

# Lecture Two – Techniques of Integration

## Section 2.1 – Integration by Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x) g(x) dx$$

**Example:**  $\int x \cos x dx$ ,  $\int x^2 e^x dx$ , and  $\int x \ln x dx$

### Integration by Parts Formula

$$\int f(x) g'(x) = f(x) g(x) - \int f'(x) g(x) dx$$

Let  $u$  and  $v$  be differentiable functions of  $x$ .

$$\int u dv = uv - \int v du$$

### Guidelines for integration by Parts

1. Let  $dv$  be the most complicated portion of the integrand that fits a basic integration formula. Let  $u$  be the remaining factor.
2. Let  $u$  be the portion of the integrand whose derivative is a function simpler than  $u$ . Let  $dv$  be the remaining factor.

### Example

Evaluate:  $\int x \cos x dx$

#### Solution

$$u = x \quad dv = \cos x dx$$

Let:

$$du = dx \quad v = \int dv = \int \cos x dx = \sin x$$

$$\begin{aligned} \int x \cos x dx &= x \sin x - \int \sin x dx \\ &= \underline{x \sin x + \cos x + C} \end{aligned}$$

$$\int u dv = uv - \int v du$$

### Example

Evaluate:  $\int \ln x \, dx$

### Solution

Let:

$$u = \ln x \quad dv = dx$$

$$du = \frac{1}{x} dx \quad v = \int dx = x$$

$$\begin{aligned} \int \ln x \, dx &= x \ln x - \int x \frac{1}{x} dx \\ &= x \ln x - \int dx \\ &= \underline{x \ln x - x + C} \end{aligned}$$

$$\int u dv = uv - \int v du$$

## Tabular Integration

### Example

Evaluate  $\int x^2 e^x dx$

### Solution

$$f(x) = x^2 \quad \text{and} \quad g(x) = e^x$$

$f(x)$ & derivatives			$\int g(x) = \int e^x$
$x^2$	$(+)$	$\rightarrow$	$e^x$
$2x$	$(-)$	$\rightarrow$	$e^x$
$2$	$(+)$	$\rightarrow$	$e^x$

It is called **tabular integration**

$$\int x^2 e^x dx = \underline{x^2 e^x - 2x e^x + 2e^x + C}$$

$$\begin{aligned} u &= x^2 & dv &= e^x dx \\ du &= 2x dx & v &= \int e^x dx = e^x \\ \int x^2 e^x dx &= x^2 e^x - 2 \int x e^x dx \end{aligned}$$

$$\begin{aligned} \text{Let: } u &= x & dv &= e^x dx \\ du &= dx & v &= \int e^x dx = e^x \end{aligned}$$

$$\begin{aligned} \int x^2 e^x dx &= x^2 e^x - 2 \int x e^x dx \\ &= x^2 e^x - 2 \left[ x e^x - \int e^x dx \right] \\ &= x^2 e^x - 2 \left( x e^x - e^x \right) + C \\ &= \underline{x^2 e^x - 2x e^x + 2e^x + C} \end{aligned}$$

### Example

Evaluate  $\int x^3 \sin x \, dx$

#### Solution

$$\int x^3 \sin x \, dx = \underline{-x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C}$$

$\int \sin x$		
+	$x^3$	$-\cos x$
-	$3x^2$	$-\sin x$
+	$6x$	$\cos x$
-	$6$	$\sin x$

### Example

Evaluate  $\int e^x \cos x \, dx$

#### Solution

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$2 \int e^x \cos x \, dx = e^x (\sin x + \cos x)$$

$$\int e^x \cos x \, dx = \underline{\frac{1}{2} e^x (\sin x + \cos x) + C}$$

		$\int \cos x \, dx$
+	$e^x$	$\sin x$
-	$e^x$	$-\cos x$
+	$e^x$	$-\int \cos x \, dx$

Let:  $u = e^x \quad dv = \cos x \, dx$   
 $du = e^x \, dx \quad v = \int \cos x \, dx = \sin x$

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

Let:  $u = e^x \quad dv = \sin x \, dx$   
 $du = e^x \, dx \quad v = \int \sin x \, dx = -\cos x$

$$\begin{aligned} \int e^x \cos x \, dx &= e^x \sin x - \int e^x \sin x \, dx \\ &= e^x \sin x - \left[ -e^x \cos x - \int (-\cos x) e^x \, dx \right] \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx \end{aligned}$$

$$\int e^x \cos x \, dx + \int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx + \int e^x \cos x \, dx$$

