Solution — Section 3.1 – Distribution of the Sample Mean / Proportion

Exercise

Assume that SAT scores are normally distributed with mean $\mu = 1518$ and standard deviation $\sigma = 325$.

a) If 1 SAT score is randomly selected, find the probability that it is less than 1500.

b) If 100 SAT scores are randomly selected; find the probability that they have a mean less than 1500.

c) If 1 SAT score is randomly selected, find the probability that it is greater than 1600.

d) If 64 SAT scores are randomly selected, find the probability that they have a mean greater than 1600.

e) If 1 SAT score is randomly selected; find the probability that it is between 1550 and 1575.

f) If 25 SAT scores are randomly selected; find the probability that they have a mean between 1550 and 1575.

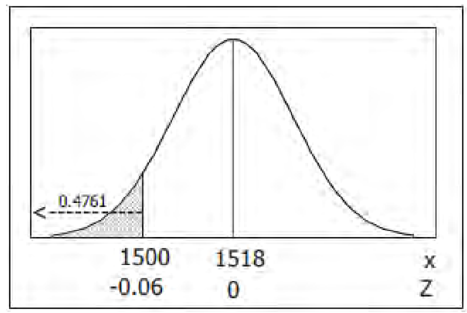
Solution

a) Normal distribution with: $\mu = 1518, \quad \sigma = 325$

$$x = 1500 \quad \rightarrow \quad z = \frac{x-\mu}{\sigma} = \frac{1500-1518}{325} = -0.06$$

$$P(x < 1500) = P(z < -0.06)$$

$$= 0.4761$$



b) Normal distribution, since the original distribution is so

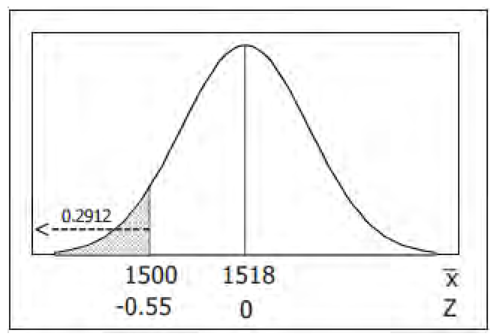
$$\mu_{\overline{x}} = \mu = 1518$$

$$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}} = \frac{325}{\sqrt{100}} = 32.5$$

$$\overline{x} = 1500 \quad \rightarrow \quad z = \frac{\overline{x}-\mu}{\sigma} = \frac{1500-1518}{32.5} = -0.55$$

$$P(x < 1500) = P(z < -0.55)$$

$$= 0.2912$$



c) Normal distribution with: $\mu = 1518, \quad \sigma = 325$

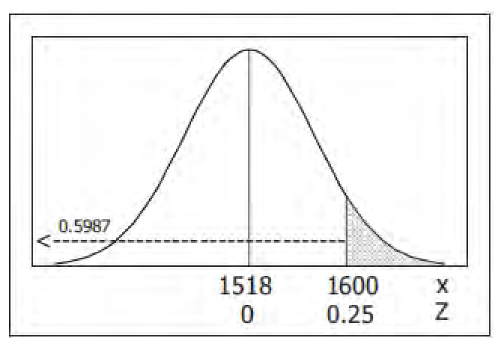
$$x = 1600 \quad \rightarrow \quad z = \frac{x-\mu}{\sigma} = \frac{1600-1518}{325} = 0.25$$

$$P(x > 1600) = P(z > 0.25)$$

$$= 1 - P(z < 0.25)$$

$$= 1 - 0.5987$$

$$= 0.4013$$



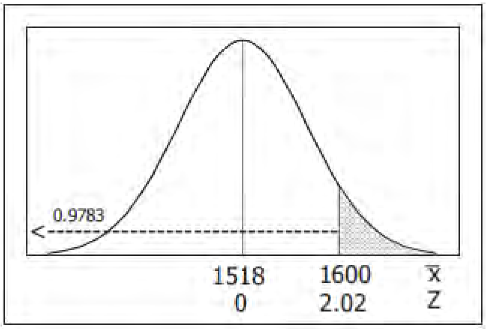
d) Normal distribution, since the original distribution is so

