Lecture Four

Section 4.1 – Matrix Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

Definition

If V and W are vector spaces, and if f is a function with domain V and codomain W, then we say that f is a transformation from V to W or that f maps V to W, which we denote by writing

$$f : V \to W$$

In the special case where $V = W$, the transformation is also called an operator on V.

Matrix Transformation

$$\begin{matrix} w_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ w_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \\ w_m = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{matrix}$$

Which we can write in matrix formation

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\vec{w} = A\vec{x}$$

Although we could view this as a linear system, we will view it instead as a transformation that maps the column vector $\vec{x}$ in $\mathbb{R}^n$ into the column vector $\vec{w}$ in $\mathbb{R}^m$ by multiplying $\vec{x}$ on the left by A. We call this a ***matrix transformation*** or ***function*** or ***mapping T*** from $\mathbb{R}^n$ to $\mathbb{R}^m$ (or ***matrix operator*** if $m = n$) and we denote it by

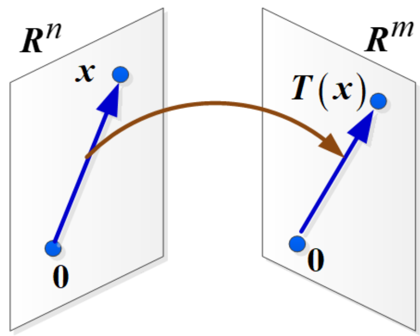
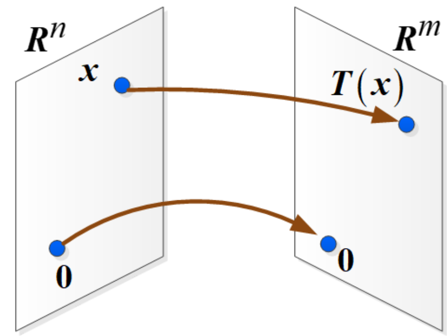
$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

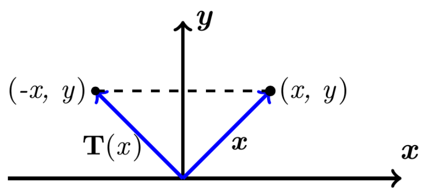
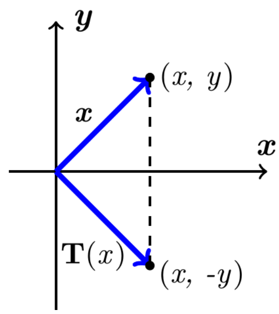
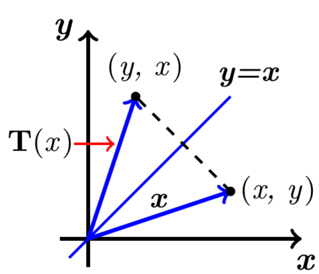
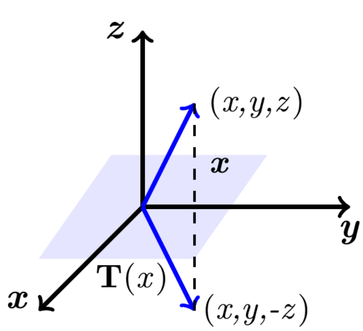
$\mathbb{R}^n$ is called the domain of T

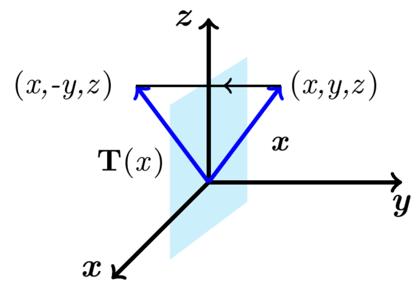
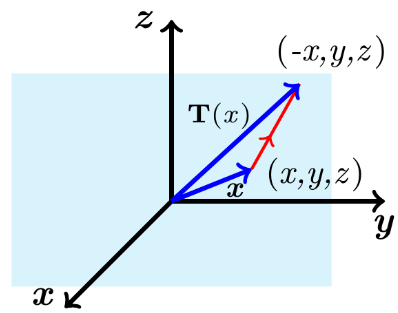
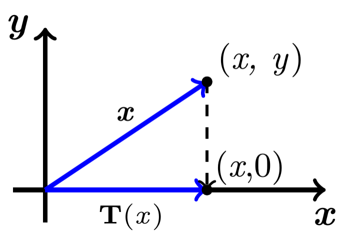
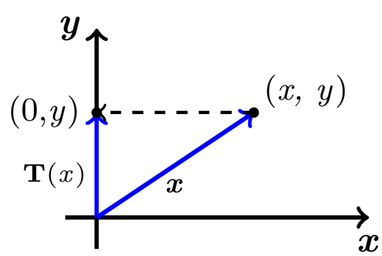
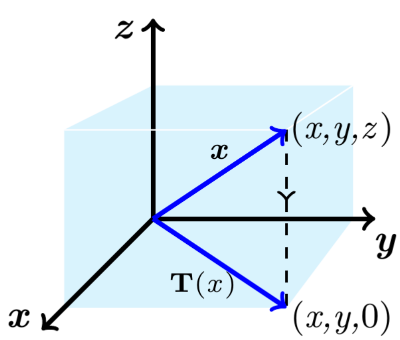
$\mathbb{R}^m$ is called the codomain of T

For $\vec{x}$ in $\mathbb{R}^n$, the vector $T(\vec{x})$ in $\mathbb{R}^m$ is called the image of $\vec{x}$ (under the action of T)

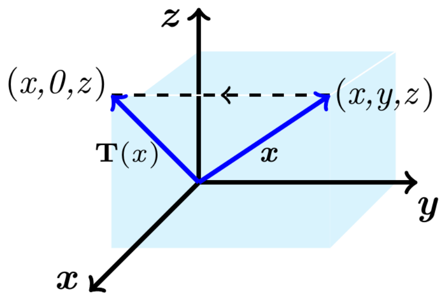
The set of all images $T(\vec{x})$ is called the range of T.

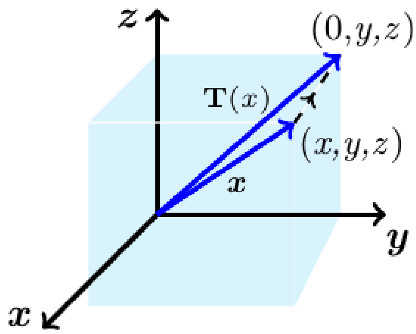
	
***T** maps vectors to vectors*	***T** maps points to points*

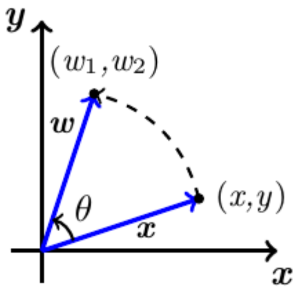
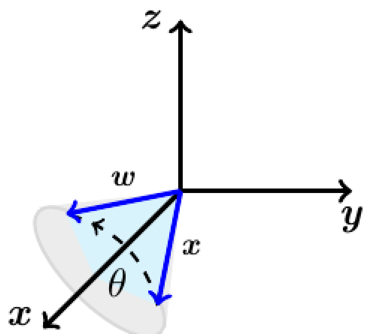
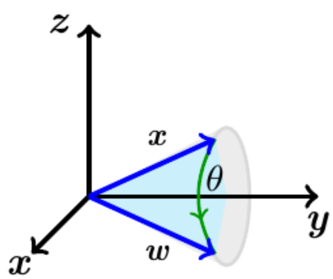
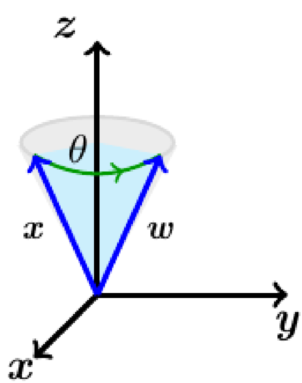
Reflection about the y-axis $T(x,y)=(-x,y)$		$T(e_1)=T(1,0)=(-1,0)$ $T(e_2)=T(0,1)=(0,1)$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection about the x-axis $T(x,y)=(x,-y)$		$T(e_1)=T(1,0)=(1,0)$ $T(e_2)=T(0,1)=(0,-1)$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about the line $y=x$ $T(x,y)=(x,-y)$		$T(e_1)=T(1,0)=(0,1)$ $T(e_2)=T(0,1)=(1,0)$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Reflection about the xy-plane $T(x,y,z)=(x,y,-z)$		$T(e_1)=T(1,0,0)=(1,0,0)$ $T(e_2)=T(0,1,0)=(0,1,0)$ $T(e_3)=T(0,0,1)=(0,0,-1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Reflection about the *xy*-plane $T(x,y,z)=(x,-y,z)$		$T(e_1)=T(1,0,0)=(1,0,0)$ $T(e_2)=T(0,1,0)=(0,-1,0)$ $T(e_3)=T(0,0,1)=(0,0,1)$	$\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$
Reflection about the *yz*-plane $T(x,y,z)=(-x,y,z)$		$T(e_1)=T(1,0,0)=(-1,0,0)$ $T(e_2)=T(0,1,0)=(0,1,0)$ $T(e_3)=T(0,0,1)=(0,0,1)$	$\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}$
Orthogonal projection on the x-axis $T(x,y)=(x,0)$		$T(e_1)=T(1,0)=(1,0)$ $T(e_2)=T(0,1)=(0,0)$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$
Orthogonal projection on the y-axis $T(x,y)=(0,y)$		$T(e_1)=T(1,0)=(0,0)$ $T(e_2)=T(0,1)=(0,1)$	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$
Orthogonal projection on the xy-Plane $T(x,y,z)=(x,y,0)$		$T(1,0,0)=(1,0,0)$ $T(0,1,0)=(0,1,0)$ $T(0,0,1)=(0,0,0)$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$

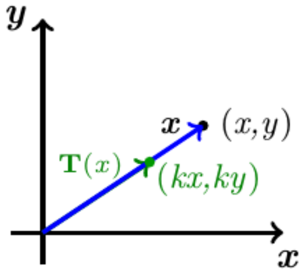
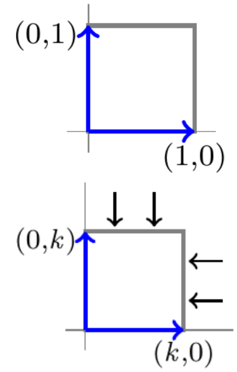
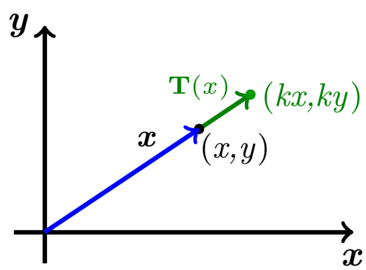
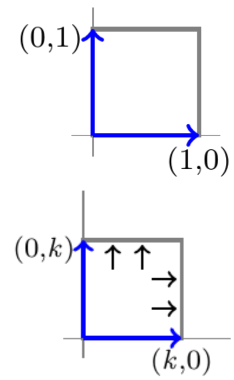
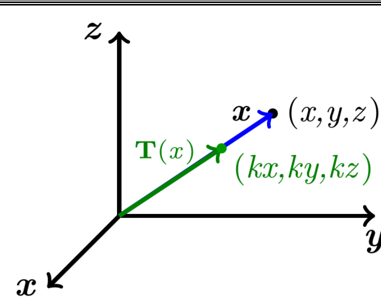
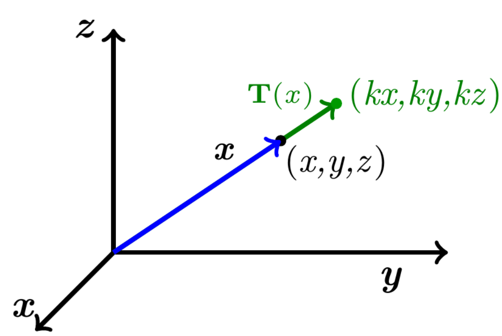
<table>
<tr>
<td>Orthogonal
projection
on the xz-Plane

$T(x,y,z)=(x,0,z)$</td>
<td>
</td>
<td>$T(1,0,0)=(1,0,0)$
$T(0,1,0)=(0,0,0)$
$T(0,0,1)=(0,0,1)$</td>
<td>$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$</td>
</tr>
<tr>
<td>Orthogonal
projection
on the yz-Plane

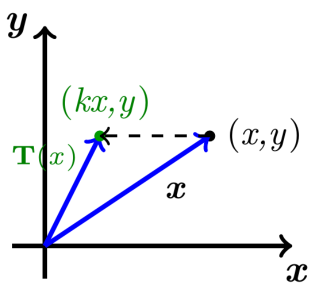
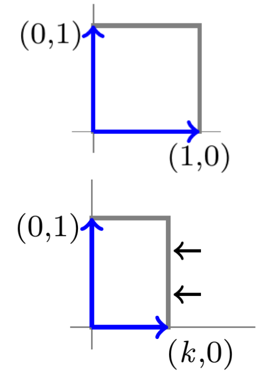
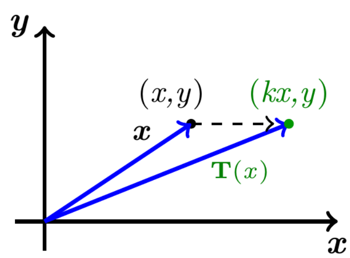
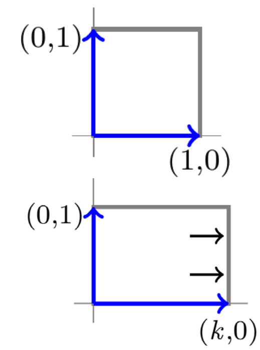
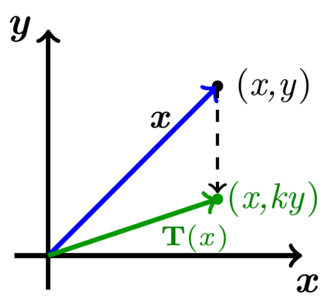
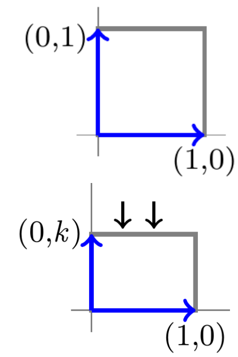
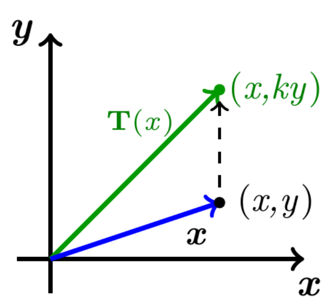
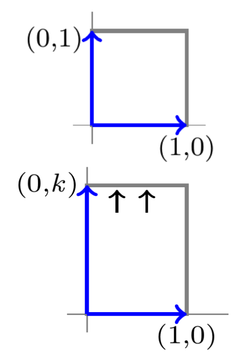
$T(x,y,z)=(0,y,z)$</td>
<td>
</td>
<td>$T(1,0,0)=(0,0,0)$
$T(0,1,0)=(0,1,0)$
$T(0,0,1)=(0,0,1)$</td>
<td>$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$</td>
</tr>
</table>

Rotation Operators			
Rotation through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
Counterclockwise rotation about the positive x-axis through an angle θ		$w_1 = x$ $w_2 = y\cos\theta - z\sin\theta$ $w_3 = y\sin\theta + z\cos\theta$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$
Counterclockwise rotation about the positive y-axis through an angle θ		$w_1 = x\cos\theta + z\sin\theta$ $w_2 = y$ $w_3 = -x\sin\theta + z\cos\theta$	$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$
Counterclockwise rotation about the positive z-axis through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$ $w_3 = z$	$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

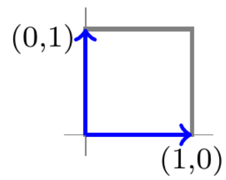
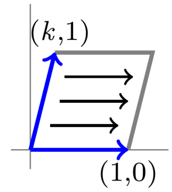
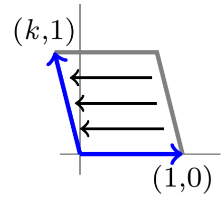
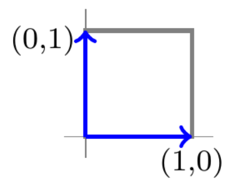
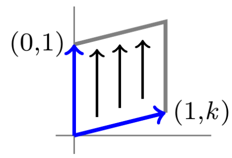
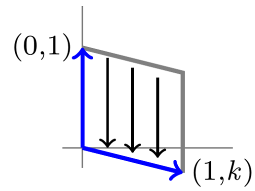
Contractions and Dilations

Contraction with factor k on $\mathbb{R}^2$ $(0 \le k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$
Dilation with factor k on $\mathbb{R}^2$ $(k > 1)$			
Contraction with factor k on $\mathbb{R}^2$ $(0 \le k \le 1)$			$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$
Dilation with factor k on $\mathbb{R}^2$ $(k \ge 1)$			

