Solution Section 1.3 – Cross Products

Exercise

Find the length and direction of $\vec{u}\times\vec{v}$ *and* $\vec{v}\times\vec{u}$: $\quad \vec{u}=2\hat{i}-2\hat{j}-\hat{k},\quad \vec{v}=\hat{i}-\hat{k}$

Solution

$$\vec{u}\times\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -2 & -1\\ 1 & 0 & -1\end{vmatrix}$$

$$\boxed{=2\hat{i}-\hat{j}+2\hat{k}}$$

Length:

$$\left|\vec{u}\times\vec{v}\right|=\sqrt{4+1+4}$$

$$\vec{u}=\hat{i}-\hat{k},\quad \vec{v}=\hat{j}+\hat{k}$$

Direction: $\boxed{\frac{1}{3}\left(2\hat{i}-\hat{j}+2\hat{k}\right)}$

$$\vec{v}\times\vec{u}=-\left(\vec{u}\times\vec{v}\right)$$

$$\boxed{=-2\hat{i}+\hat{j}-2\hat{k}}$$

Length:

$$\left|\vec{v}\times\vec{u}\right|=\sqrt{4+1+4}$$

$$\boxed{=3}$$

Direction: $\frac{1}{3}\left(-2\hat{i}+\hat{j}-2\hat{k}\right)$

Exercise

Find the length and direction of $\vec{u}\times\vec{v}$ *and* $\vec{v}\times\vec{u}$: $\quad \vec{u}=\hat{i}+\hat{j}-\hat{k},\quad \vec{v}=0$

Solution

$$\vec{u}\times\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & -1\\ 0 & 0 & 0\end{vmatrix}$$

$$\boxed{=0}$$

Length: $\boxed{=0}$

Direction: ***No direction***

$$\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v}) \quad \underline{= 0}$$

Length: $\underline{= 0}$

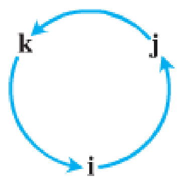
Direction: ***No direction***

Exercise

Find the length and direction of $\vec{u} \times \vec{v}$ *and* $\vec{v} \times \vec{u}$: $\quad \vec{u} = \hat{i} \times \hat{j}, \quad \vec{v} = \hat{j} \times \hat{k}$

Solution

$$\begin{aligned}\vec{u} \times \vec{v} &= (\hat{i} \times \hat{j}) \times (\hat{j} \times \hat{k}) \\ &= \hat{k} \times \hat{i} \\ &\underline{= \hat{j}}\end{aligned}$$



Length: $\underline{= 1}$

Direction: $\underline{= \hat{j}}$

$$\begin{aligned}\vec{v} \times \vec{u} &= -(\vec{u} \times \vec{v}) \\ &\underline{= -\hat{j}}\end{aligned}$$

Length: $\underline{= 1}$

Direction: $\underline{= -\hat{j}}$

Exercise

Find the length and direction of $\vec{u} \times \vec{v}$ *and* $\vec{v} \times \vec{u}$: $\quad \vec{u} = -8\hat{i} - 2\hat{j} - 4\hat{k}, \quad \vec{v} = 2\hat{i} + 2\hat{j} + \hat{k}$

Solution

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -8 & -2 & -4 \\ 2 & 2 & 1 \end{vmatrix}$$

$$\underline{= 6\hat{i} - 12\hat{k}}$$

Length:

$$\begin{aligned}|\vec{u} \times \vec{v}| &= \sqrt{36 + 144} \\ &= \sqrt{180} \\ &\underline{= 6\sqrt{5}}\end{aligned}$$

Direction: $=\frac{1}{6\sqrt{5}}\left(6\hat{i}-12\hat{k}\right)$

$$\boxed{=\frac{1}{\sqrt{5}}\hat{i}-\frac{2}{\sqrt{5}}\hat{k}}$$

$$\vec{v}\times\vec{u}=-\left(\vec{u}\times\vec{v}\right)$$

$$\boxed{=-6\hat{i}+12\hat{k}}$$

Length: $\boxed{\left|\vec{v}\times\vec{u}\right| = 6\sqrt{5}}$

Direction: $\boxed{-\frac{1}{\sqrt{5}}\hat{i}+\frac{2}{\sqrt{5}}\hat{k}}$

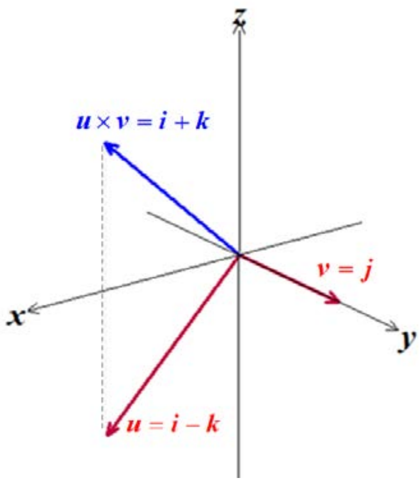
Exercise

Sketch the coordinate axes and then include the vectors $\vec{u}$, $\vec{v}$, and $\vec{u}\times\vec{v}$ as vectors starting origin for $\vec{u}=\hat{i}-\hat{k}, \quad \vec{v}=\hat{j}$

