Solution Section 1.4 – Exact Differential Equations

Exercise

Solve the differential equation $\left(2x+y\right)dx+\left(x-6y\right)dy=0$

Solution

$$\frac{\partial\psi}{\partial x}=M=2x+y \quad\Rightarrow\quad M_y=1$$

$$\frac{\partial\psi}{\partial y}=N=x-6y \quad\Rightarrow\quad N_x=1$$

$$\Rightarrow M_y=N_x$$

$$\frac{\partial\psi}{\partial x}=2x+y \quad\Rightarrow\quad \psi=\int\left(2x+y\right)dx=x^2+xy+h\left(y\right)$$

$$\psi_y=x+h'\left(y\right)=x-6y \quad\Rightarrow\quad h'\left(y\right)=-6y$$

$$h\left(y\right)=\int -6y\,dy=-3y^2$$

$$\psi\left(x,\,y\right)=\underline{x^2+xy-3y^2=C}$$

Exercise

Solve the differential equation $\left(2x+3\right)dx+\left(2y-2\right)dy=0$

Solution

$$\frac{\partial\psi}{\partial x}=M=2x+3 \quad\Rightarrow\quad M_y=2$$

$$\frac{\partial\psi}{\partial y}=N=2y-2 \quad\Rightarrow\quad N_x=2$$

$$\Rightarrow M_y=N_x$$

$$\frac{\partial\psi}{\partial x}=2x+3 \quad\Rightarrow\quad \psi=\int\left(2x+3\right)dx=x^2+3x+h\left(y\right)$$

$$\psi_y=h'\left(y\right)=2y-2 \quad\Rightarrow\quad h\left(y\right)=\int\left(2y-2\right)dy$$

$$=y^2-2y+C$$

$$\psi\left(x,\,y\right)=\underline{x^2+3x+y^2-2y=C}$$

Exercise

Solve the differential equation $(1-y\sin x)+(\cos x)y'=0$

Solution

$$\frac{\partial\psi}{\partial x}=M=1-y\sin x \quad\Rightarrow\quad M_y=-\sin x$$

$$\frac{\partial\psi}{\partial y}=N=\cos x \quad\Rightarrow\quad N_x=-\sin x$$

$$\Rightarrow M_y=N_x$$

$$\frac{\partial\psi}{\partial x}=1-y\sin x \quad\Rightarrow\quad \psi=\int(1-y\sin x)dx=x+y\cos x+h(y)$$

$$\psi_y=\cos x+h'(y)=\cos x \quad\Rightarrow\quad h'(y)=0$$

$$h(y)=C$$

$$\psi(x,y)=\underline{x+y\cos x=C}$$

Exercise

Solve the differential equation $\frac{dy}{dx}=-\frac{ax+by}{bx+cy}$

Solution

$$(bx+cy)dy=-(ax+by)dx$$

$$(ax+by)dx+(bx+cy)dy=0$$

$$\frac{\partial\psi}{\partial x}=M=ax+by \quad\Rightarrow\quad M_y=b$$

$$\frac{\partial\psi}{\partial y}=N=bx+cy \quad\Rightarrow\quad N_x=b$$

$$\Rightarrow M_y=N_x$$

$$\frac{\partial\psi}{\partial x}=ax+by \quad\Rightarrow\quad \psi=\int(ax+by)dx=\frac{1}{2}ax^2+bxy+h(y)$$

$$\psi_y=bx+h'(y)=bx+cy \quad\Rightarrow\quad h'(y)=cy$$

$$h(y)=\int cydy=\frac{1}{2}cy^2$$

$$\psi(x,y)=\frac{1}{2}ax^2+bxy+\frac{1}{2}cy^2=D$$

$$\boxed{ax^2+2bxy+cy^2=E} \qquad (E=2D)$$

Exercise

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{3x^2 + y}{3y^2 - x}$

Solution

$$\left(3x^2 + y\right)dx - \left(3y^2 - x\right)dy = 0$$

$$\left.\begin{array}{l} \dfrac{\partial \psi}{\partial x} = M = 3x^2 + y \quad \Rightarrow \quad M_y = 1 \\ \dfrac{\partial \psi}{\partial y} = N = -3y^2 + x \quad \Rightarrow \quad N_x = 1 \end{array}\right. \Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = 3x^2 + y \quad \Rightarrow \quad \psi = \int \left(3x^2 + y\right)dx = x^3 + xy + h(y)$$

$$\psi_y = x + h'(y) = -3y^2 + x \quad \Rightarrow \quad h'(y) = -3y^2$$

$$h(y) = \int -3y^2 dy = -y^3$$

$$\psi(x, y) = \underline{x^3 + xy - y^2 = C}$$

Exercise

Solve the differential equation $2xydx + \left(x^2 - 1\right)dy = 0$

Solution

$$M(x, y) = 2xy \quad N(x, y) = x^2 - 1$$

$$\frac{\partial M}{\partial y} = 2x \quad \frac{\partial N}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int 2xy\, dx = x^2 y + h(y)$$

$$\psi_y = x^2 + h'(y) = x^2 - 1$$

$$h'(y) = -1 \quad \Rightarrow \quad h(y) = -y + C$$

$$\underline{x^2 y - y = C}$$

Exercise

Find the general solution $y' = \dfrac{x^2 + y^2}{2xy}$

Solution

Let $y = xv \quad \Rightarrow \quad y' = v + xv'$

$$v + xv' = \frac{x^2 + x^2 v^2}{2x^2 v}$$

$$xv' = \frac{1+v^2}{2v} - v$$

$$x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\frac{2v}{1-v^2}dv = \frac{dx}{x}$$

$$-\int \frac{1}{1-v^2}d\left(1-v^2\right) = \int \frac{dx}{x}$$

$$-\ln\left|1-v^2\right| = \ln|x| + \ln C$$

$$\ln\frac{1}{\left|1-v^2\right|} = \ln|\mathrm{C}\,x|$$

$$\frac{1}{1-\left(\frac{y}{x}\right)^2} = \mathrm{C}\,x$$

$$\frac{x^2}{x^2-y^2} = \mathrm{C}\,x$$

$$\frac{x}{\mathrm{C}} = x^2 - y^2$$

$$\boxed{y^2 = x^2 - C_1 x}$$

Exercise

Find the general solution $\quad 2xyy' = x^2 + 2y^2$

Solution

Let $y = xv \quad \Rightarrow \quad y' = v + xv'$

$$2x(xv)(v+xv') = x^2 + 2x^2v^2 \qquad \textit{Divide both side by } x^2$$

$$2v^2 + 2xvv' = 1 + 2v^2$$

$$2xv\frac{dv}{dx} = 1$$

$$\int 2v\,dv = \int \frac{dx}{x}$$

$$v^2 = \ln x + C \qquad \left(v = \frac{y}{x}\right)$$

$$\frac{y^2}{x^2} = \ln x + C$$

$$\boxed{y^2 = x^2(\ln x + C)}$$

Exercise

Find the general solution $xy' = y + 2\sqrt{xy}$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$$x(v + xv') = vx + 2\sqrt{x^2 v}$$

$$x(v + xv') = vx + 2x\sqrt{v} \qquad \textit{Divide both side by } x$$

$$v + xv' = v + 2\sqrt{v}$$

$$x\frac{dv}{dx} = 2\sqrt{v}$$

$$\int \frac{dv}{2\sqrt{v}} = \int \frac{dx}{x}$$

$$\sqrt{v} = \ln x + C \qquad \left(v = \frac{y}{x}\right)$$

$$\sqrt{\frac{y}{x}} = \ln x + C$$

$$\frac{y}{x} = (\ln x + C)^2$$

$$\boxed{y = x(\ln x + C)^2}$$

Exercise

Find the general solution $xy^2 y' = x^3 + y^3$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$$x^3 v^2 (v + xv') = x^3 + x^3 v^3 \qquad \textit{Divide both side by } x^3$$

$$v^2 (v + xv') = 1 + v^3$$

$$v^3 + xv^2 v' = 1 + v^3$$

$$xv^2 \frac{dv}{dx} = 1$$

$$\int v^2 dv = \int \frac{dx}{x}$$

$$\frac{1}{3}v^3 = \ln x + C \qquad \left(v = \frac{y}{x}\right)$$

$$\frac{y^3}{x^3} = 3(\ln x + C)$$

$$\boxed{y^3 = 3x^3(\ln x + C)}$$

Exercise

Find the general solution $x^2 y' = xy + x^2 e^{y/x}$

Solution

Let $y = vx \Rightarrow y' = v + xv'$

$x^2(v + xv') = x^2 v + x^2 e^{vx/x}$ ***Divide both side by*** x^2

$v + xv' = v + e^v$

$x\dfrac{dv}{dx} = e^v$

$\displaystyle\int e^{-v} dv = \int \dfrac{dx}{x}$

$e^{-v} = \ln x + \ln C \qquad \left(v = \dfrac{y}{x}\right)$

$e^{-\frac{y}{x}} = \ln Cx$

$-\dfrac{y}{x} = \ln(\ln Cx)$

$\boxed{y = -x\ln(\ln Cx)}$

Exercise

Find the general solution $x^2 y' = xy + y^2$

Solution

Let $y = vx \Rightarrow y' = v + xv'$

$x^2(v + xv') = x^2 v + x^2 v^2$ ***Divide both side by*** x^2

$v + xv' = v + v^2$

$x\dfrac{dv}{dx} = v^2$

$\displaystyle\int v^{-2} dv = \int \dfrac{dx}{x}$

$-v^{-1} = \ln x + \ln C \qquad \left(v = \dfrac{y}{x}\right)$

$\dfrac{x}{y} = -\ln Cx$

$\dfrac{x}{y} = \ln\dfrac{1}{Cx}$

$\boxed{y(x) = \dfrac{x}{\ln\dfrac{1}{Cx}}}$

Exercise

Find the general solution $xyy' = x^2 + 3y^2$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$vx^2(v + xv') = x^2 + 3x^2v^2$ ***Divide both side by*** x^2

$v^2 + xvv' = 1 + 3v^2$

$xv\dfrac{dv}{dx} = 1 + 2v^2$

$\dfrac{v}{1+2v^2}dv = \dfrac{dx}{x}$

$\dfrac{1}{4}\displaystyle\int \dfrac{1}{1+2v^2}d\left(1+2v^2\right) = \int \dfrac{dx}{x}$

$\dfrac{1}{4}\ln\left(1+2v^2\right) = \ln x + \ln C$

$\ln\left(1+2v^2\right) = 4\ln \mathrm{C}\, x$

$\ln\left(1+2\dfrac{y^2}{x^2}\right) = \ln \mathrm{C}\, x^4$

$\dfrac{x^2+2y^2}{x^2} = \mathrm{C}\, x^4$

$\boxed{x^2 + 2y^2 = \mathrm{C}\, x^6}$

Exercise

Find the general solution $\left(x^2 - y^2\right)y' = 2xy$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$\left(x^2 - v^2x^2\right)(v + xv') = 2x^2v$ ***Divide both side by*** x^2

$\left(1 - v^2\right)(v + xv') = 2v$

$v + xv' = \dfrac{2v}{1-v^2}$

$x\dfrac{dv}{dx} = \dfrac{2v}{1-v^2} - v$

$x\dfrac{dv}{dx} = \dfrac{v^3+v}{1-v^2}$

$$\int \frac{1-v^2}{v^3+v}dv = \int \frac{dx}{x}$$

$$\frac{1-v^2}{v\left(v^2+1\right)} = \frac{A}{v} + \frac{Bv+C}{v^2+1} = \frac{(A+B)v^2+Cv+A}{v\left(v^2+1\right)}$$

$$\begin{cases} A+B=-1 \\ \quad A=1 \qquad \rightarrow B=-2 \end{cases}$$

$$\int \left(\frac{1}{v} - \frac{2v}{v^2+1}\right)dv = \int \frac{dx}{x}$$

$$\int \frac{1}{v}dv - \int \frac{1}{v^2+1}d\left(v^2+1\right) = \int \frac{dx}{x}$$

$$\ln|v| - \ln\left(v^2+1\right) = \ln|x| + \ln C$$

$$\ln\frac{|v|}{v^2+1} = \ln\mathrm{C}\,x$$

$$\frac{y/x}{\frac{y^2}{x^2}+1} = \mathrm{C}\,x$$

$$\frac{y}{x}\frac{x^2}{y^2+x^2} = \mathrm{C}\,x$$

$$\boxed{y(x) = \mathrm{C}\left(y^2+x^2\right)}$$

Exercise

Find the general solution $\quad xyy' = y^2 + x\sqrt{4x^2+y^2}$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$$vx^2\left(v+xv'\right) = x^2v^2 + x\sqrt{4x^2+v^2x^2}$$

$$vx^2\left(v+xv'\right) = x^2v^2 + x^2\sqrt{4+v^2} \qquad \textit{Divide both side by } x^2$$

$$v^2 + xvv' = v^2 + \sqrt{4+v^2}$$

$$xv\frac{dv}{dx} = \sqrt{4+v^2}$$

$$\int \frac{v}{\sqrt{4+v^2}}dv = \int \frac{dx}{x}$$

$$\frac{1}{2}\int \frac{1}{\sqrt{4+v^2}}\, d\left(4+v^2\right) = \int \frac{dx}{x}$$

$$\sqrt{4+v^2} = \ln x + C$$

$$\sqrt{4+\frac{y^2}{x^2}} = \ln x + C$$

$$\frac{4x^2+y^2}{x^2} = (\ln x + C)^2$$

$$\boxed{4x^2+y^2 = x^2(\ln x + C)^2}$$

Exercise

Find the general solution $\quad xy' = y + \sqrt{x^2+y^2}$

Solution

Let $y = vx \quad \Rightarrow \quad y' = v + xv'$

$$x(v+xv') = xv + \sqrt{x^2+v^2x^2}$$

$$x(v+xv') = xv + x\sqrt{1+v^2} \qquad \textit{Divide both side by } x$$

$$v + xv' = v + \sqrt{1+v^2}$$

$$x\frac{dv}{dx} = \sqrt{1+v^2}$$

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$$

$$\ln\left|v+\sqrt{1+v^2}\right| = \ln x + \ln C$$

$$\ln\left|\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}\right| = \ln \mathrm{C}x$$

$$\frac{y}{x}+\frac{1}{x}\sqrt{x^2+y^2} = \mathrm{C}x$$

$$\boxed{y+\sqrt{x^2+y^2} = \mathrm{C}x^2}$$

Exercise

Find the general solution $\quad y^2y' + 2xy^3 = 6x$

Solution

$$y' + 2xy = 6xy^{-2}$$

Let $\quad u = y^{1+2} = y^3 \quad \Rightarrow \quad y = u^{1/3}$

$$\frac{du}{dx} = 3y^2\frac{dy}{dx} \quad \Rightarrow \quad y' = \frac{1}{3}y^{-2}u' = \frac{1}{3}u^{-2/3}u'$$

$$\frac{1}{3}u^{-2/3}u' + 2xu^{1/3} = 6xu^{-2/3}$$ ***Multiply both sides by*** $3u^{2/3}$

$$u' + 6xu = 18x$$

$$e^{\int 6xdx} = e^{3x^2}$$

$$\int 18xe^{3x^2}dx = 3\int e^{3x^2}d\left(3x^2\right) = 3e^{3x^2}$$

$$u = e^{-3x^2}\left(3e^{3x^2} + C\right)$$

$$\boxed{y^3 = 3 + Ce^{-3x^2}}$$

Exercise

Find the general solution $\quad x^2y' + 2xy = 5y^4$

Solution

$$y' + 2\frac{1}{x}y = \frac{5}{x^2}y^4$$ ***Divide by*** x^2

Let $\quad u = y^{1-4} = y^{-3} \quad \Rightarrow \quad y = u^{-1/3}$

$$\frac{du}{dx} = -3y^{-4}\frac{dy}{dx} \quad \Rightarrow \quad y' = -\frac{1}{3}y^4u' = -\frac{1}{3}u^{-4/3}u'$$

$$-\frac{1}{3}u^{-4/3}u' + \frac{2}{x}u^{-1/3} = \frac{5}{x^2}u^{-4/3}$$ ***Multiply both sides by*** $-3u^{4/3}$ $\quad u = \frac{1}{x^{-6}}\left(\frac{15}{7}x^{-7} + C\right)$

$$u' - \frac{6}{x}u = -\frac{15}{x^2}$$

$$e^{\int -\frac{6}{x}dx} = e^{-6\ln x} = e^{\ln x^{-6}} = x^{-6}$$

$$\int x^{-6}\left(-\frac{15}{x^2}\right)dx = -15\int x^{-8}dx = \frac{15}{7}x^{-7}$$

$$y^{-3} = \frac{15 + 7Cx^7}{7x}$$

$$\boxed{y^3 = \frac{7x}{15+7Cx^7}}$$

Exercise

Find the general solution $\qquad 2xy' + y^3e^{-2x} = 2xy$

Solution

$$2xy' - 2xy = -e^{-2x}y^3$$

$$y' - y = -\frac{e^{-2x}}{2x}y^3 \qquad \textit{Divide by } 2x$$

Let $\quad u = y^{1-3} = y^{-2} \quad \Rightarrow \quad y = u^{-1/2}$

$$\frac{du}{dx} = -2y^{-3}\frac{dy}{dx} \quad \Rightarrow \quad y' = -\frac{1}{2}y^3u' = -\frac{1}{2}u^{-3/2}u'$$

$$-\frac{1}{2}u^{-3/2}u' - u^{-1/2} = -\frac{e^{-2x}}{2x}u^{-3/2} \qquad \textit{Multiply both sides by } -2u^{3/2}$$

$$u' + 2u = \frac{e^{-2x}}{x}$$

$$e^{\int 2dx} = e^{2x}$$

$$\int \frac{e^{-2x}}{x}e^{2x}dx = \int \frac{dx}{x} = \ln x$$

$$u = \frac{1}{e^{-2x}}(\ln x + C)$$

$$\frac{1}{y^2} = \frac{\ln x + C}{e^{2x}}$$

$$\boxed{y^2 = \frac{e^{2x}}{\ln x + C}}$$

Exercise

Find the general solution $\qquad y^2(xy' + y)\left(1+x^4\right)^{1/2} = x$

Solution

$$y^2xy' + y^3 = x\left(1+x^4\right)^{-1/2}$$

$$y' + \frac{1}{x}y = \left(1+x^4\right)^{-1/2}y^{-2} \qquad \textit{Divide both sides by } xy^2$$

