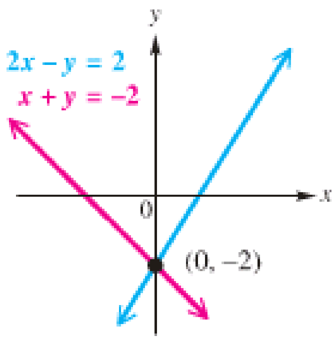
Lecture Four – Matrices

Section 4.1 – System of linear Equations

Solving Systems of Equations

1. Graphically
2. Substitution Method
3. Elimination Method

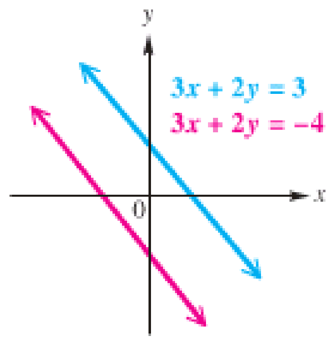
Result:



One solution (lines intersect)

Consistent

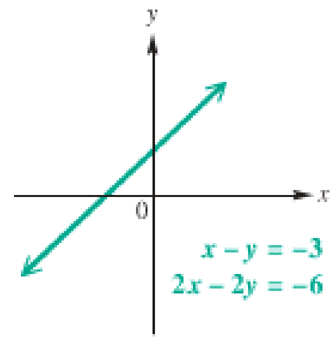
Independent



No Solution (lines //)

Inconsistent

Independent



Infinite solution

Consistent

Dependent

Substitution Method

Solve: $\begin{array}{ll} 3x+2y=11 & (1) \\ -x+y=3 & (2) \end{array}$

From $(2) \to y = x+3 \quad (3)$

$(1) \Rightarrow 3x+2(x+3)=11$

$3x+2x+6=11$

$5x+6=11$

$5x+6-6=11-6$

$5x=5$

$x=1$

From $(3) \to y=1+3=4$

Solution: $(1,\ 4)$

Elimination Method

Solve: $\begin{array}{ll} 3x-4y=1 & (1) \\ 2x+3y=12 & (2) \end{array}$

$\begin{array}{rl} -2\times) & 3x-4y=1 \\ 3\times) & 2x+3y=12 \end{array}$

$$\begin{array}{r} -6x+8y=-2 \\ 6x+9y=36 \\ \hline 17y=34 \end{array}$$

$y=\dfrac{34}{17}=2$

From $(1) \Rightarrow 3x=1+4y$

$3x=1+4(2)$

$3x=9$

$x=3$

Solution: $(3,\ 2)$

Matrices

$$\begin{array}{cc} & \text{Column} \\ & \begin{array}{ccc} C_1 & C_2 & C_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{c} Row\ 1 \to R_1 \\ Row\ 2 \to R_2 \\ Row\ 3 \to R_3 \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$

This is called Matrix (*Matrices*)

Each number in the array is an ***element*** or ***entry***

The matrix is said to be of order $m\ x\ n$

m: numbers of rows,

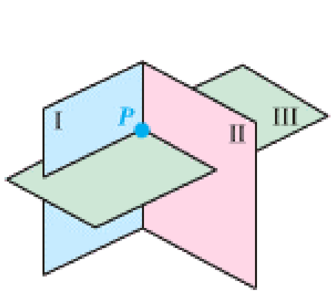
n: number of columns

When $m = n$, then matrix is said to be ***square***.

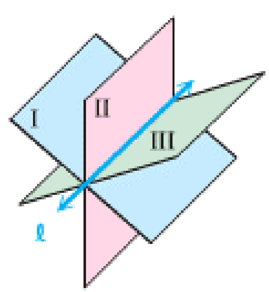
Given the system equations

$$\begin{aligned} 3x + \ \ y + 2z &= 31 \\ x + \ \ y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

The ***augmented matrix*** form is: $\left[\begin{array}{ccc|c} 3 & 1 & 2 & 31 \\ 1 & 1 & 2 & 19 \\ 1 & 3 & 2 & 25 \end{array}\right]$

