

Solution

Section 4.1 – Proving Identities

Exercise

Prove the identity $\cos \theta \cot \theta + \sin \theta = \csc \theta$

Solution

$$\begin{aligned}\cos \theta \cot \theta + \sin \theta &= \cos \theta \frac{\cos \theta}{\sin \theta} + \sin \theta \\&= \frac{\cos^2 \theta}{\sin \theta} + \sin \theta \\&= \frac{\cos^2 \theta}{\sin \theta} + \sin \theta \frac{\sin \theta}{\sin \theta} \\&= \frac{\cos^2 \theta}{\sin \theta} + \frac{\sin^2 \theta}{\sin \theta} \\&= \frac{\cos^2 \theta + \sin^2 \theta}{\sin \theta} \\&= \frac{1}{\sin \theta} \\&= \csc \theta \quad \checkmark\end{aligned}$$

Exercise

Prove the identity $\sec \theta \cot \theta - \sin \theta = \frac{\cos^2 \theta}{\sin \theta}$

Solution

$$\begin{aligned}\sec \theta \cot \theta - \sin \theta &= \frac{1}{\cos \theta} \frac{\cos \theta}{\sin \theta} - \sin \theta \\&= \frac{1}{\sin \theta} - \sin \theta \\&= \frac{1 - \sin^2 \theta}{\sin \theta} \\&= \frac{\cos^2 \theta}{\sin \theta} \quad \checkmark\end{aligned}$$

Exercise

Prove the identity $\frac{\csc \theta \tan \theta}{\sec \theta} = 1$

Solution

$$\frac{\csc \theta \tan \theta}{\sec \theta} = \csc \theta \tan \theta \frac{1}{\sec \theta}$$

$$= \frac{1}{\sin \theta} \frac{\sin \theta}{\cos \theta} \cos \theta$$

$$= 1 \quad \checkmark$$

Exercise

Prove the identity $(\sin \theta + \cos \theta)^2 = 1 + 2 \sin \theta \cos \theta$

Solution

$$(\sin \theta + \cos \theta)^2 = \sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta$$

$$= 1 + 2 \sin \theta \cos \theta \quad \checkmark$$

Exercise

Prove the identity $\sin \theta (\sec \theta + \cot \theta) = \tan \theta + \cos \theta$

Solution

$$\sin \theta (\sec \theta + \cot \theta) = \sin \theta \left(\frac{1}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \right)$$

$$= \sin \theta \frac{1}{\cos \theta} + \sin \theta \frac{\cos \theta}{\sin \theta}$$

$$= \frac{\sin \theta}{\cos \theta} + \cos \theta$$

$$= \tan \theta + \cos \theta \quad \checkmark$$

Exercise

Prove the identity $\cos \theta (\csc \theta + \tan \theta) = \cot \theta + \sin \theta$

Solution

$$\cos \theta (\csc \theta + \tan \theta) = \cos \theta \frac{1}{\sin \theta} + \cos \theta \frac{\sin \theta}{\cos \theta}$$

$$= \cot \theta + \sin \theta \quad \checkmark$$

Exercise

Prove the identity $\cos \theta (\sec \theta - \cos \theta) = \sin^2 \theta$

Solution

$$\cos \theta (\sec \theta - \cos \theta) = \cos \theta \frac{1}{\cos \theta} - \cos^2 \theta$$

$$= 1 - \cos^2 \theta$$

$$= \sin^2 \theta \quad \checkmark$$

Exercise

Prove the identity $\cot \theta + \tan \theta = \csc \theta \sec \theta$

Solution

$$\begin{aligned}\cot \theta + \tan \theta &= \frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\cos \theta} \\&= \frac{\cos^2 \theta + \sin^2 \theta}{\sin \theta \cos \theta} \\&= \frac{1}{\sin \theta \cos \theta} \\&= \frac{1}{\sin \theta} \frac{1}{\cos \theta} \\&= \csc \theta \sec \theta \quad \checkmark\end{aligned}$$

Exercise

Prove $\tan x(\cos x + \cot x) = \sin x + 1$

Solution

$$\begin{aligned}\tan x(\cos x + \cot x) &= \frac{\sin x}{\cos x} \left(\cos x + \frac{\cos x}{\sin x} \right) \\&= \cos x \frac{\sin x}{\cos x} + \frac{\sin x}{\cos x} \frac{\cos x}{\sin x} \\&= \sin x + 1 \quad \checkmark\end{aligned}$$

Exercise

Prove $\frac{1 - \cos^4 \theta}{1 + \cos^2 \theta} = \sin^2 \theta$

Solution

$$\begin{aligned}\frac{1 - \cos^4 \theta}{1 + \cos^2 \theta} &= \frac{(1 + \cos^2 \theta)(1 - \cos^2 \theta)}{1 + \cos^2 \theta} \\&= 1 - \cos^2 \theta \\&= \sin^2 \theta \quad \checkmark\end{aligned}$$

Exercise

Prove $\frac{1 - \sec x}{1 + \sec x} = \frac{\cos x - 1}{\cos x + 1}$

Solution

$$\begin{aligned}
 \frac{1 - \sec x}{1 + \sec x} &= \frac{1 - \frac{1}{\cos x}}{1 + \frac{1}{\cos x}} \\
 &= \frac{\frac{\cos x - 1}{\cos x}}{\frac{\cos x + 1}{\cos x}} \\
 &= \frac{\cos x - 1}{\cos x + 1} \quad \checkmark
 \end{aligned}$$

