Lecture Three

Section 3.1 – Inner Products

Definition

An ***inner product*** on a real vector space *V* is a function that associates a real number $\langle \boldsymbol{u},\ \boldsymbol{v} \rangle$ with each pair of vectors in *V* in such a way that the following axioms are satisfies for all vectors ***u***, ***v***, and ***w*** in *V* and all scalars *k*.

1. $\langle \boldsymbol{u},\ \boldsymbol{v} \rangle = \langle \boldsymbol{v},\ \boldsymbol{u} \rangle$ ***Symmetry axiom***
2. $\langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{w} \rangle = \langle \boldsymbol{u},\ \boldsymbol{w} \rangle + \langle \boldsymbol{v},\ \boldsymbol{w} \rangle$ ***Additivity axiom***
3. $\langle k\boldsymbol{u},\ \boldsymbol{v} \rangle = k\langle \boldsymbol{u},\ \boldsymbol{v} \rangle$ ***Homogeneity axiom***
4. $\langle \boldsymbol{v},\ \boldsymbol{v} \rangle \ge 0$ and $\langle \boldsymbol{v},\ \boldsymbol{v} \rangle = 0$ iff $\boldsymbol{v} = 0$ ***Positivity axiom***

A real vector space with an inner product is called a ***real inner product space***.

$$\boxed{\langle \boldsymbol{u},\ \boldsymbol{v} \rangle = \boldsymbol{u}\cdot\boldsymbol{v} = u_1v_1 + u_2v_2 + \cdots u_nv_n}$$

This is called the ***Euclidean inner product*** (or the ***standard inner product***)

Definition

If *V* is a real inner product space, then the norm (or length) of a vector *v* in *V* is denoted by $\|v\|$ and is defined by

$$\|\boldsymbol{v}\| = \sqrt{\langle \boldsymbol{v}.\ \boldsymbol{v} \rangle}$$

And the ***distance*** between two vectors is denoted by $d(\boldsymbol{u},\ \boldsymbol{v})$ and is defined by

$$d(\boldsymbol{u},\ \boldsymbol{v}) = \|\boldsymbol{u}-\boldsymbol{v}\| = \sqrt{\langle \boldsymbol{u}-\boldsymbol{v},\ \boldsymbol{u}-\boldsymbol{v} \rangle}$$

A vector of norm 1 is called a ***unit vector***.

Theorem

If ***u*** and ***v*** are vectors in a real inner product space *V*, and if *k* is a scalar, then:

a) $\|\boldsymbol{v}\| \ge 0$ with equality *iff* $\boldsymbol{v} = 0$

b) $\|k\boldsymbol{v}\| = |k|\|\boldsymbol{v}\|$

c) $d(\boldsymbol{u},\ \boldsymbol{v}) = d(\boldsymbol{v},\ \boldsymbol{u})$

d) $d(\boldsymbol{u},\ \boldsymbol{v}) \ge 0$ with equality *iff* $\boldsymbol{u} = \boldsymbol{v}$

Although the Euclidean inner product is the most important inner product on R^n, there are various applications in which is desirable to modify it by weighing each term differently. More precisely, if $w_1, w_2, \ldots, w_n$ are positive real numbers, which we will call weighs, and if $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and are vectors in R^n, then it can be shown that the formula

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \ldots + w_n u_n v_n$$

Defines an inner product on R^n that we call the ***weighted Euclidean inner product*** with weights $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n$

Example

Let $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$ be vectors in R^2, verify that the weighted Euclidean inner product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 3u_1 v_1 + 2u_2 v_2$ satisfies the four inner product axioms.

Solution

Axiom 1: $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 3u_1 v_1 + 2u_2 v_2 = 3v_1 u_1 + 2v_2 u_2 = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$

Axiom 2:
$$\begin{aligned}
\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle &= 3(u_1 + v_1)w_1 + 2(u_2 + v_2)w_2 \\
&= 3(u_1 w_1 + v_1 w_1) + 2(u_2 w_2 + v_2 w_2) \\
&= 3u_1 w_1 + 3v_1 w_1 + 2u_2 w_2 + 2v_2 w_2 \\
&= (3u_1 w_1 + 2u_2 w_2) + (3v_1 w_1 + 2v_2 w_2) \\
&= \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle
\end{aligned}$$

Axiom 3:
$$\begin{aligned}
\langle k\boldsymbol{u}, \boldsymbol{v} \rangle &= 3(ku_1)v_1 + 2(ku_2)v_2 \\
&= k(3u_1 v_1 + 2u_2 v_2) \\
&= k\langle \boldsymbol{u}, \boldsymbol{v} \rangle
\end{aligned}$$

Axiom 3:
$$\begin{aligned}
\langle \boldsymbol{v}, \boldsymbol{v} \rangle &= 3v_1 v_1 + 2v_2 v_2 \\
&= 3v_1^2 + 2v_2^2 \ge 0
\end{aligned}$$

$$v_1 = v_2 = 0 \quad iff \quad \boldsymbol{v} = \boldsymbol{0}$$

Exercises *Section* 3.1 – Inner Products

1. Let $\langle \boldsymbol{u},\ \boldsymbol{v} \rangle$ be the Euclidean inner product on R^2, and let $\boldsymbol{u} = (1,\ 1)$, $\boldsymbol{v} = (3,\ 2)$, $\boldsymbol{w} = (0,\ -1)$, and $k = 3$. Compute the following.

