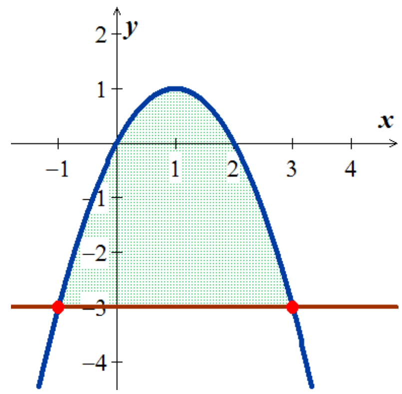
Solution Section 1.2 – Region between Curves

Exercise

Find the area of the region bounded by the graphs of $y = 2x - x^2$ and $y = -3$

Solution

$$y = -3 \quad \rightarrow \quad 2x - x^2 = -3 \Rightarrow x^2 - 2x - 3 = 0 \quad \boxed{x = -1,\ 3}$$



$$A = \int_{-1}^{3} \left[2x - x^2 - (-3)\right] dx$$

$$= \left[x^2 - \frac{x^3}{3} + 3x\right]_{-1}^{3}$$

$$= \left((3)^2 - \frac{(3)^3}{3} + 3(3)\right) - \left((-1)^2 - \frac{(-1)^3}{3} + 3(-1)\right)$$

$$= (9 - 9 + 9) - \left(1 + \frac{1}{3} - 3\right)$$

$$= \frac{32}{3}\ \ unit^2$$

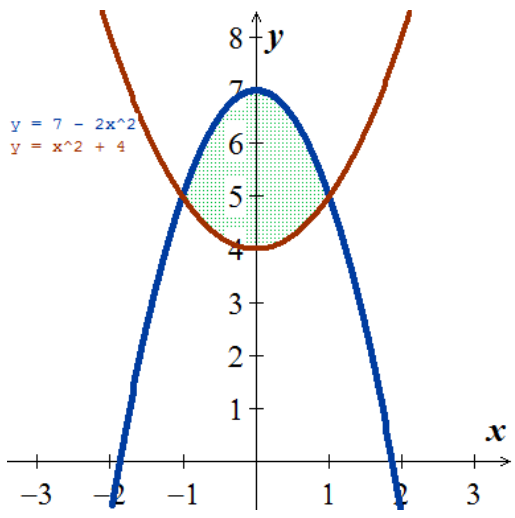
Exercise

Find the area of the region bounded by the graphs of $y = 7 - 2x^2$ and $y = x^2 + 4$

Solution

$$7 - 2x^2 = x^2 + 4$$

$$-3x^2 = -3 \quad \rightarrow \quad x^2 = 1 \Rightarrow \boxed{x = \pm 1}$$



$$A = \int_{-1}^{1} \left[\left(7 - 2x^2\right) - \left(x^2 + 4\right)\right] dx$$

$$= \int_{-1}^{1} \left(3 - 3x^2\right) dx$$

$$= \left[3x - 3\frac{x^3}{3}\right]_{-1}^{1}$$

$$= \left(3(1) - (1)^3\right) - \left(3(-1) - (-1)^3\right)$$

$$= 4\ \ unit^2$$

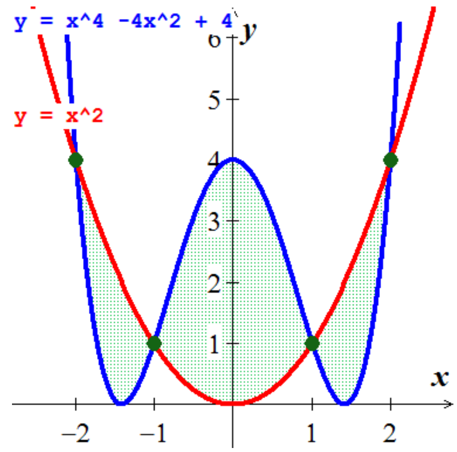
Exercise

Find the area of the region bounded by the graphs of $y = x^4 - 4x^2 + 4$ and $y = x^2$

Solution

$$x^4 - 4x^2 + 4 = x^2$$

$$x^4 - 5x^2 + 4 = 0 \quad \rightarrow \quad \boxed{x = \pm 1, \ \pm 2}$$



$$A = \int_{-2}^{-1} \left(x^2 - \left(x^4 - 4x^2 + 4\right)\right) dx + \int_{-1}^{1} \left(x^4 - 4x^2 + 4 - \left(x^2\right)\right) dx + \int_{1}^{2} \left(x^2 - \left(x^4 - 4x^2 + 4\right)\right) dx$$

$$= \int_{-2}^{-1} \left(-x^4 + 5x^2 - 4\right) dx + \int_{-1}^{1} \left(x^4 - 5x^2 + 4\right) dx + \int_{1}^{2} \left(-x^4 + 5x^2 - 4\right) dx$$

$$= \left[-\frac{x^5}{5} + \frac{5}{3}x^3 - 4x\right]_{-2}^{-1} + \left[\frac{x^5}{5} - \frac{5}{3}x^3 + 4x\right]_{-1}^{1} + \left[-\frac{x^5}{5} + \frac{5}{3}x^3 - 4x\right]_{1}^{2}$$

$$= \left[\left(-\frac{(-1)^5}{5} + \frac{5}{3}(-1)^3 - 4(-1)\right) - \left(-\frac{(-2)^5}{5} + \frac{5}{3}(-2)^3 - 4(-2)\right)\right]$$

$$+ \left[\left(\frac{(1)^5}{5} - \frac{5}{3}(1)^3 + 4(1)\right) - \left(\frac{(-1)^5}{5} - \frac{5}{3}(-1)^3 + 4(-1)\right)\right]$$

$$+ \left[\left(-\frac{(2)^5}{5} + \frac{5}{3}(2)^3 - 4(2)\right) - \left(-\frac{(1)^5}{5} + \frac{5}{3}(1)^3 - 4(1)\right)\right]$$

$$= \left(\tfrac{1}{5} - \tfrac{5}{3} + 4\right) - \left(\tfrac{32}{5} - \tfrac{40}{3} + 8\right) + \left(\tfrac{1}{5} - \tfrac{5}{3} + 4\right) - \left(-\tfrac{1}{5} + \tfrac{5}{3} - 4\right) + \left(-\tfrac{32}{5} + \tfrac{40}{3} - 8\right) - \left(-\tfrac{1}{5} + \tfrac{5}{3} - 4\right)$$

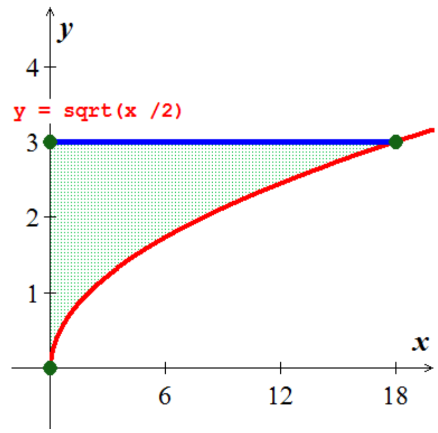
$$= 8 \ \ unit^2$$

Exercise

Find the area of the region bounded by the graphs of $x = 2y^2, \quad x = 0, \quad and \quad y = 3$

Solution

$y = 3 \quad \to \quad \boxed{x = 2y^2 = 18}$



$$A = \int_0^3 2y^2 dy$$

$$= \frac{2}{3}\left[y^3\right]_0^3$$

$$= \frac{2}{3}\left(3^3 - 0\right)$$

$$\boxed{= 18 \ \ unit^2}$$

Exercise

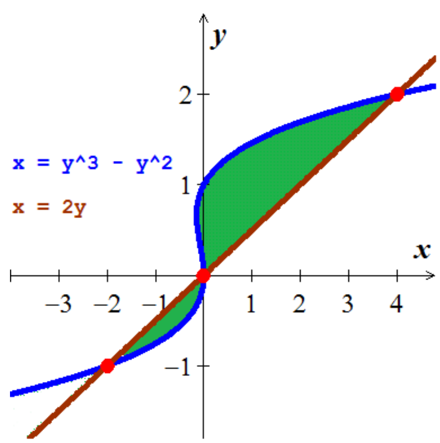
Find the area of the region bounded by the graphs of $x = y^3 - y^2 \quad and \quad x = 2y$

