

Solution **Section 2.3 – Trigonometric Substitutions**

Exercise

Evaluate the integral $\int \frac{3dx}{\sqrt{1+9x^2}}$

Solution

$$3x = \tan t \Rightarrow dx = \frac{1}{3} \sec^2 t \, dt$$

$$\sqrt{1+9x^2} = 3 \sec t$$

$$\begin{aligned} \int \frac{3dx}{\sqrt{1+9x^2}} &= \frac{1}{3} \int \frac{\sec^2 t}{3 \sec t} dt \\ &= \int \sec t \, dt \\ &= \int \sec t \frac{\sec t + \tan t}{\sec t + \tan t} dt \\ &= \int \frac{\sec^2 t + \sec t \tan t}{\sec t + \tan t} dt \\ &= \int \frac{1}{\sec t + \tan t} d(\sec t + \tan t) \\ &= \ln |\sec t + \tan t| + C \\ &= \ln \left| \sqrt{1+u^2} + u \right| + C \\ &= \ln \left| \sqrt{1+9x^2} + 3x \right| + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{x^2}{4+x^2} dx$

Solution

$$x = 2 \tan \theta \quad dx = 2 \sec^2 \theta d\theta$$

$$4+x^2 = 4 \sec^2 \theta$$

$$\begin{aligned} \int \frac{x^2}{4+x^2} dx &= \int \frac{4 \tan^2 \theta}{4 \sec^2 \theta} 2 \sec^2 \theta d\theta \\ &= 2 \int \tan^2 \theta d\theta \end{aligned}$$

$$\begin{aligned}
&= 2 \int (\sec^2 \theta - 1) d\theta & \int \sec^2 \theta d\theta = \tan \theta \\
&= 2(\tan \theta - \theta) + C \\
&= 2\left(\frac{x}{2} - \tan^{-1}\left(\frac{x}{2}\right)\right) + C \\
&= x - 2 \tan^{-1}\left(\frac{x}{2}\right) + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{x^2 \sqrt{x^2 + 1}}$

Solution

$$x = \tan \theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$dx = \sec^2 \theta d\theta$$

$$\sqrt{x^2 + 1} = \sqrt{\tan^2 \theta + 1} = \sec \theta$$

$$\begin{aligned}
\int \frac{dx}{x^2 \sqrt{x^2 + 1}} &= \int \frac{\sec^2 \theta d\theta}{\tan^2 \theta \sec \theta} \\
&= \int \frac{\sec \theta d\theta}{\tan^2 \theta} \\
&= \int \frac{\cos^2 \theta d\theta}{\sin^2 \theta \cos \theta} \\
&= \int \frac{\cos \theta d\theta}{\sin^2 \theta} \\
&= \int \sin^{-2} \theta d(\sin \theta)
\end{aligned}$$

$$= -\frac{1}{\sin \theta} + C$$

$$= -\frac{\sec \theta}{\tan \theta} + C$$

$$= -\frac{\sqrt{x^2 + 1}}{x} + C$$

Exercise

Evaluate: $\int \frac{dx}{\sqrt{x^2 + 4}}$

Solution

Let: $x = 2 \tan \theta \rightarrow dx = 2 \sec^2 \theta d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\sqrt{x^2 + 4} = 2|\sec \theta|$$

$$\begin{aligned} \int \frac{dx}{\sqrt{x^2 + 4}} &= \int \frac{2 \sec^2 \theta d\theta}{\sqrt{4 \sec^2 \theta}} \\ &= \int \frac{2 \sec^2 \theta d\theta}{2|\sec \theta|} \\ &= \int \sec \theta d\theta \\ &= \int \sec \theta \frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta} d\theta \\ &= \int \frac{\sec^2 \theta + \sec \theta \tan \theta}{\sec \theta + \tan \theta} d\theta \\ &= \int \frac{1}{\sec \theta + \tan \theta} d(\sec \theta + \tan \theta) \\ &= \ln |\sec \theta + \tan \theta| + C \\ &= \ln \left| \frac{\sqrt{x^2 + 4}}{2} + \frac{x}{2} \right| + C \end{aligned}$$

Exercise

Evaluate $\int \frac{dx}{(1 + x^2)^2}$

Solution

$$x = \tan \theta \quad 1 + x^2 = (\sec^2 \theta)^2$$

$$dx = \sec^2 \theta d\theta$$

$$\int \frac{dx}{(1 + x^2)^2} = \int \frac{\sec^2 \theta}{\sec^4 \theta} d\theta$$

$$\begin{aligned}
&= \int \frac{1}{\sec^2 \theta} d\theta \\
&= \int \cos^2 \theta d\theta \\
&= \frac{1}{2} \int (1 + \cos 2\theta) d\theta \\
&= \frac{1}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\
&= \frac{1}{2} \tan^{-1} x + \frac{1}{2} \sin \theta \cos \theta + C \\
&= \frac{1}{2} \tan^{-1} x + \frac{1}{2} \frac{x}{\sqrt{1+x^2}} \frac{1}{\sqrt{1+x^2}} + C \\
&= \frac{1}{2} \tan^{-1} x + \frac{x}{2(1+x^2)} + C
\end{aligned}$$

Exercise

Evaluate $\int \frac{dx}{\sqrt{4x^2 + 1}}$

Solution

$$\begin{aligned}
2x &= \tan \theta & \sqrt{4x^2 + 1} &= \sec \theta \\
dx &= \frac{1}{2} \sec^2 \theta d\theta \\
\int \frac{dx}{\sqrt{4x^2 + 1}} &= \frac{1}{2} \int \frac{\sec^2 \theta}{\sec \theta} d\theta \\
&= \frac{1}{2} \int \sec \theta d\theta \\
&= \int \sec \theta \frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta} d\theta \\
&= \int \frac{\sec^2 \theta + \sec \theta \tan \theta}{\sec \theta + \tan \theta} d\theta \\
&= \int \frac{1}{\sec \theta + \tan \theta} d(\sec \theta + \tan \theta) \\
&= \frac{1}{2} \ln |\sec \theta + \tan \theta| + C \\
&= \frac{1}{2} \ln \left| \sqrt{4x^2 + 1} + 2x \right| + C
\end{aligned}$$

Exercise

Evaluate $\int \frac{dx}{(x^2 + 1)^{3/2}}$

Solution

$$x = \tan \theta \quad \sqrt{x^2 + 1} = \sec \theta$$

$$dx = \sec^2 \theta \, d\theta$$

$$\int \frac{dx}{(x^2 + 1)^{3/2}} = \int \frac{\sec^2 \theta}{(\sec \theta)^3} d\theta$$

$$= \int \frac{d\theta}{\sec \theta}$$

$$= \int \cos \theta \, d\theta$$

$$= \sin \theta + C$$

$$= \frac{x}{\sqrt{x^2 + 1}} + C$$

$$\sin \theta = \frac{\tan \theta}{\sec \theta} = \frac{x}{\sqrt{x^2 + 1}}$$

Exercise

Evaluate $\int \frac{9x^3}{\sqrt{x^2 + 1}} \, dx$

Solution

$$x = \tan \theta \quad \sqrt{x^2 + 1} = \sec \theta$$

$$dx = \sec^2 \theta \, d\theta$$

$$\int \frac{9x^3}{\sqrt{x^2 + 1}} \, dx = \int \frac{9 \tan^3 \theta}{\sec \theta} (\sec^2 \theta) d\theta$$

$$= 9 \int \tan^2 \theta \tan \theta \sec \theta \, d\theta$$

$$= 9 \int (\sec^2 \theta - 1) d(\sec \theta)$$

$$= 9 \left(\frac{1}{3} \sec^3 \theta - \sec \theta \right) + C$$

$$= 3(x^2 + 1)\sqrt{x^2 + 1} - 9\sqrt{x^2 + 1} + C$$

$$= 3\sqrt{x^2+1}(x^2+1-3)+C$$

$$\underline{= 3\sqrt{x^2+1}(x^2-2)+C}$$

Exercise

Evaluate $\int \sqrt{16x^2+9} \, dx$

Solution

$$4x = 3 \tan \theta \quad \sqrt{4x^2+9} = 3 \sec \theta$$

$$4dx = 3 \sec^2 \theta \, d\theta$$

$$\int \sqrt{16x^2+9} \, dx = \int 3 \sec \theta \left(\frac{3}{4} \sec^2 \theta \right) d\theta$$

$$= \frac{9}{4} \int \sec^3 \theta \, d\theta$$

$$u = \sec x \quad dv = \sec^2 x \, dx$$

$$du = \sec x \tan x \, dx \quad v = \tan x$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \tan x (\sec x \tan x \, dx)$$

$$= \sec x \tan x - \int \tan^2 x \sec x \, dx$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$2 \int \sec^3 x \, dx = \sec x \tan x + \ln |\sec x + \tan x|$$

$$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x|$$

$$= \frac{9}{8} \sec \theta \tan \theta + \frac{9}{8} \ln |\sec \theta + \tan \theta| + C$$

$$= \frac{9}{8} \frac{\sqrt{4x^2+9}}{3} \frac{4x}{3} + \frac{9}{8} \ln \left| \frac{\sqrt{4x^2+9}}{3} + \frac{4x}{3} \right| + C$$

$$\underline{= \frac{1}{2} x \sqrt{4x^2+9} + \frac{9}{8} \ln \left| \frac{2x + \sqrt{4x^2+9}}{3} \right| + C}$$

Exercise

Evaluate $\int x\sqrt{x^2+1} \, dx$

Solution

$$\begin{aligned}\int x\sqrt{x^2+1} \, dx &= \frac{1}{2} \int (x^2+1)^{1/2} d(x^2+1) \\ &= \frac{1}{3} (x^2+1)^{3/2} + C\end{aligned}$$

OR

$$\begin{aligned}x &= \tan \theta & \sqrt{x^2+1} &= \sec \theta \\ dx &= \sec^2 \theta \, d\theta \\ \int x\sqrt{x^2+1} \, dx &= \int \tan \theta \sec^3 \theta \, d\theta \\ &= \int \sec^2 \theta \, d(\sec \theta) \\ &= \frac{1}{3} \sec^3 \theta + C \\ &= \frac{1}{3} (x^2+1)^{3/2} + C\end{aligned}$$

Exercise

Evaluate $\int \frac{\sqrt{25x^2+4}}{x^4} \, dx$

Solution

$$\begin{aligned}5x &= 2 \tan \theta & \sqrt{25x^2+4} &= 2 \sec \theta \\ 5dx &= 2 \sec^2 \theta \, d\theta \\ \int \frac{\sqrt{25x^2+4}}{x^4} \, dx &= \int \frac{2 \sec \theta}{\left(\frac{2}{5}\right)^4 \tan^4 \theta} \frac{2}{5} \sec^2 \theta \, d\theta \\ &= \frac{125}{4} \int \frac{\sec^3 \theta}{\tan^4 \theta} \, d\theta \\ &= \frac{125}{4} \int \frac{\cos \theta}{\sin^4 \theta} \, d\theta\end{aligned}$$

$$\begin{aligned}
&= \frac{125}{4} \int \sin^{-4} \theta \, d(\sin \theta) \\
&= -\frac{125}{12} \frac{1}{\sin^3 \theta} + C \\
&= -\frac{125}{12} \left(\frac{\tan \theta}{\sec \theta} \right)^3 + C \\
&= -\frac{125}{12} \left(\frac{\sqrt{25x^2 + 4}}{5x} \right)^3 + C \\
&= -\frac{(25x^2 + 4)^{3/2}}{12x^3} + C
\end{aligned}$$

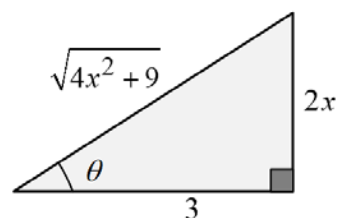
Exercise

Evaluate $\int \frac{1}{x\sqrt{4x^2 + 9}} dx$

Solution

$$\begin{aligned}
2x &= 3 \tan \theta & \sqrt{4x^2 + 9} &= 3 \sec \theta \\
dx &= \frac{3}{2} \sec^2 \theta \, d\theta
\end{aligned}$$

$$\begin{aligned}
\int \frac{1}{x\sqrt{4x^2 + 9}} dx &= \int \frac{1}{\frac{9}{2} \tan \theta \sec \theta} \left(\frac{3}{2} \sec^2 \theta \right) d\theta \\
&= \int \frac{1}{\frac{9}{2} \tan \theta \sec \theta} \left(\frac{3}{2} \sec^2 \theta \right) d\theta \\
&= \frac{1}{3} \int \frac{\sec \theta}{\tan \theta} d\theta \\
&= \frac{1}{3} \int \frac{1}{\sin \theta} d\theta \\
&= \frac{1}{3} \int \csc \theta \frac{\csc \theta + \cot \theta}{\csc \theta + \cot \theta} d\theta \\
&= \frac{1}{3} \int \frac{\csc^2 \theta + \csc \theta \cot \theta}{\csc \theta + \cot \theta} d\theta \\
&= -\frac{1}{3} \int \frac{1}{\csc \theta + \cot \theta} d(\csc \theta + \cot \theta) \\
&= -\frac{1}{3} \ln |\csc \theta + \cot \theta| + C
\end{aligned}$$



$$= -\frac{1}{3} \ln \left| \frac{\sqrt{4x^2 + 9}}{2x} + \frac{3}{2x} \right| + C$$

$$= -\frac{1}{3} \ln \left| \frac{\sqrt{4x^2 + 9} + 3}{2x} \right| + C$$

Exercise

Evaluate

$$\int \frac{1}{(x^2 + 5)^{3/2}} dx$$

Solution

$$x = \sqrt{5} \tan \theta \quad \sqrt{x^2 + 5} = \sqrt{5} \sec \theta$$

$$dx = \sqrt{5} \sec^2 \theta d\theta$$

$$\int \frac{1}{(x^2 + 5)^{3/2}} dx = \int \frac{1}{5\sqrt{5} \sec^3 \theta} (\sqrt{5} \sec^2 \theta) d\theta$$

$$= \frac{1}{5} \int \frac{1}{\sec \theta} d\theta$$

$$= \frac{1}{5} \int \cos \theta d\theta$$

$$= \frac{1}{5} \sin \theta$$

$$= \frac{1}{5} \frac{\tan \theta}{\sec \theta}$$

$$= \frac{1}{5} \frac{x}{\sqrt{5}} \frac{\sqrt{5}}{\sqrt{x^2 + 5}}$$

$$= \frac{1}{5} \frac{x}{\sqrt{x^2 + 5}} + C$$

Exercise

Evaluate the integral

$$\int \frac{x dx}{\sqrt{x^2 + 4}}$$

Solution

$$x = 2 \tan \theta \quad \sqrt{x^2 + 4} = 2 \sec \theta$$

$$dx = 2 \sec^2 \theta d\theta$$

$$\begin{aligned}
 \int \frac{x \, dx}{\sqrt{x^2 + 4}} &= \int \frac{2 \tan \theta}{2 \sec \theta} (2 \sec^2 \theta \, d\theta) \\
 &= 2 \int \tan \theta \sec \theta \, d\theta \\
 &= 2 \sec \theta \\
 &= \sqrt{x^2 + 4} + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{x^3}{\sqrt{x^2 + 4}} \, dx$

Solution

$$\begin{aligned}
 x &= 2 \tan \theta & \sqrt{x^2 + 4} &= 2 \sec \theta \\
 dx &= 2 \sec^2 \theta \, d\theta \\
 \int \frac{x^3 \, dx}{\sqrt{x^2 + 4}} &= \int \frac{8 \tan^3 \theta}{2 \sec \theta} (2 \sec^2 \theta \, d\theta) \\
 &= 8 \int \tan^2 \theta \tan \theta \sec \theta \, d\theta \\
 &= 8 \int (\sec^2 \theta - 1) d(\sec \theta) \\
 &= 8 \left(\frac{1}{3} \sec^3 \theta - \sec \theta \right) + C \\
 &= \frac{8}{3} \frac{(x^2 + 4)^{3/2}}{8} - 8 \frac{(x^2 + 4)^{1/2}}{2} + C \\
 &= \sqrt{x^2 + 4} \left(\frac{1}{3} (x^2 + 4) - 4 \right) + C \\
 &= \frac{1}{3} \sqrt{x^2 + 4} (x^2 + 4 - 12) + C \\
 &= \frac{1}{3} \sqrt{x^2 + 4} (x^2 - 8) + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{(1+4x^2)^{3/2}}$

Solution

$$x = \frac{1}{2} \tan \theta \quad \sqrt{4x^2 + 1} = \sec \theta$$

$$dx = \frac{1}{2} \sec^2 \theta \, d\theta$$

$$\begin{aligned} \int \frac{dx}{(1+4x^2)^{3/2}} &= \frac{1}{2} \int \frac{\sec^2 \theta}{(\sec^2 \theta)^{3/2}} \, d\theta \\ &= \frac{1}{2} \int \frac{\sec^2 \theta}{\sec^3 \theta} \, d\theta \\ &= \frac{1}{2} \int \frac{1}{\sec \theta} \, d\theta \\ &= \frac{1}{2} \int \cos \theta \, d\theta \\ &= \frac{1}{2} \sin \theta + C \\ &= \frac{1}{2} \frac{\tan \theta}{\sec \theta} + C \\ &= \frac{1}{2} \frac{2x}{\sqrt{4x^2 + 1}} + C \\ &= \frac{x}{\sqrt{4x^2 + 1}} + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{x^2 \sqrt{x^2 + 9}}$

Solution

$$x = 3 \tan \theta \quad \sqrt{x^2 + 9} = 3 \sec \theta$$

$$dx = 3 \sec^2 \theta \, d\theta$$

$$\begin{aligned} \int \frac{dx}{x^2 \sqrt{x^2 + 9}} &= \int \frac{3 \sec^2 \theta}{9 \tan^2 \theta (3 \sec \theta)} \, d\theta \\ &= \frac{1}{9} \int \frac{\sec \theta}{\tan^2 \theta} \, d\theta \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{9} \int \frac{1}{\cos \theta} \cdot \frac{\cos^2 \theta}{\sin^2 \theta} d\theta \\
&= \frac{1}{9} \int \frac{\cos \theta}{\sin^2 \theta} d\theta \\
&= \frac{1}{9} \int \frac{1}{\sin^2 \theta} d(\sin \theta) \\
&= -\frac{1}{9} \frac{1}{\sin \theta} + C \\
&= -\frac{1}{9} \frac{\sec \theta}{\tan \theta} + C \\
&= -\frac{1}{9} \frac{\sqrt{x^2 + 9}}{x} + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{x^2 \sqrt{x^2 + 4}}$

Solution

$$x = 2 \tan \theta \quad \sqrt{x^2 + 4} = 2 \sec \theta$$

$$dx = 2 \sec^2 \theta d\theta$$

$$\begin{aligned}
\int \frac{dx}{x^2 \sqrt{x^2 + 4}} &= \int \frac{2 \sec^2 \theta}{4 \tan^2 \theta (2 \sec \theta)} d\theta \\
&= \frac{1}{4} \int \frac{\sec \theta}{\tan^2 \theta} d\theta \\
&= \frac{1}{4} \int \frac{1}{\cos \theta} \cdot \frac{\cos^2 \theta}{\sin^2 \theta} d\theta \\
&= \frac{1}{4} \int \frac{\cos \theta}{\sin^2 \theta} d\theta \\
&= \frac{1}{4} \int \frac{1}{\sin^2 \theta} d(\sin \theta) \\
&= -\frac{1}{4} \frac{1}{\sin \theta} + C \\
&= -\frac{1}{4} \frac{\sec \theta}{\tan \theta} + C \\
&= -\frac{1}{4} \frac{\sqrt{x^2 + 4}}{x} + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{x^2 dx}{\sqrt{x^2 + 1}}$

