Solution Section 3.4 – L'Hôpital's Rule

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to -2} \dfrac{x+2}{x^2-4}$

Solution

$$\lim_{x\to -2} \frac{x+2}{x^2-4} = \frac{1}{2x}\bigg|_{x=-2} \qquad \frac{(x+2)'}{\left(x^2-4\right)'} = \frac{1}{2x}$$

$$= -\frac{1}{4}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to 1} \dfrac{x^3-1}{4x^3-x-3}$

Solution

$$\lim_{x\to 1} \frac{x^3-1}{4x^3-x-3} = \lim_{x\to 1} \frac{3x^2}{12x^2-1}$$

$$= \frac{3}{11}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to -5} \dfrac{x^2-25}{x+5}$

Solution

$$\lim_{x\to -5} \frac{x^2-25}{x+5} = \lim_{x\to -5} \frac{2x}{1}$$

$$= -10$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{t\to 0} \dfrac{\sin 5t}{2t}$

Solution

$$\lim_{t\to 0} \frac{\sin 5t}{2t} = \lim_{t\to 0} \frac{5\cos 5t}{2}$$

$$= \frac{5}{2}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{\theta \to -\pi/3} \dfrac{3\theta + \pi}{\sin\left(\theta + \frac{\pi}{3}\right)}$

Solution

$$\lim_{\theta \to -\pi/3} \frac{3\theta + \pi}{\sin\left(\theta + \frac{\pi}{3}\right)} = \lim_{\theta \to -\pi/3} \frac{3}{\cos\left(\theta + \frac{\pi}{3}\right)}$$

$$= 3$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x \to 0} \dfrac{x^2}{\ln(\sec x)}$

Solution

$$\lim_{x \to 0} \frac{x^2}{\ln(\sec x)} = \lim_{x \to 0} \frac{2x}{\frac{\sec x \tan x}{\sec x}}$$

$$= \lim_{x \to 0} \frac{2x}{\tan x} = \frac{0}{0} \qquad\qquad \frac{(2x)'}{(\tan x)'} = \frac{2}{\sec^2 x}$$

$$= \lim_{x \to 0} \frac{2}{\sec^2 x}$$

$$= \frac{2}{1}$$

$$= 2$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{\theta \to 0} \dfrac{3^{\sin\theta} - 1}{\theta}$

Solution

$$\lim_{\theta \to 0} \frac{3^{\sin\theta} - 1}{\theta} = \lim_{\theta \to 0} \frac{3^{\sin\theta} \ln(3)(\cos\theta)}{1}$$

$$= 3^{\sin 0} \ln(3)(\cos 0)$$

$$= \ln(3)$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to 0} \dfrac{3^x - 1}{2^x - 1}$

Solution

$$\lim_{x\to 0} \frac{3^x - 1}{2^x - 1} = \lim_{x\to 0} \frac{3^x \ln(3)}{2^x \ln(2)}$$

$$= \frac{3^0 \ln 3}{2^0 \ln 2}$$

$$= \frac{\ln 3}{\ln 2} \qquad = \log_2^3$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to 0^+} (\ln x - \ln \sin x)$

Solution

$$\lim_{x\to 0^+} (\ln x - \ln \sin x) = \lim_{x\to 0^+} \left(\ln \frac{x}{\sin x}\right)$$

$$= \ln \lim_{x\to 0^+} \left(\frac{x}{\sin x}\right) = \ln \frac{0}{0}$$

$$= \ln \lim_{x\to 0^+} \left(\frac{1}{\cos x}\right)$$

$$= \ln(1)$$

$$= 0$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to 0} \dfrac{\left(e^x - 1\right)^2}{x \sin x}$

Solution

$$\lim_{x\to 0} \frac{\left(e^x - 1\right)^2}{x \sin x} = \lim_{x\to 0} \frac{2\left(e^x - 1\right)e^x}{\sin x + x\cos x}$$

$$= \frac{2(1-1)}{0+0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{2e^{2x} - 2e^x}{\sin x + x \cos x}$$

$$= \lim_{x \to 0} \frac{4e^{2x} - 2e^{x}}{\cos x + \cos x - x\sin x}$$

$$= \frac{4-2}{1+1-0}$$

$$\underline{=1}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x \to \pi/2^{-}} \dfrac{1+\tan x}{\sec x}$

Solution

$$\lim_{x \to \pi/2^{-}} \frac{1+\tan x}{\sec x} = \frac{\infty}{\infty} = \lim_{x \to \pi/2^{-}} \frac{\sec^2 x}{\sec x \tan x}$$

$$= \lim_{x \to \pi/2^{-}} \frac{\sec x}{\tan x}$$

$$= \lim_{x \to \pi/2^{-}} \frac{1}{\sin x}$$

$$\underline{=1}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x \to \infty} \dfrac{4x^3 - 6x^2 + 1}{2x^3 - 10x + 3}$

Solution

$$\lim_{x \to \infty} \frac{4x^3 - 6x^2 + 1}{2x^3 - 10x + 3} \underline{= 2}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x \to 0} \dfrac{3\sin 4x}{5x}$

Solution

$$\lim_{x \to 0} \frac{3\sin 4x}{5x} = \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{12\cos 4x}{5}$$

$$\underline{= \frac{12}{5}}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to 2\pi} \dfrac{x\sin x + x^2 - 4\pi^2}{x - 2\pi}$

Solution

$$\lim_{x\to 2\pi} \frac{x\sin x + x^2 - 4\pi^2}{x - 2\pi} = \frac{0}{0}$$

$$= \lim_{x\to 2\pi} \frac{\sin x + x\cos x + 2x}{1}$$

$$= 2\pi + 4\pi$$

$$= 6\pi$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to 0} \dfrac{\tan 4x}{\tan 7x}$

Solution

$$\lim_{x\to 0} \frac{\tan 4x}{\tan 7x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{4\sec^2 4x}{7\sec^2 7x}$$

$$= \frac{4}{7}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to 0} \dfrac{\sin^2 3x}{x^2}$

Solution

$$\lim_{x\to 0} \frac{\sin^2 3x}{x^2} = \frac{0}{0}$$

$$= \lim_{x\to 0} \left(\frac{\sin 3x}{x}\right)^2 \left(\frac{3}{3}\right)^2$$

$$= 9 \lim_{x\to 0} \left(\frac{\sin 3x}{3x}\right)^2 \qquad \lim_{x\to 0} \frac{\sin ax}{ax} = 1$$

$$= 9$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to-1} \dfrac{x^3 - x^2 - 5x - 3}{x^4 + 2x^3 - x^2 - 4x - 2}$

Solution

$$\lim_{x\to-1} \frac{x^3 - x^2 - 5x - 3}{x^4 + 2x^3 - x^2 - 4x - 2} = \frac{0}{0}$$

$$= \lim_{x\to-1} \frac{3x^2 - 2x - 5}{4x^3 + 6x^2 - 2x - 4}$$

$$= \lim_{x\to-1} \frac{3x^2 - 2x - 5}{4x^3 + 6x^2 - 2x - 4} = \frac{0}{0}$$

$$= \lim_{x\to-1} \frac{6x - 2}{12x^2 + 12x - 2}$$

$$= 4$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to1} \dfrac{x^n - 1}{x - 1} \quad (n > 0)$

Solution

$$\lim_{x\to1} \frac{x^n - 1}{x - 1} = \frac{0}{0}$$

$$= \lim_{x\to1} \frac{nx^{n-1}}{1}$$

$$= n$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to1^-} (1 - x)\tan\left(\dfrac{\pi x}{2}\right)$

Solution

$$\lim_{x\to1^-} (1 - x)\tan\left(\frac{\pi x}{2}\right) = 0 \cdot \infty$$

$$= \lim_{x\to1^-} \frac{1 - x}{\cot\left(\frac{\pi x}{2}\right)}$$

$$= \lim_{x\to1^-} \frac{-1}{-\frac{\pi}{2}\csc^2\left(\frac{\pi x}{2}\right)}$$

$$= \frac{2}{\pi}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to\infty} \frac{3}{x}\csc\frac{5}{x}$

Solution

$$\lim_{x\to\infty} \frac{3}{x}\csc\frac{5}{x} = 0\cdot\infty$$

$$= 3\lim_{y\to 0}\frac{y}{\sin 5y} \qquad\qquad y = \frac{1}{x}$$

$$= 3\lim_{y\to 0}\frac{1}{5\cos 5y}$$

$$= \frac{3}{5}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to\pi/4} \dfrac{\tan x - \cot x}{x - \frac{\pi}{4}}$

Solution

$$\lim_{x\to\pi/4} \frac{\tan x - \cot x}{x - \frac{\pi}{4}} = \frac{0}{0}$$

$$= \lim_{x\to\pi/4}\frac{\sec^2 x + \csc^2 x}{1}$$

$$= 2 + 2$$

$$= 4$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to 0} \dfrac{1-\cos 3x}{8x^2}$

Solution

$$\lim_{x\to 0} \frac{1-\cos 3x}{8x^2} = \frac{0}{0}$$

$$= \lim_{x\to 0}\frac{3\sin 3x}{16x} = \frac{0}{0}$$

$$= \lim_{x\to 0}\frac{9\cos 3x}{16}$$

$$= \frac{9}{16}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to 3} \dfrac{x-1-\sqrt{x^2-5}}{x-3}$

Solution

$$\lim_{x\to 3} \frac{x-1-\sqrt{x^2-5}}{x-3} = \frac{0}{0}$$

$$= \lim_{x\to 3} \frac{1-\frac{x}{\sqrt{x^2-5}}}{1}$$

$$= 1-\frac{3}{2}$$

$$= -\frac{1}{2}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{\sqrt{8-x^2}-x}$

Solution

$$\lim_{x\to 2} \frac{x^2+x-6}{\sqrt{8-x^2}-x} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{2x+1}{\frac{-x}{\sqrt{8-x^2}}-1}$$

$$= \frac{5}{-1-1}$$

$$= -\frac{5}{2}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{h\to 0} \dfrac{\sin(x+h)-\sin x}{h}$ *x is a real number*

