Lecture Notes
Vector Analysis
MATH 332

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Contents

Chapter 1

Linear Algebra

1.1 Vectors in $\mathbb{R}^n$ and Matrix Algebra

1.1.1 Vectors

- $\mathbb{R}^n$ is the set of all ordered n-tuples of real numbers, which can be assembled as columns or as rows.

- Let $x_1, \dots, x_n$ be n real numbers. Then the **column-vector** (or just **vector**) is an ordered n-tuple of the form

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix},$$

  and the **row-vector** (also called a **covector**) is an ordered n-tuple of the form

$$\mathbf{v}^T = (v_1, v_2, \dots, v_n).$$

  The real numbers $x_1, \dots x_n$ are called the **components** of the vectors.

- The operation that converts column-vectors into row-vectors and vice versa preserving the order of the components is called the **transposition** and denoted by T. That is

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}^T = (v_1, v_2, \dots, v_n) \quad \text{and} \quad (v_1, v_2, \dots, v_n)^T = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

  Of course, for any vector $\mathbf{v}$

$$(\mathbf{v}^T)^T = \mathbf{v}.$$

- The **addition of vectors** is defined by

$$\mathbf{u}+\mathbf{v}=\begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{pmatrix},$$

and

$$\mathbf{u}+\mathbf{v}=(u_1+v_1,\dots,u_n+v_n)\,.$$

- Notice that one cannot add a column-vector and a row-vector!
- The multiplication of vectors by a real constant, called a **scalar**, is defined by

$$a\mathbf{v}=\begin{pmatrix} av_1 \\ av_2 \\ \vdots \\ av_n \end{pmatrix}, \qquad a\mathbf{v}=(av_1,\dots,av_n)\,.$$

- The vectors that have only zero elements are called **zero vectors**, that is

$$\mathbf{0}=\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \mathbf{0}^T=(0,\dots,0)\,.$$

- The set of column-vectors

$$\mathbf{e}_1=\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2=\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \cdots, \qquad \mathbf{e}_n=\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and the set of row-vectors

$$\mathbf{e}_1^T=(1,0,\dots,0), \qquad \mathbf{e}_2^T=(0,1,\dots,0), \qquad \mathbf{e}_n^T=(0,0,\dots,1)$$

are called the **standard (or canonical) bases** in $\mathbb{R}^n$.

- There is a natural **product of column-vectors and row-vectors** that assigns to a row-vector and a column-vector a real number

$$\langle \mathbf{u}^T,\mathbf{v}\rangle=(u_1,u_2,\dots,u_n)\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}=\sum_{i=1}^n u_iv_i=u_1v_1+u_2v_2+\cdots+u_nv_n\,.$$

This is the simplest instance of a more general multiplication rule for matrices which can be summarized by saying that one multiplies **row by column**.

- The product of two column-vectors and the product of two row-vectors, called the **inner product** (or the **scalar product**), is defined then by

$$(\mathbf{u}, \mathbf{v}) = (\mathbf{u}^T, \mathbf{v}^T) = \langle \mathbf{u}^T, \mathbf{v} \rangle = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + \cdots + u_n v_n \,.$$

- Finally, we define the **norm** (or the **length**) of both column-vectors and row-vectors is defined by

$$||\mathbf{v}|| = ||\mathbf{v}^T|| = \sqrt{\langle \mathbf{v}^T, \mathbf{v} \rangle} = \left(\sum_{i=1}^{n} v_i^2 \right)^{1/2} = \sqrt{v_1^2 + \cdots + v_n^2} \,.$$

1.1.2 Matrices

- A set of n^2 real numbers A_{ij}, $i, j = 1, \ldots, n$, arranged in an array that has n columns and n rows

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

  is called a **square** $n \times n$ **real matrix**.

- The set of all real square $n \times n$ matrices is denoted by $\mathrm{Mat}(n, \mathbb{R})$.

- The number A_{ij} (also called an entry of the matrix) appears in the i-th row and the j-th column of the matrix A

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1j} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2j} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ A_{i1} & A_{i2} & \cdots & \boxed{A_{ij}} & \cdots & A_{in} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nj} & \cdots & A_{nn} \end{pmatrix}$$

- **Remark.** Notice that the first index indicates the row and the second index indicates the column of the matrix.

- The matrix whose all entries are equal to zero is called the **zero matrix**.

- The **addition of matrices** is defined by

$$A + B = \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} & \cdots & A_{1n} + B_{1n} \\ A_{21} + B_{21} & A_{22} + B_{22} & \cdots & A_{2n} + B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} + B_{n1} & A_{n2} + B_{n2} & \cdots & A_{nn} + B_{nn} \end{pmatrix}$$

and the **multiplication by scalars** by

$$cA = \begin{pmatrix} cA_{11} & cA_{12} & \cdots & cA_{1n} \\ cA_{21} & cA_{22} & \cdots & cA_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ cA_{n1} & cA_{n2} & \cdots & cA_{nn} \end{pmatrix}$$

- The numbers A_{ii} are called the **diagonal entries**. Of course, there are n diagonal entries. The set of diagonal entries is called the **diagonal** of the matrix A.
- The numbers A_{ij} with $i \neq j$ are called **off-diagonal entries**; there are $n(n-1)$ off-diagonal entries.
- The numbers A_{ij} with $i < j$ are called the **upper triangular entries**. The set of upper triangular entries is called the **upper triangular part** of the matrix A.
- The numbers A_{ij} with $i > j$ are called the **lower triangular entries**. The set of lower triangular entries is called the **lower triangular part** of the matrix A.
- The number of upper-triangular entries and the lower-triangular entries is the same and is equal to $n(n-1)/2$.
- A matrix whose only non-zero entries are on the diagonal is called a **diagonal matrix**. For a diagonal matrix

$$A_{ij} = 0 \qquad \text{if} \qquad i \neq j\,.$$

- The diagonal matrix

$$A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

is also denoted by

$$A = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

- A diagonal matrix whose all diagonal entries are equal to 1

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

is called the **identity matrix**. The elements of the identity matrix are

$$I_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j\,. \end{cases}$$

- A matrix A of the form
$$A = \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{pmatrix}$$
where $*$ represents nonzero entries is called an **upper triangular matrix**. Its lower triangular part is zero, that is,
$$A_{ij} = 0 \qquad \text{if} \qquad i < j\,.$$

- A matrix A of the form
$$A = \begin{pmatrix} * & 0 & \cdots & 0 \\ * & * & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{pmatrix}$$
whose upper triangular part is zero, that is,
$$A_{ij} = 0 \qquad \text{if} \qquad i > j\,,$$
is called a **lower triangular matrix**.

- The **transpose of a matrix** A whose ij-th entry is A_{ij} is the matrix A^T whose ij-th entry is A_{ji}. That is, A^T obtained from A by switching the roles of rows and columns of A:
$$A^T = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{j1} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{j2} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ A_{1i} & A_{2i} & \cdots & \boxed{A_{ji}} & \cdots & A_{ni} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{11} & A_{2n} & \cdots & A_{jn} & \cdots & A_{nn} \end{pmatrix}$$
or
$$(A^T)_{ij} = A_{ji}\,.$$

- A matrix A is called **symmetric** if
$$A^T = A$$
and **anti-symmetric** if
$$A^T = -A\,.$$

- The number of independent entries of an anti-symmetric matrix is $n(n-1)/2$.

- The number of independent entries of a symmetric matrix is $n(n+1)/2$.

- Every matrix A can be uniquely decomposed as the sum of its diagonal part A_D, the lower triangular part A_L and the upper triangular part A_U

$$A = A_D + A_L + A_U \,.$$

- For an anti-symmetric matrix

$$A_U^T = -A_L \qquad \text{and} \qquad A_D = 0 \,.$$

- For a symmetric matrix

$$A_U^T = A_L \,.$$

- Every matrix A can be uniquely decomposed as the sum of its symmetric part A_S and its anti-symmetric part A_A

$$A = A_S + A_A \,,$$

where

$$A_S = \frac{1}{2}(A + A^T)\,, \qquad A_A = \frac{1}{2}(A - A^T)\,.$$

- The **product of matrices** is defined as follows. The ij-th entry of the product $C = AB$ of two matrices A and B is

$$C_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{in}B_{nj} \,.$$

This is again a multiplication of the "i-th row of the matrix A by the j-th column of the matrix B".

- **Theorem 1.1.1** *The product of matrices is* **associative**, *that is, for any matrices A, B, C*

$$(AB)C = A(BC)\,.$$

- **Theorem 1.1.2** *For any two matrices A and B*

$$(AB)^T = B^T A^T \,.$$

- A matrix A is called **invertible** if there is another matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I \,.$$

The matrix A^{-1} is called the **inverse** of A.

- **Theorem 1.1.3** *For any two invertible matrices A and B*

$$(AB)^{-1} = B^{-1}A^{-1}\,,$$

and

$$(A^{-1})^T = (A^T)^{-1}\,.$$

- A matrix A is called **orthogonal** if
$$A^T A = AA^T = I\,,$$
which means $A^T = A^{-1}$.

- The **trace** is a map tr : Mat$(n,\mathbb{R})$ that assigns to each matrix $A = (A_{ij})$ a real number tr A equal to the sum of the diagonal elements of a matrix
$$\operatorname{tr} A = \sum_{k=1}^{n} A_{kk}\,.$$

- **Theorem 1.1.4** *The trace has the properties*
$$\operatorname{tr}(AB) = \operatorname{tr}(BA)\,,$$
and
$$\operatorname{tr} A^T = \operatorname{tr} A\,.$$

- Obviously, the trace of an anti-symmetric matrix is equal to zero.

- Finally, we define the multiplication of column-vectors by matrices from the left and the multiplication of row-vectors by matrices from the right as follows.

- Each matrix defines a natural **left action on a column-vector** and a **right action on a row-vector**.

- For each column-vector $\mathbf{v}$ and a matrix $A = (A_{ij})$ the column-vector $\mathbf{u} = A\mathbf{v}$ is given by

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_i \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{i1} & A_{i2} & \cdots & A_{in} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_i \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} A_{11}v_1 + A_{12}v_2 + \cdots + A_{1n}v_n \\ A_{21}v_1 + A_{22}v_2 + \cdots + A_{2n}v_n \\ \vdots \\ A_{i1}v_1 + A_{i2}v_2 + \cdots + A_{in}v_n \\ \vdots \\ A_{n1}v_1 + A_{n2}v_2 + \cdots + A_{nn}v_n \end{pmatrix}$$

- The components of the vector $\mathbf{u}$ are
$$u_i = \sum_{j=1}^{n} A_{ij}v_j = A_{i1}v_1 + A_{i2}v_2 + \cdots + A_{in}v_n\,.$$

- Similarly, for a row vector $\mathbf{v}^T$ the components of the row-vector $\mathbf{u}^T = \mathbf{v}^T A$ are defined by
$$u_i = \sum_{j=1}^{n} v_j A_{ji} = v_1 A_{1i} + v_2 A_{2i} + \cdots + v_n A_{ni}\,.$$

1.1.3 Determinant

- Consider the set $\mathbb{Z}_n = \{1, 2, \ldots, n\}$ of the first n integers. A **permutation** φ of the set $\{1, 2, \ldots, n\}$ is an ordered n-tuple $(\varphi(1), \ldots, \varphi(n))$ of these numbers.

- That is, a permutation is a bijective (one-to-one and onto) function
$$\varphi : \mathbb{Z}_n \to \mathbb{Z}_n$$
that assigns to each number i from the set $\mathbb{Z}_n = \{1, \ldots, n\}$ another number $\varphi(i)$ from this set.

- An **elementary permutation** is a permutation that exchanges the order of only two numbers.

- Every permutation can be realized as a product (or a composition) of elementary permutations. A permutation that can be realized by an even number of elementary permutations is called an **even permutation**. A permutation that can be realized by an odd number of elementary permutations is called an **odd permutation**.

- **Proposition 1.1.1** *The* **parity of a permutation** *does not depend on the representation of a permutation by a product of the elementary ones.*

- That is, each representation of an even permutation has even number of elementary permutations, and similarly for odd permutations.

- The **sign of a permutation** φ, denoted by $\operatorname{sign}(\varphi)$ (or simply $(-1)^\varphi$), is defined by
$$\operatorname{sign}(\varphi) = (-1)^\varphi = \begin{cases} +1, & \text{if } \varphi \text{ is even,} \\ -1, & \text{if } \varphi \text{ is odd} \end{cases}$$

- The set of all permutations of n numbers is denoted by S_n.

- **Theorem 1.1.5** *The cardinality of this set, that is, the number of different permutations, is*
$$|S_n| = n!\,.$$

- The **determinant** is a map $\det : \operatorname{Mat}(n, \mathbb{R}) \to \mathbb{R}$ that assigns to each matrix $A = (A_{ij})$ a real number $\det A$ defined by
$$\det A = \sum_{\varphi \in S_n} \operatorname{sign}(\varphi) A_{1\varphi(1)} \cdots A_{n\varphi(n)}\,,$$
where the summation goes over all $n!$ permutations.

- The most important properties of the determinant are listed below:

Theorem 1.1.6 *1. The determinant of the product of matrices is equal to the product of the determinants:*
$$\det(AB) = \det A \det B\,.$$

2. *The determinants of a matrix A and of its transpose A^T are equal:*

$$\det A = \det A^T .$$

3. *The determinant of the inverse A^{-1} of an invertible matrix A is equal to the inverse of the determinant of A:*

$$\det A^{-1} = (\det A)^{-1}$$

4. *A matrix is invertible if and only if its determinant is non-zero.*

- The set of real invertible matrices (with non-zero determinant) is denoted by $GL(n, \mathbb{R})$. The set of matrices with positive determinant is denoted by $GL_+(n, \mathbb{R})$.
- A matrix with unit determinant is called **unimodular**.
- The set of real matrices with unit determinant is denoted by $SL(n, \mathbb{R})$.
- The set of real orthogonal matrices is denoted by $O(n)$.
- **Theorem 1.1.7** *The determinant of an orthogonal matrix is equal to either* 1 *or* -1.
- An orthogonal matrix with unit determinant (a unimodular orthogonal matrix) is called a **proper orthogonal matrix** or just a **rotation**.
- The set of real orthogonal matrices with unit determinant is denoted by $SO(n)$.
- A set G of invertible matrices forms a **group** if it is closed under taking inverse and matrix multiplication, that is, if the inverse A^{-1} of any matrix A in G belongs to the set G and the product AB of any two matrices A and B in G belongs to G.

1.1.4 Exercises

1. Show that the product of invertible matrices is an invertible matrix.
2. Show that the product of matrices with positive determinant is a matrix with positive determinant.
3. Show that the inverse of a matrix with positive determinant is a matrix with positive determinant.
4. Show that $GL(n, \mathbb{R})$ forms a group (called the **general linear group**).
5. Show that $GL_+(n, \mathbb{R})$ is a group (called the **proper general linear group**).
6. Show that the inverse of a matrix with negative determinant is a matrix with negative determinant.
7. Show that: a) the product of an even number of matrices with negative determinant is a matrix with positive determinant, b) the product of odd matrices with negative determinant is a matrix with negative determinant.
8. Show that the product of matrices with unit determinant is a matrix with unit determinant.
9. Show that the inverse of a matrix with unit determinant is a matrix with unit determinant.

10. Show that $SL(n, \mathbb{R})$ forms a group (called the **special linear group** or the **unimodular group**).
11. Show that the product of orthogonal matrices is an orthogonal matrix.
12. Show that the inverse of an orthogonal matrix is an orthogonal matrix.
13. Show that $O(n)$ forms a group (called the **orthogonal group**).
14. Show that orthogonal matrices have determinant equal to either $+1$ or -1.
15. Show that the product of orthogonal matrices with unit determinant is an orthogonal matrix with unit determinant.
16. Show that the inverse of an orthogonal matrix with unit determinant is an orthogonal matrix with unit determinant.
17. Show that $SO(n)$ forms a group (called the **proper orthogonal group** or the **rotation group**).

1.2 Vector Spaces

- A **real vector space** consists of a set E, whose elements are called **vectors**, and the set of real numbers $\mathbb{R}$, whose elements are called **scalars**. There are two operations on a vector space:

  1. **Vector addition**, $+ : E \times E \to E$, that assigns to two vectors $\mathbf{u}, \mathbf{v} \in E$ another vector $\mathbf{u} + \mathbf{v}$, and
  2. **Multiplication by scalars**, $\cdot : \mathbb{R} \times E \to E$, that assigns to a vector $\mathbf{v} \in E$ and a scalar $a \in \mathbb{R}$ a new vector $a\mathbf{v} \in E$.

  The vector addition is an **associative commutative** operation with an **additive identity**. It satisfies the following conditions:

  1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}, \quad \forall \mathbf{u}, \mathbf{v}, \in E$
  2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}), \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in E$
  3. There is a vector $\mathbf{0} \in E$, called the **zero vector**, such that for any $\mathbf{v} \in E$ there holds $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
  4. For any vector $\mathbf{v} \in E$, there is a vector $(-\mathbf{v}) \in E$, called the **opposite** of $\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

  The multiplication by scalars satisfies the following conditions:

  1. $a(b\mathbf{v}) = (ab)\mathbf{v}, \quad \forall \mathbf{v} \in E, \forall a, b\mathbb{R},$
  2. $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}, \quad \forall \mathbf{v} \in E, \forall a, b\mathbb{R},$
  3. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}, \quad \forall \mathbf{u}, \mathbf{v} \in E, \forall a\mathbb{R},$
  4. $1\,\mathbf{v} = \mathbf{v} \quad \forall \mathbf{v} \in E.$

- The zero vector is unique.

- For any $\mathbf{u}, \mathbf{v} \in E$ there is a unique vector denoted by $\mathbf{w} = \mathbf{v} - \mathbf{u}$, called the difference of $\mathbf{v}$ and $\mathbf{u}$, such that $\mathbf{u} + \mathbf{w} = \mathbf{v}$.

- For any $\mathbf{v} \in E$,
  $$0\mathbf{v} = \mathbf{0}\,, \qquad \text{and} \qquad (-1)\mathbf{v} = -\mathbf{v}\,.$$

- Let E be a real vector space and $\mathcal{A} = \{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ be a finite collection of vectors from E. A **linear combination** of these vectors is a vector
  $$a_1\mathbf{e}_1 + \cdots + a_k\mathbf{e}_k\,,$$
  where $\{a_1, \ldots, a_n\}$ are scalars.

- A finite collection of vectors $\mathcal{A} = \{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ is **linearly independent** if
  $$a_1\mathbf{e}_1 + \cdots + a_k\mathbf{e}_k = \mathbf{0}$$
  implies $a_1 = \cdots = a_k = 0$.

- A collection $\mathcal{A}$ of vectors is **linearly dependent** if it is not linearly independent.
- Two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ which are linearly dependent are also called **parallel**, denoted by $\mathbf{u}||\mathbf{v}$.
- A collection $\mathcal{A}$ of vectors is linearly independent if no vector of $\mathcal{A}$ is a linear combination of a finite number of vectors from $\mathcal{A}$.
- Let $\mathcal{A}$ be a subset of a vector space E. The **span** of $\mathcal{A}$, denoted by $\operatorname{span}\mathcal{A}$, is the subset of E consisting of all finite linear combinations of vectors from $\mathcal{A}$, i.e.

$$\operatorname{span}\mathcal{A} = \{\mathbf{v} \in E \mid \mathbf{v} = a_1\mathbf{e}_1 + \cdots + a_k\mathbf{e}_k\,,\ \mathbf{e}_i \in \mathcal{A},\ a_i \in \mathbb{R}\}\,.$$

  We say that the subset $\operatorname{span}\mathcal{A}$ is **spanned by** $\mathcal{A}$.
- **Theorem 1.2.1** *The span of any subset of a vector space is a vector space.*
- A **vector subspace** of a vector space E is a subset $S \subseteq E$ of E which is itself a vector space.
- **Theorem 1.2.2** *A subset S of E is a vector subspace of E if and only if* $\operatorname{span} S = S$.
- Span of $\mathcal{A}$ is the smallest subspace of E containing $\mathcal{A}$.
- A collection $\mathcal{B}$ of vectors of a vector space E is a **basis** of E if $\mathcal{B}$ is linearly independent and $\operatorname{span}\mathcal{B} = E$.
- A vector space E is **finite-dimensional** if it has a finite basis.
- **Theorem 1.2.3** *If the vector space E is finite-dimensional, then the number of vectors in any basis is the same.*
- The **dimension** of a finite-dimensional real vector space E, denoted by $\dim E$, is the number of vectors in a basis.
- **Theorem 1.2.4** *If $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis in E, then for every vector $\mathbf{v} \in E$ there is a unique set of real numbers $(v^i) = (v^1, \dots, v^n)$ such that*

$$\mathbf{v} = \sum_{i=1}^{n} v^i\mathbf{e}_i = v^1\mathbf{e}_1 + \cdots + v^n\mathbf{e}_n\,.$$

- The real numbers v^i, $i = 1, \dots, n$, are called the **components of the vector v with respect to the basis** $\{\mathbf{e}_i\}$.
- It is customary to denote the components of vectors by **superscripts**, which should not be confused with powers of real numbers

$$v^2 \neq (v)^2 = vv, \qquad \dots, \qquad v^n \neq (v)^n\,.$$

Examples of Vector Subspaces

- Zero subspace $\{\mathbf{0}\}$.
- **Line** with a tangent vector $\mathbf{u}$:

$$S_1 = \operatorname{span}\{\mathbf{u}\} = \{\mathbf{v} \in E \mid \mathbf{v} = t\mathbf{u},\ t \in \mathbb{R}\}\,.$$

- **Plane** spanned by two nonparallel vectors $\mathbf{u}_1$ and $\mathbf{u}_2$

$$S_2 = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\} = \{\mathbf{v} \in E \mid \mathbf{v} = t\mathbf{u}_1 + s\mathbf{u}_2,\ t, s \in \mathbb{R}\}\,.$$

- More generally, a k-**plane** spanned by a linearly independent collection of k vectors $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$

$$S_k = \operatorname{span}\{\mathbf{u}_1, \dots, \mathbf{u}_k\} = \{\mathbf{v} \in E \mid \mathbf{v} = t_1\mathbf{u}_1 + \cdots + t_k\mathbf{u}_k,\ t_1, \dots, t_k \in \mathbb{R}\}\,.$$

- An $(n-1)$-plane in an n-dimensional vector space is called a **hyperplane**.

1.2.1 Exercises

1. Show that if $\lambda\mathbf{v} = 0$, then either $\mathbf{v} = 0$ or $\lambda = 0$.
2. Prove that the span of a collection of vectors is a vector subspace.

1.3 Inner Product and Norm

- A real vector space E is called an **inner product space** if there is a function $(\cdot,\cdot) : E \times E \to \mathbb{R}$, called the **inner product**, that assigns to every two vectors $\mathbf{u}$ and $\mathbf{v}$ a real number $(\mathbf{u},\mathbf{v})$ and satisfies the conditions: $\forall \mathbf{u},\mathbf{v},\mathbf{w} \in E,\ \forall a \in \mathbb{R}$:

  1. $(\mathbf{v},\mathbf{v}) \geq 0$
  2. $(\mathbf{v},\mathbf{v}) = 0$ if and only if $\mathbf{v} = 0$
  3. $(\mathbf{u},\mathbf{v}) = (\mathbf{v},\mathbf{u})$
  4. $(\mathbf{u}+\mathbf{v},\mathbf{w}) = (\mathbf{u},\mathbf{w}) + (\mathbf{v},\mathbf{w})$
  5. $(a\mathbf{u},\mathbf{v}) = (\mathbf{u},a\mathbf{v}) = a(\mathbf{u},\mathbf{v})$

  A finite-dimensional inner product space is called a **Euclidean space**.

