Solution Section 3.1 – Proving Identities

Exercise

Prove the identity $\cos\theta\cot\theta+\sin\theta=\csc\theta$

Solution

$$\begin{aligned}
\cos\theta\cot\theta+\sin\theta &= \cos\theta\frac{\cos\theta}{\sin\theta}+\sin\theta\\
&= \frac{\cos^2\theta}{\sin\theta}+\sin\theta\\
&= \frac{\cos^2\theta}{\sin\theta}+\sin\theta\frac{\sin\theta}{\sin\theta}\\
&= \frac{\cos^2\theta}{\sin\theta}+\frac{\sin^2\theta}{\sin\theta}\\
&= \frac{\cos^2\theta+\sin^2\theta}{\sin\theta}\\
&= \frac{1}{\sin\theta}\\
&= \csc\theta
\end{aligned}$$

Exercise

Prove the identity $\sec\theta\cot\theta-\sin\theta=\dfrac{\cos^2\theta}{\sin\theta}$

Solution

$$\begin{aligned}
\sec\theta\cot\theta-\sin\theta &= \frac{1}{\cos\theta}\frac{\cos\theta}{\sin\theta}-\sin\theta\\
&= \frac{1}{\sin\theta}-\sin\theta\\
&= \frac{1-\sin^2\theta}{\sin\theta}\\
&= \frac{\cos^2\theta}{\sin\theta}
\end{aligned}$$

Exercise

Prove the identity $\dfrac{\csc\theta\tan\theta}{\sec\theta}=1$

Solution

$$\begin{aligned}
\frac{\csc\theta\tan\theta}{\sec\theta} &= \csc\theta\tan\theta\frac{1}{\sec\theta}\\
&= \frac{1}{\sin\theta}\frac{\sin\theta}{\cos\theta}\cos\theta\\
&= 1
\end{aligned}$$

Exercise

Prove the identity $(\sin\theta+\cos\theta)^2 = 1+2\sin\theta\cos\theta$

Solution

$$\begin{aligned}(\sin\theta+\cos\theta)^2 &= \sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta \\ &= 1+2\sin\theta\cos\theta\end{aligned}$$

Exercise

Prove the identity $\sin\theta(\sec\theta+\cot\theta) = \tan\theta+\cos\theta$

Solution

$$\begin{aligned}\sin\theta(\sec\theta+\cot\theta) &= \sin\theta\left(\frac{1}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right) \\ &= \sin\theta\frac{1}{\cos\theta}+\sin\theta\frac{\cos\theta}{\sin\theta} \\ &= \frac{\sin\theta}{\cos\theta}+\cos\theta \\ &= \tan\theta+\cos\theta \quad \checkmark\end{aligned}$$

Exercise

Prove the identity $\cos\theta(\csc\theta+\tan\theta) = \cot\theta+\sin\theta$

Solution

$$\begin{aligned}\cos\theta(\csc\theta+\tan\theta) &= \cos\theta\frac{1}{\sin\theta}+\cos\theta\frac{\sin\theta}{\cos\theta} \\ &= \cot\theta+\sin\theta \quad \checkmark\end{aligned}$$

Exercise

Prove the identity $\cos\theta(\sec\theta-\cos\theta) = \sin^2\theta$

Solution

$$\begin{aligned}\cos\theta(\sec\theta-\cos\theta) &= \cos\theta\frac{1}{\cos\theta}-\cos^2\theta \\ &= 1-\cos^2\theta \\ &= \sin^2\theta \quad \checkmark\end{aligned}$$

Exercise

Prove the identity $\cot\theta + \tan\theta = \csc\theta\sec\theta$

Solution

$$\cot\theta + \tan\theta = \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta}\frac{1}{\cos\theta}$$

$$= \csc\theta\sec\theta \quad \checkmark$$

Exercise

Prove $\tan x(\cos x + \cot x) = \sin x + 1$

Solution

$$\tan x(\cos x + \cot x) = \frac{\sin x}{\cos x}\left(\cos x + \frac{\cos x}{\sin x}\right)$$

$$= \cos x\frac{\sin x}{\cos x} + \frac{\sin x}{\cos x}\frac{\cos x}{\sin x}$$

$$= \sin x + 1 \quad \checkmark$$

Exercise

Prove $\dfrac{1-\cos^4\theta}{1+\cos^2\theta} = \sin^2\theta$

Solution

$$\frac{1-\cos^4\theta}{1+\cos^2\theta} = \frac{(1+\cos^2\theta)(1-\cos^2\theta)}{1+\cos^2\theta}$$

$$= 1-\cos^2\theta$$

$$= \sin^2\theta \quad \checkmark$$

Exercise

Prove $\dfrac{1-\sec x}{1+\sec x} = \dfrac{\cos x - 1}{\cos x + 1}$

Solution

$$\frac{1-\sec x}{1+\sec x} = \frac{1-\frac{1}{\cos x}}{1+\frac{1}{\cos x}}$$

$$= \frac{\frac{\cos x - 1}{\cos x}}{\frac{\cos x + 1}{\cos x}}$$

$$\boxed{= \frac{\cos x - 1}{\cos x + 1}} \quad \checkmark$$

Exercise

Prove $\quad \frac{\cos x}{1+\sin x} - \frac{1-\sin x}{\cos x} = 0$

Solution

$$\frac{\cos x}{1+\sin x} - \frac{1-\sin x}{\cos x} = \frac{\cos x}{\cos x}\frac{\cos x}{1-\sin x} - \frac{1+\sin x}{1+\sin x}\frac{1-\sin x}{\cos x}$$

$$= \frac{\cos^2 x - (1-\sin^2 x)}{\cos x(1+\sin x)}$$

$$= \frac{\cos^2 x - 1 + \sin^2 x)}{\cos x(1+\sin x)}$$

$$= \frac{1-1}{\cos x(1+\sin x)}$$

$$= \frac{0}{\cos x(1+\sin x)}$$

$$\boxed{= 0} \quad \checkmark$$

Exercise

Prove $\quad \frac{1+\cot^3 t}{1+\cot t} = \csc^2 t - \cot t$

Solution

$$\frac{1+\cot^3 t}{1+\cot t} = \frac{1+\frac{\cos^3 t}{\sin^3 t}}{1+\frac{\cos t}{\sin t}}$$

$$= \frac{\frac{\sin^3 t + \cos^3 t}{\sin^3 t}}{\frac{\sin t + \cos t}{\sin t}}$$

$$= \frac{\sin^3 t + \cos^3 t}{\sin^3 t} \cdot \frac{\sin t}{\sin t + \cos t}$$

$$= \frac{(\sin t + \cos t)(\sin^2 t - \sin t \cos t + \cos^2 t)}{\sin^2 t} \cdot \frac{1}{\sin t + \cos t}$$

$$= \frac{1 - \sin t \cos t}{\sin^2 t}$$

$$= \frac{1}{\sin^2 t} - \frac{\sin t \cos t}{\sin^2 t}$$

$$= \csc^2 t - \frac{\cos t}{\sin t}$$

$$\underline{= \csc^2 t - \cot t} \quad \checkmark$$

Exercise

Prove: $\tan x + \cot x = \sec x \csc x$

Solution

$$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$$

$$= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x}$$

$$= \frac{1}{\cos x \sin x}$$

$$= \frac{1}{\cos x}\frac{1}{\sin x}$$

$$\underline{= \sec x \ \csc x} \quad \checkmark$$

Exercise

Prove: $\dfrac{\tan x - \cot x}{\sin x \cos x} = \sec^2 x - \csc^2 x$

Solution

$$\frac{\tan x - \cot x}{\sin x \cos x} = \frac{\tan x}{\sin x \cos x} - \frac{\cot x}{\sin x \cos x}$$

$$= \tan x \frac{1}{\sin x \cos x} - \cot x \frac{1}{\sin x \cos x}$$

$$= \frac{\sin x}{\cos x}\frac{1}{\sin x \cos x} - \frac{\cos x}{\sin x}\frac{1}{\sin x \cos x}$$

$$= \frac{1}{\cos^2 x} - \frac{1}{\sin^2 x}$$

$$\underline{= \sec^2 x - \csc^2 x} \quad \checkmark$$

Exercise

Prove: $\dfrac{\sec x + \tan x}{\sec x - \tan x} = \dfrac{1 + 2\sin x + \sin^2 x}{\cos^2 x}$

Solution

$$\frac{\sec x + \tan x}{\sec x - \tan x} = \frac{\frac{1}{\cos x} + \frac{\sin x}{\cos x}}{\frac{1}{\cos x} - \frac{\sin x}{\cos x}} \frac{\cos x}{\cos x}$$

$$= \frac{\frac{1}{\cos x}\cos x + \frac{\sin x}{\cos x}\cos x}{\frac{1}{\cos x}\cos x - \frac{\sin x}{\cos x}\cos x}$$

$$= \frac{1+\sin x}{1-\sin x}$$

$$= \frac{1+\sin x}{1-\sin x}\frac{1+\sin x}{1+\sin x}$$

$$= \frac{(1+\sin x)^2}{1-\sin^2 x}$$

$$= \frac{1+2\sin x+\sin^2 x}{\cos^2 x} \quad \checkmark$$

Exercise

Prove the identity: $\sin^2 x - \cos^2 x = 2\sin^2 x - 1$

Solution

$$\sin^2 x - \cos^2 x = \sin^2 x - \left(1-\sin^2 x\right)$$

$$= \sin^2 x - 1 + \sin^2 x$$

$$= 2\sin^2 x - 1 \quad \checkmark$$

Exercise

Prove the identity: $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$

Solution

$$\sin^4 x - \cos^4 x = \left(\sin^2 x + \cos^2 x\right)\left(\sin^2 x - \cos^2 x\right)$$

$$= (1)\left(\sin^2 x - \cos^2 x\right)$$

$$= \sin^2 x - \cos^2 x \quad \checkmark$$

Exercise

Prove the identity: $\frac{\cos\alpha}{1+\sin\alpha} = \sec\alpha - \tan\alpha$

Solution

$$\frac{\cos\alpha}{1+\sin\alpha} = \frac{\cos\alpha}{1+\sin\alpha}\frac{1-\sin\alpha}{1-\sin\alpha}$$

$$= \frac{\cos\alpha - \cos\alpha\sin\alpha}{1-\sin^2\alpha}$$

$$= \frac{\cos\alpha - \cos\alpha\sin\alpha}{\cos^2\alpha}$$

$$= \frac{\cos\alpha}{\cos^2\alpha} - \frac{\cos\alpha\sin\alpha}{\cos^2\alpha}$$

$$= \frac{1}{\cos\alpha} - \frac{\sin\alpha}{\cos\alpha}$$

$$= \sec\alpha - \tan\alpha \quad \checkmark$$

Exercise

Prove the identity: $\dfrac{\sin\alpha}{1-\sin\alpha} - \dfrac{\cos\alpha}{1-\sin\alpha} = \dfrac{1-\cot\alpha}{\csc\alpha - 1}$

Solution

$$\frac{\sin\alpha}{1-\sin\alpha} - \frac{\cos\alpha}{1-\sin\alpha} = \frac{\sin\alpha - \cos\alpha}{1-\sin\alpha}$$

$$= \frac{\frac{\sin\alpha - \cos\alpha}{\sin\alpha}}{\frac{1-\sin\alpha}{\sin\alpha}}$$

$$= \frac{\frac{\sin\alpha}{\sin\alpha} - \frac{\cos\alpha}{\sin\alpha}}{\frac{1}{\sin\alpha} - \frac{\sin\alpha}{\sin\alpha}}$$

$$= \frac{1-\cot\alpha}{\csc\alpha - 1} \quad \checkmark$$

Exercise

Prove the identity: $\dfrac{\frac{1}{\tan x} + \cot x}{\frac{1}{\tan x} + \tan x} = \dfrac{2}{\sec^2 x}$

Solution

$$\frac{\frac{1}{\tan x} + \cot x}{\frac{1}{\tan x} + \tan x} = \frac{\frac{1}{\tan x} + \frac{1}{\tan x}}{\frac{1+\tan^2 x}{\tan x}}$$

$$= \frac{\frac{2}{\tan x}}{\frac{\sec^2 x}{\tan x}}$$

$$= \frac{2}{\sec^2 x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cot^2\theta+3\cot\theta-4}{\cot\theta+4}=\cot\theta-1$

Solution

$$\dfrac{\cot^2\theta+3\cot\theta-4}{\cot\theta+4}=\dfrac{(\cot\theta+4)(\cot\theta-1)}{\cot\theta+4}$$

$$=\cot\theta-1 \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin\theta}{1+\cos\theta}=\dfrac{1-\cos\theta}{\sin\theta}$

Solution

$$\dfrac{\sin\theta}{1+\cos\theta}=\dfrac{\sin\theta}{1+\cos\theta}\dfrac{1-\cos\theta}{1-\cos\theta}$$

$$=\dfrac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta}$$

$$=\dfrac{\sin\theta(1-\cos\theta)}{\sin^2\theta}$$

$$=\dfrac{1-\cos\theta}{\sin\theta} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\tan x(\csc x-\sin x)=\cos x$

Solution

$$\tan x(\csc x-\sin x)=\dfrac{\sin x}{\cos x}\left(\dfrac{1}{\sin x}-\sin x\right)$$

$$=\dfrac{\sin x}{\cos x}\left(\dfrac{1-\sin^2 x}{\sin x}\right)$$

$$=\dfrac{1}{\cos x}\left(\dfrac{\cos^2 x}{1}\right)$$

$$=\cos x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\sin x(\tan x\cos x-\cot x\cos x)=1-2\cos^2 x$

Solution

$$\sin x(\tan x\cos x-\cot x\cos x)=\sin x\cos x\left(\dfrac{\sin x}{\cos x}-\dfrac{\cos x}{\sin x}\right)$$

$$= \sin x \cos x \left(\frac{\sin^2 x - \cos^2 x}{\cos x \sin x} \right)$$

$$= 1 - \cos^2 x - \cos^2 x$$

$$\boxed{= 1 - 2\cos^2 x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $(1+\tan x)^2 + (\tan x - 1)^2 = 2\sec^2 x$

Solution

$$\begin{aligned}(1+\tan x)^2 + (\tan x - 1)^2 &= 1 + 2\tan x + \tan^2 x + 1 - 2\tan x + \tan^2 x \\ &= 2 + 2\tan^2 x \\ &= 2\left(1 + \tan^2 x\right) \\ &\boxed{= 2\sec^2 x} \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\sec x + \tan x = \dfrac{\cos x}{1 - \sin x}$

Solution

$$\begin{aligned}\sec x + \tan x &= \frac{1}{\cos x} + \frac{\sin x}{\cos x} \\ &= \frac{1+\sin x}{\cos x}\frac{1-\sin x}{1-\sin x} \\ &= \frac{1 - \sin^2 x}{\cos x(1 - \sin x)} \\ &= \frac{\cos^2 x}{\cos x(1 - \sin x)} \\ &\boxed{= \frac{\cos x}{1 - \sin x}} \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\tan x - 1}{\tan x + 1} = \dfrac{1 - \cot x}{1 + \cot x}$

Solution

$$\frac{\tan x - 1}{\tan x + 1} = \frac{\frac{1}{\cot x} - 1}{\frac{1}{\cot x} + 1}$$

$$= \frac{\frac{1-\cot x}{\cot x}}{\frac{1+\cot x}{\cot x}}$$

$$\boxed{= \frac{1-\cot x}{1+\cot x}} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $7\csc^2 x - 5\cot^2 x = 2\csc^2 x + 5$

Solution

$$\begin{aligned} 7\csc^2 x - 5\cot^2 x &= 7\csc^2 x - 5\left(\csc^2 x - 1\right) \\ &= 7\csc^2 x - 5\csc^2 x + 5 \\ &\boxed{= 2\csc^2 x + 5} \quad \checkmark \end{aligned}$$

Exercise

Prove the following equation is an identity: $1 - \frac{\cos^2 x}{1-\sin x} = -\sin x$

Solution

$$\begin{aligned} 1 - \frac{\cos^2 x}{1-\sin x} &= 1 - \frac{1-\sin^2 x}{1-\sin x} \\ &= 1 - \frac{(1-\sin x)(1+\sin x)}{1-\sin x} \\ &= 1 - (1+\sin x) \\ &= 1 - 1 - \sin x \\ &\boxed{= -\sin x} \quad \checkmark \end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{1-\cos x}{1+\cos x} = \frac{\sec x - 1}{\sec x + 1}$

Solution

$$\begin{aligned} \frac{1-\cos x}{1+\cos x} &= \frac{\frac{1}{\cos x} - \frac{\cos x}{\cos x}}{\frac{1}{\cos x} + \frac{\cos x}{\cos x}} \\ &\boxed{= \frac{\sec x - 1}{\sec x + 1}} \quad \checkmark \end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\sec x - 1}{\tan x} = \dfrac{\tan x}{\sec x + 1}$

Solution

$$\frac{\sec x - 1}{\tan x} = \frac{\sec x - 1}{\tan x}\frac{\sec x + 1}{\sec x + 1}$$

$$= \frac{\sec^2 x - 1}{\tan x(\sec x + 1)}$$

$$= \frac{\tan^2 x}{\tan x(\sec x + 1)}$$

$$= \boxed{\frac{\tan x}{\sec x + 1}} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos x}{\cos x - \sin x} = \dfrac{1}{1 - \tan x}$

Solution

$$\frac{\cos x}{\cos x - \sin x} = \frac{\frac{\cos x}{\cos x}}{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}}$$

$$= \boxed{\frac{1}{1 - \tan x}} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $(\sec x + \tan x)^2 = \dfrac{1 + \sin x}{1 - \sin x}$

Solution

$$(\sec x + \tan x)^2 = \left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right)^2$$

$$= \left(\frac{1 + \sin x}{\cos x}\right)^2$$

$$= \frac{(1 + \sin x)^2}{\cos^2 x}$$

$$= \frac{(1 + \sin x)^2}{1 - \sin^2 x}$$

$$= \frac{(1 + \sin x)^2}{(1 - \sin x)(1 + \sin x)}$$

$$= \boxed{\frac{1 + \sin x}{1 - \sin x}} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos x}{1+\tan x}-\dfrac{\sin x}{1+\cot x}=\cos x-\sin x$

Solution

$$\begin{aligned}
\frac{\cos x}{1+\tan x}-\frac{\sin x}{1+\cot x} &= \frac{\cos x}{1+\frac{\sin x}{\cos x}}-\frac{\sin x}{1+\frac{\cos x}{\sin x}} \\
&= \frac{\cos x}{\frac{\cos x+\sin x}{\cos x}}-\frac{\sin x}{\frac{\sin x+\cos x}{\sin x}} \\
&= \frac{\cos^2 x}{\cos x+\sin x}-\frac{\sin^2 x}{\sin x+\cos x} \\
&= \frac{\cos^2 x-\sin^2 x}{\cos x+\sin x} \\
&= \frac{(\cos x+\sin x)(\cos x-\sin x)}{\cos x+\sin x} \\
&= \cos x-\sin x \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\cot x+\csc x-1}{\cot x-\csc x+1}=\csc x+\cot x$

Solution

$$\begin{aligned}
\frac{\cot x+\csc x-1}{\cot x-\csc x+1} &= \frac{\cot x+\csc x-\left(\csc^2 x-\cot^2 x\right)}{\cot x-\csc x+1} \\
&= \frac{\cot x+\csc x-(\csc x-\cot x)(\csc x+\cot x)}{\cot x-\csc x+1} \\
&= \frac{(\csc x+\cot x)\left(1-(\csc x-\cot x)\right)}{\cot x-\csc x+1} \\
&= \frac{(\csc x+\cot x)(1-\csc x+\cot x)}{\cot x-\csc x+1} \\
&= \csc x+\cot x \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\tan x+\cot x}{\tan x-\cot x}=\dfrac{1}{\sin^2 x-\cos^2 x}$

Solution

$$\frac{\tan x+\cot x}{\tan x-\cot x}=\frac{\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}}{\frac{\sin x}{\cos x}-\frac{\cos x}{\sin x}}$$

$$= \frac{\frac{\sin^2 x + \cos^2 x}{\cos x \sin x}}{\frac{\sin^2 x - \cos^2 x}{\cos x \sin x}}$$

$$= \boxed{\frac{1}{\sin^2 x - \cos^2 x}} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\frac{1-\cot^2 x}{1+\cot^2 x} + 1 = 2\sin^2 x$

Solution

$$\frac{1-\cot^2 x}{1+\cot^2 x} + 1 = \frac{1-\cot^2 x + 1 + \cot^2 x}{1+\cot^2 x}$$

$$= \frac{2}{\csc^2 x}$$

$$= \boxed{2\sin^2 x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\frac{1+\cos x}{1-\cos x} - \frac{1-\cos x}{1+\cos x} = 4\cot x \csc x$

Solution

$$\frac{1+\cos x}{1-\cos x} - \frac{1-\cos x}{1+\cos x} = \frac{(1+\cos x)^2 - (1-\cos x)^2}{1-\cos^2 x}$$

$$= \frac{(1+\cos x + 1 - \cos x)(1+\cos x - 1 + \cos x)}{\sin^2 x} \qquad a^2 - b^2 = (a-b)(a+b)$$

$$= \frac{(2)(2\cos x)}{\sin^2 x}$$

$$= 4\frac{\cos x}{\sin x}\frac{1}{\sin x}$$

$$= \boxed{4\cot x \csc x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\frac{\sin^3 x - \cos^3 x}{\sin x - \cos x} = 1 + \sin x \cos x$

Solution

$$\frac{\sin^3 x - \cos^3 x}{\sin x - \cos x} = \frac{(\sin x - \cos x)\left(\sin^2 x + \sin x \cos x + \cos^2 x\right)}{\sin x - \cos x} \qquad a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$$

$$= \boxed{1 + \sin x \cos x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $1+\sec^2 x\sin^2 x=\sec^2 x$

Solution

$$1+\sec^2 x\sin^2 x=1+\frac{1}{\cos^2 x}\sin^2 x$$

$$=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}$$

$$=\frac{1}{\cos^2 x}$$

$$\underline{=\sec^2 x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{1+\csc x}{\sec x}=\cos x+\cot x$

Solution

$$\frac{1+\csc x}{\sec x}=\frac{1}{\sec x}+\frac{\csc x}{\sec x}$$

$$=\cos x+\frac{\frac{1}{\sin x}}{\frac{1}{\cos x}}$$

$$=\cos x+\frac{\cos x}{\sin x}$$

$$\underline{=\cos x+\cot x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\tan^2 x=\sec^2 x-\sin^2 x-\cos^2 x$

Solution

$$\sec^2 x-\sin^2 x-\cos^2 x=\frac{1}{\cos^2 x}-\left(\sin^2 x+\cos^2 x\right)$$

$$=\frac{1}{\cos^2 x}-1$$

$$=\frac{1-\cos^2 x}{\cos^2 x}$$

$$=\frac{\sin^2 x}{\cos^2 x}$$

$$\underline{=\tan^2 x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\frac{\sin x}{1-\cos x}+\frac{\sin x}{1+\cos x}=2\csc x$

Solution

$$\begin{aligned}\frac{\sin x}{1-\cos x}+\frac{\sin x}{1+\cos x}&=\sin x\left(\frac{1}{1-\cos x}+\frac{1}{1+\cos x}\right)\\&=\sin x\left(\frac{1+\cos x+1-\cos x}{1-\cos^2 x}\right)\\&=\sin x\left(\frac{2}{\sin^2 x}\right)\\&=\frac{2}{\sin x}\\&=\underline{2\csc x|}\quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cos^2(\alpha-\beta)-\cos^2(\alpha+\beta)=\sin^2(\alpha+\beta)-\sin^2(\alpha-\beta)$

Solution

$$\begin{aligned}\cos^2(\alpha-\beta)-\cos^2(\alpha+\beta)&=1-\sin^2(\alpha-\beta)-\left[1-\sin^2(\alpha+\beta)\right]\\&=1-\sin^2(\alpha-\beta)-1+\sin^2(\alpha+\beta)\\&=\underline{\sin^2(\alpha+\beta)-\sin^2(\alpha-\beta)|}\quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\tan x\csc x-\sec^2 x\cos x=0$

Solution

$$\begin{aligned}\tan x\csc x-\sec^2 x\cos x&=\frac{\sin x}{\cos x}\frac{1}{\sin x}-\frac{1}{\cos^2 x}\cos x\\&=\frac{1}{\cos x}-\frac{1}{\cos x}\\&=\underline{0|}\quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $(1+\tan x)^2-2\tan x=\frac{1}{(1-\sin x)(1+\sin x)}$

Solution

$$(1+\tan x)^2-2\tan x=1+2\tan x+\tan^2 x-2\tan x$$

$$= 1 + \tan^2 x$$
$$= \sec^2 x$$
$$= \frac{1}{\cos^2 x}$$
$$= \frac{1}{1 - \sin^2 x}$$
$$= \boxed{\frac{1}{(1 - \sin x)(1 + \sin x)}} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{3\csc^2 x - 5\csc x - 28}{\csc x - 4} = \dfrac{3}{\sin x} + 7$

Solution

$$\frac{3\csc^2 x - 5\csc x - 28}{\csc x - 4} = \frac{(3\csc x + 7)(\csc x - 4)}{\csc x - 4}$$
$$= 3\csc x + 7$$
$$= \boxed{\frac{3}{\sin x} + 7} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\left(\sec^2 x - 1\right)\left(\sec^2 x + 1\right) = \tan^4 x + 2\tan^2 x$

Solution

$$\left(\sec^2 x - 1\right)\left(\sec^2 x + 1\right) = \sec^4 x - 1 \qquad (a-b)(a+b) = a^2 - b^2 \quad a = \sec^2 x$$
$$= \left(\sec^2 x\right)^2 - 1$$
$$= \left(1 + \tan^2 x\right)^2 - 1$$
$$= 1 + 2\tan^2 x + \tan^4 x - 1$$
$$= \boxed{\tan^4 x + 2\tan^2 x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\csc x}{\cot x} - \dfrac{\cot x}{\csc x} = \dfrac{\sin x}{\cot x}$

Solution

$$\frac{\csc x}{\cot x} - \frac{\cot x}{\csc x} = \frac{\csc^2 x - \cot^2 x}{\cot x \csc x}$$

$$=\frac{\csc^2 x-\left(\csc^2 x-1\right)}{\cot x\csc x}$$

$$=\frac{\csc^2 x-\csc^2 x+1}{\cot x\csc x}$$

$$=\frac{1}{\cot x\frac{1}{\sin x}}$$

$$=\boxed{\frac{\sin x}{\cot x}}\quad\checkmark$$

Exercise

Prove the following equation is an identity: $\frac{1-\cos^2 x}{1+\cos x}=\frac{\sec x-1}{\sec x}$

Solution

$$\frac{1-\cos^2 x}{1+\cos x}=\frac{(1-\cos x)(1+\cos x)}{1+\cos x}$$

$$=1-\cos x$$

$$=1-\frac{1}{\sec x}$$

$$=\boxed{\frac{\sec x-1}{\sec x}}\quad\checkmark$$

Exercise

Prove the following equation is an identity: $\frac{\cos x}{1+\cos x}=\frac{\sec x-1}{\tan^2 x}$

Solution

$$\frac{\cos x}{1+\cos x}=\frac{\cos x}{1+\cos x}\frac{1-\cos x}{1-\cos x}$$

$$=\frac{\cos x-\cos^2 x}{\cos^2 x-1}$$

$$=\frac{\cos x-\cos^2 x}{\sin^2 x}\frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x}}$$

$$=\frac{\frac{1}{\cos x}-1}{\frac{\sin^2 x}{\cos^2 x}}$$

$$=\boxed{\frac{\sec x-1}{\tan^2 x}}\quad\checkmark$$

Exercise

Prove the following equation is an identity: $\frac{1-2\sin^2 x}{1+2\sin x\cos x}=\frac{\cos x-\sin x}{\cos x+\sin x}$

Solution

$$\begin{aligned}\frac{1-2\sin^2 x}{1+2\sin x\cos x} &= \frac{\cos^2 x+\sin^2 x-2\sin^2 x}{\cos^2 x+\sin^2 x+2\sin x\cos x}\\ &= \frac{\cos^2 x-\sin^2 x}{(\cos x+\sin x)^2}\\ &= \frac{(\cos x-\sin x)(\cos x+\sin x)}{(\cos x+\sin x)^2}\\ &= \frac{\cos x-\sin x}{\cos x+\sin x} \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $(\cos x-\sin x)^2+(\cos x+\sin x)^2=2$

Solution

$$\begin{aligned}(\cos x-\sin x)^2+(\cos x+\sin x)^2 &= \cos^2 x-2\sin x\cos x+\sin^2 x+\cos^2 x+2\sin x\cos x+\sin^2 x\\ &= \cos^2 x+\sin^2 x+\cos^2 x+\sin^2 x\\ &= 1+1\\ &= 2 \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\sin x}{1+\cos x}+\frac{1+\cos x}{\sin x}=2\csc x$

Solution

$$\begin{aligned}\frac{\sin x}{1+\cos x}+\frac{1+\cos x}{\sin x} &= \frac{\sin x\sin x+(1+\cos x)(1+\cos x)}{(1+\cos x)\sin x}\\ &= \frac{\sin^2 x+1+2\cos x+\cos^2 x}{(1+\cos x)\sin x}\\ &= \frac{1+1+2\cos x}{(1+\cos x)\sin x}\\ &= \frac{2+2\cos x}{(1+\cos x)\sin x}\\ &= \frac{2(1+\cos x)}{(1+\cos x)\sin x}\\ &= \frac{2}{\sin x}\\ &= 2\csc x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\sin x + \tan x}{\cot x + \csc x} = \sin x \tan x$

Solution

$$\frac{\sin x + \tan x}{\cot x + \csc x} = \frac{\sin x + \tan x}{\frac{1}{\tan x} + \frac{1}{\sin x}}$$

$$= \frac{\sin x + \tan x}{\frac{\sin x + \tan x}{\tan x \sin x}}$$

$$= (\sin x + \tan x)\frac{\tan x \sin x}{\sin x + \tan x}$$

$$\underline{= \tan x \sin x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\csc^2 x \sec^2 x = \sec^2 x + \csc^2 x$

