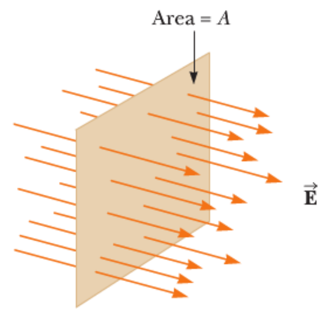
2.2 – *Gauss's Law*

Area: is a vector of an entity.



The direction of area is perpendicular to the plane of the area. To distinguish between the two possible directions (***perpendicularly in*** *and* ***perpendicularly out***) the right hand rule is used. When fingers of the right hand are wrapped along its perimeter in a counterclockwise direction, then the direction of the thumb represents the direction of the area. The direction of the area is usually the direction pointing towards you. For a closed surface, the direction of area is directed outward from the closed surface. The direction perpendicularly in is represented by a cross $(\times)$ and the direction perpendicularly out is represented by a dot $(\bullet)$.

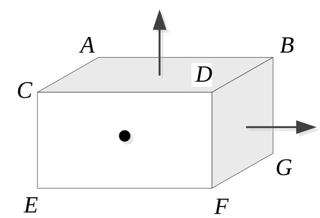
Example

Consider the cube shown, determine the direction of the area of the parallelogram.

a) ABCD – Upward (north)

b) CDEF – perpendicularly out of the paper (–)

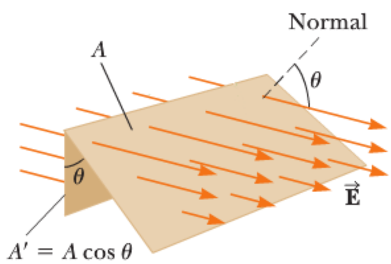
c) BDFG - East



Electric Flux (ϕ_E)

Electric flux is defined to be a measure of the amount of electric field that crosses a certain area. The electric flux crossing a small area element $d\vec{A}$ is defined to be the dot product between the electric field $(\vec{E})$ and $d\vec{E}$. In other words, the flux is equal to the product of component of the electric field perpendicular to the plane of the area (*or parallel to* $d\vec{A}$) and the area

$$\boxed{d\phi_E = \vec{E} \cdot d\vec{A}}$$



If the angle formed between $\vec{E}$ and $d\vec{A}$ is θ, then also

$$\boxed{d\phi_E = E \cdot dA \ \cos\theta}$$

$\vec{E}$: is the magnitude of the electric field

$d\vec{A}$: is the area of the small area element

θ: angle between $\vec{E}$ and $d\vec{A}$

The unit of measurement for electric flux is $\frac{N}{C}m^2$ which is defined to be ***Weber*** (***Wb***).

To find the electric flux crossing a certain area A, first the area is divided into small area elements and then the contribution to the flux from each small area element are added (*integrated*).

$$\phi_E = \int d\phi_E = \int \vec{E} \cdot d\vec{A} = \int E(\cos\theta)\,dA$$

Electric flux crossing a certain area is equal to the integral of the electric field over the area.

If the electric field is constant throughout the area and the area is flat (*i.e. the area has a fixed direction, since E = constant and θ = constant*)

$$\phi_E = \int E(\cos\theta)\,dA = E\cos(\theta)\int dA \qquad but \int dA = A$$

$$\boxed{\phi_E = EA\cos(\theta) = \vec{E} \cdot \vec{A}}$$

If $\vec{E}$ = constant and the area flat (a *plane*)

Electric flux is a scalar quantity. It can be positive (when $\theta < 90°$), zero (when $\theta = 90°$) and negative (when $90° < \theta \le 180°$)

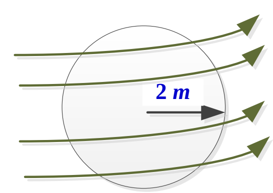
Example

In each of the following the magnitude of the electric field crossing the area is 100 N/C. Calculate the electric flux crossing the area.

