Section 1.2 – Functions

A ***set*** is a collection of objects of some type, and the objects are called ***elements*** of the set.

Notation or Terminology	***Meaning***	***Example***
$a \in S$	***a*** is an element of ***S***	$3 \in \mathbb{Z}$
$a \notin S$	***a*** is not an element of ***S***	$\frac{3}{2} \notin \mathbb{Z}$
$S \subset T$	*S* is a ***subset*** of *T* Every element of *S* is an element of *T*	$\mathbb{Z} \subset \mathbb{R}$
Constant	A letter or symbol that represents a specific element of a set.	$5,\ \sqrt{2},\ \pi$
Variable	A letter or symbol that represents any element of a set.	Let ***x*** denote any $\mathbb{R}$

Definition of a *Function*

A ***function*** is a relation between two variables such that to matches each element of a first set (called ***domain***) to an element of a second set (called ***range***) in such way that no element in the first set is assigned to two different elements in the second set.

The ***domain*** of the function is the set of all values of the independent variable for which the function is defined.

The ***range*** of the function is the set of all values taken on by the dependent variable.

The *Domain* of a Function

1. *Rational* function: $\frac{f(x)}{h(x)}$ $\Rightarrow$ ***Domain***: $h(x) \neq 0$

Example: $f(x) = \frac{1}{x-3}$ ***Domain***: $x \neq 3$

2. *Irrational* function: $\sqrt{g(x)}$ $\Rightarrow$ ***Domain***: $g(x) \geq 0$

Example: $g(x) = \sqrt{3-x} + 5$ ***Domain***: $x \leq 3$

3. Otherwise: *Domain* all real numbers

Example: $f(x) = x^3 + |x|$ ***Domain***: All real numbers, $\mathbb{R}$, or $(-\infty,\ \infty)$

(1) & (2)→ Find the domain: $f(x) = \frac{x+1}{\sqrt{x-3}}$ $\Rightarrow$ ***Domain:*** $x > 3$

$$\boxed{\begin{array}{l} ax^2+bx+c\ge 0 \;\rightarrow\; if\;\; a>0 \;\Rightarrow\; x\le x_1,\;\; x\ge x_2 \\ ax^2+bx+c\le 0 \;\rightarrow\; if\;\; a>0 \;\Rightarrow\; x_1\le x\le x_2 \end{array}}$$

Example

Let $g(x)=\dfrac{\sqrt{4+x}}{1-x}$. Find the domain of g.

Solution

$$\begin{cases} 4+x\ge 0 \Rightarrow x\ge -4 \\ 1-x\ne 0 \Rightarrow x\ne 1 \end{cases}$$

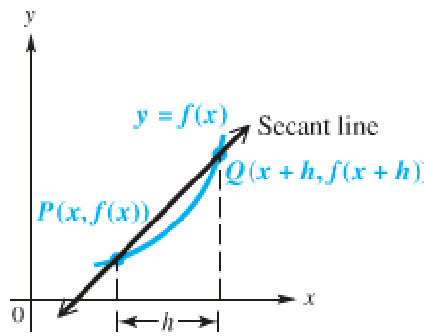
$\rightarrow\;\; \underline{[-4,\ 1)\cup(1,\ \infty)}$

Difference Quotients

$$\frac{f(x+h)-f(x)}{(x+h)-x}$$

The difference quotient is given by:

$$\frac{f(x+h)-f(x)}{h}$$



Example

For the function f given by $f(x)=2x^2-3x$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$\frac{f(x+h)-f(x)}{h}=\frac{\overbrace{2(x+h)^2-3(x+h)}^{f(x+h)}\;-\;\overbrace{(2x^2-3x)}^{f(x)}}{h}$$

