

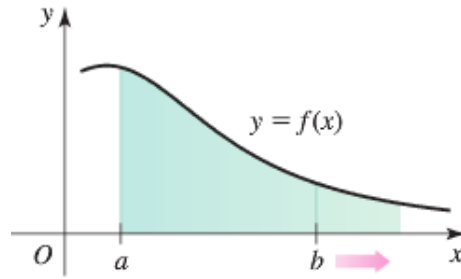
## Section 2.6 – Improper Integrals

### Definition

Integrals with infinite limits of integration are **improper integrals**.

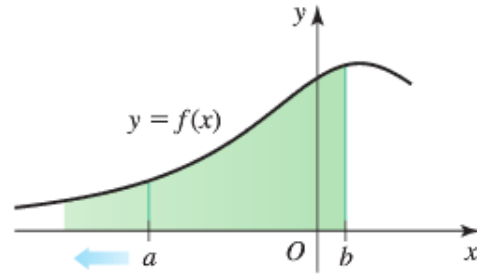
1. If  $f(x)$  is continuous on  $[a, \infty)$ , then

$$\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$$



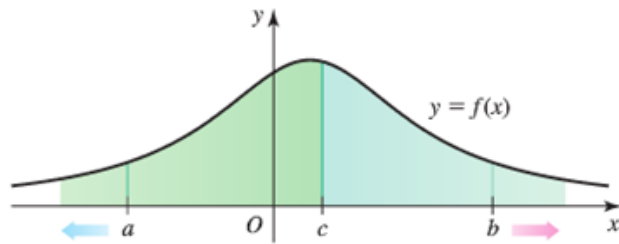
2. If  $f(x)$  is continuous on  $(-\infty, b]$ , then

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$$



3. If  $f(x)$  is continuous on  $(-\infty, \infty)$ , then

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx$$



In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

### Example

Is the area under the curve  $y = \frac{\ln x}{x^2}$  from  $x = 1$  to  $x = \infty$  finite? If so, what is its value?

### Solution

$$\begin{aligned} \int_1^b \frac{\ln x}{x^2} dx &= -\frac{1}{x} \ln x \Big|_1^b - \int_1^b \left(-\frac{1}{x}\right) \left(\frac{1}{x}\right) dx \\ &= -\left(\frac{1}{b} \ln b - \ln 1\right) + \int_1^b \frac{1}{x^2} dx \\ &= -\frac{1}{b} \ln b + \left[-\frac{1}{x}\right]_1^b \\ &= -\frac{1}{b} \ln b - \left(\frac{1}{b} - 1\right) \\ &= -\frac{1}{b} \ln b - \frac{1}{b} + 1 \end{aligned}$$

$$\begin{aligned} u &= \ln x & dv &= \frac{dx}{x^2} \\ du &= \frac{1}{x} dx & v &= -\frac{1}{x} \end{aligned}$$

$$\begin{aligned}
\int_1^{\infty} \frac{\ln x}{x^2} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x^2} dx \\
&= \lim_{b \rightarrow \infty} \left( -\frac{1}{b} \ln b - \frac{1}{b} + 1 \right) \\
&= - \lim_{b \rightarrow \infty} \left( \frac{1/b}{1} \right) - 0 + 1 \\
&= -0 + 1 \\
&= \underline{1}
\end{aligned}$$

*L'Hôpital Rule*

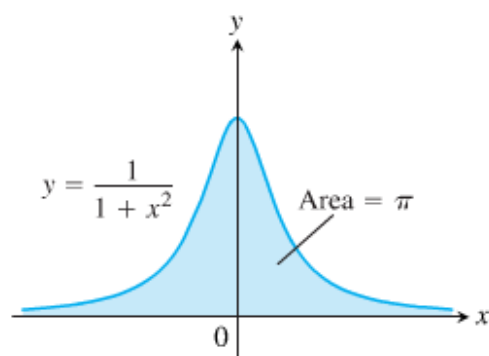
### Example

Evaluate  $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

### Solution

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{1+x^2} &= \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^{\infty} \frac{dx}{1+x^2} \\
\int_{-\infty}^0 \frac{dx}{1+x^2} &= \lim_{a \rightarrow -\infty} \int_a^0 \frac{dx}{1+x^2} \\
&= \lim_{a \rightarrow -\infty} \tan^{-1} x \Big|_a^0 \\
&= \lim_{a \rightarrow -\infty} \left( \tan^{-1} 0 - \tan^{-1} a \right) \\
&= 0 - \left( -\frac{\pi}{2} \right) \\
&= \underline{\frac{\pi}{2}}
\end{aligned}$$

$$\begin{aligned}
\int_0^{\infty} \frac{dx}{1+x^2} &= \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{1+x^2} \\
&= \lim_{b \rightarrow \infty} \tan^{-1} x \Big|_0^b \\
&= \lim_{b \rightarrow \infty} \left( \tan^{-1} b - \tan^{-1} 0 \right) \\
&= \frac{\pi}{2} - 0 \\
&= \underline{\frac{\pi}{2}}
\end{aligned}$$



