Section 3.3 – The Basics of Counting

Basic Counting Principle

The Product Rule

A procedure can be broken down into a sequence of two tasks. There are n_1 ways to do the first task and n_2 ways to do the second task. Then there are $n_1 \cdot n_2$ ways to do the procedure

Example

How many bit strings of length seven are there?

Solution

Since each of the seven bits is either a 0 or a 1, the answer is $2^7 = 128$.

Example

A new company with just two employees rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution

The procedure of assigning offices to these 2 employees consists of assigning an office to one employee, which can be done in 12 ways, then assigning an office to the second different from the office assigned to the first, which can be done in 11 ways.

By the product rule, there are $12 \cdot 11 \boxed{= 132 \ \ ways}$ to assign offices to these 2 employees.

Example

There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a computer in the center are there?

Solution

$32 \cdot 24 \boxed{= 768 \ \ ports}$

Example

How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

Solution

By the product rule, there are

$$\underbrace{26 \cdot 26 \cdot 26}_{\text{26 choices for each letter}} \cdot \underbrace{10 \cdot 10 \cdot 10}_{\text{10 choices for each digit}} = 17{,}576{,}000$$

different possible license plates.

Counting Functions

Example

How many functions are there from a set with *m* elements to a set with *n* elements?

<u>*Solution*</u>

Since a function represents a choice of one of the *n* elements of the codomain for each of the *m* elements in the domain, the product rule tells us that there are $n \cdot n \cdots n = n^m$ such functions.

Counting One-to-One Functions

Example

How many one-to-one functions are there from a set with *m* elements to one with *n* elements?

<u>***Solution***</u>

Suppose the elements in the domain are $a_1, a_2, \ldots, a_m$. There are *n* ways to choose the value of a_1 and *n* − 1 ways to choose a_2, etc. The product rule tells us that there are $n(n-1)(n-2) \cdots (n-m+1)$ such functions.

Counting Subsets of a Finite Set

Example

Use the product rule to show that the number of different subsets of a finite set *S* is $2^{|S|}$.

<u>***Solution***</u>

When the elements of S are listed in an arbitrary order, there is a one-to-one correspondence between subsets of *S* and bit strings of length |*S*|. When the *i*th element is in the subset, the bit string has a 1 in the *i*th position and a 0 otherwise. By the product rule, there are $2^{|S|}$ such bit strings, and therefore $2^{|S|}$ subsets.

Product Rule in Terms of Sets

- If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set.
- The task of choosing an element in the Cartesian product $A_1 \times A_2 \times \ldots \times A_m$ is done by choosing an element in A_1, an element in A_2, …and an element in A_m.
- By the product rule, it follows that: $|A_1 \times A_2 \times \ldots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots |A_m|$

Basic Counting Principles

Definition: ***The Sum Rule***

If a task can be done either in one of n_1 ways or in one of n_2 ways to do the second task, where none of the set of n_1 ways is the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example

The mathematics department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student.

Solution

By the sum rule it follows that there are 37 + 83 = **120** possible ways to pick a representative.

Example

A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

Solution

Since no project is on more than one list, by the sum rule there are 23 + 15 + 19 = **57** ways to choose a project.

The Sum Rule in terms of sets

The sum rule can be phrased in terms of sets.

$|A \cup B| = |A| + |B|$ as long as A and B are disjoint sets.

Or more generally, $\left|A_1 \cup A_2 \cup \ldots \cup A_m\right| = \left|A_1\right| + \left|A_2\right| + \ldots + \left|A_m\right|$ when $A_i \cup A_j = \varnothing$ for all i, j.

Example

Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels.

Solution

Use the product rule. $26 + 26 \cdot 10$ **= 286**

Subtraction Rule

Definition

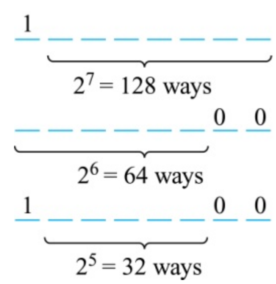
If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

Also known as, the *principle of inclusion-exclusion*: $|A \cup B| = |A| + |B| - |A \cap B|$

Example

How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution



Use the subtraction rule.

