Solution Section 2.6 – Improper Integrals

Exercise

Evaluate the integral $\int_0^{\infty} \frac{dx}{x^2+1}$

Solution

$$\int_0^{\infty} \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_0^{b} \frac{dx}{x^2+1}$$

$$= \lim_{b\to\infty} \left[\tan^{-1} x\right]_0^b$$

$$= \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} 0\right)$$

$$= \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2}$$

Exercise

Evaluate the integral $\int_0^{4} \frac{dx}{\sqrt{4-x}}$

Solution

$$\int_0^{4} \frac{dx}{\sqrt{4-x}} = \lim_{b\to 4^-} \int_0^{b} (4-x)^{-1/2}\, dx$$

$$= \lim_{b\to 4^-} \int_0^{b} -(4-x)^{-1/2}\, d(4-x)$$

$$= -2 \lim_{b\to 4^-} \left[(4-x)^{1/2}\right]_0^b$$

$$= -2 \lim_{b\to 4^-} \left[(4-b)^{1/2} - (4)^{1/2}\right]$$

$$= -2(0-2)$$

$$= 4$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{2} \frac{2dx}{x^2+4}$

Solution

$$\int_{-\infty}^{2} \frac{2dx}{x^2+4} = 2\lim_{b\to-\infty}\int_{b}^{2} \frac{dx}{x^2+2^2}$$

$$= 2\lim_{b\to-\infty} \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_b^2$$

$$= \lim_{b\to-\infty}\left[\tan^{-1}1 - \tan^{-1}\frac{b}{2}\right]$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{2}\right)$$

$$= \frac{3\pi}{4}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{xdx}{\left(x^2+4\right)^{3/2}}$

Solution

$$\int_{-\infty}^{\infty} \frac{xdx}{\left(x^2+4\right)^{3/2}} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{d\left(x^2+4\right)}{\left(x^2+4\right)^{3/2}}$$

$u = x^2 + 4 \quad \to du = 2xdx$

$$= \frac{1}{2}\left[-2\left(x^2+4\right)^{-1/2}\right]_{-\infty}^{\infty}$$

$$= -\left[\frac{1}{\sqrt{x^2+4}}\right]_{-\infty}^{\infty}$$

$$= -(0-0)$$

$$= 0$$

Exercise

Evaluate the integral $\displaystyle\int_{1}^{\infty} \frac{dx}{x\sqrt{x^2-1}}$

Solution

$$\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \int_1^2 \frac{dx}{x\sqrt{x^2-1}} + \int_2^\infty \frac{dx}{x\sqrt{x^2-1}}$$

$$= \lim_{b\to 1^+} \int_b^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{c\to\infty} \int_2^c \frac{dx}{x\sqrt{x^2-1}}$$

$$= \lim_{b\to 1^+} \left[\sec^{-1}|x|\right]_b^2 + \lim_{c\to\infty} \left[\sec^{-1}|x|\right]_2^c$$

$$= \lim_{b\to 1^+} \left(\sec^{-1} 2 - \sec^{-1} b\right) + \lim_{c\to\infty} \left(\sec^{-1} c - \sec^{-1} 2\right)$$

$$= \left(\frac{\pi}{3} - 0\right) + \left(\frac{\pi}{2} - \frac{\pi}{3}\right)$$

$$= \frac{\pi}{2}$$

Exercise

Evaluate the integral $\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx$

Solution

$$\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx = \int_{-\infty}^{0} 2xe^{-x^2}\,dx + \int_{0}^{\infty} 2xe^{-x^2}\,dx$$

$d\left(-x^2\right) = -2xdx$

$$= -\lim_{b\to-\infty} \int_b^0 e^{-x^2}\,d\left(-x^2\right) - \lim_{c\to\infty} \int_0^c e^{-x^2}\,d\left(-x^2\right)$$

$$= -\lim_{b\to-\infty} \left[e^{-x^2}\right]_b^0 - \lim_{c\to\infty} \left[e^{-x^2}\right]_0^c$$

$$= -\lim_{b\to-\infty} \left(1 - e^{-b^2}\right) - \lim_{c\to\infty} \left(e^{-c^2} - 1\right) \qquad = -(1-0) - (0-1)$$

$$= 0$$

Exercise

Evaluate the integral $\int_0^1 (-\ln x)\,dx$

Solution

$$\begin{aligned}\int_0^1 (-\ln x)\,dx &= -\lim_{b\to 0^+}\int_b^1 (\ln x)\,dx \\ &= -\lim_{b\to 0^+}\left[x\ln x - x\right]_b^1 \\ &= -\lim_{b\to 0^+}\left(\ln 1 - 1 - (b\ln b - b)\right) \\ &= -(0-1-0+0) \\ &= 1\end{aligned}$$

