

## Section 4.3 – Conservative Vector Fields

### Line Integrals of Vector Fields

Assume the vector field  $\vec{F}(x, y, z) = M(x, y, z)\hat{i} + N(x, y, z)\hat{j} + P(x, y, z)\hat{k}$  has a continuous components, and the curve  $C$  has a smooth parametrization  $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ ,  $a \leq t \leq b$ .  $\vec{r}(t)$  defines along the path  $C$  which we call the **forward direction**. At each point along the path  $C$ , the tangent vector  $\vec{T} = \frac{d\vec{r}}{ds} = \frac{\vec{v}}{|\vec{v}|}$  is a unit vector tangent to the path and pointing in this forward direction. The tangential component is given by the dot product

$$\vec{F} \cdot \vec{T} = \vec{F} \cdot \frac{d\vec{r}}{ds}$$

### Definition

Let  $\vec{F}$  be a vector field with continuous components defined along a smooth curve  $C$  parametrized by  $\vec{r}(t)$ ,  $a \leq t \leq b$ . Then the line integral of  $\vec{F}$  along  $C$  is

$$\int_C \vec{F} \cdot \vec{T} \, ds = \int_C \left( \vec{F} \cdot \frac{d\vec{r}}{ds} \right) ds = \int_C \vec{F} \cdot d\vec{r}$$

**Evaluating the Line Integral of  $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$  along  $C$ :**  $\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$

1. Express the vector field  $\vec{F}$  in terms of the parametrized curve  $C$  as  $\vec{F}(\vec{r}(t))$  by substituting the components  $x = g(t)$ ,  $y = h(t)$ ,  $z = k(t)$  of  $\vec{r}$  into the scalar components  $M(x, y, z)$ ,  $N(x, y, z)$ ,  $P(x, y, z)$  of  $\vec{F}$ .
2. Find the derivative (velocity) vector  $\frac{d\vec{r}}{dt}$ .
3. Evaluate the line integral with respect to the parameter  $t$ ,  $a \leq t \leq b$ , to obtain

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} dt$$

**Example**

Evaluate  $\int_C \vec{F} \cdot d\vec{r}$ , where  $\vec{F} = z\hat{i} + xy\hat{j} - y^2\hat{k}$  along the curve  $C$  given by

$$\vec{r}(t) = t^2\hat{i} + t\hat{j} + \sqrt{t}\hat{k} \quad 0 \leq t \leq 1.$$

**Solution**

$$\vec{F}(\vec{r}(t)) = \sqrt{t}\hat{i} + t^3\hat{j} - t^2\hat{k}$$

$$\frac{d\vec{r}}{dt} = 2t\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\hat{k}$$

$$\vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} = \left( \sqrt{t}\hat{i} + t^3\hat{j} - t^2\hat{k} \right) \cdot \left( 2t\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\hat{k} \right)$$

$$= 2t\sqrt{t} + t^3 - \frac{t^2}{2\sqrt{t}}$$

$$= 2t^{3/2} + t^3 - \frac{1}{2}t^{3/2}$$

$$= \frac{3}{2}t^{3/2} + t^3$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} dt$$

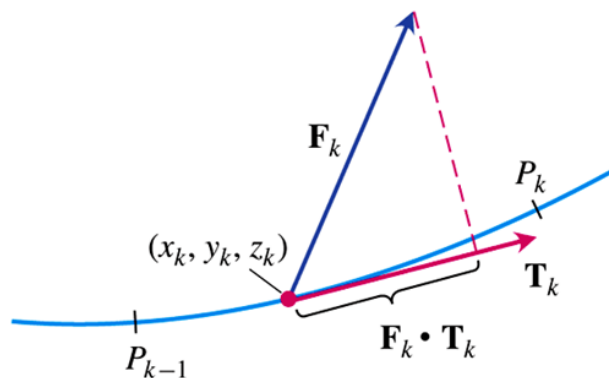
$$= \int_0^1 \left( \frac{3}{2}t^{3/2} + t^3 \right) dt$$

$$= \frac{3}{5}t^{5/2} + \frac{1}{4}t^4 \Big|_0^1$$

$$= \frac{3}{5} + \frac{1}{4}$$

$$= \frac{17}{20}$$

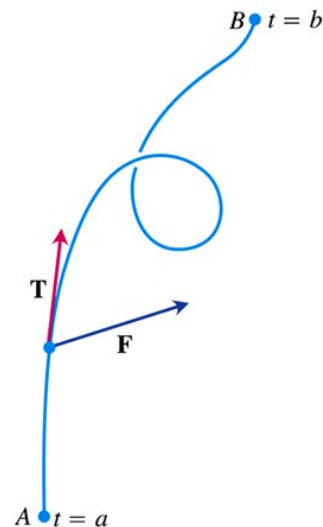
## Work Done by a Force over a Curve in Space