Solution

$y^3 - y^2 = 2y$

$y^3 - y^2 - 2y = 0$

$y\left(y^2 - y - 2\right) = 0 \quad \to \quad \boxed{y = 0, \ -1, \ 2}$



$$A = \int_{-1}^{0}\left[y^3 - y^2 - (2y)\right]dy + \int_0^2\left[2y - \left(y^3 - y^2\right)\right]dy$$

$$= \int_{-1}^{0}\left(y^3 - y^2 - 2y\right)dy + \int_0^2\left(2y - y^3 + y^2\right)dy$$

$$= \left[\frac{y^4}{4} - \frac{y^3}{3} - y^2\right]_{-1}^{0} + \left[y^2 - \frac{y^4}{4} + \frac{y^3}{3}\right]_0^2$$

$$= \left[0 - \left(\frac{1}{4} + \frac{1}{3} - 1\right)\right] + \left[\left(4 - 4 + \frac{8}{3}\right) - 0\right]$$

$$= \frac{5}{12} + \frac{8}{3}$$

$$\boxed{= \frac{37}{12} \ \ unit^2}$$

Exercise

Find the area of the region bounded by the graphs of $4x^2 + y = 4$ *and* $x^4 - y = 1$

Solution

$4x^2 + y = 4 \quad \rightarrow \quad y = 4 - 4x^2$

$x^4 - y = 1 \quad \text{and} \quad y = x^4 - 1$

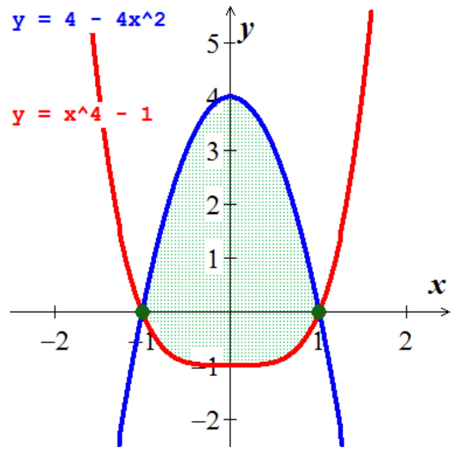
$$A = \int_{-1}^{1} \left[4 - 4x^2 - \left(x^4 - 1\right)\right] dx$$

$$= \int_{-1}^{1} \left(x^4 - 4x^2 + 5\right) dx$$

$$= \left[\frac{x^5}{5} - 4\frac{x^3}{3} + 5x\right]_{-1}^{1}$$

$$= \left(\frac{1}{5} - \frac{4}{3} + 5\right) - \left(-\frac{1}{5} + \frac{4}{3} - 5\right)$$

$$= \frac{105}{15} \ \ unit^2$$



Exercise

Find the area of the region bounded by the graphs of $x + 4y^2 = 4$ *and* $x + y^4 = 1$, *for* $x \ge 0$

Solution

$x = 4 - 4y^2 \quad x = 1 - y^4 \quad \rightarrow 4 - 4y^2 = 1 - y^4$

$y^4 - 4y^2 + 3 = 0 \quad \rightarrow \quad y^2 = 1,\ 3 \Rightarrow y = \pm 1,\ \pm\sqrt{3}$

$$\begin{cases} y = \pm 1 & \rightarrow \lfloor x = 1 - (\pm 1)^4 = 0 \rfloor \\ y = \pm\sqrt{3} & \rightarrow x = 1 - \left(\pm\sqrt{3}\right)^4 = -8 < 0 \times \end{cases}$$

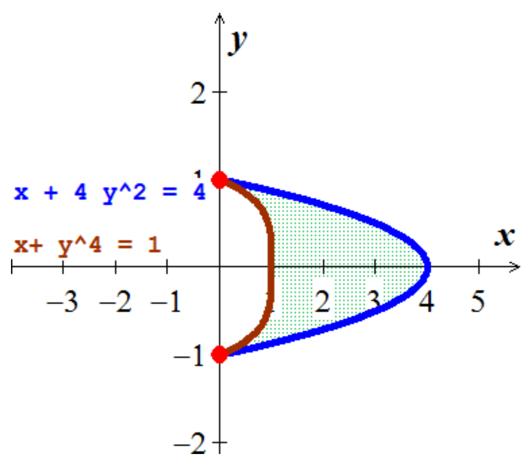
$$A = \int_{-1}^{1} \left[4 - 4y^2 - \left(1 - y^4\right)\right] dy$$

$$= \int_{-1}^{1} \left(3 - 4y^2 + y^4\right) dy$$

$$= \left[3y - 4\frac{y^3}{3} + \frac{y^5}{5}\right]_{-1}^{1}$$

$$= \left(3 - \frac{4}{3} + \frac{1}{5}\right) - \left(-3 + \frac{4}{3} - \frac{1}{5}\right)$$

$$= \frac{56}{15} \ \ unit^2$$



Exercise

Find the area of the region bounded by the graphs of $y = 2\sin x, \quad and \quad y = \sin 2x, \quad 0 \le x \le \pi$

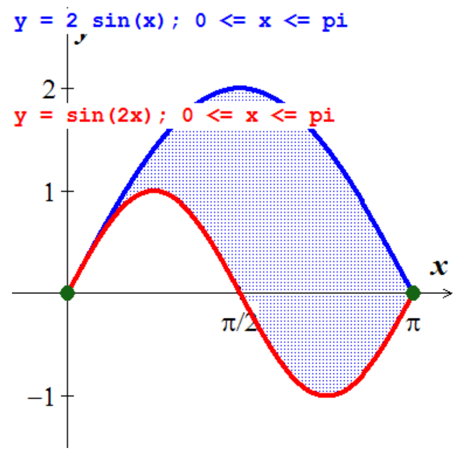
Solution

$$y = 2\sin x = \sin 2x$$

$$2\sin x = 2\sin x \cos x$$

$$2\sin x - 2\sin x \cos x = 0$$

$$2\sin x(1-\cos x) = 0 \quad \to \quad \begin{cases} \sin x = 0 & x = 0, \pi \\ \cos x = 1 & x = 0 \end{cases}$$



$$A = \int_0^{\pi} (2\sin x - \sin 2x)\,dx$$

$$= \left[-2\cos x + \frac{1}{2}\cos 2x\right]_0^{\pi}$$

$$= \left(-2(-1) + \frac{1}{2}(1)\right) - \left(-2 + \frac{1}{2}\right)$$

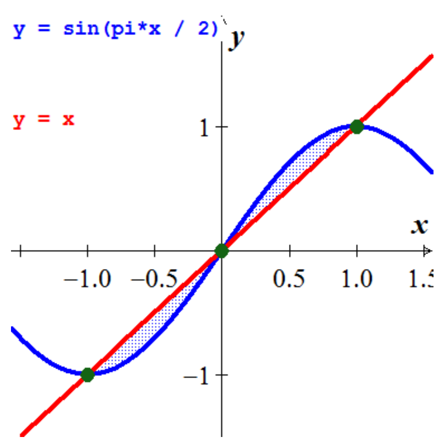
$$= 4 \ \ unit^2$$

Exercise

Find the area of the region bounded by the graphs of $y = \sin\frac{\pi x}{2} \quad and \quad y = x$

Solution

$$y = \sin\frac{\pi x}{2} = x \quad \to \quad \boxed{x = -1,\ 1}$$



$$A = \int_{-1}^{0} \left(\sin\frac{\pi x}{2} - x\right)dx + \int_0^1 \left(\sin\frac{\pi x}{2} - x\right)dx$$

$$= 2\int_0^1 \left(\sin\frac{\pi x}{2} - x\right)dx$$

$$= 2\left[-\frac{2}{\pi}\cos\frac{\pi x}{2} - \frac{x^2}{2}\right]_0^1$$

$$= 2\left[\left(0 - \frac{1}{2}\right) - \left(-\frac{2}{\pi} - 0\right)\right]$$

$$= 2\left(-\frac{1}{2} + \frac{2}{\pi}\right)$$

$$= 2\left(\frac{-\pi + 4}{2\pi}\right)$$

$$= \frac{4-\pi}{\pi} \ \ unit^2$$

Exercise

Find the area of the region bounded by the graphs of $y = x^2 + 1$ and $y = x$ for $0 \le x \le 2$