Solution

$$\lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = \frac{0}{0}$$

$$= \lim_{h\to 0} \frac{\cos(x+h)}{1}$$

$$= \cos x$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to 2} \dfrac{\sqrt[3]{3x+2}-2}{x-2}$

Solution

$$\lim_{x\to 2} \frac{\sqrt[3]{3x+2}-2}{x-2} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{(3x+2)^{-2/3}}{1}$$

$$= \frac{1}{4}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to\infty} \dfrac{3x^4-x^2}{6x^4+12}$

Solution

$$\lim_{x\to\infty} \frac{3x^4-x^2}{6x^4+12} = \frac{1}{2}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to\infty} \dfrac{4x^3-2x^2+6}{\pi x^3+4}$

Solution

$$\lim_{x\to\infty} \frac{4x^3-2x^2+6}{\pi x^3+4} = \frac{4}{\pi}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to\infty} \dfrac{8-4x^2}{3x^3+x-1}$

Solution

$$\lim_{x\to\infty} \frac{8-4x^2}{3x^3+x-1} = 0$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to\pi/2} \dfrac{2\tan x}{\sec^2 x}$

Solution

$$\lim_{x\to\pi/2} \frac{2\tan x}{\sec^2 x} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\pi/2} \frac{2\sec^2 x}{2\sec^2 x \tan x}$$

$$= \lim_{x\to\pi/2} \cot x$$

$$\boxed{= 0}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x\to 0} \frac{e^x - x - 1}{5x^2}$

Solution

$$\lim_{x\to 0} \frac{e^x - x - 1}{5x^2} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{e^x - 1}{10x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{e^x}{10}$$

$$\boxed{= \frac{1}{10}}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x\to 0} \frac{e^x - \sin x - 1}{x^4 + 8x^3 + 12x^2}$

Solution

$$\lim_{x\to 0} \frac{e^x - \sin x - 1}{x^4 + 8x^3 + 12x^2} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{e^x - \cos x}{4x^3 + 24x^2 + 24x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{e^x + \sin x}{12x^2 + 48x + 24}$$

$$\boxed{= \frac{1}{24}}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to\infty} \dfrac{e^{1/x}-1}{1/x}$

Solution

$$\lim_{x\to\infty} \frac{e^{1/x}-1}{1/x} = \frac{0}{0}$$

$$= \lim_{x\to\infty} \frac{\frac{-1}{x^2}e^{1/x}}{-1/x^2}$$

$$= \lim_{x\to\infty} e^{1/x}$$

$$\underline{=1}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to\infty} \dfrac{e^{3x}}{3e^{3x}+5}$

Solution

$$\lim_{x\to\infty} \frac{e^{3x}}{3e^{3x}+5} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{3e^{3x}}{9e^{3x}}$$

$$\underline{=\frac{1}{3}}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to\infty} \dfrac{\ln\left(3x+5e^{x}\right)}{\ln\left(7x+3e^{2x}\right)}$

Solution

$$\lim_{x\to\infty} \frac{\ln\left(3x+5e^{x}\right)}{\ln\left(7x+3e^{2x}\right)} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{3+5e^{x}}{3x+5e^{x}} \cdot \lim_{x\to\infty} \frac{7x+3e^{2x}}{7+6e^{2x}}$$

$$= \lim_{x\to\infty} \frac{5e^{x}}{3+5e^{x}} \cdot \lim_{x\to\infty} \frac{7+6e^{2x}}{12e^{2x}}$$

$$= \lim_{x\to\infty} \frac{5e^{x}}{5e^{x}} \cdot \lim_{x\to\infty} \frac{6e^{2x}}{12e^{2x}}$$

$$=1\cdot\frac{1}{2}$$

$$\boxed{=\frac{1}{2}}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim\limits_{x\to\infty}\dfrac{x^2-\ln\left(\frac{2}{x}\right)}{3x^2+2x}$

Solution

$$\lim_{x\to\infty}\frac{x^2-\ln\left(\frac{2}{x}\right)}{3x^2+2x}=\lim_{x\to\infty}\frac{2x-\left(\frac{x}{2}\right)\left(\frac{-1}{x^2}\right)}{6x+2}$$

$$=\lim_{x\to\infty}\frac{2x+\frac{1}{x}}{6x+2}$$

$$\boxed{=\frac{1}{3}}$$

Exercise

Find $\lim\limits_{x\to 1^+} x^{1/(x-1)}$

Solution

$$\lim_{x\to 1^+} x^{1/(x-1)}=1^{\infty}$$

$$f(x)=x^{1/(x-1)} \quad \Rightarrow \ln f(x)=\ln x^{1/(x-1)}=\boxed{\frac{1}{x-1}\ln x}$$

$$\lim_{x\to 1^+}\ln f(x)=\lim_{x\to 1^+}\frac{\ln x}{x-1}=\lim_{x\to 1^+}\frac{\frac{1}{x}}{1}=1$$

$$\lim_{x\to 1^+} x^{1/(x-1)}=\lim_{x\to 1^+} f(x)$$

$$=\lim_{x\to 1^+} e^{\ln f(x)}$$

$$=\lim_{x\to 1^+} e^{1}$$

$$\boxed{=e}$$

Exercise

Find $\lim\limits_{x\to e^+} (\ln x)^{1/(x-e)}$

Solution

$$\lim_{x\to e^+} (\ln x)^{1/(x-e)} = 1^{\infty}$$

$$f(x) = (\ln x)^{1/(x-e)}$$

$$\ln f(x) = \ln\left((\ln x)^{1/(x-e)}\right)$$

$$= \frac{1}{x-e}\ln(\ln x)$$

$$\lim_{x\to e^+} \ln f(x) = \lim_{x\to e^+} \frac{\ln(\ln x)}{x-e}$$

$$= \lim_{x\to e^+} \frac{\frac{1}{x\ln x}}{1}$$

$$= \lim_{x\to e^+} \frac{1}{x\ln x}$$

$$= \frac{1}{e\ln e}$$

$$= \frac{1}{e}$$

$$\lim_{x\to e^+} (\ln x)^{1/(x-e)} = \lim_{x\to e^+} f(x)$$

$$= \lim_{x\to e^+} e^{\ln f(x)}$$

$$= e^{1/e}$$

Exercise

Find $\lim\limits_{x\to\infty} (1+2x)^{1/(2\ln x)}$

Solution

$$\lim_{x\to\infty} (1+2x)^{1/(2\ln x)} = \infty^{0}$$

$$f(x) = (1+2x)^{1/(2\ln x)} \quad \Rightarrow \ln f(x) = \ln\left((1+2x)^{1/(2\ln x)}\right) = \frac{1}{2\ln x}\ln(1+2x)$$

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{\ln(1+2x)}{2\ln x}$$

$$= \lim_{x\to\infty} \frac{\frac{2}{1+2x}}{\frac{2}{x}}$$

$$= \lim_{x\to\infty} \frac{x}{1+2x}$$

$$= \frac{1}{2}$$

$$\lim_{x\to\infty} (1+2x)^{1/(2\ln x)} = \lim_{x\to\infty} e^{\ln f(x)}$$

$$= e^{1/2}$$

Exercise

Find $\lim_{x\to\infty} \left(\frac{x^2+1}{x+2}\right)^{1/x}$

Solution

$$\lim_{x\to\infty} \left(\frac{x^2+1}{x+2}\right)^{1/x} = \infty^0$$

$$f(x) = \left(\frac{x^2+1}{x+2}\right)^{1/x} \quad \Rightarrow \ln f(x) = \ln\left(\left(\frac{x^2+1}{x+2}\right)^{1/x}\right) = \frac{1}{x}\ln\left(\frac{x^2+1}{x+2}\right)$$

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{1}{x}\ln\left(\frac{x^2+1}{x+2}\right)$$

$$= \lim_{x\to\infty} \frac{\ln\left(x^2+1\right) - \ln(x+2)}{x}$$

$$= \lim_{x\to\infty} \frac{\frac{2x}{x^2+1} - \frac{1}{x+2}}{1}$$

$$= \lim_{x\to\infty} \frac{2x^2+4x-x^2-1}{\left(x^2+1\right)(x+2)}$$

$$= \lim_{x\to\infty} \frac{x^2+4x-1}{x^3+2x^2+x+2} \qquad \lim_{x\to\infty} \frac{x^2+4x-1}{x^3+2x^2+x+2} = 0$$

$$= \lim_{x\to\infty} \frac{2x+4}{3x^2+4x+1}$$

$$= \lim_{x\to\infty} \frac{2}{6x+4}$$

$$= 0$$

$$\lim_{x\to\infty}\left(\frac{x^2+1}{x+2}\right)^{1/x} = \lim_{x\to\infty} e^{\ln f(x)}$$

$$= e^0$$

$$= 1$$

Exercise

Evaluate the following limit $\lim_{t\to 2}\frac{t^3-t^2-2t}{t^2-4}$

Solution

$$\lim_{t\to 2}\frac{t^3-t^2-2t}{t^2-4} = \frac{0}{0}$$

$$= \lim_{t\to 2}\frac{3t^2-2t-2}{2t}$$

$$= \frac{3}{2}$$

Exercise

Evaluate the following limit $\lim_{x\to 0}\frac{1-\cos 6x}{2x}$

Solution

$$\lim_{x\to 0}\frac{1-\cos 6x}{2x} = \frac{0}{0}$$

$$= \lim_{x\to 0}\frac{6\sin 6x}{2}$$

$$= 0$$

Exercise

Evaluate the following limit $\lim_{x\to\infty}\frac{5x^2+2x-5}{\sqrt{x^4-1}}$

Solution

$$\lim_{x\to\infty}\frac{5x^2+2x-5}{\sqrt{x^4-1}} = \lim_{x\to\infty}\frac{5x^2}{x^2}$$

$$= 5$$

Exercise

Evaluate the following limit $\lim\limits_{\theta\to 0}\dfrac{3\sin^2 2\theta}{\theta^2}$

