

Solution **Section 2.2 - Homogeneous Equations with Constant Coefficients**

Exercise

Decide whether the equation is linear or nonlinear. For the linear equation, state whether the equation is homogenous or inhomogeneous. $t^2 y'' = 4y' - \sin t$

Solution

$$y'' - \frac{4}{t^2} y' = -\frac{\sin t}{t^2}$$

$$y'' + p(t)y' + q(t)y = g(t)$$

It is linear and inhomogeneous ($g(t) \neq 0$)

Exercise

Decide whether the equation is linear or nonlinear. For the linear equation, state whether the equation is homogenous or inhomogeneous.

$$ty'' + (\sin t)y' = 4y - \cos 5t$$

Solution

$$y'' + \left(\frac{\sin t}{t}\right)y' - \frac{4}{t}y = -\frac{\cos 5t}{t}$$

$$y'' + p(t)y' + q(t)y = g(t)$$

It is linear and inhomogeneous ($g(t) \neq 0$)

Exercise

Decide whether the equation is linear or nonlinear. For the linear equation, state whether the equation is homogenous or inhomogeneous.

$$t^2 y'' + 4yy' = 0$$

Solution

It is nonlinear ($4yy'$)

Exercise

Decide whether the equation is linear or nonlinear. For the linear equation, state whether the equation is homogenous or inhomogeneous.

$$y'' + 4y' + 7y = 3e^{-t} \sin t$$

Solution

Compare to $y'' + p(t)y' + q(t)y = g(t)$

$$\Rightarrow p(t) = 4, \quad q(t) = 7, \quad g(t) = 3e^{-t} \sin t \quad (g(t) \neq 0)$$

Hence, the equation is linear and inhomogeneous.

Exercise

Show by direct substitution that the given functions $y_1(t)$ and $y_2(t)$ are solutions of the given differential equation. Then verify by direct substitution, that any linear combination $C_1 y_1(t) + C_2 y_2(t)$ of the 2 given solutions is also a solution.

$$y'' + 4y = 0 \quad y_1(t) = \cos 2t \quad y_2(t) = \sin 2t$$

Solution

$$\begin{aligned} y &= C_1 y_1 + C_2 y_2 \\ &= C_1 \cos 2t + C_2 \sin 2t \end{aligned}$$

$$y' = -2C_1 \sin 2t + 2C_2 \cos 2t$$

$$y'' = -4C_1 \cos 2t - 4C_2 \sin 2t$$

If $C_1 y_1(t) + C_2 y_2(t)$, then

$$\begin{aligned} y'' + 4y &= -4C_1 \cos 2t - 4C_2 \sin 2t + 4(C_1 \cos 2t + C_2 \sin 2t) \\ &= -4C_1 \cos 2t - 4C_2 \sin 2t + 4C_1 \cos 2t + 4C_2 \sin 2t \\ &= 0 \end{aligned}$$

Exercise

Show by direct substitution that the given functions $y_1(t)$ and $y_2(t)$ are solutions of the given differential equation. Then verify by direct substitution, that any linear combination $C_1 y_1(t) + C_2 y_2(t)$ of the 2 given solutions is also a solution.

$$y'' - 2y' + 2y = 0; \quad y_1(t) = e^t \cos t \quad y_2(t) = e^t \sin t$$

Solution

$$y_1(t) = e^t \cos t \Rightarrow y_1'(t) = e^t \cos t - e^t \sin t = e^t (\cos t - \sin t)$$

$$\begin{aligned} \Rightarrow y_1''(t) &= e^t (\cos t - \sin t) + e^t (-\sin t - \cos t) \\ &= e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t \\ &= -2e^t \sin t \end{aligned}$$

$$\begin{aligned} y_1'' - 2y_1' + 2y_1 &= -2e^t \sin t - 2(e^t \cos t - e^t \sin t) + 2e^t \cos t \\ &= -2e^t \sin t - 2e^t \cos t + 2e^t \sin t + 2e^t \cos t \\ &= 0 \end{aligned}$$

$$y_2(t) = e^t \sin t \Rightarrow y_2'(t) = e^t \sin t + e^t \cos t$$

$$\begin{aligned} y_2''(t) &= e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t \\ &= 2e^t \cos t \end{aligned}$$

$$\begin{aligned} y_2'' - 2y_2' + 2y_2 &= 2e^t \cos t - 2(e^t \cos t + e^t \sin t) + 2e^t \sin t \\ &= 2e^t \cos t - 2e^t \cos t - 2e^t \sin t + 2e^t \sin t \\ &= 0 \end{aligned}$$

$$\text{If } y(t) = C_1 e^t \cos t + C_2 e^t \sin t$$

$$\begin{aligned} y'(t) &= C_1 e^t \cos t - C_1 e^t \sin t + C_2 e^t \sin t + C_2 e^t \cos t \\ &= (C_1 + C_2) e^t \cos t + (C_2 - C_1) e^t \sin t \end{aligned}$$

$$\begin{aligned} y''(t) &= (C_1 + C_2) e^t \cos t - (C_1 + C_2) e^t \sin t + (C_2 - C_1) e^t \sin t + (C_2 - C_1) e^t \cos t \\ &= (C_1 + C_2 + C_2 - C_1) e^t \cos t + (C_2 - C_1 - C_1 - C_2) e^t \sin t \\ &= 2C_2 e^t \cos t - 2C_1 e^t \sin t \end{aligned}$$

$$\begin{aligned} y'' - 2y' + 2y &= 2C_2 e^t \cos t - 2C_1 e^t \sin t - 2((C_1 + C_2) e^t \cos t + (C_2 - C_1) e^t \sin t) \\ &\quad + 2(C_1 e^t \cos t + C_2 e^t \sin t) \\ &= 2C_2 e^t \cos t - 2C_1 e^t \sin t - 2C_1 e^t \cos t - 2C_2 e^t \cos t + 2C_2 e^t \sin t - 2C_1 e^t \sin t \\ &\quad + 2C_1 e^t \cos t + 2C_2 e^t \sin t \\ &= 0 \end{aligned}$$

Exercise

Explain why $y_1(t)$ and $y_2(t)$ are linearly independent solutions. Calculate Wronskian and use it to explain the independence of the given solutions.

$$y'' + 9y = 0 \quad y_1(t) = \cos 3t \quad y_2(t) = \sin 3t$$

Solution

$$\begin{aligned} w(t) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix} \\ &= 3\cos^2 3t + 3\sin^2 3t \\ &= 3(\cos^2 3t + \sin^2 3t) \\ &= 3 \neq 0 \end{aligned}$$

The solutions $y_1(t)$ & $y_2(t)$ are linearly independent.

Exercise

Show that $y_1(t) = e^t$ and $y_2(t) = e^{-3t}$ form a fundamental set of solutions for $y'' + 2y' - 3y = 0$, then find a solution satisfying $y(0) = 1$ and $y'(0) = -2$.

Solution

$$\begin{aligned} y_1(t) = e^t &\Rightarrow y'' + 2y' - 3y = e^t + 2e^t - 3e^t = 0 \\ y_2(t) = e^{-3t} &\Rightarrow y'' + 2y' - 3y = 9e^{-3t} - 6e^{-3t} - 3e^{-3t} = 0 \\ y(t) = C_1 e^t + C_2 e^{-3t} &\quad y(0) = C_1 + C_2 = 1 \\ y'(t) = C_1 e^t - 3C_2 e^{-3t} &\quad y'(0) = C_1 - 3C_2 = -2 \\ \Rightarrow C_1 = \frac{1}{4} \quad C_2 = \frac{3}{4} \\ y(t) &= \frac{1}{4} e^t + \frac{3}{4} e^{-3t} \end{aligned}$$

Exercise

Find the general solution: $y'' + y' = 0$

Solution

The characteristic equation: $\lambda^2 + \lambda = 0 \rightarrow \lambda_{1,2} = 0, -1$

$$y(x) = C_1 + C_2 e^{-x}$$

Exercise

Find the general solution: $y'' - 4y = 0$

Solution

The characteristic equation: $\lambda^2 - 4 = 0 \rightarrow \lambda_{1,2} = \pm 2$

$$y(x) = C_1 e^{-2x} + C_2 e^{2x}$$

Exercise

Find the general solution: $y'' + 8y = 0$

Solution

The characteristic equation: $\lambda^2 + 8\lambda = 0 \rightarrow \lambda_{1,2} = 0, 8$

$$y(x) = C_1 + C_2 e^{8x}$$

Exercise

Find the general solution: $y'' - 36y = 0$

Solution

The characteristic equation: $\lambda^2 - 36 = 0 \rightarrow \lambda_{1,2} = \pm 6$

$$y(x) = C_1 e^{-6x} + C_2 e^{6x}$$

Exercise

Find the general solution: $y'' + 9y = 0$

Solution

The characteristic equation: $\lambda^2 + 9 = 0 \rightarrow \lambda_{1,2} = \pm 3i$

$$y(x) = C_1 \cos 3x + C_2 \sin 3x$$

Exercise

Find the general solution: $y'' + 16y = 0$

Solution

The characteristic equation: $\lambda^2 + 16 = 0 \rightarrow \lambda_{1,2} = \pm 4i$

$$\underline{y(x) = C_1 \cos 4x + C_2 \sin 4x}$$

Exercise

Find the general solution: $y'' + 25y = 0$

Solution

The characteristic equation: $\lambda^2 + 25 = 0 \rightarrow \lambda_{1,2} = \pm 5i$

$$\underline{y(x) = C_1 \cos 5x + C_2 \sin 5x}$$

Exercise

Find the general solution: $y'' - 64y = 0$

Solution

The characteristic equation: $\lambda^2 - 64 = 0 \rightarrow \lambda_{1,2} = \pm 8$

$$\underline{y(x) = C_1 e^{-8x} + C_2 e^{8x}}$$

Exercise

Find the general solution: $y'' + y' + y = 0$

Solution

The characteristic equation: $\lambda^2 + \lambda + 1 = 0$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}$$

$$\underline{y(x) = e^{-x/2} \left(C_1 \cos \frac{\sqrt{3}}{2} x + C_2 \sin \frac{\sqrt{3}}{2} x \right)}$$

Exercise

Find the general solution: $y'' + y' - y = 0$

Solution

The characteristic equation: $\lambda^2 + \lambda - 1 = 0 \rightarrow \lambda_{1,2} = \frac{-1 \pm \sqrt{5}}{2}$

$$y(x) = C_1 e^{\frac{-1-\sqrt{5}}{2}x} + C_2 e^{\frac{-1+\sqrt{5}}{2}x}$$

Exercise

Find the general solution: $y'' - y' - 2y = 0$

Solution

The characteristic equation: $\lambda^2 - \lambda - 2 = 0 \rightarrow \lambda_1 = -1, \lambda_2 = 2$

$$y(x) = C_1 e^{-x} + C_2 e^{2x}$$

Exercise

Find the general solution: $y'' - y' - 6y = 0$

Solution

The characteristic equation: $\lambda^2 - \lambda - 6 = 0 \rightarrow \lambda_{1,2} = \frac{1 \pm 5}{2} = -2, 3$

$$y(x) = C_1 e^{-2x} + C_2 e^{3x}$$

Exercise

Find the general solution: $y'' + y' - 6y = 0$

Solution

The characteristic equation: $\lambda^2 + \lambda - 6 = 0 \rightarrow \lambda_{1,2} = \frac{-1 \pm 5}{2} = -3, 2$

$$y(x) = C_1 e^{-3x} + C_2 e^{2x}$$

Exercise

Find the general solution: $y'' - y' - 11y = 0$

Solution

The characteristic equation: $\lambda^2 - \lambda - 11 = 0$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{45}}{2} = \frac{1 \pm 3\sqrt{5}}{2}$$

$$y(x) = C_1 e^{\frac{1-3\sqrt{5}}{2}x} + C_2 e^{\frac{1+3\sqrt{5}}{2}x}$$

Exercise

Find the general solution: $y'' - y' - 12y = 0$

Solution

The characteristic equation: $\lambda^2 - \lambda - 12 = 0 \Rightarrow \lambda_{1,2} = -3, 4$

$$y(t) = C_1 e^{-3t} + C_2 e^{4t}$$

Exercise

Find the general solution: $y'' + 2y' + y = 0$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda_{1,2} = -1$

$$y(t) = (C_1 + C_2 t) e^{-t}$$

Exercise

Find the general solution: $y'' + 2y' + 3y = 0$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 3 = 0$

$$\Rightarrow \lambda_{1,2} = \frac{-2 \pm 2i\sqrt{2}}{2} = -1 \pm i\sqrt{2}$$

$$y(x) = e^{-x} (C_1 \cos \sqrt{2}x + C_2 \sin \sqrt{2}x)$$

Exercise

Find the general solution: $y'' + 2y' - 3y = 0$

Solution

The characteristic equation: $\lambda^2 + 2\lambda - 3 = 0 \Rightarrow \lambda_{1,2} = 1, -3$

$$\underline{y(x) = C_1 e^{-3x} + C_2 e^x}$$

Exercise

Find the general solution: $y'' - 2y' - 3y = 0$

Solution

The characteristic equation: $\lambda^2 - 2\lambda - 3 = 0 \Rightarrow \underline{\lambda_{1,2} = -1, 3}$

$$\underline{y(x) = C_1 e^{-x} + C_2 e^{3x}}$$

Exercise

Find the general solution: $y'' - 2y' + 3y = 0$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 3 = 0$

$$\Rightarrow \lambda_{1,2} = \frac{2 \pm 2i\sqrt{2}}{2} = \underline{1 \pm i\sqrt{2}}$$

$$\underline{y(x) = e^x (C_1 \cos \sqrt{2}x + C_2 \sin \sqrt{2}x)}$$

Exercise

Find the general solution: $y'' + 2y' + 4y = 0$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 4 = 0$

$$\Rightarrow \lambda_{1,2} = \frac{-2 \pm 2i\sqrt{3}}{2} = \underline{-1 \pm i\sqrt{3}}$$

$$\underline{y(x) = e^{-x} (C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x)}$$

Exercise

Find the general solution: $y'' + 2y' - 15y = 0$

Solution

The characteristic equation: $\lambda^2 + 2\lambda - 15 = 0 \Rightarrow \lambda_{1,2} = \frac{-2 \pm 8}{2} = \underline{-5, 3}$

$$\underline{y(x) = C_1 e^{-5x} + C_2 e^{3x}}$$

Exercise

Find the general solution: $y'' + 2y' + 17y = 0$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 17 = 0 \Rightarrow \lambda_{1,2} = -1 \pm 4i$

$$\underline{y(t) = e^{-t} (C_1 \cos 4t + C_2 \sin 4t)}$$

Exercise

Find the general solution: $y'' - 2y' + 5y = 0$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 5 = 0$

$$\Rightarrow \lambda_{1,2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$\underline{y(x) = e^{-x} (C_1 \cos 2x + C_2 \sin 2x)}$$

Exercise

Find the general solution: $y'' - 3y' + 2y = 0$

Solution

The characteristic equation: $\lambda^2 - 3\lambda + 2 = 0 \rightarrow \lambda_{1,2} = 1, 2$

$$\underline{y(x) = C_1 e^x + C_2 e^{2x}}$$

Exercise

Find the general solution: $y'' + 3y' - 4y = 0$

Solution

The characteristic equation: $\lambda^2 + 3\lambda - 4 = 0 \rightarrow \lambda_{1,2} = 1, -4$

$$\underline{y(x) = C_1 e^x + C_2 e^{-4x}}$$

Exercise

Find the general solution: $y'' + 4y' - y = 0$

Solution

The characteristic equation: $\lambda^2 + 4\lambda - 1 = 0$

$$\lambda_{1,2} = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}$$

$$y(x) = C_1 e^{(-2-\sqrt{5})x} + C_2 e^{(-2+\sqrt{5})x}$$

Exercise

Find the general solution: $y'' - 4y' + 4y = 0$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda_{1,2} = 2$

$$y(t) = (C_1 + C_2 t) e^{2t}$$

Exercise

Find the general solution: $y'' + 4y' + 4y = 0$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 4 = 0 \Rightarrow \lambda_{1,2} = -2$

$$y(t) = (C_1 + C_2 t) e^{-2t}$$

Exercise

Find the general solution: $y'' - 4y' + 5y = 0$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 5 = 0$

$$\lambda_{1,2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

$$y(x) = e^{2x} (C_1 \cos x + C_2 \sin x)$$

Exercise

Find the general solution: $y'' + 4y' + 5y = 0$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 5 = 0 \rightarrow \lambda_{1,2} = \frac{-4 \pm 2i}{2} = -2 \pm i$

$$\underline{y(x) = e^{-2x} (C_1 \cos x + C_2 \sin x)}$$

Exercise

Find the general solution: $y'' + 4y' - 5y = 0$

Solution

The characteristic equation: $\lambda^2 + 4\lambda - 5 = 0 \rightarrow \lambda_{1,2} = -5, 1$

$$\underline{y(x) = C_1 e^{-5x} + C_2 e^x}$$

Exercise

Find the general solution: $y'' + 4y' + 7y = 0$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 7 = 0$

$$\lambda_{1,2} = \frac{-4 \pm 2i\sqrt{3}}{2} = -2 \pm i\sqrt{3}$$

$$\underline{y(x) = e^{-2x} (C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x)}$$

