# Solution Section 2.1 – Sequences and Summations

### Exercise

Find these terms of the sequence $\{a_n\}$, where $a_n = 2\cdot(-3)^n + 5^n$

$a)\ a_0 \quad b)\ a_1 \quad c)\ a_4 \quad d)\ a_5$

**Solution**

***a)*** $a_0 = 2\cdot(-3)^0 + 5^0$

$= 3$

***b)*** $a_1 = 2\cdot(-3)^1 + 5^1$

$= -6 + 5$

$= -1$

***c)*** $a_4 = 2\cdot(-3)^4 + 5^4$

$= 162 + 625$

$= 787$

***d)*** $a_4 = 2\cdot(-3)^5 + 5^5$

$= -486 + 3125$

$= 2639$

### Exercise

What is the term $a_8$ of the sequence $\{a_n\}$, if $a_n$ equals

$a)\ 2^{n-1} \quad b)\ 7 \quad c)\ 1+(-1)^n \quad d)\ -(2)^n$

**Solution**

***a)*** $a_8 = 2^{8-1}$

$= 128$

***b)*** $a_8 = 7$

***c)*** $a_8 = 1 + (-1)^8$

$= 2$

***d)*** $a_8 = -(2)^8$

$= -256$

### *Exercise*

What are the terms $a_0, a_1, a_2, and\ a_3$ of the sequence $\{a_n\}$, if $a_n$ equals

$a)\ 2^n + 1$ $\quad b)\ (n+1)^{n+1}$ $\quad c)\ \frac{n}{2}$ $\quad d)\ \frac{n}{2} + \frac{n}{2}$

$e)\ (-2)^n$ $\quad f)\ 3$ $\quad g)\ 7 + 4^n$ $\quad h)\ 2^n + (-2)^n$

***Solution***

**a)** $a_0 = 2^0 + 1 = 2$

$a_1 = 2^1 + 1 = 3$

$a_2 = 2^2 + 1 = 5$

$a_3 = 2^3 + 1 = 9$

**b)** $a_0 = (0+1)^{0+1} = 1$

$a_1 = (1+1)^{1+1} = 4$

$a_2 = (2+1)^{2+1} = 27$

$a_3 = (3+1)^{3+1} = 256$

**c)** $a_0 = \frac{0}{2} = 0$

$a_1 = \frac{1}{2}$

$a_2 = \frac{2}{2} = 1$

$a_3 = \frac{3}{2}$

**d)** $a_0 = \frac{0}{2} + \frac{0}{2} = 0$

$a_1 = \frac{1}{2} + \frac{1}{2} = 1$

$a_2 = \frac{2}{2} + \frac{2}{2} = 1$

$a_3 = \frac{3}{2} + \frac{3}{2} = 3$

**e)** $a_0 = (-2)^0 = 1$

$a_1 = (-2)^1 = -2$

$a_2 = (-2)^2 = 4$

$a_3 = (-2)^3 = -8$

*f)* $\underline{a_0 = 3}|$

$\underline{a_1 = 3}|$

$\underline{a_2 = 3}|$

$\underline{a_3 = 3}|$

*g)* $a_0 = 7 + 4^0 \ \underline{= 8}|$

$a_1 = 7 + 4^1 \ \underline{= 11}|$

$a_2 = 7 + 4^2 \ \underline{= 23}|$

$a_3 = 7 + 4^3 \ \underline{= 71}|$

*h)* $a_0 = 2^0 + (-2)^0 \ \underline{= 2}|$

$a_1 = 2^1 + (-2)^1 \ \underline{= 0}|$

$a_2 = 2^2 + (-2)^2 \ \underline{= 0}|$

$a_3 = 2^3 + (-2)^3 \ \underline{= 0}|$

## *Exercise*

Find at least three different sequences beginning with the terms 1, 2, 4 whose terms are generated by a simple formula or rule.

***Solution***

1. $2^{n-1} \quad \rightarrow \quad 1,\ 2,\ 4,\ 8,\ 16,\ \ldots$
2. The second pattern, $2-1=1 \quad 4-2=2$, as we see the difference to the previous increasing by value of 1.
   So, the next term $4+3=7 \quad 7+4=11$.
   Therefore; the sequence is 1, 2, 4, 7, 11, 16, …
3. 1, 2, 4, 1, 2, 4, … Repeating the terms

## *Exercise*

Find at least three different sequences beginning with the terms 3, 5, 7 whose terms are generated by a simple formula or rule.

***Solution***

One rule should be that each term is greater than the previous term by 2; the sequence would be 3, 5, 7, 9, 11, 13, . . .

Another rule could be that the $n^{th}$ old prime.

The sequence would be 3, 5, 7, 11, 13, 17, …

The sequence: 3, 5, 7, 12, 23, 43, 75, 122, 187, 273 from an equation $\frac{1}{2}\left(x^3 - 6x^2 + 15x - 4\right)$

## *Exercise*

Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions.

*a)* $a_n = 6a_{n-1}, \quad a_0 = 2$

*b)* $a_n = a_{n-1}^2, \quad a_1 = 2$

*c)* $a_n = a_{n-1} + 3a_{n-2}, \quad a_0 = 1,\ a_1 = 2$

*d)* $a_n = na_{n-1} + n^2 a_{n-2}, \quad a_0 = 1,\ a_1 = 1$

*e)* $a_n = a_{n-1} + a_{n-3}, \quad a_0 = 1,\ a_1 = 2,\ a_2 = 0$

***Solution***

***a)*** $a_n = 6a_{n-1}, \quad a_0 = 2$

$a_1 = 6a_0 = 6(2) \underline{= 12}$

$a_2 = 6a_1 = 6(12) \underline{= 72}$

$a_3 = 6a_2 = 6(72) \underline{= 432}$

$a_4 = 6a_3 = 6(432) \underline{= 2592}$

***b)*** $a_n = a_{n-1}^2, \quad a_1 = 2$

$a_2 = a_1^2 = 2^2 \underline{= 4}$

$a_3 = a_2^2 = 4^2 \underline{= 16}$

$a_4 = a_3^2 = 16^2 \underline{= 256}$

$a_5 = a_4^2 = 256^2 \underline{= 65536}$

***c)*** $a_n = a_{n-1} + 3a_{n-2}, \quad a_0 = 1,\ a_1 = 2$

$a_2 = a_1 + 3a_0 = 2 + 3(1) \underline{= 5}$

$a_3 = a_2 + 3a_1 = 5 + 3(2) \underline{= 11}$

$a_4 = a_3 + 3a_2 = 11 + 3(5) \underline{= 26}$

$a_5 = a_4 + 3a_3 = 26 + 3(11) \underline{= 59}$

**d)** $a_n = na_{n-1} + n^2 a_{n-2}, \quad a_0 = 1,\ a_1 = 1$

$a_2 = 2a_1 + 2^2 a_0 = 2(1) + 4(1) \underline{= 6}$

$a_3 = 3a_2 + 3^2 a_1 = 3(6) + 9(1) \underline{= 27}$

$a_4 = 4a_3 + 4^2 a_2 = 4(27) + 16(6) \underline{= 204}$

$a_5 = 5a_4 + 5^2 a_3 = 5(204) + 25(27) \underline{= 1695}$

**e)** $a_n = a_{n-1} + a_{n-3}, \quad a_0 = 1,\ a_1 = 2,\ a_2 = 0$

$a_3 = a_2 + a_0 = 0 + 1 \underline{= 1}$

$a_4 = a_3 + a_1 = 1 + 2 \underline{= 3}$

$a_5 = a_4 + a_2 = 3 + 0 \underline{= 3}$

$a_6 = a_5 + a_3 = 3 + 3 \underline{= 6}$

## *Exercise*

Find the first six terms of the sequence defined by each of these recurrence relations and initial conditions.

*a)* $a_n = -2a_{n-1}, \quad a_0 = -1$

*b)* $a_n = a_{n-1} - a_{n-2}, \quad a_0 = 2,\ a_1 = -1$

*c)* $a_n = 3a_{n-1}^2, \quad a_0 = 1$

*d)* $a_n = na_{n-1} + n^2 a_{n-2}, \quad a_0 = -1,\ a_1 = 0$

*e)* $a_n = a_{n-1} - a_{n-2} + a_{n-3}, \quad a_0 = 1,\ a_1 = 2,\ a_2 = 2$

***Solution***

**a)** $a_0 = -1$

$a_1 = -2a_0 \underline{= 2}$

$a_2 = -2a_1 \underline{= -4}$

$a_3 = -2a_2 \underline{= 8}$

$a_4 = -2a_3 \underline{= -16}$

$a_5 = -2a_4 \underline{= 32}$

**b)** $a_0 = 2, \quad a_1 = -1$

$a_2 = a_1 - a_0 \underline{= -3}$

$a_3 = a_2 - a_1 \underline{= -2}$

$a_4 = a_3 - a_2 \ \boxed{= 1}$

$a_5 = a_4 - a_3 \ \boxed{= 3}$

***c)*** $a_0 = 1$

$a_1 = 3a_0^2 \ \boxed{= 3}$

$a_2 = 3a_1^2 = 27 \ \boxed{= 3^3}$

$a_3 = 3a_2^2 = 2187 \ \boxed{= 3^7}$

$a_4 = 3a_3^2 = 14348907 \ \boxed{= 3^{15}}$

$a_5 = 3a_4^2 \ \boxed{= 3^{31}}$

***d)*** $a_0 = -1, \quad a_1 = 0$

$a_2 = 2a_1 + a_0^2 \ \boxed{= 1}$

$a_3 = 3a_2 + a_1^2 \ \boxed{= 3}$

$a_4 = 4a_3 + a_2^2 \ \boxed{= 13}$

$a_5 = 5a_4 + a_3^2 \ \boxed{= 74}$

***e)*** $a_0 = 1, \quad a_1 = 1, \quad a_2 = 2$

$a_3 = a_2 - a_1 + a_0 \ \boxed{= 2}$

$a_4 = a_3 - a_2 + a_1 \ \boxed{= 1}$

$a_5 = a_4 - a_3 + a_2 \ \boxed{= 1}$

## ***Exercise***

Let $a_n = 2^n + 5 \cdot 3^n \quad for \ \ n = 0, 1, 2, \ldots$

*a)* Find $a_0, a_1, a_2, a_3, and\ a_4$

*b)* Show that $a_2 = 5a_1 - 6a_0$, $a_3 = 5a_2 - 6a_1$, and $a_4 = 5a_3 - 6a_2$

*c)* Show that $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers *n* with $n \geq 2$

***Solution***

***a)*** $a_0 = 2^0 + 5 \cdot 3^0 = 1 + 5 \ \boxed{= 6}$

$a_1 = 2^1 + 5 \cdot 3^1 = 2 + 15 \ \boxed{= 17}$

$a_2 = 2^2 + 5\cdot 3^2 = 4 + 45 \ \boxed{= 49}$

$a_3 = 2^3 + 5\cdot 3^3 = 8 + 5(27) \ \boxed{= 143}$

$a_4 = 2^4 + 5\cdot 3^4 = 16 + 5(81) \ \boxed{= 421}$

***b)***

$a_2 = 5a_1 - 6a_0$

$49 \overset{?}{=} 5(17) - 6(6)$

$49 = 49 \quad \checkmark$

***Or***

$$\begin{aligned}
5a_1 - 6a_0 &= 5\left(2^1 + 5\cdot 3^1\right) - 6\left(2^0 + 5\cdot 3^0\right) \\
&= 5\cdot 2 + 5\cdot 3 - 2\cdot 3 - 2\cdot 3\cdot 5 \\
&= (5\cdot 2 - 2\cdot 3) + 5\cdot 5\cdot 3 - 5\cdot 2\cdot 3 \\
&= 2(5-3) + 5\cdot 3(5-2) \\
&= 2\cdot 2 + 5\cdot 3\cdot 3 \\
&= 2^2 + 5\cdot 3^2 \quad \checkmark
\end{aligned}$$

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$a_3 = 5a_2 - 6a_1$

$143 \overset{?}{=} 5(49) - 6(17)$

$143 = 143 \quad \checkmark$

***Or***

$$\begin{aligned}
5a_2 - 6a_1 &= 5\left(2^2 + 5\cdot 3^2\right) - 6\left(2^1 + 5\cdot 3^1\right) \\
&= 5\cdot 4 + 5\cdot 9 - 2\cdot 3\cdot 2 - 2\cdot 3\cdot 5\cdot 3 \\
&= (5\cdot 2 - 4\cdot 3) + 5\cdot 9 - 5\cdot 2\cdot 9 \\
&= 2(5-3) + 5\cdot 3(5-2) \\
&= 2\cdot 2 + 5\cdot 3\cdot 3 \\
&= 2^2 + 5\cdot 3^2 \quad \checkmark
\end{aligned}$$

$a_4 = 5a_3 - 6a_2$

$421 \overset{?}{=} 5(143) - 6(49)$

$421 = 421 \quad \checkmark$

***Or***

$$\begin{aligned}5a_3-6a_2&=5\left(2^3+5\cdot 3^3\right)-6\left(2^2+5\cdot 3^2\right)\\&=5\cdot 2^3+5^2\cdot 3^3-3\cdot 2^3-2\cdot 3^3\cdot 5\\&=5\cdot 2^3-3\cdot 2^3+5^2\cdot 3^3-2\cdot 3^3\cdot 5\\&=2^3(5-3)+5\cdot 3^3(5-2)\\&=2^4+5\cdot 3^4\quad\checkmark\end{aligned}$$

### *Exercise*

Is the sequence $\{a_n\}$ a solution of the recurrence relation $a_n=8a_{n-1}-16a_{n-2}$ if

*a)* $a_n=0$?

