Section 3.5 – Language of Hypothesis Testing

If, under a given assumption, the probability of a particular observed event is exceptionally small, we conclude that the assumption is probably not correct.

A *hypothesis* is a statement regarding a characteristic of one or more populations.

Example

In 2008, 62% of American adults regularly volunteered their time for charity work. A researcher believes that this percentage is different today

Solution

According to a study published in March, 2006 the mean length of a phone call on a cellular telephone was 3.25 minutes. A researcher believes that the mean length of a call has increased since then.

Using an old manufacturing process, the standard deviation of the amount of wine put in a bottle was 0.23 ounces. With new equipment, the quality control manager believes the standard deviation has decreased.

Caution

We test these types of statements using sample data because it is usually impossible or impractical to gain access to the entire population. If population data are available, there is no need for inferential statistics.

Example

ProCare Industries, Ltd provided a product called "Gender Choice", which, according to advertising claims, allowed couples to "increase your chances of having a girl up to 80%." Suppose we conduct an experiment with 100 couples who want to have baby girls, and they all follow the Gender Choice "easy-to-use in-home system" described in the pink package designed for girls. Assuming that Gender Choice has no effect and using only common sense and no formal statistical methods, what should we conclude about the assumption of "no effect" from Gender Choice if 100 couples using Gender Choice have 100 babies consisting of the following
  a) 52 girls
  b) 97 girls

Solution

  a) We normally expect around 50 girls in 100 births. The result of 52 girls is close to 50, so the Gender Choice product is effective. The result of 52 girls could easily occur by chance, so there isn't sufficient evidence to say that Gender Choice is effective, even though the sample proportion of girls is greater than 50%.
  b) The result of 97 girls in 100 births is extremely unlikely to occur by chance. Either an extremely rare event has occurred by chance, or Gender Choice is effective. The extremely low probability of getting 97 girls suggests that Gender Choice is effective.

Using Statistics

➢ A hypothesis is a statement or assertion about the state of nature (about the true value of an unknown population parameter):
   ✓ The accused is innocent
   ✓ $\mu = 100$

➢ Every hypothesis implies its contradiction or alternative
   ✓ The accused is guilty
   ✓ $\mu \neq 100$

➢ A hypothesis is either true or false, and you may fail to reject it or you may reject it on the basis of information
   ✓ Trial testimony and evidence
   ✓ Sample data

Making Decision

One hypothesis is maintained to be true until a decision is made to reject it as false:
   ✓ Guilt is proven "beyond a reasonable doubt"
   ✓ The alternative is highly improbable

A decision to fail to reject or reject a hypothesis may be:
   ✓ Correct
      A true hypothesis may not be rejected
         » An innocent defendant may be acquitted
      A false hypothesis may be rejected
         » A guilty defendant may be convicted

   ✓ Incorrect
      A true hypothesis may be rejected
         » An innocent defendant may be convicted
      A false hypothesis may not be rejected
         » A guilty defendant may be acquitted

Components of a Formal Hypothesis Test

Hypothesis testing is a procedure, based on sample evidence and probability, used to test statements regarding a characteristic of one or more populations.

Null Hypothesis: The ***null hypothesis***, denoted H_0, is a statement to be tested. The null hypothesis is a statement of no change, no effect or no difference and is assumed true until evidence indicates otherwise.

➢ The null hypothesis (denoted by H_0) is a statement that the value of a population parameter (such as proportion, mean, or standard deviation) is equal to some claimed value.

➢ We test the null hypothesis directly.

➢ Either reject H_0 or fail to reject H_0.

➢ $H_0 : \mu = 100$

Alternative Hypothesis: The *alternative hypothesis*, denoted H_1, is a statement that we are trying to find evidence to support.

➢ The alternative hypothesis (denoted by H_1 or H_a or H_A) is the statement that the parameter has a value that somehow differs from the null hypothesis.

➢ The symbolic form of the alternative hypothesis must use one of these symbols: $\neq, <, >$.

