Solution Section 1.1 – Introduction to System of Linear Equations

Exercise

Find a solution for x, y, z to the system of equations

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3e+2\sqrt{2}+\pi \\ 6e+5\sqrt{2}+4\pi \\ 9e+8\sqrt{2}+7\pi \end{pmatrix}$$

Solution

$$\begin{pmatrix} x+2y+3z \\ 4x+5y+6z \\ 7x+8y+9z \end{pmatrix} = \begin{pmatrix} 3e+2\sqrt{2}+\pi \\ 6e+5\sqrt{2}+4\pi \\ 9e+8\sqrt{2}+7\pi \end{pmatrix}$$

$$\Rightarrow \begin{cases} x+2y+3z = \pi+2\sqrt{2}+3e \\ 4x+5y+6z = 4\pi+5\sqrt{2}+6e \\ 7x+8y+9z = 7\pi+8\sqrt{2}+9e \end{cases}$$

Solution: $\boxed{x=\pi \quad y=\sqrt{2} \quad z=e}$

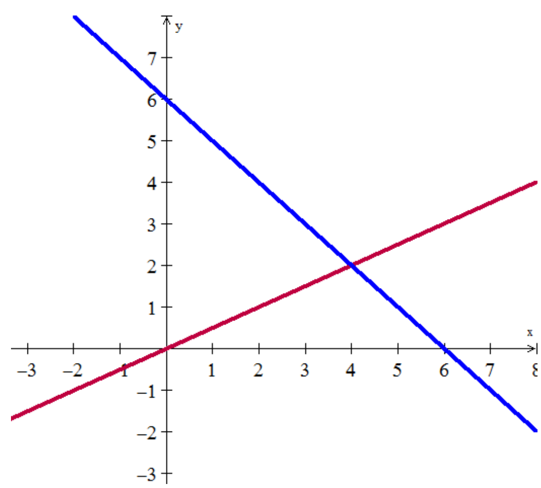
Exercise

Draw the two pictures in two planes for the equations: $x-2y=0, \quad x+y=6$

Solution

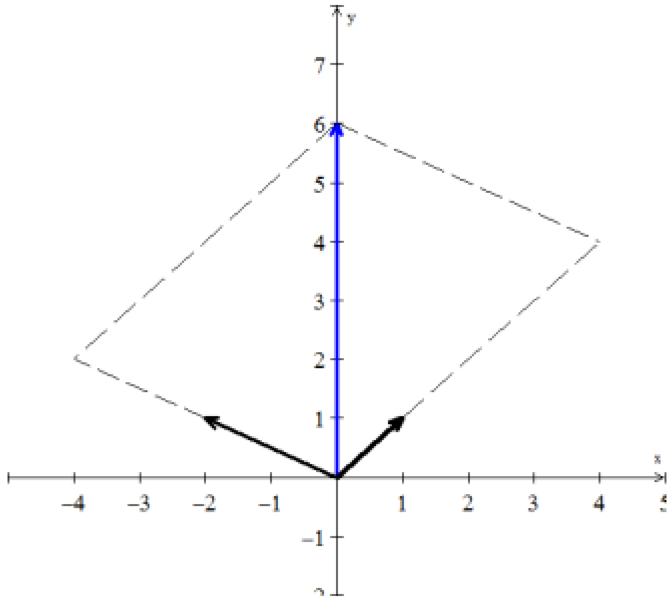
The matrix form of the 2 equations:

$$\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$



Row picture is the 2 lines from the given equations and their intersection is the point (4, 2) which is the solution for the system.

Column Picture is the column vectors $\begin{pmatrix} 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} -2 & 1 \end{pmatrix}$



$$x\begin{pmatrix}1\\1\end{pmatrix}+y\begin{pmatrix}-2\\1\end{pmatrix}=4\begin{pmatrix}1\\1\end{pmatrix}+2\begin{pmatrix}-2\\1\end{pmatrix}=\begin{pmatrix}0\\6\end{pmatrix}$$

The parallelogram show how the solution vector $\begin{pmatrix}0 & 6\end{pmatrix}$ can be written as the linear combination of the column vectors.

Exercise

Normally 4 planes in 4-dimensional space meet at a ________. Normally 4 column vectors in 4-deimensional space can combine to produce b. what combinations of $(1,\ 0,\ 0,\ 0),\quad (1,\ 1,\ 0,\ 0),\quad (1,\ 1,\ 1,\ 0),\quad (1,\ 1,\ 1,\ 1)$ produces $b=(3,\ 3,\ 3,\ 2)$?

What 4 equations for $x,\ y,\ z,\ w$ are you solving?

Solution

Normally 4 planes in 4-dimensional space meet at a ***point***.

The combination of the vectors producing b is:

$$0\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+0\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+1\begin{pmatrix}1\\1\\1\\0\end{pmatrix}+2\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}3\\3\\3\\2\end{pmatrix}$$

$$1\begin{pmatrix}1\\1\\1\\0\end{pmatrix}+2\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}3\\3\\3\\2\end{pmatrix}$$

The system of equations that satisfies the given vectors is:

$$\begin{cases} x+y+z+w=3 \\ \quad\;\; y+z+w=3 \\ \qquad\quad z+w=3 \\ \qquad\qquad\; w=2 \end{cases}$$

Exercise

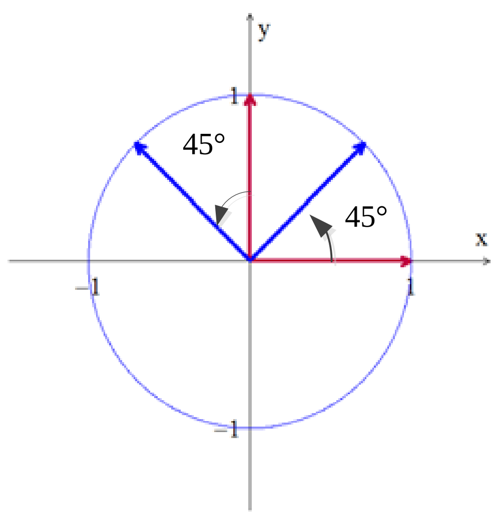
What 2 by 2 matrix A rotates every vector through 45° ?

The vector (1, 0) goes to $\left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$. The vector (0, 1) goes to $\left(-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$.

Those determine the matrix. Draw these particular vectors is the xy-plane and find A.

Solution

$$A=\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$



Exercise

What two vectors are obtained by rotating the plane vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by $30°$ (cw) ?

Write a matrix A such that for every vector v in the plane, Av is the vector obtained by rotating v clockwise by $30°$.

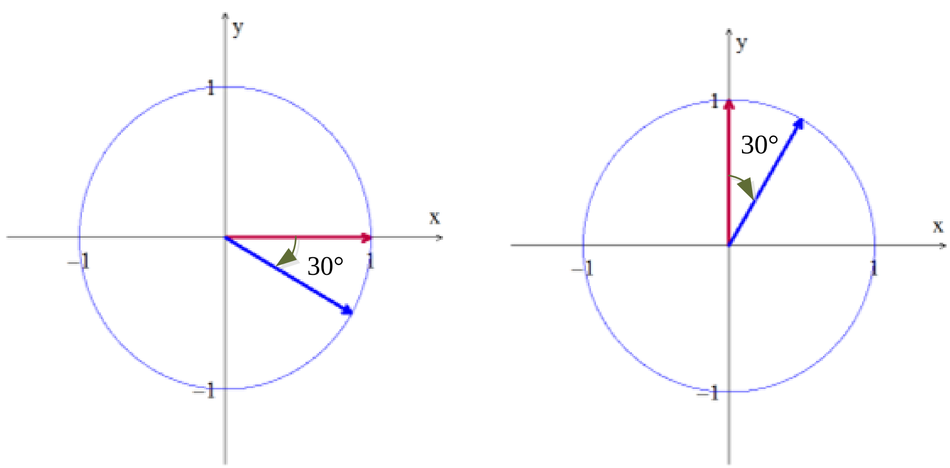
Find a matrix B such that for every 3-dimensional vector v, the vector Bv is the reflection of v through the plane $x+y+z=0$. $Hint: v=(1,\ 0,\ 0)$

Solution

Rotating the vectors by $30°$ (cw) yields:

For the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ yields to $\begin{pmatrix} \cos(-30°) \\ \sin(-30°) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}$

And for the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ yields to $\begin{pmatrix} \sin(30°) \\ \cos(30°) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$



The desired matrix is: $A = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

To get 1 from $\frac{\sqrt{3}}{2}$ is to multiply by $\frac{2}{\sqrt{3}} = 2\frac{1}{\sqrt{3}}$

The unit vector to the plane $x+y+z=0$ is $\hat{u} = \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$

$$Bv = B\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{2}{\sqrt{3}}\hat{u}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{2}{\sqrt{3}} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$B \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{\sqrt{3}} \hat{u} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \quad = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}$$

$$B \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{2}{\sqrt{3}} \hat{u} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

The solution: $\begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$

Exercise

Find a system of linear equation corresponding to the given augmented matrix

$$\begin{bmatrix} 0 & 3 & -1 & -1 & -1 \\ 5 & 2 & 0 & -3 & -6 \end{bmatrix}$$

Solution

$$\begin{cases} \quad\quad 3x_2 - x_3 - x_4 = -1 \\ 5x_1 + 2x_2 \quad\quad - 3x_4 = -6 \end{cases}$$

Exercise

Find a system of linear equation corresponding to the given augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \\ 5 & -6 & 1 & 1 \\ -8 & 0 & 0 & 3 \end{bmatrix}$$

Solution

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ -4x_1 - 3x_2 - 2x_3 = -1 \\ 5x_1 - 6x_2 + x_3 = 1 \\ -8x_1 \quad\quad\quad\quad = 3 \end{cases}$$

Exercise

Find the augmented matrix for the given system of linear equations.

$$\begin{cases} -2x_1 = 6 \\ 3x_1 = 8 \\ 9x_1 = -3 \end{cases}$$

Solution

$$\left[\begin{array}{c|c} -2 & 6 \\ 3 & 8 \\ 9 & -3 \end{array}\right]$$

Exercise

Find the augmented matrix for the given system of linear equations.

$$\begin{cases} 3x_1 - 2x_2 = -1 \\ 4x_1 + 5x_2 = 3 \\ 7x_1 + 3x_2 = 2 \end{cases}$$

Solution

$$\left[\begin{array}{cc|c} 3 & -2 & -1 \\ 4 & 5 & 3 \\ 7 & 3 & 2 \end{array}\right]$$

Exercise

Find the augmented matrix for the given system of linear equations.

$$\begin{cases} 2x_1 \qquad + 2x_3 = 1 \\ 3x_1 - x_2 + 4x_3 = 7 \\ 6x_1 + x_2 - x_3 = 0 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 1 \\ 3 & -1 & 4 & 7 \\ 6 & 1 & -1 & 0 \end{array}\right]$$

Solution   *Section* 1.2 – Gaussian Elimination

Exercise

When elimination is applied to the matrix $A = \begin{bmatrix} 3 & 1 & 0 \\ 6 & 9 & 2 \\ 0 & 1 & 5 \end{bmatrix}$

a) What are the first and second pivots?

b) What is the multiplier l_{21} in the first step (l_{21} times row 1 is subtracted from row 2)?

c) What entry in the 2, 2 position (instead of 9) would force an exchange of rows 2 and 3?

d) What is the multiplier $l_{31} = 0$, subtracting 0 times row 1 from row 3?

Solution

a) The first pivot is 3 and when 2 times row 1 is subtracted from row 2, the second pivot is revealed as 7.

$$\begin{bmatrix} 3 & 1 & 0 \\ 6 & 9 & 2 \\ 0 & 1 & 5 \end{bmatrix} \quad \begin{array}{c} \textit{subtract } 2 \textit{ times row.1} \\ \textit{from row.2} \end{array} \quad \begin{bmatrix} 3 & 1 & 0 \\ 0 & \boxed{7} & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

b) The multiplier l_{21} in the first step is $\frac{6}{3} = 2$.

c) If we reduce the entry 9 to 2, that drop of 7 in the a_{22} position would force a row exchange.

$$\begin{bmatrix} 3 & 1 & 0 \\ 6 & 9 & 2 \\ 0 & 1 & 5 \end{bmatrix} \quad \begin{array}{c} \textit{subtract } 7 \textit{ times row.1} \\ \textit{from row.2} \end{array} \quad \begin{bmatrix} 3 & 1 & 0 \\ -15 & \boxed{2} & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

d) The multiplier l_{31} is already zero because $a_{31} = 0$ and no needs row elimination.