Solution

$$x = \tan \theta \quad \sqrt{x^2 + 1} = \sec \theta$$

$$dx = \sec^2 \theta d\theta$$

$$\int \frac{x^2 dx}{\sqrt{x^2 + 1}} = \int \frac{\tan^2 \theta}{\sec \theta} \sec^2 \theta d\theta$$

$$= \int \tan^2 \theta \sec \theta d\theta$$

$$= \int (\sec^2 \theta - 1) \sec \theta d\theta$$

$$= \int \sec^3 \theta d\theta - \int \sec \theta d\theta$$

$$u = \sec \theta \quad dv = \sec^2 \theta d\theta$$

$$du = \sec \theta \tan \theta d\theta \quad v = \tan \theta$$

$$\int \sec^3 \theta d\theta = \sec \theta \tan \theta - \int \tan \theta (\sec \theta \tan \theta d\theta)$$

$$= \sec \theta \tan \theta - \int \tan^2 \theta \sec \theta d\theta$$

$$= \sec \theta \tan \theta - \int (\sec^2 \theta - 1) \sec \theta d\theta$$

$$= \sec \theta \tan \theta - \int \sec^3 \theta d\theta + \int \sec \theta d\theta$$

$$2 \int \sec^3 \theta d\theta = \sec \theta \tan \theta + \int \sec \theta d\theta$$

$$\int \sec^3 \theta d\theta = \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \int \sec \theta d\theta$$

$$\int \sec \theta d\theta = \int \sec \theta \frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta} d\theta$$

$$= \int \frac{\sec^2 \theta + \sec \theta \tan \theta}{\sec \theta + \tan \theta} d\theta$$

$$= \int \frac{1}{\sec \theta + \tan \theta} d(\sec \theta + \tan \theta)$$

$$= \ln |\sec \theta + \tan \theta| + C$$

$$\begin{aligned} \int \frac{x^2 dx}{\sqrt{x^2 + 1}} &= \int \sec^3 \theta d\theta - \int \sec \theta d\theta \\ &= \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \int \sec \theta d\theta - \int \sec \theta d\theta \\ &= \frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \int \sec \theta d\theta \\ &= \frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta| + C \\ &= \frac{1}{2} x \sqrt{x^2 + 1} - \frac{1}{2} \ln |\sqrt{x^2 + 1} + x| + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{x^3 dx}{(x^2 + a^2)^{3/2}}$

Solution

$$x = a \tan \theta \quad \sqrt{x^2 + a^2} = a \sec \theta$$

$$dx = a \sec^2 \theta d\theta$$

$$\begin{aligned} \int \frac{x^3 dx}{(x^2 + a^2)^{3/2}} &= \int \frac{a^3 \tan^3 \theta}{a^3 \sec^3 \theta} (a \sec^2 \theta) d\theta \\ &= a \int \frac{\tan^3 \theta}{\sec \theta} d\theta \\ &= a \int \frac{\sin^3 \theta}{\cos^3 \theta} \cos \theta d\theta \\ &= a \int \frac{\sin^3 \theta}{\cos^2 \theta} d\theta \\ &= a \int \sin^2 \theta \cos^{-2} \theta (\sin \theta d\theta) \\ &= -a \int (1 - \cos^2 \theta) \cos^{-2} \theta d(\cos \theta) \\ &= -a \int (\cos^{-2} \theta - 1) d(\cos \theta) \end{aligned}$$

$$\begin{aligned}
&= -a \left(-\frac{1}{\cos \theta} - \cos \theta \right) + C \\
&= a \left(\sec \theta + \frac{1}{\sec \theta} \right) + C \\
&= a \left(\frac{\sqrt{x^2 + a^2}}{a} + \frac{a}{\sqrt{x^2 + a^2}} \right) + C \\
&= \sqrt{x^2 + a^2} + \frac{a^2}{\sqrt{x^2 + a^2}} + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{(x^2 + 4)^2}$

Solution

$$x = 2 \tan \theta \quad x^2 + 4 = 4 \sec^2 \theta$$

$$dx = 2 \sec^2 \theta \, d\theta$$

$$\begin{aligned}
\int \frac{dx}{(x^2 + 4)^2} &= \int \frac{2 \sec^2 \theta}{(4 \sec^2 \theta)^2} \, d\theta \\
&= \frac{1}{8} \int \frac{\sec^2 \theta}{\sec^4 \theta} \, d\theta \\
&= \frac{1}{8} \int \frac{1}{\sec^2 \theta} \, d\theta \\
&= \frac{1}{8} \int \cos^2 \theta \, d\theta \\
&= \frac{1}{16} \int (1 + \cos 2\theta) \, d\theta \\
&= \frac{1}{16} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\
&= \frac{1}{16} (\theta + \sin \theta \cos \theta) + C \\
&= \frac{1}{16} \left(\theta + \frac{\tan \theta}{\sec \theta} \frac{1}{\sec \theta} \right) + C \\
&= \frac{1}{16} \left(\arctan \frac{x}{2} + \frac{x}{\sqrt{x^2 + 4}} \frac{2}{\sqrt{x^2 + 4}} \right) + C \\
&= \frac{1}{16} \arctan \frac{x}{2} + \frac{1}{8} \frac{x}{x^2 + 4} + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{\sqrt{4x^2 + 16}}$

Solution

$$\int \frac{dx}{\sqrt{4x^2 + 16}} = \frac{1}{2} \int \frac{dx}{\sqrt{x^2 + 4}}$$

$$x = 2 \tan \theta \quad \sqrt{x^2 + 4} = 2 \sec \theta$$

$$dx = 2 \sec^2 \theta \, d\theta$$

$$\begin{aligned} \int \frac{dx}{\sqrt{4x^2 + 16}} &= \frac{1}{2} \int \frac{2 \sec^2 \theta}{2 \sec \theta} \, d\theta \\ &= \frac{1}{2} \int \sec \theta \, d\theta \\ &= \frac{1}{2} \int \sec \theta \frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta} \, d\theta \\ &= \frac{1}{2} \int \frac{\sec^2 \theta + \sec \theta \tan \theta}{\sec \theta + \tan \theta} \, d\theta \\ &= \frac{1}{2} \int \frac{1}{\sec \theta + \tan \theta} \, d(\sec \theta + \tan \theta) \\ &= \frac{1}{2} \ln |\sec \theta + \tan \theta| + C \\ &= \frac{1}{2} \ln \left| \frac{\sqrt{x^2 + 4}}{2} + \frac{x}{2} \right| + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{x^4}{x^2 + 1} \, dx$

Solution

$$x = \tan \theta \quad x^2 + 1 = \sec^2 \theta$$

$$dx = \sec^2 \theta \, d\theta$$

$$\int \frac{x^4}{x^2 + 1} \, dx = \int \frac{\tan^4 \theta}{\sec^2 \theta} \sec^2 \theta \, d\theta$$

$$\begin{aligned}
&= \int \tan^4 \theta \, d\theta \\
&= \int \tan^2 \theta (\sec^2 \theta - 1) \, d\theta \\
&= \int \tan^2 \theta \sec^2 \theta \, d\theta - \int \tan^2 \theta \, d\theta \\
&= \int \tan^2 \theta \, d(\tan \theta) - \int (\sec^2 \theta - 1) \, d\theta \\
&= \frac{1}{3} \tan^3 \theta - \tan \theta + \theta + C \\
&= \frac{1}{3} x^3 - x + \tan^{-1} x + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{e^{2x}}{(1+e^{4x})^{3/2}} dx$

Solution

$$\begin{aligned}
\int \frac{e^{2x}}{(1+e^{4x})^{3/2}} dx &= \frac{1}{2} \int \frac{1}{\left(1+(e^{2x})^2\right)^{3/2}} d(e^{2x}) \\
&= \frac{1}{2} \int \frac{1}{(1+y^2)^{3/2}} dy
\end{aligned}$$

$$\text{Let } y = e^{2x}$$

$$\begin{aligned}
y &= \tan \theta & y^2 + 1 &= \sec^2 \theta \\
dy &= \sec^2 \theta \, d\theta
\end{aligned}$$

$$\begin{aligned}
\int \frac{e^{2x}}{(1+e^{4x})^{3/2}} dx &= \frac{1}{2} \int \frac{1}{(\sec^2 \theta)^{3/2}} \sec^2 \theta \, d\theta \\
&= \frac{1}{2} \int \frac{1}{\sec^3 \theta} \sec^2 \theta \, d\theta \\
&= \frac{1}{2} \int \frac{1}{\sec \theta} \, d\theta
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \int \cos \theta \, d\theta \\
&= \frac{1}{2} \sin \theta + C \\
&= \frac{1}{2} \frac{\tan \theta}{\sec \theta} + C \\
&= \frac{1}{2} \frac{y}{\sqrt{1+y^2}} + C \\
&= \frac{1}{2} \frac{e^{2x}}{\sqrt{1+e^{4x}}} + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{1 + \cos x}$

Solution

$$\begin{aligned}
\int \frac{dx}{1 + \cos x} &= \int \frac{1}{1 + \cos x} \frac{1 - \cos x}{1 - \cos x} \, dx \\
&= \int \frac{1 - \cos x}{1 - \cos^2 x} \, dx \\
&= \int \frac{1 - \cos x}{\sin^2 x} \, dx \\
&= \int \left(\frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x} \right) \, dx \\
&= \int \csc^2 x \, dx - \int \frac{1}{\sin^2 x} \, d(\sin x) \\
&= -\cot x + \frac{1}{\sin x} + C \\
&= -\cot x + \csc x + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{\sqrt{x^2 - a^2}}$

Solution

$$\begin{aligned}
x &= a \sec \theta & \sqrt{x^2 - a^2} &= a \tan \theta \\
dx &= a \sec \theta \tan \theta \, d\theta
\end{aligned}$$

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2 - a^2}} &= \int \frac{a \sec \theta \tan \theta}{a \tan \theta} d\theta \\
&= \int \sec \theta \frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta} d\theta \\
&= \int \frac{\sec^2 \theta + \sec \theta \tan \theta}{\sec \theta + \tan \theta} d\theta \\
&= \int \frac{1}{\sec \theta + \tan \theta} d(\sec \theta + \tan \theta) \\
&= \ln |\sec \theta + \tan \theta| + C \\
&= \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\sqrt{x^2 - 1}}{x} dx$

Solution

$$\begin{aligned}
x &= \sec \theta & \sqrt{x^2 - 1} &= \tan \theta \\
dx &= \sec \theta \tan \theta d\theta \\
\int \frac{\sqrt{x^2 - 1}}{x} dx &= \int \frac{\tan \theta}{\sec \theta} (\sec \theta \tan \theta) d\theta \\
&= \int \tan^2 \theta d\theta \\
&= \int (\sec^2 \theta - 1) d\theta \\
&= \tan \theta - \theta + C \\
&= \sqrt{x^2 - 1} - \operatorname{arcsec} \theta + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\sqrt{y^2 - 49}}{y} dy, \quad y > 7$

Solution

$$y = 7 \sec \theta \rightarrow dy = 7 \sec \theta \tan \theta d\theta$$

$$\sqrt{y^2 - 49} = 7 \tan \theta$$

$$\begin{aligned} \int \frac{\sqrt{y^2 - 49}}{y} dy &= \int \frac{(7 \tan \theta)}{7 \sec \theta} (7 \sec \theta \tan \theta) d\theta \\ &= 7 \int \tan^2 \theta d\theta \\ &= 7 \int (\sec^2 \theta - 1) d\theta \\ &= 7(\tan \theta - \theta) + C \\ &= 7 \left[\frac{\sqrt{y^2 - 49}}{7} - \sec^{-1} \left(\frac{y}{7} \right) \right] + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{5 dx}{\sqrt{25x^2 - 9}}$, $x > \frac{3}{5} = \sin^{-1} \frac{1}{2} - \sin^{-1} 0$

Solution

$$5x = 3 \sec \theta \rightarrow dx = \frac{3}{5} \sec \theta \tan \theta d\theta$$

$$\sqrt{25x^2 - 9} = 3 \tan \theta$$

$$\begin{aligned} \int \frac{5 dx}{\sqrt{25x^2 - 9}} &= \int \frac{5 \left(\frac{3}{5} \sec \theta \tan \theta d\theta \right)}{3 \tan \theta} \\ &= \int \sec \theta d\theta \\ &= \int \sec \theta \frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta} d\theta \\ &= \int \frac{\sec^2 \theta + \sec \theta \tan \theta}{\sec \theta + \tan \theta} d\theta \\ &= \int \frac{1}{\sec \theta + \tan \theta} d(\sec \theta + \tan \theta) \\ &= \ln |\sec \theta + \tan \theta| + C \\ &= \ln \left| \frac{5}{3} x + \frac{1}{3} \frac{\sqrt{25x^2 - 9}}{3} \right| + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{2dx}{x^3\sqrt{x^2-1}}, \quad x > 1$

Solution

$$x = \sec \theta \quad dx = \sec \theta \tan \theta d\theta$$

$$\sqrt{x^2-1} = \sqrt{\sec^2 \theta - 1} = \tan \theta$$

$$\int \frac{2dx}{x^3\sqrt{x^2-1}} = \int \frac{2 \sec \theta \tan \theta d\theta}{\sec^3 \theta \tan \theta}$$

$$= 2 \int \cos^2 \theta d\theta$$

$$= 2 \int \frac{1 + \cos 2\theta}{2} d\theta$$

$$= \int (1 + \cos 2\theta) d\theta$$

$$= \theta + \frac{1}{2} \sin 2\theta + C$$

$$= \theta + \sin \theta \cos \theta + C$$

$$= \sec^{-1} x + \frac{\sqrt{x^2-1}}{x^2} + C$$

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

$$x = \sec \theta = \frac{1}{\cos \theta} \Rightarrow \cos \theta = \frac{1}{x}$$

$$\sin \theta = \tan \theta \cos \theta = \sqrt{x^2-1} \left(\frac{1}{x} \right)$$

Exercise

Evaluate the integral $\int \frac{dx}{x^3 \sqrt{x^2-100}}$

Solution

$$x = 10 \sec \theta \quad \sqrt{x^2-100} = 10 \tan \theta$$

$$dx = 10 \sec \theta \tan \theta d\theta$$

$$\int \frac{dx}{x^3 \sqrt{x^2-100}} = \int \frac{10 \sec \theta \tan \theta d\theta}{10^3 \sec^3 \theta (10 \tan \theta)}$$

$$= \frac{1}{10^3} \int \frac{1}{\sec^2 \theta} d\theta$$

$$= \frac{1}{10^3} \int \cos^2 \theta d\theta$$

$$\begin{aligned}
&= \frac{1}{2 \times 10^3} \int (1 + \cos 2\theta) d\theta \\
&= \frac{1}{2 \times 10^3} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\
&= \frac{1}{2 \times 10^3} (\theta + \sin \theta \cos \theta) + C \\
&= \frac{1}{2 \times 10^3} \left(\tan^{-1} \frac{\sqrt{x^2 - 100}}{10} + \frac{\tan \theta}{\sec \theta} \frac{1}{\sec \theta} \right) + C \\
&= \frac{1}{2 \times 10^3} \left(\tan^{-1} \frac{\sqrt{x^2 - 100}}{10} + \frac{\sqrt{x^2 - 100}}{x} \frac{10}{x} \right) + C \\
&= \frac{1}{2 \times 10^3} \left(\tan^{-1} \frac{\sqrt{x^2 - 100}}{10} + \frac{10 \sqrt{x^2 - 100}}{x^2} \right) + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{x^3 dx}{x^2 - 1}$

Solution

$$\begin{aligned}
\int \frac{x^3 dx}{x^2 - 1} &= \int \left(x + \frac{x}{x^2 - 1} \right) dx \\
&= \int x dx + \int \frac{x}{x^2 - 1} dx \\
&= \int x dx + \frac{1}{2} \int \frac{d(x^2 - 1)}{x^2 - 1} \\
&= \frac{1}{2} x^2 + \frac{1}{2} \ln |x^2 - 1| + C
\end{aligned}$$

$$\begin{aligned}
&\frac{x}{x^2 - 1} = \frac{x^3}{x^3 - x} \\
&= \frac{x^3 - x + x}{x^3 - x} \\
&= 1 + \frac{x}{x^3 - x}
\end{aligned}$$

$$d(x^2 - 1) = 2x dx \Rightarrow \frac{1}{2} d(x^2 - 1) = x dx$$

Exercise

Evaluate the integral $\int \frac{(1 - x^2)^{1/2}}{x^4} dx$

Solution

$$x = \sin \theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$dx = \cos \theta d\theta$$

$$(1-x^2)^{1/2} = (1-\sin^2 \theta)^{1/2} = \cos \theta$$

$$\begin{aligned} \int \frac{(1-x^2)^{1/2}}{x^4} dx &= \int \frac{\cos \theta}{\sin^4 \theta} \cos \theta d\theta \\ &= \int \frac{\cos^2 \theta}{\sin^2 \theta} \frac{1}{\sin^2 \theta} d\theta \\ &= \int \cot^2 \theta \csc^2 \theta d\theta \\ &= -\frac{1}{3} \cot^3 \theta + C \\ &= -\frac{1}{3} \left(\frac{\cos \theta}{\sin \theta} \right)^3 + C \\ &= -\frac{1}{3} \left(\frac{\sqrt{1-x^2}}{x} \right)^3 + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\sqrt{1-(\ln x)^2}}{x \ln x} dx$

Solution

$$\ln x = \sin \theta \quad 0 < \theta \leq \frac{\pi}{2}$$

$$\frac{1}{x} dx = \cos \theta d\theta$$

$$\sqrt{1-(\ln x)^2} = \sqrt{1-\sin^2 \theta} = \cos \theta$$

$$\begin{aligned} \int \frac{\sqrt{1-(\ln x)^2}}{x \ln x} dx &= \int \frac{\cos \theta}{\sin \theta} \cos \theta d\theta \\ &= \int \frac{\cos^2 \theta}{\sin \theta} d\theta \\ &= \int \frac{1-\sin^2 \theta}{\sin \theta} d\theta \\ &= \int \frac{1}{\sin \theta} d\theta - \int \sin \theta d\theta \\ &= \int \csc \theta d\theta - \int \sin \theta d\theta \end{aligned}$$

$$= -\ln|\csc \theta + \cot \theta| + \cos \theta + C$$

$$= -\ln \left| \frac{1}{\ln x} + \frac{\sqrt{1 - (\ln x)^2}}{\ln x} \right| + \sqrt{1 - (\ln x)^2} + C$$

Exercise

Evaluate the integral $\int \sqrt{x} \sqrt{1-x} \, dx$

Solution

$$u = \sqrt{x} \rightarrow u^2 = x \Rightarrow dx = 2u \, du$$

$$u = \sin \theta \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$du = \cos \theta \, d\theta$$

$$\int \sqrt{x} \sqrt{1-x} \, dx = \int u \sqrt{1-u^2} (2u \, du)$$

$$= 2 \int u^2 \sqrt{1-u^2} \, du$$

$$\sqrt{1-u^2} = \sqrt{1-\sin^2 \theta} = \cos \theta$$

$$= 2 \int \sin^2 \theta \cos \theta \cos \theta \, d\theta$$

$$= 2 \int \sin^2 \theta \cos^2 \theta \, d\theta$$

$$\sin 2\theta = 2 \sin \theta \cos \theta \rightarrow \sin^2 2\theta = 4 \sin^2 \theta \cos^2 \theta$$

$$= \frac{1}{2} \int \sin^2 2\theta \, d\theta$$

$$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}$$

$$= \frac{1}{2} \int \frac{1 - \cos 4\theta}{2} \, d\theta$$

$$= \frac{1}{4} \int d\theta - \frac{1}{4} \int \cos 4\theta \, d\theta$$

$$= \frac{1}{4} \theta - \frac{1}{16} \sin 4\theta + C$$

$$= \frac{1}{4} \theta - \frac{1}{16} 2 \sin 2\theta \cos 2\theta + C$$

$$= \frac{1}{4} \theta - \frac{1}{8} 2 \sin \theta \cos \theta (2 \cos^2 \theta - 1) + C$$

$$= \frac{1}{4} \theta - \frac{1}{2} \sin \theta \cos^3 \theta + \frac{1}{4} \sin \theta \cos \theta + C$$

$$= \frac{1}{4} \sin^{-1} u - \frac{1}{2} u (1-u^2)^{3/2} + \frac{1}{4} u \sqrt{1-u^2} + C$$

$$= \frac{1}{4} \sin^{-1} \sqrt{x} - \frac{1}{2} \sqrt{x} (1-x)^{3/2} + \frac{1}{4} \sqrt{x} \sqrt{1-x} + C \quad |$$