Let $\quad u = y^{1+2} = y^3 \quad \Rightarrow \quad y = u^{1/3}$

$$\frac{du}{dx} = 3y^2\frac{dy}{dx} \quad \Rightarrow \quad y' = \frac{1}{3}y^{-2}u' = \frac{1}{3}u^{-2/3}u'$$

$$\frac{1}{3}u^{-2/3}u' + \frac{1}{x}u^{1/3} = \left(1+x^4\right)^{-1/2} u^{-2/3}$$ ***Multiply both sides by*** $3u^{2/3}$

$$u' + \frac{3}{x}u = 3\left(1+x^4\right)^{-1/2}$$

$$e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$$

$$\int 3\left(1+x^4\right)^{-1/2} x^3 dx = \frac{3}{4}\int \left(1+x^4\right)^{-1/2} d\left(1+x^4\right) = \frac{3}{2}\sqrt{1+x^4}$$

$$u = \frac{1}{x^3}\left(\frac{3}{2}\sqrt{1+x^4} + C\right)$$

$$\underline{y^3 = \frac{1}{x^3}\left(\frac{3}{2}\sqrt{1+x^4} + C\right)}$$

Exercise

Find the general solution $\quad 3y^2y' + y^3 = e^{-x}$

Solution

$$3y' + y = e^{-x}y^{-2}$$ ***Divide both sides by*** y^2

Let $\quad u = y^{1+2} = y^3 \quad \Rightarrow \quad y = u^{1/3}$

$$\frac{du}{dx} = 3y^2\frac{dy}{dx} \quad \Rightarrow \quad y' = \frac{1}{3}y^{-2}u' = \frac{1}{3}u^{-2/3}u'$$

$$u^{-2/3}u' + u^{1/3} = e^{-x}u^{-2/3}$$ ***Multiply both sides by*** $u^{2/3}$

$$u' + u = e^{-x}$$

$$e^{\int dx} = e^x$$

$$\int e^{-x}e^x dx = \int dx = x$$

$$u = \frac{1}{e^x}\left(x + C\right)$$

$$\underline{y^3 = e^{-x}\left(x + C\right)}$$

Exercise

Find the general solution $\quad 3xy^2y' = 3x^4 + y^3$

Solution

$$3xy^2y' - y^3 = 3x^4$$ ***Divide both sides by*** xy^2

$$3y' - \frac{1}{x}y = 3x^3 y^{-2}$$

Let $u = y^{1+2} = y^3 \quad \Rightarrow \quad y = u^{1/3} \quad \Rightarrow y' = \frac{1}{3}u^{-2/3}u'$

$$u^{-2/3}u' - \frac{1}{x}u^{1/3} = 3x^3 u^{-2/3}$$ ***Multiply both sides by*** $u^{2/3}$

$$u' - \frac{1}{x}u = 3x^3$$

$$e^{\int \frac{-1}{x}dx} = e^{-\ln x} = x^{-1}$$

$$\int 3x^3 x^{-1}dx = \int 3x^2 dx = x^3$$

$$u = x\left(x^3 + C\right)$$

$$y^3 = x^4 + Cx$$

$$\boxed{y = \sqrt[3]{x^4 + Cx}}$$

Exercise

Find the general solution $\quad xe^y y' = 2\left(e^y + x^3 e^{2x}\right)$

Solution

Let $u = e^y \quad \Rightarrow \quad y = \ln u \quad \Rightarrow y' = \frac{u'}{u}$

$$xu\frac{1}{u}u' = 2u + 2x^3 e^{2x}$$

$$u' - \frac{2}{x}u = 2x^2 e^{2x}$$

$$e^{\int \frac{-2}{x}dx} = e^{-2\ln x} = x^{-2}$$

$$\int 2x^2 e^{2x} x^{-2}dx = 2\int e^{2x}dx = e^{2x}$$

$$u = x^2\left(e^{2x} + C\right)$$

$$e^y = x^2 e^{2x} + Cx^2$$

$$\boxed{y = \ln\left(x^2 e^{2x} + Cx^2\right)}$$

Exercise

Find the general solution $\quad (2x\sin y\cos y)y' = 4x^2 + \sin^2 y$

Solution

Let $u = \sin y \quad \Rightarrow \quad u' = (\cos y)y'$

$$2xuu' = 4x^2 + u^2$$

$$u' = 2x\frac{1}{u} + \frac{1}{2x}u$$

$$u' - \frac{1}{2x}u = 2xu^{-1}$$

Let $\quad v = u^{1+1} = u^2 \quad \Rightarrow \quad u = v^{1/2}$

$$v' = 2uu' \quad \Rightarrow \quad u' = \frac{1}{2}u^{-1}v' = \frac{1}{2}v^{-1/2}v'$$

$$\frac{1}{2}v^{-1/2}v' - \frac{1}{2x}v^{1/2} = 2xv^{-1/2}$$

Multiply both sides by $2v^{1/2}$

$$v' - \frac{1}{x}v = 4x$$

$$e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

$$\int x^{-1}(4x)dx = \int 4dx = 4x$$

$$v = x(4x + C)$$

$$u^2 = 4x^2 + Cx$$

$$\boxed{\sin^2 y = 4x^2 + Cx}$$

Exercise

Find the general solution $\quad \left(x + e^y\right)y' = xe^{-y} - 1$

Solution

Let $u = e^y \quad \Rightarrow \quad y = \ln u \quad \Rightarrow \quad y' = \dfrac{u'}{u}$

$$(x+u)\frac{u'}{u} = xu^{-1} - 1$$

$$(x+u)u' = x - u$$

Let $u = vx \quad \Rightarrow \quad u' = v + xv'$

$$(x+vx)(v+xv') = x - vx$$

$$x(1+v)(v+xv') = x(1-v)$$

$$(1+v)(v+xv') = 1 - v$$

$$v + v^2 + x(1+v)v' = 1 - v$$

$$x(1+v)\frac{dv}{dx}=1-2v-v^2$$

$$\int\frac{1+v}{1-2v-v^2}dv=\int\frac{dx}{x}$$

$$-\frac{1}{2}\ln\left|1-2v-v^2\right|=\ln x+\ln C$$

$$\ln\left|1-2v-v^2\right|=-2\ln Cx \qquad v=\frac{u}{x}=\frac{e^y}{x}$$

$$\ln\left|1-2\frac{e^y}{x}-\frac{e^{2y}}{x^2}\right|=\ln(Cx)^{-2}$$

$$\frac{x^2-2xe^y-e^{2y}}{x^2}=\frac{1}{(Cx)^2}$$

$$\boxed{x^2-2xe^y-e^{2y}=C_1}$$

Exercise

Find the general solution $\left(x^2+y^2\right)dx+\left(x^2-xy\right)dy=0$

Solution

$$M(x,\,y)=x^2+y^2 \quad N(x,\,y)=x^2-xy$$

$$\frac{\partial M}{\partial y}=2y \quad \frac{\partial N}{\partial x}=2x-y \quad\Rightarrow\quad \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

$$\frac{M_y-N_x}{N}=\frac{2y-2x+y}{x^2+xy}\ \times$$

Let $y=ux \;\Rightarrow\; dy=udx+xdu$

$$-\frac{x^2+y^2}{x^2-xy}dx=udx+xdu$$

$$\left(x^2+y^2\right)dx+\left(x^2-xy\right)(udx+xdu)=0$$

$$\left(x^2+y^2\right)dx+ux(x-y)dx+x^2(x-y)du=0$$

$$\left(x^2+u^2x^2\right)dx+ux(x-ux)dx+x^2(x-ux)du=0$$

$$x^2(1+u)dx+x^3(1-u)du=0$$

$$(1+u)dx=x(u-1)du$$

$$\frac{dx}{x}=\frac{u-1}{u+1}du$$

$$\int\frac{dx}{x}=\int\left(1-\frac{2}{u+1}\right)du$$

$$\ln|x| = u - 2\ln|u+1| + \ln C$$

$$\ln|x| + \ln\left(\frac{y}{x}+1\right)^2 - \ln C = \frac{y}{x}$$

$$\ln\frac{x}{C}\left(\frac{(x+y)^2}{x^2}\right) = \frac{y}{x}$$

$$\frac{(x+y)^2}{Cx} = e^{\frac{y}{x}}$$

$$\boxed{(x+y)^2 = Cxe^{\frac{y}{x}}}$$

Exercise

Find the general solution $\quad x\frac{dy}{dx} + y = x^2y^2$

Solution

$$y' + \frac{1}{x}y = xy^2$$

Let $u = y^{1-2} = y^{-1} \quad \Rightarrow \quad y = \frac{1}{u}$

$$\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx} \quad \Rightarrow \quad y' = -y^2u' = -\frac{1}{u^2}u'$$

$$-\frac{1}{u^2}u' + \frac{1}{x}\frac{1}{u} = x\frac{1}{u^2}$$

$$u' - \frac{1}{x}u = -x \qquad \left(\times -u^2\right)$$

$$e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

$$\int -xx^{-1}dx = -x$$

$$u = x(-x+C)$$

$$\frac{1}{y} = -x^2 + Cx$$

$$\boxed{y(x) = \frac{1}{-x^2+Cx}}$$

Exercise

Solve the differential equation $\quad \left(3x^2 - 2xy + 2\right) + \left(6y^2 - x^2 + 3\right)y' = 0$

Solution

$$\frac{\partial \psi}{\partial x} = M = 3x^2 - 2xy + 2 \quad \Rightarrow \quad M_y = -2x$$

$$\frac{\partial \psi}{\partial y} = N = 6y^2 - x^2 + 3 \quad \Rightarrow \quad N_x = -2x$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = 3x^2 - 2xy + 2$$

$$\psi = \int \left(3x^2 - 2xy + 2\right) dx = x^3 - x^2 y + 2x + h(y)$$

$$\psi_y = -x^2 + h'(y) = 6y^2 - x^2 + 3 \quad \Rightarrow \quad h'(y) = 6y^2 + 3$$

$$h(y) = \int \left(6y^2 + 3\right) dy = 2y^3 + 3y$$

$$\boxed{x^3 - x^2 y + 2x + 2y^3 + 3y = C}$$

Exercise

Solve the differential equation $\left(e^x \sin y - 2y \sin x\right) dx + \left(e^x \cos y + 2\cos x\right) dy = 0$

Solution

$$\frac{\partial \psi}{\partial x} = M = e^x \sin y - 2y \sin x \quad \Rightarrow \quad M_y = e^x \cos y - 2\sin x$$

$$\frac{\partial \psi}{\partial y} = N = e^x \cos y + 2\cos x \quad \Rightarrow \quad N_x = e^x \cos y - 2\sin x$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = e^x \sin y - 2y \sin x \quad \Rightarrow \quad \psi = \int \left(e^x \sin y - 2y \sin x\right) dx = e^x \sin y + 2y \cos x + h(y)$$

$$\psi_y = e^x \cos y + 2\cos x + h'(y) = e^x \cos y + 2\cos x \quad \Rightarrow \quad h'(y) = 0 \qquad \to \boxed{h(y) = C}$$

$$\boxed{e^x \sin y + 2y \cos x = C}$$

Exercise

Solve the differential equation $\left(\frac{y}{x} + 6x\right) dx + (\ln x - 2) dy = 0, \qquad x > 0$

Solution

$$\frac{\partial \psi}{\partial x} = M = \frac{y}{x} + 6x \quad \Rightarrow \quad M_y = \frac{1}{x}$$

$$\frac{\partial \psi}{\partial y} = N = \ln x - 2 \quad \Rightarrow \quad N_x = \frac{1}{x} \qquad \Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = e^x \sin y - 2y\sin x \quad \Rightarrow \quad \psi = \int \left(\frac{y}{x} + 6x \right) dx = y\ln x + 3x^2 + h(y)$$

$$\psi_y = \ln x + h'(y) = \ln x - 2 \quad \Rightarrow \quad h'(y) = -2$$

$$h(y) = \int -2dy = -2y$$

$$\boxed{y\ln x + 3x^2 - 2y = C}$$

Exercise

Solve the differential equation $\frac{xdx}{\left(x^2+y^2\right)^{3/2}} + \frac{ydy}{\left(x^2+y^2\right)^{3/2}} = 0$

Solution

Multiply both side by $\left(x^2+y^2\right)^{3/2}$ since $x^2+y^2 \neq 0 \quad \Rightarrow \quad xdx + ydy = 0$

$$\frac{\partial \psi}{\partial x} = M = x \quad \Rightarrow \quad M_y = 0 \qquad \frac{\partial \psi}{\partial y} = N = y \quad \Rightarrow \quad N_x = 0 \qquad \Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = x \quad \Rightarrow \quad \psi = \int xdx = \frac{1}{2}x^2 + h(y)$$

$$\psi_y = h'(y) = y \quad \Rightarrow \quad h(y) = \int ydy = \frac{1}{2}y^2$$

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 = C_1$$

$$\boxed{x^2 + y^2 = C}$$

Exercise

Solve the differential equation $\left(e^{2y} - y\cos xy\right)dx + \left(2xe^{2y} - x\cos xy + 2y\right)dy = 0$

Solution

$$M(x,\ y) = e^{2y} - y\cos xy \quad N(x,\ y) = 2xe^{2y} - x\cos xy + 2y$$

$$\frac{\partial M}{\partial y} = 2e^{2y} - \cos x + xy\sin xy \quad \frac{\partial N}{\partial x} = 2e^{2y} - \cos x + xy\sin xy \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(e^{2y} - y\cos xy\right)\ dx = xe^{2y} - \sin xy + h(y)$$

$$\psi_y = 2xe^{2y} - x\cos xy + h'(y) = 2xe^{2y} - x\cos xy + 2y$$

$$h'(y) = 2y \quad \Rightarrow \quad h(y) = y^2 + C$$

$$\psi = \boxed{xe^{2y} - \sin xy + y^2 = C}$$

Exercise

Solve the differential equation $\left(3x^2 - 2xy + 2\right) + \left(6y^2 - x^2 + 3\right)y' = 0$

Solution

$$\frac{\partial\psi}{\partial x} = M = 3x^2 - 2xy + 2 \quad \Rightarrow \quad M_y = -2x$$

$$\frac{\partial\psi}{\partial y} = N = 6y^2 - x^2 + 3 \quad \Rightarrow \quad N_x = -2x$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial\psi}{\partial x} = 3x^2 - 2xy + 2 \quad \Rightarrow \quad \psi = \int \left(3x^2 - 2xy + 2\right)dx = x^3 - x^2y + 2x + h(y)$$

$$\psi_y = -x^2 + h'(y) = 6y^2 - x^2 + 3 \quad \Rightarrow \quad h'(y) = 6y^2 + 3$$

$$h(y) = \int \left(6y^2 + 3\right)dy = 2y^3 + 3y$$

$$\boxed{x^3 - x^2y + 2x + 2y^3 + 3y = C}$$

Exercise

Solve the differential equation $\left(e^x \sin y - 2y\sin x\right)dx + \left(e^x \cos y + 2\cos x\right)dy = 0$

Solution

$$\frac{\partial\psi}{\partial x} = M = e^x \sin y - 2y\sin x \quad \Rightarrow \quad M_y = e^x \cos y - 2\sin x$$

$$\frac{\partial\psi}{\partial y} = N = e^x \cos y + 2\cos x \quad \Rightarrow \quad N_x = e^x \cos y - 2\text{sinx}$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial\psi}{\partial x} = e^x \sin y - 2y\sin x$$

$$\psi = \int \left(e^x \sin y - 2y\sin x\right)dx$$

$$= e^x \sin y + 2y\cos x + h(y)$$

$$\psi_y = e^x \cos y + 2\cos x + h'(y) = e^x \cos y + 2\cos x$$

$$h'(y) = 0 \quad \to \quad \boxed{h(y) = C}$$

$$\boxed{e^x \sin y + 2y\cos x = C}$$

Exercise

Solve the differential equation $\left(\frac{y}{x}+6x\right)dx+(\ln x-2)dy=0, \qquad x>0$

Solution

$$\frac{\partial\psi}{\partial x}=M=\frac{y}{x}+6x \quad\Rightarrow\quad M_y=\frac{1}{x} \qquad \frac{\partial\psi}{\partial y}=N=\ln x-2 \quad\Rightarrow\quad N_x=\frac{1}{x} \qquad \Rightarrow M_y=N_x$$

$$\frac{\partial\psi}{\partial x}=e^x\sin y-2y\sin x \quad\Rightarrow\quad \psi=\int\left(\frac{y}{x}+6x\right)dx=y\ln x+3x^2+h(y)$$

$$\psi_y=\ln x+h'(y)=\ln x-2 \quad\Rightarrow\quad h'(y)=-2$$

$$h(y)=\int -2dy=-2y$$

$$\boxed{y\ln x+3x^2-2y=C}$$

Exercise

Solve the differential equation $\frac{xdx}{\left(x^2+y^2\right)^{3/2}}+\frac{ydy}{\left(x^2+y^2\right)^{3/2}}=0$

Solution

Multiply both side by $\left(x^2+y^2\right)^{3/2}$ since $x^2+y^2\neq 0 \quad\Rightarrow\quad xdx+ydy=0$

$$\frac{\partial\psi}{\partial x}=M=x \quad\Rightarrow\quad M_y=0$$

$$\frac{\partial\psi}{\partial y}=N=y \quad\Rightarrow\quad N_x=0 \qquad \Rightarrow M_y=N_x$$

$$\frac{\partial\psi}{\partial x}=x \quad\Rightarrow\quad \psi=\int xdx=\frac{1}{2}x^2+h(y)$$

$$\psi_y=h'(y)=y \quad\Rightarrow\quad h(y)=\int ydy=\frac{1}{2}y^2$$

$$\frac{1}{2}x^2+\frac{1}{2}y^2=C_1$$

$$\boxed{x^2+y^2=C}$$

Exercise

Find the general solution $\left(e^{2y}-y\cos xy\right)dx+\left(2xe^{2y}-x\cos xy+2y\right)dy=0$

Solution

$$M(x,\ y)=e^{2y}-y\cos xy \qquad N(x,\ y)=2xe^{2y}-x\cos xy+2y$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 2e^{2y} - \cos x + xy\sin xy\\ \frac{\partial N}{\partial x} &= 2e^{2y} - \cos x + xy\sin xy\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(e^{2y} - y\cos xy\right)\, dx = xe^{2y} - \sin xy + h(y)$$

$$\psi_y = 2xe^{2y} - x\cos xy + h'(y) = 2xe^{2y} - x\cos xy + 2y$$

$$h'(y) = 2y \quad \Rightarrow \quad h(y) = y^2$$

$$\boxed{xe^{2y} - \sin xy + y^2 = C}$$

Exercise

Find the general solution $\quad (2x-1)dx + (3y+7)dy = 0$

Solution

$$M(x,\ y) = 2x - 1 \quad N(x,\ y) = 3y + 7$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 0\\ \frac{\partial N}{\partial x} &= 0\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (2x-1)\, dx = x^2 - x + h(y)$$

$$\psi_y = h'(y) = 3y + 7 \qquad \Rightarrow \quad h(y) = \frac{3}{2}y^2 + 7$$

$$\boxed{x^2 - x + \frac{3}{2}y^2 + 7 = C}$$

Exercise

Find the general solution $\quad (5x+4y)dx + \left(4x - 8y^3\right)dy = 0$

Solution

$$M(x,\ y) = 5x + 4y \quad N(x,\ y) = 4x - 8y^3$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 4\\ \frac{\partial N}{\partial x} &= 4\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (5x+4y)\, dx = \frac{5}{2}x^2 + 4xy + h(y)$$

$$\psi_y = 4x + h'(y) = 4x - 8y^3 \quad \to h'(y) = -8y^3 \quad \Rightarrow \quad h(y) = -2y^4$$

$$\boxed{\frac{5}{2}x^2 + 4xy - 2y^4 = C}$$

Exercise

Find the general solution $\quad (\sin y - y\sin x)dx + (\cos x + x\cos y - y)dy = 0$

