Section 3.3 – Gram-Schmidt Process

Definition

A set of two or more vectors in a real inner product space is said to be ***orthogonal*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be ***orthonormal***.

Theorem

1. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n$$

2. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n$$

Proof

1. Since $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for V, every vector $\mathbf{u}$ in V can be expressed in the form

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

Let show that $c_i = \dfrac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}$ *for* $i = 1, 2, \ldots n$

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v}_i \rangle &= \langle c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n, \mathbf{v}_i \rangle \\ &= c_1\langle \mathbf{v}_1, \mathbf{v}_i \rangle + c_2\langle \mathbf{v}_2, \mathbf{v}_i \rangle + \cdots + c_n\langle \mathbf{v}_n, \mathbf{v}_i \rangle\end{aligned}$$

Since S is an orthogonal set, all of the inner products in the last equality are zero except the i^{th}, so we have

$$\langle \mathbf{u}, \mathbf{v}_i \rangle = c_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle = c_i \|\mathbf{v}_i\|^2$$

The Gram-Schmidt Process

To convert a basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_r\}$ into an orthogonal basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}$, perform the following computations:

***Step* 1**: $\boldsymbol{v}_1 = \boldsymbol{u}_1$

***Step* 2**: $\boldsymbol{v}_2 = \boldsymbol{u}_2 - \dfrac{\langle \boldsymbol{u}_2, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1$

***Step* 3**: $\boldsymbol{v}_3 = \boldsymbol{u}_3 - \dfrac{\langle \boldsymbol{u}_3, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 - \dfrac{\langle \boldsymbol{u}_3, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2$

***Step* 4**: $\boldsymbol{v}_4 = \boldsymbol{u}_4 - \dfrac{\langle \boldsymbol{u}_4, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 - \dfrac{\langle \boldsymbol{u}_4, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2 - \dfrac{\langle \boldsymbol{u}_4, \boldsymbol{v}_3 \rangle}{\|\boldsymbol{v}_3\|^2}\boldsymbol{v}_3$

To convert the orthogonal basis into an orthonormal basis $\{\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3\}$, normalize the orthogonal basis vectors. $\boxed{\boldsymbol{q}_i = \dfrac{\boldsymbol{v}_i}{\|\boldsymbol{v}_i\|}}$

Example

Assume that the vector space R^3 has the Euclidean inner product. Apply the Gram-Schmidt process to transform the basis vectors

$$u_1 = (1,\ 1,\ 1) \quad u_2 = (0,\ 1,\ 1) \quad u_3 = (0,\ 0,\ 1)$$

Into the orthogonal basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, and then normalize the orthogonal basis vectors to obtain an orthonormal basis $\{\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3\}$

Solution

$$\boldsymbol{v}_1 = \boldsymbol{u}_1 = (1,\ 1,\ 1)$$

$$\begin{aligned}\boldsymbol{v}_2 &= \boldsymbol{u}_2 - \frac{\langle \boldsymbol{u}_2, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 \\ &= (0,1,1) - \frac{0+1+1}{1^2+1^2+1^2}(1,1,1) \\ &= (0,1,1) - \frac{2}{3}(1,1,1)\end{aligned}$$

$$= \left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$\boldsymbol{v}_3 = \boldsymbol{u}_3 - proj_{\boldsymbol{v}_2}\boldsymbol{u}_3$$

$$= \boldsymbol{u}_3 - \frac{\left\langle \boldsymbol{u}_3, \boldsymbol{v}_1 \right\rangle}{\left\|\boldsymbol{v}_1\right\|^2}\boldsymbol{v}_1 - \frac{\left\langle \boldsymbol{u}_3, \boldsymbol{v}_2 \right\rangle}{\left\|\boldsymbol{v}_2\right\|^2}\boldsymbol{v}_2$$

$$= (0,0,1) - \frac{0+0+1}{1^2+1^2+1^2}(1,1,1) - \frac{0+0+\frac{1}{3}}{\left(\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{\frac{1}{3}}{\frac{2}{3}}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1}{2}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$= \left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)$$

$$\boldsymbol{q}_1 = \frac{\boldsymbol{v}_1}{\left\|\boldsymbol{v}_1\right\|} = \frac{(1,\ 1,\ 1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\boldsymbol{q}_2 = \frac{\boldsymbol{v}_2}{\left\|\boldsymbol{v}_2\right\|} = \frac{\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}{\sqrt{\left(\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2}}$$

$$= \frac{\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}{\frac{\sqrt{6}}{3}}$$

$$= \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$\boldsymbol{q}_3 = \frac{\boldsymbol{v}_3}{\left\|\boldsymbol{v}_3\right\|} = \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\sqrt{0^2+\left(-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}}$$

$$= \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\frac{\sqrt{2}}{2}}$$

$$= \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Gram-Schmidt Process (*Orthonormal*)

Suppose $v_1, \dots, v_n$ linearly independent in $\mathbb{R}^n$, construct n ***orthonormal*** $u_1, \dots, u_n$ that span the same space: span $\{u_1, \dots, u_k\}$ = span $\{v_1, \dots, v_k\}$

