Solution Section 2.1 – Integration by Parts

Exercise

Evaluate the integral $\int xe^{2x}dx$

Solution

		$\int e^{2x}\,dx$
+	x	$\frac{1}{2}e^{2x}$
–	1	$\frac{1}{4}e^{2x}$

$$\int xe^{2x}dx = \underline{\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C}$$

Let: $u = x \Rightarrow du = dx$

$$dv = e^{2x}dx \Rightarrow v = \int dv = \int e^{2x}dx = \frac{1}{2}e^{2x}$$

$$\int udv = uv - \int vdu$$

$$\int xe^{2x}dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx$$

$$= \underline{\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C}$$

Exercise

Evaluate the integral $\int x\ln x\,dx$

Solution

Let: $u = \ln x \Rightarrow du = \frac{1}{x}dx$

$$dv = xdx \Rightarrow v = \int dv = \int xdx = \frac{1}{2}x^2$$

$$\int x\ln xdx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x^2\frac{dx}{x}$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{2}\int xdx$$

$$= \underline{\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C}$$

Exercise

Evaluate the integral $\int x^3 e^x dx$

Solution

		$\int e^x\, dx$
+	x^3	e^x
−	$3x^2$	e^x
+	$6x$	e^x
−	6	e^x

$$\int x^3 e^x dx = e^x\left(x^3 - 3x^2 + 6x - 6\right) + C$$

Let: $u = x^3 \Rightarrow du = 3x^2 dx$

$$dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$\int x^3 e^x dx = x^3 e^x - \int e^x 3x^2 dx$$

$$= x^3 e^x - 3\int e^x x^2 dx$$

Let: $u = x^2 \Rightarrow du = 2x dx$

$$dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$\int e^x x^2 dx = x^2 e^x - 2\int x e^x dx$$

$$\int x^3 e^x dx = x^3 e^x - 3\left[x^2 e^x - 2\int x e^x dx\right]$$

$$= x^3 e^x - 3x^2 e^x + 6\int x e^x dx$$

Let: $u = x \Rightarrow du = dx$

$$dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x$$

$$\int x^3 e^x dx = x^3 e^x - 3x^2 e^x + 6\left[x e^x - e^x\right] + C$$

$$= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C$$

$$= e^x\left(x^3 - 3x^2 + 6x - 6\right) + C$$

Exercise

Evaluate the integral $\int \ln x^2 dx$

Solution

$$\int \ln x^2 dx = 2\int \ln x dx \qquad u = \ln x \Rightarrow du = \frac{1}{x} dx \qquad v = \int dx = x$$

$$\int \ln x^2 dx = 2\left[x\ln x - \int x \frac{1}{x} dx\right]$$

$$= 2\left[x\ln x - \int dx \right]$$

$$= 2(x\ln x - x) + C$$

$$\boxed{= 2x(\ln x - 1) + C}$$

Exercise

Evaluate the integral $\int \frac{2x}{e^x}dx$

Solution

		$\int e^{-x}\,dx$
+	$2x$	$-e^{-x}$
–	2	e^{-x}

$$\int \frac{2x}{e^x}dx \boxed{= -e^{-x}(2x+2) + C}$$

$u = 2x \Rightarrow du = 2dx$

$dv = e^{-x}dx \Rightarrow v = -e^{-x}$

$$\int \frac{2x}{e^x}dx = 2x(-e^{-x}) - \int -e^{-x}2dx$$

$$= -2xe^{-x} + 2\int e^{-x}dx$$

$$= -2xe^{-x} - 2e^{-x} + C$$

$$= -2e^{-x}(x+1) + C$$

$$\boxed{= -\frac{2(x+1)}{e^x} + C}$$

Exercise

Evaluate the integral $\int \ln(3x)dx$

Solution

$u = \ln 3x \Rightarrow du = \frac{3}{3x}dx = \frac{1}{x}dx \qquad dv = dx \Rightarrow v = x$

$$\int \ln(3x)dx = x\ln(3x) - \int x\frac{1}{x}dx$$

$$= x\ln(3x) - \int dx$$

$$= x\ln(3x) - x + C$$

$$\boxed{= x\left[\ln(3x) - 1\right] + C}$$

Exercise

Evaluate the integral $\int \frac{1}{x\ln x}dx$

Solution

$$\int \frac{1}{x\ln x}dx = \int \frac{1}{\ln x}\frac{1}{x}dx$$

$$u = \ln x \Rightarrow du = \frac{1}{x}dx$$

$$\int \frac{1}{x\ln x}dx = \int \frac{1}{u}du$$

$$= \ln u + C$$

$$\boxed{= \ln\left|\ln x\right| + C}$$

Exercise

Evaluate the integral $\int \frac{x}{\sqrt{x-1}}dx$

Solution

Let: $u = x \;\Rightarrow du = dx$

$$dv = \frac{dx}{\sqrt{x-1}} \Rightarrow v = \int (x-1)^{-1/2}d(x-1)$$

$$= \frac{(x-1)^{1/2}}{1/2}$$

$$= 2(x-1)^{1/2}$$

$$\int \frac{x}{\sqrt{x-1}}dx = 2x\sqrt{x-1} - 2\int (x-1)^{1/2}dx$$

$$= 2x\sqrt{x-1} - 2\frac{(x-1)^{3/2}}{3/2} + C$$

$$= 2x\sqrt{x-1} - \frac{4}{3}(x-1)\sqrt{x-1} + C$$

$$= \sqrt{x-1}\left[2x - \frac{4}{3}x + \frac{4}{3}\right] + C$$

$$= \sqrt{x-1}\left[\frac{6x-4x+4}{3}\right] + C$$

$$= \sqrt{x-1}\left[\frac{2x+4}{3}\right] + C$$

$$\boxed{= \frac{2}{3}\sqrt{x-1}(x+2) + C}$$

Let: $u = x-1 \;\Rightarrow\; x = u+1$

$du = dx$

$$\int \frac{x}{\sqrt{x-1}}dx = \int (u+1)u^{-1/2}du$$

$$= \int \left(u^{1/2} + u^{-1/2}\right)du$$

$$= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + C$$

$$= (x-1)^{1/2}\left(\frac{2}{3}x - \frac{2}{3} + 2\right) + C$$

$$= \sqrt{x-1}\left[\frac{2x+4}{3}\right] + C$$

$$\boxed{= \frac{2}{3}\sqrt{x-1}(x+2) + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^3 e^{x^2}}{\left(x^2+1\right)^2} dx$

Solution

Let: $u = x^2 e^{x^2} \quad \Rightarrow du = \left(2xe^{x^2} + 2xx^2 e^{x^2}\right)dx$

$$du = 2xe^{x^2}\left(1+x^2\right)dx$$

$$dv = x\left(x^2+1\right)^{-2} dx \qquad \Rightarrow v = \int x(x^2+1)^{-2} dx$$

$$= \frac{1}{2}\int (x^2+1)^{-2} d(x^2+1)$$

$$= \frac{(x^2+1)^{-1}}{-1}$$

$$= -\frac{1}{2(x^2+1)}$$

$$\int \frac{x^3 e^{x^2}}{(x^2+1)^2} dx = x^2 e^{x^2}\left(-\frac{1}{2(x^2+1)}\right) - \int -\frac{1}{2(x^2+1)} 2xe^{x^2}(x^2+1)dx$$

$$= -\frac{x^2 e^{x^2}}{2(x^2+1)} + \int xe^{x^2} dx$$

Let: $u = x^2 \quad \Rightarrow du = 2xdx$

$$\int \frac{x^3 e^{x^2}}{\left(x^2+1\right)^2} dx = -\frac{x^2 e^{x^2}}{2(x^2+1)} + \frac{1}{2}\int e^u du$$

$$= -\frac{x^2 e^{x^2}}{2(x^2+1)} + \frac{1}{2}e^u + C$$

$$= -\frac{x^2 e^{x^2}}{2(x^2+1)} + \frac{1}{2}e^{x^2} + C$$

$$= \frac{1}{2}e^{x^2}\left[-\frac{x^2}{(x^2+1)} + 1\right] + C$$

$$= \frac{1}{2}e^{x^2}\left[\frac{-x^2+x^2+1}{(x^2+1)}\right] + C$$

$$= \frac{e^{x^2}}{2(x^2+1)} + C$$

Exercise

Evaluate the integral $\int x^2 e^{-3x} dx$

Solution

$u = x^2 \Rightarrow du = 2x dx$

$dv = e^{-3x} dx \Rightarrow v = -\frac{1}{3} e^{-3x}$

$$\int x^2 e^{-3x} dx = -\frac{1}{3} x^2 e^{-3x} + \frac{2}{3} \int x e^{-3x} dx$$

$u = x \Rightarrow du = dx$

$dv = e^{-3x} dx \Rightarrow v = -\frac{1}{3} e^{-3x}$

$$\int x^2 e^{-3x} dx = -\frac{1}{3} x^2 e^{-3x} + \frac{2}{3} \left[-\frac{1}{3} x e^{-3x} + \frac{1}{3} \int e^{-3x} dx \right]$$

$$= -\frac{1}{3} x^2 e^{-3x} + \frac{2}{3} \left[-\frac{1}{3} x e^{-3x} - \frac{1}{9} e^{-3x} \right] + C$$

$$= -\frac{1}{3} x^2 e^{-3x} - \frac{2}{9} x e^{-3x} - \frac{2}{27} e^{-3x} + C$$

$$= -\frac{9x^2 + 6x + 2}{27} e^{-3x} + C$$

		$\int e^{-3x}$
+	x^2	$-\frac{1}{3} e^{-3x}$
−	$2x$	$\frac{1}{9} e^{-3x}$
+	2	$-\frac{1}{27} e^{-3x}$

$$\int x^2 e^{-3x} dx = -\frac{1}{3} x^2 e^{-3x} - \frac{2}{9} x e^{-3x} - \frac{2}{27} e^{-3x} + C$$

Exercise

Evaluate the integral $\int \theta \cos \pi\theta \, d\theta$

Solution

Let: $u = \theta \qquad dv = \cos \pi\theta \, d\theta$

$du = d\theta \qquad v = \int \cos \pi\theta \, d\theta = \frac{1}{\pi} \sin \pi\theta$

$$\int \theta \cos \pi\theta \, d\theta = \frac{\theta}{\pi} \sin \pi\theta - \int \frac{1}{\pi} \sin \pi\theta \, d\theta$$

$$= \frac{\theta}{\pi} \sin \pi\theta + \frac{1}{\pi} \frac{1}{\pi} \cos \pi\theta + C$$

$$= \frac{\theta}{\pi} \sin \pi\theta + \frac{1}{\pi^2} \cos \pi\theta + C$$

Exercise

Evaluate the integral $\int x^2 \sin x\, dx$

Solution

		$\int \sin x$
x^2	$(+)$	$-\cos x$
$2x$	$(-)$	$-\sin x$
2	$(+)$	$\cos x$
0		

$$\int x^2 \sin x dx = \underline{-x^2 \cos x + 2x \sin x + 2\cos x + C}$$

Exercise

Evaluate the integrals $\int x(\ln x)^2\, dx$

Solution

$u = \ln x \quad \to \quad x = e^u$

$du = \frac{1}{x}dx \quad \Rightarrow \quad xdu = dx \to dx = e^u du$

$$\int x(\ln x)^2\, dx = \int e^u u^2 e^u du$$

$$= \int u^2 e^{2u} du$$

$$= \frac{1}{2}u^2 e^{2u} - \frac{1}{2}ue^{2u} + \frac{1}{4}e^{2u} + C$$

$$= \frac{1}{4}e^{2u}\left(2u^2 - 2u + 1\right) + C$$

$$= \underline{\frac{1}{4}x^2\left(2(\ln x)^2 - 2\ln x + 1\right) + C}$$

	$\int e^{2u} du$
u^2	$\frac{1}{2}e^{2u}$
$2u$	$\frac{1}{4}e^{2u}$
2	$\frac{1}{8}e^{2u}$
0	

2nd Method

$u = \ln x \qquad dv = \int (x\ln x)\, dx$

$du = \frac{1}{x}dx \qquad v = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2$

$$\int x(\ln x)^2\, dx = (\ln x)\left(\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right) - \int \left(\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right)\frac{1}{x}dx$$

$u = \ln x \Rightarrow du = \frac{1}{x}dx$

$dv = xdx \Rightarrow v = \frac{1}{2}x^2$

$$\int x\ln x dx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x^2 \frac{dx}{x}$$

$$= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{4}x^2\ln x - \int \left(\frac{1}{2}x\ln x - \frac{1}{4}x\right)dx$$

$$= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{4}x^2\ln x - \left(\frac{1}{2}\left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right) - \frac{1}{8}x^2\right) + C$$

$$= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{4}x^2\ln x - \frac{1}{4}x^2\ln x + \frac{1}{8}x^2 + \frac{1}{8}x^2 + C$$

$$\boxed{= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + C}$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{2}\int xdx$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$$

3^{nd} Method

$$u = (\ln x)^2 \qquad dv = \int xdx$$

$$du = 2(\ln x)\frac{1}{x}dx \qquad v = \frac{1}{2}x^2$$

$$\int x(\ln x)^2\,dx = \frac{1}{2}x^2(\ln x)^2 - \int \frac{1}{2}x^2(2\ln x)\frac{1}{x}dx$$

$$= \frac{1}{2}x^2(\ln x)^2 - \int x\ln x dx$$

$$= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{4}x^2\ln x - \left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right) + C$$

$$= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{4}x^2\ln x - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + C$$

$$\boxed{= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + C}$$

$$u = \ln x \Rightarrow du = \frac{1}{x}dx$$

$$dv = xdx \Rightarrow v = \frac{1}{2}x^2$$

$$\int x\ln x dx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x^2\frac{dx}{x}$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{2}\int xdx$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$$

Exercise

Evaluate the integral $\int (x^2 - 2x + 1)e^{2x}dx$

Solution

		$\int e^{2x}$
+	$x^2 - 2x + 1$	$\frac{1}{2}e^{2x}$
−	$2x - 2$	$\frac{1}{4}e^{2x}$
+	2	$\frac{1}{8}e^{2x}$

$$\int (x^2 - 2x + 1)e^{2x}dx = \frac{1}{2}(x^2 - 2x + 1)e^{2x} - \frac{1}{4}(2x - 2)e^{2x} + \frac{1}{8}(2)e^{2x} + C$$

$$= \left(\frac{1}{2}x^2 - x + \frac{1}{2} - \frac{1}{2}x + \frac{1}{2} + \frac{1}{4}\right)e^{2x} + C$$

$$\boxed{= \left(\frac{1}{2}x^2 - \frac{3}{2}x + \frac{5}{4}\right)e^{2x} + C}$$

Exercise

Evaluate the integral $\int \tan^{-1} y\, dy$

Solution

Let: $u = \tan^{-1} y \quad dv = dy$

$du = \dfrac{dy}{1+y^2} \qquad v = y$

$$\int \tan^{-1} y\, dy = y\tan^{-1} y - \int \frac{ydy}{1+y^2} \qquad d\left(1+y^2\right) = 2ydy \;\rightarrow\; \tfrac{1}{2}d\left(1+y^2\right) = ydy$$

$$= y\tan^{-1} y - \int \frac{\frac{1}{2}d\left(1+y^2\right)}{1+y^2}$$

$$= y\tan^{-1} y - \tfrac{1}{2}\ln\left(1+y^2\right) + C$$

$$= y\tan^{-1} y - \ln\sqrt{1+y^2} + C$$

Exercise

Evaluate the integral $\int \sin^{-1} y\, dy$

Solution

Let: $u = \sin^{-1} y \quad dv = dy$

$du = \dfrac{dy}{\sqrt{1-y^2}} \qquad v = y$

$$\int \sin^{-1} y\, dy = y\sin^{-1} y - \int \frac{ydy}{\sqrt{1-y^2}} \qquad d\left(1-y^2\right) = -2ydy \;\rightarrow\; -\tfrac{1}{2}d\left(1-y^2\right) = ydy$$

$$= y\sin^{-1} y + \tfrac{1}{2}\int \left(1-y^2\right)^{-1/2} d\left(1-y^2\right)$$

$$= y\sin^{-1} y + \tfrac{1}{2}(2)\left(1-y^2\right)^{1/2} + C$$

$$= y\sin^{-1} y + \sqrt{1-y^2} + C$$

Exercise

Evaluate the integral $\displaystyle\int 4x\sec^2 2x\,dx$

Solution

Let: $u = 4x \quad \rightarrow du = 4 \qquad dv = \sec^2 2xdx \quad \rightarrow v = \frac{1}{2}\tan 2x$

$$\int 4x\sec^2 2x\,dx = 2x\tan 2x - \int 4\left(\frac{1}{2}\tan 2x\right)dx$$

$$= 2x\tan 2x - 2\frac{1}{2}\ln\left|\sec 2x\right| + C$$

$$\boxed{= 2x\tan 2x - \ln\left|\sec 2x\right| + C}$$

Exercise

Evaluate the integral $\displaystyle\int e^{2x}\cos 3xdx$

Solution

$$\int e^{2x}\cos 3xdx = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x - \frac{4}{9}\int e^{2x}\cos 3xdx$$

$$\int e^{2x}\cos 3xdx + \frac{4}{9}\int e^{2x}\cos 3xdx = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x$$

$$\frac{13}{9}\int e^{2x}\cos 3xdx = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x$$

$$\int e^{2x}\cos 3xdx = \boxed{\frac{e^{2x}}{13}\left(3\sin 3x + 2\cos 3x\right) + C}$$

		$\int \cos 3x\,dx$
+	e^{2x}	$\frac{1}{3}\sin 3x$
-	$2e^{2x}$	$-\frac{1}{9}\cos 3x$
+	$4e^{2x}$	$-\frac{1}{9}\int \cos 3x\,dx$

Exercise

Evaluate the integral $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$

Solution

Let: $u = \sqrt{x} \quad \rightarrow \quad du = \frac{1}{2\sqrt{x}}dx \Rightarrow 2du = \frac{1}{\sqrt{x}}dx$

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx = \int (\cos u)(2du)$$

$$= 2\int \cos u\,du$$

$$= 2\sin u + C$$

$$\boxed{= 2\sin\sqrt{x} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{(\ln x)^3}{x}\,dx$

Solution

$$\int \frac{(\ln x)^3}{x}\,dx = \int (\ln x)^3\, d(\ln x) \qquad\qquad d(\ln x) = \frac{dx}{x}$$

$$= \boxed{\frac{1}{4}(\ln x)^4 + C}$$

Exercise

Evaluate the integral $\displaystyle\int x^5 e^{x^3}\,dx$

Solution

Let: $u = x^3 \qquad dv = x^2 e^{x^3}\,dx = \frac{1}{3}\,d\left(e^{x^3}\right) \qquad d\left(e^{x^3}\right) = 3x^2 e^{x^3}\,dx$

$du = 3x^2 dx \qquad v = \frac{1}{3}e^{x^3}$

$$\int x^5 e^{x^3}\,dx = x^3\,\frac{1}{3}e^{x^3} - \int \frac{1}{3}e^{x^3}\,3x^2 dx \qquad d\left(e^{x^3}\right) = 3x^2 e^{x^3}\,dx \qquad \int udv = uv - \int vdu$$

$$= \frac{1}{3}x^3 e^{x^3} - \frac{1}{3}\int d\left(e^{x^3}\right)$$

$$= \boxed{\frac{1}{3}x^3 e^{x^3} - \frac{1}{3}e^{x^3} + C}$$

Exercise

Evaluate the integral $\displaystyle\int x^2 \ln x^3\,dx$

Solution

$$\int x^2 \ln x^3\,dx = \int 3x^2 \ln x\,dx \qquad u = \ln x \quad v = \int 3x^2 dx = x^3 \qquad \int udv = uv - \int vdu$$

$$du = \frac{1}{x}\,dx$$

$$= x^3 \ln x - \int x^2 dx$$

$$= \boxed{x^3 \ln x - \frac{1}{3}x^3 + C}$$

Exercise

Evaluate the integral $\int \ln\left(x+x^2\right)dx$

Solution

Let: $u=\ln\left(x+x^2\right) \qquad dv=dx$

$du=\frac{2x+1}{x+x^2}dx \qquad v=x$

$$\begin{aligned}
\int \ln\left(x+x^2\right)dx &= x\ln\left(x+x^2\right)-\int x\frac{2x+1}{x+x^2}dx \\
&= x\ln\left(x+x^2\right)-\int \frac{2x+1}{x(1+x)}xdx \\
&= x\ln\left(x+x^2\right)-\int \frac{2x+2-1}{1+x}dx \\
&= x\ln\left(x+x^2\right)-\int \frac{2(x+1)-1}{x+1}dx \\
&= x\ln\left(x+x^2\right)-\int \left(2-\frac{1}{x+1}\right)dx \\
&= x\ln\left(x+x^2\right)-\left(2x-\ln|x+1|\right)+C \\
&= x\ln\left(x+x^2\right)-2x+\ln|x+1|+C
\end{aligned}$$

Exercise

Evaluate the integral $\int e^{-x}\sin 4x\,dx$

Solution

		$\int \sin 4x\,dx$
+	e^{-x}	$-\frac{1}{4}\cos 4x$
-	$-e^{-x}$	$-\frac{1}{16}\sin 4x$
+	e^{-x}	$-\frac{1}{16}\int \sin 4x\,dx$

$$\int e^{-x}\sin 4x\,dx=-\frac{1}{4}e^{-x}\cos 4x-\frac{1}{16}e^{-x}\sin 4x-\frac{1}{16}\int e^{-x}\sin 4x\,dx$$

$$\left(1+\frac{1}{16}\right)\int e^{-x}\sin 4x\,dx=-\frac{1}{4}e^{-x}\cos 4x-\frac{1}{16}e^{-x}\sin 4x$$

$$\frac{17}{16}\int e^{-x}\sin 4x\,dx=-\frac{1}{16}e^{-x}\left(4\cos 4x+\sin 4x\right)$$

$$\int e^{-x}\sin 4x\,dx=-\frac{e^{-x}}{17}\left(4\cos 4x+\sin 4x\right)+C$$

Exercise

Evaluate the integral $\int e^{-2\theta}\sin 6\theta\, d\theta$

Solution

$$\int e^{-2\theta}\sin 6\theta\, d\theta = -\frac{1}{6}e^{-2\theta}\cos 6\theta - \frac{1}{18}e^{-2\theta}\sin 6\theta - \frac{1}{9}\int e^{-2\theta}\sin 6\theta\, d\theta$$

$$\left(1+\frac{1}{9}\right)\int e^{-2\theta}\sin 6\theta\, d\theta = -\frac{1}{18}e^{-2\theta}\left(3\cos 6\theta + \sin 6\theta\right)$$

$$\frac{10}{9}\int e^{-2\theta}\sin 6\theta\, d\theta = -\frac{1}{18}e^{-2\theta}\left(3\cos 6\theta + \sin 6\theta\right)$$

$$\int e^{-2\theta}\sin 6\theta\, d\theta = -\frac{e^{-2\theta}}{20}\left(3\cos 6\theta + \sin 6\theta\right) + C$$

		$\int \sin 6\theta\, d\theta$
+	$e^{-2\theta}$	$-\frac{1}{6}\cos 6\theta$
−	$-2e^{-2\theta}$	$-\frac{1}{36}\sin 6\theta$
+	$4e^{-2\theta}$	$-\frac{1}{36}\int \sin 6\theta\, d\theta$