Expansion or Compression

Compression of $\mathbb{R}^2$ in the x–direction with factor k $(0 \le k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Expansion of $\mathbb{R}^2$ in the x–direction with factor k $(k > 1)$			
Compression of $\mathbb{R}^2$ in the y–direction with factor k $(0 \le k < 1)$			$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
Expansion of $\mathbb{R}^2$ in the y–direction with factor k $(k > 1)$			

Shear

Shear of $\mathbb{R}^2$ in the x–direction with factor k $T(x, y) = (x + ky, y)$		 $(k > 0)$	 $(k < 0)$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Shear of $\mathbb{R}^2$ in the y–direction with factor k $T(x, y) = (x, y + kx)$		 $(k > 0)$	 $(k < 0)$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Orthogonal Projections on Lines through the Origin

$$T\left(e_1\right)=\begin{bmatrix}\cos^2\theta\\ \sin\theta\cos\theta\end{bmatrix}\quad T\left(e_2\right)=\begin{bmatrix}\sin\theta\cos\theta\\ \sin^2\theta\end{bmatrix}$$

$$P_0=\begin{pmatrix}\cos^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{pmatrix}$$

Example

Find the orthogonal projection of the vector $\vec{x}=(1,5)$ on the line through the origin that makes an angle of $\frac{\pi}{6}\ (=30°)$ with the x-axis

Solution

$$\begin{aligned}P_0&=\begin{pmatrix}\cos^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{pmatrix}\\
&=\begin{pmatrix}\cos^2\left(\frac{\pi}{6}\right) & \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right)\\ \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right) & \sin^2\left(\frac{\pi}{6}\right)\end{pmatrix}\\
&=\begin{pmatrix}\left(\frac{\sqrt{3}}{2}\right)^2 & \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\ \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) & \left(\frac{1}{2}\right)^2\end{pmatrix}\\
&=\begin{pmatrix}\frac{3}{4} & \frac{\sqrt{3}}{4}\\ \frac{\sqrt{3}}{4} & \frac{1}{4}\end{pmatrix}\end{aligned}$$

$$\begin{aligned}P_0\vec{x}&=\begin{pmatrix}\frac{3}{4} & \frac{\sqrt{3}}{4}\\ \frac{\sqrt{3}}{4} & \frac{1}{4}\end{pmatrix}\begin{pmatrix}1\\5\end{pmatrix}\\
&=\begin{pmatrix}\frac{3+5\sqrt{3}}{4}\\ \frac{\sqrt{3}+5}{4}\end{pmatrix}\\
&\approx\boxed{\begin{pmatrix}2.91\\1.68\end{pmatrix}}\end{aligned}$$

Example

Define a linear transformation $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$T(\vec{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

Find the images under T of $\vec{u} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\vec{u} + \vec{v} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$

Solution

$$T(\vec{u}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

$$T(\vec{v}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

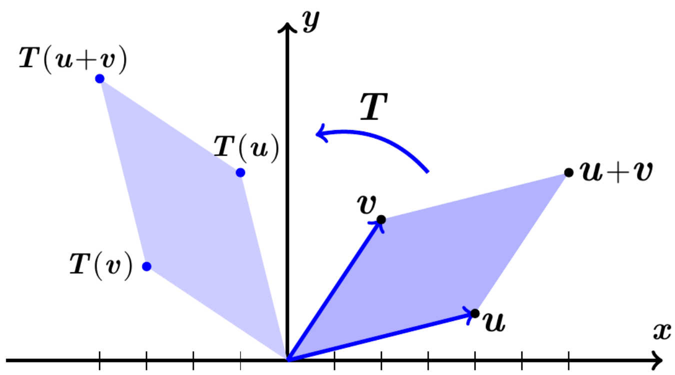
$$= \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$T(\vec{u} + \vec{v}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ 6 \end{pmatrix}$$

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$$

$$\begin{pmatrix} -4 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$



Four Fundamental Subspaces

1. The ***row space*** is $C\left(A^T\right)$, a subspace of $\mathbb{R}^n$.
2. The ***column space*** is $C(A)$, a subspace of $\mathbb{R}^m$.
3. The ***nullspace*** is $N(A)$, a subspace of $\mathbb{R}^n$.
4. The ***left nullspace*** is $N\left(A^T\right)$, a subspace of $\mathbb{R}^m$.

The Four Subspaces for R

Consider the matrix 3 by 5:

$$\begin{bmatrix} 1 & 3 & 5 & 0 & 9 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} m=3 & \quad \textit{pivot rows } 1 \textit{ and } 2 \\ n=5 & \\ r=2 & \quad \textit{pivot columns } 1 \textit{ and } 4 \end{matrix}$$

1. Rows 1 and 2 are a basis. The row space contains combination of all 3 rows.

   The ***row space*** of $\mathbb{R}$ has dimension 2 (= ***rank***).
   The dimension of the row space is r. The nonzero rows of ***R*** form a basis.

2. The ***column space*** of ***R*** has dimension $r = 2$.

   The pivot columns 1 and 4 form a basis. They are independent because they start with the r by r identity matrix.
   There are 3 special solutions:

$$\begin{aligned} C_2 &= 3C_1 && \textit{The special solution is } (-3,\ 1,\ 0,\ 0,\ 0) \\ C_3 &= 5C_1 && \textit{The special solution is } (-5,\ 0,\ 1,\ 0,\ 0) \\ C_5 &= 9C_1 + 8C_2 && \textit{The special solution is } (-9,\ 0,\ 0,\ -8,\ 1) \end{aligned}$$

   The dimension of the column space is r. The pivot columns form a basis.

3. The ***nullspace*** has dimension $n - r = 5 - 2 = 3$ (free variables). Here x_2, x_3, x_5 are free (no pivots in those columns). They yield the three special solutions to $R\vec{x} = 0$. Set a free variable to 1, and solve for x_1 and x_4.

$$s_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad s_3 = \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad s_5 = \begin{bmatrix} -9 \\ 0 \\ 0 \\ -8 \\ 1 \end{bmatrix}$$

$Rx = 0$ has the complete solution: $x = x_2 s_2 + x_3 s_3 + x_5 s_5$

The nullspace has dimension $n - r$. The special solutions form a basis.

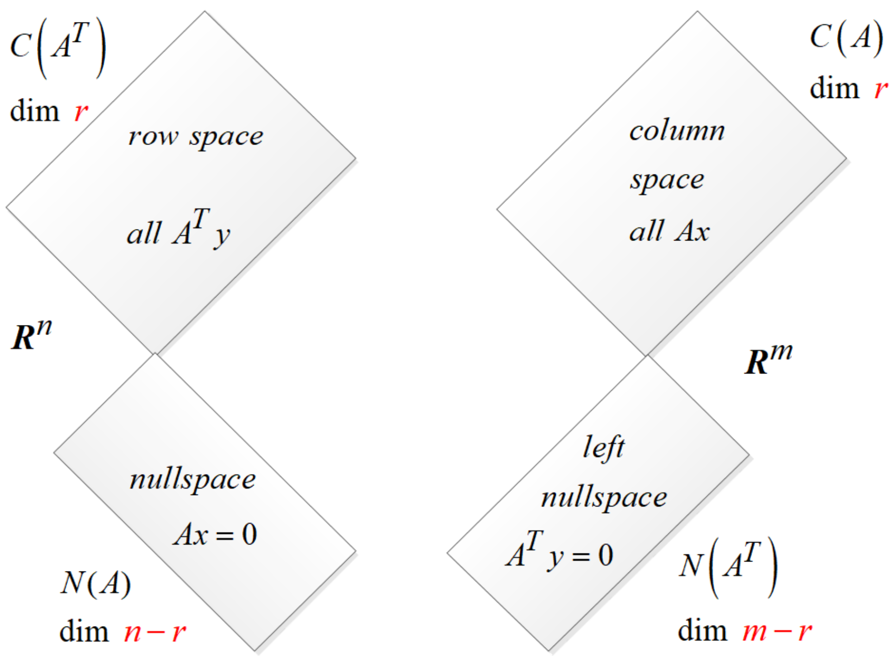
4. The ***nullspace*** of R^T has dimension $m - r = 3 - 2 = 1$

The equation $R^T y = 0$: $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 0 & 0 \\ 0 & 1 & 0 \\ 9 & 8 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} y_1 = 0 \\ y_2 = 0 \\ y_3 \ \textit{anything} \end{cases}$

The nullspace of R^T contains all vectors $y = (0,\ 0,\ y_3)$ and it is the line of the basis vector $(0,\ 0,\ 1)$.

The left nullspace has dimension m − r. The solutions are $y = (0, \ldots, y_{r+1}, \ldots, y_m)$

- In $\mathbb{R}^n$ the row space and nullspace have dimensions r and $n - r$ (adding to n)
- In $\mathbb{R}^m$ the column space and left nullspace have dimensions r and $m - r$ (total m)



The Four Subspaces for A

The subspace dimensions for A are the same as for R.

These matrices are connected by an invertible matrix E. $EA = R$ *and* $A = E^{-1}R$

1. ***A** has the same row space as **R**.* Same dimension r and same basis
   Every row of ***A*** is a combination of the rows of ***R***. Also every row of ***R*** is a combination of the rows of ***A***.

2. *The column space of **A** has dimension r.* The number of independent columns equals the number of independent rows.

3. ***A** has the same nullspace as **R**.* Dimension $n - r$ and same basis.

   (*dimension of column space*) + (*dimension of nullspace*) = *dimension of* $\boldsymbol{R}^n$

4. *The left nullspace **A*** (the nullspace of A^T) *has dimension* $m - r$.

Fundamental Theorem of Linear Algebra, (Part 1)

The column space and row space both have dimension r.

The nullspaces have dimensions $n - r$ and $m - r$.

Example

Consider $A = [1 \quad 2 \quad 3]$

A has $m = 1$, $n = 3$, and rank: $r = 1$.

The row space is a line in $\mathbb{R}^3$.

The nullspace is the plane $Ax = x_1 + 2x_2 + 3x_3 = 0$. This plane has dimension 2 (which is 3 – 1).

The columns of this is 1 by 3 matrix are in $\mathbb{R}^1$. The column space is all of $\mathbb{R}^1$.

The left nullspace contains only the zero vector.

The only solution to $A^T y = 0$ *is* $y = 0$, the only combination of the row that gives the zero row.

Thus, $N\left(A^T\right)$ *is* $\mathbb{Z}$, the zero space with dimension 0 ($m - r$). In $\mathbb{R}^m$ the dimensions (1 + 0) = 1.

Example

Consider $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

A has $m = 2$, $n = 3$, and rank: $r = 1$.

The row space is a line in $\mathbb{R}^3$.

The nullspace is the plane $x_1 + 2x_2 + 3x_3 = 0$. This plane has dimension 3 (1 + 2).

The columns are multiples of the first column (1, 1).

The left nullspace contains more than one zero vector. The solution to $A^T \vec{y} = 0$ has the solution $y = (1, \ -1)$.

The column space and nullspace are perpendicular lines in $\mathbb{R}^2$. Their dimensions are 1 and 1 = 2.

Column space = line through $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Left nullspace = line through $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Exercises Section 4.1 – Matrix Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

1. Find the standard matrix for the transformation defined by the equations

a) $\begin{cases} w_1 = 2x_1 - 3x_2 + 4x_4 \\ w_2 = 3x_1 + 5x_2 - x_4 \end{cases}$

b) $\begin{cases} w_1 = 7x_1 + 2x_2 - 8x_3 \\ w_2 = \quad\quad - x_2 + 5x_3 \\ w_3 = 4x_1 + 7x_2 - x_3 \end{cases}$

c) $\begin{cases} w_1 = x_1 \\ w_2 = x_1 + x_2 \\ w_3 = x_1 + x_2 + x_3 \\ w_4 = x_1 + x_2 + x_3 + x_4 \end{cases}$

(2 – 8) Find the standard matrix for the operator T defined by the formula

2. $T(x_1, x_2) = (2x_1 - x_2,\ x_1 + x_2)$

3. $T(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3,\ x_1 + 5x_2,\ x_3)$

4. $T(x_1, x_2, x_3) = (4x_1,\ 7x_2,\ -8x_3)$

5. $T(x_1, x_2) = (x_2,\ -x_1,\ x_1 + 3x_2,\ x_1 - x_2)$

6. $T(x_1, x_2, x_3) = (0,\ 0,\ 0,\ 0,\ 0)$

7. $T(x_1, x_2, x_3, x_4) = (x_4,\ x_1,\ x_3,\ x_2,\ x_1 - x_3)$

8. $T(x_1, x_2, x_3, x_4) = (7x_1 + 2x_2 - x_3 + x_4,\ x_2 + x_3,\ -x_1)$

(9 – 8) Plot $\vec{u} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and their images under the given transformation T

9. $T(\vec{x}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

10. $T(\vec{x}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

11. $T(\vec{x}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

12. $T(\vec{x}) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

13. $T(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Section 4.2 – General Linear Transformations

Definition

A transformation T assigns an output $T(\vec{v})$ to each input vector $\vec{v}$. The transformation is ***linear*** if it meets these requirements for all $\vec{v}$ and $\vec{w}$:

$$\begin{cases} T(c\vec{v}) = cT(\vec{v}) \\ T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w}) \end{cases}$$

We can combine both into one: $\boxed{T(c\vec{v}+d\vec{w}) = cT(\vec{v}) + dT(\vec{w})}$

Theorem

If $T: V \to W$ is a linear transformation, then:

1. $T(0) = 0$
2. $T(\vec{u}-\vec{v}) = T(\vec{u}) - T(\vec{v})$ for all $\vec{u}$ and $\vec{v}$ in V.

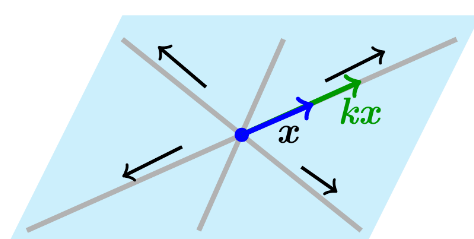
Example

If V is a vector space and k is any scalar, then the mapping $T: V \to W$ given by $T(\vec{x}) = k\vec{x}$ is a linear operator on V, for if c is any scalar and if $\vec{u}$ and $\vec{v}$ are any vectors in V, then

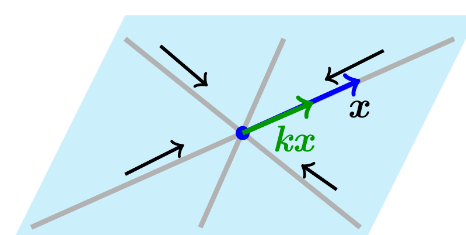
$$\begin{aligned} T(c\vec{u}) &= k(c\vec{u}) \\ &= c(k\vec{u}) \\ &= cT(\vec{u}) \end{aligned}$$

$$\begin{aligned} T(\vec{u}+\vec{v}) &= k(\vec{u}+\vec{v}) \\ &= k\vec{u} + k\vec{v} \\ &= T(\vec{u}) + T(\vec{v}) \end{aligned}$$

If $0 < k < 1$, then T is called ***contraction*** of V with factor k, and if $k > 1$, then T is called ***dilation*** of V with factor k



Dilation of V



Contraction of V

Example

Determine if the given function T is a linear transformation. Also give the domain and range of T; if T is linear, find the A such $T = f_A$. $\quad T(x, y, z) = (z - x, z - y)$

Solution

Let $\vec{u} = (x_1, y_1, z_1)$ *and* $\vec{v} = (x_2, y_2, z_2)$

$$
\begin{aligned}
T(\vec{u} + \vec{v}) &= T(x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2) \\
&= (z_1 + z_2 - (x_1 + x_2),\ z_1 + z_2 - (y_1 + y_2)) \\
&= (z_1 + z_2 - x_1 - x_2,\ z_1 + z_2 - y_1 - y_2) \\
&= (z_1 - x_1,\ z_1 - y_1) + (z_2 - x_2,\ z_2 - y_2) \\
&= T(x_1, y_1, z_1) + T(x_2, y_2, z_2) \\
&= T(\vec{u}) + T(\vec{v})
\end{aligned}
$$

$$
\begin{aligned}
T(r\vec{u}) &= T(rx_1, ry_1, rz_1) \\
&= (rz_1 - rx_1,\ rz_1 - ry_1) \\
&= r(z_1 - x_1,\ z_1 - y_1) \\
&= rT(\vec{u})
\end{aligned}
$$

Since $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(r\vec{u}) = rT(\vec{v})$

Then function T is a linear transformation.