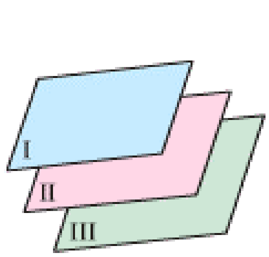


A single solution

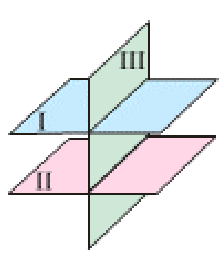


Points of a line in common

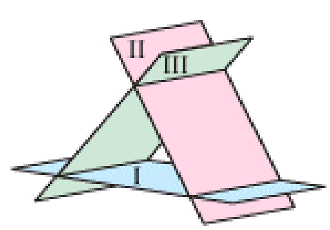
All points in common



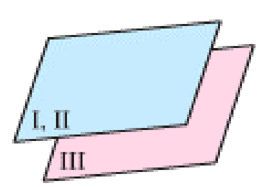
No points in common



No points in common



No points in common



No points in common

Gaussian Elimination

Example

Use the Gaussian elimination method to solve the system

$$\begin{aligned} 3x + \ \ y + 2z &= 31 \\ x + \ \ y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 3 & 1 & 2 & 31 \\ 1 & 3 & 2 & 25 \end{array}\right] \begin{array}{c} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array} \qquad \begin{array}{cccc} 3 & 1 & 2 & 31 \\ -3 & -3 & -6 & -57 \\ \hline 0 & -2 & -4 & -26 \end{array} \qquad \begin{array}{cccc} 1 & 3 & 2 & 25 \\ -1 & -1 & -2 & -19 \\ \hline 0 & 2 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & -2 & -4 & -26 \\ 0 & 2 & 0 & 6 \end{array}\right] \ -\tfrac{1}{2}R_2 \qquad \begin{array}{cccc} 0 & 1 & 2 & 13 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 2 & 0 & 6 \end{array}\right] \begin{array}{c} \\ \\ R_3 - 2R_2 \end{array} \qquad \begin{array}{cccc} 0 & 2 & 0 & 6 \\ 0 & -2 & -4 & -26 \\ \hline 0 & 0 & -4 & -20 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & -4 & -20 \end{array}\right] \begin{array}{c} \\ \\ -\tfrac{1}{4}R_3 \end{array} \qquad \begin{array}{cccc} 0 & 0 & 1 & 5 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 1 & 5 \end{array}\right] \Rightarrow \begin{array}{rl} x + y + 2z = 19 & (3) \\ y + 2z = 13 & (2) \\ z = 5 & (1) \end{array}$$

(2) $\Rightarrow y = 13 - 2z = 13 - 2(5) = 3$

(3) $\Rightarrow \ x = 19 - y - 2z = 19 - 3 - 10 = 6$

$\Rightarrow$ **(6, 3, 5)**

Gauss-Jordan Elimination

Example

Use the Gauss-Jordan method to solve the system

$$\begin{aligned} 3x + \ \ y + 2z &= 31 \\ x + \ \ y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 3 & 1 & 2 & 31 \\ 1 & 3 & 2 & 25 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 1 & 2 & 31 \\ -3 & -3 & -6 & -57 \\ \hline 0 & -2 & -4 & -26 \end{array} \qquad \begin{array}{cccc} 1 & 3 & 2 & 25 \\ -1 & -1 & -2 & -19 \\ \hline 0 & 2 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & -2 & -4 & -26 \\ 0 & 2 & 0 & 6 \end{array}\right] -\tfrac{1}{2}R_2 \qquad 0 \quad 1 \quad 2 \quad 13$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 2 & 0 & 6 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 2R_2 \end{array}$$

$$\begin{array}{cccc} 0 & 2 & 0 & 6 \\ 0 & -2 & -4 & -26 \\ \hline 0 & 0 & -4 & -20 \end{array} \qquad \begin{array}{cccc} 1 & 1 & 2 & 19 \\ 0 & -1 & -2 & -13 \\ \hline 1 & 0 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & -4 & -20 \end{array}\right] \begin{array}{l} \\ \\ -\tfrac{1}{4}R_3 \end{array} \qquad 0 \quad 0 \quad 1 \quad 5$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 1 & 5 \end{array}\right] R_2 - 2R_3 \qquad \begin{array}{cccc} 0 & 1 & 2 & 13 \\ 0 & 0 & -2 & -10 \\ \hline 0 & 1 & 0 & 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$\Rightarrow$ ***Solution*: (6, 3, 5)**

Example

Use the Gaussian elimination method to solve the system

$$\begin{aligned} 2x + \ \ y + 2z &= 4 \\ 2x + 2y \qquad &= 5 \\ 2x - \ \ y + 6z &= 2 \end{aligned}$$

<u>Solution</u>

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 4 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array}\right] \quad \tfrac{1}{2}R_1$$

$$1 \quad \frac{1}{2} \quad 1 \quad 2$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{array}$$

$$\begin{array}{cccc} 2 & 2 & 0 & 5 \\ -2 & -1 & -2 & -4 \\ \hline 0 & 1 & -2 & 1 \end{array} \qquad \begin{array}{cccc} 2 & -1 & 6 & 2 \\ -2 & -1 & -2 & -4 \\ \hline 0 & -2 & 4 & -2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 4 & -2 \end{array}\right] \begin{array}{l} \\ \\ R_3 + 2R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -2 & 4 & -2 \\ 0 & 2 & -4 & 2 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{ll} x + \frac{1}{2}y + z = 2 & (3) \\ y - 2z = 1 & (2) \\ 0 = 0 & (1) \end{array}$$

From (1): 0 = 0 is a true statement. Let ***z*** be the variable.

From (2): $\Rightarrow y = 1 + 2z$

From (3): $\Rightarrow x = -\frac{1}{2}y - z + 2$

$\Rightarrow x = -\frac{1}{2}(1 + 2z) - z + 2$

$\Rightarrow x = -\frac{1}{2} - z - z + 2$

$\Rightarrow$ x = -2z + $\frac{3}{2}$

Solution: $\boxed{\left(-2z + \frac{3}{2},\ 2z + 1,\ z\right)}$

Example

Use the Gaussian elimination method to solve the system

$$\begin{aligned} x+2y-5z&=-1\\ 2x+3y-2z&=2\\ 3x+5y-7z&=4 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -5 & -1\\ 2 & 3 & -2 & 2\\ 3 & 5 & -7 & 4 \end{array}\right] \begin{array}{l} \\ R_2-2R_1\\ R_3-3R_1 \end{array}$$

$$\begin{array}{cccc} 2 & 3 & -2 & 2\\ -2 & -4 & 10 & 2\\ \hline 0 & -1 & 8 & 4 \end{array} \qquad \begin{array}{cccc} 3 & 5 & -7 & 4\\ -3 & -6 & 15 & 3\\ \hline 0 & -1 & 8 & 7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -5 & -1\\ 0 & -1 & 8 & 4\\ 0 & -1 & 8 & 7 \end{array}\right] -R_2$$

$$0 \quad 1 \quad -8 \quad -4$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -5 & -1\\ 0 & 1 & -8 & -4\\ 0 & -1 & 8 & 7 \end{array}\right] \begin{array}{l} \\ \\ R_3+R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -1 & 8 & 7\\ 0 & 1 & -8 & -4\\ \hline 0 & 0 & 0 & 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -5 & -1\\ 0 & 1 & -8 & -4\\ 0 & 0 & 0 & 3 \end{array}\right]$$

From Row 3: 0 = 3 is a False statement.