Exercise

Evaluate the integral $\int \frac{\sqrt{x-2}}{\sqrt{x-1}} dx$

Solution

$$u = \sqrt{x-1} \rightarrow u^2 = x-1 \Rightarrow 2u du = dx$$

$$u = \sec \theta \quad 0 < \theta < \frac{\pi}{2}$$

$$du = \sec \theta \tan \theta d\theta$$

$$\int \frac{\sqrt{x-2}}{\sqrt{x-1}} dx = \int \frac{\sqrt{u^2-1}}{u} 2u du$$

$$= 2 \int \sqrt{u^2-1} du$$

$$\sqrt{u^2-1} = \sqrt{\sec^2 \theta - 1} = \tan \theta$$

$$= 2 \int \tan \theta \sec \theta \tan \theta d\theta$$

$$w = \tan \theta \quad dv = \sec \theta \tan \theta d\theta$$

$$dw = \sec^2 \theta d\theta \quad v = \sec \theta$$

$$2 \int \tan \theta \sec \theta \tan \theta d\theta = 2 \sec \theta \tan \theta - 2 \int \sec^3 \theta d\theta$$

$$= 2 \sec \theta \tan \theta - 2 \int \sec^2 \theta \sec \theta d\theta$$

$$= 2 \sec \theta \tan \theta - 2 \int (\tan^2 \theta + 1) \sec \theta d\theta$$

$$= 2 \sec \theta \tan \theta - 2 \int \tan^2 \theta \sec \theta d\theta - 2 \int \sec \theta d\theta$$

$$\int \sec \theta d\theta = \int \sec \theta \frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta} d\theta$$

$$= \int \frac{\sec^2 \theta + \sec \theta \tan \theta}{\sec \theta + \tan \theta} d\theta$$

$$= \int \frac{1}{\sec \theta + \tan \theta} d(\sec \theta + \tan \theta)$$

$$= \ln |\sec \theta + \tan \theta|$$

$$2 \int \tan^2 \theta \sec \theta d\theta = 2 \sec \theta \tan \theta - 2 \int \tan^2 \theta \sec \theta d\theta - 2 \ln |\sec \theta + \tan \theta|$$

$$4 \int \tan^2 \theta \sec \theta d\theta = 2 \sec \theta \tan \theta - 2 \ln |\sec \theta + \tan \theta|$$

$$2 \int \tan^2 \theta \sec \theta d\theta = \sec \theta \tan \theta - \ln |\sec \theta + \tan \theta|$$

$$\begin{aligned} \int \frac{\sqrt{x-2}}{\sqrt{x-1}} dx &= 2 \int \tan \theta \sec \theta \tan \theta d\theta \\ &= \sec \theta \tan \theta - \ln |\sec \theta + \tan \theta| + C \\ &= u \sqrt{u^2 - 1} - \ln \left| u + \sqrt{u^2 - 1} \right| + C \\ &= \sqrt{x-1} \sqrt{x-2} - \ln \left| \sqrt{x-1} + \sqrt{x-2} \right| + C \end{aligned}$$

Exercise

Evaluate: $\int \frac{dx}{\sqrt{4x^2 - 49}}$

Solution

$$2x = 7 \sec \theta \rightarrow dx = \frac{7}{2} \sec \theta \tan \theta d\theta$$

$$\sqrt{4x^2 - 49} = \frac{7}{2} \tan \theta$$

$$\begin{aligned} \int \frac{dx}{\sqrt{4x^2 - 49}} &= \int \frac{dx}{2\sqrt{x^2 - \left(\frac{7}{2}\right)^2}} \\ &= \frac{1}{2} \int \frac{\frac{7}{2} \sec \theta \tan \theta d\theta}{\frac{7}{2} \tan \theta} \\ &= \frac{1}{2} \int \sec \theta d\theta \\ &= \frac{1}{2} \int \sec \theta \frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta} d\theta \\ &= \frac{1}{2} \int \frac{\sec^2 \theta + \sec \theta \tan \theta}{\sec \theta + \tan \theta} d\theta \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \int \frac{1}{\sec \theta + \tan \theta} d(\sec \theta + \tan \theta) \\
&= \frac{1}{2} \ln |\sec \theta + \tan \theta| + C \\
&= \frac{1}{2} \ln \left| \frac{2x}{7} + \frac{\sqrt{4x^2 - 49}}{7} \right| + C
\end{aligned}$$

Exercise

Evaluate $\int \frac{dx}{\sqrt{x^2 - 25}}$

Solution

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2 - 25}} &= \int \frac{5 \sec \theta \tan \theta}{5 \tan \theta} d\theta & x = 5 \sec \theta & \quad \sqrt{x^2 - 25} = 5 \tan \theta \\
&= \int \sec \theta d\theta & dx = 5 \sec \theta \tan \theta d\theta & \\
&= \int \sec \theta \frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta} d\theta \\
&= \int \frac{\sec^2 \theta + \sec \theta \tan \theta}{\sec \theta + \tan \theta} d\theta \\
&= \int \frac{1}{\sec \theta + \tan \theta} d(\sec \theta + \tan \theta) \\
&= \ln |\sec \theta + \tan \theta| + C \\
&= \ln \left| \frac{x}{5} + \frac{1}{5} \sqrt{x^2 - 25} \right| + C
\end{aligned}$$

Exercise

Evaluate $\int \frac{\sqrt{x^2 - 25}}{x} dx$

Solution

$$\begin{aligned}
x &= 5 \sec \theta & \sqrt{x^2 - 25} &= 5 \tan \theta \\
dx &= 5 \sec \theta \tan \theta d\theta \\
\int \frac{\sqrt{x^2 - 25}}{x} dx &= \int \frac{5 \tan \theta}{5 \sec \theta} (5 \sec \theta \tan \theta) d\theta
\end{aligned}$$

$$\begin{aligned}
&= 5 \int \tan^2 \theta \, d\theta \\
&= 5 \int (\sec^2 \theta - 1) \, d\theta \\
&= 5(\tan \theta - \theta) + C \\
&= \sqrt{x^2 - 25} - 5 \operatorname{arcsec} \frac{x}{5} + C
\end{aligned}$$

Exercise

Evaluate $\int \frac{x^3}{\sqrt{x^2 - a^2}} \, dx$

Solution

$$\begin{aligned}
x &= a \sec \theta & \sqrt{x^2 - a^2} &= a \tan \theta \\
dx &= a \sec \theta \tan \theta \, d\theta \\
\int \frac{x^3}{\sqrt{x^2 - a^2}} \, dx &= \int \frac{a^3 \sec^3 \theta}{a \tan \theta} (a \sec \theta \tan \theta) \, d\theta \\
&= a^3 \int \sec^4 \theta \, d\theta \\
&= a^3 \int (1 + \tan^2 \theta) \sec^2 \theta \, d\theta \\
&= a^3 \int (1 + \tan^2 \theta) \, d(\tan \theta) \\
&= a^3 \left(\tan \theta + \frac{1}{3} \tan^3 \theta \right) + C \\
&= a^3 \left(\frac{\sqrt{x^2 - a^2}}{a} + \frac{1}{3} \frac{(x^2 - a^2)^{3/2}}{a^3} \right) + C \\
&= \sqrt{x^2 - a^2} \left(a^2 + \frac{1}{3} (x^2 - a^2) \right) + C \\
&= \frac{1}{3} \sqrt{x^2 - a^2} (x^2 + 2a^2) + C
\end{aligned}$$

Exercise

Evaluate $\int \frac{x^3}{\sqrt{x^2 - 25}} dx$

Solution

$$x = 5 \sec \theta \quad \sqrt{x^2 - 25} = 5 \tan \theta$$

$$dx = 5 \sec \theta \tan \theta d\theta$$

$$\begin{aligned} \int \frac{x^3}{\sqrt{x^2 - 25}} dx &= \int \frac{5^3 \sec^3 \theta}{5 \tan \theta} (5 \sec \theta \tan \theta) d\theta \\ &= 125 \int \sec^4 \theta d\theta \\ &= 125 \int (1 + \tan^2 \theta) \sec^2 \theta d\theta \\ &= 125 \int (1 + \tan^2 \theta) d(\tan \theta) \\ &= 125 \left(\tan \theta + \frac{1}{3} \tan^3 \theta \right) + C \\ &= 125 \left(\frac{\sqrt{x^2 - 25}}{5} + \frac{1}{3} \frac{(x^2 - 25)^{3/2}}{125} \right) + C \\ &= \sqrt{x^2 - 25} \left(25 + \frac{x^2 - 25}{3} \right) + C \\ &= \frac{1}{3} \sqrt{x^2 - 25} (x^2 + 50) + C \end{aligned}$$

Exercise

Evaluate $\int x^3 \sqrt{x^2 - 25} dx$

Solution

$$x = 5 \sec \theta \quad \sqrt{x^2 - 25} = 5 \tan \theta$$

$$dx = 5 \sec \theta \tan \theta d\theta$$

$$\begin{aligned} \int x^3 \sqrt{x^2 - 25} dx &= \int 5^3 \sec^3 \theta (5 \tan \theta) (5 \sec \theta \tan \theta) d\theta \\ &= 5^5 \int \sec^4 \theta \tan^2 \theta d\theta \end{aligned}$$

$$\begin{aligned}
&= 5^5 \int \sec^2 \theta (1 + \tan^2 \theta) \tan^2 \theta \, d\theta \\
&= 5^5 \int (\tan^2 \theta + \tan^4 \theta) \, d(\tan \theta) \\
&= 5^5 \left(\frac{1}{3} \tan^3 \theta + \frac{1}{5} \tan^5 \theta \right) + C \\
&= 5^5 \left(\frac{1}{3} \frac{1}{5^3} (x^2 - 25)^{3/2} + \frac{1}{5^6} (x^2 - 25)^{5/2} \right) + C \\
&= (x^2 - 25)^{3/2} \left(\frac{25}{3} + \frac{1}{5} (x^2 - 25) \right) + C \\
&= \frac{1}{15} (x^2 - 25)^{3/2} (125 + 3x^2 - 75) + C \\
&= \frac{1}{15} (x^2 - 25)^{3/2} (3x^2 + 50) + C
\end{aligned}$$

Exercise

Evaluate $\int \sqrt{5x^2 - 1} \, dx$

Solution

$$\begin{aligned}
\sqrt{5}x &= \sec \theta & \sqrt{5x^2 - 1} &= \tan \theta \\
dx &= \frac{1}{\sqrt{5}} \sec \theta \tan \theta \, d\theta \\
\int \sqrt{5x^2 - 1} \, dx &= \frac{1}{\sqrt{5}} \int \sec \theta \, d\theta \\
&= \frac{1}{\sqrt{5}} \int \sec \theta \frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta} \, d\theta \\
&= \frac{1}{\sqrt{5}} \int \frac{\sec^2 \theta + \sec \theta \tan \theta}{\sec \theta + \tan \theta} \, d\theta \\
&= \frac{1}{\sqrt{5}} \int \frac{1}{\sec \theta + \tan \theta} \, d(\sec \theta + \tan \theta) \\
&= \frac{1}{\sqrt{5}} \ln |\sec \theta + \tan \theta| + C \\
&= \frac{1}{\sqrt{5}} \ln \left| \frac{x}{\sqrt{5}} + \sqrt{5x^2 - 1} \right| + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{\sqrt{9x^2 - 25}}, \quad x > \frac{5}{3}$

Solution

$$x = \frac{5}{3} \sec \theta \quad \sqrt{9x^2 - 25} = 5 \tan \theta$$

$$dx = \frac{5}{3} \sec \theta \tan \theta \, d\theta$$

$$\begin{aligned} \int \frac{dx}{\sqrt{9x^2 - 25}} &= \int \frac{1}{5 \tan \theta} \left(\frac{5}{3} \sec \theta \tan \theta \right) d\theta \\ &= \frac{1}{3} \int \sec \theta \, d\theta \\ &= \frac{1}{3} \int \sec \theta \frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta} d\theta \\ &= \frac{1}{3} \int \frac{\sec^2 \theta + \sec \theta \tan \theta}{\sec \theta + \tan \theta} d\theta \\ &= \frac{1}{3} \int \frac{1}{\sec \theta + \tan \theta} d(\sec \theta + \tan \theta) \\ &= \frac{1}{3} \ln |\sec \theta + \tan \theta| + C \\ &= \frac{1}{3} \ln \left| \frac{3x}{5} + \frac{\sqrt{9x^2 - 25}}{5} \right| + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{x \, dx}{\sqrt{4x^2 - 1}}$

Solution

$$x = \frac{1}{2} \sec \theta \quad \sqrt{4x^2 - 1} = \tan \theta$$

$$dx = \frac{1}{2} \sec \theta \tan \theta \, d\theta$$

$$\begin{aligned} \int \frac{x \, dx}{\sqrt{4x^2 - 1}} &= \int \frac{\frac{1}{2} \sec \theta}{\tan \theta} \left(\frac{1}{2} \sec \theta \tan \theta \right) d\theta \\ &= \frac{1}{4} \int \sec^2 \theta \, d\theta \\ &= \frac{1}{4} \tan \theta + C \end{aligned}$$

$$= \frac{1}{4} \sqrt{4x^2 - 1} + C \Big|$$

Exercise

Evaluate the integral $\int \frac{dx}{\sqrt{x^2 - 81}}$

Solution

$$x = 9 \sec \theta \quad \sqrt{x^2 - 81} = 9 \tan \theta$$

$$dx = 9 \sec \theta \tan \theta \, d\theta$$

$$\begin{aligned} \int \frac{dx}{\sqrt{x^2 - 81}} &= \int \frac{9 \sec \theta \tan \theta \, d\theta}{9 \tan \theta} \\ &= \int \sec \theta \, d\theta \\ &= \int \sec \theta \frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta} \, d\theta \\ &= \int \frac{\sec^2 \theta + \sec \theta \tan \theta}{\sec \theta + \tan \theta} \, d\theta \\ &= \int \frac{1}{\sec \theta + \tan \theta} \, d(\sec \theta + \tan \theta) \\ &= \ln |\sec \theta + \tan \theta| + C \\ &= \ln \left| \frac{x}{9} + \frac{\sqrt{x^2 - 81}}{9} \right| + C \Big| \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\sqrt{x^2 - 9}}{x} \, dx$

Solution

$$x = 3 \sec \theta \quad \sqrt{x^2 - 9} = 3 \tan \theta$$

$$dx = 3 \sec \theta \tan \theta \, d\theta$$

$$\int \frac{\sqrt{x^2 - 9}}{x} \, dx = \int \frac{3 \tan \theta}{3 \sec \theta} (3 \sec \theta \tan \theta) \, d\theta$$

$$\begin{aligned}
&= 3 \int \tan^2 \theta \, d\theta \\
&= 3 \int (\sec^2 \theta - 1) \, d\theta \\
&= 3(\tan \theta - \theta) + C \\
&= \sqrt{x^2 - 9} - 3 \sec^{-1} \frac{x}{3} + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{1}{x^2 \sqrt{9x^2 - 1}} \, dx$

Solution

$$\begin{aligned}
3x &= \sec \theta & \sqrt{9x^2 - 1} &= \tan \theta \\
dx &= \frac{1}{3} \sec \theta \tan \theta \, d\theta
\end{aligned}$$

$$\begin{aligned}
\int \frac{1}{x^2 \sqrt{9x^2 - 1}} \, dx &= \int \frac{1}{\frac{1}{9} \sec^2 \theta (\tan \theta)} \cdot \frac{1}{3} \sec \theta \tan \theta \, d\theta \\
&= 3 \int \frac{1}{\sec \theta} \, d\theta \\
&= 3 \int \cos \theta \, d\theta \\
&= 3 \sin \theta + C \\
&= 3 \frac{\sqrt{9x^2 - 1}}{3x} + C \\
&= \frac{\sqrt{9x^2 - 1}}{x} + C
\end{aligned}$$

$$\sin \theta = \frac{\tan \theta}{\sec \theta}$$

Exercise

Evaluate the integral $\int \frac{dx}{(x^2 - 36)^{3/2}}$

Solution

$$\begin{aligned}
x &= 6 \sec \theta & x^2 - 36 &= 36 \tan^2 \theta \\
dx &= 6 \sec \theta \tan \theta \, d\theta
\end{aligned}$$

$$\begin{aligned}
\int \frac{dx}{(x^2 - 36)^{3/2}} &= \int \frac{6 \sec \theta \tan \theta d\theta}{(36 \tan^2 \theta)^{3/2}} \\
&= \int \frac{6 \sec \theta \tan \theta d\theta}{6^3 \tan^3 \theta} \\
&= \frac{1}{36} \int \frac{\sec \theta}{\tan^2 \theta} d\theta \\
&= \frac{1}{36} \int \frac{1}{\cos \theta} \frac{\cos^2 \theta}{\sin^2 \theta} d\theta \\
&= \frac{1}{36} \int \frac{\cos \theta}{\sin^2 \theta} d\theta \\
&= \frac{1}{36} \int \frac{1}{\sin^2 \theta} d(\sin \theta) \\
&= -\frac{1}{36} \frac{1}{\sin \theta} + C \\
&= -\frac{1}{36} \cdot \frac{x}{6} \cdot \frac{6}{\sqrt{x^2 - 36}} + C \\
&= -\frac{1}{36} \frac{x}{\sqrt{x^2 - 36}} + C
\end{aligned}$$

$$\sin \theta = \frac{\tan \theta}{\sec \theta}$$

Exercise

Evaluate the integral $\int \frac{dx}{\sqrt{36 - x^2}}$

Solution

$$\begin{aligned}
x &= 6 \sin \theta & \sqrt{36 - x^2} &= 6 \cos \theta \\
dx &= 6 \cos \theta d\theta
\end{aligned}$$

$$\begin{aligned}
\int \frac{dx}{\sqrt{36 - x^2}} &= \int \frac{6 \cos \theta d\theta}{6 \cos \theta} \\
&= \int d\theta \\
&= \theta + C \\
&= \sin^{-1} \frac{x}{6} + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \sqrt{a^2 - x^2} \, dx$

Solution

$$\begin{aligned}x &= a \sin \theta & \sqrt{a^2 - x^2} &= a \cos \theta \\dx &= a \cos \theta \, d\theta\end{aligned}$$

$$\begin{aligned}\int \sqrt{a^2 - x^2} \, dx &= \int a \cos \theta (a \cos \theta \, d\theta) \\&= a^2 \int \cos^2 \theta \, d\theta \\&= \frac{a^2}{2} \int (1 + \cos 2\theta) \, d\theta \\&= \frac{a^2}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\&= \frac{a^2}{2} \left(\arcsin \frac{x}{a} + \sin \theta \cos \theta \right) + C \\&= \frac{a^2}{2} \left(\arcsin \frac{x}{a} + \frac{x}{a} \frac{\sqrt{a^2 - x^2}}{a} \right) + C \\&= \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{1}{2} x \sqrt{a^2 - x^2} + C\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{2 - \sqrt{3x}}$

Solution

$$\begin{aligned}\text{Let } u^2 &= \sqrt{3x} \quad \rightarrow \quad u^4 = 3x \\4u^3 du &= 3dx\end{aligned}$$

$$\begin{aligned}\int \frac{dx}{2 - \sqrt{3x}} &= \frac{4}{3} \int \frac{u^3}{2 - u^2} \, du \\u &= \sqrt{2} \sin \theta & 2 - u^2 &= 2 \cos^2 \theta \\du &= \sqrt{2} \cos \theta \, d\theta \\ \int \frac{dx}{2 - \sqrt{3x}} &= \frac{4}{3} \int \frac{2\sqrt{2} \sin^3 \theta}{2 \cos^2 \theta} (\sqrt{2} \cos \theta \, d\theta)\end{aligned}$$

$$\begin{aligned}
&= \frac{8}{3} \int \frac{\sin^3 \theta}{\cos \theta} d\theta \\
&= \frac{8}{3} \int \frac{\sin^2 \theta}{\cos \theta} (\sin \theta d\theta) \\
&= -\frac{8}{3} \int \frac{(1 - \cos^2 \theta)}{\cos \theta} d(\cos \theta) \\
&= -\frac{8}{3} \int \left(\frac{1}{\cos \theta} - \cos \theta \right) d(\cos \theta) \\
&= -\frac{8}{3} \left(\ln |\cos \theta| - \frac{1}{2} \cos^2 \theta \right) + C \\
&= -\frac{8}{3} \left(\ln \left| \frac{\sqrt{2-u^2}}{\sqrt{2}} \right| - \frac{1}{2} \frac{2-u^2}{2} \right) + C \\
&= -\frac{8}{3} \left(\ln \left| \frac{\sqrt{2-\sqrt{3x}}}{\sqrt{2}} \right| - \frac{2-\sqrt{3x}}{4} \right) + C \\
&= -\frac{8}{3} \ln \left| \frac{\sqrt{2-\sqrt{3x}}}{\sqrt{2}} \right| + \frac{2}{3} (2 - \sqrt{3x}) + C \\
&= -\frac{8}{3} \ln \left(\frac{2-\sqrt{3x}}{2} \right)^{1/2} + \frac{4}{3} - \frac{2}{3} \sqrt{3x} + C \\
&= \underline{-\frac{4}{3} \ln \left(\frac{2-\sqrt{3x}}{2} \right) - \frac{2}{3} \sqrt{3x} + C_1}
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{x dx}{1 - \sqrt{x}}$

