Solution   *Section* **1.8 - Numerical Methods**

Exercise

Calculate the first five iterations of Euler's method with step $h = 0.1$ of

$$y' = ty \quad y(0) = 1$$

Solution

t	y
0.1	1.00000000
0.2	1.01000000
0.3	1.03020000
0.4	1.06110600
0.5	1.10355024

$$\frac{dy}{dt} = ty$$

$$\int \frac{dy}{y} = \int t\,dt$$

$$\ln y = \frac{1}{2}t^2 + C$$

$$y(t) = e^{t^2/2 + C}$$

$$y(0) = e^C = 1 \quad \rightarrow \quad \boxed{C = 0}$$

$$\boxed{y(t) = e^{t^2/2}}$$

Exercise

Calculate the first five iterations of Euler's method with step $h = 0.1$ of

$$z' = x - 2z \quad z(0) = 1$$

Solution

x	z
0.0	1.00000000
0.1	0.80000000
0.2	0. 65000000
0.3	0.54000000
0.4	0.46200000
0.5	0.40960000

Exercise

Calculate the first five iterations of Euler's method with step $h = 0.1$ of: $z' = 5 - z$ $z(0) = 0$

Solution

$x_0 = 0; \quad z_0 = 0$

The *first* step:

$z_1 = z_0 + h\left(5 - z_0\right) = 0 + 0.1(5 - 0) = 0.5$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

The *second* step:

$z_2 = z_1 + h\left(5 - z_1\right) = 0.5 + 0.1(5 - 0.5) = 0.95$

$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

Euler Method

t	Approx.	Exact	Difference
0.00	0.00000000	0.00000000	0.00000000
0.10	0.50000000	0.47581291	-0.02418709
0.20	0.95000000	0.90634623	-0.04365377
0.30	1.35500000	1.29590890	-0.05909110
0.40	1.71950000	1.64839977	-0.07110023
0.50	2.04755000	1.96734670	-0.08020330

$$y(t) = 5 - 5e^{-t}$$

Exercise

Given: $y' + 2xy = x$ $y(0) = 8$

a) Use a computer and Euler's method to calculate three separate approximate solutions on the interval $[0, 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

Solution

x	y
0.0	8.00000000
0.2	8.00000000
0.4	7.40000000
0.6	6.29600000
0.8	4.90496000
1.0	3.49537280
x	y
0.0	8.00000000

0.1	8.00000000
0.2	7.85000000
0.3	7.55600000
0.4	7.13264000
0.5	6.60202880
0.6	5.99182592
0.7	5.33280681
0.8	4.65621386
0.9	3.99121964
1.0	3.36280010

x	y	x	y
0.0	8.00000000		
0.05	8.00000000	0.55	6.16870319
0.10	7.96250000	0.60	5.85692451
0.15	7.88787500	0.65	5.53550904
0.20	7.77705688	0.70	5.20820096
0.25	7.63151574	0.75	4.87862689
0.30	7.45322784	0.80	4.55022987
0.35	7.24463101	0.85	4.22621148
0.40	7.00856892	0.90	3.90948351
0.45	6.74822617	0.95	3.60262999
0.50	6.46705599	1.00	3.30788014

$$y(t) = \frac{15}{2} e^{-t^2} + \frac{1}{2}$$

t	Approx.	Exact	Difference
0.00	8.00000000	8.00000000	0.00000000
0.20	8.00000000	7.70592079	-0.29407921
0.40	7.40000000	6.89107842	-0.50892158
0.60	6.29600000	5.73257245	-0.56342755
0.80	4.90496000	4.45469318	-0.45026682
1.00	3.49537280	3.25909581	-0.23627699

t	Approx.	Exact	Difference
0.00	8.00000000	8.00000000	0.00000000
0.10	8.00000000	7.92537375	-0.07462625
0.20	7.85000000	7.70592079	-0.14407921
0.30	7.55600000	7.35448389	-0.20151611
0.40	7.13264000	6.89107842	-0.24156158
0.50	6.60202880	6.34100587	-0.26102293
0.60	5.99182592	5.73257245	-0.25925347
0.70	5.33280681	5.09469796	-0.23810885
0.80	4.65621386	4.45469318	-0.20152068
0.90	3.99121964	3.83643550	-0.15478414

1.00 | 3.36280010 | 3.25909581 | -0.10370430

t	Approx.	Exact	Difference
0.00	8.00000000	8.00000000	0.00000000
0.05	8.00000000	7.98127342	-0.01872658
0.10	7.96250000	7.92537375	-0.03712625
0.15	7.88787500	7.83313428	-0.05474072
0.20	7.77705688	7.70592079	-0.07113608
0.25	7.63151574	7.54559797	-0.08591777
0.30	7.45322784	7.35448389	-0.09874395
0.35	7.24463101	7.13529429	-0.10933672
0.40	7.00856892	6.89107842	-0.11749051
0.45	6.74822617	6.62514862	-0.12307755
0.50	6.46705599	6.34100587	-0.12605012
0.55	6.16870319	6.04226366	-0.12643953
0.60	5.85692451	5.73257245	-0.12435207
0.65	5.53550904	5.41554691	-0.11996214
0.70	5.20820096	5.09469796	-0.11350300
0.75	4.87862689	4.77337119	-0.10525570
0.80	4.55022987	4.45469318	-0.09553669
0.85	4.22621148	4.14152671	-0.08468477
0.90	3.90948351	3.83643550	-0.07304801
0.95	3.60262999	3.54165879	-0.06097120
1.00	3.30788014	3.25909581	-0.04878433

Exercise

Given: $y' + 2y = 2 - e^{-4t}$ $y(0) = 1$

a) Solve the differential equation
b) Use Euler's method and Runge-Kutta methods to calculate three separate approximate solutions on the interval $[0, 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$ Plot the exact solution and approximate solutions as discrete points.

Solution

a) $e^{\int 2dt} = e^{2t}$

$$\int \left(2 - e^{-4t}\right) e^{2t} dt = \int \left(2e^{2t} - e^{-2t}\right) dt = e^{2t} + \frac{1}{2}e^{-2t}$$

$$y(t) = \frac{1}{e^{2t}}\left(e^{2t} + \frac{1}{2}e^{-2t} + C\right)$$

$$= 1 + \frac{1}{2}e^{-4t} + Ce^{-2t}$$

$$y(0) = 1 \quad \rightarrow \quad 1 = 1 + \frac{1}{2} + C \quad \Rightarrow \quad \underline{C = -\frac{1}{2}}$$

313

$$y(t) = 1 + \frac{1}{2}e^{-4t} - \frac{1}{2}e^{-2t}$$
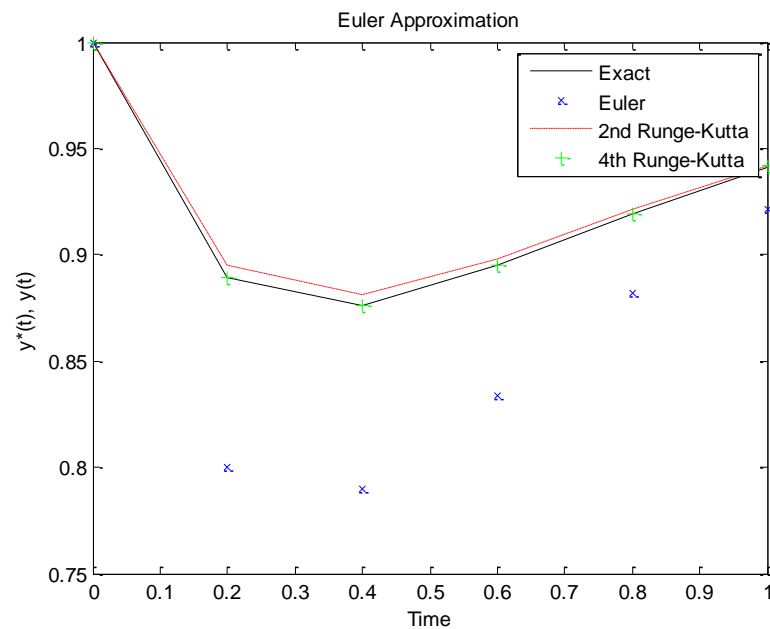
b) $h = 0.2$

