7 ADDITIONAL INTEGRATION TOPICS

EXERCISE 7-1

2. $A = \int_a^b f(x)dx$

4. $A = \int_0^b [-F(x)]dx$

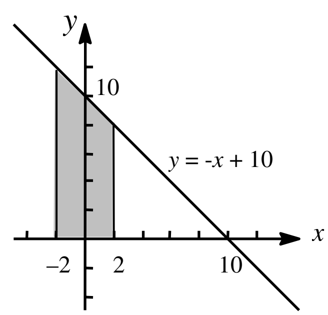
6. The area of the shaded region in Figure (C) is the same as the area of the region between the curve $y = -h(x)$ and the x-axis from $x = a$ to $x = b$ (the mirror image of the shaded region with respect to the x-axis). But the latter region is above the x-axis, and its area is given by $\int_a^b [-h(x)]dx$.

8. Area bounded by $y = -x + 10$; $y = 0$; $-2 \le x \le 2$ is given by

$$\int_{-2}^{2} (-x + 10)dx = \left[-\frac{x^2}{2} + 10x\right]_{-2}^{2}$$

$$= \left[-\frac{(2)^2}{2} + 10(2)\right] - \left[-\frac{(-2)^2}{2} + 10(-2)\right]$$

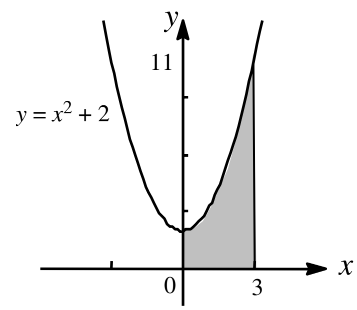
$$= [-2 + 20] - [-2 - 20] = 40$$



10. Area bounded by $y = x^2 + 2$; $y = 0$; $0 \le x \le 3$ is given by

$$\int_0^3 (x^2 + 2)dx = \left[\frac{x^3}{3} + 2x\right]_0^3$$

$$= \frac{27}{3} + 6 = 15$$

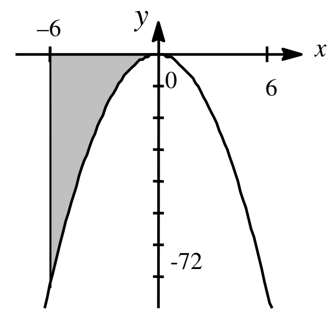


12. Area bounded by $y = -2x^2$; $y = 0$; $-6 \le x \le 0$ is given by

$$\int_{-6}^{0} -(-2x^2)dx = \int_{-6}^{0} 2x^2\, dx$$

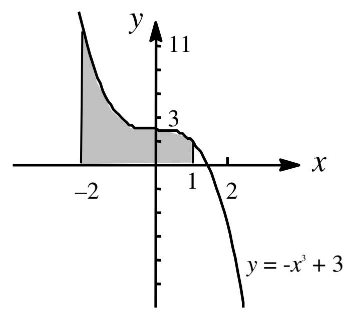
$$= \left[\frac{2x^3}{3}\right]_{-6}^{0}$$

$$= -\frac{2(-6)^3}{3} = 144$$



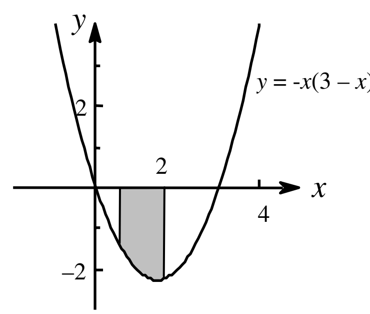
14. Area bounded by $y = -x^3 + 3$; $y = 0$; $-2 \le x \le 1$ is given by

$$\int_{-2}^{1} (-x^3 + 3)dx = \left[-\frac{x^4}{4} + 3x\right]_{-2}^{1}$$
$$= \left[-\frac{(1)^4}{4} + 3(1)\right] - \left[-\frac{(-2)^4}{4} + 3(-2)\right]$$
$$= \left[-\frac{1}{4} + 3\right] - [-4 - 6]$$
$$= 12.75$$



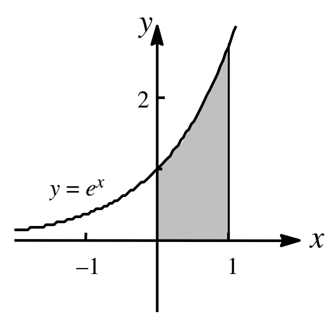
16. Area bounded by $y = -x(3 - x)$; $y = 0$; $1 \le x \le 2$ is given by

$$\int_{1}^{2} -[-x(3 - x)]dx = \int_{1}^{2} (3x - x^2)dx$$
$$= \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_{1}^{2}$$
$$= \left[\frac{3}{2}(2)^2 - \frac{1}{3}(2)^3\right] - \left[\frac{3}{2}(1)^2 - \frac{1}{3}(1)^3\right]$$
$$= \left[6 - \frac{8}{3}\right] - \left[\frac{3}{2} - \frac{1}{3}\right]$$
$$= 2.167$$



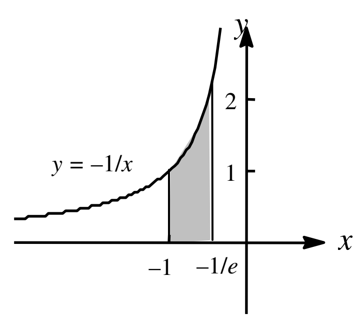
18. Area bounded by $y = e^x$; $y = 0$; $0 \le x \le 1$ is given by

$$\int_{0}^{1} e^x\,dx = [e^x]_{0}^{1} = e - 1$$
$$= 1.718$$

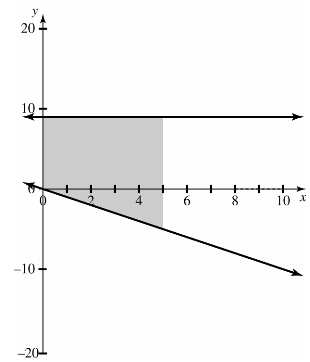


20. Area bounded by $y = -\frac{1}{x}$; $y = 0$; $-1 \le x \le -\frac{1}{e}$ is given by

$$\int_{-1}^{-1/e} \left(-\frac{1}{x}\right)dx = [-\ln|x|]_{-1}^{-1/e}$$
$$= -\ln\left(\frac{1}{e}\right) + \ln 1$$
$$= -\ln e^{-1} = \ln e = 1$$



22. Area bounded by $y = -x$; $y = 8$; $0 \le x \le 5$is given by



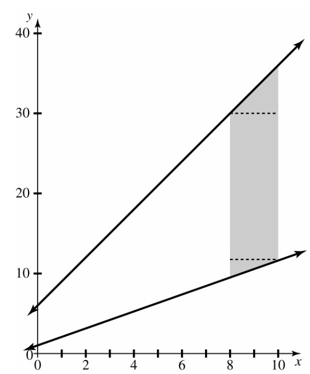
Definite integral

$$\int_0^5 \left(8-(-x)\right)dx = \int_0^5 (x+8)\,dx = \left[\frac{x^2}{2}+8x\right]_0^5$$
$$= \left(\frac{(5)^2}{2}+8(5)\right)-\left(\frac{(0)^2}{2}+8(0)\right)$$
$$= 52.5$$

Geometry

$$A_{\text{Upper Rectangle}} + A_{\text{Lower Triangle}} = (5)(8)+\frac{1}{2}(5)(5) = 52.5$$

24. Area bounded by $y = 3x + 6$; $y = \frac{1}{2}x+1$; $8 \le x \le 10$ is given by



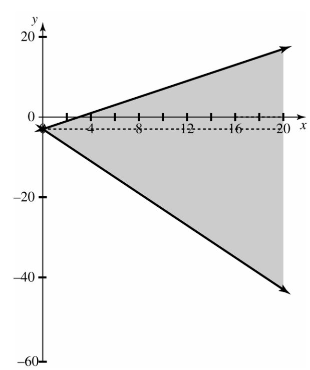
Definite integral

$$\int_8^{10}\left((3x+6)-\left(\frac{1}{2}x+1\right)\right)dx = \int_8^{10}\left(\left(\frac{5}{2}x+5\right)\right)dx = \left[\frac{5x^2}{4}+5x\right]_8^{10}$$
$$= \left(\frac{5(10)^2}{4}+5(10)\right)-\left(\frac{5(8)^2}{4}+5(8)\right) = 55$$

Geometry

$$A_{\text{Upper Triangle}} + A_{\text{Middle Rectangle}} + A_{\text{Lower Triangle}} = \frac{1}{2}(2)(6)+(2)(24)+\frac{1}{2}(2)(1) = 55$$

26. Area bounded by $y = -2x - 3$; $y = x - 3$; $0 \le x \le 20$ is given by



Definite integral

$$\int_0^{20}\left((x-3)-(-2x-3)\right)dx = \int_0^{20}(3x)\,dx = \left[\frac{3x^2}{2}\right]_0^{20} = \left(\frac{3(20)^2}{2}\right)-\left(\frac{3(0)^2}{2}\right) = 600$$

Geometry

$$A_{\text{Upper Triangle}} + A_{\text{Lower Triangle}} = \frac{1}{2}(20)(20)+\frac{1}{2}(20)(40) = 600$$

28. $A = \int_c^d [-f(x)]dx$

30. $A = \int_a^b [-f(x)]dx + \int_b^c f(x)dx$

32. $A = \int_a^b [f(x) - g(x)]dx$

34. $A = \int_b^c [g(x) - f(x)]dx + \int_c^d [f(x) - g(x)]dx$

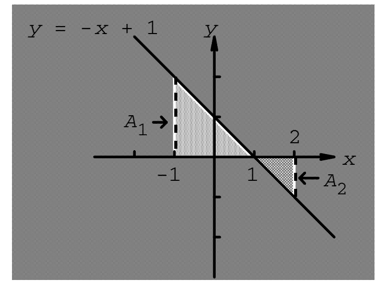
36. Find the x-intercepts b and c by solving $f(x) = 0$. Then observe that $f(x) \le 0$ on $[a, b]$, $f(x) \ge 0$ on $[b, c]$, and $f(x) \le 0$ on $[c, d]$.

Thus,

$$\text{Area} = \int_a^b [-f(x)]dx + \int_b^c f(x)dx + \int_c^d [-f(x)]dx$$

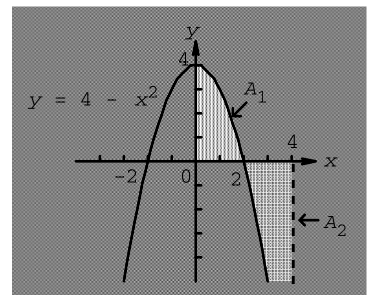
38. $A = A_1 + A_2$

$$\begin{aligned} &= \int_{-1}^{1} (-x+1)dx + \int_1^2 -(-x+1)dx \\ &= \left(-\frac{1}{2}x^2 + x\right)\Bigg|_{-1}^{1} + \left(\frac{1}{2}x^2 - x\right)\Bigg|_1^2 \\ &= \left(-\frac{1}{2}+1\right) - \left(-\frac{1}{2}-1\right) + (2-2) - \left(\frac{1}{2}-1\right) \\ &= 2.5 \end{aligned}$$



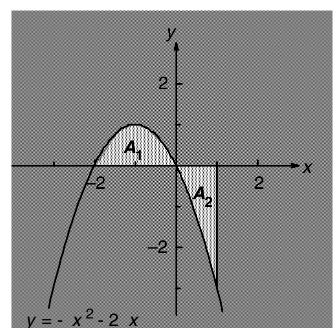
40. $A = A_1 + A_2$

$$\begin{aligned} &= \int_0^2 (4-x^2)dx + \int_2^4 -(4-x^2)dx \\ &= \left(4x - \frac{1}{3}x^3\right)\Bigg|_0^2 + \left(-4x + \frac{1}{3}x^3\right)\Bigg|_2^4 \\ &= \left(8 - \frac{8}{3}\right) + \left(-16 + \frac{64}{3}\right) - \left(-8 + \frac{8}{3}\right) \\ &= 16 \end{aligned}$$



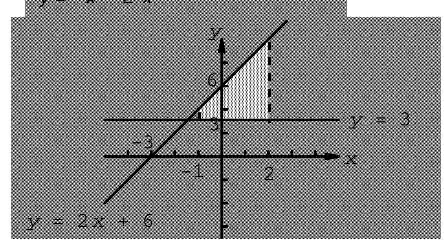
42. $A = A_1 + A_2$

$$\begin{aligned} &= \int_{-2}^{0} (-x^2 - 2x)dx + \int_0^1 -(-x^2 - 2x)dx \\ &= \left(-\frac{1}{3}x^3 - x^2\right)\Bigg|_{-2}^{0} + \left(\frac{1}{3}x^3 + x^2\right)\Bigg|_0^1 \\ &= -\left(\frac{8}{3} - 4\right) + \left(\frac{1}{3} + 1\right) = -\frac{8}{3} + 4 + \frac{1}{3} + 1 = \frac{8}{3} \\ &\approx 2.667 \end{aligned}$$



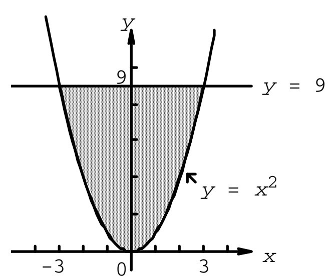
44. $A = \int_{-1}^{2} [(2x+6) - 3]dx$

$$\begin{aligned} &= \int_{-1}^{2} (2x+3)dx \\ &= (x^2 + 3x)\Big|_{-1}^{2} \\ &= 10 - (-2) = 12 \end{aligned}$$



46. $A = \int_{-3}^{3} [9 - x^2]dx$

$$\begin{aligned} &= \left(9x - \frac{1}{3}x^3\right)\Bigg|_{-3}^{3} \\ &= (18) - (-18) \\ &= 36 \end{aligned}$$

