Solution   *Section* 6.5 – Law of Sines and Cosines

Exercise

In triangle *ABC*, $B = 110°$, $C = 40°$ and $b = 18\ in$. Find the length of side *c*.

Solution

$$A = 180° - (B + C)$$
$$= 180° - 110° - 40°$$
$$\boxed{= 30°}$$

$$\frac{a}{\sin 30°} = \frac{18}{\sin 110°} \qquad \frac{a}{\sin A} = \frac{b}{\sin B}$$

$$a = \frac{18 \sin 30°}{\sin 110°}$$
$$\boxed{\approx 9.6\ \ in}$$

$$\frac{c}{\sin 40°} = \frac{18}{\sin 110°} \qquad \frac{c}{\sin C} = \frac{b}{\sin B}$$

$$c = \frac{18 \sin 40°}{\sin 110°}$$
$$\boxed{\approx 12.3\ \ in}$$

Exercise

In triangle *ABC*, $A = 110.4°$, $C = 21.8°$ and $c = 246\ in$. Find all the missing parts.

Solution

$$B = 180° - A - C$$
$$= 180° - 110.4° - 21.8°$$
$$\boxed{= 47.8°}$$

$$\frac{a}{\sin 110.4°} = \frac{246}{\sin 21.8°} \qquad \frac{a}{\sin A} = \frac{c}{\sin C}$$

$$a = \frac{246 \sin 110.4°}{\sin 21.8°}$$
$$\boxed{\approx 621\ in}$$

$$\frac{b}{47.8} = \frac{246}{\sin 21.8°} \qquad \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$b = \frac{246 \sin 47.8°}{\sin 21.8°}$$
$$\boxed{\approx 491\ in}$$

Exercise

Find the missing parts of triangle ABC if $B = 34°, C = 82°$, and $a = 5.6\ cm$.

Solution

$$\begin{aligned} A &= 180° - (B + C) \\ &= 180° - (34° + 82°) \\ &= 180° - 116° \\ &= 64° \end{aligned}$$

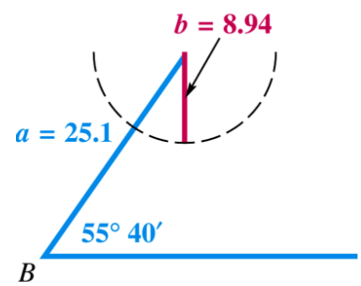
$$\begin{aligned} b &= \frac{a \sin B}{\sin A} \\ &= \frac{5.6 \sin 34°}{\sin 64°} \\ &\approx 3.5\ cm \end{aligned} \qquad \frac{b}{\sin B} = \frac{a}{\sin A}$$

$$\begin{aligned} c &= \frac{a \sin C}{\sin A} \\ &= \frac{5.6 \sin 82°}{\sin 64°} \\ &\approx 6.2\ cm \end{aligned} \qquad \frac{c}{\sin C} = \frac{a}{\sin A}$$

Exercise

Solve triangle ABC if $B = 55°40'$, $b = 8.94\ m$, and $a = 25.1\ m$.

Solution



$$\frac{\sin A}{25.1} = \frac{\sin\left(55° + \frac{40°}{60}\right)}{8.94} \qquad \frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\begin{aligned} \sin A &= \frac{25.1 \sin(55.667°)}{8.94} \\ &\approx 2.3184 > 1 \end{aligned}$$

Since $\sin A > 1$ is impossible, *no such triangle* ***exists***.

Exercise

Solve triangle ABC if $A = 55.3°$, $a = 22.8\ ft$, and $b = 24.9$ ft

Solution

$$\sin B = \frac{24.9 \sin 55.3°}{22.8} \approx 0.89787 \qquad \frac{\sin B}{b} = \frac{\sin A}{a}$$

$$B = \sin^{-1}(0.89787)$$

$$B = 63.9° \quad \text{and} \quad B = 180° - 63.9° = 116.1°$$

$C = 180° - A - B$

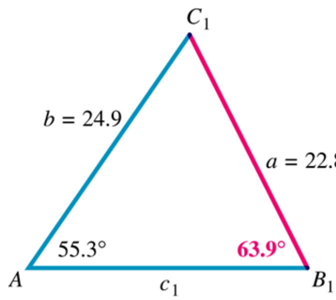
$C = 180° - 55.3° - 63.9°$

$\underline{C = 60.8°}$

$c = \dfrac{a \sin C}{\sin A}$

$= \dfrac{22.8 \sin 60.8°}{\sin 55.3°}$

$\underline{= 24.2\ ft}$



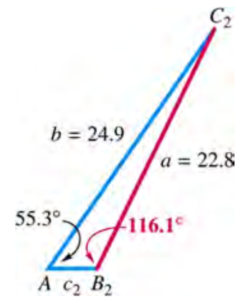
$C = 180° - 55.3° - 116.1°$

$\underline{C = 8.6°}$

$c = \dfrac{a \sin C}{\sin A}$

$= \dfrac{22.8 \sin 8.6°}{\sin 55.3°}$

$\underline{= 4.15\ ft}$



Exercise

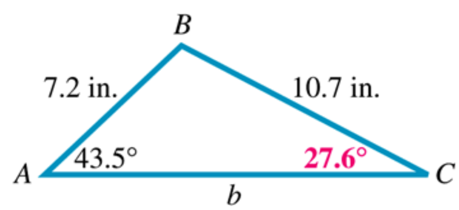
Solve triangle *ABC* given $A = 43.5°$, $a = 10.7$ in., and $c = 7.2$ in

Solution

$\sin C = \dfrac{7.2 \sin 43.5°}{10.7} \approx 0.4632 \qquad \dfrac{\sin C}{c} = \dfrac{\sin A}{a}$

$C = \sin^{-1}(0.4632)$

$C = 27.6° \quad$ *and* $\quad C = 180° - 27.6° = 152.4°$



$B = 180° - A - C$

$B = 180° - 43.5° - 27.6°$

$\underline{B = 108.9°}$

$b = \dfrac{a \sin B}{\sin A}$

$= \dfrac{10.7 \sin 108.9°}{\sin 43.5°}$

$\underline{= 14.7\ in}$

$B = 180° - 43.5° - 152.4°$

$B = -15.9°$

Is not possible

Exercise

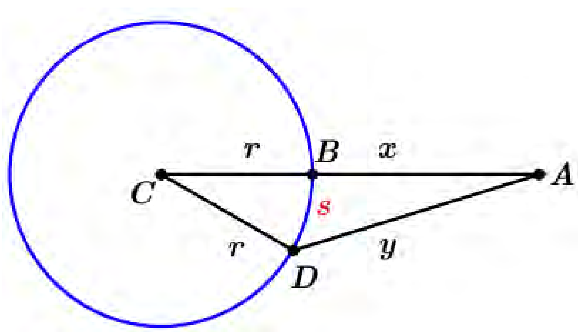
If $A = 26°,\ s = 22,\ and\ \ r = 19$ find x

Solution

$C = \theta = \dfrac{s}{r}\ rad$

$= \dfrac{22}{19} \dfrac{180°}{\pi}$

$\underline{\approx 66°}$



$$D = 180 - A - C$$
$$= 180° - 26° - 66°$$
$$\boxed{= 88°}$$

$$\frac{r+x}{\sin D} = \frac{r}{\sin A}$$

$$19 + x = \frac{19\sin 88°}{\sin 26°}$$

$$x = \frac{19\sin 88°}{\sin 26°} - 19$$
$$\boxed{\approx 24}$$

Exercise

If $a = 13$ *yd*, $b = 14$ *yd*, and $c = 15$ *yd*, find the largest angle.

Solution

$$C = \cos^{-1}\left(\frac{13^2 + 14^2 - 15^2}{2(13)(14)}\right) \qquad C = \cos^{-1}\frac{a^2+b^2-c^2}{2ab}$$

$$\boxed{\approx 67°}$$

Exercise

Solve triangle *ABC* if $b = 63.4$ *km*, and $c = 75.2$ *km*, $A = 124°\ \ 40'$

Solution

$$a = \sqrt{b^2 + c^2 - 2bc\cos A}$$
$$= \sqrt{(63.4)^2 + (75.2)^2 - 2(63.4)(75.2)\cos\left(124° + \frac{40°}{60}\right)}$$
$$\boxed{\approx 122.9\ km}$$

$$\sin B = \frac{b\sin A}{a}$$

$$\underline{|B} = \sin^{-1}\left(\frac{63.4\sin 124.67°}{122.9}\right)$$
$$\boxed{\approx 25.1°}$$

$$\underline{|C} = 180° - A - B$$
$$= 180° - 124.67° - 25.1°$$
$$\boxed{\approx 30.23°}$$

Exercise

Solve triangle *ABC* if $a = 832\ ft$, $b = 623\ ft$, and $c = 345\ ft$

Solution

$$C = \cos^{-1}\left(\frac{832^2 + 623^2 - 345^2}{2(832)(623)}\right) \qquad C = \cos^{-1}\frac{a^2 + b^2 - c^2}{2ab}$$

$$\approx 22°$$

$$\angle B = \sin^{-1}\left(\frac{623 \sin 22°}{345}\right) \qquad \sin B = \frac{b \sin C}{c}$$

$$\approx 43°$$

$$\angle A = 180° - 22° - 43°$$

$$= 115°$$

Exercise

Solve triangle *ABC* if $A = 42.3°,\ b = 12.9m,\ and\ c = 15.4m$

Solution

$$a = \sqrt{12.9^2 + 15.4^2 - 2(12.9)(15.4)\cos 42.3°} \qquad a = \sqrt{b^2 + c^2 - 2bc\cos A}$$

$$\approx 10.47\ m$$

$$\sin B = \frac{12.9 \sin 42.3°}{10.47} \qquad \sin B = \frac{b \sin A}{a}$$

$$\angle B = \sin^{-1}\left(\frac{12.9 \sin 42.3°}{10.47}\right)$$

$$\approx 56.0°$$

$$\angle C = 180° - 42.3° - 56°$$

$$= 81.7°$$

Exercise

Solve triangle *ABC* if $a = 9.47\ ft,\ b = 15.9\ ft,\ and\ c = 21.1\ ft$

Solution

$$\angle C = \cos^{-1}\left(\frac{9.47^2 + 15.9^2 - 21.1^2}{2(9.47)(15.9)}\right) \qquad C = \cos^{-1}\frac{a^2 + b^2 - c^2}{2ab}$$

$$\approx 109.9°$$

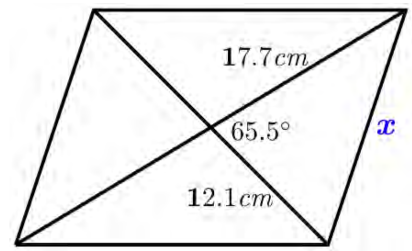
$$\sin B = \frac{15.9 \sin 109.9°}{21.1} \qquad \sin B = \frac{b \sin C}{c}$$

$$\underline{|B} = \sin^{-1}\left(\frac{15.9\sin 109.9°}{21.1}\right)$$
$$\underline{\approx 25.0° \;|}$$

$$\underline{|A} = 180° - 25° - 109.9°$$
$$\underline{= 45.1° \;|}$$

Exercise

The diagonals of a parallelogram are 24.2 *cm* and 35.4 *cm* and intersect at an angle of 65.5°. Find the length of the shorter side of the parallelogram



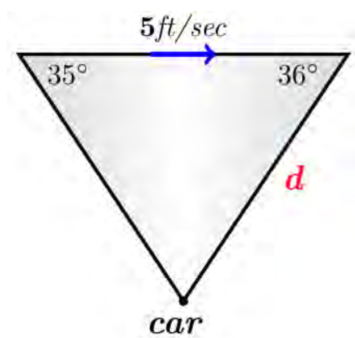
Solution

$$x = \sqrt{17.7^2 + 12.1^2 - 2(17.7)(12.1)\cos 65.5°}$$
$$\underline{= 16.8\ \ cm \;|}$$

Exercise

A man flying in a hot-air balloon in a straight line at a constant rate of 5 *feet* per *second*, while keeping it at a constant altitude. As he approaches the parking lot of a market, he notices that the angle of depression from his balloon to a friend's car in the parking lot is 35°. A minute and a half later, after flying directly over this friend's car, he looks back to see his friend getting into the car and observes the angle of depression to be 36°. At that time, what is the distance between him and his friend?

Solution



$$\angle car = 180° - 35° - 36° = 109°$$

$$\frac{d}{\sin 35°} = \frac{450}{\sin 109°}$$

$$d = \frac{450\sin 35°}{\sin 109°}$$
$$\underline{\approx 273\ \ ft \;|}$$

Exercise

A satellite is circling above the earth. When the satellite is directly above point *B*, angle *A* is 75.4°. If the distance between points *B* and *D* on the circumference of the earth is 910 *miles* and the radius of the earth is 3,960 *miles*, how far above the earth is the satellite?

Solution

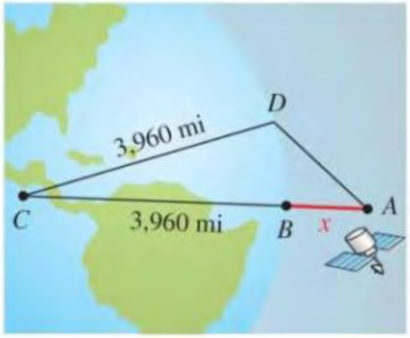
$$\theta = \frac{s}{r}$$

$C = arc\ length\ BD$ divides by *radius*

$$C = \frac{910}{3960} rad$$

$$= \frac{910}{3960} \frac{180°}{\pi}$$

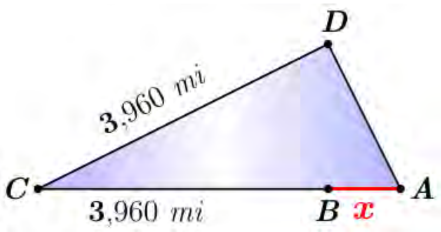
$$= 13.2°$$



$$D = 180° - (A + C)$$

$$= 180° - (75.4° + 13.2°)$$

$$\underline{= 91.4°}$$



$$\frac{CA}{\sin D} = \frac{3960}{\sin A}$$

$$\frac{x + 3960}{\sin 91.4°} = \frac{3960}{\sin 75.4°}$$

$$x + 3960 = \frac{3960 \sin 91.4°}{\sin 75.4°}$$

$$x = \frac{3960 \sin 91.4°}{\sin 75.4°} - 3960$$

$$\underline{x = 130 \ mi}$$

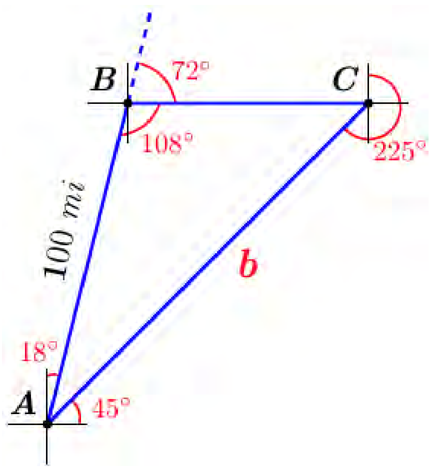
Exercise

A pilot left Fairbanks in a light plane and flew 100 *miles* toward Fort in still air on a course with bearing of 18°. She then flew due east (bearing 90°) for some time drop supplies to a snowbound family. After the drop, her course to return to Fairbanks had bearing of 225°. What was her maximum distance from Fairbanks?

Solution

From the triangle *ABC*:

$$\angle ABC = 90° + 18° = 108°$$
$$\angle ACB = 360° - 225° - 90° = 45°$$
$$\angle BAC = 90° - 18° - 45° = 27°$$



The length *AC* is the maximum distance from Fairbanks:

$$\frac{b}{\sin 108°} = \frac{100}{\sin 45°}$$

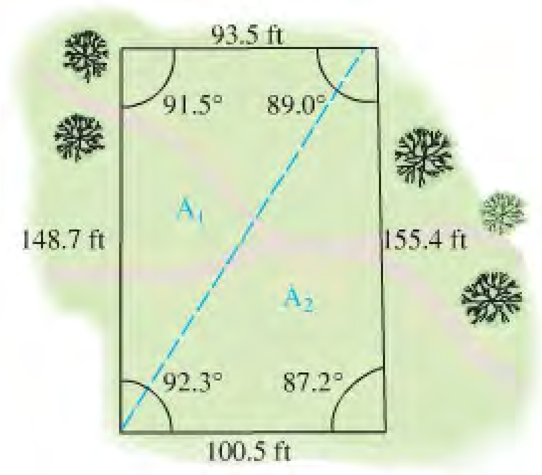
$$b = \frac{100 \sin 108°}{\sin 45°}$$

$$\underline{\approx 134.5 \ \ miles}$$

Exercise

The dimensions of a land are given in the figure. Find the area of the property in square feet.

Solution



$$A_1 = \frac{1}{2}(148.7)(93.5)\sin 91.5°$$

$$\approx 6949.3\ ft^2$$

$$A_2 = \frac{1}{2}(100.5)(155.4)\sin 87.2°$$

$$\approx 7799.5\ ft^2$$

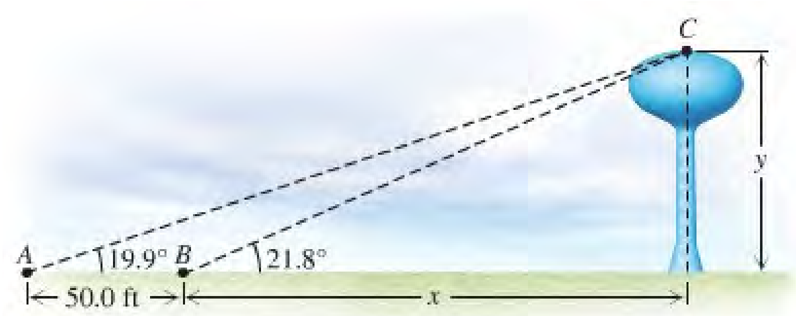
The total area $= A_1 + A_2 = 6949.3 + 7799.5$

$$= 14,748.8\ \ ft^2$$

Exercise

The angle of elevation of the top of a water tower from point *A* on the ground is 19.9°. From point *B*, 50.0 feet closer to the tower, the angle of elevation is 21.8°. What is the height of the tower?

Solution



$\angle ABC = 180° - 21.8° = 158.2°$

$\angle ACB = 180° - 19.9° - 158.2° = 1.9°$

Apply the law of sines in triangle *ABC*:

$$\frac{BC}{\sin 19.9°} = \frac{50}{\sin 1.9°}$$

$$\Rightarrow BC = \frac{50 \sin 19.9°}{\sin 1.9°} \approx 513.3$$

Using the right triangle: $\sin 21.8° = \frac{y}{BC}$

$y = 513.3 \sin 21.8°$

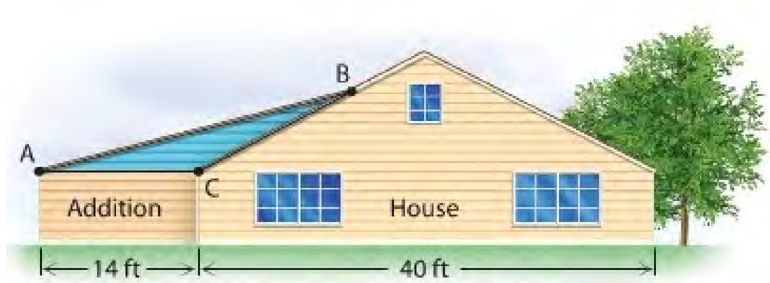
$\approx 191\ \ ft$

Or $\quad y = \dfrac{50 \tan 19.9° \tan 21.8°}{\tan 21.8° - \tan 19.9°} \approx 190.6\ \ ft$

Exercise

A 40-*feet* wide house has a roof with a 6-12 pitch (the roof rises 6 *feet* for a run of 12 *feet*). The owner plans a 14-*feet* wide addition that will have a 3-12 pitch to its roof. Find the lengths of $\overline{AB}$ and $\overline{BC}$.

Solution



$$\tan\gamma = \frac{6}{12}$$

$$\gamma = \tan^{-1}\left(\frac{6}{12}\right)$$

$$= 26.565^\circ$$

$$\tan\alpha = \frac{3}{12}$$

$$\alpha = \tan^{-1}\left(\frac{3}{12}\right)$$

$$= 14.036^\circ$$

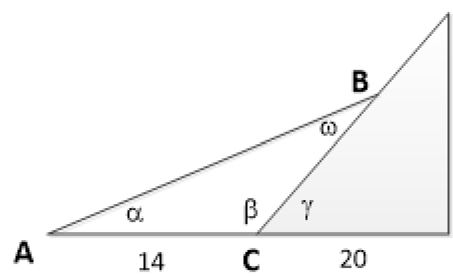
$$\beta = 180^\circ - \gamma$$

$$= 180^\circ - 26.565^\circ$$

$$= 153.435^\circ$$

$$\omega = 180^\circ - 14.036^\circ - 153.435^\circ$$

$$= 12.529^\circ$$



$$\frac{AB}{\sin 153.435^\circ} = \frac{14}{\sin 12.529^\circ}$$

$$|AB| = \frac{14\sin 153.435^\circ}{\sin 12.529^\circ}$$

$$\approx 28.9\ ft$$

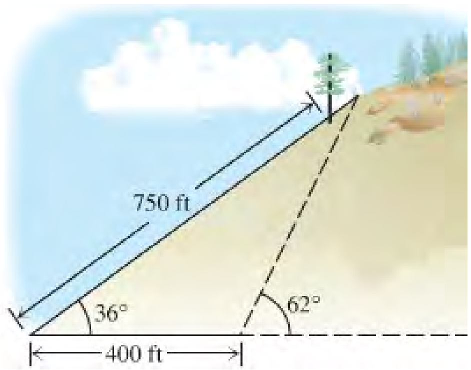
$$\frac{BC}{\sin 14.036^\circ} = \frac{14}{\sin 12.529^\circ}$$

$$|BC| = \frac{14\sin 14.036^\circ}{\sin 12.529^\circ}$$

$$\approx 15.7\ ft$$

Exercise

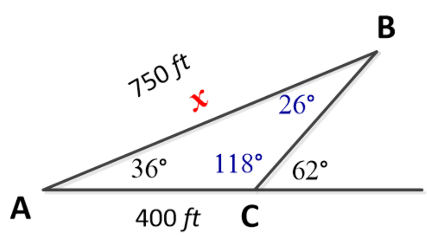
A hill has an angle of inclination of 36°. A study completed by a state's highway commission showed that the placement of a highway requires that 400 *feet* of the hill, measured horizontally, be removed. The engineers plan to leave a slope alongside the highway with an angle of inclination of 62°. Located 750 *feet* up the hill measured from the base is a tree containing the nest of an endangered hawk. Will this tree be removed in the excavation?



Solution

$\angle ACB = 180° - 62° = 118°$

$\angle ABC = 180° - 118° - 36° = 26°$



Using the law of sines:

$$\frac{x}{\sin 118°} = \frac{400}{\sin 26°}$$

$$x = \frac{400 \sin 118°}{\sin 26°}$$

$$\approx 805.7 \ ft \qquad > 750$$

Yes, the tree will have to be excavated.