Solution

$$\vec{u}\times\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 0 & -1\\ 0 & 1 & 0\end{vmatrix}$$

$$\boxed{=\hat{i}+\hat{k}}$$



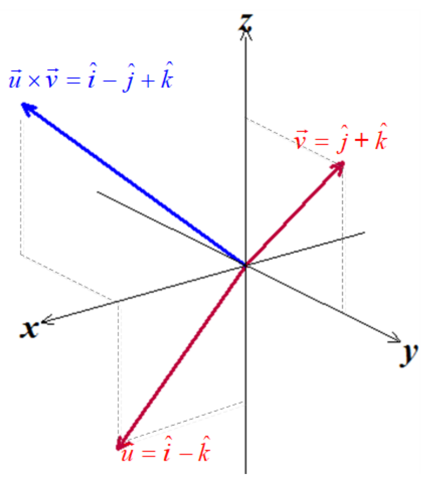
Exercise

Sketch the coordinate axes and then include the vectors $\vec{u}$, $\vec{v}$, and $\vec{u}\times\vec{v}$ as vectors starting origin for $\vec{u}=\hat{i}-\hat{k}, \quad \vec{v}=\hat{j}+\hat{k}$

Solution

$$\vec{u}\times\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 0 & -1\\ 0 & 1 & 1\end{vmatrix}$$

$$\boxed{=\hat{i}-\hat{j}+\hat{k}}$$



Exercise

Find the area of the triangle determined by the points P, Q, and R, and then find a unit vector perpendicular to plane PQR. $P(1,\ -1,\ 2),\quad Q(2,\ 0,\ -1),\quad and\ \ R(0,\ 2,\ 1)$

Solution

$$\overrightarrow{PQ} = (2-1)\hat{i} + (0+1)\hat{j} + (-1-2)\hat{k}$$
$$= \hat{i} + \hat{j} - 3\hat{k}$$

$$\overrightarrow{PR} = (0-1)\hat{i} + (2+1)\hat{j} + (1-2)\hat{k}$$
$$= -\hat{i} + 3\hat{j} - \hat{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix}$$

$$\boxed{= 8\hat{i} + 4\hat{j} + 4\hat{k}}$$

$$Area = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$$
$$= \frac{1}{2}\sqrt{8^2 + 4^2 + 4^2}$$
$$= \frac{1}{2}\sqrt{96}$$
$$\boxed{= 2\sqrt{6}}$$

$$\vec{u} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|}$$
$$= \frac{1}{4\sqrt{6}}\left(8\hat{i} + 4\hat{j} + 4\hat{k}\right)$$
$$\boxed{= \frac{1}{\sqrt{6}}\left(2\hat{i} + \hat{j} + \hat{k}\right)}$$

Exercise

Find the area of the triangle determined by the points P, Q, and R, and then find a unit vector perpendicular to plane PQR. $P(1,\ 1,\ 1),\quad Q(2,\ 1,\ 3),\quad and\ \ R(3,\ -1,\ 1)$

Solution

$$\overrightarrow{PQ} = (2-1)\hat{i} + (1-1)\hat{j} + (3-1)\hat{k}$$
$$= \hat{i} + 2\hat{k}$$

$$\overrightarrow{PR} = (3-1)\hat{i} + (-1-1)\hat{j} + (1-1)\hat{k}$$

$$= 2\hat{i} - 2\hat{j}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 2 & -2 & 0 \end{vmatrix}$$

$$= 4\hat{i} + 4\hat{j} - 2\hat{k}$$

$$Area = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$$

$$= \frac{1}{2}\sqrt{16+16+4}$$

$$= \frac{1}{2}\sqrt{36}$$

$$\underline{= 3}$$

$$\vec{u} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|}$$

$$= \frac{1}{6}\left(4\hat{i} + 4\hat{j} - 2\hat{k}\right)$$

$$\underline{= \frac{1}{3}\left(2\hat{i} + 2\hat{j} - \hat{k}\right)}$$

Exercise

Find the area of the triangle determined by the points P, Q, and R, and then find a unit vector perpendicular to plane PQR. $P(-2,\ 2,\ 0)$, $Q(0,\ 1,\ -1)$, *and* $R(-1,\ 2,\ -2)$

Solution

$$\overrightarrow{PQ} = 2\hat{i} - \hat{j} - \hat{k}$$

$$\overrightarrow{PR} = \hat{i} - 2\hat{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & -1 \\ 1 & 0 & -2 \end{vmatrix}$$

$$= 2\hat{i} + 3\hat{j} + \hat{k}$$

$$Area = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$$

$$= \frac{1}{2}\sqrt{4+9+1}$$

$$\underline{= \frac{\sqrt{14}}{2}}$$

$$\vec{u} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|}$$

$$\boxed{= \frac{1}{\sqrt{14}}\left(2\hat{i} + 3\hat{j} + \hat{k}\right)}$$

Exercise

Verify that $(\vec{u} \times \vec{v}) \bullet \vec{w} = (\vec{v} \times \vec{w}) \bullet \vec{u} = (\vec{w} \times \vec{u}) \bullet \vec{v}$ and find the volume of the parallelepiped determined by $\vec{u} = 2\hat{i}, \quad \vec{v} = 2\hat{j}, \quad and \quad \vec{w} = 2\hat{k}$

Solution

Let $\vec{u} = \langle u_1, u_2, u_3 \rangle, \quad \vec{v} = \langle v_1, v_2, v_3 \rangle, \quad \vec{w} = \langle w_1, w_2, w_3 \rangle$

$$(\vec{u} \times \vec{v}) \bullet \vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$(\vec{v} \times \vec{w}) \bullet \vec{u} = \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

$$(\vec{w} \times \vec{u}) \bullet \vec{v} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Which all have the same absolute value, by interchanging the rows the determinant does not change its absolute value.

