

Solution **Section 1.2 – Region between Curves**

Exercise

Find the area of the region bounded by the graphs of $y = 2x - x^2$ and $y = -3$

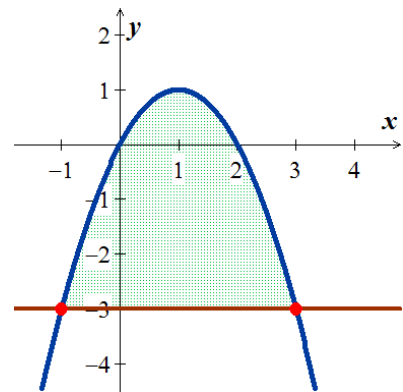
Solution

$$2x - x^2 = -3$$

$$x^2 - 2x - 3 = 0$$

$$\underline{x_{1,2} = -1, 3}$$

$$\begin{aligned} A &= \int_{-1}^3 (2x - x^2 - (-3)) dx \\ &= x^2 - \frac{x^3}{3} + 3x \Big|_{-1}^3 \\ &= \left((3)^2 - \frac{(3)^3}{3} + 3(3) \right) - \left((-1)^2 - \frac{(-1)^3}{3} + 3(-1) \right) \\ &= (9 - 9 + 9) - \left(1 + \frac{1}{3} - 3 \right) \\ &= \underline{\underline{\frac{32}{3} \text{ unit}^2}} \end{aligned}$$



Exercise

Find the area of the region bounded by the graphs of $y = 7 - 2x^2$ and $y = x^2 + 4$

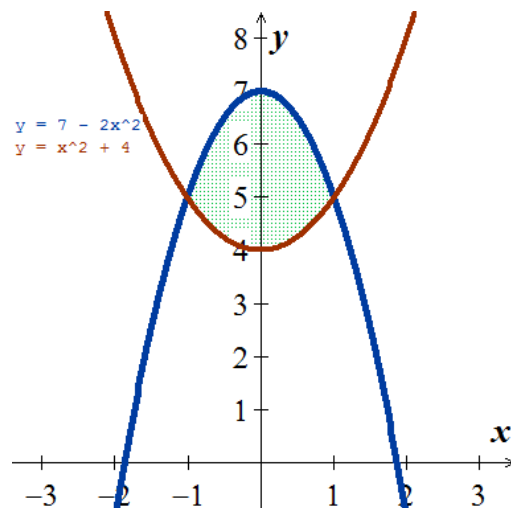
Solution

$$7 - 2x^2 = x^2 + 4$$

$$-3x^2 = -3$$

$$x^2 = 1 \Rightarrow \underline{x_{1,2} = \pm 1}$$

$$\begin{aligned} A &= \int_{-1}^1 \left[(7 - 2x^2) - (x^2 + 4) \right] dx \\ &= \int_{-1}^1 (3 - 3x^2) dx \\ &= 3x - 3\frac{x^3}{3} \Big|_{-1}^1 \end{aligned}$$



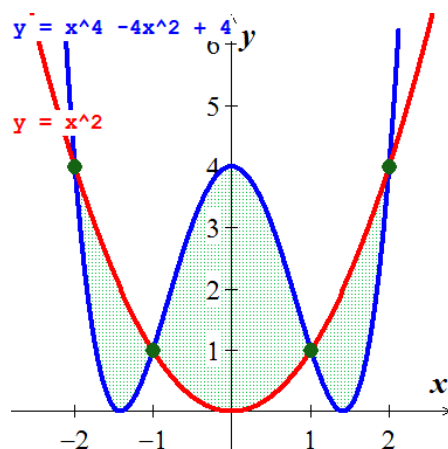
$$= \left(3 \textcolor{red}{(1)} - \textcolor{red}{(1)}^3 \right) - \left(3 \textcolor{blue}{(-1)} - \textcolor{blue}{(-1)}^3 \right)$$

$$= \underline{4 \text{ unit}^2}$$

Exercise

Find the area of the region bounded by the graphs of $y = x^4 - 4x^2 + 4$ and $y = x^2$

Solution



$$x^4 - 4x^2 + 4 = x^2$$

$$x^4 - 5x^2 + 4 = 0$$

$$x = \pm 1, \pm 2$$

$$A = \int_{-2}^{-1} \left(x^2 - (x^4 - 4x^2 + 4) \right) dx + \int_{-1}^1 \left(x^4 - 4x^2 + 4 - (x^2) \right) dx + \int_1^2 \left(x^2 - (x^4 - 4x^2 + 4) \right) dx$$

$$= \int_{-2}^{-1} \left(-x^4 + 5x^2 - 4 \right) dx + \int_{-1}^1 \left(x^4 - 5x^2 + 4 \right) dx + \int_1^2 \left(-x^4 + 5x^2 - 4 \right) dx$$

$$= \left(-\frac{x^5}{5} + \frac{5}{3}x^3 - 4x \right) \Big|_{-2}^{-1} + \left(\frac{x^5}{5} - \frac{5}{3}x^3 + 4x \right) \Big|_{-1}^1 + \left(-\frac{x^5}{5} + \frac{5}{3}x^3 - 4x \right) \Big|_1^2$$

$$= \left[\left(-\frac{\textcolor{red}{(-1)}^5}{5} + \frac{5}{3}\textcolor{red}{(-1)}^3 - 4\textcolor{red}{(-1)} \right) - \left(-\frac{\textcolor{blue}{(-2)}^5}{5} + \frac{5}{3}\textcolor{blue}{(-2)}^3 - 4\textcolor{blue}{(-2)} \right) \right]$$

$$+ \left[\left(\frac{\textcolor{red}{(1)}^5}{5} - \frac{5}{3}\textcolor{red}{(1)}^3 + 4\textcolor{red}{(1)} \right) - \left(\frac{\textcolor{blue}{(-1)}^5}{5} - \frac{5}{3}\textcolor{blue}{(-1)}^3 + 4\textcolor{blue}{(-1)} \right) \right]$$

$$+ \left[\left(-\frac{\textcolor{red}{(2)}^5}{5} + \frac{5}{3}\textcolor{red}{(2)}^3 - 4\textcolor{red}{(2)} \right) - \left(-\frac{\textcolor{blue}{(1)}^5}{5} + \frac{5}{3}\textcolor{blue}{(1)}^3 - 4\textcolor{blue}{(1)} \right) \right]$$

$$= \left(\frac{1}{5} - \frac{5}{3} + 4 \right) - \left(\frac{32}{5} - \frac{40}{3} + 8 \right) + \left(\frac{1}{5} - \frac{5}{3} + 4 \right) - \left(-\frac{1}{5} + \frac{5}{3} - 4 \right) + \left(-\frac{32}{5} + \frac{40}{3} - 8 \right) - \left(-\frac{1}{5} + \frac{5}{3} - 4 \right) \\ = 8 \text{ unit}^2$$

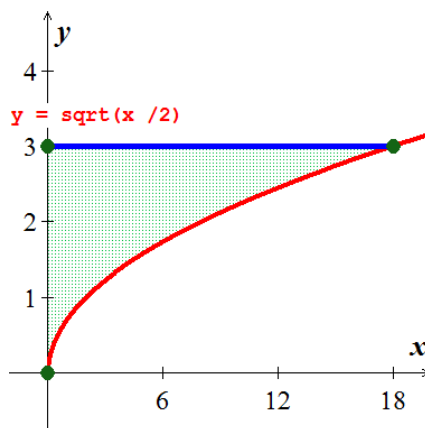
Exercise

Find the area of the region bounded by the graphs of $x = 2y^2$, $x = 0$, and $y = 3$

Solution

$$y = 3 \rightarrow x = 2y^2 = 18$$

$$A = \int_0^3 2y^2 dy \\ = \frac{2}{3} \left(y^3 \right) \Big|_0^3 \\ = \frac{2}{3} (3^3 - 0) \\ = 18 \text{ unit}^2$$



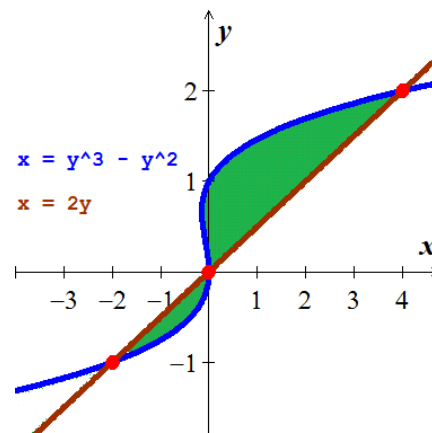
Exercise

Find the area of the region bounded by the graphs of $x = y^3 - y^2$ and $x = 2y$

Solution

$$y^3 - y^2 = 2y \\ y^3 - y^2 - 2y = 0 \\ y(y^2 - y - 2) = 0 \\ y = 0, -1, 2$$

$$A = \int_{-1}^0 (y^3 - y^2 - (2y)) dy + \int_0^2 (2y - (y^3 - y^2)) dy \\ = \int_{-1}^0 (y^3 - y^2 - 2y) dy + \int_0^2 (2y - y^3 + y^2) dy \\ = \left(\frac{y^4}{4} - \frac{y^3}{3} - y^2 \right) \Big|_{-1}^0 + \left(y^2 - \frac{y^4}{4} + \frac{y^3}{3} \right) \Big|_0^2$$



$$\begin{aligned}
 &= \left[0 - \left(\frac{1}{4} + \frac{1}{3} - 1 \right) \right] + \left[\left(4 - 4 + \frac{8}{3} \right) - 0 \right] \\
 &= \frac{5}{12} + \frac{8}{3} \\
 &= \frac{37}{12} \text{ unit}^2
 \end{aligned}$$

Exercise

Find the area of the region bounded by the graphs of $4x^2 + y = 4$ and $x^4 - y = 1$

Solution

$$4x^2 + y = 4 \rightarrow y = 4 - 4x^2$$

$$x^4 - y = 1 \rightarrow y = x^4 - 1$$

$$x^4 - 1 = 4 - 4x^2$$

$$x^4 + 4x^2 - 5 = 0$$

$$x^2 = 1, -5$$

$$x_{1,2} = \pm 1$$

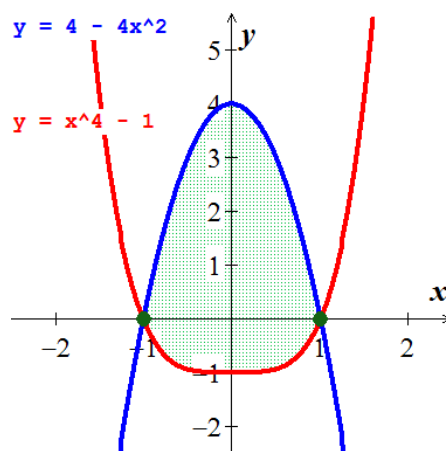
$$A = \int_{-1}^1 (4 - 4x^2 - (x^4 - 1)) dx$$

$$= \int_{-1}^1 (-x^4 - 4x^2 + 5) dx$$

$$= 2 \left(-\frac{x^5}{5} - 4\frac{x^3}{3} + 5x \right) \Big|_0^1$$

$$= 2 \left(-\frac{1}{5} - \frac{4}{3} + 5 \right)$$

$$= \frac{104}{15} \text{ unit}^2$$



Exercise

Find the area of the region bounded by the graphs of $x + 4y^2 = 4$ and $x + y^4 = 1$, for $x \geq 0$

Solution

$$x = 4 - 4y^2 \quad x = 1 - y^4$$

$$4 - 4y^2 = 1 - y^4$$

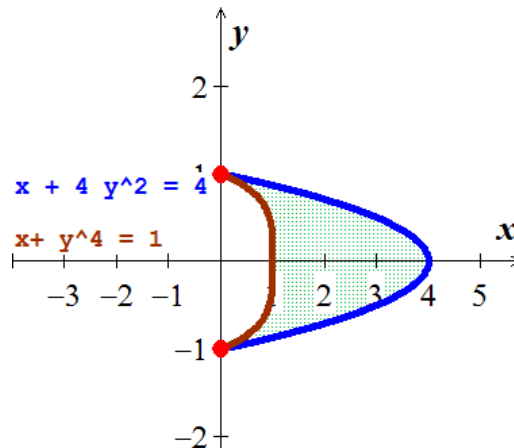
$$y^4 - 4y^2 + 3 = 0$$

$$y^2 = 1, 3$$

$$\underline{y = \pm 1, \pm \sqrt{3}}$$

$$\begin{cases} y = \pm 1 & \rightarrow \underline{x = 1 - (\pm 1)^4 = 0} \\ y = \pm \sqrt{3} & \rightarrow x = 1 - (\pm \sqrt{3})^4 = -8 < 0 \end{cases}$$

$$\begin{aligned} A &= \int_{-1}^1 \left(4 - 4y^2 - (1 - y^4) \right) dy \\ &= \int_{-1}^1 \left(3 - 4y^2 + y^4 \right) dy \\ &= 3y - 4\frac{y^3}{3} + \frac{y^5}{5} \Big|_{-1}^1 \\ &= \left(3 - \frac{4}{3} + \frac{1}{5} \right) - \left(-3 + \frac{4}{3} - \frac{1}{5} \right) \\ &= \underline{\underline{\frac{56}{15} \text{ unit}^2}}} \end{aligned}$$



Exercise

Find the area of the region bounded by the graphs of $y = 2 \sin x$, and $y = \sin 2x$, $0 \leq x \leq \pi$

Solution

$$y = 2 \sin x = \sin 2x$$

$$2 \sin x = 2 \sin x \cos x$$

$$2 \sin x - 2 \sin x \cos x = 0$$

$$2 \sin x (1 - \cos x) = 0$$

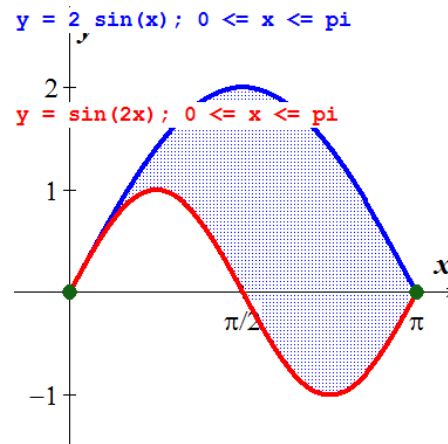
$$\begin{cases} \sin x = 0 & x = 0, \pi \\ \cos x = 1 & x = 0 \end{cases}$$

$$A = \int_0^\pi (2 \sin x - \sin 2x) dx$$

$$= -2 \cos x + \frac{1}{2} \cos 2x \Big|_0^\pi$$

$$= \left(-2(-1) + \frac{1}{2}(1) \right) - \left(-2 + \frac{1}{2} \right)$$

$$= \underline{\underline{4 \text{ unit}^2}}$$



Exercise

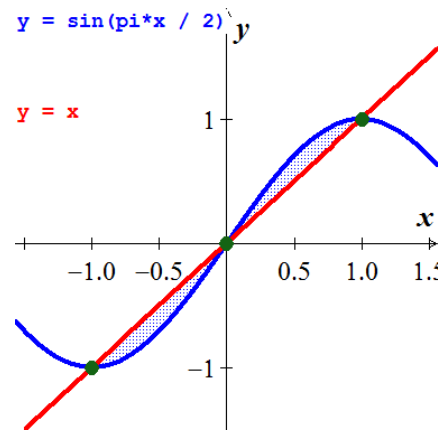
Find the area of the region bounded by the graphs of $y = \sin \frac{\pi x}{2}$ and $y = x$