Domain: $T:\ \mathbb{R}^3 \to \mathbb{R}^2$

$$
\begin{aligned}
T(x, y, z) &= (z - x, z - y) \\
&= \begin{pmatrix} -x + z \\ -y + z \end{pmatrix}
\end{aligned}
$$

$$
A = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \end{array}
$$

Example – the Zero Transformations

Let V and W be any vector spaces. The mapping $T:V\to W$ such that $T(v)=0$ for every $\vec{v}$ in V is a linear transformation called the zero transformation. To see that T is linear, observe that:

$$T\left(\vec{u}+\vec{v}\right)=0,\quad T\left(\vec{u}\right)=0,\quad T\left(\vec{v}\right)=0,\quad and\quad T\left(k\vec{u}\right)=0$$

Therefore; $\quad T\left(\vec{u}+\vec{v}\right)=T\left(\vec{u}\right)+T\left(\vec{v}\right)\quad and\quad T\left(k\vec{u}\right)=kT\left(\vec{u}\right)$

Example

Choose a fixed vector $\vec{a}=\left(1,\ 3,\ 4\right)$, and let $T(v)$ be the dot product $\vec{a}\bullet\vec{v}$:

Solution

Let $\vec{v}=\left(v_1,\ v_2,\ v_3\right)$

$$\begin{aligned}T\left(\vec{v}\right)&=\vec{a}\bullet\vec{v}\\&=\left(1,\ 3,\ 4\right)\bullet\left(v_1,\ v_2,\ v_3\right)\\&=v_1+3v_2+4v_3\end{aligned}$$

This is linear. The inputs v come from three–dimensional space, so $V=\mathbb{R}^3$. The output just numbers, so the output space is $W=\mathbb{R}^1$. We are multiplying by the row matrix A = [1, 3, 4].
Then $T(\vec{v})=A\vec{v}$

Example

Show that the length $T\left(\vec{v}\right)=\left\|\vec{v}\right\|$ is not linear.

Solution

$$\left\|\vec{v}+\vec{w}\right\|\overset{?}{=}\left\|\vec{v}\right\|+\left\|\vec{w}\right\|$$

There are not equal because the sides of a triangle satisfy an inequality $\left\|\vec{v}+\vec{w}\right\|\le\left\|\vec{v}\right\|+\left\|\vec{w}\right\|$

$$\left\|c\vec{v}\right\|\overset{?}{=}c\left\|\vec{v}\right\|$$

Not - because the length $\left\|-\vec{v}\right\|\ne-\left\|\vec{v}\right\|$

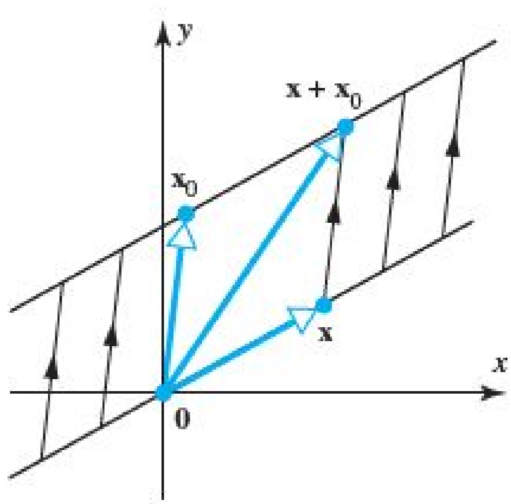
Example

If $\vec{x}_0$ is a fixed nonzero vector in $\mathbb{R}^2$, then the transformation

$$T(\vec{x}) = \vec{x} + \vec{x}_0$$

It has a geometric effect of translating each point $\vec{x}$ in a direction parallel to $\vec{x}_0$ through a distance of $\|\vec{x}_0\|$.

This cannot be a linear transformation since $T(0) = \vec{x}_0$



Theorem

Let $T : V \to W$ be the linear transformation, where *V* is finite dimensional. If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for *V*, then the image of any vector $\vec{v}$ in *V* can be expressed as

$$T(\vec{v}) = c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \cdots + c_n T(\vec{v}_n)$$

Where $c_1, c_2, \cdots, c_n$ are the coefficients required to express $\vec{v}$ as a linear combination of the vectors in *S*.

Example

Consider the basis $S=\left\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\right\}$ for ∞^3, where

$$\vec{v}_1=(1,\ 1,\ 1) \quad \vec{v}_2=(1,\ 1,\ 0) \quad \vec{v}_3=(1,\ 0,\ 0)$$

Let $T:\ \mathbb{R}^3\to\mathbb{R}^2$ be the linear transformation for which

$$T\left(\vec{v}_1\right)=(1,\ 0), \quad T\left(\vec{v}_2\right)=(2,\ -1), \quad T\left(\vec{v}_3\right)=(4,\ 3)$$

Find a formula for $T\left(\vec{x}_1, \vec{x}_2, \vec{x}_3\right)$, and then use that formula to compute $T(2,\ -3,\ 5)$

Solution

$$\left(\vec{x}_1, \vec{x}_2, \vec{x}_3\right)=c_1(1,\ 1,\ 1)+c_2(1,\ 1,\ 0)+c_3(1,\ 0,\ 0)$$

$$\begin{cases} c_1+c_2+c_3=x_1 \\ c_1+c_2 \qquad\ \ =x_2 \\ c_1 \qquad\qquad\ \ =x_3 \end{cases}$$

$$\begin{cases} c_3=x_1-x_2 \\ c_2=x_2-x_3 \\ c_1=x_3 \end{cases}$$

$$\begin{aligned}\left(\vec{x}_1, \vec{x}_2, \vec{x}_3\right)&=x_3(1,\ 1,\ 1)+\left(x_2-x_3\right)(1,\ 1,\ 0)+\left(x_1-x_2\right)(1,\ 0,\ 0)\\ &=x_3\vec{v}_1+\left(x_2-x_3\right)\vec{v}_2+\left(x_1-x_2\right)\vec{v}_3\end{aligned}$$

$$\begin{aligned}T\left(\vec{x}_1, \vec{x}_2, \vec{x}_3\right)&=x_3T\left(\vec{v}_1\right)+\left(x_2-x_3\right)T\left(\vec{v}_2\right)+\left(x_1-x_2\right)T\left(\vec{v}_3\right)\\ &=x_3(1,\ 0)+\left(x_2-x_3\right)(2,\ -1)+\left(x_1-x_2\right)(4,\ 3)\\ &=\left(4x_1-2x_2-x_3,\ 3x_1-4x_2+x_3\right)\end{aligned}$$

$$\begin{aligned}T(2,-3,5)&=(4(2)-2(-3)-5,\ 3(2)-4(-3)+5)\\ &=(9,\ 23)\end{aligned}$$

Example

T is the transformation that rotates every vector by 30°, the domain is the *xy*-plane (where the input vector $\vec{v}$ is). The range is also the *xy*-plane (where the rotated $T(\vec{v})$ is). Is the rotation linear?

Solution

Yes it is. We can rotate two vectors and add the results. The sum of rotation $T(\vec{v})+T(\vec{w})$ is the same as the rotation $T(\vec{v}+\vec{w})$ of the sum.

The whole plane is turning together, in this linear transformation.

Definition

If $T:V\to W$ is a linear transformation, then the set of vectors in *V* that *T* maps into $\vec{0}$ is called ***kernel*** of *T* and is denoted by ***ker***(*T*). The set of all vectors in *W* that are images under *T* of at least one vector in *V* is called the ***range*** of *T* and is denoted by ***R***(*T*).

Note:

Transformations have a language of their own. Where there is no matrix, we can't talk about a column space. But the idea can be rescued and used. The column space consisted of all ouputs $A\vec{v}$.

The nullspace consisted of all inputs for which $A\vec{v}-\vec{0}$. Translate those into "range" and "kernel"

Range of T = set of all outputs $T(\vec{v})$: corresponds to column space

Kernel of T = set of all outputs for which $T(\vec{v})=0$: corresponds to nullspace

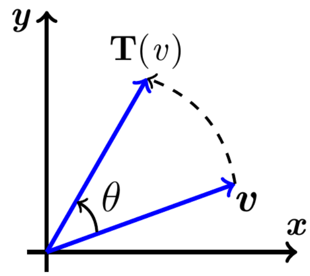
Example

Project every 3-dimensional vector down onto the *xy* plane.
The range is that plane, which contains every $T(\vec{v})$.

The kernel is the *z* axis (which projects down to zero). This projection is linear.

Example – *Kernel and Range of a Rotation*

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator that rotates each vector in the xy–plane through the angle θ. Since every vector in the xy–plane can be obtained by rotating some vector through the angle θ, it follows that $R(T) = \mathbb{R}^2$.



Moreover, the only vector that rotates into $\vec{0}$ is $\vec{0}$, so $\ker(T) = \{0\}$

Theorem

If $T: V \to W$ is a linear transformation, then:

1. The *kernel* of T is a subspace of V
2. The *range* of T is a subspace of W

Theorem

If $T: V \to W$ is a linear transformation from an n-dimensional vector space V to a vector space W, then

$$rank(T) + nullity(T) = n$$

Example

Project every 3-dimensional vector down onto horizontal plane $z = 1$.
The vector $\vec{v} = (x,\ y,\ z)$ is transformed to $T(\vec{v}) = (x,\ y,\ 1)$. This transformation is not linear, it doesn't even transform $\vec{v} = \vec{0}$ into $T(\vec{v}) = \vec{0}$.

Multiply every 3-dimensional vector by a 3 by 3 matrix A. This is definitely a linear transformation $T(\vec{v}+\vec{w}) = A(\vec{v}+\vec{w})$ which does equal $A\vec{v} + A\vec{w} = T(\vec{v}) + T(\vec{w})$

Example

Suppose A is an invertible matrix. The kernel of T is the zero vector; the range $\boldsymbol{W}$ equals the domain $\boldsymbol{V}$. Another linear transformation is multiplication by A^{-1}.

This is the inverse transformation T^{-1}, which brings every vector $T(\vec{v})$ back to $\vec{v}$:

$$T^{-1}\left(T(\vec{v})\right)=\vec{v} \quad \text{matches the matrix multiplication } A^{-1}\left(A\vec{v}\right)=\vec{v}$$

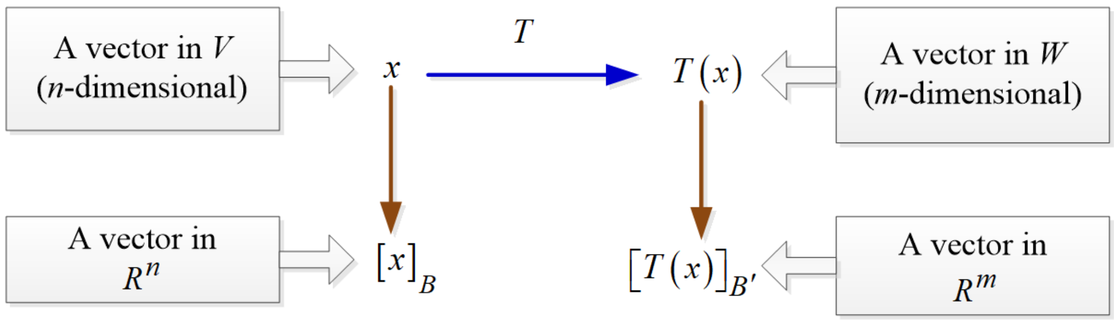
Are all linear transformation produced by matrices?

Each m by n matrix does produce a linear transformation from $V=\mathbb{R}^n$ to $W=\mathbb{R}^m$. When a linear T is described as a “rotation” or “projection” or “...” is there always a matrix hiding behind T?

The answer is yes. This is an approach to linear algebra that doesn’t start with matrices. The next section shows that we still end up with matrices.

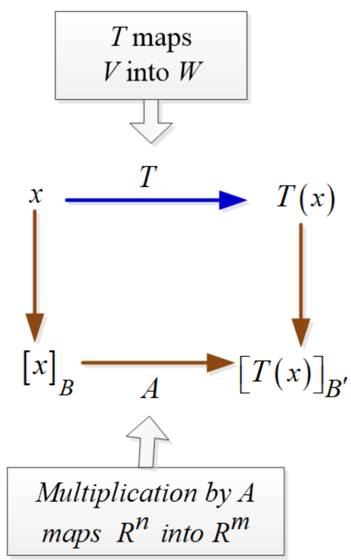
Matrices for General Linear Transformations

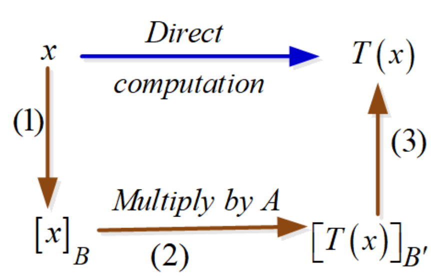
Suppose that V is an n-dimensional vector space, W is an m-dimensional vector space, and that $T: V \to W$ is a linear transformation. Suppose further that B is a basis for V, that B' is a basis for W, and that for each $\boldsymbol{x}$ in V, the coordinate matrices for $\boldsymbol{x}$ and $T(\vec{x})$ are $[\vec{x}]_B$ and $[T(\vec{x})]_{B'}$, respectively



By using matrix multiplication, we can execute the linear transformation and the following indirect procedure:

1. Compute the coordinate vector $[\vec{x}]_B$
2. Multiply $[\vec{x}]_B$ on the left by A to produce $[T(\vec{x})]_{B'}$
3. Reconstruct $T(\vec{x})$ from its coordinate vector $[T(\vec{x})]_{B'}$





$$A[\vec{x}]_B = [T(\vec{x})]_{B'}$$

Example

Let $T : P_1 \to P_2$ be the linear transformation defined by $T\left(p(x)\right) = xp(x)$

Find the matrix for T with respect to the standard bases

$$B = \left\{\vec{u}_1,\ \vec{u}_2\right\} \quad and \quad B' = \left\{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\right\}$$

Where $\vec{u}_1 = 1, \quad \vec{u}_2 = x; \quad \vec{v}_1 = 1, \quad \vec{v}_2 = x, \quad \vec{v}_3 = x^2$

Solution

$$\begin{aligned} T\left(\vec{u}_1\right) &= T(1) \\ &= x \end{aligned}$$

$$\begin{aligned} T\left(\vec{u}_2\right) &= T(x) \\ &= x(x) \\ &= x^2 \end{aligned}$$

$$\left[T\left(\vec{u}_1\right)\right]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\left[T\left(\vec{u}_2\right)\right]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The matrix for T with respect to B and B' is

$$\begin{aligned} [T]_{B,B'} &= \left[\left[T\left(\vec{u}_1\right)\right]_{B'} \middle| \left[T\left(\vec{u}_2\right)\right]_{B'}\right] \\ &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Example

Let $T : P_1 \to P_2$ be the linear transformation defined by $T\left(p(x)\right) = xp(x)$ describe in the following figure to perform the computation

$$T(a+bx) = x(a+bx) = ax + bx^2$$

$$\begin{tikzcd}
x \arrow[rr, "\text{Direct}", "\text{computation}"'] \arrow[d, "(1)"'] & & T(x) \\
{[x]_B} \arrow[rr, "\text{Multiply by } [T]_{B,B'}", "(2)"'] & & {[T(x)]_{B'}} \arrow[u, "(3)"']
\end{tikzcd}$$

$B = \{1,\ x\}$ *and* $B' = \left\{1,\ x,\ x^2\right\}$

<u>Solution</u>

Step 1: The coordinates matrix for $\vec{x} = ax + b$ relative to the basis $B = \{1,\ x\}$ is

$$[\boldsymbol{x}]_B = \begin{bmatrix} a \\ b \end{bmatrix}$$

Step 2: Multiply $[\vec{x}]_B$ by the matrix $[T]_{B,B'}$ found in previous example, we obtain

$$[T]_{B,B'}[\vec{x}]_B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ a \\ b \end{bmatrix}$$

$$= \left[T(\vec{x})\right]_{B'}$$

Step 3: Reconstructing $T(\vec{x}) = T(ax+b)$ from $\left[T(\vec{x})\right]_{B'}$ we obtain

$$T(ax+b) = 0 + ax + bx^2$$

$$= ax + bx^2$$

Exercises Section 4.2 – General Linear Transformations

1. The matrix $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ gives a shearing transformation $T(x,y)=(x,\ 3x+y)$.

What happens to (1, 0) and (2, 0) on the x-axis.
What happens to the points on the vertical lines $x = 0$ and $x = a$?.

