

Solution

Section 3.1 – Inverse Functions

Exercise

Find the inverse relation of the given set: $A = \{(-2, 2), (1, -1), (0, 4), (1, 3)\}$

Solution

$$\underline{A^{-1} = \{(2, -2), (-1, 1), (4, 0), (3, 1)\}}$$

Exercise

Find the inverse relation of the given set: $B = \{(1, -1), (2, -2), (3, -3), (4, -4)\}$

Solution

$$\underline{B^{-1} = \{(-1, 1), (-2, 2), (-3, 3), (-4, 4)\}}$$

Exercise

Find the inverse relation of the given set: $C = \{(a, -a), (b, -b), (c, -c)\}$

Solution

$$\underline{C^{-1} = \{(-a, a), (-b, b), (-c, c)\}}$$

Exercise

Find the inverse relation of the given set: $D = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4)\}$

Solution

$$\underline{D^{-1} = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4)\}}$$

Exercise

Find the inverse relation of the given set: $E = \{(-a, a), (-b, b), (-c, c), (-d, d)\}$

Solution

$$\underline{E^{-1} = \{(a, -a), (b, -b), (c, -c), (d, -d)\}}$$

Exercise

Determine whether the function is one-to-one: $f(x) = 3x - 7$

Solution

$$f(a) = f(b)$$

$$3a - 7 = 3b - 7$$

$$3a = 3b - 7 + 7$$

$$3a = 3b$$

Divide both sides by 3

$$a = b$$

\therefore The function is one-to-one

Exercise

Determine whether the function is one-to-one: $f(x) = x^2 - 9$

Solution

$$1 \neq -1$$

$$1^2 - 9 \neq (-1)^2 - 9$$

$-8 = -8 \rightarrow$ Contradict the definition

$$f(a) = f(b)$$

$$a^2 - 9 = b^2 - 9$$

$$a^2 = b^2$$

$$a = \pm b$$

\therefore The function is **not** one-to-one

Exercise

Determine whether the function is one-to-one: $f(x) = \sqrt{x}$

Solution

$$f(a) = f(b)$$

$$\sqrt{a} = \sqrt{b}$$

$$(\sqrt{a})^2 = (\sqrt{b})^2$$

Square both sides

$$a = b$$

\therefore The function is one-to-one

Exercise

Determine whether the function is one-to-one: $f(x) = \sqrt[3]{x}$

Solution

$$f(a) = f(b)$$

$$\sqrt[3]{a} = \sqrt[3]{b}$$

$$\left(\sqrt[3]{a}\right)^3 = \left(\sqrt[3]{b}\right)^3 \quad \text{cube both sides}$$

$$a = b$$

\therefore The function is one-to-one

Exercise

Determine whether the function is one-to-one: $f(x) = |x|$

Solution

$$1 \neq -1$$

$$|1| \neq |-1|$$

$$1 \neq 1 \text{ (false)}$$

\therefore The function is **not** one-to-one

Exercise

Determine whether the function is one-to-one $f(x) = \frac{2}{x+3}$

Solution

$$f(a) = f(b)$$

$$\frac{2}{a+3} = \frac{2}{b+3}$$

$$2(b+3) = 2(a+3)$$

$$b+3 = a+3$$

$$a = b$$

$\therefore f$ is one-to-one

Exercise

Determine whether the function is one-to-one $f(x) = (x-2)^3$

Solution

$$f(a) = f(b)$$

$$(a-2)^3 = (b-2)^3$$

$$\left[(a-2)^3\right]^{1/3} = \left[(b-2)^3\right]^{1/3}$$

$$a-2 = b-2$$

Add 2 on both sides

$$a = b$$

\therefore Function is one-to-one

Exercise

Determine whether the function is one-to-one $y = x^2 + 2$

Solution

$$f(a) = f(b)$$

$$a^2 + 2 = b^2 + 2$$

Subtract 2

$$a^2 = b^2$$

$$a = \pm\sqrt{b^2}$$

\therefore Function is **not** a one-to-one

The inverse function doesn't exist.

Exercise

Determine whether the function is one-to-one $f(x) = \frac{x+1}{x-3}$

Solution

$$f(a) = f(b)$$

$$\frac{a+1}{a-3} = \frac{b+1}{b-3}$$

Cross multiplication

$$(a+1)(b-3) = (b+1)(a-3)$$

$$ab - 3a + b - 3 = ab - 3b + a - 3$$

$$-4a = -4b$$

Divide by -4

$$a = b$$

\therefore Function is one-to-one

Exercise

Given that $f(x) = 5x + 8$, use composition of functions to show that $f^{-1}(x) = \frac{x-8}{5}$

Solution

$$\begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}(f(x)) \\&= f^{-1}(5x + 8) \\&= \frac{(5x + 8) - 8}{5} \\&= \frac{5x + 8 - 8}{5} \\&= \frac{5x}{5} \\&= x\end{aligned}$$

$$\begin{aligned}(f \circ f^{-1})(x) &= f(f^{-1}(x)) \\&= f\left(\frac{x-8}{5}\right) \\&= 5\left(\frac{x-8}{5}\right) + 8 \\&= x - 8 + 8 \\&= x\end{aligned}$$

Exercise

Given the function $f(x) = (x + 8)^3$

- Find $f^{-1}(x)$
- Graph f and f^{-1} in the same rectangular coordinate system
- Find the domain and the range of f and f^{-1}