Exercise

A hill has an angle of inclination of 30°. A study completed by a state's highway commission showed that the placement of a highway requires that 400 *feet* of the hill, measured horizontally, be removed. The engineers plan to leave a slope alongside the highway with an angle of inclination of 60°. Located 750 *feet* up the hill measured from the base is a tree containing the nest of an endangered hawk. Will this tree be removed in the excavation?

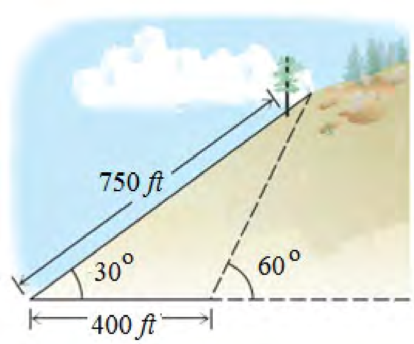
Solution

$\angle ACB = 180° - 60° = 120°$

$\angle ABC = 180° - 120° - 30° = 30°$

That implies: $|CA| = |CB|$

Using the law of sines:

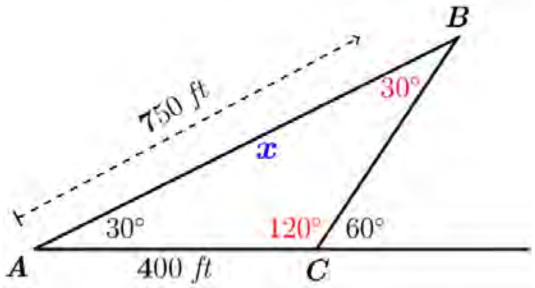


$$\frac{x}{\sin 120°} = \frac{400}{\sin 30°}$$

$$x = \frac{400 \sin 120°}{\sin 30°}$$

$$x = \frac{400 \frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

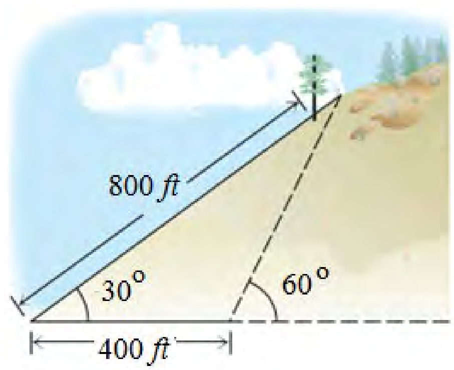
$$\underline{= 400\sqrt{3}\ ft} \quad < 750$$



∴ the tree will ***not*** have to be excavated.

Exercise

A hill has an angle of inclination of 30°. A study completed by a state's highway commission showed that the placement of a highway requires that 400 *feet* of the hill, measured horizontally, be removed. The engineers plan to leave a slope alongside the highway with an angle of inclination of 60°. Located 800 *feet* up the hill measured from the base is a tree containing the nest of an endangered hawk. Will this tree be removed in the excavation?



Solution

$$\angle ACB = 180° - 60° = 120°$$

$$\angle ABC = 180° - 120° - 30° = 30°$$

That implies: $|CA| = |CB|$

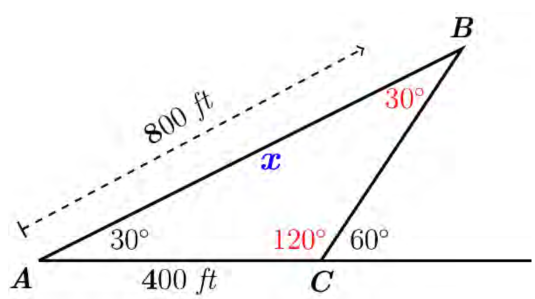
Using the law of sines:

$$\frac{x}{\sin 120°} = \frac{400}{\sin 30°}$$

$$x = \frac{400 \sin 120°}{\sin 30°}$$

$$x = \frac{400 \frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

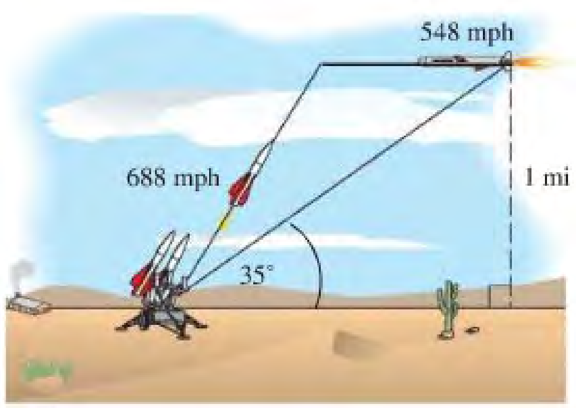
$$\underline{= 400\sqrt{3}\ ft} \quad < 800$$



∴ the tree will ***not*** have to be excavated.

Exercise

A cruise missile is traveling straight across the desert at 548 *mph* at an altitude of 1 *mile*. A gunner spots the missile coming in his direction and fires a projectile at the missile when the angle of elevation of the missile is 35°. If the speed of the projectile is 688 *mph*, then for what angle of elevation of the gun will the projectile hit the missile?



Solution

$\angle ACB = 35°$

$\angle BAC = 180° - 35° - \beta$

After ***t*** seconds;

The cruise missile distance: $548\dfrac{t}{3600}\,miles$

The Projectile distance: $688\dfrac{t}{3600}\,miles$

Using the law of sines:

$$\frac{\frac{548t}{3600}}{\sin(145° - \beta)} = \frac{\frac{688t}{3600}}{\sin 35°}$$

$$\frac{548t}{3600}\sin 35° = \frac{688t}{3600}\sin(145° - \beta)$$

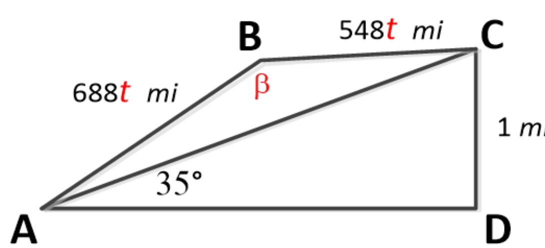
$$548\sin 35° = 688\sin(145° - \beta)$$

$$\sin(145° - \beta) = \frac{548}{688}\sin 35°$$

$$145° - \beta = \sin^{-1}\left(\frac{548}{688}\sin 35°\right)$$

$$\beta = 145° - \sin^{-1}\left(\frac{548}{688}\sin 35°\right)$$

$$\approx 117.8°$$



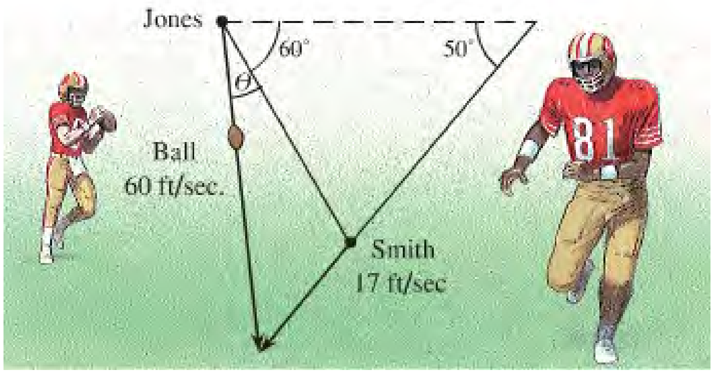
$$\Rightarrow \angle BAC = 180° - 35° - 117.8°$$

$$\approx 27.2°$$

The angle of elevation of the projectile must be $(= 35° + 27.2°)$ $62.2°$

Exercise

When the ball is snapped, Smith starts running at a 50° angle to the line of scrimmage. At the moment when Smith is at a 60° angle from Jones, Smith is running at 17 *feet/sec* and Jones passes the ball at 60 *feet/sec* to Smith. However, to complete the pass, Jones must lead Smith by the angle θ. Find θ (find θ in his head. Note that θ can be found without knowing any distances.)

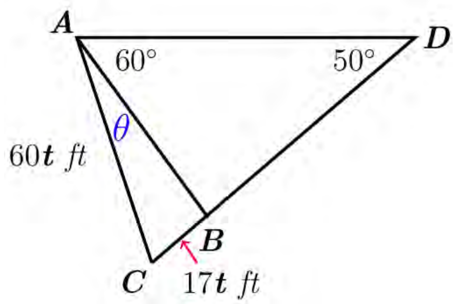


Solution

$$\angle ABD = 180° - 60° - 50°$$
$$= 70°$$

$$\angle ABC = 180° - 70°$$
$$= 110°$$

Using the law of sines:



$$\frac{17t}{\sin\theta} = \frac{60t}{\sin 110°}$$

$$\frac{17}{\sin\theta} = \frac{60}{\sin 110°}$$

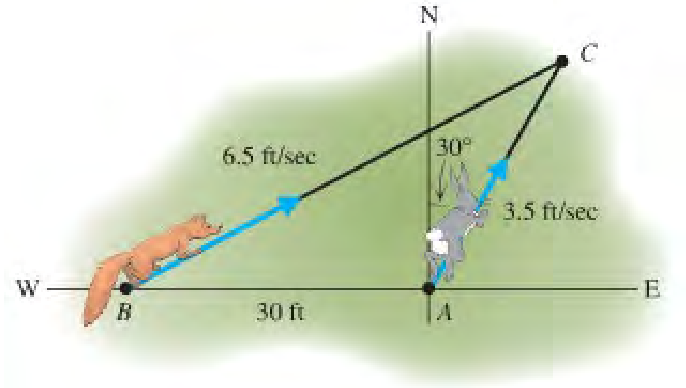
$$\sin\theta = \frac{17\sin 110°}{60}$$

$$\theta = \sin^{-1}\left(\frac{17\sin 110°}{60}\right)$$

$$= 15.4°$$

Exercise

A rabbit starts running from point *A* in a straight line in the direction 30° from the north at 3.5 *ft/sec*. At the same time a fox starts running in a straight line from a position 30 *feet* to the west of the rabbit 6.5 *ft/sec*. The fox chooses his path so that he will catch the rabbit at point *C*. In how many seconds will the fox catch the rabbit?



Solution

$$\angle BAC = 90° + 30°$$
$$= 120°$$

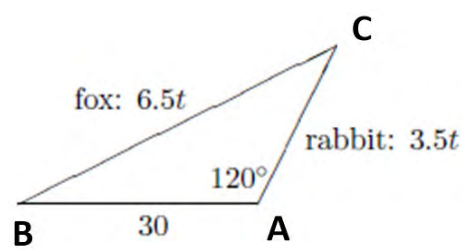
$$\frac{6.5t}{\sin 120°} = \frac{3.5t}{\sin B}$$

$$\frac{6.5}{\sin 120°} = \frac{3.5}{\sin B}$$

$$\sin B = \frac{3.5 \sin 120°}{6.5}$$

$$B = \sin^{-1}\left(\frac{3.5 \sin 120°}{6.5}\right)$$
$$\approx 28°$$

$$C = 180° - 120° - 28°$$
$$= 32°$$



$$\frac{3.5t}{\sin 28} = \frac{30}{\sin 32°}$$

$$t = \frac{30 \sin 28}{3.5 \sin 32°}$$
$$\approx 7.6 \text{ sec}$$

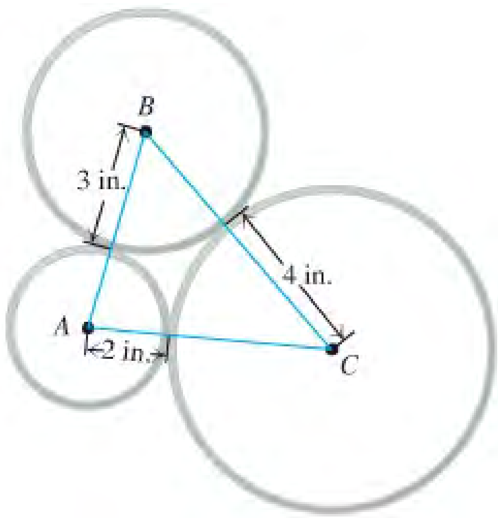
It will take 7.6 *sec*. to catch the rabbit.

Exercise

An engineer wants to position three pipes at the vertices of a triangle. If the pipes A, B, and C have radii 2 *in*, 3 *in*, and 4 *in*, respectively, then what are the measures of the angles of the triangle ABC?

Solution

$AC = 6 \quad AB = 5 \quad BC = 7$



$$A = \cos^{-1}\left(\frac{5^2+6^2-7^2}{2(5)(6)}\right)$$

$$\approx 78.5°$$

$$\frac{6}{\sin B} = \frac{7}{\sin 78.5°}$$

$$\sin B = \frac{6\sin 78.5°}{7}$$

$$B = \sin^{-1}\left(\frac{6\sin 78.5°}{7}\right)$$

$$\approx 57.1°$$

$$C = 180° - 78.5° - 57.1°$$

$$\approx 44.4°$$

Exercise

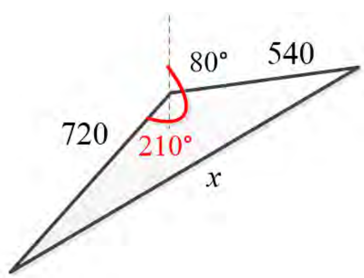
Andrea and Steve left the airport at the same time. Andrea flew at 180 *mph* on a course with bearing 80°, and Steve flew at 240 *mph* on a course with bearing 210°. How far apart were they after 3 *hr*.?

Solution

After 3 *hrs*. Steve flew: 3(240) = 720 *mph*

Andrea flew: 3(180) = 540 *mph*

$$x = \sqrt{720^2 + 540^2 - 2(720)(540)\cos 210°}$$



$$= 10\sqrt{72^2 + 54^2 + 2(72)(54)\frac{\sqrt{3}}{2}}$$

$$= 10\sqrt{(9\times 8)^2 + (9\times 6)^2 + (9\times 8)(9\times 6)\sqrt{3}}$$

$$= 90\sqrt{64 + 36 + 48\sqrt{3}}$$

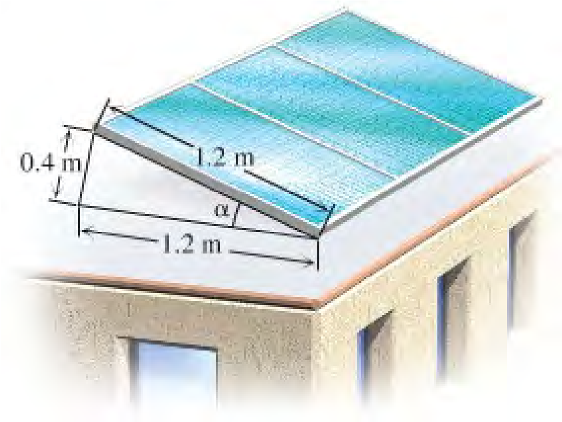
$$= 90\sqrt{100 + 48\sqrt{3}}$$

$$= 180\sqrt{25 + 12\sqrt{3}}$$

$$\approx 1218 \ \textit{miles}$$

Exercise

A solar panel with a width of 1.2 m is positioned on a flat roof. What is the angle of elevation α of the solar panel?



Solution

$$\alpha = \cos^{-1}\left(\frac{1.2^2 + 1.2^2 - 0.4^2}{2(1.2)(1.2)}\right)$$

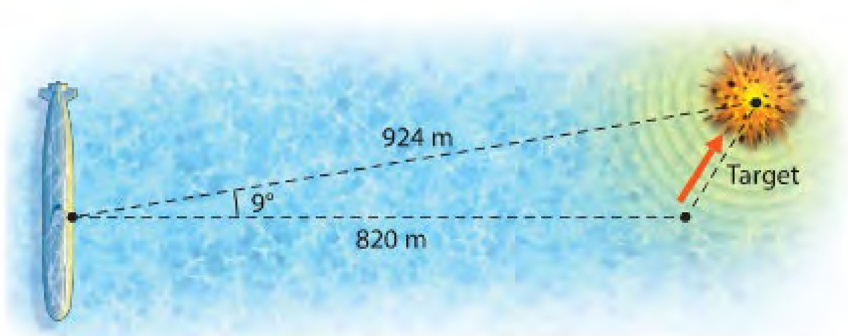
$$\approx 19.2°$$

or $\alpha = \frac{0.4}{1.2}$ $\qquad \alpha = \frac{s}{r}$

$$= \frac{1}{3} \ rad$$

Exercise

A submarine sights a moving target at a distance of 820 m. A torpedo is fired 9° ahead of the target, and travels 924 m in a straight line to hit the target. How far has the target moved from the time the torpedo is fired to the time of the hit?



Solution

$$x = \sqrt{820^2 + 924^2 - 2(820)(924)\cos 9°}$$

$$\approx 171.7 \ m$$

Exercise

A tunnel is planned through a mountain to connect points A and B on two existing roads. If the angle between the roads at point C is 28°, what is the distance from point A to B? Find $\angle CBA$ and $\angle CAB$ to the nearest tenth of a degree.

Solution

$$AB = \sqrt{5.3^2 + 7.6^2 - 2(5.3)(7.6)\cos 28°}$$

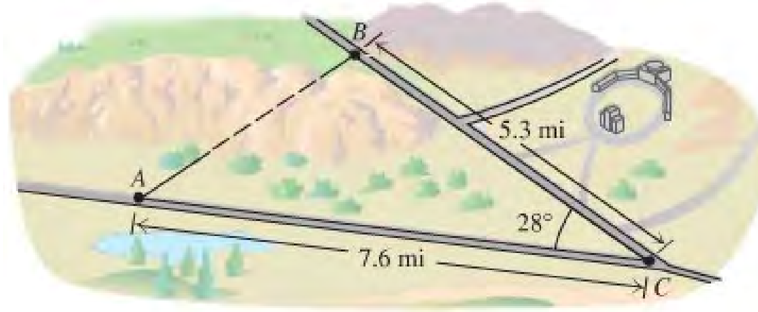
≈ 3.8

$$\angle CBA = \cos^{-1}\frac{3.8^2+5.3^2-7.6^2}{2(3.8)(5.3)}$$

$\approx 112°$

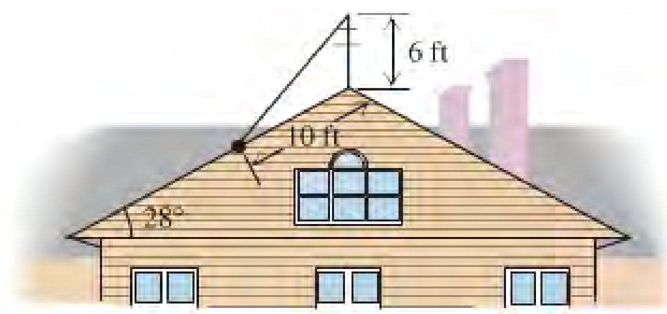
$$\angle BAC = 180° - 112° - 28°$$

$\approx 40°$



Exercise

A 6-*feet* antenna is installed at the top of a roof. A guy wire is to be attached to the top of the antenna and to a point 10 *feet* down the roof. If the angle of elevation of the roof is 28°, then what length guy wire is needed?



Solution

$\alpha = 90° - 28°$

$= 62°$

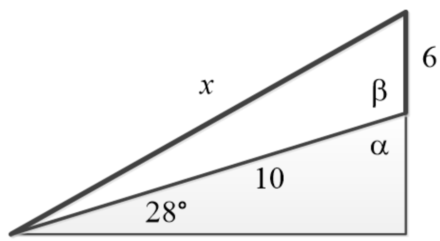
$\beta = 180° - 62°$

$= 118°$

By the cosine law,

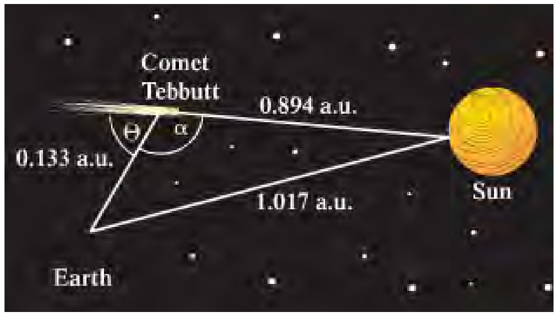
$$x = \sqrt{6^2 + 100^2 - 2(6)(10)\cos 118°}$$

$\approx 13.9\ ft$



Exercise

On June 30, 1861, Comet Tebutt, one of the greatest comets, was visible even before sunset. One of the factors that causes a comet to be extra bright is a small scattering angle θ. When Comet Tebutt was at its brightest, it was 0.133 *a.u.* from the earth, 0.894 *a.u.* from the sun, and the earth was 1.017 *a.u.* from the sun. Find the phase angle α and the scattering angle θ for Comet Tebutt on June 30, 1861. (One astronomical unit (*a.u*) is the average distance between the earth and the sub.)



Solution

By the cosine law:

$$\alpha = \cos^{-1}\left(\frac{0.133^2 + 0.894^2 - 1.017^2}{2(0.133)(0.891)}\right)$$

$$\approx 156^\circ$$

$$\theta = 180^\circ - \alpha$$

$$= 180^\circ - 156^\circ$$

$$\approx 24^\circ$$

Exercise

A human arm consists of an upper arm of 30 *cm* and a lower arm of 30 *cm*. To move the hand to the point (36, 8), the human brain chooses angle θ_1 *and* θ_2 to the nearest tenth of a degree.

Solution

$AC = 30 - 8 = 22 \qquad BC = 36$

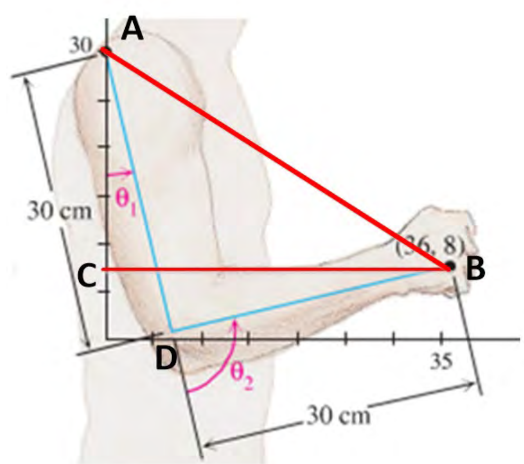
$$AB = \sqrt{AC^2 + CB^2}$$

$$= \sqrt{22^2 + 36^2}$$

$$\approx 42.19$$

$$\angle ADB = \cos^{-1}\left(\frac{AD^2 + DB^2 - AB^2}{2(AD)(DB)}\right)$$

$$= \cos^{-1}\left(\frac{30^2 + 30^2 - 42.19^2}{2(30)(30)}\right)$$



$$\approx 89.4^\circ$$

$$\theta_2 \approx 180^\circ - 89.4^\circ$$

$$\approx 90.6^\circ$$

$$\tan(\angle CAB) = \frac{BC}{AC} = \frac{36}{22}$$

$$\angle CAB = \tan^{-1}\frac{36}{22}$$

$$\approx 58.57^\circ$$

$$\frac{\sin DAB}{30} = \frac{\sin 89.4^\circ}{42.19}$$

$$\sin DAB = \frac{30\sin 89.4^\circ}{42.19}$$

$$\angle DAB = \sin^{-1}\frac{30\sin 89.4^\circ}{42.19}$$

$$\approx 45.32^\circ$$

$$\theta_1 = \angle CAB - \angle DAB$$

$$= 58.57^\circ - 45.32^\circ$$

$$\approx 13.25^\circ$$

Exercise

A forest ranger is 150 *feet* above the ground in a fire tower when she spots an angry grizzly bear east of the tower with an angle of depression of 10°. Southeast of the tower she spots a hiker with an angle of depression of 15°. Find the distance between the hiker and the angry bear.

