Section 1.7 – Sets

Introduction

A set is an unordered collection of objects, called elements or members of the set. A set is said to contain its elements. We write $a \in A$ to denote that *a* is an element of the set *A*. The notation $a \notin A$ denotes that *a* is not an element of the set *A*.

Example

Colors of a rainbow: {red, orange, yellow, green, blue, purple}

Example

States of matter {solid, liquid, gas, plasma}

Example

The set *V* of all vowels in the English alphabet can be written as: $V = \{a,\ e,\ i,\ o,\ u\}$

Example

The set *O* of odd positive integers less than 10 can be expressed by *O* = {1, 3, 5, 7, 9}

Example

The set of positive integers less than 100 can be denoted by {1, 2, 3, …, 99}

- Another way to describe a set is to use ***set builder*** notation.

For instance, the set *O* of odd positive integers less than 10 can be written as

$$O = \{x \mid x \textit{ is an odd positive integer less than } 10\}$$

Or, specifying the universe as the set of positive integers, as

$$O = \left\{x \in \mathbb{Z}^{+} \mid x \textit{ is an odd and } x < 10\right\}$$

The set of ***Natural numbers***: $\mathbb{N} = \{0,\ 1,\ 2,\ 3,\ \ldots\}$

The set of ***Integers***: $\mathbb{Z} = \{\ldots,\ -2,\ -1,\ 0,\ 1,\ 2,\ \ldots\}$

The set of ***positive integers***: $\mathbb{Z}^{+} = \{1,\ 2,\ 3,\ \ldots\}$

The set of ***Rational numbers***: $\mathbb{Q} = \left\{ \frac{p}{q} \middle| p \in \mathbb{Z},\ q \in \mathbb{Z},\ and\ q \neq 0 \right\}$

The set of ***Real numbers***: $\mathbb{R}$

The set of ***positive Real numbers***: $\mathbb{R}^{+}$

The set of ***Complex numbers***: $\mathbb{C}$

Intervals

The notations for intervals of real numbers. When *a* and *b* are real numbers with *a* < *b*, we write

$$[a,\ b] = \{x | a \le x \le b\}$$

$$[a,\ b) = \{x | a \le x < b\}$$

$$(a,\ b] = \{x | a < x \le b\}$$

$$(a,\ b) = \{x | a < x < b\}$$

$[a,\ b]$ is called ***closed interval*** from *a* to *b*.

$(a,\ b)$ is called ***open interval*** from *a* to *b*.

Definition

Two sets are equal *iff* they have the same elements. Therefore, if *A* and *B* are sets, then *A* and *B* are equal *iff* $\forall x (x \in A \leftrightarrow x \in B)$. We write $A = B$ if *A* and *B* are equal sets

Example

The set {1, 3, 5} and {3, 5, 1} are equal, because they have the same elements.

- Order of the elements of a set are listed does not matter.

  {1, 2, 3, 4, 5} = {5, 4, 3, 2, 1}

The Empty Set

There is a special set that has no elements. This set is called the ***empty set***, or ***null set***, and is denoted by $\varnothing$. The empty set can also denoted by { }.

A set with one element is called a ***singleton set***.

Venn Diagrams

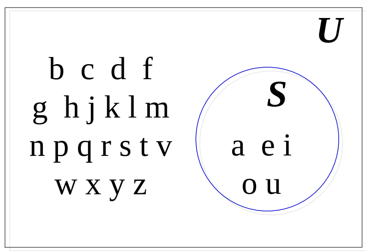
In Venn diagrams the ***universal set*** U, which contains all the objects under consideration, is represented by a rectangle.

Represents sets graphically

- ✓ The box represents the universal set
- ✓ Circles represent the set(s)

Consider set S, which is the set of all vowels in the alphabet

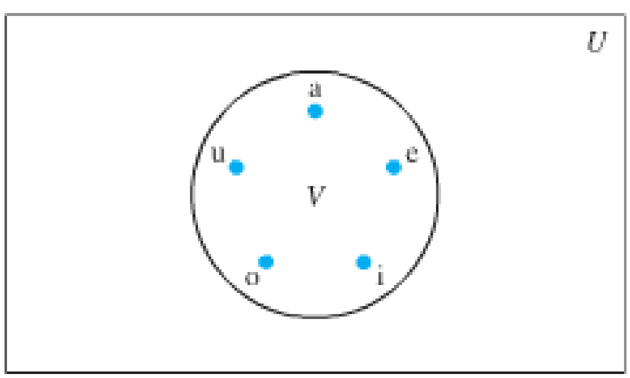
The individual elements are usually not written in a Venn diagram



Example

Draw a Venn diagram that represents *V*, the set of vowels in the English alphabet.

