Solution *Section* 4.1 – Antiderivatives, Substitution and General Power Rule

Exercise

Find indefinite integral $\int v^2 dv$

Solution

$$\int v^2 dv = \boxed{\frac{v^3}{3} + C}$$

Exercise

Find indefinite integral $\int x^{1/2} dx$

Solution

$$\int x^{1/2} dx = \boxed{\frac{2x^{3/2}}{3} + C}$$

Exercise

Find indefinite integral $\int e^{3t} dt$

Solution

$$\int e^{3t} dt = \boxed{\frac{1}{3}e^{3t} + C}$$

Exercise

Find indefinite integral $\int (6x^2 - 2e^x) dx$

Solution

$$\int (6x^2 - 2e^x) dx = \frac{6x^3}{3} - 2e^x + C$$

$$\boxed{= 2x^3 - 2e^x + C}$$

Exercise

Find indefinite integral $\int 4y^{-3}dy$

Solution

$$\int 4y^{-3}dy = 4\frac{y^{-2}}{-2} + C$$

$$\boxed{= -\frac{2}{y^2} + C}$$

Exercise

Find indefinite integral $\int (x^3 - 4x + 2)dx$

Solution

$$\int (x^3 - 4x + 2)dx = \frac{x^4}{4} - 4\frac{x^2}{2} + 2x + C$$

$$\boxed{= \frac{1}{4}x^4 - 2x^2 + 2x + C}$$

Exercise

Find indefinite integral $\int \left(3z^2 - 4z + 5\right)dz$

Solution

$$\int \left(3z^2 - 4z + 5\right)dz = 3\frac{z^3}{3} - 4\frac{z^2}{2} + 5z + C$$

$$\boxed{= z^3 - 2z^2 + 5z + C}$$

Exercise

Find indefinite integral $\int \left(x^2 - 1\right)^2 dx$

Solution

$$\int \left(x^2 - 1\right)^2 dx = \int \left(x^4 - 2x^2 + 1\right)dx \qquad (a-b)^2 = a^2 - 2ab + b^2$$

$$\boxed{= \frac{1}{5}x^5 - \frac{2}{3}x^3 + x + C}$$

Exercise

Find indefinite integral $\int \left(\sqrt[4]{x^3}+1\right)dx$

Solution

$$\int \left(\sqrt[4]{x^3}+1\right)dx = \int \left(x^{3/4}+1\right)dx$$

$$\boxed{=\frac{4}{7}x^{7/4}+x+C}$$

Exercise

Find indefinite integral $\int \sqrt{x}(x+1)dx$

Solution

$$\int \sqrt{x}(x+1)dx = \int x^{1/2}(x+1)dx$$

$$= \int \left(x^{3/2}+x^{1/2}\right)dx$$

$$\boxed{=\frac{2}{5}x^{5/2}+\frac{2}{3}x^{3/2}+C}$$

Exercise

Find the integral $\int (1+3t)t^2dt$

Solution

$$\int (1+3t)t^2dt = \int \left(t^2+3t^3\right)dt$$

$$\boxed{=\frac{1}{3}t^3+\frac{3}{4}t^4+C}$$

Exercise

Find indefinite integral $\int \frac{x^2-5}{x^2}\,dx$

Solution

$$\int \frac{x^2-5}{x^2}\,dx = \int \left(1-\frac{5}{x^2}\right)dx$$

$$= \int \left(1-5x^{-2}\right)dx$$

$$= x+5x^{-1}+C$$

$$\boxed{= x+\frac{5}{x}+C}$$

Exercise

Find indefinite integral $\int (-40x+250)dx$

Solution

$$\int (-40x+250)dx \boxed{= -20x^2+250x+C}$$

Exercise

Find the integral $\int \frac{x+2}{\sqrt{x}}dx$

Solution

$$\int \frac{x+2}{\sqrt{x}}dx = \int \left[\frac{x}{x^{1/2}}+\frac{2}{x^{1/2}}\right]dx$$

$$= \int \frac{x}{x^{1/2}}dx + \int \frac{2}{x^{1/2}}dx$$

$$= \int x^{1/2}dx + 2\int x^{-1/2}dx$$

$$= \frac{x^{3/2}}{3/2}+2\frac{x^{1/2}}{1/2}+C$$

$$\boxed{= \frac{2}{3}x^{3/2}+4x^{1/2}+C}$$

Exercise

Evaluate: $\int\left(\frac{2}{\sqrt[3]{x}}-6\sqrt{x}\right)dx$

Solution

$$\int\left(\frac{2}{\sqrt[3]{x}}-6\sqrt{x}\right)dx=\int\left(2x^{-1/3}-6x^{1/2}\right)dx$$

$$=2\frac{3}{2}x^{2/3}-6\frac{2}{3}x^{3/2}+C$$

$$=3x^{2/3}-4x^{3/2}+C$$

Exercise

Find the integral $\int(x^2-1)^3(2x)dx$

Solution

$$u=x^2-1\Rightarrow du=2xdx$$

$$\int(x^2-1)^3(2x)dx=\int u^3du$$

$$=\frac{1}{4}u^4+C$$

$$=\frac{1}{4}(x^2-1)^4+C$$

Exercise

Find the integral $\int\frac{6x}{(1+x^2)^3}dx$

Solution

$$u=1+x^2\Rightarrow du=2xdx\qquad\Rightarrow\frac{1}{2x}du=dx$$

$$\int\frac{6x}{u^3}\frac{1}{2x}du=\int 3\frac{1}{u^3}du$$

$$=3\int u^{-3}du$$

$$=3\frac{u^{-2}}{-2}+C$$

$$= -\frac{3}{2}\left(1+x^2\right)^{-2} + C$$

$$\boxed{= -\frac{3}{2}\frac{1}{\left(1+x^2\right)^2} + C}$$

Exercise

Find the integral $\int u^3\sqrt{u^4+2}\ du$

Solution

$$Let\ \ x = u^4 + 2 \ \Rightarrow dx = 4u^3 du$$

$$\Rightarrow \frac{1}{4u^3} dx = du$$

$$\int u^3\sqrt{u^4+2}\ du = \int u^3 x^{1/2}\ \frac{1}{4u^3} dx$$

$$= \frac{1}{4}\int x^{1/2}\ dx$$

$$= \frac{1}{4}\frac{2}{3}x^{3/2} + C$$

$$\boxed{= \frac{1}{6}\left(u^4+2\right)^{3/2} + C}$$

Exercise

Find the integral $\int \frac{t+2t^2}{\sqrt{t}}\ dt$

Solution

$$\int \frac{t+2t^2}{\sqrt{t}}\ dt = \int\left(\frac{t}{t^{1/2}} + 2\frac{t^2}{t^{1/2}}\right) dt$$

$$= \int \left(t^{1/2} + 2t^{3/2}\right)\ dt$$

$$= \frac{2}{3}t^{3/2} + 2\frac{2}{5}t^{5/2} + C$$

$$\boxed{= \frac{2}{3}t^{3/2} + \frac{4}{5}t^{5/2} + C}$$

Exercise

Find the integral $\displaystyle\int\left(1+\frac{1}{t}\right)^{3}\frac{1}{t^{2}}\ dt$

Solution

$$u=1+t^{-1}\Rightarrow du=-t^{-2}dt$$

$$\Rightarrow -t^{2}du=dt$$

$$\int\left(1+\frac{1}{t}\right)^{3}\frac{1}{t^{2}}\ dt=\int u^{3}\frac{1}{t^{2}}\ (-t^{2}du)$$

$$=-\int u^{3}du$$

$$=-\frac{1}{4}u^{4}+C$$

$$\boxed{=-\frac{1}{4}\left(1+\frac{1}{t}\right)^{4}+C}$$

Exercise

Find the integral $\displaystyle\int(7-3x-3x^{2})(2x+1)\ dx$

Solution

$$u=7-3x-3x^{2}\Rightarrow du=(-3-6x^{2})dx$$

$$\Rightarrow du=-3(2x^{2}+1)dx$$

$$\Rightarrow -\frac{1}{3}du=(2x^{2}+1)dx$$

$$\int(7-3x-3x^{2})(2x+1)\ dx=\int u(-\frac{1}{3})\ du$$

$$=-\frac{1}{3}\int u\ du$$

$$=-\frac{1}{6}u^{2}+C$$

$$\boxed{=-\frac{1}{6}\left(7-3x-3x^{2}\right)^{2}+C}$$

Exercise

Find the integral $\int \sqrt{x}\left(4-x^{3/2}\right)^2\, dx$

Solution

$$u = 4 - x^{3/2} \Rightarrow du = -\tfrac{3}{2}x^{1/2}dx$$

$$\Rightarrow -\tfrac{2}{3}du = \sqrt{x}dx$$

$$\int \sqrt{x}\left(4-x^{3/2}\right)^2\, dx = \int u^2\left(-\tfrac{2}{3}\right)du$$

$$= -\tfrac{2}{3}\int u^2 du$$

$$= -\tfrac{2}{9}u^3 + C$$

$$\boxed{= -\tfrac{2}{9}\left(4-x^{3/2}\right)^3 + C}$$

Exercise

Find the integral $\int \frac{1}{\sqrt{x}+\sqrt{x+1}}dx$

Solution

$$\int \frac{1}{\sqrt{x}+\sqrt{x+1}}dx = \int \frac{1}{\sqrt{x}+\sqrt{x+1}}\frac{\sqrt{x}-\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}}dx$$

