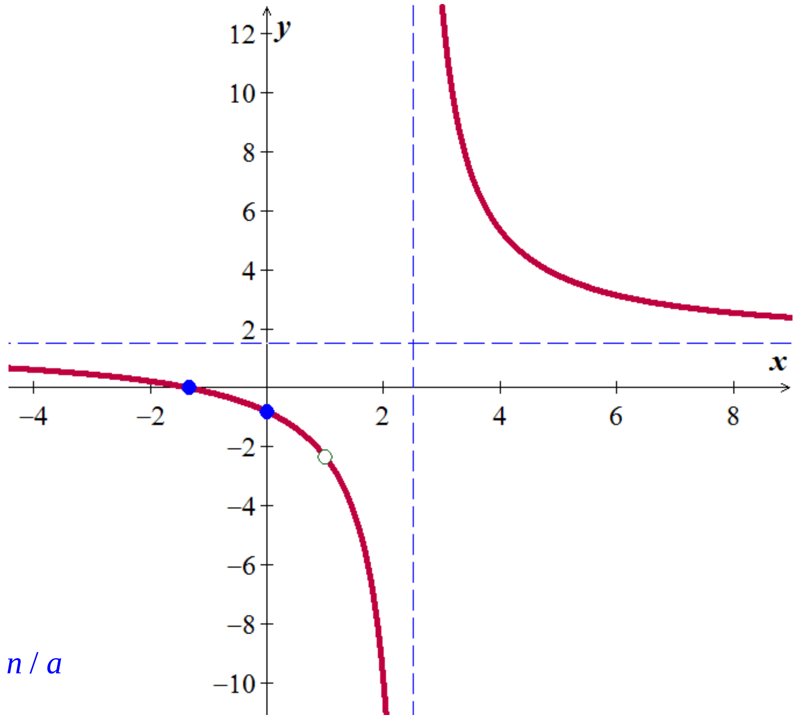
$$f(x)=\frac{3x+4}{2x-5}$$

g has a hole at $x=1$

$$f(1)=-\frac{7}{3}$$

VA: $x=\frac{5}{2}$ ***HA***: $y=0$

Hole: $\left(1,\ -\frac{7}{3}\right)$ ***Oblique asymptote***: n/a



Example

Find all asymptotes for the graph of f, if it exists

a) $f(x)=\dfrac{3x-1}{x^2-x-6}$ *b)* $f(x)=\dfrac{5x^2+1}{3x^2-4}$ *c)* $f(x)=\dfrac{2x^4-3x^2+5}{x^2+1}$

Solution

a) $f(x)=\dfrac{3x-1}{x^2-x-6}$

VA: $x=-2,\quad x=3$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

b) $f(x)=\dfrac{5x^2+1}{3x^2-4}$

$$3x^2-4=0\ \rightarrow\ 3x^2=4\ \rightarrow\ x^2=\frac{4}{3}\ \rightarrow\ \boxed{x=\pm\frac{2}{\sqrt{3}}}$$

VA: $x=\pm\frac{2}{\sqrt{3}}$ ***HA***: $y=\frac{5}{3}$

Hole: n/a ***Oblique asymptote***: n/a

c) $f(x)=\dfrac{2x^4-3x^2+5}{x^2+1}$

VA: n/a ***HA***: n/a

Hole: n/a ***Oblique asymptote***: $y=2x^2-5$

$$\begin{array}{r|l} & 2x^2-5 \\ \hline x^2+1 & 2x^4-3x^2+5 \\ & \underline{-2x^4-2x^2} \\ & \quad -5x^2+5 \end{array}$$