Solution

$$\text{Let } u^2 = \sqrt{x} \rightarrow u^4 = x$$

$$4u^3 du = dx$$

$$\begin{aligned}
\int \frac{x dx}{1 - \sqrt{x}} &= 4 \int \frac{u^4}{1 - u^2} u^3 du \\
&= 4 \int \frac{u^7}{1 - u^2} du
\end{aligned}$$

$$u = \sin \theta \quad 1 - u^2 = \cos^2 \theta$$

$$du = \cos \theta \, d\theta$$

$$\begin{aligned} \int \frac{x \, dx}{1 - \sqrt{x}} &= 4 \int \frac{\sin^7 \theta}{\cos^2 \theta} \cos \theta \, d\theta \\ &= 4 \int \frac{\sin^6 \theta}{\cos \theta} (\sin \theta \, d\theta) \\ &= -4 \int \frac{(1 - \cos^2 \theta)^3}{\cos \theta} d(\cos \theta) \\ &= -4 \int \frac{1 - 3\cos^2 \theta + 3\cos^4 \theta - \cos^6 \theta}{\cos \theta} d(\cos \theta) \\ &= -4 \int \left(\frac{1}{\cos \theta} - 3\cos \theta + 3\cos^3 \theta - \cos^5 \theta \right) d(\cos \theta) \\ &= -4 \left(\ln |\cos \theta| - \frac{3}{2} \cos^2 \theta + \frac{3}{4} \cos^4 \theta - \frac{1}{6} \cos^6 \theta \right) + C \\ &= -4 \ln |\cos \theta| + 6 \cos^2 \theta - 3 \cos^4 \theta + \frac{2}{3} \cos^6 \theta + C \\ &= -4 \ln \left| \sqrt{1 - u^2} \right| + 6(1 - u^2) - 3(1 - u^2)^2 + \frac{2}{3}(1 - u^2)^3 + C \\ &= -4 \ln (1 - \sqrt{x})^{1/2} + 6 - 6\sqrt{x} - 3(1 - \sqrt{x})^2 + \frac{2}{3}(1 - \sqrt{x})^3 + C \\ &= -2 \ln (1 - \sqrt{x}) - 6\sqrt{x} - 3(1 - 2\sqrt{x} + x) + \frac{2}{3}(1 - 3\sqrt{x} + 3x - x\sqrt{x}) \\ &= -2 \ln (1 - \sqrt{x}) - 6\sqrt{x} + 6\sqrt{x} - 3x - 2\sqrt{x} + 2x - \frac{2}{3}x\sqrt{x} \\ &= \underline{-2 \ln (1 - \sqrt{x}) - x - 2\sqrt{x} - \frac{2}{3}x\sqrt{x} + C_1} \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{\sqrt{1 - 2x^2}}$

Solution

$$\sqrt{2} \, x = \sin \theta \quad \sqrt{1 - 2x^2} = \cos \theta$$

$$dx = \frac{1}{\sqrt{2}} \cos \theta \, d\theta$$

$$\int \frac{dx}{\sqrt{1 - 2x^2}} = \frac{1}{\sqrt{2}} \int \frac{\cos \theta \, d\theta}{\cos \theta}$$

$$\begin{aligned}
 &= \frac{1}{\sqrt{2}} \theta + C \\
 &= \frac{1}{\sqrt{2}} \sin^{-1}(\sqrt{2} x) + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{x^2}{(100 - x^2)^{3/2}} dx$

Solution

$$\begin{aligned}
 x &= 10 \sin \theta & \sqrt{100 - x^2} &= 10 \cos \theta \\
 dx &= 10 \cos \theta \, d\theta
 \end{aligned}$$

$$\begin{aligned}
 \int \frac{x^2}{(100 - x^2)^{3/2}} dx &= \int \frac{100 \sin^2 \theta}{(10 \cos \theta)^3} (10 \cos \theta) \, d\theta \\
 &= \int \frac{\sin^2 \theta}{\cos^2 \theta} \, d\theta \\
 &= \int \tan^2 \theta \, d\theta \\
 &= \int (\sec^2 \theta - 1) \, d\theta \\
 &= \tan \theta - \theta + C \\
 &= \frac{x}{\sqrt{100 - x^2}} - \sin^{-1} \frac{x}{10} + C
 \end{aligned}$$

Exercise

Evaluate: $\int \frac{2dx}{\sqrt{1 - 4x^2}}$

Solution

$$\begin{aligned}
 \int \frac{2dx}{\sqrt{1 - 4x^2}} &= \int \frac{d(2x)}{\sqrt{1 - (2x)^2}} \\
 &= \sin^{-1} 2x + C
 \end{aligned}$$

Exercise

Evaluate $\int \sqrt{\frac{x}{1-x}} dx$

Solution

$$x = \sin^2 \theta \quad \sqrt{1-x} = \cos \theta$$

$$dx = 2 \sin \theta \cos \theta d\theta$$

$$\begin{aligned} \int \sqrt{\frac{x}{1-x}} dx &= \int \frac{\sin \theta}{\cos \theta} 2 \sin \theta \cos \theta d\theta \\ &= 2 \int \sin^2 \theta d\theta \\ &= \int (1 - \cos 2\theta) d\theta \\ &= \theta - \frac{1}{2} \sin 2\theta + C \\ &= \theta - \sin \theta \cos \theta + C \\ &= \underline{\arcsin \sqrt{x} - \sqrt{x}\sqrt{1-x} + C} \end{aligned}$$

Exercise

Evaluate $\int x \sqrt{2x-x^2} dx$

Solution

$$\begin{aligned} 2x - x^2 &= 1 - 1 + 2x - x^2 \\ &= 1 - (1 - 2x + x^2) \\ &= 1 - (1-x)^2 \end{aligned}$$

$$\int x \sqrt{2x-x^2} dx = \int x \sqrt{1-(1-x)^2} dx$$

$$1-x = \sin \theta \quad \sqrt{1-(1-x)^2} = \cos \theta$$

$$dx = -\cos \theta d\theta$$

$$\begin{aligned} \int x \sqrt{2x-x^2} dx &= \int (1 - \sin \theta)(\cos \theta)(-\cos \theta d\theta) \\ &= - \int (1 - \sin \theta)(\cos^2 \theta) d\theta \end{aligned}$$

$$\begin{aligned}
&= -\int \cos^2 \theta \, d\theta + \int (\sin \theta \cos^2 \theta) \, d\theta \\
&= -\frac{1}{2} \int (1 + \cos 2\theta) \, d\theta - \int \cos^2 \theta \, d(\cos \theta) \\
&= -\frac{1}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) - \frac{1}{3} \cos^3 \theta + C \\
&= -\frac{1}{2} \theta - \frac{1}{2} \sin \theta \cos \theta - \frac{1}{3} \cos^3 \theta + C \\
&= -\frac{1}{2} \arcsin(1-x) - \frac{1}{2} (1-x) \sqrt{2x-x^2} - \frac{1}{3} \left(\sqrt{2x-x^2} \right)^3 + C
\end{aligned}$$

Exercise

Evaluate $\int x^2 \sqrt{a^2 - x^2} \, dx$

Solution

$$\begin{aligned}
x &= a \sin \theta & \sqrt{a^2 - x^2} &= a \cos \theta \\
dx &= a \cos \theta \, d\theta
\end{aligned}$$

$$\begin{aligned}
\int x^2 \sqrt{a^2 - x^2} \, dx &= \int a^2 \sin^2 \theta (a \cos \theta) (a \cos \theta \, d\theta) \\
&= a^4 \int \sin^2 \theta \cos^2 \theta \, d\theta \\
&= \frac{a^4}{4} \int (1 - \cos 2\theta)(1 + \cos 2\theta) \, d\theta \\
&= \frac{1}{4} a^4 \int (1 - \cos^2 2\theta) \, d\theta \\
&= \frac{1}{4} a^4 \int \left(1 - \frac{1}{2} - \frac{1}{2} \cos 4\theta \right) \, d\theta \\
&= \frac{1}{4} a^4 \int \left(\frac{1}{2} - \frac{1}{2} \cos 4\theta \right) \, d\theta \\
&= \frac{1}{8} a^4 \left(\theta - \frac{1}{4} \sin 4\theta \right) + C \\
&= \frac{1}{8} a^4 \left(\theta - \frac{1}{2} \sin 2\theta \cos 2\theta \right) + C \\
&= \frac{1}{8} a^4 \left(\theta - \sin \theta \cos \theta (1 - 2 \sin^2 \theta) \right) + C
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{8}a^4 \left(\arcsin \frac{x}{a} - \frac{x}{a} \frac{\sqrt{a^2 - x^2}}{a} \left(1 - 2 \frac{x^2}{a^2} \right) \right) + C \\
&= \frac{1}{8}a^4 \left(\arcsin \frac{x}{a} - \frac{x \sqrt{a^2 - x^2}}{a^2} + 2 \frac{x^3 \sqrt{a^2 - x^2}}{a^4} \right) + C \\
&= \frac{1}{8}a^4 \arcsin \frac{x}{a} - \frac{1}{8}a^2 x \sqrt{a^2 - x^2} + \frac{1}{4}x^3 \sqrt{a^2 - x^2} + C
\end{aligned}$$

Exercise

Evaluate $\int \frac{x}{\sqrt{4x - x^2}} dx$

Solution

$$\begin{aligned}
4x - x^2 &= 4 - 4 + 4x - x^2 \\
&= 4 - (4 - 4x + x^2) \\
&= 4 - (2 - x)^2
\end{aligned}$$

$$\int \frac{x}{\sqrt{4x - x^2}} dx = \int \frac{x}{\sqrt{4 - (2 - x)^2}} dx$$

$$\begin{aligned}
2 - x &= 2 \sin \theta & \sqrt{4 - (2 - x)^2} &= 2 \cos \theta \\
dx &= -2 \cos \theta d\theta
\end{aligned}$$

$$\begin{aligned}
\int \frac{x}{\sqrt{4x - x^2}} dx &= \int \frac{2 - 2 \sin \theta}{2 \cos \theta} (-2 \cos \theta d\theta) \\
&= -2 \int (1 - \sin \theta) d\theta \\
&= -2(\theta + \cos \theta) + C \\
&= -2 \left(\arcsin \left(1 - \frac{1}{2}x \right) + \frac{\sqrt{4 - (2 - x)^2}}{2} \right) + C \\
&= -2 \arcsin(2 - x) - \sqrt{4x - x^2} + C
\end{aligned}$$

Exercise

Evaluate $\int \frac{x}{\sqrt{ax-x^2}} dx, \quad a > 0$

Solution

$$\begin{aligned} ax - x^2 &= \frac{a^2}{4} - \frac{a^2}{4} + ax - x^2 \\ &= \frac{a^2}{4} - \left(\frac{a^2}{4} - ax + x^2 \right) \\ &= \frac{a^2}{4} - \left(\frac{a}{2} - x \right)^2 \end{aligned}$$

$$\int \frac{x}{\sqrt{ax-x^2}} dx = \int \frac{x}{\sqrt{\frac{a^2}{4} - \left(\frac{a}{2} - x \right)^2}} dx$$

$$\frac{a}{2} - x = \frac{a}{2} \sin \theta \quad \sqrt{\frac{a^2}{4} - \left(\frac{a}{2} - x \right)^2} = \frac{a}{2} \cos \theta$$

$$dx = -\frac{a}{2} \cos \theta d\theta$$

$$\begin{aligned} \int \frac{x}{\sqrt{ax-x^2}} dx &= \int \frac{\frac{a}{2} - \frac{a}{2} \sin \theta}{\frac{a}{2} \cos \theta} \left(-\frac{a}{2} \cos \theta d\theta \right) \\ &= -\frac{a}{2} \int (1 - \sin \theta) d\theta \\ &= -\frac{a}{2} (\theta + \cos \theta) + C \\ &= -\frac{a}{2} \arcsin \left(1 - \frac{2}{a} x \right) - \sqrt{ax-x^2} + C \\ &= \frac{a}{2} \left(\arcsin \left(\frac{2}{a} x - 1 \right) + \frac{2}{a} \sqrt{ax-x^2} \right) + C \end{aligned}$$

Exercise

Evaluate $\int \frac{x^2}{\sqrt{a^2-x^2}} dx$

Solution

$$x = a \sin \theta \quad \sqrt{a^2-x^2} = a \cos \theta$$

$$dx = a \cos \theta d\theta$$

$$\begin{aligned}
\int \frac{x^2}{\sqrt{a^2 - x^2}} dx &= \int \frac{a^2 \sin^2 \theta}{a \cos \theta} (a \cos \theta d\theta) \\
&= a^2 \int \sin^2 \theta d\theta \\
&= \frac{1}{2} a^2 \int (1 - \cos 2\theta) d\theta \\
&= \frac{1}{2} a^2 \left(\theta - \frac{1}{2} \sin 2\theta \right) + C \\
&= \frac{1}{2} a^2 (\theta - \sin \theta \cos \theta) + C \\
&= \frac{1}{2} a^2 \left(\arcsin \frac{x}{a} - \frac{x}{a} \frac{\sqrt{a^2 - x^2}}{a} \right) + C \\
&= \frac{1}{2} a^2 \arcsin \left(\frac{x}{a} \right) - \frac{1}{2} x \sqrt{a^2 - x^2} + C
\end{aligned}$$

Exercise

Evaluate $\int \frac{x^2}{\sqrt{16 - x^2}} dx$

Solution

$$\begin{aligned}
x &= 4 \sin \theta & \sqrt{16 - x^2} &= 4 \cos \theta \\
dx &= 4 \cos \theta d\theta \\
\int \frac{x^2}{\sqrt{16 - x^2}} dx &= \int \frac{16 \sin^2 \theta}{4 \cos \theta} (4 \cos \theta d\theta) \\
&= 16 \int \sin^2 \theta d\theta \\
&= 8 \int (1 - \cos 2\theta) d\theta \\
&= 8 \left(\theta - \frac{1}{2} \sin 2\theta \right) + C \\
&= 8 \left(\sin^{-1} \frac{x}{4} - \sin \theta \cos \theta \right) + C \\
&= 8 \left(\sin^{-1} \frac{x}{4} - \frac{x}{4} \frac{\sqrt{16 - x^2}}{4} \right) + C \\
&= 8 \sin^{-1} \frac{x}{4} - \frac{1}{2} x \sqrt{16 - x^2} + C
\end{aligned}$$

Exercise

Evaluate $\int \frac{dx}{(16-x^2)^{3/2}}$

Solution

$$x = 4 \sin \theta \quad \sqrt{16-x^2} = 4 \cos \theta$$

$$dx = 4 \cos \theta d\theta$$

$$\begin{aligned} \int \frac{dx}{(16-x^2)^{3/2}} &= \int \frac{4 \cos \theta}{(4 \cos \theta)^3} d\theta \\ &= \frac{1}{16} \int \frac{1}{\cos^2 \theta} d\theta \\ &= \frac{1}{16} \int \sec^2 \theta d\theta \\ &= \frac{1}{16} \tan \theta + C \end{aligned}$$

Exercise

Evaluate $\int \frac{dx}{x^2 \sqrt{9-x^2}}$

Solution

$$x = 3 \sin \theta \quad \sqrt{9-x^2} = 3 \cos \theta$$

$$dx = 3 \cos \theta d\theta$$

$$\begin{aligned} \int \frac{dx}{x^2 \sqrt{9-x^2}} &= \int \frac{3 \cos \theta}{9 \sin^2 \theta (3 \cos \theta)} d\theta \\ &= \frac{1}{9} \int \csc^2 \theta d\theta \\ &= -\frac{1}{9} \cot \theta + C \\ &= -\frac{1}{9} \frac{\cos \theta}{\sin \theta} + C \\ &= -\frac{1}{9} \frac{\sqrt{9-x^2}}{\frac{3}{x}} + C \\ &= -\frac{1}{9} \frac{\sqrt{9-x^2}}{x} + C \end{aligned}$$

Exercise

Evaluate $\int \frac{dx}{x^2 \sqrt{4-x^2}}$

Solution

$$x = 2 \sin \theta \quad \sqrt{4-x^2} = 2 \cos \theta$$

$$dx = 2 \cos \theta d\theta$$

$$\begin{aligned} \int \frac{dx}{x^2 \sqrt{4-x^2}} &= \int \frac{2 \cos \theta}{4 \sin^2 \theta (2 \cos \theta)} d\theta \\ &= \frac{1}{4} \int \csc^2 \theta d\theta \\ &= -\frac{1}{4} \cot \theta + C \\ &= -\frac{1}{4} \frac{\cos \theta}{\sin \theta} + C \\ &= -\frac{1}{4} \frac{\sqrt{4-x^2}}{2} \cdot \frac{2}{x} + C \\ &= -\frac{1}{4} \frac{\sqrt{4-x^2}}{x} + C \end{aligned}$$

Exercise

Evaluate $\int \frac{4}{x^2 \sqrt{16-x^2}} dx$

Solution

$$x = 4 \sin \theta \quad \sqrt{16-x^2} = 4 \cos \theta$$

$$dx = 4 \cos \theta d\theta$$

$$\begin{aligned} \int \frac{4}{x^2 \sqrt{16-x^2}} dx &= \int \frac{16 \cos \theta}{16 \sin^2 \theta (4 \cos \theta)} d\theta \\ &= \frac{1}{4} \int \csc^2 \theta d\theta \\ &= -\frac{1}{4} \cot \theta + C \\ &= -\frac{1}{4} \frac{\cos \theta}{\sin \theta} + C \\ &= -\frac{1}{4} \frac{\sqrt{16-x^2}}{x} + C \end{aligned}$$

Exercise

Evaluate $\int \frac{x^3}{\sqrt{9-x^2}} dx$

Solution

$$x = 3 \sin \theta \quad \sqrt{9-x^2} = 3 \cos \theta$$

$$dx = 3 \cos \theta d\theta$$

$$\begin{aligned} \int \frac{x^3}{\sqrt{9-x^2}} dx &= \int \frac{27 \sin^3 \theta}{3 \cos \theta} (3 \cos \theta) d\theta \\ &= 27 \int \sin^3 \theta d\theta \\ &= 27 \int (1 - \cos^2 \theta) d(\cos \theta) \\ &= 27 \left(\cos \theta - \frac{1}{3} \cos^3 \theta \right) + C \\ &= 27 \cos \theta - 9 \cos^3 \theta + C \\ &= 27 \frac{\sqrt{9-x^2}}{3} - 9 \left(\frac{\sqrt{9-x^2}}{3} \right)^3 + C \\ &= 9\sqrt{9-x^2} - \frac{1}{3}(9-x^2)\sqrt{9-x^2} + C \\ &= \frac{1}{3}\sqrt{9-x^2} (27-9+x^2) + C \\ &= \frac{1}{3}\sqrt{9-x^2} (18+x^2) + C \end{aligned}$$

Exercise

Evaluate $\int \sqrt{25-4x^2} dx$

Solution

$$2x = 5 \sin \theta \quad \sqrt{25-4x^2} = 5 \cos \theta$$

$$dx = \frac{5}{2} \cos \theta d\theta$$

$$\begin{aligned} \int \sqrt{25-4x^2} dx &= \frac{25}{2} \int \cos^2 \theta d\theta \\ &= \frac{25}{4} \int (1 + \cos 2\theta) d\theta \end{aligned}$$

$$\begin{aligned}
&= \frac{25}{4} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\
&= \frac{25}{4} (\theta + \sin \theta \cos \theta) + C \\
&= \frac{25}{4} \left(\sin^{-1} \frac{2x}{5} + \frac{2x}{5} \frac{\sqrt{25-4x^2}}{5} \right) + C \\
&= \frac{25}{4} \sin^{-1} \frac{2x}{5} + \frac{1}{2} x \sqrt{25-4x^2} + C
\end{aligned}$$