2. A nonlinear transformation T is invertible if every $\vec{b}$ in the output space comes from exactly one x in the input space. $T(\vec{x})=\vec{b}$ always has exactly one solution. Which of these transformation (on real numbers $\vec{x}$ is invertible and what is T^{-1}? None are linear, not even T_3. When you solve $T(\vec{x})=\vec{b}$, you are inverting T:

$$T_1(\vec{x})=x^2 \quad T_2(\vec{x})=x^3 \quad T_3(\vec{x})=x+9 \quad T_4(\vec{x})=e^x \quad T_5(\vec{x})=\frac{1}{x} \quad \textit{for nonzero } x\text{'s}$$

3. If S and T are linear transformations, is $S(T(\vec{v}))$ linear or quadratic?

a) If $S(\vec{v})=\vec{v}$ and $T(\vec{v})=\vec{v}$, then $S(T(\vec{v}))=\vec{v}$ or $\vec{v}^2$?

b) $S(\vec{w}_1+\vec{w}_2)=S(\vec{w}_1)+S(\vec{w}_2)$ and $T(\vec{v}_1+\vec{v}_2)=T(\vec{v}_1)+T(\vec{v}_2)$ combine into

$$S(T(\vec{v}_1+\vec{v}_2))=S(\_\_\_\_\_)=\_\_\_\_+\_\_\_\_\_\_$$

4. Find the range and kernel (like the column space and nullspace) of T:

a) $T(v_1,\ v_2)=(v_2,\ v_1)$

b) $T(v_1,\ v_2,\ v_3)=(v_1,\ v_2)$

c) $T(v_1,\ v_2)=(0,\ 0)$

d) $T(v_1,\ v_2)=(v_1,\ v_1)$

5. M is any 2 by 2 matrix and $A=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. The transformation T is defined by $T(M)=AM$. What rules of matrix multiplication show that T is linear?

6. Which of these transformations satisfy $T(\vec{v}+\vec{w})=T(\vec{v})+T(\vec{w})$ and which satisfy $T(c\vec{v})=cT(\vec{v})$?

a) $T(\vec{v})=\dfrac{\vec{v}}{\|\vec{v}\|}$

b) $T(\vec{v})=v_1+v_2+v_3$

c) $T(\vec{v})=(v_1,\ 2v_2,\ 3v_3)$

d) $T(\vec{v})=$ largest component of $\vec{v}$.

7. Consider the basis $S=\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ for R^3, where $\vec{v}_1=(1,\ 1,\ 1)$ $\vec{v}_2=(1,\ 1,\ 0)$ $\vec{v}_3=(1,\ 0,\ 0)$ and let $T:\ \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation for which

$$T(\vec{v}_1)=(2,\ -1,\ 4),\quad T(\vec{v}_2)=(3,\ 0,\ 1),\quad T(\vec{v}_3)=(-1,\ 5,\ 1)$$

Find a formula for $T(\vec{x}_1, \vec{x}_2, \vec{x}_3)$, and then use that formula to compute $T(2,\ 4,\ -1)$

8. Consider the basis $S=\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ for R^3, where $\vec{v}_1=(1,\ 2,\ 1)$ $\vec{v}_2=(2,\ 9,\ 0)$ $\vec{v}_3=(3,\ 3,\ 4)$ and let $T:\ \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation for which

$$T(\vec{v}_1)=(1,\ 0),\quad T(\vec{v}_2)=(-1,\ 1),\quad T(\vec{v}_3)=(0,\ 1)$$

Find a formula for $T(\vec{x}_1, \vec{x}_2, \vec{x}_3)$, and then use that formula to compute $T(7,\ 13,\ 7)$

9. let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be vectors in a vector space V, and let $T: V \to R^3$ be the linear transformation for which $T(\vec{v}_1)=(1,\ -1,\ 2)$, $T(\vec{v}_2)=(0,\ 3,\ 2)$, $T(\vec{v}_3)=(-3,\ 1,\ 2)$.
Find $T(2\vec{v}_1-3\vec{v}_2+4\vec{v}_3)$

10. Let $T:\ \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operation given by the formula $T(x,\ y)=(2x-y,\ -8x+4y)$
Which of the following vectors are in $R(T)$

a) $(1,\ -4)$ ***b***) $(5,\ 0)$ ***c***) $(-3,\ 12)$

11. Let $T:\ \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operation given by the formula $T(x,\ y)=(2x-y,\ -8x+4y)$
Which of the following vectors are in $\ker(T)$

a) $(5,\ 10)$ ***b***) $(3,\ 2)$ ***c***) $(1,\ 1)$

12. Let $T:\ \mathbb{R}^4 \to \mathbb{R}^3$ be the linear operation given by the formula

$$T(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4)=(4x_1+x_2-2x_3-3x_4,\ 2x_1+x_2+x_3-4x_4,\ 6x_1-9x_3+9x_4)$$

Which of the following vectors are in $R(T)$

a) $(0,\ 0,\ 6)$ ***b***) $(1,\ 3,\ 0)$ ***c***) $(2,\ 4,\ 1)$

13. Let $T:\ \mathbb{R}^4 \to \mathbb{R}^3$ be the linear operation given by the formula

$$T(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4)=(4x_1+x_2-2x_3-3x_4,\ 2x_1+x_2+x_3-4x_4,\ 6x_1-9x_3+9x_4)$$

Which of the following vectors are in $\ker(T)$

a) $(3,\ -8,\ 2,\ 0)$ ***b***) $(0,\ 0,\ 0,\ 1)$ ***c***) $(0,\ -4,\ 1,\ 0)$

14. Determine if the given function T is a linear transformation

$$T : M_{22} \to M_{22} \quad by \quad T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2ab & 3cd \\ 0 & 0 \end{bmatrix}$$

15. Determine if the given function T is a linear transformation

$$T : M_{22} \to M_{22} \quad by \quad T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+d & 0 \\ 0 & b+c \end{bmatrix}$$

16. Determine if the given function T is a linear transformation where A is fixed 2×3 matrix

$$T : M_{22} \to M_{23} \quad by \quad T(B) = BA$$

(17 – 25) Determine if the given function T is a linear transformation. Also give the domain and range of T; if T is linear, find the A such $T = f_A$.

17. $T(x, y) = (x^2, y)$

18. $T(x, y, z) = (2x + y,\ x - y + z)$

19. $T(x, y, z) = (z - x,\ z - y)$

20. $T(x_1, x_2, x_3) = (2x_1 - x_2 + x_3,\ \ x_2 - 4x_3)$

21. $T(x_1, x_2) = (2x_1 - x_2,\ -3x_1 + x_2,\ 2x_1 - 3x_2)$

22. $T(x_1, x_2) = (x_1 + 4x_2,\ 0,\ x_1 - 3x_2,\ x_1)$

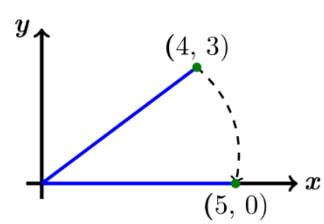
23. $T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3,\ \ x_2 - 6x_3)$

24. $T(x_1, x_2, x_3, x_4) = (x_1 + 2x_2,\ 0,\ 2x_2 + x_4,\ x_2 - x_4)$

25. $T(x_1, x_2, x_3, x_4) = 3x_1 + 4x_3 - 2x_4$

26. A Givens rotation is a linear transformation from $\mathbb{R}^n$ *to* $\mathbb{R}^n$ used in computer to create a zero entry in a vector (usually a column of a matrix). The standard matrix of a Givens rotation in $\mathbb{R}^2$ has the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}; \qquad a^2 + b^2 = 1$$



A Givens rotation in $\mathbb{R}^2$

Find a and b that $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ is rotated into $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$.

Section 4.3 – LU-Decompositions

The goal is to describe Gaussian elimination in the most useful way by looking at them closely, which are factorizations of a matrix.

The factors are triangular matrices.

The factorization that comes from elimination is $A = LU$.

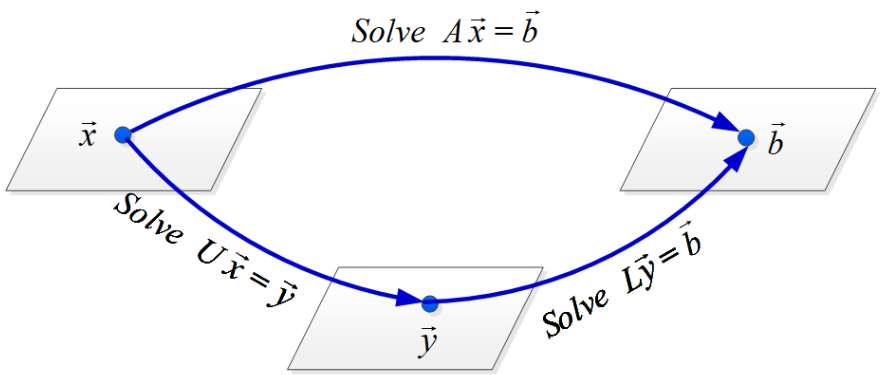
The Method of *LU*–Decomposition

Step 1: Rewrite the system $A\vec{x} = \vec{b}$ as $LU\vec{x} = \vec{b}$

Step 2: Define a new $n \times 1$ matrix $\vec{y}$ by $U\vec{x} = \vec{y}$

Step 3: Use $U\vec{x} = \vec{y}$ to rewrite $LU\vec{x} = \vec{b}$ as $L\vec{y} = \vec{b}$ and solve this system for $\vec{y}$.

Step 4: Substitute $\vec{y}$ in $U\vec{x} = \vec{y}$ and solve for $\vec{x}$.



Example

Given 2 by 2 matrix $A = \begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix}$

Find *L* and *U* and verify $A = LU$

Solution

To make ***row*** 2 ***column*** 1 is ***zero*** then we need to subtract 3 times *row* 2 from *row* 2

$$\begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix} \quad R_2 - 3R_1$$

$$\ell_{21} = -3$$

That step is $E_{21} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$ in the forward direction such that:

$$E_{21}A = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix} = U$$

The return step from U to A is $L = E_{21}^{-1}$

$$L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

Back from U to A:

$$E_{21}^{-1}U = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix}$$

$$\underline{= A}$$

Therefore; $A = LU$

Example

What matrix L and U puts A into triangular form $A = LU$ where

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad R_2 - \tfrac{1}{2}R_1 : \ell_{21}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & \tfrac{3}{2} & 1 \\ 0 & 1 & 2 \end{pmatrix} R_3 - \tfrac{2}{3}R_2 : \ell_{32}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & \tfrac{3}{2} & 1 \\ 0 & 0 & \tfrac{4}{3} \end{pmatrix} = U$$

$\ell_{21} = -\frac{1}{2} \quad \ell_{32} = -\frac{2}{3}$

The lower triangular L has all 1's on its diagonal. The multipliers ℓ_{ij} ***are below the diagonal of L with OPPOSITE sign***

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}} = \underset{L}{\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}}$$

- $\left(E_{32}E_{31}E_{21}\right)A = U$ *becomes* $A = \left(E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}\right)U$ *which is* $A = LU$

  The inverses go in opposite order.

- $\left(A = LU\right)$ This is ***elimination without row exchanges***. The *upper triangular* ***U*** has the pivots on its diagonal. The *lower triangular* ***L*** has all 1's on its diagonal.
  The multipliers ℓ_{ij} ***are below the diagonal of L***.

One Square System = *Two* Triangular Systems

Factor: into L and U, by forward elimination on A.

Solve: forward on $\vec{b}$ using L, then back substitution using U.

Solve $L\vec{c} = \vec{b}$ and then solve $U\vec{x} = \vec{c}$

Example

Forward elimination on $Ax = b$ ends at $Ux = c$

$$\begin{matrix} x + 2y = 5 \\ 4x + 9y = 21 \end{matrix} \quad \textit{becomes} \quad \begin{matrix} x + 2y = 5 \\ y = 1 \end{matrix}$$

Solution

The multiplier was 4. $\left(R_2 - 4R_1\right)$

The lower triangular system: $L\vec{c} = \vec{b}$

$$\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}[c] = \begin{bmatrix} 5 \\ 21 \end{bmatrix}$$

$$\vec{c} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

The upper triangular system: $U\vec{x} = \vec{c}$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}[x] = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

To solve 1000 equations on a PC

- Elimination on A requires about $\frac{1}{3}n^3$ multiplications and $\frac{1}{3}n^3$ subtractions.
- Each right–side needs n^2 multiplications and n^2 subtractions.

Exercises *Section* 4.3 – LU-Decompositions

1. What matrix E puts A into triangular form $EA = U$? Multiply by $E^{-1} = L$ to factor A into LU:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}$$

2. Solve $L\vec{c} = \vec{b}$ to find $\vec{c}$. Then solve $U\vec{x} = \vec{c}$ to find $\vec{x}$. What was A?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

3. Find L and U for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots

4. For which c is $A = LU$ impossible – with three pivots?

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

(5 – 14) Find an LU–decomposition of the coefficient matrix, and then use to solve the system

5. $\begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

6. $\begin{bmatrix} -5 & -10 \\ 6 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 19 \end{bmatrix}$

7. $\begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}$

8. $\begin{bmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$

9. $\begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$

10. $\begin{bmatrix} 2 & -6 & 4 \\ -4 & 8 & 0 \\ 0 & -4 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$

11. $\begin{bmatrix} 2 & -4 & 2 \\ -4 & 5 & 2 \\ 6 & -9 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}$

12. $\begin{bmatrix} 1 & -1 & 2 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$

13. $\begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 5 \\ -1 \\ 3 \\ 7 \end{bmatrix}$

14. $\begin{bmatrix} 1 & -2 & -2 & -3 \\ 3 & -9 & 0 & -9 \\ -1 & 2 & 4 & 7 \\ -3 & -6 & 26 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 1 \\ 6 \\ 0 \\ 3 \end{bmatrix}$

(15 – 24) Find an *LU* factorization matrix

15. $\begin{pmatrix} 2 & 5 \\ -3 & -4 \end{pmatrix}$

16. $\begin{pmatrix} 6 & 4 \\ 12 & 5 \end{pmatrix}$

17. $\begin{pmatrix} 3 & 1 & 2 \\ -9 & 0 & -4 \\ 9 & 9 & 14 \end{pmatrix}$

18. $\begin{pmatrix} -5 & 0 & 4 \\ 10 & 2 & -5 \\ 10 & 10 & 16 \end{pmatrix}$

19. $\begin{pmatrix} 3 & 7 & 2 \\ 6 & 19 & 4 \\ 9 & 9 & 14 \end{pmatrix}$

20. $\begin{pmatrix} 2 & 3 & 2 \\ 4 & 13 & 9 \\ -6 & 5 & 4 \end{pmatrix}$

21. $\begin{pmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ -2 & -4 & 7 & 5 \end{pmatrix}$

22. $\begin{pmatrix} 1 & 3 & 1 & 5 \\ 5 & 20 & 6 & 31 \\ -2 & -1 & -1 & -4 \\ -1 & 7 & 1 & 7 \end{pmatrix}$

23. $\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{pmatrix}$

24. $\begin{pmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{pmatrix}$

25. Let A be a lower triangular $n \times n$ matrix with nonzero entries on the diagonal. Show that A is invertible and A^{-1} is lower triangular.

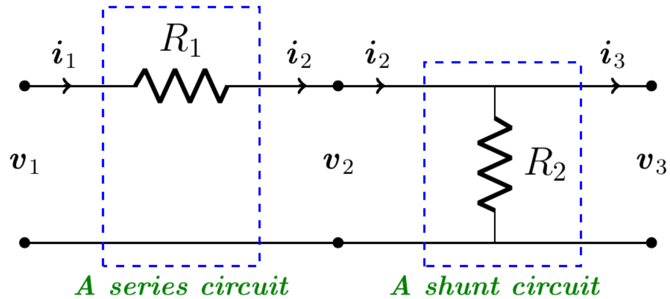
26. Let $A = LU$ be an LU factorization. Explain why A can be row reduced to U using only replacement operations.

27. Suppose an $m \times n$ matrix A admits a factorization $A = CD$ where C is $m \times 4$ and D is $4 \times n$.

a) Show that A is the sum of four outer products.

b) Let $m = 400$ *and* $n = 100$. Explain why a computer programmer might prefer to store the data from A in the form of two matrices C and D.

28. A ladder network, where two circuits are connected in series, so that the output of one circuit becomes the input of the next circuit.



The transformation $\begin{pmatrix} v_1 \\ i_1 \end{pmatrix} \longrightarrow \begin{pmatrix} v_2 \\ i_2 \end{pmatrix}$ is linear with a transfer matrix A of the ladder network.