Euler Method

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.20	0.80000000	0.88950446	0.08950446
0.40	0.79013421	0.87628378	0.08614957
0.60	0.83370122	0.89476187	0.06106065
0.80	0.88207714	0.91943284	0.03735570
1.00	0.92109384	0.94149018	0.02039633

Runge-Kutta 2nd Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.20	0.89506710	0.88950446	-0.00556264
0.40	0.88149624	0.87628378	-0.00521246
0.60	0.89823186	0.89476187	-0.00346999
0.80	0.92127837	0.91943284	-0.00184552
1.00	0.94219199	0.94149018	-0.00070181

Runge-Kutta 4th Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.20	0.88960071	0.88950446	-0.00009625
0.40	0.87638272	0.87628378	-0.00009894
0.60	0.89483774	0.89476187	-0.00007587
0.80	0.91948402	0.91943284	-0.00005118
1.00	0.94152197	0.94149018	-0.00003179

Euler Approximation

$h = 0.1$

Euler Method

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.10	0.90000000	0.92579465	0.02579465
0.20	0.85296800	0.88950446	0.03653646
0.30	0.83744150	0.87619129	0.03874979
0.40	0.83983378	0.87628378	0.03645000
0.50	0.85167737	0.88372792	0.03205055
0.60	0.86780837	0.89476187	0.02695350
0.70	0.88517490	0.90710655	0.02193165
0.80	0.90205891	0.91943284	0.01737393
0.90	0.91757091	0.93101242	0.01344151
1.00	0.93132436	0.94149018	0.01016582

Runge-Kutta 2nd Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.10	0.92648400	0.92579465	-0.00068935
0.20	0.89043763	0.88950446	-0.00093317
0.30	0.87712599	0.87619129	-0.00093470
0.40	0.87710071	0.87628378	-0.00081694
0.50	0.88437996	0.88372792	-0.00065204
0.60	0.89524226	0.89476187	-0.00048039
0.70	0.90742943	0.90710655	-0.00032288
0.80	0.91962162	0.91943284	-0.00018878
0.90	0.93109305	0.93101242	-0.00008064
1.00	0.94148757	0.94149018	0.00000260

Runge-Kutta 4th Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.10	0.92579791	0.92579465	-0.00000326
0.20	0.88950913	0.88950446	-0.00000467
0.30	0.87619629	0.87619129	-0.00000500
0.40	0.87628854	0.87628378	-0.00000476
0.50	0.88373216	0.88372792	-0.00000424
0.60	0.89476548	0.89476187	-0.00000361
0.70	0.90710953	0.90710655	-0.00000298
0.80	0.91943524	0.91943284	-0.00000240
0.90	0.93101431	0.93101242	-0.00000189
1.00	0.94149164	0.94149018	-0.00000146

Euler Approximation
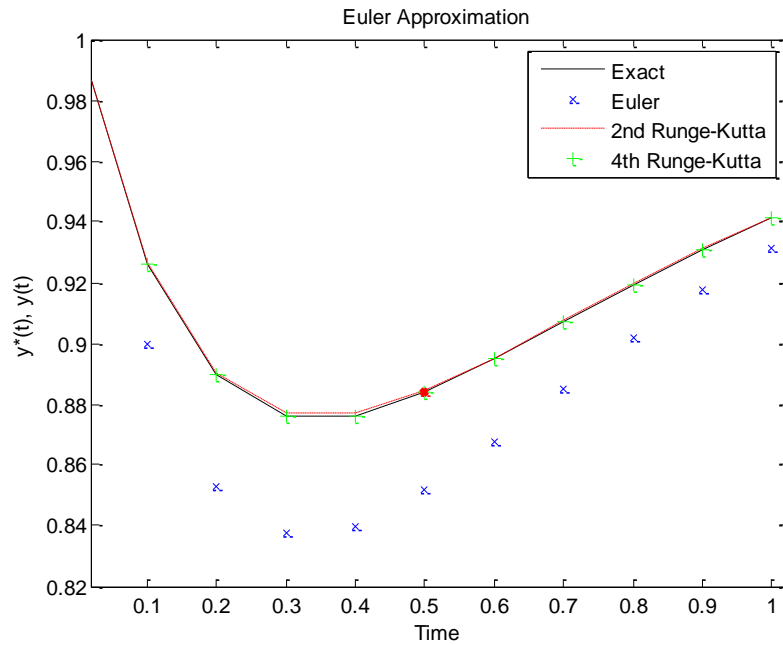
$h = 0.05$

Euler Method

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.05	0.95000000	0.95694667	0.00694667
0.10	0.91406346	0.92579465	0.01173118
0.15	0.88914111	0.90399671	0.01485559
0.20	0.87278642	0.88950446	0.01671804
0.25	0.86304133	0.88067439	0.01763306
0.30	0.85834323	0.87619129	0.01784806
0.35	0.85744919	0.87500583	0.01755664
0.40	0.85937442	0.87628378	0.01690935
0.45	0.86334216	0.87936461	0.01602246
0.50	0.86874300	0.88372792	0.01498492
0.55	0.87510193	0.88896604	0.01386410
0.60	0.88205158	0.89476187	0.01271029
0.65	0.88931053	0.90087089	0.01156037
0.70	0.89666579	0.90710655	0.01044076
0.75	0.90395871	0.91332845	0.00936974
0.80	0.91107349	0.91943284	0.00835936
0.85	0.91792803	0.92534487	0.00741684
0.90	0.92446656	0.93101242	0.00654586
0.95	0.93065372	0.93640108	0.00574736
1.00	0.93646981	0.94149018	0.00502037

Runge-Kutta 2nd Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.05	0.95703173	0.95694667	-0.00008506

316

t	Approx.	Exact	Difference
0.10	0.92593427	0.92579465	-0.00013963
0.15	0.90416803	0.90399671	-0.00017132
0.20	0.88969058	0.88950446	-0.00018612
0.25	0.88086308	0.88067439	-0.00018869
0.30	0.87637395	0.87619129	-0.00018266
0.35	0.87517663	0.87500583	-0.00017080
0.40	0.87643901	0.87628378	-0.00015523
0.45	0.87950216	0.87936461	-0.00013754
0.50	0.88384684	0.88372792	-0.00011892
0.55	0.88906627	0.88896604	-0.00010023
0.60	0.89484396	0.89476187	-0.00008208
0.65	0.90093579	0.90087089	-0.00006489
0.70	0.90715548	0.90710655	-0.00004893
0.75	0.91336281	0.91332845	-0.00003435
0.80	0.91945408	0.91943284	-0.00002123
0.85	0.92535446	0.92534487	-0.00000958
0.90	0.93101179	0.93101242	0.00000063
0.95	0.93639162	0.93640108	0.00000946
1.00	0.94147318	0.94149018	0.00001700

Runge-Kutta 4th Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.05	0.95694677	0.95694667	-0.00000011
0.10	0.92579483	0.92579465	-0.00000018
0.15	0.90399694	0.90399671	-0.00000023
0.20	0.88950472	0.88950446	-0.00000026
0.25	0.88067466	0.88067439	-0.00000027
0.30	0.87619156	0.87619129	-0.00000027
0.35	0.87500610	0.87500583	-0.00000027
0.40	0.87628404	0.87628378	-0.00000026
0.45	0.87936486	0.87936461	-0.00000025
0.50	0.88372815	0.88372792	-0.00000023
0.55	0.88896625	0.88896604	-0.00000021
0.60	0.89476207	0.89476187	-0.00000020
0.65	0.90087107	0.90087089	-0.00000018
0.70	0.90710671	0.90710655	-0.00000016
0.75	0.91332860	0.91332845	-0.00000015
0.80	0.91943297	0.91943284	-0.00000013
0.85	0.92534499	0.92534487	-0.00000011
0.90	0.93101252	0.93101242	-0.00000010
0.95	0.93640117	0.93640108	-0.00000009
1.00	0.94149026	0.94149018	-0.00000008