$$= \int \frac{\sqrt{x}-\sqrt{x+1}}{x-(x+1)}dx$$

$$= \int \frac{\sqrt{x}-\sqrt{x+1}}{-1}dx$$

$$= -\int \left(x^{1/2}-(x+1)^{1/2}\right)dx$$

$$= -\left(\tfrac{2}{3}x^{3/2} - \tfrac{2}{3}(x+1)^{3/2}\right) + C$$

$$\boxed{= \tfrac{2}{3}(x+1)^{3/2} - \tfrac{2}{3}x^{3/2} + C}$$

Exercise

Find the integral $\int \sqrt{1-x}\ dx$

Solution

$$u = 1-x$$

$$du = -dx \ \Rightarrow \ -du = dx$$

$$\int \sqrt{1-x}\ dx = \int \sqrt{u}\ (-du)$$ ***Substitute for x and dx***

$$= -\int u^{1/2}\ du$$

$$= -\frac{u^{3/2}}{3/2} + C$$

$$\boxed{= -\frac{2}{3}(1-x)^{3/2} + C}$$

Exercise

Find the integral $\int x\sqrt{x^2+4}\ dx$

Solution

$$u = x^2+4 \ \Rightarrow du = 2xdx$$

$$xdx = \frac{1}{2}du$$

$$\int \sqrt{x^2+4}\ xdx = \int u^{1/2}\ \frac{1}{2}du$$

$$= \frac{1}{2}\frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{3}u^{3/2} + C$$

$$\boxed{= \frac{1}{3}(x^2+4)^{3/2} + C}$$

Exercise

Derive the position function if a ball is thrown upward with initial velocity of 32 ft per second from an initial height of 48 ft. When does the ball hit the ground? With what velocity does the ball hit the ground?

Solution

$$s(t) = -16t^2 + 32t + 48$$

$$s(0) = 48$$

$$s'(0) = 32$$

$$s''(t) = -32$$

$$s'(t) = \int -32\,dt$$

$$= -32t + C_1$$

$$s'(0) = -32(0) + C_1 = 32$$

$$\Rightarrow C_1 = 32$$

$$s'(t) = -32t + 32$$

$$s(t) = \int (-32t + 32)\,dt$$

$$= -32\frac{t^2}{2} + 32t + C_2$$

$$s(0) = -32\frac{0^2}{2} + 32(0) + C_2 = 48 \qquad \Rightarrow C_2 = 48$$

$$s(t) = -16t^2 + 32t + 48$$

$$s(t) = -16t^2 + 32t + 48 = 0$$

$$-t^2 + 2t + 3 = 0 \;\Rightarrow t = -1,\; t = 3$$

The ball hits the ground in 3 seconds

The velocity: $v(t) = s'(t) = -32t + 32$

$$v(t = 3) = -32(3) + 32 \underline{= -64 \ ft/\sec^2}$$

Exercise

Suppose a publishing company has found that the marginal cost at a level of production of x thousand books is given by

$$\frac{dC}{dx}=\frac{50}{\sqrt{x}}$$

And that the fixed cost (the cost before the first book can be produced) is a $25,000. Find the cost function $C(x)$.

Solution

$$\frac{dC}{dx}=\frac{50}{\sqrt{x}}=50x^{-1/2}$$

$$dC=50x^{-1/2}dx$$

$$\int dC=\int 50x^{-1/2}dx$$

$$C(x)=50\frac{x^{1/2}}{1/2}+C$$

$$=50(2)x^{1/2}+C$$

$$=100\sqrt{x}+C$$

$$25000=100\sqrt{0}+C \qquad \text{Before the first } (x=0) \text{ costs } 25{,}000$$

$$\boxed{25000=C}$$

$$C(x)=100\sqrt{x}+25{,}000$$

Exercise

If the marginal cost of producing x units of a commodity is given by

$$C'(x)=0.3x^2+2x$$

And the fixed cost is $2,000, find the cost function $C(x)$ and the cost of producing 20 units.

Solution

Given: $C(0)=2{,}000$

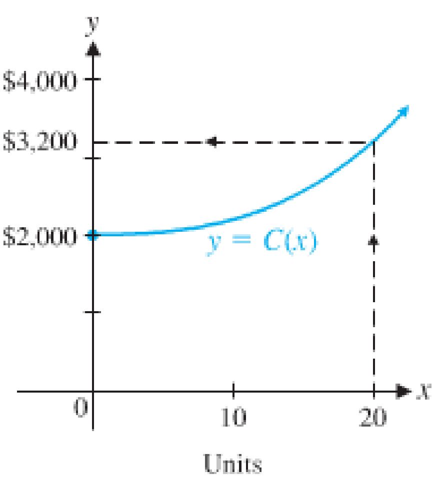
$$C(x)=\int\left(0.3x^2+2x\right)dx$$

$$=0.1x^3+x^2+K$$

$$C(0)=0.1(0)^3+(0)^2+K \quad \boxed{2{,}000=K}$$

$$C(x)=0.1x^3+x^2+2{,}000$$

$$C(20)=0.1(20)^3+(20)^2+2{,}000=\$3{,}200$$



Exercise

A satellite radio station is launching an aggressive advertising campaign in order to increase the number of daily listeners. The station currently has 27,000 daily listeners, and management expects the number of daily listeners, $S(t)$, to grow at the rate of

$$S'(t) = 60t^{1/2}$$

Listeners per day, where *t* is the number of days since the campaign began. How long should the campaign last if the station wants the number of daily listeners to grow to 41,000?

Solution

Given: $S(0) = 27,000$

$$S(t) = \int 60 \cdot t^{1/2} dt$$

$$= 40t^{3/2} + C$$

$$S(0) = 40(0)^{3/2} + C$$

$$\boxed{27,000 = C}$$

$$S(t) = 40t^{3/2} + 27,000$$

$$S(t) = 40t^{3/2} + 27,000 = 41,000$$

$$40t^{3/2} = 14,000$$

$$t^{3/2} = 350 \quad \to \quad \underline{t = (350)^{2/3} \approx 49.66}$$

The advertising campaign should last approximately 50 days.

Exercise

In 2007, U.S. consumption of renewable energy was 6.8 quadrillion Btu (or 6.8×10^{15} Btu). Since the 1960s, consumption has been growing at a rate (in quadrillion Btu's per year) given by

$$f'(t) = 0.004t + 0.062$$

Where *t* is years after 1960. Find $f(t)$ and estimate U.S. consumption of renewable energy in 2020.

Solution

From 1960 to 2007 $\Rightarrow$ 47 years

Given: $f(47) = 6.8$ quadrillion Btu

$$f(t) = \int (0.004t + 0.062) dt$$

$$= 0.002t^2 + 0.062t + C$$

$$6.8 = 0.002(47)^2 + 0.062(47) + C \quad \to \quad \underline{C = -0.532}$$

$$f(t) = 0.002t^2 + 0.062t - 0.532$$

In 2020, $t = 60$

$$f(60) = 0.002(60)^2 + 0.062(60) - 0.532 = 10.4 \text{ quadrillion Btu}$$

Exercise

The graph of the marginal revenue function from the sale of x sports watches is given in the figure.

a) Using the graph shown, describe the shape of the graph of the revenue function $R(x)$ as x increases from 0 to 1.000.

b) Find the equation of the marginal revenue function. (linear function)

c) Find the equation of the revenue function that satisfies $R(0) = 0$. Graph the revenue function over the interval [0, 1,000]. Check the shape of the graph relative to the analysis in part (*a*).

d) Find the price-demand equation and determine the price when the demand is 700 units.

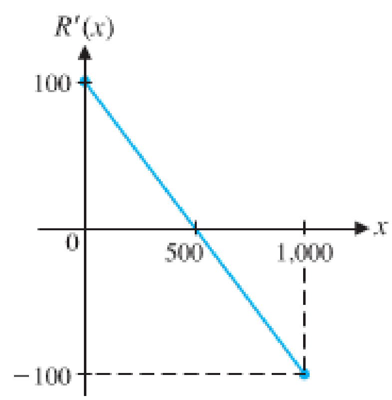
Solution

a) $R'(x) > 0$ *for* $0 < x < 500$ $\quad R(x)$ is increasing

$R'(x) < 0$ *for* $500 < x < 1,000$ $\quad R(x)$ is decreasing

Therefore, the graph of $R(x)$ has a local maximum at $x = 500$

b) The equation of $R(x)$ is $\dfrac{R'(x) - 100}{0 - 100} = \dfrac{x - 0}{500 - 0}$



$$R'(x) - 100 = (-100)\frac{x}{500}$$

$$R'(x) = -\frac{1}{5}x + 100$$

$$R'(x) = 100 - 0.2x$$

c) $R(x) = \int R'(x)\,dx = \int (100 - 0.2x)\,dx$

$$= 100x - 0.1x^2 + C$$

Given: $R(0) = 0$

$$R(0) = 100(0) - 0.1(0)^2 + C \quad \rightarrow \quad \boxed{C = 0}$$

$$R(x) = 100x - 0.1x^2$$

d) $p(x) = \dfrac{R(x)}{x} = 100 - 0.1x$

$$p(700) = 100 - 0.1(700) = 30$$

So the price is \$30 per sports watch when the demand is 700.

Exercise

The rate of change of the monthly sales of a newly released football game is given by

$$S'(t) = 500t^{1/4} \qquad S(0) = 0$$

Where t is the number of months since the game was released and $S(t)$ is the number of games sold each month. Find $S(t)$. When will monthly sales reach 20,000 games?

Solution

$$S(t) = \int S'(t)dt = \int 500t^{1/4}dt$$

$$= 500\left(\frac{4}{5}\right)t^{5/4} + C$$

$$= 400t^{5/4} + C$$

Given: $S(0) = 0$

$0 = 400(0)^{5/4} + C \quad \rightarrow \quad \boxed{C = 0}$

$S(t) = 400t^{5/4}$

$S(t) = 400t^{5/4} = 20,000$

$t^{5/4} = 50$

$t = 50^{4/5} \approx \underline{23 \ \ months}$

Exercise

If the rate of labor is given by: $\quad g(x) = 2{,}000x^{-1/3}$

And if the first 8 control units require 12,000 labor-hours, how many labor-hours, $L(x)$, will be required for the first x control units? The first 27 control units?

