Solution ***Section* 2.1 – Functions and Graphs**

Exercise

$$f(x)=\begin{cases} 2+x & \text{if } x<-4 \\ -x & \text{if } -4\le x\le 2 \\ 3x & \text{if } x>2 \end{cases}$$

Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

Solution

a) $f(-5)=2-5=-3$

b) $f(-1)=-(-1)=1$

c) $f(0)=-0=0$

d) $f(3)=3(3)=9$

Exercise

$$f(x)=\begin{cases} -2x & \text{if } x<-3 \\ 3x-1 & \text{if } -3\le x\le 2 \\ -4x & \text{if } x>2 \end{cases}$$

Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

Solution

a) $f(-5)=-2(-5)=10$

b) $f(-1)=3(-1)-1=-4$

c) $f(0)=3(0)-1=-1$

d) $f(3)=-4(3)=-12$

Exercise

$$f(x)=\begin{cases} x^3+3 & \text{if } -2\le x\le 0 \\ x+3 & \text{if } 0<x<1 \\ 4+x-x^2 & \text{if } 1\le x\le 3 \end{cases}$$

Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

Solution

a) $f(-5)=$ *doesn't exist*

b) $f(-1)=(-1)^3+3$

$\underline{=2}$

c) $f(0)=(0)^3+3$

$\underline{=3}$

d) $f(3) = 4 + (3) - (3)^2$

$\underline{=-2}$

Exercise

$$h(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & if \quad x \neq 3 \\ 6 & if \quad x = 3 \end{cases}$$

Find: $h(5),\ h(0),\ and\ h(3)$

Solution

a) $h(5) = \dfrac{5^2 - 9}{5 - 3}$

$\underline{=8}$

b) $h(0) = \dfrac{0^2 - 9}{0 - 3}$

$\underline{=3}$

c) $\underline{h(3) = 6}$

Exercise

$$f(x) = \begin{cases} 3x + 5 & if \quad x < 0 \\ 4x + 7 & if \quad x \ge 0 \end{cases}$$

Find

a) $f(0)$ *b)* $f(-2)$ *c)* $f(1)$ *d)* $f(3) + f(-3)$ *e)* Graph $f(x)$

Solution

a) $f(0) = 4(0) + 7$

$\underline{=7}$

b) $f(-2) = 3(-2) + 5$

$\underline{=-1}$

c) $f(1) = 4(1) + 7$

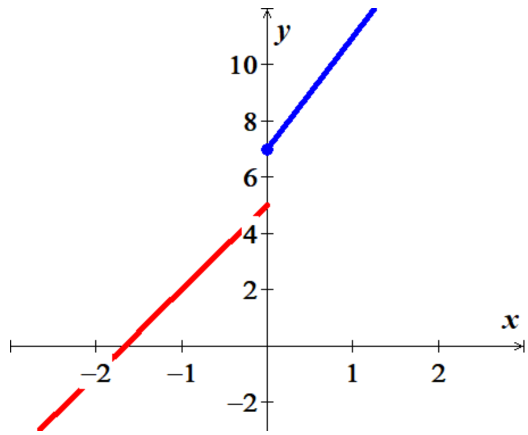
$\underline{=11}$

d) $f(3) + f(-3) = 4(3) + 7 + 3(-3) + 5$

$= 12 + 7 - 9 + 5$

$\underline{=15}$

e)



Exercise

$$f(x)=\begin{cases} 6x-1 & if \quad x<0 \\ 7x+3 & if \quad x\ge 0 \end{cases}$$ Find

a) $f(0)$ *b)* $f(-1)$ *c)* $f(4)$ *d)* $f(2)+f(-2)$ *e)* Graph $f(x)$

Solution

a) $f(0)=7(0)+3$

$=3$

b) $f(-2)=6(-1)-1$

$=-7$

c) $f(4)=7(4)+3$

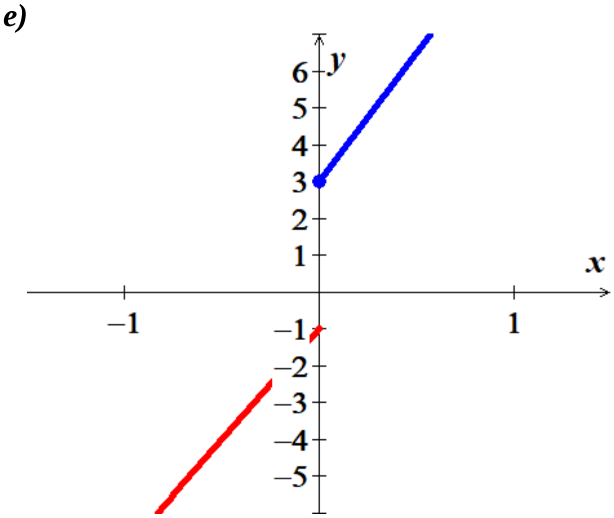
$=31$

d) $f(2)+f(-2)=7(2)+3+6(-2)-1$

$=14+3-12-1$

$=4$

e)



Exercise

$$f(x)=\begin{cases}2x+1 & if \quad x\le 1\\ 3x-2 & if \quad x>1\end{cases}$$ Find

a) $f(0)$ *b)* $f(2)$ *c)* $f(-2)$ *d)* $f(1)+f(-1)$ *e)* Graph $f(x)$

Solution

a) $f(0)=2(0)+1$

$\underline{=1}$

b) $f(2)=3(2)-2$

$\underline{=4}$

c) $f(-2)=2(-2)+1$

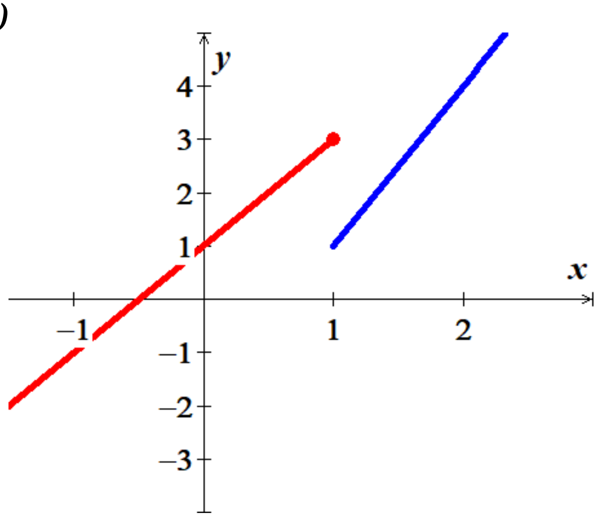
$\underline{=-3}$

d) $f(1)+f(-1)=(2(1)+1)+(2(-1)+1)$

$=2+1-2+1$

$\underline{=2}$

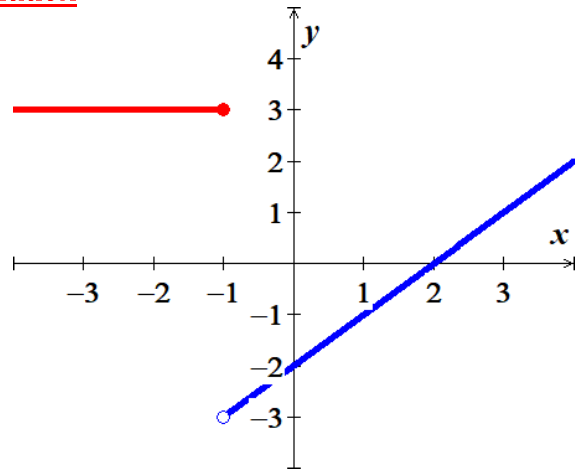
e)



Exercise

Graph the piecewise function defined by $f(x)=\begin{cases} 3 & if \quad x\le -1 \\ x-2 & if \quad x>-1 \end{cases}$

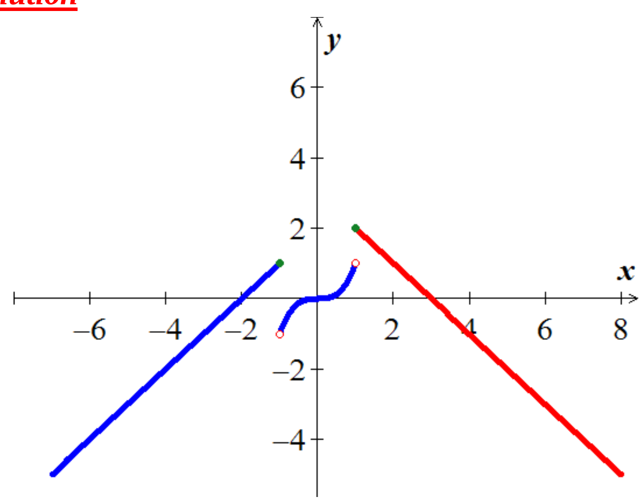
Solution



Exercise

Sketch the graph $f(x)=\begin{cases} x+2 & if & x\le -1 \\ x^3 & if & -1<x<1 \\ -x+3 & if & x\ge 1 \end{cases}$

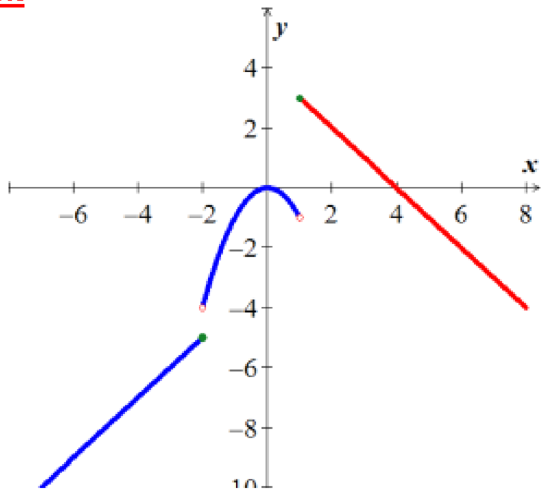
Solution



Exercise

Sketch the graph $f(x)=\begin{cases} x-3 & if \quad x\le -2 \\ -x^2 & if \quad -2<x<1 \\ -x+4 & if \quad x\ge 1 \end{cases}$

Solution



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x)=x^2-2x+3$$

Solution

Relative Maximum: *None*

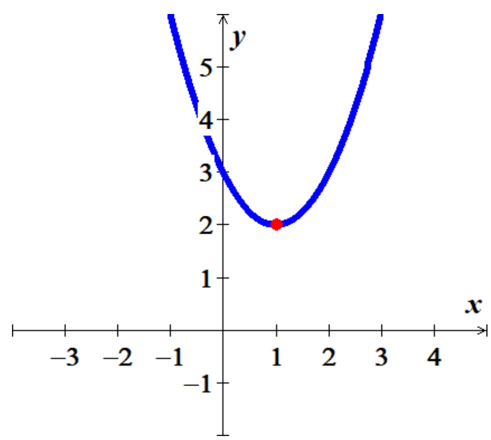
Minimum Point: $(1,\ 2)$

Increasing: $(1,\ \infty)$

Decreasing: $(-\infty,\ 1)$

Domain: $\mathbb{R}$

Range: $[2,\ \infty)$



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x) = -x^2 - 2x + 3$$

Solution

Maximum Point: $(-1,\ 4)$

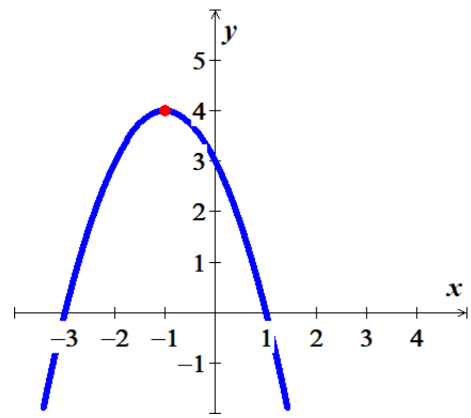
Relative Minimum: *None*

Increasing: $(-\infty,\ -1)$

Decreasing: $(-1,\ \infty)$

Domain: $\mathbb{R}$

Range: $(-\infty,\ 4]$



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x) = -x^3 + 3x^2$$

Solution

Relative Maximum: $(2,\ 4)$

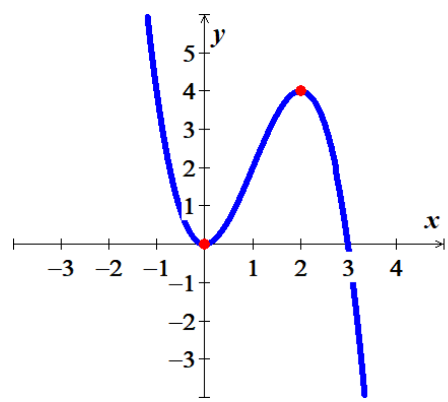
Relative Minimum: $(0,\ 0)$

Increasing: $(0,\ 2)$

Decreasing: $(-\infty,\ 0)\quad (2,\ \infty)$

Domain: $\mathbb{R}$

Range: $\mathbb{R}$



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x) = x^3 - 3x^2$$

Solution

Relative Maximum: $(0,\ 0)$

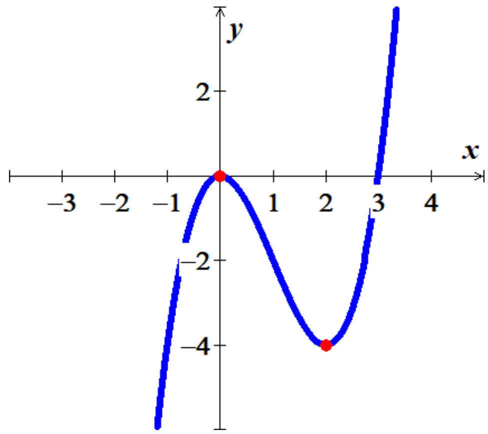
Relative Minimum: $(2,\ -4)$

Increasing: $(-\infty,\ 0)\ \ (2,\ \infty)$

Decreasing: $(0,\ 2)$

Domain: $\mathbb{R}$

Range: $\mathbb{R}$



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x) = \frac{1}{4}x^4 - 2x^2$$

Solution

Relative Maximum: $(0,\ 0)$

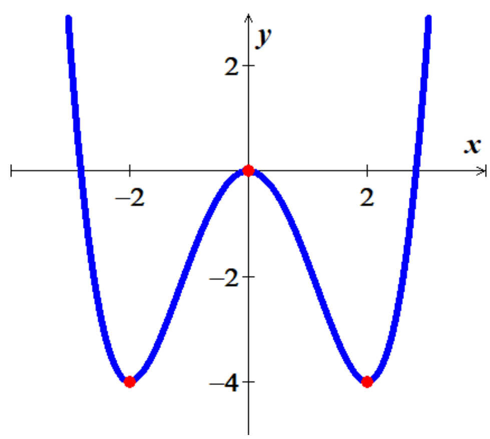
Minimum Points: $(-2,\ -4)\ \&\ (2,\ -4)$

Increasing: $(-2,\ 0) \cup (2,\ \infty)$

Decreasing: $(-\infty,\ -2) \cup (0,\ 2)$

Domain: $\mathbb{R}$

Range: $[-4,\ \infty)$



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x) = \frac{4}{81}x^4 - \frac{8}{9}x^2 + 4$$

Solution

Relative Maximum: $(0,\ 4)$

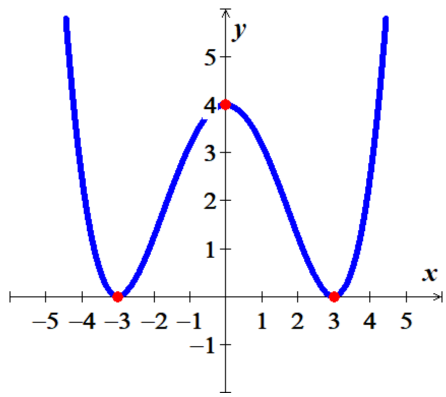
Minimum Points: $(-3,\ 0)\ \&\ (3,\ 0)$

Increasing: $(-3,\ 0) \cup (3,\ \infty)$

Decreasing: $(-\infty,\ -3) \cup (0,\ 3)$

Domain: $\mathbb{R}$

Range: $[0,\ \infty)$



Exercise

The elevation *H*, in *meters*, above sea level at which the boiling point of water is in ***t*** degrees Celsius is given by the function

$$H(t) = 1000(100-t) + 580(100-t)^2$$

At what elevation is the boiling point 99.5°.

Solution

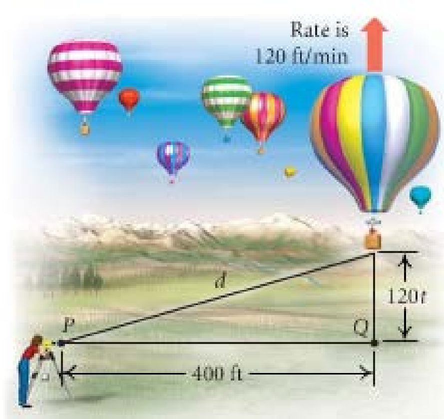
$$H(99.5) = 1000(100-99.5) + 580(100-99.5)^2$$

$$= 645\ m$$

Exercise

A hot-air balloon rises straight up from the ground at a rate of 120 *ft./min*. The balloon is tracked from a rangefinder on the ground at point P, which is 400 *ft*. from the release point Q of the balloon. Let ***d*** = the distance from the balloon to the rangefinder and ***t*** – the time, in minutes, since the balloon was released. Express ***d*** as a function of ***t***.

Solution



$$d^2 = (120t)^2 + 400^2$$

$$d = \sqrt{14400t^2 + 160000}$$

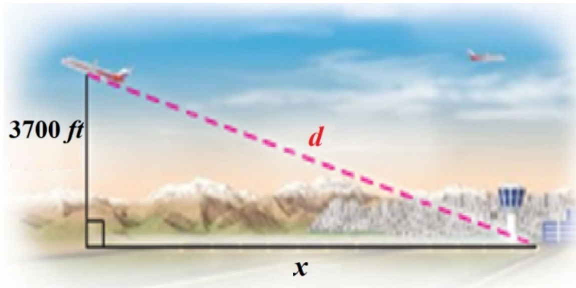
$$d = \sqrt{1600\left(9t^2 + 100\right)}$$

$$d(t) = 40\sqrt{9t^2 + 100}$$

Exercise

An airplane is flying at an altitude of 3700 *feet*. The slanted distance directly to the airport is d *feet*. Express the horizontal distance x as a function of d.

Solution



$$d^2 = (3,700)^2 + x^2$$

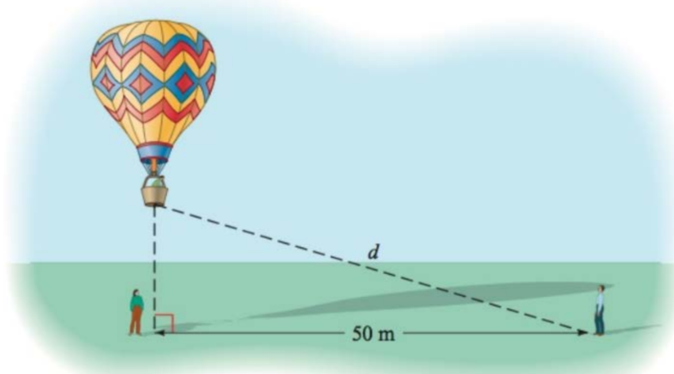
$$h^2 = d^2 - (3700)^2$$

$$\boxed{h(t) = \sqrt{d^2 - (3,700)^2}}$$

Exercise

For the first minute of flight, a hot air balloon rises vertically at a rate of 3 *m/sec*. If *t* is the time in *seconds* that the balloon has been airborne, write the distance *d* between the balloon and a point on the ground 50 *meters* from the point to lift off as a function of *t*.

Solution



$\boxed{h = 3t}$ $\qquad v = \frac{h}{t}$

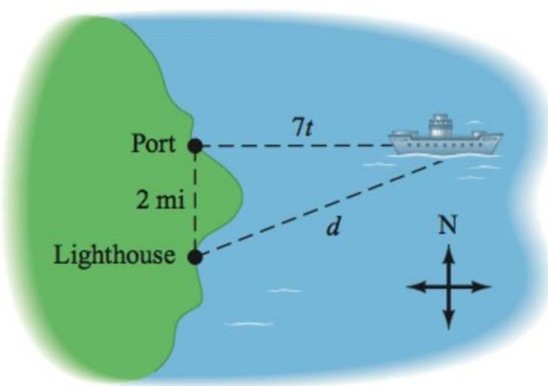
$$d^2 = h^2 + 50^2$$

$$\boxed{d(t) = \sqrt{9t^2 + 2,500}}$$

Exercise

A light house is 2 *miles* south of a port. A ship leaves port and sails east at a rate of 7 *miles* per *hour*. Express the distance *d* between the ship and the lighthouse as a function of time, given that the ship has been sailing for *t* hours.

Solution

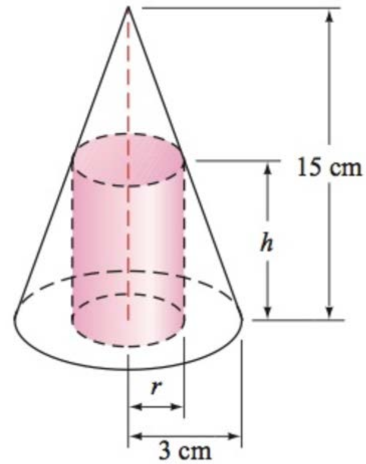


$$d^2 = 4^2 + (7t)^2$$

$$\boxed{d(t) = \sqrt{16 + 49t^2}}$$

Exercise

A cone has an altitude of 15 *cm* and a radius of 3 *cm*. A right circular cylinder of radius *r* and height *h* is inscribed in the cone. Use similar triangles to write *h* as a function of *r*.



Solution

$$\frac{15-h}{15}=\frac{r}{3}$$

$$15-h=5r$$

$$\boxed{h(r)=15-5r}$$

Exercise

Water is flowing into a conical drinking cup with an altitude of 4 *inches* am a radius of 2 *inches*.

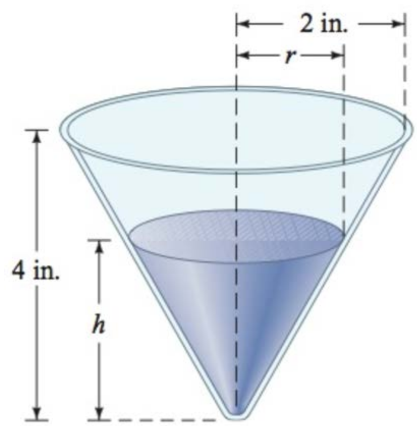
a) Write the radius *r* of the surface of the water as a function of its depth *h*.

b) Write the volume *V* of the water as a function of its depth *h*.

Solution

a) $\frac{h}{4}=\frac{r}{2}$

$$\boxed{r(h)=\frac{1}{2}h}$$



b) $Area=\pi r^2$

$$V=\frac{1}{3}\pi r^2 h$$

$$=\frac{1}{3}\pi\left(\frac{h^2}{4}\right)h$$

$$\boxed{=\frac{1}{12}\pi h^3}$$

Exercise

A water tank has the shape of a right circular cone with height 16 *feet* and radius 8 *feet*. Water is running into the tank so that the radius *r* (in *feet*) of the surface of the water is given by $r=1.5t$, where *t* is the time (in *minutes*) that the water has been running.

a) The area *A* of the surface of the water is $A=\pi r^2$. Find $A(t)$ and use it to determine the area of the surface of the water when $t=2\ \ minutes$.

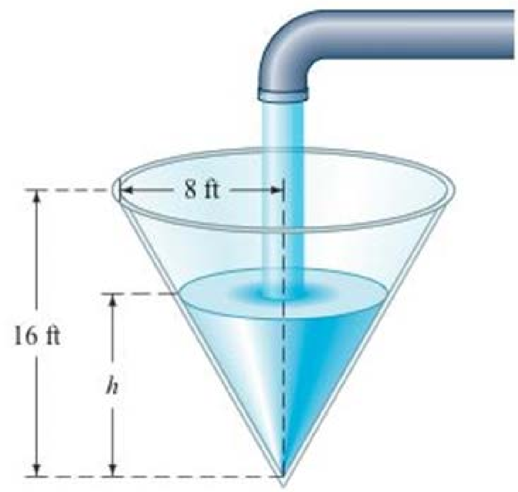
b) The volume *V* of the water is given by $V=\frac{1}{3}\pi r^2 h$. Find $V(t)$ and use it to determine the volume of the water when $t=3\ \ minutes$

Solution

c) $Area = \pi r^2$

$A(t) = \pi\left(\frac{3}{2}t\right)^2$

$= \frac{9\pi}{4}t^2$



d) $\frac{h}{16} = \frac{r}{8}$

$h = 2r$

$V(t) = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi r^2(2r)$

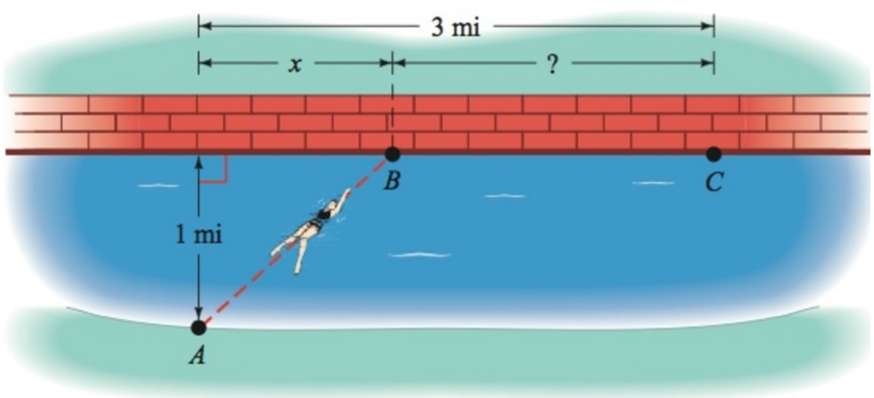
$= \frac{2}{3}\pi r^3$

$= \frac{2}{3}\pi\left(\frac{3}{2}t\right)^3$

$= \frac{9}{4}\pi t^3$

Exercise

An athlete swims from point ***A*** to point ***B*** at a rate of 2 *miles* per *hour* and runs from point ***B*** to point ***C*** at a rate of 8 *miles* per *hour*. Use the dimensions in the figure to write the time *t* required to reach point ***C*** as a function of *x*.



Solution

$Swimming\ distance = \sqrt{x^2+1}$

$t_{swim} = \frac{\sqrt{x^2+1}}{2}$ $\qquad t = \frac{d}{v}$

$Running\ distance = 3 - x$

$$t_{run} = \frac{3-x}{8} \qquad\qquad t = \frac{d}{v}$$

$$\boxed{t_{total} = \frac{\sqrt{x^2+1}}{2} + \frac{3-x}{8}}$$

Exercise

A device used in golf to estimate the distance d, in *yards*, to a hole measures the size $\boldsymbol{s}$, in *inches*, that the 7-*foot* pin appears to be in a viewfinder. Express the distance d as a function of $\boldsymbol{s}$.

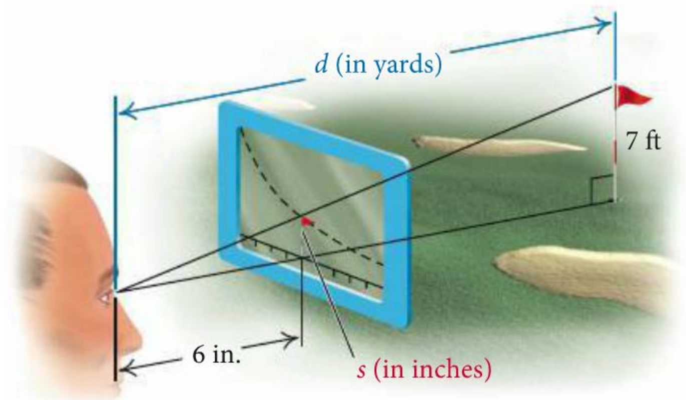
Solution

$$\frac{d}{6} = \frac{7}{s}\frac{ft}{in}$$

$$d = \frac{7}{s}\frac{ft}{in} 6in$$

$$d = \frac{42}{s}\, ft \frac{1yd}{3\,ft}$$

$$\boxed{d(s) = \frac{14}{s}}$$



Exercise

A *rhombus* is inscribed in a rectangle that is *w meters* wide with a perimeter of 40 *m*. Each vertex of the rhombus is a midpoint of a side of the rectangle. Express the area of the *rhombus* as a function of the rectangle's width.

Solution

The area of the rhombus = $\frac{1}{2}$ area of the rectangle, since each vertex of the rhombus is a midpoint of a side of the rectangle.

$$Perimter: \quad 2l + 2w = 40 \qquad \textit{Divide both sides by 2}$$

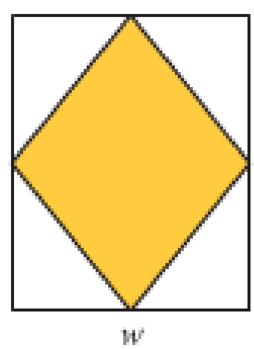
$$l + w = 20$$

$$l = 20 - w$$

$$Area\ of\ the\ rectangle = lw = (20-w)w$$

$$Area\ of\ the\ rhombus = \frac{1}{2}\left(20w - w^2\right)$$

$$\boxed{= -\frac{1}{2}w^2 + 10w}$$



Exercise

The surface area S of a right circular cylinder is given by the formula $S = 2\pi rh + 2\pi r^2$. if the height is twice the radius, find each of the following.

a) A function $S(r)$ for the surface area as a function of r.

b) A function $S(h)$ for the surface area as a function of h.

Solution

Given: $h = 2r$

a) $S = 2\pi rh + 2\pi r^2$

$$S(r) = 2\pi r(2r) + 2\pi r^2$$

$$= 4\pi r^2 + 2\pi r^2$$

$$\boxed{= 6\pi r^2}$$

b) $r = \frac{1}{2}h$

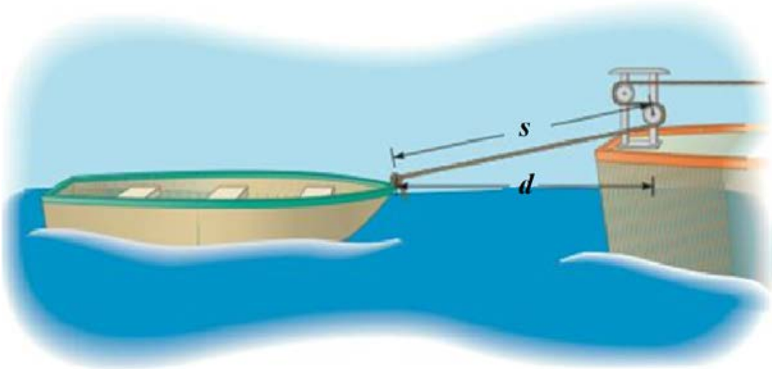
$$S(h) = 2\pi\left(\frac{1}{2}h\right)h + 2\pi\left(\frac{1}{2}h\right)^2$$

$$= \pi h^2 + \frac{1}{2}\pi h^2$$

$$\boxed{= \frac{3}{2}\pi h^2}$$

Exercise

A boat is towed by a rope that runs through a pulley that is 4 *feet* above the point where the rope is tied to the boat. The length (in *feet*) of the rope from the boat to the pulley is given by $s = 48 - t$, where t is the time in *seconds* that the boat has been in tow. The horizontal distance from the pulley to the boat is d.



a) Find $d(t)$

b) Evaluate $s(35)$ and $d(35)$

Solution

a) $s^2 = d^2 + 4^2$

$$d^2 = (48-t)^2 - 16$$

$$d(t) = \sqrt{2,304 - 96t + t^2 - 16}$$

$$= \sqrt{t^2 - 96t + 2,288}$$

b) $s(35) = 48 - 35$

$$= 13 \ \ feet$$

$$d(35) = \sqrt{(48-35)^2 - 16}$$

$$= \sqrt{13^2 - 16}$$

$$= \sqrt{153} \ \ feet$$

Exercise

The light from a lamppost casts a shadow from a ball that was dropped from a height of 22 *feet* above the ground. The distance *d*, in *feet*, the ball has dropped *t seconds* after it is released is given by $d(t) = 16t^2$. Find the distance *x*, in *feet*, of the shadow from the base of the lamppost as a function of time *t*.

