

Lecture R – Introduction to Differential Equation

Solution **Section R.1 – Derivative**

Exercise

Find the derivative of $f(t) = -3t^2 + 2t - 4$

Solution

$$\underline{f'(t) = -6t + 2}$$

Exercise

Find the derivative of $g(x) = 4\sqrt[3]{x} + 2$

Solution

$$g(x) = 4x^{1/3} + 2$$

$$g'(x) = \frac{4}{3}x^{-2/3}$$

$$\underline{= \frac{4}{3\sqrt[3]{x^2}}}$$

Exercise

Find the derivative of $f(x) = x(x^2 + 1)$

Solution

$$f(x) = x^3 + x$$

$$\underline{f'(x) = 3x^2 + 1}$$

Exercise

Find the derivative of $f(x) = \frac{2x^2 - 3x + 1}{x}$

Solution

$$f(x) = \frac{2x^2}{x} - \frac{3x}{x} + \frac{1}{x}$$

$$= 2x - 3 + \frac{1}{x}$$

$$\underline{f'(x) = 2 - \frac{1}{x^2}}$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

Exercise

Find the derivative of $f(x) = \frac{4x^3 - 3x^2 + 2x + 5}{x^2}$

Solution

$$f(x) = 4x - 3 + \frac{2}{x} + 5x^{-2} \qquad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$\begin{aligned} f'(x) &= 4 - \frac{2}{x^2} - 10x^{-3} \\ &= 4 - \frac{2}{x^2} - \frac{10}{x^3} \end{aligned}$$

Exercise

Find the derivative of $f(x) = \frac{-6x^3 + 3x^2 - 2x + 1}{x}$

Solution

$$f(x) = -6x^2 + 3x - 2 + \frac{1}{x} \qquad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$f'(x) = -12x + 3 - \frac{1}{x^2}$$

Exercise

Find the derivative of $f(x) = x\left(1 - \frac{2}{x+1}\right)$

Solution

$$f(x) = x - \frac{2x}{x+1} \qquad \left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

$$f'(x) = 1 - \frac{2}{(x+1)^2}$$

Exercise

Find the derivative of $g(s) = \frac{s^2 - 2s + 5}{\sqrt{s}}$

Solution

$$g(s) = \frac{s^2}{s^{1/2}} - 2\frac{s}{s^{1/2}} + \frac{5}{s^{1/2}}$$

$$= s^{3/2} - 2s^{1/2} + 5s^{-1/2}$$

$$\begin{aligned} g'(s) &= \frac{3}{2}s^{1/2} - 2\frac{1}{2}s^{-1/2} + 5\left(-\frac{1}{2}\right)s^{-3/2} \\ &= \frac{3}{2}s^{1/2} - s^{-1/2} - \frac{5}{2}s^{-3/2} \\ &= \frac{3}{2}\sqrt{s} - \frac{1}{\sqrt{s}} - \frac{5}{2s^{3/2}} \\ &= \frac{\frac{3}{2}\sqrt{s} - \frac{1}{\sqrt{s}} - \frac{5}{2s\sqrt{s}}}{} \end{aligned}$$

Exercise

Find the derivative of $f(x) = \frac{x+1}{\sqrt{x}}$

Solution

$$\begin{aligned} f(x) &= \frac{x}{x^{1/2}} + \frac{1}{x^{1/2}} \\ &= x^{1/2} + x^{-1/2} \\ f'(x) &= \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} \\ &= \frac{\frac{1}{2x^{1/2}} - \frac{1}{2x^{3/2}}}{} \end{aligned}$$

Exercise

Find the derivative to the following functions $y = 3x(2x^2 + 5x)$

Solution

$$\begin{aligned} y &= 6x^3 + 15x^2 \\ y' &= \underline{18x^2 + 30x} \end{aligned}$$

Exercise

Find the derivative to the following functions $y = 3(2x^2 + 5x)$

Solution

$$\begin{aligned} y &= 6x^2 + 15x \\ y' &= \underline{12x + 15} \end{aligned}$$

Exercise

Find the derivative to the following functions $y = \frac{x^2 + 4x}{5}$

Solution

$$y' = \frac{1}{5}(2x + 4)$$

Exercise

Find the derivative to the following functions $y = \frac{3x^4}{5}$

Solution

$$y' = \frac{12}{5}x^3$$

Exercise

Find the derivative to the following functions $y = \frac{x^2 - 4}{2x + 5}$

Solution

$$y' = \frac{2x^2 + 10x + 8}{(2x + 5)^2}$$

$$\frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{(ae - bd)x^2 + 2(af - dd)x + (bf - ce)}{(dx^2 + ex + f)^2}$$

Exercise

Find the derivative to the following functions $y = \frac{(1+x)(2x-1)}{x-1}$

Solution

$$\begin{aligned} y' &= \frac{(x-1) \frac{d}{dx} [(1+x)(2x-1)] - (1+x)(2x-1) \frac{d}{dx} [x-1]}{(x-1)^2} \\ &= \frac{(x-1)[(1)(2x-1) + 2(1+x)] - (1+x)(2x-1)(1)}{(x-1)^2} \\ &= \frac{(x-1)(2x-1+2+2x) - (2x-1+2x^2-x)}{(x-1)^2} \\ &= \frac{(x-1)(4x+1) - 2x+1-2x^2+x}{(x-1)^2} \end{aligned}$$

$$= \frac{4x^2 + x - 4x - 1 - 2x + 1 - 2x^2 + x}{(x-1)^2}$$

$$= \frac{2x^2 - 4x}{(x-1)^2}$$

Or

$$y = \frac{(1+x)(2x-1)}{x-1}$$

$$= \frac{2x-1+2x^2-x}{x-1}$$

$$= \frac{2x^2+x-1}{x-1}$$

$$y' = \frac{2x^2 - 4x}{(x-1)^2}$$

$$\frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{(ae-bd)x^2 + 2(af-dd)x + (bf-ce)}{(dx^2 + ex + f)^2}$$

