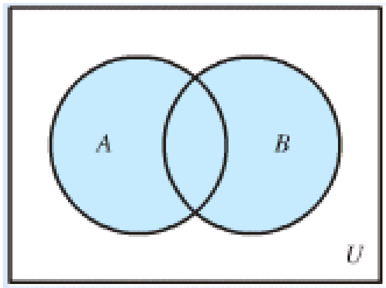
Section 1.8 – Set Operations

Union of Two Sets

Let *A* and *B* be sets, the ***union*** of the sets *A* and *B*, denoted by $A \cup B$, is the set that contains those elements that are either in *A* or in *B*, or in both.

$$A \cup B = \{x \mid x \in A \lor x \in B\}$$



Example

Let $A = \{1, 3, 5, 7, 9, 11\}$, $B = \{3, 6, 9, 12\}$. Find each set $A \cup B$

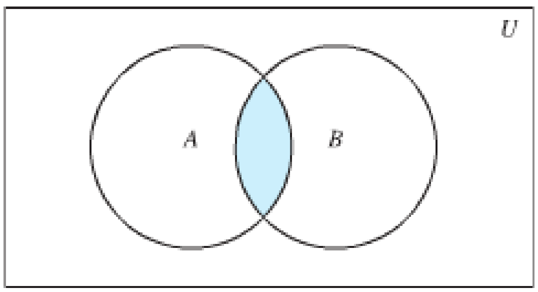
Solution

$$A \cup B = \{1, 3, 5, 6, 7, 9, 11, 12\}$$

Intersection of Two Sets

Let *A* and *B* be sets, the ***intersection*** of the sets *A* and *B*, denoted by $A \cap B$, is the set containing those elements in both *A* or in *B*.

$$A \cap B = \{x \mid x \in A \land x \in B\}$$



Example

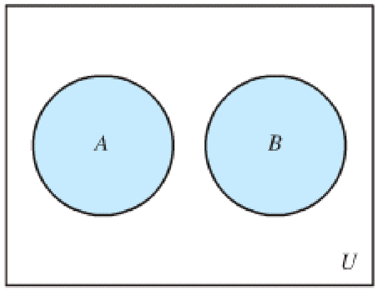
Let $A = \{3, 6, 9\}$, $B = \{2, 4, 6, 8\}$, find $A \cap B$

Solution

$$A \cap B = \{6\}$$

Disjoint Sets

For any sets *A* and *B*, if *A* and *B* are ***disjoint*** sets, then their intersection is the empty set $A \cap B = \phi$



Example

Let $A = \{1,3,5,7,9\}$, $B = \{2,4,6,8,10\}$, find $A \cap B$

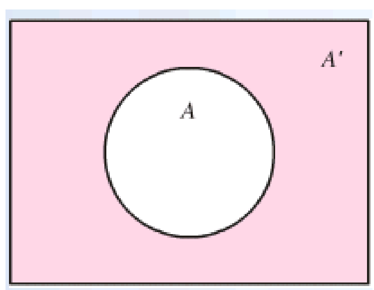
Solution

$A \cap B = \varnothing$. Therefore, *A* and *B* are disjoint.

Complement of a Set

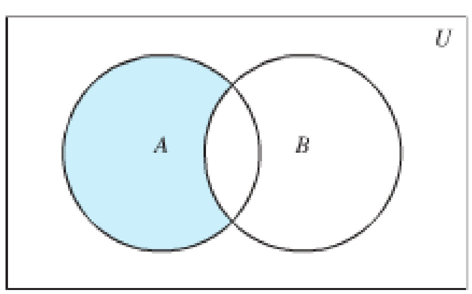
Let *A* be any set, with *U* representing the universal set, then the complement of *A*.

$$A' \text{ or } \bar{A} = \{x \mid x \notin A \text{ and } x \in U\}$$



Difference of two Sets

Let *A* and *B* be sets, the ***difference*** of *A* and *B*, denoted by $A - B$, is the set containing those elements that are *A* but not in *B*. The difference of *A* and *B* is also called the complement of *B* with respect to *A*.



Example

Find {1, 3, 5} – {1, 2, 3}

Solution

{1, 3, 5} – {1, 2, 3} = {5}

Example

What is the difference of the set of computer science majors at the school and the set of mathematics majors at the school?

Solution

The difference is the set of all computer science majors at your school are not also mathematics majors.

