Section 1.2 – Separable Equations

Separable Equation

Separable equation is an equation that can be written with its variables separated and then easily solved.

If f is independent of $y \Rightarrow y' = \frac{dy}{dx} = f(x, y)$ is separable equation if f has the form

$$f(x, y) = g(x)h(y)$$

Definition

A 1st order differential equation of the form $\frac{dy}{dx} = g(x)h(y)$ is said to be separable or to have separable variables.

$$\frac{dy}{h(y)} = g(x)dx$$

$$\frac{dy}{dx} = y^2 xe^{3x+4y} = \left(xe^{3x}\right)\left(y^2 e^{4y}\right)$$

$$\frac{dy}{dx} = y + \sin x \qquad \textit{not separable}$$

Example

At time t the sample contains $N(t)$ radioactive nuclei and is given by the differential equation:

$$N' = -\lambda N$$

This is called the ***exponential equation***.

$$N' = -\lambda N$$

$$\frac{dN}{dt} = -\lambda N \qquad \textbf{\textit{Separable equation}}$$

$$\frac{dN}{N} = -\lambda dt$$

$$\int \frac{dN}{N} = -\int \lambda dt$$

$$\ln|N| = -\lambda t + C$$

$$\begin{aligned} |N(t)| &= e^{-\lambda t + C} \\ &= e^C e^{-\lambda t} \end{aligned}$$

$$N(t) = \begin{cases} e^{C}e^{-\lambda t} & if \ \ N > 0 \\ -e^{C}e^{-\lambda t} & if \ \ N < 0 \end{cases}$$

$$N(t) = Ae^{-\lambda t} \qquad A = \begin{cases} e^{C} & if \ \ N > 0 \\ -e^{C} & if \ \ N < 0 \end{cases}$$

Example

Solve the differential equation $y' = ty^2$

Solution

$$\frac{dy}{dt} = ty^2$$

$$\frac{dy}{y^2} = tdt$$

$$\int y^{-2}dy = \int tdt$$

$$-y^{-1} = \frac{1}{2}t^2 + C$$

$$-\frac{1}{y} = \frac{t^2+2C}{2} \qquad \textit{Cross multiplication}$$

$$-\frac{2}{t^2+2C} = y$$

$$y(t) \boxed{= -\frac{2}{t^2+2C}}$$

General Method

Step 1: Establish that the equation is separate $\frac{dy}{dx} = g(x)h(y)$

Step 2: Divide both sides by $h(y)$ to separate the variables $\frac{dy}{h(y)} = g(x)dx$

Step 3: Integrate both sides $\int \frac{dy}{h(y)} = \int g(x)dx$

Step 4: Solve for the solution $y(t)$, if possible

Losing a solution

When we use separate variables, the variable divisors could be zero at a point.

Example

Find a general solution to $\frac{dy}{dx} = y^2 - 4$

Solution

$$\frac{dy}{y^2 - 4} = dx$$

$$\left(\frac{1/4}{y-2} - \frac{1/4}{y+2} \right) dy = dx \qquad y = \pm 2 \text{ Critical points}$$

$$\frac{1}{4}\left(\int \frac{dy}{y-2} - \int \frac{dy}{y+2} \right) = \int dx$$

$$\frac{1}{4}\left[\ln|y-2| - \ln|y+2| \right] = x + c_1$$

$$\ln\left| \frac{y-2}{y+2} \right| = 4x + c_2$$

$$\left| \frac{y-2}{y+2} \right| = e^{4x+c_2}$$

$$\frac{y-2}{y+2} = \pm e^{c_2} e^{4x}$$

$$y - 2 = Ce^{4x}(y+2)$$

$$y - Ce^{4x} y = 2Ce^{4x} + 2$$

$$\left(1 - Ce^{4x}\right) y = 2\left(Ce^{4x} + 1\right)$$

$$\boxed{y = 2\frac{1+Ce^{4x}}{1-Ce^{4x}}}$$

If $y = -2 \Rightarrow -2 = 2\frac{1+Ce^{4x}}{1-Ce^{4x}}$

$$-1 = \frac{1+Ce^{4x}}{1-Ce^{4x}}$$

$$-1 + Ce^{4x} = 1 + Ce^{4x} \quad \Rightarrow \qquad -1 = 1 \quad \textit{impossible}$$