No Solution or ***Inconsistent***

Exercises *Section* 4.1 – System of linear Equations

1. Use the Gauss-Jordan method to solve the system

$$\begin{cases} x-y+5z=-6 \\ 3x+3y-z=10 \\ x+3y+2z=5 \end{cases}$$

2. Use the Gauss-Jordan method to solve the system

$$\begin{cases} 2x-y+4z=-3 \\ x-2y-10z=-6 \\ 3x \qquad +4z=7 \end{cases}$$

3. Use the Gauss-Jordan method to solve the system

$$\begin{cases} 4x+3y-5z=-29 \\ 3x-7y-z=-19 \\ 2x+5y+2z=-10 \end{cases}$$

4. Use the Gauss-Jordan method to solve the system

$$\begin{cases} x+2y-3z=-15 \\ 2x-3y+4z=18 \\ -3x+y+z=1 \end{cases}$$

5. Use the Gauss-Jordan method to solve the system

$$\begin{cases} x+2y+3z=10 \\ 4x+5y+6z=11 \\ 7x+8y+9z=12 \end{cases}$$

6. Use the Gauss-Jordan method to solve the system

$$\begin{cases} 2x+y+2z=4 \\ 2x+2y \qquad =5 \\ 2x-y+6z=2 \end{cases}$$

7. Use the Gauss-Jordan method to solve the system

$$\begin{aligned} 2x-5y+3z&=1 \\ x-2y-2z&=8 \end{aligned}$$

8. At SnackMix, caramel corn worth \$2.50 per pound is mixed with honey roasted missed nuts worth \$7.50 per pound in order to get 20 lb of a mixture worth \$4.50 per pound. How much of each snack is used?

Section 4.2 – Matrix operations and Their Applications

Matrix Notation

The Matrix:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix}$$ is called the coefficient matrix of the system.

The matrix is said to be of order *m x n*
m: numbers of rows,
n: number of columns

A matrix A with *m* rows and *n* columns can be written in a general form

$$A = \left[a_{\text{ij}} \right] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

The matrix *A* above has 3 rows and 3 columns, therefore the order of the matrix *A* is (3 *x* 3)

When $m = n$, then matrix is said to be ***square***.

The numbers in a matrix are called ***entries.***

Example

$$Let \quad A = \begin{bmatrix} 5 & -2 \\ -3 & \pi \\ 1 & 6 \end{bmatrix}$$

a. What is the order of A?
3 rows and 2 columns $\Rightarrow$ A is 3 x 2

b. $a_{12} = -2$ $\qquad$ $a_{31} = 1$

Equality of Matrices

Definition of Equality of Matrices

Two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are equal if and only if they have the same order (size) $m \times n$ and if each pair corresponding elements is equal

$$a_{ij} = b_{ij} \text{ for } i = 1, 2, \ldots, m \textbf{ and } j = 1, 2, \ldots, n$$

Example

Find the values of the variables for which each statement is true, if possible.

a) $\begin{bmatrix} 2 & 1 \\ p & q \end{bmatrix} = \begin{bmatrix} x & y \\ -1 & 0 \end{bmatrix}$

$$x = 2, \ y = 1, \ \ p = -1, \ \ q = 0$$

b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$

can't be true

c) $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} w = 9 & x = 17 \\ 8 = y & -12 = z \end{bmatrix}$$

Matrix Addition and Subtraction

Given two *m x n* matrices $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ their sum is $A+B=\left[a_{ij}+b_{ij}\right]$

And their difference is $A-B=\left[a_{ij}-b_{ij}\right]$

The matrices have to be the ***same order***

Example

$$\begin{bmatrix} -4 & 3 \\ 7 & -6 \end{bmatrix}+\begin{bmatrix} 6 & -3 \\ 2 & -4 \end{bmatrix}$$

<u>*Solution*</u>

$$\begin{bmatrix} -4 & 3 \\ 7 & -6 \end{bmatrix}+\begin{bmatrix} 6 & -3 \\ 2 & -4 \end{bmatrix}=\begin{bmatrix} -4+6 & 3+(-3) \\ 7+2 & -6+(-4) \end{bmatrix}$$

$$=\begin{bmatrix} 2 & 0 \\ 9 & -10 \end{bmatrix}$$

Example

$$\begin{bmatrix} 5 & 4 \\ -3 & 7 \\ 0 & 1 \end{bmatrix}-\begin{bmatrix} -4 & 8 \\ 6 & 0 \\ -5 & 3 \end{bmatrix}$$