$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{dx}{1+x^2} &= \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^{\infty} \frac{dx}{1+x^2} \\
 &= \frac{\pi}{2} + \frac{\pi}{2} \\
 &= \pi
 \end{aligned}$$

### Example

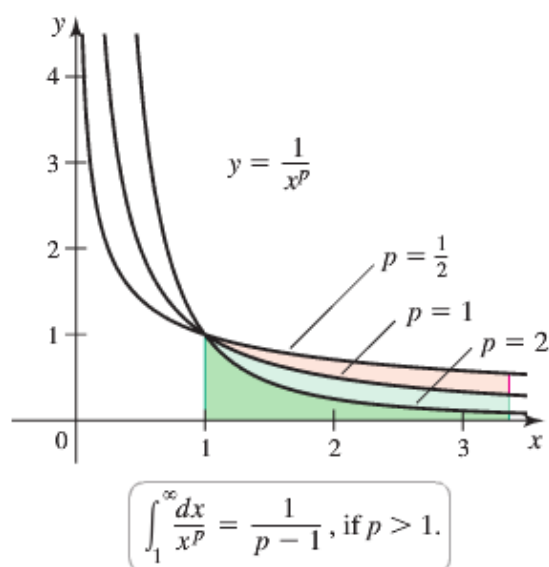
For what value of  $p$  does the integral  $\int_1^{\infty} \frac{dx}{x^p}$  converge? When the integral does converge, what is its value?

### Solution

$$\text{If } p \neq 1 \quad \int_1^b \frac{dx}{x^p} = \left. \frac{x^{-p+1}}{-p+1} \right|_1^b = \frac{1}{1-p} (b^{1-p} - 1)$$

$$\begin{aligned}
 \int_1^{\infty} \frac{dx}{x^p} &= \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x^p} \\
 &= \lim_{b \rightarrow \infty} \left[ \frac{1}{1-p} (b^{1-p} - 1) \right] \\
 &= \begin{cases} \frac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 \text{If } p = 1 \quad \int_1^{\infty} \frac{dx}{x^p} &= \int_1^{\infty} \frac{dx}{x} \\
 &= \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x} \\
 &= \lim_{b \rightarrow \infty} [\ln x]_1^b \\
 &= \lim_{b \rightarrow \infty} (\ln b - \ln 1) \\
 &= \infty
 \end{aligned}$$



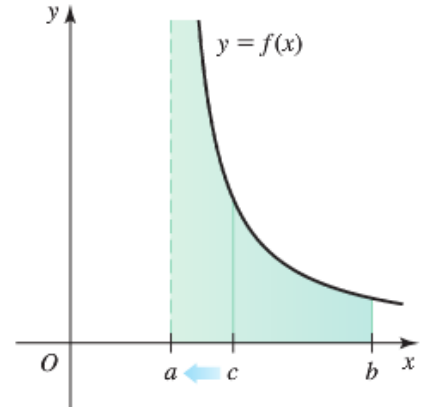
## Integrands with Vertical Asymptotes

### Definition

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals**. If the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

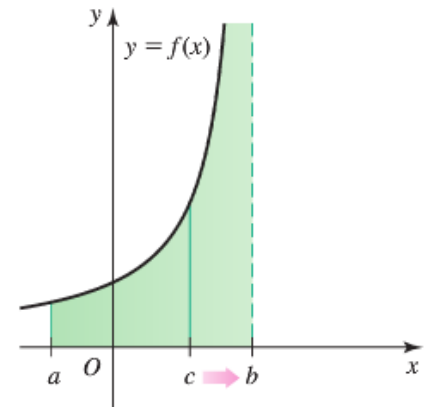
1. If  $f(x)$  is continuous on  $(a, b]$ , then

$$\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx$$



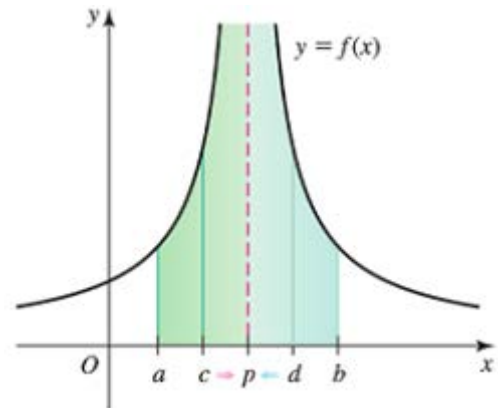
2. If  $f(x)$  is continuous on  $[a, b)$ , then

