

**Exercise**

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ :  $f(x) = e^{2x}$ ,  $a = 0$

**Solution**

$$f(x) = e^{2x} \rightarrow f(0) = e^{2(0)} = 1$$

$$f'(x) = 2e^{2x} \rightarrow f'(0) = 2$$

$$f''(x) = 4e^{2x} \rightarrow f''(0) = 4$$

$$f'''(x) = 8e^{2x} \rightarrow f'''(0) = 8$$

$$P_0(x) = f(0) = 1$$

$$\begin{aligned} P_1(x) &= f(0) + f'(0)(x-0) \\ &= 1 + 2x \end{aligned}$$

$$\begin{aligned} P_2(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 \\ &= 1 + 2x + 2x^2 \end{aligned}$$

$$\begin{aligned} P_3(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 \\ &= 1 + 2x + 2x^2 + \frac{4}{3}x^3 \end{aligned}$$

**Exercise**

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ :  $f(x) = \sin x$ ,  $a = 0$

**Solution**

$$f(x) = \sin x \rightarrow f(0) = 0$$

$$f'(x) = \cos x \rightarrow f'(0) = 1$$

$$f''(x) = -\sin x \rightarrow f''(0) = 0$$

$$f'''(x) = -\cos x \rightarrow f'''(0) = -1$$

$$P_0(x) = f(0) = 0$$

$$\begin{aligned} P_1(x) &= f(0) + f'(0)(x-0) \\ &= x \end{aligned}$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$= x$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$$= \underline{x - \frac{1}{6}x^3}$$

### Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ :  $f(x) = \ln(1+x)$ ,  $a = 0$

### Solution

$$f(x) = \ln(1+x) \rightarrow f(0) = 0$$

$$f'(x) = \frac{1}{1+x} \rightarrow f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} \rightarrow f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \rightarrow f'''(0) = 2$$

$$P_0(x) = f(0)$$

$$= \underline{0}$$

$$P_1(x) = f(0) + f'(0)(x-0)$$

$$= \underline{x}$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$= \underline{x - \frac{1}{2}x^2}$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$$= \underline{x - \frac{1}{2}x^2 + \frac{1}{3}x^3}$$

### Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ :  $f(x) = \frac{1}{x+2}$ ,  $a = 0$

### Solution

$$f(x) = (x+2)^{-1} \rightarrow f(0) = \frac{1}{2}$$

$$f'(x) = -(x+2)^{-2} \rightarrow f'(0) = -\frac{1}{4}$$

$$f''(x) = 2(x+2)^{-3} \rightarrow f''(0) = \frac{1}{4}$$

$$f'''(x) = -6(x+2)^{-4} \rightarrow f'''(0) = -\frac{3}{8}$$

$$P_0(x) = f(0)$$

$$= \frac{1}{2} \Big|$$

$$P_1(x) = f(0) + f'(0)(x-0)$$

$$= \frac{1}{2} - \frac{1}{4}x \Big|$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$= \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 \Big|$$

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$$= \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3 \Big|$$

### Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ :  $f(x) = \sqrt{1-x}$ ,  $a = 0$

### Solution

$$f(x) = (1-x)^{1/2} \rightarrow f(0) = 1$$

$$f'(x) = -\frac{1}{2}(1-x)^{-1/2} \rightarrow f'(0) = -\frac{1}{2}$$

$$f''(x) = -\frac{1}{4}(1-x)^{-3/2} \rightarrow f''(0) = -\frac{1}{4}$$

$$f'''(x) = -\frac{3}{8}(1-x)^{-5/2} \rightarrow f'''(0) = -\frac{3}{8}$$

$$P_0(x) = f(0)$$

$$= 1 \Big|$$

$$P_1(x) = f(0) + f'(0)(x-0)$$

$$= 1 - \frac{1}{2}x \Big|$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$\left. = 1 - \frac{1}{2}x - \frac{1}{8}x^2 \right|$$

$$\begin{aligned} P_3(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 \\ &= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 \end{aligned}$$

### Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ :  $f(x) = x^3$ ,  $a = 1$

### Solution

$$f(x) = x^3 \rightarrow f(1) = 1$$

$$f'(x) = 3x^2 \rightarrow f'(1) = 3$$

$$f''(x) = 6x \rightarrow f''(1) = 6$$

$$f'''(x) = 6 \rightarrow f'''(1) = 6$$

$$P_0(x) = f(a)$$

$$\left. = 1 \right|$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$\left. = 1 + 3(x-1) \right|$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\left. = 1 + 3(x-1) + 3(x-1)^2 \right|$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$\left. = 1 + 3(x-1) + 3(x-1)^2 + (x-1)^3 \right|$$

### Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ :  $f(x) = 8\sqrt{x}$ ,  $a = 1$

### Solution

$$f(x) = 8x^{1/2} \rightarrow f(1) = 8$$

$$f'(x) = 4x^{-1/2} \rightarrow f'(1) = 4$$

$$f''(x) = -2x^{-3/2} \rightarrow f''(1) = -2$$

$$f'''(x) = 3x^{-5/2} \rightarrow f'''(1) = 3$$

$$P_0(x) = f(a)$$

$$= 8$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$= 8 + 4(x-1)$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$= 8 + 4(x-1) - (x-1)^2$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= 8 + 4(x-1) - (x-1)^2 + 3(x-1)^3$$

### Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ :  $f(x) = \sin x$ ,  $a = \frac{\pi}{4}$

### Solution

$$f(x) = \sin x \rightarrow f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f'(x) = \cos x \rightarrow f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f''(x) = -\sin x \rightarrow f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f'''(x) = -\cos x \rightarrow f'''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$P_0(x) = f(a)$$

$$= \frac{\sqrt{2}}{2}$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3$$

### Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ :  $f(x) = \cos x$ ,  $a = \frac{\pi}{6}$

### Solution

$$f(x) = \cos x \rightarrow f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$f'(x) = -\sin x \rightarrow f'\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

$$f''(x) = -\cos x \rightarrow f''\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$f'''(x) = \sin x \rightarrow f'''\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$P_0(x) = f(a)$$

$$= \frac{\sqrt{3}}{2}$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right)$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{6}\right)^2$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{6}\right)^2 + \frac{1}{12}\left(x - \frac{\pi}{6}\right)^3$$

### Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ :  $f(x) = \sqrt{x}$ ,  $a = 9$

### Solution

$$f(x) = x^{1/2} \rightarrow f(9) = 3$$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \rightarrow f'(9) = \frac{1}{6}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} = -\frac{1}{4x\sqrt{x}} \rightarrow f''(9) = -\frac{1}{4 \times 3^3}$$

$$f'''(x) = \frac{3}{8}x^{-5/2} = \frac{3}{8x^2\sqrt{x}} \rightarrow f'''(9) = \frac{1}{2^3 \times 3^4}$$

$$P_0(x) = f(a)$$

$$= 3$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$= 3 + \frac{1}{6}(x-9)$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$= 3 + \frac{1}{2 \cdot 3}(x-9) - \frac{1}{2^3 \cdot 3^3}(x-9)^2$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= 3 + \frac{1}{2 \cdot 3}(x-9) - \frac{1}{2^2 \cdot 3^3}(x-9)^2 + \frac{1}{2^4 \cdot 3^5}(x-9)^3$$

### Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ :  $f(x) = \sqrt[3]{x}$ ,  $a = 8$

### Solution

$$f(x) = x^{1/3} \rightarrow f(8) = 2$$

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} \rightarrow f'(8) = \frac{1}{2^2 \times 3}$$

$$f''(x) = -\frac{2}{9}x^{-5/3} = -\frac{2}{3^2 x^{5/3}} \rightarrow f''(8) = -\frac{1}{3^2 \times 2^4}$$

$$f'''(x) = \frac{10}{3^3}x^{-8/3} = \frac{2 \cdot 5}{3^3 x^{8/3}} \rightarrow f'''(8) = \frac{5}{2^7 \times 3^3}$$

$$P_0(x) = f(a)$$

$$= 2$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$= 2 + \frac{1}{2^2 \cdot 3}(x-8)$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$= 2 + \frac{1}{2^2 \cdot 3}(x-8) - \frac{1}{2^5 \cdot 3^2}(x-8)^2$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= 2 + \frac{1}{2^2 \cdot 3}(x-8) - \frac{1}{2^5 \cdot 3^2}(x-8)^2 + \frac{1}{2^8 \cdot 3^4}(x-8)^3$$

### Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ :  $f(x) = \ln x$ ,  $a = e$

### Solution

$$f(x) = \ln x \rightarrow f(e) = 1$$

$$f'(x) = \frac{1}{x} \rightarrow f'(e) = \frac{1}{e}$$

$$f''(x) = -\frac{1}{x^2} \rightarrow f''(e) = -\frac{1}{e^2}$$

$$f'''(x) = \frac{2}{x^3} \rightarrow f'''(e) = \frac{2}{e^3}$$

$$P_0(x) = f(a)$$

$$= 1$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$= 1 + \frac{1}{e}(x-e)$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$= 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3$$

### Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ :  $f(x) = \sqrt[4]{x}$ ,  $a = 8$

### Solution

$$f(x) = x^{1/4} \rightarrow f(8) = \sqrt[4]{8}$$

$$f'(x) = \frac{1}{4}x^{-3/4} = \frac{1}{4x^{3/4}} \rightarrow f'(8 = 2^3) = \frac{1}{2^2 \times 2^{9/4}} = \frac{1}{2^4 \sqrt[4]{2}}$$

$$f''(x) = -\frac{3}{16}x^{-7/4} = -\frac{3}{2^4 x^{7/4}} \rightarrow f''(8) = -\frac{3}{2^4 2^{21/4}} = -\frac{3}{2^9 \sqrt[4]{2}}$$

$$f'''(x) = \frac{21}{2^6}x^{-11/4} = \frac{21}{2^6 x^{11/4}} \rightarrow f'''(8) = \frac{21}{2^6 \times 2^{33/4}} = \frac{21}{2^{14} \sqrt[4]{2}}$$

$$P_0(x) = f(a)$$

$$= \sqrt[4]{8}$$



$$P_1(x) = f(a) + f'(a)(x-a)$$

$$= \sqrt[4]{8} + \frac{1}{2^4 \cdot \sqrt[4]{2}}(x-8)$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$= \sqrt[4]{8} + \frac{1}{2^4 \cdot \sqrt[4]{2}}(x-8) - \frac{3}{2^{10} \cdot \sqrt[4]{2}}(x-8)^2$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= \sqrt[4]{8} + \frac{1}{2^4 \cdot \sqrt[4]{2}}(x-8) - \frac{3}{2^{10} \cdot \sqrt[4]{2}}(x-8)^2 + \frac{7}{2^{15} \cdot \sqrt[4]{2}}(x-8)^3$$

### Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ :  $f(x) = \tan^{-1}x + x^2 + 1$ ,  $a = 1$

### Solution

$$f(x) = \tan^{-1}x + x^2 + 1 \rightarrow f(1) = \frac{\pi}{4} + 2$$

$$f'(x) = \frac{1}{x^2 + 1} + 2x \rightarrow f'(1) = \frac{5}{2}$$

$$f''(x) = -\frac{2x}{(x^2 + 1)^2} + 2 \rightarrow f''(1) = -\frac{1}{2} + 2 = \frac{3}{2}$$

$$f'''(x) = -\frac{2x^2 + 2 - 8x^2}{(x^2 + 1)^3} = -\frac{2 - 2x^2}{(x^2 + 1)^3} \rightarrow f'''(1) = 0$$

$$(U^n V^m)' = U^{n-1} V^{m-1} (nUV + mUV')$$

$$P_0(x) = f(a)$$

$$= \frac{\pi}{4} + 2$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$= \frac{\pi}{4} + 2 + \frac{5}{2}(x-1)$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$= \frac{\pi}{4} + 2 + \frac{5}{2}(x-1) - \frac{3}{4}(x-1)^2$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= \frac{\pi}{4} + 2 + \frac{5}{2}(x-1) - \frac{3}{4}(x-1)^2$$

### Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ :  $f(x) = e^x$ ,  $a = \ln 2$

### Solution

$$f(x) = e^x \rightarrow f(\ln 2) = 2$$

$$f'(x) = e^x \rightarrow f'(\ln 2) = 2$$

$$f''(x) = e^x \rightarrow f''(\ln 2) = 2$$

$$f'''(x) = e^x \rightarrow f'''(\ln 2) = 2$$

$$P_0(x) = f(a)$$

$$= 2 \quad |$$

$$P_1(x) = f(a) + f'(a)(x - a)$$

$$= 2 + 2(x - \ln 2) \quad |$$

$$P_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2$$

$$= 2 + 2(x - \ln 2) + (x - \ln 2)^2 \quad |$$

$$P_3(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3$$

$$= 2 + 2(x - \ln 2) + (x - \ln 2)^2 + \frac{1}{3}(x - \ln 2)^3 \quad |$$

### Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ :  $f(x) = e^{3x}$ ;  $a = 0$

### Solution

$$f(x) = e^{3x} \rightarrow f(0) = 1$$

$$f'(x) = 3e^{3x} \rightarrow f'(0) = 3$$

$$f''(x) = 9e^{3x} \rightarrow f''(0) = 9$$

$$f'''(x) = 27e^{3x} \rightarrow f'''(0) = 27$$

$$P_0(x) = f(a)$$

$$= 1 \quad |$$

$$P_1(x) = f(a) + f'(a)(x - a)$$

$$\underline{=1+3x \mid}$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\underline{=1+3x+\frac{9}{2}x^2 \mid}$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$\underline{=1+3x+\frac{9}{2}x^2+\frac{9}{2}x^3 \mid}$$

### Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ :  $f(x) = \frac{1}{x}; \quad a = 1$

### Solution

$$f(x) = \frac{1}{x} \quad \rightarrow f(1) = 1$$

$$f'(x) = -\frac{1}{x^2} \quad \rightarrow f'(1) = -1$$

$$f''(x) = \frac{2}{x^3} \quad \rightarrow f''(1) = 2$$

$$f'''(x) = -\frac{6}{x^4} \quad \rightarrow f'''(1) = -6$$

$$P_0(x) = f(a)$$

$$= 1 - (x-1)$$

$$\underline{=1 \mid}$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$\underline{=2-x \mid}$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\underline{P_2(x) = 1 - (x-1) + (x-1)^2 \mid}$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$\underline{=1 - (x-1) + (x-1)^2 - (x-1)^3 \mid}$$

$$\underline{f(x) = \sum_{k=0}^{\infty} (-1)^k (x-1)^k \mid}$$

### Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ :  $f(x) = \cos x$ ;  $a = \frac{\pi}{2}$

### Solution

$$f(x) = \cos x \quad \rightarrow f\left(\frac{\pi}{2}\right) = 0$$

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$$f'(x) = -\sin x \quad \rightarrow f'\left(\frac{\pi}{2}\right) = -1$$

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$$f''(x) = -\cos x \quad \rightarrow f''\left(\frac{\pi}{2}\right) = 0$$

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$$f'''(x) = \sin x \quad \rightarrow f'''\left(\frac{\pi}{2}\right) = 1$$

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$$P_0(x) = f(a)$$

$$= 0$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$= -\left(x - \frac{\pi}{2}\right)$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$= -\left(x - \frac{\pi}{2}\right)$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= -\left(x - \frac{\pi}{2}\right) + \frac{1}{6}\left(x - \frac{\pi}{2}\right)^3$$

### Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ :  $f(x) = \frac{1}{x+1}$ ;  $a = 0$

### Solution

$$f(x) = \frac{1}{x+1} \quad \rightarrow f(0) = 1$$

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$$f'(x) = -\frac{1}{(x+1)^2} \quad \rightarrow f'(0) = -1$$

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$$f''(x) = \frac{2}{(x+1)^3} \quad \rightarrow f''(0) = 2$$

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$$f'''(x) = -\frac{6}{(x+1)^4} \rightarrow f'''(0) = -6$$


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$$P_0(x) = f(a)$$

$$\underline{= 1}$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$\underline{= 1-x}$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\underline{= 1-x+x^2}$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$\underline{= 1-x+x^2-x^3}$$

### Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ :  $f(x) = \tan^{-1} 4x$ ;  $a = 0$

### Solution

$$f(x) = \tan^{-1} 4x \rightarrow f(0) = 0$$


---

$$f'(x) = \frac{4}{1+16x^2} \rightarrow f'(0) = 4$$


---

$$f''(x) = -\frac{128x}{(1+16x^2)^2} \rightarrow f''(0) = 0$$


---

$$f'''(x) = -\frac{128(1+16x^2) - 2(32x)(128x)}{(1+16x^2)^3}$$

$$\rightarrow f'''(0) = -128$$

$$= \frac{6144x^2 - 128}{(1+16x^2)^3}$$


---

$$P_0(x) = f(a)$$

$$\underline{= 0}$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$\underline{= 4x}$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$= 4x \quad |$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= x - \frac{64}{3}x^3 \quad |$$

### Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ :  $f(x) = \sin 2x$ ;  $a = -\frac{\pi}{2}$

### Solution

$$f(x) = \sin 2x \quad \rightarrow f\left(-\frac{\pi}{2}\right) = 0$$

$$f'(x) = 2 \cos 2x \quad \rightarrow f'\left(-\frac{\pi}{2}\right) = -2$$

$$f''(x) = -4 \sin 2x \quad \rightarrow f''\left(-\frac{\pi}{2}\right) = 0$$

$$f'''(x) = -8 \cos 2x \quad \rightarrow f'''\left(-\frac{\pi}{2}\right) = 8$$

$$P_0(x) = f(a)$$

$$= 0 \quad |$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$= -2\left(x + \frac{\pi}{2}\right) \quad |$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$= -2\left(x + \frac{\pi}{2}\right) \quad |$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= -2\left(x + \frac{\pi}{2}\right) + \frac{4}{3}\left(x + \frac{\pi}{2}\right)^3 \quad |$$

### Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ :  $f(x) = \cosh 3x$ ;  $a = 0$

### Solution

$$f(x) = \cosh 3x \quad \rightarrow f(0) = 1$$

$$f'(x) = 3 \sinh 3x \quad \rightarrow f'(0) = 0$$

$$f''(x) = 9 \cosh 3x \quad \rightarrow f''(0) = 9$$

$$f'''(x) = 27 \sinh 3x \quad \rightarrow f'''(0) = 0$$

$$P_0(x) = f(a)$$

$$= 1$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$= 1$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$= 1 + \frac{9}{2}x^2$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= 1 + \frac{9}{2}x^2$$

### Exercise

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ :

$$f(x) = \frac{1}{4+x^2}; \quad a = 0$$

### Solution

$$f(x) = \frac{1}{4+x^2} \quad \rightarrow f(0) = \frac{1}{4}$$

$$f'(x) = -\frac{2x}{(4+x^2)^2} \quad \rightarrow f'(0) = 0$$

$$f''(x) = -\frac{8+2x^2-2(2x)^2}{(4+x^2)^3}$$

$$= -\frac{8-6x^2}{(4+x^2)^3} \quad \rightarrow f''(0) = -\frac{1}{8}$$

---


$$f'''(x) = -\frac{-12x(4+x^2) - 6x(8-6x^2)}{(4+x^2)^4} \rightarrow f'''(0) = 0$$

$$= \frac{96x - 24x^3}{(4+x^2)^4}$$


---

$$P_0(x) = f(a)$$

$$= \frac{1}{4}$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$= \frac{1}{4}$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$= \frac{1}{4} - \frac{1}{16}x^2$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= \frac{1}{4} - \frac{1}{16}x^2$$

### Exercise

Find out the third term of the Maclaurin series for the following function.  $f(x) = (1+x)^{1/3}$

### Solution

$$f(x) = (1+x)^{1/3} \rightarrow f(0) = 1$$

$$f'(x) = \frac{1}{3}(1+x)^{-2/3} \rightarrow f'(0) = \frac{1}{3}$$

$$f''(x) = -\frac{2}{9}(1+x)^{-5/3} \rightarrow f''(0) = -\frac{2}{9}$$

$$f'''(x) = \frac{10}{27}(1+x)^{-8/3} \rightarrow f'''(0) = \frac{10}{27}$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3$$

$$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3$$



### Exercise

Find out the third term of the Maclaurin series for the following function.  $f(x) = (1+x)^{-1/2}$

#### Solution

$$f(x) = (1+x)^{-1/2} \quad \rightarrow f(0) = 1$$

---

$$f'(x) = -\frac{1}{2}(1+x)^{-3/2} \quad \rightarrow f'(0) = -\frac{1}{2}$$

---

$$f''(x) = \frac{3}{4}(1+x)^{-5/2} \quad \rightarrow f''(0) = \frac{3}{4}$$

---

$$f'''(x) = -\frac{15}{8}(1+x)^{-7/2} \quad \rightarrow f'''(0) = -\frac{15}{8}$$

---

$$\begin{aligned} P_3(x) &= f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 \\ &= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 \end{aligned}$$

### Exercise

Find out the third term of the Maclaurin series for the following function.  $f(x) = \left(1 + \frac{x}{2}\right)^{-3}$

#### Solution

$$f(x) = \left(1 + \frac{x}{2}\right)^{-3} \quad \rightarrow f(0) = 1$$

---

$$f'(x) = -\frac{3}{2}\left(1 + \frac{x}{2}\right)^{-4} \quad \rightarrow f'(0) = -\frac{3}{2}$$