➢ $H_1 : \mu \neq 100$

Three ways to set up the null and alternative hypotheses

1. Equal versus not equal hypothesis (two-tailed test)

   H_0: parameter $=$ some value

   H_1: parameter $\neq$ some value

2. Equal versus less than (left-tailed test)

   H_0: parameter $=$ some value

   H_1: parameter $<$ some value

3. Equal versus greater than (right-tailed test)

   H_0: parameter $=$ some value

   H_1: parameter $>$ some value

The null hypothesis is a statement of *status quo* or *no difference* and always contains a statement of equality. The null hypothesis is assumed to be true until we have evidence to the contrary. We seek evidence that supports the statement in the alternative hypothesis.

Example

For each of the following claims, determine the null and alternative hypotheses. State whether the test is two-tailed, left-tailed or right-tailed.

a) In 2008, 62% of American adults regularly volunteered their time for charity work. A researcher believes that this percentage is different today.

b) According to a study published in March, 2006 the mean length of a phone call on a cellular telephone was 3.25 minutes. A researcher believes that the mean length of a call has increased since then.

c) Using an old manufacturing process, the standard deviation of the amount of wine put in a bottle was 0.23 ounces. With new equipment, the quality control manager believes the standard deviation has decreased.

Solution

a) In 2008, 62% of American adults regularly volunteered their time for charity work. A researcher believes that this percentage is different today.

The hypothesis deals with a population proportion, p. If the percentage participating in charity work is no different than in 2008, it will be 0.62 so the *null hypothesis is* $H_0 : p = 0.62$.

Since the researcher believes that the percentage is different today, the *alternative hypothesis is a two-tailed hypothesis:* $H_1 : p \neq 0.62$.

b) According to a study published in March, 2006 the mean length of a phone call on a cellular telephone was 3.25 minutes. A researcher believes that the mean length of a call has increased since then.

The hypothesis deals with a population mean, μ. If the mean call length on a cellular phone is no different than in 2006, it will be 3.25 minutes so the *null hypothesis is* $H_0 : \mu = 3.25$

Since the researcher believes that the mean call length has increased, the *alternative hypothesis is:* $H_1 : \mu > 3.25$, *a right-tailed test.*

c) Using an old manufacturing process, the standard deviation of the amount of wine put in a bottle was 0.23 ounces. With new equipment, the quality control manager believes the standard deviation has decreased.

The hypothesis deals with a population standard deviation, σ. If the standard deviation with the new equipment has not changed, it will be 0.23 ounces so the *null hypothesis is* $H_0 : \sigma = 0.23$.

Since the quality control manager believes that the standard deviation has decreased, the *alternative hypothesis is:* $H_1 : \sigma < 0.23$, *a left-tailed test.*

Test Statistic

Definition

The test statistic is a value used in making a decision about the null hypothesis, and is found by converting the sample statistic to a score with the assumption that the null hypothesis is true.

Test Statistic - Formulas

Test statistic for **proportion**

$$z = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}}$$

Test statistic for **mean**

$$z = \frac{\overline{x} - \mu}{\frac{\sigma}{\sqrt{n}}} \quad or \quad z = \frac{\overline{x} - \mu}{\frac{s}{\sqrt{n}}}$$

Test statistic for **standard deviation** $\quad \chi^2 = \frac{(n-1)s^2}{\sigma^2}$
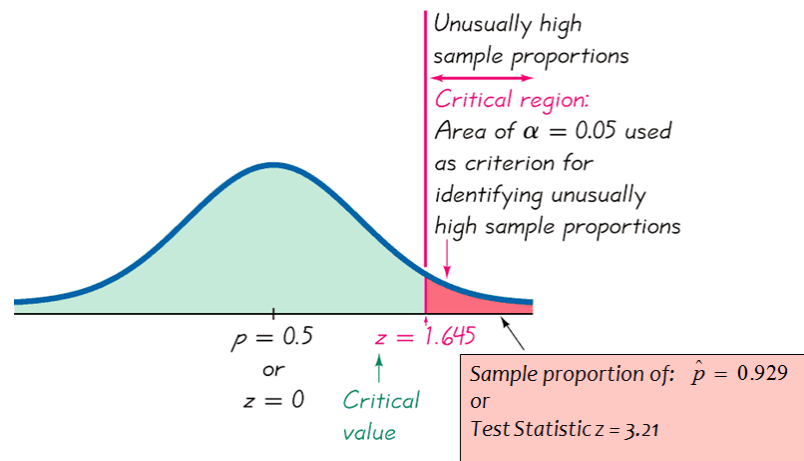
Example

