Section 4.5 – Multiple Eigenvalues Solutions

Matrix $A\ (n\times n)$ has n distinct (real or complex) eigenvalues $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$ with respective eigenvectors $\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_n$, then a general solution of the system is given by

$$\vec{x}(t)=c_1\vec{v}_1e^{\lambda_1 t}+c_2\vec{v}_2e^{\lambda_2 t}+\ \ldots+c_n\vec{v}_ne^{\lambda_n t}$$

When the characteristic equation $|A-\lambda I|=0$ doesn't have n distinct roots, and thus has at least one repeated root.

An eigenvalue is of multiplicity $k>1$ if it is a k-fold root. For each eigenvalue λ, the eigenvector equation

$$(A-\lambda I)V=0$$

has at least one nonzero solution V, so there is at least one eigenvector with λ.

Example

Find a general solution of the system

$$x'=\begin{bmatrix}9&4&0\\-6&-1&0\\6&4&3\end{bmatrix}x$$

Solution

The characteristic equation:

$$\begin{aligned}|A-\lambda I|&=\begin{vmatrix}9-\lambda&4&0\\-6&-1-\lambda&0\\6&4&3-\lambda\end{vmatrix}\\
&=(9-\lambda)(-1-\lambda)(3-\lambda)+24(3-\lambda)\\
&=(3-\lambda)\left[-9-8\lambda+\lambda^2+24\right]\\
&=(3-\lambda)\left(\lambda^2-8\lambda+15\right)\\
&=(3-\lambda)^2(5-\lambda)=0\end{aligned}$$

The distinct eigenvalues are: $\lambda_1=5,\quad \lambda_{2,3}=3\ \ (\textit{repeated})$ of multiplicity $k=2$.

For $\lambda_1=5\ \Rightarrow\ (A-5I)V_1=0$

$$\begin{pmatrix}4&4&0\\-6&-6&0\\6&4&-2\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\ \Rightarrow\ \begin{array}{l}a=-b\\ \\6a+4b-2c=0\to c=a\end{array}\ \to\ V_1=\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

For $\lambda_2=3\ \Rightarrow\ (A-3I)V_2=0$

$$\begin{pmatrix} 6 & 4 & 0 \\ -6 & -4 & 0 \\ 6 & 4 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad 3a = -2b \qquad \to V_2 = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$$

$$\text{If } a = b = 0 \quad then \quad c = 1 \qquad \to V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

V_2 and V_2 are linearly independent eigenvectors.

$$\vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} + c_3\vec{v}_3 e^{\lambda_3 t}$$

$$= c_1\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{5t} + c_2\begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} e^{3t} + c_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{3t}$$

$$\begin{cases} x_1(t) = c_1 e^{5t} + 2c_2 e^{3t} \\ x_2(t) = -c_1 e^{5t} - 3c_2 e^{3t} \\ x_3(t) = c_1 e^{5t} \qquad\qquad + c_3 e^{3t} \end{cases}$$

Defective Eigenvalues

Example

Find a general solution of the system $A = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix}$

Solution

The characteristic equation:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -3 \\ 3 & 7-\lambda \end{vmatrix}$$

$$= (1-\lambda)(7-\lambda) + 9$$

$$= \lambda^2 - 8\lambda + 16 = 0$$

The eigenvalues are: $\lambda_{1,2} = 4 \quad (multiplicity\ 2)$

For $\lambda = 4 \quad \Rightarrow \quad (A - 4I)V_1 = 0$

$$\begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad a = -b \qquad \to V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Since the eigenvalue $\lambda_{1,2} = 4$ $(multiplicity\ 2)$ has only one independent eigenvector, and hence is incomplete.

An eigenvalue λ of multiplicity $k > 1$ is called ***defective*** if it is not complete.
If λ has only $p < k$ linearly independent eigenvectors, then the number

$$d = k - p$$

of ***missing*** eigenvectors is called the defect of the defective eigenvalue λ.

Defective Multiplicity 2 Eigenvalues

1. First find a nonzero solution $\vec{v}_2$ of the equation

$$(A - \lambda I)^2 \vec{v}_2 = \vec{0} \quad \textit{such that} \quad (A - \lambda I)\vec{v}_2 = \vec{v}_1$$

is nonzero, and therefore is an eigenvectors $\vec{v}_1$ associated with λ.