a) $\phi_E = EA\cos(\theta) \qquad A = \pi r^2 = \pi 2^2 = 4\pi\ m^2$

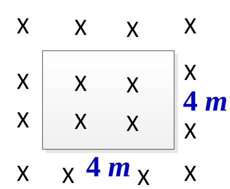
Since the direction of the area is a $(\bullet)$ or perpendicularly out $\theta = 90°$

$\phi_E = (100)(4\pi)\cos(90°)\ \underline{= 0\ \ Wb}$



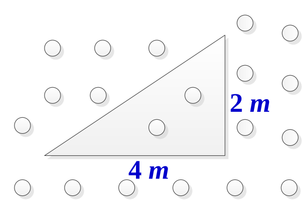
b) The direction of the area is perpendicularly out (i.e. a $(\bullet)$) while the direction of the field is perpendicularly in $(\times)$. Therefore =180°

$\phi_E = (100)(4\times 4)\cos(180°)\ \underline{= -1600\ \ Wb}$



c) The direction of both the area and field are perpendicularly out. Thus $\theta = 0°$

$\phi_E = (100)\left(\frac{1}{2}4\cdot 2\right)\cos(0°)\ \underline{= 400\ \ Wb}$



Gauss's Law

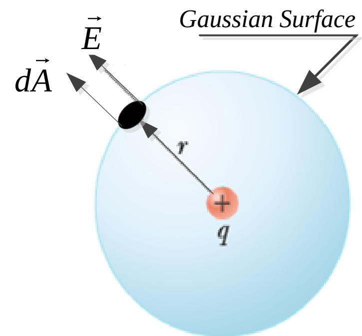
States that the total electric flux crossing a closed surface is equal to $4\pi k$ times the total charge enclosed inside the closed surface (*k Is Coulomb's constant:* $k \approx 9\times10^{9}\ Nm^{2}/C^{2}$). Mathematically Gauss's law can be written as

$$\oint \vec{E}\cdot d\vec{A} = 4\pi kq$$

closed surface

Where q is the total charge enclosed inside the closed surface.

Proof



To make the discussion easier, let's take a point charge q enclosed by a spherical surface concentric with the charge (Gauss's law applied to all shapes of the surfaces)

Let's consider a Gaussian surface or radius r with the charge q at the center. Consider a small area element $d\vec{A}$ & $\vec{E}$ are both along the position vector $\vec{r}$ of $d\vec{A}$ with respect to q.

That is $\quad d\vec{A} = dA\cdot\vec{e}_r \quad \& \quad \vec{E} = E\vec{e}_r$

$$d\phi_E = \vec{E}\cdot d\vec{A}$$

$$= E\vec{e}_r \cdot dA\,\vec{e}_r$$

$$= EdA\,\vec{e}_r\cdot\vec{e}_r \qquad \vec{e}_r\cdot\vec{e}_r = 1 \text{ because } \vec{e}_r \text{ is a unit vector}$$

$$= \vec{E}dA$$

The magnitude of the field at $d\vec{A}$ due to the point q is given by

$$E = \frac{kq}{r^2}$$

$$d\phi_E = \frac{kq}{r^2}\cdot dA$$

The total flux crossing the spherical surface is

$$\phi_E = \oint \frac{kq}{r^2}\cdot dA \qquad r \text{ is a constant over the entire surface of the sphere}$$

$$\phi_E = \frac{kq}{r^2}\int dA \qquad \int dA = 4\pi r^2 \text{ which is the surface area of a sphere of radius } r$$

$$\phi_E = \frac{kq}{r^2}4\pi r^2 = 4\pi kq \quad \surd \ \textbf{proved}$$

If there are no any changes inside a closed surface, then the total electric flux crossing the surface is zero because $q = 0$.

Application of Gauss's Law

Gauss's law is often used to get an expression for the electric field due to a symmetric distribution of charge as a function of distance.