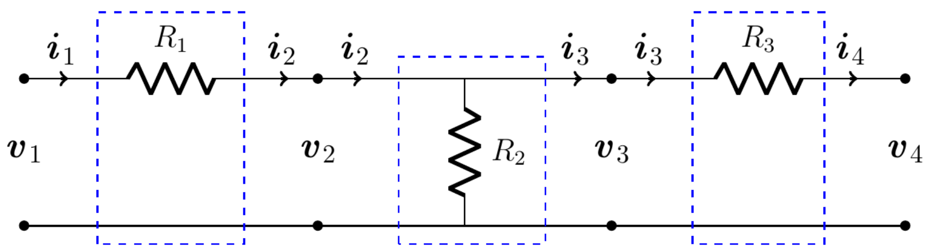
Let the transfer matrix A_1 of the series circuit is given by $\begin{pmatrix} v_2 \\ i_2 \end{pmatrix} = A_1 \begin{pmatrix} v_1 \\ i_1 \end{pmatrix}$

Let the transfer matrix A_2 of the shunt circuit is given by $\begin{pmatrix} v_3 \\ i_3 \end{pmatrix} = A_2 \begin{pmatrix} v_2 \\ i_2 \end{pmatrix}$

a) Compute the transfer matrix of the ladder network

b) Design a ladder network whose transfer matrix is $\begin{pmatrix} 1 & -8 \\ -\frac{1}{2} & 5 \end{pmatrix}$

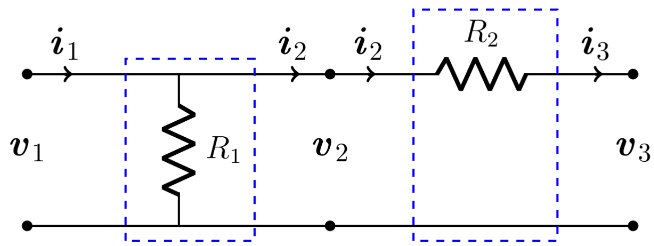
29. A ladder network, where three circuits are connected in series, so that the output of one circuit becomes the input of the next circuit.



a) Compute the transfer matrix of the ladder network

b) Design a ladder network whose transfer matrix is $\begin{pmatrix} 3 & -12 \\ -\frac{1}{3} & \frac{5}{3} \end{pmatrix}$

30. A ladder network, where two circuits are connected in series, so that the output of one circuit becomes the input of the next circuit.



a) Compute the transfer matrix of the ladder network

b) Find the values of the resistors when the input voltage is 12 volts and current is 6 amps if the output voltage is 9 volts and current is 4 amps

Section 4.4 – Eigenvalues and Eigenvectors

In many problems in science and mathematics, linear equations $A\vec{x} = \vec{b}$ come from steady state problems. Eigenvalues have their greatest importance in dynamic problems. The solution of $A\vec{x} = \lambda\vec{x}$ *or* $\frac{d\vec{x}}{dt} = A\vec{x}$ (is changing with time) has nonzero solutions. (***All matrices are square***)

Definition

Suppose A is an n x n matrix and

$$\boldsymbol{\lambda}\vec{x} = \boldsymbol{A}\vec{x}$$

The values of λ are called eigenvalues of the matrix $\boldsymbol{A}$ and the nonzero vectors $\vec{x}$ in $\mathbb{R}^n$ are called the eigenvectors corresponding to that eigenvalue (λ).

λ is the eigenvalue associated with or corresponding to the eigenvector $\vec{x}$.

- One of the meanings of the word "***eigen***" in German is "***proper***"; eigenvalues are also called ***proper values, characteristic values,*** or ***latent roots***.

Example

The vector $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 3$ since

$$\begin{aligned} A\vec{x} &= \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 6 \end{bmatrix} \\ &= 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= 3\vec{x} \end{aligned}$$

Eigenvalues and eigenvectors have a useful geometric interpretation in $\mathbb{R}^2$ and $\mathbb{R}^3$.

The equation for the *eigenvalues*

Let's rewrite the equation $\lambda\vec{x} = A\vec{x}$.

$A\vec{x} - \lambda\vec{x} = 0$ $\qquad$ λ: are the eigenvalues and not a vector

$A\vec{x} - \lambda I\vec{x} = 0$

$(A - \lambda I)\vec{x} = 0$

The matrix $A - \lambda I$ times the eigenvectors $\vec{x}$ is the zero vector.

The eigenvectors make up the nullspace of $A - \lambda I$.

Definition

The number λ is an eigenvalue of $\boldsymbol{A}$ if and only if $A - \lambda I$ is singular:

$$\boxed{\det(A - \lambda I) = 0}$$

This equation $\det(A - \lambda I) = 0$ is called ***characteristic equation*** of $\boldsymbol{A}$; the scalars satisfying this equation are the eigenvalues of $\boldsymbol{A}$. when expanding the determinant $\det(A - \lambda I)$ is a polynomial in λ of degree *n*, called the ***characteristic polynomial*** of $\boldsymbol{A}$.

Example

Find the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$

Solution

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix}$$

$$= (3-\lambda)(-\lambda) + 2$$

$$= \lambda^2 - 3\lambda + 2$$

The characteristic equation of $\boldsymbol{A}$ is:

$$\lambda^2 - 3\lambda + 2 = 0$$

The eigenvalues of $\boldsymbol{A}$ are $\lambda_1 = 1, \ \lambda_2 = 2$

Theorem

If A is an n x n triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are the entries on the main diagonal of A.

Example

Find the eigenvalues of the lower triangular matrix

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{3}{2} & 0 \\ 5 & -8 & -\frac{1}{4} \end{pmatrix}$$

Solution

The eigenvalues are: $\lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{3}{2}, \quad \textit{and} \quad \lambda_3 = -\frac{1}{4}$

Theorem

If A is an n x n matrix, the following are equivalent.

a) λ is an eigenvalue of A.

b) The system of equations $(A - \lambda I)\vec{x} = \vec{0}$ has nontrivial solutions.

c) There is a nonzero vector $\vec{x}$ in $\mathbb{R}^n$ such that $A\vec{x} = \lambda\vec{x}$.

d) λ is a real solution of the characteristic equation $\det(A - \lambda I) = 0$

Eigenvectors

To find the eigenvector $\vec{x}$, for each eigenvalue λ solve $(A-\lambda I)\vec{x}=0$ *or* $A\vec{x}=\lambda\vec{x}$

From the eigenvalues, the eigenvectors, in the form $\boldsymbol{V}_i=\begin{pmatrix} x_i \\ y_i \end{pmatrix}$, of the system can be determined by letting:

$$(A-\lambda_1 I)V_1=0 \qquad \text{and}\,(A-\lambda_2 I)V_2=0$$

Example

Find the eigenvalues and the eigenvectors of the matrix $A=\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

Solution

$$\det(A-\lambda I)=\det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix}$$

$$=(1-\lambda)(4-\lambda)-4$$

$$=\lambda^2-5\lambda+4-4$$

$$=\lambda^2-5\lambda$$

$$=\lambda(\lambda-5)=\mathbf{0}$$

The eigenvalues of $\boldsymbol{A}$ are: $\lambda_1=0 \quad \lambda_2=5$

For $\lambda_1=0$, we have:

$$(A-\lambda_1 I)V_1=0$$

$$\begin{pmatrix} 1-0 & 2 \\ 2 & 4-0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x+2y=0 \\ \end{cases}$$

$\underline{x=-2y}$

If $y=-1 \Rightarrow x=2$

Therefore, the eigenvector $V_1=\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Or $\begin{pmatrix} -2y \\ y \end{pmatrix}=y\begin{pmatrix} -2 \\ 1 \end{pmatrix} \Rightarrow V_1=\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ *or* $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

For $\lambda_2 = 5 \;\Rightarrow\; (A - \lambda_2 I)V_2 = 0$:

$$\begin{pmatrix} 1-5 & 2 \\ 2 & 4-5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow\; 2x - y = 0$$

$2x = y$

Therefore, the eigenvector $V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Power of a Matrix

Theorem

If k is a positive integer, λ is an eigenvalue of a matrix A, and $\vec{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\vec{x}$ is a corresponding eigenvector.

Example

Find the eigenvalues of A^7 for $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

Solution

$$\det(A - \lambda I) = \begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix}$$

$$= \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

The eigenvalues of A: $\lambda_1 = 1 \quad \textit{and} \quad \lambda_2 = 2$

The eigenvalues of A^7 are:

$\lambda_1 = 1^7 = 1 \quad \textit{and} \quad \lambda_2 = 2^7 = 128$

Theorem

A square matrix A is invertible *iff* $\lambda = 0$ is not an eigenvalue of A.

Summary

To solve the eigenvalue problem for an n by n matrix:

1. Compute the determinant of $A-\lambda I$. With λ subtracted along the diagonal, this determinant starts with λ^n or $-\lambda^n$. It is a polynomial in λ of degree n.
2. Find the roots of this polynomial, by solving $\det(A-\lambda I)=0$. The n roots are the n eigenvalues of A. They make $A-\lambda I$ singular.
3. For each eigenvalue λ, solve $(A-\lambda I)\vec{x}=\vec{0}$ ***to find an eigenvector x***.

Imaginary Eigenvalues

Example

Find the eigenvalues and the eigenvectors of the matrix $A=\begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}$

Solution

$$\det(A-\lambda I)=\begin{vmatrix} -2-\lambda & -1 \\ 5 & 2-\lambda \end{vmatrix}$$

$$=(-2-\lambda)(2-\lambda)+5$$

$$=\lambda^2+1=0$$

$$\Rightarrow \lambda^2=-1$$

The eigenvalues are: $\boxed{\lambda_{1,2}=\pm i}$

For $\lambda_1=i$: $(A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} -2-i & -1 \\ 5 & 2-i \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad (-2-i)x_1-y_1=0$$

$$\Rightarrow \boxed{(2+i)x_1=-y_1}$$

Therefore, the eigenvector $V_1=\begin{pmatrix} -1 \\ 2+i \end{pmatrix}$

$\lambda_1=-i: (A-\lambda_2 I)V_2=0$

$$\begin{pmatrix} -2+i & -1 \\ 5 & 2+i \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad (-2+i)x_2-y_2=0$$

$$\Rightarrow \boxed{(-2+i)x_2=y_2}$$

Therefore, the eigenvector $\boxed{V_2=\begin{pmatrix} -1 \\ 2-i \end{pmatrix}}$

Example

Find the eigenvalues and the eigenvectors of the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix}$$

$$= \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda^2 = -1$$

The eigenvalues are: $\boxed{\lambda_{1,2} = \pm i}$

The matrix $\boldsymbol{A}$ is a 90° rotation which has no real eigenvalues or eigenvectors.
No vector $A\vec{x}$ stays in the same direction as $\vec{x}$ (except the zero vector which is useless).
If we add the eigenvalues together the result is zero which is the trace of $\boldsymbol{A}$.

$\lambda_1 = i$: $\quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -ix + y = 0 \\ \end{cases}$$

$$\Rightarrow \boxed{x = -iy}$$

Therefore, the eigenvector $\boxed{V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}}$

$\lambda_2 = -i$: $\quad (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow ix + y = 0$$

$$\Rightarrow \boxed{y = -ix}$$

Therefore, the eigenvector $\boxed{V_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}}$

Exercises *Section* 4.4 – Eigenvalues and Eigenvectors

1. Find the eigenvalues and eigenvectors of $A,\ A^2,\ A^{-1},\ and\ \ A+4I$:

$$A=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix} \quad and \quad A^2=\begin{bmatrix}5 & -4\\ -4 & 5\end{bmatrix}$$

Check the trace $\lambda_1+\lambda_2$ and the determinant $\lambda_1\lambda_2$ for A and also A^2.

2. Show directly that the given vectors are eigenvectors of the given matrix. What are the corresponding eigenvalues

$$\vec{v}_1=\begin{bmatrix}1\\ -3\end{bmatrix},\quad \vec{v}_2=\begin{bmatrix}2\\ 1\end{bmatrix},\quad A=\begin{bmatrix}1 & -2\\ -3 & 6\end{bmatrix}$$

3. For which real numbers c does this matrix A have

$$A=\begin{pmatrix}2 & -c\\ -1 & 2\end{pmatrix}$$

a) Two real eigenvalues and eigenvectors.

b) A repeated eigenvalue with only one eigenvector

c) Two complex eigenvalues and eigenvectors.

4. Find the eigenvalues of ***A***, ***B***, ***AB***, and ***BA***:

$$A=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} \quad B=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \quad AB=\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix} \quad BA=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$$

a) The eigenvalues of ***AB*** (are equal to) (are not equal to) eigenvalues of ***A*** times eigenvalues of ***B***.

b) The eigenvalues of ***AB*** (are equal to) (are not equal to) eigenvalues of ***BA***.

5. When $a+b=c+d$ show that (1, 1) is an eigenvector and find both eigenvalues of

$$A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

6. The eigenvalues of A equal to the eigenvalues of A^T. This is because $\det(A-\lambda I)$ equals $\det\left(A^T-\lambda I\right)$. That is true because _____. Show by an example that the eigenvectors of A and A^T are not the same.

7. Let $A=\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}$. Compute the eigenvalues and eigenvectors of A.

8. Let $A=\begin{bmatrix}2 & -2 & 3\\ 1 & 1 & 1\\ 1 & 3 & -1\end{bmatrix}$

a) What is the characteristic polynomial for A (i.e. compute $\det(A-\lambda I)$?

b) Verify that 1 is an eigenvalue of A. What is a corresponding eigenvector?

c) What are the other eigenvalues of A?

(9 – 58) For the following matrices:

i. Find the characteristic equation.

ii. Find the eigenvalues.

iii. Find the eigenvectors.

9. $\begin{pmatrix}3 & 0\\ 8 & -1\end{pmatrix}$

10. $\begin{pmatrix}10 & -9\\ 4 & -2\end{pmatrix}$

11. $\begin{pmatrix}0 & 3\\ 4 & 0\end{pmatrix}$

12. $\begin{pmatrix}-2 & -7\\ 1 & 2\end{pmatrix}$

13. $\begin{pmatrix}12 & 14\\ -7 & -9\end{pmatrix}$

14. $\begin{pmatrix}-4 & 1\\ -2 & 1\end{pmatrix}$

15. $\begin{pmatrix}5 & 3\\ -6 & -4\end{pmatrix}$

16. $\begin{pmatrix}-2 & 3\\ 0 & -5\end{pmatrix}$

17. $\begin{pmatrix}2 & 0\\ -4 & -2\end{pmatrix}$

18. $\begin{pmatrix}1 & 2\\ 4 & 3\end{pmatrix}$

19. $\begin{pmatrix}2 & 2\\ 1 & 3\end{pmatrix}$

20. $\begin{pmatrix}-4 & 2\\ -\frac{5}{2} & 2\end{pmatrix}$

21. $\begin{pmatrix}-\frac{5}{2} & 2\\ \frac{3}{4} & -2\end{pmatrix}$

22. $\begin{pmatrix}3 & -1\\ 9 & -3\end{pmatrix}$

23. $\begin{pmatrix}-6 & 5\\ -5 & 4\end{pmatrix}$

24. $\begin{pmatrix}6 & -1\\ 5 & 2\end{pmatrix}$

25. $\begin{pmatrix}1 & 1\\ -2 & -1\end{pmatrix}$

26. $\begin{pmatrix}5 & 1\\ -2 & 3\end{pmatrix}$

27. $\begin{pmatrix}4 & 5\\ -2 & 6\end{pmatrix}$

28. $\begin{pmatrix}5 & -4\\ 2 & -1\end{pmatrix}$

29. $\begin{pmatrix}6 & -6\\ 4 & -4\end{pmatrix}$

30. $\begin{pmatrix}5 & -3\\ 2 & 0\end{pmatrix}$

31. $\begin{pmatrix}5 & -4\\ 3 & -2\end{pmatrix}$

32. $\begin{pmatrix}6 & -10\\ 2 & -3\end{pmatrix}$

33. $\begin{pmatrix}11 & -15\\ 6 & -8\end{pmatrix}$

34. $\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}$

35. $\begin{pmatrix}9 & 2\\ 2 & 6\end{pmatrix}$

36. $\begin{pmatrix}13 & 4\\ 4 & 7\end{pmatrix}$

37. $\begin{pmatrix}5 & -1\\ 3 & -1\end{pmatrix}$

38. $\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$

39. $\begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$

40. $\begin{pmatrix} -1 & 0 & 0 \\ 2 & -5 & -6 \\ -2 & 3 & 4 \end{pmatrix}$

41. $\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$

42. $\begin{pmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{pmatrix}$

43. $\begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

44. $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

45. $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix}$

46. $\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$

47. $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

48. $\begin{pmatrix} -\frac{5}{2} & 1 & 1 \\ 1 & -\frac{5}{2} & 1 \\ 1 & 1 & -\frac{5}{2} \end{pmatrix}$

49. . $\begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

50. $\begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$

51. $\begin{pmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{pmatrix}$

52. $\begin{pmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{pmatrix}$

53. $\begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ -8 & -4 & -3 \end{pmatrix}$

54. $\begin{pmatrix} -1 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & -12 & 2 \end{pmatrix}$

55. $\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$

56. $\begin{pmatrix} -6 & 4 & 4 \\ -4 & 2 & 4 \\ -10 & 8 & 4 \end{pmatrix}$

57. $\begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

58. $\begin{pmatrix} 10 & -9 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

59. Find the eigenvalues of A^9 for $A = \begin{pmatrix} 1 & 3 & 7 & 11 \\ 0 & \frac{1}{2} & 3 & 8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

60. Given: $A = \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}$. Compute A^{11}

61. Find the eigenvalues of the matrices

$$A = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0.61 & 0.52 \\ 0.39 & 0.48 \end{pmatrix}, \quad A^\infty = \begin{pmatrix} 0.57143 & 0.57143 \\ 0.42857 & 0.42857 \end{pmatrix}, \quad and \;\; B = \begin{pmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{pmatrix}$$

62. Given the matrix $\begin{bmatrix} -1 & -3 \\ -3 & 7 \end{bmatrix}$

a) Find the characteristic polynomial.
b) Find the eigenvalues
c) Find the bases for its eigenspaces
d) Graph the eigenspaces
e) Verify directly that $A\vec{v} = \lambda\vec{v}$, for all associated eigenvectors and eigenvalues.