Euler Approximation
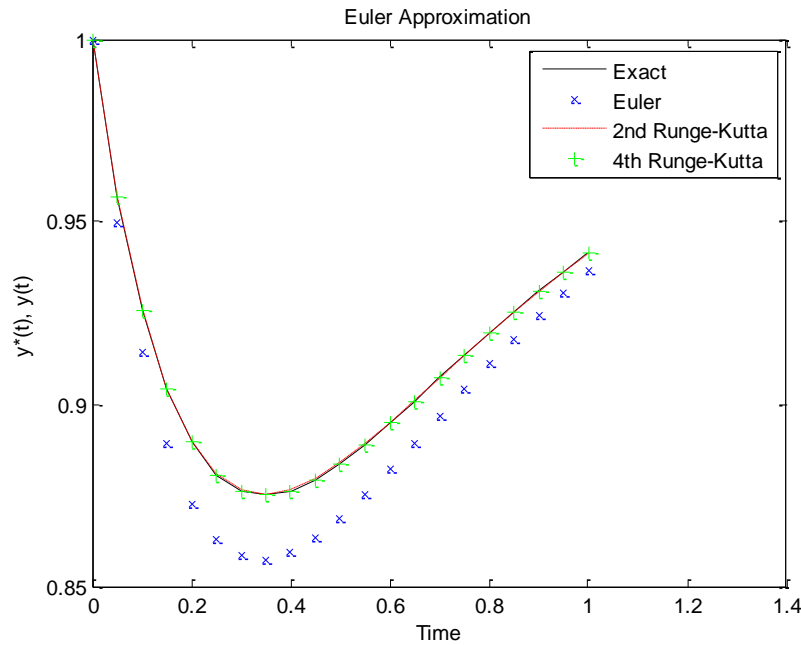
Exercise

Given: $z' - 2z = xe^{2x}$ $z(0) = 1$

a) Use a computer and Euler's method to calculate three separate approximate solutions on the interval $[0, 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a third with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

Solution

a)

Euler Method

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.20	1.40000000	1.52166119	0.12166119
0.40	2.01967299	2.40358420	0.38391121
0.60	3.00558546	3.91773797	0.91215251
0.80	4.60623367	6.53800280	1.93176913
1.00	7.24121233	11.08358415	3.84237182

Euler Method

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.10	1.20000000	1.22750977	0.02750977
0.20	1.45221403	1.52166119	0.06944716
0.30	1.77249333	1.90411415	0.13162082
0.40	2.18165556	2.40358420	0.22192865
0.50	2.70700830	3.05806706	0.35105875

0.60	3.38432406	3.91773797	0.53341391
0.70	4.26039588	5.04872396	0.78832807
0.80	5.39633906	6.53800280	1.14166374
0.90	6.87184946	8.49975469	1.62790522
1.00	8.79068763	11.08358415	2.29289652

Euler Method

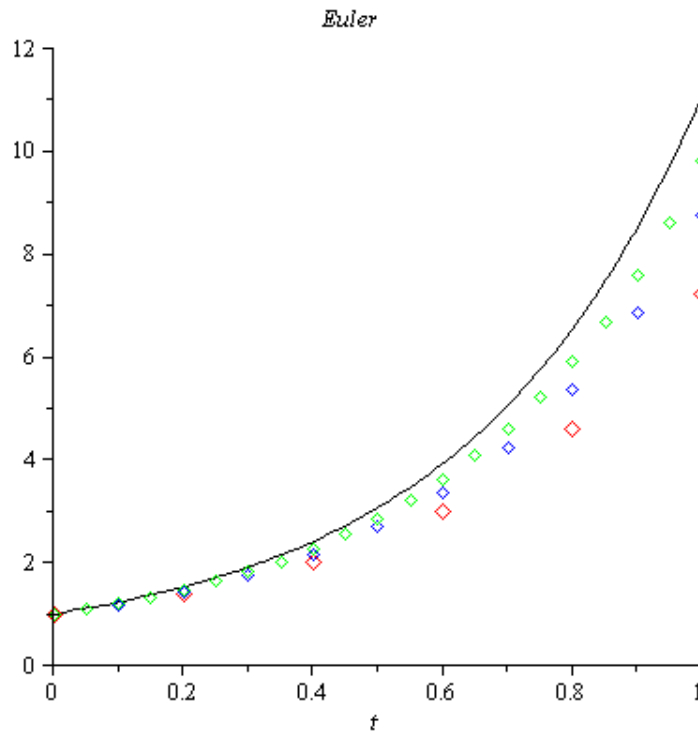| t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.05	1.10000000	1.10655238	0.00655238
0.10	1.21276293	1.22750977	0.01474684
0.15	1.34014623	1.36504472	0.02489849
0.20	1.48428480	1.52166119	0.03737639
0.25	1.64763153	1.70024381	0.05261229
0.30	1.83300369	1.90411415	0.07111045
0.35	2.04363584	2.13709506	0.09345922
0.40	2.28324010	2.40358420	0.12034410
0.45	2.55607493	2.70863793	0.15256300
0.50	2.86702349	3.05806706	0.19104356
0.55	3.22168289	3.45854614	0.23686325
0.60	3.62646574	3.91773797	0.29127223
0.65	4.08871582	4.44443559	0.35571976
0.70	4.61683955	5.04872396	0.43188441
0.75	5.22045550	5.74216412	0.52170862
0.80	5.91056439	6.53800280	0.62743841
0.85	6.69974213	7.45141089	0.75166876
0.90	7.60235911	8.49975469	0.89739558
0.95	8.63482915	9.70290431	1.06807516
1.00	9.81589205	11.08358415	1.26769209

b) $z' - 2z = xe^{2x}$

$P(x) = -2, \quad Q(x) = xe^{2x}$

$$e^{\int -2dx} = e^{-2x}$$

$$\int xe^{2x}e^{-2x}\,dx = \int x\,dx = \frac{1}{2}x^2$$

$$z(x) = \frac{1}{e^{-2x}}\left(\frac{1}{2}x^2 + C\right)$$

$$= \frac{1}{2}x^2 e^{2x} + Ce^{2x}$$

$z(0) = \underline{C = 1}$

$$\boxed{z(x) = \frac{1}{2}x^2 e^{2x} + e^{2x}}$$

Euler

Exercise

Consider the initial value problem $y' = 12 y(4 - y)$ $y(0) = 1$

Use Euler's method with step size $h = 0.04$ to sketch solution on the interval $[0, \ 2]$

Solution

$$y(t) = \frac{4}{e^{\log 3 - 48t} + 1}$$

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.04	2.44000000	2.77812333	0.33812333
0.08	4.26707200	3.75770045	-0.50937155
0.12	3.72005658	3.96254078	0.24248419
0.16	4.21993115	3.99446397	-0.22546718
0.20	3.77444588	3.99918742	0.22474154
0.24	4.18308995	3.99988085	-0.18320910
0.28	3.81546672	3.99998253	0.18451582
0.32	4.15342541	3.99999744	-0.15342797
0.36	3.84754974	3.99999962	0.15244989
0.40	4.12909852	3.99999994	-0.12909858
0.44	3.87322947	3.99999999	0.12677052
0.48	4.10891492	4.00000000	-0.10891492
0.52	3.89410430	4.00000000	0.10589570
0.56	4.09204138	4.00000000	-0.09204138
0.60	3.91125556	4.00000000	0.08874444
0.64	4.07786461	4.00000000	-0.07786461
0.68	3.92545437	4.00000000	0.07454563
0.72	4.06591460	4.00000000	-0.06591460
0.76	3.93727310	4.00000000	0.06272690
0.80	4.05582011	4.00000000	-0.05582011
0.84	3.94714987	4.00000000	0.05285013
0.88	4.04728141	4.00000000	-0.04728141
0.92	3.95542805	4.00000000	0.04457195

320

0.96	4.04005260	4.00000000	-0.04005260
1.00	3.96238159	4.00000000	0.03761841
1.04	4.03392967	4.00000000	-0.03392967
1.08	3.96823212	4.00000000	0.03176788
1.12	4.02874204	4.00000000	-0.02874204
1.16	3.97316079	4.00000000	0.02683921
1.20	4.02434630	4.00000000	-0.02434630
1.24	3.97731688	4.00000000	0.02268312
1.28	4.02062150	4.00000000	-0.02062150
1.32	3.98082411	4.00000000	0.01917589
1.36	4.01746532	4.00000000	-0.01746532
1.40	3.98378549	4.00000000	0.01621451
1.44	4.01479115	4.00000000	-0.01479115
1.48	3.98628712	4.00000000	0.01371288
1.52	4.01252558	4.00000000	-0.01252558
1.56	3.98840115	4.00000000	0.01159885
1.60	4.01060636	4.00000000	-0.01060636
1.64	3.99018815	4.00000000	0.00981185
1.68	4.00898069	4.00000000	-0.00898069
1.72	3.99169905	4.00000000	0.00830095
1.76	4.00760380	4.00000000	-0.00760380
1.80	3.99297675	4.00000000	0.00702325
1.84	4.00643771	4.00000000	-0.00643771
1.88	3.99405741	4.00000000	0.00594259
1.92	4.00545023	4.00000000	-0.00545023
1.96	3.99497153	4.00000000	0.00502847
2.00	4.00461405	4.00000000	-0.00461405

Exercise

You've seen that the error in Euler's method varies directly as the first power of the step size $\left(i.e\ E_h \approx \lambda h\right)$. This makes Euler's method an order to halve the error? How does this affect the number of required iterations?

