Solution Section 2.4 – Inhomogeneous Equations; the Method of Undetermined Coefficients

Exercise

Show that the 3 solutions $y_1 = x, \quad y_2 = x\ln x, \quad y_3 = x^2$ of the 3rd order equation $x^3y''' - x^2y'' + 2xy' - 2y = 0$ are linearly independent on an open interval $x > 0$. Then find a particular solution that satisfies the initial conditions $y(1) = 3, \quad y'(1) = 2, \quad y''(1) = 1$

Solution

$$W = \begin{vmatrix} x & x\ln x & x^2 \\ 1 & 1+\ln x & 2x \\ 0 & \frac{1}{x} & 2 \end{vmatrix}$$

$$= 2x + 2x\ln x + x - 2x - 2x\ln x$$

$$\boxed{= x \neq 0} \quad \text{since} \quad x > 0$$

$\therefore y_1, y_2, y_3$ are linearly independent.

$$y(x) = C_1x + C_2x\ln x + C_3x^2 \qquad y(1) = C_1 + C_3 = 3$$

$$y'(x) = C_1 + C_2(1+\ln x) + 2C_3x \qquad y'(1) = C_1 + C_2 + 2C_3 = 2$$

$$y''(x) = C_2\frac{1}{x} + 2C_3 \qquad y''(1) = C_2 + 2C_3 = 1$$

$$\Rightarrow \quad C_1 = 1, \quad C_2 = -3, \quad \text{and} \quad C_3 = 2$$

$$\boxed{y(x) = x - 3x\ln x + 2x^2}$$

Exercise

Find the particular solution for $y'' + 3y' + 2y = 4e^{-3t}$

Solution

$$y(t) = Ae^{-3t} \qquad \Rightarrow y' = -3Ae^{-3t}$$

$$y'' = 9Ae^{-3t}$$

$$y'' + 3y' + 2y = 4e^{-3t}$$

$$9Ae^{-3t} + 3\left(-3Ae^{-3t}\right) + 2Ae^{-3t} = 4e^{-3t}$$

$$2Ae^{-3t} = 4e^{-3t}$$

$$2A = 4$$

$$A = 2$$

The particular solution: $\boxed{y(t) = 2e^{-3t}}$

Exercise

Find the particular solution for $y'' + 6y' + 8y = -3e^{-t}$

Solution

$$y(t) = Ae^{-t}$$

$$y' = -Ae^{-t}$$

$$y'' = Ae^{-t}$$

$$y'' + 6y' + 8y = -3e^{-t}$$

$$Ae^{-t} - 6Ae^{-t} + 8Ae^{-t} = -3e^{-t}$$

$$A - 6A + 8A = -3$$

$$3A = -3 \quad \Rightarrow \quad \boxed{A = -1}$$

Therefore, the particular solution is: $\boxed{y(t) = -e^{-t}}$

Exercise

Find the particular solution for $y'' + 2y' + 5y = 12e^{-t}$

Solution

$$y(t) = Ae^{-t}$$

$$y' = -Ae^{-t}$$

$$y'' = Ae^{-t}$$

$$y'' + 2y' + 5y = 12e^{-t}$$

$$Ae^{-t} + 2\left(-Ae^{-t}\right) + 5Ae^{-t} = 12e^{-t}$$

$$4Ae^{-t} = 12e^{-t}$$

$$4A = 12 \quad \rightarrow \quad A = 3$$

The particular solution: $y(t) = 3e^{-t}$

Exercise

Find the particular solution for the given differential equation $y'' + 3y' - 18y = 18e^{2t}$

Solution

$$y(t) = Ae^{2t}$$

$$y' = 2Ae^{2t}$$

$$y'' = 4Ae^{2t}$$

$$y'' + 3y' - 18y = 18e^{2t}$$

$$4Ae^{2t} + 3\left(2Ae^{2t}\right) - 18Ae^{2t} = 18e^{2t}$$

$$4Ae^{2t} + 6Ae^{2t} - 18Ae^{2t} = 18e^{2t}$$

$$-8Ae^{2t} = 18e^{2t}$$

$$-8A = 18 \quad \rightarrow \quad A = -\frac{18}{8} = -\frac{9}{4}$$

The particular solution: $y(t) = -\frac{9}{4}e^{2t}$

Exercise

Use $y(t) = a\cos\omega t + b\sin\omega t$ to find the particular solution for $y'' + 4y = \cos 3t$

Solution

The particular solution: $y(t) = a\cos 3t + b\sin 3t$

$$y' = -3a\sin 3t + 3b\cos 3t$$
$$y'' = -9a\cos 3t - 9b\sin 3t$$

$$y'' + 4y = \cos 3t$$
$$-9a\cos 3t - 9b\sin 3t + 4\left(a\cos 3t + b\sin 3t\right) = \cos 3t$$
$$-9a\cos 3t - 9b\sin 3t + 4a\cos 3t + 4b\sin 3t = \cos 3t$$
$$-5a\cos 3t - 5b\sin 3t = \cos 3t$$

$$a = -\frac{1}{5} \qquad b = 0$$

The particular solution: $y(t) = -\frac{1}{5}\cos 3t$

Exercise

Use $y(t) = a\cos\omega t + b\sin\omega t$ to find the particular solution for $y'' + 7y' + 6y = 3\sin 2t$

Solution

The particular solution: $y(t) = a\cos 2t + b\sin 2t$

$$y' = -2a\sin 2t + 2b\cos 2t$$
$$y'' = -4a\cos 2t - 4b\sin 2t$$

$$y'' + 7y' + 6y = 3\sin 2t$$

$$-4a\cos 2t - 4b\sin 2t + 7\left(-2a\sin 2t + 2b\cos 2t\right) + 6\left(a\cos 2t + b\sin 2t\right) = 3\sin 2t$$

$$-4a\cos 2t - 4b\sin 2t - 14a\sin 2t + 14b\cos 2t + 6a\cos 2t + 6b\sin 2t = 3\sin 2t$$

$$(14b+2a)\cos 2t + (2b-14a)\sin 2t = 3\sin 2t$$

$$\begin{cases} 14b+2a=0 \\ 2b-14a=3 \end{cases} \Rightarrow a=-\frac{21}{100} \quad b=\frac{3}{100}$$

The particular solution: $\underline{y_p(t) = -\frac{21}{100}\cos 2t + \frac{3}{100}\sin 2t|}$

Exercise

Find the particular solution for $y'' + 5y' + 4y = 2 + 3t$

Solution

The particular solution: $y(t) = at + b$

$$y' = a$$
$$y'' = 0$$

$$y'' + 5y' + 4y = 2 + 3t$$
$$0 + 5a + 4(at + b) = 2 + 3t$$
$$5a + 4b + 4at = 2 + 3t$$

$$\begin{cases} 5a+4b=2 \\ 4a=3 \end{cases} \Rightarrow \begin{cases} b=-\frac{7}{16} \\ a=\frac{3}{4} \end{cases}$$

The particular solution: $\underline{y_p(t) = \frac{3}{4}t - \frac{7}{16}|}$

Exercise

Find the particular solution for $y'' + 6y' + 8y = 2t - 3$

Solution

The particular solution: $y(t) = at + b$

$$y'' + 6y' + 8y = 2t - 3$$
$$0 + 6a + 8(at + b) = 2t - 3$$
$$6a + 8at + 8b = 2t - 3$$
$$8at + 6a + 8b = 2t - 3$$

$$\begin{cases} 6a+8b=-3 \\ 8a=2 \end{cases} \Rightarrow \begin{cases} a=\frac{1}{4} \\ b=-\frac{9}{16} \end{cases}$$

The particular solution: $\underline{y_p(t) = \frac{1}{4}t - \frac{9}{16}|}$

Exercise

Find the particular solution for $y'' + 3y' + 4y = t^3$

Solution

The particular solution: $y(t) = at^3 + bt^2 + ct + d$

$$y' = 3at^2 + 2bt + c$$
$$y'' = 6at + 2b$$

$$y'' + 3y' + 4y = t^3$$
$$6at + 2b + 3\left(3at^2 + 2bt + c\right) + 4\left(at^3 + bt^2 + ct + d\right) = t^3$$
$$6at + 2b + 9at^2 + 6bt + 3c + 4at^3 + 4bt^2 + 4ct + 4d = t^3$$
$$4at^3 + (9a + 4b)t^2 + (6a + 6b + 4c)t + 2b + 3c + 4d = t^3$$

$$\begin{cases} 4a = 1 \\ 9a + 4b = 0 \\ 6a + 6b + 4c = 0 \\ 2b + 3c + 4d = 0 \end{cases} \Rightarrow \begin{cases} a = \frac{1}{4} \\ b = -\frac{9}{4}a = -\frac{9}{16} \\ c = -\frac{6}{4}a - \frac{6}{4}b = -\frac{3}{2}\frac{1}{4} + \frac{3}{2}\frac{9}{16} = \frac{15}{32} \\ d = -\frac{1}{2}b - \frac{1}{2}d = -\frac{9}{128} \end{cases}$$

The particular solution: $\boxed{y_p(t) = \frac{1}{4}t^3 - \frac{9}{16}t^2 + \frac{15}{32}t - \frac{9}{128}}$

Exercise

Find the particular solution for $y'' + 2y' + 2y = 2 + \cos 2t$

Solution

$y'' + 2y' + 2y = 2$ when $y = 1$

The particular solution: $z = Ae^{i2t}$

$$z' = (2i)Ae^{i2t}$$
$$z'' = (2i)^2 Ae^{i2t}$$

$$z'' + 2z' + 2z = e^{i2t}$$
$$(2i)^2 Ae^{i2t} + 2(2i)Ae^{i2t} + 2Ae^{i2t} = e^{i2t}$$
$$(2i)^2 A + 2(2i)A + 2A = 1$$
$$(-4 + 4i + 2)A = 1$$
$$A = \frac{1}{-2 + 4i} \cdot \frac{-2 - 4i}{-2 - 4i}$$

$$= -\frac{2}{10} - \frac{4}{10}i$$
$$= -\frac{1}{10} - \frac{1}{5}i$$

$$z_p = \left(-\frac{1}{10} - \frac{1}{5}i\right)e^{i2t}$$
$$= \left(-\frac{1}{10} - \frac{1}{5}i\right)(\cos 2t + i\sin 2t)$$
$$= -\frac{1}{10}\cos 2t + \frac{1}{5}\sin 2t + i\left(-\frac{1}{5}\cos 2t - \frac{1}{10}\sin 2t\right)$$

The general solution: $\boxed{y(t) = 1 - \frac{1}{10}\cos 2t + \frac{1}{5}\sin 2t}$

Exercise

Find the particular solution for $y'' - y = t - e^{-t}$

Solution

The characteristic eq.: $\lambda^2 - 1 = 0 \Rightarrow \lambda_1 = -1, \quad \lambda_2 = 1$

The particular solution: $y = at + b$

$$y' = a$$
$$y'' = 0$$

$$y'' - y = t$$
$$-at - b = t \quad \Rightarrow \quad \begin{cases} \boxed{a = -1}, & \boxed{b = 0} \end{cases}$$
$$y(t) = -t$$

The homogenous solution: $y_h = C_1 e^{-t} + C_2 e^{t}$

Because the inhomogeneous part of $y'' - y = e^{-t}$ is also the solution.

Therefore: $y_p = Ate^{-t}$

$$y'_p = -Ate^{-t} + Ae^{-t} = Ae^{-t}(1-t)$$
$$y''_p = -Ae^{-t} + Ate^{-t} - Ae^{-t} = Ae^{-t}(t-2)$$

$$y'' - y = e^{-t}$$
$$Ate^{-t} - 2Ae^{-t} - Ate^{-t} = e^{-t}$$
$$-2Ae^{-t} = e^{-t}$$
$$-2A = 1 \quad \Rightarrow \quad \boxed{A = -\frac{1}{2}}$$

$$y_p = -\frac{1}{2}te^{-t}$$
$$\boxed{y(t) = -t + \frac{1}{2}te^{-t}}$$

Exercise

Find the particular solution for $y'' - 2y' + y = 10e^{-2t}\cos t$

Solution

The particular solution: $y_P = e^{-2t}\left(A\cos t + B\sin t\right)$

$$y' = -2e^{-2t}\left(A\cos t + B\sin t\right) + e^{-2t}\left(-A\sin t + B\cos t\right)$$

$$= e^{-2t}\left((B-2A)\cos t - (A+2B)\sin t\right)$$

$$y'' = -2e^{-2t}\left((B-2A)\cos t - (A+2B)\sin t\right) + e^{-2t}\left((2A-B)\sin t - (A+2B)\cos t\right)$$

$$= e^{-2t}\left((3A-4B)\cos t + (4A+3B)\sin t\right)$$

$$y'' - 2y' + y = 10e^{-2t}\cos t$$

$$e^{-2t}\left((3A-4B)\cos t + (4A+3B)\sin t\right) - 2e^{-2t}\left((B-2A)\cos t - (A+2B)\sin t\right)$$
$$+ e^{-2t}\left(A\cos t + B\sin t\right) = 10e^{-2t}\cos t$$

$$\left((3A-4B-2B+4A+A)\cos t + (4A+3B+2A+4B+B)\sin t\right) = 10\cos t$$

$$\begin{cases} 8A - 6B = 10 \\ 6A + 8B = 0 \end{cases} \rightarrow A = \frac{80}{100} = \frac{4}{5} \quad B = -\frac{3}{5}$$

$$\boxed{y_P = e^{-2t}\left(\frac{4}{5}\cos t - \frac{3}{5}\sin t\right)}$$

Exercise

Find the particular solution for $y''' - 4y'' + 4y' = 5t^2 - 6t + 4t^2e^t + 3e^{5t}$

Solution

Characteristic equation: $\lambda^3 - 4\lambda^2 + 4\lambda = \lambda(\lambda-2)^2 = 0 \quad \Rightarrow \quad \lambda_1 = 0,\ \lambda_{2,3} = 2$

Homogeneous equation: $y_h = C_1 + \left(C_2 + C_3 t\right)e^{2t}$

The particular solution: $y_P = t\left(At^2 + Bt + C\right) + \left(Et^2 + Ft + G\right)e^t + He^{5t}$

$$y_P' = 3At^2 + 2Bt + C + (2Et + F)e^t + \left(Et^2 + Ft + G\right)e^t + 5He^{5t}$$

$$= 3At^2 + 2Bt + C + \left(Et^2 + (2E+F)t + F + G\right)e^t + 5He^{5t}$$

$$y_P'' = 6At + 2B + (2Et + 2E + F)e^t + \left(Et^2 + (2E+F)t + F + G\right)e^t + 25He^{5t}$$

$$= 6At + 2B + \left(Et^2 + (4E+F)t + 2E + 2F + G\right)e^t + 25He^{5t}$$

$$y''' = 6A + (2Et + 4E + F)e^t + \left(Et^2 + (4E+F)t + 2E + 2F + G\right)e^t + 125He^{5t}$$

$$= 6A + \left(Et^2 + (6E+F)t + 6E + 3F + G\right)e^t + 125He^{5t}$$

$$y''' - 4y'' + 4y' = 6A + \left(Et^2 + (6E+F)t + 6E + 3F + G\right)e^t + 125He^{5t}$$
$$- 24At - 8B - 4\left(Et^2 + (4E+F)t + 2E + 2F + G\right)e^t - 100He^{5t}$$
$$+ 12At^2 + 8Bt + 4C + 4\left(Et^2 + (2E+F)t + F + G\right)e^t + 20He^{5t}$$

$$= 12At^2 + (8B - 24A)t + 6A - 8B + 4C + \left(Et^2 + (-2E+F)t - 2E - F + G\right)e^t + 45He^{5t}$$

$$= 5t^2 - 6t + 4t^2e^t + 3e^{5t}$$

$$\begin{cases} 12A = 5 \\ 8B - 24A = -6 \\ 6A - 8B + 4C = 0 \end{cases} \rightarrow \begin{cases} A = \frac{5}{12} \\ B = \frac{1}{2} \\ C = \frac{3}{8} \end{cases} \qquad \begin{cases} E = 4 \\ F - 2E = 0 & \Rightarrow \quad F = 8 \\ -2E - F + G = 0 & \Rightarrow \quad G = 16 \end{cases} \qquad H = \frac{3}{45} = \frac{1}{15}$$

$$\boxed{y_P = \frac{5}{12}t^3 + \frac{1}{2}t^2 + \frac{3}{8}t + \left(4t^2 + 8t + 16\right)e^t + \frac{1}{15}e^{5t}}$$

Exercise

Use the complex method to find the particular solution for $y'' + 4y' + 3y = \cos 2t + 3\sin 2t$

Solution

The characteristic eq.: $\lambda^2 + 4\lambda + 3 = 0 \;\Rightarrow\; \lambda_1 = -3, \quad \lambda_2 = -1$

The homogenous solution: $y_h = e^{-t}\left(C_1 \cos t + C_2 \sin t\right)$

The particular solution: $z = Ae^{i2t}$

$$z' = (2i)Ae^{i2t}$$

$$z'' = (2i)^2 Ae^{i2t}$$

$$z'' + 4z' + 3z = e^{i2t}$$

$$(2i)^2 Ae^{i2t} + 4(2i)Ae^{i2t} + 3Ae^{i2t} = e^{i2t}$$

$$(-4 + 8i + 3)A = 1$$

$$(-1 + 8i)A = 1$$

$$A = \frac{1}{-1+8i} \cdot \frac{-1-8i}{-1-8i}$$

$$= \frac{-1-8i}{65}$$

$$= -\frac{1}{65} - i\frac{8}{65}$$

This gives the particular solution:

$$z=\left(-\frac{1}{65}-i\frac{8}{65}\right)e^{i2t}$$
$$=\left(-\frac{1}{65}-i\frac{8}{65}\right)\left(\cos 2t+i\sin 2t\right)$$
$$=\left(-\frac{1}{65}\cos 2t+\frac{8}{65}\sin 2t\right)+i\left(-\frac{8}{65}\cos 2t-\frac{1}{65}\sin 2t\right)$$

$y=-\frac{1}{65}\cos 2t+\frac{8}{65}\sin 2t$ is a solution of $y''+4y'+3y=\cos 2t$

$y=-\frac{8}{65}\cos 2t-\frac{1}{65}\sin 2t$ is a solution of $y''+4y'+3y=\sin 2t$

Therefore;

$$y(t)=-\frac{1}{65}\cos 2t+\frac{8}{65}\sin 2t+3\left(-\frac{8}{65}\cos 2t-\frac{1}{65}\sin 2t\right)$$
$$=-\frac{1}{65}\cos 2t+\frac{8}{65}\sin 2t-\frac{24}{65}\cos 2t-\frac{3}{65}\sin 2t$$
$$=-\frac{25}{65}\cos 2t+\frac{5}{65}\sin 2t$$
$$=-\frac{5}{13}\cos 2t+\frac{1}{13}\sin 2t$$

Exercise

Use the complex method to find the particular solution for $y''+4y=\cos 3t$

Solution

The particular solution: $z=Ae^{i3t}$

$$z'=(3i)Ae^{i3t}$$
$$z''=(3i)^2 Ae^{i3t}$$

$$z''+4z=\cos 3t=e^{i3t}$$
$$(3i)^2 Ae^{i3t}+4Ae^{i3t}=e^{i3t}$$
$$(-9+4)A=1 \quad\rightarrow\quad A=-\frac{1}{5}$$
$$z=-\frac{1}{5}e^{i3t}$$
$$y(t)=-\frac{1}{5}\cos 3t$$

Exercise

Find the general solution: $y''+y=2\cos x$

Solution

The characteristic equation: $\lambda^2+1=0 \quad\rightarrow\quad \lambda_{1,2}=\pm i$

$$\boxed{y_h = C_1 \cos x + C_2 \sin x}$$

$y_P = Ax\cos x + Bx\sin x$ Since y_h in functions of cosine and sine

$$y'_P = A\cos x - Ax\sin x + B\sin x + Bx\cos x$$

$$= (A+Bx)\cos x + (-Ax+B)\sin x$$

$$y''_P = B\cos x - (A+Bx)\sin x - A\sin x + (-Ax+B)\cos x$$

$$= (2B-Ax)\cos x + (-2A-Bx)\sin x$$

$$y'' + y = 2\cos x$$

$$(2B-Ax)\cos x + (-2A-Bx)\sin x + Ax\cos x + Bx\sin x = 2\cos x$$

$$2B\cos x - 2A\sin x = 2\cos x$$

$$\begin{cases} -2A = 0 \\ 2B = 2 \end{cases} \rightarrow \boxed{A = 0,\ B = 1}$$

$$\boxed{y_P = x\sin x}$$

$$\boxed{y(x) = C_1 \cos x + C_2 \sin x + x\sin x}$$

Exercise

Find the general solution for the given *DE*: $y'' + y = \cos 3x$

Solution

The characteristic equation: $\lambda^2 + 1 = 0 \rightarrow \boxed{\lambda_{1,2} = \pm i}$

$$\boxed{y_h = C_1 \cos x + C_2 \sin x}$$

$$y_P = A\cos 3x + B\sin 3x$$

$$y'_P = -3A\sin 3x + 3B\cos 3x$$

$$y''_P = -9A\cos 3x - 3B\sin 3x$$

$$y'' + y = \cos 3x$$

$$-9A\cos 3x - 3B\sin 3x + A\cos 3x + B\sin 3x = \cos 3x$$

$$\begin{cases} -8A = 1 \\ -2B = 0 \end{cases} \rightarrow \boxed{A = -\tfrac{1}{8},\ B = 0}$$

$$\boxed{y(x) = C_1 \cos x + C_2 \sin x - \tfrac{1}{8}\cos 3x}$$

Exercise

Find the general solution for the given *DE*: $y'' + y = 2x\sin x$

Solution

The characteristic equation: $\lambda^2 + 1 = 0 \rightarrow \boxed{\lambda_{1,2} = \pm i}$

$$\boxed{y_h = C_1 \cos x + C_2 \sin x}$$

$$y_P = \left(Ax^2 + Bx\right)\cos x + \left(Cx^2 + Dx\right)\sin x$$

$$y'_P = (2Ax + B)\cos x - \left(Ax^2 + Bx\right)\sin x + (2Cx + D)\sin x + \left(Cx^2 + Dx\right)\cos x$$

$$= \left(Cx^2 + 2Ax + Dx + B\right)\cos x - \left(Ax^2 + Bx - 2Cx - D\right)\sin x$$

$$y''_P = (2Cx + 2A + D)\cos x - \left(Cx^2 + 2Ax + Dx + B\right)\sin x$$
$$-(2Ax + B - 2C)\sin x - \left(Ax^2 + Bx - 2Cx - D\right)\cos x$$

$$= \left(-Ax^2 - Bx + 4Cx + 2A + 2D\right)\cos x - \left(Cx^2 + 4Ax + Dx + 2B - 2C\right)\sin x$$

$$y'' + y = 2x\sin x$$

$\cos x$	x^2	$-A + A = 0$		$\sin x$	x^2	$-C + C = 0$	
	x	$-B + 4C + B = 0$	$\boxed{C = 0}$		x	$-4A = 2$	$\boxed{A = -\frac{1}{2}}$
	x^0	$2A + 2D = 0$	$A = -D$		x^0	$2B - 2C = 0$	$\boxed{B = C = 0}$

$$\rightarrow \boxed{D = \frac{1}{2}}$$

$$\boxed{y_P = -\frac{1}{2}x^2 \cos x + \frac{1}{2}x\sin x}$$

$$\boxed{y(x) = C_1 \cos x + C_2 \sin x - \frac{1}{2}x^2 \cos x + \frac{1}{2}x\sin x}$$

Exercise

Find the general solution: $y'' - y = x^2 e^x + 5$

Solution

The characteristic equation: $\lambda^2 - 1 = 0 \rightarrow \boxed{\lambda_{1,2} = \pm 1}$

$$\boxed{y_h = C_1 e^{-x} + C_2 e^x}$$

$$y_p = \left(Ax^3 + Bx^2 + Cx\right)e^x + D$$

$$y'_p = \left(Ax^3 + Bx^2 + Cx + 3Ax^2 + 2Bx + C\right)e^x$$

$$y''_P = \left(Ax^3 + Bx^2 + 6Ax^2 + Cx + 4Bx + 6Ax + 2C + 2B\right)e^x$$

$$y'' - y = x^2e^x + 5$$

$$\left(Ax^3 + Bx^2 + 6Ax^2 + Cx + 4Bx + 6Ax + 2C + 2B - Ax^3 - Bx^2 - Cx\right)e^x - D = x^2e^x + 5$$

$$\underline{D = -5}|$$

x^2 $\quad 6A = 1 \quad \underline{A = \frac{1}{6}}|$

x $\quad 4B + 6A = 0 \quad \underline{B = -\frac{1}{4}}|$

x^0 $\quad 2C + 2B = 0 \quad \underline{C = \frac{1}{4}}|$

$$y_p = \left(\frac{1}{6}x^3 - \frac{1}{4}Bx^2 + \frac{1}{4}x\right)e^x - 5$$

$$\underline{y(x) = C_1e^{-x} + C_2e^x + \left(\frac{1}{6}x^3 - \frac{1}{4}Bx^2 + \frac{1}{4}x\right)e^x - 5}|$$

Exercise

Find the general solution for the given *DE*: $\quad y'' - y' = -3$

Solution

The characteristic equation: $\lambda^2 - \lambda = 0 \quad \rightarrow \quad \underline{\lambda_{1,2} = 0,\ 1}|$

$$\underline{y_h = C_1 + C_2e^x}|$$

$$y_P = Ax$$

$$y'_P = A$$

$$y''_P = 0$$

$$y'' - y' = -3 \quad \rightarrow \quad \underline{A = 3}|$$

$$\underline{y_P = 3x}|$$

$$\underline{y(x) = C_1 + C_2e^x - 3x}|$$

Exercise

Find the general solution for the given *DE*: $\quad y'' - y' = 2\sin x$

Solution

The characteristic equation: $\lambda^2 - \lambda = 0 \quad \rightarrow \quad \underline{\lambda_{1,2} = 0,\ 1}|$

$$\underline{y_h = C_1 + C_2e^x}|$$

$$y_P = A\cos x + B\sin x$$

$$y'_P = -A\sin x + B\cos x$$

$$y''_P = -A\cos x - B\sin x$$

$$y'' - y' = 2\sin x$$

$$-A\cos x - B\sin x - A\sin x + B\cos x = 2\sin x$$

$$\begin{cases} -A + B = 0 \\ -A - B = 2 \end{cases} \rightarrow \boxed{A = -1, \ \ B = -1}$$

$$\boxed{y_P = -\cos x - \sin x}$$

$$\boxed{y(x) = C_1 + C_2 e^x - \cos x - \sin x}$$

Exercise

Find the general solution for the given *DE*: $y'' - y' = \sin x$

Solution

The characteristic equation: $\lambda^2 - \lambda = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = 0, \ 1}$

$$\boxed{y_h = C_1 + C_2 e^x}$$

$$y_P = A\cos x + B\sin x$$

$$y'_P = -A\sin x + B\cos x$$

$$y''_P = -A\cos x - B\sin x$$

$$y'' - y' = \sin x$$

$$-A\cos x - B\sin x - A\sin x + B\cos x = \sin x$$

$$\begin{cases} -A + B = 0 \\ -A - B = 1 \end{cases} \rightarrow \boxed{A = -\frac{1}{2}, \ \ B = -\frac{1}{2}}$$

$$\boxed{y_P = -\frac{1}{2}\cos x - \frac{1}{2}\sin x}$$

$$\boxed{y(x) = C_1 + C_2 e^x - \frac{1}{2}\cos x - \frac{1}{2}\sin x}$$

Exercise

Find the general solution for the given *DE*: $y'' - y' = -8x + 3$

Solution

The characteristic equation: $\lambda^2 - \lambda = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = 0, \ 1}$

$$\boxed{y_h = C_1 + C_2 e^x}$$

$$y_P = Ax^2 + Bx$$

$$y'_P = 2Ax + B$$

$$y''_P = 2A$$

$$y'' - y' = -8x + 3$$

$$2A - 2Ax + B = -8x + 3$$

$$\begin{cases} -2A = -8 \\ 2A + B = 3 \end{cases} \rightarrow \boxed{A = 4, \quad B = -5}$$

$$\boxed{y_p = 4x^2 - 5x}$$

$$\boxed{y(x) = C_1 + C_2 e^x + 4x^2 - 5x}$$

Exercise

Find the general solution: $y'' + y = 2x + 3e^x$

Solution

The characteristic equation: $\lambda^2 + 1 = 0 \rightarrow \boxed{\lambda_{1,2} = \pm i}$

$$\boxed{y_h = C_1 \cos x + C_2 \sin x}$$

$$y_P = Ax + Be^x$$

$$y'_P = A + Be^x$$

$$y''_P = Be^x$$

$$y'' + y = 2x + 3e^x$$

$$2Be^x + Ax = 2x + 3e^x \rightarrow \boxed{A = 2, \quad B = \frac{3}{2}}$$

$$\boxed{y_p = 2x + \frac{3}{2}e^x}$$

$$\boxed{y(x) = C_1 \cos x + C_2 \sin x + 2x + \frac{3}{2}e^x}$$

Exercise

Find the general solution: $y'' - y = x^2 + e^x$

Solution

The characteristic equation: $\lambda^2 - 1 = 0 \rightarrow \boxed{\lambda_{1,2} = \pm 1}$

$$\boxed{y_h = C_1 e^{-x} + C_2 e^x}$$

$$y_P = Ax^2 + Bx + C + Dxe^x$$

$$y'_P = 2Ax + B + (D + Dx)e^x$$

$$y''_P = 2A + (2D + Dx)e^x$$

$$y'' - y = x^2 + e^x$$

$$2A + 2De^x - Ax^2 - Bx - C = x^2 + e^x$$

$$\begin{cases} -A = 1 \\ -B = 0 \\ 2A - C = 0 \\ 2D = 1 \end{cases} \rightarrow \boxed{A = -1, \ B = 0, \ C = -2, \ D = \frac{1}{2}}$$

$$\boxed{y_p = -x^2 - 2 + \frac{1}{2}xe^x}$$

$$\boxed{y(x) = C_1e^{-x} + \left(C_2 + \frac{1}{2}x\right)e^x - x^2 - 2}$$

Exercise

Find the general solution for the given *DE*: $y'' + y' = 10x^4 + 2$

Solution

The characteristic equation: $\lambda^2 + \lambda = 0 \ \rightarrow \ \boxed{\lambda_{1,2} = 0, \ -1}$

$$\boxed{y_h = C_1 + C_2e^{-x}}$$

$$y_P = Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex$$

$$y'_P = 5Ax^4 + 4Bx^3 + 3Cx^2 + 2Dx + E$$

$$y''_P = 20Ax^3 + 12Bx^2 + 6Cx + 2D$$

$$y'' + y' = 10x^4 + 2$$

$$20Ax^3 + 12Bx^2 + 6Cx + 2D + 5Ax^4 + 4Bx^3 + 3Cx^2 + 2Dx + E = 10x^4 + 2$$

$$\begin{array}{ll} x^4 & 5A = 10 \\ x^3 & 20A + 4B = 0 \\ x^2 & 12B + 3C = 0 \\ x^1 & 6C + 2D = 0 \\ x^0 & 1D + E = 2 \end{array} \rightarrow A = 2, \ B = -10, \ C = 40, \ D = -120, \ E = 242$$

$$\boxed{y_P = 2x^5 - 10x^4 + 10x^3 - 120x^2 + 242x}$$

$$\boxed{y(x) = C_1 + C_2e^{-x} + 2x^5 - 10x^4 + 10x^3 - 120x^2 + 242x}$$

Exercise

Find the general solution for the given *DE*: $y'' - y' = 5e^{x} - \sin 2x$

Solution

The characteristic equation: $\lambda^2 - \lambda = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = 0,\ 1}$

$\boxed{y_h = C_1 + C_2 e^{x}}$

$y_P = Axe^{x} + B\cos 2x + D\sin 2x$

$y'_P = (A + Ax)e^{x} - 2B\sin 2x + 2D\cos 2x$

$y''_P = (2A + Ax)e^{x} - 4B\cos 2x - 4D\sin 2x$

$y'' - y' = 5e^{x} - \sin 2x$

$(2A + Ax)e^{x} - 4B\cos 2x - 4D\sin 2x - (A + Ax)e^{x} + 2B\sin 2x - 2D\cos 2x = 5e^{x} - \sin 2x$

$Ae^{x} + (-4B - 2D)\cos 2x + (2B - 4D)\sin 2x = 5e^{x} - \sin 2x$

$$\begin{cases} A = 5 \\ -4B - 2D = 0 \\ 2B - 4D = -1 \end{cases} \quad \rightarrow \quad \boxed{A = 5,\ \ B = -\frac{1}{10},\ \ D = \frac{1}{5}}$$

$\boxed{y_P = 5xe^{x} - \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x}$

$\boxed{y(x) = C_1 + C_2 e^{x} + 5xe^{x} - \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x}$

Exercise

Find the general solution for the given *DE* $y'' + y = x\cos x - \cos x$

Solution

The characteristic equation: $\lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm i$

$y_h = C_1 \cos x + C_2 \sin x$

The particular equation: $y_p = (Ax + B)\cos x + (Cx + E)\sin x$

$y'_p = A\cos x - (Ax + B)\sin x + C\sin x + (Cx + E)\cos x$

$= (Cx + A + E)\cos x - (Ax + B + C)\sin x$

$y''_p = C\cos x - (Cx + A + E)\sin x - A\sin x - (Ax + B + C)\cos x$

$-(Ax + B)\cos x - (Cx + 2A + E)\sin x + (Ax + B)\cos x + (Cx + E)\sin x = x\cos x - \cos x$