- Number of bit strings of length eight that start with a 1 bit: $2^7 = 128$
- Number of bit strings of length eight that start with bits 00: $2^6 = 64$
- Number of bit strings of length eight that start with a 1 bit and end with bits 00 : $2^5 = 32$

Hence, the number is 128 + 64 − 32 = 160.

Example

A computer company receives 350 applications from computer graduates for a job planning a line of new Web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

Solution

Let A: majored in computer science
B: majored in business

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 220 + 147 - 51$$
$$\underline{= 316}$$

$$350 - 316 \underline{= 34}$$

We conclude that 34 of the applicants majored neither in computer science nor in business

Division Rule

Definition

There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

- ✓ Restated in terms of sets: If the finite set A is the union of n pairwise disjoint subsets each with d elements, then $n = |A|/d$.
- ✓ In terms of functions: If f is a function from A to B, where both are finite sets, and for every value $y \in B$ there are exactly d values $x \in A$ such that $f(x) = y$, then $|B| = |A|/d$.

Example

How many ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left and right neighbor?

<u>***Solution***</u>

Number the seats around the table from 1 to 4 proceeding clockwise.
There are four ways to select the person for seat 1, 3 for seat 2, 2, for seat 3, and one way for seat 4.
Thus there are 4! = 24 ways to order the four people.
But since two seatings are the same when each person has the same left and right neighbor, for every choice for seat 1, we get the same seating.
Therefore, by the division rule, there are 24/4 = 6 different seating arrangements.

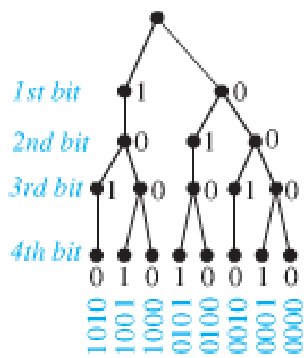
Tree Diagrams

Definition

We can solve many counting problems through the use of *tree diagrams*, where a branch represents a possible choice and the leaves represent possible outcomes.

Example

How many bit strings of length four do not have two consecutive 1s?



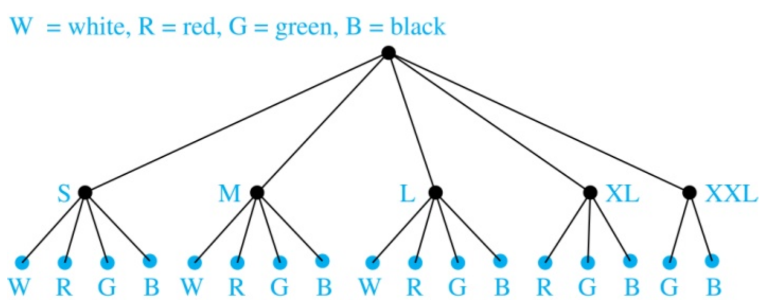
Solution

There are eight bits strings of length ***four*** without two consecutive 1s

Example

Suppose that "I Love Discrete Math" T-shirts come in five different sizes: S,M,L,XL, and XXL. Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black. What is the minimum number of stores that the campus book store needs to stock to have one of each size and color available?

Solution



The store must stock **17** T-shirts.

Exercises *Section* 3.3 – The Basics of Counting

1. There are 18 mathematics majors and 325 computer science majors at a college

a) In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?

b) In how many ways can one representative be picked who either a mathematics major or a computer science major?

2. An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?

3. A multiple-choice test contains 10 questions. There are four possible answers for each question

a) In how many ways can a student answer the questions on the test if the student answers every question?

b) In how many ways can a student answer the questions on the test if the student can leave answers blank?