2. Then from the two independent solutions

$$\vec{x}_1(t) = \vec{v}_1 e^{\lambda t} \quad \textit{and} \quad \vec{x}_2(t) = (\vec{v}_1 t + \vec{v}_2) e^{\lambda t}$$

Example

Find a general solution of the system $A = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -3 \\ 3 & 7-\lambda \end{vmatrix} = (1-\lambda)(7-\lambda) + 9$$

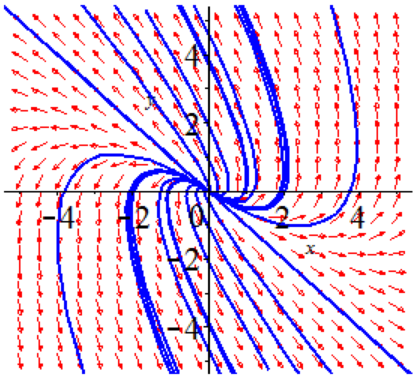
$$= \lambda^2 - 8\lambda + 16 = 0$$ The eigenvalues are: $\lambda_{1,2} = 4$ $(\textit{multiplicity } 2)$

$$(A - 4I)^2 = \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Since $(A - \lambda I)^2 \vec{v}_2 = \vec{0} \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\vec{v}_2 = \vec{0}$ and $\vec{v}_2$ is a nonzero vector, we can let $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$(A - 4I)\vec{v}_2 = \vec{v}_1 \quad \Rightarrow \quad \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} -3 \\ 3 \end{pmatrix} = \vec{v}_1}$$

$$\begin{cases} \vec{x}_1(t) = \begin{pmatrix} -3 \\ 3 \end{pmatrix} e^{4t} \\ \vec{x}_2(t) = \left(\begin{pmatrix} -3 \\ 3 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) e^{4t} \end{cases} \rightarrow \begin{cases} \vec{x}_1(t) = \begin{pmatrix} -3 \\ 3 \end{pmatrix} e^{4t} \\ \vec{x}_2(t) = \begin{pmatrix} -3t+1 \\ 3t \end{pmatrix} e^{4t} \end{cases}$$



The general solution: $\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$

$$\begin{cases} x_1(t) = (-3c_2 t + c_2 - 3c_1) e^{4t} \\ x_2(t) = (3c_2 t + 3c_1) e^{4t} \end{cases}$$

Generalized Eigenvectors

If λ is an eigenvalue of the matrix ***A***, then a rank *r* generalized eigenvector $\vec{v}$ such that

$$(A-\lambda I)^r \vec{v} = \vec{0} \quad but \quad (A-\lambda I)^{r-1}\vec{v} \neq \vec{0}$$

$$\begin{cases} \vec{x}_1(t) = \vec{v}_1 e^{\lambda t} \\ \vec{x}_2(t) = (\vec{v}_1 t + \vec{v}_2) e^{\lambda t} \\ \vec{x}_3(t) = \left(\frac{1}{2}\vec{v}_1 t^2 + \vec{v}_2 t + \vec{v}_3\right) e^{\lambda t} \\ \quad \vdots \quad\; \vdots \qquad\quad \vdots \\ \vec{x}_k(t) = \left(\frac{\vec{v}_1}{(k-1)!} t^{k-1} + \cdots + \frac{\vec{v}_{k-2}}{2!} t^2 + \vec{v}_{k-1} t + \vec{v}_k\right) e^{\lambda t} \end{cases}$$

Example

Find three linearly independent solutions of the system $\boldsymbol{x}' = \begin{bmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{bmatrix} \boldsymbol{x}$

Solution

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 1 & 2 \\ -5 & -3-\lambda & -7 \\ 1 & 0 & -\lambda \end{vmatrix}$$

$$= \lambda^2(-3-\lambda) - 7 - 2(-3-\lambda) - 5\lambda$$

$$= -\lambda^3 - 3\lambda^2 - 3\lambda - 1$$

$$= -(\lambda+1)^3 = 0$$

The eigenvalues are $\lambda_{1,2,3} = -1 \quad of \ \ multiplicity \ 3$

For $\lambda = -1 \quad \Rightarrow \quad (A+I)V = 0$

$$\begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{array}{l} a+b+2c = 0 \to b = a \\ \\ a = -c \end{array} \quad \to \ V = a\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad (a \neq 0)$$

The defect of $\lambda = -1$ is 2.

