# Solution    *Section* 3.6 – Solving Linear Recurrence Relations

### *Exercise*

Determine which of these are linear and homogeneous recurrence relations with constant coefficients. Also find the degree of those that are

*a)* $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$

*b)* $a_n = 2na_{n-1} + a_{n-2}$

*c)* $a_n = a_{n-1} + a_{n-4}$

*d)* $a_n = a_{n-1} + 2$

*e)* $a_n = a_{n-1}^2 + a_{n-2}$

*f)* $a_n = a_{n-2}$

*g)* $a_n = a_{n-1} + n$

*h)* $a_n = 3a_{n-2}$

*i)* $a_n = 3$

*j)* $a_n = a_{n-1}^2$

*k)* $a_n = a_{n-1} + 2a_{n-3}$

*l)* $a_n = \dfrac{a_{n-1}}{n}$

***Solution***

***a)*** Linear (terms $a_i$ all to the first power), has constant coefficients (3, 4 and 5), and is homogeneous (no terms are functions of just $n$); has degree 3

***b)*** Linear (terms $a_i$ all to the first power), doesn't have constant coefficients ($2n$), and is homogeneous

***c)*** Linear, homogeneous, with constant coefficients; degree 4

***d)*** Linear with constant coefficients, not homogeneous because of 2

***e)*** Not linear since $a_{n-1}^2$

***f)*** Linear, homogeneous, with constant coefficients; degree 2

***g)*** Linear but not homogeneous because of the $n$.

***h)*** Linear, homogeneous, with constant coefficients; degree 2

***i)*** Linear with constant coefficients, not homogeneous because of 3

***j)*** Not linear since $a_{n-1}^2$

***k)*** Linear, homogeneous, with constant coefficients; degree 3

***l)*** Linear with constant coefficients, not homogeneous

### *Exercise*

Solve these recurrence relations together with the initial conditions given

*a)* $a_n = 2a_{n-1} \quad for \quad n \ge 1, \ a_0 = 3$

*b)* $a_n = 5a_{n-1} - 6a_{n-2} \quad for \quad n \ge 2, \ a_0 = 1, \ a_1 = 0$

*c)* $a_n = 4a_{n-1} - 4a_{n-2} \quad for \quad n \ge 2, \ a_0 = 6, \ a_1 = 8$

*d)* $a_n = 4a_{n-2} \quad for \quad n \ge 2, \ a_0 = 0, \ a_1 = 4$

*e)* $a_n = \frac{a_{n-2}}{4} \quad for \quad n \ge 2, \ a_0 = 1, \ a_1 = 0$

*f)* $a_n = a_{n-1} + 6a_{n-2} \quad for \quad n \ge 2, \ a_0 = 3, \ a_1 = 6$

*g)* $a_n = 7a_{n-1} - 10a_{n-2} \quad for \quad n \ge 2, \ a_0 = 2, \ a_1 = 1$

*h)* $a_n = -6a_{n-1} - 9a_{n-2} \quad for \quad n \ge 2, \ a_0 = 3, \ a_1 = -3$

*i)* $a_{n+2} = -4a_{n-1} + 5a_n \quad for \quad n \ge 0, \ a_0 = 2, \ a_1 = 8$

***Solution***

***a)*** The characteristic polynomial is $r - 2 = 0 \quad \Rightarrow \quad r = 2$

The general solution: $a_n = \alpha_1 2^n$

$$3 = \alpha_1 2^0 \quad \to \quad \alpha_1 = 3$$

Therefore, the solution is $\boxed{a_n = 3 \cdot 2^n}$

***b)*** The characteristic polynomial is $r^2 - 5r + 6 = 0 \quad \Rightarrow \quad r = 2, \ 3$

The general solution: $a_n = \alpha_1 2^n + \alpha_2 3^n$

$$\begin{cases} 1 = \alpha_1 2^0 + \alpha_2 3^0 & \to \quad 1 = \alpha_1 + \alpha_2 \\ 0 = \alpha_1 2^1 + \alpha_2 3^1 & \to \quad 0 = 2\alpha_1 + 3\alpha_2 \end{cases} \quad \Rightarrow \quad \alpha_1 = 3, \quad \alpha_2 = -2$$