Example

Sketch the graph of f if $f(x)=\dfrac{3x+4}{2x-5}$

Solution

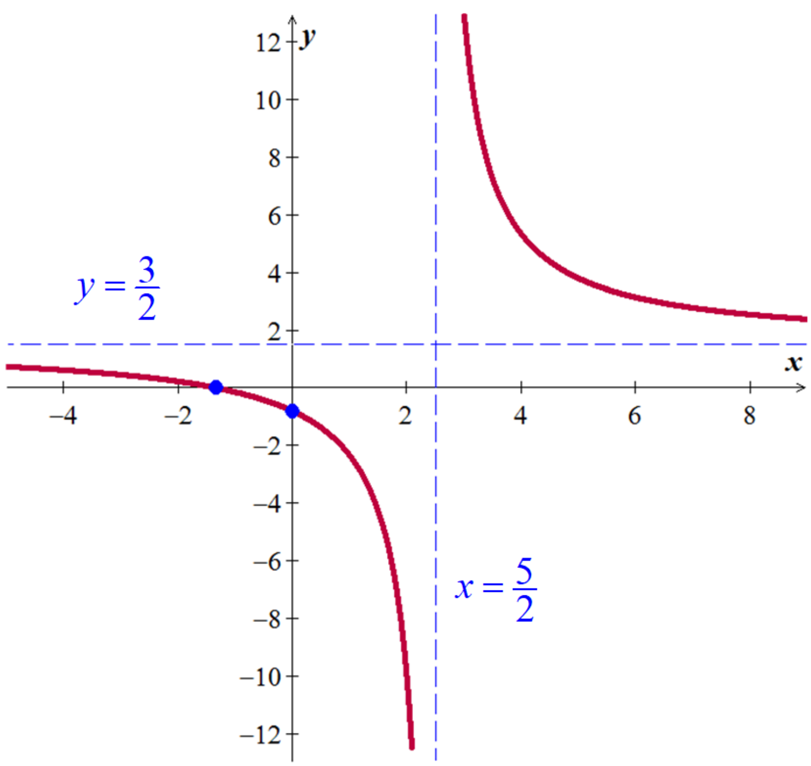
VA: $x=\frac{5}{2}$

HA: $y=-\frac{5}{3}$

Hole: n/a

Oblique asymptote: n/a

x	y
0	$-\frac{4}{5}$
$-\frac{4}{3}$	0
4	5.3



Example

Sketch the graph of f if $f(x)=\dfrac{x^2}{x^2-x-2}$

Solution

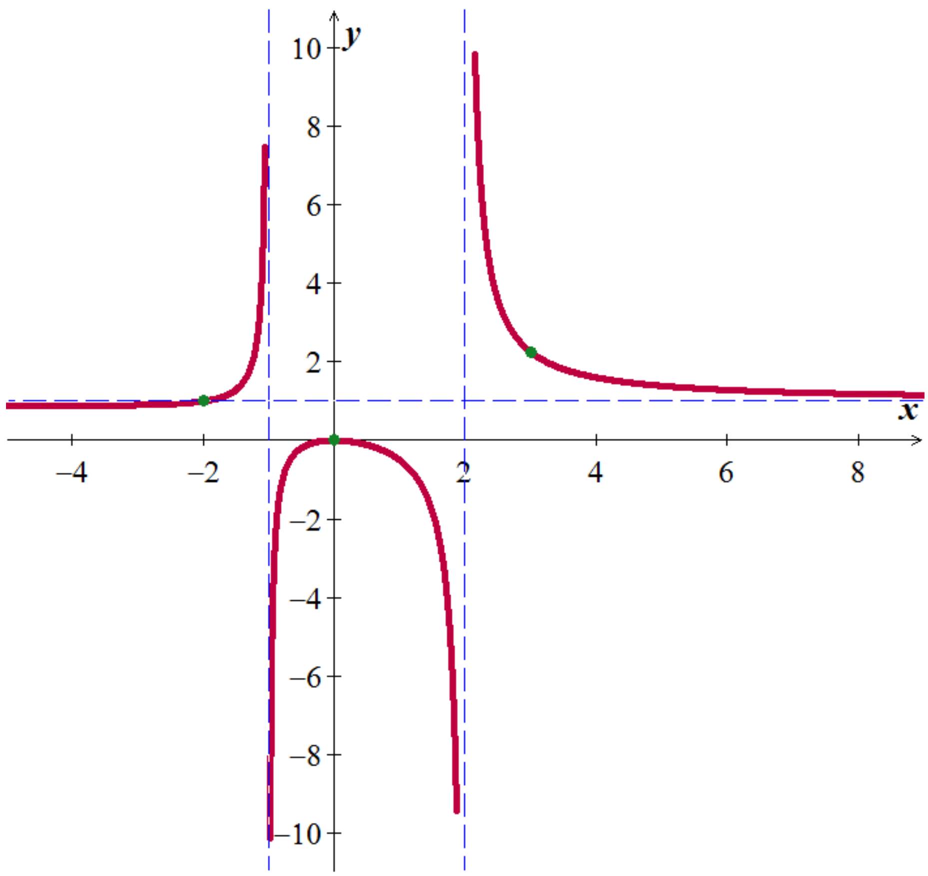
VA: $x=-1,\ 2$

HA: $y=1$

Hole: n/a

Oblique asymptote: n/a

x	**y**
0	0
–4	0.88
–2	1
3	$\frac{9}{4}$



Example

Sketch the graph of f if $f(x)=\dfrac{x-1}{x^2-x-6}$

Solution

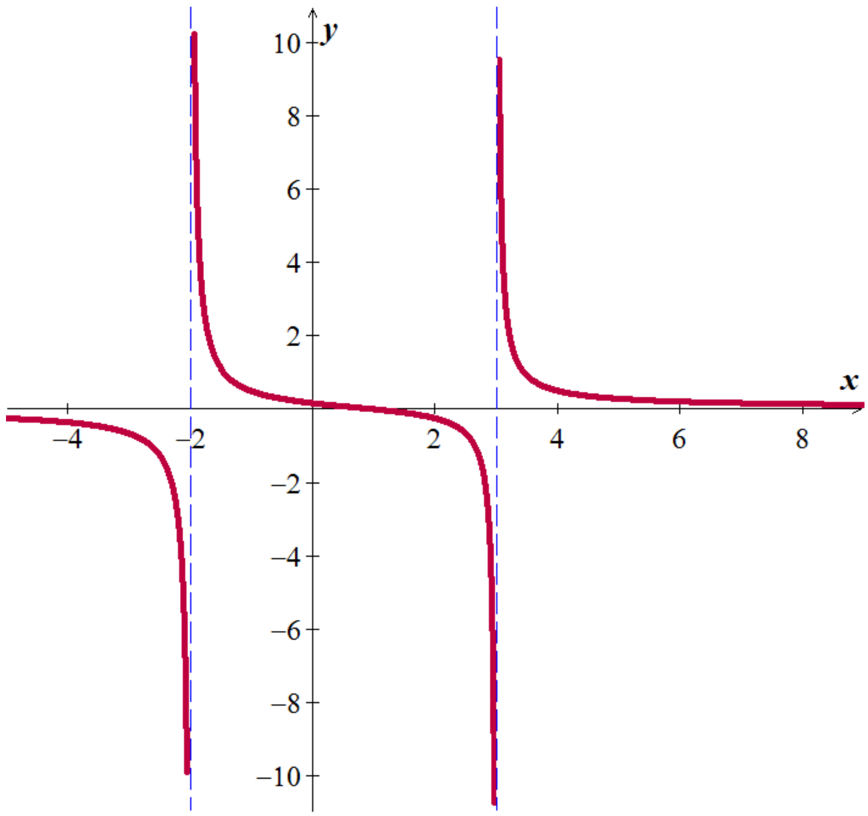
VA: $x=-2,\ 3$

HA: $y=0$

Hole: n/a

Oblique asymptote: n/a

x	y
−4	−.36
−3	−.67
0	$\frac{1}{6}$
1	0
4	.5
5	$\frac{2}{7}$



Exercises *Section* 1.4 – Rational Functions