Solution

$$\lim\limits_{\theta\to 0}\frac{3\sin^2 2\theta}{\theta^2}=\frac{0}{0}$$

$$=\lim\limits_{\theta\to 0}\frac{12\sin 2\theta\cos 2\theta}{2\theta}$$

$$=12\lim\limits_{2\theta\to 0}\frac{\sin 2\theta}{2\theta}\lim\limits_{\theta\to 0}\cos 2\theta \qquad \lim\limits_{2\theta\to 0}\frac{\sin 2\theta}{2\theta}=1$$

$$=12$$

Exercise

Evaluate the following limit $\lim\limits_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)$

Solution

$$\lim\limits_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)=\infty-\infty$$

$$=\lim\limits_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$$

$$=\lim\limits_{x\to\infty}\frac{x^2+x+1-x^2+x}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$$

$$=\lim\limits_{x\to\infty}\frac{2x}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$$

$$=\lim\limits_{x\to\infty}\frac{2x}{x+x}$$

$$=1$$

Exercise

Evaluate the following limit $\lim\limits_{\theta\to 0}2\theta\cot 3\theta$

Solution

$$\lim\limits_{\theta\to 0}2\theta\cot 3\theta=0\cdot\infty=\frac{0}{0}$$

$$=\lim\limits_{\theta\to 0}2\theta\frac{\cos 3\theta}{\sin 3\theta}$$

$$=\lim\limits_{\theta\to 0}\frac{2\theta}{\sin 3\theta}\lim\limits_{\theta\to 0}\cos 3\theta$$

$$=\frac{2}{3}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to 0} \dfrac{e^{-2x}-1+2x}{x^2}$

Solution

$$\lim\limits_{x\to 0} \frac{e^{-2x}-1+2x}{x^2} = \frac{0}{0}$$

$$= \lim\limits_{x\to 0} \frac{-2e^{-2x}+2}{2x} = \frac{0}{0}$$

$$= \lim\limits_{x\to 0} \frac{4e^{-2x}}{2}$$

$$= 2$$

Exercise

Evaluate the following limit $\lim\limits_{x\to 1} \dfrac{x^4-x^3-3x^2+5x-2}{x^3+x^2-5x+3}$

Solution

$$\lim\limits_{x\to 1} \frac{x^4-x^3-3x^2+5x-2}{x^3+x^2-5x+3} = \frac{1-1-3+5-2}{1+1-5+3} = \frac{0}{0}$$

$$= \lim\limits_{x\to 1} \frac{4x^3-3x^2-6x+5}{3x^2+2x-5} = \frac{0}{0}$$

$$= \lim\limits_{x\to 1} \frac{12x^2-6x-6}{6x+2}$$

$$= 0$$

Exercise

Evaluate the following limit $\lim\limits_{y\to 0^+} \dfrac{\ln^{10} y}{\sqrt{y}}$

Solution

$$\lim\limits_{y\to 0^+} \frac{\ln^{10} y}{\sqrt{y}} = \frac{\infty}{0}$$

$$= \lim\limits_{x\to \infty} \frac{\left(\ln \frac{1}{x}\right)^{10}}{1/\sqrt{x}} \qquad \text{Let } y = \frac{1}{x}$$

$$= \lim\limits_{x\to \infty} \frac{\left(\ln \frac{1}{x}\right)^{10}}{1/\sqrt{x}}$$

$$= \lim_{x\to\infty} \sqrt{x}(-\ln x)^{10}$$

$$\underline{= \infty}$$

Exercise

Evaluate the following limit $\displaystyle\lim_{\theta\to 0} \frac{3\sin 8\theta}{8\sin 3\theta}$

Solution

$$\lim_{\theta\to 0} \frac{3\sin 8\theta}{8\sin 3\theta} = \frac{0}{0}$$

$$= \frac{3}{8}\frac{8}{3}$$

$$\underline{=1}$$

Exercise

Evaluate the following limit $\displaystyle\lim_{x\to\infty} \frac{\ln x^{100}}{\sqrt{x}}$

Solution

$$\lim_{x\to\infty} \frac{\ln x^{100}}{\sqrt{x}} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{100(\ln x)}{\sqrt{x}}$$

$$= 100 \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}$$

$$= 200 \lim_{x\to\infty} \frac{1}{\sqrt{x}}$$

$$\underline{=0}$$

Exercise

Evaluate the following limit $\displaystyle\lim_{x\to 0} \csc x \sin^{-1} x$

Solution

$$\lim_{x\to 0} \csc x \sin^{-1} x = \lim_{x\to 0} \frac{\sin^{-1} x}{\sin x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{\cos x}$$

$$\underline{=1}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to\infty} \dfrac{\ln^3 x}{\sqrt{x}}$

Solution

$$\lim_{x\to\infty} \frac{\ln^3 x}{\sqrt{x}} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{3\ln^2 x}{x}\left(2\sqrt{x}\right) \qquad \left(\ln^3 x\right)' = \frac{3\ln^2 x}{x}$$

$$= 6\lim_{x\to\infty} \frac{\ln^2 x}{\sqrt{x}} = \frac{\infty}{\infty}$$

$$= 6\lim_{x\to\infty} \frac{2\ln x}{x}\left(2\sqrt{x}\right)$$

$$= 24\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$$

$$= 24\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}$$

$$= 48\lim_{x\to\infty} \frac{1}{\sqrt{x}}$$

$$= 0$$

Exercise

Evaluate the following limit $\lim\limits_{x\to\infty} \ln\left(\dfrac{x+1}{x-1}\right)$

Solution

$$\lim_{x\to\infty} \ln\left(\frac{x+1}{x-1}\right) = \lim_{x\to\infty} \ln(1)$$

$$= 0$$

Exercise

Evaluate the following limit $\lim\limits_{x\to 0^+} (1+x)^{\cot x}$

Solution

$$\lim_{x\to 0^+} (1+x)^{\cot x} = 1^\infty$$

$$\lim_{x\to 0^+} \ln(1+x)^{\cot x} = \lim_{x\to 0^+} (\cot x)\ln(1+x)$$

$$= \lim_{x\to 0^+} \frac{\ln(1+x)}{\tan x} = \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{\sec^2 x}$$

$$\underline{=1}$$

$$\lim_{x\to 0^+} (1+x)^{\cot x} = e^1 \underline{= e}$$

Exercise

Evaluate the following limit $\lim_{x\to \frac{\pi}{2}^+} (\sin x)^{\tan x}$

Solution

$$\lim_{x\to \frac{\pi}{2}^+} (\sin x)^{\tan x} = 1^\infty$$

$$\lim_{x\to \frac{\pi}{2}^+} \ln(\sin x)^{\tan x} = \lim_{x\to \frac{\pi}{2}^+} (\tan x)\ln(\sin x)$$

$$= \lim_{x\to \frac{\pi}{2}^+} \frac{\ln(\sin x)}{\cot x}$$

$$= \lim_{x\to \frac{\pi}{2}^+} \frac{\frac{\cos x}{\sin x}}{-\csc^2 x}$$

$$= -\lim_{x\to \frac{\pi}{2}^+} \frac{\cos x}{\sin x}\sin^2 x$$

$$= -\lim_{x\to \frac{\pi}{2}^+} (\cos x \sin x)$$

$$\underline{=0}$$

$$\lim_{x\to \frac{\pi}{2}^+} (\sin x)^{\tan x} = e^0 \underline{=1}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to\infty} \left(\sqrt{x}+1\right)^{1/x}$

Solution

$$\lim_{x\to\infty} \left(\sqrt{x}+1\right)^{1/x} = \infty^0$$

$$\lim_{x\to\infty} \ln\left(\sqrt{x}+1\right)^{1/x} = \lim_{x\to\infty} \frac{1}{x}\ln\left(\sqrt{x}+1\right)$$

$$= \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}\frac{1}{\sqrt{x}+1}}{1}$$

$$= 0$$

$$\lim_{x\to\infty} \left(\sqrt{x}+1\right)^{1/x} = e^0 = 1$$

Exercise

Evaluate the following limit $\lim\limits_{x\to 0^+} |\ln x|^x$

Solution

$$\lim_{x\to 0^+} |\ln x|^x = -\infty^0$$

$$\lim_{x\to 0^+} \ln|\ln x|^x = \lim_{x\to 0^+} x\ln|\ln x|$$

$$= \lim_{x\to 0^+} \frac{\ln|\ln x|}{\frac{1}{x}} = \frac{\infty}{\infty}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x|\ln x|}}{-\frac{1}{x^2}}$$

$$= -\lim_{x\to 0^+} \frac{x}{|\ln x|}$$

$$= 0$$

$$\lim_{x\to 0^+} |\ln x|^x = e^0 = 1$$

Exercise

Evaluate the following limit $\lim\limits_{x\to\infty} x^{1/x}$

Solution

$$\lim_{x\to\infty} x^{1/x} = \infty^0$$

$$\lim_{x\to\infty} \ln x^{1/x} = \lim_{x\to\infty} \frac{1}{x}\ln x$$

$$= \lim_{x\to\infty} \frac{\ln x}{x} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$

$$\underline{=0}$$

$$\lim_{x\to\infty} x^{1/x} = e^0 \underline{=1}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to\infty}\left(1-\frac{3}{x}\right)^x$