Solution

$$\frac{22 - 16t^2}{22} = \frac{x-12}{x}$$

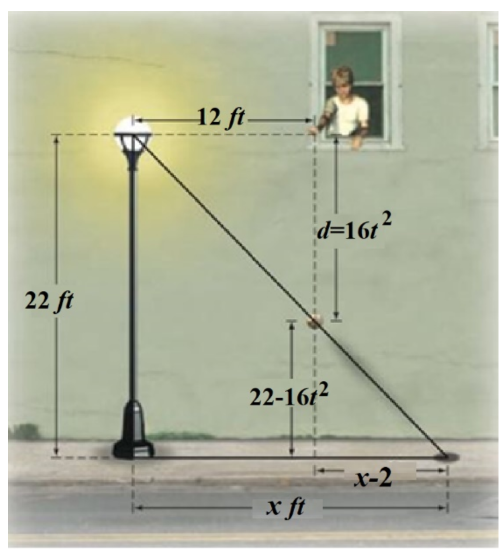
$$\left(22 - 16t^2\right)x = 22(x-12)$$

$$\left(22 - 16t^2\right)x = 22x - 264$$

$$\left(22 - 16t^2 - 22\right)x = -264$$

$$-16t^2 x = -264$$

$$x(t) = \frac{33}{2t^2}$$



Exercise

A right circular cylinder of height *h* and a radius *r* is inscribed in a right circular cone with a height of 10 *feet* and a base with radius 6 *feet*.

a) Express the height *h* of the cylinder as a function of *r*.

b) Express the volume *V* of the cylinder as a function of *r*.

c) Express the volume *V* of the cylinder as a function of *h*.

Solution

a) $\dfrac{h}{10} = \dfrac{6-r}{6}$

$$\boxed{h(r) = \frac{5}{3}(6-r)}$$

b) $V = \pi r^2 h$

$$V(r) = \frac{5}{3}\pi r^2 (6-r)$$

$$\boxed{= \frac{5}{3}\pi\left(6r^2 - r^3\right)}$$

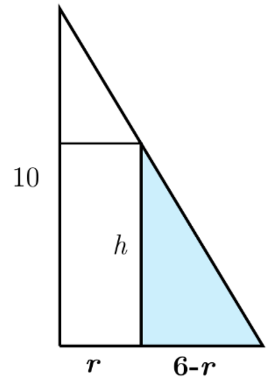
c) $\dfrac{3}{5}h = 6 - r$

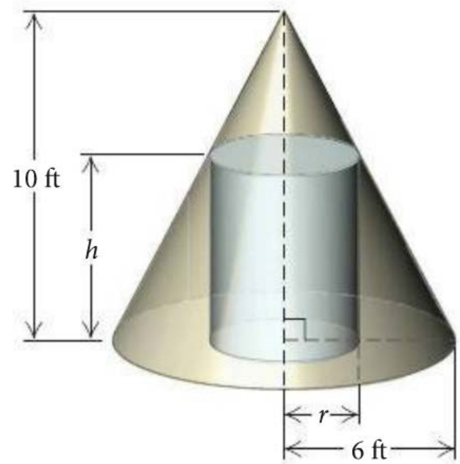
$$r = 6 - \frac{3}{5}h$$

$$V = \pi r^2 h$$

$$V(h) = \pi\left(\frac{30-3h}{5}\right)^2 h$$

$$\boxed{= \frac{1}{25}\pi h(30-3h)^2}$$





Solution Section 2.2 – Function Operations

Exercise

Find the domain: $f(x) = 7x + 4$

Solution

Domain: $(-\infty, \infty)$

Exercise

Find the domain: $f(x) = |3x - 2|$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain: $f(x) = 3x + \pi$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain: $f(x) = \sqrt{7}x + \frac{1}{2}$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain: $f(x) = -2x^2 + 3x - 5$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain: $f(x) = x^3 - 2x^2 + x - 3$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain: $f(x) = x^2 - 2x - 15$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain $f(x) = 4 - \dfrac{2}{x}$

Solution

Domain: $x \neq 0$

Exercise

Find the domain $f(x) = \dfrac{1}{x^4}$

Solution

Domain: $x \neq 0$

Exercise

Find the domain: $g(x) = \dfrac{3}{x-4}$

Solution

Domain: $x \neq 4$

Exercise

Find the domain $y = \dfrac{2}{x-3}$

Solution

Domain: $x \neq 3$

Exercise

Find the domain $\quad y=\frac{-7}{x-5}$

Solution

Domain: $\underline{x \neq 5}$

Exercise

Find the domain $\quad f(x)=\frac{x+5}{2-x}$

Solution

$2-x \neq 0$

Domain: $\underline{x \neq 2}$

Exercise

Find the domain $\quad f(x)=\frac{8}{x+4}$

Solution

$x+4 \neq 0$

Domain: $\underline{x \neq -4}$

Exercise

Find the domain $\quad f(x)=\frac{1}{x+4}$

Solution

Domain: $\underline{x \neq -4}$

Exercise

Find the domain $\quad f(x)=\frac{1}{x-4}$

Solution

Domain: $\underline{x \neq 4}$

Exercise

Find the domain $f(x)=\dfrac{3x}{x+2}$

Solution

Domain: $x \neq -2$

Exercise

Find the domain $f(x)=x-\dfrac{2}{x-3}$

Solution

Domain: $x \neq 3$

Exercise

Find the domain $f(x)=x+\dfrac{3}{x-5}$

Solution

Domain: $x \neq 5$

Exercise

Find the domain $f(x)=\dfrac{1}{2}x-\dfrac{8}{x+7}$

Solution

Domain: $x \neq -7$

Exercise

Find the domain $f(x)=\dfrac{1}{x-3}-\dfrac{8}{x+7}$

Solution

Domain: $x \neq -7,\ 3$

Exercise

Find the domain $f(x)=\dfrac{1}{x+4}-\dfrac{2x}{x-4}$

Solution

Domain: $x \neq \pm 4$

Exercise

Fib+cnd the domain $f(x)=\frac{3x^2}{x+3}-\frac{4x}{x-2}$

Solution

Domain: $\underline{x \neq -3,\ 2\ }|$

Exercise

Find the domain $f(x)=\frac{1}{x^2-2x+1}$

Solution

$x^2-2x+1 \neq 0 \qquad a+b+c=0 \;\rightarrow\; x=1,\ \frac{c}{a}$

Domain: $\underline{x \neq 1\ }|$

Exercise

Find the domain $f(x)=\frac{x}{x^2+3x+2}$

Solution

$x^2+3x+2 \neq 0 \qquad a-b+c=0 \;\rightarrow\; x=-1,\ -\frac{c}{a}$

Domain: $\underline{x \neq -1,\ -2\ }|$

Exercise

Find the domain $f(x)=\frac{x^2}{x^2-5x+4}$

Solution

$x^2-5x+4 \neq 0 \qquad a+b+c=0 \;\rightarrow\; x=1,\ \frac{c}{a}$

Domain: $\underline{x \neq 1,\ 4\ }|$

Exercise

Find the domain $f(x)=\frac{1}{x^2-4x-5}$

Solution

$x^2-4x-5 \neq 0 \qquad a-b+c=0 \;\rightarrow\; x=-1,\ -\frac{c}{a}$

Domain: $\underline{x \neq -1,\ 5\ }|$

Exercise

Find the domain $g(x)=\dfrac{2}{x^2+x-12}$

Solution

$$x^2+x-12\neq 0$$
$$(x+4)(x-3)\neq 0$$

Domain: $\boxed{x\neq -4,\ 3}$ $\boxed{(-\infty,-4)\cup(-4,3)\cup(3,\infty)}$

Exercise

Find the domain $h(x)=\dfrac{5}{\frac{4}{x}-1}$

Solution

$$x\neq 0 \qquad \frac{4}{x}-1\neq 0$$
$$\frac{4-x}{x}\neq 0$$
$$4-x\neq 0$$
$$x\neq 4$$

Domain: $\boxed{x\neq 0,\ 4}$ $\boxed{(-\infty,0)\cup(0,4)\cup(4,\infty)}$

Exercise

Find the domain $y=\sqrt{x}$

Solution

$$x\geq 0$$

Domain: $\boxed{x\geq 0}$ $\boxed{[0,\ \infty)}$

Exercise

Find the domain $f(x)=\sqrt{8-3x}$

Solution

$$8-3x\geq 0$$
$$8\geq 3x$$

Domain: $\boxed{x\leq \frac{8}{3}}$ $\boxed{\left(-\infty,\ \frac{8}{3}\right]}$

Exercise

Find the domain $y=\sqrt{4x+1}$

Solution

$4x+1\ge 0 \Rightarrow x\ge -\frac{1}{4}$

Domain: $\boxed{x\ge -\frac{1}{4}}$ $\left[-\frac{1}{4},\infty\right)$

Exercise

Find the domain $y=\sqrt{7-2x}$

Solution

$7-2x\ge 0$

$-2x\ge -7$

Domain: $\boxed{x\le \frac{7}{2}}$ $\left(-\infty,\frac{7}{2}\right]$

Exercise

Find the domain $f(x)=\sqrt{8-x}$

Solution

$8-x\ge 0$

Domain: $\boxed{x\le -8}$ $\left(-\infty,\ 8\right]$

Exercise

Find the domain $f\left(x\right)=\sqrt{3-2x}$

Solution

Domain: $\boxed{x\le \frac{3}{2}}$

Exercise

Find the domain $f\left(x\right)=\sqrt{3+2x}$

Solution

Domain: $\boxed{x\ge -\frac{3}{2}}$

Exercise

Find the domain $f(x) = \sqrt{5-x}$

Solution

Domain: $\underline{x \le 5}|$

Exercise

Find the domain $f(x) = \sqrt{x-5}$

Solution

Domain: $\underline{x \ge 5}|$

Exercise

Find the domain $f(x) = \sqrt{6-3x}$

Solution

Domain: $\underline{x \le 2}|$

Exercise

Find the domain $f(x) = \sqrt{3x-6}$

Solution

Domain: $\underline{x \ge 2}|$

Exercise

Find the domain $f(x) = \sqrt{2x+7}$

Solution

Domain: $\underline{x \ge -\frac{7}{2}}|$

Exercise

Find the domain $f(x) = \sqrt{x^2-16}$

Solution

$$x^2 - 16 = 0$$

$x^2 = 16$

$x = \pm 4$

Domain: $\underline{x \le -4 \quad x \ge 4}$

Exercise

Find the domain $\qquad f(x) = \sqrt{16 - x^2}$

Solution

$x = \pm 4$

Domain: $\underline{-4 \le x \le 4}$

Exercise

Find the domain $\qquad f(x) = \sqrt{9 - x^2}$

Solution

$x = \pm 3$

Domain: $\underline{-3 \le x \le 3}$

Exercise

Find the domain $\qquad f(x) = \sqrt{x^2 - 25}$

Solution

$x = \pm 5$

Domain: $\underline{-5 \le x \le 5}$

Exercise

Find the domain $\qquad f(x) = \sqrt{x^2 - 5x + 4}$

Solution

$x^2 - 5x + 4 \qquad\qquad a + b + c = 0 \quad \rightarrow \quad x = 1, \ \frac{c}{a}$

$x = 1, \ 4$

Domain: $\underline{x \le 1 \quad x \ge 4}$

Exercise

Find the domain $f(x) = \sqrt{x^2 + 5x + 4}$

Solution

$x^2 + 5x + 4 \qquad a - b + c = 0 \ \rightarrow \ x = -1, \ -\frac{c}{a}$

$x = -1, \ -4$

Domain: $\underline{x \le -4 \quad x \ge -1}$

Exercise

Find the domain $f(x) = \sqrt{x^2 + 3x + 2}$

Solution

$x^2 + 3x + 2 \qquad a - b + c = 0 \ \rightarrow \ x = -1, \ -\frac{c}{a}$

$x = -1, \ -2$

Domain: $\underline{x \le -2 \quad x \ge -1}$

Exercise

Find the domain $f(x) = \sqrt{x^2 - 3x + 2}$

Solution

$x^2 - 3x + 2 \qquad a + b + c = 0 \ \rightarrow \ x = 1, \ \frac{c}{a}$

$x = 1, \ 2$

Domain: $\underline{x \le 1 \quad x \ge 2}$

Exercise

Find the domain $f(x) = \sqrt{x - 4} + \sqrt{x + 1}$

Solution

$x \ge 4 \quad x \ge -1$

Domain: $\underline{x \ge 4}$

Exercise

Find the domain $f(x)=\sqrt{3-x}+\sqrt{x-2}$

Solution

$x\le 3 \quad x\ge 2$

Domain: $\underline{2\le x\le 3}$

Exercise

Find the domain $f(x)=\sqrt{1-x}+\sqrt{4-x}$

Solution

$x\le 1 \quad x\le 4$

Domain: $\underline{x\le 1}$

Exercise

Find the domain $f(x)=\sqrt{1-x}-\sqrt{x-3}$

Solution

$x\le 1 \quad x\ge 3$

Domain: $\underline{\varnothing}$

Exercise

Find the domain $f(x)=\sqrt{x+4}-\sqrt{x-1}$

Solution

$x\ge -4 \quad x\ge 1$

Domain: $\underline{x\ge 1}$

Exercise

Find the domain $f(x)=\dfrac{\sqrt{x+1}}{x}$

Solution

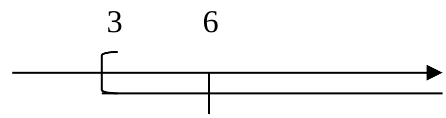
$x+1\ge 0 \qquad\qquad x\ne 0$

$x\ge -1$

Domain: $\underline{x\ge -1 \quad x\ne 0}$ $\qquad \underline{[-1,\ 0)\cup(0,\ \infty)}$

Exercise

Find the domain $g(x)=\dfrac{\sqrt{x-3}}{x-6}$



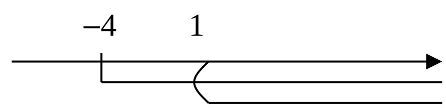
Solution

$$\rightarrow \begin{cases} x \geq 3 \\ x \neq 6 \end{cases}$$

Domain: $x \geq 3 \quad x \neq 6$ $\qquad [3,\ 6)\cup(6,\ \infty)$

Exercise

Find the domain $f(x)=\dfrac{\sqrt{x+4}}{\sqrt{x-1}}$



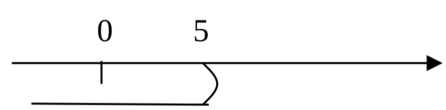
Solution

$$\rightarrow \begin{cases} x \geq -4 \\ x > 1 \end{cases}$$

Domain: $x > 1$ $\quad (1,\ \infty)$

Exercise

Find the domain $f\left(x\right)=\dfrac{\sqrt{5-x}}{x}$



Solution

$x \leq 5 \quad x \neq 0$

Domain: $x \leq 5 \quad x \neq 0$ $\quad (-\infty,\ 0)\cup(0,\ 5]$

Exercise

Find the domain $f\left(x\right)=\dfrac{x}{\sqrt{5-x}}$

Solution

Domain: $x < 5$ $\quad (-\infty,\ 5)$

Exercise

Find the domain $f(x) = \dfrac{1}{x\sqrt{5-x}}$

Solution

$x < 5 \quad x \neq 0$

Domain: $\underline{x < 5 \quad x \neq 0}$ $\underline{(-\infty,\ 0) \cup (0,\ 5)}$

Exercise

Find the domain $f(x) = \dfrac{x+1}{x^3 - 4x}$

Solution

$x^3 - 4x \neq 0$

$x\left(x^2 - 4\right) \neq 0$

Domain: $\underline{x \neq 0,\ \pm 2}$ $(-\infty, -2) \cup (-2, 0) \cup (0, 2) \cup (2, \infty)$

Exercise

Find the domain $f(x) = \dfrac{\sqrt{x+5}}{x}$

Solution

$x \geq -5 \quad x \neq 0$

Domain: $\underline{x \geq -5 \quad x \neq 0}$

Exercise

Find the domain $f(x) = \dfrac{x}{\sqrt{x+5}}$

Solution

$x > -5$

Domain: $\underline{x > -5}$

Exercise

Find the domain $f\left(x\right) = \dfrac{1}{x\sqrt{x+5}}$

Solution

$x > -5 \quad x \neq 0$

Domain: $\underline{x > -5 \quad x \neq 0}$

Exercise

Find the domain $f(x) = \dfrac{x+3}{\sqrt{x-3}}$

Solution

Domain: $\underline{x > 3}$

Exercise

Find the domain $f(x) = \dfrac{\sqrt{x+3}}{\sqrt{x-3}}$

Solution

$x \geq -3 \quad x > 3$

Domain: $\underline{x > 3}$

Exercise

Find the domain $f(x) = \dfrac{\sqrt{x-2}}{\sqrt{x+2}}$

Solution

$x \geq 2 \quad x > -2$

Domain: $\underline{x \geq 2}$

Exercise

Find the domain $f(x) = \dfrac{\sqrt{2-x}}{\sqrt{x+2}}$

Solution

$x \leq 2 \quad x > -2$

Domain: $\underline{-2 < x \leq 2}$

Exercise

Find the domain $f(x)=\dfrac{x-4}{\sqrt{x-2}}$

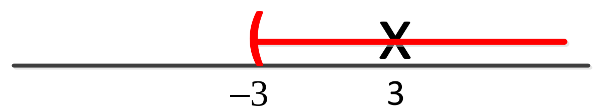
Solution

Domain: $x>2$

Exercise

Find the domain of $f(x)=\dfrac{1}{(x-3)\sqrt{x+3}}$

Solution



$x-3\neq 0 \qquad x+3>0$

$x\neq 3 \qquad x>-3$

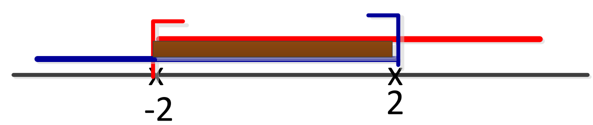
Domain: $\{x \mid x>-3 \;\; and \;\; x\neq 3\}$

$(-3,\ 3)\cup(3,\ \infty)$

Exercise

Find the domain of $f(x)=\sqrt{x+2}+\sqrt{2-x}$

Solution



$x+2\geq 0 \qquad 2-x\geq 0$

$x\geq -2 \qquad -x\geq -2\rightarrow x\leq 2$

Domain: $\{x \mid -2\leq x\leq 2\}$

Exercise

Find the domain of $f(x)=\sqrt{(x-2)(x-6)}$

Solution

$x-2\geq 0 \quad x-6\geq 0$

$x\geq 2 \quad x\geq 6$

Domain: $\{x \mid x\leq 2,\ x\geq 6\}$

$2 \qquad\qquad 6$

−	+	+
−	−	+
+	−	+

Exercise

Find the domain of $f(x)=\sqrt{x+3}-\sqrt{4-x}$

Solution

$x \ge -3 \quad x \le 4$

Domain: $\underline{-3 \le x \le 4}$

Exercise

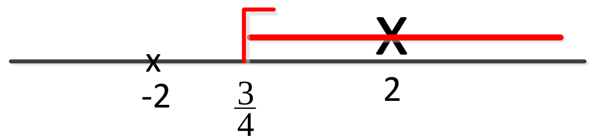
Find the domain of $f(x)=\dfrac{\sqrt{4x-3}}{x^2-4}$

Solution

$4x-3 \ge 0 \qquad x^2-4 \ne 0$

$4x \ge 3 \qquad x \ne \pm 2$

$x \ge \dfrac{3}{4}$



Domain: $\left[\dfrac{3}{4},\ 2\right)\cup(2,\ \infty)$

Exercise

Find the domain of $f(x)=\dfrac{4x}{6x^2+13x-5}$

Solution

$6x^2+13x-5 \ne 0$

$$x=\frac{-13\pm\sqrt{169+120}}{12}$$

$$=\begin{cases}\dfrac{-13-17}{12}=-\dfrac{5}{2}\\[2mm] \dfrac{-13+17}{12}=\dfrac{1}{3}\end{cases}$$

Domain: $\underline{x \ne -\dfrac{5}{2},\ \dfrac{1}{3}}$

Exercise

Find the domain of $f(x)=\dfrac{\sqrt{2x-3}}{x^2-5x+4}$

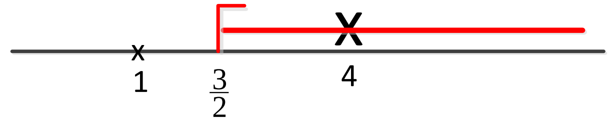
Solution

$2x-3 \ge 0 \qquad x^2-5x+4 \ne 0$

$2x \ge 3 \qquad\qquad x \ne 1,\ 4$

$x \ge \frac{3}{2}$

Domain: $x \ge \frac{3}{2},\ \ x \ne 4 \qquad \left[\frac{3}{2},\ 4\right) \cup (4,\ \infty)$



Exercise

Find the domain of $\quad f(x) = \dfrac{x^2}{\sqrt{x^2 - 5x + 4}}$

Solution

$x^2 - 5x + 4 \qquad\qquad a + b + c = 0 \;\; \rightarrow \;\; x = 1,\ \frac{c}{a}$

$x = 1,\ 4$

Domain: $x < 1 \quad x > 4$

Exercise

Find the domain of $\quad f(x) = \dfrac{x + 2}{\sqrt{x^2 + 5x + 4}}$

Solution

$x^2 + 5x + 4 \qquad\qquad a - b + c = 0 \;\; \rightarrow \;\; x = -1,\ -\frac{c}{a}$

$x = -1,\ -4$

Domain: $x < -4 \quad x > -1$

Exercise

Find the domain of $\quad f(x) = \dfrac{\sqrt{x + 2}}{\sqrt{x^2 + 3x + 2}}$

Solution

$x^2 + 3x + 2 \qquad\qquad a - b + c = 0 \;\; \rightarrow \;\; x = -1,\ -\frac{c}{a}$

$x < -2 \quad x > -1$

$\sqrt{x + 2} \;\; \rightarrow \;\; x \ge -2$

Domain: $x > -1$

Exercise

Find the domain of $f(x) = \dfrac{\sqrt{2x+3}}{x^2 - 6x + 5}$

Solution

$x^2 - 6x + 5 \qquad\qquad a + b + c = 0 \quad \to \quad x = 1, \ \frac{c}{a}$

$x \neq 1, \ 5$

$\sqrt{2x+3} \quad \to \quad x \geq -\frac{3}{2}$

Domain: $\underline{x \geq -\frac{3}{2} \quad x \neq 1, \ 5}$

Exercise

Let $f(x) = 4x - 3$ and $g(x) = 5x + 7$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\frac{f}{g}\right)(x)$

Solution

a) $(f+g)(x) = 4x - 3 + 5x + 7$

$= 9x + 4$

Domain: $\mathbb{R}$

b) $(f-g)(x) = 4x - 3 - (5x + 7)$

$= 4x - 3 - 5x - 7$

$= -x - 10$

Domain: $\mathbb{R}$

c) $(fg)(x) = (4x - 3)(5x + 7)$

$= 20x^2 + 13x - 21$

Domain: $\mathbb{R}$

d) $\left(\frac{f}{g}\right)(x) = \frac{4x - 3}{5x + 7}$

Domain: $x \neq -\frac{7}{5}$

Exercise

Let $f(x) = 2x^2 + 3$ and $g(x) = 3x - 4$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\frac{f}{g}\right)(x)$

Solution

a) $(f+g)(x) = 2x^2 + 3 + 3x - 4$

$$= 2x^2 + 3x - 1$$

Domain: $\mathbb{R}$

b) $(f-g)(x) = 2x^2 + 3 - (3x - 4)$

$$= 2x^2 + 3 - 3x + 4$$

$$= 2x^2 - x + 7$$

Domain: $\mathbb{R}$

c) $(fg)(x) = \left(2x^2 + 3\right)(3x - 4)$

$$= 6x^2 + x - 12$$

Domain: $\mathbb{R}$

d) $\left(\frac{f}{g}\right)(x) = \frac{2x^2 + 3}{3x - 4}$

Domain: $x \neq -\frac{4}{3}$

Exercise

Let $f(x) = x^2 - 2x - 3$ and $g(x) = x^2 + 3x - 2$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\frac{f}{g}\right)(x)$

Solution

a) $(f+g)(x) = x^2 - 2x - 3 + x^2 + 3x - 2$

$$= 2x^2 + x - 5$$

Domain: $\mathbb{R}$

b) $(f-g)(x) = x^2 - 2x - 3 - x^2 - 3x + 2$

$$= -5x - 1$$

Domain: $\mathbb{R}$

c) $(fg)(x) = \left(x^2 - 2x - 3\right)\left(x^2 + 3x - 2\right)$

$$= x^4 + 3x^3 - 2x^2 - 2x^3 - 6x^2 + 4x - 3x^2 - 9x + 6$$

$$= x^4 + x^3 - 11x^2 - 5x + 6$$

Domain: $\mathbb{R}$

d) $\left(\frac{f}{g}\right)(x) = \dfrac{x^2 - 2x - 3}{x^2 + 3x - 2}$

Domain: $x \neq \dfrac{-3 \pm \sqrt{17}}{2}$

Exercise

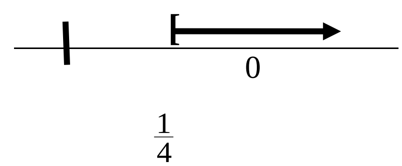
Let $f(x) = \sqrt{4x-1}$ and $g(x) = \frac{1}{x}$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\frac{f}{g}\right)(x)$

Solution

a) $(f+g)(x)$

$(f+g)(x) = \sqrt{4x-1} + \frac{1}{x}$



$4x - 1 \geq 0 \qquad x \neq 0$

$x \geq \frac{1}{4}$

Domain: $\left[\frac{1}{4}, \infty\right)$

b) $(f-g)(x)$

$(f-g)(x) = \sqrt{4x-1} - \frac{1}{x}$

$4x - 1 \geq 0 \qquad x \neq 0$

$x \geq \frac{1}{4}$

Domain: $\left[\frac{1}{4}, \infty\right)$

c) $(fg)(x) = \sqrt{4x-1}\left(\frac{1}{x}\right)$

$$= \frac{\sqrt{4x-1}}{x}$$

$4x - 1 \geq 0 \qquad\qquad x \neq 0$

$x \geq \frac{1}{4}$

Domain: $\left[\frac{1}{4}, \infty\right)$

d) $\left(\frac{f}{g}\right)(x) = \dfrac{\sqrt{4x-1}}{\frac{1}{x}}$ *Domain:* $x \neq 0$

$= x\sqrt{4x-1}$

$4x - 1 \geq 0$

$x \geq \frac{1}{4}$

Domain: $\left[\frac{1}{4}, \infty\right)$

Exercise

Given that $f(x) = x + 1$ and $g(x) = \sqrt{x+3}$

a) Find $(f+g)(x)$

b) Find the domain of $(f+g)(x)$

c) Find: $(f+g)(6)$

Solution

a) $(f+g)(x) = f(x) + g(x)$

$= x + 1 + \sqrt{x+3}$

b) $x + 3 \geq 0 \rightarrow x \geq -3$

Domain $= [-3, \infty)$

c) $(f+g)(6) = 6 + 1 + \sqrt{6+3}$

$= 10$

Exercise

Given that $f(x) = x^2 - 4$ and $g(x) = x + 2$

a) Find $(f+g)(x)$ and its domain

b) Find $(f/g)(x)$ and its domain

Solution

a) $(f+g)(x) = x^2 - 4 + x + 2$

$= x^2 + x - 2$

Domain: $\underline{\mathbb{R}}$

b) $\dfrac{f(x)}{g(x)} = \dfrac{x^2-4}{x+2}$

$x \neq -2$

Domain: $\underline{(-\infty,-2)\cup(-2,\infty)}$

Exercise

Let $f(x) = x^2 + 1$ and $g(x) = 3x + 5$. Find $(f+g)(1)$, $(f-g)(-3)$, $(fg)(5)$, and $\left(\frac{f}{g}\right)(0)$

Solution

a) $(f+g)(1) = f(1) + g(1)$

$= 1^2 + 1 + 3(1) + 5$

$\underline{= 10}$

b) $(f-g)(-3) = f(-3) - g(-3)$

$= (-3)^2 + 1 - (3(-3) + 5)$

$\underline{= 10}$

c) $(fg)(5) = f(5) \cdot g(5)$

$= \left(5^2 + 1\right) \cdot (3(5) + 5)$

$= (26) \cdot (20)$

$\underline{= 520}$

d) $\left(\frac{f}{g}\right)(0) = \dfrac{f(0)}{g(0)}$

$= \dfrac{0^2 + 1}{3(0) + 5}$

$\underline{= \dfrac{1}{5}}$

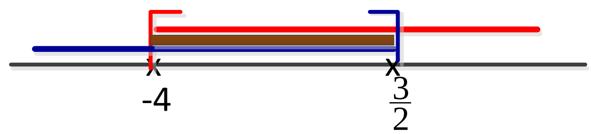
Exercise

Find $(f+g)(x),\ \ (f-g)(x),\ \ (f\cdot g)(x),\ \ and\ \ (f/g)(x)$ and the domain of

$f(x)=\sqrt{3-2x},\quad g(x)=\sqrt{x+4}$

Solution

$$f(x)+g(x)=\sqrt{3-2x}+\sqrt{x+4}$$



$3-2x\ge 0 \qquad x+4\ge 0$

$-2x\ge -3 \qquad x\ge -4$

$x\le \frac{3}{2}$

Domain: $\left\{x \mid -4\le x\le \frac{3}{2}\right\}$

$$f(x)-g(x)=\sqrt{3-2x}-\sqrt{x+4}$$

$3-2x\ge 0 \qquad x+4\ge 0$

$-2x\ge -3 \qquad x\ge -4$

$x\le \frac{3}{2}$

Domain: $\left\{x \mid -4\le x\le \frac{3}{2}\right\}$

$$\begin{aligned}(f\cdot g)(x)&=\left(\sqrt{3-2x}\right)\left(\sqrt{x+4}\right)\\&=\sqrt{(3-2x)(x+4)}\\&=\sqrt{-2x^2-5x+12}\end{aligned}$$

$3-2x\ge 0 \qquad x+4\ge 0$

$-2x\ge -3 \qquad x\ge -4$

$x\le \frac{3}{2}$

Domain: $\left\{x \mid -4\le x\le \frac{3}{2}\right\}$

$$\begin{aligned}(f/g)(x)&=\frac{\sqrt{3-2x}}{\sqrt{x+4}}\,\frac{\sqrt{x+4}}{\sqrt{x+4}}\\&=\frac{\sqrt{-2x^2-5x+12}}{x+4}\end{aligned}$$

$3-2x\ge 0 \qquad x+4> 0$

$-2x\ge -3 \qquad x> -4$

$x\le \frac{3}{2}$

Domain: $\left\{x \mid -4< x\le \frac{3}{2}\right\}$ $\qquad \left(-4,\ \frac{3}{2}\right]$

Exercise

Find $(f+g)(x)$, $(f-g)(x)$, $(f\cdot g)(x)$, *and* $(f/g)(x)$ and the domain of

$f(x)=\dfrac{2x}{x-4}$, $g(x)=\dfrac{x}{x+5}$

Solution

$$\begin{aligned}(f+g)(x)&=\frac{2x}{x-4}+\frac{x}{x+5}\\&=\frac{2x(x+5)+x(x-4)}{(x-4)(x+5)}\\&=\frac{2x^2+10x+x^2-4x}{(x-4)(x+5)}\\&=\frac{3x^2+6x}{(x-4)(x+5)}\end{aligned}$$

$x-4\neq 0 \qquad x+5\neq 0$

$x\neq 4 \qquad x\neq -5$

Domain: $\{x\mid x\neq -5,4\}$ $\qquad (-\infty,\ -5)\cup(-5,\ 4)\cup(4,\ \infty)$

$$\begin{aligned}(f-g)(x)&=\frac{2x}{x-4}-\frac{x}{x+5}\\&=\frac{2x(x+5)-x(x-4)}{(x-4)(x+5)}\\&=\frac{2x^2+10x-x^2+4x}{(x-4)(x+5)}\\&=\frac{x^2+14x}{(x-4)(x+5)}\end{aligned}$$

$x\neq 4 \qquad x\neq -5$

Domain: $\{x\mid x\neq -5,4\}$

$$\begin{aligned}(f\bullet g)(x)&=\frac{2x}{x-4}\frac{x}{x+5}\\&=\frac{2x^2}{(x-4)(x+5)}\end{aligned}$$