Exercise

Use elimination to reach upper triangular matrices ***U***. Solve by back substitution or explain why this impossible. What are the pivots (never zero)? Exchange equations when necessary. The only difference is the $-x$ in equation (3).

$$\begin{cases} x+y+z=7 \\ x+y-z=5 \\ x-y+z=3 \end{cases} \qquad \begin{cases} x+y+z=7 \\ x+y-z=5 \\ -x-y+z=3 \end{cases}$$

Solution

For the *first* system:

$$\begin{array}{lll} x+y+z=7 & \textit{subtract eqn.}1 & x+y+z=7 \\ x+y-z=5 & \textit{from eqn.}2 & 0y-2z=-2 \\ x-y+z=3 & \textit{from eqn.}3 & -2y-0z=-4 \end{array}$$

$$\begin{array}{lll} x+y+z=7 & & 1x+\ y+\ z=7 \\ x+y-z=5 & \textit{Exchange eqn.}2 & -2y-0z=-4 \\ x-y+z=3 & \textit{and eqn.}3 & -2z=-2 \end{array}$$

The solutions are: $z=1 \quad y=2 \quad x=4$ and the pivots are 1, -2, -2.

For the *second* system:

$$\begin{array}{lll} x+y+z=7 & \textit{Subtract eqn.}1 & x+y+z=7 \\ x+y-z=5 & \textit{from eqn.}2 & 0y-2z=-2 \\ -x-y+z=3 & \begin{array}{l}\textit{Add eqn.}1 \\ \textit{to eqn.}3\end{array} & 0y+2z=10 \end{array}$$

$$\begin{array}{lll} x+y+z=7 & & x+y+z=7 \\ 0y-2z=-2 & \textit{Add eqn.}2 & 0y-2z=-2 \\ 0y+2z=10 & \textit{to eqn.}3 & \boxed{0z=8} \end{array}$$

The three planes don't meet. But if we change '3' in the last equation to '-5'

$$\begin{array}{lll} x+y+z=7 & \textit{Subtract eqn.}1 & x+y+z=7 \\ x+y-z=5 & \textit{from eqn.}2 & 0y-2z=-2 \\ -x-y+z=-5 & \begin{array}{l}\textit{Add eqn.}1 \\ \textit{to eqn.}3\end{array} & 0y+2z=2 \end{array}$$

$$\begin{array}{ll} x+y+z=7 & x+y=6 \\ 0y-2z=-2 & \\ 0y+2z=10 & z=1 \end{array}$$

There are unique infinite many solutions!

The three planes now meet along a whole line.

Exercise

For which numbers a does the elimination break down (1) permanently (2) temporarily

$$\begin{aligned} ax + 3y &= -3 \\ 4x + 6y &= 6 \end{aligned}$$

Solve for x and y after fixing the second breakdown by a row change.

Solution

The matrix form is: $\begin{pmatrix} a & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$

If $a = 0$, the elimination brakes down temporarily.

$$\begin{pmatrix} 4 & 6 \\ 0 & \boxed{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$$

The system is in upper triangular form and entry row 2 column 2 is not equal to zero, therefore the system has a solution.

If $a \neq 0$,

$$\begin{pmatrix} a & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix} \qquad R_2 - \frac{4}{a}R_1$$

$$\begin{pmatrix} a & 3 \\ 0 & 6 - \frac{12}{a} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 + \frac{12}{a} \end{pmatrix}$$

$$6 - \frac{12}{a} = 0 \Rightarrow \frac{12}{a} = 6$$

$$\rightarrow \boxed{a = \frac{12}{6} = 2}$$

If $a = 2$,

$\begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \end{pmatrix}$, the system will fail and has no solution.

If $a \neq 2$;

$\begin{pmatrix} a & 3 \\ 0 & 6 - \frac{12}{a} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 + \frac{12}{a} \end{pmatrix}$, the system has a unique solution.

Exercise

Find the pivots and the solution for these four equations:

$$\begin{aligned} 2x + y \qquad\quad &= 0 \\ x + 2y + z \quad\; &= 0 \\ y + 2z + t &= 0 \\ z + 2t &= 5 \end{aligned}$$

Solution

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix} R_2 - \tfrac{1}{2}R_1$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 1.5 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix} R_3 - \tfrac{2}{3}R_2$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix} R_4 - \tfrac{3}{4}R_3$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 5 \end{pmatrix} \quad \begin{aligned} &2x = -y \Rightarrow \boxed{x = -2\tfrac{1}{2} = -1} \\ &\tfrac{3}{2}y + z = 0 \Rightarrow y = -z\tfrac{2}{3} = -(-3)\tfrac{2}{3} \to \boxed{y = 2} \\ &\tfrac{4}{3}z + t = 0 \to \tfrac{4}{3}z = -t \to \boxed{z = -4\tfrac{3}{4} = -3} \\ &\tfrac{5}{4}t = 5 \to \boxed{t = 4} \end{aligned}$$

The pivots are diagonal entries and the solution is: $(-1,\ 2,\ -3,\ 4)$

Exercise

Look for a matrix that has row sums 4 and 8, and column sums 2 and s.

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{array}{ll} a+b=4 & a+c=2 \\ c+d=8 & b+d=s \end{array}$$

The four equations are solvable only if s = ____. Then find two different matrices that have the correct row and column sums.

Solution

$$\begin{array}{rr} & a+b=4 \\ + & c+d=8 \\ \hline & a+c+b+d=12 \end{array}$$

$$2+s=12$$

$$\boxed{s=10}$$

Exercise

Three planes can fail to have an intersection point, even if no planes are parallel. The system is singular if row 3 of A is a _______ of the first two rows. Find a third equation that can't be solved together with $x+y+z=0$ and $x-2y-z=1$

Solution

The system is singular if row 3 of A is a ***linear combination*** of the first two rows.

There are many possible of a third equation that can't be solved together with $x+y+z=0$ and $x-2y-z=1$.

$$\begin{array}{lr} 3 \text{ times } 1^{st} \text{ equation} & 3x+3y+3z \\ \quad \text{minus } 2nd & -x+2y+z \\ \hline & 2x+5y+4z=1 \end{array}$$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} -1 & -2 & 3 & 1 \\ 1 & 1 & 2 & 8 \\ \hline 0 & -1 & 5 & 9 \end{array} \qquad \begin{array}{cccc} 3 & -7 & 4 & 10 \\ -3 & -3 & -6 & -24 \\ \hline 0 & -10 & -2 & -14 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right] -R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 10R_2 \end{array}$$

$$\begin{array}{cccc} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ \hline 1 & 0 & 7 & 17 \end{array} \qquad \begin{array}{cccc} 0 & -10 & -2 & -14 \\ 0 & 10 & -50 & -90 \\ \hline 0 & 0 & -52 & -104 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right] -\frac{1}{52}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 - 7R_3 \\ R_2 + 5R_3 \\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & 7 & 17 \\ 0 & 0 & -7 & -14 \\ \hline 1 & 0 & 0 & 3 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -5 & -9 \\ 0 & 0 & 5 & 10 \\ \hline 0 & 1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solution: $\boxed{(3,\ 1,\ 2)}$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x \qquad\qquad - 3w = -3 \end{cases}$$

Solution

Solution: $\boxed{(w-1,\ 2z,\ z,\ w)}$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x \qquad\qquad - 3w = -3 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ -1 & 3 & -2 & 1 \\ 3 & 4 & -7 & 10 \end{array}\right] \quad \begin{array}{l} R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 5 & -1 & 9 \\ 0 & -2 & -10 & -14 \end{array}\right] \quad 5R_3 + 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 5 & -1 & 9 \\ 0 & 0 & -52 & -52 \end{array}\right] \quad \begin{array}{rl} x + 2y + z = 8 & (3) \\ 5y - z = 9 & (2) \\ -52z = -52 & (1) \end{array}$$

$(1) \quad \Rightarrow \quad z = 1$

$(2) \quad \Rightarrow \quad 5y = 9 + 1 = 10 \rightarrow y = 2$

$(3) \quad \Rightarrow \quad x = 8 - 4 - 1 = 3$

$\therefore$ Solution: $(3,\ 2,\ 1)$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} 2u-3v+\ \ w-\ \ x+y=\ \ 0 \\ 4u-6v+2w-3x-y=-5 \\ -2u+3v-2w+2x-y=\ \ 3 \end{cases}$$

Solution

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 4 & -6 & 2 & -3 & -1 & -5 \\ -2 & 3 & -2 & 2 & -1 & 3 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3+R_1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -3 & -5 \\ 0 & 0 & -1 & 1 & 0 & 3 \end{array}\right] \qquad \begin{array}{rl} 2u-3v+w-x+y=0 & (3) \\ -x-3y=-5 & (2) \\ -w+x=3 & (1) \end{array}$$

$(2) \quad \Rightarrow \quad x=5-3y$

$(1) \quad \Rightarrow \quad w=x-3=2-3y$

$(3) \quad \Rightarrow \quad 2u=3v-2+3y+5-3y-y=3v-y+3$

$$u=\frac{3}{2}v-\frac{1}{2}y+\frac{3}{2}$$

$\therefore$ Solution: $\left(\frac{3}{2}v-\frac{1}{2}y+\frac{3}{2}, v, 2-3y, 5-3y, y\right)$

Exercise

Solve the given linear system by any method

$$\begin{cases} 2x_1+x_2+3x_3=0 \\ x_1+2x_2 \qquad =0 \\ \qquad x_2+x_3=0 \end{cases}$$

Solution

Solution: $\boxed{(0,\ 0,\ 0)}$

Exercise

Solve the given linear system by any method

$$\begin{cases} 2x+2y+4z \quad\ \ =0 \\ \quad -y-3z+w=0 \\ 3x+y+z+2w=0 \\ x+3y-2z-2w=0 \end{cases}$$

Solution

Solution: $\boxed{(-w,\ w,\ 0,\ w)}$

Exercise

Add 3 times the second row to the first of

$$\begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$$

Solution

$$E=\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$EA=\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}=\begin{bmatrix} 27 & 8 & -1 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix} \quad \xrightarrow{R_1+3R_2} \quad =\begin{bmatrix} 27 & 8 & -1 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$$

Exercise

Solve the system using Gaussian elimination $\begin{cases} 3x_1 + 2x_2 - x_3 = -15 \\ 5x_1 + 3x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 3x_3 = 11 \\ -6x_1 - 4x_2 + 2x_3 = 30 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ -6 & -4 & 2 & 30 \end{array}\right] \quad \begin{array}{c} \\ 3R_2 - 5R_1 \\ R_3 - R_1 \\ R_4 + 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & -1 & 4 & 26 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & 0 & -7 & -49 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{rl} 3x_1 + 2x_2 - x_3 = -15 & (3) \\ -x_2 + 11x_3 = 75 & (2) \\ -7x_3 = -49 & (1) \end{array}$$

$(1) \rightarrow x_3 = 7$

$(2) \rightarrow x_2 = 77 - 75 = 2$

$(1) \rightarrow 3x_1 = -15 - 4 + 7 = 12 \Rightarrow x_1 = -4$

$\therefore$ Solution: $\boxed{(-4, 2, 7)}$

Solution *Section* 1.3 – Matrices and Matrix operations

Exercise

For the matrices: $A = \begin{bmatrix} p & 0 \\ q & r \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, when does $AB = BA$

Solution

$$AB = \begin{pmatrix} p & 0 \\ q & r \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} p & p \\ q & q+r \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p & 0 \\ q & r \end{pmatrix}$$

$$= \begin{pmatrix} p+q & r \\ q & r \end{pmatrix}$$

$$AB = BA$$

$$\begin{pmatrix} p & p \\ q & q+r \end{pmatrix} = \begin{pmatrix} p+q & r \\ q & r \end{pmatrix}$$

$$\begin{cases} p = p+q \\ \boxed{p = r} \\ q+r = r \end{cases} \Rightarrow \begin{cases} \boxed{q = 0} \\ \\ q = 0 \end{cases}$$

Exercise

Find a combination $x_1 w_1 + x_2 w_2 + x_3 w_3$ that gives the zero vector:

$$w_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}; \quad w_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}; \quad w_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Those vectors are independent or dependent?
The vectors lie in a ______.
The matrix W with those columns is not invertible.

Solution

$w_1 - 2w_2 + w_3 = 0$; Therefore those vectors are dependent

The vectors lie in a plane

Exercise

The very last words say that the 5 by 5 centered difference matrix is not invertible, Write down the 5 equations $Cx = b$. Find a combination of left sides that gives zero. What combination of $b_1,\ b_2,\ b_3,\ b_4,\ b_5$ must be zero?