Exercise

Evaluate the integral $\int xe^{-4x}dx$

Solution

$$\int xe^{-4x}dx = \left(-\frac{x}{4}-\frac{1}{16}\right)e^{-4x} + C$$

		$\int e^{-4x}\, dx$
+	x	$-\frac{1}{4}e^{-4x}$
−	1	$\frac{1}{16}e^{-4x}$

Exercise

Evaluate the integral $\int x\ln(x+1)dx$

Solution

$$u = \ln(x+1) \Rightarrow du = \frac{1}{x+1}dx$$

$$dv = xdx \Rightarrow v = \frac{1}{2}x^2$$

$$\int x\ln(x+1)dx = \frac{1}{2}x^2\ln(x+1) - \frac{1}{2}\int \frac{x^2}{x+1}dx$$

$$= \frac{1}{2}x^2\ln(x+1) - \frac{1}{2}\int \left(x-1+\frac{1}{x+1}\right)dx$$

$$= \frac{1}{2}x^2\ln(x+1) - \frac{1}{2}\left(\frac{1}{2}x^2 - x + \ln(x+1)\right) + C$$

$$= \frac{1}{2}x^2\ln(x+1) - \frac{1}{4}x^2 + \frac{1}{2}x - \frac{1}{2}\ln(x+1) + C$$

$$= -\frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{2}\left(x^2-1\right)\ln(x+1) + C$$

Exercise

Evaluate the integral $\int \frac{(\ln x)^2}{x} dx$

Solution

$$\int \frac{(\ln x)^2}{x} dx = \int (\ln x)^2 d(\ln x)$$

$$= \frac{1}{3}(\ln x)^3 + C$$

Exercise

Evaluate the integral $\int \frac{xe^{2x}}{(2x+1)^2} dx$

Solution

$$u = xe^{2x} \quad \to \quad du = (2x+1)e^{2x} dx$$

$$dv = \frac{dx}{(2x+1)^2} = \frac{1}{2} \frac{d(2x+1)}{(2x+1)^2} \quad \to \quad v = -\frac{1}{2} \frac{1}{2x+1}$$

$$\int \frac{xe^{2x}}{(2x+1)^2} dx = -\frac{xe^{2x}}{4x+2} + \frac{1}{2} \int e^{2x} dx$$

$$= -\frac{x}{4x+2} e^{2x} + \frac{1}{4} e^{2x} + C$$

Exercise

Evaluate the integral $\int \frac{5x}{e^{2x}} dx$

Solution

		$\int e^{-2x}\, dx$
+	$5x$	$-\frac{1}{2}e^{-2x}$
−	5	$\frac{1}{4}e^{-2x}$

$$\int \frac{5x}{e^{2x}} dx = \int 5xe^{-2x} dx$$

$$= \left(-\frac{5}{2}x - \frac{5}{4}\right)e^{-2x} + C$$

Exercise

Evaluate the integral $\int \frac{e^{1/x}}{x^2} dx$

Solution

$$\int \frac{e^{1/x}}{x^2}dx = -\int e^{1/x} d\left(\frac{1}{x}\right)$$

$$\boxed{= -e^{1/x} + C}$$

Exercise

Evaluate the integral $\int x^5 \ln 3x\, dx$

Solution

$$u = \ln 3x \quad \rightarrow \quad du = \frac{1}{x}dx$$

$$dv = x^5\, dx \quad \rightarrow \quad v = \frac{1}{6}x^6$$

$$\int x^5 \ln 3x\, dx = \frac{1}{6}x^6 \ln 3x - \frac{1}{6}\int x^5 dx$$

$$\boxed{= \frac{1}{6}x^6 \ln 3x - \frac{1}{36}x^6 + C}$$

Exercise

Evaluate the integral $\int x\sqrt{x-5}\, dx$

Solution

Let $u = \sqrt{x-5} \quad \rightarrow \quad u^2 = x-5 \Rightarrow x = u^2 + 5$

$2udu = dx$

$$\int x\sqrt{x-5}\, dx = \int \left(u^2+5\right) u\,(2udu)$$

$$= \int \left(2u^4 + 10u^2\right) du$$

$$\boxed{= \frac{2}{5}u^5 + \frac{10}{3}u^3 + C}$$

Exercise

Evaluate the integral $\int \frac{x}{\sqrt{6x+1}} dx$

Solution

$$u = x \quad \rightarrow \quad du = dx$$

$$dv = (6x+1)^{-1/2}\, dx = \frac{1}{6}(6x+1)^{-1/2} d(6x+1) \quad \rightarrow \quad v = \frac{1}{3}(6x+1)^{1/2}$$

$$\int \frac{x}{\sqrt{6x+1}} dx = \frac{1}{3}x\sqrt{6x+1} - \frac{1}{3}\int (6x+1)^{1/2}\, dx$$

$$= \frac{1}{3}x\sqrt{6x+1} - \frac{1}{18}\int (6x+1)^{1/2}\, d(6x+1)$$

$$= \frac{1}{3}x\sqrt{6x+1} - \frac{1}{27}(6x+1)^{3/2} + C$$

Exercise

Evaluate the integral $\int x\cos x\, dx$

Solution

		$\int \cos x$
+	x	$\sin x$
–	1	$-\cos x$

$$\int x\cos x\, dx = x\sin x + \cos x + C$$

Exercise

Evaluate the integral $\int x\csc x\cot x\, dx$

Solution

$u = x \quad \to \quad du = dx$

$dv = \csc x\cot x\, dx \quad \to \quad v = -\csc x$

$$\int x\csc x\cos x\, dx = -x\csc x + \int \csc dx$$

$$= -x\csc x - \ln\left|\csc x + \cot x\right| + C$$

Exercise

Evaluate the integral $\int x^3 \sin x\, dx$

Solution

		$\int \sin x$
+	x^3	$-\cos x$
–	$3x^2$	$-\sin x$
+	$6x$	$\cos x$
–	6	$\sin x$

$$\int x^3 \sin x\, dx = -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$$

Exercise

Evaluate the integral $\int x^2 \cos x\, dx$

Solution

		$\int \cos x$
+	x^2	$\sin x$
–	$2x$	$-\cos x$
+	2	$-\sin x$

$$\int x^2 \cos x\, dx = x^2\sin x + 2x\cos x - 2\sin x + C$$

Exercise

Evaluate the integral $\displaystyle\int e^{-3x}\sin 5x\,dx$

Solution

$$\int e^{-3x}\sin 5x\,dx = -\frac{1}{5}e^{-3x}\cos 5x - \frac{3}{25}e^{-3x}\sin 5x - \frac{9}{25}\int e^{-3x}\sin 5x\,dx$$

$$\left(1+\frac{9}{25}\right)\int e^{-3x}\sin 5x\,dx = -\frac{1}{25}(5\cos 5x + 3\sin 5x)e^{-3x}$$

$$\frac{34}{25}\int e^{-3x}\sin 5x\,dx = -\frac{1}{25}(5\cos 5x + 3\sin 5x)e^{-3x}$$

$$\int e^{-3x}\sin 5x\,dx = -\frac{1}{34}(5\cos 5x + 3\sin 5x)e^{-3x} + C$$

		$\int \sin 5x$
+	e^{-3x}	$-\frac{1}{5}\cos 5x$
−	$-3e^{-3x}$	$-\frac{1}{25}\sin 5x$
+	$9e^{-3x}$	$-\int \frac{1}{25}\sin 5x$

Exercise

Evaluate the integral $\displaystyle\int e^{-3x}\sin 4x\,dx$

Solution

$$\int e^{-3x}\sin 4x\,dx = -\frac{1}{4}e^{-3x}\cos 4x - \frac{3}{16}e^{-3x}\sin 4x - \frac{9}{16}\int e^{-3x}\sin 4x\,dx$$

$$\left(1+\frac{9}{16}\right)\int e^{-3x}\sin 4x\,dx = -\frac{1}{16}(4\cos 4x + 3\sin 4x)e^{-3x}$$

$$\frac{25}{16}\int e^{-3x}\sin 4x\,dx = -\frac{1}{16}(4\cos 4x + 3\sin 4x)e^{-3x}$$

$$\int e^{-3x}\sin 4x\,dx = -\frac{1}{25}(4\cos 4x + 3\sin 4x)e^{-3x} + C$$

		$\int \sin 4x$
+	e^{-3x}	$-\frac{1}{4}\cos 4x$
−	$-3e^{-3x}$	$-\frac{1}{16}\sin 4x$
+	$9e^{-3x}$	$-\frac{1}{16}\int \sin 4x$

Exercise

Evaluate the integral $\displaystyle\int e^{4x}\cos 2x\,dx$

Solution

		$\int \cos 2x$
+	e^{4x}	$\frac{1}{2}\sin 2x$
−	$4e^{4x}$	$-\frac{1}{4}\cos 2x$
+	$16e^{4x}$	$-\frac{1}{4}\int \cos 2x$

$$\int e^{4x}\cos 2x\,dx = \frac{1}{2}e^{4x}\sin 2x + e^{4x}\cos 2x - 4\int e^{4x}\cos 2x\,dx$$

$$5\int e^{4x}\cos 2x\,dx = \frac{1}{2}(\sin 2x + 2\cos 2x)e^{4x}$$

$$\int e^{4x}\cos 2x\,dx = \frac{1}{10}(\sin 2x + 2\cos 2x)e^{4x} + C$$

Exercise

Evaluate the integral $\int e^{3x}\cos 3x\,dx$

Solution

		$\int \cos 3x$
+	e^{3x}	$\frac{1}{3}\sin 3x$
–	$3e^{3x}$	$-\frac{1}{9}\cos 3x$
+	$9e^{3x}$	$-\frac{1}{9}\int \cos 3x$

$$\int e^{3x}\cos 3x\,dx = \frac{1}{3}e^{3x}\sin 3x + \frac{1}{3}e^{3x}\cos 3x - \int e^{3x}\cos 3x\,dx$$

$$2\int e^{3x}\cos 3x\,dx = \frac{1}{3}(\sin 3x + \cos 3x)e^{3x}$$

$$\int e^{3x}\cos 3x\,dx = \underline{\frac{1}{6}(\sin 3x + \cos 3x)e^{3x} + C}$$

Exercise

Evaluate the integral $\int x^2 e^{4x}\,dx$

Solution

$$\int x^2 e^{4x}\,dx = \underline{\left(\frac{1}{4}x^2 - \frac{1}{8}x + \frac{1}{32}\right)e^{4x} + C}$$

$$\int x^n e^{ax}\,dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k n!}{(n-k)!a^{k+1}}x^{n-k}$$

Exercise

Evaluate the integral $\int x^3 e^{-3x}\,dx$

Solution

$$\int x^3 e^{-3x}\,dx = \underline{\left(-\frac{1}{3}x^3 + \frac{1}{3}x^2 - \frac{2}{9}x + \frac{2}{27}\right)e^{-3x} + C}$$

$$\int x^n e^{ax}\,dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k n!}{(n-k)!a^{k+1}}x^{n-k}$$

Exercise

Evaluate the integral $\int x^3\cos 2x\,dx$

Solution

		$\int \cos 2x$
+	x^3	$\frac{1}{2}\sin 2x$
–	$3x^2$	$-\frac{1}{4}\cos 2x$
+	$6x$	$-\frac{1}{8}\sin 2x$
–	6	$\frac{1}{16}\cos 2x$

$$\int x^3\cos 2x\,dx = \underline{\frac{1}{2}x^3\sin 2x + \frac{3}{4}x^2\cos 2x - \frac{3}{4}x\sin 2x - \frac{3}{8}\cos 2x + C}$$

Exercise

Evaluate the integral $\int x^3 \sin x\, dx$

Solution

		$\int \sin x$
$+$	x^3	$-\cos x$
$-$	$3x^2$	$-\sin x$
$+$	$6x$	$\cos x$
$-$	6	$\sin x$

$$\int x^3 \sin x\, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6\sin x + C$$

Exercise

Evaluate the integral $\int_0^{1/\sqrt{2}} 2x \sin^{-1}\left(x^2\right) dx$

Solution

$$u = \sin^{-1}\left(x^2\right) \qquad dv = 2xdx$$

$$du = \frac{2x}{\sqrt{1-x^4}}dx \qquad v = x^2$$

$$\int u dv = uv - \int v du$$

$$\int_0^{1/\sqrt{2}} 2x \sin^{-1}\left(x^2\right) dx = \left[x^2 \sin^{-1}\left(x^2\right)\right]_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} x^2 \frac{2x}{\sqrt{1-x^4}}dx \qquad d\left(1-x^4\right) = -4x^3 dx$$

$$= \left(\left(\frac{1}{\sqrt{2}}\right)^2 \sin^{-1}\left(\left(\frac{1}{\sqrt{2}}\right)^2\right) - 0\right) + \int_0^{1/\sqrt{2}} \frac{d\left(1-x^4\right)}{2\sqrt{1-x^4}}$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) + \left[\sqrt{1-x^4}\right]_0^{1/\sqrt{2}}$$

$$= \frac{1}{2}\frac{\pi}{6} + \left(\sqrt{1-\frac{1}{4}} - 1\right)$$

$$= \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

$$= \frac{\pi + 6\sqrt{3} - 12}{12}$$

Exercise

Evaluate the integral $\int_1^e x^3 \ln x dx$

Solution

$$u = \ln x \qquad v = \int x^3 dx = \frac{1}{4}x^4$$

$$du = \frac{1}{x}dx$$

$$\int_1^e x^3 \ln x dx = \left[\frac{1}{4}x^4 \ln x\right]_1^e - \frac{1}{4}\int_1^e x^4 \frac{dx}{x}$$

$$= \frac{1}{4}\left(e^4 \ln e - 1^4 \ln 1\right) - \frac{1}{4}\int_1^e x^3 dx$$

$$= \frac{e^4}{4} - \frac{1}{16}\left[x^4\right]_1^e$$

$$= \frac{e^4}{4} - \frac{1}{16}\left(e^4 - 1\right)$$

$$= \frac{4}{4}\frac{e^4}{4} - \frac{1}{16}e^4 + \frac{1}{16}$$

$$= \frac{3e^4+1}{16}$$

Exercise

Evaluate the integral $\int_0^1 x\sqrt{1-x}dx$

Solution

Let: $u = x \qquad dv = \sqrt{1-x}dx = (1-x)^{1/2}dx \qquad d(1-x) = -dx$

$du = dx \qquad v = -\int (1-x)^{1/2} d(1-x) = -\frac{2}{3}(1-x)^{2/3}$

$$\int_0^1 x\sqrt{1-x}dx = \left[x\left(-\frac{2}{3}(1-x)^{2/3}\right)\right]_0^1 - \int_0^1 -\frac{2}{3}(1-x)^{2/3}dx \qquad \int udv = uv - \int vdu$$

$$= \left[-\frac{2}{3}x(1-x)^{2/3}\right]_0^1 + \frac{2}{3}\int_0^1 (1-x)^{2/3}(-d(1-x))$$

$$= -\frac{2}{3}\left[(1)(0)^{2/3} - 0\right] - \left[\frac{2}{3}\left(\frac{2}{5}\right)(1-x)^{5/3}\right]_0^1$$

$$= -\frac{4}{15}\left[0 - (1)^{5/3}\right]$$

$$= \frac{4}{15}$$

Exercise

Evaluate the integral $\int_0^{\pi/3} x\tan^2 xdx$

Solution

$$u = x \quad \to \quad dv = \tan^2 x dx = \frac{\sin^2 x}{\cos^2 x} dx = \frac{1-\cos^2 x}{\cos^2 x} dx$$

$$du = dx \quad \to \quad v = \int \left(\frac{1}{\cos^2 x} - 1 \right) dx = \tan x - x$$

$$\int_0^{\pi/3} x \tan^2 x dx = \left[x(\tan x - x) \right]_0^{\pi/3} - \int_0^{\pi/3} (\tan x - x) dx \qquad \int u dv = uv - \int v du$$

$$= \left[\frac{\pi}{3} \left(\tan\frac{\pi}{3} - \frac{\pi}{3} \right) - 0 \right] - \left[-\ln|\cos x| - \frac{x^2}{2} \right]_0^{\pi/3}$$

$$= \frac{\pi}{3} \left(\sqrt{3} - \frac{\pi}{3} \right) + \left[\ln\left|\cos\frac{\pi}{3}\right| + \frac{1}{2} \left(\frac{\pi}{3} \right)^2 - \ln|1| - 0 \right]$$

$$= \frac{\pi}{3} \sqrt{3} - \frac{\pi^2}{9} + \ln\left|\frac{1}{2}\right| + \frac{\pi^2}{18}$$

$$= \frac{\pi}{3} \sqrt{3} - \ln 2 - \frac{\pi^2}{18}$$

Exercise

Evaluate the integral $\int_0^{\pi} x \sin x \, dx$

Solution

$$\int_0^{\pi} x \sin x \, dx = -x \cos x + \sin x \Big|_0^{\pi}$$

$$= \pi$$

		$\int \sin x \, dx$
+	x	$-\cos x$
−	1	$-\sin x$

Exercise

Evaluate the integral $\int_1^{e} \ln 2x \, dx$

Solution

$$\int_1^{e} \ln 2x \, dx = \frac{1}{2} \int_1^{e} \ln 2x \, d(2x) \qquad \int \ln x \, dx = x \ln x - x$$

$$= x \ln 2x - x \Big|_1^{e}$$

$$= e \ln 2e - e - \ln 2 + 1$$

$$= e(\ln 2 + \ln e) - e - \ln 2 + 1$$

$$= e \ln 2 - \ln 2 + 1$$

$$= (e-1)\ln 2 + 1$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/2} x\cos 2x\,dx$

Solution

		$\int \cos 2x\,dx$
+	x	$\frac{1}{2}\sin 2x$
−	1	$-\frac{1}{4}\cos 2x$

$$\int_0^{\pi/2} x\cos 2x\,dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\Big|_0^{\pi/2}$$

$$= -\frac{1}{4} - \frac{1}{4}$$

$$\underline{\left. = -\frac{1}{2} \right|}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\ln 2} xe^x\,dx$

Solution

		$\int e^x\,dx$
+	x	e^x
−	1	e^x

$$\int_0^{\ln 2} xe^x\,dx = e^x(x-1)\Big|_0^{\ln 2}$$

$$= 2(\ln 2 - 1) + 1$$

$$\underline{\left. = 2\ln 2 - 1 \right|}$$

Exercise

Evaluate the integral $\displaystyle\int_1^{e^2} x^2\ln x\,dx$

Solution

$$u = \ln x$$

$$du = \frac{1}{x}dx \qquad v = \int x^2 dx = \frac{1}{3}x^3$$

$$\int x^2\ln x\,dx = \frac{1}{3}x^3\ln x - \frac{1}{3}\int x^2 dx$$

$$\int_1^{e^2} x^2\ln x\,dx = \frac{1}{3}x^3\ln x - \frac{1}{9}x^3\Big|_1^{e^2}$$

$$= \frac{2}{3}e^6 - \frac{1}{9}e^6 + \frac{1}{9}$$

$$\underline{\left. = \frac{5}{9}e^6 + \frac{1}{9} \right|}$$

Exercise

Evaluate the integral $\displaystyle\int_0^3 xe^{x/2}dx$

Solution

$$\int_0^3 xe^{x/2}dx = (2x-4)e^{x/2}\Big|_0^3$$

$$\boxed{= 2e^{3/2}+4}$$

$$\int x^n e^{ax}\,dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k n!}{(n-k)!a^{k+1}}x^{n-k}$$

Exercise

Evaluate the integral $\displaystyle\int_0^2 x^2e^{-2x}dx$

Solution

$$\int_0^2 x^2e^{-2x}dx = \left(-\frac{1}{2}x^2+\frac{1}{2}x-\frac{1}{4}\right)e^{-2x}\Big|_0^2$$

$$=\left(-2+1-\frac{1}{4}\right)e^{-4}+\frac{1}{4}$$

$$\boxed{=\frac{1}{4}-\frac{5}{4}e^{-4}}$$

$$\int x^n e^{ax}\,dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k n!}{(n-k)!a^{k+1}}x^{n-k}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/4} x\cos 2x\,dx$

Solution

$$\int_0^{\pi/4} x\cos 2x\,dx = \frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x\Big|_0^{\pi/4}$$

$$\boxed{=\frac{\pi}{8}-\frac{1}{4}}$$

		$\int \cos 2x\,dx$
+	x	$\frac{1}{2}\sin 2x$
−	1	$-\frac{1}{4}\cos 2x$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi} x\sin 2x\,dx$

Solution

$$\int_0^{\pi} x\sin 2x\,dx = -\frac{1}{2}x\cos 2x+\frac{1}{4}\sin 2x\Big|_0^{\pi}$$

$$\boxed{=-\frac{\pi}{2}}$$

		$\int \sin 2x\,dx$
+	x	$-\frac{1}{2}\cos 2x$
−	1	$-\frac{1}{4}\sin 2x$

Exercise

Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the cure $y = e^{x}$, and the line $x = \ln 2$ about the line $x = \ln 2$

Solution

	e^{x}	
$+$	x	e^{x}
$-$	1	e^{x}

$$V = 2\pi \int_{0}^{\ln 2} (\ln 2 - x) e^{x} dx$$

$$= 2\pi \int_{0}^{\ln 2} \left(\ln 2e^{x} - xe^{x} \right) dx$$

$$= 2\pi \ln 2 \left[e^{x} \right]_{0}^{\ln 2} - 2\pi \int_{0}^{\ln 2} xe^{x} dx$$

$$= 2\pi \ln 2 \left(e^{\ln 2} - e^{0} \right) - 2\pi \left[xe^{x} - e^{x} \right]_{0}^{\ln 2}$$

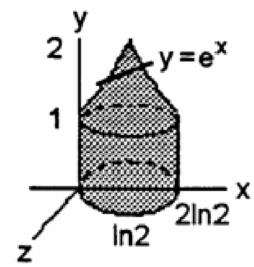
$$= 2\pi \ln 2 (2-1) - 2\pi \left[\ln 2e^{\ln 2} - e^{\ln 2} - (0-1) \right]$$

$$= 2\pi \ln 2 - 2\pi [2\ln 2 - 2 + 1]$$

$$= 2\pi \ln 2 - 4\pi \ln 2 + 2\pi$$

$$= -2\pi \ln 2 + 2\pi$$

$$= 2\pi (1 - \ln 2) \quad unit^{3}$$



Exercise

Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate aces, the cure $y = e^{-x}$, and the line $x = 1$, about

a) the line $y - axis$

b) the line $x = 1$

Solution

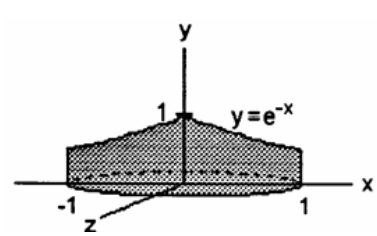
	e^{-x}	
$(+)$	x	$-e^{-x}$
$(-)$	1	e^{-x}

a) $$V = 2\pi \int_{0}^{1} xe^{-x} dx$$

$$= 2\pi \left(\left[-xe^{-x} - e^{-x} \right]_{0}^{1} \right)$$

$$= 2\pi \left(-e^{-1} - e^{-1} + 0 + 1 \right)$$

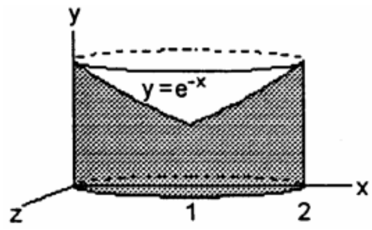
$$= 2\pi \left(-\frac{1}{e} - \frac{1}{e} + 1 \right)$$



$$= 2\pi\left(-\frac{2}{e}+1\right)$$

$$\boxed{= 2\pi - \frac{4\pi}{e}} \quad unit^3$$

b) $V = 2\pi\int_0^1 (1-x)e^{-x}dx$



$$= 2\pi\left(\int_0^1 e^{-x}dx - \int_0^1 xe^{-x}dx\right)$$

$$= 2\pi\left(\left[-e^{-x} - \left(-xe^{-x} - e^{-x}\right)\right]_0^1\right)$$

$$= 2\pi\left[e^{-x} + xe^{-x} - e^{-x}\right]_0^1$$

$$= 2\pi\left[xe^{-x}\right]_0^1$$

$$= 2\pi\left(e^{-1}\right)$$

$$\boxed{= \frac{2\pi}{e}} \quad unit^3$$

Exercise

Find the volume of the solid that is generated by the region bounded by $f(x) = e^{-x}$, $x = \ln 2$, and the coordinate axes is revolved about the y-*axis*.