Solution

a) $y = (x + 8)^3$

$$x = (y + 8)^3$$

$$(x)^{1/3} = ((y + 8)^3)^{1/3}$$

$$x^{1/3} = y + 8$$

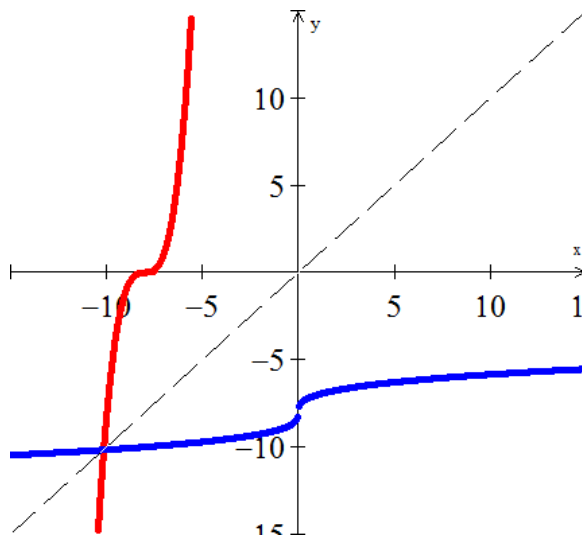
$$f^{-1}(x) = x^{1/3} - 8$$

Replace $f(x)$ with y

Interchange x and y

Subtract 8 from both sides.

b)



c) Domain of $f = \text{Range of } f^{-1}: (-\infty, \infty)$

Range of $f = \text{Domain of } f^{-1}: (-\infty, \infty)$

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = 4x; \quad g(x) = \frac{x}{4}$

Solution

$$\begin{aligned} f(g(x)) &= f\left(\frac{x}{4}\right) \\ &= 4\left(\frac{x}{4}\right) \\ &= x \end{aligned}$$

$$\begin{aligned} g(f(x)) &= g(4x) \\ &= \frac{4x}{4} \\ &= x \end{aligned}$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = 2x; \quad g(x) = \frac{1}{2x}$

Solution

$$f(g(x)) = f\left(\frac{1}{2x}\right)$$

$$= 2\left(\frac{1}{2x}\right)$$

$$\boxed{= \frac{1}{x}} \neq x$$

$\therefore f(x)$ and $g(x)$ are **not** inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = 4x - 1; \quad g(x) = \frac{x+1}{4}$

Solution

$$f(g(x)) = f\left(\frac{x+1}{4}\right)$$

$$= 4\left(\frac{x+1}{4}\right) - 1$$

$$= x + 1 - 1$$

$$\boxed{= x}$$

$$g(f(x)) = g(4x - 1)$$

$$= \frac{4x - 1 + 1}{4}$$

$$= \frac{4x}{4}$$

$$\boxed{= x}$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = \frac{1}{2}x - \frac{3}{2}; \quad g(x) = 2x + 3$

Solution

$$f(g(x)) = f(2x + 3)$$

$$= \frac{1}{2}(2x + 3) - \frac{3}{2}$$

$$= x + \frac{3}{2} - \frac{3}{2}$$

$$\boxed{= x}$$

$$g(f(x)) = g\left(\frac{1}{2}x - \frac{3}{2}\right)$$

$$= 2\left(\frac{1}{2}x - \frac{3}{2}\right) + 3$$

$$= x - 3 + 3$$

$$\underline{= x}$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other

$$f(x) = -\frac{1}{2}x - \frac{1}{2}; \quad g(x) = -2x + 1$$

Solution

$$\begin{aligned} f(g(x)) &= f(-2x + 1) \\ &= -\frac{1}{2}(-2x + 1) - \frac{1}{2} \\ &= x - \frac{1}{2} - \frac{1}{2} \\ &= \underline{\frac{1}{x} - 1} \quad \neq x \end{aligned}$$

$\therefore f(x)$ and $g(x)$ are **not** inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other

$$f(x) = 3x + 2; \quad g(x) = \frac{1}{3}(x - 2)$$

Solution

$$\begin{aligned} f(g(x)) &= f\left(\frac{x-2}{3}\right) \\ &= 3\left(\frac{x-2}{3}\right) + 2 \\ &= x - 2 + 2 \\ &= \underline{x} \end{aligned}$$

$$\begin{aligned} g(f(x)) &= g(3x + 2) \\ &= \frac{1}{3}(3x + 2 - 2) \\ &= \frac{1}{3}(3x) \\ &= \underline{x} \end{aligned}$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = \frac{5}{x+3}; \quad g(x) = \frac{5}{x} - 3$

Solution

$$\begin{aligned} f(g(x)) &= f\left(\frac{5}{x} - 3\right) \\ &= \frac{5}{\frac{5}{x} - 3 + 3} \\ &= \frac{5}{\frac{5}{x}} \\ &= 5 \frac{x}{5} \\ &= x \end{aligned}$$

$$\begin{aligned} g(f(x)) &= g\left(\frac{5}{x+3}\right) \\ &= \frac{5}{\frac{5}{x+3}} - 3 \\ &= 5\left(\frac{x+3}{5}\right) - 3 \\ &= x + 3 - 3 \\ &= x \end{aligned}$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = \frac{2x}{x+1}; \quad g(x) = \frac{-x}{x-2}$

Solution

$$\begin{aligned} f(g(x)) &= f\left(\frac{-x}{x-2}\right) \\ &= 2\left(\frac{-x}{x-2}\right) \frac{1}{\frac{-x}{x-2} + 1} \\ &= \left(\frac{-2x}{x-2}\right) \frac{x-2}{-x+x-2} \\ &= \frac{-2x}{-2} \\ &= x \end{aligned}$$

$$g(f(x)) = g\left(\frac{2x}{x+1}\right)$$

$$\begin{aligned}
&= -\left(\frac{2x}{x+1}\right) \frac{1}{\frac{2x}{x+1} - 2} \\
&= -\left(\frac{2x}{x+1}\right) \frac{x+1}{2x - 2x - 2} \\
&= \frac{-2x}{-2} \\
&= x \quad |
\end{aligned}$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = \frac{3x}{x-1}; \quad g(x) = \frac{x}{x-3}$

