

## ***Solution***      **Section 2.2 – Differentiation Rules**

### ***Exercise***

Find the derivative of  $y = \frac{1}{x^3}$

### **Solution**

$$y = x^{-3}$$

$$y' = -3x^{-3-1}$$

$$\underline{= -3x^{-4}} \quad \text{or} \quad -\frac{3}{x^4}$$

### ***Exercise***

Find the derivative of  $D_x \left( x^{4/3} \right)$

### **Solution**

$$\underline{D_x \left( x^{4/3} \right) = \frac{4}{3} x^{1/3}}$$

### ***Exercise***

Find the derivative of  $y = \sqrt{z}$

### **Solution**

$$\frac{dy}{dz} = \frac{d}{dz} \left( z^{1/2} \right)$$

$$= \frac{1}{2} z^{1/2-1}$$

$$\underline{= \frac{1}{2} z^{-1/2}} \quad \frac{1}{2z^{1/2}} \quad \frac{1}{2\sqrt{z}}$$

### ***Exercise***

Find the derivative of  $D_t (-8t)$

### **Solution**

$$\underline{D_t (-8t) = -8}$$

### ***Exercise***

Find the derivative of  $y = \frac{9}{4x^2}$

#### **Solution**

$$y = \frac{9}{4}x^{-2}$$

$$y' = \frac{9}{4}(-2)x^{-3}$$

$$= -\frac{9}{2x^3}$$

### ***Exercise***

Find the derivative of  $y = 6x^3 + 15x^2$

#### **Solution**

$$y' = 18x^2 + 30x$$

### ***Exercise***

Find the first derivative of  $y = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$

#### **Solution**

$$y' = 3(4)x^3 - 6(3)x^2 + \frac{2}{8}x + 0$$

$$= 12x^3 - 18x^2 + \frac{1}{4}x$$

### ***Exercise***

Find the derivative of  $p(t) = 12t^4 - 6\sqrt{t} + \frac{5}{t}$

#### **Solution**

$$p(t) = 12t^4 - 6t^{1/2} + 5t^{-1}$$

$$p' = 48t^3 - 3t^{-1/2} - 5t^{-2}$$

$$= 48t^3 - \frac{3}{t^{1/2}} - \frac{5}{t^2}$$

### Exercise

Find the derivative of  $f(x) = \frac{x^3 + 3\sqrt{x}}{x}$

#### Solution

$$\begin{aligned} f(x) &= \frac{x^3}{x} + 3\frac{x^{1/2}}{x} \\ &= x^2 + 3x^{-1/2} \end{aligned}$$

$$\begin{aligned} f'(x) &= 2x - \frac{3}{2}x^{-3/2} \\ &= 2x - \frac{3}{2x^{3/2}} \\ &= 2x - \frac{3}{2\sqrt{x^3}} \end{aligned}$$

### Exercise

Find the derivative of  $y = \frac{x^3 - 4x}{\sqrt{x}}$

#### Solution

$$y = \frac{x^3}{x^{1/2}} - 4\frac{x}{x^{1/2}} = x^{5/2} - 4x^{1/2}$$

$$\begin{aligned} y' &= \frac{5}{2}x^{3/2} - 4\frac{1}{2}x^{-1/2} \\ &= \frac{5}{2}x\sqrt{x} - 2\frac{2}{\sqrt{x}} \end{aligned}$$

### Exercise

Find the derivative of  $f(x) = (4x^2 - 3x)^2$

#### Solution

$$f(x) = (4x^2 - 3x)^2$$

$$= 16x^4 - 24x^3 + 9x^2$$

$$f'(x) = 64x^3 - 72x^2 + 18x$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