Solution

$$L(x) = \int g(x)dx = \int 2{,}000x^{-1/3}dx$$

$$= 2{,}000\left(\frac{3}{2}\right)x^{2/3} + C$$

$$= 3{,}000x^{2/3} + C$$

Given: $L(8) = 12{,}000$

$L(8) = 3{,}000(8)^{2/3} + C$

$C = 12{,}000 - 3{,}000(8)^{2/3} \ \underline{= 0}$

$L(x) = 3{,}000x^{2/3}$

$L(27) = 3{,}000(27)^{2/3} \ \underline{= 27{,}000}$ ***labor hours***

Exercise

The area A of a healing wound changes at a rate given approximately by

$$\frac{dA}{dt} = -4t^{-3} \qquad 1 \le t \le 10$$

Where t is time in days and $A(1) = 2\ cm^2$. What will the area of the wound be in 10 days?

Solution

$$A = \int -4t^{-3}dt$$

$$= -4\frac{t^{-2}}{-2} + C$$

$$= 2t^{-2} + C$$

$Given$: $A(1) = 2$

$$2 = 2(1)^{-2} + C \quad \rightarrow \quad 2 = 2 + C \quad \boxed{C = 0}$$

$$A(t) = 2t^{-2}$$

$$A(10) = 2(10)^{-2} = 0.02\ \ cm^2$$

Exercise

The marginal revenue (in thousands of dollars) from the sale of x gadgets is given by the following function $R'(x) = 4x\left(x^2 + 27,000\right)^{-2/3}$

a) Find the total revenue function if the revenue from 115 gadgets is \$55,581.

b) How many gadgets must be sold for a revenue of at least \$50,000.

Solution

a) $R(x) = \int R'(x)\,dx$

$$= \int 4x\left(x^2 + 27,000\right)^{-2/3} dx \qquad u = x^2 + 27,000 \quad du = 2xdx \rightarrow 4xdx = 2du$$

$$= \int u^{-2/3}(2du)$$

$$= 2\left(3u^{1/3}\right) + C$$

$$= 6\left(x^2 + 27,000\right)^{1/3} + C$$

$$R(x = 115) = 55.581$$

$$6\left(115^2+27,000\right)^{1/3}+C=55.581$$

$$C=55.581-6\left(115^2+27,000\right)^{1/3}$$

$$\boxed{C\approx -150}$$

$$R(x)=6\left(x^2+27,000\right)^{1/3}-150$$

b) $R(x)=6\left(x^2+27,000\right)^{1/3}-150=50$

$$6\left(x^2+27,000\right)^{1/3}=200$$

$$\left(x^2+27,000\right)^{1/3}=\frac{200}{6}$$

$$x^2+27,000=\left(\frac{200}{6}\right)^3$$

$$x^2=\left(\frac{200}{6}\right)^3-27,000=10037$$

$$x=\sqrt{\left(\frac{200}{6}\right)^3-27,000}\approx 100.2$$

101 gadgets must be sold to generate a revenue of at least \$50,000.

Solution Section 4.2 – Exponential and Logarithmic Integrals

Exercise

Find the integral $\int (2x+1)e^{x^2+x}dx$

Solution

$$u = x^2 + x \Rightarrow du = (2x+1)dx$$

$$\begin{aligned}\int (2x+1)e^{x^2+x}dx &= \int e^u du \\ &= e^u + C \\ &= e^{x^2+x} + C\end{aligned}$$

Exercise

Find the integral $\int \frac{1}{6x-5}dx$

Solution

$$\begin{aligned}u = 6x-5 &\Rightarrow du = 6dx \\ &\Rightarrow \tfrac{1}{6}du = dx\end{aligned}$$

$$\begin{aligned}\int \frac{1}{6x-5}dx &= \int \frac{1}{u}\frac{1}{6}du \\ &= \frac{1}{6}\int \frac{1}{u}du \\ &= \frac{1}{6}\ln|u| + C \\ &= \frac{1}{6}\ln|6x-5| + C\end{aligned}$$

Exercise

Find the integral $\displaystyle\int \frac{x^2+2x+3}{x^3+3x^2+9x+1}dx$

Solution

$$u = x^3+3x^2+9x+1 \quad \Rightarrow du = (3x^2+6x+9)dx$$

$$\Rightarrow du = 3(x^2+2x+3)dx$$

$$\Rightarrow \tfrac{1}{3}du = (x^2+2x+3)dx$$

$$\int \frac{x^2+2x+3}{x^3+3x^2+9x+1}dx = \int \frac{1}{u}\frac{1}{3}du$$

$$= \frac{1}{3}\int \frac{1}{u}du$$

$$= \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ln\left|x^3+3x^2+9x+1\right| + C$$

Exercise

Find the integral $\displaystyle\int \frac{1}{x(\ln x)^2}dx$

Solution

$$u = \ln x \Rightarrow du = \frac{1}{x}dx$$

$$\Rightarrow xdu = dx$$

$$\int \frac{1}{x(\ln x)^2}dx = \int \frac{1}{xu^2}xdu$$

$$= \int \frac{1}{u^2}du$$

$$= \int u^{-2}du$$

$$= \frac{u^{-1}}{-1} + C = -\frac{1}{u} + C$$

$$= -\frac{1}{\ln x} + C$$

Exercise

Find the integral $\int \frac{e^x}{1+e^x} dx$

Solution

$$u = 1 + e^x \Rightarrow du = e^x dx$$

$$\int \frac{1}{u} du = \ln|u| + C$$

$$= \ln(1 + e^x) + C$$

Exercise

Find the integral $\int \frac{1}{x^3} e^{1/4x^2} dx$

Solution

$$u = \frac{1}{4x^2} = \frac{1}{4} x^{-2} \Rightarrow du = \frac{1}{4}(-2x^{-3})dx$$

$$\Rightarrow du = -\frac{1}{2} x^{-3} dx$$

$$\Rightarrow -2du = \frac{1}{x^3} dx$$

$$\int e^u (-2) du = -2 \int e^u du$$

$$= -2e^u + C$$

$$= -2e^{1/4x^2} + C$$

Exercise

Find the integral $\int \frac{e^{1/\sqrt{x}}}{x^{3/2}} dx$

Solution

$$u = \frac{1}{\sqrt{x}} = x^{-1/2} \Rightarrow du = -\frac{1}{2} x^{-3/2} dx$$

$$\Rightarrow -2du = \frac{1}{x^{3/2}} dx$$

$$\int \frac{e^{1/\sqrt{x}}}{x^{3/2}}dx = \int e^{u}(-2du)$$

$$= -2\int e^{u}du$$

$$= -2e^{u} + C$$

$$= -2e^{1/\sqrt{x}} + C$$

Exercise

Find the integral $\displaystyle\int \frac{-e^{3x}}{2-e^{3x}}dx$

Solution

$$u = 2 - e^{3x} \Rightarrow du = -3e^{3x}dx$$

$$\Rightarrow \frac{du}{-3e^{3x}} = dx$$

$$\int \frac{-e^{3x}}{2-e^{3x}}dx = \int \frac{-e^{3x}}{u}\frac{du}{-3e^{3x}}$$

$$= \frac{1}{3}\int \frac{1}{u}du$$

$$= \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ln\left|2 - e^{3x}\right| + C$$

Exercise

Find the integral $\displaystyle\int (6x + e^{x})\sqrt{3x^{2} + e^{x}}\,dx$

Solution

$$u = 3x^{2} + e^{x} \Rightarrow du = (6x + e^{x})dx$$

$$\Rightarrow \frac{du}{6x+e^{x}} = dx$$

$$\int (6x+e^x)\sqrt{u}\,\frac{du}{6x+e^x} = \int u^{1/2}du$$

$$= \frac{u^{3/2}}{3/2} + C$$

$$= \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{3}(3x^2 + e^x)^{3/2} + C$$

Exercise

Find the integral $\int \frac{2(e^x - e^{-x})}{(e^x + e^{-x})^2}dx$

Solution

$$u = e^x + e^{-x} \Rightarrow du = (e^x - e^{-x})dx$$

$$\int \frac{2(e^x - e^{-x})}{(e^x + e^{-x})^2}dx = 2\int \frac{1}{u^2}du$$

$$= 2\int u^{-2}du$$

$$= 2\frac{u^{-1}}{-1} + C$$

$$= -2\frac{1}{u} + C$$

$$= -\frac{2}{e^x + e^{-x}} + C$$

Exercise

Find the integral $\int \frac{x-3}{x+3}dx$

Solution

$$\int \frac{x-3}{x+3}dx = \int \left(1 - \frac{6}{x+3}\right)dx$$

$$= x - 6\ln|x+3| + C$$

Exercise

Find the integral $\int \frac{5}{e^{-5x}+7}dx$

Solution

$$\int \frac{5}{e^{-5x}+7}\frac{e^{5x}}{e^{5x}}dx = \int \frac{5e^{5x}}{1+7e^{5x}}dx$$