*b)* $a_n=1$?

*c)* $a_n=2^n$?

*d)* $a_n=4^n$?

*e)* $a_n=n4^n$?

*f)* $a_n=2\cdot 4^n+3n4^n$?

*g)* $a_n=(-4)^n$?

*h)* $a_n=n^2 4^n$?

***<u>Solution</u>***

***a)*** Let $a_n=8a_{n-1}-16a_{n-2}=0$ We get $0=0$ which is a true statement.

$\therefore\ a_n=0$ is a solution of the recurrence relation.

***b)*** Let $a_n=8a_{n-1}-16a_{n-2}=1$

We get $1=8\cdot 1-16\cdot 1=-8$ which is a false statement.

$\therefore\ a_n=1$ is not a solution.

***c)*** Let $a_n=8a_{n-1}-16a_{n-2}=2^n$

We get $2^n=8\cdot 2^{n-1}-16\cdot 2^{n-2}=2^{n-2}(8\cdot 2-16)=0$

which is a false statement.

$\therefore\ a_n=1$ is not a solution.

***d)*** Let $a_n=8a_{n-1}-16a_{n-2}=4^n$ We get

$$4^n=8\cdot 4^{n-1}-16\cdot 4^{n-2}$$

$$= 4^{n-2}(8\cdot 4 - 16)$$

$$= 4^{n-2}\cdot(16)$$

$$= 4^{n-2}\cdot 4^2$$

$$= 4^n \text{ which is a true statement}$$

$\therefore\ a_n = 4^n$ is a solution of the recurrence relation.

***e)*** Let $a_n = 8a_{n-1} - 16a_{n-2} = n4^n$ We get

$$n4^n = 8\cdot n4^{n-1} - 16\cdot n4^{n-2}$$

$$= n4^{n-2}(8\cdot 4 - 16)$$

$$= n4^{n-2}\cdot\left(4^2\right)$$

$$= n4^n \text{ which is a true statement}$$

$\therefore\ a_n = n4^n$ is a solution of the recurrence relation.

***f)*** Let $a_n = 8a_{n-1} - 16a_{n-2} = 2\cdot 4^n + 3n4^n$ We get

$$2\cdot 4^n + 3n4^n = 8\cdot\left(2\cdot 4^{n-1} + 3(n-1)4^{n-1}\right) - 16\cdot\left(2\cdot 4^{n-2} + 3(n-2)4^{n-2}\right)$$

$$= 8\cdot 4^{n-2}\left(2\cdot 4 + 3\cdot 4(n-1) - 2\cdot\left(2 + 3(n-2)\right)\right)$$

$$= 8\cdot 4^{n-2}(8 + 12n - 12 - 4 - 6n + 12)$$

$$= 8\cdot 4^{n-2}(4 + 6n)$$

$$= 4^2 4^{n-2}(2 + 3n)$$

$$= 2\cdot 4^n + 3n\cdot 4^n \text{ which is a true statement}$$

$\therefore\ a_n = 2\cdot 4^n + 3n\cdot 4^n$ is a solution of the recurrence relation.

***g)*** Let $a_n = 8a_{n-1} - 16a_{n-2} = (-4)^n$ We get

$$(-4)^n = 8\cdot(-4)^{n-1} - 16\cdot(-4)^{n-2}$$

$$= (-4)^{n-2}\left(8\cdot(-4) - 16\right)$$

$$= (-4)^{n-2}(-48)$$

$$= (-4)^{n-2}(-16\cdot 3)$$

$$= (-4)^{n-2}(-4)^2\cdot 3$$

$$= (-4)^n\cdot 3 \text{ which is a false statement}$$

$\therefore\ a_n = 4^n$ is not a solution.

***h)*** Let $a_n = 8a_{n-1} - 16a_{n-2} = n^2 4^n$ We get

$$n^2 4^n = 8 \cdot (n-1)^2 4^{n-1} - 16 \cdot (n-2)^2 4^{n-2}$$

$$= 8 \cdot 4^{n-2}\left(\left(n^2 - 2n + 1\right) \cdot 4 - 2 \cdot \left(n^2 - 4n + 4\right)\right)$$

$$= 16 \cdot 4^{n-2}\left(2n^2 - 4n + 2 - n^2 + 4n - 4\right)$$

$$= 4^n\left(n^2 - 2\right)$$

$= n4^n$ which is a false statement

$\therefore\ a_n = n^2 4^n$ is a solution of the recurrence relation.

## *Exercise*

Is the sequence $\{a_n\}$ a solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$ if

*a)* $a_n = -n + 2$

*b)* $a_n = 5(-1)^n - n + 2$

*c)* $a_n = 3(-1)^n + 2^n - n + 2$

*d)* $a_n = 7 \cdot 2^n - n + 2$

***Solution***

***a)*** $a_{n-1} + 2a_{n-2} + 2n - 9 = -(n-1) + 2 + 2\left[-(n-2) + 2\right] + 2n - 9$

$$= -n + 1 + 2 - 2n + 4 + 4 + 2n - 9$$

$$= -n + 2$$

$$= a_n$$

***b)*** $a_{n-1} + 2a_{n-2} + 2n - 9 = 5(-1)^{n-1} - (n-1) + 2 + 2\left[5(-1)^{n-2} - (n-2) + 2\right] + 2n - 9$

$$= 5(-1)^{n-1} - n + 3 + 2\left[5(-1)^{n-2} - n + 4\right] + 2n - 9$$

$$= 5(-1)^{n-1} - n + 3 + 10(-1)^{n-2} - 2n + 8 + 2n - 9$$

$$= 5(-1)^{n-1} + 10(-1)^{n-1}(-1)^{-1} - n + 2$$

$$= 5(-1)^{n-1} - 10(-1)^{n-1} - n + 2$$

$$= -5(-1)^{n-1} - n + 2$$

$$= (-1)^1 5(-1)^{n-1} - n + 2$$

$$= 5(-1)^n - n + 2$$

$$= a_n$$

***c)*** $a_{n-1} + 2a_{n-2} + 2n - 9 = 3(-1)^{n-1} + 2^{n-1} - (n-1) + 2$

$$+2\left[3(-1)^{n-2} + 2^{n-2} - (n-2) + 2\right] + 2n - 9$$

$$= 3(-1)^{n-1} + 2^{n-1} + 6(-1)^{n-2} + 2^{n-1} - 2n + 8 + n - 6$$

$$= 3(-1)^{n-1} - 6(-1)^{n-1} + 2^n - n + 2$$

$$= -3(-1)^{n-1} + 2^n - n + 2$$

$$= 3(-1)^n + 2^n - n + 2$$

$$= a_n$$

***d)*** $a_{n-1} + 2a_{n-2} + 2n - 9 = 7 \cdot 2^{n-1} - (n-1) + 2 + 2\left[7 \cdot 2^{n-2} - (n-2) + 2\right] + 2n - 9$

$$= 7 \cdot 2^{n-1} + 7 \cdot 2^{n-1} - 2n + 8 + n - 6$$

$$= 2 \cdot 7 \cdot 2^{n-1} - n + 2$$

$$= 7 \cdot 2^n - n + 2$$

$$= a_n$$

## *Exercise*

A person deposits \$1,000.00 in an account that yields 9% interest compounded annually.

*a)* Set up a recurrence relation for the amount in the account at the end of *n* years.

*b)* Find an explicit formula for the amount in the account at the end of *n* years.

*c)* How much money will the account contain after 100 years?

***Solution***

***a)*** The amount after $n-1$ years multiplied by 1.09 to give the amount after n years, since 9% of the value must be added to account for the interest. Therefore, we have $a_n = 1.09a_{n-1}$. The initial condition is $a_0 = 1000$.

***b)*** Since multiplying by 1.09 for each year, the solution is $a_n = 1000(1.09)^n$.

***c)*** $a_{100} = 1000(1.09)^{100}$

$$\approx \$5,529,041$$

### *Exercise*

Suppose that the number of bacteria in a colony triples every hour.

*a)* Set up a recurrence relation for the number of bacteria after *n* hours have elapsed.

*b)* If 100 bacteria are used to begin new colony, how many bacteria will be in the colony in 10 hours?

***Solution***

***a)*** Since the number of bacteria triples every hour, the recurrence relation should say that the number of bacteria after *n* hours is 3 times the number of bacteria after *n* – 1 hours.

Let $a_n$ denote the number of bacteria after *n* hours, this statement translates into the recurrence relation $a_n = 3a_{n-1}$

***b)*** The initial condition is $a_0 = 100$.

$$a_n = 3 \cdot a_{n-1}$$
$$= 3^2 \cdot a_{n-2}$$
$$\vdots \quad \vdots$$
$$= 3^n \cdot a_0$$

$n = 10$

$$a_{10} = 100 \cdot 3^{10}$$
$$\underline{= 5,904,900}$$

### *Exercise*

A factory makes custom sports cars at an increasing rate. In the first month only one car is made, in the second month two cars are made, and so on, with *n* cars made in the *n*th month.

*a)* Set up a recurrence relation for the number of cars produced in the first *n* months by this factory.

*b)* How many cars are produced in the first year?

*c)* Find an explicit formula for the number of cars produced in the first *n* months by this factory

***Solution***

***a)*** Let $c_n$ be the number of cars produced in the first *n* months.

The initial condition is $c_0 = 0$.

Since *n* cars are made in the *n*th month then $c_n = c_{n-1} + n$, where $c_{n-1}$ is the first $n$–1 months

***b)*** The number of cars produced in the first year is $c_{12}$.

Plug in $n = 12$, we get

$$c_n = n + c_{n-1}$$
$$= n + (n-1) + c_{n-2}$$

$$= n + (n-1) + (n-2) + c_{n-3}$$

$$\vdots \quad \vdots$$

$$= n + (n-1) + (n-2) + \cdots + 1 + c_0 \qquad\qquad n + (n-1) + (n-2) + \cdots + 1 = \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2} + 0$$

$$= \frac{n^2 + n}{2}$$

$$c_{12} = \frac{12^2 + 12}{2}$$

$$= 78$$

***c)*** $c_n = \frac{n^2 + n}{2}$

## *Exercise*

For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence.

*a)* 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, …

*b)* 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, …

*c)* 1, 0, 2, 0, 4, 0, 8, 0, 16, 0, …

*d)* 3, 6, 12, 24, 48, 96, 192, …

*e)* 15, 8, 1, –6, –13, –20, –27, …

*f)* 3, 5, 8, 12, 17, 23, 30, 38, 47, …

*g)* 2, 16, 54, 128, 250, 432, 686, …

*h)* 2, 3, 7, 25, 121, 721, 5041, 40321, …

*i)* 3, 6, 11, 18, 27, 38, 51, 66, 83, 102, …

*j)* 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, …

*k)* 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, …

***Solution***

***a)*** We have one 1 and one 0, then two 1 and two 0, then three of each, and so on increasing the repetition by one each time. Since we have only one at the end, then we need three 1 and four 0 to continue the sequence.

***b)*** A pattern is that the positive integers are increasing order, with odd number showing once and each even number repeated.
Thus, the next terms are 9, 10, 10.

***c)*** The terms in the odd locations are the successive terms in the geometric sequence that starts with 1 and has ratio 2, and the terms in the even locations are all 0. The $n$th term is 0 if $n$ even and is $2^{(n-1)/2}$ if $n$ is odd.

Thus, the next three terms are 32, 0, 64.

***d)*** The first term is 3 and each successive term is twice the predecessor. The $n$th term is $3 \cdot 2^{n-1} \quad n > 0$.

Thus, the next three terms are 384, 768, 1536.

***e)*** The first term is 15 and each successive term is 7 less than its predecessor. The $n$th term is $15 - 7(n-1) = 22 - 7n$ .

Thus, the next three terms are –34, –41, –48.

***f)*** The first term is 3 and each successive term by adding $n$ to its predecessor.

$$3,\ 3+2,\ 5+3,\ 8+4,\ 12+5 \quad nth \quad 3+2+3+4+5+\cdots+n = 2+1+2+3+4+5+\cdots+n$$

$$= 2 + \frac{n^2+n}{2}$$

The $n$th term is $\frac{1}{2}\left(n^2+n+4\right)$.

Thus, the next three terms are 57, 68, 80.

***g)*** Since all numbers are even, then if we divide by 2 the sequence becomes: 1, 8, 27, 64, 125, 216, 343, …. This sequence appears to be $n^3$, therefore the $n$th term is $2n^3$.

Thus, the next three terms are 1024, 1458, 2000.

***h)*** The $n$th term appears to be $n! + 1$.

Thus, the next three terms are 362881, 3628801, 39916801.

***i)*** The first term is 3 then by adding 3 to the predecessor, then 5, then 7, and so on.

$$3,\ 3+3,\ 6+5,\ 11+7,\ \ldots \rightarrow\ 1+2,\ 4+2,\ 9+2,\ 16+2,\ \ldots$$

Then the $n$th term is $n^2+2$.

Thus, the next three terms are 123, 146, 171.

***j)*** This an arithmetic sequence whose difference is 4. Thus, the $n$th term is $7+4(n-1) = 4n+3$.

Thus, the next three terms are 47, 51, 55.

***k)*** This is a binary expansion of $n$. Thus, the next three terms are 1100, 1101, 1110.

# Solution Section 2.2 – Algorithms

### Exercise

List all the steps used by the Algorithm 1 to find the maximum of the list

1, 8, 12, 9, 11, 2, 14, 5, 10, 4.

**Solution**

The ***for*** loop then begins, with $i$ set equal from 2 to $n=10$ (number of the sequence).