- The inner product is often called the **dot product**, or the **scalar product**, and is denoted by
$$(\mathbf{u},\mathbf{v}) = \mathbf{u}\cdot\mathbf{v}\,.$$

- All spaces considered below are Euclidean spaces. Henceforth, E will denote an n-dimensional Euclidean space if not specified otherwise.

- The **Euclidean norm** is a function $||\cdot|| : E \to \mathbb{R}$ that assigns to every vector $\mathbf{v} \in E$ a real number $||\mathbf{v}||$ defined by
$$||\mathbf{v}|| = \sqrt{(\mathbf{v},\mathbf{v})}.$$

- The norm of a vector is also called the **length**.

- A vector with unit norm is called a **unit vector**.

- **Theorem 1.3.1** *For any* $\mathbf{u},\mathbf{v} \in E$ *there holds*
$$||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + 2(\mathbf{u},\mathbf{v}) + ||\mathbf{v}||^2\,.$$

- **Theorem 1.3.2 Cauchy-Schwarz's Inequality.** *For any* $\mathbf{u},\mathbf{v} \in E$ *there holds*
$$|(\mathbf{u},\mathbf{v})| \leq ||\mathbf{u}||\,||\mathbf{v}||\,.$$
  The equality
$$|(\mathbf{u},\mathbf{v})| = ||\mathbf{u}||\,||\mathbf{v}||$$
  holds if and only if $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel.*

- **Corollary 1.3.1 Triangle Inequality.** *For any* $\mathbf{u},\mathbf{v} \in E$ *there holds*
$$||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||\,.$$

- The **angle between two non-zero vectors u** and **v** is defined by

$$\cos\theta = \frac{(\mathbf{u}, \mathbf{v})}{||\mathbf{u}||\, ||\mathbf{v}||}, \qquad 0 \le \theta \le \pi\,.$$

  Then the inner product can be written in the form

$$(\mathbf{u}, \mathbf{v}) = ||\mathbf{u}||\, ||\mathbf{v}|| \cos\theta\,.$$

- Two non-zero vectors $\mathbf{u}, \mathbf{v} \in E$ are **orthogonal**, denoted by $\mathbf{u} \perp \mathbf{v}$, if

$$(\mathbf{u}, \mathbf{v}) = 0.$$

- A basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is called **orthonormal** if each vector of the basis is a unit vector and any two distinct vectors are orthogonal to each other, that is,

$$(\mathbf{e}_i, \mathbf{e}_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}\,.$$

- **Theorem 1.3.3** *Every Euclidean space has an orthonormal basis.*

- Let $S \subset E$ be a nonempty subset of E. We say that $\mathbf{x} \in E$ is **orthogonal** to S, denoted by $\mathbf{x} \perp S$, if $\mathbf{x}$ is orthogonal to every vector of S.

- The set

$$S^\perp = \{\mathbf{x} \in E \mid \mathbf{x} \perp S\}$$

  of all vectors orthogonal to S is called the **orthogonal complement** of S.

- **Theorem 1.3.4** *The orthogonal complement of any subset of a Euclidean space is a vector subspace.*

- Two subsets A and B of E are **orthogonal**, denoted by $A \perp B$, if every vector of A is orthogonal to every vector of B.

- Let S be a subspace of E and $S^\perp$ be its orthogonal complement. If every element of E can be uniquely represented as the sum of an element of S and an element of $S^\perp$, then E is the **direct sum** of S and $S^\perp$, which is denoted by

$$E = S \oplus S^\perp\,.$$

- The union of a basis of S and a basis of $S^\perp$ gives a basis of E.

1.3.1 Exercises

1. Show that the Euclidean norm has the following properties
   (a) $||\mathbf{v}|| \ \geq \ 0, \quad \forall \mathbf{v} \in E$;
   (b) $||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = 0$;

(c) $||a\mathbf{v}|| = |a|\,||\mathbf{v}||, \quad \forall \mathbf{v} \in E, \forall a \in \mathbb{R}$.

2. **Parallelogram Law**. Show that for any $\mathbf{u}, \mathbf{v} \in E$

$$||\mathbf{u} + \mathbf{v}||^2 + ||\mathbf{u} - \mathbf{v}||^2 = 2\left(||\mathbf{u}||^2 + ||\mathbf{v}||^2\right)$$

3. Show that any orthogonal system in E is linearly independent.

4. **Gram-Schmidt orthonormalization process.** Let $\mathcal{G} = \{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$ be a linearly independent collection of vectors. Let $O = \{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ be a new collection of vectors defined recursively by

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u}_1, \\ \mathbf{v}_j &= \mathbf{u}_j - \sum_{i=1}^{j-1} \frac{\mathbf{v}_i(\mathbf{v}_i, \mathbf{u}_j)}{||\mathbf{v}_i||^2}, \qquad 2 \le j \le k, \end{aligned}$$

and the collection $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ be defined by

$$\mathbf{e}_i = \frac{\mathbf{v}_i}{||\mathbf{v}_i||}.$$

Show that: a) O is an orthogonal system and b) $\mathcal{B}$ is an orthonormal system.

5. **Pythagorean Theorem.** Show that if $\mathbf{u} \perp \mathbf{v}$, then

$$||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 .$$

6. Let $\mathcal{B} = \{\mathbf{e}_1, \cdots \mathbf{e}_n\}$ be an orthonormal basis in E. Show that for any vector $\mathbf{v} \in E$

$$\mathbf{v} = \sum_{i=1}^{n} \mathbf{e}_i(\mathbf{e}_i, \mathbf{v})$$

and

$$||\mathbf{v}||^2 = \sum_{i=1}^{n} (\mathbf{e}_i, \mathbf{v})^2 .$$

7. Prove that the orthogonal complement of a subset S of E is a vector subspace of E.

8. Let S be a subspace in E. Prove that

$$\text{a) } E^\perp = \{\mathbf{0}\}, \qquad \text{b) } \{\mathbf{0}\}^\perp = E, \qquad \text{c) } (S^\perp)^\perp = S .$$

9. Show that the intersection of orthogonal subsets of a Euclidean space is either empty or consists of only the zero vector. That is, for two subsets A and B, if $A \perp B$, then $A \cap B = \{\mathbf{0}\}$ or $\emptyset$.

1.4 Linear Operators

- A **linear operator** on a vector space E is a mapping $\mathsf{L} : E \to E$ satisfying the condition $\forall \mathbf{u}, \mathbf{v} \in E$, $\forall a \in \mathbb{R}$,

$$\mathsf{L}(\mathbf{u}+\mathbf{v}) = \mathsf{L}(\mathbf{u}) + \mathsf{L}(\mathbf{v}) \qquad \text{and} \qquad \mathsf{L}(a\mathbf{v}) = a\mathsf{L}(\mathbf{v}).$$

- **Identity operator I** on E is defined by

$$\mathsf{I}\,\mathbf{v} = \mathbf{v}, \qquad \forall \mathbf{v} \in E$$

- **Null operator $\mathsf{0}$** $: E \to E$ is defined by

$$\mathsf{0}\mathbf{v} = \mathbf{0}, \qquad \forall \mathbf{v} \in E$$

- The vector $\mathbf{u} = \mathsf{L}(\mathbf{v})$ is the **image** of the vector $\mathbf{v}$.
- If S is a subset of E, then the set

$$\mathsf{L}(S) = \{\mathbf{u} \in E \mid \mathbf{u} = \mathsf{L}(\mathbf{v}) \text{ for some } \mathbf{v} \in S\}$$

  is the **image of the set** S and the set

$$\mathsf{L}^{-1}(S) = \{\mathbf{v} \in E \mid \mathsf{L}(\mathbf{v}) \in S\}$$

  is the **inverse image of the set** A.
- The image of the whole space E of a linear operator L is the **range** (or the **image**) of L, denoted by

$$\operatorname{Im}(\mathsf{L}) = \mathsf{L}(E) = \{\mathbf{u} \in E \mid \mathbf{u} = \mathsf{L}(\mathbf{v}) \text{ for some } \mathbf{v} \in E\}.$$

- The **kernel** $\operatorname{Ker}(\mathsf{L})$ (or the **null space**) of an operator L is the set of all vectors in E which are mapped to zero, that is

$$\operatorname{Ker}(\mathsf{L}) = \mathsf{L}^{-1}(\{\mathbf{0}\}) = \{\mathbf{v} \in E \mid \mathsf{L}(\mathbf{v}) = \mathbf{0}\}.$$

- **Theorem 1.4.1** *For any operator L the sets $\operatorname{Im}(\mathsf{L})$ and $\operatorname{Ker}(\mathsf{L})$ are vector subspaces.*
- The dimension of the kernel $\operatorname{Ker}(\mathsf{L})$ of an operator L

$$\operatorname{null}(\mathsf{L}) = \dim \operatorname{Ker}(\mathsf{L})$$

  is called the **nullity** of the operator L.
- The dimension of the range $\operatorname{Im}(\mathsf{L})$ of an operator L

$$\operatorname{rank}(\mathsf{L}) = \dim \operatorname{Ker}(\mathsf{L})$$

  is called the **rank** of the operator L.

- **Theorem 1.4.2** *For any operator* $\mathbf{L}$ *on an n-dimensional Euclidean space E*

$$\operatorname{rank}(\mathbf{L}) + \operatorname{null}(\mathbf{L}) = n$$

- The set $\mathcal{L}(E)$ of all linear operators on a vector space E is a vector space with the **addition of operators** and **multiplication by scalars** defined by

$$(\mathbf{L}_1 + \mathbf{L}_2)(\mathbf{x}) = \mathbf{L}_1(\mathbf{x}) + \mathbf{L}_2(\mathbf{x}), \qquad \text{and} \qquad (a\mathbf{L})(\mathbf{x}) = a\mathbf{L}(\mathbf{x})\,.$$

- The **product of the operators** $\mathbf{A}$ and $\mathbf{B}$ is the composition of $\mathbf{A}$ and $\mathbf{B}$.

- Since the product of operators is defined as a composition of linear mappings, it is automatically **associative**, which means that for any operators $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, there holds

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})\,.$$

- The integer powers of an operator are defined as the multiple composition of the operator with itself, i.e.

$$\mathbf{A}^0 = \mathbf{I} \qquad \mathbf{A}^1 = \mathbf{A}, \qquad \mathbf{A}^2 = \mathbf{AA}, \ldots$$

- The operator $\mathbf{A}$ on E is **invertible** if there exists an operator $\mathbf{A}^{-1}$ on E, called the **inverse** of $\mathbf{A}$, such that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}\,.$$

- **Theorem 1.4.3** *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be invertible operators. Then:*

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}\,, \qquad (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}\,.$$

- The operators $\mathbf{A}$ and $\mathbf{B}$ are **commuting** if

$$\mathbf{AB} = \mathbf{BA}$$

  and **anti-commuting** if

$$\mathbf{AB} = -\mathbf{BA}\,.$$

- The operators $\mathbf{A}$ and $\mathbf{B}$ are said to be **orthogonal** to each other if

$$\mathbf{AB} = \mathbf{BA} = \mathbf{0}\,.$$

- An operator $\mathbf{A}$ is **involutive** if

$$\mathbf{A}^2 = \mathbf{I}$$

  idempotent if

$$\mathbf{A}^2 = \mathbf{A}\,,$$

  and **nilpotent** if for some integer k

$$\mathbf{A}^k = \mathbf{0}\,.$$

Selfadjoint Operators

- The **adjoint** $\mathbf{A}^*$ of an operator $\mathbf{A}$ is defined by

$$(\mathbf{A}\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{A}^*\mathbf{v}), \qquad \forall \mathbf{u}, \mathbf{v} \in E.$$

- **Theorem 1.4.4** *For any two operators* $\mathbf{A}$ *and* $\mathbf{B}$

$$(\mathbf{A}^*)^* = \mathbf{A}, \qquad (\mathbf{A}\mathbf{B})^* = \mathbf{B}^*\mathbf{A}^*.$$

- An operator A is **self-adjoint** if

$$\mathbf{A}^* = \mathbf{A}$$

  and **anti-selfadjoint** if

$$\mathbf{A}^* = -\mathbf{A}$$

- Every operator $\mathbf{A}$ can be decomposed as the sum

$$\mathbf{A} = \mathbf{A}_S + \mathbf{A}_A$$

  of its **selfadjoint part** $\mathbf{A}_S$ and its **anti-selfadjoint part** $\mathbf{A}_A$

$$\mathbf{A}_S = \frac{1}{2}(\mathbf{A} + \mathbf{A}^*), \qquad \mathbf{A}_A = \frac{1}{2}(\mathbf{A} - \mathbf{A}^*).$$

- An operator $\mathbf{A}$ is called **unitary** if

$$\mathbf{A}\mathbf{A}^* = \mathbf{A}^*\mathbf{A} = \mathbf{I}.$$

- An operator $\mathbf{A}$ on E is called **positive**, denoted by $\mathbf{A} \geq 0$, if it is selfdadjoint and $\forall \mathbf{v} \in E$

$$(\mathbf{A}\mathbf{v}, \mathbf{v}) \geq 0.$$

Projection Operators

- Let S be a subspace of E and $E = S \oplus S^\perp$. Then for any $\mathbf{u} \in E$ there exist unique $\mathbf{v} \in S$ and $\mathbf{w} \in S^\perp$ such that

$$\mathbf{u} = \mathbf{v} + \mathbf{w}.$$

  The vector $\mathbf{v}$ is called the **projection** of $\mathbf{u}$ onto S.

- The operator $\mathbf{P}$ on E defined by

$$\mathbf{P}\mathbf{u} = \mathbf{v}$$

  is called the **projection operator** onto S.

- The operator $\mathsf{P}^\perp$ defined by
$$\mathsf{P}^\perp \mathbf{u} = \mathbf{w}$$
is the projection operator onto $S^\perp$.

- The operators P and $\mathsf{P}^\perp$ are called **complementary projections**. They have the properties:
$$\mathsf{P}^* = \mathsf{P}, \qquad (\mathsf{P}^\perp)^* = \mathsf{P}^\perp\,,$$
$$\mathsf{P} + \mathsf{P}^\perp = \mathsf{I}\,,$$
$$\mathsf{P}^2 = \mathsf{P}\,, \qquad (\mathsf{P}^\perp)^2 = \mathsf{P}^\perp\,,$$
$$\mathsf{P}\mathsf{P}^\perp = \mathsf{P}^\perp\mathsf{P} = \mathbf{0}\,.$$

- **Theorem 1.4.5** *An operator* P *is a projection if and only if* P *is idempotent and self-adjoint.*

- More generally, a collection of projections $\{\mathsf{P}_1, \dots, \mathsf{P}_k\}$ is a **complete orthogonal system of complimentary projections** if
$$\mathsf{P}_i \mathsf{P}_k = \mathbf{0} \qquad \text{if} \qquad i \neq k$$
and
$$\sum_{i=1}^{k} \mathsf{P}_i = \mathsf{P}_1 + \cdots + \mathsf{P}_k = \mathsf{I}\,.$$

- A complete orthogonal system of projections defines the orthogonal decomposition of the vector space
$$E = E_1 \oplus \cdots \oplus E_k\,,$$
where E_i is the subspace the projection P_i projects onto.

- **Theorem 1.4.6** 1. *The dimension of the subspaces E_i are equal to the ranks of the projections* P_i
$$\dim E_i = \operatorname{rank} \mathsf{P}_i\,.$$
  2. *The sum of dimensions of the vector subspaces E_i equals the dimension of the vector space E*
$$\sum_{i=1}^{n} \dim E_i = \dim E_1 + \cdots + \dim E_k = \dim E\,.$$

Spectral Decomposition Theorem

- A real number λ is called an **eigenvalue** of an operator A if there is a unit vector $\mathbf{u} \in E$ such that
$$A\mathbf{u} = \lambda \mathbf{u}\,.$$
The vector $\mathbf{u}$ is called the **eigenvector** corresponding to the eigenvalue λ.
- The span of all eigenvectors corresponding to the eigenvalue λ of an operator A is called the **eigenspace** of λ.
- The dimension of the eigenspace of the eigenvalue λ is called the **multiplicity** (also called the **geometric multiplicity**) of λ.
- An eigenvalue of multiplicity 1 is called **simple** (or **non-degenerate**).
- An eigenvalue of multiplicity greater than 1 is called **multiple** (or **degenerate**).
- The set of all eigenvalues of an operator is called the **spectrum** of the operator.
- **Theorem 1.4.7** *Let* A *be a selfadjoint operator. Then:*
  1. *The number of eigenvalues counted with multiplicity is equal to the dimension* $n = \dim E$ *of the vector space* E.
  2. *The eigenvectors corresponding to distinct eigenvalues are orthogonal to each other.*
- **Theorem 1.4.8 Spectral Decomposition of Self-Adjoint Operators.** *Let* A *be a selfadjoint operator on* E. *Then there exists an orthonormal basis* $\mathcal{B} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ *in* E *consisting of eigenvectors of* A *corresponding to the eigenvalues* $\{\lambda_1, \dots \lambda_n\}$*, and the corresponding system of orthogonal complimentary projections* $\{\mathsf{P}_1, \dots, \mathsf{P}_n\}$ *onto the one-dimensional eigenspaces* E_i,
$$\mathsf{A} = \sum_{i=1}^{n} \lambda_i \mathsf{P}_i\,.$$
The projections $\{\mathsf{P}_i\}$ *are defined by*
$$\mathsf{P}_i \mathbf{v} = \mathbf{e}_i(\mathbf{e}_i, \mathbf{v})\,.$$
and satisfy the equations
$$\sum_{i=1}^{n} \mathsf{P}_i = \mathsf{I}\,, \qquad \text{and} \qquad \mathsf{P}_i \mathsf{P}_j = \mathbf{0} \qquad \text{if } i \neq j\,.$$
- In other words, for any
$$\mathbf{v} = \sum_{i=1}^{n} \mathbf{e}_i(\mathbf{e}_i, \mathbf{v})\,,$$
we have
$$\mathsf{A}\mathbf{v} = \sum_{i=1}^{n} \lambda_i \mathbf{e}_i(\mathbf{e}_i, \mathbf{v})\,.$$

- Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function on $\mathbb{R}$. Let $\mathbf{A}$ be a selfadjoint operator on a Euclidean space E given by its spectral decomposition

$$\mathbf{A} = \sum_{i=1}^{n} \lambda_i \mathbf{P}_i \,,$$

where $\mathbf{P}_i$ are the one-dimensional projections. Then one can define a **function of the self-adjoint operator** $f(\mathbf{A})$ on E by

$$f(\mathbf{A}) = \sum_{i=1}^{n} f(\lambda_i)\mathbf{P}_i \,.$$

- The **exponential of an operator $\mathbf{A}$** is defined by

$$\exp \mathbf{A} = \sum_{k=1}^{\infty} \frac{1}{k!}\mathbf{A}^k = \sum_{i=1}^{n} e^{\lambda_i}\mathbf{P}_i$$

- **Theorem 1.4.9** *Let $\mathbf{U}$ be a unitary operator on a real vector space E. Then there exists an ani-selfadjoint operator $\mathbf{A}$ such that*

$$\mathbf{U} = \exp \mathbf{A} \,.$$

- Recall that the operators $\mathbf{U}$ and $\mathbf{A}$ satisfy the equations

$$\mathbf{U}^* = \mathbf{U}^{-1} \text{ and } \mathbf{A}^* = -\mathbf{A}.$$

- Let $\mathbf{A}$ be a self-adjoint operator with the eigenvalues $\{\lambda_1, \dots, \lambda_n\}$. Then the **trace of the operator** and the **determinant of the operator $\mathbf{A}$** are defined by

$$\operatorname{tr} \mathbf{A} = \sum_{i=1}^{n} \lambda_i \,, \qquad \det \mathbf{A} = \lambda_1 \cdots \lambda_n \,.$$

- Note that

$$\operatorname{tr} \mathbf{I} = n \,, \qquad \det \mathbf{I} = 1 \,.$$

- The trace of a projection $\mathbf{P}$ onto a vector subspace S is equal to its rank, or the dimension of the vector subspace S,

$$\operatorname{tr} \mathbf{P} = \operatorname{rank} \mathbf{P} = \dim S \,.$$

- The trace of a function of a self-adjoint operator $\mathbf{A}$ is then

$$\operatorname{tr} f(\mathbf{A}) = \sum_{i=1}^{n} f(\lambda_i) \,.$$

If there are multiple eigenvalues, then each eigenvalue should be counted with its multiplicity.

- **Theorem 1.4.10** *Let* $\mathbf{A}$ *be a self-adjoint operator. Then*

$$\det\exp\mathbf{A} = e^{\operatorname{tr}\mathbf{A}}\,.$$

- Let $\mathbf{A}$ be a positive definite operator, $\mathbf{A} > 0$. The **zeta-function** of the operator $\mathbf{A}$ is defined by

$$\zeta(s) = \operatorname{tr}\mathbf{A}^{-s} = \sum_{i=1}^{n}\frac{1}{\lambda_i^s}\,.$$

- **Theorem 1.4.11** *The zeta-functions has the properties*

$$\zeta(0) = n\,,$$

and

$$\zeta'(0) = -\log\det\mathbf{A}\,.$$

Examples

- Let $\mathbf{u}$ be a unit vector and $\mathbf{P}_\mathbf{u}$ be the projection onto the one-dimensional subspace (line) $S_\mathbf{u}$ spanned by $\mathbf{u}$ defined by

$$\mathbf{P}_\mathbf{u}\mathbf{v} = \mathbf{u}(\mathbf{u},\mathbf{v})\,.$$

The orthogonal complement $S_\mathbf{u}^\perp$ is the hyperplane with the normal $\mathbf{u}$. The operator $\mathbf{J}_\mathbf{u}$ defined by

$$\mathbf{J}_\mathbf{u} = \mathbf{I} - 2\mathbf{P}_\mathbf{u}$$

is called the **reflection operator** with respect to the hyperplane $S_\mathbf{u}^\perp$. The reflection operator is a **self-adjoint involution**, that is, it has the following properties

$$\mathbf{J}_\mathbf{u}^* = \mathbf{J}_\mathbf{u}\,, \qquad \mathbf{J}_\mathbf{u}^2 = \mathbf{I}\,.$$

The reflection operator has the eigenvalue -1 with multiplicity 1 and the eigenspace $S_\mathbf{u}$, and the eigenvalue $+1$ with multiplicity $(n-1)$ and with eigenspace $S_\mathbf{u}^\perp$.

- Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be an orthonormal system of two vectors and $\mathbf{P}_{\mathbf{u}_1,\mathbf{u}_2}$ be the projection operator onto the two-dimensional space (plane) $S_{\mathbf{u}_1,\mathbf{u}_2}$ spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$

$$\mathbf{P}_{\mathbf{u}_1,\mathbf{u}_2}\mathbf{v} = \mathbf{u}_1(\mathbf{u}_1,\mathbf{v}) + \mathbf{u}_2(\mathbf{u}_2,\mathbf{v})\,.$$

Let $\mathbf{N}_{\mathbf{u}_1,\mathbf{u}_2}$ be an operator defined by

$$\mathbf{N}_{\mathbf{u}_1,\mathbf{u}_2}\mathbf{v} = \mathbf{u}_1(\mathbf{u}_2,\mathbf{v}) - \mathbf{u}_2(\mathbf{u}_1,\mathbf{v})\,.$$

Then

$$\mathbf{N}_{\mathbf{u}_1,\mathbf{u}_2}\mathbf{P}_{\mathbf{u}_1,\mathbf{u}_2} = \mathbf{P}_{\mathbf{u}_1,\mathbf{u}_2}\mathbf{N}_{\mathbf{u}_1,\mathbf{u}_2} = \mathbf{N}_{\mathbf{u}_1,\mathbf{u}_2}$$

and

$$\mathbf{N}_{\mathbf{u}_1,\mathbf{u}_2}^2 = -\mathbf{P}_{\mathbf{u}_1,\mathbf{u}_2}\,.$$

A **rotation operator** $\mathbf{R}_{\mathbf{u}_1,\mathbf{u}_2}(\theta)$ with the angle θ in the plane $S_{\mathbf{u}_1,\mathbf{u}_2}$ is defined by

$$\mathbf{R}_{\mathbf{u}_1,\mathbf{u}_2}(\theta) = \mathbf{I} - \mathbf{P}_{\mathbf{u}_1,\mathbf{u}_2} + \cos\theta\, \mathbf{P}_{\mathbf{u}_1,\mathbf{u}_2} + \sin\theta\, \mathbf{N}_{\mathbf{u}_1,\mathbf{u}_2}\,.$$

The rotation operator is **unitary**, that is, it satisfies the equation

$$\mathbf{R}^*_{\mathbf{u}_1,\mathbf{u}_2}\mathbf{R}_{\mathbf{u}_1,\mathbf{u}_2} = \mathbf{I}\,.$$

- **Theorem 1.4.12 Spectral Decomposition of Unitary Operators on Real Vector Spaces.** *Let $\mathbf{U}$ be a unitary operator on a real vector space E. Then the only eigenvalues of $\mathbf{U}$ are $+1$ and -1 (possibly multiple) and there exists an orthogonal decomposition*

$$E = E_+ \oplus E_- \oplus V_1 \oplus \cdots \oplus V_k\,,$$

where E_+ and E_- are the eigenspaces corresponding to the eigenvalues 1 and -1, and $\{V_1, \dots, v_k\}$ are two-dimensional subspaces such that

$$\dim E = \dim E_+ + \dim E_- + 2k\,.$$

Let $\mathbf{P}_+$, $\mathbf{P}_-$, $\mathbf{P}_1, \dots, \mathbf{P}_k$ be the corresponding orthogonal complimentary system of projections, that is,

$$\mathbf{P}_+ + \mathbf{P}_- + \sum_{i=1}^{k} \mathbf{P}_i = \mathbf{I}\,.$$

Then there exists a corresponding system of operators $\mathbf{N}_1, \dots, \mathbf{N}_k$ satisfying the equations

$$\mathbf{N}_i^2 = -\mathbf{P}_i\,, \qquad \mathbf{N}_i\mathbf{P}_i = \mathbf{P}_i\mathbf{N}_i = \mathbf{N}_i\,,$$
$$\mathbf{N}_i\mathbf{P}_j = \mathbf{P}_j\mathbf{N}_i = \mathbf{0}\,, \qquad \text{if} \qquad i \neq j$$

and the angles $\theta_1, \dots \theta_k$ such that

$$\mathbf{U} = \mathbf{P}_+ - \mathbf{P}_- + \sum_{i=1}^{k}(\cos\theta_i\, \mathbf{P}_i + \sin\theta_i\, \mathbf{N}_i)\,.$$

1.4.1 Exercises

1. Prove that the range and the kernel of any operator are vector spaces.
2. Show that

$$(a\mathbf{A} + b\mathbf{B})^* = a\mathbf{A}^* + b\mathbf{B}^* \qquad \forall a, b \in \mathbb{R}\,,$$
$$(\mathbf{A}^*)^* = \mathbf{A}$$
$$(\mathbf{A}\mathbf{B})^* = \mathbf{B}^*\mathbf{A}^*$$

3. Show that for any operator $\mathbf{A}$ the operators $\mathbf{A}\mathbf{A}^*$ and $\mathbf{A} + \mathbf{A}^*$ are selfadjoint.
4. Show that the product of two selfadjoint operators is selfadjoint if and only if they commute.
5. Show that a polynomial $p(\mathbf{A})$ of a selfadjoint operator $\mathbf{A}$ is a selfadjoint operator.

6. Prove that the inverse of an invertible operator is unique.

7. Prove that an operator $\mathbf{A}$ is invertible if and only if $\operatorname{Ker} \mathbf{A} = \{\mathbf{0}\}$, that is, $\mathbf{A}\mathbf{v} = \mathbf{0}$ implies $\mathbf{v} = \mathbf{0}$.

8. Prove that for an invertible operator $\mathbf{A}$, $\operatorname{Im}(\mathbf{A}) = E$, that is, for any vector $\mathbf{v} \in E$ there is a vector $\mathbf{u} \in E$ such that $\mathbf{v} = \mathbf{A}\mathbf{u}$.

9. Show that if an operator $\mathbf{A}$ is invertible, then
$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}\,.$$

10. Show that the product $\mathbf{AB}$ of two invertible operators $\mathbf{A}$ and $\mathbf{B}$ is invertible and
$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

11. Prove that the adjoint $\mathbf{A}^*$ of any invertible operator $\mathbf{A}$ is invertible and
$$(\mathbf{A}^*)^{-1} = (\mathbf{A}^{-1})^*\,.$$

12. Prove that the inverse $\mathbf{A}^{-1}$ of a selfadjoint invertible operator is selfadjoint.

13. An operator $\mathbf{A}$ on E is called **isometric** if $\forall \mathbf{v} \in E$,
$$||\mathbf{A}\mathbf{v}|| = ||\mathbf{v}||\,.$$
Prove that an operator is unitary if and only if it is isometric.

14. Prove that unitary operators preserves inner product. That is, show that if $\mathbf{A}$ is a unitary operator, then $\forall \mathbf{u}, \mathbf{v} \in E$
$$(\mathbf{A}\mathbf{u}, \mathbf{A}\mathbf{v}) = (\mathbf{u}, \mathbf{v})\,.$$

15. Show that for every unitary operator $\mathbf{A}$ both $\mathbf{A}^{-1}$ and $\mathbf{A}^*$ are unitary.

16. Show that for any operator $\mathbf{A}$ the operators $\mathbf{A}\mathbf{A}^*$ and $\mathbf{A}^*\mathbf{A}$ are positive.

17. What subspaces do the null operator $\mathbf{0}$ and the identity operator $\mathbf{I}$ project onto?

18. Show that for any two projection operators $\mathbf{P}$ and $\mathbf{Q}$, $\mathbf{PQ} = \mathbf{0}$ if and only if $\mathbf{QP} = \mathbf{0}$.

19. Prove the following properties of orthogonal projections
$$\mathbf{P}^* = \mathbf{P}, \qquad (\mathbf{P}^\perp)^* = \mathbf{P}^\perp, \qquad \mathbf{P}^\perp + \mathbf{P} = \mathbf{I}, \qquad \mathbf{P}\mathbf{P}^\perp = \mathbf{P}^\perp\mathbf{P} = \mathbf{0}\,.$$

20. Prove that an operator is projection if and only if it is idempotent and selfadjoint.

21. Give an example of an idempotent operator in $\mathbb{R}^2$ which is not a projection.

22. Show that any projection operator $\mathbf{P}$ is positive. Moreover, show that $\forall \mathbf{v} \in E$
$$(\mathbf{P}\mathbf{v}, \mathbf{v}) \;= ||\mathbf{P}\mathbf{v}||^2\,.$$

23. Prove that the sum $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$ of two projections $\mathbf{P}_1$ and $\mathbf{P}_2$ is a projection operator if and only if $\mathbf{P}_1$ and $\mathbf{P}_2$ are orthogonal.

24. Prove that the product $\mathbf{P} = \mathbf{P}_1\mathbf{P}_2$ of two projections $\mathbf{P}_1$ and $\mathbf{P}_2$ is a projection operator if and only if $\mathbf{P}_1$ and $\mathbf{P}_2$ commute.

25. Find the eigenvalues of a projection operator.

26. Prove that the span of all eigenvectors corresponding to the eigenvalue λ of an operator $\mathbf{A}$ is a vector space.

27. Let
$$E(\lambda) = \mathrm{Ker}\,(\mathbf{A} - \lambda\mathbf{I})\,.$$
Show that: a) if λ is not an eigenvalue of $\mathbf{A}$, then $E(\lambda) = \emptyset$, and b) if λ is an eigenvalue of $\mathbf{A}$, then $E(\lambda)$ is the eigenspace corresponding to the eigenvalue λ.
28. Show that the operator $\mathbf{A} - \lambda\mathbf{I}$ is invertible if and only if λ is not an eigenvalue of the operator A.
29. Let $\mathbf{T}$ be a unitary operator. Then the operators $\mathbf{A}$ and
$$\tilde{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$$
are called **similar**. Show that the eigenvalues of similar operators are the same.
30. Show that an operator similar to a selfadjoint operator is selfadjoint and an operator similar to an anti-selfadjoint operator is anti-selfadjoint.
31. Show that all eigenvalues of a positive operator $\mathbf{A}$ are non-negative.
32. Show that the eigenvectors corresponding to distinct eigenvalues of a unitary operator are orthogonal to each other.
33. Show that the eigenvectors corresponding to distinct eigenvalues of a selfadjoint operator are orthogonal to each other.
34. Show that all eigenvalues of a unitary operator $\mathbf{A}$ have absolute value equal to 1.
35. Show that if $\mathbf{A}$ is a projection, then it can only have two eigenvalues: 1 and 0.

Chapter 2

Vector and Tensor Algebra

2.1 Metric Tensor

- Let E be a Euclidean space and $\{\mathbf{e}_i\} = \{\mathbf{e}_1, \dots \mathbf{e}_n\}$ be a basis (not necessarily orthonormal). Then each vector $\mathbf{v} \in E$ can be represented as a linear combination

$$\mathbf{v} = \sum_{i=1}^{n} v^i \mathbf{e}_i \,,$$

  where v^i, $i = 1, \dots, n$, are the components of the vector $\mathbf{v}$ with respect to the basis $\{\mathbf{e}_i\}$ (or **contravariant components** of the vector $\mathbf{v}$). We stress once again that contravariant components of vectors are denoted by upper indices (**superscripts**).

- Let $G = (g_{ij})$ be a matrix whose entries are defined by

$$g_{ij} = (\mathbf{e}_i, \mathbf{e}_j)\,.$$

  These numbers are called the **components of the metric tensor** with respect to the basis $\{\mathbf{e}_i\}$ (also called **covariant components** of the metric).

- Notice that the matrix G is symmetric, that is,

$$g_{ij} = g_{ji}\,, \qquad G^T = G.$$

- **Theorem 2.1.1** *The matrix G is invertible and*

$$\det G > 0\,.$$

- The elements of the inverse matrix $G^{-1} = (g^{ij})$ are called the **contravariant components** of the metric. They satisfy the equations

$$\sum_{j=1}^{n} g^{ij} g_{jk} = \delta^i_j\,,$$

where δ^i_j is the **Kronecker symbol** defined by

$$\delta^i_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

- Since the inverse of a symmetric matrix is symmetric, we have

$$g^{ij} = g^{ji} .$$

- In orthonormal basis

$$g_{ij} = g^{ij} = \delta_{ij} , \qquad G = G^{-1} = I .$$

- Let $\mathbf{v} \in E$ be a vector. The real numbers

$$v_i = (\mathbf{e}_i, \mathbf{v})$$

are called the **covariant components** of the vector $\mathbf{v}$. Notice that covariant components of vectors are denoted by lower indices (**subscripts**).

- **Theorem 2.1.2** *Let $\mathbf{v} \in E$ be a vector. The covariant and the contravariant components of $\mathbf{v}$ are related by*

$$v_i = \sum_{j=1}^{n} g_{ij} v^j , \qquad v^i = \sum_{j=1}^{n} g^{ij} v_j .$$

- **Theorem 2.1.3** *The metric determines the inner product and the norm by*

$$(\mathbf{u}, \mathbf{v}) = \sum_{j=1}^{n} \sum_{i=1}^{n} g_{ij} u^i v^j = \sum_{j=1}^{n} \sum_{i=1}^{n} g^{ij} u_i v_j .$$

$$||\mathbf{v}||^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} v^i v^j = \sum_{j=1}^{n} \sum_{i=1}^{n} g^{ij} v_i v_j .$$

2.2 Dual Space and Covectors

- A linear mapping $\boldsymbol{\omega} : E \to \mathbb{R}$ that assigns to a vector $\mathbf{v} \in E$ a real number $\langle \boldsymbol{\omega}, \mathbf{v} \rangle$ and satisfies the condition: $\forall \mathbf{u}, \mathbf{v} \in E$, $\forall a \in \mathbb{R}$

$$\langle \boldsymbol{\omega}, \mathbf{u} + \mathbf{v} \rangle = \langle \boldsymbol{\omega}, \mathbf{u} \rangle + \langle \boldsymbol{\omega}, \mathbf{v} \rangle \,, \qquad \text{and} \qquad \langle \boldsymbol{\omega}, a\mathbf{v} \rangle = a \langle \boldsymbol{\omega}, \mathbf{v} \rangle \,,$$

  is called a **linear functional**.

- The space of linear functionals is a vector space, called the **dual space** of E and denoted by E^*, with the addition and multiplication by scalars defined by: $\forall \boldsymbol{\omega}, \boldsymbol{\sigma} \in E^*$, $\forall \mathbf{v} \in E$, $\forall a \in \mathbb{R}$,

$$\langle \boldsymbol{\omega} + \boldsymbol{\sigma}, \mathbf{v} \rangle = \langle \boldsymbol{\omega}, \mathbf{v} \rangle + \langle \boldsymbol{\sigma}, \mathbf{v} \rangle \,, \qquad \text{and} \qquad \langle a\boldsymbol{\omega}, \mathbf{v} \rangle = a \langle \boldsymbol{\omega}, \mathbf{v} \rangle \,.$$

  The elements of the dual space E^* are also called **covectors** or **1-forms**. In keeping with tradition we will denote covectors by Greek letters.

- **Theorem 2.2.1** *The dual space E^* of a real vector space E is a real vector space of the same dimension.*

- Let $\{\mathbf{e}_i\} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis in E. A basis $\{\boldsymbol{\omega}^i\} = \{\boldsymbol{\omega}^1, \dots \boldsymbol{\omega}^n\}$ in E^* such that

$$\langle \boldsymbol{\omega}^i, \mathbf{e}_j \rangle = \delta^i_j$$

  is called the **dual basis**.

- The dual $\{\boldsymbol{\omega}^i\}$ of an orthonormal basis is also orthonormal.

- Given a dual basis $\{\boldsymbol{\omega}^i\}$ every covector σ in E^* can be represented in a unique way as

$$\boldsymbol{\sigma} = \sum_{i=1}^{n} \sigma_i \boldsymbol{\omega}^i \,,$$

  where the real numbers $(\sigma_i) = (\sigma_1, \dots, \sigma_n)$ are called the **components of the covector σ with respect to the basis $\{\omega_i\}$**.

- The advantage of using the dual basis is that it allows one to compute the components of a vector $\mathbf{v}$ and a covector σ by

$$v^i = \langle \boldsymbol{\omega}^i, \mathbf{v} \rangle$$

  and

$$\sigma_i = \langle \boldsymbol{\sigma}, \mathbf{e}_i \rangle \,.$$

  That is

$$\mathbf{v} = \sum_{i=1}^{n} \mathbf{e}_i \langle \boldsymbol{\omega}^i, \mathbf{v} \rangle$$

  and

$$\boldsymbol{\sigma} = \sum_{i=1}^{n} \langle \boldsymbol{\sigma}, \mathbf{e}_i \rangle \boldsymbol{\omega}^i \,.$$

- More generally, the action of a covector σ on a vector $\mathbf{v}$ has the form

$$\langle \boldsymbol{\sigma}, \mathbf{v} \rangle = \sum_{i=1}^{n} \langle \boldsymbol{\sigma}, \mathbf{e}_i \rangle \langle \omega^i, \mathbf{v} \rangle = \sum_{i=1}^{n} \sigma_i v^i \,.$$

- The existense of the metric allows us to define the following map

$$g : E \to E^*$$

that assigns to each vector $\mathbf{v}$ a covector $g(\mathbf{v})$ such that

$$\langle g(\mathbf{v}), \mathbf{u} \rangle = (\mathbf{v}, \mathbf{u}) \,.$$

Then

$$g(\mathbf{v}) = \sum_{i=1}^{n} (\mathbf{v}, \mathbf{e}_i) \omega^i \,.$$

In particular,

$$g(\mathbf{e}_k) = \sum_{i=1}^{n} g_{ki} \omega^i \,.$$

- Let $\mathbf{v}$ be a vector and $\boldsymbol{\sigma}$ be the corresponding covector, so $\boldsymbol{\sigma} = g(\mathbf{v})$ and $\mathbf{v} = g^{-1}(\boldsymbol{\sigma})$. Then their components are related by

$$\sigma_i = \sum_{j=1}^{n} g_{ij} v^j \,, \qquad v^i = \sum_{j=1}^{n} g^{ij} \sigma_j \,.$$

- The inverse map $g^{-1} : E^* \to E$ that assigns to each covector $\boldsymbol{\sigma}$ a vector $g^{-1}(\boldsymbol{\sigma})$ such that

$$\langle \boldsymbol{\sigma}, \mathbf{u} \rangle = (g^{-1}(\boldsymbol{\sigma}), \mathbf{u}) \,,$$

can be defined as follows. First, we define

$$g^{-1}(\omega^k) = \sum_{i=1}^{n} g^{ki} \mathbf{e}_i \,.$$

Then

$$g^{-1}(\boldsymbol{\sigma}) = \sum_{k=1}^{n} \sum_{i=1}^{n} \langle \boldsymbol{\sigma}, \mathbf{e}_k \rangle g^{ki} \mathbf{e}_i \,.$$

- The inner product on the dual space E^* is defined so that for any two covectors $\boldsymbol{\alpha}$ and $\boldsymbol{\sigma}$

$$(\alpha, \boldsymbol{\sigma}) = \langle \alpha, g^{-1}(\boldsymbol{\sigma}) \rangle = (g^{-1}(\alpha), g^{-1}(\alpha)) \,.$$

- This definition leads to

$$g^{ij} = (\omega^i, \omega^j) \,.$$

- **Theorem 2.2.2** *The inner product on the dual space E^* is determined by*

$$(\alpha, \sigma) = \sum_{i=1}^{n} \sum_{j=1}^{n} g^{ij} \alpha_i \sigma_j .$$

In particular,

$$(\omega^i, \sigma) = \sum_{j=1}^{n} g^{ij} \sigma_j$$

- The inverse map g^{-1} can be defined in terms of the inner product of covectors as

$$g^{-1}(\sigma) = \sum_{i=1}^{n} \mathbf{e}_i (\omega^i, \sigma) .$$

- Since there is a one-to-one correspondence between vectors and covectors, we can treat a vector $\mathbf{v}$ and the corresponding covector $g(\mathbf{v})$ as a single object and denote the components v^i of the vector $\mathbf{v}$ and the components of the covector $g(\mathbf{v})$ by the same letter, that is,

$$v^i = \sum_{j=1}^{n} g^{ij} v_j , \qquad v_i = \sum_{j=1}^{n} g_{ij} v^j .$$

- We call v^i the **contravaraint components** and v_i the **covariant components**. This operation is called **raising** and **lowering** an index; we use g^{ij} to raise an index and g_{ij} to lower an index.

2.2.1 Einstein Summation Convention

- In many equations of vector and tensor calculus summation over components of vectors, covectors and, more generally, tensors, with respect to a given basis frequently appear. Such a summation usually occurs on a pair of equal indices, one lower index and one upper index, and one sums over all values of indices from 1 to n. The number of summation symbols $\sum_{i=1}^{n}$ is equal to the number of pairs of repeated indices. That is why even simple equations become cumbersome and uncomfortable to work with. This lead Einstein to drop all summation signs and to adopt the following **summation convention**:

  1. In any expression there are two types of indices: **free indices** and **repeated indices**.
  2. Free indices appear only once in an expression; they are assumed to take all possible values from 1 to n. For example, in the expression

$$g_{ij} v^j$$

     the index i is a free index.

3. The position of all free indices in all terms in an equation must be the same. For example,
$$g_{ij}v^j + \alpha_i = \sigma_i$$
is a correct equation, while the equation
$$g_{ij}v^j + \alpha^i = \sigma_i$$
is a wrong equation.
4. Repeated indices appear twice in an expression. It is assumed that there is a summation over each repeated pair of indices from 1 to n. The summation over a pair of repeated indices in an expression is called the **contraction**. For example, in the expression
$$g_{ij}v^j$$
the index j is a repeated index. It actually means
$$\sum_{j=1}^{n} g_{ij}v^j\,.$$
This is the result of the contraction of the indices k and l in the expression
$$g_{ik}v^l\,.$$
5. Repeated indices are **dummy indices**: they can be replaced by any other letter (not already used in the expression) without changing the meaning of the expression. For example
$$g_{ij}v^j = g_{ik}v^k$$
just means
$$\sum_{j=1}^{n} g_{ij}v^j = g_{i1}v^1 + \cdots + g_{in}v^n,$$
no matter how the repeated index is called.
6. Indices cannot be repeated on the same level. That is, in a pair of repeated indices one index is in upper position and another is in the lower position. For example,
$$v_i v_i$$
is a wrong expression.
7. There cannot be indices occuring three or more times in any expression. For example, the expression
$$g_{ii}v^i$$
does not make sense.

- From now on we will use the Einstein summation convention. We will say that an equation is written in tensor notation.

Examples

- First, we list the equations we already obtained above

$$v_i = g_{ij}v^j\,, \qquad v^j = g^{ji}v_i\,,$$

$$(\mathbf{u},\mathbf{v}) = g_{ij}u^i v^j = u_i v^i = u^i v_i = g^{ij}u_i v_j\,,$$

$$(\boldsymbol{\alpha},\boldsymbol{\beta}) = g^{ij}\alpha_i\beta_j = \alpha^i\beta_i = \alpha_i\beta^i = g_{ij}\alpha^i\beta^j\,.$$

$$g_{ij}g^{jk} = \delta^k_i\,.$$

- A contraction of indices one of which belongs to the Kronecker symbol just renames the index. For example:

$$\delta^i_j v^j = v^i, \qquad \delta^i_j\delta^j_k = \delta^i_k\,,$$

  etc.

- The contraction of the Kronecker symbol gives

$$\delta^i_i = \sum_{i=1}^{n} 1 = n\,.$$

2.3 General Definition of a Tensor

- It should be realized that a vector is an invariant geometric object that does not depend on the basis; it exists by itself independently of the basis. The basis is just a convenient tool to represent vectors by its components. The componenets of a vector do depend on the basis. It is this transformation law of the components of a vector (and, more generally, a tensor as we will see later) that makes an n-tuple of real numbers $(v^1, \dots, v^n)$ a vector. Not every collection of n real numbers is a vector. To represent a vector, a geometric object that does not depend on the basis, these numbers should transform according to a very special rule under a change of basis.