$$=\frac{2x^2+4xh+2h^2-3x-3h-2x^2+3x}{h}$$

$$=\frac{4xh+2h^2-3h}{h}$$

$$=\frac{4xh}{h}+\frac{2h^2}{h}-\frac{3h}{h}$$

$$\underline{=4x+2h-3}$$

Piecewise-Defined Functions

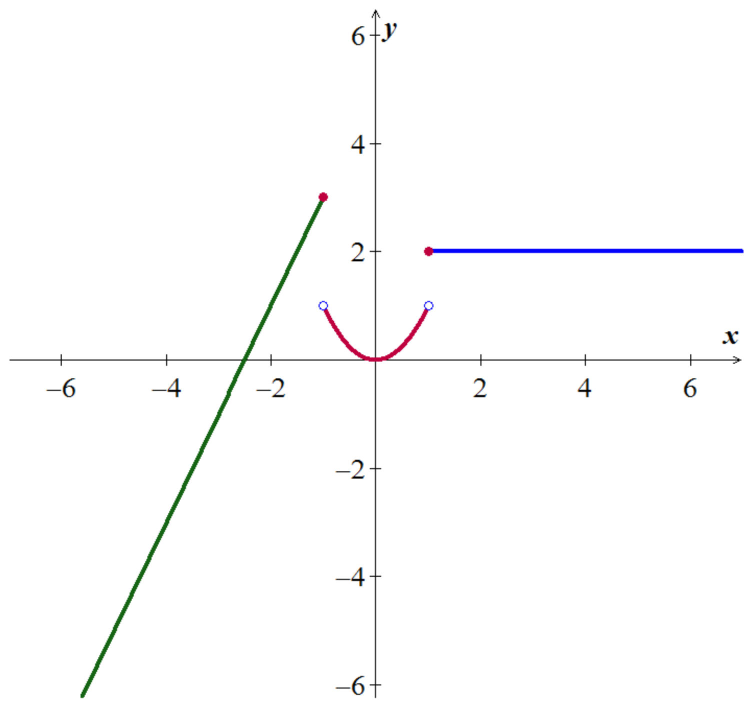
Function are sometimes described by more than one expression, we call such functions ***piecewise-defined functions***.

Example

Graph each function

$$f(x)=\begin{cases} 2x+5 & if \quad x\le -1 \\ x^2 & if \quad |x|<1 \\ 2 & if \quad x\ge 1 \end{cases}$$

Solution

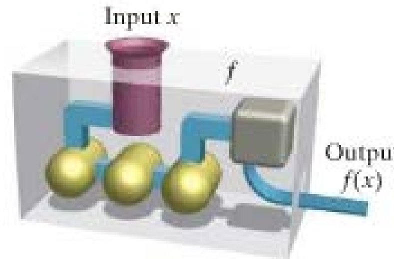


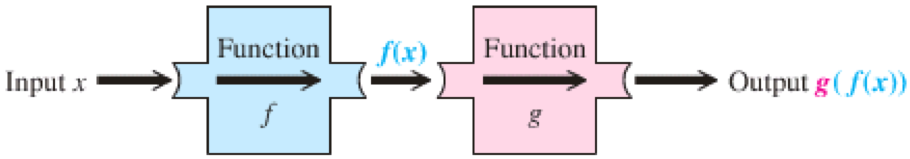
Composition of Functions

The composite function $f \circ g$, the composite of f and g, is defined as

$$\left(f \circ g\right)\left(x\right) = f\left(g\left(x\right)\right)$$

Where x is in the domain of g
and $g\left(x\right)$ is in the domain of f





Example

Let $f(x) = x^2 - 1$ *and* $g(x) = 3x + 5$

a) Find $\left(f \circ g\right)(x)$ and the domain of $f \circ g$

b) Find $\left(g \circ f\right)(x)$ and the domain of $g \circ f$

c) Find $\left(f\left(g\right)\right)(2)$ in two different ways: first using the functions f and g separately and second using the composite function $f \circ g$.

Solution

a) $\left(f \circ g\right)(x) = f\left(g(x)\right)$

$= f\left(3x + 5\right)$ ***Domain*** $: \left(3x+5\right) \to \mathbb{R}$

$= \left(\_\_\right)^2 - 1$

$= \left(3x + 5\right)^2 - 1$

$= 9x^2 + 30x + 25 - 1$

$= 9x^2 + 30x + 24$ ***Domain*** $: \left(9x^2 + 30x + 24\right) \to \mathbb{R}$

Domain of $f \circ g$: $\mathbb{R}$

b) $\left(g \circ f\right)(x) = g\left(f(x)\right)$

$= g\left(x^2 - 1\right)$ ***Domain*** $: \left(x^2 - 1\right) \to \mathbb{R}$

$= 3\left(x^2 - 1\right) + 5$

$= 3x^2 - 3 + 5$

$= 3x^2 + 2$ ***Domain*** $: \left(3x^2 + 2\right) \to \mathbb{R}$

Domain of $g \circ f$: $\mathbb{R}$

c) $g(2) = 3(2) + 5 = 11$

$$\begin{aligned}(f \circ g)(2) &= f(g(2)) \\ &= f(11) \\ &= 11^2 - 1 \\ &= 120\end{aligned}$$

$$(f \circ g)(x) = 9x^2 + 30x + 24$$

$$(f \circ g)(2) = 9(2)^2 + 30(2) + 24 \ \underline{= 120}$$

Example

Let $f(x) = x^2 - 16$ *and* $g(x) = \sqrt{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $(f \circ g)(x) = f(g(x))$