The statement of the loop is executed since 2 < 10. This is an ***if*** … ***then*** statement.

$max$**:=** 1

***for*** $i := 2$ to 10

***if max*** $< a_i$ **then** $\max := a_i$

$a_i = a_2 = 8$, since 1 < 8, then $\boldsymbol{max} := 8$

$a_i = a_3 = 12$, since 8 < 12, then $\boldsymbol{max} := 12$

$a_i = a_4 = 9$, since 12 < 9 ***is not true***, then $\boldsymbol{max} := 12$

$a_i = a_5 = 11$, since 12 < 11 ***is not true***, then $\boldsymbol{max} := 12$

$a_i = a_6 = 2$, since 12 < 2 ***is not true***, then $\boldsymbol{max} := 12$

$a_i = a_7 = 14$, since 12 < 14, then $\boldsymbol{max} := 14$

$a_i = a_8 = 5$, since 14 < 5 ***is not true***, then $\boldsymbol{max} := 14$

$a_i = a_9 = 10$, since 14 < 10 ***is not true***, then $\boldsymbol{max} := 14$

$a_i = a_{10} = 4$, since 14 < 4 ***is not true***, then $\boldsymbol{max} := 14$

Therefore ***max*** has the value $\boxed{14}$

### Exercise

Devise an algorithm that finds the sum of all the integers in a list.

**Solution**

***Procedure*** *sum* $\left\{a_1, a_2, \ldots, a_n : integers\right\}$

*sum***:=** $a_1$

***for*** $i := 2$ to $n$

$\qquad sum := sum + a_i$

***return*** *sum*{ is the sum of all the elements in the list}

### Exercise

Describe an algorithm that takes as an input a list of $n$ integers and produces as output the largest difference obtained by subtracting an integer in the list from the one following it.

**Solution**

For $i$ going from 1 through $n - 1$, compute the value of the $(i+1)^{st}$ element in the list minus the $i^{st}$ element in the list. If this is larger than the answer, reset the answer to be this value.

### Exercise

Describe an algorithm that takes as an input a list of $n$ integers in non-decreasing order and produces the list of all values that occur more than once.

**Solution**

***Procedure*** *negatives* $\{a_1, a_2, \ldots, a_n : integers\}$

$k := 0$

***for*** $i := 1$ to $n$

  ***if*** $a_i < 0$ ***then*** $k := k + 1$

***return*** $k$ { the number of negative integers in the list}

### Exercise

Describe an algorithm that takes as an input a list of $n$ integers and finds the location of the last even integer in the list or returns 0 if there are no even integers in the list.

**Solution**

***Procedure*** *last even loction* $\{a_1, a_2, \ldots, a_n : integers\}$

$k := 0$

***for*** $i := 1$ to $n$

  ***if*** $a_i$ *is even* ***then*** $k := i$

***return*** $k$ { *is the desired location* (*or* 0 *if there are no evens*)}

### Exercise

Describe an algorithm that interchanges the values of the variables $x$ and $y$, using only assignments. What is the minimum number of assignment statements needed to do this?

**Solution**

We cannot simply write $x := y$ followed by $y := x$.

$temp := x$
$x := y$
$y := temp$

## *Exercise*

List all the steps used to search for 9 in the sequence 1, 3, 4, 5, 6, 7, 9, 11 using
*a*) a linear search *b*) a binary search

***Solution***

***a)*** Note that $n$ = 8 and $x$ = 9.

***procedure*** linear_search ($x$: integer; $a_1, a_2, \ldots, a_n$ : integers)

$i := 1$

***while*** ( $i \le 8$ and $\left(i \le 8 \ \ and \ \ 9 \ne a_i\right)$

$i := i + 1$

The ***while*** loop is executed as long as $i \le 8$ and the $i^{st}$ element is not equal to 9.

$i = 1, \quad a_1 = 1; \quad 9 \ne 1$

$i = 2, \quad a_2 = 3; \quad 9 \ne 3$

$i = 3, \quad a_3 = 4; \quad 9 \ne 4$

$i = 4, \quad a_4 = 5; \quad 9 \ne 5$

$i = 5, \quad a_5 = 6; \quad 9 \ne 6$

$i = 6, \quad a_6 = 7; \quad 9 \ne 7$

$i = 7, \quad a_7 = 7; \quad 9 = 9$

Therefore the body of the loop is not executed (so $i$ is still equal to 7), and control passes beyond the loop.

***if*** $i \le n$ ***then*** $location := i$

***else*** $location := 0$

The else clause is not executed. This completes the procedure, so location has the correct value, namely 7, which indicates the location of the element $x$ in the list: 9 is the seventh element.

***b)*** ***procedure*** linear_search ($x$: integer; $a_1, a_2, \ldots, a_n$ : increasing integers)

$i := 1$

$j := 8$

***while*** $i < j$

The while step is executed, first $m = \frac{1+8}{2} = 4$

Then since $x$ (= 9) is greater than $a_4\left(= 5\right)$, the statement $i := m + 1$ is executed, so $i$ has the value 5.

$i = 4 + 1 = 5, \quad m = \frac{5+8}{2} = 6 \quad x(=9) > a_6\ (=6)$

$i = 6 + 1 = 7, \quad m = \frac{7+8}{2} = 7 \quad x(=9) > a_7\ (=9)$ fails thus $j := m$, so $j := 7$

At this point $i \not< j$, the condition $x = a_i$ is true, location is set to 7, as it should be, and the algorithm is finished.

### Exercise

Describe an algorithm that inserts an integer $x$ in the appropriate position into the list $a_1, a_2, \ldots, a_n$ of integers that are in increasing order.

***Solution***

***procedure*** insert $\left(x, a_1, a_2, \ldots, a_n : integers\right)$

$a_{n+1} := x + 1$

$i := 1$

***while*** $x > a_i$

$i := i + 1$ {***The loop ends when i is the index for x***}

***for*** $j := 0$ ***to*** $n - i$ {***Shove the rest of the list to the right***}

$a_{n-j+1} := a_{n-j}$

$a_i := x$

{$x$ has been inserted into the correct spot in the list, now of length $n + 1$}

# *Solution* — *Section* 2.3 – Divisibility and Modular Arithmetic

### *Exercise*

Does 17 divide each of these numbers?

***a***) 68  ***b***) 84  ***c***) 35  ***d***) 1001

***Solution***

***a)*** $68 = 17 \cdot 4$  *Yes*

***b)*** $84 = 17 \cdot 4 + 16$  *No*, remainder 16

***c)*** $357 = 17 \cdot 21$  *Yes*

***d)*** $1001 = 17 \cdot 58 + 15$  *No*, remainder 15

### *Exercise*

Prove that if *a* is an integer other than 0, then

***a***) $1 \text{ divides } a$  ***b***) $a \text{ divides } 0$

***Solution***

*a)* $1|a \quad \text{since} \quad a = 1 \cdot a$

*b)* $a|0 \quad \text{since} \quad 0 = a \cdot 0$

### *Exercise*

Show that if $a \mid b$ and $b \mid a$, where *a* and *b* are integers, then $a = b$ or $a = -b$.

***Solution***

Let *s* and *t* are integers such that $a = bs$ and $b = at$.

$a = bs = ats$. Since $a \neq 0$, we conclude that $st = 1$.

The only way for this to happen, since *s* and *t* are integers, is for $s = t = 1 \quad or \quad s = t = -1$.

Therefore, either $a = b$ or $a = -b$.

### *Exercise*

Show that if *a*, *b*, and *c* are integers, where $a \neq 0$ and $c \neq 0$, such that $ac \mid bc$, then $a \mid b$

***Solution***

Since $ac \mid bc \ \Rightarrow bc = (ac)t$ for some integers *t*

Since $c \neq 0$, divide both sides by *c* to obtain $b = at$ and this result to $a \mid b$ $\checkmark$

## *Exercise*

What are the quotient and remainder when

*a)* 19 is divided by 7?

*b)* $-111$ is divided by 11?

*c)* 789 is divided by 23?

*d)* 1001 is divided by 13?

*e)* 0 is divided by 19?

*f)* 3 is divided by 5?

*g)* $-1$ is divided by 3?

*h)* 4 is divided by 1?

***Solution***

**a)** $19 = 7 \cdot 2 + 5 \qquad q = 2 \quad and \quad r = 5$

**b)** $-111 = 11 \cdot (-11) + 10 \qquad q = -11 \quad and \quad r = 10$

**c)** $789 = 23 \cdot 34 + 7 \qquad q = 34 \quad and \quad r = 7$

**d)** $1001 = 13 \cdot 77 + 0 \qquad q = 77 \quad and \quad r = 0$

**e)** $0 = 19 \cdot 0 + 0 \qquad q = 0 \quad and \quad r = 0$

**f)** $3 = 5 \cdot 0 + 3 \qquad q = 0 \quad and \quad r = 3$

**g)** $-1 = 3 \cdot (-1) + 2 \qquad q = -1 \quad and \quad r = 2$

**h)** $4 = 1 \cdot 4 + 0 \qquad q = 4 \quad and \quad r = 0$

## *Exercise*

What time does a 12-hour clock read

*a)* 80 hours after it reads 11:00?

*b)* 40 hours before it reads 12:00?

*c)* 100 hours after it reads 6:00?

***Solution***

**a)** $11 - 80 \; \boldsymbol{mod} \; 12 = 11 - 8 = 7$, the clock reads 7:00.

**b)**
$$\begin{aligned} 12 - 40 \; \boldsymbol{mod} \; 12 &= -28 \; \boldsymbol{mod} \; 12 \qquad (12 - 40 = -28) \\ &= -28 + 36 \; \boldsymbol{mod} \; 12 \\ &= 8 \end{aligned}$$

The clock reads 8:00.

**c)** $6 + 100 \; \boldsymbol{mod} \; 12 = 6 + 4 = 10$, the clock reads 10:00.

## *Exercise*

What time does a 24-hour clock read

*a)* 100 hours after it reads 2:00?
*b)* 45 hours before it reads 12:00?
*c)* 168 hours after it reads 19:00?

***Solution***

**a)** $2+100\ \boldsymbol{mod}\ 24 = 2+4 = 6$, the clock reads 6:00

**b)** $12-45\ \boldsymbol{mod}\ 24 = -33\ \boldsymbol{mod}\ 24 = -33+48\ \boldsymbol{mod}\ 24 = 15$, the clock reads 15:00

**c)** $168\ \boldsymbol{mod}\ 24 = 0$, the clock reads 19:00

## *Exercise*

Suppose $a$ and $b$ are integers, $a \equiv 4 \pmod{13}$, and $b \equiv 9 \pmod{13}$. Find the integer $c$ with $0 \le c \le 12$ such that

*a)* $c \equiv 9a \pmod{13}$

*b)* $c \equiv 11b \pmod{13}$

*c)* $c \equiv a+b \pmod{13}$

*d)* $c \equiv 2a+3b \pmod{13}$

*e)* $c \equiv a^2+b^2 \pmod{13}$

*f)* $c \equiv a^3-b^3 \pmod{13}$

***Solution***

**a)** $c = 9\cdot 4\ \boldsymbol{mod}\ 13 = 36\ \boldsymbol{mod}\ 13 = 10$

**b)** $c = 11\cdot 9\ \boldsymbol{mod}\ 13 = 99\ \boldsymbol{mod}\ 13 = 8$

**c)** $c = 4+9\ \boldsymbol{mod}\ 13 = 13\ \boldsymbol{mod}\ 13 = 0$

**d)** $c = 2(4)+3(9)\ \boldsymbol{mod}\ 13 = 35\ \boldsymbol{mod}\ 13 = 9$

**e)** $c = 4^2+9^2\ \boldsymbol{mod}\ 13 = 97\ \boldsymbol{mod}\ 13 = 6$

**f)** $c = 4^3-9^3\ \boldsymbol{mod}\ 13 = -665\ \boldsymbol{mod}\ 13 = 11$ $\qquad (-665=-52\times 13+11)$

### *Exercise*

Suppose $a$ and $b$ are integers, $a \equiv 11 (\textbf{\textit{mod}}\ 19)$, and $b \equiv 3 (\textbf{\textit{mod}}\ 19)$. Find the integer $c$ with $0 \le c \le 10$ such that

*a)* $c \equiv a - b (\textbf{\textit{mod}}\ 19)$

*b)* $c \equiv 7a + 3b (\textbf{\textit{mod}}\ 19)$

*c)* $c \equiv 2a^2 + 3b^2 (\textbf{\textit{mod}}\ 19)$

*d)* $c \equiv a^3 + 4b^3 (\textbf{\textit{mod}}\ 19)$

***Solution***

***a)*** $c = 11 - 3\ \textbf{\textit{mod}}\ 19 = \underline{8}$

***b)*** $c = 7(11) + 3(3)\ \textbf{\textit{mod}}\ 19 = 86\ \textbf{\textit{mod}}\ 19 = \underline{10}$ $\qquad 7(11) + 3(3) = 86 \equiv 10 (\textbf{\textit{mod}}\ 19)$

***c)*** $2(11)^2 + 3(3)^2 = 263 \equiv 3\ (\textbf{\textit{mod}}\ 19)$

***d)*** $(11)^3 + (3)^3 = 1439 \equiv 14\ (\textbf{\textit{mod}}\ 19)$

### *Exercise*

Let $m$ be a positive integer. Show that $a\ \textbf{\textit{mod}}\ m \equiv b\ \textbf{\textit{mod}}\ m$ if $a \equiv b\ \textbf{\textit{mod}}\ m$