Let's again consider the claim that the XSORT method of gender selection increases the likelihood of having a baby girl. Preliminary results from a test of the XSORT method of gender selection involved 14 couples who gave birth to 13 girls and 1 boy. Use the given claim and the preliminary results to calculate the value of the test statistic. Use the format of the test statistic given above, so that a normal distribution is used to approximate a binomial distribution. (There are other exact methods that do not use the normal approximation.)

Solution

The claim that the XSORT method of gender selection increases the likelihood of having a baby girl results in the following null and alternative hypothesis $H_0 : p = 0.5$ and $H_1 : p > 0.5$. We work under the assumption that the null hypothesis is true with $p = 0.5$. The sample proportion of 13 girls in 14 births results in $\hat{p} = \frac{13}{14} = 0.929$. Using $p = 0.5$, $\hat{p} = 0.929$ and $n = 14$, we find the value of the test statistic as follows:

$$z = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}}$$

$$= \frac{0.929 - 0.5}{\sqrt{\frac{(0.5)(0.5)}{14}}}$$

$$= 3.21$$

Unusually high sample proportions

Critical region:
Area of $\alpha = 0.05$ used as criterion for identifying unusually high sample proportions

$p = 0.5$ or $z = 0$

$z = 1.645$
Critical value

Sample proportion of: $\hat{p} = 0.929$ or Test Statistic $z = 3.21$

We know that a z score of 3.21 is "unusual" (because it is greater than 2). It appears that in addition to being greater than 0.5, the sample proportion of 13/14 or 0.929 is significantly greater than 0.5. The figure on the next slide shows that the sample proportion of 0.929 does fall within the range of values considered to be significant because they are so far above 0.5 that they are not likely to occur by chance (assuming that the population proportion is $p = 0.5$).

Critical Region: The *critical region* (or *rejection region*) is the set of all values of the test statistic that cause us to reject the null hypothesis. For example, see the red-shaded region in the previous figure.

Significance Level: The significance level (denoted by α) is the probability that the test statistic will fall in the critical region when the null hypothesis is actually true. Common choices for α are 0.05, 0.01, and 0.10.

Critical Value: A critical value is any value that separates the critical region (where we reject the null hypothesis) from the values of the test statistic that do not lead to rejection of the null hypothesis. The critical values depend on the nature of the null hypothesis, the sampling distribution that applies, and the significance level α. The critical value of $z = 1.645$ corresponds to a significance level of $\alpha = 0.05$.

Example

Using a significant level of $\alpha = 0.05$, find the critical z value for the alternative hypothesis $H_1 : p > 0.5$ (assuming that the normal distribution can be used to approximate the binomial distribution). This alternative hypothesis is used to test the claim that the XSORT method of gender selection is effective, so that baby girls are more likely, with a proportion greater than 0.5

Solution

With $H_1 : p > 0.5$, the critical region is in the right tail. With the right tail area of 0.05, the critical value is found to be $z = 1.645$. If the right-tailed critical region is 0.05, the cumulative area to the left of the critical value is 0.95 is $z = 1.645$.
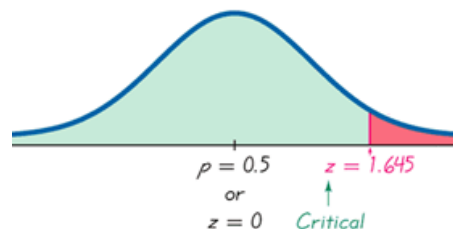

Example

Using a significant level of $\alpha = 0.05$, find the critical z value for the alternative hypothesis $H_1 : p \neq 0.5$ (assuming that the normal distribution can be used to approximate the binomial distribution).

Solution

With $H_1 : p \neq 0.5$, the significant level is 0.05, each of the two tails has an area of 0.025.