<u>*Solution*</u>

$$\begin{bmatrix} 5 & 4 \\ -3 & 7 \\ 0 & 1 \end{bmatrix}-\begin{bmatrix} -4 & 8 \\ 6 & 0 \\ -5 & 3 \end{bmatrix}=\begin{bmatrix} 5-(-4) & 4-8 \\ -3-6 & 7-0 \\ 0-(-5) & 1-3 \end{bmatrix}$$

$$=\begin{bmatrix} 9 & -4 \\ -9 & 7 \\ 5 & -2 \end{bmatrix}$$

Example

$$\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix}+\begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix}$$

<u>*Solution*</u>

$$\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix}+\begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix}=\begin{bmatrix} 5-4 & -6+6 \\ 8+8 & 9-3 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 \\ 16 & 6 \end{bmatrix}$$

Scalar Multiplication

The scalar product of a number k and a matrix A is denoted by $\boldsymbol{kA}$.

$$kA = k\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}$$

Example

Find $5\begin{bmatrix} 2 & -3 \\ 0 & 4 \end{bmatrix}$

Solution

$$5\begin{bmatrix} 2 & -3 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 2(5) & -3(5) \\ 0(5) & 4(5) \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -15 \\ 0 & 20 \end{bmatrix}$$

Example

Find $\frac{3}{4}\begin{bmatrix} 20 & 36 \\ 12 & -16 \end{bmatrix}$

Solution

$$\frac{3}{4}\begin{bmatrix} 20 & 36 \\ 12 & -16 \end{bmatrix} = \begin{bmatrix} 15 & 27 \\ 9 & -12 \end{bmatrix}$$

Example

$A = \begin{bmatrix} -4 & 1 \\ 3 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & -2 \\ 8 & 5 \end{bmatrix}$

Solution

a) $-6B = -6\begin{bmatrix} -1 & -2 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} -1(-6) & -2(-6) \\ 8(-6) & 5(-6) \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ -48 & -30 \end{bmatrix}$

b) $3A + 2B = 3\begin{bmatrix} -4 & 1 \\ 3 & 0 \end{bmatrix} + 2\begin{bmatrix} -1 & -2 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} -4(3) & 1(3) \\ 3(3) & 0(3) \end{bmatrix} + \begin{bmatrix} -1(2) & -2(2) \\ 8(2) & 5(2) \end{bmatrix}$

$$= \begin{bmatrix} -12 & 3 \\ 9 & 0 \end{bmatrix} + \begin{bmatrix} -2 & -4 \\ 16 & 10 \end{bmatrix} = \begin{bmatrix} -12-2 & 3-4 \\ 9+16 & 0+10 \end{bmatrix}$$

$$= \begin{bmatrix} -14 & -1 \\ 25 & 10 \end{bmatrix}$$

Matrix Multiplication

Product of Two Matrices

Let A be an m x n matrix and let B be an n x k matrix. To find the element in the i^{th} row and j^{th} column of the product matrix AB, multiply each element in the i^{th} row of A by the corresponding element in the j^{th} column of B, and then add these products. The product matrix AB is an m x k matrix.

Matrix ***A*** *Matrix B*

$m \times n$ $n \times k$

Inner *must be equal*

Outer: Order of AB is m x k

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$2x2 \qquad 2x2 \quad \rightarrow \quad 2x2$

$$a_{11} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & \_ \\ \_ & \_ \end{bmatrix}$$

$$a_{12} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & af+bh \\ \_ & \_ \end{bmatrix}$$

$$a_{21} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ ce+dg & \_ \end{bmatrix}$$

$$a_{22} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ \_ & cf+dh \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Example

Given: $A = \begin{bmatrix} 1 & -3 \\ 7 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 4 & -1 \end{bmatrix}$

Find ***AB*** and ***BA***.

<u>Solution</u>

$$\boldsymbol{AB} = \begin{bmatrix} 1 & -3 \\ 7 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1)+(-3)3 & 1(0)+(-3)1 & 1(-1)+(-3)4 & 1(2)+(-3)(-1) \\ 7(1)+2(3) & 7(0)+2(1) & 7(-1)+2(4) & 7(2)+2(-1) \end{bmatrix}$$

$$= \begin{bmatrix} -8 & -3 & -13 & 5 \\ 13 & 2 & 1 & 12 \end{bmatrix}$$

BA can be found since: B: $2x4$ and A: $2x2$

Note : $AB \neq BA$

Example

Given: A = $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ B = $\begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix}$

Find ***AB***.

<u>Solution</u>

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1(4)+3(1) & 1(6)+3(0) \\ 2(4)+5(1) & 2(6)+5(0) \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 6 \\ 13 & 12 \end{bmatrix}$$

Example

Given: $A=\begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix}$ $B=\begin{bmatrix} 1 & 6 \\ 3 & -5 \\ -2 & 4 \end{bmatrix}$ Find $\boldsymbol{AB}$.

<u>*Solution*</u>

$$AB=\begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 6 \\ 3 & -5 \\ -2 & 4 \end{bmatrix}$$

$$=\begin{bmatrix} 3(1)+1(3)-1(-2) & 3(6)+1(-5)-1(4) \\ 2(1)+0(3)+3(-2) & 2(6)+0(-5)+3(4) \end{bmatrix}$$

$$=\begin{bmatrix} 8 & 9 \\ -4 & 24 \end{bmatrix}$$

Example

Suppose A is a 3*x*2 matrix, while B is a 2*x*4 matrix.

a) Can the product AB be calculated

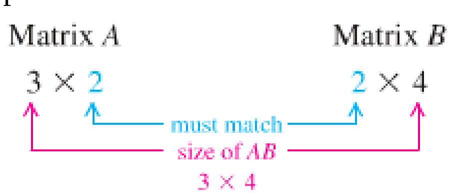
b) If AB can be calculated, what size is it?

c) Can BA be calculated?

d) If BA can be calculated, what size is it?

