Section 4.3 – LU-Decompositions

The goal is to describe Gaussian elimination in the most useful way by looking at them closely, which are factorizations of a matrix.

> ***The factors are triangular matrices.***
>
> ***The factorization that comes from elimination is*** $A = LU$ **.**

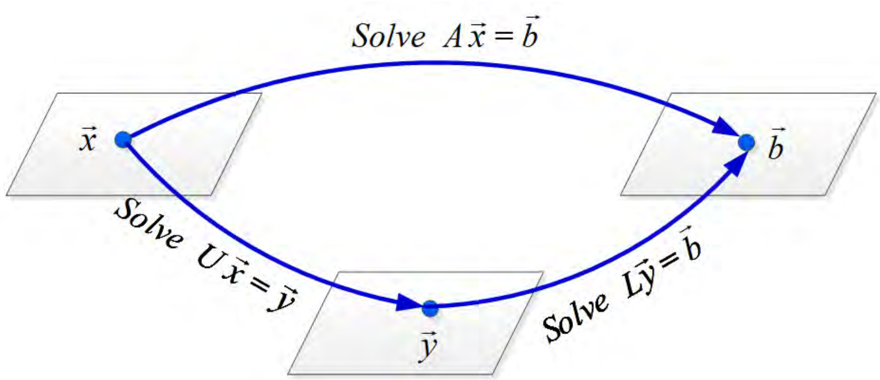
The Method of *LU*–Decomposition

Step 1: Rewrite the system $A\vec{x} = \vec{b}$ as $LU\vec{x} = \vec{b}$

Step 2: Define a new $n \times 1$ matrix $\vec{y}$ by $U\vec{x} = \vec{y}$

Step 3: Use $U\vec{x} = \vec{y}$ to rewrite $LU\vec{x} = \vec{b}$ as $L\vec{y} = \vec{b}$ and solve this system for $\vec{y}$.

Step 4: Substitute $\vec{y}$ in $U\vec{x} = \vec{y}$ and solve for $\vec{x}$.



Example

Given 2 by 2 matrix $A = \begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix}$

Find *L* and *U* and verify $A = LU$

Solution

To make ***row*** 2 ***column*** 1 is ***zero*** then we need to subtract 3 times *row* 2 from *row* 2

$$\begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix} \quad R_2 - 3R_1$$

$$\ell_{21} = -3$$

That step is $E_{21} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$ in the forward direction such that:

$$\begin{aligned} E_{21}A &= \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix} = U \end{aligned}$$

The return step from U to A is $L = E_{21}^{-1}$

$$L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

Back from U to A:

$$\begin{aligned} E_{21}^{-1}U &= \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix} \\ &= A \end{aligned}$$

Therefore; $A = LU$

Example

What matrix L and U puts A into triangular form $A = LU$ where

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad R_2 - \frac{1}{2}R_1 : \ell_{21}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 1 & 2 \end{pmatrix} R_3 - \frac{2}{3}R_2 : \ell_{32}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix} = U$$

$$\boxed{\ell_{21} = -\frac{1}{2} \qquad \ell_{32} = -\frac{2}{3}}$$

The lower triangular L has all 1's on its diagonal. The multipliers ℓ_{ij} are below the diagonal of L with OPPOSITE sign

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}} = \underset{L}{\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}}$$

- $\left(E_{32}E_{31}E_{21}\right)A = U$ *becomes* $A = \left(E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}\right)U$ *which is* $A = LU$

  The inverses go in opposite order.

- $(A = LU)$ This is ***elimination without row exchanges***. The *upper triangular* ***U*** has the pivots on its diagonal. The *lower triangular* ***L*** has all 1's on its diagonal.
  The multipliers ℓ_{ij} are below the diagonal of L.

One Square System = *Two* Triangular Systems

Factor: into L and U, by forward elimination on A.

Solve: forward on $\vec{b}$ using L, then back substitution using U.

