

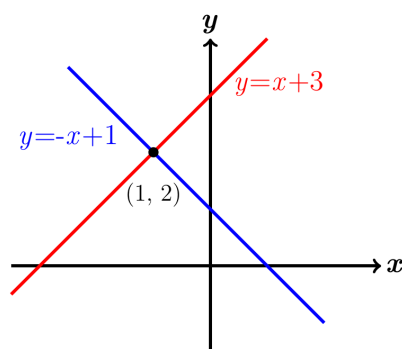
# Lecture Four      Matrices

## Section 4.1 – System of linear Equations

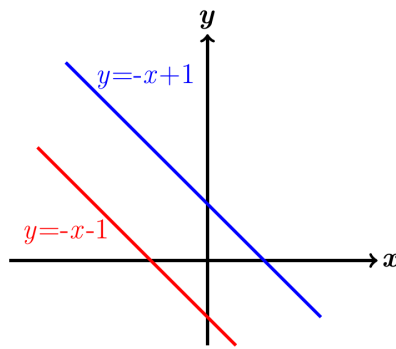
### Solving Systems of Equations

1. Graphically
2. Substitution Method
3. Elimination Method

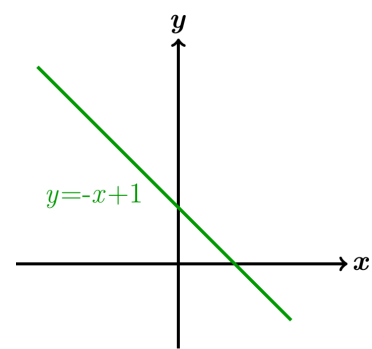
#### Graphically



**One solution (lines intersect)**  
**Consistent**  
**Independent**



**No Solution (lines //)**  
**Inconsistent**  
**Independent**



**Infinite solution**  
**Consistent**  
**Dependent**

#### Substitution Method

Solve: 
$$\begin{cases} 3x + 2y = 11 & (1) \\ -x + y = 3 & (2) \end{cases}$$

#### Solution

From (2)  $\rightarrow y = x + 3$     (3)

(1)  $\Rightarrow 3x + 2(x + 3) = 11$

$$3x + 2x + 6 = 11$$

$$5x + 6 = 11$$

$$5x + 6 - 6 = 11 - 6$$

$$5x = 5$$

$$x = 1$$

From (3)  $\rightarrow y = 1 + 3 = 4$

Solution:  $\underline{(1, 4)}$

### ***Elimination Method***

Solve:  $\begin{cases} 3x - 4y = 1 & (1) \\ 2x + 3y = 12 & (2) \end{cases}$

#### **Solution**

$$\textcolor{red}{-2\times) \quad 3x - 4y = 1}$$

$$\textcolor{red}{3\times) \quad 2x + 3y = 12}$$

$$-6x + 8y = -2$$

$$6x + 9y = 36$$

$$\hline 17y = 34$$

$$y = \frac{34}{17} = 2$$

From (1)  $\Rightarrow 3x = 1 + 4y$

$$3x = 1 + 4(\textcolor{blue}{2})$$

$$3x = 9$$

$$\textcolor{red}{x = 3}$$

Solution:  $\underline{(\textcolor{blue}{3}, \textcolor{blue}{2})}$

# Matrices

$$\begin{array}{ccc}
 & \text{Column} & \\
 & C_1 & C_2 & C_3 \\
 & \downarrow & \downarrow & \downarrow \\
 \text{Row 1} \rightarrow R_1 & a_{11} & a_{12} & a_{13} \\
 \text{Row 2} \rightarrow R_2 & a_{21} & a_{22} & a_{23} \\
 \text{Row 3} \rightarrow R_3 & a_{31} & a_{32} & a_{33}
 \end{array}
 \left[ \begin{array}{ccc}
 a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23} \\
 a_{31} & a_{32} & a_{33}
 \end{array} \right]$$

This is called Matrix (*Matrices*)

Each number in the array is an **element** or **entry**

The matrix is said to be of order  $m \times n$

$m$ : numbers of rows,

$n$ : number of columns

When  $m = n$ , then matrix is said to be **square**.

Given the system equations

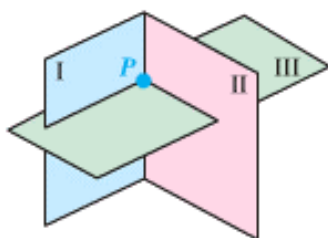
$$3x + y + 2z = 31$$

$$x + y + 2z = 19$$

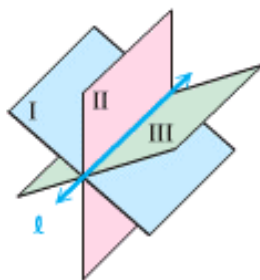
$$x + 3y + 2z = 25$$

The **augmented matrix** form is:

$$\left[ \begin{array}{ccc|c}
 3 & 1 & 2 & 31 \\
 1 & 1 & 2 & 19 \\
 1 & 3 & 2 & 25
 \end{array} \right]$$



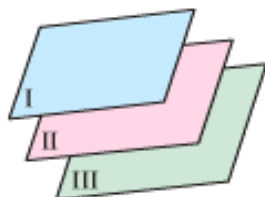
A single solution



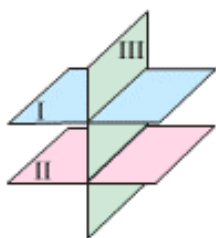
Points of a line in common



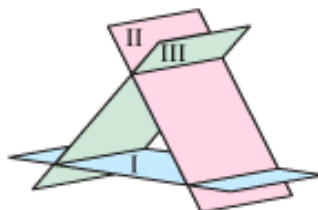
All points in common



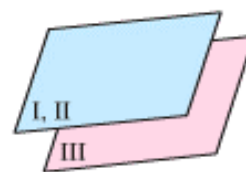
No points in common



No points in common



No points in common



No points in common

## *Gaussian Elimination*

### *Example*

Use the Gaussian elimination method to solve the system

$$3x + y + 2z = 31$$

$$x + y + 2z = 19$$

$$x + 3y + 2z = 25$$

### *Solution*

$$\left[ \begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 3 & 1 & 2 & 31 \\ 1 & 3 & 2 & 25 \end{array} \right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array} \quad \begin{array}{cccc} 3 & 1 & 2 & 31 \\ -3 & -3 & -6 & -57 \\ 0 & -2 & -4 & -26 \end{array} \quad \begin{array}{cccc} 1 & 3 & 2 & 25 \\ -1 & -1 & -2 & -19 \\ 0 & 2 & 0 & 6 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & -2 & -4 & -26 \\ 0 & 2 & 0 & 6 \end{array} \right] -\frac{1}{2}R_2 \quad \begin{array}{cccc} 0 & 1 & 2 & 13 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 2 & 0 & 6 \end{array} \right] R_3 - 2R_2 \quad \begin{array}{cccc} 0 & 2 & 0 & 6 \\ 0 & -2 & -4 & -26 \\ 0 & 0 & -4 & -20 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & -4 & -20 \end{array} \right] -\frac{1}{4}R_3 \quad \begin{array}{cccc} 0 & 0 & 1 & 5 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 1 & 5 \end{array} \right] \Rightarrow \begin{array}{l} x + y + 2z = 19 \quad (3) \\ y + 2z = 13 \quad (2) \\ z = 5 \quad (1) \end{array}$$