---

$$f''(x) = 3\left(1 + \frac{x}{2}\right)^{-5} \quad \rightarrow f''(0) = 3$$

---

$$f'''(x) = -\frac{15}{2}\left(1 + \frac{x}{2}\right)^{-6} \quad \rightarrow f'''(0) = -\frac{15}{2}$$

---

$$\begin{aligned} P_3(x) &= f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 \\ &= 1 - \frac{3}{2}x + \frac{3}{2}x^2 - \frac{5}{4}x^3 \end{aligned}$$

### Exercise

Find out the third term of the Maclaurin series for the following function.  $f(x) = (1+2x)^{-5}$

#### Solution

$$f(x) = (1+2x)^{-5} \quad \rightarrow f(0) = 1$$

$$f'(x) = -10(1+2x)^{-6} \quad \rightarrow f'(0) = -10$$

$$f''(x) = 120(1+2x)^{-7} \quad \rightarrow f''(0) = 120$$

$$f'''(x) = -1680(1+2x)^{-8} \quad \rightarrow f'''(0) = -1680$$

$$\begin{aligned} P_3(x) &= f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 \\ &= 1 - 10x + 60x^2 - 280x^3 \end{aligned}$$

### Exercise

Find the  $n$ th Maclaurin polynomial for the function  $f(x) = e^{4x}$ ,  $n = 4$

### Solution

$$f(x) = e^{4x} \quad \rightarrow \quad f(0) = 1$$

$$f'(x) = 4e^{4x} \quad \rightarrow \quad f'(0) = 4$$

$$f''(x) = 16e^{4x} \quad \rightarrow \quad f''(0) = 16$$

$$f'''(x) = 64e^{4x} \quad \rightarrow \quad f'''(0) = 64$$

$$f^{(4)}(x) = 256e^{4x} \quad \rightarrow \quad f^{(4)}(0) = 256$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$P_4(x) = 1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4$$

### Exercise

Find the  $n$ th Maclaurin polynomial for the function  $f(x) = e^{-x}$ ,  $n = 5$

### Solution

$$f(x) = e^{-x} \quad \rightarrow \quad f(0) = 1$$

$$f'(x) = -e^{-x} \quad \rightarrow \quad f'(0) = -1$$

$$f''(x) = e^{-x} \quad \rightarrow \quad f''(0) = 1$$

$$f'''(x) = -e^{-x} \rightarrow f'''(0) = -1$$

$$f^{(4)}(x) = e^{-x} \rightarrow f^{(4)}(0) = 1$$

$$f^{(5)}(x) = -e^{-x} \rightarrow f^{(5)}(0) = -1$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$\underline{P_5(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5}$$

### Exercise

Find the  $n$ th Maclaurin polynomial for the function  $f(x) = e^{-x/2}$ ,  $n = 4$

### Solution

$$f(x) = e^{-x/2} \rightarrow f(0) = 1$$

$$f'(x) = -\frac{1}{2}e^{-x/2} \rightarrow f'(0) = -\frac{1}{2}$$

$$f''(x) = \frac{1}{4}e^{-x/2} \rightarrow f''(0) = \frac{1}{4}$$

$$f'''(x) = -\frac{1}{8}e^{-x/2} \rightarrow f'''(0) = -\frac{1}{8}$$

$$f^{(4)}(x) = \frac{1}{16}e^{-x/2} \rightarrow f^{(4)}(0) = \frac{1}{16}$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\underline{P_4(x) = 1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4}$$

### Exercise

Find the  $n$ th Maclaurin polynomial for the function  $f(x) = e^{x/3}$ ,  $n = 4$

### Solution

$$f(x) = e^{x/3} \rightarrow f(0) = 1$$

$$f'(x) = \frac{1}{3}e^{x/3} \rightarrow f'(0) = \frac{1}{3}$$

$$f''(x) = \frac{1}{9}e^{x/3} \rightarrow f''(0) = \frac{1}{9}$$

$$f'''(x) = \frac{1}{27}e^{x/3} \rightarrow f'''(0) = \frac{1}{27}$$

$$f^{(4)}(x) = \frac{1}{81}e^{x/3} \rightarrow f^{(4)}(0) = \frac{1}{81}$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\underline{P_4(x) = 1 + \frac{1}{3}x + \frac{1}{18}x^2 + \frac{1}{162}x^3 + \frac{1}{1944}x^4}$$

### Exercise

Find the  $n$ th Maclaurin polynomial for the function  $f(x) = \sin x$ ,  $n = 5$

#### Solution

$$f(x) = \sin x \rightarrow f(0) = 0$$

$$f'(x) = \cos x \rightarrow f'(0) = 1$$

$$f''(x) = -\sin x \rightarrow f''(0) = 0$$

$$f'''(x) = -\cos x \rightarrow f'''(0) = -1$$

$$f^{(4)}(x) = \sin x \rightarrow f^{(4)}(0) = 0$$

$$f^{(5)}(x) = \cos x \rightarrow f^{(5)}(0) = 1$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$\underline{P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5}$$

### Exercise

Find the  $n$ th Maclaurin polynomial for the function  $f(x) = \cos \pi x$ ,  $n = 4$

#### Solution

$$f(x) = \cos \pi x \rightarrow f(0) = 1$$

$$f'(x) = -\pi \sin \pi x \rightarrow f'(0) = 0$$

$$f''(x) = -\pi^2 \cos \pi x \rightarrow f''(0) = -\pi^2$$

$$f'''(x) = \pi^3 \sin \pi x \rightarrow f'''(0) = 0$$

$$f^{(4)}(x) = \pi^4 \cos \pi x \rightarrow f^{(4)}(0) = \pi^4$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\underline{P_4(x) = 1 - \frac{\pi^2}{2}x^2 + \frac{\pi^4}{24}x^4}$$

### Exercise

Find the  $n$ th Maclaurin polynomial for the function  $f(x) = xe^x$ ,  $n = 4$

### Solution

$$f(x) = xe^x \rightarrow f(0) = 0$$

$$f'(x) = e^x + xe^x \rightarrow f'(0) = 1$$

$$f''(x) = 2e^x + xe^x \rightarrow f''(0) = 2$$

$$f'''(x) = 3e^x + xe^x \rightarrow f'''(0) = 3$$

$$f^{(4)}(x) = 4e^x + xe^x \rightarrow f^{(4)}(0) = 4$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\underline{P_4(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4}$$

### Exercise

Find the  $n$ th Maclaurin polynomial for the function  $f(x) = x^2e^{-x}$ ,  $n = 4$

### Solution

$$f(x) = x^2e^{-x} \rightarrow f(0) = 0$$

$$f'(x) = 2xe^{-x} - x^2e^{-x} \rightarrow f'(0) = 0$$

$$f''(x) = 2e^{-x} - 4xe^{-x} + x^2e^{-x} \rightarrow f''(0) = 2$$

$$f'''(x) = -6e^{-x} + 6xe^{-x} - x^2e^{-x} \rightarrow f'''(0) = -6$$

$$f^{(4)}(x) = 12e^{-x} - 8xe^{-x} + x^2e^{-x} \rightarrow f^{(4)}(0) = 12$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\underline{P_4(x) = x^2 - x^3 + \frac{1}{2}x^4}$$

### Exercise

Find the  $n$ th Maclaurin polynomial for the function  $f(x) = \frac{1}{x+1}$ ,  $n = 5$

### Solution

$$f(x) = \frac{1}{x+1} \rightarrow f(0) = 1$$

$$f'(x) = -(x+1)^{-2} \rightarrow f'(0) = -1$$

$$f''(x) = 2(x+1)^{-3} \rightarrow f''(0) = 2$$

$$f'''(x) = -6(x+1)^{-4} \rightarrow f'''(0) = -6$$

$$f^{(4)}(x) = 24(x+1)^{-5} \rightarrow f^{(4)}(0) = 24$$

$$f^{(5)}(x) = -120(x+1)^{-6} \rightarrow f^{(5)}(0) = -120$$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$\underline{P_5(x) = 1 - x + x^2 - x^3 + x^4 - x^5}$$

### Exercise

Find the  $n$ th Maclaurin polynomial for the function  $f(x) = \frac{x}{x+1}$ ,  $n = 4$

### Solution

$$f(x) = \frac{x}{x+1} = 1 - \frac{1}{x+1} \rightarrow f(0) = 0$$

$$f'(x) = (x+1)^{-2} \rightarrow f'(0) = 1$$

$$f''(x) = -2(x+1)^{-3} \rightarrow f''(0) = -2$$

$$f'''(x) = 6(x+1)^{-4} \rightarrow f'''(0) = 6$$

$$f^{(4)}(x) = -24(x+1)^{-5} \rightarrow f^{(4)}(0) = -24$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\underline{P_4(x) = x - x^2 + x^3 - x^4}$$

### Exercise

Find the  $n$ th Maclaurin polynomial for the function  $f(x) = \sec x$ ,  $n = 2$

### Solution

$$f(x) = \sec x \rightarrow f(0) = 1$$

$$f'(x) = \sec x \tan x \rightarrow f'(0) = 0$$

$$f''(x) = \sec x \tan^2 x + \sec^3 x \rightarrow f''(0) = 1$$

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$

$$\underline{P_2(x) = 1 + \frac{1}{2}x^2}$$

### Exercise

Find the  $n$ th Maclaurin polynomial for the function  $f(x) = \tan x$ ,  $n = 3$

### Solution

$$f(x) = \tan x \rightarrow f(0) = 0$$

$$f'(x) = \sec^2 x \rightarrow f'(0) = 1$$

$$f''(x) = 2 \sec^2 x \tan x \rightarrow f''(0) = 0$$

$$f'''(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x \rightarrow f'''(0) = 2$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3$$

$$\underline{P_4(x) = x + \frac{1}{3}x^3}$$

### Exercise

Find the Maclaurin series for:  $xe^x$

### Solution

$$f(x) = xe^x \rightarrow f(0) = 0$$

$$f'(x) = e^x + xe^x \rightarrow f'(0) = 1$$

$$f''(x) = 2e^x + xe^x \rightarrow f''(0) = 2$$

$$f'''(x) = 3e^x + xe^x \rightarrow f'''(0) = 3$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$f^{(n)}(x) = ne^x + xe^x \rightarrow f^{(n)}(0) = n$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$xe^x = x + x^2 + \frac{1}{2}x^3 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n-1)!} x^n$$