Solution

$$\lim_{x\to\infty}\left(1-\frac{3}{x}\right)^x = 1^\infty$$

$$\lim_{x\to\infty} \ln\left(1-\frac{3}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1-\frac{3}{x}\right)$$

$$= \lim_{x\to\infty} \frac{1}{\frac{1}{x}}\ln\left(1-\frac{3}{x}\right) = \frac{0}{0}$$

$$= \lim_{x\to\infty} \frac{3}{x^2}\frac{1}{1-\frac{3}{x}} \bullet \frac{1}{-\frac{1}{x^2}}$$

$$= -3\lim_{x\to\infty} \frac{1}{1-\frac{3}{x}}$$

$$\underline{=-3}$$

$$\lim_{x\to\infty}\left(1-\frac{3}{x}\right)^x \underline{= e^{-3}}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to\infty}\left(\frac{2}{\pi}\tan^{-1}x\right)^x$

Solution

$$\lim_{x\to\infty}\left(\frac{2}{\pi}\tan^{-1}x\right)^x = 1^\infty$$

$$\begin{aligned}
\lim_{x\to\infty}\ln\left(\frac{2}{\pi}\tan^{-1}x\right)^x &= \lim_{x\to\infty} x\ln\left(\frac{2}{\pi}\tan^{-1}x\right)\\
&= \lim_{x\to\infty}\frac{1}{\frac{1}{x}}\ln\left(\frac{2}{\pi}\tan^{-1}x\right)\\
&= \lim_{x\to\infty}\frac{1}{-\frac{1}{x^2}}\cdot\frac{\frac{2}{\pi}\frac{1}{1+x^2}}{\frac{2}{\pi}\tan^{-1}x}\\
&= -\lim_{x\to\infty}\frac{x^2}{\left(1+x^2\right)\tan^{-1}x} \qquad \lim_{x\to\infty}\frac{x^2}{1+x^2}=1\\
&= -\frac{1}{\frac{\pi}{2}}\\
&= -\frac{2}{\pi}
\end{aligned}$$

$$\lim_{x\to\infty}\left(\frac{2}{\pi}\tan^{-1}x\right)^x = e^{-\frac{2}{\pi}}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to 1}(x-1)^{\sin\pi x}$

Solution

$$\lim_{x\to 1}(x-1)^{\sin\pi x} = 0^0$$

$$\begin{aligned}
\lim_{x\to 1}\ln(x-1)^{\sin\pi x} &= \lim_{x\to 1}(\sin\pi x)\ln(x-1)\\
&= \lim_{x\to 1}\frac{\ln(x-1)}{\csc\pi x}\\
&= \lim_{x\to 1}\frac{\frac{1}{x-1}}{-\pi\csc\pi x\ \cot\pi x}\\
&= -\frac{1}{\pi}\lim_{x\to 1}\frac{1}{x-1}\frac{\sin^2\pi x}{\cos\pi x}
\end{aligned}$$

$$= -\frac{1}{\pi} \lim_{x \to 1} \frac{1}{\cos \pi x} \bullet \lim_{x \to 1} \frac{\sin^2 \pi x}{x-1}$$

$$= \frac{1}{\pi} \lim_{x \to 1} \frac{2\pi \sin \pi x \cos \pi x}{1}$$

$$= 0$$

$$\lim_{x \to 1} (x-1)^{\sin \pi x} = e^0 = 1$$

Exercise

Evaluate the following limit $\lim_{x \to \infty} \frac{2x^5 - x + 1}{5x^6 + x}$

Solution

$$\lim_{x \to \infty} \frac{2x^5 - x + 1}{5x^6 + x} = \lim_{x \to \infty} \frac{2x^5}{5x^6}$$

$$= 0$$

Exercise

Evaluate the following limit $\lim_{x \to \infty} \frac{4x^4 - \sqrt{x}}{2x^4 + x^{-1}}$

Solution

$$\lim_{x \to \infty} \frac{4x^4 - \sqrt{x}}{2x^4 + x^{-1}} = 2$$

Exercise

Evaluate the following limit $\lim_{x \to 0} \frac{1 - \cos x^n}{x^{2n}}$

Solution

$$\lim_{x \to 0} \frac{1 - \cos x^n}{x^{2n}} = \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{nx^{n-1} \sin x^n}{2nx^{2n-1}}$$

$$= \frac{1}{2} \lim_{x \to 0} \frac{\sin x^n}{x^n}$$

$$= \frac{1}{2} \lim_{x^n \to 0} \frac{\sin x^n}{x^n} \qquad \lim_{u \to 0} \frac{\sin u}{u} = 1$$

$$= \frac{1}{2}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to 0} \dfrac{1-\cos^n x}{x^2}$

Solution

$$\lim_{x\to 0} \frac{1-\cos^n x}{x^2} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{n(\sin x)\cos^{n-1} x}{2x} = \frac{0}{0}$$

$$= \frac{1}{2} \lim_{x\to 0} \frac{n(\cos x)\cos^{n-1} x - n(n-1)\sin^2 x\cos^{n-2} x}{1}$$

$$= \frac{n}{2}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to 0} \dfrac{1-\cos x^n}{x^2}$

Solution

$$\lim_{x\to 0} \frac{1-\cos x^n}{x^2} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{nx^{n-1}\sin x^n}{2x} = \frac{0}{0}$$

$$= \frac{1}{2} \lim_{x\to 0} x^{n-2}\sin x^n$$

$$= 0$$

Exercise

Evaluate the following limit $\lim\limits_{x\to 0} \dfrac{3x}{\tan 4x}$

Solution

$$\lim_{x\to 0} \frac{3x}{\tan 4x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{3}{4\sec^2 4x}$$

$$= \frac{3}{4}$$

Exercise

Evaluate the following limit $\lim_{x\to 0} \frac{\sin ax}{\sin bx}$

Solution

$$\lim_{x\to 0} \frac{\sin ax}{\sin bx} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{a\cos ax}{b\cos bx}$$

$$= \frac{a}{b}$$

Exercise

Evaluate the following limit $\lim_{x\to 2} \frac{\ln(2x-3)}{x^2-4}$

Solution

$$\lim_{x\to 2} \frac{\ln(2x-3)}{x^2-4} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{\frac{2}{2x-3}}{2x}$$

$$= \lim_{x\to 2} \frac{1}{x(2x-3)}$$

$$= \frac{1}{2}$$

Exercise

Evaluate the following limit $\lim_{x\to 0} \frac{1-\cos ax}{1-\cos bx}$

Solution

$$\lim_{x\to 0} \frac{1-\cos ax}{1-\cos bx} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{a\sin ax}{b\sin bx} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{a^2\cos ax}{b^2\cos bx}$$

$$= \frac{a^2}{b^2}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to 0} \dfrac{\sin^{-1} x}{\tan^{-1} x}$

Solution

$$\lim_{x\to 0} \frac{\sin^{-1} x}{\tan^{-1} x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{\frac{1}{1+x^2}}$$

$$= \lim_{x\to 0} \frac{1+x^2}{\sqrt{1-x^2}}$$

$$= 1$$

Exercise

Evaluate the following limit $\lim\limits_{x\to 1} \dfrac{x^{1/3}-1}{x^{2/3}-1}$

Solution

$$\lim_{x\to 1} \frac{x^{1/3}-1}{x^{2/3}-1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{\frac{1}{3}x^{-2/3}}{\frac{2}{3}x^{-1/3}}$$

$$= \frac{1}{2}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to 0} x\cot x$

Solution

$$\lim_{x\to 0} x\cot x = 0\cdot\infty$$

$$= \lim_{x\to 0} \frac{x\cos x}{\sin x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\cos x - x\sin x}{\cos x}$$

$$= 1$$

Exercise

Evaluate the following limit $\lim_{x\to 0} \frac{1-\cos x}{\ln\left(1+x^2\right)}$

Solution

$$\lim_{x\to 0} \frac{1-\cos x}{\ln\left(1+x^2\right)} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin x}{\frac{2x}{1+x^2}}$$

$$= \frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x}\left(1+x^2\right) \qquad \lim_{u\to 0} \frac{\sin u}{u} = 1$$

$$\boxed{= \frac{1}{2}}$$

Exercise

Evaluate the following limit $\lim_{x\to \pi} \frac{\sin^2 x}{x-\pi}$

Solution

$$\lim_{x\to \pi} \frac{\sin^2 x}{x-\pi} = \frac{0}{0}$$

$$= \lim_{x\to \pi} \frac{2\sin x \cos x}{1}$$

$$\boxed{= 0}$$

Exercise

Evaluate the following limit $\lim_{x\to 0} \frac{10^x - e^x}{x}$

Solution

$$\lim_{x\to 0} \frac{10^x - e^x}{x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{10^x \ln 10 - e^x}{1}$$

$$\boxed{= \ln(10) - 1}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\cos 3x}{\pi - 2x}$

Solution

$$\lim\limits_{x\to\frac{\pi}{2}} \frac{\cos 3x}{\pi - 2x} = \frac{0}{0}$$

$$= \lim\limits_{x\to\frac{\pi}{2}} \frac{-3\sin 3x}{-2}$$

$$\underline{= \frac{3}{2}}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to 1} \dfrac{\ln(ex)-1}{\sin \pi x}$

Solution

$$\lim\limits_{x\to 1} \frac{\ln(ex)-1}{\sin \pi x} = \frac{0}{0}$$

$$= \lim\limits_{x\to 1} \frac{\frac{1}{x}}{\pi \cos \pi x}$$

$$\underline{= \frac{1}{\pi}}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to\infty} x\sin\frac{1}{x}$

Solution

$$\lim\limits_{x\to\infty} x\sin\frac{1}{x} = \lim\limits_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = \frac{0}{0} \qquad y = \frac{1}{x} \xrightarrow[x\to\infty]{} 0$$

$$= \lim\limits_{y\to 0} \frac{\sin y}{y}$$

$$\underline{= 1}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to 0} \dfrac{x - \sin x}{x^3}$

Solution

$$\lim_{x\to 0}\frac{x-\sin x}{x^3}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{1-\cos x}{3x^2}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{\sin x}{6x}=\frac{0}{0}$$

$$=\frac{1}{6}$$

Exercise

Evaluate the following limit $\displaystyle\lim_{x\to 0}\frac{x-\sin x}{x-\tan x}$