Solution

$$y = \sin \frac{\pi x}{2} = x$$

$$x = \pm 1$$

$$\begin{aligned} A &= \int_{-1}^0 \left(\sin \frac{\pi x}{2} - x \right) dx + \int_0^1 \left(\sin \frac{\pi x}{2} - x \right) dx \\ &= 2 \int_0^1 \left(\sin \frac{\pi x}{2} - x \right) dx \\ &= 2 \left(-\frac{2}{\pi} \cos \frac{\pi x}{2} - \frac{x^2}{2} \right) \Big|_0^1 \\ &= 2 \left[\left(0 - \frac{1}{2} \right) - \left(-\frac{2}{\pi} - 0 \right) \right] \\ &= 2 \left(-\frac{1}{2} + \frac{2}{\pi} \right) \\ &= 2 \left(\frac{-\pi + 4}{2\pi} \right) \\ &= \frac{4 - \pi}{\pi} \text{ unit}^2 \end{aligned}$$

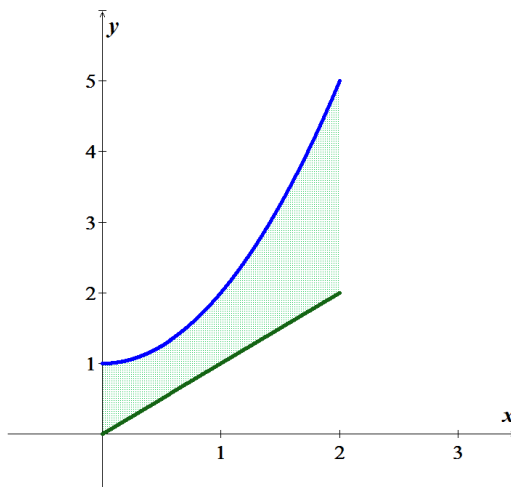


Exercise

Find the area of the region bounded by the graphs of $y = x^2 + 1$ and $y = x$ for $0 \leq x \leq 2$

Solution

$$\begin{aligned} A &= \int_0^2 [(x^2 + 1) - x] dx \\ &= \int_0^2 (x^2 - x + 1) dx \\ &= \frac{x^3}{3} - \frac{x^2}{2} + 1x \Big|_0^2 \\ &= \frac{8}{3} - 2 + 2 - 0 \\ &= \frac{8}{3} \text{ unit}^2 \end{aligned}$$



Exercise

Find the area of the region bounded by the graphs of $y = 3 - x^2$ and $y = 2x$

Solution

$$x^2 + 2x - 3 = 0$$

$$\underline{x_{1,2} = 1, -3}$$

$$A = \int_{-3}^1 \left((3 - x^2) - 2x \right) dx$$

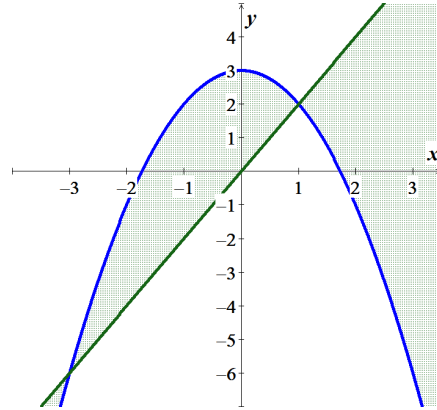
$$= \int_{-3}^1 (-x^2 - 2x + 3) dx$$

$$= -\frac{x^3}{3} - 2\frac{x^2}{2} + 3x \Big|_{-3}^1$$

$$= -\frac{1}{3} - 1 + 3 - [9 - 9 - 9]$$

$$= 11 - \frac{1}{3}$$

$$\underline{= \frac{32}{3} \text{ unit}^2}$$



Exercise

Find the area of the region bounded by the graphs of $y = x^2 - x - 2$ and x-axis

Solution

The intersection points: $x^2 - x - 2 = 0$

$$\underline{x_{1,2} = -1, 2}$$

$$A = \int_{-1}^2 \left(0 - (x^2 - x - 2) \right) dx$$

$$= -\frac{x^3}{3} + \frac{x^2}{2} + 2x \Big|_{-1}^2$$

$$= -\frac{8}{3} + 2 + 4 - \left(-\frac{1}{3} + \frac{1}{2} - 2 \right)$$

$$= \frac{10}{3} + \frac{7}{6}$$

$$\underline{= \frac{9}{2} \text{ unit}^2}$$

Exercise

Find the area between the curves $y = x^{1/2}$ and $y = x^3$

Solution

$$x^3 = x^{1/2} \quad \text{Square both sides} \rightarrow x^6 = x$$

$$x(x^5 - 1) = 0$$

$$\rightarrow \underline{x=0} \quad x^5 - 1 = 0 \Rightarrow \underline{x=1}$$

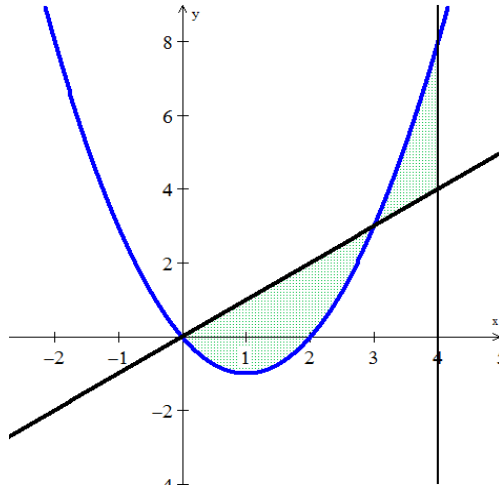
$$A = \int_0^1 (x^{1/2} - x^3) dx$$

$$= \left. \frac{2}{3} x^{3/2} - \frac{1}{4} x^4 \right|_0^1$$

$$= \frac{2}{3} - \frac{1}{4}$$

$$= \frac{8-3}{12}$$

$$= \underline{\underline{\frac{5}{12} \text{ unit}^2}}$$



Exercise

Find the area of the region bounded by the graphs of $y = x^2 - 2x$ and $y = x$ on $[0, 4]$.

Solution

$$x^2 - 2x = x \quad x^2 - 3x = 0$$

$$x(x-3) = 0$$

$$\underline{x=0, 3}$$

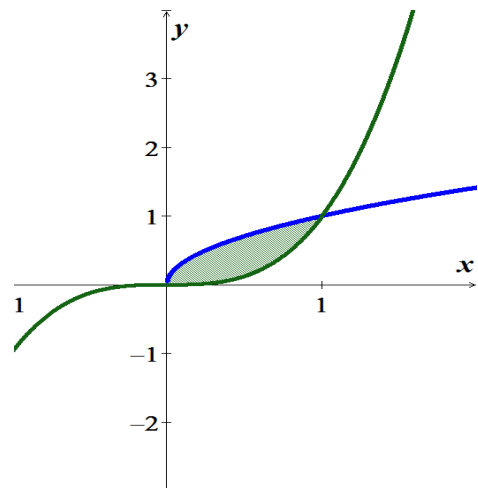
$$A = \int_0^3 (x - (x^2 - 2x)) dx + \int_3^4 (x^2 - 2x - x) dx$$

$$= \int_0^3 (-x^2 + 3x) dx + \int_3^4 (x^2 - 3x) dx$$

$$= \left(-\frac{1}{3} x^3 + \frac{3}{2} x^2 \right) \Big|_0^3 + \left(\frac{1}{3} x^3 - \frac{3}{2} x^2 \right) \Big|_3^4$$

$$= \left(-9 + \frac{27}{2} \right) + \left[\left(\frac{64}{3} - 24 \right) - \left(9 - \frac{27}{2} \right) \right]$$

$$= \left(\frac{9}{2} \right) + \left[\left(-\frac{8}{3} \right) - \left(-\frac{9}{2} \right) \right]$$



$$= \frac{9}{2} - \frac{8}{3} + \frac{9}{2}$$

$$= \frac{19}{3} \text{ unit}^2$$

Exercise

Find the area between the curves $x = 1$, $x = 2$, $y = x^3 + 2$, $y = 0$

Solution

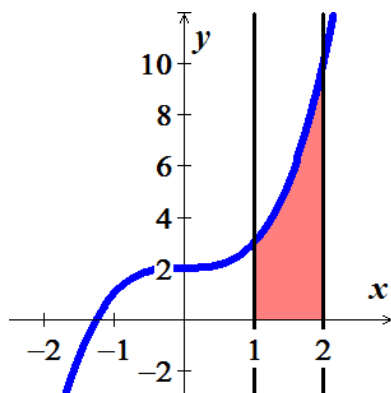
$$A = \int_1^2 (x^3 + 2 - 0) dx$$

$$= \frac{1}{4}x^4 + 2x \Big|_1^2$$

$$= (4 + 4) - \left(\frac{1}{4} + 2\right)$$

$$= 8 - \frac{9}{4}$$

$$= \frac{23}{4} \text{ unit}^2$$



Exercise

Find the area between the curves $y = x^2 - 18$, $y = x - 6$

Solution

$$x^2 - 18 = x - 6$$

$$x^2 - x - 12 = 0$$

$$x = -3, 4$$

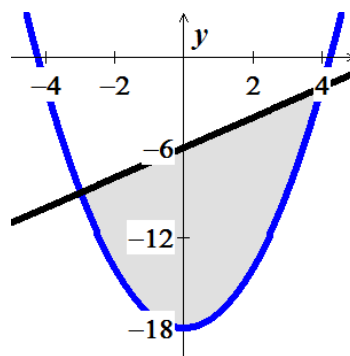
$$A = \int_{-3}^4 (x^2 - 18 - (x - 6)) dx$$

$$= \int_{-3}^4 (x^2 - x - 12) dx$$

$$= \frac{1}{3}x^3 - \frac{1}{2}x^2 - 12x \Big|_{-3}^4$$

$$= \frac{64}{3} - 8 - 48 - \left(-9 - \frac{9}{2} + 36\right)$$

$$= -\frac{104}{3} - \frac{45}{2}$$



$$\underline{= \frac{343}{6} \text{ unit}^2}$$

Exercise

Find the area between the curves $y = \sqrt{x}$, $y = x\sqrt{x}$

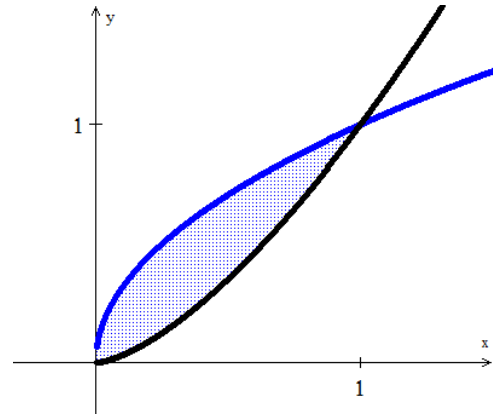
Solution

$$x\sqrt{x} = \sqrt{x} \Rightarrow (x\sqrt{x})^2 = (\sqrt{x})^2$$

$$x^2x = x \rightarrow x(x^2 - 1) = 0$$

$$\boxed{x=0} \quad x^2 - 1 = 0 \Rightarrow x = \pm 1 (\text{no negative}) \quad \boxed{x=1}$$

$$\begin{aligned} A &= \int_0^1 (\sqrt{x} - x\sqrt{x}) dx \\ &= \int_0^1 (x^{1/2} - x^{3/2}) dx \\ &= \left. \frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} \right|_0^1 \\ &= \frac{2}{3} - \frac{2}{5} \\ &= \frac{4}{15} \text{ unit}^2 \end{aligned}$$



Exercise

Find the area of the region bounded by the graphs of $f(x) = x^3 + 2x^2 - 3x$ and $g(x) = x^2 + 3x$

Solution

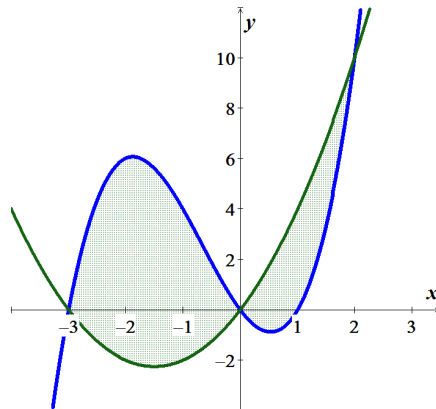
$$x^3 + 2x^2 - 3x = x^2 + 3x$$

$$x^3 + x^2 - 6x = 0$$

$$x(x^2 + x - 6) = 0$$

$$\begin{cases} x = 0 \\ x^2 + x - 6 = 0 \end{cases}$$

$$\underline{x = -3, 0, 2}$$



$$A = \int_{-3}^0 (x^3 + 2x^2 - 3x - (x^2 + 3x)) dx + \int_0^2 (x^2 + 3x - (x^3 + 2x^2 - 3x)) dx$$

$$\begin{aligned}
&= \int_{-3}^0 (x^3 + x^2 - 6x) dx + \int_0^2 (-x^3 - x^2 + 6x) dx \\
&= \left(\frac{x^4}{4} + \frac{x^3}{3} - 3x^2 \right) \Big|_{-3}^0 + \left(-\frac{x^4}{4} - \frac{x^3}{3} + 3x^2 \right) \Big|_0^2 \\
&= 0 - \left(\frac{81}{4} - 9 - 27 \right) + \left(-4 - \frac{8}{3} + 12 \right) - 0 \\
&= \underline{\underline{\frac{253}{12} \text{ unit}^2}}
\end{aligned}$$

Exercise

Find the area of the region bounded by the graphs of $y = -x^2 + 3x + 1$, $y = -x + 1$

Solution

$$y = -x^2 + 3x + 1 = -x + 1$$

$$x^2 - 4x = 0$$

$$\underline{x = 0, 4}$$

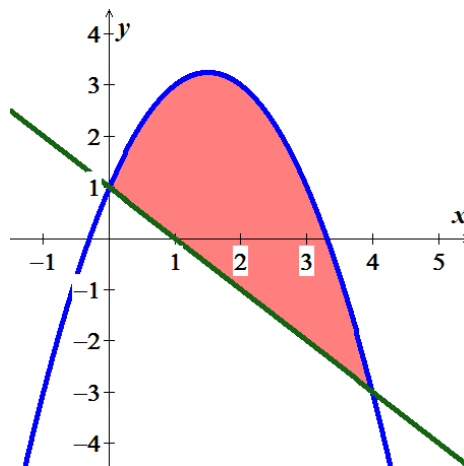
$$A = \int_0^4 (-x^2 + 3x + 1 - (-x + 1)) dx$$

$$= \int_0^4 (-x^2 + 4x) dx$$

$$= -\frac{1}{3}x^3 + 2x^2 \Big|_0^4$$

$$= -\frac{64}{3} + 32$$

$$= \underline{\underline{\frac{32}{3} \text{ unit}^2}}$$



Exercise

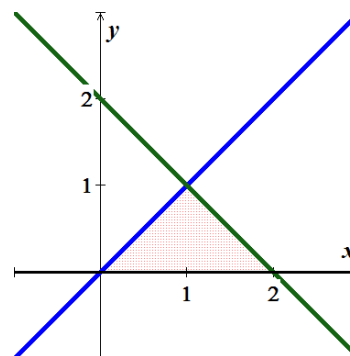
Find the area of the region bounded by the graphs of $y = x$, $y = 2 - x$, $y = 0$