$$(2) \Rightarrow y = 13 - 2z = 13 - 2(5) = 3$$

$$(3) \Rightarrow x = 19 - y - 2z = 19 - 3 - 10 = 6$$

$$\Rightarrow (6, 3, 5)$$

## Gauss-Jordan Elimination

### Example

Use the Gauss-Jordan method to solve the system

$$3x + y + 2z = 31$$

$$x + y + 2z = 19$$

$$x + 3y + 2z = 25$$

### Solution

$$\left[ \begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 3 & 1 & 2 & 31 \\ 1 & 3 & 2 & 25 \end{array} \right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array} \quad \begin{array}{ccc|c} 3 & 1 & 2 & 31 \\ -3 & -3 & -6 & -57 \\ 0 & -2 & -4 & -26 \end{array} \quad \begin{array}{ccc|c} 1 & 3 & 2 & 25 \\ -1 & -1 & -2 & -19 \\ 0 & 2 & 0 & 6 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & -2 & -4 & -26 \\ 0 & 2 & 0 & 6 \end{array} \right] -\frac{1}{2}R_2 \quad \begin{array}{ccc|c} 0 & 1 & 2 & 13 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 2 & 0 & 6 \end{array} \right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 2R_2 \end{array} \quad \begin{array}{ccc|c} 0 & 2 & 0 & 6 \\ 0 & -2 & -4 & -26 \\ 0 & 0 & -4 & -20 \end{array} \quad \begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & -1 & -2 & -13 \\ 1 & 0 & 0 & 6 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & -4 & -20 \end{array} \right] -\frac{1}{4}R_3 \quad \begin{array}{ccc|c} 0 & 0 & 1 & 5 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 1 & 5 \end{array} \right] R_2 - 2R_3 \quad \begin{array}{ccc|c} 0 & 1 & 2 & 13 \\ 0 & 0 & -2 & -10 \\ 0 & 1 & 0 & 3 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array} \right]$$

**Solution:** (6, 3, 5)

### Example

Use the Gaussian elimination method to solve the system

$$2x + y + 2z = 4$$

$$2x + 2y = 5$$

$$2x - y + 6z = 2$$

### Solution

$$\left[ \begin{array}{ccc|c} 2 & 1 & 2 & 4 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array} \right] \frac{1}{2}R_1$$
$$1 \quad \frac{1}{2} \quad 1 \quad 2$$

$$\left[ \begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array} \right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - 2R_1 \end{array}$$
$$\begin{array}{cccc} 2 & 2 & 0 & 5 \\ -2 & -1 & -2 & -4 \\ \hline 0 & 1 & -2 & 1 \end{array} \quad \begin{array}{cccc} 2 & -1 & 6 & 2 \\ -2 & -1 & -2 & -4 \\ \hline 0 & -2 & 4 & -2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 4 & -2 \end{array} \right] R_3 + 2R_2$$
$$\begin{array}{cccc} 0 & -2 & 4 & -2 \\ 0 & 2 & -4 & 2 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{ll} x + \frac{1}{2}y + z = 2 & (3) \\ y - 2z = 1 & (2) \\ 0 = 0 & (1) \end{array}$$

From (1):  $0 = 0$  is a true statement. Let  $z$  be the variable.

From (2):  $y = 1 + 2z$

From (3):  $x = -\frac{1}{2}y - z + 2$

$$x = -\frac{1}{2}(1 + 2z) - z + 2$$

$$x = -\frac{1}{2} - z - z + 2$$

$$x = -2z + \frac{3}{2}$$

**Solution:**  $\left( -2z + \frac{3}{2}, 2z + 1, z \right)$

### ***Example***

Use the Gaussian elimination method to solve the system

$$x + 2y - 5z = -1$$

$$2x + 3y - 2z = 2$$

$$3x + 5y - 7z = 4$$

### **Solution**

$$\left[ \begin{array}{ccc|c} 1 & 2 & -5 & -1 \\ 2 & 3 & -2 & 2 \\ 3 & 5 & -7 & 4 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \quad \begin{array}{ccc|c} 2 & 3 & -2 & 2 \\ -2 & -4 & 10 & 2 \\ \hline 0 & -1 & 8 & 4 \end{array} \quad \begin{array}{ccc|c} 3 & 5 & -7 & 4 \\ -3 & -6 & 15 & 3 \\ \hline 0 & -1 & 8 & 7 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & -5 & -1 \\ 0 & -1 & 8 & 4 \\ 0 & -1 & 8 & 7 \end{array} \right] \begin{array}{l} \\ -R_2 \\ \end{array} \quad \begin{array}{ccc|c} 0 & 1 & -8 & -4 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & -5 & -1 \\ 0 & 1 & -8 & -4 \\ 0 & -1 & 8 & 7 \end{array} \right] \begin{array}{l} \\ \\ R_3 + R_2 \end{array} \quad \begin{array}{ccc|c} 0 & -1 & 8 & 7 \\ 0 & 1 & -8 & -4 \\ \hline 0 & 0 & 0 & 3 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & -5 & -1 \\ 0 & 1 & -8 & -4 \\ 0 & 0 & 0 & 3 \end{array} \right]$$

From Row 3:  $0 = 3$  is a False statement.

***No Solution*** or ***Inconsistent***



## Exercises      Section 4.1 – System of linear Equations

(1 – 15) Use any method to solve the system equation (*elimination* or *substitution* method)

1. 
$$\begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$$

6. 
$$\begin{cases} 5x - 2y = 4 \\ -10x + 4y = 7 \end{cases}$$

11. 
$$\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$$

2. 
$$\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$$

7. 
$$\begin{cases} x - 4y = -8 \\ 5x - 20y = -40 \end{cases}$$

12. 
$$\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$$

3. 
$$\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$$

8. 
$$\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$$

13. 
$$\begin{cases} x - 2y = 5 \\ -10x + 2y = 4 \end{cases}$$

4. 
$$\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$$

9. 
$$\begin{cases} 2x + 10y = -14 \\ 7x - 2y = -16 \end{cases}$$

14. 
$$\begin{cases} 12x + 15y = -27 \\ 30x - 15y = -15 \end{cases}$$

5. 
$$\begin{cases} 3x + 4y = 2 \\ 2x + 5y = -1 \end{cases}$$

10. 
$$\begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$$

15. 
$$\begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$$

(16 – 27) Perform the matrix row operation (or operations) and write the new matrix.

16. 
$$\left[ \begin{array}{cc|c} 1 & 4 & 7 \\ 3 & 5 & 0 \end{array} \right] \quad R_2 - 3R_1$$

23. 
$$\left[ \begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array} \right] \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 - R_1 \end{array}$$

17. 
$$\left[ \begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -5 \end{array} \right] \quad R_2 - 2R_1$$

24. 
$$\left[ \begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 2 & 4 & 4 & 22 \\ -1 & -2 & 3 & 15 \end{array} \right] \quad \begin{array}{l} 3R_2 - 2R_1 \\ 3R_3 + R_1 \end{array}$$

18. 
$$\left[ \begin{array}{cc|c} 1 & -3 & 3 \\ 5 & 2 & 19 \end{array} \right] \quad R_2 - 5R_1$$

25. 
$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & -4 & 2 & -7 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

19. 
$$\left[ \begin{array}{cc|c} 2 & -3 & 8 \\ -6 & 9 & 4 \end{array} \right] \quad R_2 + 3R_1$$

20. 
$$\left[ \begin{array}{cc|c} 2 & 3 & 11 \\ 1 & 2 & 8 \end{array} \right] \quad 2R_2 - R_1$$

26. 
$$\left[ \begin{array}{cccc|c} 1 & -2 & 1 & 3 & -2 \\ 2 & -3 & 5 & -1 & 0 \\ 1 & 0 & 3 & 1 & -4 \\ -4 & 3 & 2 & -1 & 3 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 + 4R_1 \end{array}$$

21. 
$$\left[ \begin{array}{cc|c} 3 & 5 & -13 \\ 2 & 3 & -9 \end{array} \right] \quad 3R_2 - 2R_1$$

22. 
$$\left[ \begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 5 & 4 & 1 \end{array} \right] \quad R_3 - 5R_2$$

27. 
$$\left[ \begin{array}{cccc|c} 1 & -2 & 1 & 3 & -2 \\ -3 & 6 & -3 & -9 & 6 \\ 2 & 1 & 2 & 3 & 4 \\ 5 & 3 & 2 & -1 & -7 \end{array} \right] \quad \begin{array}{l} R_2 + 3R_1 \\ R_3 - 2R_1 \\ R_4 - 5R_1 \end{array}$$

