

Determinant

$$\det(A) = \sum_{i=1}^n (-1)^{i+j} a_{i,j} M_{i,j}$$

$$a_{1*} a_{2*} a_{3*} \dots a_{n*}$$

By: ***Fred E. Khoury***

The determinant is associated with a square matrix $(n \times n)$.

The total terms (summation) of a determinant is equal to $n!$ with n entries (elements) to each product term. Half are positive (product sign stay the same) and the other half sign has to be multiplied by negative sign (or opposite sign).

Block Method

$$\begin{vmatrix} A & 0 \\ 0 & B \end{vmatrix} = |A| \cdot |B|$$

Proof

$$\begin{aligned} \begin{vmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{vmatrix} &= a \begin{vmatrix} d & 0 & 0 \\ 0 & e & f \\ 0 & g & h \end{vmatrix} - b \begin{vmatrix} c & 0 & 0 \\ 0 & e & f \\ 0 & g & h \end{vmatrix} \\ &= ad \begin{vmatrix} e & f \\ g & h \end{vmatrix} - bc \begin{vmatrix} e & f \\ g & h \end{vmatrix} \\ &= (ad - bc) \begin{vmatrix} e & f \\ g & h \end{vmatrix} \\ &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & f \\ g & h \end{vmatrix} \\ &= |A| \cdot |B| \end{aligned}$$

Example

$$\begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & -2 & 5 \\ 0 & 0 & -2 & 7 \end{vmatrix} = \underline{8} \quad A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \underline{-2} \quad B = \begin{vmatrix} -2 & 5 \\ -2 & 7 \end{vmatrix} = \underline{-4}$$

$$\boxed{8 = (-2)(-4)}$$

Example

$$\begin{vmatrix} 1 & 2 & 2 & 0 & 0 \\ 3 & 4 & 5 & 0 & 0 \\ 6 & 7 & 8 & 0 & 0 \\ 0 & 0 & 0 & -5 & 7 \\ 0 & 0 & 0 & 3 & 4 \end{vmatrix} = \underline{-123} \quad A = \begin{vmatrix} 1 & 2 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{vmatrix} = \underline{3} \quad B = \begin{vmatrix} -5 & 7 \\ 3 & 4 \end{vmatrix} = \underline{-41}$$

$$\boxed{-123 = (3)(-41)}$$

2×2

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

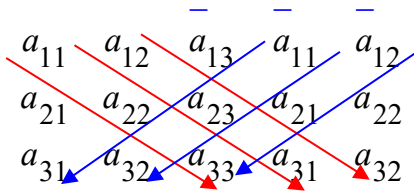
3×3

Co-Factor Method

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Diagonal Method



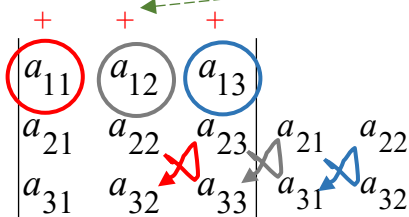
$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \quad (1)$$

$$-a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \quad (2)$$

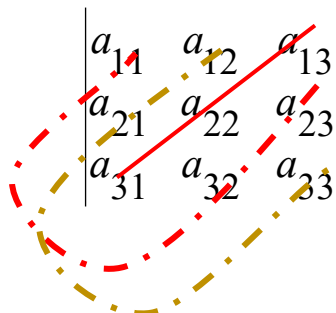
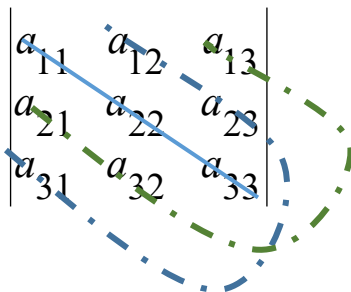
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Another Method

Plus



$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$



(Opposite sign)

4×4

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

- 1- Copy the 2nd & 3rd rows below 4th row respectively.
- 2- Copy the 1st, 2nd and 3rd column next to the 4th column respectively as is shown below

+	−	+	−			
a₁₁	a₁₂	a₁₃	a ₁₄	a ₁₁	a ₁₂	a ₁₃
a ₂₁	a ₂₂	a ₂₃	a ₂₄	a ₂₁	a ₂₂	a ₂₃
a ₃₁	a ₃₂	a ₃₃	a ₃₄	a ₃₁	a ₃₂	a ₃₃
a ₄₁	a ₄₂	a ₄₃	a ₄₄	a ₄₁	a ₄₂	a ₄₃
a ₂₁	a ₂₂	a ₂₃	a ₂₄	a ₂₁	a ₂₂	a ₂₃
a ₃₁	a ₃₂	a ₃₃	a ₃₄	a ₃₁	a ₃₂	a ₃₃

