Exercise 1

Ox and *Oy* are bisector to 2 adjacent acute angles, $\widehat{AOB}$ and $\widehat{BOC}$ where the difference is $36°$ and $\widehat{AOC} = 90°$. *Oz* is the bisector of the angle $\widehat{xOy}$. Determine the angle $\widehat{BOz}$

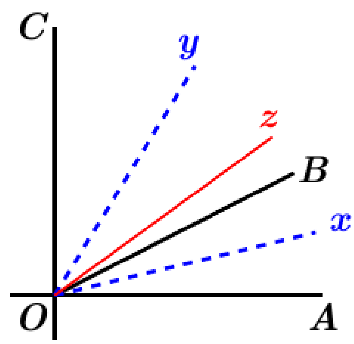
Solution

$$\begin{aligned} \widehat{BOC} - \widehat{AOB} &= 36° \\ \widehat{BOC} + \widehat{AOB} &= 90° \end{aligned}$$

$$2\ \widehat{BOC} = 126°$$

$$\widehat{BOC} = 63°$$

$$\widehat{AOB} = 27°$$



$$\widehat{xOB} = \frac{1}{2}\widehat{AOB} = \frac{27}{2}^{\circ}$$

$$\widehat{BOy} = \frac{63}{2}^{\circ}$$

$$\begin{aligned} \widehat{xOy} &= \widehat{xOB} + \widehat{BOy} \\ &= \frac{1}{2}(63° + 27°) \\ &= 45° \end{aligned}$$

$$\widehat{xOz} = \frac{45}{2}^{\circ}$$

$$\begin{aligned} \widehat{BOZ} &= \widehat{xOz} - \widehat{xOB} \\ &= \frac{1}{2}(45° - 27°) \\ &= 9° \end{aligned}$$

Exercise 2

Ox and Oy are bisector to 2 adjacent acute angles, $\widehat{AOB}$ and $\widehat{BOC}$ where the difference is $36°$. Oz is the bisector of the angle $\widehat{xOy}$. Determine the angle $\widehat{BOz}$

Solution

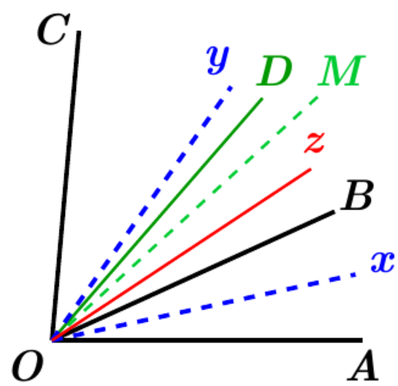
Ox is the bisector $\widehat{AOB}$ (1)

OB is the bisector $\widehat{AOD}$ (2)

OM is the bisector $\widehat{AOC}$ (3)

Oz is the bisector $\widehat{xOy}$ (4)

Oy is the bisector $\widehat{BOC}$ (5)



$\widehat{BOC} - \widehat{AOB} = 36°$

$\widehat{BOC} - \widehat{BOD} = 36°$

$\widehat{DOC} = 36°$

$$(3) \to \quad \widehat{AOM} = \frac{1}{2}\widehat{AOC} = \frac{1}{2}\left(2\widehat{AOB} + \widehat{DOC}\right) = \frac{1}{2}\left(2\widehat{AOB} + 36°\right) = \widehat{AOB} + 18°$$

$$\widehat{BOM} = \widehat{AOM} - \widehat{AOB} = \widehat{AOB} + 18° - \widehat{AOB} = 18°$$

$(1) \to \quad \widehat{BOx} = \frac{1}{2}\widehat{AOB}$

$(4) \to \quad \widehat{BOy} = \frac{1}{2}\widehat{BOC}$

$(1)+(4) \to \quad \widehat{xOy} = \frac{1}{2}\widehat{AOC}$

$(3) \to \quad \widehat{AOM} = \frac{1}{2}\widehat{AOC} = \widehat{xOy}$

$$\widehat{BOz} = \widehat{xOz} - \widehat{xOB} = \frac{1}{2}\left(\widehat{xOy} - \widehat{AOB}\right) = \frac{1}{2}\left(\widehat{AOM} - \widehat{AOB}\right) = \frac{1}{2}\widehat{BOM} = 9°$$

Exercise 3

Four consecutive half-lines (segments): OA, OB, OC, and OD formed angles such as

$$\widehat{DOA} = \widehat{COB} = 2\widehat{AOB} \qquad \text{and} \qquad \widehat{COD} = 3\widehat{AOB}$$

Calculate the angles to demonstrate that the bisectors of $\widehat{AOB}$ and $\widehat{COD}$ are in a straight line.

Solution

$\widehat{AOB} + \widehat{BOC} + \widehat{COD} + \widehat{DOA} = 360°$

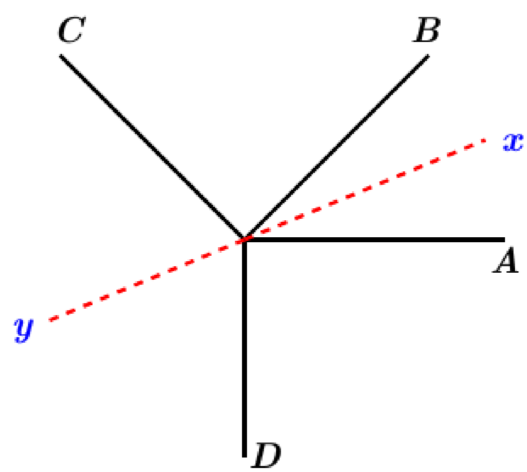
$\widehat{AOB} + 2\widehat{AOB} + 3\widehat{AOB} + 2\widehat{AOB} = 360°$

$8\widehat{AOB} = 360°$

$\boxed{\widehat{AOB} = 45°}$

$\boxed{\widehat{DOA} = \widehat{COB} = 90°}$

$\boxed{\widehat{COD} = 135°}$



Let:

Ox is the bisector $\widehat{AOB}$

Oy is the bisector $\widehat{COD}$

$$\begin{aligned}\widehat{xOy} &= \widehat{xOB} + \widehat{BOC} + \widehat{COy} \\ &= \frac{1}{2}\widehat{AOB} + 90° + \frac{1}{2}\widehat{COD} \\ &= \frac{1}{2}(45° + 135°) + 90° \\ &= \boxed{180°}\end{aligned}$$

Therefore; the bisectors of $\widehat{AOB}$ and $\widehat{COD}$ are in a straight line

Exercise 4

The segments *OA* and *OB* formed with *OX* the angles α and β.

a) Demonstrate that the bisector *OC* of the angle $\widehat{AOB}$ made with *OX* an angle $\dfrac{\alpha+\beta}{2}$.

b) Examine the cases where

   i. $\alpha+\beta=90°$

   ii. $\alpha+\beta=180°$

Solution

Given:

$$\widehat{XOA}=\alpha \quad \& \quad \widehat{XOB}=\beta$$

$$\widehat{AOC}=\frac{1}{2}\widehat{AOB}$$

$$=\frac{\beta-\alpha}{2}$$

a)

$$\widehat{XOC}=\widehat{XOA}+\widehat{AOC}$$

$$=\alpha+\frac{\beta-\alpha}{2}$$

$$=\frac{\alpha+\beta}{2}$$

b) ***i.*** If $\alpha+\beta=90°$, then

$$\widehat{XOC}=45°$$

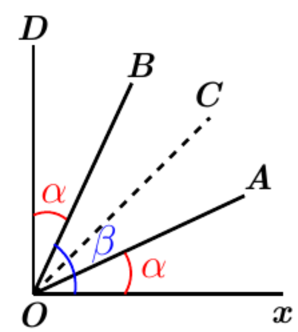
Let: $\widehat{XOD}=90°$ that implies *OC* is the bisector of $\widehat{XOD}$

