Section 3.3 – Integral Test

Nondecreasing Partial Sums

Suppose that $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \ge 0$ for all n. Then each partial sum is greater than or equal to its predecessor because $s_{n+1} = s_n + a_n$:

$$s_1 \le s_2 \le s_3 \le \cdots \le s_n \le s_{n+1} \le \cdots$$

Corollary

A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

Example

The series $\quad \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$

Solution

$$1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> \frac{2}{4} = \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> \frac{4}{8} = \frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right)}_{> \frac{8}{16} = \frac{1}{2}} + \cdots$$

The sum of the first 2 terms is $\frac{3}{2}$.

The sum of the next 2 terms is $\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

The sum of the next 4 terms is $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

The sum of 2^n terms ending with 2^{n+1} is $> \frac{2^n}{2^{n+1}} = \frac{1}{2}$

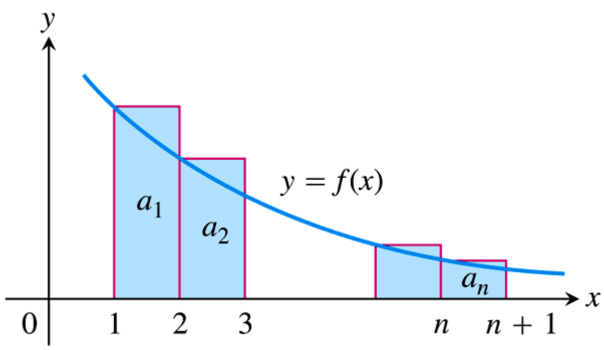
The sequence of partial sums is not bounded from above: If 2^k, the partial sum $s_n > \frac{k}{2}$.

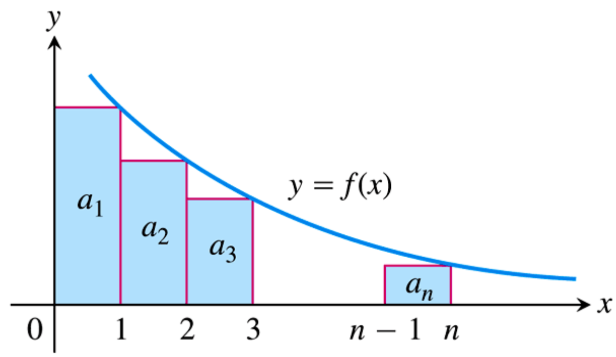
The harmonic series diverges.

The Integral Test

Theorem

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \ge N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)dx$ both converge or both diverge.





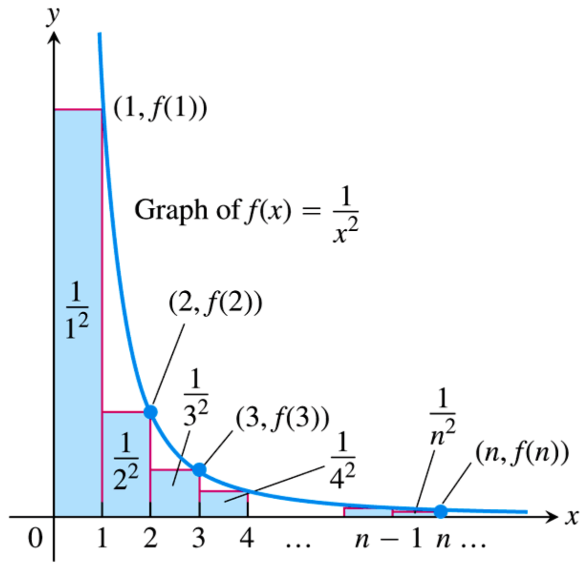
Example

Does the following series converge? $\sum_{n=1}^{\infty} \frac{1}{n^2}$

<u>***Solution***</u>

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$$

$$\begin{aligned} S_n &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} \\ &= f(1) + f(2) + f(3) + \cdots + f(n) \\ &< 1 + \int_1^n \frac{1}{x^2} dx \\ &< 1 + \lim_{b\to\infty} \left(-\frac{1}{x}\right)\Big|_1^b \\ &= 1 - \left(\frac{1}{\infty} - 1\right) \end{aligned}$$



$= 2$

Thus, the partial sums are bounded from above by 2 and the series converges.

$$\sum_{n-N}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\approx 1.64493$$

p-series

Example

Show that the ***p****-series* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$

(p a real constant) converges if $p > 1$, and diverges if $p \le 1$.