$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx$$



3. If  $f(x)$  is continuous on  $[a, b]$ , then

$$\int_a^b f(x) dx = \lim_{c \rightarrow p^-} \int_a^c f(x) dx + \lim_{d \rightarrow p^+} \int_d^b f(x) dx$$



### Example

Investigate the convergence of  $\int_0^1 \frac{1}{1-x} dx$

### Solution

$$\begin{aligned}\int_0^1 \frac{1}{1-x} dx &= \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{1-x} dx \\&= \lim_{b \rightarrow 1^-} \left[ -\ln|1-x| \right]_0^b \\&= \lim_{b \rightarrow 1^-} \left[ -\ln|1-b| + 0 \right] \\&= \underline{\underline{\infty}}\end{aligned}$$

The limit is infinite, so the integral diverges.

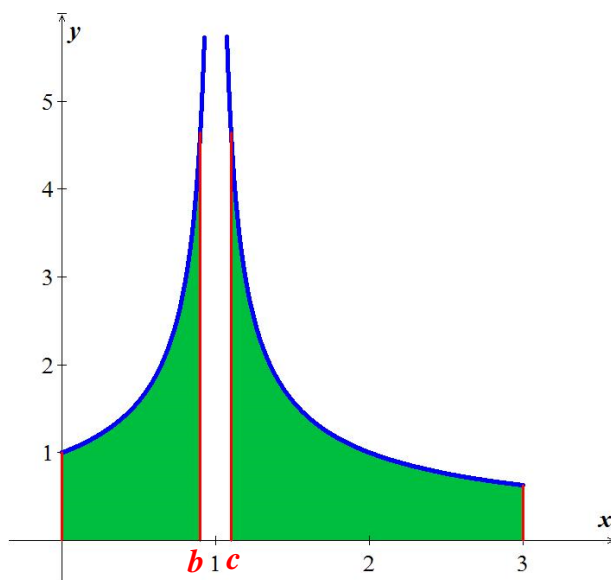
### Example

Evaluate  $\int_0^3 \frac{dx}{(x-1)^{2/3}}$

### Solution

The integrand has a vertical asymptote at  $x = 1$  and is continuous on  $[0, 1)$  and  $(1, 3]$ .

$$\begin{aligned}\int \frac{dx}{(x-1)^{2/3}} &= \int (x-1)^{-2/3} d(x-1) = 3(x-1)^{1/3} \\ \int_0^3 \frac{dx}{(x-1)^{2/3}} &= \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}} \\&= \left[ 3(x-1)^{1/3} \right]_0^{1^-} + \left[ 3(x-1)^{1/3} \right]_{1^+}^3 \\&= 3(0+1) + 3\left(\sqrt[3]{2} - 0\right) \\&= \underline{\underline{3 + 3\sqrt[3]{2}}}\end{aligned}$$



### Example

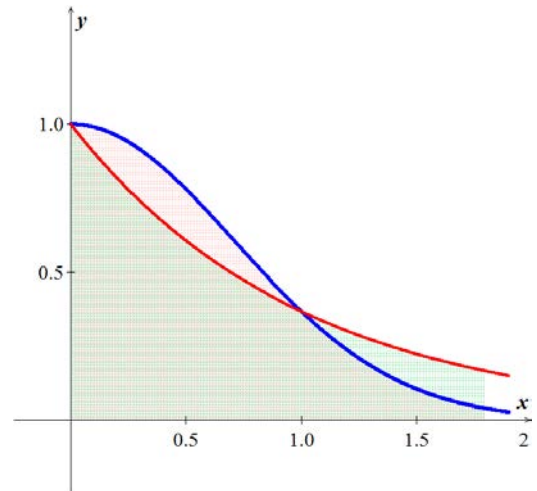
Does the integral  $\int_1^{\infty} e^{-x^2} dx$  converge?

### Solution

$$\int_1^{\infty} e^{-x^2} dx = \lim_{b \rightarrow \infty} \int_1^b e^{-x^2} dx$$

$$\int_1^b e^{-x^2} dx \leq \int_1^b e^{-x} dx = -e^{-b} + e^{-1} < e^{-1} \approx 0.36788$$

The integral converges



### Theorem – Direct Comparison Test

Let  $f$  and  $g$  be continuous on  $[a, \infty)$  with  $0 \leq f(x) \leq g(x)$  for all  $x \geq a$ . Then

1.  $\int_a^{\infty} f(x) dx$  converges if  $\int_a^{\infty} g(x) dx$  converges

2.  $\int_a^{\infty} g(x) dx$  diverges if  $\int_a^{\infty} f(x) dx$  diverges

### Theorem – Limit Comparison Test

If the positive functions  $f$  and  $g$  are continuous on  $[a, \infty)$ , and if

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty$$

Then

$$\int_a^{\infty} f(x) dx \quad \text{and} \quad \int_a^{\infty} g(x) dx$$

Both converge or both diverge

### Example

Show that  $\int_1^{\infty} \frac{dx}{1+x^2}$  converges by comparison with  $\int_1^{\infty} \frac{dx}{x^2}$ . Find and compare the two integral values.