(**1 – 21**) Determine all asymptotes of the function

1. $y = \dfrac{3x}{1-x}$
2. $y = \dfrac{x^2}{x^2+9}$
3. $y = \dfrac{x-2}{x^2-4x+3}$
4. $y = \dfrac{3}{x-5}$
5. $y = \dfrac{x^3-1}{x^2+1}$
6. $y = \dfrac{3x^2-27}{(x+3)(2x+1)}$
7. $y = \dfrac{x^3+3x^2-2}{x^2-4}$
8. $y = \dfrac{x-3}{x^2-9}$
9. $y = \dfrac{6}{\sqrt{x^2-4x}}$
10. $y = \dfrac{5x-1}{1-3x}$
11. $f(x) = \dfrac{2x-11}{x^2+2x-8}$
12. $f(x) = \dfrac{x^2-4x}{x^3-x}$
13. $f(x) = \dfrac{x-2}{x^3-5x}$
14. $f(x) = \dfrac{4x}{x^2+10x}$
15. $f(x) = \dfrac{3-x}{(x-4)(x+6)}$
16. $f(x) = \dfrac{x^3}{2x^3-x^2-3x}$
17. $f(x) = \dfrac{3x^2+5}{4x^2-3}$
18. $f(x) = \dfrac{x+6}{x^3+2x^2}$
19. $f(x) = \dfrac{x^2+4x-1}{x+3}$
20. $f(x) = \dfrac{x^2-6x}{x-5}$
21. $f(x) = \dfrac{x^3-x^2+x-4}{x^2+2x-1}$

