Section 2.3 – Probability Rules, Addition Rule and Complements

Definitions

Probability is a measure of the likelihood of a random phenomenon or chance behavior. Probability describes the long-term proportion with which a certain ***outcome*** will occur in situations with short-term uncertainty.

Use the probability applet to simulate flipping a coin 100 times. Plot the proportion of heads against the number of flips. Repeat the simulation.

Probability deals with experiments that yield random short-term results or outcomes, yet reveal long-term predictability.

The long-term proportion in which a certain outcome is observed is the probability of that outcome.

The Law of Large Numbers

As the number of repetitions of a probability experiment increases, the proportion with which a certain outcome is observed gets closer to the probability of the outcome.

Definitions

In probability, an ***experiment*** is any process that can be repeated in which the results are uncertain.

An ***event*** is any collection of results or outcomes of a procedure

A ***simple event*** is an outcome or an event that cannot be further broken down into simpler components

The ***sample space*** for a procedure consists of all possible simple events; that is, the sample space consists of ***all outcomes*** that cannot be broken down any further

Example

We use "*f*" to denote a female baby and "*m*" to denote a male baby.

Procedure	***Example of Event***	***Complete Sample Space***
Single birth	1 female (simple event)	$\{f, m\}$
3 births	2 females and 1 male (*ffm*, *fmf*, *mff*) are simple events	$\{fff, ffm, fmf, mff, mfm, mmf, mmm\}$

Notation for Probabilities

P – denotes a probability

A, B, and C – denote specific events

$P(A)$ - denotes the probability of event A occurring.

Basic Rules for Computing Probability

***Rule 1*: Relative Frequency Approximation of Probability**

Conduct (or observe) a procedure, and count the number of times event *A* actually occurs. Based on these actual results, $P(A)$ is approximated as follows:

$$P(A) = \frac{\#\ of\ \ times\ A\ occured}{\#\ of\ \ times\ procedure\ was\ repeated}$$

***Rule 2*: Classical Approach to Probability (**Requires Equally Likely Outcomes**)**

Assume that a given procedure has *n* different simple events and that each of those simple events has an equal chance of occurring. If event *A* can occur in *s* of these *n* ways, then

$$P(A) = \frac{m}{n} = \frac{\#\ of\ \ ways\ A\ can\ occur}{\#\ of\ \ different\ simple\ events} = \frac{N(E)}{N(S)}$$

***Rule 3*: Subjective Probabilities**

$P(A)$, the probability of event *A*, is estimated by using knowledge of the relevant circumstances.

Rules of probabilities

1. The probability of any event *E*, $P(E)$, must be greater than or equal to 0 and less than or equal to 1. That is, $0 \le P(E) \le 1$.
2. The sum of the probabilities of all outcomes must equal 1. That is, if the sample space

$$S = \{e_1, e_2, \ldots, e_n\} \quad then \quad P(e_1) + P(e_2) + \ \ldots + P(e_n) = 1$$

Example

Find the probability that a randomly selected car in U.S. will be in a crash this year. There were 6,511,100 cars that crashed among the 135,670,000 cars registered.

Solution

$$P(crash) = \frac{\#\ of\ cars\ that\ crashed}{Total\ \ number\ of\ cars} = \frac{6,511,100}{135,670,000} \ \underline{= 0.048}$$

Example

When studying the effect of heredity on height, we can express each individual genotype, AA, Aa, aA, and aa, on an index card and shuffle the four cards and randomly select one of them. What is the probability that we select a genotype in which the two components are different?

Solution

$$P(outcome\ with\ different\ components) = \frac{2}{4} \ \underline{= 0.5}$$

Example

What is the probability that you will get stuck in the next elevator that you ride?

Solution

There are 2 possible outcomes (stuck or not becoming stuck). But that are not equally likely, so we cannot use the classical approach.
The leaves us with a subjective estimate, in this case, say 0.0001 (equivalent to 1 chance in ten thousand). This is likely to be in general ballpark of the true probability.

Example

Find the probability that when a couple has 3 children, they will have exactly 2 boys. Assume that boys and girls are equally likely and that the gender of any child is not influenced by the gender of any other child.