Solution

$$\csc^2 x \sec^2 x = \frac{1}{\sin^2 x}\frac{1}{\cos^2 x}$$

$$= \frac{1}{\sin^2 x \cos^2 x}$$

$$= \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}$$

$$= \frac{\sin^2 x}{\sin^2 x \cos^2 x} + \frac{\cos^2 x}{\sin^2 x \cos^2 x}$$

$$= \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}$$

$$\underline{= \sec^2 x + \csc^2 x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\cos^2 x + 1 = 2\cos^2 x + \sin^2 x$

Solution

$$\cos^2 x + 1 = \cos^2 x + \cos^2 x + \sin^2 x$$

$$\underline{= 2\cos^2 x + \sin^2 x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $1-\dfrac{\cos^2 x}{1+\sin x}=\sin x$

Solution

$$\begin{aligned}1-\frac{\cos^2 x}{1+\sin x} &= 1-\frac{1-\sin^2 x}{1+\sin x}\\ &= 1-\frac{(1-\sin x)(1+\sin x)}{1+\sin x}\\ &= 1-(1-\sin x)\\ &= 1-1+\sin x\\ &= \sin x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cot^2 x=(\csc x-1)(\csc x+1)$

Solution

$$\begin{aligned}\cot^2 x &= \csc^2 x-1\\ &= (\csc x-1)(\csc x+1) \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $10\csc^2 x-6\cot^2 x=4\csc^2 x+6$

Solution

$$\begin{aligned}10\csc^2 x-6\cot^2 x &= 10\csc^2 x-6\left(\csc^2 x-1\right)\\ &= 10\csc^2 x-6\csc^2 x+6\\ &= 4\csc^2 x+6 \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\csc x+\cot x}{\tan x+\sin x}=\csc x\cot x$

Solution

$$\begin{aligned}\frac{\csc x+\cot x}{\tan x+\sin x} &= \frac{\csc x+\cot x}{\frac{1}{\cot x}+\frac{1}{\csc x}}\\ &= \frac{\csc x+\cot x}{\frac{\csc x+\cot x}{\cot x\csc x}}\\ &= \csc x+\cot x\,\frac{\cot x\csc x}{\csc x+\cot x}\\ &= \cot x\csc x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{1-\sec x}{\tan x}+\frac{\tan x}{1-\sec x}=-2\csc x$

Solution

$$\frac{1-\sec x}{\tan x}+\frac{\tan x}{1-\sec x}=\frac{(1-\sec x)(1-\sec x)+\tan^2 x}{\tan x(1-\sec x)}$$

$$=\frac{(1-\sec x)^2+\sec^2 x-1}{\tan x(1-\sec x)}$$

$$=\frac{(1-\sec x)^2+(\sec x+1)(\sec x-1)}{\tan x(1-\sec x)}$$

$$=\frac{(1-\sec x)^2-(\sec x+1)(1-\sec x)}{\tan x(1-\sec x)}$$

$$=\frac{(1-\sec x)\left[(1-\sec x)-(\sec x+1)\right]}{\tan x(1-\sec x)}$$

$$=\frac{1-\sec x-\sec x-1}{\tan x}$$

$$=\frac{-2\sec x}{\tan x}$$

$$=-2\frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}}$$

$$=-2\frac{1}{\sin x}$$

$$\underline{=-2\csc x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\csc x-\sin x=\cos x\cot x$

Solution

$$\csc x-\sin x=\frac{1}{\sin x}-\sin x$$

$$=\frac{1-\sin^2 x}{\sin x}$$

$$=\frac{\cos^2 x}{\sin x}$$

$$=\cos x\frac{\cos x}{\sin x}$$

$$\underline{=\cos x\cot x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\frac{\tan x+\sec x}{\sec x}-\frac{\tan x+\sec x}{\tan x}=-\cos x\cot x$

Solution

$$\begin{aligned}\frac{\tan x+\sec x}{\sec x}-\frac{\tan x+\sec x}{\tan x}&=\frac{(\tan x+\sec x)\tan x-\sec x(\tan x+\sec x)}{\sec x\tan x}\\&=\frac{\tan^2 x+\sec x\tan x-\sec x\tan x-\sec^2 x}{\sec x\tan x}\\&=\frac{\tan^2 x-\sec^2 x}{\sec x\tan x} \qquad 1+\tan^2\alpha=\sec^2\alpha\\&=\frac{-1}{\sec x\tan x}\\&=-\frac{1}{\sec x}\frac{1}{\tan x}\\&=\boxed{-\cos x\cot x}\ \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cot^3 x=\cot x\left(\csc^2 x-1\right)$

Solution

$$\begin{aligned}\cot^3 x&=\cot x\cot^2 x\\&=\boxed{\cot x\left(\csc^2 x-1\right)}\ \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\cot^2 x}{\csc x-1}=\frac{1+\sin x}{\sin x}$

Solution

$$\begin{aligned}\frac{\cot^2 x}{\csc x-1}&=\frac{\csc^2 x-1}{\csc x-1}\\&=\frac{(\csc x-1)(\csc x+1)}{\csc x-1}\\&=\csc x+1\\&=\frac{1}{\sin x}+1\\&=\boxed{\frac{1+\sin x}{\sin x}}\ \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cot^2 x+\csc^2 x=2\csc^2 x-1$

Solution

$$\cot^2 x + \csc^2 x = \csc^2 x - 1 + \csc^2 x$$
$$= 2\csc^2 x - 1 \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cot^2 x}{1+\csc x} = \csc x - 1$

Solution

$$\frac{\cot^2 x}{1+\csc x} = \frac{\csc^2 x - 1}{1+\csc x}$$
$$= \frac{(\csc x - 1)(\csc x + 1)}{1+\csc x}$$
$$= \csc x - 1 \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\sec^4 x - \tan^4 x = \sec^2 x + \tan^2 x$

Solution

$$\sec^4 x - \tan^4 x = \left(\sec^2 x + \tan^2 x\right)\left(\sec^2 x - \tan^2 x\right) \qquad a^2 - b^2 = (a-b)(a+b)$$
$$= \left(\sec^2 x + \tan^2 x\right)(1)$$
$$= \sec^2 x + \tan^2 x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos x}{1+\sin x} + \dfrac{1+\sin x}{\cos x} = 2\sec x$

Solution

$$\frac{\cos x}{1+\sin x} + \frac{1+\sin x}{\cos x} = \frac{\cos^2 x + (1+\sin x)^2}{(1+\sin x)\cos x}$$
$$= \frac{\cos^2 x + 1 + 2\sin x + \sin^2 x}{(1+\sin x)\cos x}$$
$$= \frac{2+2\sin x}{(1+\sin x)\cos x}$$
$$= \frac{2(1+\sin x)}{(1+\sin x)\cos x}$$
$$= \frac{2}{\cos x}$$
$$= 2\sec x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\frac{\sin x+\cos x}{\sin x-\cos x}=\frac{1+2\sin x\cos x}{2\sin^2 x-1}$

Solution

$$\frac{\sin x+\cos x}{\sin x-\cos x}=\frac{\sin x+\cos x}{\sin x-\cos x}\frac{\sin x+\cos x}{\sin x+\cos x}$$
$$=\frac{\sin^2 x+2\sin x\cos x+\cos^2 x}{\sin^2 x-\cos^2 x}$$
$$=\frac{1+2\sin x\cos x}{\sin^2 x-\left(1-\sin^2 x\right)}$$
$$=\frac{1+2\sin x\cos x}{\sin^2 x-1+\sin^2 x}$$
$$=\frac{1+2\sin x\cos x}{2\sin^2 x-1} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\frac{\csc x-1}{\csc x+1}=\frac{\cot^2 x}{\csc^2 x+2\csc x+1}$

Solution

$$\frac{\csc x-1}{\csc x+1}=\frac{\csc x-1}{\csc x+1}\frac{\csc x+1}{\csc x+1}$$
$$=\frac{\csc^2 x-1}{\csc^2 x+2\csc x+1}$$
$$=\frac{\cot^2 x}{\csc^2 x+2\csc x+1} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\csc^4 x-\cot^4 x=\csc^2 x+\cot^2 x$

Solution

$$\csc^4 x-\cot^4 x=\left(\csc^2 x+\cot^2 x\right)\left(\csc^2 x-\cot^2 x\right)$$
$$=\left(\csc^2 x+\cot^2 x\right)(1)$$
$$=\csc^2 x+\cot^2 x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\tan\left(\frac{\pi}{4}+x\right)=\cot\left(\frac{\pi}{4}-x\right)$

Solution

$$\tan\left(\frac{\pi}{4}+x\right)=\cot\left[\frac{\pi}{2}-\left(\frac{\pi}{4}+x\right)\right]$$
$$=\cot\left[\frac{\pi}{2}-\frac{\pi}{4}-x\right]$$
$$=\cot\left(\frac{\pi}{4}-x\right) \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\frac{\sin\theta}{1+\sin\theta}-\frac{\sin\theta}{1-\sin\theta}=-2\tan^2\theta$

Solution

$$\frac{\sin\theta}{1+\sin\theta}-\frac{\sin\theta}{1-\sin\theta}=\sin\theta\left[\frac{1-\sin\theta-(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}\right]$$
$$=\sin\theta\left[\frac{1-\sin\theta-1-\sin\theta}{1-\sin^2\theta}\right]$$
$$=\sin\theta\left(\frac{-2\sin\theta}{\cos^2\theta}\right)$$
$$=-2\frac{\sin^2\theta}{\cos^2\theta}$$
$$=-2\tan^2\theta \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\csc^2 x-\cos^2 x\csc^2 x=1$

Solution

$$\csc^2 x-\cos^2 x\csc^2 x=\csc^2 x\left(1-\cos^2 x\right)$$
$$=\frac{1}{\sin^2 x}\left(\sin^2 x\right)$$
$$=1 \quad \checkmark$$

Exercise

Prove the following equation is an identity: $1-2\sin^2 x=2\cos^2 x-1$

Solution

$$1-2\sin^2 x=1-2\left(1-\cos^2 x\right)$$
$$=1-2+2\cos^2 x$$
$$=2\cos^2 x-1 \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\csc^2 x - \cos x \sec x = \cot^2 x$

Solution

$$\begin{aligned}
\csc^2 x - \cos x \sec x &= \frac{1}{\sin^2 x} - \cos x \frac{1}{\cos x} \\
&= \frac{1}{\sin^2 x} - 1 \\
&= \frac{1 - \sin^2 x}{\sin^2 x} \\
&= \frac{\cos^2 x}{\sin^2 x} \\
&= \cot^2 x \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $(\sec x - \tan x)(\sec x + \tan x) = 1$

Solution

$$\begin{aligned}
(\sec x - \tan x)(\sec x + \tan x) &= \sec^2 x - \tan^2 x \\
&= 1 + \tan^2 x - \tan^2 x \\
&= 1 \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\left(1 + \tan^2 x\right)\left(1 - \sin^2 x\right) = 1$

Solution

$$\begin{aligned}
\left(1 + \tan^2 x\right)\left(1 - \sin^2 x\right) &= \sec^2 x \cos^2 x \\
&= \frac{1}{\cos^2 x} \cos^2 x \\
&= 1 \quad \checkmark
\end{aligned}$$

Solution Section 3.2 – Sum and Difference Formulas

Exercise

Write the expression as a single trigonometric function $\sin 8x \cos x - \cos 8x \sin x$

Solution

$$\sin 8x \cos x - \cos 8x \sin x = \sin(8x - x)$$
$$= \sin 7x$$

Exercise

Show that $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$

Solution

$$\sin\left(x - \frac{\pi}{2}\right) = \sin x \cos\frac{\pi}{2} - \cos x \sin\frac{\pi}{2}$$
$$= \sin x \cdot (0) - \cos x \cdot (1)$$
$$= -\cos x$$

Exercise

If $\sin A = \frac{4}{5}$ with A in QII, and $\cos B = -\frac{5}{13}$ with B in QIII, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$
d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

Solution

$\cos A = -\frac{3}{5}$ $\sin B = -\frac{12}{13}$

a)
$$\sin(A+B) = \sin A\cos B + \sin B\cos A$$
$$= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) + \left(-\frac{12}{13}\right)\left(-\frac{3}{5}\right)$$
$$= -\frac{20}{65} + \frac{36}{65}$$
$$= \frac{16}{65}$$

b)
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$= \left(-\frac{3}{5}\right)\left(-\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right)$$
$$= \frac{15}{65} + \frac{48}{65}$$
$$= \frac{63}{65}$$

c)
$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{16}{63}$$

d)
$$\sin(A-B) = \sin A\cos B - \sin B\cos A$$
$$= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) - \left(-\frac{12}{13}\right)\left(-\frac{3}{5}\right)$$
$$= -\frac{20}{65} - \frac{36}{65}$$
$$= -\frac{56}{65}$$

e)
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$
$$= \left(-\frac{3}{5}\right)\left(-\frac{5}{13}\right) + \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right)$$
$$= \frac{15}{65} - \frac{48}{65}$$
$$= -\frac{33}{65}$$

f)
$$\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)} = \frac{56}{33}$$

Exercise

If $\sin A = \frac{3}{5}$ $(A \in QII)$, and $\cos B = -\frac{12}{13}$ $(B \in QIII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

Solution

$\cos A = -\frac{4}{5}$ $\sin B = -\frac{5}{13}$

a) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$= \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) + \left(-\frac{4}{5}\right)\left(-\frac{5}{13}\right)$$

$$= \frac{-36+20}{65}$$

$$= -\frac{16}{65}$$

b) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$$= \left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right) - \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right)$$

$$= \frac{48+15}{65}$$

$$= \frac{63}{65}$$

c) $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)} = -\dfrac{16}{63}$

d) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$$= \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) - \left(-\frac{4}{5}\right)\left(-\frac{5}{13}\right)$$

$$= \frac{-36-20}{65}$$

$$= -\frac{56}{65}$$

e) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$$= \left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right) + \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right)$$

$$= \frac{48-15}{65}$$

$$= \frac{33}{65}$$

f) $\tan(A-B) = \dfrac{\sin(A-B)}{\cos(A-B)} = -\dfrac{56}{33}$

Exercise

If $\sin A = \frac{1}{\sqrt{5}}$ $(A \in QI)$, and $\tan B = \frac{3}{4}$ $(B \in QI)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

Solution

$\cos A = \sqrt{1-\frac{1}{5}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$ $\sin B = \frac{3}{5};$ $\cos B = \frac{4}{5}$

a) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$= \left(\frac{1}{\sqrt{5}}\right)\left(\frac{4}{5}\right) + \left(\frac{3}{5}\right)\left(\frac{2}{\sqrt{5}}\right)$$

$$= \frac{4+6}{5\sqrt{5}}$$

$$= \frac{10}{5\sqrt{5}}$$

$$= \frac{2}{\sqrt{5}}$$

b) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$$= \left(\frac{2}{\sqrt{5}}\right)\left(\frac{4}{5}\right) - \left(\frac{1}{\sqrt{5}}\right)\left(\frac{3}{5}\right)$$

$$= \frac{8-3}{5\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}}$$

c) $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)} = 2$

d) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

e) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$$=\left(\frac{1}{\sqrt{5}}\right)\left(\frac{4}{5}\right)-\left(\frac{3}{5}\right)\left(\frac{2}{\sqrt{5}}\right)$$
$$=\frac{4-6}{5\sqrt{5}}$$
$$=-\frac{2}{5\sqrt{5}}$$

$$=\left(\frac{2}{\sqrt{5}}\right)\left(\frac{4}{5}\right)+\left(\frac{1}{\sqrt{5}}\right)\left(\frac{3}{5}\right)$$
$$=\frac{8+3}{5\sqrt{5}}$$
$$=\frac{11}{5\sqrt{5}}$$

f) $\tan(A-B)=\dfrac{\sin(A-B)}{\cos(A-B)}=-\dfrac{2}{11}$

Exercise

If $\sin A=\frac{3}{5}\ (A\in QII)$, and $\cos B=\frac{12}{13}\ (B\in QIV)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

Solution

$\cos A=-\frac{4}{5}$ $\sin B=-\frac{5}{13}$

a) $\sin(A+B)=\sin A\cos B+\cos A\sin B$
$$=\left(\tfrac{3}{5}\right)\left(\tfrac{12}{13}\right)+\left(-\tfrac{4}{5}\right)\left(-\tfrac{5}{13}\right)$$
$$=\frac{36+20}{65}$$
$$=\frac{56}{65}$$

b) $\cos(A+B)=\cos A\cos B-\sin A\sin B$
$$=\left(-\tfrac{4}{5}\right)\left(\tfrac{12}{13}\right)-\left(\tfrac{3}{5}\right)\left(-\tfrac{5}{13}\right)$$
$$=\frac{-48-15}{65}$$
$$=-\frac{63}{65}$$

c) $\tan(A+B)=\dfrac{\sin(A+B)}{\cos(A+B)}=-\dfrac{56}{63}$

d) $\sin(A-B)=\sin A\cos B-\cos A\sin B$
$$=\left(\tfrac{3}{5}\right)\left(\tfrac{12}{13}\right)-\left(-\tfrac{4}{5}\right)\left(-\tfrac{5}{13}\right)$$
$$=\frac{36-20}{65}$$
$$=\frac{16}{65}$$

e) $\cos(A-B)=\cos A\cos B+\sin A\sin B$
$$=\left(-\tfrac{4}{5}\right)\left(\tfrac{12}{13}\right)+\left(\tfrac{3}{5}\right)\left(-\tfrac{5}{13}\right)$$
$$=\frac{-48-15}{65}$$
$$=-\frac{63}{65}$$

f) $\tan(A-B)=\dfrac{\sin(A-B)}{\cos(A-B)}=-\dfrac{16}{63}$

Exercise

If $\sin A=\frac{7}{25}\ (A\in QII)$, and $\cos B=-\frac{8}{17}\ (B\in QIII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

Solution

$\cos A=-\frac{24}{25}$ $\sin B=-\frac{5}{17}$

a) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$= \left(\frac{7}{25}\right)\left(-\frac{8}{17}\right) + \left(-\frac{24}{25}\right)\left(-\frac{5}{17}\right)$

$= \frac{-56+120}{425}$

$= \boxed{\frac{64}{425}}$

b) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$= \left(-\frac{24}{25}\right)\left(-\frac{8}{17}\right) - \left(\frac{7}{25}\right)\left(-\frac{5}{17}\right)$

$= \frac{192+35}{425}$

$= \boxed{\frac{227}{425}}$

c) $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \boxed{\frac{64}{227}}$

d) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$= \left(\frac{7}{25}\right)\left(-\frac{8}{17}\right) - \left(-\frac{24}{25}\right)\left(-\frac{5}{17}\right)$

$= \frac{-56-120}{425}$

$= \boxed{-\frac{176}{425}}$

e) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$= \left(-\frac{24}{25}\right)\left(-\frac{8}{17}\right) + \left(\frac{7}{25}\right)\left(-\frac{5}{17}\right)$

$= \frac{192-35}{425}$

$= \boxed{\frac{157}{425}}$

f) $\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)} = \boxed{-\frac{176}{157}}$

Exercise

If $\cos A = -\frac{4}{5}$ $(A \in QII)$, and $\sin B = \frac{24}{25}$ $(B \in QII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$
d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

Solution

$\sin A = \frac{3}{5}$ $\quad\quad$ $\cos B = -\frac{7}{25}$

a) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$= \left(\frac{3}{5}\right)\left(-\frac{7}{25}\right) + \left(-\frac{4}{5}\right)\left(\frac{24}{25}\right)$

$= \frac{-21-96}{125}$

$= \boxed{-\frac{117}{125}}$

b) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$= \left(-\frac{4}{5}\right)\left(-\frac{7}{25}\right) - \left(\frac{3}{5}\right)\left(\frac{24}{25}\right)$

$= \frac{28-72}{125}$

$= \boxed{-\frac{44}{125}}$

c) $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \boxed{\frac{117}{44}}$

d) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$= \left(\frac{3}{5}\right)\left(-\frac{7}{25}\right) - \left(-\frac{4}{5}\right)\left(\frac{24}{25}\right)$

$= \frac{-21+96}{125}$

$= \boxed{\frac{75}{125}}$

e) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$= \left(-\frac{4}{5}\right)\left(-\frac{7}{25}\right) + \left(\frac{3}{5}\right)\left(\frac{24}{25}\right)$

$= \frac{28+72}{125}$

$= \boxed{\frac{100}{125}}$

f) $\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)} = \boxed{\frac{75}{100}}$

Exercise

If $\cos A = \frac{15}{17}$ $(A \in QI)$, and $\cos B = -\frac{12}{13}$ $(B \in QII)$, find *a)* $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

Solution

$\sin A = \frac{8}{17}$ $\sin B = \frac{5}{13}$

a) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$= \left(\frac{8}{17}\right)\left(-\frac{12}{13}\right) + \left(\frac{15}{17}\right)\left(\frac{5}{13}\right)$$

$$= \frac{-96+75}{221}$$

$$\boxed{= -\frac{21}{221}}$$

b) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$$= \left(\frac{15}{17}\right)\left(-\frac{12}{13}\right) - \left(\frac{8}{17}\right)\left(\frac{5}{13}\right)$$

$$= \frac{-180-40}{221}$$

$$\boxed{= -\frac{220}{221}}$$

c) $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$ $\boxed{= \frac{21}{220}}$

d) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$$= \left(\frac{8}{17}\right)\left(-\frac{12}{13}\right) - \left(\frac{15}{17}\right)\left(\frac{5}{13}\right)$$

$$= \frac{-96-75}{221}$$

$$\boxed{= -\frac{171}{221}}$$

e) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$$= \left(\frac{15}{17}\right)\left(-\frac{12}{13}\right) + \left(\frac{8}{17}\right)\left(\frac{5}{13}\right)$$

$$= \frac{-180+40}{221}$$

$$\boxed{= -\frac{140}{221}}$$

f) $\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)}$ $\boxed{= \frac{171}{140}}$

Exercise

If $\sin A = -\frac{3}{5}\ (A \in QIV)$, and $\sin B = \frac{7}{25}\ (B \in QII)$, find

a) $\sin(A+B)$ *b)* $\cos(A+B)$ *c)* $\tan(A+B)$

d) $\sin(A-B)$ *e)* $\cos(A-B)$ *f)* $\tan(A-B)$

Solution

$\cos A = -\frac{4}{5}$ $\cos B = -\frac{24}{25}$

a) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$= \left(-\frac{3}{5}\right)\left(-\frac{24}{25}\right) + \left(-\frac{4}{5}\right)\left(\frac{7}{25}\right)$$

$$= \frac{72-28}{125}$$

$$\boxed{= \frac{44}{125}}$$

d) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$$= \left(-\frac{3}{5}\right)\left(-\frac{24}{25}\right) - \left(-\frac{4}{5}\right)\left(\frac{7}{25}\right)$$

$$= \frac{72+28}{125}$$

$$\boxed{= \frac{100}{125}}$$

b) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$$= \left(-\frac{4}{5}\right)\left(-\frac{24}{25}\right) - \left(-\frac{3}{5}\right)\left(\frac{7}{25}\right)$$

$$= \frac{96+21}{125}$$

e) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$$= \left(-\frac{4}{5}\right)\left(-\frac{24}{25}\right) + \left(-\frac{3}{5}\right)\left(\frac{7}{25}\right)$$

$$= \frac{96-21}{125}$$

$$= \boxed{\frac{117}{125}}$$

$$= \boxed{\frac{75}{125}}$$

c) $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \boxed{\frac{44}{117}}$

f) $\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)} = \boxed{\frac{100}{75}}$

Exercise

If $\sec A = \sqrt{5}$ with A in Q*I*, and $\sec B = \sqrt{10}$ with B in Q*I*, find $\sec(A+B)$

Solution

$$\sec(A+B) = \frac{1}{\cos(A+B)}$$

$$\sec A = \sqrt{5} \Rightarrow \cos A = \frac{1}{\sqrt{5}} \quad \sin A = \frac{2}{\sqrt{5}}$$

$$\sec B = \sqrt{10} \Rightarrow \cos B = \frac{1}{\sqrt{10}} \quad \sin B = \sqrt{1-\frac{1}{10}} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{5}}$$

$$\begin{aligned}\cos(A+B) &= \cos A\cos B - \sin A\sin B \\ &= \frac{1}{\sqrt{5}}\frac{1}{\sqrt{10}} - \frac{2}{\sqrt{5}}\frac{3}{\sqrt{10}} \\ &= \frac{1-6}{\sqrt{50}} \\ &= \frac{5}{\sqrt{50}} \\ &= \frac{5}{5\sqrt{2}} \\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

$$\boxed{\sec(A+B) = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}}$$

Exercise

Prove the identity $\quad \frac{\sin(A-B)}{\cos A\cos B} = \tan A - \tan B$

Solution

$$\begin{aligned}\frac{\sin(A-B)}{\cos A\cos B} &= \frac{\sin A\cos B - \sin B\cos A}{\cos A\cos B} \\ &= \frac{\sin A\cos B}{\cos A\cos B} - \frac{\sin B\cos A}{\cos A\cos B} \\ &= \frac{\sin A}{\cos A} - \frac{\sin B}{\cos B} \\ &= \boxed{\tan A - \tan B} \quad \checkmark\end{aligned}$$

Exercise

Prove the identity $\sec(A+B)=\dfrac{\cos(A-B)}{\cos^2 A-\sin^2 B}$

Solution

$$\sec(A+B)=\frac{1}{\cos(A+B)}$$

$$=\frac{1}{\cos A\cos B-\sin A\sin B}$$

$$=\frac{1}{\cos A\cos B-\sin A\sin B}\frac{\cos(A-B)}{\cos(A-B)}$$

$$=\frac{1}{\cos A\cos B-\sin A\sin B}\frac{\cos(A-B)}{\cos A\cos B+\sin A\sin B}$$

$$=\frac{\cos(A-B)}{\cos^2 A\cos^2 B-\sin^2 A\sin^2 B}$$

$$=\frac{\cos(A-B)}{\cos^2 A(1-\sin^2 B)-(1-\cos^2 A)\sin^2 B}$$

$$=\frac{\cos(A-B)}{\cos^2 A-\cos^2 A\sin^2 B-\sin^2 B+\cos^2 A\sin^2 B}$$

$$=\frac{\cos(A-B)}{\cos^2 A-\sin^2 B}\quad\checkmark$$

Exercise

Prove the identity $\dfrac{\cos 4\alpha}{\sin\alpha}-\dfrac{\sin 4\alpha}{\cos\alpha}=\dfrac{\cos 5\alpha}{\sin\alpha\cos\alpha}$

Solution

$$\frac{\cos 4\alpha}{\sin\alpha}-\frac{\sin 4\alpha}{\cos\alpha}=\frac{\cos 4\alpha\cos\alpha-\sin 4\alpha\sin\alpha}{\sin\alpha\cos\alpha}$$

$$=\frac{\cos(4\alpha+\alpha)}{\sin\alpha\cos\alpha}$$

$$=\frac{\cos 5\alpha}{\sin\alpha\cos\alpha}\quad\checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos(x+y)}{\cos(x-y)} = \dfrac{\cot y - \tan x}{\cot y + \tan x}$

Solution

$$\frac{\cos(x+y)}{\cos(x-y)} = \frac{\cos x\cos y - \sin x\sin y}{\cos x\cos y + \sin x\sin y}$$

$$= \frac{\frac{\cos x\cos y}{\cos x\sin y} - \frac{\sin x\sin y}{\cos x\sin y}}{\frac{\cos x\cos y}{\cos x\sin y} + \frac{\sin x\sin y}{\cos x\sin y}}$$

$$= \frac{\cot y - \tan x}{\cot y + \tan x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\cot y + \cot x}{\cot y - \cot x}$

Solution

$$\frac{\sin(x+y)}{\sin(x-y)} = \frac{\sin x\cos y + \sin y\cos x}{\sin x\cos y - \sin y\cos x}$$

$$= \frac{\frac{\sin x\cos y}{\sin x\sin y} + \frac{\sin y\cos x}{\sin x\sin y}}{\frac{\sin x\cos y}{\sin x\sin y} - \frac{\sin y\cos x}{\sin x\sin y}}$$

$$= \frac{\cot y + \cot x}{\cot y - \cot x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos(x+y)}{\cos(x-y)} = \dfrac{\cot y - \tan x}{\cot y + \tan x}$

Solution

$$\frac{\cos(x+y)}{\cos(x-y)} = \frac{\cos x\cos y - \sin x\sin y}{\cos x\cos y + \sin x\sin y}$$

$$= \frac{\frac{\cos x\cos y}{\cos x\sin y} - \frac{\sin x\sin y}{\cos x\sin y}}{\frac{\cos x\cos y}{\cos x\sin y} + \frac{\sin x\sin y}{\cos x\sin y}}$$

$$= \frac{\cot y - \tan x}{\cot y + \tan x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin(x-y)}{\sin x\cos y}=1-\cot x\tan y$

Solution

$$\frac{\sin(x-y)}{\sin x\cos y}=\frac{\sin x\cos y-\cos x\sin y}{\sin x\cos y}$$

$$=\frac{\sin x\cos y}{\sin x\cos y}-\frac{\cos x\sin y}{\sin x\cos y}$$

$$=1-\cot x\tan y \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin(x-y)}{\sin x\sin y}=\cot y-\cot x$

Solution

$$\frac{\sin(x-y)}{\sin x\sin y}=\frac{\sin x\cos y-\cos x\sin y}{\sin x\sin y}$$

$$=\frac{\sin x\cos y}{\sin x\sin y}-\frac{\cos x\sin y}{\sin x\sin y}$$

$$=\frac{\cos y}{\sin y}-\frac{\cos x}{\sin x}$$

$$=\cot y-\cot x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos(x+y)}{\cos x\sin y}=\cot y-\tan x$

Solution

$$\frac{\cos(x+y)}{\cos x\sin y}=\frac{\cos x\cos y-\sin x\sin y}{\cos x\sin y}$$

$$=\frac{\cos x\cos y}{\cos x\sin y}-\frac{\sin x\sin y}{\cos x\sin y}$$

$$=\frac{\cos y}{\sin y}-\frac{\sin x}{\cos x}$$

$$=\cot y-\tan x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin(x+y)}{\cos(x-y)} = \dfrac{1+\cot x\tan y}{\cot x+\tan y}$