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C_1$$

$$\int e^x \cos x \, dx = \underline{\frac{1}{2} e^x \sin x + \frac{1}{2} e^x \cos x + C}$$

### Example

Obtain a formula that expresses the integral  $\int \cos^n x dx$

### Solution

$$u = \cos^{n-1} x$$

$$dv = \cos x dx$$

$$\begin{aligned} \text{Let: } du &= (n-1) \cos^{n-2} x (-\sin x dx) \\ &= -(n-1) \cos^{n-2} x \sin x dx \end{aligned} \quad v = \int \cos x dx = \sin x$$

$$\int u dv = uv - \int v du$$

$$\begin{aligned} \int \cos^n x dx &= \cos^{n-1} x \sin x - \int \sin x \left( -(n-1) \cos^{n-2} x \sin x dx \right) \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x dx \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) dx \\ &= \cos^{n-1} x \sin x + (n-1) \int (\cos^{n-2} x - \cos^n x) dx \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx - (n-1) \int \cos^n x dx \end{aligned}$$

$$\int \cos^n x dx + (n-1) \int \cos^n x dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx$$

$$(1+n-1) \int \cos^n x dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx$$

$$n \int \cos^n x dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx$$

$$\int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx$$

**Example:** 
$$\int \cos^3 x dx = \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \int \cos x dx$$
  
$$= \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \sin x + C$$

## Evaluating Definite Integrals by Parts

### Example

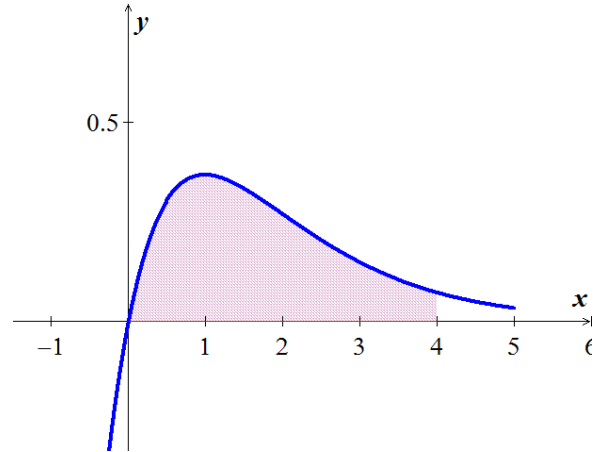
Find the area of the region bounded by the curve  $y = xe^{-x}$  and the  $x$ -axis from  $x = 0$  to  $x = 4$ .

### Solution

$$A = \int_0^4 xe^{-x} dx$$

		$\int e^{-x}$
+	$x$	$-e^{-x}$
-	$1$	$e^{-x}$

$$\begin{aligned} A &= (-x-1)e^{-x} \Big|_0^4 \\ &= -5e^{-4} + 1 \\ &\approx 0.91 \text{ unit}^2 \end{aligned}$$



### 2<sup>nd</sup> Method

$$\begin{aligned} \text{Let: } u &= x & dv &= e^{-x} dx \\ du &= dx & v &= \int e^{-x} dx = -e^{-x} \end{aligned} \quad \int u dv = uv - \int v du$$

$$\begin{aligned} \int_0^4 xe^{-x} dx &= -xe^{-x} \Big|_0^4 - \int_0^4 (-e^{-x}) dx \\ &= -[4e^{-4} - 0] + \int_0^4 e^{-x} dx \\ &= -4e^{-4} + [-e^{-x}]_0^4 \\ &= -4e^{-4} - [e^{-4} - 1] \\ &= -4e^{-4} - e^{-4} + 1 \\ &= 1 - 5e^{-4} \\ &\approx 0.91 \text{ unit}^2 \end{aligned}$$

