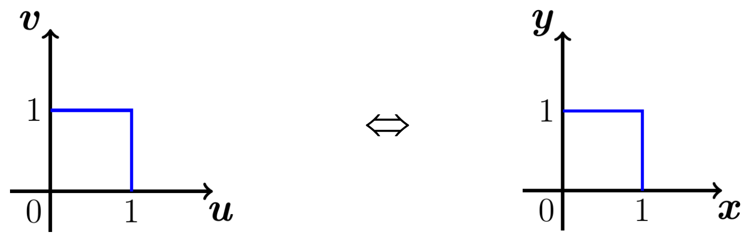
Solution Section 3.7 – Change of Variables in Multiple Integrals

Exercise

Let $S=\{0\le u\le 1,\ 0\le v\le 1\}$ be a unit square in the *uv*-plane. Find the image of *S* in the *xy*-plane under the following transformations. $T:\quad x=v,\ y=u$

Solution

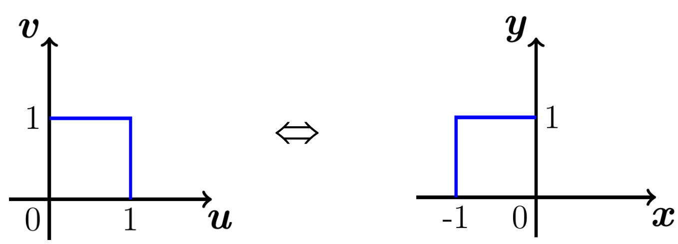


The transformation just switches the coordinates. Image *xy* is unit square.

Exercise

Let $S=\{0\le u\le 1,\ 0\le v\le 1\}$ be a unit square in the *uv*-plane. Find the image of *S* in the *xy*-plane under the following transformations. $T:\quad x=-v,\ y=u$

Solution



$T:\quad x=-v,\ y=u$

$$T=\{(x,\ y):\ \ 0\le -x\le 1\quad 0\le y\le 1\}$$
$$=\{(x,\ y):\ \ -1\le x\le 0\quad 0\le y\le 1\}$$

T is a unit square in *Q*II with one vertex at origin.

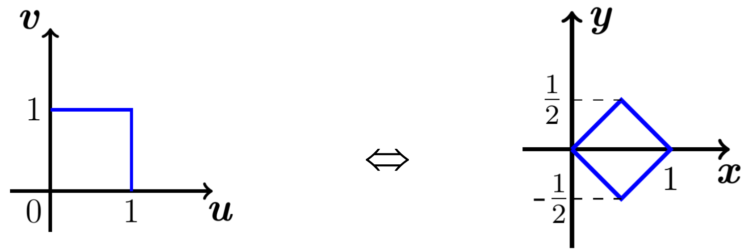
Exercise

Let $S=\{0\le u\le 1,\ 0\le v\le 1\}$ be a unit square in the *uv*-plane. Find the image of *S* in the *xy*-plane under the following transformations. $T:\quad x=\dfrac{u+v}{2},\ \ y=\dfrac{u-v}{2}$

Solution

$T:\quad x=\dfrac{u+v}{2},\ \ y=\dfrac{u-v}{2}$

$(u,\ v)$	$(x,\ y)$
$(0,\ 0)\ \to\ (1,\ 0)$	$(0,\ 0)\ \to\ \left(\frac{1}{2},\ \frac{1}{2}\right)$
$(1,\ 0)\ \to\ (1,\ 1)$	$\left(\frac{1}{2},\ \frac{1}{2}\right)\ \to\ (1,\ 0)$
$(1,\ 1)\ \to\ (0,\ 1)$	$(1,\ 0)\ \to\ \left(\frac{1}{2},\ -\frac{1}{2}\right)$



Diamond shape

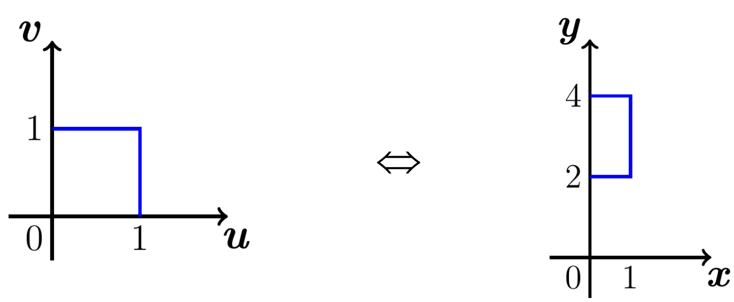
Exercise

Let $S=\{0\le u\le 1,\ 0\le v\le 1\}$ be a unit square in the uv-plane. Find the image of S in the xy-plane under the following transformations. $T:\quad x=u,\ \ y=2v+2$

Solution

$T:\quad x=u,\ \ y=2v+2$

$(u,\ v)$	$(x,\ y)$
$(0,\ 0)\ \to\ (1,\ 0)$	$(0,\ 2)\ \to\ (1,\ 2)$
$(1,\ 0)\ \to\ (1,\ 1)$	$(1,\ 2)\ \to\ (1,\ 4)$
$(1,\ 1)\ \to\ (0,\ 1)$	$(1,\ 4)\ \to\ (0,\ 4)$



Exercise

a) Solve the system $u = x - y, \quad v = 2x + y$ for *x* and *y* in terms of *u* and *v*. Then find the value of the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$

b) Find the image under the transformation $u = x - y, \ \ v = 2x + y$ of the triangular region with vertices (0, 0), (1, 1), and (1, –2) in the *xy*-plane. Sketch the transformed region in the *uv*-plane.

Solution

a)
$$\begin{cases} u = x - y \\ v = 2x + y \end{cases}$$

$$\begin{cases} x = \frac{1}{3}u + \frac{1}{3}v \\ y = -\frac{2}{3}u + \frac{1}{3}v \end{cases}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix}$$

$$= \frac{1}{9} + \frac{2}{9}$$

$$= \frac{1}{3}$$

b) From (0, 0) to (1, 1)

$$\Rightarrow \ y = x \ \rightarrow u = x - y = 0$$

From (0, 0) to (1, –2)

$$\Rightarrow \ y = -2x \ \rightarrow u = 2x + y = 0$$

From (1, 1) to (1, –2) $\Rightarrow \ x = 1$

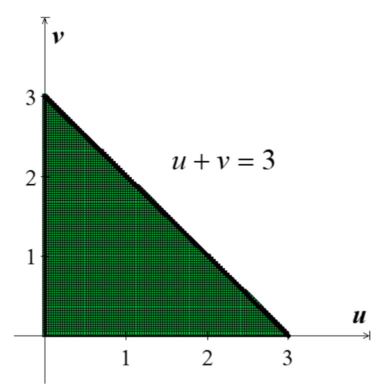
$$\rightarrow x = \frac{1}{3}u + \frac{1}{3}v = 1$$

$$u + v = 3$$

OR: $(0,\ 0) \ \rightarrow \ \begin{cases} u = 0 \\ v = 0 \end{cases}$

$(1,\ 1) \ \rightarrow \ \begin{cases} u = 0 \\ v = 3 \end{cases}$

$(1,\ -2) \ \rightarrow \ \begin{cases} u = 3 \\ v = 0 \end{cases}$



Exercise

Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy=1$, $xy=9$ and the lines $y=x$, $y=4x$. Use the transformation $x=\frac{u}{v}$, $y=uv$ with $u>0$, *and* $v>0$ to rewrite

$$\iint_R \left(\sqrt{\frac{y}{x}}+\sqrt{xy}\right)dxdy$$

As an integral over an appropriate region G in the uv-plane. Then evaluate the uv-integral over G.

Solution

$$\begin{aligned} x &= \frac{u}{v} \\ y &= uv \end{aligned} \quad\to\quad \begin{aligned} u &= xv \\ y &= xv^2 \end{aligned} \quad \begin{cases} \frac{y}{x}=v^2 \\ xy=u^2 \end{cases}$$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix}=\frac{u}{v}+\frac{u}{v}=\boxed{2\frac{u}{v}}$$

$$\begin{aligned} xy &= 1=u^2 \\ xy &= 9=u^2 \end{aligned} \quad\to\quad \begin{cases} u=1 \\ u=3 \end{cases} \qquad\qquad \begin{aligned} y=x &\Rightarrow \frac{y}{x}=1=v^2 \\ y=4x &\Rightarrow \frac{y}{x}=4=v^2 \end{aligned} \quad\to\quad \begin{cases} v=1 \\ v=2 \end{cases}$$

$$\begin{aligned}
\iint_R \left(\sqrt{\frac{y}{x}}+\sqrt{xy}\right)dxdy &= \int_1^3\int_1^2 (v+u)\frac{2u}{v}\,dvdu \\
&= 2\int_1^3\int_1^2 \left(u+\frac{u^2}{v}\right)dvdu \\
&= 2\int_1^3 \left(uv+u^2\ln v \Big|_1^2\right. du \\
&= 2\int_1^3 \left(2u+u^2\ln 2-u\right)du \\
&= 2\int_1^3 \left(u+u^2\ln 2\right)du \\
&= 2\left(\frac{1}{2}u^2+\frac{1}{3}u^3\ln 2\right. \Big|_1^3 \\
&= 2\left(\frac{9}{2}+9\ln 2-\frac{1}{2}-\frac{1}{3}\ln 2\right) \\
&= \boxed{8+\frac{52}{3}\ln 2}
\end{aligned}$$

Exercise

The area πab of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ can be found by integrating the function $f(x,y)=1$ over the region bounded by the ellipse in the *xy*-plane. Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate the integral is to use the transformation $x=au,\ y=bv$ and evaluate the transformed integral over the disk *G*: $u^2+v^2\le 1$ in the *uv*-plane. Find the area this way.

Solution

$$x=au,\ y=bv$$

$$\frac{\partial(x,y)}{\partial(u,v)}=J(u,\ v)$$

$$=\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}$$

$$=ab$$

$$u^2+v^2\le 1 \quad\to\quad -1\le u\le 1$$

$$u^2+v^2\le 1 \quad\to\quad v^2\le 1-u^2$$

$$-\sqrt{1-u^2}\le v\le \sqrt{1-u^2}$$

$$\iint_R dxdy=\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} ab\ dvdu$$

$$=ab\int_{-1}^{1}\left(\sqrt{1-u^2}+\sqrt{1-u^2}\right)du$$

$$=2ab\int_{-1}^{1}\sqrt{1-u^2}\ du$$

$$u=\sin\alpha \quad \& \quad \sqrt{1-u^2}=\cos\alpha$$

$$du=\cos\alpha\ d\alpha$$

$$\int \sqrt{1-u^2}\ dx=\int \cos^2\alpha\ d\alpha$$

$$=\frac{1}{2}\int(1+\cos 2\alpha)\ d\alpha$$

$$=\frac{1}{2}\left(\alpha+\frac{1}{2}\sin 2\alpha\right)$$

$$=\frac{1}{2}\ (\alpha+\sin\alpha\cos\alpha)$$

$$=\frac{1}{2}\left(\sin^{-1}(u)+u\sqrt{1-u^2}\right)$$

$$=2ab\,\frac{1}{2}\left(\sin^{-1}u\;+u\sqrt{1-u^2}\right.\Bigg|_{-1}^{1}$$

$$=ab\left(\sin^{-1}(1)-\sin^{-1}(-1)\right)$$

$$=ab\left(\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$\underline{=ab\pi\,}|$$

Exercise

Use the transformation $x=u+\frac{1}{2}v,\;\; y=v$ to evaluate the integral

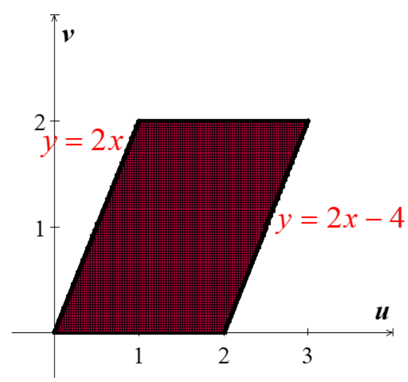
$$\int_0^2\int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2}\,dxdy$$

By first writing it as an integral over a region G in the uv-plane.