### Definition

Let  $C$  be a smooth curve parametrized by  $\mathbf{r}(t)$ ,  $a \leq t \leq b$ , and  $\vec{F}$  be a continuous force field over a region containing  $C$ . Then the **work** done in moving an object from point  $A = \vec{r}(a)$  to the point  $B = \vec{r}(b)$  along  $C$  is

$$W = \int_C \vec{F} \cdot \vec{T} \, ds = \int_a^b \vec{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} dt$$



**Different ways to write the work integral for  $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$  over the curve  $C$ :**

$$\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$$

$$W = \int_C \vec{F} \cdot \vec{T} \, ds$$

*The definition*

$$= \int_C \vec{F} \cdot d\vec{r}$$

*Vector differential form*

$$= \int_a^b \vec{F} \cdot \frac{d\vec{r}}{dt} dt$$

*Parametric vector evaluation*

$$= \int_a^b \left( M \frac{dx}{dt} + N \frac{dy}{dt} + P \frac{dz}{dt} \right) dt$$

*Parametric scalar evaluation*

$$= \int_C Mdx + Ndy + Pdz$$

*Scalar differential form*

### Example

Find the work done by the force field  $\vec{F} = (y - x^2)\hat{i} + (z - y^2)\hat{j} + (x - z^2)\hat{k}$  along the curve

$$\vec{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k} \quad 0 \leq t \leq 1, \text{ from } (0, 0, 0) \text{ to } (1, 1, 1).$$

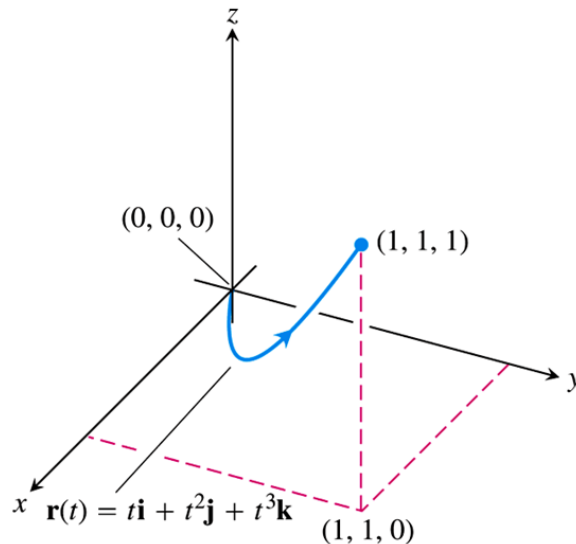
### Solution

$$\begin{aligned}\vec{F} &= (y - x^2)\hat{i} + (z - y^2)\hat{j} + (x - z^2)\hat{k} \\ &= (t^2 - t^2)\hat{i} + (t^3 - t^4)\hat{j} + (t - t^6)\hat{k} \\ &= (t^3 - t^4)\hat{j} + (t - t^6)\hat{k}\end{aligned}$$

$$\begin{aligned}\frac{d\vec{r}}{dt} &= \frac{d}{dt}(t\hat{i} + t^2\hat{j} + t^3\hat{k}) \\ &= \hat{i} + 2t\hat{j} + 3t^2\hat{k}\end{aligned}$$

$$\begin{aligned}\vec{F} \cdot \frac{d\vec{r}}{dt} &= [(t^3 - t^4)\hat{j} + (t - t^6)\hat{k}] \cdot (\hat{i} + 2t\hat{j} + 3t^2\hat{k}) \\ &= 2t(t^3 - t^4) + 3t^2(t - t^6) \\ &= 2t^4 - 2t^5 + 3t^3 - 3t^8\end{aligned}$$

$$\begin{aligned}W &= \int_0^1 \vec{F} \cdot \frac{d\vec{r}}{dt} dt \\ &= \int_0^1 (2t^4 - 2t^5 + 3t^3 - 3t^8) dt \\ &= \left. \frac{2}{5}t^5 - \frac{1}{3}t^6 + \frac{3}{4}t^4 - \frac{1}{3}t^9 \right|_0^1 \\ &= \frac{2}{5} - \frac{1}{3} + \frac{3}{4} - \frac{1}{3} \\ &= \frac{29}{60}\end{aligned}$$



### Example

Find the work done by the force field  $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$  in moving an object along the curve  $C$  parametrized by  $\vec{r}(t) = \cos(\pi t)\hat{i} + t^2\hat{j} + \sin(\pi t)\hat{k}$   $0 \leq t \leq 1$ .