Solution

$$V = 2\pi\int_0^{\ln 2} xe^{-x}\,dx \qquad V = \int_a^b 2\pi(radius)(height)\,dx \quad \textit{Shells Method}$$

		$\int e^{-x}\,dx$
+	x	$-e^{-x}$
−	1	e^{-x}

$$= 2\pi\left[e^{-x}(-x-1)\right]\Big|_0^{\ln 2}$$

$$= 2\pi\left(e^{-\ln 2}(-\ln 2 - 1) + 1\right)$$

$$= 2\pi\left(\frac{1}{2}(-\ln 2 - 1) + 1\right)$$

$$= 2\pi\left(-\frac{1}{2}\ln 2 + \frac{1}{2}\right)$$

$$\boxed{= \pi(1 - \ln 2)} \quad unit^3$$

Exercise

Find the volume of the solid that is generated by the region bounded by $f(x) = \sin x$, and the *x-axis* on $[0, \pi]$ is revolved about the *y-axis*.

Solution

$$V = \int_a^b 2\pi(\text{radius})(\text{height})\,dx \quad \textit{Shells Method}$$

$$V = 2\pi \int_0^{\pi} x \sin x \, dx$$

		$\int \sin x$
+	x	$-\cos x$
–	1	$-\sin x$

$$= 2\pi\left[-x\cos x + \sin x\right]\Big|_0^{\pi}$$

$$= 2\pi^2 \quad unit^3$$

Exercise

Find the area of the region generated when the region bounded by $y = \sin x$ and $y = \sin^{-1} x$ on the interval $\left[0, \frac{1}{2}\right]$.

Solution

$$A = \int_0^{1/2} \left(\sin^{-1} x - \sin x\right) dx$$

$$u = \sin^{-1} x \qquad du = \frac{dx}{\sqrt{1-x^2}} \qquad v = \int dx = x$$

$$= x\sin^{-1} x\Big|_0^{1/2} - \int_0^{1/2} \frac{x\,dx}{\sqrt{1-x^2}} + \cos x\Big|_0^{1/2}$$

$$= x\sin^{-1} x + \cos x\Big|_0^{1/2} + \frac{1}{2}\int_0^{1/2} \left(1-x^2\right)^{-1/2} d\left(1-x^2\right)$$

$$= x\sin^{-1} x + \cos x + \left(1-x^2\right)^{1/2}\Big|_0^{1/2}$$

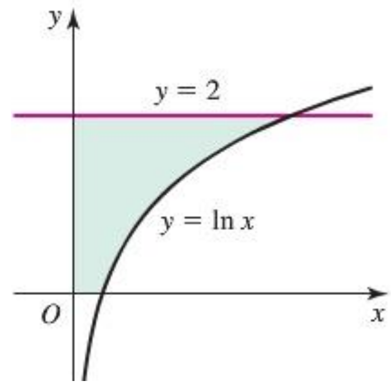
$$= \frac{1}{2}\sin^{-1}\frac{1}{2} + \cos\frac{1}{2} + \left(1-\frac{1}{4}\right)^{1/2} - 1 - 1$$

$$= \frac{\pi}{12} + \cos\frac{1}{2} + \frac{\sqrt{3}}{2} - 2 \quad unit^2$$

Exercise

Determine the area of the shaded region bounded by $y = \ln x, \quad y = 2, \quad y = 0, \quad and\ x = 0$

Solution



$$y = \ln x = 0 \quad \to \quad x = 1$$

$$y = \ln x = 2 \quad \to \quad x = e^2$$

$$A = 1 \times 2 + \int_1^2 (2 - \ln x)\, dx$$

$$= 2 + (2x - x\ln x + x)\Big|_1^2$$

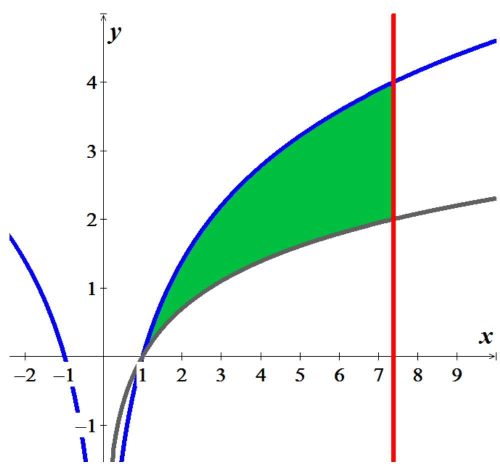
$$= 2 + 4 - 2\ln 2 + 2 - 2 - 1$$

$$\underline{= 5 - 2\ln 2} \quad unit^2$$

Exercise

Find the area between the curves $y = \ln x^2, \quad y = \ln x, \quad and \quad x = e^2$

Solution



$$y = \ln x^2 = \ln x \quad with \quad x > 0$$

$$x^2 = x \Rightarrow \underline{x = 1}$$

$$A = \int_1^{e^2} \left(\ln x^2 - \ln x\right) dx$$

$$= \int_1^{e^2} (2\ln x - \ln x)\, dx$$

$$= \int_1^{e^2} \ln x\ dx \qquad\qquad \int \ln x\ dx = x\ln x - x$$

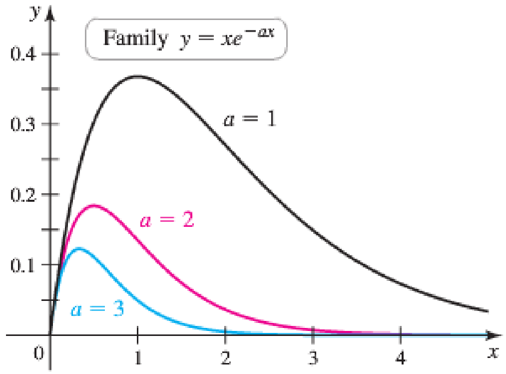
$$= (x\ln x - x)\Big|_1^{e^2}$$

$$= e^2 \ln e^2 - e^2 + 1$$

$$\underline{= e^2 + 1} \quad unit^2$$

Exercise

The curves $y = xe^{-ax}$ are shown in the figure for $a = 1,\ 2,\ and\ 3$.



a) Find the area of the region bounded by $y = xe^{-x}$ and the *x*-axis on the interval [0, 4].

b) Find the area of the region bounded by $y = xe^{-ax}$ and the *x*-axis on the interval [0, 4] where $a > 0$

c) Find the area of the region bounded by $y = xe^{-ax}$ and the *x*-axis on the interval [0, *b*]. Because this area depends on *a* and *b*, we call it $A(a,\ b)$ where $a > 0$ and $b > 0$.

d) Use part (*c*) to show that $A(1,\ \ln b) = 4A\left(2,\ \frac{1}{2}\ln b\right)$

e) Does this pattern continue? Is it true that $A(1,\ \ln b) = a^2 A\left(a,\ \frac{1}{a}\ln b\right)$

Solution

a) $$\int_0^4 xe^{-x}dx = e^{-x}(-x-1)\Big|_0^4$$

$$= e^{-4}(-5) - (-1)$$

$$= 1 - \frac{5}{e^4} \quad unit^2$$

		$\int e^{-x}\ dx$
+	x	$-e^{-x}$
−	1	e^{-x}

b) $$\int_0^4 xe^{-ax}dx = e^{-ax}\left(-\frac{1}{a}x - \frac{1}{a^2}\right)\Big|_0^4$$

$$= e^{-4a}\left(-\frac{4}{a} - \frac{1}{a^2}\right) - \left(-\frac{1}{a^2}\right)$$

$$= \frac{1}{a^2} - e^{-4a}\left(\frac{4a+1}{a^2}\right)$$

$$= \frac{1}{a^2}\left(1 - \frac{4a+1}{e^{-4a}}\right) \quad unit^2$$

		$\int e^{-ax}\ dx$
+	x	$-\frac{1}{a}e^{-ax}$
−	1	$\frac{1}{a^2}e^{-ax}$

c) $$\int_0^b xe^{-ax}dx = e^{-ax}\left(-\frac{1}{a}x - \frac{1}{a^2}\right)\Big|_0^b$$

$$= e^{-ab}\left(-\frac{b}{a}-\frac{1}{a^2}\right)-\left(-\frac{1}{a^2}\right)$$

$$= \frac{1}{a^2} - e^{-ab}\left(\frac{ab+1}{a^2}\right)$$

$$= \frac{1}{a^2}\left(1-\frac{ab+1}{e^{ab}}\right) \quad unit^2$$

d) $A(a,b)=\frac{1}{a^2}\left(1-\frac{ab+1}{e^{ab}}\right)$

$$A(1,\ \ln b)=1-\frac{\ln b+1}{e^{\ln b}}$$

$$=1-\frac{\ln b+1}{b}$$

$$A\left(2,\ \frac{1}{2}\ln b\right)=\frac{1}{4}\left(1-\frac{\ln b+1}{e^{\ln b}}\right)$$

$$=\frac{1}{4}\left(1-\frac{\ln b+1}{b}\right)$$

$$=\frac{1}{4}A(1,\ \ln b)$$

$$\therefore\quad A(1,\ \ln b)=4A\left(2,\ \frac{1}{2}\ln b\right)$$

e) $A\left(a,\ \frac{1}{a}\ln b\right)=\frac{1}{a^2}\left(1-\frac{\ln b+1}{e^{\ln b}}\right)$

$$=\frac{1}{a^2}\left(1-\frac{\ln b+1}{b}\right)$$

$$=\frac{1}{a^2}A(1,\ \ln b)$$

Yes, there is a pattern: $A(1,\ \ln b)=a^2A\left(a,\ \frac{1}{a}\ln b\right)$

Exercise

Suppose a mass on a spring that is slowed by friction has the position function $s(t)=e^{-t}\sin t$

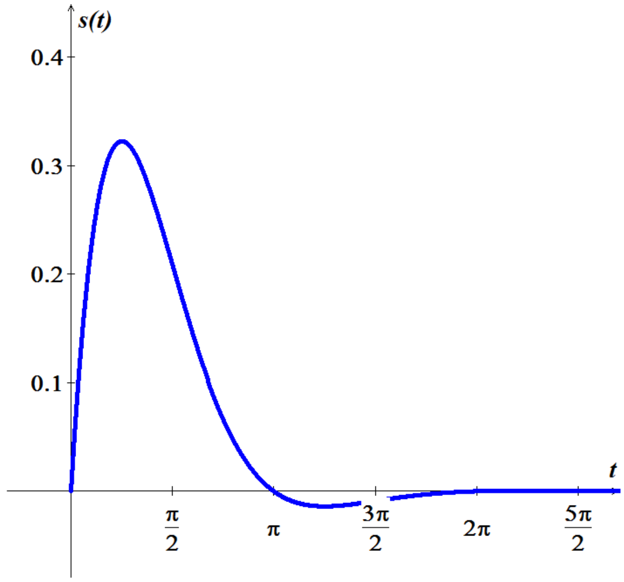
a) Graph the position function. At what times does the oscillator pass through the position $s=0$?

b) Find the average value of the position on the interval $[0,\ \pi]$.

c) Generalize part (*b*) and find the average value of the position on the interval $[n\pi,\ (n+1)\pi]$, for $n=0,\ 1,\ 2,\ \ldots$

Solution

a) $s(t)=e^{-t}\sin t=0 \qquad \sin t=0 \ \rightarrow\ t=n\pi$



b) $\int e^{-t}\sin t\,dt = -e^{-t}(\cos t+\sin t) - \int e^{-t}\sin t\,dt$

$2\int e^{-t}\sin t\,dt = -e^{-t}(\cos t+\sin t)$

		$\int \sin t$
+	e^{-t}	$-\cos t$
−	$-e^{-t}$	$-\sin t$
+	e^{-t}	$-\int \sin t\,dt$

$$\begin{aligned} Average &= \frac{1}{\pi}\int_0^{\pi} e^{-t}\sin t\,dt \\ &= -\frac{1}{2\pi}e^{-t}(\cos t-\sin t)\Big|_0^{\pi} \\ &= -\frac{1}{2\pi}\left(-e^{-\pi}-1\right) \\ &= \boxed{\frac{1}{2\pi}\left(e^{-\pi}+1\right)} \end{aligned}$$

c) $$\begin{aligned} Average &= \frac{1}{\pi}\int_{n\pi}^{(n+1)\pi} e^{-t}\sin t\,dt \\ &= -\frac{1}{2\pi}e^{-t}(\cos t-\sin t)\Big|_{n\pi}^{(n+1)\pi} \\ &= -\frac{1}{2\pi}\left(e^{-(n+1)\pi}\left(\cos((n+1)\pi)-\sin((n+1)\pi)\right)-e^{-n\pi}(\cos n\pi-\sin n\pi)\right) \\ &= -\frac{1}{2\pi}\left(e^{-(n+1)\pi}\cos((n+1)\pi)-e^{-n\pi}\cos n\pi\right) \\ &= \frac{e^{-n\pi}}{2\pi}\left(\cos n\pi - e^{-\pi}\cos(n+1)\pi\right) \\ &= \frac{e^{-n\pi}}{2\pi}\left((-1)^n - e^{-\pi}(-1)^{n+1}\right) \\ &= \boxed{(-1)^n\frac{e^{-n\pi}}{2\pi}\left(1+e^{-\pi}\right)} \end{aligned}$$

Exercise

Given the region bounded by the graphs of $y = x\sin x, \quad y = 0, \quad x = 0, \quad x = \pi$, find

a) The area of the region.

b) The volume of the solid generated by revolving the region about the *x-axis*

c) The volume of the solid generated by revolving the region about the *y-axis*

d) The centroid of the region

Solution

a) $A = \int_0^\pi x\sin x\,dx$

$= -x\cos x + \sin x\Big|_0^\pi$

$= \pi \ \ unit^2$

		$\int \sin x$
+	x	$-\cos x$
–	1	$-\sin x$

b) $V = \pi\int_0^\pi (x\sin x)^2\,dx$

$= \pi\int_0^\pi x^2\sin^2 x\,dx$

$= \frac{\pi}{2}\int_0^\pi x^2(1-\cos 2x)\,dx$

$= \frac{\pi}{2}\int_0^\pi \left(x^2 - x^2\cos 2x\right)dx$

$= \frac{\pi}{2}\left(\frac{1}{3}x^3 - \frac{1}{2}x^2\sin 2x - \frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right)_0^\pi$

$= \frac{\pi}{2}\left(\frac{1}{3}\pi^3 - \frac{\pi}{2}\right)$

$= \frac{\pi^4}{6} - \frac{\pi^2}{4} \ \ unit^3$

		$\int \cos 2x$
+	x^2	$\frac{1}{2}\sin 2x$
–	$2x$	$-\frac{1}{4}\cos 2x$
+	2	$-\frac{1}{8}\sin 2x$

c) $V = 2\pi\int_0^\pi x(x\sin x)\,dx$

$= 2\pi\int_0^\pi \left(x^2\sin x\right)dx$

$= 2\pi\left(-x^2\cos x + 2x\sin x + 2\cos x\right)_0^\pi$

$= 2\pi\left(\pi^2 - 2 - 2\right)$

$= 2\pi^3 - 8\pi \ \ unit^3$

		$\int \sin x$
+	x^2	$-\cos x$
–	$2x$	$-\sin x$
+	2	$\cos x$

d) $m = \int_0^{\pi} x \sin x \, dx = -x\cos x + \sin x \Big|_0^{\pi} = \pi$ From (***a***)

$M_x = \frac{1}{2}\int_0^{\pi} (x\sin x)^2 \, dx = \frac{1}{2}\left(\frac{\pi^3}{6} - \frac{\pi}{4}\right)$ From (***b***)

$M_y = \int_0^{\pi} x(x\sin x) \, dx = \frac{2\pi^3 - 8\pi}{2\pi} = \pi^2 - 4$ From (***c***)

$\bar{x} = \frac{M_y}{m} = \frac{\pi^2 - 4}{\pi} \approx 1.8684$

$\bar{y} = \frac{M_x}{m} = \frac{1}{\pi}\left(\frac{\pi^3}{12} - \frac{\pi}{8}\right) = \frac{\pi^2}{12} - \frac{1}{8} \approx 0.6975$

Solution Section 2.2 – Trigonometric Integrals

Exercise

Evaluate the integrals $\int \sin^4 2x \cos 2x dx$

Solution

$$d(\sin 2x) = 2\cos 2x dx \quad \Rightarrow \quad \frac{1}{2} d(\sin 2x) = \cos 2x dx$$

$$\int \sin^4 2x \cos 2x dx = \frac{1}{2}\int \sin^4 2x \; d(\sin 2x)$$

$$= \frac{1}{10}\sin^5 2x + C$$

Exercise

Evaluate the integrals $\int \sin^5 \frac{x}{2} dx$

Solution

$$\sin^5 \frac{x}{2} = \left(\sin^2 \frac{x}{2}\right)^2 \sin \frac{x}{2}$$

$$= \left(1 - \cos^2 \frac{x}{2}\right)^2 \sin \frac{x}{2}$$

$$= \left(1 - 2\cos^2 \frac{x}{2} + \cos^4 \frac{x}{2}\right) \sin \frac{x}{2}$$

$$d\left(\cos \frac{x}{2}\right) = -\frac{1}{2}\sin \frac{x}{2} dx \quad \rightarrow \quad -2d\left(\cos \frac{x}{2}\right) = \sin \frac{x}{2} dx$$

$$\int \sin^5 \frac{x}{2} dx = -2\int \left(1 - 2\cos^2 \frac{x}{2} + \cos^4 \frac{x}{2}\right) d\left(\cos \frac{x}{2}\right)$$

$$= -2\left(\cos \frac{x}{2} - \frac{2}{3}\cos^3 \frac{x}{2} + \frac{1}{5}\cos^5 \frac{x}{2}\right) + C$$

$$= -2\cos \frac{x}{2} + \frac{4}{3}\cos^3 \frac{x}{2} - \frac{2}{5}\cos^5 \frac{x}{2} + C$$

Exercise

Evaluate the integrals $\int \cos^3 2x \sin^5 2x \; dx$

Solution

$$\int \cos^3 2x \sin^5 2x \; dx = \int \left(\cos^2 2x\right) \cos 2x \sin^5 2x \; dx \qquad d(\sin 2x) = 2\cos 2x \; dx$$

$$= \int \left(1-\sin^2 2x\right)\sin^5 2x \ \left(\tfrac{1}{2} d \sin 2x\right)$$

$$= \frac{1}{2}\int \left(\sin^5 2x - \sin^7 2x\right) \ \left(d \sin 2x\right)$$

$$= \frac{1}{2}\left(\frac{1}{6}\sin^6 2x - \frac{1}{8}\sin^8 2x\right) + C$$

$$\boxed{= \frac{1}{12}\sin^6 2x - \frac{1}{16}\sin^8 2x + C}$$

Exercise

Evaluate the integrals $\int 8\cos^4 2\pi x \ dx$

Solution

$$\int 8\cos^4 2\pi x \ dx = 8\int \left(\cos 2\pi x\right)^4 dx \qquad \cos^2 \alpha = \frac{1+\cos 2\alpha}{2}$$

$$= 8\int \left(\frac{1+\cos 4\pi x}{2}\right)^2 dx$$

$$= 2\int \left(1+\cos 4\pi x\right)^2 dx$$

$$= 2\int \left(1+2\cos 4\pi x + \cos^2 4\pi x\right) dx$$

$$= 2\int dx + 4\int \cos 4\pi x dx + 2\int \cos^2 4\pi x dx$$

$$= 2x + 4\frac{1}{4\pi}\cos 4\pi x + 2\int \frac{1+\cos 8\pi x}{2} dx$$

$$= 2x + \frac{1}{\pi}\cos 4\pi x + \int \left(1+\cos 8\pi x\right) dx$$

$$= 2x + \frac{1}{\pi}\sin 4\pi x + x + \frac{1}{8\pi}\sin 8\pi x + C$$

$$\boxed{= 3x + \frac{1}{\pi}\sin 4\pi x + \frac{1}{8\pi}\sin 8\pi x + C}$$

Exercise

Evaluate the integrals $\int 16\sin^2 x\cos^2 x dx$

Solution

$$\int 16\sin^2 x\cos^2 x dx = 16\int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)dx \qquad \cos^2\alpha = \frac{1+\cos 2\alpha}{2} \quad \sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$= 4\int \left(1-\cos^2 2x\right)dx$$

$$= 4\int \left(1-\frac{1+\cos 4x}{2}\right)dx$$

$$= 4\int \frac{1-\cos 4x}{2}dx$$

$$= 2\left(x-\frac{1}{4}\sin 4x\right)+C$$

$$= 2x-\frac{1}{2}\left(2\sin 2x\cos 2x\right)+C$$

$$= 2x-\left(2\sin x\cos x\right)\left(2\cos^2 x-1\right)+C$$

$$\underline{= 2x-4\sin x\cos^3 x+2\sin x\cos x+C}$$

Exercise

Evaluate the integrals $\int \sec x\tan^2 x\, dx$

Solution

$$\int \sec x\tan^2 x\, dx = \int \sec x\tan x\tan x\, dx \qquad \begin{array}{ll} u = \tan x & dv = \sec x\tan x dx \\ du = \sec^2 x dx & v = \sec x \end{array}$$

$$\int \sec x\tan^2 x\, dx = \tan x\sec x - \int \sec x\sec^2 x\, dx$$

$$= \tan x\sec x - \int \sec x\left(1+\tan^2 x\right)\, dx$$

$$= \tan x\sec x - \left[\int \sec x\, dx + \int \sec x\tan^2 x\, dx\right]$$

$$= \tan x\sec x - \ln\left|\sec x+\tan x\right| - \int \sec x\tan^2 x\, dx$$

$$\int \sec x\tan^2 x\, dx + \int \sec x\tan^2 x\, dx = \tan x\sec x - \ln\left|\sec x+\tan x\right|$$

$$2\int \sec x\tan^2 x\, dx = \tan x\sec x - \ln\left|\sec x+\tan x\right|$$

$$\int \sec x\tan^2 x\, dx = \underline{\frac{1}{2}\tan x\sec x - \frac{1}{2}\ln\left|\sec x+\tan x\right| + C}$$