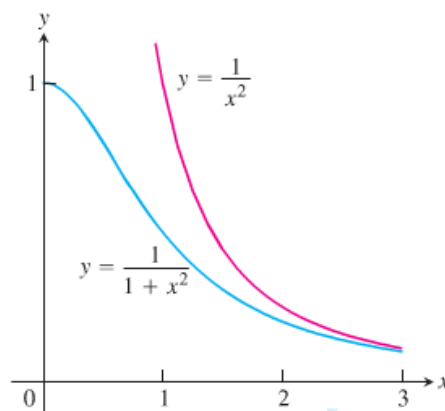
### Solution

The functions  $f(x) = \frac{1}{x^2}$  and  $g(x) = \frac{1}{1+x^2}$  are positive and continuous on  $[1, \infty)$ . Also,

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow \infty} \frac{\frac{1}{x^2}}{\frac{1}{1+x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{1+x^2}{x^2} \\ &= \underline{1}\end{aligned}$$

Therefore,  $\int_1^{\infty} \frac{dx}{1+x^2}$  converges because  $\int_1^{\infty} \frac{dx}{x^2}$  converges.

$$\begin{aligned}\int_1^{\infty} \frac{dx}{1+x^2} &= \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{1+x^2} \\ &= \lim_{b \rightarrow \infty} \left( \tan^{-1} b - \tan^{-1} 1 \right) \\ &= \frac{\pi}{2} - \frac{\pi}{4} \\ &= \underline{\frac{\pi}{4}}\end{aligned}$$



### Example

Let  $R$  be the region bounded by the graph of  $y = x^{-1}$  and the  $x$ -axis, for  $x \geq 1$ .

- What is the volume of the solid generated when  $R$  is revolved about the  $x$ -axis?
- What is the surface area of the solid generated when  $R$  is revolved about the  $x$ -axis?
- What is the volume of the solid generated when  $R$  is revolved about the  $y$ -axis?

### Solution

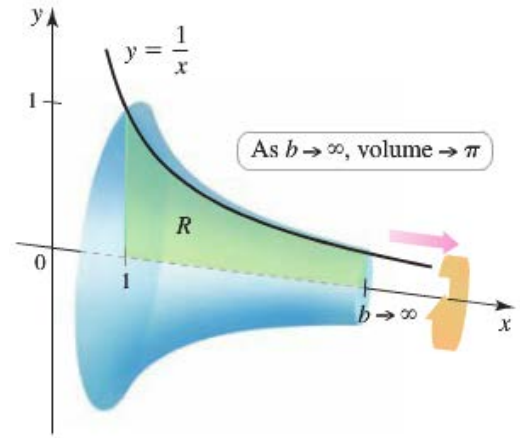
$$a) \quad V = \pi \int_1^{\infty} \frac{1}{x^2} dx$$

$$= -\pi \frac{1}{x} \Big|_1^{\infty}$$

$$= -\pi(0 - 1)$$

$$= \pi \text{ unit}^3$$

$$V = \pi \int_a^b (f(x))^2 dx$$



$$b) \quad S = 2\pi \int_1^{\infty} \frac{1}{x} \sqrt{1 + \left(-\frac{1}{x^2}\right)^2} dx$$

$$= 2\pi \int_1^{\infty} \frac{1}{x^3} \sqrt{x^4 + 1} dx$$

$$> 2\pi \int_1^{\infty} \frac{x^2}{x^3} dx \quad \sqrt{x^4 + 1} > \sqrt{x^4} = x^2$$