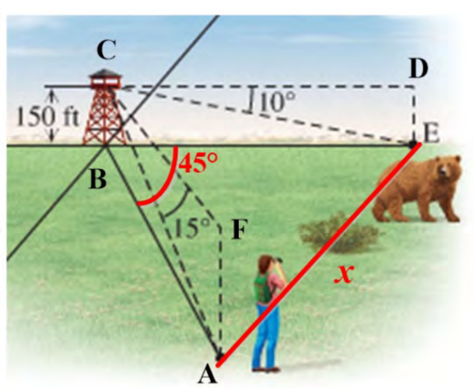
Solution

$$\angle BEC = \angle ECD = 10^\circ$$

From triangle EBC: $\tan 10^\circ = \dfrac{150}{BE}$

$$\Rightarrow BE = \frac{150}{\tan 10^\circ} \approx 850.692$$

$$\angle BAC = \angle ACF = 15^\circ$$



From triangle ABC:

$$\tan 15^\circ = \frac{150}{AB}$$

$$AB = \frac{150}{\tan 15^\circ}$$

$$\approx 559.808$$

$$x = \sqrt{AB^2 + BE^2 - 2(AB)(BE)\cos 45^\circ}$$

$$= \sqrt{559.808^2 + 850.692^2 - 2(559.808)(850.692)\cos 45^\circ}$$

$$\approx 603 \ ft$$

Exercise

Two ranger stations are on an east-west line 110 *mi* apart. A forest fire is located on a bearing $N\ 42^\circ\ E$ from the western station at A and a bearing of $N\ 15^\circ\ E$ from the eastern station at B. How far is the fire from the western station?

Solution

$$\angle BAC = 90^\circ - 42^\circ$$

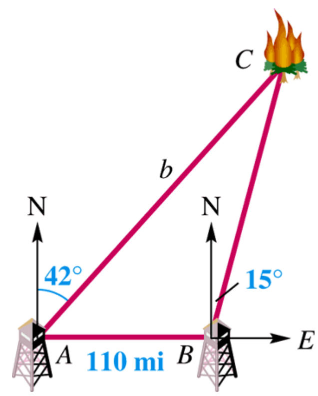
$$= 48^\circ$$

$$\angle ABC = 90^\circ + 15^\circ$$

$$= 105^\circ$$

$$\angle C = 180^\circ - 105^\circ - 48^\circ$$

$$= 27^\circ$$



$$\frac{b}{\sin 105^\circ} = \frac{110}{\sin 27^\circ} \qquad \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$b = \frac{110 \sin 105^\circ}{\sin 27^\circ}$$

$$b \approx 234 \ mi$$

The fire is about **234** *miles* from the western station.

Exercise

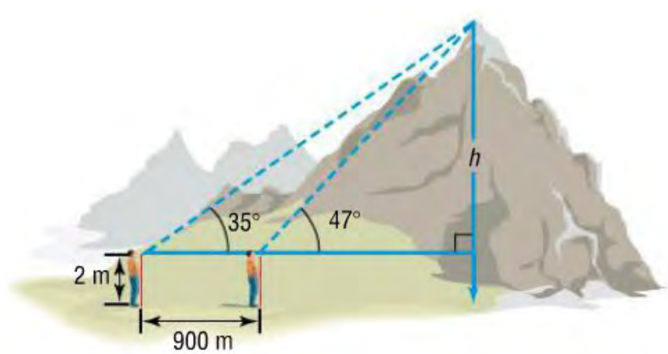
To measure the height of a mountain, a surveyor takes two sightings of the peak at a distance 900 meters apart on a direct line to the mountain. The first observation results in an angle of elevation of 47°, and the second results in an angle of elevation of 35°. If the transit is 2 *meters* high, what is the height h of the mountain?

Solution

$$h = 2 + \frac{900 \tan 35^\circ \tan 47^\circ}{\tan 47^\circ - \tan 35^\circ}$$

$$\approx 2 + 1815.86$$

$$\approx 1817.86 \text{ m}$$



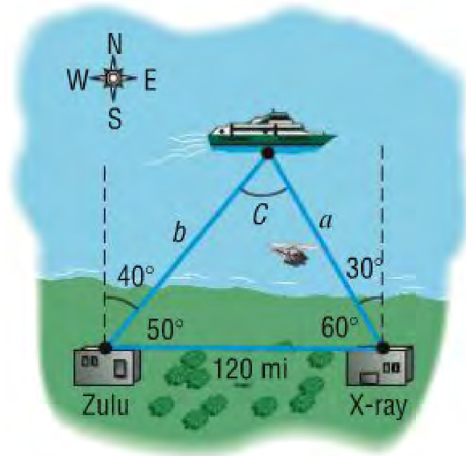
Exercise

A Station Zulu is located 120 *miles* due west of Station X-ray. A ship at sea sends an *SOS* call that is received by each station. The call to Station Zulu indicates that the bearing of the ship from Zulu is $N\ 40°\ E$. The call to Station *X*-ray indicates that the bearing of the ship from *X*-ray is $N\ 30°\ W$.

a) How far is each station from the ship?

b) If a helicopter capable of flying 200 *miles* per *hour* is dispatched from the nearest station to the ship, how long will it take to reach the ship?

Solution



a) $C = 180° - 50° - 60° = 70°$

$$b = \frac{120 \sin 60°}{\sin 70°}$$

$$\approx 110.59\ mi$$

$$a = \frac{120 \sin 50°}{\sin 70°}$$

$$\approx 97.82\ mi$$

Station Zulu is about 111 *miles* from the ship

and Station *X*-ray is about 98 *miles* from the ship.

b) ***Given***: $v = 200\ mi / hrs$

$$t = \frac{d}{v} = \frac{97.82}{200}$$

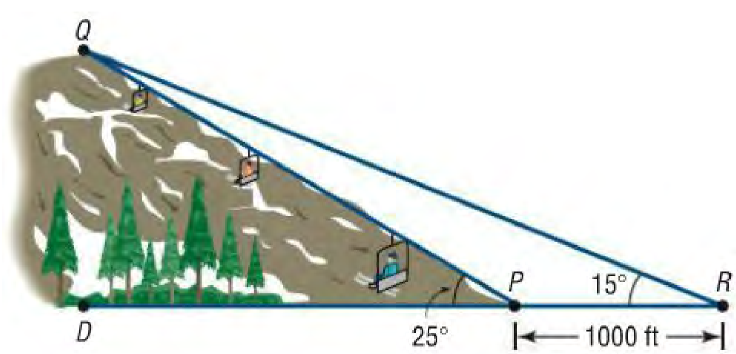
$$\approx 0.49\ hrs \quad \approx 29\ \text{min}$$

It will takes 29 *minutes* to reach Station *X*-ray.

Exercise

To find the length of the span of a proposed ski lift from *P* to *Q*, a surveyor measures $\angle DPQ$ to be $25°$ and then walks back a distance of 1000 *feet* to *R* and measures $\angle DRQ$ to be $15°$. What is the distance from *P* to *Q*.

Solution



$\angle RPQ = 180° - 25° = 155°$

$\angle PQR = 180° - 155° - 15° = 10°$

$$\frac{|PQ|}{\sin 15°} = \frac{|PR|}{\sin 10°}$$

$$|PQ| = \frac{10^3 \sin 15°}{\sin 10°}$$

$$\approx 1,490.5\ ft$$

Exercise

The highest bridge in the world is the bridge over the Royal Gorge of the Arkansas River in Colorado, sightings to the same point at water level directly under the bridge are taken from each side of the 880–foot–long bridge. How high is the bridge?

Solution

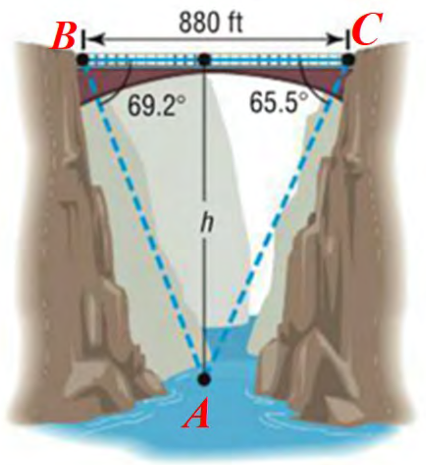
$$A = 180° - 69.2° - 65.5°$$
$$\underline{= 45.3°}$$

$$\frac{b}{\sin 65.5°} = \frac{880}{\sin 45.3°}$$

$$c = \frac{880 \sin 65.5°}{\sin 45.3°}$$
$$\underline{\approx 1126.57 \ \ ft}$$

$$\sin 69.2° = \frac{h}{c}$$

$$\boldsymbol{h} = 1126.57 \sin 69.2°$$
$$\underline{\approx 1,053.15 \ \ ft}$$



Exercise

Find the area of the triangle $b = 1, \quad c = 3, \quad A = 80°$

Solution

$$K = \frac{1}{2} bc \sin A$$
$$= \frac{1}{2}(1)(3)\sin 80°$$
$$\underline{\approx 1.48 \ \ unit^2}$$

Exercise

Find the area of the triangle $b = 4, \quad c = 1, \quad A = 120°$

Solution

$$K = \frac{1}{2} bc \sin A$$
$$= \frac{1}{2}(4)(1)\sin 120°$$
$$\underline{\approx 1.732 \ \ unit^2}$$

Exercise

Find the area of the triangle $a = 2, \quad c = 1, \quad B = 10°$

Solution

$$K = \frac{1}{2}ac\sin B$$

$$= \frac{1}{2}(2)(1)\sin 10°$$

$$\approx 0.174 \ \ unit^2$$

Exercise

Find the area of the triangle $a = 3, \quad c = 2, \quad B = 110°$

Solution

$$K = \frac{1}{2}ac\sin B$$

$$= \frac{1}{2}(3)(2)\sin 110°$$

$$\approx 2.819 \ \ unit^2$$

Exercise

Find the area of the triangle $a = 8, \quad b = 6, \quad C = 30°$

Solution

$$K = \frac{1}{2}ab\sin C$$

$$= \frac{1}{2}(8)(6)\sin 30°$$

$$= 12 \ \ unit^2$$

Exercise

Find the area of the triangle $a = 3, \quad b = 4, \quad C = 60°$

Solution

$$K = \frac{1}{2}(3)(4)\sin 60° \qquad\qquad K = \frac{1}{2}ab\sin C$$

$$\approx 5.196 \ \ unit^2$$

Exercise

Find the area of the triangle $a = 6, \quad b = 4, \quad C = 60°$

Solution

$$K = \frac{1}{2}(6)(4)\sin 60° \qquad\qquad K = \frac{1}{2}ab\sin C$$

$$\approx 10.392 \ \ unit^2$$

Exercise

Find the area of the triangle $a = 4, \quad b = 5, \quad c = 7$

Solution

$$s = \frac{1}{2}(4+5+7) \qquad\qquad s = \frac{1}{2}(a+b+c)$$

$$= 8 \ \ unit$$

$$K = \sqrt{8(8-4)(8-5)(8-7)} \qquad\qquad K = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{8(4)(3)(1)}$$

$$= \sqrt{96}$$

$$\approx 9.8 \ \ unit^2$$

Exercise

Find the area of the triangle $a = 12, \quad b = 13, \quad c = 5$

Solution

$$s = \frac{1}{2}(12+13+5) \qquad\qquad s = \frac{1}{2}(a+b+c)$$

$$= 15 \ \ unit$$

$$K = \sqrt{15(15-12)(15-13)(15-5)} \qquad\qquad K = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{15(3)(2)(10)}$$

$$= 30 \ \ unit^2$$

Exercise

Find the area of the triangle $a = 3, \quad b = 3, \quad c = 2$

Solution

$$s=\frac{1}{2}(3+3+2) \qquad s=\frac{1}{2}(a+b+c)$$

$$\underline{=4 \ \ unit}$$

$$K=\sqrt{4(4-3)(4-3)(4-2)} \qquad K=\sqrt{s(s-a)(s-b)(s-c)}$$

$$\underline{\approx 2.83 \ \ unit^2}$$

Exercise

Find the area of the triangle $a=4, \quad b=5, \quad c=3$

Solution

$$s=\frac{1}{2}(4+5+3) \qquad s=\frac{1}{2}(a+b+c)$$

$$\underline{=6 \ \ unit}$$

$$K=\sqrt{6(6-4)(6-5)(6-3)} \qquad K=\sqrt{s(s-a)(s-b)(s-c)}$$

$$=\sqrt{6(2)(1)(3)}$$

$$\underline{=6 \ \ unit^2}$$

Exercise

Find the area of the triangle $a=5, \quad b=8, \quad c=9$

Solution

$$s=\frac{1}{2}(5+8+9) \qquad s=\frac{1}{2}(a+b+c)$$

$$\underline{=11 \ \ unit}$$

$$K=\sqrt{11(11-5)(11-8)(11-9)} \qquad K=\sqrt{s(s-a)(s-b)(s-c)}$$

$$=\sqrt{11(6)(3)(2)}$$

$$=\sqrt{96}$$

$$\underline{\approx 19.9 \ \ unit^2}$$

Exercise

Find the area of the triangle $a=2, \quad b=2, \quad c=2$

Solution

$$s=\frac{1}{2}(2+2+2) \qquad s=\frac{1}{2}(a+b+c)$$

$= 3 \ \ unit$

$K = \sqrt{3(3-2)(3-2)(3-2)}$ $\qquad K = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{3}$

$\approx 1.732 \ \ unit^2$

Exercise

Find the area of the triangle $a = 4, \quad b = 3, \quad c = 6$

Solution

$s = \frac{1}{2}(4+3+6)$ $\qquad s = \frac{1}{2}(a+b+c)$

$= 6.5 \ \ unit$

$K = \sqrt{6.5(6.5-4)(6.5-3)(6.5-6)}$ $\qquad K = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{6.5(2.5)(3.5)(0.5)}$

$\approx 5.33 \ \ unit^2$

Exercise

The dimensions of a triangular lot are 100 *feet* by 50 *feet* by 75 *feet*. If the price of such land is $3 per square foot, how much does the lot cost?

Solution

$s = \frac{1}{2}(100+50+75)$ $\qquad s = \frac{1}{2}(a+b+c)$

$= 112.5 \ \ unit$

$K = \sqrt{112.5(112.5-100)(112.5-50)(112.5-75)}$ $\qquad K = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{112.5(12.5)(62.5)(37.5)}$

$\approx 1{,}815.46 \ \ ft^2$

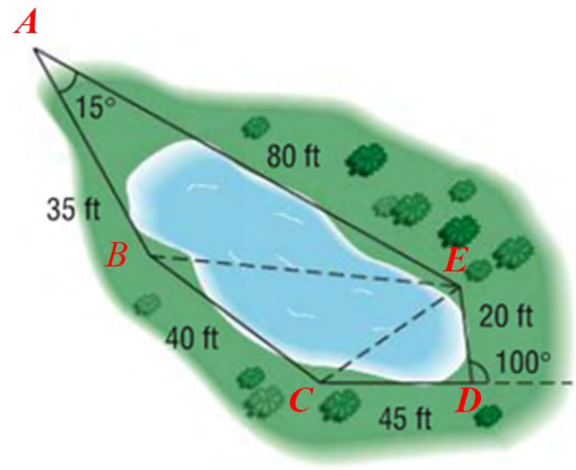
$Cost = (1{,}815.46)(\$3)$

$= \$5{,}446.38$

Exercise

To approximate the area of a lake, a surveyor walks around the perimeter of the lake. What is the approximate area of the lake?

Solution



Triangle ***ABE***:

$$|BE| = \sqrt{35^2 + 80^2 - 2(35)(80)\cos 15^\circ}$$

$$\approx 47.072\ ft$$

$$A_{\Delta ABE} = \frac{1}{2}(|AB|)(|BE|)\sin A$$

$$= \frac{1}{2}(80)(35)\sin 15^\circ$$

$$\approx 362.3\ ft^2$$

Triangle ***CDE***:

$$D = 180^\circ - 100^\circ = 80^\circ$$

$$|CE| = \sqrt{45^2 + 20^2 - 2(45)(20)\cos 80^\circ}$$

$$a = \sqrt{e^2 + b^2 - 2eb\cos A}$$

$$\approx 45.961\ ft$$

$$A_{\Delta CDE} = \frac{1}{2}(|CD|)(|DE|)\sin D$$

$$= \frac{1}{2}(45)(20)\sin 80^\circ$$

$$\approx 443.2\ ft^2$$

Triangle ***BCE***:

$$s = \frac{1}{2}(40 + 45.961 + 47.072)$$

$$\approx 66.5$$

$$A_{\Delta BCE} = \sqrt{66.5(66.5 - 40)(66.5 - 45.961)(66.5 - 47.072)}$$

$$\approx 839.6\ ft^2$$

$$A_{Total} = A_{\Delta ABE} + A_{\Delta CDE} A_{\Delta BCE}$$

$$= 362.3 + 443.2 + 839.6$$

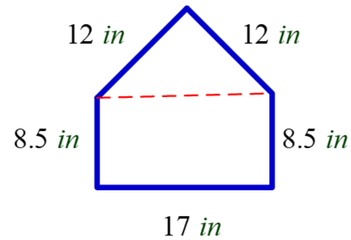
$$\approx 1,645.1\ ft^2$$

Exercise

The dimensions of home plate at any major league baseball stadium are shown. Find the area of home plate

Solution

Divide the home plate into a rectangle and triangle.



$$A_{rectangle} = 17 \times 8.5$$

$$= 144.5 \ ft^2$$

$$s = \tfrac{1}{2}(12 + 12 + 17) \qquad s = \tfrac{1}{2}(a+b+c)$$

$$= 20.5 \ unit$$

$$A_{triangle} = \sqrt{20.5(20.5-12)(20.5-12)(20.5-17)}$$

$$= \sqrt{20.5(8.5)(8.5)(3.5)}$$

$$= \sqrt{5{,}183.9375}$$

$$\approx 72 \ in^2$$

$$A_{total} = A_{\text{rectangle}} + A_{triangle}$$

$$\approx 144.5 + 72$$

$$\approx 216.5 \ in^2$$

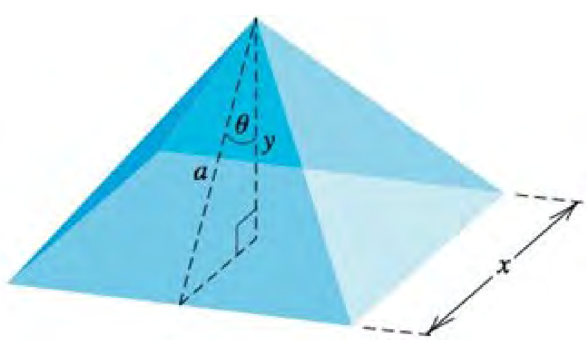
Exercise

A pyramid has a square base and congruent triangular faces. Let θ be the angle that the altitude a of a triangular face makes with the altitude y of the pyramid, and let x be the length of a side.

a) Express the total surface area S of the four faces in terms of a and θ.

b) The volume V of the pyramid equals one-third the area of the base times the altitude. Express V in terms of a and θ.

Solution



a) $\sin\theta = \dfrac{\frac{1}{2}x}{a}$

$$x = 2a\sin\theta$$

Area of one face is:

$$= \tfrac{1}{2}(base)(height) = \tfrac{1}{2}xa$$

$$= \tfrac{1}{2}(2a\sin\theta)(a)$$

$$= a\sin^2\theta$$

Total surface area: $\boxed{S = 4a\sin^2\theta}$

b) $V = \frac{1}{3}(\textit{base area})(\textit{height}) = \frac{1}{3}x^2 y$

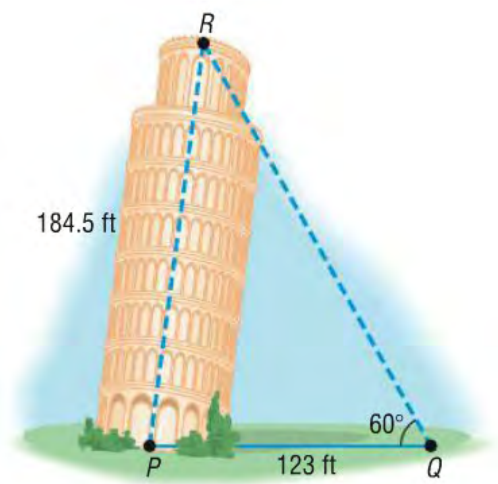
$$\cos\theta = \frac{y}{a} \quad \rightarrow \quad y = a\cos\theta$$

$$V = \frac{1}{3}(2a\sin\theta)^2(a\cos\theta)$$

$$\boxed{= \frac{4}{3}a^3\sin^2\theta\cos\theta}$$

Exercise

The famous Leaning Tower of Pisa was originally 184.5 *feet* high. At a distance of 123 *feet* from the base if the tower, the angle of elevation to the top of the tower is found to be 60°. Find the $\angle RPQ$ indicated in the figure. Also, find the perpendicular distance from R to PQ.



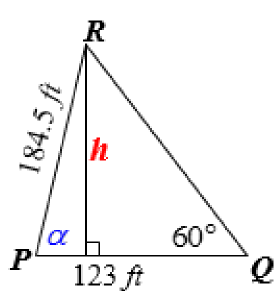
Solution

Let h: *height*

$$\frac{\sin R}{123} = \frac{\sin 60°}{184.5}$$

$$R = \sin^{-1}\left(\frac{123\sin 60°}{184.5}\right)$$

$$\boxed{\approx 35.3°}$$



$$\angle RPQ = 180° - 60° - 35.3°$$

$$\boxed{\approx 84.7°}$$

$$\sin 84.7° = \frac{h}{184.5}$$

$$h = 184.5\sin 84.7°$$

$\approx 183.71\ ft$

Exercise

If a mountaintop is viewed from a point P due south of the mountain, the angle of elevation is α. If viewed from a point Q that is d miles cast of P, the angle of elevation is β.

a) Show that the height h of the mountain is given by $h = \dfrac{d \sin\alpha \sin\beta}{\sqrt{\sin^2\alpha - \sin^2\beta}}$

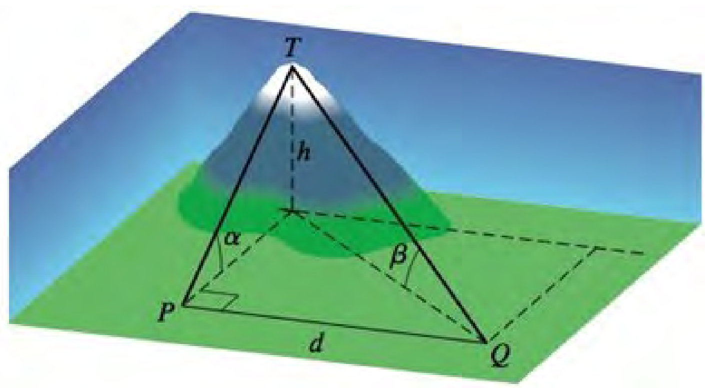
b) If $\alpha = 30°, \quad \beta = 20°, \quad and \quad d = 10\ mi$, approximate h.