To apply the method for triple eigenvalues, then

$$(A+I)^2 = \begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -1 & -3 \\ -2 & -1 & -3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$(A+I)^3 = \begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} -2 & -1 & -3 \\ -2 & -1 & -3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

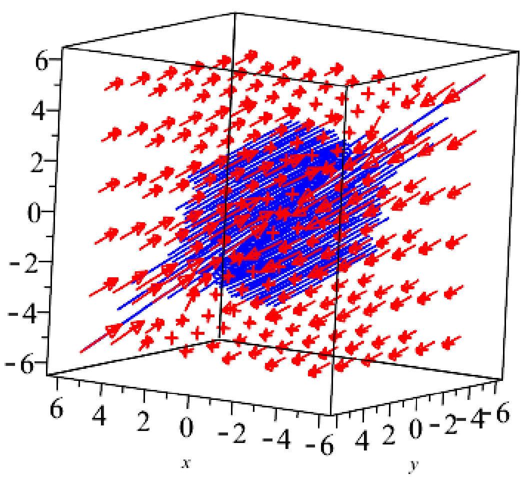
Since $(A+I)^3\vec{v}_3 = 0$, therefore any nonzero vector $\vec{v}_3 = [1 \quad 0 \quad 0]^T$ will be a solution.

$$\underline{|\vec{v}_2} = (A+I)\vec{v}_3 = \begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix}$$

$$\underline{|\vec{v}_1} = (A+I)\vec{v}_2 = \begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = \vec{v}_1 e^{-t} \\ \vec{x}_2(t) = (\vec{v}_1 t + \vec{v}_2)e^{-t} \\ \vec{x}_3(t) = \left(\tfrac{1}{2}\vec{v}_1 t^2 + \vec{v}_2 t + \vec{v}_3\right)e^{-t} \end{cases} \rightarrow \begin{cases} \vec{x}_1(t) = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} e^{-t} \\ \vec{x}_2(t) = \left(\begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} t + \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix}\right) e^{-t} \\ \vec{x}_3(t) = \left(\frac{1}{2}\begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} t^2 + \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) e^{-t} \end{cases}$$

$$\begin{cases} \vec{x}_1(t) = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} e^{-t} \\ \vec{x}_2(t) = \begin{pmatrix} -2t+1 \\ -2t-5 \\ 2t+1 \end{pmatrix} e^{-t} \\ \vec{x}_3(t) = \begin{pmatrix} -2t^2+t+1 \\ -t^2-5t \\ t^2+t \end{pmatrix} e^{-t} \end{cases}$$



The general solution:

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

$$\begin{cases} x_1(t) = \left(-c_3 t^2 + (c_3 - 2c_2)t + c_3 + c_2 - 2c_1\right)e^{-t} \\ x_2(t) = \left(-c_3 t^2 - (5c_3 + 2c_2)t - 5c_2 - 2c_1\right)e^{-t} \\ x_3(t) = \left(c_3 t^2 + (c_3 + 2c_2)t + c_2 + 2c_1\right)e^{-t} \end{cases}$$

Example

Suppose that the matrix $\boldsymbol{A}$ (6×6) has two multiplicity 3 eigenvalues $\lambda_1 = -2$ *and* $\lambda_2 = 3$ with defects 1 and 2, respectively.

Then λ_1 must have an eigenvector $\vec{u}_1$ and a length 2 chain $\{\vec{v}_1, \vec{v}_2\}$ of generalized eigenvectors.

