

Section 2.2 – Future Value of an Annuity

Annuity is any sequence of equal periodic payments.

Deposit is equal payment each interval

There are two basic types of annuities.

An **annuity due** requires that the first payment be made at the beginning of the first period.

An **ordinary annuity** requires that the first payment is made at the end of the first period. We will only deal with **ordinary annuities**.

\$100 every 6 months, rate $r = 0.06$ compounded semiannually

$$A = P \left(1 + \frac{.06}{2} \right)^{2t} = 100(1.03)^{2t}$$

Years			1		2		3	
# of Period	0	1	2	3	4	5	6	
								= 100
								= 100(1.03)
								= 100(1.03) ²
								= 100(1.03) ^{2 \cdot \frac{3}{2}} = 100(1.03)^3}
								= 100(1.03) ⁴
								= 100(1.03) ⁵
								Σ

$$\begin{aligned}
 S &= 100 + 100(1.03) + 100(1.03)^2 + 100(1.03)^3 + 100(1.03)^4 + 100(1.03)^5 \\
 &= 100(1 + 1.03 + 1.03^2 + 1.03^3 + 1.03^4 + 1.03^5) \\
 &= 100 \frac{1.03^6 - 1}{.03} \\
 &= \$646.84
 \end{aligned}$$

$$a + ar + ar^2 + ar^3 + \dots + ar^{n-1} = a \frac{r^n - 1}{r - 1}$$

$$\therefore S = R + R(1+i) + R(1+i)^2 + R(1+i)^3 + \dots + R(1+i)^{n-1}$$

$$= R \frac{(1+i)^n - 1}{1+i-1}$$

$$= R \frac{(1+i)^n - 1}{i}$$

Future Value of an Ordinary Annuity (*FV*)

$$FV = PMT \frac{(1+i)^n - 1}{i} = PMT {}_s \overline{n}|i$$

PMT: Periodic payment

i: Rate per period $i = \frac{r}{m}$

n: Number of payment

Example

What is the value of an annuity at the end of 10 years if \$1,000 is deposited every 6 months into an account earning 8% compounded semiannually? How much of this value is interest?

Solution

Given: 8% compounded semiannually $\Rightarrow i = \frac{r}{m} = \frac{.08}{2} = .04$

a) **Annuity**

$$\begin{aligned} FV &= PMT \frac{(1+i)^n - 1}{i} \\ &= 1000 \frac{(1+0.04)^{20} - 1}{0.04} \\ &\approx \$29,778.08 \end{aligned}$$

b) **How much is the interest?**

$$\text{Deposits} = 20(1000) = 20,000.00$$

$$\text{Interest} = \text{Value} - \text{Deposit}$$

$$= 29,778.08 - 20,000$$

$$= \$9,778.08$$

Sinking Funds

Sinking Fund is established for accumulating funds to meet future obligations or debts.

∴ How much I have to pay?

$$PMT = FV \frac{i}{(1+i)^n - 1} ; \quad i = \frac{r}{m}, \quad n = mt$$

Example

A bond issue is approved for building a marina in a city. The city is required to make regular payments every 3 months into a sinking fund paying 5.4% compounded quarterly. At the end of 10 years, the bond obligation will be retired with a cost of \$5,000,000.

Solution

Given: Cost = \$5,000,000.00 in 10 yrs

$$\Rightarrow n = mt = 4(10) = 40$$

$$i = \frac{r}{m} = \frac{.054}{4} = 0.0135$$

a) What should each payment be?

$$\begin{aligned} PMT &= FV \frac{i}{(1+i)^n - 1} \\ &= 5,000,000 \frac{.0135}{(1+.0135)^{40} - 1} \end{aligned}$$

$$\approx \$95,094.67 \text{ per quarter}$$

$$5000000 (.0135 / ((1+.0135) ^ 40 - 1))$$

b) How much interest is earned during the 10th year?

$$1^{\text{st}} 9 \text{ years} = t = 4(9) = 36$$

$$\begin{aligned} FV &= PMT \frac{(1+.0135)^{36} - 1}{.0135} \\ &= 95094.67 \frac{(1+.0135)^{36} - 1}{.0135} \\ &\approx 4,370,992.44 \end{aligned}$$

$$5,000,000 - 4,370,992.44 = \$629,007.56 \quad \text{after 9 years}$$

$$3 \text{ months} \Rightarrow (4) PMT$$

$$\Rightarrow (4)(95094.67) = \$380,378.68$$

$$\text{Interest} = 629,007.56 - 380,378.68 = \$248,628.88$$

Example

Experts say the baby boom generation can't count on a company pension or Social Security to provide a comfortable retirement, as their parents did. It is recommended that they start to save early and regularly. Sarah, a baby boomer, has decided to deposit \$200 each month for 20 years in an account that pays interest of 7.2% compounded monthly.

- a) How much will be in the account at the end of 20 years?
- b) Sarah believes she needs to accumulate \$130,000 in the 20-year period to have enough for retirement. What interest rate would provide that amount?