### Exercise

Find the Maclaurin series for:  $5 \cos \pi x$

#### Solution

$$f(x) = 5 \cos \pi x \rightarrow f(0) = 5$$

$$f'(x) = -5\pi \sin \pi x \rightarrow f'(0) = 0$$

$$f''(x) = -5\pi^2 \cos \pi x \rightarrow f''(0) = -5\pi^2$$

$$f'''(x) = 5\pi^3 \sin \pi x \rightarrow f'''(0) = 0$$

$$5 \cos \pi x = 5 - \frac{5\pi^2 x^2}{2!} + \frac{5\pi^4 x^4}{4!} - \frac{5\pi^6 x^6}{6!} + \dots$$

$$= 5 \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!}$$

### Exercise

Find the Maclaurin series for:  $\frac{x^2}{x+1}$

#### Solution

$$f(x) = \frac{x^2}{x+1} \rightarrow f(0) = 0$$



$$f'(x) = \frac{2x(x+1) - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2} \rightarrow f'(0) = 0$$

$$\begin{aligned} f''(x) &= \frac{(2x+2)(x+1)^2 - 2(x+1)(x^2 + 2x)}{(x+1)^4} \\ &= \frac{2x^2 + 4x + 2 - 2x^2 - 4x}{(x+1)^3} \\ &= \frac{2}{(x+1)^3} \rightarrow f''(0) = 2 \end{aligned}$$

$$f'''(x) = \frac{-6}{(x+1)^4} \rightarrow f'''(0) = -6$$

$$\begin{aligned} \frac{x^2}{x+1} &= \frac{2}{2!}x^2 - \frac{6x^3}{3!} + \frac{24x^4}{4!} - \dots \\ &= x^2 - x^3 + x^4 - \dots \end{aligned}$$

$$= \sum_{n=2}^{\infty} (-1)^n x^n$$

### Exercise

Find the Maclaurin series for:  $e^{3x+1}$

#### Solution

$$\begin{aligned} e^{3x+1} &= e \cdot e^{3x} \\ &= e \left( \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} \right) \\ &= e \sum_{n=0}^{\infty} \frac{3^n x^n}{n!} \quad (\text{for all } x) \end{aligned}$$

### Exercise

Find the Maclaurin series for:  $\cos(2x^3)$

#### Solution

$$\cos(2x^3) = 1 - \frac{(2x^3)^2}{2!} + \frac{(2x^3)^4}{4!} - \frac{(2x^3)^6}{6!} + \dots$$

$$\begin{aligned}
&= 1 - \frac{2^2 x^3}{2!} + \frac{2^4 x^{12}}{4!} - \frac{2^6 x^{18}}{6!} + \dots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n)!} x^{6n} \quad \left( \text{for all } x \right)
\end{aligned}$$

### Exercise

Find the Maclaurin series for:  $\cos(2x - \pi)$

### Solution

$$\begin{aligned}
\cos(2x - \pi) &= \cos(2x)\cos\pi + \sin(2x)\sin\pi \\
&= -\cos(2x) \\
&= -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (2x)^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 4^n}{(2n)!} x^{2n} \quad \left( \text{for all } x \right)
\end{aligned}$$

### Exercise

Find the Maclaurin series for:  $x^2 \sin\left(\frac{x}{3}\right)$

### Solution

$$\begin{aligned}
x^2 \sin\left(\frac{x}{3}\right) &= x^2 \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{x}{3}\right)^{2n+1}}{(2n+1)!} \\
&= x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{3^{2n+1} (2n+1)!} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{3^{2n+1} (2n+1)!} \quad \left( \text{for all } x \right)
\end{aligned}$$

### Exercise

Find the Maclaurin series for:  $\cos^2\left(\frac{x}{2}\right)$

### Solution

$$\begin{aligned}
\cos^2\left(\frac{x}{2}\right) &= \frac{1}{2}(1 + \cos x) \\
&= \frac{1}{2} \left( 1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \right) \\
&= \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \\
&= \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \left( \text{for all } x \right)
\end{aligned}$$

### Exercise

Find the Maclaurin series for:  $\sin x \cos x$

### Solution

$$\begin{aligned}
\sin x \cos x &= \frac{1}{2} \sin(2x) \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2x)^{2n+1} \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} 2^{2n+1} x^{2n+1} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{(2n+1)!} x^{2n+1} \quad \left( \text{for all } x \right)
\end{aligned}$$

### Exercise

Find the Maclaurin series for:  $\tan^{-1}(5x^2)$

### Solution

$$\begin{aligned}
\tan^{-1}(5x^2) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (5x^2)^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1}}{2n+1} x^{4n+2} \quad \left( \text{for } -\frac{1}{\sqrt{5}} \leq x \leq \frac{1}{\sqrt{5}} \right)
\end{aligned}$$

$5x^2 \leq 1 \rightarrow x^2 \leq \frac{1}{5} \Rightarrow -\frac{1}{\sqrt{5}} \leq x \leq \frac{1}{\sqrt{5}}$

### Exercise

Find the Maclaurin series for:  $\ln(2+x^2)$

### Solution

$$\begin{aligned}\ln(2+x^2) &= \ln 2 \left(1 + \frac{x^2}{2}\right) \\ &= \ln 2 + \ln \left(1 + \frac{x^2}{2}\right) \\ &= \ln 2 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{x^2}{2}\right)^n \qquad \frac{x^2}{2} \leq 1 \rightarrow x^2 \leq 2 \Rightarrow -\sqrt{2} \leq x \leq \sqrt{2} \\ &= \ln 2 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} \frac{x^{2n}}{2^n} \quad \left( \text{for } -\sqrt{2} \leq x \leq \sqrt{2} \right)\end{aligned}$$

### Exercise

Find the Maclaurin series for:  $\frac{1+x^3}{1+x^2}$

### Solution

$$\begin{aligned}\frac{1}{1+x} &= \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \dots \\ \frac{1}{1+x^2} &= 1 - x^2 + x^4 - x^6 + \dots \\ \frac{1+x^3}{1+x^2} &= (1+x^3)(1-x^2+x^4-x^6+\dots) \\ &= 1 - x^2 + x^4 - x^6 + \dots + x^3 - x^5 + x^7 - x^9 + \dots \\ &= 1 - x^2 + x^3 + x^4 - x^5 - x^6 + x^7 + x^8 - x^9 - \dots \\ &= 1 - x^2 + \sum_{n=2}^{\infty} (-1)^n (x^{2n-1} + x^{2n}) \quad \left( \text{for } |x| < 1 \right)\end{aligned}$$

### Exercise

Find the Maclaurin series for:  $\ln \frac{1+x}{1-x}$

### Solution

$$\begin{aligned}
\ln \frac{1+x}{1-x} &= \ln(1+x) - \ln(1-x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots - \left( -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \right) = 2x + 2\frac{x^3}{3} + \dots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \\
&= \sum_{n=0}^{\infty} \left( (-1)^n + 1 \right) \frac{x^{n+1}}{n+1} \\
&= 2 \sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1} \quad (-1 < x < 1)
\end{aligned}$$

### Exercise

Find the Maclaurin series for:  $\frac{e^{2x^2} - 1}{x^2}$

### Solution

$$\begin{aligned}
\frac{e^{2x^2} - 1}{x^2} &= \frac{1}{x^2} \left( e^{2x^2} - 1 \right) \\
&= \frac{1}{x^2} \left( 1 + 2x^2 + \frac{(2x^2)^2}{2!} + \frac{(2x^2)^3}{3!} + \dots - 1 \right) \\
&= \frac{1}{x^2} \left( 2x^2 + \frac{2^2 x^4}{2!} + \frac{2^3 x^6}{3!} + \dots \right) \\
&= 2 + \frac{2^2 x^2}{2!} + \frac{2^3 x^4}{3!} + \frac{2^4 x^6}{4!} + \dots \\
&= \sum_{n=0}^{\infty} \frac{2^{n+1}}{(n+1)!} x^{2n} \quad (\text{for all } x \neq 0)
\end{aligned}$$

### Exercise

Find the Maclaurin series for:  $\cosh x - \cos x$

### Solution

$$\cosh x - \cos x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$\begin{aligned}
& 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots - \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right) = 2\frac{x^2}{2!} + 2\frac{x^6}{6!} + \dots \\
& = \sum_{n=0}^{\infty} \left( 1 - (-1)^n \right) \frac{x^{2n}}{(2n)!} \\
& = 2 \sum_{n=0}^{\infty} \frac{x^{4n+2}}{(4n+2)!} \quad \left( \text{for all } x \right)
\end{aligned}$$

### Exercise

Find the Maclaurin series for:  $\sinh x - \sin x$

### Solution

$$\begin{aligned}
\sinh x - \sin x &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \\
&= \sum_{n=0}^{\infty} \left( 1 - (-1)^n \right) \frac{x^{2n+1}}{(2n+1)!} \\
&= 2 \sum_{n=0}^{\infty} \frac{x^{4n+3}}{(4n+3)!} \quad \left( \text{for all } x \right)
\end{aligned}$$

### Exercise

Finding Taylor and Maclaurin Series generated by  $f$  at  $x = a$ :  $f(x) = x^3 - 2x + 4$ ,  $a = 2$

### Solution

$$f(x) = x^3 - 2x + 4 \rightarrow f(2) = 8$$

$$f'(x) = 3x^2 - 2 \rightarrow f'(2) = 10$$

$$f''(x) = 6x \rightarrow f''(2) = 12$$

$$f'''(x) = 6 \rightarrow f'''(2) = 6$$

$$f^{(n)}(x) = 0 \rightarrow f^{(n)}(2) = 0 \quad (n > 3)$$

$$P_n(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \dots$$

$$\underline{x^3 - 2x + 4 = 8 + 10(x-2) + 6(x-2)^2 + (x-2)^3}$$

### Exercise

Finding Taylor and Maclaurin Series generated by  $f$  at  $x = a$ :  $f(x) = 2x^3 + x^2 + 3x - 8$ ,  $a = 1$

### Solution

$$f(x) = 2x^3 + x^2 + 3x - 8 \rightarrow f(1) = -2$$

$$f'(x) = 6x^2 + 2x + 3 \rightarrow f'(1) = 11$$

$$f''(x) = 12x + 2 \rightarrow f''(1) = 14$$

$$f'''(x) = 12 \rightarrow f'''(1) = 12$$

$$f^{(n)}(x) = 0 \rightarrow f^{(n)}(1) = 0 \quad (n \geq 4)$$

$$P_n(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \dots$$

$$\underline{2x^3 + x^2 + 3x - 8 = -2 + 11(x-1) + 7(x-1)^2 + 2(x-1)^3}$$

### Exercise

Finding Taylor and Maclaurin Series generated by  $f$  at  $x = a$ :

$$f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2, \quad a = -1$$

### Solution

$$f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2 \rightarrow f(-1) = -7$$

$$f'(x) = 15x^4 - 4x^3 + 6x^2 + 2x \rightarrow f'(-1) = 23$$

$$f''(x) = 60x^3 - 12x^2 + 12x + 2 \rightarrow f''(-1) = -82$$

$$f'''(x) = 180x^2 - 24x + 12 \rightarrow f'''(-1) = 216$$

$$f^{(4)}(x) = 360x - 24 \rightarrow f^{(4)}(-1) = -384$$

$$f^{(5)}(x) = 360 \rightarrow f^{(5)}(-1) = 360$$

$$f^{(n)}(x) = 0 \rightarrow f^{(n)}(-1) = 0 \quad (n \geq 6)$$

$$P_n(x) = f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \frac{f'''(-1)}{3!}(x+1)^3 + \frac{f^{(4)}(-1)}{4!}(x+1)^4 + \frac{f^{(5)}(-1)}{5!}(x+1)^5$$

$$\underline{3x^5 - x^4 + 2x^3 + x^2 - 2 = -7 + 23(x+1) - \frac{82}{2!}(x+1)^2 + \frac{216}{3!}(x+1)^3 - \frac{384}{4!}(x+1)^4 + \frac{360}{5!}(x+1)^5}$$