$x\neq 4 \qquad x\neq -5$

Domain: $\{x\mid x\neq -5,4\}$

$$\begin{aligned}(f/g)(x)&=\frac{2x}{x-4}\div\frac{x}{x+5}\\&=\frac{2x}{x-4}\frac{x+5}{x}\end{aligned}$$

$$= 2\frac{x+5}{x-4}$$

$x \neq 4 \qquad x \neq -5$

Domain: $\{x \mid x \neq -5, 4\}$

Exercise

Find $(f+g)(x),\ (f-g)(x),\ (f\cdot g)(x),\ and\ (f/g)(x)$ of $f(x) = x-5$ and $g(x) = x^2 - 1$

Solution

a) $(f+g)(x) = f(x) + g(x)$

$$= x - 5 + x^2 - 1$$

$$\underline{= x^2 + x - 6}$$

b) $(f-g)(x) = f(x) - g(x)$

$$= x - 5 - (x^2 - 1)$$

$$= x - 5 - x^2 + 1$$

$$\underline{= -x^2 + x - 4}$$

c) $(fg)(x) = f(x)g(x)$

$$= (x-5)(x^2-1)$$

$$= x^3 - x - 5x^2 + 5$$

$$\underline{= x^3 - 5x^2 - x + 5}$$

d) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$$\underline{= \frac{x-5}{x^2-1}}$$

Exercise

For the function *f* given by $f(x) = 9x + 5$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$f(x+h) = 9(x+h) + 5 = 9x + 9h + 5$

$$\frac{f(x+h) - f(x)}{h} = \frac{\underbrace{9x + 9h + 5}_{f(x+h)} \ - \ \underbrace{(9x+5)}_{f(x)}}{h}$$

$$= \frac{9x + 9h + 5 \ - \ 9x - 5}{h}$$

$$= \frac{9h}{h}$$

$$\underline{= 9}$$

Exercise

For the function *f* given by $f(x) = 6x + 2$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{6(x+h) + 2 - (6x+2)}{h} \\ &= \frac{6x + 6h + 2 - 6x - 2}{h} \\ &= \frac{6h}{h} \\ &\underline{= 6}\end{aligned}$$

Exercise

For the function *f* given by $f(x) = 4x + 11$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{4(x+h) + 11 - (4x+11)}{h} \\ &= \frac{4x + 4h + 11 - 4x - 11}{h} \\ &= \frac{4h}{h} \\ &\underline{= 4}\end{aligned}$$

Exercise

For the function *f* given by $f(x) = 3x - 5$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{3(x+h) - 5 - 3x + 5}{h} \\ &= \frac{3x + 3h - 5 - 3x + 5}{h} \\ &= \frac{3h}{h} \\ &\underline{= 3}\end{aligned}$$

Exercise

For the function *f* given by $f(x) = -2x - 3$, find the difference quotient $\dfrac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{-2(x+h) - 3 + 2x + 3}{h} \\ &= \frac{-2x - 2h - 3 + 2x + 3}{h} \\ &= \frac{-2h}{h} \\ &= -2\end{aligned}$$

Exercise

For the function *f* given by $f(x) = -4x + 3$, find the difference quotient $\dfrac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{-4(x+h) + 3 + 4x - 3}{h} \\ &= \frac{-4x - 4h + 3 + 4x - 3}{h} \\ &= \frac{-4h}{h} \\ &= -4\end{aligned}$$

Exercise

For the function *f* given by $f(x) = 3x - 6$, find the difference quotient $\dfrac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{3(x+h) - 6 - 3x + 6}{h} \\ &= \frac{3x + 3h - 6 - 3x + 6}{h} \\ &= \frac{3h}{h} \\ &= 3\end{aligned}$$

Exercise

For the function *f* given by $f(x) = -5x - 7$, find the difference quotient $\dfrac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{-5(x+h) - 7 + 5x + 7}{h} \\
&= \frac{-5x - 5h - 7 + 5x + 7}{h} \\
&= \frac{-5h}{h} \\
&= -5
\end{aligned}$$

Exercise

Given the function: $f(x) = 2x^2$. Find and simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}
f(x+h) &= 2(x+h)^2 \\
&= 2\left(x^2 + 2hx + h^2\right) \\
&= 2x^2 + 4hx + 2h^2
\end{aligned}$$

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{2x^2 + 4hx + 2h^2 - 2x^2}{h} \\
&= \frac{4hx + 2h^2}{h} \\
&= \frac{4hx}{h} + \frac{2h^2}{h} \\
&= 4x + 2h
\end{aligned}$$

Exercise

For the function *f* given by $f(x) = 5x^2$, find the difference quotient $\dfrac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{5(x+h)^2 - 5x^2}{h} \\
&= \frac{5\left(x^2 + 2hx + h^2\right) - 5x^2}{h}
\end{aligned}$$

$$= \frac{5x^2 + 10hx + 5h^2 - 5x^2}{h}$$

$$= \frac{10hx + 5h^2}{h}$$

$$\underline{= 10x + 5h}$$

Exercise

For the function *f* given by $f(x) = 3x^2 - 4x$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\frac{f(x+h) - f(x)}{h} = \frac{3(x+h)^2 - 4x - 3x^2 + 4x}{h}$$

$$= \frac{3\left(x^2 + 2hx + h^2\right) - 4(x+h) - 3x^2 + 4x}{h}$$

$$= \frac{3x^2 + 6hx + 3h^2 - 4x - 4h - 3x^2 + 4x}{h}$$

$$= \frac{6hx + 3h^2 - 4h}{h}$$

$$\underline{= 6x + 3h - 4}$$

Exercise

For the function *f* given by $f(x) = 2x^2 - 3x$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$f(x+h) = 2(\text{——})^2 - 3(\text{——})$$

$$= 2(x+h)^2 - 3(x+h) \qquad (a+b)^2 = a^2 + 2ab + b^2$$

$$= 2\left(x^2 + 2xh + h^2\right) - 3x - 3h$$

$$= 2x^2 + 4xh + 2h^2 - 3x - 3h$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\overbrace{2x^2 + 4xh + 2h^2 - 3x - 3h}^{f(x+h)} - \overbrace{(2x^2 - 3x)}^{f(x)}}{h}$$

$$= \frac{2x^2 + 4xh + 2h^2 - 3x - 3h - 2x^2 + 3x}{h}$$

$$= \frac{4xh + 2h^2 - 3h}{h}$$

$$=\frac{4xh}{h}+\frac{2h^2}{h}-\frac{3h}{h}$$

$$\underline{=4x+2h-3}$$

Exercise

For the function f given by $f(x)=2x^2-x-3$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$\begin{aligned} f(x+h) &= 2(x+h)^2-(x+h)-3 \\ &= 2\left(x^2+2hx+h^2\right)-x-h-3 \\ &= 2x^2+4hx+2h^2-x-h-3 \end{aligned}$$

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{2x^2+2h^2+4hx-x-h-3-\left(2x^2-x-3\right)}{h} \\ &= \frac{2x^2+2h^2+4hx-x-h-3-2x^2+x+3}{h} \\ &= \frac{2h^2+4hx-h}{h} \\ &= \frac{2h^2}{h}+\frac{4hx}{h}-\frac{h}{h} \\ &\underline{=2h+4x-1} \end{aligned}$$

Exercise

For the given function $f(x)=2x^2-x-3$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{2(x+h)^2-(x+h)-3-2x^2+x+3}{h} \\ &= \frac{2\left(x^2+2hx+h^2\right)-x-h-2x^2+x}{h} \\ &= \frac{2x^2+4hx+2h^2-h-2x^2}{h} \\ &= \frac{4hx+2h^2-h}{h} \\ &\underline{=4x+2h-1} \end{aligned}$$

Exercise

For the given function $f(x)=x^2-2x+5$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^2-2(x+h)+5-x^2+2x-5}{h} \\
&= \frac{x^2+2hx+h^2-2x-2h-x^2+2x}{h} \\
&= \frac{2hx+h^2-2h}{h} \\
&= 2x+h-2
\end{aligned}$$

Exercise

For the given function $f(x)=3x^2-2x+5$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{3(x+h)^2-2(x+h)+5-3x^2+2x-5}{h} \\
&= \frac{3\left(x^2+2hx+h^2\right)-2x-2h-3x^2+2x}{h} \\
&= \frac{3x^2+6hx+3h^2-2h-3x^2}{h} \\
&= \frac{6hx+3h^2-2h}{h} \\
&= 6x+3h-2
\end{aligned}$$

Exercise

For the given function $f(x)=-2x^2-3x+7$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{-2(x+h)^2-3(x+h)+7+2x^2+3x-7}{h} \\
&= \frac{-2\left(x^2+2hx+h^2\right)-3x-3h+2x^2+3x}{h} \\
&= \frac{-2x^2-4hx-2h^2-3h+2x^2}{h}
\end{aligned}$$

$$= \frac{-4hx - 2h^2 - 3h}{h}$$

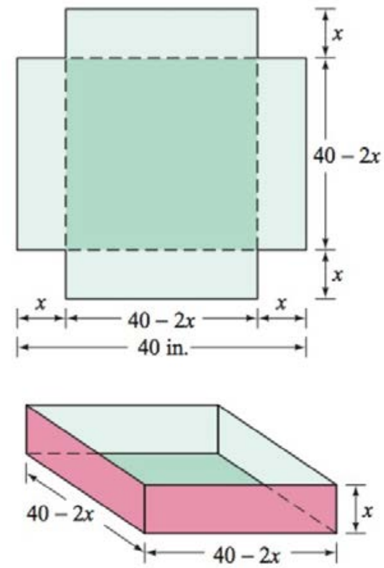
$$= -4x - 2h - 3$$

Exercise

An open box is to be made from a square piece of cardboard that measures 40 *inches* on each side, to construct the box, squares that measure x *inches* on each side are cut from each corner of the cardboard.

a) Express the volume V of the box as a function of x.

b) Determine the domain of V.



Solution

a) $V(x) = x(40 - 2x)^2$

$$= x\left(1600 - 160x + 4x^2\right)$$

$$= 4x^3 - 160x^2 + 1600x$$

b) $40 - 2x = 0$

$$x = 20$$

Domain: $\{x \mid 0 < x < 20\}$

Exercise

A child 4 *feet* tall is standing near a street lamp that is 12 *feet* high. The light from the lamp casts a shadow.

a) Find the length l of the shadow as a function of the distance d of the child from the lamppost.

b) What is the domain of the function?

c) What is the length of the shadow when the child is 8 *feet* from the base of the lamppost?

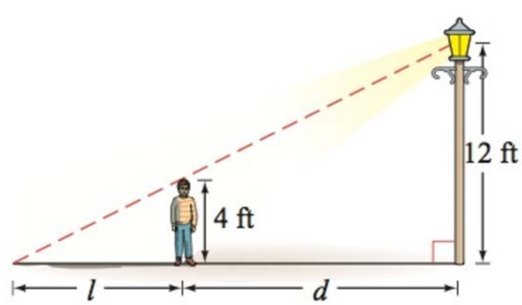
Solution

a) $\frac{l + d}{12} = \frac{l}{4}$

$$l + d = 3l$$

$$2l = d$$

$$l(d) = \frac{1}{2}d$$



b) ***Domain***: $\{x \mid 0 \le d < \infty\}$

c) ***Given***: $d = 8$

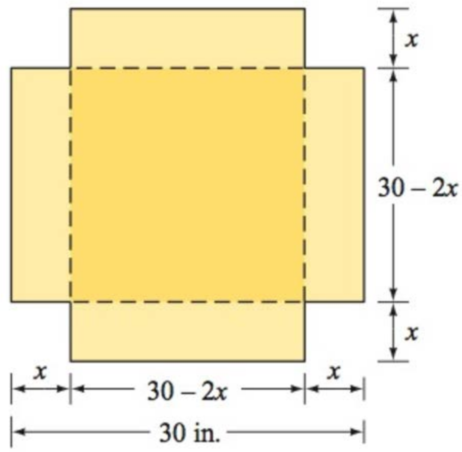
$$l = 4 \textit{ feet}$$

Exercise

An open box is to be made from a square piece of cardboard with the dimensions 30 *inches* by 30 *inches* by cutting out squares of area x^2 from each corner.

a) Express the volume *V* of the box as a function of *x*.

b) Determine the domain of *V*.



Solution

a) $V(x) = x(30-2x)^2$

$$= x\left(900 - 120x + 4x^2\right)$$

$$\underline{= 4x^3 - 120x^2 + 900x}$$

b) $30 - 2x = 0$

$\underline{x = 15}$

Domain: $\{x \mid 0 < x < 15\}$

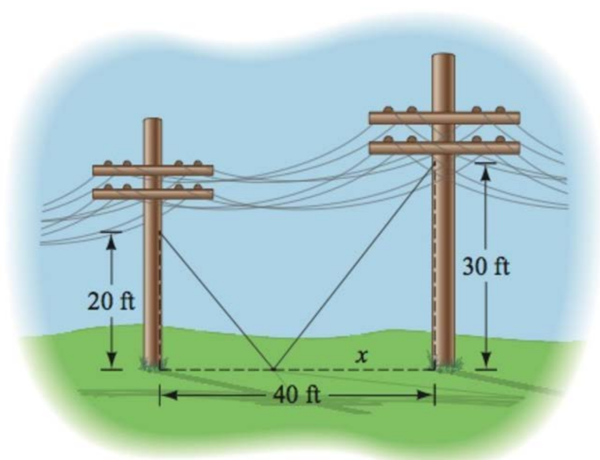
Exercise

Two guy wires are attached to utility poles that are 40 *feet* apart.

a) Find the total length of the two guy wires as a function of *x*.

b) What is the domain of this function?

Solution



a) $\ell_1 = \sqrt{(40-x)^2 + 20^2}$

$$= \sqrt{1,600 - 80x + x^2 + 400}$$

$$= \sqrt{2,000 - 80x + x^2}$$

$\ell_2 = \sqrt{x^2 + 30^2}$

$$= \sqrt{x^2 + 900}$$

$$\underline{\ell(x) = \sqrt{2,000 - 80x + x^2} + \sqrt{x^2 + 900}}$$

b) ***Domain***: $[0,\ 40]$

Exercise

A rancher has 360 *yards* of fencing with which to enclose two adjacent rectangular corrals, one for sheep and one for cattle. A river forms one side of the corrals. Suppose the width of each corral is *x* yards.

a) Express the total area of the two corrals as a function of *x*.

b) Find the domain of the function.

Solution

a) $P = 3x + l = 360$

$l = 360 - 3x$

$A = xl$

$= x(360 - 3x)$

$\boxed{A(x) = 360x - 3x^2}$

b) $x(360 - 3x) = 0$

$\boxed{x = 0}$

$360 - 3x = 0$

$3x = 360$

$\Rightarrow \boxed{x = 120}$

Domain: $\boxed{0 < x < 120}$

Exercise

A rectangle is bounded by the *x*- and *y-axis* of $y = -\frac{1}{2}x + 4$

a) Find the area of the rectangle as a function of *x*.

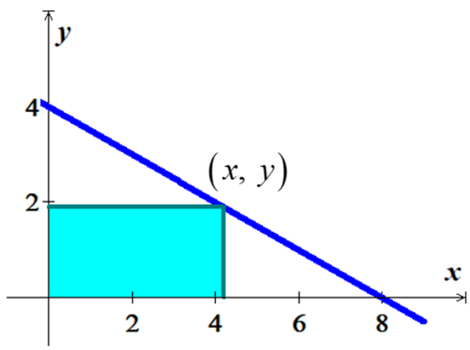
b) What is the domain of this function.

Solution

a) *Area* $= xy$

$A(x) = x\left(-\frac{1}{2}x + 4\right)$

$\boxed{= -\frac{1}{2}x^2 + 4x}$



b) $x\left(-\frac{1}{2}x + 4\right) = 0$

$x = 0 \quad x = 8$

Domain: $\boxed{0 < x < 8}$

Solution   *Section* 2.3 – Composition Functions

Exercise

Given that $f(x)=2x-5$ and $g(x)=x^2-3x+8$, find $(f\circ g)(x)$, $(g\circ f)(x)$ and their domain then find $(f\circ g)(7)$

Solution

$$f(g(x)) = f\left(x^2-3x+8\right) \qquad \textit{Domain: } (-\infty,\infty)$$

$$= 2(\text{------------}) - 5$$

$$= 2\left(2x^2-3x+8\right)-5$$

$$= 2x^2-6x+16-5$$

$$= 2x^2-6x+11 \qquad \textit{Domain: } (-\infty,\infty)$$

***Domain**: $\boxed{\mathbb{R}}$*

$$g(f(x)) = g(2x-5) \qquad \textit{Domain: } (-\infty,\infty)$$

$$= (---)^2-3(---)+8$$

$$= (2x-5)^2-3(2x-5)+8$$

$$= 4x^2-20x+25-6x+15+8$$

$$= 4x^2-26x+48 \qquad \textit{Domain: } (-\infty,\infty)$$

***Domain**: $\boxed{\mathbb{R}}$*

$$f(g(7)) = 2(7)^2-6(7)+11 \ \underline{=67}$$

Exercise

Given that $f(x)=\sqrt{x}$ and $g(x)=x-1$, find

a) $(f\circ g)(x)=f(g(x))$

b) $(g\circ f)(x)=g(f(x))$

c) $(f\circ g)(2)=f(g(2))$

Solution

a) $(f\circ g)(x)=f(g(x))$

$$= f(x-1)$$

$$\underline{=\sqrt{x-1}}$$

b) $(g \circ f)(x) = g(f(x))$

$$= g\left(\sqrt{x}\right)$$

$$\boxed{= \sqrt{x} - 1}$$

c) $(f \circ g)(2) = f(g(2))$

$$= \sqrt{x-1}$$

$$= \sqrt{2-1}$$

$$\boxed{= 1}$$

Exercise

Given that $f(x) = \dfrac{x}{x+5}$ and $g(x) = \dfrac{6}{x}$, find

a) $(f \circ g)(x) = f(g(x))$

b) $(g \circ f)(x) = g(f(x))$

c) $(f \circ g)(2) = f(g(2))$

Solution

a) $(f \circ g)(x) = f(g(x))$

$$= f\left(\frac{6}{x}\right)$$

$$= \frac{\frac{6}{x}}{\frac{6}{x}+5}$$

$$= \frac{\frac{6}{x}}{\frac{6+5x}{x}}$$

$$\boxed{= \frac{6}{6+5x}}$$

b) $(g \circ f)(x) = g(f(x))$

$$= g\left(\frac{x}{x+5}\right)$$

$$= \frac{6}{\frac{x}{x+5}}$$

$$= \frac{6(x+5)}{x}$$

$$\boxed{= \frac{6x+30}{x}}$$

c) $(f \circ g)(2) = f(g(2))$

$$= \frac{6}{6+5(2)}$$

$$= \frac{6}{16}$$

$$\boxed{= \frac{3}{8}}$$

Exercise

Find $(f \circ g)(x)$, $(g \circ f)(x)$, $f(g(-2))$ *and* $g(f(3))$: $f(x) = 2x^2 + 3x - 4$, $g(x) = 2x - 1$

Solution

$$f(g(x)) = f(2x-1)$$

$$= 2(2x-1)^2 + 3(2x-1) - 4$$

$$= 2\left(4x^2 - 4x + 1\right) + 6x - 3 - 4$$

$$= 8x^2 - 8x + 2 + 6x - 7$$

$$\boxed{= 8x^2 - 2x - 5}$$

$$g(f(x)) = g\left(2x^2 + 3x - 4\right)$$

$$= 2\left(2x^2 + 3x - 4\right) - 1$$

$$= 4x^2 + 6x - 8 - 1$$

$$\boxed{= 4x^2 + 6x - 9}$$

$$f(g(-2)) = 8(-2)^2 - 2(-2) - 5$$

$$\boxed{= 31}$$

$$g(f(3)) = 4(3)^2 + 6(3) - 9$$

$$\boxed{= 45}$$

Exercise

Find $(f \circ g)(x)$, $(g \circ f)(x)$, $f(g(-2))$ *and* $g(f(3))$: $f(x) = x^3 + 2x^2$, $g(x) = 3x$

Solution

$$f(g(x)) = f(3x)$$

$$= (3x)^3 + 2(3x)^2$$

$$= 27x^3 + 18x^2$$

$$g\left(f(x)\right) = g\left(x^3 + 2x^2\right)$$

$$= 3\left(x^3 + 2x^2\right)$$

$$= 3x^3 + 6x^2$$

$$f\left(g(-2)\right) = 27(-2)^3 + 18(-2)^2$$

$$= 288$$

$$g\left(f(3)\right) = 3(3)^3 + 6(3)^2$$

$$= 135$$

Exercise

Find $\left(f \circ g\right)(x)$, $\left(g \circ f\right)(x)$, $f\left(g(-2)\right)$ *and* $g\left(f(3)\right)$: $f(x) = |x|$, $g(x) = -7$

Solution

$$f\left(g(x)\right) = f\left(-7\right)$$

$$= |-7|$$

$$= 7$$

$$g\left(f(x)\right) = g\left(|x|\right)$$

$$= -7$$

$$f\left(g(-2)\right) = 7$$

$$g\left(f(3)\right) = -7$$

Exercise

Given $f(x) = x - 3$ *and* $g(x) = x + 3$

a) Find $\left(f \circ g\right)(x)$ and the domain of $f \circ g$

b) Find $\left(g \circ f\right)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(x+3\right)$ ***Domain***: $\mathbb{R}$

$$= \left(x-3\right) + 3$$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g\left(f(x)\right) = g(x-3)$ ***Domain***: $\mathbb{R}$

$= (x+3) - 3$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = \frac{2}{3}x$ *and* $g(x) = \frac{3}{2}x$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(\frac{3}{2}x\right)$ ***Domain***: $\mathbb{R}$

$= \frac{2}{3}\left(\frac{3}{2}x\right)$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g\left(f(x)\right) = g\left(\frac{2}{3}x\right)$ ***Domain***: $\mathbb{R}$

$= \frac{3}{2}\left(\frac{2}{3}x\right)$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = x-1$ *and* $g(x) = 3x^2 - 2x - 1$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(3x^2 - 2x - 1\right)$ ***Domain***: $\mathbb{R}$

$= 3(x-1)^2 - 2(x-1) - 1$

$$= 3\left(x^2 - 2x + 1\right) - 2x + 2 - 1$$

$$= 3x^2 - 6x + 3 - 2x + 1$$

$$\underline{= 3x^2 - 8x + 4}\,\big|$$ ***Domain***: $\mathbb{R}$

Domain: $\underline{\mathbb{R}}\,|$

b) $g\left(f(x)\right) = g(x-1)$ ***Domain***: $\mathbb{R}$

$$= 3x^2 - 2x - 1 - 1$$

$$\underline{= 3x^2 - 2x - 2}\,\big|$$ ***Domain***: $\mathbb{R}$

Domain: $\underline{\mathbb{R}}\,|$

Exercise

Given $f(x) = 3x - 2$ *and* $g(x) = x^2 - 5$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(x^2 - 5\right)$ ***Domain***: $\mathbb{R}$

$$= 3\left(x^2 - 5\right) - 2$$

$$= 3x^2 - 15 - 2$$

$$\underline{= 3x^2 - 17}\,\big|$$ ***Domain***: $\mathbb{R}$

Domain: $\underline{\mathbb{R}}\,|$

b) $g\left(f(x)\right) = g(3x - 2)$ ***Domain***: $\mathbb{R}$

$$= (3x - 2)^2 - 5$$

$$= 9x^2 - 12x + 4 - 5$$

$$\underline{= 9x^2 - 12x - 1}\,\big|$$ ***Domain***: $\mathbb{R}$

Domain: $\underline{\mathbb{R}}\,|$

Exercise

Given $f(x) = x^2 - 2$ *and* $g(x) = 4x - 3$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f(4x-3)$ ***Domain***: $\mathbb{R}$

$$= (4x-3)^2 - 2$$

$$= 16x^2 - 24x + 9 - 2$$

$$= 16x^2 - 24x + 7$$

Domain: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g\left(f(x)\right) = g\left(x^2 - 2\right)$ ***Domain***: $\mathbb{R}$

$$= 4\left(x^2 - 2\right) - 3$$

$$= 4x^2 - 8 - 3$$

$$= 4x^2 - 11$$

Domain: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = 4x^2 - x + 10$ *and* $g(x) = 2x - 7$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f(2x-7)$ ***Domain***: $\mathbb{R}$

$$= 4(2x-7)^2 - (2x-7) + 10$$

$$= 4\left(4x^2 - 28x + 49\right) - 2x + 7 + 10$$

$$= 16x^2 - 112x + 196 - 2x + 17$$

$$= 16x^2 - 114x + 213$$

Domain: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g\left(f(x)\right) = g\left(4x^2 - x + 10\right)$ ***Domain***: $\mathbb{R}$

$= 2\left(4x^2 - x + 10\right) - 7$

$= 8x^2 - 2x + 20 - 7$

$= 8x^2 - 2x + 13$ **Domain**: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = \sqrt{x}$ *and* $g(x) = x + 3$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(x + 3\right)$ ***Domain***: $\mathbb{R}$

$= \sqrt{x + 3}$ ***Domain***: $x \ge -3$

Domain: $x \ge -3$

b) $g\left(f(x)\right) = g\left(\sqrt{x}\right)$ ***Domain***: $x \ge 0$

$= \sqrt{x} + 3$ ***Domain***: $x \ge 0$

Domain: $x \ge 0$

Exercise

Given $f(x) = \sqrt{x}$ *and* $g(x) = 2 - 3x$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(2 - 3x\right)$ ***Domain***: $\mathbb{R}$

$= \sqrt{2 - 3x}$ ***Domain***: $x \le \frac{2}{3}$

Domain: $x \le \frac{2}{3}$

b) $g\left(f(x)\right) = g\left(\sqrt{x}\right)$ ***Domain***: $x \ge 0$

$= 2 - 3\sqrt{x}$ ***Domain***: $x \ge 0$

Domain: $x \ge 0$

Exercise

Given $f(x) = 3x + 2$ *and* $g(x) = \sqrt{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f(\sqrt{x})$ ***Domain***: $x \ge 0$

$= 3\sqrt{x} + 2$ ***Domain***: $x \ge 0$

Domain: $x \ge 0$

b) $g(f(x)) = g(3x+2)$ ***Domain***: $\mathbb{R}$

$= \sqrt{3x+2}$ ***Domain***: $x \ge -\frac{2}{3}$

Domain: $x \ge -\frac{2}{3}$

Exercise

Given $f(x) = x^4$ *and* $g(x) = \sqrt[4]{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f(\sqrt[4]{x})$ ***Domain***: $x \ge 0$

$= (\sqrt[4]{x})^4$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $x \ge 0$

b) $g(f(x)) = g(x^4)$ ***Domain***: $\mathbb{R}$

$= \sqrt[4]{x^4}$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = x^n$ *and* $g(x) = \sqrt[n]{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\sqrt[n]{x}\right)$ **Domain**: $\begin{cases} \text{If } n \text{ is even} & \underline{x \ge 0} \\ \text{If } n \text{ is odd} & \underline{\mathbb{R}} \end{cases}$

$= \left(\sqrt[n]{x}\right)^n$

$\underline{= x}$ **Domain**: $\mathbb{R}$

Domain: $\begin{cases} \text{If } n \text{ is even} & \underline{x \ge 0} \\ \text{If } n \text{ is odd} & \underline{\mathbb{R}} \end{cases}$

b) $g(f(x)) = g\left(x^n\right)$ **Domain**: $\mathbb{R}$

$= \sqrt[n]{x^n}$

$\underline{= x}$ **Domain**: $\mathbb{R}$

Domain: $\underline{\mathbb{R}}$

Exercise

Given $f(x) = x^2 - 3x$ *and* $g(x) = \sqrt{x+2}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\sqrt{x+2}\right)$ $x + 2 \ge 0 \Rightarrow x \ge -2$

$= \left(\sqrt{x+2}\right)^2 - 3\sqrt{x+2}$

$\underline{= x + 2 - 3\sqrt{x+2}}$ $x + 2 \ge 0 \Rightarrow x \ge -2$

Domain: $\{x \mid x \ge -2\}$

b) $g(f(x)) = g\left(x^2 - 3x\right)$ $\mathbb{R}$

$\underline{= \sqrt{x^2 - 3x + 2}}$ $x^2 - 3x + 2 \ge 0 \Rightarrow (x = 1,\ 2) \leftrightarrow x \le 1,\ x \ge 2$

Domain: $\{x \mid x \le 1,\ x \ge 2\}$

Exercise

Given $f(x)=\sqrt{x-2}$ *and* $g(x)=\sqrt{x+5}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(\sqrt{x+5}\right)$ $x+5\ge 0\Rightarrow x\ge -5$

$=\sqrt{\sqrt{x+5}-2}$ $\sqrt{x+5}-2\ge 0\Rightarrow \sqrt{x+5}\ge 2$

$x+5\ge 4$

$x\ge -1$

Domain: $\{x\mid x\ge -1\}$

b) $g(f(x))=g\left(\sqrt{x-2}\right)$ $x-2\ge 0\Rightarrow x\ge 2$

$=\sqrt{\sqrt{x-2}+5}$ $\sqrt{x-2}+5\ge 0\Rightarrow \sqrt{x-2}\ge -5$ *Always true when* $x\ge 2$

Domain: $\{x\mid x\ge 2\}$

Exercise

Given $f(x)=x^2+2$ *and* $g(x)=\sqrt{3-x}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(\sqrt{3-x}\right)$ ***Domain***: $x\le 3$

$=\left(\sqrt{3-x}\right)^2+2$

$=3-x+2$

$=5-x$ ***Domain***: $\mathbb{R}$

Domain: $x\le 3$

b) $g(f(x))=g\left(x^2+2\right)$ ***Domain***: $\mathbb{R}$

$=\sqrt{3-x^2-2}$

$=\sqrt{1-x^2}$ ***Domain***: $-1\le x\le 1$

Domain: $-1\le x\le 1$

Exercise

Given $f(x)=x^5-2$ *and* $g(x)=\sqrt[5]{x+2}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(\sqrt[5]{x+2}\right)$ **Domain**: $\mathbb{R}$

$$=\left(\sqrt[5]{x+2}\right)^5-2$$

$$=x+2-2$$

$$=x$$ **Domain**: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g(f(x))=g\left(x^5-2\right)$ **Domain**: $\mathbb{R}$

$$=\sqrt[5]{x^5-2+2}$$

$$=\sqrt[5]{x^5}$$

$$=x$$ **Domain**: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x)=1-x^2$ *and* $g(x)=\sqrt{x^2-25}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(\sqrt{x^2-25}\right)$ **Domain**: $x\le -5 \quad x\ge 5$

$$=1-\left(\sqrt{x^2-25}\right)^2$$

$$=1-\left(x^2-25\right)$$

$$=1-x^2+25$$

$$=26-x^2$$ **Domain**: $\mathbb{R}$

Domain: $x\le -5 \quad x\ge 5$

b) $g\left(f(x)\right) = g\left(1-x^2\right)$ **Domain**: $\mathbb{R}$

$$= \sqrt{\left(1-x^2\right)^2 - 25}$$

$$= \sqrt{1-2x^2+x^4-25}$$

$$\underline{= \sqrt{x^4-2x^2-24}}$$

$$x^2 = \frac{2 \pm \sqrt{4+96}}{2}$$

$$= \begin{cases} \frac{2-10}{2} = -4 \times \\ \frac{2+10}{2} = 6 \end{cases}$$

$x^2 = 6 \quad \rightarrow \quad \underline{x = \pm\sqrt{6}}$ **Domain**: $x \le -\sqrt{6} \quad x \ge \sqrt{6}$

Domain: $\underline{x \le -\sqrt{6} \quad x \ge \sqrt{6}}$

Exercise

Given $f(x) = 2x+3$ *and* $g(x) = \frac{x-3}{2}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(\frac{x-3}{2}\right)$ **Domain**: $\mathbb{R}$

$$= 2\left(\frac{x-3}{2}\right)+3$$

$$= x-3+3$$

$\underline{= x}$ **Domain**: $\mathbb{R}$

Domain: $\underline{\mathbb{R}}$

b) $g\left(f(x)\right) = g(2x+3)$ **Domain**: $\mathbb{R}$

$$= \frac{1}{2}(2x+3-3)$$

$\underline{= x}$ **Domain**: $\mathbb{R}$

Domain: $\underline{\mathbb{R}}$

Exercise

Given $f(x) = 4x - 5$ *and* $g(x) = \frac{x+5}{4}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\frac{x+5}{4}\right)$ ***Domain***: $\mathbb{R}$

