

Section 3.3 – Gram-Schmidt Process

Definition

A set of two or more vectors in a real inner product space is said to be **orthogonal** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be **orthonormal**.

Theorem

1. If $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is an orthogonal basis for an inner product space V , and if \vec{u} is any vector in V , then

$$\vec{u} = \frac{\langle \vec{u}, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 + \frac{\langle \vec{u}, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 + \dots + \frac{\langle \vec{u}, \vec{v}_n \rangle}{\|\vec{v}_n\|^2} \vec{v}_n$$

2. If $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is an orthonormal basis for an inner product space V , and if \vec{u} is any vector in V , then

$$\vec{u} = \langle \vec{u}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{u}, \vec{v}_2 \rangle \vec{v}_2 + \dots + \langle \vec{u}, \vec{v}_n \rangle \vec{v}_n$$

Proof

1. Since $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a basis for V , every vector \vec{u} in V can be expressed in the form

$$\vec{u} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n$$

Let show that $c_i = \frac{\langle \vec{u}, \vec{v}_i \rangle}{\|\vec{v}_i\|^2}$ for $i = 1, 2, \dots, n$

$$\begin{aligned} \langle \vec{u}, \vec{v}_i \rangle &= \langle c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n, \vec{v}_i \rangle \\ &= c_1 \langle \vec{v}_1, \vec{v}_i \rangle + c_2 \langle \vec{v}_2, \vec{v}_i \rangle + \dots + c_n \langle \vec{v}_n, \vec{v}_i \rangle \end{aligned}$$

Since S is an orthogonal set, all of the inner products in the last equality are zero except the i^{th} , so we have

$$\begin{aligned} \langle \vec{u}, \vec{v}_i \rangle &= c_i \langle \vec{v}_i, \vec{v}_i \rangle \\ &= c_i \|\vec{v}_i\|^2 \end{aligned}$$

The Gram-Schmidt Process

To convert a basis $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_r\}$ into an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$, perform the following computations:

$$\text{Step 1: } \vec{v}_1 = \vec{u}_1$$

$$\text{Step 2: } \vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$\text{Step 3: } \vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$\text{Step 4: } \vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3$$

To convert the orthogonal basis into an orthonormal basis $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$, normalize the orthogonal basis

vectors.
$$\vec{q}_i = \frac{\vec{v}_i}{\|\vec{v}_i\|}$$

Example

Assume that the vector space \mathbb{R}^3 has the Euclidean inner product. Apply the Gram-Schmidt process to transform the basis vectors

$$\vec{u}_1 = (1, 1, 1) \quad \vec{u}_2 = (0, 1, 1) \quad \vec{u}_3 = (0, 0, 1)$$

Into the orthogonal basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, and then normalize the ***orthogonal*** basis vectors to obtain an orthonormal basis $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$

Solution

$$\begin{aligned} \vec{v}_1 &= \vec{u}_1 \\ &= (1, 1, 1) \end{aligned}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

$$\begin{aligned}
&= (0, 1, 1) - \frac{0+1+1}{1^2+1^2+1^2} (1, 1, 1) \\
&= (0, 1, 1) - \frac{2}{3} (1, 1, 1) \\
&= \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right)
\end{aligned}$$

$$\begin{aligned}
\vec{v}_3 &= \vec{u}_3 - \text{proj}_{\vec{v}_2} \vec{u}_3 \\
&= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \\
&= (0, 0, 1) - \frac{0+0+1}{1^2+1^2+1^2} (1, 1, 1) - \frac{0+0+\frac{1}{3}}{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right) \\
&= (0, 0, 1) - \frac{1}{3} (1, 1, 1) - \frac{\frac{1}{3}}{\frac{2}{3}} \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right) \\
&= (0, 0, 1) - \frac{1}{3} (1, 1, 1) - \frac{1}{2} \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right) \\
&= \left(0, -\frac{1}{2}, \frac{1}{2} \right)
\end{aligned}$$

$$\begin{aligned}
\vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\
&= \frac{(1, 1, 1)}{\sqrt{3}} \\
&= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)
\end{aligned}$$

$$\begin{aligned}
\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\
&= \frac{\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right)}{\sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2}} \\
&= \frac{\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right)}{\frac{\sqrt{6}}{3}} \\
&= \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right)
\end{aligned}$$

$$\begin{aligned}
 \vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
 &= \frac{\left(0, -\frac{1}{2}, \frac{1}{2}\right)}{\sqrt{0^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}} \\
 &= \frac{\left(0, -\frac{1}{2}, \frac{1}{2}\right)}{\frac{\sqrt{2}}{2}} \\
 &= \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)
 \end{aligned}$$