Solution

$$J(u,\ v)=\begin{vmatrix}1 & \frac{1}{2}\\ 0 & 1\end{vmatrix}$$

$$\underline{=1}|$$



$$\begin{matrix} x=u+\frac{1}{2}v \\ y=v \end{matrix} \;\to\; \begin{matrix} u=x-\frac{1}{2}y \\ v=y \end{matrix}$$

$$x=\frac{y}{2}\to y=2x$$

$$x=\frac{y+4}{2}\to y=2x-4$$

$$0\le x\le 2$$

$x=\frac{y}{2}$	$u=x-\frac{y}{2}=\frac{y}{2}-\frac{y}{2}=0$	$u=0$
$x=\frac{y}{2}+2$	$u=x-\frac{y}{2}=\frac{y}{2}+2-\frac{y}{2}=2$	$u=2$
$y=0$	$v=0$	$v=0$
$y=2$	$v=2$	$v=2$

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3 (2x-y) e^{(2x-y)^2}\, dxdy = \int_0^2 \int_0^2 v^3 (2u) e^{4u^2}\, dudv \qquad d\left(4u^2\right) = 8udu$$

$$= \frac{1}{4} \int_0^2 v^3 dv \int_0^2 e^{4u^2} d\left(4u^2\right)$$

$$= \frac{1}{16} \left(v^4 \Big|_0^2 \quad \left(e^{4u^2} \Big|_0^2\right.\right.$$

$$= \frac{1}{16}(16)\left(e^{16}-1\right)$$

$$\boxed{= e^{16}-1}$$

Exercise

Use the transformation $x = \frac{u}{v}$, $y = uv$ to evaluate the integral

$$\int_1^2 \int_{1/y}^{y} \left(x^2+y^2\right) dxdy + \int_2^4 \int_{y/4}^{4/y} \left(x^2+y^2\right) dxdy$$

Solution

$$\begin{aligned} x &= \frac{u}{v} \\ y &= uv \end{aligned} \quad \rightarrow \quad \begin{aligned} u &= xv \\ y &= xv^2 \end{aligned}$$

$$\begin{cases} \dfrac{y}{x} = v^2 \\ xy = u^2 \end{cases}$$

$$J(u,v) = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix}$$

$$= \frac{u}{v} + \frac{u}{v}$$

$$\boxed{= \frac{2u}{v}}$$

$x = y$	$\frac{y}{x} = 1 = v^2$	$v = 1$
$x = \frac{1}{y}$	$xy = 1 = u^2$	$u = 1$
$x = \frac{4}{y}$	$xy = 4 = u^2$	$u = 2$
$x = \frac{y}{4}$	$\frac{y}{x} = 4 = v^2$	$v = 2$

$$\int_1^2\int_{1/y}^{y}\left(x^2+y^2\right)dxdy+\int_2^4\int_{y/4}^{4/y}\left(x^2+y^2\right)dxdy=\int_1^2\int_1^2\left(\frac{u^2}{v^2}+u^2v^2\right)\left(\frac{2u}{v}\right)dudv$$

$$=2\int_1^2\int_1^2\left(\frac{u^3}{v^3}+u^3v\right)dudv$$

$$=2\int_1^2\left(v^{-3}+v\right)dv\quad\left(\frac{1}{4}u^4\ \Big|_1^2\right.$$

$$=2\left(-\frac{1}{2}v^{-2}+\frac{1}{2}v^2\ \Big|_1^2\right.\left(\frac{15}{4}\right)$$

$$=\frac{15}{4}\left(-\frac{1}{4}+4-(-1+1)\right)$$

$$=\frac{15}{4}\left(\frac{15}{4}\right)$$

$$=\frac{225}{16}$$

Exercise

Find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ of the transformation

a) $x=u\cos v,\quad y=u\sin v$

b) $x=u\sin v,\quad y=u\cos v$

Solution

a) $$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\cos v & -u\sin v\\ \sin v & u\cos v\end{vmatrix}$$

$$=u\cos^2 v+u\sin^2 v$$

$$=u$$

b) $$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\sin v & u\cos v\\ \cos v & -u\sin v\end{vmatrix}$$

$$=-u\sin^2 v-u\cos^2 v$$

$$=-u$$

Exercise

Find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ of the transformation

a) $x = u\cos v, \quad y = u\sin v, \quad z = w$

b) $x = 2u - 1, \quad y = 3v - 4, \quad z = \frac{1}{2}(w-4)$

Solution

a) $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \cos v & -u\sin v & 0 \\ \sin v & u\cos v & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= u\cos^2 v + u\sin^2 v$$

$$\underline{= u}$$

b) $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{vmatrix}$$

$$\underline{= 3}$$

Exercise

Evaluate the appropriate determinant to show that the Jacobian of the transformation from Cartesian $\rho\phi\theta$–space to Cartesian xyz–space is $\rho^2\sin\phi$

Solution

$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \begin{matrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta \\ \cos\phi & -\rho\sin\phi \end{matrix}$$

$$= \rho^2\cos^2\phi\sin\phi\cos^2\theta + \rho^2\sin^3\phi\sin^2\theta + \rho^2\sin\phi\cos^2\phi\sin^2\theta + \rho^2\sin^3\phi\cos^2\theta$$

$$= \rho^2\cos^2\phi\sin\phi\left(\cos^2\theta + \sin^2\theta\right) + \rho^2\sin^3\phi\left(\sin^2\theta + \cos^2\theta\right)$$

$$= \rho^2\cos^2\phi\sin\phi + \rho^2\sin^3\phi$$

$$= \rho^2\sin\phi\left(\cos^2\phi + \sin^2\phi\right)$$

$$\underline{= \rho^2\sin\phi}$$

Exercise

How can substitutions in single definite integrals be viewed as transformations of regions? What is the Jacobian in such a case? Illustrate with an example.

Solution

Let $u = g(x) \quad \Rightarrow \quad J(x) = \frac{du}{dx} = g'(x)$

$$\int_a^b f(u)\,du = \int_{g(a)}^{g(b)} f(g(x))g'(x)\,dx$$

$g'(x)$ represents the Jacobian of the transformation $u = g(x) \quad or \quad x = g^{-1}(u)$

Exercise

Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

(***Hint***: Let $x = au,\ y = bv,\ and\ z = cw$. Then find the volume of an appropriate region in *uvw*-space)

Solution

$$J(u,v,w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

$$= abc$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$$

$$\Rightarrow \frac{(au)^2}{a^2} + \frac{(bv)^2}{b^2} + \frac{(cw)^2}{c^2} \le 1$$

$$u^2 + v^2 + w^2 \le 1$$

$$V = \frac{4\pi}{3} = \iiint\limits_G dudvdw$$

$$V = \iiint\limits_R dxdydz$$

$$= \iiint\limits_G abc\ dudvdw$$

$$= abc \iiint_G dudvdw$$

$$\boxed{= \frac{4\pi abc}{3} \quad unit^3}$$

Exercise

Use the transformation $x = u^2 - v^2, \ \ y = 2uv$ to evaluate the integral

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2} \ dydx$$

(Hint: Show that the image of the triangular region G with vertices (0, 0), (1, 0), (1, 1) in the uv-plane is the region of integration R in the xy-plane defined by the limits of integration.)

Solution

$x = u^2 - v^2, \ \ y = 2uv$

$$J(u,v) = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} \qquad J(u, \ v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= 4u^2 + 4v^2$$

$$\boxed{= 4\left(u^2 + v^2\right)}$$

$y = 2\sqrt{1-x}$	$2uv = 2\sqrt{1-u^2+v^2} \to u^2v^2 = 1-u^2+v^2$ $u^2v^2 + u^2 = 1+v^2 \Rightarrow u^2\left(v^2+1\right) = 1+v^2$	$u = \pm 1$
$y = 0$	$2uv = 0$	$u = 0, \ v = 0$
$x = 0$	$u^2 - v^2 = 0$	$u = \pm v$

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2} \ dydx = \int_0^1 \int_0^u \sqrt{\left(u^2 - v^2\right)^2 + (2uv)^2} \ \cdot \ 4\left(u^2 + v^2\right) dvdu$$

$$= 4\int_0^1 \int_0^u \sqrt{u^4 + v^4 - 2u^2v^2 + 4u^2v^2} \ \left(u^2 + v^2\right) dvdu$$

$$= 4\int_0^1 \int_0^u \sqrt{u^4 + v^4 + 2u^2v^2} \ \left(u^2 + v^2\right) dvdu$$

$$= 4\int_0^1\int_0^u \sqrt{\left(u^2+v^2\right)^2}\ \left(u^2+v^2\right)dvdu$$

$$= 4\int_0^1\int_0^u \left(u^2+v^2\right)^2\ dvdu$$

$$= 4\int_0^1\int_0^u \left(u^4+v^4+2u^2v^2\right)dvdu$$

$$= 4\int_0^1 \left(u^4v+\frac{1}{5}v^5+\frac{2}{3}u^2v^3\ \Big|_0^u\ du\right.$$

$$= 4\int_0^1 \left(u^5+\frac{1}{5}u^5+\frac{2}{3}u^5\right)du$$

$$= \frac{112}{15}\int_0^1 u^5\ du$$

$$= \frac{112}{15}\ \left(\frac{1}{6}u^6\ \Big|_0^1\right.$$

$$\boxed{= \frac{56}{45}}$$

Exercise

Evaluate $\iint\limits_R y^4 dA$; R is the region bounded by the hyperbolas $xy=1$ and $xy=4$ and the lines $\frac{y}{x}=1$, and $\frac{y}{x}=3$