### Exercise

Find the derivative of  $y = 3x(2x^2 + 5x)$

#### Solution

$$y = 6x^3 + 15x^2$$

$$\underline{y' = 18x^2 + 30x}$$

### ***Exercise***

Find the derivative of  $y = 3(2x^2 + 5x)$

### **Solution**

$$y = 6x^2 + 15x$$

$$\underline{y' = 12x + 15}$$

### ***Exercise***

Find the derivative of  $y = \frac{x^2 + 4x}{5}$

### **Solution**

$$\underline{y' = \frac{1}{5}(2x + 4)}$$

### ***Exercise***

Find the derivative of  $y = \frac{3x^4}{5}$

### **Solution**

$$\underline{y' = \frac{12}{5}x^3}$$

### ***Exercise***

Find the derivative of  $g(s) = \frac{s^2 - 2s + 5}{\sqrt{s}}$

### **Solution**

$$g(s) = \frac{s^2}{s^{1/2}} - 2\frac{s}{s^{1/2}} + \frac{5}{s^{1/2}}$$

$$= s^{3/2} - 2s^{1/2} + 5s^{-1/2}$$

$$g'(s) = \frac{3}{2}s^{1/2} - 2\frac{1}{2}s^{-1/2} + 5\left(-\frac{1}{2}\right)s^{-3/2}$$

$$= \frac{3}{2}s^{1/2} - s^{-1/2} - \frac{5}{2}s^{-3/2}$$

$$= \frac{3}{2}\sqrt{s} - \frac{1}{\sqrt{s}} - \frac{5}{2s^{3/2}}$$

$$= \frac{3}{2}\sqrt{s} - \frac{1}{\sqrt{s}} - \frac{5}{2s\sqrt{s}} \quad \Bigg|$$

### Exercise

Find the derivative of  $f(x) = \frac{x+1}{\sqrt{x}}$

### Solution

$$f(x) = \frac{x}{x^{1/2}} + \frac{1}{x^{1/2}}$$

$$= x^{1/2} + x^{-1/2}$$

$$f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}$$

$$= \frac{1}{2x^{1/2}} - \frac{1}{2x^{3/2}} \quad \Bigg|$$

### Exercise

Find the derivative of  $f(x) = 4x^{5/3} + 6x^{-3/2} - 11x$

### Solution

$$f'(x) = \frac{20}{3}x^{2/3} - 9x^{-5/2} - 11 \quad \Bigg|$$

### Exercise

Find the derivative of  $f(x) = \frac{2}{3}x^3 + \pi x^2 + 7x + 1$

### Solution

$$f'(x) = 2x^2 + 2\pi x + 7 \quad \Bigg|$$

### Exercise

Find the derivative of  $f(x) = \frac{x^5 - x^3}{15}$

### Solution

$$f(x) = \frac{1}{15}x^5 - \frac{1}{15}x^3$$

$$f'(x) = \frac{1}{3}x^4 - \frac{1}{5}x^2 \quad \Bigg|$$

**Exercise**

Find the derivative of  $f(x) = x^{1/3} + 2x^{1/4} - 3x^{1/5}$

**Solution**

$$\underline{f'(x) = \frac{1}{3}x^{-2/3} + \frac{1}{2}x^{-3/4} - \frac{3}{5}x^{-4/5} \quad |}$$

**Exercise**

Find the derivative of  $f(t) = 3\sqrt[3]{t^2} - \frac{2}{\sqrt{t^3}}$

**Solution**

$$f(t) = 3t^{2/3} - 2t^{-1/3}$$

$$\underline{f'(t) = 2t^{-1/3} + \frac{2}{3}2t^{-4/3} \quad |}$$

**Exercise**

Find the derivative of  $f(t) = \sqrt{t}\left(5 - t - \frac{1}{3}t^2\right)$

**Solution**

$$f(t) = 5t^{1/2} - t^{3/2} - \frac{1}{3}t^{5/2}$$

$$\underline{f'(t) = \frac{5}{2}t^{-1/2} - \frac{3}{2}t^{1/2} - \frac{5}{6}t^{3/2} \quad |}$$

**Exercise**

Find the derivative of  $f(x) = \frac{3}{5}x^{5/3} + \frac{5}{3}x^{-3/5}$

**Solution**

$$\underline{f'(x) = x^{2/3} - x^{-8/5} \quad |}$$

**Exercise**

Find the derivative of  $f(x) = x^{23} - x^{-23}$

**Solution**

$$\underline{f'(x) = 23x^{22} + 23x^{-24} \quad |}$$

### ***Exercise***

Find the **first** and **second** derivatives  $y = -x^3 + 3$

#### **Solution**

$$\underline{y' = -3x^2}$$

$$\underline{y'' = -6x}$$

### ***Exercise***

Find the **first** and **second** derivatives  $y = 3x^7 - 7x^3 + 21x^2$

#### **Solution**

$$\underline{y' = 21x^6 - 21x^2 + 42x}$$

$$\underline{y'' = 126x^5 - 42x + 42}$$

### ***Exercise***

Find the **first** and **second** derivatives  $y = 6x^2 - 10x - \frac{1}{x}$

#### **Solution**

$$\underline{y' = 12x - 10 + \frac{1}{x^2}}$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$y'' = 12 + \frac{-2x}{x^4}$$