Therefore, the solution is $\boxed{a_n = 3 \cdot 2^n - 2 \cdot 3^n}$

***c)*** The characteristic polynomial is $r^2 - 4r + 4 = 0 \quad \Rightarrow \quad r = 2, \ 2$

The general solution: $a_n = \alpha_1 2^n + \alpha_2 n \cdot 2^n$

$$6=\alpha_1 2^0+\alpha_2(0)2^0 \quad\to\quad 6=\alpha_1$$
$$8=\alpha_1 2^1+\alpha_2(1)2^1 \quad\to\quad 8=2\alpha_1+2\alpha_2$$
$$\Rightarrow \quad \alpha_1=6, \quad \alpha_2=-2$$

Therefore, the solution is $a_n=6\cdot 2^n-2n\cdot 2^n \boxed{=(6-2n)2^n}$

***d)*** The characteristic polynomial is $r^2-4=0 \quad\Rightarrow\quad r=\pm 2$

The general solution: $a_n=\alpha_1(-2)^n+\alpha_2 2^n$

$$0=\alpha_1(-2)^0+\alpha_2 2^0 \quad\to\quad 0=\alpha_1+\alpha_2$$
$$4=\alpha_1(-2)^1+\alpha_2 2^1 \quad\to\quad 4=-2\alpha_1+2\alpha_2$$
$$\Rightarrow \quad \alpha_1=-1, \quad \alpha_2=1$$

Therefore, the solution is $\boxed{a_n=2^n-(-2)^n}$

***e)*** The characteristic polynomial is $r^2-\frac{1}{4}=0 \quad\Rightarrow\quad r=\pm\frac{1}{2}$

The general solution: $a_n=\alpha_1\left(-\frac{1}{2}\right)^n+\alpha_2\left(\frac{1}{2}\right)^n=\alpha_1(-2)^{-n}+\alpha_2(2)^{-n}$

$$1=\alpha_1\left(-\frac{1}{2}\right)^0+\alpha_2\left(\frac{1}{2}\right)^0 \quad\to\quad 1=\alpha_1+\alpha_2$$
$$0=\alpha_1\left(-\frac{1}{2}\right)^1+\alpha_2\left(\frac{1}{2}\right)^1 \quad\to\quad 0=-\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2$$
$$\Rightarrow \quad \alpha_1=\frac{1}{2}, \quad \alpha_2=\frac{1}{2}$$

Therefore, the solution is $a_n=\frac{1}{2}\left(-\frac{1}{2}\right)^n+\frac{1}{2}\left(\frac{1}{2}\right)^n$

$$\boxed{=\left(\frac{1}{2}\right)^{n+1}-\left(-\frac{1}{2}\right)^{n+1}}$$

***f)*** The characteristic polynomial is $r^2-r-6=0 \quad\Rightarrow\quad r=-2,\ 3$

The general solution: $a_n=\alpha_1(-2)^n+\alpha_2 3^n$

$$3=\alpha_1(-2)^0+\alpha_2 3^0 \quad\to\quad 3=\alpha_1+\alpha_2$$
$$6=\alpha_1(-2)^1+\alpha_2 3^1 \quad\to\quad 6=-2\alpha_1+3\alpha_2$$
$$\Rightarrow \quad \alpha_1=\frac{3}{5}, \quad \alpha_2=\frac{12}{5}$$

Therefore, the solution is $\boxed{a_n=\frac{3}{5}(-2)^n+\frac{12}{5}3^n}$

***g)*** The characteristic polynomial is $r^2-7r+10=0 \quad\Rightarrow\quad r=2,\ 5$

The general solution: $a_n=\alpha_1 2^n+\alpha_2 5^n$

$$2=\alpha_1 2^0+\alpha_2 5^0 \quad\to\quad 2=\alpha_1+\alpha_2$$
$$1=\alpha_1 2^1+\alpha_2 5^1 \quad\to\quad 1=2\alpha_1+5\alpha_2$$
$$\Rightarrow \quad \alpha_1=3, \quad \alpha_2=-1$$