***Step* 1**: Since v_i are linearly independent ($\neq 0$), so $\|v_1\| \neq 0$ (to create a normal vector)

Let $\boxed{u_1 = \frac{v_1}{\|v_1\|} = q_1}$, then $\|u_1\| = 1$ since u_1 is orthonormal and span $\{u_1\}$ = span $\{v_1\}$

$$w_1 = v_1 \Rightarrow v_1 = \|w_1\| u_1$$

***Step* 2**: $w_2 = v_2 - (v_2.q_1)q_1$

$$\Rightarrow w_2 = v_2 - \frac{v_2.u_1}{\|v_1\|} v_1 \qquad (w_2 \perp u_1)$$

$$v_2 = \|w_2\| u_2 + (v_2.u_1)u_1 \qquad w_2 = \|w_2\| u_2$$

$$\boxed{q_2 = \frac{w_2}{\|w_2\|}}$$

***Step* 3**: $\boxed{w_3 = v_3 - (v_3.q_1)q_1 - (v_3.q_2)q_2}$

$$\boxed{q_3 = \frac{w_3}{\|w_3\|}}$$

	$u_1 = \frac{v_1}{\|v_1\|}$
$w_2 = v_2 - (v_2.u_1)u_1$	$u_2 = \frac{w_2}{\|w_2\|}$
$w_3 = v_3 - (v_3.u_1)u_1 - (v_3.u_2)u_2$	$u_3 = \frac{w_3}{\|w_3\|}$
$w_n = v_n - (v_n.u_1)u_1 - (v_n.u_2)u_2 - \dots - (v_n.u_{n-1})u_{n-1}$	$u_n = \frac{w_n}{\|w_n\|}$

Example

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of

$v_1 = (1, 1, 0, 0) \quad v_2 = (0, 1, 1, 0) \quad v_3 = (1, 0, 1, 1)$

Solution

Step 1: $u_1 = \dfrac{v_1}{\|v_1\|} = \dfrac{(1, 1, 0, 0)}{\sqrt{1^2+1^2+0+0}}$

$$= \frac{(1, 1, 0, 0)}{\sqrt{2}}$$

$$= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right)$$

Step 2: $w_2 = v_2 - (v_2.u_1)u_1$

$$= (0, 1, 1, 0) - \left[(0, 1, 1, 0).\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right)\right]\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right)$$

$$= (0, 1, 1, 0) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right)$$

$$= (0, 1, 1, 0) - \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$$

$$= \left(-\frac{1}{2}, \frac{1}{2}, 1, 0\right)$$

$$\|w_2\| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 1} = \sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{2}$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{\left(-\frac{1}{2}, \frac{1}{2}, 1, 0\right)}{\frac{\sqrt{6}}{2}}$$

$$= \frac{2}{\sqrt{6}}\left(-\frac{1}{2}, \frac{1}{2}, 1, 0\right)$$

$$= \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0\right)$$

Step 3: $v_3.u_1 = (1, 0, 1, 1).\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right) = \frac{1}{\sqrt{2}}$

$$v_3.u_2 = (1, 0, 1, 1).\left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0\right) = -\frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}} = \frac{1}{\sqrt{6}}$$

$$w_3 = v_3 - (v_3.u_1)u_1 - (v_3.u_2)u_2$$

$$= (1, 0, 1, 1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right) - \frac{1}{\sqrt{6}}\left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0\right)$$

$$= (1,\ 0,\ 1,\ 1) - \left(\frac{1}{2},\ \frac{1}{2},\ 0,\ 0\right) - \left(-\frac{1}{6},\ \frac{1}{6},\ \frac{2}{6},\ 0\right)$$

$$= \left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$u_3 = \frac{w_3}{\|w_3\|} = \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + 1^2}} \left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$= \frac{1}{\sqrt{\frac{21}{9}}} \left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$= \frac{3}{\sqrt{21}} \left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$= \left(\frac{2}{\sqrt{21}},\ -\frac{2}{\sqrt{21}},\ \frac{2}{\sqrt{21}},\ \frac{3}{\sqrt{21}}\right)$$

QR–Decomposition

Problem

If A is an $m \times n$ matrix with linearly independent column vectors, and if Q is the matrix that results by applying the Gram-Schmidt process to the column vectors of A, what relationship, if any, exists between A and Q?

To solve this problem, suppose that the column vectors of A are $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ and the orthonormal column vectors of Q are $\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_n$.