Example

Obtain an expression for the electric field due to a point charge as a function of the distance from the charge using Gauss's law.

Even though Gauss's law is applicable for any closed surface, a suitable Gaussian surface that takes advantage of the symmetry of the problem should be chosen. For the field due to a point change, a Gaussian surface that takes advantage of the following two facts should be chosen

1. The direction of the electric field due to a point charge is along the position vector of the point with respect to the charge.
2. From symmetry, the magnitude of the electric field due to a point charge on a spherical surface centered at the point charge is a constant.

A Gaussian surface that can take advantage of these two facts is a spherical surface centered at the charge. Because the direction of the area at any point along its radius or along the position vector of the point with respect to its center.

Let's apply Gauss's law over a spherical surface of radius r centered at the charge.

$$\oint \vec{E} \cdot d\vec{A} = 4\pi kq$$

$$d\vec{A} = dA\, \vec{e}_r$$

$$\vec{E} = E\vec{e}_r$$

$$\vec{E}d\vec{A} = E\vec{e}_r \cdot dA\, \vec{e}_r = EdA\left(\vec{e}_r \cdot \vec{e}_r\right) = EdA$$

$$\oint \vec{E} \cdot d\vec{A} = \oint E\ dA = 4\pi kq$$

But the electric field is constant over the entire surface

$$\oint \vec{E} \cdot d\vec{A} = E \int dA = 4\pi kq \qquad\qquad \int dA = 4\pi r^2$$

$$E\left(4\pi r^2\right) = 4\pi kq$$

$$\boxed{E = \frac{4\pi kq}{4\pi r^2} = \frac{kq}{r^2}}$$

Example

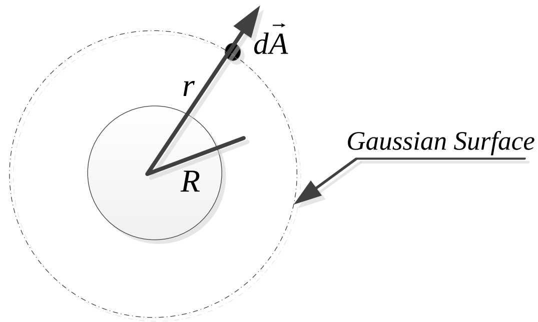
A solid sphere of radius R has a uniform charge density. The total charge in the sphere is Q.

a) Using Gauss's law, obtain an expression for the electric field at points outside the sphere.

From symmetry

1. The direction of the electric field is radial (i.e. in the direction of the position vector of the point with respect to the center of the sphere).
2. From symmetry, the magnitude of the electric field due to a point charge on a spherical surface centered at the point charge is a constant.

Taking the Gaussian surface to be a spherical surface of radius $r > R$



$$\oint \vec{E} \cdot d\vec{A} = 4\pi kQ$$

Where Q is the total charge inside the Gaussian surface (i.e. charge in the solid sphere)

$$d\vec{A} = dA\,\vec{e}_r \quad \& \quad \vec{E} = E\vec{e}_r \quad \Rightarrow \vec{E}d\vec{A} = EdA$$

But E is constant on the surface from symmetry

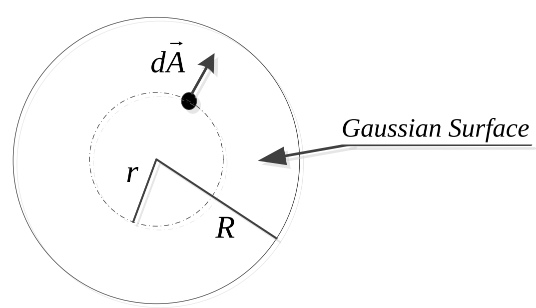
$$E\oint dA = 4\pi kQ$$

$$E 4\pi^2 = 4\pi kQ$$

$$\boxed{E = \frac{kQ}{r^2}} \quad for \quad r > R$$

b) Using Gauss's law to find an expression for the electric field for points inside the solid sphere. (assume positive charge)

Again taking the Gaussian surface to be a spherical surface of radius $r < R$ concentric with the solid sphere from symmetric, the direction of the electric field is radial and is constant on the spherical surface.