$\cancel{-2A\sin x = x\cos x - \cos x}$

The particular equation: $y_p = \left(Ax^2 + Bx + C\right)\cos x + \left(Dx^2 + Ex + F\right)\sin x$

$$y'_p = (2Ax + B)\cos x - \left(Ax^2 + Bx + C\right)\sin x + (2Dx + E)\sin x + \left(Dx^2 + Ex + F\right)\cos x$$

$$= \left(Dx^2 + (2A+E)x + B + F\right)\cos x - \left(Ax^2 + (B-2D)x + C - E\right)\sin x$$

$$y''_p = (2Dx + 2A + E)\cos x - \left(Dx^2 + (2A+E)x + B + F\right)\sin x$$

$$-(2Ax + B - 2D)\sin x - \left(Ax^2 + (B-2D)x + C - E\right)\cos x$$

$$-\left(Ax^2 + (B-4D)x - 2A + C - 2E\right)\cos x - \left(Dx^2 + (4A+E)x + 2B - 2D + F\right)\sin x$$

$$+\left(Ax^2 + Bx + C\right)\cos x + \left(Dx^2 + Ex + F\right)\sin x = x\cos x - \cos x$$

$$4Dx\cos x + (2A+2E)\cos x - 4Ax\sin x + (2D-2B)\sin x = x\cos x - \cos x$$

$$\begin{cases} 4D = 1 & \rightarrow D = \frac{1}{4} \\ 2A + 2E = -1 & E = -\frac{1}{2} \\ -4A = 0 & \rightarrow A = 0 \\ 2D - 2B = 0 & B = \frac{1}{4} \end{cases}$$

$$y(x) = C_1\cos x + C_2\sin x + \frac{1}{4}x\cos x + \frac{1}{4}x^2\sin x - \frac{1}{2}x\sin x + C\cos x + F\sin x$$

$$\boxed{y(x) = C_1\cos x + C_2\sin x + \frac{1}{4}x\cos x + \frac{1}{4}x^2\sin x - \frac{1}{2}x\sin x}$$

Exercise

Find the general solution for the given *DE*: $y'' + y = e^x\sin x$

Solution

The characteristic equation: $\lambda^2 + \lambda = \lambda(\lambda+1) = 0 \rightarrow \boxed{\lambda_{1,2} = 0,\ -1}$

$$\boxed{y_h = C_1 + C_2e^{-x}}$$

$$y_P = e^x(A\cos x + B\sin x)$$

$$y'_P = e^x(A\cos x + B\sin x - A\sin x + B\cos x)$$

$$= e^x\left((A+B)\cos x + (B-A)\sin x\right)$$

$$y''_P = e^x\left((A+B)\cos x + (B-A)\sin x - (A+B)\sin x + (B-A)\cos x\right)$$

$$y''_P = e^x(2B\cos x - 2A\sin x)$$

$y'' + y = e^x\sin x$

$$e^x(2B\cos x - 2A\sin x + A\cos x + B\sin x) = e^x \sin x$$

$$(A+2B)\cos x + (B-2A)\sin x = \sin x$$

$$\begin{cases} A+2B=0 \\ -2A+B=1 \end{cases} \qquad \Delta = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 5 \quad \Delta_A = \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} = -2 \quad \Delta_B = \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} = 1$$

$$\underline{A = -\tfrac{2}{5}, \quad B = \tfrac{1}{5}}$$

$$\underline{y_P = e^x\left(-\tfrac{2}{5}\cos x + \tfrac{1}{5}\sin x\right)}$$

$$\underline{y(x) = C_1 + C_2 e^{-x} e^x\left(-\tfrac{2}{5}\cos x + \tfrac{1}{5}\sin x\right)}$$

Exercise

Find the general solution: $\quad y'' - 4y = 4x^2$

Solution

The characteristic equation: $\lambda^2 - 4 = 0 \;\rightarrow\; \underline{\lambda_{1,2} = \pm 2}$

$$\underline{y_h = C_1 e^{-2x} + C_2 e^{2x}}$$

$$y_P = Ax^2 + Bx + C$$

$$y'_P = 2Ax + B$$

$$y''_P = 2A$$

$$y'' - 4y = 4x^2$$

$$2A - 4Ax^2 - 4Bx - 4C = x^2$$

$$x^2 \qquad -4A = 1 \qquad \rightarrow \underline{A = -\tfrac{1}{4}}$$

$$x \qquad -4B = 0 \qquad \rightarrow \underline{B = 0}$$

$$x^0 \qquad 2A - 4C = 0 \qquad \rightarrow \underline{C = -\tfrac{1}{8}}$$

$$\underline{y_P = -\tfrac{1}{4}x^2 - \tfrac{1}{8}}$$

$$\underline{y(x) = C_1 e^{-2x} + C_2 e^{2x} - \tfrac{1}{4}x^2 - \tfrac{1}{8}}$$

Exercise

Find the general solution: $\quad y'' - y' - 2y = 20\cos x$

Solution

The characteristic equation: $\lambda^2-\lambda-2=0 \quad\rightarrow\quad \underline{\lambda_{1,2}=-1,\ 2|}$

$$\underline{y_h=C_1e^{-x}+C_2e^{2x}|}$$

$$y_P=A\cos x+B\sin x$$

$$y'_P=-A\sin x+B\cos x$$

$$y''_P=-A\cos x-B\sin x$$

$$y''-y'-2y=20\cos x$$

$$-A\cos x-B\sin x+A\sin x-B\cos x-2A\cos x-2B\sin x=20\cos x$$

$$\begin{cases}\cos x & -3A-B=20\\ \sin x & A-3B=0\end{cases} \quad\rightarrow\quad \underline{A=-6,\ B=-2|}$$

$$\underline{y_P=-6\cos x-2\sin x|}$$

$$\underline{y(x)=C_1e^{-x}+C_2e^{2x}-6\cos x-2\sin x|}$$

Exercise

Find the general solution: $\quad y''-y'+\frac{1}{4}y=3+e^{x/2}$

Solution

The characteristic equation: $\lambda^2-\lambda+\frac{1}{4}=0 \quad\rightarrow\quad \underline{\lambda_{1,2}=\frac{1}{2}|}$

$$\underline{y_h=\left(C_1+C_2x\right)e^{x/2}|}$$

$$y_P=A+Bx^2e^{x/2}$$

$$y'_P=\left(2Bx+\frac{1}{2}Bx^2\right)e^{x/2}$$

$$y''_P=\left(2Bx+\frac{1}{4}Bx^2+2B\right)e^{x/2}$$

$$y''-y'+\frac{1}{4}y=3+e^{x/2}$$

$$\left(2Bx+\frac{1}{4}Bx^2+2B-2Bx-\frac{1}{2}Bx^2+\frac{1}{4}Bx^2\right)e^{x/2}+\frac{1}{4}A=3+e^{x/2}$$

$$\frac{1}{4}A=3 \quad \underline{A=12|}$$

$$x^0 \quad 2B=1 \quad \underline{B=\frac{1}{2}|}$$

$$\underline{y_P=12+\frac{1}{2}x^2e^{x/2}|}$$

$$\underline{y(x)=\left(C_1+C_2x\right)e^{x/2}+12+\frac{1}{2}x^2e^{x/2}|}$$

Exercise

Find the general solution: $y'' + y' + \frac{1}{4}y = e^{x}(\sin 3x - \cos 3x)$

Solution

The characteristic equation: $\lambda^2 + \lambda + \frac{1}{4} = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm\frac{1}{2}i}$

$$\boxed{y_h = C_1 \cos x + C_2 \sin x}$$

$$y_P = e^{x}(A\cos 3x + B\sin 3x)$$

$$\begin{aligned} y_P' &= e^{x}(A\cos 3x + B\sin 3x - 3A\sin 3x + 3B\cos 3x) \\ &= e^{x}\big((A+3B)\cos 3x + (B-3A)\sin 3x\big) \end{aligned}$$

$$\begin{aligned} y_p'' &= \big((A+3B)\cos 3x + (B-3A)\sin 3x - 3(A+3B)\sin 3x + 3(B-3A)\cos 3x\big)e^{x} \\ &= \big((-8A+6B)\cos x + (-8B-6A)\sin x\big)e^{x} \end{aligned}$$

$$y'' + y' + \frac{1}{4}y = e^{x}(\sin 3x - \cos 3x)$$

$$e^{x}\begin{cases} \cos 3x & -8A + 6B + A + 3B + \frac{1}{4}A \\ \sin 3x & -8B - 6A + B - 3A + \frac{1}{4}B \end{cases} = e^{x}(\sin 3x - \cos 3x)$$

$$\begin{cases} -\frac{27}{4}A + 9B = -1 \\ -9A - \frac{27}{4}B = 1 \end{cases} \rightarrow \begin{cases} -27A + 36B = -4 \\ -36A - 27B = 4 \end{cases} \quad \Delta = \begin{vmatrix} -27 & 36 \\ -36 & -27 \end{vmatrix} = 2{,}025 \quad \Delta_A = \begin{vmatrix} -4 & 36 \\ 4 & -27 \end{vmatrix} = -36$$

$$\boxed{A = -\frac{4}{225},\ B = -\frac{28}{225}}$$

$$\boxed{y_P = e^{x}\left(-\frac{4}{225}\cos 3x - \frac{28}{225}\sin 3x\right)}$$

$$\boxed{y(x) = C_1 \cos x + C_2 \sin x + \left(-\frac{4}{225}\cos 3x - \frac{28}{225}\sin 3x\right)e^{x}}$$

Exercise

Find the general solution: $y'' - y' - 2y = e^{3x}$

Solution

The characteristic equation: $\lambda^2 - \lambda - 2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -1,\ 2}$

$$\boxed{y_h = C_1 e^{-x} + C_2 e^{2x}}$$

$$y_P = Ae^{3x}$$

$$y'_P = 3Ae^{3x}$$

$$y''_P = 9Ae^{3x}$$

$$y'' - y' - 2y = e^{3x}$$

$$(9A - 3A - 2A)e^{3x} = e^{3x}$$

$$4A = 1 \quad \rightarrow \quad \boxed{A = \frac{1}{4}}$$

$$\boxed{y_P = \frac{1}{4}e^{3x}}$$

$$\boxed{y(x) = C_1e^{-x} + C_2e^{2x} + \frac{1}{4}e^{3x}}$$

Exercise

Find the general solution: $\quad y'' - y' - 6y = 20e^{-2x}$

Solution

The characteristic equation: $\lambda^2 - \lambda - 6 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -2,\ 3}$

$$\boxed{y_h = C_1e^{-2x} + C_2e^{3x}}$$

$$y_P = Axe^{-2x}$$

$$y'_P = (A - 2Ax)e^{-2x}$$

$$y''_P = (-4A + 4Ax)e^{-2x}$$

$$y'' - y' - 6y = 20e^{-2x}$$

$$(-4A + 4Ax - A + 2Ax - 6Ax)e^{-2x} = 20e^{-2x}$$

$$-5Ae^{-2x} = 20e^{-2x} \quad \rightarrow \quad A = -4$$

$$\boxed{y_P = -4xe^{-2x}}$$

$$\boxed{y(x) = C_1e^{-2x} + C_2e^{3x} - 4xe^{-2x}}$$

Exercise

Find the general solution: $\quad y'' + y' - 6y = 2x$

Solution

The characteristic equation: $\lambda^2 + \lambda - 6 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -3,\ 2}$

$$\boxed{y_h = C_1 e^{-3x} + C_2 e^{2x}}$$

$$y_P = Ax + B$$

$$y_P' = A$$

$$y_P'' = 0$$

$$y'' + y' - 6y = 2x$$

$$A - 6Ax - 6B = 2x$$

$$\begin{cases} x & -6A = 2 \\ x^0 & A - 6B = 0 \end{cases} \rightarrow \boxed{A = -\frac{1}{3},\ B = -\frac{1}{18}}$$

$$\boxed{y(x) = C_1 e^{-3x} + C_2 e^{2x} - \frac{1}{3}x - \frac{1}{18}}$$

Exercise

Find the general solution: $y'' - y' - 6y = e^{-x} - 7\cos x$

Solution

The characteristic equation: $\lambda^2 - \lambda - 6 = 0 \rightarrow \boxed{\lambda_{1,2} = -2,\ 3}$

$$\boxed{y_h = C_1 e^{-2x} + C_2 e^{3x}}$$

$$y_P = Ae^{-x} + B\cos x + C\sin x$$

$$y_P' = -Ae^{-x} - B\sin x + C\cos x$$

$$y_P'' = Ae^{-x} - B\cos x - C\sin x$$

$$y'' - y' - 6y = e^{-x} - 7\cos x$$

$$Ae^{-x} - B\cos x - C\sin x + Ae^{-x} + B\sin x - C\cos x - 6Ae^{-x} - 6B\cos x - 6C\sin x = e^{-x} - 7\cos x$$

$$\begin{cases} e^{-x} & -4A = 1 \\ \cos x & -7B - C = -7 \\ \sin x & B - 7C = 0 \end{cases} \rightarrow \boxed{A = -\frac{1}{4},\ B = \frac{49}{50},\ C = \frac{7}{50}}$$

$$\boxed{y_P = -\frac{1}{4}e^{-x} + \frac{49}{50}\cos x + \frac{7}{50}\sin x}$$

$$\boxed{y(x) = C_1 e^{-2x} + C_2 e^{3x} - \frac{1}{4}e^{-x} + \frac{49}{50}\cos x + \frac{7}{50}\sin x}$$

Exercise

Find the general solution: $y'' + y' + 8y = x\cos 3x + \left(10x^2 + 21x + 9\right)\sin 3x$

Solution

The characteristic equation: $\lambda^2+\lambda+8=0 \quad \rightarrow \quad \boxed{\lambda_{1,2}=-\frac{1}{2}\pm\frac{\sqrt{31}}{2}i}$

$$\boxed{y_h=e^{-x/2}\left(C_1\cos\frac{\sqrt{31}}{2}x+C_2\sin\frac{\sqrt{31}}{2}x\right)}$$

$$y_p=\left(Ax^2+Bx+C\right)\cos 3x+\left(Dx^2+Ex+F\right)\sin 3x$$

$$\begin{aligned} y_p' &= (2Ax+B)\cos 3x-3\left(Ax^2+Bx+C\right)\sin 3x+(2Dx+E)\sin 3x+3\left(Dx^2+Ex+F\right)\cos 3x \\ &= \left(3Dx^2+2Ax+3Ex+3F+B\right)\cos 3x+\left(-3Ax^2-3Bx+2Dx-3C+E\right)\sin 3x \end{aligned}$$

$$\begin{aligned} y_p'' &= (6Dx+2A+3E)\cos 3x-3\left(3Dx^2+2Ax+3Ex+3F+B\right)\sin 3x \\ &\quad +(-6Ax-3B+2D)\sin 3x+3\left(-3Ax^2-3Bx+2Dx-3C+E\right)\cos 3x \\ &= \left(-9Ax^2-9Bx+12Dx+2A-9C+6E\right)\cos 3x \\ &\quad +\left(-9Dx^2-12Ax-9Ex-9F-6B+2D\right)\sin 3x \end{aligned}$$

$$y''+y'+8y=x\cos 3x+\left(10x^2+21x+9\right)\sin 3x$$

$$\begin{cases} \cos 3x & x^2 & -A+3D=0 & (1) \\ & x & -B+12D+2A+3E=1 & (2) \\ & x^0 & 2A+B-C+6E+3F=0 & (3) \end{cases}$$

$$\begin{cases} \sin 3x & x^2 & -3A-D=10 & (4) \\ & x & -12A-3B-E+2D=21 & (5) \\ & x^0 & -6B+2D-3C+E-F=9 & (6) \end{cases}$$

$$\begin{cases} (1) & -A+3D=0 \\ (4) & 3A+D=-10 \end{cases} \quad \rightarrow \quad A=-\frac{30}{10}\boxed{=-3},\quad D=-\frac{10}{10}\boxed{=-1}$$

$$\begin{cases} (2) & -B+3E=19 \\ (5) & 3B+E=13 \end{cases} \quad \rightarrow \quad \boxed{B=\frac{20}{10}=2} \quad \boxed{E=7}$$

$$\begin{cases} (3) & -C+3F=-38 \\ (6) & -3C-F=16 \end{cases} \quad \rightarrow \quad \boxed{C=-\frac{10}{10}=-1} \quad \boxed{F=-13}$$

$$\boxed{y(x)=e^{-x/2}\left(C_1\cos\frac{\sqrt{31}}{2}x+C_2\sin\frac{\sqrt{31}}{2}x\right)+\left(-3x^2+2x-1\right)\cos 3x+\left(-x^2+7x-13\right)\sin 3x}$$

Exercise

Find the general solution: $\quad y''-y'-12y=e^{4x}$

Solution

The characteristic equation: $\lambda^2 - \lambda - 12 = 0 \quad \to \quad \boxed{\lambda_{1,2} = -3,\ 4}$

$\boxed{y_h = C_1 e^{-3x} + C_2 e^{4x}}$

$y_p = Axe^{4x}$

$y'_p = (4Ax + A)e^{4x}$

$y''_p = (16Ax + 8A)e^{4x}$

$y'' - y' - 12y = e^{4x}$

$(16Ax + 8A - 4Ax - A - 12Ax)e^{4x} = e^{4x}$

$7Ae^{4x} = e^{4x} \quad \to \quad \boxed{A = \frac{1}{7}}$

$\boxed{y_p = \frac{1}{7} xe^{4x}}$

$\boxed{y(x) = C_1 e^{-3x} + \left(C_2 + \frac{1}{7}x\right)e^{4x}}$

Exercise

Find the general solution: $y'' + 2y' = 2x + 5 - e^{-2x}$

Solution

The characteristic equation: $\lambda^2 + 2\lambda = 0 \quad \to \quad \boxed{\lambda_{1,2} = 0,\ -2}$

$\boxed{y_h = C_1 + C_2 e^{-2x}}$

$y_p = Ax^2 + Bx + Cxe^{-2x}$

$y'_p = 2Ax + B + (-2Cx + C)e^{-2x}$

$y''_p = 2A + (4Cx - 4C)e^{-2x}$

$y'' + 2y' = 2x + 5 - e^{-2x}$

$2A + 4Ax + 2B - 2Ce^{-2x} = 2x + 5 - e^{-2x}$

$4A = 2 \qquad \boxed{A = \frac{1}{2}}$

$2A + 2B = 5 \qquad \boxed{B = 2}$

$-2C = -1 \qquad \boxed{C = \frac{1}{2}}$

$y_p = \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}$

$\boxed{y(x) = C_1 + C_2 e^{-2x} + \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}}$

Exercise

Find the general solution: $y'' - 2y' = 12x - 10$

Solution

The characteristic equation: $\lambda^2 - 2\lambda = 0 \quad \rightarrow \quad \underline{\lambda_{1,2} = 0,\ 2}$

$\underline{y_h = C_1 + C_2 e^{2x}}$

$y_P = Ax^2 + Bx$

$y'_P = 2Ax + B$

$y''_P = 2A$

$$y'' - 2y' = 12x - 10$$

$$2A - 4Ax - 2B = 12x - 10$$

$$\begin{cases} -4A = 12 \\ 2A - 2B = -10 \end{cases} \rightarrow \quad \underline{A = -3,\ B = 2}$$

$\underline{y_P = -3x^2 + 2x}$

$\underline{y(x) = C_1 + C_2 e^{2x} - 3x^2 + 2x}$

Exercise

Find the general solution: $y'' + 2y' + y = \sin x + 3\cos 2x$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 1 = 0 \quad \rightarrow \quad \underline{\lambda_{1,2} = -1}$

$\underline{y_h = \left(C_1 + C_2 x\right)e^{-x}}$

$y_P = A\cos x + B\sin x + C\cos 2x + D\sin 2x$

$y'_P = -A\sin x + B\cos x - 2C\sin 2x + 2D\cos 2x$

$y''_P = -A\cos x - B\sin x - 4C\cos 2x - 4D\sin 2x$

$$y'' + 2y' + y = \sin x + 3\cos 2x$$

$$\begin{cases} \cos x & -A + 2B + A = 0 \\ \sin x & -B - 2A + B = 1 \\ \cos 2x & -4C + 4D + C = 3 \\ \sin 2x & -4D - 4C + D = 0 \end{cases} \rightarrow \quad \underline{B = 0,\ A = -\frac{1}{2}} \quad \begin{cases} -3C + 4D = 3 \\ -4C - 3D = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} -3 & 4 \\ -4 & -3 \end{vmatrix} = 25 \quad \Delta_C = \begin{vmatrix} 3 & 4 \\ 0 & -3 \end{vmatrix} = -9 \quad \Delta_D = \begin{vmatrix} -3 & 3 \\ -4 & 0 \end{vmatrix} = 12$$

$$\boxed{C = -\frac{9}{25} \quad D = \frac{12}{25}}$$

$$\boxed{y_P = -\frac{1}{2}\cos x - \frac{9}{25}\cos 2x + \frac{12}{25}\sin 2x}$$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{-x} - \frac{1}{2}\cos x - \frac{9}{25}\cos 2x + \frac{12}{25}\sin 2x}$$

Exercise

Find the general solution: $y'' - 2y' + y = 6e^x$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 1 = 0 \rightarrow \boxed{\lambda_{1,2} = 1}$

$$\boxed{y_h = \left(C_1 + C_2 x\right)e^x}$$

$$y_P = Ax^2 e^x$$

$$y'_P = \left(2Ax + Ax^2\right)e^x$$

$$y''_P = \left(2A + 4Ax + Ax^2\right)e^x$$

$$y'' - 2y' + y = 6e^x$$

$$\left(2A + 4Ax + Ax^2 - 4Ax - 2Ax^2 + Ax^2\right)e^x = 6e^x$$

$$2Ae^x = 6e^x \rightarrow A = 3$$

$$\boxed{y_P = 3x^2 e^x}$$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^x + 3x^2 e^x}$$

Exercise

Find the general solution: $y'' + 2y' + y = x^2$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 1 = 0 \rightarrow \boxed{\lambda_{1,2} = -1}$

$$\boxed{y_h = \left(C_1 + C_2 x\right)e^{-x}}$$

$$y_P = Ax^2 + Bx + C$$

$$y'_P = 2Ax + B$$

$$y''_P = 2A$$

$y'' + 2y' + y = x^2$

$2A + 4Ax + 2B + Ax^2 + Bx + C = x^2$

$$\begin{cases} x^2 & A = 1 \\ x^1 & 4A + B = 0 \\ x^0 & 2A + C = 0 \end{cases} \rightarrow \boxed{A = 1,\ B = -4,\ C = -2}$$

$$\boxed{y_P = x^2 - 4x - 2}$$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{-x} + x^2 - 4x - 2}$$

Exercise

Find the general solution: $y'' + 2y' + y = x^2 e^{-x}$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 1 = 0 \rightarrow \boxed{\lambda_{1,2} = -1}$

$$\boxed{y_h = \left(C_1 + C_2 x\right)e^{-x}}$$

$$y_p = \left(Ax^4 + Bx^3 + Cx^2\right)e^{-x}$$

$$y'_p = \left(4Ax^3 + 3Bx^2 + 2Cx - Ax^4 - Bx^3 - Cx^2\right)e^{-x}$$

$$y''_P = \left(12Ax^2 + 6Bx + 2C - 8Ax^3 - 6Bx^2 - 4Cx + Ax^4 + Bx^3 + Cx^2\right)e^{-x}$$

$y'' + 2y' + y = x^2 e^{-x}$

$$\begin{array}{lll} x^4 & A - 2A + A = 0 & \\ x^3 & -8A + B + 8A - 2B + B = 0 & \\ x^2 & 12A - 6B + C + 6B - 2C + C = 1 & A = \frac{1}{12} \\ x & 6B - 4C + 4C = 0 & B = 0 \\ x^0 & 2C = 0 & C = 0 \end{array}$$

$$\boxed{y_p = \frac{1}{12}x^4 e^{-x}}$$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{-x} + \frac{1}{12}x^4 e^{-x}}$$

Exercise

Find the general solution: $y'' - 2y' + y = x^3 + 4x$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 1 = 0 \quad \to \quad \underline{\lambda_{1,2} = 1}$

$\underline{y_h = \left(C_1 + C_2 x\right)e^x}$

$y_p = Ax^3 + Bx^2 + Cx + D$

$y'_p = 3Ax^2 + 2Bx + C$

$y''_P = 6Ax + 2B$

$y'' - 2y' + y = x^3 + 4x$

$$\begin{array}{lll} x^3 & \underline{A=1} & \\ x^2 & -6A + B = 0 & B = 6 \\ x & 6A - 4B + C = 4 & C = 22 \\ x^0 & 2B - 2C + D = 0 & D = 32 \end{array}$$

$\underline{y_p = x^3 + 6x^2 + 22x + 32}$

$\underline{y(x) = \left(C_1 + C_2 x\right)e^x + x^3 + 6x^2 + 22x + 32}$

Exercise

Find the general solution: $\quad y'' + 2y' + y = 6\sin 2x$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 1 = 0 \quad \to \quad \underline{\lambda_{1,2} = -1}$

$\underline{y_h = \left(C_1 + C_2 x\right)e^{-x}}$

$y_P = A\cos 2x + B\sin 2x$

$y'_P = -2A\sin 2x + 2B\cos 2x$

$y''_P = -4A\cos 2x - 4B\sin 2x$

$y'' + 2y' + y = 6\sin 2x$

$-4A\cos 2x - 4B\sin 2x - 4A\sin 2x + 4B\cos 2x + A\cos 2x + B\sin 2x = 6\sin 2x$

$$\begin{cases} \cos 2x & -3A + 4B = 0 \\ \sin 2x & -4A - 3B = 6 \end{cases}$$

$$\to \Delta = \begin{vmatrix} -3 & 4 \\ -4 & -3 \end{vmatrix} = 25 \quad \Delta_A = \begin{vmatrix} 0 & 4 \\ 6 & -3 \end{vmatrix} = -24 \quad \Delta_B = \begin{vmatrix} -3 & 0 \\ -4 & 6 \end{vmatrix} = -18$$

$\underline{A = -\frac{24}{25}, \ B = -\frac{18}{25}}$

$$\boxed{y_p = -\frac{24}{25}\cos 2x - \frac{18}{25}\sin 2x}$$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{-x} - \frac{24}{25}\cos 2x - \frac{18}{25}\sin 2x}$$

Exercise

Find the general solution: $\quad y'' - 2y' + y = \left(x^2 - 1\right)e^{2x} + (3x+4)e^x$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 1 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = 1}$

$$\boxed{y_h = \left(C_1 + C_2 x\right)e^x}$$

$$y_p = \left(Ax^2 + Bx + C\right)e^{2x} + \left(Dx^3 + Ex^2\right)e^x$$

$$y'_p = \left(2Ax + B + 2Ax^2 + 2Bx + 2C\right)e^{2x} + \left(3Dx^2 + 2Ex + Dx^3 + Ex^2\right)e^x$$

$$y''_p = \left(2A + 8Ax + 4B + 4Ax^2 + 4Bx + 4C\right)e^{2x} + \left(6Dx + 2E + 6Dx^2 + 4Ex + Dx^3 + Ex^2\right)e^x$$

$$y'' - 2y' + y = \left(x^2 - 1\right)e^{2x} + (3x+4)e^x$$

$$\begin{cases} e^{2x} & x^2 & \underline{A = 1} & \\ & x & 4A + B = 0 & \rightarrow \underline{B = -4} \\ & x^0 & 2A + 2B + C = -1 & \rightarrow \underline{C = 5} \end{cases}$$

$$\begin{cases} e^x & x^3 & D - 2D + D = 0 & \\ & x^2 & 6D + E - 6D - 2E + E = 0 & \\ & x & 6D + 4E - 4E = 3 & \rightarrow \underline{D = \frac{1}{2}} \\ & x^0 & 2E = 4 & \rightarrow \underline{E = 2} \end{cases}$$

$$\boxed{y_p = \left(x^2 - 4x + 5\right)e^{2x} + \left(\frac{1}{2}x^3 + 2x^2\right)e^x}$$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^x + \left(x^2 - 4x + 5\right)e^{2x} + \left(\frac{1}{2}x^3 + 2x^2\right)e^x}$$

Exercise

Find the general solution: $\quad y'' + 2y' + 2y = 5e^{6x}$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 2 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{-2 \pm 2i}{2} \ \underline{= -1 \pm i}$

$$\boxed{y_h = \left(C_1 \cos x + C_2 \sin x\right)e^{-x}}$$

$$y_P = Ae^{6x}$$

$$y'_P = 6Ae^{6x}$$

$$y''_P = 36Ae^{6x}$$

$$y'' + 2y' + 2y = 5e^{6x}$$

$$3(6A + 12A + 2A)e^{6x} = 5e^{6x} \quad \rightarrow \quad 50A = 5 \;\Rightarrow\; \boxed{A = \frac{1}{10}}$$

$$\boxed{y_P = \frac{1}{10}e^{6x}}$$

$$\boxed{y(x) = \left(C_1 \cos x + C_2 \sin x\right)e^{-x} + \frac{1}{10}e^{6x}}$$

Exercise

Find the general solution: $\quad y'' + 2y' + 2y = x^3$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 2 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{-2 \pm 2i}{2} \; \boxed{= -1 \pm i}$

$$\boxed{y_h = \left(C_1 \cos x + C_2 \sin x\right)e^{-x}}$$

$$y_P = Ax^3 + Bx^2 + Cx + D$$

$$y'_P = 3Ax^2 + 2Bx + C$$

$$y''_P = 6Ax + 2B$$

$$y'' + 2y' + 2y = x^3$$

$$6Ax + 2B + 6Ax^2 + 4Bx + 2C + 2Ax^3 + 2Bx^2 + 2Cx + 2D = x^3$$

$$\begin{array}{lll} x^3 & 2A = 1 & \rightarrow \boxed{A = \frac{1}{2}} \\ x^2 & 6A + 2B = 0 & \rightarrow \boxed{B = -\frac{3}{2}} \\ x & 6A + 4B + 2C = 0 & \rightarrow \boxed{C = \frac{3}{2}} \\ x^0 & 2B + 2C + 2D = 0 & \rightarrow \boxed{D = 0} \end{array}$$

$$\boxed{y_P = \frac{1}{2}x^3 - \frac{3}{2}x^2 + \frac{3}{2}x}$$

$$\boxed{y(x) = \left(C_1 \cos x + C_2 \sin x\right)e^{-x} + \frac{1}{2}x^3 - \frac{3}{2}x^2 + \frac{3}{2}x}$$

Exercise

Find the general solution: $y'' + 2y' + 2y = \cos x + e^{-x}$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 2 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{-2 \pm 2i}{2} \ \underline{= -1 \pm i}$

$$\underline{y_h = \left(C_1 \cos x + C_2 \sin x\right)e^{-x}}$$

$$y_P = A\cos x + B\sin x + Ce^{-x}$$

$$y_P' = -A\sin x + B\cos x - Ce^{-x}$$

$$y_P'' = -A\cos x - B\sin x + Ce^{-x}$$

$$y'' + 2y' + 2y = \cos x + e^{-x}$$

$$-A\cos x - B\sin x + Ce^{-x} - 2A\sin x + 2B\cos x - 2Ce^{-x} + 2A\cos x + 2B\sin x + 2Ce^{-x} = \cos x + e^{-x}$$

$$\begin{array}{ll} \cos x & A + 2B = 0 \\ \sin x & -2A + B = 1 \\ e^{-x} & \underline{C = 1} \end{array} \qquad \Delta = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 5 \quad \Delta_A = \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} = -2 \quad \Delta_B = \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} = 1$$

$$\underline{y_P = -\tfrac{2}{5}\cos x + \tfrac{1}{5}\sin x + e^{-x}}$$

$$\underline{y(x) = \left(C_1 \cos x + C_2 \sin x + 1\right)e^{-x} - \tfrac{2}{5}\cos x + \tfrac{1}{5}\sin x}$$

Exercise

Find the general solution: $y'' - 2y' + 2y = e^{x} \sin x$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 2 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{2 \pm 2i}{2} \ \underline{= 1 \pm i}$

$$\underline{y_h = e^{x}\left(C_1 \cos x + C_2 \sin x\right)}$$

$$y_P = xe^{x}\left(A\cos x + B\sin x\right)$$

$$y_P' = (1 + x)e^{x}\left(A\cos x + B\sin x\right) + xe^{x}\left(-A\sin x + B\cos x\right)$$

$$= \left(\left(A + (A + B)x\right)\cos x + \left(B + (B - A)x\right)\sin x\right)e^{x}$$

$$y_p'' = \left(\begin{array}{l} \left(A + (A + B)x\right)\cos x + \left(B + (B - A)x\right)\sin x + (A + B)\cos x - \left(A + (A + B)x\right)\sin x \\ \quad + (B - A)\sin x + \left(B + (B - A)x\right)\cos x \end{array}\right)e^{x}$$

$$= \left((2A + 2B + 2Bx)\cos x + (2B - 2A + -2Ax)\sin x\right)e^{x}$$

$$y'' - 2y' + 2y = e^{x}\sin x$$

$$e^{x}\begin{cases}\cos x & 2A+2B-2A+(2B-2A-2B+2A)x\\ \sin x & 2B-2A-2B+(-2A-2B+2A+2B)x\end{cases} = e^{x}\sin x$$

$$\begin{cases}2B=0\\ -2A=1\end{cases} \rightarrow \quad \boxed{A=-\frac{1}{2},\ B=0}$$

$$\boxed{y(x)=e^{x}\left(C_1\cos x+C_2\sin x\right)-\frac{1}{2}xe^{x}\cos x}$$

Exercise

Find the general solution: $\quad y''-2y'+2y=e^{2x}(\cos x-3\sin x)$

Solution

The characteristic equation: $\lambda^2-2\lambda+2=0 \quad \rightarrow \quad \lambda_{1,2}=\frac{2\pm 2i}{2}\ \boxed{=1\pm i}$

$$\boxed{y_h=e^{x}\left(C_1\cos x+C_2\sin x\right)}$$

$$y_P=e^{2x}(A\cos x+B\sin x)$$

$$y_P'=e^{2x}(2A\cos x+2B\sin x-A\sin x+B\cos x)$$

$$=e^{2x}\left((2A+B)\cos x+(2B-A)\sin x\right)$$

$$y_p''=e^{2x}\left((4A+2B)\cos x+(4B-2A)\sin x-(2A+B)\sin x+(2B-A)\cos x\right)$$

$$=e^{2x}\left((3A+4B)\cos x+(3B-4A)\sin x\right)$$

$$y''-2y'+2y=e^{2x}(\cos x-3\sin x)$$

$$e^{2x}\begin{cases}\cos x & 3A+4B-4A-2B+2A=1\\ \sin x & 3B-4A-4B+2A+2B=-3\end{cases}$$

$$\rightarrow \begin{cases}A+2B=1\\ -2A+B=-3\end{cases} \Rightarrow \boxed{A=\frac{7}{5},\ B=-\frac{1}{5}}$$

$$\boxed{y(x)=e^{x}\left(C_1\cos x+C_2\sin x\right)+e^{2x}\left(\frac{7}{5}\cos x+\frac{1}{5}\sin x\right)}$$

