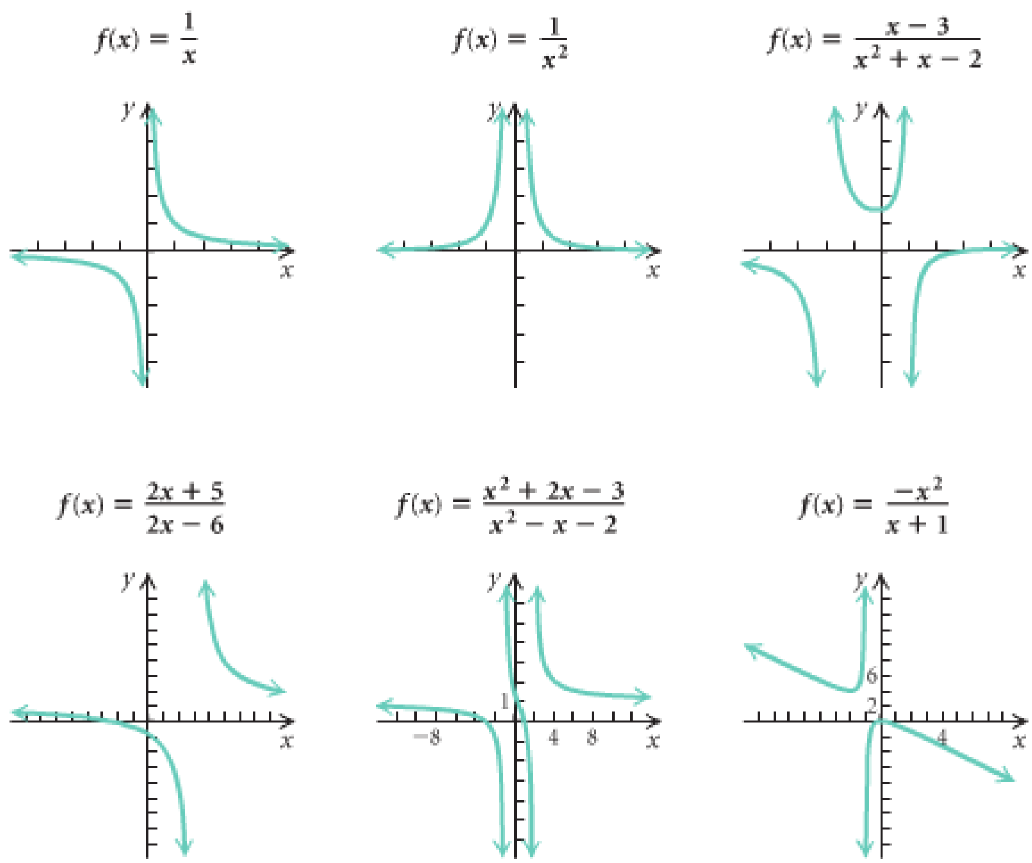
Section 3.4 – Rational Functions



Rational Function

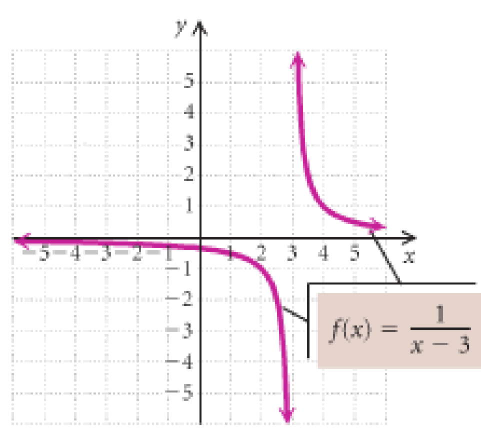
A rational function is a function f that is a quotient of two polynomials, that is,

$$f(x) = \frac{g(x)}{h(x)}$$

Where $g(x)$ and $h(x)$ are polynomials. The domain of f consists of all real numbers ***except*** the zeros of the denominator $h(x)$.