Solve $L\vec{c} = \vec{b}$ and then solve $U\vec{x} = \vec{c}$

Example

Forward elimination on $Ax = b$ ends at $Ux = c$

$$\begin{matrix} x + 2y = 5 \\ 4x + 9y = 21 \end{matrix} \quad \textit{becomes} \quad \begin{matrix} x + 2y = 5 \\ y = 1 \end{matrix}$$

Solution

The multiplier was 4. $\left(R_2 - 4R_1\right)$

The lower triangular system: $L\vec{c} = \vec{b}$

$$\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\left[c\right] = \begin{bmatrix} 5 \\ 21 \end{bmatrix}$$

$$\vec{c} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

The upper triangular system: $U\vec{x} = \vec{c}$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\left[x\right] = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

To solve 1000 equations on a PC

- ❖ Elimination on A requires about $\frac{1}{3}n^3$ multiplications and $\frac{1}{3}n^3$ subtractions.
- ❖ Each right–side needs n^2 multiplications and n^2 subtractions.

Exercises *Section* 4.3 – LU-Decompositions

1. What matrix E puts A into triangular form $EA = U$? Multiply by $E^{-1} = L$ to factor A into LU :

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}$$

2. Solve $L\vec{c} = \vec{b}$ to find $\vec{c}$. Then solve $U\vec{x} = \vec{c}$ to find $\vec{x}$. What was A?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

3. Find L and U for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots

4. For which c is $A = LU$ impossible – with three pivots?

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

(5 – 14) Find an LU–decomposition of the coefficient matrix, and then use to solve the system

5. $\begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

6. $\begin{bmatrix} -5 & -10 \\ 6 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 19 \end{bmatrix}$

7. $\begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}$

8. $\begin{bmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$

9. $\begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$

10. $\begin{bmatrix} 2 & -6 & 4 \\ -4 & 8 & 0 \\ 0 & -4 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$

11. $\begin{bmatrix} 2 & -4 & 2 \\ -4 & 5 & 2 \\ 6 & -9 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}$

12. $\begin{bmatrix} 1 & -1 & 2 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$

13. $\begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 3 \\ 7 \end{bmatrix}$

14. $\begin{bmatrix} 1 & -2 & -2 & -3 \\ 3 & -9 & 0 & -9 \\ -1 & 2 & 4 & 7 \\ -3 & -6 & 26 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \\ 3 \end{bmatrix}$

(**15 – 24**) Find an *LU* *f*actorization matrix

15. $\begin{pmatrix} 2 & 5 \\ -3 & -4 \end{pmatrix}$

16. $\begin{pmatrix} 6 & 4 \\ 12 & 5 \end{pmatrix}$

17. $\begin{pmatrix} 3 & 1 & 2 \\ -9 & 0 & -4 \\ 9 & 9 & 14 \end{pmatrix}$

18. $\begin{pmatrix} -5 & 0 & 4 \\ 10 & 2 & -5 \\ 10 & 10 & 16 \end{pmatrix}$

19. $\begin{pmatrix} 3 & 7 & 2 \\ 6 & 19 & 4 \\ 9 & 9 & 14 \end{pmatrix}$

20. $\begin{pmatrix} 2 & 3 & 2 \\ 4 & 13 & 9 \\ -6 & 5 & 4 \end{pmatrix}$

21. $\begin{pmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ -2 & -4 & 7 & 5 \end{pmatrix}$

22. $\begin{pmatrix} 1 & 3 & 1 & 5 \\ 5 & 20 & 6 & 31 \\ -2 & -1 & -1 & -4 \\ -1 & 7 & 1 & 7 \end{pmatrix}$

23. $\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{pmatrix}$

24. $\begin{pmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{pmatrix}$

25. Let *A* be a lower triangular $n \times n$ matrix with nonzero entries on the diagonal. Show that *A* is invertible and A^{-1} is lower triangular.