Exercise

Evaluate the integrals $\int \sec^2 x \tan^2 x \, dx$

Solution

$$\int \sec^2 x \tan^2 x dx = \int \tan^2 x \, d(\tan x) \qquad\qquad d(\tan x) = \sec^2 x dx$$

$$= \boxed{\frac{1}{3}\tan^3 x + C}$$

Exercise

Evaluate the integrals $\int \sec^4 x \tan^2 x \, dx$

Solution

$$\int \sec^4 x \tan^2 x dx = \int \sec^2 x \sec^2 x \tan^2 x \, dx \qquad\qquad d(\tan x) = \sec^2 x dx \quad \sec^2 x = 1 + \tan^2 x$$

$$= \int \left(1 + \tan^2 x\right) \tan^2 x \, d(\tan x)$$

$$= \int \left(\tan^2 x + \tan^4 x\right) \, d(\tan x)$$

$$= \boxed{\frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C}$$

Exercise

Evaluate the integrals $\int e^x \sec^3\left(e^x\right) \, dx$

Solution

$$u = \sec\left(e^x\right) \qquad\qquad dv = \sec\left(e^x\right) e^x dx$$

$$du = \sec\left(e^x\right)\tan\left(e^x\right) e^x dx \qquad v = \int \sec\left(e^x\right) d\left(e^x\right) = \tan\left(e^x\right)$$

$$\int e^x \sec^3\left(e^x\right) \, dx = \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec\left(e^x\right)\tan^2\left(e^x\right) e^x dx$$

$$= \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec\left(e^x\right)\left(\sec^2\left(e^x\right) - 1\right) e^x dx$$

$$= \sec\left(e^x\right)\tan\left(e^x\right) - \int \left(\sec^3\left(e^x\right) - \sec\left(e^x\right)\right) e^x dx$$

$$= \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec^3\left(e^x\right)e^x dx + \int \sec\left(e^x\right)e^x dx \qquad d\left(e^x\right) = e^x dx$$

$$= \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec^3\left(e^x\right)e^x dx + \int \sec\left(e^x\right) d\left(e^x\right)$$

$$\int \sec^3\left(e^x\right)e^x dx = \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec^3\left(e^x\right)e^x dx + \ln\left|\sec\left(e^x\right) + \tan\left(e^x\right)\right|$$

$$2\int \sec^3\left(e^x\right)e^x dx = \sec\left(e^x\right)\tan\left(e^x\right) + \ln\left|\sec\left(e^x\right) + \tan\left(e^x\right)\right| + C$$

$$\int \sec^3\left(e^x\right)e^x dx = \boxed{\tfrac{1}{2}\sec\left(e^x\right)\tan\left(e^x\right) + \tfrac{1}{2}\ln\left|\sec\left(e^x\right) + \tan\left(e^x\right)\right| + C}$$

Exercise

Evaluate $\quad \int \sin^4 x \cos^2 x \, dx$

Solution

$$\int \sin^4 x \cos^2 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 \left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{8}\int \left(1 - 2\cos 2x + \cos^2 2x\right)\left(1+\cos 2x\right) dx$$

$$= \frac{1}{8}\int \left(1 - \cos 2x - \cos^2 2x + \cos^3 2x\right) dx$$

$$= \frac{1}{8}\int \left(1 - \cos 2x - \frac{1}{2} - \frac{1}{2}\cos 4x\right) dx + \frac{1}{8}\int \cos^2 2x \cos 2x \, dx$$

$$= \frac{1}{8}\int \left(\frac{1}{2} - \cos 2x - \frac{1}{2}\cos 4x\right) dx + \frac{1}{16}\int \left(1 - \sin^2 2x\right) \, d\left(\sin 2x\right)$$

$$= \frac{1}{8}\left(\frac{1}{2}x - \frac{1}{2}\sin 2x - \frac{1}{4}\sin 4x\right) + \frac{1}{16}\sin 2x - \frac{1}{48}\sin^3 2x + C$$

$$\boxed{= \frac{1}{16}x - \frac{1}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C}$$

Exercise

Evaluate $\quad \int \tan^3 x \, \sec^4 x \, dx$

Solution

$$\int \tan^3 x \sec^4 x \, dx = \int \tan^3 x \left(1+\tan^2 x\right)\sec^2 x \, dx \qquad \sec^2 x = 1+\tan^2 x$$

$$= \int \left(\tan^3 x + \tan^5 x\right) d\left(\tan x\right) \qquad d\left(\tan x\right) = \sec^2 x dx$$

$$= \frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x + C$$

Exercise

Evaluate $\int \sin 3x \cos 7x \, dx$

Solution

$$\int \sin 3x \cos 7x \, dx = \frac{1}{2}\int \left(\sin\left(-4x\right) + \sin 10x\right) dx \qquad \sin mx \cos nx = \frac{1}{2}\left[\sin\left(m-n\right)x + \sin\left(m+n\right)x\right]$$

$$= \frac{1}{2}\int \left(-\sin 4x + \sin 10x\right) dx$$

$$= \frac{1}{2}\left(\frac{1}{4}\cos 4x - \frac{1}{10}\cos 10x\right) + C$$

$$= \frac{1}{8}\cos 4x - \frac{1}{20}\cos 10x + C$$

Exercise

Evaluate the integrals $\int \sin 2x \cos 3x \, dx$

Solution

$$\int \sin 2x \cos 3x \, dx = \frac{1}{2}\int \left(\sin 5x + \sin\left(-x\right)\right) dx \qquad \sin\alpha\cos\beta = \frac{1}{2}\left[\sin\left(\alpha+\beta\right) + \sin\left(\alpha-\beta\right)\right]$$

$$= \frac{1}{2}\int \left(\sin 5x - \sin x\right) dx$$

$$= \frac{1}{2}\left(-\frac{1}{5}\cos 5x + \cos x\right) + C$$

$$= \frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C$$

Exercise

Evaluate the integrals $\int \sin^2\theta \cos 3\theta \, d\theta$

Solution

$$\int \sin^2\theta \cos 3\theta \, d\theta = \int \frac{1-\cos 2\theta}{2} \cos 3\theta \, d\theta \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \frac{1}{2}\int \left(\cos 3\theta - \cos 2\theta \cos 3\theta\right) \, d\theta$$

$$= \frac{1}{2}\int \cos 3\theta \, d\theta - \frac{1}{2}\int \cos 2\theta \cos 3\theta \, d\theta$$

$$= \frac{1}{6}\sin 3\theta - \frac{1}{2}\int \frac{1}{2}\left(\cos(5\theta) + \cos(-\theta)\right) \, d\theta \qquad \cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

$$= \frac{1}{6}\sin 3\theta - \frac{1}{4}\left(\frac{1}{5}\sin 5\theta + \sin\theta\right) + C$$

$$= \frac{1}{6}\sin 3\theta - \frac{1}{20}\sin 5\theta - \frac{1}{4}\sin\theta + C$$

Exercise

Evaluate the integrals $\int \cos^3\theta \sin 2\theta \, d\theta$

Solution

$$\int \cos^3\theta \sin 2\theta \, d\theta = \int \cos^3\theta \left(2\sin\theta\cos\theta\right) \, d\theta \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= -2\int \cos^4\theta \, d(\cos\theta) \qquad d(\cos\theta) = -\sin\theta d\theta$$

$$= -\frac{2}{5}\cos^5\theta + C$$

Exercise

Evaluate the integrals $\int \sin\theta \sin 2\theta \sin 3\theta \, d\theta$

Solution

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$\int \sin\theta \sin 2\theta \sin 3\theta \, d\theta = \int \frac{1}{2}\left(\cos(1-2)\theta - \cos(1+2)\theta\right) sin3\theta \, d\theta$$

$$= \frac{1}{2}\int \left(\cos(-\theta) - \cos(3\theta)\right)\sin 3\theta \, d\theta$$

$$=\frac{1}{2}\int \cos\theta\ \sin 3\theta\ d\theta-\frac{1}{2}\int \cos 3\theta\ \sin 3\theta\ d\theta$$

$\sin\alpha\cos\beta=\frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$

$$=\frac{1}{4}\int (\sin 4\theta+\sin 2\theta)\ d\theta-\frac{1}{4}\int \left(\sin 6\theta+\sin(0)\right)\ d\theta$$

$$=\frac{1}{4}\left(-\frac{1}{4}\cos 4\theta-\frac{1}{2}\cos 2\theta\right)+\frac{1}{24}\cos 6\theta+C$$

$$=-\frac{1}{16}\cos 4\theta-\frac{1}{8}\cos 2\theta+\frac{1}{24}\cos 6\theta+C$$

Exercise

Evaluate the integrals $\int \frac{\sin^3 x}{\cos^4 x}\,dx$

Solution

$$\int \frac{\sin^3 x}{\cos^4 x}\,dx=\int \frac{\sin^2 x\sin x}{\cos^4 x}\,dx$$

$\cos^2\alpha+\sin^2\alpha=1$

$$=\int \frac{\left(1-\cos^2 x\right)\sin x}{\cos^4 x}\,dx$$

$$=-\int \left(\frac{1}{\cos^4 x}-\frac{\cos^2 x}{\cos^4 x}\right)d\left(\cos x\right)$$

$$=-\int \left(\cos^{-4} x-\cos^{-2} x\right)d\left(\cos x\right)$$

$$=-\left(-\frac{1}{3}\cos^{-3} x+\cos^{-1} x\right)+C$$

$$=\frac{1}{3}\frac{1}{\cos^3 x}-\frac{1}{\cos x}+C$$

$$=\frac{1}{3}\csc^3 x-\csc x+C$$

Exercise

Evaluate the integrals $\int x\cos^3 x\ dx$

Solution

$$\int x\cos^3 x dx=\int x\cos^2 x\cos x\ dx$$

$$=\int x\left(1-\sin^2 x\right)\cos x\ dx$$

$\cos^2\alpha+\sin^2\alpha=1$

$$= \int x\cos x\,dx - \int x\sin^2 x\cos x\,dx$$

$$u = x \qquad dv = \cos x dx \qquad\qquad u = x \qquad dv = \sin^2 x\cos x dx$$

$$du = dx \qquad v = \sin x \qquad\qquad du = dx \qquad v = \frac{1}{3}\sin^3 x$$

$$= x\sin x - \int \sin x\,dx - \left(\frac{1}{3}x\sin^3 x - \frac{1}{3}\int \sin^3 x\,dx\right)$$

$$= x\sin x + \cos x - \frac{1}{3}x\sin^3 x + \frac{1}{3}\int \sin^2 x\sin x\,dx$$

$$= x\sin x + \cos x - \frac{1}{3}x\sin^3 x - \frac{1}{3}\int \left(1-\cos^2 x\right)d\left(\cos x\right)$$

$$= x\sin x + \cos x - \frac{1}{3}x\sin^3 x - \frac{1}{3}\left(\cos x - \frac{1}{3}\cos^3 x\right) + C$$

$$= x\sin x + \cos x - \frac{1}{3}x\sin^3 x - \frac{1}{3}\cos x + \frac{1}{9}\cos^3 x + C$$

$$= x\sin x + \frac{2}{3}\cos x - \frac{1}{3}x\sin^3 x + \frac{1}{9}\cos^3 x + C$$

Exercise

Evaluate the integrals $\int \sin^3 x\cos^4 x\,dx$

Solution

$$\int \sin^3 x\cos^4 x\,dx = \int \sin^2 x\cos^4 x\ \sin x\,dx$$

$$= -\int \left(1-\cos^2 x\right)\cos^4 x\,d\left(\cos x\right)$$

$$= \int \left(\cos^6 x - \cos^4 x\right)\,d\left(\cos x\right)$$

$$= \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C$$

Exercise

Evaluate the integrals $\int \cos^4 x\,dx$

Solution

$$\int \cos^4 x\,dx = \frac{1}{4}\int (1+\cos 2x)^2\,dx \qquad\qquad \cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$= \frac{1}{4}\int \left(1 + 2\cos 2x + \cos^2 2x\right)dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$$

$$\boxed{= \frac{1}{4}\left(\frac{3}{2}x + \sin 2x + \frac{1}{8}\sin 4x\right) + C}$$

Exercise

Evaluate the integrals $\displaystyle\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx$

Solution

$$\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx = \int \frac{\tan^2 x \tan x}{(\sec x)^{1/2}} \frac{\sec x}{\sec x}\,dx$$

$$1 + \tan^2 \alpha = \sec^2 \alpha$$

$$= \int (\sec x)^{-3/2}\left(\sec^2 x - 1\right) d(\sec x)$$

$$= \int \left((\sec x)^{1/2} - (\sec x)^{-3/2}\right) d(\sec x)$$

$$\boxed{= \frac{2}{3}(\sec x)^{3/2} + 2(\sec x)^{-1/2} + C}$$

Exercise

Evaluate the integrals $\displaystyle\int \sec^4 3x \tan^3 3x\,dx$

Solution

$$\int \sec^4 3x \tan^3 3x\,dx = \int \sec^2 3x \tan^3 3x\ \sec^2 3x\,dx$$

$$= \frac{1}{3}\int \left(1 + \tan^2 3x\right)\tan^3 3x\,d(\tan 3x)$$

$$= \frac{1}{3}\int \left(\tan^3 3x + \tan^5 3x\right)\,d(\tan 3x)$$

$$= \frac{1}{3}\left(\frac{1}{4}\tan^4 3x + \frac{1}{6}\tan^6 3x\right) + C$$

$$\boxed{= \frac{1}{12}\tan^4 3x + \frac{1}{18}\tan^6 3x + C}$$

Exercise

Evaluate the integrals $\displaystyle\int \frac{\sec x}{\tan^2 x}\,dx$

Solution

$$\int \frac{\sec x}{\tan^2 x}\,dx = \int \frac{1}{\cos x}\frac{\cos^2 x}{\sin^2 x}\,dx$$
$$= \int \frac{\cos x}{\sin^2 x}\,dx$$
$$= \int \frac{1}{\sin^2 x}\,d(\sin x)$$
$$= -\frac{1}{\sin x} + C$$
$$= -\csc x + C$$

Exercise

Evaluate the integrals $\displaystyle\int \sin 5x \cos 4x\,dx$

Solution

$$\int \sin 5x \cos 4x\,dx = \frac{1}{2}\int (\sin x + \sin 9x)\,dx \qquad \sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$
$$= \frac{1}{2}\left(-\cos x - \frac{1}{9}\cos x9x\right) + C$$
$$= \frac{1}{2} - \cos x - \frac{1}{18}\cos x9x + C$$

Exercise

Evaluate the integrals $\displaystyle\int \sin x \cos^5 x\,dx$

Solution

$$\int \sin x \cos^5 x\,dx = -\int \cos^5 x\,d(\cos x)$$
$$= -\frac{1}{6}\cos^6 x + C$$

Exercise

Evaluate the integrals $\displaystyle\int \sin^4 x \cos^3 x\,dx$

Solution

$$\int \sin^4 x\cos^3 x\, dx = \int \sin^4 x\left(1-\sin^2 x\right)\, d(\sin x)$$
$$= \int \left(\sin^4 x - \sin^6 x\right)\, d(\sin x)$$
$$\boxed{= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C}$$

Exercise

Evaluate the integrals $\int \sin^7 2x\; \cos 2x\, dx$

Solution

$$\int \sin^7 2x\; \cos 2x\, dx = \frac{1}{2}\int \sin^7 2x\, d(\sin 2x)$$
$$\boxed{= \frac{1}{16}\sin^8 2x + C}$$

Exercise

Evaluate the integrals $\int \sin^3 2x\sqrt{\cos 2x}\, dx$

Solution

$$\int \sin^3 2x\sqrt{\cos 2x}\, dx = -\frac{1}{2}\int \left(1-\cos^2 2x\right)(\cos 2x)^{1/2}\, d(\cos 2x)$$
$$= -\frac{1}{2}\int \left((\cos 2x)^{1/2} - (\cos 2x)^{5/2}\right)\, d(\cos 2x)$$
$$= -\frac{1}{2}\left(\frac{2}{3}(\cos 2x)^{3/2} - \frac{2}{7}(\cos 2x)^{7/2}\right) + C$$
$$\boxed{= \frac{1}{7}(\cos 2x)^{7/2} - \frac{1}{3}(\cos 2x)^{3/2} + C}$$

Exercise

Evaluate the integrals $\int \frac{\cos^5\theta}{\sqrt{\sin\theta}}\, d\theta$

Solution

$$\int \frac{\cos^5\theta}{\sqrt{\sin\theta}}\, d\theta = \int (\sin\theta)^{-1/2}\left(1-\sin^2\theta\right)^2\, d(\sin\theta)$$
$$= \int (\sin\theta)^{-1/2}\left(1-2\sin^2\theta+\sin^4\theta\right)\, d(\sin\theta)$$

$$= \int \left((\sin\theta)^{-1/2} - 2(\sin\theta)^{3/2} + (\sin\theta)^{7/2} \right) d(\sin\theta)$$

$$= 2(\sin\theta)^{1/2} - \frac{1}{5}(\sin\theta)^{5/2} + \frac{2}{9}(\sin\theta)^{9/2} + C$$

Exercise

Evaluate the integrals $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\, dx$

Solution

$$\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\, dx = \int_{\pi/6}^{\pi/3} (\sin x)^{-1/2}\left(1 - \sin^2 x\right) d(\sin x)$$

$$= \int_{\pi/6}^{\pi/3} \left((\sin x)^{-1/2} - (\sin x)^{3/2} \right) d(\sin x)$$

$$= 2(\sin x)^{1/2} - \frac{2}{5}(\sin x)^{5/2} \Big|_{\pi/6}^{\pi/3}$$

$$= 2\left(\frac{\sqrt{3}}{2}\right)^{1/2} - \frac{2}{5}\left(\frac{\sqrt{3}}{2}\right)^{5/2} - 2\left(\frac{1}{2}\right)^{1/2} + \frac{2}{5}\left(\frac{1}{2}\right)^{5/2}$$

$$= \sqrt[4]{3}\sqrt{2} - \frac{3}{10}\frac{\sqrt[4]{3}}{\sqrt{2}} - \sqrt{2} + \frac{\sqrt{2}}{20}$$

$$= \frac{\sqrt{2}}{20}\left(17\sqrt[4]{3} - 19\right)$$

Exercise

Evaluate the integrals $\displaystyle\int_0^{\pi/4} \tan^4 x dx$

Solution

$$\int_0^{\pi/4} \tan^4 x dx = \int_0^{\pi/4} \tan^2 x\left(\sec^2 x - 1\right) dx$$

$$= \int_0^{\pi/4} \tan^2 x\left(\sec^2 x - 1\right) dx$$

$$= \int_0^{\pi/4} \tan^2 x \sec^2 x\, dx - \int_0^{\pi/4} \tan^2 x\, dx$$

$$= \int_0^{\pi/4} \tan^2 x\, d(\tan x) - \int_0^{\pi/4} \left(\sec^2 x - 1\right) dx$$

$$= \frac{1}{3}\tan^3 x - \tan x + x \Big|_0^{\pi/4}$$

$$= \frac{1}{3} - 1 + \frac{\pi}{4}$$

$$\underline{= \frac{\pi}{4} - \frac{2}{3}}$$

Exercise

Evaluate the integrals $\displaystyle\int_0^{\pi/2} \cos^7 x\, dx$

Solution

$$\int_0^{\pi/2} \cos^7 x\, dx = \int_0^{\pi/2} \left(\cos^2 x\right)^3\, d(\sin x)$$

$$= \int_0^{\pi/2} \left(1 - \sin^2 x\right)^3\, d(\sin x)$$

$$= \int_0^{\pi/2} \left(1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x\right)\, d(\sin x)$$

$$= \left(\sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x\right)_0^{\pi/2}$$

$$= \frac{3}{5} - \frac{1}{7}$$

$$\underline{= \frac{16}{37}}$$

Exercise

Evaluate the integrals $\displaystyle\int_0^{\pi/2} \cos^9 \theta\, d\theta$

Solution

$$\int_0^{\pi/2} \cos^9 \theta\, d\theta = \int_0^{\pi/2} \left(1 - \sin^2 x\right)^4\, d(\sin x)$$

$$= \int_0^{\pi/2} \left(1 - 4\sin^2 x + 6\sin^4 x - 4\sin^6 x + \sin^8 x\right)\, d(\sin x)$$

$$= \left(\sin x - \frac{4}{3}\sin^3 x + \frac{6}{5}\sin^5 x - \frac{4}{7}\sin^7 x + \frac{1}{9}\sin^9 x\right)_0^{\pi/2}$$

$$= 1 - \frac{4}{3} + \frac{6}{5} - \frac{4}{7} + \frac{1}{9}$$

$$\underline{= \frac{128}{315}}$$

Exercise

Evaluate the integrals $\int_0^{\pi/2} \sin^5 x\, dx$

Solution

$$\int_0^{\pi/2} \sin^5 x\, dx = \int_0^{\pi/2} \left(1-\cos^2 x\right)^2\ d\left(\cos x\right)$$

$$= \int_0^{\pi/2} \left(1-2\cos^2 x+\cos^4 x\right)\ d\left(\cos x\right)$$

$$= \left(\cos x - \frac{2}{3}\cos^3 x + \frac{1}{5}\cos^5 x\right)_0^{\pi/2}$$

$$= -1+\frac{2}{3}-\frac{1}{5}$$

$$= -\frac{8}{15}$$

Exercise

Evaluate the integrals $\int_0^{\pi/6} 3\cos^5 3x\, dx$

Solution

$$\int_0^{\pi/6} 3\cos^5 3x\, dx = \int_0^{\pi/6} 3\left(\cos^2 3x\right)^2 \cos 3x\, dx$$

$$= \int_0^{\pi/6} \left(1-\sin^2 3x\right)^2\ d\left(\sin 3x\right)$$

$$= \int_0^{\pi/6} \left(1-2\sin^2 3x+\sin^4 3x\right) d\left(\sin 3x\right)$$

$$= \left[\sin 3x - \frac{2}{3}\sin^2 3x + \frac{1}{5}\sin^4 3x\right]_0^{\pi/6}$$

$$= \sin\frac{\pi}{2} - \frac{2}{3}\sin^2\frac{\pi}{2} + \frac{1}{5}\sin^4\frac{\pi}{2} - 0$$

$$= 1-\frac{2}{3}+\frac{1}{5}$$

$$= \frac{8}{15}$$

Exercise

Evaluate the integrals $\displaystyle\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta d\theta$

Solution

$$\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta d\theta = \int_0^{\pi/2} \sin^2 2\theta \left(\cos^2 2\theta\right) \cos 2\theta d\theta \qquad d\left(\sin 2\theta\right) = 2\cos 2\theta d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^2 2\theta \left(1 - \sin^2 2\theta\right) d\left(\sin 2\theta\right)$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(\sin^2 2\theta - \sin^4 2\theta\right) d\left(\sin 2\theta\right)$$

$$= \frac{1}{2}\left[\frac{1}{3}\sin^3 2\theta - \frac{1}{5}\sin^5 2\theta\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left(\frac{1}{3}\sin^3 \pi - \frac{1}{5}\sin^5 \pi - 0\right)$$

$$\underline{= 0}$$

Exercise

Evaluate the integrals $\displaystyle\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\, dx$

Solution

$$\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\, dx = \int_0^{2\pi} \sin\frac{x}{2}\, dx \qquad \left|\sin\left(\frac{\alpha}{2}\right)\right| = \sqrt{\frac{1-\cos\alpha}{2}}$$

$$= \left[-2\cos\frac{x}{2}\right]_0^{2\pi}$$

$$= -2\left(\cos\pi - \cos 0\right)$$

$$\underline{= 2}$$

Exercise

Evaluate the integrals $\displaystyle\int_0^{\pi} \sqrt{1-\cos^2\theta}\, d\theta$