Solution



Subset

Set A is a subset of set B (written $A \subseteq B$) if and only if every element of A is also an element of B. Set A is a proper subset (written $A \subset B$) if $A \subseteq B$ and $A \neq B$

We see that $A \subseteq B$ if and only if the quantification:

$\forall x \ (x \in A \rightarrow x \in B)$ is true

Note that to show that A is not a subset of B we need only find one element $x \in A$ with $x \notin B$. Such an x is counterexample to the claim that $x \in A$ implies $x \in B$.

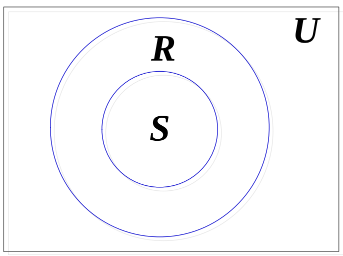
Showing that A is a Subset of B – To show that $A \subseteq B$, show that if x belong to A then a also belong to B.

Showing that A is Not a Subset of B – To show that $A \not\subseteq B$, find a single $x \in A$ such that $x \notin B$.

Example

$\{1, 2, 8\} \subseteq \{1, 2, 3, 4, 5, 6, 7\}$

Proper subsets: Venn diagram $\mathrm{S} \subset \mathrm{R}$



Example

The set of people who have taken discrete mathematics at the school is not a subset of all computer science majors at the school if there is at least one student who has taken discrete mathematics who is not a computer science major.

Theorem

For every set S

i. $\varnothing \subseteq S$ and

ii. $S \subseteq S$

Proof (*i*)

Let S be a set. To show $\varnothing \subseteq S$, we must show that $\forall x\left(x \in \varnothing \rightarrow x \in S\right)$ is true.

Because the empty set contains no elements, it follows that $x \in \varnothing$ is always false. It follows that the conditional statement $x \in \varnothing \rightarrow x \in S$ is always true, because its hypothesis is always false and a conditional statement with a false hypothesis is true. Therefore, $\forall x\left(x \in \varnothing \rightarrow x \in S\right)$ is true.

This complete the proof of (*i*) using a vacuous proof.

Showing Two Sets are Equal – To show that two sets A and B are equals, show that $A \subseteq B$ and $B \subseteq A$.

Example

We have the sets $A = \{\varnothing,\ \{a\},\ \{b\},\ \{a,b\}\}$ *and* $B = \{x \mid x \text{ is a subset of the set } \{a,b\}\ \}$

Solution

These two sets are equal, that is, $A = B$.

Note: $\{a\} \in A$ *but* $a \notin A$

The Size of a Set

Definition

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the ***cardinality*** of S. The cardinality of S is denoted by $|S|$.

- Let A be the set of odd positive integers less than 10. $|A| = 5$
- Let S be the set of of letter in English alphabet. $|S| = 26$
- The null set has no elements. $|\varnothing| = 0$

Definition

A set is said to be infinite if it is not finite.

Example: The set of positive integers is infinite.

Power Sets

Definition

Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$

Note that the empty set and the set itself are memebers of the set of subsets.

Example

What is the power set of the set {0, 1, 2}?

Solution

$$\mathcal{P}(\{0,1,2\}) = \{\varnothing,\ \{0\},\ \{1\},\ \{2\},\ \{0,1\},\ \{0,2\},\ \{1,2\},\ \{0,1,2\}\}$$

Example

What is the power set of the empty set? What is the power set of the set $\{\varnothing\}$?

Solution

$$\mathcal{P}(\varnothing) = \{\varnothing\}$$

$$\mathcal{P}(\{\varnothing\}) = \{\varnothing,\ \{\varnothing\}\}$$

Cartesian Products

Definition

The ***order n-tuple*** $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, …, and a_n as its *n*th element.

Let *A* and *B* be sets. The Cartesian product of *A* and *B*, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

Example

Let *A* represent the set of all students at a university, and let *B* represent the set of all courses offered at the university. What is the Cartesian product $A \times B$ and how can it be used?

Solution

The Cartesian product $A \times B$ consists of all the ordered pairs of the form (a, b), where *a* is a student at the university and *b* is a course offered at the university. One way to use the set $A \times B$ is to represent all possible enrollments of students in courses at the university.

Example

What is the Cartesian product *A* = {1, 2} and *B* = {*a, b, c*}?

Solution

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

Example

Show that the Cartesian product $B \times A$ is not equal to $A \times B$, where *A* = {1, 2} and *B* = {*a, b, c*}?