If $y = 2 \qquad \Rightarrow 2 = 2\frac{1+Ce^{4x}}{1-Ce^{4x}}$

$$1 - Ce^{4x} = 1 + Ce^{4x}$$

$$-Ce^{4x} = Ce^{4x} \quad \Rightarrow \quad -C = C$$

$$\boxed{y = 2 \Rightarrow C = 0}$$

Implicitly Defined Solutions

Example

Find the solutions of the equation $y' = \dfrac{e^x}{1+y}$, having initial conditions $y(0)=1$ and $y(0)=-4$

Solution

$$\frac{dy}{dx} = \frac{e^x}{1+y}$$

$$(1+y)\,dy = e^x dx$$

$$\int (1+y)\,dy = \int e^x dx$$

$$y + \tfrac{1}{2}y^2 = e^x + c$$

$$y^2 + 2y - 2\left(e^x + c\right) = 0$$

$$y(x) = \tfrac{1}{2}\left(-2 \pm \sqrt{4 + 8\left(e^x + c\right)}\right) \qquad \textit{Quadratic Formula}$$

$$= -1 \pm \sqrt{1 + 2\left(e^x + c\right)} \qquad \textit{Implicit}$$

$$y(0) = -1 + \sqrt{1 + 2\left(e^0 + c\right)} = 1$$

$$\sqrt{1 + 2(1+c)} = 2$$

$$1 + 2 + 2c = 4$$

$$2c = 1$$

$$\boxed{c = \frac{1}{2}}$$

$$y(0) = -1 - \sqrt{1 + 2\left(e^0 + c\right)} = -4$$

$$-\sqrt{1 + 2 + 2c} = -3$$

$$1 + 2 + 2c = 9$$

$$2c = 6$$

$$\boxed{c = 3}$$

$$\begin{cases} y(t) = -1 + \sqrt{2 + 2e^x} \\ y(t) = -1 - \sqrt{7 + 2e^x} \end{cases}$$

$\therefore y \neq -1$ *from* y', but it never it will be.

Explicit Solutions: $y = -1 + \sqrt{}$

Notes

1. $Q(y) = P(x) + C$ is the general solution. Typically, this is an ***implicit*** relation; we may or may not be able to solve it for y.

2. $h(y) = 0$ is a source of singular solutions:

If k is a number such that $h(k) = 0$, *then* $y = k$

Might be a ***singular solution***.

Example

Find the general solution and any singular solutions: $y' - xy^2 = x$

Solution

$$y' = x + xy^2$$

$$\frac{dy}{dx} = x\left(1 + y^2\right)$$

$$\frac{dy}{1+y^2} = xdx$$

$$\int \frac{dy}{1+y^2} = \int xdx \qquad 1 + y^2 \neq 0$$

$$\tan^{-1} y = \frac{1}{2}x^2 + C$$

$$\boxed{y = \tan\left(\frac{1}{2}x^2 + C\right)} \qquad \textit{No singular solutions}$$

Example

Find the general solution and any singular solutions: $\frac{1}{x}y' = e^x\sqrt{y+1}$

Solution

	$\int e^x$
x	e^x
1	e^x

$$\int \frac{dy}{\sqrt{y+1}} = \int xe^x dx$$

$$\int \frac{d(y+1)}{\sqrt{y+1}} = xe^x + e^x + C$$

$$\boxed{2\sqrt{y+1} = xe^x + e^x + C}$$

$$h(y) = y + 1 = 0 \quad \Rightarrow \quad y = -1 \textit{ is a singular solution}$$