Solution

Let $\begin{cases} u = xy & \rightarrow x = \frac{u}{y} \\ v = \frac{y}{x} & \rightarrow x = \frac{y}{v} \end{cases}$

$$x = \frac{y}{v} = \frac{u}{y}$$

$$y^2 = uv \quad \Rightarrow \quad \boxed{y = \sqrt{uv}}$$

$$x = \frac{\sqrt{uv}}{v}$$

$$\boxed{= \sqrt{\frac{u}{v}}}$$

$$\begin{cases} x = u^{1/2}v^{-1/2} \\ y = u^{1/2}v^{1/2} \end{cases}$$

$$J(u,\ v) = \begin{vmatrix} \frac{1}{2}u^{-1/2}v^{-1/2} & -\frac{1}{2}u^{1/2}v^{-3/2} \\ \frac{1}{2}u^{-1/2}v^{1/2} & \frac{1}{2}u^{1/2}v^{-1/2} \end{vmatrix} \qquad J(u,\ v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \frac{1}{4}v^{-1} + \frac{1}{4}v^{-1}$$

$$= \frac{1}{2v}$$

$xy = 1$	$u = xy = 1$	$u = 1$
$xy = 4$	$u = xy = 4$	$u = 4$
$\frac{y}{x} = 1$	$v = \frac{y}{x} = 1$	$v = 1$
$\frac{y}{x} = 3$	$v = \frac{y}{x} = 3$	$v = 3$

$$\iint_R y^4 dA = \int_1^4 \int_1^3 \frac{1}{2v}\left(\sqrt{uv}\right)^4 dvdu \qquad \iint_R f(x,y)\,dxdy = \iint_G f\left(g(u,v), h(u,v)\right)\left|J(u,v)\right| dudv$$

$$= \frac{1}{2}\int_1^4 u^2\ du \int_1^3 v\ dv$$

$$= \frac{1}{12}\ u^3 \Big|_1^4 \quad v^2 \Big|_1^3$$

$$= \frac{1}{12}(64-1)(9-1)$$

$$= \frac{504}{12}$$

$$= 42$$

Exercise

Evaluate $\iint_R \left(y^2 + xy - 2x^2\right)dA$; R is the region bounded by the lines $y = x$, $y = x - 3$, $y = -2x + 3$, and $y = -2x - 3$

Solution

$$\begin{cases} y - x = 0 \qquad y - x = -3 \\ y + 2x = \pm 3 \end{cases}$$

Let $\begin{cases} u = y - x \\ v = y + 2x \end{cases}$

$$\begin{aligned} y - x &= u \\ \underline{y + 2x} &= \underline{v} \end{aligned}$$

$$\rightarrow \begin{cases} x = \frac{1}{3}(v - u) \\ y = \frac{1}{3}(v + 2u) \end{cases}$$

$$J(u,\ v) = \begin{vmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{vmatrix} \qquad\qquad J(u,\ v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= -\frac{1}{3}$$

$y = x$	$\frac{1}{3}(v+2u) = \frac{1}{3}(v-u) \rightarrow 2u = -u$	$u = 0$
$y = x - 3$	$\frac{1}{3}(v+2u) = \frac{1}{3}(v-u) - 3 \rightarrow 2u = -u - 9$	$u = -3$
$y = -2x + 3$	$\frac{1}{3}(v+2u) = -\frac{2}{3}(v-u) + 3 \rightarrow v = -2v + 9$	$v = 3$
$y = -2x - 3$	$\frac{1}{3}(v+2u) = -\frac{2}{3}(v-u) - 3 \rightarrow v = -2v - 9$	$v = -3$

$$\begin{aligned} y^2 + xy - 2x^2 &= \frac{1}{9}(v+2u)^2 + \frac{1}{9}(v+2u)(v-u) - \frac{2}{9}(v-u)^2 \\ &= \frac{1}{9}\left(v^2 + 4uv + 4u^2 + v^2 + uv - 2u^2 - 2v^2 + 4uv - 2u^2\right) \\ &= uv \end{aligned}$$

$$\iint_R \left(y^2 + xy - 2x^2\right) dA = \int_{-3}^{0}\int_{-3}^{3}\left(-\frac{1}{3}\right) uv\ dvdu$$

$$= -\frac{1}{3}\int_{-3}^{0} u\ du \quad \int_{-3}^{3} v\ dv$$

$$= -\frac{1}{12}\left(u^2 \Big|_{-3}^{0}\right)\left(v^2 \Big|_{-3}^{3}\right)$$

$$= -\frac{1}{12}(-9)(9-9)$$

$$= 0$$

Exercise

Evaluate $\iint\limits_D x\, dV$; R is bounded by the planes $y-2x=0$, $y-2x=1$, $z-3y=0$, $z-3y=1$, $z-4x=0$ and $z-4x=3$

Solution

Let $\begin{cases} u=y-2x & \rightarrow & 0\le u\le 1 \\ v=z-3y & \rightarrow & 0\le v\le 1 \\ w=z-4x & \rightarrow & 0\le w\le 3 \end{cases}$

$$\begin{cases} u=y-2x & \rightarrow & y=u+2x \\ w=z-4x & \rightarrow & z=w+4x \end{cases}$$

$$v=z-3y$$

$$=w+4x-3u-6x$$

$$2x=w-3u-v$$

$$\begin{cases} x=-\frac{3}{2}u-\frac{1}{2}v+\frac{1}{2}w \\ y=-2u-v+w \\ z=-6u-2v+3w \end{cases}$$

$$J(u,\ v,\ w)=\begin{vmatrix} -2 & -1 & 1 \\ -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ -6 & -2 & 3 \end{vmatrix}$$

$$=\frac{1}{2}$$

$$\iint\limits_D x\, dV=\int_0^3\int_0^1\int_0^1\left(\frac{1}{2}\right)\frac{1}{2}(-3u-v+w)\,dudvdw$$

$$=\frac{1}{4}\int_0^3\int_0^1\left(-\frac{3}{2}u^2-vu+wu\ \Big|_0^1\ dvdw\right.$$

$$=\frac{1}{4}\int_0^3\int_0^1\left(-\frac{3}{2}-v+w\right)dvdw$$

$$=\frac{1}{4}\int_0^3\left(-\frac{3}{2}v-\frac{1}{2}v^2+wv\ \Big|_0^1\ dw\right.$$

$$=\frac{1}{4}\int_0^3\left(-\frac{3}{2}-\frac{1}{2}+w\right)dw$$

$$=\frac{1}{4}\left(-2w+\frac{1}{2}w^2\right|_0^3$$

$$=\frac{1}{4}\left(-6+\frac{9}{2}\right)$$

$$=-\frac{3}{8}$$

Exercise

Let R be the region bounded by the lines $x+y=1;\quad x+y=4;\quad x-2y=0;\quad x-2y=-4$

Evaluate the integral $\displaystyle\iint_R 3xy\,dA$

Solution

Let $\begin{cases} u=x+y \\ v=x-2y \end{cases}$

$\overline{u-v=3y} \quad \rightarrow \quad y=\frac{1}{3}(u-v)$

$$x=u-y$$

$$=u-\frac{1}{3}u+\frac{1}{3}v$$

$$=\frac{2}{3}u+\frac{1}{3}v$$

$$J(u,\ v)=\begin{vmatrix}\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3}\end{vmatrix} \qquad\qquad J(u,\ v)=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$$

$$=-\frac{1}{3}$$

$x+y=1$	$\frac{2}{3}u+\frac{1}{3}v+\frac{1}{3}u-\frac{1}{3}v=1$	$u=1$
$x+y=4$	$\frac{2}{3}u+\frac{1}{3}v+\frac{1}{3}u-\frac{1}{3}v=4$	$u=4$
$x-2y=0$	$\frac{2}{3}u+\frac{1}{3}v-\frac{2}{3}u+\frac{2}{3}v=0$	$v=0$
$x-2y=-4$	$\frac{2}{3}u+\frac{1}{3}v-\frac{2}{3}u+\frac{2}{3}v=-4$	$v=-4$

$$\iint_R 3xy\,dA=\int_1^4\int_{-4}^0 3\frac{1}{3}(2u+v)\frac{1}{3}(u-v)\left|-\frac{1}{3}\right|dv\,du$$

$$=\frac{1}{9}\int_1^4\int_{-4}^0\left(2u^2-uv-v^2\right)dv\,du$$

$$=\frac{1}{9}\int_1^4\left(2u^2v-\frac{1}{2}uv^2-\frac{1}{3}v^3\,\Big|_{-4}^{0}\;du\right.$$

$$=\frac{1}{9}\int_1^4\left(8u^2+8u-\frac{64}{3}\right)du$$

$$=\frac{8}{9}\left(\frac{1}{3}u^3+\frac{1}{2}u^2-\frac{8}{3}u\,\Big|_1^4\right.$$

$$=\frac{8}{9}\left(\frac{64}{3}+8-\frac{32}{3}-\frac{1}{3}-\frac{1}{2}+\frac{8}{3}\right)$$

$$=\frac{8}{9}\left(\frac{39}{3}+\frac{15}{2}\right)$$

$$=\frac{8}{9}\left(\frac{123}{6}\right)$$

$$=\boxed{\frac{164}{9}}$$

Exercise

Let R be the region bounded by the square with vertices (0, 1), (1, 2), (2, 1), & (1, 0).

Evaluate the integral $\iint_R (x+y)^2\sin^2(x-y)\,dA$

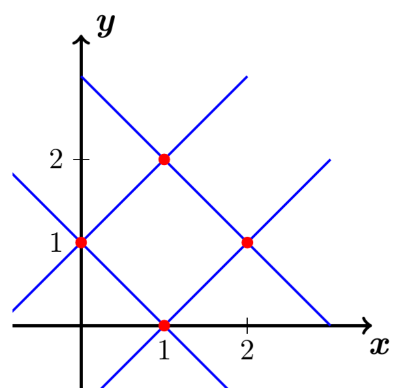
Solution

$(0,\ 1)\ \&\ (1,\ 2)\ \to m=\frac{2-1}{1-0}=1\ \Rightarrow\ \underline{y=x+1}$

$(0,\ 1)\ \&\ (1,\ 0)\ \to m=\frac{0-1}{1-0}=-1\ \Rightarrow\ \underline{y=-x+1}$

$(2,\ 1)\ \&\ (1,\ 0)\ \to m=1\ \Rightarrow\ \underline{y=x-1}$

$(2,\ 1)\ \&\ (1,\ 2)\ \to m=-1\ \Rightarrow\ \underline{y=-x+3}$



$$\begin{cases} y-x=1 & y+x=1 \\ y-x=-1 & y+x=3 \end{cases}$$

Let $\begin{cases} u=y-x \\ v=y+x \end{cases}$

$u+v=2y\ \to\ \underline{y=\frac{1}{2}u+\frac{1}{2}v}$

$x=y-u$

$$\boxed{= -\frac{1}{2}u + \frac{1}{2}v}$$

$$J(u,\ v) = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} \qquad\qquad J(u,\ v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\boxed{= -\frac{1}{2}}$$