(28 – 34) Use the Gauss-Jordan method to solve the system

$$28. \begin{cases} x - y + 5z = -6 \\ 3x + 3y - z = 10 \\ x + 3y + 2z = 5 \end{cases}$$

$$31. \begin{cases} x + 2y - 3z = -15 \\ 2x - 3y + 4z = 18 \\ -3x + y + z = 1 \end{cases}$$

$$33. \begin{cases} 2x + y + 2z = 4 \\ 2x + 2y = 5 \\ 2x - y + 6z = 2 \end{cases}$$

$$29. \begin{cases} 2x - y + 4z = -3 \\ x - 2y - 10z = -6 \\ 3x + 4z = 7 \end{cases}$$

$$32. \begin{cases} x + 2y + 3z = 10 \\ 4x + 5y + 6z = 11 \\ 7x + 8y + 9z = 12 \end{cases}$$

$$34. \begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$$

$$30. \begin{cases} 4x + 3y - 5z = -29 \\ 3x - 7y - z = -19 \\ 2x + 5y + 2z = -10 \end{cases}$$

(35 – 69) Use augmented elimination to solve linear system

$$35. \begin{cases} 2x - 5y + 3z = 1 \\ x - 2y - 2z = 8 \end{cases}$$

$$42. \begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$$

$$49. \begin{cases} 2x - 2y + z = -4 \\ 6x + 4y - 3z = -24 \\ x - 2y + 2z = 1 \end{cases}$$

$$36. \begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$$

$$43. \begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$$

$$50. \begin{cases} 9x + 3y + z = 4 \\ 16x + 4y + z = 2 \\ 25x + 5y + z = 2 \end{cases}$$

$$37. \begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$$

$$44. \begin{cases} 3x - 4y + 4z = 7 \\ x - y - 2z = 2 \\ 2x - 3y + 6z = 5 \end{cases}$$

$$51. \begin{cases} 2x - y + 2z = -8 \\ x + 2y - 3z = 9 \\ 3x - y - 4z = 3 \end{cases}$$

$$38. \begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$$

$$45. \begin{cases} x - 2y - z = 2 \\ 2x - y + z = 4 \\ -x + y + z = 4 \end{cases}$$

$$52. \begin{cases} x - 3z = -5 \\ 2x - y + 2z = 16 \\ 7x - 3y - 5z = 19 \end{cases}$$

$$39. \begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$$

$$46. \begin{cases} x + y + z = 3 \\ -y + 2z = 1 \\ -x + z = 0 \end{cases}$$

$$53. \begin{cases} x + 2y - z = 5 \\ 2x - y + 3z = 0 \\ 2y + z = 1 \end{cases}$$

$$40. \begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$$

$$47. \begin{cases} 3x + y + 3z = 14 \\ 7x + 5y + 8z = 37 \\ x + 3y + 2z = 9 \end{cases}$$

$$54. \begin{cases} x + y + z = 6 \\ 3x + 4y - 7z = 1 \\ 2x - y + 3z = 5 \end{cases}$$

$$41. \begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$$

$$48. \begin{cases} 4x - 2y + z = 7 \\ x + y + z = -2 \\ 4x + 2y + z = 3 \end{cases}$$

$$55. \begin{cases} 3x + 2y + 3z = 3 \\ 4x - 5y + 7z = 1 \\ 2x + 3y - 2z = 6 \end{cases}$$

$$56. \begin{cases} x - 3y + z = 2 \\ 4x - 12y + 4z = 8 \\ -2x + 6y - 2z = -4 \end{cases}$$

$$57. \begin{cases} 2x - 2y + z = -1 \\ x + 2y - z = 2 \\ 6x + 4y + 3z = 5 \end{cases}$$

$$58. \begin{cases} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 \\ x_2 - 3x_3 - x_4 = 0 \\ 3x_1 + 2x_3 - x_4 = 6 \\ -4x_1 + x_2 + 4x_3 + 2x_4 = -3 \end{cases}$$

$$59. \begin{cases} x_1 + x_2 + x_3 + x_4 = 5 \\ x_1 + 2x_2 - x_3 - 2x_4 = -1 \\ x_1 - 3x_2 - 3x_3 - x_4 = -1 \\ 2x_1 - x_2 + 2x_3 - x_4 = -2 \end{cases}$$

$$60. \begin{cases} 2x + 8y - z + w = 0 \\ 4x + 16y - 3z - w = -10 \\ -2x + 4y - z + 3w = -6 \\ -6x + 2y + 5z + w = 3 \end{cases}$$

$$61. \begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 + 2x_2 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

$$62. \begin{cases} 2x + 2y + 4z = 0 \\ -y - 3z + w = 0 \\ 3x + y + z + 2w = 0 \\ x + 3y - 2z - 2w = 0 \end{cases}$$

$$63. \begin{cases} 2x + z + w = 5 \\ y - w = -1 \\ 3x - z - w = 0 \\ 4x + y + 2z + w = 9 \end{cases}$$

$$64. \begin{cases} 4y + z = 20 \\ 2x - 2y + z = 0 \\ x + z = 5 \\ x + y - z = 10 \end{cases}$$

$$65. \begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x - 3w = -3 \end{cases}$$

$$66. \begin{cases} 2u - 3v + w - x + y = 0 \\ 4u - 6v + 2w - 3x - y = -5 \\ -2u + 3v - 2w + 2x - y = 3 \end{cases}$$

$$67. \begin{cases} 6x_3 + 2x_4 - 4x_5 - 8x_6 = 8 \\ 3x_3 + x_4 - 2x_5 - 4x_6 = 4 \\ 2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 + x_6 = 2 \\ 6x_1 - 9x_2 + 11x_4 - 19x_5 + 3x_6 = 1 \end{cases}$$

$$68. \begin{cases} 3x_1 + 2x_2 - x_3 = -15 \\ 5x_1 + 3x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 3x_3 = 11 \\ -6x_1 - 4x_2 + 2x_3 = 30 \end{cases}$$

$$69. \begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1 \\ 5x_3 + 10x_4 + 15x_6 = 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6 \end{cases}$$

70. At Snack Mix, caramel corn worth \$2.50 per *pound* is mixed with honey roasted missed nuts worth \$7.50 per *pound* in order to get 20 *lbs.* of a mixture worth \$4.50 per *pound*. How much of each snack is used?

## Section 4.2 – Matrix operations and Their Applications

### Matrix Notation

The Matrix:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix} \text{ is called the coefficient matrix of the system}$$

The matrix is said to be of order  $m \times n$

$m$ : numbers of rows,

$n$ : number of columns

A matrix  $A$  with  $m$  rows and  $n$  columns can be written in a general form

$$A = \begin{bmatrix} a_{ij} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

The matrix  $A$  above has 3 rows and 3 columns; therefore, the order of the matrix  $A$  is  $(3 \times 3)$

When  $m = n$ , then matrix is said to be **square**.

The numbers in a matrix are called **entries**.

### Example

$$\text{Let } A = \begin{bmatrix} 5 & -2 \\ -3 & \pi \\ 1 & 6 \end{bmatrix}$$

a. What is the order of  $A$ ?