$$u = 1+7e^{5x} \Rightarrow du = 35e^{5x}dx$$

$$\Rightarrow \frac{du}{35e^{5x}} = dx$$

$$\int \frac{5e^{5x}}{1+7e^{5x}}dx = \int \frac{5e^{5x}}{u}\frac{du}{35e^{5x}}$$

$$= \frac{1}{7}\int \frac{1}{u}du$$

$$= \frac{1}{7}\ln|u| + C$$

$$= \frac{1}{7}\ln\left|1+7e^{5x}\right| + C$$

Exercise

Find the integral $\int \frac{4x^2-3x+2}{x^2}dx$

Solution

$$\int \frac{4x^2-3x+2}{x^2}dx = \int \left(\frac{4x^2}{x^2} - \frac{3x}{x^2} + \frac{2}{x^2}\right)dx$$

$$= \int \left(4 - \frac{3}{x} + 2x^{-2}\right)dx$$

$$= 4x - 3\ln|x| - 2x^{-1} + C$$

$$= 4x - 3\ln|x| - \frac{2}{x} + C$$

Exercise

Find the integral $\int \frac{2}{e^{-x}+1}dx$

Solution

$$\int \frac{2}{e^{-x}+1}dx = \int \frac{2}{e^{-x}+1}\frac{e^x}{e^x}dx$$

$$= 2\int \frac{e^x}{1+e^x}dx$$

$$= 2\int \frac{d\left(e^x+1\right)}{1+e^x}$$

$$= 2\ln\left(e^x+1\right)+C$$

Exercise

Find the integral $\int \frac{4x^2+2x+4}{x+1}dx$

Solution

$$\int \frac{4x^2+2x+4}{x+1}dx = \int \left(4x+2+\frac{6}{x+1}\right)dx$$

$$= \int (4x-2)dx + \int \frac{6}{x+1}dx$$

$$= \int (4x-2)dx + 6\int \frac{d(x+1)}{x+1} \qquad \int \frac{d(U)}{U} = \ln|U|$$

$$= 2x^2-2x+6\ln|x+1|+C$$

Exercise

Find the indefinite integral. $\int 4xe^{x^2}\,dx$

Solution

Let $u = x^2 \rightarrow du = 2xdx$

$$\int 4xe^{x^2}\,dx = \int 2e^u(2xdx)$$

$$= \int 2e^u du$$

$$= 2e^u + C$$

$$= 2e^{x^2} + C$$

Exercise

Find the indefinite integral. $\int \frac{3x}{x^2+4}\,dx$

Solution

Let $u = x^2 + 4 \rightarrow du = 2xdx \rightarrow \frac{1}{2}du = xdx$

$$\int \frac{3x}{x^2+4}\,dx = \int \frac{3}{u}\frac{1}{2}du$$

$$= \frac{3}{2}\int \frac{1}{u}du$$

$$= \frac{3}{2}\ln|u| + C$$

$$= \frac{3}{2}\ln(x^2+4) + C$$

Exercise

Evaluate the integral $\int 12t^3 e^{-t^4}\,dt$

Solution

$$u = -t^4 \quad \rightarrow \quad du = -4t^3 dt \Rightarrow \quad -\frac{du}{4} = t^3 dt$$

$$\int 12t^3 e^{-t^4}\,dt = \int 12e^u\left(-\frac{du}{4}\right)$$

$$= -3\int e^u\,du$$

$$= -3e^u + C$$

$$= -3e^{-t^4} + C$$

$$\boxed{= -\frac{3}{e^{t^4}} + C}$$

Exercise

Evaluate the integral $\int \frac{7e^{7x}}{3+e^{7x}}dx$

Solution

Let: $u = 3 + e^{7x} \quad \rightarrow \quad du = 7e^{7x}dx$

$$\int \frac{7e^{7x}}{3+e^{7x}}dx = \int \frac{du}{u}$$

$$= \ln|u|$$

$$\boxed{= \ln\left(3 + e^{7x}\right) + C}$$

Exercise

Under certain conditions, the number of diseased cells $N(t)$ at time t increases at a rate $N'(t) = Ae^{kt}$, where A is the rate of increase at time 0 (in cells per day) and k is a constant.

a) Suppose $A = 60$, and at 4 days, the cells are growing at a rate of 180 per day. Find a formula for the number of cells after t days, given that 200 cells are present at $t = 0$.

b) Use the answer from part (a) to find the number of cells present after 9 days.

Solution

a) $N'(t) = Ae^{kt}$

$$180 = 60e^{k(4)}$$

$$3 = e^{4k}$$

$$4k = \ln 3$$

$$k = \frac{\ln 3}{4} \approx 0.27465$$

$$N'(t) = 60e^{0.27465t}$$

$$N(t) = \int N'(t)\,dt$$

$$= \int 60e^{0.27465t}\,dt$$

$$= 218.5e^{0.27465t} + C$$

$$N(t=0) = 200$$

$$218.5e^{0.27465(0)} + C = 200$$

$$218.5 + C = 200$$

$$C = 200 - 218.5 = -18.5$$

$$N(t) = 218.5e^{0.27465t} - 18.5$$

b) $N(t=9) = 218.5e^{0.27465(9)} - 18.5 = 2,569 \text{ cells}$

Solution Section 4.3 – Integration by Parts

Exercise

Find the integral $\int \ln x^2 dx$

Solution

$$\int \ln x^2 dx = 2\int \ln x dx$$

$$u = \ln x \Rightarrow du = \frac{1}{x}dx$$

$$dv = dx \Rightarrow v = x$$

$$\int \ln x^2 dx = 2\left[x\ln x - \int x\frac{1}{x}dx\right]$$

$$= 2\left[x\ln x - \int dx\right]$$

$$= 2(x\ln x - x) + C$$

$$= 2x(\ln x - 1) + C$$

Exercise

Find the integral $\int \frac{2x}{e^x}dx$

Solution

$$u = 2x \Rightarrow du = 2dx$$

$$dv = e^{-x}dx \Rightarrow v = -e^{-x}$$

$$\int \frac{2x}{e^x}dx = 2x(-e^{-x}) - \int -e^{-x}2dx$$

$$= -2xe^{-x} + 2\int e^{-x}dx$$

$$= -2xe^{-x} - 2e^{-x} + C$$

$$= -2e^{-x}(x+1) + C$$

$$= -\frac{2(x+1)}{e^x} + C$$

Exercise

Find the integral $\displaystyle\int \ln(3x)dx$

Solution

$$u = \ln 3x \Rightarrow du = \frac{3}{3x}dx = \frac{1}{x}dx$$

$$dv = dx \Rightarrow v = x$$

$$\int \ln(3x)dx = x\ln(3x) - \int x\frac{1}{x}dx$$

$$= x\ln(3x) - \int dx$$

$$= x\ln(3x) - x + C$$

$$= x\left[\ln(3x) - 1\right] + C$$

Exercise

Find the integral $\displaystyle\int \frac{1}{x\ln x}dx$

Solution

$$\int \frac{1}{x\ln x}dx = \int \frac{1}{\ln x}\frac{1}{x}dx$$

$$u = \ln x \Rightarrow du = \frac{1}{x}dx$$

$$\int \frac{1}{x\ln x}dx = \int \frac{1}{u}du$$

$$= \ln u + C$$

$$= \ln\left|\ln x\right| + C$$

Exercise

Find the integral $\displaystyle\int \frac{x}{\sqrt{x-1}}\,dx$

Solution

Let: $u = x \Rightarrow du = dx$

$$dv = \frac{dx}{\sqrt{x-1}} \Rightarrow v = \int (x-1)^{-1/2}\, d(x-1)$$

$$= \frac{(x-1)^{1/2}}{1/2}$$

$$= 2(x-1)^{1/2}$$

$$\int \frac{x}{\sqrt{x-1}}\,dx = 2x\sqrt{x-1} - 2\int (x-1)^{1/2}\,dx$$

$$= 2x\sqrt{x-1} - 2\frac{(x-1)^{3/2}}{3/2} + C$$

$$= 2x\sqrt{x-1} - \frac{4}{3}(x-1)\sqrt{x-1} + C$$

$$= \sqrt{x-1}\left[2x - \frac{4}{3}x + \frac{4}{3}\right] + C$$

$$= \sqrt{x-1}\left[\frac{6x-4x+4}{3}\right] + C$$

$$= \sqrt{x-1}\left[\frac{2x+4}{3}\right] + C$$

$$= \frac{2}{3}\sqrt{x-1}(x+2) + C$$

Exercise

Find the integral $\displaystyle\int \frac{x^3 e^{x^2}}{\left(x^2+1\right)^2}\,dx$

Solution

Let: $u = x^2 e^{x^2} \Rightarrow du = \left(2xe^{x^2} + 2xx^2e^{x^2}\right)dx$

$$du = 2xe^{x^2}\left(1+x^2\right)dx$$

$$dv = x\left(x^2+1\right)^{-2}dx \qquad \Rightarrow v = \int x(x^2+1)^{-2}dx$$

$$= \frac{1}{2}\int (x^2+1)^{-2}d(x^2+1)$$

$$= \frac{(x^2+1)^{-1}}{-1}$$

$$= -\frac{1}{2(x^2+1)}$$

$$\int \frac{x^3e^{x^2}}{(x^2+1)^2}dx = x^2e^{x^2}\left(-\frac{1}{2(x^2+1)}\right) - \int -\frac{1}{2(x^2+1)}2xe^{x^2}(x^2+1)dx$$