### Exercise

Finding Taylor and Maclaurin Series generated by  $f$  at  $x = a$ :  $f(x) = \cos\left(2x + \frac{\pi}{2}\right)$ ,  $a = \frac{\pi}{4}$

### Solution

$$f(x) = \cos\left(2x + \frac{\pi}{2}\right) \rightarrow f\left(\frac{\pi}{4}\right) = -1$$

$$f'(x) = -2\sin\left(2x + \frac{\pi}{2}\right) \rightarrow f'\left(\frac{\pi}{4}\right) = 0$$

$$f''(x) = -4\cos\left(2x + \frac{\pi}{2}\right) \rightarrow f''\left(\frac{\pi}{4}\right) = 4$$

$$f'''(x) = 8\sin\left(2x + \frac{\pi}{2}\right) \rightarrow f'''\left(\frac{\pi}{4}\right) = 0$$

$$f^{(4)}(x) = 16\cos\left(2x + \frac{\pi}{2}\right) \rightarrow f^{(4)}\left(\frac{\pi}{4}\right) = -16$$

$$f^{(5)}(x) = -32\sin\left(2x + \frac{\pi}{2}\right) \rightarrow f^{(5)}\left(\frac{\pi}{4}\right) = 0$$
$$\rightarrow f^{(2n)}\left(\frac{\pi}{4}\right) = (-1)^n 2^{2n}$$

$$\cos\left(2x + \frac{\pi}{2}\right) = -1 + \frac{4}{2!}\left(x - \frac{\pi}{4}\right)^2 - \frac{16}{4!}\left(x - \frac{\pi}{4}\right)^4 + \dots$$

$$= -1 + 2\left(x - \frac{\pi}{4}\right)^2 - \frac{2}{3}\left(x - \frac{\pi}{4}\right)^4 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} \left(x - \frac{\pi}{4}\right)^{2n}$$

### Exercise

Find the Taylor series of the functions, where is the series representation valid?  $f(x) = e^{-2x}$  about  $-1$

### Solution

$$\text{Let } t = x + 1 \Rightarrow x = t - 1$$

$$f(x) = e^{-2x} = e^{-2x-2+2} = e^{-2(x+1)+2}$$

$$= e^2 \sum_{n=0}^{\infty} \frac{(-2(x+1))^n}{n!}$$

$$= e^2 \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} (x+1)^n \quad (\text{for all } x)$$



### Exercise

Find the Taylor series of the functions, where is the series representation valid?  $f(x) = \sin x$  about  $\frac{\pi}{2}$

### Solution

$$\text{Let } y = x - \frac{\pi}{2} \Rightarrow x = y + \frac{\pi}{2}$$

$$\sin x = \sin\left(y + \frac{\pi}{2}\right) = \cos y$$

$$\begin{aligned} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} y^{2n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(x - \frac{\pi}{2}\right)^{2n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(x - \frac{\pi}{2}\right)^{2n} \end{aligned} \quad \left| \quad (\text{for all } x) \right.$$

### Exercise

Find the Taylor series of the functions, where is the series representation valid?

$$f(x) = \ln x \quad \text{in powers of } x - 3$$

### Solution

$$\text{Let } y = x - 3 \Rightarrow x = y + 3$$

$$\ln x = \ln(y + 3) = \ln 3 \left(1 + \frac{y}{3}\right)$$

$$= \ln 3 + \ln\left(1 + \frac{y}{3}\right)$$

$$= \ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(\frac{y}{3}\right)^{n+1}$$

$$= \ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \frac{(x-3)^{n+1}}{3^{n+1}}$$

$$= \ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)3^{n+1}} (x-3)^{n+1} \quad \left| \quad (0 < x \leq 6) \right.$$

### Exercise

Find the Taylor series of the functions, where is the series representation valid?

$$f(x) = \ln(2+x) \quad \text{in powers of } x-2$$

### Solution

$$\begin{aligned}\ln(2+x) &= \ln(2+x-2+2) = \ln(4+x-2) \\ &= \ln 4 \left(1 + \frac{x-2}{4}\right) \\ &= \ln 4 + \ln\left(1 + \frac{x-2}{4}\right) \\ &= \ln 4 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(\frac{x-2}{4}\right)^{n+1} \quad \frac{|x-2|}{4} < 1 \Rightarrow |x-2| < 4 \quad -4 < x-2 < 4 \quad -2 < x < 6 \\ &= \ln 4 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)4^{n+1}} (x-2)^{n+1} \quad (-2 < x \leq 6)\end{aligned}$$

### Exercise

Find the Taylor series of the functions, where is the series representation valid?

$$f(x) = e^{2x+3} \quad \text{in powers of } x+1$$

### Solution

$$\begin{aligned}e^{2x+3} &= e^{2x+2-2+3} = e^{2(x+1)+1} \\ &= e \cdot e^{2(x+1)} \\ &= e \sum_{n=0}^{\infty} \frac{(2(x+1))^n}{n!} \\ &= e \sum_{n=0}^{\infty} \frac{2^n (x+1)^n}{n!} \quad (\text{for all } x)\end{aligned}$$

### Exercise

Find the Taylor series of the functions, where is the series representation valid?

$$f(x) = \sin x - \cos x \quad \text{about } \frac{\pi}{4}$$

### Solution

$$\begin{aligned}
\sin x - \cos x &= \sin\left(x - \frac{\pi}{4} + \frac{\pi}{4}\right) - \cos\left(x - \frac{\pi}{4} + \frac{\pi}{4}\right) \\
&= \cos\left(x - \frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(x - \frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) \\
&= \cos\left(x - \frac{\pi}{4}\right)\frac{\sqrt{2}}{2} + \sin\left(x - \frac{\pi}{4}\right)\frac{\sqrt{2}}{2} - \cos\left(x - \frac{\pi}{4}\right)\frac{\sqrt{2}}{2} + \sin\left(x - \frac{\pi}{4}\right)\frac{\sqrt{2}}{2} \\
&= \sqrt{2}\sin\left(x - \frac{\pi}{4}\right) \\
&= \sqrt{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(x - \frac{\pi}{4}\right)^{2n+1} \quad \left( \text{for all } x \right)
\end{aligned}$$

### Exercise

Find the Taylor series of the functions, where is the series representation valid?

$$f(x) = \cos^2 x \quad \text{about } \frac{\pi}{8}$$

### Solution

$$\begin{aligned}
\cos^2(x) &= \frac{1}{2}(1 + \cos 2x) = \frac{1}{2}\left(1 + \cos\left(2x - \frac{\pi}{4} + \frac{\pi}{4}\right)\right) \\
&= \frac{1}{2}\left(1 + \cos\left(2x - \frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(2x - \frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)\right) \\
&= \frac{1}{2}\left(1 + \frac{\sqrt{2}}{2}\cos\left(2\left(x - \frac{\pi}{8}\right)\right) - \frac{\sqrt{2}}{2}\sin\left(2\left(x - \frac{\pi}{8}\right)\right)\right) \\
&= \frac{1}{2} + \frac{\sqrt{2}}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(2\left(x - \frac{\pi}{8}\right)\right)^{2n} - \frac{\sqrt{2}}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(2\left(x - \frac{\pi}{8}\right)\right)^{2n+1} \\
&= \frac{1}{2} + \frac{1}{2\sqrt{2}} \sum_{n=0}^{\infty} (-1)^n \left[ \frac{2^{2n}}{(2n)!} \left(x - \frac{\pi}{8}\right)^{2n} - \frac{2^{2n+1}}{(2n+1)!} \left(x - \frac{\pi}{8}\right)^{2n+1} \right] \quad (\text{for all } x)
\end{aligned}$$

### Exercise

Find the Taylor series of the functions, where is the series representation valid?

$$f(x) = \frac{x}{1+x} \quad \text{in powers of } x-1$$

### Solution

$$\begin{aligned}
\frac{x}{1+x} &= \frac{x-1+1}{1+x-1+1} = \frac{(x-1)+1}{(x-1)+2} \\
&= 1 - \frac{1}{(x-1)+2}
\end{aligned}$$

$$\begin{aligned}
&= 1 - \frac{1}{2\left(1 + \frac{x-1}{2}\right)} \\
&= 1 - \frac{1}{2} \left( 1 - \frac{x-1}{2} + \left(\frac{x-1}{2}\right)^2 - \left(\frac{x-1}{2}\right)^3 + \dots \right) \\
&= \frac{1}{2} - \frac{1}{2^2}(x-1) + \frac{1}{2^3}(x-1)^2 - \frac{1}{2^4}(x-1)^3 + \dots \qquad |x-1| < 1 \quad -1 < x-1 < 1 \\
&= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n+1}} (x-1)^n \quad \left( 0 < x < 2 \right)
\end{aligned}$$

### Exercise

Find the Taylor series of the functions, where is the series representation valid?

$$f(x) = xe^x \quad \text{in powers of } x+2$$

### Solution

$$\begin{aligned}
xe^x &= (x+2-2)e^{x+2-2} \\
&= (x+2-2)e^{-2}e^{x+2} \\
&= (x+2)e^{-2}e^{x+2} - 2e^{-2}e^{x+2} \\
&= \frac{1}{e^2}(x+2) \sum_{n=0}^{\infty} \frac{(x+2)^n}{n!} - \frac{2}{e^2} \sum_{n=0}^{\infty} \frac{(x+2)^n}{n!} \\
&= \frac{1}{e^2} \sum_{n=0}^{\infty} \frac{(x+2)^{n+1}}{n!} - \frac{2}{e^2} \sum_{n=0}^{\infty} \frac{(x+2)^n}{n!} \\
&= \frac{1}{e^2} \sum_{n=1}^{\infty} \frac{(x+2)^n}{(n-1)!} - \frac{2}{e^2} \left( 1 + \sum_{n=1}^{\infty} \frac{(x+2)^n}{n!} \right) \\
&= -\frac{2}{e^2} + \frac{1}{e^2} \sum_{n=1}^{\infty} \frac{(x+2)^n}{(n-1)!} - \frac{2}{e^2} \sum_{n=1}^{\infty} \frac{(x+2)^n}{n!} \\
&= -\frac{2}{e^2} + \frac{1}{e^2} \sum_{n=1}^{\infty} \left( \frac{1}{(n-1)!} - \frac{2}{n!} \right) (x+2)^n \quad (\text{for all } x)
\end{aligned}$$