<u>*Solution*</u>

a) Can the product AB be calculated



b) If AB can be calculated, what size is it?

The product AB is 3 *x* 4

c) Can BA be calculated?

d) If BA can be calculated, what size is it?

Can't be calculated

Example

Given: $A_{1x3} = \begin{bmatrix} 2 & 0 & 4 \end{bmatrix}$ $B_{3x1} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$ Find ***AB*** and ***BA***.

<u>Solution</u>

A B

1 *x* 3 3 *x* 1

Inner: Are equal

Outer: Order of AB is 1x1

AB = [2(1) + 0(3) + 4(7)] = [30]

BA : 3*x*1 ---- 1*x*3

$$\boldsymbol{BA} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} \begin{bmatrix} 2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1(2) & 1(0) & 1(4) \\ 3(2) & 3(0) & 3(4) \\ 7(2) & 7(0) & 7(4) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 4 \\ 6 & 0 & 12 \\ 14 & 0 & 28 \end{bmatrix}$$

Example

Given: $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 & -1 & 6 \\ 0 & 5 & 4 & 1 \end{bmatrix}$ Find ***AB*** and ***BA***.

<u>Solution</u>

a) $\boldsymbol{AB} = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 & 6 \\ 0 & 5 & 4 & 1 \end{bmatrix}$ 2*x*2 --- 2*x*4

$$= \begin{bmatrix} 1(2)+3(0) & 1(3)+3(5) & 1(-1)+3(4) & 1(6)+3(1) \\ 0(2)+2(0) & 0(3)+2(5) & 0(-1)+2(4) & 0(6)+2(1) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 18 & 11 & 9 \\ 0 & 10 & 8 & 2 \end{bmatrix}$$

b) $\boldsymbol{BA} = \begin{bmatrix} 2 & 3 & -1 & 6 \\ 0 & 5 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ + Undefined *2x4 --- 2x2 (Inner order are not equal 2, 4)*

Properties of Matrix

Addition and Scalar Multiplication

$A+B=B+A$	*Commutative Property of Addition*
$A+(B+C)=(A+B)+C$	*Associative Property of Addition*
$(kl)A=k(lA)$	*Associative Property of Scalar Multiplication*
$k(A+B)=kA+kB$	*Distributive Property*
$(k+l)A=kA+lA$	*Distributive Property*
$A+0=0+A=A$	*Additive Identity Property*
$A+(-A)=(-A)+A=0$	*Additive Inverse Property*

Multiplication

$A(BC)=(AB)C$	*Associative Property of Multiplication*
$A(B+C)=AB+AC$	*Distributive Property*
$(B+C)A=BA+CA$	*Distributive Property*

Exercises *Section* 4.2 – Matrix operations and Their Applications

1. Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$$

2. Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} x & y+3 \\ 2z & 8 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 6 & 8 \end{bmatrix}$$

3. Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} 5 & x-4 & 9 \\ 2 & -3 & 8 \\ 6 & 0 & 5 \end{bmatrix} = \begin{bmatrix} y+3 & 2 & 9 \\ z+4 & -3 & 8 \\ 6 & 0 & w \end{bmatrix}$$

4. Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} a+2 & 3b & 4c \\ d & 7f & 8 \end{bmatrix} + \begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$$

5. Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} a+11 & 12z+1 & 5m \\ 11k & 3 & 1 \end{bmatrix} + \begin{bmatrix} 9a & 9z & 4m \\ 12k & 5 & 3 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

6. $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -3 & 6 \\ -3 & 1 & -4 \end{bmatrix}$ Find: $A-B, \quad 3A+2B$

7. Find $\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + \begin{bmatrix} -6 \\ 3 \\ 12 \end{bmatrix}$

8. Find $\begin{bmatrix} 5 & 8 \\ 6 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 9 & 1 \\ 4 & 2 & 5 \end{bmatrix}$

9. Find $\begin{bmatrix} -5 & 0 \\ 4 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & 3 \end{bmatrix}$

10. $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ $\quad F = \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix}$ $\quad$ Find: $3F+2A$

11. Find $\begin{bmatrix} -4 & 3 \\ 12 & -6 \end{bmatrix} + \begin{bmatrix} 2 & -8 \\ 5 & 10 \end{bmatrix}$

12. Find $\begin{bmatrix} -5 & 6 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -3 & 2 \\ 5 & -8 \end{bmatrix}$

13. Find $\begin{bmatrix} 8 & 6 & -4 \end{bmatrix} - \begin{bmatrix} 3 & 5 & -8 \end{bmatrix}$

14. $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & -3 & 6 \\ -3 & 1 & -4 \end{bmatrix}$ $\quad$ Find: $3A + 2B$

15. Given $A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ *and* $B = \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}$. Find *AB* and *BA*.