$$= -\frac{x^2e^{x^2}}{2(x^2+1)} + \int xe^{x^2}dx$$

Let: $u = x^2 \;\Rightarrow du = 2xdx$

$$= -\frac{x^2e^{x^2}}{2(x^2+1)} + \frac{1}{2}\int e^u du$$

$$= -\frac{x^2e^{x^2}}{2(x^2+1)} + \frac{1}{2}e^u + C$$

$$= -\frac{x^2e^{x^2}}{2(x^2+1)} + \frac{1}{2}e^{x^2} + C$$

$$= \frac{1}{2}e^{x^2}\left[-\frac{x^2}{(x^2+1)} + 1\right] + C$$

$$= \frac{1}{2}e^{x^2}\left[\frac{-x^2+x^2+1}{(x^2+1)}\right] + C$$

$$= \frac{e^{x^2}}{2(x^2+1)} + C$$

Exercise

Find the integral $\int x^2 e^{-3x} dx$

Solution

$$u = x^2 \Rightarrow du = 2x dx$$

$$dv = e^{-3x} dx \Rightarrow v = -\frac{1}{3} e^{-3x}$$

$$\int x^2 e^{-3x} dx = -\frac{1}{3} x^2 e^{-3x} + \frac{2}{3} \int x e^{-3x} dx$$

$$u = x \Rightarrow du = dx$$

$$dv = e^{-3x} dx \Rightarrow v = -\frac{1}{3} e^{-3x}$$

$$\int x^2 e^{-3x} dx = -\frac{1}{3} x^2 e^{-3x} + \frac{2}{3} \left[-\frac{1}{3} x e^{-3x} + \frac{1}{3} \int e^{-3x} dx \right]$$

$$= -\frac{1}{3} x^2 e^{-3x} + \frac{2}{3} \left[-\frac{1}{3} x e^{-3x} - \frac{1}{9} e^{-3x} \right] + C$$

$$= -\frac{1}{3} x^2 e^{-3x} - \frac{2}{9} x e^{-3x} - \frac{2}{27} e^{-3x} + C$$

$$= -\frac{9x^2 + 6x + 2}{27} e^{-3x} + C$$

Solution Section 4.4 – Fundamental Theorem of Calculus

Exercise

Find the integral $\int_0^3 (2x+1)dx$

Solution

$$\int_0^3 (2x+1)dx = x^2 + x\Big|_0^3$$

$$= 3^2 + 3 - (0+0)$$

$$= 12$$

Exercise

Find the integral $\int_{-1}^4 |x-2|dx$

Solution

$$\int_{-1}^4 |x-2|dx = \int_{-1}^2 -(x-2)dx + \int_2^4 (x-2)dx$$

$$= -\tfrac{1}{2}x^2 + 2x\Big|_{-1}^2 + \left[\tfrac{1}{2}x^2 - 2x\right]_2^4$$

$$= -\tfrac{1}{2}2^2 + 2(2) - \left(-\tfrac{1}{2}(-1)^2 + 2(-1)\right) + \tfrac{1}{2}4^2 - 2(4) - \left(\tfrac{1}{2}(2)^2 - 2(2)\right)$$

$$= \frac{13}{2}$$

Exercise

Find the integral $\int_0^2 \sqrt{4-x^2}\,dx$

Solution

$$\int_0^2 \sqrt{4-x^2}\,dx = \tfrac{1}{2}x\sqrt{4-x^2} + \tfrac{4}{2}\sin^{-1}\tfrac{x}{2}$$

$\sqrt{4-x^2}$ is a semi-circle with center (0, 0) and radius = 2

Since x from 0 to 2

$$\Rightarrow \text{Area} = \frac{1}{4}(\text{Area of this circle}) = \frac{1}{4}2\pi 2^2 = 2\pi$$

Exercise

Evaluate $\int_0^1 x^2 e^x dx$

Solution

$$\int_0^1 x^2 e^x dx = x^2 e^x - 2\int xe^x dx$$

$$= x^2 e^x \Big|_0^1 - 2\left[xe^x \Big|_0^1 - \int_0^1 e^x dx \right]$$

$$= \left[x^2 e^x - 2\left(xe^x - e^x \right) \right]_0^1$$

$$= x^2 e^x - 2xe^x + 2e^x \Big|_0^1$$

$$= e^1 - 2e^1 + 2e^1 - (0 - 0 + 2e^0)$$

$$= e - 2$$

Exercise

Evaluate the integrals $\int_0^2 x(x-3)\,dx$

Solution

$$\int_0^2 x(x-3)\,dx = \int_0^2 \left(x^2 - 3x\right)dx$$

$$= \left[\frac{x^3}{3} - \frac{3x^2}{2} \right]_0^2$$

$$= \left(\frac{2^3}{3} - \frac{3(2)^2}{2} \right) - \left(\frac{0^3}{3} - \frac{3(2)^2}{2} \right)$$

$$= -\frac{10}{3}$$

Exercise

Evaluate the integrals $\int_0^4 \left(3x - \frac{x^3}{4}\right) dx$

Solution

$$\int_0^4 \left(3x - \frac{x^3}{4}\right) dx = \left[3\frac{x^2}{2} - \frac{x^4}{16}\right]_0^4$$

$$= \left(3\frac{(4)^2}{2} - \frac{(4)^4}{16}\right) - 0$$

$$\underline{= 8}$$

Exercise

Evaluate the integrals $\int_{-2}^2 \left(x^3 - 2x + 3\right) dx$

Solution

$$\int_{-2}^2 \left(x^3 - 2x + 3\right) dx = \left[\frac{x^4}{4} - x^2 + 3x\right]_{-2}^2$$

$$= \left(\frac{(2)^4}{4} - (2)^2 + 3(2)\right) - \left(\frac{(-2)^4}{4} - (-2)^2 + 3(-2)\right)$$

$$\underline{= 12}$$

Exercise

Evaluate the integrals $\int_0^1 \left(x^2 + \sqrt{x}\right) dx$

Solution

$$\int_0^1 \left(x^2 + \sqrt{x}\right) dx = \left[\frac{x^3}{3} + \frac{2}{3}x^{3/2}\right]_0^1$$

$$= \left(\frac{(1)^3}{3} + \frac{2}{3}(1)^{3/2}\right) - 0$$

$$\underline{= 1}$$

Exercise

Evaluate the integrals $\displaystyle\int_{-3}^{-1} \frac{y^5 - 2y}{y^3}\, dy$

Solution

$$\int_{-3}^{-1} \frac{y^5 - 2y}{y^3}\, dy = \int_{-3}^{-1} \left(\frac{y^5}{y^3} - \frac{2y}{y^3} \right) dy$$

$$= \int_{-3}^{-1} \left(y^2 - 2y^{-2} \right) dy$$

$$= \left[\frac{1}{3} y^3 + 2y^{-1} \right]_{-3}^{-1}$$

$$= \left(\frac{1}{3}(-1)^3 + \frac{2}{-1} \right) - \left(\frac{1}{3}(-3)^3 + \frac{2}{-3} \right)$$

$$= \frac{22}{3}$$

Exercise

Evaluate the integrals $\displaystyle\int_{1}^{8} \frac{\left(x^{1/3} + 1\right)\left(2 - x^{2/3}\right)}{x^{1/3}}\, dx$

Solution

$$\int_{1}^{8} \frac{\left(x^{1/3} + 1\right)\left(2 - x^{2/3}\right)}{x^{1/3}}\, dx = \int_{1}^{8} \frac{2x^{1/3} - x + 2 - x^{2/3}}{x^{1/3}}\, dx$$

$$= \int_{1}^{8} \left(2 - x^{2/3} + 2x^{-1/3} - x^{1/3} \right) dx$$

$$= \left[2x - \frac{3}{5} x^{5/3} + 3x^{2/3} - \frac{3}{4} x^{4/3} \right]_{1}^{8}$$

$$= \left(2(8) - \frac{3}{5}(8)^{5/3} + 3(8)^{2/3} - \frac{3}{4}(8)^{4/3} \right) - \left(2(1) - \frac{3}{5}(1)^{5/3} + 3(1)^{2/3} - \frac{3}{4}(1)^{4/3} \right)$$

$$= \left(-\frac{16}{5} \right) - \left(\frac{73}{20} \right)$$

$$= -\frac{137}{20}$$

Exercise

Evaluate the integral $\displaystyle\int_0^3 \sqrt{y+1}\, dy$

Solution

$u = y+1 \Rightarrow du = dy$

$$\int_0^3 \sqrt{y+1}\, dy = \int_0^3 u^{1/2}\, du$$

$$= \frac{2}{3}u^{3/2}\Big|_0^3$$

$$= \frac{2}{3}(y+1)^{3/2}\Big|_0^3$$

$$= \frac{2}{3}\left[(3+1)^{3/2} - (0+1)^{3/2}\right]$$

$$= \frac{2}{3}[8-1]$$

$$= \frac{14}{3}$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^{1} r\sqrt{1-r^2}\, dr$

Solution

Let $u = 1-r^2 \quad \Rightarrow \quad du = -2rdr \quad \rightarrow \quad -\frac{1}{2}du = rdr$

$$\int_{-1}^{1} r\sqrt{1-r^2}\, dr = \int_{-1}^{1} u^{1/2}\left(-\tfrac{1}{2}du\right)$$

$$= -\frac{1}{2}\frac{2}{3}u^{3/2}\Big|_{-1}^{1}$$

$$= -\frac{1}{3}\left[\left(1-r^2\right)^{3/2}\right]_{-1}^{1}$$

$$= -\frac{1}{3}\left[\left(1-(1)^2\right)^{3/2} - \left(1-(-1)^2\right)^{3/2}\right]$$

$$= -\frac{1}{3}[0-0]$$

$$= 0$$

Exercise

Evaluate the integral $\int_0^1 t^3\left(1+t^4\right)^3\, dt$

Solution

Let $u = 1+t^4 \quad \Rightarrow \quad du = 4t^3dt \quad \rightarrow \quad \frac{1}{4}du = t^3dt \quad \begin{cases} t=1 & \rightarrow u=2 \\ t=0 & \rightarrow u=1 \end{cases}$

$$\int_0^1 t^3\left(1+t^4\right)^3\, dt = \int_1^2 \frac{1}{4}u^3du$$

$$= \frac{1}{4}\left(\frac{u^4}{4}\right)_1^2$$

$$= \frac{1}{16}\left(u^4\right)_1^2$$

$$= \frac{1}{16}\left(2^4 - 1^4\right)$$

$$= \frac{15}{16}$$

Exercise

A company manufactures x HDTVs per month. The monthly marginal profit (in dollars) is given by

$$P'(x) = 165 - 0.1x \qquad 0 \le x \le 4{,}000$$

The company is currently manufacturing 1,500 HDTVs per month, but is planning to increase production. Find the change in the monthly profit if monthly production is increased to 1,600 HDTVs.