Solution

The 5 by 5 centered difference matrix is

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

The five equations $Cx = b$ are:

$$x_2 = b_1, \quad -x_1 + x_3 = b_2, \quad -x_2 + x_4 = b_3, \quad -x_3 + x_5 = b_4, \quad -x_4 = b_5.$$

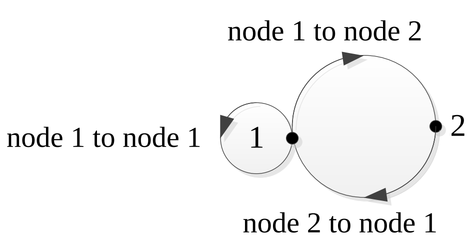
Observe that the sum of the first

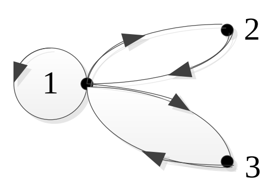
$$x_2 - x_2 + x_4 - x_4 = b_1 + b_2 + b_5$$

$$0 = b_1 + b_2 + b_5$$

Exercise

A direct graph starts with *n* nodes. There are n^2 possible edges, each edge leaves one of the *n* nodes and enters one of the *n* nodes (possibly itself). The *n* by *n* adjacency matrix has $a_{ij} = 1$ when edge leaves node *i* and enter node *j*; if no edge then $a_{ij} = 0$. Here are directed graphs and their adjacency matrices:





$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The *i*, *j* entry of A^2 is $a_{i1}a_{1j} + \ldots + a_{in}a_{nj}$.

Why does that sum count the two-step paths from *i* to any node to *j*?

The *i*, *j* entry of A^k counts *k*-steps paths:

$$\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}^2 = \begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix} \quad \begin{matrix}\textit{counts the paths}\\ \textit{with two edges}\end{matrix} \quad \begin{bmatrix}1\ to\ 2\ to\ 1, 1\ to\ 1\ to\ 1 & 1\ to\ 1\ to\ 2\\ 2\ to\ 1\ to\ 1 & 2\ to\ 1\ to\ 2\end{bmatrix}$$

List all 3-step paths between each pair of nodes and compare with A^3. When A^k has ***no zeros***, that number k is the diameter of the graph – the number of edges needed to connect the most pair of nodes. What is the diameter of the second graph?

Solution

The number $a_{ik}a_{kj}$ will be "**1**" if there is an edge from node i to k and an edge from k to j.

This is a 2-step path. The number $a_{ik}a_{kj}$ will be "**0**" if either of those edge (from node i to k and from k to j) is missing.

The sum of $a_{ik}a_{kj}$ is the number of 2-step paths leaving i and entering j.

Matrix multiplication is right for this count.

The 3-step paths are counted by A^3; we look at paths to node 2:

$$\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}^3 = \begin{bmatrix}3 & 2\\ 2 & 1\end{bmatrix} \quad \begin{matrix}\textit{counts the paths}\\ \textit{with three steps}\end{matrix} \quad \begin{bmatrix}\ldots & 1\ to\ 1\ to\ 1\ to\ 2,\ 1\ to\ 2\ to\ 1\ to\ 2\\ \ldots & 2\ to\ 1\ to\ 1\ to\ 2\end{bmatrix}$$

The A^k contain Fibonacci numbers 0, 1, 1, 2, 3, 5, 8, 13, ….

Fibonacci's rule $F_{k+2} = F_{k+1} + F_k$ show up in $(A)\left(A^k\right) = A^{k+1}$

$$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}F_{k+1} & F_k\\ F_k & F_{k-1}\end{pmatrix} = \begin{pmatrix}F_{k+2} & F_{k+1}\\ F_{k+1} & F_k\end{pmatrix} = A^{k+1}$$

There are ***13 six-step*** paths from node one to node 1.

Exercise

A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. All entries are 1. Which of these matrix operations are allowed, and what are the results?

a) AB *c)* ABD *e)* ABC *g)* $A(B+C)$

b) BA *d)* DBA *f)* $ABCD$

Solution

$$A = \begin{pmatrix}1 & 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1 & 1\end{pmatrix} \quad B = \begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix} \quad C = \begin{pmatrix}1\\ 1\\ 1\\ 1\\ 1\end{pmatrix} \quad D = \begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$$

a) $AB: (3\times 5)(5\times 3) = (3\times 3)$

$$\begin{pmatrix}1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{pmatrix}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\\1&1&1\\1&1&1\end{pmatrix}=\begin{pmatrix}5&5&5\\5&5&5\\5&5&5\end{pmatrix}$$

b) $BA:(5\times 3)(3\times 5)=(5\times 5)$

$$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{pmatrix}=\begin{pmatrix}3&3&3&3&3\\3&3&3&3&3\\3&3&3&3&3\\3&3&3&3&3\\3&3&3&3&3\end{pmatrix}$$

c) $ABD:(3\times 5)(5\times 3)(3\times 1)=(3\times 1)$

$$\begin{pmatrix}1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{pmatrix}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}5&5&5\\5&5&5\\5&5&5\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}15\\15\\15\end{pmatrix}$$

d) $DBA:(3\times 1)(5\times 3)(3\times 5)=NA$

e) $ABC:(3\times 5)(5\times 3)(5\times 1)=NA$

f) $ABCD:(3\times 5)(5\times 3)(5\times 1)(3\times 1)=NA$

g) $A(B+C):(3\times 5)((5\times 3)+(5\times 1))=NA$

Matrices B and C are not the same size.

Exercise

What rows or columns or matrices do you multiply to find.

a) The third column of *AB*?
b) The second column of *AB*?
c) The first row of *AB*?
d) The second row of *AB*?
e) The entry in row 3, column 4 of *AB*?
f) The entry in row 2, column 3 of *AB*?

Solution

a) *A* (column 3 of *B*)

b) *A* (column 2 of *B*)

c) (Row 1 of A) B

d) (Row 2 of A) B

e) (Row 3 of A) (Column 4 of B)

f) (Row 2 of A) (Column 3 of B)

Exercise

Add AB to AC and compare with $A(B+C)$:

$$A=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix}0 & 2\\ 0 & 1\end{bmatrix} \quad \text{and} \quad C=\begin{bmatrix}3 & 1\\ 0 & 0\end{bmatrix}$$

Solution

$$AB=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix}\begin{bmatrix}0 & 2\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 7\\ 0 & 7\end{bmatrix}$$

$$AC=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix}\begin{bmatrix}3 & 1\\ 0 & 0\end{bmatrix}=\begin{bmatrix}3 & 1\\ 6 & 2\end{bmatrix}$$

$$\begin{aligned}A(B+C)&=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix}\left(\begin{bmatrix}0 & 2\\ 0 & 1\end{bmatrix}+\begin{bmatrix}3 & 1\\ 0 & 0\end{bmatrix}\right)\\ &=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix}\begin{bmatrix}3 & 3\\ 0 & 1\end{bmatrix}\\ &=\begin{bmatrix}3 & 8\\ 6 & 9\end{bmatrix}\end{aligned}$$

$$\begin{aligned}AB+AC&=\begin{bmatrix}0 & 7\\ 0 & 7\end{bmatrix}+\begin{bmatrix}3 & 1\\ 6 & 2\end{bmatrix}\\ &=\begin{bmatrix}3 & 8\\ 6 & 9\end{bmatrix}\end{aligned}$$

$$\boxed{A(B+C)=AB+AC}$$

Exercise

True or False

a) If A^2 is defined then A is necessarily square.

b) If AB and BA are defined then A and B are square.

c) If AB and BA are defined then AB and BA are square.

d) If $AB = B$, then $A = I$

Solution

a) True

b) False, if A has an order m by n and B n by m: $AB : m \times m \quad BA : n \times n$

c) True; $AB : m \times m \quad BA : n \times n$

d) False, if B is the matrix of all zeros.

Exercise

a) Find a nonzero matrix A such that $A^2 = 0$

b) Find a matrix that has $A^2 \neq 0$ but $A^3 = 0$

Solution

a) A nonzero matrix A such that $A^2 = 0$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

b) A matrix that has $A^2 \neq 0$ but $A^3 = 0$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^3 = A^2 A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Exercise

Suppose you solve $Ax = b$ for three special right sides b:

$$Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

If the three solutions x_1, x_2, x_3 are the columns of a matrix X, what is A times X?

Solution

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Therefore, $Ax = I$

Exercise

Show that $(A+B)^2$ is different from $A^2 + 2AB + B^2$, when

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

Write down the correct rule for $(A+B)(A+B) = A^2 + \underline{\qquad\qquad} + B^2$

Solution

$$A + B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix}$$

$$(A+B)^2 = \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix}$$

$$2AB = 2\begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 14 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

$$=\begin{bmatrix}16 & 2\\ 3 & 0\end{bmatrix}$$

$$\begin{bmatrix}10 & 4\\ 6 & 6\end{bmatrix}\neq\begin{bmatrix}16 & 2\\ 3 & 0\end{bmatrix}$$

$$\boxed{(A+B)^2\neq A^2+2AB+B^2}$$

$$BA=\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}$$

$$A^2+AB+BA+B^2=\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}+\begin{bmatrix}7 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}+\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}$$

$$=\begin{bmatrix}10 & 4\\ 5 & 6\end{bmatrix}$$

$$\boxed{(A+B)(A+B)=A^2+\_\,AB+BA\,\_+B^2}$$

Exercise

Find the product of the 2 matrices by rows or by columns: $\begin{bmatrix}2 & 3\\ 5 & 1\end{bmatrix}\begin{bmatrix}4\\ 2\end{bmatrix}$

Solution

By rows: $\begin{bmatrix}2 & 3\\ 5 & 1\end{bmatrix}\begin{bmatrix}4\\ 2\end{bmatrix}=\begin{pmatrix}\begin{pmatrix}2 & 3\end{pmatrix}\begin{pmatrix}4\\ 2\end{pmatrix}\\ \begin{pmatrix}5 & 1\end{pmatrix}\begin{pmatrix}4\\ 2\end{pmatrix}\end{pmatrix}=\begin{pmatrix}14\\ 22\end{pmatrix}$

By columns: $\begin{pmatrix}2 & 3\\ 5 & 1\end{pmatrix}\begin{pmatrix}4\\ 2\end{pmatrix}=4\begin{pmatrix}2\\ 5\end{pmatrix}+2\begin{pmatrix}3\\ 1\end{pmatrix}=\begin{pmatrix}14\\ 22\end{pmatrix}$

Exercise

Find the product of the 2 matrices by rows or by columns: $\begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Solution

By rows: $\begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} (3 & 6)(2 & -1) \\ (6 & 12)(2 & -1) \end{pmatrix}$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

By columns: $\begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 6 \end{pmatrix} - 1\begin{pmatrix} 6 \\ 12 \end{pmatrix}$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Exercise

Find the product of the 2 matrices by rows or by columns: $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

Solution

By rows: $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} (1 & 2 & 4)(3 & 1 & 1) \\ (2 & 0 & 1)(3 & 1 & 1) \end{pmatrix}$

$$= \begin{pmatrix} 1(3) + 2(1) + 4(1) \\ 2(3) + 0(1) + 1(1) \end{pmatrix}$$

$$= \begin{pmatrix} 9 \\ 7 \end{pmatrix}$$

By columns: $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 9 \\ 7 \end{pmatrix}$$

Exercise

Find the product of the 2 matrices by rows or by columns: $\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

Solution

By rows: $\begin{pmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} (1 \quad 2 \quad 4)(2 \quad 2 \quad 3) \\ (-2 \quad 3 \quad 1)(2 \quad 2 \quad 3) \\ (-4 \quad 1 \quad 2)(2 \quad 2 \quad 3) \end{pmatrix}$

$$= \begin{pmatrix} 18 \\ 5 \\ 0 \end{pmatrix}$$

By columns: $\begin{pmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$

$$= \begin{pmatrix} 18 \\ 5 \\ 0 \end{pmatrix}$$

Exercise

Given $A = \begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix}$ $\quad B = \begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix}$ $\quad$ Find $A+B, \quad 2A, \quad and \quad -B$

Solution

$$A+B = \begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix} + \begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & -2 \\ 8 & -2 & 0 \end{bmatrix}$$

$$2A = 2\begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 6 \\ 6 & -2 & -4 \\ 0 & 0 & 8 \end{bmatrix}$$

$$-B = -\begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 3 \\ 1 & 0 & 0 \\ -8 & 2 & 4 \end{bmatrix}$$

Exercise

Given $A=\begin{bmatrix}3 & 2 & -3\\ 0 & 1 & 0\end{bmatrix}$ $\quad B=\begin{bmatrix}3 & -4\\ 0 & 1\\ 1 & 0\end{bmatrix}$ $\quad$ Find AB *and* BA *if possible*