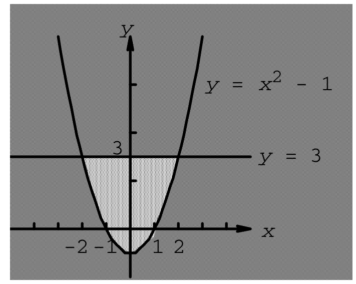


48. $A = \int_{-2}^{2} [3 - (x^2 - 1)]dx$

$= \int_{-2}^{2} (4 - x^2)dx \qquad = \left(4x - \frac{1}{3}x^3\right)\Big|_{-2}^{2}$

$= \left(\frac{16}{3}\right) - \left(-\frac{16}{3}\right)$

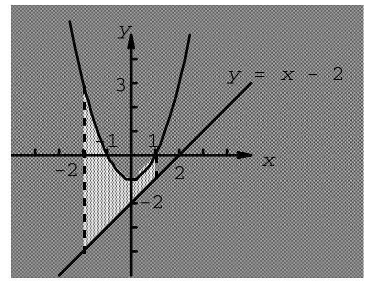
$= \frac{32}{3} \approx 10.667$



50. $A = \int_{-2}^{1} [(x^2 - 1) - (x - 2)]dx$

$= \int_{-2}^{1} (x^2 - x + 1)dx \qquad = \left(\frac{1}{3}x^3 - \frac{1}{2}x^2 + x\right)\Big|_{-2}^{1}$

$= \left(\frac{5}{6}\right) - \left(-\frac{20}{3}\right) = \frac{45}{6} = 7.5$

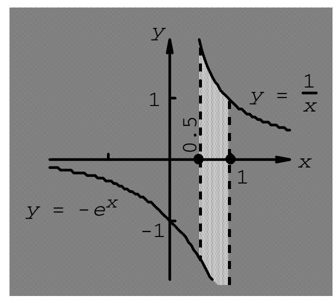


52. $A = \int_{0.5}^{1} \left[\frac{1}{x} - (-e^x)\right]dx$

$= \int_{0.5}^{1} \left(\frac{1}{x} + e^x\right)dx$

$= (\ln|x| + e^x)\Big|_{0.5}^{1}$

$= e^1 - (\ln(0.5) + e^{0.5}) \approx 1.763$



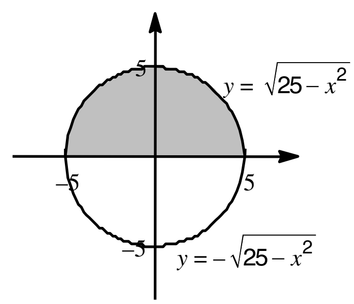
54. Area bounded by the graphs of $y = \sqrt{25 - x^2}$; $y = 0$; $-5 \le x \le 5$ is represented by the definite integral

$\int_{-5}^{5} \sqrt{25 - x^2}\, dx$ = area of the upper section of the circle with equation

$(x^2 + y^2 = 25) = \frac{1}{2}(\pi 5^2)$

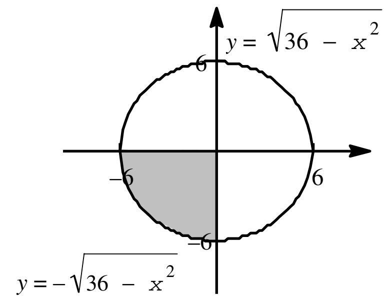
$= 12.5\pi$

$= 39.270$



56. Area bounded by the graphs of $y = -\sqrt{36 - x^2}$; $y = 0$; $-6 \le x \le 0$ is represented by the definite integral

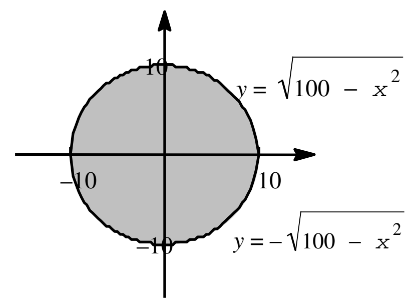
$\int_{-6}^{0} \sqrt{36 - x^2}\, dx$ = area of the section of the circle with equation $x^2 + y^2 = 36$ located in the third quadrant $= \frac{1}{4}(\pi(6)^2) = 9\pi = 28.274$.



58. Area bounded by $y = -\sqrt{100 - x^2}$; $y = \sqrt{100 - x^2}$; $-10 \le x \le 10$ is represented by the definite integral

$\int_{-10}^{10} 2\sqrt{100 - x^2}\, dx$ = area of the circle with equation

$(x^2 + y^2 = 100) = 100\pi = 314.159$.



60. The graphs of $y = 3 - 2x^2$ and $y = 2x^2 - 4x$ are shown at the right.
The x-coordinates of the points of intersection are:

$x_1 = -0.5$,

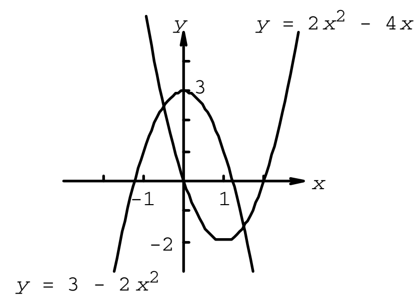
$x_2 = 1.5$.

$$A = \int_{-0.5}^{1.5} [(3 - 2x^2) - (2x^2 - 4x)]dx$$

$$= \int_{-0.5}^{1.5} (3 - 4x^2 + 4x)dx$$

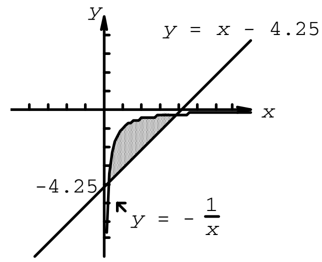
$$= \left(3x - \frac{4}{3}x^3 + 2x^2\right)\Bigg|_{-0.5}^{1.5}$$

$$= \left(3(1.5) - \frac{4}{3}(1.5)^3 + 2(1.5)^2\right) - \left(3(-0.5) - \frac{4}{3}(-0.5)^3 + 2(-0.5)^2\right) \approx 5.333$$



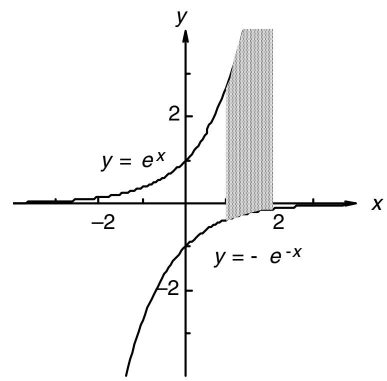
62. The graphs of $y = x - 4.25$ and $y = -\frac{1}{x}$ are shown below.
The x-coordinates of the points of intersection are: $x_1 = 0.25$, $x_2 = 4$.

$$A = \int_{0.25}^{4} \left[-\frac{1}{x} - (x - 4.25) \right] dx$$
$$= \left(-\ln|x| - \frac{1}{2}x^2 + 4.25x \right) \Bigg|_{0.25}^{4}$$
$$\approx 5.196$$



64. The graphs of $y = e^x$ and $y = -e^{-x}$ are shown at the right.

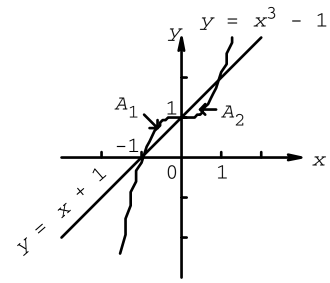
$$A = \int_{1}^{2} (e^x + e^{-x})dx$$
$$= (e^x - e^{-x})\Big|_{1}^{2}$$
$$= (e^2 - e^{-2}) - (e - e^{-1})$$
$$= \frac{e^4 - e^3 + e - 1}{e^2} = 4.903$$



66. The graphs are given at the right. To find the points of intersection, solve:

$x^3 + 1 \quad = x + 1$
$x^3 - x \quad = 0$ or $x(x^2 - 1) = 0$ or
$x(x - 1)(x + 1) = 0$
$\qquad x = -1, 0, 1$

Thus, the points of intersection are $(-1, 0)$, $(0, 1)$, and $(1, 2)$.

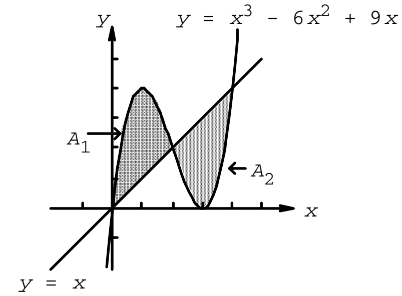


$$A = A_1 + A_2$$
$$= \int_{-1}^{0} [(x^3 + 1) - (x + 1)]dx + \int_{0}^{1} [(x + 1) - (x^3 + 1)]dx$$
$$= \int_{-1}^{0} (x^3 - x)dx + \int_{0}^{1} (x - x^3)dx$$
$$= \left(\frac{x^4}{4} - \frac{x^2}{2} \right) \Bigg|_{-1}^{0} + \left(\frac{x^2}{2} - \frac{x^4}{4} \right) \Bigg|_{0}^{1}$$
$$= 0.5$$

68. The graphs are given at the right.
To find the points of intersection, solve:

$$\begin{aligned} x^3 - 6x^2 + 9x &= x \\ x^3 - 6x^2 + 8x &= 0 \\ x(x^2 - 6x + 8) &= 0 \\ x(x-2)(x-4) &= 0 \\ x &= 0,\ x = 2,\ x = 4 \end{aligned}$$

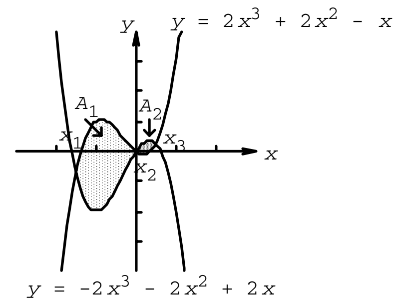
Thus, (0, 0), (2, 2), (4, 4) are the points of intersection.



$$\begin{aligned} A &= A_1 + A_2 \\ &= \int_0^2 [(x^3 - 6x^2 + 9x - (x)]dx + \int_2^4 [(x) - (x^3 - 6x^2 + 9x)]dx \\ &= \int_0^2 (x^3 - 6x^2 + 8x)dx + \int_2^4 (-x^3 + 6x^2 - 8x)dx \\ &= \left(\frac{x^4}{4} - 2x^3 + 4x^2\right)\Big|_0^2 + \left(-\frac{x^4}{4} + 2x^3 - 4x^2\right)\Big|_2^4 \\ &= (4) - (0) + (0) - (-4) = 8 \end{aligned}$$

70. The graphs are given below. The x-coordinates of the points of intersection are: $x_1 = -1.5$, $x_2 = 0$, $x_3 = 0.5$.

$$\begin{aligned} A &= A_1 + A_2 \\ &= \int_{-1.5}^0 [(2x^3 + 2x^2 - x) - (-2x^3 - 2x^2 + 2x)]dx + \int_0^{0.5} [(-2x^3 - 2x^2 + 2x) - (2x^3 + 2x^2 - x)]dx \\ &= \int_{-1.5}^0 (4x^3 + 4x^2 - 3x)dx + \int_0^{0.5} (-4x^3 - 4x^2 + 3x)dx \\ &= \left(x^4 + \frac{4x^3}{3} - \frac{3x^2}{2}\right)\Big|_{-1.5}^0 + \left(-x^4 - \frac{4x^3}{3} + \frac{3x^2}{2}\right)\Big|_0^{0.5} \\ &= 2.8125 + 0.1458\overline{3} \approx 2.958 \end{aligned}$$

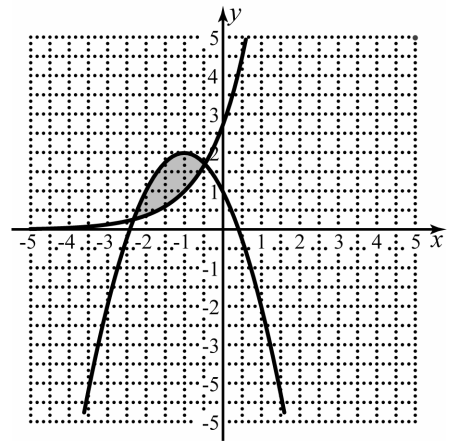


72. The graphs are given at the right. The x-coordinates of the points of intersection are: $x_1 \approx -2.316$, $x_2 \approx -0.463$.

$$A = \int_{-2.316}^{-0.463} [2 - (x+1)^2 - e^{x+1}]dx$$

$$= \left(2x - \frac{(x+1)^3}{3} - e^{x+1}\right)\Bigg|_{-2.316}^{-0.463}$$

$$\approx 1.452$$

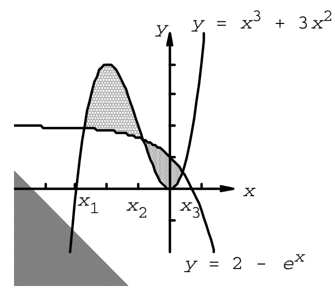


74. The graphs are given at the right. The x-coordinates of the points of intersection are: $x_1 \approx -2.743$, $x_2 \approx -0.851$, $x_3 \approx 0.392$.

$$A = A_1 + A_2$$

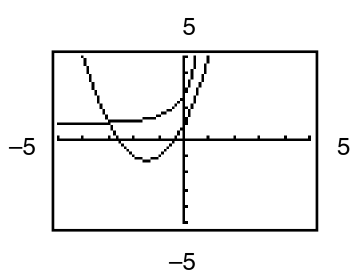
$$= \int_{-2.743}^{-0.851} [(x^3 + 3x^2) - (2 - e^x)]dx + \int_{-0.851}^{0.392} [(2 - e^x) - (x^3 + 3x^2)]dx$$

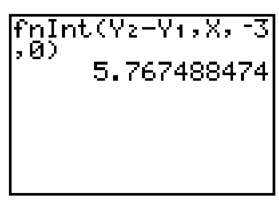
$$\approx 2.579 + 0.882 = 3.461$$



76. $y = x^2 + 3x + 1$; $y = e^{e^x}$; $-3 \le x \le 0$

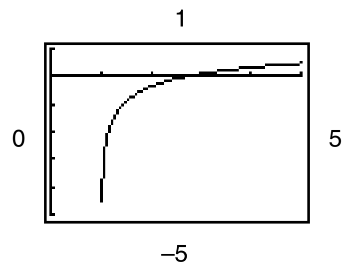
The graphs of $y_1 = x^2 + 3x + 1$ and $y_2 = e^{e^x}$ are:

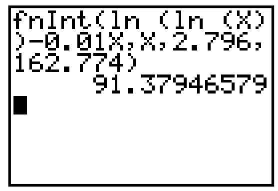




Thus, $A = \int_{-3}^{0} [e^{e^x} - (x^2 + 3x + 1)]dx \approx 5.767$

78. $y = \ln(\ln x)$; $y = 0.01x$
The graphs of $y_1 = \ln(\ln x)$ and $y_2 = 0.01x$ are:





The intersection points are: $x_1 = 2.796$ and $x_2 = 162.774$.