Solution

$$P = \int_{1,500}^{1,600} (165 - 0.1x)\,dx$$

$$= \left[165x - 0.05x^2\right]_{1,500}^{1,600}$$

$$= \left(165(1{,}600) - 0.05(1{,}600)^2\right) - \left(165(1{,}500) - 0.05(1{,}500)^2\right)$$

$$= 136{,}000 - 135{,}000$$

$$= 1{,}000$$

Increasing monthly production from 1,500 units to 1,600 units will increase the monthly profit by \$1,000.

Exercise

An amusement company maintains records for each video game installed in an arcade. Suppose that $C(t)$ and $R(t)$ represent the total accumulated costs and revenues (in thousands of dollars), respectively, t years after a particular game has been installed. Suppose also that

$$C'(t) = 2 \quad R'(t) = 9e^{-0.5t}$$

The value of t for which $C'(t) = R'(t)$ is called the ***useful life*** of the game.

a) Find the useful life of the game, to the nearest year.
b) Find the total profit accumulated during the useful life of the game.

Solution

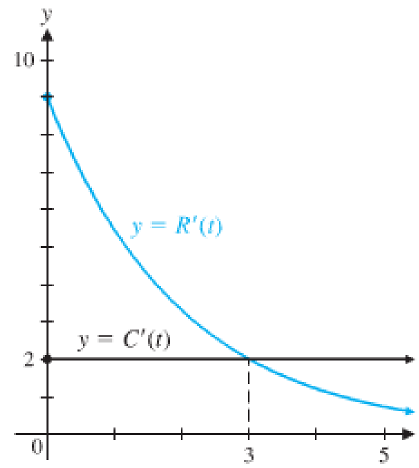
a) $R'(t) = C'(t)$

$$9e^{-0.5t} = 2$$

$$e^{-0.5t} = \frac{2}{9}$$

$$-0.5t = \ln\left(\frac{2}{9}\right) \quad \rightarrow \quad \underline{\left| t = \frac{-1}{0.5}\ln\left(\frac{2}{9}\right) \approx 3\,years \right|}$$

The game has a useful life of 3 years.



b) The total profit during the useful life of the game is

$$P(t) = \int_0^3 P'(t)\,dt$$

$$= \int_0^3 \left(R'(t) - C'(t)\right)dt$$

$$= \int_0^3 \left(9e^{-0.5t} - 2\right)dt$$

$$= \left[\frac{9}{-.5}e^{-0.5t} - 2t\right]_0^3$$

$$= \left(-18e^{-0.5(3)} - 2(3)\right) - \left(-18e^{-0.5(0)} - 2(0)\right)$$

$$= -18e^{-1.5} - 6 + 18$$

$$= 12 - 18e^{-1.5}$$

$$\approx 7.984$$

$$\underline{\left| = \$7,984 \right|}$$

Exercise

The total cost (in dollars) of printing x dictionaries is $C(x)=20,000+10x$

a) Find the average cost per unit if 1,000 dictionaries are produced.

b) Find the average value of the cost function over the interval [0, 1,000}

c) Discuss the difference between parts (***a***) and (***b***)

Solution

a) Average cost per unit: $\overline{C}(x)=\dfrac{C(x)}{x}$

$$\overline{C}(x)=\frac{20,000+10x}{x}$$

$$\overline{C}(1,000)=\frac{20,000+10(1,000)}{1,000} \quad \underline{=\$30}$$

b) Average $C(x)=\dfrac{1}{1,000}\displaystyle\int_0^{1,000}(20,000+10x)\,dx$

$$=\frac{1}{1,000}\left[20,000x+5x^2\right]_0^{1,000}$$

$$\underline{=\$25,000}$$

c) $\overline{C}(1,000)$ is the average cost per unit at a production level of 1,000 units.

Ave $C(x)$ is the average value of the total coast as production increases from 0 unit to 1,000 units.

Exercise

If the rate of labor is $g(x)=2,000\,x^{-1/3}$, then approximately how many labor–hours will be required to assemble the 9th through the 27th.

Solution

The number labor–hours to assemble the 9th through the 27$^{\text{th}}$ control units is

$$\int_8^{27} g(x)\,dx=\int_8^{27} 2,000\,x^{-1/3}\,dx$$

$$=2,000\left[\frac{3}{2}\,x^{2/3}\right]_8^{27}$$

$$=3,000\left(27^{2/3}-8^{2/3}\right)$$

$$=3,000(9-4)$$

$$\underline{=15,000\ \ \textit{labor hrs}}$$

Solution   *Section* 4.5 – The Area between Two Curves

Exercise

Find the area of the region bounded by the graphs of $y = x^2 - x - 2$ and x-axis

Solution

Determine the intersection points:

$$x^2 - x - 2 = 0 \Rightarrow \boxed{x = -1, 2}$$

$$A = \int_{-1}^{2} [0 - (x^2 - x - 2)]dx$$

$$= -\frac{x^3}{3} + \frac{x^2}{2} + 2x \Big|_{-1}^{2}$$

$$= -\frac{2^3}{3} + \frac{2^2}{2} + 2(2) - \left[-\frac{(-1)^3}{3} + \frac{(-1)^2}{2} + 2(-1) \right]$$

$$= -\frac{8}{3} + 2 + 4 - \left[\frac{1}{3} + \frac{1}{2} - 2 \right]$$

$$= 4.5 \ unit^2$$

Exercise

Find the area of the region bounded by the graphs of $f(x) = x^3 + 2x^2 - 3x$ and $g(x) = x^2 + 3x$

Solution

$$x^3 + 2x^2 - 3x = x^2 + 3x$$

$$x^3 + x^2 - 6x = 0$$

$$x(x^2 + x - 6) = 0 \Rightarrow \begin{cases} x = 0 \\ x^2 + x - 6 = 0 \end{cases}$$

$$x^2 + x - 6 = 0 \rightarrow x = -3, 2$$

$$\rightarrow x = -3, 0, 2$$

$$A = \int_{-3}^{0} (f - g)dx + \int_{0}^{2} (g - f)dx$$

$$= \int_{-3}^{0} (x^3 + 2x^2 - 3x - (x^2 + 3x))dx + \int_{0}^{2} (x^2 + 3x - (x^3 + 2x^2 - 3x))dx$$

$$= \int_{-3}^{0} (x^3 + x^2 - 6x)dx + \int_{0}^{2} (-x^3 - x^2 + 6x)dx$$

$$= \frac{x^4}{4} + \frac{x^3}{3} - 3x^2 \Big|_{-3}^{0} + \left[-\frac{x^4}{4} - \frac{x^3}{3} + 3x^2 \right]_0^2$$

$$= 0 - \left(\frac{(-3)^4}{4} + \frac{(-3)^3}{3} - 3(-3)^2 \right) + \left[\left(-\frac{2^4}{4} - \frac{2^3}{3} + 32^2 \right) - 0 \right]$$

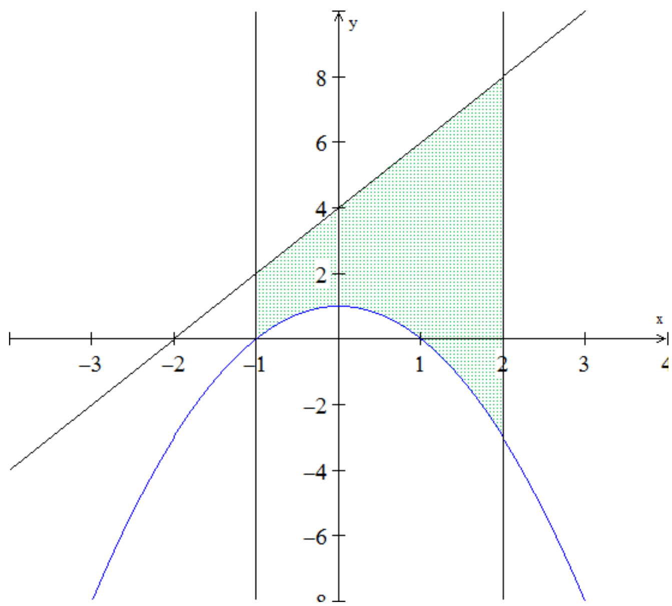
$$= \frac{253}{12}$$

$$\approx 21.083$$

Exercise

Find the area bounded by $f(x) = -x^2 + 1$, $g(x) = 2x + 4$, $x = -1$, and $x = 2$