Solution

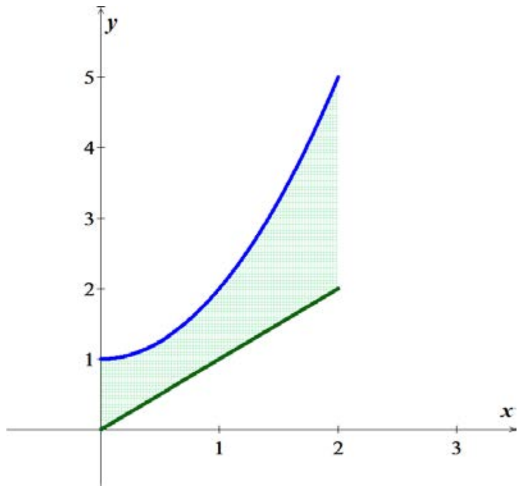
$$A = \int_0^2 [(x^2+1) - x]dx$$

$$= \int_0^2 \left(x^2 - x + 1\right)dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + 1x \Big|_0^2$$

$$= \frac{2^3}{3} - \frac{2^2}{2} + 1(2) - 0$$

$$= \frac{8}{3}\ unit^2$$



Exercise

Find the area of the region bounded by the graphs of $y = 3 - x^2$ and $y = 2x$

Solution

$$x^2 + 2x - 3 = 0 \quad \rightarrow \boxed{x = 1, -3}$$

$$A = \int_{-3}^{1} \left(\left(3 - x^2\right) - 2x\right)dx$$

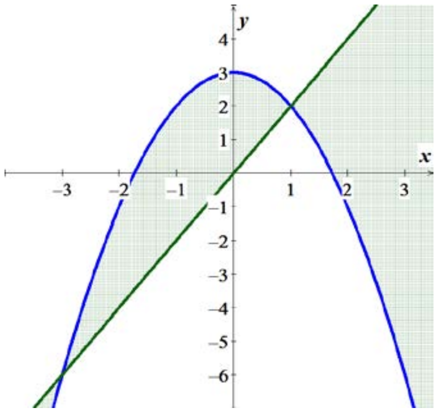
$$= \int_{-3}^{1} \left(-x^2 - 2x + 3\right)dx$$

$$= -\frac{x^3}{3} - 2\frac{x^2}{2} + 3x \Big|_{-3}^{1}$$

$$= -\frac{1^3}{3} - 1^2 + 3(1) - \left[-\frac{(-3)^3}{3} - (-3)^2 + 3(-3)\right] \quad = -\frac{1}{3} - 1 + 3 - [9 - 9 - 9]$$

$$= 11 - \frac{1}{3}$$

$$= \frac{32}{3}\ unit^2$$



Exercise

Find the area of the region bounded by the graphs of $y = x^2 - x - 2$ and x-axis

Solution

The intersection points: $x^2 - x - 2 = 0 \;\Rightarrow \boxed{x = -1, 2}$

$$A = \int_{-1}^{2} \left[0 - (x^2 - x - 2)\right] dx$$

$$= -\frac{x^3}{3} + \frac{x^2}{2} + 2x \Big|_{-1}^{2}$$

$$= -\frac{2^3}{3} + \frac{2^2}{2} + 2(2) - \left[-\frac{(-1)^3}{3} + \frac{(-1)^2}{2} + 2(-1) \right]$$

$$= -\frac{8}{3} + 2 + 4 - \left[\frac{1}{3} + \frac{1}{2} - 2 \right]$$

$$= \frac{10}{3} + \frac{7}{6}$$

$$= \frac{9}{2} \;\; unit^2$$

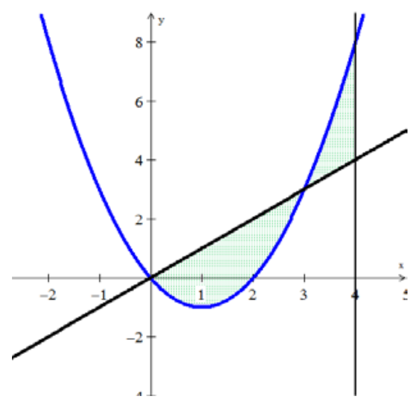
Exercise

Find the area between the curves $y = x^{1/2}$ *and* $y = x^3$

Solution

$x^3 = x^{1/2}$ *Square both sides* $\rightarrow x^6 = x$

$x\left(x^5 - 1\right) = 0 \quad \rightarrow \underline{x = 0} \quad x^5 - 1 = 0 \Rightarrow \underline{x = 1}$



$$A = \int_{0}^{1} \left(x^{1/2} - x^3\right) dx$$

$$= \frac{2}{3}x^{3/2} - \frac{1}{4}x^4 \Big|_{0}^{1}$$

$$= \frac{2}{3}1^{3/2} - \frac{1}{4}1^4 - 0$$

$$= \frac{2}{3} - \frac{1}{4}$$

$$= \frac{8-3}{12}$$

$$= \frac{5}{12} \;\; unit^2$$

Exercise

Find the area of the region bounded by the graphs of $y = x^2 - 2x$ and $y = x$ on [0, 4].

Solution

$$x^2 - 2x = x \quad x^2 - 3x = 0$$

$$x(x-3) = 0 \Rightarrow \boxed{x = 0, 3}$$

$$A = \int_0^3 \left(x - \left(x^2 - 2x\right)\right)dx + \int_3^4 \left(x^2 - 2x - x\right)dx$$

$$= \int_0^3 \left(-x^2 + 3x\right)dx + \int_3^4 \left(x^2 - 3x\right)dx$$

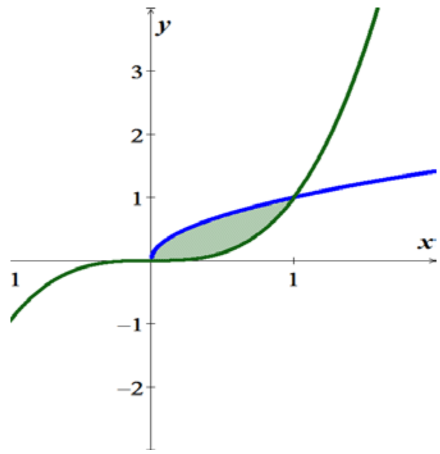
$$= \left(-\tfrac{1}{3}x^3 + \tfrac{3}{2}x^2\right)\Big|_0^3 + \left(\tfrac{1}{3}x^3 - \tfrac{3}{2}x^2\right)\Big|_3^4$$

$$= \left(-\tfrac{1}{3}3^3 + \tfrac{3}{2}3^2\right) + \left[\left(\tfrac{1}{3}4^3 - \tfrac{3}{2}4^2\right) - \left(\tfrac{1}{3}3^3 - \tfrac{3}{2}3^2\right)\right]$$

$$= \left(\tfrac{9}{2}\right) + \left[\left(-\tfrac{8}{3}\right) - \left(-\tfrac{9}{2}\right)\right]$$

$$= \tfrac{9}{2} - \tfrac{8}{3} + \tfrac{9}{2}$$

$$= \tfrac{19}{3} \ \ unit^2$$



Exercise

Find the area between the curves $x = 1, \quad x = 2, \quad y = x^3 + 2, \quad y = 0$

Solution

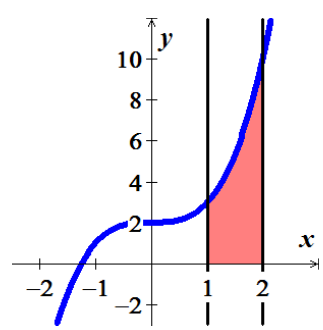
$$A = \int_1^2 \left(x^3 + 2 - 0\right)dx$$

$$= \tfrac{1}{4}x^4 + 2x\Big|_1^2$$

$$= \left(\tfrac{1}{4}2^4 + 2(2)\right) - \left(\tfrac{1}{4}1^4 + 2(1)\right)$$

$$= (8) - \left(\tfrac{9}{4}\right)$$

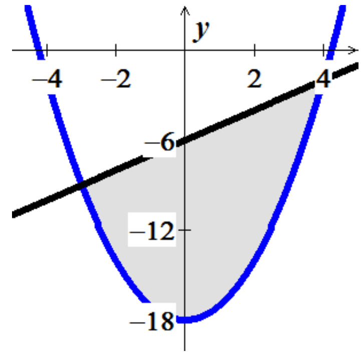
$$= \tfrac{23}{4} \ \ unit^2$$



Exercise

Find the area between the curves $y = x^2 - 18, \quad y = x - 6$

Solution



$$x^2 - 18 = x - 6$$

$$x^2 - x - 12 = 0 \rightarrow \boxed{x = -3,\ 4}$$

$$A = \int_{-3}^{4} \left(x^2 - 18 - (x-6) \right) dx$$

$$= \int_{-3}^{4} \left(x^2 - x - 12 \right) dx$$

$$= \frac{1}{3}x^3 - \frac{1}{2}x^2 - 12x \Big|_{-3}^{4}$$

$$= \left(\frac{1}{3}4^3 - \frac{1}{2}4^2 - 12(4) \right) - \left(\frac{1}{3}(-3)^3 - \frac{1}{2}(-3)^2 - 12(-3) \right)$$