$\left(\vec{u}_1 \text{ and } \vec{v}_1 \text{ are L.I}\right)$

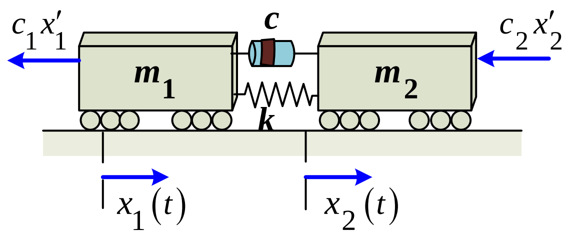
And λ_2 must have a length 3 chain $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ of generalized eigenvectors.

The six eigenvectors $\vec{u}_1, \vec{v}_1, \vec{v}_2, \vec{w}_1, \vec{w}_2, \vec{w}_3$ are then L.I and yield the following 6 independent solutions.

$$\begin{cases} \vec{x}_1(t) = \vec{u}_1 e^{-2t} \\ \vec{x}_2(t) = \vec{v}_1 e^{-2t} \\ \vec{x}_3(t) = \left(\vec{v}_1 t + \vec{v}_2\right) e^{-2t} \\ \vec{x}_4(t) = \vec{w}_1 e^{3t} \\ \vec{x}_5(t) = \left(\vec{w}_1 t + \vec{w}_2\right) e^{3t} \\ \vec{x}_6(t) = \left(\frac{1}{2}\vec{w}_1 t^2 + \vec{w}_2 t + \vec{w}_3\right) e^{3t} \end{cases}$$

Example

Two railway cars that are connected with a spring (permanently attached to both cars) and with a damper that exerts opposite forces on the two cars, of magnitude $c\left(x_1' - x_2'\right)$ proportional to their relative velocity. The two cars are also subject to frictional resistance forces $c_1 x_1'$ and $c_2 x_2'$ proportional to their respective velocities.



Let $m_1 = m_2 = c = 1$ *and* $c_1 = c_2 = k = 2$

Solution

The equations of motion:

$$\begin{cases} m_1 x_1'' = k\left(x_2 - x_1\right) - c_1 x_1' - c\left(x_1' - x_2'\right) \\ m_2 x_2'' = k\left(x_1 - x_2\right) - c_2 x_2' - c\left(x_2' - x_1'\right) \end{cases}$$

The equations can be written in the form: $Mx'' = Kx + Rx'$

where $R = \begin{bmatrix} -\left(c+c_1\right) & c \\ c & -\left(c+c_2\right) \end{bmatrix}$ is the ***resistance*** matrix.

To use the equations as a 1st-order system, let assume $x_3(t) = x_1'(t)$ *and* $x_4(t) = x_2'(t)$

$$\begin{cases} x_1' = x_3 \\ x_2' = x_4 \\ x_3' = -kx_1 + kx_2 - \left(c_1 + c\right)x_3 + cx_4 \\ x_4' = kx_1 - kx_2 + cx_3 - \left(c_2 + c\right)x_4 \end{cases} \to \begin{cases} x_1' = x_3 \\ x_2' = x_4 \\ x_3' = -2x_1 + 2x_2 - 3x_3 + x_4 \\ x_4' = 2x_1 - 2x_2 + x_3 - 3x_4 \end{cases}$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{pmatrix}$$

$$\left|A - \lambda I\right| = \begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -2 & 2 & -3-\lambda & 1 \\ 2 & -2 & 1 & -3-\lambda \end{vmatrix}$$

$$= -\lambda\left[-\lambda(-3-\lambda)^2 + 2 + 2(-3-\lambda) + \lambda\right] - 2\lambda - 2\lambda(-3-\lambda)$$

$$= -\lambda\left(-9\lambda - 6\lambda^2 - \lambda^3 - 4 - \lambda\right) + 4\lambda + 2\lambda^2$$

$$= \lambda^4 + 6\lambda^3 + 12\lambda^2 + 8\lambda$$

$$= \lambda\left(\lambda^3 + 6\lambda^2 + 12\lambda + 8\right)$$

$$= \lambda(\lambda + 2)^3 = 0$$

The eigenvalues are: $\lambda_1 = 0$ and $\lambda_{2,3,4} = -2$ (***triple***)