$$Volume = (\vec{u} \times \vec{v}) \bullet \vec{w}$$

$$= \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

$$\boxed{= 8 \ \ unit^3}$$

Exercise

Find $|\vec{v}|$, $|\vec{u}|$, $\vec{v}\bullet\vec{u}$, $\vec{u}\bullet\vec{v}$, $\vec{v}\times\vec{u}$, $\vec{u}\times\vec{v}$, $|\vec{v}\times\vec{u}|$, the angle between $\vec{v}$ and $\vec{u}$, the scalar component of $\vec{u}$ in the direction of $\vec{v}$, and the vector $proj_{\vec{v}}\,\vec{u}$

$$\vec{v}=\hat{i}+\hat{j}+2\hat{k},\quad \vec{u}=-\hat{i}-\hat{k}$$

Solution

$$|\vec{v}|=\sqrt{1+1+4}$$
$$\underline{=\sqrt{6}|}$$

$$|\vec{u}|=\sqrt{1+1}$$
$$\underline{=\sqrt{2}\ |}$$

$$\vec{v}\bullet\vec{u}=-1+0-2\ \underline{=-3\ |}$$

$$\vec{u}\bullet\vec{v}\ \underline{=-3\ |}$$

$$\vec{v}\times\vec{u}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 2\\ -1 & 0 & -1\end{vmatrix}$$
$$\underline{=-\hat{i}-\hat{j}+\hat{k}\ |}$$

$$\vec{u}\times\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 0 & -1\\ 1 & 1 & 2\end{vmatrix}$$
$$\underline{=\hat{i}+\hat{j}-\hat{k}\ |}$$

$$|\vec{v}\times\vec{u}|=\left|-\hat{i}-\hat{j}+\hat{k}\right|$$
$$=\sqrt{1+1+1}$$
$$\underline{=\sqrt{3}\ |}$$

$$\theta=\cos^{-1}\left(-\frac{3}{\sqrt{6}\sqrt{2}}\right) \qquad \theta=\cos^{-1}\frac{\vec{v}\bullet\vec{u}}{|\vec{v}||\vec{u}|}$$
$$=\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$
$$\underline{=\frac{5\pi}{6}\ |}$$

$$proj_{\vec{v}}\,\vec{u}=\frac{-3}{6}\left(\hat{i}+\hat{j}+2\hat{k}\right) \qquad proj_{\vec{v}}\,\vec{u}=\left(\frac{\vec{v}\bullet\vec{u}}{|\vec{v}|^2}\right)\vec{v}$$
$$\underline{=-\frac{1}{2}\left(\hat{i}+\hat{j}+2\hat{k}\right)\ |}$$

Exercise

Find $|\vec{v}|$, $|\vec{u}|$, $\vec{v}\bullet\vec{u}$, $\vec{u}\bullet\vec{v}$, $\vec{v}\times\vec{u}$, $\vec{u}\times\vec{v}$, $|\vec{v}\times\vec{u}|$, the angle between $\vec{v}$ and $\vec{u}$, the scalar component of $\vec{u}$ in the direction of $\vec{v}$, and the vector $proj_{\vec{v}}\vec{u}$

$$\vec{v}=2\hat{i}+\hat{j}-\hat{k},\quad \vec{u}=\hat{i}+\hat{j}-5\hat{k}$$

Solution

$$|\vec{v}|=\sqrt{4+1+1}$$

$$=\sqrt{6}$$

$$|\vec{u}|=\sqrt{1+1+25}$$

$$=\sqrt{27}$$

$$=3\sqrt{3}$$

$$\vec{v}\bullet\vec{u}=2+1+5=8$$

$$\vec{u}\bullet\vec{v}=8$$

$$\vec{v}\times\vec{u}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&1&-1\\1&1&-5\end{vmatrix}$$

$$=-4\hat{i}+9\hat{j}+\hat{k}$$

$$\vec{u}\times\vec{v}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&1&-5\\2&1&-1\end{vmatrix}$$

$$=4\hat{i}-9\hat{j}-\hat{k}$$

$$|\vec{v}\times\vec{u}|=\sqrt{16+81+1}$$

$$=\sqrt{98}$$

$$=7\sqrt{2}$$

$$\theta=\cos^{-1}\left(\frac{8}{3\sqrt{3}\sqrt{6}}\right)\qquad\qquad \theta=\cos^{-1}\frac{\vec{v}\bullet\vec{u}}{|\vec{v}||\vec{u}|}$$

$$=\cos^{-1}\left(\frac{9}{8\sqrt{2}}\right)$$

$$=0.651$$

$$proj_{\vec{v}}\vec{u}=\frac{8}{6}\left(2\hat{i}+\hat{j}-\hat{k}\right)\qquad\qquad proj_{\vec{v}}\vec{u}=\left(\frac{\vec{v}\bullet\vec{u}}{|\vec{v}|^2}\right)\vec{v}$$

$$=\frac{4}{3}\left(2\hat{i}+\hat{j}-\hat{k}\right)$$

Exercise

Find the area of the parallelogram determined by vectors $\vec{u}$ and $\vec{v}$, then the volume of the parallelepiped determined by vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$.

$$\vec{u} = \hat{i} + \hat{j} - \hat{k}, \quad \vec{v} = 2\hat{i} + \hat{j} + \hat{k}, \quad \vec{w} = -\hat{i} - 2\hat{j} + 3\hat{k}$$