Solution

$$y = x = 2 - x \rightarrow \underline{x = 1}$$

$$y = 2 - x = 0 \rightarrow \underline{x = 2}$$

$$A = \int_0^1 (x - 0) dx + \int_1^2 (2 - x - 0) dx$$



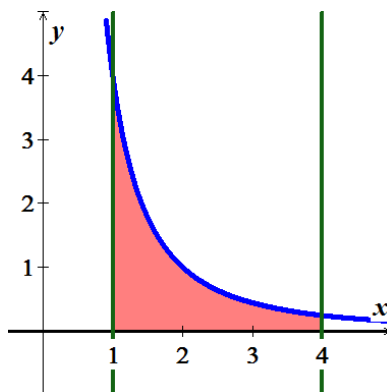
$$\begin{aligned}
 &= \frac{1}{2}x^2 \Big|_0^1 + \left(2x - \frac{1}{2}x^2\right) \Big|_1^2 \\
 &= \frac{1}{2} + 4 - 2 - 2 + \frac{1}{2} \\
 &= \underline{1 \text{ unit}^2}
 \end{aligned}$$

Exercise

Find the area of the region bounded by the graphs of $y = \frac{4}{x^2}$, $y = 0$, $x = 1$, $x = 4$

Solution

$$\begin{aligned}
 A &= \int_1^4 \frac{4}{x^2} dx \\
 &= -\frac{4}{x} \Big|_1^4 \\
 &= 4\left(-\frac{1}{4} + 1\right) \\
 &= \underline{3 \text{ unit}^2}
 \end{aligned}$$

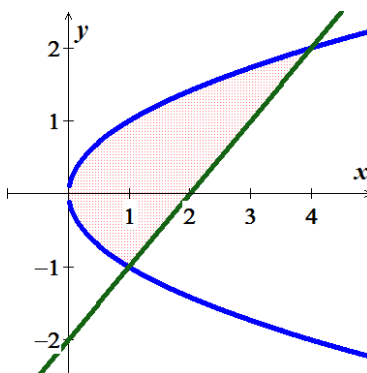


Exercise

Find the area of the region bounded by the graphs of $f(y) = y^2$, $g(y) = y + 2$

Solution

$$\begin{aligned}
 y^2 &= y + 2 \\
 y^2 - y - 2 &= 0 \\
 y &= -1, 2 \\
 A &= \int_{-1}^2 (y + 2 - y^2) dy \\
 &= \frac{1}{2}y^2 + 2y - \frac{1}{3}y^3 \Big|_{-1}^2 \\
 &= 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3} \\
 &= \underline{\frac{9}{2} \text{ unit}^2}
 \end{aligned}$$

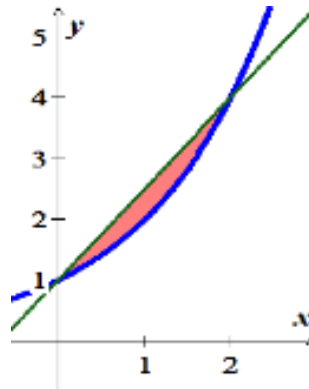


Exercise

Find the area of the region bounded by the graphs of $f(x) = 2^x$, $g(x) = \frac{3}{2}x + 1$

Solution

$$\begin{aligned}
 A &= \int_0^2 \left(\frac{3}{2}x + 1 - 2^x \right) dx \\
 &= \left. \frac{3}{4}x^2 + x - \frac{2^x}{\ln 2} \right|_0^2 \\
 &= 3 + 2 - \frac{4}{\ln 2} + \frac{1}{\ln 2} \\
 &= \underline{5 - \frac{3}{\ln 2} \text{ unit}^2}
 \end{aligned}$$



Exercise

Find the area of the region bounded by the graphs of $x = \sqrt[3]{y}$ and $x = \sqrt[5]{y}$

Solution

$$x = \sqrt[3]{y} \rightarrow y = x^3$$

$$x = \sqrt[5]{y} \rightarrow y = x^5$$

$$y = x^5 = x^3$$

$$x^5 - x^3 = 0$$

$$x^3(x^2 - 1) = 0$$

$$\underline{x = 0, \pm 1}$$

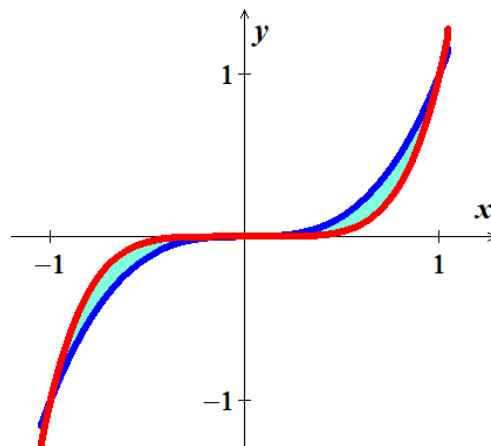
$$\text{Area} = \int_{-1}^0 (x^5 - x^3) dx + \int_0^1 (x^3 - x^5) dx$$

$$= 2 \int_0^1 (x^3 - x^5) dx$$

$$= 2 \left(\frac{1}{4}x^4 - \frac{1}{6}x^6 \right) \Big|_0^1$$

$$= 2 \left(\frac{1}{4} - \frac{1}{6} \right)$$

$$= \underline{\frac{1}{6} \text{ unit}^2}$$



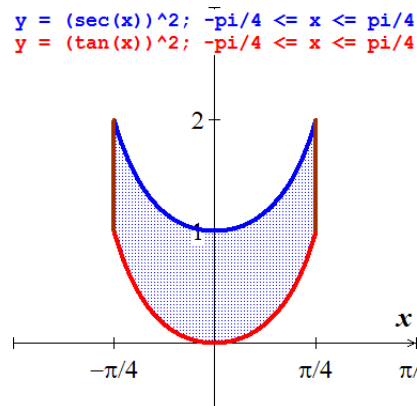
Exercise

Find the area of the region bounded by the graphs of

$$y = \sec^2 x, \quad y = \tan^2 x, \quad x = -\frac{\pi}{4}, \quad \text{and} \quad x = \frac{\pi}{4}$$

Solution

$$\begin{aligned} A &= \int_{-\pi/4}^{\pi/4} (\sec^2 x - \tan^2 x) dx \\ &= \int_{-\pi/4}^{\pi/4} (\sec^2 x - (\sec^2 x - 1)) dx \\ &= \int_{-\pi/4}^{\pi/4} dx \\ &= x \Big|_{-\pi/4}^{\pi/4} \\ &= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) \\ &= \frac{\pi}{2} \text{ unit}^2 \end{aligned}$$



Exercise

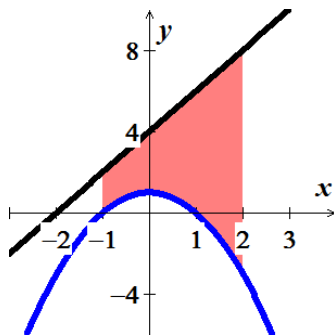
Find the area bounded by $f(x) = -x^2 + 1$, $g(x) = 2x + 4$, $x = -1$, $x = 2$

Solution

$$\begin{aligned} f \cap g &\Rightarrow -x^2 + 1 = 2x + 4 \\ &\Rightarrow x^2 + 2x + 3 = 0 \end{aligned}$$

$$\Rightarrow x = -1 \pm i\sqrt{2}$$

$$\begin{aligned} A &= \int_{-1}^2 (2x + 4 - (-x^2 + 1)) dx \\ &= \int_{-1}^2 (x^2 + 2x + 3) dx \\ &= \frac{1}{3}x^3 + x^2 + 3x \Big|_{-1}^2 \\ &= \left(\frac{8}{3} + 4 + 6\right) - \left(-\frac{1}{3} + 1 - 3\right) \\ &= 15 \text{ unit}^2 \end{aligned}$$



Exercise

Find the area of the region bounded by the graphs of $f(x) = \sqrt{x} + 3$, $g(x) = \frac{1}{2}x + 3$

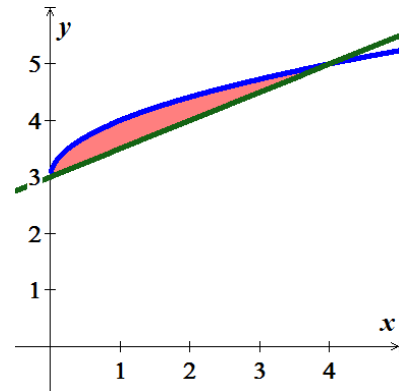
Solution

$$\sqrt{x} + 3 = \frac{1}{2}x + 3 \Rightarrow (\sqrt{x})^2 = \left(\frac{1}{2}x\right)^2$$

$$x = \frac{1}{4}x^2$$

$$x = 0, 4$$

$$\begin{aligned} A &= \int_0^4 \left(\sqrt{x} + 3 - \frac{1}{2}x - 3 \right) dx \\ &= \int_0^4 \left(x^{1/2} - \frac{1}{2}x \right) dx \\ &= \frac{2}{3}x^{3/2} - \frac{1}{4}x^2 \Big|_0^4 \\ &= \frac{16}{3} - 4 \\ &= \frac{4}{3} \text{ unit}^2 \end{aligned}$$



Exercise

Find the area of the region bounded by the graphs of $f(x) = \sqrt[3]{x-1}$, $g(x) = x-1$

Solution

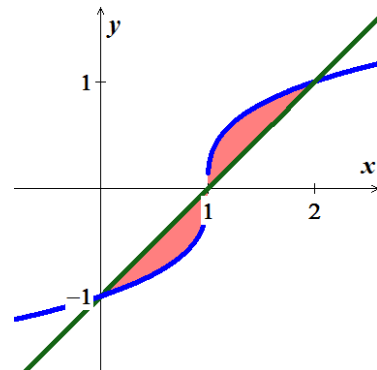
$$(\sqrt[3]{x-1})^3 = (x-1)^3$$

$$x-1 = x^3 - 3x^2 + 3x - 1$$

$$x(x^2 - 3x + 2) = 0$$

$$x = 0, 1, 2$$

$$\begin{aligned} A &= \int_0^1 \left(x-1 - \sqrt[3]{x-1} \right) dx + \int_1^2 \left(\sqrt[3]{x-1} - x+1 \right) dx \\ &= \left(\frac{1}{2}x^2 - x - \frac{3}{4}(x-1)^{4/3} \right) \Big|_0^1 + \left(\frac{3}{4}(x-1)^{4/3} - \frac{1}{2}x^2 + x \right) \Big|_1^2 \\ &= \frac{1}{2} - 1 + \frac{3}{4} + \frac{3}{4} - 2 + 2 + \frac{1}{2} - 1 \\ &= \frac{1}{2} \text{ unit}^2 \end{aligned}$$



Exercise

Find the area of the region bounded by the graphs of $f(y) = y(2 - y)$, $g(y) = -y$

Solution

$$2y - y^2 = -y$$

$$y^2 - 3y = 0$$

$$y = 0, 3$$

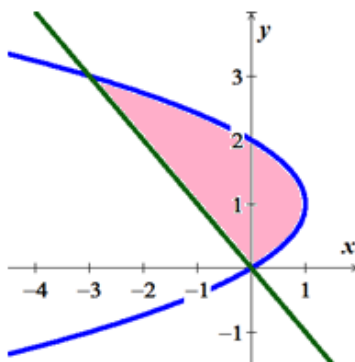
$$A = \int_0^3 (2y - y^2 + y) dy$$

$$= \int_0^3 (3y - y^2) dy$$

$$= \left. \frac{3}{2}y^2 - \frac{1}{3}y^3 \right|_0^3$$

$$= \frac{27}{2} - 9$$

$$= \frac{9}{2} \text{ unit}^2$$



Exercise

Find the area of the region bounded by the graphs of $f(y) = y^2 + 1$, $g(y) = 0$, $y = -1$, $y = 2$

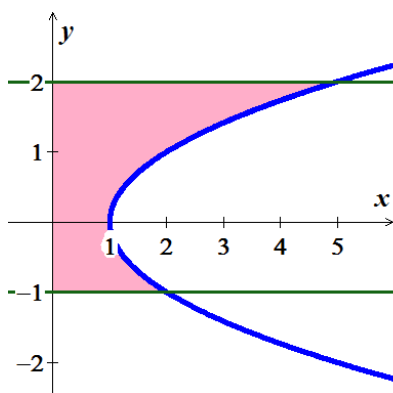
Solution

$$A = \int_{-1}^2 (y^2 + 1 - 0) dy$$

$$= \left. \frac{1}{3}y^3 + y \right|_{-1}^2$$

$$= \frac{8}{3} + 2 + \frac{1}{3} + 1$$

$$= 6 \text{ unit}^2$$

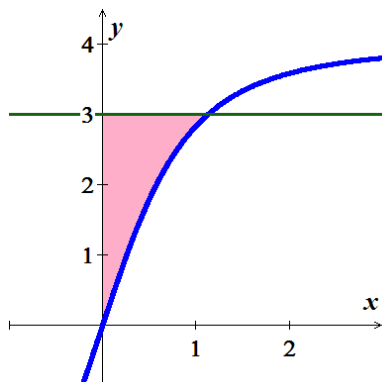


Exercise

Find the area of the region bounded by the graphs of $f(y) = \frac{y}{\sqrt{16-y^2}}$, $g(y) = 0$, $y = 3$

Solution

$$\begin{aligned}
 A &= \int_0^3 \left(\frac{y}{\sqrt{16-y^2}} - 0 \right) dy \\
 &= -\frac{1}{2} \int_0^3 (16-y^2)^{-1/2} d(16-y^2) \\
 &= -\sqrt{16-y^2} \Big|_0^3 \\
 &= -\sqrt{7} + 4 \text{ unit}^2
 \end{aligned}$$

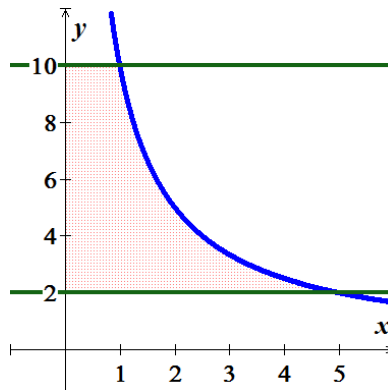


Exercise

Find the area of the region bounded by the graphs of $f(x) = \frac{10}{x}$, $x = 0$, $y = 2$, $y = 10$

Solution

$$\begin{aligned}
 y &= \frac{10}{x} \Rightarrow x = \frac{10}{y} \\
 A &= \int_2^{10} \frac{10}{y} dy \\
 &= 10 \ln y \Big|_2^{10} \\
 &= 10(\ln 10 - \ln 2) \\
 &= 10 \ln 5 \text{ unit}^2
 \end{aligned}$$