Solution

Because $E_h \approx \lambda h$ halving the step size should halve the error.

$$E \approx \lambda \left(\frac{1}{2}h\right) \approx \frac{1}{2}\lambda h \approx \frac{1}{2}E_h$$

The number of iterations is given by: $N = \dfrac{b-a}{h}$, therefore halving the step size should double the number of iterations.
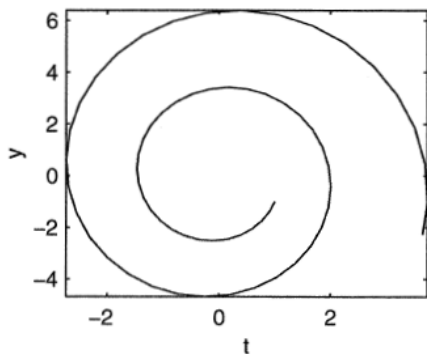
$$N = \frac{b-a}{\frac{1}{2}h} = 2\frac{b-a}{h} \approx 2N_h$$
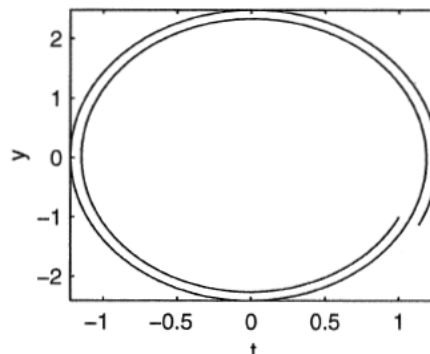
Exercise

Use Euler's method to provide an approximate solution over the given time interval using the given steps sizes. Provide a plot of v versus y for each step size

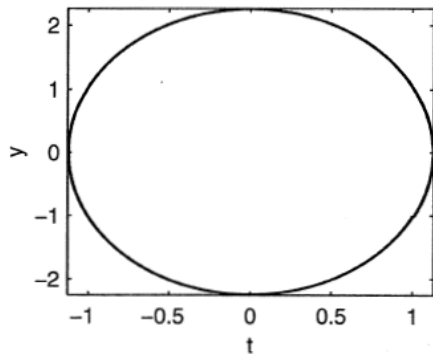$$y'' + 4y = 0, \quad y(0) = 4, \quad y'(0) = 0, \quad [0,\ 2\pi]; \quad h = 0.1,\ 0.01,\ 0.001$$

Solution

h = 0.1

h = 0.01

h = 0.001

Exercise

$z' + z = \cos x \quad z(0) = 1$

a) Use a computer and Runge-Kutta method to calculate three separate approximate solutions on the interval $[0, 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

Solution

$$z(x) = \frac{1}{2}e^{-x} + \frac{1}{2}\cos x + \frac{1}{2}\sin x$$

Runge-Kutta 2nd Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.20	0.99800666	0.99873333	0.00072667
0.40	0.98887689	0.99039969	0.00152281
0.60	0.96709749	0.96939486	0.00229738
0.80	0.92871746	0.93169588	0.00297842
1.00	0.87131508	0.87482637	0.00351128

322

Runge-Kutta 4th Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.20	0.99873272	0.99873333	0.00000061
0.40	0.99039822	0.99039969	0.00000147
0.60	0.96939245	0.96939486	0.00000241
0.80	0.93169258	0.93169588	0.00000330
1.00	0.87482232	0.87482637	0.00000405

Runge-Kutta 2nd Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.10	0.99975021	0.99983750	0.00008729
0.20	0.99855245	0.99873333	0.00018088
0.30	0.99555979	0.99583746	0.00027767
0.40	0.99002480	0.99039969	0.00037489
0.50	0.98129932	0.98176938	0.00047006
0.60	0.96883388	0.96939486	0.00056098
0.70	0.95217687	0.95282259	0.00064572
0.80	0.93097330	0.93169588	0.00072258
0.90	0.90496314	0.90575327	0.00079013
1.00	0.87397921	0.87482637	0.00084716

Runge-Kutta 4th Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.10	0.99983748	0.99983750	0.00000002
0.20	0.99873329	0.99873333	0.00000004
0.30	0.99583739	0.99583746	0.00000007
0.40	0.99039960	0.99039969	0.00000009
0.50	0.98176926	0.98176938	0.00000012
0.60	0.96939471	0.96939486	0.00000015
0.70	0.95282241	0.95282259	0.00000018
0.80	0.93169568	0.93169588	0.00000020
0.90	0.90575304	0.90575327	0.00000023
1.00	0.87482612	0.87482637	0.00000025

Runge-Kutta 2nd Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.05	0.99996876	0.99997943	0.00001067
0.10	0.99981570	0.99983750	0.00002180
0.15	0.99942531	0.99945859	0.00003328
0.20	0.99868831	0.99873333	0.00004502

```
0.25  |  0.99750164  |  0.99755858  |  0.00005694
0.30  |  0.99576852  |  0.99583746  |  0.00006894
0.35  |  0.99339836  |  0.99347931  |  0.00008094
0.40  |  0.99030682  |  0.99039969  |  0.00009287
0.45  |  0.98641574  |  0.98652039  |  0.00010465
0.50  |  0.98165315  |  0.98176938  |  0.00011623
0.55  |  0.97595326  |  0.97608078  |  0.00012752
0.60  |  0.96925639  |  0.96939486  |  0.00013847
0.65  |  0.96150896  |  0.96165799  |  0.00014903
0.70  |  0.95266344  |  0.95282259  |  0.00015915
0.75  |  0.94267832  |  0.94284709  |  0.00016877
0.80  |  0.93151803  |  0.93169588  |  0.00017785
0.85  |  0.91915289  |  0.91933924  |  0.00018635
0.90  |  0.90555903  |  0.90575327  |  0.00019423
0.95  |  0.89071835  |  0.89091981  |  0.00020146
1.00  |  0.87461836  |  0.87482637  |  0.00020801
```

Runge-Kutta 4th Order

```
   t          y           y(t)      Difference

----------------------------------------------------------
0.00  |  1.00000000  |  1.00000000  |  0.00000000
0.05  |  0.99997943  |  0.99997943  |  0.00000000
0.10  |  0.99983750  |  0.99983750  |  0.00000000
0.15  |  0.99945859  |  0.99945859  |  0.00000000
0.20  |  0.99873333  |  0.99873333  |  0.00000000
0.25  |  0.99755858  |  0.99755858  |  0.00000000
0.30  |  0.99583745  |  0.99583746  |  0.00000000
0.35  |  0.99347930  |  0.99347931  |  0.00000001
0.40  |  0.99039969  |  0.99039969  |  0.00000001
0.45  |  0.98652039  |  0.98652039  |  0.00000001
0.50  |  0.98176937  |  0.98176938  |  0.00000001
0.55  |  0.97608077  |  0.97608078  |  0.00000001
0.60  |  0.96939485  |  0.96939486  |  0.00000001
0.65  |  0.96165798  |  0.96165799  |  0.00000001
0.70  |  0.95282258  |  0.95282259  |  0.00000001
0.75  |  0.94284708  |  0.94284709  |  0.00000001
0.80  |  0.93169587  |  0.93169588  |  0.00000001
0.85  |  0.91933923  |  0.91933924  |  0.00000001
0.90  |  0.90575325  |  0.90575327  |  0.00000001
0.95  |  0.89091979  |  0.89091981  |  0.00000001
1.00  |  0.87482635  |  0.87482637  |  0.00000002
```
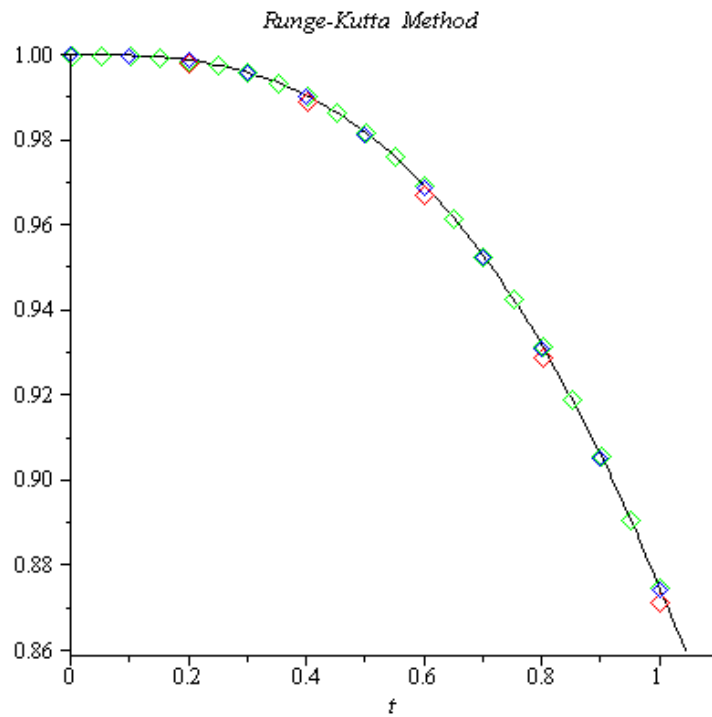
Runge-Kutta Method