Solution

$$\int_0^{\pi} \sqrt{1-\cos^2\theta}\, d\theta = \int_0^{\pi} \left|\sin\theta\right| d\theta$$

$$= \left[-\cos\theta\right]_0^{\pi}$$

$$= -\cos\pi + \cos 0$$

$$\underline{= 2}$$

Exercise

Evaluate the integrals $\displaystyle\int_0^{\pi/6} \sqrt{1+\sin x}\, dx$

Solution

$$\int_0^{\pi/6} \sqrt{1+\sin x}\, dx = \int_0^{\pi/6} \sqrt{1+\sin x}\, \frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}}\, dx$$

$$= \int_0^{\pi/6} \frac{\sqrt{1-\sin^2 x}}{\sqrt{1-\sin x}}\, dx \qquad \cos x = \sqrt{1-\sin^2 x}$$

$$= \int_0^{\pi/6} \frac{\cos x}{\sqrt{1-\sin x}}\, dx \qquad d(1-\sin x) = -\cos x dx$$

$$= -\int_0^{\pi/6} (1-\sin x)^{-1/2}\, d(1-\sin x)$$

$$= -2\left[(1-\sin x)^{1/2}\right]_0^{\pi/6}$$

$$= -2\left(\sqrt{1-\sin\tfrac{\pi}{6}} - 1\right)$$

$$= -2\left(\sqrt{1-\tfrac{1}{2}} - 1\right)$$

$$= -2\left(\frac{1}{\sqrt{2}} - 1\right)$$

$$= -2\left(\frac{\sqrt{2}}{2} - 1\right)$$

$$\underline{= 2-\sqrt{2}}$$

Exercise

Evaluate the integrals $\displaystyle\int_{-\pi}^{\pi} \left(1-\cos^2 x\right)^{3/2} dx$

Solution

$$\int_{-\pi}^{\pi}\left(1-\cos^2 x\right)^{3/2}dx=\int_{-\pi}^{\pi}\left(\sin^2 x\right)^{3/2}dx$$

$$=\int_{-\pi}^{\pi}\left|\sin^3 x\right|dx$$

$$=-\int_{-\pi}^{0}\sin^3 x\,dx+\int_{0}^{\pi}\sin^3 x\,dx \qquad \sin^2 x=1-\cos^2 x$$

$$=-\int_{-\pi}^{0}\left(1-\cos^2 x\right)\sin x\,dx+\int_{0}^{\pi}\left(1-\cos^2 x\right)\sin x\,dx \qquad d\left(\cos x\right)=-\sin x dx$$

$$=\int_{-\pi}^{0}\left(1-\cos^2 x\right)d\left(\cos x\right)-\int_{0}^{\pi}\left(1-\cos^2 x\right)d\left(\cos x\right)$$

$$=\left[\cos x-\tfrac{1}{3}\cos^3 x\right]_{-\pi}^{0}-\left[\cos x-\tfrac{1}{3}\cos^3 x\right]_{0}^{\pi}$$

$$=\left(1-\tfrac{1}{3}-\left(-1+\tfrac{1}{3}\right)\right)-\left(-1+\tfrac{1}{3}-\left(1-\tfrac{1}{3}\right)\right)$$

$$=1-\tfrac{1}{3}+1-\tfrac{1}{3}+1-\tfrac{1}{3}+1-\tfrac{1}{3}$$

$$=4-\tfrac{4}{3}$$

$$\underline{=\frac{8}{3}}$$

Exercise

Evaluate the integrals $\displaystyle\int_{\pi/4}^{\pi/2}\csc^4\theta d\theta$

Solution

$$\int_{\pi/4}^{\pi/2}\csc^4\theta d\theta=\int_{\pi/4}^{\pi/2}\left(1+\cot^2\theta\right)\csc^2\theta d\theta \qquad \csc^2\theta=1+\cot^2\theta$$

$$=\int_{\pi/4}^{\pi/2}\csc^2\theta d\theta+\int_{\pi/4}^{\pi/2}\cot^2\theta\csc^2\theta d\theta \qquad d\left(\cot\theta\right)=-\csc^2\theta d\theta$$

$$=\int_{\pi/4}^{\pi/2}\csc^2\theta d\theta-\int_{\pi/4}^{\pi/2}\cot^2\theta\, d\left(\cot\theta\right)$$

$$=\left[-\cot\theta-\tfrac{1}{3}\cot^3\theta\right]_{\pi/4}^{\pi/2}$$

$$= -\left(\cot\frac{\pi}{2} + \frac{1}{3}\cot^3\frac{\pi}{2} - \cot\frac{\pi}{4} - \frac{1}{3}\cot^3\frac{\pi}{4}\right)$$

$$= -\left(0 + \frac{1}{3}(0) - 1 - \frac{1}{3}\right)$$

$$\boxed{= \frac{4}{3}}$$

Exercise

Evaluate the integrals $\displaystyle\int_{-\pi}^{\pi} \sin 3x \sin 3x \, dx$

Solution

$$\int_{-\pi}^{\pi} \sin 3x \sin 3x \, dx = \frac{1}{2}\int_{-\pi}^{\pi} (\cos 0 - \cos 6x) \, dx \qquad \sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} (1 - \cos 6x) \, dx$$

$$= \frac{1}{2}\left[x - \frac{1}{6}\sin 6x\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left(\pi - \frac{1}{6}\sin 6\pi - \left(-\pi - \frac{1}{6}\sin(-6\pi)\right)\right)$$

$$= \frac{1}{2}(\pi + \pi)$$

$$\boxed{= \pi}$$

Exercise

Evaluate the integrals $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x \cos 7x \, dx$

Solution

$$\int_{-\pi/2}^{\pi/2} \cos x \cos 7x dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (\cos 8x + \cos(-6x))\, dx \qquad \cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} (\cos 8x + \cos 6x)\, dx$$

$$= \frac{1}{2}\left[\frac{1}{6}\sin 6x + \frac{1}{8}\sin 8x\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2}\left(\frac{1}{6}\sin(3\pi) + \frac{1}{8}\sin(4\pi) - \frac{1}{6}\sin(-3\pi) - \frac{1}{8}\sin(-4\pi)\right)$$

$$\boxed{= 0}$$

Exercise

Evaluate the integrals $\displaystyle\int_0^\pi 8\sin^4 y\cos^2 y\,dy$

Solution

$$\int_0^\pi 8\sin^4 y\cos^2 y\,dy = 8\int_0^\pi \left(\frac{1-\cos 2y}{2}\right)^2\left(\frac{1+\cos 2y}{2}\right)dy$$

$$= \int_0^\pi \left(1-2\cos 2y+\cos^2 2y\right)(1+\cos 2y)\,dy$$

$$= \int_0^\pi \left(1-2\cos 2y+\cos^2 2y+\cos 2y-2\cos^2 2y+\cos^3 2y\right)dy$$

$$= \int_0^\pi \left(1-\cos 2y-\cos^2 2y+\cos^3 2y\right)dy$$

$$= \int_0^\pi \left(1-\cos 2y-\frac{1}{2}-\frac{1}{2}\cos 4y\right)dy+\int_0^\pi \cos^2 2y\cos 2y\,dy$$

$$= \int_0^\pi \left(\frac{1}{2}-\cos 2y-\frac{1}{2}\cos 4y\right)dy+\frac{1}{2}\int_0^\pi \left(1-\sin^2 2y\right)d(\sin 2y)$$

$$= \left[\frac{1}{2}y-\frac{1}{2}\sin 2y-\frac{1}{8}\sin 4y+\frac{1}{2}\left(\sin 2y-\frac{1}{3}\sin^3 2y\right)\right]_0^\pi$$

$$\boxed{=\frac{\pi}{2}}$$

Exercise

Find the area of the region bounded by the graphs of $y=\tan x$ and $y=\sec x$ on the interval $\left[0,\ \frac{\pi}{4}\right]$

Solution

$$A=\int_0^{\pi/4}(\sec x-\tan x)\,dx$$

$$= \ln|\sec x+\tan x|+\ln|\cos x|\ \Big|_0^{\pi/4}$$

$$= \ln\left(\sqrt{2}+1\right)+\ln\frac{\sqrt{2}}{2}-0$$

$$= \ln\left(\frac{\sqrt{2}}{2}\left(\sqrt{2}+1\right)\right)$$

$$\boxed{= \ln\left(1+\frac{\sqrt{2}}{2}\right)}$$

Solution Section 2.3 – Trigonometric Substitutions

Exercise

Evaluate the integral $\displaystyle\int \frac{3dx}{\sqrt{1+9x^2}}$

Solution

$$\int \frac{3dx}{\sqrt{1+9x^2}} = \frac{1}{3}\int \frac{\sec^2 t}{3\sec t}dt$$

$$3x = \tan t \quad \Rightarrow dx = \frac{1}{3}\sec^2 t\ dt$$

$$\sqrt{1+9x^2} = 3\sec^2 t$$

$$= \int \sec t\ dt$$

$$= \ln\left|\sec t + \tan t\right| + C$$

$$= \ln\left|\sqrt{1+u^2} + u\right| + C$$

$$\boxed{= \ln\left|\sqrt{1+9x^2} + 3x\right| + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{5dx}{\sqrt{25x^2-9}}, \quad x > \frac{3}{5} = \sin^{-1}\frac{1}{2} - \sin^{-1}0$

Solution

$$\int \frac{5dx}{\sqrt{25x^2-9}} = \int \frac{5\left(\frac{3}{5}\sec\theta\tan\theta d\theta\right)}{3\tan\theta}$$

$$5x = 3\sec\theta \quad \rightarrow \quad dx = \frac{3}{5}\sec\theta\tan\theta d\theta$$

$$\sqrt{25x^2-9} = 3\tan\theta$$

$$= \int \sec\theta d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

$$\boxed{= \ln\left|\frac{5}{3}x + \frac{1}{3}\frac{\sqrt{25x^2-9}}{3}\right| + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sqrt{y^2-49}}{y}dy, \quad y > 7$

Solution

$$\int \frac{\sqrt{y^2-49}}{y}dy = \int \frac{(7\tan\theta)}{7\sec\theta}(7\sec\theta\tan\theta)d\theta$$

$$y = 7\sec\theta \quad \rightarrow \quad dy = 7\sec\theta\tan\theta d\theta$$

$$\sqrt{y^2-49} = 7\tan\theta$$

$$= 7\int \tan^2\theta\, d\theta$$

$$= 7\int \left(\sec^2\theta - 1\right) d\theta$$

$$= 7\left(\tan\theta - \theta\right) + C$$

$$\boxed{= 7\left[\frac{\sqrt{y^2 - 49}}{7} - \sec^{-1}\left(\frac{y}{7}\right)\right] + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{2dx}{x^3\sqrt{x^2-1}}, \quad x > 1$

Solution

$$\int \frac{2dx}{x^3\sqrt{x^2-1}} = \int \frac{2\sec\theta\tan\theta\, d\theta}{\sec^3\theta\tan\theta}$$

$x = \sec\theta \quad dx = \sec\theta\tan\theta\, d\theta$

$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$

$$= 2\int \cos^2\theta\, d\theta$$

$$= 2\int \frac{1+\cos 2\theta}{2} d\theta$$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= \int \left(1+\cos 2\theta\right) d\theta$$

$$= \theta + \frac{1}{2}\sin 2\theta + C$$

$$= \theta + \sin\theta\cos\theta + C$$

$$\boxed{= \sec^{-1} x + \frac{\sqrt{x^2-1}}{x^2} + C}$$

$x = \sec\theta = \frac{1}{\cos\theta} \Rightarrow \cos\theta = \frac{1}{x}$

$\sin\theta = \tan\theta\cos\theta = \sqrt{x^2-1}\left(\frac{1}{x}\right)$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^2}{4+x^2} dx$

Solution

$$\int \frac{x^2}{4+x^2} dx = \int \frac{4\tan^2\theta}{4\sec^2\theta} 2\sec^2\theta\, d\theta$$

$x = 2\tan\theta \quad dx = 2\sec^2\theta\, d\theta$

$4 + x^2 = 4 + 4\tan^2\theta = 4\sec^2\theta$

$$= 2\int \tan^2\theta\, d\theta$$

$$= 2\int \left(\sec^2\theta - 1\right) d\theta \qquad \int \sec^2\theta d\theta = \tan\theta$$

$$= 2\left(\tan\theta - \theta\right) + C$$

$$= 2\left(\frac{x}{2} - \tan^{-1}\left(\frac{x}{2}\right)\right) + C$$

$$= x - 2\tan^{-1}\left(\frac{x}{2}\right) + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+1}}$

Solution

$$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec^2\theta d\theta}{\tan^2\theta \sec\theta}$$

$$x = \tan\theta \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$dx = \sec^2\theta d\theta$$

$$\sqrt{x^2+1} = \sqrt{\tan^2\theta + 1} = \sec\theta$$

$$= \int \frac{\sec\theta d\theta}{\tan^2\theta}$$

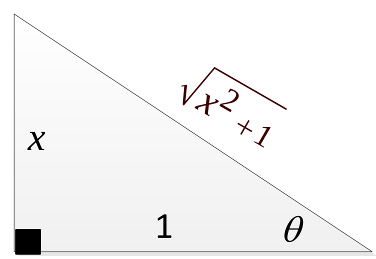
$$= \int \frac{\cos^2\theta d\theta}{\sin^2\theta \cos\theta}$$

$$= \int \frac{\cos\theta d\theta}{\sin^2\theta}$$

$$= \int \sin^{-2}\theta d\left(\sin\theta\right)$$

$$= -\frac{1}{\sin\theta} + C$$

$$= -\frac{\sqrt{x^2+1}}{x} + C$$



Exercise

Evaluate the integral $\displaystyle\int \frac{\left(1-x^2\right)^{1/2}}{x^4} dx$

Solution

$$\int \frac{\left(1-x^2\right)^{1/2}}{x^4} dx = \int \frac{\cos\theta}{\sin^4\theta} \cos\theta d\theta$$

$$x = \sin\theta \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$dx = \cos\theta d\theta$$

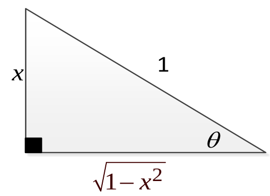
$$\left(1-x^2\right)^{1/2} = \left(1-\sin^2 x\right)^{1/2} = \cos\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta} \frac{1}{\sin^2\theta} d\theta$$

$$= \int \cot^2\theta \csc^2\theta d\theta$$

$$= -\frac{1}{3}\cot^3\theta + C$$

$$\boxed{= -\frac{1}{3}\left(\frac{\sqrt{1-x^2}}{x}\right)^3 + C}$$



Exercise

Evaluate the integral $\int \frac{x^3dx}{x^2-1}$

Solution

$$\int \frac{x^3dx}{x^2-1} = \int \left(x + \frac{x}{x^2-1}\right)dx$$

$$= \int xdx + \int \frac{x}{x^2-1}dx$$

$$= \int xdx + \frac{1}{2}\int \frac{d\left(x^2-1\right)}{x^2-1}$$

$$\boxed{= \frac{1}{2}x^2 + \frac{1}{2}\ln\left|x^2-1\right| + C}$$

$$\begin{array}{r|l} & x \\ \hline x^2-1 & x^3 \\ & \underline{x^3 - x} \\ & x \end{array}$$

$$d\left(x^2-1\right) = 2xdx \Rightarrow \frac{1}{2}d\left(x^2-1\right) = xdx$$

Exercise

Evaluate the integral $\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}dx$

Solution

$$\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}dx = \int \frac{\cos\theta}{\sin\theta}\cos\theta d\theta$$

$$= \int \frac{\cos^2\theta}{\sin\theta}d\theta$$

$$= \int \frac{1-\sin^2\theta}{\sin\theta}d\theta$$

$$= \int \frac{1}{\sin\theta}d\theta - \int \sin\theta d\theta$$

$$= \int \csc\theta d\theta - \int \sin\theta d\theta$$

$$\ln x = \sin\theta \qquad 0 < \theta \le \frac{\pi}{2}$$

$$\frac{1}{x}dx = \cos\theta d\theta$$

$$\sqrt{1-(\ln x)^2} = \sqrt{1-\sin^2\theta} = \cos\theta$$

$$= -\ln\left|\csc\theta + \cot\theta\right| + \cos\theta + C$$

$$\boxed{= -\ln\left|\frac{1}{\ln x} + \frac{\sqrt{1-(\ln x)^2}}{\ln x}\right| + \sqrt{1-(\ln x)^2} + C}$$

Exercise

Evaluate the integral $\int \sqrt{x}\ \sqrt{1-x}\ dx$

Solution

$$\int \sqrt{x}\ \sqrt{1-x}\ dx = \int u\sqrt{1-u^2}\,(2udu)$$

$u = \sqrt{x} \ \to u^2 = x \quad \Rightarrow dx = 2udu$

$u = \sin\theta \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$= 2\int u^2\sqrt{1-u^2}\,du$$

$du = \cos\theta d\theta$

$$\int \sqrt{x}\ \sqrt{1-x}\ dx = 2\int u^2\sqrt{1-u^2}\,du = 2\int \sin^2\theta\cos\theta\cos\theta d\theta$$

$\sqrt{1-u^2} = \sqrt{1-\sin^2\theta} = \cos\theta$

$$= 2\int \sin^2\theta\cos^2\theta d\theta$$

$\sin 2\theta = 2\sin\theta\cos\theta \quad \to \sin^2 2\theta = 4\sin^2\theta\cos^2\theta$

$$= \frac{1}{2}\int \sin^2 2\theta d\theta$$

$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$

$$= \frac{1}{2}\int \frac{1-\cos 4\theta}{2}\,d\theta$$

$$= \frac{1}{4}\int d\theta - \frac{1}{4}\int \cos 4\theta d\theta$$

$$= \frac{1}{4}\theta - \frac{1}{16}\sin 4\theta + C$$

$$= \frac{1}{4}\theta - \frac{1}{16}2\sin 2\theta\cos 2\theta + C$$

$$= \frac{1}{4}\theta - \frac{1}{8}2\sin\theta\cos\theta\left(2\cos^2\theta - 1\right) + C$$

$$= \frac{1}{4}\theta - \frac{1}{2}\sin\theta\cos^3\theta + \frac{1}{4}\sin\theta\cos\theta + C$$

$$= \frac{1}{4}\sin^{-1}u - \frac{1}{2}u\left(1-u^2\right)^{3/2} + \frac{1}{4}u\sqrt{1-u^2} + C$$

$$\boxed{= \frac{1}{4}\sin^{-1}\sqrt{x} - \frac{1}{2}\sqrt{x}\,(1-x)^{3/2} + \frac{1}{4}\sqrt{x}\sqrt{1-x} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sqrt{x-2}}{\sqrt{x-1}}\,dx$

Solution

$u=\sqrt{x-1} \quad \to u^2=x-1 \quad \Rightarrow 2udu=dx$

$u=\sec\theta \qquad 0<\theta<\frac{\pi}{2}$

$du=\sec\theta\tan\theta d\theta$

$\sqrt{u^2-1}=\sqrt{\sec^2\theta-1}=\tan\theta$

$$\begin{aligned}\int \frac{\sqrt{x-2}}{\sqrt{x-1}}\,dx &= \int \frac{\sqrt{u^2-1}}{u}2udu \\ &= 2\int \sqrt{u^2-1}\;du \\ &= 2\int \tan\theta\sec\theta\tan\theta d\theta\end{aligned}$$

$w=\tan\theta \qquad dv=\sec\theta\tan\theta d\theta$

$dw=\sec^2\theta d\theta \qquad v=\sec\theta$

$$\begin{aligned}2\int \tan\theta\sec\theta\tan\theta d\theta &= 2\sec\theta\tan\theta-2\int \sec^3\theta d\theta \\ &= 2\sec\theta\tan\theta-2\int \sec^2\theta\sec\theta d\theta \\ &= 2\sec\theta\tan\theta-2\int \left(\tan^2\theta+1\right)\sec\theta d\theta \\ &= 2\sec\theta\tan\theta-2\int \tan^2\theta\sec\theta d\theta-2\int \sec\theta d\theta\end{aligned}$$

$$2\int \tan^2\theta\sec\theta d\theta=2\sec\theta\tan\theta-2\int \tan^2\theta\sec\theta d\theta-2\ln\left|\sec\theta+\tan\theta\right|$$

$$4\int \tan^2\theta\sec\theta d\theta=2\sec\theta\tan\theta-2\ln\left|\sec\theta+\tan\theta\right|$$

$$2\int \tan^2\theta\sec\theta d\theta=\sec\theta\tan\theta-\ln\left|\sec\theta+\tan\theta\right|$$

$$\begin{aligned}\int \frac{\sqrt{x-2}}{\sqrt{x-1}}\,dx &= 2\int \tan\theta\sec\theta\tan\theta d\theta \\ &= \sec\theta\tan\theta-\ln\left|\sec\theta+\tan\theta\right|+C \\ &= u\sqrt{u^2-1}-\ln\left|u+\sqrt{u^2-1}\right|+C \\ &= \sqrt{x-1}\sqrt{x-2}-\ln\left|\sqrt{x-1}+\sqrt{x-2}\right|+C\end{aligned}$$

Exercise

Evaluate: $\displaystyle\int \frac{2dx}{\sqrt{1-4x^2}}$

Solution

$$\int \frac{2dx}{\sqrt{1-4x^2}} = \int \frac{du}{\sqrt{1-u^2}} \qquad\qquad u = 2x \quad \to \quad du = 2dx$$

$$= \sin^{-1} u + C$$

$$= \sin^{-1} 2x + C$$

Exercise

Evaluate: $\displaystyle\int \frac{dx}{\sqrt{4x^2-49}}$

Solution

$$\int \frac{dx}{\sqrt{4x^2-49}} = \int \frac{dx}{2\sqrt{x^2-\left(\frac{7}{2}\right)^2}} \qquad\qquad 2x = 7\sec\theta \quad \to dx = \frac{7}{2}\sec\theta\tan\theta\, d\theta$$

$$\sqrt{4x^2-49} = \frac{7}{2}\tan\theta$$

$$= \frac{1}{2}\int \frac{\frac{7}{2}\sec\theta\tan\theta d\theta}{\frac{7}{2}\tan\theta}$$

$$= \frac{1}{2}\int \sec\theta d\theta$$

$$= \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{2x}{7} + \frac{\sqrt{4x^2-49}}{7}\right| + C$$

Exercise

Evaluate: $\displaystyle\int \frac{dx}{\sqrt{x^2+4}}$

Solution

Let: $x = 2\tan\theta \quad \to dx = 2\sec^2\theta d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\sqrt{x^2+4} = 2\left|\sec\theta\right|$$

$$\int \frac{dx}{\sqrt{x^2+4}} = \int \frac{2\sec^2\theta d\theta}{\sqrt{4\sec^2\theta}}$$

$$= \int \frac{2\sec^2\theta d\theta}{2|\sec\theta|}$$

$$= \int \sec\theta d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$\boxed{= \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\left(16-x^2\right)^{3/2}}$

Solution

$$\int \frac{dx}{\left(16-x^2\right)^{3/2}} = \int \frac{4\cos\theta}{(4\cos\theta)^3} d\theta$$