Example

Find the general solution and any singular solutions: $y' = \dfrac{xy^2 - x}{y}$

Solution

$$\frac{dy}{dx} = \frac{x\left(y^2 - 1\right)}{y}$$

$$\frac{y}{y^2 - 1}dy = xdx$$

$$\frac{1}{2}\int \frac{1}{y^2 - 1}d\left(y^2 - 1\right) = \int xdx$$

$$\frac{1}{2}\ln\left|y^2 - 1\right| = \frac{1}{2}x^2 + \frac{1}{2}\ln C$$

$$\ln\left|y^2 - 1\right| - \ln C = x^2$$

$$\ln\frac{\left|y^2 - 1\right|}{C} = x^2$$

$$\frac{\left|y^2 - 1\right|}{C} = e^{x^2}$$

$$y^2 - 1 = Ce^{x^2}$$

$$\boxed{y^2 = Ce^{x^2} + 1}$$

Singular Solutions:

$$\frac{y^2 - 1}{y} = 0 \quad \Rightarrow \quad y = \pm 1$$

For $y = 1$: $\;\; if \;\; C = 0 \quad y = 1$

For $y = -1$: $\;\; No \;\; C$

No singular solution.

Example

Find the solutions to the differential equation $x' = \frac{2tx}{1+x}$, having $x(0) = 1, \ -2, \ 0$

Solution

$$\frac{dx}{dt} = \frac{2tx}{1+x}$$

$$\frac{1+x}{x}dx = 2tdt$$

$$\left(\frac{1}{x}+1\right)dx = 2tdt$$

$$\int\left(\frac{1}{x}+1\right)dx = \int 2tdt$$

$$\ln|x| + x = t^2 + c$$

For $x(0) = 1$

$$1 = 0^2 + c$$

$$c = 1$$

$$\ln|x| + x = t^2 + c \qquad x > 0$$

We can't solve for $x(t)$

$\Rightarrow$ This solution is defined as implicit.

For $x(0) = -2$

$$\ln|-2| + (-2) = 0^2 + c$$

$$c = -2 + \ln 2$$

$$\ln|x| + x = t^2 - 2 + \ln 2$$

Since the initial condition < 0, then:

$$x + \ln(-x) = t^2 - 2 + \ln 2$$

For $x(0) = 0$

$0 = 0^2 + c$ *True statement*

$y' = 0 \ \Rightarrow \ x(t) = 0$ is a solution

Exercises *Section* 1.2 – Separable Equations

Find the general solution of the differential equation.

1. $y' = xy$
2. $xy' = 2y$
3. $y' = e^{x-y}$
4. $y' = \left(1+y^2\right)e^x$
5. $y' = xy + y$
6. $y' = ye^x - 2e^x + y - 2$
7. $y' = \dfrac{x}{y+2}$
8. $y' = \dfrac{xy}{x-1}$
9. $y' = \dfrac{y^2 + ty + t^2}{t^2}$
10. $\dfrac{dy}{dx} = \dfrac{4x - x^3}{4 + y^3}$
11. $y' = \dfrac{2xy + 2x}{x^2 - 1}$
12. $\dfrac{dy}{dx} = \sin 5x$
13. $\dfrac{dy}{dx} = (x+1)^2$
14. $dx + e^{3x}dy = 0$
15. $dy - (y-1)^2\, dx = 0$
16. $x\dfrac{dy}{dx} = 4y$
17. $\dfrac{dx}{dy} = y^2 - 1$
18. $\dfrac{dy}{dx} = e^{2y}$
19. $\dfrac{dy}{dx} + 2xy^2 = 0$
20. $\dfrac{dy}{dx} = e^{3x+2y}$
21. $e^x y\dfrac{dy}{dx} = e^{-y} + e^{-2x-y}$
22. $y\ln x\dfrac{dx}{dy} = \left(\dfrac{y+1}{x}\right)^2$
23. $\dfrac{dy}{dx} = \left(\dfrac{2y+3}{4x+5}\right)^2$
24. $\csc y dx + \sec^2 x dy = 0$
25. $\sin 3x dx + 2y\cos^3 3x dy = 0$
26. $\left(e^y + 1\right)^2 e^{-y}dx + \left(e^x + 1\right)^3 e^{-x}dy = 0$
27. $x\left(1+y^2\right)^{1/2} dx = y\left(1+x^2\right)^{1/2} dy$
28. $\dfrac{dy}{dx} = y\sin x$
29. $(1+x)\dfrac{dy}{dx} = 4y$
30. $2\sqrt{x}\dfrac{dy}{dx} = \sqrt{1-y^2}$
31. $\dfrac{dy}{dx} = 3\sqrt{xy}$
32. $\dfrac{dy}{dx} = (64xy)^{1/3}$
33. $\dfrac{dy}{dx} = 2x\sec y$
34. $\left(1-x^2\right)\dfrac{dy}{dx} = 2y$
35. $(1+x)^2\dfrac{dy}{dx} = (1+y)^2$
36. $\dfrac{dy}{dx} = xy^3$
37. $y\dfrac{dy}{dx} = x\left(y^2+1\right)$
38. $y^3\dfrac{dy}{dx} = \left(y^4+1\right)\cos x$
39. $\dfrac{dy}{dx} = \dfrac{1+\sqrt{x}}{1+\sqrt{y}}$
40. $\dfrac{dy}{dx} = \dfrac{(x-1)y^5}{x^2\left(2y^3 - y\right)}$
41. $\left(x^2+1\right)(\tan y)y' = x$
42. $x^2y' = 1 - x^2 + y^2 - x^2y^2$