4. A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of the shirts are made?

5. How many different three-letter initials can people have?

6. How many different three-letter initials with none of the letters repeated can people have?

7. How many different three-letter initials are there, that begin with an A?

8. How many bit strings are there of length eight?

9. How many bit strings of length ten both begin and end with a 1?

10. How many bit strings of length n, where n is a positive integer, start and end with 1s?

11. How many strings are there of lowercase letters of length four or less, not counting the empty string?

12. How many strings are there of four lowercase letters that have the letter x in them?

13. How many positive integers between 50 and 100

a) Are divisible by 7? Which integers are these

b) Are divisible by 11? Which integers are these

c) Are divisible by 7 and 11? Which integers are these

14. How many positive integers less than 100

a) Are divisible by 7?

b) Are divisible by 7 but not by 11?

c) Are divisible by both 7 and 11?

d) Are divisible by either 7 or 11?

e) Are divisible by exactly one of 7 and 11?

f) Are divisible by neither 7 nor 11?

15. How many positive integers less than 1000

g) Are divisible by 7?
h) Are divisible by 7 but not by 11?
i) Are divisible by both 7 and 11?
j) Are divisible by either 7 or 11?
k) Are divisible by exactly one of 7 and 11?
l) Are divisible by neither 7 nor 11?
m) have distinct digits?
n) have distinct digits and are even?

16. A committee is formed consisting of one representative from each of the 50 states in the United States, where the representative from a state is either the governor or one of the two senators from that state. How many ways are there to form this committee?

17. How many license plates can be made using either three digits followed by three uppercase English letters or three uppercase English letters followed by three digits?

18. How many license plates can be made using either two uppercase English letters followed by four digits or two digits followed by four uppercase English letters?

19. How many license plates can be made using either three uppercase English letters followed by three digits or four uppercase English letters followed by two digits?

20. How many strings of eight English letter are there

a) that contain no vowels, if letters can be repeated?
b) that contain no vowels, if letters cannot be repeated?
c) that start with a vowel, if letters can be repeated?
d) that start with a vowel, if letters cannot be repeated?
e) That contain at least one vowel, if letters can be repeated?
f) That contain at least one vowel, if letters cannot be repeated?

21. How many ways are there to seat four of a group of ten people around a circular table where two seatings are considered the same when everyone has the same immediate left and immediate right neighbor?

22. In how many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people, where the bride and the groom are among these 10 people, if

a) The bride must be in the picture?
b) Both the bride and groom must be in the picture?
c) Exactly one of the bride and the groom is in the picture?

23. How many different types of homes are available if a builder offers a choice of 6 basic plans, 3 roof styles, and 2 exterior finishes?

24. A menu offers a choice of 3 salads, 8 main dishes, and 7 desserts. How many different meals consisting of one salad, one main dish, and one dessert are possible?

25. A couple has narrowed down the choice of a name for their new baby to 4 first names and 5 middle names. How many different first- and middle-name arrangements are possible?

26. An automobile manufacturer produces 8 models, each available in 7 different exterior colors, with 4 different upholstery fabrics and 5 interior colors. How many varieties of automobile are available?

27. A biologist is attempting to classify 52,000 species of insects by assigning 3 initials to each species. Is it possible to classify all the species in this way? If not, how many initials should be used?

28. How many 4-letter code words are possible using the first 10 letters of the alphabet under:

a) No letter can be repeated
b) Letters can be repeated
c) Adjacent can't be alike

29. How many 3 letters license plate without repeats

30. How many ways can 2 coins turn up heads, H, or tails, T – if the combined outcome (H, T) is to be distinguished from the outcome (T, H)?

31. How many 2-letter code words can be formed from the first 3 letters of the alphabet if no letter can be used more than once?

32. A coin is tossed with possible outcomes heads, H, or tails, T. Then a single die is tossed with possible outcomes 1, 2, 3, 4, 5, or 6. How many combined outcomes are there?

33. In how many ways can 3 coins turn up heads, H, or tails, T – if combined outcomes such as (H,T,H), (H, H, T), and (T, H, H) are to be considered different?

34. An entertainment guide recommends 6 restaurants and 3 plays that appeal to a couple.

a) If the couple goes to dinner or to a play, how many selections are possible?
b) If the couple goes to dinner and then to a play, how many combined selections are possible?