### Solution

$$\vec{F}(\vec{r}(t)) = \cos(\pi t)\hat{i} + t^2\hat{j} + \sin(\pi t)\hat{k}$$

$$\frac{d\vec{r}}{dt} = -\pi \sin(\pi t)\hat{i} + 2t\hat{j} + \pi \cos(\pi t)\hat{k}$$

$$\begin{aligned}\vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} &= (\cos(\pi t)\hat{i} + t^2\hat{j} + \sin(\pi t)\hat{k}) \cdot (-\pi \sin(\pi t)\hat{i} + 2t\hat{j} + \pi \cos(\pi t)\hat{k}) \\ &= -\pi \cos(\pi t)\sin(\pi t) + 2t^3 + \pi \cos(\pi t)\sin(\pi t) \\ &= 2t^3\end{aligned}$$

The work done is the line integral

$$\begin{aligned}W &= \int_0^1 2t^3 dt \\ &= \frac{1}{2}t^4 \Big|_0^1 \\ &= \frac{1}{2}\end{aligned}$$

## Flow integrals and Circulation for Velocity Fields

### Definitions

If  $\vec{r}(t)$  parametrizes a smooth curve  $C$  in the domain of a continuous velocity field  $\vec{F}$ , the **flow** along the curve point  $A = \vec{r}(a)$  to  $B = \vec{r}(b)$  is

$$Flow = \int_C \vec{F} \cdot \vec{T} ds$$

The integral in this case is called a **flow integral**. If the curve starts and ends at the same point, so that  $A = B$ , the flow is called the **circulation** around the curve.

### Example

A fluid's velocity field is  $\vec{F} = x\hat{i} + z\hat{j} + y\hat{k}$ . Find the flow along the helix

$$\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}, \quad 0 \leq t \leq \frac{\pi}{2}$$

### Solution

$$\vec{F} = x\hat{i} + z\hat{j} + y\hat{k}$$

$$= (\cos t)\hat{i} + t\hat{j} + (\sin t)\hat{k}$$

$$\frac{d\vec{r}}{dt} = (-\sin t)\hat{i} + (\cos t)\hat{j} + \hat{k}$$

$$\begin{aligned}\vec{F} \cdot \frac{d\vec{r}}{dt} &= ((\cos t)\hat{i} + t\hat{j} + (\sin t)\hat{k}) \cdot ((-\sin t)\hat{i} + (\cos t)\hat{j} + \hat{k}) \\ &= -\cos t \sin t + t \cos t + \sin t\end{aligned}$$

$$\text{Flow} = \int_0^{\pi/2} (-\cos t \sin t + t \cos t + \sin t) dt$$

$$\int -\cos t \sin t dt = \int \cos t d(\cos t) = \frac{1}{2} \cos^2 t$$

$$= \frac{1}{2} \cos^2 t + t \sin t + \cos t - \cos t \Big|_0^{\pi/2}$$

$$= \frac{1}{2} \cos^2 t + t \sin t \Big|_0^{\pi/2}$$

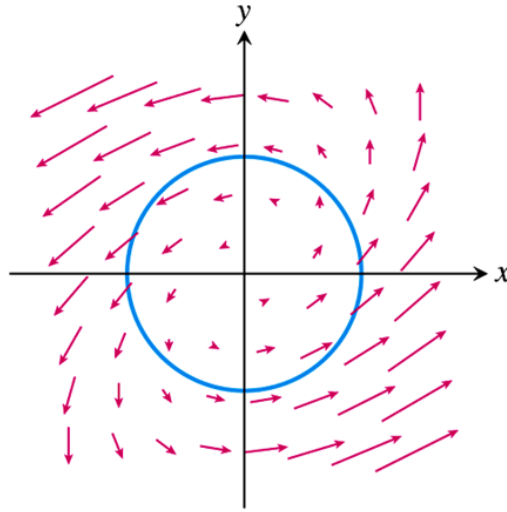
$$= \frac{\pi}{2} - \frac{1}{2}$$

		$\cos t$
+	$t$	$\rightarrow \sin t$
-	1	$\rightarrow -\cos t$

### Example

Find the circulation of the field  $\vec{F} = (x - y)\hat{i} + x\hat{j}$  around the circle

$$\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j}, \quad 0 \leq t \leq 2\pi$$