Solution

$$\lim_{x\to 0}\frac{x-\sin x}{x-\tan x}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{1-\cos x}{1-\sec^2 x}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{\sin x}{-2\sec^2 x\tan x}=\frac{0}{0}$$

$$=-\frac{1}{2}\lim_{x\to 0}\frac{\sin x}{\frac{1}{\cos^2 x}\frac{\sin x}{\cos x}}$$

$$=-\frac{1}{2}\lim_{x\to 0}\cos x$$

$$=-\frac{1}{2}$$

Exercise

Evaluate the following limit $\displaystyle\lim_{x\to 0}\frac{2-x^2-2\cos x}{x^4}$

Solution

$$\lim_{x\to 0}\frac{2-x^2-2\cos x}{x^4}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{-2x+2\sin x}{4x^3}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{-2+2\cos x}{12x^2}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{-2\sin x}{24x}=\frac{0}{0}$$

$$=-\frac{1}{12}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to 0^+} \dfrac{\sin^2 x}{\tan x - x}$

Solution

$$\lim\limits_{x\to 0^+} \frac{\sin^2 x}{\tan x - x} = \frac{0}{0}$$

$$= \lim\limits_{x\to 0^+} \frac{2\sin x\cos x}{\sec^2 x - 1} = \frac{0}{0}$$

$$= \lim\limits_{x\to 0^+} \frac{\sin 2x}{\tan^2 x} = \frac{0}{0}$$

$$= \lim\limits_{x\to 0^+} \frac{2\cos 2x}{2\tan x\sec^2 x}$$

$$= \frac{1}{0^+}$$

$$= \infty$$

Exercise

Evaluate the following limit $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\ln \sin x}{\cos x}$

Solution

$$\lim\limits_{x\to \frac{\pi}{2}} \frac{\ln \sin x}{\cos x} = \frac{0}{0}$$

$$= \lim\limits_{x\to \frac{\pi}{2}} \frac{\frac{\cos x}{\sin x}}{-\sin x}$$

$$= -\lim\limits_{x\to \frac{\pi}{2}} \frac{\cos x}{\sin^2 x}$$

$$= 0$$

Exercise

Evaluate the following limit $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\sin x}{x}$

Solution

$$\lim\limits_{x\to \frac{\pi}{2}} \frac{\sin x}{x} = \frac{1}{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to 1^-} \dfrac{\arccos x}{x-1}$

Solution

$$\lim_{x\to 1^-} \frac{\arccos x}{x-1} = \frac{0}{0}$$

$$= \lim_{x\to 1^-} \frac{-\frac{1}{\sqrt{1-x^2}}}{1}$$

$$= -\infty$$

Exercise

Evaluate the following limit $\lim\limits_{x\to\infty} x\left(2\tan^{-1}x - \pi\right)$

Solution

$$\lim_{x\to\infty} x\left(2\tan^{-1}x-\pi\right) = \infty\cdot 0$$

$$= \lim_{x\to\infty}\left(\frac{2\tan^{-1}x}{x} - \frac{\pi}{x}\right)$$

$$= \lim_{x\to\infty}\left(\frac{2\tan^{-1}x}{x}\right) - \lim_{x\to\infty}\left(\frac{\pi}{x}\right)$$

$$= 2\lim_{x\to\infty}\left(\frac{\tan^{-1}x}{x}\right)$$

$$= 2\lim_{x\to\infty}\frac{\frac{1}{1+x^2}}{1}$$

$$= 2\lim_{x\to\infty}\frac{1}{x^2}$$

$$= 0$$

Exercise

Evaluate the following limit $\lim\limits_{x\to\frac{\pi}{2}^+} x(\sec x - \tan x)$

Solution

$$\lim_{x\to\frac{\pi}{2}^+} x(\sec x-\tan x) = \infty-\infty$$

$$= \lim_{x\to\frac{\pi}{2}^+} x\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$$

$$= \lim_{x \to \frac{\pi}{2}^+} x\left(\frac{1-\sin x}{\cos x}\right) = \frac{0}{0}$$

$$= \lim_{x \to \frac{\pi}{2}^+} x\left(\frac{\cos x}{-\sin x}\right)$$

$$= 0$$

Exercise

Evaluate the following limit $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{xe^{ax}}\right)$

Solution

$$\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{xe^{ax}}\right) = \infty - \infty$$

$$= \lim_{x \to 0} \frac{e^{ax} - 1}{xe^{ax}} = \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{ae^{ax}}{(1+ax)e^{ax}}$$

$$= a$$

Exercise

Evaluate the following limit $\lim_{x \to 0^+} x^{\sqrt{x}}$

Solution

$$\lim_{x \to 0^+} x^{\sqrt{x}} = 0^0$$

$$\lim_{x \to 0^+} \ln x^{\sqrt{x}} = \lim_{x \to 0^+} \sqrt{x} \ln x$$

$$= \lim_{x \to 0^+} \frac{\ln x}{x^{-1/2}}$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}}$$

$$= -2 \lim_{x \to 0^+} \sqrt{x}$$

$$= 0$$

$$\lim_{x \to 0^+} x^{\sqrt{x}} = e^0 = 1$$

Exercise

Evaluate the following limit $\lim\limits_{x\to\pi} \dfrac{\cos x+1}{(x-\pi)^2}$

Solution

$$\lim\limits_{x\to\pi} \frac{\cos x+1}{(x-\pi)^2} = \frac{0}{0}$$

$$= \lim\limits_{x\to\pi} \frac{-\sin x}{2(x-\pi)} = \frac{0}{0}$$

$$= \lim\limits_{x\to\pi} \frac{-\cos x}{2}$$

$$= \frac{1}{2}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to 0} \dfrac{\sin x - x}{7x^3}$

Solution

$$\lim\limits_{x\to 0} \frac{\sin x - x}{7x^3} = \frac{0}{0}$$

$$= \lim\limits_{x\to 0} \frac{\cos x - 1}{21x^2} = \frac{0}{0}$$

$$= \lim\limits_{x\to 0} \frac{-\sin x}{42x} \qquad \lim\limits_{x\to 0} \frac{\sin x}{x} = 1$$

$$= -\frac{1}{42}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to\infty} \dfrac{\tan^{-1} x - \frac{\pi}{2}}{\frac{1}{x}}$

Solution

$$\lim\limits_{x\to\infty} \frac{\tan^{-1} x - \frac{\pi}{2}}{\frac{1}{x}} = \frac{\frac{\pi}{2} - \frac{\pi}{2}}{0} = \frac{0}{0}$$

$$= \lim\limits_{x\to\infty} \frac{\frac{1}{1+x^2}}{-\frac{1}{x^2}}$$

$$= -\lim\limits_{x\to\infty} \frac{x^2}{1+x^2}$$

$$= -1$$

Exercise

Evaluate the following limit $\lim\limits_{x\to 3} \dfrac{x-1-\sqrt{x^2-5}}{x-3}$

Solution

$$\lim_{x\to 3} \frac{x-1-\sqrt{x^2-5}}{x-3} = \frac{2-2}{0} = \frac{0}{0}$$

$$= \lim_{x\to 3} \frac{1-\frac{x}{\sqrt{x^2-5}}}{1}$$

$$= 1-\frac{3}{2}$$

$$\boxed{= -\frac{1}{2}}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{\sqrt{8-x^2}-x}$

Solution

$$\lim_{x\to 2} \frac{x^2+x-6}{\sqrt{8-x^2}-x} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{2x+1}{\frac{-x}{\sqrt{8-x^2}}-1}$$

$$\boxed{= -\frac{5}{2}}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to 2} \dfrac{x^2-4x+4}{\sin^2 \pi x}$

Solution

$$\lim_{x\to 2} \frac{x^2-4x+4}{\sin^2 \pi x} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{2x-4}{2\pi \sin \pi x \cos \pi x}$$

$$= \lim_{x\to 2} \frac{2x-4}{\pi \sin 2\pi x} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{2}{2\pi^2 \cos 2\pi x}$$

$$\boxed{= \frac{1}{\pi^2}}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to 2} \dfrac{(3x+2)^{1/3}-2}{x-2}$

Solution

$$\lim\limits_{x\to 2} \frac{(3x+2)^{1/3}-2}{x-2} = \frac{0}{0}$$

$$= \lim\limits_{x\to 2} \frac{(3x+2)^{-2/3}}{1}$$

$$= 8^{-2/3}$$

$$= \frac{1}{4}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to \infty} \dfrac{3x^4-x^2}{6x^4+12}$

Solution

$$\lim\limits_{x\to \infty} \frac{3x^4-x^2}{6x^4+12} = \frac{3}{6} = \frac{1}{2}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to \infty} \dfrac{4x^3-2x^2+6}{\pi x^3+4}$

Solution

$$\lim\limits_{x\to \infty} \frac{4x^3-2x^2+6}{\pi x^3+4} = \frac{4}{\pi}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to \frac{\pi}{2}^-} \frac{1}{3}(2x-\pi)\tan x$

Solution

$$\lim\limits_{x\to \frac{\pi}{2}^-} \frac{1}{3}(2x-\pi)\tan x = 0\cdot\infty$$

$$= \frac{1}{3} \lim\limits_{x\to \frac{\pi}{2}^-} \frac{\tan x}{\frac{1}{2x-\pi}} = \frac{0}{0}$$

$$=\frac{1}{3}\lim_{x\to\frac{\pi}{2}^-}\frac{\sec^2 x}{\frac{-2}{(2x-\pi)^2}}$$

$$=-\frac{1}{6}\lim_{x\to\frac{\pi}{2}^-}\frac{(2x-\pi)^2}{\cos^2 x}=\frac{0}{0}$$

$$=-\frac{1}{6}\lim_{x\to\frac{\pi}{2}^-}\frac{4(2x-\pi)}{-2\sin x\cos x}$$

$$=\frac{2}{3}\lim_{x\to\frac{\pi}{2}^-}\frac{2x-\pi}{\sin 2x}=\frac{0}{0}$$

$$=\frac{2}{3}\lim_{x\to\frac{\pi}{2}^-}\frac{2}{2\cos 2x}$$

$$=-\frac{2}{3}$$

Exercise

Evaluate the following limit $\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)$