Exercise

Find the general solution: $y'' + 4y' + 9y = 0$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 9 = 0$

$$\lambda_{1,2} = \frac{-4 \pm 2i\sqrt{5}}{2} = -2 \pm i\sqrt{5}$$

$$\underline{y(x) = e^{-2x} (C_1 \cos \sqrt{5}x + C_2 \sin \sqrt{5}x)}$$

Exercise

Find the general solution: $y'' + 5y' = 0$

Solution

The characteristic equation: $\lambda^2 + 5\lambda = 0 \rightarrow \lambda_1 = -5, \lambda_2 = 0$

$$\underline{y(x) = C_1 e^{-5x} + C_2}$$

Exercise

Find the general solution: $y'' + 5y' + 6y = 0$

Solution

The characteristic equation: $\lambda^2 + 5\lambda + 6 = 0 \rightarrow \lambda_1 = -3, \lambda_2 = -2$

$$\underline{y(x) = C_1 e^{-3x} + C_2 e^{-2x}}$$

Exercise

Find the general solution: $y'' + 6y' + 9y = 0$

Solution

The characteristic equation: $\lambda^2 + 6\lambda + 9 = 0 \rightarrow \lambda_{1,2} = -3$

$$\underline{y(x) = (C_1 + C_2 x) e^{-3x}}$$

Exercise

Find the general solution: $y'' - 6y' + 9y = 0$

Solution

The characteristic equation: $\lambda^2 - 6\lambda + 9 = 0 \Rightarrow \lambda_{1,2} = 3$

$$\underline{y(t) = (C_1 + C_2 t) e^{3t}}$$

Exercise

Find the general solution: $y'' - 6y' + 25y = 0$

Solution

The characteristic equation: $\lambda^2 - 6\lambda + 25 = 0$

$$\Rightarrow \lambda_{1,2} = \frac{6 \pm 8i}{2} = 3 \pm 4i$$

$$y(x) = e^{3x} (C_1 \cos 4x + C_2 \sin 4x)$$

Exercise

Find the general solution: $y'' + 8y' + 16y = 0$

Solution

The characteristic equation: $\lambda^2 + 8\lambda + 16 = (\lambda + 4)^2 = 0 \rightarrow \lambda_{1,2} = -4$

$$y(x) = (C_1 + C_2 x) e^{-4x}$$

Exercise

Find the general solution: $y'' + 8y' - 16y = 0$

Solution

The characteristic equation: $\lambda^2 + 8\lambda - 16 = 0$

$$\lambda_{1,2} = \frac{-8 \pm 8\sqrt{2}}{2} = -4 \pm 4\sqrt{2}$$

$$y(x) = C_1 e^{(-4-4\sqrt{2})x} + C_2 e^{(-4+4\sqrt{2})x}$$

Exercise

Find the general solution: $y'' - 9y' + 20y = 0$

Solution

The characteristic equation: $\lambda^2 - 9\lambda + 20 = 0$

$$\lambda_{1,2} = \frac{9 \pm 1}{2} = 4, 5$$

$$y(x) = C_1 e^{4x} + C_2 e^{5x}$$

Exercise

Find the general solution: $y'' - 10y' + 25y = 0$

Solution

The characteristic equation: $\lambda^2 - 10\lambda + 25 = (\lambda - 5)^2 = 0 \rightarrow \underline{\lambda_{1,2} = 5}$

$$\underline{y(x) = (C_1 + C_2 x)e^{5x}}$$

Exercise

Find the general solution: $y'' + 14y' + 49y = 0$

Solution

The characteristic equation: $\lambda^2 + 14\lambda + 49 = (\lambda + 7)^2 = 0 \Rightarrow \underline{\lambda_{1,2} = -7}$

$$\underline{y(x) = (C_1 + C_2 x)e^{-7x}}$$

Exercise

Find the general solution: $2y'' - y' - 3y = 0$

Solution

The characteristic equation: $2\lambda^2 - \lambda - 3 = 0 \rightarrow \underline{\lambda_{1,2} = -1, \frac{3}{2}}$

$$\underline{y(x) = C_1 e^{-x} + C_2 e^{3x/2}}$$

Exercise

Find the general solution: $2y'' + y' - y = 0$

Solution

The characteristic equation: $2\lambda^2 + \lambda - 1 = 0 \rightarrow \underline{\lambda_{1,2} = -1, \frac{1}{2}}$

$$\underline{y(x) = C_1 e^{-x} + C_2 e^{x/2}}$$

Exercise

Find the general solution: $2y'' + 2y' + y = 0$

Solution

The characteristic equation: $2\lambda^2 + 2\lambda + 1 = 0$

$$\lambda_{1,2} = \frac{-2 \pm 2i}{4} = \frac{1}{2} \pm \frac{1}{2}i$$

$$y(x) = e^{x/2} \left(C_1 \cos \frac{1}{2}x + C_2 \sin \frac{1}{2}x \right)$$

Exercise

Find the general solution: $2y'' + 2y' + 3y = 0$

Solution

The characteristic equation: $2\lambda^2 + 2\lambda + 3 = 0$

$$\lambda_{1,2} = \frac{-2 \pm 2i\sqrt{5}}{4} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}i$$

$$y(x) = e^{-x/2} \left(C_1 \cos \frac{\sqrt{5}}{2}x + C_2 \sin \frac{\sqrt{5}}{2}x \right)$$

Exercise

Find the general solution: $2y'' - 3y' - 2y = 0$

Solution

The characteristic equation: $2\lambda^2 - 3\lambda - 2 = 0$

$$\rightarrow \lambda_{1,2} = \frac{3 \pm 5}{4} = -\frac{1}{2}, 2$$

$$y(x) = C_1 e^{-x/2} + C_2 e^{2x}$$

Exercise

Find the general solution: $2y'' - 3y' + 4y = 0$

Solution

The characteristic equation: $2\lambda^2 - 3\lambda + 4 = 0$

$$\rightarrow \lambda_{1,2} = \frac{3 \pm i\sqrt{23}}{4} = \frac{3}{4} \pm \frac{\sqrt{23}}{4}i$$

$$y(x) = e^{3x/4} \left(C_1 \cos \frac{\sqrt{23}}{4}x + C_2 \sin \frac{\sqrt{23}}{4}x \right)$$

Exercise

Find the general solution: $2y'' - 4y' + 8y = 0$

Solution

The characteristic equation: $2\lambda^2 - 4\lambda + 8 = 0$

$$\rightarrow \lambda_{1,2} = \frac{4 \pm 4i\sqrt{3}}{4} = 1 \pm i\sqrt{3}$$

$$\underline{y(x) = e^x (C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x)}$$

Exercise

Find the general solution: $2y'' + 5y' = 0$

Solution

The characteristic equation: $2\lambda^2 + 5\lambda = \lambda(2\lambda + 5) = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = -\frac{5}{2}$

$$\underline{y(x) = C_1 + C_2 e^{-5x/2}}$$

Exercise

Find the general solution: $2y'' - 5y' - 3y = 0$

Solution

The characteristic equation: $2\lambda^2 - 5\lambda - 3 = 0$

$$\rightarrow \lambda_{1,2} = \frac{5 \pm 7}{4} = -3, \frac{1}{2}$$

$$\underline{y(x) = C_1 e^{-3x} + C_2 e^{x/2}}$$

Exercise

Find the general solution: $2y'' + 7y' - 4y = 0$

Solution

The characteristic equation: $2\lambda^2 + 7\lambda - 4 = 0$

$$\rightarrow \lambda_{1,2} = \frac{-7 \pm 9}{4} = -4, \frac{1}{2}$$

$$\underline{y(x) = C_1 e^{-4x} + C_2 e^{x/2}}$$

Exercise

Find the general solution: $3y'' + y = 0$

Solution

The characteristic equation: $3\lambda^2 + 1 = 0 \rightarrow \lambda_{1,2} = \pm \frac{1}{\sqrt{3}}i$

$$\underline{y(x) = C_1 \cos \frac{\sqrt{3}}{3}x + C_2 \sin \frac{\sqrt{3}}{3}x}$$

Exercise

Find the general solution: $3y'' - y' = 0$

Solution

The characteristic equation: $\lambda^2 + \lambda = \lambda(\lambda + 1) = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = -1$

$$\underline{y(x) = C_1 + C_2 e^{-x}}$$

Exercise

Find the general solution: $3y'' + 2y' + y = 0$

Solution

The characteristic equation: $3\lambda^2 + 2\lambda + 1 = 0$

$$\Rightarrow \lambda_{1,2} = \frac{-2 \pm \sqrt{4-12}}{6} = \underline{-\frac{1}{3} \pm i \frac{\sqrt{2}}{3}}$$

$$\underline{y(x) = e^{-x/3} \left(C_1 \cos \frac{\sqrt{2}}{3}x + C_2 \sin \frac{\sqrt{2}}{3}x \right)}$$

Exercise

Find the general solution: $3y'' + 11y' - 7y = 0$

Solution

The characteristic equation: $3\lambda^2 + 11\lambda - 7 = 0 \rightarrow \lambda_{1,2} = \frac{-11 \pm \sqrt{205}}{6}$

$$\underline{y(x) = C_1 e^{\frac{-11-\sqrt{205}}{6}x} + C_2 e^{\frac{-11+\sqrt{205}}{6}x}}$$

Exercise

Find the general solution: $3y'' - 20y' + 12y = 0$

Solution

The characteristic equation: $3\lambda^2 - 20\lambda + 12 = 0$

$$\rightarrow \lambda_{1,2} = \frac{20 \pm \sqrt{244}}{6} = \frac{10 \pm \sqrt{61}}{3}$$

$$y(x) = C_1 e^{\frac{10-\sqrt{61}}{3}x} + C_2 e^{\frac{10+\sqrt{61}}{3}x}$$

Exercise

Find the general solution: $4y'' + y' = 0$

Solution

The characteristic equation: $4\lambda^2 + \lambda = 0 \rightarrow \lambda_{1,2} = 0, -\frac{1}{4}$

$$y(x) = C_1 + C_2 e^{-x/4}$$

Exercise

Find the general solution: $4y'' + 4y' + y = 0$

Solution

The characteristic equation: $4\lambda^2 + 4\lambda + 1 = (2\lambda + 1)^2 = 0 \rightarrow \lambda_{1,2} = -\frac{1}{2}$

$$y(x) = (C_1 + C_2 x) e^{-x/2}$$

Exercise

Find the general solution: $4y'' - 4y' + y = 0$

Solution

The characteristic equation: $4\lambda^2 - 4\lambda + 1 = (2\lambda - 1)^2 = 0 \rightarrow \lambda_{1,2} = \frac{1}{2}$

$$y(x) = (C_1 + C_2 x) e^{x/2}$$

Exercise

Find the general solution: $4y'' + 4y' + 2y = 0$

Solution

The characteristic equation: $4\lambda^2 + 4\lambda + 2 = 0$

$$\rightarrow \lambda_{1,2} = \frac{-4 \pm 4i}{8} = -\frac{1}{2} \pm \frac{1}{2}i$$

$$y(x) = e^{-x/2} \left(C_1 \cos \frac{x}{2} + C_2 \sin \frac{x}{2} \right)$$

Exercise

Find the general solution: $4y'' - 4y' + y = 0$

Solution

The characteristic equation: $4\lambda^2 - 4\lambda + 1 = 0 \rightarrow \lambda_{1,2} = \frac{1}{2}$

$$y(x) = (C_1 + C_2 x) e^{x/2}$$

Exercise

Find the general solution: $4y'' - 4y' + 13y = 0$

Solution

The characteristic equation: $4\lambda^2 - 4\lambda + 13 = 0$

$$\rightarrow \lambda_{1,2} = \frac{4 \pm 8i\sqrt{3}}{8} = \frac{1}{2} \pm i\sqrt{3}$$

$$y(x) = (C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x) e^{x/2}$$

Exercise

Find the general solution: $4y'' - 8y' + 7y = 0$

Solution

The characteristic equation: $4\lambda^2 - 8\lambda + 7 = 0$

$$\rightarrow \lambda_{1,2} = \frac{8 \pm 4i\sqrt{3}}{8} = 1 \pm \frac{1}{2}i\sqrt{3}$$

$$y(x) = e^x \left(C_1 \cos \frac{\sqrt{3}}{2}x + C_2 \sin \frac{\sqrt{3}}{2}x \right)$$

Exercise

Find the general solution: $4y'' - 12y' + 9y = 0$

Solution

The characteristic equation: $4\lambda^2 - 12\lambda + 9 = 0 \rightarrow \lambda_{1,2} = \frac{12 \pm 0}{8} = \underline{\underline{\frac{3}{2}}}$

$$\underline{y(x) = (C_1 + C_2 x)e^{3x/2}}$$

Exercise

Find the general solution: $4y'' + 20y' + 25y = 0$

Solution

The characteristic equation: $4\lambda^2 + 20\lambda + 25 = (2\lambda + 5)^2 = 0 \rightarrow \lambda_{1,2} = \underline{\underline{-\frac{5}{2}}}$

$$\underline{y(x) = (C_1 + C_2 x)e^{-5x/2}}$$

Exercise

Find the general solution: $6y'' + y' - 2y = 0$

Solution

The characteristic equation: $6\lambda^2 + \lambda - 2 = 0 \rightarrow \lambda_{1,2} = \frac{-1 \pm 7}{2} = \underline{\underline{-4, 3}}$

$$\underline{y(x) = C_1 e^{-4x} + C_2 e^{3x}}$$

Exercise

Find the general solution: $6y'' + 5y' - 6y = 0$

Solution

The characteristic equation: $6\lambda^2 + 5\lambda - 6 = 0 \Rightarrow \lambda_{1,2} = -3, \frac{4}{3}$

$$\underline{y(t) = C_1 e^{-3t} + C_2 e^{4t/3}}$$

Exercise

Find the general solution: $6y'' - 7y' - 20y = 0$

Solution

The characteristic equation: $6\lambda^2 - 7\lambda - 20 = 0$

$$\Rightarrow \lambda_{1,2} = \frac{7 \pm \sqrt{529}}{12} = \frac{7 \pm 23}{12} \rightarrow \lambda_1 = -\frac{4}{3}, \lambda_2 = \frac{5}{2}$$

$$\underline{y(t) = C_1 e^{-4t/3} + C_2 e^{5t/2}}$$

Exercise

Find the general solution: $6y'' + 13y' - 5y = 0$

Solution

$$\text{The characteristic equation: } 6\lambda^2 + 13\lambda - 5 = 0 \rightarrow \lambda_{1,2} = \frac{-13 \pm 17}{12} = -\frac{5}{2}, \frac{1}{3}$$

$$\underline{y(x) = C_1 e^{-5x/2} + C_2 e^{x/3}}$$

Exercise

Find the general solution: $6y'' + 13y' + 7y = 0$

Solution

$$\text{The characteristic equation: } 6\lambda^2 + 13\lambda + 7 = 0 \rightarrow \lambda_{1,2} = -1, -\frac{7}{6}$$

$$\underline{y(x) = C_1 e^{-x} + C_2 e^{-7x/6}}$$

Exercise

Find the general solution: $6y'' - 13y' + 7y = 0$

Solution

$$\text{The characteristic equation: } 6\lambda^2 - 13\lambda + 7 = 0 \rightarrow \lambda_{1,2} = 1, \frac{7}{6}$$

$$\underline{y(x) = C_1 e^x + C_2 e^{7x/6}}$$

Exercise

Find the general solution: $8y'' - 10y' - 3y = 0$

Solution

$$\text{The characteristic equation: } 8\lambda^2 - 10\lambda - 3 = 0 \rightarrow \lambda_{1,2} = \frac{10 \pm 14}{16} = -\frac{1}{4}, \frac{3}{2}$$

$$\underline{y(x) = C_1 e^{-x/4} + C_2 e^{3x/2}}$$

Exercise

Find the general solution: $9y'' - y = 0$

Solution

The characteristic equation: $9\lambda^2 - 1 = 0 \rightarrow \lambda_{1,2} = \pm \frac{1}{3}$

$$\underline{y(x) = C_1 e^{-x/3} + C_2 e^{x/3}}$$

Exercise

Find the general solution: $9y'' + 6y' + y = 0$

Solution

The characteristic equation: $9\lambda^2 + 6\lambda + 1 = (3\lambda + 1)^2 = 0 \rightarrow \lambda_{1,2} = -\frac{1}{3}$

$$\underline{y(x) = (C_1 + C_2 x) e^{-x/3}}$$

Exercise

Find the general solution: $9y'' - 12y' + 4y = 0$

Solution

The characteristic equation: $9\lambda^2 - 12\lambda + 4 = (3\lambda - 2)^2 = 0 \rightarrow \lambda_{1,2} = \frac{2}{3}$

$$\underline{y(x) = (C_1 + C_2 x) e^{2x/3}}$$

Exercise

Find the general solution: $9y'' + 24y' + 16y = 0$

Solution

The characteristic equation: $9\lambda^2 + 24\lambda + 16 = (3\lambda + 4)^2 = 0 \rightarrow \lambda_{1,2} = -\frac{4}{3}$

$$\underline{y(x) = (C_1 + C_2 x) e^{-4x/3}}$$

Exercise

Find the general solution: $12y'' - 5y' - 2y = 0$

Solution

The characteristic equation: $12\lambda^2 - 5\lambda - 2 = 0$

$$\lambda_{1,2} = \frac{5 \pm 11}{24} = -\frac{1}{4}, \frac{2}{3}$$

$$y(x) = C_1 e^{-x/4} + C_2 e^{2x/3}$$

Exercise

Find the general solution: $16y'' - 8y' + 7y = 0$

Solution

The characteristic equation: $16\lambda^2 - 8\lambda + 7 = 0$

$$\lambda_{1,2} = \frac{8 \pm 8i\sqrt{6}}{32} = \frac{1}{4} \pm i\frac{\sqrt{6}}{4}$$

$$y(x) = e^{x/4} \left(C_1 \cos \frac{\sqrt{6}}{4} x + C_2 \sin \frac{\sqrt{6}}{4} x \right)$$