Solution



$$f \cap g \Rightarrow \quad -x^2 + 1 = 2x + 4$$

$$-x^2 - 2x - 3 = 0$$

$$x^2 + 2x + 3 = 0$$

$$\Rightarrow x = -1 \pm i\sqrt{2}$$

$$A = \int_{-1}^{2} (g - f)dx$$

$$= \int_{-1}^{2} \left(2x + 4 - \left(-x^2 + 1 \right) \right) dx$$

$$= \int_{-1}^{2} \left(x^2 + 2x + 3 \right) dx$$

$$= \frac{1}{3}x^3 + x^2 + 3x \Big|_{-1}^{2}$$

$$= \left(\frac{1}{3}(2)^3 + (2)^2 + 3(2) \right) - \left(\frac{1}{3}(-1)^3 + (-1)^2 + 3(-1) \right)$$

$$= \left(\frac{8}{3} + 4 + 6 \right) - \left(-\frac{1}{3} + 1 - 3 \right)$$

$$= \frac{8}{3} + 10 + \frac{1}{3} + 2$$

$$= 15$$

Exercise

Find the area between the curves $y = x^{1/2}$ *and* $y = x^3$

Solution

$x^3 = x^{1/2}$ ***Square both sides***

$x^6 = x$

$x^6 - x = 0$

$x\left(x^5 - 1\right) = 0$

$x = 0 \quad x^5 - 1 = 0 \Rightarrow x = 1$

$$A = \int_0^1 \left(x^{1/2} - x^3\right) dx$$

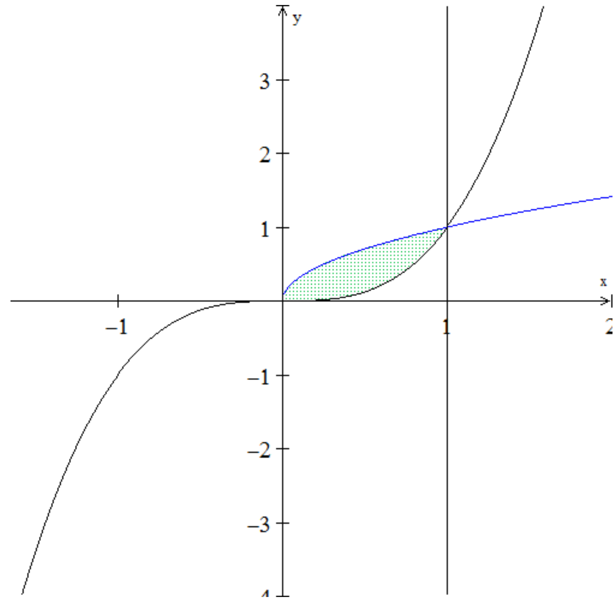
$$= \frac{2}{3}x^{3/2} - \frac{1}{4}x^4 \Big|_0^1$$

$$= \frac{2}{3}1^{3/2} - \frac{1}{4}1^4 - 0$$

$$= \frac{2}{3} - \frac{1}{4}$$

$$= \frac{8-3}{12}$$

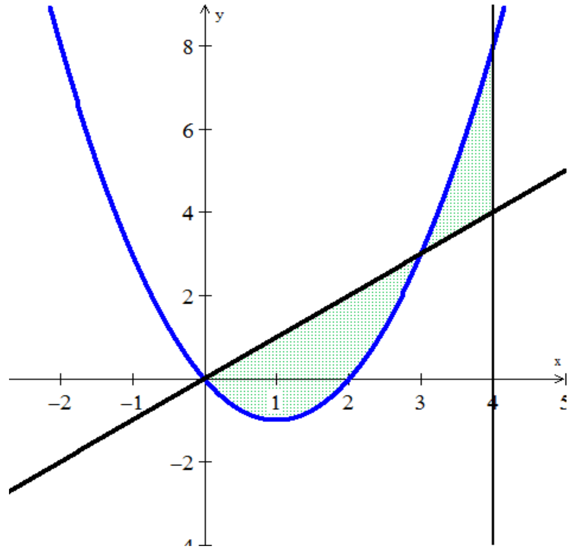
$$= \frac{5}{12}$$



Exercise

Find the area of the region bounded by the graphs of $y = x^2 - 2x$ and $y = x$ on [0, 4].

Solution



$$x^2 - 2x = x$$

$$x^2 - 3x = 0$$

$$x(x-3) = 0 \Rightarrow \boxed{x = 0, 3}$$

$$A = \int_0^3 \left(x - \left(x^2 - 2x\right)\right)dx + \int_3^4 \left(x^2 - 2x - x\right)dx$$

$$= \int_0^3 \left(-x^2 + 3x\right)dx + \int_3^4 \left(x^2 - 3x\right)dx$$

$$= \left(-\frac{1}{3}x^3 + \frac{3}{2}x^2\right)\Big|_0^3 + \left(\frac{1}{3}x^3 - \frac{3}{2}x^2\right)\Big|_3^4$$

$$= \left(-\frac{1}{3}3^3 + \frac{3}{2}3^2\right) + \left[\left(\frac{1}{3}4^3 - \frac{3}{2}4^2\right) - \left(\frac{1}{3}3^3 - \frac{3}{2}3^2\right)\right]$$

$$= \left(\frac{9}{2}\right) + \left[\left(-\frac{8}{3}\right) - \left(-\frac{9}{2}\right)\right]$$

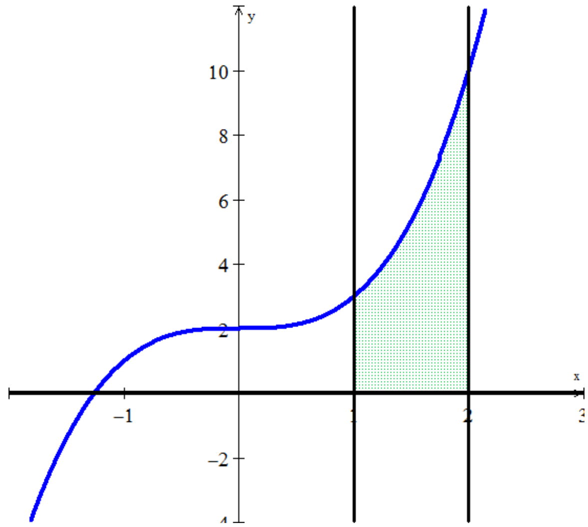
$$= \frac{9}{2} - \frac{8}{3} + \frac{9}{2}$$

$$= \frac{19}{3}$$

Exercise

Find the area between the curves $x=1, \quad x=2, \quad y=x^3+2, \quad y=0$

Solution



$$A=\int_1^2\left(x^3+2-0\right)dx$$

$$=\frac{1}{4}x^4+2x\Big|_1^2$$

$$=\left(\frac{1}{4}2^4+2(2)\right)-\left(\frac{1}{4}1^4+2(1)\right)$$

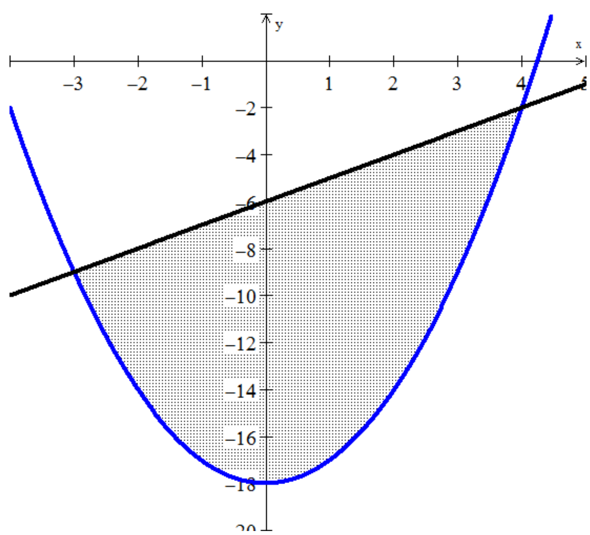
$$=(8)-\left(\frac{9}{4}\right)$$

$$=\frac{23}{4}$$

Exercise

Find the area between the curves $y=x^2-18, \quad y=x-6$

Solution



$$x^2-18=x-6$$

$$x^2-x-12=0 \rightarrow \boxed{x=-3,4}$$

$$A=\int_{-3}^{4}\left(x^2-18-(x-6)\right)dx$$

$$=\int_{-3}^{4}\left(x^2-x-12\right)dx$$

$$=\frac{1}{3}x^3-\frac{1}{2}x^2-12x\Big|_{-3}^{4}$$

$$=\left(\frac{1}{3}4^3-\frac{1}{2}4^2-12(4)\right)-\left(\frac{1}{3}(-3)^3-\frac{1}{2}(-3)^2-12(-3)\right)$$

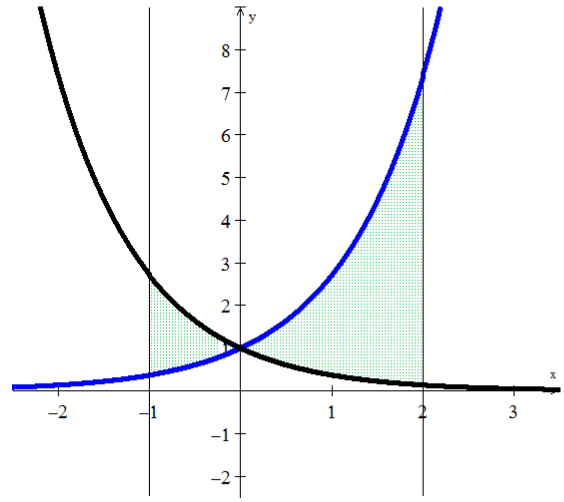
$$=\left(-\frac{104}{3}\right)-\left(\frac{45}{2}\right)$$

$$= \boxed{\frac{343}{6}}$$

Exercise

Find the area between the curves $x=-1, \quad x=2, \quad y=e^{-x}, \quad y=e^{x}$