Exercise

Given $x' = \frac{t}{x}$ $x(0) = 1$

a) Use a computer and Runge-Kutta method to calculate three separate approximate solutions on the interval $[0, 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

Solution

a)

Runge-Kutta 2th Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.20	1.01961161	1.01980390	0.00019229
0.40	1.07636229	1.07703296	0.00067067
0.60	1.16495094	1.16619038	0.00123944
0.80	1.27887002	1.28062485	0.00175483
1.00	1.41205020	1.41421356	0.00216336

Runge-Kutta 4th Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.20	1.01980437	1.01980390	-0.00000046
0.40	1.07703431	1.07703296	-0.00000135
0.60	1.16619234	1.16619038	-0.00000196

325

| 0.80 | 1.28062701 | 1.28062485 | -0.00000216 |
| 1.00 | 1.41421570 | 1.41421356 | -0.00000214 |

Runge-Kutta 2th Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.10	1.00497519	1.00498756	0.00001238
0.20	1.01975618	1.01980390	0.00004772
0.30	1.04392938	1.04403065	0.00010127
0.40	1.07686631	1.07703296	0.00016665
0.50	1.11779652	1.11803399	0.00023747
0.60	1.16588199	1.16619038	0.00030839
0.70	1.22027989	1.22065556	0.00037567
0.80	1.28018776	1.28062485	0.00043708
0.90	1.34487075	1.34536240	0.00049165
1.00	1.41367433	1.41421356	0.00053923

Runge-Kutta 4th Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.10	1.00498757	1.00498756	-0.00000001
0.20	1.01980393	1.01980390	-0.00000003
0.30	1.04403071	1.04403065	-0.00000006
0.40	1.07703304	1.07703296	-0.00000008
0.50	1.11803409	1.11803399	-0.00000010
0.60	1.16619050	1.16619038	-0.00000012
0.70	1.22065569	1.22065556	-0.00000013
0.80	1.28062498	1.28062485	-0.00000013
0.90	1.34536254	1.34536240	-0.00000013
1.00	1.41421369	1.41421356	-0.00000013

Runge-Kutta 2th Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.05	1.00124844	1.00124922	0.00000078
0.10	1.00498447	1.00498756	0.00000309
0.15	1.01118058	1.01118742	0.00000684
0.20	1.01979199	1.01980390	0.00001191
0.25	1.03075829	1.03077641	0.00001812
0.30	1.04400537	1.04403065	0.00002528
0.35	1.05944783	1.05948101	0.00003317
0.40	1.07699136	1.07703296	0.00004160
0.45	1.09653524	1.09658561	0.00005037
0.50	1.11797470	1.11803399	0.00005929
0.55	1.14120301	1.14127122	0.00006821
0.60	1.16611337	1.16619038	0.00007701
0.65	1.19260047	1.19268604	0.00008557
0.70	1.22056174	1.22065556	0.00009382
0.75	1.24989830	1.25000000	0.00010170
0.80	1.28051568	1.28062485	0.00010917
0.85	1.31232426	1.31244047	0.00011621
0.90	1.34523959	1.34536240	0.00012281
0.95	1.37918245	1.37931142	0.00012898
1.00	1.41407885	1.41421356	0.00013471

Runge-Kutta 4th Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.05	1.00124922	1.00124922	-0.00000000
0.10	1.00498756	1.00498756	-0.00000000
0.15	1.01118742	1.01118742	-0.00000000
0.20	1.01980390	1.01980390	-0.00000000
0.25	1.03077641	1.03077641	-0.00000000
0.30	1.04403065	1.04403065	-0.00000000
0.35	1.05948101	1.05948101	-0.00000000
0.40	1.07703297	1.07703296	-0.00000001
0.45	1.09658562	1.09658561	-0.00000001
0.50	1.11803400	1.11803399	-0.00000001
0.55	1.14127123	1.14127122	-0.00000001
0.60	1.16619039	1.16619038	-0.00000001
0.65	1.19268605	1.19268604	-0.00000001
0.70	1.22065557	1.22065556	-0.00000001
0.75	1.25000001	1.25000000	-0.00000001
0.80	1.28062486	1.28062485	-0.00000001
0.85	1.31244048	1.31244047	-0.00000001
0.90	1.34536241	1.34536240	-0.00000001
0.95	1.37931143	1.37931142	-0.00000001
1.00	1.41421357	1.41421356	-0.00000001

b) The equation is separable:

$$x\,dx = t\,dt$$

$$\int x\,dx = \int t\,dt$$

$$\tfrac{1}{2}x^2 = \tfrac{1}{2}t^2 + C$$
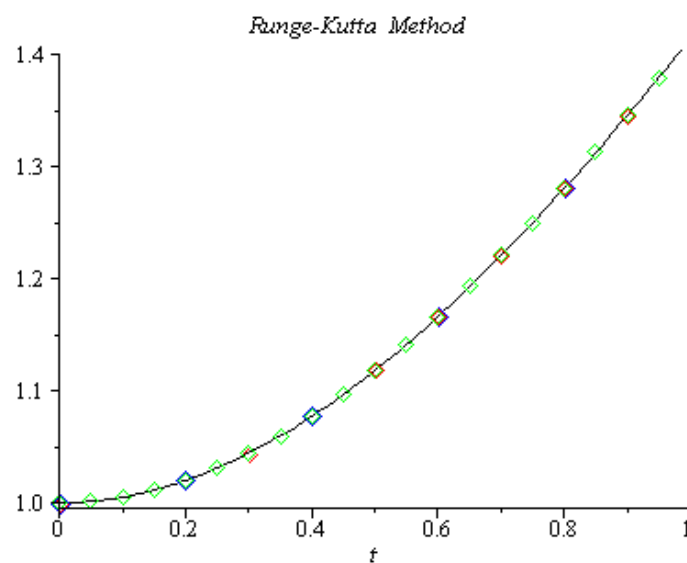
$$x^2 = t^2 + 2C$$

$$x = \sqrt{t^2 + 2C}$$

$$x(0) = \sqrt{2C}$$

$$1 = \sqrt{2C}$$

$$1 = 2C$$

$$C = \frac{1}{2}$$

$$\boxed{x(t) = \sqrt{t^2 + 1}}$$

Runge-Kutta Method

327

Exercise

Consider the initial value problem $y' = \dfrac{t}{y^2}$ $y(0) = 1$

Use Runge-Kutta method with step size $h = 0.04$ to sketch solution on the interval $[0,\ 2]$