Solution

a) Let $\overline{PT} = q \quad \& \quad \overline{QT} = p$

$\sin\alpha = \dfrac{h}{q} \ \rightarrow\ q = \dfrac{h}{\sin\alpha}$

$\sin\beta = \dfrac{h}{p} \ \rightarrow\ p = \dfrac{h}{\sin\beta}$



ΔTPQ: is right triangle at P:

$$d^2 + q^2 = p^2$$

$$d^2 + \left(\frac{h}{\sin\alpha}\right)^2 = \left(\frac{h}{\sin\beta}\right)^2$$

$$h^2\left(\frac{1}{\sin^2\beta} - \frac{1}{\sin^2\alpha}\right) = d^2$$

$$h^2\left(\frac{\sin^2\alpha - \sin^2\beta}{\sin^2\beta\sin^2\alpha}\right) = d^2$$

$$h^2 = \frac{d^2\sin^2\beta\sin^2\alpha}{\sin^2\alpha - \sin^2\beta}$$

$$h = \frac{d\ \sin\beta\sin\alpha}{\sqrt{\sin^2\alpha - \sin^2\beta}} \qquad \checkmark$$

b) **Given**: $\alpha = 30°, \quad \beta = 20°, \quad and \quad d = 10\ mi$

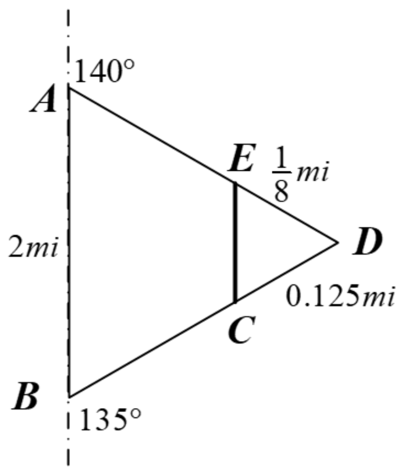
$$h = \frac{10\ \sin 20°\sin 30°}{\sqrt{\sin^2 30° - \sin^2 20°}}$$

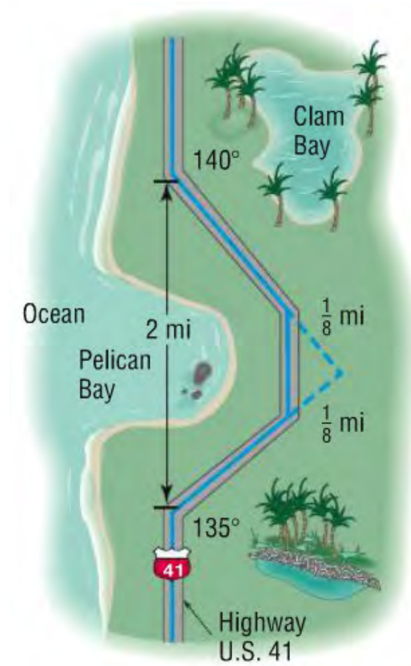
$$\approx 4.69\ \ mi$$

Exercise

A highway whose primary directions are north–south, is being constructed along the west coast of Florida. Near Naples, a bay obstructs the straight path of the road. Since the cost of a bridge is prohibitive, engineers decide to go around the bay. The path that they decide on and the measurements taken as shown in the picture. What is the length of highway needed to go around the bay?

Solution





$$A = 180° - 140°$$
$$\underline{= 40°}$$

$$B = 180° - 135°$$
$$\underline{= 45°}$$

$$D = 180° - 40° - 45°$$
$$\underline{= 95°}$$

$$\frac{BD}{\sin 40°} = \frac{2}{\sin 95°}$$

$$BD = \frac{2\sin 40°}{\sin 95°}$$
$$\underline{\approx 1.29\ mi}$$

$$BC = 1.29 - .125$$
$$\underline{\approx 1.165\ mi}$$

$$\frac{AD}{\sin 45°} = \frac{2}{\sin 95°}$$

$$AD = \frac{2\sin 45°}{\sin 95°}$$
$$\underline{\approx 1.42\ mi}$$

$$BC = 1.42 - .125$$
$$\underline{\approx 1.295\ mi}$$

$$CE = \sqrt{(0.125)^2 + (0.125)^2 - 2(0.125)^2 \cos 95°}$$
$$\underline{\approx 0.184\ mi}$$

The approximation length of highway needed to go around the bay:

$$1.165 + 1.295 + 0.184 \ \underline{\approx 2.64\ \ mi}$$

Exercise

Derive the Mollweide's formula: $\frac{a-b}{c}=\frac{\sin\left[\frac{1}{2}(A-B)\right]}{\cos\left(\frac{1}{2}C\right)}$

Solution

$$\frac{a-b}{c}=\frac{a}{c}-\frac{b}{c}$$

$$=\frac{\sin A}{\sin C}-\frac{\sin B}{\sin C}$$

$$=\frac{\sin A-\sin B}{\sin C}$$

$$=\frac{2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)}{\sin\left(2\cdot\frac{C}{2}\right)} \qquad \sin A-\sin B=2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

$$=\frac{2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)}{2\sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)} \qquad \sin 2\alpha=2\sin\alpha\cos\alpha$$

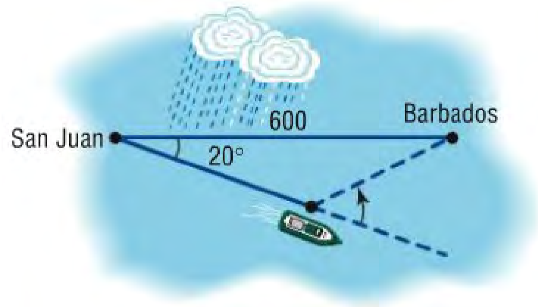
$$=\frac{\sin\left(\frac{A-B}{2}\right)}{\cos\left(\frac{C}{2}\right)}\frac{\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)}{\sin\left(\frac{C}{2}\right)} \qquad \cos A=\sin\left(\frac{\pi}{2}-A\right)$$

$$=\frac{\sin\left(\frac{A-B}{2}\right)}{\cos\left(\frac{C}{2}\right)}\frac{\sin\left(\frac{C}{2}\right)}{\sin\left(\frac{C}{2}\right)}$$

$$=\frac{\sin\left(\frac{A-B}{2}\right)}{\cos\left(\frac{C}{2}\right)} \quad \checkmark$$

Exercise

A cruise ship maintains an average speed of 15 *knots* in going from San Juan, Puerto Rico, to Barbados, West Indies, a distance of 600 *nautical miles*. To avoid a tropical storm, the captain heads out to San Juan in a direction of 20° off a direct heading to Barbados. The captain maintains the 15–knots speed for 10 *hours*, after which time the path to Barbados becomes clear of storms.



a) Through what angle should the captain turn to head directly to Barbados?

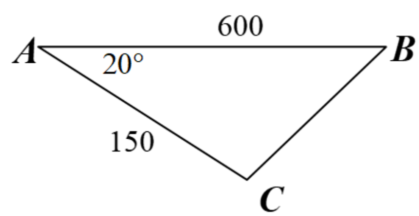
b) Once the turn is made, how long will it be before the ship reaches Barbados if the same 15–*knot* speed is maintained?

Solution

a) After 10 *hrs*., the ship travels $15\times 10=150$ *nautical miles*

$$c=\sqrt{600^2+150^2-2(600)(150)\cos 20^\circ}$$

$$\approx 461.9 \text{ nautical miles}$$



$$C=\cos^{-1}\frac{150^2+461.9^2-600^2}{2(150)(461.9)}$$

$$\approx 153.6^\circ$$

The captain needs to turn the ship through an angle of:

$$180^\circ - 153.6^\circ = 26.4^\circ$$

b) $t=\dfrac{461.9}{15}$

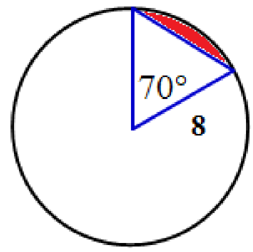
$$\approx 30.8 \text{ hrs}$$

The total time for the trip will be about $10+30.8 \approx 40.8 \text{ hrs}$

Exercise

Find the area of the segment (shaded in blue in the figure) of a circle whose radius is 8 *feet*, formed by a cebtral angle of $70°$

Solution



$$\theta = 70° \frac{\pi}{180°}$$

$$= \frac{7\pi}{18}$$

$$A_{sector} = \frac{1}{2} 8^2 \left(\frac{7\pi}{18}\right) \qquad A_{sector} = \frac{1}{2} r^2 \theta$$

$$= \frac{112\pi}{9}\ ft^2$$

$$A_{triangle} = \frac{1}{2} 8^2 \sin 70° \qquad A_{triangle} = \frac{1}{2} ab \sin\theta$$

$$\approx 30.07\ ft^2$$

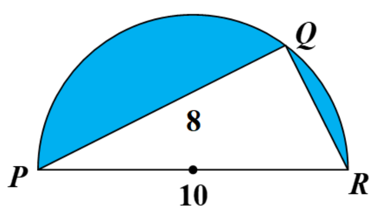
$$A_{segment} = \frac{112\pi}{9} - 30.07$$

$$\approx 9.03\ ft^2$$

Exercise

Find the area of the shaded region enclosed in a semicircle of diameter 10 *inches*. The length of the chord PQ is 8 *inches*.

Solution



$$A_{semicircle} = \frac{1}{2} \pi r^2$$

$$= \frac{25\pi}{2}\ in^2$$

$$\overline{QR} = \sqrt{10^2 - 8^2} = 6$$

$$A_{triangle} = \frac{1}{2} 6(8)$$

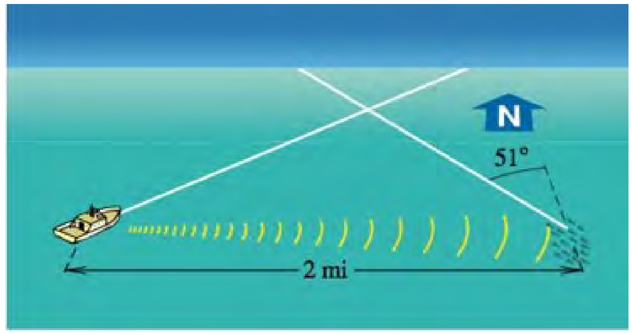
$$= 24\ in^2$$

$$A_{shaded} = \frac{25\pi}{2} - 24$$

$$= 15.27\ in^2$$

Exercise

A commercial fishing boat uses sonar equipment to detect a school of fish 2 *miles* east of the boat and traveling in the direction of $N\ 51°\ W$ at a rate of 8 mi/hr



a) The boat travels at 20 mi/hr, approximate the direction it should head to intercept the school of fish.

b) Find, to the nearest minute, the time it will take the boat to reach the fish.

Solution

a) $B = 90° - 51°$

$\quad = 39°$

Boat distance: $a = 20t$

School fish distance: $b = 8t$

$$\frac{\sin A}{a} = \frac{\sin 39°}{b}$$

$$\frac{\sin A}{20t} = \frac{\sin 39°}{8t}$$

$$A = \sin^{-1}\left(\frac{5}{2}\sin 39°\right)$$

$$\approx 14.6°$$

$90° - 14.6° = 75.4°$; the boat should travel in the (approximate) direction $N\ 75.4°\ E$

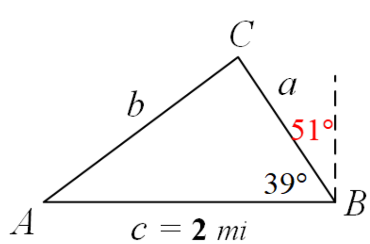
b) $C = 180° - 14.6° - 39°$

$\quad \approx 126.4°$

$$\frac{a}{\sin 14.6} = \frac{2}{\sin 126.4°}$$

$$a = \frac{2}{\sin 126.4°}\sin 14.6$$

$$\approx 1.56\ mi$$



$$t = \frac{a}{20}$$

$$\approx \frac{1.56}{20}$$

$$\approx 0.08 hr$$

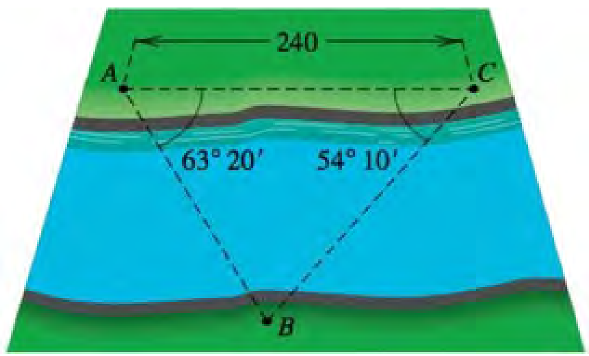
$$\approx 5\ \text{min}$$

Exercise

To find the distance between two points *A* and *B* that lie on opposite banks of a river, a surveyor lays off a line segment *AC* of length 240 *yards* along one bank and determines that the measures of $\angle BAC$ and $\angle ACB$ are $63°\ 20'$ and $54°\ 10'$, respectively.

Approximate the distance between *A* and *B*.

Solution



$$\angle B = 180° - 63°\ 20' - 54°\ 10'$$

$$= 62°\ 30'$$

$$\frac{|AB|}{\sin(54°\ 10')} = \frac{240}{\sin(62°\ 30')}$$

$$|AB| = \frac{240\sin(54°\ 10')}{\sin(62°\ 30')}$$

$$\approx 219.4\ \ yds$$

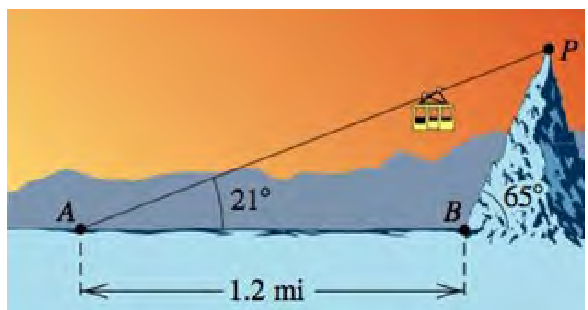
Exercise

A cable car carries passengers from a point *A*, which is 1.2 *miles* from a point *B* at the base of a mountain, to a point *P* at the top of the mountain. The angle of elevation of *P* from *A* and *B* are $21°$ and $65°$, respectively.

a) Approximate the distance between *A* and *P*.

b) Approximate the height of the mountain.

Solution



$$\angle ABP = 180° - 65° = 115°$$

$$\angle APB = 180° - 115° - 21° = 44°$$

a) $$\frac{|AP|}{\sin 115°} = \frac{1.2}{\sin 44°}$$

$$|AP| = \frac{1.2\sin 115°}{\sin 44°}$$

$$\approx 1.57\ \ mi$$

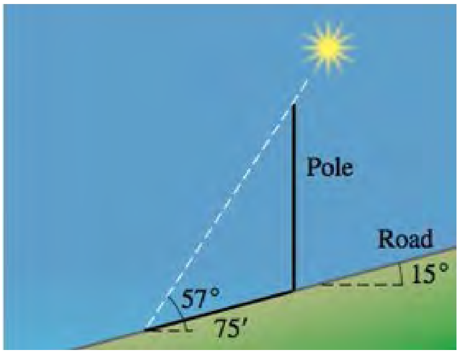
b) $$\sin 21° = \frac{h}{|AP|}$$

$$h \approx (1.57)\sin 21°$$

$$\approx 0.56\ \ mi$$

Exercise

A straight road makes an angle of 15° with the horizontal. When the angle of elevation of the sun is 57°, a vertical pole at the side of the road casts a shadow 75 *feet* long directly down the road. Approximate the length of the pole.



Solution

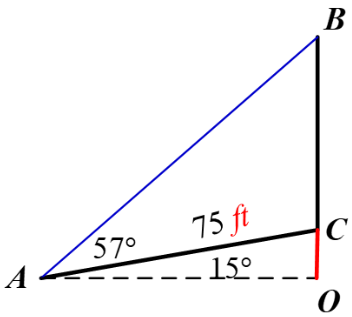
$B = 90° - 57°$

$\underline{= 33°}$

$$\frac{|BC|}{\sin(57° - 15°)} = \frac{75}{\sin 33°}$$

$$|BC| = \frac{75 \sin 42°}{\sin 33°}$$

$\underline{\approx 92.14 \ ft}$



Exercise

The angles of elevation of a balloon from two points *A* and *B* on level ground are 24° 10′ and 47° 40′, respectively. Points *A* and *B* are 8.4 *miles* apart, and the balloon is between the points, in the same vertical plane. Approximate the height of the balloon above the ground.

Solution

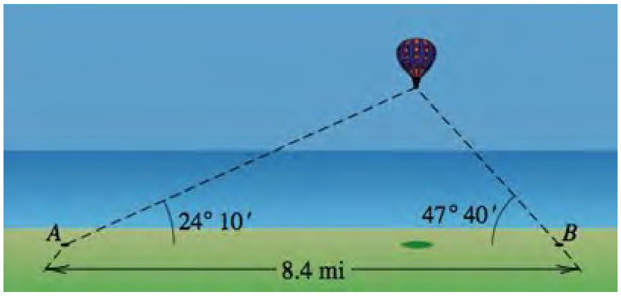
$\angle C = 180° - 24°\ 10' - 47°\ 40'$

$\underline{= 108°\ 10'}$

$$\frac{|AC|}{\sin(47°\ 40')} = \frac{84}{\sin(108°\ 10')}$$

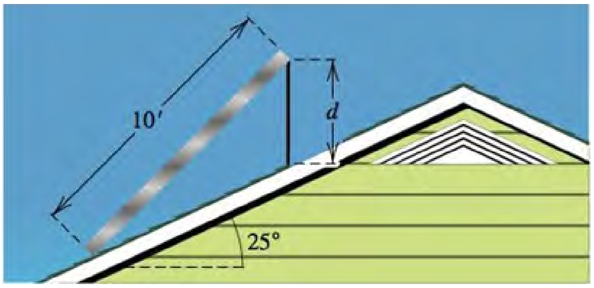
$$|AB| = \frac{84 \sin(47°\ 40')}{\sin(108°\ 10')}$$

$\underline{\approx 65.4 \ miles}$



Exercise

A solar panel 10 *feet* in width, which is to be attached to a roof that makes an angle of $25°$ with the horizontal. Approximate the length *d* of the brace that is needed for the panel to make an angle of $45°$ with the horizontal.



Solution

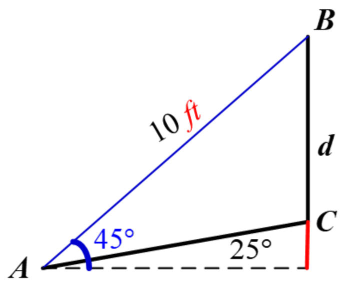
$$\angle B = 90° - 45°$$
$$= 45°$$

$$\angle C = 180° - (45° - 25°) - 45°$$
$$= 115°$$

$$\frac{d}{\sin(45° - 25°)} = \frac{10}{\sin(115°)}$$

$$d = \frac{10\ \sin(20°)}{\sin(115°)}$$

$$\approx 3.8\ ft$$

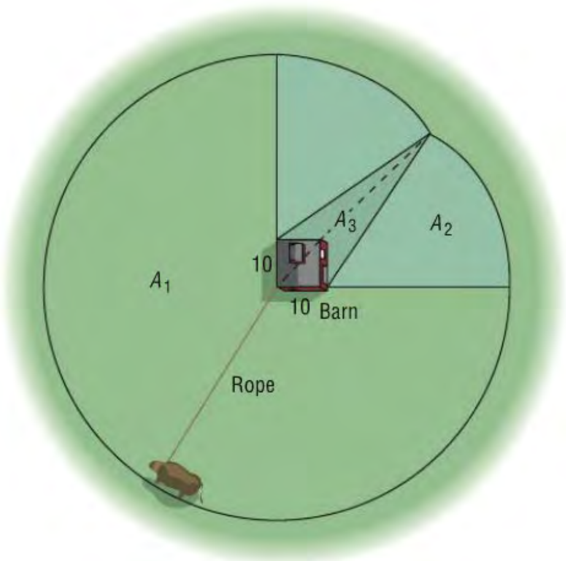


Exercise

A cow is tethered to one corner of a square barn, 10 *feet* by 10 *feet*, with a rope 100 feet long.

a) What is the maximum grazing area for the cow?

b) If the barn is rectangular, 10 *feet* by 20 *feet*, what is the maximum grazing area for the cow?



