Math 2318 – Linear Algebra **Exam 3** **Review**

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1. Find the characteristic equation, eigenvalues, and eigenvectors of $\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$

2. Find the characteristic equation, eigenvalues, and eigenvectors of $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

3. Find the eigenvalues, and eigenvectors of $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

4. Find the characteristic equation, eigenvalues, and eigenvectors of $\begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$

5. Find the characteristic equation, eigenvalues, and eigenvectors of $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$

6. Find a matrix P that diagonalizes $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

7. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, determine when A is diagonalizable, not diagonalizable. (*Hint: discriminant of the characteristic equation*)

8. Show that $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ are not similar matrices

9. Let $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ be the Euclidean inner product on R^2, and let $\boldsymbol{u} = (3, -2)$, $\boldsymbol{v} = (4, 5)$, and $k = 4$.

Verify the following for the weighted Euclidean inner product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 3u_1 v_1 + 4u_2 v_2$

a) $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$ *b)* $\langle k\boldsymbol{u}, \boldsymbol{v} \rangle = k \langle \boldsymbol{u}, \boldsymbol{v} \rangle$

10. Which of the following form orthonormal sets?

a) $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right), \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right)$ in $\boldsymbol{R}^3$

b) $\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right), \left(0, \frac{\sqrt{6}}{3}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right), \left(\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{6} \right), \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$

11. Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\boldsymbol{R}^m$.

a) $x_1 = (1,\ 1),\quad x_2 = (1,\ 2)$

b) $x_1 = (1,\ 2),\quad x_2 = (1,\ 3)$

c) $x_1 = (1,\ 2,\ 2),\quad x_2 = (2,\ 1,\ 3)$

d) $v_1 = (1,-1,-1,1),\quad v_2 = (2,1,0,1),\quad v_3 = (2,2,1,2)$

12. Find the ***QR***-decomposition of

$$a)\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad b)\ \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

13. Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$a)\ \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$b)\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$c)\ \begin{pmatrix} \cos\theta\sin\theta & -\cos\theta & -\sin^2\theta \\ \cos^2\theta & \sin\theta & -\cos\theta\sin\theta \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

14. Show that the matrix $A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ is not diagonalizable

Solution

1. $\lambda^2 - 8\lambda + 16$ *Eigenvalue*: $\lambda = 4$ *Eigenvector*: $\begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$

2. $\lambda^2 - 6\lambda + 8$ *Eigenvalue*: $\lambda = 2, 4$ *Eigenvector*: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

3. $\lambda^2 - 1$ *Eigenvalue*: $\lambda = \pm 1$ *Eigenvector*: $\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

4. $\lambda^3 - 2\lambda$ *Eigenvalue*: $\lambda = 0,\ \pm\sqrt{2}$ *Eigenvector*: $\begin{pmatrix} \frac{5}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix} \begin{pmatrix} -\frac{5}{\sqrt{2}-3} \\ \frac{1}{7}\frac{3+\sqrt{2}}{1+\sqrt{2}} \\ 1 \end{pmatrix} \begin{pmatrix} \frac{5}{3+\sqrt{2}} \\ \frac{1}{7}\frac{3-\sqrt{2}}{1-\sqrt{2}} \\ 1 \end{pmatrix}$

5. $-\lambda^3 + 4\lambda^2 - 5\lambda + 2$ *Eigenvalue*: $\lambda_{1,2} = 1,\ \lambda_3 = 2$ *Eigenvector*: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

6. $P = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

7. *diagonalizable*: $(a-d)^2 + 4bc > 0$, *not diagonalizable*: $(a-d)^2 + 4bc < 0$

8. $\det(A) = -3 \neq \det(B) = 3$

9. $a)\ -4 \quad b)\ -16$

10. *a) Orthonormal* *b) Orthonormal*

11. $a)\quad v_1 = (1,1), v_1 = \left(-\frac{1}{2}, \frac{1}{2}\right); \quad q_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), q_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$b)\quad q_1 = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), q_2 = \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

$c)\quad q_1 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right), \quad q_2 = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$d)\quad q_1 = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right), \quad q_2 = \left(\frac{3\sqrt{5}}{10}, \frac{3\sqrt{5}}{10}, \frac{\sqrt{5}}{10}, \frac{\sqrt{5}}{10}\right) \quad q_3 = \left(-\frac{\sqrt{6}}{6}, 0, \frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}\right)$

12. $a)$ $Q=\begin{pmatrix} \frac{1}{2} & -\frac{3}{\sqrt{12}} & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \end{pmatrix}$ $R=\begin{pmatrix} 2 & \frac{3}{2} & 1 \\ 0 & \frac{3}{\sqrt{12}} & \frac{2}{\sqrt{12}} \\ 0 & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$

$b)$ $Q=\begin{bmatrix} \frac{1}{2} & \frac{3\sqrt{5}}{10} & -\frac{\sqrt{6}}{6} \\ -\frac{1}{2} & \frac{3\sqrt{5}}{10} & 0 \\ -\frac{1}{2} & \frac{\sqrt{5}}{10} & \frac{\sqrt{6}}{6} \\ \frac{1}{2} & \frac{\sqrt{5}}{10} & \frac{\sqrt{6}}{3} \end{bmatrix}$ $R=\begin{pmatrix} 2 & 1 & \frac{1}{2} \\ 0 & \sqrt{5} & \frac{3\sqrt{5}}{2} \\ 0 & 0 & \frac{\sqrt{6}}{2} \end{pmatrix}$

13. $a)$ *Orthogonal* $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$b)$ *Orthogonal* $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$c)$ *Orthogonal* $\begin{pmatrix} \cos\theta\sin\theta & \cos^2\theta & \sin\theta \\ -\cos\theta & \sin\theta & 0 \\ -\sin^2\theta & -\cos\theta\sin\theta & \cos\theta \end{pmatrix}$