The Domain of a Rational Function

Example



Consider: $f(x) = \dfrac{1}{x-3}$

Find the domain and graph *f*.

Solution

$x - 3 = 0 \quad \Rightarrow \quad \boxed{x = 3}$

Thus the domain is: $\{x \mid x \neq 3\}$ *or* $(-\infty,\ 3) \cup (3,\ \infty)$

Function	Domain	
$f(x)=\frac{1}{x}$	$\{x \mid x\neq 0\}$	$(-\infty,\ 0)\cup(0,\ \infty)$
$f(x)=\frac{1}{x^2}$	$\{x \mid x\neq 0\}$	$(-\infty,\ 0)\cup(0,\ \infty)$
$f(x)=\frac{x-3}{x^2+x-2}$	$\{x \mid x\neq -2 \text{ and } x\neq 1\}$	$(-\infty,\ -2)\cup(-2,\ 1)\cup(1,\ \infty)$
$f(x)=\frac{2x+5}{2x-6}=\frac{2x+5}{2(x-3)}$	$\{x \mid x\neq 3\}$	$(-\infty,\ 3)\cup(3,\ \infty)$

Asymptotes

Vertical Asymptote (*VA*) - ***Think Domain***

The line $x=a$ is a ***vertical asymptote*** for the graph of a function f if

$$f(x)\to\infty \quad or \quad f(x)\to-\infty$$

As ***x*** approaches ***a*** from either the left or the right

Example

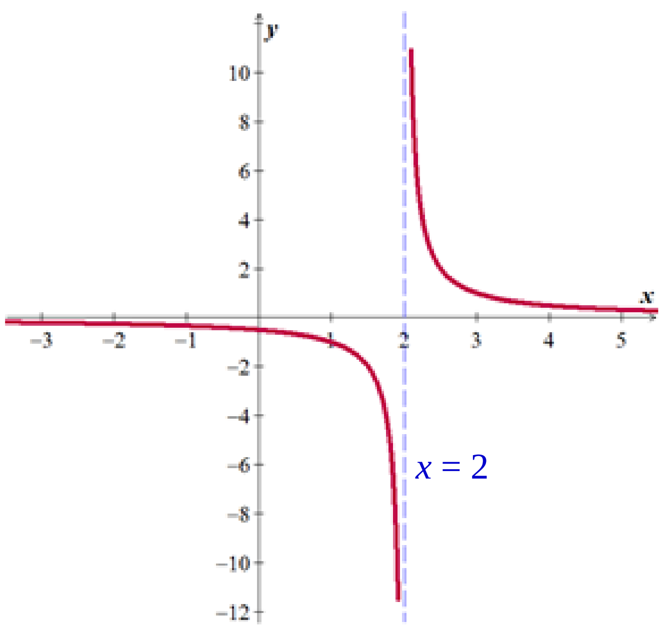
Find the vertical asymptote of $f(x)=\frac{1}{x-2}$, and sketch the graph.

Solution

VA: $x=2$

$f(x)\to\infty \quad as \quad x\to 2^+$

$f(x)\to-\infty \quad as \quad x\to 2^-$



Horizontal Asymptote (*HA*)

The line $y = c$ is a ***horizontal asymptote*** for the graph of a function f if

$$f(x) \to c \quad as \quad x \to -\infty \quad or \quad x \to -\infty$$

Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0} = \dfrac{a_n x^n}{b_m x^m}$ be a rational function.