***Solution***

Given $a\ \textbf{\textit{mod}}\ m \equiv b\ \textbf{\textit{mod}}\ m$ means that $a$ and $b$ have the same remainder $a = q_1 m + r$ and $b = q_2 m + r$ for some integer $q_1, q_2$ and $r$.

$$\begin{aligned} a - b &= q_1 m + r - q_2 m - r \\ &= (q_1 - q_2)m \end{aligned}$$

Which says that $m$ divides (is a factor). This precisely the definition of $a \equiv b\ \textbf{\textit{mod}}\ m$

### *Exercise*

Let $m$ be a positive integer. Show that $a \equiv b (\text{mod } m)$ if $a\ \textbf{mod}\ m = b\ \textbf{mod}\ m$

***Solution***

Assume that $a \equiv b\ (\textbf{\textit{mod}}\ m)$. This means that $m | a - b$, $a - b = mc \Rightarrow a = b + mc$.
Computing $a\ \textbf{\textit{mod}}\ m$, we know that $b = qm + r$ for some nonnegative $r$ less than $m$ (namely, $r \equiv b (\textbf{\textit{mod}}\ m)$). Therefore $a = qm + r + mc = (q + c)m + r$. By definition this means that $r$ must also equal $a\ \textbf{\textit{mod}}\ m$ $\checkmark$

### *Exercise*

Show that if $n$ and $k$ are positive integers, then $\lceil n/k \rceil = \left\lfloor \frac{n-1}{k} \right\rfloor + 1$

***Solution***

The quotient $\frac{n}{k}$ lies between 2 consecutive integers, let say $b-1$ *and* $b$ possibly equal to b. There exists a positive integer $b$ such that $b-1 < \frac{n}{k} \le b$. In particular $\frac{n}{k} = b$. Also since $\frac{n}{k} > b-1$ we have

$n > k(b-1) \Rightarrow n-1 \ge k(b-1)$

$\left\lfloor \frac{n-1}{k} \right\rfloor \le \frac{n-1}{k} < \frac{n}{k} \le b$ so $\left\lfloor \frac{n-1}{k} \right\rfloor < b$, therefore $\left\lfloor \frac{n-1}{k} \right\rfloor = b-1$

### *Exercise*

Evaluate these quantities

*a)* $-17$ ***mod*** $2$
*b)* $144$ ***mod*** $7$
*c)* $-101$ ***mod*** $13$
*d)* $199$ ***mod*** $19$
*e)* $13$ ***mod*** $3$
*f)* $-97$ ***mod*** $11$

***Solution***

***a)*** $-17 = 2\cdot(-9)+1$, the remainder is 1. That is, $-17$ ***mod*** $2 = 1$.
Note that we do not write $-17 = 2\cdot(-8)-1$ so $-17$ ***mod*** $2 = -1$

***b)*** $144 = 7\cdot 20 + 4$, the remainder is 4. That is, $144$ ***mod*** $7 = 4$

***c)*** $-101 = 13\cdot(-8)+3$, the remainder is 3. That is, $-101$ ***mod*** $13 = 3$

***d)*** $199 = 19\cdot 10 + 9$, the remainder is 9. That is, $199$ ***mod*** $19 = 9$

***e)*** $13 = 3\cdot 4 + 1$, the remainder is 1. That is, $13$ ***mod*** $3 = 1$

***f)*** $-97 = 11\cdot(-9)+2$, the remainder is 2. That is, $-97$ ***mod*** $11 = 2$

### *Exercise*

Find $a$ **div** $m$ and $a$ **mod** $m$ when

*a)* $a = 228$, $m = 119$
*b)* $a = 9009$, $m = 223$
*c)* $a = -10101$, $m = 333$
*d)* $a = -765432$, $m = 38271$

***Solution***

***a)*** $228 = 2 \cdot 119 + 109$

$228 \ \boldsymbol{div} \ 119 = 1 \quad and \quad 228 \ \boldsymbol{mod} \ 119 = 109$.

***b)*** $9009 = 40 \cdot 223 + 89$

$9009 \ \boldsymbol{div} \ 223 = 40 \quad and \quad 9009 \ \boldsymbol{mod} \ 223 = 89$.

***c)*** $-10101 = -31 \cdot 333 + 222$

$-10101 \ \boldsymbol{div} \ 333 = -31 \quad and \quad -10101 \ \boldsymbol{mod} \ 333 = 222$.

***d)*** $-765432 = -21 \cdot 38271 + 38259 \Rightarrow$
$-765432 \ \boldsymbol{div} \ 38271 = -11 \quad and \quad -765432 \ \boldsymbol{mod} \ 38271 = 38259$.

## *Exercise*

Find the integer $a$ such that

*a)* $a \equiv -15 (\boldsymbol{mod} \ 27) \ \ and \ \ -26 \le a \le 0$

*b)* $a \equiv 24 (\boldsymbol{mod} \ 31) \ \ and \ \ -15 \le a \le 15$

*c)* $a \equiv 99 (\boldsymbol{mod} \ 41) \ \ and \ \ 100 \le a \le 140$

*d)* $a \equiv 43 (\boldsymbol{mod} \ 23) \ \ and \ \ -22 \le a \le 0$

*e)* $a \equiv 17 (\boldsymbol{mod} \ 29) \ \ and \ \ -14 \le a \le 14$

***Solution***

***a)*** $-15$ already satisfies the inequality, the answer $a = -15$

***b)*** 24 is too large to satisfy the inequality, we subtract 31 and obtain $a = -7$

***c)*** 24 is too small to satisfy the inequality, we add 41 and obtain $a = 140$

***d)*** $a = 43 - 2 \cdot (23) = 43 - 46 = -3$

***e)*** $a = 17 - 29 = -12$

## *Exercise*

Decide whether each of these integers is congruent to 5 modulo 17.

$a)\ 37 \qquad b)\ 66 \qquad c)\ -17 \qquad d)\ -67$

***Solution***

***a)*** $37 - 3 \ \boldsymbol{mod} \ 7 = 34 \ \boldsymbol{mod} \ 7 = 6 \ne 0$, so $37 \not\equiv 3 (\boldsymbol{mod} \ 7)$

***b)*** $66 - 3 \ \boldsymbol{mod} \ 7 = 63 \ \boldsymbol{mod} \ 7 = 0$, so $37 \equiv 3 (\boldsymbol{mod} \ 7)$

***c)*** $-17 - 3 \ \boldsymbol{mod} \ 7 = -20 \ \boldsymbol{mod} \ 7 = 1 \ne 0$, so $-17 \not\equiv 3 (\boldsymbol{mod} \ 7)$

***d)*** $-67 - 3 \ \boldsymbol{mod} \ 7 = -70 \ \boldsymbol{mod} \ 7 = 0$, so $-67 \equiv 3 (\boldsymbol{mod} \ 7)$

### *Exercise*

Find each of these values.

*a)* $(-133 \textbf{ mod } 23 + 261 \textbf{ mod } 23) \textbf{ mod } 23$

*b)* $(457 \textbf{ mod } 23 \cdot 182 \textbf{ mod } 23) \textbf{ mod } 23$

*c)* $(177 \textbf{ mod } 31 + 270 \textbf{ mod } 31) \textbf{ mod } 31$

*d)* $(19^2 \textbf{\textit{ mod }} 41) \textbf{\textit{ mod }} 9$

*e)* $(32^3 \textbf{\textit{ mod }} 13)^2 \textbf{\textit{ mod }} 11$

*f)* $(99^2 \textbf{\textit{ mod }} 32)^3 \textbf{\textit{ mod }} 15$

*g)* $(3^4 \textbf{\textit{ mod }} 17)^2 \textbf{\textit{ mod }} 11$

*h)* $(19^3 \textbf{\textit{ mod }} 23)^2 \textbf{\textit{ mod }} 31$

*i)* $(89^3 \textbf{\textit{ mod }} 79)^4 \textbf{\textit{ mod }} 26$

***Solution***

***a)*** $-133 + 261 = 128 \equiv 13$

$-133 + 261 \textbf{\textit{ mod }} 23 = 128 \textbf{\textit{ mod }} 23 \underline{= 13}$ $\quad 128 = 23 \cdot (5) + 13$

***b)*** $457 \cdot 182 \textbf{\textit{ mod }} 23 = 83174 \textbf{\textit{ mod }} 23 \underline{= 6}$ $\quad 83174 = 23 \cdot (3616) + 6$

***c)*** $177 + 271 \textbf{\textit{ mod }} 31 = 448 \textbf{\textit{ mod }} 31 \underline{= 14}$ $\quad 448 = 31 \cdot (14) + 14$

***d)***
$$\begin{aligned}(19^2 \textbf{\textit{ mod }} 41) \textbf{\textit{ mod }} 9 &= (361 \textbf{\textit{ mod }} 41) \textbf{\textit{ mod }} 9 \\ &= 33 \textbf{\textit{ mod }} 9 \\ &\underline{= 6}\end{aligned}$$

***e)***
$$\begin{aligned}(32^3 \textbf{\textit{ mod }} 13)^2 \textbf{\textit{ mod }} 11 &= (32768 \textbf{\textit{ mod }} 13)^2 \textbf{\textit{ mod }} 11 \\ &= 8^2 \textbf{\textit{ mod }} 11 \\ &= 64 \textbf{\textit{ mod }} 11 \\ &\underline{= 9}\end{aligned}$$

***f)***
$$\begin{aligned}(99^2 \textbf{\textit{ mod }} 32)^3 \textbf{\textit{ mod }} 15 &= (9801 \textbf{\textit{ mod }} 32)^3 \textbf{\textit{ mod }} 15 \\ &= 9^3 \textbf{\textit{ mod }} 15 \\ &= 729 \textbf{\textit{ mod }} 15 \\ &\underline{= 9}\end{aligned}$$

***g)***
$$\begin{aligned}(3^4 \textbf{\textit{ mod }} 17)^2 \textbf{\textit{ mod }} 11 &= (81 \textbf{\textit{ mod }} 17)^2 \textbf{\textit{ mod }} 11 \\ &= 13^2 \textbf{\textit{ mod }} 11\end{aligned}$$

$= 169 \; \boldsymbol{mod} \; 11$

$\underline{= 4}$

***h)*** $\left(19^3 \; \boldsymbol{mod} \; 23\right)^2 \; \boldsymbol{mod} \; 31 = (6859 \; \boldsymbol{mod} \; 23)^2 \; \boldsymbol{mod} \; 31$

$= 5^2 \; \boldsymbol{mod} \; 31$

$= 25 \; \boldsymbol{mod} \; 31$

$\underline{= 25}$

***i)*** $\left(89^3 \; \boldsymbol{mod} \; 79\right)^4 \; \boldsymbol{mod} \; 26 = (704969 \; \boldsymbol{mod} \; 79)^4 \; \boldsymbol{mod} \; 26$

$= 52^4 \; \boldsymbol{mod} \; 26$

$= 7311616 \; \boldsymbol{mod} \; 26$

$\underline{= 0}$

# Solution Section 2.4 – Integer Representations and Algorithms

### Exercise

Convert the decimal expansion of each of these integers to a binary expansion

*a)* 321 *b)* 1023 *c)* 100632 *d)* 231 *e)* 4532

**Solution**

***a)***

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 321 | 160 | 80 | 40 | 20 | 10 | 5 | 2 | 1 | |
| 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | ← |

$321 = \boxed{(1\ 0100\ 0001)_2}$

***b)*** $1023 = 1024 - 1 = 2^{10} - 1$ 1 *less than* $(100\ \ 0000\ \ 0000)_2$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1023 | 511 | 255 | 127 | 63 | 31 | 15 | 7 | 3 | 1 | |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | ← |

$1023 = \boxed{(11\ \ 1111\ \ 1111)_2}$

***c)***

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 100632 | 50316 | 25158 | 12579 | 636289 | 3144 | 1572 | 786 | 393 | 196 | 98 | 49 | 24 |
| | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 |

| | | | | |
|---|---|---|---|---|
| 12 | 6 | 3 | 1 | |
| 0 | 0 | 1 | 1 | ← |

$100632 = \boxed{(1\ \ 1000\ \ 1001\ \ 0001\ \ 1000)_2}$

***d)***

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 231 | 115 | 57 | 28 | 14 | 7 | 3 | 1 | |
| 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | ← |

$231 = \boxed{(1110\ \ 0111)_2}$

***e)***

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4532 | 2266 | 1133 | 566 | 283 | 141 | 70 | 35 | 17 | 8 | 4 | 2 | 1 | |
| 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | ← |

$4532 = \boxed{(1\ \ 0001\ \ 1011\ \ 0100)_2}$

### *Exercise*

Convert binary the expansion of each of these integers to a decimal expansion

*a)* $(1\,1011)_2$ *b)* $(10\,1011\,0101)_2$ *c)* $(11\,1011\,1110)_2$

*d)* $(111\,1100\,0001\,1111)_2$ *e)* $(1\,1111)_2$ *f)* $(10\,0000\,0001)_2$

*g)* $(10\,0101\,0101)_2$ *h)* $(110\,1001\,0001\,0000)_2$

***Solution***

| *Decimal* | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *Binary* | 0 | 1 | 10 | 11 | 100 | 101 | 110 | 111 | 1000 | 1001 | 1010 | 1011 | 1100 | 1101 | 1110 | 1111 |