Exercise

Find the general solution: $16y'' - 12y' - 4y = 0$

Solution

The characteristic equation: $16\lambda^2 - 12\lambda - 4 = 0$

$$\lambda_{1,2} = \frac{12 \pm 20}{32} = -\frac{1}{4}, 1$$

$$y(x) = C_1 e^{-x/4} + C_2 e^x$$

Exercise

Find the general solution: $16y'' - 24y' + 9y = 0$

Solution

The characteristic equation: $16\lambda^2 - 24\lambda + 9 = (4\lambda - 3)^2 = 0 \rightarrow \lambda_{1,2} = \frac{3}{4}$

$$y(x) = (C_1 + C_2 x) e^{3x/4}$$

Exercise

Find the general solution: $25y'' + 10y' + y = 0$

Solution

The characteristic equation: $25\lambda^2 + 10\lambda + 1 = (5\lambda + 1)^2 = 0 \rightarrow \lambda_{1,2} = -\frac{1}{5}$

$$\underline{y(x) = (C_1 + C_2 x)e^{-x/5}}$$

Exercise

Find the general solution: $25y'' - 10y' + y = 0$

Solution

The characteristic equation: $25\lambda^2 - 10\lambda + 1 = (5\lambda - 1)^2 = 0 \rightarrow \lambda_{1,2} = \frac{1}{5}$

$$\underline{y(x) = (C_1 + C_2 x)e^{x/5}}$$

Exercise

Find the general solution: $35y'' - y' - 12y = 0$

Solution

The characteristic equation: $35\lambda^2 - \lambda - 12 = 0$

$$\rightarrow \lambda_{1,2} = \frac{1 \pm \sqrt{1681}}{70} = \frac{1 \pm 41}{70} \rightarrow \lambda_1 = -\frac{4}{7}, \lambda_2 = \frac{3}{5}$$

$$\underline{y(x) = C_1 e^{-4x/5} + C_2 e^{3x/5}}$$

Exercise

Find the general solution of the given higher-order differential equation: $y''' + 3y'' + 3y' + y = 0$

Solution

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3 = 0 \Rightarrow \lambda_{1,2,3} = -1$$

$$\underline{y(x) = (C_1 + C_2 x + C_3 x^2)e^{-x}}$$

Exercise

Find the general solution of the given higher-order differential equation: $y''' + 3y'' - y' - 3y = 0$

Solution

$$\lambda^3 + 3\lambda^2 - \lambda - 3 = 0$$

$$\lambda^2(\lambda + 3) - (\lambda + 3) = 0$$

$$(\lambda + 3)(\lambda^2 - 1) = 0$$

$$\lambda_{1,2,3} = -3, \pm 1$$

$$y(x) = C_1 e^{-3x} + C_2 e^{-x} + C_3 e^x$$

Exercise

Find the general solution of the given higher-order differential equation: $y^{(3)} + 3y'' - 4y = 0$

Solution

$$\lambda^3 + 3\lambda^2 - 4 = 0 \rightarrow \lambda_1 = 1$$

$$\begin{array}{c|cccc} 1 & 1 & 3 & 0 & -4 \\ & & 1 & 4 & 4 \\ \hline & 1 & 4 & 4 & 0 \end{array} \quad \lambda^2 + 4\lambda + 4 = 0 = (\lambda + 2)^2$$

$$\lambda_1 = 1, \lambda_{2,3} = -2$$

$$y(x) = C_1 e^x + (C_2 + C_3 x) e^{-2x}$$

Exercise

Find the general solution of the given higher-order differential equation: $3y''' - 19y'' + 36y' - 10y = 0$

Solution

$$3\lambda^3 - 19\lambda^2 + 36\lambda - 10 = 0 \quad \lambda_1 = \frac{1}{3}$$

$$\begin{array}{c|cccc} \frac{1}{3} & 3 & -19 & 36 & -10 \\ & & 1 & -6 & 10 \\ \hline & 3 & -18 & 30 & 0 \end{array} \rightarrow 3\lambda^2 - 18\lambda + 30 = 0$$

$$\lambda = \frac{18 \pm \sqrt{-36}}{6} \rightarrow \lambda_{2,3} = 3 \pm i$$

$$y(x) = C_1 e^{x/3} + e^{3x} (C_2 \cos x + C_3 \sin x)$$

Exercise

Find the general solution of the given higher-order differential equation: $y''' - 6y'' + 12y' - 8y = 0$

Solution

The characteristic equation: $\lambda^3 - 6\lambda^2 + 12\lambda - 8 = (\lambda - 2)^3 = 0 \Rightarrow \lambda_{1,2,3} = 2$

$$\underline{y(x) = (C_1 + C_2 x + C_3 x^2) e^{2x}}$$

Exercise

Find the general solution of the given higher-ODE: $y''' + 5y'' + 7y' + 3y = 0$

Solution

The characteristic equation: $\lambda^3 + 5\lambda^2 + 7\lambda + 3 = 0 \Rightarrow \lambda_1 = -3$

$$\begin{array}{c|cccc} -3 & 1 & 5 & 7 & 3 \\ & & -3 & -6 & -3 \\ \hline & 1 & 2 & 1 & 0 \end{array}$$

$$\lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda_{2,3} = -1$$

The general solution is: $\underline{y(x) = C_1 e^{-3x} + C_2 e^{-x} + C_3 x e^{-x}}$

Exercise

Find the general solution of the given higher-ODE: $y^{(3)} + y' - 10y = 0$

Solution

The characteristic equation:

$$\lambda^3 + \lambda - 10 = 0 \Rightarrow \lambda_1 = 2 \quad (\text{Rational Zero Theorem})$$

$$\begin{array}{c|cccc} 2 & 1 & 0 & 1 & -10 \\ & & 2 & 4 & 10 \\ \hline & 1 & 2 & 5 & 0 \end{array}$$

$$\lambda^2 + 2\lambda + 5 = 0 \Rightarrow \lambda_{2,3} = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i$$

The general solution is: $\underline{y(x) = C_1 e^{2x} + e^{-x} (C_2 \cos 2x + C_3 \sin 2x)}$

Exercise

Find the general solution of the given higher ODE: $y''' + y'' - 6y' + 4y = 0$

Solution

The characteristic equation: $\lambda^3 + \lambda^2 - 6\lambda + 4 = 0$

$$\lambda_1 = 1 \quad \left| \begin{array}{ccc|c} 1 & 1 & -6 & 4 \\ & 1 & 2 & -4 \\ \hline 1 & 2 & -4 & \boxed{0} \end{array} \right|$$

$$\lambda^2 + 2\lambda - 4 = 0 \rightarrow \lambda_{2,3} = \frac{-2 \pm 2\sqrt{5}}{2} = -1 \pm \sqrt{5}$$

$$\underline{y(x) = C_1 e^x + C_2 e^{(-1-\sqrt{5})x} + C_3 e^{(-1+\sqrt{5})x}}$$

Exercise

Find the general solution of the given higher ODE: $y''' - 6y'' - y' + 6y = 0$

Solution

The characteristic equation: $\lambda^3 - 6\lambda^2 - \lambda + 6 = 0$

$$\lambda_1 = 1 \quad \left| \begin{array}{ccc|c} 1 & 1 & -6 & -1 \\ & 1 & -5 & -6 \\ \hline 1 & -5 & -6 & \boxed{0} \end{array} \right|$$

$$\lambda^2 - 5\lambda - 6 = 0 \rightarrow \lambda_{2,3} = -1, 6$$

$$\underline{y(x) = C_1 e^{-x} + C_2 e^x + C_3 e^{6x}}$$

Exercise

Find the general solution of the given higher ODE: $y''' + 2y'' - 4y' - 8y = 0$

Solution

The characteristic equation: $\lambda^3 + 2\lambda^2 - 4\lambda - 8 = 0$

$$\lambda_1 = -2 \quad \left| \begin{array}{ccc|c} -2 & 1 & 2 & -4 \\ & -2 & 0 & 8 \\ \hline 1 & 0 & -4 & \boxed{0} \end{array} \right|$$

$$\lambda^2 - 4 = 0 \rightarrow \lambda_{2,3} = \pm 2$$

$$\underline{y(x) = (C_1 + C_2 x) e^{-2x} + C_3 e^{2x}}$$

Exercise

Find the general solution of the given higher ODE: $y''' - 7y'' + 7y' + 15y = 0$

Solution

The characteristic equation: $\lambda^3 - 7\lambda^2 + 7\lambda + 15 = 0$

$$\lambda_1 = -1 \quad \left| \begin{array}{ccc|c} -1 & 1 & -7 & 7 & 15 \\ & & -1 & 8 & -15 \\ \hline & & 1 & -8 & 15 & \underline{0} \end{array} \right|$$

$$\lambda^2 - 8\lambda + 15 = 0 \rightarrow \lambda_{2,3} = 3, 5$$

$$\underline{y(x) = C_1 e^{-x} + C_2 e^{3x} + C_3 e^{5x}}$$

Exercise

Find the general solution of the given higher ODE: $y''' + 3y'' - 4y' - 12y = 0$

Solution

The characteristic equation: $\lambda^3 + 3\lambda^2 - 4\lambda - 12 = 0$

$$\lambda^2(\lambda + 3) - 4(\lambda + 3) = 0$$

$$(\lambda + 3)(\lambda^2 - 4) = 0 \rightarrow \lambda_1 = -3, \lambda_2 = -2, \lambda_3 = 2$$

$$\underline{y(x) = C_1 e^{-3x} + C_2 e^{-2x} + C_3 e^{2x}}$$

Exercise

Find the general solution of the given higher ODE: $y''' - 4y'' - 5y' = 0$

Solution

The characteristic equation: $\lambda^3 - 4\lambda^2 - 5\lambda = \lambda(\lambda^2 - 4\lambda - 5) = 0$

$$\underline{\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = 5}$$

$$\underline{y(x) = C_1 + C_2 e^{-x} + C_3 e^{5x}}$$

Exercise

Find the general solution of the given higher ODE: $y''' - y = 0$

Solution

The characteristic equation: $\lambda^3 - 1 = (\lambda - 1)(\lambda^2 + \lambda + 1) = 0$

$$\lambda_1 = 2, \quad \lambda_{2,3} = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}$$

$$y(x) = C_1 e^{2x} + e^{-x/2} \left(C_2 \cos \frac{\sqrt{3}}{2} x + C_3 \sin \frac{\sqrt{3}}{2} x \right)$$

Exercise

Find the general solution of the given higher ODE: $y''' - 5y'' + 3y' + 9y = 0$

Solution

The characteristic equation: $\lambda^3 - 5\lambda^2 + 3\lambda + 9 = 0$

$$\lambda_1 = -1 \quad \left| \begin{array}{ccc|c} 1 & -5 & 3 & 9 \\ & -1 & 6 & -9 \\ \hline & 1 & -6 & 9 \end{array} \right| \quad \underline{0}$$

$$\lambda^2 - 6\lambda + 9 = 0 \rightarrow \lambda_{2,3} = 3, 3$$

$$y(x) = C_1 e^{-x} + (C_2 + C_3 x) e^{3x}$$

Exercise

Find the general solution of the given higher ODE: $y''' + 3y'' - 4y' - 12y = 0$

Solution

The characteristic equation: $\lambda^3 + 3\lambda^2 - 4\lambda - 12 = 0$

$$\lambda^2 (\lambda + 3) - 4(\lambda + 3) = 0$$

$$(\lambda + 3)(\lambda^2 - 4) = 0 \rightarrow \lambda_{2,3} = -3, \pm 2$$

$$y(x) = C_1 e^{-3x} + C_2 e^{-2x} + C_3 e^{2x}$$

Exercise

Find the general solution of the given higher ODE: $y''' + y'' - 2y = 0$

Solution

The characteristic equation: $\lambda^3 + \lambda^2 - 2 = 0$

$$\lambda_1 = 1 \quad \left| \begin{array}{cccc|c} 1 & 1 & 0 & -2 & \\ & 1 & 2 & 2 & \\ \hline 1 & 2 & 2 & & \underline{0} \end{array} \right|$$

$$\lambda^2 + 2\lambda + 2 = 0 \rightarrow \lambda_{2,3} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$\underline{y(x) = C_1 e^x + e^{-x} \left(C_2 \cos x + C_3 \sin x \right)}$$

Exercise

Find the general solution of the given higher ODE: $y''' - y'' - 4y = 0$

Solution

The characteristic equation: $\lambda^3 - \lambda^2 - 4 = 0$

$$\lambda_1 = 2 \quad \left| \begin{array}{cccc|c} 2 & 1 & -1 & 0 & -4 \\ & 2 & 2 & 4 & \\ \hline 1 & 1 & 2 & & \underline{0} \end{array} \right|$$

$$\lambda^2 + \lambda + 2 = 0 \rightarrow \lambda_{2,3} = \frac{-1 \pm i\sqrt{7}}{2}$$

$$\underline{y(x) = C_1 e^{2x} + e^{-x/2} \left(C_2 \cos \frac{\sqrt{7}}{2} x + C_3 \sin \frac{\sqrt{7}}{2} x \right)}$$

Exercise

Find the general solution of the given higher ODE: $y''' + 3y'' + 3y' + y = 0$

Solution

The characteristic equation: $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$

$$\lambda_1 = -1 \quad \left| \begin{array}{cccc|c} -1 & 1 & 3 & 3 & 1 \\ & -1 & -2 & -1 & \\ \hline 1 & 2 & 1 & & \underline{0} \end{array} \right|$$

$$\lambda^2 + 2\lambda + 1 = 0 \rightarrow \lambda_{2,3} = -1$$

$$\underline{y(x) = \left(C_1 + C_2 x + C_3 x^2 \right) e^{-x}}$$

Exercise

Find the general solution of the given higher ODE: $y''' - 6y'' + 12y' - 8y = 0$

Solution

The characteristic equation: $\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$

$$\lambda_1 = 2 \quad \begin{array}{c|ccc} 2 & 1 & -6 & 12 & -8 \\ & & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & \underline{0} \end{array}$$

$$\lambda^2 - 4\lambda + 4 = 0 \rightarrow \lambda_{2,3} = 2$$

$$\underline{y(x) = (C_1 + C_2 x + C_3 x^2) e^{2x}}$$

Exercise

Find the general solution of the given higher ODE: $y^{(4)} + y''' + y'' = 0$

Solution

The characteristic equation: $\lambda^4 + \lambda^3 + \lambda^2 = \lambda^2(\lambda^2 + \lambda + 1) = 0$

$$\lambda_{1,2} = 0 \quad \lambda^2 + \lambda + 1 = 0 \rightarrow \lambda_{3,4} = \frac{-1 \pm i\sqrt{3}}{2}$$

$$\underline{y(x) = C_1 + C_2 x + e^{-x/2} \left(C_3 \cos \frac{\sqrt{3}}{2} x + C_4 \sin \frac{\sqrt{3}}{2} x \right)}$$

Exercise

Find the general solution of the given higher ODE: $y^{(4)} - 2y'' + y = 0$

Solution

The characteristic equation: $\lambda^4 - 2\lambda^2 + 1 = (\lambda^2 - 1)^2 = 0$

$$\lambda^2 = 1 \quad \lambda_{1,2} = -1 \quad \lambda_{3,4} = 1$$

$$\underline{y(x) = (C_1 + C_2 x) e^{-x} + (C_3 + C_4 x) e^x}$$

Exercise

Find the general solution of the given higher ODE: $16y^{(4)} + 24y'' + 9y = 0$

Solution

The characteristic equation: $16\lambda^4 + 24\lambda^2 + 9 = (4\lambda^2 + 3)^2 = 0$

$$\lambda^2 = -\frac{3}{4} \quad \lambda_{1,2} = \pm \frac{\sqrt{3}}{2}i \quad \lambda_{3,4} = \pm \frac{\sqrt{3}}{2}i$$

$$y(x) = C_1 \cos \frac{\sqrt{3}}{2}x + C_2 \sin \frac{\sqrt{3}}{2}x + x \left(C_3 \cos \frac{\sqrt{3}}{2}x + C_4 \sin \frac{\sqrt{3}}{2}x \right)$$

Exercise

Find the general solution of the given higher ODE: $y^{(4)} - 7y'' - 18y = 0$

Solution

The characteristic equation: $\lambda^4 - 7\lambda^2 - 18 = 0$

$$\lambda^2 = \frac{7 \pm 11}{2} \quad \lambda_{1,2} = \pm 2i \quad \lambda_{3,4} = \pm 3$$

$$y(x) = C_1 \cos 2x + C_2 \sin 2x + C_3 e^{-3x} + C_4 e^{3x}$$

Exercise

Find the general solution of the given higher-order differential equation: $y^{(4)} + 2y'' + y = 0$

Solution

$$\lambda^4 + 2\lambda^2 + 1 = (\lambda^2 + 1)^2 = 0$$

$$\lambda^2 = -1 \Rightarrow \lambda = \pm i \Rightarrow \lambda_{1,2} = -i, \lambda_{3,4} = i$$

$$y(x) = (C_1 + C_2 x)e^{-ix} + (C_3 + C_4 x)e^{ix}$$

OR $y(x) = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x$

Exercise

Find the general solution of the given higher-order differential equation: $y^{(4)} + y''' + y'' = 0$

Solution

$$\lambda^4 + \lambda^3 + \lambda^2 = \lambda^2(\lambda^2 + \lambda + 1) = 0$$

$$\lambda^2 = 0 \rightarrow \lambda_{1,2} = 0 \Rightarrow \lambda = \frac{-1 \pm \sqrt{1-4}}{2} \Rightarrow \lambda_{3,4} = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}$$

$$y(x) = C_1 + C_2 x + e^{-x/2} \left(C_3 \cos\left(\frac{\sqrt{3}}{2}x\right) + C_4 \sin\left(\frac{\sqrt{3}}{2}x\right) \right)$$