(**22 – 53**) Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

22. $f(x) = \dfrac{-3x}{x+2}$
23. $f(x) = \dfrac{x+1}{x^2+2x-3}$
24. $f(x) = \dfrac{2x^2-2x-4}{x^2+x-12}$
25. $f(x) = \dfrac{-2x^2+10x-12}{x^2+x}$
26. $f(x) = \dfrac{x^2-x-6}{x+1}$
27. $f(x) = \dfrac{x^3+1}{x-2}$
28. $f(x) = \dfrac{2x^2+x-6}{x^2+3x+2}$
29. $f(x) = \dfrac{x-1}{1-x^2}$
30. $f(x) = \dfrac{x^2+x-2}{x+2}$
31. $f(x) = \dfrac{x^3-2x^2-4x+8}{x-2}$
32. $f(x) = \dfrac{2x^2-3x-1}{x-2}$
33. $f(x) = \dfrac{2x+3}{3x^2+7x-6}$
34. $f(x) = \dfrac{x^2-1}{x^2+x-6}$
35. $f(x) = \dfrac{-2x^2-x+15}{x^2-x-12}$
36. $f(x) = \dfrac{1}{x-3}$
37. $f(x) = \dfrac{-2}{x+3}$
38. $f(x) = \dfrac{x}{x+2}$
39. $f(x) = \dfrac{x-5}{x+4}$
40. $f(x) = \dfrac{2x^2-2}{x^2-9}$
41. $f(x) = \dfrac{x^2-3}{x^2+4}$
42. $f(x) = \dfrac{x^2+4}{x^2-3}$

43. $f(x) = \dfrac{x^2}{x^2 - 6x + 9}$

44. $f(x) = \dfrac{x^2 + x + 4}{x^2 + 2x - 1}$

45. $f(x) = \dfrac{2x^2 + 14}{x^2 - 6x + 5}$

46. $f(x) = \dfrac{x^2 - 4x - 5}{2x + 5}$

47. $f(x) = \dfrac{x - 3}{x^2 - 3x + 2}$

48. $f(x) = \dfrac{x^2 + 2}{x^2 + 3x + 2}$

49. $f(x) = \dfrac{x - 2}{x^2 - 3x + 2}$

50. $f(x) = \dfrac{x^2 + x}{x + 1}$

51. $f(x) = \dfrac{x^2 - 2x}{x - 2}$

52. $f(x) = \dfrac{x^2 - 3x}{x + 3}$

53. $f(x) = \dfrac{x^3 + 3x^2 - 4x + 6}{x + 2}$

(**54 – 59**) Find an equation of a rational function f that satisfies the given conditions

54. $\begin{cases} \textit{vertical asymptote}: \ x = 4 \\ \textit{horizontal asymptote}: \ y = -1 \\ x - \textit{intercept}: \ 3 \end{cases}$

55. $\begin{cases} \textit{vertical asymptote}: \ x = -4, x = 5 \\ \textit{horizontal asymptote}: \ y = \dfrac{3}{2} \\ x - \textit{intercept}: \ -2 \end{cases}$

56. $\begin{cases} \textit{vertical asymptote}: \ x = 5 \\ \textit{horizontal asymptote}: \ y = -1 \\ x - \textit{intercept}: \ 2 \end{cases}$

57. $\begin{cases} \textit{vertical asymptote}: \ x = -2, \ x = 0 \\ \textit{horizontal asymptote}: \ y = 0 \\ x - \textit{intercept}: \ 2, \quad f(3) = 1 \end{cases}$

58. $\begin{cases} \textit{vertical asymptote}: \ x = -3, \ \ x = 1 \\ \textit{horizontal asymptote}: \ y = 0 \\ x - \textit{intercept}: \ -1, \ f(0) = -2 \\ \textit{hole at} \ \ x = 2 \end{cases}$

59. $\begin{cases} \textit{vertical asymptote}: \ x = -1, \ \ x = 3 \\ \textit{horizontal asymptote}: \ y = 2 \\ x - \textit{intercept}: \ -2, \ \ 1 \\ \textit{hole}: \ \ x = 0 \end{cases}$