Exercise

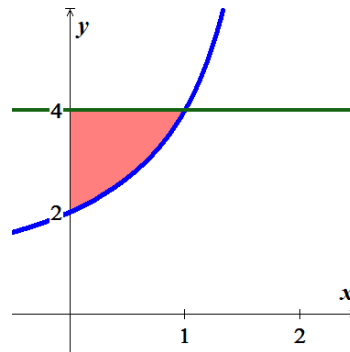
Find the area of the region bounded by the graphs of $g(x) = \frac{4}{2-x}$, $y = 4$, $x = 0$

Solution

$$\begin{aligned}
 \frac{4}{2-x} &= 4 \\
 2-x &= 1 \\
 x &= 1
 \end{aligned}$$

$$A = \int_0^1 \left(4 - \frac{4}{2-x} \right) dx$$

$$\int \frac{dx}{ax+b} = \frac{1}{a} \ln|ax+b|$$



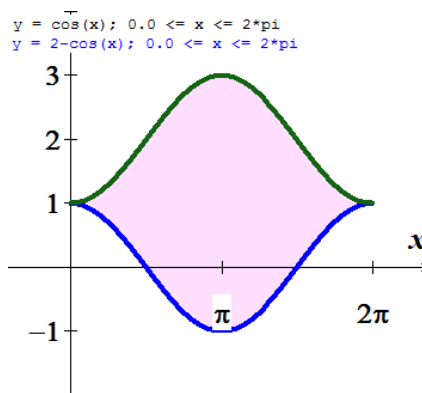
$$\begin{aligned}
 &= 4x + 4 \ln|2-x| \Big|_0^1 \\
 &= \underline{4 + 4 \ln 2 \text{ unit}^2}
 \end{aligned}$$

Exercise

Find the area of the region bounded by the graphs of $f(x) = \cos x$, $g(x) = 2 - \cos x$, $0 \leq x \leq 2\pi$

Solution

$$\begin{aligned}
 A &= \int_0^{2\pi} (2 - \cos x - \cos x) dx \\
 &= 2 \int_0^{2\pi} (1 - \cos x) dx \\
 &= 2 \left(x - \sin x \right) \Big|_0^{2\pi} \\
 &= \underline{4\pi \text{ unit}^2}
 \end{aligned}$$

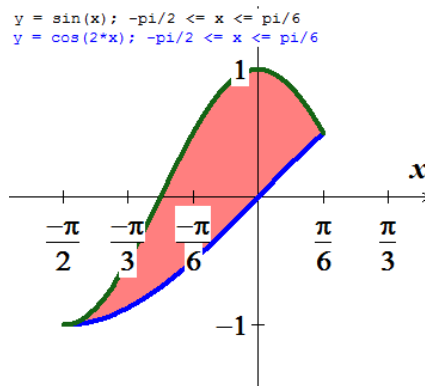


Exercise

Find the area of the region bounded by the graphs of $f(x) = \sin x$, $g(x) = \cos 2x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{6}$

Solution

$$\begin{aligned}
 A &= \int_{-\pi/2}^{\pi/6} (\cos 2x - \sin x) dx \\
 &= \frac{1}{2} \sin 2x + \cos x \Big|_{-\pi/2}^{\pi/6} \\
 &= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} \\
 &= \underline{\frac{3\sqrt{3}}{4} \text{ unit}^2}
 \end{aligned}$$

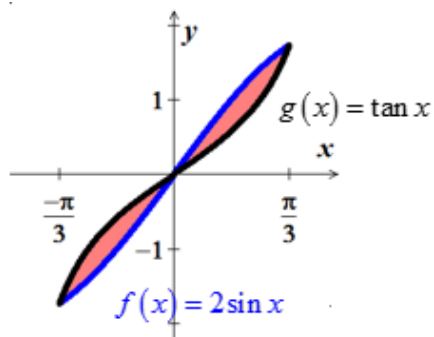


Exercise

Find the area of the region bounded by the graphs of $f(x) = 2 \sin x$, $g(x) = \tan x$, $-\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$

Solution

$$\begin{aligned} A &= 2 \int_0^{\pi/3} (2 \sin x - \tan x) dx \\ &= 2 \left(-2 \cos x + \ln |\cos x| \right) \Big|_0^{\pi/3} \\ &= 2 \left(-1 + \ln \frac{1}{2} + 2 \right) \\ &= \underline{2(1 - \ln 2) \text{ unit}^2} \end{aligned}$$



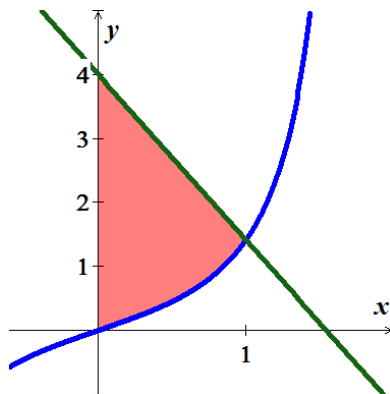
Exercise

Find the area of the region bounded by the graphs of

$$f(x) = \sec \frac{\pi x}{4} \tan \frac{\pi x}{4}, \quad g(x) = (\sqrt{2} - 4)x + 4, \quad x = 0$$

Solution

$$\begin{aligned} A &= \int_0^1 \left((\sqrt{2} - 4)x + 4 - \sec \frac{\pi x}{4} \tan \frac{\pi x}{4} \right) dx \\ &= \frac{1}{2} (\sqrt{2} - 4)x^2 + 4x - \frac{4}{\pi} \sec \frac{\pi x}{4} \Big|_0^1 \\ &= \frac{1}{2} \sqrt{2} - 2 + 4 - \frac{4}{\pi} \sqrt{2} + \frac{4}{\pi} \\ &= \underline{\frac{\sqrt{2}}{2} + 2 + \frac{4}{\pi} (1 - \sqrt{2}) \text{ unit}^2} \end{aligned}$$

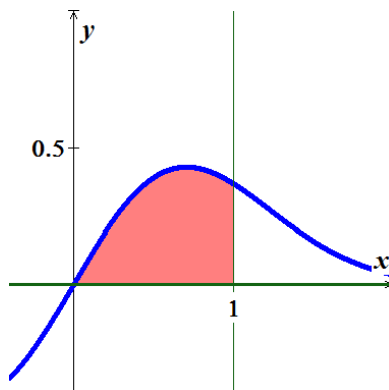


Exercise

Find the area of the region bounded by the graphs of $f(x) = xe^{-x^2}$, $y = 0$, $0 \leq x \leq 1$

Solution

$$\begin{aligned} A &= \int_0^1 x e^{-x^2} dx \\ &= -\frac{1}{2} \int_0^1 e^{-x^2} d(-x^2) \\ &= -\frac{1}{2} e^{-x^2} \Big|_0^1 \end{aligned}$$



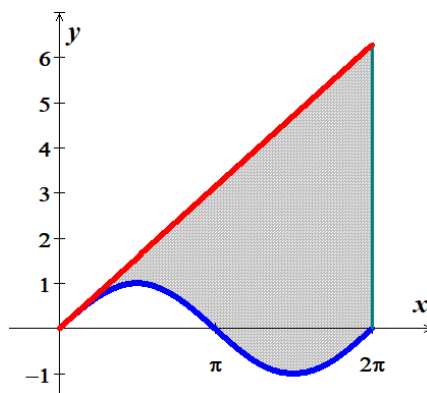
$$\begin{aligned}
 &= -\frac{1}{2}(e^{-1} - 1) \\
 &= \frac{1}{2}\left(1 - \frac{1}{e}\right) \text{ unit}^2
 \end{aligned}$$

Exercise

Find the area of the region between $y = \sin x$ and $y = x$, $0 \leq x \leq 2\pi$

Solution

$$\begin{aligned}
 A &= \int_0^{2\pi} (x - \sin x) dx \\
 &= \frac{1}{2}x^2 + \cos x \Big|_0^{2\pi} \\
 &= 2\pi^2 + 1 - 1 \\
 &= 2\pi^2 \text{ unit}^2
 \end{aligned}$$



Exercise

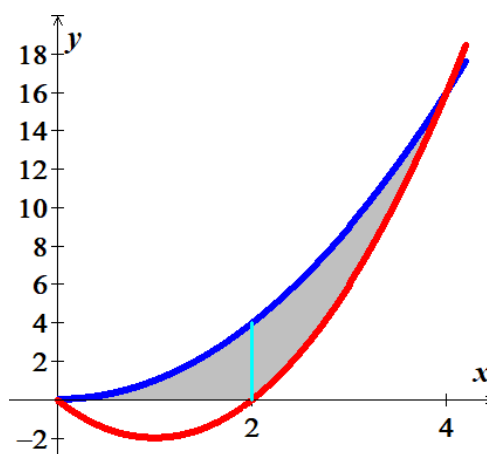
Find the area of the region bounded by $y = x^2$, $y = 2x^2 - 4x$ and $y = 0$

Solution

$$\begin{aligned}
 y = 2x^2 - 4x = x^2 &\rightarrow x^2 - 4x = 0 \\
 &\quad \underline{x = 0, 4}
 \end{aligned}$$

$$y = 2x^2 - 4x = 0 \rightarrow \underline{x = 0, 2}$$

$$\begin{aligned}
 \text{Area} &= \int_0^2 x^2 dx + \int_2^4 (x^2 - 2x^2 + 4x) dx \\
 &= \int_0^2 x^2 dx + \int_2^4 (-x^2 + 4x) dx \\
 &= \frac{1}{3}x^3 \Big|_0^2 + \left(-\frac{1}{3}x^3 + 2x^2\right) \Big|_2^4 \\
 &= \frac{8}{3} - \frac{64}{3} + 32 + \frac{8}{3} - 8 \\
 &= 8 \text{ unit}^2
 \end{aligned}$$

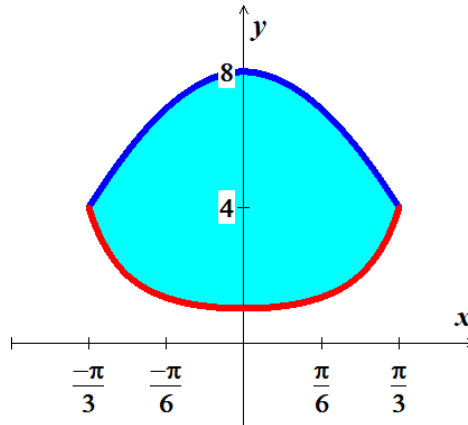


Exercise

Find the area of the region bounded by the curves and line $y = 8 \cos x$, $y = \sec^2 x$, $-\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$

Solution

$$\begin{aligned}
 \text{Area} &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (8 \cos x - \sec^2 x) dx \\
 &= 8 \sin x - \tan x \Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \\
 &= 4\sqrt{3} - \sqrt{3} + 4\sqrt{3} - \sqrt{3} \\
 &= \underline{6\sqrt{3} \text{ unit}^2}
 \end{aligned}$$



Exercise

Find the area of the region bounded by the curves and line $y^2 = 4x + 4$, $y = 4x - 16$

Solution

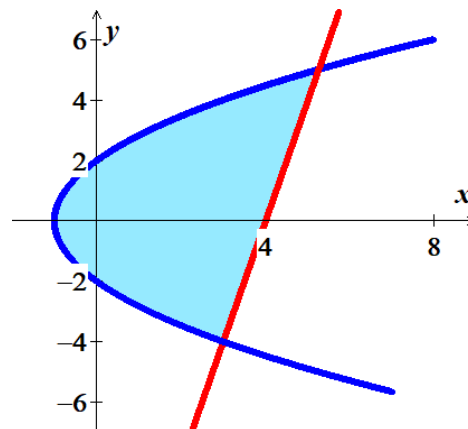
$$x = \frac{1}{4}(y^2 - 4) = \frac{1}{4}(y + 16)$$

$$y^2 - 4 = y + 16$$

$$y^2 - y - 20 = 0$$

$$y = -4, 5$$

$$\begin{aligned}
 \text{Area} &= \int_{-4}^5 \left(\frac{1}{4}y + 4 - \frac{1}{4}y^2 + 1 \right) dy \\
 &= \int_{-4}^5 \left(-\frac{1}{4}y^2 + \frac{1}{4}y + 5 \right) dy \\
 &= -\frac{1}{12}y^3 + \frac{1}{8}y^2 + 5y \Big|_{-4}^5 \\
 &= -\frac{125}{12} + \frac{25}{8} + 25 - \frac{16}{3} - 2 + 20 \\
 &= 43 - \frac{303}{24} \\
 &= \frac{729}{24} \\
 &= \underline{\frac{243}{8} \text{ unit}^2}
 \end{aligned}$$

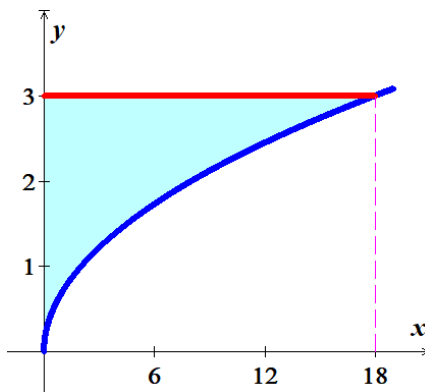


Exercise

Find the area of the region bounded by the curves and line $x = 2y^2$, $x = 0$, $y = 3$

Solution

$$\begin{aligned} \text{Area} &= \int_0^3 (2y^2) dy \\ &= \frac{2}{3} y^3 \Big|_0^3 \\ &= \underline{18 \text{ unit}^2} \end{aligned}$$



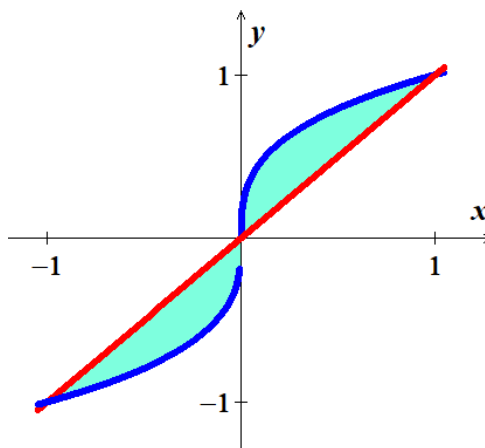
Exercise

Find the area of the region bounded by the curves: $x = y^3$ and $y = x$

Solution

$$\begin{aligned} x &= y^3 = y \\ y(y^2 - 1) &= 0 \\ y &= 0, \pm 1 \end{aligned}$$

$$\begin{aligned} \text{Area} &= \int_{-1}^0 (y^3 - y) dy + \int_0^1 (y - y^3) dy \\ &= 2 \int_0^1 (y - y^3) dy \\ &= 2 \left(\frac{1}{2} y^2 - \frac{1}{4} y^4 \right) \Big|_0^1 \\ &= 2 \left(\frac{1}{2} - \frac{1}{4} \right) \\ &= \underline{\frac{1}{2} \text{ unit}^2} \end{aligned}$$



Exercise

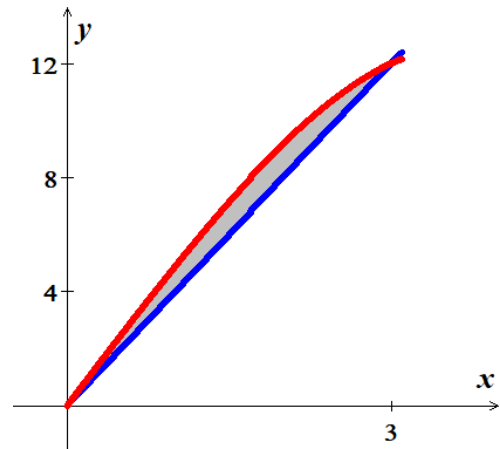
Find the area of the region in the first quadrant bounded by $y = 4x$ and $y = x\sqrt{25 - x^2}$