Exercise

Find the derivative to the following functions $y = \frac{4}{2x+1}$

Solution

$$y' = -\frac{8}{(2x+1)^2}$$

$$\left(\frac{ax+b}{cx+d} \right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative to the following functions $y = \frac{2}{(x-1)^3}$

Solution

$$y = 2(x-1)^{-3}$$

$$y' = 2(-3)(x-1)^{-4} (1)$$

$$= -\frac{6}{(x-1)^4}$$

Exercise

Find the derivative to the following functions $y = \sqrt[3]{(x+4)^2}$

Solution

$$y = (x+4)^{2/3}$$

$$y' = \frac{2}{3}(x+4)^{-1/3}$$

$$= \frac{2}{3} \frac{1}{(x+4)^{1/3}}$$

$$= \frac{2}{3 \sqrt[3]{x+4}} \quad \Bigg|$$

Exercise

Find the derivative of $f(x) = \sqrt{2t^2 + 5t + 2}$

Solution

$$f(t) = (2t^2 + 5t + 2)^{1/2}$$

$$U = 2t^2 + 5t + 2 \rightarrow U' = 4t + 5$$

$$f'(t) = \frac{1}{2}(4t+5)(2t^2 + 5t + 2)^{-1/2}$$

$$(U^n)' = nU'U^{n-1}$$

$$= \frac{1}{2} \frac{4t+5}{\sqrt{2t^2 + 5t + 2}} \quad \Bigg|$$

Exercise

Find the derivative of $f(x) = \frac{1}{(x^2 - 3x)^2}$

Solution

$$f'(x) = -\frac{2(2x-3)}{(x^2 - 3x)^3} \quad \Bigg|$$

$$\left(\frac{1}{U^n}\right)' = -\frac{nU'}{U^{n+1}}$$

Exercise

Find the derivative of $y = t^2 \sqrt{t-2}$

Solution

$$y' = 2t\sqrt{t-2} + t^2 \frac{1}{2}(t-2)^{-1/2}$$

$$f = t^2$$

$$f' = 2t$$

$$g = (t-2)^{1/2}$$

$$g' = \frac{1}{2}(t-2)^{-1/2}$$

$$= \left[2t(t-2)^{1/2} + t^2 \frac{1}{2}(t-2)^{-1/2} \right] \frac{2(t-2)^{1/2}}{2(t-2)^{1/2}}$$

$$\begin{aligned}
&= \frac{4t(t-2) + t^2}{2(t-2)^{1/2}} \\
&= \frac{4t^2 - 8t + t^2}{2\sqrt{t-2}} \\
&= \frac{5t^2 - 8t}{2\sqrt{t-2}}
\end{aligned}$$

Exercise

Find the derivative of $y = \left(\frac{-5x+6}{x^2-1} \right)^2$

Solution

$$(U^n)' = nU' U^{n-1} \quad \frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{(ae-bd)x^2 + 2(af-dd)x + (bf-ce)}{(dx^2 + ex + f)^2}$$

$$\begin{aligned}
y' &= 2 \frac{5x^2 - 12x + 5}{(x^2 - 1)^2} \left(\frac{6 - 5x}{x^2 - 1} \right) \\
&= \frac{2(5x^2 - 12x + 5)(6 - 5x)}{(x^2 - 1)^3}
\end{aligned}$$

Exercise

Find the derivative to the following functions $y = x^2 \sqrt{x^2 + 1}$

Solution

$$\begin{aligned}
y &= x^2 (x^2 + 1)^{1/2} \\
y' &= x^2 \frac{d}{dx} [(x^2 + 1)^{1/2}] + (x^2 + 1)^{1/2} \frac{d}{dx} [x^2] \\
&= x^2 \left[\frac{1}{2} (x^2 + 1)^{-1/2} (2x) \right] + (x^2 + 1)^{1/2} [2x] \\
&= x^3 (x^2 + 1)^{-1/2} + 2x (x^2 + 1)^{1/2} \\
&= \frac{(x^2 + 1)^{1/2}}{(x^2 + 1)^{1/2}} \left[x^3 (x^2 + 1)^{-1/2} + 2x (x^2 + 1)^{1/2} \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{x^3(x^2+1)^{-1/2}(x^2+1)^{1/2} + 2x(x^2+1)^{1/2}(x^2+1)^{1/2}}{(x^2+1)^{1/2}} \\
&= \frac{x^3 + 2x(x^2+1)}{(x^2+1)^{1/2}} \\
&= \frac{x^3 + 2x^3 + 2x}{\sqrt{x^2+1}} \\
&= \frac{3x^3 + 2x}{\sqrt{x^2+1}} \\
&= \frac{x(3x^2+2)}{\sqrt{x^2+1}}
\end{aligned}$$

Exercise

Find the derivative to the following functions $y = \left(\frac{x+1}{x-5}\right)^2$

Solution

$$\begin{aligned}
y' &= 2 \frac{-6}{(x-5)^2} \left(\frac{x+1}{x-5}\right) \\
&= -\frac{12(x+1)}{(x-5)^3}
\end{aligned}
\qquad
\begin{aligned}
(U^n)' &= nU' U^{n-1} & \left(\frac{ax+b}{cx+d}\right)' &= \frac{ad-bc}{(cx+d)^2}
\end{aligned}$$

Exercise

Find the derivative to the following functions $y = x^2 \sin x$

Solution

$$\begin{aligned}
y' &= 2x \sin x + x^2 \cos x \\
u &= x^2 & v &= \sin x \\
u' &= 2x & v' &= \cos x
\end{aligned}$$