Determinant =

$$\begin{aligned}
 & a_{11} (a_{22}a_{33}a_{44} + a_{24}a_{32}a_{43} + a_{23}a_{34}a_{42} - a_{24}a_{33}a_{42} - a_{22}a_{34}a_{43} - a_{23}a_{32}a_{44}) \\
 & - a_{12} (a_{23}a_{34}a_{41} + a_{21}a_{33}a_{44} + a_{24}a_{31}a_{43} - a_{21}a_{34}a_{43} - a_{23}a_{31}a_{44} - a_{24}a_{33}a_{41}) \\
 & + a_{13} (a_{24}a_{31}a_{42} + a_{22}a_{34}a_{41} + a_{21}a_{32}a_{44} - a_{22}a_{31}a_{44} - a_{24}a_{32}a_{41} - a_{21}a_{34}a_{42}) \\
 & - a_{14} (a_{21}a_{32}a_{43} + a_{22}a_{33}a_{41} + a_{23}a_{31}a_{42} - a_{21}a_{33}a_{42} - a_{22}a_{31}a_{43} - a_{23}a_{32}a_{41})
 \end{aligned}$$

+

a_{11}	a_{12}	a_{13}	a_{14}	a_{11}	a_{12}	a_{13}
a_{21}	a_{22}	a_{23}	a_{24}	a_{21}	a_{22}	a_{23}
a_{31}	a_{32}	a_{33}	a_{34}	a_{31}	a_{32}	a_{33}
a_{41}	a_{42}	a_{43}	a_{44}	a_{41}	a_{42}	a_{43}
a_{21}	a_{22}	a_{23}	a_{24}	a_{21}	a_{22}	a_{23}
a_{31}	a_{32}	a_{33}	a_{34}	a_{31}	a_{32}	a_{33}

$a_{11} (a_{22}a_{33}a_{44} + a_{24}a_{32}a_{43} + a_{23}a_{34}a_{42} - a_{24}a_{33}a_{42} - a_{22}a_{34}a_{43} - a_{23}a_{32}a_{44})$

-

a_{11}	a_{12}	a_{13}	a_{14}	a_{11}	a_{12}	a_{13}
a_{21}	a_{22}	a_{23}	a_{24}	a_{21}	a_{22}	a_{23}
a_{31}	a_{32}	a_{33}	a_{34}	a_{31}	a_{32}	a_{33}
a_{41}	a_{42}	a_{43}	a_{44}	a_{41}	a_{42}	a_{43}
a_{21}	a_{22}	a_{23}	a_{24}	a_{21}	a_{22}	a_{23}
a_{31}	a_{32}	a_{33}	a_{34}	a_{31}	a_{32}	a_{33}

$-a_{12} (a_{23}a_{34}a_{41} + a_{21}a_{33}a_{44} + a_{24}a_{31}a_{43} - a_{21}a_{34}a_{43} - a_{23}a_{31}a_{44} - a_{24}a_{33}a_{41})$

+

a_{11}	a_{12}	a_{13}	a_{14}	a_{11}	a_{12}	a_{13}
a_{21}	a_{22}	a_{23}	a_{24}	a_{21}	a_{22}	a_{23}
a_{31}	a_{32}	a_{33}	a_{34}	a_{31}	a_{32}	a_{33}
a_{41}	a_{42}	a_{43}	a_{44}	a_{41}	a_{42}	a_{43}
a_{21}	a_{22}	a_{23}	a_{24}	a_{21}	a_{22}	a_{23}
a_{31}	a_{32}	a_{33}	a_{34}	a_{31}	a_{32}	a_{33}

$+a_{13} (a_{24}a_{31}a_{42} + a_{22}a_{34}a_{41} + a_{21}a_{32}a_{44} - a_{22}a_{31}a_{44} - a_{24}a_{32}a_{41} - a_{21}a_{34}a_{42})$