Thus, $A = \int_{2.796}^{162.772} [\ln(\ln x) - 0.01x]dx \approx 91.379$

80. $\int_5^{15} R(t)dt = \int_5^{15} \left(\frac{100t}{t^2+25} + 4\right) dt = 100\int_5^{15} \frac{t}{t^2+25}\, dt + \int_5^{15} 4\, dt$

Let $u = t^2 + 25$, then $du = 2t\, dt$

$$\int \frac{t}{t^2+25}\, dt = \frac{1}{2}\int \frac{2t}{t^2+25}\, dt = \frac{1}{2}\int \frac{1}{u}\, du = \frac{1}{2}\ln|u| = \frac{1}{2}\ln(t^2 + 25)$$

Thus,

$$\begin{aligned}\int_5^{15} R(t)dt &= 50(\ln(t^2 + 25))\Big|_5^{15} + (4t)\Big|_5^{15} \\ &= 50(\ln(250) - \ln(50)) + 4(15 - 5) \\ &= 50 \ln\left(\frac{250}{50}\right) + 40 \\ &= 50 \ln(5) + 40 \approx 120\end{aligned}$$

The total production from the end of the fifth year to the end of the fifteenth year is (approximately) 120 thousand barrels.

82. To find the useful life, set $R'(t) = C'(t)$ and solve for t:

$$\begin{aligned}5te^{-0.1t^2} &= 2t \\ e^{-0.1t^2} &= \frac{2}{5} = 0.4 \\ -0.1t^2 &= \ln(0.4) \\ t^2 &= -\frac{\ln(0.4)}{0.1} \\ t &\approx 3 \text{ years}\end{aligned}$$

$$\begin{aligned}\int_0^3 [R'(t) - C'(t)]dt &= \int_0^3 [5te^{-0.1t^2} - 2t]dt \\ &= 5\int_0^3 te^{-0.1t^2}\, dt - 2\int_0^3 t\, dt\end{aligned}$$

Let $u = -0.1t^2$, then $du = -0.2t\, dt$ and

$$\int te^{-0.1t^2}\,dt = \frac{1}{-0.2}\int e^{-0.1t^2}(-0.2t)dt$$

$$= -\frac{1}{0.2}\int e^u\,du = -\frac{1}{0.2}e^u$$

$$= -\frac{1}{0.2}e^{-0.1t^2}$$

Therefore,

$$\int_0^3 [R'(t) - C'(t)]dt = -\frac{5}{0.2}(e^{-0.1t^2})\Big|_0^3 + (-t^2)\Big|_0^3$$

$$= 25(1 - e^{-0.9}) - 9$$

$$= 16 - 25e^{-0.9} \approx 5.836$$

The total profit over the useful life of the game is approximately \$5,836.

84. For 1962: $f(x) = \frac{3}{10}x + \frac{7}{10}x^2$

Index of Income Concentration $= 2\int_0^1 [x - f(x)]dx = 2\int_0^1 \left(x - \frac{3}{10}x - \frac{7}{10}x^2\right)dx$

$$= 2\int_0^1 \left(\frac{7}{10}x - \frac{7}{10}x^2\right)dx$$

$$= \frac{7}{5}\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1$$

$$= \frac{7}{5}\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{7}{30} \approx 0.233$$

For 1972: $g(x) = \frac{1}{2}x + \frac{1}{2}x^2$

Index of Income Concentration $= 2\int_0^1 [x - g(x)]dx = 2\int_0^1 \left(x - \frac{1}{2}x - \frac{1}{2}x^2\right)dx$

$$= 2\int_0^1 \left(\frac{1}{2}x - \frac{1}{2}x^2\right)dx$$

$$= \int_0^1 (x - x^2)dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \approx 0.167$$

Interpretation: Income was more equally distributed in 1972.

86. For current Lorenz curve: $f(x) = x^{2.3}$

Index of Income Concentration $= 2\int_0^1 [x - f(x)]dx = 2\int_0^1 (x - x^{2.3})dx$

$$= 2\left(\frac{x^2}{2} - \frac{x^{3.3}}{3.3}\right)\Big|_0^1$$

$$= 2\left(\frac{1}{2} - \frac{1}{3.3}\right)$$

$$= \frac{2.6}{6.6} \approx 0.394$$

For projected Lorenz curve: $g(x) = 0.4x + 0.6x^2$

Index of Income Concentration $= 2\int_0^1 [x - g(x)]dx$

$$= 2\int_0^1 (x - 0.4x - 0.6x^2)dx$$

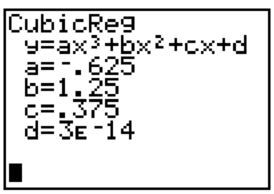
$$= 2\int_0^1 (0.6x - 0.6x^2)dx$$

$$= 1.2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1$$

$$= 1.2\left(\frac{1}{2} - \frac{1}{3}\right) = 0.2$$

Interpretation: Yes, income will be more equally distributed after the changes in the tax laws.

88. (A)



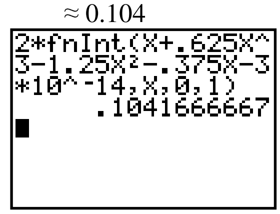
Lorenz curve:

$y = -0.625x^3 + 1.25x^2 + 0.375x + 3 \times 10^{-14}$.

(B) Index of income concentration:

$2\int_0^1 [x - f(x)]dx$

≈ 0.104



90. Area $= \int_{15}^{20} (12 + 0.006t^2)dt$

$$= (12t + 0.002t^3)\Big|_{15}^{20}$$

$$= (240 + 16) - (180 + 6.75) = 69.25$$

The total demand for wood from 1985 to 1990 is 69.25 billion cubic feet.

92. $V = \int_1^4 \frac{13}{t^{1/2}}dt = 13\int_1^4 t^{-1/2}dt = 13\left(\frac{t^{1/2}}{\frac{1}{2}}\right)\Big|_1^4$

$$= 26(2 - 1) = 26$$

Average number of words learned during the 2nd, 3rd, and 4th hours is 26.

EXERCISE 7-2

2. $\int_1^3 e^{-t}\,dt = [-e^{-t}]_1^3 = -e^{-3} - (-e^{-1}) = e^{-1} - e^{-3} = 0.32$

4. $\int_0^1 e^{3(1-t)}\,dt = \left[-\frac{1}{3}e^{3(1-t)}\right]_0^1$
$= -\frac{1}{3}e^0 - \left(-\frac{1}{3}e^3\right)$
$= \frac{1}{3}e^3 - \frac{1}{3}$
$= \frac{1}{3}(e^3 - 1) = 6.36$

6. $\int_1^{10} e^{0.07(10-t)}dt$ $= \left[-\frac{1}{0.07}e^{0.07(10-t)}\right]_1^{10}$
$= -\frac{1}{0.07}e^0 - \left(-\frac{1}{0.07}e^{0.07(9)}\right)$
$= \frac{1}{0.07}(e^{0.63} - 1) = 12.54$

8. $\int_0^{15} e^{0.05t}e^{0.06(15-t)}\,dt$ $= \int_0^{15} e^{0.05t+0.9-0.06t}\,dt$
$= e^{0.9}\int_0^{15} e^{-0.01t}\,dt = e^{0.9}\left[-\frac{1}{0.01}e^{-0.01t}\right]_0^{15}$
$= e^{0.9}\left[-\frac{1}{0.01}e^{-0.15} - \left(-\frac{1}{0.01}e^0\right)\right]$
$= \frac{1}{0.01}(e^{0.9} - e^{0.75}) = 34.26$

10. $\int_0^{25} 900e^{0.03t}e^{0.04(25-t)}\,dt = \int_0^{25} 900e^{0.03t+1-0.04t}\,dt$
$= 900e\int_0^{25} e^{-0.01t}\,dt = 900e\left[-\frac{1}{0.01}e^{-0.01t}\right]_0^{25}$
$= 900e\left[-\frac{1}{0.01}e^{-0.25} - \left(-\frac{1}{0.01}e^0\right)\right] = \frac{900e}{0.01}(1 - e^{-0.25}) = 54{,}115.36$

12. (A) and (B) are equal:

$\int_0^{10} 2{,}000e^{0.05t}e^{0.12(10-t)}dt$ $= 2{,}000\int_0^{10} e^{0.05t}e^{1.2}e^{-0.12t}\,dt$
$= 2{,}000e^{1.2}\int_0^{10} e^{-0.07t}\,dt$

So, (A) and (B) are the same.

(A) = (B) $= 2{,}000e^{1.2}\left(-\frac{e^{-0.07t}}{0.07}\right)\Big|_0^{10}$

$= \frac{2{,}000}{0.07}e^{1.2}(-e^{-0.7} + 1)$

$= \frac{2{,}000}{0.07}(e^{1.2} - e^{0.5}) \approx 47{,}754.16$

(C) $2{,}000e^{0.05}\int_0^{10} e^{0.12(10-t)}dt$ $= 2{,}000e^{0.05}\int_0^{10} e^{1.2}e^{-0.12t}\,dt$

$= 2{,}000e^{0.05}e^{1.2}\int_0^{10} e^{-0.12t}\,dt$

$= 2{,}000e^{1.25}\left(-\frac{1}{0.12}e^{-0.12t}\right)\Big|_0^{10}$

$= \frac{2{,}000}{0.12}e^{1.25}(-e^{-1.2} + 1)$

$= \frac{2{,}000}{0.12}(e^{1.25} - e^{0.05}) \approx 40{,}651.20$

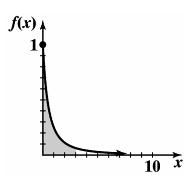
14. $f(x) = \begin{cases} \frac{1}{(x+1)^2} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

(A) Probability $(0 \leq x \leq 3)$ $= \int_0^3 \frac{1}{(x+1)^2}dx = \left(-\frac{1}{x+1}\right)\Big|_0^3$

$= -\frac{1}{4} + 1 = \frac{3}{4} = 0.75$

(B) Probability $(3 \leq x \leq 9)$ $= \int_3^9 \frac{1}{(x+1)^2}dx = \left(-\frac{1}{x+1}\right)\Big|_3^9$

$= -\frac{1}{10} + \frac{1}{4} = 0.15$

(C) 

16. We want to find d such that
Probability $(0 \leq x \leq d) = 0.5$:

$\int_0^d \frac{1}{(x+1)^2}dx = 0.5$

$\left(-\frac{1}{x+1}\right)\Big|_0^d = 0.5$

$-\frac{1}{d+1} + 1 = 0.5$

$\frac{1}{d+1} = 0.5 = \frac{1}{2}$ or $d + 1 = 2$ or $d = 1$ year

18. $f(x) = \begin{cases} 0.15e^{-0.15x} & \text{if } x \ge 0 \\ 0 & \text{otherwise} \end{cases}$

(A) Probability $(0 \le x \le 4)$ $= \int_0^4 0.15e^{-0.15x}\,dx$

$= -e^{-0.15x}\Big|_0^4 = 1 - e^{-0.6} \approx 0.45$

(B) Probability $(3 \le x \le 6)$ $= \int_3^6 0.15e^{-0.15x}dx$

$= -e^{-0.15x}\Big|_3^6$

$= -e^{-0.9} + e^{-0.45} \approx 0.23$

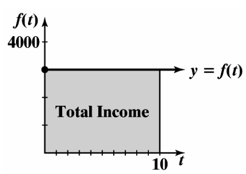
20. Probability $(x > 4)$ $= 1 -$ Probability $(0 \le x \le 4)$

$= 1 - 0.45$ (Problem 18A)

$= 0.55$

22. $f(t) = 3{,}000$

Total income $= \int_0^{10} 3{,}000\,dt = 3{,}000(10 - 0) = \$30{,}000$

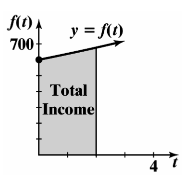
24. 

If $f(t)$ is the rate of flow of a continuous income stream, then the total income produced from 0 to 10 years is the area under the curve $y = f(t)$ from $t = 0$ to $t = 10$.

26. $f(t) = 600e^{0.06t}$

Total income $= \int_0^2 600e^{0.06t}\,dt$ $= \dfrac{600}{0.06}e^{0.06t}\Big|_0^2$

$= 10{,}000(e^{0.12} - 1) \approx \$1{,}275$

28. 

If $f(t)$ is the rate of flow of a continuous income stream, then the total income produced from 0 to 2 years is the area under the curve $y = f(t)$ from $t = 0$ to $t = 2$.

30. $f(t) = 2{,}000e^{0.06t}$

The amount in the account after 35 years is given by:

$$\int_0^{35} 2{,}000e^{0.06t}\,dt = \frac{2{,}000}{0.06}e^{0.06t}\Big|_0^{35} = \frac{100{,}000}{3}(e^{2.1} - 1) \approx \$238{,}872$$

Since $\$2{,}000 \times 35 = \$70{,}000$ was deposited into the account, the interest earned is:

$\$238{,}872 - \$70{,}000 = \$168{,}872$

32. $f(t) = 2{,}000e^{0.06t}$, $r = 0.0295$, $T = 6$.

$$FV = e^{0.0295(6)} \int_0^6 2{,}000e^{0.06t}e^{-0.0295t}\,dt$$

$$= 2{,}000e^{0.177} \int_0^6 e^{0.0305t}\,dt$$

$$= \frac{2{,}000}{0.0305} e^{0.177}(e^{0.0305t}) \Big|_0^6$$

$$= \frac{2{,}000}{0.0305} e^{0.177}(e^{0.183} - 1)$$

$$\frac{2{,}000}{0.0305}(e^{0.36} - e^{0.177}) \approx \$15{,}717.92$$

34. Total Income $= \int_0^6 2{,}000e^{0.06t}\,dt = \frac{2{,}000}{0.06}(e^{0.06t})\Big|_0^6 = \frac{100{,}000}{3}(e^{0.36} - 1) \approx \$14{,}444.31$

From Problem 32,
Interest earned = \$15,717.92 - \$14,444.31 = \$1,273.61.