Exercise

Find the derivative to the following functions $y = \frac{\sin x}{x}$

Solution

$$\begin{aligned}
y' &= \frac{x \cos x - \sin x}{x^2} \\
u &= \sin x & v &= x \\
u' &= \cos x & v' &= 1
\end{aligned}$$

Exercise

Find the derivative to the following functions $y = \frac{\cot x}{1 + \cot x}$

Solution

$$\begin{aligned}y' &= \frac{-\csc^2 x (1 + \cot x) + \csc^2 x \cot x}{(1 + \cot x)^2} \\&= \frac{-\csc^2 x - \csc^2 x \cot x + \csc^2 x \cot x}{(1 + \cot x)^2} \\&= \frac{-\csc^2 x}{(1 + \cot x)^2}\end{aligned}$$

$$u = \cot x \quad v = 1 + \cot x$$

$$u' = -\csc^2 x \quad v' = -\csc^2 x$$

Exercise

Find the derivative to the following functions $y = x^2 \sin x + 2x \cos x - 2 \sin x$

Solution

$$\begin{aligned}y' &= 2x \sin x + x^2 \cos x + 2 \cos x - 2x \sin x - 2 \cos x \\&= x^2 \cos x\end{aligned}$$

Exercise

Find the derivative to the following functions $y = x^3 \sin x \cos x$

Solution

$$\begin{aligned}y' &= (x^3)' \sin x \cos x + x^3 (\sin x)' \cos x + x^3 \sin x (\cos x)' \\&= 3x^2 \sin x \cos x + x^3 \cos^2 x - x^3 \sin^2 x\end{aligned}$$

Exercise

Find the derivative to the following functions $y = \frac{4}{\cos x} + \frac{1}{\tan x}$

Solution

$$\begin{aligned}y' &= \frac{-4 \sin x}{\cos^2 x} - \frac{\sec^2 x}{\tan^2 x} \quad \left(\frac{1}{u}\right)' = -\frac{u'}{u^2} \\&= -4 \frac{\sin x}{\cos x} \frac{1}{\cos x} - \frac{1}{\cos^2 x} \frac{\cos^2 x}{\sin^2 x} \\&= -4 \tan x \sec x - \csc^2 x\end{aligned}$$

Exercise

Find the derivative to the following functions $f(x) = x^2 e^x$

Solution

$$\begin{aligned} f'(x) &= e^x \frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(e^x) \\ &= e^x(2x) + x^2 e^x \\ &= \underline{xe^x(2+x)} \end{aligned}$$

Exercise

Find the derivative to the following functions $f(x) = \frac{e^x + e^{-x}}{2}$

Solution

$$\begin{aligned} f(x) &= \frac{e^x + e^{-x}}{2} \\ &= \frac{1}{2}(e^x + e^{-x}) \\ f'(x) &= \frac{1}{2} \left(\frac{d}{dx}[e^x] + \frac{d}{dx}[e^{-x}] \right) \\ &= \underline{\frac{1}{2}(e^x - e^{-x})} \end{aligned}$$

Exercise

Find the derivative to the following functions $f(x) = \frac{e^x}{x^2}$

Solution

$$\begin{aligned} f'(x) &= \frac{x^2 e^x - e^x(2x)}{x^4} \\ &= \frac{x^2 e^x - 2xe^x}{x^4} \\ &= \frac{xe^x(x-2)}{x^4} \\ &= \underline{\frac{e^x(x-2)}{x^3}} \end{aligned}$$

Exercise

Find the derivative to the following functions $f(x) = x^2 e^x - e^x$

Solution

$$\begin{aligned}
 f'(x) &= e^x \frac{d}{dx}[x^2] + x^2 \frac{d}{dx}[e^x] - \frac{d}{dx}[e^x] \\
 &= e^x(2x) + x^2 e^x - e^x \\
 &= \underline{e^x(x^2 + 2x - 1)}
 \end{aligned}$$

Exercise

Find the derivative to the following functions $f(x) = (1 + 2x)e^{4x}$

Solution

$$\begin{aligned}
 f'(x) &= (2)e^{4x} + (1 + 2x)(4e^{4x}) \\
 &= 2e^{4x} + (1 + 2x)(4e^{4x}) \\
 &= 2e^{4x}(1 + 2(1 + 2x)) \\
 &= 2e^{4x}(1 + 2 + 4x) \\
 &= \underline{2e^{4x}(3 + 4x)}
 \end{aligned}$$

Exercise

Find the derivative to the following functions $y = x^2 e^{5x}$

Solution

$$\begin{aligned}
 y' &= x^2(5e^{5x}) + 2x(e^{5x}) \\
 &= \underline{xe^{5x}(5x + 2)}
 \end{aligned}$$

Exercise

Find the derivative to the following functions $y = e^{x^2+1}\sqrt{5x+2}$

Solution

$$\begin{aligned}
 y &= (2x)e^{x^2+1}\sqrt{5x+2} + e^{x^2+1}\frac{5}{2\sqrt{5x+2}} \\
 &= 2xe^{x^2+1}\sqrt{5x+2}\frac{2\sqrt{5x+2}}{2\sqrt{5x+2}} + \frac{5e^{x^2+1}}{2\sqrt{5x+2}} \\
 &= \frac{4xe^{x^2+1}(5x+2)}{2\sqrt{5x+2}} + \frac{5e^{x^2+1}}{2\sqrt{5x+2}}
 \end{aligned}$$

$$= \frac{20x^2 e^{x^2+1} + 8x e^{x^2+1} + 5e^{x^2+1}}{2\sqrt{5x+2}}$$

$$= \frac{e^{x^2+1}(20x^2 + 8x + 5)}{2\sqrt{5x+2}}$$

Exercise

Find the derivative to the following functions $f(x) = \ln \sqrt[3]{x+1}$

Solution

$$f(x) = \ln(x+1)^{1/3}$$

$$= \frac{1}{3} \ln(x+1)$$

$$f'(x) = \frac{1}{3} \frac{1}{x+1}$$

$$= \frac{1}{3(x+1)}$$

Exercise

Find the derivative to the following functions $f(x) = \ln \left[x^2 \sqrt{x^2+1} \right]$