Solution

$$\begin{aligned}
f(g(x)) &= f\left(\frac{x}{x-3}\right) \\
&= 3\left(\frac{x}{x-3}\right) \frac{1}{\frac{x}{x-3} - 1} \\
&= \left(\frac{3x}{x-3}\right) \frac{x-3}{x - x + 3} \\
&= \frac{3x}{3} \\
&= x \quad |
\end{aligned}$$

$$\begin{aligned}
g(f(x)) &= g\left(\frac{3x}{x-1}\right) \\
&= \left(\frac{3x}{x-1}\right) \frac{1}{\frac{3x}{x-1} - 3} \\
&= \left(\frac{3x}{x-1}\right) \frac{x-1}{3x - 3x + 3} \\
&= \frac{3x}{3} \\
&= x \quad |
\end{aligned}$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = x^3 + 2; \quad g(x) = \sqrt[3]{x-2}$

Solution

$$\begin{aligned}
 f(g(x)) &= f(\sqrt[3]{x-2}) \\
 &= (\sqrt[3]{x-2})^3 + 2 \\
 &= x - 2 + 2 \\
 &= x
 \end{aligned}$$

$$\begin{aligned}
 g(f(x)) &= g(x^3 + 2) \\
 &= \sqrt[3]{x^3 + 2 - 2} \\
 &= \sqrt[3]{x^3} \\
 &= x
 \end{aligned}$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = (x+4)^3$; $g(x) = \sqrt[3]{x} - 4$

Solution

$$\begin{aligned}
 f(g(x)) &= f(\sqrt[3]{x} - 4) \\
 &= (\sqrt[3]{x} - 4 + 4)^3 \\
 &= (\sqrt[3]{x})^3 \\
 &= x
 \end{aligned}$$

$$\begin{aligned}
 g(f(x)) &= g((x+4)^3) \\
 &= \sqrt[3]{(x+4)^3} - 4 \\
 &= x + 4 - 4 \\
 &= x
 \end{aligned}$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = x^3 - 1$; $g(x) = \sqrt[3]{x+1}$

Solution

$$f(g(x)) = f(\sqrt[3]{x+1})$$

$$\begin{aligned}
 &= \left(\sqrt[3]{x+1}\right)^3 - 1 \\
 &= x+1-1 \\
 &= x
 \end{aligned}$$

$$\begin{aligned}
 g(f(x)) &= g(x^3 - 1) \\
 &= \sqrt[3]{x^3 - 1 + 1} \\
 &= \sqrt[3]{x^3} \\
 &= x
 \end{aligned}$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other

Exercise

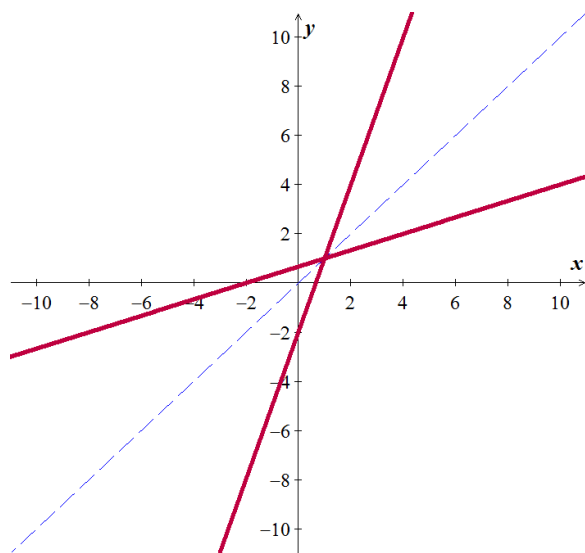
Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = 3x - 2$; $g(x) = \frac{x+2}{3}$

Solution

$$\begin{aligned}
 f(g(x)) &= f\left(\frac{x+2}{3}\right) \\
 &= 3\left(\frac{x+2}{3}\right) - 2 \\
 &= x+2-2 \\
 &= x
 \end{aligned}$$

$$\begin{aligned}
 g(f(x)) &= g(3x - 2) \\
 &= \frac{3x - 2 + 2}{3} \\
 &= \frac{3x}{3} \\
 &= x
 \end{aligned}$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other



Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other

$$f(x) = x^2 + 5, x \leq 0 \quad g(x) = -\sqrt{x-5}, x \geq 5$$

Solution

$$\begin{aligned}
 f(g(x)) &= f(-\sqrt{x-5}) \\
 &= (-\sqrt{x-5})^2 + 5
 \end{aligned}$$

$$= x - 5 + 5$$

$$= x$$

$$g(f(x)) = g(x^2 + 5)$$

$$= -\sqrt{x^2 + 5} - 5$$

$$= -\sqrt{x^2}$$

$$= -|x| \quad x \leq 0$$

$$= -(-x)$$

$$= x$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)$ and $g(x)$ are inverse functions of each other $f(x) = x^3 - 4$; $g(x) = \sqrt[3]{x+4}$

