Section 1.4 – Limits at Infinity

Notation	*Terminology*
$f(x) \to \infty$	$f(x)$ increases without bound (can be made as large positive as desired)
$f(x) \to -\infty$	$f(x)$ decreases without bound (can be made as large negative as desired)

Horizontal Asymptote (*HA*)

The line $y = b$ is a ***horizontal asymptote*** for the graph of a function f if

$$\lim_{x\to\infty} f(x) = b \quad or \quad \lim_{x\to-\infty} f(x) = b$$

Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0} = \dfrac{a_n x^n}{b_m x^m}$ be a rational function. (***Proof*** !)

1. If the degree of numerator is less than of denominator $(n < m) \Rightarrow y = 0$

$$y = \frac{2x+1}{4x^2+5} \Rightarrow \underline{y = 0}$$

2. If the degree of numerator is equal of denominator $(n = m) \Rightarrow y = \dfrac{a_n}{b_m}$

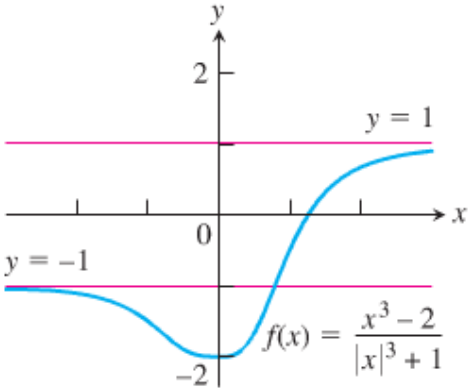
$$y = \frac{2x^2+1}{4x^2+5} \Rightarrow y = \frac{2}{4} = \underline{\frac{1}{2}}$$

3. If the degree of numerator is greater than of denominator $(n > m) \Rightarrow$ No horizontal asymptote

$$y = \frac{2x^3+1}{4x^2+5} \Rightarrow No\ \ HA$$

Example

Find the horizontal asymptotes of the graph of $f(x) = \dfrac{x^3-2}{|x|^3+1}$



Solution

For $x \geq 0$

$$\lim_{x\to\infty} \frac{x^3-2}{|x|^3+1} = \lim_{x\to\infty} \frac{x^3}{x^3}$$

$= 1$ ⌋

For $x \le 0$

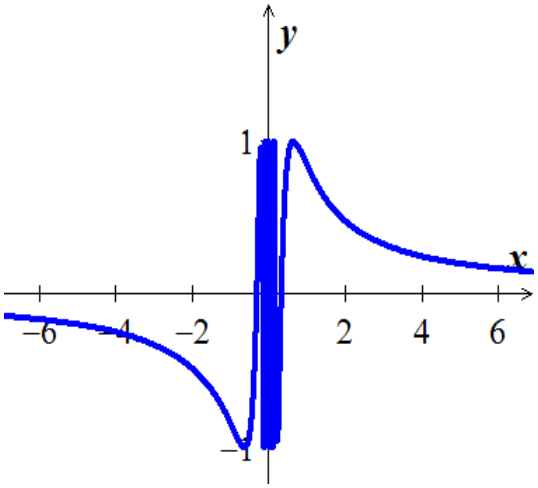
$$\lim_{x\to\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x\to-\infty} \frac{x^3}{(-x)^3}$$

$= -1$ ⌋

The ***HA*** are $y = -1$ and $y = 1$.

Example

Find $\lim_{x\to\infty} \sin\left(\frac{1}{x}\right)$



Solution

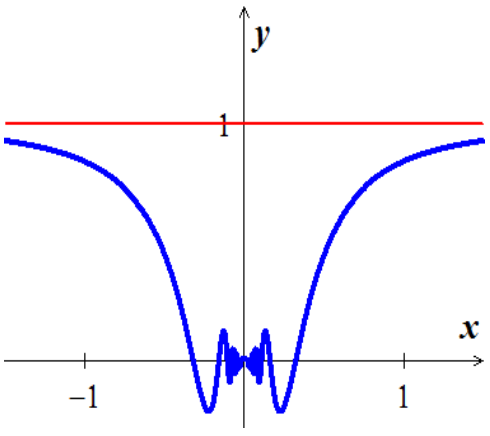
Let $t = \frac{1}{x} \Rightarrow t \to 0$ *as* $x \to \infty$

$$\lim_{x\to\infty} \sin\left(\frac{1}{x}\right) = \lim_{t\to 0} \sin t$$

$= 0$ ⌋

Example

Find $\lim_{x\to\pm\infty} x\sin\left(\frac{1}{x}\right)$



Solution

Let $t = \frac{1}{x} \Rightarrow x = \frac{1}{t}$

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{t\to 0^+} \frac{\sin t}{t}$$