3 rows and 2 columns  $\Rightarrow A$  is  $3 \times 2$

b.  $a_{12} = -2$                        $a_{31} = 1$

## Equality of Matrices

### Definition of Equality of Matrices

Two matrices  $A$  and  $B$  are equal if and only if they have the same order (size)  $m \times n$  and if each pair corresponding elements is equal

$$a_{ij} = b_{ij} \text{ for } i = 1, 2, \dots, m \text{ and } j = 1, 2, \dots, n$$

### Example

Find the values of the variables for which each statement is true, if possible.

$$a) \begin{bmatrix} 2 & 1 \\ p & q \end{bmatrix} = \begin{bmatrix} x & y \\ -1 & 0 \end{bmatrix}$$

$$x = 2, y = 1, p = -1, q = 0$$

$$b) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$$

*can't be true*

$$c) \begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} w = 9 & x = 17 \\ 8 = y & -12 = z \end{bmatrix}$$

## Matrix Addition and Subtraction

Given two  $m \times n$  matrices  $A = [a_{ij}]$  and  $B = [b_{ij}]$  their sum is  $A + B = [a_{ij} + b_{ij}]$

And their difference is  $A - B = [a_{ij} - b_{ij}]$

The matrices have to be the *same order*

**Example**

Find  $\begin{bmatrix} -4 & 3 \\ 7 & -6 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & -4 \end{bmatrix}$

**Solution**

$$\begin{bmatrix} -4 & 3 \\ 7 & -6 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} -4+6 & 3+(-3) \\ 7+2 & -6+(-4) \end{bmatrix} \\ = \begin{bmatrix} 2 & 0 \\ 9 & -10 \end{bmatrix}$$

**Example**

Find  $\begin{bmatrix} 5 & 4 \\ -3 & 7 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 8 \\ 6 & 0 \\ -5 & 3 \end{bmatrix}$

**Solution**

$$\begin{bmatrix} 5 & 4 \\ -3 & 7 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 8 \\ 6 & 0 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 5-(-4) & 4-8 \\ -3-6 & 7-0 \\ 0-(-5) & 1-3 \end{bmatrix} \\ = \begin{bmatrix} 9 & -4 \\ -9 & 7 \\ 5 & -2 \end{bmatrix}$$

**Example**

Find  $\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix}$

**Solution**

$$\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix} = \begin{bmatrix} 5-4 & -6+6 \\ 8+8 & 9-3 \end{bmatrix} \\ = \begin{bmatrix} 1 & 0 \\ 16 & 6 \end{bmatrix}$$

## ***Scalar Multiplication***

The scalar product of a number  $k$  and a matrix  $A$  is denoted by  $kA$ .

$$kA = k \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}$$

### ***Example***

Find  $5 \begin{bmatrix} 2 & -3 \\ 0 & 4 \end{bmatrix}$

### **Solution**

$$\begin{aligned} 5 \begin{bmatrix} 2 & -3 \\ 0 & 4 \end{bmatrix} &= \begin{bmatrix} 2(5) & -3(5) \\ 0(5) & 4(5) \end{bmatrix} \\ &= \begin{bmatrix} 10 & -15 \\ 0 & 20 \end{bmatrix} \end{aligned}$$

### ***Example***

Find  $\frac{3}{4} \begin{bmatrix} 20 & 36 \\ 12 & -16 \end{bmatrix}$

### **Solution**

$$\frac{3}{4} \begin{bmatrix} 20 & 36 \\ 12 & -16 \end{bmatrix} = \begin{bmatrix} 15 & 27 \\ 9 & -12 \end{bmatrix}$$

### ***Example***

Given:  $A = \begin{bmatrix} -4 & 1 \\ 3 & 0 \end{bmatrix}$        $B = \begin{bmatrix} -1 & -2 \\ 8 & 5 \end{bmatrix}$

Find:

a)  $-6B$

b)  $3A + 2B$

### **Solution**

$$\begin{aligned} \text{a) } -6B &= -6 \begin{bmatrix} -1 & -2 \\ 8 & 5 \end{bmatrix} \\ &= \begin{bmatrix} -1(-6) & -2(-6) \\ 8(-6) & 5(-6) \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 6 & 12 \\ -48 & -30 \end{bmatrix}$$

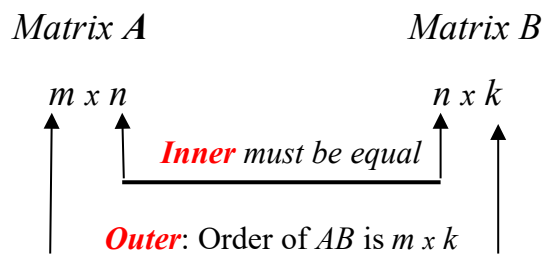
$$\begin{aligned} \mathbf{b)} \quad 3A + 2B &= 3 \begin{bmatrix} -4 & 1 \\ 3 & 0 \end{bmatrix} + 2 \begin{bmatrix} -1 & -2 \\ 8 & 5 \end{bmatrix} \\ &= \begin{bmatrix} -4(3) & 1(3) \\ 3(3) & 0(3) \end{bmatrix} + \begin{bmatrix} -1(2) & -2(2) \\ 8(2) & 5(2) \end{bmatrix} \\ &= \begin{bmatrix} -12 & 3 \\ 9 & 0 \end{bmatrix} + \begin{bmatrix} -2 & -4 \\ 16 & 10 \end{bmatrix} \\ &= \begin{bmatrix} -12-2 & 3-4 \\ 9+16 & 0+10 \end{bmatrix} \\ &= \begin{bmatrix} -14 & -1 \\ 25 & 10 \end{bmatrix} \end{aligned}$$



## Matrix Multiplication

### Product of Two Matrices

Let  $A$  be an  $m \times n$  matrix and let  $B$  be an  $n \times k$  matrix. To find the element in the  $i^{th}$  row and  $j^{th}$  column of the product matrix  $AB$ , multiply each element in the  $i^{th}$  row of  $A$  by the corresponding element in the  $j^{th}$  column of  $B$ , and then add these products. The product matrix  $AB$  is an  $m \times k$  matrix.



$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$2 \times 2$        $2 \times 2$        $\rightarrow$        $2 \times 2$

$$a_{11} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & - \\ - & - \end{bmatrix}$$

$$a_{12} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} - & af + bh \\ - & - \end{bmatrix}$$

$$a_{21} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} - & - \\ ce + dg & - \end{bmatrix}$$

$$a_{22} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} - & - \\ - & cf + dh \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

### ***Example***

Given:  $A = \begin{bmatrix} 1 & -3 \\ 7 & 2 \end{bmatrix}$   $B = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 4 & -1 \end{bmatrix}$

Find  $AB$  and  $BA$ .

### **Solution**

$$\begin{aligned} AB &= \begin{bmatrix} 1 & -3 \\ 7 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 4 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1(1) + (-3)3 & 1(0) + (-3)1 & 1(-1) + (-3)4 & 1(2) + (-3)(-1) \\ 7(1) + 2(3) & 7(0) + 2(1) & 7(-1) + 2(4) & 7(2) + 2(-1) \end{bmatrix} \\ &= \begin{bmatrix} -8 & -3 & -13 & 5 \\ 13 & 2 & 1 & 12 \end{bmatrix} \end{aligned}$$

$BA$  can be found since:  $B$ :  $2 \times 4$  and  $A$ :  $2 \times 2$

**Note:**  $AB \neq BA$

### ***Example***

Given:  $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$   $B = \begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix}$

Find  $AB$ .

### **Solution**

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1(4) + 3(1) & 1(6) + 3(0) \\ 2(4) + 5(1) & 2(6) + 5(0) \end{bmatrix} \\ &= \begin{bmatrix} 7 & 6 \\ 13 & 12 \end{bmatrix} \end{aligned}$$

### Example

Given:  $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix}$   $B = \begin{bmatrix} 1 & 6 \\ 3 & -5 \\ -2 & 4 \end{bmatrix}$  Find  $AB$ .

### Solution

$$\begin{aligned} AB &= \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 3 & -5 \\ -2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 3(1) + 1(3) - 1(-2) & 3(6) + 1(-5) - 1(4) \\ 2(1) + 0(3) + 3(-2) & 2(6) + 0(-5) + 3(4) \end{bmatrix} \\ &= \begin{bmatrix} 8 & 9 \\ -4 & 24 \end{bmatrix} \end{aligned}$$

### Example

Suppose  $A$  is a  $3 \times 2$  matrix, while  $B$  is a  $2 \times 4$  matrix.