Solution

$$f(x) = \ln(x^2) + \ln \sqrt{x^2+1} \quad \text{Product Property}$$

$$f(x) = \ln(x^2) + \ln(x^2+1)^{1/2}$$

$$f(x) = 2 \ln x + \frac{1}{2} \ln(x^2+1) \quad \text{Power Property}$$

$$f'(x) = 2 \frac{1}{x} + \frac{1}{2} \frac{2x}{x^2+1}$$

$$= \frac{2}{x} + \frac{x}{x^2+1}$$

Exercise

Find the derivative to the following functions $y = \ln \frac{x^2}{x^2+1}$

Solution

$$y = \ln x^2 - \ln x^2 + 1$$

$$y' = \frac{2x}{x^2} - \frac{2x}{x^2+1}$$

$$\underline{= \frac{2}{x} - \frac{2x}{x^2+1} \quad \Big|}$$

Exercise

Find the derivative to the following functions $y = \ln \frac{1+e^x}{1-e^x}$

Solution

$$y = \ln(1+e^x) - \ln(1-e^x)$$

$$y' = \frac{e^x}{1+e^x} - \frac{-e^x}{1-e^x}$$

$$= \frac{e^x}{1+e^x} + \frac{e^x}{1-e^x}$$

$$= \frac{e^x - e^{2x} + e^x + e^{2x}}{(1+e^x)(1-e^x)}$$

$$\underline{= \frac{2e^x}{(1+e^x)(1-e^x)} \quad \Big|}$$

Exercise

Find the derivative to the following functions $y = x \cdot 3^{x+1}$

Solution

$$y' = 3^{x+1} + x \cdot 3^{x+1} \ln 3$$

$$\underline{= 3^{x+1}(1+x \ln 3) \quad \Big|}$$

Exercise

Find the derivative to the following functions $f(t) = \frac{\log_8(t^{3/2}+1)}{t}$

Solution

$$f' = \frac{\frac{1}{\ln 8} \frac{\frac{3}{2} t^{1/2}}{t^{3/2}+1} \cdot t - \log_8(t^{3/2}+1)}{t^2}$$

$$= \frac{\frac{1}{\ln 8} \frac{\frac{3}{2} t^{3/2}}{t^{3/2}+1} - \log_8(t^{3/2}+1)}{t^2} \cdot \frac{2 \ln 8 (t^{3/2}+1)}{2 \ln 8 (t^{3/2}+1)}$$

$$= \frac{3t^{3/2} - 2(t^{3/2} + 1)(\ln 8) \log_8(t^{3/2} + 1)}{t^2(t^{3/2} + 1) \ln 8}$$

Exercise

Find the derivative of $y = \frac{2x-3}{x+1}$

Solution

$$y' = \frac{2+3}{(x+1)^2}$$

$$= \frac{5}{(x+1)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \frac{3x}{3x-2}$

Solution

$$y' = \frac{-6}{(3x-2)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \frac{x-3}{2x+5}$

Solution

$$y' = \frac{11}{(2x+5)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \frac{x^2-4}{5x^2-2}$

Solution

$$y' = \frac{2(-2+20)x}{(5x^2-2)^2}$$

$$= \frac{36x}{(5x^2-2)^2}$$

$$\left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{(cx^n+d)^2}$$

Exercise

Find the derivative of $y = \frac{3x^2 - 4}{2x^2 - 1}$

Solution

$$y' = \frac{2(-3+8)x}{(2x^2-1)^2}$$
$$= \frac{10x}{(2x^2-1)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2}$$

Exercise

Find the derivative of $y = \frac{x^2 - 4x + 1}{5x^2 - 2x - 1}$

Solution

$$y' = \frac{\begin{vmatrix} 1 & -4 \\ 5 & -2 \end{vmatrix} x^2 + 2 \begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} x + \begin{vmatrix} -4 & 1 \\ -2 & -1 \end{vmatrix}}{(5x^2 - 2x - 1)^2}$$
$$= \frac{18x^2 - 12x + 6}{(5x^2 - 2x - 1)^2}$$

$$\frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

Exercise

Find the derivative of $y = \frac{3x^2 - 4x + 2}{2x^2 + x - 1}$

Solution

$$y' = \frac{\begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} x^2 + 2 \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} x + \begin{vmatrix} -4 & 2 \\ 1 & -1 \end{vmatrix}}{(2x^2 + x - 1)^2}$$
$$= \frac{11x^2 - 14x + 6}{(2x^2 + x - 1)^2}$$

$$\frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

Exercise

Find the derivative of $f(x) = \frac{x^2 + 3}{(2x - 1)^3 (3x + 1)^4}$

Solution

$$f(x) = (x^2 + 3)(2x - 1)^{-3} (3x + 1)^{-4} \quad \left(U^m V^n W^p \right)' = U^{m-1} V^{n-1} W^{p-1} (mU'VW + nUV'W + pUVW')$$

$$f'(x) = (2x - 1)^{-4} (3x + 1)^{-5}$$

$$\begin{aligned} & \left[2x(2x - 1)(3x + 1) - 6(x^2 + 3)(3x + 1) - 12(x^2 + 3)(2x - 1) \right] \\ &= \frac{1}{(2x - 1)^4 (3x + 1)^5} \left((4x^2 - 2x)(3x + 1) - 6(3x^3 + x^2 + 9x + 3) - 12(2x^3 - x^2 + 6x - 3) \right) \end{aligned}$$

$$x^3 \quad 12 - 18 - 24$$

$$x^2 \quad 4 - 6 - 6 + 12$$

$$x \quad -2 - 54 - 72$$

$$x^0 \quad -18 + 36$$

$$f'(x) = \frac{-30x^3 + 4x^2 - 128x + 18}{(2x - 1)^4 (3x + 1)^5}$$

Exercise

Find the derivative of $f(x) = \frac{(x^3 - 3x)^3 (x^2 + 4x)^4}{(x^2 + 4x + 1)^2}$