Solution

$$\frac{\sin(x+y)}{\cos(x-y)} = \frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y+\sin x\sin y}$$

$$= \frac{\frac{\sin x\cos y}{\sin x\cos y}+\frac{\cos x\sin y}{\sin x\cos y}}{\frac{\cos x\cos y}{\sin x\cos y}+\frac{\sin x\sin y}{\sin x\cos y}}$$

$$= \frac{1+\cot x\tan y}{\cot x+\tan y} \quad \checkmark$$

Exercise

Prove the identity $\sin\left(\frac{\pi}{4}+x\right)+\sin\left(\frac{\pi}{4}-x\right)=\sqrt{2}\cos x$

Solution

$$\sin\left(\frac{\pi}{4}+x\right)+\sin\left(\frac{\pi}{4}-x\right) = \sin\frac{\pi}{4}\cos x+\sin x\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\cos x-\sin x\cos\frac{\pi}{4}$$

$$= \sin\frac{\pi}{4}\cos x+\sin\frac{\pi}{4}\cos x$$

$$= 2\sin\frac{\pi}{4}\cos x$$

$$= 2\frac{\sqrt{2}}{2}\cos x$$

$$= \sqrt{2}\cos x \quad \checkmark$$

Exercise

Prove the identity $\cos(A+B)+\cos(A-B)=2\cos A\cos B$

Solution

$$\cos(A+B)+\cos(A-B) = \cos A\cos B-\sin A\sin B+\cos A\cos B+\sin A\sin B$$

$$= \cos A\cos B+\cos A\cos B$$

$$= 2\cos A\cos B \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\sin(x-y)-\sin(y-x)=2\sin x\cos y-2\cos x\sin y$

Solution

$$\sin(x-y)-\sin(y-x) = \sin x\cos y-\sin y\cos x-(\sin y\cos x-\sin x\cos y)$$

$$= \sin x \cos y - \sin y \cos x - \sin y \cos x + \sin x \cos y$$

$$= 2\sin x \cos y - 2\sin y \cos x$$

Exercise

Prove the following equation is an identity: $\cos(x-y)+\cos(y-x) = 2\cos x \cos y + 2\sin x \sin y$

Solution

$$\cos(x-y)+\cos(y-x) = \cos x \cos y + \sin x \sin y + \cos y \cos x + \sin y \sin x$$

$$= 2\cos x \cos y + 2\sin x \sin y$$

Exercise

Prove the following equation is an identity: $\tan(x+y)\tan(x-y) = \dfrac{\tan^2 x - \tan^2 y}{1-\tan^2 x \tan^2 y}$

Solution

$$\tan(x+y)\tan(x-y) = \frac{\tan x + \tan y}{1-\tan x \tan y}\,\frac{\tan x - \tan y}{1+\tan x \tan y} \qquad (a+b)(a-b) = a^2 - b^2$$

$$= \frac{\tan^2 x - \tan^2 y}{1-\tan x^2 \tan^2 y}$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos(\alpha-\beta)}{\sin(\alpha+\beta)} = \dfrac{1-\tan\alpha\tan\beta}{\tan\alpha - \tan\beta}$

Solution

$$\frac{\cos(\alpha-\beta)}{\sin(\alpha+\beta)} = \frac{\cos\alpha\cos\beta + \sin\alpha\sin\beta}{\sin\alpha\cos\beta + \sin\beta\cos\alpha}$$

$$= \frac{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\sin\beta\cos\alpha}{\cos\alpha\cos\beta}}$$

$$= \frac{1+\tan\alpha\tan\beta}{\tan\alpha + \tan\beta}$$

Exercise

Prove the following equation is an identity: $\sec(x+y)=\dfrac{\cos x\cos y+\sin x\sin y}{\cos^2 x-\sin^2 y}$

Solution

$$\sec(x+y)=\frac{1}{\cos(x+y)}\frac{\cos(x-y)}{\cos(x-y)}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{(\cos x\cos y-\sin x\sin y)(\cos x\cos y+\sin x\sin y)}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{\cos^2 x\cos^2 y-\sin^2 x\sin^2 y}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{\cos^2 x\cos^2 y-\sin^2 x\sin^2 y}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{\cos^2 x\left(1-\sin^2 y\right)-\left(1-\cos^2 x\right)\sin^2 y}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{\cos^2 x-\cos^2 x\sin^2 y-\sin^2 y+\cos^2 x\sin^2 y}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{\cos^2 x-\sin^2 y} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\csc(x-y)=\dfrac{\sin x\cos y+\cos x\sin y}{\sin^2 x-\sin^2 y}$

Solution

$$\csc(x-y)=\frac{1}{\sin(x-y)}\frac{\sin x\cos y+\cos x\sin y}{\sin x\cos y+\cos x\sin y}$$

$$=\frac{\sin x\cos y+\cos x\sin y}{(\sin x\cos y-\cos x\sin y)(\sin x\cos y+\cos x\sin y)}$$

$$=\frac{\sin x\cos y+\cos x\sin y}{\sin^2 x\cos^2 y-\cos^2 x\sin^2 y}$$

$$=\frac{\sin x\cos y+\cos x\sin y}{\sin^2 x\left(1-\sin^2 y\right)-\left(1-\sin^2 x\right)\sin^2 y}$$

$$=\frac{\sin x\cos y+\cos x\sin y}{\sin^2 x-\sin^2 x\sin^2 y-\sin^2 y+\sin^2 x\sin^2 y}$$

$$=\frac{\sin x\cos y+\cos x\sin y}{\sin^2 x-\sin^2 y} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\tan(x+y)\tan(x-y)=\dfrac{\tan^2 x-\tan^2 y}{1-\tan^2 x\tan^2 y}$

Solution

$$\tan(x+y)\tan(x-y)=\frac{\tan x+\tan y}{1-\tan x\tan y}\,\frac{\tan x-\tan y}{1+\tan x\tan y} \qquad (a+b)(a-b)=a^2-b^2$$

$$=\boxed{\frac{\tan^2 x+\tan^2 y}{1-\tan x^2\tan^2 y}}\quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos(\alpha-\beta)}{\sin(\alpha+\beta)}=\dfrac{1-\tan\alpha\tan\beta}{\tan\alpha-\tan\beta}$

Solution

$$\frac{\cos(\alpha-\beta)}{\sin(\alpha+\beta)}=\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\sin\alpha\cos\beta+\sin\beta\cos\alpha}$$

$$=\frac{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\sin\beta\cos\alpha}{\cos\alpha\cos\beta}}$$

$$=\boxed{\frac{1+\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}}\quad \checkmark$$

Exercise

Prove the following equation is an identity: $\sec(x+y)=\dfrac{\cos x\cos y+\sin x\sin y}{\cos^2 x-\sin^2 y}$

Solution

$$\sec(x+y)=\frac{1}{\cos(x+y)}\,\frac{\cos(x-y)}{\cos(x-y)}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{(\cos x\cos y-\sin x\sin y)(\cos x\cos y+\sin x\sin y)}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{\cos^2 x\cos^2 y-\sin^2 x\sin^2 y}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{\cos^2 x\cos^2 y-\sin^2 x\sin^2 y}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{\cos^2 x\left(1-\sin^2 y\right)-\left(1-\cos^2 x\right)\sin^2 y}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{\cos^2 x-\cos^2 x\sin^2 y-\sin^2 y+\cos^2 x\sin^2 y}$$

$$=\boxed{\frac{\cos x\cos y+\sin x\sin y}{\cos^2 x-\sin^2 y}}\quad\checkmark$$

Exercise

Prove the following equation is an identity: $\csc(x-y)=\dfrac{\sin x\cos y+\cos x\sin y}{\sin^2 x-\sin^2 y}$

Solution

$$\csc(x-y)=\frac{1}{\sin(x-y)}\frac{\sin x\cos y+\cos x\sin y}{\sin x\cos y+\cos x\sin y}$$

$$=\frac{\sin x\cos y+\cos x\sin y}{(\sin x\cos y-\cos x\sin y)(\sin x\cos y+\cos x\sin y)}$$

$$=\frac{\sin x\cos y+\cos x\sin y}{\sin^2 x\cos^2 y-\cos^2 x\sin^2 y}$$

$$=\frac{\sin x\cos y+\cos x\sin y}{\sin^2 x\left(1-\sin^2 y\right)-\left(1-\sin^2 x\right)\sin^2 y}$$

$$=\frac{\sin x\cos y+\cos x\sin y}{\sin^2 x-\sin^2 x\sin^2 y-\sin^2 y+\sin^2 x\sin^2 y}$$

$$=\boxed{\frac{\sin x\cos y+\cos x\sin y}{\sin^2 x-\sin^2 y}}\quad\checkmark$$

Exercise

Prove the following equation is an identity: $\tan(x+y)+\tan(x-y)=\dfrac{2\tan x}{\cos^2 y\left(1-\tan^2 x\tan^2 y\right)}$

Solution

$$\tan(x+y)+\tan(x-y)=\frac{\tan x+\tan y}{1-\tan x\tan y}+\frac{\tan x-\tan y}{1+\tan x\tan y}$$

$$=\frac{(\tan x+\tan y)(1+\tan x\tan y)+(\tan x-\tan y)(1-\tan x\tan y)}{(1-\tan x\tan y)(1+\tan x\tan y)}$$

$$=\frac{\tan x+\tan^2 x\tan y+\tan y+\tan x\tan^2 y+\tan x-\tan^2 x\tan y-\tan y+\tan x\tan^2 y}{\left(1-\tan^2 x\tan^2 y\right)}$$

$$=\frac{2\tan x+2\tan x\tan^2 y}{\left(1-\tan^2 x\tan^2 y\right)}$$

$$=\frac{2\tan x\left(1+\tan^2 y\right)}{\left(1-\tan^2 x\tan^2 y\right)}$$

$$= \frac{2\tan x \sec^2 y}{\left(1-\tan^2 x \tan^2 y\right)}$$

$$\boxed{= \frac{2\tan x}{\cos^2 y\left(1-\tan^2 x \tan^2 y\right)}} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos(x-y)}{\cos(x+y)} = \dfrac{1+\tan x \tan y}{1-\tan x \tan y}$

Solution

$$\frac{\cos(x-y)}{\cos(x+y)} = \frac{\cos x \cos y + \sin x \sin y}{\cos x \cos y - \sin x \sin y}$$

$$= \frac{\frac{\cos x \cos y}{\cos x \cos y} + \frac{\sin x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y}{\cos x \cos y} - \frac{\sin x \sin y}{\cos x \cos y}}$$

$$\boxed{= \frac{1+\tan x \tan y}{1-\tan x \tan y}} \quad \checkmark$$

Exercise

Common household current is called ***alternating current*** because the current alternates direction within the wires. The voltage V in a typical 115-*volt* outlet can be expressed by the function $V(t) = 163\sin \omega t$ where ω is the angular speed (in *radians* per *second*) of the rotating generator at the electrical plant, and t is time measured in seconds.

a) It is essential for electric generators to rotate at precisely 60 cycles per second so household appliances and computers will function properly. Determine ω for these electric generators.

b) Determine a value of ϕ so that the graph of $V(t) = 163\cos(\omega t - \phi)$ is the same as the graph of $V(t) = 163\sin \omega t$

Solution

a) $\omega = 60 \frac{cycles}{\sec} \frac{2\pi\ rad}{cycles}$ $\boxed{= 120\pi \ \frac{rad}{\sec}}$

b) $V(t) = 163\cos(\omega t - \phi) = 163\sin \omega t$

$\cos(120\pi t)\cos\phi + \sin(120\pi t)\sin\phi = \sin 120\pi t$

$\begin{cases} \cos(120\pi t)\cos\phi = 0 \\ \sin\phi = 1 \end{cases} \rightarrow \boxed{\phi = \frac{\pi}{2}}$

Solution Section 3.3 – Double-angle Half-Angle Formulas

Exercise

Let $\sin A = -\frac{3}{5}$ with A in Q*III* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin\frac{A}{2}$ *e)* $\cos\frac{A}{2}$ *f)* $\tan\frac{A}{2}$

Solution

$\cos A = -\frac{4}{5}$ $\qquad A \in QIII \implies 180° < A < 270° \to 90° < \frac{A}{2} < 135°$

a) $\sin 2A = 2\left(-\frac{4}{5}\right)\left(-\frac{3}{5}\right)$ $\quad \sin 2A = 2\sin A\cos A$

$= \frac{24}{25}$

b) $\cos 2A = \cos^2 A - \sin^2 A$

$= \left(-\frac{4}{5}\right)^2 - \left(-\frac{3}{5}\right)^2$

$= \frac{16}{25} - \frac{9}{25}$

$= \frac{7}{25}$

c) $\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{24}{7}$

d) $\sin\frac{A}{2} = \sqrt{\frac{1}{2}\left(1+\frac{4}{5}\right)}$ $\quad \sin\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1-\cos A)}$

$= \sqrt{\frac{1}{2}\frac{9}{5}}$

$= \frac{3}{\sqrt{10}}$ $\quad \frac{3\sqrt{10}}{10}$

e) $\cos\frac{A}{2} = -\sqrt{\frac{1}{2}\left(1-\frac{4}{5}\right)}$ $\quad \cos\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1+\cos A)}$

$= -\frac{1}{\sqrt{10}}$ $\quad \frac{\sqrt{10}}{10}$

f) $\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = -3$

Exercise

Let $\sin A = \frac{3}{5}$ with A in Q*II* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin\frac{A}{2}$ *e)* $\cos\frac{A}{2}$ *f)* $\tan\frac{A}{2}$

Solution

$\cos A = -\frac{4}{5}$ $\qquad A \in QII \implies 90° < A < 180° \to 45° < \frac{A}{2} < 90°$

a) $\sin 2A = 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)$ $\quad \sin 2A = 2\sin A\cos A$

$= -\frac{24}{25}$

b) $\cos 2A = \frac{16}{25} - \frac{9}{25}$ $\quad \cos 2A = \cos^2 A - \sin^2 A$

$= \frac{7}{25}$

c) $\tan 2A = \frac{\sin 2A}{\cos 2A} = -\frac{24}{7}$

d) $\sin\frac{A}{2} = \sqrt{\frac{1}{2}\left(1+\frac{4}{5}\right)}$ $\quad \sin\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1-\cos A)}$

$= \frac{3}{\sqrt{10}}$

e) $\cos\frac{A}{2} = \sqrt{\frac{1}{2}\left(1-\frac{4}{5}\right)}$ $\quad \cos\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1+\cos A)}$

$= \frac{1}{\sqrt{10}}$

f) $\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = 3$

Exercise

Let $\cos A = \frac{3}{5}$ with A in QIV and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin\frac{A}{2}$ *e)* $\cos\frac{A}{2}$ *f)* $\tan\frac{A}{2}$

Solution

$\sin A = -\frac{4}{5}$

$A \in QIV \implies 270° < A < 360° \rightarrow 135° < \frac{A}{2} < 180°$

a) $\sin 2A = 2\left(-\frac{3}{5}\right)\left(-\frac{4}{5}\right)$ $\quad \sin 2A = 2\sin A\cos A$

$= \frac{24}{25}$

b) $\cos 2A = \frac{9}{25} - \frac{16}{25}$ $\quad \cos 2A = \cos^2 A - \sin^2 A$

$= -\frac{7}{25}$

c) $\tan 2A = \frac{\sin 2A}{\cos 2A} = -\frac{24}{7}$

d) $\sin\frac{A}{2} = \sqrt{\frac{1}{2}\left(1 - \frac{3}{5}\right)}$ $\quad \sin\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1-\cos A)}$

$= \frac{1}{\sqrt{5}}$

e) $\cos\frac{A}{2} = -\sqrt{\frac{1}{2}\left(1 + \frac{3}{5}\right)}$ $\quad \cos\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1+\cos A)}$

$= -\frac{2}{\sqrt{10}}$

f) $\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = -\frac{1}{2}$

Exercise

Let $\cos A = \frac{5}{13}$ with A in QI and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin\frac{A}{2}$ *e)* $\cos\frac{A}{2}$ *f)* $\tan\frac{A}{2}$

Solution

$\sin A = \frac{12}{13}$

$A \in QI \implies 0° < A < 90° \rightarrow 0° < \frac{A}{2} < 45°$

a) $\sin 2A = 2\left(\frac{12}{13}\right)\left(\frac{5}{13}\right)$ $\quad \sin 2A = 2\sin A\cos A$

$= \frac{120}{169}$

b) $\cos 2A = \frac{25}{169} - \frac{144}{169}$ $\quad \cos 2A = \cos^2 A - \sin^2 A$

$= -\frac{119}{169}$

c) $\tan 2A = \frac{\sin 2A}{\cos 2A} = -\frac{120}{119}$

d) $\sin\frac{A}{2} = \sqrt{\frac{1}{2}\left(1 - \frac{5}{13}\right)}$ $\quad \sin\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1-\cos A)}$

$= \sqrt{\frac{1}{2}\frac{8}{13}}$

$= \frac{2}{\sqrt{13}}$

e) $\cos\frac{A}{2} = \sqrt{\frac{1}{2}\left(1 + \frac{5}{13}\right)}$ $\quad \cos\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1+\cos A)}$

$= \frac{3}{\sqrt{13}}$

f) $\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{2}{3}$

Exercise

Let $\cos A = -\frac{12}{13}$ with A in Q*II* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin\frac{A}{2}$ *e)* $\cos\frac{A}{2}$ *f)* $\tan\frac{A}{2}$

Solution

$\sin A = \frac{5}{13}$

$90° < A < 180° \quad \to 45° < \frac{A}{2} < 90°$

a) $\sin 2A = 2\left(\frac{5}{13}\right)\left(-\frac{12}{13}\right)$ $\quad \sin 2A = 2\sin A\cos A$

$= -\frac{120}{169}$

b) $\cos 2A = \frac{144}{169} - \frac{25}{169}$ $\quad \cos 2A = \cos^2 A - \sin^2 A$

$= \frac{119}{169}$

c) $\tan 2A = \frac{\sin 2A}{\cos 2A} = -\frac{120}{119}$

d) $\sin\frac{A}{2} = \sqrt{\frac{1}{2}\left(1+\frac{12}{13}\right)}$ $\quad \sin\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1-\cos A)}$

$= \frac{5}{\sqrt{26}}$

e) $\cos\frac{A}{2} = \sqrt{\frac{1}{2}\left(1-\frac{5}{13}\right)}$ $\quad \cos\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1+\cos A)}$

$= \frac{2}{\sqrt{13}}$

f) $\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{5}{2}$

Exercise

Let $\sin A = -\frac{7}{25}$ with A in Q*III* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin\frac{A}{2}$ *e)* $\cos\frac{A}{2}$ *f)* $\tan\frac{A}{2}$

Solution

$\cos A = -\frac{24}{25}$

$180° < A < 270° \quad \to 90° < \frac{A}{2} < 135°$

a) $\sin 2A = 2\left(-\frac{7}{25}\right)\left(-\frac{24}{25}\right)$ $\quad \sin 2A = 2\sin A\cos A$

$= \frac{336}{625}$

b) $\cos 2A = \frac{576}{625} - \frac{49}{625}$ $\quad \cos 2A = \cos^2 A - \sin^2 A$

$= \frac{527}{625}$

c) $\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{336}{527}$

d) $\sin\frac{A}{2} = \sqrt{\frac{1}{2}\left(1+\frac{24}{25}\right)}$ $\quad \sin\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1-\cos A)}$

$= \sqrt{\frac{1}{2}\,\frac{49}{25}}$

$= \frac{7}{5\sqrt{2}}$

e) $\cos\frac{A}{2} = \sqrt{\frac{1}{2}\left(1-\frac{24}{25}\right)}$ $\quad \cos\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1+\cos A)}$

$= \frac{1}{5\sqrt{2}}$

f) $\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = 7$

Exercise

Let $\sin A = -\frac{24}{25}$ with A in QIV and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin\frac{A}{2}$ *e)* $\cos\frac{A}{2}$ *f)* $\tan\frac{A}{2}$

Solution

$$\cos A = \frac{7}{25}$$

$$270^\circ < A < 360^\circ \quad \to 135^\circ < \frac{A}{2} < 180^\circ$$

a) $\sin 2A = 2\left(-\frac{24}{25}\right)\left(\frac{7}{25}\right)$ $\quad \sin 2A = 2\sin A\cos A$

$= -\frac{336}{625}$

b) $\cos 2A = \frac{49}{625} - \frac{576}{625}$ $\quad \cos 2A = \cos^2 A - \sin^2 A$

$= -\frac{527}{625}$

c) $\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{336}{527}$

d) $\sin\frac{A}{2} = \sqrt{\frac{1}{2}\left(1-\frac{7}{25}\right)}$ $\quad \sin\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1-\cos A)}$

$= \frac{3}{5\sqrt{2}}$

e) $\cos\frac{A}{2} = -\sqrt{\frac{1}{2}\left(1+\frac{7}{25}\right)}$ $\quad \cos\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1+\cos A)}$

$= -\frac{4}{5}$

f) $\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = -\frac{3}{5\sqrt{2}}$

Exercise

Let $\cos A = \frac{15}{17}$ with A in QI and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin\frac{A}{2}$ *e)* $\cos\frac{A}{2}$ *f)* $\tan\frac{A}{2}$

Solution

$$\sin A = \frac{8}{17}$$

$$0^\circ < A < 90^\circ \quad \to 0^\circ < \frac{A}{2} < 45^\circ$$

a) $\sin 2A = 2\left(\frac{8}{17}\right)\left(\frac{15}{17}\right)$ $\quad \sin 2A = 2\sin A\cos A$

$= \frac{240}{289}$

b) $\cos 2A = \frac{225}{289} - \frac{64}{289}$ $\quad \cos 2A = \cos^2 A - \sin^2 A$

$= \frac{161}{289}$

c) $\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{240}{289}$

d) $\sin\frac{A}{2} = \sqrt{\frac{1}{2}\left(1-\frac{15}{17}\right)}$ $\quad \sin\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1-\cos A)}$

$= \frac{1}{\sqrt{17}}$

e) $\cos\frac{A}{2} = \sqrt{\frac{1}{2}\left(1+\frac{15}{17}\right)}$ $\quad \cos\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1+\cos A)}$

$= \frac{4}{\sqrt{17}}$

f) $\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{2}{4}$

Exercise

Let $\cos x = \frac{1}{\sqrt{10}}$ with x in Q*IV* and find $\cot 2x$

Solution

x in QIV $\Rightarrow \sin x < 0$

$$\sin x = -\sqrt{1-\cos^2 x}$$
$$= -\sqrt{1-\frac{1}{10}}$$
$$= -\sqrt{\frac{9}{10}}$$
$$= -\frac{3}{\sqrt{10}}$$

$$\cot 2x = \frac{\cos 2x}{\sin 2x}$$
$$= \frac{2\cos^2 x - 1}{2\sin x \cos x}$$
$$= \frac{2\left(\frac{1}{\sqrt{10}}\right)^2 - 1}{2\frac{1}{\sqrt{10}}\left(-\frac{3}{\sqrt{10}}\right)}$$
$$= \frac{2\frac{1}{10} - 1}{-\frac{6}{10}}$$
$$= \frac{\frac{2-10}{10}}{-\frac{6}{10}}$$
$$= \frac{-8}{-6}$$
$$= \frac{4}{3}$$

Exercise

Verify: $(\cos x - \sin x)(\cos x + \sin x) = \cos 2x$

Solution

$$(\cos x - \sin x)(\cos x + \sin x) = \cos^2 x - \sin^2 x \qquad (a+b)(a-b) = a^2 - b^2$$
$$= \cos 2x$$

Exercise

Prove: $\cot x \sin 2x = 1 + \cos 2x$

Solution

$$\begin{aligned}
\cot x \sin 2x &= \frac{\cos x}{\sin x}(2\sin x \cos x) \\
&= 2\cos^2 x \qquad\qquad \cos 2x = 2\cos^2 x - 1 \Rightarrow 2\cos^2 x = \cos 2x + 1 \\
&= \cos 2x + 1
\end{aligned}$$

Exercise

Prove: $\cot\theta = \dfrac{\sin 2\theta}{1-\cos 2\theta}$

Solution

$$\begin{aligned}
\frac{\sin 2\theta}{1-\cos 2\theta} &= \frac{2\sin\theta\cos\theta}{1-\left(1-2\sin^2\theta\right)} \\
&= \frac{2\sin\theta\cos\theta}{1-1+2\sin^2\theta} \\
&= \frac{2\sin\theta\cos\theta}{2\sin^2\theta} \\
&= \frac{\cos\theta}{\sin\theta} \\
&= \cot\theta
\end{aligned}$$

Exercise

Simplify $\cos^2 7x - \sin^2 7x$

Solution

$$\cos^2 7x - \sin^2 7x = \cos\left(2(7x)\right) \qquad \cos 2x = \cos^2 x - \sin^2 x$$

$$= \cos 14x$$

Exercise

Write $\sin 3x$ in terms of $\sin x$

Solution

$$\sin 3x = \sin(2x + x)$$

$$= \sin 2x \cos x + \cos 2x \sin x$$

$$= (2\sin x \cos x)\cos x + \left(1 - 2\sin^2 x\right)\sin x$$

$$= 2\sin x \cos^2 x + \sin x - 2\sin^3 x \qquad \cos^2 x = 1 - \sin^2 x$$

$$= 2\sin x\left(1 - \sin^2 x\right) + \sin x - 2\sin^3 x$$

$$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$$

$$= 3\sin x - 4\sin^3 x$$

Exercise

Find the values of the six trigonometric functions of θ if $\cos 2\theta = \frac{4}{5}$ *and* $90° < \theta < 180°$

Solution

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$	$\sin^2\theta = \frac{1-\cos 2\theta}{2}$
$= \frac{1+\frac{4}{5}}{2}$	$= \frac{1-\frac{4}{5}}{2}$
$= \frac{\frac{9}{5}}{2}$	$= \frac{\frac{1}{5}}{2}$
$= \frac{9}{10}$	$= \frac{1}{10}$
$\cos\theta = -\sqrt{\frac{9}{10}}$	$\sin\theta = \sqrt{\frac{1}{10}}$
$= -\frac{3}{\sqrt{10}}\frac{\sqrt{10}}{\sqrt{10}}$	$= \frac{1}{\sqrt{10}}\frac{\sqrt{10}}{\sqrt{10}}$
$= -\frac{3\sqrt{10}}{10}$	$= \frac{\sqrt{10}}{10}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\sqrt{10}}{10}}{-\frac{3\sqrt{10}}{10}} = -\frac{\sqrt{10}}{10}\frac{10}{3\sqrt{10}} = \boxed{-\frac{1}{3}}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{-\frac{1}{3}} = \boxed{-3}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{1}{\sqrt{10}}} = \boxed{\sqrt{10}}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{-\frac{3}{\sqrt{10}}} = \boxed{-\frac{\sqrt{10}}{3}}$$

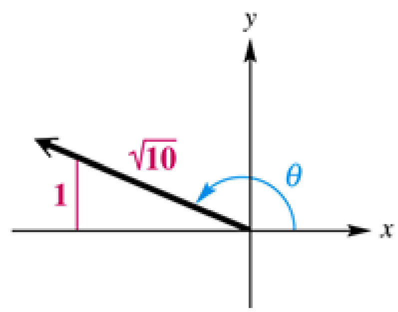
Exercise

Use a right triangle in QII to find the value of $\cos\theta$ *and* $\tan\theta$

Solution

$$r = \sqrt{10},\ y = 1$$

$$\begin{aligned} x &= -\sqrt{r^2 - y^2} \\ &= -\sqrt{\left(\sqrt{10}\right)^2 - 1^2} \\ &= -\sqrt{10-1} \\ &= -\sqrt{9} \\ &= -3 \end{aligned}$$



$$\cos\theta = \frac{-3}{\sqrt{10}}\frac{\sqrt{10}}{\sqrt{10}} = -\frac{3\sqrt{10}}{10}$$

$$\tan\theta = -\frac{1}{3}$$

Exercise

Prove the following equation is an identity: $\sin 3x = \sin x\left(3\cos^2 x - \sin^2 x\right)$

Solution

$$\begin{aligned}
\sin 3x &= \sin(x + 2x) \\
&= \sin x \cos 2x + \sin 2x \cos x \\
&= \sin x\left(\cos^2 x - \sin^2 x\right) + (2\sin x \cos x)\cos x \\
&= \sin x \cos^2 x - \sin^3 x + 2\sin x \cos^2 x \\
&= 3\sin x \cos^2 x - \sin^3 x \\
&= \sin x\left(3\cos^2 x - \sin^2 x\right)
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cos 3x = \cos^3 x - 3\cos x \sin^2 x$

Solution

$$\begin{aligned}
\cos 3x &= \cos(x + 2x) \\
&= \cos x \cos 2x - \sin x \sin 2x \\
&= \cos x\left(\cos^2 x - \sin^2 x\right) - \sin x(2\sin x \cos x) \\
&= \cos^3 x - \sin^2 x \cos x - 2\sin^2 x \cos x \\
&= \cos^3 x - 3\sin^2 x \cos x
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cos^4 x - \sin^4 x = \cos 2x$

Solution

$$\begin{aligned}
\cos^4 x - \sin^4 x &= \left(\cos^2 x - \sin^2 x\right)\left(\cos^2 x + \sin^2 x\right) \qquad (a-b)(a+b) = a^2 + b^2 \\
&= (\cos 2x)(1) \\
&= \cos 2x
\end{aligned}$$

Exercise

Prove: $\cot\theta = \dfrac{\sin 2\theta}{1-\cos 2\theta}$

Solution

$$\frac{\sin 2\theta}{1-\cos 2\theta} = \frac{2\sin\theta\cos\theta}{1-\left(1-2\sin^2\theta\right)}$$

$$= \frac{2\sin\theta\cos\theta}{1-1+2\sin^2\theta}$$

$$= \frac{2\sin\theta\cos\theta}{2\sin^2\theta}$$

$$= \frac{\cos\theta}{\sin\theta}$$

$$= \cot\theta$$

Exercise

Prove the following equation is an identity: $\sin 2x = -2\sin x \sin\left(x-\frac{\pi}{2}\right)$

Solution

$$\sin 2x = 2\sin x\cos x \qquad \cos x = \sin\left(\tfrac{\pi}{2}-x\right)$$

$$= 2\sin x\sin\left(\tfrac{\pi}{2}-x\right) \qquad \sin(-x) = -\sin x$$

$$= -2\sin x\sin\left(x-\tfrac{\pi}{2}\right)$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin 4t}{4} = \cos^3 t\sin t - \sin^3 t\cos t$