-

a_{11}	a_{12}	a_{13}	a_{14}	a_{11}	a_{12}	a_{13}
a_{21}	a_{22}	a_{23}	a_{24}	a_{21}	a_{22}	a_{23}
a_{31}	a_{32}	a_{33}	a_{34}	a_{31}	a_{32}	a_{33}
a_{41}	a_{42}	a_{43}	a_{44}	a_{41}	a_{42}	a_{43}
a_{21}	a_{22}	a_{23}	a_{24}	a_{21}	a_{22}	a_{23}
a_{31}	a_{32}	a_{33}	a_{34}	a_{31}	a_{32}	a_{33}

$-a_{14} (a_{21}a_{32}a_{43} + a_{22}a_{33}a_{41} + a_{23}a_{31}a_{42} - a_{23}a_{32}a_{41} - a_{21}a_{33}a_{42} - a_{22}a_{31}a_{43})$

2nd Method

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

Row number always 1 2 3 4 sequence to all the terms values $a_{1*}a_{2*}a_{3*}a_{4*}$

Up (Column Number) →

$a_{11}a_{22}a_{33}a_{44}$	$a_{12}a_{23}a_{34}a_{41}$	$a_{13}a_{24}a_{31}a_{42}$	$a_{14}a_{21}a_{32}a_{43}$
$a_{11}a_{23}a_{34}a_{42}$	$a_{12}a_{24}a_{31}a_{43}$	$a_{13}a_{22}a_{34}a_{41}$	$a_{14}a_{22}a_{33}a_{41}$
$a_{11}a_{24}a_{32}a_{43}$	$a_{12}a_{21}a_{33}a_{44}$	$a_{13}a_{21}a_{32}a_{44}$	$a_{14}a_{23}a_{31}a_{42}$

As for the column numbers:

1 2 3 4	2 3 4 1	3 4 1 2	4 1 2 3
1 3 4 2	↓ ↓ ↓ ↓	↓ ↓ ↓ ↓	↓ ↓ ↓ ↓
1 4 2 3	↓ ↓ ↓ ↓	↓ ↓ ↓ ↓	↓ ↓ ↓ ↓

Up (Column Number) ↓

Down (Column Number) →

$a_{11}a_{24}a_{33}a_{42}$	$a_{12}a_{21}a_{34}a_{43}$	$a_{13}a_{22}a_{31}a_{44}$	$a_{14}a_{23}a_{32}a_{41}$
$a_{11}a_{23}a_{32}a_{44}$	$a_{12}a_{24}a_{33}a_{41}$	$a_{13}a_{21}a_{34}a_{42}$	$a_{14}a_{22}a_{31}a_{43}$
$a_{11}a_{22}a_{34}a_{43}$	$a_{12}a_{23}a_{31}a_{44}$	$a_{13}a_{24}a_{32}a_{41}$	$a_{14}a_{21}a_{33}a_{42}$

As for the column numbers:

1 4 3 2	2 1 4 3	3 2 1 4	4 3 2 1
1 3 2 4	↓ ↓ ↓ ↓	↓ ↓ ↓ ↓	↓ ↓ ↓ ↓
1 2 4 3	↓ ↓ ↓ ↓	↓ ↓ ↓ ↓	↓ ↓ ↓ ↓

Down (Column Number) ↓

The red product terms have to be opposite sign (multiply by minus)

3rd Method

$$\begin{array}{ccccccc}
 a_{11} & a_{12} & a_{13} & a_{14} & a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23} & a_{24} & a_{21} & a_{22} & a_{23} \\
 a_{31} & a_{32} & a_{33} & a_{34} & a_{31} & a_{32} & a_{33} \\
 a_{41} & a_{42} & a_{43} & a_{44} & a_{41} & a_{42} & a_{43}
 \end{array}$$

$$\begin{aligned}
 &= a_{11}a_{22}(a_{33}a_{44} - a_{34}a_{43}) + a_{12}a_{23}(a_{34}a_{41} - a_{32}a_{44}) \\
 &\quad + a_{13}a_{24}(a_{31}a_{42} - a_{32}a_{41}) + a_{14}a_{21}(a_{32}a_{43} - a_{33}a_{42})
 \end{aligned}$$

$$\begin{array}{ccccccc}
 a_{11} & a_{12} & a_{13} & a_{14} & a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23} & a_{24} & a_{21} & a_{22} & a_{23} \\
 a_{31} & a_{32} & a_{33} & a_{34} & a_{31} & a_{32} & a_{33} \\
 a_{41} & a_{42} & a_{43} & a_{44} & a_{41} & a_{42} & a_{43}
 \end{array}$$