Exercise

Find the general solution of the given higher-ODE: $y^{(4)} + 4y = 0$

Solution

The characteristic equation: $\lambda^4 + 4 = 0 \Rightarrow \lambda^2 = \pm 2i \rightarrow \lambda_{1,2,3,4} = \pm \sqrt{\pm 2i}$

$$\text{Since } i = e^{\frac{\pi}{2}i} \quad -i = e^{\frac{3\pi}{2}i}$$

$$\sqrt{2i} = \left(2e^{i\pi/2}\right)^{1/2} = \sqrt{2}e^{i\pi/4} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 1 + i$$

$$\sqrt{-2i} = \left(2e^{i3\pi/2}\right)^{1/2} = \sqrt{2}e^{i3\pi/4} = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -1 + i$$

$$\lambda = \pm(\pm 1 + i) = \begin{cases} 1 + i \\ -1 + i \end{cases}$$

The general solution is: $y(x) = e^x(C_1 \cos x + C_2 \sin x) + e^{-x}(C_3 \cos x + C_4 \sin x)$

Exercise

Find the general solution of the given higher-ODE: $y^{(4)} + 2y''' + 9y'' - 2y' - 10y = 0$

Solution

The characteristic equation:

$$\lambda^4 + 2\lambda^3 + 9\lambda^2 - 2\lambda - 10 = 0 \Rightarrow \lambda_1 = 1 \quad (\text{Rational Zero Theorem})$$

$$\begin{array}{c|cccc} 1 & 1 & 2 & 9 & -2 & -10 \\ & & 1 & 3 & 12 & 10 \\ \hline & 1 & 3 & 12 & 10 & 0 \end{array} \quad \lambda^3 + 3\lambda^2 + 12\lambda + 10 = 0 \Rightarrow \lambda_2 = -1$$

$$\begin{array}{c|cccc} -1 & 1 & 3 & 12 & 10 \\ & & -1 & -2 & -10 \\ \hline & 1 & 2 & 10 & 0 \end{array} \quad \lambda^2 + 2\lambda + 10 = 0 \Rightarrow \lambda_{3,4} = -1 \pm 3i$$

The general solution is: $y(x) = C_1 e^x + C_2 e^{-x} + e^{-x}(C_3 \cos 3x + C_4 \sin 3x)$

Exercise

Find the solution of the given initial value problem $x^{(4)} - 4x^{(3)} + 7x'' - 4x' + 6x = 0$

Solution

The characteristic equation: $\lambda^4 - 4\lambda^3 + 7\lambda^2 - 4\lambda + 6 = 0 \rightarrow \lambda_{1,2,3,4} = \pm i, 2 \pm i\sqrt{2}$

$$x(t) = C_1 e^{(2+i\sqrt{2})t} + C_2 e^{(2-i\sqrt{2})t} + C_3 \cos t + C_4 \sin t$$

Exercise

Find the solution of the given initial value problem $x^{(4)} + 8x^{(3)} + 24x'' + 32x' + 16x = 0$

Solution

The characteristic equation: $\lambda^4 + 8\lambda^3 + 24\lambda^2 + 32\lambda + 16 = 0 \rightarrow \lambda_1 = -2$

$$\begin{array}{r|rrrrr} -2 & 1 & 8 & 24 & 32 & 16 \\ & & -2 & -12 & -24 & -16 \\ \hline & 1 & 6 & 12 & 8 & 0 \end{array} \quad \lambda^3 + 6\lambda^2 + 12\lambda + 8 = 0 \Rightarrow \lambda_2 = -2$$

$$\begin{array}{r|rrrr} -2 & 1 & 6 & 12 & 8 \\ & & -2 & -8 & -8 \\ \hline & 1 & 4 & 4 & 0 \end{array} \quad \lambda^2 + 4\lambda + 4 = 0 \Rightarrow \lambda_{3,4} = -2$$

The eigenvalues: $\lambda_{1,2,3,4} = -2$

$$x(t) = (C_1 + C_2 t + C_3 t^2 + C_4 t^3) e^{-2t}$$

Exercise

Find the solution of the given initial value problem $x^{(4)} - 4x'' + 16x' + 32x = 0$

Solution

The characteristic equation: $\lambda^4 - 4\lambda^2 + 16\lambda + 32 = 0 \rightarrow \lambda_1 = -2$

$$\begin{array}{r|rrrrr} -2 & 1 & 0 & -4 & 16 & 32 \\ & & -2 & 4 & 0 & -32 \\ \hline & 1 & -2 & 0 & 16 & 0 \end{array} \quad \lambda^3 - 2\lambda^2 + 16 = 0 \Rightarrow \lambda_2 = -2$$

$$\begin{array}{r|rrrr} -2 & 1 & -2 & 0 & 16 \\ & & -2 & 8 & -16 \\ \hline & 1 & -4 & 8 & 0 \end{array} \quad \lambda^2 - 4\lambda + 8 = 0 \Rightarrow \lambda_{3,4} = 2 \pm 2i$$

The eigenvalues: $\lambda = -2, -2, 2 \pm 2i$

$$x(t) = (C_1 + C_2 t)e^{-2t} + e^{2t}(C_3 \cos 2t + C_4 \sin 2t)$$

Exercise

Find the solution of the given initial value problem $x^{(4)} + 4x^{(3)} + 6x'' + 4x' + x = 0$

Solution

The characteristic equation: $\lambda^4 + 4\lambda^3 + 6\lambda^2 + 4\lambda + 1 = (\lambda + 1)^4 = 0$

$$\rightarrow \lambda_{1,2,3,4} = -1$$

$$x(t) = (C_1 + C_2 t + C_3 t^2 + C_4 t^3)e^{-t}$$

Exercise

Find the solution of the given initial value problem $y^{(4)} - y^{(3)} + y'' - 3y' - 6y = 0$

Solution

The characteristic equation: $\lambda^4 - \lambda^3 + \lambda^2 - 3\lambda - 6 = 0 \rightarrow \lambda_1 = -1$

$$\begin{array}{c|ccccc} -1 & 1 & -1 & 1 & -3 & -6 \\ & & -1 & 2 & -3 & 6 \\ \hline & 1 & -2 & 3 & -6 & 0 \end{array} \quad \lambda^3 - 2\lambda^2 + 3\lambda - 6 = 0 \Rightarrow \lambda_2 = 2$$

$$\begin{array}{c|cccc} 2 & 1 & -2 & 3 & -6 \\ & & 2 & 0 & 6 \\ \hline & 1 & 0 & 3 & 0 \end{array} \quad \lambda^2 + 3 = 0 \Rightarrow \lambda_{3,4} = \pm i\sqrt{3}$$

The eigenvalues: $\lambda = -1, 2, \pm i\sqrt{3}$

$$y(t) = C_1 e^{-t} + C_2 e^{2t} + C_3 \cos \sqrt{3}t + C_4 \sin \sqrt{3}t$$

Exercise

Find the solution of the given initial value problem $y^{(4)} + y^{(3)} - 3y'' - 5y' - 2y = 0$

Solution

The characteristic equation: $\lambda^4 + \lambda^3 - 3\lambda^2 - 5\lambda - 2 = 0 \rightarrow \lambda_1 = -1$

$$\begin{array}{c|ccccc} -1 & 1 & 1 & -3 & -5 & -2 \\ & & -1 & 0 & 3 & 2 \\ \hline & 1 & 0 & -3 & -2 & \underline{0} \end{array} \quad \lambda^3 - 3\lambda - 2 = 0 \Rightarrow \lambda_2 = 2$$

$$\begin{array}{c|cccc} 2 & 1 & 0 & -3 & -2 \\ & & 2 & 4 & 2 \\ \hline & 1 & 2 & 1 & \underline{0} \end{array} \quad \lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda_{3,4} = -1$$

The eigenvalues: $\lambda_{1,2,3} = -1, \lambda_4 = 2$

$$y(t) = (C_1 + C_2 t + C_3 t^2) e^{-t} + C_4 e^{2t}$$

Exercise

Find the solution of the given initial value problem $x^{(5)} - x^{(4)} - 2x^{(3)} + 2x'' + x' - x = 0$

Solution

The characteristic equation: $\lambda^5 - \lambda^4 - 2\lambda^3 + 2\lambda^2 + \lambda - 1 = 0 \rightarrow \lambda_1 = 1$

$$\begin{array}{c|cccccc} 1 & 1 & -1 & -2 & 2 & 1 & -1 \\ & & 1 & 0 & -2 & 0 & 1 \\ \hline & 1 & 0 & -2 & 0 & 1 & \underline{0} \end{array}$$

$$\lambda^4 - 2\lambda^2 + 1 = (\lambda^2 - 1)^2 = 0 \rightarrow \lambda^2 = 1, 1$$

The eigenvalues: $\lambda = 1, 1, 1, -1, -1$

$$x(t) = (C_1 + C_2 t + C_3 t^2) e^t + (C_4 + C_5 t) e^{-t}$$

Exercise

Find the solution of the given initial value problem $x^{(5)} + 5x^{(4)} + 10x^{(3)} + 10x'' + 5x' + x = 0$

Solution

The characteristic equation: $\lambda^5 + 5\lambda^4 + 10\lambda^3 + 10\lambda^2 + 5\lambda + 1 = (\lambda + 1)^5 = 0$

The eigenvalues: $\lambda = 1, 1, 1, -1, -1$

$$x(t) = (C_1 + C_2 t + C_3 t^2) e^t + (C_4 + C_5 t) e^{-t}$$

Exercise

Find the general solution of the given higher ODE: $y^{(5)} + 5y^{(4)} - 2y''' - 10y'' + y' + 5y = 0$

Solution

The characteristic equation: $\lambda^5 + 5\lambda^4 - 2\lambda^3 - 10\lambda^2 + \lambda + 5 = 0$

$$\underline{\lambda_1 = 1}$$

$$\begin{array}{c|cccccc} 1 & 1 & 5 & -2 & -10 & 1 & 5 \\ & & 1 & 6 & 4 & -6 & -5 \\ \hline & 1 & 6 & 4 & -6 & -5 & \underline{0} \end{array} \rightarrow \lambda^4 + 6\lambda^3 + 4\lambda^2 - 6\lambda - 5 = 0 \Rightarrow \underline{\lambda_2 = 1}$$

$$\begin{array}{c|cccccc} 1 & 1 & 6 & 4 & -6 & -5 \\ & & 1 & 7 & 11 & 5 \\ \hline & 1 & 7 & 11 & 5 & \underline{0} \end{array} \rightarrow \lambda^3 + 7\lambda^2 + 11\lambda + 5 = 0 \Rightarrow \underline{\lambda_3 = -1}$$

$$\begin{array}{c|cccccc} -1 & 1 & 7 & 11 & 5 \\ & & -1 & -6 & -5 \\ \hline & 1 & 6 & 5 & \underline{0} \end{array} \rightarrow \lambda^2 + 6\lambda + 5 = 0 \Rightarrow \underline{\lambda_{4,5} = -1, -5}$$

$$\underline{\lambda = -5, -1, -1, 1, 1}$$

$$\underline{y(x) = C_1 e^{-x} + (C_2 + C_3 x) e^{-x} + (C_4 + C_5 x) e^x}$$

Exercise

Find the general solution of the given higher ODE: $2y^{(5)} - 7y^{(4)} + 12y''' + 8y'' = 0$

Solution

The characteristic equation: $2\lambda^5 - 7\lambda^4 + 12\lambda^3 + 8\lambda^2 = \lambda^2(2\lambda^3 - 7\lambda^2 + 12\lambda + 8) = 0$

$$\underline{\lambda_{1,2} = 0, \lambda_3 = -\frac{1}{2}}$$

$$\begin{array}{c|cccccc} -\frac{1}{2} & 2 & -7 & 12 & 8 \\ & & -1 & 4 & -8 \\ \hline & 2 & -8 & 16 & \underline{0} \end{array} \rightarrow 2\lambda^2 - 8\lambda + 16 = 0 \Rightarrow \lambda_{4,5} = 2 \pm \frac{8i}{4} = \underline{2 \pm 2i}$$

$$\underline{\lambda = 0, 0, -\frac{1}{2}, 2 \pm 2i}$$

$$\underline{y(x) = C_1 + C_2 x + C_3 e^{-x/2} + e^{2x} (C_4 \cos 2x + C_5 \sin 2x)}$$

Exercise

Find the general solution of the given higher-order differential equation: $y^{(5)} - 2y^{(4)} + 17y''' = 0$

Solution

$$\lambda^5 - 2\lambda^4 + 17\lambda^3 = \lambda^3(\lambda^2 - 2\lambda + 17) = 0$$

$$\lambda^3 = 0 \rightarrow \lambda_{1,2,3} = 0 \Rightarrow \lambda = \frac{2 \pm \sqrt{4 - 68}}{2} \Rightarrow \lambda_{4,5} = 1 \pm 4i$$

$$y(x) = C_1 + C_2 x + C_3 x^2 + e^x (C_4 \cos 4x + C_5 \sin 4x)$$

Exercise

Find the solution of the given initial value problem $x^{(6)} - 5x^{(4)} + 16x^{(3)} + 36x'' - 16x' - 32x = 0$

Solution

The characteristic equation: $\lambda^6 - 5\lambda^4 + 16\lambda^3 + 36\lambda^2 - 16\lambda - 32 = 0 \rightarrow \lambda_1 = 1$

$$\begin{array}{l|cccccc} 1 & 1 & 0 & -5 & 16 & 36 & -16 & -32 \\ & & 1 & 1 & -4 & 12 & 48 & 32 \\ \hline & 1 & 1 & -4 & 12 & 48 & 32 & \underline{0} \end{array} \quad \lambda^5 + \lambda^4 - 4\lambda^3 + 12\lambda^2 + 48\lambda + 32 = 0 \rightarrow \lambda_2 = -1$$

$$\begin{array}{l|cccccc} -1 & 1 & 1 & -4 & 12 & 48 & 32 \\ & & -1 & 0 & 4 & -16 & -32 \\ \hline & 1 & 0 & -4 & 16 & 32 & \underline{0} \end{array} \quad \lambda^4 - 4\lambda^2 + 16\lambda + 32 = 0 \rightarrow \lambda_3 = -2$$

$$\begin{array}{l|cccccc} -2 & 1 & 0 & -4 & 16 & 32 \\ & & -2 & 4 & 0 & -32 \\ \hline & 1 & -2 & 0 & 16 & \underline{0} \end{array} \quad \lambda^3 - 2\lambda^2 + 16 = 0 \Rightarrow \lambda_4 = -2$$

$$\begin{array}{l|cccc} -2 & 1 & -2 & 0 & 16 \\ & & -2 & 8 & -16 \\ \hline & 1 & -4 & 8 & \underline{0} \end{array}$$

$$\lambda^2 - 4\lambda + 8 = 0 \Rightarrow \lambda_{3,4} = 2 \pm 2i$$

The eigenvalues: $\lambda = 1, -1, -2, -2, 2 \pm 2i$

$$x(t) = C_1 e^t + C_2 e^{-t} + (C_3 + C_4 t) e^{-2t} + e^{2t} (C_5 \cos 2t + C_6 \sin 2t)$$

Exercise

Find the general solution of the given higher-order differential equation: $(D^2 + 6D + 13)^2 y = 0$

Solution

The characteristic equation: $(\lambda^2 + 6\lambda + 13) = 0$

$$\Rightarrow \lambda = \frac{-6 \pm \sqrt{-16}}{2} = -3 \pm 2i \quad \text{multiplicity } k = 2$$

$$\underline{y(x) = e^{-3x} (C_1 \cos 2x + C_2 \sin 2x) + x e^{-3x} (C_3 \cos 2x + C_4 \sin 2x)}$$

Exercise

Find the general solution of the given higher-order differential equation $\lambda^3(\lambda - 1)(\lambda - 2)^3(\lambda^2 + 9) = 0$

Solution

$$\lambda^2 + 9 = 0 \Rightarrow \lambda^2 = -9 \rightarrow \lambda = \pm 3i$$

The solution: $\lambda = 0, 0, 0, 1, 2, 2, 2, \pm 3i$

$$\underline{y(x) = C_1 + C_2 x + C_3 x^2 + C_4 e^x + (C_5 + C_6 x + C_7 x^2) e^{2x} + C_8 \cos 3x + C_9 \sin 3x}$$

Exercise

Find the solution of the given initial value problem $y'' + y = 0, \quad y\left(\frac{\pi}{3}\right) = 0, \quad y'\left(\frac{\pi}{3}\right) = 2$

Solution

The characteristic equation: $\lambda^2 + 1 = 0 \rightarrow \underline{\lambda_{1,2} = \pm i}$

$$y(x) = C_1 \cos x + C_2 \sin x$$

$$y\left(\frac{\pi}{3}\right) = 0 \rightarrow \frac{1}{2} C_1 + \frac{\sqrt{3}}{2} C_2 = 0 \Rightarrow \underline{C_1 + \sqrt{3} C_2 = 0}$$

$$y'(x) = -C_1 \sin x + C_2 \cos x$$

$$y'\left(\frac{\pi}{3}\right) = 2 \rightarrow -\frac{\sqrt{3}}{2} C_1 + \frac{1}{2} C_2 = 2 \Rightarrow \underline{-\sqrt{3} C_1 + C_2 = 4}$$

$$\begin{cases} C_1 + \sqrt{3} C_2 = 0 \\ -\sqrt{3} C_1 + C_2 = 4 \end{cases} \rightarrow \begin{cases} C_1 = -\sqrt{3} C_2 \\ 3C_2 + C_2 = 4 \end{cases} \Rightarrow \underline{C_2 = 1, C_1 = -\sqrt{3}}$$

$$\underline{y(x) = -\sqrt{3} \cos x + \sin x}$$

Exercise

Find the solution of the given initial value problem $y'' + y = 0$; $y(0) = 0$, $y'(\frac{\pi}{2}) = 0$