For $\lambda_1 = 0 \quad \Rightarrow \quad (A - 0I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} c = 0 \\ d = 0 \\ -2a + 2b = 0 \to a = b \end{array} \to \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda_2 = -2 \quad \Rightarrow \quad (A + 2I)V_2 = 0$

$$\begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} 2a = -c \\ 2b = -d \\ -2a + 2b - c + d = 0 \\ 2a - 2b + c - 3d = 0 \end{matrix}$$

$$\text{Let } a = 1 \Rightarrow c = -2 \quad b = 0 \Rightarrow d = 0 \ \rightarrow \vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

$$\text{Let } a = 0 \Rightarrow c = 0 \quad b = 1 \Rightarrow d = -2 \ \rightarrow \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \end{pmatrix}$$

$$\vec{w}_1 = \vec{u}_1 + \vec{u}_2 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \\ -2 \end{pmatrix}$$

$$(A + 2I)^2 \vec{v}_2 = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -3 \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -3 \end{pmatrix}\vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow 2a_2 + 2b_2 + c_2 + d_2 = 0 \ \rightarrow \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\vec{v}_1 = (A + 2I)\vec{v}_2 = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \\ 2 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = \vec{v}_1 e^{0t} \\ \vec{x}_2(t) = \vec{w}_1 e^{-2t} \\ \vec{x}_3(t) = \vec{v}_2 e^{-2t} \\ \vec{x}_4(t) = (\vec{v}_2 t + \vec{v}_3) e^{-2t} \end{cases} \rightarrow \begin{cases} \vec{x}_1(t) = [1 \quad 1 \quad 0 \quad 0]^T \\ \vec{x}_2(t) = [1 \quad 1 \quad -2 \quad -2]^T \ e^{-2t} \\ \vec{x}_3(t) = [1 \quad -1 \quad -2 \quad 2]^T \ e^{-2t} \\ \vec{x}_4(t) = [t \quad -t \quad -2t+1 \quad 2t-1]^T \ e^{-2t} \end{cases}$$

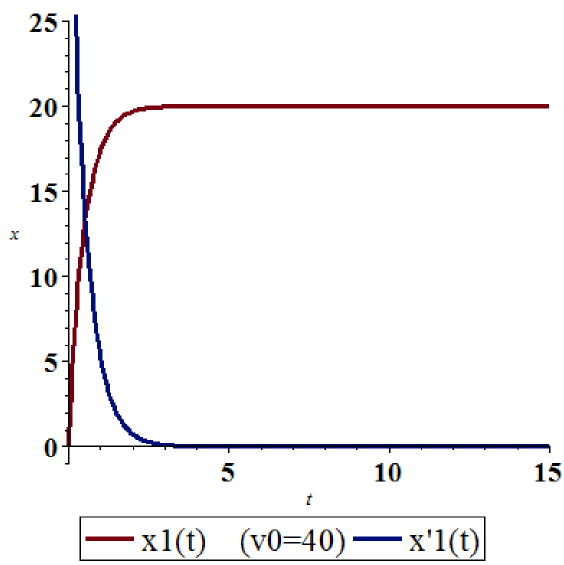
The general solution: $\vec{x}(t)=c_1\vec{x}_1(t)+c_2\vec{x}_2(t)+c_3\vec{x}_3(t)+c_4\vec{x}_4(t)$

$$\begin{cases} x_1(t)=c_1+\left(c_2+c_3+c_4t\right)e^{-2t} \\ x_2(t)=c_1+\left(c_2-c_3-c_4t\right)e^{-2t} \\ x_3(t)=\left(-2c_2-2c_3+c_4-2c_4t\right)e^{-2t} \\ x_4(t)=\left(-2c_2+2c_3-c_4+2c_4t\right)e^{-2t} \end{cases}$$

Recall that $x_3(t)=x_1'(t)$, $x_4(t)=x_2'(t)$ and since the position of the 2 cars in initial positon at rest, so $x_1(0)=x_2(0)=0$ with initial velocity of $x_1'(0)=x_2'(0)=v_0$

$$\begin{cases} x_1(0)=c_1+c_2+c_3=0 \\ x_2(0)=c_1+c_2-c_3=0 \\ x_3(0)=x_1'(0)=-2c_2-2c_3+c_4=v_0 \\ x_4(0)=x_2'(0)=-2c_2+2c_3-c_4=v_0 \end{cases} \qquad \begin{aligned} c_1&=-c_2 \\ c_3&=0 \\ c_2&=-\tfrac{1}{2}v_0 \\ c_4&=0 \end{aligned}$$

$$\begin{cases} x_1(t)=x_2(t)=\frac{1}{2}v_0\left(1-e^{-2t}\right) \\ x_1'(t)=x_2'(t)=v_0e^{-2t} \end{cases}$$