- Let $\{\mathbf{e}_i\} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ and $\{\mathbf{e}'_j\} = \{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$ be two different bases in E. Obviously, the vectors from one basis can be decomposed as a linear combination of vectors from another basis, that is

$$\mathbf{e}_i = \Lambda^j{}_i \mathbf{e}'_j\,,$$

  where $\Lambda^j{}_i$, $i, j = 1, \dots, n$, is a set of n^2 real numbers, forming the **transformation matrix** $\Lambda = (\Lambda^i{}_j)$. Of course, we also have the inverse transformation

$$\mathbf{e}'_j = \tilde\Lambda^k{}_j \mathbf{e}_k$$

  where $\tilde\Lambda^k{}_j$, $k, j = 1, \dots, n$, is another set of n^2 real numbers.

- The dual bases $\{\omega^i\}$ and $\{\omega'^j\}$ are related by

$$\omega'^i = \Lambda^i{}_j \omega^j\,, \qquad \omega^i = \tilde\Lambda^i{}_j \omega'^j\,.$$

- By using the second equation in the first and vice versa we obtain

$$\mathbf{e}_i = \tilde\Lambda^k{}_j \Lambda^j{}_i \mathbf{e}_k\,, \qquad \mathbf{e}'_j = \Lambda^i{}_k \tilde\Lambda^k{}_j \mathbf{e}'_i\,,$$

  which means that

$$\tilde\Lambda^k{}_j \Lambda^j{}_i = \delta^k_i\,, \qquad \Lambda^i{}_k \tilde\Lambda^k{}_j = \delta^i_j\,.$$

  Thus, the matrix $\tilde\Lambda = (\tilde\Lambda^i{}_j)$ is the **inverse transformation matrix**.

- In matrix notation this becomes

$$\tilde\Lambda\Lambda = I, \qquad \Lambda\tilde\Lambda = I\,,$$

  which means that the matrix Λ is invertible and

$$\tilde\Lambda = \Lambda^{-1}\,.$$

- The components of a vector $\mathbf{v}$ with respect to the basis $\{\mathbf{e}'_i\}$ are

$$v'^i = \langle \omega'^i, \mathbf{v} \rangle = \Lambda^i{}_j \langle \omega^j, \mathbf{v} \rangle\,.$$

This immediately gives

$$v'^i = \Lambda^i{}_j v^j\,.$$

This is the **transformation law of contravariant components**. It is easy to recognize this as the action of the transformation matrix on the column-vector of the vector components from the left.

- We can compute the transformation law of the components of a covector σ as follows

  $$\sigma'_i = \langle \boldsymbol{\sigma}, \mathbf{e}'_i \rangle = \tilde\Lambda^j{}_i \langle \boldsymbol{\sigma}, \mathbf{e}_j \rangle\,,$$

  which gives

  $$\sigma'_i = \tilde\Lambda^j{}_i \sigma_j\,.$$

  This is the **transformation law of covariant components**. It is the action of the inverse transformation matrix on the row-vector from the right. That is, the components of covectors are transformed with the transpose of the inverse transformation matrix!

- Now let us compute the **transformation law of the covariant components of the metric tensor** g_{ij}. By the definition we have

  $$g'_{ij} = (\mathbf{e}'_i, \mathbf{e}'_j) = \tilde\Lambda^k{}_i \tilde\Lambda^l{}_j (\mathbf{e}_k, \mathbf{e}_l)\,.$$

  This leads to

  $$g'_{ij} = \tilde\Lambda^k{}_i \tilde\Lambda^l{}_j g_{kl}\,.$$

- Similarly, the **contravariant components of the metric tensor** g^{ij} transform according to

  $$g'^{ij} = \Lambda^i{}_k \Lambda^j{}_l g^{kl}\,.$$

- The transformation law of the metric components in matrix notation reads

  $$G' = (\Lambda^{-1})^T G \Lambda^{-1}$$

  and

  $$G'^{-1} = \Lambda G^{-1} \Lambda^T\,.$$

- We denote the determinant of the covariant metric components $G = (g_{ij})$ by

  $$|g| = \det G = \det(g_{ij})\,.$$

- Taking the determinant of this equation we obtain the **transformation law of the determinant of the metric**

  $$|g'| = (\det \Lambda)^{-2} |g|\,.$$

- More generally, a set of real numbers $T^{i_1 \ldots i_p}{}_{j_1 \ldots j_q}$ is said to represent **components of a tensor of type** (p, q) (p times contravariant and q times covariant) if they transform under a change of the basis according to

$$T'^{i_1 \ldots i_p}_{j_1 \ldots j_q} = \Lambda^{i_1}{}_{l_1} \cdots \Lambda^{i_p}{}_{l_p} \tilde{\Lambda}^{m_1}{}_{j_1} \cdots \tilde{\Lambda}^{m_q}{}_{j_q} T^{l_1 \ldots l_p}_{m_1 \ldots m_q} .$$

  This is the general **transformation law of the components of a tensor of type** (p, q).

- The **rank** of the tensor of type (p, q) is the number $(p + q)$.

- A **tensor product** of a tensor A of type (p, q) and a tensor B of type (r, s) is a tensor $A \otimes B$ of type $(p + r, q + s)$ with components

$$(A \otimes B)^{i_1 \ldots i_q l_1 \ldots l_r}_{j_1 \ldots j_q k_1 \ldots k_s} = A^{i_1 \ldots i_p}_{j_1 \ldots j_q} B^{l_1 \ldots l_r}_{k_1 \ldots k_s} .$$

- The **symmetrization** of a tensor of type $(0, k)$ with components $A_{i_1 \ldots i_k}$ is another tensor of the same type with components

$$A_{(i_1 \ldots i_k)} = \frac{1}{k!} \sum_{\varphi \in S_k} A_{i_{\varphi(1)} \ldots i_{\varphi(k)}} ,$$

  where summation goes over all permutations of k indices. The symmetrization is denoted by parenthesis.

- The **antisymmetrization** of a tensor of type $(0, k)$ with components $A_{i_1 \ldots i_k}$ is another tensor of the same type with components

$$A_{[i_1 \ldots i_k]} = \frac{1}{k!} \sum_{\varphi \in S_k} \operatorname{sign}(\varphi) A_{i_{\varphi(1)} \ldots i_{\varphi(k)}} ,$$

  where summation goes over all permutations of k indices. The antisymmetrization is denoted by square brackets.

- A tensor $A_{i_1 \ldots i_k}$ is **symmetric** if

$$A_{(i_1 \ldots i_k)} = A_{i_1 \ldots i_k}$$

  and **anti-symmetric** if

$$A_{[i_1 \ldots i_k]} = A_{i_1 \ldots i_k} .$$

- Anti-symmetric tensors of type $(0, p)$ are called p**-forms**.

- Anti-symmetric tensors of type $(p, 0)$ are called p**-vectors**.

- A tensor is **isotropic** if it is a tensor product of g_{ij}, g^{ij} and δ^i_j.

- Every isotropic tensor has an even rank.

- For example, the most general isotropic tensor of rank two is

$$A^i{}_j = a\delta^i_j\,,$$

where a is a scalar, and the most general isotropic tensor of rank four is

$$A^{ij}{}_{kl} = ag^{ij}g_{kl} + b\delta^i_k\delta^j_l + c\delta^i_l\delta^j_k\,,$$

where a, b, c are scalars.

2.3.1 Orientation, Pseudotensors and Volume

- Since the transformation matrix Λ is invertible, then the determinant $\det\Lambda$ is either positive or negative. If $\det\Lambda > 0$ then we say that the bases $\{\mathbf{e}_i\}$ and $\{\mathbf{e}'_i\}$ have the **same orientation**, and if $\det\Lambda < 0$ then we say that the bases $\{\mathbf{e}_i\}$ and $\{\mathbf{e}'_i\}$ have the **opposite orientation**.

- This defines an equivalence relation on the set of all bases on E called the **orientation of the vector space** E. This equivalence relation divides the set of all bases in two equivalence classes, called the **positively oriented** and **negatively oriented** bases.

- A vector space together with a choice of what equivalence class is positively oriented is called an **oriented vector space**.

- A set of real numbers $A^{i_1\dots i_p}_{j_1\dots j_q}$ is said to represent components of a **pseudo-tensor** of type (p, q) if they transform under a change of the basis according to

$$A'^{i_1\dots i_p}_{j_1\dots j_q} = \operatorname{sign}(\det\Lambda)\Lambda^{i_1}{}_{l_1}\cdots\Lambda^{i_p}{}_{l_p}\tilde{\Lambda}^{m_1}{}_{j_1}\cdots\tilde{\Lambda}^{m_q}{}_{j_q}A^{l_1\dots l_p}_{m_1\dots m_q}\,,$$

where $\operatorname{sign}(x) = +1$ if $x > 0$ and $\operatorname{sign}(x) = -1$ if $x < 0$.

- The **Levi-Civita symbol** (also called **alternating symbol**) is defined by

$$\varepsilon_{i_1\dots i_n} = \varepsilon^{i_1\dots i_n} = \begin{cases} +1, & \text{if } (i_1,\dots,i_n) \text{ is an even permutation of } (1,\dots,n),\\ -1, & \text{if } (i_1,\dots,i_n) \text{ is an odd permutation of } (1,\dots,n),\\ 0, & \text{if two or more indices are the same}\,. \end{cases}$$

- The Levi-Civita symbols $\varepsilon_{i_1\dots i_n}$ and $\varepsilon^{i_1\dots i_n}$ do not represent tensors! They have the same values in all bases.

- **Theorem 2.3.1** *The determinant of a matrix $A = (A^i{}_j)$ can be written as*

$$\begin{aligned}\det A &= \varepsilon^{i_1\dots i_n}A^1{}_{i_1}\dots A^n{}_{i_n}\\ &= \varepsilon_{j_1\dots j_n}A^{j_1}{}_1\dots A^{j_n}{}_n\\ &= \frac{1}{n!}\varepsilon^{i_1\dots i_n}\varepsilon_{j_1\dots j_n}A^{j_1}{}_{i_1}\dots A^{j_n}{}_{i_n}\,.\end{aligned}$$

Here, as usual, a summation over all repeated indices is assumed from 1 to n.

- **Theorem 2.3.2** *There holds the identity*

$$\begin{aligned}\varepsilon^{i_1\dots i_n}\varepsilon_{j_1\dots j_n} &= \sum_{\varphi\in S_n}\operatorname{sign}(\varphi)\,\delta^{i_1}_{j_{\varphi(1)}}\cdots\delta^{i_n}_{j_{\varphi(n)}}\\ &= n!\delta^{i_1}_{[j_1}\cdots\delta^{i_n}_{j_n]}\,.\end{aligned}$$

  The contraction of this identity over k indices gives

$$\varepsilon^{i_1\dots i_{n-k}m_1\dots m_k}\varepsilon_{j_1\dots j_{n-k}m_1\dots m_k} = k!(n-k)!\delta^{i_1}_{[j_1}\cdots\delta^{i_{n-k}}_{j_{n-k}]}\,.$$

  In particular,

$$\varepsilon^{m_1\dots m_n}\varepsilon_{m_1\dots m_n} = n!\,.$$

- **Theorem 2.3.3** *The sets of real numbers $E_{i_1\dots i_n}$ and $E^{i_1\dots i_n}$ defined by*

$$E_{i_1\dots i_n} = \sqrt{|g|}\,\varepsilon_{i_1\dots i_n}$$

$$E^{i_1\dots i_n} = \frac{1}{\sqrt{|g|}}\,\varepsilon^{i_1\dots i_n}\,,$$

  where $|g| = \det(g_{ij})$, define (pseudo)-tensors of type $(0, n)$ and $(n, 0)$ respectively.

- Let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an ordered n-tuple of vectors. The **volume of the parallelepiped spanned by the vectors** $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a real number defined by

$$|\mathrm{vol}\,(\mathbf{v}_1, \dots, \mathbf{v}_n)| = \sqrt{\det((\mathbf{v}_i, \mathbf{v}_j))}\,.$$

- **Theorem 2.3.4** *Let $\{\mathbf{e}_i\}$ be a basis in E, $\{\omega^i\}$ be the dual basis, and $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a set of n vectors. Let $V = (v^i{}_j)$ be the matrix of contravariant components of the vectors $\{\mathbf{v}_j\}$*

$$v^i{}_j = \langle\omega^i, \mathbf{v}_j\rangle,$$

  and $W = (v_{ij})$ be the matrix of covariant components of the vectors $\{\mathbf{v}_j\}$

$$v_{ij} = (\mathbf{e}_i, \mathbf{v}_j) = g_{ik}v^k{}_j\,.$$

  Then the volume of the parallelepiped spanned by the vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is

$$|\mathrm{vol}\,(\mathbf{v}_1, \dots, \mathbf{v}_n)| = \sqrt{|g|}\,|\det V| = \frac{|\det W|}{\sqrt{|g|}}\,.$$

- If the vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ are linearly dependent, then

$$\mathrm{vol}\,(\mathbf{v}_1, \dots, \mathbf{v}_n) = 0\,.$$

- If the vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ are linearly independent, then the volume is a positive real number that does not depend on the orientation of the vectors.

- The **signed volume of the parallelepiped spanned by an ordered n-tuple of vectors** $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is

$$\begin{aligned} \mathrm{vol}\,(\mathbf{v}_1, \ldots, \mathbf{v}_n) &= \sqrt{|g|}\ \det V \\ &= \mathrm{sign}(\mathbf{v}_1, \ldots, \mathbf{v}_n)\ |\mathrm{vol}\,(\mathbf{v}_1, \ldots, \mathbf{v}_n)|\,,. \end{aligned}$$

The sign of the signed volume depends on the orientation of the vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$:

$$\mathrm{sign}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \mathrm{sign}(\det V) = \begin{cases} +1, & \text{if } \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \text{ is positively oriented} \\ -1, & \text{if } \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \text{ is negatively oriented} \end{cases}$$

- **Theorem 2.3.5** *The signed volume is equal to*

$$\mathrm{vol}\,(\mathbf{v}_1, \ldots, \mathbf{v}_n) = E_{i_1 \ldots i_n}\, v^{i_1}{}_1 \cdots v^{i_n}{}_n = E^{i_1 \ldots i_n}\, v_{i_1 1} \cdots v_{i_n n}\,,$$

where $v^i{}_j = \langle \omega^i, \mathbf{v}_j \rangle$ *and* $v_{ij} = (\mathbf{e}_i, \mathbf{v}_j)$.

That is why the pseudo-tensor $E_{i_1 \ldots i_n}$ is also called the **volume form**.

Exterior Product and Duality

- The volume form allows one to define the duality of k-forms and $(n-k)$-vectors as follows. For each k-form $A_{i_1 \ldots i_k}$ one assigns the **dual** $(n-k)$-vector by

$$*A^{j_1 \ldots j_{n-k}} = \frac{1}{k!} E^{j_1 \ldots j_{n-k} i_1 \ldots i_k} A_{i_1 \ldots i_k}\,.$$

Similarly, for each k-vector $A^{i_1 \ldots i_k}$ one assigns the **dual** $(n-k)$-form

$$*A_{j_1 \ldots j_{n-k}} = \frac{1}{k!} E_{j_1 \ldots j_{n-k} i_1 \ldots i_k} A^{i_1 \ldots i_k}\,.$$

- **Theorem 2.3.6** *For each k-form α there holds*

$$* * \alpha = (-1)^{k(n-k)} \alpha\,.$$

That is,

$$** = (-1)^{k(n-k)}\,.$$

- The **exterior product** of a k-form A and a m-form B is a $(k+m)$-form $A \wedge B$ defined by

$$(A \wedge B)_{i_1 \ldots i_k j_1 \ldots j_m} = \frac{(k+m)!}{k!m!} A_{[i_1 \ldots i_k} B_{j_1 \ldots j_m]}$$

Similarly, one can define the exterior product of p-vectors.

- **Theorem 2.3.7** *The exterior product is associative, that is,*

$$(A \wedge B) \wedge C = A \wedge (B \wedge C)\,.$$

- A collection $\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}\}$ of $(n-1)$ vectors defines a covector $\boldsymbol{\alpha}$ by

$$\boldsymbol{\alpha} = *(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{n-1})$$

or, in components,

$$\alpha_j = E_{ji_1 \dots i_{n-1}} v^{i_1}{}_1 \cdots v^{i_{n-1}}{}_{n-1} .$$

- **Theorem 2.3.8** *Let* $\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}\}$ *be a collection of* $(n-1)$ *vectors and* $S = \operatorname{span}\{\mathbf{v}_1, \dots \mathbf{v}_{n-1}\}$ *be the hyperplane spanned by these vectors. Let* $\mathbf{e}$ *be a unit vector orthogonal to* S *oriented in such a way that* $\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{e}\}$ *is oriented positively. Then the vector* $\mathbf{u} = g^{-1}(\alpha)$ *corresponding to the* 1*-form* $\boldsymbol{\alpha} = *(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{n-1})$ *is parallel to* $\mathbf{e}$ *(with the same orientation)*

$$\mathbf{u} = \mathbf{e}\, ||\mathbf{u}||$$

and has the norm

$$||\mathbf{u}|| = \operatorname{vol}(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{e}) .$$

- In three dimensions, i.e. when $n = 3$, this defines a binary operation $\times$, called the **vector product**, that is

$$\mathbf{u} = \mathbf{v} \times \mathbf{w} = *(\mathbf{v} \wedge \mathbf{w}) ,$$

or

$$u_j = E_{jik} v^i w^k = \sqrt{|g|}\, \varepsilon_{jik} v^i w^k .$$

2.4 Operators and Tensors

- Let $\mathbf{A}$ be an operator on E. Let $\{\mathbf{e}_i\} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis in a Euclidean space and $\{\omega^i\} = \{\omega^1, \dots, \omega^n\}$ be the dual basis in E^*. The real square matrix $A = (A^i{}_j)$, $i, j = 1, \dots, n$, defined by

$$\mathbf{A}\mathbf{e}_j = A^i{}_j \mathbf{e}_i \,,$$

  is called the **matrix of the operator A**.

- Therefore, there is a one-to-one correspondence between the operators on E and the real square $n \times n$ matrices $A = (A^i{}_j)$.

- It can be computed by

$$A^i{}_j = \langle \omega^i, \mathbf{A}\mathbf{e}_j \rangle = g^{ik}(\mathbf{e}_k, \mathbf{A}\mathbf{e}_j) \,.$$

- **Remark.** Notice that the upper index, which is the first one, indicates the row and the lower index is the second one indicating the column of the matrix. The convenience of this notation comes from the fact that all upper indices (also called **contravariant indices**) indicate the components of vectors and "belong" to the vector space E while all lower indices (called **covariant indices**) indicate components of covectors and "belong" to the dual space E^*.

- The matrix of the identity operator $\mathbf{I}$ is

$$I^i{}_j = \delta^i_j \,.$$

- For any $\mathbf{v} \in E$

$$\mathbf{v} = v^j \mathbf{e}_j, \qquad v^j = \langle \omega^j, \mathbf{v} \rangle$$

  we have

$$\mathbf{A}\mathbf{v} = A^i{}_j v^j \mathbf{e}_i \,.$$

  That is, the components u^i of the vector $\mathbf{u} = \mathbf{A}\mathbf{v}$ are given by

$$u^i = A^i{}_j v^j \,.$$

Transformation Law of Matrix of an Operator

- Under a change of the basis $\mathbf{e}_i = \Lambda^j{}_i \mathbf{e}'_j$, the matrix $A^i{}_j$ of an operator $\mathbf{A}$ transforms according to

$$A'^i{}_j = \Lambda^i{}_k A^k{}_m \tilde{\Lambda}^m{}_j \,,$$

  which in matrix notation reads

$$A' = \Lambda A \Lambda^{-1} \,.$$

- Therefore, the matrix $A = (A^i{}_j)$ of an operator $\mathbf{A}$ represents the components of a tensor of type $(1, 1)$. Conversely, such tensors naturally define linear operators on E. Thus, linear operators on E and tensors of type $(1, 1)$ can be identified.

- The determinant and the trace of the matrix of an operator are invariant under the change of the basis, that is

$$\det A' = \det A\,, \qquad \operatorname{tr} A' = \operatorname{tr} A\,.$$

- Therefore, one can define the **determinant of the operator $\mathbf{A}$** and the **trace of the operator $\mathbf{A}$** by the determinant and the trace of its matrix, that is,

$$\det \mathbf{A} = \det A\,, \qquad \operatorname{tr} \mathbf{A} = \operatorname{tr} A\,.$$

- For self-adjoint operators these definitions are consistent with the definition in terms of the eigenvalues given before.

- The matrix of the sum $\mathbf{A} + \mathbf{B}$ of two operators $\mathbf{A}$ and $\mathbf{B}$ is the sum of matrices A and B of the operatos $\mathbf{A}$ and $\mathbf{B}$.

- The matrix of a scalar multiple $c\mathbf{A}$ is equal to cA, where A is the matrix of the operator $\mathbf{A}$ and $c \in \mathbb{R}$.

Matrix of the Product of Operators

- The matrix of the product $\mathbf{C} = \mathbf{AB}$ of two operators reads

$$C^i{}_j = \langle \omega^i, \mathbf{ABe}_j \rangle = \langle \omega^i, \mathbf{Ae}_k \rangle \langle \omega^k, \mathbf{Be}_j \rangle = A^i{}_k B^k{}_j\,,$$

  which is exactly the product of matrices A and B.

- Thus, the matrix of the product $\mathbf{AB}$ of the operators $\mathbf{A}$ and $\mathbf{B}$ is equal to the product AB of matrices of these operators in the same order.

- The matrix of the inverse $\mathbf{A}^{-1}$ of an invertible operator $\mathbf{A}$ is equal to the inverse A^{-1} of the matrix A of the operator $\mathbf{A}$.