Solution

$$M(x,\ y) = \sin y - y\sin x \qquad N(x,\ y) = \cos x + x\cos y - y$$

$$\left. \begin{array}{l} \frac{\partial M}{\partial y} = \cos y - \sin x \\ \frac{\partial N}{\partial x} = -\sin x + \cos y \end{array} \right. \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (\sin y - y\sin x)\ dx = x\sin y + y\cos x + h(y)$$

$$\psi_y = x\cos y + \cos x + h'(y) = x\cos y + \cos x - y$$

$$\rightarrow h'(y) = -y \quad \Rightarrow \quad h(y) = -\frac{1}{2}y^2$$

$$\boxed{x\sin y + y\cos x - \frac{1}{2}y^2 = C}$$

Exercise

Find the general solution $\quad \left(2xy^2 - 3\right)dx + \left(2x^2y + 4\right)dy = 0$

Solution

$$M(x,\ y) = 2xy^2 - 3 \qquad N(x,\ y) = 2x^2 + 4$$

$$\left. \begin{array}{l} \frac{\partial M}{\partial y} = 4xy \\ \frac{\partial N}{\partial x} = 4xy \end{array} \right. \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(2xy^2 - 3\right)\ dx = x^2y^2 - 3x + h(y)$$

$$\psi_y = 2x^2y + h'(y) = 2x^2y + 4$$

$$\rightarrow h'(y) = 4 \quad \Rightarrow \quad h(y) = 4y$$

$$\boxed{x^2y^2 - 3x + 4y = C}$$

Exercise

Find the general solution $\left(1+\ln x+\frac{y}{x}\right)dx-(1-\ln x)dy=0$

Solution

$$M(x,\ y)=1+\ln x+\frac{y}{x}\quad N(x,\ y)=-1+\ln x$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y}&=\frac{1}{x}\\ \frac{\partial N}{\partial x}&=\frac{1}{x}\end{aligned}\right.\quad\Rightarrow\quad \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\begin{aligned}\psi&=\int\left(1+\ln x+\frac{y}{x}\right)\ dx\\ &=x+x\ln x-x+y\ln x+h(y)\\ &=x\ln x+y\ln x+h(y)\end{aligned}$$

$$\psi_y=\ln x+h'(y)=-1+\ln x$$

$$\rightarrow h'(y)=-1\quad\Rightarrow\quad h(y)=-y$$

$$\underline{x\ln x+y\ln x-y=C|}$$

Exercise

Find the general solution $\left(x-y^3+y^2\sin x\right)dx-\left(3xy^2+2y\cos x\right)dy=0$

Solution

$$M(x,\ y)=x-y^3+y^2\sin x\quad N(x,\ y)=-3xy^2-2y\cos x$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y}&=-3y^2+2y\sin x\\ \frac{\partial N}{\partial x}&=-3y^2+2y\sin x\end{aligned}\right.\quad\Rightarrow\quad \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\psi=\int\left(x-y^3+y^2\sin x\right)\ dx=\frac{1}{2}x^2-xy^3-y^2\cos x+h(y)$$

$$\psi_y=-3xy^2-2y\cos x+h'(y)=-3xy^2-2y\cos x$$

$$\rightarrow h'(y)=0\quad\Rightarrow\quad h(y)=C$$

$$\underline{\frac{1}{2}x^2-xy^3-y^2\cos x=C|}$$

Exercise

Find the general solution $\left(x^3+y^3\right)dx+3xy^2dy=0$

Solution

$M(x,\ y)=x^3+y^3 \quad N(x,\ y)=3xy^2$

$$\left.\begin{aligned}\frac{\partial M}{\partial y}&=3y^2\\ \frac{\partial N}{\partial x}&=3y^2\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\psi=\int\left(x^3+y^3\right)\,dx=\frac{1}{4}x^4+xy^3+h(y)$$

$$\psi_y=3xy^2+h'(y)=3xy^2$$

$$\to h'(y)=0 \quad\Rightarrow\quad h(y)=C$$

$$\boxed{\frac{1}{4}x^4+xy^3=C}$$

Exercise

Find the general solution $\left(3x^2y+e^y\right)dx+\left(x^3+xe^y-2y\right)dy=0$

Solution

$M(x,\ y)=3x^2y+e^y \quad N(x,\ y)=x^3+xe^y-2y$

$$\left.\begin{aligned}\frac{\partial M}{\partial y}&=3x^2+e^y\\ \frac{\partial N}{\partial x}&=3x^2+e^y\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\psi=\int\left(3x^2y+e^y\right)\,dx=x^3y+xe^y+h(y)$$

$$\psi_y=x^3+xe^y+h'(y)=x^3+xe^y-2y$$

$$\to h'(y)=-2y \quad\Rightarrow\quad h(y)=-y^2$$

$$\boxed{x^3y+xe^y-y^2=C}$$

Exercise

Find the general solution $xdy+\left(y-2xe^x-6x^2\right)dx=0$

Solution

$M(x,\ y)=y-2xe^x-6x^2 \quad N(x,\ y)=x$

$$\left.\begin{array}{l}\frac{\partial M}{\partial y}=1\\ \frac{\partial N}{\partial x}=1\end{array}\right. \Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\psi = \int\left(y-2xe^{x}-6x^{2}\right)\,dx$$

$$= xy-(2x-2)e^{x}-2x^{3}+h(y)$$

$$\psi_y = x+h'(y)=x$$

$$\rightarrow h'(y)=0 \quad \Rightarrow \quad h(y)=C$$

$$\boxed{xy-2xe^{x}+2e^{x}-2x^{3}=C}$$

Exercise

Find the general solution $\left(1-\frac{3}{y}+x\right)dy+\left(y-\frac{3}{x}+1\right)dx=0$

Solution

$$M(x,\,y)=y-\frac{3}{x}+1 \quad N(x,\,y)=1-\frac{3}{y}+x$$

$$\left.\begin{array}{l}\frac{\partial M}{\partial y}=1\\ \frac{\partial N}{\partial x}=1\end{array}\right. \Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\psi = \int\left(y-\frac{3}{x}+1\right)\,dx = xy-3\ln|x|+x+h(y)$$

$$\psi_y = x+h'(y)=1-\frac{3}{y}+x$$

$$\rightarrow h'(y)=1-\frac{3}{y} \quad \Rightarrow \quad h(y)=y-3\ln|y|$$

$$xy-3\ln|x|+x+y-3\ln|y|=C$$

$$\boxed{xy+x+y-3\ln|xy|=C}$$

Exercise

Find the general solution $\left(x^{2}y^{3}-\frac{1}{1+9x^{2}}\right)\frac{dx}{dy}+x^{3}y^{2}=0$

Solution

$$\left(x^{2}y^{3}-\frac{1}{1+9x^{2}}\right)dx+\left(x^{3}y^{2}\right)dy=0$$

$$M(x, y) = x^2y^3 - \frac{1}{1+9x^2} \quad N(x, y) = x^3y^2$$

$$\left.\begin{array}{l}\frac{\partial M}{\partial y} = 3x^2y^2 \\ \frac{\partial N}{\partial x} = 3x^2y^2\end{array}\right. \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(x^2y^3 - \frac{1}{1+9x^2} \right) dx$$

$$= \frac{1}{3}x^3y^3 - \frac{1}{3}\arctan(3x) + h(y)$$

$$\psi_y = x^3y^2 + h'(y) = x^3y^2$$

$$\rightarrow h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$\frac{1}{3}x^3y^3 - \frac{1}{3}\arctan(3x) = C_1$$

$$\boxed{x^3y^3 - \arctan(3x) = C}$$

Exercise

Find the general solution $\quad (5y - 2x)y' - 2y = 0$

Solution

$$M(x, y) = -2y \quad N(x, y) = 5y - 2x$$

$$\left.\begin{array}{l}\frac{\partial M}{\partial y} = -2 \\ \frac{\partial N}{\partial x} = -2\end{array}\right. \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (-2y)\, dx = -2xy + h(y)$$

$$\psi_y = -2x + h'(y) = 5y - 2x$$

$$\rightarrow h'(y) = 5y \quad \Rightarrow \quad h(y) = \frac{5}{2}y^2$$

$$\boxed{-2xy + \frac{5}{2}y^2 = C}$$

Exercise

Find the general solution $\quad (x - y)dx - xdy = 0$

Solution

$$M(x, y) = x - y \quad N(x, y) = -x$$

$$\left.\begin{array}{l}\frac{\partial M}{\partial y} = -1 \\ \frac{\partial N}{\partial x} = -1\end{array}\right. \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (x - y)\, dx$$

$$= \frac{1}{2}x^2 - xy + h(y)$$

$$\psi_y = -x + h'(y) = -x$$

$$\to h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$\boxed{\frac{1}{2}x^2 - xy = C}$$

Exercise

Find the general solution $(x + y)dx + xdy = 0$

Solution

$$M(x, y) = x + y \quad N(x, y) = x$$

$$\left.\begin{array}{l}\frac{\partial M}{\partial y} = 1 \\ \frac{\partial N}{\partial x} = 1\end{array}\right. \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (x + y)\, dx$$

$$= \frac{1}{2}x^2 + xy + h(y)$$

$$\psi_y = x + h'(y) = x$$

$$\to h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$\boxed{\frac{1}{2}x^2 + xy = C}$$

Exercise

Find the general solution $\frac{dy}{dx} = -\frac{2xy^2 + 1}{2x^2 y}$

Solution

$$2x^2 y\,dy = -\left(2xy^2 + 1\right)dx$$

$$\left(2xy^2 + 1\right)dx + 2x^2 y\,dy = 0$$

$$M(x,\ y) = 2xy^2 + 1 \quad N(x,\ y) = 2x^2 y$$

$$\left.\begin{array}{l}\dfrac{\partial M}{\partial y} = 4xy \\ \dfrac{\partial N}{\partial x} = 4xy\end{array}\right. \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(2xy^2 + 1\right)\,dx$$

$$= x^2 y^2 + x + h(y)$$

$$\psi_y = 2x^2 y + h'(y) = 2x^2 y$$

$$\rightarrow h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$\underline{x^2 y^2 + x = C|}$$

Exercise

Find the general solution $\left(1 + e^x y + xe^x y\right)dx + \left(xe^x + 2\right)dy = 0$

Solution

$$M(x,\ y) = 1 + e^x y + xe^x y \quad N(x,\ y) = xe^x + 2$$

$$\left.\begin{array}{l}\dfrac{\partial M}{\partial y} = e^x + xe^x \\ \dfrac{\partial N}{\partial x} = e^x + xe^x\end{array}\right. \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(1 + e^x y + xe^x y\right)\,dx$$

$$= x + e^x y + (x-1)e^x y + h(y)$$

$$= x + xe^x y + h(y)$$

$$\psi_y = xe^x + h'(y) = xe^x + 2$$

$$\rightarrow h'(y) = 2 \quad \Rightarrow \quad h(y) = 2y + C$$

$$\underline{x + xe^x y + 2y + C = 0|}$$

Exercise

Find the general solution $\left(2xy^3+1\right)dx+\left(3x^2y^2-\frac{1}{y}\right)dy=0$

Solution

$$M_y=\frac{\partial}{\partial y}\left(2xy^3+1\right)=6xy^2$$

$$N_x=\frac{\partial}{\partial x}\left(3x^2y^2-\frac{1}{y}\right)=6xy^2 \qquad \Rightarrow M_y=N_x$$

$$\psi=\int\left(2xy^3+1\right)\,dx$$

$$=x^2y^3+x+h(y)$$

$$\psi_y=3x^2y^2+h'(y)$$

$$=3x^2y^2-\frac{1}{y}$$

$$\to h'(y)=-\frac{1}{y} \quad\Rightarrow\quad h(y)=-\ln|y|+C$$

$$\boxed{x^2y^3+x-\ln|y|=C}$$

Exercise

Find the general solution $(2x+y)dx+(x-2y)dy=0$

Solution

$$M_y=\frac{\partial}{\partial y}(2x+y)=1$$

$$N_x=\frac{\partial}{\partial x}(x-2y)=1 \qquad \Rightarrow M_y=N_x$$

$$\psi=\int(2x+y)\,dx$$

$$=x^2+xy+h(y)$$

$$\psi_y=x+h'(y)=x-2y$$

$$\to h'(y)=-2y \quad\Rightarrow\quad h(y)=-y^2$$

$$\boxed{x^2+xy-y^2=C}$$

Exercise

Find the general solution $\quad e^{x}(y-x)dx+\left(1+e^{x}\right)dy=0$

Solution

$$M_y=\frac{\partial}{\partial y}\left(e^{x}(y-x)\right)=e^{x}$$
$$N_x=\frac{\partial}{\partial x}\left(1+e^{x}\right)=e^{x}$$
$$\Rightarrow\ M_y=N_x$$

$$\psi=\int\left(ye^{x}-xe^{x}\right)\,dx$$
$$=ye^{x}-(x-1)e^{x}+h(y)$$

$$\psi_y=e^{x}+h'(y)=1+e^{x}$$
$$\to h'(y)=1\quad\Rightarrow\quad h(y)=y$$

$$\boxed{ye^{x}-(x-1)e^{x}+y=C}$$

$$\boxed{y(x)=\frac{(x-1)e^{x}+C}{1+e^{x}}}$$

Exercise

Find the general solution $\quad \left(ye^{xy}-\frac{1}{y}\right)dx+\left(xe^{xy}+\frac{x}{y^2}\right)dy=0$

Solution

$$M_y=\frac{\partial}{\partial y}\left(ye^{xy}-\frac{1}{y}\right)=e^{xy}+xye^{xy}+\frac{1}{y^2}$$
$$N_x=\frac{\partial}{\partial x}\left(xe^{xy}+\frac{x}{y^2}\right)=e^{xy}+xye^{xy}+\frac{1}{y^2}$$
$$\Rightarrow\ M_y=N_x$$

$$\psi=\int\left(ye^{xy}-\frac{1}{y}\right)\,dx=e^{xy}-\frac{x}{y}+h(y)$$

$$\psi_y=xe^{xy}+\frac{x}{y^2}+h'(y)=xe^{xy}+\frac{x}{y^2}$$
$$\to h'(y)=0\quad\Rightarrow\quad h(y)=C$$

$$\boxed{e^{xy}-\frac{x}{y}=C}$$

Exercise

Find the general solution $\left(\tan x - \sin x \sin y\right)dx + \left(\cos x \cos y\right)dy = 0$

Solution

$$M(x,\ y) = \tan x - \sin x \sin y \qquad N(x,\ y) = \cos x \cos y$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= -\sin x \cos y \\ \frac{\partial N}{\partial x} &= -\sin x \cos y\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(\tan x - \sin x \sin y\right)\, dx = \ln\left|\sec x\right| + \cos x \sin y + h(y)$$

$$\psi_y = \cos x \cos y + h'(y) = \cos x \cos y$$

$$\rightarrow h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$\boxed{\ln\left|\sec x\right| + \cos x \sin y = C}$$

Exercise

Find the general solution $\left(2x^3 - xy^2 - 2y + 3\right)dx - \left(x^2 y + 2x\right)dy = 0$

Solution

$$M(x,\ y) = 2x^3 - xy^2 - 2y + 3 \qquad N(x,\ y) = -x^2 y - 2x$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= -2xy - 2 \\ \frac{\partial N}{\partial x} &= -2xy - 2\end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(2x^3 - xy^2 - 2y + 3\right)\, dx = \frac{1}{2}x^4 - \frac{1}{2}x^2 y^2 - 2xy + 3x + h(y)$$

$$\psi_y = -x^2 y - 2x + h'(y) = -x^2 y - 2x$$

$$\rightarrow h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$\boxed{\frac{1}{2}x^4 - \frac{1}{2}x^2 y^2 - 2xy + 3x = C}$$

Exercise

Find the general solution $(x+\sin y)dx+(x\cos y-2y)dy=0$

Solution

$$M(x,\ y)=x+\sin y \quad N(x,\ y)=x\cos y-2y$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y}&=\cos y\\ \frac{\partial N}{\partial x}&=\cos y\end{aligned}\right. \Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\psi=\int(x+\sin y)\ dx=\frac{1}{2}x^2+x\sin y+h(y)$$

$$\psi_y=x\cos y+h'(y)=x\cos y-2y$$

$$\to h'(y)=-2y \quad \Rightarrow \quad h(y)=-y^2$$

$$\boxed{\frac{1}{2}x^2+x\sin y-y^2=C}$$

Exercise

Find the general solution $\left(x+\dfrac{1}{\sqrt{y^2-x^2}}\right)dx+\left(1-\dfrac{x}{y\sqrt{y^2-x^2}}\right)dy=0$

Solution

$$M(x,\ y)=x+\frac{1}{\sqrt{y^2-x^2}} \quad N(x,\ y)=1-\frac{x}{y\sqrt{y^2-x^2}}$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y}&=-\frac{1}{2}(2y)\frac{1}{\sqrt{y^2-x^2}}=-\frac{y}{\sqrt{y^2-x^2}}\\ \frac{\partial N}{\partial x}&=-\frac{1}{y}\frac{1}{\sqrt{y^2-x^2}}\left(y^2-x^2+x^2\right)=-\frac{y}{\sqrt{y^2-x^2}}\end{aligned}\right. \Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\psi=\int\left(x+\frac{1}{\sqrt{y^2-x^2}}\right)dx=\frac{1}{2}x^2+\sin^{-1}\frac{x}{y}+h(y) \qquad \int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}$$

$$\psi_y=-\frac{1}{y^2\sqrt{1-\left(\frac{x}{y}\right)^2}}+h'(y)=-\frac{1}{y\sqrt{y^2-x^2}}+h'(y)=1-\frac{1}{y\sqrt{y^2-x^2}} \qquad \frac{d}{dx}(\arcsin u)=\frac{u'}{\sqrt{1-u^2}}$$

$$\to h'(y)=1 \quad \Rightarrow \quad h(y)=y$$

$$\boxed{\frac{1}{2}x^2+\sin^{-1}\frac{x}{y}+y=C}$$

Exercise

Find the general solution $\left(2x + y^2 - \cos(x+y)\right)dx + \left(2xy - \cos(x+y) - e^y\right)dy = 0$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(2x + y^2 - \cos(x+y)\right) = 2y + \sin(x+y) \\ N_x &= \frac{\partial}{\partial x}\left(2xy - \cos(x+y) - e^y\right) = 2y + \sin(x+y) \end{aligned}\right\} \Rightarrow M_y = N_x$$

$$\psi = \int \left(2x + y^2 - \cos(x+y)\right)\, dx = x^2 + xy^2 - \sin(x+y) + h(y)$$

$$\psi_y = 2xy - \cos(x+y) + h'(y) = 2xy - \cos(x+y) - e^y$$

$$\to h'(y) = -e^y \quad \Rightarrow \quad h(y) = -e^y$$

$$\boxed{x^2 + xy^2 - \sin(x+y) - e^y = C}$$

Exercise

Find the general solution $\left(\dfrac{2}{\sqrt{1-x^2}} + y\cos(xy)\right)dx + \left(x\cos(xy) - y^{-1/3}\right)dy = 0$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(\frac{2}{\sqrt{1-x^2}} + y\cos(xy)\right) = \cos(xy) - xy\sin(xy) \\ N_x &= \frac{\partial}{\partial x}\left(x\cos(xy) - y^{-1/3}\right) = \cos(xy) - xy\sin(xy) \end{aligned}\right\} \Rightarrow M_y = N_x$$

$$\psi = \int \left(x\cos(xy) - y^{-1/3}\right)\, dy = \sin(xy) - \frac{3}{2}y^{2/3} + h(x)$$

$$\psi_x = y\cos(xy) + h'(x) = \frac{2}{\sqrt{1-x^2}} + y\cos(xy)$$

$$\to h'(x) = \frac{2}{\sqrt{1-x^2}} \quad \Rightarrow \quad h(x) = 2\arcsin x$$

$$\boxed{\sin(xy) - \frac{3}{2}y^{2/3} + 2\arcsin x = C}$$

Exercise

Find the general solution $\left(2x + y\cos(xy)\right)dx + \left(x\cos(xy) - 2y\right)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}\left(2x + y\cos(xy)\right) = \cos(xy) - xy\sin(xy)$$

$$N_x = \frac{\partial}{\partial x}\left(x\cos(xy) - 2y\right) = \cos(xy) - xy\sin(xy)$$

$$\Rightarrow M_y = N_x$$

$$\psi = \int \left(2x + y\cos(xy)\right)\, dy = x^2 + \sin(xy) + h(y)$$

$$\psi_y = x\cos(xy) + h'(y) = x\cos(xy) - 2y$$

$$\to h'(y) = -2y \quad \Rightarrow \quad h(y) = -y^2$$

$$\underline{x^2 + \sin(xy) - y^2 = C}$$

Exercise

Find the general solution $\left(e^x \sin y - 3x^2\right)dx + \left(e^x \cos y + \frac{1}{3}y^{-2/3}\right)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}\left(e^x \sin y - 3x^2\right) = e^x \cos y$$

$$N_x = \frac{\partial}{\partial x}\left(e^x \cos y + \frac{1}{3}y^{-2/3}\right) = e^x \cos y$$

$$\Rightarrow M_y = N_x$$

$$\psi = \int \left(e^x \sin y - 3x^2\right)\, dy = e^x \sin y - x^3 + h(y)$$

$$\psi_y = e^x \cos y + h'(y) = e^x \cos y + \frac{1}{3}y^{-2/3}$$

$$\to h'(y) = \frac{1}{3}y^{-2/3} \quad \Rightarrow \quad h(y) = y^{1/3}$$

$$\underline{e^x \sin y - x^3 + y^{1/3} = C}$$

Exercise

Find the general solution $\left(2y\sin x\cos x - y + 2y^2 e^{xy^2}\right)dx = \left(x - \sin^2 x - 4xye^{xy^2}\right)dy$