Solution

$$f(g(x)) = f(\sqrt[3]{x+4})$$

$$= (\sqrt[3]{x+4})^3 - 4$$

$$= x + 4 - 4$$

$$= x$$

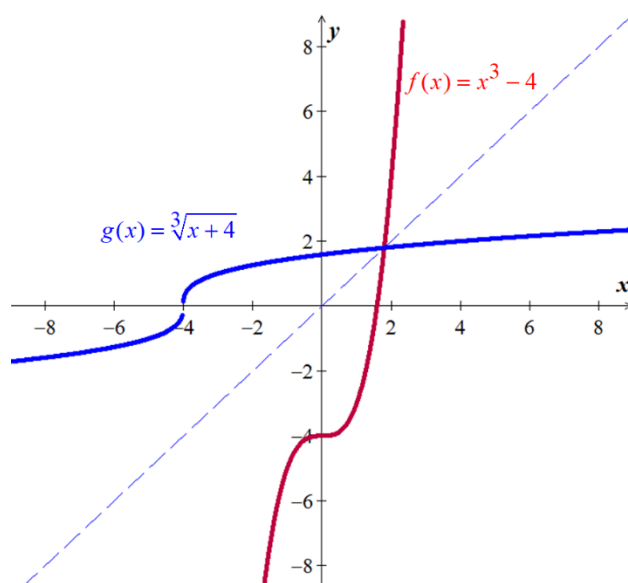
$$g(f(x)) = g(x^3 - 4)$$

$$= \sqrt[3]{x^3 - 4 + 4}$$

$$= \sqrt[3]{x^3}$$

$$= x$$

$\therefore f(x)$ and $g(x)$ are inverse functions to each other



Exercise

Find the inverse of $f(x) = (x-2)^3$

Solution

$$y = (x-2)^3$$

$$x = (y-2)^3$$

$$x^{1/3} = \left[(y-2)^3 \right]^{1/3}$$

$$x^{1/3} = y-2$$

$$\sqrt[3]{x} + 2 = y$$

$$\underline{f^{-1}(x) = \sqrt[3]{x} + 2}$$

Exercise

Find the inverse of $f(x) = \frac{x+1}{x-3}$

Solution

$$y = \frac{x+1}{x-3}$$

$$x = \frac{y+1}{y-3}$$

$$x(y-3) = y+1$$

$$xy - 3x = y+1$$

$$xy - y = 3x+1$$

$$y(x-1) = 3x+1$$

$$\underline{f^{-1}(x) = \frac{3x+1}{x-1}}$$

Exercise

Find the inverse of $f(x) = \frac{2x+1}{x-3}$

Solution

$$y = \frac{2x+1}{x-3}$$

$$x = \frac{2y+1}{y-3}$$

$$xy - 3x = 2y+1$$

$$y(x-2) = 3x+1$$

$$\underline{f^{-1}(x) = \frac{3x+1}{x-2}}$$

Exercise

Determine the domain and range of $f^{-1}: f(x) = -\frac{2}{x-1}$ (Hint: first find the domain and range of f)

Solution

$$x - 1 \neq 0 \Rightarrow x \neq 1$$

$$\text{Range of } f^{-1} = \text{Domain of } f: \mathbb{R} - \{1\} \quad (-\infty, 1) \cup (1, \infty)$$

$$\text{Domain of } f^{-1} = \text{Range of } f: \mathbb{R} - \{0\} \quad (-\infty, 0) \cup (0, \infty)$$

Exercise

Determine the domain and range of $f^{-1}: f(x) = \frac{5}{x+3}$ (Hint: first find the domain and range of f)

Solution

$$\text{Domain of } f^{-1} = \text{Range of } f: \mathbb{R} - \{0\} \quad (-\infty, 0) \cup (0, \infty)$$

$$\text{Range of } f^{-1} = \text{Domain of } f: \mathbb{R} - \{-3\} \quad (-\infty, -3) \cup (-3, \infty)$$

Exercise

Determine the domain and range of $f^{-1}: f(x) = \frac{4x+5}{3x-8}$ (Hint: first find the domain and range of f)

Solution

$$\text{Domain of } f^{-1} = \text{Range of } f: \mathbb{R} - \left\{\frac{8}{3}\right\} \quad \left(-\infty, \frac{8}{3}\right) \cup \left(\frac{8}{3}, \infty\right)$$

$$\text{Range of } f^{-1} = \text{Domain of } f: \mathbb{R} - \left\{\frac{4}{3}\right\} \quad \left(-\infty, \frac{4}{3}\right) \cup \left(\frac{4}{3}, \infty\right)$$

Exercise

For the given function $f(x) = \frac{2x}{x-1}$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

$$a) \quad f(a) = f(b)$$

$$\frac{2a}{a-1} = \frac{2b}{b-1}$$

$$2ab - 2a = 2ab - 2b$$

$$-2a = -2b$$

$$\underline{a = b} \quad \checkmark$$

$\therefore f(x)$ is one-to-one function.

$$b) \quad y = \frac{2x}{x-1}$$

$$x = \frac{2y}{y-1}$$

$$xy - x = 2y$$

$$(x-2)y = x$$

$$y = \frac{x}{x-2} = f^{-1}(x) \quad \bigg|$$

$$c) \quad \text{Domain of } f^{-1}(x) = \text{Range of } f(x): \underline{\mathbb{R} - \{1\}} \quad \bigg|$$

$$\text{Range of } f^{-1}(x) = \text{Domain of } f(x): \underline{\mathbb{R} - \{2\}} \quad \bigg|$$

Exercise

For the given function $f(x) = \frac{x}{x-2}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

$$a) \quad f(a) = f(b)$$

$$\frac{a}{a-2} = \frac{b}{b-2}$$

$$ab - 2a = ab - 2b$$

$$-2a = -2b$$

$$\underline{a = b} \quad \checkmark$$

$\therefore f(x)$ is one-to-one function.