Solution

$$AB=\begin{bmatrix}3 & 2 & -3\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}3 & -4\\ 0 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}6 & -10\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}3(3)+2(0)-3(1) & 3(-4)+2(1)-3(0)\\ 0(3)+1(0)+0(1) & 0(-4)+1(1)+0(0)\end{bmatrix}$$

$$=\begin{bmatrix}6 & -10\\ 0 & 1\end{bmatrix}$$

$$BA=\begin{bmatrix}3 & -4\\ 0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}3 & 2 & -3\\ 0 & 1 & 0\end{bmatrix}=\begin{bmatrix}9 & 2 & -9\\ 0 & 1 & 0\\ 3 & 2 & -3\end{bmatrix}$$

$$=\begin{bmatrix}3(3)-4(0) & 3(2)-4(1) & 3(-3)-4(0)\\ 0(3)+1(0) & 0(2)+1(1) & 0(-3)+1(0)\\ 1(3)+0(0) & 1(2)+0(1) & 1(-3)+0(0)\end{bmatrix}$$

$$=\begin{bmatrix}9 & 2 & -9\\ 0 & 1 & 0\\ 3 & 2 & -3\end{bmatrix}$$

Exercise

Given $A=\begin{bmatrix}5 & 3\\ -1 & 0\end{bmatrix}$ $\quad B=\begin{bmatrix}4 & -2\\ -2 & 0\\ 9 & 1\end{bmatrix}$ $\quad$ Find AB *and* BA *if possible*

Solution

$AB=$ ***Undefined***

$$BA=\begin{bmatrix}4 & -2\\ -2 & 0\\ 9 & 1\end{bmatrix}\begin{bmatrix}5 & 3\\ -1 & 0\end{bmatrix}=\begin{bmatrix}22 & 12\\ -10 & -6\\ 44 & 27\end{bmatrix}$$

Exercise

Given $A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$ $\quad$ Find AB *and* BA *if possible*

Solution

a) $AB = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix}$

b) $BA = Undefined$

Exercise

Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

Compute the following (where possible):

a) $D+E$ ***b***) $D-E$ ***c***) $5A$ ***d***) $-7C$ ***e***) $2B-C$ ***g***) $-3(D+2E)$

Solution

a) $D+E = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 6 & 5 \\ -2 & 1 & 3 \\ 7 & 3 & 7 \end{bmatrix}$

b) $D-E = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 4 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

c) $5A = 5\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 0 \\ -5 & 10 \\ 5 & 5 \end{bmatrix}$

d) $-7C = -7\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} -7 & -28 & -14 \\ -21 & -7 & -35 \end{bmatrix}$

e) $2B - C = can't\ be\ calculated$

g) $-3(D+2E) = -3\left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 12 & 2 & 6 \\ -2 & 2 & 4 \\ 8 & 2 & 6 \end{bmatrix} \right) = -3\begin{bmatrix} 13 & 7 & 8 \\ -3 & 2 & 5 \\ 11 & 4 & 10 \end{bmatrix}$

$$= -3\begin{bmatrix} 13 & 7 & 8 \\ -3 & 2 & 5 \\ 11 & 4 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} -39 & -21 & -24 \\ 9 & -6 & -15 \\ -33 & -12 & -30 \end{bmatrix}$$

Solution   Section 1.4 – Inverse Matrices - Finding A^{-1}

Exercise

Apply Gauss-Jordan method to find the inverse of this triangular "Pascal matrix"

Triangular Pascal matrix $$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$$

Solution

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_2 \\ R_4 - 3R_2 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 3 & 1 & 2 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - 3R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 3 & -3 & 1 \end{array}\right]$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}$$

- The inverse matrix A^{-1} looks like A, except odd-numbered diagonals are multiplied by -1.

Exercise

If A is invertible and $AB = AC$, prove that $B = C$

Solution

$AB = AC$ ***Multiply by*** A^{-1} ***both sides***.

$A^{-1}(AB) = A^{-1}(AC)$ ***Multiplication is associative***

$\left(A^{-1}A\right)B = \left(A^{-1}A\right)C$ $A^{-1}A = I$

$IB = IC$

$\boxed{B = C}$

Exercise

If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, find two matrices $B \neq C$ such that $AB = AC$

Solution

Let $B = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ *and* $C = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$\boxed{B \neq C \Rightarrow AB = AC}$

Exercise

If A has ***row*** 1 + ***row*** 2 = ***row*** 3, show that A is not invertible

a) Explain why $Ax=(1,\ 0,\ 0)$ can't have a solution.

b) Which right sides $(b_1,\ b_2,\ b_3)$ might allow a solution to $Ax=b$

c) What happens to ***row*** 3 in elimination?

Solution

a) Let $A_1,\ A_2,\ A_3$ be the row vectors of A and x is a solution to $Ax=(1,\ 0,\ 0)$.

Then $A_1.x=1,\ A_2.x=0,\ A_3.x=0$.

Since $\quad A_1+A_2=A_3$

Means $\quad A_1.x+A_2.x=A_3.x$

Implies $\quad 1+0=0$ a contradiction

b) If $Ax=(b_1,\ b_2,\ b_3)\Rightarrow A_1.x=b_1,\ A_2.x=b_2,\ A_3.x=b_3$

Since $\quad A_1+A_2=A_3$

$A_1.x+A_2.x=A_3.x$

$\Rightarrow b_1+b_2=b_3$

c) In the elimination matrix, the third row will be zero.

Exercise

True or false (with a counterexample if false and a reason if true):

a) A 4 by 4 matrix with a row of zeros is not invertible.

b) A matrix with 1's down the main diagonal is invertible.

c) If A is invertible then A^{-1} is invertible.

d) If A is invertible then A^2 is invertible.

Solution

a) True, because it can have at most 3 pivots.

b) False, if the matrix: $A=\begin{pmatrix}1&0&0\\0&1&1\\0&1&1\end{pmatrix}$ and only has 2 pivots, thus is not invertible.

c) True, If A is invertible then necessarily A^{-1} is invertible.

d) True, $A^2x = 0$ where x is nonzero matrix.

$$A^{-1}A^2x = \left(A^{-1}A\right)Ax = IAx = Ax = 0$$

Since A is invertible, this can only be true if x was zero to begin with. Thus A^2 must also be invertible.

Exercise

Do there exist 2 by 2 matrices A and B with real entries such that $AB - BA = I$, where I is the identity matrix?

Solution

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$BA = \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix}$$

$$\begin{aligned} AB - BA &= \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} - \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix} \\ &= \begin{pmatrix} bg-cf & af+bh-be-df \\ ce+dg-ag-ch & cf-bg \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{cases} bg - cf = 1 \\ af + bh - be - df = 0 \\ ce + dg - ag - ch = 0 \\ cf - bg = 1 \end{cases} \rightarrow \begin{cases} bg - cf = 1 \\ \underline{cf - bg = 1} \\ 0 = 2 \end{cases}$$

Therefore, $AB - BA \neq I$ for any 2 by 2 matrices.

Exercise

If B is the inverse of A^2, show that AB is the inverse of A.

Solution

Since B is the inverse of A^2 that implies: $\boxed{B = \left(A^2\right)^{-1} = (AA)^{-1} = A^{-1}A^{-1}}$

Show that AB is the inverse of A

$$\begin{aligned}(AB)A &= \left(A\left(A^{-1}A^{-1}\right)\right)A \\ &= \left(\left(AA^{-1}\right)A^{-1}\right)A \\ &= \left(IA^{-1}\right)A \\ &= A^{-1}A \\ &\boxed{= I}\end{aligned}$$

Therefore, AB is the inverse of A.

Exercise

Find and check the inverses (assuming they exist) of these block matrices.

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}$$

Solution

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix}\begin{bmatrix} I & 0 \\ A & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} I & 0 \\ C + A & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \Rightarrow C + A = 0 \to A = -C$$

$$\begin{pmatrix} I & 0 \\ C & I \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -C & I \end{pmatrix}$$

$$\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}\begin{bmatrix} E & 0 \\ F & G \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} AE & 0 \\ CE + DF & DG \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\Rightarrow \begin{cases} AE = I \\ CE + DF = 0 \\ DG = I \end{cases} \rightarrow \begin{cases} E = A^{-1} \\ \\ G = D^{-1} \end{cases}$$

$$\begin{aligned} CE + DF = 0 \rightarrow CA^{-1} + DF &= 0 \\ DF &= -CA^{-1} \\ D^{-1}DF &= -D^{-1}CA^{-1} \\ IF &= -D^{-1}CA^{-1} \\ F &= -D^{-1}CA^{-1} \end{aligned}$$

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{pmatrix}$$

$$\begin{bmatrix} 0 & I \\ I & D \end{bmatrix}\begin{bmatrix} A & I \\ I & B \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} I & B \\ A+D & I+DB \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\rightarrow \begin{cases} B = 0 \\ A + D = 0 \\ I + DB = I \end{cases} \Rightarrow \begin{cases} A = -D \\ DB = 0 \end{cases}$$

$$\begin{pmatrix} 0 & I \\ I & D \end{pmatrix}^{-1} = \begin{pmatrix} -D & I \\ I & 0 \end{pmatrix}$$

Exercise

For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

Solution

$$c = 0, \; A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 8 & 7 & 0 \end{bmatrix} \text{ (zero column 2 / row 2)}$$

$c = 2$, $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 8 & 7 & 2 \end{bmatrix}$ (equal rows)

$c = 7$, $A = \begin{bmatrix} 2 & 7 & 7 \\ 7 & 7 & 7 \\ 8 & 7 & 7 \end{bmatrix}$ (equal columns)

Exercise

Find A^{-1} and B^{-1} (if they exist) by elimination.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Solution

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \frac{1}{2}R_1 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right) \begin{array}{l} \\ \frac{2}{3}R_2 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right) \begin{array}{l} R_1 - \frac{1}{2}R_2 \\ \\ R_3 - \frac{1}{2}R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{array}\right) \begin{array}{l} \\ \\ \frac{3}{4}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right)\begin{array}{l} R_1 - \frac{1}{3}R_3 \\ R_2 - \frac{1}{3}R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right)\begin{array}{l} \frac{1}{2}R_1 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right)\begin{array}{l} \\ R_2 + R_1 \\ R_3 + R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right)\begin{array}{l} \\ \\ R_3 + R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

B^{-1} doesn't exist, and if we add the columns in B, the result is zero.

Exercise

Find A^{-1} using the Gauss-Jordan method, which has a remarkable inverse.

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 + R_2 \\ \\ \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 + R_3 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ R_3 + R_4 \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise

Find the inverse.

a) $\begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$

c) $\begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

e) $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$

f) $\begin{bmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

g) $\begin{bmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{bmatrix}$

Solution

a) $\begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{12} & \frac{1}{6} \\ \frac{1}{8} & \frac{1}{4} \end{bmatrix}$

b) $A^{-1} = \frac{1}{7-8}\begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix}$

$= -\begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix}$

$= \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$

c) $A^{-1} = \frac{1}{-15-24}\begin{bmatrix} 5 & -6 \\ -4 & -3 \end{bmatrix}$

$= -\frac{1}{39}\begin{bmatrix} 5 & -6 \\ -4 & -3 \end{bmatrix}$

$= \begin{bmatrix} -\frac{5}{39} & \frac{2}{13} \\ \frac{4}{39} & \frac{1}{13} \end{bmatrix}$

d) $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_3 - R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right] \quad R_3 - R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right] \quad -\tfrac{1}{2}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \quad \begin{array}{l} R_1 - R_3 \\ R_2 - R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

e) $$\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}^{-1} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ -2 & 2 & 0 \end{pmatrix}$$

f) $$\begin{bmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{pmatrix} \frac{\sqrt{2}}{26} & -\frac{3\sqrt{2}}{26} & 0 \\ \frac{2\sqrt{2}}{13} & \frac{\sqrt{2}}{26} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

g) $$\begin{bmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{bmatrix}^{-1} = doesn't\ exist$$ This matrix is ***singular***

Exercise

Show that A is not invertible for any values of the entries

$$A = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

Solution

Since the matrix A had zero's on its diagonals, therefore A is not invertible.

Exercise

Prove that if A is an invertible matrix and B is row equivalent to A, then B is also invertible.

Solution

Since B is row equivalent to A, there exist some elementary matrices $E_1, E_2, \ldots, E_n$ such that $B = E_n \ \cdots E_1 A$. Because $E_1, E_2, \ldots, E_n$ and A are invertible, then B is also invertible.