Exercise

Find the general solution: $\quad y''-2y'-3y=1-x^2$

Solution

The characteristic equation: $\lambda^2-2\lambda-3=0 \quad \rightarrow \quad \boxed{\lambda_{1,2}=-1,\ 3}$

$$\boxed{y_h=C_1e^{-x}+C_2e^{3x}}$$

$$y_P=Ax^2+Bx+C$$

$$y'_P = 2Ax + B$$
$$y''_P = 2A$$

$$y'' - 2y' - 3y = 1 - x^2$$

$$2A - 4Ax - 2B - 3Ax^2 - 3Bx - 3C = 1 - x^2$$

$$\begin{aligned} x^2 &\quad -3A = -1 \\ x^1 &\quad -4A - 3B = 0 \quad \to \quad \boxed{A = \tfrac{1}{3},\ B = -\tfrac{4}{9},\ C = \tfrac{5}{9}} \\ x^0 &\quad 2A - 2B - 3C = 1 \end{aligned}$$

$$\boxed{y(x) = C_1 e^{-x} + C_2 e^{3x} + \tfrac{1}{3}x^2 - \tfrac{4}{9}x + \tfrac{5}{9}}$$

Exercise

Find the general solution: $\quad y'' - 2y' - 3y = 4e^x - 9$

Solution

The characteristic equation: $\lambda^2 - 2\lambda - 3 = 0 \quad \to \quad \boxed{\lambda_{1,2} = -1,\ 3}$

$$\boxed{y_h = C_1 e^{-x} + C_2 e^{3x}}$$

$$y_P = Ae^x + B$$
$$y'_P = Ae^x$$
$$y''_P = Ae^x$$

$$y'' - 2y' - 3y = 4e^x - 9$$

$$(A - 2A - 3A)e^x - 3B = 4e^x - 9$$

$$\begin{cases} -4A = 4 \\ -3B = 9 \end{cases} \quad \to \quad \boxed{A = -1,\ B = 3}$$

$$\boxed{y(x) = C_1 e^{-x} + C_2 e^{3x} - e^x + 3}$$

Exercise

Find the general solution: $\quad y'' - 2y' - 3y = 2e^{-x}\cos x + x^2 + xe^{3x}$

Solution

The characteristic equation: $\lambda^2 - 2\lambda - 3 = 0 \quad \to \quad \boxed{\lambda_{1,2} = -1,\ 3}$

$$\boxed{y_h = C_1 e^{-x} + C_2 e^{3x}}$$

$$y_P = e^{-x}\left(A\cos x + B\sin x\right) + Cx^2 + Dx + E + \left(Fx^3 + Gx^2 + Hx\right)e^{3x}$$

➢ $y_{P1} = e^{-x}\left(A\cos x + B\sin x\right)$

$$y'_{P1} = e^{-x}\left(-A\cos x - B\sin x - A\sin x + B\cos x\right)$$
$$= e^{-x}\left((B-A)\cos x - (A+B)\sin x\right)$$

$$y''_{P2} = e^{-x}\left(-(B-A)\cos x + (A+B)\sin x - (B-A)\sin x - (A+B)\cos x\right)$$
$$= e^{-x}\left(-2B\cos x + 2A\sin x\right)$$

$$y'' - 2y' - 3y = 2e^{-x}\cos x + x^2 + xe^{3x}$$

$$e^{-x}\left(-2B\cos x + 2A\sin x - 2(B-A)\cos x + 2(A+B)\sin x - 3A\cos x - 3B\sin x\right) = 2e^{-x}\cos x$$

$$e^{-x}\left((-A-4B)\cos x + (4A-B)\sin x\right) = 2e^{-x}\cos x$$

$$\begin{cases} -A-4B=2 \\ 4A-B=0 \end{cases} \qquad \Delta = \begin{vmatrix} -1 & -4 \\ 4 & -1 \end{vmatrix} = 17 \quad \Delta_A = \begin{vmatrix} 2 & -4 \\ 0 & -1 \end{vmatrix} = -2 \quad \Delta_B = \begin{vmatrix} -1 & 2 \\ 4 & 0 \end{vmatrix} = -8$$

$$\boxed{A = -\frac{2}{17}, \quad B = -\frac{8}{17}}$$

➢ $y_{P2} = Cx^2 + Dx + E$

$$y'_{P2} = 2Cx + D$$
$$y''_{P2} = 2C$$

$$y'' - 2y' - 3y = 2e^{-x}\cos x + x^2 + xe^{3x}$$

$$2C - 4Cx - 2D - 3Cx^2 - 3Dx - 3E = x^2$$

x^2 $\quad -3C = 1 \quad \to \boxed{C = -\frac{1}{3}}$

x $\quad \frac{4}{3} - 3D = 0 \quad \to \boxed{D = \frac{4}{9}}$

x^0 $\quad -\frac{2}{3} - \frac{8}{9} - 3E = 0 \quad \to \boxed{E = -\frac{14}{27}}$

➢ $y_{P3} = \left(Fx^3 + Gx^2 + Hx\right)e^{3x}$

$$y'_{P3} = \left(3Fx^2 + 2Gx + H + 3Fx^3 + 3Gx^2 + 3Hx\right)e^{3x}$$
$$= \left(3Fx^3 + (3F+3G)x^2 + (2G+3H)x + H\right)e^{3x}$$

$$y''_{P3} = \left(9Fx^2 + (6F+6G)x + 2G + 3H + 9Fx^3 + (9F+9G)x^2 + (6G+9H)x + 3H\right)e^{3x}$$
$$= \left(9Fx^3 + (18F+9G)x^2 + (6F+12G+9H)x + 2G + 6H\right)e^{3x}$$

$$9Fx^3 + (18F+9G)x^2 + (6F+12G+9H)x + 2G + 6H - 6Fx^3 - (6F+6G)x^2$$
$$-(4G+6H)x - 2H - 3Fx^3 - 3Gx^2 - 3Hx$$

$$\left(12Fx^2+(6F+8G)x+2G+4H\right)e^{3x}$$

$$y''-2y'-3y=2e^{-x}\cos x+x^2+xe^{3x}$$

$$\begin{aligned} &x^2 \quad 12F=0 &&\to \underline{F=0} \\ &x \quad 6F+8G=1 &&\to \underline{G=\tfrac{1}{8}} \\ &x^0 \quad 2G+4H=0 &&\to \underline{H=-\tfrac{1}{16}} \end{aligned}$$

$$\underline{y_p=-\left(\tfrac{2}{17}\cos x+\tfrac{8}{17}\sin x\right)e^{-x}-\tfrac{1}{3}x^2+\tfrac{4}{9}x-\tfrac{14}{27}+\left(\tfrac{1}{8}x^2-\tfrac{1}{16}x\right)e^{3x}}$$

$$\underline{y(x)=C_1e^{-x}+C_2e^{3x}-\left(\tfrac{2}{17}\cos x+\tfrac{8}{17}\sin x\right)e^{-x}-\tfrac{1}{3}x^2+\tfrac{4}{9}x-\tfrac{14}{27}+\left(\tfrac{1}{8}x^2-\tfrac{1}{16}x\right)e^{3x}}$$

Exercise

Find the general solution: $y''-2y'+5y=25x^2+12$

Solution

The characteristic equation: $\lambda^2-2\lambda+5=0 \;\to\; \lambda_{1,2}=\frac{2\pm 4i}{2}\;\underline{=1\pm 2i}$

$$\underline{y_h=e^x\left(C_1\cos 2x+C_2\sin 2x\right)}$$

$$y_P=Ax^2+Bx+C$$

$$y'_P=2Ax+B$$

$$y''_P=2A$$

$$y''-2y'+5y=25x^2+12$$

$$\begin{aligned} &x^2 \quad 5A=25 \\ &x^1 \quad -4A+5B=0 \quad \to \quad \underline{A=5,\ B=4,\ C=-10} \\ &x^0 \quad 2A-2B-3C=12 \end{aligned}$$

$$\underline{y(x)=e^x\left(C_1\cos 2x+C_2\sin 2x\right)+5x^2+4x-10}$$

Exercise

Find the general solution: $y''-2y'+5y=e^x\cos 2x$

Solution

The characteristic equation: $\lambda^2-2\lambda+5=0 \;\to\; \lambda_{1,2}=\frac{2\pm 4i}{2}\;\underline{=1\pm 2i}$

$$\underline{y_h=e^x\left(C_1\cos 2x+C_2\sin 2x\right)}$$

$$y_P = e^x\left(Ax\cos 2x + Bx\sin 2x\right)$$

$$y'_P = e^x\left(Ax\cos 2x + Bx\sin 2x + A\cos 2x - 2Ax\sin 2x + B\sin 2x + 2Bx\cos 2x\right)$$

$$= e^x\left(\left(Ax + 2Bx + A\right)\cos 2x + \left(Bx - 2Ax + B\right)\sin 2x\right)$$

$$y''_P = e^x\begin{pmatrix}\left(Ax + 2Bx + A\right)\cos 2x + \left(Bx - 2Ax + B\right)\sin 2x + \left(A + 2B\right)\cos 2x \\ -\left(2Ax + 4Bx + 2A\right)\sin 2x + \left(B - 2A\right)\sin 2x + \left(2Bx - 4Ax + 2B\right)\cos 2x\end{pmatrix}$$

$$= e^x\left(\left(-3Ax + 4Bx + 2A + 4B\right)\cos 2x + \left(-4Ax - 3Bx - 4A + 2B\right)\sin 2x\right)$$

$$y'' - 2y' + 5y = e^x\cos 2x$$

$$\begin{array}{ll} \cos 2x & -3Ax + 4Bx + 2A + 4B - 2Ax - 4Bx - 2A + 5Ax \\ \sin 2x & -4Ax - 3Bx - 4A + 2B - 2Bx + 4Ax - 2B + 5Bx \end{array}$$

$$\rightarrow \begin{cases} 4B = 1 \\ -4A = 0 \end{cases} \Rightarrow \quad \underline{B = \frac{1}{4}} \quad \underline{A = 0}$$

$$\underline{y(x) = e^x\left(C_1\cos 2x + \left(C_2 + \frac{1}{4}x\right)\sin 2x\right)}$$

Exercise

Find the general solution: $\quad y'' - 2y' + 5y = e^x\sin x$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 5 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{2 \pm 4i}{2} \ \underline{= 1 \pm 2i}$

$$\underline{y_h = e^x\left(C_1\cos 2x + C_2\sin 2x\right)}$$

$$y_P = e^x\left(A\cos x + B\sin x\right)$$

$$y'_P = e^x\left(A\cos x + B\sin x - A\sin x + B\cos x\right)$$

$$= e^x\left(\left(A + B\right)\cos x + \left(B - A\right)\sin x\right)$$

$$y''_P = e^x\left(\left(A + B\right)\cos x + \left(B - A\right)\sin x - \left(A + B\right)\sin x + \left(B - A\right)\cos x\right)$$

$$= e^x\left(2B\cos x - 2A\sin x\right)$$

$$y'' - 2y' + 5y = e^x\sin x$$

$$\begin{array}{ll} \cos x & 2B - 2A - 2B + 5A = 0 \\ \sin x & -2A - 2B + 2A + 5B = 1 \end{array} \rightarrow \begin{cases} A = 0 \\ B = \frac{1}{3} \end{cases}$$

$$\underline{y(x) = e^x\left(C_1\cos 2x + C_2\sin 2x + \frac{1}{3}\sin x\right)}$$

Exercise

Find the general solution: $y'' + 2y' - 24y = 16 - (x+2)e^{4x}$

Solution

The characteristic equation: $\lambda^2 + 2\lambda - 24 = 0 \rightarrow \lambda_{1,2} = \frac{-2 \pm 10}{2} = -6,\ 4$

$$y_h = C_1 e^{-6x} + C_2 e^{4x}$$

$$y_P = A + \left(Bx^2 + Cx\right)e^{4x}$$

$$y'_P = \left(2Bx + C + 4Bx^2 + 4Cx\right)e^{4x}$$

$$y''_P = \left(2B + 8C + 16Bx + 16Bx^2 + 16Cx\right)e^{4x}$$

$$y'' + 2y' - 24y = 16 - (x+2)e^{4x}$$

$$-24A = 16 \qquad A = -\frac{2}{3}$$

x^2 $\quad 16B + 8B - 24B = 0$

x $\quad 16B + 16C + 8C + 4B - 24C = -1 \qquad B = -\frac{1}{20}$

x^0 $\quad 2B + 8C + 2C = -2 \qquad C = -\frac{19}{100}$

$$y_P = -\frac{2}{3} + \left(-\frac{1}{20}x^2 - \frac{19}{100}x\right)e^{4x}$$

$$y(x) = C_1 e^{-6x} + C_2 e^{4x} - \frac{2}{3} + \left(-\frac{1}{20}x^2 - \frac{19}{100}x\right)e^{4x}$$

Exercise

Find the general solution: $y'' + 3y = -48x^2 e^{3x}$

Solution

The characteristic equation: $\lambda^2 + 3 = 0 \rightarrow \lambda_{1,2} = \pm i\sqrt{3}$

$$y_h = C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x$$

$$y_P = \left(Ax^2 + Bx + C\right)e^{3x}$$

$$y'_P = \left(2Ax + B + 3Ax^2 + 3Bx + 3C\right)e^{3x}$$

$$y''_P = \left(2A + 6Ax + 3B + 6Ax + 3B + 9Ax^2 + 9Bx + 9C\right)e^{3x}$$

$$y'' + 3y = -48x^2 e^{3x}$$

$$x^2 \qquad 9A+3A=-48 \qquad \boxed{A=-4}$$

$$x \qquad 12A+9B+3B=0 \qquad \boxed{B=4}$$

$$x^0 \qquad 2A+6B+9C+3C=0 \qquad \boxed{C=-\frac{4}{3}}$$

$$\boxed{y_P=\left(-4x^2+4x-\frac{4}{3}\right)e^{3x}}$$

$$\boxed{y(x)=C_1\cos\sqrt{3}x+C_2\sin\sqrt{3}x+\left(-4x^2+4x-\frac{4}{3}\right)e^{3x}}$$

Exercise

Find the general solution: $\qquad y''-3y'=e^{3x}-12x$

Solution

The characteristic equation: $\lambda^2-3\lambda=0 \quad \to \quad \boxed{\lambda_{1,2}=0,\ 3}$

$$\boxed{y_h=C_1+C_2e^{3x}}$$

$$y_P=Axe^{3x}+Bx^2+Cx$$

$$y_P'=(A+3Ax)e^{3x}+2Bx+C$$

$$y_P''=(6A+9Ax)e^{3x}+2B$$

$$y''-3y'=e^{3x}-12x$$

$$e^{3x} \qquad 6A+9Ax-3A-9Ax=1 \qquad \boxed{A=\frac{1}{3}}$$

$$x \qquad -6B=-12 \qquad \boxed{B=2}$$

$$x^0 \qquad 2B-3C=0 \qquad \boxed{C=\frac{4}{3}}$$

$$\boxed{y_P=\frac{1}{3}xe^{3x}+2x^2+\frac{4}{3}x}$$

$$\boxed{y(x)=C_1+C_2e^{3x}+\frac{1}{3}xe^{3x}+2x^2+\frac{4}{3}x}$$

Exercise

Find the general solution: $\qquad y''+3y'=4x-5$

Solution

The characteristic equation: $\lambda^2+3\lambda=0 \quad \to \quad \boxed{\lambda_{1,2}=0,\ -3}$

$$\boxed{y_h=C_1+C_2e^{-3x}}$$

$$y_P=Ax^2+Bx$$

$$y'_P = 2Ax + B$$
$$y''_P = 2A$$
$$y'' + 3y' = 4x - 5$$

$x \qquad 6A = 4 \qquad \boxed{A = \frac{2}{3}}$

$x^0 \qquad 2A + 3B = -5 \qquad \boxed{B = -\frac{19}{9}}$

$$\boxed{y_P = \frac{2}{3}x^2 - \frac{19}{9}x}$$

$$\boxed{y(x) = C_1 + C_2 e^{-3x} + \frac{2}{3}x^2 - \frac{19}{9}x}$$

Exercise

Find the general solution: $\quad y'' - 3y' = 8e^{3x} + 4\sin x$

Solution

The characteristic equation: $\lambda^2 - 3\lambda = 0 \Rightarrow \boxed{\lambda_{1,2} = 0,\ 3}$

$$\boxed{y_h = C_1 + C_2 e^{3x}}$$

The particular equation: $y_p = Ae^{3x} + B\cos x + C\sin x$

$$y'_p = 3Ae^{3x} - B\sin x + C\cos x$$
$$y''_p = 9Ae^{3x} - B\cos x - C\sin x$$

$$y'' - 3y' = 8e^{3x} + 4\sin x$$

$$\cancel{9Ae^{3x}} - B\cos x - C\sin x \cancel{-9Ae^{3x}} + 3B\sin x - 3C\cos x = 8e^{3x} + 4\sin x$$

The particular equation: $y_p = Axe^{3x} + B\cos x + C\sin x$

$$y'_p = 3Axe^{3x} + Ae^{3x} - B\sin x + C\cos x$$
$$y''_p = 3Ae^{3x} + 9Axe^{3x} + 3Ae^{3x} - B\cos x - C\sin x$$

$$6Ae^{3x} + 9Axe^{3x} - B\cos x - C\sin x - 9Axe^{3x} - 3Ae^{3x} + 3B\sin x - 3C\cos x = 8e^{3x} + 4\sin x$$
$$3Ae^{3x} - (B + 3C)\cos x + (3B - C)\sin x = 8e^{3x} + 4\sin x$$

$$\begin{cases} 3A = 8 & \to A = \frac{8}{3} \\ -B - 3C = 0 & B = \frac{6}{5} \\ 3B - C = 4 & C = -\frac{2}{5} \end{cases}$$

$$\boxed{y(x) = C_1 + C_2 e^{3x} + \frac{8}{3}xe^{3x} + \frac{6}{5}\cos x - \frac{2}{5}\sin x}$$

Exercise

Find the general solution: $y'' + 3y' + 2y = 6$

Solution

The characteristic equation: $\lambda^2 + 3\lambda + 2 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = -1, \ -2}$

$$\boxed{y_h = C_1 e^{-x} + C_2 e^{-2x}}$$

$$y_p = A$$

$$y'_p = 0 = y''_p$$

$$y'' + 3y' + 2y = 6$$

$$2A = 6 \quad \rightarrow \quad \boxed{A = 3}$$

$$\boxed{y(x) = C_1 e^{-x} + C_2 e^{-2x} + 3}$$

Exercise

Find the general solution: $y'' + 3y' + 2y = 4x^2$

Solution

The characteristic equation: $\lambda^2 + 3\lambda + 2 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = -1, \ -2}$

$$\boxed{y_h = C_1 e^{-x} + C_2 e^{-2x}}$$

The particular equation: $y_p = ax^2 + bx + c$

$$y'_p = 2ax + b$$

$$y''_p = 2a$$

$$y''_p + 3y'_p + 2y_p = 4x^2$$

$$2a + 6ax + 3b + 2ax^2 + 2bx + 2c = 4x^2$$

$$\begin{cases} 2a = 4 & \rightarrow a = 2 \\ 6a + 2b = 0 & \rightarrow b = -6 \\ 2a + 3b + 2c = 0 & \rightarrow c = 7 \end{cases}$$

$$\boxed{y(x) = C_1 e^{-x} + C_2 e^{-2x} + 2x^2 - 6x + 7}$$

Exercise

Find the general solution: $y'' - 3y' + 2y = 5e^x$

Solution

The characteristic equation: $\lambda^2 - 3\lambda + 2 = 0 \quad \to \quad \boxed{\lambda_{1,2} = 1,\ 2}$

$$\boxed{y_h = C_1 e^x + C_2 e^{2x}}$$

$$y_p = Axe^x$$

$$y_p' = (A + Ax)e^x$$

$$y_p'' = (2A + Ax)e^x$$

$$y'' - 3y' + 2y = 5e^x$$

$$(2A + Ax - 3A - 3Ax + 2Ax)e^x = 5e^x$$

$$-Ae^x = 5e^x \quad \to \quad \boxed{A = -5}$$

$$\boxed{y_p = -5xe^x}$$

$$\boxed{y(x) = C_1 e^x + C_2 e^{2x} - 5xe^x}$$

Exercise

Find the general solution: $\quad y'' - 3y' + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$

Solution

The characteristic equation: $\lambda^2 - 3\lambda + 2 = 0 \quad \to \quad \boxed{\lambda_{1,2} = 1,\ 2}$

$$\boxed{y_h = C_1 e^x + C_2 e^{2x}}$$

$$y_p = A_1 x^2 + A_2 x + A_3 + \left(A_4 x + A_5 x^2\right)e^x + A_6 e^{3x}$$

$$y_p' = 2A_1 x + A_2 + \left(A_4 + \left(2A_5 + A_4\right)x + A_5 x^2\right)e^x + 3A_6 e^{3x}$$

$$y_p'' = 2A_1 + \left(2A_5 + 2A_4 + \left(4A_5 + A_4\right)x + A_5 x^2\right)e^x + 9A_6 e^{3x}$$

$$y'' - 3y' + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$$

$$\begin{cases} x^2 & 2A_1 = 2 & \to \boxed{A_1 = 1} \\ x & -6A_1 + 2A_2 = 0 & \to \boxed{A_2 = 3} \\ x^0 & 2A_1 - 3A_2 + 2A_3 = 0 & \to \boxed{A_3 = \frac{7}{2}} \end{cases}$$

$$x^2e^x \qquad A_5 - 3A_5 + 2A_5$$

$$xe^x \qquad 4A_5 + A_4 - 6A_5 - 3A_4 + 2A_4 = 2 \quad \rightarrow \underline{A_5 = -1}$$

$$e^x \qquad 2A_5 + 2A_4 - 3A_4 = 1 \quad \rightarrow \underline{A_4 = -3}$$

$$e^{3x} \qquad 9A_6 - 9A_6 + 2A_6 = 4 \quad \rightarrow \underline{A_6 = 2}$$

$$\underline{y_p = x^2 + 3x + \frac{7}{2} - \left(x^2 + 3x\right)e^x + 2e^{3x}}$$

$$\underline{y(x) = C_2e^{2x} + x^2 + 3x + \frac{7}{2} - \left(x^2 + 3x + C_1\right)e^x + 2e^{3x}}$$

Exercise

Find the general solution: $y'' - 3y' + 2y = 14\sin 2x - 18\cos 2x$

Solution

The characteristic equation: $\lambda^2 - 3\lambda + 2 = 0 \quad \rightarrow \quad \underline{\lambda_{1,2} = 1,\ 2}$

$$\underline{y_h = C_1e^x + C_2e^{2x}}$$

$$y_P = A\cos 2x + B\sin 2x$$

$$y'_P = -2A\sin 2x + 2B\cos 2x$$

$$y''_P = -4A\cos 2x - 4B\sin 2x$$

$$y'' - 3y' + 2y = 14\sin 2x - 18\cos 2x$$

$$-4A\cos 2x - 4B\sin 2x + 6A\sin 2x - 6B\cos 2x + 2A\cos 2x + 2B\sin 2x = 14\sin 2x - 18\cos 2x$$

$$(-2A - 6B)\cos 2x + (-2B + 6A)\sin 2x = 14\sin 2x - 18\cos 2x$$

$$\begin{cases} -2A - 6B = -18 \\ 6A - 2B = 14 \end{cases} \Rightarrow \begin{cases} A + 3B = 9 \\ 3A - B = 7 \end{cases} \rightarrow \underline{A = 3,\ \ B = 2}$$

$$\underline{y(x) = C_1e^x + C_2e^{2x} + 3\cos 2x + 2\sin 2x}$$

Exercise

Find the general solution: $y'' + 3y' + 2y = e^{-x} + e^{-2x} - x$

Solution

The characteristic equation: $\lambda^2 + 3\lambda + 2 = 0 \quad \rightarrow \quad \underline{\lambda_{1,2} = -2,\ -1}$

$$\underline{y_h = C_1e^{-x} + C_2e^{-2x}}$$

$$y_P = Axe^{-x} + Bxe^{-2x} + Cx + D$$

$$y'_P = (A - Ax)e^{-x} + (B - 2Bx)e^{-2x} + C$$

$$y''_P = (-2A + Ax)e^{-x} + (-4B + 4Bx)e^{-2x}$$

$$y'' + 3y' + 2y = e^{-x} + e^{-2x} - x$$

$$Ae^{-x} - Be^{-2x} + 3C + 2Cx + 2D = e^{-x} + e^{-2x} - x$$

$$\begin{cases} e^{-x} & \underline{A = 1} \\ e^{-2x} & -B = 1 \\ x & 2C = -1 \\ x^0 & 3C + 2D = 0 \end{cases} \rightarrow \quad \underline{B = -1,\ C = -\frac{1}{2},\ D = -\frac{3}{4}}$$

$$\underline{y_P = xe^{-x} - xe^{-2x} - \frac{1}{2}x - \frac{3}{4}}$$

$$\underline{y(x) = (C_1 + x)e^{-x} + (C_2 - x)e^{-2x} - \frac{1}{2}x - \frac{3}{4}}$$

Exercise

Find the general solution: $\quad y'' - 3y' - 10y = -3$

Solution

The characteristic equation: $\lambda^2 - 3\lambda - 10 = 0 \quad \rightarrow \quad \underline{\lambda_{1,2} = -2,\ 5}$

$$\underline{y_h = C_1e^{x} + C_2e^{2x}}$$

$$y_P = A$$

$$y'' - 3y' - 10y = -3$$

$$-10A = -3 \quad \rightarrow \quad \underline{A = \frac{3}{10}}$$

$$\underline{y(x) = C_1e^{x} + C_2e^{2x} + \frac{3}{10}}$$

Exercise

Find the general solution: $\quad y'' - 3y' - 10y = 2x - 3$

Solution

The characteristic equation: $\lambda^2 - 3\lambda - 10 = 0 \quad \rightarrow \quad \underline{\lambda_{1,2} = -2,\ 5}$

$$\underline{y_h = C_1e^{x} + C_2e^{2x}}$$

$y_P = Ax + B$

$y'_P = A$

$y''_P = 0$

$y'' - 3y' - 10y = 2x - 3$

$-3A - 10Ax - 10B = 2x - 3$

$$\begin{cases} -10A = 2 \\ -3A - 10B = -3 \end{cases} \rightarrow \boxed{A = -\frac{1}{5},\ B = \frac{4}{25}}$$

$$\boxed{y(x) = C_1 e^{x} + C_2 e^{2x} - \frac{1}{5}x + \frac{4}{25}}$$

Exercise

Find the general solution: $y'' + 3y' - 10y = 6e^{4x}$

Solution

The characteristic equation: $\lambda^2 + 3\lambda - 10 = 0 \rightarrow \boxed{\lambda_{1,2} = -5,\ 2}$

$$\boxed{y_h = C_1 e^{-5x} + C_2 e^{2x}}$$

$y_p = Ae^{4x}$

$y'_p = 4Ae^{4x}$

$y''_p = 16Ae^{4x}$

$y'' + 3y' - 10y = 6e^{4x}$

$(16A + 12A - 10A)e^{4x} = 6e^{4x}$

$8Ae^{4x} = 6e^{4x} \rightarrow \boxed{A = \frac{3}{4}}$

$$\boxed{y(x) = C_1 e^{x} + C_2 e^{2x} + \frac{3}{4}e^{4x}}$$

Exercise

Find the general solution: $y'' + 3y' - 10y = x\left(e^{x} + 1\right)$

Solution

The characteristic equation: $\lambda^2 + 3\lambda - 10 = 0 \rightarrow \boxed{\lambda_{1,2} = -5,\ 2}$

$$\boxed{y_h = C_1 e^{-5x} + C_2 e^{2x}}$$

$y_p = (Ax + B)e^{x} + Cx + D$

$$y'_p = (Ax + A + B)e^x + C$$

$$y''_p = (Ax + 2A + B)e^x$$

$$y'' + 3y' - 10y = xe^x + x$$

$$(Ax + 2A + B + 3Ax + 3A + 3B - 10Ax - 10B)e^x + 3B - 10Bx - 10C = xe^x + x$$

$$e^x \quad \begin{cases} -6A = 1 \\ 5A - 6B = 0 \end{cases} \qquad \boxed{A = -\frac{1}{6}} \qquad \boxed{B = -\frac{5}{36}}$$

$$\begin{array}{ll} x & -10C = 1 \\ x^0 & 3C + D = 0 \end{array} \rightarrow \boxed{C = -\frac{1}{10},\ D = \frac{3}{10}}$$

$$\boxed{y(x) = C_1 e^x + C_2 e^{2x} - \left(\frac{1}{6}x + \frac{5}{36}\right)e^x - \frac{1}{10}x + \frac{3}{10}}$$

Exercise

Find the general solution $\qquad y'' + 4y = 3x^3$

Solution

The characteristic equation: $\qquad \lambda^2 + 4 = 0 \rightarrow \lambda = \pm 2i$

$$\boxed{y_h = A\cos 2t + B\sin 2t}$$

$$y_p = Cx^3 + Dx^2 + Ex + F$$

$$y'_p = 3Cx^2 + 2Dx + E$$

$$y''_p = 6Cx + 2D$$

$$6Cx + 2D + 4Cx^3 + 4Dx^2 + 4Ex + 4F = 3x^3$$

$$\begin{array}{lll} x^3 & 4C = 3 & \rightarrow C = \frac{3}{4} \\ x^2 & 4D = 0 & \rightarrow D = 0 \\ x & 6C + 4E = 0 & \rightarrow E = -\frac{9}{8} \\ x^0 & 2D + 4F = 0 & \rightarrow F = 0 \end{array}$$

$$\rightarrow \boxed{y_p = \frac{3}{4}x^3 - \frac{9}{8}x}$$

$$\boxed{y(x) = A\cos 2t + B\sin 2t + \frac{3}{4}x^3 - \frac{9}{8}x}$$

Exercise

Find the general solution: $y'' + 4y = 3\sin x$

Solution

The characteristic equation: $\lambda^2 + 4 = 0 \rightarrow \boxed{\lambda_{1,2} = \pm 2i}$

$\boxed{y_h = C_1 \cos 2x + C_2 \sin 2x}$

$y_p = A\cos x + B\sin x$

$y'_p = -A\sin x + B\cos x$

$y''_p = -A\cos x - B\sin x$

$y'' + 4y = 3\sin x$

$-A\cos x - B\sin x + 4A\cos x + 4B\sin x = 3\sin x$

$$\begin{cases} 3A = 0 & \rightarrow \quad \boxed{A = 0} \\ 3B = 3 & \rightarrow \quad \boxed{B = 1} \end{cases}$$

$\boxed{y(x) = C_1 \cos 2x + C_2 \sin 2x + \sin x}$

Exercise

Find the general solution: $y'' + 4y = 3\sin 2x$

Solution

The characteristic equation: $\lambda^2 + 4 = 0 \rightarrow \boxed{\lambda_{1,2} = \pm 2i}$