Solution

$$Area = \left|\vec{u} \times \vec{v}\right| = abs\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= \left|2\hat{i} - 3\hat{j} - \hat{k}\right|$$

$$= \sqrt{4+9+1}$$

$$= \sqrt{14} \quad unit^2$$

$$Volume = (\vec{u} \times \vec{v}) \bullet \vec{w} = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ -1 & -2 & 3 \end{vmatrix}$$

$$= 1 \quad unit^3$$

Exercise

Find the area of the parallelogram determined by vectors $\vec{u}$ and $\vec{v}$, then the volume of the parallelepiped determined by vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$.

$$\vec{u} = \hat{i} + \hat{j}, \quad \vec{v} = \hat{j}, \quad \vec{w} = \hat{i} + \hat{j} + \hat{k}$$

Solution

$$Area = \left|\vec{u} \times \vec{v}\right| = \left\|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{matrix}\right\|$$

$$= \left|\hat{k}\right|$$

$$= 1 \quad unit^2$$

$$Volume = (\vec{u} \times \vec{v}) \bullet \vec{w} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1 \quad unit^3$$

Exercise

Find the volume of the parallelepiped determined by

$$\vec{u} = \hat{i} - \hat{j} + \hat{k}, \quad \vec{v} = 2\hat{i} + \hat{j} - \hat{k}, \quad and \quad \vec{w} = -\hat{i} + 2\hat{j} - \hat{k}$$

Solution

$$Volume = (\vec{u} \times \vec{v}) \bullet \vec{w}$$

$$= abs\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 2 & -1 \end{vmatrix}$$

$$= 3 \quad unit^3$$

Exercise

Find the volume of the parallelepiped determined by

$$\vec{u} = \hat{i} + \hat{j} - 2\hat{k}, \quad \vec{v} = -\hat{i} - \hat{k}, \quad and \quad \vec{w} = 2\hat{i} + 4\hat{j} - 2\hat{k}$$

Solution

$$Volume = (\vec{u} \times \vec{v}) \bullet \vec{w}$$

$$= abs\begin{vmatrix} 1 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 4 & -2 \end{vmatrix}$$

$$= 8 \quad unit^3$$

Exercise

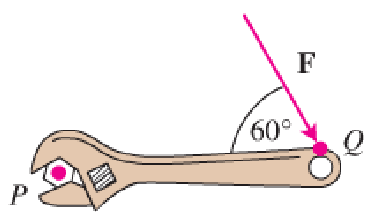
Find the magnitude of the torque force exerted by $\vec{F}$ on the bolt at P if $|\overrightarrow{PQ}| = 8\ in.$ and $|\vec{F}| = 30\ lb.$

Solution

$$|\overrightarrow{PQ} \times \vec{F}| = |\overrightarrow{PQ}|\ |\vec{F}|\ \sin 60°$$

$$= \frac{8}{12}(30)\frac{\sqrt{3}}{2}$$

$$= 10\sqrt{3}\ \ ft.lb$$



Exercise

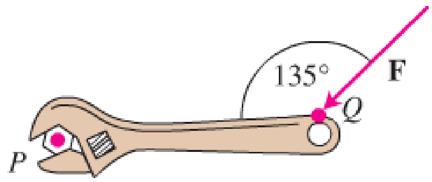
Find the magnitude of the torque force exerted by $\vec{F}$ on the bolt at P if $\left|\overrightarrow{PQ}\right| = 8\ in.$ and $\left|\vec{F}\right| = 30\ lb.$

Solution

$$\left|\overrightarrow{PQ} \times \vec{F}\right| = \left|\overrightarrow{PQ}\right|\ \left|\vec{F}\right|\ \sin 135°$$

$$= \frac{8}{12}(30)\frac{\sqrt{2}}{2}$$

$$= 10\sqrt{2}\ \ ft.lb$$



Exercise

Find the area of the parallelogram whose vertices are: $A(1,\ 0),\quad B(0,\ 1),\quad C(-1,\ 0),\quad D(0,\ -1)$

Solution

$$\overrightarrow{AB} = -\hat{\imath} + \hat{\jmath} \quad \overrightarrow{AD} = -\hat{\imath} - \hat{\jmath}$$

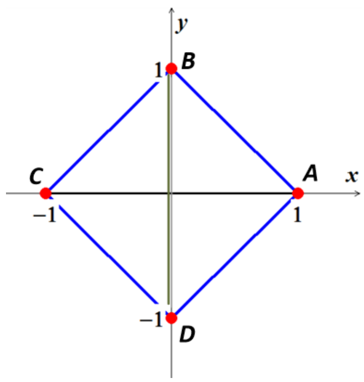
$$Area(\triangle ABD) = Area(\triangle CBD)$$

$$Area = \left|\overrightarrow{AB} \times \overrightarrow{AD}\right|$$

$$= abs\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 1 & 0 \\ -1 & -1 & 0 \end{vmatrix}$$

$$= abs\left|2\hat{k}\right|$$

$$= 2\ \ unit^2$$



Exercise

Find the area of the parallelogram whose vertices are: $A(0,\ 0),\quad B(7,\ 3),\quad C(9,\ 8),\quad D(2,\ 5)$

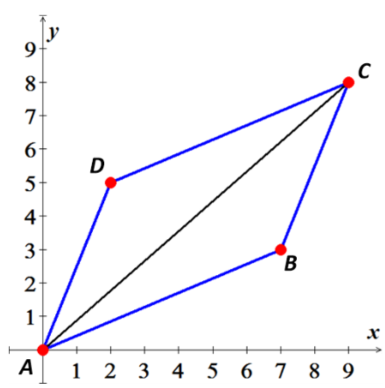
Solution

$$\overrightarrow{AB} = 7\hat{\imath} + 3\hat{\jmath} \quad \overrightarrow{AC} = 9\hat{\imath} + 8\hat{\jmath}$$

$$Area(\triangle ABC) = Area(\triangle ACD)$$

$$Area = \left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

$$= abs\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 7 & 3 & 0 \\ 9 & 8 & 0 \end{vmatrix}$$



$$= abs\left|29\hat{k}\right|$$

$$\underline{= 29 \ \ unit^2}$$

Exercise

Find the area of the parallelogram whose vertices are:

$A(-1,\ 2), \quad B(2,\ 0), \quad C(7,\ 1), \quad D(4,\ 3)$

Solution

$$\overrightarrow{AB} = 3\hat{i} - 2\hat{j} \quad \overrightarrow{AC} = 8\hat{i} - \hat{j}$$

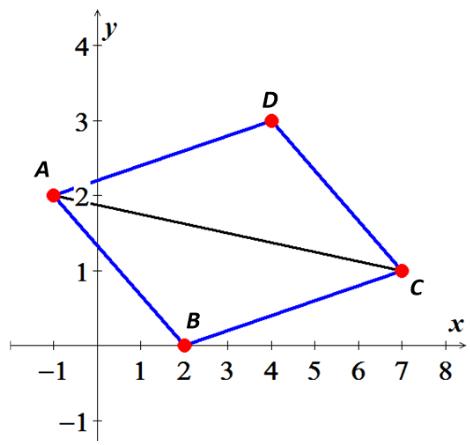
$$Area(\triangle ABC) = Area(\triangle ACD)$$

$$Area = \left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

$$= abs\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 0 \\ 8 & -1 & 0 \end{vmatrix}$$

$$= abs\left|13\hat{k}\right|$$

$$\underline{= 13 \ unit^2}$$



Exercise

Find the area of the parallelogram whose vertices are:

$A(0,\ 0,\ 0), \quad B(3,\ 2,\ 4), \quad C(5,\ 1,\ 4), \quad D(2,\ -1,\ 0)$

Solution

$$\overrightarrow{AB} = 3\hat{i} + 2\hat{j} + 4\hat{k} \quad \overrightarrow{DC} = 3\hat{i} + 2\hat{j} + 4\hat{k}$$

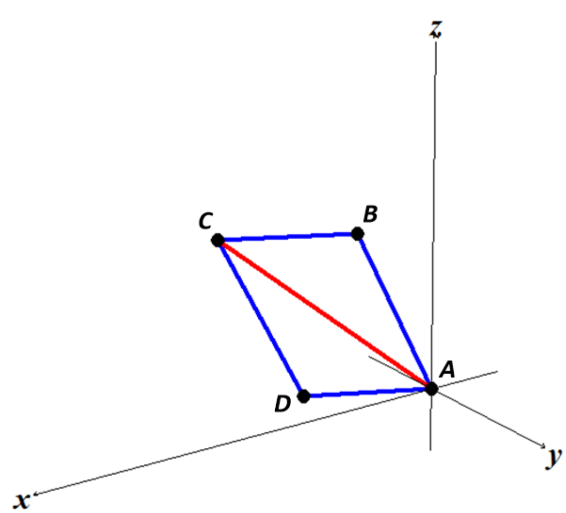
$\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$

$$\overrightarrow{AD} = 2\hat{i} - \hat{j} \quad \overrightarrow{BC} = 2\hat{i} - \hat{j}$$

$\overrightarrow{AD}$ is parallel to $\overrightarrow{BC}$

$$Area = \left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= abs\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 4 \\ 2 & -1 & 0 \end{vmatrix}$$



$$= abs\left|4\hat{i} + 8\hat{j} - 7\hat{k}\right|$$

$$= \sqrt{16 + 64 + 49}$$

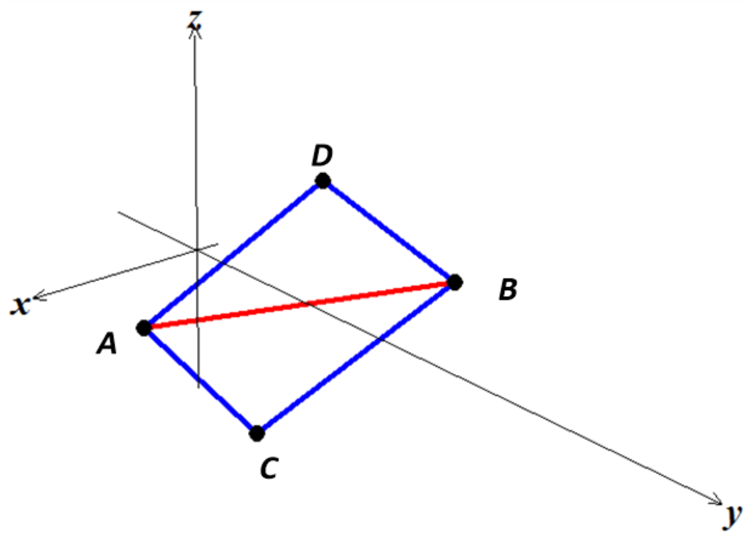
$$= \sqrt{129}\ \ unit^2$$

Exercise

Find the area of the parallelogram whose vertices are:

$$A(1,\ 0,\ -1),\quad B(1,\ 7,\ 2),\quad C(2,\ 4,\ -1),\quad D(0,\ 3,\ 2)$$

Solution



$$\overrightarrow{AC} = \hat{i} + 4\hat{j} \quad \overrightarrow{CB} = -\hat{i} + 3\hat{j} + 3\hat{k}$$

$$Area = \left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= abs\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 0 \\ -1 & 3 & 3 \end{vmatrix}$$

$$= abs\left|12\hat{i} - 3\hat{j} + 7\hat{k}\right|$$

$$= \sqrt{144 + 9 + 49}$$

$$= \sqrt{202}\ \ unit^2$$

Exercise

Find the area of the parallelogram with vertices $(1,\ 2,\ 3)$, $(1,\ 0,\ 6)$, and $(4,\ 2,\ 4)$