Diagonalization

Suppose the n by n matrix $\boldsymbol{A}$ has n linearly independent eigenvectors $x_1, \ldots, x_n$. Put them into the column of an ***eigenvector matrix P***. Then $P^{-1}AP$ is the eigenvalue matrix $\boldsymbol{A}$:

$$P^{-1}AP = \Lambda = D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Definition

A square matrix A is called ***diagonalizable*** if there is an invertible matrix P such that $P^{-1}AP$ is diagonal; the matrix P is said to ***diagonalize*** A.

Theorem

Independent x from different λ - Eigenvectors $x_1, \ldots, x_n$ that correspond to distinct (all different) eigenvalues are linearly independent. An n by n matrix that has n different eigenvalues (no repeated λ's) must be diagonalizable.

The *Jordan* Form

For every A, we want to choose M so that $M^{-1}AM$ is as nearly diagonal as possible. When A has a full set of n eigenvectors, they go into the columns of M. Then $M = P$. The matrix $P^{-1}AP$ is diagonal.

If A has $\boldsymbol{s}$ independent eigenvectors, it is similar to a matrix J that has $\boldsymbol{s}$ Jordan blocks on its diagonal. There is a matrix M such that

$$M^{-1}AM = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix} = J$$

Each block in J has one eigenvalue λ_i, one eigenvector, and 1's above the diagonal:

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

A is similar to B if they share the same Jordan form J – not otherwise.

Similar Matrices

Definition

If A and B are square matrices, then we say that ***B is similar to A*** if there exists an invertible matrix P such that $B = P^{-1}AP \quad or \quad A = PBP^{-1}$

Example

Jordan matrix J has triple eigenvalues 5, 5, 5. Its only eigenvectors are multiples of (1, 0, 0). Algebraic multiplicity 3, geometric multiplicity 1:

$$\text{If } J = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \quad then \quad J - 5I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ has rank 2.}$$

Every similar matrix $B = M^{-1}JM$ has the same triple eigenvalues 5, 5, 5. Also $B - 5I$ must have the same rank 2. Its nullspace has dimension 3 – 2 = 1. So each similar matrix B also has only one independent eigenvector.

The transpose matrix J^T has the same eigenvalues 5, 5, 5, and $J^T - 5I$ has the same rank 2. ***Jordan's theory says that*** J^T ***is similar to J***. The matrix that produces the similarity happens to be the reserve identity M:

$$J^T = M^{-1}JM \quad is \quad \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

There is one line of eigenvectors $\left(x_1,\ 0,\ 0\right)$ for J and another line $\left(0,\ 0,\ x_3\right)$ for J^T.

Example

Find Jordan form of the matrix $A = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -3 \\ 3 & 7-\lambda \end{vmatrix} = (1-\lambda)(7-\lambda) + 9$$

$$= \lambda^2 - 8\lambda + 16 = 0 \quad \text{The eigenvalues are: } \lambda_{1,2} = 4 \quad (multiplicity\ 2)$$

$$(A - 4I)^2 = \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Since $(A - \lambda I)^2 \vec{v}_2 = \vec{0} \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\vec{v}_2 = \vec{0}$ and $\vec{v}_2$ is a nonzero vector, we can let $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$(A-4I)\vec{v}_2=\vec{v}_1 \quad \Rightarrow \quad \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} -3 \\ 3 \end{pmatrix}=\vec{v}_1$$

$$Q=\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}=\begin{pmatrix} -3 & 1 \\ 3 & 0 \end{pmatrix} \quad \rightarrow \quad Q^{-1}=\begin{pmatrix} 0 & \frac{1}{3} \\ 1 & 1 \end{pmatrix}$$

$$\boldsymbol{J}=Q^{-1}AQ=\begin{pmatrix} 0 & \frac{1}{3} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} -3 & 1 \\ 3 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & \frac{1}{3} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -12 & 1 \\ 12 & 3 \end{pmatrix}$$

$$=\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$$

$\boldsymbol{J}=\boldsymbol{J}_1$ is a single 2×2 Jordan block corresponding to the single eigenvalue $\lambda=4$ of *A*.

