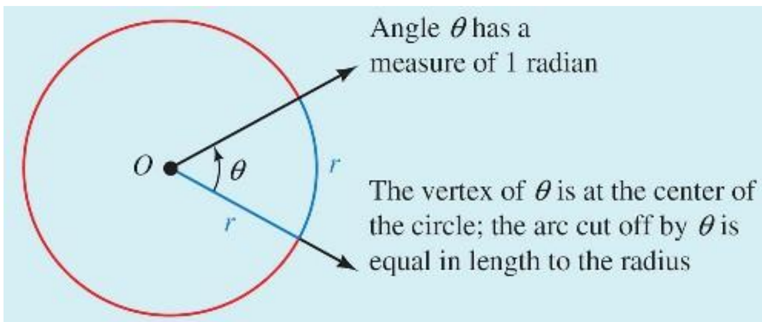
Lecture Two - Circular & Graph Functions

Section 2.1 - Radians & Degrees, Circular Functions

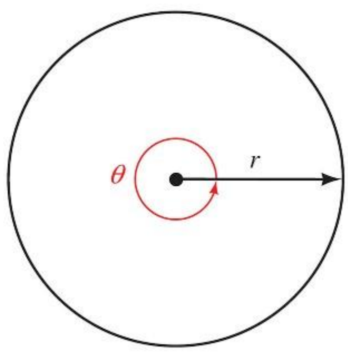
Radians

Definition

In a circle, a central angle that cuts off an arc equal in length to the radius of the circle has a measure of 1 radian (***rad***).



Degrees - Radians



θ measures one full rotation $\quad \theta = 2\pi \quad$ *The measure of θ in radians is 2π*

$$\boxed{1 = 1\ rad}$$

$$1^\circ = 1 \text{ degree}$$

If no unit of angle measure is specified, then the angle is to be measured in radians.

$$\text{Full Rotation}: \ 360^\circ = 2\pi\ rad$$

$$180^\circ = \pi\ rad$$

Converting from Degrees to Radians

$$\frac{180^\circ}{180} = \frac{\pi}{180}\ rad$$

$$\Rightarrow 1^\circ = \frac{\pi}{180}\ rad$$

Multiply a degree measure by $\frac{\pi}{180}\ rad$ and simplify to convert to radians.

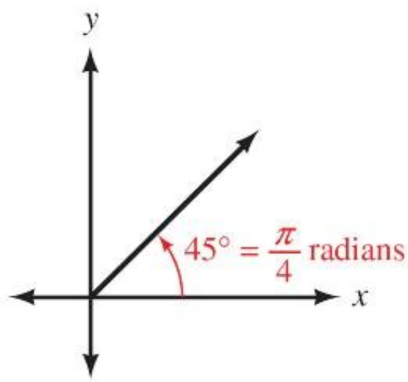
Example

Convert 45° to radians

<u>*Solution*</u>

$$45^\circ = 45\left(\frac{\pi}{180}\right) rad$$

$$= \frac{\pi}{4}\ rad$$



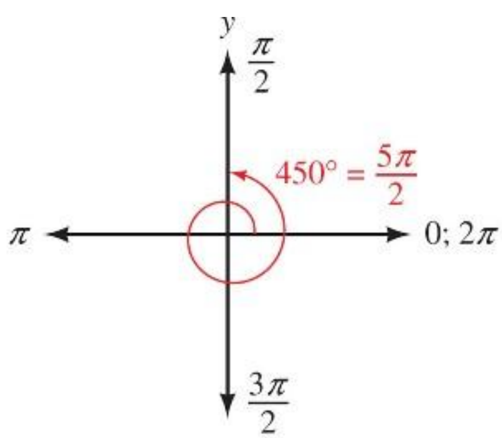
Example

Convert -450° to radians

<u>*Solution*</u>

$$-450^\circ = -450\left(\frac{\pi}{180}\right) rad$$

$$= -\frac{5\pi}{2}\ rad$$



Example

Convert 249.8° to radians

<u>*Solution*</u>

$$249.8^\circ = 249.8\left(\frac{\pi}{180}\right) rad$$

$$\approx 4.360\ rad$$

Converting from Radians to Degrees

Multiply a radian measure by $\frac{180^\circ}{\pi}$ radian and simplify to convert to degrees.

$$\frac{180^\circ}{\pi} = \frac{\pi}{\pi}\ rad$$

$$\left(\frac{180}{\pi}\right)^\circ = 1\ rad$$

Example

Convert 1 to degrees

Solution

$$\begin{aligned} 1\ rad &= 1\left(\frac{180}{\pi}\right)^\circ \\ &= 1\left(\frac{180}{3.14}\right)^\circ \\ &= 57.3^\circ \end{aligned}$$

Example

Convert $\frac{4\pi}{3}$ to degrees

Solution

$$\begin{aligned} \frac{4\pi}{3} &= \frac{4\pi}{3}\left(\frac{180}{\pi}\right)^\circ \\ &= 240^\circ \end{aligned}$$

Example

Convert-4.5 to degrees

Solution

$$\begin{aligned} -4.5 &= -4.5\left(\frac{180}{\pi}\right)^\circ \\ &\approx -257.8^\circ \end{aligned}$$

Equivalent Angle Measures in Degrees and Radians

Example

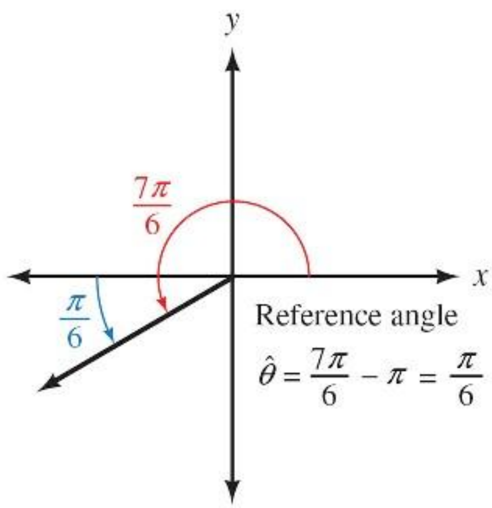
Find $\sin\frac{\pi}{6}$

Solution

$$\begin{aligned}\sin\frac{\pi}{6} &= \sin 30^\circ \\ &= \frac{1}{2}\end{aligned}$$

Example

Find $4\sin\frac{7\pi}{6}$



Solution

$$\frac{7\pi}{6} = \pi + \frac{\pi}{6}$$

$$\begin{aligned}4\sin\frac{7\pi}{6} &= 4\left(-\sin\frac{\pi}{6}\right) \\ &= 4\left(-\frac{1}{2}\right) \\ &= -2\end{aligned}$$

Example

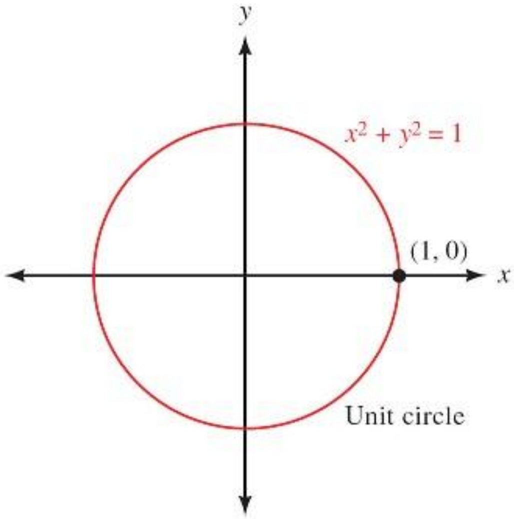
Evaluate $4\sin(2x+\pi)$ when $x = \frac{\pi}{6}$

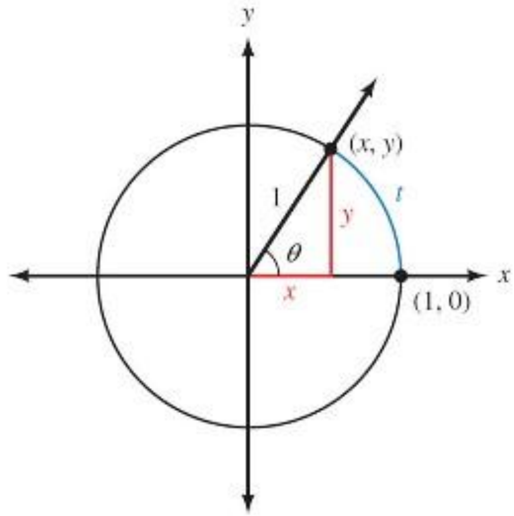
Solution

$$\begin{aligned}4\sin(2x+\pi) &= 4\sin(2\frac{\pi}{6}+\pi) \\ &= 4\sin\left(\frac{\pi}{3}+\pi\right) \\ &= -4\sin\left(\frac{\pi}{3}\right) \\ &= -4\left(\frac{\sqrt{3}}{2}\right) \\ &= -2\sqrt{3}\end{aligned}$$

Circular Functions

A unit circle has its center at the origin and a radius of 1 unit.

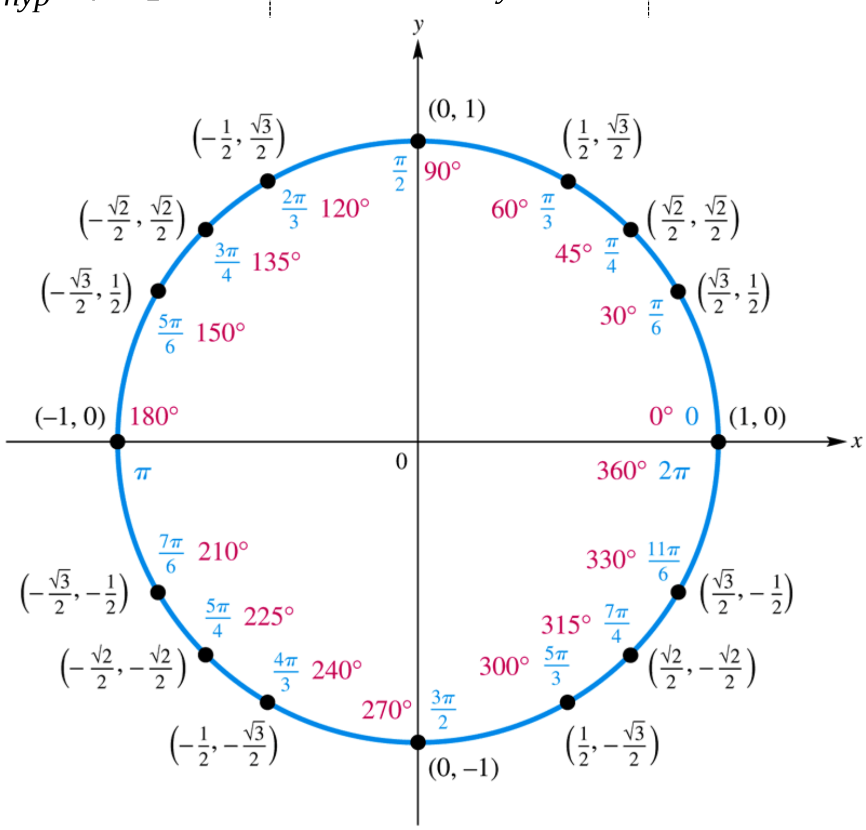




The equation of the unit circle (r = 1) is: $\boldsymbol{x^2 + y^2 = 1}$

When interpreted this way, they are called ***circular functions***.

$\sin\theta = \dfrac{opp}{hyp} = \dfrac{y}{r} = \dfrac{y}{1} = y$	$\tan\theta = \dfrac{y}{x}$	$\csc\theta = \dfrac{1}{y}$
$\cos\theta = \dfrac{adj}{hyp} = \dfrac{x}{r} = \dfrac{x}{1} = x$	$\cot\theta = \dfrac{x}{y}$	$\sec\theta = \dfrac{1}{x}$



The unit circle $x^2 + y^2 = 1$

The Unit Circle

The unit circle is symmetric with respect to the *x*-axis, *y*-axis, and the origin

Example

Find the six trigonometry functions of $\frac{5\pi}{6}$

<u>*Solution*</u>

$$\sin\frac{5\pi}{6} = y = \frac{1}{2}$$

$$\cos\frac{5\pi}{6} = x = -\frac{\sqrt{3}}{2}$$

$$\tan\frac{5\pi}{6} = \frac{\text{y}}{\text{x}} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$$

$$\cot\frac{5\pi}{6} = -\frac{1}{1/\sqrt{3}} = -\sqrt{3}$$

$$\sec\frac{5\pi}{6} = \frac{1}{\cos\frac{5\pi}{6}} = \frac{1}{-\sqrt{3}/2} = -\frac{2}{\sqrt{3}}$$

$$\csc\frac{5\pi}{6} = \frac{1}{\sin\frac{5\pi}{6}} = \frac{1}{-1/2} = -2$$

Example

Use the unit circle to find all values of *t* between 0 and 2π for which $\cos t = \frac{1}{2}$

<u>*Solution*</u>

The angles for $\cos t = \frac{1}{2}$ are $t = \frac{\pi}{3}$ *or* $60°$ and $t = \frac{5\pi}{3}$ *or* $300°$

Example

Find $\tan t$ if *t* corresponds to the point (-0.737, 0.675) on the unit circle.

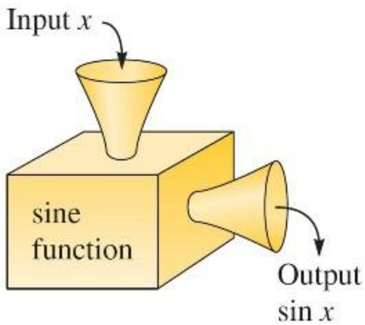
<u>*Solution*</u>

$$\tan t = \frac{y}{x}$$

$$= \frac{0.675}{-0.737}$$

$$\approx -0.916$$

Definition of the *function* is a rule that pairs each element of the domain with exactly one element from the range.

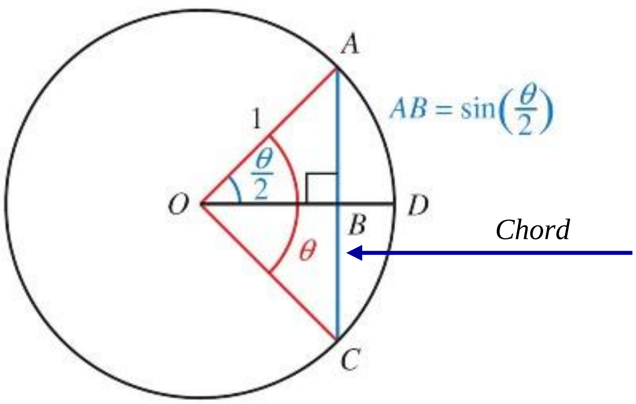


$$y = \sin x \Rightarrow y = f(x) = \sin(x)$$

Argument of the function = Angle

Value of the function = y

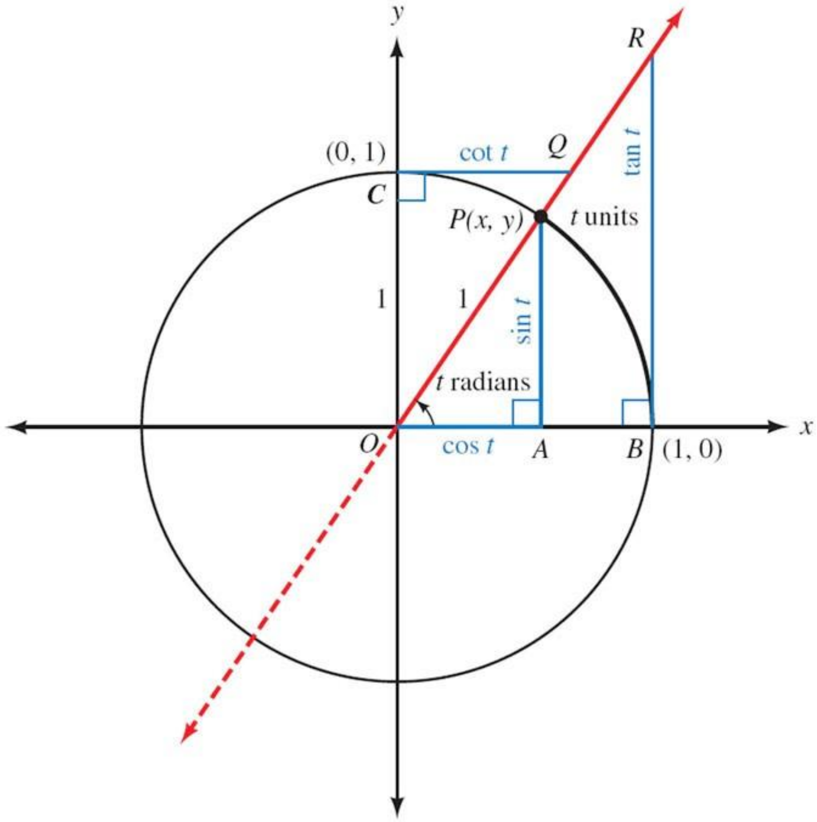
Geometric Representations



$$chord(\theta) = AC = 2AB = 2\sin\left(\frac{\theta}{2}\right)$$

Example

Describe how $\sec t$ varies as t increases from 0 to $\frac{\pi}{2}$



<u>*Solution*</u>

When $t=0,\ OR=1=OB$

$$\Rightarrow \sec t=\frac{1}{\frac{OB}{OR}}=\frac{OR}{OB}=1$$

$\rightarrow \sec t$ Will begin at a value of 1 as t increases

$\rightarrow \sec t$ Grows larger and larger

When $t=\frac{\pi}{2} \Rightarrow OP$ will be vertical

$\Rightarrow \sec t = OR$ will no longer be defined

Exercises *Section* 2.1 - Radians & Degrees, Circular Functions

1. Use a calculator to convert $256° \ 20'$ to radians to the nearest hundredth of a radian.

2. Convert $-78.4°$ to radians

3. Convert $\frac{11\pi}{6}$ to degrees

4. Convert $-\frac{5\pi}{3}$ to degrees

5. Convert $\frac{\pi}{6}$ to degrees

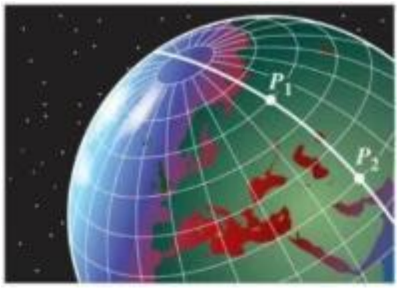
6. Use the calculator to convert 2.4 to degree measure to the nearest tenth of a degree.