- a) Can the product  $AB$  be calculated?
- b) If  $AB$  can be calculated, what size is it?
- c) Can  $BA$  be calculated?
- d) If  $BA$  can be calculated, what size is it?

### Solution

a)



b) The product  $AB$  size is  $3 \times 4$

c)

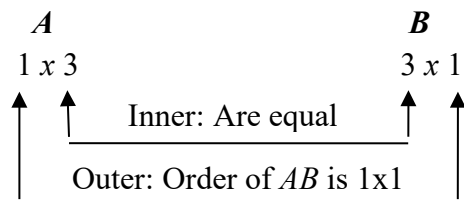


d) Can't be calculated

### Example

Given:  $A_{1 \times 3} = \begin{bmatrix} 2 & 0 & 4 \end{bmatrix}$      $B_{3 \times 1} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$     Find  $AB$  and  $BA$ .

### Solution



$$AB = [2(1) + 0(3) + 4(7)]$$
$$= [30]$$

$BA : 3 \times 1 \text{ --- } 1 \times 3$

$$BA = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} \begin{bmatrix} 2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1(2) & 1(0) & 1(4) \\ 3(2) & 3(0) & 3(4) \\ 7(2) & 7(0) & 7(4) \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 0 & 4 \\ 6 & 0 & 12 \\ 14 & 0 & 28 \end{bmatrix}$$

### Example

Given:  $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$      $B = \begin{bmatrix} 2 & 3 & -1 & 6 \\ 0 & 5 & 4 & 1 \end{bmatrix}$     Find  $AB$  and  $BA$ .

### Solution

a)  $AB = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & -1 & 6 \\ 0 & 5 & 4 & 1 \end{bmatrix}$      $2 \times 2 \text{ --- } 2 \times 4$

$$= \begin{bmatrix} 1(2) + 3(0) & 1(3) + 3(5) & 1(-1) + 3(4) & 1(6) + 3(1) \\ 0(2) + 2(0) & 0(3) + 2(5) & 0(-1) + 2(4) & 0(6) + 2(1) \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 18 & 11 & 9 \\ 0 & 10 & 8 & 2 \end{bmatrix}$$

b)  $BA = \begin{bmatrix} 2 & 3 & -1 & 6 \\ 0 & 5 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} + \text{Undefined}$      $2 \times 4 \text{ --- } 2 \times 2$  (Inner order are not equal 2, 4)

## ***Properties of Matrix***

### **Addition and Scalar Multiplication**

$$A + B = B + A \quad \text{Commutative Property of Addition}$$

$$A + (B + C) = (A + B) + C \quad \text{Associative Property of Addition}$$

$$(kl)A = k(lA) \quad \text{Associative Property of Scalar Multiplication}$$

$$k(A + B) = kA + kB \quad \text{Distributive Property}$$

$$(k + l)A = kA + lA \quad \text{Distributive Property}$$

$$A + 0 = 0 + A = A \quad \text{Additive Identity Property}$$

$$A + (-A) = (-A) + A = 0 \quad \text{Additive Inverse Property}$$

### ***Multiplication***

$$A(BC) = (AB)C \quad \text{Associative Property of Multiplication}$$

$$A(B + C) = AB + AC \quad \text{Distributive Property}$$

$$(B + C)A = BA + CA \quad \text{Distributive Property}$$

## Exercises      Section 4.2 – Matrix operations and Their Applications

(1 – 7) Find values for the variables so that the matrices are equal.

1.  $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

2.  $\begin{bmatrix} x & y+3 \\ 2z & 8 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 6 & 8 \end{bmatrix}$

3.  $\begin{bmatrix} 5 & x-4 & 9 \\ 2 & -3 & 8 \\ 6 & 0 & 5 \end{bmatrix} = \begin{bmatrix} y+3 & 2 & 9 \\ z+4 & -3 & 8 \\ 6 & 0 & w \end{bmatrix}$

4.  $\begin{bmatrix} a+2 & 3b & 4c \\ d & 7f & 8 \end{bmatrix} + \begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$

5.  $\begin{bmatrix} a+11 & 12z+1 & 5m \\ 11k & 3 & 1 \end{bmatrix} + \begin{bmatrix} 9a & 9z & 4m \\ 12k & 5 & 3 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$

6.  $\begin{bmatrix} x+2 & 3y+1 & 5z \\ 8w & 2 & 3 \end{bmatrix} + \begin{bmatrix} 3x & 2y & 5z \\ 2w & 5 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 7 & -2 \end{bmatrix}$

7.  $\begin{bmatrix} 2x-3 & y-2 & 2z+1 \\ 5 & 2w & 7 \end{bmatrix} + \begin{bmatrix} 3x-3 & y+2 & z-1 \\ -5 & 5w+1 & 3 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 9 \\ 0 & 8 & 10 \end{bmatrix}$

8. Given  $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \end{bmatrix}$   $B = \begin{bmatrix} 2 & -3 & 6 \\ -3 & 1 & -4 \end{bmatrix}$  Find:  $A - B$ ,  $3A + 2B$

9. Given  $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$   $F = \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix}$  Find:  $3F + 2A$

(10 – 22) Evaluate

10.  $\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + \begin{bmatrix} -6 \\ 3 \\ 12 \end{bmatrix}$

11.  $\begin{bmatrix} 5 & 8 \\ 6 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 9 & 1 \\ 4 & 2 & 5 \end{bmatrix}$

12.  $\begin{bmatrix} -5 & 0 \\ 4 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & 3 \end{bmatrix}$

13.  $\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix}$

14.  $\begin{bmatrix} -5 & 6 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -3 & 2 \\ 5 & -8 \end{bmatrix}$

15.  $\begin{bmatrix} 8 & 6 & -4 \end{bmatrix} - \begin{bmatrix} 3 & 5 & -8 \end{bmatrix}$

16.  $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix}$

17.  $\begin{bmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{bmatrix}$

$$18. \begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$$

$$20. \begin{bmatrix} -2 & -3 & -4 \\ 2 & -1 & 0 \\ 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix}$$

$$19. \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix}$$

$$21. \begin{bmatrix} \sqrt{2} & \sqrt{2} & -\sqrt{18} \\ \sqrt{3} & \sqrt{27} & 0 \end{bmatrix} \begin{bmatrix} 8 & -10 \\ 9 & 12 \\ 0 & 2 \end{bmatrix}$$

$$22. \begin{bmatrix} x & 2x+1 & 4 \\ 5 & x-1 & 8 \\ -2 & 3x & 2x+1 \end{bmatrix} + \begin{bmatrix} 2x-1 & -2x-1 & 4x \\ -5 & 6 & x+1 \\ -5 & 2 & -2x \end{bmatrix}$$