$$\mu_{\bar{x}} = \mu = 1518$$

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{325}{\sqrt{64}} = 40.625$$

$$\bar{x} = 1600 \quad \rightarrow \quad z = \frac{\bar{x} - \mu}{\sigma} = \frac{1600 - 1518}{40.625} \underline{= 2.02|}$$

$$P(\bar{x} > 1600) = P(z > 2.02)$$
$$= 1 - P(z < 2.02)$$
$$= 1 - 0.9783$$
$$\underline{= 0.0217|}$$

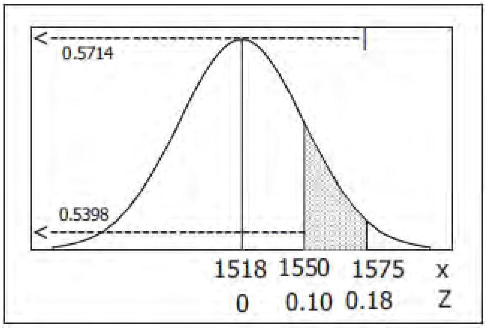


e) Normal distribution with: $\mu = 1518, \quad \sigma = 325$

$$x = 1550 \quad \rightarrow \quad z = \frac{x - \mu}{\sigma} = \frac{1550 - 1518}{325} \underline{= 0.10|}$$

$$x = 1575 \quad \rightarrow \quad z = \frac{x - \mu}{\sigma} = \frac{1575 - 1518}{325} \underline{= 0.18|}$$

$$P(1550 < x < 1575) = P(0.10 < z < 0.18)$$
$$= P(z < 0.18) - P(z < 0.10)$$
$$= 0.5714 - 0.5398$$
$$\underline{= 0.0316|}$$



f) Normal distribution, since the original distribution is so

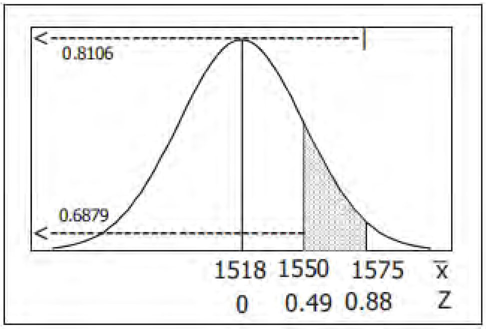
$$\mu_{\bar{x}} = \mu = 1518$$

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{325}{\sqrt{6425}} = 65$$

$$\bar{x} = 1550 \quad \rightarrow \quad z = \frac{\bar{x} - \mu}{\sigma} = \frac{1550 - 1518}{65} \underline{= 0.49|}$$

$$\bar{x} = 1575 \quad \rightarrow \quad z = \frac{\bar{x} - \mu}{\sigma} = \frac{1575 - 1518}{65} \underline{= 0.88|}$$

$$P(1550 < \bar{x} < 1575) = P(0.49 < z < 0.88)$$
$$= P(z < 0.88) - P(z < 0.49)$$
$$= 0.8106 - 0.6879$$
$$\underline{= 0.1227|}$$



Exercise

Assume that weights of mean are normally distributed with a mean of 172 lb. and a standard deviation of 29 lb.

a) Find the probability that if an individual man is randomly selected, his weight will be greater than 180 lb.

b) Find the probability that 20 randomly selected men will have a mean weight that is greater than 180 lb.