Since *OC* is the bisector of $\widehat{AOB}$, then

$$\widehat{BOD}=90°-\beta \qquad \beta=90°-\alpha$$

$$=90°-90°+\alpha$$

$$=\alpha$$



ii. If $\alpha+\beta=180°$, then

$$\widehat{XOC}=90°$$

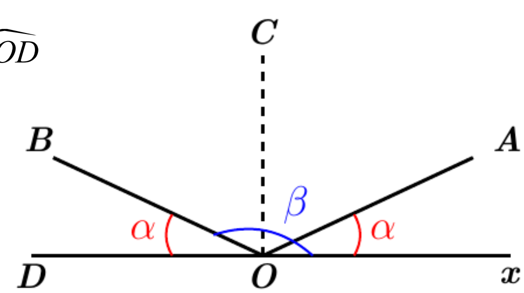
Let: $\widehat{XOD}=180°$ that implies *OC* is the bisector of $\widehat{XOD}$

Since *OC* is the bisector of $\widehat{AOB}$, then

$$\widehat{BOD}=180°-\beta \qquad \beta=180°-\alpha$$

$$=180°-180°+\alpha$$

$$=\alpha$$



Exercise 5

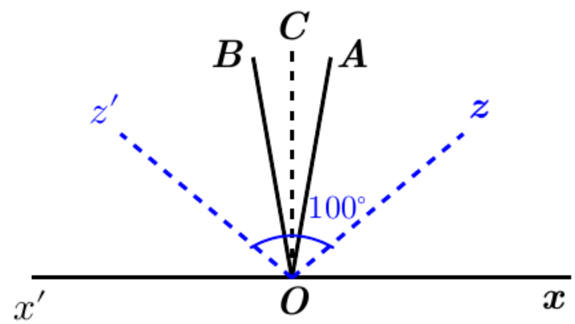
A point O takes on an infinite right $x'Ox$ be conducted the same side half-lines OA and OB, as well as the bisectors of angles $\widehat{xOA}$, $\widehat{AOB}$, and $\widehat{BOx'}$.

Calculate the angles of the figure such that the bisector of the angle $\widehat{AOB}$ is perpendicular to $x'Ox$ and the bisectors of the extreme angles formed an angle of $100°$.

Solution

Given: $\widehat{zOz'} = 100°$

$\widehat{xOC} = 90°$



OC is the bisector $\widehat{AOB}$

$\widehat{AOC} = \widehat{COB}$

Oz is the bisector $\widehat{xOA}$

$\widehat{xOz} = \widehat{zOA}$

Oz' is the bisector $\widehat{x'OB}$

$\widehat{x'Oz'} = \widehat{z'OB}$

$$\widehat{xOz} = \frac{180° - 100°}{2}$$

$$= 40°$$

$$\widehat{AOB} = 2\widehat{AOC}$$

$$= 2\left(90° - 2\widehat{xOz}\right)$$

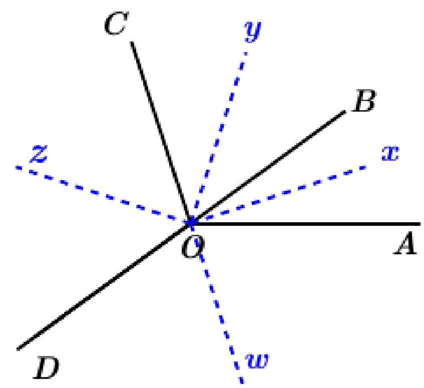
$$= 2\left(90° - 80°\right)$$

$$= 20°$$

Exercise 6

Four consecutive half-lines OA, OB, OC, and OD formed four adjacent consecutive angles which are between them like 1, 2, 3, 4.
Calculate the angles and the adjacent consecutive angles formed by their bisectors.

Solution



$$\widehat{AOB}+\widehat{BOC}+\widehat{COD}+\widehat{DOA}=360°$$

$$\widehat{AOB}+2\widehat{AOB}+3\widehat{AOB}+4\widehat{AOB}=360°$$

$$10\widehat{AOB}=360°$$

$$\widehat{AOB}=36°$$

$$\widehat{BOC}=72°$$

$$\widehat{COD}=108°$$

$$\widehat{DOA}=144°$$

$$\begin{aligned}\widehat{xOy}&=\frac{1}{2}\widehat{AOB}+\frac{1}{2}\widehat{BOC}\\&=\frac{1}{2}36°+\frac{1}{2}72°\\&=18°+36°\\&=54°\end{aligned}$$

$$\begin{aligned}\widehat{yOz}&=\frac{1}{2}\widehat{BOC}+\frac{1}{2}\widehat{COD}\\&=\frac{1}{2}72°+\frac{1}{2}108°\\&=36°+54°\\&=90°\end{aligned}$$

$$\begin{aligned}\widehat{zOw}&=\frac{1}{2}\widehat{COD}+\frac{1}{2}\widehat{DOA}\\&=\frac{1}{2}108°+\frac{1}{2}144°\\&=54°+72°\\&=126°\end{aligned}$$

$$\begin{aligned}\widehat{wOx}&=\frac{1}{2}\widehat{DOA}+\frac{1}{2}\widehat{AOB}\\&=\frac{1}{2}144°+\frac{1}{2}36°\\&=72°+18°\\&=90°\end{aligned}$$

Exercise 7

A point P is on the base BC of an isosceles triangle ABC. The two points M and N are the middle points of the segments PB and PC, respectively, which lead the perpendicular to the base BC; these perpendiculars meet AB in E, AC in F.
Demonstrate that the angle EPF is equal to A.

Solution

$\widehat{BAC} = 180° - \widehat{ABC} - \widehat{ACB}$

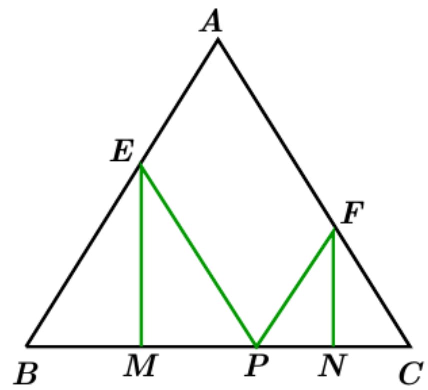
M is the middle of the segment BP and $EM \perp$ to BP, therefore

$EB = EP \quad \& \quad \widehat{EBP} = \widehat{EPB}$

N is the middle of the segment CP and $FN \perp$ to CP, therefore

$FP = FP \quad \& \quad \widehat{FPC} = \widehat{FCP}$

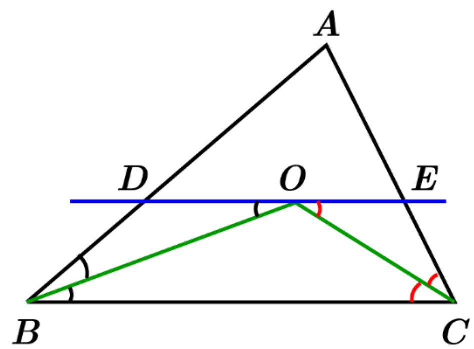
$$\begin{aligned} \widehat{EPF} &= 180° - \widehat{CPF} - \widehat{BPE} \\ &= 180° - \widehat{PFC} - \widehat{PBE} \\ &= 180° - \widehat{ABC} - \widehat{ACB} \\ &= \widehat{A} \quad \surd \end{aligned}$$