Gram-Schmidt Process (Orthonormal)

Suppose $\vec{v}_1, \dots, \vec{v}_n$ linearly independent in \mathbb{R}^n , construct n **orthonormal** $\vec{u}_1, \dots, \vec{u}_n$ that span the same space: $\text{span} \{ \vec{u}_1, \dots, \vec{u}_k \} = \text{span} \{ \vec{v}_1, \dots, \vec{v}_k \}$

Step 1: Since \vec{v}_i are linearly independent ($\neq 0$), so $\|\vec{v}_1\| \neq 0$ (to create a normal vector)

Let $\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \vec{q}_1$, then $\|\vec{u}_1\| = 1$ since \vec{u}_1 is orthonormal and $\text{span} \{ \vec{u}_1 \} = \text{span} \{ \vec{v}_1 \}$

$$\vec{w}_1 = \vec{v}_1 \Rightarrow \vec{v}_1 = \|\vec{w}_1\| \vec{u}_1$$

Step 2: $\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{q}_1) \vec{q}_1$

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{u}_1}{\|\vec{v}_1\|} \vec{v}_1 \quad (\vec{w}_2 \perp \vec{u}_1)$$

$$\vec{v}_2 = \|\vec{w}_2\| \vec{u}_2 + (\vec{v}_2 \cdot \vec{u}_1) \vec{u}_1 \quad \vec{w}_2 = \|\vec{w}_2\| \vec{u}_2$$

$$\vec{q}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

Step 3: $\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{q}_1) \vec{q}_1 - (\vec{v}_3 \cdot \vec{q}_2) \vec{q}_2$

$$\vec{q}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

	$\vec{q}_1 = \frac{\vec{v}_1}{\ \vec{v}_1\ }$
$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{q}_1) \vec{q}_1$	$\vec{q}_2 = \frac{\vec{w}_2}{\ \vec{w}_2\ }$
$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{q}_1) \vec{q}_1 - (\vec{v}_3 \cdot \vec{q}_2) \vec{q}_2$	$\vec{q}_3 = \frac{\vec{w}_3}{\ \vec{w}_3\ }$
$\vec{w}_n = \vec{v}_n - (\vec{v}_n \cdot \vec{q}_1) \vec{q}_1 - (\vec{v}_n \cdot \vec{q}_2) \vec{q}_2 - \dots - (\vec{v}_n \cdot \vec{q}_{n-1}) \vec{q}_{n-1}$	$\vec{q}_n = \frac{\vec{w}_n}{\ \vec{w}_n\ }$

Example

Use the Gram-Schmidt process to find an **orthonormal** basis for the subspaces of

$$\vec{v}_1 = (1, 1, 0, 0) \quad \vec{v}_2 = (0, 1, 1, 0) \quad \vec{v}_3 = (1, 0, 1, 1)$$

Solution

$$\begin{aligned} \text{Step 1: } \vec{q}_1 &= \frac{(1, 1, 0, 0)}{\sqrt{1^2 + 1^2 + 0 + 0}} & \vec{q}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{(1, 1, 0, 0)}{\sqrt{2}} \\ &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right) \end{aligned}$$

$$\begin{aligned} \text{Step 2: } \vec{w}_2 &= \vec{v}_2 - (\vec{v}_2 \cdot \vec{q}_1) \vec{q}_1 \\ &= (0, 1, 1, 0) - \left[(0, 1, 1, 0) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right) \right] \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right) \\ &= (0, 1, 1, 0) - \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right) \\ &= (0, 1, 1, 0) - \left(\frac{1}{2}, \frac{1}{2}, 0, 0 \right) \\ &= \left(-\frac{1}{2}, \frac{1}{2}, 1, 0 \right) \end{aligned}$$

$$\begin{aligned} \|\vec{w}_2\| &= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 1} \\ &= \frac{\sqrt{3}}{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{6}}{2} \end{aligned}$$

$$\begin{aligned} \vec{q}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\ &= \frac{\left(-\frac{1}{2}, \frac{1}{2}, 1, 0\right)}{\frac{\sqrt{6}}{2}} \\ &= \frac{2}{\sqrt{6}} \left(-\frac{1}{2}, \frac{1}{2}, 1, 0\right) \\ &= \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0\right) \end{aligned}$$