Solution

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)=\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}=\frac{0}{0}$$

$$=\lim_{x\to\infty}\frac{-\frac{1}{x^2}}{1+\frac{1}{x}}\bullet\frac{1}{-\frac{1}{x^2}}$$

$$=\lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} \qquad \lim_{x\to\infty}\frac{1}{x}=0$$

$$=1$$

Exercise

Evaluate the following limit $\lim_{x\to\frac{\pi}{2}^-}\left(\frac{\pi}{2}-x\right)\sec x$

Solution

$$\lim_{x\to\frac{\pi}{2}^-}\left(\frac{\pi}{2}-x\right)\sec x=\lim_{x\to\frac{\pi}{2}^-}\frac{\frac{\pi}{2}-x}{\cos x}=\frac{0}{0}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{-1}{-\sin x}$$

$$\underline{= 1}$$

Exercise

Evaluate the following limit $\quad \lim_{x \to \infty} \frac{e^{1/x} - 1}{\sin \frac{1}{x}}$

Solution

$$\lim_{x \to \infty} \frac{e^{1/x} - 1}{\sin \frac{1}{x}} = \frac{0}{0}$$

$$= \lim_{x \to \infty} \frac{-\frac{1}{x^2} e^{1/x}}{-\frac{1}{x^2} \cos \frac{1}{x}}$$

$$= \lim_{x \to \infty} \frac{e^{1/x}}{\cos \frac{1}{x}}$$

$$\underline{= 1}$$

Exercise

Evaluate the following limit $\quad \lim_{x \to 0^+} \sin x \sqrt{\frac{1-x}{x}}$

Solution

$$\lim_{x \to 0^+} \sin x \sqrt{\frac{1-x}{x}} = \lim_{x \to 0^+} \sin x \sqrt{\frac{x(1-x)}{x^2}}$$

$$= \lim_{x \to 0^+} \frac{\sin x}{x} \sqrt{x(1-x)} \qquad \lim_{x \to 0^+} \frac{\sin x}{x} = 1$$

$$= 1 \cdot 0$$

$$\underline{= 0}$$

Exercise

Evaluate the following limit $\quad \lim_{x \to 0} \left(\cot x - \frac{1}{x} \right)$

Solution

$$\lim_{x \to 0} \left(\cot x - \frac{1}{x} \right) = \infty - \infty$$

$$= \lim_{x\to 0}\left(\frac{\cos x}{\sin x} - \frac{1}{x}\right)$$

$$= \lim_{x\to 0}\left(\frac{x\cos x - \sin x}{x\sin x}\right) = \frac{0}{0}$$

$$= \lim_{x\to 0}\frac{\cos x - x\sin x - \cos x}{\sin x + x\cos x}$$

$$= \lim_{x\to 0}\frac{-x\sin x}{\sin x + x\cos x} = \frac{0}{0}$$

$$= \lim_{x\to 0}\frac{-\sin x - x\cos x}{\cos x + \cos x - x\sin x}$$

$$= \frac{0}{2}$$

$$\underline{= 0}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to\infty}\left(x - \sqrt{x^2+1}\right)$

Solution

$$\lim_{x\to\infty}\left(x - \sqrt{x^2+1}\right) = \infty - \infty$$

$$= \lim_{x\to\infty}\left(x - x\sqrt{1+\frac{1}{x^2}}\right)$$

$$= \lim_{x\to\infty} x\left(1 - \sqrt{1+\frac{1}{x^2}}\right) \qquad t = \frac{1}{x}\xrightarrow[x\to\infty]{} 0$$

$$= \lim_{t\to 0}\frac{1}{t}\left(1 - \sqrt{1+t^2}\right)$$

$$= \lim_{t\to 0}\frac{1 - \sqrt{1+t^2}}{t} = \frac{0}{0}$$

$$= \lim_{t\to 0}\frac{\frac{t}{\sqrt{1+t^2}}}{1}$$

$$\underline{= 0}$$

Exercise

Evaluate the following limit $\lim\limits_{\theta\to\frac{\pi}{2}^-}\left(\tan\theta - \sec\theta\right)$

Solution

$$\lim_{\theta \to \frac{\pi}{2}^-} (\tan\theta - \sec\theta) = \infty - \infty$$

$$= \lim_{\theta \to \frac{\pi}{2}^-} \left(\frac{\sin\theta}{\cos\theta} - \frac{1}{\cos\theta} \right)$$

$$= \lim_{\theta \to \frac{\pi}{2}^-} \frac{\sin\theta - 1}{\cos\theta} = \frac{0}{0}$$

$$= \lim_{\theta \to \frac{\pi}{2}^-} \frac{\cos\theta}{-\sin\theta}$$

$$\underline{= 0}$$

Exercise

Evaluate the following limit $\lim_{x \to 0^+} \ln x^{2x}$

Solution

$$\lim_{x \to 0^+} \ln x^{2x} = \lim_{x \to 0^+} 2x \ln x$$

$$= 2 \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}}$$

$$= 2 \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= -2 \lim_{x \to 0^+} x$$

$$\underline{= 0}$$

Exercise

Evaluate the following limit $\lim_{x \to 0} \ln(1 + 4x)^{3/x}$

Solution

$$\lim_{x \to 0} \ln(1 + 4x)^{3/x} = 3 \lim_{x \to 0} \frac{\ln(1 + 4x)}{x} = \frac{0}{0}$$

$$= 3 \lim_{x \to 0} \frac{\frac{4}{1 + 4x}}{1}$$

$$\underline{= 12}$$

Exercise

Evaluate the following limit $\lim\limits_{\theta\to\frac{\pi}{2}^-} \ln(\tan\theta)^{\cos\theta}$

Solution

$$\begin{aligned}
\lim_{\theta\to\frac{\pi}{2}^-} \ln(\tan\theta)^{\cos\theta} &= \lim_{\theta\to\frac{\pi}{2}^-} \cos\theta\ln(\tan\theta)\\
&= \lim_{\theta\to\frac{\pi}{2}^-} \frac{\ln(\tan\theta)}{\sec\theta}\\
&= \lim_{\theta\to\frac{\pi}{2}^-} \frac{\frac{\sec^2\theta}{\tan\theta}}{\sec\theta\tan\theta}\\
&= \lim_{\theta\to\frac{\pi}{2}^-} \frac{\sec\theta}{\tan^2\theta}\\
&= \frac{0}{\infty}\\
&= 0
\end{aligned}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to0^+} (1+x)^{\cot x}$

Solution

$$\lim_{x\to0^+} (1+x)^{\cot x} = 1^\infty$$

$$\begin{aligned}
\lim_{x\to0^+} \ln(1+x)^{\cot x} &= \lim_{x\to0^+} \cot x\ln(1+x)\\
&= \lim_{x\to0^+} \frac{\ln(1+x)}{\tan x} = \frac{0}{0}\\
&= \lim_{x\to0^+} \frac{\frac{1}{1+x}}{\sec^2 x}\\
&= \lim_{x\to0^+} \frac{1}{(1+x)\sec^2 x}\\
&= 1
\end{aligned}$$

$$\lim_{x\to0^+} (1+x)^{\cot x} = e^1 = e$$

Exercise

Evaluate the following limit $\quad \lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{\ln x}$

Solution

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{\ln x}=e^L$$

$$L=\lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^{\ln x}$$

$$=\lim_{x\to\infty}(\ln x)\ln\left(1+\frac{1}{x}\right)$$

$$=\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{\ln x}}$$

$$=\lim_{x\to\infty}\frac{-\frac{1}{x^2}}{1+\frac{1}{x}}\frac{1}{\frac{-1}{(\ln x)^2}\cdot\frac{1}{x}}$$

$$=\lim_{x\to\infty}\frac{1}{x^2}\frac{1}{\frac{x+1}{x}}x(\ln x)^2$$

$$=\lim_{x\to\infty}\frac{(\ln x)^2}{x+1}=\frac{\infty}{\infty}$$

$$=\lim_{x\to\infty}\frac{\frac{2\ln x}{x}}{1}$$

$$=2\lim_{x\to\infty}\frac{\ln x}{x}$$

$$=2\lim_{x\to\infty}\frac{\frac{1}{x}}{1}$$

$$=0$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{\ln x}=e^0=1$$

Exercise

Evaluate the following limit $\quad \lim\limits_{x\to\infty}\left(1+\frac{a}{x}\right)^{x}$

Solution

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{x}=e^L$$

$$L = \lim_{x\to\infty} \ln\left(1+\frac{a}{x}\right)^x$$

$$= \lim_{x\to\infty} x\ln\left(1+\frac{a}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{1}{-\frac{1}{x^2}}\frac{-\frac{a}{x^2}}{1+\frac{a}{x}}$$

$$= \lim_{x\to\infty} \frac{a}{1+\frac{a}{x}}$$

$$\underline{= a}$$

$$\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x \underline{= e^a}$$

Exercise

Evaluate the following limit $\lim_{x\to 0} \left(e^{5x}+x\right)^{1/x}$

Solution

$$\lim_{x\to 0} \left(e^{5x}+x\right)^{1/x} = e^L$$

$$L = \lim_{x\to 0} \ln\left(e^{5x}+x\right)^{1/x}$$

$$= \lim_{x\to 0} \frac{\ln\left(e^{5x}+x\right)}{x}$$

$$= \lim_{x\to 0} \frac{5e^{5x}+1}{e^{5x}+x}\cdot\frac{1}{1}$$

$$\underline{= 6}$$

$$\lim_{x\to 0} \left(e^{5x}+x\right)^{1/x} \underline{= e^6}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to 0}\left(e^{ax}+x\right)^{1/x}$