Solution



$$e^{x}=e^{-x}$$

$$x=-x$$

$$\Rightarrow \boxed{x=0}$$

$$A=\int_{-1}^{0}\left(e^{-x}-e^{x}\right)dx+\int_{0}^{2}\left(e^{x}-e^{-x}\right)dx$$

$$=\left(-e^{-x}-e^{x}\right)\Big|_{-1}^{0}+\left(e^{x}+e^{-x}\right)\Big|_{0}^{2}$$

$$=\left(-e^{-0}-e^{0}\right)-\left(-e^{-(-1)}-e^{-1}\right)+\left[\left(e^{2}+e^{-2}\right)-\left(e^{0}+e^{-0}\right)\right]$$

$$=(-1-1)-\left(-e-e^{-1}\right)+\left[\left(e^{2}+e^{-2}\right)-(1+1)\right]$$

$$=-2+e+e^{-1}+e^{2}+e^{-2}-2$$

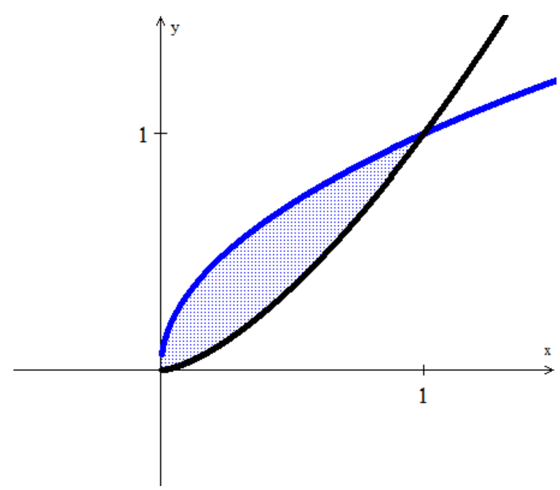
$$=e+e^{-1}+e^{2}+e^{-2}-4$$

$$\boxed{=6.61}$$

Exercise

Find the area between the curves $y=\sqrt{x}, \quad y=x\sqrt{x}$

Solution



$$x\sqrt{x}=\sqrt{x}$$

$$\left(x\sqrt{x}\right)^{2}=\left(\sqrt{x}\right)^{2}$$

$$x^{2}x=x$$

$$x\left(x^{2}-1\right)=0$$

$$\boxed{x=0} \quad x^{2}-1=0 \Rightarrow x=\pm 1 (no\ negative) \quad \boxed{x=1}$$

$$A = \int_0^1 \left(\sqrt{x} - x\sqrt{x}\right)dx$$

$$= \int_0^1 \left(x^{1/2} - x^{3/2}\right)dx$$

$$= \frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2}\Big|_0^1$$

$$= \left(\frac{2}{3}1^{3/2} - \frac{2}{5}1^{5/2}\right) - \left(\frac{2}{3}0^{3/2} - \frac{2}{5}0^{5/2}\right)$$

$$= \left(\frac{2}{3} - \frac{2}{5}\right) - 0$$

$$\boxed{= \frac{4}{15}}$$

Exercise

A company is considering a new manufacturing process in one of its plants. The new process provides substantial initial savings, with the savings declining with time *t* (in years) according to the rate-of-savings function

$$S'(t) = 100 - t^2$$

where $S'(t)$ is in thousands of dollars per year. At the same time, the cost of operating the new process increases with time *t* (in years), according to the rate-of-cost function (in thousands of dollars per year)

$$C'(t) = t^2 + \frac{14}{3}t$$

a) For how many years will the company realize savings?

b) What will be the net total savings during this period?

Solution

a) $C'(t) = S'(t)$

$$C'(t) = S'(t)$$

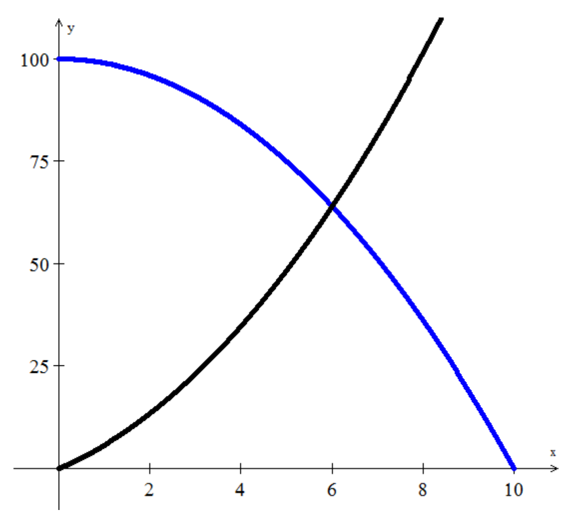
$$t^2 + \frac{14}{3}t = 100 - t^2$$

$$2t^2 + \frac{14}{3}t - 100 = 0$$

$$\rightarrow t = -\frac{25}{3} \text{ or } 6$$

$$\boxed{t = 6}$$

The company should use this type for 6 years.



b) What will be the net total savings during this period?

$$\text{Total savings } = \int_0^6 \left[\left(100-t^2\right)-\left(t^2+\frac{14}{3}t\right)\right]dt$$

$$= \int_0^6 \left[100-2t^2-\frac{14}{3}t\right]dt$$

$$= 100t-\frac{2}{3}t^3-\frac{7}{3}t^2 \Big|_0^6$$

$$= 100(6)-\frac{2}{3}(6)^3-\frac{7}{3}(6)^2-\left(100(0)-\frac{2}{3}(0)^3-\frac{7}{3}(0)^2\right)$$

$$\underline{= 372}$$

The company will save a total of \$372,000. Over the 6-year period

Exercise

Find the producers' surplus if the supply function for pork bellies is given by

$$S(x)=x^{5/2}+2x^{3/2}+50$$

Assume supply and demand are in equilibrium at $x=16$.

Solution

The equilibrium price: $\underline{p_0} = S(x=16)=16^{5/2}+2(16)^{3/2}+50\underline{=1202}$

$$\text{Producer's surplus } = \int_0^{x_0} \left[p_0 - S(x)\right]dx$$

$$= \int_0^{16} \left[1202-\left(x^{5/2}+2x^{3/2}+50\right)\right]dx$$

$$= \int_0^{16} \left[1152-x^{5/2}-2x^{3/2}\right]dx$$

$$= 1152x-\frac{2}{7}x^{7/2}-\frac{4}{5}x^{5/2} \Big|_0^{16}$$

$$= \left(1152(16)-\frac{2}{7}(16)^{7/2}-\frac{4}{5}(16)^{5/2}\right)-\left(1152(0)-\frac{2}{7}(0)^{7/2}-\frac{4}{5}(0)^{5/2}\right)$$

$$\underline{= 12,931.66}$$

The producers' surplus is 12,931.66

Exercise

Suppose the supply function for concrete is given by

$$S(q) = 100 + 3q^{3/2} + q^{5/2}$$

And that supply and demand are in equilibrium at $q = 9$. Find the producers' surplus.

Solution

$$p_0 = S(9) = 100 + 3(9)^{3/2} + (9)^{5/2} = 424$$

$$\text{Producers' surplus } = \int_0^{q_0} \left| p_0 - S(q) \right| dq$$

$$= \int_0^9 \left[424 - \left(100 + 3q^{3/2} - q^{5/2} \right) \right] dq$$

$$= \int_0^9 \left[324 - 3q^{3/2} + q^{5/2} \right] dq$$

$$= 324q - \tfrac{6}{5} q^{5/2} + \tfrac{2}{7} q^{7/2} \Big|_0^9$$

$$= 324(9) - \tfrac{6}{5}(9)^{5/2} + \tfrac{2}{7}(9)^{7/2} - (0)$$

$$= 1999.54$$

The producers' surplus is 1999.54

Exercise

Find the consumers' surplus if the demand function for grass seed is given by

$$D(x) = \frac{200}{(3x+1)^2}$$

Assuming supply and demand are in equilibrium at $x = 3$.

Solution

$$p_0 = D(x) = \frac{200}{(3(3)+1)^2} = 2$$

$$\text{Consumers' surplus } = \int_0^{x_0} \left| D(x) - p_0 \right| dx$$

$$= \int_0^3 \left| \frac{200}{(3x+1)^2} - 2 \right| dx$$

$$= \int_0^3 \frac{200}{(3x+1)^2}dx - \int_0^3 2dx \qquad u = 3x+1 \Rightarrow du = 3dx \rightarrow \frac{1}{3}du = dx$$

$$= \frac{1}{3}\int_1^{10} \frac{200}{u^2}du - \int_0^3 2dx$$

$$= \frac{200}{3}\int_1^{10} u^{-2}du - \int_0^3 2dx$$

$$= \frac{200}{3}\left[\frac{u^{-1}}{-1}\right]_1^{10} - 2x\Big|_0^3$$

$$= \frac{200}{3}\left[-\frac{1}{u}\right]_1^{10} - 2(3-0)$$

$$= \frac{200}{3}\left(-\frac{1}{10}+\frac{1}{1}\right) - 6$$

$$\underline{= 54}$$

Exercise

Find the consumers' surplus if the demand function for olive oil is given by

$$D(x) = \frac{32,000}{(2x+8)^3}$$

And if supply and demand are in equilibrium at $x = 6$.

Solution

$$p_0 = D(6) = \frac{32000}{(2(6)+8)^3} = 4$$

$$\text{Consumers' surplus } = \int_0^{x_0} \left|D(x) - p_0\right| dx$$

$$= \int_0^6 \left(\frac{32,000}{(2x+8)^3} - 4\right)dx$$

$$= \int_0^6 32,000(2x+8)^{-3}dx - \int_0^6 4dx \qquad u = 2x+8 \Rightarrow du = 2dx \rightarrow \frac{1}{2}du = dx$$

$$= 32,000\int_8^{20} u^{-3}\frac{1}{2}du - \int_0^6 4dx$$

$$= 16000.\left.\frac{u^{-2}}{-2}\right|_{8}^{20} - 4x\Big|_{0}^{6}$$

$$= -8000\left(20^{-2} - 8^{-2}\right) - 4(6)$$

$$\underline{= 81}$$