Solution

a) $A_1 = \frac{3}{4}(Area\ of\ \ the\ circle)$

$$= \frac{3}{4}\left(\pi r^2\right)$$

$$= \frac{3}{4}\left(\pi 10^4\right)$$

$$= 7{,}500\pi\ \ ft^2 \qquad \approx 23{,}561.94\ ft^2$$

$\angle ABC = 45° \quad (square) \quad AB = 10, \quad AC = 90$

$$\frac{\sin C}{10} = \frac{\sin 45°}{90}$$

$$\sin C = \frac{1}{9}\frac{\sqrt{2}}{2}$$

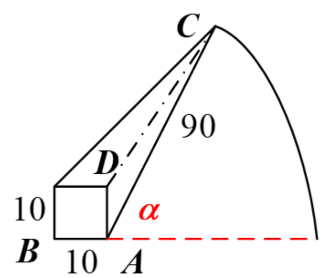
$$C = \sin^{-1}\left(\frac{\sqrt{2}}{18}\right) \approx 4.51°$$

$$\angle CAB \approx 180° - 45° - 4.51°$$

$$\approx 130.49°$$

$$\angle DAC = 130.49° - 90°$$

$$\approx 40.49°$$



$$A_3 = 2Area(ADC)$$

$$= 2 \cdot \frac{1}{2}(AD)(AC)\cos A$$

$$= (10)(90)\cos 40.49°$$

$$\approx 584.38 \ \ ft^2$$

$$\alpha = 180° - 130.49° \approx 49.51°$$

$$A_2 = \frac{1}{2}90^2\left(49.51°\frac{\pi}{180°}\right) \qquad A = \frac{1}{2}r^2\theta$$

$$\approx 3,499.66 \ \ ft^2$$

Total grazing area $\approx 23,561.94 + 584.38 + 2(3,499.66)$

$$\approx 31,146 \ \ ft^2$$

b) Let BC be the radius 90 center at B, where $B(0,\ 10)$ & $C(x,\ y)$

$$x^2 + (y-10)^2 = 90^2 \quad (\mathbf{1})$$

Let AC be radius 80 center at A, where $A(20,\ 0)$:

$$(x-20)^2 + y^2 = 80^2 \quad (\mathbf{2})$$

So the point C is the intersection of the 2 circles (**1**) & (**2**) using graphing tool:

$$C(77.705,\ 55.409)$$

$$A_1 = \frac{3}{4}(Area\ of\ \ the\ circle)$$

$$\approx 23,561.94\ ft^2$$

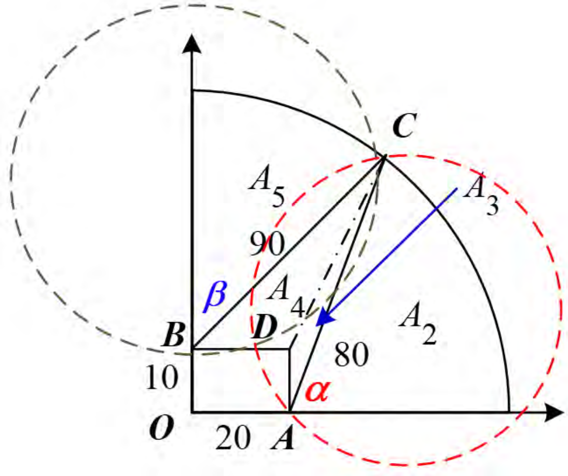
Let $D(20,\ 10)$

$$CD = \sqrt{(77.7-20)^2 + (55.4-10)^2}$$

$$\approx 73.4$$

$$\angle CAD = \cos^{-1}\frac{80^2+10^2-73.4^2}{2(80)(10)}$$

$$\approx 45.95°$$



$$A_3 = \frac{1}{2}(80)(10)\sin 45.95°$$

$$\approx 287.49\ ft^2$$

$$\alpha = 90° - 45.95°$$

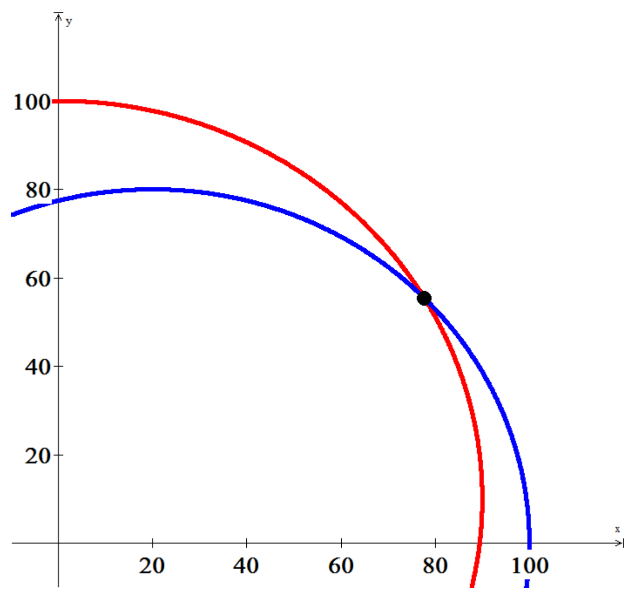
$$\approx 44.05°$$

$$A_2 = \frac{1}{2}r^2\theta = \frac{1}{2}80^2\left(44.05°\frac{\pi}{180°}\right)$$

$$\approx 2,460.22\ ft^2$$

$$\angle CBD = \cos^{-1}\frac{90^2+20^2-73.4^2}{2(90)(20)}$$

$$\approx 30.17°$$



$$A_4 = \frac{1}{2}(90)(20)\sin 30.17°$$

$$\approx 542.31\ ft^2$$

$$\alpha = 90° - 30.17°$$

$$\approx 59.83°$$

$$A_5 = \frac{1}{2}r^2\theta = \frac{1}{2}90^2\left(59.83°\frac{\pi}{180°}\right)$$

$$\approx 4,229.13\ ft^2$$

Total grazing area

$$\approx 23,561.94 + 2,460.22 + 287.49 + 542.31 + 4,229.13$$

$$\approx 31,081\ ft^2$$

Exercise

For any triangle, show that $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$ *where* $s=\frac{1}{2}(a+b+c)$

Solution

$$\cos\frac{C}{2}=\sqrt{\frac{1+\cos C}{2}}$$

$$=\sqrt{\frac{1}{2}\left(1+\frac{a^2+b^2-c^2}{2ab}\right)}$$

$$=\sqrt{\frac{2ab+a^2+b^2-c^2}{4ab}}$$

$$=\sqrt{\frac{(a+b)^2-c^2}{4ab}} \qquad x^2-y^2=(x-y)(x+y)$$

$$=\sqrt{\frac{(a+b+c)(a+b-c)}{4ab}} \qquad 2s=a+b+c \quad 2s-c=a+b$$

$$=\sqrt{\frac{2s(2s-c-c)}{4ab}}$$

$$=\sqrt{\frac{2s(2s-2c)}{4ab}}$$

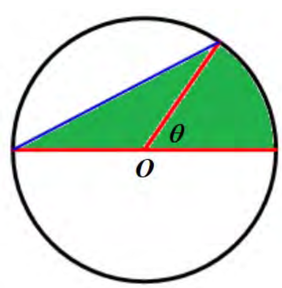
$$=\sqrt{\frac{4s(s-c)}{4ab}}$$

$$=\sqrt{\frac{s(s-c)}{ab}} \quad \checkmark$$

Exercise

The figure shows a circle of radius *r* with center at *O*. find the area *K* of the shaded region as a function of the central angle *θ*.

Solution



$$A_{sector}=\frac{1}{2}r^2\theta$$

$$A_{triangle}=\frac{1}{2}r^2\sin(\pi-\theta)$$

$$K=\frac{1}{2}r^2\theta+\frac{1}{2}r^2\sin(\pi-\theta)$$

$$=\frac{1}{2}r^2\left(\theta+\sin(\pi-\theta)\right) \qquad \sin(\pi-\theta)=\sin\theta$$

$$\boxed{=\frac{1}{2}r^2(\theta+\sin\theta)}$$

Exercise

Refer to the figure, in which a unit circle is drawn. The line segment *DB* is tangent to the circle and θ is acute.

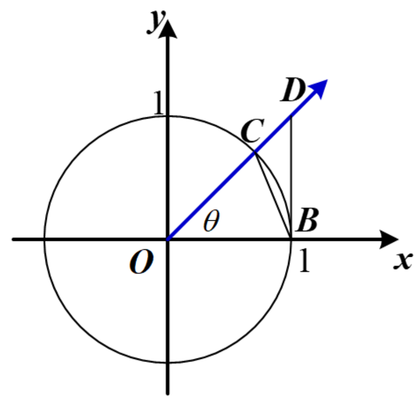
a) Express the area of ΔOBC in terms of $\sin\theta$ and $\cos\theta$.

b) Express the area of ΔOBD in terms of $\sin\theta$ and $\cos\theta$.

c) The area of the sector $\overset{\frown}{OBC}$ if the circle is $\frac{1}{2}\theta$, where θ is measured in *radians*. Use the results of part (*a*) and (*b*) and the fact that

$$Area\ \Delta OBC < Area\ \overset{\frown}{OBC} < Area\ \Delta OBD$$

To show that $1 < \dfrac{\theta}{\sin\theta} < \dfrac{1}{\cos\theta}$



Solution

a) Area of $\Delta OBC = \frac{1}{2}(1)(1)\sin\theta$

$$= \frac{1}{2}\sin\theta$$

b) $\tan\theta = \dfrac{\overline{BD}}{1} \rightarrow \overline{BD} = \tan\theta$

Area of $\Delta OBD = \frac{1}{2}\overline{OB}\times\overline{BD}$

$$= \frac{1}{2}(1)\tan\theta$$

$$= \frac{\sin\theta}{2\cos\theta}$$

c) $Area\ \Delta OBC < Area\ \overset{\frown}{OBC} < Area\ \Delta OBD$

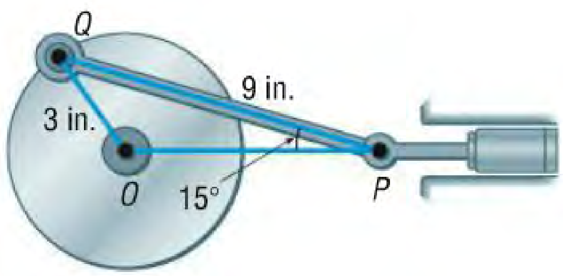
$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\frac{\sin\theta}{\cos\theta}$$

$$\sin\theta < \theta < \frac{\sin\theta}{\cos\theta} \qquad \times\frac{1}{\sin\theta}$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

Exercise

On a certain automobile, the crankshaft is 3 *inches* long and the connecting rod is 9 *inches* long. At the time when $\angle OPQ$ is 15°, how far is the piston *P* from the center *O* of the crankshaft?



Solution

$$\frac{\sin O}{9} = \frac{\sin 15^\circ}{3}$$

$$\hat{O} = \sin^{-1}\left(3\sin 15^\circ\right)$$

$$\approx 50.94^\circ$$

$$O = 50.94^\circ$$

$$Q = 180^\circ - 50.94^\circ - 15^\circ$$

$$= 114.06^\circ$$

$$\frac{q}{\sin 114.06^\circ} = \frac{3}{\sin 15^\circ}$$

$$q = \frac{3\sin 114.06^\circ}{\sin 15^\circ}$$

$$\approx 10.58\ in$$

$$O = 180^\circ - 50.94^\circ$$

$$= 129.06^\circ$$

$$Q = 180^\circ - 129.06^\circ - 15^\circ$$

$$= 35.94^\circ$$

$$\frac{q}{\sin 35.94^\circ} = \frac{3}{\sin 15^\circ}$$

$$q = \frac{3\sin 35.94^\circ}{\sin 15^\circ}$$

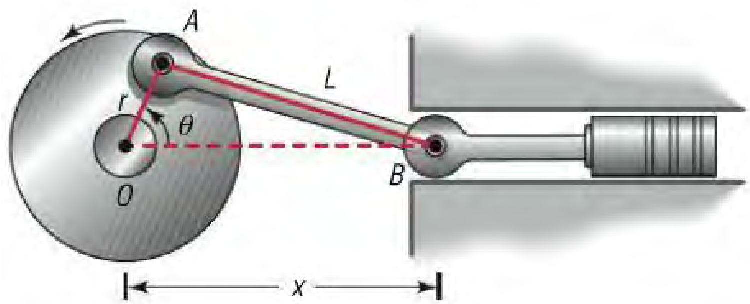
$$\approx 6.80\ in$$

The distance from the piston P to the center O of the crankshaft is approximately either 10.58 *inches* or 6.8 *inches*.

Exercise

Rod OA rotates about the fixed point O so that point A travels on a circle of radius r. Connected to point A is another rod AB of length $L > 2r$, and point B is connected to a piston. Show that the distance x between point O and point B is given by

$$x = r\cos\theta + \sqrt{r^2\cos^2\theta + L^2 - r^2}$$



Where θ is the angle of rotation of rod OA.

Solution

$$L^2 = r^2 + x^2 - 2rx\cos\theta$$ *Law of cosine*

$$x^2 - 2rx\cos\theta + r^2 - L^2 = 0$$

$$x = \frac{2r\cos\theta \pm \sqrt{4r^2\cos^2\theta - 4\left(r^2 - L^2\right)}}{2}$$ $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2r\cos\theta \pm 2\sqrt{r^2\cos^2\theta - r^2 + L^2}}{2}$$

$$\boxed{= r\cos\theta + \sqrt{r^2\cos^2\theta - r^2 + L^2}} \quad \checkmark$$

Exercise

Find the area of the segment of a circle whose radius is 5 *inches*, formed by a central angle of $40°$.

Solution

$$\theta = 40° \frac{\pi}{180°}$$

$$= \frac{2\pi}{9}$$

$$A_{\sec tor} = \frac{1}{2}5^2\left(\frac{2\pi}{9}\right) \qquad\qquad A_{\sec tor} = \frac{1}{2}r^2\theta$$

$$\boxed{= \frac{25\pi}{9}\ in^2}$$

$$A_{triangle} = \frac{1}{2}5^2\sin 40° \qquad\qquad A_{triangle} = \frac{1}{2}ab\sin\theta$$

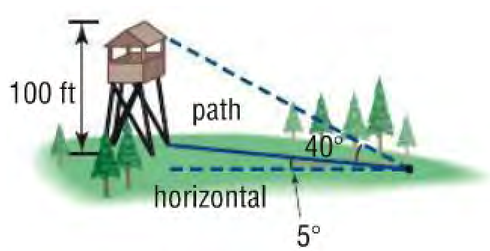
$$\boxed{= \frac{25}{2}\sin 40°\ \ in^2}$$

$$A_{segment} = \frac{25\pi}{9} - \frac{25}{2}\sin 40°$$

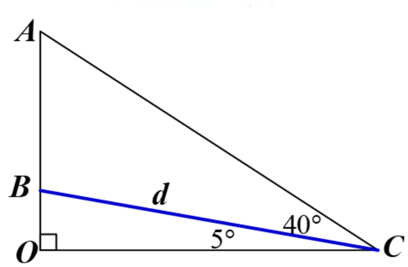
$$\boxed{\approx 0.69\ \ in^2}$$

Exercise

A forest ranger is walking on a path inclined at 5° to the horizontal directly toward a 100–*foot*–tall fire observation tower. The angle of elevation from the path to the top of the tower is 40°. How far is the ranger from the tower at this time?



Solution



$$\angle OCA = 40° + 5°$$

$$\boxed{= 45°}$$

$$A = 90° - 45°$$

$$\boxed{= 45°}$$

$$\angle ABC = B = 180° - 45° - 40°$$

$$\underline{= 95°}|$$

$$\frac{d}{\sin 45°} = \frac{100}{\sin 40°}$$

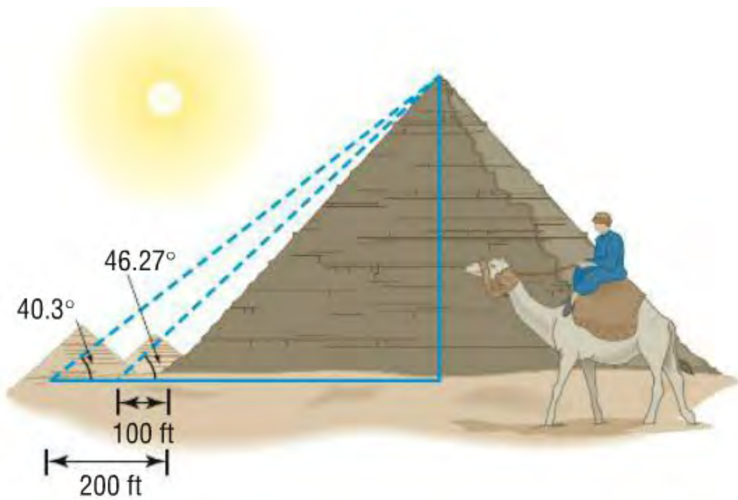
$$d = \frac{100 \sin 45°}{\sin 40°}$$

$$\underline{\approx 110.01 \ feet}|$$

The ranger distance is about **110.01** *feet* from the tower.

Exercise

One of the original Seven Wonders of the world, the Great Pyramid of Cheops was built about 2580 BC. Its original height was 480 *feet* 11 *inches*, but owing to the loss of its topmost stones, it is now shorter. Find the current height of the Great Pyramid using the information shown in the picture.



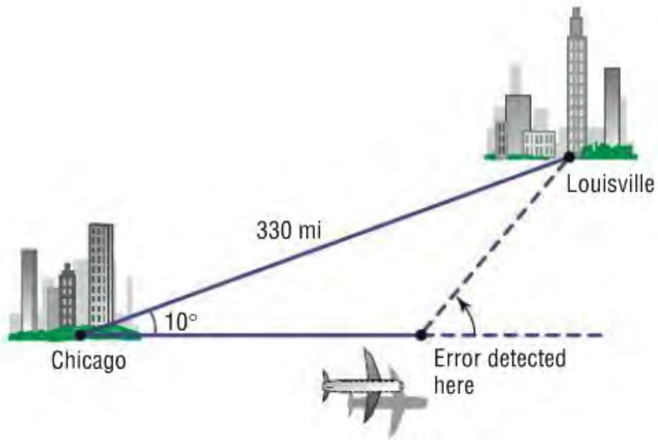
Solution

$$h = \frac{100 \tan 43.27° \tan 40.3°}{\tan 43.27° - \tan 40.3°} \qquad h = \frac{x \tan\alpha \tan\beta}{\tan\alpha - \tan\beta}$$

$$\underline{\approx 449.36 \ ft}|$$

Exercise

In attempting to fly from Chicago to Louisville, a distance of 330 *miles*, a pilot inadvertently took a course that was 10° in error.



a) If the aircraft maintains an average speed of 220 *miles* per *hours*, and if the error in direction is discovered after 15 *minutes*, through what angle should the pilot turn to head toward Louisville?

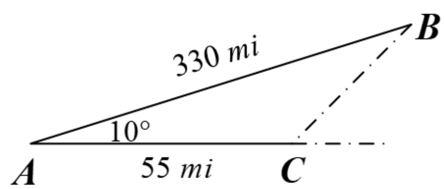
b) What new average speed should the pilot maintain so that the total time of the trip is 90 *minutes*?

Solution

a) After 15 *min*., the ship travels

$$15 \times 220 = 3,300 \times \frac{1\ hr}{60\ min}$$

$$= 55\ \ miles$$



$$c = \sqrt{330^2 + 55^2 - 2(330)(55)\cos 10°}$$

$$\approx 276\ miles$$

$$C = \cos^{-1} \frac{55^2 + 276^2 - 330^2}{2(55)(276)}$$

$$\approx 168°$$

The pilot needs to turn through an angle of:

$$180° - 1168°\ = 12°$$

b) The total time of the trip is 90 *min*

Since the pilot found that 25 *min* were used (*error*).

Then, there are $90 - 15 = 75\,min$ to complete the trip.

The plane must travel 276 miles in $75min \times \frac{1hr}{60min} = 1.25\ hrs$.

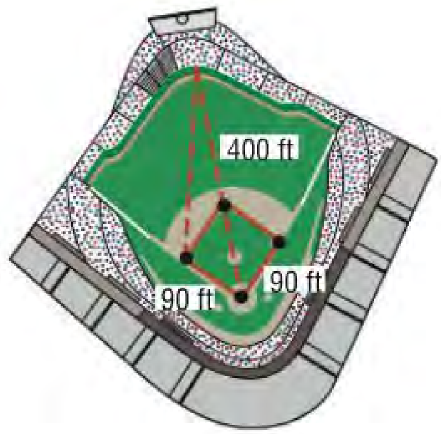
$$r = \frac{276}{1.25}$$

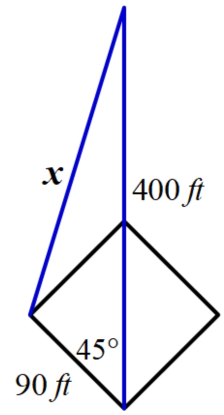
$$\approx 220.8\ \ mi / hr$$

The pilot must maintain a speed of 200.8 *mi/hr.* to complete the trip in 90 *min*.

Exercise

The distance from home plate to the fence in dead center is 400 *feet*. How far is it for the fence in dead center to third base?





Solution

$$x = \sqrt{400^2 + 90^2 - 2(400)(90)\cos 45^\circ}$$

$$= \sqrt{160000 + 8100 - 36000\sqrt{2}}$$

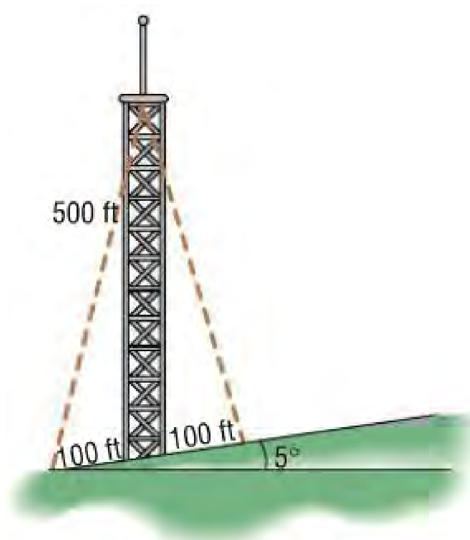
$$= 10\sqrt{1681 - 360\sqrt{2}}$$

$$\approx 342.33 \ ft$$

It is approximately 342.33 *feet* from dead center to third base.

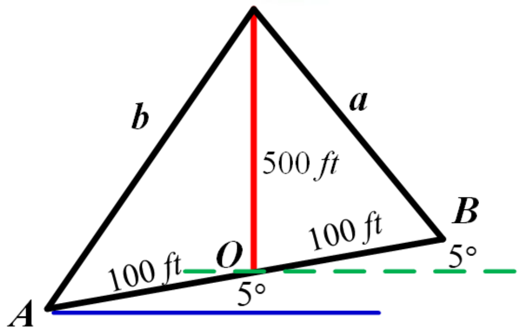
Exercise

A radio tower 500 *feet* high is located on the side of a hill with an inclination to the horizontal of 5°. How long should two guy wires be if they are to connect to the top of the tower and be secured at two points 100 *feet* directly above and directly below the base of the tower?



Solution

$$\angle BOC = 90° - 5°$$
$$= 85°$$
$$\angle AOC = 90° + 5°$$
$$= 95°$$



$$a = \sqrt{500^2 + 100^2 - 2(500)(100)\cos 85°}$$
$$\approx 501.28\ ft$$

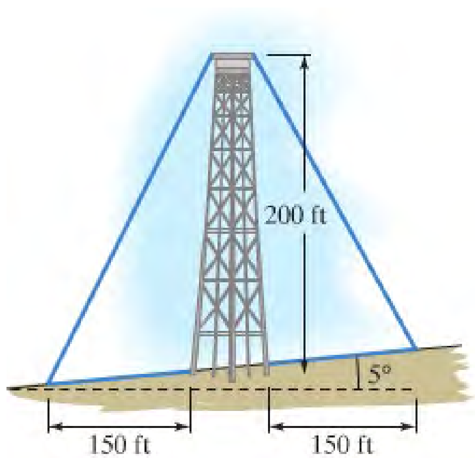
Right of the tower, it needs about **501.28** *feet* of wires

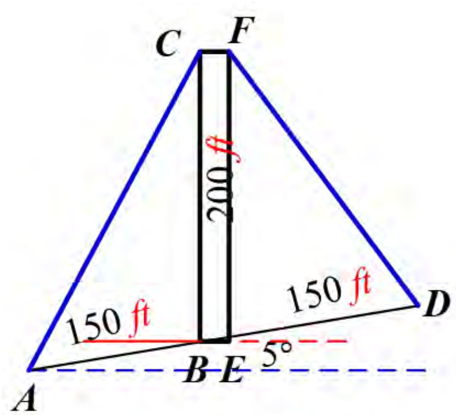
$$b = \sqrt{500^2 + 100^2 - 2(500)(100)\cos 95°}$$
$$\approx 518.38\ ft$$

Left of the tower, it needs about **518.38** *feet* of wires

Exercise

A 200-*foot* tower on the side of a hill that forms a 5° angle with the horizontal. Find the length of each of the two guy wires that are anchored 150 *feet* uphill and downhill from the tower's base and extend to the top of the tower.