$= 4\left(\frac{x+5}{4}\right) - 5$

$= x + 5 - 5$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g(f(x)) = g(4x - 5)$ ***Domain***: $\mathbb{R}$

$= \frac{1}{4}(4x - 5 + 5)$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = \frac{4}{1-5x}$ *and* $g(x) = \frac{1}{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\frac{1}{x}\right)$ ***Domain***: $x \neq 0$

$= \frac{4}{1 - 5\frac{1}{x}}$

$= \frac{4x}{x-5}$ ***Domain***: $x \neq 5$

Domain: $x \neq 0,\ 5$

b) $g(f(x)) = g\left(\frac{4}{1-5x}\right)$ ***Domain***: $x \neq \frac{1}{5}$

$= \frac{1-5x}{4}$ ***Domain***: $\mathbb{R}$

Domain: $x \neq \frac{1}{5}$

Exercise

Given $f(x)=\dfrac{1}{x-2}$ *and* $g(x)=\dfrac{x+2}{x}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f\left(g(x)\right)=f\left(\dfrac{x+2}{x}\right)$ ***Domain***: $x\neq 0$

$$=\frac{1}{\frac{x+2}{x}-2}$$

$$=\frac{1}{\frac{x+2-2x}{x}}$$

$$=\frac{x}{2-x}$$ ***Domain***: $x\neq 2$

Domain: $x\neq 0,\ 2$ $(-\infty,\ 0)\cup(0,\ 2)\cup(2,\ \infty)$

b) $g\left(f(x)\right)=g\left(\dfrac{1}{x-2}\right)$ ***Domain***: $x\neq 2$

$$=\frac{\frac{1}{x-2}+2}{\frac{1}{x-2}}$$

$$=\frac{\frac{1+2x-4}{x-2}}{\frac{1}{x-2}}$$

$$=2x-3$$ ***Domain***: $\mathbb{R}$

Domain: $x\neq 2$ $(-\infty,\ 2)\cup(2,\ \infty)$

Exercise

Given $f(x)=\dfrac{3x+5}{2}$ *and* $g(x)=\dfrac{2x-5}{3}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f\left(g(x)\right)=f\left(\dfrac{2x-5}{3}\right)$ ***Domain***: $\mathbb{R}$

$$=\frac{3\frac{2x-5}{3}+5}{2}$$

$$= \frac{2x-5+5}{2}$$

$$= \frac{2x}{2}$$

$$\underline{= x}$$

Domain: $\underline{\mathbb{R}}$

b) $g\left(f(x)\right) = g\left(\frac{3x+5}{2}\right)$ ***Domain***: $\mathbb{R}$

$$= \frac{2\frac{3x+5}{2}-5}{3}$$

$$= \frac{3x+5-5}{3}$$

$$= \frac{3x}{3}$$

$$\underline{= x}$$

Domain: $\underline{\mathbb{R}}$

Exercise

Given $f(x) = \frac{1}{1+x}$ *and* $g(x) = \frac{1-x}{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(\frac{1-x}{x}\right)$ ***Domain***: $x \neq 0$

$$= \frac{1}{1+\frac{1-x}{x}}$$

$$= \frac{x}{x+1-x}$$

$\underline{= x}$ ***Domain***: $\mathbb{R}$

Domain: $\underline{x \neq 0}$

b) $g\left(f(x)\right) = g\left(\frac{1}{x+1}\right)$ ***Domain***: $x \neq -1$

$$= \frac{1-\frac{1}{x+1}}{\frac{1}{x+1}}$$

$$= x+1-1$$

$\underline{= x}$ ***Domain***: $\mathbb{R}$

Domain: $\underline{\mathbb{R}}$

Exercise

Given $f(x)=\dfrac{x-1}{x-2}$ *and* $g(x)=\dfrac{x-3}{x-4}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f\left(g(x)\right)=f\left(\dfrac{x-3}{x-4}\right)$ ***Domain***: $x\neq 4$

$$=\dfrac{\frac{x-3}{x-4}-1}{\frac{x-3}{x-4}-2}$$

$$=\dfrac{\frac{x-3-(x-4)}{x-4}}{\frac{x-3-2(x-4)}{x-4}}$$

$$=\dfrac{x-3+x+4}{x-3-2x+8}$$

$$=\dfrac{2x+1}{-x+5}$$ ***Domain***: $x\neq 5$

Domain: $\{x\mid x\neq 4,\ 5\}$

b) $g\left(f(x)\right)=g\left(\dfrac{x-1}{x-2}\right)$ ***Domain***: $x\neq 2$

$$=\dfrac{\frac{x-1}{x-2}-3}{\frac{x-1}{x-2}-4}$$

$$=\dfrac{x-1-3(x-2)}{x-1-4(x-2)}$$

$$=\dfrac{x-1-3x+6}{x-1-4x+8}$$

$$=\dfrac{-2x+5}{-3x+7}$$ ***Domain***: $x\neq \dfrac{7}{3}$

Domain: $\left\{x\mid x\neq 2,\ \dfrac{7}{3}\right\}$

Exercise

Given $f(x)=\dfrac{6}{x-3}$ *and* $g(x)=\dfrac{1}{x}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $(f\circ g)(x)$

$$f(g(x))=f\left(\frac{1}{x}\right)$$ ***Domain***: $x\neq 0$

$$=\frac{6}{\frac{1}{x}-3}$$

$$=\frac{6}{\frac{1-3x}{x}}$$

$$=\frac{6x}{1-3x}$$ ***Domain:*** $x\neq\frac{1}{3}$

Domain: $x\neq 0,\ \frac{1}{3}$ $\qquad (-\infty,0)\cup\left(0,\frac{1}{3}\right)\cup\left(\frac{1}{3},\infty\right)$

b) $(g\circ f)(x)$

$$g(f(x))=g\left(\frac{6}{x-3}\right)$$ ***Domain***: $x\neq 3$

$$=\frac{1}{\frac{6}{x-3}}$$

$$=\frac{x-3}{6}$$ ***Domain***: $(-\infty,\infty)$

Domain: $x\neq 3$ $\qquad (-\infty,3)\cup(3,\infty)$

Exercise

Given $f(x)=\dfrac{6}{x}$ *and* $g(x)=\dfrac{1}{2x+1}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(\dfrac{1}{2x+1}\right)$ ***Domain***: $x\neq -\frac{1}{2}$

$$=\frac{6}{\frac{1}{2x+1}}$$

$$=12x+6$$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{x \neq -\frac{1}{2}}$

b) $g\left(f(x)\right) = g\left(\frac{6}{x}\right)$ ***Domain***: $x \neq 0$

$= \frac{1}{2\frac{6}{x}+1}$

$\boxed{= \frac{x}{12+x}}$ ***Domain***: $x \neq -12$

Domain: $\boxed{x \neq -12,\ 0}$

Exercise

Given $f(x) = 3x - 7$ *and* $g(x) = \frac{x+7}{3}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(\frac{x+7}{3}\right)$ ***Domain***: $\mathbb{R}$

$= 3\frac{x+7}{3} - 7$

$= x + 7 - 7$

$\boxed{= x}$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

b) $g\left(f(x)\right) = g\left(3x - 7\right)$ ***Domain***: $\mathbb{R}$

$= \frac{3x - 7 + 7}{3}$

$\boxed{= x}$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

Exercise

Given $f(x) = \frac{2x+3}{x-4}$ *and* $g(x) = \frac{4x+3}{x-2}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right)=f\left(\dfrac{4x+3}{x-2}\right)$ ***Domain***: $x\neq 2$

$$=\dfrac{2\dfrac{4x+3}{x-2}+3}{\dfrac{4x+3}{x-2}-4}$$

$$=\dfrac{8x+6+3x-6}{4x+3-4x+8}$$

$$=\dfrac{11x}{11}$$

$$=x$$ ***Domain***: $\mathbb{R}$

Domain: $x\neq 2$

b) $g\left(f(x)\right)=g\left(\dfrac{2x+3}{x-4}\right)$ ***Domain***: $x\neq 4$

$$=\dfrac{4\dfrac{2x+3}{x-4}+3}{\dfrac{2x+3}{x-4}-2}$$

$$=\dfrac{8x+12+3x-4}{2x+3-2x+8}$$

$$=\dfrac{11x}{11}$$

$$=x$$ ***Domain***: $\mathbb{R}$

Domain: $x\neq 4$

Exercise

Given $f(x)=\dfrac{2x+3}{x+4}$ *and* $g(x)=\dfrac{-4x+3}{x-2}$

a) Find $\left(f\circ g\right)(x)$ and the domain of $f\circ g$

b) Find $\left(g\circ f\right)(x)$ and the domain of $g\circ f$

Solution

a) $f\left(g(x)\right)=f\left(\dfrac{-4x+3}{x-2}\right)$ ***Domain***: $x\neq 2$

$$=\dfrac{2\dfrac{-4x+3}{x-2}+3}{\dfrac{4x+3}{x-2}+4}$$

$$=\dfrac{-8x+6+3x-6}{4x+3+4x-8}$$

$$=\dfrac{-5x}{-5}$$

$$=x$$ ***Domain***: $\mathbb{R}$

Domain: $\underline{x \neq 2}$

b) $g\left(f(x)\right) = g\left(\dfrac{2x+3}{x+4}\right)$ ***Domain***: $x \neq -4$

$$= \dfrac{-4\dfrac{2x+3}{x+4}+3}{\dfrac{2x+3}{x+4}-2}$$

$$= \dfrac{-8x-12+3x+12}{2x+3-2x-8}$$

$$= \dfrac{-5x}{-5}$$

$$\underline{= x}$$ ***Domain***: $\mathbb{R}$

Domain: $\underline{x \neq -4}$

Exercise

Given $f(x) = x+1$ *and* $g(x) = x^3 - 5x^2 + 3x + 7$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(x^3 - 5x^2 + 3x + 7\right)$ ***Domain***: $\mathbb{R}$

$$= x^3 - 5x^2 + 3x + 7 + 1$$

$$\underline{= x^3 - 5x^2 + 3x + 8}$$ ***Domain***: $\mathbb{R}$

Domain: $\underline{\mathbb{R}}$

b) $g\left(f(x)\right) = g(x+1)$ ***Domain***: $\mathbb{R}$

$$= (x+1)^3 - 5(x+1)^2 + 3(x+1) + 7$$

$$= x^3 + 3x^2 + 3x + 1 - 5\left(x^2 + 2x + 1\right) + 3x + 3 + 7$$

$$= x^3 + 3x^2 + 6x + 11 - 5x^2 - 10x - 5$$

$$\underline{= x^3 - 2x^2 - 4x + 6}$$ ***Domain***: $\mathbb{R}$

Domain: $\underline{\mathbb{R}}$

Exercise

Given $f(x) = x - 1$ *and* $g(x) = x^3 + 2x^2 - 3x - 9$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(x^3 + 2x^2 - 3x - 9\right)$ ***Domain***: $\mathbb{R}$

$= x^3 + 2x^2 - 3x - 9 - 1$

$= x^3 + 2x^2 - 3x - 10$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g(f(x)) = g(x - 1)$ ***Domain***: $\mathbb{R}$

$= (x-1)^3 + 2(x-1)^2 - (x-1) - 9$

$= x^3 - 3x^2 + 3x - 1 + 2\left(x^2 - 2x + 1\right) - 3x + 3 - 9$

$= x^3 - 3x^2 - 7 + 2x^2 - 4x + 2$

$= x^3 - x^2 - 4x - 5$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ *and* $g(x) = x^2 - 5x$: $(f \circ g)(4)$

Solution

$$(f \circ g)(4) = f(g(4))$$
$$= f(16 - 20)$$
$$= f(-4)$$
$$= -8 - 3$$
$$= -11$$

Exercise

Evaluate each composite function, where $f(x) = 2x - 3$ *and* $g(x) = x^2 - 5x$: $(g \circ f)(4)$

Solution

$$(g \circ f)(4) = g(f(4))$$

$$
\begin{aligned}
&= g(8-3) \\
&= g(5) \\
&= 25-25 \\
&\underline{=0}
\end{aligned}
$$

Exercise

Evaluate each composite function, where $f(x)=2x-3$ *and* $g(x)=x^2-5x$: $(f\circ g)(-2)$

Solution

$$
\begin{aligned}
(f\circ g)(-2) &= f(g(-2)) \\
&= f(4+10) \\
&= f(14) \\
&= 28-3 \\
&\underline{=25}
\end{aligned}
$$

Exercise

Evaluate each composite function, where $f(x)=2x-3$ *and* $g(x)=x^2-5x$: $(g\circ f)(-2)$

Solution

$$
\begin{aligned}
(g\circ f)(-2) &= g(f(-2)) \\
&= g(-4-3) \\
&= g(-7) \\
&= 49+35 \\
&\underline{=84}
\end{aligned}
$$

Exercise

Evaluate each composite function, where $f(x)=2x-3$ *and* $g(x)=x^2-5x$: $(f\circ f)(-3)$

Solution

$$
\begin{aligned}
(f\circ f)(-3) &= f(f(-3)) \\
&= f(-6-3) \\
&= f(-9) \\
&= -18-3 \\
&\underline{=-21}
\end{aligned}
$$

Exercise

Evaluate each composite function, where $f(x)=2x-3$ *and* $g(x)=x^2-5x$: $(g\circ g)(7)$

Solution

$$\begin{aligned}(g\circ g)(7)&=g(g(7))\\&=g(49-35)\\&=g(14)\\&=196-70\\&=126\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x)=2x-3$ *and* $g(x)=x^2-5x$: $(f\circ g)(\sqrt{2})$

Solution

$$\begin{aligned}(f\circ g)(\sqrt{2})&=f\left(g\left(\sqrt{2}\right)\right)\\&=f\left(2-5\sqrt{2}\right)\\&=2\left(2-5\sqrt{2}\right)-3\\&=4-10\sqrt{2}-3\\&=1-10\sqrt{2}\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x)=2x-3$ *and* $g(x)=x^2-5x$: $(g\circ f)(\sqrt{3})$

Solution

$$\begin{aligned}(g\circ f)(\sqrt{3})&=g\left(f\left(\sqrt{3}\right)\right)\\&=g\left(2\sqrt{3}-3\right)\\&=\left(2\sqrt{3}-3\right)^2-5\left(2\sqrt{3}-3\right)\\&=12-12\sqrt{3}+9-10\sqrt{3}+15\\&=36-22\sqrt{3}\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x)=2x-3$ *and* $g(x)=x^2-5x$: $(f\circ g)(2a)$

Solution

$$\begin{aligned}(f\circ g)(2a) &= f\left(g(2a)\right)\\ &= f\left(4a^2-10a\right)\\ &= 2\left(4a^2-10a\right)-3\\ &= 8a^2-20a-3\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x)=2x-3$ *and* $g(x)=x^2-5x$: $(g\circ f)(3b)$

Solution

$$\begin{aligned}(g\circ f)(3b) &= g\left(f(3b)\right)\\ &= g(6b-3)\\ &= (6b-3)^2-5(6b-3)\\ &= 36b^2-36b+9-30b+15\\ &= 36b^2-66b+24\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x)=2x-3$ *and* $g(x)=x^2-5x$: $(f\circ g)(k+1)$

Solution

$$\begin{aligned}(f\circ g)(k+1) &= f\left(g(k+1)\right)\\ &= f\left((k+1)^2-5k-5\right)\\ &= 2\left((k+1)^2-5k-5\right)-3\\ &= 2\left(k^2+2k+1\right)-10k-10-3\\ &= 2k^2+4k+2-10k-13\\ &= 2k^2-6k-11\end{aligned}$$

Exercise

Evaluate each composite function, where $f(x)=2x-3$ *and* $g(x)=x^2-5x$: $(g\circ f)(k-1)$

Solution

$$
\begin{aligned}
(g\circ f)(k-1) &= g\left(f(k-1)\right)\\
&= g(2k-2-3)\\
&= g(2k-5)\\
&= (2k-5)^2-5(2k-5)\\
&= 4k^2-20k+25-10k+25\\
&= 4k^2-30k+50
\end{aligned}
$$

Solution   *Section* 2.4 – Properties of Division

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=2x^4-x^3+7x-12; \quad p(x)=x^2-3$

Solution

$$
\begin{array}{r|l}
 & 2x^2 - x + 6 \\
\hline
x^2-3 & 2x^4 - x^3 + 0x^2 + 7x - 12 \\
 & \underline{2x^4 \qquad - 6x^2} \\
 & \quad - x^3 + 6x^2 + 7x \\
 & \quad \underline{-x^3 \qquad + 3x} \\
 & \qquad 6x^2 + 4x - 12 \\
 & \qquad \underline{6x^2 \qquad - 18} \\
 & \qquad\quad 4x + 6
\end{array}
$$

$$\boxed{Q(x)=2x^2-x+6; \quad R(x)=4x+6}$$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=3x^3+2x-4; \quad p(x)=2x^2+1$

Solution

$$
\begin{array}{r|l}
 & \frac{3}{2}x \\
\hline
2x^2+1 & 3x^3 + 0x^2 + 2x - 4 \\
 & \underline{3x^3 \qquad + \frac{3}{2}x} \\
 & \qquad \frac{1}{2}x - 4
\end{array}
$$

$$\boxed{Q(x)=\frac{3}{2}x; \quad R(x)=\frac{1}{2}x-4}$$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x) = 7x + 2; \quad p(x) = 2x^2 - x - 4$

Solution

$$
\begin{array}{r|l}
 & \frac{2}{7}x - \frac{11}{49} \\
\hline
7x + 2 & 2x^2 - x - 4 \\
 & \underline{2x^2 + \frac{4}{7}x} \\
 & \quad -\frac{11}{7}x - 4 \\
 & \quad \underline{-\frac{11}{7}x - \frac{22}{49}} \\
 & \qquad -\frac{174}{49}
\end{array}
$$

$$\underline{Q(x) = \frac{2}{7}x - \frac{11}{49}} \quad \underline{R(x) = -\frac{174}{49}}$$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x) = 9x + 4; \quad p(x) = 2x - 5$

Solution

$$
\begin{array}{r|l}
 & \frac{9}{2} \\
\hline
2x - 5 & 9x + 4 \\
 & \underline{9x - \frac{45}{2}} \\
 & \quad -\frac{37}{2}
\end{array}
$$

$$\underline{Q(x) = \frac{9}{2}; \quad R(x) = -\frac{37}{2}}$$

Exercise

Use the remainder theorem to find $f(c)$: $f(x) = x^4 - 6x^2 + 4x - 8; \quad c = -3$

Solution

$$f(-3) = (-3)^4 - 6(-3)^2 + 4(-3) - 8$$

$$\underline{= 7}$$

Exercise

Use the remainder theorem to find $f(c)$: $f(x) = x^4 + 3x^2 - 12; \quad c = -2$

Solution

$$f(-2) = (-2)^4 + 3(-2)^2 - 12$$
$$= 16$$

Exercise

Use the factor theorem to show that $x - c$ is a factor of $f(x)$: $f(x) = x^3 + x^2 - 2x + 12; \quad c = -3$

Solution

$$f(-3) = (-3)^3 + (-3)^2 - 2(-3) + 12$$
$$= 0$$

From the factor theorem; $x + 3$ is a factor of $f(x)$.

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $2x^3 - 3x^2 + 4x - 5; \quad x - 2$

Solution

$$\begin{array}{r|rrrr} 2 & 2 & -3 & 4 & -5 \\ & & 4 & 2 & 12 \\ \hline & 2 & 1 & 6 & \boxed{7} \end{array}$$

$Q(x) = 2x^2 + x + 6 \qquad R(x) = 7$

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $5x^3 - 6x^2 + 15; \quad x - 4$

Solution

$$\begin{array}{r|rrrr} 4 & 5 & -6 & 0 & 15 \\ & & 20 & 56 & 224 \\ \hline & 5 & 14 & 56 & \boxed{239} \end{array}$$

$Q(x) = 5x^2 + 14x + 56 \qquad R(x) = 239$

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $9x^3-6x^2+3x-4; \quad x-\frac{1}{3}$

Solution

$$\begin{array}{c|cccc} \frac{1}{3} & 9 & -6 & 3 & -4 \\ & & 3 & -1 & \frac{2}{3} \\ \hline & 9 & -3 & 2 & \boxed{-\frac{10}{3}} \end{array}$$

$$Q(x)=9x^2-3x+2 \qquad R(x)=-\frac{10}{3}$$

Exercise

Use the synthetic division to find $f(c)$: $f(x)=2x^3+3x^2-4x+4; \quad c=3$

Solution

$$\begin{array}{c|cccc} 3 & 2 & 3 & -4 & 4 \\ & & 9 & 36 & 93 \\ \hline & 3 & 12 & 32 & \boxed{97} \end{array}$$

$$f(3)=97$$

Exercise

Use the synthetic division to find $f(c)$: $f(x)=8x^5-3x^2+7; \quad c=\frac{1}{2}$

Solution

$$\begin{array}{c|cccccc} \frac{1}{2} & 8 & 0 & 0 & -3 & 0 & 7 \\ & & 4 & 2 & 1 & -1 & -\frac{1}{2} \\ \hline & 8 & 4 & 2 & -2 & -1 & \boxed{\frac{13}{2}} \end{array}$$

$$f\left(\frac{1}{2}\right)=\frac{13}{2}$$

Exercise

Use the synthetic division to find $f(c)$: $f(x) = x^3 - 3x^2 - 8$; $c = 1 + \sqrt{2}$

Solution

$$\begin{array}{r|cccc}
1+\sqrt{2} & 3 & -3 & 0 & -8 \\
 & & 3+3\sqrt{2} & 6+3\sqrt{2} & 12+9\sqrt{2} \\
\hline
 & 3 & 3\sqrt{2} & 6+3\sqrt{2} & \boxed{4+9\sqrt{2}}
\end{array}$$

$$f\left(1+\sqrt{2}\right) = 4 + 9\sqrt{2}$$

Exercise

Use the synthetic division to show that c is a zero of $f(x)$: $f(x) = 3x^4 + 8x^3 - 2x^2 - 10x + 4$; $c = -2$

Solution

$$\begin{array}{r|ccccc}
-2 & 3 & 8 & -2 & -10 & 4 \\
 & & -6 & -4 & 12 & -4 \\
\hline
 & 3 & 2 & -6 & 2 & \boxed{0}
\end{array}$$

$$f(-2) = 0$$

Exercise

Use the synthetic division to show that c is a zero of $f(x)$: $f(x) = 27x^4 - 9x^3 + 3x^2 + 6x + 1$; $c = -\frac{1}{3}$

Solution

$$\begin{array}{r|ccccc}
-\frac{1}{3} & 27 & -9 & 3 & 6 & 1 \\
 & & -9 & 6 & -3 & -1 \\
\hline
 & 27 & -18 & 9 & 3 & \boxed{0}
\end{array}$$

$$f\left(-\tfrac{1}{3}\right) = 0$$

Exercise

Find all values of k such that $f(x)$ is divisible by the given linear polynomial:

$f(x) = kx^3 + x^2 + k^2x + 3k^2 + 11$; $x + 2$

Solution

$$\begin{array}{r|cccc} -2 & k & 1 & k^2 & 3k^2+11 \\ & & -2k & 4k-2 & -2k^2-8k+4 \\ \hline & k & 1-2k & k^2+4k-2 & k^2-8k+15 \end{array}$$

$k^2-8k+15=0 \Rightarrow \boxed{k=3,\ 5}$

Exercise

Find all solutions of the equation: $x^3-x^2-10x-8=0$

Solution

possibilities for $\frac{c}{d}: \pm\left\{\frac{8}{1}\right\} = \pm\{1,\ 2,\ 4,\ 8\}$

$$\begin{array}{r|cccc} -1 & 1 & -1 & -10 & -8 \\ & & -1 & 2 & 8 \\ \hline & 1 & -2 & -8 & 0 \end{array} \rightarrow \ x^2-2x-8=0$$

The solutions are: $x=-1,\ -2,\ 4$

Exercise

Find all solutions of the equation: $x^3+x^2-14x-24=0$

Solution

possibilities for $\frac{c}{d}: \pm\left\{\frac{24}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24\}$

$$\begin{array}{r|cccc} -2 & 1 & 1 & -14 & -24 \\ & & -2 & 2 & 24 \\ \hline & 1 & -1 & -12 & 0 \end{array} \rightarrow \ x^2-x-12=0$$

The solutions are: $x=-2,\ -3,\ 4$

Exercise

Find all solutions of the equation: $2x^3-3x^2-17x+30=0$

Solution

possibilities for $\frac{c}{d}: \pm\left\{\frac{30}{2}\right\} = \pm\left\{1,\ 2,\ 3,\ 5,\ 6,\ 10,\ 15,\ 30,\ \frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ \frac{15}{2}\right\}$

$$\begin{array}{c|cccc} 2 & 2 & -3 & -17 & 30 \\ & & 4 & 2 & -30 \\ \hline & 2 & 1 & -15 & \boxed{0} \end{array} \rightarrow \quad 2x^2 + x - 15 = 0$$

The solutions are: $\boxed{x = 2, \ -3, \ \frac{5}{2}}$

Exercise

Find all solutions of the equation: $\quad 12x^3 + 8x^2 - 3x - 2 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{2}{12}\right\} = \pm\left\{1, \ 2, \ \frac{1}{2}, \ \frac{1}{3}, \ \frac{2}{3}, \ \frac{1}{2}, \ \frac{1}{6}, \ \frac{1}{12}\right\}$

$$\begin{array}{c|cccc} \frac{1}{2} & 12 & 8 & -3 & -2 \\ & & 6 & 7 & 2 \\ \hline & 12 & 14 & 4 & \boxed{0} \end{array} \rightarrow \quad 12x^2 + 14x + 4 = 0$$

The solutions are: $\boxed{x = \frac{1}{2}, \ -\frac{1}{2}, \ -\frac{2}{3}}$

Exercise

Find all solutions of the equation: $\quad x^3 + x^2 - 6x - 8 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{8}{1}\right\} = \pm\{1, \ 2, \ 4, \ 8\}$

$$\begin{array}{c|cccc} -2 & 1 & 1 & -6 & -8 \\ & & -2 & 2 & 8 \\ \hline & 1 & -1 & -4 & \boxed{0} \end{array} \rightarrow \quad x^2 - x - 4 = 0$$

$$x = \frac{1 \pm \sqrt{1+16}}{2}$$

The solutions are: $\boxed{x = -2, \ \frac{1 \pm \sqrt{17}}{2}}$

Exercise

Find all solutions of the equation: $\quad x^3 - 19x - 30 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{30}{1}\right\} = \pm\{1, \ 2, \ 3, \ 5, \ 6, \ 15, \ 30\}$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -19 & -30 \\ & & -2 & 4 & 30 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array} \rightarrow \quad x^2 - 2x - 15$$

$$x = \frac{2 \pm \sqrt{4+60}}{2}$$

$$= \begin{cases} \frac{2-8}{2} = -3 \\ \frac{2+8}{2} = 5 \end{cases}$$

The solutions are: $\underline{x = -2, \ -3, \ 5}$

Exercise

Find all solutions of the equation: $\quad 2x^3 + x^2 - 25x + 12 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{12}{2}\right\} = \pm\left\{1, \ 2, \ 3, \ 4, \ 6, \ 12, \ \frac{1}{2}, \ \frac{3}{2}\right\}$

$$\begin{array}{r|rrrr} 3 & 2 & 1 & -25 & 12 \\ & & 6 & 21 & -12 \\ \hline & 2 & 7 & -4 & \boxed{0} \end{array} \rightarrow \quad 2x^2 + 7x - 4$$

$$x = \frac{-7 \pm \sqrt{49+32}}{4}$$

$$= \begin{cases} \frac{-7-9}{4} = -4 \\ \frac{-7+9}{4} = \frac{1}{2} \end{cases}$$

The solutions are: $\underline{x = -4, \ \frac{1}{2}, \ 3}$

Exercise

Find all solutions of the equation: $\quad 3x^3 + 11x^2 - 6x - 8 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{8}{3}\right\} = \pm\left\{1, \ 2, \ 4, \ 8, \ \frac{1}{3}, \ \frac{2}{3}, \ \frac{4}{3}, \ \frac{8}{3}\right\}$

$$\begin{array}{r|rrrr} 1 & 3 & 11 & -6 & -8 \\ & & 3 & 14 & 8 \\ \hline & 3 & 14 & 8 & \boxed{0} \end{array} \rightarrow \quad 3x^2 + 14x + 8$$

$$x = \frac{-14 \pm \sqrt{196-96}}{6}$$

$$= \begin{cases} \frac{-14-10}{6} = -4 \\ \frac{-14+10}{6} = -\frac{2}{3} \end{cases}$$

The solutions are: $\underline{x = -4,\ -\frac{2}{3},\ 1}$

Exercise

Find all solutions of the equation: $2x^3 + 9x^2 - 2x - 9 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{9}{2}\right\} = \pm\left\{1,\ 3,\ 9,\ \frac{1}{2},\ \frac{3}{2},\ \frac{9}{2}\right\}$

$$\begin{array}{r|rrrr} 1 & 2 & 9 & -2 & -9 \\ & & 2 & 11 & 9 \\ \hline & 2 & 11 & 9 & \boxed{0} \end{array} \quad \to \quad 2x^2 + 11x + 9$$

$\underline{x = -1,\ -\frac{9}{2}}$ $\qquad a - b + c = 0 \quad \to \quad x = -1,\ -\frac{c}{a}$

The solutions are: $\underline{x = -\frac{9}{2},\ -1,\ 1}$

Exercise

Find all solutions of the equation: $x^3 + 3x^2 - 6x - 8 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{8}{1}\right\} = \pm\{1,\ 2,\ 4,\ 8\}$

$$\begin{array}{r|rrrr} -1 & 1 & 3 & -6 & -8 \\ & & -1 & -2 & 8 \\ \hline & 1 & 2 & -8 & \boxed{0} \end{array} \to \quad x^2 + 2x - 8 = 0$$

$$x = \frac{-2 \pm \sqrt{4+32}}{2}$$

$$= \begin{cases} \frac{-2-6}{2} = -4 \\ \frac{-2+6}{2} = 2 \end{cases}$$

The solutions are: $\underline{x = -4,\ -1,\ 2}$

Exercise

Find all solutions of the equation: $3x^3 - x^2 - 6x + 2 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{2}{3}\right\} = \pm\left\{1,\ 2,\ \frac{1}{3},\ \frac{2}{3}\right\}$

$$\begin{array}{c|cccc} \frac{1}{3} & 3 & -1 & -6 & 2 \\ & & 1 & 0 & -2 \\ \hline & 3 & 0 & -6 & \boxed{0} \end{array} \quad \to \quad 3x^2 - 6 = 0$$

$x^2 = 2$

The solutions are: $\underline{x = \frac{1}{3},\ \pm\sqrt{2}}$

Exercise

Find all solutions of the equation: $x^3 - 8x^2 + 8x + 24 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{24}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24\}$

$$\begin{array}{c|cccc} 6 & 1 & -8 & 8 & 24 \\ & & 6 & -12 & -24 \\ \hline & 1 & -2 & -4 & \boxed{0} \end{array} \quad \to \quad x^2 - 2x - 4 = 0$$

$$x = \frac{2 \pm \sqrt{4+16}}{2}$$

$$= \frac{2 \pm 2\sqrt{5}}{2}$$

The solutions are: $\underline{x = 6,\ 1 \pm \sqrt{5}}$

Exercise

Find all solutions of the equation: $x^3 - 7x^2 - 7x + 69 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{69}{1}\right\} = \pm\{1,\ 3,\ 23,\ 69\}$

$$\begin{array}{c|cccc} -3 & 1 & -7 & -7 & 69 \\ & & -3 & 30 & -69 \\ \hline & 1 & -10 & 23 & \boxed{0} \end{array} \quad \to \quad x^2 - 10x + 23 = 0$$

$$x=\frac{10\pm\sqrt{100-92}}{2}$$
$$=\frac{10\pm 2\sqrt{2}}{2}$$

The solutions are: $\underline{x=-3,\ 5\pm\sqrt{2}}$

Exercise

Find all solutions of the equation: $x^3-3x-2=0$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{2}{1}\right\}=\pm\{1,\ 2\}$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -3 & -2 \\ & & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array} \quad \to \quad x^2-x-2=0$$