$= 1$ ⌋

$$\lim_{x\to-\infty} x\sin\left(\frac{1}{x}\right) = \lim_{t\to 0^-} \frac{\sin t}{t}$$

$= 1$ ⌋

Example

Find the horizontal asymptote of $y = 2 + \dfrac{\sin x}{x}$

Solution

Since $0 \le \left|\dfrac{\sin x}{x}\right| \le \left|\dfrac{1}{x}\right|$

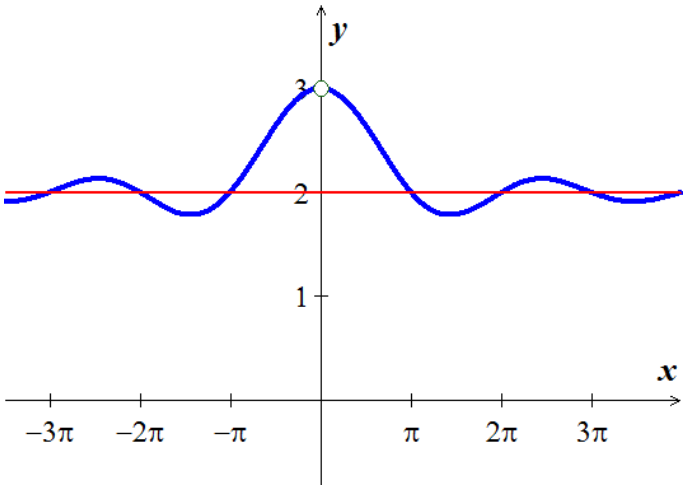
$$\lim_{x\to\pm\infty}\left|\frac{1}{x}\right| = 0$$

$$\lim_{x\to\pm\infty}\frac{\sin x}{x} = 0$$

$$\lim_{x\to\pm\infty}\left(2 + \frac{\sin x}{x}\right) = 2 + 0$$

$$= 2$$

***HA*:** $y = 2$



Example

Find $\lim_{x\to\infty}\left(x - \sqrt{x^2+16}\right)$

Solution

$$\lim_{x\to\infty}\left(x - \sqrt{x^2+16}\right) = \lim_{x\to\infty}\left(x - \sqrt{x^2+16}\right)\cdot\frac{x + \sqrt{x^2+16}}{x + \sqrt{x^2+16}} \qquad (a-b)(a+b) = a^2 - b^2$$

$$= \lim_{x\to\infty}\frac{x^2 - \left(x^2+16\right)}{x + \sqrt{x^2+16}}$$

$$= \lim_{x\to\infty}\frac{x^2 - x^2 - 16}{x + \sqrt{x^2+16}}$$

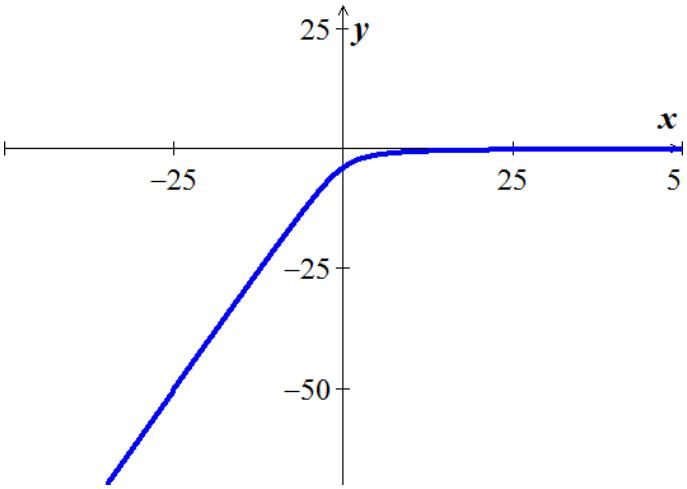
$$= \lim_{x\to\infty}\frac{-16}{x + \sqrt{x^2+16}}$$