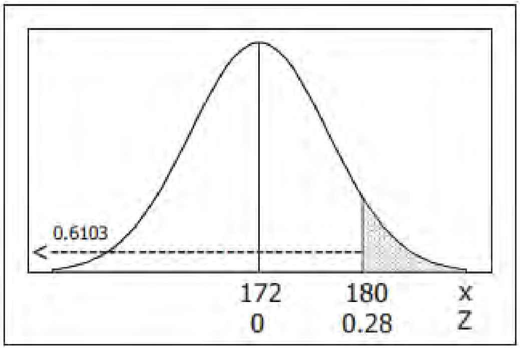
c) If 20 mean have a mean weight greater than 180 lb., the total weight exceeds the 3500 lb. safe capacity of a particular water taxi. Based on the proceeding results, is this safety concern? Why or why not?

Solution

a) Normal distribution with: $\mu = 172, \quad \sigma = 29$

$$x = 180 \quad \rightarrow \quad z = \frac{x-\mu}{\sigma} = \frac{180-172}{29} \underline{= 0.28|}$$

$$\begin{aligned} P(x > 180) &= P(z > 0.28) \\ &= 1 - P(z < 0.28) \\ &= 1 - 0.6103 \\ &\underline{= 0.3897|} \end{aligned}$$



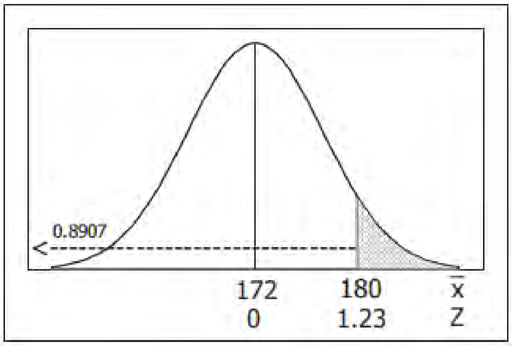
b) Normal distribution, since the original distribution is so

$$\mu_{\overline{x}} = \mu = 172$$

$$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}} = \frac{29}{\sqrt{20}} = 6.48$$

$$\overline{x} = 180 \quad \rightarrow \quad z = \frac{\overline{x}-\mu}{\sigma} = \frac{180-172}{6.48} \underline{= 1.23|}$$

$$\begin{aligned} P(\overline{x} > 180) &= P(z > 1.23) \\ &= 1 - P(z < 1.23) \\ &= 1 - 0.8907 \\ &\underline{= 0.1093|} \end{aligned}$$



c) Yes. A capacity of 20 is not appropriate when the passengers are all adult men, since a 10.93% probability of overloading is too much of a risk.

Exercise

Membership requires an IQ score above 131.5. Nine candidates take IQ tests, and their summary results indicated that their mean IQ score is 133, (IQ scores are normally distributed with mean of 100 and a standard deviation of 15.)

a) If 1 person is randomly selected from the general population, find the probability of getting someone with an IQ score of at least 133.

b) If 9 people are randomly selected, find the probability that their mean IQ score is at least 133.

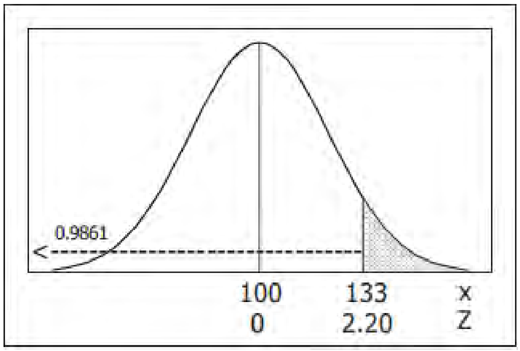
c) Although the summary results are available, the individual IQ test scores have been lost. Can it be concluded that all 9 candidates have IQ scores above 131.5 so that they are all eligible for membership?