16. Find $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix}$

17. Find $\begin{bmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{bmatrix}$

18. Find $\begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix} . \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$

19. Find $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix} . \begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix}$

20. Find $\begin{bmatrix} -2 & -3 & -4 \\ 2 & -1 & 0 \\ 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix}$

21. Find $\begin{bmatrix} \sqrt{2} & \sqrt{2} & -\sqrt{18} \\ \sqrt{3} & \sqrt{27} & 0 \end{bmatrix} \begin{bmatrix} 8 & -10 \\ 9 & 12 \\ 0 & 2 \end{bmatrix}$

22. A contractor builds three kinds of houses, models *A*, *B*, and *C*, with a choice of two styles, Spanish and contemporary. Matrix *P* shows the number of each kind of house planned for a new 100-home subdivision. The amounts for each of the exterior materials depend primarily on the style of the house. These amounts are shown in matrix *Q*. (concrete is in cubic yards, lumber in units of 1000 board feet, brick in 1000s, and shingles in units of 100 ft^2.) Matrix *R* gives the cost in dollars for each kind of material.

a) What is the total cost of these materials for each model?

b) How much of each of four kinds of material must be ordered

c) What is the total cost for exterior materials?

23. Mitchell Fabricators manufactures three styles of bicycle frames in its two plants. The following table shows the number of each style produced at each plant

	Mountain Bike	Racing Bike	Touring Bike
North Plant	150	120	100
South Plant	180	90	130

a) Write a 2 *x* 3 matrix ***A*** that represents the information in the table

b) The manufacturer increased production of each style by 20%. Find a Matrix ***M*** that represents the increased production figures.

c) Find the matrix ***A*** **+** ***M*** and tell what it represents

24. Sal's Shoes and Fred's Footwear both have outlets in California and Arizona. Sal's sells shoes for $80, sandals for $40, and boots for $120. Fred's prices are $60, $30, and $150 for shoes, sandals and boots, respectively. Half of all sales in California stores are shoes, 1/4 are sandals, and 1/4 are boots. In Arizona the fractions are 1/5 shoes, 1/5 are sandals, and 3/5 are boots.

a) Write a 2 x 3 matrix called P representing prices for the two stores and three types of footwear.

b) Write a 2 x 3 matrix called F representing fraction of each type of footwear sold in each state.

c) Only one of the two products PF and FP is meaningful. Determine which one it is, calculate the product, and describe what the entries represent.

Section 4.3 – Multiplicative Inverses of Matrices

Identity Matrix

The $n \times n$ identity matrix with 1's on the main diagonal and 0's elsewhere and is denoted by

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2x2} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3x3} \qquad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{I} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

The Multiplicative Identity Matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Then $AI = IA = A$

Example

$$A = \begin{bmatrix} 4 & -7 \\ -3 & 2 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

<u>*Solution*</u>

$$AI = \begin{bmatrix} 4 & -7 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4(1)-7(0) & 4(0)-7(1) \\ -3(1)+2(0) & -3(0)+2(1) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -7 \\ -3 & 2 \end{bmatrix} = A$$

$$= A$$

Multiplicative inverse of a matrix

Multiplicative inverse of a matrix A_{nxn} and $A^{-1}{}_{nxn}$ if exists, then:

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

Example

Show that B is Multiplicative inverse of A.

A = $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ B = $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

Solution

$$\boldsymbol{A.B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2(1)-1(1) & 2(-1)+1(2) \\ 1(1)+1(-1) & 1(-1)+1(2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

$$\text{BA} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

$\therefore$ B is multiplicative inverse of a matrix A: $B = A^{-1}$

Finding Inverse matrix

To find inverse matrix using Gauss-Jordan method:

$$\left[A \middle| I\right] \rightarrow \left[I \middle| A^{-1}\right] \qquad \textit{where } A^{-1} \textit{ read as "}A \textit{ inverse"}$$

For 2 by 2 matrices (*only*)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

If $ad - bc = 0$, then A^{-1} doesn't exist

Example

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix} \Rightarrow A^{-1} = ?$$

<u>*Solution*</u>

$$A^{-1} = \frac{1}{(-1)(4) - (-2)(3)}\begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \Rightarrow A^{-1} = ?$$

<u>*Solution*</u>

$$A^{-1} = \frac{1}{(3)(1) - (-2)(-1)}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

To find inverse matrix using Gauss-Jordan method:

$$\left[A \middle| I\right] \rightarrow \left[I \middle| A^{-1}\right] \qquad where\ A^{-1}\ read\ as\ "A\ inverse"$$

Example

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix} \qquad \text{Find } A^{-1}$$

<u>*Solution*</u>

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ R_2 + R_1 \\ R_3 - R_1 \end{matrix}$$

$$\begin{array}{cccccc} -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 \\ \hline 0 & 2 & 5 & 1 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 1 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & -1 & 0 & 0 \\ \hline 0 & -1 & -2 & -1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right] \tfrac{1}{2}R_2$$

$$\begin{array}{cccccc} 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right] R_3 + R_2$$

$$\begin{array}{cccccc} 0 & -1 & -2 & -1 & 0 & 1 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ \hline 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}\right] 2R_3$$

$$\begin{array}{cccccc} 0 & 0 & 1 & -1 & 1 & 2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right] \begin{matrix} R_1 - 2R_3 \\ R_2 - \frac{5}{2}R_3 \\ \\ \end{matrix}$$

$$\begin{array}{cccccc} 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{5}{2} & \frac{5}{2} & -\frac{5}{2} & -5 \\ \hline 0 & 1 & 0 & 3 & -2 & -5 \end{array} \qquad \begin{array}{cccccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 2 & -2 & -4 \\ \hline 1 & 0 & 0 & 3 & -2 & -4 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 1 & 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right]$$

$$\Rightarrow \qquad A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

Solving a System Using A^{-1}

To solve the matrix equation $AX = B$.