Solution

$$f(x) = (x^3 - 3x)^3 (x^2 + 4x)^4 (x^2 + 4x + 1)^{-2}$$

$$\left(U^m V^n W^p \right)' = U^{m-1} V^{n-1} W^{p-1} (mU'VW + nUV'W + pUVW')$$

$$\begin{aligned} f'(x) &= (x^3 - 3x)^2 (x^2 + 4x)^3 (x^2 + 4x + 1)^{-3} \left[3(3x^2 - 3)(x^2 + 4x)(x^2 + 4x + 1) \right. \\ &\quad \left. + 4(x^3 - 3x)(2x + 4)(x^2 + 4x + 1) - 2(x^3 - 3x)(x^2 + 4x)(2x + 4) \right] \end{aligned}$$

$$\begin{aligned} f'(x) &= (x^3 - 3x)^2 (x^2 + 4x)^3 (x^2 + 4x + 1)^{-3} \left[(9x^2 - 9)(x^4 + 8x^3 + 9x^2 + 4x) \right. \\ &\quad \left. + (4x^3 - 12x)(2x^3 + 12x^2 + 18x + 4) + (-2x^3 + 6x)(2x^3 + 12x^2 + 16x) \right] \end{aligned}$$

x^6	9+8-4
x^5	72+48-24
x^4	81-9+72-24-32+12
x^3	36-72+16-144+72
x^2	-81-216+96
x^1	-36-48

$$f'(x) = \frac{\left(13x^6 + 96x^5 + 100x^4 - 92x^3 - 201x^2 - 84x\right)\left(x^3 - 3x\right)^2\left(x^2 + 4x\right)^3}{\left(x^2 + 4x + 1\right)^3}$$

Solution Section R.2 – Integration

Exercise

Find each indefinite integral. $\int \frac{x+2}{\sqrt{x}} dx$

Solution

$$\begin{aligned}\int \frac{x+2}{\sqrt{x}} dx &= \int \left[\frac{x}{x^{1/2}} + \frac{2}{x^{1/2}} \right] dx \\&= \int \frac{x}{x^{1/2}} dx + \int \frac{2}{x^{1/2}} dx \\&= \int x^{1/2} dx + 2 \int x^{-1/2} dx \\&= \frac{x^{3/2}}{3/2} + 2 \frac{x^{1/2}}{1/2} + C \\&= \frac{2}{3} x^{3/2} + 4x^{1/2} + C\end{aligned}$$

Exercise

Find each indefinite integral $\int 4y^{-3} dy$

Solution

$$\begin{aligned}\int 4y^{-3} dy &= 4 \frac{y^{-2}}{-2} + C \\&= -\frac{2}{y^2} + C\end{aligned}$$

Exercise

Find each indefinite integral $\int (x^3 - 4x + 2) dx$

Solution

$$\int (x^3 - 4x + 2) dx = \frac{1}{4} x^4 - 2x^2 + 2x + C$$

Exercise

Find each indefinite integral $\int \left(\sqrt[4]{x^3} + 1 \right) dx$

Solution

$$\int \left(x^{3/4} + 1 \right) dx = \underline{\frac{4}{7} x^{7/4} + x + C}$$

Exercise

Find each indefinite integral $\int \sqrt{x}(x+1) dx$

Solution

$$\begin{aligned} \int x^{1/2}(x+1) dx &= \int \left(x^{3/2} + x^{1/2} \right) dx \\ &= \underline{\frac{2}{5} x^{5/2} + \frac{2}{3} x^{3/2} + C} \end{aligned}$$

Exercise

Find each indefinite integral $\int (1+3t)t^2 dt$

Solution

$$\int \left(t^2 + 3t^3 \right) dt = \underline{\frac{1}{3} t^3 + \frac{3}{4} t^4 + C}$$

Exercise

Find each indefinite integral $\int \frac{x^2-5}{x^2} dx$

Solution

$$\begin{aligned} \int \frac{x^2-5}{x^2} dx &= \int \left(1 - \frac{5}{x^2} \right) dx \\ &= \int \left(1 - 5x^{-2} \right) dx \\ &= \underline{x + \frac{5}{x} + C} \end{aligned}$$

Exercise

Find each indefinite integral $\int (-40x + 250) dx$

Solution

$$\int (-40x + 250) dx = \underline{-20x^2 + 250x + C}$$

Exercise

Find each indefinite integral $\int (7 - 3x - 3x^2)(2x + 1) dx$

Solution

$$\begin{aligned} \int (7 - 3x - 3x^2)(2x + 1) dx &= \int (14x + 7 - 6x^2 - 3x - 6x^3 - 3x^2) dx \\ &= \int (-6x^3 - 9x^2 + 11x + 7) dx \\ &= \underline{-\frac{3}{2}x^4 - 3x^3 + \frac{11}{2}x^2 + 7x + C} \end{aligned}$$

Exercise

Evaluate the integral $\int xe^{2x} dx$

Solution

Let: $u = x \Rightarrow du = dx$

$$dv = e^{2x} dx \Rightarrow v = \int dv = \int e^{2x} dx = \frac{1}{2}e^{2x}$$

$$\int u dv = uv - \int v du$$

$$\begin{aligned} \int xe^{2x} dx &= \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x} dx \\ &= \frac{1}{2}xe^{2x} - \frac{1}{2} \frac{1}{2}e^{2x} + C \\ &= \underline{\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C} \end{aligned}$$