$$b) \quad y = \frac{x}{x-2}$$

$$x = \frac{y}{y-2}$$

$$xy - 2x = y$$

$$(x-1)y = 2x$$

$$\underline{f^{-1}(x) = \frac{2x}{x-1} \mid}$$

c) Domain of $f^{-1}(x)$ = Range of $f(x)$: $\underline{\mathbb{R} - \{2\} \mid}$

Range of $f^{-1}(x)$ = Domain of $f(x)$: $\underline{\mathbb{R} - \{1\} \mid}$

Exercise

For the given function $f(x) = \frac{x+1}{x-1}$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\frac{a+1}{a-1} = \frac{b+1}{b-1}$$

$$ab - a + b - 1 = ab - b + a - 1$$

$$-2a = -2b$$

$$\underline{a = b} \quad \checkmark$$

$\therefore f(x)$ is one-to-one function.

b) $y = \frac{x+1}{x-1}$

$$x = \frac{y+1}{y-1}$$

$$xy - x = y + 1$$

$$(x-1)y = x+1$$

$$\underline{f^{-1}(x) = \frac{x+1}{x-1} \mid}$$

c) Domain of $f^{-1}(x)$ = Range of $f(x)$: $\underline{\mathbb{R} - \{1\} \mid}$

Range of $f^{-1}(x)$ = Domain of $f(x)$: $\underline{\mathbb{R} - \{1\} \mid}$

Exercise $f(x) = \frac{2x+1}{x+3}$

For the given function

- a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\frac{2a+1}{a+3} = \frac{2b+1}{b+3}$$

$$2ab + 6a + b + 3 = 2ab + 6b + a + 3$$

$$5a = 5b$$

$$\underline{a = b} \quad \checkmark$$

$\therefore f(x)$ is one-to-one function.

b) $y = \frac{2x+1}{x+3}$

$$x = \frac{2y+1}{y+3}$$

$$xy + 3x = 2y + 1$$

$$(x-2)y = -3x+1$$

$$\underline{f^{-1}(x) = \frac{-3x+1}{x-2}} \quad \checkmark$$

c) Domain of $f^{-1}(x)$ = Range of $f(x)$: $\underline{\mathbb{R} - \{-3\}}$

Range of $f^{-1}(x)$ = Domain of $f(x)$: $\underline{\mathbb{R} - \{2\}}$

Exercise

For the given function $f(x) = \frac{3x-1}{x-2}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\frac{3a-1}{a-2} = \frac{3b-1}{b-2}$$

$$3ab - 6a - b + 2 = 3ab - 6b - a + 2$$

$$-5a = -5b$$

$$\underline{a = b} \quad \checkmark$$

$\therefore f(x)$ is one-to-one function.

$$b) \quad y = \frac{3x-1}{x-2}$$

$$x = \frac{3y-1}{y-2}$$

$$xy - 2x = 3y - 1$$

$$(x-3)y = 2x-1$$

$$\underline{f^{-1}(x) = \frac{2x-1}{x-3} \mid}$$

$$c) \quad \text{Domain of } f^{-1}(x) = \text{Range of } f(x): \quad \underline{\mathbb{R} - \{2\} \mid}$$

$$\text{Range of } f^{-1}(x) = \text{Domain of } f(x): \quad \underline{\mathbb{R} - \{3\} \mid}$$

Exercise

For the given function $f(x) = \frac{3x-2}{x+4}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

$$a) \quad f(a) = f(b)$$

$$\frac{3a-2}{a+4} = \frac{3b-2}{b+4}$$

$$3ab + 12a - 2b - 8 = 3ab + 12b - 2a - 8$$

$$14a = 14b$$

$$\underline{a = b \mid} \quad \checkmark$$

$\therefore f(x)$ is one-to-one function.

$$b) \quad y = \frac{3x-2}{x+4}$$

$$x = \frac{3y-2}{y+4}$$

$$xy + 4x = 3y - 2$$

$$(x-3)y = -4x-2$$

$$\underline{f^{-1}(x) = \frac{-4x-2}{x-3} \mid}$$

$$c) \quad \text{Domain of } f^{-1}(x) = \text{Range of } f(x): \quad \underline{\mathbb{R} - \{-4\} \mid}$$

$$\text{Range of } f^{-1}(x) = \text{Domain of } f(x): \quad \underline{\mathbb{R} - \{3\} \mid}$$

Exercise

For the given function $f(x) = \frac{-3x-2}{x+4}$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\frac{-3a-2}{a+4} = \frac{-3b-2}{b+4}$$

$$-3ab - 12a - 2b - 8 = -3ab - 12b - 2a - 8$$

$$-10a = -10b$$

$$\underline{a = b} \quad \checkmark$$

$\therefore f(x)$ is one-to-one function.

b) $y = \frac{-3x-2}{x+4}$

$$x = \frac{-3y-2}{y+4}$$

$$xy + 4x = -3y - 2$$

$$(x+3)y = -4x-2$$

$$\underline{f^{-1}(x) = \frac{-4x-2}{x+3}}$$

c) Domain of $f^{-1}(x) = \text{Range of } f(x): \underline{\mathbb{R} - \{-4\}}$

Range of $f^{-1}(x) = \text{Domain of } f(x): \underline{\mathbb{R} - \{-3\}}$

Exercise

For the given function $f(x) = \sqrt{x-1} \quad x \geq 1$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\sqrt{a-1} = \sqrt{b-1}$$

$$(\sqrt{a-1})^2 = (\sqrt{b-1})^2$$

$$a-1 = b-1$$

$$\underline{a=b} \quad \checkmark$$

$\therefore f(x)$ is one-to-one function.