Therefore, the solution is $\boxed{a_n=3\cdot 2^n-5^n}$

***h)*** The characteristic polynomial is $r^2+6r+9=0 \quad \Rightarrow \quad r=-3,\ -3$

The general solution: $a_n=\alpha_1(-3)^n+\alpha_2 n(-3)^n$

$$\left.\begin{array}{ll} 3=\alpha_1(-3)^0+\alpha_2(0)(-3)^0 & \rightarrow \quad 3=\alpha_1 \\ -3=\alpha_1(-3)^1+\alpha_2(1)(-3)^1 & \rightarrow \quad -3=-3\alpha_1+-3\alpha_2 \end{array}\right\} \Rightarrow \quad \alpha_1=3, \quad \alpha_2=-2$$

Therefore, the solution is $a_n=3\cdot(-3)^n-2n(-3)^n=(3-2n)(-3)^n$

***i)*** The characteristic polynomial is $r^2+4r-5=0 \quad \Rightarrow \quad r=-5,\ 1$

The general solution: $a_n=\alpha_1(-5)^n+\alpha_2 1^n=\alpha_1(-5)^n+\alpha_2$

$$\left.\begin{array}{ll} 2=\alpha_1(-5)^0+\alpha_2 & \rightarrow \quad 2=\alpha_1+\alpha_2 \\ 8=\alpha_1(-5)^1+\alpha_2 & \rightarrow \quad 8=-5\alpha_1+\alpha_2 \end{array}\right\} \Rightarrow \quad \alpha_1=-1, \quad \alpha_2=3$$

Therefore, the solution is $a_n=-(-5)^n+3$

### *Exercise*

How many different messages can be transmitted in $n$ microseconds using three different signals if one signal requires 1 microsecond for transmittal, the other two signals require 2 microseconds each for transmittal, and a signal in a message is followed immediately by the next signal?

***Solution***

The model is the recurrence relation $a_n=a_{n-1}+a_{n-2}+a_{n-2}=a_{n-1}+2a_{n-2}$ with $a_0=a_1=1$

The characteristic polynomial is $r^2-r-2=0$

So, the roots are $-1$, and 2

The general solution: $a_n=\alpha_1(-1)^n+\alpha_2 2^n$

Plugging in initial conditions gives

$$\left.\begin{array}{ll} 1=\alpha_1(-1)^0+\alpha_2 2^0 & \rightarrow \quad 1=\alpha_1+\alpha_2 \\ 1=\alpha_1(-1)^1+\alpha_2 2^1 & \rightarrow \quad 1=-\alpha_1+2\alpha_2 \end{array}\right. \quad \Rightarrow \quad \alpha_1=\frac{1}{3}, \quad \alpha_2=\frac{2}{3}$$

Therefore, the solution is in $n$ microseconds $a_n=\frac{1}{3}(-1)^n+\frac{2}{3}2^n$ messages can be transmitted.

### *Exercise*

In how many ways can a $2\times n$ rectangular checkerboard be tiled using $1\times 2$ and $2\times 2$ pieces?

***Solution***

Let $t_n$ be the number of ways like to tile a $2\times n$ board with $1\times 2$ and $2\times 2$ pieces. To obtain the recurrence relation, imagine what tiles are placed at the left-hand end of the board. We can place a $2\times 2$ tile there, leaving a $2\times(n-2)$ board to be tiled, which of course can be done in $t_{n-2}$ ways.

We can place a $1\times 2$ tile at the edge, oriented vertically, leaving $2\times(n-1)$ board to be tiled, which of course can be done in $t_{n-1}$ ways.

Finally, we can place two $1\times 2$ tiles horizontally, one above the other, leaving a $2\times(n-2)$ board to be tiled, which of course can be done in $t_{n-2}$ ways. These 3 possibilities are disjoint.