### Exercise

Find the  $n$ th Taylor polynomial centered at  $c$  for the function  $f(x) = \frac{2}{x}$ ,  $n = 3$ ,  $c = 1$

#### Solution

$$f(x) = \frac{2}{x} \rightarrow f(1) = 2$$

$$f'(x) = -\frac{2}{x^2} \rightarrow f'(1) = -2$$

$$f''(x) = \frac{4}{x^3} \rightarrow f''(1) = 4$$

$$f'''(x) = -\frac{12}{x^4} \rightarrow f'''(1) = -12$$

$$P_3(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3$$

$$P_3(x) = 2 - 2(x-1) + 2(x-1)^2 - 2(x-1)^3$$

### Exercise

Find the  $n$ th Taylor polynomial centered at  $c$  for the function  $f(x) = \frac{1}{x^2}$ ,  $n = 4$ ,  $c = 2$

#### Solution

$$f(x) = \frac{1}{x^2} \rightarrow f(2) = \frac{1}{4}$$

$$f'(x) = -\frac{2}{x^3} \rightarrow f'(2) = -\frac{1}{4}$$

$$f''(x) = \frac{6}{x^4} \rightarrow f''(2) = \frac{3}{8}$$

$$f'''(x) = -\frac{24}{x^5} \rightarrow f'''(2) = -\frac{3}{4}$$

$$f^{(4)}(x) = \frac{120}{x^6} \rightarrow f^{(4)}(2) = \frac{15}{8}$$

$$P_4(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \frac{f^{(4)}(c)}{4!}(x-c)^4$$

$$P_4(x) = \frac{1}{4} - \frac{1}{4}(x-2) + \frac{3}{16}(x-2)^2 - \frac{1}{8}(x-2)^3 + \frac{5}{64}(x-2)^4$$

### Exercise

Find the  $n$ th Taylor polynomial centered at  $c$  for the function  $f(x) = \sqrt{x}$ ,  $n = 3$ ,  $c = 4$

#### Solution

$$f(x) = x^{1/2} \rightarrow f(4) = 2$$

$$f'(x) = \frac{1}{2}x^{-1/2} \rightarrow f'(4) = \frac{1}{4}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} \rightarrow f''(4) = -\frac{1}{4} \frac{1}{(2^2)^{3/2}} = -\frac{1}{32}$$

$$f'''(x) = \frac{3}{8}x^{-5/2} \rightarrow f'''(4) = \frac{3}{256}$$

$$P_3(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3$$

$$P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 \quad \Big|$$

### Exercise

Find the  $n$ th Taylor polynomial centered at  $c$  for the function  $f(x) = \sqrt[3]{x}$ ,  $n = 3$ ,  $c = 8$

#### Solution

$$f(x) = x^{1/3} \rightarrow f(8) = 2$$

$$f'(x) = \frac{1}{3}x^{-2/3} \rightarrow f'(8) = \frac{1}{12}$$

$$f''(x) = -\frac{2}{9}x^{-5/3} \rightarrow f''(8) = -\frac{1}{144}$$

$$f'''(x) = \frac{10}{27}x^{-8/3} \rightarrow f'''(8) = \frac{5}{3456}$$

$$P_3(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3$$

$$P_3(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20,736}(x-8)^3 \quad \Big|$$

### Exercise

Find the  $n$ th Taylor polynomial centered at  $c$  for the function  $f(x) = \ln x$ ,  $n = 4$ ,  $c = 2$

#### Solution

$$f(x) = \ln x \rightarrow f(2) = \ln 2$$

$$f'(x) = \frac{1}{x} \rightarrow f'(2) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{x^2} \rightarrow f''(2) = -\frac{1}{4}$$

$$f'''(x) = \frac{2}{x^3} \rightarrow f'''(2) = \frac{1}{4}$$

$$f^{(4)}(x) = -\frac{6}{x^4} \rightarrow f^{(4)}(2) = -\frac{3}{8}$$

$$P_4(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + \frac{f^{(4)}(c)}{4!}(x-c)^4$$

$$P_4(x) = \ln 2 + \frac{1}{2} \frac{1}{4}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \frac{1}{64}(x-2)^4 \quad \Big|$$

### Exercise

Find the  $n$ th Taylor polynomial centered at  $c$  for the function  $f(x) = x^2 \cos x$ ,  $n = 2$ ,  $c = \pi$

#### Solution

$$f(x) = x^2 \cos x \rightarrow f(\pi) = -\pi^2$$

$$f'(x) = 2x \cos x - x^2 \sin x \rightarrow f'(\pi) = -2\pi$$

$$f''(x) = 2 \cos x - 4x \sin x - x^2 \cos x \rightarrow f''(\pi) = -2 + \pi^2$$

$$P_2(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2$$

$$= -\pi^2 - 2\pi(x-\pi) + \frac{\pi^2-2}{2}(x-\pi)^2 \quad \Big|$$

### Exercise

Find the  $n$ th-order Taylor polynomial centered at  $c$  for the function  $f(x) = \sin 2x$ ;  $n = 3$ ,  $c = 0$

#### Solution

$$f(x) = \sin 2x \rightarrow f(0) = 0$$

$$f'(x) = 2 \cos 2x \rightarrow f'(0) = 2$$

$$f''(x) = -4 \sin 2x \rightarrow f''(0) = 0$$

$$f'''(x) = -8 \cos 2x \rightarrow f'''(0) = -8$$

$$P(x) = 2x - \frac{8}{3!}x^3$$

$$= 2x - \frac{1}{3!}(2x)^3 \quad \Big|$$

### Exercise

Find the  $n$ th-order Taylor polynomial centered at  $c$  for the function  $f(x) = \cos x^2$ ;  $n = 2$ ,  $c = 0$

### Solution

$$f(x) = \cos x^2 \quad \rightarrow f(0) = 1$$

---

$$f'(x) = -2x \sin x^2 \quad \rightarrow f'(0) = 0$$

---

$$f''(x) = -2 \sin x^2 - 4x^2 \cos x^2 \quad \rightarrow f''(0) = 0$$

---

$$\underline{P(x) = 1}$$

### Exercise

Find the  $n$ th-order Taylor polynomial centered at  $c$  for the function  $f(x) = e^{-x}$ ;  $n = 2$ ,  $c = 0$

### Solution

$$f(x) = e^{-x} \quad \rightarrow f(0) = 1$$

---

$$f'(x) = -e^{-x} \quad \rightarrow f'(0) = -1$$

---

$$f''(x) = e^{-x} \quad \rightarrow f''(0) = 1$$

---

$$\underline{P(x) = 1 - x - \frac{1}{2}x^2}$$

### Exercise

Find the  $n$ th-order Taylor polynomial centered at  $c$  for the function  $f(x) = \cos x$ ;  $n = 2$ ,  $c = \frac{\pi}{4}$

### Solution

$$f(x) = \cos x \quad \rightarrow f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

---

$$f'(x) = -\sin x \quad \rightarrow f'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

---

$$f''(x) = -\cos x \quad \rightarrow f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

---

$$\underline{P(x) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2}$$



### Exercise

Find the  $n$ th-order Taylor polynomial centered at  $c$  for the function  $f(x) = \ln x$ ;  $n = 2$ ,  $c = 1$

### Solution

$$f(x) = \ln x \quad \rightarrow f(1) = 0$$

$$f'(x) = \frac{1}{x} \quad \rightarrow f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \quad \rightarrow f''(1) = -1$$

$$P(x) = x - 1 - \frac{1}{2}(x-1)^2$$

### Exercise

Find the  $n$ th-order Taylor polynomial centered at  $c$  for the function  $f(x) = \sinh 2x$ ;  $n = 4$ ,  $c = 0$

### Solution

$$f(x) = \sinh 2x \quad \rightarrow f(0) = 0$$

$$f'(x) = 2 \cosh 2x \quad \rightarrow f'(0) = 2$$

$$f''(x) = 4 \sinh 2x \quad \rightarrow f''(0) = 0$$

$$f'''(x) = 8 \cosh 2x \quad \rightarrow f'''(0) = 8$$

$$f^{(iv)}(x) = 16 \sinh 2x \quad \rightarrow f^{(iv)}(0) = 0$$

$$P(x) = 2x - \frac{8}{3!}x^3$$
$$= 2x - \frac{1}{6}(2x)^3$$

### Exercise

Find the  $n$ th-order Taylor polynomial centered at  $c$  for the function  $f(x) = \cosh x$ ;  $n = 3$ ,  $c = \ln 2$

### Solution

$$f(x) = \cosh x \quad \rightarrow f(\ln 2) = \frac{e^{\ln 2} + e^{-\ln 2}}{2}$$
$$= \frac{1}{2} \left( 2 + \frac{1}{2} \right)$$

$$\underline{= \frac{5}{4}} \quad \Big|$$

$$f'(x) = \sinh x \quad \rightarrow f'(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2}$$

$$= \frac{1}{2} \left( 2 - \frac{1}{2} \right)$$

$$\underline{= \frac{3}{4}} \quad \Big|$$

$$f''(x) = \cosh x \quad \rightarrow f''(\ln 2) = \frac{e^{\ln 2} + e^{-\ln 2}}{2}$$

$$= \frac{1}{2} \left( 2 + \frac{1}{2} \right)$$

$$\underline{= \frac{5}{4}} \quad \Big|$$

$$f'''(x) = \sinh x \quad \rightarrow f'''(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2}$$

$$= \frac{1}{2} \left( 2 - \frac{1}{2} \right)$$

$$\underline{= \frac{3}{4}} \quad \Big|$$

$$P(x) = \frac{5}{4} + \frac{3}{4}(x - \ln 2) + \frac{5}{4} \frac{1}{2!}(x - \ln 2)^2 + \frac{3}{4} \frac{1}{3!}(x - \ln 2)^3$$

$$\underline{= \frac{5}{4} + \frac{3}{4}(x - \ln 2) + \frac{5}{8}(x - \ln 2)^2 + \frac{1}{8}(x - \ln 2)^3} \quad \Big|$$

### ***Exercise***

Find the sums of the series  $1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \dots$

### **Solution**

$$1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \dots = 1 + \left(x^2\right)^1 + \frac{\left(x^2\right)^2}{2!} + \frac{\left(x^2\right)^3}{3!} + \frac{\left(x^2\right)^4}{4!} + \dots$$

$$\underline{= e^{x^2}} \quad \Big|$$

### ***Exercise***

Find the sums of the series  $1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \frac{x^8}{9!} + \dots$