From the Normal Distribution Table, $z = -1.96$ and $z = 1.96$ (right side)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
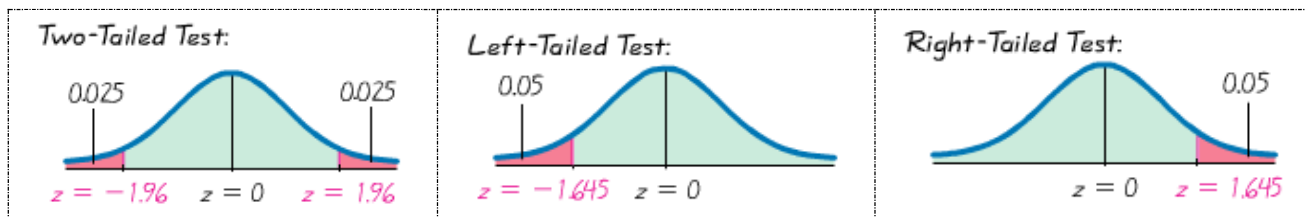
P-Value

The **P-value** (or **p-value** or **probability value**) is the probability of getting a value of the test statistic that is **at least as extreme** as the one representing the sample data, assuming that the null hypothesis is true.

Critical region in the **left** tail:	P-value = area to the **left** of the test statistic
Critical region in the **right** tail:	P-value = area to the **right** of the test statistic
Critical region in **two** tails:	P-value = **twice** the area in the tail beyond the test statistic

The null hypothesis is rejected if the P-value is very small, such as 0.05 or less.
Here is a memory tool useful for interpreting the P-value:
  ✓ If the P is low, the null must go.
  ✓ If the P is high, the null will fly.

Caution

Don't confuse a *P*-value with a proportion *p*. Know this distinction:
P-value = probability of getting a test statistic at least as extreme as the one representing sample data
p = population proportion

Example

Consider the claim that with the XSORT method of gender selection, the likelihood of having a baby girl is different from $p = 0.5$, and use the test statistic $z = 3.21$ found from 13 girls in 14 births. First determine whether the given conditions result in a critical region in the right tail, left tail, or two tails, then use Figure 8-5 to find the *P*-value. Interpret the *P*-value.

Solution

The claim that the likelihood of having a baby girl is different from $p = 0.5$ can be expressed as $p \neq 0.5$ so the critical region is in two tails. Using Figure 8-5 to find the *P*-value for a two-tailed test, we see that the *P*-value is *twice* the area to the right of the test statistic $z = 3.21$. From Normal Distribution Table, the area to the right of $z = 3.21$ is 0.0007.
In this case, the *P*-value is twice the area to the right of the test statistic, so we have:

$$P\text{–value} = 2 \times 0.0007 = 0.0014$$

➢ The *P*-value is 0.0014 (or 0.0013 if greater precision is used for the calculations). The small *P*-value of 0.0014 shows that there is a very small chance of getting the sample results that led to a test statistic of $z = 3.21$. This suggests that with the XSORT method of gender selection, the likelihood of having a baby girl is different from 0.5.

## Types of Hypothesis Tests: Two-tailed, Left-tailed, Right-tailed

The *tails* in a distribution are the extreme regions bounded by critical values.
Determinations of *P*-values and critical values are affected by whether a critical region is in two tails, the left tail, or the right tail. It therefore becomes important to correctly characterize a hypothesis test as two-tailed, left-tailed, or right-tailed.
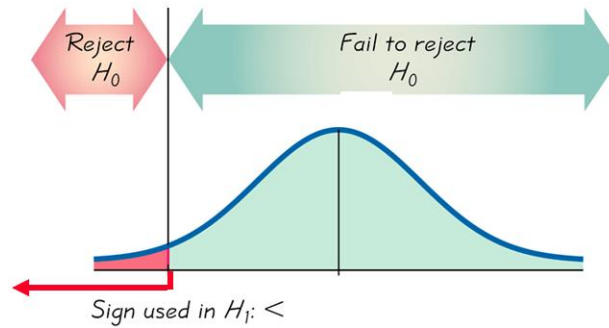
$$H_0 : \ = \quad \& \quad H_1 : \ \neq$$