Exercise

Prove $\frac{\cos x}{1 + \sin x} - \frac{1 - \sin x}{\cos x} = 0$

Solution

$$\begin{aligned}
 \frac{\cos x}{1 + \sin x} - \frac{1 - \sin x}{\cos x} &= \frac{\cos x}{\cos x} \frac{\cos x}{1 + \sin x} - \frac{1 + \sin x}{1 + \sin x} \frac{1 - \sin x}{\cos x} \\
 &= \frac{\cos^2 x - (1 - \sin^2 x)}{\cos x(1 + \sin x)} \\
 &= \frac{\cos^2 x - 1 + \sin^2 x}{\cos x(1 + \sin x)} \\
 &= \frac{1 - 1}{\cos x(1 + \sin x)} \\
 &= \frac{0}{\cos x(1 + \sin x)} \\
 &= 0 \quad \checkmark
 \end{aligned}$$

Exercise

Prove $\frac{1 + \cot^3 t}{1 + \cot t} = \csc^2 t - \cot t$

Solution

$$\begin{aligned}
 \frac{1 + \cot^3 t}{1 + \cot t} &= \frac{1 + \frac{\cos^3 t}{\sin^3 t}}{1 + \frac{\cos t}{\sin t}} \\
 &= \frac{\frac{\sin^3 t + \cos^3 t}{\sin^3 t}}{\frac{\sin t + \cos t}{\sin t}} \\
 &= \frac{\sin^3 t + \cos^3 t}{\sin^3 t} \cdot \frac{\sin t}{\sin t + \cos t}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{(\sin t + \cos t)(\sin^2 t - \sin t \cos t + \cos^2 t)}{\sin^2 t} \cdot \frac{1}{\sin t + \cos t} \\
&= \frac{1 - \sin t \cos t}{\sin^2 t} \\
&= \frac{1}{\sin^2 t} - \frac{\sin t \cos t}{\sin^2 t} \\
&= \csc^2 t - \frac{\cos t}{\sin t} \\
&= \csc^2 t - \cot t \quad \checkmark
\end{aligned}$$

Exercise

Prove: $\tan x + \cot x = \sec x \csc x$

Solution

$$\begin{aligned}
\tan x + \cot x &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \\
&= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} \\
&= \frac{1}{\cos x \sin x} \\
&= \frac{1}{\cos x} \cdot \frac{1}{\sin x} \\
&= \sec x \csc x \quad \checkmark
\end{aligned}$$

Exercise

Prove: $\frac{\tan x - \cot x}{\sin x \cos x} = \sec^2 x - \csc^2 x$

Solution

$$\begin{aligned}
\frac{\tan x - \cot x}{\sin x \cos x} &= \frac{\tan x}{\sin x \cos x} - \frac{\cot x}{\sin x \cos x} \\
&= \tan x \frac{1}{\sin x \cos x} - \cot x \frac{1}{\sin x \cos x} \\
&= \frac{\sin x}{\cos x} \frac{1}{\sin x \cos x} - \frac{\cos x}{\sin x} \frac{1}{\sin x \cos x} \\
&= \frac{1}{\cos^2 x} - \frac{1}{\sin^2 x} \\
&= \sec^2 x - \csc^2 x \quad \checkmark
\end{aligned}$$

Exercise

Prove: $\frac{\sec x + \tan x}{\sec x - \tan x} = \frac{1 + 2 \sin x + \sin^2 x}{\cos^2 x}$

Solution

$$\begin{aligned}\frac{\sec x + \tan x}{\sec x - \tan x} &= \frac{\frac{1}{\cos x} + \frac{\sin x}{\cos x}}{\frac{1}{\cos x} - \frac{\sin x}{\cos x}} \frac{\cos x}{\cos x} \\&= \frac{\frac{1}{\cos x} \cos x + \frac{\sin x}{\cos x} \cos x}{\frac{1}{\cos x} \cos x - \frac{\sin x}{\cos x} \cos x} \\&= \frac{1 + \sin x}{1 - \sin x} \\&= \frac{1 + \sin x}{1 - \sin x} \frac{1 + \sin x}{1 + \sin x} \\&= \frac{(1 + \sin x)^2}{1 - \sin^2 x} \\&= \frac{1 + 2 \sin x + \sin^2 x}{\cos^2 x} \quad \checkmark\end{aligned}$$

Exercise

Prove the identity: $\sin^2 x - \cos^2 x = 2 \sin^2 x - 1$

Solution

$$\begin{aligned}\sin^2 x - \cos^2 x &= \sin^2 x - (1 - \sin^2 x) \\&= \sin^2 x - 1 + \sin^2 x \\&= 2 \sin^2 x - 1 \quad \checkmark\end{aligned}$$

Exercise

Prove the identity: $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$

Solution

$$\begin{aligned}\sin^4 x - \cos^4 x &= (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x) \\&= (1)(\sin^2 x - \cos^2 x) \\&= \sin^2 x - \cos^2 x \quad \checkmark\end{aligned}$$

Exercise

Prove the identity: $\frac{\cos \alpha}{1 + \sin \alpha} = \sec \alpha - \tan \alpha$

Solution

$$\begin{aligned}\frac{\cos \alpha}{1 + \sin \alpha} &= \frac{\cos \alpha}{1 + \sin \alpha} \frac{1 - \sin \alpha}{1 - \sin \alpha} \\&= \frac{\cos \alpha - \cos \alpha \sin \alpha}{1 - \sin^2 \alpha} \\&= \frac{\cos \alpha - \cos \alpha \sin \alpha}{\cos^2 \alpha} \\&= \frac{\cos \alpha}{\cos^2 \alpha} - \frac{\cos \alpha \sin \alpha}{\cos^2 \alpha} \\&= \frac{1}{\cos \alpha} - \frac{\sin \alpha}{\cos \alpha} \\&= \sec \alpha - \tan \alpha \quad \checkmark\end{aligned}$$