$y - x = 1$	$\frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}u - \frac{1}{2}v = 1$	$u = 1$
$y - x = -1$	$\frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}u - \frac{1}{2}v = -1$	$u = -1$
$x + y = 1$	$\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}u + \frac{1}{2}v = 1$	$v = 1$
$x + y = 3$	$\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}u + \frac{1}{2}v = 3$	$v = 3$

$$\iint\limits_R (x+y)^2 \sin^2(x-y)\,dA = \int_{-1}^{1}\int_{1}^{3} \left(-\frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}u + \frac{1}{2}v\right)^2 \sin^2\left(-\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}u - \frac{1}{2}v\right) dvdu$$

$$= \int_{-1}^{1}\int_{1}^{3} v^2 \sin^2(-u)\,dvdu$$

$$= \int_{-1}^{1} \frac{1}{2}(1 - \cos 2u)\,du \qquad \left(\frac{1}{3}v^3 \,\Big|_1^3\right.$$

$$= \frac{1}{6}(27 - 1) \quad \left(u - \frac{1}{2}\sin 2u \,\Big|_{-1}^{1}\right.$$

$$= \frac{13}{3}\left(1 - \frac{1}{2}\sin 2 + 1 - \frac{1}{2}\sin 2\right)$$

$$\boxed{= \frac{13}{3}(2 - \sin 2)}$$

Exercise

Evaluate $\iiint_D yz\ dV$ D is bounded by the planes: $x + 2y = 1,\ \ x + 2y = 2,\ x - z = 0,\ x - z = 2,$ $2y - z = 0,$ and $2y - z = 3$

Solution

Let $\begin{cases} u = x + 2y \\ \left.\begin{aligned} v &= x - z \\ w &= 2y - s \end{aligned}\right\} \to v - w = x - 2y \end{cases}$

$$\begin{cases} u = x + 2y \\ v - w = x - 2y \end{cases} \rightarrow u + v - w = 2x$$

$$2y = \frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}w - v + u$$

$$z = \frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}w - v$$

$$\begin{cases} x = \frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}w \\ y = \frac{1}{4}u - \frac{1}{4}v + \frac{1}{4}w \\ z = \frac{1}{2}u - \frac{1}{2}v - \frac{1}{2}w \end{cases}$$

$$J(u,\ v,\ w) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} - \frac{1}{16} + \frac{1}{16} + \frac{1}{16}$$

$$= \frac{1}{4}$$

$x + 2y = 1$	$\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}w + \frac{1}{2}u - \frac{1}{2}v + \frac{1}{2}w = 1$	$u = 1$
$x + 2y = 2$	$\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}w + \frac{1}{2}u - \frac{1}{2}v + \frac{1}{2}w = 2$	$u = 2$
$x - z = 0$	$\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}w - \frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}w = 0$	$v = 0$
$x - z = 2$	$\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}w - \frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}w = 2$	$v = 2$
$2y - z = 0$	$\frac{1}{2}u - \frac{1}{2}v + \frac{1}{2}w - \frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}w = 0$	$w = 0$
$2y - z = 3$	$\frac{1}{2}u - \frac{1}{2}v + \frac{1}{2}w - \frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}w = 3$	$w = 3$

$$\iiint_D yz\ dV = \int_1^2 \int_0^2 \int_0^3 \frac{1}{8}(u - v + w)(u - v - w)\frac{1}{4}\ dwdvdu$$

$$= \frac{1}{32}\int_1^2 \int_0^2 \int_0^3 \left((u - v)^2 - w^2\right)\ dwdvdu$$

$$= \frac{1}{32}\int_1^2 \int_0^2 \left(\left(u^2 - 2uv + v^2\right)w - \frac{1}{3}w^3\ \Big|_0^3\right)\ dvdu$$

$$= \frac{1}{32} \int_1^2 \int_0^2 \left(3u^2 - 6uv + 3v^2 - 9\right) \, dvdu$$

$$= \frac{1}{32} \int_1^2 \left(3u^2 v - 3uv^2 + v^3 - 9v \,\right|_0^2 \, du$$

$$= \frac{1}{32} \int_1^2 \left(6u^2 - 12u + 8 - 18\right) \, du$$

$$= \frac{1}{32} \int_1^2 \left(6u^2 - 12u - 10\right) \, du$$

$$= \frac{1}{32} \left(2u^3 - 6u^2 - 10u \,\right|_1^2$$

$$= \frac{1}{16}(8 - 12 - 10 - 1 + 3 + 5)$$

$$\boxed{= -\frac{7}{16}}$$

Exercise

Evaluate $\iint\limits_R xy \, dA$; R is the square with vertices $(0,\ 0)$, $(1,\ 1)$, $(2,\ 0)$, and $(1,\ -1)$

Solution

$(0,\ 0)$ & $(1,\ 1)$ $\to y = x \quad \Rightarrow y - x = 0$

$(0,\ 0)$ & $(2,\ 0)$ $\to y = 0$

$(0,\ 0)$ & $(1,\ -1)$ $\to y = -x \quad \Rightarrow y + x = 0$

$(1,\ 1)$ & $(2,\ 0)$ $\to y = -x + 2 \quad \Rightarrow y + x = 2$

$(1,\ 1)$ & $(1,\ -1)$ $\to x = 1$

$(2,\ 0)$ & $(1,\ -1)$ $\to y = x - 2 \quad \Rightarrow y - x = -2$

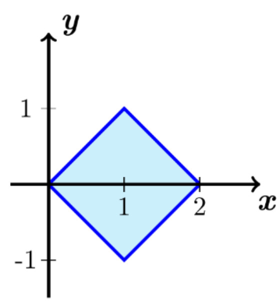
Let $\begin{cases} u = y - x \\ v = y + x \end{cases}$

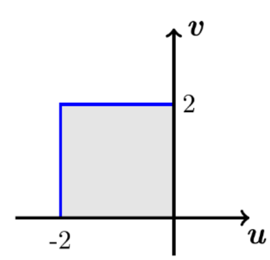
$$u + v = 2y \quad \to \quad \boxed{y = \frac{1}{2}(u + v)}$$

$$x = v - y$$

$$= v - \frac{1}{2}u - \frac{1}{2}v$$

$$\boxed{= \frac{1}{2}v - \frac{1}{2}u}$$





$$\begin{cases} x = -\frac{1}{2}u + \frac{1}{2}v \\ y = \frac{1}{2}u + \frac{1}{2}v \end{cases}$$

$y - x = 0$	$\frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}u - \frac{1}{2}v = 0$	$u = 0$
$y - x = -2$	$\frac{1}{2}u + \frac{1}{2}v + \frac{1}{2}u - \frac{1}{2}v = -2$	$u = -2$
$y + x = 0$	$\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}u + \frac{1}{2}v = 0$	$v = 0$
$y + x = 2$	$\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}u + \frac{1}{2}v = 2$	$v = 2$

$$J(u,\ v) = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} \qquad J(u,\ v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= -\frac{1}{2}$$

$$\iint_R xy\ dA = \int_0^2 \int_{-2}^2 \left(-\frac{1}{2}u + \frac{1}{2}v\right)\left(\frac{1}{2}u + \frac{1}{2}v\right)\left|-\frac{1}{2}\right| du dv$$

$$= \frac{1}{8}\int_0^2 \int_{-2}^2 (v-u)(v-u)\, du dv$$

$$= \frac{1}{8}\int_0^2 \int_{-2}^2 \left(v^2 - u^2\right) du dv$$

$$= \frac{1}{8}\int_0^2 \left(v^2 u - \frac{1}{3}u^3 \ \Big|_{-2}^{2}\right. dv$$

$$= \frac{1}{8}\int_0^2 \left(2v^2 - \frac{8}{3} + 2v^2 - \frac{8}{3}\right) dv$$

$$= \frac{1}{8}\int_0^2 \left(4v^2 - \frac{16}{3}\right) dv$$

$$= \frac{1}{2}\ \left(\frac{1}{3}v^3 - \frac{4}{3}v\ \Big|_0^2\right.$$

$$= \frac{1}{6}(8-8)$$

$$= 0$$

Exercise

Evaluate $\iint\limits_R x^2 y\, dA$; $R = \{(x,\ y):\ \ 0 \le x \le 2,\quad x \le y \le x+4\}$

Solution

$y = x \quad \to \quad y - x = 0$

$y = x+4 \quad \to \quad y - x = 4$

Let $\begin{cases} u = x \\ v = y - x \end{cases}$ $\qquad y = u + v$

$\begin{cases} x = u \\ y = u + v \end{cases}$

$J(u,\ v) = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$ $\qquad J(u,\ v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$= 1$

$x = 0$		$u = 0$
$x = 2$		$u = 2$
$y - x = 0$	$u + v - u = 0$	$v = 0$
$y - x = 4$	$u + v - u = 4$	$v = 4$

$$\iint\limits_R x^2 y\, dA = \int_0^4 \int_0^2 u^2 (u+v)|1|\ dudv$$

$$= \int_0^4 \int_0^2 \left(u^3 + vu^2\right) dudv$$

$$= \int_0^4 \left(\frac{1}{4}u^4 + \frac{1}{3}vu^3\ \Big|_0^2\right) dv$$

$$= \int_0^4 \left(4 + \frac{8}{3}v\right) dv$$

$$= 4v + \frac{4}{3}v^2\ \Big|_0^4$$

$$= 16 + \frac{64}{3}$$

$$= \frac{112}{3}$$

Exercise

Evaluate $\iint_R x^2\sqrt{x+2y}\ dA$; $R=\left\{(x,\ y):\ \ 0\le x\le 2,\ \ -\frac{x}{2}\le y\le 1-x\right\}$

Solution

$$y=-\frac{1}{2}x \quad\to\quad 2y+x=0$$

$$y=1-x \quad\to\quad y+x=1$$

Let $\begin{cases} u=\frac{1}{2}x \\ v=y+x \end{cases}$

$$\begin{cases} x=2u \\ y=-2u+v \end{cases}$$

$$J(u,\ v)=\begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} \qquad\qquad J(u,\ v)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$=2\,\rfloor$$

$x=0$	$2u=0$	$u=0$
$x=2$	$2u=2$	$u=1$
$y+x=1$	$v=1$	$v=1$
$y=-\frac{1}{2}x$	$-2u+v=-u$	$v=u$

$$\iint_R x^2\sqrt{x+2y}\ dA=\int_0^1\int_u^1 4u^2\sqrt{2u-4u+2v}\,|2|\ dvdu$$

$$=4\int_0^1\int_u^1 u^2(-2u+2v)^{1/2}\ d(-2u+2v)\,du$$

$$=\frac{8}{3}\int_0^1 u^2(-2u+2v)^{3/2}\ \Big|_u^1\ du$$

$$=\frac{8}{3}\int_0^1 u^2\left((-2u+2)^{3/2}-0\right)du$$

$$=\frac{16\sqrt{2}}{3}\int_0^1 u^2(1-u)^{3/2}\ du$$

Let $w=1-u \ \to\ dw=-du$

$u=1-w$

$$= -\frac{16\sqrt{2}}{3}\int_0^1 (1-w)^2\, w^{3/2}\, dw$$

$$= -\frac{16\sqrt{2}}{3}\int_0^1 \left(1-2w+w^2\right) w^{3/2}\, dw$$

$$= -\frac{16\sqrt{2}}{3}\int_0^1 \left(w^{3/2}-2w^{5/2}+w^{7/2}\right) dw$$

$$= -\frac{16\sqrt{2}}{3}\left(\frac{2}{5}(1-u)^{5/2}-\frac{4}{7}(1-u)^{7/2}+\frac{2}{9}(1-u)^{9/2}\right)\Bigg|_0^1$$

$$= -\frac{16\sqrt{2}}{3}\left(-\frac{2}{5}+\frac{4}{7}-\frac{2}{9}\right)$$

$$= -\frac{16\sqrt{2}}{3}\left(\frac{-126+184-70}{315}\right)$$

$$= -\frac{16\sqrt{2}}{3}\left(-\frac{16}{315}\right)$$

$$= \frac{256\sqrt{2}}{945}$$

Exercise

Evaluate $\iint\limits_R xy\, dA$; where R is bounded by the ellipse $9x^2+4y^2=36$.