Solution

The characteristic equation: $\lambda^2 + 1 = 0 \rightarrow \lambda_{1,2} = \pm i$

$$y(x) = C_1 \cos x + C_2 \sin x$$

$$y(0) = 0 \rightarrow C_1 = 0$$

$$y'(x) = -C_1 \sin x + C_2 \cos x$$

$$y'(\frac{\pi}{2}) = 0 \rightarrow C_2 = 0$$

$$y(x) = C_2 \sin x$$

Exercise

Find the solution of the given initial value problem $y'' + y' = 0$; $y(0) = 2$, $y'(0) = 1$

Solution

The characteristic equation: $\lambda^2 + \lambda = 0 \rightarrow \lambda_{1,2} = 0, -1$

$$y(x) = C_1 + C_2 e^{-x}$$

$$y(0) = 2 \rightarrow C_1 + C_2 = 2$$

$$y'(x) = -C_2 e^{-x}$$

$$y'(0) = 1 \rightarrow C_2 = -1$$

$$C_1 + C_2 = 2 \rightarrow C_1 = 3$$

$$y(x) = 3e^{-4x}$$

Exercise

Find the general solution: $y'' - y' - 2y = 0$; $y(0) = -1$, $y'(0) = 2$

Solution

The characteristic equation: $\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda_1 = 2; \lambda_2 = -1$

$$y(t) = C_1 e^{2t} + C_2 e^{-t} \quad y(0) = C_1 + C_2 = -1$$

$$y'(t) = 2C_1 e^{2t} - C_2 e^{-t} \quad y'(0) = 2C_1 - C_2 = 2$$

$$\underline{C_1 = \frac{1}{3} \quad C_2 = -\frac{4}{3}}$$

$$\underline{y(t) = \frac{1}{3}e^{2t} - \frac{4}{3}e^{-t}}$$

Exercise

Find the solution of the given initial value problem $y'' + y' + 2y = 0$, $y(0) = 0$, $y'(0) = 0$

Solution

The characteristic equation: $\lambda^2 + \lambda + 2 = 0 \rightarrow \underline{\lambda_{1,2} = \frac{-1 \pm i\sqrt{7}}{2}}$

$$y(x) = e^{-x/2} \left(C_1 \cos \frac{\sqrt{7}}{2}x + C_2 \sin \frac{\sqrt{7}}{2}x \right)$$

$$y(0) = 0 \rightarrow \underline{C_1 = 0}$$

$$y'(x) = e^{-x/2} \left(-\frac{1}{2}C_1 \cos \frac{\sqrt{7}}{2}x - \frac{1}{2}C_2 \sin \frac{\sqrt{7}}{2}x - \frac{\sqrt{7}}{2}C_1 \sin \frac{\sqrt{7}}{2}x + \frac{\sqrt{7}}{2}C_2 \cos \frac{\sqrt{7}}{2}x \right)$$

$$y'(0) = 0 \rightarrow -\frac{1}{2}C_1 + \frac{\sqrt{7}}{2}C_2 = 0 \Rightarrow \underline{C_2 = 0}$$

$$\underline{y(x) = 0}$$

Exercise

Find the solution of the given initial value problem $y'' + 2y' + y = 0$; $y(0) = 1$, $y'(0) = -3$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0 \rightarrow \underline{\lambda_{1,2} = -1}$

$$\underline{y(x) = (C_1 + C_2 x)e^{-x}}$$

$$y(0) = 1 \rightarrow \underline{C_1 = 1}$$

$$y'(x) = (C_2 - C_1 - C_2 x)e^{-x}$$

$$y'(0) = -3 \rightarrow C_2 - C_1 = -3 \Rightarrow \underline{C_2 = -2}$$

$$\underline{y(x) = (1 - 2x)e^{-x}}$$

Exercise

Find the solution of the given initial value problem $y'' - 2y' + y = 0$; $y(0) = 5$, $y'(0) = 10$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0 \rightarrow \lambda_{1,2} = 1$

$$y(x) = (C_1 + C_2 x)e^x$$

$$y(0) = 5 \rightarrow C_1 = 5$$

$$y'(x) = (C_2 + C_1 + C_2 x)e^x$$

$$y'(0) = 10 \rightarrow C_2 + C_1 = 10 \Rightarrow C_2 = 5$$

$$y(x) = 5(1+x)e^x$$

Exercise

Find the solution of the given initial value problem $y'' - 2y' - 2y = 0$; $y(0) = 0$, $y'(0) = 3$

Solution

The characteristic equation: $\lambda^2 - 2\lambda - 2 = 0 \rightarrow \lambda_{1,2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$

$$y(x) = C_1 e^{(1-\sqrt{3})x} + C_2 e^{(1+\sqrt{3})x}$$

$$y(0) = 0 \rightarrow C_1 + C_2 = 0$$

$$y'(x) = (1-\sqrt{3})C_1 e^{(1-\sqrt{3})x} + (1+\sqrt{3})C_2 e^{(1+\sqrt{3})x}$$

$$y'(0) = 3 \rightarrow (1-\sqrt{3})C_1 + (1+\sqrt{3})C_2 = 3$$

$$\begin{cases} C_1 + C_2 = 0 \rightarrow C_1 = -C_2 \\ (1-\sqrt{3})C_1 + (1+\sqrt{3})C_2 = 3 \end{cases} \rightarrow C_1(1-\sqrt{3}-1-\sqrt{3}) = 3 \quad C_1 = -\frac{\sqrt{3}}{2}, C_2 = \frac{\sqrt{3}}{2}$$

$$y(x) = -\frac{\sqrt{3}}{2} e^{(1-\sqrt{3})x} + \frac{\sqrt{3}}{2} e^{(1+\sqrt{3})x}$$

Exercise

Find the solution of the given initial value problem $y'' - 2y' + 2y = 0$; $y(0) = 1$, $y(\pi) = 1$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 2 = 0 \rightarrow \lambda_{1,2} = \frac{2 \pm 2i}{2} = 1 \pm i$

$$y(x) = e^x (C_1 \cos x + C_2 \sin x)$$

$$y(0) = 1 \rightarrow \underline{C_1 = 1}$$

$$y(\pi) = 1 \rightarrow -e^\pi C_1 = 1 \quad \underline{C_1 = -e^{-\pi}}$$

$$C_1 = -e^{-\pi} \neq 1$$

There is No solution the *ODE* under the given conditions.

Exercise

Find the solution of the given initial value problem. $y'' - 2y' - 3y = 0$; $y(0) = 2$, $y'(0) = -3$

Solution

The characteristic equation: $\lambda^2 - 2\lambda - 3 = 0 \Rightarrow \lambda_1 = -1$; $\lambda_2 = 3$

$$y(t) = C_1 e^{-t} + C_2 e^{3t}$$

$$y(0) = C_1 + C_2 = 2$$

$$y'(t) = -C_1 e^{-t} + 3C_2 e^{3t}$$

$$y'(0) = -C_1 + 3C_2 = -3$$

$$\begin{cases} C_1 + C_2 = 2 \\ -C_1 + 3C_2 = -3 \end{cases} \rightarrow C_1 = \frac{9}{4} \quad C_2 = -\frac{1}{4}$$

$$\underline{y(t) = \frac{9}{4} e^{-t} - \frac{1}{4} e^{3t}}$$

Exercise

Find the solution of the given initial value problem $y'' + 2y' - 8y = 0$; $y(0) = 3$, $y'(0) = -12$

Solution

The characteristic equation: $\lambda^2 + 2\lambda - 8 = 0 \rightarrow \underline{\lambda_{1,2} = -4, 2}$

$$\underline{y(x) = C_1 e^{-4x} + C_2 e^{2x}}$$

$$y(0) = 3 \rightarrow C_1 + C_2 = 3$$

$$y'(x) = -4C_1 e^{-4x} + 2C_2 e^{2x}$$

$$y'(0) = -12 \rightarrow -4C_1 + 2C_2 = -12$$

$$\begin{cases} C_1 + C_2 = 3 \\ -4C_1 + 2C_2 = -12 \end{cases} \rightarrow \underline{C_1 = 3, C_2 = 0}$$

$$\underline{y(x) = 3e^{-4x}}$$

Exercise

Find the general solution: $y'' - 2y' + 17y = 0$; $y(0) = -2, y'(0) = 3$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 17 = 0$

$$\Rightarrow \lambda_{1,2} = 1 \pm 4i$$

$$y(t) = e^t (C_1 \cos 4t + C_2 \sin 4t)$$

$$y(t) = e^t (C_1 \cos 4t + C_2 \sin 4t) \rightarrow y(0) = \boxed{C_1 = -2}$$

$$y'(t) = e^t (C_1 \cos 4t + C_2 \sin 4t) + e^t (-4C_1 \sin 4t + 4C_2 \cos 4t)$$

$$\Rightarrow y'(0) = C_1 + 4C_2 = 3$$

$$\Rightarrow \boxed{C_2 = \frac{5}{4}}$$

$$\underline{y(t) = e^t \left(-2 \cos 4t + \frac{5}{4} \sin 4t \right)}$$

Exercise

Find the general solution: $y'' + 2\sqrt{2}y' + 2y = 0$; $y(0) = 1, y'(0) = 0$

Solution

The characteristic equation: $\lambda^2 + 2\sqrt{2}\lambda + 2 = (\lambda + \sqrt{2})^2 = 0 \Rightarrow \underline{\lambda_{1,2} = \pm\sqrt{2}i}$

$$\underline{y(t) = C_1 \cos \sqrt{2}t + C_2 \sin \sqrt{2}t}$$

$$y(0) = 1 \rightarrow \underline{C_1 = 1}$$

$$y'(t) = -\sqrt{2}C_1 \sin \sqrt{2}t + \sqrt{2}C_2 \cos \sqrt{2}t$$

$$y'(0) = 0 \rightarrow \sqrt{2}C_2 = 0 \Rightarrow \underline{C_2 = 0}$$

$$\underline{y(t) = \cos \sqrt{2}t}$$

Exercise

Find the general solution: $y'' + 3y' - 10y = 0$; $y(0) = 4$, $y'(0) = -2$

Solution

The characteristic equation: $\lambda^2 + 3\lambda + 10 = 0 \rightarrow \lambda_{1,2} = -5, 2$

$$y(x) = C_1 e^{-5x} + C_2 e^{2x}$$

$$y(0) = 4 \rightarrow C_1 + C_2 = 4$$

$$y' = -5C_1 e^{-5x} + 2C_2 e^{2x}$$

$$y'(0) = -2 \rightarrow -5C_1 + 2C_2 = -2$$

$$\begin{cases} C_1 + C_2 = 4 \\ -5C_1 + 2C_2 = -2 \end{cases} \rightarrow C_1 = \frac{10}{7}, C_2 = \frac{18}{7}$$

$$y(x) = \frac{10}{7} e^{-5x} + \frac{18}{7} e^{2x}$$

Exercise

Find the solution of the given initial value problem $y'' + 4y = 0$; $y(0) = 0$, $y(\pi) = 0$

Solution

The characteristic equation: $\lambda^2 + 4 = 0 \rightarrow \lambda_{1,2} = \pm 2i$

$$y(x) = C_1 \cos 2x + C_2 \sin 2x$$

$$y(0) = 0 \rightarrow C_1 = 0$$

$$y(\pi) = 0 \rightarrow C_2 = 0$$

$$y(x) = C_2 \sin 2x$$

Exercise

Find the solution of the given initial value problem $y'' + 4y = 0$; $y\left(\frac{\pi}{4}\right) = -2$, $y'\left(\frac{\pi}{4}\right) = 1$

Solution

The characteristic equation: $\lambda^2 + 4 = 0 \rightarrow \lambda_{1,2} = \pm 2i$

$$y(x) = C_1 \cos 2x + C_2 \sin 2x$$

$$y\left(\frac{\pi}{4}\right) = -2 \rightarrow \underline{C_2 = -2}$$

$$y' = -2C_1 \sin 2x + 2C_2 \cos 2x$$

$$y'\left(\frac{\pi}{4}\right) = 1 \rightarrow -2C_1 = 1 \Rightarrow \underline{C_1 = -\frac{1}{2}}$$

$$\underline{y(x) = -2\cos 2x - \frac{1}{2}\sin 2x}$$

Exercise

Find the solution of the given initial value problem $y'' + 4y' + 2y = 0$; $y(0) = -1$, $y'(0) = 2$

Solution

$$\text{The characteristic equation: } \lambda^2 + 4\lambda + 2 = 0 \rightarrow \lambda_{1,2} = \frac{-4 \pm 2\sqrt{2}}{2} = \underline{-2 \pm \sqrt{2}}$$

$$y(x) = C_1 e^{-(2+\sqrt{2})x} + C_2 e^{(-2+\sqrt{2})x}$$

$$y(0) = -1 \rightarrow \underline{C_1 + C_2 = -1}$$

$$y'(x) = -(2+\sqrt{2})C_1 e^{-(2+\sqrt{2})x} + (-2+\sqrt{2})C_2 e^{(-2+\sqrt{2})x}$$

$$y'(0) = 2 \rightarrow \underline{(-2-\sqrt{2})C_1 + (-2+\sqrt{2})C_2 = 2}$$

$$\begin{cases} C_1 + C_2 = -1 \\ (-2-\sqrt{2})C_1 + (-2+\sqrt{2})C_2 = 2 \end{cases}$$

$$\rightarrow \Delta = \begin{vmatrix} 1 & 1 \\ -2-\sqrt{2} & -2+\sqrt{2} \end{vmatrix} = 2\sqrt{2} \quad \Delta_{C_c} = \begin{vmatrix} -1 & 1 \\ 2 & -2+\sqrt{2} \end{vmatrix} = -\sqrt{2}$$

$$\underline{C_1 = -\frac{1}{2}, \quad C_2 = -\frac{1}{2}}$$

$$\underline{y(x) = -\frac{1}{2}e^{-(2+\sqrt{2})x} - \frac{1}{2}e^{(-2+\sqrt{2})x}}$$

Exercise

Find the solution of the given initial value problem $y'' - 4y' + 3y = 0$, $y(0) = 1$, $y'(0) = \frac{1}{3}$

Solution

$$\text{The characteristic equation: } \lambda^2 - 4\lambda + 3 = 0 \rightarrow \underline{\lambda_{1,2} = 1, 3}$$

$$\underline{y(x) = C_1 e^x + C_2 e^{3x}}$$

$$y(0)=1 \rightarrow C_1 + C_2 = 1$$

$$y'(x) = C_1 e^x + 3C_2 e^{3x}$$

$$y'(0) = \frac{1}{3} \rightarrow C_1 + 3C_2 = \frac{1}{3}$$

$$\begin{cases} C_1 + C_2 = 1 \\ C_1 + 3C_2 = \frac{1}{3} \end{cases} \rightarrow C_2 = -\frac{1}{3} \quad C_1 = \frac{4}{3}$$

$$\underline{y(x) = \frac{4}{3}e^x - \frac{1}{3}e^{3x}}$$

Exercise

Find the solution of the given initial value problem $y'' - 4y' + 4y = 0$; $y(1)=1$, $y'(1)=1$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0 \rightarrow \underline{\lambda_{1,2} = 2}$

$$\underline{y(x) = (C_1 + C_2 x)e^{2x}}$$

$$y(1)=1 \rightarrow (C_1 + C_2)e^2 = 1 \Rightarrow C_1 + C_2 = e^{-2}$$

$$y'(x) = (C_2 + 2C_1 + 2C_2 x)e^{2x}$$

$$y'(1)=1 \rightarrow (2C_1 + 3C_2)e^2 = 1 \Rightarrow 2C_1 + 3C_2 = e^{-2}$$

$$\begin{cases} C_1 + C_2 = e^{-2} \\ 2C_1 + 3C_2 = e^{-2} \end{cases} \rightarrow \Delta = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1 \quad \Delta_{C_1} = \begin{vmatrix} e^{-2} & 1 \\ e^{-2} & 3 \end{vmatrix} = 2e^{-2}$$

$$\Rightarrow \underline{C_1 = 2e^{-2}} \quad \underline{C_2 = -e^{-2}}$$

$$y(x) = (2e^{-2} - e^{-2}x)e^{-x}$$

$$\underline{= 2e^{-x-2} - xe^{-x-2}}$$

Exercise

Find the solution of the given initial value problem $y'' + 4y' + 4y = 0$, $y(0)=1$, $y'(0)=3$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0 \rightarrow \underline{\lambda_{1,2} = -2}$

$$\underline{y(x) = (C_1 + C_2 x)e^{-2x}}$$

$$y(0) = 1 \rightarrow \underline{C_1 = 1}$$

$$y'(x) = (C_2 - 2C_1 - 2C_2 x)e^{-2x}$$

$$y'(0) = 3 \rightarrow C_2 - 2 = 3 \Rightarrow \underline{C_2 = 5}$$

$$\underline{y(x) = (1 + 5x)e^{-2x}}$$

Exercise

Find the solution of the given initial value problem: $y'' - 4y' + 5y = 0$; $y(0) = 1$, $y'(0) = 5$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 5 = 0 \Rightarrow \lambda_{1,2} = 2 \pm i$

$$y(x) = e^{2x}(C_1 \cos x + C_2 \sin x)$$

$$y(x) = e^{2x}(C_1 \cos x + C_2 \sin x) \Rightarrow y(0) = \boxed{C_1 = 1}$$

$$y'(x) = 2e^{2x}(C_1 \cos x + C_2 \sin x) + e^{2x}(-C_1 \sin x + C_2 \cos x)$$

$$\Rightarrow y'(0) = 2C_1 + C_2 = 5 \Rightarrow \boxed{C_2 = 3}$$

$$\underline{y(x) = e^{2x}(\cos x + 3\sin x)}$$

Exercise

Find the solution of the given initial value problem $y'' + 4y' + 5y = 0$; $y(0) = 1$, $y'(0) = 0$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 5 = 0 \rightarrow \lambda_{1,2} = \frac{-4 \pm 2i}{2} = -2 \pm i$

$$\underline{y(x) = e^{-2x}(C_1 \cos x + C_2 \sin x)}$$

$$y(0) = 1 \rightarrow \underline{C_1 = 1}$$

$$y'(x) = e^{-2x}(-2C_1 \cos x - 2C_2 \sin x - C_1 \sin x + C_2 \cos x)$$

$$y'(0) = 0 \rightarrow -2C_1 + C_2 = 0 \Rightarrow \underline{C_2 = 2}$$

$$\underline{y(x) = e^{-2x}(\cos x + 2\sin x)}$$

Exercise

Find the solution of the given initial value problem $y'' + 4y' + 5y = 0$; $y\left(\frac{\pi}{2}\right) = \frac{1}{2}$, $y'\left(\frac{\pi}{2}\right) = -2$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 5 = 0 \rightarrow \lambda_{1,2} = \frac{-4 \pm 2i}{2} = -2 \pm i$

$$y(x) = e^{-2x} (C_1 \cos x + C_2 \sin x)$$

$$y\left(\frac{\pi}{2}\right) = \frac{1}{2} \rightarrow e^{-\pi} C_2 = \frac{1}{2} \Rightarrow C_2 = \frac{e^\pi}{2}$$

$$y'(x) = e^{-2x} (-2C_1 \cos x - 2C_2 \sin x - C_1 \sin x + C_2 \cos x)$$

$$y'\left(\frac{\pi}{2}\right) = -2 \rightarrow e^{-\pi} \left(-2\frac{e^\pi}{2} - C_1\right) = -2$$

$$\Rightarrow 1 + e^{-\pi} C_1 = 2 \rightarrow C_1 = e^\pi$$

$$y(x) = e^{-2x} \left(e^\pi \cos x + \frac{1}{2} e^\pi \sin x\right) \\ = e^{\pi-2x} \left(\cos x + \frac{1}{2} \sin x\right)$$