$$\begin{aligned}
 &= -a_{13}a_{22}(a_{31}a_{44} - a_{34}a_{41}) - a_{12}a_{21}(a_{34}a_{43} - a_{33}a_{44}) \\
 &\quad - a_{11}a_{24}(a_{33}a_{42} - a_{32}a_{43}) - a_{14}a_{23}(a_{32}a_{41} - a_{31}a_{42})
 \end{aligned}$$

$$\begin{array}{ccccccc}
 a_{11} & a_{12} & a_{13} & a_{14} & a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23} & a_{24} & a_{21} & a_{22} & a_{23} \\
 a_{31} & a_{32} & a_{33} & a_{34} & a_{31} & a_{32} & a_{33} \\
 a_{41} & a_{42} & a_{43} & a_{44} & a_{41} & a_{42} & a_{43}
 \end{array}$$

$$\begin{aligned}
 &= a_{11}a_{23}(a_{34}a_{42} - a_{32}a_{44}) + a_{12}a_{24}(a_{31}a_{43} - a_{33}a_{41}) \\
 &\quad + a_{13}a_{21}(a_{32}a_{44} - a_{34}a_{42}) + a_{14}a_{22}(a_{33}a_{41} - a_{31}a_{43})
 \end{aligned}$$

$$\begin{vmatrix}
 a_{11} & a_{12} & a_{13} & a_{14} \\
 a_{21} & a_{22} & a_{23} & a_{24} \\
 a_{31} & a_{32} & a_{33} & a_{34} \\
 a_{41} & a_{42} & a_{43} & a_{44}
 \end{vmatrix}$$

$$\begin{aligned}
 &= a_{11}a_{22}a_{33}a_{44} - a_{11}a_{22}a_{34}a_{43} + a_{12}a_{23}a_{34}a_{41} - a_{12}a_{23}a_{32}a_{44} + a_{13}a_{24}a_{31}a_{42} \\
 &\quad - a_{13}a_{24}a_{32}a_{41} + a_{14}a_{21}a_{32}a_{43} - a_{14}a_{21}a_{33}a_{42} - a_{13}a_{22}a_{31}a_{44} + a_{13}a_{22}a_{34}a_{41} \\
 &\quad - a_{12}a_{21}a_{34}a_{43} + a_{12}a_{21}a_{33}a_{44} - a_{11}a_{24}a_{33}a_{42} + a_{11}a_{24}a_{32}a_{43} - a_{14}a_{23}a_{32}a_{41} \\
 &\quad + a_{14}a_{23}a_{31}a_{42} + a_{11}a_{23}a_{34}a_{42} - a_{11}a_{23}a_{32}a_{44} + a_{12}a_{24}a_{31}a_{43} - a_{12}a_{24}a_{33}a_{41} \\
 &\quad + a_{13}a_{21}a_{32}a_{44} - a_{13}a_{21}a_{34}a_{42} + a_{14}a_{22}a_{33}a_{41} - a_{14}a_{22}a_{31}a_{43}
 \end{aligned}$$

$$\begin{vmatrix} 2 & 6 & 2 & 2 \\ 2 & 5 & 2 & 2 \\ 1 & 3 & 8 & 9 \\ 1 & 3 & 2 & -3 \end{vmatrix} = 2 \begin{vmatrix} 1 & 3 & 1 & 1 \\ 2 & 5 & 2 & 2 \\ 1 & 3 & 8 & 9 \\ 1 & 3 & 2 & -3 \end{vmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{matrix} \\
= 2(7) \begin{vmatrix} 1 & 3 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & \frac{8}{7} \\ 0 & 0 & 1 & -4 \end{vmatrix} \\
= 14 \begin{vmatrix} 1 & 3 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & \frac{8}{7} \\ 0 & 0 & 1 & -4 \end{vmatrix} \begin{matrix} \\ \\ R_4 - R_3 \end{matrix} \\
= 14 \begin{vmatrix} 1 & 3 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & \frac{8}{7} \\ 0 & 0 & 0 & -\frac{36}{7} \end{vmatrix} \\
= 14(1)(-1)(1)\left(-\frac{36}{7}\right) \\
= \underline{72}$$

