Section 1.4 – Equations and Application

Linear Equations

A ***linear equation*** in one variable is an equation that is equivalent to one of the form $mx + b = 0$

Equation-Solving Principles

Addition Principle: If $a = b$ is true $\Rightarrow$ $a + c = b + c$

Multiplication Principle: If $a = b$ is true $\Rightarrow$ $ac = bc$

Solve the following equations

a) $x - 2 = 3$

$$x - 2 + 2 = 3 + 2$$
$$x = 5$$

b) $\frac{x}{2} = 3$

$$2\frac{x}{2} = (2)3$$
$$x = 6$$

Solve: $2x - 5 + 8 = 3x + 2(2 - 3x)$

$$2x - 5 + 8 = 3x + 4 - 6x$$
$$2x + 3 = 4 - 3x$$
$$2x + 3 - 3 + 3x = 4 - 3x - 3 + 3x$$
$$5x = 1$$

Divide both sides by 5

$$x = \frac{1}{5}$$

The Zero-Product Principle:

If $ab = 0$. then $a = 0$ or $b = 0$.

Solve $6x^2 + 7x = 3$

$$6x^2 + 7x - 3 = 0$$
$$(3x - 1)(2x + 3) = 0$$

$$3x - 1 = 0 \qquad 2x + 3 = 0$$
$$3x = 1 \qquad 2x = -3$$
$$x = \frac{1}{3} \qquad x = -\frac{3}{2}$$

Quadratic Formula

$$ax^2+bx+c=0 \quad \Rightarrow \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Solve $x^2-4x-5=0 \qquad \Rightarrow \; a=1, \;\; b=-4, \;\; c=-5$

$$\begin{aligned} x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(-5)}}{2(1)} \\ &= \frac{4\pm\sqrt{16+20}}{2} \\ &= \frac{4\pm\sqrt{36}}{2} \\ &= \frac{4\pm 6}{2} \end{aligned}$$

$$x=\frac{4+6}{2} \qquad\qquad x=\frac{4-6}{2}$$

$$=\frac{10}{2} \qquad\qquad =\frac{-2}{2}$$

$$=5 \qquad\qquad =-1$$

Solve $x^2+1=4x$

$$x^2-4x+1=0$$

$$\begin{aligned} x &= \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(1)}}{2(1)} \\ &= \frac{4\pm\sqrt{16-4}}{2} \\ &= \frac{4\pm\sqrt{12}}{2} \\ &= \frac{4\pm 2\sqrt{3}}{2} \\ &= \frac{2\left(2\pm\sqrt{3}\right)}{2} \\ &= 2\pm\sqrt{3} \end{aligned}$$

Equations with Fractions

Solve $\frac{r}{10}-\frac{2}{15}=\frac{3r}{20}-\frac{1}{5}$

10	2 5
15	3 5
20	2 2 5
5	5
	2 2 3 5 = 60

$(60)\frac{r}{10}-(60)\frac{2}{15}=(60)\frac{3r}{20}-(60)\frac{1}{5}$

$6r-8=9r-12$

$6r-8+8-9r=9r-12+8-9r$

$-3r=-4$

$r=\frac{-4}{-3}=\frac{4}{3}$

Solve $\frac{2}{x-3}+\frac{1}{x}=\frac{6}{x(x-3)}$ $\qquad x-3\neq 0$

Conditions: $x\neq 0,\ 3$

$x(x-3)\frac{2}{x-3}+x(x-3)\frac{1}{x}=x(x-3)\frac{6}{x(x-3)}$

$2x+x-3=6$

$3x=9$

$x=3$

Solve $\frac{1}{x-2}-\frac{3x}{x-1}=\frac{2x+1}{x^2-3x+2}$ $\qquad cond.\ x\neq 1,\ 2$

$(x-2)(x-1)\frac{1}{x-2}-(x-2)(x-1)\frac{3x}{x-1}=(x-2)(x-1)\frac{2x+1}{x^2-3x+2}$