$\underline{x=-1,\ 2}$ $\qquad a-b+c=0 \ \to\ x=-1,\ -\frac{c}{a}$

The solutions are: $\underline{x=-1,\ -1,\ 2}$

Exercise

Find all solutions of the equation: $x^3-2x+1=0$

Solution

possibilities for $\frac{c}{d}:\pm\{1\}$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -2 & 1 \\ & & 1 & 1 & 1 \\ \hline & 1 & 1 & -1 & \boxed{0} \end{array} \quad \to \quad x^2+x-1=0$$

$$x=\frac{-1\pm\sqrt{5}}{2}$$

The solutions are: $\underline{x=1,\ \frac{-1\pm\sqrt{5}}{2}}$

Exercise

Find all solutions of the equation: $x^3 - 2x^2 - 11x + 12 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{12}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -11 & 12 \\ & & 1 & -1 & 12 \\ \hline & 1 & -1 & -12 & \boxed{0} \end{array} \quad \to \quad x^2 - x - 12 = 0$$

$$x = \frac{1 \pm \sqrt{1+48}}{2}$$

$$= \begin{cases} \frac{1-7}{2} = -3 \\ \frac{1+7}{2} = 4 \end{cases}$$

The solutions are: $\underline{x = -3,\ 1,\ 4}$

Exercise

Find all solutions of the equation: $x^3 - 2x^2 - 7x - 4 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{4}{1}\right\} = \pm\{1,\ 2,\ 4\}$

$$\begin{array}{r|rrrr} -1 & 1 & -2 & -7 & -4 \\ & & -1 & 3 & 4 \\ \hline & 1 & -3 & -4 & \boxed{0} \end{array} \quad \to \quad x^2 - 3x - 4 = 0$$

$\underline{x = -1,\ 4}$ $\qquad a - b + c = 0 \quad \to \quad x = -1,\ -\frac{c}{a}$

The solutions are: $\underline{x = -1,\ -1,\ 4}$

Exercise

Find all solutions of the equation: $x^3 - 10x - 12 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{12}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -10 & -12 \\ & & -2 & 4 & 12 \\ \hline & 1 & -2 & -6 & \boxed{0} \end{array} \quad \rightarrow \quad x^2 - 2x - 6 = 0$$

$$x = \frac{2 \pm \sqrt{4+24}}{2}$$

$$= \frac{2 \pm 2\sqrt{7}}{2}$$

The solutions are: $x = -2, \ 1 \pm \sqrt{7}$

Exercise

Find all solutions of the equation: $x^3 - 5x^2 + 17x - 13 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{13}{1}\right\} = \pm\{1, \ 13\}$

$$\begin{array}{r|rrrr} 1 & 1 & -5 & 17 & -13 \\ & & 1 & -4 & 13 \\ \hline & 1 & -4 & 13 & \boxed{0} \end{array} \quad \rightarrow \quad x^2 - 4x + 13 = 0$$

$$x = \frac{4 \pm \sqrt{16-52}}{2}$$

$$= \frac{4 \pm 6i}{2}$$

The solutions are: $x = 1, \ 2 \pm 3i$

Exercise

Find all solutions of the equation: $6x^3 + 25x^2 - 24x + 5 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{5}{6}\right\} = \pm\left\{1, \ 5, \ \frac{1}{2}, \ \frac{5}{2}, \ \frac{1}{3}, \ \frac{5}{3}, \ \frac{1}{6}, \ \frac{5}{6}\right\}$

$$\begin{array}{r|rrrr} -5 & 6 & 25 & -24 & 5 \\ & & -30 & 25 & -5 \\ \hline & 6 & -5 & 1 & \boxed{0} \end{array} \quad \rightarrow \quad 6x^2 - 5x + 1 = 0$$

$$x = \frac{5 \pm \sqrt{25-24}}{12}$$

$$= \begin{cases} \frac{5-1}{12} = \frac{1}{3} \\ \frac{5+1}{12} = \frac{1}{2} \end{cases}$$

The solutions are: $\underline{x = -5,\ \frac{1}{3},\ \frac{1}{2}}$

Exercise

Find all solutions of the equation: $8x^3 + 18x^2 + 45x + 27 = 0$

Solution

$$\textit{possibilities}: \ \pm\left\{\frac{27}{8}\right\} = \pm\left\{\frac{1,\ 3,\ 9,\ 27}{1,\ 2,\ 4,\ 8}\right\}$$

$$= \pm\left\{1,\ 3,\ 9,\ 27,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{3}{2},\ \frac{3}{4},\ \frac{3}{8},\ \frac{9}{2},\ \frac{9}{4},\ \frac{9}{8},\ \frac{27}{2},\ \frac{27}{4},\ \frac{27}{8}\right\}$$

$$\begin{array}{r|rrrr} -\frac{3}{4} & 8 & 18 & 45 & 27 \\ & & -6 & -9 & -27 \\ \hline & 8 & 12 & 36 & \boxed{0} \end{array} \quad \to \ 8x^2 + 12x + 36 = 0$$

The solutions are: $\underline{x = -\frac{3}{4},\ -\frac{3}{4} \pm i\frac{3\sqrt{7}}{4}}$

Exercise

Find all solutions of the equation: $3x^3 - x^2 + 11x - 20 = 0$

Solution

$$\textit{possibilities}: \ \pm\left\{\frac{20}{3}\right\} = \pm\left\{\frac{1,\ 2,\ 4,\ 5,\ 10,\ 20}{1,\ 3}\right\}$$

$$= \pm\left\{1,\ 2,\ 4,\ 5,\ 10,\ 20,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3},\ \frac{5}{3},\ \frac{10}{3},\ \frac{20}{3}\right\}$$

$$\begin{array}{r|rrrr} \frac{4}{3} & 3 & -1 & 11 & -20 \\ & & 4 & 4 & 20 \\ \hline & 3 & 3 & 15 & \boxed{0} \end{array} \quad \to \ 3x^2 + 3x + 15 = 0$$

$$x = \frac{-3 \pm \sqrt{9 - 180}}{6}$$

$$= \frac{-3 \pm 3\sqrt{19}}{6}$$

The solutions are: $\underline{x = \frac{4}{3},\ -\frac{1}{2} \pm i\frac{\sqrt{19}}{2}}$

Exercise

Find all solutions of the equation: $x^4 - x^3 - 9x^2 + 3x + 18 = 0$

Solution

possibilities for $\frac{c}{d}: \pm\left\{\frac{18}{1}\right\} = \pm\{1,\ 2,\ 3,\ 6,\ 9,\ 18\}$

$$\begin{array}{r|rrrrr} -2 & 1 & -1 & -9 & 3 & 18 \\ & & -2 & 6 & 6 & -18 \\ \hline 3 & 1 & -3 & -3 & 9 & 0 \\ & & 3 & 0 & -9 & \\ \hline & 1 & 0 & -3 & 0 & \end{array}$$

$\rightarrow x^3 - 3x^2 - 3x + 9 = 0 \ \rightarrow\ \pm\left\{\frac{9}{1}\right\} = \pm\{1,\ 3,\ 9\}$

$\rightarrow x^2 - 3 = 0 \Rightarrow \underline{x = \pm\sqrt{3}}$

The solutions are: $x = -2,\ 3,\ \pm\sqrt{3}$

Exercise

Find all solutions of the equation: $2x^4 - 9x^3 + 9x^2 + x - 3 = 0$

Solution

possibilities for $\frac{c}{d}: \pm\left\{\frac{3}{2}\right\} = \pm\left\{1,\ 3,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$$\begin{array}{r|rrrrr} 1 & 2 & -9 & 9 & 1 & -3 \\ & & 2 & -7 & 2 & 3 \\ \hline 1 & 2 & -7 & 2 & 3 & 0 \\ & & 2 & -5 & -3 & \\ \hline & 2 & -5 & -3 & 0 & \end{array}$$

$\rightarrow 2x^3 - 7x^2 + 2x + 3 = 0 \ \rightarrow\ \pm\left\{\frac{3}{2}\right\} = \pm\left\{1,\ 3,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$\rightarrow 2x^2 - 5x - 3 = 0$

$$x = \frac{5 \pm \sqrt{25 + 24}}{4}$$

$$= \begin{cases} \frac{5-7}{4} = -\frac{1}{2} \\ \frac{5+7}{4} = 3 \end{cases}$$

The solutions are: $x = 1,\ 1,\ -\frac{1}{2},\ 3$

Exercise

Find all solutions of the equation: $6x^4+5x^3-17x^2-6x=0$

Solution

$$x\left(6x^3+5x^2-17x-6\right)=0 \quad\to\quad \underline{x=0}$$

$$possibilities:\ \pm\left\{\frac{6}{6}\right\}=\pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{3},\ \frac{2}{3}\right\}$$

$$\begin{array}{r|rrrr} -2 & 6 & 5 & -17 & -6 \\ & & -12 & 14 & 6 \\ \hline & 6 & -7 & -3 & \boxed{0} \end{array} \quad\to\quad 6x^2-7x-3=0$$

$$x=\frac{7\pm\sqrt{49+72}}{12}$$

$$=\begin{cases}\frac{7-11}{12}=-\frac{1}{3}\\ \frac{7+11}{12}=\frac{3}{2}\end{cases}$$

The solutions are: $x=0,\ -2,\ -\frac{1}{3},\ \frac{3}{2}$

Exercise

Find all solutions of the equation: $x^4-2x^2-16x-15=0$

Solution

$$possibilities:\ \pm\left\{\frac{15}{1}\right\}=\pm\{1,\ 3,\ 5,\ 15\}$$

$$\begin{array}{r|rrrrr} -1 & 1 & 0 & -2 & -16 & -15 \\ & & -1 & 1 & 1 & 15 \\ \hline 3 & 1 & -1 & -1 & -15 & 0 \\ & & 3 & 6 & 15 & \\ \hline & 1 & 2 & 5 & 0 & \end{array}$$

$\to\ x^3-x^2-x-15=0\ \to\ \pm\left\{\frac{15}{1}\right\}=\pm\{1,\ 3,\ 5,\ 15\}$

$\to\ x^2+2x+5=0$

$$x=\frac{-2\pm\sqrt{-16}}{2}$$

$$=-1\pm 2i$$

The solutions are: $x=-1,\ 3,\ -1\pm 2i$

Exercise

Find all solutions of the equation: $\quad x^4 - 2x^3 - 5x^2 + 8x + 4 = 0$

Solution

$$possibilities: \ \pm\left\{\frac{4}{1}\right\} = \pm\{1,\ 2,\ 4\}$$

$$\begin{array}{r|rrrrr}
2 & 1 & -2 & -5 & 8 & 4 \\
 & & 2 & 0 & -10 & -4 \\ \hline
-2 & 1 & 0 & -5 & -2 & 0 \\
 & & -2 & 4 & 2 & \\ \hline
 & 1 & -2 & -1 & 0 &
\end{array}
\begin{array}{l}
\\ \\
\to x^3 - 5x - 2 = 0 \ \to \ \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\} \\
\\
\to x^2 - 2x - 1 = 0
\end{array}$$

$$x = \frac{2 \pm \sqrt{8}}{2}$$

$$= \frac{2 \pm 2\sqrt{2}}{2}$$

The solutions are: $\boxed{x = -2,\ 2,\ 1 \pm \sqrt{2}}$

Exercise

Find all solutions of the equation: $\quad 2x^4 - 17x^3 + 4x^2 + 35x - 24 = 0$

Solution

$$possibilities: \ \pm\left\{\frac{24}{2}\right\} = \pm\left\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24,\ \frac{1}{2},\ \frac{3}{2}\right\}$$

$$\begin{array}{r|rrrrr}
1 & 2 & -17 & 4 & 35 & -24 \\
 & & 2 & -15 & -11 & 24 \\ \hline
1 & 2 & -15 & -11 & 24 & 0 \\
 & & 2 & -13 & 24 & \\ \hline
 & 2 & -13 & -24 & 0 &
\end{array}
\begin{array}{l}
\\ \\
\to 2x^3 - 15x^2 - 11x + 24 = 0 \ \to \ \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\} \\
\\
\to 2x^2 - 13x - 24 = 0
\end{array}$$

$$x = \frac{13 \pm \sqrt{169 + 192}}{4}$$

$$= \begin{cases} \dfrac{13 - 19}{4} = -\dfrac{3}{2} \\[2mm] \dfrac{13 + 19}{4} = 8 \end{cases}$$

The solutions are: $\boxed{x = -\frac{3}{2},\ 1,\ 1,\ 8}$

Exercise

Find all solutions of the equation: $x^4 + x^3 - 3x^2 - 5x - 2 = 0$

Solution

possibilities : $\pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$

$$\begin{array}{r|rrrrr} -1 & 1 & 1 & -3 & -5 & -2 \\ & & -1 & 0 & 3 & 2 \\ \hline -1 & 1 & 0 & -3 & -2 & 0 \\ & & -1 & 1 & 2 & \\ \hline & 1 & -1 & -2 & 0 & \end{array}$$

$\to x^3 - 3x - 2 = 0 \to \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$

$\to x^2 - x - 2 = 0$

$$x = \frac{1 \pm \sqrt{9}}{2}$$

$$= \begin{cases} \frac{1-3}{2} = -1 \\ \frac{1+3}{2} = 2 \end{cases}$$

The solutions are: $x = -1,\ -1,\ -1,\ 2$

Exercise

Find all solutions of the equation: $6x^4 - 17x^3 - 11x^2 + 42x = 0$

Solution

$$x\left(6x^3 - 17x^2 - 11x + 42\right) = 0$$

$x = 0 \quad 6x^3 - 17x^2 - 11x + 42 = 0$

possibilities : $\pm\left\{\frac{42}{6}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ 7,\ 14,\ 21,\ 42,\ \frac{1}{2},\ \frac{3}{2},\ \frac{7}{2},\ \frac{21}{2},\ \frac{1}{3},\ \frac{2}{3},\ \frac{7}{3},\ \frac{14}{3},\ \frac{1}{6},\ \frac{7}{6},\ \frac{21}{6}\right\}$

$$\begin{array}{r|rrrr} 2 & 6 & -17 & -11 & 42 \\ & & 12 & -10 & -42 \\ \hline & 6 & -5 & -21 & \boxed{0} \end{array}$$

$\to 6x^2 - 5x - 21 = 0$

$$x = \frac{5 \pm \sqrt{25 + 504}}{12}$$

$$= \begin{cases} \frac{5-23}{12} = -\frac{3}{2} \\ \frac{5+23}{12} = \frac{7}{3} \end{cases}$$

The solutions are: $x = -\frac{3}{2},\ 0,\ 2,\ \frac{7}{3}$

Exercise

Find all solutions of the equation: $x^4 - 5x^2 - 2x = 0$

Solution

$$x\left(x^3 - 5x - 2\right) = 0$$

$$x = 0 \quad x^3 - 5x - 2 = 0$$

possibilities : $\pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -5 & -2 \\ & & -2 & 4 & 2 \\ \hline & 1 & -2 & -1 & \boxed{0} \end{array} \rightarrow x^2 - 2x - 1 = 0$$

$$x = \frac{2 \pm \sqrt{4+4}}{2}$$

$$= \frac{2 \pm 2\sqrt{2}}{2}$$

The solutions are: $x = -2,\ 0,\ 1 \pm \sqrt{2}$

Exercise

Find all solutions of the equation: $3x^4 - 4x^3 - 11x^2 + 16x - 4 = 0$

Solution

possibilities : $\pm\left\{\frac{4}{3}\right\} = \pm\left\{1,\ 2,\ 4,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3}\right\}$

$$\begin{array}{r|rrrrr} 1 & 3 & -4 & -11 & 16 & -4 \\ & & 3 & -1 & -12 & 4 \\ \hline 2 & 3 & -1 & -12 & 4 & 0 \\ & & 6 & 10 & -4 & \\ \hline & 3 & 5 & -2 & 0 & \end{array} \rightarrow 3x^3 - x^2 - 12x + 4 = 0 \rightarrow \pm\left\{\frac{4}{3}\right\} = \pm\left\{1,\ 2,\ 4,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3}\right\}$$

$$\rightarrow 3x^2 + 5x - 2 = 0$$

$$x = \frac{-5 \pm \sqrt{25+24}}{6}$$

$$= \begin{cases} \frac{-5-7}{6} = -2 \\ \frac{-5+7}{6} = \frac{1}{3} \end{cases}$$

The solutions are: $x = -2,\ \frac{1}{3},\ 1,\ 2$

Exercise

Find all solutions of the equation: $6x^4 + 23x^3 + 19x^2 - 8x - 4 = 0$

Solution

possibilities : $\pm\left\{\frac{4}{6}\right\} = \pm\left\{1,\ 2,\ 4,\ \frac{1}{6},\ \frac{2}{3},\ \frac{2}{3}\right\}$

$$\begin{array}{r|rrrrr}
-2 & 6 & 23 & 19 & -8 & -4 \\
 & & -12 & -22 & 6 & 4 \\
\hline
-2 & 6 & 11 & -3 & -2 & 0 \\
 & & -12 & 2 & 2 & \\
\hline
 & 6 & -1 & -1 & 0 &
\end{array}$$

$\rightarrow 6x^3 + 11x^2 - 3x - 2 = 0 \rightarrow \pm\left\{\frac{2}{6}\right\} = \pm\left\{1,\ 2,\ \frac{1}{6},\ \frac{1}{3}\right\}$

$\rightarrow 6x^2 - x - 1 = 0$

$$x = \frac{1 \pm \sqrt{25}}{12}$$

$$= \begin{cases} \frac{1-5}{12} = -\frac{1}{3} \\ \frac{1+5}{12} = \frac{1}{2} \end{cases}$$

The solutions are: $\underline{x = -2,\ -2,\ -\frac{1}{3},\ \frac{1}{2}}$

Exercise

Find all solutions of the equation: $4x^4 - 12x^3 + 3x^2 + 12x - 7 = 0$

Solution

possibilities : $\pm\left\{\frac{7}{4}\right\} = \pm\left\{1,\ 7,\ \frac{1}{2},\ \frac{7}{2},\ \frac{1}{4},\ \frac{7}{4}\right\}$

$$\begin{array}{r|rrrrr}
1 & 4 & -12 & 3 & 12 & -7 \\
 & & 4 & -8 & -5 & 7 \\
\hline
-1 & 4 & -8 & -5 & 7 & 0 \\
 & & -4 & 12 & -7 & \\
\hline
 & 4 & -12 & 7 & 0 &
\end{array}$$

$\rightarrow 4x^3 - 8x^2 - 5x + 7 = 0 \rightarrow \pm\left\{\frac{7}{4}\right\} = \pm\left\{1,\ 7,\ \frac{1}{2},\ \frac{7}{2},\ \frac{1}{4},\ \frac{7}{4}\right\}$

$\rightarrow 4x^2 - 12x + 7 = 0$

$$x = \frac{12 \pm \sqrt{144 - 112}}{8}$$

$$= \frac{12 \pm 4\sqrt{2}}{8}$$

The solutions are: $\underline{x = -1,\ 1,\ \frac{3 \pm \sqrt{2}}{2}}$

Exercise

Find all solutions of the equation: $2x^4 - 9x^3 - 2x^2 + 27x - 12 = 0$

Solution

$$\text{possibilities}: \ \pm\left\{\frac{12}{2}\right\} = \pm\left\{1,\ 2,\ 3,\ 4,\ 6,\ 12,\ \frac{1}{2},\ \frac{3}{2}\right\}$$

$$\begin{array}{r|rrrrr}
4 & 2 & -9 & -2 & 27 & -12 \\
 & & 8 & -4 & -24 & 12 \\
\hline
\frac{1}{2} & 2 & -1 & -6 & 3 & 0 \\
 & & 1 & 0 & -3 & \\
\hline
 & 2 & 0 & -6 & 0 &
\end{array}$$

$\rightarrow 2x^3 - x^2 - 6x + 3 = 0 \ \rightarrow \ \pm\left\{\frac{3}{2}\right\} = \pm\left\{1,\ 3,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$\rightarrow 2x^2 - 6 = 0$

$x^2 = 3$

The solutions are: $\boxed{x = \frac{1}{2},\ 4,\ \pm\sqrt{3}}$

Exercise

Find all solutions of the equation: $2x^4 - 19x^3 + 51x^2 - 31x + 5 = 0$

Solution

$$\text{possibilities}: \ \pm\left\{\frac{5}{2}\right\} = \pm\left\{1,\ 5,\ \frac{1}{2},\ \frac{5}{2}\right\}$$

$$\begin{array}{r|rrrrr}
5 & 2 & -19 & 51 & -31 & 5 \\
 & & 10 & -45 & 30 & -5 \\
\hline
\frac{1}{2} & 2 & -9 & 6 & -1 & 0 \\
 & & 1 & -4 & 1 & \\
\hline
 & 2 & -8 & 2 & 0 &
\end{array}$$

$\rightarrow 2x^3 - 9x^2 + 6x - 1 = 0 \ \rightarrow \ \pm\left\{\frac{1}{2}\right\}$

$\rightarrow 2x^2 - 8x + 2 = 0$

$$x = \frac{8 \pm \sqrt{64 - 16}}{4}$$

$$= \frac{8 \pm 4\sqrt{3}}{4}$$

The solutions are: $\boxed{x = \frac{1}{2},\ 5,\ 2 \pm \sqrt{3}}$

Exercise

Find all solutions of the equation: $4x^4 - 35x^3 + 71x^2 - 4x - 6 = 0$

Solution

$$\text{possibilities}: \ \pm\left\{\frac{6}{4}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{4},\ \frac{3}{4}\right\}$$

$$\begin{array}{r|rrrrr}
3 & 4 & -35 & 71 & -4 & -6 \\
 & & 12 & -69 & 6 & 6 \\
\hline
-\frac{1}{4} & 4 & -23 & 2 & 2 & 0 \\
 & & -1 & 6 & -2 & \\
\hline
 & 4 & -24 & 8 & 0 &
\end{array}$$

$\rightarrow 4x^3 - 23x^2 + 2x + 2 = 0 \rightarrow \pm\left\{\frac{2}{4}\right\} = \pm\left\{1,\ 2,\ \frac{1}{2},\ \frac{1}{4}\right\}$

$\rightarrow 4x^2 - 24x + 8 = 0$

$x^2 - 6x + 2 = 0$

$x = \dfrac{6 \pm \sqrt{36-8}}{2}$

$= \dfrac{6 \pm 2\sqrt{7}}{4}$

The solutions are: $\boxed{x = -\frac{1}{4},\ 3,\ 3 \pm \sqrt{7}}$

Exercise

Find all solutions of the equation: $2x^4 + 3x^3 - 4x^2 - 3x + 2 = 0$

Solution

possibilities : $\pm\left\{\frac{2}{2}\right\} = \pm\left\{1,\ 2,\ \frac{1}{2}\right\}$

$$\begin{array}{r|rrrrr}
1 & 2 & 3 & -4 & -3 & 2 \\
 & & 2 & 5 & 1 & -2 \\
\hline
-1 & 2 & 5 & 1 & -2 & 0 \\
 & & -2 & -3 & 2 & \\
\hline
 & 2 & 3 & -2 & 0 &
\end{array}$$

$\rightarrow 2x^3 - 23x^2 + 2x - 2 = 0 \rightarrow \pm\left\{\frac{2}{2}\right\} = \pm\left\{1,\ 2,\ \frac{1}{2}\right\}$

$\rightarrow 2x^2 + 3x - 2 = 0$

$x = \dfrac{-3 \pm \sqrt{9+16}}{4}$

$= \begin{cases} \dfrac{-3-5}{4} = -2 \\ \dfrac{-3+5}{4} = \dfrac{1}{2} \end{cases}$

The solutions are: $\boxed{x = -2,\ \ -1,\ \frac{1}{2},\ 1}$

Exercise

Find all solutions of the equation: $x^4 + 3x^3 - 30x^2 - 6x + 56 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{56}{1}\right\} = \pm\{1,\ 2,\ 4,\ 7,\ 8,\ 14,\ 28,\ 56\}$

$$\begin{array}{r|rrrrr}
4 & 1 & 3 & -3 & -6 & 56 \\
 & & 4 & 28 & -8 & -56 \\
\hline
-7 & 1 & 7 & -2 & -14 & 0 \\
 & & -7 & 0 & 14 & \\
\hline
 & 1 & 0 & -2 & &
\end{array}$$

$\rightarrow \quad x^3 + 7x^2 - 2x - 14 = 0 \quad \Rightarrow \quad \frac{c}{d} = \pm\left\{\frac{14}{1}\right\} = \pm\{1,\ 2,\ 7,\ 14\}$

$\rightarrow \quad x^2 - 2 = 0$

$\Rightarrow \underline{x = \pm\sqrt{2}}$

The solutions are: $\boxed{x = 4,\ -7,\ \pm\sqrt{2}}$

Exercise

Find all solutions of the equation: $\quad 3x^5 - 10x^4 - 6x^3 + 24x^2 + 11x - 6 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{6}{3}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$

$$\begin{array}{r|rrrrrr}
-1 & 3 & -10 & -6 & 24 & 11 & -6 \\
 & & -3 & 13 & -7 & -17 & 6 \\
\hline
-1 & 3 & -13 & 7 & 17 & -6 & 0 \\
 & & -3 & 16 & -23 & 6 & \\
\hline
2 & 3 & -16 & 23 & -6 & 0 & \\
 & & 6 & 20 & 6 & & \\
\hline
 & 3 & -10 & 3 & 0 & &
\end{array}$$

$x^4 - 13x^3 + 7x^2 + 17x - 6 = 0 \rightarrow \pm\left\{\frac{6}{3}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$

$3x^3 - 16x^2 + 26x - 6 = 0 \rightarrow \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$

$3x^2 - 10x + 3 = 0$

$$x = \frac{10 \pm \sqrt{100 - 36}}{6}$$

$$= \begin{cases} \frac{10-8}{6} = \frac{1}{3} \\ \frac{10+8}{6} = 3 \end{cases}$$

The solutions are: $\boxed{x = -1,\ -1,\ \frac{1}{3},\ 2,\ 3}$

Exercise

Find all solutions of the equation: $\quad 6x^5 + 19x^4 + x^3 - 6x^2 = 0$

Solution

$x^2\left(6x^3 + 19x^2 + x - 6\right) = 0 \quad \rightarrow \quad \underline{x = 0,\ 0}$

$6x^3+19x^2+x-6=0$

possibilities for $\frac{c}{d}:\pm\left\{\frac{6}{6}\right\}=\pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{3},\ \frac{2}{3}\right\}$

$$\begin{array}{r|rrrr} -3 & 6 & 19 & 1 & -6 \\ & & -18 & -3 & 6 \\ \hline & 6 & 1 & -2 & \boxed{0} \end{array} \quad 6x^2+x-2=0$$

$$x=\frac{-1\pm\sqrt{1+48}}{12}$$

$$=\begin{cases} \frac{-1-7}{12}=-\frac{2}{3} \\ \frac{-1+7}{12}=\frac{1}{2} \end{cases}$$

The solutions are: $\underline{x=0,\ 0,\ -\frac{2}{3},\ -3,\ \frac{1}{2}}$

Exercise

Find all solutions of the equation: $x^5+5x^4+10x^3+10x^2+5x+1=0$

Solution

$x^5+5x^4+10x^3+10x^2+5x+1=(x+1)^5=0$

possibilities for $\frac{c}{d}:\pm\{1\}$

$$\begin{array}{r|rrrrrr} -1 & 1 & 5 & 10 & 10 & 5 & 1 \\ & & -1 & -4 & -6 & -4 & -1 \\ \hline -1 & 1 & 4 & 6 & 4 & 1 & 0 \\ & & -1 & -3 & -3 & -1 & \\ \hline -1 & 1 & 3 & 3 & 1 & 0 & \\ & & -1 & -2 & -1 & & \\ \hline & 1 & 2 & 1 & 0 & & \end{array}$$

$\rightarrow\ x^4+4x^3+6x^2+4x+1=0\rightarrow\pm\{1\}$

$\rightarrow\ x^3+3x^2+3x+1=0\rightarrow\pm\{1\}$

$\rightarrow\ x^2+2x+1=0$

$x^2+2x+1=(x+1)^2$

The solutions are: $\underline{x=-1,\ -1,\ -1,\ -1,\ -1}$

Exercise

Find all solutions of the equation: $x^5 - x^4 - 7x^3 + 7x^2 + 12x - 12 = 0$

Solution

possibilities for $\frac{c}{d}$: $\pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$$\begin{array}{r|rrrrrr}
1 & 1 & -1 & -7 & 7 & 12 & -12 \\
 & & 1 & 0 & -7 & 0 & 12 \\ \hline
2 & 1 & 0 & -7 & 0 & 12 & 0 \\
 & & 2 & 4 & -6 & -12 & \\ \hline
-2 & 1 & 2 & -3 & -6 & 0 & \\
 & & -2 & 0 & 6 & & \\ \hline
 & 1 & 0 & -3 & 0 & &
\end{array}$$

$\rightarrow x^4 - 7x^2 - 12 = 0 \rightarrow \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$\rightarrow x^3 + 2x^2 - 3x - 6 = 0 \rightarrow \pm\{1,\ 2,\ 3,\ 6\}$

$\rightarrow x^2 - 3 = 0$

$x^2 = 3$

The solutions are: $\underline{x = -2,\ 1,\ 2,\ \pm\sqrt{3}}$

Exercise

Find all solutions of the equation: $x^5 - 2x^3 - 8x = 0$

Solution

$x\left(x^4 - 2x^2 - 8\right) = 0$

$\underline{x = 0}$

$x^4 - 2x^2 - 8 = 0$

$x^2 = \dfrac{2 \pm \sqrt{4+32}}{2}$

$= \begin{cases} \dfrac{2-6}{2} = -2 \\ \dfrac{2+6}{2} = 4 \end{cases}$

$\begin{cases} x^2 = -2 & \rightarrow & x = \pm i\sqrt{2} \\ x^2 = 4 & \rightarrow & x = \pm 2 \end{cases}$

The solutions are: $\underline{x = 0,\ \pm 2,\ \pm i\sqrt{2}}$

Exercise

Find all solutions of the equation: $x^5 - 32 = 0$

Solution

possibilities for $\frac{c}{d}: \pm\{1,\ 2,\ 4,\ 8,\ 16,\ 32\}$

2	1	0	0	0	0	−32
		2	4	8	16	32
	1	2	4	8	16	0

$\to x^4 + 2x^3 + 4x^2 + 8x + 16 = 0 \to \pm\{1,\ 2,\ 4,\ 8,\ 16\}$

$x^4 + 2x^3 + 4x^2 + 8x + 16 = 0$

Cannot be solved using rational zero theorem.
Therefore; using program

The solutions are: $x = 2,\ \dfrac{-1-\sqrt{5} \pm i\sqrt{2}\sqrt{5-\sqrt{5}}}{2},\ \dfrac{-1+\sqrt{5} \pm i\sqrt{2}\sqrt{5-\sqrt{5}}}{2}$

Exercise

Find all solutions of the equation: $3x^6 - 10x^5 - 29x^4 + 34x^3 + 50x^2 - 24x - 24 = 0$

Solution

possibilities for $\frac{c}{d}: \pm\left\{\frac{24}{3}\right\} = \pm\left\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3},\ \frac{8}{3}\right\}$

1	3	−10	−29	34	50	−24	−24
		3	−7	−36	−2	48	24
−1	3	−7	−36	−2	48	24	0
		−3	10	26	−24	−24	
−2	3	−10	−26	24	24	0	
		−6	32	−12	−24		
$-\frac{2}{3}$	3	−16	6	12	0		
		−2	12	−12			
	3	−18	18	0			

$\to 3x^5 - 7x^4 - 36x^3 - 2x^2 + 48x + 24 = 0$

$\to 3x^4 - 10x^3 - 26x^2 + 24x + 24 = 0$

$\to 3x^3 - 16x^2 + 12x - 12 = 0$

$\to 3x^2 - 18x + 18 = 0$

$x^2 - 6x + 6 = 0$

$$x = \frac{6 \pm \sqrt{36-24}}{2}$$

$$= \frac{6 \pm 2\sqrt{3}}{2}$$

The solutions are: $x = -2,\ -1,\ 1,\ -\frac{2}{3},\ 3 \pm \sqrt{3}$

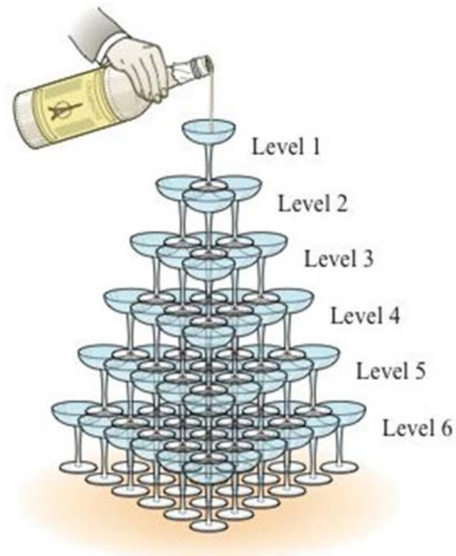
Exercise

Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T(k)=\frac{1}{6}\left(k^3+3k^2+2k\right)$$

Where k is the number of levels in the pyramid. If 220 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

Solution



$$\frac{1}{6}\left(k^3+3k^2+2k\right)=220$$

$$k^3+3k^2+2k-1,320=0$$

$$\begin{array}{r|rrrr} 10 & 1 & 3 & 2 & -1320 \\ & & 10 & 130 & 1320 \\ \hline & 1 & 13 & 132 & 0 \end{array} \quad \rightarrow \quad k^2+13k+132=0$$

$$k=\frac{-13\pm\sqrt{-359}}{2} \quad \mathbb{C}$$

The are **10** levels in the pyramid.