Exercise

Prove the identity: $\frac{\sin \alpha}{1 - \sin \alpha} - \frac{\cos \alpha}{1 - \sin \alpha} = \frac{1 - \cot \alpha}{\csc \alpha - 1}$

Solution

$$\begin{aligned}\frac{\sin \alpha}{1 - \sin \alpha} - \frac{\cos \alpha}{1 - \sin \alpha} &= \frac{\sin \alpha - \cos \alpha}{1 - \sin \alpha} \\&= \frac{\sin \alpha - \cos \alpha}{\frac{1 - \sin \alpha}{\sin \alpha}} \\&= \frac{\sin \alpha - \cos \alpha}{\frac{1}{\sin \alpha} - \frac{\sin \alpha}{\sin \alpha}} \\&= \frac{1 - \cot \alpha}{\csc \alpha - 1} \quad \checkmark\end{aligned}$$

Exercise

Prove the identity: $\frac{\frac{1}{\tan x} + \cot x}{\frac{1}{\tan x} + \tan x} = \frac{2}{\sec^2 x}$

Solution

$$\begin{aligned}
\frac{\frac{1}{\tan x} + \cot x}{\frac{1}{\tan x} + \tan x} &= \frac{\frac{1}{\tan x} + \frac{1}{\tan x}}{\frac{1 + \tan^2 x}{\tan x}} \\
&= \frac{\frac{2}{\tan x}}{\frac{\sec^2 x}{\tan x}} \\
&= \frac{2}{\sec^2 x} \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\cot^2 \theta + 3\cot \theta - 4}{\cot \theta + 4} = \cot \theta - 1$

Solution

$$\begin{aligned}
\frac{\cot^2 \theta + 3\cot \theta - 4}{\cot \theta + 4} &= \frac{(\cot \theta + 4)(\cot \theta - 1)}{\cot \theta + 4} \\
&= \cot \theta - 1 \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\sin \theta}{1 + \cos \theta} = \frac{1 - \cos \theta}{\sin \theta}$

Solution

$$\begin{aligned}
\frac{\sin \theta}{1 + \cos \theta} &= \frac{\sin \theta}{1 + \cos \theta} \frac{1 - \cos \theta}{1 - \cos \theta} \\
&= \frac{\sin \theta(1 - \cos \theta)}{1 - \cos^2 \theta} \\
&= \frac{\sin \theta(1 - \cos \theta)}{\sin^2 \theta} \\
&= \frac{1 - \cos \theta}{\sin \theta} \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\tan x(\csc x - \sin x) = \cos x$

Solution

$$\tan x(\csc x - \sin x) = \frac{\sin x}{\cos x} \left(\frac{1}{\sin x} - \sin x \right)$$

$$\begin{aligned}
&= \frac{\sin x}{\cos x} \left(\frac{1 - \sin^2 x}{\sin x} \right) \\
&= \frac{1}{\cos x} \left(\frac{\cos^2 x}{1} \right) \\
&= \cos x \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\sin x(\tan x \cos x - \cot x \cos x) = 1 - 2\cos^2 x$

Solution

$$\begin{aligned}
\sin x(\tan x \cos x - \cot x \cos x) &= \sin x \cos x \left(\frac{\sin x}{\cos x} - \frac{\cos x}{\sin x} \right) \\
&= \sin x \cos x \left(\frac{\sin^2 x - \cos^2 x}{\cos x \sin x} \right) \\
&= 1 - \cos^2 x - \cos^2 x \\
&= 1 - 2\cos^2 x \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $(1 + \tan x)^2 + (\tan x - 1)^2 = 2\sec^2 x$

Solution

$$\begin{aligned}
(1 + \tan x)^2 + (\tan x - 1)^2 &= 1 + 2\tan x + \tan^2 x + 1 - 2\tan x + \tan^2 x \\
&= 2 + 2\tan^2 x \\
&= 2(1 + \tan^2 x) \\
&= 2\sec^2 x \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\sec x + \tan x = \frac{\cos x}{1 - \sin x}$

Solution

$$\begin{aligned}
\sec x + \tan x &= \frac{1}{\cos x} + \frac{\sin x}{\cos x} \\
&= \frac{1 + \sin x}{\cos x} \cdot \frac{1 - \sin x}{1 - \sin x}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1 - \sin^2 x}{\cos x(1 - \sin x)} \\
&= \frac{\cos^2 x}{\cos x(1 - \sin x)} \\
&= \frac{\cos x}{1 - \sin x} \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\tan x - 1}{\tan x + 1} = \frac{1 - \cot x}{1 + \cot x}$

Solution

$$\begin{aligned}
\frac{\tan x - 1}{\tan x + 1} &= \frac{\frac{1}{\cot x} - 1}{\frac{1}{\cot x} + 1} \\
&= \frac{\frac{1 - \cot x}{\cot x}}{\frac{1 + \cot x}{\cot x}} \\
&= \frac{1 - \cot x}{1 + \cot x} \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $7 \csc^2 x - 5 \cot^2 x = 2 \csc^2 x + 5$