$$\text{Step 3: } \vec{v}_3 \cdot \vec{q}_1 = (1, 0, 1, 1) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right) \\ = \frac{1}{\sqrt{2}} \Big|$$

$$\vec{v}_3 \cdot \vec{q}_2 = (1, 0, 1, 1) \cdot \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0 \right) \\ = -\frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}} \\ = \frac{1}{\sqrt{6}} \Big|$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{q}_1) \vec{q}_1 - (\vec{v}_3 \cdot \vec{q}_2) \vec{q}_2 \\ = (1, 0, 1, 1) - \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right) - \frac{1}{\sqrt{6}} \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0 \right) \\ = (1, 0, 1, 1) - \left(\frac{1}{2}, \frac{1}{2}, 0, 0 \right) - \left(-\frac{1}{6}, \frac{1}{6}, \frac{2}{6}, 0 \right) \\ = \left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}, 1 \right) \Big|$$

$$\vec{q}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|} \\ = \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + 1^2}} \left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}, 1 \right) \\ = \frac{1}{\sqrt{\frac{21}{9}}} \left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}, 1 \right) \\ = \frac{3}{\sqrt{21}} \left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}, 1 \right) \\ = \left(\frac{2}{\sqrt{21}}, -\frac{2}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{3}{\sqrt{21}} \right) \Big|$$

The **orthonormal** basis:

$$\left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right), \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0 \right), \left(\frac{2}{\sqrt{21}}, -\frac{2}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{3}{\sqrt{21}} \right) \right\}$$

QR-Decomposition

Problem

If A is an $m \times n$ matrix with linearly independent column vectors, and if Q is the matrix that results by applying the Gram-Schmidt process to the column vectors of A , what relationship, if any, exists between A and Q ?

To solve this problem, suppose that the column vectors of A are $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ and the orthonormal column vectors of Q are $\vec{q}_1, \vec{q}_2, \dots, \vec{q}_n$.

$$\begin{aligned}\vec{u}_1 &= \langle \vec{u}_1, \vec{q}_1 \rangle \vec{q}_1 + \langle \vec{u}_1, \vec{q}_2 \rangle \vec{q}_2 + \dots + \langle \vec{u}_1, \vec{q}_n \rangle \vec{q}_n \\ \vec{u}_2 &= \langle \vec{u}_2, \vec{q}_1 \rangle \vec{q}_1 + \langle \vec{u}_2, \vec{q}_2 \rangle \vec{q}_2 + \dots + \langle \vec{u}_2, \vec{q}_n \rangle \vec{q}_n \\ &\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \\ \vec{u}_n &= \langle \vec{u}_n, \vec{q}_1 \rangle \vec{q}_1 + \langle \vec{u}_n, \vec{q}_2 \rangle \vec{q}_2 + \dots + \langle \vec{u}_n, \vec{q}_n \rangle \vec{q}_n\end{aligned}$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle & \dots & \langle \vec{u}_n, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle & \dots & \langle \vec{u}_n, \vec{q}_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \langle \vec{u}_n, \vec{q}_n \rangle \end{bmatrix}$$

The equation $A = QR$ is a factorization of A into the product of a matrix Q with orthonormal column vectors and an invertible upper triangular matrix R . We call it the **QR-decomposition of A** .

Theorem

If A is an $m \times n$ matrix with linearly independent column vectors, then A can be factored as

$$A = QR$$

Where Q is an $m \times n$ matrix with orthonormal column vectors, and R is an $n \times n$ invertible upper triangular matrix.

Example

Find the QR -decomposition of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Solution

The column vectors of are

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

From the previous example

$$\vec{q}_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \quad \vec{q}_2 = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right) \quad \vec{q}_3 = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$\begin{aligned} R &= \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle & \langle \vec{u}_3, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle & \langle \vec{u}_3, \vec{q}_2 \rangle \\ 0 & 0 & \langle \vec{u}_3, \vec{q}_3 \rangle \end{bmatrix} \\ &= \begin{bmatrix} (1)\frac{1}{\sqrt{3}} + (1)\frac{1}{\sqrt{3}} + (1)\frac{1}{\sqrt{3}} & 0 + (1)\frac{1}{\sqrt{3}} + (1)\frac{1}{\sqrt{3}} & 0 + 0 + (1)\frac{1}{\sqrt{3}} \\ 0 & 0\left(\frac{-2}{\sqrt{6}}\right) + (1)\frac{1}{\sqrt{6}} + (1)\frac{1}{\sqrt{6}} & 0\left(\frac{-2}{\sqrt{6}}\right) + 0\frac{1}{\sqrt{6}} + (1)\frac{1}{\sqrt{6}} \\ 0 & 0 & 0 + (0)\frac{-1}{\sqrt{2}} + (1)\frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} &= \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \\ \mathbf{A} &= \mathbf{Q} \mathbf{R} \end{aligned}$$