$\boxed{y_h = C_1 \cos 2x + C_2 \sin 2x}$

$y_p = Ax\cos 2x + Bx\sin 2x$

$y'_p = A\cos 2x - 2Ax\sin 2x + B\sin 2x + 2Bx\cos 2x$

$= (A + 2Bx)\cos 2x + (B - 2Ax)\sin 2x$

$y''_p = 2B\cos 2x - 2A\sin 2x - 2(A + 2Bx)\sin 2x + 2(B - 2Ax)\cos 2x$

$= (4B - 4Ax)\cos 2x + (-4A - 4Bx)\sin 2x$

$y'' + 4y = 3\sin 2x$

$\cos 2x$	x	$-4A + 4A$	$\sin 2x$	x	$-4B + 4B$
	x^0	$4B = 0$		x^0	$-4A = 3$

$\rightarrow \boxed{A = -\frac{3}{4}}$

$\boxed{y_p = -\frac{3}{4} x\cos 2x}$

$\boxed{y(x) = C_1 \cos 2x + C_2 \sin 2x - \frac{3}{4} x\cos 2x}$

Exercise

Find the general solution: $y'' + 4y = 4\cos x + 3\sin x - 8$

Solution

The characteristic equation: $\lambda^2 + 4 = 0 \rightarrow \boxed{\lambda_{1,2} = \pm 2i}$

$$\boxed{y_h = C_1 \cos 2x + C_2 \sin 2x}$$

$$y_p = A\cos x + B\sin x + C$$

$$y_p' = -A\sin x + B\cos x$$

$$y_p'' = -A\cos x - B\sin x$$

$$y'' + 4y = 4\cos x + 3\sin x - 8$$

$$\begin{array}{lll} \cos x & -A + 4A = 4 & \boxed{A = \frac{4}{3}} \\ \sin x & -B + 4B = 3 & \boxed{B = 1} \\ & 4C = -8 & \boxed{C = -2} \end{array}$$

$$\boxed{y_p = \frac{4}{3}\cos x + \sin x - 2}$$

$$\boxed{y(x) = C_1 \cos 2x + C_2 \sin 2x + \frac{4}{3}\cos x + \sin x - 2}$$

Exercise

Find the general solution: $y'' - 4y = \left(x^2 - 3\right)\sin 2x$

Solution

The characteristic equation: $\lambda^2 - 4 = 0 \rightarrow \boxed{\lambda_{1,2} = \pm 2}$

$$\boxed{y_h = C_1 e^{-2x} + C_2 e^{2x}}$$

$$y_p = \left(Ax^2 + Bx + C\right)\cos 2x + \left(Dx^2 + Ex + F\right)\sin 2x$$

$$y_p' = \left(2Ax + B + 2Dx^2 + 2Ex + 2F\right)\cos 2x + \left(2Dx + E - 2Ax^2 - 2Bx - 2C\right)\sin 2x$$

$$\begin{aligned} y_p'' &= \left(2A + 4Dx + 2E + 4Dx + 2E - 4Ax^2 - 4Bx - 4C\right)\cos 2x \\ &\quad + \left(2D - 4Ax - 2B - 4Ax - 2B - 4Dx^2 - 4Ex - 4F\right)\sin 2x \\ &= \left(-4Ax^2 + 8Dx - 4Bx + 2A - 4C + 4E\right)\cos 2x + \left(-4Dx^2 - 8Ax - 4Ex + 2D - 4B - 4F\right)\sin 2x \end{aligned}$$

$$y'' - 4y = \left(x^2 - 3\right)\sin 2x$$

$\cos 2x \quad x^2 \qquad -4A-4A=0 \qquad \boxed{A=0}$

$\qquad\qquad x \qquad 8D-4B-4B=0 \qquad D-B=0$

$\qquad\qquad x^0 \qquad 2A-4C+4E-4C=0 \quad E-2C=0$

$\sin 2x \quad x^2 \qquad -4D-4D=1 \qquad \boxed{D=-\frac{1}{8}}$

$\qquad\qquad x \qquad -8A-4E-4E=0 \qquad \boxed{E=0}$

$\qquad\qquad x^0 \qquad 2D-4B-8F=-3 \quad B+2F=\frac{11}{16}$

$B=D=\boxed{-\frac{1}{8}} \quad C=\frac{1}{2}E=\boxed{0} \quad \boxed{F=\frac{13}{32}}$

$$\boxed{y_p=-\frac{1}{8}x\cos 2x+\left(-\frac{1}{8}x^2+\frac{13}{32}\right)\sin 2x}$$

$$\boxed{y(x)=C_1e^{-2x}+C_2e^{2x}-\frac{1}{8}x\cos 2x+\left(-\frac{1}{8}x^2+\frac{13}{32}\right)\sin 2x}$$

Exercise

Find the general solution: $\quad y''+4y'+4y=2x+6$

Solution

The characteristic equation: $\lambda^2+4\lambda+4=0 \quad\rightarrow\quad \boxed{\lambda_{1,2}=-2}$

$$\boxed{y_h=\left(C_1+C_2x\right)e^{-2x}}$$

$y_P=Ax+B$

$\quad y'_P=A$

$\quad y''_P=0$

$y''+4y'+4y=2x+6$

$\quad 4A+4Ax+4B=2x+6$

$\qquad x \qquad 4A=2 \qquad \rightarrow \boxed{A=\frac{1}{2}}$

$\qquad x^0 \quad 4A+4B=6 \quad \rightarrow \boxed{B=1}$

$y_P=\frac{1}{2}x+1$

$$\boxed{y(x)=\left(C_1+C_2x\right)e^{-2x}+\frac{1}{2}x+1}$$

Exercise

Find the general solution: $\quad y''+4y'+5y=5x+e^{-x}$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 5 = 0 \quad \to \quad \lambda_{1,2} = \frac{-4 \pm 2i}{2} \underline{= -2 \pm i}$

$$\underline{y_h = e^{-2x}\left(C_1 \cos x + C_2 \sin x\right)}$$

$$y_p = Ax + B + Ce^{-x}$$

$$y_p' = A - Ce^{-x}$$

$$y_p'' = Ce^{-x}$$

$$y'' + 4y' + 5y = 5x + e^{-x}$$

$$Ce^{-x} + 4A - 4Ce^{-x} + 5Ax + 5B + 5Ce^{-x} = 5x + e^{-x}$$

$x \qquad 5A = 5 \qquad \to \underline{A = 1}$

$x^0 \qquad 4A + 5B = 0 \qquad \to \underline{B = -\frac{4}{5}}$

$e^{-x} \qquad 2C = 1 \qquad \to \underline{C = \frac{1}{2}}$

$$\underline{y_p = x - \frac{4}{5} + \frac{1}{2}e^{-x}}$$

$$\underline{y(x) = e^{-2x}\left(C_1 \cos x + C_2 \sin x\right) + x - \frac{4}{5} + \frac{1}{2}e^{-x}}$$

Exercise

Find the general solution: $\quad y'' + 4y' + 5y = 2e^{-2x} + \cos x$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 5 = 0 \quad \to \quad \lambda_{1,2} = \frac{-4 \pm 2i}{2} \underline{= -2 \pm i}$

$$\underline{y_h = e^{-2x}\left(C_1 \cos x + C_2 \sin x\right)}$$

$$y_p = Ae^{-2x} + B\cos x + C\sin x$$

$$y_p' = -2Ae^{-x} - B\sin x + C\cos x$$

$$y_p'' = 4Ae^{-x} - B\cos x - C\sin x$$

$$y'' + 4y' + 5y = 2e^{-2x} + \cos x$$

$$4Ae^{-x} - B\cos x - C\sin x - 8Ae^{-x} - 4B\sin x + 4C\cos x + 5Ae^{-x} + 5B\cos x + 5C\sin x = 2e^{-2x} + \cos x$$

$\cos x \qquad 4B + 4C = 1 \qquad \underline{B = \frac{1}{8}}$

$\sin x \qquad -4B + 4C = 0 \qquad \underline{C = \frac{1}{8}}$

$e^{-2x} \qquad A = 2 \qquad \to \underline{A = 2}$

$$\boxed{y_P = 2e^{-2x} + \frac{1}{8}\cos x + \frac{1}{8}\sin x}$$

$$\boxed{y(x) = e^{-2x}\left(C_1 \cos x + C_2 \sin x\right) + 2e^{-2x} + \frac{1}{8}\cos x + \frac{1}{8}\sin x}$$

Exercise

Find the general solution: $\quad y'' + 5y' = 15x^2$

Solution

The characteristic equation: $\lambda^2 + 5\lambda = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = 0, \ -5}$

$$\boxed{y_h = C_1 + C_2 e^{-5x}}$$

$$y_P = Ax^3 + Bx^2 + Cx$$

$$y'_P = 3Ax^2 + 2Bx + C$$

$$y''_P = 6Ax + 2B$$

$$y'' + 5y' = 15x^2$$

$$6Ax + 2B + 15Ax^2 + 10Bx + 5C = 15x^2$$

$$15A = 15 \quad \rightarrow \quad \boxed{A = 1}$$

$$6A + 10B = 0 \quad \rightarrow \quad \boxed{B = -\frac{3}{5}}$$

$$2B + 5C = 0 \quad \rightarrow \quad \boxed{C = \frac{6}{25}}$$

$$\Rightarrow \boxed{y_P = x^3 - \frac{3}{5}x^2 + \frac{6}{25}x}$$

$$\boxed{y(x) = C_1 + C_2 e^{-5x} + x^3 - \frac{3}{5}x^2 + \frac{6}{25}x}$$

Exercise

Find the general solution: $\quad y'' - 5y' = 2x^3 - 4x^2 - x + 6$

Solution

The characteristic equation: $\lambda^2 - 5\lambda = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = 0, \ 5}$

$$\boxed{y_h = C_1 + C_2 e^{5x}}$$

$$y_P = Ax^4 + Bx^3 + Cx^2 + Dx$$

$$y'_P = 4Ax^3 + 3Bx^2 + 2Cx + D$$

$$y''_P = 12Ax^2 + 6Bx + 2C$$

$y'' - 5y' = 2x^3 - 4x^2 - x + 6$

$$12Ax^2 + 6Bx + 2C - 20Ax^3 - 15Bx^2 - 10Cx - 5D = 2x^3 - 4x^2 - x + 6$$

x^3	$-20A = 2$	$\boxed{A = -\frac{1}{10}}$
x^2	$12A - 15B = -4$	$\boxed{B = \frac{14}{75}}$
x	$6B - 10C = -1$	$\boxed{C = \frac{53}{250}}$
x^0	$2C - 5D = 6$	$\boxed{D = -\frac{697}{625}}$

$$\boxed{y_P = -\frac{1}{10}x^4 + \frac{14}{75}x^3 + \frac{53}{250}x^2 - \frac{697}{625}x}$$

$$\boxed{y(x) = C_1 + C_2 e^{5x} - \frac{1}{10}x^4 + \frac{14}{75}x^3 + \frac{53}{250}x^2 - \frac{697}{625}x}$$

Exercise

Find the general solution: $y'' + 6y' + 8y = 3e^{-2x} + 2x$

Solution

The characteristic equation: $\lambda^2 + 6\lambda + 8 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -2, \ -4}$

$$\boxed{y_h = C_1 e^{-2x} + C_2 e^{-4x}}$$

$$y_P = Axe^{-2x} + Bx + C$$

$$y'_P = (A - 2Ax)e^{-2x} + B$$

$$y''_P = (-4A + 4Ax)e^{-2x}$$

$y'' + 6y' + 8y = 3e^{-2x} + 2x$

$$(-4A + 4Ax + 6A - 12Ax + 8Ax)e^{-2x} + 6B + 8Bx + 8C = 3e^{-2x} + 2x$$

e^{-2x}	$2A = 3$	$\rightarrow \boxed{A = \frac{3}{2}}$
x	$8B = 2$	$\boxed{B = \frac{1}{4}}$
x^0	$6B + 8C = 0$	$\boxed{C = -\frac{3}{16}}$

$$\boxed{y_P = \frac{3}{2}xe^{-2x} + \frac{1}{4}x - \frac{3}{16}}$$

$$\boxed{y(x) = C_1 e^{-2x} + C_2 e^{-4x} + \frac{3}{2}xe^{-2x} + \frac{1}{4}x - \frac{3}{16}}$$

Exercise

Find the general solution: $y'' - 6y' + 9y = e^{3x}$

Solution

The characteristic equation: $\lambda^2 - 6\lambda + 9 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = 3}$

$$\boxed{y_h = \left(C_1 + C_2 x\right)e^{3x}}$$

$$y_p = Ax^2 e^{3x}$$

$$y_p' = \left(2Ax + 3Ax^2\right)e^{3x}$$

$$y_p'' = \left(2A + 12Ax + 9Ax^2\right)e^{x}$$

$$y'' - 6y' + 9y = e^{3x}$$

$$\left(2A + 12Ax + 9Ax^2 - 12Ax - 18Ax^2 + 9Ax^2\right)e^{3x} = e^{3x}$$

$$2Ae^{3x} = e^{3x} \quad \rightarrow \quad \boxed{A = \frac{1}{2}}$$

$$\boxed{y(x) = \left(C_1 + C_2 x + \frac{1}{2}\right)e^{3x}}$$

Exercise

Find the general solution: $y'' + 6y' + 9y = -xe^{4x}$

Solution

The characteristic equation: $\lambda^2 + 6\lambda + 9 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -3}$

$$\boxed{y_h = \left(C_1 + C_2 x\right)e^{-3x}}$$

$$y_p = (Ax + B)e^{4x}$$

$$y_p' = (A + 4Ax + 4B)e^{4x}$$

$$y_p'' = (16Ax + 8A + 16B)e^{4x}$$

$$y'' + 6y' + 9y = -xe^{4x}$$

$$(16Ax + 8A + 16B + 6A + 24Ax + 24B + 9Ax + 9B)e^{4x} = -xe^{4x}$$

$$x \qquad 49A = -1 \qquad \boxed{A = -\frac{1}{49}}$$

$$x^0 \qquad 14A + 25B = 0 \qquad \boxed{B = \frac{2}{175}}$$

$$\boxed{y_p = \left(-\frac{1}{49}x + \frac{2}{175}\right)e^{4x}}$$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{-3x} + \left(-\frac{1}{49}x + \frac{2}{175}\right)e^{4x}}$$

Exercise

Find the general solution $\quad y'' + 6y' + 13y = e^{-3x}\cos 2x$

Solution

$$\lambda^2 + 6\lambda + 13 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{-6 \pm \sqrt{-16}}{2} = -3 \pm 2i$$

$$\boxed{y_h = e^{-3x}\left(C_1 \cos 2x + C_2 \sin 2x\right)}$$

$$y_p = e^{-3x}\left(Ax\cos 2x + Bx\sin 2x\right)$$

$$y'_p = e^{-3x}\left(A\cos 2x - 2Ax\sin 2x + B\sin 2x + 2Bx\cos 2x - 3Ax\cos 2x - 3Bx\sin 2x\right)$$

$$= e^{-3x}\left((A - 3Ax + 2Bx)\cos 2x + (B - 2Ax - 3Bx)\sin 2x\right)$$

$$y''_p = e^{-3x}\begin{pmatrix}(-3A + 2B)\cos 2x + (-2A - 3B)\sin 2x - (2A - 6Ax + 4Bx)\sin 2x + (2B - 4Ax - 6Bx)\cos 2x \\ -3(A - 3Ax + 2Bx)\cos 2x - 3(B - 2Ax - 3Bx)\sin 2x\end{pmatrix}$$

$$= e^{-3x}\left((-6A + 4B + 5Ax - 12Bx)\cos 2x + (-4A - 6B + 12Ax + 5Bx)\sin 2x\right)$$

$$(-6A + 4B + 2Ax - 12Bx)\cos 2x + (-4A - 6B + 12Ax + 5Bx)\sin 2x + 6(A - 3Ax + 2Bx)\cos 2x$$

$$+6(B - 2Ax - 3Bx)\sin 2x + 13Ax\cos 2x + 13Bx\sin 2x = e^{-3x}\cos 2x$$

$$\begin{cases} \sin 2x & 13Bx + 6B - 12Ax - 18Bx - 4A - 6B + 12Ax + 5Bx = 0 \\ \cos 2x & 13Ax + 6A - 18Ax + 12Bx - 6A + 4B + 5Ax - 12Bx = 1 \end{cases}$$

$$\begin{cases} -4A = 0 & \rightarrow \boxed{A = 0} \\ 4B = 1 & \rightarrow \boxed{B = \frac{1}{4}} \end{cases} \qquad y_p = \frac{1}{4}xe^{-3x}\sin 2x$$

$$\boxed{y(x) = e^{-3x}\left(C_1 \cos 2x + C_2 \sin 2x + \frac{1}{4}x\sin 2x\right)}$$

Exercise

Find the general solution: $\quad y'' - 7y' = -3$

Solution

The characteristic equation: $\lambda^2 - 7\lambda = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = 0,\ 7}$

$$\boxed{y_h = C_1 + C_2 e^{7x}}$$

$$y_P = Ax^2 + Bx$$

$$y'_P = 2Ax + B$$

$$y''_P = 2A$$

$$y'' - 7y' = -3$$

$$2A - 14Ax - 7B = -3$$

$$\rightarrow \begin{cases} -14A = 0 & \boxed{A = 0} \\ 2A - 7B = -3 & \boxed{B = \frac{3}{7}} \end{cases}$$

$$\boxed{y_p = \frac{3}{7}x}$$

$$\boxed{y(x) = C_1 + C_2 e^{7x} + \frac{3}{7}x}$$

Exercise

Find the general solution: $y'' + 7y' = 42x^2 + 5x + 1$

Solution

The characteristic equation: $\lambda^2 + 7\lambda = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = 0, \ -7}$

$$\boxed{y_h = C_1 + C_2 e^{-7x}}$$

$$y_p = Ax^3 + Bx^2 + Cx$$

$$y_p' = 3Ax^2 + 2Bx + C$$

$$y_p'' = 6Ax + 2B$$

$$y'' + 7y' = 42x^2 + 5x + 1$$

$$6Ax + 2B + 21Ax^2 + 14Bx + 7C = 42x^2 + 5x + 1$$

$$21A = 42 \quad \rightarrow \quad \boxed{A = 2}$$

$$6A + 14B = 0 \quad \rightarrow \quad \boxed{B = -\frac{6}{7}}$$

$$2B + 7C = 0 \quad \rightarrow \quad \boxed{C = \frac{12}{49}}$$

$$\boxed{y_p = 2x^3 - \frac{6}{7}x^2 + \frac{12}{25}x}$$

$$\boxed{y(x) = C_1 + C_2 e^{-7x} + 2x^3 - \frac{6}{7}x^2 + \frac{12}{25}x}$$

Exercise

Find the general solution $y'' + 8y = 5x + 2e^{-x}$

Solution

The characteristic equation: $\lambda^2 + 8 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm 2i\sqrt{2}$

$$y_h = C_1 \cos\left(2\sqrt{2}x\right) + C_2 \sin\left(2\sqrt{2}x\right)$$

$$y_p = A + Bx + Ce^{-x}$$

$$y'_p = B - Ce^{-x}$$

$$y''_p = Ce^{-x}$$

$$y'' + 8y = 5x + 2e^{-x}$$

$$Ce^{-x} + 8A + 8Bx + 8Ce^{-x} = 5x + 2e^{-x}$$

$$8A = 0 \quad \rightarrow A = 0$$

$$8B = 5 \quad \rightarrow B = \frac{5}{8}$$

$$9C = 2 \quad C = \frac{2}{9}$$

$$\boxed{y(x) = C_1 \cos\left(2\sqrt{2}x\right) + C_2 \sin\left(2\sqrt{2}x\right) + \frac{5}{8}x + \frac{2}{9}e^{-x}}$$

Exercise

Find the general solution: $\quad y'' - 8y' + 20y = 100x^2 - 26xe^x$

Solution

The characteristic equation: $\lambda^2 - 8\lambda + 20 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{8 \pm 4i}{2} \boxed{= 4 \pm 2i}$

$$\boxed{y_h = \left(C_1 \cos 2x + C_2 \sin 2x\right)e^{4x}}$$

$$y_P = (Ax + B)e^x + Cx^2 + Dx + E$$

$$y'_P = (Ax + A + B)e^x + 2Cx + D$$

$$y''_P = (Ax + 2A + B)e^x + 2C$$

$$y'' - 8y' + 20y = 100x^2 - 26xe^x$$

$$e^x \quad x \qquad A - 8A + 20A = -26 \qquad \boxed{A = -2}$$

$$x^0 \quad 2A + B - 8A - 8B + 20B = 0 \qquad \boxed{B = -\frac{12}{13}}$$

$$x^2 \qquad 20C = 100 \qquad \boxed{C = 5}$$

$$x \qquad -16C + 20D = 0 \qquad \boxed{D = 4}$$

$$x^0 \quad 2C - 8D + 20E = 0 \qquad \boxed{E = \frac{11}{10}}$$

$$\boxed{y_P = \left(-2x - \frac{12}{13}\right)e^x + 5x^2 + 4x + \frac{11}{10}}$$

$$\boxed{y(x) = \left(C_1 \cos 2x + C_2 \sin 2x\right)e^{4x} + \left(-2x - \frac{12}{13}\right)e^x + 5x^2 + 4x + \frac{11}{10}}$$

Exercise

Find the general solution: $y'' - 9y = 54$

Solution

The characteristic equation: $\lambda^2 - 9 = 0 \quad \rightarrow \quad \lambda_{1,2} = \pm 3$

$y_h = C_1 e^{-3x} + C_2 e^{3x}$

$y_P = A$

$y_P' = y_P'' = 0$

$y'' - 9y = 54$

$-9A = 54 \quad \rightarrow \quad A = -6$

$y(x) = C_1 e^{-3x} + C_2 e^{3x} - 6$

Exercise

Find the general solution: $y'' + 9y = x^2 \cos 3x + 4\sin x$

Solution

The characteristic equation: $\lambda^2 + 9 = 0 \quad \rightarrow \quad \lambda_{1,2} = \pm 3i$

$y_h = C_1 \cos 3x + C_2 \sin 3x$

$y_P = \left(Ax^3 + Bx^2 + Cx\right)\cos 3x + \left(Dx^3 + Ex^2 + Fx\right)\sin 3x + G\cos x + H\sin x$

- $y_{P1} = \left(Ax^3 + Bx^2 + Cx\right)\cos 3x$

$y_{P1}' = \left(3Ax^2 + 2Bx + C\right)\cos 3x - \left(3Ax^3 + 3Bx^2 + 3Cx\right)\sin 3x$

$y_{P1}'' = \left(-9Ax^3 - 9Bx^2 - 9Cx + 6Ax + 2B\right)\cos 3x - \left(18Ax^2 + 12Bx + 6C\right)\sin 3x$

$y'' + 9y$

$\left(-9Ax^3 - 9Bx^2 - 9Cx + 6Ax + 2B + 9Ax^3 + 9Bx^2 + 9Cx\right)\cos 3x - \left(18Ax^2 + 12Bx + 6C\right)\sin 3x$

$(6Ax + 2B)\cos 3x - \left(18Ax^2 + 12Bx + 6C\right)\sin 3x$

- $y_{P2} = \left(Dx^3 + Ex^2 + Fx\right)\sin 3x$

$y_{P2}' = \left(3Dx^2 + 2Ex + F\right)\sin 3x + \left(3Dx^3 + 3Ex^2 + 3Fx\right)\cos 3x$

$y_{P2}'' = \left(6Dx + 2E - 9Dx^3 - 9Ex^2 - 9Fx\right)\sin 3x + 2\left(9Dx^2 + 6Ex + 3F\right)\cos 3x$

$$y'' + 9y$$

$$\left(6Dx + 2E - 9Dx^3 - 9Ex^2 - 9Fx + 9Dx^3 + 9Ex^2 + 9Fx\right)\sin 3x + 2\left(9Dx^2 + 6Ex + 3F\right)\cos 3x$$

$$(6Dx + 2E)\sin 3x + \left(18Dx^2 + 12Ex + 6F\right)\cos 3x$$

$$y'' + 9y = x^2 \cos 3x + 4\sin x$$

$$\left(6Ax + 2B + 18Dx^2 + 12Ex + 6F\right)\cos 3x - \left(18Ax^2 + 12Bx + 6C - 6Dx - 2E\right)\sin 3x = x^2 \cos 3x$$

$\cos 3x$ $\quad x^2 \quad 18D = 1 \quad \to \boxed{D = \frac{1}{18}}$

$\quad x \quad 6A + 12E = 0 \quad A + 2E = 0$

$\quad x^0 \quad 2B + 6F = 0 \quad B + 3F = 0$

$\sin 3x$ $\quad x^2 \quad -18A = 0 \quad \to \boxed{A = 0}$

$\quad x \quad 12B - 6D = 0 \quad \to \boxed{B = \frac{1}{36}}$

$\quad x^0 \quad 6C - 2E = 0$

$A + 2E = 0 \quad \to \quad \boxed{E = 0}$

$B + 3F = 0 \quad \to \quad \boxed{F = -\frac{1}{108}}$

$6C - 2E = 0 \quad \to \quad \boxed{C = 0}$

$$\boxed{y_P = \frac{1}{36}x^2 \cos 3x + \left(\frac{1}{18}x^3 - \frac{1}{108}x\right)\sin 3x}$$

- $y_{P3} = G\cos x + H\sin x$

  $y'_{P3} = -G\sin x + H\cos x$

  $y''_{P3} = -G\cos x - H\sin x$

  $y'' + 9y = x^2 \cos 3x + 4\sin x$

  $-G\cos x - H\sin x + 9G\cos x + 9H\sin x = 4\sin x$

  $\cos x \quad 8G = 0 \quad \to \boxed{G = 0}$

  $\sin x \quad 8H = 4 \quad \to \boxed{H = \frac{1}{2}}$

  $\boxed{y_{P3} = \frac{1}{2}\sin x}$

$$\boxed{y(x) = C_1 \cos 3x + C_2 \sin 3x + \frac{1}{36}x^2 \cos 3x + \left(\frac{1}{18}x^3 - \frac{1}{108}x\right)\sin 3x + \frac{1}{2}\sin x}$$

Exercise

Find the general solution: $y'' + 10y' + 25y = 14e^{-5x}$

Solution

The characteristic equation: $\lambda^2 + 10\lambda + 25 = 0 \rightarrow \lambda_{1,2} = -5$

$$y_h = \left(C_1 + C_2 x\right)e^{-5x}$$

$$y_P = Ax^2 e^{-5x}$$

$$y'_P = \left(2Ax - 5Ax^2\right)e^{-5x}$$

$$y''_P = \left(2A - 20Ax + 25Ax^2\right)e^{-5x}$$

$$y'' + 10y' + 25y = 14e^{-5x}$$

$$\left(2A - 20Ax + 25Ax^2 + 20Ax - 50Ax^2 + 25Ax^2\right)e^{-5x} = 14e^{-5x}$$

$$2A = 14 \quad \rightarrow \quad A = 7$$

$$y(x) = \left(C_1 + C_2 x + 7x^2\right)e^{-5x}$$

Exercise

Find the general solution: $y'' - 10y' + 25y = 30x + 3$

Solution

The characteristic equation: $\lambda^2 - 10\lambda + 25 = 0 \rightarrow \lambda_{1,2} = 5$

$$y_h = \left(C_1 + C_2 x\right)e^{5x}$$

$$y_P = Ax + B$$

$$y'_P = A$$

$$y''_P = 0$$

$$y'' - 10y' + 25y = 30x + 3$$

$$x \qquad 25A = 30 \qquad A = \frac{6}{5}$$

$$x^0 \qquad -10A + 20B = 0 \qquad B = \frac{3}{5}$$

$$y_P = \frac{6}{5}x + \frac{3}{5}$$

$$y(x) = \left(C_1 + C_2 x\right)e^{5x} + \frac{6}{5}x + \frac{3}{5}$$

Exercise

Find the general solution: $y'' - 16y = 2e^{4x}$

Solution

The characteristic equation: $\lambda^2 - 16 = 0 \quad \to \quad \boxed{\lambda_{1,2} = \pm 4}$

$$\boxed{y_h = C_1 e^{-4x} + C_2 e^{4x}}$$

$$y_P = Axe^{4x}$$

$$y'_P = (A + 4Ax)e^{4x}$$

$$y''_P = (8A + 16Ax)e^{4x}$$

$$y'' - 16y = 2e^{4x}$$

$$x \qquad 16A - 16A$$

$$x^0 \qquad 8A - 16A = 2 \qquad \boxed{A = -\frac{1}{4}}$$

$$\boxed{y_P = -\frac{1}{4}xe^{4x}}$$

$$\boxed{y(x) = C_1 e^{-4x} + C_2 e^{4x} - \frac{1}{4}xe^{4x}}$$

Exercise

Find the general solution: $y'' + 25y = 6\sin x$

Solution

The characteristic equation: $\lambda^2 + 25 = 0 \quad \to \quad \boxed{\lambda_{1,2} = \pm 5i}$

$$\boxed{y_h = C_1 \cos 5x + C_2 \sin 5x}$$

$$y_P = A\cos x + B\sin x$$

$$y'_P = -A\sin x + B\cos x$$

$$y''_P = -A\cos x - B\sin x$$

$$y'' + 25y = 6\sin x$$

$$\cos x \qquad -A + 25A = 0 \qquad \boxed{A = 0}$$

$$\sin x \qquad -B + 25B = 6 \qquad \boxed{B = \frac{1}{4}}$$

$$\boxed{y_P = \frac{1}{4}\sin x}$$

$$\boxed{y(x) = C_1 \cos 5x + C_2 \sin 5x + \frac{1}{4}\sin x}$$

Exercise

Find the general solution $y'' + 25y = 20\sin 5x$

Solution

The characteristic equation: $\lambda^2 + 25 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm 5i$

$\boxed{y_h = C_1 \cos 5x + C_2 \sin 5x}$

The particular equation: $y_p = A\cos 5x + B\sin 5x$

$$y'_p = -5A\sin 5x + 5B\cos 5x$$

$$y''_p = -25A\cos 5x - 25B\sin 5x$$

$$-25A\cos x - 25B\sin x + 25A\cos x + 25B\sin x = 20\sin 5x$$

$\cancel{0 = 20\sin 5x}$

The particular equation: $y_p = Ax\cos 5x + Bx\sin 5x$

$$y'_p = A\cos 5x - 5Ax\sin 5x + B\sin 5x + 5Bx\cos 5x = (A + 5Bx)\cos 5x + (B - 5Ax)\sin 5x$$

$$\begin{aligned} y''_p &= 5B\cos 5x - 5(A + 5Bx)\sin 5x - 5A\sin 5x + 5(B - 5Ax)\cos 5x \\ &= 10B\cos 5x - 25Ax\cos 5x - 10A\sin 5x - 25Bx\sin 5x \end{aligned}$$

$$10B\cos 5x - 25Ax\cos 5x - 10A\sin 5x - 25Bx\sin 5x + 25Ax\cos 5x + 25Bx\cos 5x = 20\sin 5x$$

$$10B\cos 5x - 10A\sin 5x = 20\sin 5x$$

$$\begin{cases} 10B = 0 & \rightarrow B = 0 \\ -10A = 20 & \rightarrow A = -2 \end{cases}$$

$\boxed{y(x) = C_1 \cos 5x + C_2 \sin 5x - 2x\cos 5x}$

Exercise

Find the general solution: $\frac{1}{4}y'' + y' + y = x^2 - 2x$

Solution

The characteristic equation: $\frac{1}{4}\lambda^2 + \lambda + 1 = \left(\frac{1}{2}\lambda + 1\right)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -2}$

$\boxed{y_h = \left(C_1 + C_2 x\right)e^{-2x}}$

$$y_P = Ax^2 + Bx + C$$

$$y'_P = 2Ax + B$$

$$y''_P = 2A$$

$$\frac{1}{4}y'' + y' + y = x^2 - 2x$$

$$\begin{array}{lll} x^2 & & \boxed{A=1} \\ x & 2A+B=-2 & \boxed{B=-4} \\ x^0 & \frac{1}{2}A+B+C=0 & \boxed{C=\frac{7}{2}} \end{array}$$

$$\boxed{y_P = x^2-4x+\frac{7}{2}}$$

$$\boxed{y(x)=\left(C_1+C_2x\right)e^{-2x}+x^2-4x+\frac{7}{2}}$$

Exercise

Find the general solution: $2y''-5y'+2y=-6e^{x/2}$

Solution

The characteristic equation: $2\lambda^2-5\lambda+2=0 \rightarrow \boxed{\lambda_{1,2}=\frac{1}{2},\ 2}$

$$\boxed{y_h=C_1e^{x/2}+C_2e^{2x}}$$

$$y_P=\left(Ax^2+Bx\right)e^{x/2}$$

$$y'_P=\left(2Ax+B+\frac{1}{2}Ax^2+\frac{1}{2}Bx\right)e^{x/2}$$

$$y''_P=\left(2A+Ax+\frac{1}{2}B+Ax+\frac{1}{2}B+\frac{1}{4}Ax^2+\frac{1}{4}Bx\right)e^{x/2}$$

$$=\left(\frac{1}{4}Ax^2+2Ax+\frac{1}{4}Bx+2A+B\right)e^{x/2}$$

$$2y''-5y'+2y=-6e^{x/2}$$

$$\left(\frac{1}{2}Ax^2+4Ax+\frac{1}{2}Bx+4A+2B-10Ax-5B-\frac{5}{2}Ax^2-\frac{5}{2}Bx+2Ax^2+2Bx\right)e^{x/2}=-6e^{x/2}$$

$$(-6Ax+4A-3B)e^{x/2}=-6e^{x/2}$$

$$\begin{cases} -6A=0 & \rightarrow \boxed{A=0} \\ 4A-3B=-6 & \rightarrow \boxed{B=2} \end{cases}$$

$$\boxed{y_P=2xe^{x/2}}$$

$$\boxed{y(x)=C_1e^{x/2}+C_2e^{2x}+2xe^{x/2}}$$

Exercise

Find the general solution: $2y''-7y'+5y=-29$

Solution

The characteristic equation: $2\lambda^2-7\lambda+5=0 \rightarrow \boxed{\lambda_{1,2}=1,\ \frac{5}{2}}$

$$\boxed{y_h = C_1 e^{x} + C_2 e^{5x/2}}$$

$y_P = A$

$y_P' = y_P'' = 0$

$2y'' - 7y' + 5y = -29$

$5A = -29 \quad \rightarrow \quad \boxed{A = -\frac{29}{5}}$

$$\boxed{y_P = -\frac{29}{5}}$$

$$\boxed{y(x) = C_1 e^{x} + C_2 e^{5x/2} - \frac{29}{5}}$$

Exercise

Find the general solution: $\quad 4y'' + 9y = 15$

Solution

The characteristic equation: $\quad 4\lambda^2 + 9 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm\frac{3}{2}}$

$$\boxed{y_h = C_1 e^{-3x/2} + C_2 e^{3x/2}}$$

$y_P = A$

$y_P' = y_P'' = 0$

$4y'' + 9y = 15$

$9A = 15 \quad \rightarrow \quad \boxed{A = \frac{15}{9}}$

$$\boxed{y_P = \frac{15}{9}}$$

$$\boxed{y(x) = C_1 e^{-3x/2} + C_2 e^{3x/2} + \frac{15}{9}}$$

Exercise

Find the general solution: $\quad 4y'' - 4y' - 3y = \cos 2x$

Solution

The characteristic equation: $\quad 4\lambda^2 - 4\lambda - 3 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{4 \pm 8}{8} \boxed{= -\frac{1}{2},\ \frac{3}{2}}$

$$\boxed{y_h = C_1 e^{-x/2} + C_2 e^{3x/2}}$$

$y_P = A\cos 2x + B\sin 2x$

$y_P' = -2A\sin 2x + 2B\cos 2x$

$y_P'' = -4A\cos 2x - 4B\sin 2x$

$$4y'' - 4y' - 3y = \cos 2x$$

$$\begin{array}{lll} \cos 2x & -16A - 8B - 3A = 1 \\ \sin 2x & -16B + 8A - 3B = 0 \end{array} \to \begin{cases} -19A - 8B = 1 \\ 8A - 19B = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} -19 & -8 \\ 8 & -19 \end{vmatrix} = 425 \quad \Delta_A = \begin{vmatrix} 1 & -8 \\ 0 & -19 \end{vmatrix} = -19 \quad \Delta_B = \begin{vmatrix} -19 & 1 \\ 8 & 0 \end{vmatrix} = -8$$

$$\boxed{A = -\frac{19}{425}, \quad B = -\frac{8}{425}}$$

$$\boxed{y_P = -\frac{19}{425}\cos 2x - \frac{8}{425}\sin 2x}$$

$$\boxed{y(x) = C_1 e^{-x/2} + C_2 e^{3x/2} - \frac{19}{425}\cos 2x - \frac{8}{425}\sin 2x}$$