43. $xy' + 4y = 0$

44. $\left(x^2+1\right)y' + 2xy = 0$

45. $\dfrac{y'}{\left(x^2+1\right)y} = 3$

46. $y + e^x y' = 0$

47. $\dfrac{dx}{dt} = 3xt^2$

48. $x\dfrac{dy}{dx} = \dfrac{1}{y^3}$

49. $\dfrac{dy}{dx} = \dfrac{x}{y^2\sqrt{x+1}}$

50. $\dfrac{dx}{dt} - x^3 = x$

51. $\dfrac{dy}{dx} = \dfrac{x}{ye^{x+2y}}$

52. $\dfrac{dy}{dx} = \dfrac{\sec^2 y}{1+x^2}$

53. $x\dfrac{dv}{dx} = \dfrac{1-4v^2}{3v}$

54. $\dfrac{dy}{dx} = 3x^2\left(1+y^2\right)^{3/2}$

55. $\dfrac{1}{y}dy + ye^{\cos x}\sin x dx = 0$

56. $\left(x + xy^2\right)dx + e^{x^2} y dy = 0$

Find the exact solution of the initial value problem. Indicate the interval of existence.

57. $y' = \dfrac{y}{x}$, $y(1) = -2$

58. $y' = -\dfrac{2t\left(1+y^2\right)}{y}$, $y(0) = 1$

59. $y' = \dfrac{\sin x}{y}$, $y\left(\frac{\pi}{2}\right) = 1$

60. $4tdy = \left(y^2 + ty^2\right)dt$, $y(1) = 1$

61. $y' = \dfrac{1-2t}{y}$, $y(1) = -2$

62. $y' = y^2 - 4$, $y(0) = 0$

63. $\dfrac{dy}{dx} = \dfrac{3x^2+4x+2}{2(y-1)}$, $y(0) = -1$

64. $y' = \dfrac{x}{1+2y}$, $y(-1) = 0$

65. $\left(e^{2y} - y\right)\cos x\dfrac{dy}{dx} = e^y \sin 2x$, $y(0) = 0$

66. $\dfrac{dy}{dx} = e^{-x^2}$, $y(3) = 5$

67. $\dfrac{dy}{dx} + 2y = 1$, $y(0) = \dfrac{5}{2}$

68. $\sqrt{1-y^2}dx - \sqrt{1-x^2}dy = 0$, $y(0) = \dfrac{\sqrt{3}}{2}$

69. $\left(1+x^4\right)dy + x\left(1+4y^2\right)dx = 0$, $y(1) = 0$

70. $\dfrac{1}{t^2}\dfrac{dy}{dt} = y$, $y(0) = 1$

71. $\dfrac{dy}{dt} = -y^2 e^{2t}$; $y(0) = 1$

72. $\dfrac{dy}{dt} - (2t+1)y = 0$; $y(0) = 2$

73. $\dfrac{dy}{dt} + 4ty^2 = 0$; $y(0) = 1$

74. $\dfrac{dy}{dx} = ye^x$; $y(0) = 2e$

75. $\dfrac{dy}{dx} = 3x^2\left(y^2+1\right)$; $y(0) = 1$

76. $2y\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2-16}}$; $y(5) = 2$

77. $\dfrac{dy}{dx} = 4x^3 y - y$; $y(1) = -3$

78. $\dfrac{dy}{dx} + 1 = 2y$; $y(1) = 1$

79. $(\tan x)\dfrac{dy}{dx} = y$; $y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