Example

Let *A* be the set of positive integers greater than 10 (with universal set the set of all positive integers). Find $\overline{A}$

Solution

$$\overline{A} = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10\}$$

Set Identities

Identity	Name
$A \cap U = A$ $A \cup \varnothing = A$	***Identity laws***
$A \cup U = U$ $A \cap \varnothing = \varnothing$	***Domination laws***
$A \cup A = A$ $A \cap A = A$	***Idempotent laws***
$\overline{\left(\overline{A}\right)} = A$	***Complementation laws***
$A \cup B = B \cup A$ $A \cap B = B \cap A$	***Commutative laws***
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap (C)$	***Associative laws***
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	***Distributive laws***
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	***De Morgan's laws***
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	***Absorption laws***
$A \cup \overline{A} = U$ $A \cap \overline{A} = \varnothing$	***Complement laws***

Example

Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution

1. We need to show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

   Suppose that $x \in \overline{A \cap B} \quad \Rightarrow \quad x \notin A \cap B$ (by the definition of complement)

   Using the definition of the intersection, we see that the proposition $\neg\left((x \in A) \wedge (x \in B)\right)$ is true.

   $\neg(x \in A)$ *or* $\neg(x \in B)$ ***By applying De Morgan's law of the proposition***

   $x \notin A$ *or* $x \notin B$ ***Using the definition of the negation of proposition***

   $x \in \overline{A}$ *or* $x \in \overline{B}$ ***Using the complement of a set***

   $x \in \overline{A} \cup \overline{B}$ ***Using the definition of union***

   $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

2. We need to show that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

   Suppose that $x \in \overline{A} \cup \overline{B} \quad \Rightarrow \quad x \in \overline{A}$ *or* $x \in \overline{B}$ (by the definition of union)

   $x \notin A$ *or* $x \notin B$ ***Using the definition of the complement***

$\neg(x \in A) \ \vee \ \neg(x \in B)$ ***True***

$\neg(x \in A) \ \wedge \ \neg(x \in B)$ ***By applying De Morgan's law of the proposition***

$\neg(x \in A \cap B)$ ***Using the definition of the intersection***

$x \in \overline{A \cap B}$ ***Using the definition of complement***

That shows that $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$

Therefore; $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Example

Use set builder notation and logical equivalences to establish the first De Morgan law $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Solution

$\overline{A \cap B} = \{x \mid x \notin A \cap B\}$ ***By definition of complement***

$= \{x \mid \neg(x \in (A \cap B))\}$

$= \{x \mid \neg(x \in A \ \wedge \ x \in B)\}$ ***By definition of complement***

$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$ ***By the first De Morgan law for logical equivalences***

$= \{x \mid x \notin A \ \vee \ x \notin B\}$ ***By definition of does not belong symbol***

$= \{x \mid x \in \bar{A} \ \vee \ x \in \bar{B}\}$ ***By definition of complement***

$= \{x \mid x \in \bar{A} \cup \bar{B}\}$ ***By definition of union***

$= \bar{A} \cup \bar{B}$

Example

Use a membership table to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution

A Membership Table for the Distributive Property

A	**B**	**C**	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	**1**	**1**	**1**	**1**	**1**	**1**	**1**
1	**1**	**0**	**1**	**1**	**1**	**0**	**1**
1	**0**	**1**	**1**	**1**	**0**	**1**	**1**
1	**0**	**0**	**0**	**0**	**0**	**0**	**0**
0	**1**	**1**	**1**	**0**	**0**	**0**	**0**
0	**1**	**0**	**1**	**0**	**0**	**0**	**0**
0	**0**	**1**	**1**	**0**	**0**	**0**	**0**
0	**0**	**0**	**0**	**0**	**0**	**0**	**0**

