Lecture Three

Section 3.1 – Quadratic Functions and Models

Quadratic Function

A function f is a ***quadratic function*** if $f(x)=ax^2+bx+c$

Vertex of a Parabola

The ***vertex*** of the graph of $f(x)$ is

V_x or $x_V=-\frac{b}{2a}$

V_y or $y_V=f\left(-\frac{b}{2a}\right)$

Vertex Point $\left(-\frac{b}{2a},\ f\left(-\frac{b}{2a}\right)\right)$

$f(x)=x^2-4x-2$

$x=-\frac{b}{2a}=-\frac{-4}{2(1)}=2$

$y=f\left(-\frac{b}{2a}\right)=f(2)$

$=(2)^2-4(2)-2$

$=-6$

Vertex point: $\boxed{(2,-6)}$

Axis of Symmetry: $x=V_x=-\frac{b}{2a}$

Axis of Symmetry: $x=2$

Minimum or Maximum Point

If $a>0\Rightarrow f(x)$ has a ***minimum*** point

If $a<0\Rightarrow f(x)$ has a ***maximum*** point

@ vertex point $\left(V_x,V_y\right)$

Minimum point @ $(2,-6)$

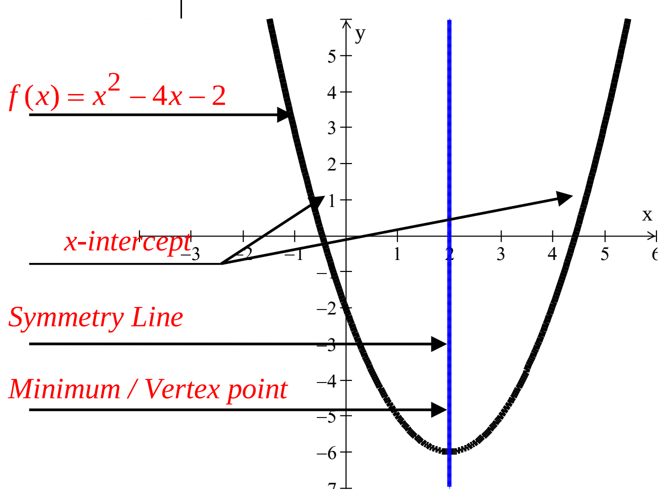
Range

If $a>0\Rightarrow\left[V_y,\ \infty\right)$

If $a<0\Rightarrow\left(-\infty,\ V_y\right]$

$[-6,\infty)$

Domain: $(-\infty,\infty)$



Example

For the graph of the function $f(x) = -x^2 - 2x + 8$

a. **Find the vertex point**

$$x = -\frac{-2}{2(-1)} = -1$$

$$y = f\left(-1\right) = -(-1)^2 - 2(-1) + 8 = 9$$

Vertex point (–1, 9)

b. **Find the line of symmetry:** $x = -1$

c. **State whether there is a maximum or minimum value *and* find that value**

Minimum point, value (–1, 9)

d. **Find the *x*-intercept**

$x = -4,\ 2$

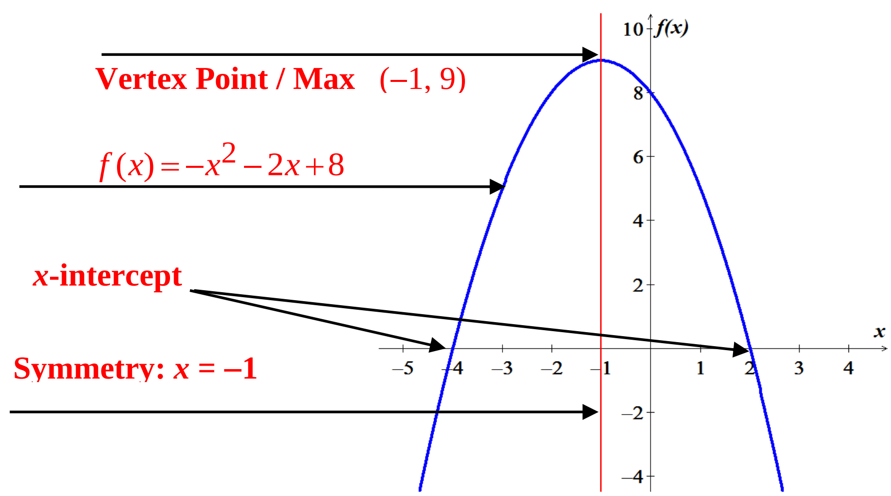
e. **Find the y-intercept**

$y = 8$

f. **Find the range and the domain of the function.**

Range: (–∞, 9]   *Domain*: (–∞, ∞)

g. **Graph the function and label, show part *a thru d* on the plot below**



h. **On what intervals is the function increasing? Decreasing?**

Increasing: (–∞,–1)   *Decreasing*: (–1, ∞)

Example

Find the axis and vertex of the parabola having equation $f(x) = 2x^2 + 4x + 5$

Solution

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{4}{2(2)} \\ &= -1 \end{aligned}$$

Axis of the parabola: $x = -1$

$$\begin{aligned} y &= f(-1) \\ &= 2(-1)^2 + 4(-1) + 5 \\ &= 3 \end{aligned}$$

Vertex point: $\boxed{(-1, 3)}$

Maximizing Area

You have 120 ft of fencing to enclose a rectangular region. Find the dimensions of the rectangle that maximize the enclosed area. What is the maximum area?