Solution

$$S = \{bbb,\ bbg,\ bgb,\ gbb,\ bgg,\ gbg,\ ggb,\ ggg\}$$

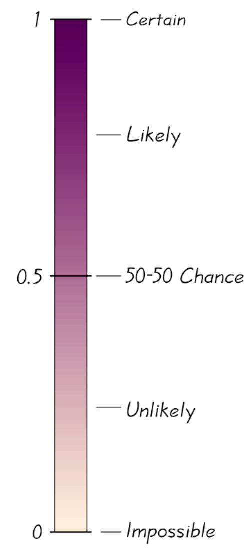
$$P(2\ boys) = \frac{3}{8} = 0.375$$

Probability Limits

Always express a probability as a fraction or decimal number between 0 and 1.

- ✓ The probability of an impossible event is 0.
- ✓ The probability of an event that is certain to occur is 1.
- ✓ For any event *A*, the probability of *A* is between 0 and 1 inclusive.

That is, $0 \le P(A) \le 1$.



Example

If a year is selected at random, find the probability that Thanksgiving Day will be

a) On a Wednesday
b) On a Thursday

Solution

a) $P(On\ a\ Wednesday) = 0$

b) It is certain that Thanksgiving Day will be on a Thursday. When an event certain to occur:

$$P(On\ a\ Thursday) = 1$$

Addition Rule and Complements

Definition

A ***compound event*** is any event combining 2 or more simple events

Notation for Addition Rule

$P(A \text{ or } B) = P(\text{in a single trial, event } A \text{ occurs or event } B \text{ occurs or they both occur})$

Example

If 1 subject is randomly selected from the 98 subjects given the polygraph, find the probability of selecting a subject who had a positive test result or lied.

	No (Did Not Lie)	***Yes*** (Lied)
Positive test result	15 (*false positive*)	42 (*true positive*)
Negative test result	32 (*true negative*)	9 (*false negative*)

Solution

There are 66 subjects who had a positive test result or lied.

$P(\text{positive test result or lied}) = \frac{66}{98} = 0.673$

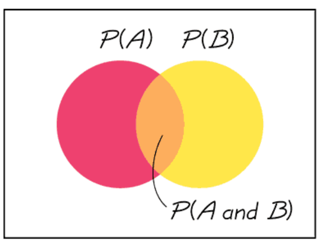
- When finding the probability that event *A* occurs or event *B* occurs, find the total number of ways *A* can occur and the number of ways *B* can occur, but *find that total in such a way that no outcome is counted more than once*.

Formal Addition Rule

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

where *P*(*A* and *B*) denotes the probability that *A* and *B* both occur at the same time as an outcome in a trial of a procedure.

Total Area = 1

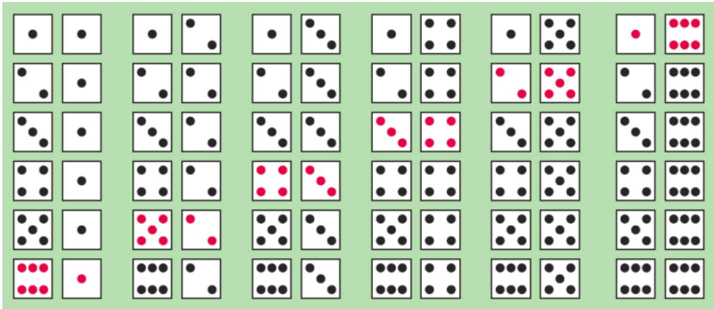


Intuitive Addition Rule

To find $P(A \text{ or } B)$, find the sum of the number of ways event A can occur and the number of ways event B can occur, *adding in such a way that every outcome is counted only once*. $P(A \text{ or } B)$ is equal to that sum, divided by the total number of outcomes in the sample space.

Example

Suppose that a pair of dice are thrown. Let E = "the first die is a two" and let F = "the sum of the dice is less than or equal to 5". Find $P(E$ ***or*** $F)$



Solution

$$P(E)=\frac{N(E)}{N(S)}=\frac{6}{36}$$

$$P(F)=\frac{N(F)}{N(S)}=\frac{10}{36}$$

$$P(E \text{ and } F)=\frac{N(E \text{ and } F)}{N(S)}=\frac{3}{36}$$

$$P(E \text{ or } F)=P(E)+P(F)-P(E \text{ and } F)$$

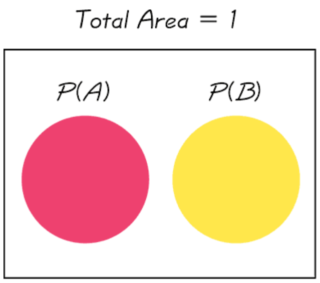
$$=\frac{6}{36}+\frac{10}{36}-\frac{3}{36}$$

$$\boxed{=\frac{13}{36}}$$

Disjoint or Mutually Exclusive

Definition

Events A and B are ***disjoint*** (or ***mutually exclusive***) if they cannot occur at the same time. (That is, disjoint events do not overlap.)