Solution

$$\begin{aligned}
7 \csc^2 x - 5 \cot^2 x &= 7 \csc^2 x - 5(\csc^2 x - 1) \\
&= 7 \csc^2 x - 5 \csc^2 x + 5 \\
&= 2 \csc^2 x + 5 \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $1 - \frac{\cos^2 x}{1 - \sin x} = -\sin x$

Solution

$$\begin{aligned}
1 - \frac{\cos^2 x}{1 - \sin x} &= 1 - \frac{1 - \sin^2 x}{1 - \sin x} \\
&= 1 - \frac{(1 - \sin x)(1 + \sin x)}{1 - \sin x}
\end{aligned}$$

$$= 1 - (1 + \sin x)$$

$$= 1 - 1 - \sin x$$

$$= -\sin x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\frac{1 - \cos x}{1 + \cos x} = \frac{\sec x - 1}{\sec x + 1}$

Solution

$$\begin{aligned} \frac{1 - \cos x}{1 + \cos x} &= \frac{\frac{1}{\cos x} - \frac{\cos x}{\cos x}}{\frac{1}{\cos x} + \frac{\cos x}{\cos x}} \\ &= \frac{\sec x - 1}{\sec x + 1} \quad \checkmark \end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\sec x - 1}{\tan x} = \frac{\tan x}{\sec x + 1}$

Solution

$$\begin{aligned} \frac{\sec x - 1}{\tan x} &= \frac{\sec x - 1}{\tan x} \frac{\sec x + 1}{\sec x + 1} \\ &= \frac{\sec^2 x - 1}{\tan x (\sec x + 1)} \\ &= \frac{\tan^2 x}{\tan x (\sec x + 1)} \\ &= \frac{\tan x}{\sec x + 1} \quad \checkmark \end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\cos x}{\cos x - \sin x} = \frac{1}{1 - \tan x}$

Solution

$$\begin{aligned} \frac{\cos x}{\cos x - \sin x} &= \frac{\frac{\cos x}{\cos x}}{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}} \\ &= \frac{1}{1 - \tan x} \quad \checkmark \end{aligned}$$

Exercise

Prove the following equation is an identity: $(\sec x + \tan x)^2 = \frac{1 + \sin x}{1 - \sin x}$

Solution

$$\begin{aligned}(\sec x + \tan x)^2 &= \left(\frac{1}{\cos x} + \frac{\sin x}{\cos x} \right)^2 \\&= \left(\frac{1 + \sin x}{\cos x} \right)^2 \\&= \frac{(1 + \sin x)^2}{\cos^2 x} \\&= \frac{(1 + \sin x)^2}{1 - \sin^2 x} \\&= \frac{(1 + \sin x)^2}{(1 - \sin x)(1 + \sin x)} \\&= \frac{1 + \sin x}{1 - \sin x} \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\cos x}{1 + \tan x} - \frac{\sin x}{1 + \cot x} = \cos x - \sin x$

Solution

$$\begin{aligned}\frac{\cos x}{1 + \tan x} - \frac{\sin x}{1 + \cot x} &= \frac{\cos x}{1 + \frac{\sin x}{\cos x}} - \frac{\sin x}{1 + \frac{\cos x}{\sin x}} \\&= \frac{\cos x}{\frac{\cos x + \sin x}{\cos x}} - \frac{\sin x}{\frac{\sin x + \cos x}{\sin x}} \\&= \frac{\cos^2 x}{\cos x + \sin x} - \frac{\sin^2 x}{\sin x + \cos x} \\&= \frac{\cos^2 x - \sin^2 x}{\cos x + \sin x} \\&= \frac{(\cos x + \sin x)(\cos x - \sin x)}{\cos x + \sin x} \\&= \cos x - \sin x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\cot x + \csc x - 1}{\cot x - \csc x + 1} = \csc x + \cot x$

Solution

$$\begin{aligned}\frac{\cot x + \csc x - 1}{\cot x - \csc x + 1} &= \frac{\cot x + \csc x - (\csc^2 x - \cot^2 x)}{\cot x - \csc x + 1} \\&= \frac{\cot x + \csc x - (\csc x - \cot x)(\csc x + \cot x)}{\cot x - \csc x + 1} \\&= \frac{(\csc x + \cot x)(1 - (\csc x - \cot x))}{\cot x - \csc x + 1} \\&= \frac{(\csc x + \cot x)(1 - \csc x + \cot x)}{\cot x - \csc x + 1} \\&= \csc x + \cot x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\tan x + \cot x}{\tan x - \cot x} = \frac{1}{\sin^2 x - \cos^2 x}$

Solution

$$\begin{aligned}\frac{\tan x + \cot x}{\tan x - \cot x} &= \frac{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}}{\frac{\sin x}{\cos x} - \frac{\cos x}{\sin x}} \\&= \frac{\frac{\sin^2 x + \cos^2 x}{\cos x \sin x}}{\frac{\sin^2 x - \cos^2 x}{\cos x \sin x}} \\&= \frac{1}{\sin^2 x - \cos^2 x} \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{1 - \cot^2 x}{1 + \cot^2 x} + 1 = 2 \sin^2 x$

Solution

$$\begin{aligned}\frac{1 - \cot^2 x}{1 + \cot^2 x} + 1 &= \frac{1 - \cot^2 x + 1 + \cot^2 x}{1 + \cot^2 x} \\&= \frac{2}{\csc^2 x} \\&= 2 \sin^2 x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{1+\cos x}{1-\cos x} - \frac{1-\cos x}{1+\cos x} = 4\cot x \csc x$