Solution

$$u^2+v^2=1 \quad \rightarrow \quad v=\pm\sqrt{1-u^2}$$

$$-1\le u\le 1$$

$$\frac{x^2}{4}+\frac{y^2}{9}=1 \quad \rightarrow \quad \left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2=1$$

$$\begin{cases} \frac{x}{2}=u & \rightarrow \quad x=2u \\ \frac{y}{3}=v & \rightarrow \quad y=3v \end{cases}$$

$$J(u,\ v)=\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix}$$

$$=6$$

$$\iint_R xy\ dA = \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} (2u)(3v)|6|\ dvdu$$

$$= 18\int_{-1}^{1} u\left(v^2 \Big|_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\right. du$$

$$= 36\int_{-1}^{1} u\left(1-u^2\right)du$$

$$= 36\int_{-1}^{1} \left(u-u^3\right)du$$

$$= 36\left(\frac{1}{2}u^2 - \frac{1}{4}u^4 \Big|_{-1}^{1}\right.$$

$$= 36\left(\frac{1}{2}-\frac{1}{4}-\frac{1}{2}+\frac{1}{4}\right)$$

$$= 0$$

Exercise

Evaluate $= \int_0^1\int_{2u-2}^{2u} \sqrt{u+u-v}\,|1|\ dvdu$

Solution

$x = y, \quad x = y+2$

$0 \le x \le 1$

$$\begin{cases} u = x \\ v = x+y \end{cases} \rightarrow \begin{cases} x = u \\ y = -u+v \end{cases}$$

$$J(u,\ v) = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}$$

$$= 1$$

$x=0$		$u=0$
$x=1$		$u=1$
$x=y$	$u=-u+v$	$v=2u$
$x=y+2$	$u=-u+v+2$	$v=2u-2$

$$\int_0^1\int_y^{y+2} \sqrt{x-y}\ dxdy = \int_0^1\int_{2u-2}^{2u} \sqrt{u+u-v}\,|1|\ dvdu$$

$$= -\int_0^1 \int_{2u-2}^{2u} (2u-v)^{1/2}\ d(2u-v)\,du$$

$$= -\frac{2}{3}\int_0^1 (2u-v)^{3/2}\ \Big|_{2u-2}^{2u}\ du$$

$$= \frac{2}{3}\int_0^1 (2)^{3/2}\,du$$

$$\underline{= \frac{4\sqrt{2}}{3}}$$

Exercise

Evaluate $\iint\limits_R \sqrt{y^2-x^2}\ dA$; where R is the diamond bounded by $y-x=0,\ \ y-x=2,\ \ y+x=0$, and $y+x=2$

Solution

$$\begin{cases} u = y-x \\ v = y+x \end{cases} \quad \to \quad \begin{cases} x = -\frac{1}{2}u + \frac{1}{2}v \\ y = \frac{1}{2}u + \frac{1}{2}v \end{cases}$$

$$J(u,\ v) = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$\underline{= -\frac{1}{2}}$$

$y-x=0$	$\frac{1}{2}u+\frac{1}{2}v+\frac{1}{2}u-\frac{1}{2}v=0$	$u=0$
$y-x=2$	$\frac{1}{2}u+\frac{1}{2}v+\frac{1}{2}u-\frac{1}{2}v=2$	$u=2$
$y+x=0$	$\frac{1}{2}u+\frac{1}{2}v-\frac{1}{2}u+\frac{1}{2}v=0$	$v=0$
$y+x=2$	$\frac{1}{2}u+\frac{1}{2}v-\frac{1}{2}u+\frac{1}{2}v=2$	$v=2$

$$\iint\limits_R \sqrt{y^2-x^2}\ dA = \int_0^2\int_0^2 \left|-\frac{1}{2}\right| \sqrt{\left(\frac{1}{2}u+\frac{1}{2}v\right)^2 - \left(-\frac{1}{2}u+\frac{1}{2}v\right)^2}\ dvdu$$

$$= \frac{1}{8}\int_0^2\int_0^2 \sqrt{u^2+2uv+v^2-u^2+2uv-v^2}\ dvdu$$

$$=\frac{1}{4}\int_0^2\int_0^2 \sqrt{uv}\ dvdu$$

$$=\frac{1}{4}\int_0^2\int_0^2 \frac{1}{u}(uv)^{1/2}\ d(uv)\,du$$

$$=\frac{1}{6}\int_0^2 \frac{1}{u}\ (uv)^{3/2}\ \Big|_0^2\ du$$

$$=\frac{\sqrt{2}}{3}\int_0^2 \frac{1}{u}(u)^{3/2}\,du$$

$$=\frac{\sqrt{2}}{3}\int_0^2 (u)^{1/2}\,du$$

$$=\frac{2\sqrt{2}}{9}\ (u)^{3/2}\ \Big|_0^2$$

$$\boxed{=\frac{16}{9}}$$

Exercise

Evaluate $\iint\limits_R \left(\frac{y-x}{y+2x+1}\right)^4\ dA$; where R is the parallelogram bounded by $y-x=1,\ \ y-x=2,$ $y+2x=0$, and $y+2x=4$

Solution

Let $\begin{cases} u=y-x \\ v=y+2x \end{cases} \rightarrow \begin{cases} x=-\frac{1}{3}u+\frac{1}{3}v \\ y=\frac{2}{3}u+\frac{1}{3}v \end{cases}$

$$J(u,\ v)=\begin{vmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix}$$

$$\boxed{=-1}$$

$y-x=1$	$\frac{2}{3}u+\frac{1}{3}v+\frac{1}{3}u-\frac{1}{3}v=1$	$u=1$
$y-x=2$	$\frac{2}{3}u+\frac{1}{3}v+\frac{1}{3}u-\frac{1}{3}v=2$	$u=2$
$y+2x=0$	$\frac{2}{3}u+\frac{1}{3}v-\frac{2}{3}u+\frac{2}{3}v=0$	$v=0$
$y+2x=4$	$\frac{2}{3}u+\frac{1}{3}v-\frac{2}{3}u+\frac{2}{3}v=4$	$v=4$

$$\iint_R \left(\frac{y-x}{y+2x+1}\right)^4 dA = \int_0^4 \int_1^2 |-1| \left(\frac{\frac{1}{3}(2u+v+u-v)}{\frac{1}{3}(2u+v-2u+2v)+1}\right)^4 dudv$$

$$= \int_0^4 \int_1^2 \left(\frac{u}{v+1}\right)^4 dudv$$

$$= \int_0^4 (v+1)^{-4} d(3v+1) \quad \int_1^2 u^4\, du$$

$$= -\frac{1}{3}\left((v+1)^{-3}\right|_0^4 \quad \left(\frac{1}{5}u^5\right|_1^2$$

$$= -\frac{1}{15}\left(\frac{1}{125}-1\right)(32)$$

$$= -\frac{32}{15}\left(-\frac{124}{125}\right)$$

$$= \frac{3,968}{1,875}$$

Exercise

Evaluate $\iint_R e^{xy}\, dA$; where R is the region bounded by $xy=1$, $xy=4$, $\frac{y}{x}=1$, and $\frac{y}{x}=3$

Solution

Let $\begin{cases} u = xy \\ v = \frac{y}{x} \end{cases} \rightarrow \begin{array}{l} y = \frac{u}{x} \\ v = \frac{u}{x^2} \end{array}$

$$\begin{cases} x = \sqrt{\frac{u}{v}} \\ y = \sqrt{uv} \end{cases}$$

$$J(u,\, v) = \begin{vmatrix} \frac{1}{2\sqrt{uv}} & -\frac{\sqrt{u}}{v^{3/2}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{vmatrix}$$

$$= \frac{1}{4v} + \frac{1}{4v}$$

$$= \frac{1}{2v}$$

$xy=1$	$\sqrt{\frac{u}{v}}\sqrt{uv}=1$	$u=1$

$xy=4$	$\sqrt{\frac{u}{v}}\sqrt{uv}=4$	$u=4$
$\frac{y}{x}=1$	$\sqrt{\frac{v}{u}}\sqrt{uv}=1$	$v=1$
$\frac{y}{x}=3$	$\sqrt{\frac{v}{u}}\sqrt{uv}=3$	$v=3$

$$\iint_R e^{xy}\ dA=\int_1^3\int_1^4\left|\frac{1}{2v}\right|e^{\sqrt{\frac{u}{v}}\sqrt{uv}}\ dudv$$

$$=\frac{1}{2}\int_1^3\frac{1}{v}dv\ \int_1^4 e^u\ du$$

$$=\frac{1}{2}\ln v\Big|_1^3\quad\left(e^u\Big|_1^4\right.$$

$$\boxed{=\frac{\ln 3}{2}\left(e^4-e\right)}$$

Exercise

Evaluate $\iint_R xy\ dA$; where R is the region bounded by the hyperbolas $xy=1$, $xy=4$, $y=1$, and $y=3$

Solution

Let $\begin{cases}u=xy\\ v=y\end{cases}\to\quad x=\frac{u}{v}\quad y=v$

$$J(u,\ v)=\begin{vmatrix}\frac{1}{v} & -\frac{u}{v^2}\\ 0 & 1\end{vmatrix}$$

$$\boxed{=\frac{1}{v}}$$

$xy=1$	$u=1$
$xy=4$	$u=4$
$y=1$	$v=1$
$y=3$	$v=3$

$$\iint_R xy\ dA=\int_1^3\int_1^4\left|\frac{1}{v}\right|u\ dudv$$

$$= \int_1^3 \frac{1}{v} dv \int_1^4 u\ du$$

$$= \ln v \Big|_1^3 \quad \frac{1}{2}\left(u^2 \Big|_1^4\right.$$

$$= \frac{15}{2}\ln 3$$

Exercise

Evaluate $\iint_R (x-y)\sqrt{x-2y}\ dA$; where R is the triangular region bounded by $y=0,\ x-2y=0,$ and $x-y=1$