Solution

$$\langle 1,\ 0,\ 6\rangle - \langle 1,\ 2,\ 3\rangle = \langle 0,\ -2,\ 3\rangle$$

$$\langle 4,\ 2,\ 4\rangle - \langle 1,\ 2,\ 3\rangle = \langle 3,\ 0,\ 1\rangle$$

$$Area = \left|\langle 0,\ -2,\ 3\rangle \times \langle 3,\ 0,\ 1\rangle\right|$$

$$= \left\|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2 & 3 \\ 3 & 0 & 1 \end{matrix}\right\|$$

$$= \left|\langle -2,\ 9,\ 6\rangle\right|$$

$$= \sqrt{4 + 81 + 36}$$

$$= 11\ \ unit^2$$

Exercise

Find the area of the parallelogram with vertices $(1,\ 0,\ 3)$, $(5,\ 0,\ -1)$, and $(0,\ 2,\ -2)$

Solution

$$\langle 5,\ 0,\ -1\rangle - \langle 1,\ 0,\ 3\rangle = \langle 4,\ 0,\ -4\rangle$$

$$\langle 0,\ 2,\ -2\rangle - \langle 1,\ 0,\ 3\rangle = \langle -1,\ 2,\ -5\rangle$$

$$Area = \left|\langle 4,\ 0,\ -4\rangle \times \langle -1,\ 2,\ -5\rangle\right|$$

$$= \left\| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 0 & 4 \\ -1 & 2 & -5 \end{matrix} \right\|$$

$$= \left|\langle 8,\ 24,\ 8\rangle\right|$$

$$= \sqrt{64 + 576 + 64}$$

$$= \sqrt{64(1+9+1)}$$

$$= 8\sqrt{11}\ \ unit^2$$

Exercise

Find the area of the triangle whose vertices are: $A(0,\ 0)$, $B(-2,\ 3)$, $C(3,\ 1)$

Solution

$$\overrightarrow{AB} = -2\hat{i} + 3\hat{j} \quad \overrightarrow{AC} = 3\hat{i} + \hat{j}$$

$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= \left(\frac{1}{2}\right) abs \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 3 & 0 \\ 3 & 1 & 0 \end{vmatrix}$$

$$= \frac{1}{2}\left|-11\hat{k}\right|$$

$$= \frac{11}{2}\ unit^2$$

Exercise

Find the area of the triangle whose vertices are: $A(-1,\ -1),\quad B(3,\ 3),\quad C(2,\ 1)$

Solution

$$\overrightarrow{AB} = 4\hat{i} + 4\hat{j} \quad \overrightarrow{AC} = 3\hat{i} + 2\hat{j}$$

$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= \frac{1}{2}\left\|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 3 & 2 & 0 \end{matrix}\right\|$$

$$= \frac{1}{2}\left|-4\hat{k}\right|$$

$$= 2\ unit^2$$

Exercise

Find the area of the triangle whose vertices are: $A(1,\ 0,\ 0),\quad B(0,\ 0,\ 2),\quad C(0,\ 0,\ -1)$

Solution

$$\overrightarrow{AB} = -\hat{i} + 2\hat{k} \quad \overrightarrow{AC} = -\hat{i} - \hat{k}$$

$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= \frac{1}{2}\left\|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & 2 \\ -1 & 0 & -1 \end{matrix}\right\|$$

$$= \frac{1}{2}\left|3\hat{k}\right|$$

$$= \frac{3}{2}\ unit^2$$

Exercise

Find the area of the triangle whose vertices are: $A(0,\ 0,\ 0),\quad B(-1,\ 1,\ -1),\quad C(3,\ 0,\ 3)$

Solution

$$\overrightarrow{AB} = -\hat{i} + \hat{j} - \hat{k} \quad \overrightarrow{AC} = 3\hat{i} + 3\hat{k}$$

$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= \frac{1}{2}\left\|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -1 \\ 3 & 0 & 3 \end{vmatrix}\right\|$$

$$= \frac{1}{2}\left|3\hat{i} - 3\hat{k}\right|$$

$$= \frac{1}{2}\sqrt{9+9}$$

$$= \frac{3\sqrt{2}}{2}\ unit^2$$

Exercise

Find the volume of the parallelepiped if four of its eight vertices are:

$A(0,\ 0,\ 0),\quad B(1,\ 2,\ 0),\quad C(0,\ -3,\ 2),\quad D(3,\ -4,\ 5)$

Solution

$$\overrightarrow{AB} = \hat{i} + 2\hat{j} \quad \overrightarrow{AC} = -3\hat{j} + 2\hat{k} \quad \overrightarrow{AD} = 3\hat{i} - 4\hat{j} + 5\hat{k}$$

$$\left(\overrightarrow{AB} \times \overrightarrow{AC}\right) \bullet \overrightarrow{AD} = \begin{vmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 3 & -4 & 5 \end{vmatrix}$$

$$= 5$$

$$Volume = \left|\left(\overrightarrow{AB} \times \overrightarrow{AC}\right) \bullet \overrightarrow{AD}\right|$$

$$= 5\ unit^3$$

Exercise

Let $\vec{u} = \langle 2,\ 4,\ -5\rangle$ and $\vec{v} = \langle -6,\ 10,\ 2\rangle$

a) Compute $\vec{u} - 3\vec{v}$

b) Compute $\left|\vec{u} + \vec{v}\right|$

c) Find the unit vector with the same direction as $\vec{u}$

d) Find a vector parallel to $\vec{v}$ with length 20.