Exercise

Evaluate the integral $\int_{-1}^4 \frac{dx}{\sqrt{|x|}}$

Solution

$$\begin{aligned}\int_{-1}^4 \frac{dx}{\sqrt{|x|}} &= \lim_{b\to 0^-}\int_{-1}^b \frac{dx}{\sqrt{-x}} + \lim_{c\to 0^+}\int_c^4 \frac{dx}{\sqrt{x}} \\ &= \lim_{b\to 0^-}\left[-2\sqrt{-x}\right]_{-1}^b + \lim_{c\to 0^+}\left[2\sqrt{x}\right]_c^4 \\ &= \lim_{b\to 0^-}\left(-2\sqrt{-b}+2\right) + \lim_{c\to 0^+}\left(2\sqrt{4}-2\sqrt{c}\right) \\ &= 2+4 \\ &= 6\end{aligned}$$

Exercise

Evaluate the integral $\int_0^\infty e^{-3x}\,dx$

Solution

$$\int_0^\infty e^{-3x}\,dx = -\frac{1}{3}e^{-3x}\Big|_0^\infty$$

$$= -\frac{1}{3}\left(e^{-\infty} - 1\right)$$

$$\boxed{= \frac{1}{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Solution

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{1+x^2} &= \tan^{-1} x \Big|_{-\infty}^{\infty} \\ &= \tan^{-1}\infty - \tan^{-1}(-\infty) \\ &= \frac{\pi}{2} + \frac{\pi}{2} \\ &= \pi\end{aligned}$$

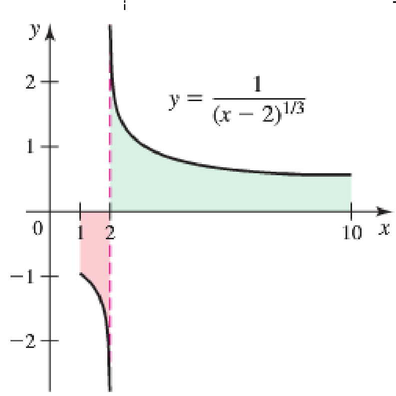
Exercise

Evaluate the integral $\displaystyle\int_{1}^{10} \frac{dx}{(x-2)^{1/3}}$

Solution

$$\begin{aligned}\int_1^{10} (x-2)^{-1/3}\, dx &= \frac{3}{2}(x-2)^{2/3}\Big|_1^{10} \\ &= \frac{3}{2}\left(8^{2/3} - (-1)^{2/3}\right) \\ &= \frac{3}{2}(4-1) \\ &= \frac{9}{2}\end{aligned}$$

$$\begin{aligned}\int_1^{10} (x-2)^{-1/3}\, dx &= \int_1^{2} (x-2)^{-1/3}\, dx + \int_2^{10} (x-2)^{-1/3}\, dx \\ &= \frac{3}{2}(x-2)^{2/3}\Big|_1^{2} + (x-2)^{2/3}\Big|_2^{10} \\ &= \frac{3}{2}\left(0 - (-1)^{2/3}\right) + \frac{3}{2}\left(8^{2/3} - 0\right) \\ &= \frac{3}{2}(-1+4) \\ &= \frac{9}{2}\end{aligned}$$



Exercise

Evaluate the integral $\int_1^{\infty} \frac{dx}{x^2}$

Solution

$$\int_1^{\infty} \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^{\infty}$$