1. If the degree of numerator is less than of denominator $(n < m) \Rightarrow$ y = 0

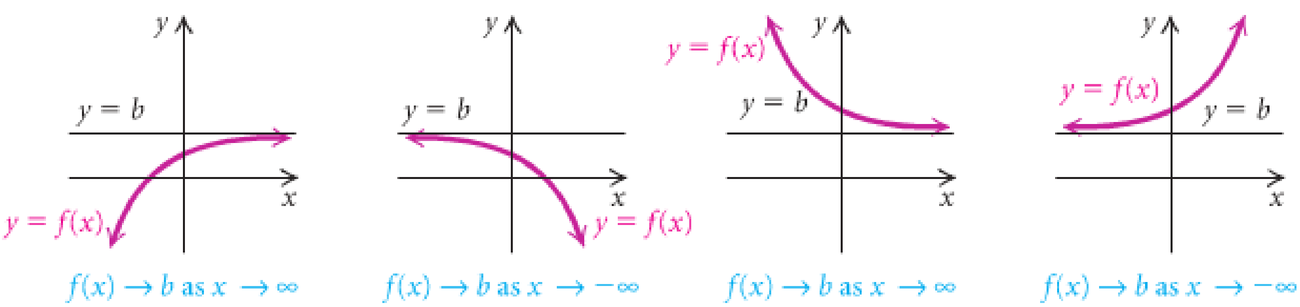
$$y = \frac{2x+1}{4x^2+5} \quad \Rightarrow \boxed{y = 0}$$

2. If the degree of numerator is equal of denominator $(n = m) \Rightarrow y = \dfrac{a_n}{b_m}$

$$y = \frac{2x^2+1}{4x^2+5} \quad \Rightarrow \boxed{y = \frac{2}{4} = \frac{1}{2}}$$

3. If the degree of numerator is greater than of denominator $(n > m) \Rightarrow$ No horizontal asymptote

$$y = \frac{2x^3+1}{4x^2+5} \quad \Rightarrow No \;\; HA$$



Example

Determine the horizontal asymptote of $f(x) = \dfrac{-7x^4 - 10x^2 + 1}{11x^4 + x - 2}$.

Solution

$$f(x) = \frac{-7x^4 - 10x^2 + 1}{11x^4 + x - 2} \to \frac{-7x^4}{11x^4} = -\frac{7}{11}$$

Therefore, the horizontal asymptote (***HA***) is: $\boxed{y = -\frac{7}{11}}$

Example

Find the vertical and the horizontal asymptote for the graph of f , if it exists

a) $f(x)=\dfrac{3x-1}{x^2-x-6}$

b) $f(x)=\dfrac{5x^2+1}{3x^2-4}$

c) $f(x)=\dfrac{2x^4-3x^2+5}{x^2+1}$

Solution

a) $f(x)=\dfrac{3x-1}{x^2-x-6}$

$$x^2-x-6=0 \;\rightarrow\; x=-2,\ 3$$

VA: $x=-2,\quad x=3$

HA: $y=0$

b) $f(x)=\dfrac{5x^2+1}{3x^2-4}$

$$3x^2-4=0 \;\rightarrow\; 3x^2=4 \;\rightarrow\; x^2=\frac{4}{3} \;\rightarrow\; \boxed{x=\pm\frac{2}{\sqrt{3}}}$$

VA: $x=-\dfrac{2}{\sqrt{3}},\quad x=\dfrac{2}{\sqrt{3}}$

HA: $y=\dfrac{5}{3}$

c) $f(x)=\dfrac{2x^4-3x^2+5}{x^2+1}$

$$x^2+1=0 \;\rightarrow\; x^2=-1$$

VA: *n/a*

HA: *n/a*

Slant or *Oblique* Asymptotes

When the degree of the numerator is one greater than the degree of the numerator, the graph has a slant or oblique asymptote and it is a line $y = ax + b, \quad a \neq 0$. To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

$$y = \frac{3x^2 - 1}{x + 2}$$

$$\begin{array}{r|l} & \quad\quad 3x - 6 \\ \hline x + 2 & 3x^2 + 0x - 1 \\ & \underline{3x^2 + 6x} \\ & \quad\quad -6x - 1 \\ & \quad\quad \underline{-6x - 12} \\ & \quad\quad\quad R = 11 \end{array}$$