### Solution

$$\begin{aligned}\vec{F} &= (x - y)\hat{i} + x\hat{j} \\ &= (\cos t - \sin t)\hat{i} + (\cos t)\hat{j}\end{aligned}$$

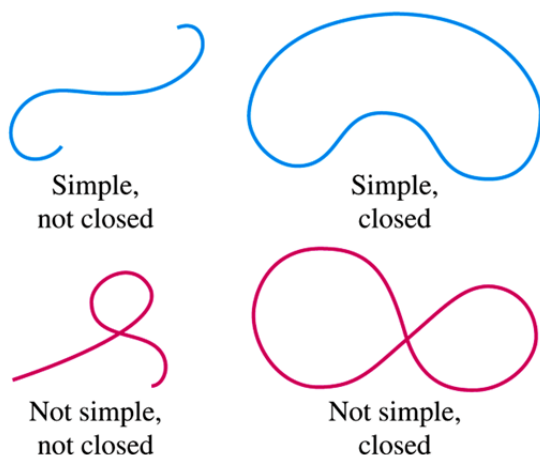
$$\frac{d\vec{r}}{dt} = (-\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\begin{aligned}\vec{F} \cdot \frac{d\vec{r}}{dt} &= ((\cos t - \sin t)\hat{i} + (\cos t)\hat{j}) \cdot ((-\sin t)\hat{i} + (\cos t)\hat{j}) \\ &= -\cos t \sin t + \sin^2 t + \cos^2 t \\ &= 1 - \cos t \sin t\end{aligned}$$

$$\begin{aligned}\text{Circulation} &= \int_0^{2\pi} (1 - \cos t \sin t) dt \\ &= t + \frac{1}{2} \cos^2 t \Big|_0^{2\pi} \\ &= 2\pi + \frac{1}{2} - \frac{1}{2} \\ &= 2\pi\end{aligned}$$

## Flux across a Simple Plane Curve

A curve in the  $xy$ -plane is simple if it does not cross itself. When a curve starts and ends at the same point, it is a **closed curve** or **loop**.

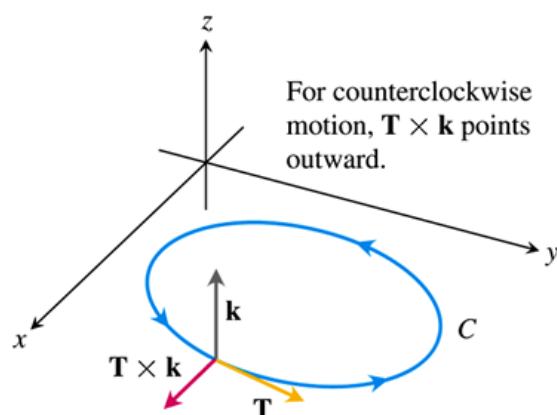
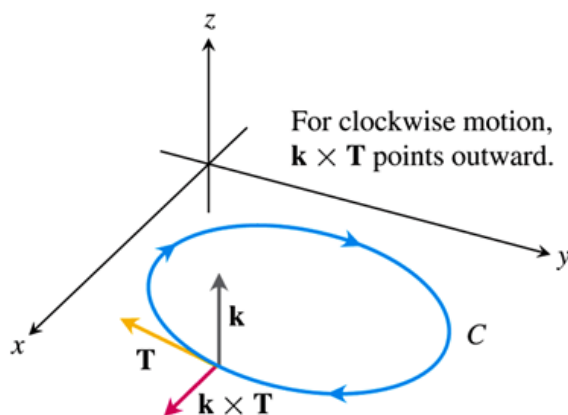


### Definition

If  $C$  is a smooth simple closed curve in the domain of a continuous velocity field in