**a)** $(1\,1011)_2 = 1+2^1+2^3+2^4$

$= 1+2+8+16$

$= 27$

**b)** $(10\,1011\,0101)_2 = 1+2^2+2^4+2^5+2^7+2^9$

$= 1+4+16+32+128+512$

$= 693$

**c)** $(11\,1011\,1110)_2 = 2^1+2^2+2^3+2^4+2^5+2^7+2^8+2^9$

$= 958$

**d)** $(111\,1100\,0001\,1111)_2 = 1+2^1+2^2+2^3+2^4+2^{10}+2^{11}+2^{12}+2^{13}+2^{14}$

$= 31775$

**e)** $(1\,1111)_2 = 1+2^1+2^2+2^3+2^4$

$= 1+2+8+16$

$= 31$

**f)** $(10\,0000\,0001)_2 = 1+2^9$

$= 1+512$

$= 513$

**g)** $(10\,0101\,0101)_2 = 2^9+2^6+2^4+2^2+1 = 597$

**h)** $(110\,1001\,0001\,0000)_2 = 2^{14}+2^{13}+2^{11}+2^8+\ 2^4 = 26896$

### *Exercise*

Convert the binary expansion of each of these integers to an octal expansion *a)* $(1111\ 0111)_2$

*b)* $(1010\ 1010\ 1010)_2$

*c)* $(111\ 0111\ 0111\ 0111)_2$

*d)* $(101\ 0101\ 0101\ 0101)_2$

***Solution***

***a)*** $(1111\ 0111)_2 = (11\ 110\ 111)_2 \ \underline{=(367)_8}|$

***b)*** $(1010\ 1010\ 1010)_2 = (101\ 010\ 101\ 010)_2 = \underline{(5252)_8}|$

***c)*** $(111\ 0111\ 0111\ 0111)_2 = (111\ 011\ 101\ 110\ 111)_2 = \underline{(73567)_8}|$

***d)*** $(101\ 0101\ 0101\ 0101)_2 = (101\ 010\ 101\ 010\ 101)_2 = \underline{(52525)_8}|$

## Exercise

Convert the octal expansion of each of these integers to a binary expansion

*a)* $(572)_8$  *b)* $(1604)_8$  *c)* $(423)_8$  *d)* $(2417)_8$  *e)* $(73567)_8$  *f)* $(52525)_8$

**Solution**

| **Octal** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Binary** | 0 | 1 | 10 | 11 | 100 | 101 | 110 | 111 | 1000 | 1001 | 1010 | 1011 | 1100 | 1101 | 1110 | 1111 |

**a)** $\begin{array}{c:c:c} 5_8 & 7_8 & 2_8 \\ \hline 101_2 & 111_2 & 010_2 \end{array} \quad \Rightarrow (572)_8 = \underline{(1\ 0111\ 1010)_2}$

**b)** $\begin{array}{c:c:c:c} 1_8 & 6_8 & 0_8 & 4_8 \\ \hline 1_2 & 110_2 & 000_2 & 100_2 \end{array} \quad \Rightarrow (1604)_8 = \underline{(11\ 1000\ 0100)_2}$

**c)** $\begin{array}{c:c:c} 4_8 & 2_8 & 3_8 \\ \hline 100_2 & 010_2 & 011_2 \end{array} \quad \Rightarrow (423)_8 = \underline{(1\ 0001\ 0011)_2}$

**d)** $\begin{array}{c:c:c:c:c} 7_8 & 3_8 & 5_8 & 6_8 & 7_8 \\ \hline 111_2 & 011_2 & 101_2 & 110_2 & 111_2 \end{array} \quad \Rightarrow (73567)_8 = \underline{(111\ 0111\ 0111\ 0111)_2}$

**e)** $\begin{array}{c:c:c:c:c} 5_8 & 2_8 & 5_8 & 2_8 & 5_8 \\ \hline 101_2 & 010_2 & 101_2 & 010_2 & 101_2 \end{array} \quad \Rightarrow (52525)_8 = \underline{(101\ 0101\ 0101\ 0101)_2}$

## Exercise

Convert the hexadecimal expansion of each of these integers to a binary expansion

*a)* $(80E)_{16}$  *b)* $(135AB)_{16}$  *c)* $(ABBA)_{16}$

*d)* $(DEFACED)_{16}$  *e)* $(BADFACED)_{16}$  *f)* $(ABCDEF)_{16}$

**Solution**

| **Hexadecimal** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | A | B | C | D | E | F |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Binary** | 0 | 1 | 10 | 11 | 100 | 101 | 110 | 111 | 1000 | 1001 | 1010 | 1011 | 1100 | 1101 | 1110 | 1111 |

**a)** $\begin{array}{c:c:c} 8_{16} & 0_{16} & E_{16} \\ \hline 1000_2 & 0000_2 & 1110_2 \end{array} \quad \Rightarrow (80E)_{16} = \underline{(1000\ 0000\ 1110)_2}$

**b)** $\begin{array}{c:c:c:c:c} 1_{16} & 3_{16} & 5_{16} & A_{16} & B_{16} \\ \hline 0001_2 & 0011_2 & 0101_2 & 1010_2 & 1011_2 \end{array}$

$$\Rightarrow (135AB)_{16} = \underline{(0001\ 0011\ 0101\ 1010\ 1011)_2}$$

**c)**

$$\begin{array}{c|c|c|c} A_{16} & B_{16} & B_{16} & A_{16} \\ \hline 1010_2 & 1011_2 & 1011_2 & 1010_2 \end{array} \Rightarrow (ABBA)_{16} = \underline{(1010\ 1011\ 1011\ 1010)_2}$$

**d)**

$$\begin{array}{c|c|c|c|c|c|c} D_{16} & E_{16} & F_{16} & A_{16} & C_{16} & E_{16} & D_{16} \\ \hline 1101_2 & 1110_2 & 1111_2 & 1010_2 & 1100_2 & 1110_2 & 1101_2 \end{array}$$

$$\Rightarrow (DEFACED)_{16} = \underline{(1101\ 1110\ 1111\ 1010\ 1100\ 1110\ 1101)_2}$$

**e)**

$$\begin{array}{c|c|c|c|c|c|c|c} B_{16} & A_{16} & D_{16} & F_{16} & A_{16} & C_{16} & E_{16} & D_{16} \\ \hline 1011_2 & 1010_2 & 1101_2 & 1111_2 & 1010_2 & 1100_2 & 1110_2 & 1101_2 \end{array}$$

$$\Rightarrow (BADFACED)_{16} = \underline{(1011\ 1010\ 1101\ 1111\ 1010\ 1100\ 1110\ 1101)_2}$$

**f)**

$$\begin{array}{|c|c|c|c|c|c|} A_{16} & B_{16} & C_{16} & D_{16} & E_{16} & F_{16} \\ \hline 1010_2 & 1011_2 & 1100_2 & 1101_2 & 1110_2 & 1111_2 \end{array}$$

$$\Rightarrow (ABCDEF)_{16} = \underline{(1010\ 1011\ 1100\ 1101\ 1110\ 1111)_2}$$

## *Exercise*

Show that the binary expansion of a positive integer can be obtained from its hexadecimal expansion by translating each hexadecimal digit into a block of four binary digits.

***Solution***

Let $(\ldots h_2 h_1 h_0)_{16}$ be the hexadecimal expansion of a positive integer. The value of that integer is

$$h_0 + h_1 \cdot 16 + h_2 \cdot 16^2 + \cdots = h_0 + h_1 \cdot 2^4 + h_2 \cdot 2^8 + \cdots$$

If we replace each hexadecimal digit $h_i$ by its binary expansion $(b_{i3} b_{i2} b_{i1} b_{i0})_2$, then

$$h_i = b_{i0} + 2b_{i1} + 4b_{i2} + 8b_{i3}$$

Therefore the value of the entire number is

$$\begin{aligned} &b_{00} + 2b_{01} + 4b_{02} + 8b_{03} + (b_{10} + 2b_{11} + 4b_{12} + 8b_{13}) \cdot 2^4 \\ &\qquad + (b_{20} + 2b_{21} + 4b_{22} + 8b_{23}) \cdot 2^8 + \cdots \\ &= b_{00} + 2b_{01} + 4b_{02} + 8b_{03} + 2^4 b_{10} + 2^5 b_{11} + 2^6 b_{12} + 2^7 b_{13} \\ &\qquad + 2^8 b_{20} + 2^9 b_{21} + 2^{10} b_{22} + 2^{11} b_{23} + \cdots \end{aligned}$$

Which is the value of the binary expansion $(\cdots b_{23} b_{22} b_{21} b_{20} b_{13} b_{12} b_{11} b_{10} b_{03} b_{02} b_{01} b_{00})_2$

## *Exercise*

Show that the binary expansion of a positive integer can be obtained from its octal expansion by translating each octal digit into a block of three binary digits.

***Solution***

Let $(\ldots d_2 d_1 d_0)_8$ be the octal expansion of a positive integer. The value of that integer is

$d_0 + d_1 \cdot 8 + d_2 \cdot 8^2 + \cdots = d_0 + d_1 \cdot 2^2 + d_2 \cdot 2^6 + \cdots$

If we replace each octal digit $d_i$ by its binary expansion $(b_{i2} b_{i1} b_{i0})_2$, then

$d_i = b_{i0} + 2b_{i1} + 4b_{i2}$

Therefore the value of the entire number is

$$b_{00} + 2b_{01} + 4b_{02} + (b_{10} + 2b_{11} + 4b_{12}) \cdot 2^3 + (b_{20} + 2b_{21} + 4b_{22}) \cdot 2^6 + \cdots$$

$$= b_{00} + 2b_{01} + 4b_{02} + 2^3 b_{10} + 2^4 b_{11} + 2^5 b_{12} + 2^6 b_{20} + 2^7 b_{21} + 2^8 b_{22} + \cdots$$

Which is the value of the binary expansion $(\cdots b_{22} b_{21} b_{20} b_{12} b_{11} b_{10} b_{02} b_{01} b_{00})_2$

## *Exercise*

Explain how to convert from binary to base 64 expansions and from base 64 expansions to binary expansions and from octal to base 64 expansions and from base 64 expansions to octal expansions

***Solution***

$64 = 2^8 = 8^2$, in base 64 we need 64 symbols, from 0 to up to something representing 63. Corresponding to each such symbol would be a binary string of 6 digits, from 000000 for 0 to 001010 for ***a***, 100011 for ***z***, 100100 for ***A***, 111101 for ***Z***, for 111110 for @, and 111111 for $.

To translate from binary to base 64, we group the binary digits from the right in groups of 6 and use the list of correspondences to replace each 6 bits by one base-64 digits.

To convert from base 64 to binary, we just replace each base-64 digit by its corresponding 6 bits.

For conversion between octal and base 64, we change the binary strings in the table to octal strings, replacing each 6-bit string by its 2-digit octal equivalent, and then follow the same procedures as above, interchanging base-64 digits and 2-digits strings of octal digits.

### *Exercise*

Find the sum and product of each of these pairs of numbers. Express your answers as a base 3 expansions

*a)* $(112)_3$, $(210)_3$ *b)* $(2112)_3$, $(12021)_3$

*c)* $(20001)_3$, $(1111)_3$ *d)* $(120021)_3$, $(2002)_3$

**Solution**

**a)**

$$\begin{array}{cccccc} & & 1 & 1 & 2 \\ + & & 2 & 1 & 0 \\ \hline & 1 & 0 & 2 & 2 \end{array} \qquad \begin{array}{ccccccc} & & & & 1 & 1 & 2 \\ \times & & & & 2 & 1 & 0 \\ \hline & & & & 0 & 0 & 0 \\ & & & 1 & 1 & 2 & \\ & 1 & 0 & 0 & 1 & & \\ \hline & 1 & 0 & 1 & 2 & 2 & 0 \end{array} \qquad \begin{array}{ccccc} & & 1 & 1 & \\ \hdashline & & 1 & 1 & 2 \\ \times & & & & 2 \\ \hline & 1 & 0 & 0 & 1 \end{array}$$

**b)**

$$\begin{array}{cccccc} & & 2 & 1 & 1 & 2 \\ + & 1 & 2 & 0 & 2 & 1 \\ \hline & 2 & 1 & 2 & 1 & 0 \end{array} \qquad \begin{array}{cccccccccc} & & & & & & 2 & 1 & 1 & 2 \\ & & & \times & & 1 & 2 & 0 & 2 & 1 \\ \hline & & & & & & 2 & 1 & 1 & 2 \\ & & & & 1 & 2 & 0 & 0 & 1 & \\ & & & & & & & 0 & & \\ & & 1 & 2 & 0 & 0 & 1 & & & \\ & & 2 & 1 & 1 & 2 & & & & \\ \hline & 1 & 1 & 1 & 0 & 2 & 0 & 1 & 2 & 2 \end{array} \qquad \begin{array}{cccccc} & 1 & 1 & 1 & 1 & \\ \hdashline & & 2 & 1 & 1 & 2 \\ \times & & & & 0 & 2 \\ \hline & 1 & 2 & 0 & 0 & 1 \end{array}$$

**c)**

$$\begin{array}{cccccc} & 2 & 0 & 0 & 0 & 1 \\ + & & 1 & 1 & 1 & 1 \\ \hline & 2 & 1 & 1 & 1 & 2 \end{array} \qquad \begin{array}{ccccccccc} & & & & 2 & 0 & 0 & 0 & 1 \\ & & \times & & & 1 & 1 & 1 & 1 \\ \hline & & & & 2 & 0 & 0 & 0 & 1 \\ & & & 2 & 0 & 0 & 0 & 1 & \\ & & 2 & 0 & 0 & 0 & 1 & & \\ & 2 & 0 & 0 & 0 & 1 & & & \\ \hline & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ \hline \end{array}$$

**d)**

$$\begin{array}{ccccccc} & & & & 1 & 1 & \\ & 1 & 2 & 0 & 0 & 2 & 1 \\ + & & & 2 & 0 & 0 & 2 \\ \hline & 1 & 2 & 2 & 1 & 0 & 0 \end{array} \qquad \begin{array}{ccccccccccc} & & & & & 1 & 2 & 0 & 0 & 2 & 1 \\ & & & \times & & & & 2 & 0 & 0 & 2 \\ \hline & & & & 1 & 0 & 1 & 0 & 1 & 1 & 2 \\ & & & & 0 & 0 & 0 & 0 & 0 & 0 & \\ & & & 0 & 0 & 0 & 0 & 0 & 0 & & \\ & 1 & 0 & 1 & 0 & 1 & 1 & 2 & & & \\ \hline & 1 & 0 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 2 \end{array}$$

### Exercise

Find the sum and product of each of these pairs of numbers. Express your answers as an octal expansion.