Solution

$$y(t) = 3^{1/3}\left(\tfrac{1}{2}t^2 + \tfrac{1}{3}\right)^{1/3}$$

Runge-Kutta 4th Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.04	1.00079936	1.00079936	-0.00000000
0.08	1.00318981	1.00318981	-0.00000000
0.12	1.00714877	1.00714877	-0.00000000
0.16	1.01263957	1.01263957	-0.00000000
0.20	1.01961283	1.01961282	-0.00000000
0.24	1.02800822	1.02800822	-0.00000000
0.28	1.03775651	1.03775651	-0.00000000
0.32	1.04878166	1.04878166	-0.00000001
0.36	1.06100297	1.06100297	-0.00000001
0.40	1.07433708	1.07433707	-0.00000001
0.44	1.08869975	1.08869974	-0.00000001
0.48	1.10400743	1.10400742	-0.00000001
0.52	1.12017855	1.12017854	-0.00000001
0.56	1.13713450	1.13713449	-0.00000001
0.60	1.15480036	1.15480035	-0.00000001
0.64	1.17310545	1.17310544	-0.00000001
0.68	1.19198361	1.19198360	-0.00000001
0.72	1.21137336	1.21137335	-0.00000001
0.76	1.23121787	1.23121787	-0.00000001
0.80	1.25146496	1.25146495	-0.00000001
0.84	1.27206683	1.27206682	-0.00000001
0.88	1.29297992	1.29297991	-0.00000001
0.92	1.31416464	1.31416463	-0.00000001
0.96	1.33558509	1.33558508	-0.00000001
1.00	1.35720882	1.35720881	-0.00000001
1.04	1.37900650	1.37900650	-0.00000001
1.08	1.40095174	1.40095173	-0.00000001
1.12	1.42302075	1.42302075	-0.00000001
1.16	1.44519217	1.44519216	-0.00000001
1.20	1.46744679	1.46744678	-0.00000001
1.24	1.48976740	1.48976739	-0.00000001
1.28	1.51213855	1.51213854	-0.00000001
1.32	1.53454641	1.53454640	-0.00000001
1.36	1.55697860	1.55697859	-0.00000001
1.40	1.57942403	1.57942403	-0.00000001
1.44	1.60187281	1.60187281	-0.00000001
1.48	1.62431609	1.62431608	-0.00000001

```
1.52  |  1.64674596  |  1.64674596  |  -0.00000001
1.56  |  1.66915540  |  1.66915539  |  -0.00000001
1.60  |  1.69153812  |  1.69153811  |  -0.00000001
1.64  |  1.71388854  |  1.71388853  |  -0.00000001
1.68  |  1.73620170  |  1.73620169  |  -0.00000001
1.72  |  1.75847320  |  1.75847319  |  -0.00000001
1.76  |  1.78069914  |  1.78069913  |  -0.00000001
1.80  |  1.80287607  |  1.80287606  |  -0.00000000
1.84  |  1.82500094  |  1.82500094  |  -0.00000000
1.88  |  1.84707109  |  1.84707109  |  -0.00000000
1.92  |  1.86908417  |  1.86908417  |  -0.00000000
1.96  |  1.89103813  |  1.89103813  |  -0.00000000
2.00  |  1.91293119  |  1.91293118  |  -0.00000000
```

Exercise

Consider the initial value problem $y' - y = -\frac{1}{2}e^{t/2}\sin 5t + 5e^{t/2}\cos 5t \quad y(0) = 0$

Use Runge-Kutta method with step size $h = 0.05$ to sketch solution on the interval $[0,\ 5]$

Solution

$$e^{\int -dt} = e^{-t}$$

$$\int\left(-\frac{1}{2}e^{t/2}\sin 5t + 5e^{t/2}\cos 5t\right)e^{-t}dt = -\frac{1}{2}\int\left(e^{-t/2}\sin 5t\right)dt + 5\int\left(e^{-t/2}\cos 5t\right)dt$$

		$\int\sin 5t$
+	$e^{-t/2}$	$-\frac{1}{5}\cos 5t$
−	$-\frac{1}{2}e^{-t/2}$	$-\frac{1}{25}\sin 5t$
+	$\frac{1}{4}e^{-t/2}$	

		$\int\cos 5t$
+	$e^{-t/2}$	$\frac{1}{5}\sin 5t$
−	$-\frac{1}{2}e^{-t/2}$	$-\frac{1}{25}\cos 5t$
+	$\frac{1}{4}e^{-t/2}$	

$$\int\left(e^{-t/2}\sin 5t\right)dt = \left(-\frac{1}{5}\cos 5t - \frac{1}{50}\sin 5t\right)e^{-t/2} - \frac{1}{100}\int\left(e^{-t/2}\sin 5t\right)dt$$

$$\frac{101}{100}\int\left(e^{-t/2}\sin 5t\right)dt = -\frac{1}{50}(10\cos 5t + \sin 5t)e^{-t/2}$$

$$\int\left(e^{-t/2}\sin 5t\right)dt = -\frac{2}{101}(10\cos 5t + \sin 5t)e^{-t/2}$$

$$\int\left(e^{-t/2}\cos 5t\right)dt = e^{-t/2}\left(\frac{1}{5}\sin 5t - \frac{1}{50}\cos 5t\right) - \frac{1}{100}\int\left(e^{-t/2}\cos 5t\right)dt$$

$$\frac{101}{100}\int\left(e^{-t/2}\cos 5t\right)dt = \frac{1}{50}e^{-t/2}(10\sin 5t - \cos 5t)$$

$$\int \left(e^{-t/2} \cos 5t \right) dt = \frac{2}{101} e^{-t/2} \left(10 \sin 5t - \cos 5t \right)$$

$$\int \left(-\frac{1}{2} e^{t/2} \sin 5t + 5 e^{t/2} \cos 5t \right) e^{-t} dt = \left(\frac{10}{101} \cos 5t + \frac{1}{101} \sin 5t + \frac{100}{101} \sin 5t - \frac{10}{101} \cos 5t \right) e^{-t/2}$$

$$= e^{-t/2} \sin 5t$$

$$y(t) = e^t \left(e^{-t/2} \sin 5t + C \right)$$

$$= e^{t/2} \sin 5t + C e^t$$

$$y(0) = 0 \quad \rightarrow \quad \underline{C = 0}$$

$$\underline{y(t) = e^{t/2} \sin 5t}$$

Euler Method

t	Approx.	Exact	Difference
0.00	0.00000000	0.00000000	0.00000000
0.05	0.25000000	0.25366702	0.00366702
0.10	0.50451846	0.50400621	-0.00051225
0.15	0.74778853	0.73472762	-0.01306091
0.20	0.96397872	0.92996926	-0.03400946
0.25	1.13821003	1.07534045	-0.06286957
0.30	1.25756378	1.15892383	-0.09863994
0.35	1.31201510	1.17216954	-0.13984556
0.40	1.29522788	1.11061839	-0.18460949
0.45	1.20515309	0.97439874	-0.23075435
0.50	1.04438175	0.76845344	-0.27592831
0.55	0.82021731	0.50246840	-0.31774891
0.60	0.54444836	0.19049209	-0.35395627
0.65	0.23282095	-0.14974538	-0.38256633
0.70	-0.09577085	-0.49778510	-0.40201424
0.75	-0.42033859	-0.83161681	-0.41127822
0.80	-0.71904181	-1.12901665	-0.40997485
0.85	-0.97054890	-1.36896715	-0.39841824
0.90	-1.15543496	-1.53307240	-0.37763744
0.95	-1.25752830	-1.60687699	-0.34934869
1.00	-1.26511659	-1.58099885	-0.31588225
1.05	-1.17192748	-1.45199341	-0.28006593
1.10	-0.97780916	-1.22287990	-0.24507073
1.15	-0.68905162	-0.90327826	-0.21422665
1.20	-0.31830941	-0.50912823	-0.19081882
1.25	0.11588941	-0.06198694	-0.17787635
1.30	0.59003789	0.41207112	-0.17796677
1.35	1.07691137	0.88390140	-0.19300997
1.40	1.54713288	1.32300854	-0.22412434
1.45	1.97095749	1.69944069	-0.27151680
1.50	2.32017202	1.98574597	-0.33442605
1.55	2.56999372	2.15886828	-0.41112544
1.60	2.70084550	2.20185727	-0.49898823