Example

	No (Did Not Lie)	***Yes*** (Lied)
Positive test result	15 (*false positive*)	42 (*true positive*)
Negative test result	32 (*true negative*)	9 (*false negative*)

a) Consider the procedure of randomly selecting 1 of the 98 subjects. Determine whether the following event are disjoint:

A: Getting a subject with a negative test result.

B: Getting a subject who did not lie.

b) Assuming that 1 subject is randomly selected from the 98 that were tested, find the probability of selecting a subject who had a negative test result or did not lie.

c) Assuming that one of the 98 test results summarized in the table above is randomly selected, find the probability that it is a positive test result.

Solution

a) There are 41 subjects with negative test results and are 47 subjects who did lie. The event of getting a subject with a negative test result and getting a subject who did not lie can occur at the same time, there are 32 subjects. Therefore, the events are not disjoint.

b) $P(\textit{negative test result or did not lie}) = \frac{56}{98} \underline{= 0.571}$

c) $P(\textit{positive result}) = \frac{\#\ \textit{of positive results}}{\textit{Total number of results}} = \frac{42+15}{15+42+32+9} = \frac{57}{98} \underline{= 0.582}$

Complementary Events

The complement of event *A*, denoted by $\bar{A}$, consists of all outcomes in which the event *A* ***does not*** occur.

$P(A)$ and $P(\bar{A})$ are disjoint

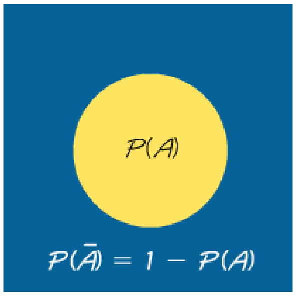
It is impossible for an event and its complement to occur at the same time.

Rule of Complementary Events

$$P(A)+P(\bar{A})=1$$

$$P(\bar{A})=1-P(A)$$

$$P(A)=1-P(\bar{A})$$



Example

FBI data show that 62.4% of murders are cleared by arrests. We can express the probability of a murder being cleared by an arrest as $P(cleared)=0.624$. For a randomly selected murder, find $P(\overline{cleared})$

Solution

$$\begin{aligned} P(\overline{cleared}) &= 1-P(cleared) \\ &= 1-.624 \\ &= 0.376 \end{aligned}$$

Example

A typical question on a SAT test requires the test taker to select one of five possible choices: A, B, C, D, or E. Because only one answer is correct, if you make a random guess, your probability of being correct is $\frac{1}{5}$ *or* 0.2. Find the probability of making a random guess and not being correct (or being incorrect).

Solution

$$P(\text{not guessing the correct answer}) = P(\overline{correct}) = \frac{4}{5} = 0.8$$

or $\quad P(\overline{correct}) = 1-P(correct) = 1-\frac{1}{5} = \frac{4}{5} = 0.8$

Odds

Definition

The ***actual odds against*** event A occurring are the ratio $\frac{P(\bar{A})}{P(A)}$, usually expressed in the form of a:b (or "a to b"), where a and b are integers having no common factors.

The ***actual odds in favor*** of event A occurring are the ratio $\frac{P(A)}{P(\bar{A})}$, which is the reciprocal of the actual odds against the event. If the odds against A are a:b, then the odds in favor of A are b:a.

The ***payoff odds*** against event A occurring are the ratio of the net profit (if you win) to the amount bet.

$$payoff\ \ odds\ against\ event\ A = \frac{net\ profit}{amount\ bet}$$

Example

If you bet \$5 on the number 13 in roulette, your probability of winning is $\frac{1}{38}$ and the payoff odds are given by the casino as 35:1.

a) Find the actual odds against the outcome of 13.
b) How much net profit would you make if you win by betting on 13?
c) If the casino was not operating for profit, and the payoff odds were changed to match the actual odds against 13, how much would you win if the outcome were 13?