- **Theorem 2.4.1** *The algebra $\mathcal{L}(E)$ of linear operators on E is isomorphic to the algebra* $\mathrm{Mat}(n, \mathbb{R})$ *of real square $n \times n$ matrices.*

Matrix of the Adjoint Operator

- For the adjoint operator $\mathbf{A}^*$ we have

$$(\mathbf{e}_i, \mathbf{A}^* \mathbf{e}_j) = (\mathbf{Ae}_i, \mathbf{e}_j) = (\mathbf{e}_j, \mathbf{Ae}_i)\,.$$

  Therefore, the **matrix of the adjoint operator** is

$$A^{*k}{}_j = g^{ki}(\mathbf{e}_i, \mathbf{A}^* \mathbf{e}_j) = g^{ki} A^l{}_i g_{lj}\,.$$

  In matrix notation this reads

$$A^* = G^{-1} A^T G\,.$$

- Thus, the matrix of a self-adjoint operator **A** satisfies the equation

$$A^k{}_j = g^{ki}A^l{}_i g_{lj} \qquad \text{or} \qquad g_{ik}A^k{}_j = A^l{}_i g_{lj} \,,$$

  which in matrix notation reads

$$A = G^{-1}A^T G \,, \qquad \text{or} \qquad GA = A^T G \,.$$

- The matrix of a unitary operator **A** satisfies the equation

$$g^{ki}A^l{}_i g_{lj} A^j{}_m = \delta^k_m \qquad \text{or} \qquad A^l{}_i g_{lj} A^j{}_m = g_{im} \,,$$

  which in matrix notation has the form

$$G^{-1}A^T G A = I \,, \qquad \text{or} \qquad A^T G A = G \,.$$

2.5 Vector Algebra in $\mathbb{R}^3$

- We denote the standard orthonormal basis in $\mathbb{R}^3$ by

$$\mathbf{e}_1 = \mathbf{i}\,, \qquad \mathbf{e}_2 = \mathbf{j}\,, \qquad \mathbf{e}_3 = \mathbf{k}\,,$$

  so that

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}\,.$$

- Each vector $\mathbf{v}$ is decomposed as

$$\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}\,.$$

  The components are computed by

$$v_1 = \mathbf{v}\cdot\mathbf{i}\,, \qquad v_2 = \mathbf{v}\cdot\mathbf{j}\,, \qquad v_3 = \mathbf{v}\cdot\mathbf{k}\,.$$

- The norm of the vector

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}\,.$$

- Scalar product is defined by

$$\mathbf{v}\cdot\mathbf{u} = v_1u_1 + v_2u_2 + v_3u_3\,.$$

- The angle between vectors

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||}\,.$$

- The orthogonal decomposition of a vector $\mathbf{v}$ with respect to a given unit vector $\mathbf{u}$ is

$$\mathbf{v} = \mathbf{v}_{||} + \mathbf{v}_{\perp}\,,$$

  where

$$v_{||} = \mathbf{u}(\mathbf{u}\cdot\mathbf{v})\,, \qquad v_{\perp} = \mathbf{v} - \mathbf{u}(\mathbf{u}\cdot\mathbf{v})\,.$$

- We denote the Cartesian coordinates in $\mathbb{R}^3$ by

$$x^1 = x\,, \qquad x^2 = y\,, \qquad x^3 = z\,.$$

  The **radius vector** (the **position vector**) is

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}\,.$$

- The **parametric equation of a line** parallel to a vector $\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{u}\,,$$

  where $\mathbf{r}_0 = x_0\,\mathbf{i} + y_0\,\mathbf{j} + z_0\,\mathbf{k}$ is a fixed vector and t is a real parameter. In components,

$$x = x_0 + at\,, y = y_0 + bt\,, z = z_0 + ct\,.$$

  The **non-parametric equation of a line** (if a, b, c are non-zero) is

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}\,.$$

- The **parametric equation of a plane** spanned by two non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$ is
$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{u} + s\mathbf{v}\,,$$
where t and s are real parameters.

- A vector $\mathbf{n}$ that is perpendicular to both vectors $\mathbf{u}$ and $\mathbf{v}$ is **normal** to the plane.

- The **non-parametric equation** of a plane with the normal $\mathbf{n} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ is
$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$$
or
$$a(x - x_0) + b(y - y_0) + (z - z_0) = 0\,,$$
which can also be written as
$$ax + by + cz = d\,,$$
where
$$d = ax_0 + by_0 + cz_0\,.$$

- The **positive (right-handed) orientation of a plane** is defined by the **right hand (or counterclockwise) rule**. That is, if $\mathbf{u}_1$ and $\mathbf{u}_2$ span a plane then we orient the plane by saying which vector is the first and which is the second. The orientation is positive if the rotation from $\mathbf{u}_1$ to $\mathbf{u}_2$ is counterclockwise and negative if it is clockwise. A plane has two sides. The positive side of the plane is the side with the positive orientation, the other side has the negative (left-handed) orientation.

- The **vector product** of two vectors is defined by
$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\,,$$
or, in components,
$$w^i = \varepsilon^{ijk} u_j v_k = \frac{1}{2}\varepsilon^{ijk}(u_j v_k - u_k v_j)\,.$$

- The vector products of the basis vectors are
$$\mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk}\mathbf{e}_k\,.$$

- If $\mathbf{u}$ and $\mathbf{v}$ are two nonzero nonparallel vectors, then the vector $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ is orthogonal to both vectors, $\mathbf{u}$ and $\mathbf{v}$, and, hence, to the plane spanned by these vectors. It defines a normal to this plane.

- The **area of the parallelogram spanned by two vectors $\mathbf{u}$ and $\mathbf{v}$** is
$$\text{area}(\mathbf{u}, \mathbf{v}) = |\mathbf{u} \times \mathbf{v}| = ||\mathbf{u}||\;||\mathbf{v}|| \sin\theta\,.$$

- The **signed volume of the parallelepiped spanned by three vectors u**, **v** and **w** is
$$\mathrm{vol}(\mathbf{u},\mathbf{v},\mathbf{w}) = \mathbf{u}\cdot(\mathbf{v},\mathbf{w}) = \det\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \varepsilon^{ijk}u_i v_j w_k\,.$$
The signed volume is also called the **scalar triple product** and denoted by
$$[\mathbf{u},\mathbf{v},\mathbf{w}] = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})\,.$$

- The signed volume is zero if and only if the vectors are linearly dependent, that is, coplanar.

- For linearly independent vectors its sign depends on the orientation of the triple of vectors $\{\mathbf{u},\mathbf{v}\mathbf{w}\}$
$$\mathrm{vol}(\mathbf{u},\mathbf{v},\mathbf{w}) = \mathrm{sign}\,(\mathbf{u},\mathbf{v},\mathbf{w})|\mathrm{vol}(\mathbf{u},\mathbf{v},\mathbf{w})|\,,$$
where
$$\mathrm{sign}\,(\mathbf{u},\mathbf{v},\mathbf{w}) = \begin{cases} 1 & \text{if } \{\mathbf{u},\mathbf{v},\mathbf{w}\} \text{ is positively oriented} \\ -1 & \text{if } \{\mathbf{u},\mathbf{v},\mathbf{w}\} \text{ is negatively oriented} \end{cases}$$

- The scalar triple product is linear in each argument, anti-symmetric
$$[\mathbf{u},\mathbf{v},\mathbf{w}] = -[\mathbf{v},\mathbf{u},\mathbf{w}] = -[\mathbf{u},\mathbf{w},\mathbf{v}] = -[\mathbf{w},\mathbf{v},\mathbf{u}]$$
cyclic
$$[\mathbf{u},\mathbf{v},\mathbf{w}] = [\mathbf{v},\mathbf{w},\mathbf{u}] = [\mathbf{w},\mathbf{u},\mathbf{v}]\,.$$
It is normalized so that
$$[\,\mathbf{i}\,,\,\mathbf{j}\,,\,\mathbf{k}\,] = 1\,.$$

- The orthogonal decomposition of a vector **v** with respect to a unit vector **u** can be written in the form
$$\mathbf{v} = \mathbf{u}(\mathbf{u}\cdot\mathbf{v}) - \mathbf{u}\times(\mathbf{u}\times\mathbf{v})\,.$$

- The Levi-Civita symbol in three dimensions
$$\varepsilon_{ijk} = \varepsilon^{ijk} = \begin{cases} +1 & \text{if } (i,j,k) = (1,2,3), (2,3,1), (3,1,2) \\ -1 & \text{if } (i,j,k) = (2,1,3), (3,2,1), (1,3,2) \\ 0 & \text{otherwise} \end{cases}$$
has the following properties:
$$\varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{ikj} = -\varepsilon_{kji}$$
$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$$

$$
\begin{aligned}
\varepsilon_{ijk}\varepsilon^{mnl} &= 6\delta^m_{[i}\delta^n_j\delta^l_{k]} \\
&= \delta^m_i\delta^n_j\delta^l_k + \delta^m_j\delta^n_k\delta^l_i + \delta^m_k\delta^n_i\delta^l_j - \delta^m_i\delta^n_k\delta^l_j - \delta^m_j\delta^n_i\delta^l_k - \delta^m_k\delta^n_j\delta^l_i \\
\varepsilon_{ijk}\varepsilon^{mnk} &= 2\delta^m_{[i}\delta^n_{j]} = \delta^m_i\delta^n_j - \delta^m_j\delta^n_i \\
\varepsilon_{ijk}\varepsilon^{mjk} &= 2\delta^m_i \\
\varepsilon_{ijk}\varepsilon^{ijk} &= 6
\end{aligned}
$$

- This leads to many vector identities that express double vector product in terms of scalar product. For example,

$$
\begin{aligned}
&\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w} \\
&\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) + \mathbf{v}\times(\mathbf{w}\times\mathbf{u}) + \mathbf{w}\times(\mathbf{u}\times\mathbf{v}) = \mathbf{0} \\
&(\mathbf{u}\times\mathbf{v})\times(\mathbf{w}\times\mathbf{n}) = \mathbf{v}[\mathbf{u},\mathbf{w},\mathbf{n}] - \mathbf{u}[\mathbf{v},\mathbf{w},\mathbf{n}] \\
&(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{w}\times\mathbf{n}) = (\mathbf{u}\cdot\mathbf{w})(\mathbf{v}\cdot\mathbf{n}) - (\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{w})
\end{aligned}
$$

Chapter 3

Geometry

3.1 Geometry of Euclidean Space

- The set $\mathbb{R}^n$ can be viewed geometrically as a set of points, that we will denote by P, Q, etc. With each point P we associate an ordered n-tuple of real numbers $(x^i_P) = (x^1_P, \dots, x^n_P)$, called the **coordinates** of the point P. The assignment of n-tuples of real numbers to the points in space should be bijective. That is, different points are assigned different n-tuples, and for every n-tuple there is a point in space with such coordinates. Such a map is called a **coordinate system**.

- A space $\mathbb{R}^n$ with a coordinate system is a **Euclidean space** if the **distance** between any two points P and Q is determined by

$$d(P, Q) = \sqrt{\sum_{i=1}^{n} (x^i_P - x^i_Q)^2}\,.$$

  Such coordinate system is called **Cartesian**.

- The point O with the zero coordinates $(0, \dots, 0)$ is called the **origin** of the Cartesian coordinate system.

- In $\mathbb{R}^n$ it is convenient to associate vectors with points in space. With each point P with Cartesian coordinates $(x^1, \dots, x^n)$ in $\mathbb{R}^n$ we associate the column-vector $\mathbf{r} = (x^i)$ with the components equal to the Cartesian coordinates of the point P. We say that this vector **points from the origin** O **to the point** P; it has its tail at the point O and its tip at the point P. This vector is often called the **radius vector**, or the **position vector**, of the point P and denoted by $\mathbf{r}$.

- Similarly, with every two points P and Q with the coordinates (x^i_P) and (x^i_Q) we associate the vector

$$\mathbf{u}_{PQ} = \mathbf{r}_Q - \mathbf{r}_P = (x^i_Q - x^i_P)$$

  that **points from the point** P **to the point** Q.

- Obviously, the Euclidean distance is given by

$$d(P, Q) = ||\,\mathbf{r}_P - \mathbf{r}_Q||\,.$$

- The standard (orthonormal) basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$ are the unit vectors that connect the origin O with the points $\{(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\}$ that have only one nonzero coordinate which is equal to 1.

- The one-dimensional subspaces L_i spanned by a single basis vector $\mathbf{e}_i$,

$$L_i = \operatorname{span}\{\mathbf{e}_i\} = \{P \mid \mathbf{r}_P = t\mathbf{e}_i, t \in \mathbb{R}\},$$

  are the lines called the **coordinate axes**. There are n coordinate axes; they are mutually orthogonal and intersect at only one point, the origin O.

- The two-dimensional subspaces P_{ij} spanned by a couple of basis vectors $\mathbf{e}_i$ and $\mathbf{e}_j$,

$$P_{ij} = \operatorname{span}\{\mathbf{e}_i, \mathbf{e}_j\} = \{P \mid \mathbf{r}_P = t\mathbf{e}_i + s\mathbf{e}_j, t, s \in \mathbb{R}\},$$

  are the planes called the **coordinate planes**. There are $n(n-1)/2$ coordinate planes; the coordinate planes are mutually orthogonal and intersect along the coordinate axes.

- Let a and b be real numbers such that $a < b$. The set $[a, b]$ is a closed interval in $\mathbb{R}$. A **parametrized curve** C in $\mathbb{R}^n$ is a map $C : [a, b] \to \mathbb{R}^n$ which assigns a point in $\mathbb{R}^n$

$$C : \ \mathbf{r}(t) = (x^i(t))$$

  to each real number $t \in [a, b]$.

- The positive **orientation** of the curve C is determined by the standard orientation of $\mathbb{R}$, that is, by the direction of increasing values of the parameter t.

- The point $\mathbf{r}(a)$ is the **initial point** and the point $\mathbf{r}(b)$ is the **endpoint** of the curve.

- The curve $(-C)$ is the parametrized curve with the opposite orientation. If the curve C is parametrized by $\mathbf{r}(t)$, $a \le t \le b$, then the curve $(-C)$ is parametrized by

$$(-C) : \ \mathbf{r}(-t + a + b)\,.$$

- The **boundary** ∂C **of the curve** C consists of two points C_0 and C_1 corresponding to $\mathbf{r}(a)$ and $\mathbf{r}(b)$, that is,

$$\partial C = C_1 - C_0\,.$$

- A curve C is **continuous** if all the functions $x^i(t)$ are continuous for any t on $[a, b]$.

- Let a_1, a_2 and b_1, b_2 be real numbers such that
$$a_1 < b_1, \qquad \text{and} \qquad a_2 < b_2 .$$
The set $D = [a_1, b_1] \times [a_2, b_2]$ is a **closed rectangle** in the plane $\mathbb{R}^2$.

- A **parametrized surface** S in $\mathbb{R}^n$ is a map $S : D \to \mathbb{R}^n$ which assigns a point
$$S : \ \mathbf{r}(u) = (x^i(u))$$
in $\mathbb{R}^n$ to each point $u = (u^1, u^2)$ in the rectangle D.

- The positive **orientation** of the surface S is determined by the positive orientation of the standard basis in $\mathbb{R}^2$. The surface $(-S)$ is the surface with the opposite orientation.

- The **boundary** ∂S **of the surface** S consists of four curves $S_{(1),0}$, $S_{(1),1}$, $S_{(2),0}$, and $S_{(2),1}$ parametrized by $\mathbf{r}(a_1, v)$, $\mathbf{r}(b_1, v)$, $\mathbf{r}(u, a_2)$ and $\mathbf{r}(u, b_2)$ respectively. Taking into account the orientation, the boundary of the surface S is
$$\partial S = S_{(2),0} + S_{(1),1} - S_{(2),1} - S_{(1),0} .$$

- Let $a_1, \dots, a_k$ and $b_1, \dots, b_k$ be real numbers such that
$$a_i < b_i , \qquad i = 1, \dots, k.$$
The set $D = [a_1, b_1] \times \cdots \times [a_k, b_k]$ is called a **closed** k**-rectangle** in $\mathbb{R}^k$. In particular, the set $[0, 1]^k = [0, 1] \times \cdots \times [0, 1]$ is the **standard** k**-cube**.

- Let $D = [a_1, b_1] \times \cdots \times [a_k, b_k]$ be a closed rectangle in $\mathbb{R}^k$. A **parametrized** k**-dimensional surface** S in $\mathbb{R}^n$ is a continuous map $S : D \to \mathbb{R}^n$ which assigns a point
$$S : \ \mathbf{r}(u) = (x^i(u))$$
in $\mathbb{R}^n$ to each point $u = (u^1, \dots, u^k)$ in the rectangle D.

- A $(n - 1)$-dimensional surface is called the **hypersurface**. A non-parametrized hypersurface can be described by a single equation
$$F(x) = F(x^1, \dots, x^n) = 0 ,$$
where $F : \mathbb{R}^n \to \mathbb{R}$ is a real-valued function of n coordinates.

- The **boundary** ∂S of S consists of $(k - 1)$-surfaces, $S_{(i),0}$ and $S_{(i),1}$, $i = 1, \dots, k$, called the **faces** of the k-surface S. Of course, a k-surface S has $2k$ faces. The face $S_{(i),0}$ is parametrized by
$$S_{(i),0} : \ \mathbf{r}(u^1, \dots, u^{i-1}, a_i, u^{i+1}, \dots, u^k) ,$$
where the i-th parameter u^i is fixed at the initial point, i.e. $u^i = a_i$, and the face $S_{(i),0}$ is parametrized by
$$S_{(i),1} : \ \mathbf{r}(u^1, \dots, u^{i-1}, b_i, u^{i+1}, \dots, u^k) ,$$
where the i-th parameter u^i is fixed at the endpoint, i.e. $u^i = b_i$.

- The boundary of the surface S is defined by

$$\partial S = \sum_{i=1}^{k} (-1)^i (S_{(i),0} - S_{(i),1}) .$$

- Let $S_1, \dots, S_m$ be parametrized k-surfaces. A formal sum

$$S = \sum_{i=1}^{m} a_i S_i$$

with integer coefficients $a_1, \dots, a_m$, is called a k-**chain**. Usually (but not always) the integers a_i are equal to 1, (-1) or 0.

- The product of any k-chain S with zero is called the zero chain

$$0S = 0 .$$

- The addition of k-chains and multiplication by integers is defined by

$$\sum_{i=1}^{m} a_i S_i + \sum_{i=1}^{m} b_i S_i = \sum_{i=1}^{m} (a_i + b_i) S_i ,$$

$$b\left(\sum_{i=1}^{m} a_i S_i\right) = \sum_{i=1}^{m} (b a_i) S_i .$$

- The **boundary** of a k-chain S is an $(k-1)$-chain ∂S defined by

$$\partial\left(\sum_{i=1}^{m} a_i S_i\right) = \sum_{i=1}^{m} a_i \partial S_i .$$

- **Theorem 3.1.1** *For any k-chain S there holds*

$$\partial(\partial S) = 0 .$$

3.2 Basic Topology of $\mathbb{R}^n$

- Let P_0 be a point in a Euclidean space $\mathbb{R}^n$ and $\varepsilon > 0$ be a positive real number. The **open ball** $B_\varepsilon(P_0)$ or radius ϵ with the center at P_0 is the set of all points whose distance from the point P_0 is less than ε, that is,

$$B_\varepsilon(P_0) = \{P \mid d(P, P_0) < \varepsilon\}.$$

- A **neighborhood** of a point P_0 is any set that contains an open ball centered at P_0.
- Let S be a subset of a Euclidean space $\mathbb{R}^n$. A point P is an **interior point** of S if there is a neighborhood of P that lies completely in S.
- A point P is an **exterior** point of S if there is a neighborhood of P that lies completely outside of S.
- A point P is a **boundary point** of S if it is neither an interior nor an exterior point. If P is a boundary point of S, then every neighborhood of P contains points in S and points not in S.
- The set of boundary points of S is called the **boundary** of S, denoted by ∂S.
- The set of all interior points of S is called the **interior** of S, denoted by S^o.
- A set S is called **open** if every point of S is an interior point of S, that is, $S = S^o$.
- A set S is **closed** if it contains all its boundary points, that is, $S = S^o \cup \partial S$.
- Henceforth, we will consider only open sets and call them **regions** of space.
- A region S is called **connected** (or arc-wise connected) if for any two points P and Q in S there is an arc joining P and Q that lies within S.
- A connected region, that is a connected open set, is called a **domain**.
- A domain S is said to be **simply-connected** if every closed curve lying within S can be continuously deformed to a point in the domain without any part of the curve passing through regions outside the domain.
- A domain is simply connected if for any closed curve lying in the domain there can be found a surface within the domain that has that curve as its boundary.
- A domain is said to be **star-shaped** if there is a point P in the domain such that for any other point in the domain the entire line segment joining these two points lies in the domain.

3.3 Curvilinear Coordinate Systems

- We say that a function $f(x) = f(x^1, \ldots, x^n)$ is **smooth** if it has continuous partial derivatives of all orders.

- Let P be a point with Cartesian coordinates (x^i). Suppose that we assign another n-tuple of real numbers $(q^i) = (q^1, \ldots, q^n)$ to the point P, so that

$$x^i = f^i(q)\,,$$

where $f^i(q) = f^i(q^1, \ldots, q^n)$ are smooth functions of the variables x^i. We will call this a **change of coordinates**.

- The matrix

$$J = \left(\frac{\partial x^i}{\partial q^j}\right)$$

is called the **Jacobian matrix**. The determinant of this matrix is called the **Jacobian**.

- A point P_0 at which the Jacobian matrix is invertible, that is, the Jacobian is not zero, $\det J \neq 0$, is called a **nonsingular point** of the new coordinate system (q^i).

- **Theorem 3.3.1 (Inverse Function Theorem)** *In a neighborhood of any nonsingular point P_0 the change of coordinates is invertible.*

That is, if $x^i = f^i(q)$ and

$$\det\left(\frac{\partial x^i}{\partial q^j}\right)\Bigg|_{P_0} \neq 0\,,$$

then for all points sufficiently close to P_0 there exist n smooth functions

$$q^i = h^i(x) = h^i(x^1, \ldots, x^n)\,,$$

of the variables (x^i) such that

$$f^i(h^1(x), \ldots, h^n(x)) = x^i\,, \qquad h^i(f^1(q), \ldots, f^n(q)) = q^i\,.$$

The Jacobian matrix of the inverse transformation is the inverse matrix of the Jacobian matrix of direct transformation, i.e.

$$\frac{\partial x^i}{\partial q^j}\frac{\partial q^j}{\partial x^k} = \delta^i_k\,, \qquad \textit{and} \qquad \frac{\partial q^i}{\partial x^j}\frac{\partial x^j}{\partial q^k} = \delta^i_k\,.$$

- The curves C_i along which only one coordinate is varied, while all other are fixed, are called the **coordinate curves**, that is,

$$x^i = x^i(q_0^1, \ldots, q_0^{i-1}, q^i, q_0^{i+1}, \ldots, q_0^n)\}\,.$$

- The vectors
$$\mathbf{e}_i = \frac{\partial\,\mathbf{r}}{\partial q^i}$$
are tangent to the coordinate curves.

- The surfaces S_{ij} along which only two coordinates are varied, while all other are fixed, are called the **coordinate surfaces**, that is,
$$x^i = x^i(q_0^1, \ldots, q_0^{i-1}, q^i, q_0^{i+1}, \ldots, q_0^{j-1}, q^j, q_0^{j+1}, \ldots, q_0^n)\}\,.$$

- **Theorem 3.3.2** *For each point P there are n coordinate curves that pass through P. The set of tangent vectors $\{\mathbf{e}_i\}$ to these coordinate curves is linearly independent and forms a basis.*

- The basis $\{\mathbf{e}_i\}$ is not necessarily orthonormal.

- The metric tensor is defined as usual by
$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = \sum_{k=1}^{n} \frac{\partial x^k}{\partial q^i}\frac{\partial x^k}{\partial q^j}\,.$$

- The dual basis of 1-forms is defined by
$$\boldsymbol{\omega}^i = dq^i = \frac{\partial q^i}{\partial x^j}\boldsymbol{\sigma}^j\,,$$
where $\boldsymbol{\sigma}^j$ is the standard dual basis.

- The vector
$$d\,\mathbf{r} = \mathbf{e}_i\; dq^i = \frac{\partial\,\mathbf{r}}{\partial q^i}\; dq^i$$
is called the **infinitesimal displacement**.