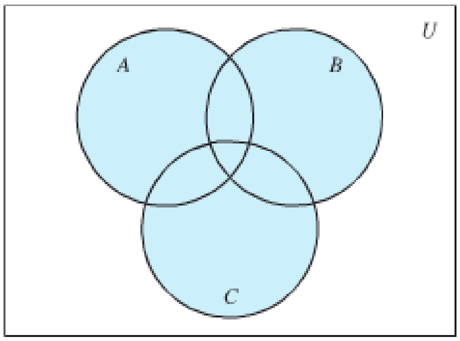
Example

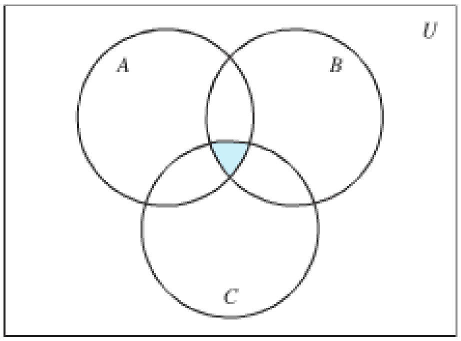
Let *A*, *B*, and *C* be sets. Show that $\overline{A\cup(B\cap C)}=(\bar{C}\cup\bar{B})\cap\bar{A}$

Solution

$$\begin{aligned}\overline{A\cup(B\cap C)} &= \bar{A}\cap\left(\overline{B\cap C}\right) && \textit{By the first De Morgan law}\\ &= \bar{A}\cap\left(\bar{B}\cup\bar{C}\right) && \textit{By the second De Morgan law}\\ &= \left(\bar{B}\cup\bar{C}\right)\cap\bar{A} && \textit{By the commutative law for intersection}\\ &= \left(\bar{C}\cup\bar{B}\right)\cap\bar{A} && \textit{By the commutative law for union}\end{aligned}$$

Generalized Unions and Intersections





$$A\cup(B\cup C)=(A\cup B)\cup C$$
$$A\cap(B\cap C)=(A\cap B)\cap C$$

Example

Let A = {0, 2, 4, 6, 8}, B = {0, 1, 2, 3, 4}, and C = {0, 3, 6, 9}. What are $A\cup B\cup C$ and $A\cap B\cap C$

Solution

$$A\cup B\cup C=\{0,\ 1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 9\}$$
$$A\cap B\cap C=\{0\}$$

Definition

The ***union*** of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

$$A_1\cup A_2\cup\cdots\cup A_n=\bigcup_{i=1}^{n}A_i$$

Definition

The ***intersection*** of a collection of sets is the set that contains those elements that are members of at all the sets in the collection.

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

For i = 1, 2, …, let $A_i = \{i,\ i+1,\ i+2,\ \ldots\}$. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i,\ i+1,\ i+2,\ \ldots\} = \{1,\ 2,\ 3,\ \ldots\}$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i,\ i+1,\ i+2,\ \ldots\} = \{n,\ n+1,\ n+2,\ \ldots\} = A_n$$

Exercises *Section* 1.8 – Set Operations

1. Let A be the set of students who live within one mile of school and let B be the set of students who walk to classes. Describe the students in each of these sets.

a) $A \cap B$

b) $A \cup B$

c) $A - B$

d) $B - A$

2. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$

a) $A \cup B$

b) $A \cap B$

c) $A - B$

d) $B - A$

3. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$

a) $A \cup B$

b) $A \cap B$

c) $A - B$

d) $B - A$

4. Prove the domination laws by showing that

a) $A \cup U = U$

b) $A \cap U = A$

c) $A \cup \varnothing = A$

d) $A \cap \varnothing = \varnothing$

5. Prove the complement laws by showing that

a) $A \cup \overline{A} = U$

b) $A \cap \overline{A} = \varnothing$

6. Show that

a) $A - \varnothing = A$

b) $\varnothing - A = \varnothing$

7. Prove the absorption law by showing that if A and B are sets, then

a) $A \cap (A \cup B) = A$

b) $A \cup (A \cap B) = A$

8. Show that if *A*, *B*, and *C* are sets, then $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$

9. Let *A* and *B* be sets. Show that

a) $(A \cap B) \subseteq A$

b) $A \subseteq (A \cup B)$

c) $(A - B) \subseteq A$

d) $A \cap (B - A) = \varnothing$

e) $A \cup (B - A) = A \cup B$

10. Draw the Venn diagrams for each of these combinations of the sets *A*, *B*, and *C*.

a) $A \cap (B - C)$

b) $(A \cap B) \cup (A \cap C)$

c) $(A \cap \overline{B}) \cup (A \cap \overline{C})$

d) $\overline{A} \cap \overline{B} \cap \overline{C}$

e) $(A - B) \cup (A - C) \cup (B - C)$

11. Show that $A \oplus B = (A \cup B) - (A \cap B)$

12. Show that $A \oplus B = (A - B) \cup (B - A)$