Solution

a) Normal distribution with: $\mu = 100, \quad \sigma = 15$

$$x = 133 \quad \to \quad z = \frac{x-\mu}{\sigma} = \frac{133-100}{15} \underline{= 2.20}$$

$$P(x > 133) = P(z > 2.20)$$
$$= 1 - P(z < 2.20)$$
$$= 1 - 0.9861$$
$$\underline{= 0.0139}$$



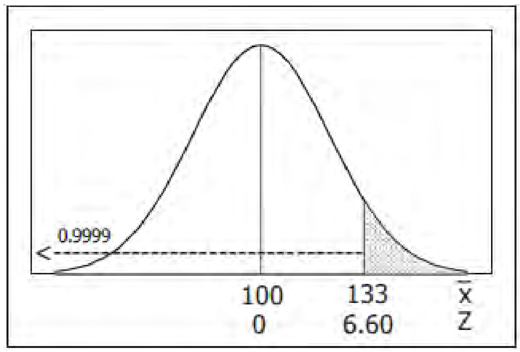
b) Normal distribution, since the original distribution is so

$$\mu_{\bar{x}} = \mu = 100$$

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{15}{\sqrt{9}} = 5$$

$$\bar{x} = 133 \quad \to \quad z = \frac{\bar{x}-\mu}{\sigma} = \frac{133-100}{5} \underline{= 6.60}$$

$$P(\bar{x} > 133) = P(z > 6.60)$$
$$= 1 - P(z < 6.60)$$
$$= 1 - 0.9999$$
$$\underline{= 0.0001}$$



c) No. Even though the mean score is 133, some of the individual scores may be below 131.5.

Exercise

For women aged 18-24, systolic blood pressures (in mm Hg) are normally distributed with a mean of 114.8 and a standard deviation of 13.1. Hypertension is commonly defined as a systolic blood pressure above 140.

a) If a woman between the ages of 18 and 24 is randomly selected, find the probability that her systolic blood pressure is greater than 140.

b) If 4 women in that age bracket are randomly selected, find the probability that their mean systolic blood pressure is greater than 140.

c) Given that part (*b*) involves a sample size that is not larger than 30, why can the central limit theorem be used?

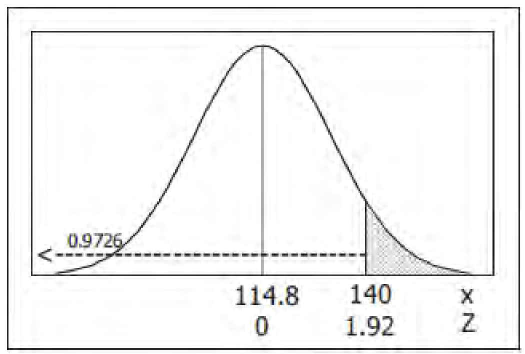
d) If a physician is given a report stating that 4 women have a mean systolic, blood pressure below 140, can she conclude that none of the women have hypertension (with a blood pressure greater than 140)?

Solution

a) Normal distribution with: $\mu = 114.8, \quad \sigma = 13.1$

$$x = 140 \quad \rightarrow \quad z = \frac{x-\mu}{\sigma} = \frac{140-114.8}{13.1} = 1.92$$

$$P(x > 140) = P(z > 1.92)$$
$$= 1 - P(z < 1.92)$$
$$= 1 - 0.9726$$
$$= 0.0274$$



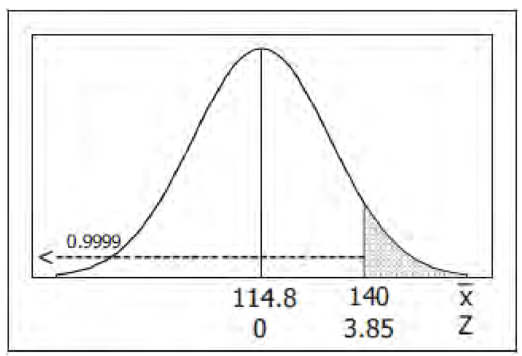
b) Normal distribution, since the original distribution is so

$$\mu_{\overline{x}} = \mu = 114.8$$

$$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}} = \frac{13.1}{\sqrt{4}} = 6.55$$

$$\overline{x} = 140 \quad \rightarrow \quad z = \frac{\overline{x}-\mu}{\sigma} = \frac{140-100}{6.55} = 3.85$$

$$P(\overline{x} > 140) = P(z > 3.85)$$
$$= 1 - P(z < 3.85)$$
$$= 1 - 0.9999$$
$$= 0.0001$$



c) Since the original distribution is normal, the Central Limit Theorem can be used in part (b) even though the sample size does not exceed 30.

d) The mean can be less than 140 when one or more of the values is greater than 140.