$x = 4\sin\theta \qquad \sqrt{16-x^2} = 4\cos\theta$

$dx = 4\cos\theta d\theta$

$$= \frac{1}{16}\int \frac{1}{\cos^2\theta} d\theta$$

$$= \frac{1}{16}\int \sec^2\theta \, d\theta$$

$$\boxed{= \frac{1}{16}\tan\theta + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\left(1+x^2\right)^2}$

Solution

$$\int \frac{dx}{\left(1+x^2\right)^2} = \int \frac{\sec^2\theta}{\sec^4\theta} d\theta$$

$x = \tan\theta \qquad 1+x^2 = \left(\sec^2\theta\right)^2$

$dx = \sec^2\theta \, d\theta$

$$= \int \frac{1}{\sec^2\theta} d\theta$$

$$= \int \cos^2\theta \, d\theta$$

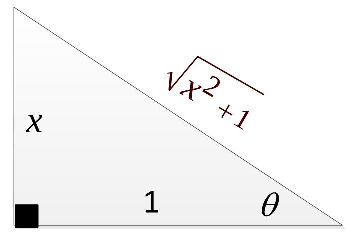
$$= \frac{1}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{2}\tan^{-1} x + \frac{1}{2}\sin\theta\cos\theta + C$$

$$= \frac{1}{2}\tan^{-1} x + \frac{1}{2}\frac{x}{\sqrt{1+x^2}}\frac{1}{\sqrt{1+x^2}} + C$$

$$\boxed{= \frac{1}{2}\tan^{-1} x + \frac{x}{2\left(1+x^2\right)} + C}$$



Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2+4}}$

Solution

$$\int \frac{dx}{\sqrt{x^2+4}} = \int \frac{2\sec^2\theta}{2\sec\theta}\, d\theta$$

$x = 2\tan\theta \qquad \sqrt{x^2+4} = 2\sec\theta$

$dx = 2\sec^2\theta \, d\theta$

$$= \int \sec\theta \, d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C$$

$$= \ln\left(\sqrt{x^2+4} + x\right) - \ln 2 + C$$

$$\boxed{= \ln\left(\sqrt{x^2+4} + x\right) + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}}$

Solution

$$\int \frac{dx}{x^2\sqrt{9-x^2}} = \int \frac{3\cos\theta}{9\sin^2\theta(3\cos\theta)}\, d\theta$$

$x = 3\sin\theta \qquad \sqrt{9-x^2} = 3\cos\theta$

$dx = 3\cos\theta d\theta$

$$= \frac{1}{9}\int \csc^2\theta\, d\theta$$

$$= -\frac{1}{9}\cot\theta + C \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{9-x^2}}{3}\cdot\frac{3}{x}$$

$$\boxed{= -\frac{1}{9}\frac{\sqrt{9-x^2}}{x} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{4x^2+1}}$

Solution

$2x = \tan\theta \qquad \sqrt{4x^2+1} = \sec\theta$

$dx = \frac{1}{2}\sec^2\theta\, d\theta$

$$\int \frac{dx}{\sqrt{4x^2+1}} = \frac{1}{2}\int \frac{\sec^2\theta}{\sec\theta}d\theta$$

$$= \frac{1}{2}\int \sec\theta\, d\theta$$

$$= \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C$$

$$\boxed{= \frac{1}{2}\ln\left|\sqrt{4x^2+1} + 2x\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\left(x^2+1\right)^{3/2}}$

Solution

$x = \tan\theta \qquad \sqrt{x^2+1} = \sec\theta$

$dx = \sec^2\theta\, d\theta$

$$\int \frac{dx}{\left(x^2+1\right)^{3/2}} = \int \frac{\sec^2\theta}{(\sec\theta)^3}d\theta$$

$$= \int \frac{d\theta}{\sec\theta}$$

$$= \int \cos\theta\, d\theta$$

$$= \sin\theta + C \qquad \sin\theta = \frac{\tan\theta}{\sec\theta} = \frac{x}{\sqrt{x^2+1}}$$

$$\boxed{= \frac{x}{\sqrt{x^2+1}} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{4}{x^2\sqrt{16-x^2}}\,dx$

Solution

$$\int \frac{4}{x^2\sqrt{16-x^2}}\,dx = \int \frac{16\cos\theta}{16\sin^2\theta(4\cos\theta)}\,d\theta$$

$x = 4\sin\theta \qquad \sqrt{16-x^2} = 4\cos\theta$

$dx = 4\cos\theta d\theta$

$$= \frac{1}{4}\int \csc^2\theta\,d\theta$$

$$= -\frac{1}{4}\cot\theta + C$$

Exercise

Evaluate $\displaystyle\int \frac{x^3}{\sqrt{9-x^2}}\,dx$

Solution

$$\int \frac{x^3}{\sqrt{9-x^2}}\,dx = \int \frac{27\sin^3\theta}{3\cos\theta}(3\cos\theta)\,d\theta$$

$x = 3\sin\theta \qquad \sqrt{9-x^2} = 3\cos\theta$

$dx = 3\cos\theta d\theta$

$$= 27\int \sin^3\theta\,d\theta$$

$$= 27\int \left(1-\cos^2\theta\right)\,d(\cos\theta)$$

$$= 27\left(\cos\theta - \frac{1}{3}\cos^3\theta\right) + C$$

$$= 27\cos\theta - 9\cos^3\theta + C$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2-25}}$

Solution

$$\int \frac{dx}{\sqrt{x^2-25}} = \int \frac{5\sec\theta\tan\theta}{5\tan\theta}\,d\theta$$

$x = 5\sec\theta \qquad \sqrt{x^2-25} = 5\tan\theta$

$dx = 5\sec\theta\tan\theta\,d\theta$

$$= \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{x}{5} + \frac{1}{5}\sqrt{x^2-25}\right| + C$$

Exercise

Evaluate $\displaystyle\int \frac{\sqrt{x^2-25}}{x}\, dx$

Solution

$$\int \frac{\sqrt{x^2-25}}{x}\, dx = \int \frac{5\tan\theta}{5\sec\theta}(5\sec\theta\tan\theta)\, d\theta \qquad x = 5\sec\theta \qquad \sqrt{x^2-25} = 5\tan\theta$$

$$dx = 5\sec\theta\tan\theta\, d\theta$$

$$= 5\int \tan^2\theta\, d\theta$$

$$= 5\int \left(\sec^2\theta - 1\right)\, d\theta$$

$$= 5(\tan\theta - \theta) + C$$

$$\boxed{= \sqrt{x^2-25} - 5\,\text{arcsec}\,\frac{x}{5} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x^3}{\sqrt{x^2-25}}\, dx$

Solution

$$\int \frac{x^3}{\sqrt{x^2-25}}\, dx = \int \frac{5^3\sec^3\theta}{5\tan\theta}(5\sec\theta\tan\theta)\, d\theta \qquad x = 5\sec\theta \qquad \sqrt{x^2-25} = 5\tan\theta$$

$$dx = 5\sec\theta\tan\theta\, d\theta$$

$$= 125\int \sec^4\theta\, d\theta$$

$$= 125\int \left(1+\tan^2\theta\right)\sec^2\theta\, d\theta$$

$$= 125\int \left(1+\tan^2\theta\right)\, d(\tan\theta)$$

$$= 125\left(\tan\theta + \frac{1}{3}\tan^3\theta\right) + C$$

$$= 125\left(\frac{\sqrt{x^2-25}}{5} + \frac{1}{3}\frac{\left(x^2-25\right)^{3/2}}{125}\right) + C$$

$$= \sqrt{x^2-25}\left(25 + \frac{x^2-25}{3}\right) + C$$

$$\boxed{= \frac{1}{3}\sqrt{x^2-25}\left(x^2+50\right) + C}$$

Exercise

Evaluate $\displaystyle\int x^3\sqrt{x^2-25}\,dx$

Solution

$$\int x^3\sqrt{x^2-25}\,dx=\int 5^3\sec^3\theta(5\tan\theta)(5\sec\theta\tan\theta)\,d\theta$$

$x=5\sec\theta \qquad \sqrt{x^2-25}=5\tan\theta$

$dx=5\sec\theta\tan\theta\,d\theta$

$$\begin{aligned}
&=5^5\int\sec^4\theta\tan^2\theta\,d\theta\\
&=5^5\int\sec^2\theta\left(1+\tan^2\theta\right)\tan^2\theta\,d\theta\\
&=5^5\int\left(\tan^2\theta+\tan^4\theta\right)\,d(\tan\theta)\\
&=5^5\left(\tfrac{1}{3}\tan^3\theta+\tfrac{1}{5}\tan^5\theta\right)+C\\
&=5^5\left(\frac{1}{3}\frac{1}{5^3}\left(x^2-25\right)^{3/2}+\frac{1}{5^6}\left(x^2-25\right)^{5/2}\right)+C\\
&=\left(x^2-25\right)^{3/2}\left(\frac{25}{3}+\frac{1}{5}\left(x^2-25\right)\right)+C\\
&=\frac{1}{15}\left(x^2-25\right)^{3/2}\left(125+3x^2-75\right)+C\\
&=\frac{1}{15}\left(x^2-25\right)^{3/2}\left(3x^2+50\right)+C
\end{aligned}$$

Exercise

Evaluate $\displaystyle\int x\sqrt{x^2+1}\,dx$

Solution

$$\int x\sqrt{x^2+1}\,dx=\frac{1}{2}\int\left(x^2+1\right)^{1/2}\,d\left(x^2+1\right)$$

$$=\frac{1}{3}\left(x^2+1\right)^{3/2}+C$$

$$\int x\sqrt{x^2+1}\,dx=\int\tan\theta\sec^3\theta\,d\theta$$

$x=\tan\theta \qquad \sqrt{x^2+1}=\sec\theta$

$dx=\sec^2\theta\,d\theta$

$$\begin{aligned}
&=\int\sec^2\theta\,d(\sec\theta)\\
&=\frac{1}{3}\sec^3\theta+C\\
&=\frac{1}{3}\left(x^2+1\right)^{3/2}+C
\end{aligned}$$

Exercise

Evaluate $\displaystyle\int \frac{9x^3}{\sqrt{x^2+1}}\,dx$

Solution

$$\int \frac{9x^3}{\sqrt{x^2+1}}\,dx = \int \frac{9\tan^3\theta}{\sec\theta}\left(\sec^2\theta\right)d\theta$$

$x = \tan\theta \qquad \sqrt{x^2+1} = \sec\theta$

$dx = \sec^2\theta\, d\theta$

$$= 9\int \tan^2\theta \tan\theta \sec\theta\, d\theta$$

$$= 9\int \left(\sec^2\theta - 1\right)\, d\left(\sec\theta\right)$$

$$= 9\left(\frac{1}{3}\sec^3\theta - \sec\theta\right) + C$$

$$= 3\left(x^2+1\right)\sqrt{x^2+1} - 9\sqrt{x^2+1} + C$$

$$= 3\sqrt{x^2+1}\left(x^2+1-3\right) + C$$

$$= 3\sqrt{x^2+1}\left(x^2-2\right) + C$$

Exercise

Evaluate $\displaystyle\int_0^{\sqrt{3}/2} \frac{x^2}{\left(1-x^2\right)^{3/2}}\,dx$

Solution

$$\int_0^{\sqrt{3}/2} \frac{x^2}{\left(1-x^2\right)^{3/2}}\,dx = \int_0^{\sqrt{3}/2} \frac{\sin^2\theta}{\cos^3\theta}\left(\cos\theta\right)d\theta$$

$x = \sin\theta \qquad \sqrt{1-x^2} = \cos\theta$

$dx = \cos\theta\, d\theta$

$$= \int_0^{\sqrt{3}/2} \tan^2\theta\, d\theta$$

$$= \int_0^{\sqrt{3}/2} \left(\sec^2\theta - 1\right)\, d\theta$$

$$= \left(\tan\theta - \theta\right)\Big|_0^{\sqrt{3}/2}$$

$$= \left(\frac{x}{\sqrt{1-x^2}} - \arcsin x\right)\Bigg|_0^{\sqrt{3}/2}$$

$$= \frac{\sqrt{3}}{2}\frac{1}{\sqrt{1-\frac{3}{4}}} - \frac{\pi}{3}$$

$$\boxed{= \sqrt{3} - \frac{\pi}{3}}$$

Exercise

Evaluate $\displaystyle\int_0^{\sqrt{3}/2} \frac{1}{\left(1-x^2\right)^{5/2}}\,dx$

Solution

$$\int_0^{\sqrt{3}/2} \frac{1}{\left(1-x^2\right)^{5/2}}\,dx = \int_0^{\sqrt{3}/2} \frac{1}{\cos^5\theta}\cos\theta\; d\theta$$

$x = \sin\theta \qquad \sqrt{1-x^2} = \cos\theta$

$dx = \cos\theta d\theta$

$$= \int_0^{\sqrt{3}/2} \sec^4\theta\; d\theta$$

$$= \int_0^{\sqrt{3}/2} \left(1+\tan^2\theta\right)\sec^2\theta\; d\theta$$

$$= \int_0^{\sqrt{3}/2} \left(1+\tan^2\theta\right)\; d\left(\tan\theta\right)$$

$$= \tan\theta + \frac{1}{3}\tan^3\theta \Big|_0^{\sqrt{3}/2}$$

$$= \frac{x}{\sqrt{1-x^2}} + \frac{1}{3}\frac{x^3}{\left(1-x^2\right)^{3/2}} \Bigg|_0^{\sqrt{3}/2}$$

$$= \frac{\sqrt{3}}{2}\frac{1}{\sqrt{1-\frac{3}{4}}} + \frac{\sqrt{3}}{8}\frac{1}{\left(\frac{1}{4}\right)^{3/2}}$$

$$= \sqrt{3} + \sqrt{3}$$

$$\boxed{= 2\sqrt{3}}$$

Exercise

Evaluate $\displaystyle\int_0^3 \frac{x^3}{\sqrt{x^2+9}}\,dx$

Solution

$$\int_0^3 \frac{x^3}{\sqrt{x^2+9}}\,dx = \int_0^3 \frac{27\tan^3\theta}{3\sec\theta}3\sec^2\theta\,d\theta$$

$x = 3\tan\theta \qquad \sqrt{x^2+9} = 3\sec\theta$

$dx = 3\sec^2\theta\,d\theta$

$$= 27\int_0^3 \tan^2\theta\tan\theta\sec\theta\,d\theta$$

$$= 27\int_0^3 \left(\sec^2\theta - 1\right)\,d(\sec\theta)$$

$$= 27\left(\frac{1}{3}\sec^3\theta - \sec\theta\right)\Big|_0^3$$

$$= 9\sqrt{x^2+9}\left(\frac{x^2+9}{27} - 1\right)\Big|_0^3$$

$$= \frac{1}{3}\sqrt{x^2+9}\left(x^2 - 18\right)\Big|_0^3$$

$$= -9\sqrt{2} + 18$$

Exercise

Evaluate $\displaystyle\int_0^{3/5} \sqrt{9-25x^2}\,dx$

Solution

$$\int_0^{3/5} \sqrt{9-25x^2}\,dx = \frac{9}{5}\int_0^{3/5} \cos^2\theta\,d\theta$$

$5x = 3\sin\theta \qquad \sqrt{9-25x^2} = 3\cos\theta$

$dx = \frac{3}{5}\cos\theta\,d\theta$

$$= \frac{9}{10}\int_0^{3/5} (1+\cos 2\theta)\,d\theta$$

$$= \frac{9}{10}\left(\theta + \frac{1}{2}\sin 2\theta\right)_0^{3/5}$$

$\sin 2\theta = 2\sin\theta\cos\theta = 2\frac{5x}{3}\frac{5\sqrt{9-25x^2}}{3}$

$$= \frac{9}{10}\left(\arcsin\frac{5x}{3} + \frac{25}{9}x\sqrt{9-25x^2}\right)_0^{3/5}$$

$$= \frac{9\pi}{20}$$

Exercise

Evaluate $\displaystyle\int_4^6 \frac{x^2}{\sqrt{x^2-9}}\,dx$

Solution

$$\int_4^6 \frac{x^2}{\sqrt{x^2-9}}\,dx = \int_4^6 \frac{9\sec^2\theta}{3\tan\theta}\left(3\sec\theta\tan\theta\right)d\theta$$

$x = 3\sec\theta \qquad \sqrt{x^2-9} = 3\tan\theta$

$dx = 3\sec\theta\tan\theta\ d\theta$

$$= 9\int_4^6 \sec^3\theta\ d\theta$$

$$= \frac{9}{2}\Big[\sec\theta\tan\theta + \ln\left|\sec\theta+\tan\theta\right|\Big]_4^6$$

$$= \frac{9}{2}\left[\frac{x}{3}\frac{\sqrt{x^2-9}}{3} + \ln\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{3}\right|\right]_4^6$$

$$= \frac{9}{2}\left(2\sqrt{3} + \ln\left(2+\sqrt{3}\right) - \frac{4\sqrt{7}}{9} - \ln\left(\frac{4+\sqrt{7}}{3}\right)\right)$$

$$= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\ln\left(\frac{6+3\sqrt{3}}{4+\sqrt{7}}\right)$$

$u=\sec x \qquad dv=\sec^2 x dx$

$du=\sec x\tan x dx \qquad v=\tan x$

$$\int \sec^3 x dx = \sec x\tan x - \int \tan x\left(\sec x\tan x dx\right)$$

$$= \sec x\tan x - \int \tan^2 x\sec x dx$$

$$= \sec x\tan x - \int\left(\sec^2 x - 1\right)\sec x\ dx$$

$$= \sec x\tan x - \int \sec^3 x\ dx + \int \sec x\ dx$$

$$2\int \sec^3 x\ dx = \sec x\tan x + \ln\left|\sec x+\tan x\right|$$

$$\int \sec^3 x\ dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln\left|\sec x+\tan x\right|$$

Exercise

Evaluate $\displaystyle\int_{\sqrt{3}}^2 \frac{\sqrt{x^2-3}}{x}dx$

Solution

$$\int_{\sqrt{3}}^2 \frac{\sqrt{x^2-3}}{x}dx = \int_{\sqrt{3}}^2 \frac{\sqrt{3}\tan\theta}{\sqrt{3}\sec\theta}\left(\sqrt{3}\sec\theta\tan\theta\right)d\theta$$

$x = \sqrt{3}\sec\theta \qquad \sqrt{x^2-3} = \sqrt{3}\tan\theta$

$dx = \sqrt{3}\sec\theta\tan\theta\ d\theta$

$$= \sqrt{3}\int_{\sqrt{3}}^2 \tan^2\theta\ d\theta$$

$$= \sqrt{3}\int_{\sqrt{3}}^2 \left(\sec^2\theta - 1\right)\ d\theta$$

$$= \sqrt{3}\left(\tan\theta - \theta\right)\Big|_{\sqrt{3}}^2$$

$$= \sqrt{3}\left(\frac{\sqrt{x^2-3}}{\sqrt{3}} - \operatorname{arcsec}\frac{x}{\sqrt{3}}\right)\Bigg|_{\sqrt{3}}^2$$

$$= \sqrt{3}\left(\frac{1}{\sqrt{3}} - \frac{\pi}{6}\right)$$

$$\boxed{= 1 - \frac{\pi\sqrt{3}}{6}}$$

Exercise

Evaluate $\displaystyle\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2}\,dx$

Solution

$$\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2}\,dx = \int_1^4 \frac{\sqrt{(x+2)^2-9}}{x+2}\,dx$$

$x+2 = 3\sec\theta \qquad \sqrt{(x+2)^2-9} = 3\tan\theta$

$dx = 3\sec\theta\tan\theta\; d\theta$

$$= \int_1^4 \frac{3\tan\theta}{3\sec\theta}\left(3\sec\theta\tan\theta\right)\; d\theta = 3\int_1^4 \tan^2\theta\; d\theta$$

$$= 3\int_1^4 \left(\sec^2\theta - 1\right)\; d\theta$$

$$= 3\left(\tan\theta - \theta\right)\Big|_1^4$$

$$= \sqrt{(x+2)^2-9} - 3\sec^{-1}\left(\frac{x+2}{3}\right)\Big|_1^4$$

$$= \sqrt{27} - 3\sec^{-1}(2) + 3\sec^{-1}(1)$$

$$\boxed{= 3\sqrt{3} - \pi}$$

$\theta = \sec^{-1}\left(\frac{x+2}{3}\right)$

$$\begin{cases} x = 4 \quad \to \quad \theta = \sec^{-1}(2) = \frac{\pi}{3} \\ x = 1 \quad \to \quad \theta = \sec^{-1}(1) = 0 \end{cases}$$

$$= 3\left(\tan\theta - \theta\right)\Big|_0^{\pi/3}$$

$$\boxed{= 3\sqrt{3} - \pi}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}}$

Solution

$$\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1}\frac{x}{3}\right]_0^{3/2}$$

$$\boxed{= \frac{\pi}{6}}$$

Exercise

Evaluate the integral $\displaystyle\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^x dx}{\left(1+e^{2x}\right)^{3/2}}$

Solution

$$\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^x dx}{\left(1+e^{2x}\right)^{3/2}} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{\tan\theta}{\left(\sec^2\theta\right)^{3/2}} \frac{\sec^2\theta}{\tan\theta} d\theta$$

$$= \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{\sec^2\theta}{\sec^3\theta} d\theta$$

$$= \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{1}{\sec\theta} d\theta$$

$$= \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \cos\theta d\theta$$

$$= \sin\theta \Big|_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)}$$

$$= \sin\left(\tan^{-1}(3/4)\right) - \sin\left(\tan^{-1}(4/3)\right)$$

$$= \frac{4}{5} - \frac{3}{5}$$

$$= \frac{1}{5}$$

$e^x = \tan\theta \to x = \ln(\tan\theta)$

$dx = \frac{\sec^2\theta}{\tan\theta} d\theta$

$\tan^{-1}\left(\frac{3}{4}\right) < \theta < \tan^{-1}\left(\frac{4}{3}\right)$

$1 + e^{2x} = 1 + \tan^2\theta = \sec^2\theta$

Exercise

Evaluate the integral $\displaystyle\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}}$

Solution

$$\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}} = \int_0^{\pi/4} \frac{e^{\tan\theta}\sec^2\theta}{e^{\tan\theta}\sec\theta} d\theta$$

$$= \int_0^{\pi/4} \sec\theta d\theta$$

$$= \Big[\ln\left|\sec\theta + \tan\theta\right|\Big]_0^{\pi/4}$$

$y = e^{\tan\theta} \quad 1 \le y \le e \to \quad 0 \le \theta = \tan^{-1}(\ln y) \le \frac{\pi}{4}$

$dy = e^{\tan\theta}\sec^2\theta d\theta$

$\sqrt{1+(\ln y)^2} = \sqrt{1+\tan^2\theta} = \sec\theta$

$$= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln\left|\sec 0 + \tan 0\right|$$

$$\boxed{= \ln\left(1+\sqrt{2}\right)}$$

Exercise

Evaluate $\displaystyle\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2-1}}$

Solution

Let: $u = 3y \quad \Rightarrow \quad du = 3dy \quad \rightarrow \quad \frac{du}{3} = dy$

$$\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2-1}} = \int_{-2/3}^{-\sqrt{2}/3} \frac{\frac{du}{3}}{\frac{u}{3}\sqrt{u^2-1}}$$

$$= \int_{-2/3}^{-\sqrt{2}/3} \frac{du}{u\sqrt{u^2-1}}$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right|$$

$$= \sec^{-1}|3y| \Big|_{-2/3}^{-\sqrt{2}/3}$$

$$= \sec^{-1}\left|-\sqrt{2}\right| - \sec^{-1}|-2|$$

$$= \frac{\pi}{4} - \frac{\pi}{3}$$

$$\boxed{= -\frac{\pi}{12}}$$

Exercise

Evaluate $\displaystyle\int_0^2 \sqrt{1+4x^2}\,dx$

Solution

$$\int_0^2 \sqrt{1+4x^2}\,dx = \frac{1}{2}\int_0^2 \sec^3\theta \; d\theta$$

$2x = \tan\theta \qquad \sqrt{1+4x^2} = \sec\theta$

$dx = \frac{1}{2}\sec^2\theta d\theta$

$$\int \sec^3 x dx = \sec x \tan x - \int \tan x\left(\sec x \tan x dx\right)$$

$u = \sec x \qquad dv = \sec^2 x dx$

$du = \sec x \tan x dx \qquad v = \tan x$

$$= \sec x \tan x - \int \tan^2 x \sec x dx$$

$$= \sec x \tan x - \int \left(\sec^2 x - 1\right)\sec x \; dx$$

$$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$2\int \sec^3 x\, dx = \sec x \tan x + \ln\left|\sec x + \tan x\right|$$

$$\int \sec^3 x\, dx = \underline{\frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right|}$$

$$\int_0^2 \sqrt{1+4x^2}\,dx = \frac{1}{4}\left(\sec\theta \tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right)\Big|_0^2$$

$$= \frac{1}{4}\left(2x\sqrt{1+4x^2} + \ln\left|2x + \sqrt{1+4x^2}\right|\right)\Big|_0^2$$

$$= \frac{1}{4}\left(4\sqrt{17} + \ln\left|4+\sqrt{17}\right|\right)$$

$$= \sqrt{17} + \frac{1}{4}\ln\left(4+\sqrt{17}\right)$$

Exercise

Consider the region bounded by the graph $y = \sqrt{x \tan^{-1} x}$ and $y = 0$ for $0 \le x \le 1$. Find the volume of the solid formed by revolving this region about the x-axis.