## Formula

Evaluate  $\int x^n e^{ax} dx$

		$\int e^{ax}$
+	$x^n$	$\frac{1}{a} e^{ax}$
-	$nx^{n-1}$	$\frac{1}{a^2} e^{ax}$
+	$n(n-1)x^{n-2}$	$\frac{1}{a^3} e^{ax}$
-	$n(n-1)(n-2)x^{n-3}$	$\frac{1}{a^4} e^{ax}$
	$\vdots \vdots$	$\vdots \vdots$

$$\int x^n e^{ax} dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a^2} x^{n-1} e^{ax} + \frac{n(n-1)}{a^3} x^{n-2} e^{ax} - \frac{n(n-1)(n-2)}{a^4} x^{n-3} e^{ax} + \dots$$

$$= e^{ax} \sum_{k=0}^n (-1)^k \cdot \frac{n!}{(n-k)!} \cdot \frac{1}{a^{k+1}} \cdot x^{n-k}$$

## Exercises      Section 2.1 – Integration by Parts

(1 – 92)    Evaluate the integrals

1.  $\int x \ln x \, dx$

2.  $\int \ln x^2 \, dx$

3.  $\int \ln(3x) \, dx$

4.  $\int \frac{1}{x \ln x} \, dx$

5.  $\int x(\ln x)^2 \, dx$

6.  $\int x^2 (\ln x)^2 \, dx$

7.  $\int \frac{(\ln x)^3}{x} \, dx$

8.  $\int x^2 \ln x^3 \, dx$

9.  $\int \ln(x + x^2) \, dx$

10.  $\int x \ln(x + 1) \, dx$

11.  $\int \frac{(\ln x)^2}{x} \, dx$

12.  $\int x^5 \ln 3x \, dx$

13.  $\int x^5 \ln x \, dx$

14.  $\int \ln(x + 1) \, dx$

15.  $\int \frac{\ln x}{x^{10}} \, dx$

16.  $\int x e^{2x} \, dx$

17.  $\int x^3 e^x \, dx$

18.  $\int \frac{2x}{e^x} \, dx$

19.  $\int \frac{x^3 e^{x^2}}{(x^2 + 1)^2} \, dx$

20.  $\int x^2 e^{-3x} \, dx$

21.  $\int (x^2 - 2x + 1) e^{2x} \, dx$

22.  $\int x^5 e^{-x^3} \, dx$

23.  $\int x e^{-4x} \, dx$

24.  $\int \frac{x e^{2x}}{(2x + 1)^2} \, dx$

25.  $\int \frac{5x}{e^{2x}} \, dx$

26.  $\int \frac{e^{1/x}}{x^2} \, dx$

27.  $\int x^2 e^{4x} \, dx$

28.  $\int x^3 e^{-3x} \, dx$

29.  $\int x^4 e^x \, dx$

30.  $\int x^3 e^{4x} \, dx$

31.  $\int (x + 1)^2 e^x \, dx$

32.  $\int 2x e^{3x} \, dx$

33.  $\int x^2 \sin x \, dx$

34.  $\int \theta \cos \pi \theta \, d\theta$

35.  $\int 4x \sec^2 2x \, dx$

36.  $\int x^3 \sin x \, dx$

37.  $\int (x^3 - 2x) \sin 2x \, dx$

38.  $\int x^2 \sin 2x \, dx$

39.  $\int x^2 \sin(1 - x) \, dx$

40.  $\int x \sin x \cos x \, dx$

41.  $\int x \cos x \, dx$

42.  $\int x \csc x \cot x \, dx$

43.  $\int x^2 \cos x \, dx$

44.  $\int x^3 \cos 2x \, dx$

45.  $\int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx$

$$\begin{aligned}
46. & \int x \sinh x \, dx \\
47. & \int x^2 \cosh x \, dx \\
48. & \int e^{2x} \cos 3x \, dx \\
49. & \int e^{-3x} \sin 5x \, dx \\
50. & \int e^{-x} \sin 4x \, dx \\
51. & \int e^{-2\theta} \sin 6\theta \, d\theta \\
52. & \int e^{-3x} \sin 4x \, dx \\
53. & \int e^{4x} \cos 2x \, dx \\
54. & \int e^{3x} \cos 3x \, dx \\
55. & \int e^{3x} \cos 2x \, dx \\
56. & \int e^x \sin x \, dx \\
57. & \int e^{-2x} \sin 3x \, dx \\
58. & \int \frac{x}{\sqrt{x-1}} \, dx \\
59. & \int x\sqrt{x-5} \, dx \\
60. & \int \frac{x}{\sqrt{6x+1}} \, dx \\
61. & \int \frac{x}{2\sqrt{x+2}} \, dx \\
62. & \int \frac{2x^2 - 3x}{(x-1)^3} \, dx
\end{aligned}$$