$$= \lim_{x\to\infty}\frac{\frac{-16}{x}}{\frac{x}{x} + \sqrt{\frac{x^2}{x^2} + \frac{16}{x^2}}}$$

$$= \lim_{x \to \infty} \frac{-\frac{16}{x}}{1+\sqrt{1+\frac{16}{x^2}}}$$

$$= \frac{0}{1+\sqrt{1+0}}$$

$$= 0$$



Slant or *Oblique* Asymptotes

When the degree of the numerator is one greater than the degree of the numerator, the graph has a ***slant*** or ***oblique*** asymptote and it is a line $y = ax + b, \quad a \neq 0$. To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

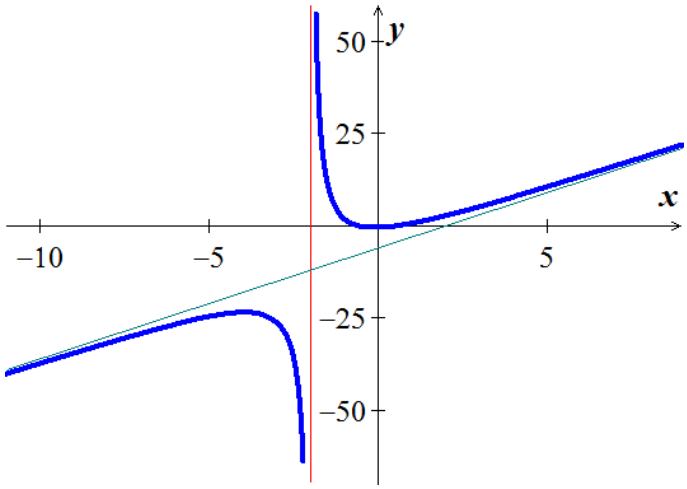
Example

$$y = \frac{3x^2 - 1}{x + 2}$$

$$\begin{array}{r} 3x - 6 \\ x+2 \overline{)\,3x^2 + 0x - 1} \\ \underline{3x^2 + 6x} \\ -6x - 1 \\ \underline{-6x - 12} \\ R = 11 \end{array}$$

$$y = \frac{3x^2 - 1}{x + 2}$$

$$\underline{= (3x - 6) + \frac{11}{x + 2}}$$



The ***oblique asymptote*** is the line **$y = 3x - 6$**

Example

Find the horizontal and vertical asymptotes of the curve $y = \frac{x+3}{x+2}$

Solution

HA: $y \to \frac{x}{x} = 1 \quad \Rightarrow \quad \underline{y = 1}$

VA: $x + 2 = 0 \quad \Rightarrow \quad \underline{x = -2}$

Example

Find the horizontal and vertical asymptotes of the curve $f(x) = -\frac{8}{x^2 - 4}$

Solution

HA: $y \to \lim_{x \to \infty} -\frac{8}{x^2} = 0 \quad \Rightarrow \quad \underline{y = 0}$

VA: $x^2 - 4 = 0 \quad \Rightarrow \quad \underline{x = \pm 2}$

$$\lim_{x \to 2^+} f(x) = -\infty \quad \textit{and} \quad \lim_{x \to 2^-} f(x) = \infty$$

Infinite Limits

The limit has a value of infinity or minus infinity, such a function $f(x) = \frac{1}{x}$. It is convenient to describe the behavior of f by saying that $f(x)$ approaches ∞ as $x \to 0^+$.

Definition

We say $\lim_{x \to 0^+} f(x) = \infty$

That $\lim_{x \to 0^+} \frac{1}{x}$ doesn't exist because $\frac{1}{x}$ becomes arbitrary large and positive as $x \to 0^+$.

We say $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x} = -\infty$

That $\lim_{x \to 0^-} \frac{1}{x}$ doesn't exist because $\frac{1}{x}$ becomes arbitrary large and negative as $x \to 0^-$.