Exercise

Engineers must consider the breadths of male heads when designing motorcycle helmets. Men have head breaths that are normally distributed with a mean of 6.0 in. and a standard deviation of 1.0 in.

a) If one male is randomly selected, find the probability that his head breadth is less than 6.2 in.

b) The Safeguard Helmet Company plans an initial production run of 100 helmets. Find the probability that 100 randomly selected men have a mean head breadth less than 6.2 in.

c) The production manager sees the result from part (b) and reasons that all helmets should be made for men with head breadths less than 6.2 in., because they would fit all but a few men. What is wrong with that reasoning?

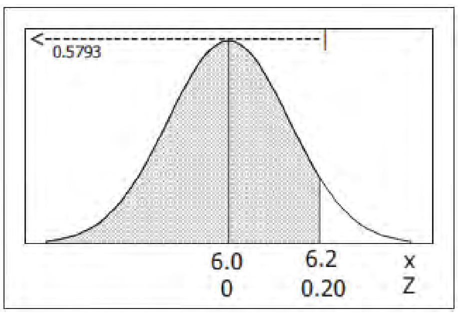
Solution

a) Normal distribution with: $\mu = 6.0, \quad \sigma = 1.0$

$$x = 6.2 \quad \rightarrow \quad z = \frac{x-\mu}{\sigma} = \frac{6.2-6.0}{1.0} \underline{= 0.20}$$

$$P(x < 6.2) = P(z < 0.20)$$

$$\underline{= 0.5793}$$



b) Normal distribution, since the original distribution is so

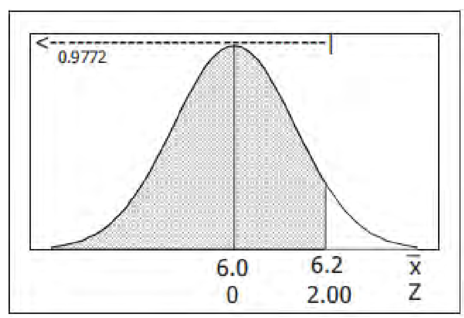
$$\mu_{\overline{x}} = \mu = 6.0$$

$$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}} = \frac{1.0}{\sqrt{100}} = 0.10$$

$$\overline{x} = 6.2 \quad \rightarrow \quad z = \frac{\overline{x}-\mu}{\sigma} = \frac{6.2-6.0}{0.10} \underline{= 2.0}$$

$$P(\overline{x} < 6.2) = P(z < 2.00)$$

$$\underline{= 0.9772}$$



c) Probabilities concerning means do not apply to individuals, It is the information from part (a) that is relevant, since the helmets will be worn by one man at a time – and that indicates that the proportion of men with head breadth greater than 6.2 inches is 1 – 0.5793 = 0.4207 = 42.07%.

Exercise

Currently, quarters have weighs that are normally distributed with a mean 5,670 g and a standard deviation of 0.062 g. A vending machine is configured to accept only those quarters with weights between 5.550 g and 5.790 g.

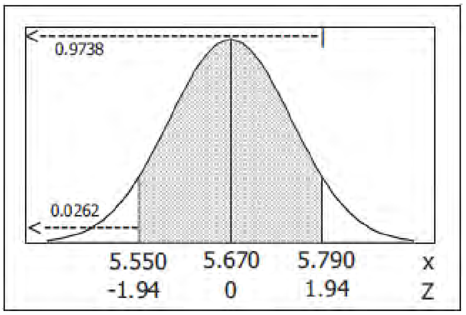
a) If 280 different quarters are inserted into the vending machine, what is the expected number of rejected quarter?

b) If 280 different quarters are inserted into the vending machine, what is the probability that the mean falls between the limits of 5.550 g and 5.790 g?

c) If you own the vending machine, which result would concern you more? The result from part (*a*) or the result from part (*b*)? Why?