Solution

a) With odds: $P(13) = \frac{1}{38}$ and $P(not\ 13) = \frac{37}{38}$

Actual odds against 13 $\frac{P(not\ 13)}{P(13)} = \frac{37/38}{1/38} = \frac{37}{1}$ or 37:1

b) Because the payoffs odds against 13 are 35:1, we have:

35:1 = (*net profit*) : (*amount bet*)

So there is a \$35 profit for each \$1 bet. For \$5 bet, the net profit is $5 \times 35 = \$175$.
The winning bettor would collect \$175 plus the original \$5 bet. That is, the total amount collected would be \$180, for the net profit of \$175.

c) If the casino were not operating for profit, the payoff odds would be equal to the actual odds against the outcome of 13, or 37:1. So there is a net profit of \$37 for each \$1 bet.
For a \$5 bet the net profit would be \$185. (The casino makes its profit by paying only \$175 instead of the \$185 that would be paid with a roulette game that is fair instead of favoring the casino.)

Rounding Off Probabilities

When expressing the value of a probability, either give the ***exact*** *fraction* or decimal or round off final decimal results to three significant digits. (*Suggestion*: When a probability is not a simple fraction such as $\frac{2}{3}$ or $\frac{5}{9}$, expresses it as a decimal so that the number can be better understood.)

- ✓ The probability of $\frac{1}{3}$ can be left as a fraction, or rounded to 0.333. (Do ***not*** round to 0.3)
- ✓ The probability of $\frac{2}{4}$ can be expressed as $\frac{1}{2}$ or 0.5; because is exact, there is no need to express as 0.500.
- ✓ $\frac{1941}{3405} = 0.570$

Exercises *Section* 2.3 – Probability Rules, Addition Rule and Complements

1. Based on recent results, the probability of someone in the U.S. being injured while using sports or recreation equipment is $\frac{1}{500}$ (based on data from Statistical Abstract of the U.S.). What does it mean when we say that the probability is $\frac{1}{500}$? Is such an injury unusual?

2. When a baby is born, there is approximately a 50–50 chance that the baby is a girl. Indicate the degree of likelihood as a probability value between 0 and 1.

3. When a rolling a single die, there are 6 chances in 36 that the outcome is a 7. Indicate the degree of likelihood as a probability value between 0 and 1.

4. Identify probability values

a) What is the probability of an event that is certain to occur?
b) What is the probability of an impossible event?
c) A sample space consists of 10 separate events that are equally likely. What is the probability of each?
d) On a true/false test, what is the probability of answering a question correctly if you make a random guess?
e) On a multiple-choice test with five possible answers for each question, what is the probability of answering a question correctly if you make a random guess?

5. When a couple has 3 children, find the probability of each event.

a) There is exactly one girl. *b)* There are exactly 2 girls. *c)* All are girls

6. The 110th Congress of the U.S. included 84 male Senators and 16 female Senators. If one of these Senators is randomly selected, what is the probability that a woman is selected? Does this probability agree with a claim that men and women have the same chance of being elected as Senators?

7. When Mendel conducted his famous genetics experiments with peas, one sample of offspring consisted of 428 green peas and 152 yellow peas. Based on those results, estimate the probability of getting an offspring pea that is green, Is the result reasonably close to the expected value of $\frac{3}{4}$, as claimed by Mendel?

8. A single fair die is rolled. Find the probability of each event

a) Getting a 2
b) Getting an odd number
c) Getting a number less than 5
d) Getting a number greater than 2
e) Getting any number except 3

9. A jar contains 3 white, 4 orange, 5 yellow, and 8 black marbles. If a marble is drawn at random, find the probability that it is the following.

a) White *b)* Orange *c)* Yellow *d)* Black *e)* Not black

10. The student sitting next to you in class concludes that the probability of the ceiling falling down on both of you before class ends is 1/2, because there are two possible outcomes - the ceiling will fall or not fall. What is wrong with this reasoning?

11. Let consider rolling 2 dice. Find the probabilities of the following events

a) ***E*** = Sum of 5 turns up

b) ***F*** = a sum that is a prime number greater than 7 turns up

12. A poll was conducted preceding an election to determine the relationship between voter persuasion concerning a controversial issue and the area of the city in which the voter lives. Five hundred registered voters were interviewed from three areas of the city. The data are shown below. Compute the probability of having no opinion on the issue or living in the inner city.