7. In navigation, distance is not usually measured along a straight line, but along a great circle because the Earth is round. The formula to determine the great circle distance between two points $P_1\left(LT_1, LN_1\right)$ and $P_2\left(LT_2, LN_2\right)$ whose coordinates are given as latitudes and longitudes involves the expression

$$\sin\left(LT_1\right)\sin\left(LT_2\right)+\cos\left(LT_1\right)\cos\left(LT_2\right)\cos\left(LN_1-LN_2\right)$$

To use this formula, the latitudes and longitudes must be entered as angles in radians. However, most GPS units give these coordinates in degrees and minutes. To use this formula thus requires converting from degrees to radians.

Evaluate this expression for $P_1(N \ 32° \ 22.108', W \ 64° \ 41.178')$ and $P_2(N \ 13° \ 0.4809', W \ 59° \ 29.263')$ corresponding to Bermuda and Barbados, respectively.



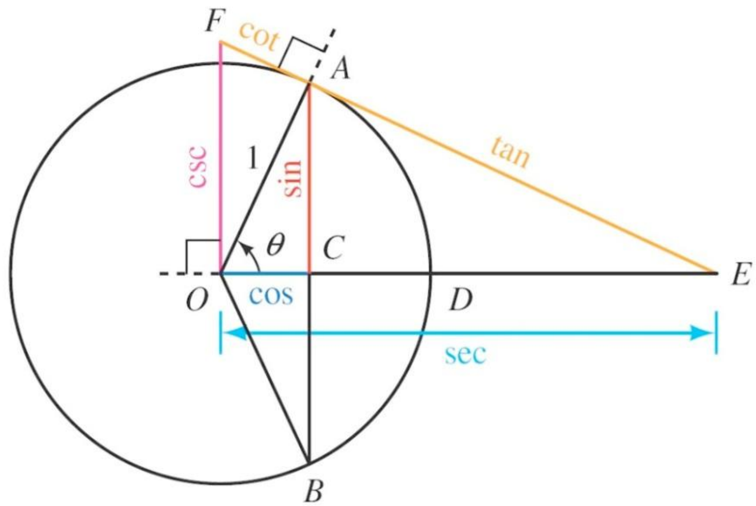
8. If the angle θ is in standard position and the terminal side of θ intersects the unit circle at the point $\left(-\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\right)$

9. Find the exact values of $\sin\frac{3\pi}{2}$, $\cos\frac{3\pi}{2}$, *and* $\tan\frac{3\pi}{2}$

10. Use reference angles and degree/radian conversion to find exact value of $\cos\frac{2\pi}{3}$

11. Evaluate $\sin\frac{13\pi}{6}$. Identify the function, the argument of the function, and the function value.

12. Show why $OF = \csc\theta$



13. Evaluate $\sin\dfrac{9\pi}{4}$. Identify the function, the argument of the function, and the value of the function.

14. The function is the sine function, $\dfrac{9\pi}{4}$ is the argument, and $\dfrac{1}{\sqrt{2}}$ is the value of the function

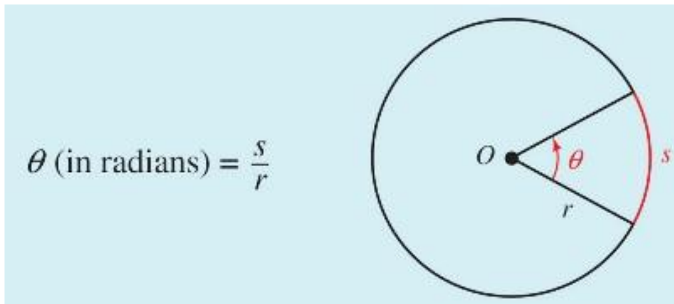
15. Evaluate: $\cot 2.37$

Section 2.2 – Arc Length and Sector Area

Arc Length

Definition

If a central angle θ, in a circle of a radius r, cuts off an arc of length s, then the measure of θ, in radians is:



$\theta r = \frac{s}{r} r$

$$s = r\theta \qquad (\theta \textit{ in radians})$$

> ***Note***:
>
> When applying the formula $s = r\theta$, the value of ***must*** be in radian.

Example

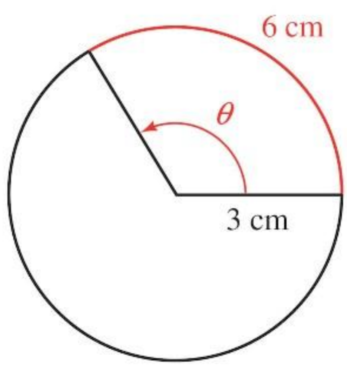
A central angle θ in a circle of radius 3 *cm* cuts off an arc of length 6 *cm*. What is the radian measure of θ.

<u>*Solution*</u>

$\theta = \frac{s}{r}$

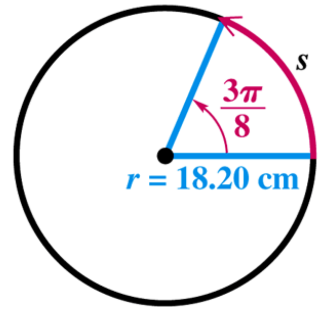
$= \frac{6\ cm}{3\ cm}$

$= 2\ rad$



Example

A circle has radius 18.20 *cm*. Find the length of the arc intercepted by a central angle with measure $\frac{3\pi}{8}$ radians.



<u>*Solution*</u>

Given: $\theta = \frac{3\pi}{8}\ rad, \quad r = 18.20\ cm$

$$s = r\theta$$
$$= 18.20\left(\frac{3\pi}{8}\right)cm$$
$$\approx 21.44\ cm$$

Example

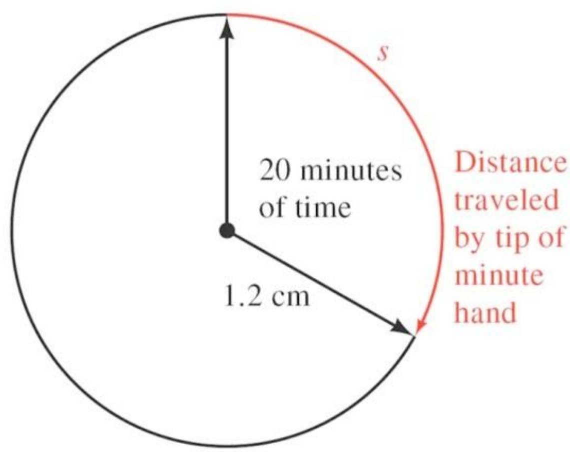
The minute hand of a clock is 1.2 cm long. To two significant digits, how far does the tip of the minute hand move in 20 minutes?

<u>*Solution*</u>

Given: $r = 1.2$ cm

One complete rotation = 1 *hour* = 60 *minutes* = 2π

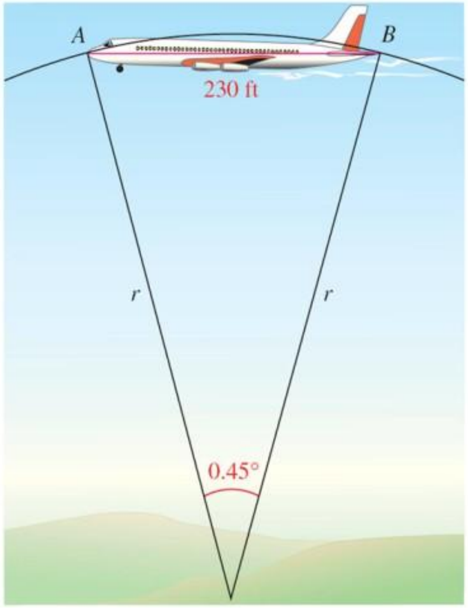
$$\Rightarrow \frac{\theta}{2\pi} = \frac{20}{60}$$
$$\Rightarrow \theta = \frac{20}{60}2\pi$$
$$\Rightarrow \theta = \frac{2\pi}{3}$$
$$s = r\theta$$
$$= 1.2\frac{2\pi}{3}$$
$$\approx 2.5\,cm$$



Example

A person standing on the earth notices that a 747 jet flying overhead subtends an angle 0.45°. If the length of the jet is 230 *ft.*, find its altitude to the nearest thousand feet.

<u>*Solution*</u>



$$s = r\theta$$

$$\Rightarrow r = \frac{s}{\theta}$$

$$= \frac{230}{0.45\left(\frac{\pi}{180}\right)}$$

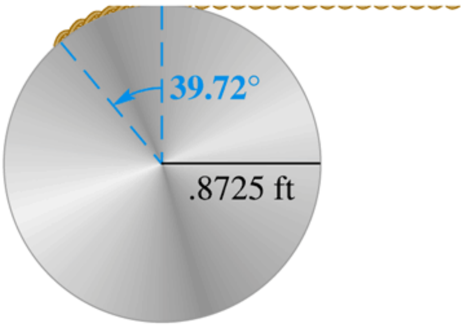
$$= \frac{230(180)}{0.45\pi}$$

$$= 29{,}000\ ft$$

Example

A rope is being wound around a drum with radius 0.8725 ft. How much rope will be wound around the drum if the drum is rotated through an angle of 39.72°?

<u>*Solution*</u>



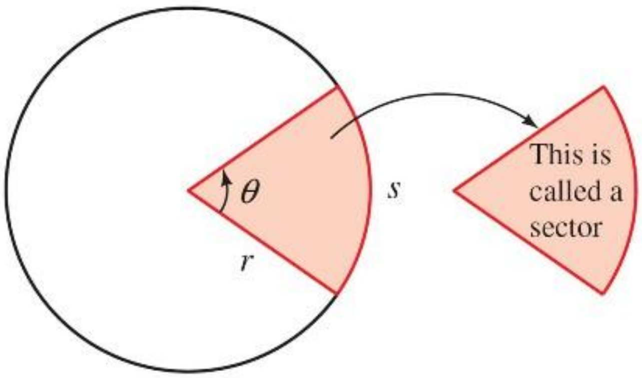
$$s = r\theta$$

$$= 0.8725\left(39.72\frac{\pi}{180}\right)$$

$$\approx 0.6049\ feet$$

Area of a Sector

A sector of a circle is a portion of the interior of a circle intercepted by a central angle.



$$\begin{array}{lcccl} \textit{Area of sector} & \rightarrow & \dfrac{A}{\pi r^2} = \dfrac{\theta}{2\pi} & \leftarrow & \textit{Central angle } \theta \\ \textit{Area of circle} & \rightarrow & & \leftarrow & \textit{One full rotation} \end{array}$$

$$\frac{A}{\pi r^2}\pi r^2 = \frac{\theta}{2\pi}\pi r^2$$

$$A = \frac{1}{2}r^2\theta$$

Definition

If θ (in radians) is a central angle in a circle with radius r, then the area of the sector formed by an angle θ is given by

$$A = \frac{1}{2}r^2\theta \qquad (\boldsymbol{\theta} \textbf{ in radians})$$

Example

Find the area of the sector formed by a central angle of 1.4 radians in a circle of radius 2.1 meters

<u>*Solution*</u>

Given: $r = 2.1$ m
$\theta = 1.4$

$$A = \frac{1}{2}r^2\theta$$

$$= \frac{1}{2}(2.1)^2(1.4)$$

$$= 3.1\, m^2$$

Example

If the sector formed by a central angle of 15° has an area of $\frac{\pi}{3}\text{cm}^2$, find the radius of a circle.

<u>Solution</u>

Given: $\theta = 15° \frac{\pi}{180} = \frac{\pi}{12}$

$A = \frac{\pi}{3}$

$$A = \frac{1}{2}r^2\theta$$

$$\frac{\pi}{3} = \frac{1}{2}r^2 \frac{\pi}{12}$$

$$\frac{24}{\pi}\frac{\pi}{3} = \frac{1}{2}r^2 \frac{\pi}{12}\frac{24}{\pi}$$

$$8 = r^2$$

$$r = \sqrt{8}$$

$$r = 2\sqrt{2}\ cm$$

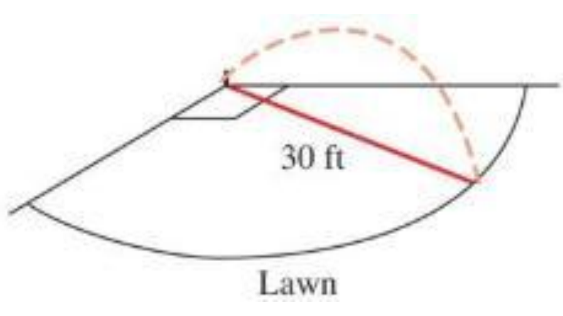
Example

A lawn sprinkler located at the corner of a yard is set to rotate 90° and project water out 30.0 ft. To three significant digits, what area of lawn is watered by the sprinkler?

<u>Solution</u>

Given: $\theta = 90° = \frac{\pi}{2}$

$r = 30\,ft$



$$A = \frac{1}{2}r^2\theta$$

$$= \frac{1}{2}(30)^2\frac{\pi}{2}$$

$$= 707\ ft^2$$

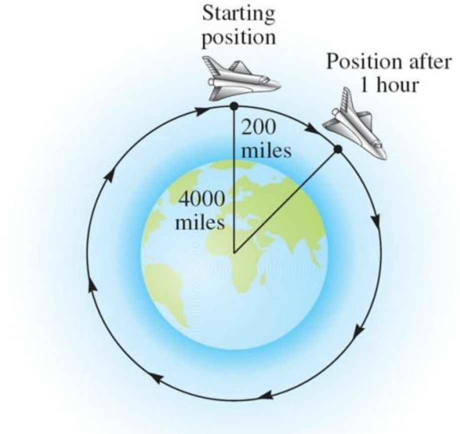
Exercises *Section* 2.2 – Arc Length and Sector Area

1. The minute hand of a clock is 1.2 cm long. How far does the tip of the minute hand travel in 40 minutes?

2. Find the radian measure if angle θ, if θ is a central angle in a circle of radius $r = 4$ inches, and θ cuts off an arc of length $s = 12\pi$ inches.

3. Give the length of the arc cut off by a central angle of 2 radians in a circle of radius 4.3 inches

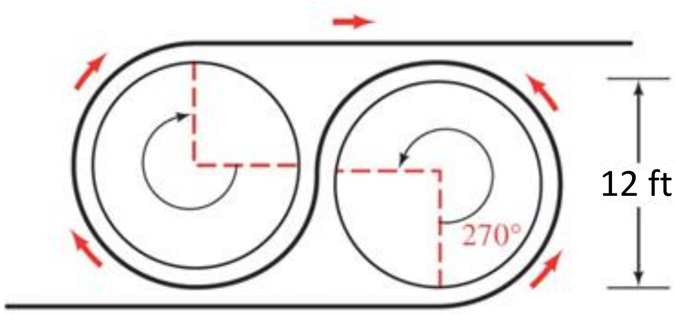
4. A space shuttle 200 miles above the earth is orbiting the earth once every 6 hours. How long, in hours, does it take the space shuttle to travel 8,400 miles? (Assume the radius of the earth is 4,000 miles.) Give both the exact value and an approximate value for your answer.



5. The pendulum on a grandfather clock swings from side to side once every second. If the length of the pendulum is 4 feet and the angle through which it swings is 20°. Find the total distance traveled in 1 minute by the tip of the pendulum on the grandfather clock.

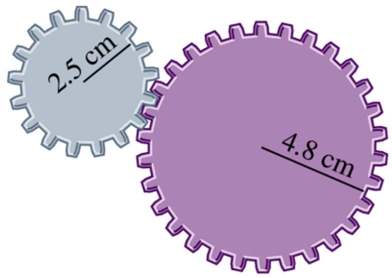
6. Reno, Nevada is due north of Los Angeles. The latitude of Reno is 40°, while that of Los Angeles is 34° N. The radius of Earth is about 4000 mi. Find the north-south distance between the two cities.

7. The first cable railway to make use of the figure-eight drive system was a Sutter Street Railway. Each drive sheave was 12 feet in diameter. Find the length of cable riding on one of the drive sheaves.

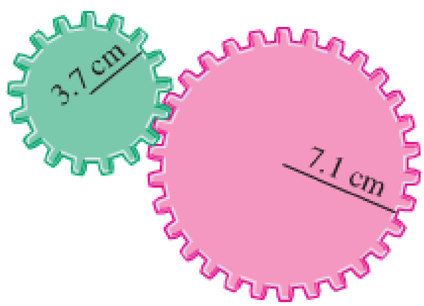


8. The diameter of a model of George Ferris's Ferris wheel is 250 feet, and θ is the central angle formed as a rider travels from his or her initial position P_0 to position P_1. Find the distance traveled by the rider if $\theta = 45°$ and if $\theta = 105°$.