Solution

$$\begin{aligned}\frac{1+\cos x}{1-\cos x} - \frac{1-\cos x}{1+\cos x} &= \frac{(1+\cos x)^2 - (1-\cos x)^2}{1-\cos^2 x} \\ &= \frac{(1+\cos x+1-\cos x)(1+\cos x-1+\cos x)}{\sin^2 x} & a^2 - b^2 &= (a-b)(a+b) \\ &= \frac{(2)(2\cos x)}{\sin^2 x} \\ &= 4 \frac{\cos x}{\sin x} \frac{1}{\sin x} \\ &= 4\cot x \csc x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\sin^3 x - \cos^3 x}{\sin x - \cos x} = 1 + \sin x \cos x$

Solution

$$\begin{aligned}\frac{\sin^3 x - \cos^3 x}{\sin x - \cos x} &= \frac{(\sin x - \cos x)(\sin^2 x + \sin x \cos x + \cos^2 x)}{\sin x - \cos x} & a^3 - b^3 &= (a-b)(a^2 + ab + b^2) \\ &= 1 + \sin x \cos x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $1 + \sec^2 x \sin^2 x = \sec^2 x$

Solution

$$\begin{aligned}1 + \sec^2 x \sin^2 x &= 1 + \frac{1}{\cos^2 x} \sin^2 x \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{1 + \csc x}{\sec x} = \cos x + \cot x$

Solution

$$\begin{aligned}\frac{1 + \csc x}{\sec x} &= \frac{1}{\sec x} + \frac{\csc x}{\sec x} \\&= \cos x + \frac{\frac{1}{\sin x}}{\frac{1}{\cos x}} \\&= \cos x + \frac{\cos x}{\sin x} \\&= \cos x + \cot x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\tan^2 x = \sec^2 x - \sin^2 x - \cos^2 x$

Solution

$$\begin{aligned}\sec^2 x - \sin^2 x - \cos^2 x &= \frac{1}{\cos^2 x} - (\sin^2 x + \cos^2 x) \\&= \frac{1}{\cos^2 x} - 1 \\&= \frac{1 - \cos^2 x}{\cos^2 x} \\&= \frac{\sin^2 x}{\cos^2 x} \\&= \tan^2 x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\sin x}{1 - \cos x} + \frac{\sin x}{1 + \cos x} = 2 \csc x$

Solution

$$\begin{aligned}\frac{\sin x}{1 - \cos x} + \frac{\sin x}{1 + \cos x} &= \sin x \left(\frac{1}{1 - \cos x} + \frac{1}{1 + \cos x} \right) \\&= \sin x \left(\frac{1 + \cos x + 1 - \cos x}{1 - \cos^2 x} \right) \\&= \sin x \left(\frac{2}{\sin^2 x} \right)\end{aligned}$$

$$\begin{aligned}
 &= \frac{2}{\sin x} \\
 &= 2 \csc x \quad \checkmark
 \end{aligned}$$

Exercise

Prove the following equation is an identity: $\cos^2(\alpha - \beta) - \cos^2(\alpha + \beta) = \sin^2(\alpha + \beta) - \sin^2(\alpha - \beta)$

Solution

$$\begin{aligned}
 \cos^2(\alpha - \beta) - \cos^2(\alpha + \beta) &= 1 - \sin^2(\alpha - \beta) - [1 - \sin^2(\alpha + \beta)] \\
 &= 1 - \sin^2(\alpha - \beta) - 1 + \sin^2(\alpha + \beta) \\
 &= \sin^2(\alpha + \beta) - \sin^2(\alpha - \beta) \quad \checkmark
 \end{aligned}$$

Exercise

Prove the following equation is an identity: $\tan x \csc x - \sec^2 x \cos x = 0$

Solution

$$\begin{aligned}
 \tan x \csc x - \sec^2 x \cos x &= \frac{\sin x}{\cos x} \frac{1}{\sin x} - \frac{1}{\cos^2 x} \cos x \\
 &= \frac{1}{\cos x} - \frac{1}{\cos x} \\
 &= 0 \quad \checkmark
 \end{aligned}$$

Exercise

Prove the following equation is an identity: $(1 + \tan x)^2 - 2 \tan x = \frac{1}{(1 - \sin x)(1 + \sin x)}$

Solution

$$\begin{aligned}
 (1 + \tan x)^2 - 2 \tan x &= 1 + 2 \tan x + \tan^2 x - 2 \tan x \\
 &= 1 + \tan^2 x \\
 &= \sec^2 x \\
 &= \frac{1}{\cos^2 x} \\
 &= \frac{1}{1 - \sin^2 x} \\
 &= \frac{1}{(1 - \sin x)(1 + \sin x)} \quad \checkmark
 \end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{3\csc^2 x - 5\csc x - 28}{\csc x - 4} = \frac{3}{\sin x} + 7$

Solution

$$\begin{aligned}\frac{3\csc^2 x - 5\csc x - 28}{\csc x - 4} &= \frac{(3\csc x + 7)(\csc x - 4)}{\csc x - 4} \\ &= 3\csc x + 7 \\ &= \frac{3}{\sin x} + 7 \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $(\sec^2 x - 1)(\sec^2 x + 1) = \tan^4 x + 2\tan^2 x$

Solution

$$\begin{aligned}(\sec^2 x - 1)(\sec^2 x + 1) &= \sec^4 x - 1 & (a-b)(a+b) &= a^2 - b^2 \quad a = \sec^2 x \\ &= (\sec^2 x)^2 - 1 \\ &= (1 + \tan^2 x)^2 - 1 \\ &= 1 + 2\tan^2 x + \tan^4 x - 1 \\ &= \tan^4 x + 2\tan^2 x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\csc x}{\cot x} - \frac{\cot x}{\csc x} = \frac{\sin x}{\cot x}$