Solution

$$B = 90° + 5° = 95°$$

$$b = \sqrt{200^2 + 150^2 - 2(200)(150)\cos 95°} \qquad b^2 = a^2 + c^2 - 2ac\cos B$$
$$\approx 260.2\ ft$$

$$E = 90° - 5° = 85°$$

$$e = \sqrt{200^2 + 150^2 - 2(200)(150)\cos 85°} \qquad e^2 = d^2 + f^2 - 2df\cos E$$
$$\approx 239.3\ ft$$

Exercise

When the angle of elevation of the sun is 62°, a telephone pole that is tilted at an angle of 8° directly away from the sun casts a shadow 20 *feet* long. Determine the length of the pole.

Solution

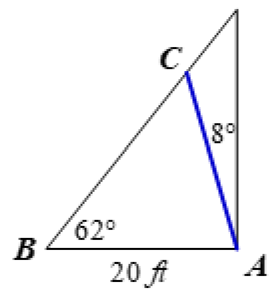
$A = 90^\circ - 8^\circ \ \underline{= 82^\circ|}$

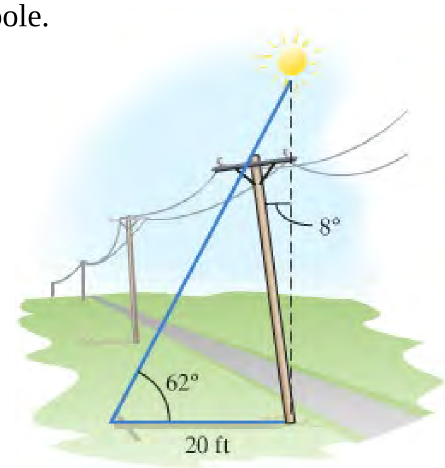
$C = 180^\circ - 82^\circ - 62^\circ \ \underline{= 36^\circ|}$

$\dfrac{|AC|}{\sin 62^\circ} = \dfrac{20}{\sin 82^\circ}$

$|AC| = \dfrac{20 \sin 62^\circ}{\sin 82^\circ}$

$\underline{\approx 30.0\ ft|}$



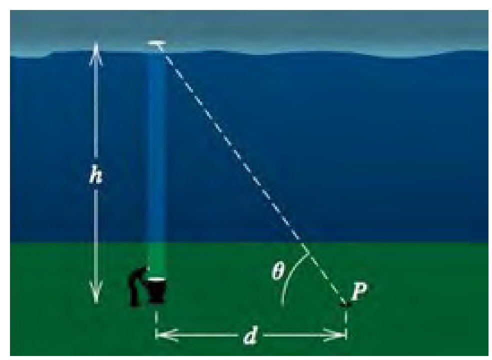


Exercise

To measure the height h of a cloud cover, a meteorology student directs a spotlight vertically upward from the ground. From a point P on level ground that is d meters from the spotlight, the angle of elevation θ of the light image on the clouds is then measured.

a) Express h in terms of d and θ

b) Approximate h if $d = 1000\ m$ and $\theta = 59^\circ$



Solution

a) $\tan\theta = \dfrac{h}{d}$

$\underline{h = d\tan\theta\ |}$

b) **Given**: $d = 1000\ m$ and $\theta = 59^\circ$

$h = 1,000 \tan 59^\circ$

$\underline{\approx 1,664.3\ m\ |}$

Exercise

A hot–air balloon is rising vertically. From a point on level ground 125 *feet* from the point directly under the passenger compartment, the angle of elevation to the balloon changes from 19.2° to 31.7°. How far does the balloon rise during this period?

Solution

$$\tan 19.2^\circ = \frac{y}{125}$$

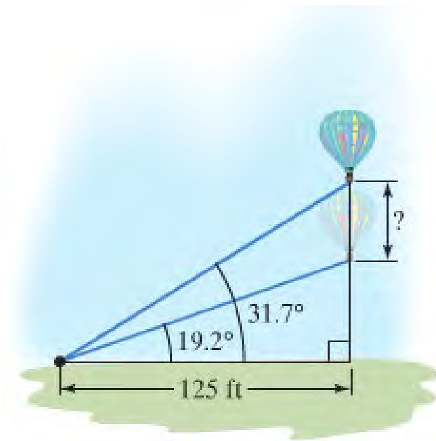
$$\underline{y = 125\tan 19.2^\circ}|$$

$$\tan 31.7^\circ = \frac{x+y}{125}$$

$$x = 125\tan 31.7^\circ - y$$

$$= 125\tan 31.7^\circ - 125\tan 19.2^\circ$$

$$\approx 33.7\ ft$$



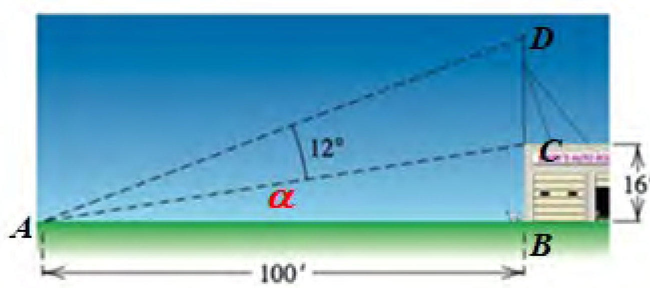
Exercise

A *CB* antenna is located on the top of a garage that is 16 *feet* tall. From a point on level ground that is 100 *feet* from a point directly below the antenna, the antenna subtends an angle of 12°. Approximate the length of the antenna.

Solution

$$\alpha = \tan^{-1}\frac{16}{100}$$

$$\approx 9.09^\circ$$



$$\tan\left(12^\circ + 9.09^\circ\right) = \frac{16 + |CD|}{100}$$

$$|CD| = 100\tan\left(12^\circ + 9.09^\circ\right) - 26$$

$$\approx 38.567\ ft$$

The length of the antenna is ≈ 39 *feet*

Exercise

A tsunami is a tidal wave caused by an earthquake beneath the sea. These waves can be more than 100 *feet* in height and can travel at great speeds. Engineers sometimes represent such waves by trigonometric expressions of the form $y = a\cos bt$ and use these representations to estimate the effectiveness of sea walls. Suppose that a wave has height $h = 50\ ft$ and period time 30 *minutes* and is traveling at the rate of $180\ ft\,/\,\sec$

a) Let $(x,\ y)$ be a point on the wave represented in the figure. Express *y* as a function of *t* if $y = 25\ ft$ when $t = 0$.

b) The wave length *L* is the distance between two successive crests of the wave. Approximate *L* in *feet*.

Solution

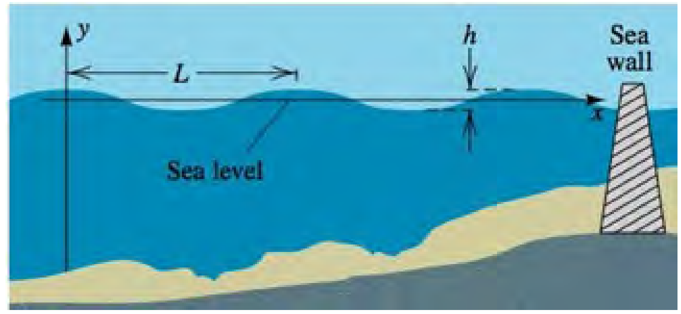
a) ***Given***: $y = a\cos bt \quad |A| = h, \quad P = L$

$a = |A| \boxed{= h}$

$P = \dfrac{2\pi}{b} = L$

$\boxed{b = \dfrac{2\pi}{L}}$

$\boxed{y = h\cos\dfrac{2\pi}{L}t}$



Given: $y = 25\ ft$ when $t = 0$

$25 = h\cos 0$

$\boxed{h = 25\ ft}$

$\boxed{y(t) = 25\cos\dfrac{2\pi}{L}t}$

Exercise

Two fire–lookout stations are 20 *miles* apart, with station *B* directly east of station *A*. Both stations spot fire on a mountain to the north. The bearing from station *A* to the fire is N50°E. The bearing from station *B* to the fire is N36°W. How far is the fire from station *A*?

Solution

$C = 180° - 40° - 54°$

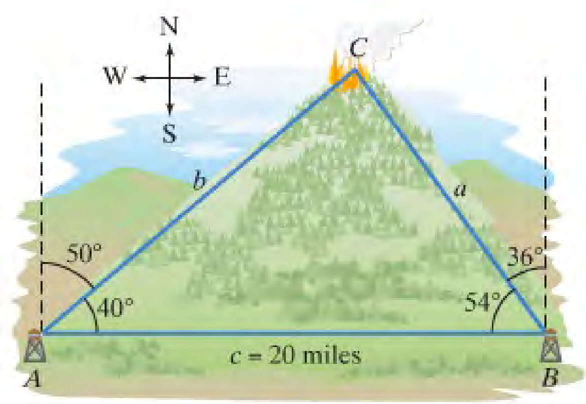
$\boxed{= 86°}$

$\dfrac{b}{\sin 54°} = \dfrac{20}{\sin 86°}$

$b = \dfrac{20\sin 54°}{\sin 86°}$

$\boxed{\approx 16.2\ miles}$

The fire is approximately 16.2 *miles* from station *A*.



Exercise

A 1200–yard–long sand beach and an oils platform in the ocean. The angle made with the platform from one end of the beach is 85° and from the other end is 76°. Find the distance of the oil platform from each end of the beach.

Solution

$C = 180° - 85° - 76°$

$\underline{= 19°}$

$\dfrac{b}{\sin 85°} = \dfrac{1200}{\sin 19°}$

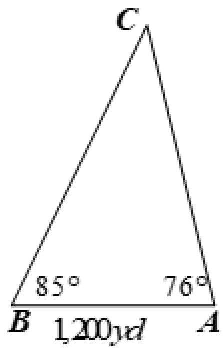
$b = \dfrac{1200 \sin 85°}{\sin 19°}$

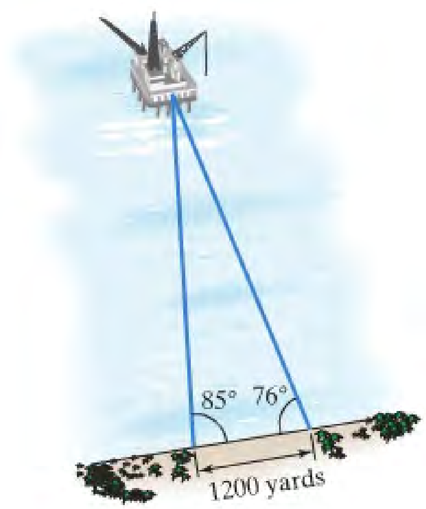
$\underline{\approx 3671.8 \ yds}$

$\dfrac{a}{\sin 76°} = \dfrac{1200}{\sin 19°}$

$a = \dfrac{1200 \sin 75°}{\sin 19°}$

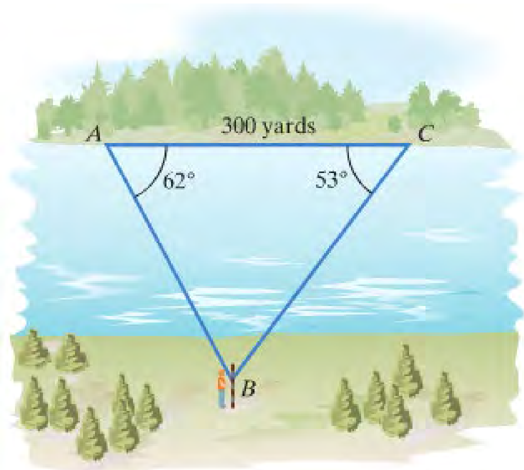
$\underline{\approx 3576.4 \ yds}$





Exercise

A surveyor needs to determine the distance between two points that lie on opposite banks of a river. 300 *yards* are measured along one bank. The angle from each end of this line segment to a point on the opposite bank are 62° and 53°. Find the distance between *A* and *B*.



Solution

$$B = 180° - 62° - 53°$$
$$\underline{= 65°}$$

$$\frac{|AB|}{\sin 53°} = \frac{300}{\sin 65°}$$

$$|AB| = \frac{300\sin 53°}{\sin 65°}$$
$$\underline{\approx 264.4 \ \ yds}$$

Exercise

A pine tree growing on a hillside makes a 75° angle with the hill. From a point 80 *feet* up the hill, the angle of elevation to the top of the tree is 62° and the angle of depression to the bottom is 23°. Find the height of the tree.

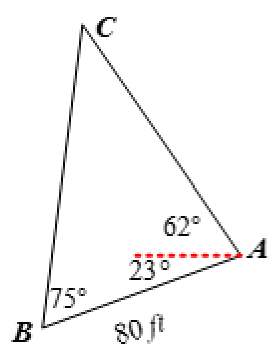
Solution

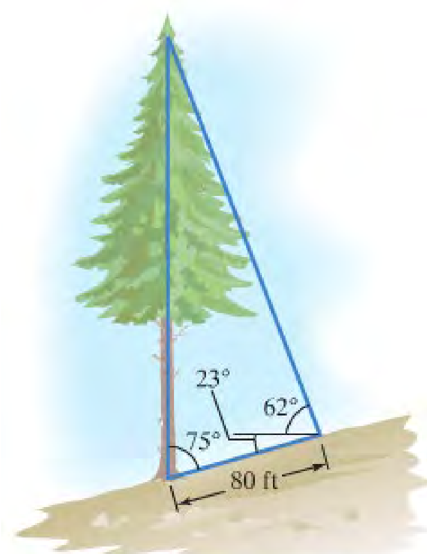
$$A = 62° + 23°$$
$$\underline{= 85°}$$

$$C = 180° - 75° - 85°$$
$$\underline{= 20°}$$

$$\frac{|BC|}{\sin 85°} = \frac{80}{\sin 20°}$$

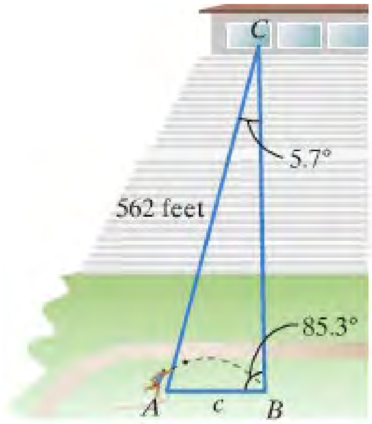
$$|BC| = \frac{80\sin 85°}{\sin 20°}$$
$$\underline{\approx 233 \ \ ft}$$





Exercise

The shot of a hot-put ring is tossed from *A* lands at *B*. Using modern electronic equipment, the distance of the toss can be measured without the use of measuring tapes. When the shot lands at *B*, an electronic transmitter placed at *B* sends a signal to a device in the official's booth above the track. The device determines the angles *B* and *C*. At a track meet, the distance from the official' booth to the shot-ring is 562 *feet*. If $B = 85.3°$ and $C = 5.7°$, determine the length of the toss.



Solution

$$\frac{c}{\sin 5.7°} = \frac{562}{\sin 85.3°}$$

$$c = \frac{562 \sin 5.7°}{\sin 85.3°}$$

$$\approx 56 \ ft$$

Exercise

A pier forms an 85° angle with a straight shore. At a distance of 100 *feet* from a pier, the line of sight to the tip forms a 37° angle. Find the length of the pier.

Solution

$$B = 180° - 85°$$

$$= 95°$$

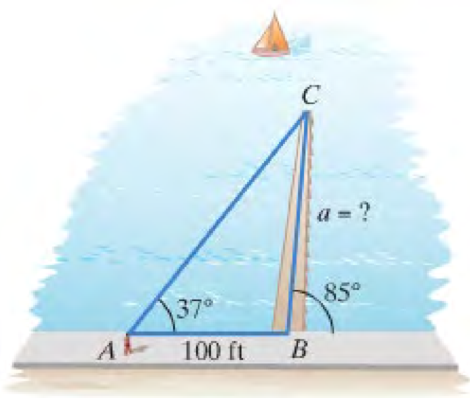
$$C = 180° - 37° - 95°$$

$$= 48°$$

$$\frac{a}{\sin 37°} = \frac{100}{\sin 48°}$$

$$a = \frac{100 \sin 37°}{\sin 48°}$$

$$\approx 81.0 \ ft$$



Exercise

A leaning wall is inclined 6° from the vertical. At a distance of 40 *feet* from the wall, the angle of elevation to the top is 22°.Find the height of the wall.

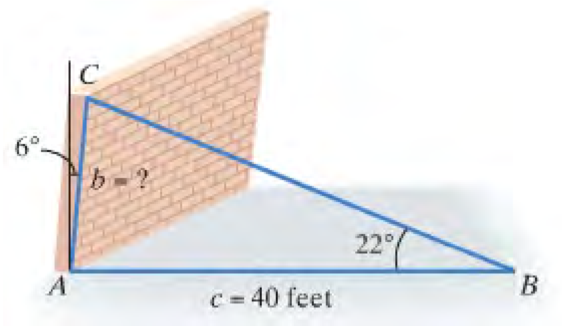
Solution

$$A = 90° - 6°$$

$$= 84°$$

$$C = 180° - 84° - 22°$$

$$= 74°$$



$$\frac{b}{\sin 22^\circ} = \frac{40}{\sin 74^\circ}$$

$$b = \frac{40 \sin 22^\circ}{\sin 74^\circ}$$

$$\approx 15.6 \ ft$$

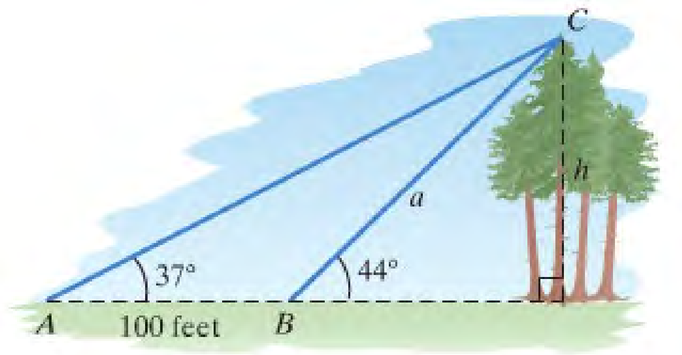
Exercise

Redwood trees are hundreds of feet tall. The height of one of these is represented by *h*.

a) Find the height of the tree.

b) Find *a*.

Solution



a) $h = \frac{100 \tan 44^\circ \tan 37^\circ}{\tan 44^\circ - \tan 37^\circ}$

$\approx 343.0 \ ft$

b) $\sin 44^\circ = \frac{h}{a}$

$a = \frac{343}{\sin 44^\circ}$

$\approx 493.8 \ ft$

Exercise

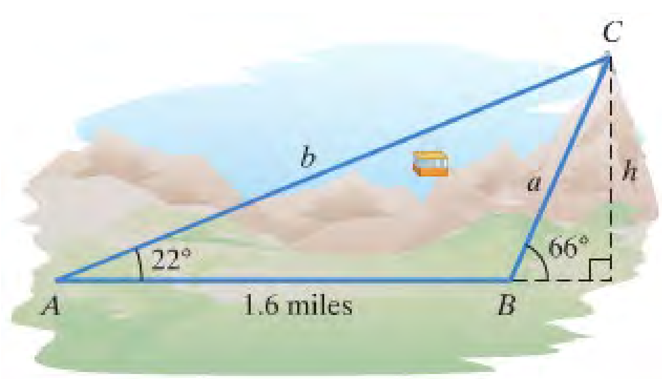
A carry cable car that carries passengers from *A* to *C*. Point *A* is 1.6 *miles* from the base of the mountain. The angles of elevation from *A* and *B* to the mountain's peak are 22° and 66°, respectively.

a) Find the height of the mountain.

b) Determine the distance covered by the cable car.

c) Find *a*.

Solution



a) $h = \frac{1.6 \ \tan 66^\circ \tan 22^\circ}{\tan 66^\circ - \tan 22^\circ}$

$\approx 0.788 \ mi$

b) $\sin 22^\circ = \frac{h}{b}$

$b = \frac{0.788}{\sin 22^\circ}$

$\approx 2.1042 \ mi$

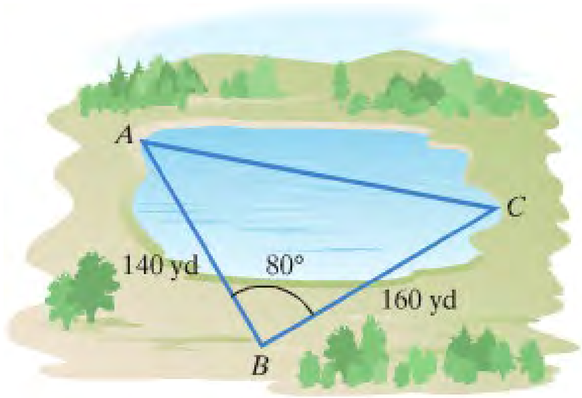
c) $\sin 66^\circ = \frac{h}{a}$

$$b = \frac{1.2687}{\sin 66°}$$

$$\approx 1.389 \ mi$$

Exercise

Find the distance across the lake from *A* to *C*.



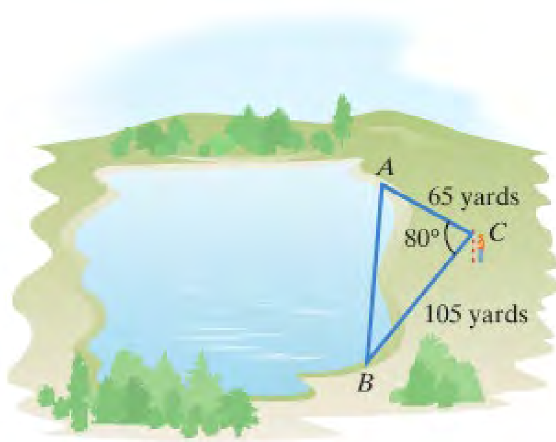
Solution

$$b = \sqrt{160^2 + 140^2 - 2(160)(140)\cos 80°} \qquad b^2 = a^2 + c^2 - 2ac\cos B$$

$$\approx 193 \ yd$$

Exercise

To find the distance across a protected cove at a lake, a surveyor makes the measurements. Find the distance from *A* to *B*.



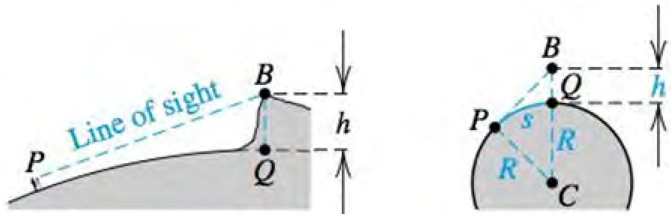
Solution

$$c = \sqrt{105^2 + 65^2 - 2(105)(65)\cos 80°} \qquad c^2 = a^2 + b^2 - 2ab\cos C$$

$$\approx 113 \ yd$$

Exercise

A surveyor using a transit, sights the edge B of a bluff, as shown in the left of the figure. Because of the curvature of Earth, the true elevation $\boldsymbol{h}$ of the bluff is larger than that measured by the surveyor. A cross-sectional schematic view of Earth is shown in the right part of the figure.



a) If s is the length of arc PQ and R is the distance from P to the center C of Earth, express h in terms of R and s.

b) If $R = 4{,}000\ mi$ and $s = 50\ mi$, estimate the elevation of the bluff in feet.