(23 – 33) Find  $AB$  and  $BA$ , if possible

$$23. A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}$$

$$29. A = \begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$

$$24. A = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix} \quad B = \begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix}$$

$$30. A = \begin{pmatrix} 2 & 4 \\ 0 & -1 \\ -3 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 0 & -2 \\ -2 & 6 & 2 \end{pmatrix}$$

$$25. A = \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix}$$

$$26. A = \begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix}$$

$$31. A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix}$$

$$27. A = \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}$$

$$32. A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

$$28. A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

$$33. A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$34. \text{ Given } A = \begin{bmatrix} -3 & 4 \\ 2 & -3 \\ -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 1 \\ 1 & -2 \\ 3 & -4 \end{bmatrix}, \text{ Find}$$

$$a) A + B$$

$$c) 3A$$

$$e) 2A + 3B$$

$$g) AB$$

$$b) A - B$$

$$d) -2B$$

$$f) A^2$$

$$h) BA$$

35. Given  $A = \begin{bmatrix} 2 & -2 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}$   $B = \begin{bmatrix} -1 & 8 \\ 2 & -2 \\ -4 & 3 \end{bmatrix}$ , Find

a)  $A + B$

c)  $3A$

e)  $2A + 3B$

g)  $AB$

b)  $A - B$

d)  $-2B$

f)  $A^2$

h)  $BA$

36. Given  $A = \begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}$   $B = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$ , Find

a)  $A + B$

c)  $3A$

e)  $2A + 3B$

g)  $AB$

b)  $A - B$

d)  $-2B$

f)  $A^2$

h)  $BA$

37. Given  $A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}$   $B = \begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix}$ , Find

a)  $A + B$

c)  $3A$

e)  $2A + 3B$

g)  $AB$

b)  $A - B$

d)  $-2B$

f)  $A^2$

h)  $BA$

38. Given  $A = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$   $B = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$   $C = \begin{pmatrix} 4 & 3 & 2 \\ -1 & 2 & 1 \end{pmatrix}$   $D = \begin{pmatrix} -2 & 3 \\ 2 & -1 \\ 3 & 2 \end{pmatrix}$ , Find

a)  $4A - 2B$

d)  $2A - 3B$

g)  $A^2$

j)  $CA$

b)  $3A + C$

e)  $AB$

h)  $B^3$

k)  $CD$

c)  $3A + B$

f)  $BA$

i)  $AC$

l)  $DC$

39. Given  $A = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$   $B = \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$   $C = \begin{pmatrix} 1 & 4 & 5 \\ -2 & 3 & 4 \\ -1 & 0 & -2 \end{pmatrix}$   $D = \begin{pmatrix} 2 & 4 & -2 \\ 0 & 3 & 5 \\ -3 & 1 & 1 \end{pmatrix}$ , Find

a)  $4A - 2B$

d)  $2A - 3B$

g)  $A^2$

j)  $CB$

b)  $3A + C$

e)  $AB$

h)  $B^3$

k)  $CD$

c)  $3A + B$

f)  $BA$

i)  $AC$

l)  $DC$

40. A contractor builds three kinds of houses, models  $A$ ,  $B$ , and  $C$ , with a choice of two styles, Spanish and contemporary. Matrix  $P$  shows the number of each kind of house planned for a new 100-home subdivision. The amounts for each of the exterior materials depend primarily on the style of the house. These amounts are shown in matrix  $Q$ . (concrete is in *cubic yards*, lumber in units of 1000 board *feet*, brick in 1000s, and shingles in units of 100  $ft^2$ .) Matrix  $R$  gives the cost in dollars for each kind of material.



- a) What is the total cost of these materials for each model?
- b) How much of each of four kinds of material must be ordered
- c) What is the total cost for exterior materials?

41. Mitchell Fabricators manufactures three styles of bicycle frames in its two plants. The following table shows the number of each style produced at each plant

|                    | <i>Mountain Bike</i> | <i>Racing Bike</i> | <i>Touring Bike</i> |
|--------------------|----------------------|--------------------|---------------------|
| <i>North Plant</i> | 150                  | 120                | 100                 |
| <i>South Plant</i> | 180                  | 90                 | 130                 |

- a) Write a  $2 \times 3$  matrix  $A$  that represents the information in the table
  - b) The manufacturer increased production of each style by 20%. Find a Matrix  $M$  that represents the increased production figures.
  - c) Find the matrix  $A + M$  and tell what it represents
42. Sal's Shoes and Fred's Footwear both have outlets in California and Arizona. Sal's sells shoes for \$80, sandals for \$40, and boots for \$120. Fred's prices are \$60, \$30, and \$150 for shoes, sandals and boots, respectively. Half of all sales in California stores are shoes,  $1/4$  are *sandals*, and  $1/4$  are *boots*. In Arizona, the fractions are  $1/5$  *shoes*,  $1/5$  are *sandals*, and  $3/5$  are *boots*.
- a) Write a  $2 \times 3$  matrix called  $P$  representing prices for the two stores and three types of footwear.
  - b) Write a  $2 \times 3$  matrix called  $F$  representing fraction of each type of footwear sold in each state.
  - c) Only one of the two products  $PF$  and  $FP$  is meaningful. Determine which one it is, calculate the product, and describe what the entries represent.

## Section 4.3 – Multiplicative Inverses of Matrices

### Identity Matrix

The  $n \times n$  identity matrix with 1's on the main diagonal and 0's elsewhere and is denoted by

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2 \times 2} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3} \quad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

### The Multiplicative Identity Matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Then  $AI = IA = A$

### Example

$$A = \begin{bmatrix} 4 & -7 \\ -3 & 2 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

### Solution

$$\begin{aligned} AI &= \begin{bmatrix} 4 & -7 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4(1) - 7(0) & 4(0) - 7(1) \\ -3(1) + 2(0) & -3(0) + 2(1) \end{bmatrix} \\ &= \begin{bmatrix} 4 & -7 \\ -3 & 2 \end{bmatrix} = A \\ &= A \end{aligned}$$

## Multiplicative inverse of a matrix

Multiplicative inverse of a matrix  $A_{n \times n}$  and  $A^{-1}_{n \times n}$  if exists, then:

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

### *Example*

Show that  $B$  is Multiplicative inverse of  $A$ .

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

### *Solution*

$$\begin{aligned} A \cdot B &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2(1) - 1(1) & 2(-1) + 1(2) \\ 1(1) + 1(-1) & 1(-1) + 1(2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I \end{aligned}$$

$$\begin{aligned} BA &= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I \end{aligned}$$

$\therefore B$  is multiplicative inverse of a matrix  $A$ :  $B = A^{-1}$

## ***Finding Inverse matrix***

To find inverse matrix using Gauss-Jordan method:

$$\left[ A | I \right] \rightarrow \left[ I | A^{-1} \right] \quad \text{where } A^{-1} \text{ read as "A inverse"}$$

**For 2 by 2 matrices (*only*)**

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{aligned} A^{-1} &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \begin{bmatrix} \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\ \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix} \end{aligned}$$

If  $ad - bc = 0$ , then  $A^{-1}$  doesn't exist

## ***Example***

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix} \Rightarrow A^{-1} = ?$$

## **Solution**

$$\begin{aligned} A^{-1} &= \frac{1}{(-1)(4) - (-2)(3)} \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix} \end{aligned}$$

***Example***

$$A = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \Rightarrow A^{-1} = ?$$

***Solution***

$$\begin{aligned} A^{-1} &= \frac{1}{(3)(1) - (-2)(-1)} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \end{aligned}$$

To find inverse matrix using Gauss-Jordan method:

$$\left[ A | I \right] \rightarrow \left[ I | A^{-1} \right] \quad \text{where } A^{-1} \text{ read as "A inverse"}$$

### Example

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix} \quad \text{Find } A^{-1}$$

### Solution

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{ccc|ccc} -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \end{array} \quad \begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & -1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array} \right] 2R_3 + R_2$$

$$\begin{array}{ccc|ccc} 0 & -2 & -4 & -2 & 0 & 2 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array} \right] \begin{array}{l} R_1 - 2R_3 \\ R_2 - 5R_3 \\ \end{array}$$

$$\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 2 & -2 & -4 \\ 0 & 0 & 0 & 3 & -2 & -4 \end{array} \quad \begin{array}{ccc|ccc} 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & 0 & -5 & 5 & -5 & -10 \\ 0 & 2 & 0 & 6 & -4 & -10 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 2 & 0 & 6 & -4 & -10 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array} \right] \frac{1}{2}R_2$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 1 & 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array} \right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

### Example

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix} \quad \text{Find } A^{-1}$$

### Solution

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{cccccc|cccccc} -1 & 2 & 3 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 1 & 0 & 0 & -1 & 0 & -2 & -1 & 0 & 0 \\ \hline 0 & 2 & 5 & 1 & 1 & 0 & 0 & -1 & -2 & -1 & 0 & 1 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array} \right] \frac{1}{2}R_2$$

$$0 \quad 1 \quad \frac{5}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad 0$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array} \right] R_3 + R_2$$

$$\begin{array}{cccccc|cccccc} 0 & -1 & -2 & -1 & 0 & 1 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ \hline 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array} \right] 2R_3$$

$$0 \quad 0 \quad 1 \quad -1 \quad 1 \quad 2$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array} \right] \begin{array}{l} R_1 - 2R_3 \\ R_2 - \frac{5}{2}R_3 \end{array}$$

$$\begin{array}{cccccc|cccccc} 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & -\frac{5}{2} & \frac{5}{2} & -\frac{5}{2} & -5 & 0 & 0 & -2 & 2 & -2 & -4 \\ \hline 0 & 1 & 0 & 3 & -2 & -5 & 1 & 0 & 0 & 3 & -2 & -4 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 1 & 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array} \right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

## Solving a System Using $A^{-1}$

To solve the matrix equation  $AX = B$ .