Solution

$$\begin{aligned}\frac{\csc x}{\cot x} - \frac{\cot x}{\csc x} &= \frac{\csc^2 x - \cot^2 x}{\cot x \csc x} \\ &= \frac{\csc^2 x - (\csc^2 x - 1)}{\cot x \csc x} \\ &= \frac{\csc^2 x - \csc^2 x + 1}{\cot x \csc x} \\ &= \frac{1}{\cot x \frac{1}{\sin x}} \\ &= \frac{\sin x}{\cot x} \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{1 - \cos^2 x}{1 + \cos x} = \frac{\sec x - 1}{\sec x}$

Solution

$$\begin{aligned}\frac{1 - \cos^2 x}{1 + \cos x} &= \frac{(1 - \cos x)(1 + \cos x)}{1 + \cos x} \\&= 1 - \cos x \\&= 1 - \frac{1}{\sec x} \\&= \frac{\sec x - 1}{\sec x} \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\cos x}{1 + \cos x} = \frac{\sec x - 1}{\tan^2 x}$

Solution

$$\begin{aligned}\frac{\cos x}{1 + \cos x} &= \frac{\cos x}{1 + \cos x} \frac{1 - \cos x}{1 - \cos x} \\&= \frac{\cos x - \cos^2 x}{\cos^2 x - 1} \\&= \frac{\cos x - \cos^2 x}{\sin^2 x} \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x}} \\&= \frac{\frac{1}{\cos x} - 1}{\frac{\sin^2 x}{\cos^2 x}} \\&= \frac{\sec x - 1}{\tan^2 x} \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{1 - 2\sin^2 x}{1 + 2\sin x \cos x} = \frac{\cos x - \sin x}{\cos x + \sin x}$

Solution

$$\begin{aligned}\frac{1 - 2\sin^2 x}{1 + 2\sin x \cos x} &= \frac{\cos^2 x + \sin^2 x - 2\sin^2 x}{\cos^2 x + \sin^2 x + 2\sin x \cos x} \\&= \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2}\end{aligned}$$

$$\begin{aligned}
&= \frac{(\cos x - \sin x)(\cos x + \sin x)}{(\cos x + \sin x)^2} \\
&= \frac{\cos x - \sin x}{\cos x + \sin x} \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $(\cos x - \sin x)^2 + (\cos x + \sin x)^2 = 2$

Solution

$$\begin{aligned}
(\cos x - \sin x)^2 + (\cos x + \sin x)^2 &= \cos^2 x - 2\sin x \cos x + \sin^2 x + \cos^2 x + 2\sin x \cos x + \sin^2 x \\
&= \cos^2 x + \sin^2 x + \cos^2 x + \sin^2 x \\
&= 1 + 1 \\
&= 2 \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x} = 2 \csc x$

Solution

$$\begin{aligned}
\frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x} &= \frac{\sin x \sin x + (1 + \cos x)(1 + \cos x)}{(1 + \cos x) \sin x} \\
&= \frac{\sin^2 x + 1 + 2\cos x + \cos^2 x}{(1 + \cos x) \sin x} \\
&= \frac{1 + 1 + 2\cos x}{(1 + \cos x) \sin x} \\
&= \frac{2 + 2\cos x}{(1 + \cos x) \sin x} \\
&= \frac{2(1 + \cos x)}{(1 + \cos x) \sin x} \\
&= \frac{2}{\sin x} \\
&= 2 \csc x \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\sin x + \tan x}{\cot x + \csc x} = \sin x \tan x$

Solution

$$\begin{aligned}\frac{\sin x + \tan x}{\cot x + \csc x} &= \frac{\sin x + \tan x}{\frac{1}{\tan x} + \frac{1}{\sin x}} \\&= \frac{\sin x + \tan x}{\frac{\sin x + \tan x}{\tan x \sin x}} \\&= (\sin x + \tan x) \frac{\tan x \sin x}{\sin x + \tan x} \\&= \tan x \sin x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\csc^2 x \sec^2 x = \sec^2 x + \csc^2 x$

Solution

$$\begin{aligned}\csc^2 x \sec^2 x &= \frac{1}{\sin^2 x} \frac{1}{\cos^2 x} \\&= \frac{1}{\sin^2 x \cos^2 x} \\&= \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} \\&= \frac{\sin^2 x}{\sin^2 x \cos^2 x} + \frac{\cos^2 x}{\sin^2 x \cos^2 x} \\&= \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \\&= \sec^2 x + \csc^2 x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cos^2 x + 1 = 2\cos^2 x + \sin^2 x$

Solution

$$\begin{aligned}\cos^2 x + 1 &= \cos^2 x + \cos^2 x + \sin^2 x \\&= 2\cos^2 x + \sin^2 x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $1 - \frac{\cos^2 x}{1 + \sin x} = \sin x$

Solution

$$\begin{aligned} 1 - \frac{\cos^2 x}{1 + \sin x} &= 1 - \frac{1 - \sin^2 x}{1 + \sin x} \\ &= 1 - \frac{(1 - \sin x)(1 + \sin x)}{1 + \sin x} \\ &= 1 - (1 - \sin x) \\ &= 1 - 1 + \sin x \\ &= \sin x \quad \checkmark \end{aligned}$$

Exercise

Prove the following equation is an identity: $\cot^2 x = (\csc x - 1)(\csc x + 1)$