$= f(\sqrt{x})$ ***Domain*** $: (\sqrt{x}) \to \boldsymbol{x} \ge 0$

$= (\sqrt{x})^2 - 16$

$= x - 16$ ***Domain*** $: (x - 16) \to \mathbb{R}$

Domain of $f \circ g$: $\boldsymbol{x} \ge 0$

b) $(g \circ f)(x) = g(f(x))$

$= g(x^2 - 16)$ ***Domain*** $: (x^2 - 1) \to \mathbb{R}$

$= \sqrt{x^2 - 16}$ ***Domain*** $: \left(\sqrt{x^2 - 16}\right) \to |x| \ge 4$

Domain of $g \circ f$: $|x| \ge 4$ *or* $(-\infty, -4] \cup [4, \infty)$

Even and Odd Functions

Given the function $f(x)$ then find $f(-x)$ and simplify:

- If $f(-x) = f(x) \Rightarrow f$ is ***even***, or
- If $f(-x) = -f(x) \Rightarrow f$ is ***odd***
- ***Neither***

Example

Decide whether each function is even, odd, or neither

a) $f(x) = 8x^4 - 3x^2$

$$\begin{aligned} f(-x) &= 8(-x)^4 - 3(-x)^2 \\ &= 8x^4 - 3x^2 \\ &= f(x) \end{aligned}$$

Function is *Even*

b) $f(x) = 6x^3 - 9x$

$$\begin{aligned} f(-x) &= 6(-x)^3 - 9(-x) \\ &= -6x^3 + 9x \\ &= -\left(6x^3 - 9x\right) \\ &= -f(x) \end{aligned}$$