$$= 2\pi \int_1^{\infty} \frac{1}{x} dx$$

$$= 2\pi (\ln x) \Big|_1^{\infty}$$

$$= \infty \text{ unit}^2$$

$$S = 2\pi \int_a^b f(x) \sqrt{1 + f'(x)^2} dx$$

$$c) \quad V = 2\pi \int_1^{\infty} x \frac{1}{x} dx$$

$$= 2\pi x \Big|_1^{\infty}$$

$$= \infty \text{ unit}^3$$

$$V = 2\pi \int_a^b x \cdot f(x) dx \quad (\text{Shell method})$$



## Exercises      Section 2.6 – Improper Integrals

(1 – 81)      Evaluate the integrals

1.  $\int_0^{\infty} \frac{dx}{x^2 + 1}$

2.  $\int_0^4 \frac{dx}{\sqrt{4-x}}$

3.  $\int_{-\infty}^2 \frac{2dx}{x^2 + 4}$

4.  $\int_{-\infty}^{\infty} \frac{xdx}{(x^2 + 4)^{3/2}}$

5.  $\int_1^{\infty} \frac{dx}{x\sqrt{x^2 - 1}}$

6.  $\int_{-\infty}^{\infty} 2xe^{-x^2} dx$

7.  $\int_0^1 (-\ln x) dx$

8.  $\int_{-1}^4 \frac{dx}{\sqrt{|x|}}$

9.  $\int_0^{\infty} e^{-3x} dx$

10.  $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

11.  $\int_1^{10} \frac{dx}{(x-2)^{1/3}}$

12.  $\int_1^{\infty} \frac{dx}{x^2}$

13.  $\int_0^{\infty} \frac{dx}{(x+1)^3}$

14.  $\int_{-\infty}^0 e^x dx$

15.  $\int_1^{\infty} 2^{-x} dx$

16.  $\int_{-\infty}^0 \frac{dx}{\sqrt[3]{2-x}}$

17.  $\int_{4/\pi}^{\infty} \frac{1}{x^2} \sec^2\left(\frac{1}{x}\right) dx$

18.  $\int_{e^2}^{\infty} \frac{dx}{x \ln^p x} \quad p > 1$

19.  $\int_0^{\infty} \frac{p}{\sqrt[5]{p^2 + 1}} dp$

20.  $\int_{-1}^1 \ln y^2 dy$

21.  $\int_{-2}^6 \frac{dx}{\sqrt{|x-2|}}$

22.  $\int_0^{\infty} xe^{-x} dx$

23.  $\int_0^1 x \ln x dx$

24.  $\int_0^1 x \ln x dx$

25.  $\int_1^{\infty} \frac{\ln x}{x^2} dx$

26.  $\int_1^{\infty} (1-x)e^x dx$

27.  $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx$

28.  $\int_0^1 \frac{dx}{\sqrt[3]{x}}$

29.  $\int_1^{\infty} \frac{3}{\sqrt[3]{x}} dx$

30.  $\int_1^{\infty} \frac{4}{\sqrt[4]{x}} dx$

31.  $\int_0^2 \frac{dx}{x^3}$

32.  $\int_1^{\infty} \frac{dx}{x^3}$

33.  $\int_1^{\infty} \frac{6}{x^4} dx$

34.  $\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)}$

35.  $\int_{-\infty}^0 xe^{-4x} dx$

36.  $\int_0^{\infty} x e^{-x/3} dx$
37.  $\int_0^{\infty} x^2 e^{-x} dx$
38.  $\int_0^{\infty} e^{-x} \cos x dx$
39.  $\int_4^{\infty} \frac{1}{x(\ln x)^3} dx$
40.  $\int_1^{\infty} \frac{\ln x}{x} dx$
41.  $\int_{-\infty}^{\infty} \frac{4}{16+x^2} dx$
42.  $\int_0^{\infty} \frac{x^3}{(x^2+1)^2} dx$
43.  $\int_0^{\infty} \frac{1}{e^x + e^{-x}} dx$
44.  $\int_0^{\infty} \frac{e^x}{1+e^x} dx$
45.  $\int_0^{\infty} \cos \pi x dx$
46.  $\int_0^{\infty} \sin \frac{x}{2} dx$
47.  $\int_1^{\infty} \frac{dx}{(x+1)^9}$
48.  $\int_1^{\infty} \frac{3x-1}{4x^3-x^2} dx$
49.  $\int_{-\infty}^{\infty} \frac{4}{x^2+16} dx$
50.  $\int_{-\infty}^{-1} \frac{dx}{(x-1)^4}$
51.  $\int_0^{\infty} x e^{-x} dx$
52.  $\int_0^{\infty} \frac{6x}{1+x^6} dx$
53.  $\int_0^2 \frac{dx}{\sqrt[3]{|x-1|}}$
54.  $\int_{-1}^1 \frac{dx}{x^2+2x+5}$
55.  $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+5}$
56.  $\int_0^{\infty} \cos x dx$
57.  $\int_2^{\infty} \frac{\cos\left(\frac{\pi}{x}\right)}{x^2} dx$
58.  $\int_{-\infty}^a \sqrt{e^x} dx$
59.  $\int_0^{\infty} \frac{e^x}{e^{2x}+1} dx$
60.  $\int_1^{\infty} \frac{dx}{x(x+1)}$
61.  $\int_1^{\infty} \frac{dx}{x^2(x+1)}$
62.  $\int_1^{\infty} \frac{3x^2+1}{x^3+x} dx$
63.  $\int_1^{\infty} \frac{1}{x^2} \sin \frac{\pi}{x} dx$
64.  $\int_2^{\infty} \frac{dx}{(x+2)^2}$
65.  $\int_1^{\infty} \frac{\tan^{-1} x}{x^2+1} dx$
66.  $\int_{-3}^1 \frac{dx}{(2x+6)^{2/3}}$
67.  $\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$
68.  $\int_0^{\ln 3} \frac{e^x}{(e^x-1)^{2/3}} dx$
69.  $\int_1^2 \frac{dx}{\sqrt{x-1}}$
70.  $\int_{-1}^1 \frac{dx}{x^2}$
71.  $\int_0^2 \frac{dx}{(x-1)^2}$
72.  $\int_{-1}^2 \frac{dx}{(x-1)^2}$
73.  $\int_1^{\infty} \frac{dx}{x \sqrt{x^2-1}}$
74.  $\int_0^{\infty} x e^{-x^2} dx$