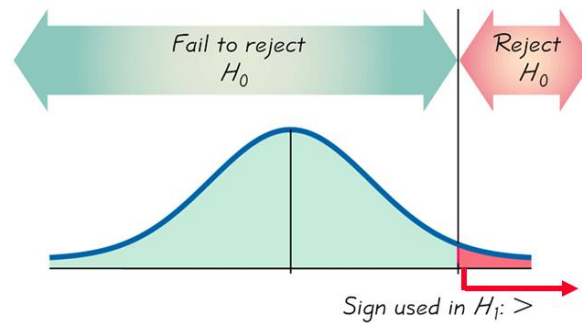α is divided equally between the two tails of the critical region

Two–tailed test: The critical region is the two extreme regions (tails) under the curve
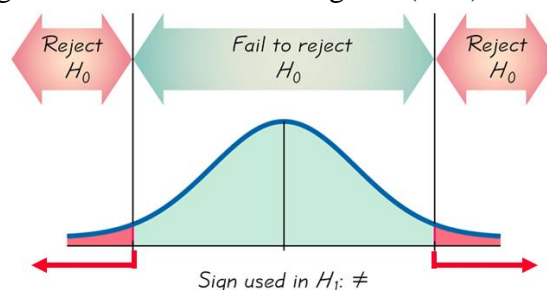
Left–tailed test: The critical region is in the extreme region (tail) under the curve

Right–tailed test: The critical region is in the extreme right region (tail) under the curve

Conclusions in Hypothesis Testing

We always test the null hypothesis. The initial conclusion will always be one of the following:
    1. Reject the null hypothesis.
    2. Fail to reject the null hypothesis.

Decision Criterion

P-value method:
Using the significance level α:
    If P-value $\leq \alpha$, **reject** H_0.

    If P-value $> \alpha$, **fail to reject** H_0.

Traditional method:
    If the test statistic falls within the critical region, **reject** H_0.

    If the test statistic does not fall within the critical region, **fail to reject** H_0.

Another option:
    Instead of using a significance level such as 0.05, simply identify the P-value and leave the decision to the reader.

Confidence Intervals:

A confidence interval estimate of a population parameter contains the likely values of that parameter.

If a confidence interval does not include a claimed value of a population parameter, reject that claim.

Example

Suppose a geneticist claims that the XSORT method of gender selection increases the likelihood of a baby girl. This claim of $p > 0.5$ becomes the alternative hypothesis, while the null hypothesis becomes $p = 0.5$. Further suppose that the sample evidence causes us to reject the null hypothesis of $p = 0.5$. State the conclusion in simple, nontechnical terms.

Solution

Because the original claim does not contain equality, it becomes the alternative hypothesis. Because we reject the null hypothesis, we conclude "There is sufficient evidence to support the claim that the XSORT method of gender selection increases the likelihood of a baby girl.

Type I and Type II Errors

Type I error: The mistake of rejecting the null hypothesis when it is actually true. The symbol α (alpha) is used to represent the probability of a type I error.

Type II error: The mistake of failing to reject the null hypothesis when it is actually false. The symbol β (beta) is used to represent the probability of a type II error.

		Reality	
		H_0 is True	H_1 is True
Conclusion	Do Not (Fail to) Reject H_0	Correct Conclusion	Type **II** Error $P(\text{type II error}) = \beta$
	Reject H_0	Type **I** Error $P(\text{type I error}) = \alpha$	Correct Conclusion

Four Outcomes from Hypothesis Testing

1. Reject the null hypothesis when the alternative hypothesis is true. This decision would be correct.
2. Do not reject the null hypothesis when the null hypothesis is true. This decision would be correct.
3. Reject the null hypothesis when the null hypothesis is true. This decision would be incorrect. This type of error is called a Type I error.
4. Do not reject the null hypothesis when the alternative hypothesis is true. This decision would be incorrect. This type of error is called a Type II error.