$$\begin{aligned}
63. & \int \frac{x^2 + 3x + 4}{\sqrt[3]{2x+1}} \, dx \\
64. & \int \frac{x}{\sqrt{x+1}} \, dx \\
65. & \int \frac{x^5}{\sqrt{1-2x^3}} \, dx \\
66. & \int x\sqrt{1-3x} \, dx \\
67. & \int \sin(\ln x) \, dx \\
68. & \int \tan^{-1} y \, dy \\
69. & \int \sin^{-1} y \, dy \\
70. & \int x \tan^{-1} x \, dx \\
71. & \int \sinh^{-1} x \, dx \\
72. & \int \tan^{-1} 3x \, dx \\
73. & \int \cos^{-1}\left(\frac{x}{2}\right) \, dx \\
74. & \int x \sec^{-1} x \, dx \\
75. & \int_{-1}^0 2x^2 \sqrt{x+1} \, dx \\
76. & \int_0^{1/\sqrt{2}} x \tan^{-1} x^2 \, dx \\
77. & \int_1^e x^2 \ln x \, dx \\
78. & \int_{-1}^{\ln 2} \frac{3t}{e^t} \, dt
\end{aligned}$$

$$\begin{aligned}
79. & \int_{\pi}^{2\pi} \cot \frac{x}{3} \, dx \\
80. & \int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2) \, dx \\
81. & \int_1^e x^3 \ln x \, dx \\
82. & \int_0^1 x\sqrt{1-x} \, dx \\
83. & \int_0^{\pi/3} x \tan^2 x \, dx \\
84. & \int_0^{\pi} x \sin x \, dx \\
85. & \int_1^e \ln 2x \, dx \\
86. & \int_0^{\pi/2} x \cos 2x \, dx \\
87. & \int_0^{\ln 2} x e^x \, dx \\
88. & \int_1^{e^2} x^2 \ln x \, dx \\
89. & \int_0^3 x e^{x/2} \, dx \\
90. & \int_0^2 x^2 e^{-2x} \, dx \\
91. & \int_0^{\pi/4} x \cos 2x \, dx \\
92. & \int_0^{\pi} x \sin 2x \, dx \\
93. & \int_1^4 e^{\sqrt{x}} \, dx
\end{aligned}$$



(94 – 98) Use integration by parts to establish the reduction formula

$$94. \int x^n \sin x \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx$$

$$95. \int x^n e^{ax} \, dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx, \quad a \neq 0$$

$$96. \int (\ln x)^n \, dx = x (\ln x)^n - n \int (\ln x)^{n-1} \, dx$$

$$97. \int_a^b \left( \int_x^b f(t) \, dt \right) dx = \int_a^b (x-a) f(x) \, dx$$

$$98. \int \sqrt{1-x^2} \, dx = \frac{1}{2} x \sqrt{1-x^2} + \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} \, dx$$

$$99. \text{ Find the indefinite integral: } \int 5x^n \ln ax \, dx \quad a \neq 0, n \neq -1$$

100. Find the volume of the solid generated by the region bounded by  $f(x) = x \ln x$ , and the  $x$ -axis on  $[1, e^2]$  is revolved about the  $y$ -axis.

101. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve  $y = e^x$ , and the line  $x = \ln 2$  about the line  $x = \ln 2$

102. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve  $y = e^{-x}$ , and the line  $x = 1$ , about  
a) the line  $y$ -axis  
b) the line  $x = 1$

103. Find the volume of the solid that is generated by the region bounded by  $f(x) = e^{-x}$ ,  $x = \ln 2$ , and the coordinate axes is revolved about the  $y$ -axis.

104. Find the volume of the solid that is generated by the region bounded by  $f(x) = e^{-x}$ , and the  $x$ -axis on  $[1, \ln 2]$  is revolved about the line  $x = \ln 2$ .