Solution

$$\begin{aligned} y &= x\sqrt{25 - x^2} = 4x \\ x &= 0 \mid 25 - x^2 = 16 \end{aligned}$$

$$x^2 = 9 \rightarrow \underline{x=3} \mid (\in QI)$$

$$\begin{aligned} \text{Area} &= \int_0^3 \left(x\sqrt{25-x^2} - 4x \right) dx \\ &= -\frac{1}{2} \int_0^3 \left(25-x^2 \right)^{1/2} d\left(25-x^2 \right) - \int_0^3 4x \, dx \\ &= \left(-\frac{1}{3} \left(25-x^2 \right)^{3/2} - 2x^2 \right) \Big|_0^3 \\ &= -\frac{1}{3} (64 - 125) - 18 \\ &= \frac{61}{3} - 18 \\ &= \frac{7}{3} \text{ unit}^2 \end{aligned}$$



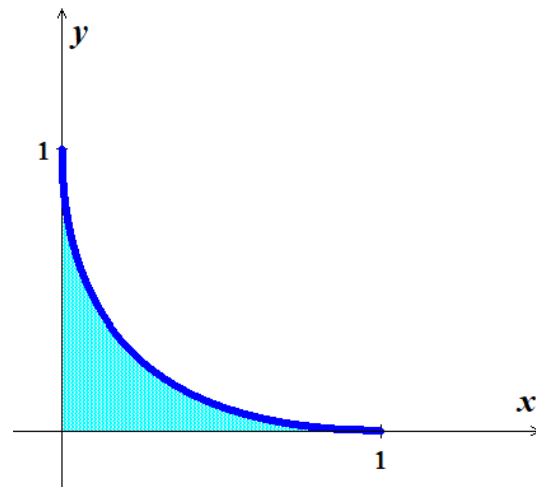
Exercise

Find the area of the region in the first quadrant bounded by the curve $\sqrt{x} + \sqrt{y} = 1$

Solution

$$\begin{aligned} \sqrt{y} &= 1 - \sqrt{x} \\ y &= (1 - \sqrt{x})^2 = 0 \\ \underline{x=1} \mid \end{aligned}$$

$$\begin{aligned} \text{Area} &= \int_0^1 (1 - \sqrt{x})^2 \, dx \\ &= \int_0^1 (1 - 2\sqrt{x} + x) \, dx \\ &= x - \frac{4}{3}x^{3/2} + \frac{1}{2}x^2 \Big|_0^1 \\ &= 1 - \frac{4}{3} + \frac{1}{2} \\ &= \frac{1}{6} \text{ unit}^2 \end{aligned}$$



Exercise

Find the area of the region in the first quadrant bounded by $y = \frac{x}{6}$ and $y = 1 - \left| \frac{x}{2} - 1 \right|$

Solution

$$\frac{x}{2} - 1 = 0 \Rightarrow \underline{x = 2}$$

$$y = 1 - \frac{x}{2} + 1 = \frac{x}{6}$$

$$x\left(\frac{1}{6} + \frac{1}{2}\right) = 2 \rightarrow \underline{x = 3}$$

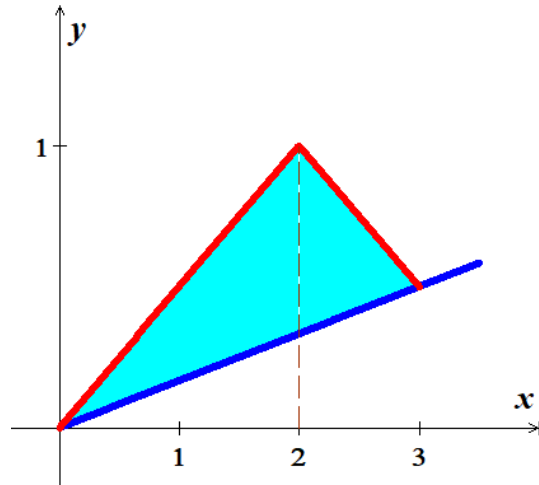
$$Area = \int_0^2 \left(1 + \frac{x}{2} - 1 - \frac{x}{6}\right) dx + \int_2^3 \left(1 - \frac{x}{2} + 1 - \frac{x}{6}\right) dx$$

$$= \int_0^2 \left(\frac{1}{3}x\right) dx + \int_2^3 \left(2 - \frac{2}{3}x\right) dx$$

$$= \frac{1}{6}x^2 \Big|_0^2 + \left(2x - \frac{1}{3}x^2\right) \Big|_2^3$$

$$= \frac{2}{3} + 6 - 3 - 4 + \frac{4}{3}$$

$$= \underline{1 \text{ unit}^2}$$



Exercise

Find the area of the region in the first quadrant bounded by $y = x^p$ and $y = \sqrt[p]{x}$ where $p = 100$ and $p = 1000$

Solution

$$y = x^p = \sqrt[p]{x}$$

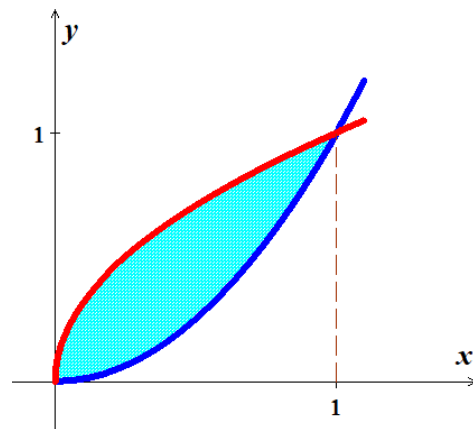
$$\underline{x = 0, 1}$$

$$Area = \int_0^1 \left(x^{1/p} - x^p\right) dx$$

$$= \frac{p}{p+1} x^{\frac{p+1}{p}} - \frac{1}{p+1} x^{p+1} \Big|_0^1$$

$$= \frac{p}{p+1} - \frac{1}{p+1}$$

$$= \underline{\frac{p-1}{p+1} \text{ unit}}$$



For $p = 100$

$$\underline{Area_{100} = \frac{99}{101} \text{ unit}^2}$$

For $p = 1000$

$$\underline{Area_{1000} = \frac{999}{1001} \text{ unit}^2}$$

Exercise

Consider the functions $y = \frac{x^2}{a}$ and $y = \sqrt{\frac{x}{a}}$, where $a > 0$. Find $A(a)$, the area of the region between the curves.

Solution

$$y = \frac{x^2}{a} = \sqrt{\frac{x}{a}}$$

$$\frac{x^4}{a^2} = \frac{x}{a}$$

$$\frac{x}{a^2}(x^3 - a) = 0$$

$$\underline{x = 0, \sqrt[3]{a}}$$

$$\begin{aligned} Area &= \int_0^{\sqrt[3]{a}} \left(\sqrt{\frac{x}{a}} - \frac{x^2}{a} \right) dx \\ &= \frac{2}{3\sqrt{a}} x^{3/2} - \frac{1}{3a} x^3 \bigg|_0^{\sqrt[3]{a}} \\ &= \frac{2}{3} - \frac{1}{3} \\ &= \frac{1}{3} \text{ unit}^2 \end{aligned}$$

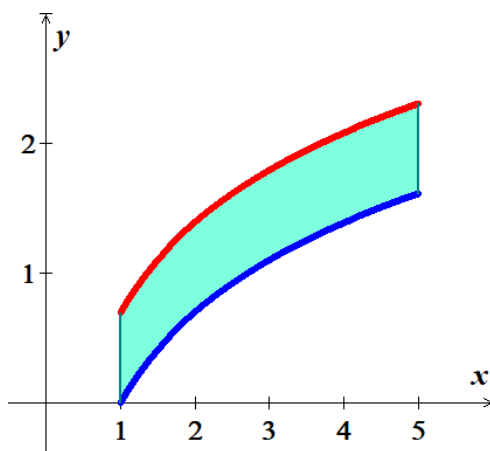
Exercise

Find the area between the curves $y = \ln x$ and $y = \ln 2x$ from $x = 1$ to $x = 5$

Solution

$$Area = \int_1^5 (\ln 2x - \ln x) dx$$

$$\begin{aligned}
&= \int_1^5 (\ln 2 + \ln x - \ln x) dx \\
&= \int_1^5 (\ln 2) dx \\
&= (\ln 2)x \Big|_1^5 \\
&= (\ln 2)(5-1) \\
&= 4 \ln 2 \\
&= \ln 2^4 \\
&= \ln 16 \text{ unit}^2
\end{aligned}$$



Exercise

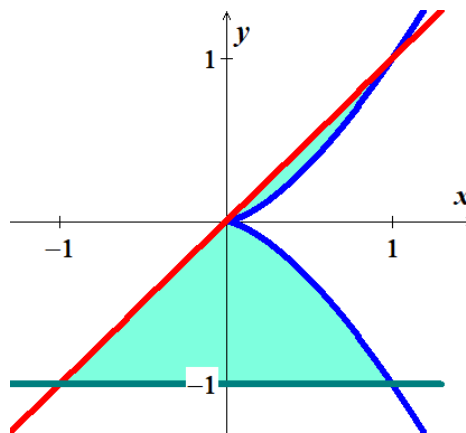
Find the total area of the region enclosed by the curve $x = y^{2/3}$ and lines $x = y$ and $y = -1$

Solution

$$x = y^{2/3} = y$$

$$y = 0, 1$$

$$\begin{aligned}
\text{Area} &= \int_{-1}^0 (y^{2/3} - y) dy + \int_0^1 (y^{2/3} - y) dy \\
&= \left(\frac{3}{5} y^{5/3} - \frac{1}{2} y^2 \right) \Big|_{-1}^0 + \left(\frac{3}{5} y^{5/3} - \frac{1}{2} y^2 \right) \Big|_0^1 \\
&= \frac{3}{5} + \frac{1}{2} + \frac{3}{5} - \frac{1}{2} \\
&= \frac{6}{5} \text{ unit}^2
\end{aligned}$$



Exercise

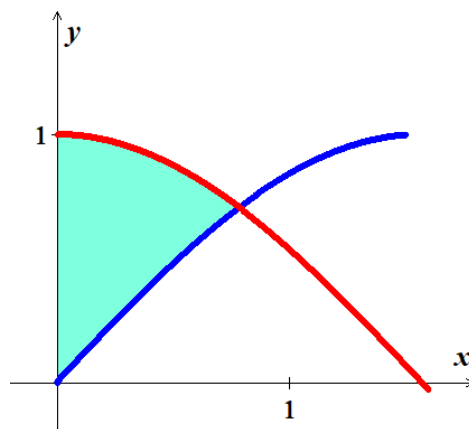
Find the area of the “triangular region in the first quadrant bounded on the left by the y -axis and on the right by the curves $\sin x$ and $\cos x$

Solution

$$\sin x = \cos x$$

$$x = \frac{\pi}{4}$$

$$\begin{aligned}
 \text{Area} &= \int_0^{\pi/4} (\cos x - \sin x) dx \\
 &= \sin x + \cos x \Big|_0^{\pi/4} \\
 &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1 \\
 &= \sqrt{2} - 1 \text{ unit}^2
 \end{aligned}$$



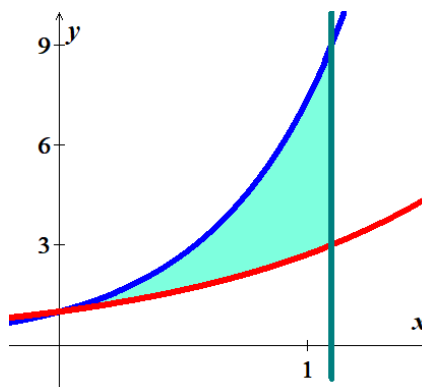
Exercise

Find the area of the “triangular region in the first quadrant bounded above by the curve $y = e^{2x}$, below by the curve $y = e^x$, and on the right by the line $x = \ln 3$

Solution

$$y = e^{2x} = e^x \rightarrow x = 0$$

$$\begin{aligned}
 \text{Area} &= \int_0^{\ln 3} (e^{2x} - e^x) dx \\
 &= \frac{1}{2} e^{2x} - e^x \Big|_0^{\ln 3} \\
 &= \frac{1}{2} e^{2 \ln 3} - e^{\ln 3} - \frac{1}{2} + 1 \\
 &= \frac{1}{2} e^{\ln 9} - 3 + \frac{1}{2} \\
 &= \frac{9}{2} - \frac{5}{2} \\
 &= 2 \text{ unit}^2
 \end{aligned}$$

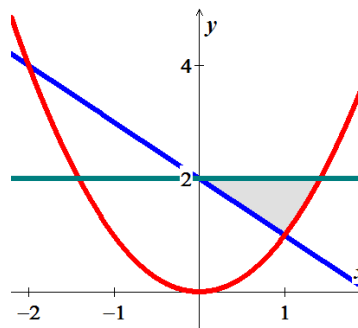


Exercise

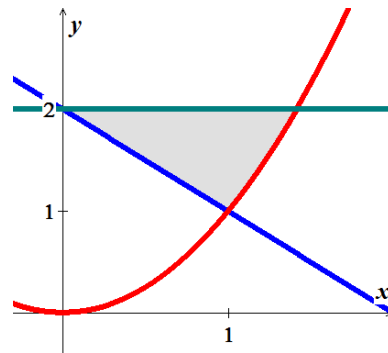
Find the area of the triangular region bounded on the left by $x + y = 2$, on the right by $y = x^2$, and above by $y = 2$

Solution

$$\begin{aligned}
 y &= x^2 = 2 - x \\
 x^2 + x - 2 &= 0 \rightarrow x = -2, 1 \\
 y &= x^2 = 2 \rightarrow x = \sqrt{2}, -\sqrt{2} \\
 y &= 2 - x = 2 \rightarrow x = 0
 \end{aligned}$$



$$\begin{aligned}
 \text{Area} &= \int_0^1 (2 - (2 - x)) dx + \int_1^{\sqrt{2}} (2 - x^2) dx \\
 &= \frac{1}{2}x^2 \Big|_0^1 + \left(2x - \frac{1}{3}x^3 \right) \Big|_1^{\sqrt{2}} \\
 &= \frac{1}{2} + 2\sqrt{2} - \frac{2\sqrt{2}}{3} - 2 + \frac{1}{3} \\
 &= \frac{1 - 2\sqrt{2}}{3} - \frac{3}{2} + 2\sqrt{2} \\
 &= \frac{2 - 4\sqrt{2} - 9 + 12\sqrt{2}}{6} \\
 &= \frac{8\sqrt{2} - 7}{6} \text{ unit}^2
 \end{aligned}$$



Exercise

Find the extreme values of $f(x) = x^3 - 3x^2$ and find the area of the region enclosed by the graph of f and the x -axis.