Solution

Let $\begin{cases} u = x - 2y \\ v = y \end{cases} \rightarrow \begin{matrix} x = u + 2v \\ y = v \end{matrix}$

$$J(u,\ v) = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= 1$$

$y=0$		$v=0$
$x-2y=0$		$u=0$
$x-y=1$	$u+2v-v=1$	$v=1-u$

$$\iint_R (x-y)\sqrt{x-2y}\ dA = \int_0^1 \int_0^{1-u} (u+v)\sqrt{u}\ dvdu$$

$$= \int_0^1 \int_0^{1-u} \left(u^{3/2} + vu^{1/2}\right) dvdu$$

$$= \int_0^1 \left(u^{3/2}v + \frac{1}{2}u^{1/2}v^2 \Big|_0^{1-u}\right. du$$

$$= \int_0^1 \left(u^{3/2}(1-u) + \frac{1}{2}u^{1/2}\left(1-2u+u^2\right)\right) du$$

$$= \int_0^1 \left(u^{3/2} - u^{5/2} + \frac{1}{2}u^{1/2} - u^{3/2} + \frac{1}{2}u^{5/2}\right) du$$

$$= \int_0^1 \left(\frac{1}{2}u^{1/2} - \frac{1}{2}u^{5/2}\right) du$$

$$= \frac{1}{3}u^{3/2} - \frac{1}{7}u^{7/2} \Big|_0^1$$

$$= \frac{1}{3} - \frac{1}{7}$$

$$= \frac{4}{21}$$

Exercise

Evaluate $\iiint_D xy\ dV$: D is bounded by the planes: $y-x=0,\ \ y-x=2,\ \ z-y=0,\ \ z-y=2,$ $z=0,$ and $z=3$

Solution

Let: $\begin{cases} u = y - x \\ v = z - y \\ w = z \end{cases} \quad \rightarrow \begin{cases} x = -u - v + w \\ y = -v + w \end{cases}$

$$J(u,\ v,\ w) = \begin{vmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1$$

$y-x=0$	$u=0$
$y-x=2$	$u=2$
$z-y=0$	$v=0$
$z-y=2$	$v=2$
$z=0$	$w=0$
$z=3$	$w=3$

$$\iiint_D xydV = \int_0^3 \int_0^2 \int_0^2 (w-u-v)(w-v)\ dudvdw$$

$$= \int_0^3 \int_0^2 \int_0^2 \left(w^2 - 2vw - uw + uv + v^2\right)\ dudvdw$$

$$= \int_0^3 \int_0^2 \left(w^2u - 2vwu - \frac{1}{2}wu^2 + \frac{1}{2}vu^2 + v^2u \Big|_0^2\ dvdw\right.$$

$$= \int_0^3 \int_0^2 \left(2w^2 - 4vw - 2w + 2v + 2v^2\right) dv dw$$

$$= \int_0^3 \left(2w^2 v - 2wv^2 - 2wv + v^2 + \frac{2}{3}v^3 \Big|_0^2 dw\right.$$

$$= \int_0^3 \left(4w^2 - 12w + \frac{28}{3}\right) dw$$

$$= \frac{4}{3}w^3 - 6w^2 + \frac{28}{3}w \Big|_0^3$$

$$= 36 - 54 + 28$$

$$\underline{= 10}$$

Exercise

Evaluate $\iiint_D dV$: D is bounded by the planes: $y - 2x = 0$, $y - 2x = 1$, $z - 3y = 0$, $z - 3y = 1$, $z - 4x = 0$, and $z - 4x = 3$

Solution

Let: $\begin{cases} u = y - 2x \\ v = z - 3y \\ w = z - 4x \end{cases} \rightarrow \begin{cases} x = \frac{1}{2}y - \frac{1}{2}u \\ z = v + 3y \\ \end{cases}$

$$w = v + 3y - 2y + 2u$$

$$\begin{cases} y = -2u - v + w \\ x = -\frac{3}{2}u - \frac{1}{2}v + \frac{1}{2}w \\ z = -6u - 2v + 3w \end{cases}$$

$$J(u,\ v,\ w) = \begin{vmatrix} -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ -2 & -1 & 1 \\ -6 & -2 & 3 \end{vmatrix}$$

$$= \frac{9}{2} + 3 + 2 - 3 - 3 - 3$$

$$\underline{= \frac{1}{2}}$$

$y - 2x = 0$	$u = 0$
$y - 2x = 1$	$u = 1$
$z - 3y = 0$	$v = 0$

$z-3y=1$	$v=1$
$z-x=0$	$w=0$
$z-4x=3$	$w=3$

$$\iiint_D dV = \int_0^3 dw \int_0^1 dv \int_0^1 \frac{1}{2}\, du$$

$$= \frac{1}{2}(3)(1)(1)$$

$$= \frac{3}{2}$$

Exercise

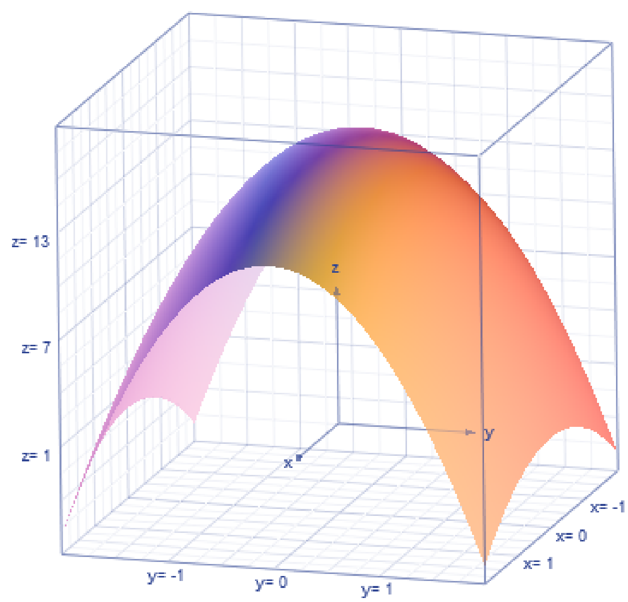
Evaluate $\iiint_D z\, dV$: D is bounded by the paraboloid $z = 16 - x^2 - 4y^2$ and the xy-plane.

Solution

$$z = 16 - x^2 - 4y^2 = 0$$

$$x^2 + 4y^2 = 16 \quad \rightarrow \quad \left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

Let: $\begin{cases} u = \frac{x}{4} \\ v = \frac{y}{2} \\ w = z \end{cases} \quad \begin{cases} x = 4u \\ y = 2v \\ z = w \end{cases}$



$$u^2 + v^2 = 1 \quad \rightarrow \quad v = \pm\sqrt{1-u^2}$$

$$-1 \le u \le 1$$

$$w = z = 16 - 16u^2 - 16v^2$$

$$J(u,\ v,\ w) = \begin{vmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 8$$

$$\iiint_D z\, dV = \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \int_0^{16-16u^2-16v^2} 8w\, dw dv du$$

$$= 4 \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} w^2 \Big|_0^{16-16u^2-16v^2} dv du$$

$$= 4\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} 16^2\left(1-\left(u^2+v^2\right)\right)^2 dvdu$$

$$= 1,024\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \left(1-2\left(u^2+v^2\right)+\left(u^2+v^2\right)^2\right)dvdu$$

$$= 1,024\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \left(1-2u^2-2v^2+u^4+2u^2v^2+v^4\right)dvdu$$

$$= 1,024\int_{-1}^{1}\left(v-2u^2v-\frac{2}{3}v^3+u^4v+\frac{2}{3}u^2v^3+\frac{1}{5}v^5 \right|_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} du$$

$$= 1,024\int_{-1}^{1}\left(\left(1-2u^2+u^4\right)v-\frac{2}{3}\left(1-u^2\right)v^3+\frac{1}{5}v^5 \right|_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} du$$

$$= 2,048\int_{-1}^{1}\left(\left(1-u^2\right)^2\left(1-u^2\right)^{1/2}-\frac{2}{3}\left(1-u^2\right)^{5/2}+\frac{1}{5}\left(1-u^2\right)^{5/2}\right)du$$

$$= 2,048\left(\frac{8}{15}\right)\int_{-1}^{1}\left(1-u^2\right)^{5/2} du$$

$$u=\sin t \quad \rightarrow \quad du=\cos t dt$$

$$\begin{cases} u=1=\sin t & \rightarrow t=\frac{\pi}{2} \\ u=-1=\sin t & \rightarrow t=-\frac{\pi}{2} \end{cases}$$

$$= 2,048\left(\frac{8}{15}\right)\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^5 t\cos t\ dt$$

$$= 2,048\left(\frac{8}{15}\right)\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^6 t\ dt$$

$$= \frac{2,048}{15}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(1+\cos 2t\right)^3\ dt$$

$$= \frac{2,048}{15}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(1+3\cos 2t+3\cos^2 2t+\cos^3 2t\right)\ dt$$

$$= \frac{2,048}{15} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{5}{2} + 3\cos 2t + \frac{3}{2}\cos 4t + \cos^2 2t \left(\sin 2t \right) \right) dt$$

$$\int \cos^2 2t \left(\sin 2t \right) dt = -\frac{1}{2} \int \cos^2 2t \ d\left(\cos 2t \right)$$

$$= -\frac{1}{6} \cos^3 2t$$

$$= \frac{2,048}{15} \left(\frac{5}{2}t + \frac{3}{2}\sin 2t + \frac{3}{8}\sin 4t - \frac{1}{6}\cos^3 2t \right. \Bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= \frac{2,048}{15} \left(\frac{15\pi}{4} + \frac{1}{6} - \frac{5\pi}{4} - \frac{1}{6} \right)$$

$$= \frac{2,048}{15} \left(\frac{5\pi}{2} \right)$$

$$\boxed{= \frac{1,024\pi}{3}}$$

Exercise

Evaluate $\iiint_D dV$: D is bounded by the upper half of the ellipsoid $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$ and the xy-plane.