$\vec{F} = M(x, y)\hat{i} + N(x, y)\hat{j}$  in the plane, and if  $\vec{n}$  is the outward-pointing unit normal vector on  $C$ , the flux of  $\vec{F}$  across  $C$  is

$$\text{Flux of } \vec{F} \text{ across } C = \int_C \vec{F} \cdot \vec{n} \, ds$$



$$\begin{aligned} \vec{n} &= \vec{T} \times \hat{k} \\ &= \left( \frac{dx}{ds} \hat{i} + \frac{dy}{ds} \hat{j} \right) \times \hat{k} \\ &= \frac{dy}{ds} \hat{i} - \frac{dx}{ds} \hat{j} \end{aligned}$$

$$\vec{F} \cdot \vec{n} = M(x, y) \frac{dy}{ds} - N(x, y) \frac{dx}{ds}$$



## Calculating Flux Across a Smooth Closed Plane Curve

$$\left( \text{Flux of } \vec{F} = M\hat{i} + N\hat{j} \text{ across } C \right) = \oint_C Mdy - Ndx$$

The integral can be evaluated from any smooth parametrization  $x = g(t)$ ,  $y = h(t)$ ,  $a \leq t \leq b$ , that traces  $C$  counterclockwise exactly once.

### Example

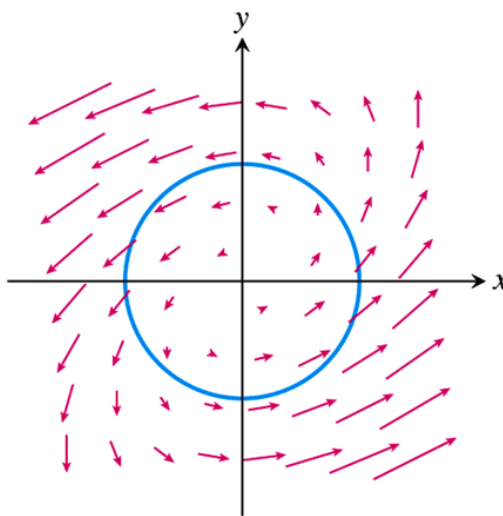
Find the flux of  $\vec{F} = (x - y)\hat{i} + x\hat{j}$  across the circle  $x^2 + y^2 = 1$  in the  $xy$ -plane. (The vector field and curve)

### Solution

The parametrization  $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j}$ ,  $0 \leq t \leq 2\pi$  traces the circle counterclockwise exactly once.

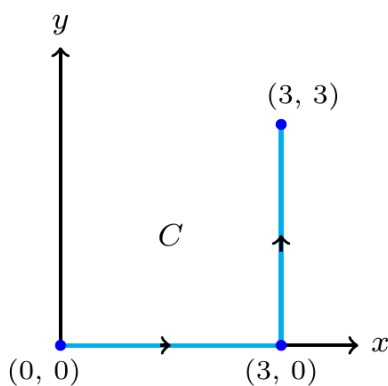
$$\begin{aligned} M = x - y = \cos t - \sin t, & \quad dy = d(\sin t) = \cos t \, dt \\ N = x = \cos t, & \quad dx = d(\cos t) = -\sin t \, dt \end{aligned}$$

$$\begin{aligned} \text{Flux} &= \int_C Mdy - Ndx \\ &= \int_0^{2\pi} (\cos^2 t - \sin t \cos t + \cos t \sin t) dt \\ &= \int_0^{2\pi} \cos^2 t \, dt \\ &= \int_0^{2\pi} \left( \frac{1}{2} + \frac{1}{2} \cos 2t \right) dt \\ &= \left( \frac{1}{2}t + \frac{1}{4} \sin 2t \right) \Big|_0^{2\pi} \\ &= \pi \end{aligned}$$

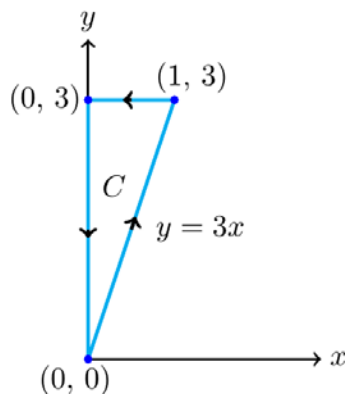


## Exercises      Section 4.3 – Conservative Vector Fields

- Find the gradient field of the function  $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$
- Find the gradient field of the function  $f(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2}$
- Find the gradient field of the function  $f(x, y, z) = e^z - \ln(x^2 + y^2)$
- Find the line integral of  $\int_C (x - y) dx$  where  $C: x = t, \quad y = 2t + 1, \quad \text{for } 0 \leq t \leq 3$
- Find the line integral of  $\int_C (x^2 + y^2) dy$  where  $C$  is