e) Compute $\vec{u} \bullet \vec{v}$ and the angle between $\vec{u}$ and $\vec{v}$.

f) Compute $\vec{u} \times \vec{v}$, $\vec{v} \times \vec{u}$

g) Find the area of the triangle with vertices $(0,\ 0,\ 0)$, $(2,\ 4,\ -5)$, and $(-6,\ 10,\ 2)$

Solution

a) $\boldsymbol{u} - 3\boldsymbol{v} = \langle 2,\ 4,\ -5\rangle - 3\langle -6,\ 10,\ 2\rangle$

$$= \langle 2,\ 4,\ -5\rangle - \langle -18,\ 30,\ 6\rangle$$

$$\boxed{= \langle 20,\ -26,\ -11\rangle}$$

b) $$|\boldsymbol{u}+\mathbf{v}| = |\langle 2,\ 4,\ -5\rangle + \langle -6,\ 10,\ 2\rangle|$$

$$= |\langle -4,\ 14,\ -3\rangle|$$

$$= \sqrt{16+196+9}$$

$$\boxed{= \sqrt{221}}$$

c) unit vector of $\boldsymbol{u} = \langle 2,\ 4,\ -5\rangle$

$$\frac{\vec{u}}{|\vec{u}|} = \frac{\langle 2,\ 4,\ -5\rangle}{\sqrt{4+16+25}}$$

$$= \frac{\langle 2,\ 4,\ -5\rangle}{\sqrt{45}}$$

$$\boxed{= \frac{1}{3\sqrt{5}}\langle 2,\ 4,\ -5\rangle}$$

d) $$|\vec{v}| = |\langle -6,\ 10,\ 2\rangle|$$

$$= \sqrt{36+100+4}$$

$$= \sqrt{140}$$

$$\boxed{= 2\sqrt{35}}$$

The desired vector parallel to $\vec{v}$ with length 20 is:

$$= \frac{20}{2\sqrt{35}}\langle -6,\ 10,\ 2\rangle$$

$$\boxed{= \frac{20}{\sqrt{35}}\langle -3,\ 5,\ 1\rangle}$$

e) $$\vec{u}\bullet\vec{v} = \langle 2,\ 4,\ -5\rangle \bullet \langle -6,\ 10,\ 2\rangle$$

$$= -12+40-10$$

$$\boxed{= 18}$$

$$\theta = \cos^{-1}\frac{18}{3\sqrt{5}\ 2\sqrt{35}} \qquad\qquad \theta = \cos^{-1}\frac{\vec{u}\bullet\vec{v}}{|\vec{u}|\ |\vec{v}|}$$

$$\boxed{= \cos^{-1}\frac{3}{5\sqrt{7}}} \qquad \boxed{\approx 76.9^\circ}$$

f) $$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 4 & -5 \\ -6 & 10 & 2 \end{vmatrix}$$

$$\boxed{= 58\hat{i} + 26\hat{j} + 44\hat{k}}$$

$$\boxed{\vec{v}\times\vec{u} = \langle -58,\ -26,\ -44\rangle}$$

g) Area of the triangle with vertices $(0,\ 0,\ 0)$, $(2,\ 4,\ -5)$, and $(-6,\ 10,\ 2)$

$$Area = \frac{1}{2}|\vec{u}\times\vec{v}|$$

$$= \frac{1}{2}\sqrt{58^2 + 26^2 + 44^2}$$

$$= \frac{1}{2}\sqrt{5,976}$$

$$\underline{= 3\sqrt{166}\ \ unit^2}$$

Exercise

Find a unit vector normal to the vectors $\langle 2,\ -6,\ 9\rangle$ and $\langle -1,\ 0,\ 6\rangle$

Solution

$$\vec{N} = \langle 2,\ -6,\ 9\rangle \times \langle -1,\ 0,\ 6\rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -6 & 9 \\ -1 & 0 & 6 \end{vmatrix}$$

$$= \langle -36,\ -21,\ -6\rangle$$

$$\left|\vec{N}\right| = \sqrt{36^2 + 21^2 + 6^2}$$

$$= \sqrt{1,773}$$

$$\underline{= 3\sqrt{197}}$$

$$\text{Unit normal } = \frac{1}{3\sqrt{197}}\langle -36,\ -21,\ -6\rangle$$

$$\underline{= -\frac{1}{\sqrt{197}}\langle 12,\ 7,\ 2\rangle}$$

Exercise

Find the angle between $\langle 2,\ 0,\ -2\rangle$ and $\langle 2,\ 2,\ 0\rangle$ using the dot product then the cross product.