Area of city	***Favor***	***Oppose***	***No Opinion***
East	30	40	55
North	25	45	50
Inner	95	65	85

13. Suppose a single fair die is rolled. Use the sample space $S = \{1,2,3,4,5,6\}$ and give the probability of each event.

a) E: the die shows an even number

b) F: the die show a number less than 10

c) G: the die shows an 8

14. A solitaire game was played 500 times. Among the 500 trials, the game was won 77 times. (The results are from the solitaire game, and the Vegas rules of "draw 3" with $52 bet and a return of $5 per card are used). Based on these results, find the odds against winning.

15. A roulette wheel has 38 slots. One slot is 0, another is 00, and the others are numbered 1 through 36, respectively. You place a bet that the outcome is an odd number.

a) What is your probability of winning?

b) What are the actual odds against winning?

c) When you bet that the outcome is an odd number, the payoff odds are 1:1. How much profit do you make if you bet $18 and win?

d) How much profit would you make on the $18 bet if you could somehow convince the casino to change its payoff odds so that they are the same as the actual odds against winning?

16. Women have a 0.25% rate of red/green color blindness. If a woman is randomly selected, what is the probability that she does not have red/green color blindness?

17. A pew Research center poll showed that 79% of Americans believe that it is morally wrong to not report all income on tax returns. What is the probability that an American does not have that belief?

18. When the author observed a sobriety checkpoint conducted by the Dutchess County Sheriff Department, he saw that 676 drivers were screened and 6 were arrested for driving while intoxicated. Based on those results, we can estimate that $P(I) = 0.00888$, where I denotes the event of screening a driver and getting someone who is intoxicated. What does $P(\bar{I})$ denote and what is its value?

19. Use the polygraph test data

	No (Did Not Lie)	***Yes*** (Lied)
Positive test result	15 (*false positive*)	42 (*true positive*)
Negative test result	32 (*true negative*)	9 (*false negative*)

a) If one of the test subjects is randomly selected, find the probability that the subject had a positive test result or did not lie

b) If one of the test subjects is randomly selected, find the probability that the subject did not lie

c) If one of the test subjects is randomly selected, find the probability that the subject had a true negative test result

d) If one of the test subjects is randomly selected, find the probability that the subject had a negative test result or lied.

20. Use the data

Was the challenge to the call successful?		
	Yes	***No***
Men	201	288
Women	126	224

a) If S denotes the event of selecting a successful challenge, find $P(\bar{S})$

b) If M denotes the event of selecting a challenge made by a man, find $P(\bar{M})$

c) Find the probability that the selected challenge was made by a man or was successful.

d) Find the probability that the selected challenge was made by a woman or was successful.

e) Find $P(\textit{challenge was made by a man or was not successful})$

f) Find $P(\textit{challenge was made by a woman or was not successful})$

21. Refer to the table below

Age

	18 – 21	22 – 29	30 – 39	40 – 49	50 – 59	60 and over
Responded	73	255	245	136	138	202
Refused	11	20	33	16	27	49

a) What is the probability that the selected person refused to answer? Does that probability value suggest that refusals are a problem for pollsters? Why or why not?

b) A pharmaceutical company is interested in opinions of the elderly, because they are either receiving Medicare or will receive it soon. What is the probability that the selected subject is someone 60 and over who responded?

c) What is the probability that the selected person responded or is in the 18–21 age bracket?

d) What is the probability that the selected person refused or is over 59 years of age?

e) A market researcher is interested in responses, especially from those between the ages of 22 and 39, because they are the people more likely to make purchases. Find the probability that a selected responds or is aged between the ages of 22 and 39.

f) A market researcher is not interested in refusals or subjects below 22 years of age or over 59. Find the probability that the selected person refused to answer or is below 22 or is older than 59.

22. Two dice are rolled. Find the probabilities of the following events.

a) The first die is 3 or the sum is 8

b) The second die is 5 or the sum is 10.