Solution

$$\lim_{x\to 0}\left(e^{ax}+x\right)^{1/x}=e^{L}$$

$$L=\lim_{x\to 0}\ \ln\left(e^{ax}+x\right)^{1/x}$$

$$=\lim_{x\to 0}\frac{\ln\left(e^{ax}+x\right)}{x}$$

$$=\lim_{x\to 0}\frac{ae^{ax}+1}{e^{ax}+x}\bullet\frac{1}{1}$$

$$\underline{=a+1}$$

$$\lim_{x\to 0}\left(e^{ax}+x\right)^{1/x}\ \underline{=e^{a+1}}$$

Exercise

Evaluate the following limit $\lim\limits_{x\to 0}\left(2^{ax}+x\right)^{1/x}$

Solution

$$\lim_{x\to 0}\left(2^{ax}+x\right)^{1/x}=e^{L}$$

$$L=\lim_{x\to 0}\ \ln\left(2^{ax}+x\right)^{1/x}$$

$$=\lim_{x\to 0}\frac{\ln\left(2^{ax}+x\right)}{x}$$

$$=\lim_{x\to 0}\frac{a2^{ax}\ln 2+1}{2^{ax}+x}\bullet\frac{1}{1}$$

$$\underline{=a\ln 2+1}$$

$$\lim_{x\to 0}\left(2^{ax}+x\right)^{1/x}=e^{a\ln 2+1}$$

$$=e\ e^{a\ln 2}$$

$$\underline{=e\cdot 2^{a}}$$

Exercise

Evaluate the following limit $\lim\limits_{x \to 0^+} (\tan x)^x$

Solution

$$\lim_{x \to 0^+} (\tan x)^x = e^L$$

$$\begin{aligned}
L &= \lim_{x \to 0^+} \ln (\tan x)^x \\
&= \lim_{x \to 0^+} x \ln (\tan x) \\
&= \lim_{x \to 0^+} \frac{\ln (\tan x)}{\frac{1}{x}} \\
&= \lim_{x \to 0^+} \frac{\sec^2 x}{\tan x} \cdot \frac{1}{-\frac{1}{x^2}} \\
&= -\lim_{x \to 0^+} \frac{x^2 \cos x}{\sin x \cos^2 x} \\
&= -\lim_{x \to 0^+} \frac{x^2}{\sin x \cos x} \\
&= -\lim_{x \to 0^+} \frac{x}{\sin x} \cdot \lim_{x \to 0^+} \frac{x}{\cos x} \\
&= -(1) \cdot (0) \\
&= 0
\end{aligned}$$

$$\lim_{x \to 0^+} (\tan x)^x = e^0 = 1$$

Exercise

The functions $f(x) = \left(x^x\right)^x$ *and* $g(x) = x^{\left(x^x\right)}$ are different functions. For example, $f(3) = 19{,}683$ and $g(3) \approx 7.6 \times 10^{12}$. Determine whether $\lim\limits_{x \to 0^+} f(x)$ and $\lim\limits_{x \to 0^+} g(x)$ are intermediate forms and evaluate the limits.

Solution

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(x^x\right)^x$$

$$\lim_{x \to 0^+} \ln \left(x^x\right)^x = \lim_{x \to 0^+} x \ln \left(x^x\right)$$

$$= \lim_{x\to 0^+} x^2 \ln x$$

$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^2}}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{2}{x^3}}$$

$$= -\lim_{x\to 0^+} \frac{x^2}{2}$$

$$= 0$$

$$\lim_{x\to 0^+} \left(x^x\right)^x = e^0 = 1$$

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} x^{\left(x^x\right)}$$

$$\lim_{x\to 0^+} \left(x^x\right) = e^L$$

$$L = \lim_{x\to 0^+} \ln\left(x^x\right)$$

$$= \lim_{x\to 0^+} x \ln x$$

$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to 0^+} (-x)$$

$$= 0$$

$$\lim_{x\to 0^+} \left(x^x\right) = e^0 = 1$$

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} x^{\left(x^x\right)}$$

$$= \lim_{x\to 0^+} 0^1$$

$$= 0$$

Exercise

Consider the function $g(x)=\left(1+\frac{1}{x}\right)^{x+a}$. show that if $0\le a<\frac{1}{2}$, then $g(x)\to e$ from below as $x\to\infty$; if $\frac{1}{2}\le a<1$, then $g(x)\to e$ from above as $x\to\infty$

Solution

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x+a}=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{a}$$
$$=e\bullet 1$$
$$=e$$

$$\lim_{x\to\infty}\ln g(x)=\ln 2=1$$

It suffices to determine whether $\ln g(x)-1$ is $(+)$ or $(-)$ as $x\to\infty$

Consider

$$\lim_{x\to\infty}x(\ln g(x)-1)=\lim_{x\to\infty}x\left[(x+a)\ln\left(1+\frac{1}{x}\right)-1\right]$$

Let $t=\frac{1}{x}\to 0 \quad\Rightarrow x=\frac{1}{t}$

$$\lim_{x\to\infty}x\left[(x+a)\ln\left(1+\frac{1}{x}\right)-1\right]=\lim_{t\to0}\frac{1}{t}\left[\left(\frac{1}{t}+a\right)\ln(1+t)-1\right]$$
$$=\lim_{t\to0}\frac{(1+at)\ln(1+t)-t}{t^2}=\frac{0}{0}$$
$$=\lim_{t\to0}\frac{a\ln(1+t)+\frac{1+at}{1+t}-1}{2t}$$
$$=\lim_{t\to0}\frac{a(1+t)\ln(1+t)+1+at-1-t}{2t(1+t)}$$
$$=\lim_{t\to0}\frac{a(1+t)\ln(1+t)+(a-1)t}{2t+2t^2}=\frac{0}{0}$$
$$=\lim_{t\to0}\frac{a\ln(1+t)+a+a-1}{2+4t}$$
$$=\lim_{t\to0}\frac{a\ln(1+t)+2a-1}{2+4t}$$
$$=\frac{2a-1}{2}$$
$$=a-\frac{1}{2}$$

When $a>\frac{1}{2} \quad\Rightarrow\quad g(x)>e$ *as* $x\to\infty$

If $0 \le a < \frac{1}{2} \quad \Rightarrow \quad g(x) < e \ \ as \ \ x \to \infty$

Exercise

Let $f(x) = (a+x)^x$, where $a > 0$

a) What is the domain of *f* (in terms of *a*)?

b) Describe the end behavior of *f* (near the left boundary of its domain and as $x \to \infty$).

c) Compute f'.

d) Show that *f* has a single local minimum at the point *z* that satisfies $(z+a)\ln(z+a) + z = 0$

e) Describe how $f(z)$ varies as *a* increases.

Solution

a) $a + x \ge 0 \quad \Rightarrow \quad Domain: \ \ [-a, \ \infty)$

b) $\displaystyle\lim_{x \to -a^+} (a+x)^x = e^L$

$$\begin{aligned} L &= \lim_{x \to -a^+} \ln(a+x)^x \\ &= \lim_{x \to -a^+} x\ln(a+x) \\ &= \lim_{x \to -a^+} \frac{\ln(a+x)}{\frac{1}{x}} \\ &= \lim_{x \to -a^+} \frac{\frac{1}{a+x}}{-\frac{1}{x^2}} \\ &= \lim_{x \to -a^+} \frac{-x^2}{a+x} \\ &= \frac{-a^2}{0^-} \\ &= \infty \end{aligned}$$

$$\lim_{x \to -a^+} (a+x)^x = e^\infty = \infty$$

c) $\ln f(x) = \ln(a+x)^x$

$$(\ln f(x))' = (x\ln(a+x))'$$

$$\frac{f'(x)}{f(x)} = \ln(a+x) + \frac{x}{a+x}$$

$$f'(x) = \left(\ln(a+x) + \frac{x}{a+x}\right)(a+x)^x$$

$$= (a+x)^x \ln(a+x) + x(a+x)^{x-1}$$

d) $f'(x) = \left(\ln(a+x) + \dfrac{x}{a+x}\right)(a+x)^x = 0$

$\ln(a+x) + \dfrac{x}{a+x} = 0$

$(a+x)\ln(a+x) + x = 0$

Let $z = a$

$(z+a)\ln(z+a) + z = 0$

$\ln(a+z) + \dfrac{z}{a+z} = 0$

$\ln(a+z) + \dfrac{z+a-a}{a+z} = 0$

$\ln(a+z) + 1 - \dfrac{a}{a+z} = 0$

$\ln(a+z) + -\dfrac{a}{a+z} = -1$

As z increases left side increases.

e) As $a \to \infty \;\Rightarrow z \to -\infty \;\Rightarrow f(x) \to 0$

Solution Section 3.6 – Newton's Method

Exercise

Use Newton's method to estimate the on real solution of $x^3+3x+1=0$.
Start with $x_0=0$ and then find x_2

Solution

$$y=x^3+3x+1 \quad\rightarrow\quad y'=3x^2+3$$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{x_n^3+3x_n+1}{3x_n^2+3}$$

$$x_0=0$$

$$x_1=x_0-\frac{x_0^3+3x_0+1}{3x_0^2+3}=0-\frac{0+3(0)+1}{3(0)+3}=\underline{-\frac{1}{3}}$$

$$x_2=x_1-\frac{x_1^3+3x_1+1}{3x_1^2+3}=-\frac{1}{3}-\frac{\left(-\frac{1}{3}\right)^3+3\left(-\frac{1}{3}\right)+1}{3\left(-\frac{1}{3}\right)+3}=\underline{-0.3222}$$

Exercise

Use Newton's method to estimate the on real solution of $x^4+x-3=0$. Start with $x_0=-1$ for the left-hand zero and with $x_0=1$ for the zero on the right. Then, in each case, find x_2