Solution

$$\left(\frac{x}{3} \right)^2 + \left(\frac{y}{2} \right)^2 + z^2 = 1$$

Let: $\begin{cases} u = \frac{x}{3} \\ v = \frac{y}{2} \\ w = z \end{cases}$ $\quad \begin{cases} x = 3u \\ y = 2v \\ z = w \end{cases}$

$$J\left(u, \ v, \ w \right) = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\boxed{= 6}$$

$$u^2 + v^2 + w^2 = 1$$

$$-\sqrt{1 - v^2 - w^2} \le u \le \sqrt{1 - v^2 - w^2}$$

$$-\sqrt{1 - w^2} \le v \le \sqrt{1 - w^2} \quad \& \quad 0 \le w \le 1$$

$$\iiint_D dV = \int_0^1 \int_{-\sqrt{1-w^2}}^{\sqrt{1-w^2}} \int_{-\sqrt{1-v^2-w^2}}^{\sqrt{1-v^2-w^2}} (6)\ dudvdw$$

$$= 6 \int_0^1 \int_{-\sqrt{1-w^2}}^{\sqrt{1-w^2}} u \Big|_{-\sqrt{1-v^2-w^2}}^{\sqrt{1-v^2-w^2}} dvdw$$

$$= 12 \int_0^1 \int_{-\sqrt{1-w^2}}^{\sqrt{1-w^2}} \sqrt{1-\left(v^2+w^2\right)}\ dvdw$$

$$v^2 + w^2 = r^2 \quad \to \quad 0 \le r \le 1 \quad 0 \le \theta \le \pi$$

$$= 12 \int_0^{\pi} \int_0^1 \sqrt{1-r^2}\ r\ drd\theta$$

$$= -6 \int_0^{\pi} d\theta \quad \int_0^1 \left(1-r^2\right)^{1/2} d\left(1-r^2\right)$$

$$= -6\pi \frac{2}{3} \left(1-r^2\right)^{3/2} \Big|_0^1$$

$$\underline{= 4\pi}$$

Exercise

Evaluate $\iiint_D xz\ dV : D$ is bounded by the planes: $y = x,\ \ y = x+2,\ \ x-z=0,\ \ z = x+3,\ \ z = 0,$ and $z = 4$

Solution

$y - x = 0 \quad y - x = 2$

$z - x = 0 \quad z - x = 3$

Let: $\begin{cases} u = y - x \\ v = z - x \\ w = z \end{cases} \quad \to \begin{cases} y = u - v + w \\ x = -v + w \end{cases}$

$$J(u,\ v,\ w) = \begin{vmatrix} 0 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\underline{= 1}$$

$y = x$	$u - v + w = w - v$	$u = 0$

$y = x + 2$	$u - v + w = w - v + 2$	$u = 2$
$x - z = 0$	$w - v - w = 0$	$v = 0$
$z = x + 3$	$w + v - w = 3$	$v = 3$
$z = 0$		$w = 0$
$z = 4$		$w = 4$

$$\iiint_D xz\,dV = \int_0^2 \int_0^3 \int_0^4 (w-v)(w)\ dwdvdu$$

$$= \int_0^2 du \ \int_0^3 \int_0^4 \left(w^2 - vw\right)\ dwdv$$

$$= 2\int_0^3 \left(\frac{1}{3}w^3 - \frac{1}{2}vw^2\ \Big|_0^4\ dv\right.$$

$$= 2\int_0^3 \left(\frac{64}{3} - 8v\right)dv$$

$$= 2\ \left(\frac{64}{3}v - 4v^2\ \Big|_0^3\right.$$

$$= 18(64 - 36)$$

$$\underline{= 56\ |}$$

Exercise

Let R be the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$ are real numbers.

Find the area of R.

Solution

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Let: $\begin{cases} u = \frac{x}{a} & \rightarrow\ x = au \\ v = \frac{y}{b} & \rightarrow\ y = bv \end{cases}$

$$J(u,\ v) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}$$

$$\underline{= ab\ |}$$

$$u^2 + v^2 = 1 \quad \rightarrow \quad u = \sqrt{1 - v^2}$$

$$-\sqrt{1 - v^2} \le u \le \sqrt{1 - v^2} \quad \& \quad -1 \le v \le 1$$

Since, $u^2+v^2=1$ is a unit circle, then the area $\pi r^2=\pi$. Therefore, the area of the ellipse is $ab\pi$

$$\iint_R dA=\int_{-1}^{1}\int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} ab\ dudv$$

$$=ab\int_{-1}^{1}\left(u\ \Big|_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}}\right) dv$$

$$=2ab\int_{-1}^{1}\sqrt{1-v^2}\ dv$$

$$v=\sin t \quad\to\quad dv=\cos t dt$$

$$\begin{cases} v=1=\sin t & \to t=\frac{\pi}{2} \\ v=-1=\sin t & \to t=-\frac{\pi}{2} \end{cases}$$

$$=2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\ dt$$

$$=ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\cos 2t)\ dt$$

$$=ab\left(t+\frac{1}{2}\sin 2t\ \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right.$$

$$=ab\left(\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$\underline{ab\pi}$$

Exercise

Let R be the region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>0$ and $b>0$ are real numbers.

Evaluates $\iint_R |xy|\,dA$

Solution

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$$

Let: $\begin{cases} u = \dfrac{x}{a} & \to \ x = au \\ v = \dfrac{y}{b} & \to \ y = bv \end{cases}$

$$J(u,\ v) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}$$

$$= ab$$

$$u^2 + v^2 = 1 \quad \to \quad u = \sqrt{1-v^2}$$

$$-\sqrt{1-v^2} \le u \le \sqrt{1-v^2} \quad \& \quad -1 \le v \le 1$$

$$\iint_R |xy|\, dA = \int_{-1}^{1} \int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} |abuv|\, ab\ dudv$$

$$= 4a^2b^2 \int_0^1 \int_0^{\sqrt{1-v^2}} uv\ dudv$$

$$= 2a^2b^2 \int_0^1 v\left(u^2 \Big|_0^{\sqrt{1-v^2}}\right.\ dv$$

$$= 2a^2b^2 \int_0^1 v\left(1-v^2\right) dv$$

$$= 2a^2b^2 \int_0^1 \left(v-v^3\right) dv$$

$$= 2a^2b^2 \left(\frac{1}{2}v^2 - \frac{1}{4}v^4\right. \Big|_0^1$$

$$= 2a^2b^2 \left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \frac{1}{2}a^2b^2$$

Exercise

Let R be the region bounded by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$ are real numbers.

Find the center of mass of the upper half of R $(y \ge 0)$ assuming it has a constant density.

Solution

Let: $\begin{cases} u = \dfrac{x}{a} & \rightarrow\ x = au \\ v = \dfrac{y}{b} & \rightarrow\ y = bv \end{cases}$

$$J(u,\ v) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}$$

$$\underline{= ab\ |}$$

$u^2 + v^2 = 1 \quad \rightarrow \quad v = \sqrt{1-u^2}$

$0 \le v \le \sqrt{1-u^2} \quad \& \quad -1 \le u \le 1$

Since, $u^2 + v^2 = 1$ is a unit circle, then the area $\pi r^2 = \pi$. Therefore, the area of the ellipse is $ab\pi$

Mass of the upper half is given by:

$$m = \frac{1}{2}\pi ab$$

By symmetry, $\boxed{\overline{x} = 0}$

$$\overline{y} = \frac{2}{\pi ab}\int_{-1}^{1}\int_{0}^{\sqrt{1-u^2}} aby\ dvdu$$

$$= \frac{2}{\pi}\int_{-1}^{1}\int_{0}^{\sqrt{1-u^2}} bv\ dvdu$$

$$= \frac{b}{\pi}\int_{-1}^{1} v^2 \Big|_{0}^{\sqrt{1-u^2}}\ du$$

$$= \frac{b}{\pi}\int_{-1}^{1} \left(1-u^2\right) du$$

$$= \frac{b}{\pi}\left(u - \frac{1}{3}u^3\right)\Big|_{-1}^{1}$$

$$= \frac{2b}{\pi}\left(1-\frac{1}{3}\right)$$

$$\boxed{= \frac{4b}{3\pi}}$$

$\therefore$ the center of mass of the upper half of R is $\left(0,\ \dfrac{4b}{3\pi}\right)$

Exercise

Let R be the region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>0$ and $b>0$ are real numbers.

Find the average square of the distance between points of R and the origin.

Solution

Let: $\begin{cases} u=\frac{x}{a} & \to\ x=au \\ v=\frac{y}{b} & \to\ y=bv \end{cases}$

The distance between points of R and the origin is $d=\sqrt{x^2+y^2}$

$$d=\sqrt{a^2u^2+b^2v^2}$$

$$J(u,\ v)=\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}$$

$$=ab$$

$$u^2+v^2=1 \quad\to\quad v=\sqrt{1-u^2}$$

$$0\le v\le\sqrt{1-u^2} \quad \& \quad -1\le u\le 1$$

Average square of the distance is:

$$avg=\frac{1}{\pi ab}\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} ab\left(a^2u^2+b^2v^2\right)\,dvdu$$

$$=\frac{1}{\pi}\int_{-1}^{1}\left(a^2u^2v+\frac{1}{3}b^2v^3\right.\Bigg|_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\,du$$

$$=\frac{2}{\pi}\int_{-1}^{1}\left(a^2u^2\left(1-u^2\right)^{1/2}+\frac{1}{3}b^2\left(1-u^2\right)^{3/2}\right)du$$

$$u=\sin t \quad\to\quad du=\cos t dt$$

$$\begin{cases} u=1=\sin t & \to t=\frac{\pi}{2} \\ u=-1=\sin t & \to t=-\frac{\pi}{2} \end{cases}$$

$$\int u^2\left(1-u^2\right)^{1/2}du=\int \sin^2 t\cos t\cos t\ dt$$

$$=\frac{1}{4}\int\left(1-\cos 2t\right)\left(1+\cos 2t\right)dt$$

$$=\frac{1}{4}\int\left(1-\cos^2 2t\right)dt$$

$$= \frac{1}{4}\int\left(\frac{1}{2} - \frac{1}{2}\cos 4t\right)dt$$

$$= \frac{1}{8}\left(t - \frac{1}{4}\sin 4t\right)$$

$$\int\left(1-u^2\right)^{3/2}du = \int \cos^4 t\ dt$$

$$= \frac{1}{4}\int (1+\cos 2t)^2\ dt$$

$$= \frac{1}{4}\int\left(1 + 2\cos 2t + \cos^2 2t\right)dt$$

$$= \frac{1}{4}\int\left(\frac{3}{2} + 2\cos 2t + \frac{1}{2}\cos 4t\right)dt$$

$$= \frac{1}{4}\left(\frac{3}{2}t + \sin 2t + \frac{1}{8}\sin 4t\right)$$

$$= \frac{2}{\pi}\left(\frac{a^2}{8}\left(t - \frac{1}{4}\sin 4t\right) + \frac{b^2}{12}\left(\frac{3}{2}t + \sin 2t + \frac{1}{8}\sin 4t\right)\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{4}{\pi}\left(\frac{a^2}{8}\frac{\pi}{2} + \frac{b^2}{8}\frac{\pi}{2}\right)$$

$$= \frac{a^2+b^2}{4}$$

Exercise

Let R be the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$ are real numbers.

Find the average distance between points in the upper half of R and the x-axis.

Solution

Let: $\begin{cases} u = \frac{x}{a} & \rightarrow\ x = au \\ v = \frac{y}{b} & \rightarrow\ y = bv \end{cases}$

The distance between points in the upper half of R and the x-axis is $d = y = bv$

$$J(u,\ v) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}$$

$$= ab$$

$$u^2 + v^2 = 1 \quad \rightarrow \quad v = \sqrt{1-u^2}$$

$0 \le v \le \sqrt{1-u^2} \quad \& \quad -1 \le u \le 1$

$$avg = \frac{2}{\pi ab} \int_{-1}^{1} \int_{0}^{\sqrt{1-u^2}} ab^2 v \, dvdu$$

$$= \frac{b}{\pi} \int_{-1}^{1} v^2 \Big|_{0}^{\sqrt{1-u^2}} du$$

$$= \frac{b}{\pi} \int_{-1}^{1} \left(1-u^2\right) du$$

$$= \frac{2b}{\pi} \left(u - \frac{1}{3}u^3 \right|_{0}^{1}$$

$$= \frac{2b}{\pi} \left(1 - \frac{1}{3}\right)$$

$$= \frac{4b}{3\pi}$$

Exercise

Let *D* be the region bounded by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, where $a > 0$, $b > 0$ and $c > 0$ are real numbers. Find the Volume of *D*.