Exercise

Find the general solution: $9y'' - 6y' + y = 9xe^{x/3}$

Solution

The characteristic equation: $9\lambda^2 - 6\lambda + 1 = 0 \quad \to \quad \boxed{\lambda_{1,2} = \frac{1}{3}}$

$$\boxed{y_h = \left(C_1 + C_2 x\right)e^{x/3}}$$

$$y_P = \left(Ax^3 + Bx^2\right)e^{x/3}$$

$$y'_P = \left(3Ax^2 + 2Bx + \frac{1}{3}Ax^3 + \frac{1}{3}Bx^2\right)e^{x/3}$$

$$y''_P = \left(6Ax + 2B + Ax^2 + \frac{2}{3}Bx + Ax^2 + \frac{2}{3}Bx + \frac{1}{9}Ax^3 + \frac{1}{9}Bx^2\right)e^{x/3}$$

$$= \left(\frac{1}{9}Ax^3 + 2Ax^2 + \frac{1}{9}Bx^2 + 6Ax + \frac{4}{3}Bx + 2B\right)e^{x/3}$$

$$9y'' - 6y' + y = 9xe^{x/3}$$

$$\left(Ax^3 + 18Ax^2 + Bx^2 + 54Ax + 12Bx + 18B - 18Ax^2 - 12Bx - 2Ax^3 - 2Bx^2 + Ax^3 + Bx^2\right)e^{x/3} = 9xe^{x/3}$$

$$(54Ax + 18B)e^{x/3} = 9xe^{x/3}$$

$$\begin{array}{lll} x & 54A = 9 & \to \boxed{A = \frac{1}{6}} \\ x^0 & 18B = 0 & \to \boxed{B = 0} \end{array}$$

$$\boxed{y_P = \frac{1}{6}x^3 e^{x/3}}$$

$$\boxed{y(x) = \left(C_1 + C_2 x + \frac{1}{6}x^3\right)e^{x/3}}$$

Exercise

Find the general solution $\qquad y''' + y'' = 8x^2$

Solution

The characteristic equation: $\lambda^3 + \lambda^2 = \lambda^2(\lambda+1) = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = 0,\ \lambda_3 = -1}$

$$\boxed{y_h = C_1 + C_2 x + C_3 e^{-x}}$$

$$y_p = Ax^4 + Bx^3 + Cx^2$$

$$y'_p = 4Ax^3 + 3Bx^2 + 2Cx$$

$$y''_p = 12Ax^2 + 6Bx + 2C$$

$$y'''_p = 24Ax + 6B$$

$y''' + y'' = 8x^2$

$x^2 \qquad 12A = 8 \qquad \boxed{A = \frac{2}{3}}$

$x \qquad 24A + 6B = 0 \qquad \boxed{B = -\frac{8}{3}}$

$x^0 \qquad 6B + 2C = 0 \qquad \boxed{C = 8}$

$$\boxed{y_p = \frac{2}{3}x^4 - \frac{8}{3}x^3 + 8x^2}$$

$$\therefore \ \boxed{y(x) = C_1 + C_2 x + C_3 e^{-x} + \frac{2}{3}x^4 - \frac{8}{3}x^3 + 8x^2}$$

Exercise

Find the general solution $\qquad y''' - y'' - 4y' + 4y = 5 - e^x + e^{2x}$

Solution

The characteristic equation: $\lambda^3 - \lambda^2 - 4\lambda + 4 = 0 \quad \rightarrow \quad \boxed{\lambda_1 = 1}$

$$\begin{array}{c|cccc} 1 & 1 & -1 & -4 & 4 \\ & & 1 & 0 & 4 \\ \hline & 1 & 0 & -4 & 0 \end{array} \qquad \rightarrow \lambda^2 - 4 = 0 \quad \Rightarrow \quad \lambda_{2,3} = \pm 2$$

$$\boxed{y_h = C_1 e^{2x} + C_2 e^x + C_3 e^{2x}}$$

$$y_p = A + Bxe^x + Cxe^{2x}$$

$$y'_p = (Bx + B)e^x + (2Cx + C)e^{2x}$$

$$y''_p = (Bx + 2B)e^x + (4Cx + 4C)e^{2x}$$

$$y_p''' = (Bx+3B)e^x + (8Cx+12C)e^{2x}$$

$$y''' - y'' - 4y' + 4y = 5 - e^x + e^{2x}$$

$$4A = 5 \qquad \boxed{A = \frac{5}{4}}$$

$$e^x \begin{cases} x & B - B - 4B + 4B \\ x^0 & 3B - 2B - 4B = -1 \end{cases} \qquad \boxed{B = -\frac{1}{3}}$$

$$e^{2x} \begin{cases} x & \\ x^0 & 12C - 4C - 4C = 1 \end{cases} \qquad \boxed{C = \frac{1}{4}}$$

$$\boxed{y_p = \frac{5}{4} - \frac{1}{3}xe^x + \frac{1}{4}xe^{2x}}$$

$$\therefore \ \boxed{y(x) = C_1 e^{2x} + C_2 e^x + C_3 e^{2x} + \frac{5}{4} - \frac{1}{3}xe^x + \frac{1}{4}xe^{2x}}$$

Exercise

Find the general solution $\qquad y^{(3)} + y'' = 3e^x + 4x^2$

Solution

$$\lambda^3 + \lambda^2 = 0 \qquad \lambda^2(\lambda+1) = 0 \ \rightarrow \ \boxed{\lambda_{1,2} = 0, \ \lambda_3 = -1}$$

$$\boxed{y_h = C_1 + C_2 x + C_3 e^{-x}}$$

$$y_p = Ae^x + x^2\left(Bx^2 + Cx + D\right)$$

$$= Ae^x + Bx^4 + Cx^3 + Dx^2$$

$$y_p' = Ae^x + 4Bx^3 + 3Cx^2 + 2Dx$$

$$y_p'' = Ae^x + 12Bx^2 + 6Cx + 2D$$

$$y_p''' = Ae^x + 24Bx + 6C$$

$$Ae^x + 24Bx + 6C + Ae^x + 12Bx^2 + 6Cx + 2D = 3e^x + 4x^2$$

$$2A=3 \qquad A=\frac{3}{2}$$

$$12B=4 \qquad B=\frac{1}{3}$$

$$24B+6C=0 \qquad C=-\frac{4}{3}$$

$$6C+2D=0 \qquad D=4$$

$$y_p = 3e^x + \frac{1}{3}x^4 - \frac{4}{3}x^3 + 4x^2$$

$$\therefore \ \boxed{y(x) = C_1 + C_2 x + C_3 e^{-x} + 3e^x + \frac{1}{3}x^4 - \frac{4}{3}x^3 + 4x^2}$$

Exercise

Find the general solution $\quad y''' + 2y'' + y' = 10$

Solution

The characteristic equation: $\lambda^3 + 2\lambda^2 + \lambda = 0 \quad \to \quad \boxed{\lambda_{1,2,3} = 0,\ -1,\ -1}$

$$\boxed{y_h = C_1 + \left(C_2 + C_3 x\right)e^{-x}}$$

$$y_p = Ax$$

$$y'_p = A$$

$$y''_p = y'''_p = 0$$

$$y''' + 2y'' + y' = 10$$

$$\boxed{A = 10}$$

$$\boxed{y_p = 10x}$$

$$\boxed{y(x) = C_1 + \left(C_2 + C_3 x\right)e^{-x} + 10x}$$

Exercise

Find the general solution $\quad y''' - 2y'' - 4y' + 8y = 6xe^{2x}$

Solution

The characteristic equation: $\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$

$$\lambda^2(\lambda - 2) - 4(\lambda - 2) = 0$$

$$\left(\lambda^2 - 4\right)(\lambda - 2) = 0 \quad \to \quad \boxed{\lambda_{1,2,3} = 2,\ 2,\ -2}$$

$$\boxed{y_h = C_1 e^{-2x} + \left(C_2 + C_3 x\right)e^{2x}}$$

$$y_p = \left(Ax^3 + Bx^2\right)e^{2x}$$

$$y'_p = \left(2Ax^3 + 3Ax^2 + 2Bx^2 + 2Bx\right)e^{2x}$$

$$y''_p = \left(4Ax^3 + 12Ax^2 + 6Ax + 4Bx^2 + 8Bx + 2B\right)e^{2x}$$

$$y'''_p = \left(8Ax^3 + 36Ax^2 + 36Ax + 6A + 8Bx^2 + 24Bx + 12B\right)e^{2x}$$

$$y''' - 2y'' - 4y' + 8y = 6xe^{2x}$$

$$x^3 \qquad 8A-8A-8A+8A$$

$$x^2 \qquad 36A-24A-12A+8B-8B-8B+8B$$

$$x \qquad 36A-12A+24B-16B-8B=6 \qquad 24A=6 \;\rightarrow\; \boxed{A=\frac{1}{4}}$$

$$x^0 \qquad 6A+12B-4B=0 \qquad 6A+8B=0 \;\rightarrow\; \boxed{B=-\frac{3}{16}}$$

$$\boxed{y_p=\left(\frac{1}{4}x^3-\frac{3}{16}x^2\right)e^{2x}}$$

$$\boxed{y(x)=C_1e^{-2x}+\left(C_2+C_3x\right)e^{2x}+\left(\frac{1}{4}x^3-\frac{3}{16}x^2\right)e^{2x}}$$

Exercise

Find the general solution: $\quad y^{(3)}-3y''+3y'-y=3e^x$

Solution

The characteristic equation: $\lambda^3-3\lambda^2+3\lambda-1=(\lambda-1)^3=0 \;\rightarrow\; \boxed{\lambda_{1,2,3}=1}$

$$\boxed{y_h=\left(C_1+C_2x+C_3x^2\right)e^x}$$

$$y_p=Ax^3e^x$$

$$y'_p=\left(Ax^3+3Ax^2\right)e^x$$

$$y''_p=\left(Ax^3+6Ax^2+6Ax\right)e^x$$

$$y'''_p=\left(Ax^3+9Ax^2+18Ax+6A\right)e^x$$

$$y^{(3)}-3y''+3y'-y=3e^x$$

$$6Ae^x=3e^x \;\rightarrow\; \boxed{A=\frac{1}{2}}$$

$$\boxed{y(x)=\left(C_1+C_2x+C_3x^2\right)e^x+\frac{1}{2}x^3e^x}$$

Exercise

Find the general solution $\quad y'''-3y''+3y'-y=x-4e^x$

Solution

The characteristic equation: $\lambda^3-3\lambda^2+3\lambda-1=(\lambda-1)^3=0 \;\rightarrow\; \boxed{\lambda_{1,2,3}=1}$

$$\boxed{y_h=\left(C_1+C_2x+C_3x^2\right)e^x}$$

$$y_p = Ax + B + Cx^3e^x$$

$$y'_p = A + \left(Cx^3 + 3Cx^2\right)e^x$$

$$y''_p = \left(Cx^3 + 6Cx^2 + 6Cx\right)e^x$$

$$y'''_p = \left(Cx^3 + 9Cx^2 + 18Cx + 6C\right)e^x$$

$$y''' - 3y'' + 3y' - y = x - 4e^x$$

$$\begin{cases} x & \boxed{-A = 1} \\ x^0 & 3A - B = 0 \end{cases} \qquad \boxed{B = -3}$$

$$\begin{array}{ll} x^3 & C - 3C + 3C - C = 0 \\ x^2 & 9C - 18C + 9C = 0 \\ x & 18C - 18C = 0 \\ x^0 & 6C = -4 \quad \to \quad \boxed{C = -\frac{2}{3}} \end{array}$$

$$\boxed{y_p = -x - 3 - \frac{2}{3}x^3e^x}$$

$$\boxed{y(x) = \left(C_1 + C_2x + C_3x^2\right)e^x - x - 3 - \frac{2}{3}x^3e^x}$$

Exercise

Find the general solution: $y^{(3)} - 4y'' + y' + 6y = 4\sin 2x$

Solution

The characteristic equation: $\lambda^3 - 4\lambda^2 + \lambda + 6 = 0 \quad \to \quad \boxed{\lambda_1 = -1}$

$$\begin{array}{r|rrrr} -1 & 1 & -4 & 1 & 6 \\ & & -1 & 5 & -6 \\ \hline & 1 & -5 & 6 & 0 \end{array} \qquad \lambda^2 - 5\lambda + 6 = 0 \quad \to \quad \boxed{\lambda_{2,3} = 2,\ 3}$$

$$\boxed{y_h = C_1e^{-x} + C_2e^{2x} + C_3e^{3x}}$$

$$y_p = A\cos 2x + B\sin 2x$$

$$y'_p = -2A\sin 2x + 2B\cos 2x$$

$$y''_p = -4A\cos 2x - 4B\sin 2x$$

$$y'''_p = 8A\sin 2x - 8B\cos 2x$$

$$y^{(3)} - 4y'' + y' + 6y = 4\sin 2x$$

$$8A\sin 2x - 8B\cos 2x + 16A\cos 2x + 16B\sin 2x - 2A\sin 2x + 2B\cos 2x + 6A\cos 2x + 6B\sin 2x = 4\sin 2x$$

$$\begin{cases} \cos 2x & 22A - 6B = 0 \\ \sin 2x & 6A + 22B = 4 \end{cases}$$

$$A = \frac{24}{520} = \frac{3}{65} \qquad B = \frac{88}{520} = \frac{11}{65}$$

$$y_p = \frac{3}{65}\cos 2x + \frac{11}{65}\sin 2x$$

$$y(x) = C_1 e^{-x} + C_2 e^{2x} + C_3 e^{3x} + \frac{3}{65}\cos 2x + \frac{11}{65}\sin 2x$$

Exercise

Find the general solution for the given differential equation $\quad y''' - 3y'' + 3y' - y = e^x - x + 16$

Solution

The characteristic equation: $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3 = 0 \;\Rightarrow\; \lambda_{1,2,3} = 1$

$$y_h = \left(C_1 + C_2 x + C_3 x^2\right)e^x$$

The particular equation: $y_p = A + Bx + \left(Cx^3 + Ex^4\right)e^x$

$$y'_p = B + \left(3Cx^2 + 4Ex^3 + Cx^3 + Ex^4\right)e^x = B + \left(3Cx^2 + (4E + C)x^3 + Ex^4\right)e^x$$

$$\begin{aligned} y''_p &= \left(6Cx + 3(4E + C)x^2 + 4Ex^3\right)e^x + \left(3Cx^2 + (4E + C)x^3 + Ex^4\right)e^x \\ &= \left(6Cx + (12E + 6C)x^2 + (8E + C)x^3 + Ex^4\right)e^x \end{aligned}$$

$$\begin{aligned} y'''_p &= \left(6C + (24E + 12C)x + (24E + 3C)x^2 + 4Ex^3\right)e^x \\ &\qquad + \left(6Cx + (12E + 6C)x^2 + (8E + C)x^3 + Ex^4\right)e^x \\ &= \left(6C + (24E + 18C)x + (36E + 9C)x^2 + (12E + C)x^3 + Ex^4\right)e^x \end{aligned}$$

$$\left(6C + (24E + 18C)x + (36E + 9C)x^2 + (12E + C)x^3 + Ex^4\right)e^x - 3\left(6Cx + (12E + 6C)x^2 + (8E + C)x^3\right.$$
$$\left. + Ex^4\right)e^x + 3B + 3\left(3Cx^2 + (4E + C)x^3 + Ex^4\right)e^x - A - Bx - \left(Cx^3 + Ex^4\right)e^x \;\underline{= e^x - x + 16}$$

$$(6C + 24Ex)e^x - A + 3B - Bx \;\underline{= e^x - x + 16}$$

$$\begin{cases} 6C = 1 & \rightarrow C = \frac{1}{6} \\ 24E = 0 & \rightarrow E = 0 \\ -B = -1 & \rightarrow B = 1 \\ 3B - A = 16 & \rightarrow A = -13 \end{cases}$$

$$\boxed{y_p = x - 13 + \frac{1}{6}x^3 e^x}$$

$$\boxed{y(x) = \left(C_1 + C_2 x + C_3 x^2\right)e^x + \frac{1}{6}x^3 e^x + x - 13}$$

Exercise

Find the general solution $\quad y''' - 6y'' = 3 - \cos x$

Solution

The characteristic equation: $\lambda^3 - 6\lambda^2 = 0 \quad \to \quad \boxed{\lambda_{1,2,3} = 0,\ 0,\ 6}$

$$\boxed{y_h = C_1 + C_2 x + C_3 e^{6x}}$$

$$y_p = Ax^2 + B\cos x + C\sin x$$

$$y'_p = 2Ax - B\sin x + C\cos x$$

$$y''_p = 2A - B\cos x - C\sin x$$

$$y'''_p = B\sin x - C\cos x$$

$$y''' - 6y'' = 3 - \cos x$$

$$\begin{array}{rll} & -12A = 3 & \boxed{A = -\frac{1}{4}} \\ \cos x & -C + 6B = -1 & \boxed{B = -\frac{6}{37}} \\ \sin x & B + 6C = 0 & \boxed{C = \frac{1}{37}} \end{array}$$

$$\boxed{y_p = -\frac{1}{4}x^2 - \frac{6}{37}\cos x + \frac{1}{37}\sin x}$$

$$\boxed{y(x) = C_1 + C_2 x + C_3 e^{6x} - \frac{1}{4}x^2 - \frac{6}{37}\cos x + \frac{1}{37}\sin x}$$

Exercise

Find the general solution $\quad y^{(3)} - 6y'' + 11y' - 6y = 2xe^{-x}$

Solution

The characteristic equation: $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \quad \to \quad \boxed{\lambda_1 = 1}$

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 11 & -6 \\ & & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array} \qquad \lambda^2 - 5\lambda + 6 = 0 \quad \to \quad \boxed{\lambda_{2,3} = 2,\ 3}$$

$$\boxed{y_h = C_1 e^x + C_2 e^{2x} + C_3 e^{3x}}$$

$$y_p = (Ax+B)e^{-x}$$

$$y'_p = (-Ax+A-B)e^{-x}$$

$$y''_p = (Ax-2A+B)e^{-x}$$

$$y'''_p = (-Ax+3A-B)e^{-x}$$

$$y^{(3)} - 6y'' + 11y' - 6y = 2xe^{-x}$$

$$(-24Ax+26A-24B)e^{-x} = 2xe^{-x}$$

$$\begin{cases} -24A = 2 & \rightarrow \boxed{A = -\frac{1}{12}} \\ 26A - 24B = 0 & \rightarrow \boxed{B = -\frac{13}{144}} \end{cases}$$

$$\boxed{y_p = \left(-\frac{1}{12}x - \frac{13}{144}\right)e^{-x}}$$

$$\boxed{y(x) = C_1 e^{x} + C_2 e^{2x} + C_3 e^{3x} - \frac{1}{12}xe^{-x} - \frac{13}{144}e^{-x}}$$

Exercise

Find the general solution: $y''' + 8y'' = -6x^2 + 9x + 2$

Solution

The characteristic equation: $\lambda^3 + 8\lambda^2 = \lambda^2(\lambda+8) = 0 \quad \Rightarrow \quad \lambda_{1,2} = 0,\ \lambda_3 = -8$

$$y_h = \left(C_1 + C_2 x\right)e^{0} + e^{-8x} = C_1 + C_2 x + e^{-8x}$$

The particular equation: $y_p = Ax^4 + Bx^3 + Cx^2$

$$y'_p = 4Ax^3 + 3Bx^2 + 2Cx$$

$$y''_p = 12Ax^2 + 6Bx + 2C$$

$$y''_p = 24Ax + 6B$$

$$y''' + 8y'' = -6x^2 + 9x + 2$$

$$24Ax + 6B + 96Ax^2 + 48Bx + 16C = -6x^2 + 9x + 2$$

$$\begin{cases} 96A = -6 & \rightarrow A = -\frac{1}{16} \\ 24A + 48B = 9 & \rightarrow B = \frac{1}{96}\left(9 + \frac{3}{2}\right) = \frac{21}{108} = \frac{7}{32} \\ 6B + 16C = 2 & \rightarrow C = \frac{1}{16}\left(2 - \frac{21}{16}\right) = \frac{11}{256} \end{cases}$$

$$\boxed{y(x) = C_1 + C_2 x + e^{-8x} - \frac{1}{16}x^4 + \frac{7}{32}x^3 + \frac{11}{256}x^2}$$

Exercise

Find the general solution: $y^{(4)} + y'' = 3x^2 + 4\sin x - 2\cos x$

Solution

The characteristic equation: $\lambda^4 + 2\lambda^2 = 0 \quad \rightarrow \quad \lambda_{1,2} = 0 \quad \lambda_{3,4} = \pm 2i$

$$y_h = C_1 + C_2 x + C_3 \cos x + C_4 \sin x$$

$$y_p = Ax^4 + Bx^3 + Cx^2 + Ex\cos x + Fx\sin x$$

$$\begin{aligned} y'_p &= 4Ax^3 + 3Bx^2 + 2Cx + E\cos x - Ex\sin x + F\sin x + Fx\cos x \\ &= 4Ax^3 + 3Bx^2 + 2Cx + (E + Fx)\cos x + (F - Ex)\sin x \end{aligned}$$

$$\begin{aligned} y''_p &= 12Ax^2 + 6Bx + 2C + F\cos x - (E + Fx)\sin x - E\sin x + (F - Ex)\cos x \\ &= 12Ax^2 + 6Bx + 2C + (2F - Ex)\cos x - (2E + Fx)\sin x \end{aligned}$$

$$\begin{aligned} y'''_p &= 24Ax + 6B - E\cos x - (2F - Ex)\sin x - F\sin x - (2E + Fx)\cos x \\ &= 24Ax + 6B - (3E + Fx)\cos x - (3F - Ex)\sin x \end{aligned}$$

$$\begin{aligned} y^{(4)} &= 24A - F\cos x + (3E + Fx)\sin x + E\sin x - (3F - Ex)\cos x \\ &= 24A - (4F - Ex)\cos x + (4E + Fx)\sin x \end{aligned}$$

$$y^{(4)} + y'' = 3x^2 + 4\sin x - 2\cos x$$

$$\begin{cases} x^2 & 12A = 3 & \rightarrow A = \frac{1}{4} \\ x & 6B = 0 & \rightarrow B = 0 \\ x^0 & 24A + 2C = 0 & \rightarrow C = -3 \end{cases}$$

$$\begin{cases} \cos x & x & E - E & \\ & x^0 & -4F + 2F = -2 & \rightarrow F = 1 \\ \sin x & x & F - F & \\ & x^0 & 4E - 2E = 4 & \rightarrow E = 2 \end{cases}$$

$$y_p = \frac{1}{4}x^4 - 3x^2 + 2x\cos x + x\sin x$$

$$y(x) = C_1 + C_2 x + C_3 \cos x + C_4 \sin x + \frac{1}{4}x^4 - 3x^2 + 2x\cos x + x\sin x$$

Exercise

Find the general solution $\quad y^{(4)}+2y''+y=(x-2)^2$

Solution

The characteristic equation: $\lambda^4+2\lambda^2+1=\left(\lambda^2+1\right)^2=0 \ \rightarrow\ \lambda_{1,2}=i \quad \lambda_{3,4}=-i$

$$\boxed{y_h=C_1\cos x+C_2\sin x+C_3x\cos x+C_4x\sin x}$$

$$y_p=Ax^2+Bx+C$$

$$y'_p=2Ax+B$$

$$y''_p=2A$$

$$y'''_p=y_p^{(4)}=0$$

$$y^{(4)}+2y''+y=x^2-4x+4$$

$x^2 \qquad \boxed{A=1}$

$x \qquad \boxed{B=-4}$

$x^0 \quad 4A+C=4 \quad \boxed{C=0}$

$$\boxed{y_p=x^2-4x}$$

$$\boxed{y(x)=C_1\cos x+C_2\sin x+C_3x\cos x+C_4x\sin x+x^2-4x}$$

Exercise

Find the general solution $\quad y^{(4)}-y''=4x+2xe^{-x}$

Solution

The characteristic equation: $\lambda^4-\lambda^2=\lambda^2\left(\lambda^2-1\right)=0 \ \rightarrow\ \boxed{\lambda_{1,2,3,4}=0,\ 0,\ \pm 1}$

$$\boxed{y_h=C_1+C_2x+C_3e^{-x}+C_4e^{x}}$$

$$y_p=Ax^3+\left(Bx^2+Cx\right)e^{-x}$$

$$y'_p=3Ax^2+\left(2Bx+C-Bx^2-Cx\right)e^{-x}$$

$$y''_p=6Ax+\left(Bx^2-4Bx+Cx+2B-2C\right)e^{-x}$$

$$y'''_p=6A+\left(2Bx-4B+C-Bx^2+4Bx-Cx-2B+2C\right)e^{-x}$$

$$=6A+\left(-Bx^2+6Bx-Cx-6B+3C\right)e^{-x}$$

$$y_p^{(4)}=\left(Bx^2-6Bx+Cx+6B-3C-2Bx+6B-C\right)e^{-x}$$

$$= \left(Bx^2 - 8Bx + Cx + 12B - 4C\right)e^{-x}$$

$y^{(4)} - y'' = 4x + 2xe^{-x}$

$$-6A = 4 \quad \rightarrow \quad \boxed{A = -\frac{2}{3}}$$

$$\begin{cases} e^{-x} & x^2 & -B - B = 0 & \\ & x & -8B + C + 4B - C = 2 & B = -\frac{1}{2} \\ & x^0 & 12B - 4C - 2B + 2C = 0 & C = -\frac{5}{2} \end{cases}$$

$$\boxed{y_p = -\frac{2}{3}x^3 + \left(-\frac{1}{2}x^2 - \frac{5}{2}x\right)e^{-x}}$$

$$\boxed{y(x) = C_1 + C_2x + C_3e^{-x} + C_4e^x - \frac{2}{3}x^3 - \left(\frac{1}{2}x^2 + \frac{5}{2}x\right)e^{-x}}$$

Exercise

Find the general solution $\left(D^2 + D - 2\right)y = 2x - 40\cos 2x$

Solution

The characteristic equation: $\lambda^2 + \lambda - 2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = 1, \ -2}$

$$\boxed{y_h = C_1e^x + C_2e^{-2x}}$$

$$y_p = Ax + B + C\cos 2x + D\sin 2x$$

$$y_p' = A - 2C\sin 2x + 2D\cos 2x$$

$$y_p'' = -4C\cos 2x - 4D\sin 2x$$

$y'' + y' - 2y = 2x - 40\cos 2x$

$$-4C\cos 2x - 4D\sin 2x + A - 2C\sin 2x + 2D\cos 2x - 2Ax - 2B - 2C\cos 2x - 2D\sin 2x = 2x - 40\cos 2x$$

$$\begin{cases} x & -2A = 2 & \boxed{A = -1} \\ x^0 & A - 2B = 0 & \boxed{B = -\frac{1}{2}} \\ \cos 2x & -6C + 2D = -40 & \boxed{C = \frac{240}{40} = 6} \\ \sin 2x & -2C - 6D = 0 & \boxed{D = -\frac{80}{40} = -2} \end{cases}$$

$$\boxed{y_p = -x - \frac{1}{2} + 6\cos 2x - 2\sin 2x}$$

$$\boxed{y(x) = C_1e^x + C_2e^{-2x} - x - \frac{1}{2} + 6\cos 2x - 2\sin 2x}$$

Exercise

Find the general solution $\left(D^2-3D+2\right)y=2\sin x$

Solution

The characteristic equation: $\lambda^2+\lambda-2=0 \quad \rightarrow \quad \underline{\lambda_{1,2}=1,\ 2|}$

$$\underline{y_h=C_1e^x+C_2e^{2x}|}$$

$$y_p=A\cos x+B\sin x$$

$$y_p'=-A\sin x+B\cos x$$

$$y_p''=-A\cos x-B\sin x$$

$$y''-3y'+2y=2\sin x$$

$$-A\cos x-B\sin x+3A\sin x-3B\cos x+2A\cos x+2B\sin x=2\sin x$$

$$\begin{cases} \cos x & A-3B=0 & \underline{A=\frac{6}{10}=\frac{3}{5}|} \\ \sin x & 3A+B=2 & \underline{B=\frac{2}{10}=\frac{1}{5}|} \end{cases}$$

$$\underline{y_p=\frac{3}{5}\cos x+\frac{1}{5}\sin x|}$$

$$\underline{y(x)=C_1e^x+C_2e^{2x}+\frac{3}{5}\cos x+\frac{1}{5}\sin x|}$$

Exercise

Find the general solution $(D-2)^3\left(D^2+9\right)y=x^2e^{2x}+x\sin 3x$

Solution

$$(\lambda-2)^3\left(\lambda^2+9\right)=0 \quad \rightarrow \quad \underline{\lambda_{1,2,3}=2;\ \ \lambda_{4,5}=\pm 3i|}$$

$$\underline{y_h=\left(C_1+C_2x+C_3x^2\right)e^{2x}+C_4\cos 3x++C_5\sin 3x|}$$

$$y_p=\left(Ax^2+Bx+C\right)e^{2x}+\left(Dx^2+Ex\right)\cos 3x+\left(Fx^2+Gx\right)\sin 3x$$

$$\left(D^3-6D^2+12D-8\right)\left(D^2+9\right)y=\left(D^5-6D^4+21D^3-62D^2+108D-72\right)y$$

$$=y^{(5)}-6y^{(4)}+21y^{(3)}-62y''+108y'-72y$$

$$y_{p1}=\left(Ax^5+Bx^4+Cx^3\right)e^{2x} \qquad x^3\left(Ax^2+Bx+C\right)$$

$$y_{p1}'=\left(2Ax^5+5Ax^4+2Bx^4+2Cx^3+4Bx^3+3Cx^2\right)e^{2x}$$

$$y''_{p1}=\left(10Ax^4+20Ax^3+8Bx^3+6Cx^2+12Bx^2+6Cx+4Ax^5+10Ax^4+4Bx^4+4Cx^3+8Bx^3+6Cx^2\right)e^{2x}$$
$$=\left(4Ax^5+20Ax^4+4Bx^4+20Ax^3+16Bx^3+4Cx^3+12Cx^2+12Bx^2+6Cx\right)e^{2x}$$

$$y^{(3)}_{p1}=\left(\begin{array}{l}20Ax^4+80Ax^3+16Bx^3+60Ax^2+48Bx^2+12Cx^2+24Cx+24Bx+6C\\+8Ax^5+40Ax^4+8Bx^4+40Ax^3+32Bx^3+8Cx^3+24Cx^2+24Bx^2+12Cx\end{array}\right)e^{2x}$$
$$=\left(8Ax^5+60Ax^4+8Bx^4+120Ax^3+48Bx^3+8Cx^3+60Ax^2+72Bx^2+36Cx^2+24Bx+36Cx+6C\right)e^{2x}$$

$$y^{(4)}_{p1}=\left(\begin{array}{l}16Ax^5+120Ax^4+16Bx^4+240Ax^3+96Bx^3+16Cx^3+120Ax^2+144Bx^2+72Cx^2+48Bx\\+72Cx+12C+40Ax^4+240Ax^3+32Bx^3+360Ax^2+144Bx^2+24Cx^2+120Ax+144Bx+36Cx\\+24B+12C\end{array}\right)e^{2x}$$
$$=\left(\begin{array}{l}16Ax^5+160Ax^4+16Bx^4+480Ax^3+128Bx^3+16Cx^3+480Ax^2+288Bx^2+96Cx^2\\+120Ax+192Bx+144Cx+24B+48C\end{array}\right)e^{2x}$$

$$y^{(5)}_{p1}=\left(\begin{array}{l}32Ax^5+320Ax^4+32Bx^4+960Ax^3+256Bx^3+32Cx^3+960Ax^2+576Bx^2+192Cx^2\\+240Ax+384Bx+288Cx+48B+96C+80Ax^4+640Ax^3+64Bx^3+1440Ax^2+384Bx^2\\+48Cx^2+960Ax+576Bx+192Cx+120A+192B+144C\end{array}\right)e^{2x}$$
$$=\left(\begin{array}{l}32Ax^5+400Ax^4+32Bx^4+1600Ax^3+320Bx^3+32Cx^3+2400Ax^2+960Bx^2+240Cx^2\\+1200Ax+960Bx+480Cx+120A+240B+240C\end{array}\right)e^{2x}$$

e^{2x} $\quad y^{(5)}-6y^{(4)}+21y^{(3)}-62y''+108y'-72y=x^2e^{2x}$

	x^5	x^4	x^3	x^2	x	x^0
	32	400	1600	2400	1200	120
	−96	−960	−2880	−2880	−720	
	168	1260	2520	1260		
A	−248	−1240	−1240			
	216	540				
	−72					
		32	320	960	960	240
		−96	−768	−1728	−1152	−144
		168	1008	1512	504	
B		−248	−992	−744		
		216	432			
		−72				
			32	240	480	240
			−96	−576	−864	−288
			168	756	756	126
C			−248	−744	372	
			216	324		
			−72			

$$\to\quad\begin{array}{lcl}x^5 & 0=0 & \\ x^4 & 0=0 & \\ x^3 & 0=0 & \\ x^2 & 780A=1 & A=\frac{1}{780}\\ x & 480A+312B=0 & B=-\frac{1}{507}\\ x^0 & 120A+96B+78C=0 & C=\frac{1}{2197}\end{array}$$

$$\Rightarrow\quad A=\frac{1}{780};\quad B=-\frac{1}{507};\quad C=\frac{1}{2197}\quad\to\quad \underline{y_{p1}=\left(\frac{1}{780}x^5-\frac{1}{507}x^4+\frac{1}{2197}x^3\right)e^{2x}}$$

$$y_{p2}=\left(Dx^2+Ex\right)\cos 3x+\left(Fx^2+Gx\right)\sin 3x$$

$$y'_{p2}=\left(3Fx^2+3Gx+2Dx+E\right)\cos 3x+\left(-3Dx^2-3Ex+2Fx+G\right)\sin 3x$$

$$y''_{p2}=\left(-9Dx^2-9Ex+12Fx+6G+2D\right)\cos 3x+\left(-9Fx^2-9Gx-12Dx-9E+2F\right)\sin 3x$$

$$y^{(3)}{}_{p2}=\left(-27Fx^2-27Gx-54Dx-36E+18F\right)\cos 3x+\left(27Dx^2+27Ex-54Fx-27G-18D\right)\sin 3x$$

$$y^{(4)}{}_{p2}=\left(81Dx^2+81Ex-216Fx-108G-108D\right)\cos 3x+\left(81Fx^2+81Gx+216Dx+135E-108F\right)\sin 3x$$

$$y^{(5)}{}_{p2}=\left(243Fx^2+243Gx+810Dx-108E-540F\right)\cos 3x+\left(-243Dx^2-243Ex+810Fx+405G+540D\right)\sin 3x$$

$$y^{(5)}-6y^{(4)}+21y^{(3)}-62y''+108y'-72y=x\sin 3x$$

$\cos 3x$	x^2	x	x^0
D	$\begin{matrix}-486\\558\\-72\end{matrix}$	$\begin{matrix}810\\-1134\\216\end{matrix}$	$\begin{matrix}648\\-124\end{matrix}$
E		$\begin{matrix}-486\\558\\-72\end{matrix}$	$\begin{matrix}-162\\-756\\108\end{matrix}$
F	$\begin{matrix}243\\-567\\324\end{matrix}$	$\begin{matrix}1296\\-744\end{matrix}$	$\begin{matrix}-540\\378\end{matrix}$
G		$\begin{matrix}243\\-567\\324\end{matrix}$	$\begin{matrix}648\\-372\end{matrix}$

$$\rightarrow\quad \begin{array}{ll} x^2 & 0=0 \\ x & -108D+552F=0 \\ x^0 & 524D-810E-162F+276G=0 \end{array}$$