9. Two gears are adjusted so that the smaller gear drives the larger one. If the smaller gear rotates through an angle of 225°, through how many degrees will the larger gear rotate?

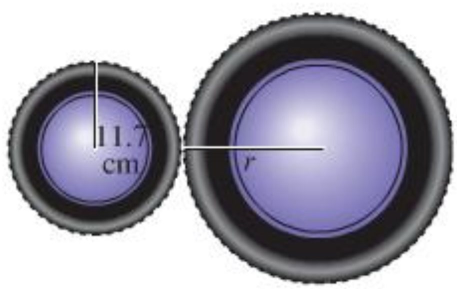


10. Two gears are adjusted so that the smaller gear drives the larger one. If the smaller gear rotates through an angle of 300°, through how many degrees will the larger rotate?



11. The rotation of the smaller wheel causes the larger wheel to rotate. Through how many degrees will the larger wheel rotate if the smaller one rotates through 60.0°?

12. Find the radius of the larger wheel if the smaller wheel rotates 80° when the larger wheel rotates 50°.

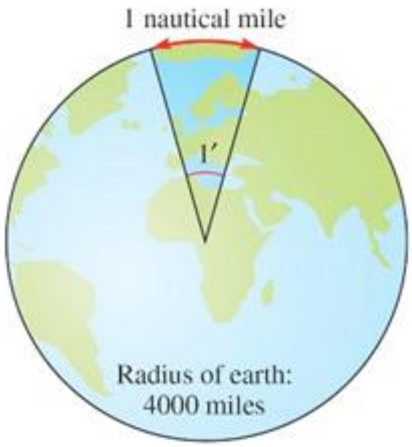


13. Los Angeles and New York City are approximately 2,500 miles apart on the surface of the earth. Assuming that the radius of the earth is 4,000 miles, find the radian measure of the central angle with its vertex at the center of the earth that has Los Angeles on one side and New York City in the other side.

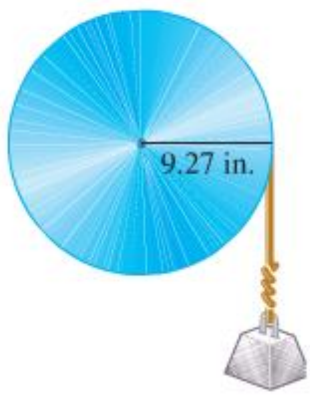
14. Find the number of regular (statute) miles in 1 nautical mile to the nearest hundredth of a mile. (Use 4,000 miles for the radius of the earth).

15. If two ships are 20 nautical miles apart on the ocean, how many statute miles apart are they?

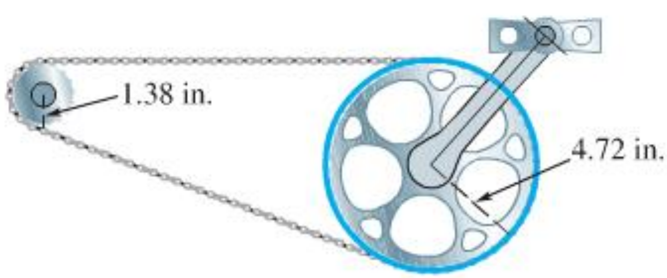
16. If a central angle with its vertex at the center of the earth has a measure of 1′, then the arc on the surface of the earth that is cut off by this angle (knows as the great circle distance) has a measure of 1 nautical mile.



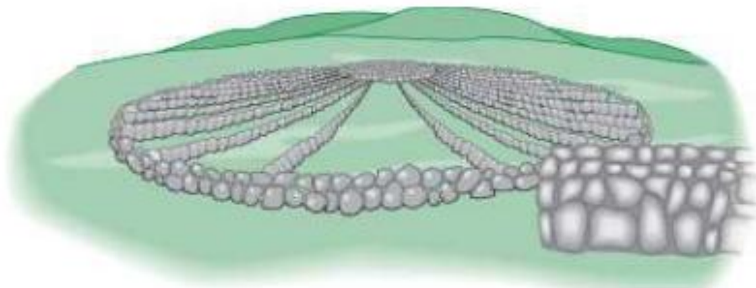
17. How many inches will the weight rise if the pulley is rotated through an angle of 71° 50′? Through what angle, to the nearest minute, must the pulley be rotated to raise the weight 6 in?



18. The figure shows the chain drive of a bicycle. How far will the bicycle move if the pedals are rotated through 180°? Assume the radius of the bicycle wheel is 13.6 in.



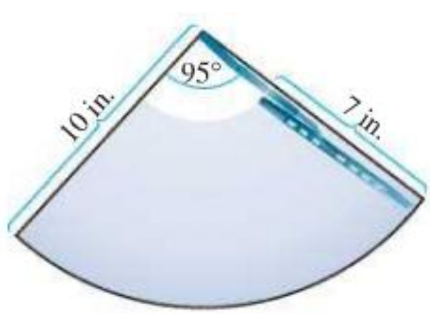
19. The circular of a Medicine Wheel is 2500 yrs old. There are 27 aboriginal spokes in the wheel, all equally spaced.



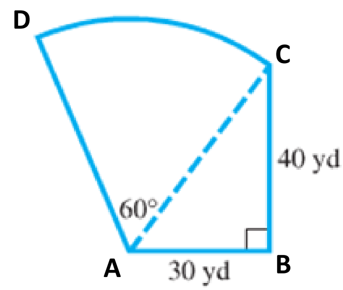
a) Find the measure of each central angle in degrees and in radians.
b) The radius measure of each of the wheel is 76.0 ft, find the circumference.
c) Find the length of each arc intercepted by consecutive pairs of spokes.
d) Find the area of each sector formed by consecutive spokes,

20. Find the radius of the pulley if a rotation of 51.6° raises the weight 11.4 cm.

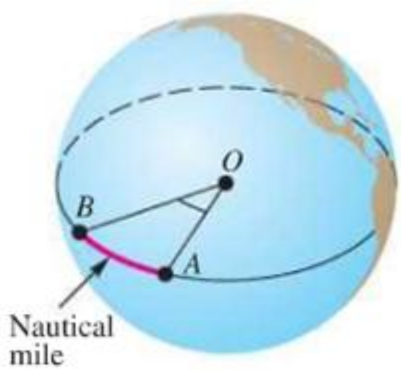
21. The total arm and blade of a single windshield wiper was 10 in. long and rotated back and forth through an angle of 95°. The shaded region in the figure is the portion of the windshield cleaned by the 7-in. wiper blade. What is the area of the region cleaned?



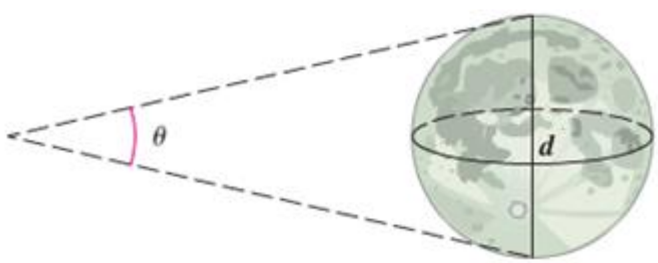
22. A frequent problem in surveying city lots and rural lands adjacent to curves of highways and railways is that of finding the area when one or more of the boundary lines is the arc of the circle. Find the area of the lot.



23. Nautical miles are used by ships and airplanes. They are different from statue miles, which equal 5280 ft. A nautical mile is defined to be the arc length along the equator intercepted by a central angle AOB of 1 min. If the equatorial radius is 3963 mi, use the arc length formula to approximate the number of statute miles in 1 nautical mile.



24. The distance to the moon is approximately 238,900 mi. Use the arc length formula to estimate the diameter d of the moon if angle θ is measured to be 0.5170°.



Section 2.3 – Linear and Angular Velocities

The most intuitive measure of the rate at which the rider is traveling around the wheel is what we call ***linear velocity***.

Another way to specify how fast the rider is traveling around the wheel is with what we call ***angular velocity***.

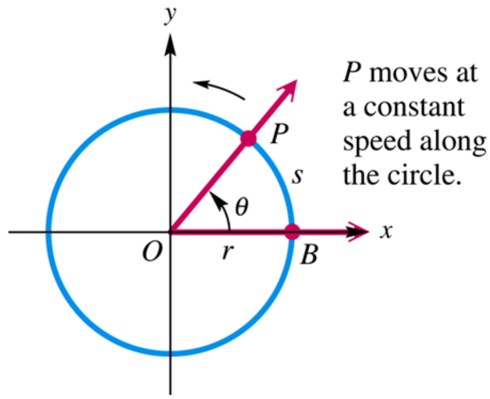
Linear Speed

Definition

If P is a point on a circle of radius r, and P moves a distance s on the circumference of the circle in an amount of time t, then the ***linear velocity***, v, of P is given by the formula

$$speed = \frac{distance}{time}$$

$$v = \frac{s}{t}$$



Example

A point on a circle travels 5 cm in 2 sec. Find the linear velocity of the point.

<u>*Solution*</u>

Given: $s = 5cm$

$t = 2\ sec$

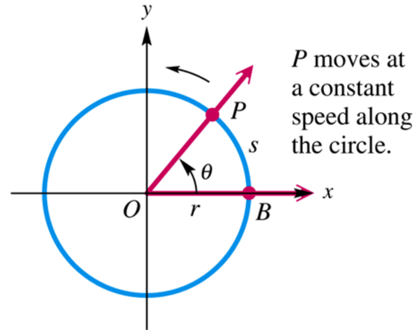
$$v = \frac{s}{t} = \frac{5\ cm}{2\ sec}$$
$$= 2.5\ cm/sec$$

Angular Speed

Definition

If P is a point moving with uniform circular motion on a circle of radius r, and the line from the center of the circle through P sweeps out a central angle θ in an amount of time t, then the *angular velocity*, ω (omega), of P is given by the formula

$$\omega = \frac{\theta}{t} \qquad \textit{where } \theta \textit{ is measured in radians}$$



Example

A point on a circle rotates through $\frac{3\pi}{4}$ radians in 3 sec. Give the angular velocity of the point.

<u>*Solution*</u>

Given: $\theta = \frac{3\pi}{4}\ rad$

$t = 3sec$

$$\omega = \frac{\frac{3\pi}{4}\ rad}{3\ sec}$$

$$= \frac{\pi}{4}\ rad / sec$$

Example

A bicycle wheel with a radius of 13.0 *in*. turns with an angular velocity of 3 radians per seconds. Find the distance traveled by a point on the bicycle tire in 1 minute.

<u>*Solution*</u>

Given: $r = 13.0\ in.$

$\omega = 3\text{rad}/sec$

$t = 1\ min = 60\ sec.$

$$\omega = \frac{\theta}{t}$$

$$\omega t = \theta \qquad s = r\theta \ \Rightarrow \theta = \frac{s}{r}$$

$$\omega t = \frac{s}{r}$$

$$s = \omega tr$$

$$= 3 \times 60 \times 13$$

$$= 2{,}340\ inches$$

$$or\ \frac{2{,}340}{12} = 195\,ft$$

Relationship between the Two Velocities

If $s = r\theta$

$$\frac{s}{t} = \frac{r\theta}{t}$$

$$\frac{s}{t} = r\frac{\theta}{t}$$

$$v = r\omega$$

$$v = r\frac{\theta}{t}$$

Linear and Angular Velocity

If a point is moving with uniform circular motion on a circle of radius r, then the linear velocity v and angular velocity ω of the point are related by the formula

$$v = r\omega$$

Example

A phonograph record is turning at 45 revolutions per minute (*rpm*). If the distance from the center of the record to a point on the edge of the record is 3 inches, find the angular velocity and the linear velocity of the point in feet per minute.

<u>*Solution*</u>

$$\begin{aligned} \omega &= 45\ rpm \\ &= 45\ \frac{rev}{min} \qquad\qquad 1\ revolution = 2\pi rad \\ &= 45\ \frac{rev}{min}\ \frac{2\pi\ rad}{1\ rev} \\ &= 90\pi\ rad\ /\ min \end{aligned}$$

$$\begin{aligned} v &= r\omega \\ &= (3\ in.)\left(90\pi \frac{rad}{min}\right) \\ &= 270\pi \frac{in}{min} \\ &= 848\ in\ /\ min \end{aligned}$$

$$v = 848\ \frac{in}{min}\frac{1\,ft}{12in}$$

$$v = 70.7\ ft\ /\ \text{min}$$

Example

Suppose that P is on a circle with radius 10 cm, and ray OP is rotating with angular speed $\frac{\pi}{18}\ rad/\sec$.

a) Find the angle generated by P in 6 seconds

b) Find the distance traveled by P along the circle in 6 seconds.

c) Find the linear speed of P in cm per sec.

<u>*Solution*</u>

a) $\theta = \omega t$

$$\boxed{\theta = \frac{\pi}{18}.6 = \frac{\pi}{3}\ rad}$$

b) $s = r\theta$

$$\boxed{s = 10\left(\frac{\pi}{3}\right) = \frac{10\pi}{3}\ cm}$$

c) $v = \frac{s}{t}$

$$v = \frac{\frac{10\pi}{3}}{6}$$

$$= \frac{10\pi}{18}$$

$$= \frac{5\pi}{9}\ cm/\sec$$

Example

A belt runs a pulley of radius 6 cm at $80\ rev/\min$.

a) Find the angular speed of the pulley in radians per sec.

b) Find the linear speed of the belt in cm per sec.

<u>*Solution*</u>

a) $\omega = 80\frac{rev}{\min}\cdot\frac{1\min}{60\sec}\cdot\frac{2\pi}{1rev}$

$$= \frac{8\pi}{3}\ rad/\sec$$

b) $v = r\omega$

$$= 6\left(\frac{8\pi}{3}\right)$$

$$\approx 50\ cm/\sec$$

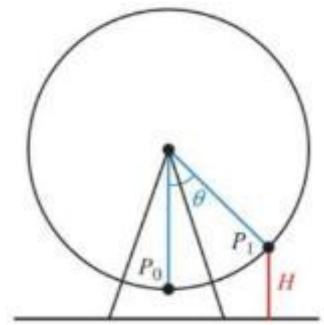
Example

The diameter of the Ferris wheel is 250 *ft*, the distance from the ground to the bottom of the wheel is 14 *ft*, and one complete revolution takes 20 minutes, find

a. The linear velocity, in miles per hour, of a person riding on the wheel.

b. The height of the rider in terms of the time *t*, where *t* is measured in minutes.

<u>Solution</u>



Given: $\theta = 1\ rev = 2\pi\ rad$

$t = 20\ min.$

$r = \frac{D}{2} = \frac{250}{2} = 125\,ft$

a. $\omega = \frac{\theta}{t}$ *or* $\boxed{v = \frac{r\theta}{t}}$

$= \frac{2\pi}{20}$

$= \frac{\pi}{10}\ rad\,/\,min$

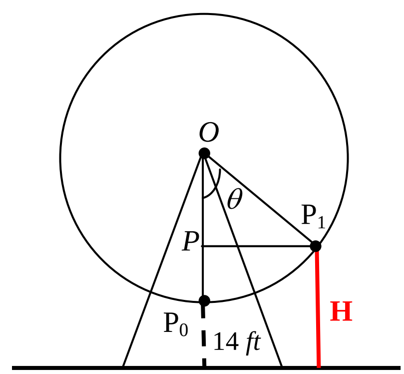
$v = r\omega$

$= (125\,ft)\left(\frac{\pi}{10} rad\,/\,\min\right)$

$\approx 39.27\ ft\,/\min$

$v \approx 39.27\ \frac{ft}{\min}\ \frac{60\min}{1hr}\ \frac{1mile}{5{,}280\,ft}$

$\approx 0.45\ mi\,/\,hr$



b. $\cos\theta = \frac{OP}{OP_1}$

$= \frac{OP}{125}$

$OP = 125\cos\theta$

$H = PP_0 + 14$

$= OP_0 - OP + 14$

$= 125 - 125\cos\theta + 14$

$= 139 - 125\cos\theta$

$\omega = \frac{\theta}{t}$

$\theta = \omega\, t$

$\theta = \frac{\pi}{10}\ t$

$H = 139 - 125\cos\left(\frac{\pi}{10}t\right)$

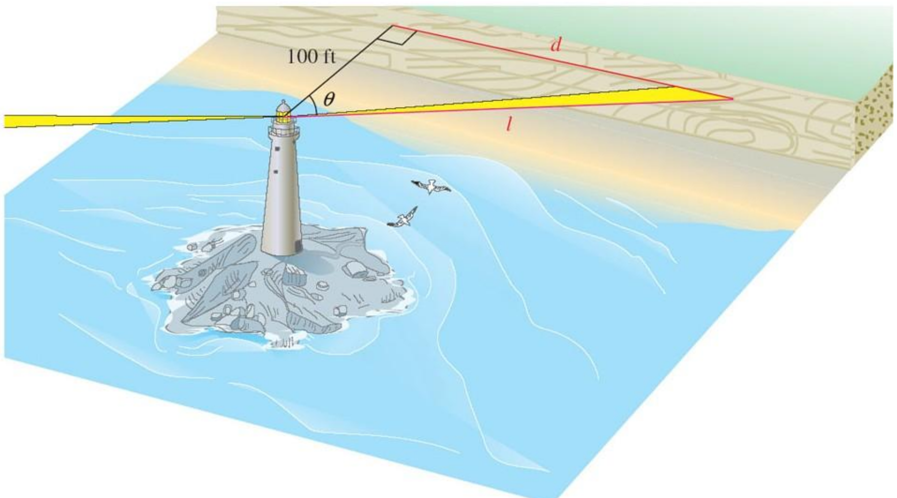
Exercises *Section* 2.3 – Linear and Angular Velocities

1. Find the linear velocity of a point moving with uniform circular motion, if $s = 12$ cm and $t = 2$ sec.

2. Find the distance s covered by a point moving with linear velocity $v = 55$ mi/hr and $t = 0.5$ hr.

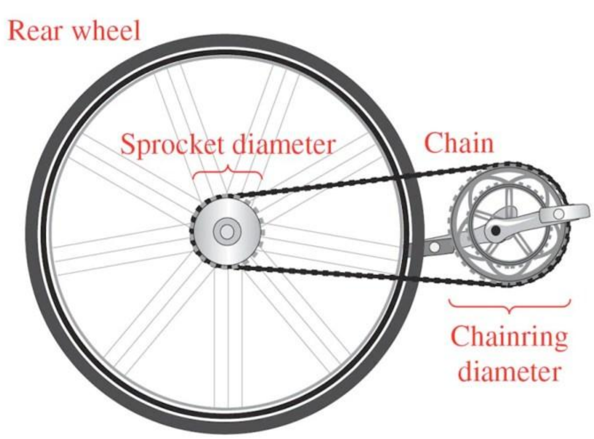
3. Point $\boldsymbol{P}$ sweeps out central angle $\theta = 12\pi$ as it rotates on a circle of radius r with $t = 5\pi$ sec. Find the angular velocity of point $\boldsymbol{P}$.