Example

Assume that we are conducting a hypothesis test of the claim that a method of gender selection increases the likelihood of a baby girl, so that the probability of a baby girls is $p > 0.5$. Here are the null and alternative hypotheses: $H_0 : p = 0.5$, and $H_1 : p > 0.5$.

   a) Identify a type I error.
   b) Identify a type II error.

Solution

   a) A type I error is the mistake of rejecting a true null hypothesis, so this is a type I error: Conclude that there is sufficient evidence to support $p > 0.5$, when in reality $p = 0.5$.
   b) A type II error is the mistake of failing to reject the null hypothesis when it is false, so this is a type II error: Fail to reject $p = 0.5$ (and therefore fail to support $p > 0.5$) when in reality $p > 0.5$.

Example

For each of the following claims, explain what it would mean to make a Type I error. What would it mean to make a Type II error?

 a) In 2008, 62% of American adults regularly volunteered their time for charity work. A researcher believes that this percentage is different today.
 b) According to a study published in March, 2006 the mean length of a phone call on a cellular telephone was 3.25 minutes. A researcher believes that the mean length of a call has increased since then.

Solution

 a) In 2008, 62% of American adults regularly volunteered their time for charity work. A researcher believes that this percentage is different today.
 A ***Type I error*** is made if the researcher concludes that $p \neq 0.62$ when the true proportion of Americans 18 years or older who participated in some form of charity work is currently 62%.
 A ***Type II error*** is made if the sample evidence leads the researcher to believe that the current percentage of Americans 18 years or older who participated in some form of charity work is still 62% when, in fact, this percentage differs from 62%.

 b) According to a study published in March, 2006 the mean length of a phone call on a cellular telephone was 3.25 minutes. A researcher believes that the mean length of a call has increased since then.
 A ***Type I error*** occurs if the sample evidence leads the researcher to conclude that $\mu > 3.25$ when, in fact, the actual mean call length on a cellular phone is still 3.25 minutes.
 A ***Type II error*** occurs if the researcher fails to reject the hypothesis that the mean length of a phone call on a cellular phone is 3.25 minutes when, in fact, it is longer than 3.25 minutes.

Controlling Type I and Type II Errors

 * For any fixed a, an increase in the sample size n will cause a decrease in b.
 * For any fixed sample size n, a decrease in a will cause an increase in b. Conversely, an increase in a will cause a decrease in b.
 * To decrease both a and b, increase the sample size.

$\alpha = P$ (Type I Error)
$\quad = P$ (rejecting H_0 when H_0 is true)

$B = P$ (Type II Error)
$\quad = P$ (not rejecting H_0 when H_1 is true)

The probability of making a Type I error, α, is chosen by the researcher *before* the sample data is collected.

The level of significance, α, is the probability of making a Type I error.

Comprehensive Hypothesis Test

A confidence interval estimate of a population parameter contains the likely values of that parameter. We should therefore reject a claim that the population parameter has a value that is not included in the confidence interval.

Table 8-2	Confidence Level for Confidence Interval	
	Two-Tailed Test	One-Tailed Test
Significance	0.01	99%	98%
Level for	0.05	95%	90%
Hypothesis	0.10	90%	80%
Test			

Definition

The ***power of a hypothesis test*** is the probability $(1 - \beta)$ of rejecting a false null hypothesis. The value of the power is computed by using a particular significance level α and a particular value of the population parameter that is an alternative to the value assumed true in the null hypothesis.
That is, the power of the hypothesis test is the probability of supporting an alternative hypothesis that is true.

Power and the Design of Experiments

Just as 0.05 is a common choice for a significance level, a power of at least 0.80 is a common requirement for determining that a hypothesis test is effective. (Some statisticians argue that the power should be higher, such as 0.85 or 0.90.) When designing an experiment, we might consider how much of a difference between the claimed value of a parameter and its true value is an important amount of difference. When designing an experiment, a goal of having a power value of at least 0.80 can often be used to determine the minimum required sample size.

Exercises   *Section* 3.5 – Language of Hypothesis Testing

1. Bottles of Bayer aspirin are labeled with a statement that the tablets each contain 325 mg of aspirin. A quality control manager claims that a large sample of data can be used to support the claim that the mean amount of aspirin in the tablets is equal to 325 mg, as the label indicates. Can a hypothesis test be used to support that claim? Why or Why not?