$$b) \quad y = \sqrt{x-1}$$

$$x = \sqrt{y-1}$$

$$x^2 = y-1$$

$$y = x^2 + 1$$

$$\underline{f^{-1}(x) = x^2 + 1 \quad x \geq 0}$$

$$c) \quad \text{Domain of } f(x) = \text{Range of } f^{-1}(x): \quad \underline{[1, \infty)}$$

$$\text{Range of } f(x) = \text{Domain of } f^{-1}(x): \quad \underline{[0, \infty)}$$

Exercise

For the given function $f(x) = \sqrt{2-x} \quad x \leq 2$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

$$a) \quad f(a) = f(b)$$

$$\sqrt{2-a} = \sqrt{2-b}$$

$$(\sqrt{2-a})^2 = (\sqrt{2-b})^2$$

$$2-a = 2-b$$

$$\underline{a=b} \quad \checkmark$$

$\therefore f(x)$ is one-to-one function.

$$b) \quad y = \sqrt{2-x}$$

$$x = \sqrt{2-y}$$

$$x^2 = 2-y$$

$$y = 2-x^2$$

$$\underline{f^{-1}(x) = 2 - x^2 \quad x \geq 0}$$

c) Domain of $f(x)$ = Range of $f^{-1}(x)$: $\underline{(-\infty, 2]}$

Range of $f(x)$ = Domain of $f^{-1}(x)$: $\underline{[0, \infty)}$

Exercise

For the given function $f(x) = x^2 + 4x \quad x \geq -2$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

$$\begin{aligned} x_{\text{vertex}} &= -\frac{4}{2} \\ &= -2 \end{aligned}$$

$$\begin{aligned} f(-2) &= 4 - 8 \\ &= -4 \end{aligned}$$

$$\text{Vertex} = (-2, -4)$$

a) Since, $f(x)$ is a restricted function with $x \geq -2$.

$x = -2$ is the line symmetry, therefore; $f(x)$ is one-to-one function.

b) $y = x^2 + 4x$

$$x = y^2 + 4y$$

$$y^2 + 4y - x = 0$$

$$y = \frac{-4 \pm \sqrt{16 + 4x}}{2}$$

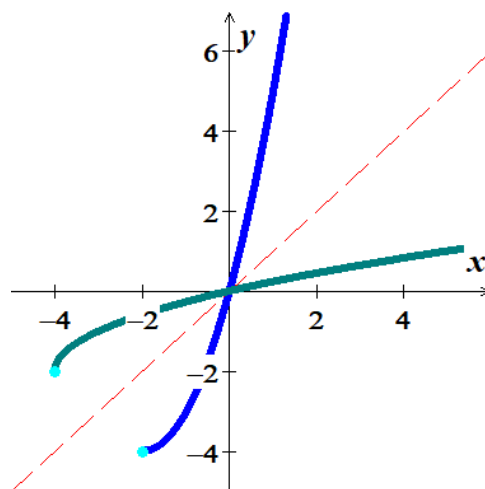
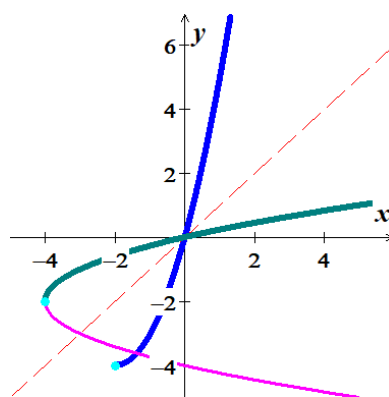
$$= \frac{-4 \pm 2\sqrt{4 + x}}{2}$$

$$= -2 + \sqrt{x + 4}$$

$$\underline{f^{-1}(x) = -2 + \sqrt{x + 4} \quad x \geq 0}$$

c) Domain of $f(x)$ = Range of $f^{-1}(x)$: $\underline{[-2, \infty)}$

Range of $f(x)$ = Domain of $f^{-1}(x)$: $\underline{[-4, \infty)}$



Exercise

For the given function $f(x) = 3x + 5$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$3a + 5 = 3b + 5$$

$$3a = 3b$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

b) $y = 3x + 5$

$$x = 3y + 5$$

$$x - 5 = 3y$$

$$\frac{x-5}{3} = y$$

$$\underline{f^{-1}(x) = \frac{x-5}{3}}$$

Interchange x and y

Solve for y

c) Domain of $f^{-1} = \text{Range of } f: \mathbb{R}$

Range of $f^{-1} = \text{Domain of } f: \mathbb{R}$

Exercise

For the given function $f(x) = \frac{1}{3x-2}$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\frac{1}{3a-2} = \frac{1}{3b-2}$$

$$3b - 2 = 3a - 2$$

$$3b = 3a$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

$$b) \quad y = \frac{1}{3x-2}$$

$$x = \frac{1}{3y-2}$$

Interchange x and y

$$x(3y-2) = 1$$

Solve for y

$$3xy - 2x = 1$$

$$3xy = 1 + 2x$$

$$\underline{f^{-1}(x) = \frac{1+2x}{3x}}$$

$$c) \quad \text{Domain of } f^{-1} = \text{Range of } f: \underline{\mathbb{R} - \left\{\frac{2}{3}\right\}}$$

$$\text{Range of } f^{-1} = \text{Domain of } f: \underline{\mathbb{R} - \{0\}}$$