Solution

$$f'(x) = 3x^2 - 6x = 0$$

$$3x(x - 2) = 0 \rightarrow \underline{x = 0, 2} \quad (CN)$$

$$\underline{f(0) = 0}$$

$$\underline{f(2) = -4}$$

$f(x)$ has a relative minimum at $(2, -4)$

$f(x)$ has a relative maximum at $(0, 0)$

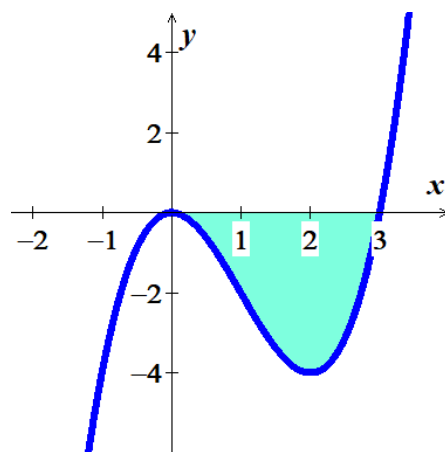
$$f(x) = x^2(x - 3) = 0 \rightarrow \underline{x = 0, 3}$$

$$\text{Area} = - \int_0^3 (x^3 - 3x^2) dx$$

$$= - \frac{1}{4}x^4 + x^3 \Big|_0^3$$

$$= -\frac{81}{4} + 27$$

$$= \frac{27}{4} \text{ unit}^2$$



Exercise

Determine the area of the shaded region in

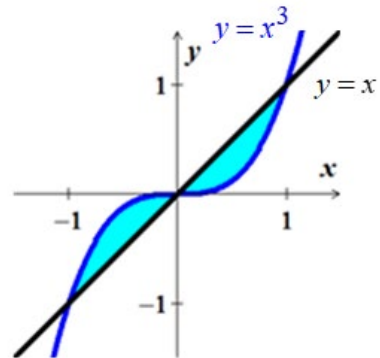
Solution

$$y = x^3 = x$$

$$x(x^2 - 1) = 0$$

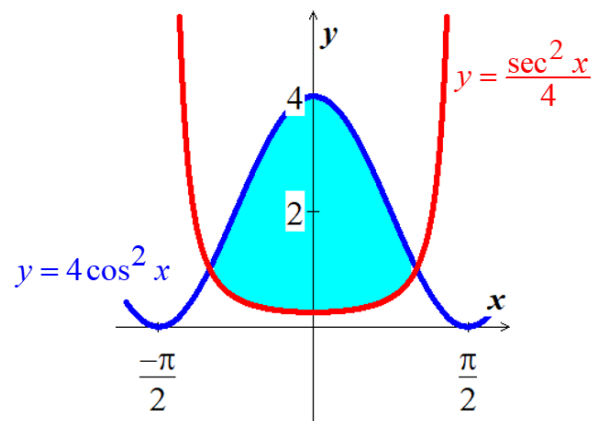
$$\therefore x = 0, \pm 1$$

$$\begin{aligned} \text{Area} &= \int_{-1}^0 (x^3 - x) dx + \int_0^1 (x - x^3) dx \\ &= \left. \frac{1}{4}x^4 - \frac{1}{2}x^2 \right|_{-1}^0 + \left. \left(\frac{1}{2}x^2 - \frac{1}{4}x^4 \right) \right|_0^1 \\ &= -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} \\ &= \frac{1}{2} \text{ unit}^2 \end{aligned}$$



Exercise

Determine the area of the shaded region in



Solution

$$y = \frac{\sec^2 x}{4} = 4\cos^2 x$$

$$\cos^4 x = \frac{1}{16}$$

$$\cos x = \pm \frac{1}{2}$$

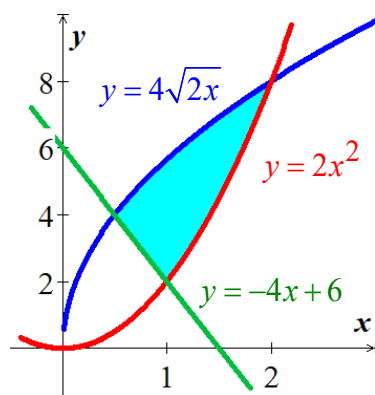
$$x = \pm \frac{\pi}{3}$$

By the symmetry;

$$\begin{aligned}
 \text{Area} &= 2 \int_0^{\pi/3} \left(4\cos^2 x - \frac{1}{4}\sec^2 x \right) dx \\
 &= 2 \int_0^{\pi/3} \left(2 + 2\cos 2x - \frac{1}{4}\sec^2 x \right) dx \\
 &= 2 \left(2x + \sin 2x - \frac{1}{4}\tan x \right) \Big|_0^{\pi/3} \\
 &= 2 \left(\frac{2\pi}{3} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4} \right) \\
 &= \underline{\underline{\frac{4\pi}{3} + \frac{\sqrt{3}}{2} \text{ unit}^2}}
 \end{aligned}$$

Exercise

Determine the area of the shaded region in



Solution

$$y = 4\sqrt{2x} = -4x + 6$$

$$(4\sqrt{2x})^2 = (-4x + 6)^2$$

$$32x = 16x^2 - 48x + 36$$

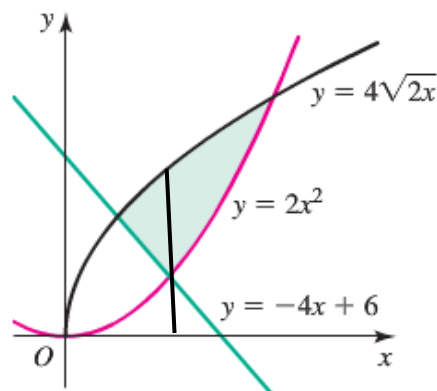
$$16x^2 - 80x + 36 = 0 \rightarrow x = \frac{1}{2}, \frac{9}{2}$$

$$y = 4\sqrt{2x} = 2x^2 \rightarrow (4\sqrt{2x})^2 = (2x^2)^2$$

$$32x = 4x^4 \rightarrow 4x(x^3 - 8) = 0 \rightarrow x = 2, \frac{1}{2}$$

$$y = 2x^2 = -4x + 6 \rightarrow x^2 + 2x - 3 = 0$$

$$\rightarrow x = 1, -3$$



$$\begin{aligned}
 \text{Area} &= \int_{1/2}^1 \left(4\sqrt{2x} - (-4x + 6) \right) dx + \int_1^2 \left(4\sqrt{2x} - 2x^2 \right) dx \\
 &= \left(\frac{8\sqrt{2}}{3} x^{3/2} + 2x^2 - 6x \right) \Big|_{1/2}^1 + \left(\frac{8\sqrt{2}}{3} x^{3/2} - \frac{2}{3} x^3 \right) \Big|_1^2 \\
 &= \left(\frac{8\sqrt{2}}{3} + 2 - 6 - \frac{8\sqrt{2}}{3} \frac{1}{2\sqrt{2}} - \frac{1}{2} + 3 \right) + \left(\frac{32}{3} - \frac{16}{3} - \frac{8\sqrt{2}}{3} + \frac{2}{3} \right) \\
 &= -1 - \frac{4}{3} - \frac{1}{2} + 6 \\
 &= \frac{19}{6} \text{ unit}^2
 \end{aligned}$$

Exercise

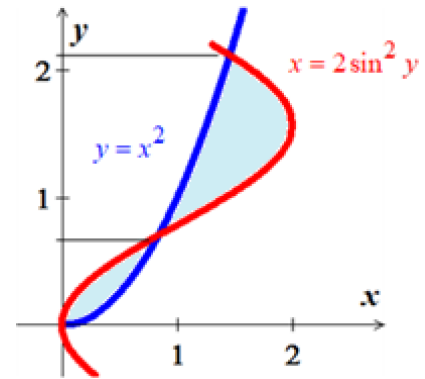
Determine the area of the shaded region in

Solution

From the graph the intersection are:

$$y = 0, \quad y \approx .705, \quad y \approx 2.12$$

$$\begin{aligned}
 A &= \int_0^{.705} \left(\sqrt{y} - 2\sin^2 y \right) dy + \int_{.705}^{2.12} \left(2\sin^2 y - \sqrt{y} \right) dy \\
 &= \int_0^{.705} \left(y^{1/2} - 1 + \cos 2y \right) dy + \int_{.705}^{2.12} \left(1 - \cos 2y - y^{1/2} \right) dy \\
 &= \left(\frac{2}{3} y^{3/2} - y + \frac{1}{2} \sin 2y \right) \Big|_0^{.705} + \left(y - \frac{1}{2} \sin 2y - \frac{2}{3} y^{3/2} \right) \Big|_{.705}^{2.12} \\
 &= \frac{2}{3} (.705)^{3/2} - 0.705 + \frac{1}{2} \sin(1.41) + 2.12 - \frac{1}{2} \sin(4.24) - \frac{2}{3} (2.12)^{3/2} - .705 + \frac{1}{2} \sin(1.41) + \frac{2}{3} (.705)^{3/2} \\
 &\approx .8738 \text{ unit}^2
 \end{aligned}$$



Exercise

Determine the area of the shaded regions between $y = \sin x$ and $y = \sin 2x$, for $0 \leq x \leq \pi$

Solution

$$y = \sin x = \sin 2x$$

$$\sin x = 2 \sin x \cos x$$

$$\sin x(2 \cos x - 1) = 0$$

$$\sin x = 0 \rightarrow x = 0, \pi$$

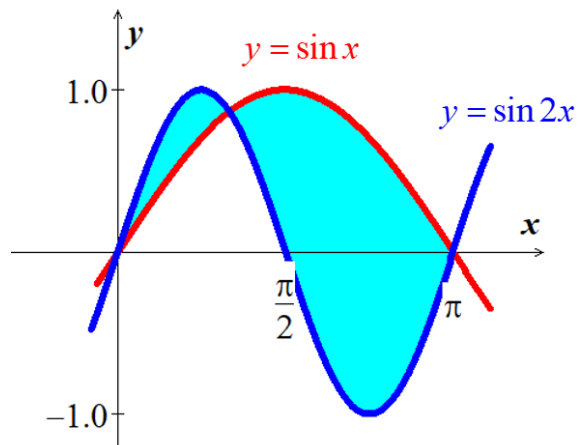
$$\cos x = \frac{1}{2} \rightarrow x = \frac{\pi}{3}$$

$$A = \int_0^{\pi/3} (\sin 2x - \sin x) dx + \int_{\pi/3}^{\pi} (\sin x - \sin 2x) dx$$

$$= \left(-\frac{1}{2} \cos 2x + \cos x \right) \Big|_0^{\pi/3} + \left(-\cos x + \frac{1}{2} \cos 2x \right) \Big|_{\pi/3}^{\pi}$$

$$= \left(\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - 1 \right) + \left(1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} \right)$$

$$= \frac{5}{2} \text{ unit}^2$$



Exercise

Determine the area of the shaded region bounded by the curve $x^2 = y^4(1 - y^3)$

Solution

$$x^2 = y^4(1 - y^3)$$

$$x = y^2 \sqrt{1 - y^3}$$

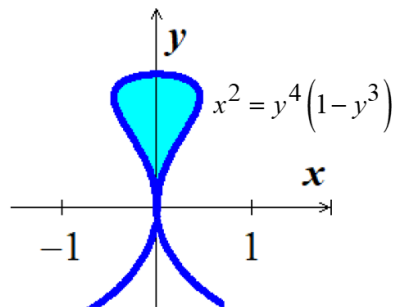
Since it is symmetric about y-axis, then

$$A = 2 \int_0^1 y^2 \sqrt{1 - y^3} dy$$

$$= -\frac{2}{3} \int_0^1 (1 - y^3)^{1/2} d(1 - y^3)$$

$$= -\frac{4}{9} (1 - y^3)^{3/2} \Big|_0^1$$

$$= \frac{4}{9} \text{ unit}^2$$



Exercise

Determine the area of the region bounded by the curves

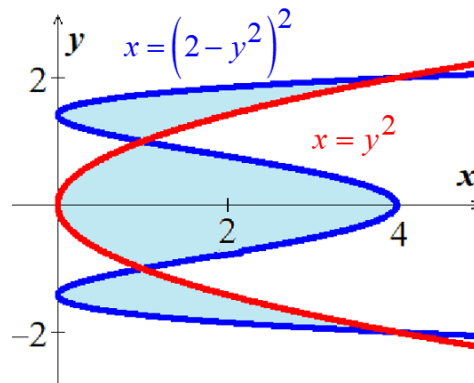
$$x = y^2 \text{ and } x = (2 - y^2)^2$$

Solution

$$x = y^2 = 4 - 4y^2 + y^4$$

$$y^4 - 5y^2 + 4 = 0$$

$$\begin{cases} y^2 = 1 & \rightarrow y = \pm 1 \\ y^2 = 4 & \rightarrow y = \pm 2 \end{cases}$$



$$Area = \int_{-2}^{-1} \left(y^2 - (2 - y^2)^2 \right) dy + \int_{-1}^1 \left((2 - y^2)^2 - y^2 \right) dy + \int_1^2 \left(y^2 - (2 - y^2)^2 \right) dy$$

$$= \int_{-2}^{-1} (5y^2 - 4 - y^4) dy + \int_{-1}^1 (y^4 - 5y^2 + 4) dy + \int_1^2 (5y^2 - 4 - y^4) dy$$

$$= \left(\frac{5}{3}y^3 - 4y - \frac{1}{5}y^5 \right) \Big|_{-2}^{-1} + \left(\frac{1}{5}y^5 - \frac{5}{3}y^3 + 4y \right) \Big|_{-1}^1 + \left(\frac{5}{3}y^3 - 4y - \frac{1}{5}y^5 \right) \Big|_1^2$$

$$= \left(-\frac{5}{3} + 4 + \frac{1}{5} + \frac{40}{3} - 8 - \frac{32}{5} \right) + \left(\frac{1}{5} - \frac{5}{3} + 4 + \frac{1}{5} - \frac{5}{3} + 4 \right) + \left(\frac{40}{3} - 8 - \frac{32}{5} - \frac{5}{3} + 4 + \frac{1}{5} \right)$$

$$= \frac{35}{3} - 4 - \frac{31}{5} + \frac{2}{5} - \frac{10}{3} + 8 + \frac{35}{3} - 4 - \frac{31}{5}$$

$$= 20 - 12$$

$$= 8 \text{ unit}^2$$

Exercise

Find the area of the region bounded by the curves and line

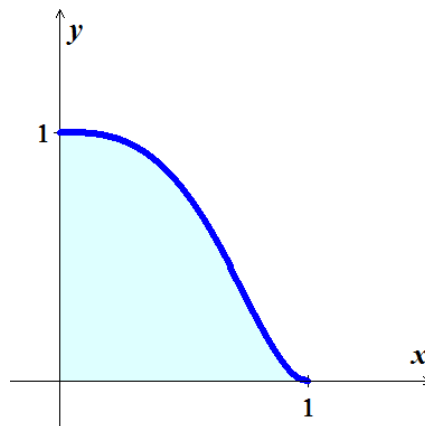
$$x^3 + \sqrt{y} = 1, \quad x = 0, \quad y = 0, \quad \text{for } 0 \leq x \leq 1$$

Solution

$$\sqrt{y} = 1 - x^3$$

$$y = (1 - x^3)^2$$

$$Area = \int_0^1 (1 - 2x^3 + x^6) dx$$



$$\begin{aligned}
 &= x - \frac{1}{2}x^4 + \frac{1}{7}x^7 \Big|_0^1 \\
 &= 1 - \frac{1}{2} + \frac{1}{7} \\
 &= \frac{9}{14} \text{ unit}^2
 \end{aligned}$$