Therefore, our recurrence relation is $t_n = t_{n-1} + 2t_{n-2}$

The initial conditions are $t_0 = t_1 = 1$, since there is only one way to tile as $2\times 0$ board and $2\times 1$ board.

This recurrence relation has characteristic roots $-1$ and 2.

So, the general solution is $t_n = \alpha_1 (-1)^n + \alpha_2 2^n$

Plugging in initial conditions gives

$$\begin{cases} 1 = \alpha_1 (-1)^0 + \alpha_2 2^0 & \rightarrow \quad 1 = \alpha_1 + \alpha_2 \\ 1 = \alpha_1 (-1)^1 + \alpha_2 2^1 & \rightarrow \quad 1 = -\alpha_1 + 2\alpha_2 \end{cases} \Rightarrow \quad \alpha_1 = \frac{1}{3}, \quad \alpha_2 = \frac{2}{3}$$

Therefore, the solution is $a_n = \frac{1}{3}(-1)^n + \frac{2}{3}\cdot 2^n$

$$= \frac{(-1)^n}{3} + \frac{2^{n+1}}{3}$$

### *Exercise*

Find the solution to $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3} \quad for \quad n \ge 3,\ a_0 = 3,\ a_1 = 6\ and\ a_2 = 0$

***Solution***

$a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = 0$

The characteristic polynomial is $r^3 - 2r^2 - r + 2 = 0$

That implies to: $r^2(r-2) - (r-2) = (r-2)\left(r^2 - 1\right) = 0$

So, the roots are 1, –1, and 2

The general solution:

$$a_n = \alpha_1 1^n + \alpha_2 (-1)^n + \alpha_3 2^n$$

$$= \alpha_1 + \alpha_2 (-1)^n + \alpha_3 2^n$$

Plugging in initial conditions gives

$$3 = \alpha_1 + \alpha_2 (-1)^0 + \alpha_3 2^0 \quad \rightarrow \quad 3 = \alpha_1 + \alpha_2 + \alpha_3$$

$$6 = \alpha_1 + \alpha_2 (-1)^1 + \alpha_3 2^1 \quad \rightarrow \quad 6 = \alpha_1 - \alpha_2 + 2\alpha_3$$

$$\Rightarrow \ \alpha_1 = 6, \quad \alpha_2 = -2, \quad \alpha_3 = -1$$

$$0 = \alpha_1 + \alpha_2 (-1)^2 + \alpha_3 2^2 \quad \rightarrow \quad 0 = \alpha_1 + \alpha_2 + 4\alpha_3$$

Therefore, the solution is $a_n = 6 - 2(-1)^n - 2^n$

## *Exercise*

Find the solution to $a_n = 7a_{n-2} + 6a_{n-3}$ *with* $a_0 = 9,\ a_1 = 10$ *and* $a_2 = 32$

***Solution***

This is a third-degree recurrence relation.

The characteristic polynomial is $r^3 - 7r - 6 = 0$

By the rational root test, the possible rational roots are $\pm\left\{\frac{6}{1}\right\} = \pm\{1,\ 2,\ 3,\ 6\}$

We find that $r = -1$ (using calculator).

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array} \qquad Q(x) = r^2 - r - 6 = (r+2)(r-3)$$

$$r^3 - 6r^2 + 12r - 8 = (r+1)(r+2)(r-3) = 0$$

So, the roots are $-2, -1$, and 3.