Exercise

Evaluate $\int e^x \sqrt{1-e^{2x}} dx$

Solution

$$e^x = \sin \theta \quad \sqrt{1-e^{2x}} = \cos \theta$$

$$e^x dx = \cos \theta d\theta$$

$$\begin{aligned}
\int e^x \sqrt{1-e^{2x}} dx &= \int \cos^2 \theta d\theta \\
&= \frac{1}{2} \int (1 + \cos 2\theta) d\theta \\
&= \frac{1}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\
&= \frac{1}{2} (\theta + \sin \theta \cos \theta) + C \\
&= \frac{1}{2} \left(\arcsin e^x + e^x \sqrt{1-e^{2x}} \right) + C
\end{aligned}$$

Exercise

Evaluate $\int \frac{\sqrt{1-x}}{\sqrt{x}} dx$

Solution

$$\sqrt{x} = \sin \theta \rightarrow x = \sin^2 \theta \quad \sqrt{1-x} = \cos \theta$$

$$dx = 2 \sin \theta \cos \theta d\theta$$

$$\begin{aligned}
\int \frac{\sqrt{1-x}}{\sqrt{x}} dx &= \int \frac{\cos \theta}{\sin \theta} (2 \sin \theta \cos \theta) d\theta \\
&= 2 \int \cos^2 \theta d\theta
\end{aligned}$$

$$\begin{aligned}
&= \int (1 + \cos 2\theta) d\theta \\
&= \theta + \frac{1}{2} \sin 2\theta + C \\
&= \theta + 2 \sin \theta \cos \theta + C \\
&= \arcsin \sqrt{x} + 2\sqrt{x}\sqrt{1-x} + C \quad |
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{y dy}{\sqrt{16-y^2}}$

Solution

$$\begin{aligned}
y &= 4 \sin \theta & \sqrt{16-y^2} &= 4 \cos \theta \\
dy &= 4 \cos \theta d\theta
\end{aligned}$$

$$\begin{aligned}
\int \frac{y dy}{\sqrt{16-y^2}} &= \int \frac{4 \sin \theta}{4 \cos \theta} (4 \cos \theta d\theta) \\
&= 4 \int \sin \theta d\theta \\
&= -4 \cos \theta + C \\
&= -\sqrt{16-y^2} + C \quad |
\end{aligned}$$

OR

$$\begin{aligned}
\int \frac{y dy}{\sqrt{16-y^2}} &= -\frac{1}{2} \int (16-y^2)^{-1/2} d(16-y^2) \\
&= -(16-y^2)^{1/2} + C \\
&= -\sqrt{16-y^2} + C \quad |
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{x^3}{\sqrt{4-x^2}} dx$

Solution

$$\begin{aligned}
x &= 2 \sin \theta & \sqrt{4-x^2} &= 2 \cos \theta \\
dx &= 2 \cos \theta d\theta
\end{aligned}$$

$$\begin{aligned}
\int \frac{x^3}{\sqrt{4-x^2}} dx &= \int \frac{8\sin^3 \theta}{2\cos \theta} (2\cos \theta d\theta) \\
&= 8 \int \sin^2 \theta \sin \theta d\theta \\
&= -8 \int (1 - \cos^2 \theta) d(\cos \theta) \\
&= -8 \left(\cos \theta - \frac{1}{3} \cos^3 \theta \right) + C \\
&= -8 \frac{\sqrt{4-x^2}}{2} + \frac{8}{3} \frac{(4-x^2)\sqrt{4-x^2}}{8} + C \\
&= \frac{1}{3} \sqrt{4-x^2} (-12 + 4 - x^2) + C \\
&= -\frac{1}{3} \sqrt{4-x^2} (x^2 + 8) + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{\sqrt{4-x^2}}$

Solution

$$\begin{aligned}
x &= 2 \sin \theta & \sqrt{4-x^2} &= 2 \cos \theta \\
dx &= 2 \cos \theta d\theta
\end{aligned}$$

$$\begin{aligned}
\int \frac{1}{\sqrt{4-x^2}} dx &= \int \frac{1}{2\cos \theta} (2\cos \theta d\theta) \\
&= \int d\theta \\
&= \theta + C \\
&= \sin^{-1} \frac{x}{2} + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{(1-x^2)^{3/2}}$

Solution

$$x = \sin \theta \quad \sqrt{1-x^2} = \cos \theta$$

$$dx = \cos \theta d\theta$$

$$\begin{aligned} \int \frac{dx}{(1-x^2)^{3/2}} &= \int \frac{\cos \theta d\theta}{\cos^3 \theta} \\ &= \int \frac{d\theta}{\cos^2 \theta} \\ &= \int \sec^2 \theta d\theta \\ &= \tan \theta + C \\ &= \frac{\sin \theta}{\cos \theta} + C \\ &= \frac{x}{\sqrt{1-x^2}} + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{(1-x^2)^{5/2}}{x^8} dx$

Solution

$$x = \sin \theta \quad 1-x^2 = \cos^2 \theta$$

$$dx = \cos \theta d\theta$$

$$\begin{aligned} \int \frac{(1-x^2)^{5/2}}{x^8} dx &= \int \frac{(\cos^2 \theta)^{5/2}}{\sin^8 \theta} \cos \theta d\theta \\ &= \int \frac{\cos^5 \theta}{\sin^8 \theta} \cos \theta d\theta \\ &= \int \frac{\cos^6 \theta}{\sin^6 \theta} \cdot \frac{1}{\sin^2 \theta} d\theta \\ &= \int \cot^6 \theta \csc^2 \theta d\theta \\ &= - \int \cot^6 \theta d(\cot \theta) \\ &= -\frac{1}{7} \cot^7 \theta + C \end{aligned}$$

$$\begin{aligned}
 &= -\frac{1}{7} \left(\frac{\cos \theta}{\sin \theta} \right)^7 + C \\
 &= -\frac{1}{7} \left(\frac{\sqrt{1-x^2}}{x} \right)^7 + C
 \end{aligned}$$

Exercise

Evaluate $\int \frac{dx}{\sqrt{3-2x-x^2}}$

Solution

$$\begin{aligned}
 3-2x-x^2 &= 3+1-1-2x-x^2 \\
 &= 4-(1+2x+x^2) \\
 &= 4-(1+x)^2
 \end{aligned}$$

$$\int \frac{dx}{\sqrt{3-2x-x^2}} = \int \frac{dx}{\sqrt{4-(1+x)^2}}$$

$$\begin{aligned}
 x+1 &= 2 \sin \theta & \sqrt{4-(1+x)^2} &= 2 \cos \theta \\
 dx &= 2 \cos \theta \, d\theta
 \end{aligned}$$

$$\begin{aligned}
 \int \frac{dx}{\sqrt{3-2x-x^2}} &= \int \frac{2 \cos \theta}{2 \cos \theta} \, d\theta \\
 &= \int d\theta \\
 &= \theta + C \\
 &= \sin^{-1} \left(\frac{x+1}{2} \right) + C
 \end{aligned}$$

Exercise

Evaluate $\int \frac{1}{x^4+4x^2+4} \, dx$

Solution

$$\begin{aligned}
 x &= \sqrt{2} \tan \theta & x^2+2 &= 2 \sec^2 \theta \\
 dx &= \sqrt{2} \sec^2 \theta \, d\theta
 \end{aligned}$$

$$\begin{aligned}
\int \frac{1}{x^4 + 4x^2 + 4} dx &= \int \frac{dx}{(x^2 + 2)^2} \\
&= \int \frac{\sqrt{2} \sec^2 \theta}{4 \sec^4 \theta} d\theta \\
&= \frac{\sqrt{2}}{4} \int \cos^2 \theta d\theta \\
&= \frac{\sqrt{2}}{8} \int (1 + \cos 2\theta) d\theta \\
&= \frac{\sqrt{2}}{8} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\
&= \frac{\sqrt{2}}{8} (\theta + \sin \theta \cos \theta) + C \\
&= \frac{\sqrt{2}}{8} \left(\theta + \frac{\tan \theta}{\sec \theta} \frac{1}{\sec \theta} \right) + C \\
&= \frac{\sqrt{2}}{8} \left(\arctan \frac{x}{\sqrt{2}} + \frac{x\sqrt{2}}{x^2 + 2} \right) + C
\end{aligned}$$

Exercise

Evaluate $\int \frac{x^3 + x + 1}{x^4 + 2x^2 + 1} dx$

Solution

$$x = \tan \theta \quad x^2 + 1 = \sec^2 \theta$$

$$dx = \sec^2 \theta d\theta$$

$$\begin{aligned}
\int \frac{x^3 + x + 1}{x^4 + 2x^2 + 1} dx &= \int \frac{x^3 + x}{x^4 + 2x^2 + 1} dx + \int \frac{1}{(x^2 + 1)^2} dx \\
&= \frac{1}{4} \int \frac{1}{x^4 + 2x^2 + 1} d(x^4 + 2x^2 + 1) + \int \frac{1}{\sec^2 \theta} d\theta \\
&= \frac{1}{4} \ln(x^2 + 1)^2 + \int \cos^2 \theta d\theta \\
&= \frac{1}{2} \ln(x^2 + 1) + \frac{1}{2} \int (1 + \cos 2\theta) d\theta \\
&= \frac{1}{2} \ln(x^2 + 1) + \frac{1}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \ln(x^2 + 1) + \frac{1}{2} (\theta + \sin \theta \cos \theta) + C \\
&= \frac{1}{2} \ln(x^2 + 1) + \frac{1}{2} \left(\theta + \frac{\tan \theta}{\sec \theta} \frac{1}{\sec \theta} \right) + C \\
&= \frac{1}{2} \ln(x^2 + 1) + \frac{1}{2} \left(\arctan x + \frac{x}{x^2 + 1} \right) + C
\end{aligned}$$

Exercise

Evaluate $\int \operatorname{arcsec} 2x \, dx \quad x > \frac{1}{2}$

Solution

$$u = \operatorname{arcsec} 2x \quad dv = dx$$

$$du = \frac{dx}{x\sqrt{4x^2 - 1}} \quad v = x$$

$$\int \operatorname{arcsec} 2x \, dx = x \operatorname{arcsec} 2x - \int \frac{dx}{\sqrt{4x^2 - 1}}$$

$$2x = \sec \theta \quad \sqrt{4x^2 - 1} = \tan \theta$$

$$dx = \frac{1}{2} \sec \theta \tan \theta \, d\theta$$

$$= x \operatorname{arcsec} 2x - \frac{1}{2} \int \frac{\sec \theta \tan \theta}{\tan \theta} \, d\theta$$

$$= x \operatorname{arcsec} 2x - \frac{1}{2} \int \sec \theta \, d\theta$$

$$= x \operatorname{arcsec} 2x - \frac{1}{2} \ln |\sec \theta + \tan \theta| + C$$

$$= x \operatorname{arcsec} 2x - \frac{1}{2} \ln \left| 2x + \sqrt{4x^2 - 1} \right| + C$$

Exercise

Evaluate $\int x \arcsin x \, dx$

Solution

$$u = \arcsin x \quad dv = x \, dx$$

$$du = \frac{dx}{\sqrt{1 - x^2}} \quad v = \frac{1}{2} x^2$$

$$\int x \arcsin x \, dx = \frac{1}{2} x^2 \arcsin x - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} \, dx$$

$$x = \sin \theta \quad \sqrt{1-x^2} = \cos \theta$$

$$dx = \cos \theta \, d\theta$$

$$= \frac{1}{2} x^2 \arcsin x - \frac{1}{2} \int \frac{\sin^2 \theta}{\cos \theta} \cos \theta \, d\theta$$

$$= \frac{1}{2} x^2 \arcsin x - \frac{1}{2} \int \sin^2 \theta \, d\theta$$

$$= \frac{1}{2} x^2 \arcsin x - \frac{1}{4} \int (1 - \cos 2\theta) \, d\theta$$

$$= \frac{1}{2} x^2 \arcsin x - \frac{1}{4} \left(\theta - \frac{1}{2} \sin 2\theta \right) + C$$

$$= \frac{1}{2} x^2 \arcsin x - \frac{1}{4} (\theta - \sin \theta \cos \theta) + C$$

$$= \frac{1}{2} x^2 \arcsin x - \frac{1}{4} \left(\arcsin x - \frac{1}{x\sqrt{1-x^2}} \right) + C$$

$$= \frac{1}{4} (2x^2 - 1) \arcsin x + \frac{1}{4} \frac{1}{x\sqrt{1-x^2}} + C$$

Exercise

Evaluate $\int_0^2 \sqrt{1+4x^2} \, dx$

Solution

$$2x = \tan \theta \quad \sqrt{1+4x^2} = \sec \theta$$

$$dx = \frac{1}{2} \sec^2 \theta \, d\theta$$

$$\int_0^2 \sqrt{1+4x^2} \, dx = \frac{1}{2} \int_0^2 \sec^3 \theta \, d\theta$$

$$u = \sec x \quad dv = \sec^2 x \, dx$$

$$du = \sec x \tan x \, dx \quad v = \tan x$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \tan x (\sec x \tan x \, dx)$$

$$= \sec x \tan x - \int \tan^2 x \sec x \, dx$$

$$\begin{aligned}
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx \\
&= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx
\end{aligned}$$

$$2 \int \sec^3 x \, dx = \sec x \tan x + \ln |\sec x + \tan x|$$

$$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x|$$

$$\begin{aligned}
\int_0^2 \sqrt{1+4x^2} \, dx &= \frac{1}{4} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \Big|_0^2 \\
&= \frac{1}{4} \left(2x\sqrt{1+4x^2} + \ln \left| 2x + \sqrt{1+4x^2} \right| \right) \Big|_0^2 \\
&= \frac{1}{4} (4\sqrt{17} + \ln |4 + \sqrt{17}|) \\
&= \sqrt{17} + \frac{1}{4} \ln (4 + \sqrt{17})
\end{aligned}$$

Exercise

Evaluate $\int_0^3 \frac{x^3}{\sqrt{x^2+9}} \, dx$

Solution

$$x = 3 \tan \theta \quad \sqrt{x^2 + 9} = 3 \sec \theta$$

$$dx = 3 \sec^2 \theta \, d\theta$$

$$\begin{aligned}
\int_0^3 \frac{x^3}{\sqrt{x^2+9}} \, dx &= \int_0^3 \frac{27 \tan^3 \theta}{3 \sec \theta} 3 \sec^2 \theta \, d\theta \\
&= 27 \int_0^3 \tan^2 \theta \tan \theta \sec \theta \, d\theta \\
&= 27 \int_0^3 (\sec^2 \theta - 1) d(\sec \theta) \\
&= 27 \left(\frac{1}{3} \sec^3 \theta - \sec \theta \right) \Big|_0^3 \\
&= 9\sqrt{x^2+9} \left(\frac{x^2+9}{27} - 1 \right) \Big|_0^3
\end{aligned}$$

$$= \frac{1}{3} \sqrt{x^2 + 9} (x^2 - 18) \Big|_0^3$$

$$= \underline{-9\sqrt{2} + 18}$$

Exercise

Evaluate the integral $\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^x dx}{(1 + e^{2x})^{3/2}}$

Solution

$$e^x = \tan \theta \rightarrow x = \ln(\tan \theta)$$

$$dx = \frac{\sec^2 \theta}{\tan \theta} d\theta$$

$$\tan^{-1}\left(\frac{3}{4}\right) < \theta < \tan^{-1}\left(\frac{4}{3}\right)$$

$$\begin{aligned} \int_{\ln(3/4)}^{\ln(4/3)} \frac{e^x dx}{(1 + e^{2x})^{3/2}} &= \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{\tan \theta}{(\sec^2 \theta)^{3/2}} \frac{\sec^2 \theta}{\tan \theta} d\theta \\ &= \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{\sec^2 \theta}{\sec^3 \theta} d\theta \\ &= \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{1}{\sec \theta} d\theta \\ &= \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \cos \theta d\theta \\ &= \sin \theta \Big|_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \\ &= \sin\left(\tan^{-1}(4/3)\right) - \sin\left(\tan^{-1}(3/4)\right) \\ &= \frac{4}{5} - \frac{3}{5} \\ &= \underline{\frac{1}{5}} \end{aligned}$$

Exercise

Evaluate the integral $\int_1^e \frac{e^x dx}{(1+e^{2x})^{3/2}}$

Solution

$$\int_1^e \frac{e^x dx}{(1+e^{2x})^{3/2}} = \int_1^e \frac{1}{\left(1+(e^x)^2\right)^{3/2}} d(e^x) \quad \text{Let } y = e^x$$

$$= \int_1^e \frac{1}{(1+y^2)^{3/2}} dy$$

$$y = \tan \theta \quad \sqrt{y^2 + 1} = \sec \theta$$

$$dy = \sec^2 \theta d\theta$$

$$\int_1^e \frac{e^x dx}{(1+e^{2x})^{3/2}} = \int_1^e \frac{1}{(\sec^2 \theta)^{3/2}} \sec^2 \theta d\theta$$

$$= \int_1^e \frac{1}{\sec^3 \theta} \sec^2 \theta d\theta$$

$$= \int_1^e \frac{1}{\sec \theta} d\theta$$

$$= \int_1^e \cos \theta d\theta$$

$$= \sin \theta \Big|_1^e$$

$$= \frac{\tan \theta}{\sec \theta} \Big|_1^e$$

$$= \frac{y}{\sqrt{1+y^2}} \Big|_1^e$$

$$= \frac{e^x}{\sqrt{1+e^{2x}}} \Big|_1^e$$

$$= \frac{e^e}{\sqrt{1+e^{2e}}} - \frac{e}{\sqrt{1+e^2}}$$

Exercise

Evaluate the integral $\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}}$

Solution

$$y = e^{\tan \theta} \quad 1 \leq y \leq e \rightarrow 0 \leq \theta = \tan^{-1}(\ln y) \leq \frac{\pi}{4}$$

$$dy = e^{\tan \theta} \sec^2 \theta d\theta$$

$$\sqrt{1+(\ln y)^2} = \sqrt{1+\tan^2 \theta} = \sec \theta$$

$$\begin{aligned} \int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}} &= \int_0^{\pi/4} \frac{e^{\tan \theta} \sec^2 \theta}{e^{\tan \theta} \sec \theta} d\theta \\ &= \int_0^{\pi/4} \sec \theta d\theta \\ &= \left(\ln |\sec \theta + \tan \theta| \right) \Big|_0^{\pi/4} \\ &= \ln \left| \sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right| - \ln |\sec 0 + \tan 0| \\ &= \ln(1 + \sqrt{2}) \end{aligned}$$

Exercise

Evaluate the integral $\int_{1/2}^{1/4} \frac{dy}{y\sqrt{1+(\ln y)^2}}$

Solution

$$\text{Let } x = \ln y \rightarrow y = e^x$$

$$dy = e^x dx$$

$$\begin{aligned} \int_{1/2}^{1/4} \frac{dy}{y\sqrt{1+(\ln y)^2}} &= \int_{1/2}^{1/4} \frac{e^x dx}{e^x \sqrt{1+x^2}} \\ &= \int_{1/2}^{1/4} \frac{dx}{\sqrt{1+x^2}} \end{aligned}$$

$$x = \tan \theta \quad \sqrt{x^2 + 1} = \sec \theta$$

$$dx = \sec^2 \theta d\theta$$

$$\begin{aligned}
\int_{1/2}^{1/4} \frac{dy}{y\sqrt{1+(\ln y)^2}} &= \int_{1/2}^{1/4} \frac{\sec^2 \theta \, d\theta}{\sec \theta} \\
&= \int_{1/2}^{1/4} \sec \theta \, d\theta \\
&= \int_{1/2}^{1/4} \sec \theta \frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta} d\theta \\
&= \int_{1/2}^{1/4} \frac{\sec^2 \theta + \sec \theta \tan \theta}{\sec \theta + \tan \theta} d\theta \\
&= \int_{1/2}^{1/4} \frac{1}{\sec \theta + \tan \theta} d(\sec \theta + \tan \theta) \\
&= \ln(\sec \theta + \tan \theta) \Big|_{1/2}^{1/4} \\
&= \ln\left(\sqrt{x^2 + 1} + x\right) \Big|_{1/2}^{1/4} \\
&= \ln\left(\sqrt{\ln^2 y + 1} + \ln y\right) \Big|_{1/2}^{1/4} \\
&= \ln \left| \sqrt{\left(\ln \frac{1}{4}\right)^2 + 1} + \ln \frac{1}{4} \right| - \ln \left| \sqrt{\left(\ln \frac{1}{2}\right)^2 + 1} + \ln \frac{1}{2} \right| \\
&= \ln \left| \sqrt{(-\ln 4)^2 + 1} - \ln 4 \right| - \ln \left| \sqrt{(-\ln 2)^2 + 1} - \ln 2 \right| \\
&= \ln \left| \frac{\sqrt{(\ln 4)^2 + 1} - \ln 4}{\sqrt{(\ln 2)^2 + 1} - \ln 2} \right|
\end{aligned}$$