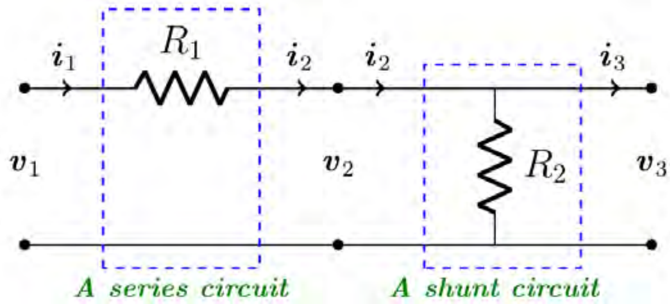
26. Let $A = LU$ be an LU factorization. Explain why *A* can be row reduced to *U* using only replacement operations.

27. Suppose an $m \times n$ matrix A admits a factorization $A = CD$ where C is $m \times 4$ and D is $4 \times n$.

a) Show that A is the sum of four outer products.

b) Let $m = 400$ *and* $n = 100$. Explain why a computer programmer might prefer to store the data from A in the form of two matrices C and D.

28. A ladder network, where two circuits are connected in series, so that the output of one circuit becomes the input of the next circuit.



The transformation $\begin{pmatrix} v_1 \\ i_1 \end{pmatrix} \longrightarrow \begin{pmatrix} v_2 \\ i_2 \end{pmatrix}$ is linear with a transfer matrix A of the ladder network.

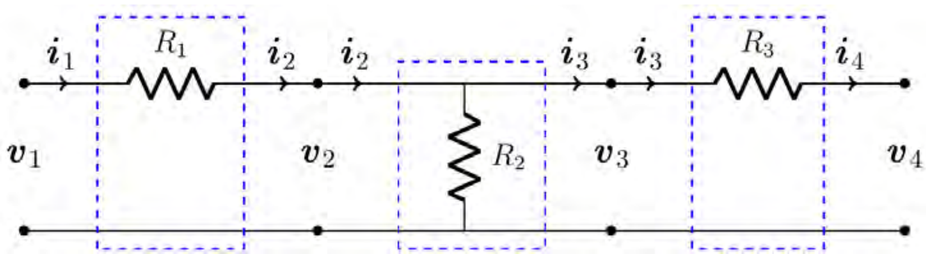
Let the transfer matrix A_1 of the series circuit is given by $\begin{pmatrix} v_2 \\ i_2 \end{pmatrix} = A_1 \begin{pmatrix} v_1 \\ i_1 \end{pmatrix}$

Let the transfer matrix A_2 of the shunt circuit is given by $\begin{pmatrix} v_3 \\ i_3 \end{pmatrix} = A_2 \begin{pmatrix} v_2 \\ i_2 \end{pmatrix}$

a) Compute the transfer matrix of the ladder network

b) Design a ladder network whose transfer matrix is $\begin{pmatrix} 1 & -8 \\ -\frac{1}{2} & 5 \end{pmatrix}$

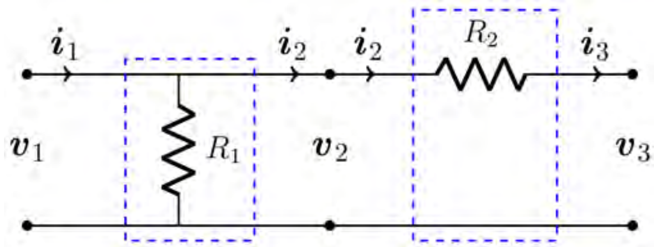
29. A ladder network, where three circuits are connected in series, so that the output of one circuit becomes the input of the next circuit.



a) Compute the transfer matrix of the ladder network

b) Design a ladder network whose transfer matrix is $\begin{pmatrix} 3 & -12 \\ -\frac{1}{3} & \frac{5}{3} \end{pmatrix}$

30. A ladder network, where two circuits are connected in series, so that the output of one circuit becomes the input of the next circuit.



a) Compute the transfer matrix of the ladder network

b) Find the values of the resistors when the input voltage is 12 volts and current is 6 amps if the output voltage is 9 volts and current is 4 amps