Solution

$$\left(2y\sin x\cos x - y + 2y^2 e^{xy^2}\right)dx - \left(x - \sin^2 x - 4xye^{xy^2}\right)dy = 0$$

$$M(x,\, y) = 2y\sin x\cos x - y + 2y^2 e^{xy^2} \qquad N(x,\, y) = -x + \sin^2 x + 4xye^{xy^2}$$

$$\frac{\partial M}{\partial y} = 2\sin x\cos x - 1 + 4ye^{xy^2} + 4xy^3 e^{xy^2}$$

$$\frac{\partial N}{\partial x} = -1 + 2\sin x\cos x + 4ye^{xy^2} + 4xy^3 e^{xy^2}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(2y\sin x\cos x - y + 2y^2 e^{xy^2}\right) dx$$

$$= \int (2y\sin x)\, d(\sin x) - xy + 2e^{xy^2}$$

$$= y\sin^2 x - xy + 2e^{xy^2}$$

$$\psi_y = \sin^2 x - x + 4xye^{xy^2} + h'(y) = -x + \sin^2 x + 4xye^{xy^2}$$

$$\to h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$\underline{y\sin^2 x - xy + 2e^{xy^2} = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$x^2y^3 + x\left(1 + y^2\right)y' = 0, \qquad \mu(x,\, y) = \frac{1}{xy^3}$$

Solution

$$M_y = \frac{\partial}{\partial y}\left(x^2y^3\right) = 3x^2y^2$$

$$N_x = \frac{\partial}{\partial x}\left(x + xy^2\right) = 1 + y^2$$

$$\Rightarrow M_y \neq N_x$$

$$x^2y^3\left(\frac{1}{xy^3}\right) + x\left(1 + y^2\right)\left(\frac{1}{xy^3}\right)y' = 0$$

$$x + \left(\frac{1 + y^2}{y^3}\right)\frac{dy}{dx} = 0 \quad \Rightarrow \quad \left(\frac{1 + y^2}{y^3}\right)dy = -xdx$$

$$\int \left(y^{-3} + \frac{1}{y}\right)dy = -\int xdx$$

$$-\frac{1}{2}y^{-2} + \ln|y| = -\frac{1}{2}x^2 + C_0$$

$$\underline{x^2 - y^{-2} + \ln|y| = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$y^2 - xy + \left(x^2\right)y' = 0, \qquad \mu(x, y) = \frac{1}{xy^2}$$

Solution

$$M_y = \frac{\partial}{\partial y}\left(y^2 - xy\right) = 2y - x \qquad N_x = \frac{\partial}{\partial x}\left(x^2\right) = 2x \quad \Rightarrow M_y \neq N_x$$

$$\left(y^2 - xy\right)\left(\frac{1}{xy^2}\right) + \left(x^2\right)\left(\frac{1}{xy^2}\right)y' = 0$$

$$\left(\frac{1}{x} - \frac{1}{y}\right) + \left(\frac{x}{y^2}\right)y' = 0$$

$$M_y = \frac{\partial}{\partial y}\left(\frac{1}{x} - \frac{1}{y}\right) = \frac{1}{y^2}$$

$$\Rightarrow M_y = N_x$$

$$N_x = \frac{\partial}{\partial x}\left(\frac{x}{y^2}\right) = \frac{1}{y^2}$$

$$\frac{\partial \psi}{\partial x} = \frac{1}{x} - \frac{1}{y} \quad \Rightarrow \quad \psi = \int \left(\frac{1}{x} - \frac{1}{y}\right)dx = \ln|x| - \frac{x}{y} + h(y)$$

$$\psi_y = \frac{x}{y^2} + h'(y) = \frac{x}{y^2}$$

$$\Rightarrow \quad h'(y) = 0 \quad \rightarrow h(y) = C$$

$$\ln|x| - \frac{x}{y} = C$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$x^2y^3 - y + x\left(1 + x^2y^2\right)y' = 0, \qquad \mu(x, y) = \frac{1}{xy}$$

Solution

$$M_y = \frac{\partial}{\partial y}\left(x^2y^3 - y\right) = 3y^2 - 1 \qquad N_x = \frac{\partial}{\partial x}\left(x + x^3y^2\right) = 1 + 3x^2y^2 \quad \Rightarrow M_y \neq N_x$$

$$\left(x^2y^3 - y\right)\left(\frac{1}{xy}\right) + x\left(1 + x^2y^2\right)\left(\frac{1}{xy}\right)y' = 0$$

$$\left(xy^2 - \frac{1}{x}\right) + \left(\frac{1}{y} + x^2y\right)y' = 0$$

$$M_y = \frac{\partial}{\partial y}\left(xy^2 - \frac{1}{x}\right) = 2xy$$
$$N_x = \frac{\partial}{\partial x}\left(\frac{1}{y} + x^2 y\right) = 2xy \qquad \Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = xy^2 - \frac{1}{x} \quad \Rightarrow \quad \psi = \int \left(xy^2 - \frac{1}{x}\right) dx = \frac{1}{2}x^2 y^2 - \ln|x| + h(y)$$

$$\psi_y = x^2 y + h'(y) = \frac{1}{y} + x^2 y$$

$$\Rightarrow \quad h'(y) = \frac{1}{y} \quad \rightarrow h(y) = \ln|y|$$

$$\boxed{\frac{1}{2}x^2 y^2 - \ln x + \ln y = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$\left(\frac{\sin y}{y} - 2e^{-x}\sin x\right)dx + \left(\frac{\cos y + 2e^{-x}\cos x}{y}\right)dy = 0, \qquad \mu(x, y) = ye^x$$

Solution

$$M_y = \frac{\partial}{\partial y}\left(\frac{\sin y}{y} - 2e^{-x}\sin x\right) = \frac{y\cos y - \sin y}{y^2}$$

$$N_x = \frac{\partial}{\partial x}\left(\frac{\cos y + 2e^{-x}\cos x}{y}\right) = \frac{1}{y}\left(-2e^{-x}\cos x - 2e^{-x}\sin x\right)$$

$$\Rightarrow M_y \neq N_x$$

$$\left(ye^x\right)\left(\frac{\sin y}{y} - 2e^{-x}\sin x\right)dx + \left(ye^x\right)\left(\frac{\cos y + 2e^{-x}\cos x}{y}\right)dy = 0$$

$$\left(e^x \sin y - 2y\sin x\right)dx + \left(e^x\cos y + 2\cos x\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(e^x \sin y - 2y\sin x\right) = e^x\cos y - 2\sin x$$
$$N_x = \frac{\partial}{\partial x}\left(e^x\cos y + 2\cos x\right) = e^x\cos y - 2\sin x \qquad \Rightarrow M_y = N_x$$

$$\psi = \int \left(e^x \sin y - 2y\sin x\right)dx$$

$$= e^x \sin y + 2y\cos x + h(y)$$

$$\psi_y = e^x\cos y + 2\cos x + h'(y)$$

$$= e^{x}\cos y + 2\cos x$$

$$\Rightarrow \quad h'(y) = 0 \quad \to h(y) = C$$

$$\boxed{e^{x}\sin y + 2y\cos x = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$(x+2)\sin y dx + x\cos y dy = 0, \qquad \mu(x, y) = xe^{x}$$

Solution

$$\begin{aligned} M_y &= \frac{\partial}{\partial y}\left((x+2)\sin y\right) = (x+2)\cos y \\ N_x &= \frac{\partial}{\partial x}(x\cos y) = -x\sin y \end{aligned} \quad \Rightarrow M_y \neq N_x$$

$$\left(xe^{x}\right)(x+2)\sin y dx + \left(xe^{x}\right)x\cos y dy = 0$$

$$\left(x^2+2x\right)e^{x}\sin y dx + x^2e^{x}\cos y dy = 0$$

$$\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(\left(x^2+2x\right)e^{x}\sin y\right) = \left(x^2+2x\right)e^{x}\cos y \\ N_x &= \frac{\partial}{\partial x}\left(x^2e^{x}\cos y\right) = \left(2xe^{x}+x^2\right)e^{x}\cos y \end{aligned} \quad \Rightarrow M_y = N_x$$

$$\psi = \int\left(x^2e^{x}\cos y\right)dy = x^2e^{x}\sin y + h(x)$$

$$\psi_x = \left(x^2+2x\right)e^{x}\sin y + h'(x) = \left(x^2+2x\right)e^{x}\sin y$$

$$\Rightarrow \quad h'(x) = 0 \ \to \ h(x) = C$$

$$\boxed{x^2e^{x}\sin y = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$\left(x^2+y^2-x\right)dx - ydy = 0, \qquad \mu(x, y) = \frac{1}{x^2+y^2}$$

Solution

$$M_y = \frac{\partial}{\partial y}\left(x^2 + y^2 - x\right) = 2y$$
$$N_x = \frac{\partial}{\partial x}(-y) = 0$$
$$\Rightarrow M_y \neq N_x$$

$$\frac{1}{x^2 + y^2}\left(x^2 + y^2 - x\right)dx - \frac{y}{x^2 + y^2}dy = 0$$

$$\left(1 - \frac{x}{x^2 + y^2}\right)dx - \frac{y}{x^2 + y^2}dy = 0$$

$$M_y = \left(1 - \frac{x}{x^2 + y^2}\right) = \frac{2xy}{\left(x^2 + y^2\right)^2}$$
$$N_x = \left(\frac{-y}{x^2 + y^2}\right) = \frac{2xy}{\left(x^2 + y^2\right)^2}$$
$$\Rightarrow M_y = N_x$$

$$\frac{d\psi}{dx} = 1 - \frac{x}{x^2 + y^2}$$

$$\psi = \int\left(1 - \frac{x}{x^2 + y^2}\right)dx$$
$$= \int dx - \frac{1}{2}\int \frac{1}{x^2 + y^2}d\left(x^2 + y^2\right)$$
$$= x - \frac{1}{2}\ln\left(x^2 + y^2\right) + h(y)$$

$$\psi_y = -\frac{y}{x^2 + y^2} + h'(y)$$
$$= -\frac{y}{x^2 + y^2}$$

$$h'(y) = 0 \quad \rightarrow \quad h(y) = C$$

$$x - \frac{1}{2}\ln\left(x^2 + y^2\right) = C$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation $(2y - 6x)dx + \left(3x - 4x^2y^{-1}\right)dy = 0,$ $\quad \mu(x, y) = xy^2$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}(2y-6x) = 2 \\ N_x &= \frac{\partial}{\partial x}\left(3x-4x^2y^{-1}\right) = 3-\frac{8x}{y} \end{aligned}\right\} \Rightarrow M_y \neq N_x$$

$$xy^2(2y-6x)dx + xy^2\left(3x-4x^2y^{-1}\right)dy = 0$$

$$\left(2xy^3-6x^2y^2\right)dx + \left(3x^2y^2-4x^3y\right)dy = 0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(2xy^3-6x^2y^2\right) = 6xy^2-12x^2y \\ N_x &= \frac{\partial}{\partial x}\left(3x^2y^2-4x^3y\right) = 6xy^2-12x^2y \end{aligned}\right\} \Rightarrow M_y = N_x$$

$$\psi = \int\left(2xy^3-6x^2y^2\right)dx$$

$$= x^2y^3-2x^3y^2+h(y)$$

$$\psi_y = 3x^2y^2-4x^3y+h'(y)$$

$$= 3x^2y^2-4x^3y$$

$$h'(y)=0 \quad \to \quad \boxed{h(y)=C}$$

$$\boxed{x^2y^3-2x^3y^2=C}$$

Exercise

Find the general solution of the homogenous equation $\left(x^2+y^2\right)dx-2xydy=0$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(x^2+y^2\right) = 2y \\ N_x &= \frac{\partial}{\partial x}(-2xy) = -2y \end{aligned}\right\} \Rightarrow M_y \neq N_x$$

$$\frac{M_y-N_x}{N} = \frac{2y+2y}{-2xy} = -\frac{4y}{2xy} = -\frac{2}{x}$$

$$\frac{d\mu}{dx} = -\mu\frac{2}{x} \quad \Rightarrow \quad \int\frac{d\mu}{\mu} = -2\int\frac{dx}{x}$$

$$\ln\mu = -2\ln x$$

$$\ln\mu = \ln x^{-2} \quad \Rightarrow \quad \boxed{\mu = \frac{1}{x^2}}$$

$$\frac{1}{x^2}\left(x^2+y^2\right)dx-\frac{1}{x^2}2xydy=0 \qquad \Rightarrow \quad \left(1+\frac{y^2}{x^2}\right)dx-\frac{2y}{x}dy=0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(1+\frac{y^2}{x^2}\right)=\frac{2y}{x^2} \\ N_x &= \frac{\partial}{\partial x}\left(-\frac{2y}{x}\right)=\frac{2y}{x^2}\end{aligned}\right\} \Rightarrow M_y=N_x$$

$$\psi=\int\left(1+\frac{y^2}{x^2}\right)dx$$

$$=x-\frac{y^2}{x}+h(y)$$

$$\psi_y=-\frac{2y}{x}+h'(y)=-\frac{2y}{x}$$

$$h'(y)=0 \quad \Rightarrow \quad h(y)=C$$

$$x-\frac{y^2}{x}=C \qquad\qquad \textit{multiply by } x$$

$$\boxed{x^2-y^2=Cx}$$

Exercise

Find the general solution of the homogenous equation $\quad (x+y)dx+(y-x)dy=0$

Solution

$$(x+y)dx=-(y-x)dy$$

$$\frac{dy}{dx}=\frac{x+y}{x-y}$$

$$=\frac{\frac{x+y}{x}}{\frac{x-y}{x}}$$

$$=\frac{1+\frac{y}{x}}{1-\frac{y}{x}}$$

$$\frac{dy}{dx}=\frac{1+v}{1-v}=x\frac{dv}{dx}+v$$

$$x\frac{dv}{dx}=\frac{1+v}{1-v}-v=\frac{1+v^2}{1-v}$$

$$\frac{1-v}{1+v^2}dv=\frac{dx}{x}$$

$$\int \frac{dx}{x} = \int \frac{1}{1+v^2} dv - \int \frac{v}{1+v^2} dv$$

$$\int \frac{1}{a^2 + x^2} dx = \arctan \frac{x}{a}$$

$$\ln x = \arctan v - \frac{1}{2} \int \frac{1}{1+v^2} d\left(1+v^2\right)$$

$$\ln x + C = \arctan v - \frac{1}{2} \ln\left(1+v^2\right)$$

$$\ln x + C = \arctan \frac{y}{x} - \frac{1}{2} \ln\left(1+\frac{y^2}{x^2}\right)$$

$$\boxed{\arctan \frac{y}{x} - \frac{1}{2} \ln\left(1+\frac{y^2}{x^2}\right) - \ln x = C}$$

Exercise

Find the general solution of the homogenous equation $\dfrac{dy}{dx} = \dfrac{y\left(x^2+y^2\right)}{xy^2 - 2x^3}$

Solution

$$\frac{dy}{dx} = \frac{y}{x} \frac{\frac{x^2+y^2}{x^2}}{\frac{y^2-2x^2}{x^2}}$$

$$= \frac{y}{x} \frac{1+\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)^2 - 2}$$

$$= v\frac{1+v^2}{v^2-2} = x\frac{dv}{dx} + v$$

$$\frac{v+v^3}{v^2-2} - v = x\frac{dv}{dx}$$

$$x\frac{dv}{dx} = \frac{v+v^3-v^3+2v}{v^2-2} = \frac{3v}{v^2-2}$$

$$\int \frac{dx}{x} = \int \frac{v^2-2}{3v} dv = \frac{1}{3}\int \left(v - \frac{2}{v}\right) dv$$

$$3\ln x + C = \frac{1}{2}v^2 - 2\ln v$$

$$3\ln x + C = \frac{1}{2}\frac{y^2}{x^2} - 2\ln \frac{y}{x}$$

$$3\ln x + C = \frac{1}{2}\frac{y^2}{x^2} - 2(\ln y - \ln x)$$

$$6\ln x + C = \frac{y^2}{x^2} - 4\ln y + 4\ln x$$

$$\boxed{\frac{y^2}{x^2} - 4\ln y - 2\ln x = C}$$

Exercise

Find an integrating factor and solve the given equation $\left(3x^2y + 2xy + y^3\right)dx + \left(x^2 + y^2\right)dy = 0$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(3x^2y + 2xy + y^3\right) = 3x^2 + 2x + 3y^2 \\ N_x &= \frac{\partial}{\partial x}\left(x^2 + y^2\right) = 2x \end{aligned}\right. \quad \Rightarrow \ M_y \neq N_x$$

$$\frac{M_y - N_x}{N} = \frac{3x^2 + 2x + 3y^2 - 2x}{x^2 + y^2} = 3$$

$$\frac{d\mu}{dx} = 3\mu \quad \Rightarrow \quad \int \frac{d\mu}{\mu} = 3\int dx$$

$$\ln\mu = 3x \quad \Rightarrow \quad \mu = e^{3x}$$

$$e^{3x}\left(3x^2y + 2xy + y^3\right)dx + e^{3x}\left(x^2 + y^2\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left[e^{3x}\left(3x^2y + 2xy + y^3\right)\right] = e^{3x}\left(3x^2 + 2x + 3y^2\right)$$

$$N_x = \frac{\partial}{\partial x}e^{3x}\left(x^2 + y^2\right) = 3e^{3x}\left(x^2 + y^2\right) + 2xe^{3x} = e^{3x}\left(3x^2 + 3y^2 + 2x\right)$$

$$\Rightarrow \ M_y = N_x$$

$$\psi = \int\left(e^{3x}\left(3x^2y + 2xy + y^3\right)\right)dx = +h(y)$$

$$= e^{3x}\left(x^2y + \tfrac{2}{3}xy + \tfrac{1}{3}y^3 - \tfrac{2}{3}xy - \tfrac{2}{9}y + \tfrac{2}{9}y\right) + h(y)$$

$$= e^{3x}\left(x^2y + \tfrac{1}{3}y^3\right) + h(y)$$

	$\int e^{3x}$
$3x^2y + 2xy + y^3$	$\frac{1}{3}e^{3x}$
$6xy + 2y$	$\frac{1}{9}e^{3x}$
$6y$	$\frac{1}{27}e^{3x}$

$$\psi_y = e^{3x}\left(x^2 + y^2\right) + h'(y) = e^{3x}\left(x^2 + y^2\right)$$

$$h'(y) = 0 \ \Rightarrow \ h(y) = C$$

$$\boxed{e^{3x}\left(x^2y+\frac{1}{3}y^3\right)=C}$$

Exercise

Find an integrating factor and solve the given equation $\quad dx+\left(\frac{x}{y}-\sin y\right)dy=0$