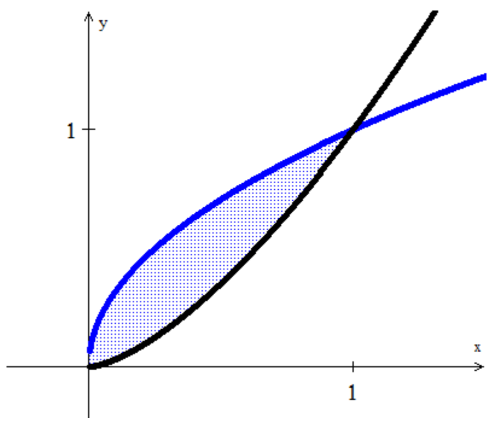
$$= \left(-\frac{104}{3} \right) - \left(\frac{45}{2} \right)$$

$$= \frac{343}{6}\ unit^2$$

Exercise

Find the area between the curves $y = \sqrt{x}, \quad y = x\sqrt{x}$

Solution



$$x\sqrt{x} = \sqrt{x} \quad \Rightarrow \quad \left(x\sqrt{x} \right)^2 = \left(\sqrt{x} \right)^2$$

$$x^2 x = x \quad \rightarrow \quad x\left(x^2 - 1 \right) = 0$$

$$\boxed{x = 0} \quad x^2 - 1 = 0 \Rightarrow x = \pm 1 (no\ negative) \quad \boxed{x = 1}$$

$$A = \int_0^1 \left(\sqrt{x} - x\sqrt{x} \right) dx$$

$$= \int_0^1 \left(x^{1/2} - x^{3/2} \right) dx$$

$$= \frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} \Big|_0^1$$

$$= \left(\frac{2}{3}1^{3/2} - \frac{2}{5}1^{5/2} \right) - \left(\frac{2}{3}0^{3/2} - \frac{2}{5}0^{5/2} \right)$$

$$= \left(\frac{2}{3} - \frac{2}{5} \right) - 0$$

$$= \frac{4}{15}\ unit^2$$

Exercise

Find the area of the region bounded by the graphs of $f(x) = x^3 + 2x^2 - 3x$ and $g(x) = x^2 + 3x$

Solution

$$x^3 + 2x^2 - 3x = x^2 + 3x \quad \to \quad x^3 + x^2 - 6x = 0$$

$$x(x^2 + x - 6) = 0 \Rightarrow \begin{cases} x = 0 \\ x^2 + x - 6 = 0 \end{cases} \to \boxed{x = -3,\ 0,\ 2}$$

$$A = \int_{-3}^{0} (f - g)dx + \int_{0}^{2} (g - f)dx$$

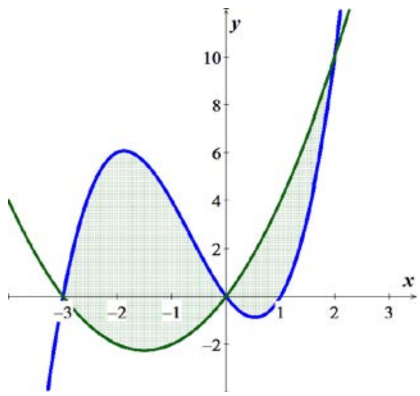
$$= \int_{-3}^{0} (x^3 + 2x^2 - 3x - (x^2 + 3x))dx + \int_{0}^{2} (x^2 + 3x - (x^3 + 2x^2 - 3x))dx$$

$$= \int_{-3}^{0} (x^3 + x^2 - 6x)dx + \int_{0}^{2} (-x^3 - x^2 + 6x)dx$$

$$= \frac{x^4}{4} + \frac{x^3}{3} - 3x^2 \Big|_{-3}^{0} + \left[-\frac{x^4}{4} - \frac{x^3}{3} + 3x^2 \right]_0^2$$

$$= 0 - \left(\frac{(-3)^4}{4} + \frac{(-3)^3}{3} - 3(-3)^2 \right) + \left[\left(-\frac{2^4}{4} - \frac{2^3}{3} + 32^2 \right) - 0 \right]$$

$$= \frac{253}{12}\ unit^2 \qquad \approx 21.083$$



Exercise

Find the area of the region bounded by the graphs of $y = -x^2 + 3x + 1, \quad y = -x + 1$

Solution

$$y = -x^2 + 3x + 1 = -x + 1 \ \to\ x^2 - 4x = 0 \ \Rightarrow\ x = 0,\ 4$$

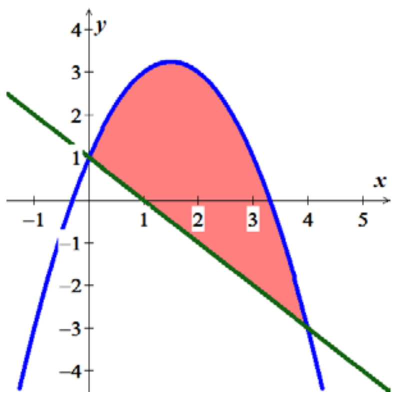
$$A = \int_{0}^{4} \left[-x^2 + 3x + 1 - (-x + 1) \right] dx$$

$$= \int_{0}^{4} \left(-x^2 + 4x \right) dx$$

$$= -\frac{1}{3}x^3 + 2x^2 \Big|_{0}^{4}$$

$$= -\frac{64}{3} + 32$$

$$= \frac{32}{3}\ unit^2$$



Exercise

Find the area of the region bounded by the graphs of $\quad y = x, \quad y = 2 - x, \quad y = 0$

Solution

$y = x = 2 - x \;\rightarrow\; \underline{x = 1}$

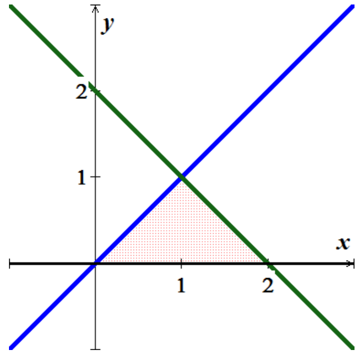
$y = 2 - x = 0 \;\rightarrow\; \underline{x = 2}$

$$A = \int_0^1 (x - 0)\, dx + \int_1^2 (2 - x - 0)\, dx$$

$$= \frac{1}{2}x^2 \Big|_0^1 + \left(2x - \frac{1}{2}x^2\right)\Big|_1^2$$

$$= \frac{1}{2} + 4 - 2 - 2 + \frac{1}{2}$$

$$\underline{= 1 \;\; unit^2}$$



Exercise

Find the area of the region bounded by the graphs of $\quad y = \frac{4}{x^2}, \quad y = 0, \quad x = 1, \quad x = 4$

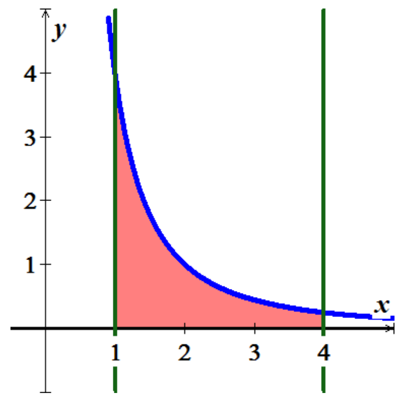
Solution

$$A = \int_1^4 \frac{4}{x^2}\, dx$$

$$= -\frac{4}{x}\Big|_1^4$$

$$= 4\left(-\frac{1}{4} + 1\right)$$

$$\underline{= 3 \;\; unit^2}$$



Exercise

Find the area of the region bounded by the graphs of

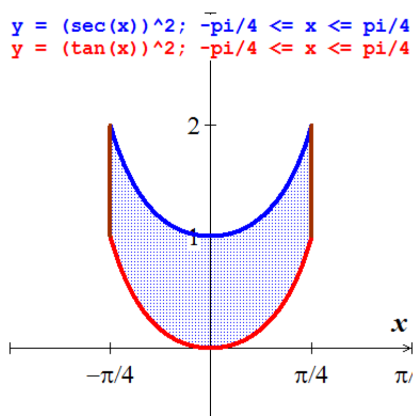
$$y = \sec^2 x, \quad y = \tan^2 x, \quad x = -\frac{\pi}{4}, \quad and \quad x = \frac{\pi}{4}$$

Solution

$$A = \int_{-\pi/4}^{\pi/4} \left(\sec^2 x - \tan^2 x\right) dx$$

$$= \int_{-\pi/4}^{\pi/4} \left(\sec^2 x - \left(\sec^2 x - 1\right)\right) dx$$

$$= \int_{-\pi/4}^{\pi/4} dx$$



$$= x\Big|_{-\pi/4}^{\pi/4}$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right)$$

$$= \frac{\pi}{2}\ \ unit^2$$

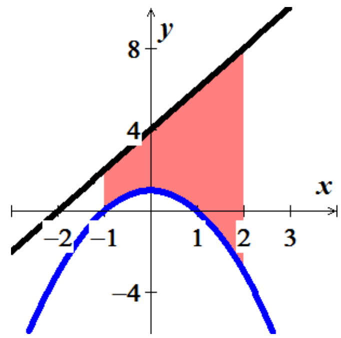
Exercise

Find the area bounded by $f(x) = -x^2 + 1,\ g(x) = 2x + 4,\ x = -1,\ x = 2$

Solution

$$f \cap g \Rightarrow \quad -x^2 + 1 = 2x + 4$$

$$x^2 + 2x + 3 = 0 \quad \Rightarrow x = -1 \pm i\sqrt{2}$$



$$A = \int_{-1}^{2} (g - f)dx$$

$$= \int_{-1}^{2} \left(2x + 4 - \left(-x^2 + 1\right)\right)dx$$

$$= \int_{-1}^{2} \left(x^2 + 2x + 3\right)dx$$

$$= \frac{1}{3}x^3 + x^2 + 3x\Big|_{-1}^{2}$$

$$= \left(\frac{1}{3}(2)^3 + (2)^2 + 3(2)\right) - \left(\frac{1}{3}(-1)^3 + (-1)^2 + 3(-1)\right)$$

$$= \left(\frac{8}{3} + 4 + 6\right) - \left(-\frac{1}{3} + 1 - 3\right)$$

$$= \frac{8}{3} + 10 + \frac{1}{3} + 2$$

$$= 15\ \ unit^2$$

Exercise

Find the area of the region bounded by the graphs of $\quad f(x) = \sqrt{x} + 3, \quad g(x) = \frac{1}{2}x + 3$

