# *Section* 5.7 – Mathematical Induction

If $n$ is a positive integer and we let $P_n$ denote the mathematical statement $(xy)^n = x^n y^n$, we obtained the following ***infinite sequence*** of statements:

$$\begin{array}{ll} \text{Statement } P_1: & (xy)^1 = x^1 y^1 \\ \text{Statement } P_2: & (xy)^2 = x^2 y^2 \\ \text{Statement } P_3: & (xy)^3 = x^3 y^3 \\ \quad\vdots & \quad\vdots \\ \text{Statement } P_n: & (xy)^n = x^n y^n \\ \quad\vdots & \quad\vdots \end{array}$$

## *Principle* of Mathematical Induction

If with each positive integer $n$ there is associated a statement $P_n$ then all the statements $P_n$ are true, provided the following two conditions are satisfied.

**1)** $P_1$ is true.

**2)** Whenever $k$ is a positive integer such that $P_k$ is true, then $P_{k+1}$ is also true.

### *Steps in Applying the Principle of Mathematical Induction*

**1)** Show that $P_1$ is true.

**2)** Assume that $P_k$ is true, and then prove that $P_{k+1}$ is true.

### *Example*

Use the mathematical induction to prove that for every positive integer $n$, the sum of the first $n$ positive integers is:

$$\frac{n(n+1)}{2}$$

***Solution***

**(1)** For $n = 1 \Rightarrow \frac{1(1+1)}{2} = 1$

$1 = 1$ $\checkmark$

Hence $P_1$ is true.

**(2)** Assume that $P_k$ is true.

Thus, the induction hypothesis is: $1+2+3+\ldots+k = \frac{k(k+1)}{2}$

For $k+1$: $1+2+3+\ldots+k+(k+1) \overset{?}{=} \frac{(k+1)((k+1)+1)}{2}$

$$\begin{aligned}1+2+3+\ldots+k+(k+1) &= (1+2+3+\ldots+k)+(k+1)\\ &= \frac{k(k+1)}{2}+(k+1) && \textit{Induction hypothesis}\\ &= \frac{k(k+1)+2(k+1)}{2}\\ &= \frac{(k+1)(k+2)}{2} && \textit{Factor out } \mathbf{k+1}\\ &= \frac{(k+1)((k+1)+1)}{2} \quad \checkmark && \textit{Change form of } \mathbf{k+2}\end{aligned}$$

This shows that $P_{k+1}$ is also true.

$\therefore$ By the mathematical induction, the proof is completed

***Example***

Prove that for every positive integer $n$,

$$1^2 + 3^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

***Solution***

**(1)** For $\boldsymbol{n = 1} \Rightarrow 1^2 = \frac{1(2(1)-1)(2(1)+1)}{3}$

$$1 = \frac{3}{3}$$

$1 = 1$ ✓ hence $P_1$ is true.

**(2)** $1^2 + 3^2 + \ldots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$

For $k+1$:

$$1^2 + 3^2 + \ldots + (2k-1)^2 + (2(k+1)-1)^2 \overset{?}{=} \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$$

$$1^2 + 3^2 + \ldots + (2k-1)^2 + (2(k+1)-1)^2 \overset{?}{=} \frac{(k+1)(2k+1)(2k+3)}{3}$$

$$1^2 + 3^2 + \ldots + (2k-1)^2 + [2k+2-1]^2 = \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

$$= \frac{k(2k-1)(2k+1) + 3(2k+1)^2}{3}$$

$$= \frac{(2k+1)[k(2k-1) + 3(2k+1)]}{3}$$

$$= \frac{(2k+1)(2k^2 - k + 6k + 3)}{3}$$

$$= \frac{(2k+1)(2k^2 + 5k + 3)}{3}$$

$$= \frac{(2k+1)(k+1)(2k+3)}{3} \checkmark$$

This shows that $P_{k+1}$ is also true.