The General Cayley-Hamilton Theorem

Every diagonalizable matrix A satisfies its characteristic equation $p(\lambda)=|A-\lambda I|=0 \ \ (p(A)=0)$. Using Jordan normal form to show that this is true whether or not *A* is diagonalizable.

If $\boldsymbol{J}=Q^{-1}AQ \quad \Rightarrow \quad p(A)=Q^{-1}p(\boldsymbol{J})Q$

If the Jordan blocks $\boldsymbol{J}_1, \boldsymbol{J}_2, \ldots, \boldsymbol{J}_s$ have sizes $k_1, k_2, \ldots, k_s$ that is $\boldsymbol{J}_i \ (k_i \times k_i)$ matrix and the corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$ *respectively*, then

$$p(\lambda)=(\lambda_1-\lambda)^{k_1}(\lambda_2-\lambda)^{k_2}\ \ldots(\lambda_s-\lambda)^{k_s}$$

$$\rightarrow \qquad p(\boldsymbol{J})=(\lambda_1 I-\boldsymbol{J})^{k_1}(\lambda_2 I-\boldsymbol{J})^{k_2}\ \ldots(\lambda_s I-\boldsymbol{J})^{k_s}$$

$p(\boldsymbol{J})$ has the same block-diagonal structure as $\boldsymbol{J}$ itself

$$(\lambda_i I-\boldsymbol{J}_i)^{k_i}=\begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & -1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}^{k_i}$$

Exercises Section 4.5 – Multiple Eigenvalues Solutions

Find the general solutions

1. $x' = \begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix} x$

2. $x' = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} x$

3. $x' = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} x$

4. $x' = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix} x$

5. $x' = \begin{bmatrix} 2 & 0 & 0 \\ -7 & 9 & 7 \\ 0 & 0 & 2 \end{bmatrix} x$

6. $x' = \begin{bmatrix} 25 & 12 & 0 \\ -18 & -5 & 0 \\ 6 & 6 & 13 \end{bmatrix} x$

7. $x' = \begin{bmatrix} -3 & 0 & -4 \\ -1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix} x$

8. $x' = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix} x$

9. $x' = \begin{bmatrix} 0 & 0 & 1 \\ -5 & -1 & -5 \\ 4 & 1 & -2 \end{bmatrix} x$

10. $x' = \begin{bmatrix} 39 & 8 & -16 \\ -36 & -5 & 16 \\ 72 & 16 & -29 \end{bmatrix} x$

11. $x' = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} x$

12. $x' = \begin{bmatrix} -1 & -4 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} x$

13. The characteristic equation of the coefficient matrix ***A*** of the system

$$x' = \begin{bmatrix} 3 & -4 & 1 & 0 \\ 4 & 3 & 0 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{bmatrix} x \qquad \text{is} \quad p(\lambda) = \left(\lambda^2 - 6\lambda + 25\right)^2 = 0$$

Therefore, ***A*** has the repeated complex pair $3 \pm 4i$ of eigenvalues. First show that the complex vectors $\vec{v}_1 = \begin{bmatrix} 1 & i & 0 & 0 \end{bmatrix}^T$ *and* $\vec{v}_2 = \begin{bmatrix} 0 & 0 & 1 & i \end{bmatrix}^T$ form a length 2 chain $\left\{\vec{v}_1, \vec{v}_2\right\}$ associated with the eigenvalue $\lambda = 3 - 4i$. Then calculate the real and imaginary parts of the complex-valued solutions

$$\vec{v}_1 e^{\lambda t} \quad \textit{and} \quad \left(\vec{v}_1 t + \vec{v}_2\right) e^{\lambda t}$$

To find four independent real-valued solutions of $x' = Ax$