Solution

$$ydx+(x-y\sin y)dy=0 \qquad \textit{Multiply by } \boldsymbol{y} \textit{ both sides}$$

$$M_y=\frac{\partial}{\partial y}(y)=1; \quad N_x=\frac{\partial}{\partial x}(x-y\sin y)=1; \quad M_y=N_x$$

$$\frac{\partial\psi}{\partial x}=2x+y^2 \quad\Rightarrow\quad \psi=\int ydx=xy+h(y)$$

$$\psi_y=x+h'(y)=x-y\sin y \quad\Rightarrow\quad h'(y)=-y\sin y$$

$$h(y)=-\int y\sin ydy=y\cos y-\sin y$$

	$\int -\sin y$
y	$\cos y$
1	$\sin y$

$$\boxed{xy+y\cos y-\sin y=C}$$

Exercise

Find an integrating factor and solve the given equation $\quad e^x dx+\left(e^x\cot y+2y\csc y\right)dy=0$

Solution

$$\left.\begin{aligned} M_y&=\frac{\partial}{\partial y}\left(e^x\right)=0 \\ N_x&=\frac{\partial}{\partial x}\left(e^x\cot y+2y\csc y\right)=e^x \end{aligned}\right\} \Rightarrow M_y\neq N_x$$

$$e^x dx+\left(e^x\frac{\cos y}{\sin y}+2y\frac{1}{\sin y}\right)dy=0 \qquad \textit{Multiply by } \boldsymbol{\sin y} \textit{ both sides}$$

$$(\sin y)e^x dx+(\sin y)\left(e^x\frac{\cos y}{\sin y}+2y\frac{1}{\sin y}\right)dy=0$$

$$e^x\sin ydx+\left(e^x\cos y+2y\right)dy=0$$

$$\left.\begin{aligned} M_y&=\frac{\partial}{\partial y}\left(e^x\sin y\right)=e^x\cos y \\ N_x&=\frac{\partial}{\partial x}\left(e^x\cos y+2y\right)=e^x\cos y \end{aligned}\right\} \Rightarrow M_y=N_x$$

$$\psi=\int\left(e^x\sin y\right)dx=e^x\sin y+h(y)$$

$$\psi_y = e^x \cos y + h'(y) = e^x \cos y + 2y$$

$$\rightarrow\ h'(y) = 2y \quad \Rightarrow\ h(y) = y^2$$

$$\psi(x,\ y) = e^x \sin y + y^2 = C$$

$$\boxed{e^x \sin y + y^2 = C}$$

Exercise

Find an integrating factor and solve the given equation $\left(3x + \frac{6}{y}\right)dx + \left(\frac{x^2}{y} + 3\frac{y}{x}\right)dy = 0$

Solution

$$xy\left(3x + \frac{6}{y}\right)dx + xy\left(\frac{x^2}{y} + 3\frac{y}{x}\right)dy = 0$$

$$\left(3x^2 y + 6x\right)dx + \left(x^3 + 3y^2\right)dy = 0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(3x^2 y + 6x\right) = 3x^2 \\ N_x &= \frac{\partial}{\partial x}\left(x^3 + 3y^2\right) = 3x^2 \end{aligned}\right. \quad \Rightarrow M_y = N_x$$

$$\psi = \int \left(3x^2 y + 6x\right)dx = x^3 y + 3x^2 + h(y)$$

$$\psi_y = x^3 + h'(y) = x^3 + 3y^2$$

$$h'(y) = 3y^2 \quad \Rightarrow\ h(y) = y^3$$

$$\boxed{x^3 y + 3x^2 + y^3 = C}$$

Exercise

Find an integrating factor and solve the given equation $\left(x + 3x^3 \sin y\right)dx + \left(x^4 \cos y\right)dy = 0$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(x + 3x^3 \sin y\right) = 3x^3 \cos y \\ N_x &= \frac{\partial}{\partial x}\left(x^4 \cos y\right) = 4x^3 \cos y \end{aligned}\right. \quad \Rightarrow M_y \ne N_x$$

$$\frac{M_y - N_x}{N} = \frac{3x^3 \cos y - 4x^3 \cos y}{x^4 \cos y} = -\frac{1}{x}$$

$$\mu = e^{-\int \frac{1}{x}dx} = e^{-\ln x} \underline{= \frac{1}{x}|} \qquad \mu = e^{\int \frac{M_y - N_x}{N}dx}$$

$$\frac{1}{x}\left(x + 3x^3 \sin y\right)dx + \frac{1}{x}\left(x^4 \cos y\right)dy = 0$$

$$\left(1 + 3x^2 \sin y\right)dx + \left(x^3 \cos y\right)dy = 0$$

$$\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(1 + 3x^2 \sin y\right) = 3x^2 \cos y \\ N_x &= \frac{\partial}{\partial x}\left(x^3 \cos y\right) = 3x^2 \cos y \end{aligned} \quad \Rightarrow M_y = N_x$$

$$\psi = \int \left(1 + 3x^2 \sin y\right)dx = x + x^3 \sin y + h(y)$$

$$\psi_y = -x^3 \cos y + h'(y) = -x^3 \cos y$$

$$h'(y) = 0 \;\Rightarrow\; h(y) = C$$

$$\underline{x + x^3 \sin y = C|}$$

Exercise

Find an integrating factor and solve the given equation $\left(2x^2 + y\right)dx + \left(x^2 y - x\right)dy = 0$

Solution

$$\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(2x^2 + y\right) = 1 \\ N_x &= \frac{\partial}{\partial x}\left(x^2 y - x\right) = 2xy - 1 \end{aligned} \quad \Rightarrow M_y \neq N_x$$

$$\frac{M_y - N_x}{N} = \frac{1 - 2xy + 1}{x^2 y - x} = \frac{2(1 - xy)}{x(xy - 1)} = -\frac{2}{x}$$

$$\mu = e^{-\int \frac{2}{x}dx} = e^{-2\ln x} \underline{= x^{-2}|} \qquad \mu = e^{\int \frac{M_y - N_x}{N}dx}$$

$$x^{-2}\left(2x^2 + y\right)dx + x^{-2}\left(x^2 y - x\right)dy = 0$$

$$\left(2 + \frac{y}{x^2}\right)dx + \left(y - \frac{1}{x}\right)dy = 0$$

$$\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(2 + \frac{y}{x^2}\right) = \frac{1}{x^2} \\ N_x &= \frac{\partial}{\partial x}\left(y - \frac{1}{x}\right) = \frac{1}{x^2} \end{aligned} \quad \Rightarrow M_y = N_x$$

$$\psi = \int \left(2 + \frac{y}{x^2}\right) dx$$

$$= 2x - \frac{y}{x} + h(y)$$

$$\psi_y = -\frac{1}{x} + h'(y) = y - \frac{1}{x}$$

$$h'(y) = y \Rightarrow h(y) = \frac{1}{2} y^2$$

$$\boxed{2x - \frac{y}{x} + \frac{1}{2} y^2 = C}$$

Exercise

Find an integrating factor and solve the given equation $\left(3x^2 + y\right) dx + \left(x^2 y - x\right) dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}\left(3x^2 + y\right) = 1$$
$$N_x = \frac{\partial}{\partial x}\left(x^2 y - x\right) = 2xy - 1 \qquad \Rightarrow M_y \ne N_x$$

$$\frac{M_y - N_x}{N} = \frac{1 - 2xy + 1}{x^2 y - x}$$

$$= \frac{2(1 - xy)}{x(xy - 1)}$$

$$= -\frac{2}{x}$$

$$e^{-\int \frac{2}{x} dx} = e^{-2\ln x} \boxed{= x^{-2}}$$

$$x^{-2}\left(3x^2 + y\right) dx + x^{-2}\left(x^2 y - x\right) dy = 0$$

$$\left(3 + \frac{y}{x^2}\right) dx + \left(y - \frac{1}{x}\right) dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(3 + \frac{y}{x^2}\right) = \frac{1}{x^2}$$
$$N_x = \frac{\partial}{\partial x}\left(y - \frac{1}{x}\right) = \frac{1}{x^2} \qquad \Rightarrow M_y = N_x$$

$$\psi = \int \left(3 + \frac{y}{x^2}\right) dx$$

$$= 3x - \frac{y}{x} + h(y)$$

$$\psi_y = -\frac{1}{x} + h'(y)$$

$$= y - \frac{1}{x}$$

$$h'(y) = y \ \Rightarrow \ h(y) = \frac{1}{2}y^2$$

$$\boxed{3x - \frac{y}{x} + \frac{1}{2}y^2 = C}$$

Exercise

Find an integrating factor and solve the given equation $\left(2y^2 + 2y + 4x^2\right)dx + (2xy + x)dy = 0$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(2y^2 + 2y + 4x^2\right) = 4y + 2 \\ N_x &= \frac{\partial}{\partial x}(2xy + x) = 2y + 1 \end{aligned}\right\} \Rightarrow M_y \neq N_x$$

$$\frac{M_y - N_x}{N} = \frac{4y + 2 - 2y - 1}{x(2y+1)} = \frac{2y+1}{x(2y+1)} = \frac{1}{x}$$

$$\mu = e^{\int \frac{1}{x}dx} = e^{\ln x} = \boxed{x} \qquad \mu = e^{\int \frac{M_y - N_x}{N}dx}$$

$$x\left(2y^2 + 2y + 4x^2\right)dx + x(2xy + x)dy = 0$$

$$\left(2xy^2 + 2xy + 4x^3\right)dx + \left(2x^2y + x^2\right)dy = 0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(2xy^2 + 2xy + 4x^3\right) = 4xy + 2x \\ N_x &= \frac{\partial}{\partial x}\left(2x^2y + x^2\right) = 4xy + 2x \end{aligned}\right\} \Rightarrow M_y = N_x$$

$$\psi = \int \left(2xy^2 + 2xy + 4x^3\right)dx$$

$$= x^2y^2 + x^2y + x^4 + h(y)$$

$$\psi_y = 2x^2y + x^2 + h'(y) = 2x^2y + x^2$$

$$h'(y) = 0 \ \Rightarrow \ h(y) = C$$

$$\boxed{x^2y^2 + x^2y + x^4 = C}$$

Exercise

Find an integrating factor and solve the given equation $\left(x^4 - x + y\right)dx - xdy = 0$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(x^4 - x + y\right) = 1 \\ N_x &= \frac{\partial}{\partial x}(-x) = -1 \end{aligned}\right. \quad \Rightarrow\ M_y \neq N_x$$

$$\frac{M_y - N_x}{N} = -\frac{2}{x}$$

$$\mu = e^{-\int \frac{2}{x}dx} = e^{-2\ln x} \underline{= x^{-2}|} \qquad \mu = e^{\int \frac{M_y - N_x}{N}dx}$$

$$x^{-2}\left(x^4 - x + y\right)dx - x^{-2}xdy = 0$$

$$\left(x^2 - x^{-1} + yx^{-2}\right)dx - x^{-1}dy = 0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(x^2 - x^{-1} + yx^{-2}\right) = x^{-2} \\ N_x &= \frac{\partial}{\partial x}\left(-x^{-1}\right) = x^{-2} \end{aligned}\right. \quad \Rightarrow\ M_y = N_x$$

$$\begin{aligned} \psi &= \int\left(x^2 - x^{-1} + yx^{-2}\right)dx \\ &= \frac{1}{3}x^3 - \ln|x| - \frac{y}{x} + h(y) \end{aligned}$$

$$\begin{aligned} \psi_y &= -\frac{1}{x} + h'(y) \\ &= -\frac{1}{x} \end{aligned}$$

$$h'(y) = 0 \ \Rightarrow\ h(y) = C$$

$$\underline{\frac{1}{3}x^3 - \ln|x| - \frac{y}{x} = C|}$$

Exercise

Find an integrating factor and solve the given equation $(2xy)dx + \left(y^2 - 3x^2\right)dy = 0$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}(2xy) = 2x \\ N_x &= \frac{\partial}{\partial x}\left(y^2 - 3x^2\right) = -6x \end{aligned}\right. \quad \Rightarrow\ M_y \neq N_x$$

$$\frac{N_x - M_y}{M} = \frac{-8x}{2xy} = -\frac{4}{y}$$

$$\mu = e^{-\int \frac{4}{y}dy} = e^{-4\ln y} \boxed{= y^{-4}} \qquad\qquad \mu = e^{\int \frac{N_x - M_y}{M}dy}$$

$$y^{-4}(2xy)dx + y^{-4}\left(y^2 - 3x^2\right)dy = 0$$

$$\left(2xy^{-3}\right)dx + \left(y^{-2} - 3x^2y^{-4}\right)dy = 0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(2xy^{-3}\right) = -6xy^{-4} \\ N_x &= \frac{\partial}{\partial x}\left(y^{-2} - 3x^2y^{-4}\right) = -6xy^{-4} \end{aligned}\right. \Rightarrow M_y = N_x$$

$$\psi = \int \left(2xy^{-3}\right)dx = x^2y^{-3} + h(y)$$

$$\psi_y = -3x^2y^{-4} + h'(y) = y^{-2} - 3x^2y^{-4}$$

$$h'(y) = \frac{1}{y^2} \Rightarrow h(y) = -\frac{1}{y}$$

$$\boxed{x^2y^{-3} - \frac{1}{y} = C}$$

Exercise

Solve the given initial-value problem $\qquad \dfrac{dy}{dx} = \dfrac{xy^2 - \cos x \sin x}{y\left(1 - x^2\right)}, \quad y(0) = 2$

Solution

$$\left(xy^2 - \cos x \sin x\right)dx - y\left(1 - x^2\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(xy^2 - \cos x \sin x\right) = 2xy \qquad N_x = \frac{\partial}{\partial x}\left(-y + yx^2\right) = 2xy \qquad \Rightarrow \boxed{M_y = N_x}$$

$$\begin{aligned} \psi &= \int \left(xy^2 - \cos x \sin x\right)dx \\ &= \int \left(xy^2 - \frac{1}{2}\sin 2x\right)dx \\ &= \frac{1}{2}x^2y^2 + \frac{1}{4}\cos 2x + h(y) \end{aligned}$$

$$\psi_y = x^2y + h'(y) = -y + yx^2$$

$$h'(y) = -y \Rightarrow h(y) = -\frac{1}{2}y^2$$

$$\psi = \frac{1}{2}x^2y^2 + \frac{1}{4}\cos 2x - \frac{1}{2}y^2 = C$$

$$y(0) = 2 \quad \Rightarrow \quad \frac{1}{4} - \frac{1}{2}(4) = C \rightarrow \boxed{C = -\frac{7}{4}}$$

$$\frac{1}{2}x^2y^2 + \frac{1}{4}\cos 2x - \frac{1}{2}y^2 = -\frac{7}{4}$$

$$\boxed{2x^2y^2 + \cos 2x - 2y^2 = -7} \qquad\qquad 2x^2y^2 + 2\cos^2 x - 1 - 2y^2 = -7$$

$$x^2y^2 + \cos^2 x - y^2 = -3$$

Exercise

Solve the given initial-value problem $\quad (x+y)^2\,dx + \left(2xy + x^2 - 1\right)dy, \quad y(1) = 1$

Solution

$$M_y = \frac{\partial}{\partial y}(x+y)^2 = 2(x+y)$$

$$N_x = \frac{\partial}{\partial x}\left(2xy + x^2 - 1\right) = 2y + 2x \qquad \Rightarrow M_y = N_x$$

$$\psi = \int\left(x^2 + 2xy + y^2\right)dx$$

$$= \frac{1}{3}x^3 + x^2y + xy^2 + h(y)$$

$$\psi_y = x^2 + 2xy + h'(y)$$

$$= 2xy + x^2 - 1$$

$$h'(y) = -1 \quad \Rightarrow \quad h(y) = -y$$

$$\psi = \frac{1}{3}x^3 + x^2y + xy^2 - y = C$$

$$y(1) = 1 \quad \Rightarrow \quad \frac{1}{3} + 1 + 1 - 1 = C \quad \rightarrow \quad C = \frac{4}{3}$$

$$\frac{1}{3}x^3 + x^2y + xy^2 - y = \frac{4}{3}$$

Exercise

Solve the given initial-value problem $\quad \left(e^x + y\right)dx + \left(2 + x + ye^y\right)dy, \quad y(0) = 1$

Solution

$$M_y = \frac{\partial}{\partial y}\left(e^x + y\right) = 1$$

$$N_x = \frac{\partial}{\partial x}\left(2 + x + ye^y\right) = 1 \qquad \Rightarrow M_y = N_x$$

$$\psi = \int \left(e^x + y\right)dx$$

$$= e^x + xy + h(y)$$

$$\psi_y = x + h'(y)$$

$$= 2 + x + ye^y$$

$$h'(y) = 2 + ye^y$$

$$h(y) = 2y + e^y(y-1)$$

$$e^x + xy + 2y + e^y(y-1) = C$$

$$y(0) = 1 \quad \Rightarrow \quad 1 + 2 = C \quad \rightarrow \quad C = 3$$

$$e^x + xy + 2y + e^y(y-1) = 3$$

Exercise

Solve the given initial-value problem $\qquad (2x - y)dx + (2y - x)dy, \quad y(1) = 3$