$\sin 3x$	x^2	x	x^0
D	$\begin{matrix}-243\\567\\-324\end{matrix}$	$\begin{matrix}-1296\\744\end{matrix}$	$\begin{matrix}540\\-378\end{matrix}$
E		$\begin{matrix}-243\\567\\-324\end{matrix}$	$\begin{matrix}486\\558\end{matrix}$
F	$\begin{matrix}-486\\558\\-72\end{matrix}$	$\begin{matrix}810\\-1134\\216\end{matrix}$	$\begin{matrix}648\\-124\end{matrix}$
G		$\begin{matrix}-486\\558\\-72\end{matrix}$	$\begin{matrix}405\\-567\\108\end{matrix}$

$$\rightarrow\quad \begin{array}{ll} x^2 & 0=0 \\ x & -552D-108F=1 \\ x^0 & 162D+1044E+524F-54G=0 \end{array}$$

$$\Rightarrow\quad D=-\frac{23}{13{,}182};\quad E=-\frac{251}{114{,}244};\quad F=-\frac{3}{8788}\quad G=\frac{1379}{514{,}098}$$

$$y_{p2}=\left(-\frac{23}{13{,}182}x^2-\frac{251}{114{,}244}x\right)\cos 3x+\left(-\frac{3}{8788}x^2+\frac{1379}{514{,}098}x\right)\sin 3x$$

$$y(x)=\left(C_1+C_2x+C_3x^2+\frac{1}{780}x^5-\frac{1}{507}x^4+\frac{1}{2197}x^3\right)e^{2x}$$
$$-\left(+\frac{23}{13{,}182}x^2+\frac{251}{114{,}244}x+C_4\right)\cos 3x+\left(-\frac{3}{8788}x^2+\frac{1379}{514{,}098}x+C_5\right)\sin 3x$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + y = \cos x; \quad y(0) = 1, \quad y'(0) = -1$$

Solution

The characteristic equation: $\lambda^2 + 1 = 0 \quad \to \quad \lambda_{1,2} = \pm i$

$y_h = C_1 \cos x + C_2 \sin x$

$y_P = Ax\cos x + Bx\sin x$

$$y'_P = A\cos x - Ax\sin x + B\sin x + Bx\cos x$$
$$= (A + Bx)\cos x + (B - Ax)\sin x$$
$$y''_P = B\cos x - A\sin x - (A + Bx)\sin x + (B - Ax)\cos x$$
$$= (2B - Ax)\cos x - (2A + Bx)\sin x$$

$y'' + y = \cos x$

$(2B - Ax)\cos x - (2A + Bx)\sin x + Ax\cos x + Bx\sin x = \cos x$

$2B\cos x - 2A\sin x = \cos x$

$$\begin{cases} 2B = 1 \\ -2A = 0 \end{cases} \quad \to \quad A = 0, \ B = \frac{1}{2}$$

$y_P = \frac{1}{2}x\sin x$

$y(x) = C_1 \cos x + C_2 \sin x + \frac{1}{2}x\sin x$

$$y(0) = 1 \quad \to \quad C_1 = 1$$

$$y'(x) = -C_1 \sin x + C_2 \cos x + \frac{1}{2}(\sin x + x\cos x)$$

$$y'(0) = -1 \quad \to \quad C_2 = -1$$

$y(x) = \cos x - \sin x + \frac{1}{2}x\sin x$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + y' = x \ ; \quad y(1) = 0, \quad y'(1) = 1$$

Solution

The characteristic equation: $\lambda^2 + \lambda = 0 \quad \to \quad \lambda_{1,2} = 0, \ 1$

$y_h = C_1 + C_2 e^x$

$y_P = Ax^2 + Bx$

$$y'_P = 2Ax + B$$
$$y''_P = 2A$$

$$y'' + y' = x$$

$$\begin{cases} x & 2A = 1 \\ x^0 & 2A + B = 0 \end{cases} \rightarrow \boxed{A = \frac{1}{2},\ B = -1}$$

$$\boxed{y_P = \frac{1}{2}x^2 - x}$$

$$y(x) = C_1 + C_2 e^x + \frac{1}{2}x^2 - x$$

$$y(1) = 0 \quad \rightarrow C_1 + C_2 e + \frac{1}{2} - 1 = 0 \quad \boxed{C_1 + eC_2 = \frac{1}{2}}$$

$$y(x) = C_2 e^x + x - 1$$

$$y'(1) = 1 \quad \rightarrow \quad eC_2 = 1 \quad \boxed{C_2 = \frac{1}{e}}$$

$$C_1 + e\frac{1}{e} = \frac{1}{2} \quad \rightarrow \quad \boxed{C_1 = \frac{1}{2}}$$

$$y(x) = \frac{1}{2} + \frac{1}{e}e^x + \frac{1}{2}x^2 - x$$
$$\boxed{= \frac{1}{2} + e^{x-1} + \frac{1}{2}x^2 - x}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + y' = -x\ ; \quad y(0) = 1, \quad y'(0) = 0$$

Solution

The characteristic equation: $\lambda^2 + \lambda = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = 0,\ 1}$

$$\boxed{y_h = C_1 + C_2 e^x}$$

$$y_P = Ax^2 + Bx$$
$$y'_P = 2Ax + B$$
$$y''_P = 2A$$

$$y'' + y' = -x$$

$$\begin{cases} x & 2A = -1 \\ x^0 & 2A + B = 0 \end{cases} \rightarrow \boxed{A = -\frac{1}{2},\ B = 1}$$

$$\boxed{y_P = -\frac{1}{2}x^2 + x}$$

$y(x) = C_1 + C_2 e^x - \frac{1}{2}x^2 + x$

$y(0) = 1 \quad \rightarrow \quad C_1 + C_2 = 1$

$y(x) = C_2 e^x - x + 1$

$y'(0) = 0 \quad \rightarrow \quad \boxed{C_2 = -1} \quad \Rightarrow \boxed{C_1 = 2}$

$\boxed{y(x) = 2 - e^x - \frac{1}{2}x^2 + x}$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + y = 8\cos 2t - 4\sin t \qquad y\left(\frac{\pi}{2}\right) = -1, \quad y'\left(\frac{\pi}{2}\right) = 0$$

Solution

The characteristic equation: $\lambda^2 + 1 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = \pm i}$

$\boxed{y_h = C_1 \cos t + C_2 \sin t}$

The particular equation: $y_p = At\cos t + Bt\sin t + C\cos 2t + E\sin 2t$

$y'_p = A\cos t - At\sin t + B\sin t + Bt\cos t - 2C\sin 2t + 2E\cos 2t$

$y''_p = -A\sin t - A\sin t - At\cos t + B\cos t + B\cos t - Bt\sin t - 4C\cos 2t - 4E\sin 2t$

$-2A\sin t + 2B\cos t - At\cos t - Bt\sin t - 4C\cos 2t - 4E\sin 2t + At\cos t + Bt\sin t + C\cos 2t + E\sin 2t$

$= 8\cos 2t - 4\sin t$

$$\begin{cases} -2A = -4 & \rightarrow A = 2 \\ 2B = 0 & \rightarrow B = 0 \\ -3C = 8 & \rightarrow C = -\frac{8}{3} \\ -3E = 0 & \rightarrow E = 0 \end{cases}$$

$\boxed{y_p = 2t\cos t - \frac{8}{3}\cos 2t}$

$\boxed{y(x) = C_1 \cos t + C_2 \sin t + 2t\cos t - \frac{8}{3}\cos 2t}$

$y\left(\frac{\pi}{2}\right) = -1 \quad \rightarrow \quad C_2 + \frac{8}{3} = -1 \Rightarrow \boxed{C_2 = -\frac{11}{3}}$

$y'(x) = -C_1 \sin t + C_2 \cos t + 2\cos t - 2t\sin t + \frac{16}{3}\sin 2t$

$y'\left(\frac{\pi}{2}\right) = 0 \quad \rightarrow \quad -C_1 - 2\frac{\pi}{2} = 0 \Rightarrow \boxed{C_1 = -\pi}$

$\boxed{y(x) = -\pi\cos t - \frac{11}{3}\sin t + 2t\cos t - \frac{8}{3}\cos 2t}$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - y' - 2y = 4x^2; \quad y(0) = 1, \quad y'(0) = 4$$

Solution

$\lambda^2 - \lambda - 2 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = -1,\ 2}$

$\boxed{y_h = C_1 e^{-x} + C_2 e^{2x}}$

$y_p = Ax^2 + Bx + C$

$y'_p = 2Ax + B$

$y''_p = 2A$

$y'' - y' - 2y = 4x^2$

$2A - 2Ax - B - 2Ax^2 - 2Bx - 2C = 4x^2$

$$\begin{cases} x^2 & -2A = 4 & \rightarrow \boxed{A = -2} \\ x & -2A - 2B = 0 & \rightarrow \boxed{B = 2} \\ x^0 & 2A - B - 2C = 0 & \rightarrow \boxed{C = -3} \end{cases}$$

$\boxed{y_p = -2x^2 + 2x - 3}$

$y(x) = C_1 e^{-x} + C_2 e^{2x} - 2x^2 + 2x - 3$

$y(0) = 1 \quad \rightarrow C_1 + C_2 - 3 = 1 \quad \Rightarrow \quad \boxed{C_1 + C_2 = 4}$

$y' = -C_1 e^{-x} + 2C_2 e^{2x} - 4x + 2$

$y'(0) = 4 \quad \rightarrow -C_1 + 2C_2 + 2 = 4 \quad \Rightarrow \quad -C_1 + 2C_2 = 2$

$$\begin{cases} C_1 + C_2 = 4 \\ -C_1 + 2C_2 = 2 \end{cases} \quad \rightarrow \quad \boxed{C_1 = 2,\ C_2 = 2}$$

$\boxed{y(x) = 2e^{-x} + 2e^{2x} - 2x^2 + 2x - 3}$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - y' - 2y = e^{3x}; \quad y(1) = 2, \quad y'(1) = 1$$

Solution

$\lambda^2 - \lambda - 2 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = -1,\ 2}$

$$\boxed{y_h = C_1 e^{-x} + C_2 e^{2x}}$$

$$y_p = Ae^{3x}$$

$$y'_p = 3Ae^{3x}$$

$$y''_p = 9Ae^{3x}$$

$$y'' - y' - 2y = e^{3x}$$

$$9A - 3A - 2A = 1 \quad \rightarrow \quad \boxed{A = \frac{1}{4}}$$

$$\boxed{y_p = \frac{1}{4}e^{3x}}$$

$$y(x) = C_1 e^{-x} + C_2 e^{2x} + \frac{1}{4}e^{3x}$$

$$y(1) = 2 \quad \rightarrow C_1 e^{-1} + C_2 e^2 + \frac{1}{4}e^3 = 2 \quad \Rightarrow \quad \frac{1}{e}C_1 + e^2 C_2 = 2 - \frac{1}{4}e^3$$

$$y'(x) = -C_1 e^{-x} + 2C_2 e^{2x} + \frac{3}{4}e^{3x}$$

$$y'(1) = 1 \quad \rightarrow -\frac{1}{e}C_1 + 2e^2 C_2 + \frac{3}{4}e^3 = 1 \quad \Rightarrow \quad -C_1 + 2e^3 C_2 = e - \frac{3}{4}e^4$$

$$\begin{cases} C_1 + e^3 C_2 = 2e - \frac{1}{4}e^4 \\ -C_1 + 2e^3 C_2 = e - \frac{3}{4}e^4 \end{cases} \quad \rightarrow \quad 3e^3 C_2 = 3e - e^4$$

$$3e^3 C_2 = 3e - e^4 \quad \rightarrow \quad \boxed{C_2 = e^{-2} - \frac{1}{3}e}$$

$$C_1 = 2e - \frac{1}{4}e^4 - e^3\left(e^{-2} - \frac{1}{3}e\right) \quad \rightarrow \quad \boxed{C_1 = e - \frac{1}{12}e^4}$$

$$\boxed{y(x) = \left(e - \frac{1}{12}e^4\right)e^{-x} + \left(e^{-2} - \frac{1}{3}e\right)e^{2x} + \frac{1}{4}e^{3x}}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - y' - 2y = e^{3x}\ ; \qquad y(0) = 1, \quad y'(0) = 2$$

Solution

$$\lambda^2 - \lambda - 2 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = -1,\ 2}$$

$$\boxed{y_h = C_1 e^{-x} + C_2 e^{2x}}$$

$$y_p = Ae^{3x}$$

$$y'_p = 3Ae^{3x}$$

$$y_p'' = 9Ae^{3x}$$

$$y'' - y' - 2y = e^{3x}$$

$$9A - 3A - 2A = 1 \quad \rightarrow \quad \boxed{A = \frac{1}{4}}$$

$$\boxed{y_p = \frac{1}{4}e^{3x}}$$

$$y(x) = C_1 e^{-x} + C_2 e^{2x} + \frac{1}{4}e^{3x}$$

$$y(0) = 1 \quad \rightarrow C_1 + C_2 + \frac{1}{4} = 1 \quad \Rightarrow \ C_1 + C_2 = \frac{3}{4}$$

$$y'(x) = -C_1 e^{-x} + 2C_2 e^{2x} + \frac{3}{4}e^{3x}$$

$$y'(0) = 2 \quad \rightarrow -C_1 + 2C_2 + \frac{3}{4} = 2 \quad \Rightarrow \ -C_1 + 2C_2 = \frac{5}{4}$$

$$\begin{cases} C_1 + C_2 = \frac{3}{4} \\ -C_1 + 2C_2 = \frac{5}{4} \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 3 \quad \Delta_{C_1} = \begin{vmatrix} \frac{3}{4} & 1 \\ \frac{5}{4} & 2 \end{vmatrix} = \frac{1}{4} \quad \Delta_{C_2} = \begin{vmatrix} 1 & \frac{3}{4} \\ -1 & \frac{5}{4} \end{vmatrix} = 2$$

$$\boxed{C_1 = \frac{1}{12} \quad C_2 = \frac{2}{3}}$$

$$\boxed{y(x) = \frac{1}{12}e^{-x} + \frac{2}{3}e^{2x} + \frac{1}{4}e^{3x}}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - y' - 2y = e^{3x}; \quad y(0) = 2, \quad y'(0) = 1$$

Solution

$$\lambda^2 - \lambda - 2 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = -1, \ 2}$$

$$\boxed{y_h = C_1 e^{-x} + C_2 e^{2x}}$$

$$y_p = Ae^{3x}$$

$$y_p' = 3Ae^{3x}$$

$$y_p'' = 9Ae^{3x}$$

$$y'' - y' - 2y = e^{3x}$$

$$9A - 3A - 2A = 1 \quad \rightarrow \quad \boxed{A = \frac{1}{4}}$$

$$\boxed{y_p = \frac{1}{4}e^{3x}}$$

$$y(x) = C_1 e^{-x} + C_2 e^{2x} + \frac{1}{4}e^{3x}$$

$$y(0) = 2 \quad \rightarrow C_1 + C_2 + \frac{1}{4} = 2 \quad \Rightarrow \; C_1 + C_2 = \frac{7}{4}$$

$$y'(x) = -C_1 e^{-x} + 2C_2 e^{2x} + \frac{3}{4}e^{3x}$$

$$y'(0) = 1 \quad \rightarrow -C_1 + 2C_2 + \frac{3}{4} = 1 \quad \Rightarrow \; -C_1 + 2C_2 = \frac{1}{4}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 3 \quad \Delta_{C_1} = \begin{vmatrix} \frac{7}{4} & 1 \\ \frac{1}{4} & 2 \end{vmatrix} = \frac{13}{4} \quad \Delta_{C_2} = \begin{vmatrix} 1 & \frac{7}{4} \\ -1 & \frac{1}{4} \end{vmatrix} = 2$$

$$\boxed{C_1 = \frac{13}{12} \quad C_2 = \frac{2}{3}}$$

$$\boxed{y(x) = \frac{13}{12}e^{-x} + \frac{2}{3}e^{2x} + \frac{1}{4}e^{3x}}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + 2y' + y = 2\cos t \; ; \quad y(0) = 3, \quad y'(0) = 0$$

Solution

$$\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = -1}$$

$$\boxed{y_h = \left(C_1 + C_2 t\right)e^{-t}}$$

$$y_p = A\cos t + B\sin t$$

$$y_p' = -A\sin t + B\cos t$$

$$y_p'' = -A\cos t - B\sin t$$

$$y'' + 2y' + y = 2\cos t$$

$$\begin{cases} \cos t & -A + 2B + A = 2 \\ \sin t & -B - 2A + B = 0 \end{cases} \quad \rightarrow \quad \boxed{A = 0, \; B = 1}$$

$$\boxed{y_p(t) = \sin t}$$

$$y(t) = \left(C_1 + C_2 t\right)e^{-t} + \sin t$$

$$y(0) = 3 \quad \rightarrow \quad \boxed{C_1 = 3}$$

$$y' = \left(C_2 - C_1 - C_2 t\right)e^{-t} + \cos t$$

$$y'(0) = 0 \quad \rightarrow C_2 - C_1 + 1 = 0 \quad \Rightarrow \underline{C_2 = 2|}$$

$$\underline{y(t) = (3 + 2t)e^{-t} + \sin t|}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - 2y' + y = t^3; \quad y(0) = 1, \quad y'(0) = 0$$

Solution

The homogeneous eq.: $y'' - 2y' + y = 0$

The characteristic eq.: $\lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda_{1,2} = 1$

$$y_h = \left(C_1 + C_2 t\right)e^t$$

$$y_p = at^3 + bt^2 + ct + d$$

$$y_p' = 3at^2 + 2bt + c$$

$$y_p'' = 6at + 2b$$

$$\begin{aligned} y'' - 2y' + y &= 6at + 2b - 6at^2 - 4bt - 2c + at^3 + bt^2 + ct + d \\ &= at^3 + (b - 6a)t^2 + (6a - 4b + c)t + 2b - 2c + d \end{aligned}$$

$$\begin{cases} a = 1 & \\ b - 6a = 0 & \Rightarrow \quad b = 6 \\ 6a - 4b + c = 0 & \Rightarrow \quad c = 18 \\ 2b - 2c + d = 0 & \Rightarrow \quad d = 24 \end{cases}$$

The particular solution is: $y_p = t^3 + 6t^2 + 18t + 24$

The general solution: $y = \left(C_1 + C_2 t\right)e^t + t^3 + 6t^2 + 18t + 24$

$$y(0) = \left(C_1 + C_2(0)\right)e^{(0)} + (0)^3 + 6(0)^2 + 18(0) + 24$$

$$1 = C_1 + 24$$

$$y' = C_2 e^t \left(C_1 + C_2 t\right)e^t + 3t^2 + 12t + 18$$

$$y'(0) = C_2 e^{(0)} \left(C_1 + C_2(0)\right)e^{(0)} + 3(0)^2 + 12(0) + 18$$

$$0 = C_2 + C_1 + 18$$

$$\underline{C_1 = -23 \quad C_2 = 5|}$$

The general solution: $\underline{y(t) = (-23 + 5t)e^t + t^3 + 6t^2 + 18t + 24|}$

Exercise

Find the solution of the given initial value problem

$$y'' - 2y' + y = -3 - x + x^2 \;;\quad y(0) = -2, \quad y'(0) = 1$$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 1 = 0 \rightarrow \boxed{\lambda_{1,2} = 1}$

$$\boxed{y_h = (C_1 + C_2 x)e^x}$$

$$y_p = Ax^2 + Bx + C$$

$$y'_p = 2Ax + B$$

$$y''_p = 2A$$

$$y'' - 2y' + y = -3 - x + x^2$$

$$2A - 4Ax - 2B + Ax^2 + Bx + C = -3 - x + x^2$$

$$\begin{cases} x^2 & \boxed{A = 1} \\ x & -4A + B = -1 \quad \rightarrow \boxed{B = 3} \\ x^0 & 2A - 2B + C = -3 \quad \rightarrow \boxed{C = 1} \end{cases}$$

$$\boxed{y_p = x^2 + 3x + 1}$$

$$y(x) = (C_1 + C_2 x)e^x + x^2 + 3x + 1$$

$$y(0) = -2 \quad \rightarrow C_1 + 1 = -2 \quad \boxed{C_1 = -3}$$

$$y' = (C_2 + C_1 + C_2 x)e^x + 2x + 3$$

$$y'(0) = 1 \quad \rightarrow C_2 - 3 + 3 = 1 \quad \boxed{C_2 = 1}$$

$$\boxed{y(x) = (x - 3)e^x + x^2 + 3x + 1}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - 2y' + 2y = x + 1; \quad y(0) = 3, \quad y'(0) = 0$$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 2 = 0 \rightarrow \lambda_{1,2} = \frac{2 \pm 2i}{2} \boxed{= 1 \pm i}$

$$\boxed{y_h = e^x (C_1 \cos x + C_2 \sin x)}$$

$$y_P = Ax + B$$

$$y'_P = A$$

$$y''_P = 0$$

$$y'' - 2y' + 2y = x + 1$$

$$-2A + 2Ax + 2B = x + 1$$

$$\begin{cases} 2A = 1 \\ -2A + 2B = 1 \end{cases} \rightarrow \quad \boxed{A = \frac{1}{2},\ B = 1}$$

$$y_P = \frac{1}{2}x + 1$$

$$y(x) = e^x\left(C_1 \cos x + C_2 \sin x\right) + \frac{1}{2}x + 1$$

$$y(0) = 3 \quad \rightarrow C_1 + 1 = 3 \quad \boxed{C_1 = 2}$$

$$y'(x) = e^x\left(C_1 \cos x + C_2 \sin x - C_1 \sin x + C_2 \cos x\right) + \frac{1}{2}$$

$$y'(0) = 0 \quad \rightarrow C_1 + C_2 + \frac{1}{2} = 0 \quad \Rightarrow \boxed{C_2 = -\frac{5}{2}}$$

$$\boxed{y(x) = e^x\left(2\cos x - \frac{5}{2}\sin x\right) + \frac{1}{2}x + 1}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + 2y' + 2y = \sin 3x; \quad y(0) = 2, \quad y'(0) = 0$$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 2 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{-2 \pm 2i}{2} = \boxed{-1 \pm i}$

$$\boxed{y_h = e^{-x}\left(C_1 \cos x + C_2 \sin x\right)}$$

$$y_P = A\cos 3x + B\sin 3x$$

$$y_P' = -3A\sin 3x + 3B\cos 3x$$

$$y_P'' = -9A\cos 3x - 9B\sin 3x$$

$$y'' + 2y' + 2y = \sin 3x$$

$$-9A\cos 3x - 9B\sin 3x - 6A\sin 3x + 6B\cos 3x + 2A\cos 3x + 2B\sin 3x = \sin 3x$$

$$\begin{cases} \cos 3x & -7A + 6B = 0 \\ \sin 3x & -6A - 7B = 1 \end{cases}$$

$$\rightarrow \Delta = \begin{vmatrix} -7 & 6 \\ -6 & -7 \end{vmatrix} = 85 \quad \Delta_A = \begin{vmatrix} 0 & 6 \\ 1 & -7 \end{vmatrix} = -6 \quad \Delta_B = \begin{vmatrix} -7 & 0 \\ -6 & 1 \end{vmatrix} = -7$$

$$\boxed{A = -\frac{6}{85},\ B = -\frac{7}{85}}$$

$$\boxed{y_P = -\frac{6}{85}\cos 3x - \frac{7}{85}\sin 3x}$$

$$y(x) = e^{-x}\left(C_1 \cos x + C_2 \sin x\right) - \frac{6}{85}\cos 3x - \frac{7}{85}\sin 3x$$

$$y(0)=2 \quad \to C_1 - \frac{6}{85} = 2 \quad \boxed{C_1 = \frac{176}{85}}$$

$$y'(x) = e^{-x}\left(-C_1 \sin x + C_2 \cos x - C_1 \cos x - C_2 \sin x\right) + \frac{18}{85}\sin 3x - \frac{21}{85}\cos 3x$$

$$y'(0)=0 \quad \to C_2 - C_1 - \frac{21}{85} = 0 \quad \boxed{C_2 = \frac{106}{85}}$$

$$\boxed{y(x) = e^{-x}\left(\frac{176}{85}\cos x + \frac{106}{85}\sin x\right) - \frac{6}{85}\cos 3x - \frac{7}{85}\sin 3x}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + 2y' + 2y = 2\cos 2t; \quad y(0) = -2, \quad y'(0) = 0$$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 2 = 0 \quad \to \quad \lambda_{1,2} = \frac{-2 \pm 2i}{2} \; \boxed{= -1 \pm i}$

The homogenous solution: $y_h = e^{-t}\left(C_1 \cos t + C_2 \sin t\right)$

The particular solution: $z = Ae^{i2t}$

$$z' = (2i)Ae^{i2t}$$

$$z'' = (2i)^2 Ae^{i2t}$$

$$z'' + 2z' + 2z = 2e^{i2t}$$

$$(2i)^2 Ae^{i2t} + 2(2i)Ae^{i2t} + 2Ae^{i2t} = 2e^{i2t}$$

$$(-4 + 4i + 2)A = 2$$

$$(-2 + 4i)A = 2$$

$$\begin{aligned} A &= \frac{2}{-2+4i} \cdot \frac{-2-4i}{-2-4i} \\ &= \frac{-4-8i}{20} \\ &= \frac{-4}{20} - \frac{8i}{20} \\ &= -\frac{1}{5} - \frac{2}{5}i \end{aligned}$$

This gives the particular solution:

$$\begin{aligned} z &= \left(-\frac{1}{5} - \frac{2}{5}i\right)e^{i2t} \\ &= \left(-\frac{1}{5} - \frac{2}{5}i\right)(\cos 2t + i\sin 2t) \\ &= -\frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t - i\left(\frac{1}{5}\sin 2t + \frac{2}{5}\cos 2t\right) \end{aligned}$$

The real part of this solution $\left(-\frac{1}{5}\cos 2t+\frac{2}{5}\sin 2t\right)$ is the particular solution of the system.