$$y = \frac{3x^2 - 1}{x + 2} = 3x - 6 + \frac{11}{x + 2}$$

The ***oblique asymptote*** is the line ***y* = 3*x* – 6**

Example

Find all the asymptotes of $\quad f(x) = \dfrac{2x^2 - 3x - 1}{x - 2}$

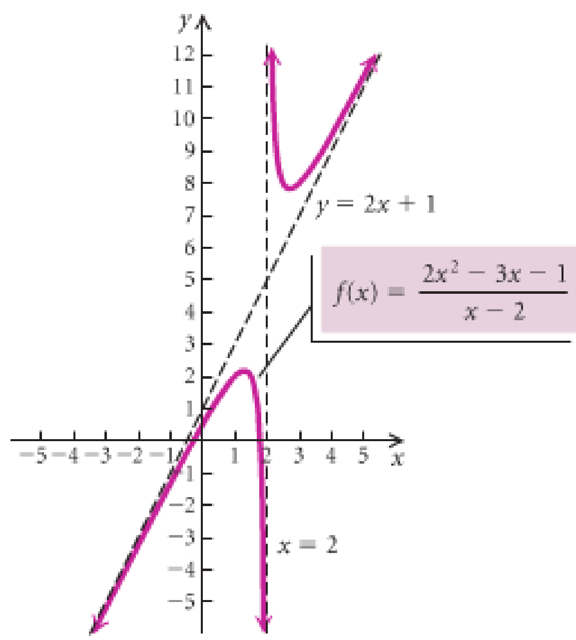
Solution

$$\begin{array}{r|l} & \quad\quad 2x + 1 \\ \hline x - 2 & 2x^2 - 3x - 1 \\ & \underline{-2x^2 + 4x} \\ & \quad\quad x - 1 \\ & \quad\quad \underline{-x + 2} \\ & \quad\quad\quad 1 \end{array}$$

$$f(x) = \frac{2x^2 - 3x - 1}{x - 2} = (2x + 1) + \frac{1}{x - 2}$$

The ***oblique asymptote*** is the line $y = 2x + 1$

VA:: $x = 2$



Graph That Has a ***Hole***

Example

Sketch the graph of g if $g(x)=\dfrac{(3x+4)(x-1)}{(2x-5)(x-1)}$

Solution

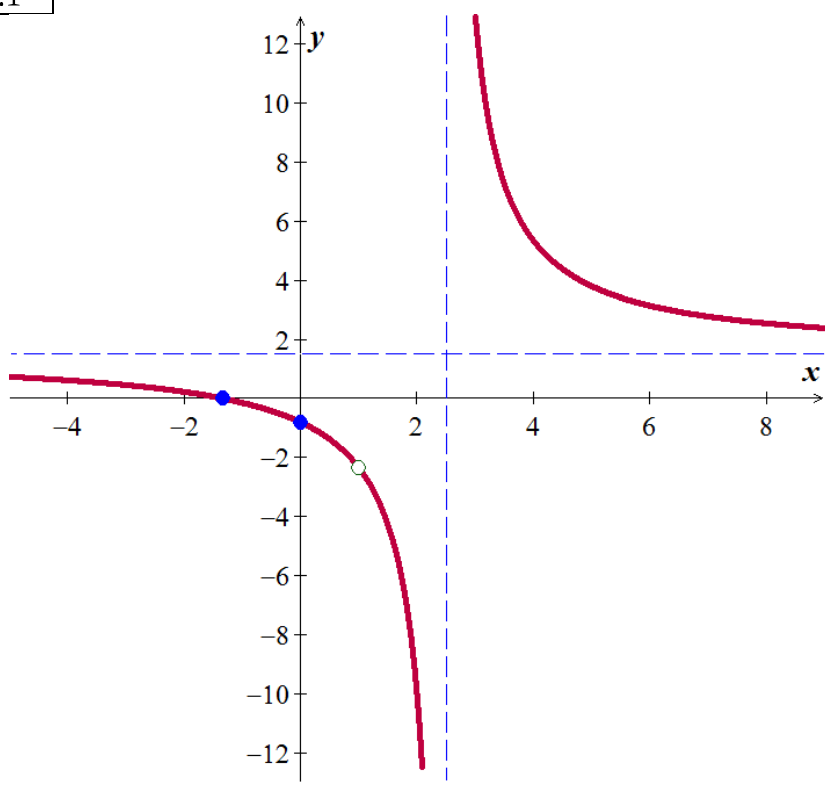
$$g(x)=\frac{(3x+4)(x-1)}{(2x-5)(x-1)}=\frac{3x+4}{2x-5}=f(x)$$