Solution

$$y=x^4+x-3 \quad\rightarrow\quad y'=4x^3+1$$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{x_n^4+x_n-3}{4x_n^3+1}$$

$$\boxed{x_0=-1}$$

$$x_1=x_0-\frac{x_0^4+x_0-3}{4x_0^3+1}=-1-\frac{(-1)^4+(-1)-3}{4(-1)^3+1}=\underline{-2}$$

$$x_2=x_1-\frac{x_1^4+x_1-3}{4x_1^3+1}=-2-\frac{(-2)^4+(-2)-3}{4(-2)^3+1}=\underline{-1.64516}$$

$\boxed{x_0 = 1}$

$$\underline{x_1} = x_0 - \frac{x_0^4 + x_0 - 3}{4x_0^3 + 1} = 1 - \frac{(1)^4 + (1) - 3}{4(1)^3 + 1} = \underline{\frac{6}{5}}$$

$$\underline{x_2} = x_1 - \frac{x_1^4 + x_1 - 3}{4x_1^3 + 1} = \frac{6}{5} - \frac{\left(\frac{6}{5}\right)^4 + \left(\frac{6}{5}\right) - 3}{4\left(\frac{6}{5}\right)^3 + 1} = \underline{1.16542}$$

Exercise

Use Newton's method to estimate the on real solution of $2x - x^2 + 1 = 0$. Start with $x_0 = 0$ for the left-hand zero and with $x_0 = 2$ for the zero on the right. Then, in each case, find x_2

Solution

$$y = 2x - x^2 + 1 \quad \rightarrow \quad y' = 2 - 2x$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{2x_n - x_n^2 + 1}{2 - 2x_n}$$

$\boxed{x_0 = 0}$

$$\underline{x_1} = x_0 - \frac{2x_0 - x_0^2 + 1}{2 - 2x_0} = 0 - \frac{2(0) - (0)^2 + 1}{2 - 2(0)} = \underline{-0.5}$$

$$\underline{x_2} = x_1 - \frac{2x_1 - x_1^2 + 1}{2 - 2x_1} = -0.5 - \frac{2(-0.5) - (-0.5)^2 + 1}{2 - 2(-0.5)} = \underline{-0.41667}$$

$\boxed{x_0 = 2}$

$$\underline{x_1} = x_0 - \frac{2x_0 - x_0^2 + 1}{2 - 2x_0} = 2 - \frac{2(2) - (2)^2 + 1}{2 - 2(2)} = \underline{2.5}$$

$$\underline{x_2} = x_1 - \frac{2x_1 - x_1^2 + 1}{2 - 2x_1} = 2.5 - \frac{2(2.5) - (2.5)^2 + 1}{2 - 2(2.5)} = \underline{2.41667}$$

Exercise

Use Newton's method to estimate the on real solution of $x^4 - 2 = 0$. Start with $x_0 = 1$ and then find x_2

Solution

$$y = x^4 - 2 \quad \to \quad y' = 4x^3$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - 2}{4x_n^3}$$

$$\boxed{x_0 = 1}$$

$$x_1 = x_0 - \frac{x_0^4 - 2}{4x_0^3} = 1 - \frac{(1)^4 - 2}{4(1)^3} = \underline{1.25}$$

$$x_2 = x_1 - \frac{x_1^4 - 2}{4x_1^3} = 1.25 - \frac{(1.25)^4 - 2}{4(1.25)^3} \approx \underline{1.1935}$$

Exercise

Use the Newton's method to approximate the roots to ten digits of $f(x) = 3x^3 - 4x^2 + 1$

Solution

By inspection: $x_1 = 1$ (***root***)

$$\begin{array}{c|cccc} 1 & 3 & -4 & 0 & 1 \\ & & 3 & -1 & -1 \\ \hline & 3 & -1 & -1 & 0 \end{array} \quad \to \quad 3x^2 - x - 1$$

$$f(x) = (x-1)\left(3x^2 - x - 1\right)$$

We apply Newton's method to $g(x) = 3x^2 - x - 1$

$g(0) = -1 \quad g(1) = 1 \quad g'(x) = 6x - 1$

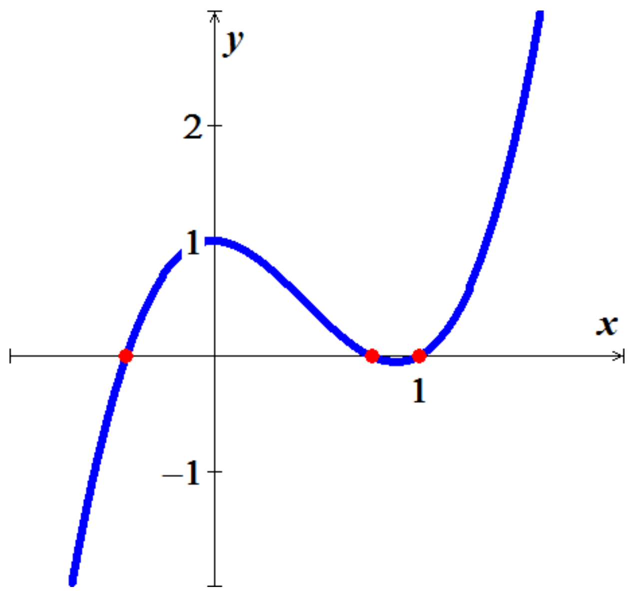
n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	0.0000000000	–1	–1	–1
1	–1	3.000000	–7.000000	–0.5714285714
2	–0.5714285714	0.5510204082	–4.4285714286	–0.4470046083
3	–0.4470046083	0.0464439678	–3.6820276497	–0.4343909149
4	–0.4343909149	0.0004773158	–3.6063454894	–0.4342585605

5	–0.4342585605	0.0000000525	–3.6055513629	–0.4342585459
6	–0.4342585459	0.0000000000	–3.6055512755	–0.4342585459

$x_2 \approx -0.4342585459$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	1	1	5	0.8
1	0.8000000000	0.1200000000	3.8000000000	0.7684210526
2	0.7684210526	0.0029916898	3.6105263158	0.7675924505
3	0.7675924505	0.0000020597	3.6055547030	0.7675918792
4	0.7675918792	–0.0000000000	3.6055512755	0.7675918792

$x_3 \approx 0.7675918792$



Exercise

Use the Newton's method to approximate the roots to ten digits of $f(x) = e^{-2x} + 2e^{x} - 6$

Solution

$f'(x) = -2e^{-2x} + 2e^{x}$

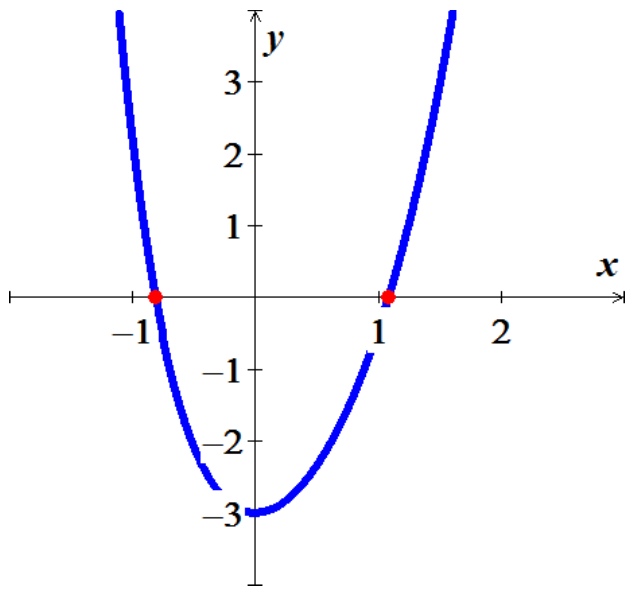
n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	–1.0000000000	2.1248149813	–14.0423533155	–0.8486852642
1	–0.8486852642	0.3155271886	–10.0631909420	–0.8173306780
2	–0.8173306780	0.0109389885	–9.3722247034	–0.8161635070

3	–0.8161635070	0.0000145618	–9.3472814463	–0.8161619491
4	–0.8161619491	0.0000000000	–9.3472481901	–0.8161619491

$x \approx -0.8161619491$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	1.0000000000	–0.4281010598	5.1658930904	1.0828706774
1	1.0828706774	0.0209547377	5.6769600505	1.0791794875
2	1.0791794875	0.0000433190	5.6534997356	1.0791718252
3	1.0791718252	0.0000000002	5.6534511061	1.0791718251

$x \approx 1.0791718251$



Exercise

Use the Newton's method to approximate the roots to ten digits of $f(x) = 2x^5 - 6x^3 - 4x + 2$

Solution

$f(x) = 10x^4 - 18x^2 - 4$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	0.0000000000	2.0000000000	–4.0000000000	0.5000000000
1	0.5000000000	–0.6875000000	–7.8750000000	0.4126984127
2	0.4126984127	–0.0485945125	–6.7756706818	0.4055265009
3	0.4055265009	–0.0003088207	–6.6896876153	0.4054803372
4	0.4054803372	–0.0000000128	–6.6891368363	0.4054803353

$x \approx 0.4054803353$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	−2.0000000000	−6.0000000000	84.0000000000	−1.9285714286
1	−1.9285714286	−0.6062020289	67.3894731362	−1.9195759282
2	−1.9195759282	−0.0087501134	65.4495366742	−1.9194422357
3	−1.9194422357	−0.0000019108	65.4209537375	−1.9194422065
4	−1.9194422065	0.0000000000	65.4209474938	−1.9194422065

$x \approx -1.9194422065$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	2.0000000000	10.0000000000	84.0000000000	1.8809523810
1	1.8809523810	1.6364873659	57.4894829058	1.8524865245
2	1.8524865245	0.0789319071	51.9953690888	1.8509684681
3	1.8509684681	0.0002159404	51.7110173775	1.8509642921
4	1.8509642921	0.0000000016	51.7102363689	1.8509642921

$x \approx 1.8509642921$