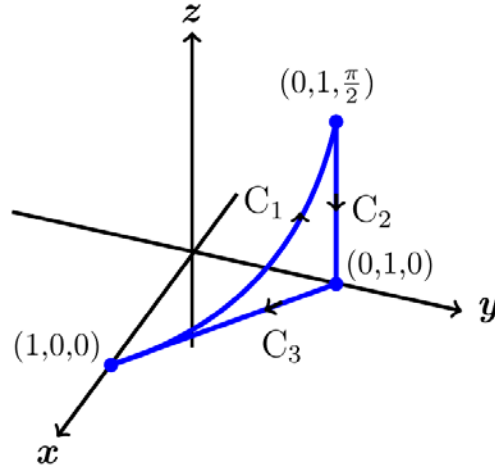
- Find the line integral of  $\int_C \sqrt{x + y} \, dx$  where  $C$  is



- Find the work done by the force field  $\vec{F} = xy\hat{i} + y\hat{j} - yz\hat{k}$  over the curve  $\vec{r}(t) = t\hat{i} + t^2\hat{j} + t\hat{k}, \quad 0 \leq t \leq 1.$

8. Find the work done by the force field  $\vec{F} = 2y\hat{i} + 3x\hat{j} + (x + y)\hat{k}$  over the curve  
 $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + \frac{t}{6}\hat{k}, \quad 0 \leq t \leq 2\pi$
9. Find the work done by the force field  $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$  over the curve  
 $\vec{r}(t) = (\sin t)\hat{i} + (\cos t)\hat{j} + t\hat{k}, \quad 0 \leq t \leq 2\pi.$
10. Find the work required to move an object with given force field  $\vec{F} = \langle -y, z, x \rangle$  on the path consisting of the line segments from  $(0, 0, 0)$  to  $(0, 1, 0)$  followed by the line segment from  $(0, 1, 0)$  to  $(0, 1, 4)$
11. Find the work required to move an object with given force field  $\vec{F} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}$  on the path  
 $\vec{r}(t) = \langle t^2, 3t^2, -t^2 \rangle$  for  $1 \leq t \leq 2$
12. Evaluate  $\int_C \vec{F} \cdot \vec{T} \, ds$  for the vector field  $\vec{F} = x^2\hat{i} - y\hat{j}$  along the curve  $x = y^2$  from  $(4, 2)$  to  $(1, -1)$
13. Find the circulation and flux of the fields  $\vec{F}_1 = x\hat{i} + y\hat{j}$  and  $\vec{F}_2 = -y\hat{i} + x\hat{j}$  around and across each of the following curves.  
 a) The circle  $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j}, \quad 0 \leq t \leq 2\pi$   
 b) The ellipse  $\vec{r}(t) = (\cos t)\hat{i} + (4 \sin t)\hat{j}, \quad 0 \leq t \leq 2\pi$
14. Find the circulation and flux of the fields  $\vec{F}_1 = 2x\hat{i} - 3y\hat{j}$  and  $\vec{F}_2 = 2x\hat{i} + (x - y)\hat{j}$  across the circle  $\vec{r}(t) = (a \cos t)\hat{i} + (a \sin t)\hat{j}, \quad 0 \leq t \leq 2\pi$
15. Find a field  $\vec{F} = M(x, y)\hat{i} + N(x, y)\hat{j}$  in the  $xy$ -plane with the property that at each point  $(x, y) \neq (0, 0)$ ,  $\vec{F}$  points toward the origin and  $|\vec{F}|$  is  
 a) The distance from  $(x, y)$  to the origin  
 b) Inversely proportional to the distance from  $(x, y)$  to the origin.  
 (The field is undefined at  $(0, 0)$ .)
16. A fluid's velocity field is  $\vec{F} = -4xy\hat{i} + 8y\hat{j} + 2\hat{k}$ . Find the flow along the curve  
 $\vec{r}(t) = t\hat{i} + t^2\hat{j} + \hat{k}, \quad 0 \leq t \leq 2$

17. A fluid's velocity field is  $\vec{F} = x^2\hat{i} + yz\hat{j} + y^2\hat{k}$ . Find the flow along the curve  $\vec{r}(t) = 3t\hat{j} + 4t\hat{k}$ ,  $0 \leq t \leq 1$
18. Find the circulation of  $\vec{F} = 2x\hat{j} + 2z\hat{j} + 2y\hat{k}$  around the closed path consisting of the following three curves traversed in the direction of increasing  $t$ .