Exercise 8

Given the triangle ABC and the bisectors BO and CO of the angles of the base, where the point O is the intersection of the 2 bisectors. A line DOE passes through the point O parallel to base BC.
Prove that $DE = DB + CE$

Solution



CO is the bisector of $\widehat{BCE} \;\Rightarrow\; \widehat{BCO} = \widehat{OCE}$

$OE \parallel BC \;\Rightarrow\; \widehat{COE} = \widehat{BOC}$

$\therefore\; \widehat{EOC} = \widehat{OCE} \;\rightarrow\; \underline{OE = EC}$

Similar; BO is the bisector of $\widehat{DBC} \;\Rightarrow\; \widehat{DBO} = \widehat{OBC}$

$DO \parallel BC \;\Rightarrow\; \widehat{DOB} = \widehat{OBC}$

$\therefore\; \widehat{DOB} = \widehat{OBC} \;\rightarrow\; \underline{DO = DB}$

$DE = DO + OE$

$\underline{= DB + CE}$

Exercise 9

A right triangle ABC at A with a height AH. We drop perpendiculars HE and HD from H to sides AB and AC respectively.

a) Prove that $DE = AH$

b) Prove that AM is perpendicular to DE, where M is the middle point of BC.

c) Prove that MN (N is the middle point of AB) and the segment Bx (parallel to DE) are intersect on AH.

d) Prove that AM and HD are intersect on Bx.

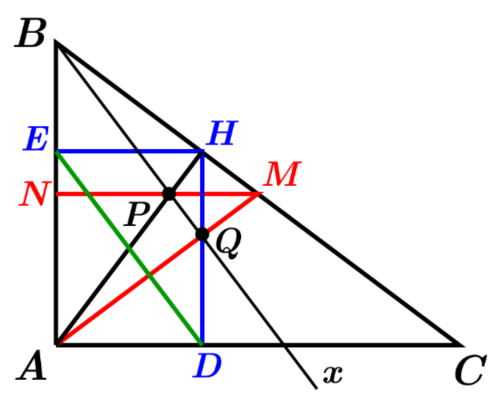
Solution

a) The triangles AEH and ADH are right triangles and angle A is right angle.
Then $AEHD$ is a rectangle.
Therefore, $DE = AH$

b) A middle point of a hypotenuse of a right triangle is the center of the circle of that triangle.
Therefore, $MC = MA = MB$

That implies to: $\widehat{MAC} = \widehat{MCA}$

From the rectangle $ADHE$: $\widehat{EAH} = \widehat{EDH}$



$\widehat{EAH} + \widehat{HAM} + \widehat{MAC} = 90°$ $\qquad$ $\widehat{HAM} + \widehat{MAC} = \widehat{HAC}$

$\widehat{EAH} + \widehat{HAC} = 90°$

$\widehat{EAH} + 90° - \widehat{MCA} = 90°$

$\widehat{EAH} = \widehat{MCA} = \widehat{EDH} = \widehat{MAC}$

$\widehat{ADE} + \widehat{EDH} = 90°$

$\widehat{ADE} + \widehat{MAD} = 90°$

Therefore, AM is perpendicular to DE.

c) N is the middle point of $AB \Rightarrow NA = NB$

Bx parallel to $DE \Rightarrow \widehat{ABx} = \widehat{AED} = \widehat{EDH} = \widehat{EAH}$

Let point P the intersection of Bx and AH. Since $\widehat{ABP} = \widehat{BAP}$, then the triangle BPA is an isosceles. PN is the perpendicular to AB as well MN. Which gives us that points M, P, N are on the same line.
Therefore, segment MN and AH intersect at point P.

d) Let Point Q be the intersection of AM and Bx.

$\widehat{ABQ} = \widehat{BAH}$ $\quad$ & $\quad$ $\widehat{BAQ} = \widehat{ABH}$

Then, the triangles BHA and BQA are equivalent, therefore $AQ \perp BQ$ with hypotenuse AB.

$HQ \,//\, AB$, line HQ has to be perpendicular to AC.

AM and HD are intersecting on Bx at Q.

Exercise 10

Given an isosceles triangle ABC with a peak at A. Extend base BC the length $CD = AB$, then extend AB of a length $BE = \frac{1}{2}BC$, at the end draw a line EHF, H is the middle point of BC and F is located on AD.

a) Prove that $\widehat{ADB} = \frac{1}{2}\widehat{ABC}$

b) Prove that $EA = HD$

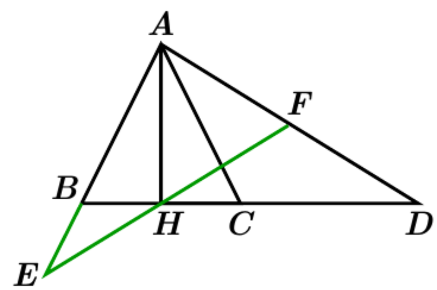
c) Prove that $FA = FD = FH$

d) Calculate the value of the angles $\widehat{AFH}$ and $\widehat{ADB}$ where $\widehat{BAC} = 58°$.

Solution

a) Triangle ABC is isosceles, then $\widehat{ABC} = \widehat{ACB}$

Since, $CD = AB = AC$, then $\widehat{CAD} = \widehat{ADC}$



$$2\widehat{ADC} = 180° - \widehat{ACD}$$

$$2\widehat{ADC} = 180° - \left(180° - \widehat{ACB}\right)$$

$$2\widehat{ADC} = \widehat{ACB}$$

$$\widehat{ADB} = \frac{1}{2}\widehat{ACB}$$

$$= \frac{1}{2}\widehat{ABC}$$

b) $BE = \frac{1}{2}BC$ $\quad$ H the middle point of BC

$= HC$

$CD = AB$

$HC + CD = BE + AB$

$EA = HD$ ✓

c) $\widehat{ADH} = \frac{1}{2}\widehat{ABD}$

$$= \frac{1}{2}\left(180° - \widehat{HBE}\right)$$

$$= \frac{1}{2}\left(180° - 180° + 2\widehat{BHE}\right)$$

$$= \widehat{BHE}$$

$\Rightarrow$ $FD = FH$

$\widehat{AHF} = 90° - \widehat{FHD}$

$= 90° - \widehat{ADH}$ $\quad$ $(\Delta\ HDA)$

$= 90° - \left(90° - \widehat{HAF}\right)$

$= \widehat{HAF}$

$\Rightarrow \quad \boxed{FA = FH}$

$FA = FD = FH$ ✓

d) $\widehat{BAC} = 58°$

$$\widehat{ADB} = \frac{1}{2}\widehat{ACB}$$
$$= \frac{1}{2}\left(\frac{1}{2}\left(180° - \widehat{BAC}\right)\right)$$
$$= \frac{1}{4}(180° - 58°)$$
$$= \frac{122}{4}^{\circ}$$
$$\boxed{= \frac{61}{2}^{\circ}} \qquad \boxed{= 30.5°}$$

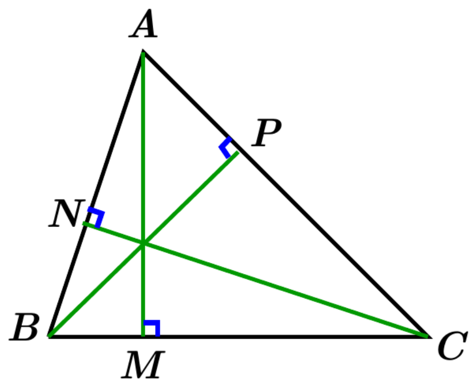
Triangle *AFH* is isosceles then,

$$\widehat{AFH} = 180° - \widehat{HFD}$$
$$= 180° - \left(180° - 2\widehat{FDH}\right)$$
$$= 2\widehat{FDH}$$
$$= 2\frac{61°}{2}$$
$$\boxed{= 61°}$$

Exercise 11

Demonstrate that the heights of a triangle share the angles of triangle that equal to each other.