∴ By the mathematical induction, the proof is completed

### *Example*

Prove that 2 is a factor of $n^2 + 5n$ for every positive integer $n$,

***Solution***

**(1)** For $\boldsymbol{n = 1} \Rightarrow n^2 + 5n = 1^2 + 5(1)$

$$= 6$$

$$= 2.3 \quad \surd$$

Thus, 2 is a factor of $n^2 + 5n$ for $n = 1$; hence $P_1$ is true.

**(2)** 2 is a factor of $k^2 + 5k \Leftrightarrow k^2 + 5k = 2p$

is 2 a factor of $(k+1)^2 + 5(k+1)$?

$$\begin{aligned}(k+1)^2 + 5(k+1) &= k^2 + 2k + 1 + 5k + 5 \\ &= k^2 + 5k + 2k + 6 \\ &= \left(k^2 + 5k\right) + 2(k+3) \\ &= 2p + 2(k+3) \\ &= 2.(p + k + 3) \quad \surd\end{aligned}$$

Thus, 2 is a factor of the last expression; hence $P_{k+1}$ is also true.

∴ By the mathematical induction, the proof is completed

### *Steps in Applying the Extended Principle of Mathematical Induction*

1. Show that $P_1$ is true.
2. Assume that $P_k$ is true with $k \ge j$, and then prove that $P_{k+1}$ is true.

## *Example*

Let $a$ be a nonzero real number such that $a > -1$. Prove that $(1+a)^n > 1+na$ for every integer $n \ge 2$.

***Solution***

For $\boldsymbol{n = 1} \Rightarrow (1+a)^1 > 1+(1)a \Rightarrow P_1$ is false.

***Step* 1**. For $\boldsymbol{n = 2} \Rightarrow (1+a)^2 \overset{?}{>} 1+(2)a$

$$1+2a+a^2 > 1+a \quad \surd$$

$\Rightarrow P_2$ is true.

***Step* 2**. Assume that $P_k$ is true $(1+a)^k > 1+ka$

We need to prove that $P_{k+1}$ is true, that is $(1+a)^{k+1} > 1+(k+1)a$

$$\begin{aligned}(1+a)^{k+1} &= (1+a)^k(1+a)^1\\ &> (1+ka)(1+a)\end{aligned}$$

$$\begin{aligned}(1+ka)(1+a) &= 1+a+ka+ka^2\\ &= 1+(a+ka)+ka^2\\ &= 1+a(k+1)+ka^2\\ &> 1+(k+1)a\end{aligned}$$

$$\begin{aligned}(1+a)^{k+1} &> (1+ka)(1+a)\\ &> 1+(k+1)a \quad \surd\end{aligned}$$

Thus, $P_{k+1}$ is also true.

∴ By the mathematical induction, the proof is completed

# *Exercises* *Section* 5.7 – Mathematical Induction

**1.** Find all positive integers $n$ for which the given statement is not true

$a)\ \ 3^n > 6n \qquad b)\ \ 3^n > 2n+1 \qquad c)\ \ 2^n > n^2 \qquad d)\ \ n! > 2n$

**2.** Prove that the statement is true for every positive integer $n$. $2+4+6+\ldots+2n = n(n+1)$

**3.** Prove that the statement is true for every positive integer $n$. $1+3+5+\ldots+(2n-1) = n^2$

**4.** Prove that the statement is true for every positive integer $n$. $2+7+12+\ldots+(5n-3) = \frac{1}{2}n(5n-1)$

**(5 – 35)** Prove that the statement is true by the mathematical induction

**5.** $1+2\cdot 2+3\cdot 2^2+\ldots+n\cdot 2^{n-1} = 1+(n-1).2^n$

**6.** $1^2+2^2+3^2+\ldots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$

**7.** $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\ldots+\dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$

**8.** $\dfrac{1}{2}+\dfrac{1}{2^2}+\dfrac{1}{2^3}+\ldots+\dfrac{1}{2^n} = 1-\dfrac{1}{2^n}$

**9.** $\dfrac{1}{1\cdot 4}+\dfrac{1}{4\cdot 7}+\dfrac{1}{7\cdot 10}+\cdots+\dfrac{1}{(3n-2)\cdot(3n+1)} = \dfrac{n}{3n+1}$