$$75. \int_{-\infty}^{\infty} x e^{-x^2} dx$$

$$76. \int_{-\infty}^{\infty} \frac{x}{x^2 + 1} dx$$

$$77. \int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2}$$

$$78. \int_{-\infty}^{\infty} \frac{dx}{x^2 + 6x + 12}$$

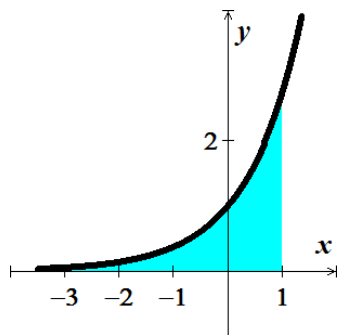
$$79. \int \frac{dx}{2 - \sqrt{3x}}$$

$$80. \int \theta \cos(2\theta + 1) d\theta$$

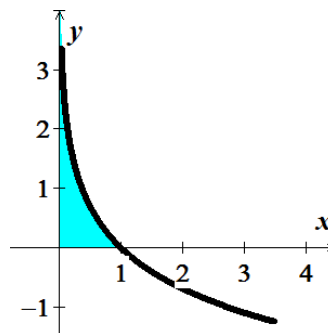
$$81. \int \sqrt{x} \sqrt{1 + \sqrt{x}} dx$$

(82 – 85) Find the area of the unbounded shaded region

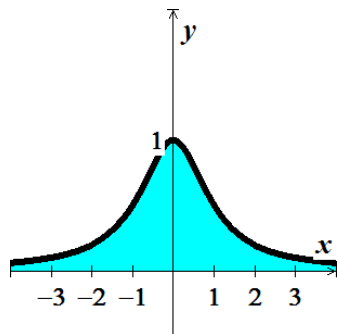
$$82. y = e^x, \quad -\infty < x \leq 1$$



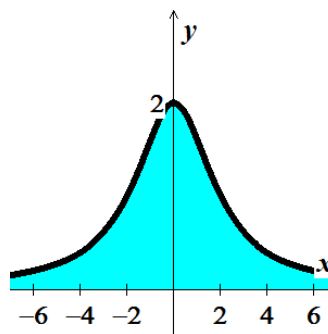
$$83. y = -\ln x$$



$$84. y = \frac{1}{x^2 + 1}$$



$$85. y = \frac{8}{x^2 + 4}$$



86. Find the area of the region  $R$  between the graph of  $f(x) = \frac{1}{\sqrt{9 - x^2}}$  and the  $x$ -axis on the interval  $(-3, 3)$  (if it exists)

87. Find the volume of the region bounded by  $f(x) = (x^2 + 1)^{-1/2}$  and the  $x$ -axis on the interval  $[2, \infty)$  is revolved about the  $x$ -axis.