Solution

$$\sqrt{x} + 3 = \frac{1}{2}x + 3 \ \Rightarrow\ \left(\sqrt{x}\right)^2 = \left(\frac{1}{2}x\right)^2$$

$$x = \frac{1}{4}x^2 \to x = 0,\ 4$$

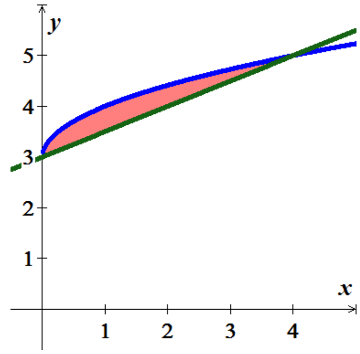
$$A=\int_0^4\left(\sqrt{x}+3-\frac{1}{2}x-3\right)dx$$

$$=\int_0^4\left(x^{1/2}-\frac{1}{2}x\right)dx$$

$$=\frac{2}{3}x^{3/2}-\frac{1}{4}x^2\Big|_0^4$$

$$=\frac{16}{3}-4$$

$$=\frac{4}{3}\ unit^2$$



Exercise

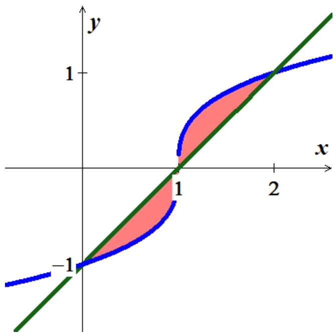
Find the area of the region bounded by the graphs of $f(x)=\sqrt[3]{x-1},\quad g(x)=x-1$

Solution

$$\left(\sqrt[3]{x-1}\right)^3=(x-1)^3$$

$$x-1=x^3-3x^2+3x-1$$

$$x\left(x^2-3x+2\right)=0\ \rightarrow\ x=0,\ 1,\ 2$$



$$A=\int_0^1\left(x-1-\sqrt[3]{x-1}\right)dx+\int_1^2\left(\sqrt[3]{x-1}-x+1\right)dx$$

$$=\left[\frac{1}{2}x^2-x-\frac{3}{4}(x-1)^{4/3}\right]_0^1+\left[\frac{3}{4}(x-1)^{4/3}-\frac{1}{2}x^2+x\right]_1^2$$

$$=\frac{1}{2}-1+\frac{3}{4}+\frac{3}{4}-2+2+\frac{1}{2}-1$$

$$=\frac{1}{2}\ unit^2$$

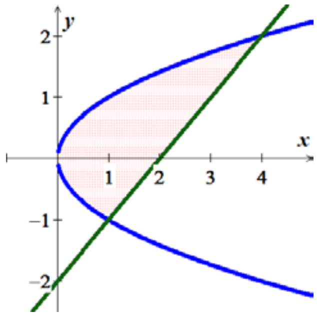
Exercise

Find the area of the region bounded by the graphs of $f(y)=y^2,\quad g(y)=y+2$

Solution

$$y^2=y+2\ \Rightarrow\ y^2-y-2=0\ \rightarrow\ y=-1,\ 2$$

$$A=\int_{-1}^2\left(y+2-y^2\right)dy$$



$$= \frac{1}{2}y^2 + 2y - \frac{1}{3}y^3 \Big|_{-1}^{2}$$

$$= 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}$$

$$= \frac{9}{2} \ \ unit^2$$

Exercise

Find the area of the region bounded by the graphs of $f(y) = y(2-y), \quad g(y) = -y$

Solution

$$2y - y^2 = -y \quad \Rightarrow y^2 - 3y = 0 \ \rightarrow \ y = 0, \ 3$$

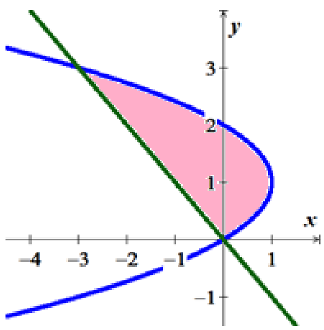
$$A = \int_0^3 \left(2y - y^2 + y\right) dy$$

$$= \int_0^3 \left(3y - y^2\right) dy$$

$$= \frac{3}{2}y^2 - \frac{1}{3}y^3 \Big|_0^3$$

$$= \frac{27}{2} - 9$$

$$= \frac{9}{2} \ \ unit^2$$



Exercise

Find the area of the region bounded by the graphs of $f(y) = y^2 + 1, \quad g(y) = 0, \quad y = -1, \quad y = 2$

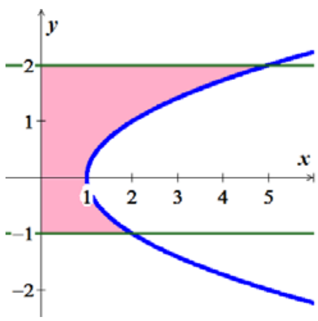
Solution

$$A = \int_{-1}^{2} \left(y^2 + 1 - 0\right) \ dy$$

$$= \frac{1}{3}y^3 + y \Big|_{-1}^{2}$$

$$= \frac{8}{3} + 2 + \frac{1}{3} + 1$$

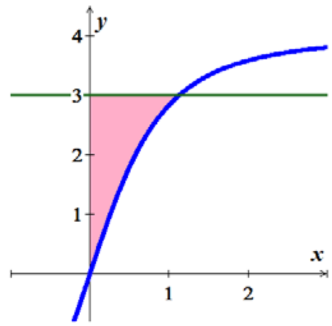
$$= 6 \ \ unit^2$$



Exercise

Find the area of the region bounded by the graphs of $f(y)=\dfrac{y}{\sqrt{16-y^2}}, \quad g(y)=0, \quad y=3$

Solution



$$A=\int_0^3\left(\frac{y}{\sqrt{16-y^2}}-0\right)dy$$

$$=-\frac{1}{2}\int_0^3\left(16-y^2\right)^{-1/2}\,d\left(16-y^2\right)$$

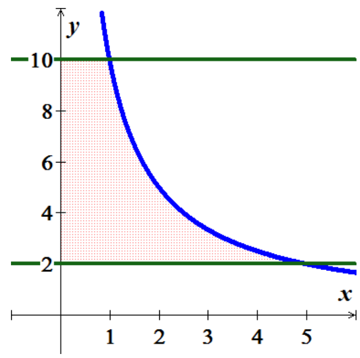
$$=-\sqrt{16-y^2}\,\Big|_0^3$$

$$\underline{=-\sqrt{7}+4\ \ unit^2}$$

Exercise

Find the area of the region bounded by the graphs of $f(x)=\dfrac{10}{x}, \quad x=0, \quad y=2, \quad y=10$

Solution



$$y=\frac{10}{x}\ \Rightarrow\ x=\frac{0}{y}$$

$$A=\int_2^{10}\frac{10}{y}\,dy$$

$$=10\ln y\,\Big|_2^{10}$$

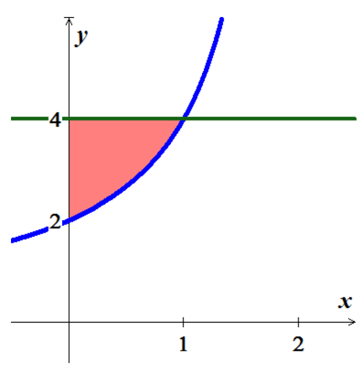
$$=10(\ln 10-\ln 2)$$

$$\underline{=10\ln 5\ \ unit^2}$$

Exercise

Find the area of the region bounded by the graphs of $g(x)=\dfrac{4}{2-x}, \quad y=4, \quad x=0$