**10.** $\dfrac{4}{5}+\dfrac{4}{5^2}+\dfrac{4}{5^3}+\cdots+\dfrac{4}{5^n} = 1-\dfrac{1}{5^n}$

**11.** $1^3+2^3+3^3+\cdots+n^3 = \dfrac{n^2(n+1)^2}{4}$

**12.** $3+3^2+3^3+\ldots+3^n = \frac{3}{2}\left(3^n-1\right)$

**13.** $x^{2n}+x^{2n-1}y+\cdots+xy^{2n-1}+y^{2n} = \dfrac{x^{2n+1}-y^{2n+1}}{x-y}$

**14.** $5\cdot 6+5\cdot 6^2+5\cdot 6^3+\cdots+5\cdot 6^n = 6\left(6^n-1\right)$

**15.** $7\cdot 8+7\cdot 8^2+7\cdot 8^3+\cdots+7\cdot 8^n = 8\left(8^n-1\right)$

**16.** $3+6+9+\cdots+3n = \dfrac{3n(n+1)}{2}$

**17.** $5+10+15+\cdots+5n = \dfrac{5n(n+1)}{2}$

**18.** $1+3+5+\cdots+(2n-1) = n^2$

**19.** $4+7+10+\cdots+(3n+1)=\dfrac{n(3n+5)}{2}$

**20.** $2+4+6+\cdots+2(n-1)+2n=n(n+1)$

**21.** $1+(1+2)+(1+2+3)+\ \cdots\ +(1+2+\cdots+n)=\dfrac{n(n+1)(n+2)}{6} \quad =\displaystyle\sum_{k=1}^{n}\left(\sum_{i=1}^{k} i\right)$

**22.** $1+2+3+\ \cdots\ +n < \dfrac{(2n+3)^2}{7}$

**23.** $\dfrac{1}{2n} \le \dfrac{1\cdot 3\cdot 5\cdots(2n-3)\cdot(2n-1)}{2\cdot 4\cdot 6\cdots(2n-2)\cdot(2n)}$

**24.** $\dfrac{2n+1}{2n+2} \le \dfrac{\sqrt{n+1}}{\sqrt{n+2}}$

**25.** $n! < n^n \quad \textit{for} \quad n>1$

**26.** For every positive integer *n*. $\quad n<2^n$

**27.** For every positive integer *n*. $\quad$ 3 is a factor of $n^3-n+3$

**28.** For every positive integer *n*. $\quad$ 4 is a factor of $5^n-1$

**29.** $\left(a^m\right)^n = a^{mn}$ (*a* and *m* are constant)

**30.** $2^n > 2n \quad \textit{if} \quad n \ge 3$

**31.** If $0<a<1$, then $a^n < a^{n-1}$

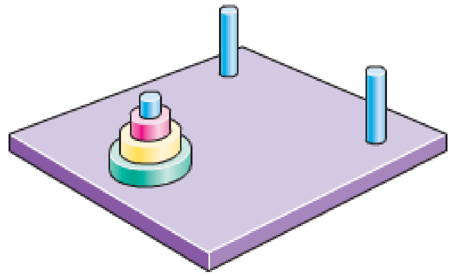
**32.** If $n\ge 4$, then $n! > 2^n$

**33.** $3^n > 2n+1 \quad \textit{if} \quad n\ge 2$

**34.** $2^n > n^2 \quad \textit{for} \quad n>4$

**35.** $4^n > n^4 \quad \textit{for} \quad n\ge 5$

**36.** A pile of *n* rings, each smaller than the one below it, is on a peg on board. Two other pegs are attached to the board. In the game called the Tower of Hanoi puzzle, all the rings must moved one at a time, to a different peg with no ring ever placed on top of a smaller ring.



Find the least number of moves that would be required.
Prove your result by mathematical induction.