Solution

a) Normal distribution with: $\mu = 5.67, \quad \sigma = 0.062$



$$x = 5.55 \quad \to \quad z = \frac{x-\mu}{\sigma} = \frac{5.55-5.67}{0.062} = -1.94$$

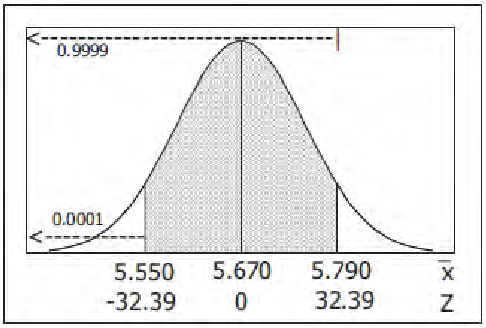
$$x = 5.79 \quad \to \quad z = \frac{x-\mu}{\sigma} = \frac{5.79-5.67}{0.062} = 1.94$$

$$\begin{aligned} P(5.55 < x < 5.79) &= P(-1.94 < z < 1.94) \\ &= P(z < 1.94) - P(z < -1.94) \\ &= 0.9738 - 0.0262 \\ &= 0.9476 \end{aligned}$$

If 0.9476 of the quarters are accepted, then 1 – 0.9476 = 0.0524 of the quarters are rejected.

For 280 quarters, we expect $(0.0524)(280) = 14.7$ of them rejected.

b) Normal distribution, since the original distribution is so



$$\mu_{\overline{x}} = \mu = 5.67$$

$$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}} = \frac{0.062}{\sqrt{280}} = 0.00371$$

$$\overline{x} = 5.55 \quad \to \quad z = \frac{\overline{x}-\mu}{\sigma} = \frac{5.55-5.67}{.00371} = -32.39$$

$$\overline{x} = 5.79 \quad \to \quad z = \frac{\overline{x}-\mu}{\sigma} = \frac{5.79-5.67}{.00371} = 32.39$$

$$\begin{aligned} P(5.55 < \overline{x} < 5.79) &= P(-32.39 < z < 32.39) \\ &= P(z < 32.39) - P(z < -32.39) \\ &= 0.9999 - .0001 \\ &= 0..9998 \end{aligned}$$

c) Probabilities concerning means do not apply to individuals, It is the information from part (a) that is relevant, since the vending machine deals with quarters one at a time.

Exercise

The annual precipitation amounts in a certain mountain range are normally distributed with a mean of 101 inches, and a standard deviation of 12 inches. What is the probability that the mean annual precipitation during 36 randomly picked years will be less than 103.8 inches?

Solution

Given: $x = 103.8; \quad \mu = 101$

$$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}} = \frac{12}{\sqrt{36}} = 2$$

$$z = \frac{x-\mu}{\sigma} = \frac{103.8-101}{2} = 1.4$$

$$P(x < 103.8) = P(z < 1.4)$$

$$= 0.9192$$

Exercise

The annual precipitation amounts in a certain mountain range are normally distributed with a mean of 72 inches, and a standard deviation of 14 inches. What is the probability that the mean annual precipitation during 49 randomly picked years will be less than 74.8 inches?

Solution

Given: $x = 74.8; \quad \mu = 72$

$$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}} = \frac{14}{\sqrt{49}} = 2$$

$$z = \frac{x-\mu}{\sigma} = \frac{74.8-72}{2} = 1.4$$

$$P(x < 74.8) = P(z < 1.4)$$

$$= 0.9192$$

Exercise

The weights of the fish in a certain lake are normally distributed with a mean of 13 lb. and a standard deviation of 6. If 4 fish are randomly selected, what is the probability that the mean weight will be between 10.6 and 16.6 lb.?