$$\underline{= 12 - \frac{2}{x^3}}$$

### ***Exercise***

Find the **first** and **second** derivatives  $f(x) = \frac{1}{2}x^4 + \pi x^3 - 7x + 1$

#### **Solution**

$$\underline{f'(x) = 2x^3 + 3\pi x^2 - 7}$$

$$\underline{f''(x) = 6x^2 + 6\pi x}$$

### Exercise

Find the *first* and *second* derivatives  $y = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$

#### Solution

$$\underline{y' = 12x^3 - 18x^2 + \frac{x}{4}}$$

$$\underline{y'' = 36x^2 - 36x + \frac{1}{4}}$$

### Exercise

Find the *first* and *second* derivatives  $y = (2x - 3)(1 - 5x)$

#### Solution

$$y = -10x^2 + 17x - 3$$

$$\underline{y' = -20x + 17}$$

$$\underline{y'' = -20}$$

### Exercise

Find the derivative  $f(x) = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$ ,  $f^{(4)}(x)$

#### Solution

$$\underline{f^{(4)}(x) = 3(4!) = 72}$$

### Exercise

Find the derivative  $f(x) = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$ ,  $f^{(5)}(x)$

#### Solution

$$\underline{f^{(5)}(x) = 0}$$

### Exercise

Find the derivative  $f(x) = 2x^6 + 4x^4 - x + 2$ ,  $f^{(6)}(x)$

#### Solution

$$f^{(6)}(x) = 2(6!)$$



$$\underline{= 1,440}$$

### Exercise

Find the derivative  $f(x) = 4x^5 + 4x^4 + x^2 - 2$ ,  $f^{(5)}(x)$

### Solution

$$f^{(5)}(x) = 4(5!) \\ \underline{= 480}$$

### Exercise

Find the derivative  $f(x) = 4x^5 + 4x^4 + x^2 - 2$ ,  $f^{(6)}(x)$

### Solution

$$f^{(6)}(x) = \underline{0}$$

### Exercise

Find the derivative  $f(x) = 4x^4 - 2x^3 + x + 2$ ,  $f^{(4)}(x)$

### Solution

$$f^{(4)}(x) = 4(4!) \\ \underline{= 96}$$

### Exercise

Find an equation for the line perpendicular to the tangent to the curve  $y = x^3 - 4x + 1$  at the point (2, 1).

### Solution

$$y' = 3x^2 - 4$$

$$m = y'|_{x=2} = 3(2)^2 - 4 = 8$$

$$\underline{m_1 = -\frac{1}{8}}$$

$$y = -\frac{1}{8}(x - 2) + 1$$

$$\underline{y = -\frac{1}{8}x - \frac{3}{4}}$$

$$y = m(x - x_1) + y_1$$

### Exercise

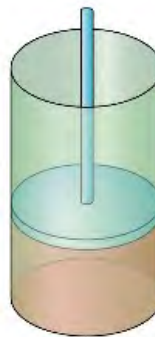
If gas in a cylinder is maintained at a constant temperature  $T$ , the pressure  $P$  is related to the volume  $V$  by a formula of the form

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2}$$

In which  $a$ ,  $b$ ,  $n$ , and  $R$  are constants. Find  $\frac{dP}{dV}$

### Solution

$$\begin{aligned}\frac{dP}{dV} &= \frac{d}{dV} \left( \frac{nRT}{V - nb} \right) - \frac{d}{dV} \left( \frac{an^2}{V^2} \right) \\ &= -nRT \frac{(V - nb)'}{(V - nb)^2} - an^2 \left( -\frac{2V}{V^4} \right) \\ &= -nRT \frac{1}{(V - nb)^2} + an^2 \left( \frac{2}{V^3} \right) \\ &= \underline{-\frac{nRT}{(V - nb)^2} + \frac{2an^2}{V^3}}\end{aligned}$$