The total charge enclosed inside the Gaussian surface is $q = \frac{4}{3}\pi r^3 \rho$

Where ρ is the charge density and $\rho = \dfrac{Q}{\frac{4}{3}\pi R^3}$

$$q = \frac{4}{3}\pi r^3 \frac{Q}{\frac{4}{3}\pi R^3} = \frac{r^3}{R^3}Q$$

$$\oint \vec{E}\cdot d\vec{A} = 4\pi k\left(\frac{r^3}{R^3}Q\right) \qquad\qquad \vec{E}\cdot d\vec{A} = EdA$$

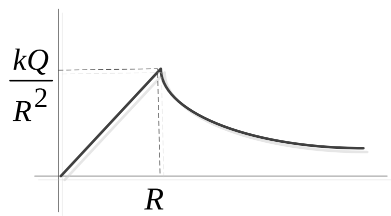
$$\oint EdA = 4\pi k\frac{r^3}{R^3}Q$$

$$E\oint dA = 4\pi k\frac{r^3}{R^3}Q$$

$$E4\pi r^2 = 4\pi k\frac{r^3}{R^3}Q$$

$$E = \frac{kQ}{R^3}r \quad for \quad r < R$$

c) Plot the graph of E versus r

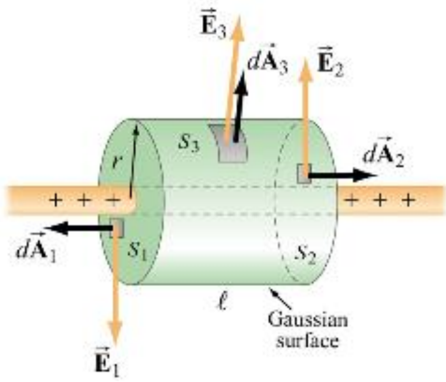


Example

Obtain an expression for the electric field due to an infinitely long uniformly charged (*charge density* λ) thin rod (*assume positive charge*)

Since it is infinitely long, any point in the wire can be assumed to be the mid-point. Thus the *x*-component of the field due to any charge in the wire will be cancelled by the *x*-component of the field of a charge on the other side as shown. Thus, the *x*-component of the field at any point is zero. In other words, the direction of the field is radically perpendicular to the wire. Again because of symmetry, all points at the same ⊥ distance from the wire will have the same magnitude for the electric field.

Now let's take our Gaussian surface to be a cylinder of radius *r* concentric with the wire as shown.



The electric flux of the end faces are zero because the area is perpendicular to the field. On the surface, the area element $d\vec{A}$ & $\vec{E}$ have the same directions

$$\vec{E}\cdot d\vec{A} - EdA$$

$$\phi_E = \oint \vec{E}\cdot d\vec{A} = \int_{end\ force=0} \vec{E}d\vec{A} + \int_{curved\ surface} \vec{E}d\vec{A} = 4\pi kq$$

$$\phi_E = \int_{curved\ surface} \vec{E}d\vec{A} = \int_{curved\ surface} EdA = E\int_{curved\ surface} dA$$

(Because *E* is constant over the cylindrical surface)

$$\int dA = area\ of\ \ curve\ surface = 2\pi rL$$

$$\phi_E = E2\pi rL = 4\pi kq$$

If the charge density of the wire is λ. Then,

$$\lambda = \frac{q}{L} \Rightarrow q = \lambda L$$

$$2\pi rEL = 4\pi k\lambda L$$

$$\boxed{E = \frac{2k\lambda}{r}}$$

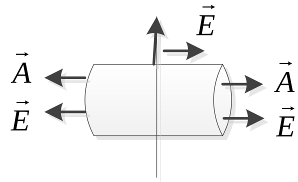
(magnitude of the electric field due to a uniformly charged infinitely long wire, its direction is radically perpendicular from the wire)

Example

Obtain an expression for the electric field due to a uniformly charged infinite plane (*assume a positive charge*).