*a)* $(763)_8,\ (147)_8$ *b)* $(6001)_8,\ (272)_8$

*c)* $(1111)_8,\ (777)_8$ *d)* $(54321)_8,\ (3456)_8$

## Solution

**a)**

$$\begin{array}{cccc} & 1 & 1 & \\ & 7 & 6 & 3 \\ + & 1 & 4 & 7 \\ \hline 1 & 1 & 3 & 2 \end{array}$$

$$\begin{array}{cccccc} & & & 7 & 6 & 3 \\ & \times & & 1 & 4 & 7 \\ \hline & & 5 & 4 & 4 & 5 \\ & 3 & 7 & 1 & 4 & \\ & 7 & 6 & 3 & & \\ \hline 1 & 4 & 4 & 3 & 0 & 5 \end{array}$$

**b)**

$$\begin{array}{ccccc} & 6 & 0 & 0 & 1 \\ + & & 2 & 7 & 2 \\ \hline & 6 & 2 & 7 & 3 \end{array}$$

$$\boxed{(6001)_8 + (272)_8 = 6273}$$

$6001 = 6\cdot 8^3 + 1 = 3073$

$272 = 2\cdot 8^2 + 7\cdot 8 + 2 = 186$

$6001 \cdot 272 = 3073 \cdot 186 = 571{,}578$

$$\begin{aligned} 571{,}578 &= 8\times 71447 + 2 \\ 71447 &= 8\times 8930 + 7 \\ 8930 &= 8\times 1116 + 2 \\ 1116 &= 8\times 139 + 4 \\ 139 &= 8\times 17 + 3 \\ 17 &= 8\times 2 + 1 \\ 2 & \end{aligned}$$

$$\boxed{(6001)_8 \cdot (272)_8 = 2{,}134{,}272}$$

**c)**

$$\begin{array}{ccccc} & 1 & 1 & 1 & \\ & 1 & 1 & 1 & 1 \\ + & & 7 & 7 & 7 \\ \hline & 2 & 1 & 1 & 0 \end{array}$$

$$\boxed{(1111)_8 + (777)_8 = 2110}$$

$(1111)_8 = 1\cdot 8^3 + 1\cdot 8^2 + 1\cdot 8 + 1 = 585$

$(777)_8 = 7\cdot 8^2 + 7\cdot 8 + 7 = 511$

$(1111)_8 \cdot (777)_8 = (585)(511) = 298{,}935$

$$\begin{aligned} 298935 &= 8\times 37366 + 7 \\ 37366 &= 8\times 4670 + 6 \\ 4670 &= 8\times 583 + 6 \\ 583 &= 8\times 72 + 7 \\ 72 &= 8\times 9 + 0 \\ 9 &= 8\times 1 + 1 \\ 1 & \end{aligned}$$

$$\boxed{(1111)_8 \cdot (777)_8 = 1{,}107{,}667}$$

***d)***

$$\begin{array}{r} 5\ 4\ 3\ 2\ 1 \\ +\quad 3\ 4\ 5\ 6 \\ \hline 5\ 7\ 7\ 7\ 7 \end{array}$$

$$\boxed{(54321)_8 + (3456)_8 = 57,777}$$

$(54321)_8 = 5 \cdot 8^4 + 4 \cdot 8^3 + 3 \cdot 8^2 + 2 \cdot 8 + 1 = 22,737$

$(3456)_8 = 3 \cdot 8^3 + 4 \cdot 8^2 + 5 \cdot 8 + 6 = 1838$

$(54321)_8 \cdot (3456)_8 = (22,737)(1838) = 41,790,606$

$41790606 = 8 \times 5223825 + 6$

$5223825 = 8 \times 652978 + 1$

$652978 = 8 \times 81622 + 2$

$81622 = 8 \times 10202 + 6$

$10202 = 8 \times 1275 + 2$

$1275 = 8 \times 159 + 3$

$159 = 8 \times 19 + 7$

$19 = 8 \times 2 + 3$

$2$

$$\boxed{(54321)_8 \cdot (3456)_8 = 237,326,216}$$

## *Exercise*

Find the sum and product of each of these pairs of numbers. Express your answers as a hexadecimal expansion.

*a)* $(1AB)_{16},\ (BBC)_{16}$ *b)* $(20CBA)_{16},\ (A01)_{16}$

*c)* $(ABCDE)_{16},\ (1111)_{16}$ *d)* $(E0000E)_{16},\ (BAAA)_{16}$

***Solution***

| ***Decimal*** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ***Hexadecimal*** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | A | B | C | D | E | F |

***a)*** $1AB = 1 * 16^2 + 10 * 16 + 11 = 427$

$BBC = 11 * 16^2 + 11 * 16 + 12 = 3004$

$1AB + BBC = 427 + 3004$

$= 3431$

$3431 = 16 \times 214 + 7$

$214 = 16 \times 14 + 6$

$14$ $14 = D$

$$\boxed{1AB + BBC = D67}$$

$(1AB) \times (BBC) = (427)(3004)$

$= 1,282,708$

$1282708 = 16 \times 80169 + 4$

$80169 = 16 \times 5010 + 9$

$5010 = 16 \times 313 + 2$

$313 = 16 \times 19 + 9$

$19 = 16 \times 1 + 3$

$1$

$$\boxed{(1AB) \times (BBC) = 139,294}$$

***b)*** $(20CBA)_{16} = 2*16^4 + 0 + 12*16^2 + 11*16 + 10 = 134{,}330$

$(A01)_{16} = 10*16^2 + 0*16 + 12 = 2{,}561$

$$(20CBA)_{16} + (A01)_{16} = 134{,}330 + 2{,}561$$
$$= 136{,}891$$

$136891 = 16 \times 8555 + 11 \qquad 11 = B$
$8555 = 16 \times 534 + 11 \qquad 11 = B$
$534 = 16 \times 33 + 6$
$33 = 16 \times 2 + 1$
$2$

$\underline{(20CBA)_{16} + (A01)_{16} = 21{,}6BB}|$

$$(20CBA)_{16} \times (A01)_{16} = (134{,}330)(2{,}561)$$
$$= 344{,}019{,}130$$

$344019130 = 16 \times 21501195 + 10 \qquad 10 = A$
$21501195 = 16 \times 1343824 + 11 \qquad 11 = B$
$1343824 = 16 \times 83989 + 0$
$83989 = 16 \times 5249 + 5$
$5249 = 16 \times 328 + 1$
$328 = 16 \times 20 + 8$
$20 = 16 \times 1 + 4$
$1$

$\underline{(20CBA)_{16} \times (A01)_{16} = 14{,}815{,}0BA}|$

***c)*** $(ABCDE)_{16} = 10*16^4 + 11*16^4 + 12*16^2 + 13*16 + 14 = 703{,}710$

$(1111)_{16} = 1*16^3 + 1*16^2 + 1*16 + 1 = 4369$

$$(ABCDE)_{16} + (1111)_{16} = 703{,}710 + 4369$$
$$= 708{,}079$$

$708079 = 16 \times 44254 + 15 \qquad 15 = F$
$44254 = 16 \times 2765 + 14 \qquad 14 = E$
$2765 = 16 \times 172 + 13 \qquad 13 = D$
$172 = 16 \times 10 + 12 \qquad 12 = C$
$10 \qquad 10 = A$

$\underline{(ABCDE)_{16} + (1111)_{16} = AC{,}DEF}|$

$$\left(ABCDE\right)_{16} \times \left(1111\right)_{16} = (703{,}710)(4369)$$
$$= 3{,}074{,}508{,}990$$

$3074508990 = 16 \times 192156811 + 14 \quad 14 = E$
$192156811 = 16 \times 12009800 + 11 \quad 11 = B$
$12009800 = 16 \times 750612 + 8$
$750612 = 16 \times 46913 + 4$
$46913 = 16 \times 2932 + 1$
$2932 = 16 \times 183 + 4$
$183 = 16 \times 11 + 7$
$11 \quad 11 = B$

$$\boxed{\left(ABCDE\right)_{16} \times \left(1111\right)_{16} = B7{,}414{,}8BE}$$

***d)*** $\left(E0000E\right)_{16} = 14 * 16^5 + 14 = 14{,}680{,}078$

$\left(BAAA\right)_{16} = 11 * 16^3 + 10 * 16^2 + 10 * 16 + 10 = 47{,}786$

$$\left(E0000E\right)_{16} + \left(BAAA\right)_{16} = 14{,}680{,}078 + 47{,}786$$
$$= 14{,}727{,}864$$

$14727864 = 16 \times 920491 + 8$
$920491 = 16 \times 57530 + 11 \quad 11 = B$
$57530 = 16 \times 3595 + 10 \quad 10 = A$
$3595 = 16 \times 224 + 11 \quad 11 = B$
$224 = 16 \times 14 + 0$
$14 \quad 14 = E$

$$\boxed{\left(E0000E\right)_{16} + \left(BAAA\right)_{16} = E0B, AB8}$$

$$\left(E0000E\right)_{16} \left(BAAA\right)_{16} = (14{,}680{,}078)(47{,}786)$$
$$= 701{,}502{,}207{,}308$$

$701502207308 = 16 \times 43843887956 + 12 \quad 12 = C$
$43843887956 = 16 \times 2740242997 + 4$
$2740242997 = 16 \times 171265187 + 5$
$171265187 = 16 \times 10704074 + 3$
$10704074 = 16 \times 669004 + 10 \quad 10 = A$
$669004 = 16 \times 41812 + 12 \quad 12 = C$
$41812 = 16 \times 2613 + 4$
$2613 = 16 \times 163 + 5$
$163 = 16 \times 10 + 3$
$10 \quad 10 = A$

$$\left(E0000E\right)_{16} \times \left(BAAA\right)_{16} = \underline{A{,}354{,}CA3{,}54C}$$

# Solution Section 2.5 – Primes and Greatest Common Divisors

### Exercise

Determine whether each of these integers is prime.

*a)* 21 *b)* 29 *c)* 71 *d)* 97 *e)* 111

*f)* 143 *g)* 19 *h)* 27 *i)* 93 *j)* 101

*k)* 107 *l)* 113

**Solution**

The numbers: 29, 71, 97, 19, 101, 107, and 113 are primes.

Not Prime: $21 = 3 \cdot 7$ $\quad 111 = 3 \cdot 37$ $\quad 143 = 13 \cdot 11$ $\quad 27 = 3^3$ $\quad 93 = 3 \cdot 31$

### Exercise

Find the prime factorization of each these integers.

*a)* 88 *b)* 126 *c)* 729 *d)* 1001 *e)* 1111

*f)* 909,090 *g)* 39 *h)* 81 *i)* 101 *j)* 143

*k)* 289 *l)* 899

**Solution**

***a)*** $88 = 2^3 \cdot 11$

***b)*** $126 = 2 \cdot 3^2 \cdot 7$

***c)*** $729 = 3^6$

***d)*** $1001 = 11 \cdot 91$

***e)*** $1111 = 11 \cdot 101$

***f)*** $909090 = 2 \cdot 5 \cdot 9 \cdot 91 \cdot 111$

***g)*** $39 = 3 \cdot 13$

***h)*** $81 = 3^4$

***i)*** $101 = 101$ $\quad$ (***Prime***)

***j)*** $143 = 11 \cdot 13$

***k)*** $289 = 17^2$

***l)*** $899 = 29 \cdot 31$

### *Exercise*

Find the prime factorization of 10!

***Solution***

$10! = 3628800$

$10! = (2 \cdot 5)\,!$

### *Exercise*

Show that if $a^m + 1$ is composite if *a* and *m* are integers greater than 1 and *m* is odd. [*Hint*: Show that *x* + 1 is a factor of the polynomial $a^m + 1$ if *m* is odd]

***Solution***

Since *m* is odd, then we can factor $a^m + 1 = (a+1)\left(a^{m-1} - a^{m-2} + a^{m-3} - \cdots - 1\right)$

Because *a* and *m* are both greater than 1, we know that $1 < a + 1 < a^m + 1$ . This provides a factoring of $a^m + 1$ into proper factors, so $a^m + 1$ is composite.

### *Exercise*

Show that if $2^m + 1$ is an odd prime, then $m = 2^n$ for some nonnegative integer *n*. [*Hint*: First show the polynomial identity $x^m + 1 = \left(x^k + 1\right)\left(x^{k(t-1)} - x^{k(t-2)} + \cdots - x^k + 1\right)$ holds, where $m = kt$ and *t* is odd]

***Solution***

Assume $y = x^k$ , then the claimed identity is

$$\left(y^t + 1\right) = (y+1)\left(y^{t-1} - y^{t-2} + y^{t-3} - \cdots - y + 1\right)$$

By multiplying out the right-hand side and noticing the "telescoping" that occurs.
Let show that *m* is a power of 2 that is only prime factor is 2.