Solution

$$\frac{\sin 4t}{4} = \frac{1}{4}(2\sin 2t\cos 2t)$$

$$= \frac{1}{2}(2\sin t\cos t)\left(\cos^2 t - \sin^2 t\right)$$

$$= \sin t\cos t\left(\cos^2 t - \sin^2 t\right)$$

$$= \sin t\cos^3 t - \cos t\sin^3 t$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos 2x}{\sin^2 x} = \csc^2 x - 2$

Solution

$$\frac{\cos 2x}{\sin^2 x} = \frac{1-2\sin^2 x}{\sin^2 x}$$

$$= \frac{1}{\sin^2 x} - \frac{2\sin^2 x}{\sin^2 x}$$

$$\underline{= \csc^2 x - 2|}$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos 2x + \cos 2y}{\sin x + \cos y} = 2\cos y - 2\sin x$

Solution

$$\frac{\cos 2x + \cos 2y}{\sin x + \cos y} = \frac{2\cos\left(\frac{2x+2y}{2}\right)\cos\left(\frac{2x-2y}{2}\right)}{\sin x + \cos y}$$

$$= \frac{2\cos(x+y)\cos(x-y)}{\sin x + \cos y}$$

$$= \frac{2(\cos x\cos y - \sin x\sin y)(\cos x\cos y + \sin x\sin y)}{\sin x + \cos y}$$

$$= 2\frac{\cos^2 x\cos^2 y - \sin^2 x\sin^2 y}{\sin x + \cos y}$$

$$= 2\frac{\left(1-\sin^2 x\right)\cos^2 y - \sin^2 x\left(1-\cos^2 y\right)}{\sin x + \cos y}$$

$$= 2\frac{\cos^2 y - \sin^2 x\cos^2 y - \sin^2 x + \sin^2 x\cos^2 y}{\sin x + \cos y}$$

$$= 2\frac{\cos^2 y - \sin^2 x}{\sin x + \cos y}$$

$$= 2\frac{(\cos y - \sin x)(\cos y + \sin x)}{\sin x + \cos y}$$

$$= 2(\cos y - \sin x)$$

$$\underline{= 2\cos y - 2\sin x|}$$

$$\frac{\cos 2x + \cos 2y}{\sin x + \cos y} = \frac{\cos^2 x - \sin^2 x + \cos^2 y - \sin^2 y}{\sin x + \cos y}$$

$$= \frac{1-\sin^2 x - \sin^2 x + \cos^2 y - \left(1-\cos^2 y\right)}{\sin x + \cos y}$$

$$= \frac{1 - 2\sin^2 x + \cos^2 y - 1 + \cos^2 y}{\sin x + \cos y}$$

$$= \frac{2\cos^2 y - 2\sin^2 x}{\sin x + \cos y}$$

$$= 2\frac{\cos^2 y - \sin^2 x}{\sin x + \cos y}$$

$$= 2\frac{(\cos y - \sin x)(\cos y + \sin x)}{\sin x + \cos y}$$

$$= 2(\cos y - \sin x)$$

$$= 2\cos y - 2\sin x$$

Exercise

Prove the following equation is an identity: $\frac{\cos 2x}{\cos^2 x} = \sec^2 x - 2\tan^2 x$

Solution

$$\frac{\cos 2x}{\cos^2 x} = \frac{1 - 2\sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} - \frac{2\sin^2 x}{\cos^2 x}$$

$$= \sec^2 x - 2$$

Exercise

Prove the following equation is an identity: $\sin 4x = (4\sin x \cos x)\left(2\cos^2 x - 1\right)$

Solution

$$\sin 4x = \sin(2(2x))$$

$$= 2\sin 2x \cos 2x$$

$$= 2(2\sin x \cos x)\left(2\cos^2 x - 1\right)$$

$$= (4\sin x \cos x)\left(2\cos^2 x - 1\right)$$

Exercise

Prove the following equation is an identity: $\cos 4x = \cos^4 x - 6\sin^2 x\cos^2 x + \sin^4 x$

Solution

$$\begin{aligned}
\cos 4x &= \cos\left(2(2x)\right) \\
&= \cos^2 2x - \sin^2 2x \\
&= (\cos 2x)^2 - (\sin 2x)^2 \\
&= \left(\cos^2 x - \sin^2 x\right)^2 - (2\sin x\cos x)^2 \\
&= \cos^4 x - 2\sin^2 x\cos^2 x - \sin^4 x - 4\sin^2 x\cos^2 x \\
&= \cos^4 x - 6\sin^2 x\cos^2 x - \sin^4 x
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cos 2y = \dfrac{1-\tan^2 y}{1+\tan^2 y}$

Solution

$\cos 2y = \cos^2 y - \sin^2 y$ $= \dfrac{\cos^2 y - \sin^2 y}{1}$ $= \dfrac{\cos^2 y - \sin^2 y}{\cos^2 y + \sin^2 y}$ $= \dfrac{\frac{\cos^2 y}{\cos^2 y} - \frac{\sin^2 y}{\cos^2 y}}{\frac{\cos^2 y}{\cos^2 y} + \frac{\sin^2 y}{\cos^2 y}}$ $= \dfrac{1-\tan^2 y}{1+\tan^2 y}$	$\dfrac{1-\tan^2 y}{1+\tan^2 y} = \dfrac{1-\frac{\sin^2 y}{\cos^2 y}}{1+\frac{\sin^2 y}{\cos^2 y}}$ $= \dfrac{\frac{\cos^2 y - \sin^2 y}{\cos^2 y}}{\frac{\cos^2 y + \sin^2 y}{\cos^2 y}}$ $= \dfrac{\cos^2 y - \sin^2 y}{\cos^2 y + \sin^2 y}$ $= \dfrac{\cos^2 y - \sin^2 y}{1}$ $= \cos^2 y - \sin^2 y$

Exercise

Prove the following equation is an identity: $\tan^2 x(1+\cos 2x) = 1-\cos 2x$

Solution

$$\begin{aligned}
\tan^2 x(1+\cos 2x) &= \frac{\sin^2 x}{\cos^2 x}\left(1+2\cos^2 x-1\right) \\
&= \frac{\sin^2 x}{\cos^2 x}\left(2\cos^2 x\right) \\
&= 2\sin^2 x \\
&= 1-1+2\sin^2 x \\
&= 1-\left(1-2\sin^2 x\right) \\
&= \underline{1-\cos 2x}|
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos 2x}{\sin^2 x} = 2\cot^2 x - \csc^2 x$

Solution

$$\begin{aligned}
\frac{\cos 2x}{\sin^2 x} &= \frac{\cos^2 x - \sin^2 x}{\sin^2 x} \\
&= \frac{\cos^2 x}{\sin^2 x} - \frac{\sin^2 x}{\sin^2 x} \\
&= \cot^2 x - 1 \qquad\qquad \cot^2 x + 1 = \csc^2 x \\
&= \cot^2 x + \cot^2 x - \csc^2 x \\
&= \underline{2\cot^2 x - \csc^2 x}|
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\tan x + \cot x = 2\csc 2x$

Solution

$$\begin{aligned}
\tan x + \cot x &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \\
&= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} \\
&= \frac{1}{\cos x \sin x} \\
&= \frac{1}{\frac{1}{2}\sin 2x}
\end{aligned}$$

$$= 2\frac{1}{\sin 2x}$$

$$\underline{= 2\csc 2x|}$$

Exercise

Prove the following equation is an identity: $\tan 2x = \dfrac{2}{\cot x - \tan x}$

Solution

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

$$= \frac{2\frac{\tan x}{\tan x}}{\frac{1}{\tan x} - \frac{\tan^2 x}{\tan x}}$$

$$\underline{= \frac{2}{\cot x - \tan x}|}$$

Exercise

Prove the following equation is an identity: $\dfrac{1 - \tan x}{1 + \tan x} = \dfrac{1 - \sin 2x}{\cos 2x}$

Solution

$$\frac{1 - \tan x}{1 + \tan x} = \frac{1 - \frac{\sin x}{\cos x}}{1 + \frac{\sin x}{\cos x}}$$

$$= \frac{\frac{\cos x - \sin x}{\cos x}}{\frac{\cos x + \sin x}{\cos x}}$$

$$= \frac{\cos x - \sin x}{\cos x + \sin x}\frac{\cos x - \sin x}{\cos x - \sin x}$$

$$= \frac{\cos^2 x - 2\cos x \sin x + \sin^2 x}{\cos^2 x + \sin^2 x}$$

$$\underline{= \frac{1 - \sin 2x}{\cos 2x}|}$$

Exercise

Prove the following equation is an identity: $\sin 2\alpha \sin 2\beta = \sin^2(\alpha + \beta) - \sin^2(\alpha - \beta)$

Solution

$$\sin 2\alpha \sin 2\beta = (2\sin\alpha\cos\alpha)(2\sin\beta\cos\beta)$$

$$= (2\sin\alpha\cos\beta)(2\sin\beta\cos\alpha)$$

$$=\left(2\tfrac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]\right)\left(2\tfrac{1}{2}\left[\sin(\beta+\alpha)+\sin(\beta-\alpha)\right]\right)$$

$$=\left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right)\left(\sin(\alpha+\beta)-\sin(\alpha-\beta)\right)$$

$$\boxed{=\sin^2(\alpha+\beta)-\sin^2(\alpha-\beta)}$$

Exercise

Prove the following equation is an identity: $\cos^2(A-B)-\cos^2(A+B)=\sin 2A\ \sin 2B$

Solution

$$\begin{aligned}\cos^2(A-B)-\cos^2(A+B)&=\left(\cos(A-B)-\cos(A+B)\right)\left(\cos(A-B)+\cos(A+B)\right)\\&=(2\sin A\sin B)(2\cos A\cos B)\\&=(2\sin A\cos A)(2\sin B\cos B)\\&=\boxed{\sin 2A\ \sin 2B}\end{aligned}$$

Exercise

Use half-angle formulas to find the exact value of $\sin 105°$

Solution

$$\begin{aligned}\sin 105°&=\sin\frac{210°}{2}\\&=\sqrt{\frac{1-\cos 210°}{2}} \qquad\qquad \textit{reference}: 210°-180°=30°\\&=\sqrt{\frac{1+\cos 30°}{2}}\\&=\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}\\&=\sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}}\\&=\sqrt{\frac{2+\sqrt{3}}{4}}\\&=\boxed{\frac{\sqrt{2+\sqrt{3}}}{2}}\end{aligned}$$

Exercise

Find the exact of $\tan 22.5°$

Solution

$$\tan 22.5° = \tan\frac{45°}{2}$$

$$= \frac{1-\cos 45°}{\sin 45°}$$

$$= \frac{1-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$$

$$= \frac{\frac{2-\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$$

$$= \frac{2-\sqrt{2}}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}} - \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \sqrt{2}-1$$

Exercise

Given: $\cos x = \frac{2}{3}$, $\frac{3\pi}{2} < x < 2\pi$, find $\cos\frac{x}{2}$, $\sin\frac{x}{2}$, and $\tan\frac{x}{2}$

Solution

$$\frac{3\pi}{2} < x < 2\pi \Rightarrow \frac{3\pi}{4} < \frac{x}{2} < \pi \Rightarrow x \in QII$$

$\cos\frac{x}{2} = -\sqrt{\frac{1+\cos x}{2}}$	$\sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}}$	$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$
$= -\sqrt{\frac{1+\frac{2}{3}}{2}}$	$= \sqrt{\frac{1-\frac{2}{3}}{2}}$	$= \frac{\frac{\sqrt{6}}{6}}{-\frac{\sqrt{30}}{6}}$
$= -\sqrt{\frac{1}{2}\frac{3+2}{3}}$	$= \sqrt{\frac{1}{2}\frac{3-2}{3}}$	$= -\frac{\sqrt{6}}{\sqrt{30}}\frac{\sqrt{30}}{\sqrt{30}}$
$= -\sqrt{\frac{5}{6}}$	$= \sqrt{\frac{1}{6}}$	$= -\frac{6\sqrt{5}}{30}$
$= -\frac{\sqrt{5}}{\sqrt{6}}\frac{\sqrt{6}}{\sqrt{6}}$	$= \frac{1}{\sqrt{6}}\frac{\sqrt{6}}{\sqrt{6}}$	$= -\frac{\sqrt{5}}{5}$
$= -\frac{\sqrt{30}}{6}$	$= \frac{\sqrt{6}}{6}$	

Exercise

Prove the identity $\quad 2\csc x\cos^2\frac{x}{2}=\frac{\sin x}{1-\cos x}$

Solution

$$\begin{aligned}2\csc x\cos^2\frac{x}{2}&=2\frac{1}{\sin x}\frac{1+\cos x}{2}\\&=\frac{1+\cos x}{\sin x}\frac{1-\cos x}{1-\cos x}\\&=\frac{1-\cos^2 x}{\sin x(1-\cos x)}\\&=\frac{\sin^2 x}{\sin x(1-\cos x)}\\&=\boxed{\frac{\sin x}{1-\cos x}}\end{aligned}$$

$(a-b)(a+b)=a^2-b^2$

Exercise

Prove the identity $\quad \tan\frac{\alpha}{2}=\sin\alpha+\cos\alpha\cot\alpha-\cot\alpha$

Solution

$$\begin{aligned}\tan\frac{\alpha}{2}&=\frac{1-\cos\alpha}{\sin\alpha}\\&=\frac{\sin^2\alpha+\cos^2\alpha-\cos\alpha}{\sin\alpha}\\&=\frac{\sin^2\alpha}{\sin\alpha}+\frac{\cos^2\alpha}{\sin\alpha}-\frac{\cos\alpha}{\sin\alpha}\\&=\sin\alpha+\cos\alpha\frac{\cos\alpha}{\sin\alpha}-\cot\alpha\\&=\boxed{\sin\alpha+\cos\alpha\cot\alpha-\cot\alpha}\end{aligned}$$

$1=\sin^2\alpha+\cos^2\alpha$

Exercise

Prove the following equation is an identity: $\quad \sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)=\frac{\sin^2 x}{4}$

Solution

$$\begin{aligned}\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)&=\frac{1-\cos x}{2}\cdot\frac{1+\cos x}{2}\\&=\frac{1-\cos^2 x}{4}\\&=\boxed{\frac{\sin^2 x}{4}}\end{aligned}$$

$(a-b)(a+b)=a^2+b^2$

$\cos^2\alpha+\sin^2\alpha=1$

Exercise

Prove the following equation is an identity: $\tan\frac{x}{2}+\cot\frac{x}{2}=2\csc x$

Solution

$$\tan\frac{x}{2}+\cot\frac{x}{2}=\tan\frac{x}{2}+\frac{1}{\tan\frac{x}{2}}$$

$$=\frac{\sin x}{1+\cos x}+\frac{\sin x}{1-\cos x}$$

$$=\sin x\frac{(1-\cos x)+(1+\cos x)}{1-\cos^2 x}$$

$$=\sin x\frac{2}{\sin^2 x}$$

$$=\frac{2}{\sin x}$$

$$=2\csc x$$

Exercise

Prove the following equation is an identity: $2\sin^2\left(\frac{x}{2}\right)=\frac{\sin^2 x}{1+\cos x}$

Solution

$$2\sin^2\left(\frac{x}{2}\right)=2\frac{1-\cos x}{2}$$

$$=1-\cos x\cdot\frac{1+\cos x}{1+\cos x}$$

$$=\frac{1-\cos^2 x}{1+\cos x}$$

$$=\frac{\sin^2 x}{1+\cos x}$$

Exercise

Prove the following equation is an identity: $\tan^2\left(\frac{x}{2}\right)=\frac{\sec x+\cos x-2}{\sec x-\cos x}$

Solution

$$\tan^2\left(\frac{x}{2}\right)=\frac{1-\cos x}{1+\cos x}$$

$\tan^2\alpha=\frac{1-\cos 2\alpha}{1+\cos 2\alpha}$; $x=2\alpha$; $\tan^2\frac{x}{2}=\frac{1-\cos x}{1+\cos x}$

$$=\frac{1-\cos x}{1+\cos x}\frac{1-\cos x}{1-\cos x}$$

$$=\frac{1-2\cos x+\cos^2 x}{1-\cos^2 x}\ \frac{\frac{1}{\cos x}}{\frac{1}{\cos x}}$$

$$= \frac{\frac{1-2\cos x+\cos^2 x}{\cos x}}{\frac{1-\cos^2 x}{\cos x}}$$

$$= \frac{\frac{1}{\cos x} - \frac{2\cos x}{\cos x} + \frac{\cos^2 x}{\cos x}}{\frac{1}{\cos x} - \frac{\cos^2 x}{\cos x}}$$

$$= \boxed{\frac{\sec x - 2 + \cos x}{\sec x - \cos x}}$$

$$\frac{\sec x + \cos x - 2}{\sec x - \cos x} = \frac{\frac{1}{\cos x} + \cos x - 2}{\frac{1}{\cos x} - \cos x}$$

$$= \frac{\frac{1+\cos^2 x - 2\cos x}{\cos x}}{\frac{1-\cos^2 x}{\cos x}}$$

$$= \frac{(1-\cos x)^2}{(1-\cos x)(1+\cos x)}$$

$$= \frac{1-\cos x}{1+\cos x} \qquad \tan^2\alpha = \frac{1-\cos 2\alpha}{1+\cos 2\alpha}; \quad x = 2\alpha; \quad \tan^2\frac{x}{2} = \frac{1-\cos x}{1+\cos x}$$

$$= \boxed{\tan^2\left(\frac{x}{2}\right)}$$

Exercise

Prove the following equation is an identity: $\sec^2\left(\frac{x}{2}\right) = \frac{2\sec x + 2}{\sec x + 2 + \cos x}$

Solution

$$\sec^2\left(\frac{x}{2}\right) = \frac{1}{\cos^2\left(\frac{x}{2}\right)} \qquad \cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}} \Rightarrow \cos^2\left(\frac{\alpha}{2}\right) = \frac{1+\cos\alpha}{2}$$

$$= \frac{1}{\frac{1+\cos x}{2}}$$

$$= \frac{2}{1+\cos x}\frac{1+\cos x}{1+\cos x}$$

$$= \frac{2+2\cos x}{1+2\cos x+\cos^2 x}$$

$$= \frac{2+2\cos x}{1+2\cos x+\cos^2 x}\,\frac{\frac{1}{\cos x}}{\frac{1}{\cos x}}$$

$$=\frac{\frac{2}{\cos x}+2\frac{\cos x}{\cos x}}{\frac{1}{\cos x}+\frac{2\cos x}{\cos x}+\frac{\cos^2 x}{\cos x}}$$

$$=\boxed{\frac{2\sec x+2}{\sec x+2+\cos x}}$$

Exercise

Prove the following equation is an identity: $\dfrac{1-\sin^2\left(\frac{x}{2}\right)}{1+\sin^2\left(\frac{x}{2}\right)}=\dfrac{1+\cos x}{3-\cos x}$

Solution

$$\frac{1-\sin^2\left(\frac{x}{2}\right)}{1+\sin^2\left(\frac{x}{2}\right)}=\frac{1-\frac{1-\cos x}{2}}{1+\frac{1-\cos x}{2}}$$

$$=\frac{\frac{2-1-\cos x}{2}}{\frac{2+1-\cos x}{2}}$$

$$=\boxed{\frac{1-\cos x}{3-\cos x}}$$

Exercise

Prove the following equation is an identity: $\dfrac{1-\cos^2\left(\frac{x}{2}\right)}{1-\sin^2\left(\frac{x}{2}\right)}=\dfrac{1-\cos x}{1+\cos x}$

Solution

$$\frac{1-\cos^2\left(\frac{x}{2}\right)}{1-\sin^2\left(\frac{x}{2}\right)}=\frac{1-\frac{1+\cos x}{2}}{1-\frac{1-\cos x}{2}}$$

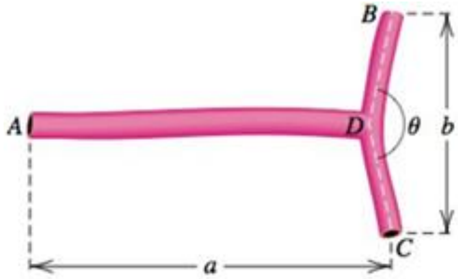
$$=\frac{\frac{2-(1+\cos x)}{2}}{\frac{2-(1-\cos x)}{2}}$$

$$=\frac{\frac{2-1-\cos x}{2}}{\frac{2-1+\cos x}{2}}$$

$$=\boxed{\frac{1-\cos x}{1+\cos x}}$$

Exercise

A common form of cardiovascular branching is bifurcation, in which an artery splits into two smaller blood vessels. The bifurcation angle θ is the angle formed by the two smaller arteries. The line through A and D bisects θ and is perpendicular to the line through B and C.



a) Show that the length l of the artery from A to B is given by $l = a + \frac{b}{2}\tan\frac{\theta}{4}$.

b) Estimate the length l from the three measurements $a = 10\ mm$, $b = 6\ mm$, and $\theta = 156°$.

Solution

a) $\tan\frac{\theta}{2} = \dfrac{\frac{b}{2}}{a - |AD|}$

$$|AD| = a - \frac{b}{2}\frac{1}{\tan\frac{\theta}{2}}$$

$$\sin\frac{\theta}{2} = \frac{b}{2}\frac{1}{|DB|} \Rightarrow |DB| = \frac{b}{2}\frac{1}{\sin\frac{\theta}{2}}$$

$$l = |AD| + |DB|$$

$$= a - \frac{b}{2}\frac{1}{\tan\frac{\theta}{2}} + \frac{b}{2}\frac{1}{\sin\frac{\theta}{2}}$$

$$= a + \frac{b}{2}\left(\frac{1}{\sin\frac{\theta}{2}} - \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right)$$

$$= a + \frac{b}{2}\left(\frac{1-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right) \qquad \tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}$$

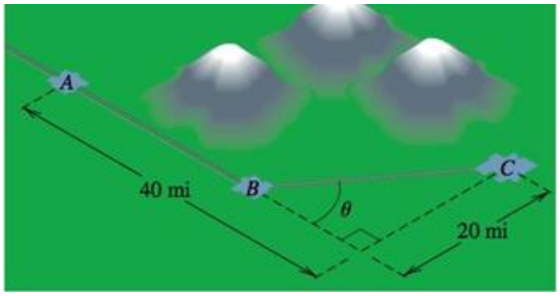
$$\boxed{= a + \frac{b}{2}\tan\frac{\theta}{4}}$$

b) **Given**: $a = 10\ mm,\ b = 6\ mm,\ \theta = 156°$

$$\boxed{l = 10 + \frac{6}{2}\tan\frac{156°}{4} = 10 + 3\tan 39° \approx 12.43\ mm}$$

Exercise

A proposed rail road route through three towns located at points *A*, *B*, and *C*. At *B*, the track will turn toward *C* at an angle θ.



a) Show that the total distance *d* from *A* to *C* is given by $d = 20\tan\frac{1}{2}\theta + 40$

b) Because of mountains between *A* and *C*, the turning point *B* must be at least 20 *miles* from *A*.Is there a route that avoids the mountains and measures exactly 50 *miles*?

Solution

a) $d = |AB| + |BC|$

$$\tan\theta = \frac{20}{40 - |AB|} \quad \to \quad |AB| = 40 - \frac{20}{\tan\theta}$$

$$\sin\theta = \frac{20}{|BC|} \quad \to \quad |BC| = \frac{20}{\sin\theta}$$

$$\begin{aligned} d &= 40 - \frac{20}{\tan\theta} + \frac{20}{\sin\theta} \\ &= 40 + 20\left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right) \\ &= 40 + 20\left(\frac{1-\cos\theta}{\sin\theta}\right) \qquad \tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} \\ &= 40 + 20\tan\frac{\theta}{2} \end{aligned}$$

b) $50 = 40 + 20\tan\frac{\theta}{2}$

$$20\tan\frac{\theta}{2} = 10 \quad \to \quad \frac{\theta}{2} = \tan^{-1}\frac{1}{2} \approx 25.565° \quad \Rightarrow \quad \underline{\theta = 53.13°}$$

$$|AB| = 40 - \frac{20}{\tan 53.13°} \approx 25$$

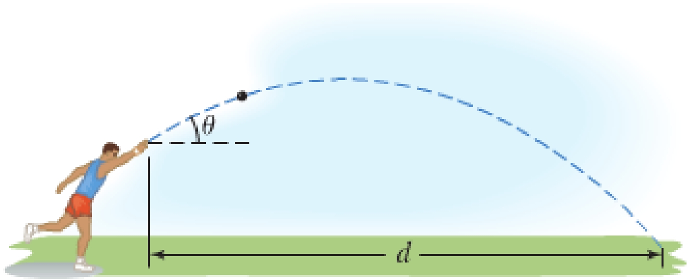
Yes, point *B* is 25 *miles* from *A*.

Exercise

Throwing events in track and field include the shot put, the discus throw, the hammer throw, and the javelin throw. The distance that the athlete can achieve depends on the initial speed of the object thrown and the angle above the horizontal at which the object leaves the hand. This angle is represented by θ. The distance, d, in *feet*, that the athlete throws is modeled by the formula

$$d = \frac{v_0^2}{16}\sin\theta\cos\theta$$

In which v_0 is the initial speed of the object thrown, in *feet* per *second*, and θ is the angle, in *degrees*, at which the object leaves the hand.



a) Use the identity to express the formula so that it contains the since function only.

b) Use the formula from part (*a*) to find the angle, θ, that produces the maximum distance, d, for a given initial speed, v_0.

Solution

a) $d = \frac{v_0^2}{16}\sin\theta\cos\theta$ $\qquad \sin 2\theta = 2\sin\theta\cos\theta$

$$= \frac{v_0^2}{16}\frac{1}{2}\sin 2\theta$$

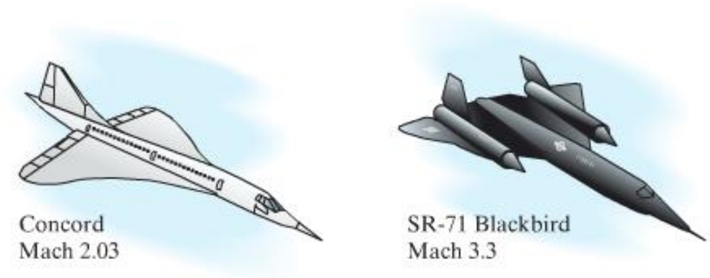
$$= \frac{v_0^2}{32}\sin 2\theta$$

b) The maximum value of a sine function is 1 at $\frac{\pi}{2}$ on the interval $[0,\ 2\pi]$

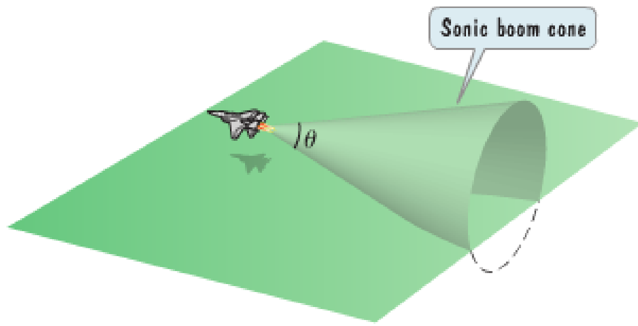
$$2\theta = \frac{\pi}{2} \Rightarrow \boxed{\theta = \frac{\pi}{4}}$$

Exercise

The speed of a supersonic aircraft is usually represented by a Mach number. A Mach number is the speed of the aircraft, in *miles* per *hour*, divided by the speed of sound, approximately 740 *mph*. Thus, a plane flying at twice the speed of sound has a speed, *M*, of Mach 2.



If an aircraft has a speed greater than Mach 1, a sonic boom is heard, created by sound waves that form a cone with a vertex angle θ.



The relationship between the cone's vertex angle θ, and the Mach speed, *M*, of an aircraft that is flying faster than the speed of sound is given by

$$\sin\frac{\theta}{2} = \frac{1}{M}$$

a) If $\theta = \frac{\pi}{6}$, determine the Mach speed, *M*, of the aircraft. Express the speed as an exact value and as decimal to the nearest tenth.

b) If $\theta = \frac{\pi}{4}$, determine the Mach speed, *M*, of the aircraft. Express the speed as an exact value and as decimal to the nearest tenth.