$$= -\left(\frac{1}{\infty} - 1\right)$$

$$= -(0-1)$$

$$\underline{=1}$$

Exercise

Evaluate the integral $\int_0^{\infty} \frac{dx}{(x+1)^3}$

Solution

$$\int_0^{\infty} (x+1)^{-3} dx = -\frac{2}{(x+1)^2}\Big|_0^{\infty}$$

$$= -2\left(\frac{1}{\infty} - 1\right)$$

$$= -2(0-1)$$

$$\underline{= 2}$$

Exercise

Evaluate the integral $\int_{-\infty}^{0} e^x \, dx$

Solution

$$\int_{-\infty}^{0} e^x \, dx = e^x\Big|_{-\infty}^{0}$$

$$= \left(1 - e^{-\infty}\right)$$

$$\underline{=1}$$

Exercise

Evaluate the integral $\displaystyle\int_1^{\infty} 2^{-x}\ dx$

Solution

$$\int_1^{\infty} 2^{-x}\ dx = -\int_1^{\infty} 2^{-x}\ d(-x) \qquad\qquad \int a^x dx = \frac{a^x}{\ln a}$$

$$= -\frac{2^{-x}}{\ln 2}\Big|_1^{\infty}$$

$$= -\frac{1}{\ln 2}\left(0-\frac{1}{2}\right)$$

$$= \frac{1}{2\ln 2}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{0} \frac{dx}{\sqrt[3]{2-x}}$

Solution

$$\int_{-\infty}^{0} \frac{dx}{\sqrt[3]{2-x}} = -\int_{-\infty}^{0} (2-x)^{-1/3}\ d(2-x)$$

$$= -\frac{3}{2}(2-x)^{2/3}\Big|_{-\infty}^{0}$$

$$= -\frac{3}{2}\left(2^{2/3}-\infty\right)$$

$$= \infty \quad \textit{diverges}$$

Exercise

Evaluate the integral $\displaystyle\int_{4/\pi}^{\infty} \frac{1}{x^2}\sec^2\left(\frac{1}{x}\right)dx$

Solution

$$\int_{4/\pi}^{\infty} \frac{1}{x^2}\sec^2\left(\frac{1}{x}\right)\ dx = -\int_{4/\pi}^{\infty} \sec^2\left(\frac{1}{x}\right)\ d\left(\frac{1}{x}\right) \qquad\qquad d\left(\frac{1}{x}\right) = -\frac{1}{x^2}dx$$

$$= -\tan\left(\frac{1}{x}\right)\Big|_{4/\pi}^{\infty}$$

$$= -\left(\tan 0 - \tan\frac{\pi}{4}\right)$$

$$\underline{= 1}$$

Exercise

Evaluate the integral $\displaystyle\int_{e^2}^{\infty} \frac{dx}{x\ln^p x} \quad p > 1$

Solution

$$\int_{e^2}^{\infty} \frac{dx}{x\ln^p x} = \int_{e^2}^{\infty} (\ln x)^{-p}\ d(\ln x)$$

$$= \frac{1}{1-p}(\ln x)^{1-p}\Big|_{e^2}^{\infty}$$

$$= \frac{1}{1-p}\left((\ln x)^{-\infty} - \left(\ln e^2\right)^{1-p}\right)$$

$$= \frac{-1}{1-p}2^{1-p}$$

$$\boxed{= \frac{1}{(p-1)2^{p-1}}}$$

Exercise

Evaluate the integral $\displaystyle\int_{0}^{\infty} \frac{p}{\sqrt[5]{p^2+1}}\, dp$

Solution

$$\int_{0}^{\infty} \frac{p}{\sqrt[5]{p^2+1}}\, dp = \frac{1}{2}\int_{0}^{\infty} \left(p^2+1\right)^{-1/5}\ d\left(p^2+1\right) \qquad d\left(p^2+1\right) = 2pdp$$

$$= \frac{5}{8}\left(p^2+1\right)^{4/5}\Big|_{0}^{\infty}$$

$$\underline{= \infty} \quad \textit{diverges}$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^{1} \ln y^2\ dy$

Solution

$$\int_{-1}^{1} \ln y^2 \, dy = 2\int_{0}^{1} \ln y^2 \, dy$$

$$\int \ln x^2 dx = 2\int \ln x dx$$

$$u = \ln x \Rightarrow du = \frac{1}{x} dx$$

$$v = \int dx = x$$

$$= 2\left[x \ln x - \int dx \right]$$

$$= 2(x \ln x - x)$$

$$\int_{-1}^{1} \ln y^2 \, dy = 4\left(y \ln y - y \right) \Big|_{0}^{1}$$

$$= 4(-1-0)$$

$$= -4$$

Exercise

Evaluate the integral $\displaystyle\int_{-2}^{6} \frac{dx}{\sqrt{|x-2|}}$

Solution

$$\int_{-2}^{6} \frac{dx}{\sqrt{|x-2|}} = \int_{-2}^{2} \frac{dx}{\sqrt{2-x}} + \int_{2}^{6} \frac{dx}{\sqrt{x-2}}$$