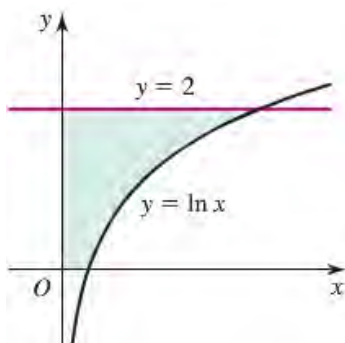
105. Find the volume of the solid that is generated by the region bounded by  $f(x) = \sin x$ , and the  $x$ -axis on  $[0, \pi]$  is revolved about the  $y$ -axis.

**106.** Find the area of the region generated when the region bounded by  $y = \sin x$  and  $y = \sin^{-1} x$  on the interval  $\left[0, \frac{1}{2}\right]$ .

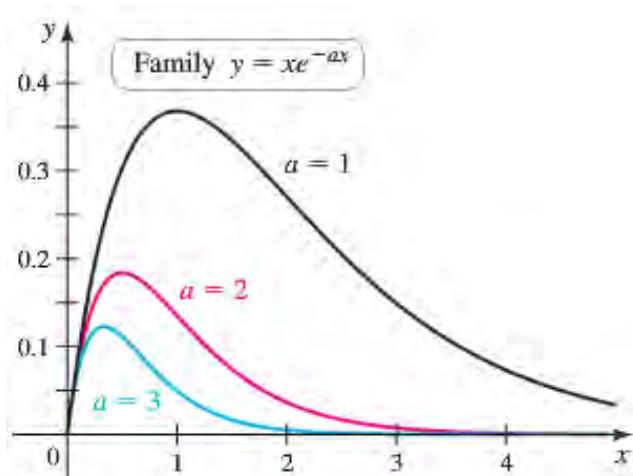
**107.** Find the area between the curves  $y = \ln x^2$ ,  $y = \ln x$ , and  $x = e^2$

**108.** Determine the area of the shaded region bounded by

$$y = \ln x, \quad y = 2, \quad y = 0, \quad \text{and} \quad x = 0$$



**109.** The curves  $y = xe^{-ax}$  are shown in the figure for  $a = 1, 2$ , and  $3$ .



- Find the area of the region bounded by  $y = xe^{-x}$  and the  $x$ -axis on the interval  $[0, 4]$ .
- Find the area of the region bounded by  $y = xe^{-ax}$  and the  $x$ -axis on the interval  $[0, 4]$  where  $a > 0$
- Find the area of the region bounded by  $y = xe^{-ax}$  and the  $x$ -axis on the interval  $[0, b]$ . Because this area depends on  $a$  and  $b$ , we call it  $A(a, b)$  where  $a > 0$  and  $b > 0$ .
- Use part (c) to show that  $A(1, \ln b) = 4A\left(2, \frac{1}{2} \ln b\right)$
- Does this pattern continue? Is it true that  $A(1, \ln b) = a^2 A\left(a, \frac{1}{a} \ln b\right)$

- 110.** Suppose a mass on a spring that is slowed by friction has the position function  $s(t) = e^{-t} \sin t$
- Graph the position function. At what times does the oscillator pass through the position  $s = 0$ ?
  - Find the average value of the position on the interval  $[0, \pi]$ .
  - Generalize part (b) and find the average value of the position on the interval  $[n\pi, (n+1)\pi]$ , for  $n = 0, 1, 2, \dots$
- 111.** Given the region bounded by the graphs of  $y = x \sin x$ ,  $y = 0$ ,  $x = 0$ ,  $x = \pi$ , find
- The area of the region.
  - The volume of the solid generated by revolving the region about the  $x$ -axis
  - The volume of the solid generated by revolving the region about the  $y$ -axis
  - The centroid of the region
- 112.** The region  $R$  is bounded by the curve  $y = \ln x$  and the  $x$ -axis on the interval  $[1, e]$ . Find the volume of the solid that is generated when  $R$  is revolved in the following ways
- About the  $x$ -axis
  - About the  $y$ -axis
  - About the line  $x = 1$
  - About the line  $y = 1$
- 113.** A string stretched between the two points  $(0, 0)$  and  $(2, 0)$  is plucked by displacing the string  $h$  units at its midpoint. The motion of the string is modeled by a **Fourier Sine series** whose coefficients are given by

$$b_n = h \int_0^1 x \sin \frac{n\pi x}{2} dx + h \int_1^2 (-x + 2) \sin \frac{n\pi x}{2} dx$$

Find  $b_n$