- The arc length, called the **interval**, is determined by
$$ds^2 = \|d\,\mathbf{r}\,\|^2 = d\,\mathbf{r}\,\cdot d\,\mathbf{r} = g_{ij}\; dq^i dq^j\,.$$

- The volume of the parallelepiped spanned by the vectors $\{\mathbf{e}_1 dq^1, \ldots, \mathbf{e}_n dq^n\}$, called the **volume element**, is
$$dV = \sqrt{|g|}\; dq^1 \cdots dq^n\,,$$
where, as usual, $|g| = \det(g_{ij})$.

- A coordinate system is called **orthogonal** if the vectors $\partial\,\mathbf{r}\,/\partial q^i$ are mutually orthogonal. The norms of these vectors
$$h_i = \left\|\frac{\partial\,\mathbf{r}}{\partial q^i}\right\|$$
are called the **scale factors**.

- Then one can introduce the **orthonormal basis** $\{\mathbf{e}_i\}$ by

$$\mathbf{e}_i = \left\| \frac{\partial \mathbf{r}}{\partial q^i} \right\|^{-1} \frac{\partial \mathbf{r}}{\partial q^i} = \frac{1}{h_i} \frac{\partial \mathbf{r}}{\partial q^i} .$$

- For an orthonormal system the vector components are (there is no difference between contravariant and covariant components)

$$v_i = v^i = \mathbf{v} \cdot \mathbf{e}_i .$$

- Then the interval has the form

$$ds^2 = \sum_{i=1}^{n} h_i^2 (dq^i)^2 .$$

- The volume element in orthogonal coordinate system is

$$dV = h_1 \cdots h_n \, dq^1 \cdots dq^n .$$

3.3.1 Change of Coordinates

- Let (q^i) and $(q' j)$ be two curvilinear coordinate systems. Then they should be related by a smooth invertible transformation

$$q'^i = f^i(q) = f^i(q^1, \dots, q^n) , \qquad q^i = h^i(q') = f^i(q'^1, \dots, q'^n) ,$$

such that

$$f^i(h(q')) = q'^i , \qquad h^i(f(q)) = q^i .$$

- The Jacobian matrices are related by

$$\frac{\partial q'^i}{\partial q^j} \frac{\partial q^j}{\partial q'^k} = \delta^i_k , \qquad \frac{\partial q^i}{\partial q'^j} \frac{\partial q'^j}{\partial q^k} = \delta^i_k ,$$

so that the matrix $\left(\frac{\partial q'^i}{\partial q^j}\right)$ is inverse to $\left(\frac{\partial q^i}{\partial q'^j}\right)$.

- The basis vectors in these coordinate systems are

$$\mathbf{e}'_i = \frac{\partial \mathbf{r}}{\partial q'^i} , \qquad \mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial q^i} .$$

Therefore, they are related by a linear transformation

$$\mathbf{e}'_i = \frac{\partial q^j}{\partial q'^i} \mathbf{e}_j , \qquad \mathbf{e}_j = \frac{\partial q'^i}{\partial q^j} \mathbf{e}'_j .$$

They have the same orientation if the Jacobian of the change of coordinates is positive and oppositive orientation if the Jacobian is negative.

- Thus, a set of real numbers $T^{i_1\dots i_p}{}_{j_1\dots j_q}$ is said to represent **components of a tensor of type** (p,q) (p times contravariant and q times covariant) if they transform under a change of coordinates according to

$$T'^{i_1\dots i_p}_{j_1\dots j_q} = \frac{\partial q'^{i_1}}{\partial q^{l_1}}\cdots\frac{\partial q'^{i_p}}{\partial q^{l_p}}\frac{\partial q^{m_1}}{\partial q'^{j_1}}\cdots\frac{\partial q^{m_q}}{\partial q'^{j_q}}T^{l_1\dots l_p}_{m_1\dots m_q}\,.$$

  This is the general **transformation law of the components of a tensor of type** (p,q) with respect to a change of curvilinear coordinates.

- A pseudo-tensor has an additional factor equal to the sign of the Jacobian, that is, the components of a pseudo-tensor of type (p,q) transform as

$$T'^{i_1\dots i_p}_{j_1\dots j_q} = \operatorname{sign}\left[\det\left(\frac{\partial q'^{i_1}}{\partial q^{l_1}}\right)\right]\frac{\partial q'^{i_1}}{\partial q^{l_1}}\cdots\frac{\partial q'^{i_p}}{\partial q^{l_p}}\frac{\partial q^{m_1}}{\partial q'^{j_1}}\cdots\frac{\partial q^{m_q}}{\partial q'^{j_q}}T^{l_1\dots l_p}_{m_1\dots m_q}\,.$$

  This is the general **transformation law of the components of a pseudo-tensor of type** (p,q) with respect to a change of curvilinear coordinates.

3.3.2 Examples

- The **polar coordinates** in $\mathbb{R}^2$ are introduced by

$$x^1 = \rho\cos\varphi\,,\qquad x^2 = \rho\sin\varphi\,,$$

  where $\rho \ge 0$ and $0 \le \varphi < 2\pi$. The Jacobian matrix is

$$J = \begin{pmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{pmatrix}.$$

  The Jacobian is

$$\det J = \rho\,.$$

  Thus, the only singular point of the polar coordinate system is the origin $\rho = 0$. At all nonsingular points the change of variables is invertible and we have

$$\rho = \sqrt{(x^1)^2 + (x^2)^2}\,,\qquad \varphi = \cos^{-1}\left(\frac{x^1}{\rho}\right) = \sin^{-1}\left(\frac{x^2}{\rho}\right).$$

  The coordinate curves of ρ are half-lines (rays) going through origin with the slope $\tan\varphi$. The coordinate curves of φ are circles with the radius ρ centered at the origin.

- The **cylindrical coordinates** in $\mathbb{R}^3$ are introduced by

$$x^1 = \rho\cos\varphi\,,\qquad x^2 = \rho\sin\varphi\,,\qquad x^3 = z$$

  where $\rho \ge 0$, $0 \le \varphi < 2\pi$ and $z \in \mathbb{R}$. The Jacobian matrix is

$$J = \begin{pmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The Jacobian is

$$\det J = \rho \,.$$

Thus, the only singular point of the cylindrical coordinate system is the origin $\rho = 0$. At all nonsingular points the change of variables is invertible and we have

$$\rho = \sqrt{(x^1)^2 + (x^2)^2}\,, \qquad \varphi = \cos^{-1}\left(\frac{x^1}{\rho}\right) = \sin^{-1}\left(\frac{x^2}{\rho}\right), \qquad z = x^3\,.$$

The coordinate curves of ρ are horizontal half-lines in the plane $z = \text{const}$ going through the z-axis. The coordinate curves of φ are circles in the plane $z = \text{const}$ of radius ρ centered at the z axis. The coordinate curves of z are vertical lines. The coordinate surfaces of ρ, φ are horizontal planes. The coordinate surfaces of ρ, z are vertical half-planes going through the z-axis. The coordinate surfaces of φ, z are vertical cylinders centered at the origin.

- The **spherical coordinates** in $\mathbb{R}^3$ are introduced by

$$x^1 = r\sin\theta\cos\varphi\,, \qquad x^2 = r\sin\theta\sin\varphi\,, \qquad x^3 = r\cos\theta$$

where $r \geq 0$, $0 \leq \varphi < 2\pi$ and $0 \leq \theta \leq \pi$. The Jacobian matrix is

$$J = \begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}.$$

The Jacobian is

$$\det J = r^2\sin\theta\,.$$

Thus, the singular points of the spherical coordinate system are the points where either $r = 0$, which is the origin, or $\theta = 0$ or $\theta = \pi$, which is the whole z-axis. At all nonsingular points the change of variables is invertible and we have

$$\begin{aligned} r &= \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}\,, \\ \varphi &= \cos^{-1}\left(\frac{x^1}{\rho}\right) = \sin^{-1}\left(\frac{x^2}{\rho}\right), \\ \theta &= \cos^{-1}\left(\frac{x^3}{r}\right), \end{aligned}$$

where $\rho = \sqrt{(x^1)^2 + (x^2)^2}$.

The coordinate curves of r are half-lines going through the origin. The coordinate curves of φ are circles of radius $r\sin\theta$ centered at the z axis. The coordinate curves of θ are vertical half-circles of radius r centered at the origin. The coordinate surfaces of r, φ are half-cones around the z-axis going through the origin. The coordinate surfaces of r, θ are vertical half-planes going through the z-axis. The coordinates surfaces of φ, θ are spheres of radius r centered at the origin.

3.4 Vector Functions of a Single Variable

- A vector-valued function is a map $\mathbf{v} : [a, b] \to E$ from an interval $[a, b]$ of real numbers to a vector space E that assigns a vector $\mathbf{v}(t)$ to each real number $t \in [a, b]$.

- We say that a vector valued function $\mathbf{v}(t)$ has a **limit** $\mathbf{v}_0$ as $t \to t_0$, denoted by

$$\lim_{t\to t_0} \mathbf{v}(t) = \mathbf{v}_0$$

  if

$$\lim_{t\to t_0} ||\mathbf{v}(t) - \mathbf{v}_0|| = 0\,.$$

- A vector valued function $\mathbf{v}(t)$ is **continuous** at t_0 if

$$\lim_{t\to t_0} \mathbf{v}(t) = \mathbf{v}(t_0)\,.$$

  A vector valued function $\mathbf{v}(t)$ is continuous on the interval $[a, b]$ if it is continuous at every point t of this interval.

- A vector valued function $\mathbf{v}(t)$ is **differentiable** at t_0 if there exists the limit

$$\lim_{h\to 0} \frac{\mathbf{v}(t_0 + h) - \mathbf{v}(t_0)}{h}\,.$$

  If this limit exists it is called the **derivative** of the function $\mathbf{v}(t)$ at t_0 and denoted by

$$\mathbf{v}'(t_0) = \frac{d\mathbf{v}}{dt} = \lim_{h\to 0} \frac{\mathbf{v}(t_0 + h) - \mathbf{v}(t_0)}{h}\,.$$

  If the function $\mathbf{v}(t)$ is differentiable at every t in an interval $[a, b]$, then it is called differentiable on that interval.

- Let $\{\mathbf{e}_i\}$ be a constant basis in E that does not depend on t. Then a vector valued function $\mathbf{v}(t)$ is represented by its components

$$\mathbf{v}(t) = v^i(t)\mathbf{e}_i$$

  and the derivative of $\mathbf{v}$ can be computed componentwise

$$\frac{d\mathbf{v}}{dt} = \frac{dv^i}{dt}\,\mathbf{e}_i\,.$$

- The derivative is a linear operation, that is,

$$\frac{d}{dt}(\mathbf{u} + \mathbf{v}) = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt}\,, \qquad \frac{d}{dt}(c\mathbf{v}) = c\frac{d\mathbf{v}}{dt}\,,$$

  where c is a scalar constant.

- More generally, the derivative satisfies the product rules
$$\frac{d}{dt}(f\mathbf{v}) = f\frac{d\mathbf{v}}{dt} + \frac{df}{dt}\mathbf{v}\,,$$
$$\frac{d}{dt}(\mathbf{u},\mathbf{v}) = \left(\frac{d\mathbf{u}}{dt},\mathbf{v}\right) + \left(\mathbf{u},\frac{d\mathbf{v}}{dt}\right)$$
Similarly for the exterior product
$$\frac{d}{dt}\boldsymbol{\omega}\wedge\boldsymbol{\sigma} = \frac{d\omega}{dt}\wedge\boldsymbol{\sigma} + \omega\wedge\frac{d\boldsymbol{\sigma}}{dt}$$
By taking the dual of this equation we obtain in $\mathbb{R}^3$ the product rule for the vector product
$$\frac{d}{dt}\mathbf{u}\times\mathbf{v} = \frac{d\mathbf{u}}{dt}\times\mathbf{v} + \mathbf{u}\times\frac{d\mathbf{v}}{dt}$$

- **Theorem 3.4.1** *The derivative $\mathbf{v}'(t)$ of a vector valued function $\mathbf{v}(t)$ with the constant norm is orthogonal to $\mathbf{v}(t)$. That is, if $\|\mathbf{v}(t)\| = \text{const}$, then for any t*
$$(\mathbf{v}'(t),\mathbf{v}(t)) = 0\,.$$

3.5 Geometry of Curves

- Let $\mathbf{r} = \mathbf{r}(t)$ be a parametrized curve.
- A curve
$$\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{u}$$
is a **straight line** parallel to the vector $\mathbf{u}$ passing through the point $\mathbf{r}_0$.
- Let $\mathbf{u}$ and $\mathbf{v}$ be two orthonormal vectors. Then the curve
$$\mathbf{r} = \mathbf{r}_0 + a(\cos t\,\mathbf{u} + \sin t\,\mathbf{v})$$
is a **circle** of radius a with the center at $\mathbf{r}_0$ in the plane spanned by the vectors $\mathbf{u}$ and $\mathbf{v}$.
- Let $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be an orthonormal triple of vectors. Then the curve
$$\mathbf{r} = \mathbf{r}_0 + b(\cos t\,\mathbf{u} + \sin t\,\mathbf{v}) + at\,\mathbf{w}$$
is the **helix** of radius b with the axis passing through the point $\mathbf{r}_0$ and parallel to $\mathbf{w}$.
- The vertical distance between the coils of the helix, equal to $2\pi|a|$, is called the **pitch**.
- Let (q^i) be a curvilinear coordinate system. Then a curve can be described by $q^i = q^i(t)$, which in Cartesian coordinates becomes $\mathbf{r} = \mathbf{r}(q(t))$.
- The derivative
$$\frac{d\mathbf{r}}{dt} = \frac{\partial \mathbf{r}}{\partial q^i}\frac{dq^i}{dt}$$
of the vector valued function $\mathbf{r}(t)$ is called the **tangent vector**. If $\mathbf{r}(t)$ represents the position of a particle at the time t, then $\mathbf{r}'$ is the **velocity** of the particle.
- The norm
$$\left\|\frac{d\mathbf{r}}{dt}\right\| = \sqrt{g_{ij}(q(t))\frac{dq^i}{dt}\frac{dq^j}{dt}}$$
of the velocity is called the **speed**. Here, as usual
$$g_{ij} = \frac{\partial \mathbf{r}}{\partial q^i} \cdot \frac{\partial \mathbf{r}}{\partial q^j} = \sum_{k=1}^{n} \frac{\partial x^k}{\partial q^i}\frac{\partial x^k}{\partial q^j}$$
is the metric tensor in the coordinate system (q^i).
- We will say that a curve $\mathbf{r} = \mathbf{r}(t)$ is **smooth** if:
a) it has continuous derivatives of all orders,
b) there are no self-intersections, and
c) the speed is non-zero, i.e. $\|\mathbf{r}'(t)\| \neq 0$, at every point on the curve.

- For a curve $\mathbf{r} : [a,b] \to \mathbb{R}^n$, the possibility that $\mathbf{r}(a) = \mathbf{r}(b)$ is allowed. Then it is called a **closed curve**. A closed curve does not have a boundary.

- A curve consisting of a finite number of smooth arcs joined together without self-intersections is called **piece-wise smooth**, or just **regular**.

- For each regular curve there is a **natural parametrization**, or the **unit-speed** parametrization with a natural parameter s such that

$$\left\| \frac{d\mathbf{r}}{ds} \right\| = 1 \, .$$

- The **orientation** of a parametrized curve is determined by the direction of increasing parameter. The point $\mathbf{r}(a)$ is called the initial point and the point $\mathbf{r}(b)$ is called the endpoint.

- Nonparametric curves are not oriented.

- The **unit tangent** is determined by

$$\mathbf{T} = \left\| \frac{d\mathbf{r}}{dt} \right\|^{-1} \frac{d\mathbf{r}}{dt} \, .$$

For the natural parametrization the tangent is the unit tangent, i.e.

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\partial \mathbf{r}}{\partial q^i} \frac{dq^i}{ds} \, .$$

- The norm of the displacement vector $d\mathbf{r} = \mathbf{r}' dt$

$$ds = \left\| \frac{d\mathbf{r}}{dt} \right\| dt = \|d\mathbf{r}\| = \sqrt{g_{ij}(q(t)) \frac{dq^i}{dt} \frac{dq^j}{dt}} \, dt \, .$$

is called the **length element**.

- The **length** of a smooth curve $\mathbf{r} : [a,b] \to \mathbb{R}^n$ is defined by

$$L = \int_C ds = \int_a^b \left\| \frac{d\mathbf{r}}{dt} \right\| dt = \int_a^b \sqrt{g_{ij}(q(t)) \frac{dq^i}{dt} \frac{dq^j}{dt}} \, dt$$

For the natural parametrization the length of the curve is simply

$$L = b - a \, .$$

That is why, the parameter s is nothing but the **length of the arc** of the curve from the initial point $\mathbf{r}(a)$ to the current point $\mathbf{r}(t)$

$$s(t) = \int_a^t d\tau \left\| \frac{d\mathbf{r}}{d\tau} \right\| \, .$$

This means that
$$\frac{ds}{dt} = \left\| \frac{d\,\mathbf{r}}{dt} \right\| ,$$
and
$$\frac{d\,\mathbf{r}}{dt} = \frac{ds}{dt}\frac{d\,\mathbf{r}}{ds} .$$

- The second derivative
$$\mathbf{r}'' = \frac{d^2\,\mathbf{r}}{dt^2} = \frac{d}{dt}\left(\frac{\partial\,\mathbf{r}}{\partial q^i}\right)\frac{dq^i}{dt} + \frac{\partial\,\mathbf{r}}{\partial q^i}\frac{d^2 q^i}{dt^2} .$$
is called the **acceleration**.

- In the natural parametrization this gives the natural rate of change of the unit tangent
$$\frac{d\,\mathbf{T}}{ds} = \frac{d^2\,\mathbf{r}}{ds^2} = \frac{d}{ds}\left(\frac{\partial\,\mathbf{r}}{\partial q^i}\right)\frac{dq^i}{ds} + \frac{\partial\,\mathbf{r}}{\partial q^i}\frac{d^2 q^i}{ds^2} .$$

- The norm of this vector is called the **curvature** of the curve
$$\kappa = \left\| \frac{d\,\mathbf{T}}{ds} \right\| = \left\| \frac{d\,\mathbf{r}}{dt} \right\|^{-1} \left\| \frac{d\,\mathbf{T}}{dt} \right\| .$$
The **radius of curvature** is defined by
$$\rho = \frac{1}{\kappa} .$$

- The normalized rate of change of the unit tangent defines the **principal normal**
$$\mathbf{N} = \rho\,\frac{d\,\mathbf{T}}{ds} = \left\| \frac{d\,\mathbf{T}}{dt} \right\|^{-1} \frac{d\,\mathbf{T}}{dt} .$$

- The unit tangent and the principal normal are orthogonal to each other. They form an orthonormal system.

- **Theorem 3.5.1** *For any smooth curve $\mathbf{r} = \mathbf{r}(t)$, the acceleration $\mathbf{r}''$ lies in the plane spanned by the vectors $\mathbf{T}$ and $\mathbf{N}$. The orthogonal decomposition of $\mathbf{r}''$ with respect to $\mathbf{T}$ and $\mathbf{N}$ has the form*
$$\mathbf{r}'' = \frac{d\,\|\mathbf{r}'\|}{dt}\,\mathbf{T} + \kappa\,\|\mathbf{r}'\|^2\,\mathbf{N} .$$

- The vector
$$\frac{d\,\mathbf{N}}{ds} + \kappa\,\mathbf{T}$$
is orthogonal to both vectors $\mathbf{T}$ and $\mathbf{N}$, and hence, to the plane spanned by these vectors. In a general space $\mathbb{R}^n$ this vector could be decomposed with respect to a basis in the $(n-2)$-dimensional subspace orthogonal to this plane. We will restrict below to the case $n = 3$.

- In $\mathbb{R}^3$ one defines the **binormal**

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} .$$

  Then the triple $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a right-handed orthonormal system called a **moving frame**.

- By using the orthogonal decomposition of the acceleration one can obtain an alternative formula for the curvature of a curve in $\mathbb{R}^3$ as follows. We compute

$$\mathbf{r}' \times \mathbf{r}'' = \kappa \, ||\mathbf{r}'||^3 \mathbf{B} .$$

  Therefore,

$$\kappa = \frac{||\mathbf{r}' \times \mathbf{r}''||}{||\mathbf{r}'||^3} .$$

- The scalar quantity

$$\tau = \mathbf{B} \cdot \frac{d\mathbf{N}}{ds} = -\mathbf{N} \cdot \frac{d\mathbf{B}}{ds}$$

  is called the **torsion** of the curve.

- **Theorem 3.5.2 (Frenet-Serret Equations)** *For any smooth curve in $\mathbb{R}^3$ there hold*

$$\begin{aligned} \frac{d\mathbf{T}}{ds} &= \kappa \mathbf{N} \\ \frac{d\mathbf{N}}{ds} &= -\kappa \mathbf{T} + \tau \mathbf{B} \\ \frac{d\mathbf{B}}{ds} &= -\tau \mathbf{N} . \end{aligned}$$

- **Theorem 3.5.3** *Any two curves in $\mathbb{R}^3$ with identical curvature and torsion are congruent.*

3.6 Geometry of Surfaces

- Let S be a parametrized surface. It can be described in general curvilinear coordinates by $q^i = q^i(u, v)$, where $u \in [a, b]$ and $v \in [c, d]$. Then $\mathbf{r} = \mathbf{r}(q(u, v))$.

- The parameters u and v are called the **local coordinates** on the surface.

- The curves $\mathbf{r}(u, v_0)$ and $\mathbf{r}(u_0, v)$, with one coordinate being fixed, are called the **coordinate curves**.

- The tangent vectors to the coordinate curves

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \frac{\partial \mathbf{r}}{\partial q^i}\frac{\partial q^i}{\partial u} \qquad \text{and} \qquad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial \mathbf{r}}{\partial q^i}\frac{\partial q^i}{\partial v}$$

  are **tangent** to the surface.

- A surface is **smooth** if:
  a) $\mathbf{r}(u, v)$ has continuous partial derivatives of all orders,
  b) the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ are non-zero and linearly independent,
  c) there are no self-intersections.

- It is allowed that $\mathbf{r}(a, v) = \mathbf{r}(b, v)$ and $\mathbf{r}(u, c) = \mathbf{r}(u, d)$.

- A plane T_P spanned by the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ at a point P on a smooth surface S is called the **tangent plane**.

- A surface is smooth if the tangent plane is well defined, that is, the tangent vectors are linearly independent (nonparallel), which means that it does not degenerate to a line or a point at every point of the surface.

- A surface is **piece-wise smooth** if it consists of a finite number of smooth pieces joined together.

- The orientation of the surface is achived by cutting it in small pieces and orienting the small pieces separately. If this can be made consistenly for the whole surface, then it is called **orientable**.

- The boundary ∂S of the surface $\mathbf{r} = \mathbf{r}(u, v)$, where $u \in [a, b], v \in [c, d]$ consists of the curves $\mathbf{r}(a, v)$, $\mathbf{r}(b, v)$, $\mathbf{r}(u, c)$ and $\mathbf{r}(u, d)$. A surface without boundary is called **closed**.

- **Remark.** There are non-orientable smooth surfaces.