5×5

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} =$$

$$\begin{aligned} & a_{11}a_{22}a_{33}a_{44}a_{55} + a_{11}a_{22}a_{34}a_{45}a_{53} + a_{11}a_{22}a_{35}a_{43}a_{54} + a_{11}a_{23}a_{35}a_{44}a_{52} + a_{11}a_{23}a_{32}a_{45}a_{54} + a_{11}a_{23}a_{34}a_{42}a_{55} \\ & -a_{11}a_{22}a_{33}a_{45}a_{54} - a_{11}a_{22}a_{34}a_{43}a_{55} - a_{11}a_{22}a_{35}a_{44}a_{53} - a_{11}a_{23}a_{35}a_{42}a_{54} - a_{11}a_{23}a_{32}a_{44}a_{55} - a_{11}a_{23}a_{34}a_{45}a_{52} \\ & + a_{11}a_{24}a_{32}a_{43}a_{55} + a_{11}a_{24}a_{33}a_{45}a_{52} + a_{11}a_{24}a_{35}a_{42}a_{53} + a_{11}a_{25}a_{34}a_{43}a_{52} + a_{11}a_{25}a_{32}a_{44}a_{53} + a_{11}a_{25}a_{33}a_{42}a_{54} \\ & -a_{11}a_{24}a_{32}a_{45}a_{53} - a_{11}a_{24}a_{33}a_{42}a_{55} - a_{11}a_{24}a_{35}a_{43}a_{52} - a_{11}a_{25}a_{34}a_{42}a_{53} - a_{11}a_{25}a_{32}a_{43}a_{54} - a_{11}a_{25}a_{33}a_{44}a_{52} \\ & + a_{12}a_{21}a_{35}a_{44}a_{53} + a_{12}a_{21}a_{33}a_{45}a_{54} + a_{12}a_{21}a_{34}a_{43}a_{55} + a_{12}a_{23}a_{31}a_{44}a_{55} + a_{12}a_{23}a_{34}a_{45}a_{51} + a_{12}a_{23}a_{35}a_{41}a_{54} \\ & -a_{12}a_{21}a_{35}a_{43}a_{54} - a_{12}a_{21}a_{33}a_{44}a_{55} - a_{12}a_{21}a_{34}a_{45}a_{53} - a_{12}a_{23}a_{31}a_{45}a_{54} - a_{12}a_{23}a_{34}a_{41}a_{55} - a_{12}a_{23}a_{35}a_{44}a_{51} \\ & + a_{12}a_{24}a_{35}a_{43}a_{51} + a_{12}a_{24}a_{31}a_{45}a_{53} + a_{12}a_{24}a_{33}a_{41}a_{55} + a_{12}a_{25}a_{31}a_{43}a_{54} + a_{12}a_{25}a_{33}a_{44}a_{51} + a_{12}a_{25}a_{34}a_{41}a_{53} \\ & -a_{12}a_{24}a_{35}a_{41}a_{53} - a_{12}a_{24}a_{31}a_{43}a_{55} - a_{12}a_{24}a_{33}a_{45}a_{51} - a_{12}a_{25}a_{31}a_{44}a_{53} - a_{12}a_{25}a_{33}a_{41}a_{54} - a_{12}a_{25}a_{34}a_{43}a_{51} \\ & + a_{13}a_{21}a_{32}a_{44}a_{55} + a_{13}a_{21}a_{34}a_{45}a_{52} + a_{13}a_{21}a_{35}a_{42}a_{54} + a_{13}a_{22}a_{35}a_{44}a_{51} + a_{13}a_{22}a_{31}a_{45}a_{54} + a_{13}a_{22}a_{34}a_{41}a_{55} \\ & -a_{13}a_{21}a_{32}a_{45}a_{54} - a_{13}a_{21}a_{34}a_{42}a_{55} - a_{13}a_{21}a_{35}a_{44}a_{52} - a_{13}a_{22}a_{35}a_{41}a_{54} - a_{13}a_{22}a_{31}a_{44}a_{55} - a_{13}a_{22}a_{34}a_{45}a_{51} \\ & + a_{13}a_{24}a_{31}a_{42}a_{55} + a_{13}a_{24}a_{32}a_{45}a_{51} + a_{13}a_{24}a_{35}a_{41}a_{52} + a_{13}a_{25}a_{31}a_{44}a_{52} + a_{13}a_{25}a_{32}a_{41}a_{54} + a_{13}a_{25}a_{34}a_{42}a_{51} \\ & -a_{13}a_{24}a_{31}a_{45}a_{52} - a_{13}a_{24}a_{32}a_{41}a_{55} - a_{13}a_{24}a_{35}a_{42}a_{51} - a_{13}a_{25}a_{31}a_{42}a_{54} - a_{13}a_{25}a_{32}a_{44}a_{51} - a_{13}a_{25}a_{34}a_{41}a_{52} \\ & + a_{14}a_{21}a_{35}a_{43}a_{52} + a_{14}a_{21}a_{32}a_{45}a_{53} + a_{14}a_{21}a_{33}a_{42}a_{55} + a_{14}a_{22}a_{31}a_{43}a_{55} + a_{14}a_{22}a_{33}a_{45}a_{51} + a_{14}a_{22}a_{35}a_{41}a_{53} \\ & -a_{14}a_{21}a_{35}a_{42}a_{53} - a_{14}a_{21}a_{32}a_{43}a_{55} - a_{14}a_{21}a_{33}a_{45}a_{52} - a_{14}a_{22}a_{31}a_{45}a_{53} - a_{14}a_{22}a_{33}a_{41}a_{55} - a_{14}a_{22}a_{35}a_{43}a_{51} \\ & + a_{14}a_{23}a_{35}a_{42}a_{51} + a_{14}a_{23}a_{31}a_{45}a_{52} + a_{14}a_{23}a_{32}a_{41}a_{55} + a_{14}a_{25}a_{31}a_{42}a_{53} + a_{14}a_{25}a_{32}a_{43}a_{51} + a_{14}a_{25}a_{33}a_{41}a_{52} \\ & -a_{14}a_{23}a_{35}a_{41}a_{52} - a_{14}a_{23}a_{31}a_{42}a_{55} - a_{14}a_{23}a_{32}a_{45}a_{51} - a_{14}a_{25}a_{31}a_{43}a_{52} - a_{14}a_{25}a_{32}a_{41}a_{53} - a_{14}a_{25}a_{33}a_{42}a_{51} \end{aligned}$$