4. Find an equation that expresses $\boldsymbol{l}$ in terms of time t. Find $\boldsymbol{l}$ when t is 0.5 sec, 1.0 sec, and 1.5 sec. (assume the light goes through one rotation every 4 seconds.)



5. Find the angular velocity, in radians per minute, associated with given 7.2 *rpm*.

6. When Lance Armstrong blazed up Mount Ventoux in the 2002 tour, he was equipped with a 150-millimeter-diameter chainring and a 95-millimeter-diameter sprocket. Lance is known for maintaining a very high cadence, or pedal rate. The sprocket and rear wheel rotate at the same rate, and the diameter of the rear wheel is 700 mm. If he was pedaling at a rate of 90 revolutions per minute, find his speed in kilometers per hour. (1 km = 1,000,000 mm or 10^6 mm)

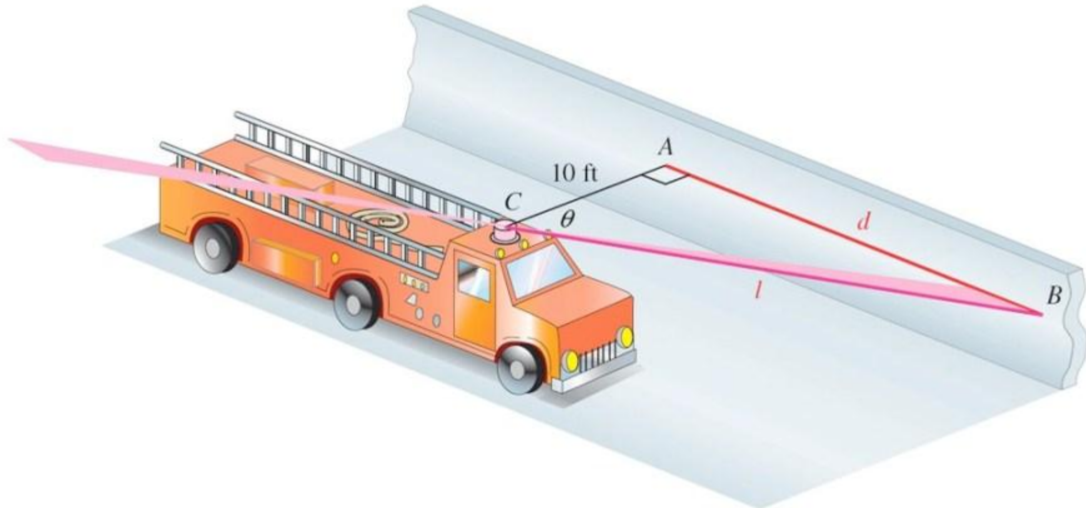


7. A Ferris wheel has a radius 50.0 ft. A person takes a seat and then the wheel turns $\frac{2\pi}{3}$ *rad* .

a) How far is the person above the ground?

b) If it takes 30 sec for the wheel to turn $\frac{2\pi}{3}$ *rad* , what is the angular speed of the wheel?

8. A fire truck parked on the shoulder of a freeway next to a long block wall. The red light on the top of the truck is 10 feet from the wall and rotates through a complete revolution every 2 seconds. Find the equations that give the lengths d and ℓ in terms of time.



9. Suppose that point P is on a circle with radius 60 cm, and ray OP is rotating with angular speed $\frac{\pi}{12}$ radian per sec.

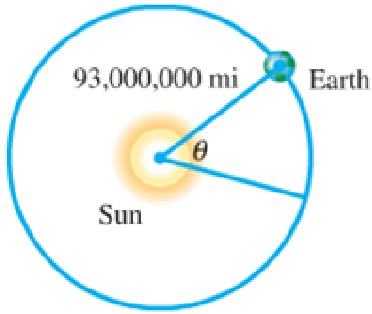
a) Find the angle generated by P in 8 sec.

b) Find the distance traveled by P along the circle in 8 sec.

c) Find the linear speed of P in 8 sec.

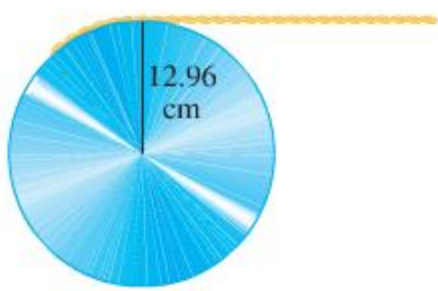
10. Tires of a bicycle have radius 13 in. and are turning at the rate of 215 revolutions per min. How fast is the bicycle traveling in miles per hour? (*Hint*: 1 mi = 5280 ft.)

11. Earth travels about the sun in an orbit that is almost circular. Assume that the orbit is a circle with radius 93,000,000 mi. Its angular and linear speeds are used in designing solar-power facilities.
   a) Assume that a year is 365 days, and find the angle formed by Earth's movement in one day.
   b) Give the angular speed in radians per hour.
   c) Find the linear speed of Earth in miles per hour.

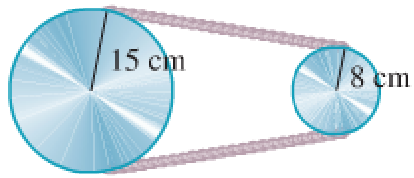


12. Earth revolves on its axis once every 24 hr. Assuming that earth's radius is 6400 km, find the following.
   a) Angular speed of Earth in radians per day and radians per hour.
   b) Linear speed at the North Pole or South Pole
   c) Linear speed ar a city on the equator

13. The pulley has a radius of 12.96 cm. Suppose it takes 18 sec for 56 cm of belt to go around the pulley.
   a) Find the linear speed of the belt in cm per sec.
   b) Find the angular speed of the pulley in rad per sec.



14. The two pulleys have radii of 15 cm and 8 cm, respectively. The larger pulley rotates 25 times in 36 sec. Find the angular speed of each pulley in rad per sec.



15. A thread is being pulled off a spool at the rate of 59.4 cm per sec. Find the radius of the spool if it makes 152 revolutions per min.

16. A railroad track is laid along the arc of a circle of radius 1800 ft. The circular part of the track subtends a central angle of 40°. How long (in seconds) will it take a point on the front of a train traveling 30 mph to go around this portion of the track?

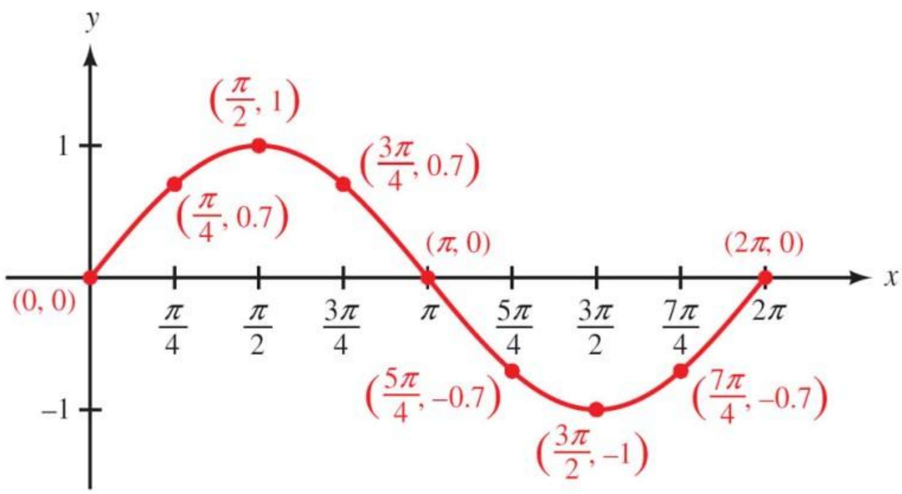
17. A 90-horsepower outboard motor at full throttle will rotate it propeller at exactly 5000 revolutions per min. Find the angular speed of the propeller in radians per second.

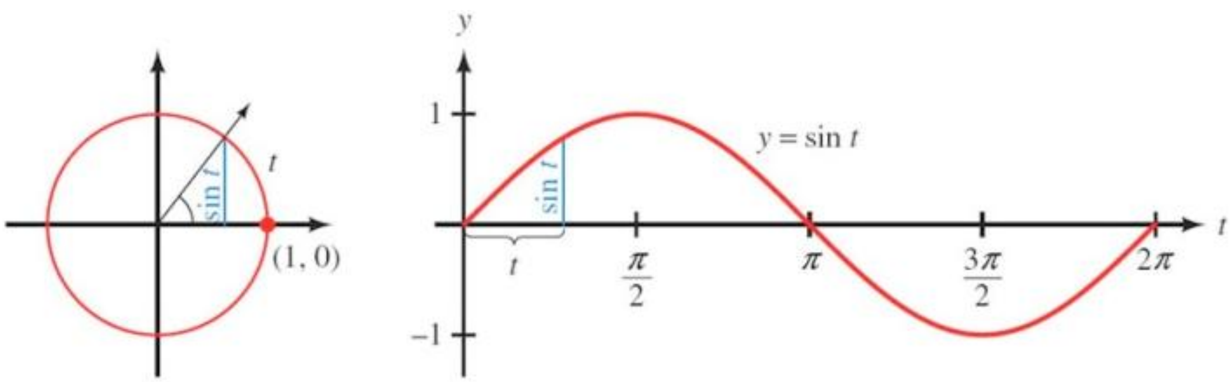
18. The shoulder joint can rotate at 25 rad/min. If a golfer's arm is straight and the distance from the shoulder to the club head is 5.00 ft, find the linear speed of the club head from the shoulder rotation

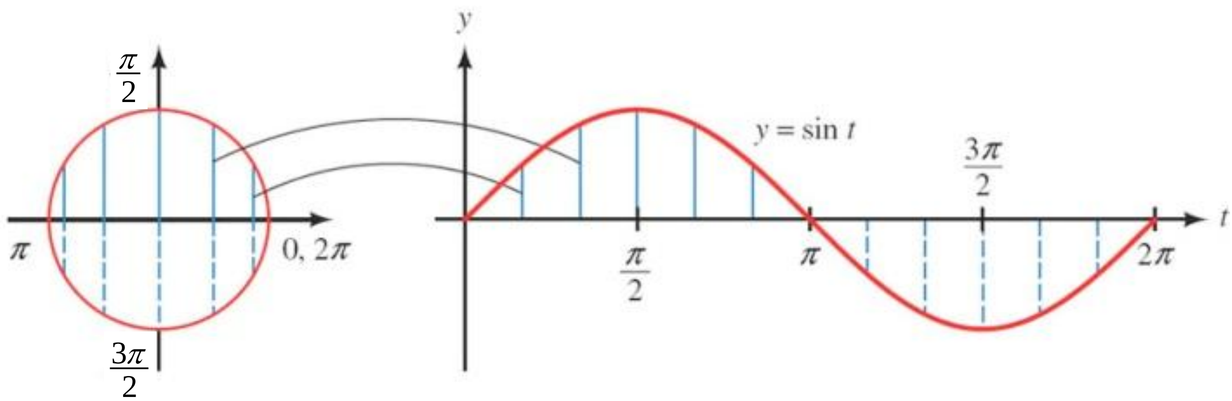
Section 2.4 – Translation of Trigonometric Functions

The *Sine* Graph

Graphing the function: $y = \sin x$



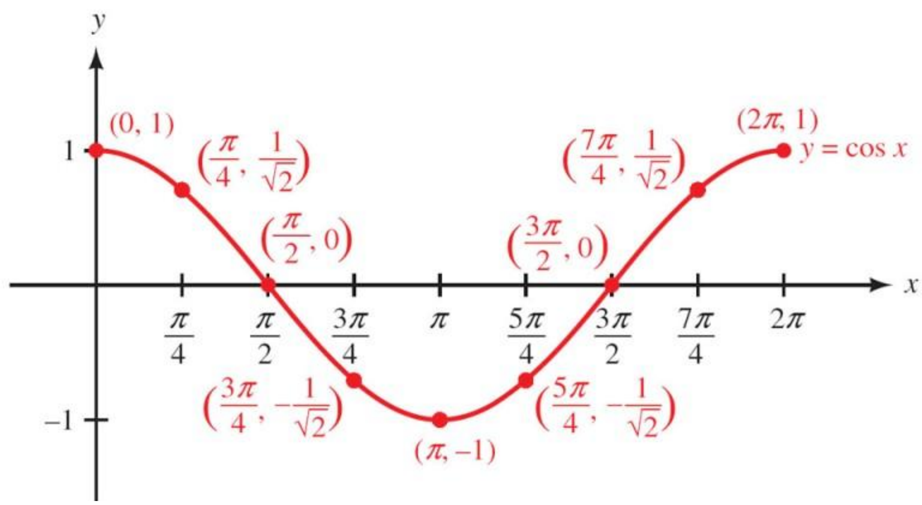




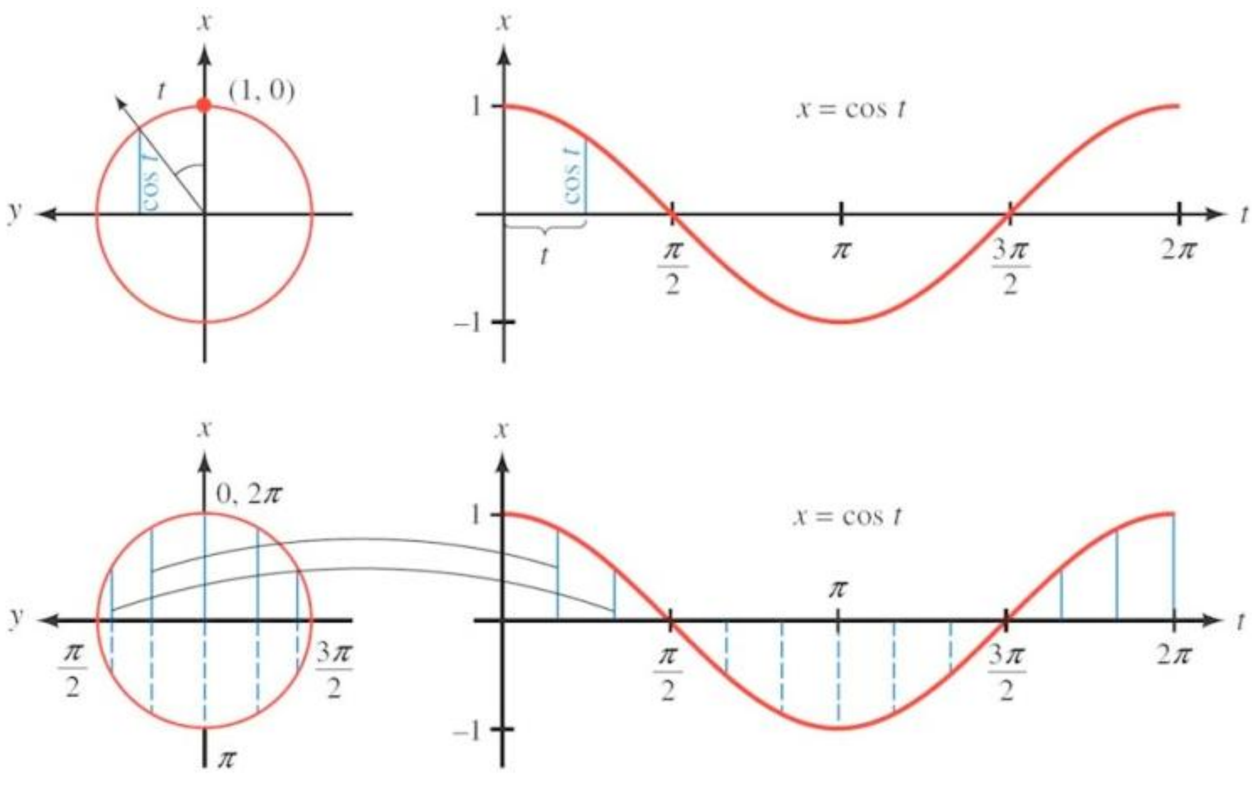
Range: $-1 \le y \le 1$ $\boxed{-1 \le \sin x \le 1}$

The *Cosine* Graph

Graphing the function: $y = \cos x$



Unit Circle **(*rotated*)**



Amplitude

If the greatest value of y is M and the least value of y is m, then the amplitude of the graph of y is defined to be

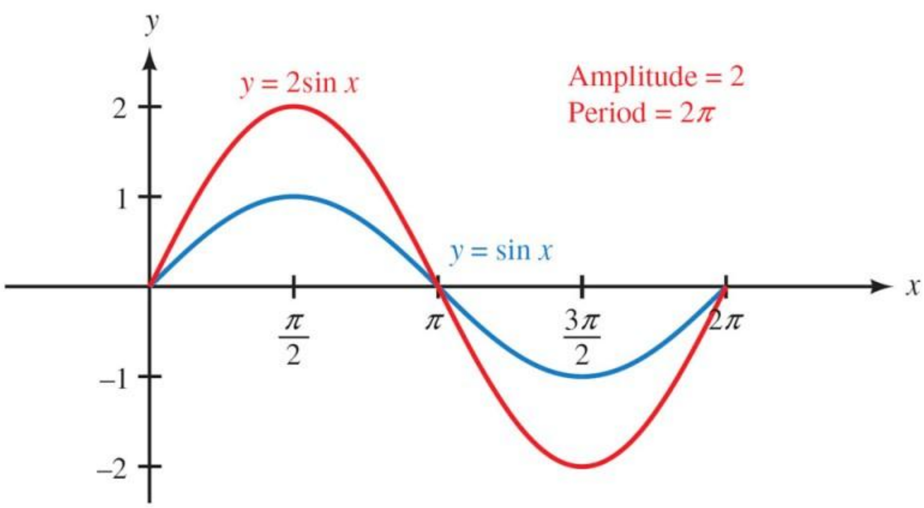
$$A = \frac{1}{2}|M - m|$$

The amplitude is $|A|$.

