Section 4.4 – *Bernoulli Trials & Binomial Distributions*

Bernoulli Trials (Jacob 1654-1705)

- show *or* not, False *or* True
  E *or* E′ $\Rightarrow$ 2 possible outcomes $\Rightarrow$ Bernoulli Experiment *or* trial
  Success (S) or Failure (F)

Probability of success: $P(S) = p$

Probability of Failure: $P(F) = 1 - p = q$ $\qquad\Rightarrow p + q = 1$

Example

Find p and q for a single roll of a fair die, where a success is a number divisible by 3 turning up
Divisible by 3: $\{3, 6\}$

<u>*Solution*</u>

$$p = \frac{2}{6} = \frac{1}{3} \Rightarrow q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$$

Bernoulli Trials

1. Only 2 outcomes are possible or each trial
2. Probability of success = p and P(*Failure*) = $q \to p + q = 1$
3. All trials are independent

Example

If we roll a fair die five times and identify a success in a single roll with a 1 turning up, what is the probability of the sequence FSSSF occurring?

<u>*Solution*</u>

$$P(FSSSF) = p^3q^2 = \left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 \approx 0.003$$

$$Outcome\text{: 3 out of 5 } \Rightarrow C_{5,3} = 10$$

$$P(exactly\ 3\ success) = C_{5,3}\,p^3q^2$$

Probability of *x* success in *n* Bernoulli Trials

The probability of exactly x success in n independent repeated Bernoulli trials, with the probability of success of each trial p (and of failure q), is

$$P(x\ success) = C_{n,x}\, p^{x} q^{n-x}$$

Example

If a fair die is rolled five times, what is the probability of rolling

a) Exactly one 3?

b) At least one 3?

<u>*Solution*</u>

a) Exactly one 3?

$$P(exactly\ 1\text{-}3's \to x=1) = C_{5,1}\left(\frac{1}{6}\right)^{1}\left(\frac{5}{6}\right)^{4}$$

$$\approx .402$$

b) At least one 3?

$$\begin{aligned} P(x \ge 1) &= P(x=1) + P(x=2) + \cdots + P(x=5) \\ &= 1 - P(x<1) \\ &= 1 - P(x=0) \\ &\approx 1 - .402 \\ &\approx .598 \end{aligned}$$

Binomial Formula: ***Brief Review***

$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2+2ab+b^2$$
$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

$$(a+b)^n = C_{n,0}a^n + C_{n,1}a^{n-1}b + C_{n,2}a^{n-2}b^2 + \cdots + C_{n,n}b^n$$

Binomial Distribution

In 3 Trials

Simple Event	***Pr of E***	X_3	
FFF	q^3	0	q^3
FFS	q^2p	1	
FSF	q^2p	1	$3q^2p$
SFF	q^2p	1	
FSS	p^2q	2	
SFS	p^2q	2	$3p^2q$
SSF	p^2q	2	
SSS	p^3	3	p^3
			$(p+q)^3$

1- $0 \le P(X_3 = x) \le 1 \qquad \therefore X \in \{0,1,2,3\}$

2- $1 = 1^3 = (p+q)^3 = C_{3,0}q^3 + C_{3,1}q^2p + C_{3,2}qp^2 + C_{3,3}p^3$ **$1 = p + q$**

$$= q^3 + 3q^2p + 3qp^2 + p^3$$
$$= P(X_3 = 0) + P(X_3 = 1) + P(X_3 = 2) + P(X_3 = 3)$$

Binomial Distribution

$$\Rightarrow P(X_n = x) = P(x \text{ success in } n \text{ trials}) = C_{n,x}\, p^x q^{n-x}$$

Example

Suppose a fair die is rolled two times and a success on a single roll is considered to be rolling a number divisible by 3.