Example

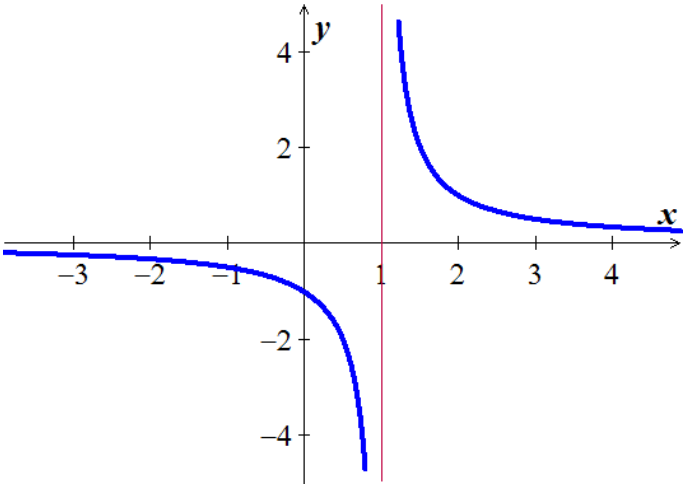
Find $\lim_{x \to 1^+} \frac{1}{x-1}$ *and* $\lim_{x \to 1^-} \frac{1}{x-1}$

Solution

As $x \to 1^+ \quad \Rightarrow \quad x - 1 \to 0^+$

$$\lim_{x \to 1^+} \frac{1}{x-1} = \infty$$

$$\lim_{x \to 1^-} \frac{1}{x-1} = -\infty$$



Examples

- $$\lim_{x\to 2}\frac{(x-2)^2}{x^2-4}=\lim_{x\to 2}\frac{(x-2)^2}{(x-2)(x+2)}$$
$$=\lim_{x\to 2}\frac{(x-2)}{(x+2)}$$
$$=\frac{0}{4}$$
$$=0$$

- $$\lim_{x\to 2}\frac{x-2}{x^2-4}=\lim_{x\to 2}\frac{x-2}{(x-2)(x+2)}$$
$$=\lim_{x\to 2}\frac{1}{x+2}$$
$$=\frac{1}{4}$$

- $$\lim_{x\to 2^+}\frac{x-3}{x^2-4}=\lim_{x\to 2^+}\frac{x-3}{(x-2)(x+2)}$$
$$=-\infty$$

- $$\lim_{x\to 2^-}\frac{x-3}{x^2-4}=\lim_{x\to 2^-}\frac{x-3}{(x-2)(x+2)}$$
$$=\infty$$

- $$\lim_{x\to 2}\frac{x-3}{x^2-4}=\lim_{x\to 2}\frac{x-3}{(x-2)(x+2)}$$
$$=\textit{doesn't exist}$$