Exercise

Determine if the given matrix has an inverse, and find the inverse if it exists. Check your answer by multiplying $A \cdot A^{-1} = I$

a) $\begin{bmatrix} 2 & 3 \\ -3 & -5 \end{bmatrix}$ *b)* $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}$

Solution

a) $2(-5) - 3(-3) = -10 + 9 = -1$

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} -5 & -3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ -3 & -2 \end{bmatrix}$$

$$AA^{-1} = \begin{pmatrix} 2 & 3 \\ -3 & -5 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

b) $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \quad R_3 - 2R_1$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 3 & 3 & -2 & 0 & 1 \end{array}\right] \quad R_3 - 3R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & * & * & * \end{array}\right]$$

The inverse matrix doesn't exist

Exercise

Show that the inverse of $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is $\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$

Solution

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\theta)\cos(-\theta) - (\sin\theta)\sin(-\theta) & (\cos\theta)\sin(-\theta) - (\sin\theta)\cos(-\theta) \\ (-\sin\theta)\cos(-\theta) - (\cos\theta)\sin(-\theta) & (-\sin\theta)\sin(-\theta) + (\cos\theta)\cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\theta + \sin\theta\sin\theta & -\cos\theta\sin\theta - \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin\theta\sin\theta + \cos\theta\cos\theta \end{bmatrix} \qquad \begin{cases} \cos(-\theta) = \cos\theta & (\mathit{even}) \\ \sin(-\theta) = -\sin\theta & (\mathit{odd}) \end{cases}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

Exercise

Solve $Lc = b$ to find $\boldsymbol{c}$. Then solve $Ux = c$ to find $\boldsymbol{x}$. What was A?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Solution

$$Lc = b$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

$$\rightarrow \begin{cases} \boxed{c_1 = 4} \\ c_1 + c_2 = 5 \Rightarrow \underline{c_2} = 5 - 4 = 1 \\ c_1 + c_2 + c_3 = 6 \Rightarrow \underline{c_3} = 6 - 4 - 1 = 1 \end{cases} \qquad c = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

$$Ux = c$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x + y + z = 4 \\ y + z = 1 \\ z = 1 \end{cases} \rightarrow \begin{cases} x = 3 \\ y = 0 \end{cases} \qquad x = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

$$Lc = b \Rightarrow LUx = b$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}}_{b}$$

Exercise

Find L and U for the symmetric matrix

$$A=\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on a, b, c, d to get $A=LU$ with four pivots

Solution

$$A=\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

Exercise

Determine whether the given matrix is invertible

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Solution

The matrix is a diagonal matrix with nonzero entries on the diagonal, so it is invertible.

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}^{-1}=\begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Exercise

Find A^2, A^{-2}, *and* A^{-k} by inspection $A=\begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}$

Solution

$$A^2=\begin{bmatrix}1^2 & 0\\ 0 & (-2)^2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}$$

$$A^{-2}=\begin{bmatrix}1^{-2} & 0\\ 0 & (-2)^{-2}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & \frac{1}{4}\end{bmatrix}$$

$$A^{-k}=\begin{bmatrix}1^{-k} & 0\\ 0 & (-2)^{-k}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & \frac{1}{(-2)^k}\end{bmatrix}$$

Exercise

Find A^2, A^{-2}, *and* A^{-k} by inspection $A=\begin{bmatrix}\frac{1}{2} & 0 & 0\\ 0 & \frac{1}{3} & 0\\ 0 & 0 & \frac{1}{4}\end{bmatrix}$

Solution

$$A^2=\begin{bmatrix}\left(\frac{1}{2}\right)^2 & 0 & 0\\ 0 & \left(\frac{1}{3}\right)^2 & 0\\ 0 & 0 & \left(\frac{1}{4}\right)^2\end{bmatrix}=\begin{bmatrix}\frac{1}{4} & 0 & 0\\ 0 & \frac{1}{9} & 0\\ 0 & 0 & \frac{1}{16}\end{bmatrix}$$

$$A^{-2}=\begin{bmatrix}\left(\frac{1}{2}\right)^{-2} & 0 & 0\\ 0 & \left(\frac{1}{3}\right)^{-2} & 0\\ 0 & 0 & \left(\frac{1}{4}\right)^{-2}\end{bmatrix}=\begin{bmatrix}4 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 16\end{bmatrix}$$

$$A^{-k}=\begin{bmatrix}\left(\frac{1}{2}\right)^{-k} & 0 & 0\\ 0 & \left(\frac{1}{3}\right)^{-k} & 0\\ 0 & 0 & \left(\frac{1}{4}\right)^{-k}\end{bmatrix}=\begin{bmatrix}2^k & 0 & 0\\ 0 & 3^k & 0\\ 0 & 0 & 4^k\end{bmatrix}$$

Exercise

Find $A^2,\ A^{-2},\ and\ A^{-k}$ by inspection $A=\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Solution

$$A^2=\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$A^{-2}=\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{16} & 0 & 0 \\ 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$A^{-k}=\begin{bmatrix} (-2)^{-k} & 0 & 0 & 0 \\ 0 & (-4)^{-k} & 0 & 0 \\ 0 & 0 & (-3)^{-k} & 0 \\ 0 & 0 & 0 & (2)^{-k} \end{bmatrix}$$

Exercise

Decide whether the given matrix is symmetric $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

Solution

Not *symmetric*, since $a_{12}\neq a_{21}\quad (1\neq -1)$

Exercise

Decide whether the given matrix is symmetric $\begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{bmatrix}$

Solution

Symmetric

Exercise

Decide whether the given matrix is symmetric $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

Solution

Not *symmetric*, since $a_{13} = 1 \neq 3 = a_{31}$

Exercise

Find all values of the unknown constant(s) in order for A to be symmetric

$$A = \begin{bmatrix} 2 & a-2b+2c & 2a+b+c \\ 3 & 5 & a+c \\ 0 & -2 & 7 \end{bmatrix}$$

Solution

$$\begin{cases} a-2b+2c=3 \\ 2a+\ b+c=0 \\ a \qquad +c=-2 \end{cases} \qquad \to a=11, \quad b=9, \quad c=-13$$

Exercise

Find a diagonal matrix A that satisfies the given condition $A^{-2} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^{-2} = \begin{pmatrix} a^{-2} & 0 & 0 \\ 0 & b^{-2} & 0 \\ 0 & 0 & c^{-2} \end{pmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\to \begin{cases} a^{-2}=9 & \Rightarrow & a=\pm 9^{-1/2}=\pm\frac{1}{3} \\ b^{-2}=4 & \Rightarrow & b=\pm 2^{-1/2}=\pm\frac{1}{2} \\ c^{-2}=1 & \Rightarrow & c=\pm 1^{-1/2}=\pm 1 \end{cases}$$

$$A = \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A = \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A = \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \cdots \quad A = \begin{pmatrix} \pm\frac{1}{3} & 0 & 0 \\ 0 & \pm\frac{1}{2} & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$$

Exercise

Let A be an $n \times n$ symmetric matrix

a) Show that A^2 is symmetric

b) Show that $2A^2 - 3A + I$ is symmetric

Solution

a) The property of the transpose states that $(AB)^T = B^T A^T$

$$\begin{aligned}\left(A^2\right)^T &= (AA)^T \\ &= A^T A^T \\ &= \left(A^T\right)^2 \qquad \textit{A is symmetric} \\ &= A^2\end{aligned}$$

b) $$\begin{aligned}\left(2A^2 - 3A + I\right)^T &= 2\left(A^2\right)^T - 3(A)^T + (I)^T \\ &= 2\left(A^T\right)^2 - 3A^T + (I)^T \qquad \textit{A and I are symmetric} \\ &= 2A^2 - 3A + I \quad \textbf{Symmetric}\end{aligned}$$

Exercise

Prove if $A^T A = A$, then A is symmetric and $A = A^2$

Solution

If $A^T A = A$, then

$$\begin{aligned}A^T &= \left(A^T A\right)^T \\ &= A^T \left(A^T\right)^T \\ &= A^T A \\ &= A\end{aligned}$$

So A is symmetric.

Since $A = A^T$

$$AA = A^T A \qquad A^T A = A$$

$$A^2 = A$$

Exercise

A square matrix A is called ***skew-symmetric*** if $A^T = -A$. Prove

a) If A is an invertible skew-symmetric matrix, then A^{-1} is skew-symmetric.

b) If A and B are skew-symmetric matrices, then so are A^T, $A+B$, $A-B$, *and* kA for any scalar k.

c) Every square matrix A can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

$\left[\text{Hint: Note the identity } A = \frac{1}{2}\left(A+A^T\right) + \frac{1}{2}\left(A-A^T\right)\right]$

Solution

a)
$$\left(A^{-1}\right)^T = \left(A^T\right)^{-1}$$
$$= (-A)^{-1} \qquad \textit{skew-symmetric}$$
$$= -A^{-1}$$

$\therefore$ A^{-1} is also skew-symmetric

b) Let A and B are skew-symmetric matrices

$$\left(A^T\right)^T = (-A)^T = -A^T$$

$$(A+B)^T = A^T + B^T = -A - B = -(A+B)$$

$$(A-B)^T = A^T - B^T = -A + B = -(A-B)$$

$$(kA)^T = k(A)^T = k(-A) = -kA$$

c) We need to prove from the hint that $\frac{1}{2}\left(A+A^T\right)$ is symmetric and $\frac{1}{2}\left(A-A^T\right)$ is skew-symmetric

$$\frac{1}{2}\left(A+A^T\right)^T = \frac{1}{2}\left(A^T + \left(A^T\right)^T\right)$$
$$= \frac{1}{2}\left(A+A^T\right) \qquad \textit{Thus } \tfrac{1}{2}\left(A+A^T\right) \textbf{\textit{ is symmetric}}$$

$$\frac{1}{2}\left(A-A^T\right)^T = \frac{1}{2}\left(A^T - \left(A^T\right)^T\right)$$
$$= \frac{1}{2}\left(A^T - A\right)$$
$$= -\frac{1}{2}\left(A-A^T\right) \qquad \textit{Thus } \tfrac{1}{2}\left(A-A^T\right) \textbf{\textit{ is skew-symmetric}}$$

Exercise

Suppose R is rectangular (m by n) and A is symmetric (m by m)

a) Transpose $R^T AR$ to show its symmetric

b) Show why $R^T R$ has no negative numbers on its diagonal.

Solution

a) $\left(R^T AR\right)^T = \left(\left(R^T A\right)R\right)^T$

$$= R^T \left(R^T A\right)^T$$

$$= R^T A^T \left(R^T\right)^T$$

$$= R^T AR$$

b) $\left(R^T R\right)_{jj} = \left(column\ j\ of\ R\right).\left(column\ j\ of\ R\right)$

= ***Product of the diagonal entry by itself.***

= length squared of column j.

Exercise

If L is a lower-triangular matrix, then $\left(L^{-1}\right)^T$ is _______Triangular

Solution

$\left(L^{-1}\right)^T$ is ***upper*** triangular.

L^{-1} is a lower-triangular because L is.

The transpose carries the lower-triangular matrices to the upper-triangular (and vice versa).

Exercise

True or False

a) The block matrix $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ is automatically symmetric

b) If A and B are symmetric then their product is symmetric

c) If A is not symmetric then A^{-1} is not symmetric

d) When A, B, C are symmetric, the transpose of ABC is CBA.

e) The transpose of a diagonal matrix is a diagonal.

f) The transpose of an upper triangular matrix is an upper triangular matrix.

g) The sum of an upper triangular matrix and a lower triangular matrix is a diagonal matrix.

h) All entries of a symmetric matrix are determined by the entries occurring on and above the main diagonal.

i) All entries of an upper triangular matrix are determined by the entries occurring on and above the main diagonal.

j) The inverse of an invertible lower triangular matrix is an upper triangular matrix.

k) A diagonal matrix is invertible if and only if all of its diagonal entries are positive.

l) The sum of a diagonal matrix and a lower triangular matrix is a lower triangular matrix.

m) A matrix that is both symmetric and upper triangular must be a diagonal matrix.

n) If A and B are $n\times n$ matrices such that $A+B$ is symmetric, then A and B are symmetric.

o) If A and B are $n\times n$ matrices such that $A+B$ is upper triangular, then A and B are upper triangular.

p) If A^2 is a symmetric matrix, then A is a symmetric matrix.

q) If kA is a symmetric matrix for some $k\neq 0$, then A is a symmetric matrix.

Solution

a) False: $\left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right)$

b) False $\underset{A}{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}}\underset{B}{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}=\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

c) True by definition.

d) True $(ABC)^T=C^T(AB)^T=C^TB^TA^T=CBA$ Since $A^T=A,\ B^T=B,\ C^T=C$

e) True Since a diagonal matrix must be square and have zeros off the main diagonal, its transpose is also diagonal.

f) False The transpose of an upper triangular matrix is lower triangular.

g) False $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}+\begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix}$

h) True The entries above the main diagonal determine the entries below the main diagonal in a symmetric matrix.

i) True in an upper triangular matrix, the series below the main diagonal are all zeros.

j) False The inverse of an invertible lower triangular matrix is lower triangular.

k) False The diagonal entries may be negative, as long as they are nonzero.

l) True Adding a diagonal matrix to a lower triangular matrix will not create nonzero entries above the main diagonal.

m) True Since the entries below the main diagonal must be zero, so also must be the entries above the main diagonal.

n) False $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ which is symmetric

o) False $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 0 & 4 \end{bmatrix}$ which is upper triangular.

p) False $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

q) True $(kA)^T = kA$ then

$(kA)^T - kA = 0$

$kA^T - kA = 0$

$k\left(A^T - A\right) = 0$ since $k \neq 0$ then $A^T = A$

Therefore, A is a symmetric matrix

Exercise

Find 2 by 2 symmetric matrices $A = A^T$ with these properties

a) A is not invertible

b) A is invertible but cannot be factored into LU (row exchanges needed)

c) A can be factored into LDL^T but not into LL^T (because of negative D)

Solution

a) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

b) $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ only need a *zero* in the diagonal.

c) $A = LDL^T$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} a & 0 \\ a & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a & a \\ a & a+d \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}}_{LL^T} \to \begin{cases} a = 1 \\ d = 1 \end{cases}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \to D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Exercise

A group of matrices includes AB and A^{-1} if it includes A and B. "Products and inverses stay in the group." Which of these sets are groups?
Lower triangular matrices L with 1's on the diagonal, symmetric matrices S, positive matrices M, diagonal invertible matrices D, permutation matrices P, matrices with $Q^T = Q^{-1}$. ***Invent two more matrix groups***.