Solution



$$\frac{4}{2-x}=4\ \Rightarrow\ 2-x=1\ \rightarrow\ \underline{x=1}$$

$$A=\int_0^1\left(4-\frac{4}{2-x}\right)dx \qquad\qquad \int\frac{dx}{ax+b}=\frac{1}{a}\ln|ax+b|$$

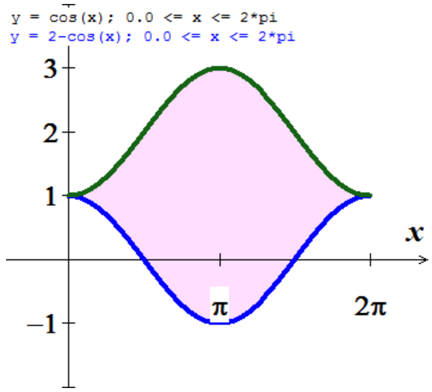
$$=4x+4\ln|2-x|\,\Big|_0^1$$

$$\underline{=4+4\ln 2\ \ unit^2}$$

Exercise

Find the area of the region bounded by the graphs of $f(x)=\cos x, \quad g(x)=2-\cos x, \quad 0\le x\le 2\pi$

Solution



$$A=\int_0^{2\pi}(2-\cos x-\cos x)\,dx$$

$$=2\int_0^{2\pi}(1-\cos x)\,dx$$

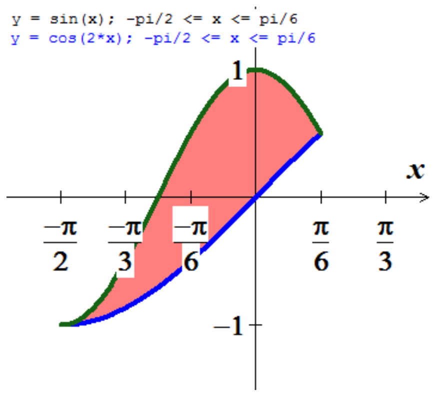
$$=2(x-\sin x)\Big|_0^{2\pi}$$

$$\underline{=4\pi}$$

Exercise

Find the area of the region bounded by the graphs of $f(x)=\sin x, \quad g(x)=\cos 2x, \quad -\frac{\pi}{2}\le x\le \frac{\pi}{6}$

Solution



$$A=\int_{-\pi/2}^{\pi/6}(\cos 2x-\sin x)\,dx$$

$$=\frac{1}{2}\sin 2x+\cos x\Big|_{-\pi/2}^{\pi/6}$$

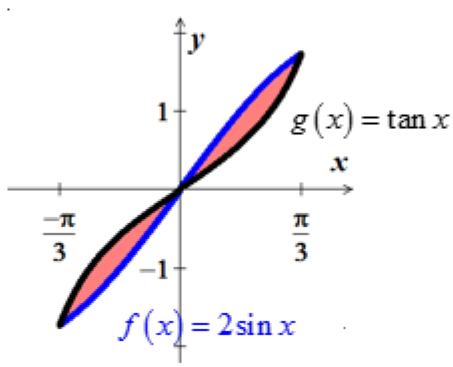
$$=\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{2}$$

$$\underline{=\frac{3\sqrt{3}}{4}}$$

Exercise

Find the area of the region bounded by the graphs of $f(x)=2\sin x, \quad g(x)=\tan x, \quad -\frac{\pi}{3}\le x\le \frac{\pi}{3}$

Solution



$$A=2\int_0^{\pi/3}(2\sin x-\tan x)\,dx$$

$$=2\left(-2\cos x+\ln|\cos x|\right)\Big|_0^{\pi/3}$$

$$=2\left(-1+\ln\frac{1}{2}+2\right)$$

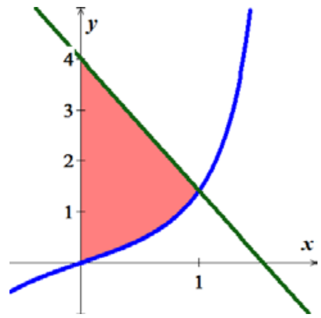
$$\underline{=2(1-\ln 2)}$$

Exercise

Find the area of the region bounded by the graphs of

$$f(x) = \sec\frac{\pi x}{4}\tan\frac{\pi x}{4}, \quad g(x) = (\sqrt{2} - 4)x + 4, \quad x = 0$$

Solution



$$A = \int_0^1 \left((\sqrt{2} - 4)x + 4 - \sec\frac{\pi x}{4}\tan\frac{\pi x}{4} \right) dx$$

$$= \frac{1}{2}(\sqrt{2} - 4)x^2 + 4x - \frac{4}{\pi}\sec\frac{\pi x}{4} \Big|_0^1$$

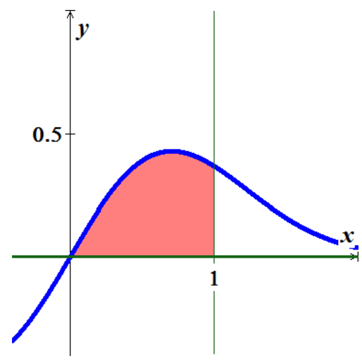
$$= \frac{1}{2}\sqrt{2} - 2 + 4 - \frac{4}{\pi}\sqrt{2} + \frac{4}{\pi}$$

$$= \frac{\sqrt{2}}{2} + 2 + \frac{4}{\pi}(1 - \sqrt{2})$$

Exercise

Find the area of the region bounded by the graphs of $f(x) = xe^{-x^2}, \quad y = 0, \quad 0 \le x \le 1$

Solution



$$A = \int_0^1 xe^{-x^2} dx$$

$$= -\frac{1}{2}\int_0^1 e^{-x^2} d\left(-x^2\right)$$

$$= -\frac{1}{2}e^{-x^2} \Big|_0^1$$

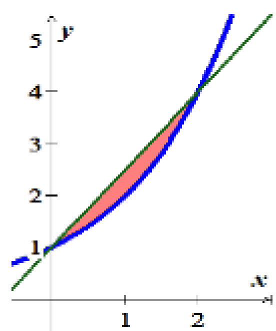
$$= -\frac{1}{2}\left(e^{-1} - 1\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{e}\right)$$

Exercise

Find the area of the region bounded by the graphs of $f(x) = 2^x, \quad g(x) = \frac{3}{2}x + 1$

Solution



$$A = \int_0^2 \left(\frac{3}{2}x + 1 - 2^x \right) dx$$

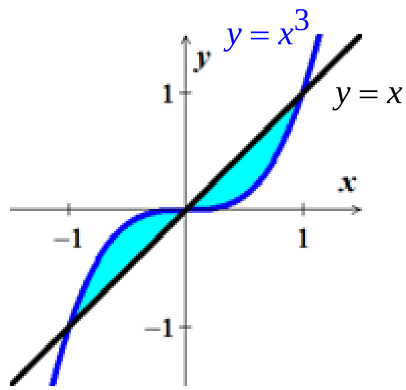
$$= \frac{3}{4}x^2 + x - \frac{2^x}{\ln 2} \Big|_0^2$$

$$= 3 + 2 - \frac{4}{\ln 2} + \frac{1}{\ln 2}$$

$$\underline{= 5 - \frac{3}{\ln 2}}$$

Exercise

Determine the area of the shaded region in



Solution

$$y = x^3 = x \quad \rightarrow \quad x\left(x^2 - 1\right) = 0 \quad \therefore x \underline{= 0, \ \pm 1}$$

$$Area = \int_{-1}^{0} \left(x^3 - x\right) dx + \int_{0}^{1} \left(x - x^3\right) dx$$

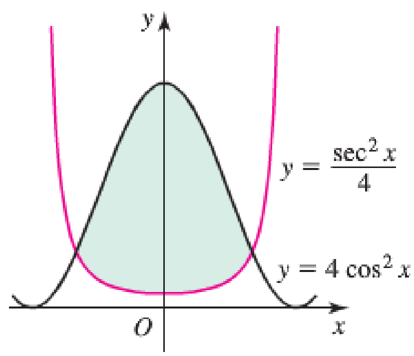
$$= \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_{-1}^{0} + \left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_{0}^{1}$$

$$= -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4}$$

$$\underline{= \frac{1}{2} \ unit^2}$$

Exercise

Determine the area of the shaded region in



Solution

$$y = \frac{\sec^2 x}{4} = 4\cos^2 x \quad \rightarrow \quad \cos^4 x = \frac{1}{16}$$

$$\cos x = \pm\frac{1}{2} \quad \rightarrow \quad \underline{x = \pm\frac{\pi}{3}}$$