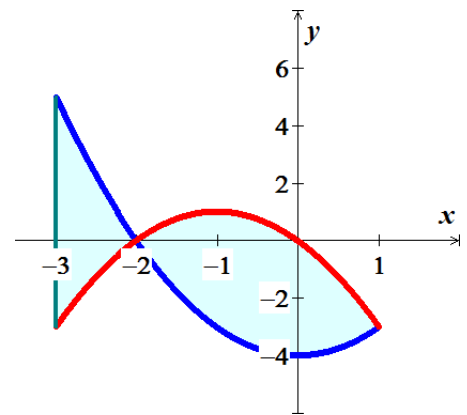
Exercise

Determine the area of the shaded regions: $y = x^2 - 4$, $y = -x^2 - 2x$, $-3 \leq x \leq 1$

Solution

$$y = x^2 - 4, \quad y = -x^2 - 2x$$

$$\begin{aligned}
 \text{Area} &= \int_{-3}^{-2} (x^2 - 4 + x^2 + 2x) dx + \int_{-2}^1 (-x^2 - 2x - x^2 + 4) dx \\
 &= \int_{-3}^{-2} (2x^2 + 2x - 4) dx + \int_{-2}^1 (-2x^2 - 2x + 4) dx \\
 &= \left. \frac{2}{3}x^3 + x^2 - 4x \right|_{-3}^{-2} + \left. \left(-\frac{2}{3}x^3 - x^2 + 4x \right) \right|_{-2}^1 \\
 &= -\frac{16}{3} + 4 + 8 + 18 - 9 - 12 - \frac{2}{3} - 1 + 4 - \frac{16}{3} + 4 + 8 \\
 &= 24 - \frac{34}{3} \\
 &= \frac{38}{3} \text{ unit}^2
 \end{aligned}$$



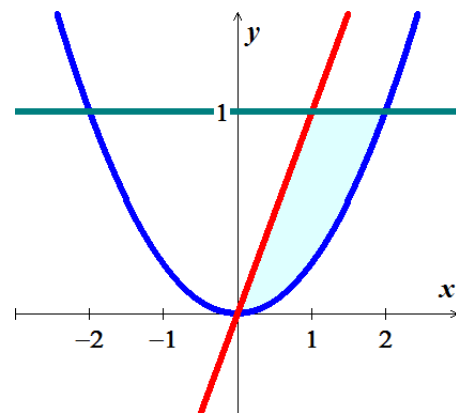
Exercise

Determine the area of the shaded regions: $y = \frac{1}{4}x^2$, $y = x$, $y = 1$

Solution

$$y = \frac{1}{4}x^2 \rightarrow x = 2\sqrt{y}$$

$$\begin{aligned}
 \text{Area} &= \int_0^1 (2y^{1/2} - y) dy \\
 &= \left. \frac{4}{3}y^{3/2} - \frac{1}{2}y^2 \right|_0^1
 \end{aligned}$$



$$= \frac{4}{3} - \frac{1}{2}$$

$$= \frac{5}{6} \text{ unit}^2$$

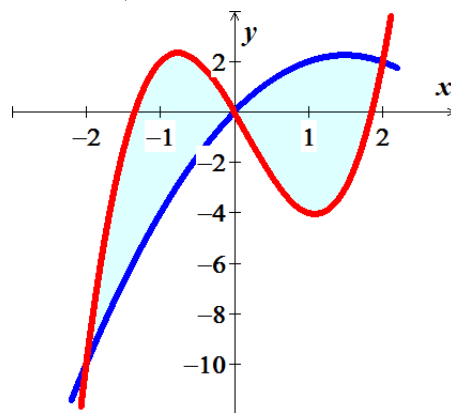
Exercise

Determine the area of the shaded regions: $y = -x^2 + 3x$, $y = 2x^3 - x^2 - 5x$, $-2 \leq x \leq 2$

Solution

$$y = -x^2 + 3x, \quad y = 2x^3 - x^2 - 5x$$

$$\begin{aligned} \text{Area} &= \int_{-2}^0 (2x^3 - x^2 - 5x + x^2 - 3x) dx + \int_0^2 (-x^2 + 3x - 2x^3 + x^2 + 5x) dx \\ &= \int_{-2}^0 (2x^3 - 8x) dx + \int_0^2 (-2x^3 + 8x) dx \\ &= \frac{1}{2}x^4 - 4x^2 \Big|_{-2}^0 + \left(-\frac{1}{2}x^4 + 4x^2 \right) \Big|_0^2 \\ &= -8 + 16 - 8 + 16 \\ &= 16 \text{ unit}^2 \end{aligned}$$



Exercise

Determine the area of the shaded regions:

$$y = 4 - x^2, \quad y = -x + 2, \quad -2 \leq x \leq 3$$

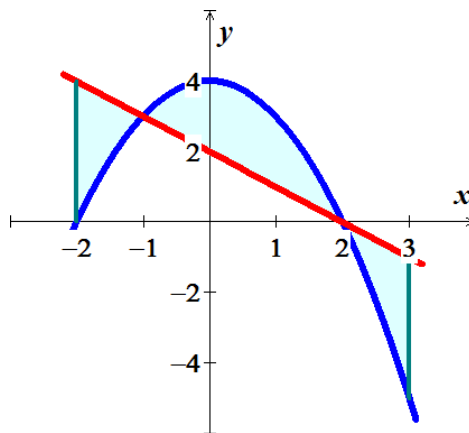
Solution

$$y = 4 - x^2, \quad y = -x + 2$$

$$y = 4 - x^2 = -x + 2$$

$$x^2 - x - 2 = 0$$

$$x = -1, 2$$



$$\begin{aligned} \text{Area} &= \int_{-2}^{-1} (-x + 2 - 4 + x^2) dx + \int_{-1}^2 (4 - x^2 + x - 2) dx + \int_2^3 (-x + 2 - 4 + x^2) dx \\ &= \int_{-2}^{-1} (-x - 2 + x^2) dx + \int_{-1}^2 (-x^2 + x + 2) dx + \int_2^3 (-x - 2 + x^2) dx \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2}x^2 - 2x + \frac{1}{3}x^3 \Big|_{-2}^{-1} + \left(-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x \right) \Big|_{-1}^2 + \left(-\frac{1}{2}x^2 - 2x + \frac{1}{3}x^3 \right) \Big|_2^3 \\
&= -\frac{1}{2} + 2 - \frac{1}{3} + 2 - 4 + \frac{8}{3} - \frac{8}{3} + 2 + 4 - \frac{1}{3} - \frac{1}{2} + 2 - \frac{9}{2} - 6 + 9 + 2 + 4 - \frac{8}{3} \\
&= -\frac{10}{3} - \frac{9}{2} + 16 \\
&= \frac{49}{6} \text{ unit}^2
\end{aligned}$$

Exercise

Determine the area of the shaded regions: $y = \frac{1}{3}x^3 - x$, $y = \frac{1}{3}x$, $-2 \leq x \leq 3$

Solution

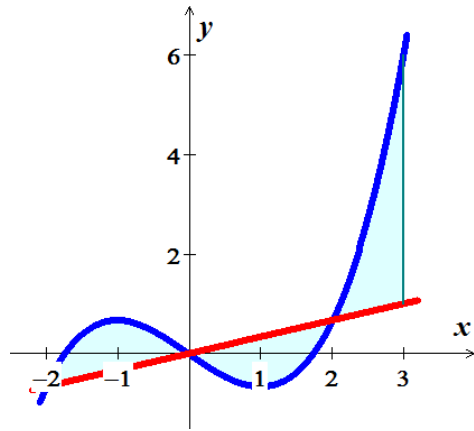
$$y = \frac{1}{3}x^3 - x, \quad y = \frac{1}{3}x$$

$$y = \frac{1}{3}x^3 - x = \frac{1}{3}x$$

$$x^3 - 4x = 0$$

$$x(x^2 - 4) = 0$$

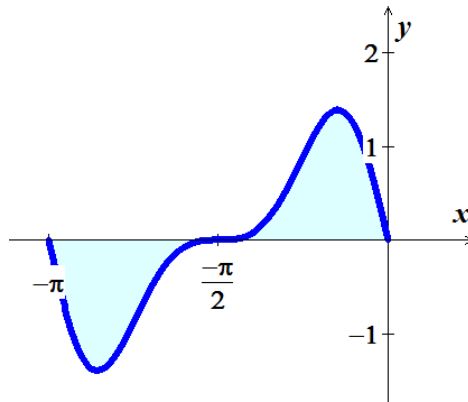
$$x = 0, \pm 2$$



$$\begin{aligned}
\text{Area} &= \int_{-2}^0 \left(\frac{1}{3}x^3 - x - \frac{1}{3}x \right) dx + \int_0^2 \left(\frac{1}{3}x - \frac{1}{3}x^3 + x \right) dx + \int_2^3 \left(\frac{1}{3}x^3 - x - \frac{1}{3}x \right) dx \\
&= \int_{-2}^0 \left(\frac{1}{3}x^3 - \frac{4}{3}x \right) dx + \int_0^2 \left(\frac{4}{3}x - \frac{1}{3}x^3 \right) dx + \int_2^3 \left(\frac{1}{3}x^3 - \frac{4}{3}x \right) dx \\
&= \frac{1}{12}x^4 - \frac{2}{3}x^2 \Big|_{-2}^0 + \left(\frac{2}{3}x^2 - \frac{1}{12}x^4 \right) \Big|_0^2 + \left(\frac{1}{12}x^4 - \frac{2}{3}x^2 \right) \Big|_2^3 \\
&= -\frac{4}{3} + \frac{8}{3} + \frac{8}{3} - \frac{4}{3} + \frac{27}{4} - 6 - \frac{4}{3} + \frac{8}{3} \\
&= \frac{27}{4} - 2 \\
&= \frac{19}{4} \text{ unit}^2
\end{aligned}$$

Exercise

Determine the area of the shaded regions: $y = \frac{\pi}{2} \cos x \sin(\pi + \pi \sin x) \quad -\pi \leq x \leq 0$



Solution

$$d(\pi + \pi \sin x) = \pi \cos x \, dx$$

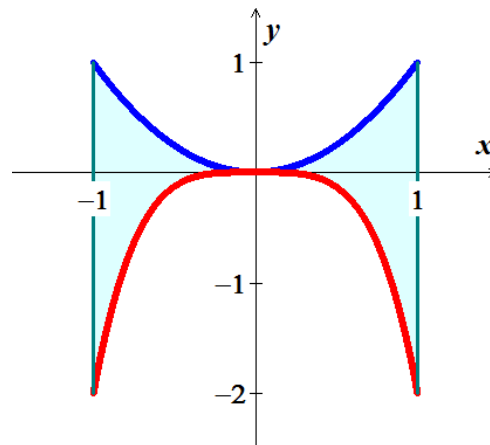
$$\begin{aligned} \text{Area} &= \int_{-\pi}^{-\pi/2} \left(-\frac{\pi}{2} \cos x \sin(\pi + \pi \sin x) \right) dx + \int_{-\pi/2}^0 \left(\frac{\pi}{2} \cos x \sin(\pi + \pi \sin x) \right) dx \\ &= 2 \int_{-\pi/2}^0 \left(\frac{1}{2} \sin(\pi + \pi \sin x) \right) d(\pi + \pi \sin x) \\ &= -\cos(\pi + \pi \sin x) \Big|_{-\pi/2}^0 \\ &= -\cos \pi + \cos 0 \\ &= 2 \text{ unit}^2 \end{aligned}$$

Exercise

Determine the area of the shaded regions: $y = x^2$, $y = -2x^4$, $-1 \leq x \leq 1$

Solution

$$\begin{aligned} \text{Area} &= \int_{-1}^1 (x^2 + 2x^4) dx \\ &= \frac{1}{3}x^3 + \frac{2}{5}x^5 \Big|_{-1}^1 \\ &= \frac{1}{3} + \frac{2}{5} + \frac{1}{3} + \frac{2}{5} \\ &= \frac{2}{3} + \frac{4}{5} \\ &= \frac{22}{15} \text{ unit}^2 \end{aligned}$$

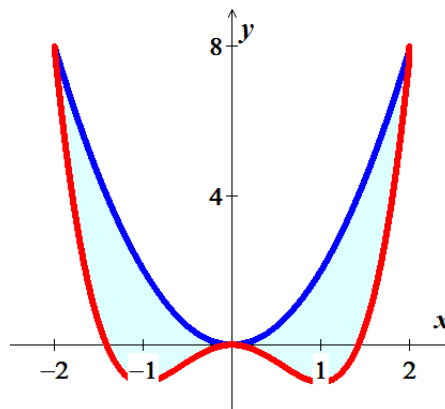


Exercise

Determine the area of the shaded regions: $y = 2x^2$, $y = x^4 - 2x^2$, $-2 \leq x \leq 2$

Solution

$$\begin{aligned}
 \text{Area} &= \int_{-2}^2 (2x^2 - x^4 + 2x^2) dx \\
 &= \int_{-2}^2 (4x^2 - x^4) dx \\
 &= \left. \frac{4}{3}x^3 - \frac{1}{5}x^5 \right|_{-2}^2 \\
 &= \frac{32}{3} - \frac{32}{5} + \frac{32}{3} - \frac{32}{5} \\
 &= \frac{64}{3} - \frac{64}{5} \\
 &= \frac{128}{15} \text{ unit}^2
 \end{aligned}$$



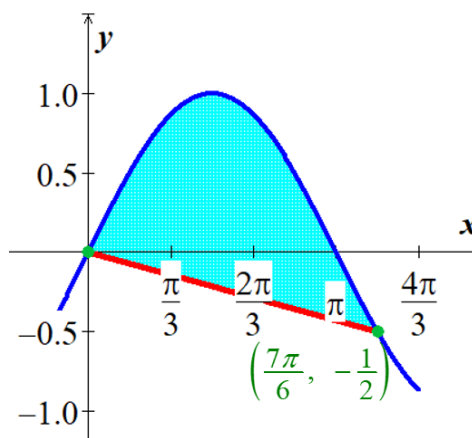
Exercise

Find the area between the graph of $y = \sin x$ and the line segment joining the points $(0, 0)$ and $(\frac{7\pi}{6}, -\frac{1}{2})$.

Solution

$$\begin{aligned}
 \text{Line: } y &= \frac{-\frac{1}{2}}{\frac{7\pi}{6}} \left(x - \frac{7\pi}{6} \right) - \frac{1}{2} \\
 &= -\frac{3}{7\pi} \left(x - \frac{7\pi}{6} \right) - \frac{1}{2} \\
 &= -\frac{3}{7\pi} x
 \end{aligned}$$

$$\begin{aligned}
 A &= \int_0^{7\pi/6} \left(\sin x + \frac{3}{7\pi} x \right) dx \\
 &= -\cos x + \frac{3}{14\pi} x^2 \Big|_0^{7\pi/6} \\
 &= \frac{\sqrt{3}}{2} + \frac{7\pi}{24} + 1 \text{ unit}^2
 \end{aligned}$$



Exercise

The surface of a machine part is the region between the graphs of $y_1 = |x|$ and $y_2 = 0.08x^2 + k$

- Find k where the parabola is tangent to the graph of y_1
- Find the area of the surface of the machine part.