80. $e^{-2t}\dfrac{dy}{dt} = \dfrac{1+e^{-2t}}{y}$, $y(0) = 0$

81. $\dfrac{dy}{dt} = y\cos t + y$, $y(0) = 2$

82. $\dfrac{dy}{dt} = \dfrac{t+2}{y}$, $y(0) = 2$

83. $x\frac{dy}{dx} - y = 2x^2y;\quad y(1) = 1$

84. $\frac{dy}{dx} = 2xy^2 + 3x^2y^2;\quad y(1) = -1$

85. $\frac{dy}{dx} = 6e^{2x-y};\quad y(0) = 0$

86. $2\sqrt{x}\frac{dy}{dx} = \cos^2 y;\quad y(4) = \frac{\pi}{4}$

87. $y' + 3y = 0;\quad y(0) = -3$

88. $2y' - y = 0;\quad y(-1) = 2$

89. $2xy - y' = 0;\quad y(1) = 3$

90. $y\frac{dy}{dx} - \sin x = 0;\quad y\left(\frac{\pi}{2}\right) = -2$

91. $\frac{dy}{dt} = \frac{1}{y^2};\quad y(1) = 2$

92. $y' + \frac{1}{y+1} = 0;\quad y(1) = 0$

93. $y' + e^y t = e^y \sin t;\quad y(0) = 0$

94. $y' - 2ty^2 = 0;\quad y(0) = -1$

95. $\frac{dy}{dx} = 1 + y^2;\quad y\left(\frac{\pi}{4}\right) = -1$

96. $\frac{dy}{dt} = t - ty^2;\quad y(0) = \frac{1}{2}$

97. $3y^2\frac{dy}{dt} + 2t = 1;\quad y(-1) = -1$

98. $e^x y' + (\cos y)^2 = 0;\quad y(0) = \frac{\pi}{4}$

99. $(2y - \sin y)y' + x = \sin x;\quad y(0) = 0$

100. $e^y y' + \frac{x}{y+1} = \frac{2}{y+1};\quad y(1) = 2$

101. $(\ln y)y' + x = 1;\quad y(3) = e$

102. $y' = x^3(1 - y);\quad y(0) = 3$

103. $y' = \left(1 + y^2\right)\tan x;\quad y(0) = \sqrt{3}$

104. $\frac{1}{2}\frac{dy}{dx} = \sqrt{1+y}\,\cos x;\quad y(\pi) = 0$

105. $x^2\frac{dy}{dx} = \frac{4x^2 - x - 2}{(x+1)(y+1)};\quad y(1) = 1$

106. $\frac{1}{\theta}\frac{dy}{d\theta} = \frac{y\sin\theta}{y^2+1}\quad y(\pi) = 1$

107. $x^2dx + 2ydy = 0;\quad y(0) = 2$

108. $\frac{1}{t}\frac{dy}{dt} = 2\cos^2 y;\quad y(0) = \frac{\pi}{4}$

109. $\frac{dy}{dx} = 8x^3e^{-2y};\quad y(1) = 0$

110. $\frac{dy}{dx} = x^2(1 + y);\quad y(0) = 3$

111. $\sqrt{y}dx + (1 + x)dy = 0;\quad y(0) = 1$

112. $\frac{dy}{dx} = 6y^2x,\quad y(1) = \frac{1}{25}$

113. $\frac{dy}{dx} = \frac{3x^2 + 4x - 4}{2y - 4},\quad y(1) = 3$

114. $y' = e^{-y}(2x - 4)\quad y(5) = 0$

115. $\frac{dr}{d\theta} = \frac{r^2}{\theta},\quad r(1) = 2$

116. $\frac{dy}{dt} = e^{y-t}\left(1 + t^2\right)\sec y,\quad y(0) = 0$