a) Write the function defining the distribution

b) Construct a table for the distribution

c) Construct a histogram for the distribution

<u>Solution</u>

a) 3 & 6 are divisible by 3 $\Rightarrow p = \frac{2}{6} = \frac{1}{3}$

$p = \frac{1}{3},\ \ q = \frac{2}{3}$

$(Rolled\ twice) \rightarrow n = 2$

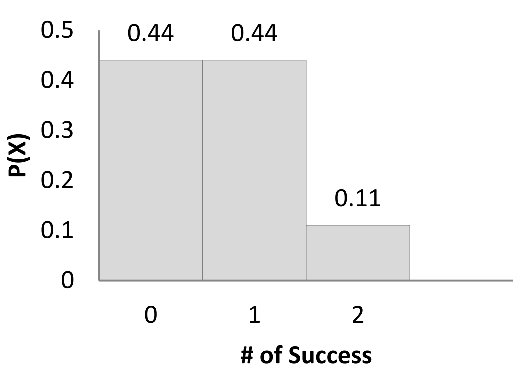
Probability function for the binomial distribution

$$P(X) = P(x \text{ success in } 2 \text{ trials}) = C_{2,x}\left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{2-x} \qquad x \in \{0,1,2\}$$

b) Distribution Table

x	P(x)
0	$C_{2,0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{2-0} = \frac{4}{9} = .44$
1	$C_{2,1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^{2-1} = 2\frac{1}{3}\frac{2}{3} = .44$
2	$C_{2,2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^{2-2} = .11$

c) Histogram



Expected Value

The expected value is denoted by: $E(X)=x_1 p_1 + x_2 p_2 + \cdots + x_n p_n$

E: called Mean of Random Variable X

Standard Deviation: $\sigma = \sqrt{(x_1-\mu)^2 p_1 + (x2-\mu)^2 p_2 + \cdots}$

Mean: $\mu = np$

Standard Deviation: $\sigma = \sqrt{npq}$

Example

Suppose a fair die is rolled two times and a success on a single roll is considered to be rolling a number divisible by 3. Compute the mean and standard deviation

Solution

$n=2 \rightarrow p=\frac{1}{3}, q=\frac{2}{3}$

$\mu = np = 2\frac{1}{3} = \frac{2}{3} \approx .67$

$\sigma = \sqrt{npq} = \sqrt{2\frac{1}{3}\frac{2}{3}} \approx .67$

Example

The probability of recovering after a particular type of operation is 0.5. Let us investigate the binomial distribution involving four patients undergoing this operation

a) Write the function defining the distribution
b) Construct a table for the distribution
c) Construct a histogram for the distribution

<u>Solution</u>

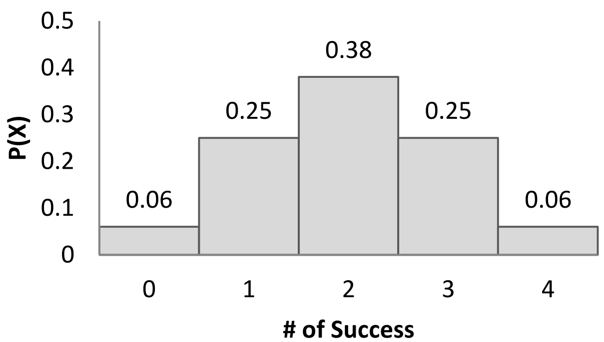
$$n = 4 \rightarrow p = .5, \quad q = .5$$

a) $P(X) = C_{4,x}(.5)^{x}(.5)^{4-x} = C_{4,x}(.5)^{4}$

b) Distribution table

x	**P(*x*)**	
0	.06	$P(x=0) = C_{4,0}(.5)^{0}(.5)^{4-0}$
1	.25	$P(x=1) = C_{4,1}(.5)^{1}(.5)^{4-1}$
2	.38	$P(x=2) = C_{4,2}(.5)^{2}(.5)^{4-2}$
3	.25	$P(x=3) = C_{4,3}(.5)^{3}(.5)^{4-3}$
4	.06	$P(x=4) = C_{4,4}(.5)^{4}(.5)^{4-4}$

c) Histogram



d) $\mu = np = 4(.5) = 2$

$\sigma = \sqrt{npq} = \sqrt{4(.5)(.5)} = 1$

Exercises *Section 4.4 – Bernoulli Trials & Binomial Distributions*

1. If a baseball player has a batting average of 0.350, what is the probability that the player will get the following number of hits in the next four times at bat?

a) Exactly 2 hits

b) At least 2 hits.