- In $\mathbb{R}^3$ one can define the **unit normal vector** to the surface by

$$\mathbf{n} = ||\mathbf{r}_u \times \mathbf{r}_v||^{-1}\, \mathbf{r}_u \times \mathbf{r}_v \,.$$

  Notice that

$$||\mathbf{r}_u \times \mathbf{r}_v|| = \sqrt{||\mathbf{r}_u||^2 ||\mathbf{r}_v||^2 - (\mathbf{r}_u \cdot \mathbf{r}_v)^2}\,.$$

In components,

$$(\mathbf{r}_u \times \mathbf{r}_v)_i = \sqrt{g}\varepsilon_{ilm}\frac{\partial q^l}{\partial u}\frac{\partial q^m}{\partial v}\,,$$

$$\mathbf{r}_u \cdot \mathbf{r}_u = g_{ij}\frac{\partial q^i}{\partial u}\frac{\partial q^j}{\partial u}\,, \qquad \mathbf{r}_v \cdot \mathbf{r}_v = g_{ij}\frac{\partial q^i}{\partial v}\frac{\partial q^j}{\partial v}\,, \qquad \mathbf{r}_u \cdot \mathbf{r}_v = g_{ij}\frac{\partial q^i}{\partial u}\frac{\partial q^j}{\partial v}\,.$$

- The sign of the normal is determined by the orientation of the surface.
- For a smooth surface the unit normal vector field **n** varies smoothly over the surface.
- The normal to a closed surface in $\mathbb{R}^3$ is usually oriented in the **outward** direction.
- In $\mathbb{R}^3$ a surface can also be described by a single equation

$$F(x, y, z) = 0\,.$$

This equation does not prescribe the orientation though. Then

$$\frac{\partial F}{\partial x^i}\frac{\partial x^i}{\partial u} = 0\,, \qquad \frac{\partial F}{\partial x^i}\frac{\partial x^i}{\partial v} = 0\,.$$

The unit normal vector is then

$$\mathbf{n} = \pm\frac{\mathbf{grad}\,F}{||\,\mathbf{grad}\,F||}\,.$$

The sign here is fixed by the choice of the orientation. In components,

$$n_i = \frac{\partial F}{\partial x^i}\,.$$

- Let $\mathbf{r}(u, v)$ be a surface, $u = u(t), v = v(t)$ be a curve in the rectangle $D = [a, b] \times [c, d]$, and $\mathbf{r}((u(t), v(t))$ be the image of that curve on the surface S. Then the arc length of this curve is

$$dl = \left\|\frac{d\,\mathbf{r}}{dt}\right\| dt\,,$$

or

$$dl^2 = ||d\,\mathbf{r}||^2 = h_{ab}\,du^a\,du^b\,,$$

where $u^1 = u$ and $u^2 = v$, and

$$h_{ab} = g_{ij}\frac{\partial q^i}{\partial u^a}\frac{\partial q^j}{\partial u^b}\,,$$

is the **induced metric** on the surface and the indices a, b take only the values $1, 2$. In more detail,

$$dl^2 = h_{11}\,du^2 + 2h_{12}\,du\,dv + h_{22}\,dv^2\,,$$

and

$$h_{11} = \mathbf{r}_u \cdot \mathbf{r}_u\,, \qquad h_{12} = h_{21} = \mathbf{r}_u \cdot \mathbf{r}_v\,, \qquad h_{22} = \mathbf{r}_v \cdot \mathbf{r}_v\,.$$

- The area of a plane spanned by the vectors $\mathbf{r}_u\, du$ and $\mathbf{r}_v\, dv$ is called the **area element**
$$dS = \sqrt{h}\, du\, dv\,,$$
where
$$h = \det h_{ab} = \|\mathbf{r}_u\|^2\, \|\mathbf{r}_v\|^2 - (\mathbf{r}_u \cdot \mathbf{r}_v)^2\,.$$

- In $\mathbb{R}^3$ the area element can also be written as
$$dS = \|\mathbf{r}_u \times \mathbf{r}_v\|\, du\, dv$$

- The area element of a surface in $\mathbb{R}^3$ parametrized by
$$x = u, \qquad y = v, \qquad z = f(u, v)\,,$$
is
$$dS = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy\,.$$

- The area element of a surface in $\mathbb{R}^3$ described by one equation $F(x, y, z) = 0$ is
$$\begin{aligned} dS &= \left|\frac{\partial F}{\partial z}\right|^{-1} \|\mathbf{grad}\, F\|\; dx\, dy \\ &= \left|\frac{\partial F}{\partial z}\right|^{-1} \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}\, dx\, dy \end{aligned}$$
if $\partial F/\partial z \neq 0$.

- The area of a surface S described by $\mathbf{r} : D \to \mathbb{R}^n$, where $D = [a, b] \times [c, d]$, is
$$S = \int_S dS = \int_a^b \int_c^d \sqrt{h}\, du\, dv\,.$$

- Let S be a parametrized hypersurface defined by $\mathbf{r} = \mathbf{r}(u) = \mathbf{r}(u^1, \dots, \mathbf{r}^{n-1})$. The tangent vectors to the hypersurface are
$$\frac{\partial \mathbf{r}}{\partial u^a}\,,$$
where $a = 1, 2, \dots, (n - 1)$.

  The tangent space at a point P on the hypersurface is the hyperplane equal to the span of these vectors
$$T = \text{span}\left\{\frac{\partial \mathbf{r}}{\partial u^a}, \dots, \frac{\partial \mathbf{r}}{\partial u^a}\right\}\,.$$

  The unit normal to the hypersurface S at the point P is the unit vector $\mathbf{n}$ orthogonal to T.

If the hypersurface is described by a single equation

$$F(x) = F(x^1, \dots, x^n) = 0\,,$$

then the normal is

$$\mathbf{n} = \pm \frac{\mathbf{grad}\, F}{\|\,\mathbf{grad}\, F\|}\,.$$

The sign here is fixed by the choice of the orientation. In components,

$$(dF)_i = \frac{\partial F}{\partial x^i}\,.$$

The arc length of a curve on the hypersurface is

$$dl^2 = h_{ab} du^a du^b\,,$$

where

$$h_{ab} = g_{ij} \frac{\partial q^i}{\partial u^a} \frac{\partial q^j}{\partial u^b}\,.$$

The area element of the hypersurface is

$$dS = \sqrt{h}\, du^1 \dots du^{n-1}\,.$$

Chapter 4

Vector Analysis

4.1 Vector Functions of Several Variables

- The set $\mathbb{R}^n$ is the set of ordered n-tuples of real numbers $x = (x^1, \ldots, x^n)$. We call such n-tuples **points** in the space $\mathbb{R}^n$. Note that the points in space (although related to vectors) are not vectors themselves!
- Let $D = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a closed rectangle in $\mathbb{R}^n$.
- A **scalar field** is a scalar-valued function of n variables. In other words, it is a map $f : D \to \mathbb{R}$, which assigns a real number $f(x) = f(x^1, \ldots, x^n)$ to each point $x = (x^1, \ldots, x^n)$ of D.
- The hypersurfaces defined by
$$f(x) = c\,,$$
where c is a constant, are called **level surfaces** of the scalar field f.
- The level surfaces do not intersect.
- A **vector field** is a vector-valued function of n variables; it is a map $\mathbf{v} : D \to \mathbb{R}^n$ that assigns a vector $\mathbf{v}(x)$ to each point $x = (x^1, \ldots, x^n)$ in D.
- A **tensor field** is a tensor-valued function on D.
- Let $\mathbf{v}$ be a vector field. A point x_0 in $\mathbb{R}^n$ such that $\mathbf{v}(x_0) = \mathbf{0}$ is called a **singular point** (or a **critical point**) of the vector field $\mathbf{v}$. A point x is called **regular** if it is not singular, that is, if $\mathbf{v}(x) \neq \mathbf{0}$.
- In a neighborhood of a regular point of a vector field $\mathbf{v}$ there is a family of parametrized curves $\mathbf{r}(t)$ such that at each point the vector $\mathbf{v}$ is tangent to the curves, that is,
$$\frac{d\,\mathbf{r}}{dt} = f\mathbf{v}\,,$$
where f is a scalar field. Such curves are called the **flow lines**, or **stream lines**, or characteristic curves, of the vector field $\mathbf{v}$.

- Flow lines do not intersect.
- No flow lines pass through a singular point.
- The flow lines of a vector field $\mathbf{v} = v^i(x)\mathbf{e}_i$ can be found from the differential equations
$$\frac{dx^1}{v^1} = \cdots = \frac{dx^n}{v^n}\,.$$

4.2 Directional Derivative and the Gradient

- Let P_0 be a point and $\mathbf{u}$ be a unit vector. Then $\mathbf{r}(s) = \mathbf{r}_0 + s\mathbf{u}$ is the equation of the oriented line passing through P_0 with the unit tangent $\mathbf{u}$.

- Let $f(x)$ be a scalar field. Then the derivative
$$\frac{d}{ds} f(x(s))\Big|_{s=0}$$
at $s = 0$ is called the **directional derivative** of f at the point P_0 in the direction of $\mathbf{u}$ and denoted by
$$\nabla_{\mathbf{u}} f = \frac{d}{ds} f(x(s))\Big|_{s=0} .$$

- The directional derivatives in the direction of the basis vectors $\mathbf{e}_i$ are the partial derivatives
$$\nabla_{\mathbf{e}_i} f = \frac{\partial f}{\partial x^i} ,$$
which are also denoted by
$$\partial_i f = \frac{\partial f}{\partial x^i} .$$

- More generally, let $\mathbf{r}(s)$ be a parametrized curve in the natural parametrization and $\mathbf{u} = d\mathbf{r}/ds$ be the unit tangent. Then
$$\nabla_{\mathbf{u}} f = \frac{\partial f}{\partial x^i} \frac{dx^i}{ds}$$
In curvilinear coordinates
$$\nabla_{\mathbf{u}} f = \frac{\partial f}{\partial q^i} \frac{dq^i}{ds} .$$

- The covector (1-form) field with the components $\partial f/\partial q^i$ is denoted by
$$df = \frac{\partial f}{\partial x^i} dx^i = \frac{\partial f}{\partial q^i} dq^i .$$

- Therefore, the 1-forms dq^i form a basis in the dual space of covectors.

- The vector field corresponding to the 1-form df is called the **gradient** of the scalar field f and denoted by
$$\mathbf{grad}\, f = \nabla f = g^{ij} \frac{\partial f}{\partial q^i} \mathbf{e}_j .$$

- The directional derivative is simply the action of the covector df on the vector $\mathbf{u}$ (or the inner product of the vectors $\mathbf{grad}\, f$ and $\mathbf{u}$)
$$\nabla_{\mathbf{u}} f = \langle df, \mathbf{u} \rangle = (\mathbf{grad}\, f, \mathbf{u}) .$$

- Therefore,
$$\nabla_{\mathbf{u}} f = \|\,\mathbf{grad}\, f\|\ \cos\theta\,,$$
where θ is the angle between the gradient and the unit tangent $\mathbf{u}$.

- Gradient of a scalar field points in the direction of the maximum rate of increase of the scalar field.

- The maximum value of the directional derivative at a fixed point is equal to the norm of the gradient
$$\max_{\mathbf{u}} \nabla_{\mathbf{u}} f = \|\,\mathbf{grad}\, f\|\,.$$

- The minimum value of the directional derivative at a fixed point is equal to the negative norm of the gradient
$$\min_{\mathbf{u}} \nabla_{\mathbf{u}} f = -\|\,\mathbf{grad}\, f\|\,.$$

- Let f be a scalar field and P_0 be a point where $\mathbf{grad}\, f \neq 0$. Let $\mathbf{r} = \mathbf{r}(s)$ be a curve passing through P with the unit tangent $\mathbf{u} = d\,\mathbf{r}\,/ds$. Suppose that the directional derivative vanishes, $\nabla_{\mathbf{u}} f = 0$. Then the unit tangent $\mathbf{u}$ is orthogonal to the gradient $\mathbf{grad}\, f$ at P. The set of all such curves forms a level surface $f(x) = c$, where $c = f(P_0)$. The gradient $\mathbf{grad}\, f$ is orthogonal to the tangent plane to the this surface at P_0.

- **Theorem 4.2.1** *For any smooth scalar field f there is a level surface $f(x) = c$ passing through every point where the gradient of f is non-zero,* $\mathbf{grad}\, f \neq 0$. *The gradient* $\mathbf{grad}\, f$ *is orthogonal to this surface at this point.*

- A vector field $\mathbf{v}$ is called **conservative** if there is a scalar field f such that
$$\mathbf{v} = \mathbf{grad}\, f\,.$$
The scalar field f is called the **scalar potential** of $\mathbf{v}$.

4.3 Exterior Derivative

- Recall that antisymmetric tensors of type $(0, k)$ are called the k-forms, and the antisymmetric tensors of type $(k, 0)$ are called k-vectors. We denote the space of all k-forms by Λ_k and the space of all k-vectors by Λ^k.
- The exterior derivative d is an operator
$$d : \Lambda_k \to \Lambda_{k+1} ,$$
that assigns a $(k + 1)$-form to each k-form. It is defined as follows.
- A scalar field can be also called a 0-form. The exterior derivative of a zero form f is a 1-form
$$df = \frac{\partial f}{\partial q^i}\, dx^i$$
with components
$$(df)_i = \frac{\partial f}{\partial q^i} .$$
- The exterior derivative of a 1-form is a 2-form $d\boldsymbol{\sigma}$ defined by
$$(d\boldsymbol{\sigma})_{ij} = \frac{\partial \sigma_j}{\partial q^i} - \frac{\partial \sigma_i}{\partial q^j} .$$
- The exterior derivative of a k-form $\boldsymbol{\sigma}$ is a $(k + 1)$-form $d\boldsymbol{\sigma}$ with components
$$(d\boldsymbol{\sigma})_{i_1 i_2 \dots i_{k+1}} = (k+1)\frac{\partial}{\partial q^{[i_1}}\sigma_{i_2 \dots i_{k+1}]} = \sum_{\varphi \in S_{k+1}} \operatorname{sign}(\varphi)\frac{\partial}{\partial q^{i_{\varphi(1)}}}\sigma_{i_{\varphi(2)} \dots i_{\varphi(k+1)}} .$$
- **Theorem 4.3.1** *The exterior derivative of a k-form is a $(k + 1)$-form.*
- The exterior derivative plays the role of the gradient for k-forms.
- **Theorem 4.3.2** *The exterior derivative has the property*
$$d^2 = 0$$
- Recall that the duality operator $*$ assigns a $(n - k)$-vector to each k-form and an $(n - k)$-form to each k-vector:
$$* : \Lambda_k \to \Lambda^{n-k} , \qquad * : \Lambda^k \to \Lambda_{n-k} .$$
- Therefore, one can define the operator
$$*d : \Lambda_k \to \Lambda^{n-k-1} ,$$
which assigns a $(n - k - 1)$-vector to each k-form by
$$(*d\boldsymbol{\sigma})^{i_1 \dots i_{n-k-1}} = \frac{1}{k!} g^{-1/2} \varepsilon^{i_1 \dots i_{n-k-1} j_1 j_2 \dots j_{k+1}} \frac{\partial}{\partial q^{j_1}} \sigma_{j_2 \dots j_{k+1}}$$

- We can also define the operator

$$*d* : \Lambda^k \to \Lambda^{k-1}$$

acting on k-vectors, which assigns a $(k-1)$ vector to a k-vector.

- **Theorem 4.3.3** *For any k-vector A with components $A^{i_1 \dots i_k}$ there holds*

$$(*d*A)^{i_1 \dots i_{k-1}} = (-1)^{nk+1} g^{-1/2} \frac{\partial}{\partial q^j} \left(g^{1/2} A^{j i_1 \dots i_{k-1}} \right) .$$

- The operator $*d*$ plays the role of the divergence of k-vectors.

- **Theorem 4.3.4** *The operator $*d*$ has the property*

$$(*d*)^2 = 0$$

- Let G denote the operator that converts k-vectors to k-forms,

$$G : \Lambda^k \to \Lambda_k .$$

That is, if $A^{j_1 \cdots j_k}$ are the components of a k-vector, then the corresponding k-form $\boldsymbol{\sigma} = GA$ has components

$$\sigma_{i_1 \dots i_k} = g_{i_1 j_1} \dots g_{i_k j_k} A^{j_1 \dots j_k} .$$

- Then the operator

$$G * d : \Lambda_k \to \Lambda_{n-k-1}$$

assigns a $(n-k-1)$-form to each k-form by

$$(G * d\boldsymbol{\sigma})_{i_1 \dots i_{n-k-1}} = \frac{1}{k!} g^{-1/2} g_{i_1 m_1} \cdots g_{i_{n-k-1} m_{n-k-1}} \varepsilon^{m_1 \dots m_{n-k-1} j_1 j_2 \dots j_{k+1}} \frac{\partial}{\partial q^{j_1}} \sigma_{j_2 \dots j_{k+1}}$$

- The operator $*d$ plays the role of the curl of k-forms.

- Further, we can define the operator

$$\delta = G * dG* : \Lambda_k \to \Lambda_{k-1} ,$$

which assigns a $(k-1)$-form to each k-form.

- **Theorem 4.3.5** *For any k-form A with components $\sigma_{i_1 \dots i_k}$ there holds*

$$(\delta\boldsymbol{\sigma})_{i_1 \dots i_{k-1}} = (-1)^{nk+1} g^{-1/2} g_{i_1 j_1} \cdots g_{i_{k-1} j_{k-1}} \frac{\partial}{\partial q^j} \left(g^{1/2} g^{jp} g^{j_1 m_1} \cdots g^{j_{k-1} m_{k-1}} \sigma_{p m_1 \dots m_{k-1}} \right) .$$

- The operator δ plays the role of the divergence of k-forms.

- **Theorem 4.3.6** *The operator δ has the property*

$$\delta^2 = 0 .$$

- Therefore the operator

  $$L = d\delta + \delta d$$

  assigns a k-form to each k-form, that is,

  $$L : \Lambda_k \to \Lambda_k \,.$$

- This operator plays the role of the Laplacian of k-forms.
- A k-form $\boldsymbol{\sigma}$ is called **closed** if $d\boldsymbol{\sigma} = 0$.
- A k-form $\boldsymbol{\sigma}$ is called **exact** if there is a $(k-1)$-form $\boldsymbol{\alpha}$ such that $\boldsymbol{\sigma} = d\boldsymbol{\alpha}$.
- A 1-form σ corresponding to conservative vector field $\mathbf{v}$ is exact, that is, $\sigma = df$.
- Every exact k-form is closed.

4.4 Divergence

- The **divergence** of a vector field $\mathbf{v}$ is a scalar field defined by

$$\operatorname{div}\mathbf{v} = (-1)^{n+1} * d * \mathbf{v}\,,$$

which in local coordinates becomes

$$\operatorname{div}\mathbf{v} = g^{-1/2}\frac{\partial}{\partial q^i}(g^{1/2}v^i)\,.$$

where $g = \det g_{ij}$.

- **Theorem 4.4.1** *For any vector field* $\mathbf{v}$ *the divergence* $\operatorname{div}\mathbf{v}$ *is a scalar field.*

- The divergence of a covector field $\boldsymbol{\sigma}$ is

$$\operatorname{div}\boldsymbol{\sigma} = g^{-1/2}\frac{\partial}{\partial q^i}(g^{1/2}g^{ij}\sigma_j)\,.$$

- In Cartesian coordinates this gives simply

$$\operatorname{div}\mathbf{v} = \partial_i v^i\,.$$

- A vector field $\mathbf{v}$ is called **solenoidal** if

$$\operatorname{div}\mathbf{v} = 0\,.$$

- The 2-form $*\mathbf{v}$ dual to a solenoidal vector field $\mathbf{v}$ is closed, that is, $d * \mathbf{v} = 0$.

Physical Interpretation of Divergence

- The divergence of a vector field is the net outflux of the vector field per unit volume.

4.5 Curl

- Recall that the operator $*d$ assigns a $(n-k-1)$-vector to a k-form. In case $n = 3$ and $k = 1$ this operator assigns a vector to a 1-form. This enables one to define the **curl operator** in $\mathbb{R}^3$, which assigns a vector to a covector by

$$\mathbf{curl}\,\boldsymbol{\sigma} = *d\boldsymbol{\sigma}\,,$$

  or, in components,

$$(\mathbf{curl}\,\boldsymbol{\sigma})^i = g^{-1/2}\varepsilon^{ijk}\frac{\partial}{\partial q^j}\sigma_k = g^{-1/2}\det\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial q^1} & \dfrac{\partial}{\partial q^2} & \dfrac{\partial}{\partial q^3} \\ \sigma_1 & \sigma_2 & \sigma_3 \end{vmatrix}\,.$$

- We can also define the curl of a vector field $\mathbf{v}$ by

$$(\mathbf{curl}\,\mathbf{v})^i = g^{-1/2}\varepsilon^{ijk}\frac{\partial}{\partial q^j}(g_{km}v^m)\,.$$

- In Cartesian coordinates we have simply

$$(\mathbf{curl}\,\boldsymbol{\sigma})^i = \varepsilon^{ijk}\partial_j\sigma_k\,.$$

  This can also be written in the form

$$\mathbf{curl}\,\boldsymbol{\sigma} = \det\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ \sigma_1 & \sigma_2 & \sigma_3 \end{vmatrix}$$

- A vector field $\mathbf{v}$ in $\mathbb{R}^3$ is called **irrotational** if

$$\mathbf{curl}\,\mathbf{v} = \mathbf{0}\,.$$

- The one-form $\boldsymbol{\sigma}$ corresponding to an irrotational vector field $\mathbf{v}$ is closed, that is $d\boldsymbol{\sigma} = 0$.

- Each conservative vector field is irrotational.

- Let $\mathbf{v}$ be a vector field. If there is a vector field $\mathbf{A}$ such that

$$\mathbf{v} = \mathbf{curl}\,\mathbf{A}\,,$$

  when $\mathbf{A}$ is called the **vector potential** of $\mathbf{v}$.

- If $\mathbf{v}$ has a vector potential, then it is solenoidal.

- If $\mathbf{A}$ is a vector potential for $\mathbf{v}$, then the 2-form $*\mathbf{v}$ dual to $\mathbf{v}$ is exact, that is, $*\mathbf{v} = d\boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is the 1-form corresponding to $\mathbf{A}$.

Physical Interpretation of the Curl

- The curl of a vector field measures its tendency to swirl; it is the swirl per unit area.

4.6 Laplacian

- The scalar **Laplace operator** (or the **Laplacian**) is the map $\Delta : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ that assigns a scalar field to a scalar field. It is defined by

$$\Delta f = \text{div } \mathbf{grad}\, f = g^{-1/2}\partial_i g^{1/2} g^{ij}\partial_j f\,.$$

- In Cartesian coordinates it is simply

$$\Delta f = \partial^i \partial_i f\,.$$

- The Laplacian of a 1-form (covector field) $\boldsymbol{\sigma}$ is defined as follows. First, one obtains a 2-form $d\boldsymbol{\sigma}$ by the exterior derivative. Then one take the dual of this 2-form to get a $(n-2)$-form $*d\boldsymbol{\sigma}$. Then one acts by exterior derivative to get a $(n-1)$-form $d * d\boldsymbol{\sigma}$, and, finally, by taking the dual again one gets the 1-form $*d*d\boldsymbol{\sigma}$. Similarly, reversing the order of operations one gets the 1-form $d*d*\boldsymbol{\sigma}$. The Laplacian is the sum of these 1-forms, i.e.

$$\Delta\boldsymbol{\sigma} = -(G * dG * d + dG * dG*)\boldsymbol{\sigma}\,.$$

  The expression of this Laplacian in components is too complicated, in general.