Example

Identify the amplitude of the graph and then sketch the graph: $y = 2\sin x$ *for* $0 \le x \le 2\pi$

Amplitude: $A = 2$



Note:

If $A > 0$, then the graph of $y = A\sin x$ and $y = A\cos x$ will have amplitude A and range [-A, A].

Period

For any function $y = f(x)$, the smallest positive number p for which

$$f(x+p) = f(x) \qquad \text{for all } x$$

is called the period of $f(x)$

The least possible positive value of p is the period of the function.

Many things in daily life repeat with a predictable pattern, such as weather, tides, and hours of daylight.

This periodic graph represents a normal heartbeat.

The graphs $y = A\sin Bx$ and $y = A\cos Bx$ $\qquad \rightarrow Period = \dfrac{2\pi}{|B|}$

One cycle: $0 \le \arg ument \le 2\pi$

$$0 \le Bx \le 2\pi$$

Example

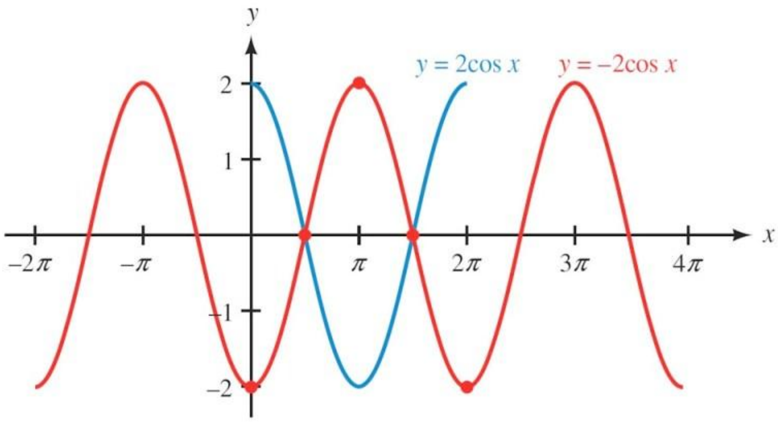
$y = \sin x \qquad \rightarrow P = 2\pi$

$y = \sin 2x \qquad \rightarrow P = \dfrac{2\pi}{2} = \pi$

$y = \sin 3x \qquad \rightarrow P = \dfrac{2\pi}{3}$

$y = \cos\frac{1}{2}x \qquad \rightarrow P = \dfrac{2\pi}{\frac{1}{2}} = 4\pi$

Reflecting About the *x*-axis



Note: The graphs of $y = A\sin x$ and $y = A\cos x$ will be reflected about *x*-axis if $A < 0$. The amplitude will be $|A|$.

Even and *Odd* Functions

Definition

An *even function* is a function for which $f(-x) = f(x)$

An *odd function* is a function for which $f(-x) = -f(x)$

Even Functions	***Odd Functions***
$y = \cos(\theta)$, $y = \sec(\theta)$	$y = \sin(\theta)$, $y = \csc(\theta)$ $y = \tan(\theta)$, $y = \cot(\theta)$
Graphs are symmetric about the y-axis	*Graphs are symmetric about the origin*

Vertical Translations

For $d > 0$, $\quad y = f(x) + d \Rightarrow$ The graph shifted up d units

$y = f(x) - d \Rightarrow$ The graph shifted down d units

Example

Sketch the graph $y = -3 - 2\sin \pi x$

Amplitude: $A = 2$

Period: $P = \dfrac{2\pi}{\pi} = 2$

Vertical Shifting: $y = -3$ *Down 3 units*

Phase shift

If we add a term to the argument of the function, the graph will be translated in a ***horizontal direction***.

In the function $y = f(x - c)$, the expression $x - c$ is called the ***argument***.

$$\boxed{\phi = -\frac{C}{B}}$$

Example

Graph $y = \sin\left(x + \frac{\pi}{2}\right), \quad if \quad -\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

Amplitude: $A = 1$

Period: $P = \frac{2\pi}{1} = 2\pi$

$x + \frac{\pi}{2} = 0 \to x = -\frac{\pi}{2}$

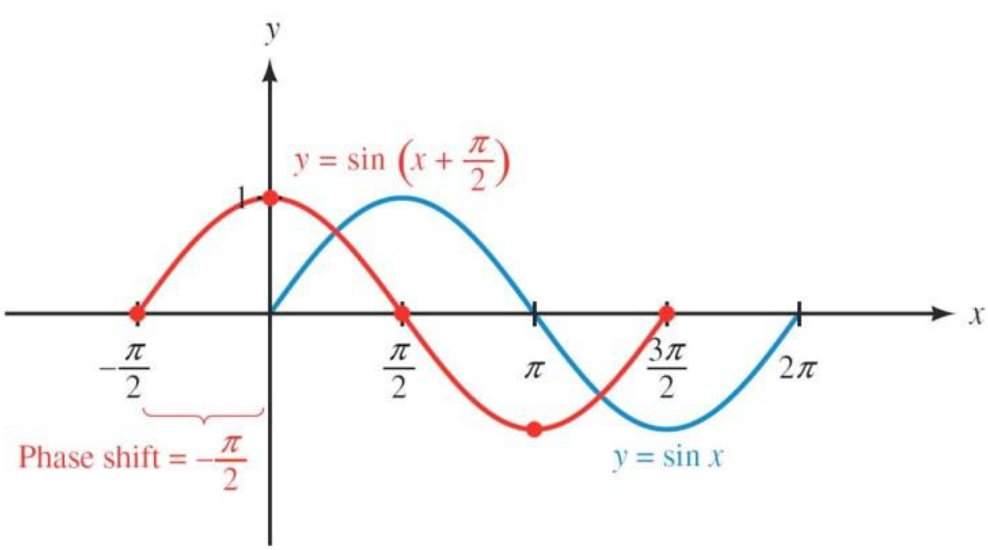
Phase Shift: $\phi = -\frac{\pi}{2}$

$0 \le \arg ument \le 2\pi$

$0 \le x + \frac{\pi}{2} \le 2\pi$

$-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

x	x	x	$y = \sin\left(x + \frac{\pi}{2}\right)$
$\phi + 0$	$-\frac{\pi}{2} + 0$	$-\frac{\pi}{2}$	0
$\phi + \frac{1}{4}P$	$-\frac{\pi}{2} + \frac{1}{2}\pi$	0	1
$\phi + \frac{1}{2}P$	$-\frac{\pi}{2} + \frac{3}{2}\pi$	$\frac{\pi}{2}$	0
$\phi + \frac{3}{4}P$	$-\frac{\pi}{2} + \frac{3}{4}\pi$	π	-1
$\phi + P$	$-\frac{\pi}{2} + 2\pi$	$\frac{3\pi}{2}$	0



Graphing the *Sine* and *Cosine* Functions

The graphs of $y = k + A\sin(Bx + C)$ and $y = k + A\cos(Bx + C)$, where B > 0, will have the following characteristics:

Amplitude $= |A|$ Period: $P = \frac{2\pi}{B}$

Phase Shift: $\phi = -\frac{C}{B}$ Vertical translation: $y = k$

If $A < 0$ the graph will be reflected about the x-axis

Example

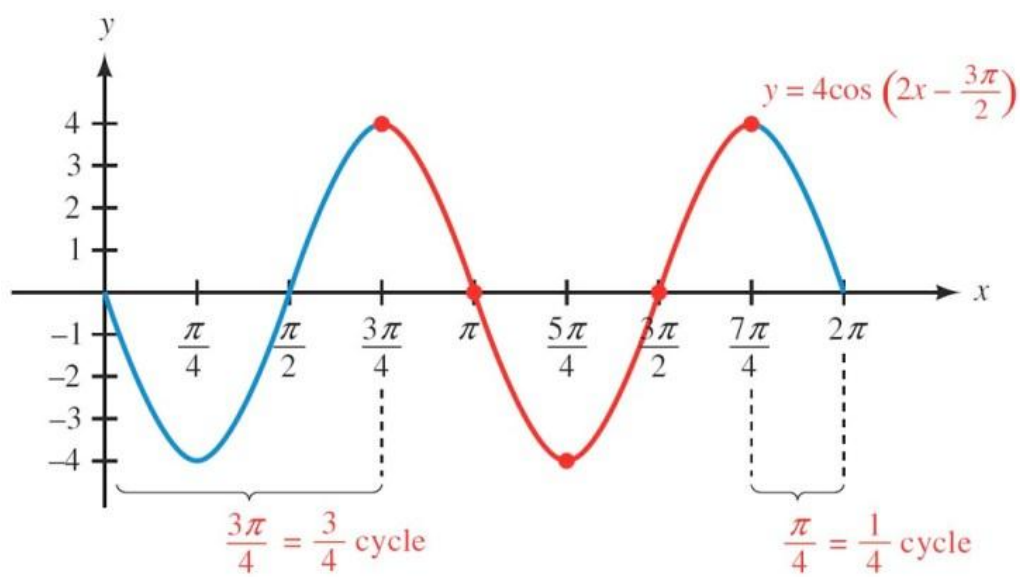
Graph $y = 4\cos\left(2x - \frac{3\pi}{2}\right)$ *for* $0 \le x \le 2\pi$

Amplitude: $A = 4$

Period: $P = \frac{2\pi}{2} = \pi$

Phase Shift: $\phi = \frac{\frac{3\pi}{2}}{2} = \frac{3\pi}{4}$

x	x	$y = 4\cos\left(2x - \frac{3\pi}{2}\right)$
$\frac{3\pi}{4} + 0$	$\frac{3\pi}{4}$	4
$\frac{3\pi}{4} + \frac{1}{4}\pi$	π	0
$\frac{3\pi}{4} + \frac{1}{2}\pi$	$\frac{5\pi}{4}$	-4
$\frac{3\pi}{4} + \frac{3}{4}\pi$	$\frac{3\pi}{2}$	0
$\frac{3\pi}{4} + \pi$	$\frac{7\pi}{4}$	4



Example

Graph one complete cycle $y = 3 - 5\sin\left(\pi x + \frac{\pi}{4}\right)$

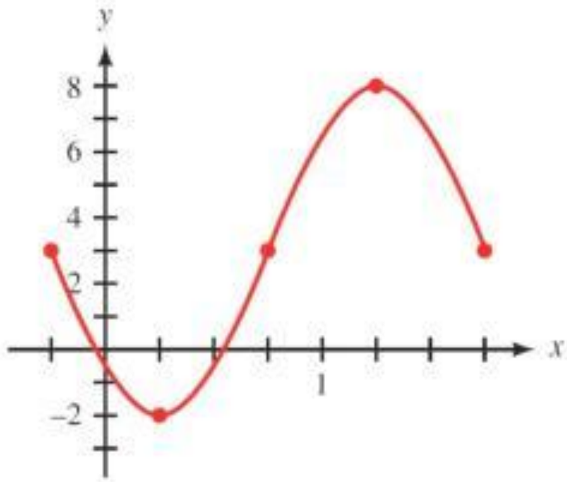
Amplitude: $A = 5$

Period: $P = \frac{2\pi}{\pi} = 2$

Phase Shift: $\phi = -\frac{\frac{\pi}{4}}{\pi} = -\frac{1}{4}$

VT: $y = 3$

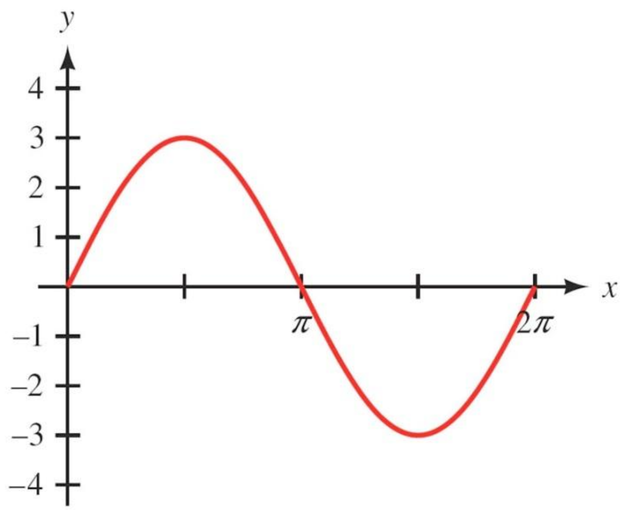
x	x	$y = 3 - 5\sin\left(\pi x + \frac{\pi}{4}\right)$
$-\frac{1}{4} + 0$	$-\frac{1}{4}$	3
$-\frac{1}{4} + \frac{1}{2}$	$\frac{1}{4}$	-2
$-\frac{1}{4} + 1$	$\frac{3}{4}$	3
$-\frac{1}{4} + \frac{3}{2}$	$\frac{5}{4}$	8
$-\frac{1}{4} + 2$	$\frac{7}{4}$	3



Finding the *Sine* and *Cosine* Functions from the Graph

Example

Find an equation $y = k + A\sin(Bx + C)$ or $y = k + A\cos(Bx + C)$ to match the graph



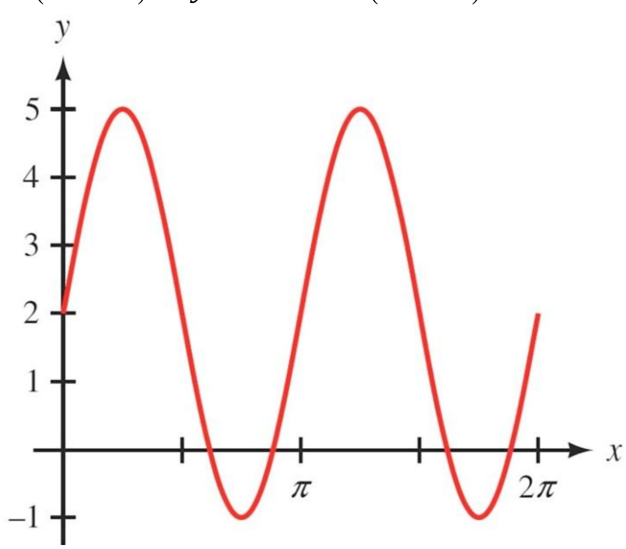
Amplitude = 3

$B = \frac{2\pi}{P} = \frac{2\pi}{2\pi} = 1$

$$y = 3\sin x \qquad 0 \le x \le 2\pi$$

Example

Find an equation $y = k + A\sin(Bx + C)$ or $y = k + A\cos(Bx + C)$ to match the graph



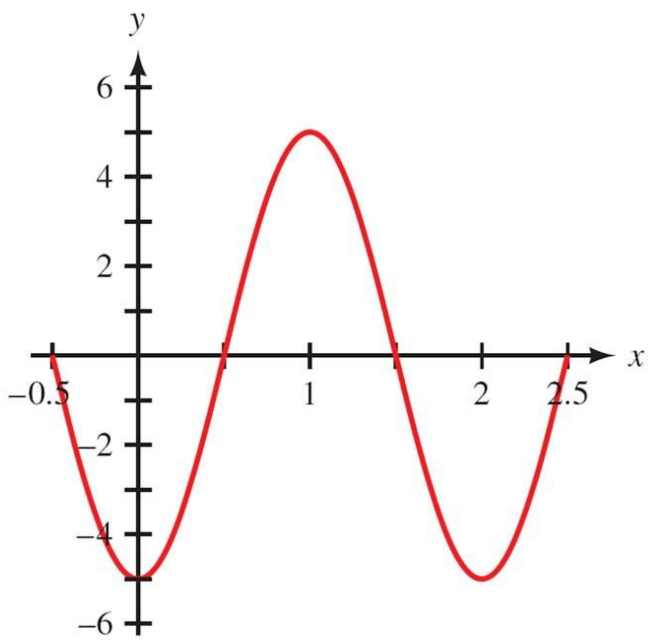
B = $\frac{2\pi}{\pi} = 2$

Amplitude = 3

$$y = 2 + 3\sin 2x \qquad 0 \le x \le 2\pi$$

Example

Find an equation $y = k + A\sin(Bx + C)$ or $y = k + A\cos(Bx + C)$ to match the graph