$$\begin{aligned}
& +a_{15}a_{21}a_{32}a_{43}a_{54} + a_{15}a_{21}a_{33}a_{44}a_{52} + a_{15}a_{21}a_{34}a_{42}a_{53} + a_{15}a_{22}a_{34}a_{43}a_{51} + a_{15}a_{22}a_{33}a_{41}a_{54} + a_{15}a_{22}a_{31}a_{44}a_{53} \\
& -a_{15}a_{21}a_{32}a_{44}a_{53} - a_{15}a_{21}a_{33}a_{42}a_{54} - a_{15}a_{21}a_{34}a_{43}a_{52} - a_{15}a_{22}a_{34}a_{41}a_{53} - a_{15}a_{22}a_{33}a_{44}a_{51} - a_{15}a_{22}a_{31}a_{43}a_{54} \\
& +a_{15}a_{23}a_{31}a_{42}a_{54} + a_{15}a_{23}a_{32}a_{44}a_{51} + a_{15}a_{23}a_{34}a_{41}a_{52} + a_{15}a_{24}a_{33}a_{42}a_{51} + a_{15}a_{24}a_{31}a_{43}a_{52} + a_{15}a_{24}a_{32}a_{41}a_{53} \\
& -a_{15}a_{23}a_{31}a_{44}a_{52} - a_{15}a_{23}a_{32}a_{41}a_{54} - a_{15}a_{23}a_{34}a_{42}a_{51} - a_{15}a_{24}a_{33}a_{41}a_{52} - a_{15}a_{24}a_{31}a_{42}a_{53} - a_{15}a_{24}a_{32}a_{43}a_{51}
\end{aligned}$$

+	-	+	-	+
a_{11}	a_{12}	a_{13}	a_{14}	a_{15}
a_{21}	a_{22}	a_{23}	a_{24}	a_{25}
a_{31}	a_{32}	a_{33}	a_{34}	a_{35}
a_{41}	a_{42}	a_{43}	a_{44}	a_{45}
a_{51}	a_{52}	a_{53}	a_{54}	a_{55}

The Row subscript (first) always will be $123\cdots n$

$$\begin{array}{cccc} a_{1*} a_{2*} a_{3*} \cdots & a_{1*} a_{2*} a_{3*} \cdots & a_{1*} a_{2*} a_{3*} \cdots & \\ -a_{1*} a_{2*} a_{3*} \cdots & -a_{1*} a_{2*} a_{3*} \cdots & -a_{1*} a_{2*} a_{3*} \cdots & \end{array}$$