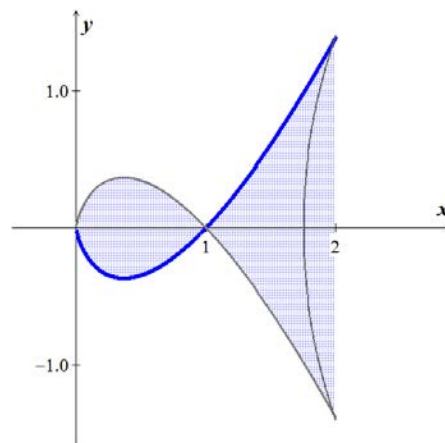
88. Find the volume of the region bounded by  $f(x) = \sqrt{\frac{x+1}{x^3}}$  and the  $x$ -axis on the interval  $[1, \infty)$  is revolved about the  $x$ -axis .
89. Find the volume of the region bounded by  $f(x) = (x+1)^{-3}$  and the  $x$ -axis on the interval  $[0, \infty)$  is revolved about the  $y$ -axis .
90. Find the volume of the region bounded by  $f(x) = \frac{1}{\sqrt{x} \ln x}$  and the  $x$ -axis on the interval  $[2, \infty)$  is revolved about the  $x$ -axis .
91. Find the volume of the region bounded by  $f(x) = \frac{\sqrt{x}}{\sqrt[3]{x^2+1}}$  and the  $x$ -axis on the interval  $[0, \infty)$  is revolved about the  $x$ -axis .
92. Find the volume of the region bounded by  $f(x) = (x^2 - 1)^{-1/4}$  and the  $x$ -axis on the interval  $(1, 2]$  is revolved about the  $y$ -axis .
93. Find the volume of the region bounded by  $f(x) = \tan x$  and the  $x$ -axis on the interval  $\left[0, \frac{\pi}{2}\right)$  is revolved about the  $x$ -axis .
94. Find the volume of the region bounded by  $f(x) = -\ln x$  and the  $x$ -axis on the interval  $(0, 1]$  is revolved about the  $x$ -axis .
95. Find the volume of the solid generated by revolving the region bounded by the graphs of  $y = xe^{-x}$ ,  $y = 0$ , and  $x = 0$  about the  $x$ -axis .

96. The region between the  $x$ -axis and the curve

$$f(x) = \begin{cases} 0, & x = 0 \\ x \ln x, & 0 < x \leq 2 \end{cases}$$

is revolved about the  $x$ -axis to generate the solid.

Find the volume of the solid.



(97 – 98) Consider the region satisfying the inequalities

- a) Find the area of the region
- b) Find the volume of the solid generated by revolving the region about the  $x$ -axis .
- c) Find the volume of the solid generated by revolving the region about the  $y$ -axis .

97.  $y \leq e^{-x}, \quad y \geq 0, \quad x \geq 0$

98.  $y \leq \frac{1}{x^2}, \quad y \geq 0, \quad x \geq 1$

99. Find the perimeter of the hypocycloid of four cusps  $x^{2/3} + y^{2/3} = 4$

100. Find the arc length of the graph  $y = \sqrt{16 - x^2}$  over the interval  $[0, 4]$

101. The region bounded by  $(x - 2)^2 + y^2 = 1$  is revolved about the  $y$ -axis to form a torus. Find the surface area of the torus.

102. Find the surface area formed by revolving the graph  $y = 2e^{-x}$  on the interval  $[0, \infty)$  about the  $x$ -axis

103. The magnetic potential  $P$  at a point on the axis of a circular coil is given by

$$P = \frac{2\pi N I r}{k} \int_c^\infty \frac{1}{(r^2 + x^2)^{3/2}} dx$$

Where  $N, I, r, k$ , and  $c$  are constants. Find  $P$ .

104. A “semi-infinite” uniform rod occupies the nonnegative  $x$ -axis . The rod has a linear density  $\delta$ , which means that a segment of length  $dx$  has a mass of  $\delta dx$  . A particle of mass  $M$  is located at the point  $(-a, 0)$  . The gravitational force  $F$  that the rod exerts on the mass is given by

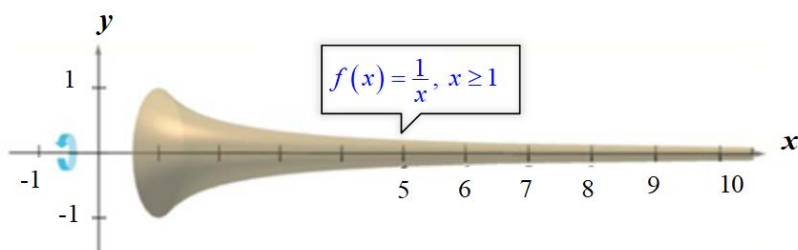
$$F = \int_0^\infty \frac{GM\delta}{(a+x)^2} dx$$

Where  $G$  is the gravitational constant. Find  $F$ .

105. Let  $R$  be the region bounded by the graph of  $f(x) = x^{-p}$  and the  $x$ -axis

- a) Let  $S$  be the solid generated when  $R$  is revolved about the  $x$ -axis . For what values of  $p$  is the volume of  $S$  finite for  $0 < x \leq 1$  ?
- b) Let  $S$  be the solid generated when  $R$  is revolved about the  $y$ -axis . For what values of  $p$  is the volume of  $S$  finite for  $0 < x \leq 1$  ?
- c) Let  $S$  be the solid generated when  $R$  is revolved about the  $x$ -axis . For what values of  $p$  is the volume of  $S$  finite for  $x \geq 1$  ?
- d) Let  $S$  be the solid generated when  $R$  is revolved about the  $y$ -axis . For what values of  $p$  is the volume of  $S$  finite for  $x \geq 1$  ?