Solution

From the symmetry the component of the electric field parallel to the plane will be zero at any point. Those direction of the field is ⊥ to the plane everywhere.



Let choose a cylinder surface of base area A as out Gaussian surface.
The electric flux on the surface is zero because the area and the field are perpendicular to each other.

$$\oint \vec{E}\cdot d\vec{A} = \int_{curved\ surface=0} \vec{E}dA + \int_{left\ end} \vec{E}dA + \int_{right\ end} \vec{E}dA$$

On the ends the electric fields area are parallel

$$\vec{E}\cdot d\vec{A} - EdA$$

$$\oint \vec{E}\cdot d\vec{A} = \int_{left\ end} \vec{E}dA + \int_{right\ end} \vec{E}dA = 2\int EdA$$

Again from symmetry the electric field at point at the same perpendicular distance from the plane is constant

$$\phi_E = 2\int EdA = 2E\int dA \quad \& \quad \int dA = A$$

$$\phi_E = 2EA = 4\pi kq$$

$$E = \frac{2\pi kq}{A}$$

$$\frac{q}{A} = charge\ density = \sigma$$

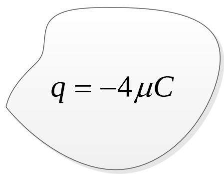
$$\boxed{E = 2\pi k\sigma}$$

Therefore the electric field due to a uniformly charged infinite plan is constant everywhere its direction is perpendicular to the plane.

Example

Calculate the flux crossing the following Gaussian surface.

a)



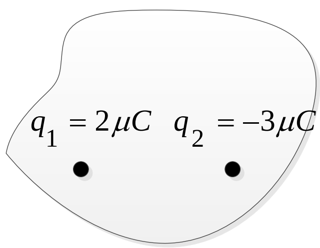
$$\phi_E = 4\pi kq \qquad q = -4\times10^{-6}C$$

$$= 4\pi\left(9\times10^{9}\right)\left(-4\times10^{-6}\right)$$

$$\boxed{= -4.52\times10^{5}\ Wb}$$

---

b)



$$q = q_1 + q_2 = 2\times10^{-6}C - 3\times10^{-6}C = -10^{-6}$$

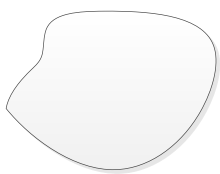
$$\phi_E = 4\pi kq$$

$$= 4\pi\left(9\times10^{9}\right)\left(-10^{-6}\right)$$

$$\boxed{= -1.13\times10^{5}\ Wb}$$

---

c)



$$\phi_E = 0 \quad \text{since} \quad q = 0$$

Example

A solid sphere of radius 0.1m is uniformly charged and has a charge of $2\times10^{-3}C$.

a) Calculate the strength of the field at a distance 0.2m from the center of the sphere.

Given: $r = 0.2m, \quad R = 0.1m, \quad Q = 2\times10^{-3}$

Since $r > R$

$$E = \frac{kQ}{r^2} = \frac{\left(9\times10^{9}\right)\left(2\times10^{-3}\right)}{(0.2)^2}\ \boxed{= 4.5\times10^{8}}$$

b) Calculate the strength of the field at a distance of 0.01m from the center of the sphere.