### **Solution**

$$1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \frac{x^8}{9!} + \dots = \frac{1}{x} \left( x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \dots \right)$$

$$= \frac{1}{x} \sinh x$$

$$= \begin{cases} \frac{e^x - e^{-x}}{2x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

### Exercise

Find the sums of the series  $x^3 - \frac{x^9}{3! \times 4} + \frac{x^{15}}{5! \times 16} - \frac{x^{21}}{7! \times 64} + \frac{x^{27}}{9! \times 256} - \dots$

### Solution

$$x^3 - \frac{x^9}{3! \times 4} + \frac{x^{15}}{5! \times 16} - \frac{x^{21}}{7! \times 64} + \frac{x^{27}}{9! \times 256} - \dots = 2 \left( \frac{x^3}{2} - \frac{1}{3!} \left( \frac{x^3}{2} \right)^3 + \frac{1}{5!} \left( \frac{x^3}{2} \right)^5 - \frac{1}{7!} \left( \frac{x^3}{2} \right)^7 + \dots \right)$$

$$\underline{= 2 \sin \left( \frac{x^3}{2} \right)} \quad \text{for all } x$$

### Exercise

Use the geometric series  $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ , for  $|x| < 1$ , to determine the Maclaurin series and the interval of convergence for the following function

$$f(x) = \frac{1}{1-x^2}$$

### Solution

The Maclaurin series for  $f(x) = \frac{1}{1-x^2}$  is  $\sum_{k=0}^{\infty} x^{2k}$

By the *Root test*:

$$\sqrt[k]{x^{2k}} = x^2 < 1$$

$$-1 < x < 1$$

At  $x = -1$ ,

the series is  $\sum (-1)^{2k} = \sum 1$  which diverges

At  $x = 1$ ,

the series is  $\sum (1)^{2k} = \sum 1$  which diverges

The *interval* of convergence is the real line  $(-1, 1)$

### Exercise

Use the geometric series  $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ , for  $|x| < 1$ , to determine the Maclaurin series and the interval of convergence for the following function

$$f(x) = \frac{1}{1+x^3}$$

### Solution

The Maclaurin series for  $f(x) = \frac{1}{1+x^3}$  is

$$\sum_{k=0}^{\infty} (-x)^{3k} = \sum_{k=0}^{\infty} (-1)^k x^{3k}$$

By the *Root test*:

$$\sqrt[k]{|(-x)^{3k}|} = x^3 < 1$$
$$-1 < x < 1$$

At  $x = -1$ ,

the series is  $\sum (1)^{3k} = \sum 1$  which diverges

At  $x = 1$ ,

the series is  $\sum (-1)^{3k} = \sum (-1)^k$  which diverges absolutely (*harmonic*)

The *interval* of convergence is the real line  $(-1, 1)$

### Exercise

Use the geometric series  $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ , for  $|x| < 1$ , to determine the Maclaurin series and the interval of convergence for the following function

$$f(x) = \frac{1}{1+5x}$$

### Solution

The Maclaurin series for  $f(x) = \frac{1}{1+5x}$  is

$$\sum_{k=0}^{\infty} (-5x)^k = \sum_{k=0}^{\infty} (-5)^k x^k$$

By the *Root test*:

$$\sqrt[k]{|(-5x)^k|} = |5x| < 1$$

$$-\frac{1}{5} < x < \frac{1}{5}$$

At  $x = -\frac{1}{5}$ ,

the series is  $\sum (1)^k = \sum 1$  which diverges

At  $x = \frac{1}{5}$ ,

the series is  $\sum (-1)^k$  which diverges absolutely (*harmonic*)

The *interval* of convergence is the real line  $\left(-\frac{1}{5}, \frac{1}{5}\right)$

### ***Exercise***

Use the geometric series  $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ , for  $|x| < 1$ , to determine the Maclaurin series and the interval of convergence for the following function

$$f(x) = \frac{10}{1+x}$$

### **Solution**

The Maclaurin series for  $f(x) = \frac{10}{1+x}$  is

$$\sum_{k=0}^{\infty} 10(-x)^k$$

By the *Root test*:

$$\sqrt[k]{|(-x)^k|} = |x| < 1$$

$$-1 < x < 1$$

At  $x = -1$ ,

the series is  $\sum 10(1)^k = \sum 10$  which diverges

At  $x = 1$ ,

the series is  $\sum 10(-1)^k$  which diverges absolutely (*harmonic*)

The interval of convergence is the real line  $(-1, 1)$

### Exercise

Use the geometric series  $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ , for  $|x| < 1$ , to determine the Maclaurin series and the interval of convergence for the following function

$$f(x) = \frac{1}{(1-10x)^2}$$

### Solution

$$\frac{1}{1-10x} = \sum_{k=0}^{\infty} (10x)^k$$

$$\frac{1}{10} \cdot \frac{1}{1-10x} = \frac{1}{10} \sum_{k=0}^{\infty} (10x)^k$$

$$\left( \frac{1}{10} \cdot \frac{1}{1-10x} \right)' = \frac{1}{(1-10x)^2}$$

Thus, the Maclaurin series for  $f(x)$

$$\begin{aligned} \left( \frac{1}{10} \sum_{k=0}^{\infty} (10x)^k \right)' &= \frac{1}{10} \sum_{k=0}^{\infty} 10k (10x)^{k-1} \\ &= \sum_{k=0}^{\infty} k (10x)^{k-1} \end{aligned}$$

$$L = \lim_{k \rightarrow \infty} \left| \frac{k(10x)^{k-1}}{(k+1)(10x)^k} \right| \qquad L = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right|$$

$$= \left| (10x)^{-1} \right| < 1$$

$$|x| < \frac{1}{10} \rightarrow -\frac{1}{10} < x < \frac{1}{10}$$

At  $x = -\frac{1}{10}$ ,

the series is  $\sum k(-1)^k$  which diverges absolutely

At  $x = \frac{1}{10}$ ,

the series is  $\sum k$  which diverges

The *interval* of convergence is the real line  $\left(-\frac{1}{10}, \frac{1}{10}\right)$

### Exercise

Use the geometric series  $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ , for  $|x| < 1$ , to determine the Maclaurin series and the interval of convergence for the following function

$$f(x) = \ln(1-4x)$$

### Solution

$$\begin{aligned} -4 \int \frac{dx}{1-4x} &= \int \frac{d(1-4x)}{1-4x} \\ &= \ln(1-4x) \end{aligned}$$

$$\begin{aligned} \int \frac{dx}{1-4x} &= -\frac{1}{4} \ln(1-4x) \\ &= -\frac{1}{4} f(x) \end{aligned}$$

$$\int \left( -\frac{1}{4} \sum_{k=0}^{\infty} (4x)^k \right) = - \sum_{k=0}^{\infty} \frac{1}{k+1} (4x)^{k+1}$$

$$f(x) = - \sum_{k=0}^{\infty} \frac{1}{k+1} (4x)^{k+1}$$

$$\begin{aligned} L &= \lim_{k \rightarrow \infty} \left| \frac{(4x)^{k+2}}{k+2} \cdot \frac{k+1}{(4x)^{k+1}} \right| & L &= \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| \\ &= \lim_{k \rightarrow \infty} \frac{k+1}{k+2} |4x| \\ &= |4x| < 1 \\ |x| < \frac{1}{4} &\rightarrow -\frac{1}{4} < x < \frac{1}{4} \end{aligned}$$

At  $x = -\frac{1}{4}$ ,

the series is  $f(x) = \ln 2$  which converges

At  $x = \frac{1}{4}$ ,

the series is  $f(x) = \ln(0) = -\infty$  which diverges

The *interval* of convergence is the real line  $\left[-\frac{1}{4}, \frac{1}{4}\right)$

## Exercise

The limit  $\lim_{n \rightarrow \infty} \frac{n!}{\sqrt{2\pi} n^{n+1/2} e^{-n}} = 1$  that is the relative error in the approximation  $n! \approx \sqrt{2\pi} n^{n+1/2} e^{-n}$

Approaches zero as  $n$  increases. That is  $n!$  grows at a rate comparable to  $\sqrt{2\pi} n^{n+1/2} e^{-n}$ . This result, known as Stirling's Formula, is often very useful in applied mathematics and statistics. Prove it by carrying out the following steps.

a) Use the identity  $\ln(n!) = \sum_{j=1}^n \ln j$  and the increasing nature of  $\ln$  to show that if  $n \geq 1$ ,

$$\int_0^n \ln x \, dx < \ln(n!) < \int_1^{n+1} \ln x \, dx$$

And hence that  $n \ln n - n < \ln(n!) < (n+1) \ln(n+1) - n$

b) If  $c_n = \ln(n!) - \left(n + \frac{1}{2}\right) \ln n + n$ , show that

$$\begin{aligned} c_n - c_{n+1} &= \left(n + \frac{1}{2}\right) \ln\left(\frac{n+1}{n}\right) - 1 \\ &= \left(n + \frac{1}{2}\right) \ln\left(\frac{1 + \frac{1}{2n+1}}{1 - \frac{1}{2n+1}}\right) - 1 \end{aligned}$$

c) Use the Maclaurin series for  $\ln \frac{1+t}{1-t}$  to show that

$$\begin{aligned} 0 < c_n - c_{n+1} &< \frac{1}{3} \left( \frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^4} + \frac{1}{(2n+1)^6} + \dots \right) \\ &= \frac{1}{12} \left( \frac{1}{n} - \frac{1}{n+1} \right) \end{aligned}$$

and therefore that  $\{c_n\}$  is decreasing and  $\{c_n - \frac{1}{12n}\}$  is increasing. Hence conclude that  $\lim_{n \rightarrow \infty} c_n = c$

exists, and that

$$\lim_{n \rightarrow \infty} \frac{n!}{n^{n+1/2} e^{-n}} = \lim_{n \rightarrow \infty} e^{c_n} = e^c$$