Solution

a) At $\theta = \frac{\pi}{6}$

$$\begin{aligned}\sin\frac{\theta}{2} &= \sqrt{\frac{1}{2}(1-\cos\theta)} \\ &= \sqrt{\frac{1}{2}\left(1-\cos\frac{\pi}{6}\right)} \\ &= \sqrt{\frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right)} \\ &= \sqrt{\frac{2-\sqrt{3}}{4}} \\ &= \frac{1}{2}\sqrt{2-\sqrt{3}} = \frac{1}{M}\end{aligned}$$

$$M = \frac{2}{\sqrt{2-\sqrt{3}}}\frac{\sqrt{2-\sqrt{3}}}{\sqrt{2-\sqrt{3}}}$$

$$= \frac{2\sqrt{2-\sqrt{3}}}{2-\sqrt{3}}\frac{2+\sqrt{3}}{2+\sqrt{3}}$$

$$\boxed{= 2\left(2+\sqrt{3}\right)\sqrt{2-\sqrt{3}}} \quad \approx 3.9$$

b) At $\theta = \frac{\pi}{4}$

$$\sin\frac{\theta}{2} = \sqrt{\frac{1}{2}(1-\cos\theta)}$$

$$= \sqrt{\frac{1}{2}\left(1-\cos\frac{\pi}{4}\right)}$$

$$= \sqrt{\frac{1}{2}\left(1-\frac{\sqrt{2}}{2}\right)}$$

$$= \sqrt{\frac{2-\sqrt{2}}{4}}$$

$$= \frac{1}{2}\sqrt{2-\sqrt{2}} = \frac{1}{M}$$

$$M = \frac{2}{\sqrt{2-\sqrt{2}}}\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2-\sqrt{2}}}$$

$$= \frac{2\sqrt{2-\sqrt{2}}}{2-\sqrt{2}}\frac{2+\sqrt{2}}{2+\sqrt{2}}$$

$$= \frac{2\left(2+\sqrt{2}\right)\sqrt{2-\sqrt{2}}}{2}$$

$$\boxed{= \left(2+\sqrt{2}\right)\sqrt{2-\sqrt{2}}} \quad \approx 2.6$$

Solutions   Section 3.4 – Solving Trigonometric Equations

Exercise

Find all solutions of the equation: $\sin x = \frac{\sqrt{2}}{2}$

Solution

$$\sin x = \frac{\sqrt{2}}{2} \Rightarrow \hat{x} = \sin^{-1}\frac{\sqrt{2}}{2} = 45^\circ \qquad x \in QI,\ QII$$

$$x = 45^\circ \qquad \to \boxed{x = 45^\circ + 360^\circ k}$$

$$x = 180^\circ - 45^\circ = 135^\circ \quad \to \boxed{x = 135^\circ + 360^\circ k}$$

Exercise

Find all solutions of the equation: $\cos x = -\frac{\pi}{3}$

Solution

$$\cos x = -\frac{\pi}{3} < -1 \text{ has no solution } \left([-1,1]\right)$$

Exercise

Find all solutions of the equation: $2\cos\theta - \sqrt{3} = 0$

Solution

$$2\cos\theta = \sqrt{3} \Rightarrow \cos\theta = \frac{\sqrt{3}}{2} \qquad \theta \in QI,\ QIV$$

$$\theta = \frac{\pi}{6} \qquad \to \boxed{\theta = \frac{\pi}{6} + 2\pi k}$$

$$\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6} \quad \to \boxed{\theta = \frac{11\pi}{6} + 2\pi k}$$

Exercise

Find all solutions of the equation: $2\cos 2\theta - \sqrt{3} = 0$

Solution

$$2\cos 2\theta = \sqrt{3} \Rightarrow \cos 2\theta = \frac{\sqrt{3}}{2} \qquad \theta \in QI,\ QIV$$

$$2\theta = \frac{\pi}{6} \qquad \to \boxed{\theta = \frac{\pi}{12} + \pi n}$$

$$2\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6} \quad \to \boxed{\theta = \frac{11\pi}{12} + \pi n}$$

Exercise

Find all solutions of the equation: $\sqrt{3}\tan\frac{1}{3}x=1$

Solution

$$\tan\frac{1}{3}x=\frac{1}{\sqrt{3}} \quad\Rightarrow\quad \frac{1}{3}x=\frac{\pi}{6}\ or\ \frac{7\pi}{6}$$

$$\frac{1}{3}x=\frac{\pi}{6}+\pi n \quad\rightarrow\quad \boxed{x=\frac{\pi}{2}+3\pi n}$$

Exercise

Find all solutions of the equation: $\cos\left(4x-\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$

Solution

$$\cos\left(4x-\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}=\cos\frac{\pi}{4}$$

$$\rightarrow 4x-\frac{\pi}{4}=\frac{\pi}{4}+2\pi k \Rightarrow 4x=\frac{\pi}{2}+2\pi k \rightarrow \boxed{x=\frac{\pi}{4}+\frac{\pi}{2}k}$$

$$\cos\left(4x-\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}=\cos\frac{7\pi}{4}$$

$$\rightarrow 4x-\frac{\pi}{4}=\frac{7\pi}{4}+2\pi k \Rightarrow 4x=2\pi+2\pi k \rightarrow \boxed{x=\frac{\pi}{2}k}$$

Exercise

Find all solutions of the equation: $(\cos\theta-1)(\sin\theta+1)=0$

Solution

$\cos\theta-1=0$	$\sin\theta+1=0$
$\cos\theta=1$	$\sin\theta=-1$
$\boxed{\theta=0^\circ+360^\circ k}$	$\boxed{\theta=270^\circ+360^\circ k}$

Exercise

Find all solutions of the equation: $\cot^2 x-3=0$

Solution

$$\cot^2 x=3 \Rightarrow \cot x=\pm\sqrt{3}$$

$$\boxed{x=\frac{\pi}{6}+2\pi k,\ \frac{5\pi}{6}+2\pi k,\ \frac{7\pi}{6}+2\pi k,\ \frac{11\pi}{6}+2\pi k} \qquad \text{Or } \boxed{x=\frac{\pi}{6}+\pi n,\ \frac{5\pi}{6}+\pi n}$$

Exercise

Find all solutions of the equation: $\cos x + 1 = 2\sin^2 x$

Solution

$$\cos x + 1 = 2\left(1 - \cos^2 x\right)$$

$$\cos x + 1 = 2 - 2\cos^2 x$$

$$\cos x + 1 - 2 + 2\cos^2 x = 0$$

$$2\cos^2 x + \cos x - 1 = 0$$

$\cos x = -1$ $\boxed{x = \pi + 2\pi n}$	$\cos x = \frac{1}{2}$ $\boxed{x = \frac{\pi}{3} + 2\pi n; \quad x = \frac{5\pi}{3} + 2\pi n}$

Exercise

Find all solutions of the equation: $\cos(\ln x) = 0$

Solution

$$\cos(\ln x) = 0 \to \begin{cases} \ln x = \frac{\pi}{2} + 2\pi k \\ \ln x = \frac{3\pi}{2} + 2\pi k \end{cases} \to \ln x = \frac{\pi}{2} + \pi n \Rightarrow \boxed{x = e^{\pi/2 + \pi n}}$$

Exercise

Find the solutions of the equation that are in the interval $[0, 2\pi)$: $2\sin^2 x = 1 - \sin x$

Solution

$$2\sin^2 x + \sin x - 1 = 0$$

$\sin x = -1$ $\boxed{x = \frac{3\pi}{2}}$	$\sin x = \frac{1}{2}$ $\boxed{x = \frac{\pi}{6}; \quad x = \frac{5\pi}{6}}$

Exercise

Find the solutions of the equation that are in the interval $[0, 2\pi)$: $\tan^2 x \sin x = \sin x$

Solution

$$\tan^2 x \sin x - \sin x = 0$$

$$\sin x\left(\tan^2 x - 1\right) = 0$$

$\sin x = 0$ $\boxed{x = 0; \quad x = \pi}$	$\tan^2 x - 1 = 0 \Rightarrow \tan^2 x = 1$ $\tan x = \pm 1$ $\boxed{x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}}$

Exercise

Find the solutions of the equation that are in the interval $[0,\ 2\pi)$: $1 - \sin x = \sqrt{3}\cos x$

Solution

$$(1 - \sin x)^2 = \left(\sqrt{3}\cos x\right)^2$$

$$1 - 2\sin x + \sin^2 x = 3\cos^2 x$$

$$1 - 2\sin x + \sin^2 x = 3\left(1 - \sin^2 x\right)$$

$$1 - 2\sin x + \sin^2 x = 3 - 3\sin^2 x$$

$$1 - 2\sin x + \sin^2 x - 3 + 3\sin^2 x = 0$$

$$4\sin^2 x - 2\sin x - 2 = 0$$

$\sin x = 1$	$\sin x = -\frac{1}{2}$	
$x = \frac{\pi}{2} \to (\mathit{check})$ $1 - \sin\frac{\pi}{2} = \sqrt{3}\cos\frac{\pi}{2}$ $1 - (1) = \sqrt{3}(0)$ $0 = 0$	$x = \frac{7\pi}{6}$ $1 - \sin\frac{7\pi}{6} = \sqrt{3}\cos\frac{7\pi}{6}$ $1 - \left(-\frac{1}{2}\right) = \sqrt{3}\left(-\frac{\sqrt{3}}{2}\right)$ $\frac{3}{2} = -\frac{3}{2}$	$x = \frac{11\pi}{6}$ $1 - \sin\frac{11\pi}{6} = \sqrt{3}\cos\frac{11\pi}{6}$ $1 - \left(-\frac{1}{2}\right) = \sqrt{3}\left(\frac{\sqrt{3}}{2}\right)$ $\frac{3}{2} = \frac{3}{2}$

The solutions are: $\boxed{x = \frac{\pi}{2}, \frac{11\pi}{6}}$

Exercise

Find the solutions of the equation that are in the interval $[0,\ 2\pi)$: $\sin x + \cos x \cot x = \csc x$

Solution

$$\sin x + \cos x \frac{\cos x}{\sin x} = \frac{1}{\sin x} \qquad \textit{Multiply by } \sin x \textit{ both sides } (\sin x \neq 0)$$

$$\sin^2 x + \cos^2 x = 1$$

$1 = 1$ ***(True)***

The solutions are: $x \in [0,\ 2\pi)$ except 0 and π.

Exercise

Find the solutions of the equation that are in the interval $[0,\ 2\pi)$: $2\sin^3 x+\sin^2 x-2\sin x-1=0$

Solution

$$\sin^2 x(2\sin x+1)-(2\sin x+1)=0$$

Factor by grouping

$$(2\sin x+1)\left(\sin^2 x-1\right)=0$$

$2\sin x+1=0$ $\sin x=-\frac{1}{2}$ $\boxed{x=\frac{7\pi}{6},\ \frac{11\pi}{6}}$	$\sin^2 x=1$ $\sin x=\pm 1$ $\boxed{x=\frac{\pi}{2},\ \frac{3\pi}{2}}$

Exercise

Find the solutions of the equation that are in the interval $[0,\ 2\pi)$: $2\tan x\csc x+2\csc x+\tan x+1=0$

Solution

$$2\tan x\csc x+\tan x+2\csc x+1=0$$

$$\tan x(2\csc x+1)+(2\csc x+1)=0$$

$$(2\csc x+1)(\tan x+1)=0$$

$2\csc x+1=0$ $\csc x=-\frac{1}{2}=\frac{1}{\sin x}$ $\sin x=-2\ (impossible)$	$\tan x=-1$ $\boxed{x=\frac{3\pi}{4},\ \frac{7\pi}{4}}$

Exercise

Solve $2\cos\theta+\sqrt{3}=0$ $\quad$ *if* $0^\circ\le\theta<360^\circ$

Solution

$$2\cos\theta=-\sqrt{3}$$

$$\cos\theta=-\frac{\sqrt{3}}{2}\ \Rightarrow\ \hat{\theta}=\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$\boxed{\theta=150^\circ,\ 210^\circ}$$

Exercise

Solve $5\cos t + \sqrt{12} = \cos t \qquad if \;\; 0 \le t < 2\pi$

Solution

$5\cos t - \cos t = -\sqrt{12}$

$\sqrt{12} = \sqrt{4.3} = 2\sqrt{3}$

$4\cos t = -2\sqrt{3}$

$4\cos t = -2\sqrt{3}$

$\cos t = -\dfrac{2\sqrt{3}}{4}$

$\cos t = -\dfrac{\sqrt{3}}{2} \quad \Rightarrow \quad t = \cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

$\boxed{\theta = \dfrac{5\pi}{6}, \;\; \dfrac{7\pi}{6}}$

Exercise

Solve $\tan\theta - 2\cos\theta\tan\theta = 0 \qquad if \;\; 0° \le \theta < 360°$

Solution

$\tan\theta(1 - 2\cos\theta) = 0$

$\tan\theta = 0$

$\theta = 0°, \; 180°$

$1 - 2\cos\theta = 0$

$1 = 2\cos\theta$

$\cos\theta = \dfrac{1}{2} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\dfrac{1}{2}\right)$

$\theta = 60°, \; 300°$

$\boxed{\theta = 0°, \; 60°, \; 180°, \; 300°}$

Exercise

Solve $2\sin^2\theta - 2\sin\theta - 1 = 0 \qquad if \;\; 0° \le \theta < 360°$

Solution

$$\sin\theta = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{12}}{4}$$

$$= \frac{2 \pm 2\sqrt{3}}{4}$$

$$= \frac{1 \pm \sqrt{3}}{2}$$

$\hat{\theta} = \sin^{-1}\left(\frac{1-\sqrt{3}}{2}\right) = -21.47°$ $\qquad$ $\sin\theta = \frac{1+\sqrt{3}}{2} = 1.366 > 1$

$\theta = 360° - 21.47° \; \underline{= 338.53°}$

$\theta = 180° + 21.47° \; \underline{= 201.47°}$

Exercise

Solve $\cos\left(A - \frac{\pi}{9}\right) = -\frac{1}{2}$

Solution

$-\frac{1}{2}$ *is negative* $\rightarrow$ *cosine is in QII or QIII.*

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$
$$\cos\left(A - \frac{\pi}{9}\right) = -\frac{1}{2} = \cos\frac{2\pi}{3}$$
$$A - \frac{\pi}{9} = \frac{2\pi}{3} + 2\pi k$$
$$A = \frac{2\pi}{3} + \frac{\pi}{9} + 2\pi k$$
$$\boxed{A = \frac{7\pi}{9} + 2\pi k}$$

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{4\pi}{3}$$
$$\cos\left(A - \frac{\pi}{9}\right) = -\frac{1}{2} = \cos\frac{4\pi}{3}$$
$$A - \frac{\pi}{9} = \frac{4\pi}{3} + 2\pi k$$
$$A = \frac{4\pi}{3} + \frac{\pi}{9} + 2\pi k$$
$$\boxed{A = \frac{13\pi}{9} + 2\pi k}$$

Exercise

Solve: $4\cos\theta - 3\sec\theta = 0 \;\; if \;\; 0° \le \theta < 360°$

Solution

$$4\cos\theta - 3\frac{1}{\cos\theta} = 0 \qquad \boxed{\cos\theta \ne 0}$$
$$4\cos\theta\cos\theta - 3\frac{1}{\cos\theta}\cos\theta = 0$$
$$4\cos^2\theta - 3 = 0$$
$$4\cos^2\theta = 3$$
$$\cos^2\theta = \frac{3}{4}$$
$$\cos\theta = \pm\frac{\sqrt{3}}{2} \Rightarrow \theta = \cos^{-1}\left(\pm\frac{\sqrt{3}}{2}\right)$$

The solutions are: $\boxed{\theta = 30°,\ 150°,\ 210°,\ 330°}$

Exercise

Solve: $2\sin^2 x - \cos x - 1 = 0 \quad if \quad 0 \le x < 2\pi$

Solution

$$2\left(1-\cos^2 x\right) - \cos x - 1 = 0$$

$$2 - 2\cos^2 x - \cos x - 1 = 0$$

$$-2\cos^2 x - \cos x + 1 = 0$$

$\cos x = -1 \qquad\qquad \cos x = \frac{1}{2}$

$x = \pi \qquad\qquad x = \frac{\pi}{3},\ \frac{5\pi}{3}$

The solutions are: $\boxed{x = \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}}$

Exercise

Solve: $\sin\theta - \sqrt{3}\cos\theta = 1 \qquad if \quad 0° \le \theta < 360°$

Solution

$$\sin\theta - 1 = -\sqrt{3}\cos\theta$$

$$(\sin\theta - 1)^2 = \left(-\sqrt{3}\cos\theta\right)^2$$

$$\sin^2\theta - 2\sin\theta + 1 = 3\cos^2\theta \qquad\qquad \cos^2\theta = 1 - \sin^2\theta$$

$$\sin^2\theta - 2\sin\theta + 1 = 3\left(1 - \sin^2\theta\right)$$

$$\sin^2\theta - 2\sin\theta + 1 = 3 - 3\sin^2\theta$$

$$\sin^2\theta - 2\sin\theta + 1 - 3 + 3\sin^2\theta = 0$$

$$4\sin^2\theta - 2\sin\theta - 2 = 0$$

$\sin\theta = 1 \quad \Rightarrow \quad \underline{\theta = 90°} \qquad \sin\theta = -\frac{1}{2} \quad \Rightarrow \quad \underline{\theta = 210°,\ 330°}$

Check

$\theta = 90°$	$\theta = 210°$	$\theta = 330°$
$\sin 90° - \sqrt{3}\cos 90° \overset{?}{=} 1$	$\sin 210° - \sqrt{3}\cos 210° \overset{?}{=} 1$	$\sin 330° - \sqrt{3}\cos 330° \overset{?}{=} 1$
$1 - \sqrt{3}(0) \overset{?}{=} 1$	$-\frac{1}{2} - \sqrt{3}\left(-\frac{\sqrt{3}}{2}\right) \overset{?}{=} 1$	$-\frac{1}{2} - \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) \overset{?}{=} 1$
$1 = 1$	$-\frac{1}{2} + \frac{3}{2} \overset{?}{=} 1$	$-\frac{1}{2} - \frac{3}{2} \overset{?}{=} 1$
	$1 = 1$	$-2 \ne 1$
		(False statement)

The solutions are: $\boxed{90°,\ 210°}$

Exercise

Solve: $7\sin^2\theta - 9\cos 2\theta = 0 \qquad if \;\; 0° \le \theta < 360°$

Solution

$$7\sin^2\theta - 9\left(1 - 2\sin^2\theta\right) = 0 \qquad\qquad \cos^2\theta = 1 - 2\sin^2\theta$$

$$7\sin^2\theta - 9 + 18\sin^2\theta = 0$$

$$25\sin^2\theta - 9 = 0$$

$$25\sin^2\theta = 9$$

$$\sin^2\theta = \frac{9}{25} \quad \Rightarrow \quad \sin\theta = \pm\frac{3}{5}$$

$$\hat{\theta} = \sin^{-1}\left(\frac{3}{5}\right) \approx 36.87°$$

$$\theta \approx 36.87° \qquad \theta \approx 180° - 36.87° \approx 143.13°$$

$$\theta \approx 180° + 36.87° \approx 216.87° \qquad \theta \approx 360° - 36.87° \approx 323.13°$$

The solutions are: $\boxed{36.87°,\ 143.13°,\ 216.87°,\ 323.13°}$

Exercise

Solve: $2\cos^2 t - 9\cos t = 5 \qquad if \;\; 0 \le t < 2\pi$

Solution

$$2\cos^2 t - 9\cos t - 5 = 0$$

$$(2\cos t + 1)(\cos t - 5) = 0$$

$$2\cos t + 1 = 0 \qquad \cos t - 5 = 0$$

$$\cos t = -\frac{1}{2} \qquad \cos t = 5$$

$$\cos t = -\frac{1}{2} \qquad \cos t = 5$$

$$\hat{t} = \cos^{-1}\left(-\frac{1}{2}\right) \qquad \textit{No solution}$$

$$\hat{t} = \frac{\pi}{3}$$

Negative sign $\rightarrow$ *cosine is in QII or QIII*

$$t = \pi - \frac{\pi}{3} \qquad t = \pi + \frac{\pi}{3}$$

$$t = \frac{2\pi}{3} \qquad t = \frac{4\pi}{3}$$

The solutions are: $\boxed{\frac{2\pi}{3},\ \frac{4\pi}{3}}$

Exercise

Solve $\sin\theta\tan\theta = \sin\theta \quad if \;\; 0° \le \theta < 360°$

Solution

$$\sin\theta\tan\theta - \sin\theta = 0$$

$$\sin\theta(\tan\theta - 1) = 0$$

$\sin\theta = 0 \qquad\qquad \tan\theta - 1 = 0$

$\theta = 0°,\ 180° \qquad\quad \tan\theta = 1$

$\theta = 45°,\ 225°$

The solutions are: $\boxed{0°,\ 45°,\ 180°,\ 225°}$

Exercise

Solve $\tan^2 x + \tan x - 2 = 0 \quad if \;\; 0 \le x < 2\pi$

Solution

$$\tan^2 x + \tan x - 2 = 0$$

$\tan x = 1 \qquad\qquad \tan x = -2$

$x = \frac{\pi}{4}, \frac{5\pi}{4} \qquad \hat{x} = \tan^{-1}(2) \approx 1.107 \qquad x \in QII,\ QIV$

$x = 2.034,\ 5.176$

The solutions are: $\boxed{\frac{\pi}{4},\ \frac{5\pi}{4},\ 2.034,\ 5.176}$

Exercise

Solve $\tan x + \sqrt{3} = \sec x \qquad if \;\; 0 \le x < 2\pi$

Solution

$$\left(\tan x + \sqrt{3}\right)^2 = (\sec x)^2$$

$$\tan^2 x + 2\sqrt{3}\tan x + 3 = \sec^2 x$$

$$\tan^2 x + 2\sqrt{3}\tan x + 3 = 1 + \tan^2 x$$

$$\tan^2 x + 2\sqrt{3}\tan x + 3 - 1 - \tan^2 x = 0$$

$$2\sqrt{3}\tan x + 2 = 0$$

$$2\sqrt{3}\tan x = -2$$

$$\tan x = -\frac{2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

$$x = \frac{5\pi}{6} \quad or \quad x = \frac{11\pi}{6}$$

$\tan\frac{5\pi}{6}+\sqrt{3}\overset{?}{=}\sec\frac{5\pi}{6}$ $-\frac{\sqrt{3}}{3}+\sqrt{3}\overset{?}{=}-\frac{2\sqrt{3}}{3}$ $\frac{2\sqrt{3}}{3}\neq-\frac{2\sqrt{3}}{3}$ ***False***	$\tan\frac{11\pi}{6}+\sqrt{3}\overset{?}{=}\sec\frac{11\pi}{6}$ $-\frac{\sqrt{3}}{3}+\sqrt{3}\overset{?}{=}\frac{2\sqrt{3}}{3}$ $\frac{2\sqrt{3}}{3}=\frac{2\sqrt{3}}{3}$

The solutions are: $\boxed{\frac{11\pi}{6}}$

Exercise

Solve $2\cos\theta+\sqrt{3}=0 \qquad if \;\; 0^\circ \le \theta < 360^\circ$

Solution

$$2\cos\theta=-\sqrt{3}$$

$$\cos\theta=-\frac{\sqrt{3}}{2} \quad\Rightarrow\quad \hat{\theta}=\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$\boxed{\theta=150^\circ,\;\; 210^\circ}$$

Exercise

Solve $5\cos t+\sqrt{12}=\cos t \qquad if \;\; 0\le t<2\pi$

Solution

$$5\cos t-\cos t=-\sqrt{12}$$

$$\sqrt{12}=\sqrt{4.3}=2\sqrt{3}$$

$$4\cos t=-2\sqrt{3}$$

$$4\cos t=-2\sqrt{3}$$

$$\cos t=-\frac{2\sqrt{3}}{4}$$

$$\cos t=-\frac{\sqrt{3}}{2} \quad\Rightarrow\quad t=\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$\boxed{\theta=\frac{5\pi}{6},\;\; \frac{7\pi}{6}}$$

Exercise

Solve $\tan\theta - 2\cos\theta\tan\theta = 0 \qquad if \;\; 0° \le \theta < 360°$

Solution

$$\tan\theta\left(1-2\cos\theta\right)=0$$

$\tan\theta = 0$ $\qquad$ $1-2\cos\theta = 0$

$\theta = 0°,\ 180°$ $\qquad$ $1 = 2\cos\theta$

$$\cos\theta = \frac{1}{2} \quad\Rightarrow\quad \theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = 60°,\ 300°$$

$$\boxed{\theta = 0°,\ 60°,\ 180°,\ 300°}$$

Exercise

Solve $2\sin^2\theta - 2\sin\theta - 1 = 0 \qquad if \;\; 0° \le \theta < 360°$

Solution

$$\sin\theta = \frac{-(-2)\pm\sqrt{(-2)^2-4(2)(-1)}}{2(2)}$$

$$= \frac{2\pm\sqrt{12}}{4}$$

$$= \frac{2\pm 2\sqrt{3}}{4}$$

$$= \frac{1\pm\sqrt{3}}{2}$$

$\hat{\theta} = \sin^{-1}\left(\frac{1-\sqrt{3}}{2}\right) = -21.47°$ $\qquad$ $\sin\theta = \frac{1+\sqrt{3}}{2} = 1.366 > 1$

$$\theta = 360° - 21.47° \ \underline{= 338.53°}|$$

$$\theta = 180° + 21.47° \ \underline{= 201.47°}|$$

Exercise

Solve $\cos\left(A-\frac{\pi}{9}\right) = -\frac{1}{2}$

Solution

$-\frac{1}{2}$ *is negative* $\rightarrow$ *cosine is in QII or QIII.*

$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ $\qquad$ $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{4\pi}{3}$

$\cos\left(A-\frac{\pi}{9}\right) = -\frac{1}{2} = \cos\frac{2\pi}{3}$ $\qquad$ $\cos\left(A-\frac{\pi}{9}\right) = -\frac{1}{2} = \cos\frac{4\pi}{3}$

$A - \frac{\pi}{9} = \frac{2\pi}{3} + 2\pi k$

$A = \frac{2\pi}{3} + \frac{\pi}{9} + 2\pi k$

$\boxed{A = \frac{7\pi}{9} + 2\pi k}$

$A - \frac{\pi}{9} = \frac{4\pi}{3} + 2\pi k$

$A = \frac{4\pi}{3} + \frac{\pi}{9} + 2\pi k$

$\boxed{A = \frac{13\pi}{9} + 2\pi k}$

Exercise

Solve: $4\cos\theta - 3\sec\theta = 0 \;\; if \;\; 0° \le \theta < 360°$

Solution

$$4\cos\theta - 3\frac{1}{\cos\theta} = 0 \qquad \boxed{\cos\theta \ne 0}$$

$$4\cos\theta\cos\theta - 3\frac{1}{\cos\theta}\cos\theta = 0$$

$$4\cos^2\theta - 3 = 0$$

$$4\cos^2\theta = 3$$

$$\cos^2\theta = \frac{3}{4}$$

$$\cos\theta = \pm\frac{\sqrt{3}}{2} \Rightarrow \theta = \cos^{-1}\left(\pm\frac{\sqrt{3}}{2}\right)$$

The solutions are: $\boxed{\theta = 30°,\; 150°,\; 210°,\; 330°}$

Exercise

Solve: $2\sin^2 x - \cos x - 1 = 0 \;\; if \;\; 0 \le x < 2\pi$

Solution

$$2\left(1 - \cos^2 x\right) - \cos x - 1 = 0$$

$$2 - 2\cos^2 x - \cos x - 1 = 0$$

$$-2\cos^2 x - \cos x + 1 = 0$$

$$\cos x = -1 \qquad\qquad \cos x = \frac{1}{2}$$

$$x = \pi \qquad\qquad x = \frac{\pi}{3},\; \frac{5\pi}{3}$$

The solutions are: $\boxed{x = \frac{\pi}{3},\; \pi,\; \frac{5\pi}{3}}$

Exercise

Solve: $\sin\theta - \sqrt{3}\cos\theta = 1$ $\quad$ *if* $0° \le \theta < 360°$

Solution

$$\sin\theta - 1 = -\sqrt{3}\cos\theta$$

$$(\sin\theta - 1)^2 = \left(-\sqrt{3}\cos\theta\right)^2$$

$$\sin^2\theta - 2\sin\theta + 1 = 3\cos^2\theta \qquad \cos^2\theta = 1 - \sin^2\theta$$

$$\sin^2\theta - 2\sin\theta + 1 = 3\left(1 - \sin^2\theta\right)$$

$$\sin^2\theta - 2\sin\theta + 1 - 3 + 3\sin^2\theta = 0$$

$$4\sin^2\theta - 2\sin\theta - 2 = 0$$

$\sin\theta = 1$ $\qquad\qquad$ $\sin\theta = -\frac{1}{2}$

$\theta = 90°$ $\qquad\qquad$ $\theta = 210°,\ 330°$

Check

$\theta = 90°$	$\theta = 210°$	$\theta = 330°$
$\sin 90° - \sqrt{3}\cos 90° \overset{?}{=} 1$ $1 - \sqrt{3}(0) \overset{?}{=} 1$ $1 = 1$	$\sin 210° - \sqrt{3}\cos 210° \overset{?}{=} 1$ $-\frac{1}{2} - \sqrt{3}\left(-\frac{\sqrt{3}}{2}\right) \overset{?}{=} 1$ $-\frac{1}{2} + \frac{3}{2} \overset{?}{=} 1$ $1 = 1$	$\sin 330° - \sqrt{3}\cos 330° \overset{?}{=} 1$ $-\frac{1}{2} - \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) \overset{?}{=} 1$ $-\frac{1}{2} - \frac{3}{2} \overset{?}{=} 1$ $-2 \ne 1$ ***(False statement)***

The solutions are: $\boxed{90°,\ 210°}$

Exercise

Solve: $7\sin^2\theta - 9\cos 2\theta = 0$ $\quad$ *if* $0° \le \theta < 360°$

Solution

$$7\sin^2\theta - 9\left(1 - 2\sin^2\theta\right) = 0 \qquad \cos^2\theta = 1 - 2\sin^2\theta$$

$$7\sin^2\theta - 9 + 18\sin^2\theta = 0$$

$$25\sin^2\theta - 9 = 0$$

$$25\sin^2\theta = 9$$

$$\sin^2\theta = \frac{9}{25} \quad \Rightarrow \quad \sin\theta = \pm\frac{3}{5}$$

$$\hat{\theta} = \sin^{-1}\left(\frac{3}{5}\right) \approx 36.87°$$

$$\theta \approx 36.87^\circ \qquad \theta \approx 180^\circ - 36.87^\circ \approx 143.13^\circ$$
$$\theta \approx 180^\circ + 36.87^\circ \approx 216.87^\circ \qquad \theta \approx 360^\circ - 36.87^\circ \approx 323.13^\circ$$

The solutions are: $\boxed{36.87^\circ,\ 143.13^\circ,\ 216.87^\circ,\ 323.13^\circ}$

Exercise

Solve: $2\cos^2 t - 9\cos t = 5 \qquad if\ \ 0 \le t < 2\pi$

Solution

$$2\cos^2 t - 9\cos t - 5 = 0$$
$$(2\cos t + 1)(\cos t - 5) = 0$$
$$2\cos t + 1 = 0 \qquad \cos t - 5 = 0$$
$$\cos t = -\frac{1}{2} \qquad \cos t = 5$$
$$\cos t = -\frac{1}{2} \qquad \cos t = 5$$
$$\hat{t} = \cos^{-1}\left(-\frac{1}{2}\right) \qquad \textit{No solution}$$
$$\hat{t} = \frac{\pi}{3}$$

Negative sign $\rightarrow$ *cosine is in QII or QIII*

$$t = \pi - \frac{\pi}{3} \qquad t = \pi + \frac{\pi}{3}$$
$$t = \frac{2\pi}{3} \qquad t = \frac{4\pi}{3}$$

The solutions are: $\boxed{\frac{2\pi}{3},\ \frac{4\pi}{3}}$

Exercise

Solve $\sin\theta\tan\theta = \sin\theta \qquad if\ \ 0^\circ \le \theta < 360^\circ$

Solution

$$\sin\theta\tan\theta - \sin\theta = 0$$
$$\sin\theta(\tan\theta - 1) = 0$$
$$\sin\theta = 0 \qquad \tan\theta - 1 = 0$$
$$\theta = 0^\circ,\ 180^\circ \qquad \tan\theta = 1$$
$$\theta = 45^\circ,\ 225^\circ$$

The solutions are: $\boxed{0^\circ,\ 45^\circ,\ 180^\circ,\ 225^\circ}$

Exercise

Solve $\tan^2 x + \tan x - 2 = 0 \quad if \quad 0 \le x < 2\pi$

Solution

$$\tan^2 x + \tan x - 2 = 0$$

$\tan x = 1 \qquad\qquad \tan x = -2$

$x = \frac{\pi}{4}, \frac{5\pi}{4} \qquad\qquad \hat{x} = \tan^{-1}(2) \approx 1.107 \qquad\qquad x \in QII,\ QIV$

$x = 2.034,\ 5.176$

The solutions are: $\boxed{\frac{\pi}{4},\ \frac{5\pi}{4},\ 2.034,\ 5.176}$

Exercise

Solve $\tan x + \sqrt{3} = \sec x \qquad\qquad if \quad 0 \le x < 2\pi$

Solution

$$\left(\tan x + \sqrt{3}\right)^2 = (\sec x)^2$$

$$\tan^2 x + 2\sqrt{3}\tan x + 3 = \sec^2 x$$

$$\tan^2 x + 2\sqrt{3}\tan x + 3 = 1 + \tan^2 x$$

$$\tan^2 x + 2\sqrt{3}\tan x + 3 - 1 - \tan^2 x = 0$$

$$2\sqrt{3}\tan x + 2 = 0$$

$$2\sqrt{3}\tan x = -2$$

$$\tan x = -\frac{2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

$$x = \frac{5\pi}{6} \quad or \quad x = \frac{11\pi}{6}$$

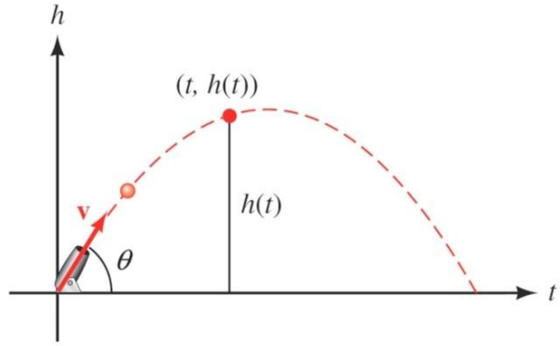
$\tan\frac{5\pi}{6} + \sqrt{3} \stackrel{?}{=} \sec\frac{5\pi}{6}$	$\tan\frac{11\pi}{6} + \sqrt{3} \stackrel{?}{=} \sec\frac{11\pi}{6}$
$-\frac{\sqrt{3}}{3} + \sqrt{3} \stackrel{?}{=} -\frac{2\sqrt{3}}{3}$	$-\frac{\sqrt{3}}{3} + \sqrt{3} \stackrel{?}{=} \frac{2\sqrt{3}}{3}$
$\frac{2\sqrt{3}}{3} \ne -\frac{2\sqrt{3}}{3}$	$\frac{2\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$
False	

The solutions are: $\boxed{\frac{11\pi}{6}}$

Exercise

If a projectile (such as a bullet) is fired into the air with an initial velocity $\boldsymbol{v}$ at an angle of elevation θ, then the height h of the projectile at time t is given by:

$$h(t) = -16t^2 + vt\sin\theta$$



a) Give the equation for the height, if $\boldsymbol{v}$ is 600 *ft./sec* and θ = 45°.

b) Use the equation in part (*a*) to find the height of the object after $\sqrt{3}$ seconds.

c) Find the angle of elevation of θ of a rifle barrel, if a bullet fired at 1,500 ft./sec takes 3 seconds to reach a height of 750 feet. Give your answer in the nearest of a degree.