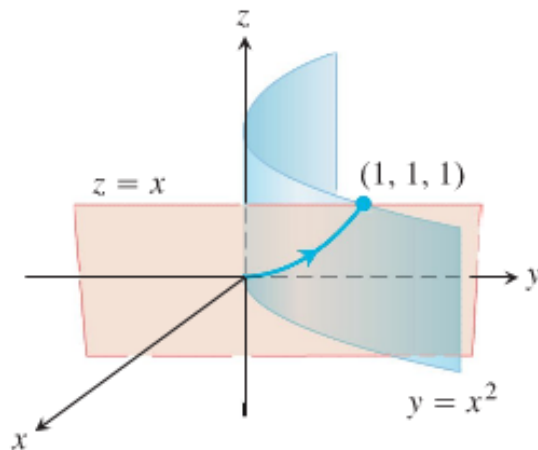


$$C_1 : \vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}, \quad 0 \leq t \leq \frac{\pi}{2}$$

$$C_2 : \vec{r}(t) = \hat{j} + \frac{\pi}{2}(1-t)\hat{k}, \quad 0 \leq t \leq 1$$

$$C_3 : \vec{r}(t) = t\hat{i} + (1-t)\hat{j}, \quad 0 \leq t \leq 1$$

19. The field  $\vec{F} = xy\hat{i} + y\hat{j} - yz\hat{k}$  is the velocity field of a flow in space. Find the flow from  $(0, 0, 0)$  to  $(1, 1, 1)$  along the curve of intersection of the cylinder  $y = x^2$  and the plane  $z = x$ .  
(*Hint:* Use  $t = x$  as the parameter.)



20. Find the work required to move an object with given force field  $\vec{F} = \langle -y, z, x \rangle$  on the path consisting of the line segments from  $(0, 0, 0)$  to  $(0, 1, 0)$  followed by the line segment from  $(0, 1, 0)$  to  $(0, 1, 4)$

21. Find the work required to move an object with given force field  $\vec{F} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}$  on the path

$$\vec{r}(t) = \langle t^2, 3t^2, -t^2 \rangle \text{ for } 1 \leq t \leq 2$$

22. Evaluate  $\int_C (x - y)dx + (x + y)dy$  counterclockwise around the triangle with vertices  $(0, 0)$ ,  $(1, 0)$  and  $(0, 1)$

- (23–28) Evaluate the line integral  $\int_C \vec{F} \cdot d\vec{r}$  for the vector fields  $\vec{F}$  and curves  $C$ .

23.  $\vec{F} = \nabla(x^2 y)$ ;  $C: \vec{r}(t) = \langle 9 - t^2, t \rangle$ , for  $0 \leq t \leq 3$

24.  $\vec{F} = \nabla(xyz)$ ;  $C: \vec{r}(t) = \langle \cos t, \sin t, \frac{t}{\pi} \rangle$ , for  $0 \leq t \leq \pi$

25.  $\vec{F} = \langle x, -y \rangle$ ;  $C$  is the square with vertices  $(\pm 1, \pm 1)$  with counterclockwise orientation.

26.  $\vec{F} = \langle y, z, -x \rangle$ ;  $C: \vec{r}(t) = \langle \cos t, \sin t, 4 \rangle$ , for  $0 \leq t \leq 2\pi$

27.  $\vec{F} = \langle y^2, x \rangle$ ; where  $C$  is the arc of the parabola  $x = 4 - y^2$  from  $(-5, -3)$  to  $(0, 2)$

28.  $\vec{F} = \langle x^2 + y^2, 4x + y^2 \rangle$ ; where  $C$  is the straight line segment from  $(6, 3)$  to  $(6, 0)$

- (29–34) Evaluate the line integral  $\int_C \vec{F} \cdot \vec{T} ds$  for the vector fields  $\vec{F}$  and curves  $C$ .