Function is *Odd*

c) $f(x) = 3x^2 + 5x$

$$\begin{aligned} f(-x) &= 3(-x)^2 + 5(-x) \\ &= 3x^2 - 5x \end{aligned}$$

Function is *Neither*

Exercises *Section* 1.2 –Functions

(**1 – 80**) Find the Domain

1. $f(x)=7x+4$

2. $f(x)=\left|3x-2\right|$

3. $f(x)=3x+\pi$

4. $f(x)=\sqrt{7}x+\frac{1}{2}$

5. $f(x)=-2x^2+3x-5$

6. $f(x)=x^3-2x^2+x-3$

7. $f(x)=x^2-2x-15$

8. $f(x)=4-\frac{2}{x}$

9. $f(x)=\frac{1}{x^4}$

10. $g(x)=\frac{3}{x-4}$

11. $y=\frac{2}{x-3}$

12. $y=\frac{-7}{x-5}$

13. $f(x)=\frac{x+5}{2-x}$

14. $f(x)=\frac{8}{x+4}$

15. $f(x)=\frac{1}{x+4}$

16. $f(x)=\frac{1}{x-4}$

17. $f(x)=\frac{3x}{x+2}$

18. $f(x)=x-\frac{2}{x-3}$

19. $f(x)=x+\frac{3}{x-5}$

20. $f(x)=\frac{1}{2}x-\frac{8}{x+7}$

21. $f(x)=\frac{1}{x-3}-\frac{8}{x+7}$

22. $f(x)=\frac{1}{x+4}-\frac{2x}{x-4}$

23. $f(x)=\frac{3x^2}{x+3}-\frac{4x}{x-2}$

24. $f(x)=\frac{1}{x^2-2x+1}$

25. $f(x)=\frac{x}{x^2+3x+2}$

26. $f(x)=\frac{x^2}{x^2-5x+4}$

27. $f(x)=\frac{1}{x^2-4x-5}$

28. $g(x)=\frac{2}{x^2+x-12}$

29. $h(x)=\frac{5}{\frac{4}{x}-1}$

30. $y=\sqrt{x}$

31. $f(x)=\sqrt{8-3x}$

32. $y=\sqrt{4x+1}$

33. $y=\sqrt{7-2x}$

34. $f(x)=\sqrt{8-x}$

35. $f(x)=\sqrt{3-2x}$

36. $f(x)=\sqrt{3+2x}$

37. $f(x)=\sqrt{5-x}$

38. $f(x)=\sqrt{x-5}$

39. $f(x)=\sqrt{6-3x}$

40. $f(x)=\sqrt{3x-6}$

41. $f(x)=\sqrt{2x+7}$

42. $f(x)=\sqrt{x^2-16}$

43. $f(x)=\sqrt{16-x^2}$

44. $f(x)=\sqrt{9-x^2}$

45. $f(x)=\sqrt{x^2-25}$

46. $f(x)=\sqrt{x^2-5x+4}$

47. $f(x)=\sqrt{x^2+5x+4}$

48. $f(x)=\sqrt{x^2+3x+2}$

49. $f(x)=\sqrt{x^2-3x+2}$

50. $f(x)=\sqrt{x-4}+\sqrt{x+1}$

51. $f(x)=\sqrt{3-x}+\sqrt{x-2}$

52. $f(x)=\sqrt{1-x}+\sqrt{4-x}$

53. $f(x)=\sqrt{1-x}-\sqrt{x-3}$

54. $f(x)=\sqrt{x+4}-\sqrt{x-1}$

55. $f(x)=\frac{\sqrt{x+1}}{x}$

56. $g(x)=\frac{\sqrt{x-3}}{x-6}$

57. $f(x)=\frac{\sqrt{x+4}}{\sqrt{x-1}}$

58. $f(x)=\frac{\sqrt{5-x}}{x}$

59. $f(x)=\frac{x}{\sqrt{5-x}}$

60. $f(x)=\dfrac{1}{x\sqrt{5-x}}$

61. $f(x)=\dfrac{x+1}{x^3-4x}$

62. $f(x)=\dfrac{\sqrt{x+5}}{x}$

63. $f(x)=\dfrac{x}{\sqrt{x+5}}$

64. $f(x)=\dfrac{1}{x\sqrt{x+5}}$

65. $f(x)=\dfrac{x+3}{\sqrt{x-3}}$

66. $f(x)=\dfrac{\sqrt{x+3}}{\sqrt{x-3}}$

67. $f(x)=\dfrac{\sqrt{x-2}}{\sqrt{x+2}}$

68. $f(x)=\dfrac{\sqrt{2-x}}{\sqrt{x+2}}$

69. $f(x)=\dfrac{x-4}{\sqrt{x-2}}$

70. $f(x)=\dfrac{1}{(x-3)\sqrt{x+3}}$

71. $f(x)=\sqrt{x+2}+\sqrt{2-x}$

72. $f(x)=\sqrt{(x-2)(x-6)}$

73. $f(x)=\sqrt{x+3}-\sqrt{4-x}$

74. $f(x)=\dfrac{\sqrt{4x-3}}{x^2-4}$

75. $f(x)=\dfrac{4x}{6x^2+13x-5}$

76. $f(x)=\dfrac{\sqrt{2x-3}}{x^2-5x+4}$

77. $f(x)=\dfrac{x^2}{\sqrt{x^2-5x+4}}$

78. $f(x)=\dfrac{x+2}{\sqrt{x^2+5x+4}}$

79. $f(x)=\dfrac{\sqrt{x+2}}{\sqrt{x^2+3x+2}}$

80. $f(x)=\dfrac{\sqrt{2x+3}}{x^2-6x+5}$

(**81 – 97**) Find and simplify the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ for the given function

81. $f(x)=9x+5$

82. $f(x)=6x+2$

83. $f(x)=4x+11$

84. $f(x)=3x-5$

85. $f(x)=-2x-3$

86. $f(x)=-4x+3$

87. $f(x)=3x-6$

88. $f(x)=-5x-7$

89. $f(x)=2x^2$

90. $f(x)=5x^2$

91. $f(x)=3x^2-4x$

92. $f(x)=2x^2-3x$

93. $f(x)=2x^2-x-3$

94. $f(x)=x^2-2x+5$

95. $f(x)=3x^2-2x+5$

96. $f(x)=-2x^2-3x+7$

97. $f(x)=\sqrt{x-3}$

98. Let $f(x)=4x-3$ and $g(x)=5x+7$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\dfrac{f}{g}\right)(x)$

99. Let $f(x)=2x^2+3$ and $g(x)=3x-4$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\dfrac{f}{g}\right)(x)$

100. Let $f(x)=x^2-2x-3$ and $g(x)=x^2+3x-2$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\dfrac{f}{g}\right)(x)$

101. Let $f(x)=\sqrt{4x-1}$ and $g(x)=\dfrac{1}{x}$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\dfrac{f}{g}\right)(x)$