B = $\frac{2\pi}{2} = \pi$

Amplitude = 5

$$y = -5\cos \pi x \quad -0.5 \le x \le 2.5$$

Or

Phase shift = $-0.5 = -\frac{C}{B}$

$$0.5 = \frac{C}{\pi}$$

$$0.5\pi = C$$

$$y = -5\sin(\pi x + \frac{\pi}{2}) \qquad -0.5 \le x \le 2.5$$

Exercises *Section* 2.4 – Translation of Trigonometric Functions

Find the amplitude, the period, any vertical translation, and any phase shift of

1. $y = 2\sin(x - \pi)$

2. $y = \frac{2}{3}\sin\left(x + \frac{\pi}{2}\right)$

3. $y = 4\cos\left(\frac{1}{2}x + \frac{\pi}{2}\right)$

4. $y = \frac{1}{2}\sin\left(\frac{1}{2}x + \pi\right)$

5. $y = 3\cos\frac{\pi}{2}\left(x - \frac{1}{2}\right)$

6. $y = -\cos\pi\left(x - \frac{1}{3}\right)$

7. $y = 2 - \sin\left(3x - \frac{\pi}{5}\right)$

8. $y = -\frac{2}{3}\sin\left(3x - \frac{\pi}{2}\right)$

9. $y = -1 + \frac{1}{2}\cos(2x - 3\pi)$

10. $y = 2 - \frac{1}{3}\cos\left(\pi x + \frac{3\pi}{2}\right)$

11. $y = \frac{5}{2} - 3\cos\left(\pi x - \frac{\pi}{4}\right)$

12. $y = \cos\frac{1}{2}x$

13. $y = -3 + \sin\left(\pi x + \frac{\pi}{2}\right)$

14. $y = \frac{2}{3} - \frac{4}{3}\cos(3x - \pi)$

Find the amplitude, the period, the phase shift, and the vertical translation and sketch the graph of the equation

15. $y = 2\sin\left(x - \frac{\pi}{3}\right)$

16. $y = 4\cos\left(x - \frac{\pi}{4}\right)$

17. $y = -\sin(3x + \pi) - 1$

18. $y = \cos(2x - \pi) + 2$

19. $y = \sin\left(\frac{1}{2}x - \frac{\pi}{3}\right)$

20. $y = 5\sin\left(3x - \frac{\pi}{2}\right)$

21. $y = 3\cos\left(\frac{1}{2}x - \frac{\pi}{4}\right)$

22. $y = -5\cos\left(\frac{1}{3}x + \frac{\pi}{6}\right)$

23. $y = -2\sin(2\pi x + \pi)$

24. $y = -2\sin(2x - \pi) + 3$

25. $y = 3\cos(x + 3\pi) - 2$

26. $y = 5\cos(2x + 2\pi) + 2$

27. $y = -4\sin(3x - \pi) - 3$

Graph

28. $y = 2\sin(-\pi x)$ *for* $-3 \le x \le 3$

29. $y = 4\cos\left(-\frac{2}{3}x\right)$ *for* $-\frac{15\pi}{4} \le x \le \frac{15\pi}{4}$

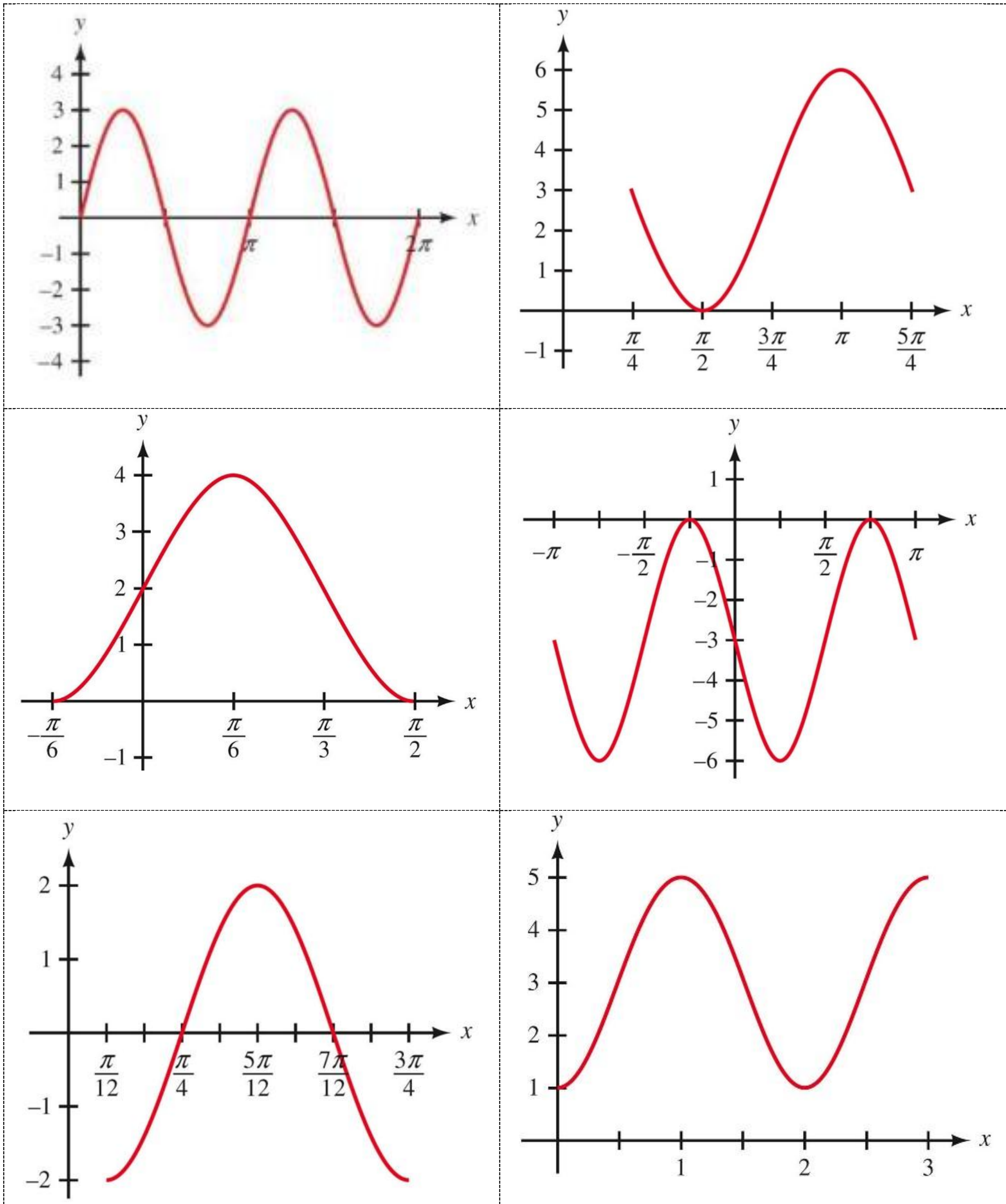
30. $y = \cos\left(x - \frac{\pi}{6}\right)$ for one complete cycle

31. $y = \frac{2}{3} - \frac{4}{3}\cos(3x - \pi)$ for one complete cycle

32. $y = -3 + \sin\left(\pi x + \frac{\pi}{2}\right)$ for one complete cycle

33. $y = -1 + 2\sin(4x + \pi)$ over two periods.

34. Find an equation $y = k + A\sin(Bx + C)$ or $y = k + A\cos(Bx + C)$ to match the graph



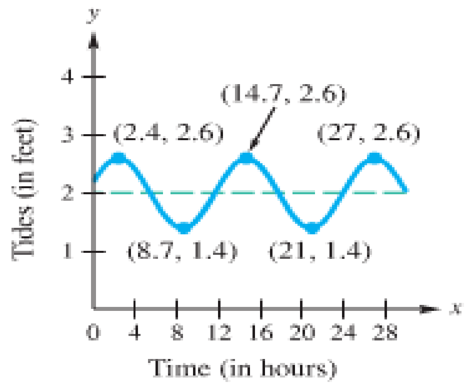
35. The figure shows a function *f* that models the tides in feet at Clearwater Beach, *x* hours after midnight starting on Aug. 26,

a) Find the time between high tides.

b) What is the difference in water levels between high tide and low tide?

c) The tides can be modeled by $f(x) = 0.6\cos[0.511x - 2.4] + 2$

Estimate the tides when $x = 10$.

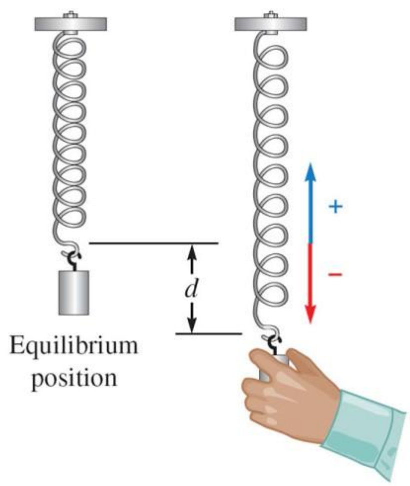


36. The maximum afternoon temperature in a given city might be modeled by $t = 60 - 30\cos\frac{\pi x}{6}$

Where *t* represents the maximum afternoon temperature in month *x*, with $x = 0$ representing January, $x = 1$ representing February, and so on.. Find the maximum afternoon temperature to the nearest degree for each month.

a) Jan. *b)* Apr. *c)* May. *d)* Jun. *e)* Oct.

37. A mass attached to a spring oscillates upward and downward. The length *L* of the spring after *t* seconds is given by the function $L = 15 - 3.5\cos(2\pi t)$, where *L* is measured in cm.



a) Sketch the graph of this function for $0 \le t \le 5$

b) What is the length the spring when it is at equilibrium?

c) What is the length the spring when it is shortest?

d) What is the length the spring when it is longest?

38. The diameter of the Ferris wheel is 250 *ft*, the distance from the ground to the bottom of the wheel is 14 *ft*. We found the height of a rider on that Ferris wheel was given by the function:

$$H = 139 - 125\cos\left(\frac{\pi}{10}t\right)$$

Where t is the number of minutes from the beginning of a ride. Graph a complete cycle of this function.

Section 2.5 – Other Trigonometric Functions

Vertical Asymptote

A ***vertical asymptote*** is a vertical line that the graph approaches but does not intersect, while function values increase or decrease without bound as x-values get closer and closer to the line.

Graphing the *Tangent* and *Cotangent* Functions

The graphs of $y = k + A\tan(Bx + C)$ and $y = k + A\cot(Bx + C)$, where $B > 0$, will have the following characteristics:

No *Amplitude*

$Period = \frac{\pi}{|B|}$

Phase Shift $= -\frac{C}{B}$

Vertical translation = k

One cycle: $0 \le argument \le \pi$ $\boxed{or}$ $-\frac{\pi}{2} < argument \le \frac{\pi}{2}$

Tangent Functions

Domain: $\left\{x \mid x \neq (2n+1)\frac{\pi}{2},\ \textit{where } n \in \mathbb{Z}\right\}$

Range: $(-\infty,\ \infty)$

- The graph is discontinuous at values of x of the form $x = (2n+1)\frac{\pi}{2}$ and has ***vertical asymptotes*** at these values.
- Its x-intercepts are of the form $x = n\pi$.
- Its period is π.
- Its graph has no amplitude, since there are no minimum or maximum values.
- The graph is symmetric with respect to the origin, so the function is an odd function. For all x in the domain, tan(–x) = – tan(x).

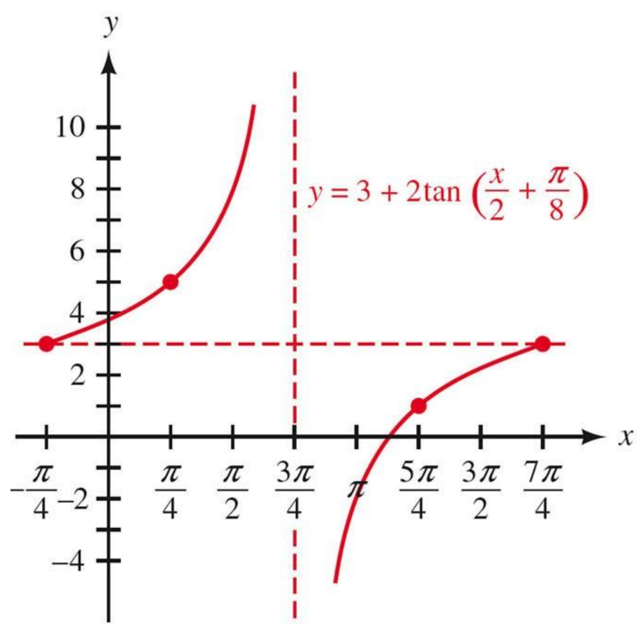
Example

Graph one complete cycle $y = 3 + 2\tan\left(\frac{x}{2} + \frac{\pi}{8}\right)$

Period $P = \frac{\pi}{1/2} = 2\pi$

Phase shift: $\phi = -\frac{\frac{\pi}{8}}{1/2} = -\frac{\pi}{4}$

x	x	$y = 3 + 2\tan\left(\frac{x}{2} + \frac{\pi}{8}\right)$
$-\frac{\pi}{4} + 0$	$-\frac{\pi}{4}$	3
$-\frac{\pi}{4} + \frac{1}{2}\pi$	$\frac{\pi}{4}$	5
$-\frac{\pi}{4} + \pi$	$\frac{3\pi}{4}$	∞
$-\frac{\pi}{4} + \frac{3}{2}\pi$	$\frac{5\pi}{4}$	1
$-\frac{\pi}{4} + 2\pi$	$\frac{7\pi}{4}$	3



Cotangent Functions

Domain: $\{x \mid x \neq n\pi, \ where\ n \in \mathbb{Z}\}$

Range: $(-\infty, \ \infty)$

- The graph is discontinuous at values of x of the form $x = n\pi$ and has ***vertical asymptotes*** at these values.
- Its x-intercepts are of the form $x = (2n+1)\frac{\pi}{2}$.
- Its period is π.
- Its graph has no amplitude, since there are no minimum or maximum values.
- The graph is symmetric with respect to the origin, so the function is an odd function. For all x in the domain, $\cot(-x) = -\cot(x)$.

Example

Graph two complete cycles $y = -2 - \cot\left(x - \frac{\pi}{4}\right)$

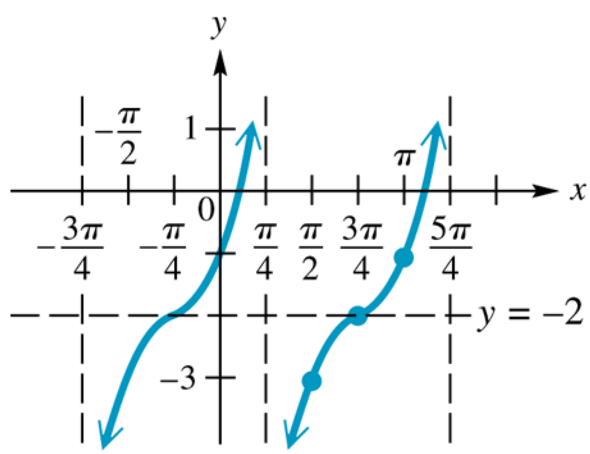
Period $P = \frac{\pi}{1} = \pi$

Phase shift: $\phi = -\frac{-\frac{\pi}{4}}{1} = \frac{\pi}{4}$

VA: $x = \frac{\pi}{4}, \ \frac{\pi}{4} + \pi = \frac{5\pi}{4}$

Vertical Translation: y = -2

x	$y = -2 - \cot\left(x - \frac{\pi}{4}\right)$
$\frac{\pi}{4}$	- ∞
$\frac{\pi}{2}$	- 3
$\frac{3\pi}{4}$	- 2
π	3
$\frac{5\pi}{4}$	∞



Graphing the *Secant* Function

Domain: $\left\{ x \mid x \neq (2n+1)\frac{\pi}{2},\ where\ n \in \mathbb{Z} \right\}$

Range: $(-\infty,\ -1] \cup [1,\ \infty)$

- The graph is discontinuous at values of x of the form $x = (2n+1)\frac{\pi}{2}$ and has ***vertical asymptotes*** at these values.
- There are no x-intercepts.
- Its period is 2π.
- Its graph has no amplitude, since there are no minimum or maximum values.
- The graph is symmetric with respect to the y-axis, so the function is an even function. For all x in the domain, $sec(-x) = \sec(x)$.