63. Given the matrix $\begin{bmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{bmatrix}$

a) Find the characteristic polynomial.
b) Find the eigenvalues
c) Find the bases for its eigenspaces
d) Graph the eigenspaces
e) Verify directly that $A\vec{v} = \lambda\vec{v}$, for all associated eigenvectors and eigenvalues.

64. Explain why a 2×2 matrix can have at most two distinct eigenvalues. Explain why an $n\times n$ matrix can have at most *n* distinct eigenvalues.

65. Construct an example of a 2×2 matrix with only one distinct eigenvalue.

66. Let λ be an eigenvalue of an invertible matrix *A*. Show that λ^{-1} is an eigenvalue of A^{-1}.

67. Show that if A^2 is the zero matrix, then the only eigenvalue of *A* is 0.

68. Show that λ is an eigenvalue of *A* if and only if λ is an eigenvalue of A^T.

69. For $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, find one eigenvalue, without calculation. Justify your answer.

70. For $A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$, find one eigenvalue, and two linearly independent eigenvectors, without calculation. Justify your answer.

71. Consider an $n\times n$ matrix *A* with the property that the row sums all equal the same number ***s***. Show that ***s*** is an eigenvalue of *A*.

72. Consider an $n\times n$ matrix *A* with the property that the column sums all equal the same number ***s***. Show that ***s*** is an eigenvalue of *A*.

73. Let A be the matrix of the linear transformation T on $\mathbb{R}^2$

T: reflects points across some line through the origin.

Without writing A, find an eigenvalue of A and describe the eigenspace.

74. Let A be the matrix of the linear transformation T on $\mathbb{R}^2$

T: reflects points about some line through the origin.

Without writing A, find an eigenvalue of A and describe the eigenspace.

75. Show that if $\vec{v}$ is an eigenvector of the matrix product AB and $B\vec{v} \neq \vec{0}$, then $B\vec{v}$ is an eigenvector of BA

76. Explain and demonstrate that the eigenspace of a matrix A corresponding to some eigenvalue λ is a subspace.

77. If λ is an eigenvalue of the matrix A, prove that λ^2 is an eigenvalue of A^2.

Section 4.5 – Diagonalization

When $\vec{x}$ is an eigenvector, multiplication by $\boldsymbol{A}$ is just multiplication by a single number: $A\vec{x} = \lambda\vec{x}$.
The matrix $\boldsymbol{A}$ turns into a diagonal matrix $\boldsymbol{\Lambda}$ when we use the eigenvectors property.

Diagonalization

Suppose the n by n matrix $\boldsymbol{A}$ has n linearly independent eigenvectors $\vec{x}_1, \ldots, \vec{x}_n$. Put them into the column of an ***eigenvector matrix P***. Then $P^{-1}AP$ is the eigenvalue matrix $\boldsymbol{\Lambda}$:

$$P^{-1}AP = \Lambda = D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Example

The projection matrix $A = \begin{pmatrix} .5 & .5 \\ .5 & .5 \end{pmatrix}$ has $\lambda_{1,2} = 0$ *and* 1

Solution

For $\lambda_1 = 0 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \frac{1}{2}x + \frac{1}{2}y = 0$$

$x = -y$

Therefore, $V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

For $\lambda_2 = 1 \quad \Rightarrow \quad (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad -\frac{1}{2}x + \frac{1}{2}y = 0$$

$x = y$

Therefore, $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The eigenvectors are: $(-1,\ 1)$ & $(1,\ 1)$ that are the value of P.

$$P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$P^{-1} = -\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underset{P^{-1}}{\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}} \quad \underset{A}{\begin{pmatrix} .5 & .5 \\ .5 & .5 \end{pmatrix}} \quad \underset{P}{\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}} \quad \underset{=}{=} \quad \underset{D}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}$$

Definition

A square matrix A is called ***diagonalizable*** if there is an invertible matrix P such that $P^{-1}AP$ is diagonal; the matrix P is said to ***diagonalize*** A.

Theorem

Independent x from different λ - Eigenvectors $\vec{x}_1, \ldots, \vec{x}_n$ that correspond to distinct (all different) eigenvalues are linearly independent. An n by n matrix that has n different eigenvalues (no repeated λ's) must be diagonalizable.

Proof

Suppose $c_1\vec{x}_1 + c_2\vec{x}_2 = 0$ (1)

$$\begin{pmatrix} c_1\vec{x}_1 & c_2\vec{x}_2 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0$$

$$c_1\lambda_1\vec{x}_1 + c_2\lambda_2\vec{x}_2 = 0 \quad (2)$$

Multiply (1) by λ_2, that implies to

$$c_1\lambda_2\vec{x}_1 + c_2\lambda_2\vec{x}_2 = 0 \quad (3)$$

(2) − (3)

$$c_1\lambda_1\vec{x}_1 + c_2\lambda_2\vec{x}_2 - \left(c_1\lambda_2\vec{x}_1 + c_2\lambda_2\vec{x}_2\right) = 0$$

$$c_1\lambda_1\vec{x}_1 + c_2\lambda_2\vec{x}_2 - c_1\lambda_2\vec{x}_1 - c_2\lambda_2\vec{x}_2 = 0$$

$$c_1\lambda_1\vec{x}_1 - c_1\lambda_2\vec{x}_1 = 0$$

$$c_1\left(\lambda_1 - \lambda_2\right)\vec{x}_1 = 0$$

Since $\vec{x}_i \neq 0$ and λ's are different $\lambda_1 - \lambda_2 \neq 0$, we forced $\underline{c_1 = 0}|$

Similarly; Multiply (1) by λ_1, that implies to $c_1\lambda_1\vec{x}_1 + c_2\lambda_1\vec{x}_2 = 0$ (4)

(2) − (4)

$$c_1\lambda_1\vec{x}_1 + c_2\lambda_2\vec{x}_2 - c_1\lambda_1\vec{x}_1 - c_2\lambda_1\vec{x}_2 = 0$$

$$c_2\left(\lambda_2 - \lambda_1\right)\vec{x}_2 = 0 \quad \Rightarrow \underline{c_2 = 0}|$$

Therefore, $\vec{x}_1$ *and* $\vec{x}_2$ must be independent.

Theorem

If $\vec{v}_1, \ldots, \vec{v}_n$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, then $\left\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\right\}$ is linearly independent set.

Theorem

If an n x n matrix $\boldsymbol{A}$ has n distinct eigenvalues, then the following are equivalent:

a) $\boldsymbol{A}$ is diagonalizable

b) $\boldsymbol{A}$ has n linearly independent eigenvectors.

Example

Given the Markov matrix $A = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}$

Solution

$$\left|A - \lambda I\right| = \begin{vmatrix} .8-\lambda & .3 \\ .2 & .7-\lambda \end{vmatrix}$$

$$= (.8-\lambda)(.7-\lambda) - .06$$

$$= \lambda^2 - 1.5\lambda + .56 - .06$$

$$= \lambda^2 - 1.5\lambda + .5 = 0$$

The eigenvalues are: $\underline{\lambda_1 = 1, \quad \lambda_2 = .5}|$

For $\lambda_1 = 1$, we have: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -.2 & .3 \\ .2 & -.3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad -.2x + .3y = 0$$

$\Rightarrow \underline{2x = 3y}|$

Therefore, the eigenvector $\underline{V_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}}|$

For $\lambda_2 = .5$, we have: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} .3 & .3 \\ .2 & .2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad .3x + .3y = 0$$

$\Rightarrow \underline{x = -y}|$

Therefore, the eigenvector $\underline{V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}|$

The eigenvector matrix is given by:

$$P = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}$$

$$P^{-1} = -\frac{1}{5}\begin{pmatrix} -1 & -1 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{3}{5} \end{pmatrix}$$

$$\overset{P}{\begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}}\overset{D}{\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}}\overset{P^{-1}}{\begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{3}{5} \end{pmatrix}} = \begin{pmatrix} 3 & \frac{1}{2} \\ 2 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{3}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{8}{10} & \frac{3}{10} \\ \frac{2}{10} & \frac{7}{10} \end{pmatrix}$$

$$= \underset{A}{\begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}}$$

Eigenvalues of AB and A + B

An eigenvalue of ***A*** times an eigenvalue of ***B*** usually does not give an eigenvalue of ***AB***.

$$\lambda_A \lambda_B \neq \lambda_{AB}$$

Commuting matrices share eigenvectors: Suppose ***A*** and ***B*** can be diagonalized. They share the eigenvector matrix ***P*** if and only if $AB = BA$.

Matrix Powers A^k

$$A^2 = PDP^{-1}PDP^{-1}$$
$$= PD^2P^{-1}$$

$$A^k = \left(PDP^{-1}\right)\cdots\left(PDP^{-1}\right)$$
$$= PD^kP^{-1}$$

The eigenvector matrix for A^k is still S, and the eigenvalue matrix is A^k. The eigenvectors don't change, and the eigenvalues are taken to the k^{th} power. When A is diagonalized, $A^k\vec{u}_0$ is easy.

Here are steps (taken from Fibonacci):

1. Find the eigenvalues of A and look for n independent eigenvectors.
2. Write $\vec{u}_0$ as a combination $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ of the eigenvectors.
3. Multiply each eigenvector $\vec{v}_i$ *by* $\left(\lambda_i\right)^k$. Then

$$\vec{u}_k = A_k\vec{u}_0$$
$$= c_1\left(\lambda_1\right)^k\vec{v}_1 + \cdots + c_n\left(\lambda_n\right)^k\vec{v}_n$$

Example

Compute A^k *where* $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

Solution

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix}$$
$$= (1-\lambda)(2-\lambda) = 0$$

The eigenvalues are: $\lambda_{1,2} = 1,\ 2$

For $\lambda_1 = 1 \Rightarrow \left(A-\lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow y = 0$$

$$\Rightarrow V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow \quad \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \underline{x = y}$$

$$\Rightarrow \quad \underline{V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

The eigenvector matrix is given by:

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \Rightarrow \quad D^k = \begin{pmatrix} 1^k & 0 \\ 0 & 2^k \end{pmatrix}$$

$$\begin{aligned} A^k &= PD^kP^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^k \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2^k \\ 0 & 2^k \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{pmatrix} \end{aligned}$$

Similar Matrices

Definition

If A and B are square matrices, then we say that ***B* is similar to *A*** if there exists an invertible matrix P such that $B = P^{-1}AP \quad or \quad A = PBP^{-1}$

- Similar matrices B and $M^{-1}AM$ have the same eigenvalues. If $\vec{x}$ is an eigenvector of A then $M^{-1}\vec{x}$ is an eigenvector of $B = M^{-1}AM$.

Proof

Since $B = M^{-1}AM \Rightarrow A = MBM^{-1}$

Suppose $A\vec{x} = \lambda\vec{x}$:

$$MBM^{-1}\vec{x} = \lambda\vec{x}$$

$$BM^{-1}\vec{x} = \lambda M^{-1}\vec{x}$$

The eigenvalue of B is the same λ. The eigenvector is now $M^{-1}x$

Example

The projection $A = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ is similar to $D = P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Choose $M = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$; the similar matrix $M^{-1}AM \ \ is \ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Also choose $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$; the similar matrix $M^{-1}AM \ \ is \ \begin{bmatrix} .5 & -.5 \\ -.5 & .5 \end{bmatrix}$

These matrices $M^{-1}AM$ all have the same eigenvalues 1 and 0.

***Every 2 by 2 matrix with those eigenvalues is similar to A*.**

The eigenvectors change with M.

Example

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is similar to every matrix $B = \begin{bmatrix} cd & d^2 \\ -c^2 & -cd \end{bmatrix}$ except $B = 0$.

These matrices B all have zero determinant (like A). T
hey all have rank one (like A). Their trace is $cd - cd = 0$.

Their eigenvalues are 0 and 0 (like A).

Choose $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with* $ad - cd = 1$ and $B = M^{-1}AM$

Connections between similar matrices A and B:

Not Changed	***Changed***
Eigenvalues	Eigenvectors
Trace and determinant	Nullspace
Rank	Column space
Number of independent	Row space
eigenvectors	Left nullspace
Jordan form	Singular values

Example

Jordan matrix J has triple eigenvalues 5, 5, 5. Its only eigenvectors are multiples of (1, 0, 0). Algebraic multiplicity 3, geometric multiplicity 1:

$$\text{If } J = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \quad \textit{then} \quad J - 5I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ has rank 2.}$$

Every similar matrix $B = M^{-1}JM$ has the same triple eigenvalues 5, 5, 5. Also $B - 5I$ must have the same rank 2. Its nullspace has dimension 3 − 2 = 1. So each similar matrix B also has only one independent eigenvector.

The transpose matrix J^T has the same eigenvalues 5, 5, 5, and $J^T - 5I$ has the same rank 2. ***Jordan's theory says that J^T is similar to J***. The matrix that produces the similarity happens to be the reserve identity M:

$$J^T = M^{-1}JM \quad \textit{is} \quad \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

There is one line of eigenvectors $\left(x_1,\ 0,\ 0\right)$ for J and another line $\left(0,\ 0,\ x_3\right)$ for J^T.

Fibonacci Numbers

Every new Fibonacci number is the sum of the two previous F's.

The ***sequence*** 0, 1, 1, 2, 3, 5, 8, 13, …. comes from $F_{k+2}=F_{k+1}+F_k$

Problem

Find the Fibonacci number F_{100}

We can apply the rule one step at a time, or just use Linear algebra.

Let consider the matrix equation: $u_{k+1}=Au_k$. Fibonacci rule gave us a two-step rule for scalars.

Let $\vec{u}_k=\begin{pmatrix}F_{k+1}\\ F_k\end{pmatrix}$, the rule $\begin{matrix}F_{k+2}=F_{k+1}+F_k\\ F_{k+1}=F_{k+1}\end{matrix}$ becomes $\vec{u}_{k+1}=\begin{bmatrix}1&1\\1&0\end{bmatrix}\vec{u}_k$.

Every step multiplies by $A=\begin{pmatrix}1&1\\1&0\end{pmatrix}$, after 100 steps we reach $\vec{u}_{100}=\begin{pmatrix}F_{101}\\ F_{100}\end{pmatrix}$

$$\vec{u}_0=\begin{pmatrix}1\\0\end{pmatrix},\quad \vec{u}_1=\begin{pmatrix}1\\1\end{pmatrix},\quad \vec{u}_2=\begin{pmatrix}2\\1\end{pmatrix},\quad \vec{u}_3=\begin{pmatrix}3\\2\end{pmatrix},\quad \ldots\quad \vec{u}_{100}=\begin{pmatrix}F_{101}\\ F_{100}\end{pmatrix}$$

$$\begin{aligned}|A-\lambda I|&=\begin{vmatrix}1-\lambda&1\\1&-\lambda\end{vmatrix}\\&=-\lambda(1-\lambda)-1\\&=\lambda^2-\lambda-1\end{aligned}$$

The characteristic equation is $\lambda^2-\lambda-1=0$ and the solutions are

$$\lambda_1=\frac{1+\sqrt{5}}{2}\approx 1.618\quad and\quad \lambda_2=\frac{1-\sqrt{5}}{2}\approx -.618$$

For $\lambda_1\ \Rightarrow\ (A-\lambda_1 I)\vec{v}_1=0$

$$\begin{pmatrix}1-\lambda_1&1\\1&-\lambda_1\end{pmatrix}\begin{pmatrix}x_1\\y_1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad\to\quad x_1-\lambda_1 y_1=0$$

$$\boxed{x_1=\lambda_1 y_1}$$

$$\Rightarrow\ \boxed{\vec{v}_1=\begin{pmatrix}\lambda_1\\1\end{pmatrix}}$$

For $\lambda_2\ \Rightarrow\ (A-\lambda_2 I)\vec{v}_2=0$

$$\begin{pmatrix} 1-\lambda_2 & 1 \\ 1 & -\lambda_2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x_1 - \lambda_2 y_1 = 0$$

$$\boxed{x_1 = \lambda_2 y_1}$$

$$\Rightarrow \boxed{\vec{v}_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}}$$

The eigenvector matrix is given by:

$$S = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$$

The combination of these eigenvectors that give $\vec{u}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\lambda_1 - \lambda_2}\left(\begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} - \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}\right)$$

$$\boxed{= \frac{\vec{v}_1 - \vec{v}_2}{\lambda_1 - \lambda_2}}$$

$$\vec{u}_{100} = \frac{(\lambda_1)^{100}\,\vec{v}_1 - (\lambda_2)^{100}\,\vec{v}_2}{\lambda_1 - \lambda_2}$$

$$F_{100} = \frac{1}{\lambda_1 - \lambda_2}\left[(\lambda_1)^{100} - (\lambda_2)^{100}\right]$$

$$= \frac{1}{\frac{1+\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{100} - \left(\frac{1+\sqrt{5}}{2}\right)^{100}\right]$$

$$\boxed{\approx 2.54 \times 10^{20}}$$

The *Jordan* Form

For every A, we want to choose M so that $M^{-1}AM$ is as nearly diagonal as possible. When A has a full set of n eigenvectors, they go into the columns of M. Then $M = P$. The matrix $P^{-1}AP$ is diagonal.