Calculus: Applying the Gram-Schmidt Process

We can apply the Gram-Schmidt orthogonalization procedure to generate some polynomials that are orthonormal on the interval $x \in [-1, 1]$ with inner product

$$\langle p, q \rangle = \int_{-1}^1 p(x)q(x)dx$$

Example

Apply the Gram-Schmidt orthonormalization process to the basis $B = \{1, x, x^2\}$ in \mathbb{P}_2 using the inner product

Solution

$$B = \{1, x, x^2\}$$

$$\text{Let } \vec{u}_1 = 1, \quad \vec{u}_2 = x, \quad \vec{u}_3 = x^2$$

$$\underline{\vec{v}_1 = \vec{u}_1 = 1}$$

$$\begin{aligned} \langle \vec{v}_1, \vec{v}_1 \rangle &= \int_{-1}^1 dx \\ &= x \Big|_{-1}^1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \langle \vec{u}_2, \vec{v}_1 \rangle &= \int_{-1}^1 x dx \\ &= \frac{1}{2}x^2 \Big|_{-1}^1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \\ &= x - \frac{0}{2}(1) \\ &= x \end{aligned}$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^1 x^2 dx$$

$$= \frac{1}{3} x^3 \Big|_{-1}^1$$

$$= \frac{2}{3} \Big|$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_{-1}^1 x^2 \, dx$$

$$= \frac{1}{3} x^3 \Big|_{-1}^1$$

$$= \frac{2}{3} \Big|$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^1 x^3 \, dx$$

$$= \frac{1}{4} x^4 \Big|_{-1}^1$$

$$= 0 \Big|$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= x^2 - \frac{3}{2}(x)(0) - \frac{1}{2} \frac{2}{3}$$

$$= x^2 - \frac{1}{3} \Big|$$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_{-1}^1 \left(x^2 - \frac{1}{3} \right)^2 \, dx$$

$$= \int_{-1}^1 \left(x^4 - \frac{2}{3} x^2 + \frac{1}{9} \right) \, dx$$

$$= \left(\frac{1}{5} x^5 - \frac{2}{9} x^3 + \frac{1}{9} x \right) \Big|_{-1}^1$$

$$= \frac{1}{5} - \frac{2}{9} + \frac{1}{9} + \frac{1}{5} - \frac{2}{9} + \frac{1}{9}$$

$$= \frac{8}{45} \Big|$$

$$\bar{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{1}{\sqrt{2}} \Big|$$

$$\begin{aligned}\vec{q}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} \\ &= \frac{x}{\sqrt{2/3}} \\ &= \frac{\sqrt{3}}{\sqrt{2}}x\end{aligned}$$

$$\begin{aligned}\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\ &= \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3} \right) \\ &= \frac{3\sqrt{5}}{2\sqrt{2}} \left(x^2 - \frac{1}{3} \right)\end{aligned}$$

The **orthonormal** basis is $\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \frac{1}{2}\sqrt{\frac{5}{2}}(3x^2-1) \right\}$

Exercises Section 3.3 – Gram-Schmidt Process

(1 – 14) Use the Gram-Schmidt process to find an *orthonormal* basis for the subspaces of \mathbb{R}^m .