Solution

$$\begin{aligned} \cot^2 x &= \csc^2 x - 1 \\ &= (\csc x - 1)(\csc x + 1) \quad \checkmark \end{aligned}$$

Exercise

Prove the following equation is an identity: $10\csc^2 x - 6\cot^2 x = 4\csc^2 x + 6$

Solution

$$\begin{aligned} 10\csc^2 x - 6\cot^2 x &= 10\csc^2 x - 6(\csc^2 x - 1) \\ &= 10\csc^2 x - 6\csc^2 x + 6 \\ &= 4\csc^2 x + 6 \quad \checkmark \end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\csc x + \cot x}{\tan x + \sin x} = \csc x \cot x$

Solution

$$\frac{\csc x + \cot x}{\tan x + \sin x} = \frac{\csc x + \cot x}{\frac{1}{\cot x} + \frac{1}{\csc x}}$$

$$\begin{aligned}
&= \frac{\csc x + \cot x}{\csc x + \cot x} \cdot \frac{\cot x \csc x}{\cot x \csc x} \\
&= \csc x + \cot x \cdot \frac{\cot x \csc x}{\csc x + \cot x} \\
&= \cot x \csc x \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{1 - \sec x}{\tan x} + \frac{\tan x}{1 - \sec x} = -2 \csc x$

Solution

$$\begin{aligned}
\frac{1 - \sec x}{\tan x} + \frac{\tan x}{1 - \sec x} &= \frac{(1 - \sec x)(1 - \sec x) + \tan^2 x}{\tan x(1 - \sec x)} \\
&= \frac{(1 - \sec x)^2 + \sec^2 x - 1}{\tan x(1 - \sec x)} \\
&= \frac{(1 - \sec x)^2 + (\sec x + 1)(\sec x - 1)}{\tan x(1 - \sec x)} \\
&= \frac{(1 - \sec x)^2 - (\sec x + 1)(1 - \sec x)}{\tan x(1 - \sec x)} \\
&= \frac{(1 - \sec x)[(1 - \sec x) - (\sec x + 1)]}{\tan x(1 - \sec x)} \\
&= \frac{1 - \sec x - \sec x - 1}{\tan x} \\
&= \frac{-2 \sec x}{\tan x} \\
&= -2 \frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}} \\
&= -2 \frac{1}{\sin x} \\
&= -2 \csc x \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\csc x - \sin x = \cos x \cot x$

Solution

$$\csc x - \sin x = \frac{1}{\sin x} - \sin x$$

$$\begin{aligned}
&= \frac{1 - \sin^2 x}{\sin x} \\
&= \frac{\cos^2 x}{\sin x} \\
&= \cos x \frac{\cos x}{\sin x} \\
&= \cos x \cot x \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\tan x + \sec x}{\sec x} - \frac{\tan x + \sec x}{\tan x} = -\cos x \cot x$

Solution

$$\begin{aligned}
\frac{\tan x + \sec x}{\sec x} - \frac{\tan x + \sec x}{\tan x} &= \frac{(\tan x + \sec x)\tan x - \sec x(\tan x + \sec x)}{\sec x \tan x} \\
&= \frac{\tan^2 x + \sec x \tan x - \sec x \tan x - \sec^2 x}{\sec x \tan x} \\
&= \frac{\tan^2 x - \sec^2 x}{\sec x \tan x} & 1 + \tan^2 \alpha = \sec^2 \alpha \\
&= \frac{-1}{\sec x \tan x} \\
&= -\frac{1}{\sec x} \frac{1}{\tan x} \\
&= -\cos x \cot x \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cot^3 x = \cot x (\csc^2 x - 1)$

Solution

$$\begin{aligned}
\cot^3 x &= \cot x \cot^2 x \\
&= \cot x (\csc^2 x - 1) \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\cot^2 x}{\csc x - 1} = \frac{1 + \sin x}{\sin x}$

Solution

$$\begin{aligned}
\frac{\cot^2 x}{\csc x - 1} &= \frac{\csc^2 x - 1}{\csc x - 1} \\
&= \frac{(\csc x - 1)(\csc x + 1)}{\csc x - 1}
\end{aligned}$$

$$\begin{aligned}
 &= \csc x + 1 \\
 &= \frac{1}{\sin x} + 1 \\
 &= \frac{1 + \sin x}{\sin x} \quad \checkmark
 \end{aligned}$$

Exercise

Prove the following equation is an identity: $\cot^2 x + \csc^2 x = 2\csc^2 x - 1$

Solution

$$\begin{aligned}
 \cot^2 x + \csc^2 x &= \csc^2 x - 1 + \csc^2 x \\
 &= 2\csc^2 x - 1 \quad \checkmark
 \end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\cot^2 x}{1 + \csc x} = \csc x - 1$

Solution

$$\begin{aligned}
 \frac{\cot^2 x}{1 + \csc x} &= \frac{\csc^2 x - 1}{1 + \csc x} \\
 &= \frac{(\csc x - 1)(\csc x + 1)}{1 + \csc x} \\
 &= \csc x - 1 \quad \checkmark
 \end{aligned}$$

Exercise

Prove the following equation is an identity: $\sec^4 x - \tan^4 x = \sec^2 x + \tan^2 x$

Solution

$$\begin{aligned}
 \sec^4 x - \tan^4 x &= (\sec^2 x + \tan^2 x)(\sec^2 x - \tan^2 x) & a^2 - b^2 &= (a - b)(a + b) \\
 &= (\sec^2 x + \tan^2 x)(1) \\
 &= \sec^2 x + \tan^2 x \quad \checkmark
 \end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\cos x}{1 + \sin x} + \frac{1 + \sin x}{\cos x} = 2 \sec x$