$$\begin{aligned}
\boldsymbol{u}_1 &= \langle \boldsymbol{u}_1, \boldsymbol{q}_1 \rangle \boldsymbol{q}_1 + \langle \boldsymbol{u}_1, \boldsymbol{q}_2 \rangle \boldsymbol{q}_2 + \cdots + \langle \boldsymbol{u}_1, \boldsymbol{q}_n \rangle \boldsymbol{q}_n \\
\boldsymbol{u}_2 &= \langle \boldsymbol{u}_2, \boldsymbol{q}_1 \rangle \boldsymbol{q}_1 + \langle \boldsymbol{u}_2, \boldsymbol{q}_2 \rangle \boldsymbol{q}_2 + \cdots + \langle \boldsymbol{u}_2, \boldsymbol{q}_n \rangle \boldsymbol{q}_n \\
\vdots \quad & \quad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
\boldsymbol{u}_n &= \langle \boldsymbol{u}_n, \boldsymbol{q}_1 \rangle \boldsymbol{q}_1 + \langle \boldsymbol{u}_n, \boldsymbol{q}_2 \rangle \boldsymbol{q}_2 + \cdots + \langle \boldsymbol{u}_n, \boldsymbol{q}_n \rangle \boldsymbol{q}_n
\end{aligned}$$

$$R = \begin{bmatrix} \langle \boldsymbol{u}_1, \boldsymbol{q}_1 \rangle & \langle \boldsymbol{u}_2, \boldsymbol{q}_1 \rangle & \cdots & \langle \boldsymbol{u}_n, \boldsymbol{q}_1 \rangle \\ 0 & \langle \boldsymbol{u}_2, \boldsymbol{q}_2 \rangle & \cdots & \langle \boldsymbol{u}_n, \boldsymbol{q}_2 \rangle \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \langle \boldsymbol{u}_n, \boldsymbol{q}_n \rangle \end{bmatrix}$$

The equation $A = QR$ is a factorization of A into the product of a matrix Q with orthonormal column vectors and an invertible upper triangular matrix R. We call it the ***QR-decomposition of A***.

Theorem

If A is an $m \times n$ matrix with linearly independent column vectors, then A can be factored as

$$A = QR$$

Where Q is an $m \times n$ matrix with orthonormal column vectors, and R is an $n \times n$ invertible upper triangular matrix.

Example

Find the QR-decomposition of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Solution

The column vectors of are

$$\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

From the previous example

$$\boldsymbol{q}_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \quad \boldsymbol{q}_2 = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right) \quad \boldsymbol{q}_3 = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$R = \begin{bmatrix} \langle \boldsymbol{u}_1, \boldsymbol{q}_1 \rangle & \langle \boldsymbol{u}_2, \boldsymbol{q}_1 \rangle & \langle \boldsymbol{u}_3, \boldsymbol{q}_1 \rangle \\ 0 & \langle \boldsymbol{u}_2, \boldsymbol{q}_2 \rangle & \langle \boldsymbol{u}_3, \boldsymbol{q}_2 \rangle \\ 0 & 0 & \langle \boldsymbol{u}_3, \boldsymbol{q}_3 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} (1)\frac{1}{\sqrt{3}} + (1)\frac{1}{\sqrt{3}} + (1)\frac{1}{\sqrt{3}} & 0 + (1)\frac{1}{\sqrt{3}} + (1)\frac{1}{\sqrt{3}} & 0 + 0 + (1)\frac{1}{\sqrt{3}} \\ 0 & 0\left(\frac{-2}{\sqrt{6}}\right) + (1)\frac{1}{\sqrt{6}} + (1)\frac{1}{\sqrt{6}} & 0\left(\frac{-2}{\sqrt{6}}\right) + 0\frac{1}{\sqrt{6}} + (1)\frac{1}{\sqrt{6}} \\ 0 & 0 & 0 + (0)\frac{-1}{\sqrt{2}} + (1)\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\underset{A}{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}} = \underset{Q}{\begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}} \cdot \underset{R}{\begin{pmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}}$$

Exercises *Section* 3.3 – Gram-Schmidt Process

1. Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\boldsymbol{R}^m$.

a) $\boldsymbol{u}_1 = (1,\ -3),\quad \boldsymbol{u}_2 = (2,\ 2)$

b) $\boldsymbol{u}_1 = (1,\ 0),\quad \boldsymbol{u}_2 = (3,\ -5)$

c) $\{(1,\ 1,\ 1),\ (-1,\ 1,\ 0),\ (1,\ 2,\ 1)\}$

d) $\{(1,\ 1,\ 1),\ (0,\ 1,\ 1),\ (0,\ 0,\ 1)\}$

e) $\{(1,\ 1,\ 1,\ 1),\ (1,\ 2,\ 1,\ 0),\ (1,\ 3,\ 0,\ 0)\}$

f) $\{(0,\ 2,\ -1,\ 1),\ (0,\ 0,\ 1,\ 1),\ (-2,\ 1,\ 1,\ -1)\}$

g) $\boldsymbol{u}_1 = (1,\ 0,\ 0),\quad \boldsymbol{u}_2 = (3,\ 7,\ -2),\quad \boldsymbol{u}_3 = (0,\ 4,\ 1)$

h) $\boldsymbol{u}_1 = (0,2,1,0),\quad \boldsymbol{u}_2 = (1,-1,0,0),\quad \boldsymbol{u}_3 = (1,2,0,-1),\quad \boldsymbol{u}_4 = (1,0,0,1)$

2. Find the ***QR***-decomposition of

a) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

3. Verify that the Cauchy-Schwarz inequality holds for the given vectors using the Euclidean inner product

$\boldsymbol{u} = (0,\ -2,\ 2,\ 1),\quad \boldsymbol{v} = (-1,\ -1,\ 1,\ 1)$