Exercise

Find the solution of the given initial value problem $y'' - 4y' - 5y = 0$; $y(1) = 0$, $y'(1) = 2$

Solution

The characteristic equation: $\lambda^2 - 4\lambda - 5 = 0 \rightarrow \lambda_{1,2} = -1, 5$

$$y(x) = C_1 e^{-x} + C_2 e^{5x}$$

$$y(1) = 0 \rightarrow C_1 e^{-1} + C_2 e^5 = 0$$

$$y'(x) = -C_1 e^{-x} + 5C_2 e^{5x}$$

$$y'(1) = 2 \rightarrow e^{-1} C_1 + 5e^5 C_2 = 2$$

$$\begin{cases} e^{-1} C_1 + e^5 C_2 = 0 \\ -e^{-1} C_1 + 5e^5 C_2 = 2 \end{cases} \rightarrow \Delta = \begin{vmatrix} e^{-1} & e^5 \\ -e^{-1} & 5e^5 \end{vmatrix} = 6e^4 \quad \Delta_{C_1} = \begin{vmatrix} 0 & e^5 \\ 2 & 5e^5 \end{vmatrix} = -2e^5$$

$$\Rightarrow C_1 = -\frac{1}{3}e, \quad C_2 = \frac{1}{3}e^{-5}$$

$$y(x) = -\frac{1}{3}e^{1-x} + \frac{1}{3}e^{-5x-5}$$

Exercise

Find the solution of the given initial value problem $y'' - 4y' - 5y = 0$, $y(-1) = 3$, $y'(-1) = 9$

Solution

The characteristic equation: $\lambda^2 - 4\lambda - 5 = 0 \rightarrow \lambda_{1,2} = -1, 5$

$$y(x) = C_1 e^{-x} + C_2 e^{5x}$$

$$y(-1) = 3 \rightarrow C_1 e + C_2 e^{-5} = 3$$

$$y'(x) = -C_1 e^{-x} + 5C_2 e^{5x}$$

$$y'(-1) = 9 \rightarrow -C_1 e + 5C_2 e^{-5} = 9$$

$$\begin{cases} eC_1 + e^{-5}C_2 = 3 \\ -eC_1 + 5e^{-5}C_2 = 9 \end{cases} \rightarrow \Delta = \begin{vmatrix} e & e^{-5} \\ -e & 5e^{-5} \end{vmatrix} = 6e^{-4} \quad \Delta_{C_1} = \begin{vmatrix} 3 & e^{-5} \\ 9 & 5e^{-5} \end{vmatrix} = 6e^{-5}$$

$$\Rightarrow C_1 = e^{-1}, C_2 = 2e^5$$

$$y(x) = e^{-1-x} + 2e^{5x+5}$$

Exercise

Find the solution of the given initial value problem $y'' - 4y' + 9y = 0$, $y(0) = 0$, $y'(0) = -8$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 9 = 0 \rightarrow \lambda_{1,2} = \frac{4 \pm 2i\sqrt{5}}{2} = 2 \pm i\sqrt{5}$

$$y(t) = e^{2t} (C_1 \cos \sqrt{5}t + C_2 \sin \sqrt{5}t)$$

$$y(0) = 0 \rightarrow C_1 = 0$$

$$y' = e^{2t} (2C_1 \cos \sqrt{5}t + 2C_2 \sin \sqrt{5}t - \sqrt{5}C_1 \sin \sqrt{5}t + \sqrt{5}C_2 \cos \sqrt{5}t)$$

$$y'(0) = -8 \rightarrow -\sqrt{5}C_2 = -8 \Rightarrow C_2 = \frac{8}{\sqrt{5}}$$

$$y(t) = \frac{8}{\sqrt{5}} e^{2t} \sin \sqrt{5}t$$

Exercise

Find the solution of the given initial value problem. $y'' - 4y' + 13y = 0$; $y(0) = -1$, $y'(0) = 2$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 13 = 0$

$$\Rightarrow \lambda_{1,2} = \frac{4 \pm \sqrt{16 - 52}}{2} = 2 \pm 3i$$

$$\underline{y(x) = e^{2x} (C_1 \cos 3x + C_2 \sin 3x)}$$

$$y(0) = e^0 (C_1 \cos(0) + C_2 \sin(0)) \rightarrow \underline{C_1 = -1}$$

$$y'(x) = 2e^{2x} (C_1 \cos 3x + C_2 \sin 3x) + e^{2x} (-3C_1 \sin 3x + 3C_2 \cos 3x)$$

$$y'(0) = 2C_1 + 3C_2 = 2 \Rightarrow \underline{C_2 = \frac{2 - 2(-1)}{3} = \frac{4}{3}}$$

$$\underline{y(x) = e^{2x} \left(-\cos 3x + \frac{4}{3} \sin 3x \right)}$$

Exercise

Find the solution of the given initial value problem $y'' - 5y' + 6y = 0$; $y(1) = e^2$, $y'(1) = 3e^2$

Solution

The characteristic equation: $\lambda^2 - 5\lambda + 6 = 0 \rightarrow \underline{\lambda_{1,2} = 2, 3}$

$$y(x) = C_1 e^{2x} + C_2 e^{3x}$$

$$\underline{y(1) = e^2} \rightarrow C_1 e^2 + C_2 e^3 = e^2 \Rightarrow \underline{C_1 + eC_2 = 1}$$

$$y'(x) = 2C_1 e^{2x} + 3C_2 e^{3x}$$

$$\underline{y'(1) = 3e^2} \rightarrow 2C_1 e^2 + 3C_2 e^3 = 3e^2 \Rightarrow \underline{2C_1 + 3eC_2 = 3}$$

$$\underline{-2 \times} \begin{cases} C_1 + eC_2 = 1 \\ 2C_1 + 3eC_2 = 3 \end{cases} \rightarrow eC_2 = 1 \quad \underline{C_2 = e^{-1}; C_1 = 0}$$

$$\underline{y(x) = e^{3x-1}}$$

Exercise

Find the solution of the given initial value problem $y'' + 6y' + 5y = 0$, $y(1) = 0$, $y'(0) = 3$

Solution

The characteristic equation: $\lambda^2 + 6\lambda + 5 = 0 \rightarrow \underline{\lambda_{1,2} = -1, -5}$

$$y(x) = C_1 e^{-x} + C_2 e^{-5x}$$

$$y(1)=0 \rightarrow C_1 e^{-1} + C_2 e^{-5} = 0 \Rightarrow C_1 e^4 + C_2 = e^5$$

$$y'(x) = -C_1 e^{-x} - 5C_2 e^{-5x}$$

$$y'(0)=3 \rightarrow -C_1 - 5C_2 = 3$$

$$\begin{cases} e^4 C_1 + C_2 = e^5 \\ C_1 + 5C_2 = -3 \end{cases}$$

$$\rightarrow \Delta = \begin{vmatrix} e^4 & 1 \\ 1 & 5 \end{vmatrix} = 5e^4 - 1 \quad \Delta_{C_1} = \begin{vmatrix} e^5 & 1 \\ -3 & 5 \end{vmatrix} = 5e^5 + 3 \quad \Delta_{C_2} = \begin{vmatrix} e^4 & e^5 \\ 1 & -3 \end{vmatrix} = -3e^4 - e^5$$

$$C_1 = \frac{5e^5 + 3}{5e^4 - 1}; \quad C_2 = -\frac{3e^4 + e^5}{5e^4 - 1}$$

$$y(x) = \frac{5e^5 + 3}{5e^4 - 1} e^{-x} - \frac{3e^4 + e^5}{5e^4 - 1} e^{-5x}$$

Exercise

Find the solution of the given initial value problem $y'' - 6y' + 5y = 0; \quad y(0) = 3, \quad y'(0) = 11$

Solution

The characteristic equation: $\lambda^2 - 6\lambda + 5 = 0 \rightarrow \lambda_{1,2} = 1, 5$

$$y(x) = C_1 e^x + C_2 e^{5x}$$

$$y(0)=3 \rightarrow C_1 + C_2 = 3$$

$$y'(x) = C_1 e^x + 5C_2 e^{5x}$$

$$y'(0)=11 \rightarrow C_1 + 5C_2 = 11$$

$$\begin{cases} C_1 + C_2 = 3 \\ C_1 + 5C_2 = 11 \end{cases} \rightarrow C_2 = 2; C_1 = 1$$

$$y(x) = e^x + 2e^{5x}$$

Exercise

Find the solution of the given initial value problem $y'' - 6y' + 9y = 0, \quad y(0) = 2, \quad y'(0) = \frac{25}{3}$

Solution

The characteristic equation: $\lambda^2 - 6\lambda + 9 = 0 \rightarrow \lambda_{1,2} = 3$

$$\underline{y(x) = (C_1 + C_2 x)e^{3x}}$$

$$y(0) = 2 \rightarrow \underline{C_1 = 2}$$

$$y'(x) = (C_2 + 3C_1 + 3C_2 x)e^{3x}$$

$$y'(0) = \frac{25}{3} \rightarrow C_2 + 3C_1 = \frac{25}{3} \quad \underline{C_2 = \frac{7}{3}}$$

$$\underline{y(x) = \left(2 + \frac{7}{3}x\right)e^{3x}}$$

Exercise

Find the solution of the given initial value problem $y'' - 6y' + 9y = 0$; $y(0) = 0$, $y'(0) = 5$

Solution

$$\text{The characteristic equation: } \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0 \rightarrow \underline{\lambda_{1,2} = 3}$$

$$y(x) = (C_1 + C_2 x)e^{3x}$$

$$y(0) = 0 \rightarrow \underline{C_1 = 0}$$

$$y'(x) = (C_2 + 3C_1 + 3C_2 x)e^{3x}$$

$$y'(0) = 5 \rightarrow 3C_1 + C_2 = 5 \Rightarrow \underline{C_2 = 5}$$

$$\underline{y(x) = 5xe^{3x}}$$

Exercise

Find the solution of the given initial value problem $y'' + 6y' + 9y = 0$; $y(0) = 2$, $y'(0) = -2$

Solution

$$\text{The characteristic equation: } \lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0 \rightarrow \underline{\lambda_{1,2} = -3}$$

$$y(x) = (C_1 + C_2 x)e^{-3x}$$

$$y(0) = 2 \rightarrow \underline{C_1 = 2}$$

$$y'(x) = (C_2 - 3C_1 - 3C_2 x)e^{-3x}$$

$$y'(0) = -2 \rightarrow C_2 - 6 = -2 \Rightarrow \underline{C_2 = 4}$$

$$\underline{y(x) = (2 + 4x)e^{-3x}}$$

Exercise

Find the solution of the given initial value problem $y'' + 8y' - 9y = 0$; $y(1) = 2$, $y'(1) = 0$

Solution

The characteristic equation: $\lambda^2 + 8\lambda - 9 = 0 \rightarrow \lambda_{1,2} = 1, -9$

$$y(x) = C_1 e^{-9x} + C_2 e^x$$

$$y(1) = 2 \rightarrow C_1 e^{-9} + C_2 e = 2 \Rightarrow C_1 + e^{10} C_2 = 2e^9$$

$$y'(x) = -9C_1 e^{-9x} + C_2 e^x$$

$$y'(1) = 0 \rightarrow -9C_1 e^{-9} + C_2 e = 0 \Rightarrow C_2 = 9e^{-10} C_1$$

$$C_1 + e^{10} (9e^{-10} C_1) = 2e^9 \Rightarrow C_1 = \frac{e^9}{5} \quad C_2 = \frac{9}{5e}$$

$$y(x) = \frac{1}{5} e^{9-9x} + \frac{9}{5} e^{x-1}$$

Exercise

Find the solution of the given initial value problem. $y'' - 8y' + 17y = 0$; $y(0) = 4$, $y'(0) = -1$

Solution

The characteristic equation: $\lambda^2 - 8\lambda + 17 = 0$

$$\Rightarrow \lambda_{1,2} = \frac{8 \pm \sqrt{64 - 68}}{2} = \frac{8 \pm i2}{2} = 4 \pm i$$

$$y(x) = e^{4x} (C_1 \cos x + C_2 \sin x)$$

$$y(0) = e^0 (C_1 \cos(0) + C_2 \sin(0)) \rightarrow 4 = C_1$$

$$y'(x) = 4e^{4x} (C_1 \cos x + C_2 \sin x) + e^{4x} (-C_1 \sin x + C_2 \cos x)$$

$$y'(0) = 4C_1 + C_2 = -1 \Rightarrow C_2 = -1 - 16 = -17$$

$$y(x) = e^{4x} (4 \cos x - 17 \sin x)$$

Exercise

Find the solution of the given initial value problem $y'' - 9y = 0$, $y(0) = 2$, $y'(0) = -1$

Solution

The characteristic equation: $\lambda^2 - 9 = 0 \rightarrow \lambda_{1,2} = \pm 3$

$$y(x) = C_1 e^{-3x} + C_2 e^{3x}$$

$$y(0) = 2 \rightarrow C_1 + C_2 = 2$$

$$y'(x) = -3C_1 e^{-3x} + 3C_2 e^{3x}$$

$$y'(0) = -1 \rightarrow -3C_1 + 3C_2 = -1$$

$$\begin{cases} C_1 + C_2 = 2 \\ -3C_1 + 3C_2 = -1 \end{cases} \rightarrow C_1 = \frac{7}{6}, C_2 = \frac{5}{6}$$

$$y(x) = \frac{7}{6} e^{-3x} + \frac{5}{6} e^{3x}$$

Exercise

Find the solution of the given initial value problem $y'' - 10y' + 25y = 0$, $y(0) = 1$, $y'(1) = 0$

Solution

The characteristic equation: $\lambda^2 - 10\lambda + 25 = (\lambda - 5)^2 = 0 \rightarrow \lambda_{1,2} = 5$

$$y(x) = (C_1 + C_2 x) e^{5x}$$

$$y(0) = 1 \rightarrow C_1 = 1$$

$$y'(x) = (C_2 + 5C_1 + 5C_2 x) e^{5x}$$

$$y'(1) = 0 \rightarrow (6C_2 + 5C_1) e^5 = 0 \Rightarrow 6C_2 + 5C_1 = 0 \Rightarrow C_2 = -\frac{5}{6}$$

$$y(x) = \left(1 - \frac{5}{6}x\right) e^{5x}$$

Exercise

Find the general solution: $y'' + 10y' + 25y = 0$; $y(0) = 2$, $y'(0) = -1$

Solution

The characteristic equation: $\lambda^2 + 10\lambda + 25 = 0$

$$\Rightarrow \lambda_{1,2} = -5$$

$$y(t) = (C_1 + C_2 t) e^{-5t}$$

$$y(t) = (C_1 + C_2 t)e^{-5t} \quad y(0) = \boxed{C_1 = 2}$$

$$y' = C_2 e^{-5t} - 5(C_1 + C_2 t)e^{-5t} \quad y'(0) = C_2 - 5C_1 = -1 \Rightarrow \boxed{C_2 = 9}$$

$$\underline{y(t) = (2 + 9t)e^{-5t}}$$

Exercise

Find the general solution: $y'' + 11y' + 24y = 0$; $y(0) = 0$, $y'(0) = -7$

Solution

The characteristic equation: $\lambda^2 + 11\lambda + 24 = 0 \rightarrow \lambda_{1,2} = \frac{-11 \pm 5}{2} \quad \lambda_{1,2} = -8, -3$

$$y(x) = C_1 e^{-8x} + C_2 e^{-3x}$$

$$y(0) = 0 \rightarrow \Rightarrow \boxed{C_1 + C_2 = 0}$$

$$y' = -8C_1 e^{-8x} - 3C_2 e^{-3x}$$

$$y'(0) = -7 \rightarrow \boxed{-8C_1 - 3C_2 = -7}$$

$$\begin{cases} C_1 + C_2 = 0 \\ -8C_1 - 3C_2 = -7 \end{cases} \rightarrow \boxed{C_1 = \frac{7}{5}, C_2 = -\frac{7}{5}}$$

$$\underline{y(x) = \frac{7}{5}e^{-8x} - \frac{7}{5}e^{-3x}}$$

Exercise

Find the solution of the given initial value problem $y'' + 12y = 0$, $y(0) = 0$, $y'(0) = 1$