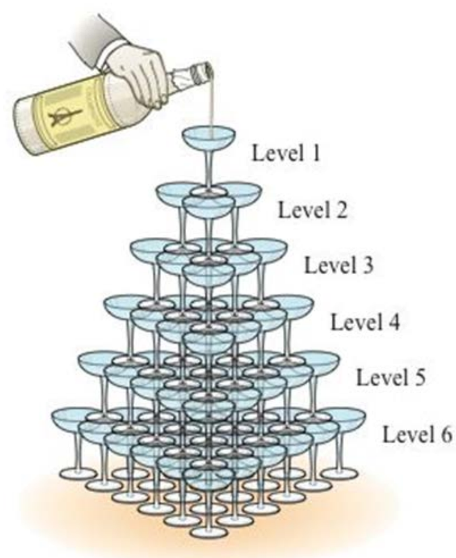
Exercise

Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T(k)=\frac{1}{6}\left(2k^3+3k^2+k\right)$$

Where k is the number of levels in the pyramid. If 140 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

Solution



$$\frac{1}{6}\left(2k^3+3k^2+k\right)=150$$

$$2k^3+3k^2+k-840=0$$

$$\begin{array}{r|rrrr} 7 & 2 & 3 & 1 & -840 \\ & & 14 & 119 & 840 \\ \hline & 2 & 17 & 120 & 0 \end{array} \quad \rightarrow \quad 2k^2+17k+120=0$$

$$k=\frac{-17\pm\sqrt{-671}}{4} \quad \mathbb{C}$$

The are **7** levels in the pyramid.

Exercise

A carbon dioxide cartridge for a paintball rifle has the shape of a right circular cylinder with a hemisphere at each end. The cylinder is 4 *inches* long, and the volume of the cartridge is $2\pi \ \ in^3$.
The common interior radius of the cylinder and the hemispheres is denoted by x. Estimate the length of the radius x.

Solution

Volume of the Cartridge = 2 × (Volume of Hemisphere) + Volume of Cylinder

$$Volume\ of\ \ Sphere = \frac{4}{3}\pi x^3$$

$$Volume\ of\ \ Cylinder = 4\pi x^2$$

$$Volume\ of\ \ Cartridge = \frac{4}{3}\pi x^3 + 4\pi x^2$$

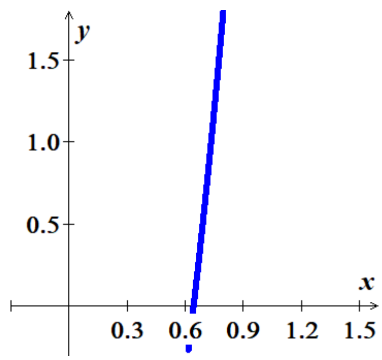
$$2\pi = \frac{4}{3}\pi x^3 + 4\pi x^2$$

$$2x^3 + 6x^2 = 3$$

$$2x^3 + 6x^2 - 3 = 0$$

Using Graph:

$\underline{x \approx 0.64 \ \ in.}$

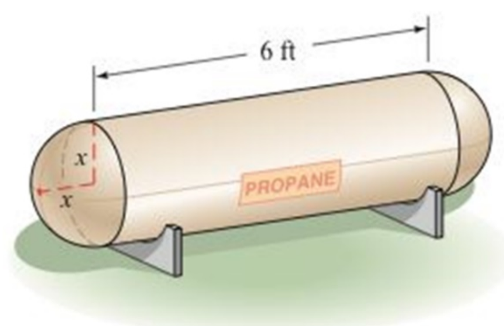


Exercise

A propane tank has the shape of a circular cylinder with a hemisphere at each end. The cylinder is 6 *feet* long and the volume of the tank is $9\pi \ \ ft^3$. Find the length of the radius x.

Solution

Volume of the Cartridge = 2 × (Volume of Hemisphere) + Volume of Cylinder



$$Volume\ of\ \ Sphere = \frac{4}{3}\pi x^3$$

$$Volume\ of\ \ Cylinder = 6\pi x^2$$

$$Volume\ of\ \ Cartridge = \frac{4}{3}\pi x^3 + 6\pi x^2$$

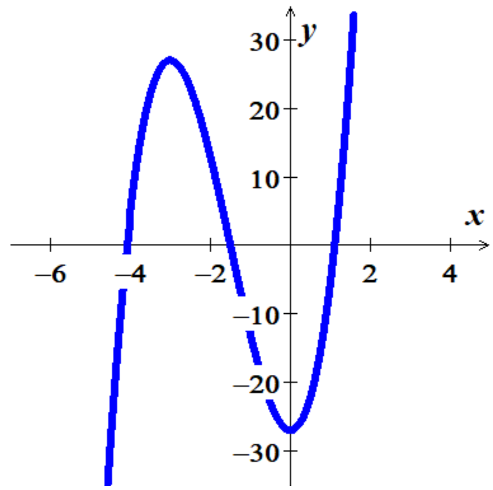
$$9\pi = \frac{4}{3}\pi x^3 + 6\pi x^2$$

$$27 = 4x^3 + 18x^2$$

$$4x^3 + 18x^2 - 27 = 0$$

$$possibilities\ for\ \ \frac{c}{d} : \pm\left\{\frac{27}{4}\right\} = \pm\left\{1,\ 3,\ 9,\ 27,\ \frac{1}{2},\ \frac{3}{2},\ \frac{9}{2},\ \frac{27}{2},\ \frac{1}{4},\ \frac{3}{4},\ \frac{9}{4},\ \frac{27}{4}\right\}$$

$$\begin{array}{r|rrrr} -\frac{3}{2} & 4 & 18 & 0 & -27 \\ & & -6 & -18 & 27 \\ \hline & 4 & 12 & -18 & 0 \end{array} \rightarrow \quad 4x^2+12x-18=0$$



$$2x^2+6x-9=0$$

$$x=\frac{-6\pm\sqrt{36+72}}{4}$$

$$=\frac{-6\pm 6\sqrt{3}}{4}$$

$$=\frac{-3\pm 3\sqrt{3}}{2}$$

$$x=-\frac{3}{2},\ \frac{-3-3\sqrt{3}}{2},\ \frac{-3+3\sqrt{3}}{2}$$

$\therefore$ the length of the radius x is $\frac{-3+3\sqrt{3}}{2}\approx 1.1$ *foot*

Exercise

A cube measures n inches on each edge. If a slice 2 *inches* thick is cut from one face of the cube, the resulting solid has a volume of $567\ in^3$. Find n.

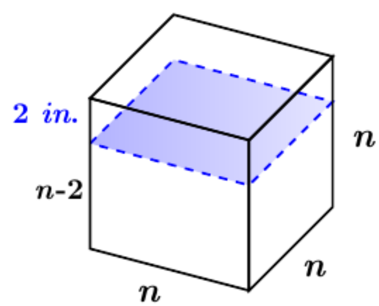
Solution

$$Volume=n^2(n-2)$$

$$n^3-2n^2=567$$

$$n^3-2n^2-567=0$$

$$possibilities\ for\ \frac{c}{d} := \pm\{1,\ 3,\ 7,\ 9,\ 21,\ 27,\ 63,\ 81,\ 189,\ 567\}$$



$$\begin{array}{r|rrrr} 9 & 1 & -2 & 0 & -567 \\ & & 9 & 63 & 567 \\ \hline & 1 & 7 & 63 & 0 \end{array} \rightarrow \quad n^2+7n+63=0$$

$$n=\frac{-7\pm\sqrt{49-252}}{2}$$

$$=\frac{-7\pm i\sqrt{203}}{2}$$

$\therefore\ \ n=9$

Exercise

A cube measures n inches on each edge. If a slice 1 *inch* thick is cut from one face of the cube and then a slice 3 inches thick is cut from another face of the cube, the resulting solid has a volume of $1560\ in^3$. Find the dimensions of the original cube.

Solution

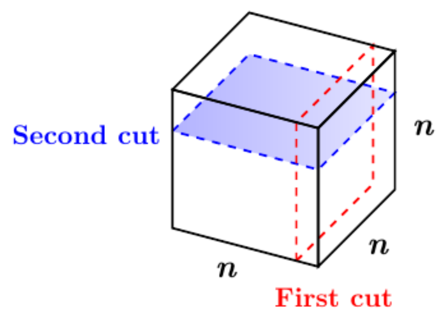
$Volume = n(n-1)(n-3)$

$n^3 - 4n^2 + 3n = 1560$

$n^3 - 4n^2 + 3n - 1560 = 0$

$possibilities\ for\ \frac{c}{d} := \pm\left\{\begin{matrix} 1,\ 2,\ 4,\ 5,\ 6,\ 8,\ 10,\ 12,\ 13,\ 15,\ 20,\ 24,\ 26,\ 30,\ 39, \\ 40,\ 52,\ 60,\ 65,\ 78,\ 104,\ 120,\ 130,\ 156,\ 195,\ 260,\ 312,\ 390,\ 780,\ 1560 \end{matrix}\right\}$

$$\begin{array}{r|rrrr} 13 & 1 & -4 & 3 & -1560 \\ & & 13 & 117 & 1560 \\ \hline & 1 & 9 & 120 & 0 \end{array} \quad \to \quad n^2 + 9n + 120 = 0$$



$n = \dfrac{-9 \pm \sqrt{81-480}}{2}$

$= \dfrac{-9 \pm i\sqrt{399}}{2}$ X

$\therefore\ \underline{n = 13}$

Exercise

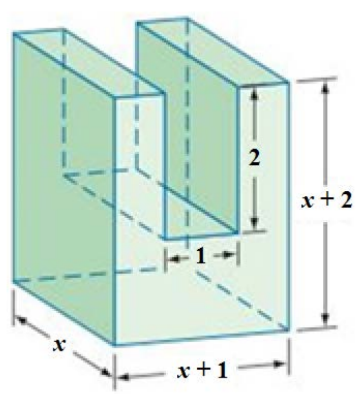
For what value of x will the volume of the following solid be $112\ in^3$

Solution

Volume of the bottom portion $= x^2(x+1)$

Volume of one side portion $= 2x\left(\frac{1}{2}x\right)$

$= x^2$



Total Volume $= x^2(x+1) + 2x^2$

$112 = x^3 + 3x^2$

$x^3 + 3x^2 - 112 = 0$

$possibilities\ for\ \frac{c}{d} := \pm\{1,\ 2,\ 4,\ 8,\ 14,\ 28,\ 56,\ 112\}$

$$\begin{array}{c|cccc} 4 & 1 & 3 & 0 & -112 \\ & & 4 & 28 & 112 \\ \hline & 1 & 7 & 28 & 0 \end{array} \rightarrow \quad x^2 + 7x + 28 = 0$$

$$x = \frac{-7 \pm \sqrt{49-112}}{2}$$

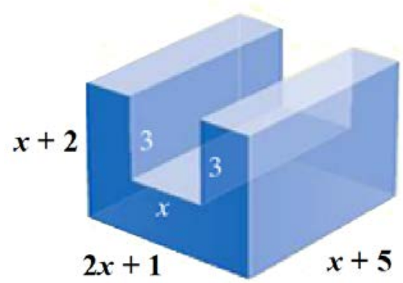
$$= \frac{-7 \pm 3i\sqrt{7}}{2} \quad \times$$

$$\therefore \quad \boxed{x = 4}$$

Exercise

For what value of x will the volume of the following solid be $208 \ in^3$

Solution



Volume of the bottom portion $= (2x+1)(x+5)(x+2-3)$

$$= \left(2x^2+11x+5\right)(x-1)$$

$$= 2x^3+11x^2+5x-2x^2-11x-5$$

$$= 2x^3+9x^2-6x-5$$

Volume of one side portion $= (3)\frac{1}{2}(2x+1-x)(x+5)$

$$= \frac{3}{2}(x+1)(x+5)$$

$$= \frac{3}{2}\left(x^2+6x+5\right)$$

Total Volume $= 2x^3+9x^2-6x-5+2\left(\frac{3}{2}\right)\left(x^2+6x+5\right)$

$$208 = 2x^3+9x^2-6x-5+3x^2+18x+15$$

$$2x^3+12x^2+12x-198=0$$

$$x^3+6x^2+6x-99=0$$

possibilities for $\frac{c}{d} := \pm\{1,\ 3,\ 9,\ 11,\ 33,\ 99\}$

$$\begin{array}{c|cccc} 3 & 1 & 6 & 6 & -99 \\ & & 3 & 27 & 99 \\ \hline & 1 & 9 & 33 & 0 \end{array} \rightarrow \quad x^2 + 9x + 33 = 0$$

$$x = \frac{-9 \pm \sqrt{81-132}}{2}$$

$$= \frac{-9 \pm i\sqrt{51}}{2} \quad \times$$

$$\therefore \quad \boxed{x = 3}$$

Exercise

The length of rectangular box is 1 *inch* more than twice the height of the box, and the width is 3 *inches* more than the height. If the volume of the box is $126\ in^3$, find the dimensions of the box.

Solution

$$Volume = x(2x+1)(x+3)$$

$$2x^3 + 7x^2 + 3x = 126$$

$$2x^3 + 7x^2 + 3x - 126 = 0$$

$$possibilities\ for\ \frac{c}{d} := \pm\left\{\frac{126}{2}\right\}$$

$$= \pm\left\{1,\ 2,\ 3,\ 6,\ 9,\ 14,\ 21,\ 42,\ 63,\ 126,\ \frac{1}{2},\ \frac{3}{2},\ \frac{9}{2},\ \frac{21}{2},\ \frac{63}{2}\right\}$$

$$\begin{array}{r|rrrr} 3 & 2 & 7 & 3 & -126 \\ & & 6 & 39 & 126 \\ \hline & 2 & 13 & 42 & 0 \end{array} \quad \rightarrow \quad 2x^2 + 13x + 42 = 0$$

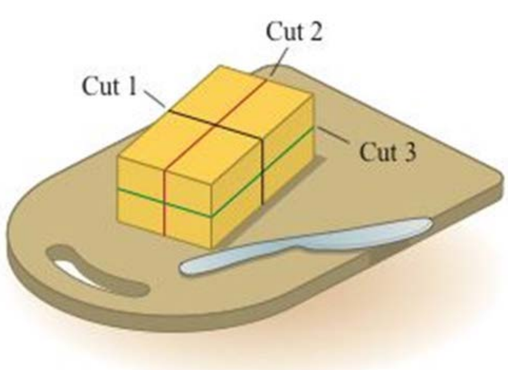
$$x = \frac{-13 \pm \sqrt{169 - 336}}{4}$$

$$= \frac{-13 \pm i\sqrt{167}}{4} \quad \times$$

$$\therefore \quad \underline{x = 3}$$

Exercise

One straight cut through a thick piece of cheese produces two pieces. Two straight cuts can produce a maximum of four pieces. Two straight cuts can produce a maximum of four pieces. Three straight cuts can produce a maximum of eight pieces.



You might be inclined to think that every additional cut double number of pieces. However, for four straight cuts, you get a maximum of 15 pieces. The maximum number of pieces P that can be produced by n straight cuts is given by

$$P(n) = \frac{n^3 + 5n + 6}{6}$$

a) Determine number of pieces that can be produces by five straight cuts.

b) What is the fewest number of straight cuts that are needed to produce 64 pieces?

Solution

a) $P(5) = \dfrac{5^3 + 25 + 6}{6}$

$\underline{= 26}$

b) $\dfrac{n^3 + 5n + 6}{6} = 64$

$n^3 + 5n + 6 = 384$

$n^3 + 5n - 378 = 0$

possibilities for $\frac{c}{d} := \pm\{378\}$

$= \pm\{1,\ 2,\ 3,\ 6,\ 7,\ 9,\ 14,\ 18,\ 21,\ 27,\ 42,\ 54,\ 63,\ 126,\ 189,\ 378\}$

$$\begin{array}{r|rrrr} 7 & 1 & 0 & 5 & -378 \\ & & 7 & 49 & 378 \\ \hline & 1 & 7 & 54 & 0 \end{array} \quad \rightarrow \quad n^2 + 7n + 54 = 0$$

$n = \dfrac{-7 \pm \sqrt{49 - 216}}{2}$

$= \dfrac{-7 \pm i\sqrt{167}}{2}$ X

$\therefore\ \underline{n = 7}$

Exercise

The number of ways one can select three cards from a group of *n* cards (the order of the selection matters), where $n \ge 3$, is given by $P(n) = n^3 - 3n^2 + 2n$. For a certain card trick, a magician has determined that there are exactly 504 *ways* to choose three cards from a given group. How many cards are in the group?

Solution

$P(n) = n^3 - 3n^2 + 2n = 504$

$n^3 - 3n^2 + 2n - 504 = 0$

possibilities for $\frac{c}{d} := \pm\{504\}$

$$= \pm\left\{\begin{array}{l} 1,\ 2,\ 3,\ 4,\ 6,\ 7,\ 8,\ 9,\ 12,\ 14,\ 18,\ 21, \\ 24,\ 28,\ 36,\ 42,\ 56,\ 63,\ 72,\ 84,\ 126,\ 168,\ 252,\ 504 \end{array}\right\}$$

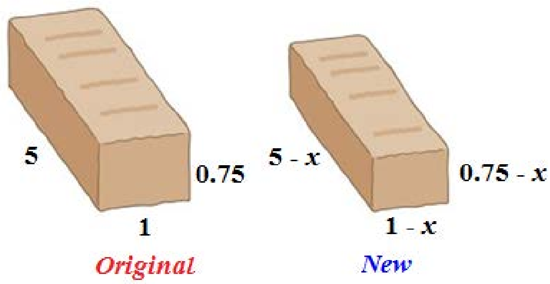
$$\begin{array}{r|rrrr} 9 & 1 & -3 & 2 & -504 \\ & & 9 & 54 & 504 \\ \hline & 1 & 6 & 56 & 0 \end{array} \quad \rightarrow \quad n^2 + 6n + 56 = 0$$

$$n = \frac{-6 \pm \sqrt{36 - 224}}{2}$$
$$= -3 \pm i\sqrt{47} \quad \times$$

$\therefore \quad \underline{n = 9}$

Exercise

A nutrition bar in the shape of a rectangular solid measure 0.75 *in*. by 1 *in*. by 5 *inches*.



To reduce costs, the manufacturer has decided to decrease each of the dimensions of the nutrition bar by *x inches*, what value of *x* will produce a new bar with a volume that is $0.75 \ in^3$ less than the present bar's volume.

Solution

$$V_{original} = (5)(1)\left(\frac{3}{4}\right)$$
$$= \frac{15}{4}$$

$$V_{new} = (5-x)(1-x)\left(\frac{3}{4} - x\right) \qquad \left(x < \frac{3}{4}\right)$$

$$\left(5 - 6x + x^2\right)\left(\frac{3-4x}{4}\right) = \frac{15}{4} - \frac{3}{4}$$

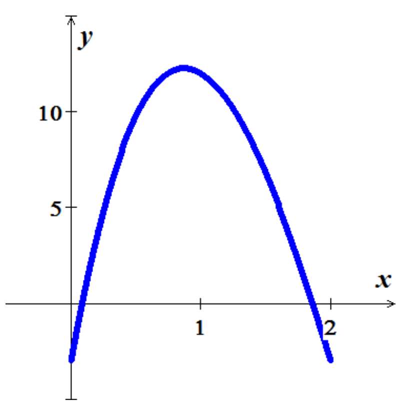
$$15 - 20x - 18x + 24x^2 + 3x^2 - 4x^3 = 4(3)$$

$$4x^3 - 27x^2 + 38x - 3 = 0$$

From graph table:

```
0.08200   -0.06334
0.08400    0.00386
```

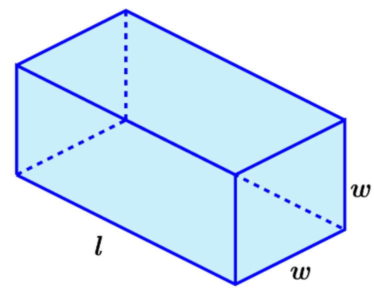
$\underline{x \approx 0.083 \ \ in.}$



Exercise

A rectangular box is square on two ends and has length plus girth of 81 *inches*. (Girth: distance *around* the box). Determine the possible lengths l $(l > w)$ of the box if its volume is $4900\ \ in^3$.

Solution



$$81 = l + 4w$$
$$l = 81 - 4w$$
$$V = lw^2$$
$$= (81-4w)w^2$$

$$-4w^3 + 81w^2 = 4900$$
$$4w^3 - 81w^2 + 4900 = 0$$

$$\textit{possibilities for } \frac{c}{d} := \pm\left\{\frac{4900}{4}\right\} = \pm\left\{\begin{matrix}1,\ 2,\ 4,\ 7,\ 10,\ 14,\ 20,\ 28,\ 49, \\ 100,\ 175,\ 245,\ 350,\ 490,\ 700,\ 1225,\ 2450,\ 4900, \cdots\end{matrix}\right\}$$

$$\begin{array}{r|rrrr} 14 & 4 & -81 & 0 & 4900 \\ & & 56 & -350 & -4900 \\ \hline & 4 & -25 & -350 & 0 \end{array} \rightarrow \ 4w^2 - 25w - 350 = 0$$

$$w = \frac{25 \pm \sqrt{625 + 5600}}{8}$$
$$= \frac{25 \pm 5\sqrt{249}}{8}$$
$$= \begin{cases} \dfrac{25 - 5\sqrt{249}}{8} < 0 \\ \dfrac{25 + 5\sqrt{249}}{8} \approx 13 \end{cases}$$

$$l = 81 - 4(14) = 25$$
$$l = 81 - 4(13) = 29$$

∴ the possible lengths l are around **25** *in*. ***or*** **29** *in*.

Solution *Section* 2.5 – Polynomial Functions

Exercise

Determine the end behavior of the graph of the polynomial function $f(x)=5x^3+7x^2-x+9$

Solution

Leading term: $5x^3$ with 3^{rd} degree (n is ***odd***)

$x\to-\infty \ \Rightarrow \ f(x)\to-\infty$ $f(x)$ falls left

$x\to\infty \ \Rightarrow \ f(x)\to\infty$ $f(x)$ rises right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x)=11x^3-6x^2+x+3$

Solution

Leading term: $11x^3$ with 3^{rd} degree (n is ***odd***)

$x\to-\infty \ \Rightarrow \ f(x)\to-\infty$ $f(x)$ falls left

$x\to\infty \ \Rightarrow \ f(x)\to\infty$ rises right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x)=-11x^3-6x^2+x+3$

Solution

Leading term: $-11x^3$ with 3^{rd} degree (n is ***odd***)

$x\to-\infty \ \Rightarrow \ f(x)\to\infty$ $f(x)$ rises left

$x\to\infty \ \Rightarrow \ f(x)\to-\infty$ $f(x)$ falls right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x)=2x^3+3x^2-23x-42$

Solution

Leading term: $2x^3$ with 3^{rd} degree (n is ***odd***)

$x\to-\infty \ \Rightarrow \ f(x)\to-\infty$ $f(x)$ falls left

$x\to\infty \ \Rightarrow \ f(x)\to\infty$ $f(x)$ rises right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = 5x^4 + 7x^2 - x + 9$

Solution

Leading term: $5x^4$ with 4^{rd} degree (n is ***even***)

$x \to -\infty \;\Rightarrow\; f(x) \to \infty$ $\quad f(x)$ rises left

$x \to \infty \;\Rightarrow\; f(x) \to \infty$ $\quad f(x)$ rises right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = 11x^4 - 6x^2 + x + 3$

Solution

Leading term: $11x^4$ with 4^{rd} degree (n is ***even***)

$x \to -\infty \;\Rightarrow\; f(x) \to \infty$ $\quad f(x)$ rises left

$x \to \infty \;\Rightarrow\; f(x) \to \infty$ $\quad f(x)$ rises right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = -5x^4 + 7x^2 - x + 9$

Solution

Leading term: $-5x^4$ with 4^{rd} degree (n is ***even***)

$x \to -\infty \;\Rightarrow\; f(x) \to -\infty$ $\quad f(x)$ falls left

$x \to \infty \;\Rightarrow\; f(x) \to -\infty$ $\quad f(x)$ falls right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = -11x^4 - 6x^2 + x + 3$

Solution

Leading term: $-11x^4$ with 4^{rd} degree (n is ***even***)

$x \to -\infty \;\Rightarrow\; f(x) \to -\infty$ $\quad f(x)$ falls left

$x \to \infty \;\Rightarrow\; f(x) \to -\infty$ $\quad f(x)$ falls right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = 5x^5 - 16x^2 - 20x + 64$

Solution

Leading term: $5x^5$ with 5^{th} degree (n is ***odd***)

$x \to -\infty \;\Rightarrow\; f(x) \to -\infty$ $\qquad$ $f(x)$ falls left

$x \to \infty \;\Rightarrow\; f(x) \to \infty$ $\qquad$ $f(x)$ rises right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = -5x^5 - 16x^2 - 20x + 64$

Solution

Leading term: $-5x^5$ with 5^{th} degree (n is ***odd***)

$x \to -\infty \;\Rightarrow\; f(x) \to \infty$ $\qquad$ $f(x)$ rises left

$x \to \infty \;\Rightarrow\; f(x) \to -\infty$ $\qquad$ $f(x)$ falls right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = -3x^6 - 16x^3 + 64$

Solution

Leading term: $-3x^6$ with 6^{th} degree (n is ***even***)

$x \to -\infty \;\Rightarrow\; f(x) \to -\infty$ $\qquad$ $f(x)$ falls left

$x \to \infty \;\Rightarrow\; f(x) \to -\infty$ $\qquad$ $f(x)$ falls right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x) = 3x^6 - 16x^3 + 4$

Solution

Leading term: $3x^6$ with 6^{th} degree (n is ***even***)

$x \to -\infty \;\Rightarrow\; f(x) \to \infty$ $\qquad$ $f(x)$ rises left

$x \to \infty \;\Rightarrow\; f(x) \to \infty$ $\qquad$ $f(x)$ rises right

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x)=x^3-x-1;$ *between* 1 *and* 2

Solution

$$f(1)=(1)^3-(1)-1$$
$$\underline{=-1\,|}$$
$$f(2)=(2)^3-(2)-1$$
$$\underline{=5\,|}$$

Since $f(1)$ *and* $f(2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 1 and 2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x)=x^3-4x^2+2;$ *between* 0 *and* 1

Solution

$$f(0)=(0)^3-4(0)^2+2$$
$$\underline{=2\,|}$$
$$f(1)=(1)^3-4(1)^2+2$$
$$\underline{=-1\,|}$$

Since $f(0)$ *and* $f(1)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 0 and 1.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x)=2x^4-4x^2+1;$ *between* -1 *and* 0

Solution

$$f(-1)=2(-1)^4-4(-1)^2+1$$
$$\underline{=-1\,|}$$
$$f(0)=2(0)^4-4(0)^2+1$$
$$\underline{=1\,|}$$

Since $f(0)$ *and* $f(-1)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between -1 and 0.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = x^4 + 6x^3 - 18x^2$; *between* 2 *and* 3

Solution

$$f(2) = (2)^4 + 6(2)^3 - 18(2)^2$$
$$\underline{= -8}$$
$$f(3) = (3)^4 + 6(3)^3 - 18(3)^2$$
$$\underline{= 81}$$

Since $f(2)$ *and* $f(3)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 2 and 3.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = x^3 + x^2 - 2x + 1$; *between* -3 *and* -2

Solution

$$f(-3) = (-3)^3 + (-3)^2 - 2(-3) + 1$$
$$\underline{= -11}$$
$$f(-2) = (-2)^3 + (-2)^2 - 2(-2) + 1$$
$$\underline{= 1}$$

Since $f(-3)$ *and* $f(-2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between -2 and -3.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = x^5 - x^3 - 1$; *between* 1 *and* 2

Solution

$f(1) = (1)^5 - (1)^3 - 1$

$\quad \underline{= -1}$

$f(2) = (2)^5 - (2)^3 - 1$

$\quad \underline{= 23}$

Since $f(1)$ *and* $f(2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 1 and 2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = 3x^3 - 10x + 9;$ *between* -3 *and* -2

Solution

$f(-3) = 3(-3)^3 - 10(-3) + 9$

$\quad \underline{= -42}$

$f(-2) = 3(-2)^3 - 10(-2) + 9$

$\quad \underline{= 5}$

Since $f(-3)$ *and* $f(-2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between –3 and –2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = 3x^3 - 8x^2 + x + 2;$ *between* 2 *and* 3

Solution

$f(2) = 3(2)^3 - 8(2)^2 + (2) + 2$

$\quad \underline{= -4}$

$f(3) = 3(3)^3 - 8(3)^2 + (3) + 2$

$\quad \underline{= 14}$

Since $f(2)$ *and* $f(3)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 2 and 3.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = 3x^3 - 8x^2 + x + 2;\quad between\ \ 1\ and\ \ 2$

Solution

$$f(1) = 3(1)^3 - 8(1)^2 + (1) + 2$$

$$= -2$$

$$f(2) = 3(2)^3 - 8(2)^2 + (2) + 2$$

$$= -4$$

Since $f(1)$ *and* $f(2)$ have same signs.

Therefore, ***cannot be determined***.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3;\quad between\ \ 0\ and\ \ 1$

Solution

$$f(0) = (0)^5 + 2(0)^4 - 6(0)^3 + 2(0) - 3$$

$$= -3$$

$$f(1) = (1)^5 + 2(1)^4 - 6(1)^3 + 2(1) - 3$$

$$= -4$$

Since $f(0)$ *and* $f(1)$ have same signs.