Solution

$$\begin{cases} M_y = \frac{\partial}{\partial y}(2x - y) = -1 \\ N_x = \frac{\partial}{\partial x}(2y - x) = -1 \end{cases} \Rightarrow M_y = N_x$$

$$\psi = \int (2x - y)dx$$

$$= x^2 - xy + h(y)$$

$$\psi_y = -x + h'(y) = 2y - x$$

$$h'(y) = 2y \quad \rightarrow h(y) = y^2$$

$$x^2 - xy + y^2 = C$$

$$y(1) = 3 \quad \Rightarrow \quad 1 - 3 + 9 = C \quad \rightarrow \quad C = 7$$

$$\boxed{x^2 - xy + y^2 = 7}$$

$$y^2 - xy + x^2 - 7 = 0 \quad \rightarrow \quad y = \frac{x \pm \sqrt{x^2 - 4x^2 + 28}}{2} = \frac{x \pm \sqrt{-3x^2 + 28}}{2}$$

$$since \;\; y(1) = 3 \quad \rightarrow \quad y = \frac{1}{2}\left(1 \pm \sqrt{-3(1)^2 + 28}\right) = \frac{1}{2}(1 \pm 5) \begin{cases} \frac{1+5}{2} = 3 \\ \cancel{\frac{1-5}{2} = -2} \end{cases}$$

$$\boxed{y(x) = \frac{x+\sqrt{-3x^2+28}}{2}} \qquad |x|<\sqrt{\frac{28}{3}}$$

Exercise

Solve the given initial-value problem $\left(9x^2+y-1\right)dx-(4y-x)dy, \quad y(1)=0$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(9x^2+y-1\right)=1 \\ N_x &= \frac{\partial}{\partial x}(x-4y)=1 \end{aligned}\right. \Rightarrow \boxed{M_y = N_x}$$

$$\psi = \int\left(9x^2+y-1\right)dx$$

$$= 3x^3+xy-x+h(y)$$

$$\psi_y = x+h'(y) = x-4y$$

$$h'(y)=-4y \quad \to h(y)=-2y^2$$

$$3x^3+xy-x-2y^2=C$$

$$y(1)=0 \quad \Rightarrow \quad 3-1=C \quad \to \quad \boxed{C=2}$$

$$\boxed{-2y^2+xy+3x^3-x=2}$$

$$-2y^2+xy+3x^3-x-2=0 \quad \to \quad y=\frac{-x\pm\sqrt{x^2+24x^3-8x-16}}{-4}$$

$$since\ \ y(1)=0 \quad \to \quad y=-\frac{1}{4}\left(-1\pm\sqrt{1+24-8-16}\right)=-\frac{1}{4}(-1\pm 1)\begin{cases} -\frac{-1+1}{4}=0 \\ \cancel{\frac{-1-1}{-4}=\frac{1}{2}} \end{cases}$$

$$\boxed{y(x)=\frac{x-\sqrt{x^2+24x^3-8x-16}}{4}}$$

Exercise

Solve $\qquad \dfrac{dy}{dx}=\dfrac{xy^2-\cos x\sin x}{y\left(1-x^2\right)}, \quad y(0)=2$

Solution

$$\left(xy^2-\cos x\sin x\right)dx-y\left(1-x^2\right)dy=0$$

$$\left(xy^2-\cos x\sin x\right)dx+y\left(x^2-1\right)dy=0$$

$$M(x,\ y)=xy^2-\cos x\sin x \quad N(x,\ y)=y\left(x^2-1\right)$$

$$\frac{\partial M}{\partial y} = 2xy$$
$$\frac{\partial N}{\partial x} = 2xy \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(xy^2 - \cos x \sin x\right)\, dx$$
$$= \int xy^2\, dx - \int \sin x\, d(\sin x)$$
$$= \frac{1}{2}x^2y^2 - \frac{1}{2}\sin^2 x + h(y)$$

$$\psi_y = x^2y + h'(y) = x^2y - y$$

$$h'(y) = -y \quad \Rightarrow \quad h(y) = -\frac{1}{2}y^2$$

$$\boxed{\frac{1}{2}x^2y^2 - \frac{1}{2}\sin^2 x - \frac{1}{2}y^2 = C}$$

$$y(0) = 2 \quad \rightarrow \boxed{-2 = C}$$

$$\boxed{\frac{1}{2}x^2y^2 - \frac{1}{2}\sin^2 x - \frac{1}{2}y^2 = -2}$$

Exercise

Solve the given initial-value problem $\quad \frac{dy}{dx} = (-2x + y)^2 - 7, \quad y(0) = 0$

Solution

Let $u = -2x + y \quad \Rightarrow \quad \frac{du}{dx} = -2 + \frac{dy}{dx}$

$$\frac{du}{dx} + 2 = u^2 - 7$$

$$\frac{du}{dx} = u^2 - 9$$

$$\int \frac{du}{u^2 - 9} = \int dx$$

$$\frac{1}{u^2 - 9} = \frac{A}{u-3} + \frac{B}{u+3}$$

$$Au + 3A + Bu - 3B = 0 \quad \begin{cases} A + B = 0 \\ 3A - 3B = 1 \end{cases} \rightarrow A = \frac{1}{6}, \quad B = -\frac{1}{6}$$

$$\frac{1}{6}\int \left(\frac{1}{u-3} - \frac{1}{u+3}\right) du = \int dx$$

$$\frac{1}{6}\left(\ln|u-3| - \ln|u+3|\right) = x + C$$

$$\ln\left|\frac{u-3}{u+3}\right| = 6x + C$$

$$\frac{u-3}{u+3} = e^{6x+C} = Ae^{6x}$$

$$u-3=Aue^{6x}+3Ae^{6x}$$

$$u=\frac{3+3Ae^{6x}}{1-Ae^{6x}}=-2x+y$$

$$y=2x+\frac{3+3Ae^{6x}}{1-Ae^{6x}} \qquad y(0)=0$$

$$0=\frac{3+3A}{1-A} \rightarrow \underline{A=-1}$$

$$\boxed{y(x)=2x+\frac{3\left(1-e^{6x}\right)}{1+e^{6x}}}$$

Exercise

Solve the given initial-value problem $(2y-x)y'-y+2x=0, \quad y(1)=0$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}(2y-x)=2 \\ N_x &= \frac{\partial}{\partial x}(-y+2x)=2 \end{aligned}\right. \Rightarrow \underline{M_y=N_x}$$

$$\psi=\int(-y+2x)dx=x^2-xy+h(y)$$

$$\psi_y=-x+h'(y)=2y-x \quad \Rightarrow h'(y)=2y \quad \rightarrow h(y)=y^2$$

$$x^2-xy+y^2=C$$

$$y(1)=0 \quad \rightarrow 1=C$$

$$x^2-xy+y^2=1$$

$$y^2-xy+x^2-1=0$$

$$y=\frac{x\pm\sqrt{x^2-4x^2+4}}{2}=\frac{x\pm\sqrt{4-3x^2}}{2}$$

Since $y(1)=0 \quad \rightarrow \boxed{y(x)=\frac{x-\sqrt{4-3x^2}}{2}}$

Exercise

Solve the given initial-value problem $\left(x+y^3\right)y'+y+x^3=0, \quad y(0)=-2$

Solution

$$M_y = \frac{\partial}{\partial y}\left(y + x^3\right) = 1$$

$$N_x = \frac{\partial}{\partial x}\left(x + y^3\right) = 1 \quad \Rightarrow \underline{M_y = N_x}|$$

$$\psi = \int \left(y + x^3\right)dx = xy + \frac{1}{4}x^4 + h(y)$$

$$\psi_y = x + h'(y) = x + y^3$$

$$h'(y) = y^3 \quad \to h(y) = \frac{1}{4}y^4$$

$$xy + \frac{1}{4}x^4 + \frac{1}{4}y^4 = C$$

$$y(0) = -2 \quad \to \quad \underline{4 = C}|$$

$$\underline{xy + \frac{1}{4}x^4 + \frac{1}{4}y^4 = 4}|$$

Exercise

Solve the given initial-value problem $\quad y' = \left(3x^2 + 1\right)\left(y^2 + 1\right), \quad y(0) = 1$

Solution

$$\frac{1}{y^2 + 1}y' - \left(3x^2 + 1\right) = 0$$

$$M_y = \frac{\partial}{\partial y}\left(-3x^2 - 1\right) = 0$$

$$N_x = \frac{\partial}{\partial x}\left(\frac{1}{y^2 + 1}\right) = 0 \quad \Rightarrow M_y = N_x$$

$$\psi = \int \left(-3x^2 - 1\right)dx = -x^3 - x + h(y)$$

$$\psi_y = h'(y) = \frac{1}{y^2 + 1} \quad \to h(y) = \tan^{-1} y$$

$$-x^3 - x + \tan^{-1} y = C$$

$$y(0) = 1 \quad \to \tan^{-1} 1 = C \ \Rightarrow \ \underline{C = \frac{\pi}{4}}|$$

$$-x^3 - x + \tan^{-1} y = \frac{\pi}{4}$$

$$\tan^{-1} y = x^3 + x + \frac{\pi}{4}$$

$$\underline{y(x) = \tan\left(x^3 + x + \frac{\pi}{4}\right)}|$$

Exercise

Solve the given initial-value problem $\left(y^3+\cos t\right)y' = 2+y\sin t, \quad y(0)=-1$

Solution

$$\left(y^3+\cos t\right)y' - (2+y\sin t) = 0$$

$$\left.\begin{array}{l} M_y = \frac{\partial}{\partial y}(-2-y\sin t) = -\sin t \\ N_t = \frac{\partial}{\partial t}\left(y^3+\cos t\right) = -\sin t \end{array}\right\} \Rightarrow \boxed{M_y = N_t}$$

$$\psi = \int (-2-y\sin t)\,dt$$

$$= -2t + y\cos t + h(y)$$

$$\psi_y = \cos t + h'(y) = y^3 + \cos t$$

$$h'(y) = y^3 \quad \rightarrow h(y) = \frac{1}{4}y^4$$

$$-2t + y\cos t + \frac{1}{4}y^4 = C$$

$$y(0) = -1 \quad \rightarrow -1 + \frac{1}{4} = C \Rightarrow \boxed{C = -\frac{3}{4}}$$

$$\boxed{-2t + y\cos t + \frac{1}{4}y^4 = -\frac{3}{4}}$$

Exercise

Solve the given initial-value problem $\left(y^3-t^3\right)y' = 3t^2y+1, \quad y(-2)=-1$

Solution

$$\left(y^3-t^3\right)y' - \left(3t^2y+1\right) = 0$$

$$\left.\begin{array}{l} M_y = \frac{\partial}{\partial y}\left(-3t^2y-1\right) = -3t^2 \\ N_t = \frac{\partial}{\partial t}\left(y^3-t^3\right) = -3t^2 \end{array}\right\} \Rightarrow \boxed{M_y = N_t}$$

$$\psi = \int \left(-3t^2y-1\right)dt$$

$$= -t^3y - t + h(y)$$

$$\psi_y = -t^3 + h'(y)$$

$$= y^3 - t^3$$

$$h'(y)=y^3 \quad \rightarrow h(y)=\frac{1}{4}y^4$$

$$-t^3y-t+\frac{1}{4}y^4=C$$

$$y(-2)=-1 \quad -8+2+\frac{1}{4}=C \quad \Rightarrow \quad \boxed{C=-\frac{23}{4}}$$

$$\boxed{-t^3y-t+\frac{1}{4}y^4=-\frac{23}{4}}$$

Exercise

Solve the given initial-value problem $\quad \left(e^{2y}+t^2y\right)y'+ty^2+\cos t=0, \quad y\left(\frac{\pi}{2}\right)=0$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(ty^2+\cos t\right)=2yt \\ N_t &= \frac{\partial}{\partial t}\left(e^{2y}+t^2y\right)=2ty \end{aligned}\right\} \Rightarrow \boxed{M_y=N_t}$$

$$\begin{aligned} \psi &= \int\left(ty^2+\cos t\right)dt \\ &= \frac{1}{2}t^2y^2+\sin t+h(y) \end{aligned}$$

$$\begin{aligned} \psi_y &= t^2y+h'(y) \\ &= t^2y+e^{2y} \end{aligned}$$

$$h'(y)=e^{2y} \quad \rightarrow h(y)=\frac{1}{2}e^{2y}$$

$$\frac{1}{2}t^2y^2+\sin t+\frac{1}{2}e^{2y}=C$$

$$y\left(\frac{\pi}{2}\right)=0 \quad 1+\frac{1}{2}=C \Rightarrow \boxed{C=\frac{3}{2}}$$

$$\boxed{\frac{1}{2}t^2y^2+\sin t+\frac{1}{2}e^{2y}=\frac{3}{2}}$$

Exercise

Solve the given initial-value problem $\quad y'=-\dfrac{y\cos(ty)+1}{t\cos(ty)+2ye^{y^2}}, \quad y(\pi)=0$

Solution

$$\left(t\cos(ty)+2ye^{y^2}\right)y'+\left(y\cos(ty)+1\right)=0$$

$$M_y = \frac{\partial}{\partial y}(y\cos ty + 1) = \cos ty - ty\sin ty$$

$$N_t = \frac{\partial}{\partial t}\left(t\cos ty + 2ye^{y^2}\right) = \cos ty - ty\sin ty$$

$$\Rightarrow \underline{M_y = N_t}$$

$$\psi = \int (y\cos ty + 1)\,dt$$

$$= \sin ty + t + h(y)$$

$$\psi_y = t\cos ty + h'(y)$$

$$= t\cos ty + 2ye^{y^2}$$

$$h'(y) = 2ye^{y^2} \quad \to h(y) = e^{y^2}$$

$$\sin ty + t + e^{y^2} = C$$

$$y(\pi) = 0 \quad \Rightarrow \quad \underline{C = -\pi - 1}$$

$$\underline{\sin ty + t + e^{y^2} = \pi + 1}$$

Exercise

Solve the given initial-value problem $\left(2ty + \frac{1}{y}\right)y' + y^2 = 1, \quad y(1) = 1$

Solution

$$\left(2ty + \frac{1}{y}\right)y' + y^2 - 1 = 0$$

$$M_y = \frac{\partial}{\partial y}\left(y^2 - 1\right) = 2y$$

$$N_t = \frac{\partial}{\partial t}\left(2ty + \frac{1}{y}\right) = 2y$$

$$\Rightarrow \underline{M_y = N_t}$$

$$\psi = \int \left(y^2 - 1\right)dt$$

$$= ty^2 - t + h(y)$$

$$\psi_y = t\cos ty + h'(y)$$

$$= 2ty + h'(y)$$

$$= 2ty + \frac{1}{y}$$

$$h'(y) = \frac{1}{y} \quad \to h(y) = \ln y$$

$$ty^2 - t + \ln|y| = C$$

$$y(1) = 1 \quad \Rightarrow \quad \underline{C = 0}$$

$$\boxed{ty^2 - t + \ln|y| = 0}$$

Exercise

Solve the given initial-value problem $\left(ye^x + 1\right)dx + \left(e^x - 1\right)dy = 0 \quad y(1) = 1$

Solution

$$\left.\begin{array}{l} M_y = \frac{\partial}{\partial y}\left(ye^x + 1\right) = e^x \\ N_x = \frac{\partial}{\partial x}\left(e^x - 1\right) = e^x \end{array}\right\} \Rightarrow M_y = N_x$$

$$\psi = \int \left(ye^x + 1\right)dx$$

$$= ye^x + x + h(y)$$

$$\psi_y = e^x + h'(y) = e^x - 1$$

$$\rightarrow h'(y) = -1 \Rightarrow h(y) = -y$$

$$\underline{ye^x + x - y = C}$$

$$y(1) = 1 \quad \rightarrow \quad \underline{C = e}$$

$$\boxed{ye^x + x - y = e}$$

Exercise

Solve the given initial-value problem $2xy^2 + 4 = 2\left(3 - x^2 y\right)y' \quad y(-1) = 8$

Solution

$$2xy^2 + 4 - 2\left(3 - x^2 y\right)y' = 0$$

$$\left.\begin{array}{l} M = 2xy^2 + 4 \Rightarrow M_y = 4xy \\ N = -6 + 2x^2 y \Rightarrow N_x = 4xy \end{array}\right\} \rightarrow M_y = N_x$$

$$\psi = \int \left(2xy^2 + 4\right) dx \qquad \qquad \psi = \int M dx$$

$$= x^2 y^2 + 4x + h(y)$$

$$\psi_y = 2x^2 y + h'(y)$$

$$= 2x^2 y - 6$$

$$h'(y) = -6 \quad \rightarrow \quad h(y) = -6y$$

$$\boxed{x^2 y^2 + 4x - 6y = C}$$

$$y(-1) = 8 \quad \rightarrow \quad 64 - 4 - 48 = C \quad \boxed{C = 12}$$

$$\boxed{x^2 y^2 + 4x - 6y = 12}$$

Exercise

Solve the given initial-value problem $\quad y' + \frac{4}{x} y = x^3 y^2 \quad y(2) = -1$

Solution

Let $u = y^{1-2} = y^{-1} \quad \Rightarrow \quad y = \frac{1}{u}$

$$\frac{du}{dx} = -\frac{1}{y^2} \frac{dy}{dx} \quad \Rightarrow \quad u' = -\frac{1}{y^2} y'$$

$$y' = -y^2 u' = -\frac{1}{u^2} u'$$

$$y' + \frac{4}{x} y = x^3 y^2$$

$$-\frac{1}{u^2} u' + \frac{4}{x} \frac{1}{u} = x^3 \frac{1}{u^2}$$

$$u' - \frac{4}{x} u = -x^3$$

$$e^{\int -\frac{4}{x} dx} = e^{-4 \ln x} = x^{-4}$$

$$\int -x^3 x^{-4} dx = -\int \frac{1}{x} dx = -\ln x$$

$$u = x^4 \left(-\ln x + C \right)$$

$$y = \frac{1}{x^4 (C - \ln x)} \qquad y = \frac{1}{u}$$

$$y(2) = -1 \quad \rightarrow -1 = \frac{1}{16(C - \ln 2)} \quad \boxed{C = \ln 2 - \frac{1}{16}}$$

$$\boxed{y(x) = \frac{1}{x^4 \left(\ln 2 - \frac{1}{16} - \ln x \right)}}$$

Exercise

Solve the given initial-value problem $y' = 5y + e^{-2x}y^{-2}$ $y(0) = 2$

Solution

$$y^2y' - 5y^3 = e^{-2x}$$

Let $u = y^3 \Rightarrow y = u^{1/3}$

$$u' = 3y^2y' \Rightarrow y' = \frac{1}{3}u^{-2/3}u'$$

$$\frac{1}{3}u' - 5u = e^{-2x}$$

$$u' - 15u = 3e^{-2x}$$

$$e^{\int -15dx} = e^{-15x}$$

$$\int 3e^{-2x}e^{-15x}dx = 3\int e^{-17x}dx = -\frac{3}{17}e^{-17x}$$

$$u = e^{15x}\left(-\frac{3}{17}e^{-17x} + C\right) \qquad u = y^3$$

$$y^3 = e^{15x}C - \frac{3}{17}e^{-2x}$$

$$y(0) = 2 \rightarrow 8 = C - \frac{3}{17} \quad \boxed{C = \frac{139}{17}}$$

$$\boxed{y(x) = \left(\frac{139e^{15x} - 3e^{-2x}}{17}\right)^{1/3}}$$

Exercise

Solve the given initial-value problem $6y' - 2y = xy^4$ $y(0) = -2$

Solution

$$6y^{-4}y' - 2y^{-3} = x$$

Let $u = y^{-3} \Rightarrow y = u^{-1/3}$

$$y' = -\frac{1}{3}u^{-4/3}u'$$

$$6u^{4/3}\left(-\frac{1}{3}u^{-4/3}\right)u' - 2u = x$$

$$-2u' - 2u = x$$

$$u' + u = -\frac{1}{2}x$$

$$e^{\int dx} = e^x$$

$$-\int \frac{1}{2} x e^{x} dx = -\frac{1}{2}(x-1)e^{x}$$

$$u = \frac{1}{e^{x}}\left(-\frac{1}{2}(x-1)e^{x} + C\right) \qquad u = y^{-3}$$

$$y^{-3} = \frac{C}{e^{x}} - \frac{1}{2}(x-1)$$

$$y(0) = -2 \quad \rightarrow \quad -\frac{1}{8} = C + \frac{1}{2} \quad \boxed{C = -\frac{5}{8}}$$

$$\frac{1}{y^{3}} = -\frac{5}{8}e^{-x} - \frac{1}{2}(x-1)$$

$$= -\frac{5e^{-x} - 4x + 4}{8}$$

$$\boxed{y(x) = -\frac{2}{\left(5e^{-x} - 4x + 4\right)^{1/3}}}$$

Exercise

Solve the given initial-value problem $\quad y' + \frac{y}{x} - \sqrt{y} = 0 \quad y(1) = 0$