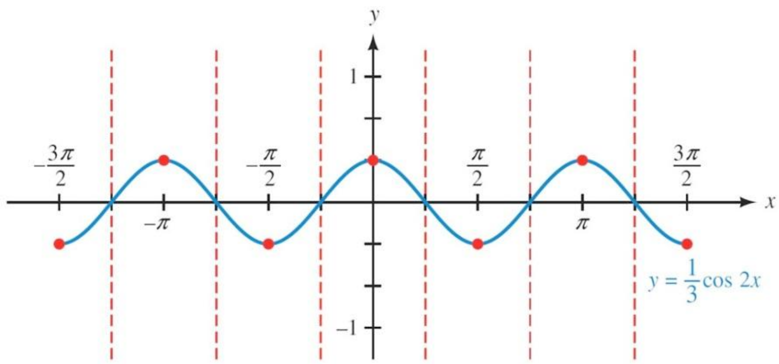
Example

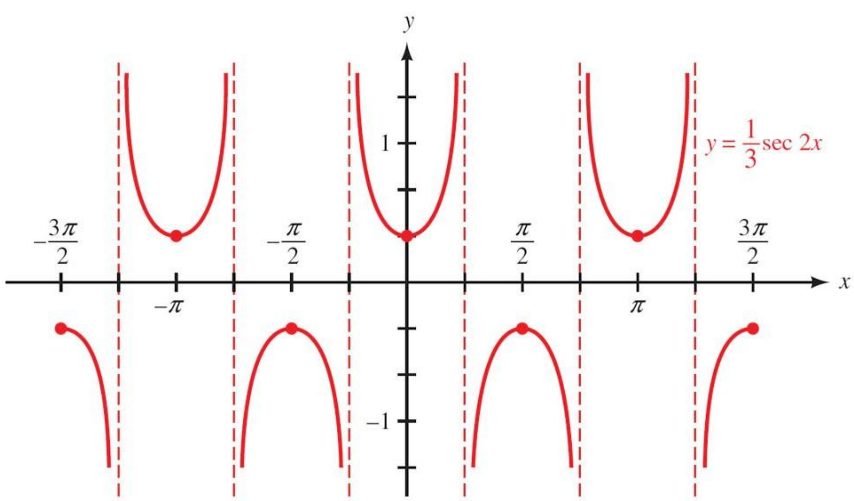
Graph $y = \frac{1}{3}\sec 2x$ *for* $-\frac{3\pi}{2} \le x \le \frac{3\pi}{2}$

Period $= \frac{2\pi}{2} = \pi$

First, graph $y = \frac{1}{3}\cos 2x$

x	$y = \frac{1}{3}\cos 2x$
0	$\frac{1}{3}$
$\frac{\pi}{4}$	0
$\frac{\pi}{2}$	$-\frac{1}{3}$
$\frac{3\pi}{4}$	0
π	$\frac{1}{3}$





Graphing the *Cosecant* Function

Domain: $\{x \mid x \neq n\pi,\ where\ n \in \mathbb{Z}\}$

Range: $(-\infty,\ -1] \cup [1,\ \infty)$

- The graph is discontinuous at values of x of the form $x = n\pi$ and has ***vertical asymptotes*** at these values.
- There are no x-intercepts.
- Its period is 2π.
- Its graph has no amplitude, since there are no minimum or maximum values.
- The graph is symmetric with respect to the *origin*, so the function is an odd function. For all x in the domain $\csc(-x) = -\csc(x)$.

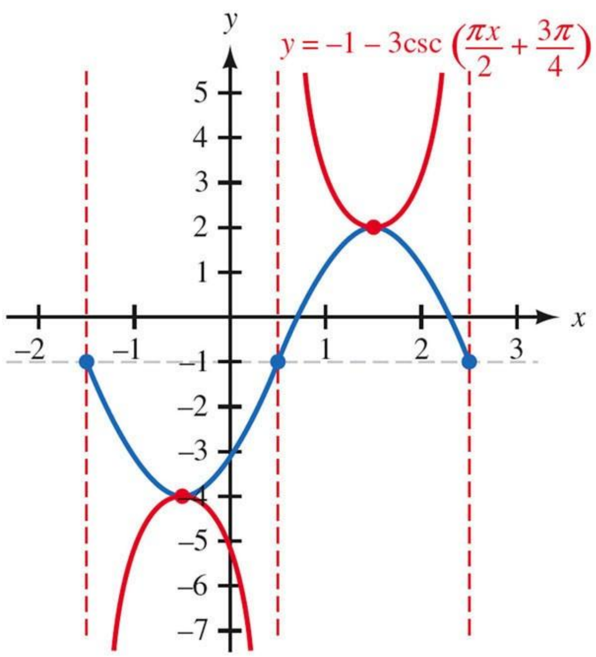
Example

Graph one complete cycle $y = -1 - 3\csc\left(\frac{\pi x}{2} + \frac{3\pi}{4}\right)$

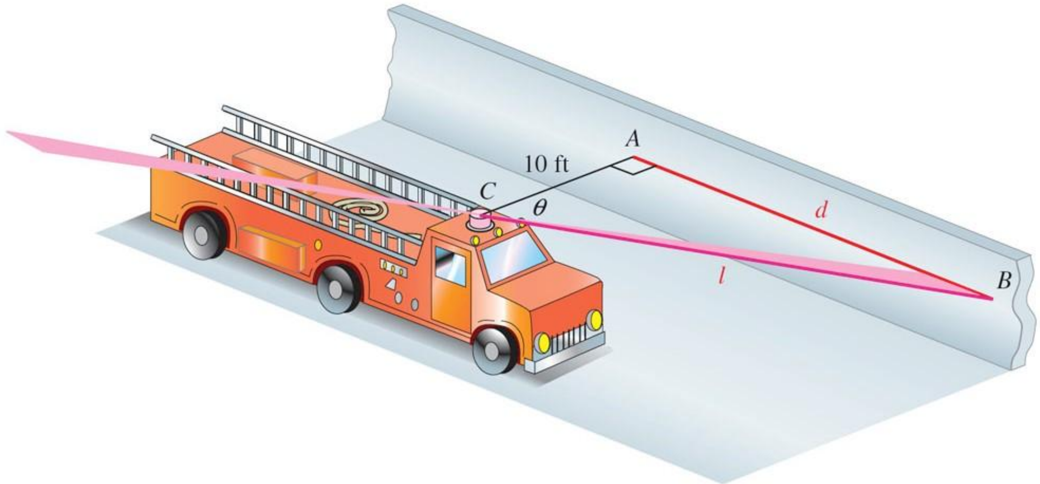
Period $= \frac{2\pi}{\pi/2} = 4$

Phase shift: $\phi = -\frac{\frac{3\pi}{4}}{\frac{\pi}{2}} = -\frac{3}{2}$

	x	$y = -1 - 3\sin\left(\frac{\pi x}{2} + \frac{3\pi}{4}\right)$
$-\frac{3}{2} + 0$	$-\frac{3}{2}$	-1
$-\frac{3}{2} + 1$	$-\frac{1}{2}$	-4
$-\frac{3}{2} + 2$	$\frac{1}{2}$	-1
$-\frac{3}{2} + 3$	$\frac{3}{2}$	2
$-\frac{3}{2} + 4$	$\frac{5}{2}$	-1



Example



A fire truck parked on the shoulder of a freeway next to a long block wall. The red light on the top is 10 feet from the wall and rotates through one complete revolution every 2 seconds. Graph the function that gives the length d in terms of time t from t = 0 to t = 2.

$$\omega = \frac{\theta}{t} = \frac{2\pi}{2} = \pi \ rad / sec$$

$$\tan\theta = \frac{d}{10} \quad \rightarrow d = 10\tan\theta$$

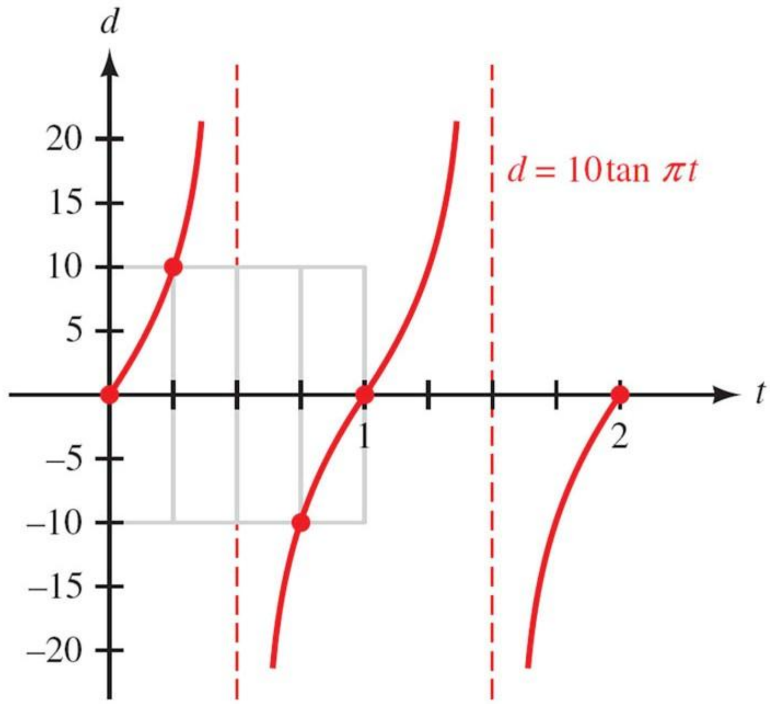
$$d(t) = 10\tan \pi t$$

Period $= \frac{\pi}{\pi} = 1$

One cycle: $0 \le \pi t \le \pi$

$0 \le t \le 1$

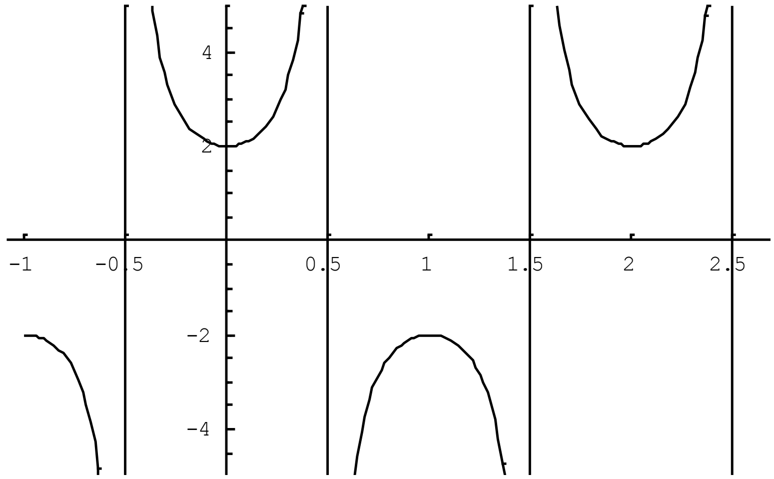
t	$d = 10\tan \pi t$
0	0
$\frac{1}{4}$	10
$\frac{1}{2}$	∞
$\frac{3}{4}$	-10
1	0



Finding the *Secant* and *Cosecant* Functions from the Graph

Example

Find an equation $y = k + A\sec(Bx + C)$ or $y = k + A\csc(Bx + C)$ to match the graph



<u>*Solution*</u>

For cosine:

A = 2 $\qquad P = 2 = \frac{2\pi}{B} \Rightarrow \underline{\left| B = \frac{2\pi}{2} = \pi \right|}$ $\qquad$ *Phase shift* $= -\frac{C}{B} = 0 \Rightarrow \boxed{C = 0}$

$y = 2\sec(\pi x)$ from –1 to 2.5.

Exercises *Section* 2.5 – Other Trigonometric Functions

1. Graph one complete cycle $y = 4\csc x$

2. Graph $y = 3\tan x$ *for* $-\pi \le x \le \pi$

3. Graph one complete cycle $y = \frac{1}{2}\cot(-2x)$

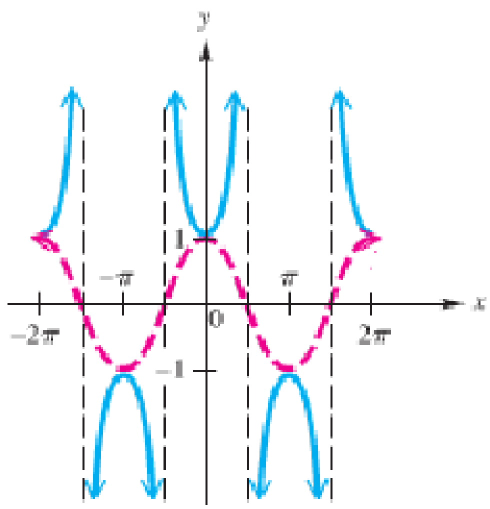
4. Graph over a 2-period interval $y = 1 - 2\cot 2\left(x + \frac{\pi}{2}\right)$

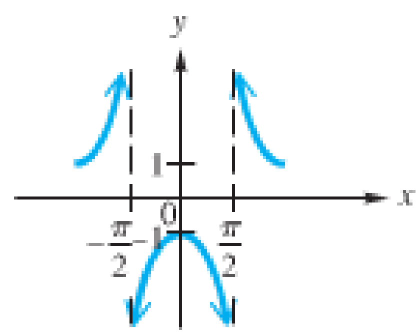
5. Graph over a 2-period interval $y = \frac{2}{3}\tan\left(\frac{3}{4}x - \pi\right) - 2$

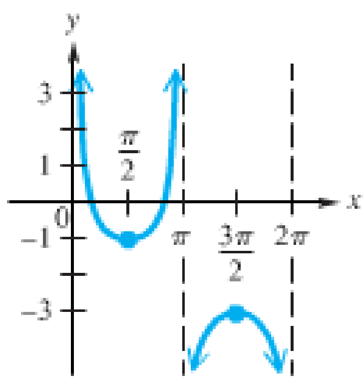
6. Graph over a one-period interval $y = 1 - \frac{1}{2}\csc\left(x - \frac{3\pi}{4}\right)$

7. Graph over a one-period interval $y = 2 + \frac{1}{4}\sec\left(\frac{1}{2}x - \pi\right)$

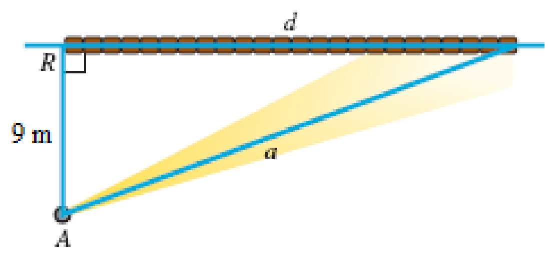
8. Find an equation to match the graph



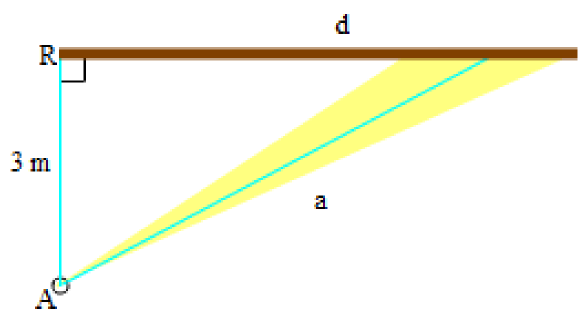




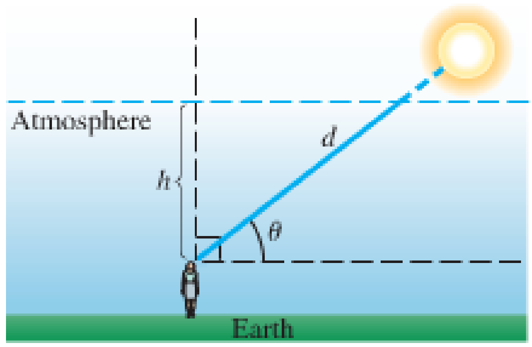
9. A rotating beacon is located at point A next to a long wall. The beacon is 9 m from the wall. The distance $\boldsymbol{a}$ is given by $a = 9\left|\sec 2\pi t\right|$, where t is time measured in seconds since the beacon started rotating. (When t = 0, the beacon is aimed at point R.) Find $\boldsymbol{a}$ for t = 0.45



10. A rotating beacon is located 3 m south of point R on an east-west wall. d, the length of the light display along the wall from R, is given by $d = 3\tan 2\pi t$, where t is time measured in seconds since the beacon started rotating. (When t = 0, the beacon is aimed at point R. When the beacon is aimed to the right of R, the value of d is positive; d is negative if the beacon is aimed to the left of R.) Find a for t = 0.8



11. The shortest path for the sun's rays through Earth's atmosphere occurs when the sun is directly overhead. Disregarding the curvature of Earth, as the sun moves lower on the horizon, the distance that sunlight passes through the atmosphere increases by a factor of $\csc\theta$, where θ is the angle of elevation of the sun. This increased distance reduces both the intensity of the sun and the amount of ultraviolet light that reached Earth's surface.

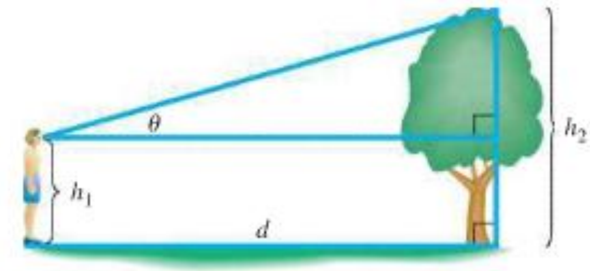


a) Verify that $d = h\csc\theta$

b) Determine θ when $d = 2h$

c) The atmosphere filters out the ultraviolet light that causes skin to burn, Compare the difference between sunbathing when $\theta = \frac{\pi}{2}$ and when $\theta = \frac{\pi}{3}$. Which measure gives less ultraviolet light?

12. Let a person whose eyes are h_1 feet from the ground stand ***d*** feet from an object h_1 feet tall, where $h_2 > h_1$ feet. Let θ be the angle of elevation to the top of the object.



a) Show that $d = (h_2 - h_1)\cot\theta$

b) Let $h_2 = 55$ and $h_1 = 5$. Graph ***d*** for the interval $0 < \theta \leq \frac{\pi}{2}$

Section 2.6 - Inverse Trigonometry Functions

Definition

A ***function*** is a rule or correspondence that pairs each element of the domain with exactly one element from the range. That is, a function is a set of ordered pairs in which no two different ordered pairs have the same first coordinate.