Exercise

Evaluate the integral $\int x \ln x dx$

Solution

Let: $u = \ln x \Rightarrow du = \frac{1}{x} dx$

$dv = x dx \Rightarrow v = \int dv = \int x dx = \frac{1}{2} x^2$

$$\begin{aligned} \int x \ln x dx &= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x^2 \frac{dx}{x} \\ &= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x dx \\ &= \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C \end{aligned}$$

Exercise

Evaluate the integral $\int x^2 \sin x dx$

Solution

$$\int x^2 \sin x dx = \underline{-x^2 \cos x - 2x \sin x + 2 \cos x + C}$$

$\int \sin x$		
x^2	$(+)$	$-\cos x$
$2x$	$(-)$	$-\sin x$
2	$(+)$	$\cos x$

Exercise

Evaluate the integral $\int (x^2 - 2x + 1) e^{2x} dx$

Solution

$$\begin{aligned} \int (x^2 - 2x + 1) e^{2x} dx &= \frac{1}{2} (x^2 - 2x + 1) e^{2x} - \frac{1}{4} (2x - 2) e^{2x} + \frac{1}{8} (2) e^{2x} + C \\ &= \left(\frac{1}{2} x^2 - x + \frac{1}{2} - \frac{1}{2} x + \frac{1}{2} + \frac{1}{4} \right) e^{2x} + C \\ &= \underline{\left(\frac{1}{2} x^2 - \frac{3}{2} x + \frac{5}{4} \right) e^{2x} + C} \end{aligned}$$

$\int e^{2x}$		
$+$	$x^2 - 2x + 1$	$\frac{1}{2} e^{2x}$
$-$	$2x - 2$	$\frac{1}{4} e^{2x}$
$+$	2	$\frac{1}{8} e^{2x}$

Exercise

Evaluate the integral $\int e^{2x} \cos 3x dx$

Solution

		$\int \cos 3x$
+	e^{2x}	$\frac{1}{3} \sin 3x$
-	$\frac{1}{2} e^{2x}$	$-\frac{1}{9} \cos 3x$
+	$\frac{1}{4} e^{2x}$	

$$\int e^{2x} \cos 3x dx = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x - \frac{9}{4} \int e^{2x} \cos 3x dx$$

$$\int e^{2x} \cos 3x dx + \frac{9}{4} \int e^{2x} \cos 3x dx = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x + C_1$$

$$\frac{13}{4} \int e^{2x} \cos 3x dx = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x + C_1$$

$$\frac{4}{13} \frac{13}{4} \int e^{2x} \cos 3x dx = \frac{4}{13} \frac{1}{2} e^{2x} \cos 3x + \frac{4}{13} \frac{3}{4} e^{2x} \sin 3x + \frac{4}{13} C_1$$

$$\int e^{2x} \cos 3x dx = \underline{\frac{e^{2x}}{13} (2 \cos 3x + 3 \sin 3x) + C}$$

Exercise

Find the general solution of the differential equation $y' = 2t + 3$

Solution

$$\int dy = \int (2t + 3) dt$$

$$\underline{y(t) = t^2 + 3t + C}$$

Exercise

Find the general solution of the differential equation $y' = 3t^2 + 2t + 3$

Solution

$$\int dy = \int (3t^2 + 2t + 3) dt$$

$$\underline{y(t) = t^3 + t^2 + 3t + C}$$

Exercise

Find the general solution of the differential equation $y' = \sin 2t + 2 \cos 3t$

Solution

$$\int dy = \int (\sin 2t + 2 \cos 3t) dt$$

$$\underline{y(t) = -\frac{1}{2} \cos 2t + \frac{2}{3} \sin 3t + C}$$

Exercise

Find the general solution of the differential equation: $y' = x^3(3x^4 + 1)^2$

Solution

$$\int x^3(3x^4 + 1)^2 dx$$

$$u = 3x^4 + 1 \Rightarrow du = 12x^3 dx$$

$$\begin{aligned} \int x^3(3x^4 + 1)^2 dx &= \int \frac{1}{12} u^2 du \\ &= \frac{1}{12} \frac{(3x^4 + 1)^3}{3} + C \\ &= \frac{1}{36} (3x^4 + 1)^3 + C \end{aligned}$$

$$\underline{y(x) = \frac{1}{36} (3x^4 + 1)^3 + C}$$

Exercise

Find the general solution of the differential equation: $y' = 5x\sqrt{x^2 - 1}$

Solution

$$\int 5x\sqrt{x^2 - 1} dx$$

$$u = x^2 - 1 \Rightarrow du = 2x dx$$

$$\int 5x(x^2 - 1)^{1/2} dx$$

$$= 5 \int u^{1/2} \frac{1}{2} du$$

Substitute for x and dx

$$= \frac{5}{2} \int u^{1/2} du$$

$$\begin{aligned}
&= \frac{5}{2} \frac{u^{3/2}}{3/2} + C \\
&= \frac{5}{3} u^{3/2} + C \\
&= \frac{5}{3} (x^2 - 1)^{3/2} + C
\end{aligned}$$

Exercise

Find the general solution of the differential equation: $y' = x\sqrt{x^2 + 4}$

Solution

$$\begin{aligned}
u = x^2 + 4 &\Rightarrow du = 2x dx \\
x dx &= \frac{1}{2} du \\
\int \sqrt{x^2 + 4} \, x dx &= \int u^{1/2} \frac{1}{2} du \\
&= \frac{1}{2} \frac{u^{3/2}}{3/2} + C \\
&= \frac{1}{3} u^{3/2} + C \\
&= \frac{1}{3} (x^2 + 4)^{3/2} + C \\
y(x) &= \frac{1}{3} (x^2 + 4)^{3/2} + C
\end{aligned}$$