$$= -\int_{-2}^{2} (2-x)^{-1/2} \, d(2-x) + \int_{2}^{6} (x-2)^{-1/2} \, d(x-2)$$

$$= -2\sqrt{2-x} \Big|_{-2}^{2} + 2\sqrt{x-2} \Big|_{2}^{6}$$

$$= -2(0-2) + 2(2-0)$$

$$= 8$$

Exercise

Evaluate $\displaystyle\int_0^{\infty} xe^{-x}\,dx$

Solution

$$\int_0^{\infty} xe^{-x}\,dx = -xe^{-x} - e^{-x}\Big|_0^{\infty}$$

$$= 0 - (-1)$$

$$= 1$$

		$\int e^{-x}$
+	x	$-e^{-x}$
−	1	e^{-x}

Exercise

Evaluate $\displaystyle\int_0^{1} x\ln x\,dx$

Solution

$u = \ln x \quad dv = x\,dx$

$du = \dfrac{dx}{x} \quad v = \dfrac{1}{2}x^2$

$$\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$$

$$\int_0^1 x\ln x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2\Big|_0^1$$

$$= -\frac{1}{4}$$

Exercise

Evaluate $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\,dx$

Solution

$u = \ln x \quad dv = \dfrac{1}{x^2}\,dx$

$du = \dfrac{dx}{x} \quad v = -\dfrac{1}{x}$

$$\int \frac{\ln x}{x^2}\,dx = -\frac{1}{x}\ln x + \int \frac{1}{x^2}\,dx$$

$$= -\frac{1}{x}\ln x - \frac{1}{x}$$

$$\int_1^{\infty} \frac{\ln x}{x^2}\, dx = -\frac{1}{x}(\ln x + 1)\Big|_1^{\infty}$$

$$\boxed{= 1}$$

Exercise

Evaluate $\displaystyle\int_1^{\infty} (1-x)e^{-x}\, dx$

Solution

$$\int_1^{\infty} (1-x)e^{-x}\, dx = \left[-e^{-x} - (-x-1)e^{-x}\right]_1^{\infty}$$

$$= \left[xe^{-x}\right]_1^{\infty}$$

$$= 0 - e^{1}$$

$$\boxed{= \frac{1}{e}}$$

Exercise

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\, dx$

Solution

$$\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\, dx = \int_{-\infty}^{\infty} \frac{d\left(e^x\right)}{1+\left(e^x\right)^2}$$

$$= \arctan e^x \Big|_{-\infty}^{\infty} \qquad = \arctan\infty - \arctan 0$$

$$\boxed{= \frac{\pi}{2}}$$

Exercise

Evaluate $\displaystyle\int_0^{1} \frac{dx}{\sqrt[3]{x}}$

Solution

$$\int_0^1 x^{-1/3}\,dx = \frac{3}{2}x^{2/3}\Big|_0^1$$

$$\underline{= \frac{3}{2}}$$

Exercise

Evaluate $\displaystyle\int_1^{\infty} 4x^{-1/4}\,dx$

Solution

$$\int_1^{\infty} 4x^{-1/4}\,dx = \frac{16}{3}x^{3/4}\Big|_1^{\infty}$$

$$\underline{= \infty} \qquad \textit{Diverges}$$

Exercise

Evaluate $\displaystyle\int_0^2 \frac{dx}{x^3}$

Solution

$$\int_0^2 \frac{dx}{x^3} = -\frac{1}{2x^2}\Big|_0^2$$

$$= -\frac{1}{8} + \infty$$

$$\underline{= \infty} \qquad \textit{Diverges}$$

Exercise

Evaluate $\displaystyle\int_1^{\infty} \frac{dx}{x^3}$

Solution

$$\int_1^{\infty} \frac{dx}{x^3} = -\frac{1}{2x^2}\Big|_1^{\infty}$$

$$\underline{= \frac{1}{2}}$$

Exercise

Evaluate $\displaystyle\int_1^{\infty} 6x^{-4}\, dx$

Solution

$$\int_1^{\infty} 6x^{-4}\, dx = -2\frac{1}{x^3}\Big|_1^{\infty}$$

$$= 2$$

Exercise

Evaluate $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x}\,(x+1)}$

Solution

$u = \sqrt{x} \ \rightarrow \ u^2 = x$

$dx = 2udu$

$$\int_0^{\infty} \frac{dx}{\sqrt{x}\,(x+1)} = \int_0^{\infty} \frac{2u}{u\left(u^2+1\right)} du$$