The ***inverse*** of function is found by interchanging the coordinates in each ordered pair that is an element of the function

Inverse Function Notation

if $y = f(x)$ is one-to-one function, then the inverse of f is also a function and can be denoted by

$$y = f^{-1}(x)$$

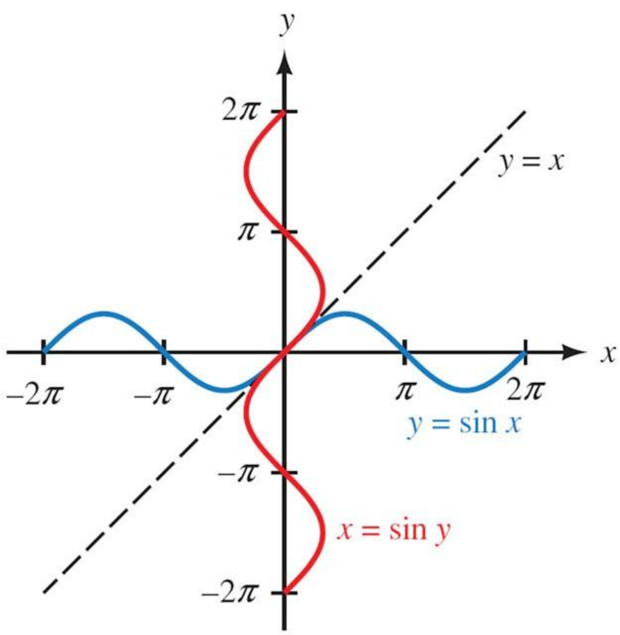
The inverse Sine Relation

To find the inverse of $y = \sin x$

1. Interchange x and y $\quad \rightarrow x = \sin y$

To graph $x = \sin y$

1. Graph $y = \sin x$
2. Draw the line $y = x$
3. Reflect $y = \sin x$ about the line $y = x$

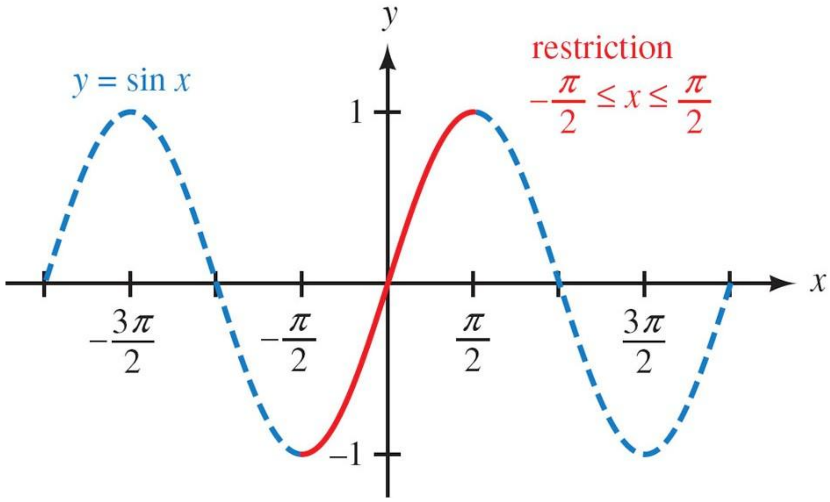


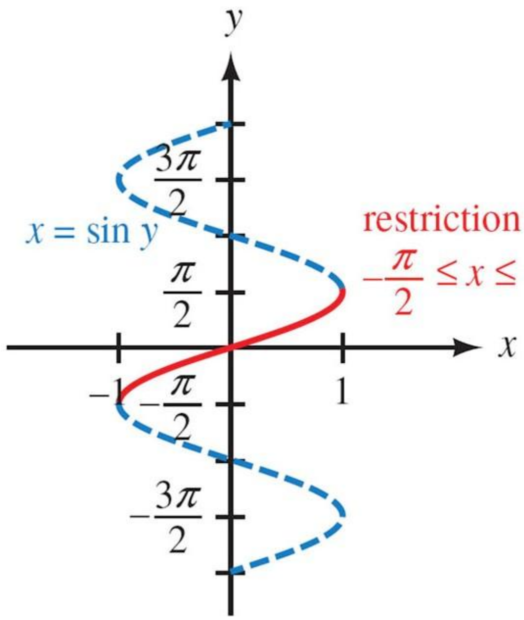
The Inverse *Sine* Function

Notation

The notation used to indicate the inverse *sine* function is as follow:

Notation	***Meaning***
$y=\sin^{-1}x$ *or* $y=\arcsin x$	$x=\sin y$ *and* $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$



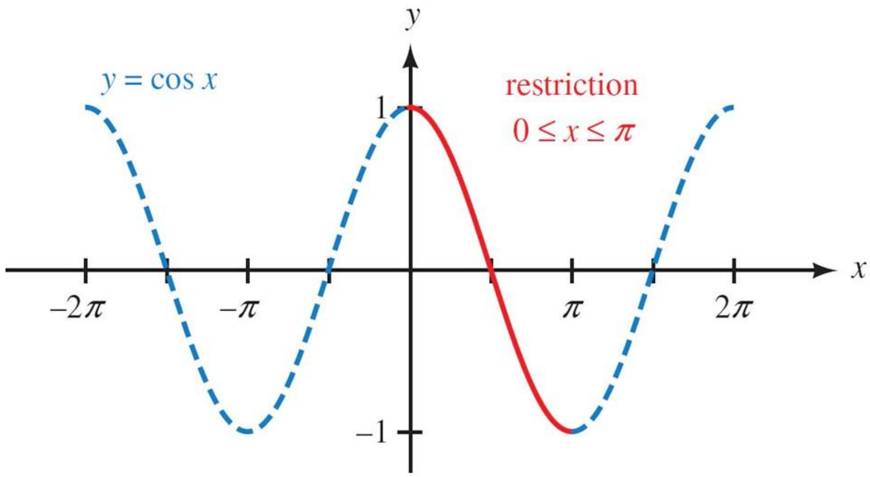


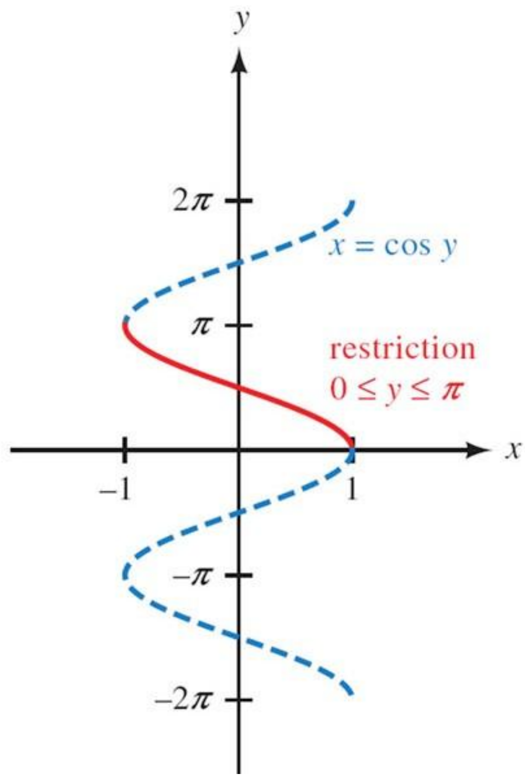
The Inverse *Cosine* Function

Notation

The notation used to indicate the inverse *cosine* function is as follow:

Notation	***Meaning***
$y = \cos^{-1} x$ *or* $y = \arccos x$	$x = \cos y$ *and* $0 \le y \le \pi$



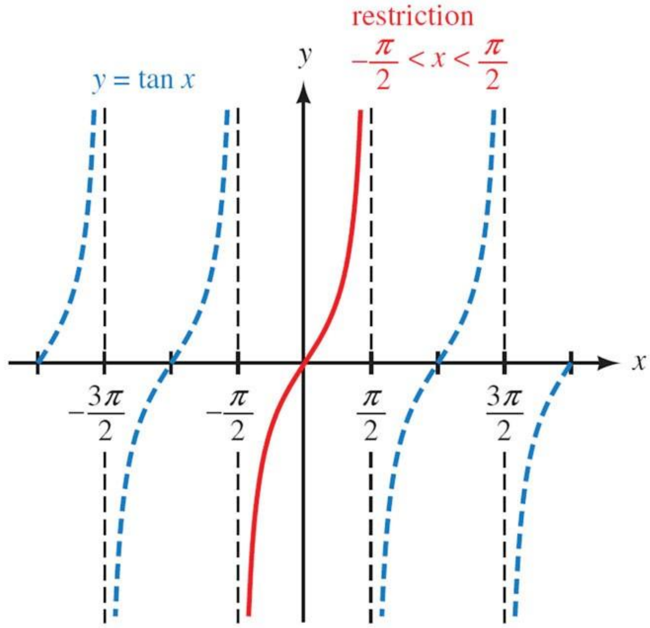


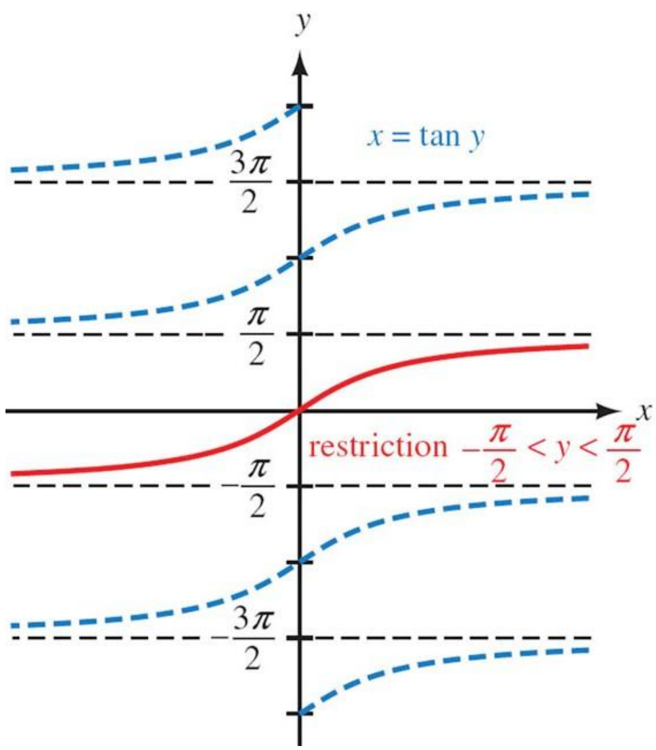
The Inverse *Tangent* Function

Notation

The notation used to indicate the inverse *tangent* function is as follow:

Notation	***Meaning***
$y=\tan^{-1}x$ *or* $y=\arctan x$	$x=\tan y$ *and* $-\frac{\pi}{2}<y<\frac{\pi}{2}$





Example

Evaluate in radians without using a calculator or tables.

a. $\sin^{-1}\frac{1}{2}$

$$-\frac{\pi}{2} \le angle \le \frac{\pi}{2} \Rightarrow \sin\frac{\pi}{6} = \frac{1}{2}$$

$$\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$

b. $\arccos\left(-\frac{\sqrt{3}}{2}\right)$

$$0 < angle < \pi \Rightarrow \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$$

$$\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$$

c. $\tan^{-1}(-1)$

$$-\frac{\pi}{2} < angle < \frac{\pi}{2} \Rightarrow \tan\left(-\frac{\pi}{4}\right) = -1$$

$$\tan^{-1}(-1) = -\frac{\pi}{4}$$

Example

Use a calculator to evaluate each expression to the nearest tenth of a degree

a. $\arcsin(0.5075)$

$$\arcsin(0.5075) = 30.5°$$

b. $\arcsin(-0.5075)$

$$\arcsin(-0.5075) = -30.5°$$

c. $\cos^{-1}(0.6428)$

$$\cos^{-1}(0.6428) = 50.0°$$

d. $\cos^{-1}(-0.6428)$

$$\cos^{-1}(-0.6428) = 130.0°$$

e. $\arctan(4.474)$

$$\arctan(4.474) = 77.4°$$

f. $\arctan(-4.474)$

$$\arctan(-4.474) = -77.4°$$

Example

Simplify $3|\sec\theta|$ if $\theta = \tan^{-1}\frac{x}{3}$ for some real number x.

Solution

$$\theta = \tan^{-1}\frac{x}{3} \to -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2} \Rightarrow \cos\theta > 0$

$$\Rightarrow \sec\theta > 0$$

$$3|\sec\theta| = 3\sec\theta$$

Example

Evaluate each expression

a. $\sin\left(\sin^{-1}\frac{1}{2}\right)$

$$\sin\left(\sin^{-1}\frac{1}{2}\right) = \sin\left(\frac{\pi}{6}\right) \qquad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \to \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$= \frac{1}{2}$$

b. $\sin^{-1}\sin(135°)$

$$\sin(135°) = \sin(180° - 135°)$$

$$= \sin(45°)$$

$$= \frac{\sqrt{2}}{2}$$

$$\sin^{-1}\sin(135°) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= 45°$$

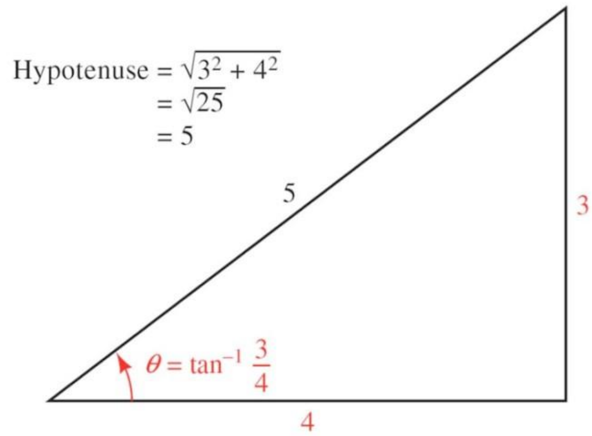
Example

Simplify $\tan^{-1}(\tan x)$ if $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$$\tan^{-1}(\tan x) = x$$

Example

Evaluate $\sin\left(\tan^{-1}\frac{3}{4}\right)$ without using a calculator



Solution

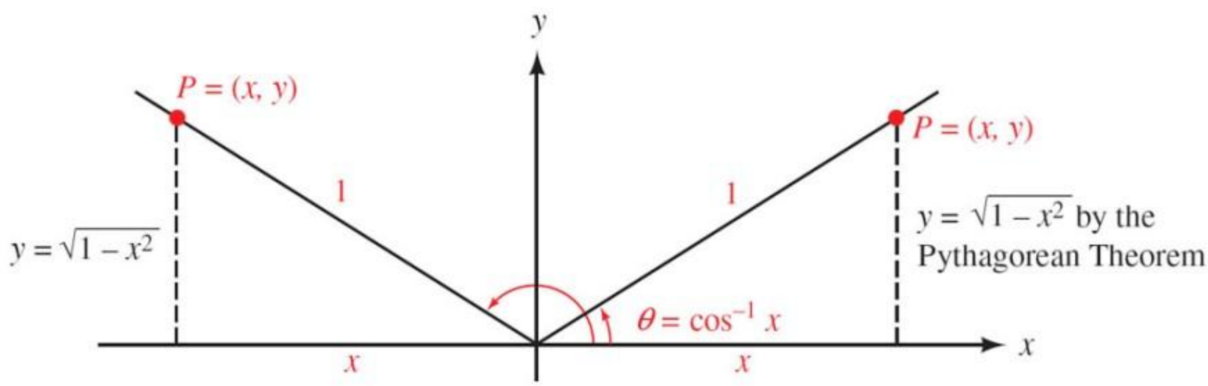
$$\theta = \tan^{-1}\frac{3}{4} \Rightarrow \tan\theta = \frac{3}{4} \to 0° < \theta < 90°$$

$$\sin\left(\tan^{-1}\frac{3}{4}\right) = \sin\theta$$
$$= \frac{3}{5}$$

Example

Evaluate $\sin(\cos^{-1}x)$ as an equivalent expression in x only

Solution



$$\sin(\theta) = \frac{y}{r}$$
$$= \frac{\sqrt{1-x^2}}{1}$$
$$= \sqrt{1-x^2}$$

$$\sin(\cos^{-1}x) = \sin\theta$$
$$= \sqrt{1-x^2}$$

Exercises *Section* 2.6 - Inverse Trigonometry Functions

1. Evaluate without using a calculator: $\cos\left(\cos^{-1}\frac{3}{5}\right)$

2. Evaluate without using a calculator: $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

3. Evaluate without using a calculator: $\tan\left(\cos^{-1}\frac{3}{5}\right)$

4. Evaluate without using a calculator: $\sin\left(\cos^{-1}\frac{1}{\sqrt{5}}\right)$

5. Evaluate without using a calculator: $\cos\left(\sin^{-1}\frac{1}{2}\right)$

6. Evaluate without using a calculator: $\sin\left(\sin^{-1}\frac{3}{5}\right)$

7. Evaluate without using a calculator: $\cos\left(\tan^{-1}\frac{3}{4}\right)$

8. Evaluate without using a calculator: $\tan\left(\sin^{-1}\frac{3}{5}\right)$

9. Evaluate without using a calculator: $\sec\left(\cos^{-1}\frac{1}{\sqrt{5}}\right)$

10. Evaluate without using a calculator: $\cot\left(\tan^{-1}\frac{1}{2}\right)$

11. Write an equivalent expression that involves x only for $\cos\left(\cos^{-1}x\right)$

12. Write an equivalent expression that involves x only for $\tan\left(\cos^{-1}x\right)$

13. Write an equivalent expression that involves x only for $\csc\left(\sin^{-1}\frac{1}{x}\right)$