Solution

The lower triangular matrices L with 1's on the diagonal form a group.
Clearly the product of two is a third. The Gauss-Jordan method shows that the inverse of one is another.
The symmetric matrices don't form a group. An example of the 2 symmetric matrices A and B whose product is not symmetric

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \quad AB = \begin{bmatrix} 2 & 4 & 5 \\ 1 & 2 & 3 \\ 3 & 5 & 6 \end{bmatrix}$$

The positive matrices do not form a group.

$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad M^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, the inverse is not symmetric.

The diagonal invertible matrices form a group.
The permutation matrices form a group.
The matrices with $Q^T = Q^{-1}$ form a group. If A and B are two matrices, then so are AB and A^{-1}, as

$$(AB)^T = B^T A^T = B^{-1}A^{-1} = (AB)^{-1}$$

$$\left(A^{-1}\right)^T = \left(A^T\right)^{-1} = A^{-1}$$

There are many more matrix groups. For example, given two, the block matrices $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ form a third as A ranges over the first group and B ranges over the second.
Another example is the set of all products cP where c is a nonzero scalar and P is a permutation matrix of given size.

Exercise

Write $A=\begin{bmatrix}1 & 2\\ 4 & 9\end{bmatrix}$ as the product EH of an elementary row operation matrix E and a symmetric matrix H.

Solution

$$A=EH$$
$$E^{-1}A=E^{-1}EH$$
$$E^{-1}A=H$$

An elementary row operation matrix has the form $E=\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}$

The inverse is: $E^{-1}=\begin{pmatrix}1 & 0\\ -x & 1\end{pmatrix}$

$$H=\begin{pmatrix}1 & 0\\ -x & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ 4 & 9\end{pmatrix}=\begin{pmatrix}1 & 2\\ -x+4 & -2x+9\end{pmatrix}$$

Since matrix H is symmetric, therefore:

$$\Rightarrow -x+4=2\rightarrow \boxed{x=2}$$

$$\begin{pmatrix}1 & 2\\ 4 & 9\end{pmatrix}=\underbrace{\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}}_{Elementary}\ \underbrace{\begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix}}_{Symmetric}$$

Exercise

When is the product of two symmetric matrices symmetric? Explain your answer.

Solution

AB is symmetric *iff* $AB=(AB)^T$

$$\begin{aligned}AB&=(AB)^T\\ &=B^T A^T && \textit{A and B are symmetric}\\ &=BA\end{aligned}$$

AB is symmetric iff A and B commute

Exercise

Express $\left((AB)^{-1}\right)^T$ in terms of $\left(A^{-1}\right)^T$ and $\left(B^{-1}\right)^T$

Solution

$$\left((AB)^{-1}\right)^T = \left(B^{-1}A^{-1}\right)^T$$

$$= \left(A^{-1}\right)^T \left(B^{-1}\right)^T$$

Exercise

Find the transpose of the given matrix: $\begin{bmatrix} 8 & -1 \\ 3 & 5 \\ -2 & 5 \\ 1 & 2 \\ -3 & -5 \end{bmatrix}$

Solution

$$A^T = \begin{bmatrix} 8 & 3 & -2 & 1 & -3 \\ -1 & 5 & 5 & 2 & -5 \end{bmatrix}$$

Exercise

For the given matrix, compute A^T, $\left(A^T\right)^{-1}$, A^{-1}, and $\left(A^{-1}\right)^T$, then compare $\left(A^T\right)^{-1}$ and $\left(A^{-1}\right)^T$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

Solution

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_1 - 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 + R_3 \\ R_1 - 2R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \qquad \left(A^T\right)^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 2 & 1 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\left(A^{-1}\right)^T = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

Solution   Section 1.6 – The Properties of Determinants

Exercise

Verify that $\det(AB) = \det(A)\det(B)$ when: $A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{pmatrix}$

Solution

$$AB = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{pmatrix}$$

$$\det(AB) = \begin{vmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{vmatrix}\begin{matrix} 9 & -1 \\ 31 & 1 \\ 10 & 0 \end{matrix} = -170$$

$$\det(A) = \begin{vmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 10$$

$$\det(B) = \begin{vmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{vmatrix} = -17$$

$$\det(AB) = \det(A)\det(B) \quad = -170 \quad \checkmark$$

Exercise

For which value(s) of $\boldsymbol{k}$ does A fail to be invertible? $A = \begin{bmatrix} k-3 & -2 \\ -2 & k-2 \end{bmatrix}$

Solution

For $\boldsymbol{A}$ to have an invertible the determinant cannot be equal to zero. To ***fail*** det(A) = 0.

$$|A| = \begin{vmatrix} k-3 & -2 \\ -2 & k-2 \end{vmatrix} = 0$$

$$(k-3)(k-2) - 4 = 0$$

$$k^2 - 5k + 6 - 4 = 0$$

$$k^2 - 5k + 2 = 0 \Rightarrow \boxed{k = \frac{5 \pm \sqrt{17}}{2}}$$

Exercise

Without directly evaluating, show that $\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$

Solution

$$\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \overset{R_2+R_1}{=} \begin{vmatrix} a+b+c & b+c+a & c+b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$$

It is equal to zero, since first row and third row are proportional.

$$\begin{vmatrix} a+b+c & b+c+a & c+b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \underset{R_3 - \frac{1}{a+b+c}R_1}{=} \begin{vmatrix} a+b+c & b+c+a & c+b+a \\ a & b & c \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Exercise

If the entries in every row of A add to zero, solve $\boldsymbol{Ax}$ = 0 to prove det A = 0. If those entries add to one, show that det $(A - I) = 0$. Does this mean det $A = I$?

Solution

If $\boldsymbol{x} = (1, 1, \dots, 1)$, then $\boldsymbol{Ax}$ = the sums of the rows of $\boldsymbol{A}$. Since every row of A add to zero, that implies $\boldsymbol{Ax} = 0$. Since A has non-zero nullspace, it is not invertible and det $A = 0$. If the entries in every row of A sum to one, then the entries in every row of $A - \mathrm{I}$ sum to zero. A – I has a non-zero nullspace and det $(A - \mathrm{I}) = 0$. This does not mean that det $A = \mathrm{I}$.

Example: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ every row of A add to zero $\Rightarrow \det A = -1 \neq 1 = \det I$

Exercise

Does $\det(AB) = \det(BA)$ in general?

a) True or false if $\boldsymbol{A}$ and $\boldsymbol{B}$ are square n x n matrices?

b) True or false if $\boldsymbol{A}$ is m x n and B is n x m with $m \neq n$?

Solution

a) Matrices A and B are square matrices, then by the property:

$$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$$

Therefore it is true for any $\boldsymbol{A}$ and $\boldsymbol{B}$ square matrices.

b) False, example if $A=\begin{pmatrix}1\\1\end{pmatrix}$ $\quad B=\begin{pmatrix}1 & 1\end{pmatrix}$

$$\det AB=\det\left[\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix}\right]=\det\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}=0$$

$$\det AB=\det\left[\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right]=\det(2)=2$$

Exercise

True or false, with a reason if true or a counterexample if false:

a) The determinant of $I+A$ is 1 + det ***A***.

b) The determinant of ABC is $|A||B||C|$.

c) The determinant of 4A is $4|A|$

d) The determinant of $AB-BA$ is zero. (try an example)

e) If A is not invertible then AB is not invertible.

f) The determinant of $A-B$ equals to det A – det B.

Solution

a) False, if $A=\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\Rightarrow\det(I+A)=\det\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}=0$

$$\det A=1\Rightarrow 1+\det A=1+1=2\ \underline{\neq\det(I+A)}$$

b) True, $\det(ABC)=\det(A)\det(BC)=\det(A)\det(B)\det(C)$.

c) False, in general $\det(4A)=4^n\det(A)$ if A is n x n.

d) False, $A=\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$, $\quad B=\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$

$$\begin{aligned}AB-BA&=\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}-\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\\&=\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}-\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\\&=\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\end{aligned}$$

$$\det(AB-BA)=\begin{vmatrix}0 & 1\\-1 & 0\end{vmatrix}=1\neq 0$$

e) False, any matrix is invertible, iff its determinant is nonzero. So det A = 0 which

$\det(AB) = \det(A)\det(B) = 0$. Therefore, AB can't be invertible.

f) $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow |A| = 0, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow |B| = -1$

$$\left.\begin{array}{l} \underline{\det(A) - \det(B)} = 0 - (-1) \ = 1 \\ \underline{\det(A - B)} = \det\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \ = -1 \end{array}\right. \Rightarrow \det(A - B) \neq \det(A) - \det(B)$$

Exercise

Use row operations to show the 3 by 3 "Vandermonde determinant" is

$$\det\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b-a)(c-a)(c-b)$$

Solution

$$\det\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = \det\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$= \det\begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix} \quad factor\,(b-a)$$

$$= (b-a)\det\begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & c-a & c^2-a^2 \end{bmatrix} \quad R_2 - (c-a)R_2$$

$$(c-a)(c+a) - (b+a)(c-a) = (c-a)(c+a-b-a)$$

$$= (b-a)\det\begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & (c-a)(c-b) \end{bmatrix} \quad \textit{Multiply the main diagonal by } (b - a)$$

$$\underline{= (b-a)(c-a)(c-b)}$$

Exercise

The inverse of a 2 by 2 matrix seems to have determinant = 1:

$$\det A^{-1} = \det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{ad-bc}{ad-bc} = 1$$

What is wrong with this calculation? What is the correct $\det A^{-1}$

Solution

The $\det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $(ad - bc)$ it is part of the determinant and it is not the solution.

$$\det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\det\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad-bc}\frac{1}{ad-bc}(ad-bc)$$

$$\boxed{= \frac{1}{ad-bc}}$$

Exercise

A *Hessenberg* matrix is a triangular matrix with one extra diagonal. Use cofactors of row 1 to show that the 4 by 4 determinant satisfies Fibonacci's rule $|H_4| = |H_3| + |H_2|$. The same rule will continue for all sizes $|H_n| = |H_{n-1}| + |H_{n-2}|$. Which Fibonacci number is $|H_n|$?

$$H_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad H_3 = \begin{bmatrix} 2 & 1 & \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad H_4 = \begin{bmatrix} 2 & 1 & & \\ \mathbf{1} & 2 & \mathbf{1} & \\ \mathbf{1} & 1 & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & 1 & \mathbf{1} & \mathbf{2} \end{bmatrix}$$

Solution

$$|H_4| = 2C_{11} + 1C_{12}$$

The cofactor C_{11} for H_4 is the determinant $|H_3|$.

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

The cofactor $C_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$

$$= -\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= -\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} \qquad \left|H_2\right| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= -\left|H_3\right| + \left|H_2\right|$$

$$\left|H_4\right| = 2C_{11} + 1C_{12}$$

$$= 2\left|H_3\right| - \left|H_3\right| + \left|H_2\right|$$

$$= \left|H_3\right| + \left|H_2\right|$$

The actual number: $\left|H_2\right| = 3, \quad \left|H_3\right| = 5, \quad H_4 = 8$.

Since $\left|H_n\right|$ follows Fibonacci's rule $\left|H_{n-1}\right| + \left|H_{n-2}\right|$, it must be $\left|H_n\right| = F_{n+2}$.