Exercise

Find the general solution of the differential equation: $y' = (2x + 1)e^{x^2 + x}$

Solution

$$\begin{aligned}
\int dy &= \int (2x + 1)e^{x^2 + x} dx & u = x^2 + x &\Rightarrow du = (2x + 1) dx \\
\int dy &= \int e^u du \\
y &= e^u + C \\
y(x) &= e^{x^2 + x} + C
\end{aligned}$$

Exercise

Find the general solution of the differential equation: $y' = \frac{1}{6x-5}$

Solution

$$\int dy = \int \frac{1}{6x-5} dx$$

$$\int dy = \frac{1}{6} \int \frac{1}{6x-5} d(6x-5)$$

$$\underline{y(x) = \frac{1}{6} \ln|6x-5| + C}$$

Exercise

Find the general solution of the differential equation: $y' = \frac{x^2+2x+3}{x^3+3x^2+9x+1}$

Solution

$$\int dy = \int \frac{x^2+2x+3}{x^3+3x^2+9x+1} dx$$

$$u = x^3 + 3x^2 + 9x + 1 \quad du = 3(x^2 + 2x + 3) dx$$

$$\int dy = \frac{1}{3} \int \frac{du}{u}$$

$$y(x) = \frac{1}{3} \ln|u| + C$$

$$\underline{y(x) = \frac{1}{3} \ln|x^3 + 3x^2 + 9x + 1| + C}$$

Exercise

Find the general solution of the differential equation: $y' = \frac{1}{x(\ln x)^2}$

Solution

$$\int dy = \int \frac{1}{x(\ln x)^2} dx$$

$$u = \ln x \quad du = \frac{dx}{x}$$

$$\int dy = \int \frac{1}{u^2} du$$

$$y = -\frac{1}{u} + C$$

$$\underline{y(x) = -\frac{1}{\ln x} + C}$$

Exercise

Evaluate the integrals $\int_{-2}^2 (x^3 - 2x + 3) dx$

Solution

$$\begin{aligned}\int_{-2}^2 (x^3 - 2x + 3) dx &= \left[\frac{x^4}{4} - x^2 + 3x \right]_{-2}^2 \\ &= \left(\frac{(2)^4}{4} - (2)^2 + 3(2) \right) - \left(\frac{(-2)^4}{4} - (-2)^2 + 3(-2) \right) \\ &= \underline{12}\end{aligned}$$

Exercise

Evaluate the integrals $\int_0^1 (x^2 + \sqrt{x}) dx$

Solution

$$\begin{aligned}\int_0^1 (x^2 + \sqrt{x}) dx &= \left[\frac{x^3}{3} + \frac{2}{3} x^{3/2} \right]_0^1 \\ &= \left(\frac{(1)^3}{3} + \frac{2}{3} (1)^{3/2} \right) - 0 \\ &= \underline{1}\end{aligned}$$

Exercise

Evaluate the integrals $\int_0^{\pi/3} 4 \sec u \tan u \, du$

Solution

$$\begin{aligned}\int_0^{\pi/3} 4 \sec u \tan u \, du &= 4 \sec u \Big|_0^{\pi/3} \\ &= 4 \left(\sec \frac{\pi}{3} - \sec 0 \right) \\ &= 4(2 - 1) \\ &= \underline{4}\end{aligned}$$

Exercise

Evaluate the integrals $\int_{\pi/4}^{3\pi/4} \csc \theta \cot \theta d\theta$

Solution

$$\begin{aligned}\int_{\pi/4}^{3\pi/4} \csc \theta \cot \theta d\theta &= -\csc \theta \Big|_{\pi/4}^{3\pi/4} \\&= -\left(\csc \frac{3\pi}{4} - \csc \frac{\pi}{4}\right) \\&= -(\sqrt{2} - \sqrt{2}) \\&= \underline{0}\end{aligned}$$

Exercise

Evaluate the integrals $\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt$

Solution

$$\begin{aligned}\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt &= \int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \pi t^{-2}\right) dt \\&= \left[4\tan t - \pi t^{-1}\right]_{-\pi/3}^{-\pi/4} \\&= \left(4\tan\left(-\frac{\pi}{4}\right) - \pi\left(-\frac{4}{\pi}\right)\right) - \left(4\tan\left(-\frac{\pi}{3}\right) - \pi\left(-\frac{3}{\pi}\right)\right) \\&= (4(-1) + 4) - (4(-\sqrt{3}) + 3) \\&= -(-4\sqrt{3} + 3) \\&= \underline{4\sqrt{3} - 3}\end{aligned}$$

Exercise

Evaluate the integrals $\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} dy$

Solution

$$\begin{aligned}\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} dy &= \int_{-3}^{-1} \left(\frac{y^5}{y^3} - \frac{2y}{y^3}\right) dy \\&= \int_{-3}^{-1} (y^2 - 2y^{-2}) dy\end{aligned}$$

$$\begin{aligned}
&= \left[\frac{1}{3} y^3 + 2y^{-1} \right]_{-3}^{-1} \\
&= \left(\frac{1}{3} (-1)^3 + \frac{2}{-1} \right) - \left(\frac{1}{3} (-3)^3 + \frac{2}{-3} \right) \\
&= \frac{22}{3}
\end{aligned}$$

Exercise

Evaluate the integrals $\int_1^8 \frac{(x^{1/3} + 1)(2 - x^{2/3})}{x^{1/3}} dx$

Solution

$$\begin{aligned}
\int_1^8 \frac{(x^{1/3} + 1)(2 - x^{2/3})}{x^{1/3}} dx &= \int_1^8 \frac{2x^{1/3} - x + 2 - x^{2/3}}{x^{1/3}} dx \\
&= \int_1^8 (2 - x^{2/3} + 2x^{-1/3} - x^{1/3}) dx \\
&= \left[2x - \frac{3}{5} x^{5/3} + 3x^{2/3} - \frac{3}{4} x^{4/3} \right]_1^8 \\
&= \left(2(8) - \frac{3}{5} (8)^{5/3} + 3(8)^{2/3} - \frac{3}{4} (8)^{4/3} \right) - \left(2(1) - \frac{3}{5} (1)^{5/3} + 3(1)^{2/3} - \frac{3}{4} (1)^{4/3} \right) \\
&= \left(-\frac{16}{5} \right) - \left(\frac{73}{20} \right) \\
&= -\frac{137}{20}
\end{aligned}$$