### Exercise

Show that if  $(a, f(a))$  is any point on the graph of  $f(x) = x^2$ , then the slope of the tangent line at that point is  $m = 2a$

### Solution

$$\begin{aligned}m = f'(a) &= \lim_{x \rightarrow a} \frac{x^2 - a^2}{x - a} \\ &= \lim_{x \rightarrow a} \frac{(x - a)(x + a)}{x - a} \\ &= \lim_{x \rightarrow a} (x + a) \\ &= \underline{2a}\end{aligned}$$

### Exercise

Show that if  $(a, f(a))$  is any point on the graph of  $f(x) = bx^2 + cx + d$ , then the slope of the tangent line at that point is  $m = 2ab + c$

### Solution

$$\begin{aligned}
m = f'(a) &= \lim_{h \rightarrow 0} \frac{b(a+h)^2 + c(a+h) + d - ba^2 - ca - d}{h} \\
&= \lim_{h \rightarrow 0} \frac{ba^2 + 2abh + bh^2 + ch - ba^2}{h} \\
&= \lim_{h \rightarrow 0} \frac{2abh + bh^2 + ch}{h} \\
&= \lim_{h \rightarrow 0} (2ab + bh + c) \\
&= \underline{2ab + c}
\end{aligned}$$

### Exercise

Let  $f(x) = x^2$

- Show that  $\frac{f(x) - f(y)}{x - y} = f'\left(\frac{x+y}{2}\right)$ , for all  $x \neq y$
- Is this property true for  $f(x) = ax^2$ , where  $a$  is a nonzero real number?
- Give a geometrical interpretation of this property.
- Is this property true for  $f(x) = ax^3$ ?

### Solution

**a)**  $f'(x) = 2x$

$$\begin{aligned}
\frac{f(x) - f(y)}{x - y} &= \frac{x^2 - y^2}{x - y} \\
&= \frac{(x - y)(x + y)}{x - y} \\
&= \underline{x + y}
\end{aligned}$$

$$\begin{aligned}
f'\left(\frac{x+y}{2}\right) &= 2\left(\frac{x+y}{2}\right) \\
&= \underline{x + y}
\end{aligned}$$

$$\frac{f(x) - f(y)}{x - y} = f'\left(\frac{x+y}{2}\right), \text{ for all } x \neq y$$

**b)**  $f(x) = ax^2 \rightarrow f'(x) = 2ax$

$$\begin{aligned}
f'\left(\frac{x+y}{2}\right) &= 2a\left(\frac{x+y}{2}\right) \\
&= \underline{a(x+y)}
\end{aligned}$$

$$\begin{aligned}\frac{f(x) - f(y)}{x - y} &= \frac{ax^2 - ay^2}{x - y} \\ &= \frac{a(x - y)(x + y)}{x - y} \\ &= a(x + y) \quad | \end{aligned}$$

$$\frac{f(x) - f(y)}{x - y} = f'\left(\frac{x + y}{2}\right), \text{ for all } x \neq y$$

**c)** Line thru  $(x, f(x))$  and  $(y, f(y))$  is parallel to the tangent line and midpoint is between  $x$  and  $y$ .

**d)**  $f(x) = ax^3 \rightarrow f'(x) = 3ax^2$

$$\begin{aligned}f'\left(\frac{x + y}{2}\right) &= 3a\left(\frac{x + y}{2}\right)^2 \\ &= \frac{3}{4}a(x + y)^2 \quad | \end{aligned}$$

$$\begin{aligned}\frac{f(x) - f(y)}{x - y} &= \frac{ax^3 - ay^3}{x - y} \\ &= \frac{a(x - y)(x^2 + xy + y^2)}{x - y} \\ &= a(x^2 + xy + y^2) \quad | \end{aligned}$$

$$x^2 + xy + y^2 \neq (x + y)^2$$

$$\frac{f(x) - f(y)}{x - y} \neq f'\left(\frac{x + y}{2}\right) \quad (\text{No})$$