29.  $\vec{F} = \langle x, y \rangle$  on the parabola  $\vec{r}(t) = \langle 4t, t^2 \rangle$   $0 \leq t \leq 1$

30.  $\vec{F} = \langle -y, x \rangle$  on the semicircle  $\vec{r}(t) = \langle 4 \cos t, 4 \sin t \rangle$   $0 \leq t \leq \pi$

31.  $\vec{F} = \langle y, x \rangle$  on the line segment from  $(1, 1)$  to  $(5, 10)$

32.  $\vec{F} = \langle -y, x \rangle$  on the parabola  $y = x^2$  from  $(0, 0)$  to  $(1, 1)$

33.  $\vec{F} = \frac{\langle x, y \rangle}{(x^2 + y^2)^{3/2}}$  on the curve  $\vec{r}(t) = \langle t^2, 3t^2 \rangle$   $1 \leq t \leq 2$

34.  $\vec{F} = \frac{\langle x, y \rangle}{x^2 + y^2}$  on the line  $\vec{r}(t) = \langle t, 4t \rangle$   $1 \leq t \leq 10$

(35–45) Find the work required to move an object on the given oriented curve

35.  $\vec{F} = \langle y, -x \rangle$  on the path consisting of the line segment from  $(1, 2)$  to  $(0, 0)$  followed by the line segment from  $(0, 0)$  to  $(0, 4)$

36.  $\vec{F} = \langle x, y \rangle$  on the path consisting of the line segment from  $(-1, 0)$  to  $(0, 8)$  followed by the line segment from  $(0, 8)$  to  $(2, 8)$

37.  $\vec{F} = \langle x^2, -xy \rangle$  on runs from  $(1, 0)$  to  $(0, 1)$  along the unit circle and then from  $(0, 1)$  to  $(0, 0)$  along the  $y$ -axis.

38.  $\vec{F} = \langle y, x \rangle$  on the parabola  $y = 2x^2$  from  $(0, 0)$  to  $(2, 8)$

39.  $\vec{F} = \langle y, -x \rangle$  on the line  $y = 10 - 2x$  from  $(1, 8)$  to  $(3, 4)$

40.  $\vec{F} = \langle x, y, z \rangle$  on the tilted ellipse  $\vec{r}(t) = \langle 4 \cos t, 4 \sin t, 4 \cos t \rangle$   $0 \leq t \leq 2\pi$

41.  $\vec{F} = \langle -y, x, z \rangle$  on the helix  $\vec{r}(t) = \langle 2 \cos t, 2 \sin t, \frac{t}{2\pi} \rangle$   $0 \leq t \leq 2\pi$

42.  $\vec{F} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}$  on the line segment from  $(1, 1, 1)$  to  $(10, 10, 10)$

43.  $\vec{F} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}$  on the path  $\vec{r}(t) = \langle t^2, 3t^2, -t^2 \rangle$ ,  $1 \leq t \leq 2$

44.  $\vec{F} = \frac{\langle x, y \rangle}{(x^2 + y^2)^{3/2}}$  over the plane curve  $\vec{r}(t) = \langle e^t \cos t, e^t \sin t \rangle$  from the point  $(1, 0)$  to the point  $(e^{2\pi}, 0)$  by using the parametrization of the curve to evaluate the work integral

45.  $\vec{F} = \frac{\langle x, y, z \rangle}{x^2 + y^2 + z^2}$  on the line segment from  $(1, 1, 1)$  to  $(8, 4, 2)$

46. Let  $C$  be the circle of radius 2 centered at the origin with counterclockwise orientation

a) Give the unit outward vector at any point  $(x, y)$  on  $C$ .

b) Find the normal component of the vector field  $\vec{F} = 2\langle y, -x \rangle$  at any point on  $C$ .

c) Find the normal component of the vector field  $\vec{F} = \frac{\langle x, y \rangle}{x^2 + y^2}$  at any point on  $C$ .

**47.** Find the flow of the field  $\vec{F} = \nabla(x^2 z e^y)$

- a) Once around the ellipse  $C$  in which the plane  $x + y + z = 1$  intersects the cylinder  $x^2 + z^2 = 25$ , clockwise as viewed from the positive  $y$ -axis.
- b) Along the curved boundary of the helicoid  $\vec{r}(r, \theta) = (r \cos \theta)\hat{i} + (r \sin \theta)\hat{j} + \theta\hat{k}$  from  $(1, 0, 0)$  to  $(1, 0, 2\pi)$