102. Find $(f+g)(x),\ (f-g)(x),\ (f\cdot g)(x),\ \textit{and}\ (f/g)(x)$ and the domain of

$f(x)=\sqrt{3-2x},\quad g(x)=\sqrt{x+4}$

103. Find $(f+g)(x),\ (f-g)(x),\ (f\cdot g)(x),\ \textit{and}\ (f/g)(x)$ and the domain of

$f(x)=\dfrac{2x}{x-4},\quad g(x)=\dfrac{x}{x+5}$

104. Let $f(x)=\sqrt{4x-1}$ and $g(x)=\dfrac{1}{x}$. Find each of the following and give the domain

e) $(f+g)(x)$ *f)* $(f-g)(x)$ *g)* $(fg)(x)$ *h)* $\left(\dfrac{f}{g}\right)(x)$

105. Given that $f(x)=x+1$ and $g(x)=\sqrt{x+3}$

a) Find $(f+g)(x)$

b) Find the domain of $(f+g)(x)$

c) Find: $(f+g)(6)$

106. Given that $f(x)=x^2-4$ and $g(x)=x+2$

a) Find $(f+g)(x)$ and its domain

b) Find $(f/g)(x)$ and its domain

107. Find $(f\circ g)(x),\ (g\circ f)(x),\ f(g(-2))\ \textit{and}\ g(f(3))$

$f(x)=2x^2+3x-4,\quad g(x)=2x-1$

108. Find $(f\circ g)(x),\ (g\circ f)(x),\ f(g(-2))\ \textit{and}\ g(f(3))$

$f(x)=x^3+2x^2,\quad g(x)=3x$

109. Find $(f\circ g)(x),\ (g\circ f)(x),\ f(g(-2))\ \textit{and}\ g(f(3))$

$f(x)=|x|,\quad g(x)=-7$

(**110 – 139**) For the given function; find:

a) Find $(f \circ g)(x)$ and the ***domain*** of $f \circ g$

b) Find $(g \circ f)(x)$ and the ***domain*** of $g \circ f$

110. $f(x) = x - 3$ *and* $g(x) = x + 3$

111. $f(x) = \frac{2}{3}x$ *and* $g(x) = \frac{3}{2}x$

112. $f(x) = x - 1$ *and* $g(x) = 3x^2 - 2x - 1$

113. $f(x) = 3x - 2$ *and* $g(x) = x^2 - 5$

114. $f(x) = x^2 - 2$ *and* $g(x) = 4x - 3$

115. $f(x) = 4x^2 - x + 10$ *and* $g(x) = 2x - 7$

116. $f(x) = \sqrt{x}$ *and* $g(x) = x + 3$

117. $f(x) = \sqrt{x}$ *and* $g(x) = 2 - 3x$

118. $f(x) = 3x + 2$ *and* $g(x) = \sqrt{x}$

119. $f(x) = x^4$ *and* $g(x) = \sqrt[4]{x}$

120. $f(x) = x^n$ *and* $g(x) = \sqrt[n]{x}$

121. $f(x) = x^2 - 3x$ *and* $g(x) = \sqrt{x+2}$

122. $f(x) = \sqrt{x-2}$ *and* $g(x) = \sqrt{x+5}$

123. $f(x) = x^2 + 2$ *and* $g(x) = \sqrt{3-x}$

124. $f(x) = x^5 - 2$ *and* $g(x) = \sqrt[5]{x+2}$

125. $f(x) = 1 - x^2$ *and* $g(x) = \sqrt{x^2 - 25}$

126. $f(x) = 2x + 3$ *and* $g(x) = \frac{x-3}{2}$

127. $f(x) = 4x - 5$ *and* $g(x) = \frac{x+5}{4}$

128. $f(x) = \frac{4}{1-5x}$ *and* $g(x) = \frac{1}{x}$

129. $f(x) = \frac{1}{x-2}$ *and* $g(x) = \frac{x+2}{x}$

130. $f(x) = \frac{1}{1+x}$ *and* $g(x) = \frac{1-x}{x}$

131. $f(x) = \frac{3x+5}{2}$ *and* $g(x) = \frac{2x-5}{3}$

132. $f(x) = \frac{x-1}{x-2}$ *and* $g(x) = \frac{x-3}{x-4}$

133. $f(x) = \frac{6}{x-3}$ *and* $g(x) = \frac{1}{x}$

134. $f(x) = \frac{6}{x}$ *and* $g(x) = \frac{1}{2x+1}$

135. $f(x) = 3x - 7$ *and* $g(x) = \frac{x+7}{3}$

136. $f(x) = \frac{2x+3}{x-4}$ *and* $g(x) = \frac{4x+3}{x-2}$

137. $f(x) = \frac{2x+3}{x+4}$ *and* $g(x) = \frac{-4x+3}{x-2}$

138. $f(x) = x + 1$ *and* $g(x) = x^3 - 5x^2 + 3x + 7$

139. $f(x) = x - 1$ *and* $g(x) = x^3 + 2x^2 - 3x - 9$

140. Given that $f(x) = 2x - 5$ and $g(x) = x^2 - 3x + 8$, find $(f \circ g)(x)$, $(g \circ f)(x)$ and their domain then find $(f \circ g)(7)$

141. Given that $f(x) = \sqrt{x}$ and $g(x) = x - 1$, find

a) $(f \circ g)(x) = f(g(x))$

b) $(g \circ f)(x) = g(f(x))$

c) $(f \circ g)(2) = f(g(2))$

142. Given that $f(x)=\frac{x}{x+5}$ and $g(x)=\frac{6}{x}$, find

a) $(f\circ g)(x)=f(g(x))$

b) $(g\circ f)(x)=g(f(x))$

c) $(f\circ g)(2)=f(g(2))$

(**143 – 167**) Determine whether f is even, odd, or neither

143. $f(x)=3x^4+2x^2-5$

144. $f(x)=8x^3-3x^2$

145. $f(x)=\sqrt{x^2+4}$

146. $f(x)=3x^2-5x+1$

147. $f(x)=\sqrt[3]{x^3-x}$

148. $f(x)=|x|-3$

149. $f(x)=x^3-\frac{1}{x}$

150. $f(x)=-x^3+2x$

151. $f(x)=x^5-2x^3$

152. $f(x)=.5x^4-2x^2+6$

153. $f(x)=.75x^2+|x|+4$

154. $f(x)=x^3-x+9$

155. $f(x)=x^4-5x+8$

156. $f(x)=x^3+x$

157. $g(x)=x^2-x$

158. $h(x)=2x^2+x^4$

159. $f(x)=2x^2+x^4+1$

160. $f(x)=\frac{1}{5}x^6-3x^2$

161. $f(x)=x\sqrt{1-x^2}$

162. $f(x)=x^2\sqrt{1-x^2}$

163. $f(x)=5x^7-6x^3-2x$

164. $f(x)=5x^6-3x^2-7$

165. $f(x)=x^2+6$

166. $f(x)=7x^3-x$

167. $h(x)=x^5+1$

168. $f(x)=\begin{cases}2+x & if\ \ x<-4\\ -x & if\ \ -4\le x\le 2\\ 3x & if\ \ x>2\end{cases}$ Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

169. $f(x)=\begin{cases}-2x & if\ \ x<-3\\ 3x-1 & if\ \ -3\le x\le 2\\ -4x & if\ \ x>2\end{cases}$ Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

170. $f(x)=\begin{cases} x^3+3 & if \quad -2\le x\le 0 \\ x+3 & if \quad 0<x<1 \\ 4+x-x^2 & if \quad 1\le x\le 3 \end{cases}$ Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

171. $h(x)=\begin{cases} \dfrac{x^2-9}{x-3} & if \quad x\ne 3 \\ 6 & if \quad x=3 \end{cases}$ Find: $h(5),\ h(0),\ and\ h(3)$

172. Graph the piecewise function defined by $f(x)=\begin{cases} 3 & if \quad x\le -1 \\ x-2 & if \quad x>-1 \end{cases}$

173. Sketch the graph $f(x)=\begin{cases} x+2 & if \quad x\le -1 \\ x^3 & if \quad -1<x<1 \\ -x+3 & if \quad x\ge 1 \end{cases}$

174. Sketch the graph $f(x)=\begin{cases} x-3 & if \quad x\le -2 \\ -x^2 & if \quad -2<x<1 \\ -x+4 & if \quad x\ge 1 \end{cases}$