Exercise

Evaluate $\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2 - 1}}$

Solution

Let: $u = 3y \Rightarrow du = 3dy \rightarrow \frac{du}{3} = dy$

$$\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2 - 1}} = \int_{-2/3}^{-\sqrt{2}/3} \frac{\frac{du}{3}}{\frac{u}{3}\sqrt{u^2 - 1}}$$

$$\begin{aligned}
&= \int_{-2/3}^{-\sqrt{2}/3} \frac{du}{u\sqrt{u^2-1}} \\
&= \sec^{-1}|3y| \Big|_{-2/3}^{-\sqrt{2}/3} \\
&= \sec^{-1}|\sqrt{2}| - \sec^{-1}|-2| \\
&= \frac{\pi}{4} - \frac{\pi}{3} \\
&= -\frac{\pi}{12}
\end{aligned}$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \sec^{-1} \left| \frac{x}{a} \right|$$

Exercise

Evaluate the integral $\int_0^{\sqrt{3}/2} \frac{4}{9+4x^2} dx$

Solution

$$x = \frac{3}{2} \tan \theta \quad 9 + 4x^2 = 9 \sec^2 \theta$$

$$dx = \frac{3}{2} \sec^2 \theta d\theta$$

$$\begin{aligned}
\int_0^{\sqrt{3}/2} \frac{4}{9+4x^2} dx &= \int_0^{\sqrt{3}/2} \frac{4}{9 \sec^2 \theta} \left(\frac{3}{2} \sec^2 \theta d\theta \right) \\
&= \frac{2}{3} \int_0^{\sqrt{3}/2} d\theta \\
&= \frac{2}{3} \theta \Big|_0^{\sqrt{3}/2} \\
&= \frac{2}{3} \arctan \left(\frac{2}{3} x \right) \Big|_0^{\sqrt{3}/2} \\
&= \frac{2}{3} \left(\arctan \left(\frac{\sqrt{3}}{3} \right) - \arctan(0) \right) \\
&= \frac{2}{3} \arctan \frac{1}{\sqrt{3}} \\
&= \frac{2}{3} \frac{\pi}{6} \\
&= \frac{\pi}{9}
\end{aligned}$$

Exercise

Evaluate the integral $\int_{1/12}^{1/4} \frac{dx}{\sqrt{x}(1+4x)}$

Solution

$$\text{Let } u = \sqrt{x} \rightarrow x = u^2$$

$$dx = 2u du$$

$$\begin{aligned} \int_{1/12}^{1/4} \frac{dx}{\sqrt{x}(1+4x)} &= \int_{1/12}^{1/4} \frac{2u du}{u(1+4u^2)} \\ &= \int_{1/12}^{1/4} \frac{d(2u)}{1+(2u)^2} \\ &= \arctan 2u \Big|_{1/12}^{1/4} \\ &= \arctan 2\sqrt{x} \Big|_{1/12}^{1/4} \\ &= \arctan 1 - \arctan \frac{1}{\sqrt{3}} \\ &= \frac{\pi}{4} - \frac{\pi}{6} \\ &= \frac{\pi}{12} \end{aligned}$$

Exercise

Evaluate the integral $\int_{8\sqrt{2}}^{16} \frac{dx}{\sqrt{x^2-64}}$

Solution

$$x = 8 \sec \theta \quad \sqrt{x^2 - 64} = 8 \tan \theta$$

$$dx = 8 \sec \theta \tan \theta d\theta$$

$$\begin{aligned} \int_{8\sqrt{2}}^{16} \frac{dx}{\sqrt{x^2-64}} &= \int_{8\sqrt{2}}^{16} \frac{8 \sec \theta \tan \theta d\theta}{8 \tan \theta} \\ &= \int_{8\sqrt{2}}^{16} \sec \theta d\theta \end{aligned}$$

$$\begin{aligned}
&= \int_{8\sqrt{2}}^{16} \sec \theta \frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta} d\theta \\
&= \int_{8\sqrt{2}}^{16} \frac{\sec^2 \theta + \sec \theta \tan \theta}{\sec \theta + \tan \theta} d\theta \\
&= \int_{8\sqrt{2}}^{16} \frac{1}{\sec \theta + \tan \theta} d(\sec \theta + \tan \theta) \\
&= \ln |\sec \theta + \tan \theta| \Big|_{8\sqrt{2}}^{16} \\
&= \ln \left| \frac{x}{8} + \frac{\sqrt{x^2 - 64}}{8} \right| \Big|_{8\sqrt{2}}^{16} \\
&= \ln \left| 2 + \frac{\sqrt{16^2 - 64}}{8} \right| - \ln \left| \sqrt{2} + \frac{\sqrt{128 - 64}}{8} \right| \\
&= \ln \left| 2 + \frac{8\sqrt{4-1}}{8} \right| - \ln(\sqrt{2} + 1) \\
&= \ln(2 + \sqrt{3}) - \ln(\sqrt{2} + 1) \\
&= \ln \left(\frac{2 + \sqrt{3}}{1 + \sqrt{2}} \right)
\end{aligned}$$

Exercise

Evaluate the integral $\int_{\sqrt{2}}^2 \frac{\sqrt{x^2 - 1}}{x} dx$

Solution

$$\begin{aligned}
x &= \sec \theta & \sqrt{x^2 - 1} &= \tan \theta \\
dx &= \sec \theta \tan \theta d\theta \\
\int_{\sqrt{2}}^2 \frac{\sqrt{x^2 - 1}}{x} dx &= \int_{\sqrt{2}}^2 \frac{\tan \theta}{\sec \theta} \sec \theta \tan \theta d\theta \\
&= \int_{\sqrt{2}}^2 \tan^2 \theta d\theta \\
&= \int_{\sqrt{2}}^2 (\sec^2 \theta - 1) d\theta
\end{aligned}$$

$$= \tan \theta - \theta \Big|_{\sqrt{2}}^2$$

$$x = 2 = \sec \theta \quad \rightarrow \quad \theta = \arccos \frac{1}{2} = \frac{\pi}{3}$$

$$x = \sqrt{2} = \sec \theta \quad \rightarrow \quad \theta = \arccos \frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

$$= \tan \theta - \theta \Big|_{\pi/4}^{\pi/3}$$

$$= \tan \frac{\pi}{3} - \frac{\pi}{3} - \tan \frac{\pi}{4} + \frac{\pi}{4}$$

$$= \sqrt{3} - 1 + \frac{\pi}{12}$$

Exercise

Evaluate $\int_4^6 \frac{x^2}{\sqrt{x^2 - 9}} dx$

Solution

$$x = 3 \sec \theta \quad \sqrt{x^2 - 9} = 3 \tan \theta$$

$$dx = 3 \sec \theta \tan \theta d\theta$$

$$\int_4^6 \frac{x^2}{\sqrt{x^2 - 9}} dx = \int_4^6 \frac{9 \sec^2 \theta}{3 \tan \theta} (3 \sec \theta \tan \theta) d\theta$$

$$= 9 \int_4^6 \sec^3 \theta d\theta$$

$$u = \sec x \quad dv = \sec^2 x dx$$

$$du = \sec x \tan x dx \quad v = \tan x$$

$$\int \sec^3 x dx = \sec x \tan x - \int \tan x (\sec x \tan x dx)$$

$$= \sec x \tan x - \int \tan^2 x \sec x dx$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx$$

$$= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$$

$$2 \int \sec^3 x dx = \sec x \tan x + \ln |\sec x + \tan x|$$

$$\int \sec^3 x dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x|$$

$$\begin{aligned}
&= \frac{9}{2} \left[\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right]_4^6 \\
&= \frac{9}{2} \left[\frac{x \sqrt{x^2 - 9}}{3} + \ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| \right]_4^6 \\
&= \frac{9}{2} \left(2\sqrt{3} + \ln(2 + \sqrt{3}) - \frac{4\sqrt{7}}{9} - \ln \left(\frac{4 + \sqrt{7}}{3} \right) \right) \\
&= \underline{9\sqrt{3} - 2\sqrt{7} + \frac{9}{2} \ln \left(\frac{6 + 3\sqrt{3}}{4 + \sqrt{7}} \right)}
\end{aligned}$$

Exercise

Evaluate $\int_{\sqrt{3}}^2 \frac{\sqrt{x^2 - 3}}{x} dx$

Solution

$$\begin{aligned}
x &= \sqrt{3} \sec \theta & \sqrt{x^2 - 3} &= \sqrt{3} \tan \theta \\
dx &= \sqrt{3} \sec \theta \tan \theta d\theta \\
\int_{\sqrt{3}}^2 \frac{\sqrt{x^2 - 3}}{x} dx &= \int_{\sqrt{3}}^2 \frac{\sqrt{3} \tan \theta}{\sqrt{3} \sec \theta} (\sqrt{3} \sec \theta \tan \theta) d\theta \\
&= \sqrt{3} \int_{\sqrt{3}}^2 \tan^2 \theta d\theta \\
&= \sqrt{3} \int_{\sqrt{3}}^2 (\sec^2 \theta - 1) d\theta \\
&= \sqrt{3} (\tan \theta - \theta) \Big|_{\sqrt{3}}^2 \\
&= \sqrt{3} \left(\frac{\sqrt{x^2 - 3}}{\sqrt{3}} - \operatorname{arcsec} \frac{x}{\sqrt{3}} \right) \Big|_{\sqrt{3}}^2 \\
&= \sqrt{3} \left(\frac{1}{\sqrt{3}} - \frac{\pi}{6} \right) \\
&= \underline{1 - \frac{\pi\sqrt{3}}{6}}
\end{aligned}$$

Exercise

Evaluate $\int_0^{\sqrt{3}/2} \frac{x^2}{(1-x^2)^{3/2}} dx$

Solution

$$x = \sin \theta \quad \sqrt{1-x^2} = \cos \theta$$

$$dx = \cos \theta d\theta$$

$$\begin{aligned} \int_0^{\sqrt{3}/2} \frac{x^2}{(1-x^2)^{3/2}} dx &= \int_0^{\sqrt{3}/2} \frac{\sin^2 \theta}{\cos^3 \theta} (\cos \theta) d\theta \\ &= \int_0^{\sqrt{3}/2} \tan^2 \theta d\theta \\ &= \int_0^{\sqrt{3}/2} (\sec^2 \theta - 1) d\theta \\ &= (\tan \theta - \theta) \Big|_0^{\sqrt{3}/2} \\ &= \left(\frac{x}{\sqrt{1-x^2}} - \arcsin x \right) \Big|_0^{\sqrt{3}/2} \\ &= \frac{\sqrt{3}}{2} \frac{1}{\sqrt{1-\frac{3}{4}}} - \frac{\pi}{3} \\ &= \sqrt{3} - \frac{\pi}{3} \end{aligned}$$

Exercise

Evaluate $\int_0^{\sqrt{3}/2} \frac{1}{(1-x^2)^{5/2}} dx$

Solution

$$x = \sin \theta \quad \sqrt{1-x^2} = \cos \theta$$

$$dx = \cos \theta d\theta$$

$$\int_0^{\sqrt{3}/2} \frac{1}{(1-x^2)^{5/2}} dx = \int_0^{\sqrt{3}/2} \frac{1}{\cos^5 \theta} \cos \theta d\theta$$

$$\begin{aligned}
&= \int_0^{\sqrt{3}/2} \sec^4 \theta \, d\theta \\
&= \int_0^{\sqrt{3}/2} (1 + \tan^2 \theta) \sec^2 \theta \, d\theta \\
&= \int_0^{\sqrt{3}/2} (1 + \tan^2 \theta) \, d(\tan \theta) \\
&= \tan \theta + \frac{1}{3} \tan^3 \theta \Big|_0^{\sqrt{3}/2} \\
&= \frac{x}{\sqrt{1-x^2}} + \frac{1}{3} \frac{x^3}{(1-x^2)^{3/2}} \Big|_0^{\sqrt{3}/2} \\
&= \frac{\sqrt{3}}{2} \frac{1}{\sqrt{1-\frac{3}{4}}} + \frac{\sqrt{3}}{8} \frac{1}{\left(\frac{1}{4}\right)^{3/2}} \\
&= \sqrt{3} + \sqrt{3} \\
&= 2\sqrt{3}
\end{aligned}$$

Exercise

Evaluate $\int_0^{3/5} \sqrt{9-25x^2} \, dx$

Solution

$$\begin{aligned}
5x &= 3 \sin \theta & \sqrt{9-25x^2} &= 3 \cos \theta \\
dx &= \frac{3}{5} \cos \theta \, d\theta \\
\int_0^{3/5} \sqrt{9-25x^2} \, dx &= \frac{9}{5} \int_0^{3/5} \cos^2 \theta \, d\theta \\
&= \frac{9}{10} \int_0^{3/5} (1 + \cos 2\theta) \, d\theta \\
&= \frac{9}{10} \left(\theta + \frac{1}{2} \sin 2\theta \right) \Big|_0^{3/5} \\
&= \frac{9}{10} \left(\theta + \sin \theta \cos \theta \right) \Big|_0^{3/5}
\end{aligned}$$

$$\begin{aligned}
&= \frac{9}{10} \left(\arcsin \frac{5x}{3} + 2 \cdot \frac{5x}{3} \cdot \frac{5\sqrt{9-25x^2}}{3} \right) \Big|_0^{3/5} \\
&= \frac{9}{10} \left(\arcsin \frac{5x}{3} + \frac{25}{9} x \sqrt{9-25x^2} \right) \Big|_0^{3/5} \\
&= \frac{9\pi}{20}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}}$

Solution

$$\begin{aligned}
\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}} &= \sin^{-1} \frac{x}{3} \Big|_0^{3/2} \\
&= \frac{\pi}{6}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

Solution

$$\begin{aligned}
\int_0^3 \frac{dx}{\sqrt{9-x^2}} &= \sin^{-1} \frac{x}{3} \Big|_0^3 \\
&= \sin^{-1} 1 - \sin^{-1} 0 \\
&= \frac{\pi}{2}
\end{aligned}$$

Exercise

Evaluate $\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2} dx$

Solution

$$\begin{aligned}
x+2 &= 3 \sec \theta & \sqrt{(x+2)^2-9} &= 3 \tan \theta \\
dx &= 3 \sec \theta \tan \theta d\theta
\end{aligned}$$

$$\begin{aligned}
\int_1^4 \frac{\sqrt{x^2 + 4x - 5}}{x + 2} dx &= \int_1^4 \frac{\sqrt{(x+2)^2 - 9}}{x+2} dx \\
&= \int_1^4 \frac{3 \tan \theta}{3 \sec \theta} (3 \sec \theta \tan \theta) d\theta \\
&= 3 \int_1^4 \tan^2 \theta d\theta \\
&= 3 \int_1^4 (\sec^2 \theta - 1) d\theta \\
&= 3 (\tan \theta - \theta) \Big|_1^4 \\
&= \sqrt{(x+2)^2 - 9} - 3 \sec^{-1} \left(\frac{x+2}{3} \right) \Big|_1^4 \\
&= \sqrt{27} - 3 \sec^{-1}(2) + 3 \sec^{-1}(1) \\
&= \underline{3\sqrt{3} - \pi}
\end{aligned}$$

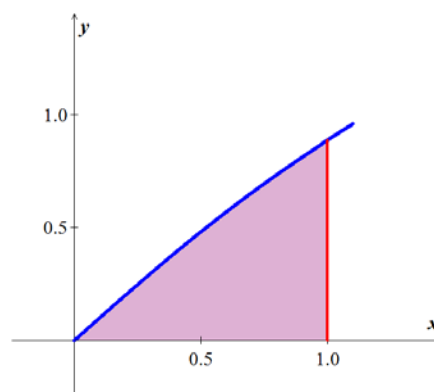
Exercise

Consider the region bounded by the graph $y = \sqrt{x \tan^{-1} x}$ and $y = 0$ for $0 \leq x \leq 1$. Find the volume of the solid formed by revolving this region about the x -axis.

Solution

$$\begin{aligned}
V &= \pi \int_0^1 \left(\sqrt{x \tan^{-1} x} \right)^2 dx \\
&= \pi \int_0^1 x \tan^{-1} x dx \\
V &= \pi \left(\frac{1}{2} \left[x^2 \tan^{-1} x \right]_0^1 - \frac{1}{2} \int_0^1 \frac{x^2}{1+x^2} dx \right) \\
&= \frac{\pi}{2} \left(\left(1 \tan^{-1} 1 - 0 \right) - \int_0^1 \left(1 - \frac{1}{1+x^2} \right) dx \right) \\
&= \frac{\pi}{2} \left(\frac{\pi}{4} - \int_0^1 dx + \int_0^1 \frac{1}{1+x^2} dx \right)
\end{aligned}$$

$$\begin{aligned}
u &= \tan^{-1} x & dv &= x dx \\
du &= \frac{1}{x^2 + 1} dx & v &= \frac{1}{2} x^2
\end{aligned}$$



$$\begin{aligned}
&= \frac{\pi}{2} \left(\frac{\pi}{4} - [x]_0^1 + \left[\tan^{-1} x \right]_0^1 \right) \\
&= \frac{\pi}{2} \left(\frac{\pi}{4} - 1 + \tan^{-1} 1 \right) \\
&= \frac{\pi}{2} \left(\frac{\pi}{4} - 1 + \frac{\pi}{4} \right) \\
&= \frac{\pi}{2} \left(\frac{\pi}{2} - 1 \right) \\
&= \frac{\pi^2}{4} - \frac{\pi}{2}
\end{aligned}$$

Exercise

Use two approach to show that the area of a cap (or segment) of a circle of radius r subtended by an angle θ is given by

$$A_{seg} = \frac{1}{2} r^2 (\theta - \sin \theta)$$

- Find the area using geometry (no calculus).
- Find the area using calculus

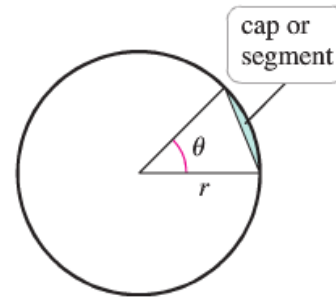
Solution

- Area of a segment (*cap*) = Area of a sector **minus** Area of the isosceles triangle

The area of a sector: $A = \frac{1}{2} \theta r^2$

Area of the isosceles triangle: $A = \frac{1}{2} r^2 \sin \theta$

$$\begin{aligned}
A_{seg} &= \frac{1}{2} r^2 \theta - \frac{1}{2} r^2 \sin \theta \\
&= \frac{1}{2} r^2 (\theta - \sin \theta)
\end{aligned}$$

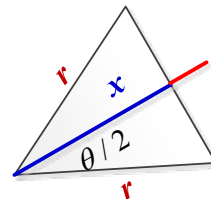


- $0 \leq \theta \leq \pi \rightarrow 0 \leq \frac{\theta}{2} \leq \frac{\pi}{2}$

$$x = r \cos \frac{\alpha}{2} \rightarrow dx = -\frac{1}{2} r \sin \frac{\alpha}{2} d\alpha$$

$$\sqrt{r^2 - x^2} = r \sin \frac{\alpha}{2}$$

$$\begin{aligned}
A_{cap} &= 2 \int_{r \cos \theta/2}^r \sqrt{r^2 - x^2} dx \\
&= 2 \int_{\theta}^0 \left(r \sin \frac{\alpha}{2} \right) \left(-\frac{1}{2} r \sin \frac{\alpha}{2} \right) d\alpha
\end{aligned}$$



$$\begin{aligned}
&= r^2 \int_0^\theta \left(\sin^2 \frac{\alpha}{2} \right) d\alpha \\
&= \frac{1}{2} r^2 \int_0^\theta (1 - \cos \alpha) d\alpha \\
&= \frac{1}{2} r^2 (\alpha - \sin \alpha) \Big|_0^\theta \\
&= \frac{1}{2} r^2 (\theta - \sin \theta)
\end{aligned}$$

Exercise

A lune is a crescent-shaped region bounded by the arcs of two circles. Let C_1 be a circle of radius 4 centered at the origin. Let C_2 be a circle of radius 3 centered at the point $(2, 0)$. Find the area of the lune that lies inside C_1 and outside C_2 .