Exercise

Evaluate the determinant:

a) $\begin{vmatrix} -1 & 7 \\ -8 & -3 \end{vmatrix}$

b) $\begin{vmatrix} a-3 & 5 \\ -3 & a-2 \end{vmatrix}$

c) $\begin{vmatrix} k-1 & 2 \\ 4 & k-3 \end{vmatrix}$

d) $\begin{vmatrix} c & -4 & 3 \\ 2 & 1 & c^2 \\ 4 & c-1 & 2 \end{vmatrix}$

e) $\begin{vmatrix} 1 & 0 & 3 \\ 4 & 0 & -1 \\ 2 & 8 & 6 \end{vmatrix}$

f) $\begin{vmatrix} x & -3 & 9 \\ 2 & 4 & x+1 \\ 1 & x^2 & 3 \end{vmatrix}$

g) $\begin{vmatrix} -3 & 1 & 2 \\ 6 & 2 & 1 \\ -9 & 1 & 2 \end{vmatrix}$

h) $\begin{vmatrix} 1 & 4 & -3 & 1 \\ 2 & 0 & 6 & 3 \\ 4 & -1 & 2 & 5 \\ 1 & 0 & -2 & 4 \end{vmatrix}$

i) $\begin{vmatrix} 1 & 3 & 2 & -1 \\ 4 & 3 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 5 & 1 & 3 & -2 \end{vmatrix}$

j) $\begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 3 & 1 & -2 & 0 \\ 1 & 0 & 3 & -3 \end{vmatrix}$

Solution

a) $\begin{vmatrix} -1 & 7 \\ -8 & -3 \end{vmatrix} = (-1)(-3)-(7)(-8) \underline{=59|}$

b) $\begin{vmatrix} a-3 & 5 \\ -3 & a-2 \end{vmatrix} = (a-3)(a-2)+15$

$= a^2-5a+6+15$

$\underline{= a^2-5a+21|}$

c) $\begin{vmatrix} k-1 & 2 \\ 4 & k-3 \end{vmatrix} = (k-1)(k-3)-8$

$= k^2-4k+3-8$

$\underline{= k^2-4k-5|}$

d) $\begin{vmatrix} c & -4 & 3 \\ 2 & 1 & c^2 \\ 4 & c-1 & 2 \end{vmatrix} = 2c-16c^2+6c-6-12-c^4+c^3+16 \ \underline{= -c^4+c^3-16c^2+8c-2|}$

e) $\begin{vmatrix} 1 & 0 & 3 \\ 4 & 0 & -1 \\ 2 & 8 & 6 \end{vmatrix} = 0+0+96-0+8-0 \ \underline{=104|}$

f) $\begin{vmatrix} x & -3 & 9 \\ 2 & 4 & x+1 \\ 1 & x^2 & 3 \end{vmatrix} = 12x-3(x+1)+18x^2-36-x^3(x+1)+18$

$= 12x-3x-3+18x^2-36-x^4-x^3+18$

$\underline{= -x^4-x^3+18x^2+9x-21|}$

g) $\begin{vmatrix} -3 & 1 & 2 \\ 6 & 2 & 1 \\ -9 & 1 & 2 \end{vmatrix} = -12-9+12+36+3-12 \ \underline{=18|}$

h) $\begin{vmatrix} 1 & 4 & -3 & 1 \\ 2 & 0 & 6 & 3 \\ 4 & -1 & 2 & 5 \\ 1 & 0 & -2 & 4 \end{vmatrix} \underline{=275|}$

i) $\begin{vmatrix} 1 & 3 & 2 & -1 \\ 4 & 3 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 5 & 1 & 3 & -2 \end{vmatrix} = 0$ Since row 3 has zero.

j) $\begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 3 & 1 & -2 & 0 \\ 1 & 0 & 3 & -3 \end{vmatrix} = (2)(-1)(-2)(-3) \underline{= -12}$

Exercise

Find all the values of λ for which det($\boldsymbol{A}$) = 0: $A = \begin{bmatrix} \lambda - 1 & -2 \\ 1 & \lambda - 4 \end{bmatrix}$

Solution

$$\begin{vmatrix} \lambda - 1 & -2 \\ 1 & \lambda - 4 \end{vmatrix} = (\lambda - 1)(\lambda - 4) + 2$$

$$= \lambda^2 - 5\lambda + 4 + 2$$

$$= \lambda^2 - 5\lambda + 6 = 0$$

Solve for λ.

$$\boxed{\lambda = -1,\ 6}$$

Exercise

Find all the values of λ for which det($\boldsymbol{A}$) = 0: $A = \begin{bmatrix} \lambda - 6 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 4 & \lambda - 4 \end{bmatrix}$

Solution

$$\begin{vmatrix} \lambda - 6 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 4 & \lambda - 4 \end{vmatrix} = \lambda(\lambda - 6)(\lambda - 4) + 4(\lambda - 6)$$

$$= \lambda\left(\lambda^2 - 10\lambda + 24\right) + 4\lambda - 24$$

$$= \lambda^3 - 10\lambda^2 + 24\lambda + 4\lambda - 24$$

$$= \lambda^3 - 10\lambda^2 + 28\lambda - 24$$

$$\lambda^3 - 10\lambda^2 + 28\lambda - 24 = 0 \to \boxed{\lambda = 2,\ 2,\ 6}$$

Exercise

Prove that if a square matrix $\boldsymbol{A}$ has a column of zeros, then det($\boldsymbol{A}$) = 0

Solution

Consider a 3 by 3 matrix with a zero column, however to find the determinant we can interchange any column of that matrix; therefore:

$$\boldsymbol{A}=[a_{ij}]=\begin{bmatrix}0 & a_{12} & a_{13}\\ 0 & a_{22} & a_{23}\\ 0 & a_{32} & a_{33}\end{bmatrix}$$

By definition, the determinant of $\boldsymbol{A}$ using the cofactor:

$$|A|=a_{11}\begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}-a_{12}\begin{vmatrix}a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix}+a_{13}\begin{vmatrix}a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix}$$

$$=0\begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}-a_{12}\begin{vmatrix}0 & a_{23}\\ 0 & a_{33}\end{vmatrix}+a_{13}\begin{vmatrix}0 & a_{22}\\ 0 & a_{32}\end{vmatrix}$$

$$\underline{=0}$$

Exercise

With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:

$$\begin{vmatrix}A & B\\ 0 & D\end{vmatrix}=|A||D| \quad \textit{but} \quad \begin{vmatrix}A & B\\ C & D\end{vmatrix}\neq|A||D|-|C||B|$$

a) Why is the first statement true? Somehow B doesn't enter.

b) Show by example that equality fails (as shown) when C enters.

c) Show by example that the answer $\det(AD-CB)$ is also wrong.

Solution

a) If we don't pick any 0 entries, then the first two columns are picked from $\boldsymbol{A}$ and the last two rows are from D. We can't pick any columns or rows from B, because there aren't any left.

b) $\left|\begin{array}{cc:cc}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \hdashline 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\end{array}\right|=1.$

and $A=\begin{vmatrix}0 & 1\\ 0 & 0\end{vmatrix}=0,\quad B=\begin{vmatrix}0 & 0\\ 1 & 0\end{vmatrix}=0,\quad C=\begin{vmatrix}0 & 0\\ 1 & 0\end{vmatrix}=0,\quad D=\begin{vmatrix}0 & 1\\ 0 & 0\end{vmatrix}=0$

c) Use the example from part (*b*): $1\neq 0$

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D| - |C||B|$$

Exercise

Show that the value of the following determinant is independent of θ.

$$\begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta - \cos\theta & \sin\theta + \cos\theta & 1 \end{vmatrix}$$

Solution

$$\begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta - \cos\theta & \sin\theta + \cos\theta & 1 \end{vmatrix} = \begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix}$$

$$= \sin^2\theta + \cos^2\theta$$

$$= \sin^2\theta - \left(-\cos^2\theta\right)$$

$$= 1$$

Therefore, the determinant is independent of θ.

Exercise

Show that the matrices $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ *and* $B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$ commute if and only if $\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$

Solution

$$AB = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} d & e \\ 0 & f \end{pmatrix} = \begin{pmatrix} ad & ae+bf \\ 0 & cf \end{pmatrix}$$

$$BA = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} da & db+ec \\ 0 & fc \end{pmatrix}$$

$$AB = BA \quad \Rightarrow \quad \begin{pmatrix} ad & ae+bf \\ 0 & cf \end{pmatrix} = \begin{pmatrix} da & db+ec \\ 0 & fc \end{pmatrix}$$

Iff $ae + bf = db + ec$

$$\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = b(d-f) - e(a-c) = bd - bf - ea + ec = 0$$

$$\boxed{bd + ec = bf + ae} \quad \checkmark$$

Exercise

Show that $\det(A) = \frac{1}{2}\begin{vmatrix} tr(A) & 1 \\ tr(A^2) & tr(A) \end{vmatrix}$ for every 2×2 matrix A.

Solution

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow tr(A) = a + d$

$$A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} \Rightarrow tr(A^2) = a^2 + bc + bc + d^2$$

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{aligned} \frac{1}{2}\begin{vmatrix} tr(A) & 1 \\ tr(A^2) & tr(A) \end{vmatrix} &= \frac{1}{2}\begin{vmatrix} a+d & 1 \\ a^2 + bc + bc + d^2 & a+d \end{vmatrix} \\ &= \frac{1}{2}\left[(a+d)^2 - \left(a^2 + bc + bc + d^2\right)\right] \\ &= \frac{1}{2}\left(a^2 + 2ad + d^2 - a^2 - bc - bc - d^2\right) \\ &= \frac{1}{2}(2ad - 2bc) \\ &= ad - bc \\ &= \det(A) \end{aligned}$$

Exercise

What is the maximum number of zeros that a 4×4 matrix can have without a zero determinant? Explain your reasoning.

Solution

The maximum number of zeros that a 4×4 matrix can have without a zero determinant is 12 zeros. If the main diagonal has nonzero entries and the rest are zero, then the determinant of the matrix is equal to the product of the main diagonal entries.

Exercise

Evaluate *det* ***A***, *det* ***E***, and *det* (***AE***). Then verify that (*det* ***A***)(*det* ***E***) = *det*(***AE***)

$$A = \begin{bmatrix} 4 & 1 & 3 \\ 0 & -2 & 0 \\ 3 & 1 & 5 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & & \\ & 3 & \\ & & 1 \end{bmatrix}$$

Solution

$$\det(A) = \begin{vmatrix} 4 & 1 & 3 \\ 0 & -2 & 0 \\ 3 & 1 & 5 \end{vmatrix} = -40 + 18 \underline{= -22}$$

$$\det(E) = \begin{vmatrix} 1 & & \\ & 3 & \\ & & 1 \end{vmatrix} \underline{= 3}$$

$$AE = \begin{bmatrix} 4 & 1 & 3 \\ 0 & -2 & 0 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 3 \\ 0 & -6 & 0 \\ 3 & 3 & 5 \end{bmatrix}$$

$$\det(AE) = \begin{vmatrix} 4 & 3 & 3 \\ 0 & -6 & 0 \\ 3 & 3 & 5 \end{vmatrix} = -120 + 54 \underline{= -66}$$

$$\det(A)\det(E) = (-22)(3) \underline{= -66}$$

$$\det(A)\det(E) = \det(AE) \qquad \checkmark$$

Exercise

Show that $\begin{bmatrix} \sin^2\alpha & \sin^2\beta & \sin^2\gamma \\ \cos^2\alpha & \cos^2\beta & \cos^2\gamma \\ 1 & 1 & 1 \end{bmatrix}$ is not invertible for any values of α, β, γ

Solution

$$\begin{vmatrix} \sin^2\alpha & \sin^2\beta & \sin^2\gamma \\ \cos^2\alpha & \cos^2\beta & \cos^2\gamma \\ 1 & 1 & 1 \end{vmatrix} = \sin^2\alpha\cos^2\beta + \sin^2\beta\cos^2\gamma + \sin^2\gamma\cos^2\alpha - \sin^2\gamma\cos^2\beta - \sin^2\alpha\cos^2\gamma - \cos^2\alpha\sin^2\beta$$

$$= \sin^2\alpha\left(\cos^2\beta - \cos^2\gamma\right) + \cos^2\alpha\left(\sin^2\gamma - \sin^2\beta\right) + \sin^2\beta\cos^2\gamma - \sin^2\gamma\cos^2\beta$$

$$= \sin^2\alpha\left(\cos^2\beta - \cos^2\gamma\right) + \cos^2\alpha\left(1 - \cos^2\gamma - 1 + \cos^2\beta\right) + \left(1 - \cos^2\beta\right)\cos^2\gamma - \left(1 - \cos^2\gamma\right)\cos^2\beta$$

$$= \sin^2\alpha\left(\cos^2\beta - \cos^2\gamma\right) + \cos^2\alpha\left(\cos^2\beta - \cos^2\gamma\right) + \cos^2\gamma - \cos^2\gamma\cos^2\beta - \cos^2\beta + \cos^2\gamma\cos^2\beta$$

$$= \left(\sin^2\alpha + +\cos^2\alpha\right)\left(\cos^2\beta - \cos^2\gamma\right) + \cos^2\gamma - \cos^2\beta$$

$$= \cos^2\beta - \cos^2\gamma + \cos^2\gamma - \cos^2\beta$$

$\underline{= 0}$ Therefore, this matrix in not invertible.