Solution

$$y' + \frac{1}{x}y = y^{1/2}$$

$$y^{-1/2}y' + \frac{1}{x}y^{1/2} = 1$$

Let $u = y^{1/2} \quad \Rightarrow \quad y = u^{2}$

$$y' = 2u\,u'$$

$$u^{-1}(2uu') + \frac{1}{x}u = 1$$

$$2u' + \frac{1}{x}u = 1$$

$$u' + \frac{1}{2x}u = \frac{1}{2}$$

$$e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\ln x} = x^{1/2}$$

$$\int \frac{1}{2}x^{1/2}dx = \frac{1}{3}x^{3/2}$$

$$u = \frac{1}{x^{1/2}}\left(\frac{1}{3}x^{3/2} + C\right)$$

$$y^{1/2} = \frac{1}{3}x + Cx^{-1/2}$$

$y(1)=0 \quad \to\ 0 = C + \frac{1}{3} \quad \boxed{C = -\frac{1}{3}}$

$$\boxed{y(x) = \left(\frac{1}{3}x + \frac{1}{3}x^{-1/2}\right)^2}$$

Exercise

Solve the given initial-value problem $\quad xyy' + 4x^2 + y^2 = 0 \quad y(2) = -7$

Solution

$$xyy' = -4x^2 - y^2$$

$$\frac{y}{x}y' = -4 - \left(\frac{y}{x}\right)^2$$

Let $\frac{y}{x} = v \quad \to y = xv \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v\left(v + x\frac{dv}{dx}\right) = -4 - v^2$$

$$vx\frac{dv}{dx} = -4 - 2v^2$$

$$-\frac{v}{4+2v^2}dv = \frac{dx}{x}$$

$$-\frac{1}{4}\int \frac{1}{4+2v^2}d\left(4+2v^2\right) = \int \frac{1}{x}dx$$

$$-\frac{1}{4}\ln\left(4+2v^2\right) = \ln x + \ln C_1$$

$$\ln\left(4+2v^2\right)^{-\frac{1}{4}} = \ln C_1 x$$

$$\left(4+2v^2\right)^{-\frac{1}{4}} = C_1 x$$

$$4 + 2\left(\frac{y}{x}\right)^2 = Cx^{-4}$$

$$\frac{2y^2}{x^2} = \frac{C}{x^4} - 4$$

$$y^2 = \frac{1}{2}\frac{C - 4x^4}{x^2}$$

$y(2) = -7 \quad \to 49 = \frac{1}{2}\frac{C-64}{4} \quad \to\ \boxed{C = 456}$

$$y^2 = \frac{1}{2}\frac{456 - 4x^4}{x^2}$$

$$y^2 = \frac{228 - 2x^4}{x^2}$$

$$y = \pm\frac{\sqrt{228 - 2x^4}}{x}$$

Since the given initial $y(2) = -7$

$$\boxed{y = -\frac{\sqrt{228 - 2x^4}}{x}}$$

Exercise

Solve the given initial-value problem $\quad xy' = y(\ln x - \ln y) \quad y(1) = 4$

Solution

$$y' = \frac{y}{x}\ln\frac{x}{y}$$

Let $\frac{y}{x} = v \quad \rightarrow y = xv \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + xv' = v\ln\frac{1}{v}$$

$$x\frac{dv}{dx} = -v(1 + \ln v)$$

$$\int\frac{dv}{v(1 + \ln v)} = -\int\frac{dx}{x}$$

$$\int\frac{d(1 + \ln v)}{1 + \ln v} = -\int\frac{dx}{x}$$

$$\ln(1 + \ln v) = -\ln x + \ln C$$

$$\ln\left(1 + \ln\frac{y}{x}\right) = \ln\frac{C}{x}$$

$$1 + \ln\frac{y}{x} = \frac{C}{x}$$

$$y(1) = 4 \quad \rightarrow \quad \boxed{C = 1 + \ln 4}$$

$$\ln\frac{y}{x} = \frac{1 + \ln 4}{x} - 1$$

$$\boxed{y(x) = xe^{\frac{1 + \ln 4}{x} - 1}}$$

Exercise

Solve the given initial-value problem $y'-(4x-y+1)^2=0 \quad y(0)=2$

Solution

Let $v=4x-y \;\Rightarrow\; v'=4-y'$

$y'=4-v'$

$4-v'-(v+1)^2=0$

$v'=4-(v+1)^2$

$\quad =-v^2-2v+3$

$\int \frac{dv}{v^2+2v-3}=-\int dx$

$\frac{1}{(v-1)(v+3)}=\frac{A}{v-1}+\frac{B}{v+3}$

$Av+3A+Bv-B=1$

$\begin{cases} A+B=0 \\ 3A-B=1 \end{cases} \rightarrow \underline{A=\frac{1}{4},\;\; B=-\frac{1}{4}}$

$\int \frac{dv}{(v-1)(v+3)}=-\int dx$

$\frac{1}{4}\int\left(\frac{1}{v-1}-\frac{1}{v+3}\right)=-\int dx$

$\frac{1}{4}\left(\ln(v-1)-\ln(v+3)\right)=-x+C_1$

$\ln\frac{v-1}{v+3}=C-4x$

$\ln\left|\frac{4x-y-1}{4x-y+3}\right|=C-4x$

$y(0)=2 \;\rightarrow\; \underline{C=\ln 3}$

$\frac{4x-y-1}{4x-y+3}=e^{\ln 3-4x}$

$4x-y-1=3(4x-y+3)e^{-4x}$

$(4x-y-1)e^{4x}=12x-3y+9$

$3y-ye^{4x}=12x+9+(1-4x)e^{4x}$

$$\underline{y(x)=\frac{12x+9+(1-4x)e^{4x}}{3-e^{4x}}}$$

Exercise

Solve the given initial-value problem $\left(e^{t+y}+2y\right)y'+\left(e^{t+y}+3t^2\right)=0, \quad y(0)=0$

Solution

$$M_y = \frac{\partial}{\partial y}\left(e^{t+y} + 3t^2\right) = e^{t+y}$$
$$N_t = \frac{\partial}{\partial t}\left(e^{t+y} + 2y\right) = e^{t+y}$$
$$\Rightarrow \underline{M_y = N_t}$$

$$\psi = \int\left(e^{t+y} + 3t^2\right)dt$$
$$= e^{t+y} + t^3 + h(y)$$

$$\psi_y = e^{t+y} + h'(y) = e^{t+y} + 2y$$

$$h'(y) = 2y \quad \rightarrow h(y) = y^2$$

$$e^{t+y} + t^3 + y^2 = C$$

$$y(0) = 0 \quad \Rightarrow \quad \underline{C = 1}$$

$$\underline{e^{t+y} + t^3 + y^2 = 1}$$

Exercise

Solve the given initial-value problem $\quad (4y + 2x - 5)dx + (6y + 4x - 1)dy, \quad y(-1) = 2$

Solution

$$M_y = \frac{\partial}{\partial y}(4y + 2x - 5) = 4$$
$$N_x = \frac{\partial}{\partial x}(6y + 4x - 1) = 4$$
$$\Rightarrow \underline{M_y = N_x}$$

$$\psi = \int(4y + 2x - 5)dx$$
$$= 4xy + x^2 - 5x + h(y)$$

$$\psi_y = 4x + h'(y)$$
$$= 6y + 4x - 1$$

$$\Rightarrow h'(y) = 6y - 1 \quad \rightarrow h(y) = 3y^2 - y$$

$$\psi = 4xy + x^2 - 5x + 3y^2 - y = C$$

$$y(-1) = 2 \quad \Rightarrow \quad 4(-1)(2) + 1 + 5 + 12 - 2 = C \quad \rightarrow \quad \underline{C = 8}$$

$$\underline{4xy + x^2 - 5x + 3y^2 - y = 8}$$

Exercise

Solve the given initial-value problem $\left(ye^{xy}-\frac{1}{y}\right)dx+\left(xe^{xy}+\frac{x}{y^2}\right)dy=0 \quad y(1)=1$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(ye^{xy}-\frac{1}{y}\right)=(1+xy)e^{xy}+\frac{1}{y^2} \\ N_x &= \frac{\partial}{\partial x}\left(xe^{xy}+\frac{x}{y^2}\right)=(1+xy)e^{xy}+\frac{1}{y^2}\end{aligned}\right\} \Rightarrow M_y=N_x$$

$$\psi=\int\left(ye^{xy}-\frac{1}{y}\right)dx=e^{xy}-\frac{x}{y}+h(y)$$

$$\psi_y=xe^{xy}+\frac{x}{y^2}+h'(y)=xe^{xy}+\frac{x}{y^2}$$

$$\to h'(y)=0 \Rightarrow h(y)=C$$

$$\boxed{e^{xy}-\frac{x}{y}=C}$$

$$y(1)=1 \quad \to \underline{C=e-1}$$

$$\boxed{e^{xy}-\frac{x}{y}=e-1}$$

Exercise

Solve the given initial-value problem $(2y\ln t-t\sin y)y'+\frac{1}{t}y^2+\cos y=0, \quad y(2)=0$

Solution

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(\frac{1}{t}y^2+\cos y\right)=\frac{2y}{t}-\sin y \\ N_t &= \frac{\partial}{\partial t}(2y\ln t-t\sin y)=\frac{2y}{t}-\sin y\end{aligned}\right\} \Rightarrow \boxed{M_y=N_t}$$

$$\psi=\int\left(\frac{1}{t}y^2+\cos y\right)dt$$

$$=y^2\ln|t|+t\cos y+h(y)$$

$$\psi_y=2y\ln t-t\sin y+h'(y)=2y\ln t-t\sin y$$

$$\Rightarrow h'(y)=0 \qquad \to h(y)=C$$

$$y^2\ln|t|+t\cos y+C=0$$

$$y(2)=0 \quad \Rightarrow \underline{C=-2}$$

$$\boxed{y^2 \ln|t| + t\cos y - 2 = 0}$$

Exercise

Solve the given initial-value problem $\quad (\tan y - 2)dx + \left(x\sec^2 y + \frac{1}{y}\right)dy = 0 \quad y(0) = 1$

Solution

$$M_y = \frac{\partial}{\partial y}(\tan y - 2) = \sec^2 y$$

$$N_x = \frac{\partial}{\partial x}\left(x\sec^2 y + \frac{1}{y}\right) = \sec^2 y \qquad \Rightarrow M_y = N_x$$

$$\psi = \int (\tan y - 2)dx$$

$$= x\tan y - 2x + h(y)$$

$$\psi_y = x\sec^2 y + h'(y)$$

$$= x\sec^2 y + \frac{1}{y}$$

$$\rightarrow h'(y) = \frac{1}{y} \Rightarrow h(y) = \ln|y|$$

$$\boxed{x\tan y - 2x + \ln|y| = C}$$

$$y(0) = 1 \quad \rightarrow \boxed{C = 0}$$

$$\boxed{x\tan y - 2x + \ln y = 0}$$

Exercise

Solve the given initial-value problem $\quad 2xy - 9x^2 + \left(2y + x^2 + 1\right)\frac{dy}{dx} = 0 \quad y(0) = -3$

Solution

$$M = 2xy - 9x^2 \Rightarrow M_y = 2x$$

$$N = 2y + x^2 + 1 \Rightarrow N_x = 2x \qquad \rightarrow M_y = N_x$$

$$\psi = \int \left(2xy - 9x^2\right) dx \qquad \psi = \int M dx$$

$$= x^2 y - 3x^3 + h(y)$$

$$\psi_y = x^2 + h'(y)$$

$$= 2y + x^2 + 1$$

$$h'(y) = 2y + 1 \quad \to \quad h(y) = y^2 + y$$

$$\underline{x^2y - 3x^3 + y^2 + y = C}$$

$$y(0) = -3 \quad \to 9 - 3 = C \quad \underline{C = 6}$$

$$\underline{x^2y - 3x^3 + y^2 + y = 6}$$

Exercise

Solve the given initial-value problem $\quad \dfrac{2t}{t^2+1}y - 2t + \left(2 - \ln\left(t^2+1\right)\right)\dfrac{dy}{dt} = 0 \quad y(5) = 0$

Solution

$$\left.\begin{array}{l} M = \dfrac{2t}{t^2+1}y - 2t \quad \Rightarrow \quad M_y = \dfrac{2t}{t^2+1} \\ N = 2 - \ln\left(t^2+1\right) \quad \Rightarrow \quad N_t = \dfrac{2t}{t^2+1} \end{array}\right. \quad \to \quad M_y = N_t$$

$$\psi = \int \left(\frac{2t}{t^2+1}y - 2t\right) dt \qquad\qquad \psi = \int M dt$$

$$= y\int \frac{1}{t^2+1} d\left(t^2+1\right) - t^2$$

$$= \ln\left(t^2+1\right)y - t^2 + h(y)$$

$$\psi_y = \ln\left(t^2+1\right) + h'(y)$$

$$= 2 - \ln\left(t^2+1\right)$$

$$h'(y) = -2 \quad \to \quad h(y) = -2y$$

$$\ln\left(t^2+1\right)y - t^2 - 2y = C$$

$$y(5) = 0 \quad \to \quad \underline{C = -25}$$

$$\left(\ln\left(t^2+1\right) - 2\right)y - t^2 = -25$$

$$\underline{y(t) = \frac{t^2 - 25}{\ln\left(t^2+1\right) - 2}}$$

Exercise

Solve the given initial-value problem $\quad 3y^3e^{3xy}-1+\left(2ye^{3xy}+3xy^2e^{3xy}\right)y'=0 \quad y(0)=1$

Solution

$$M=3y^3e^{3xy}-1 \qquad \Rightarrow \quad M_y=9y^2e^{3xy}+9xy^3e^{3xy}$$

$$N=2ye^{3xy}+3xy^2e^{3xy} \quad \Rightarrow \quad N_x=6y^2e^{3xy}+3y^2e^{3xy}+9xy^3e^{3xy}$$

$$\rightarrow \quad M_y=N_x$$

$$\psi=\int\left(3y^3e^{3xy}-1\right)\,dx \qquad\qquad \psi=\int Mdx$$

$$=y^2e^{3xy}-x+h(y)$$

$$\psi_y=2ye^{3xy}+3xy^2e^{3xy}+h'(y)$$

$$=2ye^{3xy}+3xy^2e^{3xy}$$

$$\Rightarrow h'(y)=0 \quad \rightarrow \quad h(y)=C$$

$$y^2e^{3xy}-x=C$$

$$y(0)=1 \quad \rightarrow \quad \underline{C=1}$$

$$\boxed{y^2e^{3xy}-x=1}$$

Exercise

Solve the given initial-value problem $\quad 2xydx+\left(1+x^2\right)dy=0\ ; \quad y(2)=-5$

Solution

$$M=2xy \quad \Rightarrow \quad M_y=2x$$

$$N=1+x^2 \quad \Rightarrow \quad N_x=2x$$

$$\rightarrow \quad M_y=N_x$$

$$\psi=\int(2xy)\,dx=x^2y+h(y)$$

$$\psi_y=x^2+h'(y)=1+x^2$$

$$\Rightarrow h'(y)=1 \quad \rightarrow \quad h(y)=y$$

$$x^2y+y=C$$

$$y\left(x^2+1\right)=C$$

$$y(2) = -5 \quad \rightarrow -20 - 5 = C \quad \Rightarrow \underline{C = -25}$$

$$\boxed{y(x) = -\frac{25}{1 + x^2}}$$

Exercise

Solve the given initial-value problem $\frac{dy}{dx} = -\frac{2x\cos y + 3x^2 y}{x^3 - x^2 \sin y - y}$; $y(0) = 2$

Solution

$$\left(x^3 - x^2 \sin y - y\right)dy = -\left(2x\cos y + 3x^2 y\right)dx$$

$$\left(2x\cos y + 3x^2 y\right)dx + \left(x^3 - x^2 \sin y - y\right)dy = 0$$

$$\begin{aligned} M &= 2x\cos y + 3x^2 y \quad \Rightarrow \quad M_y = -2x\sin y + 3x^2 \\ N &= x^3 - x^2 \sin y - y \quad \Rightarrow \quad N_x = 3x^2 - 2x\sin y \end{aligned} \quad \rightarrow \quad M_y = N_x$$

$$\psi = \int \left(2x\cos y + 3x^2 y\right)\, dx = x^2 \cos y + x^3 y + h(y)$$

$$\psi_y = -x^2 \sin y + x^3 + h'(y) = x^3 - x^2 \sin y - y$$

$$\Rightarrow h'(y) = -y \quad \rightarrow \quad h(y) = -\frac{1}{2}y^2$$

$$x^2 \cos y + x^3 y - \frac{1}{2}y^2 = C$$

$$y(0) = 2 \quad \rightarrow \quad \underline{-2 = C}$$

$$\boxed{x^2 \cos y + x^3 y - \frac{1}{2}y^2 = -2}$$

Exercise

Find an integrating factor of the form $x^n y^m$ and solve the equation

$$\left(2y^2 - 6xy\right)dx + \left(3xy - 4x^2\right)dy = 0$$

Solution

$$x^n y^m \left(2y^2 - 6xy\right)dx + x^n y^m \left(3xy - 4x^2\right)dy = 0$$

$$\left(2x^n y^{m+2} - 6x^{n+1} y^{m+1}\right)dx + \left(3x^{n+1} y^{m+1} - 4x^{n+2} y^m\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(2x^n y^{m+2} - 6x^{n+1}y^{m+1}\right) = 2(m+2)x^n y^{m+1} - 6(m+1)x^{n+1}y^m$$

$$N_x = \frac{\partial}{\partial x}\left(3x^{n+1}y^{m+1} - 4x^{n+2}y^m\right) = 3(n+1)x^n y^{m+1} - 4(n+2)x^{n+1}y^m$$