It is the combination of the second subscript number (column). a_{ij}

$a_{1*} a_{2*} a_{3*} a_{4*} \cdots a_{n*}$ where $***\cdots*$ the ways that we can arrange 1 *through* n

$$****: \begin{array}{cccccc} 1234 & 1324 & 1423 & 2134 & 2314 & 2413 \\ 1243 & 1342 & 1432 & 2143 & 2341 & 2431 \end{array} \cdots$$

First write the first sequence, then when you switch the last 2 numbers (34 to 43) change the sign to negative. However, see below for more when you have to do that.

$$***: \begin{array}{ccc} 123 & 213 & 312 \\ 132 & 231 & 321 \end{array} = 3! \text{ terms}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{array}{ccc} a_{11}a_{22}a_{33} & +a_{12}a_{23}a_{31} & +a_{13}a_{21}a_{32} \\ -a_{11}a_{23}a_{32} & -a_{12}a_{21}a_{33} & -a_{13}a_{22}a_{31} \end{array}$$

$$****: \begin{array}{cccccccccccc} 1234 & 1324 & 1423 & 2134 & 2314 & 2413 & 3124 & 3214 & 3412 & 4123 & 4213 & 4312 \\ 1243 & 1342 & 1432 & 2143 & 2341 & 2431 & 3142 & 3241 & 3421 & 4132 & 4231 & 4321 \end{array} = 4! \text{ terms}$$

Change sign when the subscript
is value is increasing

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{array}{cccc} a_{11}a_{22}a_{33}a_{44} & -a_{12}a_{23}a_{34}a_{41} & a_{13}a_{24}a_{31}a_{42} & -a_{14}a_{21}a_{32}a_{43} \\ -a_{11}a_{22}a_{34}a_{43} & a_{12}a_{23}a_{31}a_{44} & -a_{13}a_{24}a_{32}a_{41} & a_{14}a_{21}a_{33}a_{42} \\ a_{11}a_{23}a_{34}a_{42} & -a_{12}a_{24}a_{31}a_{43} & a_{13}a_{22}a_{34}a_{41} & -a_{14}a_{22}a_{33}a_{41} \\ -a_{11}a_{23}a_{32}a_{44} & a_{12}a_{24}a_{33}a_{41} & -a_{13}a_{22}a_{31}a_{44} & a_{14}a_{22}a_{31}a_{43} \\ a_{11}a_{24}a_{32}a_{43} & -a_{12}a_{21}a_{33}a_{44} & a_{13}a_{21}a_{32}a_{44} & -a_{14}a_{23}a_{31}a_{42} \\ -a_{11}a_{24}a_{33}a_{42} & a_{12}a_{21}a_{34}a_{43} & -a_{13}a_{21}a_{34}a_{42} & a_{14}a_{23}a_{32}a_{41} \end{array}$$

*****: 12345 12453 12534 13245 13452 13524 14235 14325 14523 15234 15342 15423
12354 12435 12543 13254 13452 13542 14253 14352 14532 14243 15324 15432