Solution

Given: $x_1 = 10.6; \quad x_2 = 16.6; \quad \mu = 13$

$$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}} = \frac{6}{\sqrt{4}} = 3$$

$$z_1 = \frac{x_1-\mu}{\sigma} = \frac{10.6-13}{3} = -0.8 \qquad z_2 = \frac{x_2-\mu}{\sigma} = \frac{16.6-13}{3} = 1.2$$

$$P(10.6 < x < 16.6) = P(-0.8 < z < 1.2)$$
$$= P(z = 1.2) - P(z = -0.8)$$
$$= .8849 - .2119$$
$$\underline{= 0.6730}$$

Exercise

For women aged 18-24, systolic blood pressures (in mm Hg) are normally distributed with a mean of 114.8 and a standard deviation of 13.1. If 23 women aged 18-24 are randomly selected, find the probability that their mean systolic blood pressure is between 119 and 122.

Solution

Given: $x_1 = 119$; $x_2 = 122$; $\mu = 114.8$

$$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}} = \frac{114.8}{\sqrt{23}} = 23.937$$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{119 - 114.8}{23.937} = 0.18 \qquad z_2 = \frac{x_2 - \mu}{\sigma} = \frac{122 - 114.8}{23.937} = 0.3$$

$$P(119 < x < 122) = P(0.18 < z < 0.3)$$
$$= P(z = 0.3) - P(z = 0.18)$$
$$= 0.6179 - 0.5714$$
$$\underline{= 0.0465}$$

Exercise

A study of the amount of time it takes a mechanic to rebuild the transmission for 2005 Chevy shows that the mean is 8.4 hours and the standard deviation is 1.8 hours. If 40 mechanics are randomly selected, find the probability that their mean rebuild time

a) Exceeds 8.7 hours.

b) Exceeds 8.1 hours.

Solution

a) Given: $x = 8.7$; $\mu = 8.4$

$$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}} = \frac{1.8}{\sqrt{40}} = 0.285$$

$$z = \frac{x - \mu}{\sigma} = \frac{8.7 - 8.4}{0.285} = 1.05$$

$$P(x > 8.7) = P(z > 1.05)$$
$$= 1 - P(z < 1.05)$$
$$= 1 - 0.8531$$
$$\underline{= 0.1469}$$

b) ***Given***: $x = 8.1; \quad \mu = 8.4$

$$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}} = \frac{1.8}{\sqrt{40}} = 0.285$$

$$z = \frac{x-\mu}{\sigma} = \frac{8.1-8.4}{0.285} = -1.05$$

$$\begin{aligned} P(x > 8.1) &= P(z > -1.05) \\ &= 1 - P(z < -1.05) \\ &= 1 - 0.1469 \\ &= 0.8531 \end{aligned}$$

Exercise

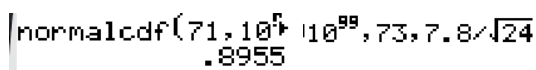
A final exam in Math 160 has a mean of 73 with standard deviation 7.8. If 24 students are randomly selected, find the probability that the mean of their test scores is greater than 71.

Solution

$$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}} = \frac{7.8}{\sqrt{24}} = 0.159$$

$$z = \frac{x-\mu}{\sigma} = \frac{71-73}{0.159} = -1.25$$

$$\begin{aligned} P(x > 71) &= P(z > -1.25) \\ &= 1 - P(z < -1.25) \\ &= 0.8955 \end{aligned}$$