VA: $x=\frac{5}{2}$

HA: $y=\frac{3}{2}$

The only different between the graphs that g has a ***hole*** at $x=1\to f(1)=-\frac{7}{3}$

x	*y*
–4	.6
1.3	0
0	–.8
4	5.3
6	3.1



Exercises Section 3.4 – Rational Functions

Find the vertical and horizontal asymptotes (if any) of

1. $y=\dfrac{3x}{1-x}$

2. $y=\dfrac{x^2}{x^2+9}$

3. $y=\dfrac{x-2}{x^2-4x+3}$

4. $y=\dfrac{3}{x-5}$

5. $y=\dfrac{x^3-1}{x^2+1}$

6. $y=\dfrac{3x^2-27}{(x+3)(2x+1)}$

7. $y=\dfrac{x^3+3x^2-2}{x^2-4}$

8. $y=\dfrac{x-3}{x^2-9}$

9. $y=\dfrac{6}{\sqrt{x^2-4x}}$

10. $y=\dfrac{5x-1}{1-3x}$

11. $f(x)=\dfrac{2x-11}{x^2+2x-8}$

12. $f(x)=\dfrac{x^2-4x}{x^3-x}$

13. $f(x)=\dfrac{x-2}{x^3-5x}$

Determine all asymptotes of the function

14. $f(x)=\dfrac{4x}{x^2+10x}$

15. $f(x)=\dfrac{3-x}{(x-4)(x+6)}$

16. $f(x)=\dfrac{x^3}{2x^3-x^2-3x}$

17. $f(x)=\dfrac{3x^2+5}{4x^2-3}$

18. $f(x)=\dfrac{x+6}{x^3+2x^2}$

19. $f(x)=\dfrac{x^2+4x-1}{x+3}$

20. $f(x)=\dfrac{x^2-6x}{x-5}$

21. $f(x)=\dfrac{x^3-x^2+x-4}{x^2+2x-1}$

22. $f(x)=\dfrac{-3x}{x+2}$

23. $f(x)=\dfrac{x+1}{x^2+2x-3}$

24. $f(x)=\dfrac{2x^2-2x-4}{x^2+x-12}$

25. $f(x)=\dfrac{-2x^2+10x-12}{x^2+x}$

26. $f(x)=\dfrac{x^2-x-6}{x+1}$

27. $f(x)=\dfrac{x^3+1}{x-2}$

28. $f(x)=\dfrac{2x^2+x-6}{x^2+3x+2}$

29. $f(x)=\dfrac{x-1}{1-x^2}$

30. $f(x)=\dfrac{x^2+x-2}{x+2}$

31. $f(x)=\dfrac{x^3-2x^2-4x+8}{x-2}$

32. Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} \text{vertical asymptote: } x=4 \\ \text{horizontal asymptote: } y=-1 \\ x-\text{intercept: } 3 \end{cases}$$

33. Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} \text{vertical asymptote: } x=-3, x=1 \\ \text{horizontal asymptote: } y=0 \\ x-\text{intercept: } -1,\ f(0)=-2 \\ \text{hole at } x=2 \end{cases}$$

34. Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote:\ x=-4, x=5 \\ horizontal\ asymptote:\ y=\dfrac{3}{2} \\ x-intercept:\ -2 \end{cases}$$