- $\boldsymbol{X}$: matrix of the variables
- $\boldsymbol{A}$: Coefficient matrix
- $\boldsymbol{B}$: Constant matrix

$$AX = B$$

$$A^{-1}(AX) = A^{-1}B \qquad \textit{Multiply both side by } A^{-1}$$

$$\left(A^{-1}A\right)X = A^{-1}B \qquad \textit{Associate property}$$

$$IX = A^{-1}B \qquad \textit{Multiplicative inverse property}$$

$$X = A^{-1}B \qquad \textit{Identity property}$$

Example

Solve the system using A^{-1}

$$\begin{cases} x + 2z = 6 \\ -x + 2y + 3z = -5 \\ x - y = 6 \end{cases} \qquad \text{Given } A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

<u>*Solution*</u>

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} \underset{=}{=} \underbrace{\begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix}}_{B}$$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} . \begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix} = \begin{bmatrix} 3(6)-2(-5)-4(6) \\ 3(6)-2(-5)-5(6) \\ -1(6)+1(-5)+2(6) \end{bmatrix} = \begin{bmatrix} 18+10-24 \\ 18+10-30 \\ -6-5+12 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

Solution: $\{(4, -2, 1)\}$

Example

Use the inverse of the coefficient matrix to solve the linear system

$$\begin{aligned} 2x - 3y &= 4 \\ x + 5y &= 2 \end{aligned}$$

<u>*Solution*</u>

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \end{bmatrix} \qquad B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

The solution of the system is $(2, 0)$

Exercise *Section* 4.3 – Multiplicative Inverses of Matrices

1. Show that B is Multiplicative inverse of A

$$A=\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \qquad B=\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

2. $A=\begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix} \Rightarrow A^{-1}=?$

3. $A=\begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix} \Rightarrow A^{-1}=?$

4. Find the inverse of $A=\begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$

5. Find the inverse of $A=\begin{bmatrix} a & b \\ 3 & 3 \end{bmatrix}$

6. Find the inverse of $A=\begin{bmatrix} -2 & a \\ 4 & a \end{bmatrix}$

7. Find the inverse of $A=\begin{bmatrix} 4 & 4 \\ b & a \end{bmatrix}$

8. Find A^{-1} *if* $A=\begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

9. Find A^{-1}, where $A=\begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{bmatrix}$

10. Find A^{-1}, where $A=\begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

11. Find A^{-1}, where $A=\begin{bmatrix} -2 & 5 & 3 \\ 4 & -1 & 3 \\ 7 & -2 & 5 \end{bmatrix}$

12. Solve the system using A^{-1} $\begin{cases} x \quad\quad +2z=6 \\ -x+2y+3z=-5 \\ x-y \quad\quad =6 \end{cases}$ Given $A^{-1}=\begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

13. Solve the system using A^{-1} $\begin{aligned} x+2y+5z&=2 \\ 2x+3y+8z&=3 \\ -x+\ \ y+2z&=3 \end{aligned}$

a) Write the linear system as a matrix equation in the form *AX*= *B*

b) Solve the system using the inverse that is given for the coefficient matrix

$$\textit{the inverse of } \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{bmatrix} \textit{ is } \begin{bmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{bmatrix}$$

14. Solve the system using A^{-1} $\begin{aligned} x-y+z&=8 \\ 2y-z&=-7 \\ 2x+3y\quad &=1 \end{aligned}$

a) Write the linear system as a matrix equation in the form *AX*= *B*

b) Solve the system using the inverse that is given for the coefficient matrix

$$\textit{the inverse is } \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

15. Use the inverse of the coefficient matrix to solve the linear system

$$\begin{aligned} x\qquad\quad +z&=-1 \\ 2x-2y-z&=5 \\ 3x\qquad\qquad &=6 \end{aligned}$$

Section 4.4 – Determinants and Cramer's Rule

Determinant of a 2 *x* 2 Matrix

Determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and is define as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example

Let $A = \begin{bmatrix} -3 & 4 \\ 6 & 8 \end{bmatrix}$. Find $|A|$

<u>*Solution*</u>

$$\begin{aligned} |A| &= \begin{vmatrix} -3 & 4 \\ 6 & 8 \end{vmatrix} \\ &= -3(8) - 4(6) \\ &= -48 \end{aligned}$$

Evaluate: $\begin{vmatrix} 2 & -3 \\ -4 & 1 \end{vmatrix}$ = 2(1) – (-3)(-4)

= 2 – 12

= -10

$$\boldsymbol{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Minor

For a square matrix $A = \left[a_{ij}\right]$, the minor M_{ij}. Of an element a_{ij} is the determinant of the matrix formed by deleting the i^{th} row and the j^{th} column of A.