Solution

Let: $\begin{cases} u = \frac{x}{a} & \rightarrow \ x = au \\ v = \frac{y}{b} & \rightarrow \ y = bv \\ w = \frac{z}{c} & \rightarrow \ z = cw \end{cases}$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \rightarrow \quad u^2 + v^2 + w^2 = 1$$

$$J(u,\ v) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

$$= abc$$

Since, $u^2 + v^2 + w^2 = 1$ is a unit sphere, then the volume $\frac{4\pi r^2}{3} = \frac{4\pi}{3}$. Therefore, the volume of the ellipsoid is $\frac{4}{3}abc\pi$

Exercise

Let D be the region bounded by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, where $a>0$, $b>0$ and $c>0$ are real numbers. Evaluates $\iint_V |xyz|\,dV$

Solution

Let: $\begin{cases} u=\frac{x}{a} & \rightarrow\ x=au \\ v=\frac{y}{b} & \rightarrow\ y=bv \\ w=\frac{z}{c} & \rightarrow\ z=cw \end{cases}$

$$J(u,\ v)=\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

$$\underline{=abc\ |}$$

$$u^2+v^2+w^2=1 \quad \rightarrow \quad w=\sqrt{1-u^2-v^2}$$

$$-\sqrt{1-u^2-v^2}\le w\le\sqrt{1-u^2-v^2}$$

$$-\sqrt{1-u^2}\le v\le\sqrt{1-u^2} \quad \& \quad -1\le u\le 1$$

$$\iint_R |xyz|\,dA=\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\int_{-\sqrt{1-u^2-v^2}}^{\sqrt{1-u^2-v^2}}(aubvcw)abc\ \ dwdvdu$$

$$=4a^2b^2c^2\int_0^1\int_0^{\sqrt{1-u^2}} uv\left\{w^2\ \Big|_0^{\sqrt{1-u^2-v^2}}\right.\ dvdu$$

$$=4a^2b^2c^2\int_0^1\int_0^{\sqrt{1-u^2}}\left(uv-u^3v-uv^3\right)dvdu$$

$$=4a^2b^2c^2\int_0^1\left(\frac{1}{2}\left(u-u^3\right)v^2-\frac{1}{4}uv^4\ \Big|_0^{\sqrt{1-u^2}}\right.\ du$$

$$=a^2b^2c^2\int_0^1\left(2u\left(1-u^2\right)\left(1-u^2\right)-u\left(1-u^2\right)^2\right)du$$

$$=a^2b^2c^2\int_0^1\left(1-u^2\right)^2(2u-u)\,du$$

$$= a^2b^2c^2 \int_0^1 u\left(1-u^2\right)^2 du$$

$$= -\frac{1}{2}a^2b^2c^2 \int_0^1 \left(1-u^2\right)^2 d\left(1-u^2\right)$$

$$= -\frac{1}{6}a^2b^2c^2 \left(1-u^2\right)^3 \Big|_0^1$$

$$= \frac{1}{6}a^2b^2c^2$$

Exercise

Let D be the region bounded by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, where $a>0$, $b>0$ and $c>0$ are real numbers. Find the center of mass of the upper half of D $(z \ge 0)$ assuming it has a constant density.

Solution

Let: $\begin{cases} u = \frac{x}{a} & \rightarrow x = au \\ v = \frac{y}{b} & \rightarrow y = bv \\ w = \frac{z}{c} & \rightarrow z = cw \end{cases}$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \rightarrow \quad u^2+v^2+w^2=1$$

$$J(u,\ v) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

$$= abc$$

Since, $u^2+v^2+w^2=1$ is a unit sphere, then the volume $\frac{4\pi r^2}{3}=\frac{4\pi}{3}$. Therefore, the volume of the ellipsoid is $\frac{4}{3}abc\pi$

$m = \frac{1}{2}\frac{4}{3}abc\pi$ (upper half of D)

$$m = \frac{2}{3}abc\pi$$

By symmetry $\overline{x} = \overline{y} = 0$

$$\bar{z} = \frac{3}{2\pi abc}\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\int_{0}^{\sqrt{1-u^2-v^2}} (abc)(cw)\ dwdvdu$$

$$= \frac{3c}{4\pi}\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \left(w^2 \Big|_0^{\sqrt{1-u^2-v^2}}\right. dvdu$$

$$= \frac{3c}{4\pi}\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \left(1-u^2-v^2\right)dvdu$$

$$= \frac{3c}{2\pi}\int_{-1}^{1} \left(\left(1-u^2\right)v-\frac{1}{3}v^3 \Big|_0^{\sqrt{1-u^2}}\right. du$$

$$= \frac{3c}{2\pi}\int_{-1}^{1} \left(\left(1-u^2\right)^{3/2}-\frac{1}{3}\left(1-u^2\right)^{3/2}\right)du$$

$$= \frac{c}{\pi}\int_{-1}^{1} \left(1-u^2\right)^{3/2} du$$

$$u = \sin t \quad \rightarrow \quad du = \cos t dt$$

$$\begin{cases} u = 1 = \sin t & \rightarrow t = \frac{\pi}{2} \\ u = -1 = \sin t & \rightarrow t = -\frac{\pi}{2} \end{cases}$$

$$\int \left(1-u^2\right)^{3/2} du = \int \cos^4 t\ dt$$

$$= \frac{1}{4}\int \left(1+\cos 2t\right)^2 dt$$

$$= \frac{1}{4}\int \left(1+2\cos 2t+\cos^2 2t\right)dt$$

$$= \frac{1}{4}\int \left(\frac{3}{2}+2\cos 2t+\frac{1}{2}\cos 4t\right)dt$$

$$= \frac{1}{4}\left(\frac{3}{2}t+\sin 2t+\frac{1}{8}\sin 4t\right)$$

$$= \frac{c}{4\pi}\left(\frac{3}{2}t+\sin 2t+\frac{1}{8}\sin 4t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right.$$

$$= \frac{c}{4\pi}\left(2\frac{3}{2}\frac{\pi}{2}\right)$$

$$= \frac{3c}{8}$$

Exercise

Let D be the region bounded by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, where $a>0$, $b>0$ and $c>0$ are real numbers. Find the average square of the distance between points of D and the origin.

Solution

Let: $\begin{cases} u=\frac{x}{a} & \to\ x=au \\ v=\frac{y}{b} & \to\ y=bv \\ w=\frac{z}{c} & \to\ z=cw \end{cases}$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad\to\quad u^2+v^2+w^2=1$$

$$J(u,\ v)=\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

$$=abc$$

The distance between a point on D and the origin is

$$d=\sqrt{x^2+y^2+z^2}$$

$$=\sqrt{a^2u^2+b^2v^2+c^2w^2}$$

Since, $u^2+v^2+w^2=1$ is a unit sphere, then the volume $\frac{4\pi r^2}{3}=\frac{4\pi}{3}$. Therefore, the volume of the ellipsoid is $\frac{4}{3}abc\pi$

$m=\frac{1}{2}\frac{4}{3}abc\pi$ (upper half of D)

$$m=\frac{2}{3}abc\pi$$

$$avg=\frac{1}{2}\frac{3}{2\pi abc}\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\int_{-\sqrt{1-u^2-v^2}}^{\sqrt{1-u^2-v^2}}\left(a^2u^2+b^2v^2+c^2w^2\right)(abc)\ dwdvdu$$

$$=\frac{3}{4\pi}\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\int_{-\sqrt{1-u^2-v^2}}^{\sqrt{1-u^2-v^2}}\left(a^2u^2+b^2v^2+c^2w^2\right)\ dwdvdu$$

Let $u=r\cos\theta \quad v=r\sin\theta$

$$avg=\frac{3}{4\pi}\int_{0}^{2\pi}\int_{0}^{1}\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}}\left(a^2r^2\cos^2\theta+b^2r^2\sin^2\theta+c^2z^2\right)\ r\ dzdrd\theta$$

$$=\frac{3}{2\pi}\int_0^{2\pi}\int_0^1\left(\left(a^2\cos^2\theta+b^2\sin^2\theta\right)r^3z+\frac{1}{3}rc^2z^3\ \Big|_0^{\sqrt{1-r^2}}\ drd\theta\right.$$

$$=\frac{3}{2\pi}\int_0^{2\pi}\int_0^1\left(\left(a^2\cos^2\theta+b^2\sin^2\theta\right)r^3\left(1-r^2\right)^{1/2}+\frac{1}{3}rc^2\left(1-r^2\right)^{3/2}\right)drd\theta$$

$$r=\sin t \quad\rightarrow\quad dr=\cos t dt$$

$$\begin{cases} r=1=\sin t & \rightarrow t=\frac{\pi}{2} \\ r=0=\sin t & \rightarrow t=0 \end{cases}$$

$$\int_0^1 r^3\left(1-r^2\right)^{1/2}dr=\int_0^{\frac{\pi}{2}}\sin^3 t\cos^2 t dt$$

$$=-\int_0^{\frac{\pi}{2}}\left(1-\cos^2 t\right)\cos^2 t\ d\left(\cos t\right)$$

$$=-\int_0^{\frac{\pi}{2}}\left(\cos^2 t-\cos^4 t\right)d\left(\cos t\right)$$

$$=\frac{1}{5}\cos^5 t-\frac{1}{3}\cos^3 t\ \Big|_0^{\frac{\pi}{2}}$$

$$=-\frac{1}{5}+\frac{1}{3}$$

$$=\frac{2}{15}$$

$$=\frac{1}{5\pi}\int_0^{2\pi}\left(a^2\cos^2\theta+b^2\sin^2\theta\right)d\theta\ -\ \frac{c^2}{4\pi}\int_0^{2\pi}d\theta\ \int_0^1\left(1-r^2\right)^{3/2}d\left(1-r^2\right)$$

$$=\frac{1}{10\pi}\int_0^{2\pi}\left(a^2\left(1+\cos 2\theta\right)+b^2\left(1-\cos 2\theta\right)\right)d\theta\ -\ \frac{c^2}{5}\left(1-r^2\right)^{5/2}\ \Big|_0^1$$

$$=\frac{1}{10\pi}\left(a^2\left(\theta+\frac{1}{2}\sin 2\theta\right)+b^2\left(\theta-\frac{1}{2}\sin 2\theta\right)\ \Big|_0^{2\pi}\ +\ \frac{c^2}{5}\right.$$

$$=\frac{1}{10\pi}\left(2\pi a^2+2\pi b^2\right)+\frac{c^2}{5}$$

$$=\frac{a^2+b^2+c^2}{5}$$