Solution



Consider the 2 right triangles APB and ANC, which they have the same angle A.

Therefore, $\widehat{ABP} = \widehat{ACN}$.

Similar, consider the 2 right triangles BPC and AMC, which they have the same angle C.

Therefore, $\widehat{MAC} = \widehat{CBP}$.

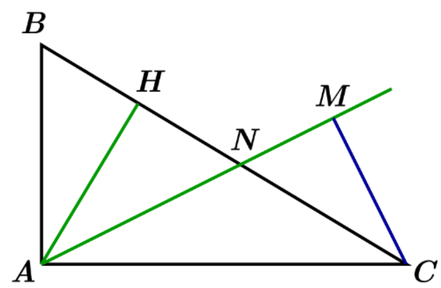
Similar, consider the 2 right triangles BNC and AMB, which they have the same angle B.

Therefore, $\widehat{BCN} = \widehat{BAM}$.

Exercise 12

A right triangle ABC at A where $AB < AC$, drop a perpendicular AH from A to the hypotenuse BC where $HN = HB$. From C drops a perpendicular CM at AN. Demonstrate that BC is the bisector of the angle $\widehat{ACM}$.

Solution



Consider the 2 right triangles ABC and ABH with a common angle B, then

$\widehat{BAH} = \widehat{ACB}$

Given: $HN = HB$, then $\widehat{HAN} = \widehat{BAH} = \widehat{ACB}$

$$\widehat{NAC} = 90° - \widehat{HAB} - \widehat{HAN}$$
$$= 90° - 2\widehat{HCA}$$

Consider the 2 right triangles AHN and CMC, where $\widehat{HNA} = \widehat{MNC}$

Therefore, $\widehat{HAN} = \widehat{NCM}$

Since $\widehat{HAN} = \widehat{ACB}$

Then $\widehat{ACB} = \widehat{MCB}$

Therefore, BC is the bisector of the angle $\widehat{ACM}$

Exercise 13

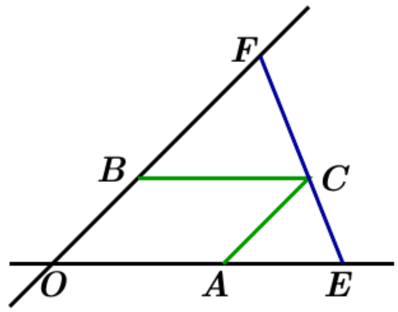
On the sides of an angle that it takes the length OA and OB, so that $OA+OB=\ell$ (is given) and construct a parallelogram $OABC$. What is the place of the summit C of parallelogram?

Solution

Let segment OE extension of segment OA such that $OE=\ell$
Let segment OF extension of segment OB such that $OF=\ell$

Then, the triangle OEF is an isosceles.

$\widehat{OEF}=\widehat{OFE}=90^\circ-\frac{1}{2}\widehat{EOF}$



$OA+OB=\ell$

$\begin{cases} OA+AC=\ell \\ OA+AE=\ell \end{cases} \Rightarrow AC=AE$

$\begin{cases} OB+BC=\ell \\ OB+BF=\ell \end{cases} \Rightarrow BC=BF$

$\widehat{OEF}=\widehat{OFE}=\widehat{FCB}=\widehat{ACE}$

Therefore, the point C, E, and F are aligned.

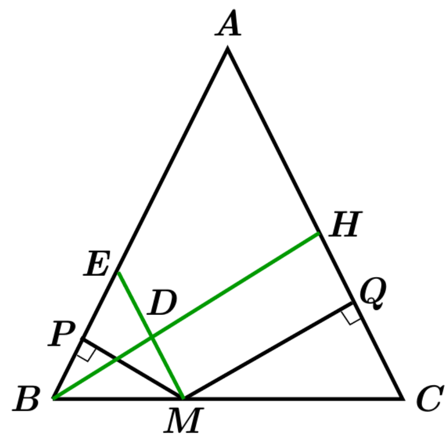
Exercise 14

Demonstrate that the sum of distances from a point M on the base BC of an isosceles triangle ABC to the sides equal a constant.

Solution

The shortest distance from a point to a line is the perpendicular from that point to the line. Therefore, let:

$MP \perp AB$
$MQ \perp AC$



Let $BH \perp AC$ (Shortest distance from B to side AC.)

Let D be the point of intersection ME and BH.

Let $ME \parallel AC$
Where the point E is the intersection of the lines MD and AB.

Since $MD \parallel AC$ then $\widehat{DMB} = \widehat{ACB}$

Since triangle ABC is an isosceles
$\widehat{DMB} = \widehat{ACB} = \widehat{PBM}$

The right triangles BPM and BDM at P & D and have the same hypotenuse, then
$\Rightarrow \quad |MP| = |BD|$

$MD \parallel HQ$ and DH & $MQ \perp HQ$
$\Rightarrow \quad |MQ| = |DH|$

$$\begin{aligned} |MP| + |MQ| &= |BD| + |DH| \\ &= |BH| \\ &= \textbf{\textit{constant}} \end{aligned}$$

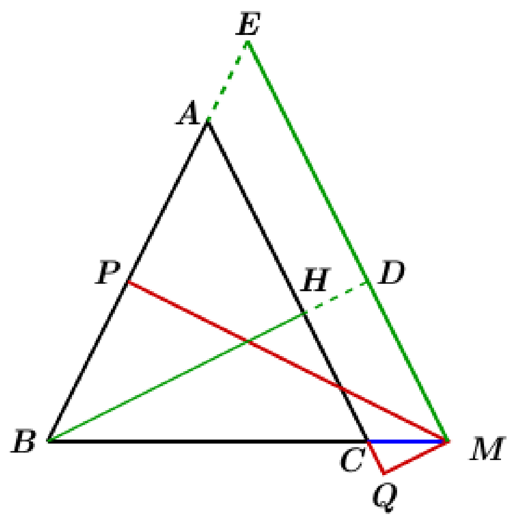
Therefore; the sum of distances from a point M on the base BC of an isosceles triangle ABC to the sides equal a constant.

Exercise 15

Demonstrate that the difference of distances from a point M taken on the extension of the base BC of an isosceles triangle ABC to the sides equal a constant.

Solution

The shortest distance from a point to a line is the perpendicular from that point to the line. Therefore, let:



$MP \perp AB$
$MQ \perp AC$

Let $BH \perp AC$ (Shortest distance from B to side AC.)

Let D be the point of intersection ME and BH.

Let $ME \parallel AC$
Where the point E is the intersection of the extensions of the lines MD and AB.