Solution

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

$$B \times A = \{(a, 1), (a, 2), (b, 1), (b\ 2), (c, 1), (c, 2)\}$$

$$\Rightarrow A \times B \neq B \times A$$

Definition

The ***Cartesian product*** of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of ordered *n*-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$. In other words,

$$A_1 \times A_2 \times \ldots \times A_n = \left\{ (a_1, a_2, \ldots, a_n) \middle| a_i \in A_i \ \ for\ i = 1, 2, \ldots, n \right\}$$

Example

What is the Cartesian product $A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$

Solution

$$\begin{aligned} A \times B \times C = \{&(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), \\ &(1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\} \end{aligned}$$

Example

Suppose that $A = \{1, 2\}$, find A^2 and A^3

Solution

$$A^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

$$A^3 = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$$

Example

What are the ordered pairs in the less than or equal relation, which contains (a, b) if $a \le b$, on the set $\{0, 1, 2, 3\}$?

Solution

The ordered pairs in ***R*** are:

(0, 0), (0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)

Using Set Notation with Quantifiers

For example $\forall x \in S\left(P(x)\right)$ denotes

Universal quantification of $P(x)$ over all elements in the S

Shorthand for $\forall x\left(x \in S \to P(x)\right)$

$\exists x \in S\left(P(x)\right)$ denotes

Existential quantification of $P(x)$ over all elements in the S

Shorthand for $\exists x\left(x \in S \wedge P(x)\right)$

Example

What do the statements $\forall x \in \boldsymbol{R}\left(x^2 \ge 0\right)$ and $\exists x \in \boldsymbol{Z}\left(x^2 = 1\right)$ mean?

Solution

The statement $\forall x \in \boldsymbol{R}\left(x^2 \ge 0\right)$ states that for every real numbers *x*, $x^2 \ge 0$.

This statement can be expressed as "The square of every real number is nonnegative." This is a true statement.

The statement $\exists x \in \boldsymbol{Z}\left(x^2 = 1\right)$ states that there exists an integer *x*, $x^2 = 1$.

This statement can be expressed as "The is an integer whose square is 1." This is also a true statement because $x = 1$ *or* -1 such an integer.

Exercises *Section* 1.7 – Sets

1. List the members of these sets

a) $\left\{x \middle| x \text{ is a real number such that } x^2 = 1\right\}$

b) $\left\{x \middle| x \text{ is a positive integer less than } 12\right\}$

c) $\left\{x \middle| x \text{ is the square of an integer and } x < 100\right\}$

d) $\left\{x \middle| x \text{ is an integer such that } x^2 = 2\right\}$

2. Determine whether each these pairs of sets are equal.

a) $\{1,3,3,3,5,5,5,5,5\}, \quad \{5,3,1\}$

b) $\{\{1\}\}, \quad \{1, \{1\}\}$

c) $\varnothing, \quad \{\varnothing\}$

3. For each of the following sets, determine whether 2 is an element of that set.

a) $\left\{x \in \mathbb{R} \middle| x \text{ is an integer greater than } 1\right\}$

b) $\left\{x \in \mathbb{R} \middle| x \text{ is the square of an integer}\right\}$

c) $\{2, \{2\}\}$

d) $\{\{2\}, \{\{2\}\}\}$

e) $\{\{2\}, \{2, \{2\}\}\}$

f) $\{\{\{2\}\}\}$

4. Determine whether each of these statements is true or false

a) $0 \in \varnothing$

b) $\varnothing \in \{0\}$

c) $\{0\} \subset \varnothing$

d) $\varnothing \subset \{0\}$

e) $\{0\} \in \{0\}$

f) $\{0\} \subset \{0\}$

g) $\{\varnothing\} \subseteq \{\varnothing\}$

h) $x \in \{x\}$

i) $\{x\} \subseteq \{x\}$

j) $\{x\} \in \{x\}$

k) $\{x\} \in \{\{x\}\}$

l) $\varnothing \subseteq \{x\}$

m) $\varnothing \in \{x\}$

5. Use a Venn Diagram to illustrate the relationships $A \subset B$ and $B \subset C$.

6. Use a Venn Diagram to illustrate the relationships $A \subset B$ and $A \subset C$.

7. Suppose that *A*, *B*, and *C* are sets such that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$

8. What is the cardinality of each of these sets?

a) $\{a\}$

b) $\{\{a\}\}$

c) $\{a, \{a\}\}$

d) $\{a, \{a\}, \{a, \{a\}\}\}$

9. How many elements does each of these sets have where *a* and *b* are distinct elements?

a) $\mathcal{P}(\{a, b, \{a, b\}\})$

b) $\mathcal{P}(\{\varnothing, a, \{a\}, \{\{a\}\}\})$

c) $\mathcal{P}(\mathcal{P}(\varnothing))$

10. What is the Cartesian product $A \times B \times C$, where *A* is the set of all airlines and *B* and *C* are both the set of all cities in the United States? Give an example of how this Cartesian product can be used.

11. What is the Cartesian product $A \times B$, where *A* is the set of all courses offered by the mathematics department and *B* is the set of mathematics professors at this university? Give an example of how this Cartesian product can be used.

12. Let *A* be a set. Show that $\varnothing \times A = A \times \varnothing = \varnothing$