$x-1-3x(x-2)=2x+1$

$x-1-3x^2+6x-2x-1=0$

$-3x^2+5x-2=0$

$3x^2-5x+2=0$

$(x-1)(3x-2)=0$

$x-1=0$ $\qquad 3x-2=0$

$x=1$ $\qquad x=\frac{2}{3}$

Solution: $x=\frac{2}{3}$

Slopes and Equations of Lines

Slope of a line *(Definition)*

The slope of a line is defined as the vertical change (the *rise*) over the horizontal change (the *run*) as one travels along the line.

$$slope: \quad m = \frac{change\ in\ y}{change\ in\ x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Find the slope of the line through each pair point

a) $(7,6)$ *and* $(-4,5)$

$$m = \frac{5-6}{-4-7}$$

$$= \frac{-1}{-11}$$

$$= \frac{1}{11}$$

b) $(5,-3)$ *and* $(-2,-3)$

$$m = \frac{-3+3}{-2-5}$$

$$= \frac{0}{-7}$$

$$= 0$$

c) $(2,-4)$ *and* $(2,3)$

$$m = \frac{3+4}{2-2}$$

$= \frac{7}{0}$ Which is undefined $\rightarrow$ line is vertical.

Equations of a Line

$$y = mx + b$$

This *linear equation* is called the *slope-intercept form* of the equation of a line.

Point-Slope Form

$$y - y_1 = m(x - x_1)$$

Example

Find the equation of the line through $(0,\ -3)$ with slope $\frac{3}{4}$

Solution

$$y - y_1 = m(x - x_1)$$

$$y + 3 = \frac{3}{4}(x - 0)$$

$$y + 3 = \frac{3}{4}x$$

$$y = \frac{3}{4}x - 3$$

$$(4)y = (4)\frac{3}{4}x - (4)3$$

$$4y = 3x - 12$$

$$4y - 3x = -12$$

$$3x - 4y = 12$$

Example

Find the equation of the line that passes through the point $(3,\ -7)$ and has slope $\frac{5}{4}$

Solution

$$y - y_1 = m(x - x_1)$$

$$y + 7 = \frac{5}{4}(x - 3)$$

$$y + 7 = \frac{5}{4}x - \frac{15}{4}$$

$$y + 7 - 7 = \frac{5}{4}x - \frac{15}{4} - 7 \qquad -\frac{15}{4} - 7 = -\frac{15}{4} - 7\frac{4}{4} = -\frac{15}{4} - \frac{28}{4} = -\frac{43}{4}$$

$$y = \frac{5}{4}x - \frac{43}{4}$$

Parallel Lines (//)

Two lines are parallel if and only if they have the same slope, or they are both vertical. $m_1 = m_2$

Example

Find the equation of the line that passes through the point $(3,\ 5)$ and is parallel to the line $2x+5y=4$

Solution

$$2x+5y=4$$

$$5y=-2x+4$$

$$y=-\frac{2}{5}x+\frac{4}{5}$$

$$m_1 = m_2$$

$$Slope: m=-\frac{2}{5}$$

$$y-y_1=m\left(x-x_1\right)$$

$$y-5=-\frac{2}{5}(x-3)$$

$$y-5=-\frac{2}{5}x+\frac{6}{5}$$

$$y-5+5=-\frac{2}{5}x+\frac{6}{5}+5$$

$$y=-\frac{2}{5}x+\frac{31}{5}$$

Perpendicular Lines ($\perp$)

Two lines are perpendicular if and only if the product of their slope is -1. $m_1 \cdot m_2 = -1$

Example

Find the slope of the line L perpendicular to the line having the equation $5x-y=4$

Solution

$$5x-y=4$$

$5x-4=y \rightarrow$ Slope = 5

Slope of the line L = $-\frac{1}{5}$

Linear Functions and Applications

Linear Function

A relationship f defined by

$$y = f(x) = mx + b$$

For real numbers m and b, is a ***linear function***

Example

Let $g(x) = -4x + 5$***. Find*** $g(3)$***,*** $g(0)$***,*** $g(-2)$***, and*** $g(b)$