Solution

$$V = \pi \int_0^1 \left(\sqrt{x \tan^{-1} x}\right)^2 dx$$

$$= \pi \int_0^1 x \tan^{-1} x dx$$

$$u = \tan^{-1} x \qquad dv = xdx$$

$$du = \frac{1}{x^2+1}dx \qquad v = \frac{1}{2}x^2$$

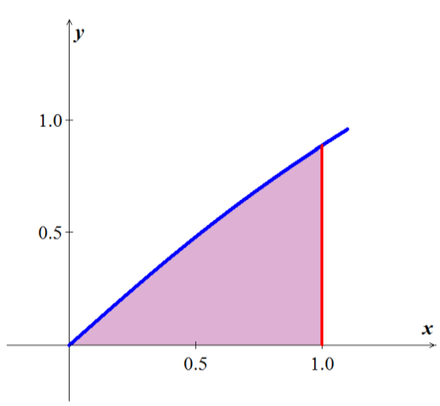
$$V = \pi\left(\frac{1}{2}\left[x^2 \tan^{-1} x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}dx\right)$$

$$= \frac{\pi}{2}\left(\left(1\tan^{-1}1 - 0\right) - \int_0^1\left(1 - \frac{1}{1+x^2}\right)dx\right)$$

$$= \frac{\pi}{2}\left(\frac{\pi}{4} - \int_0^1 dx + \int_0^1 \frac{1}{1+x^2}dx\right)$$

$$= \frac{\pi}{2}\left(\frac{\pi}{4} - \left[x\right]_0^1 + \left[\tan^{-1} x\right]_0^1\right)$$

$$= \frac{\pi}{2}\left(\frac{\pi}{4} - 1 + \tan^{-1} 1\right)$$



$$= \frac{\pi}{2}\left(\frac{\pi}{4} - 1 + \frac{\pi}{4}\right)$$

$$= \frac{\pi}{2}\left(\frac{\pi}{2} - 1\right)$$

$$= \frac{\pi^2}{4} - \frac{\pi}{2}$$

Exercise

Use two approach to show that the area of a cap (or segment) of a circle of radius r subtended by an angle θ is given by

$$A_{seg} = \frac{1}{2} r^2 (\theta - \sin\theta)$$

a) Find the area using geometry (no calculus).

b) Find the area using calculus

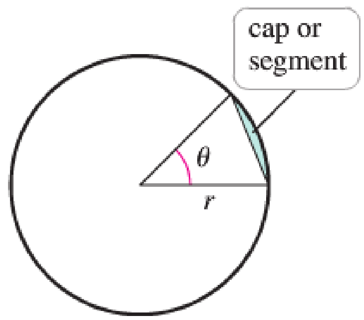
Solution

a) Area of a segment (*cap*) = Area of a sector ***minus*** Area of the isosceles triangle

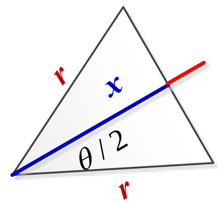
The area of a sector: $A = \frac{1}{2}\theta r^2$

Area of the isosceles triangle: $A = \frac{1}{2} r^2 \sin\theta$

$$A_{seg} = \frac{1}{2} r^2 \theta - \frac{1}{2} r^2 \sin\theta = \frac{1}{2} r^2 (\theta - \sin\theta)$$



b) $0 \le \theta \le \pi \quad \to \quad 0 \le \frac{\theta}{2} \le \frac{\pi}{2}$



$$x = r\cos\frac{\alpha}{2} \quad \to \quad dx = -\frac{1}{2} r \sin\frac{\alpha}{2}\, d\alpha$$

$$\sqrt{r^2 - x^2} = r\sin\frac{\alpha}{2}$$

$$A_{cap} = 2\int_{r\cos\theta/2}^{r} \sqrt{r^2 - x^2}\, dx$$

$$= 2\int_{\theta}^{0} \left(r\sin\frac{\alpha}{2}\right)\left(-\frac{1}{2} r\sin\frac{\alpha}{2}\right) d\alpha$$

$$= r^2 \int_0^{\theta} \left(\sin^2\frac{\alpha}{2}\right) d\alpha$$

$$= \frac{1}{2} r^2 \int_0^{\theta} (1 - \cos\alpha)\, d\alpha$$

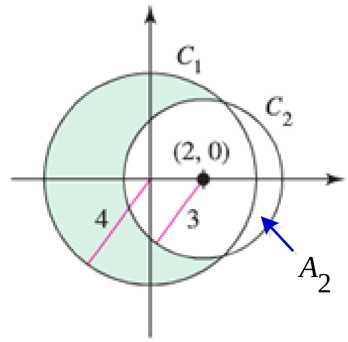
$$= \frac{1}{2} r^2 (\alpha - \sin\alpha)\Big|_0^{\theta}$$

$$= \frac{1}{2} r^2 (\theta - \sin\theta)$$

Exercise

A lune is a crescent-shaped region bounded by the arcs of two circles. Let C_1 be a circle of radius 4 centered at the origin. Let C_2 be a circle of radius 3 centered at the point $(2,\ 0)$. Find the area of the lune that lies inside C_1 and outside C_2.

Solution



$$C_1 \quad \to \quad x^2+y^2=16 \ \Rightarrow \ y^2=16-x^2$$

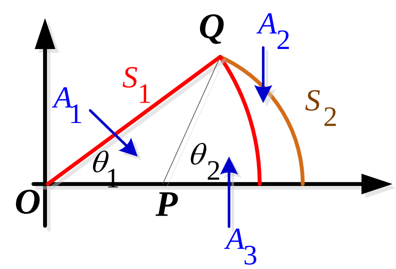
$$C_2 \quad \to \quad (x-2)^2+y^2=9 \ \Rightarrow \ y^2=9-(x-2)^2$$

$$16-x^2=9-x^2+4x-4$$

$$11=4x \quad \to \quad \underline{x=\frac{11}{4}} \ \Rightarrow \ \underline{y=\pm\frac{\sqrt{135}}{4}=\pm\frac{3\sqrt{15}}{4}}$$

For ***sector*** C_1: $\quad \theta_1=\tan^{-1}\frac{y}{x}=\tan^{-1}\frac{3\sqrt{15}}{11}$

$$\text{Area: } S_1=\frac{1}{2}r^2\theta_1=8\tan^{-1}\left(\frac{3\sqrt{15}}{11}\right)$$



For ***sector*** C_2: $\quad x_2=\frac{11}{4}-2=\frac{3}{4}$

$$\theta_2=\tan^{-1}\frac{y}{x_2}=\tan^{-1}\sqrt{15}$$

$$\text{Area: } S_2=\frac{1}{2}r_2^2\theta_2=\frac{9}{2}\tan^{-1}\left(\sqrt{15}\right)$$

$OQ=4, \quad PQ=3, \quad OP=2$

$$Area(\Delta APQ)=A_1=\frac{1}{2}(4)(2)\sin\theta_1=4\frac{y}{4}=\underline{\frac{3\sqrt{15}}{4}}$$

$$A_2=S_2-S_1+A_1$$

$$=\underline{\frac{9}{2}\tan^{-1}\left(\sqrt{15}\right)-8\tan^{-1}\left(\frac{3\sqrt{15}}{11}\right)+\frac{3\sqrt{15}}{4}}$$

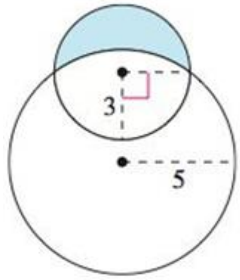
$$A_{lune}=A_{C_1}-A_{C_2}+2A_2$$

$$=16\pi-9\pi+9\tan^{-1}\left(\sqrt{15}\right)-16\tan^{-1}\left(\frac{3\sqrt{15}}{11}\right)+\frac{3\sqrt{15}}{2}$$

$$=\underline{7\pi+9\tan^{-1}\left(\sqrt{15}\right)-16\tan^{-1}\left(\frac{3\sqrt{15}}{11}\right)+\frac{3\sqrt{15}}{2}} \quad \underline{\approx 26.66} \quad unit^2$$

Exercise

The crescent-shaped region bounded by two circles forms a lune. Find the area of the lune given that the radius of the smaller circle is 3 and the radius of the larger circle is 5.



Solution

Large Circle: $x^2+y^2=25 \quad \to y=\sqrt{25-x^2}$

Small Circle: $r=3 \quad \to \quad y=\sqrt{25-9}\ \underline{=4}$

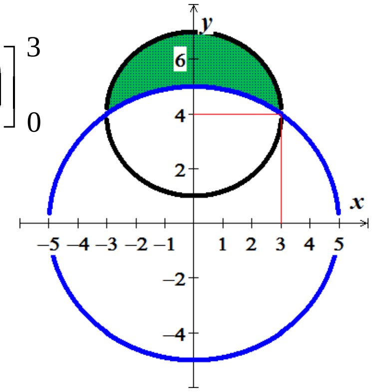
$$x^2+(y-4)^2=9 \quad \to y=4+\sqrt{9-x^2}$$

$$A=2\int_0^3\left(4+\sqrt{9-x^2}-\sqrt{25-x^2}\right)dx$$

$$=2\left[4x+\frac{1}{2}\left(9\arcsin\left(\frac{x}{3}\right)+x\sqrt{9-x^2}\right)-\frac{1}{2}\left(25\arcsin\left(\frac{x}{5}\right)+x\sqrt{25-x^2}\right)\right]_0^3$$

$$=2\left[12+\frac{1}{2}\left(9\frac{\pi}{2}\right)-\frac{1}{2}\left(25\arcsin\left(\frac{3}{5}\right)+12\right)\right]$$

$$\boxed{=12+\frac{9\pi}{2}-25\ \arcsin\left(\frac{3}{5}\right)}$$



Exercise

The surface of a machine part is the region between the graphs of $y=|x|$ *and* $x^2+(y-k)^2=25$

a) Find k when the circle is tangent to the graph of $y=|x|$

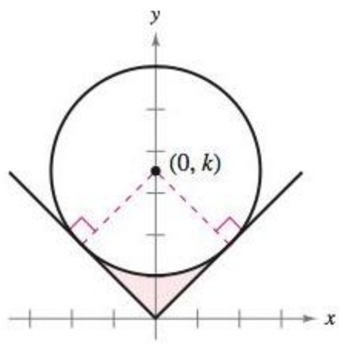
b) Find the area of the surface of the machine part.

c) Find the area of the surface of the machine part as a function of the radius r of the circle.

Solution

a) $x^2+(y-k)^2=25 \quad \to \quad \underline{r=5}$

$k^2=5^2+5^2=50 \quad \to \quad \boxed{k=5\sqrt{2}}$



b) $Area = area\ square - \frac{1}{4}(area\ circle)$

$$=5^2-\frac{1}{4}\pi 5^2$$

$$\boxed{=25\left(1-\frac{\pi}{4}\right)\ unit^2}$$

c) $Area = area\ square - \frac{1}{4}(area\ circle)$

$$=r^2-\frac{1}{4}\pi r^2$$

$$\boxed{=r^2\left(1-\frac{\pi}{4}\right)\ unit^2}$$

Exercise

Consider the function $f(x)=\left(9+x^2\right)^{-1/2}$ and the region $\boldsymbol{R}$ on the interval [0, 4].

a) Find the area of *R*.

b) Find the volume of the solid generated when *R* is revolved about the *x-axis*.

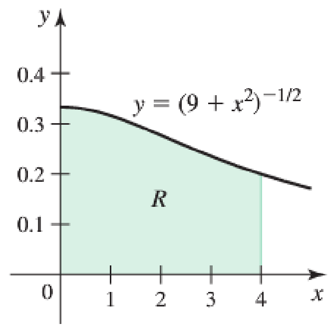
c) Find the volume of the solid generated when *R* is revolved about the *y-axis*.

Solution

a)
$$A=\int_0^4 \frac{dx}{\sqrt{9+x^2}}$$

$x=3\tan\theta \ \rightarrow\ dx=3\sec^2\theta\,d\theta$

$\sqrt{9+x^2}=3\sec\theta$



$$=\int_0^4 \frac{3\sec^2\theta\,d\theta}{3\sec\theta}$$

$$=\int_0^4 \sec\theta\,d\theta$$

$$=\ln\left|\sec\theta+\tan\theta\right|\Big|_0^4$$

$$=\ln\left|\frac{\sqrt{9+x^2}}{3}+\frac{x}{3}\right|\Bigg|_0^4$$

$$=\ln\left(\frac{5}{3}+\frac{4}{3}\right)-0$$

$$=\ln 3 \quad unit^2$$

b)
$$V=\pi\int_0^4 \frac{dx}{9+x^2}$$

$x=3\tan\theta \ \rightarrow\ dx=3\sec^2\theta\,d\theta$

$9+x^2=9\sec^2\theta$

$$=\pi\int_0^4 \frac{3\sec^2\theta\,d\theta}{9\sec^2\theta}$$

$$=\frac{\pi}{3}\int_0^4 d\theta$$

$$=\frac{\pi}{3}\theta\Big|_0^4$$

$$=\frac{\pi}{3}\tan^{-1}\frac{x}{3}\Big|_0^4$$

$$=\frac{\pi}{3}\tan^{-1}\frac{4}{3} \quad unit^3$$

c)
$$V=2\pi\int_0^4 \frac{x}{\sqrt{9+x^2}}dx$$

$d\left(9+x^2\right)=2x\,dx$

$$= \pi \int_0^4 \left(9 + x^2\right)^{-1/2} d\left(9 + x^2\right)$$

$$= 2\pi \left(9 + x^2\right)^{1/2} \Big|_0^4$$

$$= 2\pi (5 - 3)$$

$$\underline{= 4\pi \,|} \quad unit^3$$

Exercise

A total of $\boldsymbol{Q}$ is distributed uniformly on a line segment of length $2L$ along the y-*axis*. The x-component of the electric field at a point $(a,\ 0)$ is given by

$$E_x = \frac{kQa}{2L} \int_{-L}^{L} \frac{dy}{\left(a^2 + y^2\right)^{3/2}}$$

Where k is a physical constant and $a > 0$

a) Confirm that $E_x(a) = \dfrac{kQ}{a\sqrt{a^2 + L^2}}$

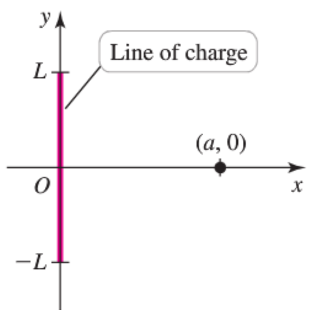
b) Letting $\rho = \dfrac{Q}{2L}$ be the charge density on the line segment, show that if $L \to \infty$, *then* $E_x = \dfrac{2k\rho}{a}$

Solution

a) $E_x = \dfrac{kQa}{2L} \displaystyle\int_{-L}^{L} \frac{dy}{\left(a^2 + y^2\right)^{3/2}}$

$y = a\tan\theta \ \to dy = a\sec^2\theta \ d\theta$

$\sqrt{a^2 + y^2} = a\sec\theta$



$$= \frac{kQa}{2L} \int_{-L}^{L} \frac{a\sec^2\theta \ d\theta}{a^3\sec^3\theta}$$

$$= \frac{kQ}{2aL} \int_{-L}^{L} \frac{d\theta}{\sec\theta}$$

$$= \frac{kQ}{2aL} \int_{-L}^{L} \cos\theta d\theta$$

$$= \frac{kQ}{2aL} \sin\theta \ \Big|_{-L}^{L}$$

$$= \frac{kQ}{2aL} \left(\frac{y}{\sqrt{a^2 + y^2}} \right) \Big|_{-L}^{L}$$

$$= \frac{kQ}{2aL}\left(\frac{2L}{\sqrt{a^2+L^2}}\right)$$

$$\boxed{= \frac{kQ}{a\sqrt{a^2+L^2}}}$$

b) Let $\rho = \frac{Q}{2L} \quad \rightarrow \quad Q = 2\rho L$

$$E_x(a) = \frac{kQa}{2L} \lim_{L\to\infty} \int_{-L}^{L} \frac{dy}{\left(a^2+y^2\right)^{3/2}}$$

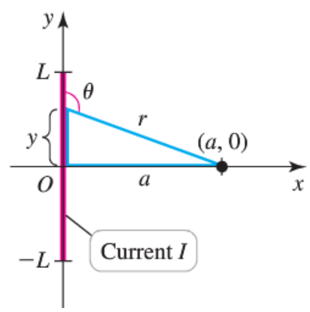
$$= \frac{kQa}{2L} \lim_{L\to\infty}\left(\frac{2L}{a^2\sqrt{a^2+L^2}}\right)$$

$$= k\rho a \ \frac{2}{a^2}$$

$$\boxed{= \frac{2k\rho}{a}}$$

Exercise

A long, straight wire of length $2L$ on the *y-axis* carries a current I. according to the Biot-Savart Law, the magnitude of the field due to the current at a point $(a,\ 0)$ is given by



$$B(a) = \frac{\mu_0 I}{4\pi}\int_{-L}^{L} \frac{\sin\theta}{r^2}\,dy$$

Where μ_0 is a physical constant, $a > 0$, and θ, r, and y are related to the figure

a) Show that the magnitude of the magnetic field at $(a,\ 0)$ is

$$B(a) = \frac{\mu_0 IL}{2\pi a\sqrt{a^2+L^2}}$$

b) What is the magnitude of the magnetic field at $(a,\ 0)$ due to an infinitely long wire $(L \to \infty)$?

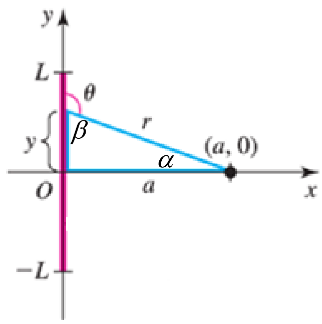
Solution

a) $\beta = \pi - \theta \quad \& \quad \alpha + \beta = \frac{\pi}{2}$

$$\sin\theta = \sin(\pi - \beta) = \sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha = \frac{a}{r}$$

$$r^2 = y^2 + a^2$$

$$\frac{\sin\theta}{r^2} = \frac{a}{r^3} = \frac{a}{\left(a^2+y^2\right)^{3/2}}$$



$$B(a)=\frac{\mu_0 I}{4\pi}\int_{-L}^{L}\frac{\sin\theta}{r^2}dy$$

$$y=a\tan u \quad\rightarrow\quad dy=a\sec^2 u\ du$$

$$\sqrt{a^2+y^2}=a\sec u$$

$$=\frac{\mu_0 I}{4\pi}\int_{-L}^{L}\frac{a}{\left(a^2+y^2\right)^{3/2}}dy$$

$$=\frac{\mu_0 I}{2\pi}\int_{0}^{L}\frac{a^2\sec^2 u\ du}{a^3\sec^3 u}$$

$$=\frac{\mu_0 I}{2a\pi}\int_{0}^{L}\frac{1}{\sec u}du$$

$$=\frac{\mu_0 I}{2a\pi}\int_{0}^{L}\cos u\ du$$

$$=\frac{\mu_0 I}{2a\pi}\sin u\Big|_0^L$$

$$=\frac{\mu_0 I}{2a\pi}\frac{y}{\sqrt{a^2+y^2}}\Big|_0^L$$

$$\boxed{=\frac{\mu_0 IL}{2a\pi\sqrt{a^2+L^2}}}$$

b) $$\lim_{L\to\infty}B(a)=\lim_{L\to\infty}\frac{\mu_0 IL}{2a\pi\sqrt{a^2+L^2}}$$

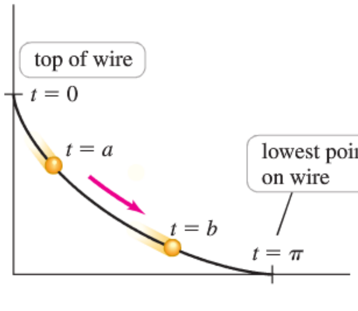
$$=\frac{\mu_0 I}{2a\pi}\lim_{L\to\infty}\frac{L}{\sqrt{a^2+L^2}}$$

$$\lim_{L\to\infty}\frac{L}{\sqrt{a^2+L^2}}=\lim_{L\to\infty}\frac{L}{\sqrt{L^2}}=1$$

$$\boxed{=\frac{\mu_0 I}{2a\pi}}$$

Exercise

The cycloid is the curve traced by a point on the rim of a rolling wheel. Imagine a wire shaped like an inverted cycloid.



A bead sliding down this wire without friction has some remarkable properties. Among all wire shapes, the cycloid is the shape that produces the fastest descent time. It can be shown that the descent time between any two points $0 \le a < b \le \pi$ on the curve is

$$descent\ time = \int_a^b \sqrt{\frac{1-\cos t}{g(\cos a - \cos t)}}\, dt$$

Where g is the acceleration due to gravity, $t = 0$ corresponds to the top of the wire, and $t = \pi$ corresponds to the lowest point on the wire.

a) Find the descent time on the interval $[a,\ b]$.

b) Show that when $b = \pi$, the descent time is the same for all values of a; that is, the descent time to the bottom of the wire is the same for all starting points.