2. If a true-false test with 10 questions is given, what is the probability of scoring

a) Exactly 70% just by guessing?

b) 70% or better just by guessing?

3. If 60% of the electorate supports the mayor, what is the probability that in a random sample of 10 voters, fewer than half support her?

4. Each year a company selects a number of employees for a management training program given by nearby university. On the average, 70% of those sent complete the program. Out of 7 people sent by the company, what is the probability that

a) Exactly 5 complete the program?

b) 5 or more complete the program?

5. If the probability of a new employee in a fast-food chain still being with the company at the end of 1 year is 0.6, what is the probability that out of 8 newly hired people?

a) 5 will still be with the company after 1 year?

b) 5 or more will still be with the company after 1 year?

6. A manufacturing process produces, on the average, 6 defective items out of 100. To control quality, each day a sample of 10 completed items is selected at random and inspected. If the sample produces more than 2 defective items, then the whole day's output is inspected and the manufacturing process is reviewed. What is the probability of this happening, assuming that the process is still producing 6% defective items?

7. A manufacturing process produces, on the average, 3% defective items. The company ships 10 items in each box and wishes to guarantee no more than 1 defective item per box. If this guarantee accompanies each box, what is the probability that the box will fail to satisfy the guarantee?

8. A manufacturing process produces, on the average, 5 defective items out of 100. To control quality, each day a random sample of 6 completed items is selected and inspected. If a success on a single trial (inspection of 1 item) is finding the item defective, then the inspection of each of the 6 items in the sample constitutes a binomial experiment, which has a binomial distribution.

a) Write the function defining the distribution

b) Construct a table and histogram for the distribution.

c) Compute the mean and standard deviation.

9. Each year a company selects 5 employees for a management training program given at a nearly university. On the average, 40% of those sent complete the course in the top 10% of their class. If we consider an employee finishing in the top 10% of the class a success in a binomial experiment, then for the 5 employee entering the program there exists a binomial distribution involving P(x success out of 5).

a) Write the function defining the distribution

b) Construct a table and histogram for the distribution.

c) Compute the mean and standard deviation.

10. A person with tuberculosis is given a chest *x*-ray. Four tuberculosis *x*-ray specialists examine each *x*-ray independently. If each specialist can detect tuberculosis 80% of the time when it is present, what is the probability that at least 1 of the specialists will detect tuberculosis in this person?

11. A pharmaceutical laboratory claims that a drug it produces causes serious side effects in 20 people out of 1,000 on the average. To check this claim, a hospital administers the drug to 10 randomly chosen patients and finds that 3 suffer from serious side effects. If the laboratory's claims are correct, what is the probability of the hospital obtaining these results?

12. The probability that brown-eyed parents, both with the recessive gene for blue, will have a child with brown eye is .75. If such parents have 5 children, what is the probability that they will have

a) All blue-eyed children?

b) Exactly 3 children with brown eyes?

c) At least 3 children with brown eyes?

13. The probability of gene mutation under a given level of radiation is 3×10^{-5}. What is the probability of the occurrence of at least 1 gene mutation if 10^5 genes are expected to this level of radiation?

14. If the probability of a person contracting influenza on exposure is .6 consider the binomial distribution for a family of 6 that has been exposed.

a) Write the function defining the distribution.

b) Compute the mean and standard deviation.

15. The probability that a given drug will produce a serious side effect in a person using the drug is .02. In the binomial distribution for 450 people using the drug, what are the mean and standard deviation?

16. An opinion poll based on a small sample can be unrepresentative of the population. To see why, let us assume that 40% of the electorate favors a certain candidate. If a random sample of 7 is asked their preference, what is the probability that a majority will favor this candidate?

17. A multiple choice test is given with 5 choices only one is correct, for each of 5 questions. Answering each of the 5 questions by guessing constitutes a binomial experiment with an associated binomial distribution

a) Write the function defining the distribution.

b) Compute the mean and standard deviation.

18. Suppose a die is rolled 4 times.

a) Find the probability distribution for the number of times 1 is rolled.

b) What is the expected number of times 1 is rolled