23. One card is drawn from an ordinary of 52 cards. Find the probabilities of drawing the following cards

a) A 9 or 10

b) A red card or a 3

c) A 9 or a black 10

d) A heart or a black card

e) A face card or a diamond

24. One card is drawn from an ordinary of 52 cards. Find the probabilities of drawing the following cards

a) Less than a 4 (count aces as ones)

b) A diamond or a 7

c) A black card or an ace

d) A heart or a jack

e) A red card or a face card

25. Pam invites relatives to a party: her mother, 2 aunts, 3 uncles, 2 brothers, 1 male cousin, and 4 female cousins. If the chances of any one guest first equally likely, find the probabilities that the first guest to arrive is as follows.

a) A brother or an uncle

b) A brother or a cousin

c) A brother or her mother

d) An uncle or a cousin

e) A male or a cousin

f) A female or a cousin

26. Suppose $P(E)=0.26$, $P(F)=0.41$, and $P(E\cap F)=0.16$. Find the following

a) $P(E\cup F)$ *b)* $P(E'\cap F)$ *c)* $P(E\cap F')$ *d)* $P(E'\cup F')$

27. Suppose $P(E)=0.42$, $P(F)=0.35$, and $P(E\cup F)=0.59$. Find the following

a) $P(E'\cap F')$ *b)* $P(E'\cup F')$ *c)* $P(E'\cup F)$ *d)* $P(E\cap F')$

28. From survey involving 1,000 people in the certain city, it was found that 500 people had tried a certain brand of diet cola, 600 had tried a certain brand of regular cola, and 200 had tried both types of cola. If a resident of the city is selected at random, what is the empirical probability that

a) The resident has not tried either cola? What are the empirical odds for this event?

b) The resident has tried the diet or has not tried the regular cola? What are the empirical odds against this event?

29. In a poll, respondents were asked whether they had ever been in a car accident. 329 respondents indicated that they had been in a car accident and 322 respondents said that they had not been in a car accident. If one of these respondents is randomly selected, what is the probability of getting someone who has been in a car accident?

30. Refer to the table which summarizes the results of testing for a certain disease

	Positive Test Result	***Negative Test Result***
Subject has the disease	114	5
Subject does not have the disease	12	177

If one of the results is randomly selected, what is the probability that it is a false negative (test indicates the person does not have the disease when in fact they do)?. What is the probability suggest about the accuracy of the test?

31. In a certain town, 2% of people commute to work by bicycle. If a person is selected randomly from the town, what are the odds against selecting someone who commutes by bicycle?

32. Suppose you are playing a game of chance, if you bet \$4 on a certain event, you will collect \$176 (including your \$4 bet) if you win. Find the odds used for determining the payoff.

33. The odds in favor of a particular horse winning a race are 4:5.

a) Find the probability of the horse winning.
b) Find the odds against the horse winning.

34. Consider the sample space of equally likely events for the rolling of a single fair die.

a) What is the probability of rolling an odd number **and** a prime number?
b) What is the probability of rolling an odd number **or** a prime number?

35. Suppose that 2 fair Dice are rolled

a) What is the probability of that a sum of 2 or 3 turns up?
b) What is the probability of that both dice turn up the same or that a sum greater than 8 turns up?

36. A single card is drawn from an ordinary of 52 cards. Calculate the probabilities of and odds for each event

a) A face card or a club is drawn
b) A king or a heart is drawn
c) A black card or an ace is drawn
d) A heart or a number less than 7 (count an ace as 1) is drawn.

37. What is the probability of getting at least 1 black card in a 7-card hand dealt from a standard 52-card deck?

38. What is the probability that a number selected at random from the first 600 positive integers is (exactly) divisible by 6 or 9?

39. What is the probability that a number selected at random from the first 1,000 positive integers is (exactly) divisible by 6 or 8?

40. From a survey involving 1,000 students at a large university, a market research company found that 750 students owned stereos, 450 owned cars, and 350 owned cars and stereos. If a student at the university is selected at random, what is the (empirical) probability that

a) The student owns either a car or a stereo?

b) The student owns neither a car nor a stereo?

41. In order to test a new car, an automobile manufacturer wants to select 4 employees to test drive the car for 1 year. If 12 management and 8 union employees volunteer to be test drivers and the selection is made at random, what is the probability that at least 1 union employee is selected.

42. A shipment of 60 inexpensive digital watches, including 9 that are defective, is sent to a department store. The receiving department selects 10 at random for testing and rejects the whole shipment if 1 or more in the sample are found defective. What is the probability that the shipment will be rejected?

43. If you bet \$5 on the number 13 in roulette, your probability of winning is $\frac{1}{38}$ and the payoff odds are given by the casino as 35:1.

a) Find the actual odds against the outcome of 13.

b) How much net profit would you make if you win by betting on 13?

c) If the casino was not operating for profit, and the payoff odds were changed to match the actual odds against 13, how much would you win if the outcome were 13?