36. Clothing store: $f(t) = 12{,}000$, $r = 0.04$, $T = 10$.

$$FV = e^{0.04(10)} \int_0^{10} 12{,}000e^{-0.04t}dt = 12{,}000e^{0.4} \int_0^{10} e^{-0.04t}dt$$

$$= \frac{12{,}000e^{0.4}}{-0.04}(e^{-0.04t})\Big|_0^{10} = -300{,}000\, e^{0.4}(e^{-0.4} - 1) = -300{,}000(1 - e^{0.4}) \approx \$147{,}547.41$$

Computer store: $g(t) = 10{,}000e^{0.05t}$, $r = 0.04$, $T = 10$.

$$FV = e^{0.04(10)} \int_0^{10} 10{,}000e^{0.05t}e^{-0.04t}\,dt = 10{,}000e^{0.4} \int_0^{10} e^{0.01t}\,dt$$

$$= \frac{10{,}000e^{0.4}}{0.01}(e^{0.01t})\Big|_0^{10} = 1{,}000{,}000\, e^{0.4}(e^{0.1} - 1)$$

$$= 1{,}000{,}000(e^{0.5} - e^{0.4}) \approx \$\ 156{,}896.57$$

The computer store is the better investment.

38. Bond: $P = \$10{,}000$, $r = 0.0375$, $T = 5$.
$FV = 10{,}000e^{0.0375(5)} = 10{,}000e^{0.1875} \approx \$12{,}062.30$
Business: $f(t) = 2{,}250$, $r = 0.0375$, $T = 5$.

$$FV = e^{0.0375(5)} \int_0^5 2{,}250e^{-0.0375t}\,dt = 2{,}250e^{0.375} \int_0^5 e^{-0.0375t}\,dt$$

$$= \frac{2{,}250e^{0.1875}}{-0.0375}(e^{-0.0375t})\Big|_0^5 = -60{,}000e^{0.1875}(e^{-0.1875} - 1)$$

$$= -60{,}000(1 - e^{0.1875}) \approx \$12{,}373.81$$

The business is the better investment.

40. $f(t) = 1{,}000e^{0.03t}$, $r = 0.0765$, $T = 12$.

$$FV = e^{0.0765(12)}\int_0^{12} 1{,}000e^{0.03t}e^{-0.0765t}\,dt$$

$$= 1{,}000e^{0.918}\int_0^{12} e^{-0.0465t}\,dt = \frac{1{,}000e^{0.918}}{-0.0465}\left(e^{-0.0465t}\right)\Big|_0^{12}$$

$$= -\frac{10{,}000{,}000}{465}e^{0.918}(e^{-0.558} - 1)$$

$$= \frac{10{,}000{,}000}{465}(e^{0.918} - e^{0.36}) \approx \$23{,}031$$

The relationship between present value (PV) and future value (FV) at a continuously compounded interest rate r (expressed as a decimal) for t years is:

$$FV = PVe^{rt} \text{ or } PV = FVe^{-rt}$$

Thus, we have:

$$PV = 23{,}031e^{-0.0765(12)} = 23{,}031e^{-0.918} \approx 9{,}197$$

Thus, the single deposit should be \$9,197.

42. $f(t) = ke^{ct}$, rate r (expressed as a decimal), years T:

$$FV = e^{rT}\int_0^{T} ke^{ct}e^{-rt}\,dt = ke^{rT}\int_0^{T} e^{(c-r)t}\,dt,\ c \neq r$$

$$= \frac{ke^{rT}}{(c-r)}\left(e^{(c-r)t}\right)\Big|_0^{T}$$

$$= \frac{ke^{rT}}{c-r}\left(e^{(c-r)T} - 1\right)$$

$$= \frac{k}{c-r}\left(e^{cT} - e^{rT}\right)$$

44. $D(x) = 200 - 0.02x$, $\bar{p} = 120$

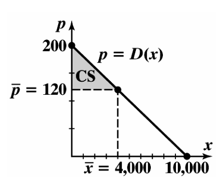
First, find $\bar{x}$: $120 = 200 - 0.02\bar{x}$

$\bar{x} = 4{,}000$

$$CS = \int_0^{4{,}000} [200 - 0.02x - 120]\,dx = \int_0^{4{,}000} (80 - 0.02x)\,dx$$

$$= (80x - 0.01x^2)\Big|_0^{4{,}000} \approx \$160{,}000$$

46.



The shaded area is the consumers' surplus and represents the total savings to consumers who are willing to pay more than \$120 for a product but are still able to buy the product for \$120.

48. $p = S(x) = 15 + 0.1x + 0.003x^2,\ \overline{p} = 55.$

First find $\overline{x}$: $55 = 15 + 0.1\overline{x} + 0.003\overline{x}^2$

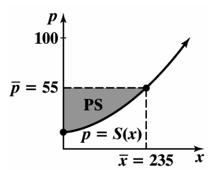
$$0.003\overline{x}^2 + 0.1\overline{x} - 40 = 0$$

$$\overline{x} = \frac{-0.1+\sqrt{(0.1)^2+0.48}}{0.006} = \frac{-0.1+0.7}{0.006} \approx 100$$

$$PS = \int_0^{100} [55 - (15 + 0.1x + 0.003x^2)]dx$$

$$= \int_0^{100} (40 - 0.1x - 0.003x^2)dx$$

$$= \left(40x - \frac{0.1x^2}{2} - \frac{0.003}{3}x^3\right)\Bigg|_0^{100} = \$2,500$$

50. 

The area of the region PS is the producers' surplus and represents the total gain to producers who are willing to supply units at a lower price than \$55 but are still able to supply the product at \$55.

52. $p = D(x) = 25 - 0.004x^2$; $p = S(x) = 5 + 0.004x^2$

Equilibrium price: $D(x) = S(x)$

$$25 - 0.004x^2 = 5 + 0.004x^2$$

$$0.008x^2 = 20$$

$$x^2 = 2,500$$

$$x = 50$$

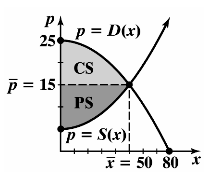
Thus, $\overline{x} = 50$ and $\overline{p} = 25 - 0.004(50)^2 = 15.$

$$CS = \int_0^{50} [(25 - 0.004x^2) - 15]dx = \int_0^{50} (10 - 0.004x^2)dx$$

$$= \left(10x - (0.004)\frac{x^3}{3}\right)\Bigg|_0^{50}$$

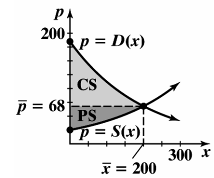
$$\approx \$333$$

$$PS = \int_0^{50} [15 - (5 + 0.004x^2)]dx = \int_0^{50} (10 - 0.004x^2)dx$$



$$= \left(10x - (0.004)\left(\frac{x^3}{3}\right)\right)\Bigg|_0^{50}$$

$$\approx \$333$$



54. $D(x) = 185e^{-0.005x}$ and $S(x) = 25e^{0.005x}$

Equilibrium price: $D(x) = S(x)$

$$185e^{-0.005x} = 25e^{0.005x}$$
$$e^{0.01x} = \frac{185}{25} = 7.4$$
$$0.01x = \ln(7.4)$$
$$x = 100\ln(7.4) \approx 200$$

Thus, $\bar{p} = 25e^{0.005(200)} = 25e \approx 68.$

$$CS = \int_0^{200} [185e^{-0.005x} - 68]dx = \left(\frac{185e^{-0.005x}}{-0.005} - 68x\right)\Bigg|_0^{200}$$
$$= -37{,}000e^{-1} - 13{,}600 + 37{,}000 \approx \$9{,}788$$

$$PS = \int_0^{200} [68 - 25e^{0.005x}]dx = \left(68x - \frac{25e^{0.005x}}{0.005}\right)\Bigg|_0^{200}$$
$$= 13{,}600 - 5{,}000e + 5{,}000 \approx \$5{,}009$$

56. $D(x) = 190 - 0.2x$; $S(x) = 25e^{0.005x}$

Equilibrium price: $D(x) = S(x)$

$$190 - 0.2x = 25e^{0.005x}$$

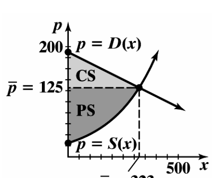
Using a graphing utility, we find that

$$\bar{x} \approx 323$$

Thus, $\bar{p} = 190 - (0.2)(323) \approx 125$

$$CS = \int_0^{323} [190 - 0.2x - 125]dx = \int_0^{323} (65 - 0.2x)dx$$
$$= (65x - 0.1x^2)\Big|_0^{323}$$
$$\approx \$10{,}562$$

$$PS = \int_0^{323} [125 - 25e^{0.005x}]dx = \left(125x - \frac{25e^{0.005x}}{0.005}\right)\Bigg|_0^{323} \approx \$20{,}236$$



58. $D(x) = 185e^{-0.005x}$; $S(x) = 20 + 0.002x^2$

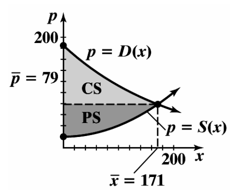
Equilibrium price: $D(x) = S(x)$

Using a graphing utility, we find that $\bar{x} \approx 171.2$

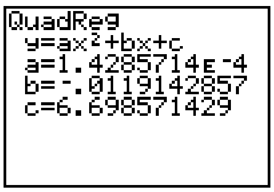
Thus, $\bar{p} = 20 + 0.002(171.2)^2 \approx 79$

$$CS = \int_0^{171.2} [185e^{-0.005x} - 79]dx = \left(-\frac{185e^{-0.005x}}{0.005} - 79x\right)\Bigg|_0^{171.2} \approx \$7{,}756$$

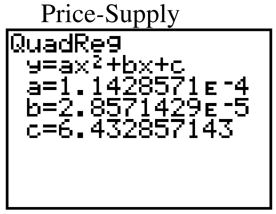
$$PS = \int_0^{171.2} [79 - (20 + 0.002x^2)]dx = \int_0^{171.2} (59 - 0.002x^2)dx = \left(59x - \frac{0.002x^3}{3}\right)\Bigg|_0^{171.2} \approx \$6{,}756$$



60. (A) Price-Demand

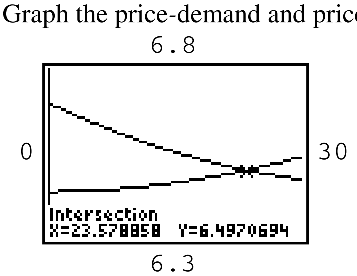


Price-Supply



$p = D(x)$ $p = S(x)$

Graph the price-demand and price-supply models and find their point of intersection.



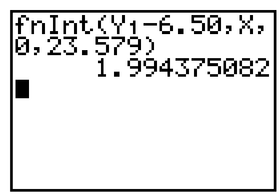
Equilibrium quantity $\bar{x} = 23.579$

Equilibrium price $\bar{p} = 6.50$

(B) Let $D(x)$ be the quadratic regression model in part (A).

Consumers' surplus: $CS = \int_0^{23.579} [D(x) - 6.50]dx$

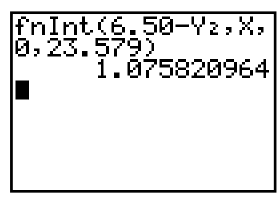
≈ 1.994 or \$1,994



Let $S(x)$ be the quadratic regression model in part (A).

Producers' surplus: $PS = \int_0^{23.579} [6.50 - S(x)]dx$

≈ 1.076 or \$1,076



EXERCISE 7-3

2. $\int xe^{4x}\,dx$

Let $u = x$ and $dv = e^{4x}\,dx$. Then $du = dx$ and $v = \frac{e^{4x}}{4}$.

$$\int xe^{4x}\,dx = \frac{xe^{4x}}{4} - \int \frac{e^{4x}}{4}dx = \frac{1}{4}xe^{4x} - \frac{1}{4}\int e^{4x}\,dx = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + C$$

4. $\int x^3 \ln x\,dx$

Let $u = \ln x$ and $dv = x^3\,dx$. Then $du = \frac{1}{x}\,dx$ and $v = \frac{x^4}{4}$.

$$\int x^3 \ln x\,dx = \frac{1}{4}x^4 \ln x - \int \frac{x^3}{4}\,dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$$

6. $\int (5x - 7)(x - 1)^4\,dx$

The better choice is $u = 5x - 7$, $dv = (x - 1)^4\,dx$

The alternative is $u = (x - 1)^4$, $dv = (5x - 7)dx$, which will lead to an integral of the form $\int (x - 1)^3(5x - 7)^2\,dx$.

Let $u = 5x - 7$ and $dv = (x - 1)^4\,dx$. Then $du = 5\,dx$ and $v = \frac{1}{5}(x-1)^5$.