- $X$ : matrix of the variables
- $A$ : Coefficient matrix
- $B$ : Constant matrix

$$AX = B$$

$$A^{-1}(AX) = A^{-1}B \quad \text{Multiply both side by } A^{-1}$$

$$(A^{-1}A)X = A^{-1}B \quad \text{Associate property}$$

$$IX = A^{-1}B \quad \text{Multiplicative inverse property}$$

$$X = A^{-1}B \quad \text{Identity property}$$

---

### Example

Solve the system using  $A^{-1}$

$$\begin{array}{rcrcrcrcrcrl} x & & & + & 2z & = & 6 \\ -x & + & 2y & + & 3z & = & -5 \\ x & - & y & & & = & 6 \end{array}$$

$$\text{Given } A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

### Solution

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix}$$

$A \quad X = B$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix} = \begin{bmatrix} 3(6)-2(-5)-4(6) \\ 3(6)-2(-5)-5(6) \\ -1(6)+1(-5)+2(6) \end{bmatrix} = \begin{bmatrix} 18+10-24 \\ 18+10-30 \\ -6-5+12 \end{bmatrix}$$
$$= \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

**Solution:**  $\{(4, -2, 1)\}$



### ***Example***

Use the inverse of the coefficient matrix to solve the linear system

$$2x - 3y = 4$$

$$x + 5y = 2$$

### **Solution**

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

The solution of the system is  $(2, 0)$

## Exercise

## Section 4.3 – Multiplicative Inverses of Matrices

Show that  $B$  is Multiplicative inverse of  $A$

1.  $A = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$

2.  $A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \quad \& \quad B = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$

Find the inverse, if exists, of

3.  $A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$

14.  $A = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$

25.  $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

4.  $A = \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$

15.  $A = \begin{pmatrix} -6 & 9 \\ 2 & -3 \end{pmatrix}$

26.  $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{bmatrix}$

5.  $A = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$

16.  $A = \begin{pmatrix} -2 & 7 \\ 0 & 2 \end{pmatrix}$

27.  $A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

6.  $A = \begin{bmatrix} a & b \\ 3 & 3 \end{bmatrix}$

17.  $A = \begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$

28.  $A = \begin{bmatrix} -2 & 5 & 3 \\ 4 & -1 & 3 \\ 7 & -2 & 5 \end{bmatrix}$

7.  $A = \begin{bmatrix} -2 & a \\ 4 & a \end{bmatrix}$

18.  $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

29.  $A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{pmatrix}$

8.  $A = \begin{bmatrix} 4 & 4 \\ b & a \end{bmatrix}$

19.  $A = \begin{pmatrix} 2 & 1 \\ a & a \end{pmatrix}$

30.  $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{pmatrix}$

9.  $A = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$

20.  $A = \begin{pmatrix} b & 3 \\ b & 2 \end{pmatrix}$

31.  $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

10.  $A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$

21.  $A = \begin{pmatrix} 1 & a \\ 3 & a \end{pmatrix}$

32.  $A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & -1 \\ 3 & 1 & 2 \end{pmatrix}$

11.  $A = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$

22.  $A = \begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix}$

12.  $A = \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

23.  $A = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$

13.  $A = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$

24.  $A = \begin{pmatrix} -3 & \frac{1}{2} \\ 6 & -1 \end{pmatrix}$

$$33. \quad A = \begin{pmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$$

$$34. \quad A = \begin{pmatrix} -3 & 1 & -1 \\ 1 & -4 & -7 \\ 1 & 2 & 5 \end{pmatrix}$$

$$35. \quad A = \begin{pmatrix} 1 & 1 & -3 \\ 2 & -4 & 1 \\ -5 & 7 & 1 \end{pmatrix}$$

$$36. \quad A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{pmatrix}$$

$$37. \quad A = \begin{bmatrix} -2 & -3 & 4 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & -6 & 1 \\ -2 & -2 & 5 & 1 \end{bmatrix}$$

$$38. \quad A = \begin{bmatrix} 1 & -14 & 7 & 38 \\ -1 & 2 & 1 & -2 \\ 1 & 2 & -1 & -6 \\ 1 & -2 & 3 & 6 \end{bmatrix}$$

$$39. \quad A = \begin{bmatrix} 10 & 20 & -30 & 15 \\ 3 & -7 & 14 & -8 \\ -7 & -2 & -1 & 2 \\ 4 & 4 & -3 & 1 \end{bmatrix}$$

State the conditions under which  $A^{-1}$  exists. Then find a formula for  $A^{-1}$

$$40. \quad A = [x]$$

$$41. \quad A = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$$

$$42. \quad A = \begin{bmatrix} 0 & 0 & x \\ 0 & y & 0 \\ z & 0 & 0 \end{bmatrix}$$

$$43. \quad A = \begin{bmatrix} x & 1 & 1 & 1 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{bmatrix}$$

$$44. \quad \text{Solve the system using } A^{-1} \quad \begin{cases} x & + 2z = 6 \\ -x + 2y + 3z = -5 \\ x - y & = 6 \end{cases} \quad \text{Given } A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

45. Solve the system using  $A^{-1}$

$$\begin{cases} x + 2y + 5z = 2 \\ 2x + 3y + 8z = 3 \\ -x + y + 2z = 3 \end{cases}$$

a) Write the linear system as a matrix equation in the form  $AX = B$

b) Solve the system using the inverse that is given for the coefficient matrix

$$\text{the inverse of } \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{bmatrix} \text{ is } \begin{bmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{bmatrix}$$

46. Solve the system using  $A^{-1}$

$$\begin{cases} x - y + z = 8 \\ 2y - z = -7 \\ 2x + 3y = 1 \end{cases}$$

a) Write the linear system as a matrix equation in the form  $AX = B$

b) Solve the system using the inverse that is given for the coefficient matrix

$$\text{the inverse is } \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

(47–75) Use the *inverse* of the coefficient matrix to solve the linear system

47.  $\begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$

58.  $\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$

67.  $\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$

48.  $\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$

59.  $\begin{cases} x - 2y = 5 \\ -10x + 2y = 4 \end{cases}$

68.  $\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$

49.  $\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$

60.  $\begin{cases} 12x + 15y = -27 \\ 30x - 15y = -15 \end{cases}$

50.  $\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$

61.  $\begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$

69.  $\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$

51.  $\begin{cases} 3x + 4y = 2 \\ 2x + 5y = -1 \end{cases}$

62.  $\begin{cases} -2x + 3y = 4 \\ -3x + 4y = 5 \end{cases}$

70.  $\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$

52.  $\begin{cases} 5x - 2y = 4 \\ -10x + 4y = 7 \end{cases}$

63.  $\begin{cases} x - 2y = 6 \\ 4x + 3y = 2 \end{cases}$

53.  $\begin{cases} x - 4y = -8 \\ 5x - 20y = -40 \end{cases}$

64.  $\begin{cases} 2x - 3y = 7 \\ 4x + y = -7 \end{cases}$

71.  $\begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$

54.  $\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$

65.  $\begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$

72.  $\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$

55.  $\begin{cases} 2x + 10y = -14 \\ 7x - 2y = -16 \end{cases}$

66.  $\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$

73.  $\begin{cases} x - 2y - z = 2 \\ 2x - y + z = 4 \\ -x + y + z = 4 \end{cases}$