Solution

a) $$\int_a^b \sqrt{\frac{1-\cos t}{g(\cos a - \cos t)}}\, dt = \int_a^b \sqrt{\frac{(1-\cos t)(1+\cos t)}{g(\cos a - \cos t)(1+\cos t)}}\, dt$$

$$= \frac{1}{\sqrt{g}}\int_a^b \sqrt{\frac{\left(1-\cos^2 t\right)}{\cos a + (\cos a - 1)\cos t - \cos^2 t}}\, dt$$

$$= \frac{1}{\sqrt{g}}\int_a^b \frac{\sin t}{\sqrt{\cos a + \left(\frac{\cos a - 1}{2}\right)^2 - \left(\frac{\cos a - 1}{2}\right)^2 + (\cos a - 1)\cos t - \cos^2 t}}\, dt$$

$$= \frac{1}{\sqrt{g}}\int_a^b \frac{\sin t}{\sqrt{\cos a + \left(\frac{\cos a - 1}{2}\right)^2 - \left(\left(\frac{\cos a - 1}{2}\right) - \cos t\right)^2}}\, dt$$

Let: $v = \sqrt{\cos a + \left(\frac{\cos a - 1}{2}\right)^2}$

$= \frac{1}{2}\sqrt{4\cos a + \cos^2 a - 2\cos a + 1}$

$= \frac{1}{2}(\cos a + 1)$

$$\frac{\cos a - 1}{2} - \cos t = v\sin\theta \quad \to \quad \sin t\ dt = v\cos\theta\ d\theta$$

$$\sqrt{v - \left(\left(\frac{\cos a - 1}{2}\right) - \cos t\right)^2} = v\cos\theta$$

$$= \frac{1}{\sqrt{g}}\int_a^b \frac{v\cos\theta}{v\cos\theta}\, d\theta$$

$$= \frac{1}{\sqrt{g}}\theta\, \Big|_a^b$$

$$\theta = \sin^{-1}\left(\frac{\cos a - 1 - 2\cos t}{2}\, \frac{2}{1+\cos a}\right)$$

$$= \frac{1}{\sqrt{g}} \sin^{-1}\left(\frac{\cos a - 1 - 2\cos t}{1+\cos a}\right) \Bigg|_a^b$$

$$= \frac{1}{\sqrt{g}} \left(\sin^{-1}\left(\frac{\cos a - 1 - 2\cos b}{1+\cos a}\right) - \sin^{-1}(-1)\right)$$

$$\boxed{= \frac{1}{\sqrt{g}} \left(\sin^{-1}\left(\frac{\cos a - 1 - 2\cos b}{1+\cos a}\right) + \frac{\pi}{2}\right)}$$

b) $$\frac{1}{\sqrt{g}} \left(\sin^{-1}\left(\frac{\cos a - 1 - 2\cos b}{1+\cos a}\right) + \frac{\pi}{2}\right)\Bigg|_{b=\pi} = \frac{1}{\sqrt{g}} \left(\sin^{-1}\left(\frac{\cos a - 1 + 2}{1+\cos a}\right) + \frac{\pi}{2}\right)$$

$$= \frac{1}{\sqrt{g}} \left(\sin^{-1}(1) + \frac{\pi}{2}\right)$$

$$\boxed{= \frac{\pi}{\sqrt{g}}}$$

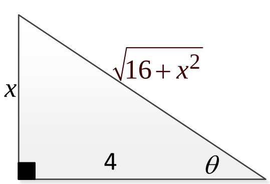
Exercise

Find the area of the region bounded by the curve $f(x) = \left(16 + x^2\right)^{-3/2}$ and the x-$axis$ on the interval $[0,\ 3]$

Solution

$$A = \int_0^3 \frac{dx}{\left(16+x^2\right)^{3/2}} \qquad x = 4\tan\theta \quad \to dx = 4\sec^2\theta\, d\theta \qquad 16 + x^2 = 16\sec^2\theta$$

$$= \int_0^3 \frac{4\sec^2\theta d\theta}{\left(16\sec^2\theta\right)^{3/2}}$$



$$= \int_0^3 \frac{4\sec^2\theta}{4^3\sec^3\theta} d\theta$$

$$= \frac{1}{16}\int_0^3 \cos\theta\, d\theta$$

$$= \frac{1}{16}\sin\theta \Big|_0^3$$

$$= \frac{1}{16} \frac{x}{\sqrt{16+x^2}} \Bigg|_0^3$$

$$= \frac{1}{16}\left(\frac{3}{5} - 0\right)$$

$$\boxed{= \frac{3}{80} \quad unit^2}$$

Exercise

Find the length of the curve $y = ax^2$ from $x = 0$ to $x = 10$, where $a > 0$ is a real number.

Solution

$$1+(y')^2 = 1+(2ax)^2$$

$$L = \int_0^{10} \sqrt{1+4a^2x^2}\ dx$$

$$= \int_0^{10} 2a\sqrt{\frac{1}{4a^2}+x^2}\ dx \qquad\qquad x=\frac{1}{2a}\tan\theta \quad \frac{1}{4a^2}+x^2=\frac{1}{4a^2}\sec^2\theta$$

$$= \int_0^{10} 2a\frac{1}{2a}\sec\theta\frac{1}{4a^2}\sec^2\theta\ d\theta \qquad\qquad dx = \frac{1}{4a^2}\sec^2\theta\ d\theta$$

$$= \frac{1}{2a}\int_0^{10} \sec^3\theta\ d\theta \qquad\qquad \begin{array}{ll} u=\sec x & dv=\sec^2 xdx \\ du=\sec x\tan xdx & v=\tan x \end{array}$$

$$\int \sec^3 xdx = \sec x\tan x - \int \tan x(\sec x\tan xdx)$$

$$= \sec x\tan x - \int \tan^2 x\sec xdx$$

$$= \sec x\tan x - \int \left(\sec^2 x - 1\right)\sec x\ dx$$

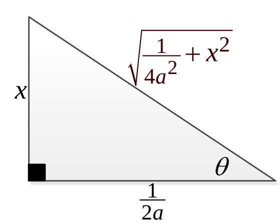
$$= \sec x\tan x - \int \sec^3 x\ dx + \int \sec x\ dx$$

$$\int \sec^3 x\ dx + \int \sec^3 x\ dx = \sec x\tan x - \int \sec^3 x\ dx + \int \sec^3 x\ dx + \int \sec x\ dx$$

$$2\int \sec^3 x\ dx = \sec x\tan x + \int \sec x\ dx$$

$$= \sec x\tan x + \ln\left|\sec x + \tan x\right|$$

$$= \frac{1}{4a}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right)\Big|_0^{10}$$



$$= \frac{1}{4a}\left(2a\sqrt{\frac{1}{4a^2}+x^2}(2ax) + \ln\left|\sqrt{1+4a^2x^2} + 2ax\right|\right)\Big|_0^{10}$$

$$= \frac{1}{4a}\left((2ax)\sqrt{1+4a^2x^2} + \ln\left|\sqrt{1+4a^2x^2} + 2ax\right|\right)\Big|_0^{10}$$

$$\boxed{= \frac{1}{4a}\left((20a)\sqrt{1+400a^2} + \ln\left|\sqrt{1+400a^2} + 20a\right|\right)}$$

Exercise

Find the arc length of the graph of $f(x)=\frac{1}{2}x^2$ from $x=0$ to $x=1$

Solution

$$1+(f')^2=1+x^2$$

$$L=\int_0^1\sqrt{1+x^2}\,dx \qquad x=\tan\theta \quad \sqrt{x^2+1}=\sec\theta \quad dx=\sec^2\theta\,d\theta$$

$$=\int_0^1\sec^3\theta\,d\theta$$

$$=\frac{1}{2}\left(\sec\theta\tan\theta+\ln\left|\sec\theta+\tan\theta\right|\right)\Big|_0^1$$

$$=\frac{1}{2}\left(x\sqrt{x^2+1}+\ln\left|x+\sqrt{x^2+1}\right|\right)\Big|_0^1$$

$$\boxed{=\frac{1}{2}\left(\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right)}$$

Exercise

A projectile is launched from the ground with an initial speed V at an angle θ from the horizontal. Assume that the x-*axis* is the horizontal ground and y is the height above the ground. Neglecting air resistance and letting g be the acceleration due to gravity, it can be shown that the trajectory of the projectile is given by

$$y=-\frac{1}{2}kx^2+y_{max} \quad where \quad k=\frac{g}{(V\cos\theta)^2}$$

$$and \qquad y_{max}=\frac{(V\sin\theta)^2}{2g}$$

a) Note that the high point of the trajectory occurs at $(0,\ y_{max})$. If the projectile is on the ground at $(-a,\ 0)$ and $(a,\ 0)$, what is a?

b) Show that the length of the trajectory (arc length) is $2\int_0^a\sqrt{1+k^2x^2}\,dx$

c) Evaluate the arc length integral and express your result in the terms of V, g, and θ.

d) For fixed value of V and g, show that the launch angle θ that maximizes the length of the trajectory satisfies $(\sin\theta)\ln(\sec\theta+\tan\theta)=1$

Solution

a) At $(\pm a,\ 0)\to\ y=0=-\frac{1}{2}ka^2+y_{max}$

$$a^2=\frac{2}{k}y_{max} \quad\Rightarrow\quad \boxed{a=\sqrt{\frac{2y_{max}}{k}}}$$

b) $y' = -kx \;\Rightarrow\; 1+(y')^2 = 1+k^2x^2$

$$L = \int_{-a}^{a} \sqrt{1+k^2x^2}\,dx$$ since $y(x)$ is an even function

$$= 2\int_0^a \sqrt{1+k^2x^2}\,dx$$

c) $L = 2\int_0^a \sqrt{1+k^2x^2}\,dx$ $\qquad x = \frac{1}{k}\tan\theta \;\Rightarrow\; dx = \frac{1}{k}\sec^2\theta\,d\theta; \quad 1+k^2x^2 = \sec^2\theta$

$$= 2\int_0^a \frac{1}{k}\sec\theta\sec^2\theta\,d\theta$$

$$= \frac{2}{k}\int_0^a \sec^3\theta\,d\theta$$

$$= \frac{1}{k}\left(\sec\theta\tan\theta + \ln\left|\sec\theta+\tan\theta\right|\right)\Big|_0^a$$

$$= \frac{1}{k}\left(\sqrt{1+k^2x^2}\,(kx) + \ln\left|\sqrt{1+k^2x^2}+kx\right|\right)\Big|_0^a$$

$$= \frac{1}{k}\left(ak\sqrt{1+k^2a^2} + \ln\left|\sqrt{1+k^2a^2}+ka\right|\right)$$

$$a = \sqrt{\frac{2}{k}\frac{(V\sin\theta)^2}{2g}} = \frac{V\sin\theta}{\sqrt{g\dfrac{g}{(V\cos\theta)^2}}} = \frac{V^2}{g}\sin\theta\cos\theta$$

$$k = \frac{g}{(V\cos\theta)^2}$$

$$ak = \tan\theta$$

$$L(\theta) = \frac{(V\cos\theta)^2}{g}\left(\tan\theta\sqrt{1+\tan^2\theta} + \ln\left|\sqrt{1+\tan^2\theta}+\tan\theta\right|\right)$$

$$= \frac{V^2\cos^2\theta}{g}\left(\tan\theta\sec\theta + \ln\left|\sec\theta+\tan\theta\right|\right)$$

$$= \frac{V^2}{g}\sin\theta + \frac{V^2}{g}\cos^2\theta\ln\left|\sec\theta+\tan\theta\right|$$

$$= \frac{V^2}{g}\left(\sin\theta + \cos^2\theta\sinh^{-1}(\tan\theta)\right)$$

d) $$L'(\theta) = \frac{V^2}{g}\left(\cos\theta - 2\cos\theta\sin\theta\sinh^{-1}(\tan\theta) + \cos^2\theta\frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\right)$$

$$= \frac{V^2}{g}\left(\cos\theta - 2\cos\theta\sin\theta\sinh^{-1}(\tan\theta) + \cos^2\theta\sec\theta\right)$$

$$= \frac{2V^2\cos\theta}{g}\left(1 - \sin\theta\sinh^{-1}(\tan\theta)\right) = 0$$

$$\sin\theta\sinh^{-1}(\tan\theta) = 1$$

$$\sin\theta\ln(\sec\theta+\tan\theta) = 1 \quad \checkmark$$

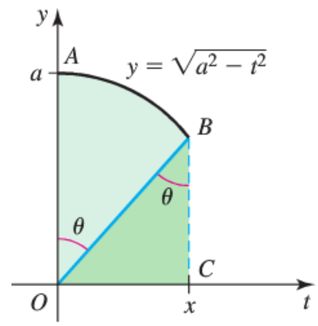
Exercise

Let $F(x)=\int_0^x \sqrt{a^2-t^2}\,dt$. The figure shows that $F(x) = \text{area of sector } OAB + \text{area of triangle } OBC$

a) Use the figure to prove that $F(x)=\dfrac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\dfrac{x\sqrt{a^2-x^2}}{2}$

b) Conclude that $\displaystyle\int \sqrt{a^2-x^2}\,dx=\dfrac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\dfrac{x\sqrt{a^2-x^2}}{2}+C$

Solution



a) Area of sector OAB is $\frac{1}{2}\theta a^2$

From the triangle OBC: $\sin\theta=\dfrac{x}{a} \quad \rightarrow \quad \theta=\sin^{-1}\dfrac{x}{a}$

$$|BC|=\sqrt{a^2-x^2}$$

Area of sector OAB is $\frac{1}{2}a^2\sin^{-1}\dfrac{x}{a}$

Area of triangle OBC: $\frac{1}{2}x\sqrt{a^2-x^2}$

$F(x) = \text{area of sector } OAB + \text{area of triangle } OBC$

$$=\frac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\frac{x\sqrt{a^2-x^2}}{2}$$

b) $$\frac{d}{dx}\left(\frac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\frac{x\sqrt{a^2-x^2}}{2}+C\right)=\frac{a^2}{2}\frac{\frac{1}{a}}{\sqrt{1-\left(\frac{x}{a}\right)^2}}+\frac{1}{2}\sqrt{a^2-x^2}-\frac{1}{2}\frac{x^2}{\sqrt{a^2-x^2}}$$

$$=\frac{1}{2}\frac{a^2}{\sqrt{a^2-x^2}}+\frac{1}{2}\frac{a^2-2x^2}{\sqrt{a^2-x^2}}$$

$$=\frac{1}{2}\frac{2a^2-2x^2}{\sqrt{a^2-x^2}}$$

$$=\frac{a^2-x^2}{\sqrt{a^2-x^2}}$$

$$=\sqrt{a^2-x^2}$$

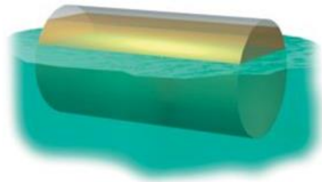
By the antiderivative:

$$\int \sqrt{a^2-x^2}\,dx=\frac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\frac{x\sqrt{a^2-x^2}}{2}+C \quad \checkmark$$

Exercise

A sealed barrel of oil (weighing 48 pounds per cubic foot) is floating in seawater (weighing 64 pounds per cubic foot). The barrel is not completely full of oil. With the barrel lying on its side, the top 0.2 *foot* of the barrel is empty.

Compare the fluid forces against one end of the barrel from the inside and from the outside.



Solution

$$x^2+y^2=1 \quad \to \quad 2x = 2\sqrt{1-y^2}$$

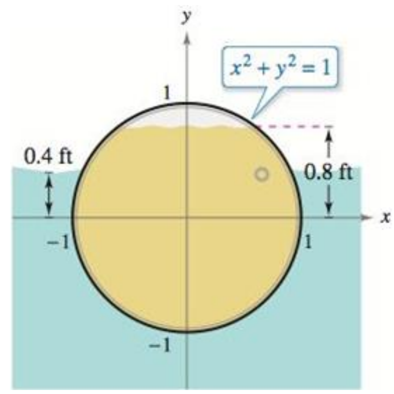
$$F_{inside} = 48\int_{-1}^{0.8}(0.8-y)(2)\sqrt{1-y^2}\,dy \qquad\qquad F = w\int_c^d h(y)L(y)\,dy$$

$$= 76.8\int_{-1}^{0.8}\sqrt{1-y^2}\,dy - 96\int_{-1}^{0.8} y\sqrt{1-y^2}\,dy$$

$$= 76.8\int_{-1}^{0.8}\sqrt{1-y^2}\,dy + 48\int_{-1}^{0.8}\left(1-y^2\right)^{1/2}\,d\left(1-y^2\right) \qquad y=\sin\theta \quad \sqrt{1-y^2}=\cos\theta \quad dy=\cos\theta\, d\theta$$

$$= 76.8\int_{-1}^{0.8}\cos^2\theta\,d\theta + 32\left(1-y^2\right)^{3/2}\Big|_{-1}^{0.8}$$

$$= 38.4\int_{-1}^{0.8}(1+\cos 2\theta)\,d\theta + 32(0.16)^{3/2}$$



$$= 38.4\left(\theta+\frac{1}{2}\sin 2\theta\right)\Big|_{-1}^{0.8} + 32(0.4)^3$$

$$= 38.4\left(\arcsin y + y\sqrt{1-y^2}\right)\Big|_{-1}^{0.8} + 2.048$$

$$= 38.4\left(\arcsin 0.8 + 0.32 + \frac{\pi}{2}\right) + 2.048$$

$$\boxed{\approx 121.3 \ \ lbs}$$

$$F_{outside} = 64\int_{-1}^{0.4}(0.4-y)(2)\sqrt{1-y^2}\,dy \qquad\qquad F = w\int_c^d h(y)L(y)\,dy$$

$$= 51.2\int_{-1}^{0.4}\sqrt{1-y^2}\,dy - 128\int_{-1}^{0.4} y\sqrt{1-y^2}\,dy$$

$$= 25.6\left(\arcsin y + y\sqrt{1-y^2}\right)\Big|_{-1}^{0.4} + \frac{128}{3}\left(1-y^2\right)^{3/2}\Big|_{-1}^{0.4}$$

$$\boxed{\approx 93.0 \ \ lbs}$$

Exercise

The axis of a storage tank in the form of a right circular cylinder is horizontal. The radius and length of the tank are 1 *meter* and 3 *meters*, respectively.

a) Determine the volume of fluid in the tank as a function of its depth *d*.

b) Graph the function in part (a).

c) Design a dip stick for the tank with markings of $\frac{1}{4}, \frac{1}{2}, \text{ and } \frac{3}{4}$

d) Fluid is entering the tank at a rate of $\frac{1}{4}\ m^3/min$. Determine the rate of change of the depth of the fluid as a function of its depth *d*.

e) Graph the function in part (*d*).\When will the rate of change of the depth be minimum?

Solution

a) Consider the center at $(0,\ 1)$: $x^2+(y-1)^2=1 \ \rightarrow x=\sqrt{1-(y-1)^2}$

The depth: $0 \le d \le 2$

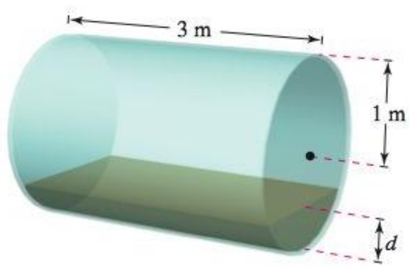
$$V=\int_0^d (3)\left(2\sqrt{1-(y-1)^2}\right)dy$$

$$=6\int_0^d \sqrt{1-(y-1)^2}\ d(y-1)$$

$$y-1=\sin\theta \qquad \sqrt{1-(y-1)^2}=\cos\theta$$

$$d(y-1)=\cos\theta d\theta$$

$$=6\int_0^d \cos^2\theta\ d\theta$$



$$=3\int_0^d (1+\cos 2\theta)\ d\theta$$

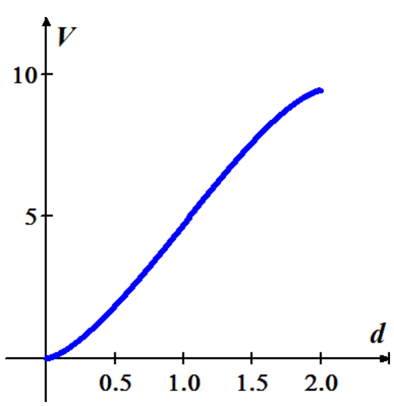
$$=3\left(\theta+\frac{1}{2}\sin 2\theta\right)_0^d$$

$$=3(\theta+\sin\theta\cos\theta)\Big|_0^d$$

$$=3\left(\arcsin(y-1)+(y-1)\sqrt{1-(y-1)^2}\right)_0^d$$

$$\boxed{=3\arcsin(d-1)+3(d-1)\sqrt{2d-d^2}+\frac{3\pi}{2}}$$

b)



c) The full tank holds $3\pi\ m^3$

A dip stick for the tank with markings of $\frac{1}{4},\ \frac{1}{2},\ and\ \frac{3}{4}$

The horizontal lines are: $y=\frac{3\pi}{4},\quad y=\frac{3\pi}{2},\quad y=\frac{9\pi}{4}$

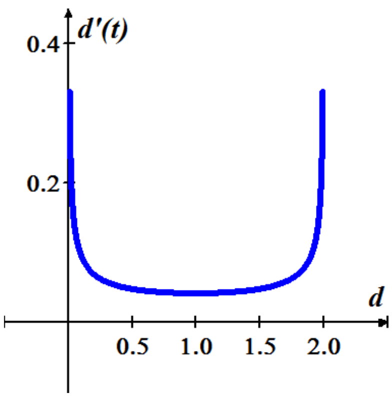
Intersect the curve at $\underline{d=0.596,\quad d=1.0,\quad d=1.404}$

d) $V=6\int_0^d \sqrt{1-(y-1)^2}\,dy \quad\to\quad \frac{dV}{dt}=\frac{dV}{dd}\frac{dd}{dt}$

$$\frac{dV}{dt}=6\sqrt{1-(d-1)^2}\cdot d'(t)=\frac{1}{4}$$

$$\boxed{d'(t)=\frac{1}{24\sqrt{1-(d-1)^2}}}$$

e)



From the graph, the minimum occurs at $d=1$, which is the widest part of the tank.

Exercise

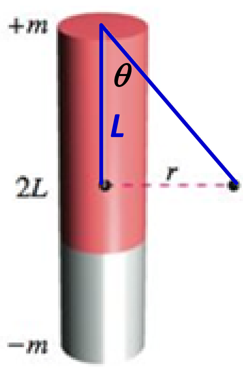
The field strength H of a magnet of length $2L$ on a particle r units from the center of the magnet is

$$H=\frac{2mL}{\left(r^2+L^2\right)^{3/2}}$$

Where $\pm m$ are the poles of the magnet.

Find the average field strength as the particle moves from 0 to R units from the center by evaluating the integral

$$\frac{1}{R}\int_0^R \frac{2mL}{\left(r^2+L^2\right)^{3/2}}dr$$



Solution

$r=L\tan\theta \quad\to\quad dr=L\sec^2\theta d\theta$

$r^2+L^2=L^2\tan^2\theta+L^2\ \underline{=L^2\sec^2\theta}$

$$\frac{1}{R}\int_0^R \frac{2mL}{\left(r^2+L^2\right)^{3/2}}\,dr = \frac{1}{R}\int_0^R \frac{2mL}{(L\sec\theta)^3}\,L\sec^2\theta\; d\theta$$

$$= \frac{2m}{RL}\int_0^R \frac{1}{\sec\theta}\; d\theta$$

$$= \frac{2m}{RL}\int_0^R \cos\theta\; d\theta$$

$$= \frac{2m}{RL}\sin\theta\Big|_0^R$$

$$= \frac{2m}{RL}\frac{r}{\sqrt{r^2+L^2}}\Bigg|_0^R$$

$$\underline{= \frac{2m}{L\,\sqrt{R^2+L^2}}\Big|}$$