Substitute into the integration by parts formula:

$$\int (5x - 7)(x - 1)^4dx = \frac{1}{5}(5x-7)(x - 1)^5 - \int (x - 1)^5\,dx = \frac{1}{5}(5x-7)(x - 1)^5 - \frac{1}{6}(x - 1)^6 + C$$

8. $\int (x - 1)e^{-x}\,dx$

Let $u = x - 1$ and $dv = e^{-x}\,dx$. Then $du = dx$ and $v = -e^{-x}$.

$$\int (x - 1)e^{-x}\,dx = -(x - 1)e^{-x} + \int e^{-x}\,dx = -(x - 1)e^{-x} - e^{-x} + C = -xe^{-x} + e^{-x} - e^{-x} + C = -xe^{-x} + C$$

10. $\int xe^{-x^2}dx$

Let $u = -x^2$, then $du = -2x\,dx$.

$$\int xe^{-x^2}dx = \int e^{-x^2}\frac{-2}{-2}x\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C$$

12. $\int_0^1 (x + 1)e^x\,dx$

Let $u = x + 1$ and $dv = e^x\,dx$. Then $du = dx$ and $v = e^x$.

$$\int (x + 1)e^x\,dx = (x + 1)e^x - \int e^x\,dx = (x + 1)e^x - e^x + C = xe^x + e^x - e^x + C = xe^x + C$$

Thus, $\int_0^1 (x + 1)e^x\,dx = (xe^x)\Big|_0^1 = e - 0 = e \approx 2.7183$

14. $\int_1^2 \ln\left(\frac{x}{2}\right) dx$

Let $u = \ln\left(\frac{x}{2}\right)$ and $dv = dx$. Then $du = \frac{1}{x/2}\cdot\frac{1}{2}\,dx = \frac{1}{x}\,dx$ and $v = x$.

$$\int \ln\left(\frac{x}{2}\right) dx = x\ln\left(\frac{x}{2}\right) - \int dx = x\ln\left(\frac{x}{2}\right) - x + C$$

Thus, $\int_1^2 \ln\left(\frac{x}{2}\right) dx$

$$= \left(x\ln\left(\frac{x}{2}\right) - x\right)\Bigg|_1^2$$

$$= (2\ln(1) - 2) - \left(\ln\left(\frac{1}{2}\right) - 1\right)$$

$$= -2 - \ln\left(\frac{1}{2}\right) + 1 = -1 + \ln 2 \approx -0.3069$$

16. $\int \frac{x^2}{x^3+5}\,dx = \int (x^3+5)^{-1}x^2\,dx$

Let $u = x^3 + 5$, then $du = 3x^2\,dx$.

$$\int \frac{x^2}{x^3+5}\,dx = \int (x^3+5)^{-1}x^2\,dx = \int (x^3+5)^{-1}\,\frac{3}{3}x^2\,dx$$

$$= \frac{1}{3}\int u^{-1}\,du$$

$$= \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ln(x^3+5) + C \quad (x > 0)$$

18. $\int \frac{e^x}{e^x+1}\,dx$

Let $u = e^x + 1$, then $du = e^x\,dx$.

$$\int \frac{e^x}{e^x+1}\,dx = \int (e^x+1)^{-1}\,e^x\,dx = \int u^{-1}\,du$$

$$= \ln|u| + C$$

$$= \ln(e^x+1) + C$$

20. $\int \frac{\ln x}{\sqrt{x}}\,dx = \int (\ln x)x^{-1/2}\,dx$

Let $u = \ln x$ and $dv = x^{-1/2}\,dx$. Then $du = \frac{1}{x}\,dx$ and $v = 2x^{1/2}$.

$$\int \frac{\ln x}{\sqrt{x}}\,dx = 2(\ln x)x^{1/2} - \int 2x^{-1/2}\,dx = 2x^{1/2}\ln x - 4x^{1/2} + C$$

22. $\int (x+2)(x-1)^2\,dx$

Let $u = x + 2$, $dv = (x - 1)^2\,dx$. Then $du = dx$ and $v = \frac{1}{3}(x - 1)^3$. Therefore

$$\begin{aligned}
\int (x+2)(x-1)^2\,dx &= (x+2)\frac{(x-1)^3}{3} - \int \frac{(x-1)^3}{3}\,dx \\
&= (x+2)\frac{(x-1)^3}{3} - \frac{1}{3}\cdot\frac{(x-1)^4}{4} + C \\
&= \frac{(x+2)(x-1)^3}{3} - \frac{(x-1)^4}{12} + C \\
&= \frac{x^4}{4} - \frac{3x^2}{2} + 2x - \frac{3}{4} + C \\
&= \frac{x^4}{4} - \frac{3x^2}{2} + 2x + C \qquad \left(-\frac{3}{4} \text{ is incoporated into } C\right)
\end{aligned}$$

$$\int (x+2)(x-1)^2\,dx = \int \left(x^3 - 3x + 2\right)dx = \frac{x^4}{4} - \frac{3x^2}{2} + 2x + C$$

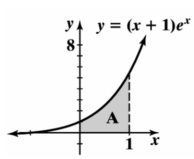
24. $\int (5x-1)(x+2)^2\,dx$

Let $u = 5x + 1$, $dv = (x + 2)^2\,dx$. Then $du = 5dx$ and $v = \frac{1}{3}(x + 2)^3$. Therefore

$$\begin{aligned}
\int (5x-1)(x+2)^2\,dx &= (5x-1)\frac{(x+2)^3}{3} - \int \frac{5(x+2)^3}{3}dx \\
&= (5x-1)\frac{(x+2)^3}{3} - \frac{5}{3}\cdot\frac{(x+2)^4}{4} + C \\
&= \frac{(5x-1)(x+2)^3}{3} - \frac{5(x+2)^4}{12} + C \\
&= \frac{5x^4}{4} + \frac{19x^3}{3} + 8x^2 - 4x - \frac{28}{3} + C \\
&= \frac{5x^4}{4} + \frac{19x^3}{3} + 8x^2 - 4x + C \qquad \left(-\frac{28}{3} \text{ is incoporated into } C\right)
\end{aligned}$$

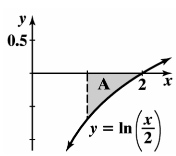
$$\int (5x-1)(x+2)^2\,dx = \int \left(5x^3 + 19x^2 + 16x - 4\right)dx = \frac{5x^4}{4} + \frac{19x^3}{3} + 8x^2 - 4x + C$$

26.



The integral represents the area between the curve $y = (x + 1)e^x$ and the x axis from $x = 0$ to $x = 1$.

28.



The integral represents the negative of the area between the curve $y = \ln\left(\frac{x}{2}\right)$ and the x axis from $x = 1$ to $x = 2$.

30. $\int x^3 e^x \, dx$

Let $u = x^3$ and $dv = e^x \, dx$. Then $du = 3x^2 \, dx$ and $v = e^x$.

$\int x^3 e^x \, dx = x^3 e^x - \int 3x^2 e^x \, dx = x^3 e^x - 3\int x^2 e^x \, dx$

$\int x^2 e^x \, dx$ can be computed by using integration-by-parts again.

Let $u = x^2$ and $dv = e^x \, dx$. Then $du = 2x \, dx$ and $v = e^x$.

$\int x^2 e^x \, dx = x^2 e^x - 2\int xe^x \, dx$

$\int xe^x \, dx$ can be computed by using integration-by-parts again.

Let $u = x$ and $dv = e^x \, dx$. Then $du = dx$ and $v = e^x$.

$\int xe^x \, dx = xe^x - \int e^x \, dx = xe^x - e^x + C_1$

Thus,

$$\begin{aligned}\int x^3 e^x \, dx &= x^3 e^x - 3[x^2 e^x - 2(xe^x - e^x + C_1)] \\ &= x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C \\ &= (x^3 - 3x^2 + 6x - 6)e^x + C\end{aligned}$$

32. $\int \ln(ax)dx, \; a > 0$

Let $u = \ln(ax)$ and $dv = dx$. Then $du = \frac{1}{x} dx$ and $v = x$.

$\int \ln(ax)dx = x \ln(ax) - \int dx = x \ln(ax) - x + C$

34. $\int_1^2 x^3 e^{x^2} \, dx$

Let $t = x^2$, then $dt = 2x \, dx$.

$\int x^3 e^{x^2} \, dx = \int x^2 e^{x^2} \frac{2}{2} x \, dx = \frac{1}{2} \int te^t \, dt$

To compute $\int te^t \, dt$ we use integration-by-parts.

Let $u = t$ and $dv = e^t \, dt$. Then $du = dt$ and $v = e^t$.

$\int te^t \, dt = te^t - \int e^t \, dt = te^t - e^t + C_1.$

Thus,

$$\begin{aligned}\int x^3 e^{x^2} \, dx &= \frac{1}{2}(te^t - e^t + C_1) \\ &= \frac{1}{2}(x^2 e^{x^2} - e^{x^2}) + C\end{aligned}$$

and

$$\int_1^2 x^3 e^{x^2} \, dx = \left(\frac{1}{2}\left(x^2 e^{x^2} - e^{x^2} \right) \right) \Big|_1^2$$

$$= \frac{1}{2}(4e^4 - e^4) - \frac{1}{2}(e - e)$$
$$= \frac{3}{2}e^4 \approx 81.8972$$

36. $\int_0^2 \ln(4 - x)dx$

Let $u = \ln(4 - x)$ and $dv = dx$. Then $du = -\frac{1}{4-x}dx$ and $v = x$.

$$\begin{aligned}
\int \ln(4 - x)dx &= x\ln(4 - x) + \int \frac{x}{4-x}dx \\
&= x\ln(4 - x) + \int \frac{(x-4+4)}{4-x}dx \\
&= x\ln(4 - x) + \int \frac{-(4-x)}{4-x}dx + \int \frac{4}{4-x}dx \\
&= x\ln(4 - x) - \int dx + 4\int \frac{1}{4-x}dx \\
&= x\ln(4 - x) - x - 4\ln|4 - x| + C
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^2 \ln(4 - x)dx &= (x\ln(4 - x) - x - 4\ln(4 - x))\Big|_0^2 \\
&= (2\ln 2 - 2 - 4\ln 2) - (-4\ln 4) \\
&= -2 - 2\ln 2 + 4\ln 4 \\
&= -2 - 2\ln 2 + 8\ln 2 \\
&= -2 + 6\ln 2 \approx 2.1589
\end{aligned}$$

38. $\int xe^{x+1}dx$

Let $u = x$ and $dv = e^{x+1}dx$. Then $du = dx$ and $v = e^{x+1}$.

$$\int xe^{x+1}dx = xe^{x+1} - \int e^{x+1}dx = xe^{x+1} - e^{x+1} + C$$

40. $\int x\ln(1 + x)dx$

Let $u = \ln(1 + x)$ and $dv = x\,dx$. Then $du = \frac{1}{1+x}dx$ and $v = \frac{1}{2}x^2$.

$$\begin{aligned}
\int x\ln(1 + x)dx &= \frac{1}{2}x^2\ln(1 + x) - \frac{1}{2}\int \frac{x^2}{1+x}dx \\
&= \frac{1}{2}x^2\ln(1 + x) - \frac{1}{2}\int \frac{x^2-1+1}{1+x}dx \\
&= \frac{1}{2}x^2\ln(1 + x) - \frac{1}{2}\int \frac{x^2-1}{1+x} - \frac{1}{2}\int \frac{1}{1+x}dx \\
&= \frac{1}{2}x^2\ln(1 + x) - \frac{1}{2}\int \frac{(x-1)(x+1)}{1+x}dx - \frac{1}{2}\ln(1 + x)
\end{aligned}$$

$$= \frac{1}{2}x^2 \ln(1+x) - \frac{1}{2}\int (x-1)dx - \frac{1}{2}\ln(1+x)$$
$$= \frac{1}{2}x^2 \ln(1+x) - \frac{1}{2}\left(\frac{1}{2}(x-1)^2\right) - \frac{1}{2}\ln(1+x) + C$$
$$= \frac{1}{2}(x^2-1)\ln(1+x) - \frac{1}{4}(x-1)^2 + C$$

Note: The answer could also be given in the following equivalent form:

$$\int x\ln(1+x)dx$$
$$= \frac{1}{2}(1+x)^2\ln(1+x) - (1+x)\ln(1+x) + (1+x) - \frac{1}{4}(1+x)^2 + C$$

42. $\int \frac{\ln\left(1+\sqrt{x}\right)}{\sqrt{x}}dx$

Let $t = \sqrt{x} = x^{1/2}$, then $dt = \frac{1}{2}x^{-1/2}dx$.

$$\int \frac{\ln\left(1+\sqrt{x}\right)}{\sqrt{x}}dx = \int \ln(1+\sqrt{x})\,\frac{2}{2}x^{-1/2}dx = 2\int \ln(1+t)dt$$

To compute $\int \ln(1+t)dt$ we use integration-by-parts.

Let $u = \ln(1+t)$ and $dv = dt$. Then $du = \frac{1}{1+t}dt$ and $v = t$.

$$\int \ln(1+t)dt = t\ln(1+t) - \int \frac{t}{1+t}dt$$
$$= t\ln(1+t) - \int \frac{t+1-1}{1+t}dt$$
$$= t\ln(1+t) - \int \left(1-\frac{1}{1+t}\right)dt$$
$$= t\ln(1+t) - t + \ln(1+t) + C_1$$
$$= \sqrt{x}\ln(1+\sqrt{x}) - \sqrt{x} + \ln(1+\sqrt{x}) + C_1$$
$$= (1+\sqrt{x})\ln(1+\sqrt{x}) - \sqrt{x} + C_1$$

Thus, $\int \frac{\ln\left(1+\sqrt{x}\right)}{\sqrt{x}}dx = 2(1+\sqrt{x})\ln(1+\sqrt{x}) - 2\sqrt{x} + C$

44. $\int x(\ln x)^2\,dx$

Let $u = (\ln x)^2$ and $dv = x\,dx$. Then $du = 2\frac{\ln x}{x}dx$ and $v = \frac{1}{2}x^2$.

$$\int x(\ln x)^2\,dx = \frac{1}{2}x^2(\ln x)^2 - \int x(\ln x)dx$$

To compute $\int x(\ln x)dx$ we use integration-by-parts.