Given: $r = 0.01m, \quad R = 0.1m, \quad Q = 2\times10^{-3}$

Since $r < R$

$$E = \frac{kQ}{R^3}r = \frac{\left(9\times10^{9}\right)\left(2\times10^{-3}\right)}{(0.1)^3}(0.01)\ \boxed{= 1.8\times10^{8}}$$

Example

Calculate the electric field at a distance of 0.03*m* from a uniformly charged infinite plane of charge density $2\ C/m^2$

Solution

$$E = 2\pi k\sigma \qquad \sigma = 2\ C/m^2$$

$$= 2\pi\left(9\times10^9\right)(2)$$

$$\underline{= 113\times10^{11}\ N/C}$$

Example

Calculate the electric field at a point perpendicular distance 2mm from an infinitely long thin rod charge density $2\ \mu C/m^2$

Solution

Given: $r = 0.002\ m, \quad \lambda = 2\times10^{-6}C$

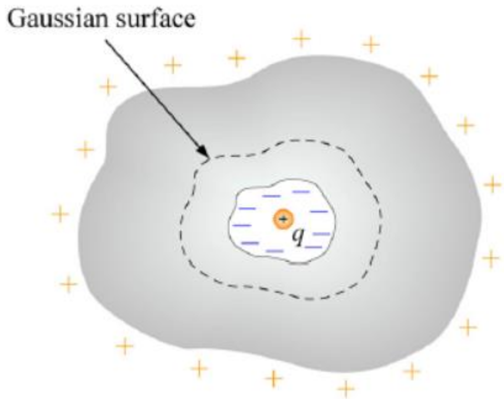
$$E = \frac{k\lambda}{2\pi r}$$

$$= \frac{\left(9\times10^9\right)\left(2\times10^{-6}\right)}{2\pi(0.002)}$$

$$\underline{= 1.4\times10^{6}\ N/C}$$

Conductors in Electrostatic Equilibrium

Conductors are said to be in electrostatic equilibrium, if the free charges are at rest. The fact that the charges are at rest indicate that the electric field inside a conductor in electrostatic equilibrium must be zero, because if there was a non-zero field the free charges would be moving.



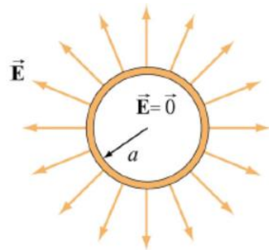
Now let's consider a Gaussian surface inside a conductor in electrostatic equilibrium. Gausses law implies that

$$\oint \vec{E} \cdot d\vec{A} = 4\pi kq$$

But $\quad \vec{E} = 0$ everywhere inside

$$\oint \vec{E} \cdot dA = 0 = 4\pi kq \quad \Rightarrow \quad q = 0$$

Therefore there is no net excess charge inside a conductor in electrostatic equilibrium.
If there is an excess charge, it must reside on the surface of the conductor since the charges are not moving on the surface of the conductor, the component of the electric field just outside the conductor in a direction parallel to the surface must be zero. Thus, the electric field just outside a conductor must be perpendicular to the surface of the conductor.



To find the magnitude of the electric field just outside the conductor in terms of the charge density (σ) on the surface, let's use a small cylindrical surface as shown. The flux of the surface cylinder inside the conductor is zero because $E = 0$ inside. The flux on the curved surface outside is zero because the electric field is parallel to the surface (*or* $\perp$ *to the area*). The only contribution to the flux comes from the end face outside the conductor.

$$\oint \vec{E}\cdot d\vec{A} = \underbrace{\int \vec{E}\cdot d\vec{A}}_{\text{outside end}} = \int E dA$$

If the area is taken to be very small E can be taken to be constant

$$E\int dA = EA = 4\pi kq$$

$$\Rightarrow \; E = 4\pi k\frac{q}{A}$$ *Where q is the charge enclosed inside the Gaussian surface*

$$\frac{q}{A} = \sigma$$ *Charge density on the surface*

$$\boxed{E = 4\pi k\sigma}$$

Magnitude of electric field just outside a conductor its direction is $\perp$ to the surface of the conductor, in electrostatic equilibrium