Solution

$$a) \quad y'_1 = 1 \quad y'_2 = 0.16x$$

$$0.16x = 1$$

$$x = 6.25$$

$$y_1 = y_2$$

$$6.25 = 0.08(6.25)^2 + k$$

$$k = 6.25 - 0.08(6.25)^2$$

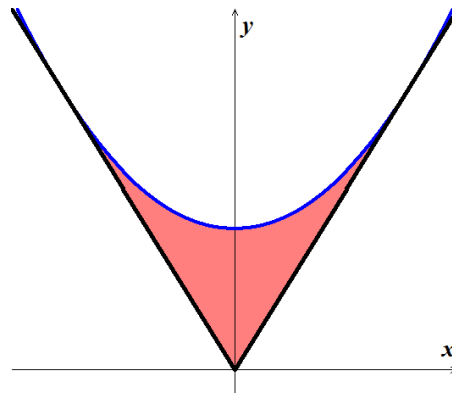
$$= 3.125$$

$$b) \quad A = 2 \int_0^{6.25} (y_2 - y_1) dx$$

$$= 2 \int_0^{6.25} (0.08x^2 + 3.125 - x) dx$$

$$= 2 \left(\frac{0.08}{3} x^3 + 3.125x - \frac{1}{2} x^2 \right) \Big|_0^{6.25}$$

$$\approx 13.02083 \text{ unit}^2$$



Exercise

Find the area of the regions R_1 and R_2 (separately) shown in the figure, which are formed by the graphs of

$$y = 16 - x^2 \quad \text{and} \quad y = 5x - 8$$

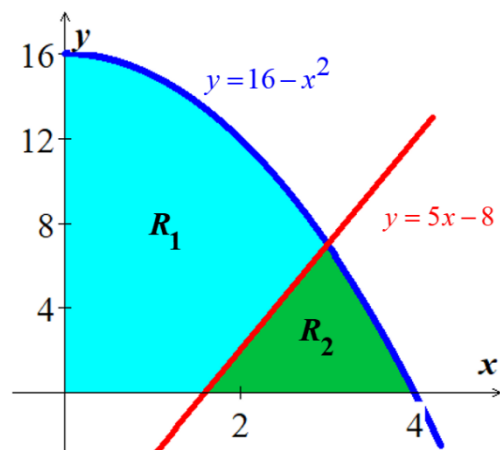
Solution

$$y = 5x - 8 = 0 \rightarrow x = \frac{8}{5}$$

$$y = 16 - x^2 = 5x - 8$$

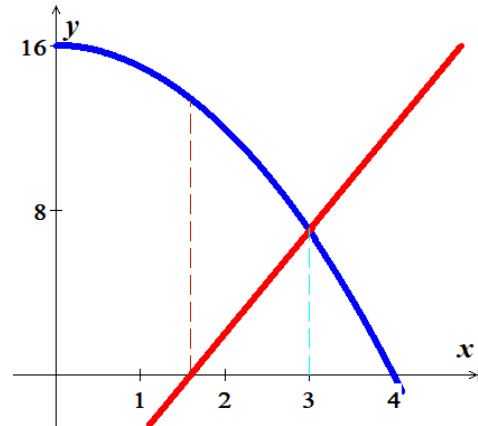
$$x^2 + 5x - 24 = 0 \rightarrow x = 3, \quad x = -8$$

Region R_1 :



$$\begin{aligned}
 \text{Area} &= \int_0^{8/5} (16 - x^2) dx + \int_{8/5}^3 (16 - x^2 - 5x + 8) dx \\
 &= 16x - \frac{1}{3}x^3 \Big|_0^{8/5} + \left(24x - \frac{1}{3}x^3 - \frac{5}{2}x^2 \right) \Big|_{8/5}^3 \\
 &= \frac{128}{5} - \frac{512}{375} + 72 - 9 - \frac{45}{2} - \frac{192}{5} + \frac{512}{375} - \frac{64}{10} \\
 &= \frac{341}{10} \text{ unit}^2
 \end{aligned}$$

$y = 5x - 8$



Region R_2 :

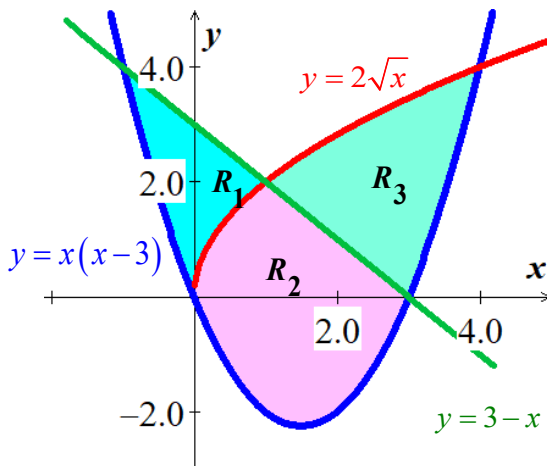
$$\begin{aligned}
 \text{Area} &= \int_{8/5}^3 (5x - 8) dx + \int_3^4 (16 - x^2) dx \\
 &= \left(\frac{5}{2}x^2 - 8x \right) \Big|_{8/5}^3 + \left(16x - \frac{1}{3}x^3 \right) \Big|_3^4 \\
 &= \frac{45}{2} - 24 - \frac{8}{5} + \frac{64}{5} + 64 - \frac{64}{3} - 48 - 9 \\
 &= \frac{257}{30} \text{ unit}^2
 \end{aligned}$$

Exercise

Find the area of the regions R_1 , R_2 and R_3 (separately) shown in the figure, which are formed by the graphs of $y = 2\sqrt{x}$, $y = 3 - x$, and $y = x(x - 3)$

Solution

$$\begin{aligned}
 y &= x^2 - 3x = 3 - x \\
 x^2 - 2x - 3 &= 0 \rightarrow \underline{x = -1, 3} \\
 y &= x^2 - 3x = 2\sqrt{x} \\
 \text{from graph} &\rightarrow \underline{x = 0, 4} \\
 y &= 3 - x = 2\sqrt{x} \\
 9 - 6x + x^2 &= 4x \\
 x^2 - 10x + 9 &= 0 \rightarrow \underline{x = 1, 9}
 \end{aligned}$$



Region R_1 :

$$\begin{aligned}
 \text{Area} &= \int_{-1}^0 (3-x-x^2+3x) dx + \int_0^1 (3-x-2\sqrt{x}) dx \\
 &= 3x + x^2 - \frac{1}{3}x^3 \Big|_{-1}^0 + \left(3x - \frac{1}{2}x^2 - \frac{4}{3}x^{3/2} \right) \Big|_0^1 \\
 &= 3 - 1 - \frac{1}{3} + 3 - \frac{1}{2} - \frac{4}{3} \\
 &= \underline{\underline{\frac{17}{6} \text{ unit}^2}}
 \end{aligned}$$

Region R_2 :

$$\begin{aligned}
 \text{Area} &= \int_0^1 (2\sqrt{x}-x^2+3x) dx + \int_1^3 (3-x-x^2+3x) dx \\
 &= \frac{4}{3}x^{3/2} - \frac{1}{3}x^3 + \frac{3}{2}x^2 \Big|_0^1 + \left(3x + x^2 - \frac{1}{3}x^3 \right) \Big|_1^3 \\
 &= \frac{4}{3} - \frac{1}{3} + \frac{3}{2} + 9 + 9 - 9 - 3 - 1 + \frac{1}{3} \\
 &= \underline{\underline{\frac{47}{6} \text{ unit}^2}}
 \end{aligned}$$

Region R_3 :

$$\begin{aligned}
 \text{Area} &= \int_1^3 (2\sqrt{x}-3+x) dx + \int_3^4 (2\sqrt{x}-x^2+3x) dx \\
 &= \frac{4}{3}x^{3/2} - 3x + \frac{1}{2}x^2 \Big|_1^3 + \left(\frac{4}{3}x^{3/2} - \frac{1}{3}x^3 + \frac{3}{2}x^2 \right) \Big|_3^4 \\
 &= 4\sqrt{3} - 9 + \frac{9}{2} - \frac{4}{3} + 3 - \frac{1}{2} + \frac{32}{3} - \frac{64}{3} + 24 - 4\sqrt{3} + 9 - \frac{27}{2} \\
 &= \underline{\underline{\frac{11}{2} \text{ unit}^2}}
 \end{aligned}$$

Exercise

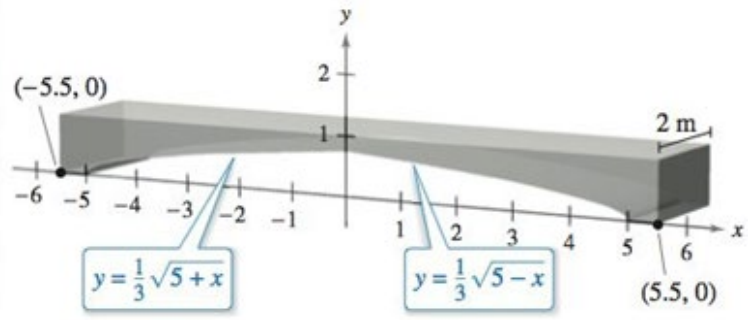
Concrete sections for a new building have the dimensions (in meters) and shape shown in figure

- Find the area of the face of the section superimposed on the rectangular coordinate system.
- Find the volume of concrete in one of the sections by multiplying the area in part (a) by 2 meters.
- One cubic meter of concrete weighs 5,000 pounds. Find the weight of the section.

Solution

$$a) \quad A = 2 \int_0^5 \left(1 - \frac{1}{3}\sqrt{5+x} \right) dx + 2 \int_5^{5.5} (1-0) dx$$

$$\begin{aligned}
&= 2 \left(x + \frac{2}{9}(5-x)^{3/2} \right) \Big|_0^5 + 2x \Big|_5^{5.5} \\
&= 2 \left(5 - \frac{2}{9}5^{3/2} \right) + 2(5.5 - 5) \\
&= 10 - \frac{20\sqrt{5}}{9} + 1 \\
&= \left(11 - \frac{20\sqrt{5}}{9} \right) m^2
\end{aligned}$$



b) $V = 2A$

$$= \left(22 - \frac{40\sqrt{5}}{9} \right) m^3$$

c) $W = 5,000V$

$$= \left(11 - \frac{20\sqrt{5}}{9} \right) \times 10^4 \text{ lb}$$

Exercise

A Lorenz curve is given by $y = L(x)$, where $0 \leq x \leq 1$ represents the lowest fraction of the population of a society in terms of wealth and $0 \leq y \leq 1$ represents the fraction of the total wealth that is owned by that fraction of the society. For example, the Lorenz curve in the figure shows that $L(0.5) = 0.2$, which means that the lowest 0.5 (50%) of the society owns 0.2 (20%) of the wealth.

a) A Lorenz curve $y = L(x)$ is accompanied by the line $y = x$, called the **line of perfect equality**.

Explain why this line is given the name.

b) Explain why a Lorenz curve satisfies the conditions $L(0) = 0$, $L(1) = 1$, and $L'(x) \geq 0$ on $[0, 1]$

c) Graph the Lorenz curves $L(x) = x^p$ corresponding to $p = 1.1, 1.5, 2, 3, 4$. Which value of p corresponds to the **most** equitable distribution of wealth (closest to the line of perfect equality)? Which value of p corresponds to the **least** equitable distribution of wealth? Explain.

d) The information in the Lorenz curve is often summarized in a single measure called the **Gini index**, which is defined as follows. Let A be the area of the region between $y = x$ and $y = L(x)$ and Let B be the area of the region between $y = L(x)$ and the x -axis. Then the Gini index is $G = \frac{A}{A+B}$.

$$\text{Show that } G = 2A = 1 - 2 \int_0^1 L(x) dx.$$

e) Compute the Gini index for the cases $L(x) = x^p$ and $p = 1.1, 1.5, 2, 3, 4$.

f) What is the smallest interval $[a, b]$ on which values of the Gini index lie, for $L(x) = x^p$ with $p \geq 1$? Which endpoints of $[a, b]$ correspond to the least and most equitable distribution of wealth?

g) Consider the Lorenz curve described by $L(x) = \frac{5x^2}{6} + \frac{x}{6}$. Show that it satisfies the conditions

$L(0) = 0$, $L(1) = 1$, and $L'(x) \geq 0$ on $[0, 1]$. Find the Gini index for this function.

Solution

- a) Let the point $N = (a, a)$ on the curve $y = x$ would represent the notion that the lowest $p\%$ of the society owns $p\%$ of the wealth, which would represent a form of equality.
- b) The function must be increasing and concave up because the poorest $p\%$ cannot own more than $p\%$ of the wealth.

c) $y = x^{1.1}$ is closet to $y = x$, and $y = x^4$ is furthest from $y = x$

d) Since, $B = \int_0^1 L(x) dx$ and $A + B = \frac{1}{2}$

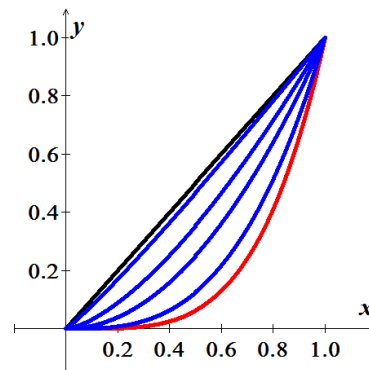
$$\text{Then } A = \frac{1}{2} - B = \frac{1}{2} - \int_0^1 L(x) dx$$

$$G = \frac{A}{A+B}$$

$$= \frac{A}{\frac{1}{2}}$$

$$= 2A$$

$$= 1 - 2 \int_0^1 L(x) dx \quad \checkmark$$



e) For $L(x) = x^p$

$$\begin{aligned} G &= 1 - 2 \int_0^1 x^p dx \\ &= 1 - \frac{2}{p+1} \left(x^{p+1} \right) \Big|_0^1 \\ &= 1 - \frac{2}{p+1} \\ &= \frac{p-1}{p+1} \end{aligned}$$

P	1.1	1.5	2	3	4
G	$\frac{1}{21}$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{5}$

f) For $p = 1$

$$\begin{aligned} G &= \frac{p-1}{p+1} \\ &= 0 \end{aligned}$$

$$\lim_{p \rightarrow \infty} \frac{p-1}{p+1} = 1, \text{ the largest value of } G \text{ approaches } 1.$$

$$g) \quad L(x) = \frac{5x^2}{6} + \frac{x}{6} \quad \rightarrow L(0) = 0, \quad L(1) = 1$$

$$L'(x) = \frac{5}{3}x + \frac{1}{6} > 0 \quad x \in [0, 1]$$

$$L''(x) = \frac{5}{3} > 0$$

The Gini index is:

$$\begin{aligned} G &= 1 - 2 \int_0^1 \left(\frac{5x^2}{6} + \frac{x}{6} \right) dx \\ &= 1 - 2 \left(\frac{5x^3}{18} + \frac{x^2}{12} \right) \Big|_0^1 \\ &= 1 - 2 \left(\frac{5}{18} + \frac{1}{12} \right) \\ &= 1 - \frac{5}{9} - \frac{1}{6} \\ &= \frac{5}{18} \end{aligned}$$