Let $u = \ln x$ and $dv = x\,dx$. Then $du = \frac{1}{x}dx$ and $v = \frac{x^2}{2}$.

$$\int x(\ln x)dx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C_1$$

Thus,

$$\int x(\ln x)^2\,dx = \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + C$$
$$= \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{x^2}{4} + C$$

46. $\int x(\ln x)^3\,dx$

Let $u = (\ln x)^3$ and $dv = x\,dx$. Then $du = \frac{3(\ln x)^2}{x}dx$ and $v = \frac{1}{2}x^2\,dx$.

$$\int x(\ln x)^3\,dx = \frac{1}{2}x^2(\ln x)^3 - \frac{3}{2}\int x(\ln x)^2\,dx$$

From Problem 44,

$$\int x(\ln x)^2\,dx = \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{x^2}{4} + C,$$

and hence

$$\int x(\ln x)^3\,dx = \frac{1}{2}x^2(\ln x)^3 - \frac{3}{4}x^2(\ln x)^2 + \frac{3}{4}x^2\ln x - \frac{3}{8}x^2 + C_1.$$

48. $\int_1^e \ln(x^4)dx = \int_1^e 4\ln x\,dx = 4\int_1^e \ln x\,dx$

Let $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x}dx$ and $v = x$.

$$\int_1^e \ln x\,dx = \;= x\ln x\,\Big]_1^e - \int_1^e (x)\left(\frac{1}{x}\right)dx$$
$$= e\ln e - 1\ln 1 - \int_1^e dx$$
$$= e - (e - 1) = 1$$

Therefore $\int_1^e \ln(x^4)dx = 4(1) = 4.$

50.
$$\int_1^2 \ln(xe^x)dx = \int_1^2 (\ln x + \ln e^x)dx$$
$$= \int_1^2 (\ln x + x)dx$$
$$= \int_1^2 \ln x\,dx + \int_1^2 x\,dx$$

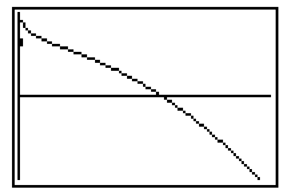
From problem 48 above, $\int_1^2 \ln x\,dx$
$$= x\ln x\,\Big]_1^2 - \int_1^2 x\left(\frac{1}{x}\right)dx$$
$$= 2\ln 2 - 1\ln 1 - \int_1^2 dx$$
$$= 2\ln 2 - (2 - 1)$$
$$= 2\ln 2 - 1$$

Thus

$$\int_1^2 \ln(xe^x)dx = 2\ln 2 - 1 + \left[\frac{x^2}{2}\right]_1^2 = 2\ln 2 - 1 + \left(\frac{2^2}{2} - \frac{1^2}{2}\right)$$
$$= 2\ln 2 - 1 + \frac{3}{2} = 2\ln 2 + \frac{1}{2}$$

52. $y = 6 - x^2 - \ln x,\ 1 \le x \le 4$

$y = 0$ at $x \approx 2.275$



$$A = \int_1^{2.275} (6 - x^2 - \ln x)dx + \int_{2.275}^4 [-(6 - x^2 - \ln x)]dx$$
$$= \int_1^{2.275} (6 - x^2 - \ln x)dx + \int_{2.275}^4 (-6 + x^2 + \ln x)dx$$

Now, $\int \ln x\, dx$ is found using integration-by-parts.

Let $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x}dx$ and $v = x$.

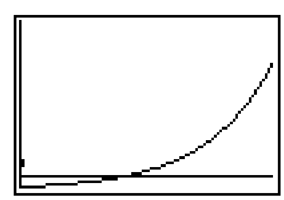
$$\int \ln x\, dx = x\ln x - \int\left(\frac{1}{x}\right)x\, dx = x\ln x - x + C$$

Thus,

$$A = \left(6x - \frac{x^3}{3} - x\ln x + x\right)\Bigg|_1^{2.275} + \left(-6x + \frac{x^3}{3} + x\ln x - x\right)\Bigg|_{2.275}^4$$
$$\approx (10.130 - 6.667) + (-1.121 + 10.130) = 12.47$$

54. $y = xe^x + x - 6,\ 0 \le x \le 3$

$y = 0$ at $x \approx 1.293$



$$A = \int_0^{1.293} [-(xe^x + x - 6)]dx + \int_{1.293}^3 (xe^x + x - 6)dx$$
$$= \int_0^{1.293} (-xe^x - x + 6)dx + \int_{1.293}^3 (xe^x + x - 6)dx$$

Now, $\int xe^x\, dx$ is found using integration-by-parts.

Let $u = x$ and $dv = e^x\, dx$. Then, $du = dx$ and $v = e^x$.

$$\int xe^x\, dx = xe^x - \int e^x\, dx = xe^x - e^x + C$$

Thus,

$$A = \left(-xe^x + e^x - \frac{x^2}{2} + 6x\right)\Bigg|_0^{1.293} + \left(xe^x - e^x + \frac{x^2}{2} - 6x\right)\Bigg|_{1.293}^{3}$$

$\approx (5.854 - 1) + (26.671 + 5.854) \approx 37.38$

56. The total production over the first 12 months is given by the definite integral:

$$\int_0^{12} 10te^{-0.1t}dt = 10\int_0^{12} te^{-0.1t}dt$$

To calculate this integral we use integration-by-parts.

Let $u = t$ and $dv = e^{-0.1t}dt$. Then $du = dt$ and $v = \frac{1}{-0.1}e^{-0.1t} = -10e^{-0.1t}$

$$\int te^{-0.1t}dt = -10te^{-0.1t} + 10\int e^{-0.1t}dt = -10te^{-0.1t} + 10\left(\frac{e^{-0.1t}}{-0.1}\right) + C$$

$$= -10te^{-0.1t} - 100e^{-0.1t} + C$$

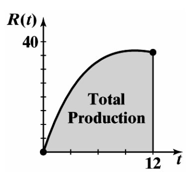
Thus,

$$\int_0^{12} 10te^{-0.1t}dt = 10(-10te^{-0.1t} - 100e^{-0.1t})\Big|_0^{12}$$

$$= 10[(-120e^{-1.2} - 100e^{-1.2}) - (-100)]$$

$$= 1{,}000 - 2{,}200e^{-1.2} \approx 337$$

To the nearest thousand, the total production is 337 thousand barrels.

58. 

The total production for the first year of operation (in thousands of barrels) is the same as the area under the rate of production function, $R(t) = 10te^{-0.1t}$, from $t = 0$ to $t = 12$.

60. From page 425, Future Value = $e^{rT}\int_0^T f(t)e^{-rt}\,dt$. Now r=0.0415,

$T = 4$, $f(t) = 1{,}000 - 250t$. Thus,

$$FV = e^{(0.0415)(4)}\int_0^4 (1{,}000 - 250t)e^{-0.0415t}\,dt$$

$$= e^{0.166}\int_0^4 1{,}000e^{-0.0415t}\,dt - 250e^{0.166}\int_0^4 te^{-0.0415t}\,dt.$$

We calculate the second integral using integration-by-parts.

Let $u = t$, $dv = e^{-0.0415t}\,dt$. Then $du = dt$ and $v = \dfrac{e^{-0.0415t}}{-0.0415}$.

$$\int te^{-0.0415t}\,dt = -\frac{1}{0.0415}te^{-0.0415t} + \frac{1}{0.0415}\int e^{-0.0415t}\,dt$$
$$= -\frac{1}{0.0415}te^{-0.0415t} - \frac{1}{(0.0415)^2}e^{-0.0415t} + C$$

Thus, we have:

$$FV = -\frac{1{,}000e^{0.166}}{0.0415}\left(e^{-0.0415t}\right)\Big|_0^4 - 250e^{0.166}\left(-\frac{te^{-0.0415t}}{0.0415} - \frac{e^{-0.0415t}}{(0.0415)^2}\right)\Bigg|_0^4$$

$$= \frac{1{,}000e^{0.166}}{0.0415}\left(1-e^{-0.166}\right) - 250e^{0.166}\left[\left(-\frac{4e^{-0.166}}{0.0415} - \frac{e^{-0.166}}{(0.0415)^2}\right) - \left(-\frac{1}{(0.0415)^2}\right)\right]$$

$$= \frac{1{,}000}{0.0415}\left(e^{0.166}-1\right) - 250\left(-\frac{4}{0.0415} - \frac{1}{(0.0415)^2} + \frac{e^{0.166}}{(0.0415)^2}\right) \approx \$2{,}235.74$$

Total income = $\int_0^4 (1{,}000 - 250t)dt = (1{,}000t - 125t^2)\Big|_0^4 = \$2{,}000$

Interest earned $\approx$ \$2,235.74 – \$2,000 = \$235.74.

62. Index of Income Concentration $= 2\int_0^1 (x - x^2e^{x-1})dx$

$$= 2\int_0^1 x\,dx - 2\int_0^1 x^2e^{x-1}dx$$

We calculate the second integral using integration-by-parts.
Let $u = x^2$, $dv = e^{x-1}\,dx$. Then $du = 2x\,dx$, $v = e^{x-1}$.

$$\int x^2e^{x-1}\,dx = x^2e^{x-1} - 2\int xe^{x-1}dx$$

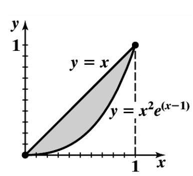
To compute $\int xe^{x-1}dx$ we use integration-by-parts again:

Let $u = x$ and $dv = e^{x-1}dx$. Then $du = dx$ and $v = e^{x-1}$.

$$\int xe^{x-1}dx = xe^{x-1} - \int e^{x-1}dx = xe^{x-1} - e^{x-1} + C$$

Therefore, $2\int_0^1 x\,dx - 2\int_0^1 x^2e^{x-1}dx$ $= 2\left(\dfrac{x^2}{2}\right)\Big|_0^1 - 2(x^2e^{x-1} - 2xe^{x-1} + 2e^{x-1})\Big|_0^1$

$$= 1 - 2[(1 - 2 + 2) - (2e^{-1})]$$
$$= 1 - 2 + 4e^{-1} = 4e^{-1} - 1 \approx 0.472$$

64. 

The area bounded by $y = x$ and the Lorenz curve $y = x^2e^{(x-1)}$ divided by the area under the curve $y = x$ from $x = 0$ to $x = 1$ is the index of income concentration. It is a measure of the concentration of income—the closer to zero, the closer to all the income being equally distributed; the closer to one, the closer to all the income being concentrated in a few hands.

66. $S'(t) = 350 \ln(t+1)$, $S(0) = 0$

$$S(t) = \int 350 \ln(t+1)dt = 350\int \ln(t+1)dt$$

Let $u = \ln(t+1)$ and $dv = dt$. Then $du = \frac{1}{t+1}dt$ and $v = t$.

$$\begin{aligned}\int \ln(t+1)dt &= t\ln(t+1) - \int \frac{t}{t+1}dt \\ &= t\ln(t+1) - \int \frac{t+1-1}{t+1}dt \\ &= t\ln(t+1) - \int \left(1-\frac{1}{t+1}\right)dt \\ &= t\ln(t+1) - (t - \ln(t+1)) + C \\ &= t\ln(t+1) - t + \ln(t+1) + C\end{aligned}$$

Now,

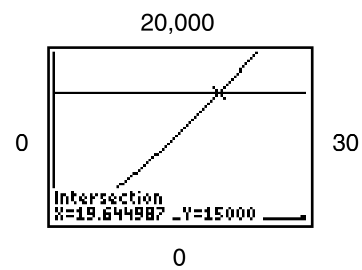
$$\begin{aligned}S(t) &= 350(t\ln(t+1) + \ln(t+1) - t) + C \\ &= 350((t+1)\ln(t+1) - t) + C\end{aligned}$$

Since $S(0) = 0$, we have $C = 0$ and hence

$$S(t) = 350((t+1)\ln(t+1) - t).$$

To find how long the company will continue to manufacture games, solve $S(t) = 15{,}000$ for t.

The company will manufacture games for 20 months.



68. $p = S(x) = 5\ln(x+1)$; $\overline{p} = \$26$. To find $\overline{x}$, solve

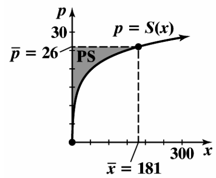
$$\begin{aligned}5\ln(\overline{x}+1) &= 26 \\ \ln(\overline{x}+1) &= \frac{26}{5} = 5.2 \\ \overline{x}+1 &= e^{5.2} \\ \overline{x} &= e^{5.2} - 1 \approx 181 \text{ (to the nearest higher unit)}\end{aligned}$$

$$\begin{aligned}PS = \int_0^{181} [\overline{p} - S(x)]dx &= \int_0^{181} (26 - 5\ln(x+1))dx \\ &= \int_0^{181} 26\,dx - 5\int_0^{181} \ln(x+1)dx\end{aligned}$$

From Problem 66, $\int \ln(x+1)dx = (x+1)\ln(x+1) - x + C$, and hence

$$PS = 26(x)\Big|_0^{181} - 5((x+1)\ln(x+1) - x)\Big|_0^{181}$$

$$= 4{,}706 - 5(182\ln(182) - 181) \approx \$875$$

70. 

The area bounded by the price-supply equation, $p = 5\ln(x + 1)$, and the price equation, $y = \bar{p} = 26$, from $x = 0$ to $x = \bar{x} = 181$, represents the producers' surplus. This is the amount gained by producers who are willing to sell for less.

72. $R(t) = te^{-0.2t}$

Total amount $= \int_0^{10} R(t)dt = \int_0^{10} te^{-0.2t}\,dt$

Let $u = t$ and $dv = e^{-0.2t}\,dt$. Then $du = dt$ and $v = \dfrac{e^{-0.2t}}{-0.2} = -5e^{-0.2t}$.

$$\begin{aligned}\int te^{-0.2t}\,dt &= -5te^{-0.2t} + 5\int e^{-0.2t}\,dt \\ &= -5te^{-0.2t} + 5\left(\frac{e^{-0.2t}}{-0.2}\right) + C \\ &= -5te^{-0.2t} - 25e^{-0.2t} + C\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^{10} R(t)dt &= (-5te^{-0.2t} - 25e^{-0.2t})\Big|_0^{10} \\ &= (-50e^{-2} - 25e^{-2}) - (-25) \\ &= 25 - 75e^{-2} = 25(1 - 3e^{-2}) \approx 14.85\end{aligned}$$

74. $N'(t) = (t + 10)e^{-0.1t}$, $0 \le t \le 15$; $N(0) = 0$

$N(t) - N(0) = \int_0^t N'(x)dx = \int_0^t (x + 10)e^{-0.1x}\,dx$

$$\begin{aligned}N(t) = \int_0^t (x + 10)e^{-0.1x}\,dx &= \int_0^t xe^{-0.1x}\,dx + 10\int_0^t e^{-0.1x}\,dx \\ &= \int_0^t xe^{-0.1x}\,dx + 10\left(\frac{e^{-0.1x}}{-0.1}\right)\Big|_0^t \\ &= \int_0^t xe^{-0.1x}\,dx + 10\left(\frac{e^{-0.1t}}{-0.1} + \frac{1}{0.1}\right) \\ &= \int_0^t xe^{-0.1x}\,dx + 100(1 - e^{-0.1t})\end{aligned}$$

To compute the last integral we use integration-by-parts.