a) $\langle \boldsymbol{u},\ \boldsymbol{v} \rangle$ *c)* $\langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{w} \rangle$ *e)* $d(\boldsymbol{u},\ \boldsymbol{v})$

b) $\langle k\boldsymbol{v},\ \boldsymbol{w} \rangle$ *d)* $\|\boldsymbol{v}\|$ *f)* $\|\boldsymbol{u}-k\boldsymbol{v}\|$

2. Let $\langle \boldsymbol{u},\ \boldsymbol{v} \rangle$ be the Euclidean inner product on R^2, and let $\boldsymbol{u} = (1,\ 1)$, $\boldsymbol{v} = (3,\ 2)$, $\boldsymbol{w} = (0,\ -1)$ and $k = 3$. Compute the following for the weighted Euclidean inner product $\langle \boldsymbol{u},\ \boldsymbol{v} \rangle = 2u_1v_1 + 3u_2v_2$.

a) $\langle \boldsymbol{u},\ \boldsymbol{v} \rangle$ *c)* $\langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{w} \rangle$ *e)* $d(\boldsymbol{u},\ \boldsymbol{v})$

b) $\langle k\boldsymbol{v},\ \boldsymbol{w} \rangle$ *d)* $\|\boldsymbol{v}\|$ *f)* $\|\boldsymbol{u}-k\boldsymbol{v}\|$

3. Let $\langle \boldsymbol{u},\ \boldsymbol{v} \rangle$ be the Euclidean inner product on R^2, and let $\boldsymbol{u} = (3,\ -2)$, $\boldsymbol{v} = (4,\ 5)$, $\boldsymbol{w} = (-1,\ 6)$, and $k = -4$. Verify the following.

a) $\langle \boldsymbol{u},\ \boldsymbol{v} \rangle = \langle \boldsymbol{v},\ \boldsymbol{u} \rangle$

b) $\langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{w} \rangle = \langle \boldsymbol{u},\ \boldsymbol{w} \rangle + \langle \boldsymbol{v},\ \boldsymbol{w} \rangle$

c) $\langle \boldsymbol{u},\ \boldsymbol{v}+\boldsymbol{w} \rangle = \langle \boldsymbol{u},\ \boldsymbol{v} \rangle + \langle \boldsymbol{u},\ \boldsymbol{w} \rangle$

d) $\langle k\boldsymbol{u},\ \boldsymbol{v} \rangle = k\langle \boldsymbol{u},\ \boldsymbol{v} \rangle$

e) $\langle \boldsymbol{0},\ \boldsymbol{v} \rangle = \langle \boldsymbol{v},\ \boldsymbol{0} \rangle = 0$

4. Let $\langle \boldsymbol{u},\ \boldsymbol{v} \rangle$ be the Euclidean inner product on R^2, and let $\boldsymbol{u} = (3,\ -2)$, $\boldsymbol{v} = (4,\ 5)$, $\boldsymbol{w} = (-1,\ 6)$, and $k = -4$. Verify the following for the weighted Euclidean inner product $\langle \boldsymbol{u},\ \boldsymbol{v} \rangle = 4u_1v_1 + 5u_2v_2$

a) $\langle \boldsymbol{u},\ \boldsymbol{v} \rangle = \langle \boldsymbol{v},\ \boldsymbol{u} \rangle$

b) $\langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{w} \rangle = \langle \boldsymbol{u},\ \boldsymbol{w} \rangle + \langle \boldsymbol{v},\ \boldsymbol{w} \rangle$

c) $\langle \boldsymbol{u},\ \boldsymbol{v}+\boldsymbol{w} \rangle = \langle \boldsymbol{u},\ \boldsymbol{v} \rangle + \langle \boldsymbol{u},\ \boldsymbol{w} \rangle$

d) $\langle k\boldsymbol{u},\ \boldsymbol{v} \rangle = k\langle \boldsymbol{u},\ \boldsymbol{v} \rangle$

e) $\langle \boldsymbol{0},\ \boldsymbol{v} \rangle = \langle \boldsymbol{v},\ \boldsymbol{0} \rangle = 0$

5. Let $\boldsymbol{u} = (u_1,\ u_2)$ and $\boldsymbol{v} = (v_1,\ v_2)$. Show that the following are inner product on R^3 by verifying that the inner product axioms hold. $\langle \boldsymbol{u},\ \boldsymbol{v} \rangle = 3u_1v_1 + 5u_2v_2$

6. Show that the following identity holds for the vectors in any inner product space

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 + \|\boldsymbol{u}-\boldsymbol{v}\|^2 = 2\|\boldsymbol{u}\|^2 + 2\|\boldsymbol{v}\|^2$$