If A has $\boldsymbol{s}$ independent eigenvectors, it is similar to a matrix J that has $\boldsymbol{s}$ Jordan blocks on its diagonal. There is a matrix M such that

$$M^{-1}AM = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix} = J$$

Each block in J has one eigenvalue λ_i, one eigenvector, and 1's above the diagonal:

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

A is similar to B if they share the same Jordan form J – not otherwise.

Exercises Section 4.5 – Diagonalization

1. The Lucas numbers are like Fibonacci numbers except they start with $L_1 = 1$ *and* $L_2 = 3$. Following the rule $L_{k+2} = L_{k+1} + L_k$. The next Lucas numbers are 4, 7, 11, 18. Show that the Lucas number $L_{100} = \lambda_1^{100} + \lambda_2^{100}$.

2. Find all eigenvector matrices *S* that diagonalize *A* (rank 1) to give $S^{-1}AS = \Lambda$:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

What is A^n? Which matrices *B* commute with *A* (so that *AB* = *BA*)

(3 – 6) Determine whether the matrix is diagonalizable

3. $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$

4. $\begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$

5. $\begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$

6. $\begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

(7 – 26) Determine if the matrices are diagonalizable. If so, find a matrix ***P*** that diagonalizes *A* and determine $P^{-1}AP$.

7. $A = \begin{pmatrix} -14 & 12 \\ -20 & 17 \end{pmatrix}$

8. $A = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}$

9. $A = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}$

10. $A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$

11. $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$

12. $A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 3 & 2 \end{pmatrix}$

13. $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$

14. $A = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{pmatrix}$

15. $A = \begin{pmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

16. $A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$

17. $A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & -2 \\ 1 & 3 & 1 \end{pmatrix}$

18. $A = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{pmatrix}$

19. $A = \begin{pmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{pmatrix}$

20. $A = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

21. $A = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

22. $A = \begin{pmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{pmatrix}$

23. $A = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}$

24. $A = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

25. $A = \begin{pmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

26. $A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}$

27. The 4 by 4 triangular Pascal matrix P_L and its inverse (alternating diagonals) are

$$P_L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \quad \textit{and} \quad P_L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

Check that P_L and P_L^{-1} have the same eigenvalues. Find a diagonal matrix D with alternating signs that gives $P_L^{-1} = D^{-1}P_L D$, so P_L is similar to P_L^{-1}. Show that $P_L D$ with columns of alternating signs is its own inverse.

Since P_L and P_L^{-1} are similar they have the same Jordan form J. Find J by checking the number of independent eigenvectors of P_L with $\lambda = 1$.

28. These Jordan matrices have eigenvalues 0, 0, 0, 0. They have two eigenvectors (one from each block). But the block sizes don't match and they are not similar:

$$J = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad and \quad K = \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

For any matrix M compare JM with MK. If they are equal show that M is not invertible. Then $M^{-1}JM = K$ is Impossible; J is not similar to K.

29. If $\boldsymbol{x}$ is in the nullspace of A show that $M^{-1}\boldsymbol{x}$ is in the nullspace of $M^{-1}AM$.
The nullspaces of A and $M^{-1}AM$ have the same (vectors) (basis) (dimension)

30. Prove that A^T is always similar to A (λ's are the same):

a) For one Jordan block J_i, find M_i so that $M_i^{-1}J_iM_i = J_i^T$.

b) For any J with blocks J_i, build M_0 from blocks so that $M_0^{-1}JM_0 = J^T$.

c) For any $A = MJM^{-1}$: Show that A^T is similar to J^T and so to J and so to A.

31. Why are these statements all true?

a) If A is similar to B then A^2 is similar to B^2.

b) A^2 and B^2 can be similar when A and B are not similar.

c) $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ is similar to $\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}$

d) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ is not similar to $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

e) If we exchange rows 1 and 2 of A, and then exchange columns 1 and 2 the eigenvalues stay the same. In this case M=?

32. If an n x n matrix A has all eigenvalues $\lambda = 0$ prove that A^n is the zero matrix.

33. If A is similar to A^{-1}, must all the eigenvalues equal to 1 or −1?.

(34 – 42) Determine whether the *two matrices* are similar matrices

34. $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}$

35. $A = \begin{pmatrix} 4 & -1 \\ 2 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 4 & 1 \\ 2 & 4 \end{pmatrix}$

36. $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

37. $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}1&2&3\\0&4&5\\0&0&6\end{pmatrix}$

38. $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}4&0&0\\7&1&0\\8&9&6\end{pmatrix}$

39. $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}4&7&0\\0&1&0\\8&9&6\end{pmatrix}$

40. $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}1&2&0\\3&4&5\\0&7&6\end{pmatrix}$

41. $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}1&2&0\\3&4&0\\0&5&6\end{pmatrix}$

42. $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}1&0&0\\2&5&1\\3&1&5\end{pmatrix}$

43. Prove that two similar matrices have the same determinant. Explain geometrically why this is reasonable.

44. Prove that two similar matrices have the same characteristic polynomial and thus the same eigenvalues. Explain geometrically why this is reasonable.

45. Suppose that A is a matrix. Suppose that the linear transformation associated to A has two linearly independent eigenvectors. Prove that A is similar to a diagonal matrix.

46. Prove that if A is a 2×2 matrix that has two distinct eigenvalues, then A is similar to a diagonal matrix.

47. Suppose that the characteristic polynomial of a matrix has a double root. Is it true that the matrix has two linearly independent eigenvectors? Consider the example $\begin{pmatrix}0&0\\1&0\end{pmatrix}$. Is it true that matrices with equal characteristic polynomial are necessarily similar?

48. Show that the given matrix is not diagonalizable. $\begin{bmatrix}1&1\\-1&-1\end{bmatrix}$

49. Determine if the given matrix is diagonalizable. If, so, find matrices S and $\Lambda(D)$ such that the given matrix equals $S\Lambda S^{-1}$

a) $\begin{bmatrix}3&3\\4&2\end{bmatrix}$

b) $\begin{bmatrix}1&0&1\\0&2&0\\-1&0&-1\end{bmatrix}$

50. A is a 5×5 matrix with *two* eigenvalues. One eigenspace is *three*–dimensional, and the other eigenspace is *two*–dimensional. Is A diagonalizable? Why?

51. A is a 3×3 matrix with *two* eigenvalues. Each eigenspace is *one*–dimensional. Is A diagonalizable? Why?

52. A is a 4×4 matrix with *three* eigenvalues. One eigenspace is *one*–dimensional, and one of the other eigenspace is *two*–dimensional. Is it possible that A is *not* diagonalizable? Justify your answer?

53. A is a 7×7 matrix with *three* eigenvalues. One eigenspace is *two*–dimensional, and one of the other eigenspace is *three*–dimensional. Is it possible that A is *not* diagonalizable? Justify your answer?

54. Show that if A is diagonalizable and invertible, then so is A^{-1}.

55. Show that if A has n linearly independent eigenvectors, then so does A^T.

56. A factorization $A = PDP^{-1}$ is not unique. Demonstrate this for the matrix $A = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix}$ with $D_1 = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$, find a matrix P_1 such that $A = P_1 D_1 P_1^{-1}$.

57. Construct a nonzero 2×2 matrix that is invertible but not diagonalizable.

58. Construct a nonzero 2×2 matrix that is diagonalizable but not invertible.

59. What are the matrices that are similar to themselves only?

60. For any scalars a, b, and c, show that

$$A = \begin{pmatrix} b & c & a \\ c & a & b \\ a & b & c \end{pmatrix}, \quad B = \begin{pmatrix} c & a & b \\ a & b & c \\ b & c & a \end{pmatrix}, \quad C = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$$

are similar.

Moreover, if $BC = CB$, then A has two zero eigenvalues.

(61 – 64) For positive integer $k \ge 2$, compute

61. $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}^k$

62. $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^k$

63. $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^k$

64. $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^k$

65. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Show that A^k is similar to A fro every positive integer k. It is true more generally for any matrix with all eigenvalues equal to 1.

66. Can a matrix be similar to two different diagonal matrices?

67. Prove that if A is diagonalizable, then A^T is diagonalizable.

68. Prove that if the eigenvalues of a diagonalizable matrix A are all ± 1, then the matrix is equal to its inverse.

69. Prove that if A is diagonalizable with n real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $|A| = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$

70. If x is a real number, then we can define e^x by the series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

In similar way, If X is a square matrix, then we can define e^X by the series

$$e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \frac{1}{4!}X^4 + \cdots$$

Evaluate e^X, where X is the indicated square matrix.

a) $X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

b) $X = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

c) $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

d) $X = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

Section 4.6 – Orthogonal Diagonalization

Definition

A square matrix $\boldsymbol{A}$ is called orthogonally diagonalizable if there is an orthogonal matrix P such that $P^{-1}AP\left(=P^{T}AP\right)$ is diagonal; the matrix P is said to orthogonally diagonalize A.

$$P^{T}AP=D$$

We say that A is orthogonally diagonalizable and that P orthogonally diagonalizes A.

Theorem

If $\boldsymbol{A}$ is an n x n matrix, then the following are equivalent.

a) $\boldsymbol{A}$ is orthogonally diagonalizable

b) $\boldsymbol{A}$ has an orthonormal set of n eigenvectors.

c) $\boldsymbol{A}$ is symmetric.

Theorem

If $\boldsymbol{A}$ is symmetric matrix, then:

a) The eigenvalues of $\boldsymbol{A}$ are all real numbers.

b) Eigenvectors from different eigenspaces are orthogonal.

Example

Find an orthogonal matrix P that diagonalizes

$$A=\begin{pmatrix}4&2&2\\2&4&2\\2&2&4\end{pmatrix}$$

Solution

$$\begin{aligned}\det\left(A-\lambda I\right)&=\begin{pmatrix}4-\lambda&2&2\\2&4-\lambda&2\\2&2&4-\lambda\end{pmatrix}\\&=\left(4-\lambda\right)^{3}+8+8-4\left(4-\lambda\right)-4\left(4-\lambda\right)-4\left(4-\lambda\right)\\&=64-48\lambda+12\lambda^{2}-\lambda^{3}+16-12\left(4-\lambda\right)\\&=64-48\lambda+12\lambda^{2}-\lambda^{3}+16-48+12\lambda\\&=-\lambda^{3}+12\lambda^{2}-36\lambda+32=0\end{aligned}$$

The eigenvalues are: $\boxed{\lambda_{1,2} = 2 \quad and \quad \lambda_3 = 8}$

For $\lambda_{1,2} = 2$, we have: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad 2x_1 + 2y_1 + 2z_1 = 0$$

$$\Rightarrow \quad \boxed{x_1 + y_1 + z_1 = 0}$$

If $z_1 = 0 \quad \Rightarrow \quad \boxed{x_1 = -y_1}$

Therefore, the eigenvector $\boxed{V_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}$

If $y_1 = 0 \quad \Rightarrow \quad \boxed{x_1 = -z_1}$

Therefore, the eigenvector $\boxed{V_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$

For $\lambda_3 = 8$, we have: $(A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} -4x_3 + 2y_3 + 2z_3 = 0 \\ 2x_3 - 4y_3 + 2z_3 = 0 \\ 2x_3 + 2y_3 - 4z_3 = 0 \end{cases}$$

$$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \begin{matrix} \\ 2R_2 + R_1 \\ 2R_3 + R_1 \end{matrix}$$

$$\begin{pmatrix} -4 & 2 & 2 \\ 0 & -6 & 6 \\ 0 & 6 & -6 \end{pmatrix} \begin{matrix} 3R_1 + R_2 \\ \\ R_3 + R_2 \end{matrix}$$

$$\begin{pmatrix} -12 & 0 & 12 \\ 0 & -6 & 6 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} -\frac{1}{12}R_1 \\ -\frac{1}{6}R_2 \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \to x_3 - z_3 = 0 \\ \to y_3 - z_3 = 0 \end{matrix}$$

$$\Rightarrow \underline{x_3 = y_3 = z_3}$$

Therefore the eigenvector $V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(-1,\ 1,\ 0)}{\sqrt{(-1)^2+1^2+0}}$$

$$= \frac{(-1,\ 1,\ 0)}{\sqrt{2}}$$

$$= \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$\vec{w}_2 = v_2 - \left(\vec{v}_2 \bullet \vec{u}_1\right)\vec{u}_1$$

$$= (-1,\ 0,\ 1) - \left[(-1,\ 0,\ 1)\bullet\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)\right]\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= (-1,\ 0,\ 1) - \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= \left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{\left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}}$$

$$= \frac{\left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)}{\sqrt{\frac{6}{4}}}$$

$$= \frac{2}{\sqrt{6}}\left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)$$

$$= \left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)$$

$$\begin{aligned}\vec{w}_3 &= \vec{v}_3 - \left(\vec{v}_3 \bullet \vec{u}_1\right)\vec{u}_1 - \left(\vec{v}_3 \bullet \vec{u}_2\right)\vec{u}_2 \\ &= (1,\ 1,\ 1) - (0)\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) - (0)\left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right) \\ &= (1,\ 1,\ 1)\end{aligned}$$

$$\begin{aligned}\vec{u}_3 &= \frac{\vec{w}_3}{\left\|\vec{w}_3\right\|} \\ &= \frac{1}{\sqrt{1^2+1^2+1^2}}(1,\ 1,\ 1) \\ &= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\end{aligned}$$

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \qquad (\mathit{Orthogonal})$$

$$\begin{aligned}P^T AP &= \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} & 0 \\ -\frac{2}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{4}{\sqrt{6}} \\ \frac{8}{\sqrt{3}} & \frac{8}{\sqrt{3}} & \frac{8}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}\end{aligned}$$

Spectral Decomposition

The spectral decomposition of A is:

$$A = \lambda_1 \vec{u}_1 \vec{u}_1^T + \lambda_2 \vec{u}_2 \vec{u}_2^T + \cdots + \lambda_n \vec{u}_n \vec{u}_n^T$$

Example

The matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}$$

$$= (1-\lambda)(-2-\lambda) - 4$$

$$= \lambda^2 + \lambda - 6 = 0$$

The eigenvalues are: $\lambda_1 = -3 \quad and \quad \lambda_2 = 2$

For $\lambda_1 = -3$: $\left(A - \lambda_1 I\right)\vec{v}_1 = 0$

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad 2x_1 = -y_1$$

Therefore, the eigenvector $\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

For $\lambda_2 = 2$: $\left(A - \lambda_2 I\right)\vec{v}_2 = 0$

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x_1 = 2y_1$$

Therefore, the eigenvector $\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

The corresponding eigenvectors are: $\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad and \quad \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \frac{(1, -2)}{\sqrt{1^2 + (-2)^2}}$$

$$= \left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$$

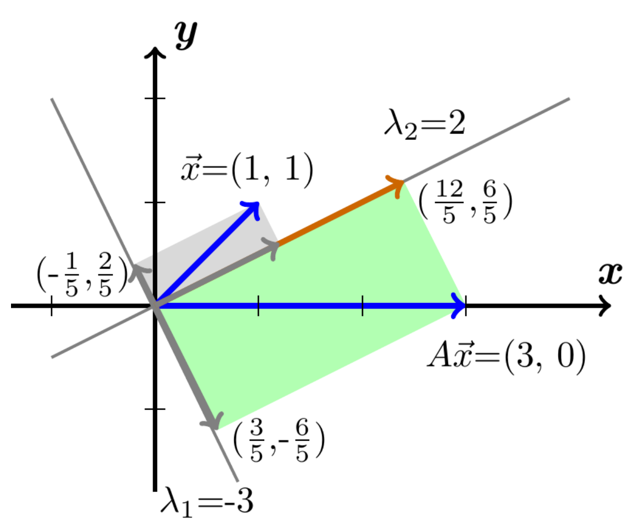
$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \bullet \vec{u}_1)\vec{u}_1$$
$$= (2,\ 1) - \left[(2,\ 1) \bullet \left(\frac{1}{\sqrt{5}},\ -\frac{2}{\sqrt{5}}\right)\right]\left(\frac{1}{\sqrt{5}},\ -\frac{2}{\sqrt{5}}\right)$$
$$= (2,\ 1) - (0)\left(\frac{1}{\sqrt{5}},\ -\frac{2}{\sqrt{5}}\right)$$
$$= (2,\ 1)$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$
$$= \frac{(2,\ 1)}{\sqrt{5}}$$
$$= \left(\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right)$$

$$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} = \lambda_1 \vec{u}_1 \vec{u}_1^T + \lambda_2 \vec{u}_2 \vec{u}_2^T$$
$$= -3\begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix} + 2\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$
$$= -3\begin{pmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{pmatrix} + 2\begin{pmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