The general solution:

$$a_n = \alpha_1 (-2)^n + \alpha_2 (-1)^n + \alpha_3 3^n$$

Plugging in initial conditions gives

$$a_0 = 9 = \alpha_1 (-2)^0 + \alpha_2 (-1)^0 + \alpha_3 3^0 \quad \rightarrow \quad 9 = \alpha_1 + \alpha_2 + \alpha_3$$

$$a_1 = 10 = \alpha_1 (-2)^1 + \alpha_2 (-1)^1 + \alpha_3 3^1 \quad \rightarrow \quad 10 = -2\alpha_1 - \alpha_2 + 3\alpha_3$$

$$a_2 = 32 = \alpha_1 (-2)^2 + \alpha_2 (-1)^2 + \alpha_3 3^2 \quad \rightarrow \quad 32 = 4\alpha_1 + \alpha_2 + 9\alpha_3$$

The solution to the system of equations is $\alpha_1 = -3,\ \alpha_2 = 8$ *and* $\alpha_3 = 4$

Therefore, the specific solution is $a_n = -3(-2)^n + 8(-1)^n + 4 \cdot 3^n$

### *Exercise*

Find the solution to $a_n = 5a_{n-2} - 4a_{n-4}$ *with* $a_0 = 3,\ a_1 = 2,\ a_2 = 6 \textit{ and } a_3 = 8$

***Solution***

This is a fourth-degree recurrence relation.

The characteristic polynomial is $r^4 - 5r^2 - 4 = 0$

That implies to: $\left(r^2 - 1\right)\left(r^2 - 4\right) = (r-1)(r+1)(r-2)(r+2) = 0$

So, the roots are 1, –1, 2, –2

The general solution: $a_n = \alpha_1 + \alpha_2(-1)^n + \alpha_3 2^n + \alpha_4(-2)^n$

Plugging in initial conditions gives

$$3 = \alpha_1 + \alpha_2(-1)^0 + \alpha_3 2^0 + \alpha_4(-2)^0 \quad \rightarrow \quad 3 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

$$10 = \alpha_1 + \alpha_2(-1)^1 + \alpha_3 2^1 + \alpha_4(-2)^1 \quad \rightarrow \quad 10 = \alpha_1 - \alpha_2 + 2\alpha_3 - 2\alpha_4$$

$$6 = \alpha_1 + \alpha_2(-1)^2 + \alpha_3 2^2 + \alpha_4(-2)^2 \quad \rightarrow \quad 6 = \alpha_1 + \alpha_2 + 4\alpha_3 + 4\alpha_4$$

$$8 = \alpha_1 + \alpha_2(-1)^3 + \alpha_3 2^3 + \alpha_4(-2)^3 \quad \rightarrow \quad 8 = \alpha_1 - \alpha_2 + 8\alpha_3 - 8\alpha_4$$

The solution to the system of equations is $\alpha_1 = \alpha_2 = \alpha_3 = 1 \quad \textit{and} \quad \alpha_4 = 0$

Therefore, the solution is $\boxed{a_n = 1 + (-1)^n + 2^n}$

### *Exercise*

Find the recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ *with* $a_0 = -5,\ a_1 = 4 \textit{ and } a_2 = 88$

***Solution***

This is a third-degree recurrence relation.

The characteristic polynomial is $r^3 - 6r^2 + 12r - 8 = 0$

By the rational root test, the possible rational roots are $\pm 1,\ \pm 2,\ \pm 4,\ \pm 8$

We find that $r = 2$ (using calculator).

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 12 & -8 \\ & & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array} \qquad Q(x) = r^2 - 4r + 4 = (r-2)^2$$

$$r^3 - 6r^2 + 12r - 8 = (r-2)^3 = 0$$

Hence the only root is 2, with multiplicity 3.

The general solution: $\boxed{a_n = \alpha_1 2^n + \alpha_2 n \cdot 2^n + \alpha_3 n^2 \cdot (-2)^n}$

Plugging in initial conditions gives

$$\boxed{-5 = a_0 = \alpha_1}$$

$$4 = a_1 = 2\alpha_1 + 2\alpha_2 + 2\alpha_3 \qquad \alpha_1 + \alpha_2 + \alpha_3 = 2 \rightarrow \alpha_2 + \alpha_3 = 7$$

$$88 = a_2 = 4\alpha_1 + 8\alpha_2 + 16\alpha_3 \qquad \alpha_1 + 2\alpha_2 + 4\alpha_3 = 22 \rightarrow 2\alpha_2 + 4\alpha_3 = 27$$

$$\rightarrow \begin{cases} \alpha_2 + \alpha_3 = 7 \\ 2\alpha_2 + 4\alpha_3 = 27 \end{cases} \Rightarrow \boxed{\alpha_2 = \frac{1}{2}} \quad \boxed{\alpha_3 = \frac{13}{2}}$$