Solution

a) $h(t) = -16t^2 + 600t\sin 45°$

$$= -16t^2 + 600t\frac{\sqrt{2}}{2}$$

$$= -16t^2 + 300\sqrt{2}\,t$$

b) $h\left(t = \sqrt{3}\right) = -16\left(\sqrt{3}\right)^2 + 300\sqrt{2}\,\sqrt{3}$

$$\approx 686.8\ ft$$

c) $h(t) = -16t^2 + vt\sin\theta$

$$750 = -16(3)^2 + 1500(3)\sin\theta$$

$$750 = -144 + 4500\sin\theta$$

$$750 + 144 = 4500\sin\theta$$

$$\frac{894}{4500} = \sin\theta$$

$$\theta = \sin^{-1}\left(\frac{894}{4500}\right) \approx 11.5°$$

Solution Section 3.5 – Inverse Trigonometric Functions

Exercise

Find the exact value of the expression whenever it is defined: $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Solution

$$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$

Exercise

Find the exact value of the expression whenever it is defined: $\arccos\left(\frac{\sqrt{2}}{2}\right)$

Solution

$$\arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

Exercise

Find the exact value of the expression whenever it is defined: $\arctan\left(-\frac{\sqrt{3}}{3}\right)$

Solution

$$\arctan\left(-\frac{\sqrt{3}}{3}\right) = -\arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$$

Exercise

Find the exact value of the expression whenever it is defined: $\sin\left[\arcsin\left(-\frac{3}{10}\right)\right]$

Solution

$$\alpha = \arcsin\left(-\frac{3}{10}\right) \Rightarrow \sin\alpha = -\frac{3}{10}$$

$$\sin\left[\arcsin\left(-\frac{3}{10}\right)\right] = -\frac{3}{10}$$

Exercise

Find the exact value of the expression whenever it is defined: $\tan\left[\arctan(14)\right]$

Solution

$$\tan\left[\arctan(14)\right] = 14$$

Exercise

Find the exact value of the expression whenever it is defined: $\sin\left[\sin^{-1}\left(\frac{2}{3}\right)\right]$

Solution

$$\sin\left[\sin^{-1}\left(\frac{2}{3}\right)\right] = \frac{2}{3}$$

Exercise

Find the exact value of the expression whenever it is defined: $\cos^{-1}\left[\cos\left(\frac{5\pi}{6}\right)\right]$

Solution

$$\cos^{-1}\left[\cos\left(\frac{5\pi}{6}\right)\right] = \frac{5\pi}{6} \qquad 0 \le \frac{5\pi}{6} \le \pi$$

Exercise

Find the exact value of the expression whenever it is defined: $\tan^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right]$

Solution

$$\tan^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right] = -\frac{\pi}{6} \qquad -\frac{\pi}{2} \le -\frac{\pi}{6} \le \frac{\pi}{2}$$

Exercise

Find the exact value of the expression whenever it is defined: $\arcsin\left[\sin\left(-\frac{\pi}{2}\right)\right]$

Solution

$$\arcsin\left[\sin\left(-\frac{\pi}{2}\right)\right] = -\frac{\pi}{2} \qquad -\frac{\pi}{2} \le -\frac{\pi}{2} \le \frac{\pi}{2}$$

Exercise

Find the exact value of the expression whenever it is defined: $\arccos\left[\cos(0)\right]$

Solution

$$\arccos\left[\cos(0)\right] = 0 \qquad 0 \le 0 \le \pi$$

Exercise

Find the exact value of the expression whenever it is defined: $\arctan\left[\tan\left(-\frac{\pi}{4}\right)\right]$

Solution

$$\tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right] = -\frac{\pi}{4} \qquad -\frac{\pi}{2} \le -\frac{\pi}{4} \le \frac{\pi}{2}$$

Exercise

Find the exact value of the expression whenever it is defined: $\sin\left[\arcsin\left(\frac{1}{2}\right) + \arccos 0\right]$

Solution

$$\sin\left[\arcsin\left(\frac{1}{2}\right) + \arccos 0\right] = \sin\left(\frac{\pi}{6} + 0\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

Exercise

Find the exact value of the expression whenever it is defined: $\cos\left[\arctan\left(-\frac{3}{4}\right) - \arcsin\frac{4}{5}\right]$

Solution

$$\cos\left[\arctan\left(-\frac{3}{4}\right) - \arcsin\frac{4}{5}\right] = \cos(\alpha - \beta)$$

$$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$\alpha = \arctan\left(-\frac{3}{4}\right) \Rightarrow \tan\alpha = -\frac{3}{4}$ $r = \sqrt{3^2 + 4^2} = 5$ $\sin\alpha = -\frac{3}{5} \qquad \cos\alpha = \frac{4}{5}$	$\beta = \arcsin\frac{4}{5} \Rightarrow \sin\beta = \frac{4}{5}$ $\Rightarrow \cos\beta = \frac{3}{5}$

$$\cos\left[\arctan\left(-\frac{3}{4}\right) - \arcsin\frac{4}{5}\right] = \frac{4}{5}\frac{3}{5} + \left(-\frac{3}{5}\right)\frac{4}{5} = 0$$

Exercise

Find the exact value of the expression whenever it is defined: $\tan\left[\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)\right]$

Solution

$$\tan\left[\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)\right] = \tan\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

Exercise

Find the exact value of the expression whenever it is defined: $\sin\left[2\arccos\left(-\frac{3}{5}\right)\right]$

Solution

$$\sin\left[2\arccos\left(-\frac{3}{5}\right)\right]=\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\alpha = \arccos\left(-\frac{3}{5}\right) \to \cos\alpha = -\frac{3}{5}$$

$$\sin\alpha = \frac{3}{5}$$

$$\sin\left[2\arccos\left(-\frac{3}{5}\right)\right]=2\frac{3}{5}\left(-\frac{3}{5}\right)\ \underline{=-\frac{18}{25}}$$

Exercise

Find the exact value of the expression whenever it is defined: $\cos\left[2\sin^{-1}\left(\frac{15}{17}\right)\right]$

Solution

$$\cos\left[2\sin^{-1}\left(\frac{15}{17}\right)\right]=\cos 2\alpha = 1-2\sin^2\alpha$$

$$\alpha = \sin^{-1}\left(\frac{15}{17}\right) \to \sin\alpha = \frac{15}{17}$$

$$\cos\left[2\sin^{-1}\left(\frac{15}{17}\right)\right]=1-2\left(\frac{15}{17}\right)^2$$

$$=1-\frac{450}{289}$$

$$\underline{=-\frac{161}{289}}$$

Exercise

Find the exact value of the expression whenever it is defined: $\tan\left[2\tan^{-1}\left(\frac{3}{4}\right)\right]$

Solution

$$\tan\left[2\tan^{-1}\left(\frac{3}{4}\right)\right]=\tan 2\alpha \Rightarrow \alpha = \tan^{-1}\left(\frac{3}{4}\right) \quad \to \quad \tan\alpha = \frac{3}{4}$$

$$\tan\left[2\tan^{-1}\left(\frac{3}{4}\right)\right]=\frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$=\frac{2\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^2}$$

$$= \frac{\frac{3}{2}}{1-\frac{9}{16}}$$

$$= \frac{3}{2}\frac{16}{7}$$

$$\boxed{= \frac{24}{7}}$$

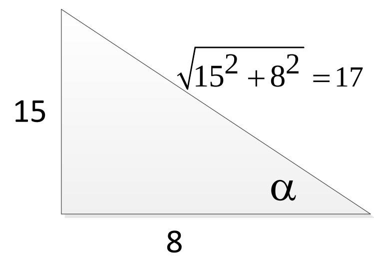
Exercise

Find the exact value of the expression whenever it is defined: $\cos\left[\frac{1}{2}\tan^{-1}\left(\frac{8}{15}\right)\right]$

Solution

$$\cos\left[\frac{1}{2}\tan^{-1}\left(\frac{8}{15}\right)\right] = \cos\left(\frac{1}{2}\alpha\right)$$

$$\rightarrow \alpha = \tan^{-1}\left(\frac{8}{15}\right) \Rightarrow \tan\alpha = \frac{8}{15}$$



$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1}{2}(1+\cos\alpha)}$$

$$= \sqrt{\frac{1}{2}\left(1+\frac{8}{17}\right)}$$

$$= \sqrt{\frac{25}{34}}$$

$$\boxed{= \frac{5}{\sqrt{34}}} \quad or \quad \frac{5\sqrt{34}}{34}$$

Exercise

Evaluate without using a calculator: $\cos\left(\cos^{-1}\frac{3}{5}\right)$

Solution

$$\cos\left(\cos^{-1}\frac{3}{5}\right) \boxed{= \frac{3}{5}}$$

Exercise

Evaluate without using a calculator: $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

Solution

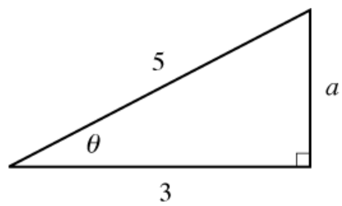
$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) \boxed{= \frac{5\pi}{6}}$$

Exercise

Evaluate without using a calculator: $\tan\left(\cos^{-1}\frac{3}{5}\right)$

Solution

$\tan\left(\cos^{-1}\frac{3}{5}\right)$



$5^2 = 3^2 + a^2 \to a = 4$

$\tan\left(\cos^{-1}\frac{3}{5}\right) = \tan\theta$

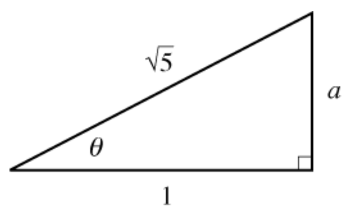
$\qquad = \boxed{\frac{4}{3}}$

Exercise

Evaluate without using a calculator: $\sin\left(\cos^{-1}\frac{1}{\sqrt{5}}\right)$

Solution

$\sin\left(\cos^{-1}\frac{1}{\sqrt{5}}\right)$



$\left(\sqrt{5}\right)^2 = 1^2 + a^2 \to a^2 = 5 - 1 \to a = 2$

$\sin\left(\cos^{-1}\frac{1}{\sqrt{5}}\right) = \sin\theta$

$\qquad = \boxed{\frac{2}{\sqrt{5}}}$

Exercise

Evaluate without using a calculator: $\cos\left(\sin^{-1}\frac{1}{2}\right)$

Solution

$\cos\left(\sin^{-1}\frac{1}{2}\right)$

$\sin\frac{\pi}{6} = \frac{1}{2} \Rightarrow \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$\cos\left(\sin^{-1}\frac{1}{2}\right) = \cos\frac{\pi}{6}$

$\qquad = \boxed{\frac{\sqrt{3}}{2}}$

Exercise

Evaluate without using a calculator: $\sin\left(\sin^{-1}\frac{3}{5}\right)$

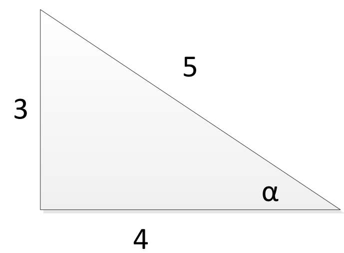
Solution

$$\sin\left(\sin^{-1}\frac{3}{5}\right) = \underline{\frac{3}{5}|}$$

Exercise

Evaluate without using a calculator: $\cos\left(\tan^{-1}\frac{3}{4}\right)$

Solution



$$\alpha = \tan^{-1}\frac{3}{4} \Rightarrow \tan\alpha = \frac{3}{4}$$

$$r = \sqrt{3^2 + 4^2} = 5$$

$$\Rightarrow \boxed{\cos\alpha = \frac{4}{5}}$$

Exercise

Evaluate without using a calculator: $\tan\left(\sin^{-1}\frac{3}{5}\right)$

Solution

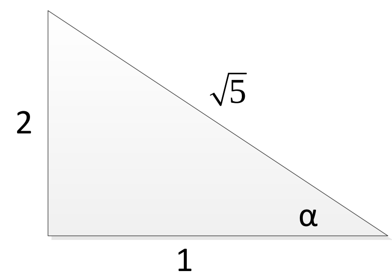
$$\sin\alpha = \frac{3}{5}$$

$$\boxed{\tan\left(\sin^{-1}\frac{3}{5}\right) = \frac{3}{4}}$$

Exercise

Evaluate without using a calculator: $\sec\left(\cos^{-1}\frac{1}{\sqrt{5}}\right)$

Solution



$$\alpha = \cos^{-1}\frac{1}{\sqrt{5}} \quad \rightarrow \quad \cos\alpha = \frac{1}{\sqrt{5}}$$

$$\underline{|\sec\alpha} = \frac{1}{\cos\alpha} = \frac{1}{\frac{1}{\sqrt{5}}} = \underline{\sqrt{5}|}$$

Exercise

Evaluate without using a calculator: $\cot\left(\tan^{-1}\frac{1}{2}\right)$

Solution

$$\alpha = \tan^{-1}\frac{1}{2} \Rightarrow \tan\alpha = \frac{1}{2}$$

$$\boxed{\cot\alpha = \frac{1}{\tan\alpha} = 2}$$

Exercise

Write an equivalent expression that involves x only for $\cos\left(\cos^{-1}x\right)$

Solution

$$\alpha = \cos^{-1}x \Rightarrow \cos\alpha = x$$

$$\boxed{\cos\left(\cos^{-1}x\right) = \cos\alpha = x}$$

Exercise

Write an equivalent expression that involves x only for $\tan\left(\cos^{-1}x\right)$

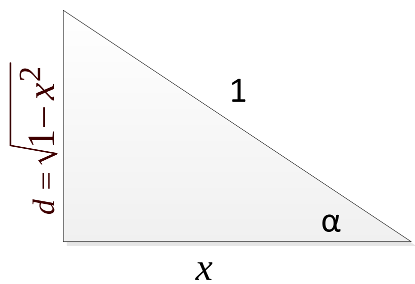
Solution

$$\alpha = \cos^{-1}x \Rightarrow \cos\alpha = x = \frac{x}{1}$$

$$x^2 + d^2 = 1 \Rightarrow d^2 = 1 - x^2$$

$$d = \sqrt{1-x^2}$$

$$\boxed{\tan\left(\cos^{-1}x\right) = \tan\alpha = \frac{\sqrt{1-x^2}}{x}}$$



Exercise

Write an equivalent expression that involves x only for $\csc\left(\sin^{-1}\frac{1}{x}\right)$

Solution

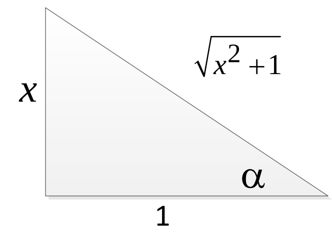
$$\alpha = \sin^{-1}\frac{1}{x} \Rightarrow \sin\alpha = \frac{1}{x}$$

$$\boxed{\csc\left(\sin^{-1}x\right) = \csc\alpha = \frac{1}{\sin\alpha} = x}$$

Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\sin\left(\tan^{-1} x\right)$

Solution



$$\sin\left(\tan^{-1} x\right) = \sin\alpha \Rightarrow \alpha = \tan^{-1} x \to \tan\alpha = x$$

$$\sin\left(\tan^{-1} x\right) = \frac{x}{\sqrt{x^2+1}}$$

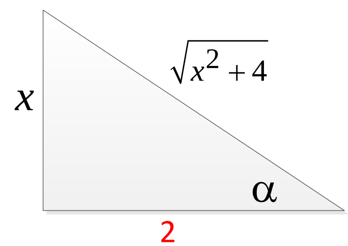
Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\sec\left(\sin^{-1}\frac{x}{\sqrt{x^2+4}}\right)$

Solution

$$\alpha = \sin^{-1}\frac{x}{\sqrt{x^2+4}} \Rightarrow \sin\alpha = \frac{x}{\sqrt{x^2+4}}$$

$$\sqrt{\left(\sqrt{x^2+4}\right)^2 - x^2} = \sqrt{x^2+4-x^2} = \sqrt{4} = 2$$



$$\sec\left(\sin^{-1}\frac{x}{\sqrt{x^2+4}}\right) = \frac{1}{\cos\alpha}$$

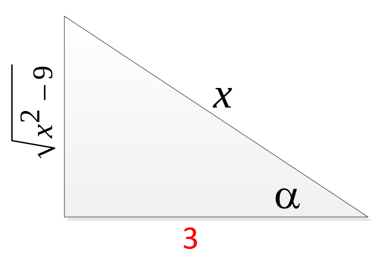
$$= \boxed{\frac{2}{\sqrt{x^2+4}}}$$

Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\cot\left(\sin^{-1}\frac{\sqrt{x^2-9}}{x}\right)$

Solution

$$\alpha = \sin^{-1}\frac{\sqrt{x^2-9}}{x} \Rightarrow \sin\alpha = \frac{\sqrt{x^2-9}}{x}$$

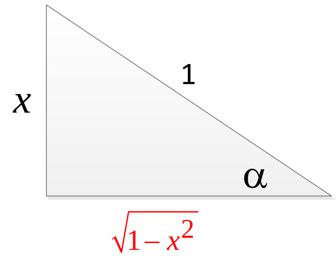


$$\cot\left(\sin^{-1}\frac{\sqrt{x^2-9}}{x}\right) = \cot\alpha = \boxed{\frac{3}{\sqrt{x^2-9}}}$$

Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\sin\left(2\sin^{-1} x\right)$

Solution



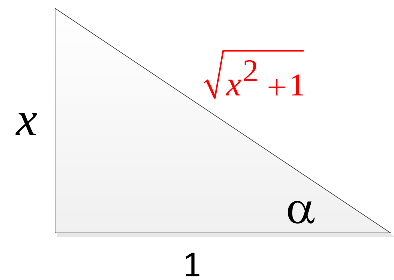
$$\alpha = \sin^{-1} x \to \sin\alpha = x$$

$$\begin{aligned} \sin\left(2\sin^{-1} x\right) &= \sin 2\alpha \\ &= 2\sin\alpha\cos\alpha \\ &= \boxed{2x\sqrt{1-x^2}} \end{aligned}$$

Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\cos\left(2\tan^{-1} x\right)$

Solution



$$\alpha = \tan^{-1} x \to \tan\alpha = x$$

$$\begin{aligned} \cos\left(2\tan^{-1} x\right) &= \cos(2\alpha) \\ &= 2\cos^2\alpha - 1 \\ &= 2\left(\frac{1}{\sqrt{x^2+1}}\right)^2 - 1 \\ &= \frac{2}{x^2+1} - 1 \\ &= \boxed{\frac{-x^2+1}{x^2+1}} \end{aligned}$$

Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\cos\left(\frac{1}{2}\arccos x\right)$

Solution

$$\alpha = \arccos x \Rightarrow \cos\alpha = x$$

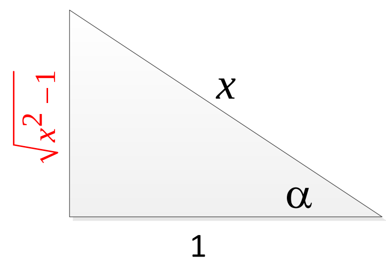
$$\begin{aligned} \cos\left(\frac{1}{2}\arccos x\right) = \cos\left(\frac{\alpha}{2}\right) &== \sqrt{\frac{1+\cos\alpha}{2}} \\ &= \boxed{\sqrt{\frac{1+x}{2}}} \end{aligned}$$

Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\tan\left(\frac{1}{2}\cos^{-1}\frac{1}{x}\right)$

Solution

$$\alpha = \cos^{-1}\frac{1}{x} \quad \Rightarrow \quad \cos\alpha = \frac{1}{x}$$



$$\tan\left(\frac{1}{2}\cos^{-1}\frac{1}{x}\right) = \tan\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{\sin\alpha}$$

$$= \frac{1-\frac{1}{x}}{\frac{\sqrt{x^2-1}}{x}} = \frac{\frac{x-1}{x}}{\frac{\sqrt{x^2-1}}{x}}$$

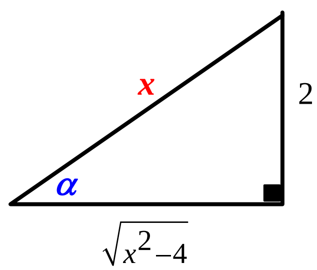
$$= \frac{x-1}{\sqrt{x^2-1}}$$

Exercise

Write the expression as an algebraic expression in x: $\sec\left(\tan^{-1}\frac{2}{\sqrt{x^2-4}}\right) \quad x > 0$

Solution

$$\tan\alpha = \frac{2}{\sqrt{x^2-4}}$$



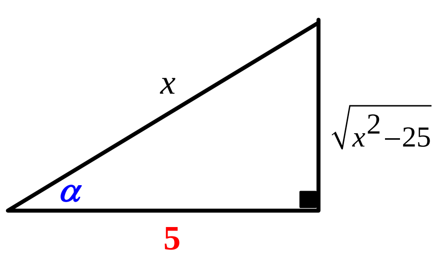
$$\sec\alpha = \frac{x}{\sqrt{x^2-4}}$$

Exercise

Write the expression as an algebraic expression in x: $\sec\left(\sin^{-1}\frac{\sqrt{x^2-25}}{x}\right) \quad x > 0$

Solution

$$\sin\alpha = \frac{\sqrt{x^2-25}}{x}$$



$$\sec\alpha = \frac{x}{5}$$

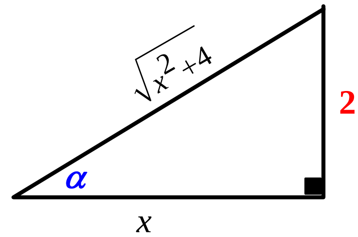
Exercise

Write the expression as an algebraic expression in x: $\sin\left(\cos^{-1}\dfrac{x}{\sqrt{x^2+4}}\right) \quad x>0$

Solution

$$\cos\alpha = \frac{x}{\sqrt{x^2+4}}$$

$$\boxed{\sin\alpha = \frac{2}{\sqrt{x^2+4}}}$$



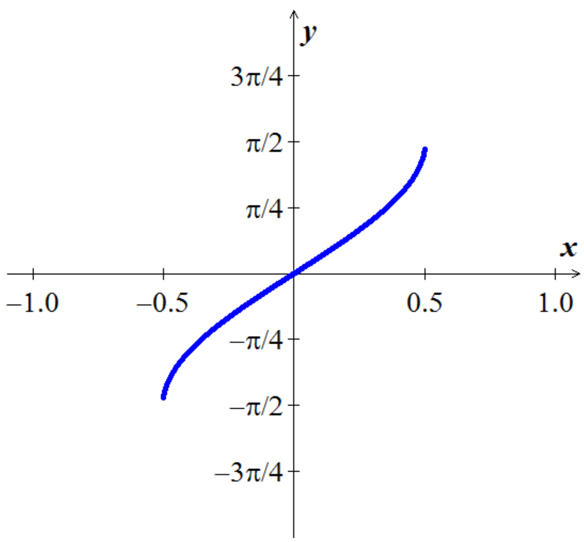
Exercise

Sketch he graph of the equation: $y=\sin^{-1}2x$

Solution

$$-\tfrac{\pi}{2} \le y \le \tfrac{\pi}{2} \quad \textit{and} \quad -1 \le 2x \le 1$$

$$-\frac{1}{2} \le x \le \frac{1}{2}$$



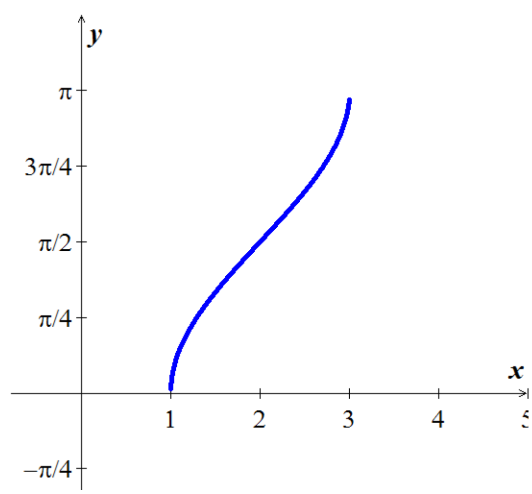
Exercise

Sketch he graph of the equation: $y=\sin^{-1}(x-2)+\frac{\pi}{2}$

Solution

$$-\tfrac{\pi}{2}+\tfrac{\pi}{2} \le y \le \tfrac{\pi}{2}+\tfrac{\pi}{2} \quad \textit{and} \quad -1 \le x-2 \le 1$$

$$0 \le y \le \pi \quad \textit{and} \quad 1 \le x \le 3$$



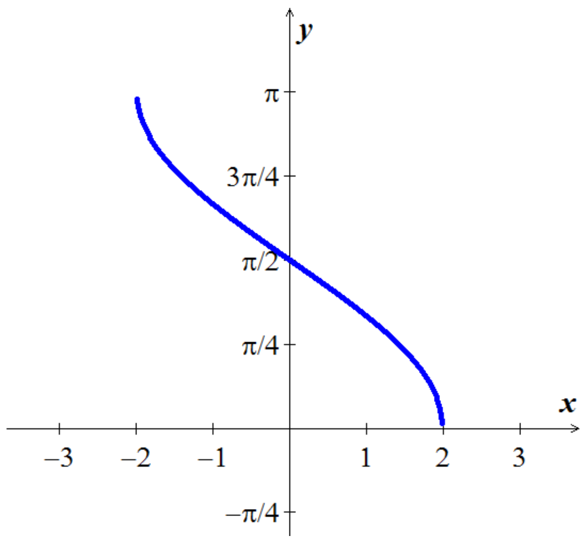
Exercise

Sketch he graph of the equation: $y = \cos^{-1}\frac{1}{2}x$

Solution

$$0 \le y \le \pi \quad and \quad -1 \le \frac{1}{2}x \le 1$$

$$-2 \le x \le 2$$



Exercise

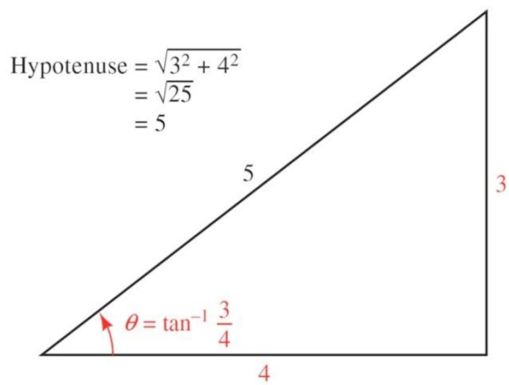
Evaluate $\sin\left(\tan^{-1}\frac{3}{4}\right)$ without using a calculator

Solution

$$\theta = \tan^{-1}\frac{3}{4} \Rightarrow \tan\theta = \frac{3}{4} \rightarrow 0° < \theta < 90°$$

$$\sin\left(\tan^{-1}\frac{3}{4}\right) = \sin\theta$$

$$= \frac{3}{5}$$



Exercise

Evaluate $\sin(\cos^{-1}x)$ as an equivalent expression in x only

Solution

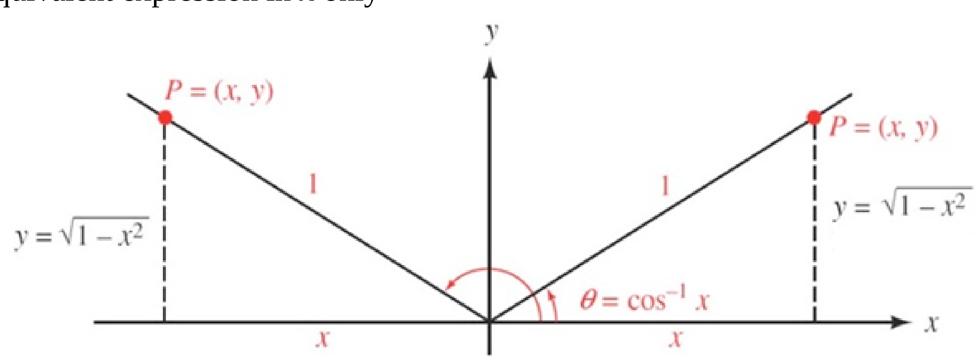
$$\sin(\theta) = \frac{y}{r}$$

$$= \frac{\sqrt{1-x^2}}{1}$$

$$= \sqrt{1-x^2}$$

$$\sin(\cos^{-1}x) = \sin\theta$$

$$= \sqrt{1-x^2}$$



Solution   Section 3.6 – Polar Coordinates

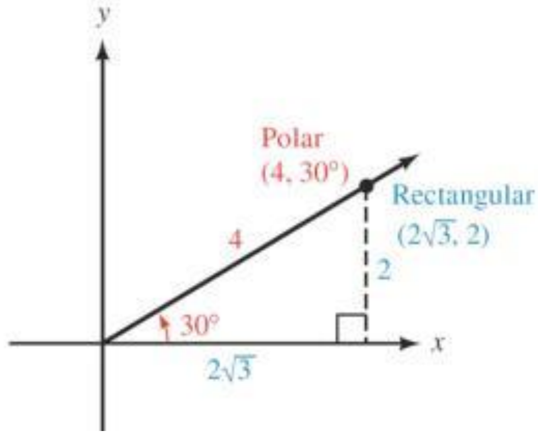
Exercise

Convert to rectangular coordinates. $(4,\ 30°)$

Solution

$$x = r\cos\theta$$
$$= 4\cos 30°$$
$$= 4\left(\frac{\sqrt{3}}{2}\right)$$
$$= 2\sqrt{3}$$

$$y = r\sin\theta$$
$$= 4\sin 30°$$
$$= 4\left(\frac{1}{2}\right)$$
$$= 2$$



The point $(2\sqrt{3},\ 2)$ in rectangular coordinates is equivalent to $(4,\ 30°)$ in polar coordinates.