By the symmetry;

$$Area = 2\int_{0}^{\pi/3} \left(4\cos^2 x - \frac{1}{4}\sec^2 x\right) dx$$

$$= 2\int_{0}^{\pi/3} \left(2 + 2\cos 2x - \frac{1}{4}\sec^2 x\right) dx$$

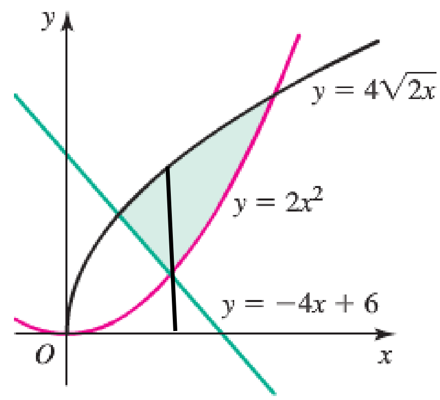
$$= 2\left[2x + \sin 2x - \frac{1}{4}\tan x\right]_{0}^{\pi/3}$$

$$= 2\left(\frac{2\pi}{3} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4}\right)$$

$$\underline{= \frac{4\pi}{3} + \frac{\sqrt{3}}{2} \ unit^2}$$

Exercise

Determine the area of the shaded region in



Solution

$$y=4\sqrt{2x}=-4x+6 \quad \rightarrow \quad \left(4\sqrt{2x}\right)^2=(-4x+6)^2$$

$$32x=16x^2-48x+36$$

$$16x^2-80x+36=0 \quad \rightarrow \quad x=\frac{1}{2}, \ \cancel{\frac{9}{2}}$$

$$y=4\sqrt{2x}=2x^2 \quad \rightarrow \quad \left(4\sqrt{2x}\right)^2=\left(2x^2\right)^2$$

$$32x=4x^4 \ \rightarrow \ 4x\left(x^3-8\right)=0 \ \rightarrow \ x=2, \ \cancel{0}$$

$$y=2x^2=-4x+6 \quad \rightarrow \quad x^2+2x-3=0 \quad \rightarrow \quad x=1, \ \cancel{-3}$$

$$Area=\int_{1/2}^{1}\left(4\sqrt{2x}-(-4x+6)\right)dx+\int_{1}^{2}\left(4\sqrt{2x}-2x^2\right)dx$$

$$=\left(\frac{8\sqrt{2}}{3}x^{3/2}+2x^2-6x\right)\Big|_{1/2}^{1}+\left(\frac{8\sqrt{2}}{3}x^{3/2}-\frac{2}{3}x^3\right)\Big|_{1}^{2}$$

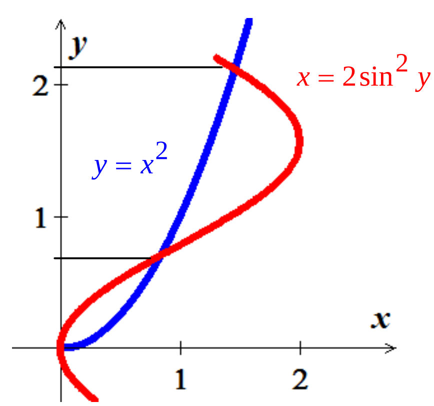
$$=\left(\frac{8\sqrt{2}}{3}+2-6-\frac{8\sqrt{2}}{3}\frac{1}{2\sqrt{2}}-\frac{1}{2}+3\right)+\left(\frac{32}{3}-\frac{16}{3}-\frac{8\sqrt{2}}{3}+\frac{2}{3}\right)$$

$$=-1-\frac{4}{3}-\frac{1}{2}+6$$

$$=\boxed{\frac{19}{6} \ unit^2}$$

Exercise

Determine the area of the shaded region in



Solution

From the graph the intersection are: $y=0, \ \ y\approx .705, \ \ y\approx 2.12$

$$A=\int_0^{.705}\left(\sqrt{y}-2\sin^2 y\right)dy+\int_{.705}^{2.12}\left(2\sin^2 y-\sqrt{y}\right)dy$$

$$=\int_0^{.705}\left(y^{1/2}-1+\cos 2y\right)dy+\int_{.705}^{2.12}\left(1-\cos 2y-y^{1/2}\right)dy$$

$$=\left(\frac{2}{3}y^{3/2}-y+\frac{1}{2}\sin 2y\right)\Big|_0^{.705}+\left(y-\frac{1}{2}\sin 2y-\frac{2}{3}y^{3/2}\right)\Big|_{.705}^{2.12}$$

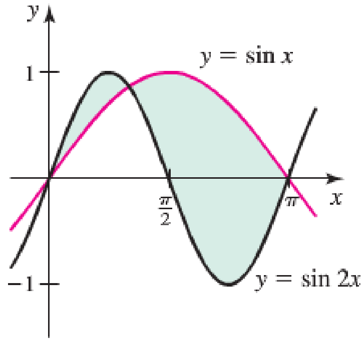
$$=\frac{2}{3}(.705)^{3/2}-0.705+\frac{1}{2}\sin(1.41)+2.12-\frac{1}{2}\sin(4.24)-\frac{2}{3}(2.12)^{3/2}-.705+\frac{1}{2}\sin(1.41)+\frac{2}{3}(.705)^{3/2}$$

$$\approx \boxed{.8738 \ \ unit^2}$$

Exercise

Determine the area of the shaded regions between $y = \sin x$ *and* $y = \sin 2x$, *for* $0 \le x \le \pi$

Solution



$y = \sin x = \sin 2x$

$\sin x = 2\sin x \cos x \quad \rightarrow \quad \sin x(2\cos x - 1) = 0$

$\sin x = 0 \quad \rightarrow \underline{x = 0,\ \pi}|$

$\cos x = \frac{1}{2} \quad \rightarrow \underline{x = \frac{\pi}{3}}|$

$$A = \int_0^{\pi/3} (\sin 2x - \sin x)\,dx + \int_{\pi/3}^{\pi} (\sin x - \sin 2x)\,dx$$

$$= \left[-\frac{1}{2}\cos 2x + \cos x\right]_0^{\pi/3} + \left[-\cos x + \frac{1}{2}\cos 2x\right]_{\pi/3}^{\pi}$$

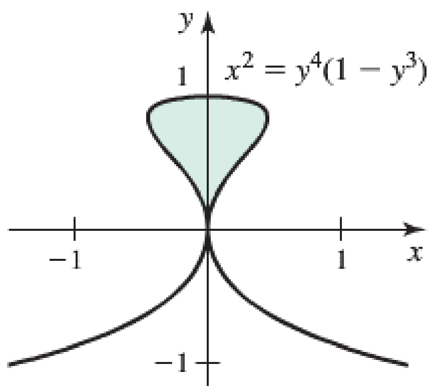
$$= \left(\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - 1\right) + \left(1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4}\right)$$

$$= \frac{5}{2}\ unit^2$$

Exercise

Determine the area of the shaded region bounded by the curve $x^2 = y^4\left(1 - y^3\right)$

Solution



$x^2 = y^4\left(1 - y^3\right) \rightarrow x = y^2\sqrt{1 - y^3}$

Since it is symmetric about y-axis, then

$$A = 2\int_0^1 y^2\sqrt{1 - y^3}\ dy$$

$$= -\frac{2}{3}\int_0^1 \left(1 - y^3\right)^{1/2}\ d\left(1 - y^3\right)$$

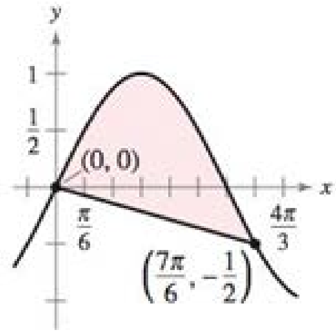
$$= -\frac{4}{9}\left(1 - y^3\right)^{3/2}\Big|_0^1$$

$$= \frac{4}{9}\ unit^2$$

Exercise

Find the area between the graph of $y = \sin x$ and the line segment joining the points $(0,\ 0)$ and $\left(\frac{7\pi}{6},\ -\frac{1}{2}\right)$.

Solution



$$\text{Line: } y = \frac{-\frac{1}{2}}{\frac{7\pi}{6}}\left(x - \frac{7\pi}{6}\right) - \frac{1}{2}$$

$$= -\frac{3}{7\pi}\left(x - \frac{7\pi}{6}\right) - \frac{1}{2}$$

$$\underline{= -\frac{3}{7\pi}x}$$

$$A = \int_0^{7\pi/6}\left(\sin x + \frac{3}{7\pi}x\right)dx$$

$$= -\cos x + \frac{3}{14\pi}x^2 \Big|_0^{7\pi/6}$$

$$\underline{= \frac{\sqrt{3}}{2} + \frac{7\pi}{24} + 1}$$

Exercise

The surface of a machine part is the region between the graphs of $y_1 = |x|$ and $y_2 = 0.08x^2 + k$

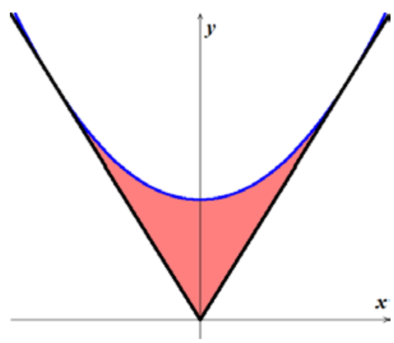
a) Find k where the parabola is tangent to the graph of y_1

b) Find the area of the surface of the machine part.