Solution

The characteristic equation: $\lambda^2 + 12 = 0 \rightarrow \lambda_{1,2} = \pm 2i\sqrt{3}$

$$y(x) = C_1 \cos 2\sqrt{3}x + C_2 \sin 2\sqrt{3}x$$

$$y(0) = 0 \rightarrow \boxed{C_1 = 0}$$

$$y'(x) = -2\sqrt{3}C_1 \sin 2\sqrt{3}x + 2\sqrt{3}C_2 \cos 2\sqrt{3}x$$

$$y'(0) = 1 \rightarrow 2\sqrt{3}C_2 = 1 \Rightarrow \boxed{C_2 = \frac{\sqrt{3}}{3}}$$

$$\underline{y(x) = \frac{\sqrt{3}}{3} \sin 2\sqrt{3}x}$$

Exercise

Find the solution of the given initial value problem $y'' + 16y = 0$, $y(\pi) = 2$, $y'(0) = -2$

Solution

The characteristic equation: $\lambda^2 + 16 = 0 \rightarrow \lambda_{1,2} = \pm 4i$

$$y(x) = C_1 \cos 4x + C_2 \sin 4x$$

$$y(0) = 2 \rightarrow C_1 = 2$$

$$y'(x) = -4C_1 \sin 4x + 4C_2 \cos 4x$$

$$y'(0) = -2 \rightarrow 4C_2 = -2 \Rightarrow C_2 = -\frac{1}{2}$$

$$y(x) = 2\cos 4x - \frac{1}{2}\sin 4x$$

Exercise

Find the solution of the given initial value problem $y'' + 16y = 0$, $y\left(\frac{\pi}{2}\right) = -10$, $y'\left(\frac{\pi}{2}\right) = 3$

Solution

The characteristic equation: $\lambda^2 + 16 = 0 \rightarrow \lambda_{1,2} = \pm 4i$

$$y(x) = C_1 \cos 4x + C_2 \sin 4x$$

$$y\left(\frac{\pi}{2}\right) = -10 \rightarrow C_1 = -10$$

$$y' = -4C_1 \sin 4x + 4C_2 \cos 4x$$

$$y'\left(\frac{\pi}{2}\right) = 3 \rightarrow 4C_2 = 3 \Rightarrow C_2 = \frac{3}{4}$$

$$y(x) = -10\cos 4x + \frac{3}{4}\sin 4x$$

Exercise

Find the solution of the given initial value problem $y'' + 16y = 0$, $y(\pi) = 2$, $y'(0) = -2$

Solution

The characteristic equation: $\lambda^2 + 16 = 0 \rightarrow \lambda_{1,2} = \pm 4i$

$$y(x) = C_1 \cos 4x + C_2 \sin 4x$$

$$y(\pi) = 2 \rightarrow C_1 = 2$$

$$y'(x) = -4C_1 \sin 4x + 4C_2 \cos 4x$$

$$y'(0) = -2 \rightarrow 4C_2 = -2 \Rightarrow C_2 = -\frac{1}{2}$$

$$y(x) = 2\cos 4x - \frac{1}{2}\sin 4x$$

Exercise

Find the general solution: $y'' + 25y = 0$; $y(0) = 1$, $y'(0) = -1$

Solution

The characteristic equation: $\lambda^2 + 25 = 0$

$$\Rightarrow \lambda_{1,2} = \pm 5i$$

$$y(t) = e^{0t} (C_1 \cos 5t + C_2 \sin 5t)$$

$$y(t) = C_1 \cos 5t + C_2 \sin 5t$$

$$y(t) = C_1 \cos 5t + C_2 \sin 5t \rightarrow y(0) = C_1 = 1$$

$$y'(t) = -5C_1 \sin 5t + 5C_2 \cos 5t$$

$$\Rightarrow y'(0) = 5C_2 = -1 \rightarrow C_2 = -\frac{1}{5}$$

$$y(t) = 5\cos 5t - \frac{1}{5}\sin 5t$$

Exercise

Find the solution of the given initial value problem $2y'' - 2y' + y = 0$; $y(-\pi) = 1$, $y'(-\pi) = -1$

Solution

The characteristic equation: $2\lambda^2 - 2\lambda + 1 = 0 \rightarrow \lambda_{1,2} = \frac{2 \pm 2i}{4} = \frac{1}{2} \pm \frac{1}{2}i$

$$y(x) = e^{x/2} \left(C_1 \cos \frac{x}{2} + C_2 \sin \frac{x}{2} \right)$$

$$y(-\pi) = 1 \rightarrow e^{-\pi/2} (-C_2) = 1 \Rightarrow C_2 = -e^{\pi/2}$$

$$y'(x) = e^{x/2} \left(\frac{1}{2} C_1 \cos \frac{x}{2} + \frac{1}{2} C_2 \sin \frac{x}{2} - \frac{1}{2} C_1 \sin \frac{x}{2} + \frac{1}{2} C_2 \cos \frac{x}{2} \right)$$

$$y'(-\pi) = -1 \rightarrow e^{-\pi/2} \left(\frac{1}{2} e^{\pi/2} + \frac{1}{2} C_1 \right) = -1$$

$$\frac{1}{2} + \frac{1}{2} e^{-\pi/2} C_1 = -1 \rightarrow C_1 = -3e^{\pi/2}$$

$$y(x) = e^{x/2} \left(-3e^{\pi/2} \cos \frac{x}{2} - e^{\pi/2} \sin \frac{x}{2} \right) \\ = -e^{(x+\pi)/2} \left(3 \cos \frac{x}{2} + \sin \frac{x}{2} \right)$$

Exercise

Find the solution of the given initial value problem $3y'' + y' - 14y = 0$, $y(0) = 2$, $y'(0) = -1$

Solution

The characteristic equation: $3\lambda^2 + \lambda - 14 = 0 \rightarrow \lambda_{1,2} = \frac{-1 \pm 13}{6} = -\frac{7}{3}, 2$

$$y(x) = C_1 e^{-7x/3} + C_2 e^{2x}$$

$$y(0) = 2 \rightarrow C_1 + C_2 = 2$$

$$y'(x) = -\frac{7}{3}C_1 e^{-7x/3} + 2C_2 e^{2x}$$

$$y'(0) = -1 \rightarrow -\frac{7}{3}C_1 + 2C_2 = -1$$

$$\begin{cases} C_1 + C_2 = 2 \\ -7C_1 + 6C_2 = -3 \end{cases} \rightarrow C_2 = \frac{11}{13}, C_1 = \frac{15}{13}$$

$$y(x) = \frac{11}{13} e^{-7x/3} + \frac{15}{13} e^{2x}$$

Exercise

Find the solution of the given initial value problem $3y'' + 2y' - 8y = 0$, $y(0) = -6$, $y'(0) = -18$

Solution

The characteristic equation: $3\lambda^2 + 2\lambda - 8 = 0 \rightarrow \lambda_{1,2} = \frac{-2 \pm 10}{6} = -2, \frac{4}{3}$

$$y(x) = C_1 e^{-2x} + C_2 e^{4x/3}$$

$$y(0) = -6 \rightarrow C_1 + C_2 = -6$$

$$y' = -2C_1 e^{-2x} + \frac{4}{3}C_2 e^{4x/3}$$

$$y'(0) = -18 \rightarrow -2C_1 + \frac{4}{3}C_2 = -18$$

$$\begin{cases} C_1 + C_2 = -6 \\ -3C_1 + 2C_2 = -27 \end{cases} \rightarrow C_1 = \frac{15}{5} = 3, C_2 = -9$$

$$y(x) = 3e^{-2x} - 9e^{4x/3}$$

Exercise

Find the solution of the given initial value problem $4y'' - 4y' + y = 0, \quad y(0) = 4, \quad y'(0) = 4$

Solution

The characteristic equation: $4\lambda^2 - 4\lambda + 1 = (2\lambda - 1)^2 = 0 \rightarrow \lambda_{1,2} = \frac{1}{2}$

$$y(x) = (C_1 + C_2 x)e^{x/2}$$

$$y(0) = 4 \rightarrow C_1 = 4$$

$$y'(x) = \left(C_2 + \frac{1}{2}C_1 + \frac{1}{2}C_2 x\right)e^{x/2}$$

$$y'(0) = 4 \rightarrow C_2 + \frac{1}{2}C_1 = 4 \Rightarrow C_2 = 2$$

$$y(x) = 2(1+x)e^{x/2}$$

Exercise

Find the solution of the given initial value problem $4y'' - 4y' + y = 0; \quad y(1) = -4, \quad y'(1) = 0$

Solution

The characteristic equation: $4\lambda^2 - 4\lambda + 1 = (2\lambda - 1)^2 = 0 \rightarrow \lambda_{1,2} = \frac{1}{2}$

$$y(x) = (C_1 + C_2 x)e^{x/2}$$

$$y(1) = -4 \rightarrow (C_1 + C_2)e^{1/2} = -4$$

$$y'(x) = \left(C_2 + \frac{1}{2}C_1 + \frac{1}{2}C_2 x\right)e^{x/2}$$

$$y'(1) = 0 \rightarrow \left(C_2 + \frac{1}{2}C_1 + \frac{1}{2}C_2\right)e^{1/2} = 0 \Rightarrow C_1 + 3C_2 = 0$$

$$\begin{cases} C_1 + C_2 = -4e^{-1/2} \\ C_1 + 3C_2 = 0 \end{cases} \rightarrow C_2 = 2e^{-1/2} \quad C_1 = -6e^{-1/2}$$

$$y(x) = (-6e^{-1/2} + 2e^{-1/2}x)e^{x/2}$$

$$= 2(x-3)e^{(x-1)/2}$$

Exercise

Find the solution of the given initial value problem $4y'' - 4y' - 3y = 0, \quad y(0) = 1, \quad y'(0) = 5$

Solution

The characteristic equation: $4\lambda^2 - 4\lambda - 3 = 0 \rightarrow \lambda_{1,2} = \frac{4 \pm 8}{8} = -\frac{1}{2}, \frac{3}{2}$

$$y(x) = C_1 e^{-x/2} + C_2 e^{3x/2}$$

$$y(0) = 1 \rightarrow C_1 + C_2 = 1$$

$$y'(x) = -\frac{1}{2}C_1 e^{-x/2} + \frac{3}{2}C_2 e^{3x/2}$$

$$y'(0) = 5 \rightarrow -\frac{1}{2}C_1 + \frac{3}{2}C_2 = 5$$

$$\begin{cases} C_1 + C_2 = 1 \\ -C_1 + 3C_2 = 10 \end{cases} \rightarrow C_2 = \frac{11}{4} \quad C_1 = -\frac{7}{4}$$

$$y(x) = -\frac{7}{4}e^{-x/2} + \frac{11}{4}e^{3x/2}$$

Exercise

Find the solution of the given initial value problem $4y'' + 4y' + 5y = 0, \quad y(\pi) = 1, \quad y'(\pi) = 0$

Solution

The characteristic equation: $4\lambda^2 + 4\lambda + 5 = 0 \rightarrow \lambda_{1,2} = \frac{-4 \pm 8i}{8} = -\frac{1}{2} \pm i$

$$y(x) = e^{-x/2} (C_1 \cos x + C_2 \sin x)$$

$$y(\pi) = 1 \rightarrow -C_1 e^{-\pi/2} = 1 \quad C_1 = -e^{\pi/2}$$

$$y'(x) = e^{-x/2} \left(-C_1 \sin x + C_2 \cos x - \frac{1}{2}C_1 \cos x - \frac{1}{2}C_2 \sin x \right)$$

$$y'(\pi) = 0 \rightarrow \left(-C_2 + \frac{1}{2}C_1 \right) e^{-\pi/2} = 0$$

$$C_2 = \frac{1}{2}C_1 = -\frac{1}{2}e^{\pi/2}$$

$$y(x) = e^{-x/2} \left(-e^{\pi/2} \cos x - \frac{1}{2}e^{\pi/2} \sin x \right)$$

$$= -\frac{1}{2}e^{(\pi-x)/2} (2\cos x + \sin x)$$

Exercise

Find the solution of the given initial value problem $4y'' + 4y' + 17y = 0, \quad y(0) = -1, \quad y'(0) = 2$

Solution

The characteristic equation: $4\lambda^2 + 4\lambda + 17 = 0$

$$\rightarrow \lambda_{1,2} = \frac{-4 \pm 16i}{8} = -\frac{1}{2} \pm 2i$$

$$y(x) = e^{-x/2} (C_1 \cos 2x + C_2 \sin 2x)$$

$$y(0) = -1 \rightarrow -1 = C_1$$

$$y'(x) = e^{-x/2} \left(-\frac{1}{2} C_1 \cos 2x - \frac{1}{2} C_2 \sin 2x - 2C_1 \sin 2x + 2C_2 \cos 2x \right)$$

$$y'(0) = 2 \rightarrow 2 = \frac{1}{2} + 2C_2 \Rightarrow C_2 = \frac{3}{4}$$

$$y(x) = e^{-x/2} \left(-\cos 2x + \frac{3}{4} \sin 2x \right)$$

Exercise

Find the solution of the given initial value problem $4y'' - 5y' = 0, \quad y(-2) = 0, \quad y'(-2) = 7$

Solution

$$\text{The characteristic equation: } 4\lambda^2 - 5\lambda = 0 \rightarrow \lambda_{1,2} = 0, \frac{5}{4}$$

$$y(t) = C_1 + C_2 e^{5t/4}$$

$$y(-2) = 0 \rightarrow C_1 + C_2 e^{-5/2} = 0$$

$$y' = \frac{5}{4} C_2 e^{5t/4}$$

$$y'(-2) = 7 \rightarrow \frac{5}{4} C_2 e^{-5/2} = 7 \Rightarrow C_2 = \frac{28}{5} e^{5/2}$$

$$C_1 = -\frac{28}{5}$$

$$y(t) = -\frac{28}{5} + \frac{28}{5} e^{5t/4}$$

Exercise

Find the solution of the given initial value problem $4y'' + 12y' + 9y = 0, \quad y(0) = 2, \quad y'(0) = 1$

Solution

$$\text{The characteristic equation: } 4\lambda^2 + 12\lambda + 9 = (2\lambda + 3)^2 = 0 \rightarrow \lambda_{1,2} = -\frac{3}{2}$$

$$y(x) = (C_1 + C_2 x) e^{-3x/2}$$

$$y(0) = 2 \rightarrow C_1 = 2$$

$$y'(x) = \left(C_2 - \frac{3}{2}C_1 - \frac{3}{2}C_2 x \right) e^{-3x/2}$$

$$y'(0) = 1 \rightarrow C_2 - \frac{3}{2}C_1 = 1 \Rightarrow \underline{C_2 = 4}$$

$$\underline{y(x) = (2 + 4x)e^{-3x/2}}$$

Exercise

Find the solution of the given initial value problem $4y'' + 24y' + 37y = 0$, $y(\pi) = 1$, $y'(\pi) = 0$

Solution

The characteristic equation: $4\lambda^2 + 24\lambda + 37 = 0$

$$\rightarrow \lambda_{1,2} = \frac{-24 \pm 4i}{8} = -3 \pm \frac{1}{2}i$$

$$y(t) = e^{-3t} \left(C_1 \cos \frac{1}{2}t + C_2 \sin \frac{1}{2}t \right)$$

$$y(\pi) = 1 \rightarrow e^{-3\pi} C_2 = 1 \Rightarrow \underline{C_2 = e^{3\pi}}$$

$$y' = e^{-3t} \left(-3C_1 \cos \frac{1}{2}t - 3C_2 \sin \frac{1}{2}t - \frac{1}{2}C_1 \sin \frac{1}{2}t + \frac{1}{2}C_2 \cos \frac{1}{2}t \right)$$

$$y'(\pi) = 0 \rightarrow e^{-3\pi} \left(-3e^{3\pi} - \frac{1}{2}C_1 \right) = 0 \Rightarrow \underline{C_1 = -6e^{3\pi}}$$

$$y(t) = e^{-3t} \left(-6e^{3\pi} \cos \frac{1}{2}t + e^{3\pi} \sin \frac{1}{2}t \right)$$

$$\underline{= -6e^{3(\pi-t)} \cos \frac{t}{2} + e^{3(\pi-t)} \sin \frac{t}{2}}$$

Exercise

Find the solution of the given initial value problem $9y'' + y = 0$; $y\left(\frac{\pi}{2}\right) = 4$, $y'\left(\frac{\pi}{2}\right) = 0$

Solution

The characteristic equation: $9\lambda^2 + 1 = 0 \rightarrow \underline{\lambda_{1,2} = \pm \frac{1}{3}i}$

$$\underline{y(x) = C_1 \cos \frac{x}{3} + C_2 \sin \frac{x}{3}}$$

$$y\left(\frac{\pi}{2}\right) = 4 \rightarrow \frac{\sqrt{3}}{2}C_1 + \frac{1}{2}C_2 = 4$$

$$y'(x) = -\frac{1}{3}C_1 \sin \frac{x}{3} + \frac{1}{3}C_2 \cos \frac{x}{3}$$

$$y'\left(\frac{\pi}{2}\right) = 0 \rightarrow -\frac{1}{2}C_1 + \frac{\sqrt{3}}{2}C_2 = 0$$

$$\begin{cases} \sqrt{3}C_1 + C_2 = 8 \\ -C_1 + \sqrt{3}C_2 = 0 \end{cases} \quad \Delta = \begin{vmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{vmatrix} = 4 \quad \Delta_1 = \begin{vmatrix} 8 & 1 \\ 0 & \sqrt{3} \end{vmatrix} = 8\sqrt{3} \quad \Delta_2 = \begin{vmatrix} \sqrt{3} & 8 \\ -1 & 0 \end{vmatrix} = 8$$

$$\underline{C_1 = 2\sqrt{3} \quad C_2 = 2}$$

$$\underline{y(x) = 2\sqrt{3} \cos \frac{x}{3} + 2 \sin \frac{x}{3}}$$

Exercise

Find the solution of the given initial value problem $9y'' + \pi^2 y = 0; \quad y(3) = 2, \quad y'(3) = -\pi$