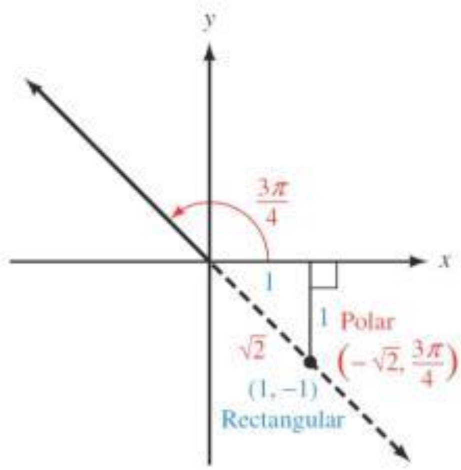
Exercise

Convert to rectangular coordinates $\left(-\sqrt{2},\ \frac{3\pi}{4}\right)$.

Solution

$$x = -\sqrt{2}\cos\frac{3\pi}{4}$$
$$= -\sqrt{2}\left(-\frac{1}{\sqrt{2}}\right)$$
$$= 1$$

$$y = -\sqrt{2}\sin\frac{3\pi}{4}$$
$$= -\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)$$
$$= -1$$



The point $(1,\ -1)$ in rectangular coordinates is equivalent to $\left(-\sqrt{2},\ \frac{3\pi}{4}\right)$ in polar coordinates.

Exercise

Convert to rectangular coordinates $(3,\ 270°)$.

Solution

$$x = 3\cos 270°$$
$$= 3(0)$$
$$= 0$$

$$y = 3\sin 270°$$
$$= 3(-1)$$
$$= -3$$

Exercise

Convert to rectangular coordinates $\left(2,\ 60°\right)$

Solution

$$x = 2\cos 60°$$
$$= 2\left(\frac{1}{2}\right)$$
$$= 0$$

$$y = 2\sin 60°$$
$$= 2\frac{\sqrt{3}}{2}$$
$$= \sqrt{3}$$

$$\boxed{\left(1,\ \sqrt{3}\right)}$$

Exercise

Convert to rectangular coordinates $\left(\sqrt{2},\ -225°\right)$

Solution

$$x = \sqrt{2}\cos\left(-225°\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}}\right) = -1$$

$$y = \sqrt{2}\sin\left(-225°\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 1$$

$$\boxed{\left(-1,\ 1\right)}$$

Exercise

Convert to rectangular coordinates $\left(4\sqrt{3},\ -\frac{\pi}{6}\right)$

Solution

$$x = 4\sqrt{3}\cos\left(-\frac{\pi}{6}\right) = 4\sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = 6$$

$$y = 4\sqrt{3}\sin\left(-\frac{\pi}{6}\right) = 4\sqrt{3}\left(-\frac{1}{2}\right) = -2\sqrt{3} \qquad \Rightarrow \boxed{\left(6,\ -2\sqrt{3}\right)}$$

Exercise

Change the polar coordinates to rectangular coordinates $\left(-2,\ \frac{7\pi}{6}\right)$

Solution

$$x = -2\cos\left(\frac{7\pi}{6}\right) = -2\left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

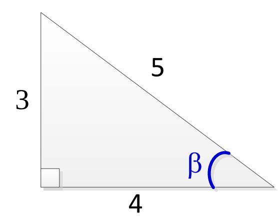
$$y = -2\sin\left(\frac{7\pi}{6}\right) = -2\left(-\frac{1}{2}\right) = 1 \qquad \Rightarrow \boxed{\left(\sqrt{3},\ 1\right)}$$

Exercise

Change the polar coordinates to rectangular coordinates $\left(6,\ \arctan\frac{3}{4}\right)$

Solution

$$\arctan\frac{3}{4} = \beta \quad \Rightarrow \quad \tan\beta = \frac{3}{4}$$



$$x = 2\cos\beta = 2\left(\frac{4}{5}\right) = \frac{8}{5}$$

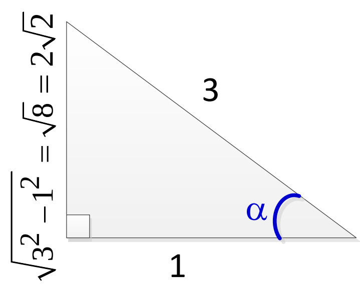
$$y = 2\sin\beta = 2\left(\frac{3}{5}\right) = \frac{6}{5} \qquad \Rightarrow \boxed{\left(\frac{8}{5},\ \frac{6}{5}\right)}$$

Exercise

Change the polar coordinates to rectangular coordinates $\left(10,\ \arccos\left(-\frac{1}{3}\right)\right)$

Solution

$$\arccos\left(-\frac{1}{3}\right) = \alpha \quad \Rightarrow \quad \cos\alpha = -\frac{1}{3} \quad (QII)$$



$$x = 10\cos\alpha = 10\left(-\frac{1}{3}\right) = -\frac{10}{3}$$

$$y = 10\sin\alpha = 10\left(\frac{2\sqrt{2}}{3}\right) = \frac{20\sqrt{2}}{3} \qquad \Rightarrow \boxed{\left(-\frac{10}{3},\ \frac{20\sqrt{2}}{3}\right)}$$

Exercise

Convert to polar coordinates $(3,\ 3)$.

Solution

$$(3,\ 3) \to \begin{cases} r = \sqrt{3^2+3^2} = \sqrt{18} = 3\sqrt{2} \\ \hat{\theta} = \tan^{-1}\left(\frac{3}{3}\right) = \tan^{-1}(1) = 45° \end{cases}$$

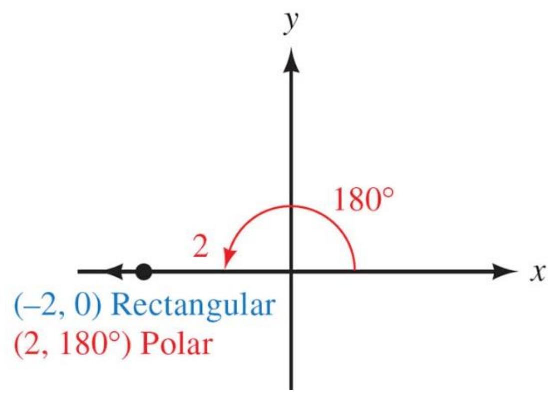
The angle is in quadrant I; therefore, $\boxed{\theta = 45°}$

$$\boxed{(3,\ 3) \ = \ \left(3\sqrt{2},\ 45°\right)}$$

Exercise

Convert to polar coordinates $(-2,\ 0)$.

Solution



$$r = \pm\sqrt{4+0}$$
$$= \pm\ 2$$

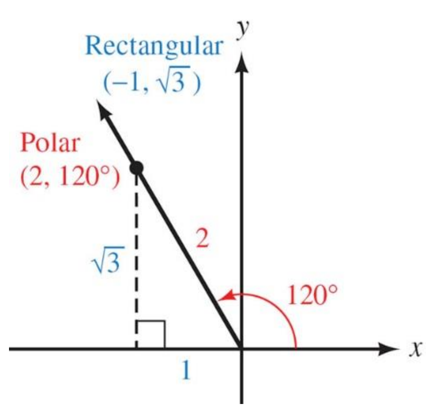
$$\theta = \tan^{-1}\frac{0}{-2}$$
$$= 0°$$

The point $r = 2, \quad \theta = 180°$

Exercise

Convert to polar coordinates $(-1,\ \sqrt{3})$.

Solution



$$r = \pm\sqrt{1+3}$$
$$= \pm\ 2$$

$$\theta = \tan^{-1}\frac{\sqrt{3}}{-1}$$
$$= 120°$$

The point $r = 2, \quad \theta = 120°$

Exercise

Convert to polar coordinates $\left(-3,\ -3\right)$ $r \ge 0$ $0° \le \theta < 360°$

Solution

$$\left(-3,\ -3\right) \to \begin{cases} r = \sqrt{(-3)^2 + (-3)^2} = 3\sqrt{2} \\ \hat{\theta} = \tan^{-1}\left(\frac{3}{3}\right) = \tan^{-1}(1) = 45° \end{cases}$$

The angle is in quadrant III; therefore, $\lfloor \theta = 180° + 45° \underline{= 225°} \rfloor$

$$\boxed{\left(-3,\ -3\right) = \left(3\sqrt{2},\ 225°\right)}$$

Exercise

Convert to polar coordinates $\left(2,\ -2\sqrt{3}\right)$ $r \ge 0$ $0° \le \theta < 360°$

Solution

$$\left(2,\ -2\sqrt{3}\right) \to \begin{cases} r = \sqrt{2^2 + \left(-2\sqrt{3}\right)^2} = 4 \\ \hat{\theta} = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \tan^{-1}\left(\sqrt{3}\right) = 60° \end{cases}$$

The angle is in quadrant IV; therefore, $\lfloor \theta = 360° - 60° \underline{= 300°} \rfloor$

$$\boxed{\left(2,\ -2\sqrt{3}\right) = \left(4,\ 300°\right)}$$

Exercise

Convert to polar coordinates $\left(-2,\ 0\right)$ $r \ge 0$ $0 \le \theta < 2\pi$

Solution

$$\left(-2,\ 0\right) \to \begin{cases} r = \sqrt{(-2)^2 + 0^2} = 2 \\ \hat{\theta} = \tan^{-1}\left(\frac{0}{2}\right) = 0 \Rightarrow \theta = \pi \end{cases}$$

$$\boxed{\left(-2,\ 0\right) = \left(2,\ \pi\right)}$$

Exercise

Convert to polar coordinates $\left(-1,\ -\sqrt{3}\right)$ $r \ge 0$ $0 \le \theta < 2\pi$

Solution

$$\left(-1,\ -\sqrt{3}\right) \to \begin{cases} r = \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2} = 2 \\ \hat{\theta} = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3} \end{cases}$$

The angle is in quadrant III; therefore, $\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$

$$\boxed{\left(-1,\ -\sqrt{3}\right) = \left(2,\ \frac{4\pi}{3}\right)}$$

Exercise

Change the rectangular coordinates to polar coordinates $\left(7,\ -7\sqrt{3}\right)$ $r > 0$ $0 \le \theta < 2\pi$

Solution

$$\left(7,\ -7\sqrt{3}\right) \to \begin{cases} r = \sqrt{(7)^2 + \left(-7\sqrt{3}\right)^2} = \sqrt{196} = 14 \\ \hat{\theta} = \tan^{-1}\left(\frac{7\sqrt{3}}{7}\right) = \frac{\pi}{3} \end{cases}$$

The angle is in quadrant IV; therefore, $\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

$$\boxed{\left(-7,\ -7\sqrt{3}\right) = \left(14,\ \frac{5\pi}{3}\right)}$$

Exercise

Change the rectangular coordinates to polar coordinates $\left(-2\sqrt{2},\ -2\sqrt{2}\right)$ $r > 0$ $0 \le \theta < 2\pi$

Solution

$$\left(-2\sqrt{2},\ -2\sqrt{2}\right) \to \begin{cases} r = \sqrt{\left(-2\sqrt{2}\right)^2 + \left(-2\sqrt{2}\right)^2} = 4 \\ \hat{\theta} = \tan^{-1}\left(\frac{-2\sqrt{2}}{-2\sqrt{2}}\right) = \frac{\pi}{4} \end{cases}$$

The angle is in quadrant III; therefore, $\theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$

$$\boxed{\left(-2\sqrt{2},\ -2\sqrt{2}\right) = \left(4,\ \frac{5\pi}{4}\right)}$$

Exercise

The point $(0,\ -3)$ in rectangular coordinates is equivalent to $(3,\ 270°)$ in polar coordinates.

Solution

$$r = \sqrt{0+(-3)^2} = 3$$

$$\hat{\theta} = \tan^{-1}\frac{0}{3} = 90°$$

The point $(3,\ 270°)$

Exercise

The point $(1,\ -1)$ in rectangular coordinates is equivalent to $\left(-\sqrt{2},\ \frac{3\pi}{4}\right)$ in polar coordinates.

Solution

$$r = \sqrt{(1)^2+(-1)^2} = \sqrt{2}$$

$$\hat{\theta} = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$\theta \in QIV \quad \to \quad \theta = \frac{7\pi}{4}$$

$$\left(\sqrt{2},\ \frac{7\pi}{4}\right) \quad \Leftrightarrow \quad \left(-\sqrt{2},\ \frac{3\pi}{4}\right)$$

Exercise

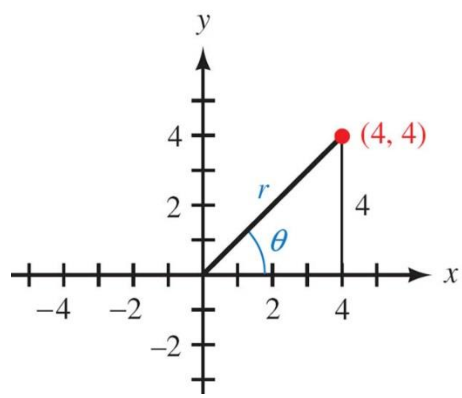
A point lies at (4, 4) on a rectangular coordinate system. Give its address in polar coordinates $(r,\ \theta)$

Solution

$$r = \sqrt{4^2+4^2} = \sqrt{32} = 4\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{4}{4}\right) = \tan^{-1}(1) = 45°$$

$$\left(4\sqrt{2},\ 45°\right)$$



Exercise

Write the equation in rectangular coordinates $r^2 = 4$

Solution

$$r^2 = 4$$

$$x^2 + y^2 = 4$$

Exercise

Write the equation in rectangular coordinates $r = 6\cos\theta$

Solution

$$r = 6\cos\theta$$

$$r = 6\frac{x}{r}$$

$$r^2 = 6x$$

$$\boxed{x^2 + y^2 = 6x}$$

Exercise

Write the equation in rectangular coordinates $r^2 = 4\cos 2\theta$

Solution

$$r^2 = 4\left(\cos^2\theta - \sin^2\theta\right) \qquad \cos\theta = \frac{x}{r} \quad \sin\theta = \frac{y}{r}$$

$$= 4\left(\left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2\right)$$

$$= 4\left(\frac{x^2}{r^2} - \frac{y^2}{r^2}\right)$$

$$= 4\left(\frac{x^2 - y^2}{r^2}\right)$$

$$r^4 = 4\left(x^2 - y^2\right) \qquad r^2 = x^2 + y^2$$

$$\boxed{\left(x^2 + y^2\right)^4 = 4x^2 - 4y^2}$$

Exercise

Write the equation in rectangular coordinates $r\left(\cos\theta - \sin\theta\right) = 2$

Solution

$$r\left(\cos\theta - \sin\theta\right) = 2 \qquad \cos\theta = \frac{x}{r} \quad \sin\theta = \frac{y}{r}$$

$$r\left(\frac{x}{r} - \frac{y}{r}\right) = 2$$

$$r\left(\frac{x - y}{r}\right) = 2$$

$$\boxed{x - y = 2}$$

Exercise

Write the equation in rectangular coordinates $r^2 = 4\sin 2\theta$

Solution

$$r^2 = 4\sin 2\theta \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= 4(2\sin\theta\cos\theta) \qquad \cos\theta = \frac{x}{r} \quad \sin\theta = \frac{y}{r}$$

$$= 8\left(\frac{y}{r}\right)\left(\frac{x}{r}\right)$$

$$= 8\frac{xy}{r^2}$$

$$r^4 = 8xy \qquad r^2 = x^2 + y^2$$

$$\boxed{\left(x^2 + y^2\right)^2 = 8xy}$$

Exercise

Find an equation in x and y that has the same graph as polar equation. $r\sin\theta = -2$

Solution

$$r\sin\theta = -2 \qquad y = r\sin\theta$$

$$\boxed{y = -2}$$

Exercise

Find an equation in x and y that has the same graph as polar equation. $\theta = \frac{\pi}{4}$

Solution

$$\tan\theta = \tan\frac{\pi}{4}$$

$$\frac{y}{x} = 1$$

$$\boxed{y = x}$$

Exercise

Find an equation in x and y that has the same graph as polar $r^2\left(4\sin^2\theta - 9\cos^2\theta\right) = 36$

Solution

$$r^2\left(4\sin^2\theta - 9\cos^2\theta\right) = 36 \qquad \cos\theta = \frac{x}{r} \quad \sin\theta = \frac{y}{r}$$

$$r^2\left(4\frac{y^2}{r^2} - 9\frac{x^2}{r^2}\right) = 36$$

$$r^2\left(\frac{4y^2-9x^2}{r^2}\right)=36$$

$$\boxed{4y^2-9x^2=36}$$

Exercise

Find an equation in x and y that has the same graph as polar $\quad r^2\left(\cos^2\theta+4\sin^2\theta\right)=16$

Solution

$$r^2\left(\cos^2\theta+4\sin^2\theta\right)=16 \qquad \cos\theta=\frac{x}{r} \quad \sin\theta=\frac{y}{r}$$

$$r^2\left(\frac{x^2}{r^2}+4\frac{y^2}{r^2}\right)=16$$

$$r^2\left(\frac{x^2+4y^2}{r^2}\right)=16$$

$$\boxed{x^2+4y^2=16}$$

Exercise

Find an equation in x and y that has the same graph as polar $\quad r(\sin\theta-2\cos\theta)=6$

Solution

$$r(\sin\theta-2\cos\theta)=6 \qquad \cos\theta=\frac{x}{r} \quad \sin\theta=\frac{y}{r}$$

$$r\left(\frac{y}{r}-2\frac{x}{r}\right)=6$$

$$r\left(\frac{y-2x}{r}\right)=6$$

$$\boxed{y-2x=6}$$

Exercise

Find an equation in x and y that has the same graph as polar $\quad r\left(\sin\theta+r\cos^2\theta\right)=1$

Solution

$$r\left(\sin\theta+r\cos^2\theta\right)=1 \qquad \cos\theta=\frac{x}{r} \quad \sin\theta=\frac{y}{r}$$

$$r\left(\frac{y}{r}+r\frac{x^2}{r^2}\right)=1$$

$$r\left(\frac{y}{r}+\frac{x^2}{r}\right)=1$$

$$r\left(\frac{y+x^2}{r}\right)=1$$

$$\boxed{y+x^2=1}$$

Exercise

Find an equation in x and y that has the same graph as polar $\quad r=8\sin\theta-2\cos\theta$

Solution

$$r=8\sin\theta-2\cos\theta \qquad \cos\theta=\frac{x}{r} \quad \sin\theta=\frac{y}{r}$$

$$r=8\frac{y}{r}-2\frac{x}{r}$$

$$r^2=8y-2x \qquad r^2=x^2+y^2$$

$$\boxed{x^2+y^2=8y-2x}$$

Exercise

Find an equation in x and y that has the same graph as polar $\quad r=\tan\theta$

Solution

$$r=\tan\theta$$

$$x^2+y^2=\frac{y^2}{x^2}$$

$$x^4+x^2y^2=y^2$$

$$\boxed{\sqrt{x^2+y^2}=\frac{y}{x}}$$

Exercise

Find a polar equation that has the same graph as the equation in x and y. $\quad y^2=6x$

Solution

$$y^2=6x \qquad x=r\cos\theta \quad y=r\sin\theta$$

$$(r\sin\theta)^2=6(r\cos\theta)$$

$$r^2\sin^2\theta=6r\cos\theta$$

$$\boxed{r=6\frac{\cos\theta}{\sin^2\theta}}$$

Exercise

Find a polar equation that has the same graph as the equation in x and y. $xy = 8$

Solution

$xy = 8$ $\qquad x = r\cos\theta \quad y = r\sin\theta$

$(r\cos\theta)(r\sin\theta) = 8$

$$\boxed{r^2 = \frac{8}{\cos\theta\sin\theta}}$$

Exercise

Find a polar equation that has the same graph as the equation in x and y. $(x+2)^2 + (y-3)^2 = 13$

Solution

$x^2 + 4x + 4 + y^2 - 6y + 9 = 13$

$x^2 + 4x + y^2 - 6y = 13 - 9 - 4$

$x^2 + 4x + y^2 - 6y = 0$

$x^2 + y^2 = 6y - 4x$ $\qquad x = r\cos\theta \quad y = r\sin\theta$

$r^2 = 6r\sin\theta - 4r\cos\theta$

$r^2 = r(6\sin\theta - 4\cos\theta)$ $\qquad$ ***Divide by r***

$$\boxed{r = 6\sin\theta - 4\cos\theta}$$

Exercise

Find a polar equation that has the same graph as the equation in x and y. $y^2 - x^2 = 4$

Solution

$y^2 - x^2 = 4$ $\qquad x = r\cos\theta \quad y = r\sin\theta$

$r^2\sin^2\theta - r^2\cos^2\theta = 4$

$r^2\left(\sin^2\theta - \cos^2\theta\right) = 4$ $\qquad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$r^2(-\cos 2\theta) = 4$

$$\boxed{r^2 = -\frac{4}{\cos 2\theta}}$$

Exercise

Write the equation in polar coordinates $x + y = 5$

Solution

$$r\cos\theta + r\sin\theta = 5 \qquad\qquad x = r\cos\theta \quad y = r\sin\theta$$

$$r(\cos\theta + \sin\theta) = 5$$

$$\boxed{r = \frac{5}{\cos\theta + \sin\theta}}$$

Exercise

Write the equation in polar coordinates $x^2 + y^2 = 9$

Solution

$$x^2 + y^2 = 9 \qquad\qquad r^2 = x^2 + y^2$$

$$\boxed{r^2 = 9}$$

Exercise

Write the equation in polar coordinates $x^2 + y^2 = 4x$

Solution

$$r^2 = 4r\cos\theta$$

$$\frac{r^2}{r} = \frac{4r\cos\theta}{r}$$

$$\boxed{r = 4\cos\theta}$$

Exercise

Write the equation in polar coordinates $y = -x$

Solution

$$y = -x \qquad\qquad x = r\cos\theta \quad y = r\sin\theta$$

$$r\sin\theta = -r\cos\theta$$

$$\boxed{\sin\theta = -\cos\theta}$$

Exercise

Write the equation in polar coordinates $x + y = 4$

Solution

$$r\cos\theta + r\sin\theta = 4 \qquad\qquad x = r\cos\theta \quad y = r\sin\theta$$

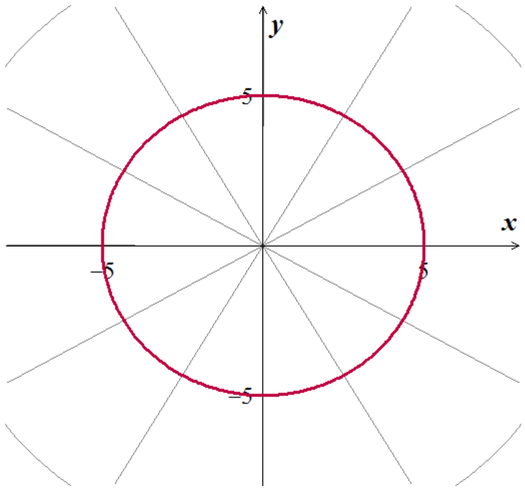
$$r(\cos\theta + \sin\theta) = 4$$

$$\boxed{r = \frac{4}{\cos\theta + \sin\theta}}$$

Exercise

Sketch the graph of the polar equation $r = 5$

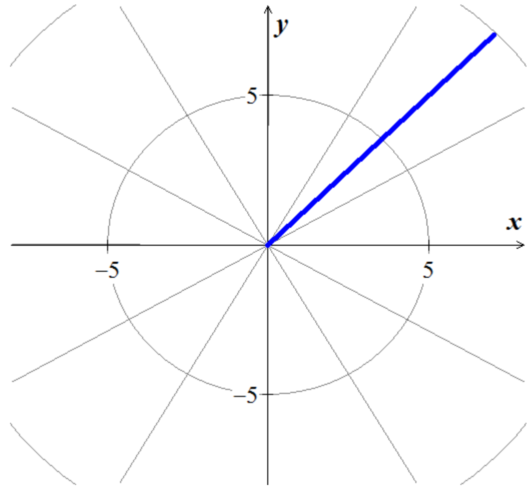
Solution



Exercise

Sketch the graph of the polar equation $\theta = \frac{\pi}{4}$

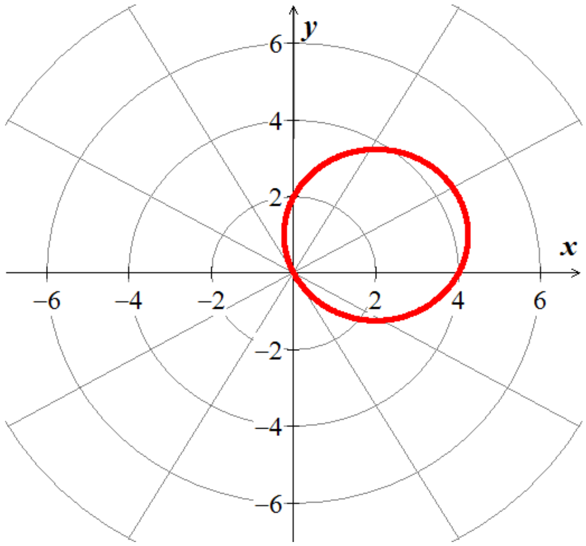
Solution



Exercise

Sketch graph $r = 4\cos\theta + 2\sin\theta$

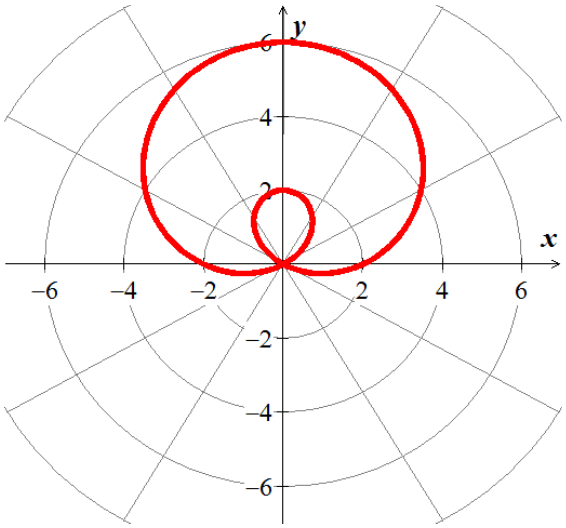
Solution



Exercise

Sketch the graph of the polar $r = 2 + 4\sin\theta$

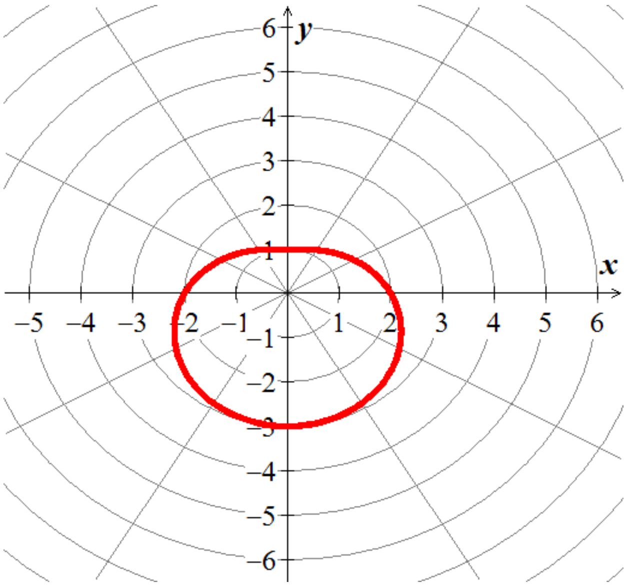
Solution



Exercise

Sketch the graph $r = 2 - \cos\theta$

Solution

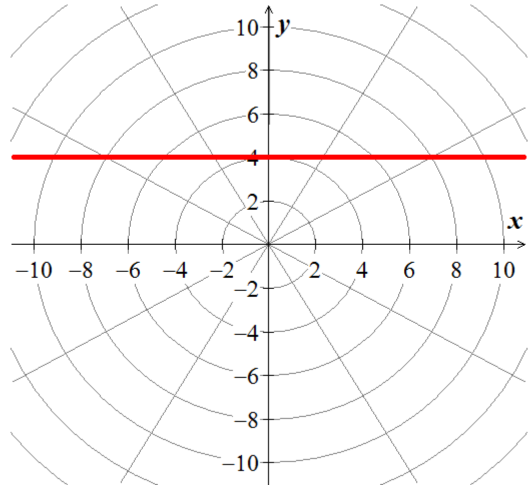


Exercise

Sketch the graph $r = 4\csc\theta$

Solution

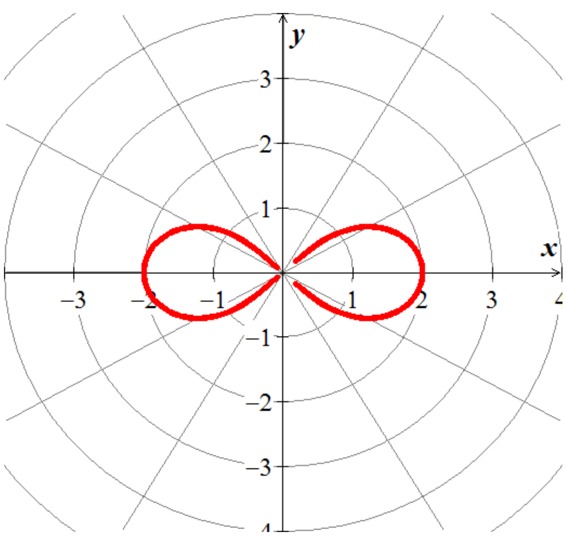
$$r = 4\csc\theta = \frac{4}{\sin\theta} \Rightarrow r\sin\theta = 4 = y$$