Exercise

For the given function $f(x) = \frac{3x+2}{2x-5}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

$$a) \quad f(a) = f(b)$$

$$\frac{3a+2}{2a-5} = \frac{3b+2}{2b-5}$$

$$6ab - 15a + 4b - 10 = 6ab - 15b + 4a - 10$$

$$19a = 19b$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

$$b) \quad y = \frac{3x+2}{2x-5}$$

$$x = \frac{3y+2}{2y-5}$$

Interchange x and y

$$2xy - 5x = 2y + 2$$

Solve for y

$$(2x-3)y = 5x+2$$

$$\underline{f^{-1}(x) = \frac{5x+2}{2x-3}}$$

$$c) \quad \text{Domain of } f^{-1} = \text{Range of } f: \underline{\mathbb{R} - \left\{\frac{5}{2}\right\}}$$

$$\text{Range of } f^{-1} = \text{Domain of } f: \underline{\mathbb{R} - \left\{\frac{3}{2}\right\}}$$

Exercise

For the given function $f(x) = \frac{4x}{x-2}$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\frac{4a}{a-2} = \frac{4b}{b-2}$$

$$4ab - 8a = 4ab - 8b$$

$$-8a = -8b$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

b) $y = \frac{4x}{x-2}$

$$x = \frac{4y}{y-2}$$

$$xy - 2x = 4y$$

$$(x-4)y = 2x$$

$$f^{-1}(x) = \frac{2x}{x-4} \quad \Big|$$

c) Domain of $f^{-1} = \text{Range of } f: \underline{\mathbb{R} - \{2\}}$

Range of $f^{-1} = \text{Domain of } f: \underline{\mathbb{R} - \{4\}}$

Exercise

For the given function $f(x) = 2 - 3x^2; \quad x \leq 0$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$2 - 3a^2 = 2 - 3b^2$$

$$-3a^2 = -3b^2$$

$$a^2 = b^2$$

$$a = b \quad \text{since } x \leq 0$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

b) $y = 2 - 3x^2$

$$x = 2 - 3y^2$$

$$3y^2 = 2 - x$$

$$y^2 = \frac{2-x}{3}$$

$$\underline{f^{-1}(x) = -\sqrt{\frac{2-x}{3}}} \quad \text{Since } x < 0$$

c) Domain of f^{-1} = Range of f : \mathbb{R}

Range of f^{-1} = Domain of f : \mathbb{R}

Exercise

For the given function $f(x) = 2x^3 - 5$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$2a^3 - 5 = 2b^3 - 5$$

$$a^3 = b^3$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

b) $y = 2x^3 - 5$

$$y + 5 = 2x^3$$

$$\frac{y+5}{2} = x^3$$

$$x = \sqrt[3]{\frac{y+5}{2}}$$

$$\underline{f^{-1}(x) = \sqrt[3]{\frac{x+5}{2}}}$$

c) Domain of f^{-1} = Range of f : \mathbb{R}

Range of f^{-1} = Domain of f : \mathbb{R}

Exercise

For the given function $f(x) = \sqrt{3-x}$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$(\sqrt{3-a})^2 = (\sqrt{3-b})^2$$

$$3-a = 3-b$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

b) $y = \sqrt{3-x} \quad y \geq 0$

$$y = \sqrt{3-x}$$

$$y^2 = 3-x$$

$$x = 3 - y^2 \quad x \geq 0$$

$$\underline{f^{-1}(x) = 3 - x^2}$$

c) Domain of f^{-1} = Range of f : $\underline{(-\infty, 3]}$

Range of f^{-1} = Domain of f : $\underline{[0, \infty)}$

Exercise

For the given function $f(x) = \sqrt[3]{x} + 1$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\sqrt[3]{a} + 1 = \sqrt[3]{b} + 1$$

$$(\sqrt[3]{a})^3 = (\sqrt[3]{b})^3$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

b) $y = \sqrt[3]{x} + 1$

$$y = \sqrt[3]{x} + 1$$

$$y - 1 = \sqrt[3]{x}$$

$$(y - 1)^3 = x$$

$$\underline{f^{-1}(x) = (x - 1)^3}$$

c) Domain of f^{-1} = Range of f : \mathbb{R}

Range of f^{-1} = Domain of f : \mathbb{R}

Exercise

For the given function $f(x) = (x^3 + 1)^5$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$(a^3 + 1)^5 = (b^3 + 1)^5$$

$$a^3 + 1 = b^3 + 1$$

$$a^3 = b^3$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

b) $y = (x^3 + 1)^5$

$$y = (x^3 + 1)^5$$

$$\sqrt[5]{y} = x^3 + 1$$

$$\sqrt[5]{y} - 1 = x^3$$

$$x = \sqrt[3]{\sqrt[5]{y} - 1}$$

$$\underline{f^{-1}(x) = \sqrt[3]{\sqrt[5]{x} - 1}}$$

c) Domain of f^{-1} = Range of f : \mathbb{R}

Range of f^{-1} = Domain of f : \mathbb{R}

Exercise

For the given function $f(x) = x^2 - 6x$; $x \geq 3$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$a^2 - 6a = b^2 - 6b$$

$$a^2 - b^2 = 6a - 6b$$

$$(a - b)(a + b) = 6(a - b)$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

b) $y = x^2 - 6x$

$$x^2 - 6x - y = 0$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-y)}}{2(1)}$$

$$= \frac{6 \pm 4\sqrt{9+y}}{2}$$

$$= 3 \pm \sqrt{9+y}$$

Since $x \geq 3 \Rightarrow$ we can select $x = 3 + \sqrt{y+9}$

$$\therefore f^{-1}(x) = 3 + \sqrt{x+9}$$

c) Domain of f^{-1} = Range of f : $\mathbb{R} : \geq 3$

Range of f^{-1} = Domain of f : ≥ -9

Exercise

For the given function $f(x) = (x - 2)^3$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$(a - 2)^3 = (b - 2)^3$$