56.  $\begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$

57.  $\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$

## Section 4.4 – Determinants

### Determinant of a 2 x 2 Matrix

Determinant of the matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is denoted  $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$  and is define as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

#### *Example*

Let  $A = \begin{bmatrix} -3 & 4 \\ 6 & 8 \end{bmatrix}$ . Find  $|A|$

#### *Solution*

$$\begin{aligned} |A| &= \begin{vmatrix} -3 & 4 \\ 6 & 8 \end{vmatrix} \\ &= -3(8) - 4(6) \\ &= \underline{-48} \end{aligned}$$

#### *Example*

Evaluate:  $\begin{vmatrix} 2 & -3 \\ -4 & 1 \end{vmatrix}$

#### *Solution*

$$\begin{aligned} \begin{vmatrix} 2 & -3 \\ -4 & 1 \end{vmatrix} &= 2(1) - (-3)(-4) \\ &= 2 - 12 \\ &= \underline{-10} \end{aligned}$$

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

## Minor

For a square matrix  $A = [a_{ij}]$ , the minor  $M_{ij}$  of an element  $a_{ij}$  is the determinant of the matrix formed by deleting the  $i^{\text{th}}$  row and the  $j^{\text{th}}$  column of  $A$ .

$$\text{Cofactor: } A_{ij} = (-1)^{i+j} M_{ij}$$

$$\begin{aligned} |A| &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{aligned}$$

## Example

$$A = \begin{pmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{pmatrix} \text{ Find the determinant of } A.$$

## Solution

$$\begin{aligned} |A| &= \begin{vmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{vmatrix} \\ &= -8 \begin{vmatrix} -6 & 7 \\ -3 & 5 \end{vmatrix} - 0 \begin{vmatrix} 4 & 7 \\ -1 & 5 \end{vmatrix} + 6 \begin{vmatrix} 4 & -6 \\ -1 & -3 \end{vmatrix} \\ &= -8(-30 - (-21)) - 0 + 6(-12 - 6) \\ &= -8(-9) + 6(-18) \\ &= \underline{-36} \end{aligned}$$

## Determinant Using Diagonal Method

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \quad (1)$$

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

$$a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} \quad (2)$$

$$\text{Determinant: } D = (1) - (2)$$

### Example

Evaluate  $\begin{vmatrix} 2 & -3 & -2 \\ -1 & -4 & -3 \\ -1 & 0 & 2 \end{vmatrix}$

### Solution

$$\begin{vmatrix} 2 & -3 & -2 \\ -1 & -4 & -3 \\ -1 & 0 & 2 \end{vmatrix} \begin{array}{cc} 2 & -3 \\ -1 & -4 \\ -1 & 0 \end{array} = 2(-4)(2) + (-3)(-3)(-1) + (-2)(-1)(0) - (-2)(-4)(-1) - (2)(-3)(0) - (-3)(-1)(2) \\ = -23$$

### Example

Evaluate  $\begin{vmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{vmatrix}$

### Solution

$$\begin{vmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{vmatrix} \begin{array}{cc} -8 & 0 \\ 4 & -6 \\ -1 & -3 \end{array} = (-8)(-6)(5) + 0(7)(-1) + 6(4)(-3) - 6(-6)(-1) - (-8)(7)(-3) - 0(4)(5) \\ = -36$$

***Example***

Evaluate  $\begin{vmatrix} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{vmatrix}$

***Solution***

$$\begin{vmatrix} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{vmatrix} \begin{matrix} x & 0 \\ 2 & x \\ -3 & x \end{matrix} = x^2 + 0 - 2x - (3x) - x^4 - 0$$
$$= -x^4 + x^2 - 5x$$



## Exercises      Section 4.4 – Determinants

(1 – 34)    Evaluate

1.  $\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix}$

2.  $\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix}$

3.  $\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix}$

4.  $\begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix}$

5.  $\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix}$

6.  $\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix}$

7.  $\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix}$

8.  $\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix}$

9.  $\begin{vmatrix} 5 & 3 \\ -2 & 3 \end{vmatrix}$

10.  $\begin{vmatrix} -4 & -1 \\ 5 & 6 \end{vmatrix}$

11.  $\begin{vmatrix} \sqrt{3} & -2 \\ -3 & \sqrt{3} \end{vmatrix}$

12.  $\begin{vmatrix} \sqrt{7} & 6 \\ -3 & \sqrt{7} \end{vmatrix}$

13.  $\begin{vmatrix} \sqrt{5} & 3 \\ -2 & 2 \end{vmatrix}$

14.  $\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{vmatrix}$

15.  $\begin{vmatrix} \frac{1}{5} & \frac{1}{6} \\ -6 & -5 \end{vmatrix}$

16.  $\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{2} & \frac{3}{4} \end{vmatrix}$

17.  $\begin{vmatrix} x & x^2 \\ 4 & x \end{vmatrix}$

18.  $\begin{vmatrix} x & x^2 \\ x & 9 \end{vmatrix}$

19.  $\begin{vmatrix} x^2 & x \\ -3 & 2 \end{vmatrix}$

20.  $\begin{vmatrix} x+2 & 6 \\ x-2 & 4 \end{vmatrix}$

21.  $\begin{vmatrix} x+1 & -6 \\ x+3 & -3 \end{vmatrix}$

22.  $\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & -5 \\ 2 & 5 & -1 \end{vmatrix}$

23.  $\begin{vmatrix} 4 & 0 & 0 \\ 3 & -1 & 4 \\ 2 & -3 & 6 \end{vmatrix}$

24.  $\begin{vmatrix} 3 & 1 & 0 \\ -3 & -4 & 0 \\ -1 & 3 & 5 \end{vmatrix}$

25.  $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & -4 & 5 \end{vmatrix}$

26.  $\begin{vmatrix} x & 0 & -1 \\ 2 & 1 & x^2 \\ -3 & x & 1 \end{vmatrix}$

27.  $\begin{vmatrix} x & 1 & -1 \\ x^2 & x & x \\ 0 & x & 1 \end{vmatrix}$

28.  $\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix}$

29.  $\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix}$

30.  $\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix}$

31.  $\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix}$

32.  $\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix}$

33.  $\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix}$

34.  $\begin{vmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix}$

**(35 – 46)** Solve for  $x$

$$35. \begin{vmatrix} x & 3 \\ 2 & 1 \end{vmatrix} = 12$$

$$36. \begin{vmatrix} x & 1 \\ 2 & x \end{vmatrix} = -1$$

$$37. \begin{vmatrix} 3 & x \\ x & 4 \end{vmatrix} = -13$$

$$38. \begin{vmatrix} x & 2 \\ 3 & x \end{vmatrix} = x$$

$$39. \begin{vmatrix} 4 & 6 \\ -2 & x \end{vmatrix} = 32$$

$$40. \begin{vmatrix} x+2 & -3 \\ x+5 & -4 \end{vmatrix} = 3x-5$$

$$41. \begin{vmatrix} x+3 & -6 \\ x-2 & -4 \end{vmatrix} = 28$$

$$42. \begin{vmatrix} x & -3 \\ -1 & x \end{vmatrix} \geq 0$$

$$43. \begin{vmatrix} 2 & x & 1 \\ 1 & 2 & -1 \\ 3 & 4 & -2 \end{vmatrix} = -6$$

$$44. \begin{vmatrix} 1 & x & -3 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix} = 8$$

$$45. \begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix} = 39$$

$$46. \begin{vmatrix} x & 0 & 0 \\ 7 & x & 1 \\ 7 & 2 & 1 \end{vmatrix} = -1$$