Solution

$$C_1 \rightarrow x^2 + y^2 = 16 \Rightarrow y^2 = 16 - x^2$$

$$C_2 \rightarrow (x-2)^2 + y^2 = 9 \Rightarrow y^2 = 9 - (x-2)^2$$

$$16 - x^2 = 9 - x^2 + 4x - 4$$

$$11 = 4x \rightarrow x = \frac{11}{4}$$

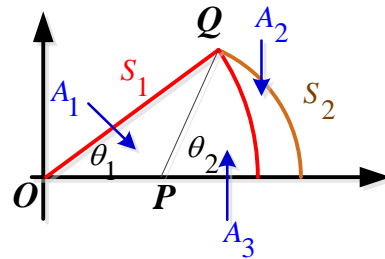
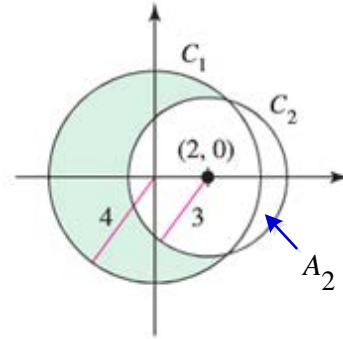
$$\begin{aligned}
y &= \pm \frac{\sqrt{135}}{4} \\
&= \pm \frac{3\sqrt{15}}{4}
\end{aligned}$$

$$\begin{aligned}
\text{For sector } C_1: \theta_1 &= \tan^{-1} \frac{y}{x} \\
&= \tan^{-1} \frac{3\sqrt{15}}{11}
\end{aligned}$$

$$\begin{aligned}
\text{Area: } S_1 &= \frac{1}{2} r^2 \theta_1 \\
&= 8 \tan^{-1} \left(\frac{3\sqrt{15}}{11} \right)
\end{aligned}$$

$$\text{For sector } C_2: x_2 = \frac{11}{4} - 2 = \frac{3}{4}$$

$$\theta_2 = \tan^{-1} \frac{y}{x_2}$$



$$= \tan^{-1} \sqrt{15}$$

$$\begin{aligned} \text{Area: } S_2 &= \frac{1}{2} r_2^2 \theta_2 \\ &= \frac{9}{2} \tan^{-1}(\sqrt{15}) \end{aligned}$$

$$OQ = 4, \quad PQ = 3, \quad OP = 2$$

$$\begin{aligned} \text{Area}(\triangle APQ) &= A_1 = \frac{1}{2}(4)(2) \sin \theta_1 \\ &= 4 \frac{y}{4} \\ &= \frac{3\sqrt{15}}{4} \end{aligned}$$

$$\begin{aligned} A_2 &= S_2 - S_1 + A_1 \\ &= \frac{9}{2} \tan^{-1}(\sqrt{15}) - 8 \tan^{-1}\left(\frac{3\sqrt{15}}{11}\right) + \frac{3\sqrt{15}}{4} \end{aligned}$$

$$\begin{aligned} A_{lune} &= A_{C_1} - A_{C_2} + 2A_2 \\ &= 16\pi - 9\pi + 9 \tan^{-1}(\sqrt{15}) - 16 \tan^{-1}\left(\frac{3\sqrt{15}}{11}\right) + \frac{3\sqrt{15}}{2} \\ &= 7\pi + 9 \tan^{-1}(\sqrt{15}) - 16 \tan^{-1}\left(\frac{3\sqrt{15}}{11}\right) + \frac{3\sqrt{15}}{2} \text{ unit}^2 \\ &\approx 26.66 \text{ unit}^2 \end{aligned}$$

Exercise

The crescent-shaped region bounded by two circles forms a lune. Find the area of the lune given that the radius of the smaller circle is 3 and the radius of the larger circle is 5.

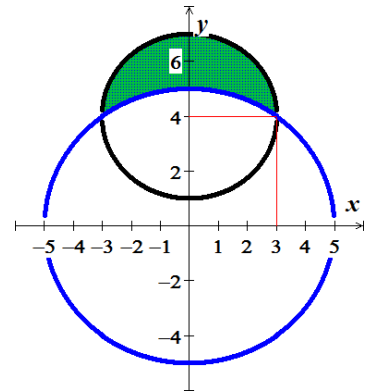
Solution

$$\text{Large Circle: } x^2 + y^2 = 25 \rightarrow y = \sqrt{25 - x^2}$$

$$\text{Small Circle: } r = 3 \rightarrow y = \sqrt{25 - 9} = 4$$

$$x^2 + (y - 4)^2 = 9 \rightarrow y = 4 + \sqrt{9 - x^2}$$

$$\begin{aligned} A &= 2 \int_0^3 \left(4 + \sqrt{9 - x^2} - \sqrt{25 - x^2} \right) dx \\ &= 2 \left[4x + \frac{1}{2} \left(9 \arcsin\left(\frac{x}{3}\right) + x\sqrt{9 - x^2} \right) - \frac{1}{2} \left(25 \arcsin\left(\frac{x}{5}\right) + x\sqrt{25 - x^2} \right) \right]_0^3 \end{aligned}$$



$$= 2 \left[12 + \frac{1}{2} \left(9 \frac{\pi}{2} \right) - \frac{1}{2} \left(25 \arcsin \left(\frac{3}{5} \right) + 12 \right) \right]$$

$$\underline{= 12 + \frac{9\pi}{2} - 25 \arcsin \left(\frac{3}{5} \right)}$$

Exercise

The surface of a machine part is the region between the graphs of $y = |x|$ and $x^2 + (y - k)^2 = 25$

- Find k when the circle is tangent to the graph of $y = |x|$
- Find the area of the surface of the machine part.
- Find the area of the surface of the machine part as a function of the radius r of the circle.

Solution

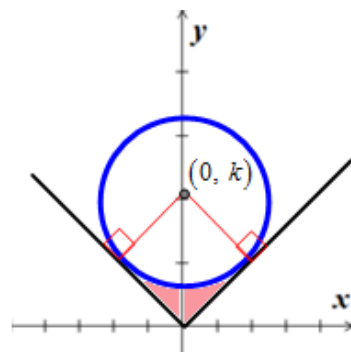
$$a) \quad x^2 + (y - k)^2 = 25 \rightarrow \underline{r = 5}$$

$$k^2 = 5^2 + 5^2 = 50 \rightarrow \underline{k = 5\sqrt{2}}$$

$$b) \quad \text{Area} = \text{area square} - \frac{1}{4}(\text{area circle})$$

$$= 5^2 - \frac{1}{4} \pi 5^2$$

$$\underline{= 25 \left(1 - \frac{\pi}{4} \right) \text{ unit}^2}$$



$$c) \quad \text{Area} = \text{area square} - \frac{1}{4}(\text{area circle})$$

$$= r^2 - \frac{1}{4} \pi r^2$$

$$\underline{= r^2 \left(1 - \frac{\pi}{4} \right) \text{ unit}^2}$$

Exercise

Consider the function $f(x) = (9 + x^2)^{-1/2}$ and the region R on the interval $[0, 4]$.

- Find the area of R .
- Find the volume of the solid generated when R is revolved about the x -axis.
- Find the volume of the solid generated when R is revolved about the y -axis.

Solution

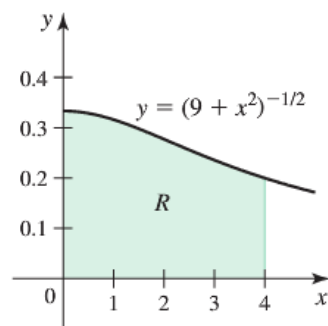
$$a) \quad A = \int_0^4 \frac{dx}{\sqrt{9 + x^2}}$$

$$= \int_0^4 \frac{3 \sec^2 \theta \, d\theta}{3 \sec \theta}$$

$$x = 3 \tan \theta \rightarrow dx = 3 \sec^2 \theta \, d\theta$$

$$\sqrt{9 + x^2} = 3 \sec \theta$$

$$\begin{aligned}
&= \int_0^4 \sec \theta \, d\theta \\
&= \ln |\sec \theta + \tan \theta| \Big|_0^4 \\
&= \ln \left| \frac{\sqrt{9+x^2}}{3} + \frac{x}{3} \right| \Big|_0^4 \\
&= \ln \left(\frac{5}{3} + \frac{4}{3} \right) - 0 \\
&= \ln 3 \text{ unit}^2
\end{aligned}$$



b) $x = 3 \tan \theta \rightarrow dx = 3 \sec^2 \theta \, d\theta$

$$9 + x^2 = 9 \sec^2 \theta$$

$$\begin{aligned}
V &= \pi \int_0^4 \frac{dx}{9 + x^2} \\
&= \pi \int_0^4 \frac{3 \sec^2 \theta \, d\theta}{9 \sec^2 \theta} \\
&= \frac{\pi}{3} \int_0^4 d\theta \\
&= \frac{\pi}{3} \theta \Big|_0^4 \\
&= \frac{\pi}{3} \tan^{-1} \frac{x}{3} \Big|_0^4 \\
&= \frac{\pi}{3} \tan^{-1} \frac{4}{3} \text{ unit}^3
\end{aligned}$$

c) $V = 2\pi \int_0^4 \frac{x}{\sqrt{9+x^2}} dx$

$$\begin{aligned}
&= \pi \int_0^4 (9+x^2)^{-1/2} d(9+x^2) \\
&= 2\pi (9+x^2)^{1/2} \Big|_0^4 \\
&= 2\pi (5-3) \\
&= 4\pi \text{ unit}^3
\end{aligned}$$

$$d(9+x^2) = 2x dx$$

Exercise

A total of Q is distributed uniformly on a line segment of length $2L$ along the y -axis. The x -component of the electric field at a point $(a, 0)$ is given by

$$E_x = \frac{kQa}{2L} \int_{-L}^L \frac{dy}{(a^2 + y^2)^{3/2}}$$

Where k is a physical constant and $a > 0$

a) Confirm that $E_x(a) = \frac{kQ}{a\sqrt{a^2 + L^2}}$

b) Letting $\rho = \frac{Q}{2L}$ be the charge density on the line segment, show that if $L \rightarrow \infty$, then $E_x = \frac{2k\rho}{a}$

Solution

a) $E_x = \frac{kQa}{2L} \int_{-L}^L \frac{dy}{(a^2 + y^2)^{3/2}}$

$$y = a \tan \theta \rightarrow dy = a \sec^2 \theta d\theta$$

$$\sqrt{a^2 + y^2} = a \sec \theta$$

$$= \frac{kQa}{2L} \int_{-L}^L \frac{a \sec^2 \theta d\theta}{a^3 \sec^3 \theta}$$

$$= \frac{kQ}{2aL} \int_{-L}^L \frac{d\theta}{\sec \theta}$$

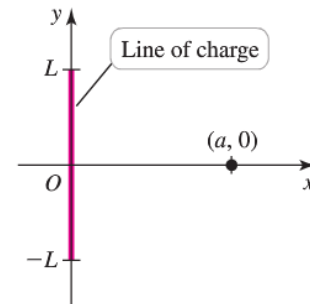
$$= \frac{kQ}{2aL} \int_{-L}^L \cos \theta d\theta$$

$$= \frac{kQ}{2aL} \sin \theta \Big|_{-L}^L$$

$$= \frac{kQ}{2aL} \left(\frac{y}{\sqrt{a^2 + y^2}} \right) \Big|_{-L}^L$$

$$= \frac{kQ}{2aL} \left(\frac{2L}{\sqrt{a^2 + L^2}} \right)$$

$$= \frac{kQ}{a\sqrt{a^2 + L^2}}$$



b) Let $\rho = \frac{Q}{2L} \rightarrow Q = 2\rho L$

$$\begin{aligned}
 E_x(a) &= \frac{kQa}{2L} \lim_{L \rightarrow \infty} \int_{-L}^L \frac{dy}{(a^2 + y^2)^{3/2}} \\
 &= \frac{kQa}{2L} \lim_{L \rightarrow \infty} \left(\frac{2L}{a^2 \sqrt{a^2 + L^2}} \right) \\
 &= k\rho a \frac{2}{a^2} \\
 &= \frac{2k\rho}{a}
 \end{aligned}$$

Exercise

A long, straight wire of length $2L$ on the y -axis carries a current I . according to the Biot-Savart Law, the magnitude of the field due to the current at a point $(a, 0)$ is given by

$$B(a) = \frac{\mu_0 I}{4\pi} \int_{-L}^L \frac{\sin \theta}{r^2} dy$$

Where μ_0 is a physical constant, $a > 0$, and θ, r , and y are related to the figure

- a) Show that the magnitude of the magnetic field at $(a, 0)$ is $B(a) = \frac{\mu_0 IL}{2\pi a \sqrt{a^2 + L^2}}$
- b) What is the magnitude of the magnetic field at $(a, 0)$ due to an infinitely long wire ($L \rightarrow \infty$)?

Solution

a) $\beta = \pi - \theta$ & $\alpha + \beta = \frac{\pi}{2}$

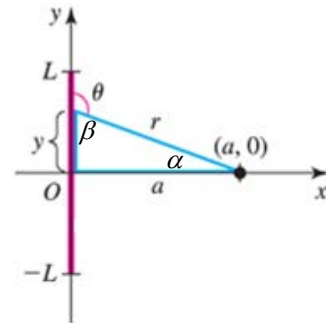
$$\sin \theta = \sin(\pi - \beta) = \sin\left(\frac{\pi}{2} + \alpha\right) = \cos \alpha = \frac{a}{r}$$

$$r^2 = y^2 + a^2$$

$$\frac{\sin \theta}{r^2} = \frac{a}{r^3} = \frac{a}{(a^2 + y^2)^{3/2}}$$

$$\begin{aligned}
 B(a) &= \frac{\mu_0 I}{4\pi} \int_{-L}^L \frac{\sin \theta}{r^2} dy \\
 &= \frac{\mu_0 I}{4\pi} \int_{-L}^L \frac{a}{(a^2 + y^2)^{3/2}} dy
 \end{aligned}$$

$$\begin{aligned}
 y &= a \tan u & \sqrt{a^2 + y^2} &= a \sec u \\
 dy &= a \sec^2 u \, du
 \end{aligned}$$

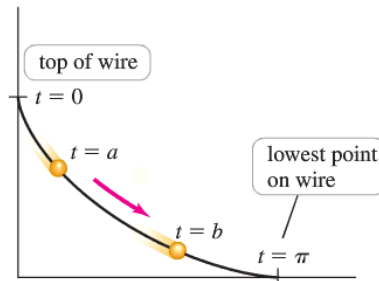


$$\begin{aligned}
&= \frac{\mu_0 I}{2\pi} \int_0^L \frac{a^2 \sec^2 u \, du}{a^3 \sec^3 u} \\
&= \frac{\mu_0 I}{2a\pi} \int_0^L \frac{1}{\sec u} du \\
&= \frac{\mu_0 I}{2a\pi} \int_0^L \cos u \, du \\
&= \frac{\mu_0 I}{2a\pi} \sin u \Big|_0^L \\
&= \frac{\mu_0 I}{2a\pi} \frac{y}{\sqrt{a^2 + y^2}} \Big|_0^L \\
&= \frac{\mu_0 IL}{2a\pi \sqrt{a^2 + L^2}}
\end{aligned}$$

$$\begin{aligned}
b) \quad \lim_{L \rightarrow \infty} B(a) &= \lim_{L \rightarrow \infty} \frac{\mu_0 IL}{2a\pi \sqrt{a^2 + L^2}} \\
&= \frac{\mu_0 I}{2a\pi} \lim_{L \rightarrow \infty} \frac{L}{\sqrt{a^2 + L^2}} \qquad \lim_{L \rightarrow \infty} \frac{L}{\sqrt{a^2 + L^2}} = \lim_{L \rightarrow \infty} \frac{L}{\sqrt{L^2}} = 1 \\
&= \frac{\mu_0 I}{2a\pi}
\end{aligned}$$

Exercise

The cycloid is the curve traced by a point on the rim of a rolling wheel. Imagine a wire shaped like an inverted cycloid.



A bead sliding down this wire without friction has some remarkable properties. Among all wire shapes, the cycloid is the shape that produces the fastest descent time. It can be shown that the descent time between any two points $0 \leq a < b \leq \pi$ on the curve is

$$\text{descent time} = \int_a^b \sqrt{\frac{1 - \cos t}{g(\cos a - \cos t)}} \, dt$$

Where g is the acceleration due to gravity, $t = 0$ corresponds to the top of the wire, and $t = \pi$ corresponds to the lowest point on the wire.

a) Find the descent time on the interval $[a, b]$.

b) Show that when $b = \pi$, the descent time is the same for all values of a ; that is, the descent time to the bottom of the wire is the same for all starting points.

Solution

$$\begin{aligned}
 a) \quad \int_a^b \sqrt{\frac{1 - \cos t}{g(\cos a - \cos t)}} dt &= \int_a^b \sqrt{\frac{(1 - \cos t)(1 + \cos t)}{g(\cos a - \cos t)(1 + \cos t)}} dt \\
 &= \frac{1}{\sqrt{g}} \int_a^b \sqrt{\frac{(1 - \cos^2 t)}{\cos a + (\cos a - 1)\cos t - \cos^2 t}} dt \\
 &= \frac{1}{\sqrt{g}} \int_a^b \frac{\sin t}{\sqrt{\cos a + \left(\frac{\cos a - 1}{2}\right)^2 - \left(\frac{\cos a - 1}{2}\right)^2 + (\cos a - 1)\cos t - \cos^2 t}} dt \\
 &= \frac{1}{\sqrt{g}} \int_a^b \frac{\sin t}{\sqrt{\cos a + \left(\frac{\cos a - 1}{2}\right)^2 - \left(\left(\frac{\cos a - 1}{2}\right) - \cos t\right)^2}} dt \\
 \text{Let: } v &= \sqrt{\cos a + \left(\frac{\cos a - 1}{2}\right)^2} \\
 &= \frac{1}{2} \sqrt{4 \cos a + \cos^2 a - 2 \cos a + 1} \\
 &= \frac{1}{2} (\cos a + 1) \\
 \frac{\cos a - 1}{2} - \cos t &= v \sin \theta \rightarrow \sin t dt = v \cos \theta d\theta \\
 \sqrt{v - \left(\left(\frac{\cos a - 1}{2}\right) - \cos t\right)^2} &= v \cos \theta \\
 &= \frac{1}{\sqrt{g}} \int_a^b \frac{v \cos \theta}{v \cos \theta} d\theta \\
 &= \frac{1}{\sqrt{g}} \theta \Big|_a^b \\
 \theta &= \sin^{-1} \left(\frac{\cos a - 1 - 2 \cos t}{2} \cdot \frac{2}{1 + \cos a} \right) \\
 &= \frac{1}{\sqrt{g}} \sin^{-1} \left(\frac{\cos a - 1 - 2 \cos t}{1 + \cos a} \right) \Big|_a^b
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{g}} \left(\sin^{-1} \left(\frac{\cos a - 1 - 2 \cos b}{1 + \cos a} \right) - \sin^{-1}(-1) \right) \\
&= \frac{1}{\sqrt{g}} \left(\sin^{-1} \left(\frac{\cos a - 1 - 2 \cos b}{1 + \cos a} \right) + \frac{\pi}{2} \right) \Big|
\end{aligned}$$

$$\begin{aligned}
b) \quad \frac{1}{\sqrt{g}} \left(\sin^{-1} \left(\frac{\cos a - 1 - 2 \cos b}{1 + \cos a} \right) + \frac{\pi}{2} \right) \Big|_{b=\pi} &= \frac{1}{\sqrt{g}} \left(\sin^{-1} \left(\frac{\cos a - 1 + 2}{1 + \cos a} \right) + \frac{\pi}{2} \right) \\
&= \frac{1}{\sqrt{g}} \left(\sin^{-1}(1) + \frac{\pi}{2} \right) \\
&= \frac{\pi}{\sqrt{g}} \Big|
\end{aligned}$$

Exercise

Find the area of the region bounded by the curve $f(x) = (16 + x^2)^{-3/2}$ and the x -axis on the interval $[0, 3]$

Solution

$$A = \int_0^3 \frac{dx}{(16 + x^2)^{3/2}}$$

$$x = 4 \tan \theta \rightarrow dx = 4 \sec^2 \theta d\theta$$

$$16 + x^2 = 16 \sec^2 \theta$$

$$= \int_0^3 \frac{4 \sec^2 \theta d\theta}{(16 \sec^2 \theta)^{3/2}}$$

$$= \int_0^3 \frac{4 \sec^2 \theta}{4^3 \sec^3 \theta} d\theta$$

$$= \frac{1}{16} \int_0^3 \cos \theta d\theta$$

$$= \frac{1}{16} \sin \theta \Big|_0^3$$

$$= \frac{1}{16} \frac{\tan \theta}{\sec \theta} \Big|_0^3$$

$$= \frac{1}{16} \frac{x}{\sqrt{16 + x^2}} \Big|_0^3$$

$$= \frac{1}{16} \left(\frac{3}{5} - 0 \right)$$

$$= \frac{3}{80} \text{ unit}^2$$

Exercise

Find the length of the curve $y = ax^2$ from $x = 0$ to $x = 10$, where $a > 0$ is a real number.