Solution

$$P = 2l + 2w$$

$$120 = 2l + 2w$$

$$60 = l + w \qquad \to \boxed{l = 60 - w}$$

$$\begin{aligned} A &= lw \\ &= (60 - w)w \\ &= 60w - w^2 \\ &= -w^2 + 60w \end{aligned}$$

Vertex: $w = -\frac{60}{2(-1)} = 30$

$\to l = 60 - w = 30$

$A = lw = (30)(30)\ \boxed{= 900\ ft^2}$

Example

A stone mason has enough stones to enclose a rectangular patio with 60 *ft* of stone wall. If the house forms one side of the rectangle, what is the maximum area that the mason can enclose? What should the dimensions of the patio be in order to yield this area?

Solution

$$P = l + 2w = 60 \quad \Rightarrow \boxed{l = 60 - 2w}$$

$$\begin{aligned} A &= lw \\ &= (60 - 2w)w \\ &= 60w - 2w^2 \\ &= -2w^2 + 60w \end{aligned}$$

$$\begin{aligned} w &= -\frac{b}{2a} \\ &= -\frac{60}{2(-2)} \\ &= 15\, ft \end{aligned}$$

$$\Rightarrow l = 60 - 2w = 60 - 2(15) = 30\, ft$$

$$Area = (15)(30) = 450\, ft^2$$

Position Function (*Projectile Motion*)

Example

A model rocket is launched with an initial velocity of 100 ft/sec from the top of a hill that is 20 ft high. Its height *t* seconds after it has been launched is given by the function $s(t) = -16t^2 + 100t + 20$. Determine the time at which the rocket reaches its maximum height and find the maximum height.

Solution

$$\begin{aligned} t &= -\frac{b}{2a} \\ &= -\frac{100}{2(-16)} \\ &= 3.125 \text{ sec} \end{aligned}$$

$$s(t = 3.125) = -16(3.125)^2 + 100(3.125) + 20 = 176.25\, ft$$

Exercises *Section* 3.1 – Quadratic Functions and Models

1. Give the vertex, axis, domain, and range. Then, graph the function $f(x) = x^2 + 6x + 5$

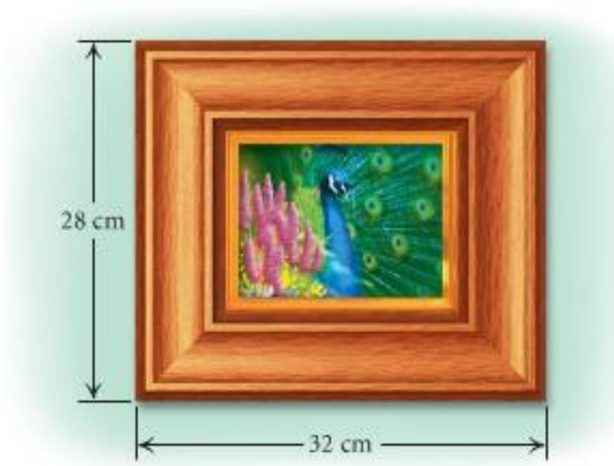
2. Give the vertex, axis, domain, and range. Then, graph the function $f(x) = -x^2 - 6x - 5$

3. Give the vertex, axis, domain, and range. Then, graph the function $f(x) = x^2 - 4x + 2$

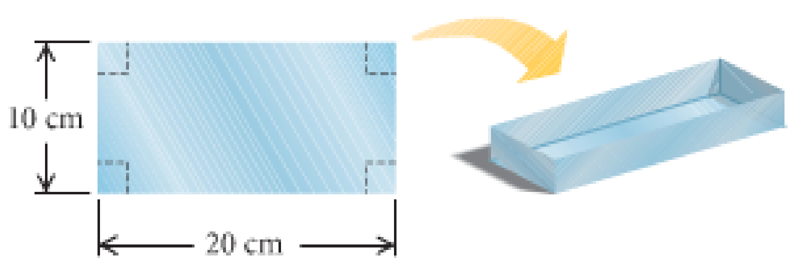
4. Give the vertex, axis, domain, and range. Then, graph the function $f(x) = -2x^2 + 16x - 26$

5. You have 600 *ft.* of fencing to enclose a rectangular plot that borders on a river. If you do not fence the side along the river, find the length and width of the plot that will maximize the area. What is the largest area that can be enclosed?

6. A picture frame measures 28 cm by 32 cm and is of uniform width. What is the width of the frame if 192 cm^2 of the picture shows?



7. An open box is made from a 10-cm by 20-cm of tin by cutting a square from each corner and folding up the edges. The area of the resulting base is 96 cm^2. What is the length of the sides of the squares?



8. A fourth-grade class decides to enclose a rectangular garden, using the side of the school as one side of the rectangle. What is the maximum area that the class can enclose with 32 ft. of fence? What should the dimensions of the garden be in order to yield this area?

9. A rancher needs to enclose two adjacent rectangular corrals, one for cattle and one for sheep. If a river forms one side of the corrals and 240 yd of fencing is available, what is the largest total area that can be enclosed?

10. A Norman window is a rectangle with a semicircle on top. Sky Blue Windows is designing a Norman window that will require 24 ft of trim on the outer edges. What dimensions will allow the maximum amount of light to enter a house?

A Norman window

11. A frog leaps from a stump 3.5 ft. high and lands 3.5 ft. from the base of the stump.

It is determined that the height of the frog as a function of its distance, x, from the base of the stump is given by the function $h(x) = -0.5x^2 + 0.75x + 3.5$ where h is in feet.

a) How high is the frog when its horizontal distance from the base of the stump is 2 ft.?
b) At what two distances from the base of the stump after is jumped was the frog 3.6 ft. above the ground?
c) At what distance from the base did the frog reach its highest point?
d) What was the maximum height reached by the frog?