Solution

The characteristic equation: $9\lambda^2 + \pi^2 = 0 \rightarrow \underline{\lambda_{1,2} = \pm \frac{\pi}{3}i}$

$$\underline{y(x) = C_1 \cos \frac{\pi}{3}x + C_2 \sin \frac{\pi}{3}x}$$

$$\underline{y(3) = 2 \rightarrow C_1 = -2}$$

$$y'(x) = -\frac{\pi}{3}C_1 \sin \frac{\pi}{3}x + \frac{\pi}{3}C_2 \cos \frac{\pi}{3}x$$

$$\underline{y'(3) = -\pi \rightarrow -\frac{\pi}{3}C_2 = -\pi \Rightarrow C_2 = 3}$$

$$\underline{y(x) = -2 \cos \frac{\pi}{3}x + 3 \sin \frac{\pi}{3}x}$$

Exercise

Find the solution of the given initial value problem $9y'' - 6y' + y = 0; \quad y(3) = -2, \quad y'(3) = -\frac{5}{3}$

Solution

The characteristic equation: $9\lambda^2 - 6\lambda + 1 = 0 = (3\lambda - 1)^2 \rightarrow \underline{\lambda_{1,2} = \frac{1}{3}}$

$$\underline{y(x) = (C_1 + C_2 x)e^{x/3}}$$

$$\underline{y(3) = -2 \rightarrow (C_1 + 3C_2)e = -2}$$

$$y'(x) = (C_2 + \frac{1}{3}C_1 + \frac{1}{3}C_2 x)e^{x/3}$$

$$\underline{y'(3) = -\frac{5}{3} \rightarrow (2C_2 + \frac{1}{3}C_1)e = -\frac{5}{3}}$$

$$\begin{cases} C_1 + 3C_2 = -\frac{2}{e} \\ C_1 + 6C_2 = -\frac{5}{e} \end{cases} \quad \Delta = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \quad \Delta_1 = \begin{vmatrix} -\frac{2}{e} & 3 \\ -\frac{5}{e} & 6 \end{vmatrix} = \frac{3}{e} \quad \Delta_2 = \begin{vmatrix} 1 & -\frac{2}{e} \\ 1 & -\frac{5}{e} \end{vmatrix} = -\frac{3}{e}$$

$$\underline{C_1 = \frac{1}{e} \quad C_2 = -\frac{1}{e}}$$

$$\underline{y(x) = \frac{1}{e}(1-x)e^{x/3}}$$

Exercise

Find the solution of the given initial value problem $9y'' + 6y' + 2y = 0$; $y(3\pi) = 0$, $y'(3\pi) = \frac{1}{3}$

Solution

The characteristic equation: $9\lambda^2 + 6\lambda + 2 = 0$

$$\rightarrow \lambda_{1,2} = \frac{-6 \pm 6i}{18} = -\frac{1}{3} \pm \frac{1}{3}i$$

$$\underline{y(x) = \left(C_1 \cos \frac{x}{3} + C_2 \sin \frac{x}{3} \right) e^{-x/3}}$$

$$y(3\pi) = 0 \rightarrow -C_1 e^{-\pi} = 0 \Rightarrow \underline{C_1 = 0}$$

$$y'(x) = \left(-\frac{1}{3}C_1 \sin \frac{x}{3} + \frac{1}{3}C_2 \cos \frac{x}{3} - \frac{1}{3}C_1 \cos \frac{x}{3} - \frac{1}{3}C_2 \sin \frac{x}{3} \right) e^{-x/3}$$

$$y'(3\pi) = -\frac{5}{3} \rightarrow \left(-\frac{1}{3}C_2 \right) e^{-\pi} = \frac{1}{3} \Rightarrow \underline{C_2 = -e^\pi}$$

$$\underline{y(x) = -e^\pi \sin \frac{x}{3} e^{-x/3}}$$

Exercise

Find the solution of the given initial value problem $9y'' - 12y' + 4y = 0$, $y(0) = -1$, $y'(0) = 1$

Solution

The characteristic equation: $9\lambda^2 - 12\lambda + 4 = (3\lambda - 2)^2 = 0 \rightarrow \underline{\lambda_{1,2} = \frac{2}{3}}$

$$y(x) = (C_1 + C_2 x) e^{2x/3}$$

$$y(0) = -1 \rightarrow \underline{C_1 = -1}$$

$$y'(x) = \left(C_2 + \frac{2}{3}C_1 + \frac{2}{3}C_2 x \right) e^{2x/3}$$

$$y'(0) = 1 \rightarrow C_2 + \frac{2}{3}C_1 = 1 \Rightarrow \underline{C_2 = \frac{5}{3}}$$

$$\underline{y(x) = \left(-1 + \frac{5}{3}x \right) e^{2x/3}}$$

Exercise

Find the solution of the given initial value problem $12y'' + 5y' - 2y = 0$, $y(0) = 1$, $y'(0) = -1$

Solution

The characteristic equation: $12\lambda^2 + 5\lambda - 2 = 0 \rightarrow \lambda_{1,2} = \frac{-5 \pm 11}{24} = \underline{-\frac{2}{3}, \frac{1}{4}}$

$$y(x) = C_1 e^{2x/3} + C_2 e^{x/4}$$

$$y(0) = 1 \rightarrow C_1 + C_2 = 1$$

$$y'(x) = \frac{2}{3}C_1 e^{2x/3} + \frac{1}{4}C_2 e^{x/4}$$

$$y'(0) = -1 \rightarrow \frac{2}{3}C_1 + \frac{1}{4}C_2 = -1$$

$$\begin{cases} C_1 + C_2 = 1 \\ 8C_1 + 3C_2 = -12 \end{cases} \rightarrow \underline{C_1 = -3, C_2 = 4}$$

$$\underline{y(x) = -3e^{2x/3} + 4e^{x/4}}$$

Exercise

Find the solution of the given initial value problem $16y'' - 8y' + y = 0$; $y(0) = -4$, $y'(0) = 3$

Solution

The characteristic equation: $16\lambda^2 - 8\lambda + 1 = 0 \rightarrow \lambda_{1,2} = \frac{8 \pm 0}{32} = \underline{\frac{1}{4}}$

$$y(x) = (C_1 + C_2 x) e^{x/4}$$

$$y(0) = -4 \rightarrow \underline{C_1 = -4}$$

$$y'(x) = (C_2 + \frac{1}{4}C_1 + \frac{1}{4}C_2 x) e^{x/4}$$

$$y'(0) = 3 \rightarrow C_2 - 1 = 3 \Rightarrow \underline{C_2 = 4}$$

$$\underline{y(x) = (-4 + 4x) e^{x/4}}$$

Exercise

Find the solution of the given initial value problem

$$25y'' + 20y' + 4y = 0; \quad y(5) = 4e^{-2}, \quad y'(5) = -\frac{3}{5}e^{-2}$$

Solution

The characteristic equation: $25\lambda^2 + 20\lambda + 4 = 0 \rightarrow \lambda_{1,2} = \frac{-20 \pm 0}{50} = -\frac{2}{5}$

$$y(x) = (C_1 + C_2 x)e^{-2x/5}$$

$$y(5) = 4e^{-2} \rightarrow (C_1 + 5C_2)e^{-2} = 4e^{-2} \Rightarrow \underline{C_1 + 5C_2 = 4}$$

$$y'(x) = \left(C_2 - \frac{2}{5}C_1 - \frac{2}{5}C_2 x\right)e^{-2x/5}$$

$$y'(5) = -\frac{3}{5}e^{-2} \rightarrow \left(C_2 - \frac{2}{5}C_1 - 2C_2\right)e^{-2} = -\frac{3}{5}e^{-2} \Rightarrow \underline{2C_1 + 5C_2 = 3}$$

$$\begin{cases} C_1 + 5C_2 = 4 \\ 2C_1 + 5C_2 = 3 \end{cases} \rightarrow \Delta = \begin{vmatrix} 1 & 5 \\ 2 & 5 \end{vmatrix} = -5 \quad \Delta_1 = \begin{vmatrix} 4 & 5 \\ 3 & 5 \end{vmatrix} = 5$$

$$\underline{C_1 = -1, \quad C_2 = 1}$$

$$\underline{y(x) = (-1 + x)e^{-2x/5}}$$

Exercise

Find the solution of the given initial value problem

$$y''' + 12y'' + 36y' = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = -7$$

Solution

$$\lambda^3 + 12\lambda^2 + 36\lambda = 0$$

$$\lambda(\lambda + 6)^2 = 0 \rightarrow \underline{\lambda_1 = 0, -6, -6}$$

$$y(x) = C_1 + (C_2 + C_3 x)e^{-6x}$$

$$y(0) = 0 \rightarrow C_1 + C_2 = 0$$

$$y'(x) = (C_3 - 6C_2 - 6C_3 x)e^{-6x}$$

$$y'(0) = 1 \rightarrow C_3 - 6C_2 = 1$$

$$y'' = (-12C_3 + 36C_2 + 36C_3 x)e^{-6x}$$

$$y''(0) = -7 \rightarrow -12C_3 + 36C_2 = -7$$

$$\begin{cases} C_3 - 6C_2 = 1 \\ -12C_3 + 36C_2 = -7 \end{cases} \rightarrow C_3 = \frac{1}{6}, \quad C_2 = -\frac{5}{36}, \quad C_1 = \frac{5}{36}$$

$$\underline{y(x) = \frac{5}{36} - \frac{5}{36}e^{-6x} + \frac{1}{6}xe^{-6x}}$$

Exercise

Find the solution of the given initial value problem

$$y''' + 2y'' - 5y' - 6y = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1$$

Solution

The characteristic equation: $\lambda^3 + 2\lambda^2 - 5\lambda - 6 = 0 \rightarrow \lambda_1 = -1$

$$\begin{array}{c|cccc} -1 & 1 & 2 & -5 & -6 \\ & & -1 & -1 & 6 \\ \hline & 1 & 1 & -6 & 0 \end{array} \quad \lambda^2 + \lambda - 6 = 0 \Rightarrow \lambda_3 = -3, \lambda_4 = 2$$

$$y(x) = C_1 e^{-3x} + C_2 e^{-x} + C_3 e^{2x}$$

$$y(0) = 0 \rightarrow C_1 + C_2 + C_3 = 0$$

$$y'(x) = -3C_1 e^{-3x} - C_2 e^{-x} + 2C_3 e^{2x}$$

$$y'(0) = 0 \rightarrow -3C_1 - C_2 + 2C_3 = 0$$

$$y''(x) = 9C_1 e^{-3x} + C_2 e^{-x} + 4C_3 e^{2x}$$

$$y''(0) = 1 \rightarrow 9C_1 + C_2 + 4C_3 = 1$$

$$\begin{cases} C_1 + C_2 + C_3 = 0 \\ -3C_1 - C_2 + 2C_3 = 0 \\ 9C_1 + C_2 + 4C_3 = 1 \end{cases} \rightarrow \Delta = \begin{vmatrix} 1 & 1 & 1 \\ -3 & -1 & 2 \\ 9 & 1 & 4 \end{vmatrix} = 30 \quad \Delta_{C_1} = \begin{vmatrix} 0 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix} = 3 \quad \Delta_{C_2} = \begin{vmatrix} 1 & 0 & 1 \\ -3 & 0 & 2 \\ 9 & 1 & 4 \end{vmatrix} = 5$$

$$C_1 = \frac{1}{10}, \quad C_2 = \frac{1}{6}, \quad C_3 = -\frac{4}{15}$$

$$y(x) = \frac{1}{10} e^{-3x} + \frac{1}{6} e^{-x} - \frac{4}{15} e^{2x}$$

Exercise

The roots of the characteristic equation of a certain differential equation are:

$$3, -5, 0, 0, 0, 0, -5, 2 \pm 3i \text{ and } 2 \pm 3i$$

Write a general solution of this homogeneous differential equation.

Solution

$$\text{For } \lambda = 0, 0, 0, 0 \Rightarrow y_1 = C_1 + C_2 x + C_3 x^2 + C_4 x^3$$

$$\text{For } \lambda = 3 \Rightarrow y_2 = C_5 e^{3x}$$

$$\text{For } \lambda = -5, -5 \Rightarrow y_3 = C_6 e^{-5x} + C_7 x e^{-5x}$$

For $\lambda = 2 \pm 3i, 2 \pm 3i$

$$\Rightarrow y_4 = e^{2x} (C_8 \cos 3x + C_9 \sin 3x) + x e^{-5x} (C_{10} \cos 3x + C_{11} \sin 3x)$$

$$y(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 e^{3x} + C_6 e^{-5x} + C_7 x e^{-5x} + e^{2x} (C_8 \cos 3x + C_9 \sin 3x) + x e^{-5x} (C_{10} \cos 3x + C_{11} \sin 3x)$$

Exercise

$y(x) = C_1 e^{2x} + C_2 e^{-2x} + C_3 \cos 2x + C_4 \sin 2x$ is the general solution of a homogeneous equation. What is the equation?

Solution

$$\lambda_1 = 2, \quad \lambda_2 = -2, \quad \lambda_{3,4} = \pm 2i$$

$$(\lambda - 2)(\lambda + 2)(\lambda - 2i)(\lambda + 2i) = 0$$

$$(\lambda^2 - 4)(\lambda^2 + 4) = 0$$

$$\lambda^4 - 16 = 0 \Rightarrow \underline{y^{(4)} - 16y = 0}$$

Exercise

Show that the second differential equation $y'' + 4y = 0$

a) Has no solution to the boundary value $y(0) = 0, \quad y(\pi) = 1$

b) There are infinitely many solutions to the boundary value $y(0) = 0, \quad y(\pi) = 0$

Solution

The characteristic equation: $\lambda^2 + 4 = 0 \rightarrow \underline{\lambda_{1,2} = \pm 2i}$

$$\underline{y(x) = C_1 \cos 2x + C_2 \sin 2x}$$

a) $y(0) = 0 \rightarrow \underline{C_1 = 0}$

$y(\pi) = 1 \rightarrow \underline{C_1 = 1}$

Therefore, there is no solution since ~~$C_1 = 0 = 1$~~

b) $y(0) = 0 \rightarrow \underline{C_1 = 0}$

$y(\pi) = 0 \rightarrow \underline{C_1 = 0}$

$\underline{y(x) = C_2 \sin 2x}$

\therefore There are infinite many solutions.

Exercise

Show that the general solution of the equation/ $y'' + Py' + Qy = 0$

(where P and Q are constant) approaches 0 as $x \rightarrow \infty$ if and only if P and Q are both positive.

Solution

$$\lambda^2 + P\lambda + Q = 0$$

$$\text{The solutions: } \lambda_{1,2} = \frac{-P \pm \sqrt{P^2 - 4Q}}{2}$$

$$\text{If } P^2 - 4Q < 0 \rightarrow P < 2\sqrt{Q} \quad (P \text{ \& } Q \text{ are positives})$$

$$\lambda_{1,2} = -\frac{P}{2} \pm i \frac{\sqrt{4Q - P^2}}{2}$$

$$y(x) = e^{-Px/2} \left(C_1 \cos \frac{1}{2} \sqrt{4Q - P^2} x + C_2 \sin \frac{1}{2} \sqrt{4Q - P^2} x \right)$$

$$\lim_{x \rightarrow \infty} y(x) = \lim_{x \rightarrow \infty} \left[e^{-Px/2} \left(C_1 \cos \frac{1}{2} \sqrt{4Q - P^2} x + C_2 \sin \frac{1}{2} \sqrt{4Q - P^2} x \right) \right]$$

$$= 0$$

$$\lim_{x \rightarrow \infty} \left(e^{-Px/2} \right) = \lim_{x \rightarrow \infty} \left(\frac{1}{e^{Px/2}} \right) = \frac{1}{\infty} = 0 \quad (P > 0)$$

$$\text{If } P^2 - 4Q = 0 \rightarrow \lambda_{1,2} = -\frac{1}{2}P$$

$$y(x) = (C_1 + C_2 x) e^{-Px/2}$$

$$\lim_{x \rightarrow \infty} y(x) = \lim_{x \rightarrow \infty} (C_1 + C_2 x) e^{-Px/2} = 0$$

$$\text{If } P^2 - 4Q > 0 \rightarrow \lambda_{1,2} = \frac{-P \pm \sqrt{P^2 - 4Q}}{2}$$

$$y(x) = C_1 e^{\frac{-P - \sqrt{P^2 - 4Q}}{2} x} + C_2 e^{\frac{-P + \sqrt{P^2 - 4Q}}{2} x}$$

$$\sqrt{P^2 - 4Q} < \sqrt{P^2} = P \rightarrow \frac{-P + \sqrt{P^2 - 4Q}}{2} < 0$$

$$\lim_{x \rightarrow \infty} e^{\frac{-P + \sqrt{P^2 - 4Q}}{2} x} = 0$$

$$\lim_{x \rightarrow \infty} y(x) = \lim_{x \rightarrow \infty} \left(C_1 e^{\frac{-P - \sqrt{P^2 - 4Q}}{2} x} + C_2 e^{\frac{-P + \sqrt{P^2 - 4Q}}{2} x} \right) = 0$$