Solution

$$\langle 2,\ 0,\ -2\rangle \bullet \langle 2,\ 2,\ 0\rangle = 4$$

$$|\langle 2,\ 0,\ -2\rangle| = \sqrt{4+4}\ \underline{= 2\sqrt{2}}$$

$$|\langle 2,\ 2,\ 0\rangle|\ \underline{= 2\sqrt{2}}$$

$$\theta = \cos^{-1}\frac{4}{\left(2\sqrt{2}\right)\left(2\sqrt{2}\right)} \qquad\qquad \theta = \cos^{-1}\frac{\vec{u}\bullet\vec{v}}{|\vec{u}|\ |\vec{v}|}$$

$$= \cos^{-1}\frac{1}{2}$$

$$\boxed{= \frac{\pi}{3}}$$

$$\langle 2,\ 0,\ -2\rangle \times \langle 2,\ 2,\ 0\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -2 \\ 2 & 2 & 0 \end{vmatrix}$$

$$= \langle 4,\ -4,\ 4\rangle$$

$$|\langle 4,\ -4,\ 4\rangle| = \sqrt{16+16+16}$$

$$= 4\sqrt{3}$$

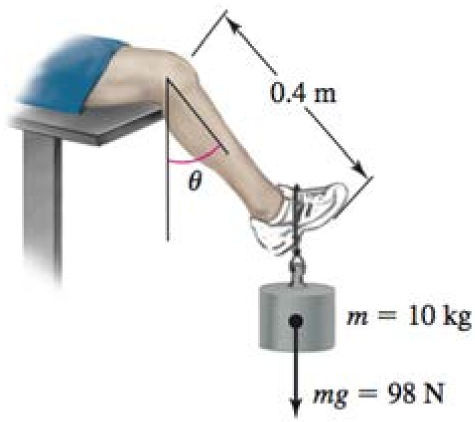
$$\theta = \sin^{-1}\frac{4\sqrt{3}}{\left(2\sqrt{2}\right)^2}$$

$$= \sin^{-1}\frac{\sqrt{3}}{2}$$

$$\boxed{= \frac{\pi}{3}}$$

Exercise

You do leg lifts with 10-*kg* weight attached to your foot, so the resulting force is $mg \approx 98\ N$ directed vertically downward. If the distance from your knee to the weight is 0.4*m* and her lower leg makes an angle of θ to the vertical, find the magnitude of the torque about your knee as your leg is lifted (as a function of θ).



a) What is the minimum and maximum magnitude of the torque?
b) Does the direction of the torque change as your leg is lifted?

Solution

a) $T(\theta) = (.4)(98)\sin\theta$ $\qquad$ $T(\theta) = |r||F|\sin\theta$

$$= \frac{392}{10}\sin\theta$$

$$= \frac{196}{5}\sin\theta \quad (N-m)$$

The maximum torque is $\frac{196}{5} = 39.2$ when $\sin\theta = 1 \quad \Rightarrow \quad \theta = 90°$

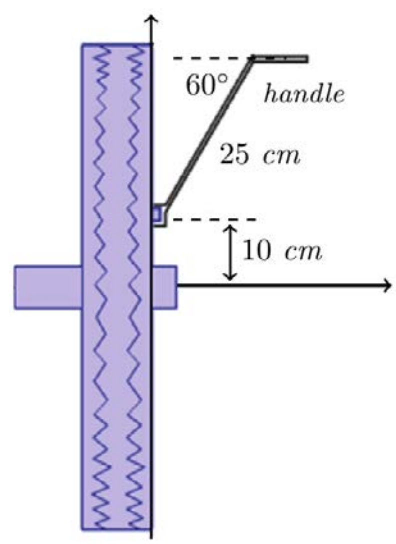
The minimum torque is 0 when $\sin\theta = 0 \quad \Rightarrow \quad \theta = 0°$

b) The direction of the torque does not change as the knee is lifted

Exercise

An automobile wheel has center at the origin and axle along the *y*-axis. One of the retaining nuts holding the wheel is at position $P_0(0,\ 0,\ 10)$. (Distances are measured in *cm*.) A bent tire wrench with arm 25 *cm* long and inclined at an angle of 60° to the direction of its handle is fitted to the nut in an upright direction. If the horizontal force $\vec{F} = 500\hat{i}\ \ (N)$ is applied to the handle of the wrench, what is its torque on the nut? What part (component) of this torque is effective in trying to rotate the nut about its horizontal axis? What is the effective torque trying to rotate the wheel?

Solution

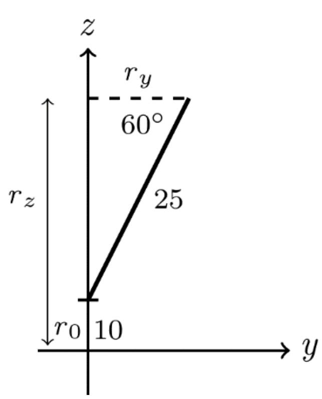


$$r_0 = 10\hat{k}$$

$$\vec{r} = r_y\hat{i} + r_z\hat{k}$$

$$= 25\cos 60°\,\hat{j} + (10 + 25\sin 60°)\,\hat{k}$$

$$= \frac{25}{2}\hat{j} + \left(10 + \frac{25\sqrt{3}}{2}\right)\hat{k}$$



$$T = (r - r_0) \times \vec{F}$$

$$= \left(\frac{25}{2}\hat{j} + \frac{25\sqrt{3}}{2}\hat{k}\right) \times (500\hat{i})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \frac{25}{2} & \frac{25\sqrt{3}}{2} \\ 500 & 0 & 0 \end{vmatrix}$$

$$\boxed{= 6,250\sqrt{3}\,\hat{j} - 6,250\hat{k}}$$

Since the torque is effective in turning horizontally that implies $T = 6,250\sqrt{3}\ \ N\text{-}cm$.

The effective torque $(r_0 = 0)$:

$$\left(10 + \frac{25\sqrt{3}}{2}\right)\hat{k} \times 500\hat{i} \ \boxed{= \left(5,000 + 6,250\sqrt{3}\right)\hat{j}} \quad \textit{Joule}$$