Exercises *Section* 1.4 – Limits at Infinity

(1 – 8) Find the limit as $x \to \infty$ and as $x \to -\infty$ of

1. $h(x) = \dfrac{-5 + \dfrac{7}{x}}{3 - \dfrac{1}{x^2}}$

2. $f(x) = \dfrac{2x+3}{5x+7}$

3. $f(x) = \dfrac{2x^3+7}{x^3 - x^2 + x + 7}$

4. $f(x) = \dfrac{x+1}{x^2+3}$

5. $f(x) = \dfrac{7x^3}{x^3 - 3x^2 + 6x}$

6. $f(x) = \dfrac{9x^4 + x}{2x^4 + 5x^2 - x + 6}$

7. $f(x) = \dfrac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x}$

(8 – 60) Evaluate the limits

8. $\lim\limits_{x\to\infty} x^{12}$

9. $\lim\limits_{x\to-\infty} 3x^9$

10. $\lim\limits_{x\to-\infty} x^{-8}$

11. $\lim\limits_{x\to-\infty} x^{-9}$

12. $\lim\limits_{x\to-\infty} 2x^{-6}$

13. $\lim\limits_{x\to\infty} \left(3x^{12} - 9x^7\right)$

14. $\lim\limits_{x\to-\infty} \left(3x^7 + x^2\right)$

15. $\lim\limits_{x\to-\infty} \left(-2x^{16} + 2\right)$

16. $\lim\limits_{x\to-\infty} \left(2x^{-6} + 4x^5\right)$

17. $\lim\limits_{x\to-\infty} \dfrac{\cos x}{3x}$

18. $\lim\limits_{x\to\infty} \dfrac{x + \sin x}{2x + 7 - 5\sin x}$

19. $\lim\limits_{x\to\infty} \sqrt{\dfrac{8x^2 - 3}{2x^2 + x}}$

20. $\lim\limits_{x\to-\infty} \left(\dfrac{x^2 + x - 1}{8x^2 - 3}\right)^{1/3}$

21. $\lim\limits_{x\to\infty} \dfrac{2\sqrt{x} + x^{-1}}{3x - 7}$

22. $\lim\limits_{x\to\infty} \dfrac{x^{-1} + x^{-4}}{x^{-2} + x^{-3}}$

23. $\lim\limits_{x\to-\infty} \dfrac{4 - 3x^3}{\sqrt{x^6 + 9}}$

24. $\lim\limits_{x\to\infty} \left(\sqrt{x^2 + 3x} - \sqrt{x^2 - 2x}\right)$

25. $\lim\limits_{x\to-\infty} \left(\sqrt{x^2 + 3} + x\right)$

26. $\lim\limits_{x\to\infty} \dfrac{2x - 3}{4x + 10}$

27. $\lim\limits_{x\to\infty} \dfrac{x^4 - 1}{x^5 + 2}$

28. $\lim\limits_{x\to-\infty} \left(-3x^3 + 5\right)$

29. $\lim\limits_{x\to\infty} \left(e^{-2x} + \dfrac{2}{x}\right)$

30. $\lim\limits_{x\to\infty} \dfrac{1}{\ln x + 1}$

31. $\lim_{x\to\infty}\left(3+\frac{10}{x^2}\right)$

32. $\lim_{x\to\infty}\left(5+\frac{1}{x}+\frac{10}{x^2}\right)$

33. $\lim_{x\to\infty}\frac{4x^2+2x+3}{x^2}$

34. $\lim_{x\to\infty}\left(5+\frac{100}{x}+\frac{\sin^4 x^3}{x^2}\right)$

35. $\lim_{\theta\to\infty}\frac{\cos\theta}{\theta^2}$

36. $\lim_{\theta\to\infty}\frac{\cos\theta^5}{\sqrt{\theta}}$

37. $\lim_{x\to\infty}\frac{4x}{20x+1}$

38. $\lim_{x\to-\infty}\frac{4x}{20x+1}$

39. $\lim_{x\to\infty}\frac{3x^2-7}{x^2+5x}$

40. $\lim_{x\to-\infty}\frac{3x^2-7}{x^2+5x}$

41. $\lim_{x\to\infty}\frac{6x^2-9x+8}{3x^2+2}$

42. $\lim_{x\to-\infty}\frac{6x^2-9x+8}{3x^2+2}$

43. $\lim_{x\to\infty}\frac{4x^2-7}{8x^2+5x+2}$

44. $\lim_{x\to-\infty}\frac{4x^2-7}{8x^2+5x+2}$

45. $\lim_{x\to\infty}\frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4}$

46. $\lim_{x\to-\infty}\frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4}$

47. $\lim_{x\to\infty}\frac{3x^4+3x^3-36x^2}{x^4-25x^2+144}$

48. $\lim_{x\to-\infty}\frac{3x^4+3x^3-36x^2}{x^4-25x^2+144}$

49. $\lim_{x\to\infty}16x^2\left(4x^2-\sqrt{16x^4+1}\right)$

50. $\lim_{x\to-\infty}16x^2\left(4x^2-\sqrt{16x^4+1}\right)$

51. $\lim_{x\to\infty}\frac{x-1}{x^{2/3}-1}$

52. $\lim_{x\to-\infty}\frac{x-1}{x^{2/3}-1}$

53. $\lim_{x\to\infty}\frac{\sqrt{x^2+2x+6}-3}{x-1}$

54. $\lim_{x\to\infty}\frac{\left|1-x^2\right|}{x(x+1)}$

55. $\lim_{x\to\infty}\left(\sqrt{|x|}-\sqrt{|x-1|}\right)$

56. $\lim_{x\to\infty}\frac{\tan^{-1}x}{x}$

57. $\lim_{x\to\infty}\frac{\cos x}{e^{3x}}$

58. $\lim_{x\to0}\frac{2e^x+10e^{-x}}{e^x+e^{-x}}$

59. $\lim_{x\to\infty}\frac{2e^x+10e^{-x}}{e^x+e^{-x}}$

60. $\lim_{x\to-\infty}\frac{2e^x+10e^{-x}}{e^x+e^{-x}}$

(61 – 64) Graph the rational function and include the equations of the asymptotes