Exercise

Sketch the graph $r^2 = 4\cos 2\theta$

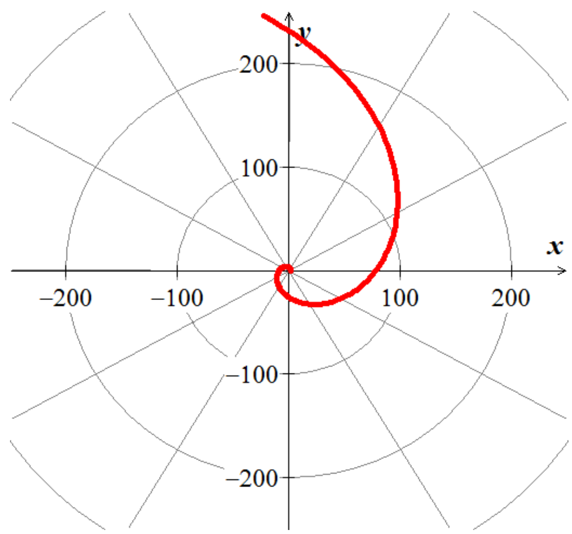
Solution



Exercise

Sketch the graph $r = 2^{\theta} \quad \theta \ge 0$

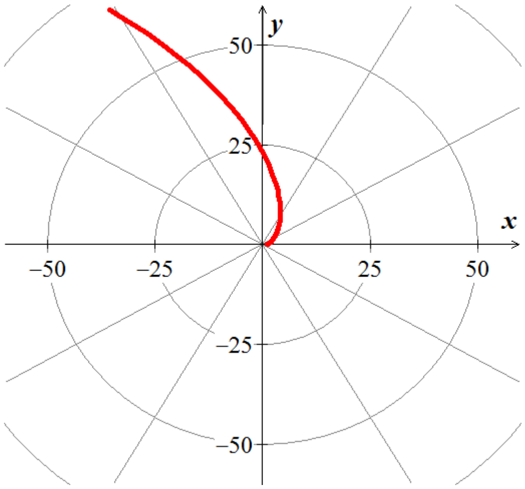
Solution



Exercise

Sketch the graph of the polar equation $r = e^{2\theta} \quad \theta \ge 0$

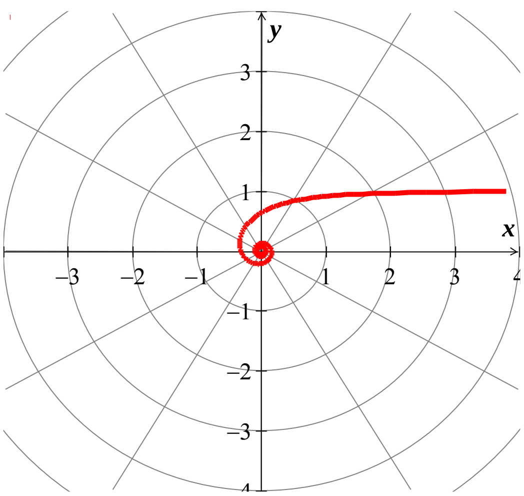
Solution

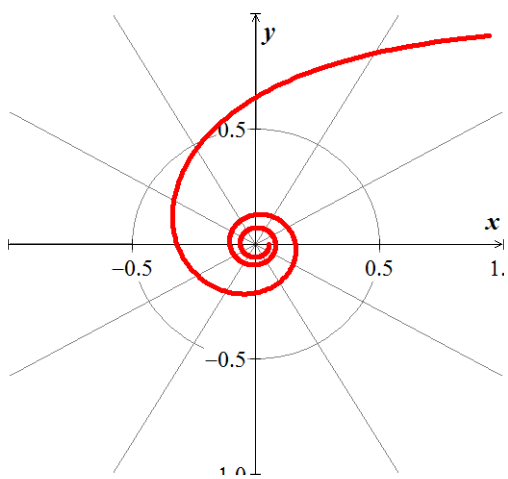


Exercise

Sketch the graph of the polar equation $r\theta = 1 \quad \theta > 0$

Solution

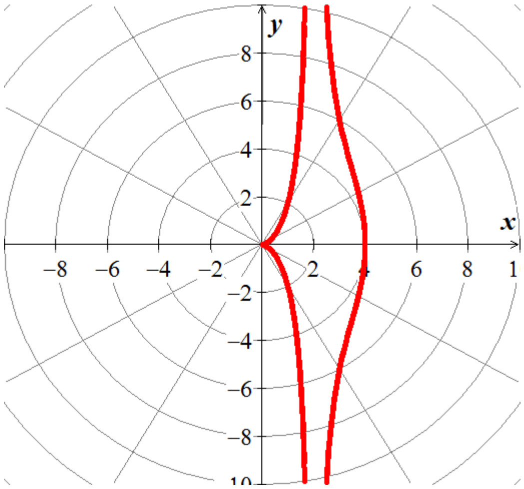




Exercise

Sketch the graph of the polar equation $\qquad r = 2 + 2\sec\theta$

Solution



Solution Section 3.7 – Trigonometric Form

Exercise

Write $-\sqrt{3}+i$ in trigonometric form. (Use radian measure)

Solution

$$-\sqrt{3}+i \Rightarrow \begin{cases} x=-\sqrt{3} \\ y=1 \end{cases}$$

$$r=\sqrt{\left(-\sqrt{3}\right)^2+1^2}=2$$

$$\tan\theta=\frac{y}{x}=\frac{1}{-\sqrt{3}}=-\frac{\sqrt{3}}{3}$$

The reference angle for θ is $\frac{\pi}{6}$ and the angle is in quadrant II.

Therefore, $\theta=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$

$$\boxed{-\sqrt{3}+i=2\ cis\frac{5\pi}{6}}$$

Exercise

Write $3-4i$ in trigonometric form.

Solution

$$3-4i \Rightarrow \begin{cases} r=\sqrt{3^2+(-4)^2}=5 \\ \hat{\theta}=\tan^{-1}\left(\frac{4}{3}\right)\approx 53^\circ \end{cases}$$

The angle is in quadrant *II*; therefore, $\theta=180^\circ-53^\circ=127^\circ$

$$\boxed{3-4i=5\ cis127^\circ}$$

Exercise

Write $-21-20i$ in trigonometric form.

Solution

$$-21-20i \Rightarrow \begin{cases} r=\sqrt{(-21)^2+(-20)^2}=29 \\ \hat{\theta}=\tan^{-1}\left(\frac{20}{21}\right)\approx 43.6^\circ \end{cases}$$

The angle is in quadrant *III*; therefore, $\theta=180^\circ+43.6^\circ=223.6^\circ$

$$\boxed{-21-20i=29\ cis223.6^\circ}$$

Exercise

Write $11+2i$ in trigonometric form.

Solution

$$11+2i \Rightarrow \begin{cases} r=\sqrt{11^2+2^2}=\sqrt{125}=5\sqrt{5} \\ \hat{\theta}=\tan^{-1}\left(\frac{2}{11}\right)\approx 10.3^\circ \end{cases}$$

The angle is in quadrant *I*; therefore, $\underline{\theta = 10.3^\circ}$

$$\underline{11+2i = 5\sqrt{5}\ cis 10.3^\circ}$$

Exercise

Write $\sqrt{3}-i$ in trigonometric form.

Solution

$$r=\sqrt{3+1}=2$$

$$\hat{\theta}=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\approx 30^\circ \xrightarrow{QIV} \theta = 360^\circ - 30^\circ = 330^\circ$$

$$\sqrt{3}-i \ \underline{= 2\ cis 330^\circ}$$

Exercise

Write $1-\sqrt{3}i$ in trigonometric form.

Solution

$$r=\sqrt{1+3}=2$$

$$\hat{\theta}=\tan^{-1}\left(\sqrt{3}\right)\approx 60^\circ \xrightarrow{QIV} \theta = 360^\circ - 60^\circ = 300^\circ$$

$$1-\sqrt{3}i \ \underline{= 2\ cis 300^\circ}$$

Exercise

Write $9\sqrt{3}+9i$ in trigonometric form.

Solution

$$r=9\sqrt{3+1}=18$$

$$\theta=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\approx 30^\circ$$

$$9\sqrt{3}+9i \ \underline{= 18\ cis 30^\circ}$$

Exercise

Write $-2+3i$ in trigonometric form.

Solution

$$r=\sqrt{4+9}=\sqrt{13}$$

$$\hat{\theta}=\tan^{-1}\left(\frac{3}{2}\right)\approx 56.31° \xrightarrow{QII} \theta=180°-56.31°=123.69°$$

$$-2+3i \underline{=\sqrt{13}\ cis123.69°}$$

Exercise

Write $4\left(\cos 30°+i\sin 30°\right)$ in standard form.

Solution

$$4\left(\cos 30°+i\sin 30°\right)=4\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)$$

$$\underline{=2\sqrt{3}+2i}$$

Exercise

Write $\sqrt{2}\ cis\frac{7\pi}{4}$ in standard form.

Solution

$$\sqrt{2}\ cis\frac{7\pi}{4}=\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$$

$$=\sqrt{2}\left(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)$$

$$\underline{=1-i}$$

Exercise

Write $3cis210°$ in standard form.

Solution

$$3cis210°=3\left(\cos 210°+i\sin 210°\right)$$

$$\underline{=-\frac{3\sqrt{3}}{2}-\frac{3}{2}i}$$

Exercise

Write $4\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$ in standard form.

Solution

$$4\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)=4\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)$$
$$=2\sqrt{2}-2i\sqrt{2}$$

Exercise

Write $4cis\frac{\pi}{2}$ in standard form.

Solution

$$4cis\frac{\pi}{2}=4\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$
$$=4i$$

Exercise

Find the quotient $\frac{20cis(75°)}{4cis(40°)}$. Write the result in rectangular form.

Solution

$$\frac{20cis(75°)}{4cis(40°)}=\frac{20}{4}cis(75°-40°)$$
$$=5cis(35°)$$
$$=5(\cos 35°+i\sin 35°)$$
$$=4.1+2.87i$$

Exercise

Divide $z_1=1+i\sqrt{3}$ *by* $z_2=\sqrt{3}+i$. Write the result in rectangular form.

Solution

$$\frac{z_1}{z_2}=\frac{1+i\sqrt{3}}{\sqrt{3}+i}$$

$$=\frac{1+i\sqrt{3}}{\sqrt{3}+i}\,\frac{\sqrt{3}-i}{\sqrt{3}-i}$$

$$=\frac{\sqrt{3}-i+3i-\sqrt{3}\,i^2}{3+1}$$

$\boxed{or}$ $1+i\sqrt{3}:\begin{cases} r=\sqrt{1^2+\left(\sqrt{3}\right)^2} \\ \theta=\tan^{-1}\frac{\sqrt{3}}{1}=\frac{\pi}{3}\end{cases}$

$$\sqrt{3}+i:\begin{cases} r=\sqrt{\left(\sqrt{3}\right)^2+1^2} \\ \theta=\tan^{-1}\frac{1}{\sqrt{3}}=\frac{\pi}{6}\end{cases}$$

$$\frac{z_1}{z_2}=\frac{2cis\frac{\pi}{3}}{2cis\frac{\pi}{6}}$$

$$=\frac{2}{2}cis\left(\frac{\pi}{3}-\frac{\pi}{6}\right)$$

$$= \frac{2\sqrt{3} + 2i}{4} \qquad\qquad = \frac{2}{2} cis\left(\frac{\pi}{6}\right)$$

$$= \frac{2\sqrt{3}}{4} + \frac{2i}{4} \qquad\qquad = cis\left(\frac{\pi}{6}\right)$$

$$= \frac{\sqrt{3}}{2} + \frac{i}{2} \qquad = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$$

Exercise

Find $(1+i)^8$ and express the result in rectangular form.

Solution

$$1+i \;\Rightarrow\; \begin{cases} r = \sqrt{1^2 + 1^2} = \sqrt{2} \\ \theta = \tan^{-1} 1 = \frac{\pi}{4} \end{cases} \qquad \rightarrow\; 1+i = \sqrt{2}\,cis\,\frac{\pi}{4}$$

$$(1+i)^8 = \left(\sqrt{2}\,cis\,\frac{\pi}{4}\right)^8$$

$$= \left(\sqrt{2}\right)^8 cis\left[8\left(\frac{\pi}{4}\right)\right]$$

$$= 16\,cis\,2\pi$$

$$= 16\left(\cos 2\pi + i\sin 2\pi\right)$$

$$= 16(1 + i0)$$

$$= 16$$

Exercise

Find $(1+i)^{10}$ and express the result in rectangular form.

Solution

$$(1+i)^{10} = \left(\sqrt{2}\,cis\,\frac{\pi}{4}\right)^{10}$$

$$= \left(\sqrt{2}\right)^{10} cis\left[10\left(\frac{\pi}{4}\right)\right]$$

$$= 32\,cis\,\frac{5\pi}{2}$$

$$= 32\,cis\,\frac{\pi}{2}$$

$$= 32\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= 32(0 + i)$$

$$= 32i$$

Exercise

Find and express the result in rectangular form $(1-i)^5$

Solution

$$r=\sqrt{1+1}=\sqrt{2}$$

$$\hat{\theta}=\tan^{-1}1=45^\circ \xrightarrow{QIV} \theta=360^\circ-45^\circ=315^\circ$$

$$\begin{aligned}(1-i)^5 &= \left(\sqrt{2}\ cis315^\circ\right)^5\\ &= 4\sqrt{2}\left(\ cis(5\times 315^\circ)\right)\\ &= 4\sqrt{2}\left(\ cis(1575)\right) \qquad\qquad 1575^\circ-4\times 360^\circ=135^\circ\\ &= 4\sqrt{2}\left(\cos 135^\circ+i\sin 135^\circ\right)\\ &= 4\sqrt{2}\left(-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)\\ &= -4+4i\end{aligned}$$

Exercise

Find and express the result in rectangular form $\left(1-\sqrt{5}i\right)^8$

Solution

$$r=\sqrt{1+5}=\sqrt{6}$$

$$\hat{\theta}=\tan^{-1}\sqrt{5}\approx 66^\circ \xrightarrow{QIV} \theta=360^\circ-66^\circ=294^\circ$$

$$\begin{aligned}\left(1-\sqrt{5}i\right)^8 &= \left(\sqrt{6}cis294^\circ\right)^8\\ &= \left(\sqrt{6}\right)^8\left(cis2352^\circ\right) \qquad\qquad 2352^\circ-6\times 360^\circ=192^\circ\\ &= 1296\left(\cos 192^\circ+i\sin 192^\circ\right)\\ &= 1296(-.978-0.208i)\\ &= -1267.488-269.568\ i\end{aligned}$$

Exercise

Find and express the result in rectangular form $(3cis80^\circ)^3$

Solution

$$\begin{aligned}(3cis80^\circ)^3 &= 3^3(cis240^\circ)\\ &= 27\left(\cos 240^\circ+i\sin 240^\circ\right)\\ &= 27\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\\ &= -\frac{27}{2}-i\frac{27\sqrt{3}}{2}\end{aligned}$$

Exercise

Find and express the result in rectangular form $\left(\sqrt{3}cis10^\circ\right)^6$

Solution

$$\left(\sqrt{3}cis10^\circ\right)^6 = 27\left(cis60^\circ\right)$$
$$= 27\left(\cos 60^\circ + i\sin 60^\circ\right)$$
$$\boxed{= \frac{27}{2} + i\frac{27\sqrt{3}}{2}}$$

Exercise

Find and express the result in rectangular form $\left(\sqrt{2} - i\right)^6$

Solution

$$r = \sqrt{2+1} = \sqrt{3}$$
$$\hat{\theta} = \tan^{-1}\frac{1}{\sqrt{2}} \approx 35.26^\circ \to \theta = 360^\circ - 35.26^\circ = 324.74^\circ$$
$$\left(\sqrt{2} - i\right)^6 = \left(\sqrt{3}\ cis324.74^\circ\right)^6$$
$$= 27\left(cis1948.44^\circ\right) \qquad 1948.44^\circ - 5\times 360^\circ = 148.44^\circ$$
$$= 27\left(\cos 148.44^\circ + i\sin 148.44^\circ\right)$$
$$\boxed{= -23 + 14.142i}$$

Exercise

Find and express the result in rectangular form $\left(4cis40^\circ\right)^6$

Solution

$$\left(4cis40^\circ\right)^6 = 4^6\left(cis\left(6\times 40^\circ\right)\right)$$
$$= 4^6\left(\cos 240^\circ + i\sin 240^\circ\right)$$
$$= 4096\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$
$$\boxed{= -2048 + 2048\ i\sqrt{3}}$$

Exercise

Find and express the result in rectangular form $\left(2cis30^\circ\right)^5$

Solution

$$\left(2cis30^\circ\right)^5 = 2^5 cis\left(5(30^\circ)\right)$$

$$= 32\left(\cos 150° + i\sin 150°\right)$$

$$= 32\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$\boxed{= -16\sqrt{3} + 16i}$$

Exercise

Find and express the result in rectangular form $\left(\frac{1}{2}cis72°\right)^5$

Solution

$$\left(\frac{1}{2}cis72°\right)^5 = \frac{1}{2^5}cis\left(5\times 72°\right)$$

$$= \frac{1}{32}cis\left(\cos 360° + i\sin 360°\right)$$

$$\boxed{= \frac{1}{32}}$$

Exercise

Find fifth roots of $z = 1 + i\sqrt{3}$ and express the result in rectangular form.

Solution

$$1 + i\sqrt{3} \Rightarrow \begin{cases} r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2 \\ \theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60° \end{cases}$$

$$\left(1 + i\sqrt{3}\right)^{1/5} = \left(2cis60°\right)^{1/5}$$

$$= \sqrt[5]{2}\left(cis\,\frac{60°}{5} + \frac{360°k}{5}\right)$$

$$= \sqrt[5]{2}\ cis\left(12° + 72°k\right)$$

$$If\ \ k = 0 \Rightarrow \sqrt[5]{2}\ cis\left(12° + 72°0\right) = \boxed{\sqrt[5]{2}\ cis12°}$$

$$If\ \ k = 1 \Rightarrow \sqrt[5]{2}\ cis\left(12° + 72°.(1)\right) = \boxed{\sqrt[5]{2}\ cis84°}$$

$$If\ \ k = 2 \Rightarrow \sqrt[5]{2}\ cis\left(12° + 72°.(2)\right) = \boxed{\sqrt[5]{2}\ cis156°}$$

$$If\ \ k = 3 \Rightarrow \sqrt[5]{2}\ cis\left(12° + 72°.(3)\right) = \boxed{\sqrt[5]{2}\ cis228°}$$

$$If\ \ k = 4 \Rightarrow \sqrt[5]{2}\ cis\left(12° + 72°.(4)\right) = \boxed{\sqrt[5]{2}\ cis300°}$$

Exercise

Find the fourth roots of $z = 16cis60°$

Solution

$$\sqrt[4]{z} = \sqrt[4]{16}\ cis\left(\frac{60°}{4} + \frac{360°}{4}k\right)$$
$$= 2cis(15° + 90°k)$$

$$If\ \ k = 0 \Rightarrow 2\ cis(15° + 90°(0)) = \underline{2cis15°|}$$

$$If\ \ k = 1 \Rightarrow 2\ cis(15° + 90°(1)) = \underline{2cis105°|}$$

$$If\ \ k = 2 \Rightarrow 2\ cis(15° + 90°(2)) = \underline{2cis195°|}$$

$$If\ \ k = 3 \Rightarrow 2\ cis(15° + 90°(3)) = \underline{2cis285°|}$$

Exercise

Find the fourth roots of $\sqrt{3} - i$

Solution

$$r = \sqrt{3+1} = 2$$

$$\hat{\theta} = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6} \quad \xrightarrow{QIV} \quad \theta = \frac{11\pi}{6}$$

$$\sqrt[4]{\sqrt{3} - i} = \sqrt[4]{2\ cis\frac{11\pi}{6}}$$
$$= \sqrt[4]{2}\ cis\left(\frac{1}{4}\frac{11\pi}{6} + \frac{2\pi k}{4}\right)$$
$$= \sqrt[4]{2}\ cis\left(\frac{11\pi}{24} + \frac{\pi k}{2}\right)$$

$$k = 0 \Rightarrow \sqrt[4]{2}\ cis\left(\frac{11\pi}{24} + 0\right) = \underline{\sqrt[4]{2}\ cis\frac{11\pi}{24}|}$$

$$k = 1 \Rightarrow \sqrt[4]{2}\ cis\left(\frac{11\pi}{24} + \frac{\pi}{2}\right) = \underline{\sqrt[4]{2}\ cis\frac{23\pi}{24}|}$$

$$k = 2 \Rightarrow \sqrt[4]{2}\ cis\left(\frac{11\pi}{24} + \pi\right) = \underline{\sqrt[4]{2}\ cis\frac{35\pi}{24}|}$$

$$k = 3 \Rightarrow \sqrt[4]{2}\ cis\left(\frac{11\pi}{24} + \frac{3\pi}{2}\right) = \underline{\sqrt[4]{2}\ cis\frac{47\pi}{24}|}$$

Exercise

Find the fourth roots of $4 - 4\sqrt{3}i$

Solution

$$r = 4\sqrt{1+3} = 8$$

$$\hat{\theta}=\tan^{-1}\sqrt{3}=\frac{\pi}{3} \xrightarrow{QIV} \theta=\frac{5\pi}{3}$$

$$\sqrt[4]{4-4\sqrt{3}i}=\sqrt[4]{8\ cis\frac{5\pi}{3}}$$

$$=\sqrt[4]{8}\ cis\left(\frac{5\pi}{12}+\frac{\pi k}{2}\right)$$

$$k=0\Rightarrow\sqrt[4]{8}\ cis\left(\frac{5\pi}{12}+0\right)=\boxed{\sqrt[4]{8}\ cis\frac{5\pi}{12}}$$

$$k=1\Rightarrow\sqrt[4]{8}\ cis\left(\frac{5\pi}{12}+\frac{\pi}{2}\right)=\boxed{\sqrt[4]{8}\ cis\frac{11\pi}{12}}$$

$$k=2\Rightarrow\sqrt[4]{8}\ cis\left(\frac{5\pi}{12}+\pi\right)=\boxed{\sqrt[4]{8}\ cis\frac{17\pi}{12}}$$

$$k=3\Rightarrow\sqrt[4]{8}\ cis\left(\frac{5\pi}{12}+\frac{3\pi}{2}\right)=\boxed{\sqrt[4]{8}\ cis\frac{23\pi}{12}}$$

Exercise

Find the fourth roots of $-16i$

Solution

$$r=16;\quad \theta=\frac{3\pi}{2}$$

$$\sqrt[4]{-16i}=\sqrt[4]{16\ cis\frac{3\pi}{2}}$$

$$=2\ cis\left(\frac{3\pi}{8}+\frac{\pi k}{2}\right)$$

$$k=0\Rightarrow 2\ cis\left(\frac{3\pi}{8}+0\right)=\boxed{2\ cis\frac{3\pi}{8}}$$

$$k=1\Rightarrow 2\ cis\left(\frac{3\pi}{8}+\frac{\pi}{2}\right)=\boxed{2\ cis\frac{7\pi}{8}}$$

$$k=2\Rightarrow 2\ cis\left(\frac{3\pi}{8}+\pi\right)=\boxed{2\ cis\frac{11\pi}{8}}$$

$$k=1\Rightarrow 2\ cis\left(\frac{3\pi}{8}+\frac{3\pi}{2}\right)=\boxed{2\ cis\frac{15\pi}{8}}$$

Exercise

Find the cube roots of 27.

Solution

$$\sqrt[3]{27}=\left(27cis0^\circ\right)^{1/3}$$

$$=\sqrt[3]{27}\ cis\left(\frac{0^\circ}{3}+\frac{360^\circ}{3}k\right)$$

$$=3\ cis\left(0^\circ+120^\circ k\right)$$

$$k=0 \Rightarrow z=3\ cis\left(0°+120°(0)\right)=\underline{2cis0°}$$

$$k=1 \Rightarrow z=3\ cis\left(0°+120°(1)\right)=\underline{2cis120°}$$

$$k=2 \Rightarrow z=3\ cis\left(0°+120°(2)\right)=\underline{2cis240°}$$

Exercise

Find the cube roots of $8-8i$

Solution

$$r=8\sqrt{1+1}=8\sqrt{2}$$

$$\hat{\theta}=\tan^{-1}1=\frac{\pi}{4} \quad \xrightarrow{QIV} \quad \theta=\frac{7\pi}{4}$$

$$\sqrt[3]{8-8i}=\sqrt[3]{8\sqrt{2}\ cis\frac{7\pi}{4}}$$

$$=2\sqrt[3]{2}\ cis\left(\frac{7\pi}{12}+\frac{2\pi k}{3}\right)$$

$$k=0 \Rightarrow z=2\sqrt[3]{2}\ cis\left(\frac{7\pi}{12}+0\right)=\underline{2\sqrt[3]{2}\ cis\frac{7\pi}{12}}$$

$$k=1 \Rightarrow z=2\sqrt[3]{2}\ cis\left(\frac{7\pi}{12}+\frac{2\pi}{3}\right)=\underline{2\sqrt[3]{2}\ cis\frac{15\pi}{12}}$$

$$k=2 \Rightarrow z=2\sqrt[3]{2}\ cis\left(\frac{7\pi}{12}+\frac{4\pi}{3}\right)=\underline{2\sqrt[3]{2}\ cis\frac{23\pi}{12}}$$

Exercise

Find the cube roots of -8

Solution

$$r=8; \quad \theta=\frac{3\pi}{2}$$

$$\sqrt[3]{-8}=\sqrt[3]{8\ cis\frac{3\pi}{2}}$$

$$=2\ cis\left(\frac{\pi}{2}+\frac{2\pi k}{3}\right)$$

$$k=0 \Rightarrow z=2\ cis\left(\frac{\pi}{2}+0\right)=\underline{2\ cis\frac{\pi}{2}}$$

$$k=1 \Rightarrow z=2\ cis\left(\frac{\pi}{2}+\frac{2\pi}{3}\right)=\underline{2\ cis\frac{7\pi}{6}}$$

$$k=2 \Rightarrow z=2\ cis\left(\frac{\pi}{2}+\frac{4\pi}{3}\right)=\underline{2\ cis\frac{11\pi}{6}}$$

Exercise

Find all complex number solutions of $x^3+1=0$.

Solution

$$x^3+1=0 \Rightarrow x^3=-1$$

$$-1 \Rightarrow \begin{cases} r=\sqrt{(-1)^2+0^2}=1 \\ \theta=\tan^{-1}\left(\frac{0}{-1}\right)=\pi \end{cases}$$

$$x^3=-1=1\ cis\pi$$

$$x=(1\ cis\pi)^{1/3}$$

$$=(1)^{1/3}\ cis\left(\frac{\pi}{3}+\frac{2\pi}{3}k\right)$$

$$=cis\left(\frac{\pi}{3}+\frac{2\pi}{3}k\right)$$

$$If\ \ k=0 \Rightarrow x=cis\left(\frac{\pi}{3}+\frac{2\pi}{3}(0)\right)=cis\,\frac{\pi}{3} \qquad x=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}=\frac{1}{2}+i\frac{\sqrt{3}}{2}$$

$$If\ \ k=1 \Rightarrow x=cis\left(\frac{\pi}{3}+\frac{2\pi}{3}(1)\right)=cis\left(\frac{3\pi}{3}\right)=cis\pi \qquad x=\cos\pi+i\sin\pi=-1$$

$$If\ \ k=2 \Rightarrow x=cis\left(\frac{\pi}{3}+\frac{2\pi}{3}(2)\right)=cis\,\frac{5\pi}{3} \qquad x=\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}=\frac{1}{2}-i\frac{\sqrt{3}}{2}$$