The spectral decomposition about the image of the vector $\vec{x} = (1,\ 1)$

$$A\vec{x} = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= -3\begin{pmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= -3\begin{pmatrix} -\frac{1}{5} \\ \frac{2}{5} \end{pmatrix} + 2\begin{pmatrix} \frac{6}{5} \\ \frac{3}{5} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{3}{5} \\ -\frac{6}{5} \end{pmatrix} + \begin{pmatrix} \frac{12}{5} \\ \frac{6}{5} \end{pmatrix}$$
$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$



Example

Consider a 2 by 2 symmetric matrix $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

Solution

The eigenvalues are:

$$\det(A-\lambda I) = \begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix}$$

$$= (a-\lambda)(c-\lambda) - b^2$$

$$\boxed{= \lambda^2 - (a+c)\lambda + ac - b^2 = 0}$$

$$\lambda = \frac{(a+c) \pm \sqrt{(a+c)^2 - 4\left(ac-b^2\right)}}{2} \qquad \therefore (a+c)^2 - 4\left(ac-b^2\right) > 0$$

The eigenvectors are:

For $\lambda_1 \Rightarrow (A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} a-\lambda_1 & b \\ b & c-\lambda_1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} (a-\lambda_1)x_1 + by_1 = 0 & (1) \\ bx_1 + (c-\lambda_1)y_1 = 0 \end{cases}$$

$$(1) \Rightarrow \boxed{by_1 = (\lambda_1 - a)x_1}$$

$$\boxed{V_1 = \begin{pmatrix} b \\ \lambda_1 - a \end{pmatrix}}$$

For $\lambda_2 \Rightarrow (A-\lambda_2 I)V_2 = 0$

$$\begin{pmatrix} a-\lambda_2 & b \\ b & c-\lambda_2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} (a-\lambda_2)x_2 + by_2 = 0 \\ bx_2 + (c-\lambda_2)y_2 = 0 & (2) \end{cases}$$

$$(2) \Rightarrow \boxed{bx_2 = (\lambda_2 - c)y_2}$$

$$\boxed{V_2 = \begin{pmatrix} \lambda_2 - c \\ b \end{pmatrix}}$$

$$\lambda_1+\lambda_2=\frac{(a+c)-\sqrt{(a+c)^2-4\left(ac-b^2\right)}+(a+c)+\sqrt{(a+c)^2-4\left(ac-b^2\right)}}{2}$$

$$=\frac{2(a+c)}{2}$$

$$\underline{=a+c}\,\rfloor$$

$$V_1\bullet V_2=b\left(\lambda_1-a\right)+b\left(\lambda_2-c\right)$$

$$=b\left(\lambda_1+\lambda_2-a-c\right)$$

$$=b(a+c-a-c)$$

$$\underline{=0}\,\rfloor$$

Therefore, these eigenvectors are perpendicular.

Theorem

Orthogonal Eigenvectors: Eigenvectors of a real symmetric matrix (when they correspond to different λ′s) are always perpendicular.

Proof

Suppose $A\vec{x} = \lambda_1 \vec{x}$, $A\vec{y} = \lambda_2 \vec{y}$ and $A = A^T$.

The dot products of the first equation with y and the second with x:

$$\begin{aligned}
\left(\lambda_1 \vec{x}\right)^T \vec{y} &= \left(A\vec{x}\right)^T \vec{y} \\
&= \vec{x}^T A^T \vec{y} \\
&= \vec{x}^T A\vec{y} \\
&= \vec{x}^T \lambda_2 \vec{y}
\end{aligned}$$

$$\Rightarrow \quad \vec{x}^T \lambda_1 \vec{y} = \vec{x}^T \lambda_2 \vec{y}$$

Since $\lambda_1 \neq \lambda_2$, this proves that $\vec{x}^T \vec{y} = 0$.

The eigenvector $\vec{x}$ $\left(\text{for } \lambda_1\right)$ is perpendicular to the eigenvector $\vec{y}$ $\left(\text{for } \lambda_2\right)$

Example

Find the λ′s and v′s for this symmetric matrix with trace zero: $A = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$

Solution

$$\begin{aligned}
\det\left(A - \lambda I\right) &= \begin{vmatrix} -3-\lambda & 4 \\ 4 & 3-\lambda \end{vmatrix} \\
&= \left(-3-\lambda\right)\left(3-\lambda\right) - 16 \\
&= -9 + \lambda^2 - 16 \\
&= \lambda^2 - 25 = 0
\end{aligned}$$

The eigenvalues are: $\lambda_1 = -5 \quad \lambda_2 = 5$

The eigenvectors are:

For $\lambda_1 = -5 \ \Rightarrow \ \left(A - \lambda_1 I\right)\vec{v}_1 = 0$

$$\begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad 2x_1 + 4y_1 = 0$$

$$\Rightarrow \quad x_1 = -2y_1$$

$$\Rightarrow \quad \boxed{\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = 5 \quad \Rightarrow \quad \left(A - \lambda_2 I\right)\vec{v}_2 = 0$

$$\begin{pmatrix} -8 & 4 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad 4x_2 - 2y_2 = 0$$

$$\Rightarrow \quad \boxed{2x_2 = y_2}$$

$$\Rightarrow \quad \boxed{\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

$$\begin{aligned} \vec{v}_1 \bullet \vec{v}_2 &= (-2)(1) + (1)(2) \\ &= -2 + 2 \\ &\boxed{= 0} \end{aligned}$$

Thus, the eigenvectors are perpendicular.

The unit vector of the eigenvectors by dividing by their length $\sqrt{2^2 + 1^2} = \sqrt{5}$

The eigenvectors $\vec{v}_1$ *and* $\vec{v}_2$ are the columns of Q.

$$Q = \frac{\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}}{\sqrt{5}}$$

$$Q^{-1} = Q^T = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\begin{aligned} A &= QDQ^T \\ &= \frac{1}{\sqrt{5}}\left(\frac{1}{\sqrt{5}}\right)\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -5 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 10 & 5 \\ -5 & 10 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} \quad \checkmark \end{aligned}$$

- Every symmetric matrix A has a complete set of orthogonal eigenvectors:

$$A = PDP^{-1} \quad \Rightarrow \quad A = QDQ^T$$

Complex Eigenvalues of Real Matrices

For real matrices, complex λ's and x's come in "conjugate pairs"

$$\text{if} \quad Ax = \lambda x \quad \text{then} \quad A\bar{x} = \bar{\lambda}\bar{x}$$

Example

Given $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Solution

The eigenvalues of A:

$$\det(A - \lambda I) = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix}$$

$$= (\cos\theta - \lambda)(\cos\theta - \lambda) + \sin^2\theta$$

$$= \cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta$$

$$= \lambda^2 - 2\lambda\cos\theta + 1 = 0$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \frac{2\cos\theta \pm \sqrt{4\left(\cos^2\theta - 1\right)}}{2} \qquad \cos^2\theta + \sin^2\theta = 1 \to \cos^2\theta - 1 = -\sin^2\theta$$

$$= \frac{2\cos\theta \pm 2\sqrt{-\sin^2\theta}}{2}$$

$$= \cos\theta \pm i\sin\theta$$

The *eigenvalues* are conjugate to each other.

For $\lambda_1 = \cos\theta + i\sin\theta$: $\quad (A - \lambda_1 I)\vec{v}_1 = 0$

$$\begin{pmatrix} \cos\theta - (\cos\theta + i\sin\theta) & -\sin\theta \\ \sin\theta & \cos\theta - (\cos\theta + i\sin\theta) \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad i\sin\theta x_1 = \sin\theta y_1$$

$$\Rightarrow \quad x_1 = iy_1$$

The eigenvectors:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\rightarrow \quad \boxed{\vec{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}}$$

$$A\vec{v}_1 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$= (\cos\theta + i\sin\theta) \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$A\vec{v}_2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= (\cos\theta - i\sin\theta) \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$|\lambda| = \sqrt{\cos^2\theta + \sin^2\theta}$$

$$\boxed{= 1}$$

This fact holds for the eigenvalues of every orthogonal matrix.

Theorem – Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.

a) A is invertible

b) $A\vec{x} = \vec{0}$ has only the trivial solution

c) The reduced row echelon form of A is I_n

d) A is expressible as a product of elementary matrices

e) $A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ matrix $\vec{b}$

f) $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ matrix $\vec{b}$

g) $\det(A) \neq 0$

h) The column vectors of A are linearly independent

i) The row vectors of A are linearly independent

j) The column vectors of A span $\mathbb{R}^n$

k) The row vectors of A span $\mathbb{R}^n$

l) The column vectors of A form a basis for $\mathbb{R}^n$

m) The row vectors of A form a basis for $\mathbb{R}^n$

n) A has a rank n.

o) A has nullity 0.

p) The orthogonal complement of the null space of A is $\mathbb{R}^n$

q) The orthogonal complement of the row space of A is $\{0\}$

r) The range of T_A is $\mathbb{R}^n$

s) T_A is one-to-one.

t) $\lambda = 0$ is not an eigenvalue of A.

u) $A^T A$ is invertible,

Exercises *Section* 4.6 – Orthogonal Diagonalization

(**1 – 10**) Determine whether the matrix *is* orthogonal

1. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

2. $\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

3. $\begin{pmatrix} \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$

4. $\begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$

5. $\begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}$

6. $\begin{pmatrix} -4 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 4 \end{pmatrix}$

7. $\begin{pmatrix} 4 & -1 & -4 \\ -1 & 0 & -17 \\ 1 & 4 & -1 \end{pmatrix}$

8. $\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \end{pmatrix}$

9. $\begin{pmatrix} \frac{\sqrt{2}}{3} & 0 & \frac{\sqrt{5}}{2} \\ 0 & \frac{2\sqrt{5}}{5} & 0 \\ -\frac{\sqrt{2}}{6} & -\frac{\sqrt{5}}{5} & \frac{1}{2} \end{pmatrix}$

10. $\begin{pmatrix} \frac{1}{8} & 0 & 0 & \frac{3\sqrt{7}}{8} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{3\sqrt{7}}{8} & 0 & 0 & \frac{1}{8} \end{pmatrix}$

(**11 – 24**)Find a matrix *P* that orthogonally diagonalizes *A*, and determine $P^{-1}AP$

11. $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

12. $A = \begin{pmatrix} 6 & 2\sqrt{3} \\ 2\sqrt{3} & 7 \end{pmatrix}$

13. $A = \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}$

14. $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

15. $A = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$

16. $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

17. $A = \begin{pmatrix} 0 & 10 & 10 \\ 10 & 5 & 0 \\ 10 & 0 & -5 \end{pmatrix}$

18. $A=\begin{pmatrix}0&3&0\\3&0&4\\0&4&0\end{pmatrix}$

19. $A=\begin{pmatrix}2&2&-2\\2&-1&4\\-2&4&-1\end{pmatrix}$

20. $A=\begin{pmatrix}-2&0&-36\\0&-3&0\\-36&0&-23\end{pmatrix}$

21. $A=\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix}$

22. $A=\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}$

23. $A=\begin{pmatrix}-7&24&0&0\\24&7&0&0\\0&0&-7&24\\0&0&24&7\end{pmatrix}$

24. $A=\begin{pmatrix}3&1&0&0\\1&3&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$

25. Find the eigenvalues of A and B and check the Orthogonality of their first two eigenvectors. Graph these eigenvectors to see the discrete sines and cosines:

$$A=\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}\qquad B=\begin{bmatrix}1&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&1\end{bmatrix}$$

The $-1, 2, -1$ pattern in both matrices is a "second derivative. Then $A\vec{x}=\lambda\vec{x}$ and $B\vec{x}=\lambda\vec{x}$ are like $\frac{d^2\vec{x}}{dt^2}=\lambda\vec{x}\ \frac{d^2x}{dt^2}=\lambda x$. This has eigenvectors $x=\sin kt$ and $x=\cos kt$ that are the bases for Fourier series. The matrices lead to "discrete sines" and "discrete cosines" that are the bases for the discrete Fourier Transform. This DFT is absolutely central to all areas of digital signal processing.

26. Suppose $A\vec{x}=\lambda\vec{x}$ and $A\vec{y}=0\vec{y}$ and $\lambda \neq 0$. Then y is in the nullspace and $\vec{x}$ is in the column space. They are perpendicular because ________. (why are these subspaces orthogonal?) If the second eigenvalue is a nonzero number β, apply this argument to $A-\beta I$. The eigenvalue moves to zero and the eigenvectors stay the same – so they are perpendicular.

27. Which of these classes of matrices do A and B belong to: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\qquad B=\frac{1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$$

Which of these factorizations are possible for A and B: LU, QR, ADP^{-1}, QDQ^T?

28. *True* or *false*. Give a reason or a counterexample.

a) A matrix with real eigenvalues and eigenvectors is symmetric.
b) A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.
c) The inverse of a symmetric matrix is symmetric
d) The eigenvector matrix S of a symmetric matrix is symmetric.
e) A complex symmetric matrix has real eigenvalues.
f) If A is symmetric, then e^{iA} is symmetric.
g) If A is Hermitian, then e^{iA} is Hermitian.
h) An $n \times n$ matrix that is orthogonally diagonalizable must be symmetric.
i) If $A^T = A$ and if vectors $\vec{u}$ and $\vec{v}$ satisfy $A\vec{u} = 3\vec{u}$ and $A\vec{v} = 4\vec{v}$, then $\vec{u} \bullet \vec{v} = 0$
j) An $n \times n$ symmetric matrix has n distinct real eigenvalues.
k) For nonzero $\vec{v}$ in $\mathbb{R}^n$, the matrix $\vec{v}\vec{v}^T$ is called a projection matrix.
l) Every symmetric matrix is orthogonally diagonalizable
m) If $B = PDP^T$, where $P^T = P^{-1}$ and D is a diagonal matrix, then B is a symmetric matrix.
n) An orthogonal matrix is orthogonally diagonalizable.
o) The dimension of an eigenspace of a symmetric matrix equals the multiplicity of the corresponding eigenvalue.

29. Find a symmetric matrix $\begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$ that has a negative eigenvalue.

a) How do you know it must have a negative pivot?
b) How do you know it can't have two negative eigenvalues?

30. Prove that A is any $m \times n$ matrix, then $A^T A$ has an orthonormal set of n eigenvectors

31. Construct a 3 by 3 matrix A with no zero entries whose columns are mutually perpendicular. Compute $A^T A$. Why is it a diagonal matrix?

32. Assuming that $b \neq 0$, find a matrix that orthogonally diagonalizes $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$

33. Suppose A is a symmetric $n \times n$ matrix and B is any $n \times m$ matrix. Show that $B^T AB$, $B^T B$, and BB^T are symmetric matrices.

34. Show that if A is an $n \times n$ symmetric matrix, then $(A\vec{x}) \bullet \vec{y} = \vec{x} \bullet (A\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$.

35. Suppose A is invertible and orthogonally diagonalizable. Explain why A^{-1} is also orthogonally diagonalizable.

36. Suppose A and B are both orthogonally diagonalizable and $AB = BA$. Explain why AB is also orthogonally diagonalizable.

37. Let $A = PDP^{-1}$, where P is orthogonal and D is diagonal, let λ be an eigenvalue of A of multiplicity k. Then λ appears k times on the diagonal of D. Explain why the dimension of the eigenspace for λ is k.

38. Suppose $A = PUP^{-1}$, where P is orthogonal and U is an upper triangular. Show that if A is symmetric, then U is symmetric and hence is actually a diagonal matrix.

39. Let $\vec{u}$ be a unit vector in $\mathbb{R}^n$, and let $B = \vec{u}\vec{u}^T$.

a) Given $\vec{x} \in \mathbb{R}^n$, compute $B\vec{x}$ and show that $B\vec{x}$ is the orthogonal projection of $\vec{x}$ onto $\vec{u}$.

b) Show that B is a symmetric matrix and $B^2 = B$.

c) Show that $\vec{u}$ is an eigenvector of B. What is the corresponding eigenvalue?

40. Let B be an $n \times n$ symmetric matrix such that $B^2 = B$. Any such matrix is called a ***projection matrix*** (or an *orthogonal projection matrix*). Given any $\vec{y} \in \mathbb{R}^n$, let $\hat{y} = B\vec{y}$ and $\vec{z} = \vec{y} - \hat{y}$.

a) Show that $\vec{z}$ is orthogonal to $\hat{y}$.

b) Let W be the column space of B. Show that $\vec{y}$ is the sum of a vector in W and a vector in $W^{\perp}$. Why does this prove that $B\vec{y}$ is the orthogonal projection of $\vec{y}$ onto the column space of B?