Therefore, ***cannot be determined***.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 2x^3 + 3x^2 - 23x - 42,\quad a = 3,\quad b = 4$

Solution

$$P(3) = 54 + 27 - 69 - 42$$

$$= -30$$

$$P(4) = 128 + 48 - 92 - 42$$

$$= 90$$

Since $P(3)$ *and* $P(4)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 3 and 4.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 4x^3 - x^2 - 6x + 1, \quad a = 0, \quad b = 1$

Solution

$P(0) \underline{= 1}$

$P(1) = 4 - 1 - 6 + 1$

$\underline{= -2}$

Since $P(0)$ *and* $P(1)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 0 and 1.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 3x^3 + 7x^2 + 3x + 7, \quad a = -3, \quad b = -2$

Solution

$P(-3) = -81 + 63 - 9 + 7$

$\underline{= -20}$

$P(-2) = -24 + 28 - 6 + 7$

$\underline{= 5}$

Since $P(-3)$ *and* $P(-2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between –3 and –2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 2x^3 - 21x^2 - 2x + 25, \quad a = 1, \quad b = 2$

Solution

$P(1) = 2 - 21 - 2 + 25$

$\underline{= 4}$

$$P(2) = 16 - 84 - 4 + 25$$

$$\boxed{= -47}$$

Since $P(1)$ *and* $P(2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 1 and 2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 4x^4 + 7x^3 - 11x^2 + 7x - 15, \quad a = 1, \quad b = \frac{3}{2}$

Solution

$$P(1) = 4 + 7 - 11 + 7 - 15$$

$$\boxed{= -8}$$

$$P\left(\frac{3}{2}\right) = 81 + \frac{189}{8} - \frac{99}{4} + \frac{21}{2} - 15$$

$$= 66 + \frac{189 - 198 + 84}{8}$$

$$= 66 + \frac{75}{8}$$

$$\boxed{= \frac{603}{8}}$$

Since $P(1)$ *and* $P\left(\frac{3}{2}\right)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 1 and $\frac{3}{2}$.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = 5x^3 - 16x^2 - 20x + 64, \quad a = 3, \quad b = \frac{7}{2}$

Solution

$$P(3) = 135 - 144 - 60 + 64$$

$$\boxed{= -5}$$

$$P\left(\frac{7}{2}\right) = \frac{1715}{8} - 196 - 70 + 64$$

$$= \frac{1715}{8} - 202$$

$$\boxed{= \frac{99}{8}}$$

Since $P(3)$ *and* $P\left(\frac{7}{2}\right)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 3 and $\frac{7}{2}$.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x)=x^4-x^2-x-4, \quad a=1, \quad b=2$

Solution

$$P(1)=1-1-1-4$$
$$\underline{=-5}$$
$$P(2)=16-4-2-4$$
$$\underline{=6}$$

Since $P(1)$ *and* $P(2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 1 and 2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x)=x^3-x-8, \quad a=2, \quad b=3$

Solution

$$P(2)=8-2-8$$
$$\underline{=-2}$$
$$P(3)=27-3-8$$
$$\underline{=16}$$

Since $P(2)$ *and* $P(3)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 2 and 3.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x)=x^3-x-8, \quad a=0, \quad b=1$

Solution

$$\underline{P(0)=-8}$$

$P(1) = 1 - 1 - 8$

$\quad = -8$

Since $P(0)$ *and* $P(1)$ have same sign.

Therefore, ***cannot be determined***.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $P(x) = x^3 - x - 8, \quad a = 2.1, \quad b = 2.2$

Solution

$$P(2.1) = P\left(\frac{21}{10}\right)$$

$$= \frac{9261}{1000} - \frac{21}{10} - 8$$

$$= \frac{9261 - 2100 - 8000}{1000}$$

$$= -\frac{839}{1,000}$$

$$P(2.2) = P\left(\frac{2.2}{10}\right)$$

$$= \frac{10,648}{1000} - \frac{22}{10} - 8$$

$$= \frac{10,648 - 2,200 - 8,000}{1000}$$

$$= \frac{448}{1,000}$$

$$= 0.448$$

Since $P(2.1)$ *and* $P(2.2)$ have opposite signs.

Therefore, the polynomial ***has a real zero*** between 2.1 and 2.2.

Solution Section 2.6 – Rational Functions

Exercise

Determine all asymptotes of the function: $y = \dfrac{3x}{1-x}$

Solution

VA: $x=1$ ***HA***: $y=-3$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \dfrac{x^2}{x^2+9}$

Solution

VA: n/a $x^2+9 \neq 0$ ***HA***: $y=1$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \dfrac{x-2}{x^2-4x+3}$

Solution

$$x^2-4x+3=0 \implies x=1,\ 3$$

$$y=\frac{x}{x^2} \to 0$$

VA: $x=1,\ \ x=3$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \dfrac{3}{x-5}$

Solution

VA: $x = 5$ ***HA***: $y = 0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \dfrac{x^3 - 1}{x^2 + 1}$

Solution

VA: *none*

HA: *none*

Hole: n/a

Oblique asymptote: $y = x$

$$\begin{array}{r} x \\ x^2+1 \overline{\smash{\big)} x^3 - 1} \\ \underline{-x^3 - x} \\ -x - 1 \end{array}$$

$$y = x - \frac{x+1}{x^2+1}$$

Exercise

Determine all asymptotes of the function: $y = \dfrac{x^3 + 3x^2 - 2}{x^2 - 4}$

Solution

VA: $x = \pm 2$

HA: n/a

Hole: n/a

Oblique asymptote: $y = x + 3$

$$\begin{array}{r} x + 3 \\ x^2 - 4 \overline{\smash{\big)} x^3 + 3x^2 - 2} \\ \underline{-x^3 \quad + 4x} \\ 3x^2 + 4x - 2 \\ \underline{-3x^2 \qquad + 12} \\ 4x + 10 \end{array}$$

$$y = x + 3 + \frac{4x+10}{x^2-4}$$

Exercise

Determine all asymptotes of the function: $y = \dfrac{3x^2 - 27}{(x+3)(2x+1)}$

Solution

$$y = \frac{3x^2 - 27}{(x+3)(2x+1)} = \frac{3\left(x^2 - 9\right)}{(x+3)(2x+1)} = \frac{3(x+3)(x-3)}{(x+3)(2x+1)} = \frac{3(x-3)}{(2x+1)}$$

VA: $x = -3, \ -\frac{1}{2}$ ***HA***: $y = \frac{3}{2}$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \dfrac{x-3}{x^2-9}$

Solution

$$x^2 - 9 = 0 \rightarrow \boxed{x = \pm 3}$$

$$y = \frac{x-3}{(x-3)(x+3)}$$

$$= \frac{1}{x+3}$$

VA: $x = 3$ ***HA***: $y = 0$

Hole: $x = 3 \rightarrow y = \frac{1}{6}$ ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \dfrac{6}{\sqrt{x^2-4x}}$

Solution

$$x^2 - 4x = 0$$

$$\Rightarrow x(x-4) = 0 \rightarrow \boxed{x = 0, 4}$$

VA: $x = 0,\ x = 4$ ***HA***: $y = 0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \dfrac{5x-1}{1-3x}$

Solution

VA: $x = \frac{1}{3}$ ***HA***: $y = -\frac{5}{3}$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $f(x) = \dfrac{2x-11}{x^2+2x-8}$

Solution

VA: $x = 2,\ x = -4$ ***HA***: $y = 0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $f(x)=\dfrac{x^2-4x}{x^3-x}$

Solution

$$f(x)=\frac{x(x-4)}{x\left(x^2-1\right)}$$

$$=\frac{x-4}{x^2-1}$$

VA: $x=-1,\ x=1$ ***HA***: $y=0$

Hole: $x=0\rightarrow y=4$ ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $f(x)=\dfrac{x-2}{x^3-5x}$

Solution

VA: $x=0,\ x=\pm\sqrt{5}$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{4x}{x^2+10x}$

Solution

$x^2+10x=0 \quad\rightarrow\quad x=0,\ -10$

Domain: $(-\infty,\ -10)\cup(-10,\ 0)\cup(0,\ \infty)$

$$f(x)=\frac{4x}{x(x+10)}$$

$$=\frac{4}{x+10}$$

VA: $x=-10$ ***HA***: $y=0$

Hole: $x=0 \quad\rightarrow\quad y=\frac{4}{10}\Rightarrow$ *hole* $\left(0,\ \frac{2}{5}\right)$ ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{3-x}{(x-4)(x+6)}$

Solution

VA: $x = -6$ and $x = 4$ $\qquad$ HA: $y = 0$

$Hole$: n/a $\qquad$ $Oblique\ asymptote$: n/a

Exercise

Determine all asymptotes of the function $f(x) = \dfrac{x^3}{2x^3 - x^2 - 3x}$

Solution

$$2x^3 - x^2 - 3x = x\left(2x^2 - x - 3\right) = 0 \quad \rightarrow \quad x = 0, \ -1, \ \frac{3}{2}$$

$$f(x) = \frac{x^3}{2x^3 - x^2 - 3x}$$

$$= \frac{x^3}{x\left(2x^2 - x - 3\right)}$$

$$= \frac{x^2}{2x^2 - x - 3}$$

VA: $x = -1$ and $x = \frac{3}{2}$ $\qquad$ HA: $y = \frac{1}{2}$

$Hole$: $x = 0 \quad \rightarrow \quad y = 0 \Rightarrow hole \ (0, \ 0)$ $\qquad$ $Oblique\ asymptote$: n/a

Exercise

Determine all asymptotes of the function $f(x) = \dfrac{3x^2 + 5}{4x^2 - 3}$

Solution

$$4x^2 - 3 = 0 \quad \rightarrow \quad x = \pm\frac{\sqrt{3}}{2}$$

$Domain$: $\left(-\infty, \ -\frac{\sqrt{3}}{2}\right) \cup \left(-\frac{\sqrt{3}}{2}, \ \frac{\sqrt{3}}{2}\right) \cup \left(\frac{\sqrt{3}}{2}, \ \infty\right)$

VA: $x = -\frac{\sqrt{3}}{2}$ and $x = \frac{\sqrt{3}}{2}$ $\qquad$ HA: $y = \frac{3}{4}$

$Hole$: n/a $\qquad$ $Oblique\ asymptote$: n/a

Exercise

Determine all asymptotes of the function $f(x) = \dfrac{x + 6}{x^3 + 2x^2}$

Solution

$$x^3 + 2x^2 = x^2(x + 2) = 0 \quad \rightarrow \quad x = 0, -2$$

Domain: $(-\infty,\ -2)\cup(-2,\ 0)\cup(0,\ \infty)$

VA: $x=0$ *and* $x=2$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^2+4x-1}{x+3}$

Solution

VA: $x=-3$

HA: n/a

Hole: n/a

Oblique asymptote: $y=x+1$

$$\begin{array}{r|l} & x+1 \\ \hline x+3 & x^2+4x-1 \\ & \underline{-x^2-3x} \\ & \qquad x-1 \\ & \qquad \underline{-x-3} \\ & \qquad\quad -4 \end{array}$$

$$f(x)=\frac{x^2+4x-1}{x+3}=x+1-\frac{4}{x+3}$$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^2-6x}{x-5}$

Solution

$x-5=0 \quad\to\quad x=5$

Domain: $(-\infty,\ 5)\cup(5,\ \infty)$

$$\begin{array}{r|l} & x-1 \\ \hline x-5 & x^2-6x \\ & \underline{-x^2+5x} \\ & \qquad -x \\ & \qquad \underline{x-5} \\ & \qquad -5 \end{array}$$

$$\begin{aligned} f(x)&=\frac{x^2-6x}{x-5} \\ &=x-1-\frac{5}{x-5} \end{aligned}$$

VA: $x=5$

HA: N/A

Hole: N/A

Oblique asymptote: $y=x-1$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^3-x^2+x-4}{x^2+2x-1}$

Solution

$$x^2+2x-1=0 \quad\to\quad x=-1\pm\sqrt{2}$$

Domain: $\left(-\infty,\ -1-\sqrt{2}\right)\cup\left(-1-\sqrt{2},\ -1+\sqrt{2}\right)\cup\left(-1+\sqrt{2},\ \infty\right)$

$$\begin{aligned} f(x) &= \frac{x^3-x^2+x-4}{x^2+2x-1} \\ &= x-3+\frac{8x-7}{x^2+2x-1} \end{aligned}$$

$$\begin{array}{r|l} & x-3 \\ \hline x^2+2x-1 & x^3-x^2+x-4 \\ & \underline{-x^3-2x^2+x} \\ & -3x^2+2x-4 \\ & \underline{3x^2+6x-3} \\ & 8x-7 \end{array}$$

VA: $x=-1\pm\sqrt{2}$

HA: n/a

Hole: n/a

Oblique asymptote: $y=x-3$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{4x}{x^2+10x}$

Solution

$x^2+10x=0 \quad\to\quad x=0,\ -10$ **Domain**: $(-\infty,\ -10)\cup(-10,\ 0)\cup(0,\ \infty)$

$$f(x)=\frac{4x}{x(x+10)}=\frac{4}{x+10}$$

VA: $x=-10$ **HA**: $y=0$

Hole: $x=0 \quad\to\quad y=\frac{4}{10}\Rightarrow$ hole $\left(0,\ \frac{2}{5}\right)$

Oblique asymptote: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{3-x}{(x-4)(x+6)}$

Solution

Domain: $(-\infty,\ -6)\cup(-6,\ 4)\cup(4,\ \infty)$

VA: $x=-6$ *and* $x=4$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^3}{2x^3-x^2-3x}$

Solution

$$2x^3-x^2-3x=x\left(2x^2-x-3\right)=0 \quad \to \quad x=0,\ -1,\ \tfrac{3}{2}$$

Domain: $(-\infty,\ -1)\cup(-1,\ 0)\cup\left(0,\ \tfrac{3}{2}\right)\cup\left(\tfrac{3}{2},\ \infty\right)$

$$f(x)=\frac{x^3}{2x^3-x^2-3x}=\frac{x^3}{x\left(2x^2-x-3\right)}=\frac{x^2}{2x^2-x-3}$$

VA: $x=-1$ *and* $x=\frac{3}{2}$ ***HA***: $y=\frac{1}{2}$

Hole: $x=0 \quad \to \quad y=0 \Rightarrow$ *hole* $(0,\ 0)$

Oblique asymptote: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{3x^2+5}{4x^2-3}$

Solution

$4x^2-3=0 \quad \to \quad x=\pm\frac{\sqrt{3}}{2}$ ***Domain***: $\left(-\infty,\ -\frac{\sqrt{3}}{2}\right)\cup\left(-\frac{\sqrt{3}}{2},\ \frac{\sqrt{3}}{2}\right)\cup\left(\frac{\sqrt{3}}{2},\ \infty\right)$

VA: $x=-\frac{\sqrt{3}}{2}$ *and* $x=\frac{\sqrt{3}}{2}$ ***HA***: $y=\frac{3}{4}$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x+6}{x^3+2x^2}$

Solution

$x^3+2x^2=x^2(x+2)=0 \quad \to \quad x=0,-2$ ***Domain***: $(-\infty,\ -2)\cup(-2,\ 0)\cup(0,\ \infty)$

VA: $x=0$ *and* $x=2$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^2+4x-1}{x+3}$

Solution

$x+3=0 \quad \to \quad x=-3$ ***Domain***: $(-\infty,\ -3)\cup(-3,\ \infty)$

$$\begin{array}{r|l} & x+1 \\ \hline x+3 & x^2+4x-1 \\ & \underline{-x^2-3x} \\ & \quad x-1 \\ & \quad \underline{-x-3} \\ & \quad\ -4 \end{array}$$

$$f(x)=\frac{x^2+4x-1}{x+3}=x+1-\frac{4}{x+3}$$

VA: $x=-3$ ***HA***: n/a

Hole: n/a ***Oblique asymptote***: $y=x+1$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^2-6x}{x-5}$

Solution

$x-5=0 \quad \to \quad x=5$ ***Domain***: $(-\infty,\ 5)\cup(5,\ \infty)$

$$\begin{array}{r|l} & x-1 \\ \hline x-5 & x^2-6x \\ & \underline{-x^2+5x} \\ & \quad -x \\ & \quad \underline{x-5} \\ & \quad -5 \end{array}$$

$$f(x)=\frac{x^2-6x}{x-5}=x-1-\frac{5}{x-5}$$

VA: $x=5$

HA: N/A

Hole: N/A

Oblique asymptote: $y=x-1$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^3-x^2+x-4}{x^2+2x-1}$

Solution

$x^2+2x-1=0 \quad \to \quad x=-1\pm\sqrt{2}$

Domain: $\left(-\infty,\ -1-\sqrt{2}\right)\cup\left(-1-\sqrt{2},\ -1+\sqrt{2}\right)\cup\left(-1+\sqrt{2},\ \infty\right)$

$$\begin{array}{r}
x-3 \\
x^2+2x-1 \overline{\big)\, x^3-x^2+x-4} \\
\underline{-x^3-2x^2+x} \\
-3x^2+2x-4 \\
\underline{3x^2+6x-3} \\
8x-7
\end{array}$$

VA: $x=-1\pm\sqrt{2}$

HA: N/A

Hole: N/A

Oblique asymptote: $y=x-3$

$$f(x)=\frac{x^3-x^2+x-4}{x^2+2x-1}$$

$$=x-3+\frac{8x-7}{x^2+2x-1}$$

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

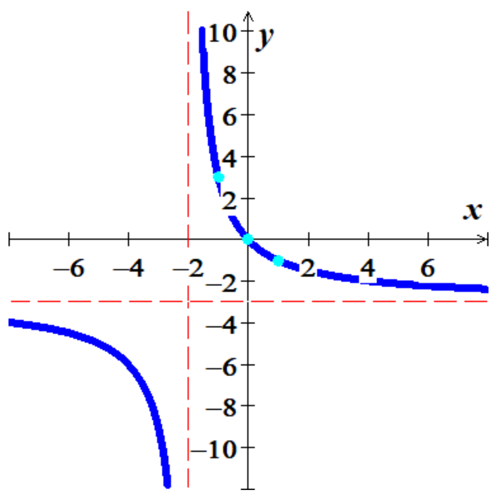
$$f(x)=\frac{-3x}{x+2}$$

Solution

VA: $x=-2$ **HA**: $y=-3$

Hole: n/a **OA**: n/a

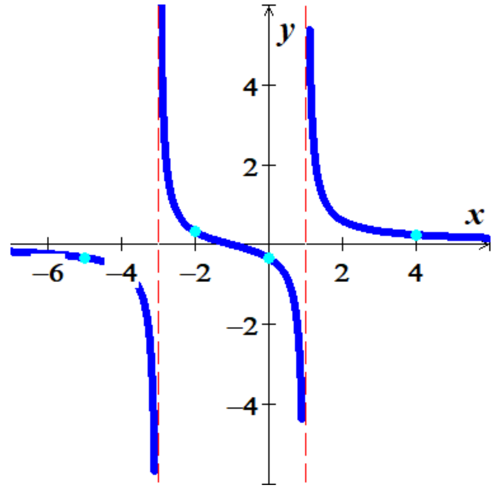
x	y
0	0
1	–1
–1	3



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x+1}{x^2+2x-3}$$



Solution

VA: $x=1,\ x=-3$ ***HA***: $y=0$

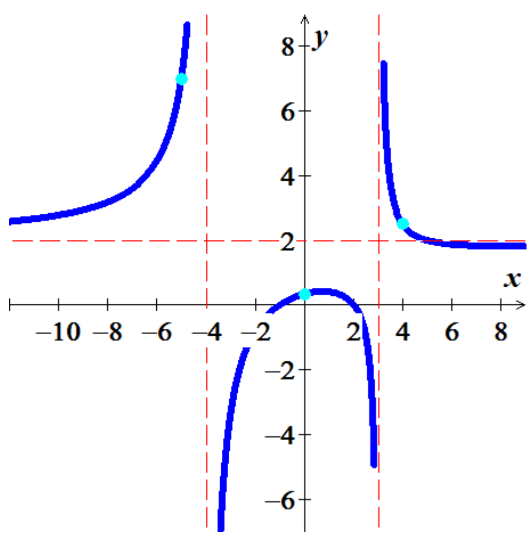
Hole: n/a ***Oblique asymptote***: n/a

x	y
−5	−0.33
−2	0.33
0	−1/3
4	0.24

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{2x^2-2x-4}{x^2+x-12}$$



Solution

VA: $x=-4,\ 3$ ***HA***: $y=2$

Hole: n/a ***OA***: n/a

x	y
−5	7
−2	−0.8
0	1/3
4	2.5
5	2

Exercise

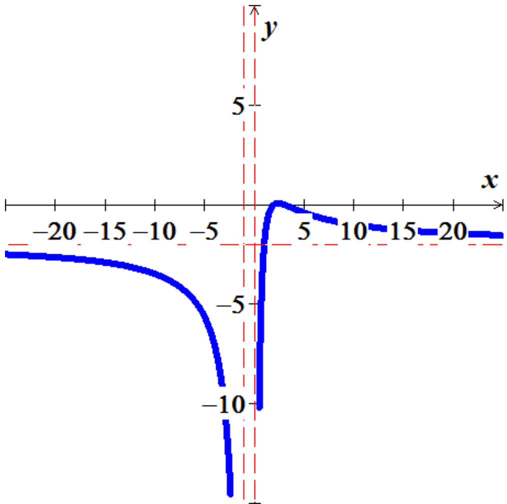
Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

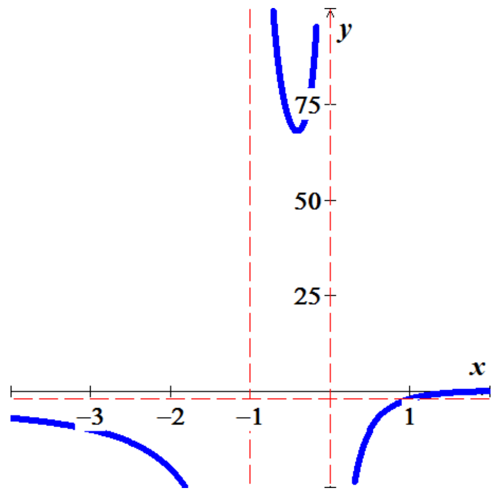
$$f(x) = \frac{-2x^2 + 10x - 12}{x^2 + x}$$

Solution

VA: $x = -1,\ 0$ ***HA***: $y = -2$

Hole: n/a ***OA***: n/a





Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x) = \frac{x^2 - x - 6}{x + 1}$$

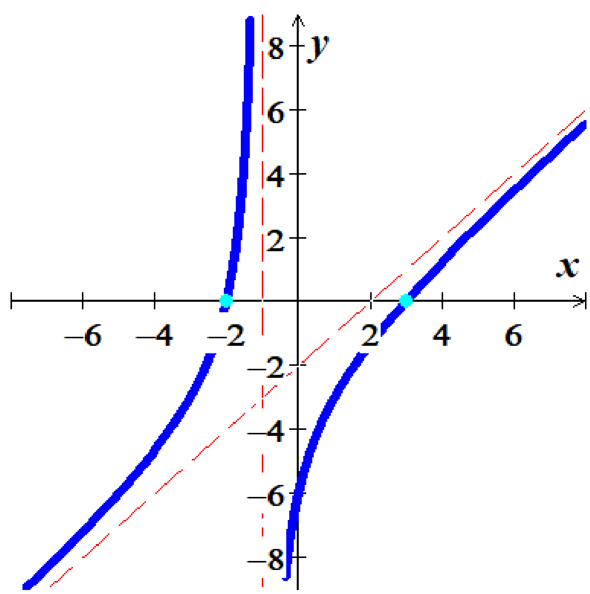
Solution

$$\begin{array}{r|l} & x - 2 \\ \hline x+1 & x^2 - x - 6 \\ & \underline{x^2 + x} \\ & \quad -2x - 6 \\ & \quad \underline{-2x - 2} \\ & \qquad \boxed{-4} \end{array}$$

VA: $x = -1$ ***HA***: n/a

Hole: n/a ***OA***: $y = x - 2$

x	y
2	0
–2	0
0	–6



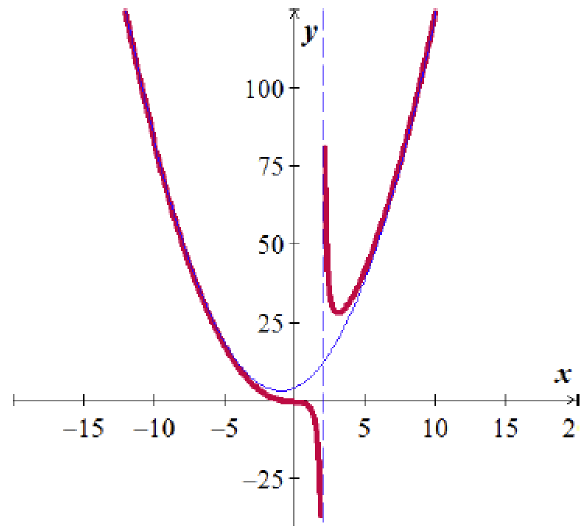
Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x) = \frac{x^3+1}{x-2}$$

Solution

$$\begin{array}{r|l} & x^2+2x+4 \\ \hline x-2 & x^3-1 \\ & \underline{x^3-2x^2} \\ & \quad 2x^2 \\ & \quad \underline{2x^2-4x} \\ & \qquad 4x-1 \\ & \qquad \underline{4x-8} \\ & \qquad\quad 7 \end{array}$$



VA: $x=2$ ***HA***: n/a

Hole: n/a ***OA***: $y = x^2+2x+4$

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x) = \frac{2x^2+x-6}{x^2+3x+2}$$

Solution

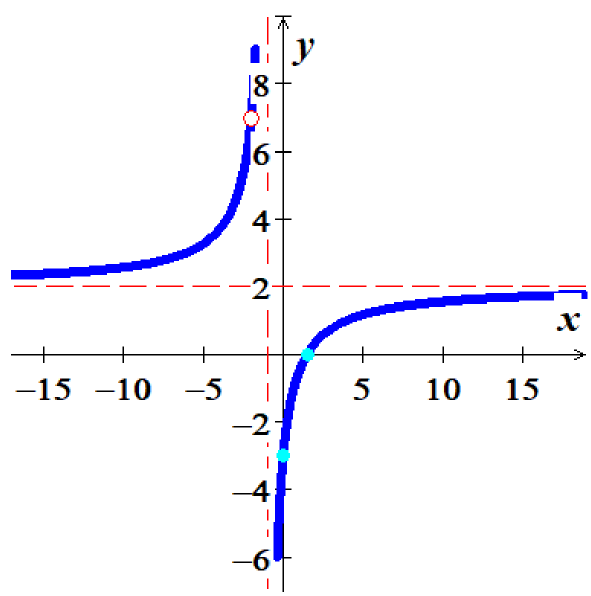
$$f(x) = \frac{(2x-3)(x+2)}{(x+1)(x+2)}$$

$$= \frac{2x-3}{x+1}$$

VA: $x=-1$ ***HA***: $y=2$

Hole: $(-2,\ 7)$ ***OA***: n/a

x	y
0	−3
$-\frac{3}{2}$	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{x-1}{1-x^2}$$

Solution

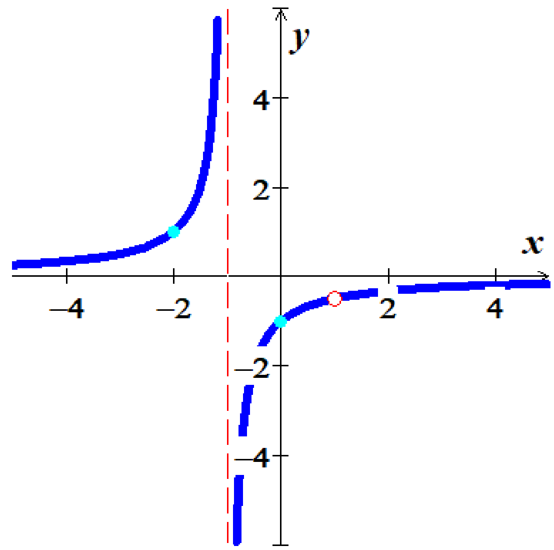
$$f(x)=\frac{x-1}{(x+1)(1-x)}$$

$$=-\frac{1}{x+1}$$

VA: $x=-1$ ***HA***: $y=0$

Hole: $\left(1,\ -\frac{1}{2}\right)$ ***OA***: n/a

x	***y***
0	–1
–2	1



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{x^2+x-2}{x+2}$$

Solution

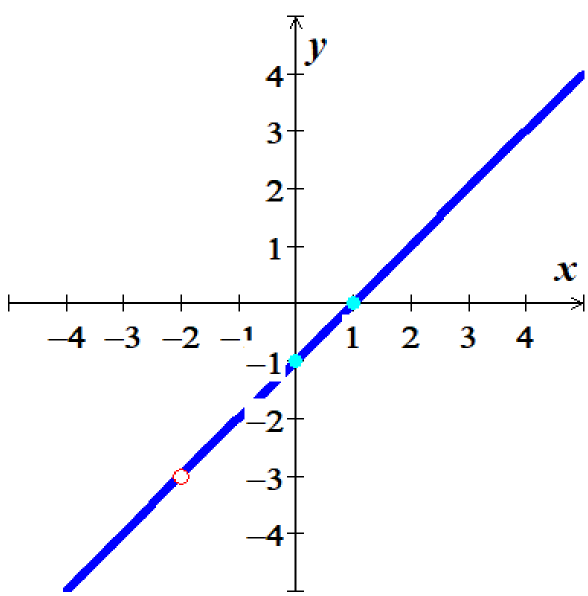
$$f(x)=\frac{(x+2)(x-1)}{x+2}$$

$$=x-1$$

VA: n/a ***HA***: n/a

Hole: $(-2,\ -3)$ ***OA***: n/a

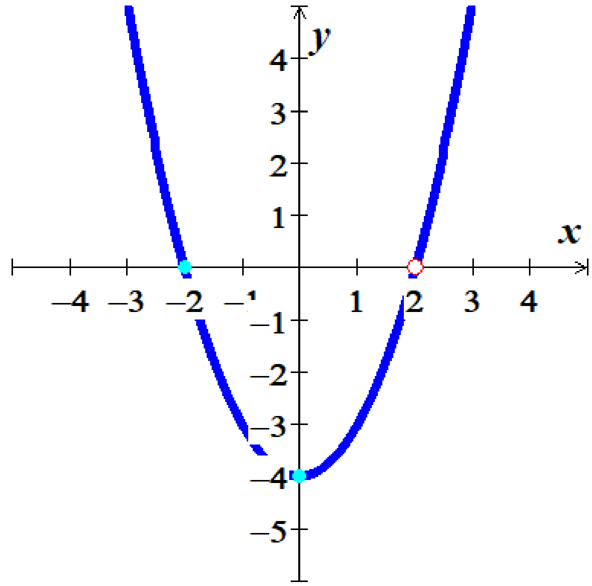
x	***y***
0	–1
1	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{x^3-2x^2-4x+8}{x-2}$$



Solution

$$f(x)=\frac{\left(x^2-4\right)(x-2)}{x-2}$$

$$=x^2-4$$

VA: n/a ***HA***: n/a

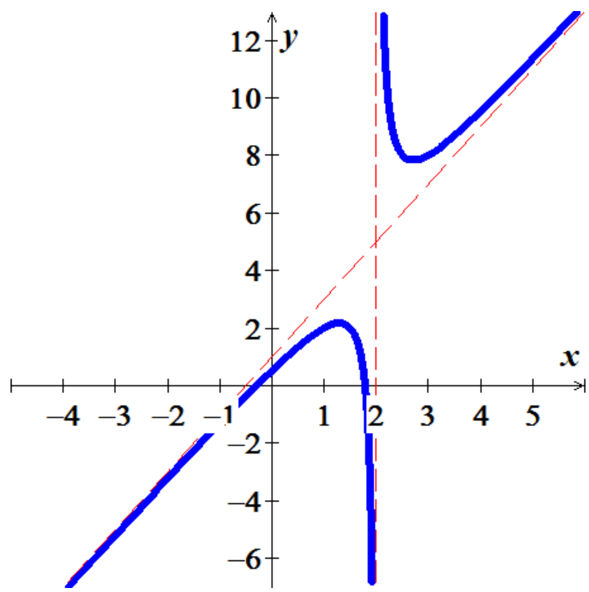
Hole: $(2,\ 0)$ ***OA***: n/a

x	y
0	−4
−2	0

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{2x^2-3x-1}{x-2}$$



Solution

$$\begin{array}{r|l} & 2x+1 \\ \hline x-2 & 2x^2-3x-1 \\ & \underline{-2x^2+4x} \\ & \qquad x-1 \\ & \qquad \underline{-x+2} \\ & \qquad\quad 1 \end{array}$$

$$f(x)=\frac{2x^2-3x-1}{x-2}$$

$$=(2x+1)+\frac{1}{x-2}$$

VA: $x=2$ ***HA***: $y=1$

Hole: n/a ***OA***: $y=2x+1$

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{2x+3}{3x^2+7x-6}$$

Solution

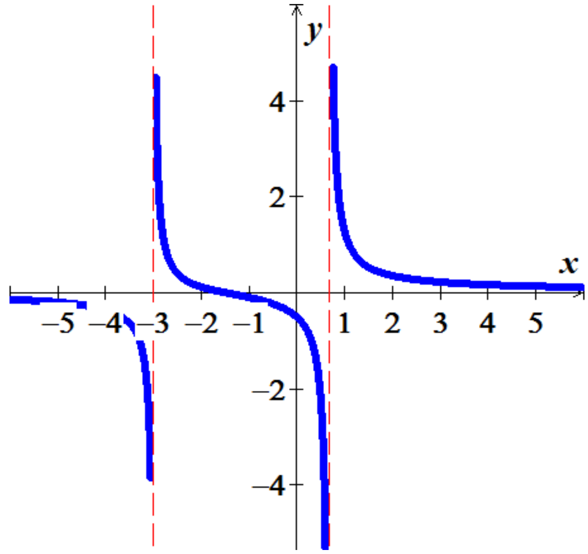
$3x^2+7x-6=0 \quad \Rightarrow \quad x=-3,\ \frac{2}{3}$