- The components expression for this is

$$\begin{aligned}(\Delta\mathbf{v})^i \quad &= \quad \Big\{g^{ij}\frac{\partial}{\partial q^j}g^{-1/2}\frac{\partial}{\partial q^k}g^{1/2} - g^{-1/2}\frac{\partial}{\partial q^j}g^{1/2}g^{pi}g^{qj}\frac{\partial}{\partial q^p}g_{qk} \\ &\quad +g^{-1/2}\frac{\partial}{\partial q^j}g^{1/2}g^{pj}g^{qi}\frac{\partial}{\partial q^p}g_{qk}\Big\}v^k\,.\end{aligned}$$

- Of course, in Cartesian coordinates this simpifies significantly

$$(\Delta\mathbf{v})^i = \partial^j\partial_j v^i\,.$$

- In $\mathbb{R}^3$ it can be written as

$$\Delta\mathbf{v} = \mathbf{grad}\ \text{div}\,\mathbf{v} - \mathbf{curl}\ \mathbf{curl}\,\mathbf{v}\,.$$

Interpretation of the Laplacian

- The Laplacian Δ measures the difference between the value of a scalar field $f(P)$ at a point P and the average of f around this point.

4.7 Differential Vector Identities

- The identities below that involve the vector product and the curl apply only for $\mathbb{R}^3$. Other formulas are valid for arbitrary $\mathbb{R}^n$ in arbitrary coordinate systems:

$$
\begin{aligned}
&\mathbf{grad}\,(fh) = (\,\mathbf{grad}\,f)h + f\,\mathbf{grad}\,h\\
&\mathrm{div}\,(f\mathbf{v}) = (\,\mathbf{grad}\,f)\cdot\mathbf{v} + f\,\mathrm{div}\,\mathbf{v}\\
&\mathbf{grad}\,f(h(x)) = \frac{df}{dh}\,\mathbf{grad}\,h\\
&\mathbf{curl}\,(f\mathbf{v}) = (\,\mathbf{grad}\,f)\times\mathbf{v} + f\,\mathbf{curl}\,\mathbf{v}\\
&\mathrm{div}\,(\mathbf{u}\times\mathbf{v}) = (\,\mathbf{curl}\,\mathbf{u})\cdot\mathbf{v} - \mathbf{u}\cdot(\,\mathbf{curl}\,\mathbf{v})\\
&\mathbf{curl}\;\mathbf{grad}\,f = \mathbf{0}\\
&\mathrm{div}\;\mathbf{curl}\,\mathbf{v} = 0\\
&\mathrm{div}\,(\,\mathbf{grad}\,f\times\,\mathbf{grad}\,h) = 0
\end{aligned}
$$

- Let $\mathbf{e}_i$ be the standard basis in $\mathbb{R}^n$, x^i be the Cartesian coordinates, $\mathbf{r} = x^i\mathbf{e}_i$ be the position (radius) vector field and $r = \|\,\mathbf{r}\,\| = \sqrt{x_i x^i}$. Scalar fields that depend only on r and vector fields that depend on $\mathbf{x}$ and r are called **radial fields**. Below $\mathbf{a}$ is a constant vector field.

$$
\begin{aligned}
&\mathrm{div}\;\mathbf{r} = n\\
&\mathbf{curl}\;\mathbf{r} = 0\\
&\mathbf{grad}\,(\mathbf{a}\cdot\,\mathbf{r}\,) = \mathbf{a}\\
&\mathbf{curl}\,(\mathbf{a}\times\,\mathbf{r}\,) = 2\mathbf{a}\\
&\mathbf{grad}\,r = \frac{\mathbf{r}}{r}\\
&\mathbf{grad}\,f(r) = \frac{df}{dr}\,\frac{\mathbf{r}}{r}\\
&\mathbf{grad}\,\frac{1}{r} = -\frac{\mathbf{r}}{r^3}\\
&\mathbf{grad}\,r^k = kr^{k-2}\,\mathbf{r}\\
&\Delta f(r) = f'' + \frac{(n-1)}{r}f'\\
&\Delta r^k = k(k+n-2)r^{k-2}\\
&\Delta\frac{1}{r^{n-2}} = 0
\end{aligned}
$$

- Some useful formulas when working with radial fields are

$$
\partial_i x^k = \delta^k_i\,, \qquad \delta^i_i = n\,.
$$

4.8 Orthogonal Curvilinear Coordinate Systems in $\mathbb{R}^3$

- Let (q^1, q^2, q^3) be an orthogonal coordinate system in $\mathbb{R}^3$ and $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ be the corresponding orthonormal basis

  $$\hat{\mathbf{e}}_i = \frac{1}{h_i}\frac{\partial \mathbf{r}}{\partial q^i} .$$

  where

  $$h_i = \left\| \frac{\partial \mathbf{r}}{\partial q^i} \right\|$$

  are the scale factors.

  Then for any vector $\mathbf{v} = v^i \hat{\mathbf{e}}_i$ the contravariant and the covariant components coincide

  $$v^i = v_i = \hat{\mathbf{e}}_i \cdot \mathbf{v} .$$

- The displacement vector, the interval and the volume element in the orthogonal coordinate system are

  $$\begin{aligned} d\,\mathbf{r} &= h_1\hat{\mathbf{e}}_1 + h_2\hat{\mathbf{e}}_2 + h_3\hat{\mathbf{e}}_3 , \\ ds^2 &= h_1^2(dq^1)^2 + h_2^2(dq^2)^2 + h_3^2(dq^3)^2 , \\ dV &= h_1h_2h_3\; dq^1dq^2dq^3 . \end{aligned}$$

- The differential operators introduced above take the following form

  $$\mathbf{grad}\, f = \hat{\mathbf{e}}_1\frac{1}{h_1}\frac{\partial}{\partial q^1} f + \hat{\mathbf{e}}_2\frac{1}{h_2}\frac{\partial}{\partial q^2} f + \hat{\mathbf{e}}_3\frac{1}{h_3}\frac{\partial}{\partial q^3} f$$

  $$\operatorname{div}\mathbf{v} = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial q^1}(h_2h_3v_1) + \frac{\partial}{\partial q^2}(h_3h_1v_2) + \frac{\partial}{\partial q^3}(h_1h_2v_3)\right\}$$

  $$\begin{aligned} \mathbf{curl}\,\mathbf{v} &= \hat{\mathbf{e}}_1\frac{1}{h_2h_3}\left[\frac{\partial}{\partial q^2}(h_3v_3) - \frac{\partial}{\partial q^3}(h_2v_2)\right] \\ &+ \hat{\mathbf{e}}_2\frac{1}{h_3h_1}\left[\frac{\partial}{\partial q^3}(h_1v_1) - \frac{\partial}{\partial q^1}(h_3v_3)\right] \\ &+ \hat{\mathbf{e}}_3\frac{1}{h_1h_2}\left[\frac{\partial}{\partial q^1}(h_2v_2) - \frac{\partial}{\partial q^2}(h_1v_1)\right] \end{aligned}$$

  $$\Delta f = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial q^1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial q^1}\right) + \frac{\partial}{\partial q^1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial q^1}\right) + \frac{\partial}{\partial q^1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial q^1}\right)\right\} f$$

- **Cylindrical coordinates:**

  $$d\,\mathbf{r} = d\rho\,\hat{\mathbf{e}}_\rho + \rho d\varphi\,\hat{\mathbf{e}}_\varphi + dz\,\hat{\mathbf{e}}_z$$

  $$ds^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2$$

$$dV = \rho\, d\rho\, d\varphi\, dz$$

$$\mathbf{grad}\, f = \hat{\mathbf{e}}_\rho \partial_\rho f + \hat{\mathbf{e}}_\varphi \frac{1}{\rho} \partial_\varphi f + \hat{\mathbf{e}}_z \partial_z f$$

$$\operatorname{div} \mathbf{v} = \frac{1}{\rho} \partial_\rho (\rho v_\rho) + \frac{1}{\rho} \partial_\varphi v_\varphi + \partial_z v_z$$

$$\mathbf{curl}\, \mathbf{v} = \frac{1}{\rho} \begin{vmatrix} \hat{\mathbf{e}}_\rho & \rho \hat{\mathbf{e}}_\varphi & \hat{\mathbf{e}}_z \\ \partial_\rho & \partial_\varphi & \partial_z \\ v_\rho & \rho v_\varphi & v_z \end{vmatrix}$$

$$\Delta f = \frac{1}{\rho} \partial_\rho \left(\rho\, \partial_\rho\, f \right) + \frac{1}{\rho^2} \partial_\varphi^2\, f + \partial_z^2\, f$$

- **Spherical coordinates:**

$$d\,\mathbf{r} = dr\, \hat{\mathbf{e}}_r + r d\theta\, \hat{\mathbf{e}}_\theta + r \sin\theta\, d\varphi\, \hat{\mathbf{e}}_\varphi$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2$$

$$dV = r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$\mathbf{grad}\, f = \hat{\mathbf{e}}_r \partial_r f + \hat{\mathbf{e}}_\theta \frac{1}{r} \partial_\theta f + \hat{\mathbf{e}}_\varphi \frac{1}{r \sin\theta} \partial_\varphi f$$

$$\operatorname{div} \mathbf{v} = \frac{1}{r^2} \partial_r (r^2 v_r) + \frac{1}{r \sin\theta} \partial_\theta (\sin\theta\, v_\theta) + \frac{1}{r \sin\theta} \partial_\varphi v_\varphi$$

$$\mathbf{curl}\, \mathbf{v} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\mathbf{e}}_r & r \hat{\mathbf{e}}_\theta & r \sin\theta\, \hat{\mathbf{e}}_\varphi \\ \partial_r & \partial_\theta & \partial_\varphi \\ v_r & r v_\theta & r \sin\theta\, v_\varphi \end{vmatrix}$$

$$\Delta f = \frac{1}{r^2} \partial_r (r^2 \partial_r f) + \frac{1}{r^2 \sin\theta} \partial_\theta (\sin\theta \partial_\theta f) + \frac{1}{r^2 \sin^2\theta} \partial_\varphi^2 f$$

Chapter 5

Integration

5.1 Line Integrals

- Let C be a smooth curve described by $\mathbf{r}(t)$, where $t \in [a, b]$. The **length of the curve** is defined by
$$L = \int_C ds = \int_a^b \left\| \frac{d\,\mathbf{r}}{dt} \right\| \, dt\,.$$
- Let f be a scalar field. Then the **line integral of the scalar field** f is
$$\int_C f \, ds = \int_a^b f(x(t)) \left\| \frac{d\,\mathbf{r}}{dt} \right\| \, dt\,.$$
- If $\mathbf{v}$ is a vector field, then the **line integral of the vector field v** along the curve C is defined by
$$\int_C \mathbf{v} \cdot d\,\mathbf{r} = \int_a^b \mathbf{v}(x(t)) \cdot \frac{d\,\mathbf{r}}{dt} \, dt\,.$$
- In components, the line integral of a vector field takes the form
$$\int_C \mathbf{v} \cdot d\,\mathbf{r} = \int_C v_i dq^i = \int_C \left(v_1 dq^1 + \cdots + v_n dq^n \right) ,$$
where $v_i = g_{ij} v^j$ are the covariant components of the vector field.
- The expression
$$\boldsymbol{\sigma} = v_i dq^i = v_1 dq^1 + \cdots + v_n dq^n$$
is called a **differential** 1**-form**. Each covector naturally defines a differential form. That is why it is also called a 1**-form**.
- If C is a closed curve, then the line integral of a vector field is denoted by
$$\oint_C \mathbf{v} \cdot d\,\mathbf{r}$$
and is called the **circulation of the vector field v** about the closed curve C.

5.2 Surface Integrals

- Let S be a smooth parametrized surface described by $\mathbf{r} : D \to \mathbb{R}^n$, where $D = [a,b] \times [c,d]$. The **surface integral of a scalar field** f is

$$\int_S f \, dS = \int_a^b \int_c^d f(x(u,v)) \sqrt{h} \, du \, dv \,,$$

  where $u^1 = u, u^2 = v$, $h = \det h_{ab}$ and h_{ab} is the induced metric on the surface

$$h_{ab} = g_{ij} \frac{\partial q^i}{\partial u^a} \frac{\partial q^i}{\partial u^b} \,.$$

- Let A be an antisymmetric tensor field of type $(0,2)$ with components A_{ij}. It naturally defines the **differential** 2**-form**

$$\begin{aligned} \boldsymbol{\alpha} &= \sum_{i<j} A_{ij} \, dq^i \wedge dq^j \\ &= A_{12} dq^1 \wedge dq^2 + \cdots + A_{1n} dq^1 \wedge dq^n \\ &\quad + A_{23} dq^2 \wedge dq^3 + \cdots + A_{2n} dq^2 \wedge dq^n \\ &\quad + \cdots + A_{n-1,n} \, dq^{n-1} \wedge dq^n \,. \end{aligned}$$

- Then the **surface integral of a** 2**-form** α is defined by

$$\int_S \alpha = \int_S \sum_{i<j} A_{ij} \, dq^i \wedge dq^j = \int_a^b \int_c^d \sum_{i<j} A_{ij} J^{ij} \, du \, dv \,,$$

  where

$$J^{ij} = \frac{\partial q^i}{\partial u} \frac{\partial q^j}{\partial v} - \frac{\partial q^j}{\partial u} \frac{\partial q^i}{\partial v} \,.$$

- In $\mathbb{R}^3$ every 2-form defines a dual vector. Therefore, one can integrate vectors over a surface. Let $\mathbf{v}$ be a vector field in $\mathbb{R}^3$. Then the dual two form is

$$A_{ij} = \sqrt{g} \, \varepsilon_{ijk} v^k \,,$$

  or

$$A_{12} = \sqrt{g} \, v^3 \,, \qquad A_{13} = -\sqrt{g} \, v^2 \,, \qquad A_{23} = \sqrt{g} \, v^1 \,.$$

  Ttherefore,

$$\boldsymbol{\alpha} = \sqrt{g} \left(v^3 dq^1 \wedge dq^2 - v^2 dq^1 \wedge dq^3 + v^1 dq^2 \wedge dq^3 \right) \,.$$

  Then the **surface integral of the vector field v**, called the **total flux of the vector field through the surface**, is

$$\int_S \boldsymbol{\alpha} = \int_S \mathbf{v} \cdot \mathbf{n} \, dS = \int_a^b \int_c^d [\mathbf{v}, \mathbf{r}_u, \mathbf{r}_v] \, du \, dv \,,$$

where

$$\mathbf{n} = \|\mathbf{r}_u \times \mathbf{r}_v\|^{-1}\,\mathbf{r}_u \times \mathbf{r}_v$$

is the unit normal to the surface and

$$[\mathbf{v}, \mathbf{r}_u, \mathbf{r}_v] = \mathrm{vol}\,(\mathbf{v}, \mathbf{r}_u, \mathbf{r}_v) = \sqrt{g}\varepsilon_{ijk} v^i \frac{\partial q^j}{\partial u}\frac{\partial q^k}{\partial v}\,.$$

- Similarly, the integrals of a **differential** k**-form**

$$\boldsymbol{\alpha} = \sum_{i_1<\cdots<i_k} A_{i_1\ldots i_k}\, dq^{i_1} \wedge \cdots \wedge dq^{i_k}$$

with components $A_{i_1\ldots i_k}$ over a k-dimensional surface $q^i = q^i(u^1, \ldots, u^k)$, $u^i \in [a_i, b_i]$ are defined by

$$\int_S \boldsymbol{\alpha} = \int_{a_k}^{b_k} \cdots \int_{a_1}^{b_1} \sum_{i_1<\cdots<i_k} A_{i_1\ldots i_k} J^{i_1\ldots i_k}\, du^1 \cdots du^k\,,$$

where

$$J^{i_1\ldots i_k} = k!\frac{\partial q^{[i_1}}{\partial u^1} \cdots \frac{\partial q^{i_k]}}{\partial u^k}\,.$$

- The surface integral over a closed surface S without boundary is denoted by

$$\oint_S \boldsymbol{\alpha}\,.$$

- In the case $k = n-1$ we obtain the integral of a $(n-1)$-form $\boldsymbol{\alpha}$ over a hypersurface

$$\int_S \boldsymbol{\alpha} = \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \sum_{i_1<\cdots<i_{n-1}} A_{i_1\ldots i_{n-1}} J^{i_1\ldots i_{n-1}}\, du^1 \cdots du^{n-1}\,.$$

Let $\mathbf{n}$ be the unit vector orthogonal to the hypersurface and $\mathbf{v} = *\boldsymbol{\alpha}$ be the vector field dual to the $(n-1)$-form $\boldsymbol{\alpha}$. Then

$$\int_S \boldsymbol{\alpha} = \int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_1}^{b_1} \mathbf{v}\cdot\mathbf{n}\sqrt{h}\, du^1 \cdots du^{n-1}\,.$$

This defines the **total flux of the vector field v through the hypersurface** S.

The normal can be determined by

$$\sqrt{h}\, n_j = \frac{1}{(n-1)!}\sqrt{g}\varepsilon_{j i_1\ldots i_{n-1}} \frac{\partial q^{i_1}}{\partial u^1} \cdots \frac{\partial q^{i_{n-1}}}{\partial u^{n-1}}\,.$$

5.3 Volume Integrals

- Let $D = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a domain in $\mathbb{R}^n$ described in local coordinates (q^i) by
$$q^i \in [a_i, b_i]\,.$$

- The **volume element** in general curvilinear coordinates is
$$dV = \sqrt{g}\; dq^1 \cdots dq^n\,,$$
where $g = \det(g_{ij})$.

- The **volume of the region** D is
$$V = \int_D dV = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \sqrt{g}\; dq^1 \cdots dq^n\,.$$

- The **volume integral of a scalar field** $f(x)$ is
$$\int_D f\; dV = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x(q))\,\sqrt{g}\; dq^1 \cdots dq^n\,.$$

5.4 Fundamental Integral Theorems

5.4.1 Fundamental Theorem of Line Integrals

- **Theorem 5.4.1** *Let C be a smooth curve parametrized by* $\mathbf{r}(t)$*,* $t \in [a, b]$*. Then for any scalar field f (a* 0*-form)*

$$\int_C df = \int_C \frac{\partial f}{\partial q^i} dq^i = \int_C \mathbf{grad}\, f \cdot d\,\mathbf{r} = f(x(b)) - f(x(a))\,.$$

- The line integral of a conservative vector field does not depend on the interior of the curve but only on the endpoints of the curve.

- **Corollary 5.4.1** *The circulation of a smooth conservative vector field over a closed smooth curve is zero,*

$$\oint_C \mathbf{grad}\, f \cdot d\,\mathbf{r} = 0\,.$$

5.4.2 Green's Theorem

- **Theorem 5.4.2** *Let x and y be the Cartesian coordinates in* $\mathbb{R}^2$*. Let U be a bounded region in* $\mathbb{R}^2$ *with the boundary ∂U, which is a closed curve oriented couterclockwise. Then for any* 1*-form* $\boldsymbol{\alpha} = A_i\, dx^i = A_1 dx + A_2 dy$

$$\int_U \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) dx\, dy = \oint_{\partial U} (A_1 dx + A_2 dy)\ .$$

5.4.3 Stokes's Theorem

- **Theorem 5.4.3** *Let S be a bounded surface in* $\mathbb{R}^3$ *with the boundary ∂S oriented consistently with the the surface S. Then for any vector field* $\mathbf{v}$

$$\int_S \mathbf{curl}\, \mathbf{v} \cdot \mathbf{n}\, dS = \oint_{\partial S} \mathbf{v} \cdot d\,\mathbf{r}\ .$$

5.4.4 Gauss's Theorem

- **Theorem 5.4.4** *Let D be a bounded domain in* $\mathbb{R}^3$ *with the boundary ∂D oriented by an outward normal. Then for any vector field* $\mathbf{v}$

$$\int_D \operatorname{div} \mathbf{v}\, dV = \oint_{\partial D} \mathbf{v} \cdot \mathbf{n}\, dS$$

5.4.5 General Stokes's Theorem

- **Theorem 5.4.5** *Let S be a bounded smooth k-dimensional surface in $\mathbb{R}^n$ with the boundary ∂S, which is a closed $(k-1)$-dimensional surface oriented consistently with S. Let*
$$\boldsymbol{\alpha} = \sum_{i_1<\cdots<i_{k-1}} A_{i_1\ldots i_{k-1}} dq^{i_1} \wedge \cdots \wedge dq^{i_{k-1}}$$
be a smooth $(k-1)$-form. Then
$$\int_S d\boldsymbol{\alpha} = \oint_{\partial S} \boldsymbol{\alpha}\,.$$

- In components this formula takes the form
$$\int_S \frac{\partial A_{i_1\ldots i_{k-1}}}{\partial q^j} \frac{\partial q^{[j}}{\partial u^1} \frac{\partial q^{i_1}}{\partial u^2} \cdots \frac{\partial q^{i_{k-1}]}}{\partial u^k}\, du^1 \cdots du^k = \int_{\partial S} A_{i_1\ldots i_{k-1}} \frac{\partial q^{[i_1}}{\partial u^1} \cdots \frac{\partial q^{i_{k-1}]}}{\partial u^{k-1}}\, du^1 \cdots du^{k-1}\,.$$

Chapter 6

Potential Theory

6.1 Simply Connected Domains

6.2 Conservative Vector Fields

6.2.1 Scalar Potential

6.3 Irrotational Vector Fields

6.4 Solenoidal Vector Fields

6.4.1 Vector Potential

6.5 Laplace Equation

6.5.1 Harmonic Functions

6.6 Poisson Equation

6.6.1 Dirac Delta Function

6.6.2 Point Sources

6.6.3 Dirichlet Problem

6.6.4 Neumann Problem

6.6.5 Green's Functions

6.7 Fundamental Theorem of Vector Analysis

Chapter 7

Basic Concepts of Differential Geometry

7.1 Manifolds

7.2 Differential Forms

7.2.1 Exterior Product

7.2.2 Exterior Derivative

7.3 Integration of Differential Forms

7.4 General Stokes's Theorem

7.5 Tensors in General Curvilinear Coordinate Systems

7.5.1 Covariant Derivative

Chapter 8

Applications

8.1 Mechanics

8.1.1 Inertia Tensor

8.1.2 Angular Momentum Tensor

8.2 Elasticity

8.2.1 Strain Tensor

8.2.2 Stress Tensor

8.3 Fluid Dynamics

8.3.1 Continuity Equation

8.3.2 Tensor of Momentum Flux Density

8.3.3 Euler's Equations

8.3.4 Rate of Deformation Tensor

8.3.5 Navier-Stokes Equations

8.4 Heat and Diffusion Equations

8.5 Electrodynamics

8.5.1 Tensor of Electromagnetic Field

8.5.2 Maxwell Equations

8.5.3 Scalar and Vector Potentials

8.5.4 Wave Equations

8.5.5 D'Alambert Operator

8.5.6 Energy-Momentum Tensor

8.6 Basic Concepts of Special and General Relativity

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Notation