61. $y = \dfrac{1}{2x+4}$ **62.** $y = \dfrac{2x}{x+1}$ **63.** $y = \dfrac{x^2}{x-1}$ **64.** $y = \dfrac{x^3+1}{x^2}$

65. Let $f(x) = \dfrac{x^2-5x+6}{x^2-2x}$

a) Analyze $\lim\limits_{x\to 0^-} f(x)$, $\lim\limits_{x\to 0^+} f(x)$, $\lim\limits_{x\to 2^-} f(x)$, and $\lim\limits_{x\to 2^+} f(x)$

b) Does the graph of f have any vertical asymptotes? Explain?

(66 – 85) Find the vertical, horizontal, hole, and oblique asymptotes (if any) of

66. $y = \dfrac{3x}{1-x}$

67. $y = \dfrac{x^2}{x^2+9}$

68. $y = \dfrac{x-2}{x^2-4x+3}$

69. $y = \dfrac{5x-1}{1-3x}$

70. $y = \dfrac{3}{x-5}$

71. $y = \dfrac{x^3-1}{x^2+1}$

72. $y = \dfrac{3x^2-27}{(x+3)(2x+1)}$

73. $y = \dfrac{x^3+3x^2-2}{x^2-4}$

74. $y = \dfrac{x-3}{x^2-9}$

75. $y = \dfrac{6}{\sqrt{x^2-4x}}$

76. $f(x) = \dfrac{4x^3+1}{1-x^3}$

77. $f(x) = \dfrac{x+1}{\sqrt{9x^2+x}}$

78. $f(x) = 1-e^{-2x}$

79. $f(x) = \dfrac{1}{\ln x^2}$

80. $f(x) = \dfrac{1}{\tan^{-1} x}$

81. $f(x) = \dfrac{2x^2+6}{2x^2+3x-2}$

82. $f(x) = \dfrac{3x^2+2x-1}{4x+1}$

83. $f(x) = \dfrac{9x^2+4}{(2x-1)^2}$

84. $f(x) = \dfrac{1+x-2x^2-x^3}{x^2+1}$

85. $f(x) = \dfrac{x(x+2)^3}{3x^2-4x}$

(85 – 142) Find the limits

86. $\lim\limits_{x\to 0} \dfrac{x^2-4x+4}{x^3+5x^2-14x}$

87. $\lim\limits_{x\to 2} \dfrac{x^2-4x+4}{x^3+5x^2-14x}$

88. $\lim\limits_{x\to a} \dfrac{x^2-a^2}{x^4-a^4}$

89. $\lim\limits_{x\to 0} \dfrac{(x+h)^2-x^2}{h}$

90. $\lim\limits_{h\to 0} \dfrac{(x+h)^2-x^2}{h}$

91. $\lim\limits_{x\to 1} \dfrac{1-\sqrt{x}}{1-x}$

92. $\lim\limits_{x\to 0} \dfrac{\frac{1}{2+x}-\frac{1}{2}}{x}$

93. $\lim\limits_{x\to 1} \dfrac{x^{1/3}-1}{\sqrt{x}-1}$

94. $\lim\limits_{x\to 64} \dfrac{x^{2/3}-16}{\sqrt{x}-8}$

95. $\lim_{x\to 0} \dfrac{\tan(2x)}{\tan(\pi x)}$

96. $\lim_{x\to \pi^-} \csc x$

97. $\lim_{x\to \pi} \sin\left(\frac{x}{2} + \sin x\right)$

98. $\lim_{x\to \pi} \cos^2(x - \tan x)$

99. $\lim_{x\to 0} \dfrac{8x}{3\sin x - x}$

100. $\lim_{x\to 0} \dfrac{\cos 2x - 1}{\sin x}$

101. $\lim_{x\to -\infty} \dfrac{4 - 3x^3}{\sqrt{x^6 + 9}}$

102. $\lim_{x\to -\infty} \dfrac{x^2 - 4x + 8}{3x^3}$

103. $\lim_{x\to -\infty} \dfrac{2x^2 + 3}{5x^2 + 7}$

104. $\lim_{x\to \infty} \dfrac{x^4 + x^3}{12x^3 + 128}$

105. $\lim_{x\to -\infty} \dfrac{2 + \sqrt{x}}{2 - \sqrt{x}}$

106. $\lim_{x\to \infty} \dfrac{2 + \sqrt{x}}{2 - \sqrt{x}}$

107. $\lim_{x\to -\infty} \dfrac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$