2. In the preliminary results from couples using the Gender Choice method of gender selection to increase the likelihood of having a baby girl, 20 couples used the Gender Choice method with the result that 8 of them had baby girls and 12 had baby boys. Given that the sample proportion of girls is $\frac{8}{20}$ *or* 0.4, can the sample data support the claim that the proportion of girls is greater than 0.5? Can any sample proportion less than 0.5 be used to support a claim that the population proportion is greater than 0.5?

3. Express the null hypothesis H_0 and alternative hypothesis H_1 in symbolic form. Be sure to use the correct symbol $(\mu,\ p,\ \sigma)$ for indicated parameter

    a) The mean annual income of employees who took a statistics course is greater than $60,000.
    b) The proportion of people aged 18 to 25 who currently use illicit drugs is equal to 0.20 (or 20%).
    c) The standard deviation of human body temperatures is equal to 0.62°F.
    d) The majority of college students have credit cards.
    e) The proportion of homes with fire extinguishers is 0.80.
    f) The mean weight of plastic discarded by households in one week is less than 1 kg.

4. Assume that the normal distribution applies and find the critical z values.

    a) Two-tailed test: $\alpha = 0.01$.
    b) Right-tailed test: $\alpha = 0.02$.
    c) Left-tailed test: $\alpha = 0.10$.
    d) $\alpha = 0.05$; H_1 is $p \neq 0.4$
    e) $\alpha = 0.01$; H_1 is $p > 0.5$
    f) $\alpha = 0.005$; H_1 is $p < 0.8$
    g) $\alpha = 0.05$ for two-tailed test
    h) $\alpha = 0.05$ for left-tailed test
    i) $\alpha = 0.08$; H_1 is $\mu \neq 3.25$

5. The claim is that the proportion of peas with yellow pods is equal to 0.25 (or 25%). The sample statistics from one of Mendel's experiments include 580 peas with 152 of them having yellow pods. Find the value of the test statistic z using $z = \dfrac{\hat{p} - p}{\sqrt{\dfrac{pq}{n}}}$

6. The claim is that less than $\frac{1}{2}$ of adults in U.S. have carbon monoxide detectors. A KRC Research survey of 1005 adults resulted in 462 who have carbon monoxide detectors. Find the value of the test statistic z using $z = \dfrac{\hat{p} - p}{\sqrt{\dfrac{pq}{n}}}$

7. The claim is that more than 25% of adults prefer Italian food as their favorite ethnic food. A Harris Interactive survey of 1122 adults resulted in 314 who say that Italian food is their favorite ethnic food. Find the value of the test statistic z using $z = \dfrac{\hat{p} - p}{\sqrt{\dfrac{pq}{n}}}$

8. Find *P*-value by using a 0.05 significance level and state the conclusion about the null hypothesis. (reject the null hypothesis or fail to reject the null hypothesis)
   a) The test statistic in a left-tailed test is $z = -1.25$
   b) The test statistic in a right-tailed test is $z = 2.50$
   c) The test statistic in a two-tailed test is $z = 1.75$
   d) With $H_1 : p \neq 0.707$, the test statistic is $z = -2.75$
   e) With $H_1 : p > \frac{1}{4}$, the test statistic is $z = 2.30$
   f) With $H_1 : p < 0.777$, the test statistic is $z = -2.95$

9. The percentage of nonsmokers exposed to secondhand smoke is equal to 41%. Identify the type I error and type II error.

10. The percentage of Americans who believe that life exists only on earth is equal to 20%. Identify the type I error and type II error.

11. The percentage of college students who consume alcohol is greater than 70%. Identify the type I error and type II error.

12. An entomologist writes an article in a scientific journal which claims that fewer than 13 in 10,000 male fireflies are unable to produce light due to a genetic mutation. Use the parameter p, the true proportion of fireflies unable to produce light. Express the null hypothesis and the alternative hypothesis in symbolic form. $(\mu, \, p, \, \sigma)$