Thus, the general solution is:

$$y=e^{-t}\left(C_1\cos t+C_2\sin t\right)-\frac{1}{5}\cos 2t+\frac{2}{5}\sin 2t$$

The initial condition: $y(0)=-2$

$$-2=e^{-0}\left(C_1\cos 0+C_2\sin 0\right)-\frac{1}{5}\cos 0+\frac{2}{5}\sin 0$$

$$-2=C_1-\frac{1}{5}\quad\Rightarrow\quad \boxed{C_1=-\frac{9}{5}}$$

$$y'=e^{-t}\left(-C_1\sin t+C_2\cos t\right)-e^{-t}\left(C_1\cos t+C_2\sin t\right)+\frac{2}{5}\sin 2t+\frac{4}{5}\cos 2t$$

$$=-C_1e^{-t}\sin t+C_2e^{-t}\cos t-C_1e^{-t}\cos t-C_2e^{-t}\sin t+\frac{2}{5}\sin 2t+\frac{4}{5}\cos 2t$$

$$=\left(C_2-C_1\right)e^{-t}\cos t-\left(C_1+C_2\right)e^{-t}\sin t+\frac{2}{5}\sin 2t+\frac{4}{5}\cos 2t$$

The initial condition: $y'(0)=0$

$$0=\left(C_2-C_1\right)e^{-0}\cos 0-\left(C_1+C_2\right)e^{-0}\sin 0+\frac{2}{5}\sin 0+\frac{4}{5}\cos 0$$

$$0=C_2-C_1+\frac{4}{5}$$

$$C_2=C_1-\frac{4}{5}=-\frac{9}{5}-\frac{4}{5}=-\frac{13}{5}$$

The general solution: $\quad y(t)=e^{-t}\left(-\frac{9}{5}\cos t-\frac{13}{5}\sin t\right)-\frac{1}{5}\cos 2t+\frac{2}{5}\sin 2t$

Exercise

Find the solution of the given initial value problem

$$y''-2y'-3y=2e^{x}-10\sin x\ ;\quad y(0)=2,\quad y'(0)=4$$

Solution

The characteristic equation: $\lambda^2-2\lambda-3=0 \quad\to\quad \lambda_{1,2}=-1,\ 3$

$$y_h=C_1e^{-x}+C_2e^{3x}$$

$$y_p=Ae^{x}+B\cos x+C\sin x$$

$$y_p'=Ae^{x}-B\sin x+C\cos x$$

$$y_p''=Ae^{x}-B\cos x-C\sin x$$

$$y''-2y'-3y=2e^{x}-10\sin x$$

$$\begin{cases} e^x & A-2A-3A=2 \quad \rightarrow \boxed{A=-\frac{1}{2}} \\ \cos x & -B-2C-3B=0 \\ \sin x & -C+2B-3C=-10 \end{cases}$$

$$\begin{cases} -4B-2C=0 \\ 2B-4C=-10 \end{cases} \rightarrow \boxed{B=-\frac{20}{20}=-1} \quad \boxed{C=2}$$

$$\boxed{y_p=-\frac{1}{2}e^x-\cos x+2\sin x}$$

$$y(x)=C_1e^{-x}+C_2e^{3x}-\frac{1}{2}e^x-\cos x+2\sin x$$

$$y(0)=2 \quad \rightarrow C_1+C_2-\frac{1}{2}-1=2 \quad \Rightarrow C_1+C_2=\frac{7}{2}$$

$$y'=-C_1e^{-x}+3C_2e^{3x}-\frac{1}{2}e^x+\sin x+2\cos x$$

$$y'(0)=4 \quad \rightarrow -C_1+3C_2-\frac{1}{2}+2=4 \quad \Rightarrow -C_1+3C_2=\frac{5}{2}$$

$$\begin{cases} 2C_1+2C_2=7 \\ -2C_1+6C_2=5 \end{cases} \rightarrow \boxed{C_1=\frac{32}{16}=2} \quad \boxed{C_2=\frac{24}{16}=\frac{3}{2}}$$

$$\boxed{y(x)=2e^{-x}+\frac{3}{2}e^{3x}-\frac{1}{2}e^x-\cos x+2\sin x}$$

Exercise

Find the solution of the given initial value problem

$$y''+2y'+10y=4+26x+6x^2+10x^3 \;;\quad y(0)=2, \quad y'(0)=9$$

Solution

The characteristic equation: $\lambda^2+2\lambda+10=0 \rightarrow \boxed{\lambda_{1,2}=-1\pm 3i}$

$$\boxed{y_h(x)=e^{-x}\left(C_1\cos 3x+C_2\sin 3x\right)}$$

$$y_p=Ax^3+Bx^2+Cx+D$$

$$y'_p=3Ax^2+2Bx+C$$

$$y''_p=6Ax+2B$$

$$y''+2y'+10y=4+26x+6x^2+10x^3$$

$$\begin{cases} x^3 & 10A = 10 & \rightarrow \boxed{A = 1} \\ x^2 & 6A + 10B = 6 & \rightarrow \boxed{B = 0} \\ x & 6A + 4B + 10C = 26 & \rightarrow \boxed{C = 2} \\ x^0 & 2B + 2C + 10D = 4 & \rightarrow \boxed{D = 0} \end{cases}$$

$$\boxed{y_p = x^3 + 2x}$$

$$y(x) = e^{-x}\left(C_1 \cos 3x + C_2 \sin 3x\right) + x^3 + 2x$$

$$y(0) = 2 \quad \rightarrow \quad \boxed{C_1 = 2}$$

$$y' = e^{-x}\left(-C_1 \cos 3x - C_2 \sin 3x - 3C_1 \sin 3x + 3C_2 \cos 3x\right) + 3x^2 + 2$$

$$y'(0) = 9 \quad \rightarrow \quad -C_1 + 3C_2 + 2 = 9 \quad \Rightarrow \quad \boxed{C_2 = 3}$$

$$\boxed{y(x) = e^{-x}\left(2\cos 3x + 3\sin 3x\right) + x^3 + 2x}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - 2y' + 10y = 6\cos 3t - \sin 3t \;; \quad y(0) = 2, \quad y'(0) = -8$$

Solution

$$\lambda^2 - 2\lambda + 10 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = 1 \pm 3i}$$

$$\boxed{y_h(t) = e^{t}\left(C_1 \cos 3t + C_2 \sin 3t\right)}$$

$$y_p = A\cos 3t + B\sin 3t$$

$$y_p' = -3A\sin 3t + 3B\cos 3t$$

$$y_p'' = -9A\cos 3t - 9B\sin 3t$$

$$y'' - 2y' + 10y = 6\cos 3t - \sin 3t$$

$$\begin{cases} \cos 3t & -9A - 6B + 10A = 6 \\ \sin 3t & -9B + 6A + 10B = -1 \end{cases} \rightarrow \begin{cases} A - 6B = 6 \\ 6A + B = -1 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & -6 \\ 6 & 1 \end{vmatrix} = 37 \quad \Delta_A = \begin{vmatrix} 6 & -6 \\ -1 & 1 \end{vmatrix} = 0 \quad \Delta_B = \begin{vmatrix} 1 & 6 \\ 6 & -1 \end{vmatrix} = -37$$

$$\boxed{A = 0 \quad B = -1}$$

$$\boxed{y_p(t) = -\sin 3t}$$

$$y(t) = e^{t}\left(C_1 \cos 3t + C_2 \sin 3t\right) - \sin 3t$$

$$y(0)=2 \quad \to \quad \boxed{C_1=2}$$

$$y'=e^t\left(C_1\cos 3t+C_2\sin 3t-3C_1\sin 3t+3C_2\cos 3t\right)-3\cos 3t$$

$$y'(0)=-8 \quad \to C_1+3C_2-3=-8 \quad \Rightarrow \quad \boxed{C_2=-\frac{7}{3}}$$

$$\boxed{y(t)=e^t\left(2\cos 3t-\frac{7}{3}\sin 3t\right)-\sin 3t}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y''+3y'+2y=e^x; \quad y(0)=0, \quad y'(0)=3$$

Solution

The characteristic equation: $\lambda^2+3\lambda+2=0 \quad \to \quad \boxed{\lambda_{1,2}=-2,\ -1}$

$$\boxed{y_h=C_1e^{-x}+C_2e^{-2x}}$$

$$y_P=Ae^x$$

$$y_P'=Ae^x$$

$$y_P''=Ae^x$$

$$y''+3y'+2y=e^x$$

$$(A+3A+2A)e^x=e^x \qquad \to \quad \boxed{A=\frac{1}{6}}$$

$$\boxed{y_P=\frac{1}{6}e^x}$$

$$y(x)=C_1e^{-x}+C_2e^{-2x}+\frac{1}{6}e^x$$

$$y(0)=0 \quad \to C_1+C_2=-\frac{1}{6}$$

$$y'(x)=-C_1e^{-x}-2C_2e^{-2x}+\frac{1}{6}e^x$$

$$y'(0)=3 \quad \to -C_1-2C_2+\frac{1}{6}=3 \quad \Rightarrow \quad C_1+2C_2=-\frac{17}{6}$$

$$\begin{cases} C_1+C_2=-\frac{1}{6} \\ C_1+2C_2=\frac{17}{6} \end{cases} \to \Delta=\begin{vmatrix}1 & 1\\ 1 & 2\end{vmatrix}=1 \quad \Delta_{C_1}=\begin{vmatrix}-\frac{1}{6} & 1\\ -\frac{17}{6} & 2\end{vmatrix}=\frac{5}{2} \quad \Delta_{C_2}=\begin{vmatrix}1 & -\frac{1}{6}\\ 1 & -\frac{17}{6}\end{vmatrix}=-\frac{8}{3}$$

$$\boxed{C_1=\frac{5}{2},\ C_2=-\frac{8}{3}}$$

$$\boxed{y(x)=\frac{5}{2}e^{-x}-\frac{8}{3}e^{-2x}+\frac{1}{6}e^x}$$

Exercise

Find the general solution $\quad y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x \quad y(0) = 1 \quad y'(0) = 2$

Solution

$$\lambda^2 - 3\lambda + 2 = 0 \quad \rightarrow \quad \lambda_{1,2} = 1,\ 2$$

$$y_h = C_1 e^x + C_2 e^{2x}$$

$$y_p = Ae^{-x} + B\cos 3x + C\sin 3x$$

$$y_p' = -Ae^{-x} - 3B\sin 3x + 3C\cos 3x$$

$$y_p'' = Ae^{-x} - 9B\cos 3x - 9C\sin 3x$$

$$Ae^{-x} - 9B\cos 3x - 9C\sin 3x + 3Ae^{-x} + 9B\sin 3x - 9C\cos 3x + 2Ae^{-x} + 2B\cos 3x + 2C\sin 3x = 3e^{-x} - 10\cos 3x$$

$$6A = 3 \qquad A = \frac{1}{2}$$

$$-7B - 9C = -10 \qquad B = \frac{7}{13}$$

$$9B - 7C = 0 \qquad C = \frac{9}{13}$$

$$\Delta = \begin{vmatrix} 7 & 9 \\ 9 & -7 \end{vmatrix} = -130 \quad \Delta_B = \begin{vmatrix} 10 & 9 \\ 0 & -7 \end{vmatrix} = -70 \quad \Delta_C = \begin{vmatrix} 7 & 10 \\ 9 & 0 \end{vmatrix} = -90$$

$$\therefore\ \underline{y(x) = C_1 e^x + C_2 e^{2x} + \frac{1}{2}e^{-x} + \frac{7}{13}\cos 3x + \frac{9}{13}\sin 3x}$$

$$y(0) = C_1 + C_2 + \frac{1}{2} + \frac{7}{13} = 1$$

$$y'(x) = C_1 e^x + 2C_2 e^{2x} - \frac{1}{2}e^{-x} - \frac{21}{13}\sin 3x + \frac{27}{13}\cos 3x \ \rightarrow\ y'(0) = C_1 + 2C_2 - \frac{1}{2} + \frac{27}{13} = 2$$

$$\begin{cases} C_1 + C_2 = -\frac{1}{26} \\ C_1 + 2C_2 = \frac{11}{26} \end{cases} \quad \underline{C_2 = \frac{12}{26} = \frac{6}{13}} \quad \underline{C_1 = -\frac{1}{26} - \frac{6}{13} = -\frac{1}{2}}$$

$$\therefore\ \underline{y(x) = -\frac{1}{2}e^x + \frac{6}{13}e^{2x} + \frac{1}{2}e^{-x} + \frac{7}{13}\cos 3x + \frac{9}{13}\sin 3x}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + 4y = -2\ ; \quad y\left(\frac{\pi}{8}\right) = \frac{1}{2}, \quad y'\left(\frac{\pi}{8}\right) = 2$$

Solution

The characteristic equation: $\lambda^2 + 4 = 0 \ \rightarrow\ \underline{\lambda_{1,2} = \pm 2i}$

$$\underline{y_h = C_1\cos 2x + C_2\sin 2x}$$

$y_P = A$

$y'_P = y''_P = 0$

$y'' + 4y = -2$

$4A = -2 \quad \rightarrow \quad \boxed{A = -\frac{1}{2}}$

$\boxed{y_P = -\frac{1}{2}}$

$y(x) = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{2}$

$y\left(\frac{\pi}{8}\right) = \frac{1}{2} \quad \rightarrow \frac{\sqrt{2}}{2}C_1 + \frac{\sqrt{2}}{2}C_2 - \frac{1}{2} = \frac{1}{2} \quad \Rightarrow \sqrt{2}C_1 + \sqrt{2}C_2 = 2$

$y'(x) = -2C_1 \sin 2x + 2C_2 \cos 2x$

$y\left(\frac{\pi}{8}\right) = 2 \quad \rightarrow \quad -\sqrt{2}C_1 + \sqrt{2}C_2 = 2$

$$\begin{cases} \sqrt{2}C_1 + \sqrt{2}C_2 = 2 \\ -\sqrt{2}C_1 + \sqrt{2}C_2 = 2 \end{cases} \rightarrow \quad \boxed{C_2 = \sqrt{2},\ C_1 = 0}$$

$\boxed{y(x) = \sqrt{2}\sin 2x - \frac{1}{2}}$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + 4y = 2x\ ; \quad y(0) = 1, \quad y'(0) = 2$$

Solution

The characteristic equation: $\lambda^2 + 4 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 2i}$

$\boxed{y_h = C_1 \cos 2x + C_2 \sin 2x}$

$y_P = Ax + B$

$y'_P = A$

$y''_P = 0$

$y'' + 4y = 2x$

$4Ax + B = 2x$

$$\begin{cases} 4A = 2 \\ B = 0 \end{cases} \rightarrow \quad \boxed{A = \frac{1}{2},\ B = 0}$$

$\boxed{y_P = \frac{1}{2}x}$

$y(x) = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{2}x$

$$y(0)=1 \quad \to \quad \boxed{C_1 = 1}$$

$$y'(x) = -2C_1 \sin 2x + 2C_2 \cos 2x + \frac{1}{2}$$

$$y'(0)=2 \quad \to 2C_2 + \frac{1}{2} = 2 \quad \boxed{C_2 = \frac{3}{4}}$$

$$\boxed{y(x) = \cos 2x + \frac{3}{4}\sin 2x + \frac{1}{2}x}$$

Exercise

Find the solution of the given initial value problem

$$y'' - 4y' + 4y = e^x \;; \quad y(0)=2, \quad y'(0)=0$$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 4 = 0 \quad \to \quad \boxed{\lambda_{1,2} = 2}$

$$\boxed{y_h = (C_1 + C_2 x)e^{2x}}$$

$$y_p = Ae^x$$

$$y'_p = Ae^x$$

$$y''_p = Ae^x$$

$$y'' - 4y' + 4y = e^x$$

$$\to \boxed{A = 1}$$

$$\boxed{y_p = e^x}$$

$$y(x) = (C_1 + C_2 x)e^{2x} + e^x$$

$$y(0)=2 \quad \to C_1 + 1 = 2 \quad \Rightarrow \boxed{C_1 = 1}$$

$$y' = (C_2 + 2C_1 + 2C_2 x)e^{2x} + e^x$$

$$y'(0)=0 \quad \to C_2 + 2C_1 + 1 = 0 \quad \Rightarrow \boxed{C_2 = -3}$$

$$\boxed{y(x) = (1-3x)e^{2x} + e^x}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - 4y' + 8y = x^3; \quad y(0)=2, \quad y'(0)=4$$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 8 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{4 \pm 4i}{2} \underline{= 2 \pm 2i}$

$$\underline{y_h = e^{2x}\left(C_1 \cos 2x + C_2 \sin 2x\right)}$$

$$y_P = Ax^3 + Bx^2 + Cx + D$$

$$y'_P = 3Ax^2 + 2Bx + C$$

$$y''_P = 6Ax + 2B$$

$$y'' - 4y' + 8y = x^3$$

$$\begin{cases} x^3 & 8A = 1 \\ x^2 & -12A + 8B = 0 \\ x & 6A - 8B + 8C = 0 \\ x^0 & 2B - 4C + 8D = 0 \end{cases} \rightarrow \underline{A = \tfrac{1}{8},\ B = \tfrac{3}{16},\ C = \tfrac{3}{32},\ D = 0}$$

$$\underline{y_P = \tfrac{1}{8}x^3 + \tfrac{3}{16}x^2 + \tfrac{3}{32}x}$$

$$y(x) = e^{2x}\left(C_1 \cos 2x + C_2 \sin 2x\right) + \tfrac{1}{8}x^3 + \tfrac{3}{16}x^2 + \tfrac{3}{32}x$$

$$y(0) = 2 \quad \rightarrow \quad \underline{C_1 = 2}$$

$$y'(x) = e^{2x}\left(2C_1 \cos 2x + 2C_2 \sin 2x - 2C_1 \sin 2x + 2C_2 \cos 2x\right) + \tfrac{3}{8}x^2 + \tfrac{3}{8}x + \tfrac{3}{32}$$

$$y'(0) = 4 \quad \rightarrow \quad 2C_1 + 2C_2 + \tfrac{3}{32} = 4 \quad \underline{C_2 = -\tfrac{3}{64}}$$

$$\underline{y(x) = e^{2x}\left(2\cos 2x - \tfrac{3}{64}\sin 2x\right) + \tfrac{1}{8}x^3 + \tfrac{3}{16}x^2 + \tfrac{3}{32}x}$$

Exercise

Find the general solution $\quad y'' + 4y = \sin^2 2t; \quad x\left(\frac{\pi}{8}\right) = 0 \quad x'\left(\frac{\pi}{8}\right) = 0$

Solution

Characteristic Eqn.: $\lambda^2 + 4 = 0 \quad \Rightarrow \quad \underline{\lambda_{1,2} = \pm 2i}$

$$\underline{y_h = C_1 \cos 2t + C_2 \sin 2t}$$

$$y'' + 4y = \tfrac{1}{2} - \tfrac{1}{2}\cos 4t$$

$$y_p = A + B\cos 4t + D\sin 4t$$

$$y'_p = -4B\sin 4t + 4D\cos 4t$$

$$y''_p = -16B\cos 4t - 16D\sin 4t$$

$$-16B\cos 4t - 16D\sin 4t + 4A + 4B\cos 4t + 4D\sin 4t = \frac{1}{2} - \frac{1}{2}\cos 4t$$

$$\begin{cases} & 4A = \frac{1}{2} & \boxed{A = \frac{1}{8}} \\ \cos 4t & -12B = -\frac{1}{2} & \boxed{B = \frac{1}{24}} \\ \sin 4t & -12D = 0 & \boxed{D = 0} \end{cases}$$

$$\boxed{y_p = \frac{1}{8} + \frac{1}{24}\cos 4t}$$

$$y(t) = C_1\cos 2t + C_2\sin 2t + \frac{1}{8} + \frac{1}{24}\cos 4t$$

$$x\left(\frac{\pi}{8}\right) = 0 \quad \to \quad \frac{1}{\sqrt{2}}C_1 + \frac{1}{\sqrt{2}}C_2 + \frac{1}{8} = 0 \quad \Rightarrow \quad \boxed{C_1 + C_2 = -\frac{\sqrt{2}}{8}}$$

$$y' = -2C_1\sin 2t + 2C_2\cos 2t - \frac{1}{6}\sin 4t$$

$$x\left(\frac{\pi}{8}\right) = 0 \quad \to \quad -\sqrt{2}C_1 + \sqrt{2}C_2 - \frac{1}{6} = 0 \quad \Rightarrow \quad \boxed{C_1 - C_2 = \frac{\sqrt{2}}{12}}$$

$$\begin{cases} C_1 + C_2 = -\frac{\sqrt{2}}{8} \\ C_1 - C_2 = \frac{\sqrt{2}}{12} \end{cases} \quad \to \quad C_1 = -\frac{\sqrt{2}}{48}, \quad C_2 = -\frac{5\sqrt{2}}{48}$$

The ***general*** solution:

$$\boxed{y(t) = -\frac{5\sqrt{2}}{48}\cos 2t - \frac{\sqrt{2}}{48}\sin 2t + \frac{1}{8} + \frac{1}{24}\cos 4t}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + 4y' + 4y = (3 + x)e^{-2x} \ ; \quad y(0) = 2, \quad y'(0) = 5$$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 4 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = -2}$

$$\boxed{y_h = (C_1 + C_2x)e^{-2x}}$$

$$y_P = \left(Ax^3 + Bx^2\right)e^{-2x}$$

$$y'_P = \left(-2Ax^3 - 2Bx^2 + 3Ax^2 + 2Bx\right)e^{-2x}$$

$$y''_P = \left(4Ax^3 + 4Bx^2 - 6Ax^2 - 4Bx - 6Ax^2 - 4Bx + 6Ax + 2B\right)e^{-2x}$$

$$= \left(4Ax^3 + 4Bx^2 - 12Ax^2 - 8Bx + 6Ax + 2B\right)e^{-2x}$$

$$y'' + 4y' + 4y = (3 + x)e^{-2x}$$

$$x^3 \qquad 4A-8A+4A=0$$

$$x^2 \qquad 4B-12A-8B+12A+4B=0$$

$$x \qquad -8B+6A+8B=1 \qquad \rightarrow \quad \boxed{A=\frac{1}{6},\ B=\frac{3}{2}}$$

$$x^0 \qquad 2B=3$$

$$\boxed{y_P=\left(\frac{1}{6}x^3+\frac{3}{2}x^2\right)e^{-2x}}$$

$$y(x)=\left(C_1+C_2x+\frac{3}{2}x^2+\frac{1}{6}x^3\right)e^{-2x}$$

$$y(0)=2 \quad \rightarrow \quad \boxed{C_1=2}$$

$$y'(x)=\left(C_2+3x+\frac{1}{2}x^2-2C_1-2C_2x-3x^2-\frac{1}{3}x^3\right)e^{-2x}$$

$$y'(0)=5 \quad \rightarrow C_2-2C_1=5 \quad \boxed{C_2=9}$$

$$\therefore\ \boxed{y(x)=\left(2+9x+\frac{3}{2}x^2+\frac{1}{6}x^3\right)e^{-2x}}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y''+4y'+4y=4-t; \qquad y(0)=-1, \quad y'(0)=0$$

Solution

The characteristic *eqn.* : $\lambda^2+4\lambda+4=0 \ \Rightarrow\ \boxed{\lambda_{1,2}=-2}$

$$\boxed{y_h=\left(C_1+C_2t\right)e^{-2t}}$$

$$y_p=at+b$$

$$y'_p=a$$

$$y''_p=0$$

$$y''+4y'+4y=4a+4at+4b=4at+4a+4b$$

$$\begin{cases} 4a=-1 & \Rightarrow \quad a=-\frac{1}{4} \\ 4a+4b=4 & \Rightarrow \quad b=\frac{1}{4} \end{cases}$$

The particular solution is: $y_p=-\frac{1}{4}t+\frac{1}{4}$

The general solution is: $y(t)=\left(C_1+C_2t\right)e^{-2t}-\frac{1}{4}t+\frac{1}{4}$

$$y(0)=\left(C_1+C_2 0\right)e^{-2(0)}-\frac{1}{4}0+\frac{1}{4}$$

$$-1=C_1+\frac{1}{4} \quad \Rightarrow \quad \boxed{C_1=-\frac{5}{4}}$$

$$y'(t) = C_2 e^{-2t} - 2\left(C_1 + C_2 t\right)e^{-2t} - \frac{1}{4}$$

$$0 = C_2 - 2C_1 - \frac{1}{4}$$

$$C_2 = -\frac{10}{4} + \frac{1}{4} \quad \Rightarrow \quad \boxed{C_2 = -\frac{9}{4}}$$

The general solution is: $\boxed{y(t) = \left(C_1 + C_2 t\right)e^{-2t} - \frac{1}{4}t + \frac{1}{4}}$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - 4y' - 5y = 4e^{-2t}; \quad y(0) = 0, \quad y'(0) = -1$$

Solution

The *characteristic* eq.: $\lambda^2 - 4\lambda - 5 = 0 \quad \Rightarrow \lambda_1 = 5, \lambda_2 = -1$

$$y_h = C_1 e^{5t} + C_2 e^{-t}$$

$$y_p = Ae^{-2t}$$

$$y' = -2Ae^{-2t}$$

$$y'' = 4Ae^{-2t}$$

$$y'' - 4y' - 5y = 4e^{-2t}$$

$$4Ae^{-2t} + 8Ae^{-2t} - 5Ae^{-2t} = 4e^{-2t}$$

$$7A = 4$$

$$A = \frac{4}{7}$$

$$y_p = \frac{4}{7}e^{-2t}$$

$$y = C_1 e^{5t} + C_2 e^{-t} + \frac{4}{7}e^{-2t}$$

$$y(0) = C_1 e^{5(0)} + C_2 e^{-(0)} + \frac{4}{7}e^{-2(0)}$$

$$0 = C_1 + C_2 + \frac{4}{7}$$

$$C_1 + C_2 = -\frac{4}{7} \qquad \textbf{(1)}$$

$$y' = 5C_1 e^{5t} - C_2 e^{-t} - \frac{8}{7}e^{-2t}$$

$$y'(0) = 5C_1 e^{5(0)} - C_2 e^{-(0)} - \frac{8}{7}e^{-2(0)}$$

$$-1 = 5C_1 - C_2 - \frac{8}{7}$$

$$5C_1 - C_2 = \frac{1}{7} \qquad \textbf{(2)}$$

$$\boxed{C_1 = -\frac{1}{14}} \text{ and } \boxed{C_2 = -\frac{1}{2}}$$

$$\boxed{y(t) = -\frac{1}{14}e^{5t} - \frac{1}{2}e^{-t} + \frac{4}{7}e^{-2t}}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + 4y' + 5y = 35e^{-4x} \; ; \quad y(0) = -3, \quad y'(0) = 1$$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 5 = 0 \;\Rightarrow\; \lambda_{1,2} = \frac{-4 \pm 2i}{2} = \boxed{-2 \pm i}$

$$\boxed{y_h = \left(C_1 \cos x + C_2 \sin x\right)e^{-2x}}$$

$$y_P = Ae^{-4x}$$

$$y'_P = -4Ae^{-4x}$$

$$y''_P = 16Ae^{-4x}$$

$$y'' + 4y' + 5y = 35e^{-4x}$$

$$16A - 16A + 5A = 35 \quad \to \quad \boxed{A = 7}$$

$$\boxed{y_P = 7e^{-4x}}$$

$$y(x) = \left(C_1 \cos x + C_2 \sin x\right)e^{-2x} + 7e^{-4x}$$

$$y(0) = -3 \quad \to C_1 + 7 = -3 \quad \boxed{C_1 = -10}$$

$$y'(x) = \left(-C_1 \sin x + C_2 \cos x - 2C_1 \cos x - 2C_2 \sin x\right)e^{-2x} - 28e^{-4x}$$

$$y'(0) = 1 \quad \to \; C_2 - 2C_1 - 28 = 1 \quad \boxed{C_2 = 9}$$

$$\boxed{y(x) = \left(-10\cos x + 9\sin x\right)e^{-2x} + 7e^{-4x}}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + 4y' + 8y = \sin t \; ; \quad y(0) = 1, \quad y'(0) = 0$$

Solution

$$\lambda^2 + 4\lambda + 8 = 0 \quad \Rightarrow \boxed{\lambda_{1,2} = -2 \pm 2i}$$

$$\boxed{y_h = \left(C_1 \cos 2t + C_2 \sin 2t\right)e^{-2t}}$$

$$y_p = A\cos t + B\sin t$$

$$y_p = -A\sin t + B\cos t$$

$$y_p = -A\cos t - B\sin t$$

$$y'' + 4y' + 8y = \sin t$$

$$\begin{cases} \cos t & -A + 4B + 8A = 0 \\ \sin t & -B - 4A + 8B = 1 \end{cases} \rightarrow \begin{cases} 7A + 4B = 0 \\ -4A + 7B = 1 \end{cases}$$

$$\Delta = \begin{vmatrix} 7 & 4 \\ -4 & 7 \end{vmatrix} = 65 \quad \Delta_A = \begin{vmatrix} 0 & 4 \\ 1 & 7 \end{vmatrix} = -4 \quad \Delta_B = \begin{vmatrix} 7 & 0 \\ -4 & 1 \end{vmatrix} = 7$$

$$A = -\frac{4}{65}, \quad B = \frac{7}{65}$$

$$y_p = -\frac{4}{65}\cos t + \frac{7}{65}\sin t$$

$$y(t) = \left(C_1 \cos 2t + C_2 \sin 2t\right)e^{-2t} - \frac{4}{65}\cos t + \frac{7}{65}\sin t$$

$$y(0) = 1 \quad \rightarrow C_1 - \frac{4}{65} = 1 \quad \Rightarrow C_1 = \frac{69}{65}$$

$$y' = \left(-2C_1 \sin 2t + 2C_2 \cos 2t - 2C_1 \cos 2t - 2C_2 \sin 2t\right)e^{-2t} + \frac{4}{65}\sin t + \frac{7}{65}\cos t$$

$$y(0) = 0 \quad \rightarrow 2C_2 - \frac{138}{65} + \frac{7}{65} = 0 \quad \Rightarrow C_2 = \frac{131}{130}$$

$$y(t) = \left(\frac{69}{65}\cos 2t + \frac{130}{131}\sin 2t\right)e^{-2t} - \frac{4}{65}\cos t + \frac{7}{65}\sin t$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - 4y' - 12y = 3e^{5t} \;;\quad y(0) = \frac{18}{7}, \quad y'(0) = -\frac{1}{7}$$

Solution

$$\lambda^2 - 4\lambda - 12 = 0 \quad \Rightarrow \lambda_{1,2} = -2,\ 6$$

$$y_h = C_1 e^{-2t} + C_2 e^{6t}$$

$$y_p = Ae^{5t}$$

$$y'_p = 5Ae^{5t}$$

$$y''_p = 25Ae^{5t}$$

$$y'' - 4y' - 12y = 3e^{5t}$$

$$25A-20A-12A=3 \quad \rightarrow \quad \boxed{A=-\frac{3}{7}}$$

$$\boxed{y_p=-\frac{3}{7}e^{5t}}$$

$$y(t)=C_1e^{-2t}+C_2e^{6t}-\frac{3}{7}e^{5t}$$

$$y(0)=\frac{18}{7} \quad \rightarrow C_1+C_2-\frac{3}{7}=\frac{18}{7} \quad \Rightarrow \ C_1+C_2=3$$

$$y'=-2C_1e^{-2t}+6C_2e^{6t}-\frac{15}{7}e^{5t}$$

$$y'(0)=-\frac{1}{7} \quad \rightarrow -2C_1+6C_2-\frac{15}{7}=-\frac{1}{7} \quad \Rightarrow -C_1+3C_2=1$$

$$\Delta=\begin{vmatrix}1 & 1\\ -1 & 3\end{vmatrix}=4 \quad \Delta_{C_1}=\begin{vmatrix}3 & 1\\ 1 & 3\end{vmatrix}=8 \quad \Delta_{C_2}=\begin{vmatrix}1 & 3\\ -1 & 1\end{vmatrix}=4$$

$$\boxed{C_1=2 \quad C_2=1}$$

$$\boxed{y(t)=2e^{-2t}+e^{6t}-\frac{3}{7}e^{5t}}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y''-4y'-12y=\sin 2t \ ; \quad y(0)=0, \quad y'(0)=0$$

Solution

$$\lambda^2-4\lambda-12=0 \quad \Rightarrow \ \boxed{\lambda_{1,2}=-2,\ 6}$$

$$\boxed{y_h=C_1e^{-2t}+C_2e^{6t}}$$

$$y_p=A\cos 2t+B\sin 2t$$

$$y_p'=-2A\sin 2t+2B\cos 2t$$

$$y_p''=-4A\cos 2t-4B\sin 2t$$

$$y''-4y'-12y=\sin 2t$$

$$\begin{cases}\cos 2t & -4A-8B-12A=0\\ \sin 2t & -4B+8A-12B=1\end{cases} \quad \rightarrow \begin{cases}2A+B=0\\ 8A-16B=1\end{cases}$$

$$\Delta=\begin{vmatrix}2 & 1\\ 8 & -16\end{vmatrix}=-40 \quad \Delta_A=\begin{vmatrix}0 & 1\\ 1 & -16\end{vmatrix}=-1 \quad \Delta_B=\begin{vmatrix}2 & 0\\ 8 & 1\end{vmatrix}=2$$

$$\boxed{A=\frac{1}{40} \quad B=-\frac{1}{20}}$$

$$\boxed{y_p=\frac{1}{40}\cos 2t-\frac{1}{20}\sin 2t}$$

$$y(t)=C_1e^{-2t}+C_2e^{6t}+\frac{1}{40}\cos 2t-\frac{1}{20}\sin 2t$$

$$y(0)=0 \quad \rightarrow \quad \Rightarrow C_1+C_2=-\frac{1}{40}$$

$$y'=-2C_1e^{-2t}+6C_2e^{6t}-\frac{1}{20}\sin 2t-\frac{1}{10}\cos 2t$$

$$y'(0)=0 \quad \rightarrow -2C_1+6C_2-\frac{1}{10}=0 \quad \Rightarrow \quad -C_1+3C_2=\frac{1}{20}$$

$$\Delta=\begin{vmatrix}1 & 1\\ -1 & 3\end{vmatrix}=4 \quad \Delta_{C_1}=\begin{vmatrix}-\frac{1}{40} & 1\\ \frac{1}{20} & 3\end{vmatrix}=-\frac{1}{8} \quad \Delta_{C_2}=\begin{vmatrix}1 & -\frac{1}{40}\\ -1 & \frac{1}{20}\end{vmatrix}=\frac{1}{40}$$

$$\boxed{C_1=-\frac{1}{32} \quad C_2=\frac{1}{160}}$$

$$\boxed{y(t)=-\frac{1}{32}e^{-2t}+\frac{1}{160}e^{6t}+\frac{1}{40}\cos 2t-\frac{1}{20}\sin 2t}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y''-5y'=t-2 \quad y(0)=0, \quad y'(0)=2$$

Solution

The characteristic equation: $\lambda^2-5\lambda=0 \quad \Rightarrow \quad \lambda_1=0,\ \lambda_2=5$

$$\boxed{y_h=C_1+C_2e^{5t}}$$

The particular equation: $y_p=At+Bt^2 \quad \Rightarrow \quad y'_p=A+2Bt \ \rightarrow\ y''_p=2B$

$$2B-5A-10Bt=t-2$$

$$\begin{cases}-10B=1 & \rightarrow B=-\frac{1}{10}\\ 2B-5A=-2 & \rightarrow A=\frac{1}{5}\left(-\frac{1}{5}+2\right)=\frac{9}{25}\end{cases}$$

$$\Rightarrow \quad \boxed{y_p=-\frac{1}{10}t^2+\frac{9}{25}t}$$

$$y(t)=C_1+C_2e^{5t}-\frac{1}{10}t^2+\frac{9}{25}t$$

$$y(0)=0 \quad \Rightarrow \quad C_1+C_2=0 \Rightarrow \boxed{C_1=-C_2}$$

$$y'(t)=5C_2e^{5t}-\frac{1}{5}t+\frac{9}{25}$$

$$y'(0)=2 \quad \Rightarrow \quad 5C_2+\frac{9}{25}=2$$

$$\boxed{C_2=\frac{1}{5}\left(2-\frac{9}{25}\right)=\frac{41}{125}} \qquad \boxed{C_1=-\frac{41}{125}}$$

$$\boxed{y(t)=-\frac{41}{125}+\frac{41}{125}e^{5t}-\frac{1}{10}t^2+\frac{9}{25}t}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + 5y' - 6y = 10e^{2x} \ ; \quad y(0) = 1, \quad y'(0) = 1$$

Solution

The characteristic equation: $\lambda^2 + 5\lambda - 6 = 0 \Rightarrow \boxed{\lambda_{1,2} = 1, \ -6}$

$$\boxed{y_h = C_1 e^x + C_2 e^{-6x}}$$

$$y_P = Ae^{2x}$$

$$y'_P = 2Ae^{2x}$$

$$y''_P = 4Ae^{2x}$$

$$y'' + 5y' - 6y = 10e^{2x}$$

$$4A + 10A - 6A = 10 \quad \rightarrow \quad \boxed{A = \frac{5}{4}}$$

$$\boxed{y_P = \frac{5}{4}e^{2x}}$$

$$y(x) = C_1 e^x + C_2 e^{-6x} + \frac{5}{4}e^{2x}$$

$$y(0) = 1 \quad \rightarrow C_1 + C_2 + \frac{5}{4} = 1 \quad \Rightarrow \ C_1 + C_2 = -\frac{1}{4}$$

$$y'(x) = C_1 e^x - 6C_2 e^{-6x} + \frac{5}{2}e^{2x}$$

$$y'(0) = 1 \quad \rightarrow \ C_1 - 6C_2 + \frac{5}{2} = 1 \quad C_1 - 6C_2 = -\frac{3}{2}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 1 & -6 \end{vmatrix} = -7 \quad \Delta_{C_1} = \begin{vmatrix} -\frac{1}{4} & 1 \\ -\frac{3}{2} & -6 \end{vmatrix} = 3 \quad \Delta_{C_2} = \begin{vmatrix} 1 & -\frac{1}{4} \\ 1 & -\frac{3}{2} \end{vmatrix} = -\frac{5}{4}$$

$$\boxed{C_1 = -\frac{3}{7}, \ C_2 = \frac{5}{28}}$$

$$\boxed{y(x) = -\frac{3}{7}e^x + \frac{5}{28}e^{-6x} + \frac{5}{4}e^{2x}}$$

Exercise

Find the solution of the given initial value problem

$$y'' + 6y' + 10y = 22 + 20x \ ; \quad y(0) = 2, \quad y'(0) = -2$$

Solution

The characteristic equation: $\lambda^2 + 6\lambda + 10 = 0 \ \rightarrow \ \boxed{\lambda_{1,2} = -3 \pm i}$

$$\boxed{y_h = e^{-3x}\left(C_1 \cos x + C_2 \sin x\right)}$$

$$y_p = Ax + B$$

$$y'_p = A$$

$$y_p = 0$$

$$y'' + 6y' + 10y = 22 + 20x$$

$$\begin{cases} x & 10A = 20 & \rightarrow \boxed{A = 2} \\ x^0 & 6A + 10B = 22 & \rightarrow \boxed{B = 1} \end{cases}$$

$$\boxed{y_p = 2x + 1}$$

$$y(x) = e^{-3x}\left(C_1 \cos x + C_2 \sin x\right) + 2x + 1$$

$$y(0) = 2 \quad \rightarrow C_1 + 1 = 2 \quad \Rightarrow \boxed{C_1 = 1}$$

$$y' = e^{-3x}\left(-3C_1 \cos x - 3C_2 \sin x - C_1 \sin x + C_2 \cos x\right) + 2$$

$$y'(0) = -2 \quad \rightarrow \; -3 + C_2 + 2 = -2 \quad \Rightarrow \boxed{C_2 = -1}$$

$$\therefore \; \boxed{y(x) = e^{-3x}(\cos x - \sin x) + 2x + 1}$$

Exercise

Find the solution of the given initial value problem

$$y'' + 7y' + 12y = -2\cos 2x + 36\sin 2x \; ; \quad y(0) = -3, \quad y'(0) = 3$$