108. $\lim_{x\to \infty} \dfrac{\frac{1}{x} + \frac{1}{x^4}}{\frac{1}{x^2} - \frac{1}{x^3}}$

109. $\lim_{x\to \infty} \dfrac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}}$

110. $\lim_{x\to 2^+} \ln(x - 2)$

111. $\lim_{x\to 1} x^2 \ln\left(2 - \sqrt{x}\right)$

112. $\lim_{\theta\to 0^+} \sqrt{\theta} e^{\cos\frac{\pi}{\theta}}$

113. $\lim_{x\to \infty} \dfrac{2x - 3}{5x + 6}$

114. $\lim_{x\to \infty} \dfrac{2x^2 - 3}{5x^2 + 6}$

115. $\lim_{x\to \infty} \dfrac{2x - 3}{5x^3 + 6}$

116. $\lim_{x\to \infty} \dfrac{1}{5x^2 - 3x + 6}$

117. $\lim_{\theta\to 0} \dfrac{\theta \cot 4\theta}{\sin^2\theta \cot^2 2\theta}$

118. $\lim_{x\to 0^+} \dfrac{\sqrt{x^2 + 4x + 5} - \sqrt{5}}{x}$

119. $\lim_{x\to 2} \dfrac{x^4 - 16}{x - 2}$

120. $\lim_{x\to 2} \dfrac{x^3 - 8}{x - 2}$

121. $\lim_{x\to -\infty} \dfrac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4}$

122. $\lim_{x\to -\infty} \dfrac{\sqrt{x^2 + 1}}{x + 1}$

123. $\lim_{x\to \infty} \dfrac{\sqrt{x^2 + 1}}{x + 1}$

124. $\lim_{x\to \infty} \dfrac{x - 3}{\sqrt{4x^2 + 25}}$

125. $\lim_{x\to -\infty} \dfrac{4 - 3x^3}{\sqrt{x^6 + 9}}$

126. $\lim_{x\to \infty} \dfrac{x^4 - x}{15x^3 + 4}$

127. $\lim_{x\to \infty} \dfrac{x + \sin x + 2\sqrt{x}}{x + \sin x}$

128. $\lim_{x\to \infty} \dfrac{x^{2/3} - x^{-1}}{x^{2/3} + \cos^2 x}$

129. $\lim_{x\to \infty} \dfrac{\sin 2x}{x}$

130. $\lim_{x\to 0} \dfrac{\sin 5x}{3x}$

131. $\lim_{x\to -\infty} \dfrac{\cos x}{2x}$

132. $\lim_{x\to -\infty} \left(\dfrac{x^2 + x - 1}{8x^2 - 3}\right)^{1/3}$

133. $\lim_{x\to -1} \dfrac{\sqrt{x^2 + 8} - 3}{x + 1}$

134. $\lim_{x\to -\infty} \left(\dfrac{1 - x^3}{x^2 + 7x}\right)^5$

135. $\lim_{x\to \infty} \sqrt{\dfrac{x^2 - 5x}{x^3 + x - 2}}$

136. $\lim_{x\to \infty} \dfrac{2\sqrt{x} + x^{-1}}{3x - 7}$

137. $\lim_{x\to -5^-} \dfrac{3x}{2x + 10}$

138. $\lim_{x\to -8^+} \dfrac{3x}{x + 8}$

139. $\lim_{x\to 0} \dfrac{-1}{x^2(x + 1)}$

140. $\lim_{x\to 7} \dfrac{4}{(x - 7)^2}$

141. $\lim_{x\to 0} \dfrac{1}{x^{2/3}}$

142. $\lim_{x\to -\infty} \left(x + \sqrt{x^2 - 4x + 2}\right)$