Solution

$$\begin{aligned}\frac{\cos x}{1 + \sin x} + \frac{1 + \sin x}{\cos x} &= \frac{\cos^2 x + (1 + \sin x)^2}{(1 + \sin x)\cos x} \\&= \frac{\cos^2 x + 1 + 2\sin x + \sin^2 x}{(1 + \sin x)\cos x} \\&= \frac{2 + 2\sin x}{(1 + \sin x)\cos x} \\&= \frac{2(1 + \sin x)}{(1 + \sin x)\cos x} \\&= \frac{2}{\cos x} \\&= 2\sec x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\sin x + \cos x}{\sin x - \cos x} = \frac{1 + 2\sin x \cos x}{2\sin^2 x - 1}$

Solution

$$\begin{aligned}\frac{\sin x + \cos x}{\sin x - \cos x} &= \frac{\sin x + \cos x}{\sin x - \cos x} \cdot \frac{\sin x + \cos x}{\sin x + \cos x} \\&= \frac{\sin^2 x + 2\sin x \cos x + \cos^2 x}{\sin^2 x - \cos^2 x} \\&= \frac{1 + 2\sin x \cos x}{\sin^2 x - (1 - \sin^2 x)} \\&= \frac{1 + 2\sin x \cos x}{\sin^2 x - 1 + \sin^2 x} \\&= \frac{1 + 2\sin x \cos x}{2\sin^2 x - 1} \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\csc x - 1}{\csc x + 1} = \frac{\cot^2 x}{\csc^2 x + 2 \csc x + 1}$

Solution

$$\begin{aligned}\frac{\csc x - 1}{\csc x + 1} &= \frac{\csc x - 1}{\csc x + 1} \frac{\csc x + 1}{\csc x + 1} \\ &= \frac{\csc^2 x - 1}{\csc^2 x + 2 \csc x + 1} \\ &= \frac{\cot^2 x}{\csc^2 x + 2 \csc x + 1} \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\csc^4 x - \cot^4 x = \csc^2 x + \cot^2 x$

Solution

$$\begin{aligned}\csc^4 x - \cot^4 x &= (\csc^2 x + \cot^2 x)(\csc^2 x - \cot^2 x) \\ &= (\csc^2 x + \cot^2 x)(1) \\ &= \csc^2 x + \cot^2 x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\tan\left(\frac{\pi}{4} + x\right) = \cot\left(\frac{\pi}{4} - x\right)$

Solution

$$\begin{aligned}\tan\left(\frac{\pi}{4} + x\right) &= \cot\left[\frac{\pi}{2} - \left(\frac{\pi}{4} + x\right)\right] \\ &= \cot\left(\frac{\pi}{2} - \frac{\pi}{4} - x\right) \\ &= \cot\left(\frac{\pi}{4} - x\right) \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\sin \theta}{1 + \sin \theta} - \frac{\sin \theta}{1 - \sin \theta} = -2 \tan^2 \theta$

Solution

$$\frac{\sin \theta}{1 + \sin \theta} - \frac{\sin \theta}{1 - \sin \theta} = \sin \theta \left[\frac{1 - \sin \theta - (1 + \sin \theta)}{(1 + \sin \theta)(1 - \sin \theta)} \right]$$

$$\begin{aligned}
&= \sin \theta \left[\frac{1 - \sin \theta - 1 - \sin \theta}{1 - \sin^2 \theta} \right] \\
&= \sin \theta \left(\frac{-2 \sin \theta}{\cos^2 \theta} \right) \\
&= -2 \frac{\sin^2 \theta}{\cos^2 \theta} \\
&= -2 \tan^2 \theta \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\csc^2 x - \cos^2 x \csc^2 x = 1$

Solution

$$\begin{aligned}
\csc^2 x - \cos^2 x \csc^2 x &= \csc^2 x (1 - \cos^2 x) \\
&= \frac{1}{\sin^2 x} (\sin^2 x) \\
&= 1 \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $1 - 2 \sin^2 x = 2 \cos^2 x - 1$

Solution

$$\begin{aligned}
1 - 2 \sin^2 x &= 1 - 2(1 - \cos^2 x) \\
&= 1 - 2 + 2 \cos^2 x \\
&= 2 \cos^2 x - 1 \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\csc^2 x - \cos x \sec x = \cot^2 x$

Solution

$$\begin{aligned}
\csc^2 x - \cos x \sec x &= \frac{1}{\sin^2 x} - \cos x \frac{1}{\cos x} \\
&= \frac{1}{\sin^2 x} - 1
\end{aligned}$$

$$\begin{aligned}
&= \frac{1 - \sin^2 x}{\sin^2 x} \\
&= \frac{\cos^2 x}{\sin^2 x} \\
&= \cot^2 x \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $(\sec x - \tan x)(\sec x + \tan x) = 1$

Solution

$$\begin{aligned}
(\sec x - \tan x)(\sec x + \tan x) &= \sec^2 x - \tan^2 x \\
&= 1 + \tan^2 x - \tan^2 x \\
&= 1 \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $(1 + \tan^2 x)(1 - \sin^2 x) = 1$

Solution

$$\begin{aligned}
(1 + \tan^2 x)(1 - \sin^2 x) &= \sec^2 x \cos^2 x \\
&= \frac{1}{\cos^2 x} \cos^2 x \\
&= 1 \quad \checkmark
\end{aligned}$$