Solution

$$g(x) = -4x + 5$$
$$g(\text{—}) = -4(\text{—}) + 5$$

$$\begin{aligned} g(3) &= -4(3) + 5 \\ &= -7 \end{aligned}$$

$$\begin{aligned} g(0) &= -4(0) + 5 \\ &= 5 \end{aligned}$$

$$\begin{aligned} g(-2) &= -4(-2) + 5 \\ &= 13 \end{aligned}$$

$$g(b) = -4b + 5$$

Cost Analysis

Definition

In a cost function of the form $C(x) = mx + b$ is the *linear cost function*

m: Represents the marginal cost per item

b: Fixed cost.

Example

The marginal cost to make x hatches of a prescription medication is $10 per batch, while the cost to produce 100 batches is $1500. Find the cost function $C(x)$, given that it is linear.

<u>*Solution*</u>

Since the cost function is linear $\Rightarrow C(x) = mx + b$

The marginal cost = slope $\Rightarrow m = 10$

$$C(x) = 10x + b$$
$$1500 = 10(100) + b$$
$$1500 = 1000 + b$$
$$b = 500$$

$$C(x) = 10x + 500$$

Break-Even Analysis

$R(x)$ *Revenue*

$P(x)$ *Profit*

$C(x)$ *Cost*

x *Units*

p *Price per unit*

$$R(x) = p \cdot x$$

$$\boxed{P(x) = R(x) - C(x)}$$

The number of units at which revenue just equals cost $\left(R(x) = C(x)\right)$ is the ***break-even quantity***.
The corresponding ordered pair gives the ***break-even point***.

Example

A firm producing poultry feed finds that the total cost $C(x)$ in dollars of producing and selling x units is given by

$$C(x) = 20x + 100$$

Management plans to charge \$24 per unit for the feed.

a) How many units must be sold for the firm to break even?

The revenue: $R(x) = px$
$= 24x$

Break-even: $R(x) = C(x)$

$$24x = 20x + 100$$
$$4x = 100$$
$$x = 25$$

b) What is the profit if 100 units of feed are sold?

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 24x - (20x + 100) \\ &= 24x - 20x - 100 \\ &= 4x - 100 \end{aligned}$$

$$\begin{aligned} P(100) &= 4(100) - 100 \\ &= 300 \end{aligned}$$

c) How many units must be sold to produce a profit of \$900?

$$900 = 4x - 100$$
$$1000 = 4x$$
$$x = 250$$

Exercises *Section* 1.4 – Equations and Application

1. Suppose that Greg, manager of a giant supermarket chain, has studied the supply and demand for watermelons. He has noticed that the demand increases as the price decreases. He has determined that the quantity (in thousands) demanded weekly q, and the price (in dollars) per watermelons, p, are related by the linear function.

$$p = D(q) = 9 - 0.75q \quad \text{Demand function}$$

a) Find the demand at a price of $5.25 per watermelon and at a price of $3.75 per watermelon.

b) Greg also noticed that the supply of watermelons decreased as the price decreased. Price p and supply q are related by the linear function

$$p = S(q) = 0.75q \qquad \text{Demand function}$$

Find the supply at a price of $5.25 per watermelon and at a price of $3.00 per watermelon.

c) Use the algebra to find the equilibrium quantity for the watermelon in example 2

2. In Recent years, the percentage of the U.S. population age 18 and older who smoke has decreased at a roughly constant rate, from 23.3% in 2000 to 20.9% in 2004.

a) Find the equation describing this linear relationship.

b) One of the objectives of Healthy People 2010 (a campaign of the U.S. Department of Health and Human Services) is to reduce the percentage of U.S. adults to smoke to 12% or less by the year 2010. If this decline in smoking continues at the same rate, will they meet this objective

3. The number of African Americans earning doctorate degrees has risen at an approximately constant rate from 1987 to 2005. The linear equation $y = 63.6x + 787$, where x represents the number of years since 1987, can be used to estimate the annual number of African Americans earning doctorate degrees. Determine this number in 2006.