$$a - 2 = b - 2$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

$$b) \quad y = (x-2)^3$$

$$x = (y-2)^3$$

$$x^{1/3} = \left[(y-2)^3 \right]^{1/3}$$

$$x^{1/3} = y-2$$

$$\sqrt[3]{x} + 2 = y$$

$$\therefore \underline{f^{-1}(x) = \sqrt[3]{x} + 2}$$

c) Domain of f^{-1} = Range of f : \mathbb{R}

Range of f^{-1} = Domain of f : \mathbb{R}

Exercise

For the given function $f(x) = \frac{x+1}{x-3}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

$$a) \quad f(a) = f(b)$$

$$\frac{a+1}{a-3} = \frac{b+1}{b-3}$$

$$ab - 3a + b - 3 = ab - 3b + a - 3$$

$$-4a = -4b$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

$$b) \quad y = \frac{x+1}{x-3}$$

$$x = \frac{y+1}{y-3}$$

$$x(y-3) = y+1$$

$$xy - 3x = y+1$$

$$xy - y = 3x+1$$

$$y(x-1) = 3x+1$$

$$y = \frac{3x+1}{x-1} = f^{-1}(x)$$

c) Domain of f^{-1} = Range of f : $\mathbb{R} - \{3\}$

Range of f^{-1} = Domain of f : $\mathbb{R} - \{1\}$

Exercise

For the given function $f(x) = \frac{2x+1}{x-3}$

- Is $f(x)$ one-to-one function
- Find $f^{-1}(x)$, if it exists
- Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\frac{2a+1}{a-3} = \frac{2b+1}{b-3}$$

$$2ab - 6a + b - 3 = 2ab - 6b + a - 3$$

$$-7a = -7b$$

$$a = b$$

$\therefore f(x)$ is **1-1** & $f^{-1}(x)$ exists

b) $y = \frac{2x+1}{x-3}$

$$x = \frac{2y+1}{y-3}$$

$$xy - 3x = 2y + 1$$

$$y(x-2) = 3x+1$$

$$y = \frac{3x+1}{x-2} = f^{-1}(x)$$

c) Domain of f^{-1} = Range of f : $\mathbb{R} - \{3\}$

Range of f^{-1} = Domain of f : $\mathbb{R} - \{2\}$

Exercise

The function $w(x) = 2x + 24$ can be used to convert a U.S. women's shoe size into an Italian women's shoe size. Determine the function $w^{-1}(x)$ that can use to convert an Italian women's shoe size to its equivalent U.S. shoe size.

Solution

$$x = 2w^{-1}(x) + 24$$

$$2w^{-1}(x) = x - 24$$

$$\underline{w^{-1}(x) = \frac{1}{2}x - 12}$$



Exercise

The function $m(x) = 1.3x - 4.7$ can be used to convert a U.S. men's shoe size into an U.K. women's shoe size. Determine the function $m^{-1}(x)$ that can used to convert an U.K. men's shoe size to its equivalent U.S. shoe size.

Solution

$$x = 1.3m^{-1}(x) - 4.7$$

$$1.3m^{-1}(x) = x + 4.7$$

$$\frac{13}{10}m^{-1}(x) = x + \frac{47}{10}$$

$$\underline{m^{-1}(x) = \frac{10}{13}x + \frac{47}{13}}$$

$$\underline{w^{-1}(x) = \frac{1}{2}x - 12}$$

Exercise

A catering service use the function $c(x) = \frac{300 + 12x}{x}$ to determine the amount, in *dollars*, it charges per person for a sit-down dinner, where x is the number of people in attendance.

a) Find $c(30)$ and explain what it represents

b) Find $c^{-1}(x)$

c) Use $c^{-1}(x)$ to determine how many people attended a dinner for which the cost per person was \$15.00

Solution

$$\begin{aligned}
 a) \quad c(30) &= \frac{300 + 12(30)}{30} \\
 &= \frac{30 + 36}{3} \\
 &= \frac{66}{3} \\
 &= \$22 \quad |
 \end{aligned}$$

Catering service will charge \$22 per person to a sit-down dinner.

$$\begin{aligned}
 b) \quad cx &= 300 + 12x \\
 (c - 12)x &= 300 \\
 c^{-1}(x) &= \frac{300}{x - 12} \quad |
 \end{aligned}$$

$$\begin{aligned}
 c) \quad c^{-1}(15) &= \frac{300}{15 - 12} \\
 &= \frac{300}{3} \\
 &= 100 \quad |
 \end{aligned}$$

Exercise

A landscaping service use the function $c(x) = \frac{600 + 140x}{x}$ to determine the amount, in *dollars*, it charges per tree to deliver, where x is the number of trees.

- Find $c(5)$ and explain what it represents
- Find $c^{-1}(x)$
- Use $c^{-1}(x)$ to determine how many trees were delivered for which the cost per tree was \$160.00

Solution

$$\begin{aligned}
 d) \quad c(5) &= \frac{600 + 140(5)}{5} \\
 &= \frac{1,300}{5} \\
 &= \$260 \quad |
 \end{aligned}$$

Landscaping service will charge \$260 per tree to deliver.

$$\begin{aligned}
 e) \quad y &= \frac{600 + 140x}{x} \\
 x &= \frac{600 + 140y}{y} \\
 xy &= 600 + 140y \\
 (x - 140)y &= 600
 \end{aligned}$$

$$\underline{c^{-1}(x) = \frac{600}{x-140} \mid}$$

$$\begin{aligned} f) \quad c^{-1}(160) &= \frac{600}{160-140} \\ &= \frac{600}{20} \\ &= \underline{30 \mid} \end{aligned}$$