Since $MD \parallel AC$ then $\widehat{DMB} = \widehat{ACB}$

Since triangle ABC is an isosceles
$\widehat{DMB} = \widehat{ACB} = \widehat{PBM}$

The right triangles BPM and BDM at P & D and have the same hypotenuse, then

$\Rightarrow \quad |MP| = |BD|$

$MD \parallel HQ$ and DH & $MQ \perp HQ$

$\Rightarrow \quad |MQ| = |DH|$

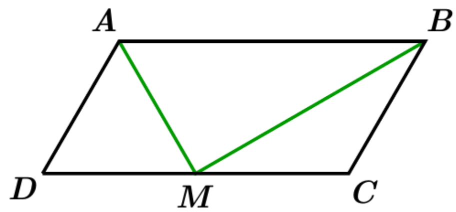
$$\begin{aligned}|MP| - |MQ| &= |BD| - |DH| \\ &= |BH| \\ &= \textbf{constant}\end{aligned}$$

Therefore; the difference of distances from a point M taken on the extension of the base BC of an isosceles triangle ABC to the sides equal a constant.

Exercise 16

Consider a parallelogram $ABCD$ in which $CD = 2AD$. In the joint A and B the middle M of BC. Prove that the angle $\widehat{AMB}$ is a right angle.

Solution



Since the point M is the middle of side BC, then

$MD = MC = \frac{1}{2}CD$

$\Rightarrow \quad MD = AD = BC$

Therefore; the triangles ADM and BCM are isosceles at D and C respectively.
Which implies that $MA = MB$

Let O be the middle point of the side AB, and $OA = OB = AD$

O and M are middle of the parallelogram $ABCD$, that implies
$OM = BC = AD$

$\Rightarrow \quad OA = OB = OM$

The triangle MAB inscribed in a circle with center at O and diameter AB, that will imply that is a right triangle at the point M.

Or

$\widehat{AMD} = \frac{1}{2}\left(180° - \widehat{MDA}\right)$

$\widehat{BMC} = \frac{1}{2}\left(180° - \widehat{MCB}\right)$

$\widehat{ADM} + \widehat{MCB} = 180°$

$\widehat{DMA} + \widehat{AMB} + \widehat{BMC} = 180°$

$$\begin{aligned}\widehat{AMB} &= 180° - \left(\widehat{BMC} + \widehat{DMA}\right)\\ &= 180° - \left(90° - \frac{1}{2}\widehat{MDA} + 90° - \frac{1}{2}\widehat{MCB}\right)\\ &= \frac{1}{2}\left(\widehat{MDA} + \widehat{MCB}\right)\\ &= \frac{1}{2}(180°)\\ &\underline{= 90°}\end{aligned}$$

Exercise 17

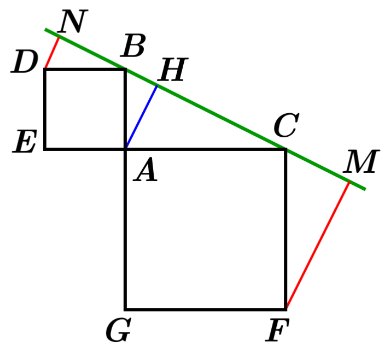
From the sides *AB* and *AC* of a right triangle *ABC* at *A*, draw two squares *ABDE* and *ACFG*. Then lead *DN* and *FM* perpendicular to the line *BC*.

a) Prove that $DN + FM = BC$

b) Prove that the points *D, A, F* on a straight line.

c) Prove that the lines *DE* and *FG* contribute on the extension of the height *AH*.

Solution



a) Let consider the 2 right triangles *DNB* & *BHA* at points *N* & *H* respectively, with $DB = AB$. Then

$$\widehat{HAB} = 90° - \widehat{ABH}$$
$$= 90° - \left(90° - \widehat{NBD}\right)$$
$$= \widehat{NBD}$$
$$\Rightarrow \quad \widehat{BDN} = \widehat{ABH}$$

∴ The 2 triangles are equals, which implies that $DN = BH$

Similar, for the 2 right triangles *CMF* & *AHC* at points *M* & *H* respectively, with $AC = CF$. Then

$$\widehat{HAC} = 90° - \widehat{ACH}$$
$$= 90° - \left(90° - \widehat{MCF}\right)$$
$$= \widehat{MCF}$$
$$\Rightarrow \quad \widehat{ACH} = \widehat{CFM}$$

∴ The 2 triangles are equals, which implies that $FM = HC$

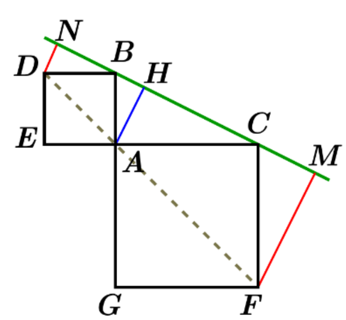
$$DN + FM = BH + HC$$
$$= BC \quad \checkmark$$

b) Since *ABDE* is a square, then $\widehat{BAD} = 45°$

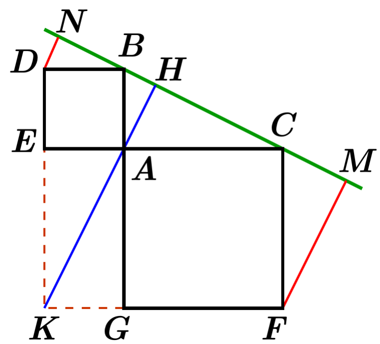
And *ACFG* is a square, then $\widehat{CAF} = 45°$

$$\widehat{DAF} = \widehat{DAB} + \widehat{BAC} + \widehat{CAF}$$
$$= 45° + 90° + 45°$$
$$= 180°$$

∴ The points *D*, *A*, & *F* are on a straight line.



c) Let the point K be the intersection of the extension of the sides DE and FG.
Which will result of $GKEA$ is a rectangle with $AE = GK \quad \& \quad EK = AG$



Consider the 2 right triangles BAC & KGA at points A & G respectively with $AE = AB = GK$

$\widehat{ACB} = \widehat{GAK} = \widehat{ACH}$

From the right triangle AHC:

$\widehat{HAC} + \widehat{ACH} = 90°$

$\rightarrow \quad \widehat{HAC} + \widehat{KAG} = 90°$

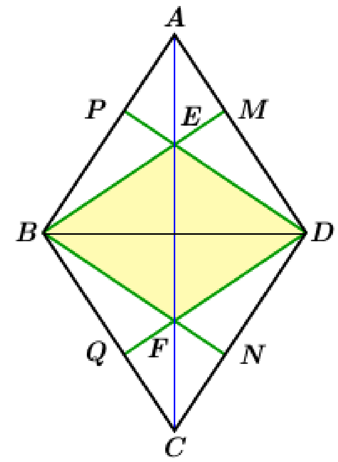
$$\begin{aligned}\widehat{HAC} + \widehat{CAG} + \widehat{KAG} &= \left(\widehat{HAC} + \widehat{KAG}\right) + \widehat{CAG} \\ &= 90° + 90° \\ &= 180°\end{aligned}$$

∴ The points K, A, & H are on a straight line.

Exercise 18

Given a diamond *ABCD*; the peak *B* and *D*, the same the perpendiculars *BM, BN, DP, DQ* on opposite sides. These perpendiculars are intersected at *E* and *F*.

Demonstrate that the angles of the quadrilateral *BFDE* are equals to the diamond and which is a diamond itself.