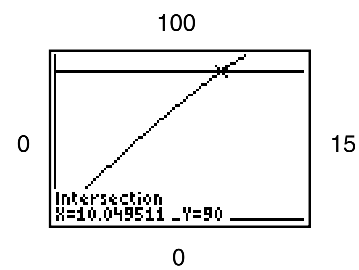
Let $u = x$ and $dv = e^{-0.1x}\,dx$. Then $du = dx$ and $v = \dfrac{e^{-0.1x}}{-0.1} = -10e^{-0.1x}$. Thus,

$$\int xe^{-0.1x}\,dx = -10xe^{-0.1x} + 10\int e^{-0.1x}\,dx$$
$$= -10xe^{-0.1x} + 10\left(\frac{e^{-0.1x}}{-0.1}\right) + C$$
$$= -10xe^{-0.1x} - 100e^{-0.1x} + C$$

Therefore,

$$N(t) = (-10xe^{-0.1x} - 100e^{-0.1x})\Big|_0^t + 100(1 - e^{-0.1t})$$
$$= (-10te^{-0.1t} - 100e^{-0.1t}) - (-100) + 100(1 - e^{-0.1t})$$
$$N(t) = -10te^{-0.1t} - 200e^{-0.1t} + 200$$

To find how long it will take a student to achieve the 90 words per minute level, solve
$N(t) = 90$:



It will take 10 weeks.

By the end of the course, a student should be able to type
$N(15) = -10(15)e^{-1.5} - 200e^{-1.5} + 200 \approx 122$ words per minute.

EXERCISE 7-4

2. Use Formula 10 with $a = b = 1$:
$$\int \frac{1}{x^2(1+x)}\,dx = -\frac{1}{x} + \ln\left|\frac{1+x}{x}\right| + C$$

4. Use Formula 19 with $a = 5$, $b = 2$, $c = 2$, $d = 1$:
$$\int \frac{x}{(5+2x)^2(2+x)}\,dx = -\frac{5}{2}\cdot\frac{1}{5+2x} - 2\ln\left|\frac{2+x}{5+2x}\right| + C$$

6. Use Formula 27 with $a = 16$ and $b = 1$:
$$\int \frac{1}{x\sqrt{16+x}}\,dx = \frac{1}{4}\ln\left|\frac{\sqrt{16+x}-4}{\sqrt{16+x}+4}\right| + C$$

8. Use Formula 31 with $a = 3$:
$$\int \frac{\sqrt{9-x^2}}{x}\,dx = \sqrt{9-x^2} - 3\ln\left|\frac{3+\sqrt{9-x^2}}{x}\right| + C$$

10. Use Formula 45 with $a = 4$:
$$\int \frac{1}{x^2\sqrt{x^2-16}}\,dx = \frac{\sqrt{x^2-16}}{16x} + C$$

12. Use Formula 51 with $n = 3$:

$$\int x^3 \ln x\, dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$$

14. Use Formula 48 with $a = 3$, $c = 5$, $d = 2$:

$$\int \frac{1}{5+2e^{3x}}\, dx = \frac{x}{5} - \frac{1}{15}\ln|5 + 2e^{3x}| + C$$

16. Use Formula 6 with $a = 6$, $b = 1$:

$$\begin{aligned}\int_2^6 \frac{x}{(6+x)^2}\, dx &= \left(\ln|6+x| + \frac{6}{6+x}\right)\Big|_2^6 \\ &= \left(\ln 12 + \frac{1}{2}\right) - \left(\ln 8 + \frac{3}{4}\right) \\ &= \ln 12 - \ln 8 + \frac{1}{2} - \frac{3}{4} \\ &= \ln\left(\frac{12}{8}\right) - \frac{1}{4} = \ln\left(\frac{3}{2}\right) - \frac{1}{4} \approx 0.1555\end{aligned}$$

18. Use Formula 16 with $a = 3$, $b = 1$, $c = 1$, $d = 1$:

$$\begin{aligned}\int_0^7 \frac{x}{(3+x)(1+x)}\, dx &= \frac{1}{2}(3\ln|3 + x| - \ln|1 + x|)\Big|_0^7 \\ &= \frac{1}{2}(3 \ln 10 - \ln 8) - \frac{1}{2}(3 \ln 3) \approx 0.7662\end{aligned}$$

20. Use Formula 40 with $a = 4$:

$$\begin{aligned}\int_4^5 \sqrt{x^2-16}\, dx &= \left(\frac{1}{2}\left(x\sqrt{x^2-16} - 16\ln\left|x+\sqrt{x^2-16}\right|\right)\right)\Big|_4^5 \\ &= \frac{1}{2}(15 - 16 \ln 8) - \frac{1}{2}(-16 \ln 4) \\ &= \frac{15}{2} - 8 \ln 2 \approx 1.9548\end{aligned}$$

22. $\int x^2\sqrt{9x^2-1}\, dx$

Let $u = 3x$, then $du = 3\, dx$ and

$$\begin{aligned}\int x^2\sqrt{9x^2-1}\, dx &= \int \frac{9}{9} x^2 \sqrt{9x^2-1}\ \frac{3}{3} dx \\ &= \frac{1}{27}\int u^2\sqrt{u^2-1}\, du\end{aligned}$$

Use Formula 41 with $a = 1$ for the integral on the right hand side:

$$\int u^2\sqrt{u^2-1}\,du = \frac{1}{8}\left[u(2u^2-1)\sqrt{u^2-1}-\ln\left|u+\sqrt{u^2-1}\right|\right] + C$$

Therefore, by substituting $3x$ for u, we have:

$$\int x^2\sqrt{9x^2-1}\,dx = \frac{1}{216}\left[3x(18x^2-1)\sqrt{9x^2-1}-\ln\left|3x+\sqrt{9x^2-1}\right|\right] + C$$

24. $\int x\sqrt{x^4-16}\,dx$

Let $u = x^2$, then $du = 2x\,dx$, and

$$\begin{aligned}\int x\sqrt{x^4-16}\,dx &= \int \sqrt{x^4-16}\,\frac{2}{2}x\,dx\\ &= \frac{1}{2}\int \sqrt{u^2-16}\,du\end{aligned}$$

Using Formula 40 with $a = 4$, we have:

$$\int \sqrt{u^2-16}\,du = \frac{1}{2}\left(u\sqrt{u^2-16}-16\ln\left|u+\sqrt{u^2-16}\right|\right) + C$$

Therefore,

$$\int x\sqrt{x^4-16}\,dx = \frac{1}{4}\left(x^2\sqrt{x^4-16}-16\ln\left|x^2+\sqrt{x^4-16}\right|\right) + C$$

26. $\int \frac{x^2}{\sqrt{x^6+4}}\,dx$

Let $u = x^3$, then $du = 3x^2\,dx$, and

$$\int \frac{x^2}{\sqrt{x^6+4}}\,dx = \int \frac{1}{\sqrt{x^6+4}}\,\frac{3}{3}x^2\,dx = \frac{1}{3}\int \frac{1}{\sqrt{u^2+4}}\,du$$

Using Formula 36 with $a = 2$, we have:

$$\begin{aligned}\int \frac{x^2}{\sqrt{x^6+4}}\,dx &= \frac{1}{3}\ln|u + \sqrt{u^2+4}\,| + C\\ &= \frac{1}{3}\ln|x^3 + \sqrt{x^6+4}\,| + C\end{aligned}$$

28. $\int \frac{\sqrt{x^4+4}}{x}\,dx$

Let $u = x^2$, then $du = 2x\,dx$, and

$$\int \frac{\sqrt{x^4+4}}{x}\,dx = \int \frac{\sqrt{x^4+4}}{x^2}\,\frac{2}{2}x\,dx = \frac{1}{2}\int \frac{\sqrt{u^2+4}}{u}\,du$$

Using Formula 34 with $a = 2$, we have:

$$\int \frac{\sqrt{u^2+4}}{u}\,du = \sqrt{u^2+4} - 2\ln\left|\frac{2+\sqrt{u^2+4}}{u}\right| + C$$

Thus,

$$\int \frac{\sqrt{x^4+4}}{x}\,dx = \frac{1}{2}\sqrt{x^4+4} - \ln\left|\frac{2+\sqrt{x^4+4}}{x^2}\right| + C$$

30. $\displaystyle\int \frac{e^x}{(4+e^x)^2(2+e^x)}\,dx$

Let $u = e^x$, then $du = e^x\,dx$ and

$$\int \frac{e^x}{(4+e^x)^2(2+e^x)}\,dx = \int \frac{1}{(4+u)^2(2+u)}\,du$$

Now, using Formula 18 with $a = 4$, $b = 1$, $c = 2$, $d = 1$, we have (after substituting e^x for u):

$$\int \frac{e^x}{(4+e^x)^2(2+e^x)}\,dx = \frac{1}{2}\cdot\frac{1}{4+e^x} + \frac{1}{4}\ln\left|\frac{2+e^x}{4+e^x}\right| + C$$

32. $\displaystyle\int \frac{1}{x\ln x\sqrt{4+\ln x}}\,dx$

Let $u = \ln x$, then $du = \frac{1}{x}\,dx$ and

$$\int \frac{1}{x\ln x\sqrt{4+\ln x}}\,dx = \int \frac{1}{\ln x\sqrt{4+\ln x}}\cdot\frac{1}{x}\,dx = \int \frac{1}{u\sqrt{4+u}}\,du$$

Using Formula 27 with $a = 4$, $b = 1$, we have:

$$\int \frac{1}{u\sqrt{4+u}}\,du = \frac{1}{2}\ln\left|\frac{\sqrt{4+u}-2}{\sqrt{4+u}+2}\right| + C$$

Therefore,

$$\int \frac{1}{x\ln x\sqrt{4+\ln x}}\,dx = \frac{1}{2}\ln\left|\frac{\sqrt{4+\ln x}-2}{\sqrt{4+\ln x}+2}\right| + C$$

34. $\displaystyle\int x^2e^{-4x}dx$

From Formula 47 with $n = 2$, $a = -4$, we have:

$$\int x^2e^{-4x}dx = \frac{x^2e^{-4x}}{-4} - \frac{2}{-4}\int xe^{-4x}dx$$

From Formula 47 with $n = 1$, $a = -4$, we have:

$$\int xe^{-4x}dx = \frac{xe^{-4x}}{-4} - \frac{1}{-4}\int e^{-4x}dx = -\frac{1}{4}xe^{-4x} - \frac{1}{16}e^{-4x} + C_1$$

Therefore,

$$\int x^2e^{-4x}dx = -\frac{1}{4}x^2e^{-4x} - \frac{1}{8}xe^{-4x} - \frac{1}{32}e^{-4x} + C$$

36. $\int x^3e^{2x}dx$

From Formula 47 with $n = 3$, $a = 2$, we have:

$$\int x^3e^{2x}dx = \frac{x^3e^{2x}}{2} - \frac{3}{2}\int x^2e^{2x}dx$$

From Formula 47 with $n = 2$, $a = 2$, we have:

$$\int x^2e^{2x}dx = \frac{x^2e^{2x}}{2} - \frac{2}{2}\int xe^{2x}dx$$

From Formula 47 with $n = 1$, $a = 2$, we have:

$$\int xe^{2x} = \frac{xe^{2x}}{2} - \frac{1}{2}\int e^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C_1$$

Therefore,

$$\int x^2e^{2x}dx = \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C_2$$

and finally

$$\int x^3e^{2x}dx = \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{8}e^{2x} + C$$

38. $\int (\ln x)^4\,dx$

From Formula 52, with $n = 4$ we have:

$$\int (\ln x)^4\,dx = x(\ln x)^4 - 4\int (\ln x)^3\,dx$$

$$\int (\ln x)^3\,dx = x(\ln x)^3 - 3\int (\ln x)^2\,dx$$

$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2\int (\ln x)dx$$

$$\int (\ln x)dx = x(\ln x) - \int dx = x\ln x - x + C$$

Therefore,

$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2x(\ln x) + 2x + C_1$$

$$\int (\ln x)^3\,dx = x(\ln x)^3 - 3x(\ln x)^2 + 6x(\ln x) - 6x + C_2$$

and finally

$$\int (\ln x)^4\,dx = x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24x\ln x + 24x + C$$

40. $\int_3^5 x^2\sqrt{x^2-9}\,dx$

From Formula 41 with $a = 3$, we have:

$$\int_3^5 x^2\sqrt{x^2-9}\,dx = \frac{1}{8}\left[x(2x^2-9)\sqrt{x^2-9}-81\ln\left|x+\sqrt{x^2-9}\right|\right]\Bigg|_3^5$$

$$= \frac{1}{8}[(820 - 81 \ln 9) - (-81 \ln 3)]$$

$$= \frac{1}{8}[820 - 81 \ln 3]$$

$$= \frac{205}{2} - \frac{81}{8}\ln 3 \approx 91.377$$

42. $\displaystyle\int_2^4 \frac{x}{(x^2-1)^2}\,dx$

Let $u = x^2 - 1$, then $du = 2x\,dx$, and

$$\int \frac{x}{(x^2-1)^2}\,dx = \int \frac{1}{(x^2-1)^2}\,\frac{2}{2}x\,dx = \int \frac{1}{u^2}\cdot\frac{1}{2}\,du$$

$$= \frac{1}{2}\int u^{-2}\,du$$

$$= -\frac{1}{2}u^{-1} + C$$

$$= -\frac{1}{2}(x^2 - 1)^{-1} + C$$

Thus,

$$\int_2^4 \frac{x}{(x^2-1)^2}\,dx = \left(-\frac{1}{2}(x^2-1)^{-1}\right)\Bigg|_2^4 = -\frac{1}{2}(15)^{-1} + \frac{1}{2}(3)^{-1}$$

$$= -\frac{1}{30} + \frac{1}{6} = \frac{4}{30} = \frac{2}{15}$$

44. $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$

Let $u = \ln x$, then $du = \frac{1}{x}\,dx$, and

$$\int \frac{(\ln x)^2}{x}\,dx = \int (\ln x)^2\,\frac{1}{x}\,dx = \int u^2\,du$$

$$= \frac{1}{3}u^3 + C$$

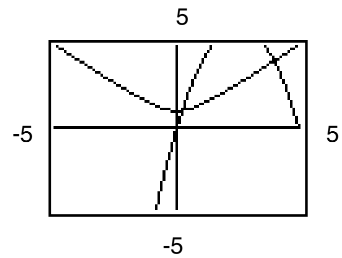
$$= \frac{1}{3}(\ln x)^3 + C$$

46. $\displaystyle\int \frac{x^2}{\sqrt{x^2-1}}\,dx$

From Formula 44 with $a = 1$, we have:

$$\int \frac{x^2}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\left(x\sqrt{x^2-1} + \ln\left|x + \sqrt{x^2-1}\right|\right) + C$$