Solution

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^p}\,dx &= \lim_{b\to\infty}\left(\frac{x^{-p+1}}{-p+1}\Big|_1^b\right) \\
&= \frac{1}{1-p}\lim_{b\to\infty}\left[b^{-p+1} - 1\right] \\
&= \frac{1}{1-p}\lim_{b\to\infty}\left[\frac{1}{b^{p-1}} - 1\right] \qquad \boldsymbol{p > 1} \\
&= \frac{1}{1-p}(0-1) \\
&= \frac{1}{p-1}
\end{aligned}$$

The series ***converges*** when $p > 1$.

if $p \le 1$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{1}{1-p}\lim_{b\to\infty}\left(b^{1-p} - 1\right)$$

$$= \infty$$

The series ***diverges***.

Example

Does the following series converge $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+1}$

Solution

$$f(x)=\frac{1}{x^2+1} \quad \to \quad \int_1^{\infty}\frac{1}{x^2+1}dx=\lim_{b\to\infty}\left[\arctan x\right]_1^b$$

$$=\lim_{b\to\infty}\left(\arctan b-\arctan 1\right)$$

$$=\frac{\pi}{2}-\frac{\pi}{4}$$

$$=\frac{\pi}{4}$$

The series ***converges***, but we do not know the value of its sum

Bounds for the Remainder in the Integral Test

Suppose $\{a_n\}$ is a sequence of positive terms with $a_k = f(k)$, where f is a continuous positive decreasing function of x for all $x \geq n$, and that $\sum a_n$ converges to S. Then the remainder $R_n = S - s_n$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

Example

Estimate the sum of the series $\sum \frac{1}{n^2}$ with $n = 10$

Solution

$$\int_n^{\infty} \frac{1}{x^2}dx = \lim_{b\to\infty} \left(-\frac{1}{x}\right|_n^b$$

$$= -\lim_{b\to\infty}\left(\frac{1}{b} - \frac{1}{n}\right)$$

$$= \frac{1}{n}$$

$$s_{10} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{100} \approx 1.5497677$$

$$s_{10} + \frac{1}{11} \leq S \leq s_{10} + \frac{1}{10}$$

$$1.5497677 + \frac{1}{11} \leq S \leq 1.5497677 + \frac{1}{10}$$

$$1.64067679 \leq S \leq 1.6497677$$

If we approximate the sum S by the midpoint of this interval, then

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \approx 1.645222$$

The error in this approximation is less than half the length of the interval, so the error is less than 0.005.

Exercises Section 3.3 – Integral Test

(**1 – 22**) Use the ***Integral Test*** to determine if the series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{0.2}}$

2. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+4}$

3. $\displaystyle\sum_{n=1}^{\infty}e^{-2n}$

4. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$

5. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{e^{n/3}}$

6. $\displaystyle\sum_{n=1}^{\infty}\frac{n-4}{n^2-2n+1}$

7. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln n}$

8. $\displaystyle\sum_{k=1}^{\infty}\frac{k}{\sqrt{k^2+4}}$

9. $\displaystyle\sum_{k=1}^{\infty}ke^{-2k^2}$

10. $\displaystyle\sum_{k=0}^{\infty}\frac{1}{\sqrt{k+8}}$

11. $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k(\ln k)\ln(\ln k)}$

12. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^2+1}$

13. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+1}$

14. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$

15. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$

16. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{1/4}}$

17. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^5}$

18. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{e^n}$

19. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{e^n}$

20. $\displaystyle\sum_{k=1}^{\infty}\frac{|\sin k|}{k^2}$

21. $\displaystyle\sum_{k=1}^{\infty}\frac{k}{\left(k^2+1\right)^3}$

22. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{\sqrt[3]{k+10}}$

(**23 – 28**) Use the ***p-series Test*** to determine if the series converge or diverge.

23. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^9}$

24. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^6}$

25. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^{1/9}}$

26. $\displaystyle\sum_{k=1}^{\infty}k^{-2}$

27. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{\sqrt[4]{k}}$

28. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{\sqrt[4]{16k^2}}$

(29 – 54) Determine if the series converge or diverge

29. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^8}$

30. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3^k}$

31. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{5/2}}$

32. $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

33. $\displaystyle\sum_{n=0}^{\infty} \frac{10}{n^2+9}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n+1)(3n+4)}$

35. $\displaystyle\sum_{n=1}^{\infty} e^{-n}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

37. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{-8}{n}$

39. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$

40. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{4^n+3}$

41. $\displaystyle\sum_{n=1}^{\infty} n\sin\frac{1}{n}$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{10}}$

43. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(n-2)^4}$

44. $\displaystyle\sum_{n=2}^{\infty} \frac{n^e}{n^\pi}$

45. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k+3^k}{4^k}$

46. $\displaystyle\sum_{n=1}^{\infty} 2n^{-3/2}$

47. $\displaystyle\sum_{k=0}^{\infty} \frac{k}{\sqrt{k^2+4}}$

48. $\displaystyle\sum_{k=2}^{\infty} \frac{4}{k\ln^2 k}$

49. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$

50. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$

51. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$

52. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^\pi}$

53. $1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\frac{1}{4\sqrt{4}}+\frac{1}{5\sqrt{5}}+\cdots$

54. $1+\frac{1}{\sqrt[3]{4}}+\frac{1}{\sqrt[3]{9}}+\frac{1}{\sqrt[3]{16}}+\frac{1}{\sqrt[3]{25}}+\cdots$

55. Consider the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$, where p is a real number.

a) Use the Integral Test to determine the values of p for which this series converges.

b) Does this series converge faster for $p=2$ or $p=3$? Explain.

56. Consider the series $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)\,(\ln(\ln k))^{p}}$, where p is a real number.

a) For what values of p does this series converge?

b) Which he following series converge faster? Explain.

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^{2}} \quad \text{or} \quad \sum_{k=2}^{\infty} \frac{1}{k(\ln k)\,(\ln(\ln k))^{2}}$$

57. Consider a wedding cake of infinite height, each layer of which is a right circular cylinder of height 1. The bottom layer of the cake has a radius of 1, the second layer has a radius of $\frac{1}{2}$, the third layer has a radius of $\frac{1}{3}$, and the n^{th} layer has a radius of $\frac{1}{n}$.

a) To determine how much frosting is needed to cover the cake, find the area of the lateral (vertical sides of the wedding cake. What is the area of the horizontal surfaces of the cake?

b) Determine the volume of the cake.

c) Comment on your answer to parts (*a*) and (*b*)

58. The Riemann zeta function is the subject of extensive research and is associated with several renowned unsolved problems. Is its defined by $\zeta(x) = \sum_{k=1}^{\neq} \frac{1}{k^{x}}$, when x is a real number, the zeta function becomes a p-series. For even positive integers ρ, the value of $\zeta(\rho)$ is known exactly. For example,

$$\sum_{k=1}^{\neq} \frac{1}{k^{2}} = \frac{\pi^{2}}{6}, \quad \sum_{k=1}^{\neq} \frac{1}{k^{4}} = \frac{\pi^{4}}{90}, \quad \textit{and} \quad \sum_{k=1}^{\neq} \frac{1}{k^{6}} = \frac{\pi^{6}}{945}, \quad \ldots$$

a) Use the estimation techniques to approximate $\zeta(3)$ and $\zeta(5)$ (whose values are not known exactly) with a remainder less than 10^{-3}.

b) Determine the sum of the reciprocals of the squares of the odd positive integers by rearranging the terms of the series $(x=2)$ without changing the value of the series.

59. Consider a set of identical dominoes that are 2 *inches* long. The dominoes are stacked on top of each other with their long edges aligned so that each domino overhangs the one beneath is as far as possible

a) If there are n dominoes in the stack, what is the greatest distance that the top domino can be made to overhang the bottom domino? (*Hint*: Put the n^{th} domino beneath the previous $n-1$ dominoes.)

b) If we allow for infinitely many dominoes in the stack, what is the greatest distance that the top domino can be made to overhang the bottom domino?

60. A theorem states that the sequence of prime numbers $\{p_k\}$ satisfies $\lim\limits_{k\to\infty} \dfrac{p_k}{k\ln k}$.

Show that $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\ln k}$ diverges, which implies that the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{p_k}$

(A prime number is a positive integer number that is divisible only by 1 and itself).