Solution

From the right triangles *BPD* & *BMD*, that implies $\widehat{MBD} = \widehat{PDB}$

$\Rightarrow \quad \widehat{EBD} = \widehat{EDB}$

Similar, from the right triangles *BND* & *BQD*, that implies $\widehat{NBD} = \widehat{QDB}$

$\Rightarrow \quad \widehat{FBD} = \widehat{FDB}$

$\widehat{EBD} + \widehat{DBF} = \widehat{EDB} + \widehat{BDF}$

$\widehat{EBF} = \widehat{EDF}$

Since, $AC \perp BD$, then $EF \perp BD$

The 2 triangles *EBF* & *EDF* have *EF* as a common side and $\widehat{EBF} = \widehat{EDF}$, then

$\widehat{BEF} = \widehat{DEF} = \widehat{BFE} = \widehat{DFE}$

$\widehat{BED} = \widehat{BFD}$

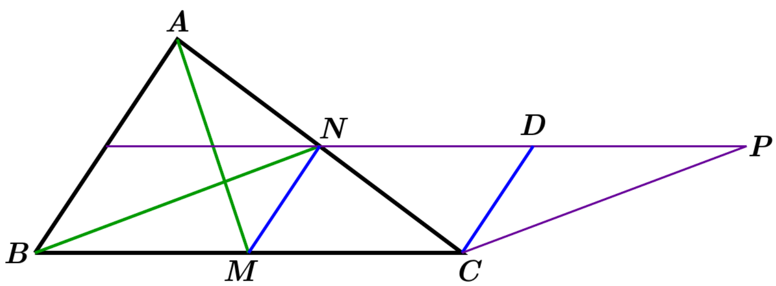
Therefore; the angles of the quadrilateral *BFDE* are equals to the diamond and which is a diamond itself.

Exercise 19

In a triangle ABC, we trace the median AM and BN and from N a parallel to BC, from C a parallel to BN; that the two sides intersect at a point P. Let D be the middle point of the segment PN.

Prove that CD is parallel to MN.

Solution



Since the points M & N are middle of the sides BC & AC of the triangle ABC, then $MN \parallel AB$

Given: $NP \parallel MC$

$BN \parallel CP$

Since M & D are the middle points of the segments BC and NP respectively, then $BN \parallel CP \parallel MD$

Therefore, $BNPC$ is a parallelogram, and $MC = ND$.
Since $MC = ND$ & $MN = CD$

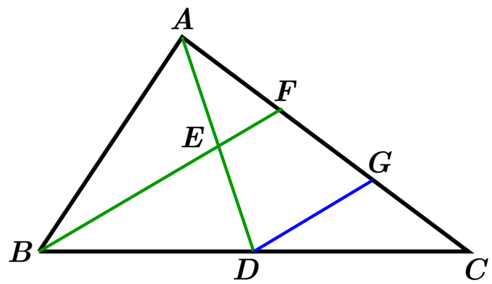
Therefore; $MCDC$ is a parallelogram which implies to CD parallel to MN

Exercise 20

The median AD of a given triangle ABC to the side BC. The same the median BE to the side AD which intersect AC at a point F.

Prove that where $AF = \frac{1}{3} AC$

Solution



Let DG be parallel to segment BEF.

Given: E is the middle point of the segment $AD \Rightarrow AE = ED$

D is the middle point of the segment $BC \Rightarrow BD = DC$

Since $EF \parallel DG$, and $AE = ED$, that implies $AF = FG$

Consider the triangles CDG and CBF:

$EF \parallel DG$, and $CD = DB$, that implies $GC = FG$

That will imply to: $AF = FG = GC$

$$\begin{aligned} AC &= AF + FG + GC \\ &= 3AF \end{aligned}$$

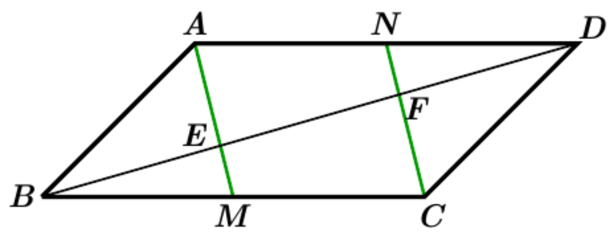
Therefore; $\boxed{AF = \frac{1}{3} AC}$

Exercise 21

In a parallelogram $ABCD$, from the points peak A and C joint the middle of opposite sides at M and N respectively.

Prove that the diagonal BD is divided in three equal parts.

Solution



M is the middle point of the segment BC, then $BM = CM$

N is the middle point of the segment AD, then $NA = ND$

From these, implies that $AM \parallel CN$.

From the triangles BEM & BCF, and since $ME \parallel CF$

It will give us that $BE = EF$

From the triangles DFN & DEA, and since $AE \parallel FN$

It will give us that $\Rightarrow \quad DF = EF$

Therefore, $BE = EF = DF$

$$BD = BE + EF + FD$$

$$= 3BE$$

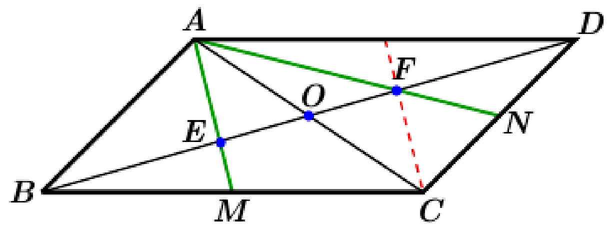
Therefore; the diagonal BD is divided in three equal parts

Exercise 22

In a parallelogram $ABCD$, from the point peak A, extend to the middle of sides BC and DC at M and N respectively.

Prove that the diagonal BD is divided in three equal parts.

Solution



Let a point E be the intersection of the segments AM & BD.
Let a point F be the intersection of the segments AN & BD.

Le O be the intersection of the both diagonal AC & BD.
From the triangles BEM & BCF, and since $ME \parallel CF$

$$\Rightarrow \quad BE = EF$$

Similar, $\Rightarrow \quad DF = EF$

$$BO = OF \quad \rightarrow \quad OE = OF$$

$$\begin{aligned} BO &= BE + EO \\ &= BE + \frac{1}{2}BE \\ &= \frac{3}{2}BE \end{aligned}$$

$$\begin{aligned} BE &= \frac{2}{3}BO \\ &= \frac{2}{3}\left(\frac{1}{2}BD\right) \\ &= \frac{1}{3}BD \end{aligned}$$

$$\begin{aligned} DF &= \frac{2}{3}DO \\ &= \frac{2}{3}\left(\frac{1}{2}BD\right) \\ &= \frac{1}{3}BD \end{aligned}$$

Therefore; the diagonal BD is divided in three equal parts

Exercise 23

Consider a triangle ABC with a bisector AF of the angle A. by F, we lead FE parallel to AB, and by E we lead ED parallel to BC.