Solution

a) Right triangle CPB: $\cos\theta = \dfrac{R}{R+h}$

$$R + h = \frac{R}{\cos\theta} = R\sec\theta$$

$$h = R\sec\theta - R$$

From the arc: $\theta = \dfrac{s}{R}$

$$h = R\sec\frac{s}{R} - R$$

b) ***Given***: $R = 4{,}000\ mi$, $s = 50\ mi$

$$h = 4{,}000\sec\frac{50}{4{,}000} - 4{,}000$$

$$\approx 1{,}650\ \ ft$$

Exercise

Shown in the figure is a design for a rain gutter.

a) Express the volume V as a function of θ.

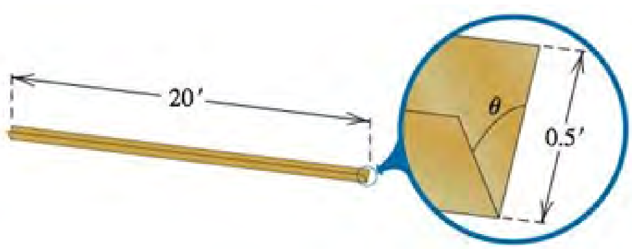
b) Approximate the acute angle θ that results in a volume of $2\ ft^3$

Solution

a) $Volume = 20 \times (Area\ of\ \ the\ sector\text{ - }triangle)$

$$= 20\times\left(\frac{1}{2}(0.5)^2\sin\theta\right)$$

$$= 2.5\ \sin\theta$$



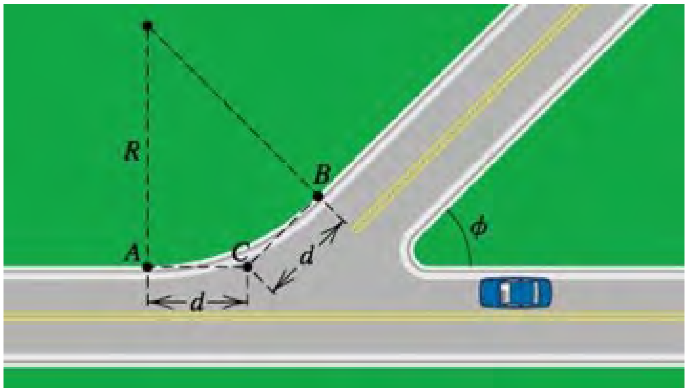
b) $\frac{5}{2}\sin\theta = 2$

$$\theta = \sin^{-1}\left(\frac{4}{5}\right)$$

$$\approx 53.13°$$

Exercise

A highway engineer is designing curbing for a street at an intersection where two highways meet at an angle ϕ, as shown in the figure, the curbing between points *A* and *B* is to be constructed using a circle that is tangent to the highway at these two points.



a) Show that the relationship between the radius *R* of the circle and the distance *d* in the figure is given by the equation $d = R\tan\frac{\phi}{2}$.

b) If $\phi = 45°$ and $d = 20\ ft$, approximate *R* and the length of the curbing.

Solution

a) $\tan\frac{\phi}{2} = \frac{d}{R}$

$$\boxed{d = R\tan\frac{\phi}{2}}$$

b) Given: $\phi = 45°$, $d = 20\ ft$

$$R = \frac{d}{\tan\frac{\phi}{2}}$$

$$= \frac{20}{\tan\frac{45°}{2}}$$

$$\boxed{\approx 48.28\ ft}$$

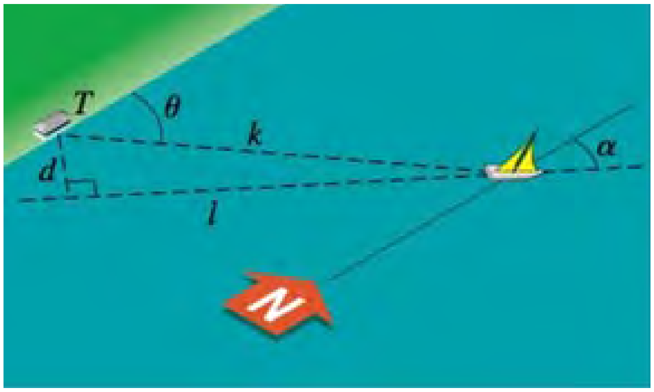
Length of the curbing (s) $= R\phi$

$$= 48.28 \times \frac{\pi}{4}$$

$$\boxed{\approx 37.92\ ft}$$

Exercise

A sailboat is following a straight line course l. (Assume that the shoreline is parallel to the north-south line.) The shortest distance from a tracking station T to the course is d miles. As the boat sails, the tracking station records its distance k from T and its direction θ with respect to T. Angle α specifies the direction of the sailboat.



a) Express α in terms of d, k, and θ.

b) Estimate α to the nearest degree if $d = 50\ mi$, $k = 210\ mi$, and $\theta = 53.4°$

Solution

a) $\beta = \theta - \alpha$

$\sin\beta = \dfrac{d}{k} \quad \rightarrow \quad \beta = \sin^{-1}\dfrac{d}{k}$

$\beta = \theta - \alpha = \sin^{-1}\dfrac{d}{k}$

$\alpha = \theta - \sin^{-1}\dfrac{d}{k}$

b) ***Given***: $d = 50\ mi,\ k = 210\ mi,\ \theta = 53.4°$

$\alpha = 53.4° - \sin^{-1}\dfrac{50}{210}$

$\approx 39.6°$

Exercise

An art critic whose eye level is 6 *feet* above the floor views a painting that is 10 *feet* in height and is mounted 4 *feet* above the floor.

a) If the critic is standing x *feet* from the wall, express the viewing angle θ in terms of x.

b) Use the addition formula for the tangent to show that $\theta = \tan^{-1}\left(\dfrac{10x}{x^2 - 16}\right)$

c) For what value of x is $\theta = 45°$?

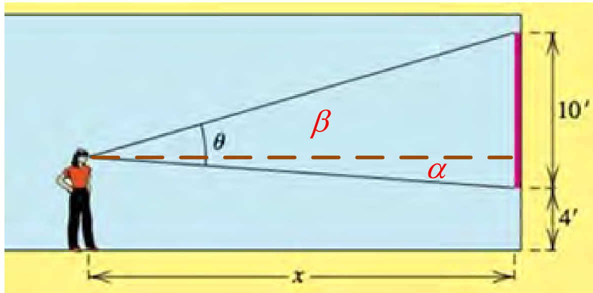
Solution

a) $\tan\alpha = \dfrac{2}{x} \quad \rightarrow \quad \alpha = \tan^{-1}\dfrac{2}{x}$

$\tan\beta = \dfrac{8}{x} \quad \rightarrow \quad \underline{\beta = \tan^{-1}\dfrac{8}{x}}$

$\theta = \alpha + \beta$

$\quad = \tan^{-1}\dfrac{8}{x} + \tan^{-1}\dfrac{2}{x}$



b) $\tan\theta = \tan(\alpha+\beta)$

$\tan\theta = \dfrac{\frac{2}{x}+\frac{8}{x}}{1-\frac{16}{x^2}}$

$\quad = \dfrac{10x}{x^2-16}$

$\theta = \tan^{-1}\left(\dfrac{10x}{x^2-16}\right)$

c) $\tan 45° = \dfrac{10x}{x^2-16} = 1$

$x^2 - 16 = 10x$

$x^2 - 10x - 16 = 0 \qquad x = \dfrac{10 \pm \sqrt{100+64}}{2} = \dfrac{10 \pm \sqrt{164}}{2} = \dfrac{10 \pm 2\sqrt{41}}{2}$

$x = 5 + \sqrt{41}\ \ ft$

Exercise

When an individual is walking, the magnitude *F* of the vertical force of one foot on the ground can be described by

$$F = A(\cos bt - a\cos 3bt), \qquad \text{where } t \text{ is time in seconds, } A > 0,\ b > 0 \text{ and } 0 < a < 1$$

a) Show that $F = 0$, when $t = -\dfrac{\pi}{2b}$ and $t = \dfrac{\pi}{2b}$. (the time $t = -\dfrac{\pi}{2b}$ corresponds to the moment when the foot first touches the ground and the weight of the body is being supported by the other foot.)

b) The maximum force occurs when $3a\sin 3bt = \sin bt$.

If $a = \dfrac{1}{3}$, find the solutions of this equation for the interval $-\dfrac{\pi}{2b} < t < \dfrac{\pi}{2b}$.

c) If $a = \dfrac{1}{3}$, express the maximum force in terms of *A*.

Solution

a) $F = A(\cos bt - a\cos 3bt) = 0 \quad \Rightarrow \quad \cos bt = a\cos 3bt$

Since $a \neq 0$, $\cos bt = \cos 3bt = 0$

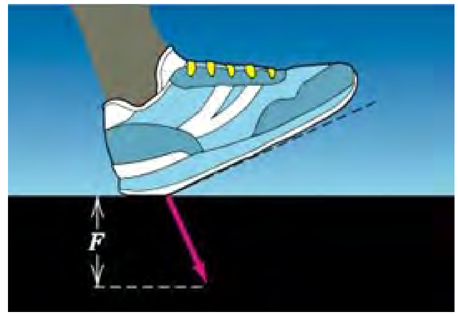
$\cos bt = 0$

$$bt = \pm\frac{\pi}{2}$$

$$\boxed{t = \pm\frac{\pi}{2b}}$$

$$\cos 3bt = 0$$

$$\rightarrow\ 3bt = \pm\frac{\pi}{2} + 2k\pi$$



b) ***Given***: $a = \frac{1}{3}$

$$3a\sin 3bt = 3\frac{1}{3}\sin 3bt$$

$$= \boxed{\sin 3bt = \sin bt}$$

$$\sin 3bt = \sin bt \rightarrow \begin{cases} \boxed{t = 0} \\ bt = \pm\frac{\pi}{4} & \boxed{t = \pm\frac{\pi}{4b}} \\ 3bt = \pm\frac{\pi}{4} \end{cases}$$

c) $F = A\left(\cos bt - \frac{1}{3}\cos 3bt\right)$

$$= A\left(\cos b\frac{\pi}{4b} - \frac{1}{3}\cos 3b\frac{\pi}{4b}\right)$$

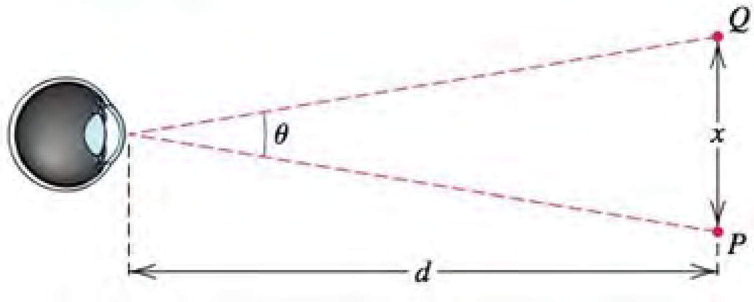
$$= A\left(\cos\frac{\pi}{4} - \frac{1}{3}\cos\frac{3\pi}{4}\right)$$

$$= A\left(\frac{\sqrt{2}}{2} + \frac{1}{3}\frac{\sqrt{2}}{2}\right)$$

$$\boxed{= \frac{2\sqrt{2}}{3}A}$$

Exercise

The human eye can distinguish between two distant points P and Q provided the angle of resolution θ is not too small. Suppose P and Q are x units apart and are d units from the eye.



a) Express x in terms of d and θ.

b) For a person with normal vision, the smallest distinguishable angle of resolution is about 0.0005 *radian*. If a pen 6 *inches* long is viewed by such an individual at a distance of d *feet*, for what values of d will be the end points of the pen be distinguishable?

Solution

a) $\tan\frac{\theta}{2} = \frac{\frac{x}{2}}{d}$

$$\boxed{x = 2d\tan\frac{\theta}{2}}$$

b) $d = \frac{x}{2}\frac{1}{\tan\frac{\theta}{2}}$

$$= 3\frac{1}{\tan\frac{.0005}{2}}\frac{1\ ft}{12\ in}$$

$$\boxed{\approx 999.99\ ft}$$

$$\boxed{d \le 1{,}000\ ft}$$

Exercise

A satellite S circles a planet at a distance d *miles* from the planet's surface. The portion of the planet's surface that is visible from the satellite is determined by the angle θ.

a) Assuming that the planet is spherical in shape, express d in terms of θ and the radius r of the planet.

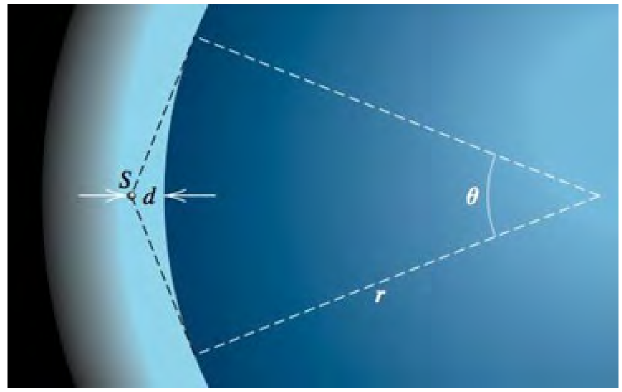
b) Approximate θ for a satellite 300 *miles* from the surface of Earth, using $r = 4{,}000\ mi$.

Solution

a) $\cos\frac{\theta}{2} = \frac{r}{r+d}$

$$r + d = \frac{r}{\cos\frac{\theta}{2}}$$

$$\boxed{d = r\left(\sec\frac{\theta}{2} - 1\right)}$$



b) ***Given***: $r = 4,000\ mi$ & $d = 300\ mi$

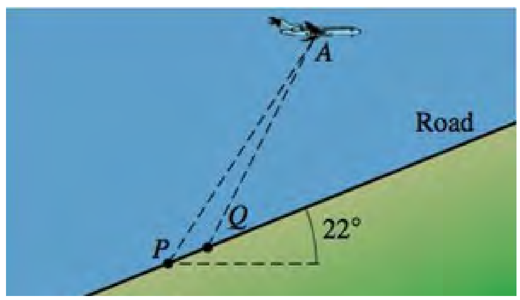
$$\frac{\theta}{2} = \cos^{-1}\frac{r}{r+d}$$

$$\theta = 2\cos^{-1}\frac{4,000}{4,300}$$

$$\boxed{\approx 43.06^\circ}$$

Exercise

A straight road makes an angle of 22° with the horizontal. From a certain point *P* on the road, the angle of elevation of an airplane at point *A* is 57°. At the same instant, form another point *Q*, 100 *meters* farther up the road, the angle of elevation is 63°. The points *P*, *Q*, and *A* lie in the same vertical plane.



Approximate the distance from *P* to the airplane.

Solution

$$\angle APQ = 57^\circ - 22^\circ$$
$$\boxed{= 35^\circ}$$

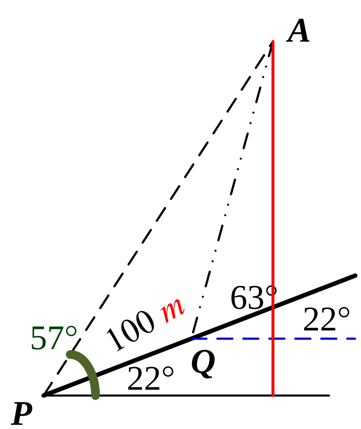
$$\angle AQP = 180^\circ - (63^\circ - 22^\circ)$$
$$\boxed{= 139^\circ}$$

$$\angle QAP = 180^\circ - 139^\circ - 35^\circ$$
$$\boxed{= 6^\circ}$$

$$\frac{|PA|}{\sin\angle AQP} = \frac{100}{\sin\angle QAP}$$

$$|PA| = \frac{100\ \sin 139^\circ}{\sin 6^\circ}$$

$$\boxed{\approx 627.64\ m}$$



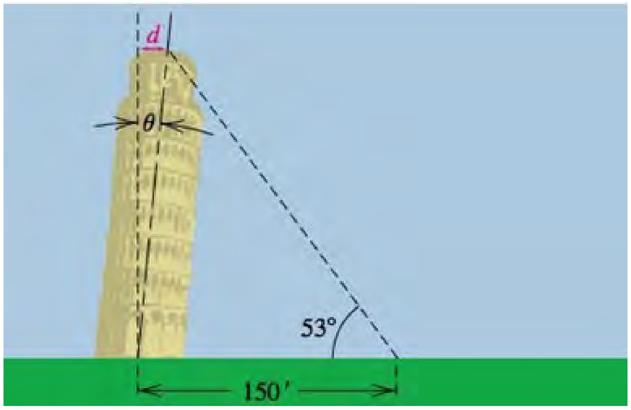
Exercise

The leaning tower of Pisa was originally perpendicular to the ground and 179 *feet* tall. Because of sinking into the earth, it now leans at a certain angle θ from the perpendicular. When the top of the tower is viewed from a point 150 *feet* from the center of its base, the angle of elevation is $53°$.

a) Approximate the angle θ.

b) Approximate the distance *d* that the center of the top the tower has moved from the perpendicular.

Solution



a) $\sin 53° = \dfrac{179}{x}$

$$x = \frac{179}{\sin 53°}$$
$$\approx 224.13$$

$$y = \sqrt{224.13^2 + 150^2 - 2(150)(224.13)\cos 53°}$$
$$\approx 179.635$$

$$\cos\theta = \frac{179}{179.635}$$

$$\theta = \cos^{-1}\frac{179}{179.635}$$
$$\approx 4.8°$$

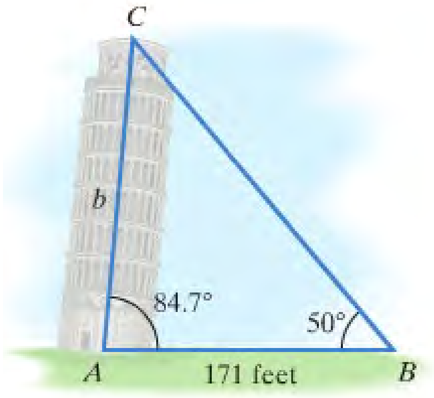
b) $\tan\theta = \dfrac{d}{179}$

$$d = 179\tan 5°$$
$$\approx 15.6\ ft$$

Exercise

The leaning Tower of Pisa in Italy leans at an angle of about 84.7°, 171 *feet* from the base of the tower, the angle of elevation to the top is 50°. Find the distance from the base to the top of the tower.

Solution



$$C = 180° - 84.7° - 50°$$
$$= 45.3°$$

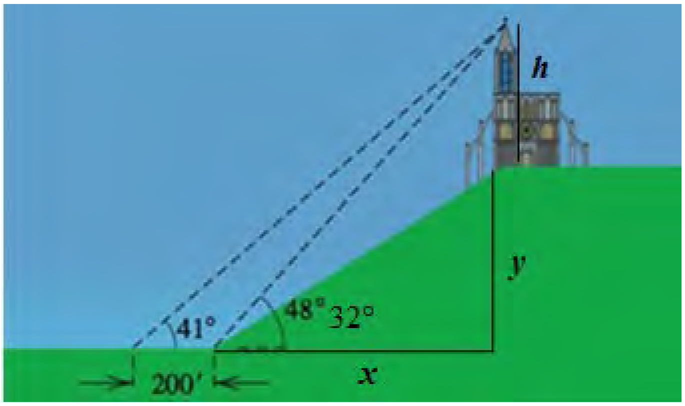
$$\frac{b}{\sin 50°} = \frac{171}{\sin 45.3°}$$

$$b = \frac{171\sin 50°}{\sin 45.3°}$$

$$\approx 184\ ft$$

Exercise

A cathedral is located on a hill. When the top of the spire is viewed from the base of the hill, the angle of elevation is $48°$. When it is viewed at a distance of 200 *feet* from the base of the hill, the angle is $41°$. The hill rises at an angle of $32°$.



Approximate the height of the cathedral.

Solution

$$h + y = \frac{200\ \tan 48° \tan 41°}{\tan 48° - \tan 41°}$$

$$\approx 800.114$$

$$\tan 32° = \frac{y}{x} \quad \rightarrow \quad x = \frac{y}{\tan 32°}$$

$$\tan 48° = \frac{h+y}{x} \quad \rightarrow \quad x = \frac{h+y}{\tan 48°}$$

$$\frac{y}{\tan 32°} = \frac{h+y}{\tan 48°}$$

$$y = \frac{800.114 \tan 32°}{\tan 48°}$$

$$\approx 450.172$$

$$h \approx 800.114 - 450.172$$

$$\approx 350\ \ ft$$

Exercise

A helicopter hovers at an altitude that is 1,000 *feet* above a mountain peak of altitude 5,210 *feet*. A second, taller peak is viewed from both the mountaintop and the helicopter. From the helicopter, the angle of depression is $43°$, and from the mountaintop, the angle of elevation is $18°$.

a) Approximate the distance from peak to peak.
b) Approximate the altitude of the taller peak.

Solution

a) $\angle BAC = 90° - 43° = 47°$

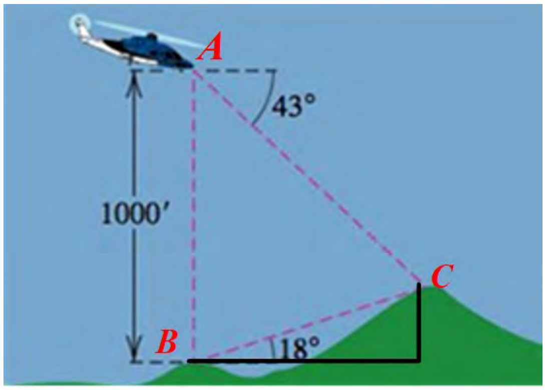
$\angle ABC = 90° - 18° = 72°$

$\angle ACB = 180° - 72° - 47° = 61°$

$$\frac{|BC|}{\sin 47°} = \frac{1000}{\sin 61°}$$

$$|BC| = \frac{1000 \sin 47°}{\sin 61°}$$

$$\approx 836.2 \ ft$$



b) $\sin 18° = \dfrac{h}{|BC|}$

$h = |BC| \sin 18°$

$\approx 258.4 \ ft$

Exercise

The volume V of the right triangular prism shown in the figure is $\frac{1}{3}Bh$, where B is the area of the base and h is the height of the prism.

a) Approximate h.
b) Approximate V.