Suppose to the contrary that *m* has an odd prime factor *t* and $m = kt$ , where *k* is a positive integer. Letting $x = 2$ in the identity given in the hint, we have $2^m + 1 = \left(2^k + 1\right)(\cdots)$. Because $2^k + 1 > 1$ and the prime $2^m + 1$ can have no proper factor greater than 1, we must have $2^m + 1 = 2^k + 1$, so $m = k$ and $t = 1$ contradicting the fact that *t* is prime. This completes the proof by contradiction.

### *Exercise*

Which positive integers less than 12 are relatively prime to 12?

***Solution***

By inspection with mental arithmetic, the greatest common divisors of the numbers from 1 to 11 with 12 whose ***gcd*** is 1, are 1, 5, 7, and 11. These are so few since 12 had many factors – in particular, both 2 and 3.

### *Exercise*

Which positive integers less than 30 are relatively prime to 30?

***Solution***

The prime factors of 30 are 2, 3, and 5.

Thus we are looking for positive integers less than 30 that have none of these prime factors. Since the smallest prime number other than these is 7, and $7^2$ is already greater than 30, in fact only primes (and the number 1) will satisfy this condition.

Therefore the answer is 1, 7, 11, 13, 17, 18, 23, and 29.

### *Exercise*

Determine whether the integers in each of these sets are pairwise relatively prime.

*a)* $21,\ 34,\ 55$ *b)* 14, 17, 85 *c)* 25, 41, 49, 64 *d)* 17, 18, 19, 23

*e)* 11, 15, 19 *f)* 14, 15, 21 *g)* 12, 17, 31, 37 *h)* 7, 8, 9, 11

***Solution***

***a)*** $21 = 3 \cdot 7, \quad 34 = 2 \cdot 17, \quad 55 = 5 \cdot 11$ These are pairwise relatively prime

***b)*** $85 = 5 \cdot 17$ These are not pairwise relatively prime

***c)*** $25 = 5^2, \quad 41\, \textit{is prime}, \quad 49 = 7^2, \quad 64 = 2^6$ These are pairwise relatively prime

***d)*** $17,\ 19,\ \textit{and}\ \ 23\ \ \textit{are prime} \quad 18 = 2 \cdot 3^2$ These are pairwise relatively prime

***e)*** $11\ \ \textit{and}\ \ 19\ \ \textit{are prime} \quad 15 = 3 \cdot 5$ These are pairwise relatively prime

***f)*** $14 = 2 \cdot 7 \quad \textit{and} \quad 21 = 3 \cdot 7$ These are not pairwise relatively prime

***g)*** $17,\ 31,\ \textit{and}\ \ 37\ \ \textit{are prime} \quad 12 = 2^2 \cdot 3$ These are pairwise relatively prime

***h)*** $7\ \ \textit{and}\ \ 11\ \ \textit{are prime} \quad 8 = 2^3 \quad 9 = 3^2$ These are pairwise relatively prime

### *Exercise*

We call a positive integer ***perfect*** if it equals the sum of its positive divisors other than itself

*a)* Show that 6 and 28 are perfect.

*b)* Show that $2^{p-1}\left(2^p-1\right)$ is a perfect number when $2^p-1$ is prime

## Solution

***a)*** Since $6=1+2+3$, and these three summands are the only proper divisors of 6, we conclude that 6 is perfect.
$28=1+2+4+7+14$ are also the only proper divisors of 28

***b)*** We need to find all proper divisors of $2^{p-1}\left(2^p-1\right)$. Certainly all the numbers

$1,\ 2,\ 4,\ 8,\ \ldots,\ 2^{p-1}$ are proper divisors, and their sum is $2^p-1$ (geometric series). Also each of these divisors times $2^p-1$ is also a divisor, and all but the last is proper. Again adding up this geometric series we find a sum of $2^{p-1}\left(2^p-1\right)$. There are no other proper divisors. Therefore the sum of all the divisors is

$$\begin{aligned}\left(2^p-1\right)+\left(2^p-1\right)\left(2^{p-1}-1\right) &= \left(2^p-1\right)\left(1+2^{p-1}-1\right)\\ &= \left(2^p-1\right)2^{p-1}\end{aligned}$$

Which is our original number. Therefore this number is perfect.

### *Exercise*

Show that if $2^n-1$ is prime, then *n* is prime. *Hint*: Use the identity
$$2^{ab}-1=\left(2^a-1\right)\cdot\left(2^{a(b-1)}+2^{a(b-2)}+\cdots+2^a+1\right)$$

## Solution

We will prove the assertion by proving its contrapositive.
Suppose that *n* is not prime. Then by definition $n=ab$ for some integers *a* and *b* each greater than 1.
Since $a>1,\ 2^a-1$, the first factor in the suggested identity, is greater than 1. The second factor is also greater than 1.
Thus $2^n-1=2^b-1$ is the product of 2 integers each greater than 1, so it is not prime.

### *Exercise*

Determine whether each of these integers is prime, verifying some of Mersenne's claims

*a)* $2^7-1$ *b)* $2^9-1$ *c)* $2^{11}-1$ *d)* $2^{13}-1$

***Solution***

***a)*** $2^7-1=127$. 2, 3, 5, 7, 11 are not factors of 127, since $\sqrt{127}<13$, therefore 127 is prime.

***b)*** $2^9-1=511=7\cdot 73$ So this number is not prime.

***c)*** $2^{11}-1=2047=23\cdot 89$ So this number is not prime.

***d)*** $2^{13}-1=8191$.

Since $\sqrt{8191}<97$

then 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, and 89 are not factors of 8191, therefore 8191 is prime.

### *Exercise*

What are the greatest common divisors of these pairs of integers?

*a)* $2^2\cdot 3^3\cdot 5^5,\ \ 2^5\cdot 3^3\cdot 5^2$

*b)* $2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13,\ \ 2^{11}\cdot 3^9\cdot 11\cdot 17^{14}$

*c)* $17,\ \ 17^{17}$

*d)* $2^2\cdot 7,\ \ 5^3\cdot 13$

*e)* $0,\ \ 5$

*f)* $2\cdot 3\cdot 5\cdot 7,\ \ 2\cdot 3\cdot 5\cdot 7$

*g)* $3^7\cdot 5^3\cdot 7^3,\ \ 2^{11}\cdot 3^5\cdot 5^9$

*h)* $11\cdot 13\cdot 17,\ \ 2^9\cdot 3^7\cdot 5^5\cdot 7^3$

*i)* $23^{31},\ \ 23^{17}$

*j)* $41\cdot 43\cdot 53,\ \ 41\cdot 43\cdot 53$

*k)* $1111,\ \ 0$

***Solution***

***a)*** $2^2\cdot 3^3\cdot 5^2$

***b)*** $2\cdot 3\cdot 5$

***c)*** $17$

***d)*** $1$

***e)*** $5$

***f)*** $2\cdot 3\cdot 5\cdot 7$

***g)*** $3^5 \cdot 5^3$

***h)*** $1$

***i)*** $23^{17}$

***j)*** $41 \cdot 43 \cdot 53$

***k)*** $1111$

## *Exercise*

What is the least common multiple of each pair

*a)* $2^2 \cdot 3^3 \cdot 5^5,\ \ 2^5 \cdot 3^3 \cdot 5^2$

*b)* $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13,\ \ 2^{11} \cdot 3^9 \cdot 11 \cdot 17^{14}$

*c)* $17,\ \ 17^{17}$

*d)* $2^2 \cdot 7,\ \ 5^3 \cdot 13$

*e)* $0,\ \ 5$

*f)* $2 \cdot 3 \cdot 5 \cdot 7,\ \ 2 \cdot 3 \cdot 5 \cdot 7$

*g)* $3^7 \cdot 5^3 \cdot 7^3,\ \ 2^{11} \cdot 3^5 \cdot 5^9$

*h)* $11 \cdot 13 \cdot 17,\ \ 2^9 \cdot 3^7 \cdot 5^5 \cdot 7^3$

*i)* $23^{31},\ \ 23^{17}$

*j)* $41 \cdot 43 \cdot 53,\ \ 41 \cdot 43 \cdot 53$

*k)* $1111,\ \ 0$

***Solution***

***a)*** $2^5 \cdot 3^3 \cdot 5^5$

***b)*** $2^{11} \cdot 3^9 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17^{14}$

***c)*** $17^{17}$

***d)*** $2^2 \cdot 5^3 \cdot 7 \cdot 13$

***e)*** *Undefined*

***f)*** $2 \cdot 3 \cdot 5 \cdot 7$

***g)*** $2^{11} \cdot 3^5 \cdot 5^9 \cdot 7^3$

***h)*** $2^9 \cdot 3^7 \cdot 5^5 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17$

***i)*** $23^{31}$

***j)*** $41 \cdot 43 \cdot 53$

***k)*** *Undefined*

## *Exercise*

Find gcd(1000, 625) and lcm(1000, 625) and verify that $\gcd(100,\ 625) \cdot lcm(100,\ 625) = 1000 \cdot 625$

***Solution***

$1000 = 2^3 \cdot 5^3$

$625 = 5^5$

$\gcd(1000, 625) = 5^3 = 125$

$lcm(1000, 625) = 2^3 \cdot 5^4 = 5000$

Therefore, $125 \cdot 5000 = 625000 = 1000 \cdot 625$

## Exercise

Find *gcd*(92928, 123552) and lcm(92928, 123552) and verify that

$gcd(92928,\ 123552) \cdot lcm(92928,\ 123552) = 92928 \bullet 123552$

**Solution**

$92928 = 2^8 \cdot 3 \cdot 11^2$

$123552 = 2^5 \cdot 3^3 \cdot 11 \cdot 13$

$\gcd(92928,\ 123552) = 2^5 \cdot 3 \cdot 11 = 1056$

$lcm(92928,\ 123552) = 2^8 \cdot 3^3 \cdot 11^2 \cdot 13 = 10{,}872{,}576$

$$\gcd(92928,\ 123552) \cdot lcm(92928,\ 123552) = \left(2^5 \cdot 3 \cdot 11\right)\left(2^8 \cdot 3^3 \cdot 11^2 \cdot 13\right)$$

$$= 2^{13} \cdot 3^4 \cdot 11^3 \cdot 13$$

$$(92928)(123552) = \left(2^8 \cdot 3 \cdot 11^2\right)\left(2^5 \cdot 3^3 \cdot 11 \cdot 13\right) = 2^{13} \cdot 3^4 \cdot 11^3 \cdot 13$$

$$\gcd(92928,\ 123552) \cdot lcm(92928,\ 123552) = 92928 \bullet 123552 \ \underline{= 11{,}481{,}440{,}256}$$

## Exercise

Use the Euclidean algorithm to find

*a)* $\gcd(1,\ 5)$ *b)* $\gcd(100,\ 101)$ *c)* $\gcd(123,\ 277)$ *d)* $\gcd(1529,\ 14039)$

*e)* $\gcd(1529,\ 14038)$ *f)* $\gcd(12,\ 18)$ *g)* $\gcd(111,\ 201)$ *h)* $\gcd(1001,\ 1331)$

*i)* $\gcd(12345,\ 54321)$ *j)* $\gcd(1000,\ 5040)$ *k)* $\gcd(9888,\ 6060)$

**Solution**

***a)*** $5 = 1 \cdot 5 + 0$

$\gcd(1, 5) = \gcd(1, 0) = 1$

***b)*** $101 = 100 \cdot 1 + 1$

$1 = 1 \cdot 1 + 0$

$\gcd(100, 101) = \gcd(100, 1) = \gcd(1, 0) = 1$

***c)*** $277 = 123 \cdot 2 + 31$

$123 = 31 \cdot 3 + 30$

$31 = 30 \cdot 1 + 1$

$30 = 1 \cdot 30 + 0$

$\gcd(123, 277) = \gcd(123, 31) = \gcd(31, 30) = \gcd(30, 1) = \gcd(1, 0) = 1$

***d)*** $14039 = 1529 \cdot 9 + 278$

$1529 = 278 \cdot 5 + 139$

$278 = 139 \cdot 2 + 0$

$\gcd(1529, 14039) = \gcd(1529, 278) = \gcd(278, 139) = \gcd(139, 0) = 139$

***e)*** $14038 = 1529 \cdot 9 + 277$

$1529 = 277 \cdot 5 + 144$

$277 = 144 \cdot 1 + 133$

$144 = 133 \cdot 1 + 11$

$133 = 11 \cdot 12 + 1$

$11 = 1 \cdot 11 + 0$

$\gcd(1529, 14038) = \gcd(1529, 277) = \gcd(277, 144) = \gcd(144, 133) = \gcd(133, 11)$

$= \gcd(11, 1) = \gcd(1, 0) = 1$

***f)*** $18 = 12 \cdot 1 + 6$

$12 = 6 \cdot 2 + 0$

$\gcd(12, 18) = \gcd(12, 6) = 6$

***g)*** $201 = 111 \cdot 1 + 90$

$111 = 90 \cdot 1 + 21$

$90 = 21 \cdot 4 + 6$

$21 = 6 \cdot 3 + 3$

$6 = 3 \cdot 2 + 0$

$\gcd(111, 201) = \gcd(111, 90) = \gcd(90, 21) = \gcd(21, 6) = \gcd(6, 3) = \gcd(3, 0) = 3$

***h)*** $1331 = 1001 \cdot 1 + 330$

$1001 = 330 \cdot 3 + 11$

$330 = 11 \cdot 30 + 0$

$\gcd(1001, 1331) = \gcd(1001, 330) = \gcd(330, 11) = \gcd(11, 0) = 11$

***i)*** $54321 = 12345 \cdot 4 + 4941$

$12345 = 4941 \cdot 2 + 2463$

$4941 = 2463 \cdot 2 + 15$

$2463 = 15 \cdot 164 + 3$

$15 = 3 \cdot 5 + 0$

$\gcd(12345, 54321) = \gcd(12345, 4941) = \gcd(4941, 2463) = \gcd(2463, 15)$

$= \gcd(15, 3) = \gcd(3, 0) = 3$

***j)*** $5040 = 1000 \cdot 5 + 40$

$1000 = 40 \cdot 25 + 0$

$\gcd(1000, 5040) = \gcd(1000, 40) = \gcd(40, 0) = 40$

***k)*** $9888 = 6060 \cdot 1 + 3828$

$6060 = 3828 \cdot 1 + 2232$

$3828 = 2232 \cdot 1 + 1596$

$2232 = 1596 \cdot 1 + 636$
$1596 = 636 \cdot 2 + 324$
$636 = 324 \cdot 1 + 312$
$324 = 312 \cdot 1 + 12$
$312 = 12 \cdot 26 + 0$

$$\gcd(9888,\ 6060) = \gcd(6060,\ 3828) = \gcd(3828,\ 2232) = \gcd(2232,\ 1596) = \gcd(1596,\ 636)$$
$$= \gcd(636,\ 324) = \gcd(324,\ 312) = \gcd(312,\ 12) = \gcd(12,\ 0) = 12$$

## *Exercise*

Prove that the product of any three consecutive integers is divisible by 6.