For the equation to be exact, then

$$2(m+2)x^n y^{m+1} - 6(m+1)x^{n+1}y^m = 3(n+1)x^n y^{m+1} - 4(n+2)x^{n+1}y^m$$

$$2(m+2)y - 6(m+1)x = 3(n+1)y - 4(n+2)x$$

$$\begin{cases} 2m+4 = 3n+3 \\ 3m+3 = 2n+4 \end{cases} \rightarrow \begin{cases} 2m-3n=-1 \\ 3m-2n=1 \end{cases} \Rightarrow \underline{m=1, \quad n=1}$$

$$xy\left(2y^2 - 6xy\right)dx + xy\left(3xy - 4x^2\right)dy = 0$$

$$\left(2xy^3 - 6x^2y^2\right)dx + \left(3x^2y^2 - 4x^3y\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(2xy^3 - 6x^2y^2\right) = 6xy^2 - 12x^2y$$

$$N_x = \frac{\partial}{\partial x}\left(3x^2y^2 - 4x^3y\right) = 6xy^2 - 12x^2y$$

$$\Rightarrow M_y = N_x$$

$$\psi = \int\left(2xy^3 - 6x^2y^2\right)dx$$

$$= x^2y^3 - 2x^3y^2 + h(y)$$

$$\psi_y = 3x^2y^2 - 4x^3y + h'(y)$$

$$= 3x^2y^2 - 4x^3y$$

$$h'(y) = 0 \Rightarrow h(y) = C$$

$$\underline{x^2y^3 - 2x^3y^2 = C}$$

Exercise

Find an integrating factor of the form $x^n y^m$ and solve the equation

$$(12+5xy)dx + \left(6xy^{-1} + 3x^2\right)dy = 0$$

Solution

$$x^n y^m(12+5xy)dx + x^n y^m\left(6xy^{-1} + 3x^2\right)dy = 0$$

$$\left(12x^n y^m + 5x^{n+1}y^{m+1}\right)dx + \left(6x^{n+1}y^{m-1} + 3x^{n+2}y^m\right)dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(12x^n y^m + 5x^{n+1}y^{m+1}\right) = 12mx^n y^{m-1} + 5(m+1)x^{n+1}y^m$$

$$N_x = \frac{\partial}{\partial x}\left(6x^{n+1}y^{m-1} + 3x^{n+2}y^m\right) = 6(n+1)x^n y^{m-1} + 3(n+2)x^{n+1}y^m$$

For the equation to be exact, then

$$12mx^n y^{m-1} + 5(m+1)x^{n+1}y^m = 6(n+1)x^n y^{m-1} + 3(n+2)x^{n+1}y^m$$

$$12m + 5(m+1)xy = 6(n+1) + 3(n+2)xy$$

$$\begin{cases} 12m = 6n+6 \\ 5m+5 = 3n+6 \end{cases} \rightarrow \begin{cases} 2m - n = 1 \\ 5m - 6n = 1 \end{cases} \Rightarrow \underline{n = 3, \ \ m = 2}$$

$$x^3y^2(12 + 5xy)dx + x^3y^2\left(6xy^{-1} + 3x^2\right)dy = 0$$

$$\left(12x^3y^2 + 5x^4y^3\right)dx + \left(6x^4y + 3x^5y^2\right)dy = 0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(12x^3y^2 + 5x^4y^3\right) = 24x^3y + 15x^4y^2 \\ N_x &= \frac{\partial}{\partial x}\left(6x^4y + 3x^5y^2\right) = 24x^3y + 15x^4y^2 \end{aligned}\right. \Rightarrow M_y = N_x$$

$$\psi = \int \left(12x^3y^2 + 5x^4y^3\right)dx$$

$$= 3x^4y^2 + x^5y^3 + h(y)$$

$$\psi_y = 6x^4y + 3x^5y^2 + h'(y)$$

$$= 6x^4y + 3x^5y^2$$

$$h'(y) = 0 \ \Rightarrow \ h(y) = C$$

$$\underline{3x^4y^2 + x^5y^3 = C}$$

Exercise

Find an integrating factor of the form $x^n y^m$ and solve the equation

$$\left(3y + 4xy^2\right)dx + \left(2x + 3x^2y\right)dy = 0$$

Solution

$$x^n y^m\left(3y + 4xy^2\right)dx + x^n y^m\left(2x + 3x^2y\right)dy = 0$$

$$\left(3x^n y^{m+1} + 4x^{n+1}y^{m+2}\right)dx + \left(2x^{n+1}y^m + 3x^{n+2}y^{m+1}\right)dy = 0$$

$$M = 3x^n y^{m+1} + 4x^{n+1}y^{m+2} \ ; \quad N = 2x^{n+1}y^m + 3x^{n+2}y^{m+1}$$

$$M_y = \frac{\partial}{\partial y}\left(3x^n y^{m+1} + 4x^{n+1}y^{m+2}\right) = 3(m+1)x^n y^m + 4(m+2)x^{n+1}y^{m+1}$$

$$N_x = \frac{\partial}{\partial x}\left(2x^{n+1}y^m + 3x^{n+2}y^{m+1}\right) = 2(n+1)x^n y^m + 3(n+2)x^{n+1}y^{m+1}$$

For the equation to be exact, then

$$3(m+1)x^n y^m + 4(m+2)x^{n+1}y^{m+1} = 2(n+1)x^n y^m + 3(n+2)x^{n+1}y^{m+1}$$

$$3(m+1) + 4(m+2)xy = 2(n+1) + 3(n+2)xy$$

$$\begin{cases} 3m+3 = 2n+2 \\ 4m+8 = 3n+6 \end{cases} \rightarrow \begin{cases} 3m-2n = -1 \\ 4m-3n = -2 \end{cases}$$

$$\underline{m} = \frac{\begin{vmatrix} -1 & -2 \\ -2 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 4 & -3 \end{vmatrix}} = \frac{-1}{-1} \underline{=1|} \qquad \underline{|n} = \frac{\begin{vmatrix} 3 & -1 \\ 4 & -2 \end{vmatrix}}{-1} \underline{=2|}$$

$$x^2y\left(3y + 4xy^2\right)dx + x^2y\left(2x + 3x^2y\right)dy = 0$$

$$\left(3x^2y^2 + 4x^3y^3\right)dx + \left(2x^3y + 3x^4y^2\right)dy = 0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(3x^2y^2 + 4x^3y^3\right) = 6x^2y + 12x^3y^2 \\ N_x &= \frac{\partial}{\partial x}\left(2x^3y + 3x^4y^2\right) = 6x^2y + 12x^3y^2 \end{aligned}\right. \Rightarrow M_y = N_x$$

$$\psi = \int \left(3x^2y^2 + 4x^3y^3\right)dx$$

$$= x^3y^2 + x^4y^3 + h(y)$$

$$\psi_y = 2x^3y + 3x^4y^2 + h'(y) = 2x^3y + 3x^4y^2$$

$$h'(y) = 0 \Rightarrow h(y) = C$$

$$\underline{x^3y^2 + x^4y^3 = C|}$$

Exercise

Find the general solution by using either Bernoulli $\frac{dy}{dx} - 5y = -\frac{5}{2}xy^3$

Solution

$y^3 \Rightarrow n = 3$

Let $u = y^{1-3} = y^{-2} = \frac{1}{y^2} \quad \Rightarrow \quad y = \frac{1}{\sqrt{u}}$

$$\frac{du}{dx}=-2y^{-3}\frac{dy}{dx} \quad\Rightarrow\quad -\frac{1}{2}y^3\frac{du}{dx}=\frac{dy}{dx}$$

$$\frac{dy}{dx}-5y=-\frac{5}{2}xy^3$$

$$-\frac{1}{2}u^{-3/2}\frac{du}{dx}-5u^{-1/2}=-\frac{5}{2}xu^{-3/2} \qquad \times\ -2u^{3/2}$$

$$\frac{du}{dx}+10u=5x$$

$$e^{\int 10dx}=e^{10x}$$

$$\int 5xe^{10x}dx=\left(\frac{1}{2}x-\frac{1}{20}\right)e^{10x}$$

		$\int e^{10x}$
+	$5x$	$\frac{1}{10}e^{10x}$
–	5	$\frac{1}{100}e^{10x}$

$$u=\frac{1}{e^{10x}}\left(\left(\frac{1}{2}x-\frac{1}{20}\right)e^{10x}+C\right)$$

$$\boxed{\frac{1}{y^2}=\frac{1}{2}x-\frac{1}{20}+Ce^{-10x}}$$

Exercise

Find the general solution by using either Bernoulli $\frac{dy}{dx}+\frac{y}{x}=x^2y^2$

Solution

$$y^2 \ \Rightarrow\ n=2$$

Let $u=y^{1-2}=\frac{1}{y} \quad\Rightarrow\quad y=\frac{1}{u}$

$$\frac{du}{dx}=-\frac{1}{y^2}\frac{dy}{dx} \quad\Rightarrow\quad -\frac{1}{u^2}\frac{du}{dx}=\frac{dy}{dx}$$

$$\frac{dy}{dx}+\frac{y}{x}=x^2y^2$$

$$-\frac{1}{u^2}\frac{du}{dx}+\frac{1}{xu}=x^2\frac{1}{u^2} \qquad \times\ -u^2$$

$$\frac{du}{dx}-\frac{1}{x}u=-x^2$$

$$e^{\int -\frac{1}{x}dx}=e^{-\ln|x|}=\frac{1}{x}$$

$$\int x^2\frac{1}{x}dx=\frac{1}{2}x^2$$

$$u=x\left(\frac{1}{2}x^2+C_1\right)$$

$$\frac{1}{y}=\frac{1}{2}x^3+C_1x$$

$$\boxed{y(x) = \frac{2}{x^3 + Cx}}$$

Exercise

Find the general solution by using either Bernoulli $\frac{dy}{dx} - y = e^{2x}y^3$

Solution

$$y^3 \Rightarrow n = 3$$

Let $u = y^{1-3} = y^{-2} = \frac{1}{y^2} \quad \Rightarrow \quad y = \frac{1}{\sqrt{u}} = u^{-1/2}$

$$\frac{du}{dx} = -2y^{-3}\frac{dy}{dx} \quad \Rightarrow \quad -\frac{1}{2}y^3\frac{du}{dx} = \frac{dy}{dx}$$

$$\frac{dy}{dx} - y = e^{2x}y^3$$

$$-\frac{1}{2}u^{-3/2}\frac{du}{dx} - u^{-1/2} = e^{2x}u^{-3/2} \qquad \times\ -2u^{3/2}$$

$$\frac{du}{dx} + 2u = -2e^{2x}$$

$$e^{\int 2dx} = e^{2x}$$

$$\int -2e^{2x}e^{2x}dx = -\frac{1}{2}e^{4x}$$

$$u = e^{-2x}\left(-\frac{1}{2}e^{4x} + C\right)$$

$$\boxed{y^{-2} = -\frac{1}{2}e^{2x} + Ce^{-2x}} \qquad y = \pm\sqrt{\frac{-2}{e^{2x} + Ce^{-2x}}}$$

Exercise

Find the general solution by using either Bernoulli $\frac{dy}{dx} + \frac{y}{x-2} = 5(x-2)y^{1/2}$

Solution

$$y^{1/2} \Rightarrow n = \frac{1}{2}$$

Let $u = y^{1-\frac{1}{2}} = y^{1/2} \quad \Rightarrow \quad y = u^2 \qquad \to \frac{dy}{dx} = 2u\frac{du}{dx}$

$$\frac{dy}{dx} + \frac{1}{x-2}y = 5(x-2)y^{1/2}$$

$$2u\frac{du}{dx} + \frac{1}{x-2}u^2 = 5(x-2)u \qquad \times\ \frac{1}{2u}$$

$$\frac{du}{dx} + \frac{1}{2}\frac{1}{x-2}u = \frac{5}{2}(x-2)$$

$$e^{\frac{1}{2}\int \frac{1}{x-2}dx} = e^{\frac{1}{2}\ln|x-2|} = \sqrt{x-2}$$

$$\int \frac{1}{2}(x-2)\sqrt{x-2}\ dx = \int \frac{1}{2}(x-2)^{3/2}\ dx = \frac{1}{5}(x-2)^{5/2}$$

$$u = \frac{1}{\sqrt{x-2}}\left(\frac{1}{5}(x-2)^{5/2} + C\right)$$

$$y^{1/2} = \frac{1}{5}(x-2)^{3/2} + \frac{C}{\sqrt{x-2}}$$

$$\boxed{y(x) = \left(\frac{1}{5}(x-2)^{3/2} + \frac{C}{\sqrt{x-2}}\right)^2}$$

Exercise

Find the general solution by using either Bernoulli $\frac{dy}{dx} + y = e^x y^{-2}$

Solution

$$y^{-2} \Rightarrow n = -2$$

Let $u = y^{1+2} = y^3 \quad \Rightarrow \quad y = u^{1/3}$

$$\to \frac{dy}{dx} = \frac{1}{3}u^{-2/3}\frac{du}{dx}$$

$$\frac{dy}{dx} + y = e^x y^{-2}$$

$$\frac{1}{3}u^{-2/3}\frac{du}{dx} + u^{1/3} = e^x u^{-2/3} \qquad \times\ 3u^{2/3}$$

$$\frac{du}{dx} + 3u = 3e^x$$

$$e^{\int 3dx} = e^{3x}$$

$$\int 3e^x e^{3x}\ dx = \frac{3}{4}e^{4x}$$

$$u = e^{-3x}\left(\frac{3}{4}e^{4x} + C\right)$$

$$\boxed{y^3 = \frac{3}{4}e^x + Ce^{-3x}}$$

Exercise

Find the general solution by using either Bernoulli $\frac{dy}{dx}+y^3x+y=0$

Solution

$$\frac{dy}{dx}+y=-xy^3$$

$$y^3 \;\Rightarrow\; n=3$$

Let $u=y^{1-3}=y^{-2}=\frac{1}{y^2} \quad\Rightarrow\quad y=\frac{1}{\sqrt{u}}=u^{-1/2}$

$$\frac{dy}{dx}=-\frac{1}{2}u^{-3/2}\frac{du}{dx}$$

$$-\frac{1}{2}u^{-3/2}\frac{du}{dx}+u^{-1/2}=-xu^{-3/2} \qquad \times\ -2u^{3/2}$$

$$\frac{du}{dx}-2u=2x$$

$$e^{\int -2dx}=e^{-2x}$$

$$\int 2xe^{-2x}dx=\left(-x-\frac{1}{2}\right)e^{-2x}$$

		$\int e^{-2x}$
+	$2x$	$-\frac{1}{2}e^{-2x}$
−	2	$\frac{1}{4}e^{-2x}$

$$u=e^{2x}\left(\left(-x-\frac{1}{2}\right)e^{-2x}+C\right)$$

$$\boxed{\frac{1}{y^2}=-x-\frac{1}{2}+Ce^{2x}} \qquad y(x)=\pm\frac{1}{\sqrt{Ce^{2x}-x-\frac{1}{2}}}$$

Exercise

Find the general solution by using homogeneous equations. $\left(xy+y^2\right)dx-x^2dy=0$

Solution

$$\frac{dy}{dx}=\frac{xy+y^2}{x^2}=\frac{y}{x}+\left(\frac{y}{x}\right)^2$$

Let $v=\frac{y}{x}\to y=xv \quad\Rightarrow\quad y'=v+xv'$

$$v+xv'=v+v^2$$

$$x\frac{dv}{dx}=v^2$$

$$\int\frac{dv}{v^2}=\int\frac{dx}{x}$$

$$-\frac{1}{v}=\ln|x|+C$$

$$-\frac{x}{y}=\ln|x|+C$$

$$\boxed{y(x)=-\frac{x}{\ln|x|+C}}$$

Exercise

Find the general solution by using homogeneous equations. $\left(x^2+y^2\right)dx+2xydy=0$

Solution

$$\frac{dy}{dx}=-\frac{x^2+y^2}{2xy}$$

$$=-\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right)$$

Let $v=\frac{y}{x}\rightarrow y=xv \quad\Rightarrow\quad y'=v+xv'$

$$v+xv'=-\frac{1}{2}v-\frac{1}{2v}$$

$$xv'=-\frac{3}{2}v-\frac{1}{2v}$$

$$x\frac{dv}{dx}=-\left(\frac{3v^2+1}{2v}\right)$$

$$\int\frac{2v}{3v^2+1}dv=-\int\frac{1}{x}dx$$

$$\frac{1}{3}\ln\left(3v^2+1\right)=-\ln|x|+\ln C_1$$

$$\ln\left(3v^2+1\right)=-3\ln\left|C_1x\right|$$

$$3v^2+1=\frac{C}{|x|^3}$$

$$3\frac{y^2}{x^2}=\frac{C}{|x|^3}-1$$

$$3xy^2=C-x^3$$

$$\boxed{3xy^2+x^3=C}$$

Exercise

Find the general solution by using homogeneous equations. $\left(y^2 - xy\right)dx + x^2 dy = 0$

Solution

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2}$$

$$= \frac{y}{x} - \left(\frac{y}{x}\right)^2$$

Let $v = \frac{y}{x} \to y = xv \quad \Rightarrow \quad y' = v + xv'$

$$v + xv' = v - v^2$$

$$x\frac{dv}{dx} = -v^2$$

$$-\int \frac{1}{v^2} dv = \int \frac{1}{x} dx$$

$$\frac{1}{v} = \ln|x| + C$$

$$\frac{x}{y} = \ln|x| + C$$

$$\boxed{y(x) = \frac{x}{\ln|x| + C}}$$

Exercise

Find the general solution by using homogeneous equations. $\frac{dy}{d\theta} = \frac{1}{\theta}\left(\theta \sec\left(\frac{y}{\theta}\right) + y\right)$

Solution

$$\frac{dy}{d\theta} = \sec\left(\frac{y}{\theta}\right) + \frac{y}{\theta}$$

Let $v = \frac{y}{\theta} \to y = \theta v \quad \Rightarrow \quad y' = v + \theta v'$

$$v + \theta v' = \sec v + v$$

$$\int \cos v \, dv = \int \frac{d\theta}{\theta}$$

$$\sin v = \ln|\theta| + C$$

$$\frac{y}{\theta} = \arcsin\left(\ln|\theta| + C\right)$$

$$\boxed{y(\theta) = \theta \arcsin\left(\ln|\theta| + C\right)}$$

Exercise

Find the general solution by using homogeneous equations. $\frac{dy}{dx} = \frac{y(\ln y - \ln x + 1)}{x}$

Solution

$$\frac{dy}{dx} = \frac{y}{x}\left(\ln\frac{y}{x} + 1\right)$$

Let $v = \frac{y}{x} \to y = xv \quad \Rightarrow \quad y' = v + xv'$

$$v + xv' = v\ln|v| + v$$

$$x\frac{dv}{dx} = v\ln|v|$$

$$\frac{dv}{v\ln|v|} = \frac{dx}{x}$$

$$\int \frac{1}{\ln|v|}\, d(\ln v) = \int \frac{dx}{x}$$

$$\ln|\ln v| = \ln|x| + \ln C$$

$$\ln|\ln v| = \ln|Cx|$$

$$\ln v = Cx$$

$$v = e^{Cx} = \frac{y}{x}$$

$$\boxed{y(x) = xe^{Cx}}$$