Solution Section 1.7 – Properties of Determinants: Cramer's Rule

Exercise

Use Cramer's Rule with ratios $\dfrac{\det B_j}{\det A}$ to solve $A\boldsymbol{x} = b$. Also find the inverse matrix $A^{-1} = \dfrac{C^T}{\det A}$. Why is the solution $\boldsymbol{x}$ is the first part the same as column 3 of A^{-1}? Which cofactors are involved in computing that column $\boldsymbol{x}$?

$$Ax = b \quad is \quad \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Find the volumes of the boxes whose edges are columns of $\boldsymbol{A}$ and then rows of A^{-1}.

Solution

$$|A| = \begin{vmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{vmatrix} = 2$$

$$|B_1| = \begin{vmatrix} 0 & 6 & 2 \\ 0 & 4 & 2 \\ 1 & 9 & 0 \end{vmatrix} = 4 \qquad |B_2| = \begin{vmatrix} 2 & 0 & 2 \\ 1 & 0 & 2 \\ 5 & 1 & 0 \end{vmatrix} = -2 \qquad |B_1| = \begin{vmatrix} 2 & 6 & 0 \\ 1 & 4 & 0 \\ 5 & 9 & 1 \end{vmatrix} = 2$$

$$x = \tfrac{4}{2} = 2; \quad y = \tfrac{-2}{2} = -1; \quad z = \tfrac{2}{2} = 1$$

The solution is: $(2,\ -1,\ 1)$

$$C_{11} = \begin{vmatrix} 4 & 2 \\ 9 & 0 \end{vmatrix} = -18 \quad C_{12} = -\begin{vmatrix} 1 & 2 \\ 5 & 0 \end{vmatrix} = 10 \quad C_{13} = \begin{vmatrix} 1 & 4 \\ 5 & 9 \end{vmatrix} = -11$$

$$C_{21} = -\begin{vmatrix} 6 & 2 \\ 9 & 0 \end{vmatrix} = 18 \quad C_{22} = \begin{vmatrix} 2 & 2 \\ 5 & 0 \end{vmatrix} = -10 \quad C_{23} = -\begin{vmatrix} 2 & 6 \\ 5 & 9 \end{vmatrix} = 12$$

$$C_{31} = \begin{vmatrix} 6 & 2 \\ 4 & 2 \end{vmatrix} = 4 \quad C_{32} = -\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = -2 \quad C_{33} = \begin{vmatrix} 2 & 6 \\ 1 & 4 \end{vmatrix} = 2$$

$$C = \begin{pmatrix} -18 & 10 & -11 \\ 18 & -10 & 12 \\ 4 & -2 & 2 \end{pmatrix} \Rightarrow C^T = \begin{pmatrix} -18 & 18 & 4 \\ 10 & -10 & -2 \\ -11 & 12 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{C^T}{\det A} = \frac{1}{2}\begin{pmatrix} -18 & 18 & 4 \\ 10 & -10 & -2 \\ -11 & 12 & 2 \end{pmatrix} = \begin{pmatrix} -9 & 9 & 2 \\ 5 & -5 & -1 \\ -\frac{11}{2} & 6 & 1 \end{pmatrix}$$

The solution $\boldsymbol{x}$ is the third column of A^{-1} because $\boldsymbol{b}$ = (0, 0, 1) is the third column of I.

The volume of the boxes whose edges are columns of $\boldsymbol{A}$ = det($\boldsymbol{A}$) = 2.

Since $\left|A^T\right| = |A|$. The box from rows of A^{-1} has volume $\left|A^{-1}\right| = \frac{1}{|A|}\ \underline{= \frac{1}{2}}$

Exercise

Verify that $\det(AB) = \det(BA)$ and determine whether the equality $\det(A+B) = \det(A) + \det(B)$ holds

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

Solution

$$AB = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{bmatrix} \qquad \det(AB) = \begin{vmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{vmatrix} = -170$$

$$BA = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 6 \\ 17 & 11 & 4 \\ 10 & 5 & 2 \end{bmatrix} \qquad \det(BA) = \begin{vmatrix} -1 & -3 & 6 \\ 17 & 11 & 4 \\ 10 & 5 & 2 \end{vmatrix} = -170$$

Thus, $\boxed{\det(AB) = \det(BA)}$

$$\det(A) = \begin{vmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 10$$

$$\det(B) = \begin{vmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{vmatrix} = -17$$

$$A + B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 10 & 5 & 2 \\ 5 & 0 & 3 \end{bmatrix} \qquad \det(A+B) = \begin{vmatrix} 3 & 0 & 3 \\ 10 & 5 & 2 \\ 5 & 0 & 3 \end{vmatrix} = -30$$

$$\det(A) + \det(B) = 10 - 17 = -7$$

$$\neq \underline{\det(A+B)}$$

Exercise

Verify that $\det(kA) = k^n \det(A) \quad A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}, \quad k = 2$

Solution

$$\det(A) = \begin{vmatrix} -1 & 2 \\ 3 & 4 \end{vmatrix} = -10$$

$$\begin{aligned} \det(2A) &= \begin{vmatrix} -2 & 4 \\ 6 & 8 \end{vmatrix} \\ &= -40 \\ &= 4(-10) \\ &= 2^2(-10) \\ &= k^2 \det(A) \end{aligned}$$

Exercise

Verify that $\det(kA) = k^n \det(A) \quad A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{bmatrix}, \quad k = -2$

Solution

$$\det(A) = \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{vmatrix} = 56$$

$$\begin{aligned} \det(-2A) &= \begin{vmatrix} -4 & 2 & -6 \\ -6 & -4 & -2 \\ -2 & -8 & -0 \end{vmatrix} \\ &= -448 \\ &= (-2)^3(56) \\ &= k^3 \det(A) \end{aligned}$$

Exercise

Verify that $\det(kA) = k^n \det(A)$ $\quad A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix}, \quad k = 3$

Solution

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{vmatrix} = -7$$

$$\begin{aligned} \det(3A) &= \begin{vmatrix} 3 & 3 & 3 \\ 0 & 6 & 9 \\ 0 & 3 & -6 \end{vmatrix} \\ &= -189 \\ &= 3^3(-7) \\ &= k^3 \det(A) \end{aligned}$$

Exercise

Solve by using Cramer's rule

a) $\begin{cases} 7x - 2y = 3 \\ 3x + y = 5 \end{cases}$

b) $\begin{cases} 4x + 5y \qquad = 2 \\ 11x + y + 2z = 3 \\ x + 5y + 2z = 1 \end{cases}$

c) $\begin{cases} x - 4y + z = 6 \\ 4x - y + 2z = -1 \\ 2x + 2y - 3z = -20 \end{cases}$

d) $\begin{cases} -x_1 - 4x_2 + 2x_3 + x_4 = -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 = 14 \\ -x_1 + x_2 + 3x_3 + x_4 = 11 \\ -x_1 - 2x_2 + x_3 - 4x_4 = -4 \end{cases}$

e) $\begin{cases} 2x - y + z = -1 \\ 3x + 4y - z = -1 \\ 4x - y + 2z = -1 \end{cases}$

Solution

a) $\begin{cases} 7x - 2y = 3 \\ 3x + y = 5 \end{cases}$

$$D = \begin{vmatrix} 7 & -2 \\ 3 & 1 \end{vmatrix} = 13 \qquad D_x = \begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} = 13 \qquad D_y = \begin{vmatrix} 7 & 3 \\ 3 & 5 \end{vmatrix} = 26$$

$$x = \frac{D_x}{D} = \frac{13}{13} = 1 \qquad y = \frac{D_y}{D} = \frac{26}{13} = 2$$

Solution: $\boxed{(1,\ 2)}$

b) $\begin{cases} 4x+5y \qquad = 2 \\ 11x+y+2z=3 \\ x+5y+2z=1 \end{cases}$

$D = \begin{vmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} = -132$ $\qquad D_x = \begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} = -36$

$D_y = \begin{vmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} = -24$ $\qquad D_z = \begin{vmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix} = 12$

$x = \dfrac{D_x}{D} = \dfrac{-36}{-132} = \dfrac{3}{11}$ $\quad y = \dfrac{D_y}{D} = \dfrac{-24}{-132} = \dfrac{2}{11}$ $\quad z = \dfrac{D_z}{D} = \dfrac{12}{-132} = -\dfrac{1}{11}$

Solution: $\boxed{\left(\frac{3}{11}, \frac{2}{11}, -\frac{1}{11}\right)}$

c) $\begin{cases} x-4y+z=6 \\ 4x-y+2z=-1 \\ 2x+2y-3z=-20 \end{cases}$

$D = \begin{vmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{vmatrix} = 3-16+8+2-4-48 = -55$

$D_x = \begin{vmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{vmatrix} = 18+160-2-20-24+12 = 144$

$D_y = \begin{vmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{vmatrix} = 3+24-80+2+40+72 = 61$

$D_z = \begin{vmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{vmatrix} = 20+8+48+12+2-320 = -230$

$x = \dfrac{D_x}{D} = -\dfrac{144}{55}$, $\quad y = \dfrac{D_y}{D} = -\dfrac{61}{55}$, $\quad z = \dfrac{D_z}{D} = \dfrac{-230}{-55} = \dfrac{46}{11}$

Solution: $\left(-\frac{144}{55}, -\frac{61}{55}, \frac{46}{11}\right)$

d) $\begin{cases} -x_1 - 4x_2 + 2x_3 + x_4 = -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 = 14 \\ -x_1 + x_2 + 3x_3 + x_4 = 11 \\ -x_1 - 2x_2 + x_3 - 4x_4 = -4 \end{cases}$

$D = -423 \quad D_{x_1} = -2115 \quad D_{x_2} = -3384 \quad D_{x_3} = -1269 \quad D_{x_4} = 423$

$\underline{x_1 = \frac{D_{x_1}}{D} = \frac{-2115}{-423} = 5}$ $\qquad$ $\underline{x_2 = \frac{D_{x_2}}{D} = \frac{-3384}{-423} = 8}$

$\underline{x_3 = \frac{D_{x_3}}{D} = \frac{-1269}{-423} = 3}$ $\qquad$ $\underline{x_4 = \frac{D_{x_4}}{D} = \frac{423}{-423} = -1}$

Solution: $\boxed{(5,\ 8,\ 3,\ -1)}$

e) $\begin{cases} 2x - y + z = -1 \\ 3x + 4y - z = -1 \\ 4x - y + 2z = -1 \end{cases}$

$$D = \begin{vmatrix} 2 & -1 & 1 \\ 3 & 4 & -1 \\ 4 & -1 & 2 \end{vmatrix} = 16 + 4 - 3 - 16 - 2 + 6 \underline{= 5}$$

$$D_x = \begin{vmatrix} -1 & -1 & 1 \\ -1 & 4 & -1 \\ -1 & -1 & 2 \end{vmatrix} = -8 - 1 + 1 + 4 + 1 - 2 \underline{= -5}$$

$$D_y = \begin{vmatrix} 2 & -1 & 1 \\ 3 & -1 & -1 \\ 4 & -1 & 2 \end{vmatrix} = -4 + 4 - 3 + 4 - 2 + 6 \underline{= 5}$$

$$D_z = \begin{vmatrix} 2 & -1 & -1 \\ 3 & 4 & -1 \\ 4 & -1 & -1 \end{vmatrix} = -8 + 4 + 3 + 16 - 2 - 3 \underline{= 10}$$

$\underline{x = \frac{D_x}{D}} = \frac{-5}{5} = \underline{-1}$, $\quad$ $\underline{y = \frac{D_y}{D}} = \frac{5}{5} = \underline{1}$, $\quad$ $\underline{z = \frac{D_z}{D}} = \frac{10}{5} = \underline{2}$

$\therefore$ Solution: $\underline{(-1,\ 1,\ 2)}$

Exercise

Show that the matrix A is invertible for all values of θ, then find A^{-1} using $A^{-1} = \frac{1}{\det(A)} adj(A)$

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

$$\det(A) = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1 \qquad \Rightarrow \boldsymbol{A} \text{ is invertible}$$

$$C_{11} = \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} = \cos\theta; \quad C_{12} = -\begin{vmatrix} -\sin\theta & 0 \\ 0 & 1 \end{vmatrix} = \sin\theta; \quad C_{13} = \begin{vmatrix} -\sin\theta & \cos\theta \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{21} = -\begin{vmatrix} \sin\theta & 0 \\ 0 & 1 \end{vmatrix} = -\sin\theta; \quad C_{22} = \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} = \cos\theta; \quad C_{23} = -\begin{vmatrix} \cos\theta & \sin\theta \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{31} = \begin{vmatrix} \sin\theta & 0 \\ \cos\theta & 0 \end{vmatrix} = 0; \quad C_{32} = -\begin{vmatrix} \cos\theta & 0 \\ -\sin\theta & 0 \end{vmatrix} = 0; \quad C_{33} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = 1$$

$$adj(A) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

$$= \frac{1}{1}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$