Exercise

Evaluate: $\int_0^1 (2t + 3)^3 dt$

Solution

$$\begin{aligned}
\int_0^1 (2t + 3)^3 dt &= \int_0^1 u^3 \frac{1}{2} du & u = 2t + 3 \Rightarrow du = 2dt \rightarrow \frac{du}{2} = dt \\
&= \frac{1}{2} \int_0^1 u^3 du \\
&= \frac{1}{2} \frac{u^4}{4} \Big|_0^1 \\
&= \frac{1}{8} (2t + 3)^4 \Big|_0^1
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{8} \left[(2(1) + 3)^4 - (2(0) + 3)^4 \right] \\
&= \frac{1}{8} (5^4 - 3^4) \\
&= 68
\end{aligned}$$

Exercise

Evaluate the integral $\int_{-1}^1 r\sqrt{1-r^2} \, dr$

Solution

$$\text{Let } u = 1 - r^2 \Rightarrow du = -2rdr \rightarrow -\frac{1}{2}du = rdr$$

$$\begin{aligned}
\int_{-1}^1 r\sqrt{1-r^2} \, dr &= \int_{-1}^1 u^{1/2} \left(-\frac{1}{2}du\right) \\
&= -\frac{1}{2} \frac{2}{3} u^{3/2} \Big|_{-1}^1 \\
&= -\frac{1}{3} \left[(1-r^2)^{3/2} \right]_{-1}^1 \\
&= -\frac{1}{3} \left[(1-(1)^2)^{3/2} - (1-(-1)^2)^{3/2} \right] \\
&= -\frac{1}{3} (0-0) \\
&= 0
\end{aligned}$$

Exercise

Find the general solution of $F'(x) = 4x + 2$, and find the particular solution that satisfies the initial condition $F(1) = 8$.

Solution

$$\begin{aligned}
F(x) &= \int (4x + 2) dx \\
&= 2x^2 + 2x + C
\end{aligned}$$

$$F(x) = 2(1)^2 + 2(1) + C = 8$$

$$2 + 2 + C = 8$$

$$C = 4$$

$$F(x) = 2x^2 + 2x + 4$$

Exercise

Find the general solution of the differential equation: $y' = t \cos 3t$

Solution

$$u = t \rightarrow du = dt$$

$$dv = \cos 3t \rightarrow v = \frac{1}{3} \sin 3t$$

$$\begin{aligned} y &= \frac{1}{3} t \sin 3t - \frac{1}{3} \int \sin 3t dt \\ &= \frac{1}{3} t \sin 3t - \frac{1}{3} \frac{1}{3} \cos 3t + C \end{aligned}$$

$$\underline{y(t) = \frac{1}{3} t \sin 3t - \frac{1}{9} \cos 3t + C}$$

Exercise

A ball is thrown into the air from an initial height of 6 m with an initial velocity of 120 m/s. What will be the maximum height of the ball and at what time will this event occur?

Solution

$$\frac{dv}{dt} = -g \Rightarrow dv = -g dt$$

$$v(t) = -gt + C_1$$

$$v(t=0) = -g(0) + C_1 = 120$$

$$C_1 = 120$$

$$v(t) = -9.8t + 120$$

$$\frac{dx}{dt} = v \Rightarrow dx = v dt$$

$$x(t) = \int (-9.8t + 120) dt$$

$$= -4.9t^2 + 120t + C_2$$

$$x(0) = -4.9(0)^2 + 120(0) + C_2 = 6$$

$$C_2 = 6$$

$$x(t) = -4.9t^2 + 120t + 6$$

$$v(t) = -9.8t + 120 = 0 \rightarrow t = \frac{120}{9.8} = 12.24 \text{ sec}$$

$$x(t=12.24) = -4.9(12.24)^2 + 120(12.24) + 6$$

$$\underline{= 740.69 \text{ m}}$$

Exercise

Derive the position function if a ball is thrown upward with initial velocity of 32 *ft* per *second* from an initial height of 48 *ft*. When does the ball hit the ground? With what velocity does the ball hit the ground?

Solution

$$s(t) = -16t^2 + 32t + 48$$

$$s(0) = 48$$

$$s''(t) = -32$$

$$\begin{aligned} s'(t) &= \int -32 dt \quad s'(0) = 32 \\ &= -32t + C_1 \end{aligned}$$

$$\begin{aligned} s'(0) &= -32(0) + C_1 = 32 \\ &\Rightarrow C_1 = 32 \end{aligned}$$

$$s'(t) = -32t + 32$$

$$\begin{aligned} s(t) &= \int (-32t + 32) dt \\ &= -32 \frac{t^2}{2} + 32t + C_2 \end{aligned}$$

$$s(0) = -32 \frac{0^2}{2} + 32(0) + C_2 = 48 \quad \Rightarrow C_2 = 48$$

$$s(t) = -16t^2 + 32t + 48$$

$$s(t) = -16t^2 + 32t + 48 = 0$$

$$-t^2 + 2t + 3 = 0 \Rightarrow t = -1, t = 3$$

The ball hits the ground in **3** seconds

The velocity: $v(t) = s'(t) = -32t + 32$

$$v(t = 3) = -32(3) + 32 = \underline{-64 \text{ ft/sec}^2}$$