Solution

a) $y_1' = 1 \quad y_2' = 0.16x \quad \Rightarrow 0.16x = 1 \ \rightarrow \ \underline{x = \frac{1}{0.16} = 6.25}$

$y_1 = y_2$



$6.25 = 0.08(6.25)^2 + k$

$k = 6.25 - 0.08(6.25)^2 \ \underline{= 3.125}$

b) $A = 2\int_0^{6.25}\left(y_2 - y_1\right)dx$

$$= 2\int_0^{6.25}\left(0.08x^2 + 3.125 - x\right)dx$$

$$= 2\left(\frac{.08}{3}x^3 + 3.125x - \frac{1}{2}x^2\right)\Big|_0^{6.25}$$

$$\underline{\approx 13.02083 \ \ unit^2}$$

Exercise

Concrete sections for a new building have the dimensions (in meters) and shape shown in figure

a) Find the area of the face of the section superimposed on the rectangularcoordinate system.
b) Find the volume of concrete in one of the sections by multiplying the area in part (*a*) by 2 *meters*.
c) One cubic meter of concrete weighs 5,000 *pounds*. Find the weight of the section.

Solution

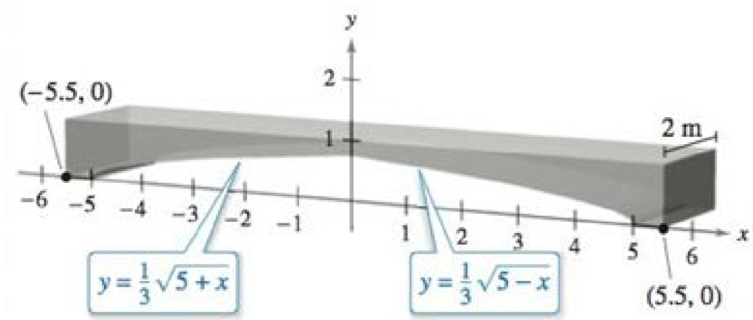
a) $A=2\int_0^5\left(1-\frac{1}{3}\sqrt{5+x}\right)dx+2\int_5^{5.5}(1-0)\,dx$

$=2\left[x+\frac{2}{9}(5-x)^{3/2}\right]_0^5+2x\Big|_5^{5.5}$

$=2\left(5-\frac{2}{9}5^{3/2}\right)+2(5.5-5)$

$=10-\frac{20\sqrt{5}}{9}+1$

$=11-\frac{20\sqrt{5}}{9}\ \ m^2$



b) $V=2A=22-\frac{40\sqrt{5}}{9}\ \ m^3$

c) $W=5{,}000V=\left(11-\frac{20\sqrt{5}}{9}\right)\times 10^4\ \ lb$

Exercise

A Lorenz curve is given by $y=L(x)$, where $0\le x\le 1$ represents the lowest fraction of the population of a society in terms of wealth and $0\le y\le 1$ represents the fraction of the total wealth that is owned by that fraction of the society. For example, the Lorenz curve in the figure shows that $L(0.5)=0.2$, which means that the lowest 0.5 (50%) of the society owns 0.2 (20%) of the wealth.

a) A Lorenz curve $y=L(x)$ is accompanied by the line $y=x$, called the ***line of perfect equality***. Explain why this line is given the name.
b) Explain why a Lorenz curve satisfies the conditions $L(0)=0,\ L(1)=1,\ and\ L'(x)\ge 0\quad on\ [0,\ 1]$
c) Graph the Lorenz curves $L(x)=x^p$ corresponding to *p* = 1.1, 1.5, 2, 3, 4. Which value of *p* corresponds to the ***most*** equitable distribution of wealth (closest to the line of perfect equality)? Which value of *p* corresponds to the ***least*** equitable distribution of wealth? Explain.
d) The information in the Lorenz curve is often summarized in a single measure called the ***Gini index***, which is defined as follows. Let *A* be the area of the region between $y=x$ and $y=L(x)$ and Let *B* be the area of the region between $y=L(x)$ and the *x*-axis. Then the Gini index is $G=\frac{A}{A+B}$.

Show that $G = 2A = 1 - 2\int_0^1 L(x)\,dx$.

e) Compute the Gini index for the cases $L(x) = x^p$ and p = 1.1, 1.5, 2, 3, 4.

f) What is the smallest interval [*a*, *b*] on which values of the Gini index lie, for $L(x) = x^p$ with $p \ge 1$? Which endpoints of [*a*, *b*] correspond to the least and most equitable distribution of wealth?

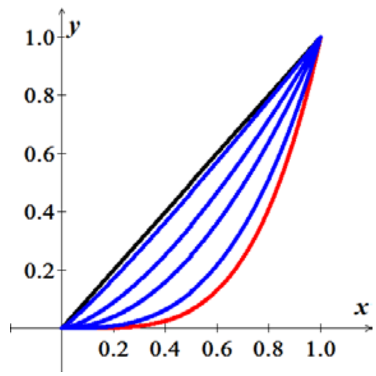
g) Consider the Lorenz curve described by $L(x) = \frac{5x^2}{6} + \frac{x}{6}$. Show that it satisfies the conditions $L(0) = 0,\ L(1) = 1,\ and\ L'(x) \ge 0\quad on\ [0,\ 1]$. Find the Gini index for this function.

Solution

a) Let the point $N = (a,\ a)$ on the curve $y = x$ would represent the notion that the lowest *p*% of the society owns *p*% of the wealth, which would represent a form of equality.

b) The function must be increasing and concave up because the poorest *p*% cannot own more than *p*% of the wealth.

c) $y = x^{1.1}$ is closet to $y = x$, and $y = x^4$ is furthest from $y = x$



d) Since, $B = \int_0^1 L(x)\,dx$ and $A + B = \frac{1}{2}$

Then $A = \frac{1}{2} - B = \frac{1}{2} - \int_0^1 L(x)\,dx$

$$G = \frac{A}{A+B} = \frac{A}{\frac{1}{2}} = 2A = 1 - 2\int_0^1 L(x)\,dx \quad \checkmark$$

e) For $L(x) = x^p$

$$G = 1 - 2\int_0^1 x^p\,dx$$

$$= 1 - \frac{2}{p+1}\left(x^{p+1}\right)\Big|_0^1$$

$$= 1 - \frac{2}{p+1}$$

$$\boxed{= \frac{p-1}{p+1}}$$

P	1.1	1.5	2	3	4
G	$\frac{1}{21}$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{5}$

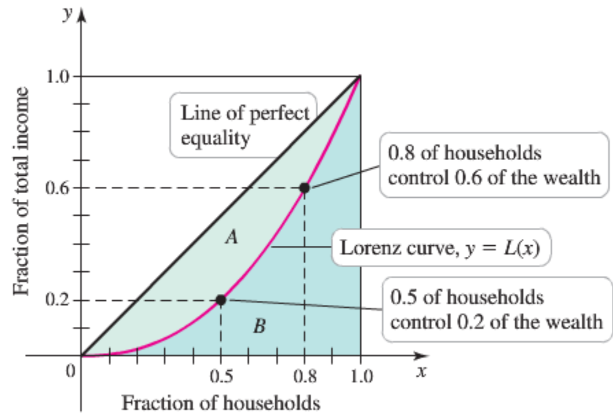
f) For $p = 1 \quad \rightarrow \quad \boxed{G = \frac{p-1}{p+1} = 0}$

$\lim\limits_{p\to\infty} \frac{p-1}{p+1}\ \boxed{= 1}$, the largest value of *G* approaches 1.

g) $L(x) = \frac{5x^2}{6} + \frac{x}{6} \qquad \rightarrow L(0) = 0, \quad L(0) = 1$

$$L'(x) = \frac{5}{3}x + \frac{1}{6} > 0 \quad x \in [0,\ 1]$$

$$L''(x) = \frac{5}{3} > 0$$



The Gini index is:

$$\begin{aligned} G &= 1 - 2\int_0^1 \left(\frac{5x^2}{6} + \frac{x}{6} \right) dx \\ &= 1 - 2\left(\frac{5x^3}{18} + \frac{x^2}{12} \right)\Bigg|_0^1 \\ &= 1 - 2\left(\frac{5}{18} + \frac{1}{12} \right) \\ &= 1 - \frac{5}{9} - \frac{1}{6} \\ &= \frac{5}{18} \end{aligned}$$