Exercise

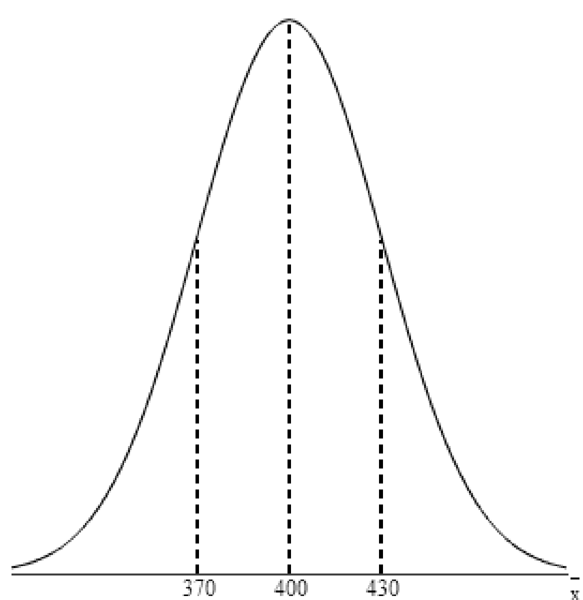
The sampling distribution of the sample mean shown in the graph.

a) What is the value of $\mu_{\overline{x}}$?

b) What is the value of $\sigma_{\overline{x}}$?

c) If the sample size is $n = 9$, what is likely true about the shape of the population?

d) If the sample size is $n = 9$, what is the standard deviation of the population for which the sample was drawn?



Solution

a) $\mu_{\overline{x}} = 400$

b) $\sigma_{\overline{x}} = 430 - 400 = 30$

c) The shape of the population is normal

d) $\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}} \quad \Rightarrow \quad 30 = \frac{\sigma}{\sqrt{9}}$

$\sigma = 3(30) = 90$

Exercise

A sample random of size $n = 81$ is obtained from a population with $\mu = 77$ *and* $\sigma = 18$.

a) Describe the sampling distribution of $\overline{x}$

b) What is $P(\overline{x} > 79.6)$?

c) What is $P(\overline{x} \le 72.5)$?

d) What is $P(75.1 < \overline{x} < 80.9)$?

Solution

a) The distribution is approximately normal

$\mu_{\overline{x}} = \mu = 77$

$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}} = \frac{18}{\sqrt{81}} = 2$

b) $z = \frac{\overline{x} - \mu}{\sigma_{\overline{x}}} = \frac{79.6 - 77}{2} = 1.30$

$P(\overline{x} > 79.6) = 1 - P(z = 1.30) = 0.0968$

c) $z = \frac{\overline{x} - \mu}{\sigma_{\overline{x}}} = \frac{72.5 - 77}{2} = -2.25$

$P(\overline{x} > 72.5) = P(z = -2.25) = 0.0122$

d) $z_1 = \frac{\overline{x} - \mu}{\sigma_{\overline{x}}} = \frac{75.1 - 77}{2} = -0.95 \qquad z_2 = \frac{80.9 - 77}{2} = 1.95$

$$\begin{aligned} P(75.1 < \overline{x} < 80.9) &= P\left(z_2 = 1.95\right) - P\left(z_1 = -0.95\right) \\ &= .9744 - .1711 \\ &= 0.8033 \end{aligned}$$

Exercise

The reading speed of second grade students is approximately normal, with a mean of 90 words per minute (*wpm*) and a standard deviation of 10 *wpm*.

a) What is the probability a randomly selected student will read more than 95 wpm?

b) What is the probability that a random sample of 11 second grade students results in a mean reading rate of more than 95 wpm?

c) What is the probability that a random sample of 22 second grade students results in a mean reading rate of more than 95 wpm?

d) What effect does increasing the sample size have on the probability?

Solution

a) $z = \dfrac{\overline{x} - \mu}{\sigma} = \dfrac{95 - 90}{10} = 0.5$

$P(\overline{x} > 95) = 1 - P(z = .5) \ \underline{= 0.3085|}$

b) $z = \dfrac{\overline{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{95 - 90}{10/\sqrt{11}} \ \underline{= 1.66|}$

$P(\overline{x} > 95) = 1 - P(z = 1.66) \ \underline{= 0.0486|}$

c) $z = \dfrac{\overline{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{95 - 90}{10/\sqrt{22}} \ \underline{= 2.35|}$

$P(\overline{x} > 95) = 1 - P(z = 2.35) \ \underline{= 0.0095|}$

d) Increasing the sample size decreases the probability because $\sigma_{\overline{x}}$ increases as *n* increases.