Prove that $AE = BD$

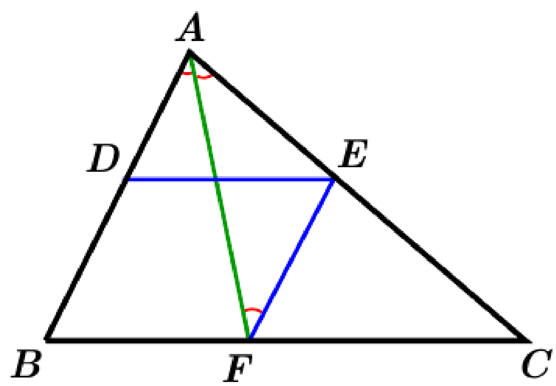
Solution

Given: $\widehat{EAF} = \widehat{FAB}$

Since $FE \,//\, AB$, then

$$\widehat{FEC} = \widehat{BAE} = 2\widehat{EAF}$$

$$\begin{aligned}\widehat{AEF} &= 180° - \widehat{FEC} \\ &= 180° - 2\widehat{EAF}\end{aligned}$$



Consider the triangle AEF:

$$\widehat{EAF} + \widehat{EFA} + \widehat{AEF} = 180°$$

$$\widehat{EAF} + \widehat{EFA} + 180° - 2\widehat{EAF} = 180°$$

$$\widehat{EFA} - \widehat{EAF} = 0°$$

$$\widehat{EFA} = \widehat{EAF}$$

∴ Triangle AEF is isosceles

$\Rightarrow$ $\underline{AE = EF}$

Given $DE \,//\, BF$ & $FE \,//\, DB$

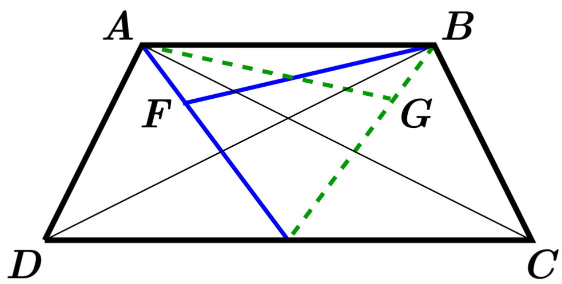
$FEDB$ is a parallelogram;

Then, $\underline{EF = DB = AE}$

Exercise 24

Given an isosceles trapezoid $ABCD$ $\left(AD = BC\right)$ with diagonals AC and BD. The bisector of angles $\widehat{DAB}$ and $\widehat{DBA}$ intersect in F, and the bisector of angles $\widehat{CBA}$ and $\widehat{CAB}$ intersect in G.
Demonstrate that FG is parallel to AB

Solution



Consider the 2 triangles ABD & ABC:

- Both has the AB as common
- $AD = BC$

That implies to: $\widehat{ABD} = \widehat{CAB}$

Since BF is the bisector of the angle $\widehat{ABD}$

$\widehat{ABF} = \widehat{FBD}$

$$\Rightarrow \quad \widehat{ABF} = \frac{1}{2}\widehat{ABD}$$

$$= \frac{1}{2}\widehat{CAB}$$

$$= \frac{1}{2}\left(2\widehat{BAG}\right)$$

$$= \widehat{BAG}$$

$\underline{\widehat{ABF} = \widehat{BAG}}\,|$

From the 2 triangles AFB & AGB

- Both has the AB as common
- $\widehat{ABF} = \widehat{BAG}$

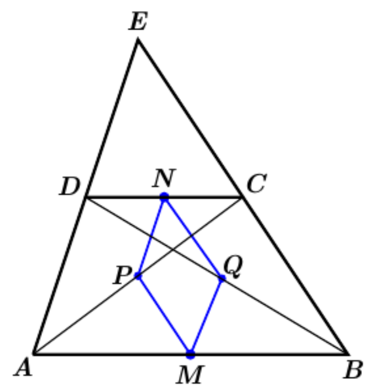
$\underline{FG \,//\, AB}\,|$

Exercise 25

Let M and N be the middle points of the bases AB and CD of a trapezoid $ABCD$.
Let P and Q be the middle points of the diagonals AC and BD respectively.

Demonstrate that the angles $\widehat{M}$ and $\widehat{N}$ of quadrilateral $MNPQ$ are equals to the angle formed by extending the sides not parallel to BC and AD, where intersect at point E.

Solution



Since N is the mid-point of the side DC, and P is the mid-point of the side AC, then

$$\Rightarrow \quad NP \parallel AD$$

Since M is the mid-point of the side AB, and Q is the mid-point of the side DB, then

$$\Rightarrow \quad QM \parallel AD$$

$$\therefore \quad NP \parallel QM \parallel AE$$

Since N is the mid-point of the side DC, and Q is the mid-point of the side DB, then

$$\Rightarrow \quad NQ \parallel CB$$

Since M is the mid-point of the side AB, and P is the mid-point of the side AC, then

$$\Rightarrow \quad MP \parallel CB$$

$$\therefore \quad NQ \parallel MP \parallel BE$$

$$\rightarrow \begin{cases} NP \parallel QM \\ NQ \parallel PM \end{cases}$$

$$\therefore \quad \widehat{PMQ} = \widehat{PNQ}$$

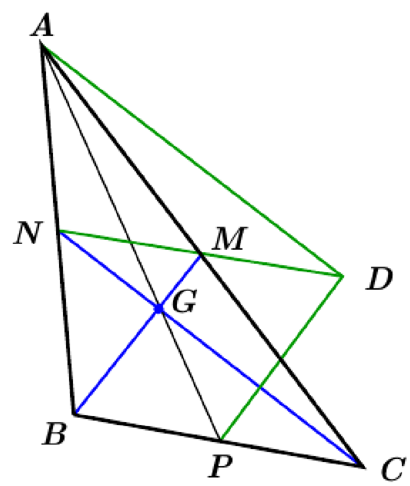
Exercise 26

In a triangle *ABC*, the medians segment *BM* and *CN* intersect in right angles and the measurement are 3 and 6 units respectively.

1. Construct a geometrical to the triangle *ABC*.
2. In the trace of third median *AP* which leads *MN* extension such the distance $MD = MN$, which lead to the segments *AD* and *PD*. Calculate *AD* and *DP*.
3. What is the natural of the triangle APD?

Solution

1. Since *M* and *N* are the middle point of the sides *AC* & *AB*, then



$$BG = \frac{2}{3}BM$$
$$= \frac{2}{3}(3)$$
$$= 2 \quad units$$

Similar,

$$CG = \frac{2}{3}CN$$
$$= \frac{2}{3}(6)$$
$$= 4 \quad units$$

Wish, we lead to: $GM = 1$ & $GN = 2$

We can construct 2 perpendicular lines intersect at a point *G*, then we use to measure the distance from the point *G* to get the points *B*, *C*, *M*, & *N*.

By extending the segment *BN* and *CM* with equal distance and which it will intersect at point *A*.

2. Since $ND \parallel BC$ & $MD = MN$

The parallelogram *BPDM*, $BP = MD = MN$

Then $PD = MB = 3 \;\; units$

$AD \parallel CN$ and *M* is the intersection of the diagonals of the parallelogram *ADCN*, then

$AD = CN = 6 \;\; units$

3. $PD \parallel BN$ & $MB \perp CN$, then $PD \perp CN$

$AD \parallel CN$ & $PD \perp CN$, then $AD \perp PD$

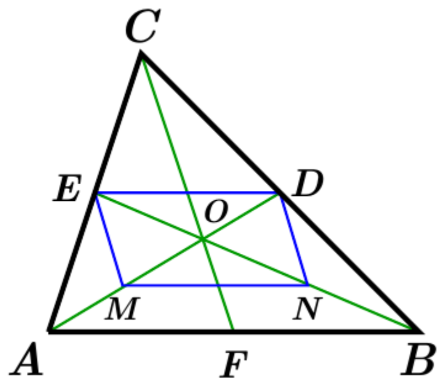
Therefore; the triangle *ADP* is right triangle at point *D*.