1.65	2.69988727	2.10527280	-0.59461447
1.70	2.56219211	1.86817786	-0.69401425
1.75	2.29147346	1.49863486	-0.79283860
1.80	1.90029341	1.01364791	-0.88664550
1.85	1.40971217	0.43852637	-0.97118580
1.90	0.84837261	-0.19431898	-1.04269159
1.95	0.25104980	-0.84709881	-1.09814861
2.00	-0.34326837	-1.47880270	-1.13553433
2.05	-0.89366994	-2.04767466	-1.15400472
2.10	-1.35984868	-2.51386357	-1.15401490
2.15	-1.70472418	-2.84208687	-1.13736269
2.20	-1.89698837	-3.00413660	-1.10714824
2.25	-1.91341049	-2.98105850	-1.06764801
2.30	-1.74073987	-2.76484685	-1.02410698
2.35	-1.37706328	-2.35952135	-0.98245807
2.40	-0.83250141	-1.78148483	-0.94898341
2.45	-0.12916591	-1.05910196	-0.92993604
2.50	0.69965839	-0.23148617	-0.93114456
2.55	1.61109302	0.65346778	-0.95762524
2.60	2.55494455	1.54171751	-1.01322704
2.65	3.47657170	2.37623761	-1.10033409
2.70	4.32018626	3.10053857	-1.21964769
2.75	5.03239764	3.66233161	-1.37006604
2.80	5.56578542	4.01711092	-1.54867451
2.85	5.88227111	4.13141845	-1.75085267
2.90	5.95606196	3.98556468	-1.97049728
2.95	5.77595524	3.57560239	-2.20035285
3.00	5.34682192	2.91438791	-2.43243401
3.05	4.69013206	2.03161484	-2.65851722
3.10	3.84343922	0.97276538	-2.87067384
3.15	2.85880347	-0.20300879	-3.06181226
3.20	1.80020034	-1.42599446	-3.22619481
3.25	0.74003061	-2.61986419	-3.35989480
3.30	-0.24509055	-3.70625192	-3.46116137
3.35	-1.07903060	-4.60969444	-3.53066383
3.40	-1.69104556	-5.26263929	-3.57159374
3.45	-2.02058926	-5.61020009	-3.58961083
3.50	-2.02171162	-5.61434002	-3.59262841
3.55	-1.66674125	-5.25718495	-3.59044370
3.60	-0.94897881	-4.54320809	-3.59422929
3.65	0.11582328	-3.50008777	-3.61591105
3.70	1.48935225	-2.17811492	-3.66746718
3.75	3.11207493	-0.64811430	-3.76018923
3.80	4.90601419	1.00206320	-3.90395099
3.85	6.77885700	2.67232428	-4.10653272
3.90	8.62919773	4.25615056	-4.37304717
3.95	10.35263755	5.64712789	-4.70550966
4.00	11.84839105	6.74580367	-5.10258738
4.05	13.02600082	7.46644758	-5.55955324
4.10	13.81173327	7.74327526	-6.06845801
4.15	14.15422757	7.53570683	-6.61852074

```
4.20 | 14.02899591 |   6.83227210 |  -7.19672381
4.25 | 13.44142635 |   5.65284162 |  -7.78858473
4.30 | 12.42801840 |   4.04895406 |  -8.37906435
4.35 | 11.05568119 |   2.10212134 |  -8.95355985
4.40 |  9.41904025 |  -0.07988319 |  -9.49892344
4.45 |  7.63582435 |  -2.36861780 | -10.00444215
4.50 |  5.84053115 |  -4.62218308 | -10.46271423
4.55 |  4.17669135 |  -6.69366883 | -10.87036018
4.60 |  2.78815793 |  -8.44035671 | -11.22851463
4.65 |  1.80993130 |  -9.73312476 | -11.54305605
4.70 |  1.35908774 | -10.46545870 | -11.82454643
4.75 |  1.52640052 | -10.56146949 | -12.08787001
4.80 |  2.36922910 |  -9.98235001 | -12.35157911
4.85 |  3.90619931 |  -8.73077509 | -12.63697440
4.90 |  6.11410924 |  -6.85285609 | -12.96696533
4.95 |  8.92737431 |  -4.43739883 | -13.36477314
5.00 | 12.24017767 |  -1.61237440 | -13.85255207
```

Runge-Kutta 2nd Order

t	Approx.	Exact	Difference
0.00	0.00000000	0.00000000	0.00000000
0.05	0.25225923	0.25366702	0.00140779
0.10	0.50126928	0.50400621	0.00273693
0.15	0.73083289	0.73472762	0.00389473
0.20	0.92517466	0.92996926	0.00479460
0.25	1.06997848	1.07534045	0.00536197
0.30	1.15338415	1.15892383	0.00553968
0.35	1.16687699	1.17216954	0.00529254
0.40	1.10600766	1.11061839	0.00461072
0.45	0.97088706	0.97439874	0.00351168
0.50	0.76641304	0.76845344	0.00204040
0.55	0.50220045	0.50246840	0.00026796
0.60	0.19220390	0.19049209	-0.00171182
0.65	-0.14595823	-0.14974538	-0.00378715
0.70	-0.49195043	-0.49778510	-0.00583466
0.75	-0.82389011	-0.83161681	-0.00772670
0.80	-1.11967729	-1.12901665	-0.00933937
0.85	-1.35840643	-1.36896715	-0.01056072
0.90	-1.52177383	-1.53307240	-0.01129857
0.95	-1.59538951	-1.60687699	-0.01148748
1.00	-1.56990449	-1.58099885	-0.01109436
1.05	-1.44187115	-1.45199341	-0.01012226
1.10	-1.21426784	-1.22287990	-0.00861206
1.15	-0.89663646	-0.90327826	-0.00664180
1.20	-0.50480467	-0.50912823	-0.00432356
1.25	-0.06018891	-0.06198694	-0.00179803
1.30	0.41129806	0.41207112	0.00077306
1.35	0.88068514	0.88390140	0.00321626
1.40	1.31765160	1.32300854	0.00535694

1.45	1.69241026	1.69944069	0.00703043
1.50	1.97765283	1.98574597	0.00809314
1.55	2.15043546	2.15886828	0.00843282
1.60	2.19387987	2.20185727	0.00797740
1.65	2.09857119	2.10527280	0.00670161
1.70	1.86354684	1.86817786	0.00463102
1.75	1.49679192	1.49863486	0.00184294
1.80	1.01518384	1.01364791	-0.00153593
1.85	0.44386158	0.43852637	-0.00533521
1.90	-0.18496945	-0.19431898	-0.00934953
1.95	-0.83374872	-0.84709881	-0.01335009
2.00	-1.46170451	-1.47880270	-0.01709819
2.05	-2.02731452	-2.04767466	-0.02036014
2.10	-2.49094092	-2.51386357	-0.02292265
2.15	-2.81747925	-2.84208687	-0.02460762
2.20	-2.97885110	-3.00413660	-0.02528550
2.25	-2.95617232	-2.98105850	-0.02488618
2.30	-2.74144026	-2.76484685	-0.02340659
2.35	-2.33860692	-2.35952135	-0.02091443
2.40	-1.76393735	-1.78148483	-0.01754748
2.45	-1.04559354	-1.05910196	-0.01350842
2.50	-0.22243099	-0.23148617	-0.00905518
2.55	0.65795501	0.65346778	-0.00448723
2.60	1.54184603	1.54171751	-0.00012852
2.65	2.37254558	2.37623761	0.00369203
2.70	3.09387721	3.10053857	0.00666136
2.75	3.65382846	3.66233161	0.00850315
2.80	4.00811352	4.01711092	0.00899739
2.85	4.12342111	4.13141845	0.00799734
2.90	3.98012211	3.98556468	0.00544257
2.95	3.57423517	3.57560239	0.00136722
3.00	2.91848542	2.91438791	-0.00409752
3.05	2.04234183	2.03161484	-0.01072699
3.10	0.99097829	0.97276538	-0.01821290
3.15	-0.17682977	-0.20300879	-0.02617901
3.20	-1.39179241	-1.42599446	-0.03420205
3.25	-2.57802748	-2.61986419	-0.04183671
3.30	-3.65760857	-3.70625192	-0.04864335
3.35	-4.55547762	-4.60969444	-0.05421682
3.40	-5.20442486	-5.26263929	-0.05821444
3.45	-5.54981858	-5.61020009	-0.06038151
3.50	-5.55376757	-5.61434002	-0.06057245
3.55	-5.19841904	-5.25718495	-0.05876591
3.60	-4.48813526	-4.54320809	-0.05507283
3.65	-3.45035150	-3.50008777	-0.04973627
3.70	-2.13499213	-2.17811492	-0.04312279
3.75	-0.61240881	-0.64811430	-0.03570549
3.80	1.03010243	1.00206320	-0.02803923
3.85	2.69305372	2.67232428	-0.02072944
3.90	4.27054658	4.25615056	-0.01439602
3.95	5.65676237	5.64712789	-0.00963448

```
4.00  |   6.75278040  |   6.74580367  |  -0.00697673
4.05  |   7.47330147  |   7.46644758  |  -0.00685388
4.10  |   7.75283900  |   7.74327526  |  -0.00956374
4.15  |   7.55095199  |   7.53570683  |  -0.01524515
4.20  |   6.85613350  |   6.83227210  |  -0.02386140
4.25  |   5.68803555  |   5.65284162  |  -0.03519394
4.30  |   4.09780169  |   4.04895406  |  -0.04884764
4.35  |   2.16638901  |   2.10212134  |  -0.06426767
4.40  |   0.00088437  |  -0.07988319  |  -0.08076755
4.45  |  -2.27105067  |  -2.36861780  |  -0.09756712
4.50  |  -4.50834452  |  -4.62218308  |  -0.11383856
4.55  |  -6.56491101  |  -6.69366883  |  -0.12875783
4.60  |  -8.29879815  |  -8.44035671  |  -0.14155855
4.65  |  -9.58153975  |  -9.73312476  |  -0.15158501
4.70  | -10.30711803  | -10.46545870  |  -0.15834067
4.75  | -10.39994039  | -10.56146949  |  -0.16152910
4.80  |  -9.82126597  |  -9.98235001  |  -0.16108404
4.85  |  -8.57358877  |  -8.73077509  |  -0.15718632
4.90  |  -6.70259041  |  -6.85285609  |  -0.15026568
4.95  |  -4.29641205  |  -4.43739883  |  -0.14098678
5.00  |  -1.48215502  |  -1.61237440  |  -0.13021938
```