48. $f(x) = \sqrt{1+x^2}$, $g(x) = 5x - x^2$



The graphs of f and g are shown at the right. The x-coordinates of the points of intersection are:
$x_1 = 0.21$, $x_2 = 3.97$

$$A = \int_{0.21}^{3.97} \left[(5x - x^2) - \sqrt{1+x^2}\,\right]dx$$

$$= \int_{0.21}^{3.97} (5x - x^2)dx - \int_{0.21}^{3.97} \sqrt{1+x^2}\,dx$$

For the second integral we use Formula 32 with $a = 1$:

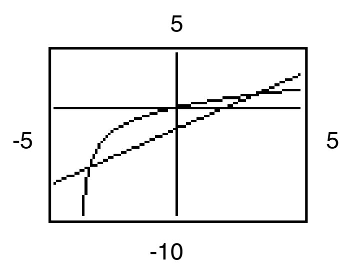
$$\int \sqrt{1+x^2}\,dx = \frac{1}{2}\left(x\sqrt{x^2+1} + \ln\left|x + \sqrt{x^2+1}\right|\right) + C$$

Thus,

$$A = \left(\frac{5}{2}x^2 - \frac{1}{3}x^3\right)\Bigg|_{0.21}^{3.97} - \frac{1}{2}\left(x\sqrt{x^2+1} + \ln\left|x + \sqrt{x^2+1}\right|\right)\Bigg|_{0.21}^{3.97}$$

$$\approx (18.545 - 0.107) - (9.170 - 0.212) = 9.48$$

50. $f(x) = \dfrac{x}{\sqrt{x+4}}$, $g(x) = x - 2$



The graphs of f and g are shown at the right. The x-coordinates of the points of intersection are:
$x_1 = -3.59$, $x_2 = 3.19$

$$A = \int_{-3.59}^{3.19} \left[\frac{x}{\sqrt{x+4}} - (x-2)\right]dx$$

$$= \int_{-3.59}^{3.19} \frac{x}{\sqrt{x+4}}\,dx - \int_{-3.59}^{3.19} (x - 2)dx$$

For the first integral, we use Formula 25 with $a = 4$ and $b = 1$:

$$\int \frac{x}{\sqrt{x+4}}\,x = \frac{2(x-8)}{3}\sqrt{x+4} + C, \text{ and therefore}$$

$$A = \left(\frac{2}{3}(x-8)\sqrt{x+4}\right)\Bigg|_{-3.59}^{3.19} - \frac{1}{2}(x - 2)^2\Bigg|_{-3.59}^{3.19}$$

$$\approx (-8.598 + 4.947) + (-0.708 + 15.624) \approx 11.27$$

52. Find $\bar{x}$, the supply when the price $\bar{p} = 20$:

$$20 = \frac{10\bar{x}}{300 - \bar{x}}$$
$$6{,}000 - 20\bar{x} = 10\bar{x}$$
$$30\bar{x} = 6{,}000$$
$$\bar{x} = 200$$

Producers' surplus:

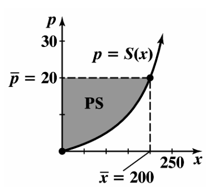
$$PS = \int_0^{\bar{x}} [\bar{p} - S(x)]dx = \int_0^{200} \left[20 - \frac{10x}{300 - x}\right] dx$$
$$= \int_0^{200} \left[\frac{6{,}000 - 20x - 10x}{300 - x}\right] dx$$
$$= \int_0^{200} \left(\frac{6{,}000 - 30x}{300 - x}\right) dx$$

Use Formula 20 with $a = 6{,}000$, $b = -30$, $c = 300$, $d = -1$. Thus,

$$PS = \left[\frac{-30x}{-1} + \frac{-6{,}000 + 9{,}000}{1}\ln|300 - x|\right]_0^{200}$$
$$= (6{,}000 + 3{,}000 \ln 100) - (3{,}000 \ln 300)$$
$$= 6{,}000 + 3{,}000 \ln\left(\frac{100}{300}\right) = 6{,}000 + 3{,}000 \ln\left(\frac{1}{3}\right)$$
$$= 6{,}000 - 3{,}000 \ln 3 \approx 2{,}704$$

Thus, the producers' surplus is $2,704.

54.



The shaded region represents the producers' surplus.

56. $C'(x) = \dfrac{65 + 20x}{1 + 0.4x}$, $C(0) = 11{,}000$

$$C(x) = \int \frac{65 + 20x}{1 + 0.4x}\, dx$$

Use Formula 20 with $a = 65$, $b = 20$, $c = 1$, $d = 0.4$:

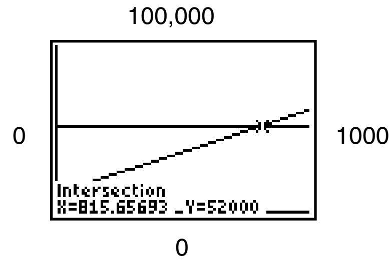
$$C(x) = \frac{20x}{0.4} + \frac{26 - 20}{0.16}\ln|1 + 0.4x| + C_1$$

Since $C(0) = 11{,}000$, we have $C_1 = 11{,}000$ and therefore,

$$C(x) = 50x + 37.5\ln(1 + 0.4x) + 11{,}000$$

To find the production level that produces a cost of $52,000 per week, we need to solve:
$52{,}000 = 50x + 37.5\ln(1 + 0.4x) + 11{,}000$ for x.

The production level is 816 DVD players.



For a production level of 700 DVD players per week, the cost is:
$C(700) = 50(700) + 37.5 \ln(1 + 0.4(700)) + 11{,}000$
$\approx \$46{,}211$

58. $FV = e^{rT}\int_0^T f(t)e^{-rt}\,dt$

Now, $r = 0.037$, $T = 5$, $f(t) = 200t$.

$$FV = e^{(0.037)5}\int_0^5 200te^{-0.037t}\,dt = 200e^{0.185}\int_0^5 te^{-0.037t}\,dt$$

To evaluate the integral, use Formula 47 with $n = 1$ and $a = -0.037$:

$$\begin{aligned}\int te^{-0.037t}\,dt &= \frac{te^{-0.037t}}{-0.037} - \frac{1}{-0.037}\int e^{-0.037t}\,dt\\ &= \frac{-te^{-0.037t}}{0.037} + \frac{1}{0.037}\left(\frac{e^{-0.037t}}{-0.037}\right) + C\\ &= \frac{-te^{-0.037t}}{0.037} - \frac{e^{-0.0.37t}}{(0.037)^2} + C\end{aligned}$$

Thus,

$$FV = 200e^{0.185}\left(\frac{-te^{-0.037t}}{0.037} - \frac{e^{-0.037t}}{(0.037)^2}\right)\Bigg|_0^5$$

$$= 200e^{0.185}\left[\left(\frac{-(5)e^{-0.037(5)}}{0.037} - \frac{e^{-0.037(5)}}{(0.037)^2}\right) - \left(\frac{-(0)e^{-0.037(0)}}{0.037} - \frac{e^{-0.037(0)}}{(0.037)^2}\right)\right]$$

$$= 200e^{0.185}\left(\frac{-5e^{-0.185}}{0.037} - \frac{e^{-0.185}}{(0.037)^2} + \frac{1}{(0.037)^2}\right)$$

$$= \frac{-1000e^{0.185-0.185}}{0.037} - \frac{200e^{0.185-0.185}}{(0.037)^2} + \frac{200e^{0.185}}{(0.037)^2}$$

$$= \frac{-1000e^{0}}{0.037} - \frac{200e^{0}}{(0.037)^2} + \frac{200e^{0.185}}{(0.037)^2}$$

$$= \frac{-1000(0.037) - 200 + 200e^{0.185}}{(0.037)^2} \approx \$2{,}661.57$$

$$\text{Total income} = \int_0^5 200t\,dt = 100t^2\Big|_0^5 = 2{,}500$$

Thus, Interest Earned = 2,661.57 - 2,500 $\approx$ \$161.57

60. Index of Income Concentration:

$$2\int_0^1 [x - f(x)]dx = 2\int_0^1 \left[x - \frac{1}{2}x^2\sqrt{1+3x}\right]dx$$

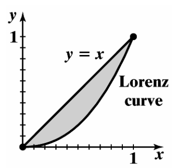
$$= 2\int_0^1 x\,dx - \int_0^1 x^2\sqrt{1+3x}\,dx$$

For the second integral, use Formula 23 with $a = 1$ and $b = 3$:

$$2\int_0^1 [x - f(x)]dx = x^2\Big|_0^1 - \left[\frac{2(135x^2 - 36x + 8)}{2,835}\sqrt{(1+3x)^3}\right]_0^1$$

$$= 1 - \left[\frac{1,712}{2,835} - \frac{16}{2,835}\right]$$

$$= 1 - \frac{1,696}{2,835} = \frac{1,139}{2,835} \approx 0.4018$$

62. 

As the area bounded by the two curves gets larger, the Lorenz curve moves away from $y = x$ and the distribution of income approaches perfect inequality — one individual would have all of the wealth and the rest would have none.

64. $D(x) = \dfrac{50}{\sqrt{100+6x}}$

Average price $= \dfrac{1}{250-50}\displaystyle\int_{50}^{250} \frac{50}{\sqrt{100+6x}}\,dx$

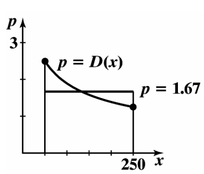
$= \dfrac{1}{4}\displaystyle\int_{50}^{250} \frac{1}{\sqrt{100+6x}}\,dx$

For this integral use Formula 24 with $a = 100$ and $b = 6$. Then

$$\text{Average price} = \overline{p} = \frac{1}{4}\left[\frac{2\sqrt{100+6x}}{6}\right]_{50}^{250}$$

$$= \frac{1}{12}\left(\sqrt{1,600} - \sqrt{400}\right)$$

$$= \frac{1}{12}(40 - 20) = \frac{20}{12} \approx \$1.67$$

66. 

The area under the price-demand curve, $y = p(x)$, is the same as the area under the average price line, $y = 1.67$, from $x = 50$ to $x = 250$.

68. $R'(x) = \dfrac{x}{\sqrt{1+2x}}$, $R(0) = 0$

$$R(x) = \int \frac{x}{\sqrt{1+2x}}\,dx = \frac{2(2x-2)}{12}\sqrt{1+2x} + C$$

(From Formula 25 with $a = 1$ and $b = 2$).

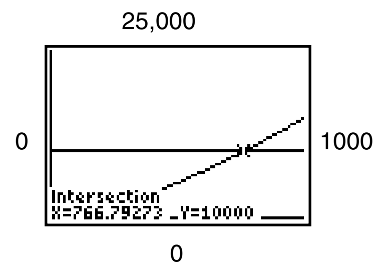
$R(0) = 0 = -\dfrac{1}{3} + C$ and hence $C = \dfrac{1}{3}$.

Thus, $R(x) = \dfrac{1}{3}(x - 1)\sqrt{1+2x} + \dfrac{1}{3}$

The number of calculators that must be sold to produce \$10,000 in revenue: 767

For $x = 1{,}000$,

$$R(1{,}000) = \frac{1}{3}\,999\sqrt{2{,}001} + \frac{1}{3} \approx \$14{,}896$$



70. $C(t) = t\sqrt{24-t}$, $0 \le t \le 24$

Average concentration $= \dfrac{1}{24-0}\displaystyle\int_0^{24} t\sqrt{24-t}\,dt$

$$= \frac{1}{24}\left[\frac{2(-3t-48)}{15}\sqrt{(24-t)^3}\right]_0^{24}$$

(Formula 22 with $a = 24$, $b = -1$)

$$= \frac{1}{24}\left[(0) - \left(\frac{-96}{15}\sqrt{(24)^3}\right)\right]$$

$$= \frac{96\sqrt{24}}{15} = \frac{32\sqrt{24}}{5} = \frac{64\sqrt{6}}{5} \approx 31.35 \text{ ppm}$$

72. $f(t) = \dfrac{500}{2+3e^{-t}}$, $[0, 10]$

Average $= \dfrac{1}{10-0}\displaystyle\int_0^{10} \frac{500}{2+3e^{-t}}\,dt$

$$= 50\int_0^{10} \frac{1}{2+3e^{-t}}\,dt$$

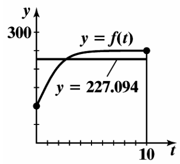
(Formula 48 with $a = -1$, $c = 2$, $d = 3$)

$$= 50\left[\frac{t}{2} + \frac{1}{2}\ln\left|2+3e^{-t}\right|\right]_0^{10}$$

$$= 50\left[\left(5 + \frac{1}{2}\ln(2+3e^{-10})\right) - \left(\frac{1}{2}\ln 5\right)\right]$$

$= 250 + 25\ln(2 + 3e^{-10}) - 25\ln 5 \approx 227.094$ or 227,094 voters.

74.



The area under voter equation, $y = f(t)$,
is the same as the area under the line representing the average number of voters, $y = 227.094$, over the interval [0, 10].