Exercise 27

Inside the triangle ABC, the median AD, BE, and CF intersect at a point O. We take M the middle point of the segment OA, N the middle point of segment OB.

Show that $DEMN$ is a parallelogram.

Solution



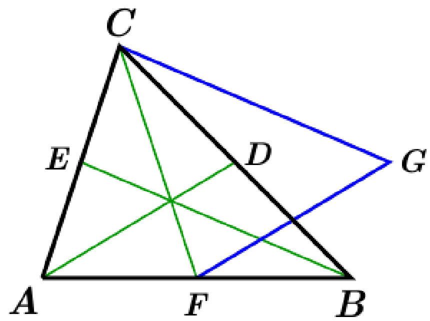
DE & MN are parallel to AB and equals to $\frac{1}{2}|AB|$

That implies to $ME \parallel DN$.

Therefore; $DEMN$ is a parallelogram

Exercise 28

Inside the triangle ABC, the median AD, BE, and CF intersect at a point O. From the point F, draw FG parallel to AD and are equals, then joint A to G.



Show that $CG = BE$.

Solution

Given: $FG \parallel AD$ & $FG = AD$

Then, the quadrilateral $AFGD$ is a parallelogram which it results to $DG \parallel AF$ & $DG = AF$.

$\rightarrow$ $DG \parallel BF$

Since F is the mid-point of the side AB, then $AF = DG = FB$.

Then, the quadrilateral $BFDG$ is a parallelogram which it results to $FD \parallel BG$ & $DF = GB$.

So, $FD \parallel BG \parallel CE$

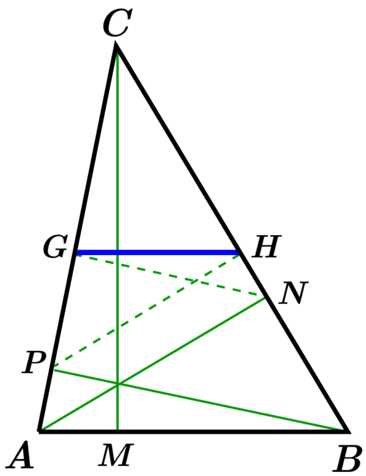
Given that D & F are midpoints, then $DF = \frac{1}{2}AC = CE$

And $CE \parallel BG$ & $DF = CE$, then $BGCE$ is a parallelogram.

Therefore, $CG = BE$

Exercise 29

The height of a triangle ABC (each of the three lines passing through a vertex of the triangle and perpendicular to the opposite side to this vertex) are AN, BP, CM.

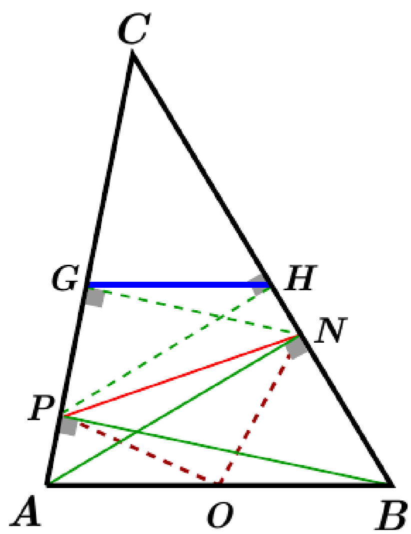


From P, let PH perpendicular to BC, same from N, let NG perpendicular to AC.

Show that GH is parallel to AB.

Solution

Let the point O be the middle of the segment AB.
Then O is the center of the 2 triangles ANB & APB.



The triangle OBN is isosceles, implies to $\widehat{ONB} = \widehat{OBN}$

The triangle OPA is isosceles, implies to $\widehat{OPA} = \widehat{OAP}$

$$\widehat{PON} = 180° - \left(\widehat{NOB} + \widehat{POA}\right)$$

$$= 180° - \left(180° - 2\widehat{NBO} + 180° - 2\widehat{OAP}\right)$$

$$= 2\widehat{B} + 2\widehat{A} - 180°$$

Consider the triangle PON with $OP = ON$, then

$$\widehat{OPN} = \widehat{ONP}$$

$$\widehat{OPN} = \frac{1}{2}\left(180° - \widehat{PON}\right)$$

$$= \frac{1}{2}\left(180° - 2\widehat{B} - 2\widehat{A} + 180°\right)$$

$$= 180° - \widehat{B} - \widehat{A}$$

$$\widehat{APN} = \widehat{APO} + \widehat{OPN}$$

$$= \widehat{A} + 180° - \widehat{B} - \widehat{A}$$

$$= 180° - \widehat{B}$$

$$\widehat{CPN} = 180° - \widehat{APN}$$

$$= 180° - 180° + \widehat{B}$$

$= \widehat{B}$

From the 2 right triangles *CHP* & *CGN*

$\widehat{HPC} = \widehat{GNC}$

$\widehat{GHN} = 180° - \widehat{HGN} - \widehat{HNG}$

$\qquad = 180° - \widehat{HGN} - \widehat{CPH}$

$180° - \widehat{GHC} = 180° - \widehat{HGN} - \widehat{CPH}$

$\widehat{GHC} = \widehat{HGN} + \widehat{CPH}$

Let *Q* be the middle point of the segment *PN*.
Since *PGN* & *PHN* are right triangle with the same hypothesis.
Then, the triangles *HQN* & *GQN* are isosceles.

$\widehat{H} = \widehat{N} \quad \& \quad \widehat{G} = \widehat{P}$

$\widehat{GQH} = 180° - \left(180° - 2\widehat{P} + 180° - 2\widehat{N}\right)$

$\qquad = 2\widehat{P} + 2\widehat{N} - 180°$

Since $QG = QH \quad \Rightarrow \quad \widehat{QGH} = \widehat{QHG}$

$\widehat{QGH} = \frac{1}{2}\left(180° - \widehat{GQH}\right)$

$\qquad = \frac{1}{2}\left(180° - 2\widehat{P} - 2\widehat{N} + 180°\right)$

$\qquad = 180° - \widehat{P} - \widehat{N}$

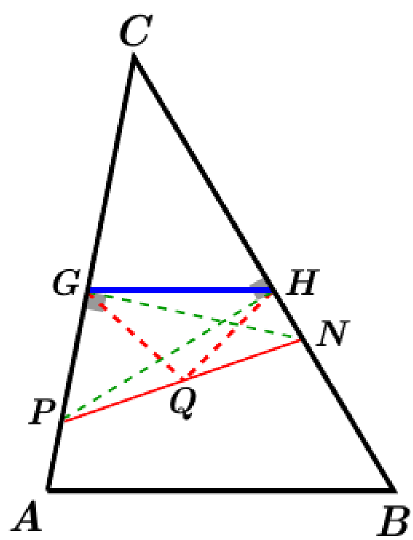
$\widehat{HGN} = \widehat{QGH} - \widehat{QGN}$

$\qquad = 180° - \widehat{P} - \widehat{N} - 90° + \widehat{QGP}$

$\qquad = 90° - \widehat{P} - \widehat{N} + \widehat{P}$

$\qquad = 90° - \widehat{N}$

$\qquad = \widehat{NPH}$



$\widehat{CHG} = \widehat{HGN} + \widehat{CPH}$

$\qquad = \widehat{NPH} + \widehat{CPH}$

$\qquad = \widehat{NPC}$

$\qquad = \widehat{B}$

Therefore, $GH \parallel AB$