## Solution

$$a) \quad \ln(k-1) < \int_{k-1}^k \ln x \, dx < \ln k, \quad k = 1, 2, 3, \dots$$

$$\begin{aligned} n \ln n - n &= \int_0^n \ln x \, dx < \ln(n!) < \int_1^{n+1} \ln x \, dx \\ &= (n+1) \ln(n+1) - n - 1 \end{aligned}$$



$$< (n+1) \ln(n+1)$$

**b)** If  $c_n = \ln(n!) - \left(n + \frac{1}{2}\right) \ln n + n$ , then

$$\begin{aligned}
 c_n - c_{n+1} &= \ln(n!) - \left(n + \frac{1}{2}\right) \ln n + n - \left[ \ln((n+1)!) - \left(n + 1 + \frac{1}{2}\right) \ln(n+1) + n + 1 \right] \\
 &= \ln(n!) - \left(n + \frac{1}{2}\right) \ln n - \ln((n+1)!) + \left(n + \frac{1}{2} + 1\right) \ln(n+1) - 1 \\
 &= \ln(n!) - \ln((n+1)!) - \left(n + \frac{1}{2}\right) \ln n^{n+1/2} + \left(n + \frac{1}{2}\right) \ln(n+1) + \ln(n+1) - 1 \\
 &= \ln\left(\frac{n!}{(n+1)!}\right) - \left(n + \frac{1}{2}\right) \ln \frac{n}{n+1} + \ln(n+1) - 1 \\
 &= \ln\left(\frac{1}{n+1}\right) - \left(n + \frac{1}{2}\right) \ln \frac{n}{n+1} + \ln(n+1) - 1 \\
 &= -\ln(n+1) + \left(n + \frac{1}{2}\right) \ln\left(\frac{n}{n+1}\right)^{-1} + \ln(n+1) - 1 \\
 &= \left(n + \frac{1}{2}\right) \ln\left(\frac{n+1}{n}\right) - 1 \\
 &= \left(n + \frac{1}{2}\right) \ln\left(\frac{2n+2}{2n}\right) - 1 \\
 &= \left(n + \frac{1}{2}\right) \ln\left(\frac{\frac{2n+2}{2n+1}}{\frac{2n}{2n+1}}\right) - 1 \\
 &= \left(n + \frac{1}{2}\right) \ln\left(\frac{\frac{2n+1+1}{2n+1}}{\frac{2n+1-1}{2n+1}}\right) - 1 \\
 &= \left(n + \frac{1}{2}\right) \ln\left(\frac{1 + \frac{1}{2n+1}}{1 - \frac{1}{2n+1}}\right) - 1
 \end{aligned}$$

**c)**  $\ln \frac{1+t}{1-t} = \ln(1+t) - \ln(1-t)$

$$\begin{aligned}
 &= t - \frac{t^2}{2} + \frac{t^3}{3} - \dots + t + \frac{t^2}{2} + \frac{t^3}{3} - \dots \\
 &= 2\left(t + \frac{t^3}{3} + \frac{t^5}{5} + \dots\right) \quad \text{for } -1 < t < 1
 \end{aligned}$$

$$0 < c_n - c_{n+1} = \left(\frac{2n+1}{2}\right) \ln\left(\frac{1 + \frac{1}{2n+1}}{1 - \frac{1}{2n+1}}\right) - 1$$

$$\ln \frac{1+t}{1-t} = 2\left(t + \frac{t^3}{3} + \frac{t^5}{5} + \dots\right)$$

$$\begin{aligned}
&= \frac{1}{2}(2n+1)(2) \left( \frac{1}{2n+1} + \frac{1}{3} \left( \frac{1}{2n+1} \right)^3 + \frac{1}{5} \left( \frac{1}{2n+1} \right)^5 + \dots \right) - 1 \\
&= 1 + \frac{1}{3(2n+1)^2} + \frac{1}{5(2n+1)^4} + \dots - 1 \\
&= \frac{1}{3(2n+1)^2} + \frac{1}{5(2n+1)^4} + \frac{1}{7(2n+1)^6} + \dots \\
&< \frac{1}{3} \left( \frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^4} + \dots \right) \quad \text{Geometric series } S_n = a \frac{1}{1-r} \\
&= \frac{1}{3} \cdot \frac{1}{(2n+1)^2} \cdot \frac{1}{1 - \frac{1}{(2n+1)^2}} \\
&= \frac{1}{3} \cdot \frac{1}{(2n+1)^2} \cdot \frac{1}{1 - \frac{1}{(2n+1)^2}} \\
&= \frac{1}{3} \cdot \frac{1}{(2n+1)^2 - 1} \\
&= \frac{1}{3} \cdot \frac{1}{4n^2 + 4n} \\
&= \frac{1}{12} \cdot \frac{1}{n(n+1)} \\
&= \frac{1}{12} \left( \frac{1}{n} - \frac{1}{n+1} \right) \Big|
\end{aligned}$$

These inequalities imply that  $\{c_n\}$  is decreasing and  $\{c_n - \frac{1}{12n}\}$  is increasing.

Thus  $\{c_n\}$  is bounded below by  $c_1 - \frac{1}{12} = 1 - \frac{1}{12} = \frac{11}{12}$   $\left( c_1 = \ln(1!) - \left(1 + \frac{1}{2}\right) \ln 1 + 1 \right)$

So  $\lim_{n \rightarrow \infty} c_n = c$  exists.

Since  $e^{c_n} = \frac{n!}{n^{(n+1/2)} e^{-n}}$ , we have

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{n!}{n^{(n+1/2)} e^{-n}} &= \lim_{n \rightarrow \infty} e^{c_n} \\
&= e^c \Big| \text{ exists.}
\end{aligned}$$

### Exercise

Suppose you want to approximate  $\sqrt[3]{128}$  to within  $10^{-4}$  of the exact value.

- a) Use a Taylor polynomial for  $f(x) = (125 + x)^{1/3}$  centered at 0.
- b) Use a Taylor polynomial for  $f(x) = x^{1/3}$  centered at 125.
- c) Compare the two approaches. Are they equivalent?

### Solution

$$\sqrt[3]{128} \approx 5.03968419958 \quad |$$

a)  $a = 0$

$$f(x) = (125 + x)^{1/3} \quad \rightarrow \quad f(0) = (125)^{1/3} = 5$$

---

$$\rightarrow f'(0) = \frac{1}{3}(125)^{-2/3}$$

$$f'(x) = \frac{1}{3}(125 + x)^{-2/3} \quad = \frac{1}{3}(5^3)^{-2/3} = \frac{1}{3}(5)^{-2} \\ = \frac{1}{75}$$

---

$$f''(x) = -\frac{2}{9}(125 + x)^{-5/3} \quad \rightarrow \quad f''(0) = -\frac{2}{9}(5^3)^{-5/3} = -\frac{2}{9} \frac{1}{5^5} \\ = -\frac{2}{28,125}$$

---

$$f(x) = 5 + \frac{1}{75}x - \frac{1}{28,125}x^2$$

$$125 + x = 128 \quad \Rightarrow \quad \underline{x = 3}$$

$$f(3) = 5 + \frac{1}{75}(3) - \frac{1}{28,125}(9) \\ = 5 + \frac{1}{25} - \frac{1}{3,125} \\ = 5 + .04 - .00032 \\ \approx \underline{5.03968} \quad |$$

b)  $a = 125 = 5^3$

$$f(x) = x^{1/3} \quad \rightarrow \quad f(125) = 5$$

---

$$f'(x) = \frac{1}{3}x^{-2/3} \quad \rightarrow \quad f'(0) = \frac{1}{3}(5^3)^{-2/3} \\ = \frac{1}{75}$$

---

---


$$f''(x) = -\frac{2}{9}x^{-5/3} \quad \rightarrow f''(1) = -\frac{2}{9}(5^3)^{-5/3}$$

$$= -\frac{2}{28,125}$$


---

$$f(x) = 5 + \frac{1}{75}(x-125) - \frac{1}{28,125}(x-125)^2$$

$$f(128) = 5 + \frac{1}{75}(3) - \frac{1}{28,125}(3)^2$$

$$= 5 + \frac{1}{25} - \frac{1}{3,125}$$

$$= 5 + .04 - .00032$$

$$\approx 5.03968$$

c) Both the results from part (a) and (b) are the same since they are just shifting.

### Exercise

Consider the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

- Use the definition of the derivative to show that  $f'(0) = 0$
- Assume the fact that  $f^k(0) = 0$  for  $k = 1, 2, 3, \dots$ . Write the Taylor series for  $f$  centered at 0.
- Explain why the Taylor series for  $f$  does not converge to  $f$  for  $x \neq 0$

### Solution

$$a) \quad f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \rightarrow 0} \frac{e^{-1/x^2} - 0}{x}$$

$$= \lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x} = \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{2}{x^3} e^{-1/x^2}}{1}$$

$$= \lim_{x \rightarrow 0} \frac{2e^{-1/x^2}}{x^3} = \frac{0}{0}$$

$$\text{Let } y = \frac{1}{x^2} \Rightarrow x = \frac{1}{\sqrt{y}}$$

$$x \rightarrow 0 \Rightarrow y \rightarrow \infty$$

$$\begin{aligned} f'(0) &= \lim_{y \rightarrow \infty} \frac{e^{-y}}{\frac{1}{\sqrt{y}}} \\ &= \lim_{y \rightarrow \infty} \frac{\sqrt{y}}{e^y} \\ &= 0 \end{aligned}$$

b) **Given:**  $f^k(0) = 0$

Since the Taylor series centered at 0 has only one term  $f(x) = f(0) = 0$  and  $f^k(0) = 0$  (derivatives are equal to 0).

Therefore; the Taylor series is zero.

c) It does not converge to  $f(x)$  because when  $x \neq 0$ ,  $f(x) \neq 0$

### Exercise

Teams  $A$  and  $B$  go into sudden death overtime after playing to a tie. The teams alternate possession of the ball and the first team to score wins. Each team has a  $\frac{1}{6}$  chance of scoring when it has the ball, with Team  $A$  having the ball first.

a) The probability that Team  $A$  ultimately wins is  $\sum_{k=0}^{\infty} \frac{1}{6} \left(\frac{5}{6}\right)^{2k}$ . Evaluate this series.

b) The expected number of rounds (possessions by either team) required for the overtime to end is

$$\frac{1}{6} \sum_{k=1}^{\infty} k \left(\frac{5}{6}\right)^{k-1}. \text{ Evaluate this series.}$$

### Solution

a)  $\sum_{k=0}^{\infty} \frac{1}{6} \left(\frac{5}{6}\right)^{2k} = \frac{1}{6} \sum_{k=0}^{\infty} \left(\frac{25}{36}\right)^k$

It is a *Geometric* series with  $r = \frac{25}{36} < 1$ , then

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{6} \left(\frac{5}{6}\right)^{2k} &= \frac{1}{6} \cdot \frac{1}{1 - \frac{25}{36}} \\ &= \frac{1}{6} \cdot \frac{36}{11} \end{aligned}$$

$$\underline{= \frac{6}{11} \mid}$$

**b)** Using the series  $\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$

$$\left( \sum_{k=1}^{\infty} x^k \right)' = \left( \frac{x}{1-x} \right)'$$

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

Let  $x = \frac{5}{6}$

$$\frac{1}{6} \sum_{k=1}^{\infty} k \left( \frac{5}{6} \right)^{k-1} = \frac{1}{6} \frac{1}{\left( 1 - \frac{5}{6} \right)^2}$$

$$= \frac{1}{6} \frac{1}{\left( \frac{1}{6} \right)^2}$$

$$\underline{= 6 \mid}$$