Cofactor: $A_{ij} = (-1)^{i+j} M_{ij}$

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example

$$|A| = \begin{vmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{vmatrix} = -8\begin{vmatrix} -6 & 7 \\ -3 & 5 \end{vmatrix} - 0\begin{vmatrix} 4 & 7 \\ -1 & 5 \end{vmatrix} + 6\begin{vmatrix} 4 & -6 \\ -1 & -3 \end{vmatrix}$$

$= -8(-30 - (-21)) - 0 + 6(-12 - 6)$

$= -8(-9) + 6(-18)$

$= -36$

Determinant Using Diagonal Method

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} \qquad a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \quad (1)$$

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} \qquad a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} \quad (2)$$

Determinant: $\boldsymbol{D = (1) - (2)}$

Example

$$\left|\begin{array}{ccc} 2 & -3 & -2 \\ -1 & -4 & -3 \\ -1 & 0 & 2 \end{array}\right| \begin{array}{cc} 2 & -3 \\ -1 & -4 \\ -1 & 0 \end{array} = 2(-4)(2) + (-3)(-3)(-1) + (-2)(-1)(0) - (-2)(-4)(-1) - (2)(-3)(0) - (-3)(-1)(2)$$

$$\underline{= -23|}$$

Example

$$\left|\begin{array}{ccc} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{array}\right| \begin{array}{cc} -8 & 0 \\ 4 & -6 \\ -1 & -3 \end{array} = (-8)(-6)(-5) + 0(7)(-1) + 6(4)(-3) - 6(-6)(-1) - (-8)(7)(-3) - 0(4)(5)$$

$$\underline{= -36|}$$

Example

Evaluate $\left|\begin{array}{ccc} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{array}\right|$

<u>*Solution*</u>

$$\left|\begin{array}{ccc} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{array}\right| \begin{array}{cc} x & 0 \\ 2 & x \\ -3 & x \end{array} = x^2 + 0 - 2x - (3x) - x^4 - 0$$

$$\underline{= -x^4 + x^2 - 5x|}$$

Cramer's Rule

Given:

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

If $D \neq 0 \qquad x = \dfrac{D_x}{D} \qquad y = \dfrac{D_y}{D}$

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \qquad D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \qquad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Example

Use Cramer's rule to solve the system

$$5x + 7y = -1$$
$$6x + 8y = 1$$

Solution

$$D = \begin{vmatrix} 5 & 7 \\ 6 & 8 \end{vmatrix} = -2$$

$$D_x = \begin{vmatrix} -1 & 7 \\ 1 & 8 \end{vmatrix} = -15$$

$$D_y = \begin{vmatrix} 5 & -1 \\ 6 & 1 \end{vmatrix} = 11$$

$$x = \frac{D_x}{D} = \frac{-15}{-2} = \frac{15}{2}$$

$$y = \frac{D_y}{D} = \frac{11}{-2} = -\frac{11}{2}$$

Solution: $\boxed{\left(\frac{15}{2},\ -\frac{11}{2}\right)}$

$$D_x = \begin{matrix} b_1 & a_{12} & a_{13} & b_1 & a_{12} \\ b_2 & a_{22} & a_{23} & b_2 & a_{22} \\ b_3 & a_{32} & a_{33} & b_3 & a_{32} \end{matrix}$$

$$D_x = b_1 a_{22} a_{33} + a_{12} a_{23} b_3 + a_{13} b_2 a_{32} - a_{13} a_{22} b_3 - b_1 a_{23} a_{32} - a_{12} b_2 a_{33}$$

$$D_y = \begin{matrix} a_{11} & b_1 & a_{13} & a_{11} & b_1 \\ a_{21} & b_2 & a_{23} & a_{21} & b_2 \\ a_{31} & b_3 & a_{33} & a_{31} & b_3 \end{matrix}$$

$$D_z = \begin{matrix} a_{11} & a_{12} & b_1 & a_{11} & a_{12} \\ a_{21} & a_{22} & b_2 & a_{21} & a_{22} \\ a_{31} & a_{32} & b_3 & a_{31} & a_{32} \end{matrix}$$

$$x = \frac{D_x}{D} \qquad y = \frac{D_y}{D} \qquad z = \frac{D_z}{D}$$

Example

Use Cramer's rule to solve the system

$x - 3y + 7z = 13$
$x + y + z = 1$
$x - 2y + 3z = 4$

<u>*Solution*</u>

$$D = \begin{vmatrix} 1 & -3 & 7 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = -10 \qquad D_x = \begin{vmatrix} 13 & -3 & 7 \\ 1 & 1 & 1 \\ 4 & -2 & 3 \end{vmatrix} = 20$$

$$D_y = \begin{vmatrix} 1 & 13 & 7 \\ 1 & 1 & 1 \\ 1 & 4 & 3 \end{vmatrix} = -6 \qquad D_z = \begin{vmatrix} 1 & -3 & 13 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{vmatrix} = -24$$

$$x = \frac{20}{-10} = -2$$

$$y = \frac{-6}{-10} = \frac{3}{5}$$

$$z = \frac{-24}{-10} = \frac{12}{5}$$

Solution: $\boxed{\left(-2, \frac{3}{5}, \frac{12}{5}\right)}$

Exercises *Section* 4.4 – Determinants and Cramer's Rule

Evaluate

1. $\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix}$

2. $\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix}$

3. $\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix}$

4. $\begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix}$

5. $\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix}$

6. $\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix}$

7. $\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix}$

8. $\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix}$

9. $\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix}$

10. $\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix}$

11. $\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix}$

12. Use Cramer's rule to solve the system $\begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$

13. Use Cramer's rule to solve the system $\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$

14. Use Cramer's rule to solve the system $\begin{cases} 3x + 2y - z = 4 \\ 3x - 2y + z = 5 \\ 4x - 5y - z = -1 \end{cases}$

15. Find the quadratic function $f(x) = ax^2 + bx + c$ for which $f(1) = -10,\quad f(-2) = -31,\quad f(2) = -19$. What is the function?