Solution

The characteristic equation: $\lambda^2 + 7\lambda + 12 = 0 \rightarrow \boxed{\lambda_{1,2} = -4, \; -3}$

$$\boxed{y_h = C_1 e^{-4x} + C_2 e^{-3x}}$$

$$y_p = A\cos 2x + B\sin 2x$$

$$y'_p = -2A\sin 2x + 2B\cos 2x$$

$$y_p = -4A\cos 2x - 4B\sin 2x$$

$$y'' + 7y' + 12y = -2\cos 2x + 36\sin 2x$$

$$\begin{cases} \cos 2x & -4A + 14B + 12A = -2 & \rightarrow 4A + 7B = -1 \\ \sin 2x & -4B - 14A + 12B = 36 & \rightarrow -7A + 4B = 18 \end{cases}$$

$$A = -\frac{130}{65} \boxed{= -2} \qquad B = \frac{65}{65} \boxed{= 1}$$

$$\boxed{y_p = -2\cos 2x + \sin 2x}$$

$$y(x) = C_1 e^{-4x} + C_2 e^{-3x} - 2\cos 2x + \sin 2x$$

$y(0)=-3 \quad \rightarrow C_1+C_2-2=-3 \quad \Rightarrow \quad C_1+C_2=-1$

$y'=-4C_1e^{-4x}-3C_2e^{-3x}+4\sin 2x+2\cos 2x$

$y'(0)=3 \quad \rightarrow \quad -4C_1-3C_2+2=3 \quad \Rightarrow \quad -4C_1-3C_2=1$

$$\begin{cases} C_1+C_2=-1 \\ -4C_1-3C_2=1 \end{cases} \rightarrow \quad \underline{C_1=2} \quad \underline{C_2=-3}$$

$\underline{y(x)=2e^{-4x}-3e^{-3x}-2\cos 2x+\sin 2x}$

Exercise

Find the solution of the given initial value problem

$$y''+8y'+7y=10e^{-2x} \; ; \quad y(0)=-2, \quad y'(0)=10$$

Solution

The characteristic equation: $\lambda^2+8\lambda+7=0 \quad \rightarrow \quad \underline{\lambda_{1,2}=-1, \ -7}$

$\underline{y_h=C_1e^{-x}+C_2e^{-7x}}$

$y_p=Ae^{-2x}$

$y'_p=-2Ae^{-2x}$

$y''_p=4Ae^{-2x}$

$y''+8y'+7y=10e^{-2x}$

$(4-16+7)Ae^{-2x}=10e^{-2x} \quad \rightarrow \quad \underline{A=-2}$

$\underline{y_p=-2e^{-2x}}$

$y(x)=C_1e^{-x}+C_2e^{-7x}-2e^{-2x}$

$y(0)=-2 \quad \rightarrow C_1+C_2-2=-2 \quad \Rightarrow \underline{C_1+C_2=0}$

$y'=-C_1e^{-x}-7C_2e^{-7x}+4e^{-2x}$

$y'(0)=10 \quad \rightarrow -C_1-7C_2+4=10 \quad \Rightarrow \quad \underline{-C_1-7C_2=6}$

$$\begin{cases} C_1+C_2=0 \\ C_1+7C_2=-6 \end{cases} \rightarrow \quad C_1=\frac{6}{6}\underline{=1} \quad \underline{C_1=-1}$$

$\underline{y(x)=e^{-x}-e^{-7x}-2e^{-2x}}$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' + 9y = \sin 2x; \quad y(0) = 1, \quad y'(0) = 0$$

Solution

The characteristic equation: $\lambda^2 + 9 = 0 \rightarrow \boxed{\lambda_{1,2} = \pm 3i}$

$\boxed{y_h = C_1 \cos 3x + C_2 \sin 3x}$

$y_P = A\cos 2x + B\sin 2x$

$y'_P = -2A\sin 2x + 2B\cos 2x$

$y''_P = -4A\cos 2x - 4B\sin 2x$

$y'' + 9y = \sin 2x$

$-4A\cos 2x - 4B\sin 2x + 9A\cos 2x + 9B\sin 2x = \sin 2x$

$$\begin{cases} \cos 2x & 5A = 0 \\ \sin 2x & 5B = 1 \end{cases} \rightarrow \boxed{A = 0,\ B = \frac{1}{5}}$$

$y_P = \frac{1}{5}\sin 2x$

$y(x) = C_1 \cos 3x + C_2 \sin 3x + \frac{1}{5}\sin 2x$

$y(0) = 1 \rightarrow \boxed{C_1 = 1}$

$y'(x) = -3C_1 \sin 3x + 3C_2 \cos 3x + \frac{2}{5}\cos 2x$

$y'(0) = 0 \rightarrow 3C_2 + \frac{2}{5} = 0 \quad \boxed{C_2 = -\frac{2}{15}}$

$\boxed{y(x) = \cos 3x - \frac{2}{15}\sin 3x + \frac{1}{5}\sin 2x}$

Exercise

Find the general solution that satisfy the given initial conditions

$$y'' - 64y = 16; \quad y(0) = 1, \quad y'(0) = 0$$

Solution

The characteristic equation: $\lambda^2 - 64 = 0 \Rightarrow \boxed{\lambda_{1,2} = \pm 8}$

$\boxed{y_h = C_1 e^{-8x} + C_2 e^{8x}}$

The particular equation: $y_p = A \Rightarrow y'_p = y''_p = 0$

$-64A = 16 \Rightarrow A = -\frac{1}{4} \Rightarrow \boxed{y_p = -\frac{1}{4}}$

$y(x) = C_1 e^{-8x} + C_2 e^{8x} - \frac{1}{4}$

$$y(0)=1 \quad \to \quad C_1 + C_2 - \frac{1}{4} = 1$$

$$y' = -8C_1 e^{-8x} + 8C_2 e^{8x}$$

$$y'(0)=0 \quad \to \quad -8C_1 + 8C_2 = 0$$

$$\begin{cases} C_1 + C_2 = \frac{5}{4} \\ -C_1 + C_2 = 0 \end{cases} \quad \to \quad C_2 = \frac{5}{8} = C_1$$

$$\boxed{y(x) = \frac{5}{8}e^{-8x} + \frac{5}{8}e^{8x} - \frac{1}{4}}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$2y'' + 3y' - 2y = 14x^2 - 4x + 11\ ; \quad y(0)=0, \quad y'(0)=0$$

Solution

The characteristic equation: $2\lambda^2 + 3\lambda - 2 = 0 \quad \to \quad \lambda_{1,2} = \frac{-3 \pm 5}{4} = \boxed{-2,\ \frac{1}{2}}$

$$\boxed{y_h = C_1 e^{-2x} + C_2 e^{x/2}}$$

$$y_P = Ax^2 + Bx + C$$

$$y_P' = 2Ax + B$$

$$y_P'' = 2A$$

$$2y'' + 3y' - 2y = 14x^2 - 4x + 11$$

$$\begin{array}{ll} x^2 & -2A = 14 \\ x & 6A - 2B = -4 \\ x^0 & 4A + 3B - 2C = 11 \end{array} \quad \to \quad \boxed{A=-7,\ B=-19,\ C=-37}$$

$$\boxed{y_P = -7x^2 - 19x - 37}$$

$$y(x) = C_1 e^{-2x} + C_2 e^{x/2} - 7x^2 - 19x - 37$$

$$y(0)=0 \quad \to \quad C_1 + C_2 = 37$$

$$y'(x) = -2C_1 e^{-2x} + \frac{1}{2}C_2 e^{x/2} - 14x - 19$$

$$y'(0)=0 \quad \to \quad -2C_1 + \frac{1}{2}C_2 = 19$$

$$\begin{cases} C_1 + C_2 = 37 \\ -4C_1 + C_2 = 38 \end{cases} \quad \to \quad \boxed{C_1 = -\frac{1}{5},\ C_2 = \frac{186}{5}}$$

$$\boxed{y(x) = -\frac{1}{5}e^{-2x} + \frac{186}{5}e^{x/2} - 7x^2 - 19x - 37}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$5y'' + y' = -6x\ ; \quad y(0) = 0, \quad y'(0) = -10$$

Solution

The characteristic equation: $5\lambda^2 + \lambda = 0 \rightarrow \underline{\lambda_{1,2} = 0, \ -\frac{1}{5}}$

$$\underline{y_h = C_1 + C_2 e^{-x/5}}$$

$$y_P = Ax^2 + Bx$$

$$y'_P = 2Ax + B$$

$$y''_P = 2A$$

$$5y'' + y' = -6x$$

$$\begin{cases} x & 2A = -6 \\ x^0 & 10A + B = 0 \end{cases} \rightarrow \underline{A = -3, \ B = 30}$$

$$\underline{y_P = -3x^2 + 30x}$$

$$y(x) = C_1 + C_2 e^{-x/5} - 3x^2 + 30x$$

$$y(0) = 0 \quad \rightarrow \quad C_1 + C_2 = 0$$

$$y(x) = -\frac{1}{5} C_2 e^{-x/5} - 6x + 30$$

$$y'(0) = -10 \quad \rightarrow -\frac{1}{5} C_2 + 30 = -10 \quad \underline{C_2 = 200}$$

$$C_1 = -C_2 \quad \rightarrow \quad \underline{C_1 = -200}$$

$$\underline{y(x) = 200e^{-x/5} - 200 + 30x - 3x^2}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$x'' + 9x = 10\cos 2t; \quad x(0) = x'(0) = 0$$

Solution

The characteristic equation: $\lambda^2 + 9 = 0 \;\rightarrow\; \boxed{\lambda_{1,2} = \pm 3i}$

$$\boxed{x_h = C_1 \cos 3t + C_2 \sin 3t}$$

$$x_P = A\cos 2t + B\sin 2t$$

$$x'_P = -2A\sin 2t + 2B\cos 2t$$

$$x''_P = -4A\cos 2t - 4B\sin 2t$$

$$x'' + 9x = 10\cos 2t$$

$$-4A\cos 2t - 4B\sin 2t + 9A\cos 2t + 9B\sin 2t = 10\cos 2t$$

$$\begin{cases} \cos 2t & 5A = 10 \\ \sin 2t & 5B = 0 \end{cases} \rightarrow \boxed{A = 2,\ B = 0}$$

$$\boxed{x_P = 2\cos 2t}$$

$$x(t) = C_1 \cos 3t + C_2 \sin 3t + 2\cos 2t$$

$$x(0) = 0 \quad \rightarrow \quad \boxed{C_1 = -2}$$

$$x'(t) = -3C_1 \sin 3t + 3C_2 \cos 3t - 4\sin 2t$$

$$x'(0) = 0 \quad \rightarrow \quad \boxed{C_2 = 0}$$

$$\boxed{x(t) = -2\cos 3t + 2\cos 2t}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$x'' + 4x = 5\sin 3t\ ; \quad x(0) = x'(0) = 0$$

Solution

The characteristic equation: $\lambda^2 + 4 = 0 \;\rightarrow\; \boxed{\lambda_{1,2} = \pm 2i}$

$$\boxed{x_h = C_1 \cos 2t + C_2 \sin 2t}$$

$$x_p = A\cos 3t + B\sin 3t$$

$$x'_P = -3A\sin 3t + 3B\cos 3t$$

$$x''_P = -9A\cos 3t - 9B\sin 3t$$

$$x'' + 4x = 5\sin 3t$$

$$-9A\cos 3t - 9B\sin 3t + 4A\cos 3t + 4B\sin 3t = 5\sin 3t$$

$$\begin{cases} \cos 3t & -5A=0 \\ \sin 3t & -5B=5 \end{cases} \rightarrow \boxed{A=0,\ B=-1}$$

$$\boxed{x_P = -\sin 3t}$$

$$x(t) = C_1\cos 2t + C_2\sin 2t - \sin 3t$$

$$x(0)=0 \quad \rightarrow \quad \boxed{C_1 = 0}$$

$$x'(t) = -2C_1\sin 2t + 2C_2\cos 2t - 3\cos 3t$$

$$x'(0)=0 \quad \rightarrow \quad \boxed{C_2 = \frac{3}{2}}$$

$$\boxed{x(t) = \frac{3}{2}\sin 2t - \sin 3t}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$x'' + 100x = 225\cos 5t + 300\sin 5t\ ; \quad x(0)=375, \quad x'(0)=0$$

Solution

The characteristic equation: $\lambda^2 + 100 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 10i}$

$$\boxed{x_h = C_1\cos 10t + C_2\sin 10t}$$

$$x_p = A\cos 5t + B\sin 5t$$

$$x'_P = -5A\sin 5t + 5B\cos 5t$$

$$x''_P = -25A\cos 5t - 25B\sin 5t$$

$$x'' + 100x = 225\cos 5t + 300\sin 5t$$

$$-25A\cos 5t - 25B\sin 5t + 100\cos 5t + 100\sin 100t = 225\cos 5t + 300\sin 5t$$

$$\begin{cases} \cos 5t & 75A=225 \\ \sin 5t & 75B=300 \end{cases} \rightarrow \boxed{A=3,\ B=4}$$

$$\boxed{x_p = 3\cos 5t + 4\sin 5t}$$

$$x(t) = C_1\cos 10t + C_2\sin 10t + 3\cos 5t + 4\sin 5t$$

$$x(0)=375 \quad \rightarrow C_1 + 3 = 375 \quad \boxed{C_1 = 372}$$

$$x'(t) = -10C_1\sin 10t + 10C_2\cos 10t - 15\sin 5t + 20\cos 5t$$

$$x'(0)=0 \quad \rightarrow \ 10C_2 + 20 = 0 \quad \boxed{C_2 = -2}$$

$$\boxed{x(t) = 372\cos 10t - 2\sin 10t + 3\cos 5t + 4\sin 5t}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$x'' + 25x = 90\cos 4t \ ; \quad x(0) = 0, \quad x'(0) = 90$$

Solution

The characteristic equation: $\lambda^2 + 25 = 0 \ \rightarrow \ \boxed{\lambda_{1,2} = \pm 5i}$

$\boxed{x_h = C_1 \cos 5t + C_2 \sin 5t}$

$x_p = A\cos 4t + B\sin 4t$

$x'_p = -4A\sin 4t + 4B\cos 4t$

$x''_p = -16A\cos 4t - 16B\sin 4t$

$x'' + 25x = 90\cos 4t$

$-16A\cos 4t - 16B\sin 4t + 25A\cos 4t + 25B\sin 4t = 90\cos 4t$

$$\begin{cases} \cos 4t & 9A = 90 \\ \sin 4t & 9B = 0 \end{cases} \rightarrow \boxed{A = 10, \ B = 0}$$

$\boxed{x_p = 10\cos 4t}$

$x(t) = C_1 \cos 5t + C_2 \sin 5t + 10\cos 4t$

$x(0) = 0 \ \rightarrow \ \boxed{C_1 = -10}$

$x'(t) = -5C_1 \sin 5t + 5C_2 \cos 5t - 40\sin 4t$

$x'(0) = 90 \ \rightarrow \ 5C_2 = 90 \quad \boxed{C_2 = 18}$

$\boxed{x(t) = -10\cos 5t + 18\sin 5t + 10\cos 4t}$

Exercise

Find the solution of the given initial value problem

$$y^{(3)} - y' = 4e^{-x} + 3e^{2x} \ ; \quad y(0) = 0, \quad y'(0) = -1, \quad y''(0) = 2$$

Solution

The characteristic equation: $\lambda^3 - \lambda = \lambda\left(\lambda^2 - 1\right) = 0 \ \rightarrow \ \boxed{\lambda_{1,2,3} = 0, \ \pm 1}$

$\boxed{y_h = C_1 + C_2 e^{-x} + C_3 e^{x}}$

$y_p = Axe^{-x} + Be^{2x}$

$y'_p = (A - Ax)e^{-x} + 2Be^{2x}$

$y''_p = (-2A + Ax)e^{-x} + 4Be^{2x}$

$$y'''_p = (3A - Ax)e^{-x} + 8Be^{2x}$$

$$y^{(3)} - y' = 4e^{-x} + 3e^{2x}$$

$$(2A)e^{-x} + 6Be^{2x} = 4e^{-x} + 3e^{2x}$$

$$\begin{cases} 2A = 4 & \to \underline{A = 2} \\ 6B = 3 & \to \underline{B = \frac{1}{2}} \end{cases}$$

$$\underline{y_p = 2xe^{-x} + \frac{1}{2}e^{2x}}$$

$$y(x) = C_1 + C_2 e^{-x} + C_3 e^x + 2xe^{-x} + \frac{1}{2}e^{2x}$$

$$y(0) = 0 \quad \to C_1 + C_2 + C_3 + \frac{1}{2} = 0 \quad \Rightarrow \underline{C_1 + C_2 + C_3 = -\frac{1}{2}}$$

$$y' = -C_2 e^{-x} + C_3 e^x + 2e^{-x} - 2xe^{-x} + e^{2x}$$

$$y'(0) = -1 \quad \to -C_2 + C_3 + 2 + 1 = -1 \quad \Rightarrow \underline{-C_2 + C_3 = -4}$$

$$y'' = C_2 e^{-x} + C_3 e^x - 4e^{-x} + 2xe^{-x} + 2e^{2x}$$

$$y''(0) = 2 \quad \to C_2 + C_3 - 4 + 2 = 2 \quad \Rightarrow \underline{C_2 + C_3 = 4}$$

$$\begin{cases} C_1 + C_2 + C_3 = -\frac{1}{2} \\ -C_2 + C_3 = -4 \\ C_2 + C_3 = 4 \end{cases} \to \underline{C_3 = 0} \quad \underline{C_2 = 4} \quad \underline{C_1 = -\frac{9}{2}}$$

$$\underline{y(x) = -\frac{9}{2} + 4e^{-x} + 2xe^{-x} + \frac{1}{2}e^{2x}}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y^{(3)} + y'' = x + e^{-x} \;; \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1$$

Solution

The characteristic equation: $\lambda^3 + \lambda^2 = 0 \quad \to \quad \underline{\lambda_{1,2,3} = 0,\ 0,\ -1}$

$$\underline{y_h = C_1 + C_2 x + C_3 e^{-x}}$$

$$y_p = Ax^3 + Bx^2 + Cxe^{-x}$$

$$y'_p = 3Ax^2 + 2Bx + (C - Cx)e^{-x}$$

$$y''_p = 6Ax + 2B + (-2C + Cx)e^{-x}$$

$$y'''_p = 6A + (3C - Cx)e^{-x}$$

$$y^{(3)} + y'' = x + e^{-x}$$

$$6Ax + 2B + 6A + Ce^{-x} = x + e^{-x}$$

$$\begin{cases} e^x & C = 1 \\ x & 6A = 1 \\ x^0 & 2B + 6A = 0 \end{cases} \quad \rightarrow \quad \underline{A = \frac{1}{6},\ B = -\frac{1}{2},\ C = 1|}$$

$$\underline{y_p = \frac{1}{6}x^3 - \frac{1}{2}x^2 + xe^{-x}|}$$

$$y(x) = C_1 + C_2 x + C_3 e^{-x} + \frac{1}{6}x^3 - \frac{1}{2}x^2 + xe^{-x}$$

$$y(0) = 1 \quad \rightarrow \quad C_1 + C_3 = 1$$

$$y'(x) = C_2 - C_3 e^{-x} + \frac{1}{2}x^2 - x + (1 - x)e^{-x}$$

$$y'(0) = 0 \quad \rightarrow \quad C_2 - C_3 = -1$$

$$y''(x) = C_3 e^{-x} + x - 1 + (-2 + x)e^{-x}$$

$$y''(0) = 1 \quad \rightarrow \quad C_3 - 1 - 2 = 1 \quad \Rightarrow \quad \underline{C_3 = 4|}$$

$$\begin{cases} C_1 + C_3 = 1 \\ C_2 - C_3 = -1 \end{cases} \quad \rightarrow \quad \underline{C_1 = -3,\ C_2 = 3|}$$

$$\underline{y(x) = -3 + 3x + 4e^{-x} + \frac{1}{6}x^3 - \frac{1}{2}x^2 + xe^{-x}|}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y^{(3)} - 2y'' + y' = 1 + xe^x; \quad y(0) = y'(0) = 0, \quad y''(0) = 1$$

Solution

The characteristic equation: $\lambda^3 - 2\lambda^2 + \lambda = \lambda\left(\lambda^2 - 2\lambda + 1\right) = 0 \quad \rightarrow \quad \underline{\lambda_{1,2,3} = 0,\ 1,\ 1|}$

$$\underline{y_h = C_1 + \left(C_2 + C_3 x\right)e^x|}$$

$$y_p = Ax^2 + Bx + \left(Cx^3 + Dx^2\right)e^x$$

$$y'_p = 2Ax + B + \left(2Dx + (3C + D)x^2 + Cx^3\right)e^x$$

$$y''_p = 2A + \left(2Dx + (3C + D)x^2 + Cx^3 + 2D + (6C + 2D)x + 3Cx^2\right)e^x$$

$$= 2A + \left(2D + (6C+4D)x + (6C+D)x^2 + Cx^3\right)e^x$$

$$y_P''' = \left(2D + (6C+4D)x + (6C+D)x^2 + Cx^3 + (6C+4D) + (12C+2D)x + 3Cx^2\right)e^x$$

$$= \left(6C + 6D + (18C+6D)x + (9C+D)x^2 + Cx^3\right)e^x$$

$$y^{(3)} - 2y'' + y' = 1 + xe^x$$

$$\left(6C + 2D + (6C+4D)x\right)e^x - 4A + 2Ax + B = 1 + xe^x$$

$$\begin{cases} e^x & 6C + 2D = 0 \\ & 6C = 1 \\ x & 2A = 0 \\ x^0 & B - 4A = 1 \end{cases} \quad \rightarrow \quad \underline{A = 0,\ B = 1,\ C = \tfrac{1}{6},\ D = -\tfrac{1}{2}}$$

$$\underline{y_p = x + \left(\tfrac{1}{6}x^3 - \tfrac{1}{2}x^2\right)e^x}$$

$$y(x) = C_1 + \left(C_2 + C_3 x\right)e^x + x + \left(\tfrac{1}{6}x^3 - \tfrac{1}{2}x^2\right)e^x$$

$$y(0) = 0 \quad \rightarrow \quad C_1 + C_2 = 0$$

$$y'(x) = \left(C_2 + C_3 + C_3 x\right)e^x + 1 + \left(-x + \tfrac{1}{6}x^3\right)e^x$$

$$y'(0) = 0 \quad \rightarrow \quad C_2 + C_3 = -1$$

$$y''(x) = \left(C_2 + 2C_3 + C_3 x\right)e^x + \left(-x + \tfrac{1}{6}x^3 - 1 + \tfrac{1}{2}x^2\right)e^x$$

$$y''(0) = 1 \quad \rightarrow \quad C_2 + 2C_3 - 1 = 1 \quad \Rightarrow C_2 + 2C_3 = 2$$

$$\begin{cases} C_1 + C_2 = 0 \\ C_2 + C_3 = -1 \\ C_2 + 2C_3 = 2 \end{cases} \quad \rightarrow \quad \underline{C_3 = 3,\ C_2 = -4,\ C_1 = 4}$$

$$\underline{y(x) = 4 + x + \left(-4 + 3x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3\right)e^x}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y^{(4)} - y = 5\ ; \qquad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0$$

Solution

The characteristic equation: $\lambda^4 - 1 = 0 \ \rightarrow \lambda^2 = \pm 1 \quad \underline{\lambda_{1,2,3,4} = \pm 1,\ \pm i}$

$$\boxed{y_h = C_1 e^{-x} + C_2 e^{x} + C_3 \cos x + C_4 \sin x}$$

$$y_p = A$$

$$y'_p = y''_p = y'''_p = y^{(4)}{}_p = 0$$

$$y^{(4)} - y = 5 \quad \rightarrow \quad \boxed{-A = 5}$$

$$\boxed{y_p = -5}$$

$$y(x) = C_1 e^{-x} + C_2 e^{x} + C_3 \cos x + C_4 \sin x + 5$$

$$y(0) = 0 \quad \rightarrow \quad C_1 + C_2 + C_3 = -5$$

$$y'(x) = -C_1 e^{-x} + C_2 e^{x} - C_3 \sin x + C_4 \cos x$$

$$y'(0) = 0 \quad \rightarrow \quad -C_1 + C_2 + C_4 = 0$$

$$y''(x) = C_1 e^{-x} + C_2 e^{x} - C_3 \cos x - C_4 \sin x$$

$$y''(0) = 0 \quad \rightarrow \quad C_1 + C_2 - C_3 = 0$$

$$y'''(x) = -C_1 e^{-x} + C_2 e^{x} + C_3 \sin x - C_4 \cos x$$

$$y^{(3)}(0) = 0 \quad \rightarrow \quad -C_1 + C_2 - C_4 = 0$$

$$\begin{aligned} C_1 + C_2 + C_3 &= -5 \\ C_1 + C_2 - C_3 &= 0 \\ -C_1 + C_2 + C_4 &= 0 \\ -C_1 + C_2 - C_4 &= 0 \end{aligned} \quad \rightarrow \quad \boxed{C_1 = \tfrac{5}{4},\ C_2 = \tfrac{5}{4},\ C_3 = \tfrac{5}{2},\ C_4 = 0}$$

$$\boxed{y(x) = \tfrac{5}{4} e^{-x} + \tfrac{5}{4} e^{x} + \tfrac{5}{2} \cos x - 5}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y^{(4)} - 4y'' = x^2 \ ; \quad y(0) = y'(0) = 1, \quad y''(0) = y^{(3)}(0) = -1$$

Solution

The characteristic equation: $\lambda^4 - 4\lambda^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2,3,4} = 0,\ 0,\ \pm 2}$

$$\boxed{y_h = C_1 + C_2 x + C_3 e^{-2x} + C_4 e^{2x}}$$

$$y_p = Ax^4 + Bx^3 + Cx^2$$

$$y'_p = 4Ax^3 + 3Bx^2 + 2Cx$$

$$y''_p = 12Ax^2 + 6Bx + 2C$$

$$y'''_p = 24Ax + 6B$$

$$y_p^{(4)} = 24A$$

$$y^{(4)} - 4y'' = x^2$$

$$24A - 48Ax^2 - 24Bx - 8C = x^2$$

$$\begin{cases} -48A = 1 \\ 24B = 0 \\ 24A - 8C = 0 \end{cases} \rightarrow \boxed{A = -\frac{1}{48},\ B = 0,\ C = -\frac{1}{16}}$$

$$\boxed{y_p = \frac{1}{48}x^4 + \frac{1}{16}x^2}$$

$$y(x) = C_1 + C_2 x + C_3 e^{-2x} + C_4 e^{2x} - \frac{1}{48}x^4 - \frac{1}{16}x^2$$

$$y(0) = 1 \quad \rightarrow \quad C_1 + C_3 + C_4 = 1$$

$$y'(x) = C_2 - 2C_3 e^{-2x} + 2C_4 e^{2x} - \frac{1}{12}x^3 - \frac{1}{8}x$$

$$y'(0) = 1 \quad \rightarrow \quad C_2 - 2C_3 + 2C_4 = 1$$

$$y''(x) = 4C_3 e^{-2x} + 4C_4 e^{2x} - \frac{1}{4}x^2 - \frac{1}{8}$$

$$y''(0) = -1 \quad \rightarrow \quad 4C_3 + 4C_4 - \frac{1}{8} = -1 \quad 4C_3 + 4C_4 = -\frac{7}{8}$$

$$y^{(3)}(x) = -8C_3 e^{-2x} + 8C_4 e^{2x} - \frac{1}{2}x$$

$$y^{(3)}(0) = -1 \quad \rightarrow \quad -8C_3 + 8C_4 = -1$$

$$\begin{aligned} C_1 + C_3 + C_4 &= 1 \\ C_2 - 2C_3 + 2C_4 &= 1 \\ 4C_3 + 4C_4 &= -\frac{7}{8} \\ -8C_3 + 8C_4 &= -1 \end{aligned} \quad \rightarrow \quad \boxed{C_1 = \frac{117}{96},\ C_2 = \frac{5}{4},\ C_3 = -\frac{3}{64},\ C_4 = -\frac{11}{64}}$$

$$\boxed{y(x) = \frac{117}{96} + \frac{5}{4}x - \frac{3}{64}e^{-2x} - \frac{11}{64}e^{2x} - \frac{1}{48}x^4 - \frac{1}{16}x^2}$$

Exercise

Find the general solution that satisfy the given initial conditions

$$y^{(4)} - y''' = x + e^x; \qquad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0$$

Solution

The characteristic equation: $\lambda^4 - \lambda^3 = 0 \quad \rightarrow \quad \underline{\lambda_{1,2,3,4} = 0,\ 0,\ 0,\ 1}$

$\underline{y_h = C_1 + C_2 x + C_3 x^2 + C_4 e^x}$

$y_p = Ax^4 + Bx^3 + Cxe^x$

$y'_p = 4Ax^3 + 3Bx^2 + (Cx + C)e^x$

$y''_p = 12Ax^2 + 6Bx + (Cx + 2C)e^x$

$y'''_p = 24Ax + 6B + (Cx + 3C)e^x$

$y_p^{(4)} = 24A + (Cx + 4C)e^x$

$y^{(4)} - y''' = x + e^x$

$$\begin{array}{ll} x & -24A = 1 \\ x^0 & 24A - 6B = 0 \end{array} \quad \rightarrow \quad \underline{A = -\frac{1}{24},\ \ B = -\frac{1}{6}}$$

$$e^x \quad \begin{array}{ll} x & C - C = 0 \\ x^0 & 4C - 3C = 1 \end{array} \quad \rightarrow \quad \underline{C = 1}$$

$\underline{y_p = -\frac{1}{24}x^4 - \frac{1}{6}x^3 + xe^x}$

$y(x) = C_1 + C_2 x + C_3 x^2 + C_4 e^x - \frac{1}{24}x^4 - \frac{1}{6}x^3 + xe^x$

$y(0) = 0 \quad \rightarrow \quad C_1 + C_4 = 0$

$y'(x) = C_2 + 2C_3 x + C_4 e^x - \frac{1}{6}x^3 - \frac{1}{2}x^2 + (1 + x)e^x$

$y'(0) = 0 \quad \rightarrow \quad C_2 + C_4 + 1 = 0$

$y''(x) = 2C_3 + C_4 e^x - \frac{1}{2}x^2 - x + (2 + x)e^x$

$y''(0) = 0 \quad \rightarrow \quad 2C_3 + C_4 = -2$

$y'''(x) = C_4 e^x - x - 1 + (3 + x)e^x$

$y'''(0) = 0 \quad \rightarrow \quad \underline{C_4 = -2}$

$2C_3 + C_4 = -2 \quad \rightarrow \quad \underline{C_3 = 0}$

$$C_2 + C_4 + 1 = 0 \quad \to \quad \boxed{C_2 = 1}$$

$$C_1 + C_4 = 0 \quad \to \quad \boxed{C_1 = 2}$$

$$\boxed{y(x) = 2 + x - 2e^x - \frac{1}{24}x^4 - \frac{1}{6}x^3 + xe^x}$$

Exercise

If k and b are positive constants, then find the general solution of $\quad y'' + k^2 y = \sin bx$

Solution

The characteristic equation: $\quad \lambda^2 + k^2 = 0 \quad \to (k > 0) \quad \boxed{\lambda_{1,2} = \pm ki}$

$$\boxed{y_h = C_1 \cos kx + C_2 \sin kx}$$

If $k \neq b$

$$y_p = A\cos bx + B\sin bx$$

$$y'_p = -bA\sin bx + bB\cos bx$$

$$y''_p = -b^2 A\cos bx - b^2 B\sin bx$$

$$y'' + k^2 y = \sin bx$$

$$-b^2 A\cos bx - b^2 B\sin bx + kA\cos bx + kB\sin bx = \sin bx$$

$$\left(k - b^2\right)A\cos bx + \left(k - b^2\right)B\sin bx = \sin bx$$

$$\begin{cases} \left(k - b^2\right)A = 0 \\ \left(k - b^2\right)B = 1 \end{cases} \quad \to \quad \boxed{A = 0} \quad \boxed{B = \frac{1}{k - b^2}}$$

$$\boxed{y_p = \frac{1}{k - b^2}\sin bx}$$

$$\boxed{y(x) = C_1 \cos kx + C_2 \sin kx + \frac{1}{k - b^2}\sin bx}$$