VA: $x=-3$ *and* $x=\frac{2}{3}$

HA: $y=0$

Hole: n/a

OA: n/a



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{x^2-1}{x^2+x-6}$$

Solution

$x^2+x-6=0 \quad \Rightarrow \quad \underline{x=-3,\ 2}|$

VA: $x=-3$ *and* $x=2$ ***HA***: $y=1$

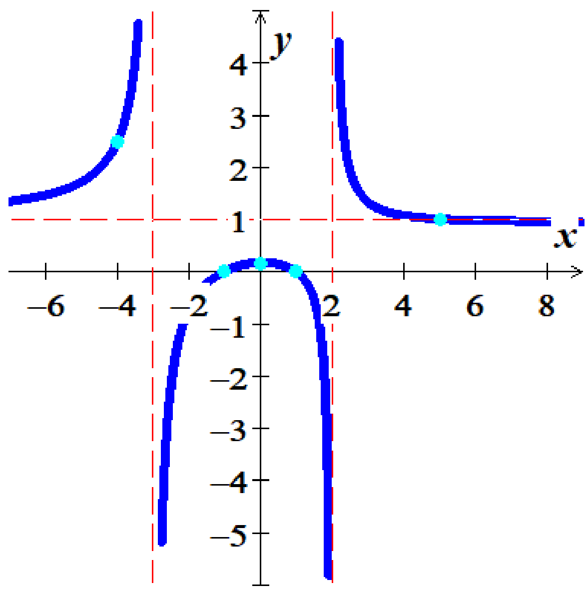
Hole: n/a ***OA:*** n/a

$1=\frac{x^2-1}{x^2+x-6}$

$x^2+x-6=x^2-1$

$\underline{x=5}|$

x	*y*
0	$\frac{1}{6}$
5	1
±1	0
-4	$\frac{5}{2}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{-2x^2 - x + 15}{x^2 - x - 12}$$

Solution

$x^2 - x - 12 = 0 \quad \Rightarrow \quad \underline{x = -3,\ 4}$

Domain: $(-\infty, -3) \cup (-3, 4) \cup (4, \infty)$

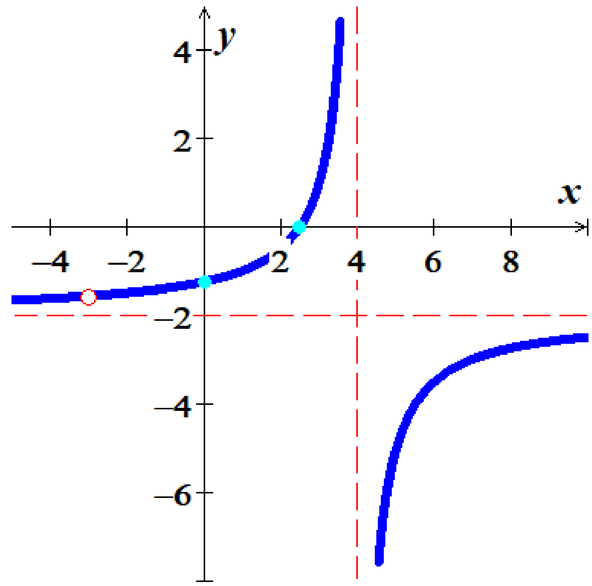
$$f(x) = \frac{(-2x+5)(x+3)}{(x-4)(x+3)}$$

$$= \frac{-2x+5}{x-4}$$

VA: $x = 4$ ***HA***: $y = -2$

Hole: $\left(-3,\ -\frac{11}{7}\right)$ ***OA***: n/a

x	y
0	$-\frac{5}{4}$
$\frac{5}{2}$	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

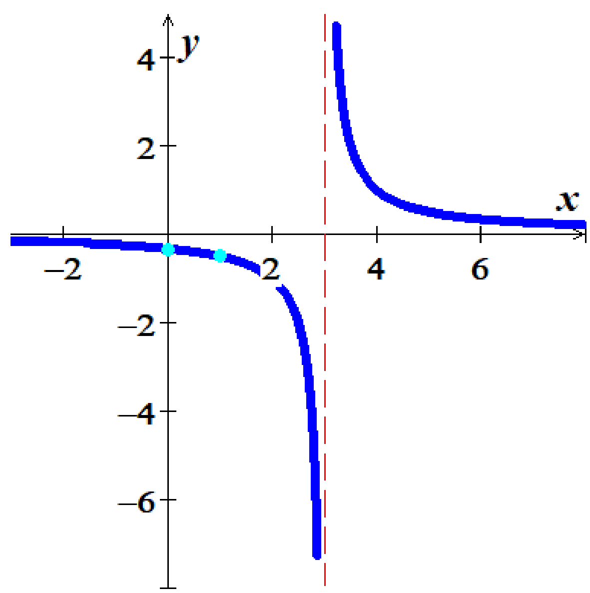
$$f(x) = \frac{1}{x-3}$$

Solution

VA: $x = 3$ ***HA***: $y = 0$

Hole: n/a ***OA***: n/a

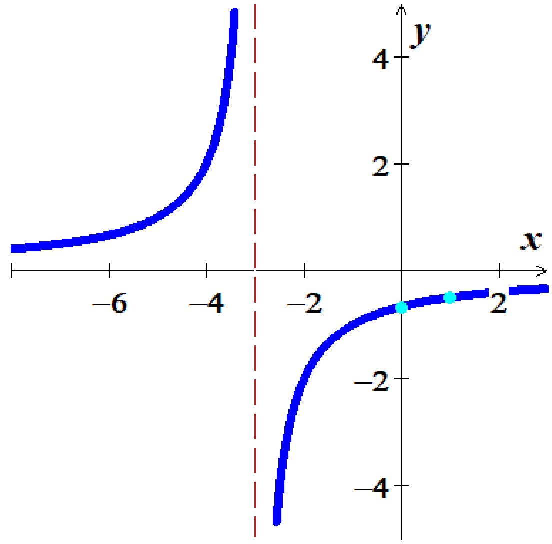
x	y
0	$-\frac{1}{3}$
1	$-\frac{1}{2}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{-2}{x+3}$$



Solution

VA: $x=-3$ ***HA***: $y=0$

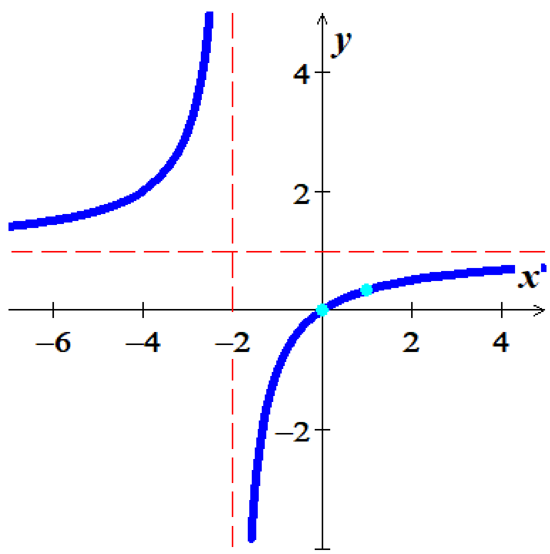
Hole: n/a ***OA:*** n/a

x	y
0	$-\frac{2}{3}$
1	$-\frac{1}{2}$

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x}{x+2}$$



Solution

VA: $x=-2$ ***HA***: $y=1$

Hole: n/a ***OA:*** n/a

x	y
0	0
1	$\frac{1}{3}$

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

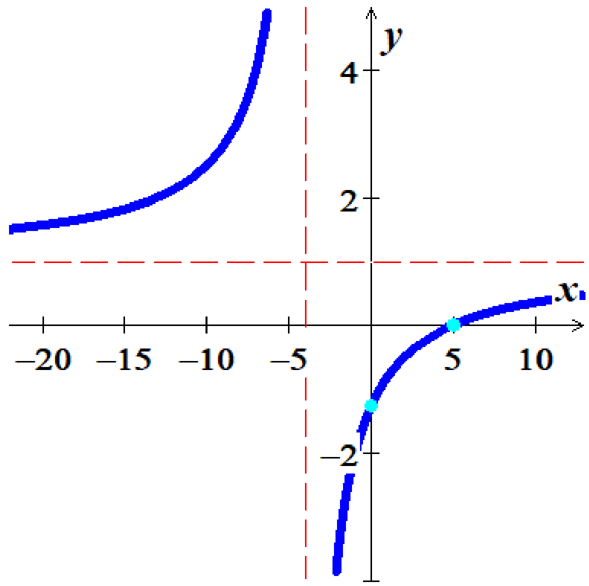
$$f(x)=\frac{x-5}{x+4}$$

Solution

VA: $x=-4$ ***HA***: $y=1$

Hole: n/a ***OA:*** n/a

x	***y***
0	$-\frac{5}{4}$
5	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{2x^2-2}{x^2-9}$$

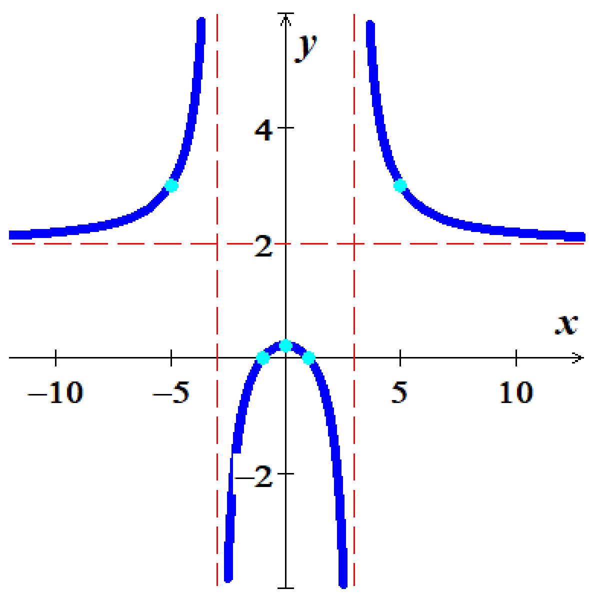
Solution

$x^2=9 \quad \rightarrow \quad \underline{x=\pm 3}$

VA: $x=\pm 3$ ***HA***: $y=2$

Hole: n/a ***OA:*** n/a

x	***y***
0	$\frac{2}{9}$
± 1	0
± 5	3



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

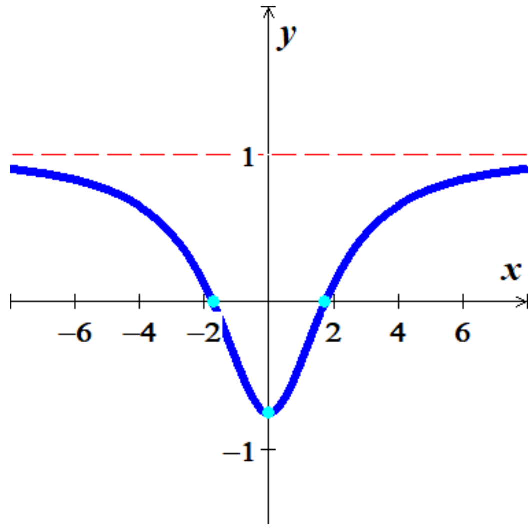
$$f(x)=\frac{x^2-3}{x^2+4}$$

Solution

VA: n/a **HA**: $y=1$

Hole: n/a **OA:** n/a

x	y
0	$-\frac{3}{4}$
$\pm\sqrt{3}$	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2+4}{x^2-3}$$

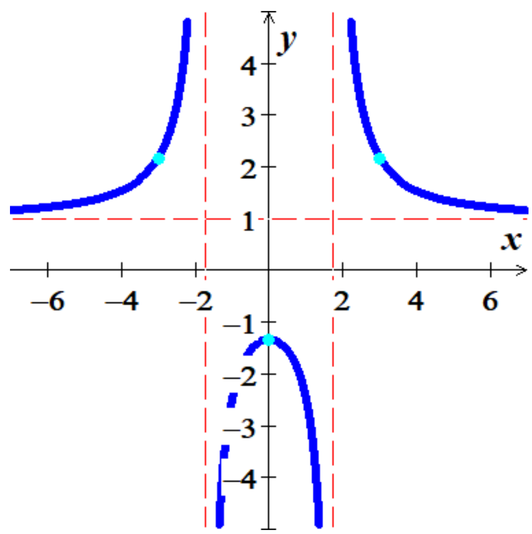
Solution

$x^2-3=0 \quad\rightarrow\quad x=\pm\sqrt{3}$

VA: $x=\pm\sqrt{3}$ **HA**: $y=1$

Hole: n/a **OA:** n/a

x	y
0	$-\frac{4}{3}$
± 3	$\frac{13}{6}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2}{x^2-6x+9}$$

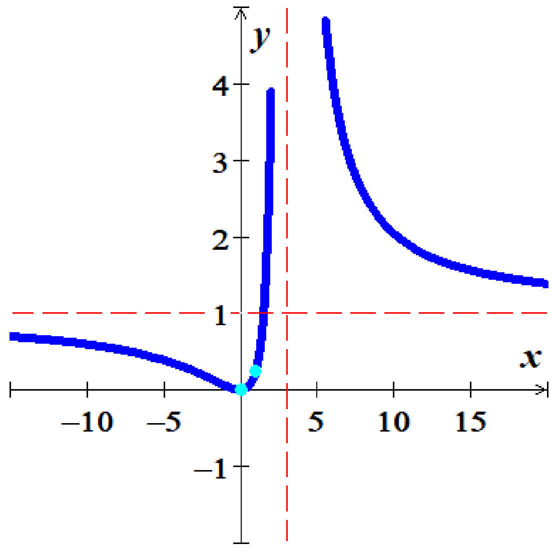
Solution

$x^2-6x+9=0 \quad \to \quad x=3$

VA: $x=3$ ***HA***: $y=1$

Hole: n/a ***OA:*** n/a

x	y
0	0
1	$\frac{1}{4}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2+x+4}{x^2+2x-1}$$

Solution

$x^2+2x-1=0$

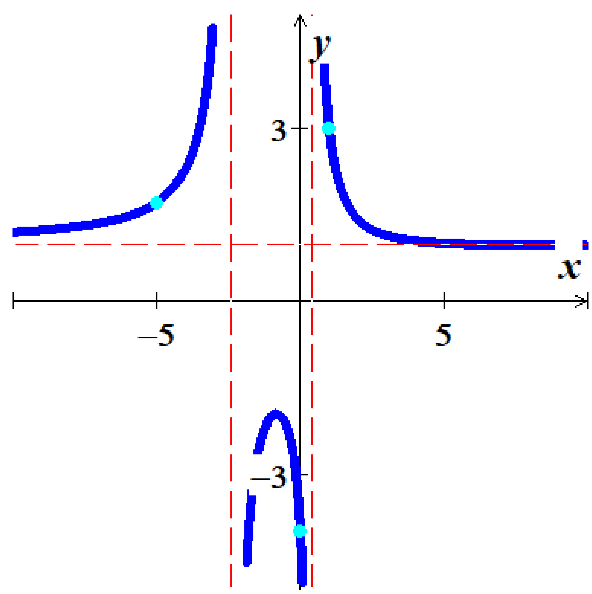
$x=\frac{-2\pm\sqrt{8}}{2}$

$\quad=-1\pm\sqrt{2}$

VA: $x=-1\pm\sqrt{2}$ ***HA***: $y=1$

Hole: n/a ***OA:*** n/a

x	y
0	–4
1	3
–5	$\frac{12}{7}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

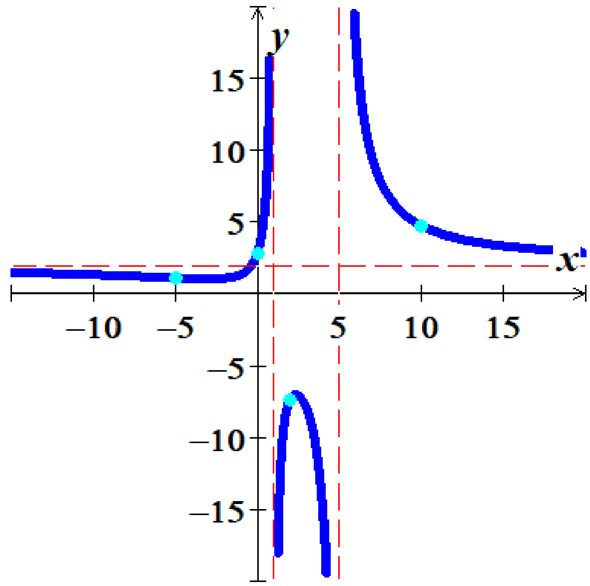
$$f(x)=\frac{2x^2+14}{x^2-6x+5}$$

Solution

VA: $x=1,\ 5$ ***HA***: $y=2$

Hole: n/a ***OA:*** n/a

x	y
0	$\frac{14}{5}$
2	$-\frac{22}{3}$
–5	$\frac{16}{15}$
10	$\frac{214}{45}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2-4x-5}{2x+5}$$

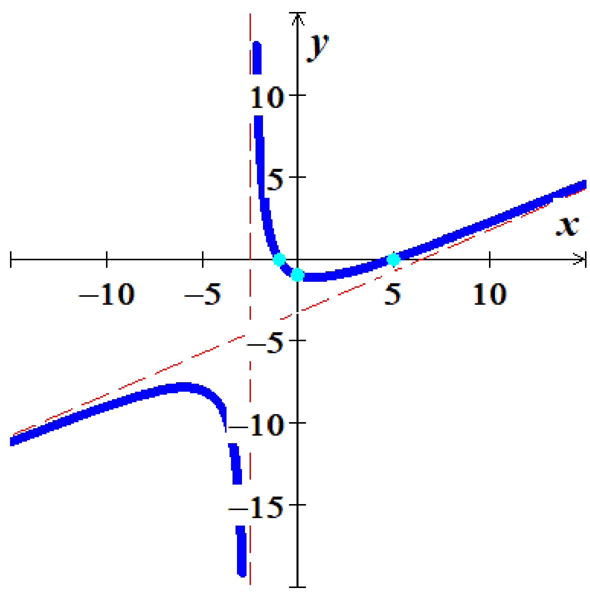
Solution

$$\begin{array}{r|l} & \frac{1}{2}x-\frac{13}{4} \\ \hline 2x+5 & x^2-4x-5 \\ & x^2+\frac{5}{2}x \\ \hline & -\frac{13}{2}x-5 \end{array}$$

VA: $x=-\frac{5}{2}$ ***HA***: n/a

Hole: n/a ***OA:*** $y=\frac{1}{2}x-\frac{13}{2}$

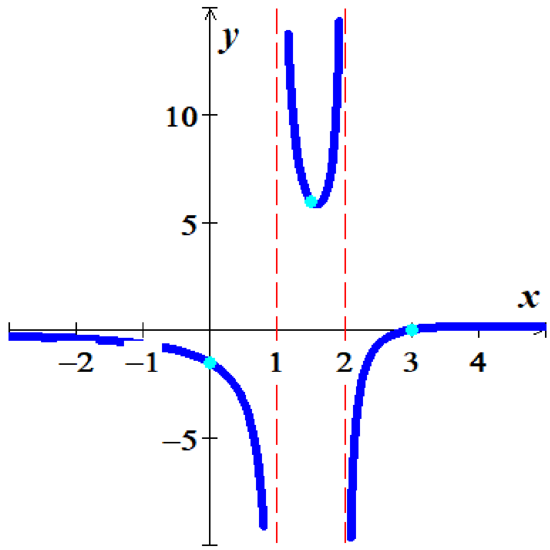
x	y
0	–1
–1, 5	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x-3}{x^2 - 3x + 2}$$



Solution

$x^2 - 3x + 2 \quad \to \quad \underline{x = 1,\ 2}$

VA: $x = 1,\ 2$ ***HA***: $y = 0$

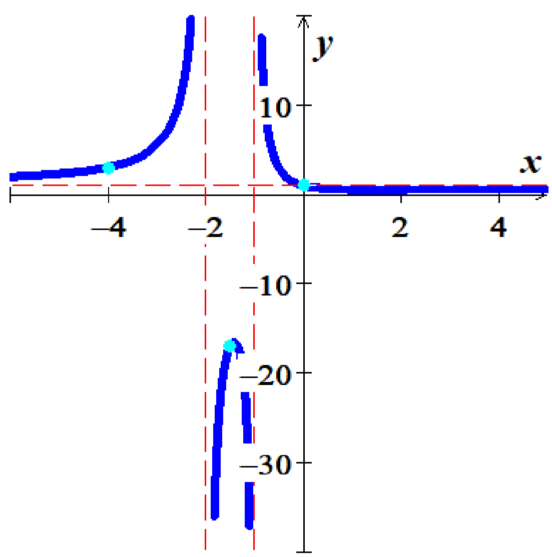
Hole: n/a ***OA:*** n/a

x	y
0	$-\frac{3}{2}$
3	0
$\frac{3}{2}$	6

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x^2 + 2}{x^2 + 3x + 2}$$



Solution

$x^2 + 3x + 2 \quad \to \quad \underline{x = -1,\ -2}$

VA: $x = -1,\ -2$ ***HA***: $y = 1$

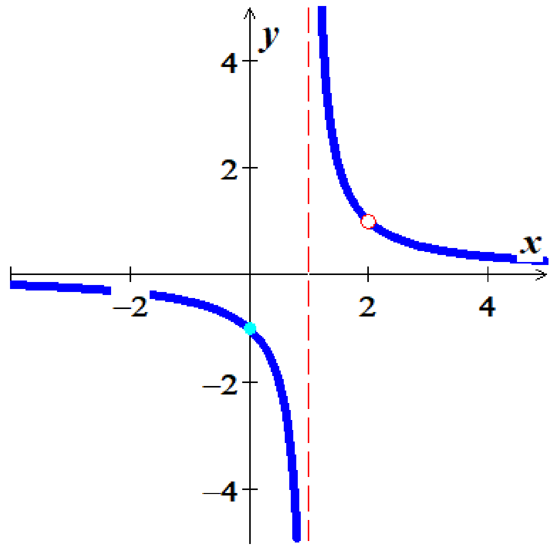
Hole: n/a ***OA:*** n/a

x	y
0	1
$-\frac{3}{2}$	-17
-4	3

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x-2}{x^2-3x+2}$$



Solution

$x^2 - 3x + 2 \quad \rightarrow \quad \underline{x = 1,\ 2|}$

$$f(x) = \frac{x-2}{(x-2)(x-1)}$$

$$= \frac{1}{x-1}$$

VA: $x = 1$ ***HA***: $y = 0$

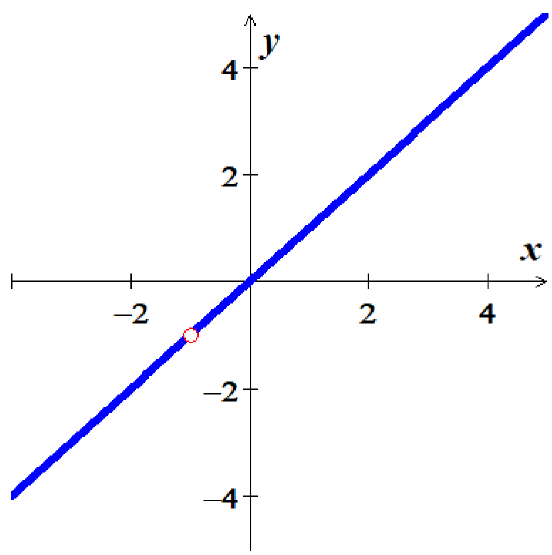
Hole: $(2,\ 1)$ ***OA:*** n/a

x	**y**
0	−1

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x^2+x}{x+1}$$



Solution

$$f(x) = \frac{x(x+1)}{x+1}$$

$$\underline{= x\,|}$$

VA: n/a ***HA***: n/a

Hole: $(-1,\ -1)$ ***OA:*** n/a

Hole: $\left(-3,\ -\frac{11}{7}\right)$ ***OA:*** n/a

x	**y**
0	0

Exercise

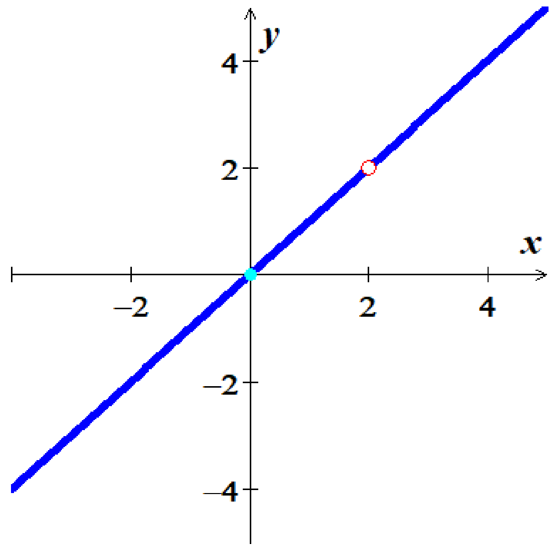
Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x^2 - 2x}{x - 2}$$

Solution

$$f(x) = \frac{x(x-2)}{x-2}$$

$$\underline{= x\,|}$$



VA: n/a ***HA***: n/a

Hole: $(2,\ 2)$ ***OA:*** n/a

Hole: $\left(-3,\ -\frac{11}{7}\right)$ ***OA:*** n/a

x	***y***
0	0

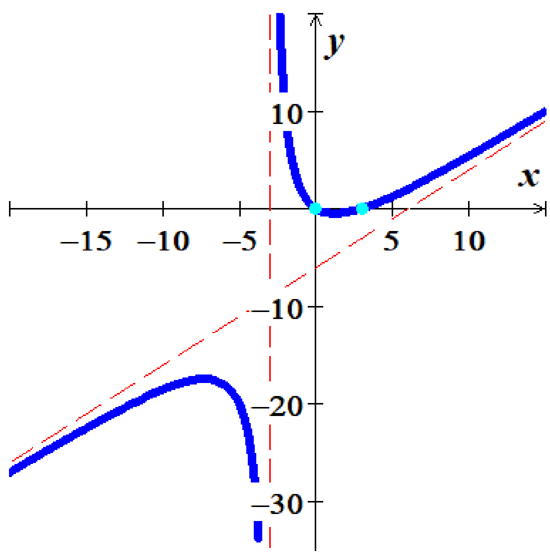
Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x^2 - 3x}{x + 3}$$

Solution

$$\begin{array}{r|l} & x - 6 \\ \hline x+3 & x^2 - 3x \\ & \underline{x^2 + 3x} \\ & -6x - 5 \end{array}$$



VA: $x = -3$ ***HA***: n/a

Hole: n/a ***OA:*** $y = x - 6$

x	***y***
0	0
3	0

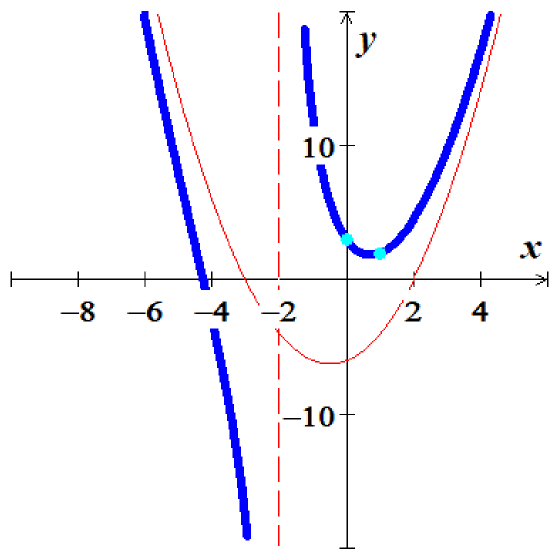
Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^3+3x^2-4x+6}{x+2}$$

Solution

$$\begin{array}{r|l} & x^2+x-6 \\ \hline x+2 & x^3+3x^2-4x+6 \\ & \underline{x^3+2x^2} \\ & x^2-4x \\ & \underline{x^2+2x} \\ & -6x+6 \end{array}$$



VA: $x=-2$ ***HA***: n/a

Hole: n/a ***OA:*** $y=x^2+x-6$

x	***y***
0	3
1	2

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} \textit{vertical asymptote}: \ x=4 \\ \textit{horizontal asymptote}: \ y=-1 \\ x-\textit{intercept}: \ 3 \end{cases}$$

Solution

Vertical Asymptote: $f(x)=\dfrac{}{x-4}$

Horizontal Asymptote: $f(x)=\dfrac{-x+a}{x-4}$

x-intercept: $f(x=3)=\dfrac{-3+a}{3-4}=0 \quad \Rightarrow \quad \underline{a=3}$

$$\boxed{f(x)=\frac{-x+3}{x-4}}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote: \ x=-4, x=5 \\ horizontal\ asymptote: \ y=\frac{3}{2} \\ x-intercept: \ -2 \end{cases}$$

Solution

Vertical Asymptote: $f(x)=\frac{}{(x+4)(x-5)}$

Horizontal Asymptote: $f(x)=\frac{3}{2}\frac{(x+a)(x+b)}{(x+4)(x-5)}$

x-intercept: $f(x=-2)=\frac{3}{2}\frac{(-2+a)(-2+b)}{}$

$$0=(-2+a)(-2+b)$$

$$\boxed{a=b=2}$$

$$f(x)=\frac{3}{2}\frac{(x-2)^2}{x^2-x-20}$$

$$=\frac{3x^2-12x+12}{2x^2-2x-40}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote: \ x=5 \\ horizontal\ asymptote: \ y=-1 \\ x-intercept: \ 2 \end{cases}$$

Solution

Vertical Asymptote: $f(x)=\frac{}{x-5}$

x-intercept : $f(x)=\frac{x-2}{x-5}$

Horizontal Asymptote: $f(x)=-\frac{x-2}{x-5}$

$$f(x)=-\frac{x-2}{x-5}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote: \ x=-2,\ x=0 \\ horizontal\ asymptote: \ y=0 \\ x-intercept: \ 2, \quad f(3)=1 \end{cases}$$

Solution

Vertical Asymptote: $\quad f(x)=\dfrac{}{x(x+2)}$

x-intercept : $\quad f(x)=\dfrac{x-2}{x(x+2)}$

Horizontal Asymptote: $\quad f(x)=\dfrac{a(x-2)}{x(x+2)}$

$f(3)=1 \quad \rightarrow \dfrac{a(1)}{(3)(5)}=1 \quad \Rightarrow \underline{a=15}$

$$f(x)=\frac{15x-30}{x^2+2x}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote: \ x=-3,\ x=1 \\ horizontal\ asymptote: \ y=0 \\ x-intercept: \ -1, \quad f(0)=-2 \\ hole: \ x=2 \end{cases}$$

Solution

Vertical Asymptote: $\quad f(x)=\dfrac{}{(x+3)(x-1)}$

x-intercept : $\quad f(x)=\dfrac{(x+1)}{(x+3)(x-1)}$

Horizontal Asymptote: $\quad f(x)=\dfrac{a(x+1)}{(x+3)(x-1)}$

$f(0)=-2 \quad \rightarrow \dfrac{a}{-3}=-2 \quad \Rightarrow \underline{a=6}$

Hole at $x=2$**:** $\quad f(x)=\dfrac{6(x+1)(x-2)}{\left(x^2+2x-3\right)(x-2)}$

$$f(x)=\frac{6x^2-6x-12}{x^3-7x+6}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote:\ x=-1,\ \ x=3 \\ horizontal\ asymptote:\ y=2 \\ x-intercept:\ \ -2,\ \ 1 \\ \ \ hole:\ \ x=0 \end{cases}$$

Solution

Vertical Asymptote: $\quad f(x)=\dfrac{}{(x+1)(x-3)}$

Horizontal Asymptote: $\quad f(x)=\dfrac{2}{(x+1)(x-3)}$

x-intercept : $\quad f(x)=\dfrac{2(x+2)(x-1)}{(x+1)(x-3)}$

Hole at $x=0$**:** $\quad f(x)=\dfrac{2x(x+2)(x-1)}{x(x+1)(x-3)}$

$$f(x)=\frac{2x^3+2x^2-4x}{x^3-2x^2-3x}$$