Solution

a) **Given:** $PMT = 200$ $m = 12$ $r = 0.072$

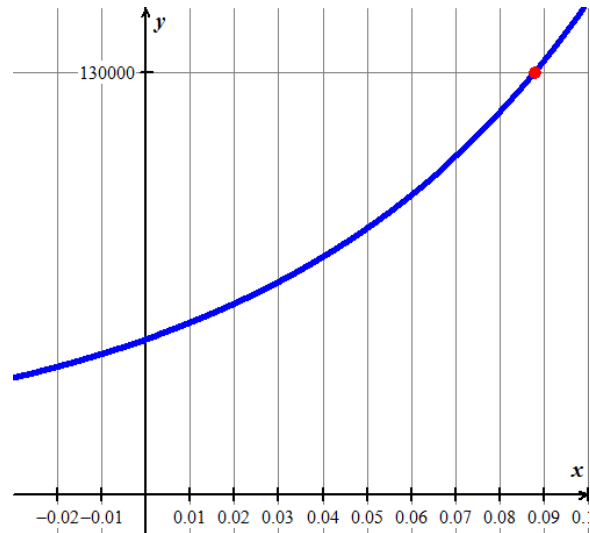
$$i = \frac{r}{m} = \frac{.072}{12}$$

$$n = mt = 12(20) = 240$$

$$\begin{aligned} FV &= PMT \left[\frac{(1+i)^n - 1}{i} \right] \\ &= 200 \left[\frac{\left(1 + \frac{.072}{12}\right)^{240} - 1}{\frac{.072}{12}} \right] \\ &\approx \$106,752.47 \end{aligned}$$

b) $130000 = 200 \left[\frac{\left(1 + \frac{r}{12}\right)^{240} - 1}{\frac{r}{12}} \right]$

$$130000 = 200 \frac{12}{r} \left[\left(\frac{12+r}{12}\right)^{240} - 1 \right]$$
$$\frac{130000}{2400} = \frac{1}{r} \left[\left(\frac{12+r}{12}\right)^{240} - 1 \right]$$
$$\frac{325}{6} = \frac{1}{r} \left[\left(\frac{12+r}{12}\right)^{240} - 1 \right]$$



Using a calculator or program; the annual interest rate is 8.79%.

Exercises ***Section 2.2 – Future Value of an Annuity***

1. Recently, Guaranty Income Life offered an annuity that pays 6.65% compounded monthly. If \$500 is deposited into this annuity every month, how much is in the account after 10 years? How much of this is interest?
2. Recently, USG Annuity Life offered an annuity that pays 4.25% compounded monthly. If \$1,000 is deposited into this annuity every month, how much is in the account after 15 years? How much of this is interest?
3. In order to accumulate enough money for a down payment on a house, a couple deposits \$300 per month into an account paying 6% compounded monthly. If payments are made at the end of each period, how much money will be in the account in 5 years?
4. A self-employed person has a Keogh retirement plan. (This type of plan is free of taxes until money is withdrawn.) If deposits of \$7,500 are made each year into an account paying 8% compounded annually, how much will be in the account after 20 years?
5. Sun America recently offered an annuity that pays 6.35% compounded monthly. What equal monthly deposit should be made into this annuity in order to have \$200,000 in 15 years?
6. Recently, The Hartford offered an annuity that pays 5.5% compounded monthly. What equal monthly deposit should be made into this annuity in order to have \$100,000 in 10 years?
7. Compu-bank, an online banking service, offered a money market account with an APY of 4.86%.
 - a) If interest is compounded monthly, what is the equivalent annual nominal rate?
 - b) If you wish to have \$10,000 in the account after 4 years, what equal deposit should you make each month?
8. American Express's online banking division offered a money market account with an APY of 5.65%.
 - a) If interest is compounded monthly, what is the equivalent annual nominal rate?
 - b) If you wish to have \$1,000,000 in the account after 8 years, what equal deposit should you make each month?
9. Find the future value of an annuity due if payments of \$500 are made at the beginning of each quarter for 7 years, in an account paying 6% compounded quarterly.
10. A 45 year-old man puts \$2500 in a retirement account at the end of each quarter until he reaches the age of 60, then makes no further deposits. If the account pays 6% interest compounded quarterly, how much will be in the account when the man retires at age 65?

11. A father opened a savings account for his daughter on the day she was born, depositing \$1000. Each year on her birthday he deposits another \$1000, making the last deposit on her 21st birthday. If the account pays 5.25% interest compounded annually, how much is in the account at the end of the day on his daughter's 21st birthday? How much interest has been earned?
12. You deposits \$10,000 at the beginning of each year for 12 years in an account paying 5% compounded annually. Then you put the total amount on deposit in another account paying 6% compounded semi-annually for another 9 years. Find the final amount on deposit after the entire 21-year period.
13. You need \$10,000 in 8 years.
 - a) What amount should be deposit at the end of each quarter at 8% compounded quarterly so that he will have his \$10,000?
 - b) Find your quarterly deposit if the money is deposited at 6% compounded quarterly.
14. You want to have a \$20,000 down payment when you buy a car in 6 years. How much money must you deposit at the end of each quarter in an account paying 3.2% compounded quarterly so that you will have the down payment you desire?
15. You sell a land and then you will be paid a lump sum of \$60,000 in 7 years. Until then, the buyer pays 8% simple interest quarterly.
 - a) Find the amount of each quarterly interest payment on the \$60,000
 - b) The buyer sets up a sinking fund so that enough money will be present to pay off the \$60,000. The buyer will make semiannual payments into the sinking fund; the account pays 6% compounded semiannually. Find the amount of each payment into the fund.