***Solution***

Consider the product $n(n+1)(n+2)$ for some integer *n*.

Since every second integer is even (divisible by 2), then this product is divisible by 2.
Since every third integer is divisible by 3, then this product is divisible by 3.
Therefore, this product has both 2 and 3 in its prime factorization and is therefore divisible by $2 \cdot 3 = 6$

## *Exercise*

Show that if *a, b*, and *m* are integers such that $m \ge 2$ and $a \equiv b(\textbf{mod}\ m)$, then $\gcd(a,\ m) = \gcd(b,\ m)$

***Solution***

From $a \equiv b(\textbf{mod}\ m)$ we know that $b = a + sm$ for some integer *s*. If *d* is a common divisor of *a* and *m*, then it divides the right-hand side of this equation, so it also divides *b*. We can rewrite the equation as $a = b - sm$, and they by similar reasoning, we see that every common divisor of *b* and *m* is also a divisor of *a*.

This shows that the set of common divisors of *a* and *m* is equal to the set of common divisors of *b* and *m*, so certainly $\gcd(a, m) = \gcd(b, m)$

## *Exercise*

Prove or disprove that $n^2 - 79n + 1601$ is prime whenever *n* is a positive integer.

***Solution***

Using calculator or spread sheet because it is hard to get started:

| $n^2 - 79n + 1601$ | |
|---|---|
| $n = 1$ | 1523 |
| $n = 2$ | 1447 |
| $n = 3$ | 1373 |
| $n = 4$ | 1301 |
| $n = 5$ | 1231 |
| $n = 6$ | 1163 |

All the values are prime. This may lead us to believe that the propositions is true, but it gives no clue as to how to prove it.
If we let *n* = 1601, then

$$1601^2 - 79(1601) + 1601 = 1601(1601 - 79 + 1) = 1601 \cdot 1523.$$

So we got a counterexample and the proposition is false.

The smallest $n$ for which this expression is not prime is $n = 80$; this gives the value $1681 = 41 \cdot 41$

# ***Solution*** *Section* 2.6 – Applications of Congruences

### *Exercise*

Find the memory locations assigned by the hashing function $h(k) = k$ **mod** 97 to the records of customers with Social Security numbers?

| | | | |
|---|---|---|---|
| *a)* 034567981 | *b)* 183211232 | *c)* 220195744 | *d)* 987255335 |
| *e)* 104578690 | *f)* 432222187 | *g)* 372201919 | *h)* 501338753 |

***Solution***

***a)*** $034567981 \ \boldsymbol{mod} \ 97 = 91$

***b)*** $183211232 \ \boldsymbol{mod} \ 97 = 57$

***c)*** $220195744 \ \boldsymbol{mod} \ 97 = 21$

***d)*** $987255335 \ \boldsymbol{mod} \ 97 = 5$

***e)*** $104578690 \ \boldsymbol{mod} \ 97 = 80$

***f)*** $432222187 \ \boldsymbol{mod} \ 97 = 81$

***g)*** $372201919 \ \boldsymbol{mod} \ 97 = 18$

***h)*** $501338753 \ \boldsymbol{mod} \ 97 = 73$

### *Exercise*

A parking lot has 31 visitor spaces, numbered from 0 to 30. Visitors are assigned parking spaces using the hashing function $h(k) = k$ **mod** 31, where $k$ is the number formed from the first three digits on a visitor's license plate.

*a)* Which spaces are assigned by the hashing function to cars that have these first three digits on their license plates: 317, 918, 007, 100, 111, 310

*b)* Describe a procedure visitors should follow to find a free parking space, when the space they are assigned is occupied.

***Solution***

***a)*** $317 \ \boldsymbol{mod} \ 31 = 7$

$918 \ \boldsymbol{mod} \ 31 = 19$

$007 \ \boldsymbol{mod} \ 31 = 7$

$100 \ \boldsymbol{mod} \ 31 = 7$

$111 \ \boldsymbol{mod} \ 31 = 18$

$310 \ \boldsymbol{mod} \ 31 = 0$

***b)*** Take the next available space, where the next space is computed by adding 1 to the space number and pretending that 30 + 1 = 0.

### Exercise

Find the sequence of pseudorandom numbers generated by the linear congruential generator

*a)* $x_{n+1} = \left(3x_n + 2\right)$ **mod** 13 with seed $x_0 = 1$.

*b)* $x_{n+1} = \left(4x_n + 1\right)$ **mod** 7 with seed $x_0 = 3$.

### Solution

***a)*** Given $x_0 = 1$, the $x_1 = \left(3x_0 + 2\right)\boldsymbol{mod}\ 13 = \left(3\cdot 1 + 2\right)\boldsymbol{mod}\ 13 = 5\ \boldsymbol{mod}\ 13 = 5$

$x_2 = \left(3\cdot 5 + 2\right)\boldsymbol{mod}\ 13 = 17\ \boldsymbol{mod}\ 13 = 4$

$x_3 = \left(3\cdot 4 + 2\right)\boldsymbol{mod}\ 13 = 14\ \boldsymbol{mod}\ 13 = 1$

The sequence keep continue to repeat 1, 5, 4, 1, 5, 4, …

***b)*** Given $x_0 = 3$, the $x_1 = \left(4x_0 + 1\right)\boldsymbol{mod}\ 7 = \left(4\cdot 3 + 1\right)\boldsymbol{mod}\ 7 = 13\ \boldsymbol{mod}\ 7 = 6$

$x_2 = \left(4\cdot 6 + 1\right)\boldsymbol{mod}\ 7 = 25\ \boldsymbol{mod}\ 7 = 4$

$x_3 = \left(4\cdot 4 + 1\right)\boldsymbol{mod}\ 7 = 17\ \boldsymbol{mod}\ 7 = 3$

The sequence keep continue to repeat 3, 6, 4, 3, 6, 4, …

### Exercise

Find the sequence of pseudorandom numbers generated by using the pure multiplicative generator $x_{n+1} = 3x_n$ **mod** 11 with seed $x_0 = 2$.

### Solution

$x_1 = 3x_0\ \textbf{mod}\ 11 = 3\cdot 2\ \textbf{mod}\ 11 = 6$

$x_2 = 3x_1\ \textbf{mod}\ 11 = 3\cdot 6\ \textbf{mod}\ 11 = 18\ \textbf{mod}\ 11 = 7$

$x_3 = 3x_2\ \textbf{mod}\ 11 = 3\cdot 7\ \textbf{mod}\ 11 = 21\ \textbf{mod}\ 11 = 10$

$x_4 = 3x_3\ \textbf{mod}\ 11 = 3\cdot 10\ \textbf{mod}\ 11 = 30\ \textbf{mod}\ 11 = 8$

$x_5 = 3x_4\ \textbf{mod}\ 11 = 3\cdot 8\ \textbf{mod}\ 11 = 24\ \textbf{mod}\ 11 = 2$

Since $x_5 = x_0$, the sequence repeats forever: 2, 6, 7, 10, 8, 2, 6, 7, 10, 8, …

### Exercise

The first nine digits of the ISBN-10 of the European version of the fifth edition of this book are $0-07-119881$. What is the check digit for that book?

### Solution

Let *d* be the check digit.

$1\cdot 0 + 2\cdot 0 + 3\cdot 7 + 4\cdot 1 + 5\cdot 1 + 6\cdot 9 + 7\cdot 8 + 8\cdot 8 + 9\cdot 1 + 10\cdot d \equiv 0\left(\boldsymbol{mod}\ 11\right)$

$213+10 \cdot d \equiv 0(\boldsymbol{mod}\ 11)$

So $213 \equiv 4(\boldsymbol{mod}\ 11)$ *and* $10 \equiv -1(\boldsymbol{mod}\ 11)$

This is equivalent to: $4-d \equiv 0(\boldsymbol{mod}\ 11)$ or $\underline{d=4}$

## Exercise

The ISBN-10 of the sixth edition of Elementary Number Theory and Its Applications is $0-321-500Q1-8$, where $Q$ is a digit. Find the value of $Q$.

***Solution***

$$1 \cdot 0+2 \cdot 3+3 \cdot 2+4 \cdot 1+5 \cdot 5+6 \cdot 0+7 \cdot 0+8 \cdot Q+9 \cdot 1+10 \cdot 8 \equiv 0(\boldsymbol{mod}\ 11)$$

$$130+8Q \equiv 0(\boldsymbol{mod}\ 11)$$

$$8Q \equiv -130(\boldsymbol{mod}\ 11) \equiv 2(\boldsymbol{mod}\ 11) \qquad -130 \equiv (-12 \cdot 11+2)(\boldsymbol{mod}\ 11)$$

$8Q \equiv 2(\boldsymbol{mod}\ 11)$ Since $24 \equiv 2(\boldsymbol{mod}\ 11)$

Therefore $8Q=24$

This is equivalent to: $\underline{Q=3}$

## Exercise

The USPS sells money orders identified by 11-digit number $x_1, x_2, \ldots, x_{11}$. The first ten digits identify the money order: $x_{11}$ is a check digit that satisfies $x_{11}=x_1+x_2+\cdots+x_{10}$ **mod** $9$. Find the check digit for the USPS money orders that have identification number that start with these ten digits

*a)* 7555618873 *b)* 6966133421 *c)* 8018927435 *d)* 3289744134

*e)* 74051489623 *f)* 88382013445 *g)* 56152240784 *h)* 66606631178

***Solution***

***a)*** $(7+5+5+5+6+1+8+8+7+3)\ \boldsymbol{mod}\ 9=55\ \boldsymbol{mod}\ 9=1$

***b)*** $(6+9+6+6+1+3+3+4+2+1)\ \boldsymbol{mod}\ 9=41\ \boldsymbol{mod}\ 9=5$

***c)*** $(8+0+1+8+9+2+7+4+3+5)\ \boldsymbol{mod}\ 9=47\ \boldsymbol{mod}\ 9=2$

***d)*** $(3+2+8+9+7+4+4+1+3+4)\ \boldsymbol{mod}\ 9=45\ \boldsymbol{mod}\ 9=0$

***e)*** $(7+4+0+5+1+4+8+9+6+2+3)\ \boldsymbol{mod}\ 9=49\ \boldsymbol{mod}\ 9=4$

***f)*** $(8+8+3+8+2+0+1+3+4+4+5)\ \boldsymbol{mod}\ 9=46\ \boldsymbol{mod}\ 9=1$

***g)*** $(5+6+1+5+2+2+4+0+7+8+4)\ \boldsymbol{mod}\ 9=44\ \boldsymbol{mod}\ 9=8$

***h)*** $(6+6+6+0+6+6+3+1+1+7+8)\ \boldsymbol{mod}\ 9=50\ \boldsymbol{mod}\ 9=5$

### *Exercise*

Determine which single digit errors are detected by the USPS money order code.

***Solution***

If one digit change to a value not congruent to it modulo 9, then the modular equivalence implied by the equation in the preamble will no longer hold. Therefore all single digit errors are detected except for the substitution of a 9 for a 0 or vice versa.

### *Exercise*

Determine which transposition errors are detected by the USPS money order code.

***Solution***

Because the first ten digits are added, any transposition error involving them will go undetected. The sum of the first ten digits will be the same for the transposed number as it is for the correct number.

Suppose that the last digit is transposed with another digit; without loss of generality; we can assume it's the tenth digit and that $x_{10} \neq x_{11}$ .

Then the correct equation will be

$$x_{11} \equiv x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} \left(\boldsymbol{mod}\ 9\right) \qquad (1)$$

But the equation resulting from the error will read

$$x_{10} \equiv x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{11} \left(\boldsymbol{mod}\ 9\right) \qquad (2)$$

Subtract equations $(2)$ & $(1)$

$$x_{11} - x_{10} \equiv x_{10} - x_{11} \left(\boldsymbol{mod}\ 9\right)$$

$$2x_{11} \equiv 2x_{10} \left(\boldsymbol{mod}\ 9\right) \qquad \textit{Divide by } 2 \textit{ both sides since } 2 \textit{ is prime}$$

$$x_{11} \equiv x_{10} \left(\boldsymbol{mod}\ 9\right) \quad \text{Which is false}$$

The check equation will fail.

Therefore, we conclude that transposition errors involving the eleventh digits are detected.