Runge-Kutta 4th Order

t	Approx.	Exact	Difference
0.00	0.00000000	0.00000000	0.00000000
0.05	0.25366730	0.25366702	-0.00000028
0.10	0.50400675	0.50400621	-0.00000054
0.15	0.73472836	0.73472762	-0.00000075
0.20	0.92997015	0.92996926	-0.00000089
0.25	1.07534142	1.07534045	-0.00000096
0.30	1.15892478	1.15892383	-0.00000095
0.35	1.17217037	1.17216954	-0.00000084
0.40	1.11061902	1.11061839	-0.00000064
0.45	0.97439909	0.97439874	-0.00000035
0.50	0.76845344	0.76845344	0.00000001
0.55	0.50246799	0.50246840	0.00000042
0.60	0.19049122	0.19049209	0.00000086
0.65	-0.14974669	-0.14974538	0.00000131
0.70	-0.49778684	-0.49778510	0.00000175
0.75	-0.83161894	-0.83161681	0.00000213
0.80	-1.12901910	-1.12901665	0.00000245
0.85	-1.36896982	-1.36896715	0.00000267
0.90	-1.53307519	-1.53307240	0.00000279
0.95	-1.60687978	-1.60687699	0.00000279
1.00	-1.58100151	-1.58099885	0.00000266
1.05	-1.45199583	-1.45199341	0.00000242
1.10	-1.22288197	-1.22287990	0.00000207
1.15	-0.90327990	-0.90327826	0.00000164
1.20	-0.50912938	-0.50912823	0.00000115
1.25	-0.06198757	-0.06198694	0.00000063

1.30	0.41207099	0.41207112	0.00000013
1.35	0.88390173	0.88390140	-0.00000033
1.40	1.32300925	1.32300854	-0.00000070
1.45	1.69944165	1.69944069	-0.00000096
1.50	1.98574704	1.98574597	-0.00000108
1.55	2.15886931	2.15886828	-0.00000103
1.60	2.20185808	2.20185727	-0.00000081
1.65	2.10527321	2.10527280	-0.00000041
1.70	1.86817772	1.86817786	0.00000014
1.75	1.49863402	1.49863486	0.00000084
1.80	1.01364626	1.01364791	0.00000165
1.85	0.43852385	0.43852637	0.00000252
1.90	-0.19432240	-0.19431898	0.00000342
1.95	-0.84710311	-0.84709881	0.00000430
2.00	-1.47880780	-1.47880270	0.00000510
2.05	-2.04768043	-2.04767466	0.00000577
2.10	-2.51386985	-2.51386357	0.00000628
2.15	-2.84209346	-2.84208687	0.00000659
2.20	-3.00414329	-3.00413660	0.00000669
2.25	-2.98106506	-2.98105850	0.00000656
2.30	-2.76485306	-2.76484685	0.00000621
2.35	-2.35952701	-2.35952135	0.00000566
2.40	-1.78148978	-1.78148483	0.00000496
2.45	-1.05910610	-1.05910196	0.00000414
2.50	-0.23148944	-0.23148617	0.00000327
2.55	0.65346536	0.65346778	0.00000242
2.60	1.54171585	1.54171751	0.00000165
2.65	2.37623657	2.37623761	0.00000104
2.70	3.10053793	3.10053857	0.00000064
2.75	3.66233110	3.66233161	0.00000051
2.80	4.01711023	4.01711092	0.00000068
2.85	4.13141725	4.13141845	0.00000119
2.90	3.98556265	3.98556468	0.00000203
2.95	3.57559921	3.57560239	0.00000319
3.00	2.91438329	2.91438791	0.00000462
3.05	2.03160857	2.03161484	0.00000627
3.10	0.97275731	0.97276538	0.00000808
3.15	-0.20301874	-0.20300879	0.00000995
3.20	-1.42600625	-1.42599446	0.00001179
3.25	-2.61987770	-2.61986419	0.00001351
3.30	-3.70626694	-3.70625192	0.00001502
3.35	-4.60971069	-4.60969444	0.00001625
3.40	-5.26265642	-5.26263929	0.00001712
3.45	-5.61021769	-5.61020009	0.00001760
3.50	-5.61435770	-5.61434002	0.00001767
3.55	-5.25720229	-5.25718495	0.00001735
3.60	-4.54322475	-4.54320809	0.00001666
3.65	-3.50010344	-3.50008777	0.00001567
3.70	-2.17812940	-2.17811492	0.00001448
3.75	-0.64812749	-0.64811430	0.00001319
3.80	1.00205127	1.00206320	0.00001193

```
3.85  |  2.67231345  |  2.67232428  |  0.00001083
3.90  |  4.25614054  |  4.25615056  |  0.00001002
3.95  |  5.64711827  |  5.64712789  |  0.00000962
4.00  |  6.74579394  |  6.74580367  |  0.00000973
4.05  |  7.46643715  |  7.46644758  |  0.00001043
4.10  |  7.74326349  |  7.74327526  |  0.00001177
4.15  |  7.53569307  |  7.53570683  |  0.00001376
4.20  |  6.83225573  |  6.83227210  |  0.00001637
4.25  |  5.65282207  |  5.65284162  |  0.00001955
4.30  |  4.04893088  |  4.04895406  |  0.00002318
4.35  |  2.10209420  |  2.10212134  |  0.00002714
4.40  |  -0.07991446  |  -0.07988319  |  0.00003128
4.45  |  -2.36865321  |  -2.36861780  |  0.00003542
4.50  |  -4.62222247  |  -4.62218308  |  0.00003939
4.55  |  -6.69371186  |  -6.69366883  |  0.00004303
4.60  |  -8.44040289  |  -8.44035671  |  0.00004618
4.65  |  -9.73317349  |  -9.73312476  |  0.00004873
4.70  |  -10.46550929  |  -10.46545870  |  0.00005060
4.75  |  -10.56152123  |  -10.56146949  |  0.00005174
4.80  |  -9.98240218  |  -9.98235001  |  0.00005217
4.85  |  -8.73082706  |  -8.73077509  |  0.00005196
4.90  |  -6.85290732  |  -6.85285609  |  0.00005123
4.95  |  -4.43744898  |  -4.43739883  |  0.00005015
5.00  |  -1.61242331  |  -1.61237440  |  0.00004891
```


Euler & Runge-Kutta Approximation