1. $\vec{u}_1 = (1, -3), \vec{u}_2 = (2, 2)$
2. $\vec{u}_1 = (1, 0), \vec{u}_2 = (3, -5)$
3. $\{(1, 1, 1), (-1, 1, 0), (1, 2, 1)\}$
4. $\{(1, 1, 1), (0, 1, 1), (0, 0, 1)\}$
5. $\{(1, 1, 1), (0, 2, 1), (1, 0, 3)\}$
6. $\{(2, 2, 2), (1, 0, -1), (0, 3, 1)\}$
7. $\{(1, -1, 0), (0, 1, 0), (2, 3, 1)\}$
8. $\{(3, 0, 4), (-1, 0, 7), (2, 9, 11)\}$
9. $\{(1, 1, 1, 1), (1, 2, 1, 0), (1, 3, 0, 0)\}$
10. $\{(0, 2, -1, 1), (0, 0, 1, 1), (-2, 1, 1, -1)\}$
11. $\vec{u}_1 = (1, 0, 0), \vec{u}_2 = (3, 7, -2), \vec{u}_3 = (0, 4, 1)$
12. $\vec{u}_1 = (1, 1, 1, 1), \vec{u}_2 = (1, 2, 4, 5), \vec{u}_3 = (1, -3, -4, -2)$
13. $\vec{u}_1 = (1, 1, 1, 1), \vec{u}_2 = (1, 1, 2, 4), \vec{u}_3 = (1, 2, -4, -3)$
14. $\vec{u}_1 = (0, 2, 1, 0), \vec{u}_2 = (1, -1, 0, 0), \vec{u}_3 = (1, 2, 0, -1), \vec{u}_4 = (1, 0, 0, 1)$

(15 – 26) Use the Gram-Schmidt process to find an *orthogonal* basis for the subspaces of \mathbb{R}^m .

15. $\{(1, 1, 0), (1, 2, 0), (0, 1, 2)\}$
16. $\{(1, -2, 2), (2, 2, 1), (2, -1, -2)\}$
17. $\{(1, 0, 0), (1, 1, 1), (1, 1, -1)\}$
18. $\{(4, -3, 0), (1, 2, 0), (0, 0, 4)\}$
19. $\{(0, 1, 2), (2, 0, 0), (1, 1, 1)\}$
20. $\{(0, 1, 1), (1, 1, 0), (1, 0, 1)\}$
21. $\{(1, 2, -2), (1, 0, -4), (5, 2, 0), (1, 1, -1)\}$
22. $\{(-3, 1, 2), (1, 1, 1), (2, 0, -1), (1, -3, 2)\}$
23. $\{(2, 1, 1), (0, 3, -1), (3, -4, -2), (-1, -1, 3)\}$
24. $\vec{u}_1 = (1, 1, 0, -1), \vec{u}_2 = (1, 3, 0, 1), \vec{u}_3 = (4, 2, 2, 0)$
25. $\vec{u}_1 = (1, 1, 1, 1), \vec{u}_2 = (1, 1, 2, 4), \vec{u}_3 = (1, 2, -4, -3)$

26. $\{(3, 4, 0, 0), (-1, 1, 0, 0), (2, 1, 0, -1), (0, 1, 1, 0)\}$

27. Find the **QR**-decomposition of

a) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

28. Verify that the Cauchy-Schwarz inequality holds for the given vectors using the Euclidean inner product

$$\vec{u} = (0, -2, 2, 1), \quad \vec{v} = (-1, -1, 1, 1)$$

(29 – 33) Apply the Gram-Schmidt **orthonormalization** process in $\mathbb{C}^0[-1, 1]$ spanned by the functions, using the inner product

29. $f_1(x) = x + 2, \quad f_2(x) = x^2 - 3x + 4$

30. $f_1(x) = x, \quad f_2(x) = x^3, \quad f_3(x) = x^5$

31. $f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x) = \frac{1}{2}(3x^2 - 1)$

32. $f_1(x) = 1, \quad f_2(x) = \sin \pi x, \quad f_3(x) = \cos \pi x$

33. $f_1(x) = \sin \pi x, \quad f_2(x) = \sin 2\pi x, \quad f_3(x) = \sin 3\pi x$

34. For $\mathbb{P}_3[x]$, define the inner product over \mathbb{R} as

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x) dx$$

a) If $f(x) = 1$ is a unit vector in $\mathbb{P}_3[x]$?

b) Find an orthonormal basis for the subspace spanned by x and x^2 .

c) Complete the basis in part (b) to an orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product.

d) Is

$$[f, g] = \int_0^1 f(x)g(x) dx$$

Also, an inner product for $\mathbb{P}_3[x]$

e) Find a pair of vectors \vec{v} and \vec{w} such that

$$\langle \vec{v}, \vec{w} \rangle = 0 \quad \text{but} \quad [\vec{v}, \vec{w}] \neq 0$$

f) Is the basis found in part (c) an orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product in part (d)?