- 106.** The solid formed by revolving (about the  $x$ -axis) the unbounded region lying between the graph of  $f(x) = \frac{1}{x}$  and the  $x$ -axis ( $x \geq 1$ ) is called **Gabriel's Horn**.



Show that this solid has a finite volume and an infinite surface area.

- 107.** Water is drained from a 3000-gal tank at a rate that starts at 100 gal/hr. and decreases continuously by 5% /hr. If the drain left open indefinitely, how much water drains from the tank? Can a full tank be emptied at this rate?

**108.** Let  $I(a) = \int_0^{\infty} \frac{dx}{(1+x^a)(1+x^2)}$ , where  $a$  is a real number.

- a) Evaluate  $I(a)$  and show that its value is independent of  $a$ .

(**Hint:** split the integral into two integrals over  $[0, 1]$  and  $[1, \infty)$ ; then use a change of variables to convert the second integral into an integral over  $[0, 1]$ .)

- b) Let  $f$  be any positive continuous function on  $\left[0, \frac{\pi}{2}\right]$

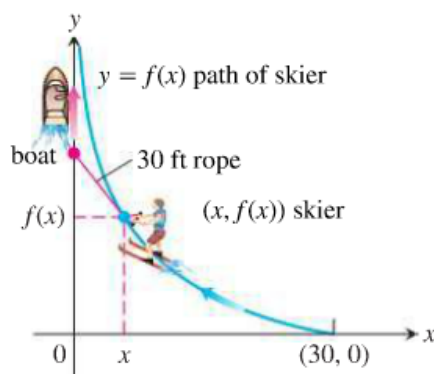
Evaluate  $\int_0^{\pi/2} \frac{f(\cos x)}{f(\cos x) + f(\sin x)} dx$

(**Hint:** Use the identity  $\cos\left(\frac{\pi}{2} - x\right) = \sin x$ )

- 109.** Let  $R$  be the region bounded by  $y = \ln x$ , the  $x$ -axis, and the line  $x = a$ , where  $a > 1$ .

- a) Find the volume  $V_1(a)$  of the solid generated when  $R$  is revolved about the  $x$ -axis (as a function of  $a$ ).
- b) Find the volume  $V_2(a)$  of the solid generated when  $R$  is revolved about the  $y$ -axis (as a function of  $a$ ).
- c) Graph  $V_1$  and  $V_2$ . For what values of  $a > 1$  is  $V_1(a) > V_2(a)$ ?

- 110.** Let  $R$  be the region bounded by the graph of  $f(x) = x^{-p}$  and the  $x$ -axis, for  $x \geq 1$ . Let  $V_1$  and  $V_2$  be the volumes of the solids generated when  $R$  is revolved about the  $x$ -axis and the  $y$ -axis, respectively, if they exist.
- For what values of  $p$  (if any) is  $V_1 = V_2$ ?
  - Repeat part (a) on the interval  $(0, 1]$ .
- 111.** Let  $R_1$  be the region bounded by the graph of  $y = e^{-ax}$  and the  $x$ -axis on the interval  $[0, b]$  where  $a > 0$  and  $b > 0$ . Let  $R_2$  be the region bounded by the graph of  $y = e^{-ax}$  and the  $x$ -axis on the interval  $[b, \infty)$ . Let  $V_1$  and  $V_2$  be the volumes of the solids generated when  $R_1$  and  $R_2$  are revolved about the  $x$ -axis. Find and graph the relationship between  $a$  and  $b$  for which  $V_1 = V_2$ .
- 112.** Suppose that a boat is positioned at the origin with a water skier tethered to the boat at the point  $(30, 0)$  on a rope 30 feet long. As the boat travels along the positive  $y$ -axis, the skier is pulled behind the boat along an unknown path  $y = f(x)$ , as shown



a) Show that  $f'(x) = \frac{-\sqrt{900 - x^2}}{x}$

(Hint: Assume that the skier is always pointed directly at the boat and the rope is on line is on a line tangent to the path  $y = f(x)$ .)

b) Solve the equation in part (a) for  $f(x)$ , using  $f(30) = 0$

- 113.** Many chemical reactions are the result of the interaction of 2 molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If  $a$  is the amount of substance  $A$  and  $b$  is the substance  $B$  at time  $t = 0$ , and if  $x$  is the amount of product at time  $t$ , then the rate of formation of  $x$  may be given by the differential equation

$$\frac{dx}{dt} = k(a-x)(b-x) \quad \text{or} \quad \frac{1}{(a-x)(b-x)} \frac{dx}{dt} = k$$

Where  $k$  is a constant for the reaction. Integrate both sides of this equation to obtain a relation between  $x$  and  $t$ .

a) If  $a = b$

b) If  $a \neq b$

Assume in each case that  $x = 0$  when  $t = 0$