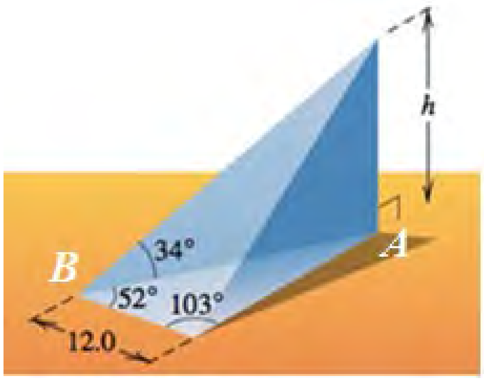
Solution

a) $\angle A = 180° - 103° - 52° = 25°$

$$\frac{|AB|}{\sin 103°} = \frac{12}{\sin 25°}$$

$$|AB| = \frac{12 \sin 103°}{\sin 25°}$$

$$\approx 27.7$$



$\tan 34° = \dfrac{h}{|AB|}$

$h = 27.7 \tan 34°$

≈ 18.7

b) $B = \frac{1}{2}(27.7)(12)\sin 52° \approx 131$

$V = \frac{1}{3}Bh$

$= \frac{1}{3}(131)(18.7)$

$\approx 816 \ \ unit^3$

Exercise

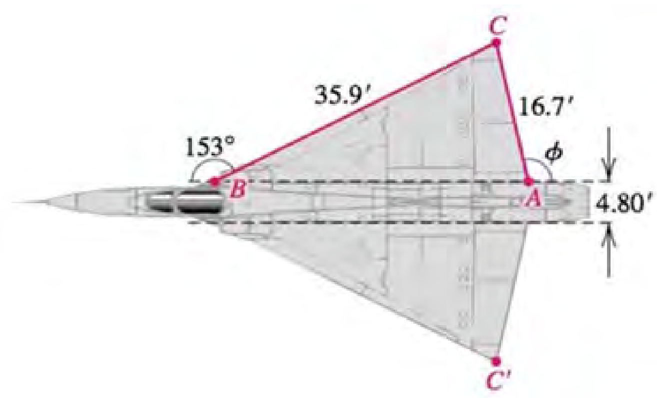
Shown in the figure is a plan for the top of a wing of a jet fighter.

a) Approximate angle ϕ.

b) If the fuselage is 4.80 *feet* wide, approximate the wing span CC'.

c) Approximate the area of the triangle *ABC.*

Solution



a) $\angle ABC = 180° - 153°$

$= 27°$

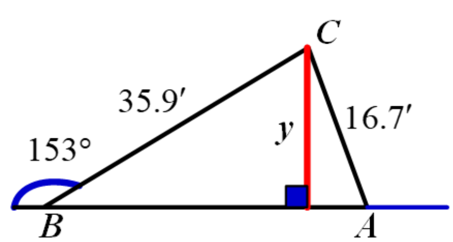
$\frac{\sin A}{35.9} = \frac{\sin 27°}{16.7}$

$A = \sin^{-1}\left(\frac{35.9}{16.7}\sin 27°\right)$

$\approx 77.4°$

$\phi = 180° - 77.4°$

$\approx 102.6°$



b) $\sin A = \frac{y}{16.7}$

$y = 16.7\sin 77.4°$

≈ 16.3

$CC' = 2(16.3) + 4.8$

$\approx 37.4 \ \ ft$

c) $\angle ACB = 180° - 27° - 77.4°$

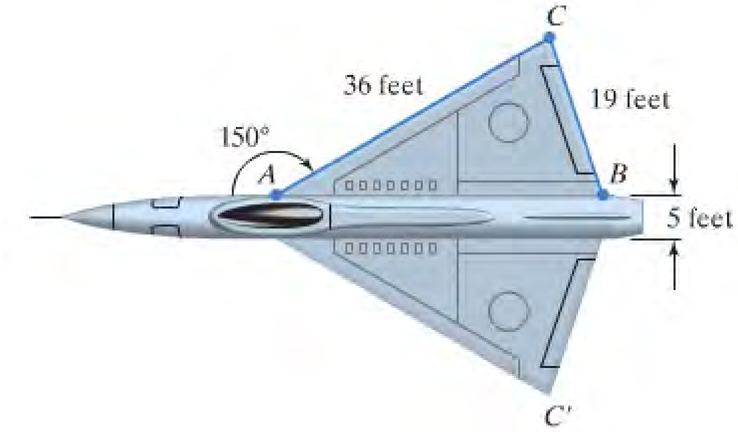
$= 75.6°$

$Area = \frac{1}{2}(35.9)(16.7)\sin 75.6°$

$\approx 290.35 \ \ ft^2$

Exercise

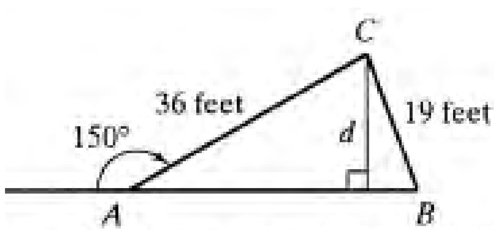
Shown in the figure is a plan for the top of a wing of a jet fighter. The fuselage is 5 *feet* wide. Find the wing span CC'



Solution

$$\angle BAC = 180° - 150°$$
$$= 30°$$

$$\sin A = \frac{d}{36}$$
$$d = 36 \sin 30°$$
$$= 36\left(\frac{1}{2}\right)$$
$$= 18\ ft$$

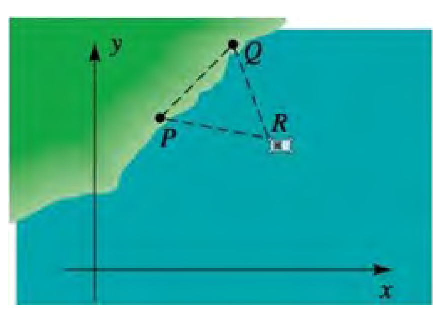


$$CC' = 2(18) + 5$$
$$= 41\ \ ft$$

Exercise

Computer software for surveyors makes use of coordinate systems to locate geographic positions. An offshore oil well at point *R* is viewed from points *P* and *A* and $\angle QPR$ and $\angle RQP$ are found to be $55°\ 50'$ and $65°\ 22'$, respectively. If points *P* and *Q* have coordinates $(1487.7,\ 3452.8)$ and $(3145.8,\ 5127.5)$, respectively. Approximate the coordinates of *R*.

Solution



Find: $R(x,\ y)$

$$|PQ| = \sqrt{(3{,}145.8 - 1{,}487.7)^2 + (5{,}127.5 - 3{,}452.8)^2}$$
$$\approx 2{,}356.6$$

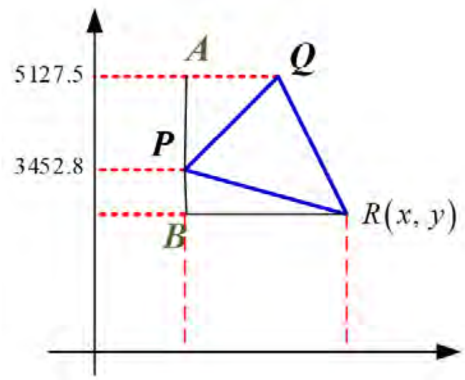
$$\angle PRQ = 180° - 65°\ 22' - 55°\ 50'$$
$$= 180° - 121°\ 12'$$

$$\underline{= 58.8°}$$

$$\frac{|PR|}{\sin(65.37°)} = \frac{2,356.6}{\sin(58.8°)}$$

$$|PR| = \frac{2,356.6 \sin(65.4°)}{\sin(58.8°)}$$

$$\underline{\approx 2,505.0}$$



$$|AP| = 5127.5 - 3452.8$$

$$\underline{= 1,647.7}$$

$$\angle APQ = \cos^{-1}\frac{|AP|}{|PQ|}$$

$$= \cos^{-1}\frac{1,647.7}{2,356.6}$$

$$\underline{\approx 45.6°}$$

$$\angle BPR \approx 180° - 45.6° - 55.83°$$

$$\underline{\approx 78.57°}$$

$$\sin 78.57° = \frac{|BR|}{|PR|}$$

$$|BR| = 2,505 \sin 78.57°$$

$$\underline{\approx 2,455.3}$$

$$x = x_P + |BR|$$

$$= 1487.7 + 2455.3$$

$$\underline{\approx 3,943}$$

$$\cos 78.57° = \frac{|PB|}{|PR|}$$

$$|PB| = 2,505 \cos 78.57°$$

$$\underline{\approx 496.4}$$

$$y = y_P - |PB|$$

$$= 3452.8 - 496.4$$

$$\underline{\approx 2,956.4}$$

$$\underline{R(3,943,\ \ 2,946.4)}$$

Exercise

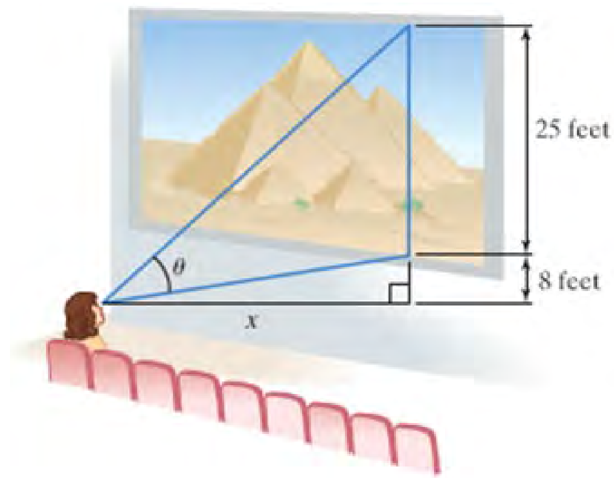
Your movie theater has a 25–*foot*–high screen located 8 *feet* above your eye level. If you sit too close to the screen, your viewing angle is too small resulting in a distorted picture. By contrast, if you sit too far back, the image is quite small, diminishing the movie's visual impact. If you sit *x feet* back from the screen, your viewing angle θ, is giving by

$$\theta = \tan^{-1}\frac{33}{x} - \tan^{-1}\frac{8}{x}$$

Find the viewing angle, in radians, at distances of 5 *feet*, 10 *feet*, 15 *feet*, 25 *feet*, and 25 *feet*.

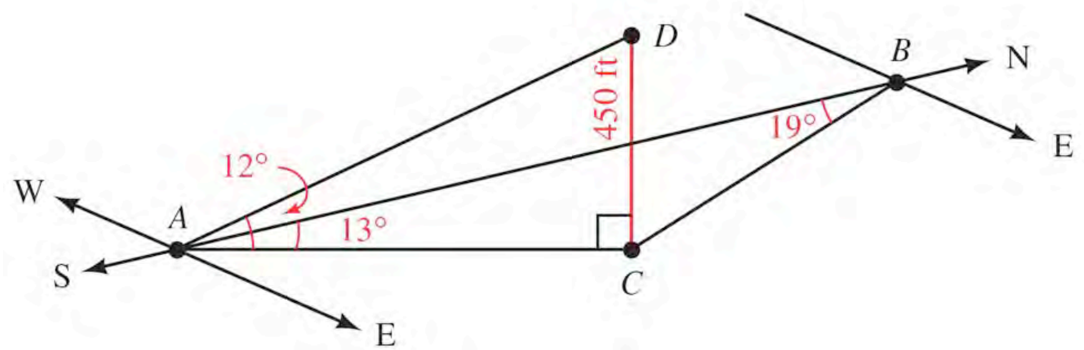
Solution

x	$\theta = \tan^{-1}\frac{33}{x} - \tan^{-1}\frac{8}{x}$
5	$\theta = \tan^{-1}\frac{33}{5} - \tan^{-1}\frac{8}{5}$ $\approx 0.408\ rad$
10	$\theta = \tan^{-1}\frac{33}{10} - \tan^{-1}\frac{8}{10}$ $\approx 0.602\ rad$
15	$\theta = \tan^{-1}\frac{33}{15} - \tan^{-1}\frac{8}{15}$ $\approx 0.654\ rad$
20	$\theta = \tan^{-1}\frac{33}{20} - \tan^{-1}\frac{8}{20}$ $\approx 0.645\ rad$
25	$\theta = \tan^{-1}\frac{33}{25} - \tan^{-1}\frac{8}{25}$ $\approx 0.613\ rad$



Exercise

A hot-air balloon is flying over a dry lake when the wind stops blowing. The balloon comes to a stop 450 *feet* above the ground at point *D*. A jeep following the balloon runs out of gas at point *A*. The nearest service station is due north of the jeep at point *B*. The bearing of the balloon from the jeep at *A* is N 13° E, while the bearing of the balloon from the service station at *B* is S 19° E. If the angle of elevation of the balloon from *A* is 12°, how far will the people in the jeep have to walk to reach the service station at point *B*?



Solution

$$\tan 12° = \frac{DC}{AC}$$

$$\begin{aligned} AC &= \frac{DC}{\tan 12°} \\ &= \frac{450}{\tan 12°} \\ &\approx 2{,}117\ \ ft \end{aligned}$$

$$\begin{aligned} \angle ACB &= 180° - (13° + 19°) \\ &= 148° \end{aligned}$$

Using triangle *ABC*

$$\frac{AB}{\sin 148°} = \frac{2117}{\sin 19°}$$

$$\begin{aligned} AB &= \frac{2117 \sin 148°}{\sin 19°} \\ &\approx 3{,}446\ \ ft \end{aligned}$$

Exercise

Derive the formula: $\dfrac{a-b}{a+b}=\dfrac{\tan\left[\frac{1}{2}(A-B)\right]}{\tan\left[\frac{1}{2}(A+B)\right]}$

Solution

$$\frac{a-b}{a+b}=\frac{\frac{a-b}{c}}{\frac{a+b}{c}}$$

$$=\frac{a-b}{c}\div\frac{a+b}{c}$$

$$=\left(\frac{a}{c}-\frac{b}{c}\right)\div\left(\frac{a}{c}+\frac{b}{c}\right) \qquad \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}\ \rightarrow \frac{a}{c}=\frac{\sin A}{\sin C}\quad \frac{b}{c}=\frac{\sin B}{\sin C}$$

$$=\left(\frac{\sin A}{\sin C}-\frac{\sin B}{\sin C}\right)\div\left(\frac{\sin A}{\sin C}+\frac{\sin B}{\sin C}\right)$$

$$=\left(\frac{\sin A-\sin B}{\sin C}\right)\div\left(\frac{\sin A+\sin B}{\sin C}\right)$$

$$=\frac{\sin A-\sin B}{\sin A+\sin B} \qquad \sin x-\sin y=2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) \qquad \sin x+\sin y=2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$=\frac{2\cos\left(\frac{1}{2}(A+B)\right)\sin\left(\frac{1}{2}(A-B)\right)}{2\sin\left(\frac{1}{2}(A+B)\right)\cos\left(\frac{1}{2}(A-B)\right)}$$

$$=\frac{\sin\left(\frac{1}{2}(A-B)\right)}{\cos\left(\frac{1}{2}(A-B)\right)}\cdot\frac{\cos\left(\frac{1}{2}(A+B)\right)}{\sin\left(\frac{1}{2}(A+B)\right)}$$

$$=\frac{\tan\left[\frac{1}{2}(A-B)\right]}{\tan\left[\frac{1}{2}(A+B)\right]}$$

Exercise

For any triangle, show that $\sin\dfrac{C}{2}=\sqrt{\dfrac{(s-a)(s-b)}{ab}}$ *where* $s=\frac{1}{2}(a+b+c)$

Solution

$$\sin\frac{C}{2}=\sqrt{\frac{1}{2}(1-\cos C)}$$

$$=\sqrt{\frac{1}{2}\left(1-\frac{a^2+b^2-c^2}{2ab}\right)}$$

$$=\sqrt{\frac{2ab-a^2-b^2+c^2}{4ab}}$$

$$= \sqrt{\frac{-\left(a^2 + b^2 - 2ab - c^2\right)}{4ab}}$$

$$= \sqrt{\frac{-\left((a-b)^2 - c^2\right)}{4ab}} \qquad x^2 - y^2 = (x-y)(x+y)$$

$$= \sqrt{\frac{-(a-b+c)(a-b-c)}{4ab}} \qquad 2s-b=a+c \quad 2s-a=b+c$$

$$= \sqrt{\frac{(2s-b-b)(2s-a-a)}{4ab}}$$

$$= \sqrt{\frac{4(s-b)(s-a)}{4ab}}$$

$$= \sqrt{\frac{(s-b)(s-a)}{ab}} \quad \checkmark$$

Exercise

Prove the identity $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$

Solution

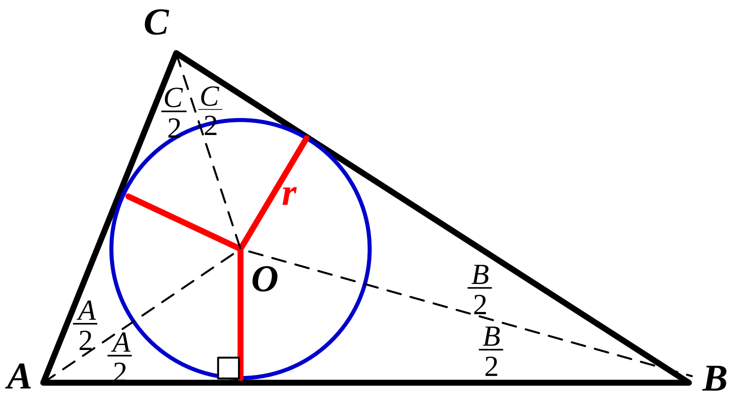
$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{1}{a}\frac{b^2 + c^2 - a^2}{2bc} + \frac{1}{b}\frac{a^2 + c^2 - b^2}{2ac} + \frac{1}{c}\frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{b^2 + c^2 - a^2 + a^2 + c^2 - b^2 + a^2 + b^2 - c^2}{2abc}$$

$$= \frac{a^2 + b^2 + c^2}{2abc} \quad \checkmark$$

Exercise

The lines that bisect each angle of a triangle meet in a single point O, and perpendicular distance r from O to each side of the triangle is the same. The circle with center at O and radius r is called the inscribed circle of the triangle.



a) Show that $r = \dfrac{c\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}}$

b) Show that $\cot\frac{C}{2} = \dfrac{s-c}{r}$ where $s = \frac{1}{2}(a+b+c)$

c) Show that $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \dfrac{s}{r}$

d) Show that the area K of triangle ABC is $K = rs$, where $s = \frac{1}{2}(a+b+c)$.

e) Show that $r = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$

Solution

a) $\Delta AOB \quad \rightarrow \quad \angle AOB = \pi - \left(\frac{A}{2} + \frac{B}{2}\right)$

$$\sin\frac{A}{2} = \frac{r}{\overline{OA}} \quad \Rightarrow \quad \overline{OA} = \frac{r}{\sin\frac{A}{2}}$$

$$\frac{\overline{OA}}{\sin\frac{B}{2}} = \frac{c}{\sin(\angle AOB)}$$

$$\sin(\angle AOB) = \sin\left(\pi - \left(\frac{A}{2} + \frac{B}{2}\right)\right) \qquad \sin(\pi - \alpha) = \sin\alpha$$

$$= \sin\left(\frac{A}{2} + \frac{B}{2}\right)$$

$$= \sin\left(\frac{A+B}{2}\right) \qquad \sin\alpha = \cos\left(\frac{1}{2} - \alpha\right)$$

$$= \cos\left(\frac{\pi}{2} - \frac{A+B}{2}\right)$$

$$= \cos\left(\frac{\pi - (A+B)}{2}\right) \qquad C = \pi - (A+B)$$

$$= \cos\left(\frac{C}{2}\right)$$

$$\frac{r}{\sin\frac{A}{2}}\frac{1}{\sin\frac{B}{2}} = \frac{c}{\cos\left(\frac{C}{2}\right)}$$

$$\boxed{r = \frac{c\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}}} \quad \checkmark$$

b) $\cot\frac{C}{2} = \dfrac{\cos\frac{C}{2}}{\sin\frac{C}{2}}$

From part (*a*): $\cos\frac{C}{2} = \dfrac{c\sin\frac{A}{2}\sin\frac{B}{2}}{r}$

$$\begin{aligned}
\sin\frac{C}{2} &= \sqrt{\frac{1}{2}(1-\cos C)} \\
&= \sqrt{\frac{1}{2}\left(1-\frac{a^2+b^2-c^2}{2ab}\right)} \\
&= \sqrt{\frac{2ab-a^2-b^2+c^2}{4ab}} \\
&= \sqrt{\frac{-\left(a^2+b^2-2ab-c^2\right)}{4ab}} \\
&= \sqrt{\frac{-\left((a-b)^2-c^2\right)}{4ab}} && x^2-y^2=(x-y)(x+y) \\
&= \sqrt{\frac{-(a-b+c)(a-b-c)}{4ab}} && 2s-b=a+c \quad 2s-a=b+c \\
&= \sqrt{\frac{(2s-b-b)(2s-a-a)}{4ab}} \\
&= \sqrt{\frac{4(s-b)(s-a)}{4ab}} \\
&= \sqrt{\frac{(s-b)(s-a)}{ab}}
\end{aligned}$$

$$\cot\frac{C}{2} = \frac{\frac{c\sin\frac{A}{2}\sin\frac{B}{2}}{r}}{\sqrt{\frac{(s-b)(s-a)}{ab}}}$$

$$= \frac{c}{r}\frac{\sqrt{ab}}{\sqrt{(s-b)(s-a)}}\sin\frac{A}{2}\sin\frac{B}{2} \qquad \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \quad \& \quad \sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}$$

$$=\frac{c}{r}\frac{\sqrt{ab}}{\sqrt{(s-b)(s-a)}}\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-c)}{ac}}$$

$$=\frac{c}{r}\sqrt{\frac{ab}{abc^2}}\sqrt{\frac{(s-a)(s-b)(s-c)^2}{(s-b)(s-a)}}$$

$$=\frac{c}{r}\frac{1}{c}(s-c)$$

$$\boxed{=\frac{s-c}{r}}\quad\checkmark$$

c) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s-a}{r}+\frac{s-b}{r}+\frac{s-c}{r}$

$$=\frac{3s-a-b-c}{r}$$

$$=\frac{3s-(a+b+c)}{r} \qquad 2s=a+b+c$$

$$=\frac{3s-2s}{r}$$

$$\boxed{=\frac{s}{r}}\quad\checkmark$$

d) $K=Area(\Delta AOB)+Area(\Delta AOC)+Area(\Delta BOC)$

$$Area(\Delta AOB)=\frac{1}{2}(height)(base)=\frac{1}{2}rc$$

$$Area(\Delta AOC)=\frac{1}{2}(height)(base)=\frac{1}{2}rb$$

$$Area(\Delta BOC)=\frac{1}{2}(height)(base)=\frac{1}{2}ra$$

$$K=\frac{1}{2}rc+\frac{1}{2}rb+\frac{1}{2}ra$$

$$=\frac{1}{2}r(a+b+c) \qquad 2s=a+b+c$$

$$=\frac{1}{2}r(2s)$$

$$\boxed{=rs}\quad\checkmark$$

e) $K=\sqrt{s(s-a)(s-b)(s-c)}$

$$rs=\sqrt{s(s-a)(s-b)(s-c)}$$

$$r=\frac{1}{s}\sqrt{s(s-a)(s-b)(s-c)}$$

$$=\sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2}}$$

$$\boxed{=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}}\quad\checkmark$$