Solution

$$1 + (y')^2 = 1 + (2ax)^2$$

$$L = \int_0^{10} \sqrt{1 + 4a^2 x^2} \, dx$$

$$= \int_0^{10} 2a \sqrt{\frac{1}{4a^2} + x^2} \, dx$$

$$x = \frac{1}{2a} \tan \theta \quad \frac{1}{4a^2} + x^2 = \frac{1}{4a^2} \sec^2 \theta$$

$$dx = \frac{1}{4a^2} \sec^2 \theta \, d\theta$$

$$= \int_0^{10} 2a \frac{1}{2a} \sec \theta \frac{1}{4a^2} \sec^2 \theta \, d\theta$$

$$= \frac{1}{2a} \int_0^{10} \sec^3 \theta \, d\theta$$

$$u = \sec x \quad dv = \sec^2 x \, dx$$

$$du = \sec x \tan x \, dx \quad v = \tan x$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \tan x (\sec x \tan x \, dx)$$

$$= \sec x \tan x - \int \tan^2 x \sec x \, dx$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$\int \sec^3 x \, dx + \int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec^3 x \, dx + \int \sec x \, dx$$

$$2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx$$

$$= \sec x \tan x + \ln |\sec x + \tan x|$$

$$\begin{aligned}
&= \frac{1}{4a} \left(\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right) \Bigg|_0^{10} \\
&= \frac{1}{4a} \left(2a \sqrt{\frac{1}{4a^2} + x^2} (2ax) + \ln \left| \sqrt{1 + 4a^2 x^2} + 2ax \right| \right) \Bigg|_0^{10} \\
&= \frac{1}{4a} \left((2ax) \sqrt{1 + 4a^2 x^2} + \ln \left| \sqrt{1 + 4a^2 x^2} + 2ax \right| \right) \Bigg|_0^{10} \\
&= \frac{1}{4a} \left((20a) \sqrt{1 + 400a^2} + \ln \left| \sqrt{1 + 400a^2} + 20a \right| \right) \Bigg|_0^{10}
\end{aligned}$$

Exercise

Find the arc length of the graph of $f(x) = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$

Solution

$$1 + (f')^2 = 1 + x^2$$

$$L = \int_0^1 \sqrt{1 + x^2} \, dx$$

$$\begin{aligned}
x &= \tan \theta & \sqrt{x^2 + 1} &= \sec \theta \\
dx &= \sec^2 \theta \, d\theta
\end{aligned}$$

$$= \int_0^1 \sec^3 \theta \, d\theta$$

$$= \frac{1}{2} \left(\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right) \Bigg|_0^1$$

$$= \frac{1}{2} \left(x \sqrt{x^2 + 1} + \ln \left| x + \sqrt{x^2 + 1} \right| \right) \Bigg|_0^1$$

$$= \frac{1}{2} \left(\sqrt{2} + \ln(1 + \sqrt{2}) \right)$$

Exercise

A projectile is launched from the ground with an initial speed V at an angle θ from the horizontal. Assume that the x -axis is the horizontal ground and y is the height above the ground. Neglecting air resistance and letting g be the acceleration due to gravity, it can be shown that the trajectory of the projectile is given by

$$y = -\frac{1}{2}kx^2 + y_{\max} \quad \text{where } k = \frac{g}{(V \cos \theta)^2}$$

$$\text{and} \quad y_{\max} = \frac{(V \sin \theta)^2}{2g}$$

- a) Note that the high point of the trajectory occurs at $(0, y_{\max})$. If the projectile is on the ground at $(-a, 0)$ and $(a, 0)$, what is a ?
- b) Show that the length of the trajectory (arc length) is $2 \int_0^a \sqrt{1 + k^2 x^2} \, dx$
- c) Evaluate the arc length integral and express your result in the terms of V , g , and θ .
- d) For fixed value of V and g , show that the launch angle θ that maximizes the length of the trajectory satisfies $(\sin \theta) \ln(\sec \theta + \tan \theta) = 1$

Solution

a) At $(\pm a, 0) \rightarrow y = 0 = -\frac{1}{2}ka^2 + y_{\max}$

$$a^2 = \frac{2}{k} y_{\max} \Rightarrow a = \sqrt{\frac{2y_{\max}}{k}}$$

b) $y' = -kx \Rightarrow 1 + (y')^2 = 1 + k^2 x^2$

$$L = \int_{-a}^a \sqrt{1 + k^2 x^2} \, dx \quad \text{since } y(x) \text{ is an even function}$$

$$= 2 \int_0^a \sqrt{1 + k^2 x^2} \, dx$$

c) $L = 2 \int_0^a \sqrt{1 + k^2 x^2} \, dx$

$$x = \frac{1}{k} \tan \theta \Rightarrow dx = \frac{1}{k} \sec^2 \theta \, d\theta$$

$$1 + k^2 x^2 = \sec^2 \theta$$

$$= 2 \int_0^a \frac{1}{k} \sec \theta \sec^2 \theta \, d\theta$$

$$\begin{aligned}
&= \frac{2}{k} \int_0^a \sec^3 \theta \, d\theta \\
&= \frac{1}{k} \left(\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right) \Big|_0^a \\
&= \frac{1}{k} \left(\sqrt{1+k^2 x^2} (kx) + \ln \left| \sqrt{1+k^2 x^2} + kx \right| \right) \Big|_0^a \\
&= \frac{1}{k} \left(ak\sqrt{1+k^2 a^2} + \ln \left| \sqrt{1+k^2 a^2} + ka \right| \right)
\end{aligned}$$

$$\begin{aligned}
a &= \sqrt{\frac{2}{k} \frac{(V \sin \theta)^2}{2g}} \\
&= \frac{V \sin \theta}{\sqrt{g \frac{g}{(V \cos \theta)^2}}} \\
&= \frac{V^2}{g} \sin \theta \cos \theta
\end{aligned}$$

$$k = \frac{g}{(V \cos \theta)^2}$$

$$ak = \tan \theta$$

$$\begin{aligned}
L(\theta) &= \frac{(V \cos \theta)^2}{g} \left(\tan \theta \sqrt{1 + \tan^2 \theta} + \ln \left| \sqrt{1 + \tan^2 \theta} + \tan \theta \right| \right) \\
&= \frac{V^2 \cos^2 \theta}{g} (\tan \theta \sec \theta + \ln |\sec \theta + \tan \theta|) \\
&= \frac{V^2}{g} \sin \theta + \frac{V^2}{g} \cos^2 \theta \ln |\sec \theta + \tan \theta| \\
&= \frac{V^2}{g} \left(\sin \theta + \cos^2 \theta \sinh^{-1}(\tan \theta) \right)
\end{aligned}$$

$$\begin{aligned}
d) \quad L'(\theta) &= \frac{V^2}{g} \left(\cos \theta - 2 \cos \theta \sin \theta \sinh^{-1}(\tan \theta) + \cos^2 \theta \frac{\sec^2 \theta}{\sqrt{1 + \tan^2 \theta}} \right) \\
&= \frac{V^2}{g} \left(\cos \theta - 2 \cos \theta \sin \theta \sinh^{-1}(\tan \theta) + \cos^2 \theta \sec \theta \right) \\
&= \frac{2V^2 \cos \theta}{g} \left(1 - \sin \theta \sinh^{-1}(\tan \theta) \right) = 0
\end{aligned}$$

$$\sin \theta \sinh^{-1}(\tan \theta) = 1$$

$$\sin \theta \ln(\sec \theta + \tan \theta) = 1 \quad \checkmark$$

Exercise

Let $F(x) = \int_0^x \sqrt{a^2 - t^2} dt$. The figure shows that $F(x) = \text{area of sector } OAB + \text{area of triangle } OBC$

a) Use the figure to prove that $F(x) = \frac{a^2 \sin^{-1}\left(\frac{x}{a}\right)}{2} + \frac{x\sqrt{a^2 - x^2}}{2}$

b) Conclude that $\int \sqrt{a^2 - x^2} dx = \frac{a^2 \sin^{-1}\left(\frac{x}{a}\right)}{2} + \frac{x\sqrt{a^2 - x^2}}{2} + C$

Solution

a) Area of sector OAB is $\frac{1}{2}\theta a^2$

From the triangle OBC : $\sin \theta = \frac{x}{a} \rightarrow \theta = \sin^{-1} \frac{x}{a}$

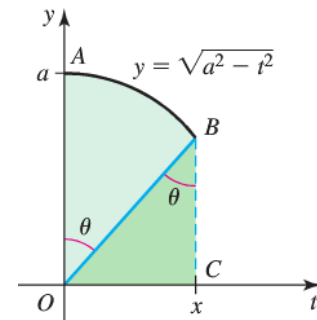
$$|BC| = \sqrt{a^2 - x^2}$$

Area of sector OAB is $\frac{1}{2}a^2 \sin^{-1} \frac{x}{a}$

Area of triangle OBC : $\frac{1}{2}x\sqrt{a^2 - x^2}$

$F(x) = \text{area of sector } OAB + \text{area of triangle } OBC$

$$= \frac{a^2 \sin^{-1}\left(\frac{x}{a}\right)}{2} + \frac{x\sqrt{a^2 - x^2}}{2}$$



$$\begin{aligned} b) \quad \frac{d}{dx} \left(\frac{a^2 \sin^{-1}\left(\frac{x}{a}\right)}{2} + \frac{x\sqrt{a^2 - x^2}}{2} + C \right) &= \frac{a^2}{2} \frac{\frac{1}{a}}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} + \frac{1}{2} \sqrt{a^2 - x^2} - \frac{1}{2} \frac{x^2}{\sqrt{a^2 - x^2}} \\ &= \frac{1}{2} \frac{a^2}{\sqrt{a^2 - x^2}} + \frac{1}{2} \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}} \\ &= \frac{1}{2} \frac{2a^2 - 2x^2}{\sqrt{a^2 - x^2}} \\ &= \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} \\ &= \sqrt{a^2 - x^2} \end{aligned}$$

By the antiderivative:

$$\int \sqrt{a^2 - x^2} dx = \frac{a^2 \sin^{-1}\left(\frac{x}{a}\right)}{2} + \frac{x\sqrt{a^2 - x^2}}{2} + C \quad \checkmark$$

Exercise

A sealed barrel of oil (weighing 48 pounds per cubic foot) is floating in seawater (weighing 64 pounds per cubic foot). The barrel is not completely full of oil. With the barrel lying on its side, the top 0.2 foot of the barrel is empty.

Compare the fluid forces against one end of the barrel from the inside and from the outside.

Solution

$$x^2 + y^2 = 1 \rightarrow 2x = 2\sqrt{1 - y^2}$$

$$F_{inside} = 48 \int_{-1}^{0.8} (0.8 - y)(2)\sqrt{1 - y^2} dy$$

$$= 76.8 \int_{-1}^{0.8} \sqrt{1 - y^2} dy - 96 \int_{-1}^{0.8} y\sqrt{1 - y^2} dy$$

$$= 76.8 \int_{-1}^{0.8} \sqrt{1 - y^2} dy + 48 \int_{-1}^{0.8} (1 - y^2)^{1/2} d(1 - y^2)$$

$$y = \sin \theta \quad \sqrt{1 - y^2} = \cos \theta$$

$$dy = \cos \theta d\theta$$

$$= 76.8 \int_{-1}^{0.8} \cos^2 \theta d\theta + 32(1 - y^2)^{3/2} \Big|_{-1}^{0.8}$$

$$= 38.4 \int_{-1}^{0.8} (1 + \cos 2\theta) d\theta + 32(0.16)^{3/2}$$

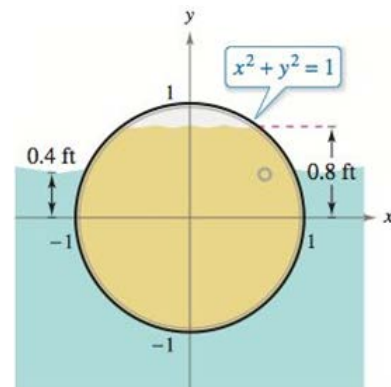
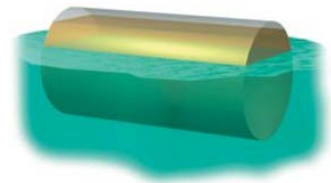
$$= 38.4 \left(\theta + \frac{1}{2} \sin 2\theta \right) \Big|_{-1}^{0.8} + 32(0.4)^3$$

$$= 38.4 \left(\arcsin y + y\sqrt{1 - y^2} \right) \Big|_{-1}^{0.8} + 2.048$$

$$= 38.4 \left(\arcsin 0.8 + 0.32 + \frac{\pi}{2} \right) + 2.048$$

$$\approx 121.3 \text{ lbs}$$

$$F = w \int_c^d h(y)L(y) dy$$



$$F_{outside} = 64 \int_{-1}^{0.4} (0.4 - y)(2)\sqrt{1 - y^2} dy$$

$$= 51.2 \int_{-1}^{0.4} \sqrt{1 - y^2} dy - 128 \int_{-1}^{0.4} y\sqrt{1 - y^2} dy$$

$$F = w \int_c^d h(y)L(y) dy$$

$$= 25.6 \left(\arcsin y + y\sqrt{1-y^2} \right) \Big|_{-1}^{0.4} + \frac{128}{3} (1-y^2)^{3/2} \Big|_{-1}^{0.4}$$

$$\approx \underline{93.0 \text{ lbs}}$$

Exercise

The axis of a storage tank in the form of a right circular cylinder is horizontal. The radius and length of the tank are 1 meter and 3 meters, respectively.

- Determine the volume of fluid in the tank as a function of its depth d .
- Graph the function in part (a).
- Design a dip stick for the tank with markings of $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$
- Fluid is entering the tank at a rate of $\frac{1}{4} \text{ m}^3/\text{min}$. Determine the rate of change of the depth of the fluid as a function of its depth d .
- Graph the function in part (d). When will the rate of change of the depth be minimum?

Solution

a) Consider the center at $(0, 1)$: $x^2 + (y-1)^2 = 1 \rightarrow x = \sqrt{1-(y-1)^2}$

The depth: $0 \leq d \leq 2$

$$V = \int_0^d (3) \left(2\sqrt{1-(y-1)^2} \right) dy$$

$$= 6 \int_0^d \sqrt{1-(y-1)^2} d(y-1)$$

$$= 6 \int_0^d \cos^2 \theta d\theta$$

$$= 3 \int_0^d (1 + \cos 2\theta) d\theta$$

$$= 3 \left(\theta + \frac{1}{2} \sin 2\theta \right) \Big|_0^d$$

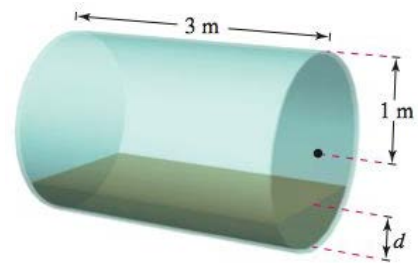
$$= 3 \left(\theta + \sin \theta \cos \theta \right) \Big|_0^d$$

$$= 3 \left(\arcsin(y-1) + (y-1)\sqrt{1-(y-1)^2} \right) \Big|_0^d$$

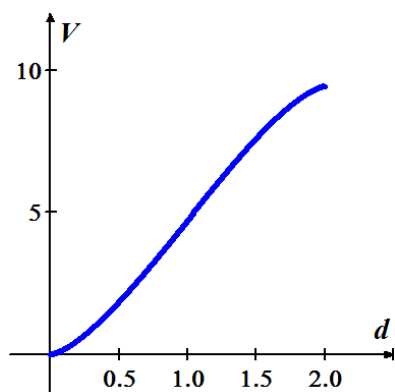
$$\underline{= 3 \arcsin(d-1) + 3(d-1)\sqrt{2d-d^2} + \frac{3\pi}{2}}$$

$$y-1 = \sin \theta \quad \sqrt{1-(y-1)^2} = \cos \theta$$

$$d(y-1) = \cos \theta d\theta$$



b)



c) The full tank holds $3\pi \text{ m}^3$

A dip stick for the tank with markings of $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$

The horizontal lines are: $y = \frac{3\pi}{4}$, $y = \frac{3\pi}{2}$, $y = \frac{9\pi}{4}$

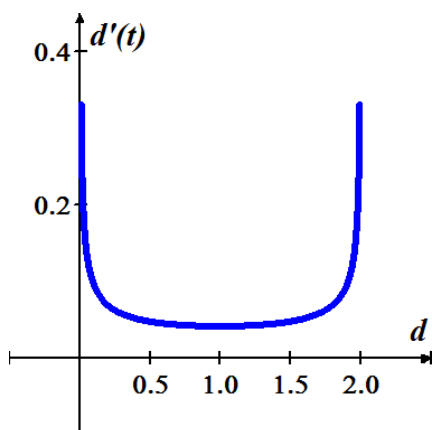
Intersect the curve at $d = 0.596$, $d = 1.0$, $d = 1.404$

$$d) \quad V = 6 \int_0^d \sqrt{1 - (y-1)^2} \, dy \rightarrow \frac{dV}{dt} = \frac{dV}{dd} \frac{dd}{dt}$$

$$\frac{dV}{dt} = 6\sqrt{1 - (d-1)^2} \cdot d'(t) = \frac{1}{4}$$

$$d'(t) = \frac{1}{24\sqrt{1 - (d-1)^2}}$$

e)



From the graph, the minimum occurs at $d = 1$, which is the widest part of the tank.

Exercise

The field strength H of a magnet of length $2L$ on a particle r units from the center of the magnet is

$$H = \frac{2mL}{(r^2 + L^2)^{3/2}}$$

Where $\pm m$ are the poles of the magnet.

Find the average field strength as the particle moves from 0 to R units from the center by evaluating the integral

$$\frac{1}{R} \int_0^R \frac{2mL}{(r^2 + L^2)^{3/2}} dr$$

Solution

$$r = L \tan \theta \rightarrow dr = L \sec^2 \theta d\theta$$

$$r^2 + L^2 = L^2 \tan^2 \theta + L^2 = L^2 \sec^2 \theta \Big|$$

$$\frac{1}{R} \int_0^R \frac{2mL}{(r^2 + L^2)^{3/2}} dr = \frac{1}{R} \int_0^R \frac{2mL}{(L \sec \theta)^3} L \sec^2 \theta d\theta$$

$$= \frac{2m}{RL} \int_0^R \frac{1}{\sec \theta} d\theta$$

$$= \frac{2m}{RL} \int_0^R \cos \theta d\theta$$

$$= \frac{2m}{RL} \sin \theta \Big|_0^R$$

$$= \frac{2m}{RL} \frac{r}{\sqrt{r^2 + L^2}} \Big|_0^R$$

$$= \frac{2m}{L \sqrt{R^2 + L^2}} \Big|$$