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} =$$

$a_{11} a_{22} a_{33} a_{44} a_{55}$	$+a_{12} a_{21} a_{35} a_{44} a_{53}$ <i>starts 2-1, down</i>
$-a_{11} a_{22} a_{33} a_{45} a_{54}$	$-a_{12} a_{21} a_{35} a_{43} a_{54}$
$+a_{11} a_{22} a_{34} a_{45} a_{53}$	$+a_{12} a_{21} a_{34} a_{43} a_{55}$
$-a_{11} a_{22} a_{34} a_{43} a_{55}$	$-a_{12} a_{21} a_{34} a_{45} a_{53}$
$+a_{11} a_{22} a_{35} a_{43} a_{54}$	$+a_{12} a_{21} a_{33} a_{45} a_{54}$
$-a_{11} a_{22} a_{35} a_{44} a_{53}$	$-a_{12} a_{21} a_{33} a_{44} a_{55}$
$+a_{11} a_{23} a_{35} a_{44} a_{52}$	$+a_{12} a_{23} a_{34} a_{45} a_{51}$
$-a_{11} a_{23} a_{35} a_{42} a_{54}$	$-a_{12} a_{23} a_{34} a_{41} a_{55}$
$+a_{11} a_{23} a_{32} a_{45} a_{54}$	$+a_{12} a_{23} a_{35} a_{41} a_{54}$
$-a_{11} a_{23} a_{32} a_{44} a_{55}$	$-a_{12} a_{23} a_{35} a_{44} a_{51}$
$+a_{11} a_{23} a_{34} a_{42} a_{55}$	$+a_{12} a_{23} a_{31} a_{44} a_{55}$
$-a_{11} a_{23} a_{34} a_{45} a_{52}$	$-a_{12} a_{23} a_{31} a_{45} a_{54}$
$+a_{11} a_{24} a_{32} a_{43} a_{55}$	$+a_{12} a_{24} a_{33} a_{41} a_{55}$
$-a_{11} a_{24} a_{32} a_{45} a_{53}$	$-a_{12} a_{24} a_{33} a_{45} a_{51}$
$+a_{11} a_{24} a_{33} a_{45} a_{52}$	$+a_{12} a_{24} a_{35} a_{43} a_{51}$
$-a_{11} a_{24} a_{33} a_{42} a_{55}$	$-a_{12} a_{24} a_{35} a_{41} a_{53}$
$+a_{11} a_{24} a_{35} a_{42} a_{53}$	$+a_{12} a_{24} a_{31} a_{45} a_{53}$
$-a_{11} a_{24} a_{35} a_{43} a_{52}$	$-a_{12} a_{24} a_{31} a_{43} a_{55}$
$+a_{11} a_{25} a_{34} a_{43} a_{52}$	$+a_{12} a_{25} a_{31} a_{43} a_{54}$
$-a_{11} a_{25} a_{34} a_{42} a_{53}$	$-a_{12} a_{25} a_{31} a_{44} a_{53}$
$+a_{11} a_{25} a_{32} a_{44} a_{53}$	$+a_{12} a_{25} a_{33} a_{44} a_{51}$
$-a_{11} a_{25} a_{32} a_{43} a_{54}$	$-a_{12} a_{25} a_{33} a_{41} a_{54}$
$+a_{11} a_{25} a_{33} a_{42} a_{54}$	$+a_{12} a_{25} a_{34} a_{41} a_{53}$
$-a_{11} a_{25} a_{33} a_{44} a_{52}$	$-a_{12} a_{25} a_{34} a_{43} a_{51}$

Graphically

$\begin{vmatrix} \textcircled{a_{11}} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$	$\begin{aligned} &+a_{11}a_{22}a_{33}a_{44} \\ &+a_{11}a_{23}a_{34}a_{42} \\ &+a_{11}a_{24}a_{32}a_{43} \end{aligned}$	$\begin{vmatrix} \textcircled{a_{11}} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$	$\begin{aligned} &-a_{11}a_{22}a_{34}a_{43} \\ &-a_{11}a_{23}a_{32}a_{44} \\ &-a_{11}a_{24}a_{33}a_{42} \end{aligned}$
$\begin{vmatrix} a_{11} & \textcircled{a_{12}} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$	$\begin{aligned} &+a_{12}a_{23}a_{31}a_{44} \\ &+a_{12}a_{24}a_{33}a_{41} \\ &+a_{12}a_{21}a_{34}a_{43} \end{aligned}$	$\begin{vmatrix} a_{11} & \textcircled{a_{12}} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$	$\begin{aligned} &-a_{12}a_{23}a_{34}a_{41} \\ &-a_{12}a_{24}a_{31}a_{43} \\ &-a_{12}a_{21}a_{33}a_{44} \end{aligned}$
$\begin{vmatrix} a_{11} & a_{12} & \textcircled{a_{13}} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$	$\begin{aligned} &+a_{13}a_{24}a_{31}a_{42} \\ &+a_{13}a_{22}a_{34}a_{41} \\ &+a_{13}a_{21}a_{32}a_{44} \end{aligned}$	$\begin{vmatrix} a_{11} & a_{12} & \textcircled{a_{13}} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$	$\begin{aligned} &-a_{13}a_{24}a_{32}a_{41} \\ &-a_{13}a_{22}a_{31}a_{44} \\ &-a_{13}a_{21}a_{34}a_{42} \end{aligned}$
$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \textcircled{a_{14}} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$	$\begin{aligned} &+a_{14}a_{21}a_{33}a_{42} \\ &+a_{14}a_{22}a_{31}a_{43} \\ &+a_{14}a_{23}a_{32}a_{41} \end{aligned}$	$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \textcircled{a_{14}} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$	$\begin{aligned} &-a_{14}a_{21}a_{32}a_{43} \\ &-a_{14}a_{22}a_{33}a_{41} \\ &-a_{14}a_{23}a_{31}a_{42} \end{aligned}$