Therefore, the solution: $a_n = -5 \cdot 2^n + \frac{1}{2} n \cdot 2^n + \frac{13}{2} n^2 \cdot (-2)^n$

$$\boxed{= -5 \cdot 2^n + n \cdot 2^{n-1} + 13n^2 \cdot (-2)^{n-1}}$$

***Exercise***

Find the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ *with* $a_0 = 5,\ a_1 = -9$ *and* $a_2 = 15$

***Solution***

This is a third-degree recurrence relation.

The characteristic polynomial is $r^3 + 3r^2 + 3r + 1 = 0$

$r^3 + 3r^2 + 3r + 1 = 0 = (r+1)^3 = 0$

Hence the only root is $-1$, with multiplicity 3.

The general solution: $\boxed{a_n = \alpha_1 (-1)^n + \alpha_2 n \cdot (-1)^n + \alpha_3 n^2 \cdot (-1)^n}$

Plugging in initial conditions gives

$$\boxed{5 = a_0 = \alpha_1}$$

$$a_1 = -9 = -\alpha_1 - \alpha_2 - \alpha_3 \rightarrow \alpha_2 + \alpha_3 = 9 - \alpha_1 = 4$$

$$a_2 = 15 = \alpha_1 + 2\alpha_2 + 4\alpha_3 \rightarrow 2\alpha_2 + 4\alpha_3 = 15 - \alpha_1 = 10$$

$$\rightarrow \begin{cases} \alpha_2 + \alpha_3 = 4 \\ 2\alpha_2 + 4\alpha_3 = 10 \end{cases} \Rightarrow \boxed{\alpha_2 = 3} \quad \boxed{\alpha_3 = 1}$$

Therefore, the specific solution is $a_n = 5(-1)^n + 3n \cdot (-1)^n + n^2 \cdot (-1)^n$

$$\boxed{= \left(n^2 + 3n + 5\right)(-1)^n}$$

### *Exercise*

Find the general form of the solutions of the recurrence relation $a_n = 8a_{n-2} - 16a_{n-4}$

***Solution***

This is a fourth-degree recurrence relation.

The characteristic polynomial is $r^4 - 8r^2 + 16 = \left(r^2 - 4\right)^2$

$\left(r^2 - 4\right)^2 = (r-2)^2 (r+2)^2 = 0$

The roots are –2 and 2, each with multiplicity 2.

The general solution: $\quad a_n = \alpha_1 2^n + \alpha_2 n \cdot 2^n + \alpha_3 (-2)^n + \alpha_4 n \cdot (-2)^n$

### *Exercise*

What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots 1, 1, 1, 1, –2, –2, –2, 3, 3, –4?

***Solution***

There are 4 distinct roots, so $t = 4$. The multiplicities are 4, 3, 2, and 1.

The general solution:

$$\begin{aligned} a_n &= \left(\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2 + \alpha_{1,3} n^3\right) + \left(\alpha_{2,0} + \alpha_{2,1} n + \alpha_{2,2} n^2\right)(-2)^n \\ &\quad + \left(\alpha_{3,0} + \alpha_{3,1} n\right) 3^n + \alpha_{4,0} (-4)^n \end{aligned}$$

### *Exercise*

What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots $-1, -1, -1, 2, 2, 5, 5, 7$?

***Solution***

There are 4 distinct roots, so $t = 4$. The multiplicities are 3, 2, 2, and 1.

The general solution:

$$a_n = \left(\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2\right)(-1)^n + \left(\alpha_{2,0} + \alpha_{2,1} n\right) 2^n + \left(\alpha_{3,0} + \alpha_{3,1} n\right) 5^n + \alpha_{4,0} 7^n$$