$$= 2\int_0^{\infty} \frac{1}{u^2+1} du$$

$$= 2\arctan\sqrt{x}\Big|_0^{\infty}$$

$$= 2\left(\frac{\pi}{2} - 0\right)$$

$$= \pi$$

Exercise

Evaluate $\displaystyle\int_{-\infty}^{0} xe^{-4x}\, dx$

Solution

$$\int_{-\infty}^{0} xe^{-4x}\, dx = \left(-\frac{x}{4} - \frac{1}{16}\right)e^{-4x}\Big|_{-\infty}^{0}$$

$$\int x^n e^{ax}\, dx = e^{ax} \sum_{k=0}^{n} \frac{(-1)^k\, n!}{a^{k+1}(n-k)!} x^{n-k}$$

$$= -\frac{1}{16} - \infty$$

$$= -\infty$$ ***Diverges***

Exercise

Evaluate $\displaystyle\int_0^{\infty} xe^{-x/3}\,dx$

Solution

$$\int_0^{\infty} xe^{-x/3}\,dx = (-3x-9)e^{-x/3}\Big|_0^{\infty}$$

$$\underline{= 9}$$

$$\int x^n e^{ax}\,dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k\,n!}{a^{k+1}(n-k)!}x^{n-k}$$

Exercise

Evaluate $\displaystyle\int_0^{\infty} x^2e^{-x}dx$

Solution

$$\int_0^{\infty} x^2e^{-x}dx = \left(-x^2-2x-2\right)e^{-x}\Big|_0^{\infty}$$

$$\underline{= 2}$$

$$\int x^n e^{ax}\,dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k\,n!}{a^{k+1}(n-k)!}x^{n-k}$$

Exercise

Evaluate $\displaystyle\int_0^{\infty} e^{-x}\cos x\,dx$

Solution

		$\int \cos x$
+	e^{-x}	$\sin x$
−	$-e^{-x}$	$-\cos x$
+	e^{-x}	$-\int \cos x$

$$\int e^{-x}\cos x\,dx = e^{-x}(\sin x-\cos x) - \int e^{-x}\cos x\,dx$$

$$2\int e^{-x}\cos x\,dx = e^{-x}(\sin x-\cos x)$$

$$\int_0^{\infty} e^{-x}\cos x\,dx = \frac{1}{2}e^{-x}(\sin x-\cos x)\Big|_0^{\infty}$$

$$\underline{= \frac{1}{2}}$$

Exercise

Evaluate $\displaystyle\int_4^{\infty} \frac{1}{x(\ln x)^3}\,dx$

Solution

$$\int_4^{\infty} \frac{1}{x(\ln x)^3}\,dx = \int_4^{\infty} (\ln x)^{-3}\,d(\ln x)$$

$$= -\frac{1}{2}\frac{1}{(\ln x)^2}\Bigg|_4^{\infty}$$

$$= \frac{1}{2}\left(0 - \frac{1}{(\ln 4)^2}\right)$$

$$= \frac{1}{2(\ln 4)^2}$$

Exercise

Evaluate $\displaystyle\int_1^{\infty} \frac{\ln x}{x}\,dx$

Solution

$$\int_1^{\infty} \frac{\ln x}{x}\,dx = \int_1^{\infty} \ln x\,d(\ln x)$$

$$= \frac{1}{2}(\ln x)^2\Big|_1^{\infty}$$

$$= \infty \quad \textit{diverges}$$

Exercise

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{4}{16+x^2}\,dx$

Solution

$$\int_{-\infty}^{\infty} \frac{4}{16+x^2}\,dx = \arctan\left(\frac{x}{4}\right)\Big|_{-\infty}^{\infty}$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$$

$$= \pi$$

Exercise

Evaluate $\displaystyle\int_0^\infty \frac{x^3}{\left(x^2+1\right)^2}dx$

Solution

$$\frac{x^3}{\left(x^2+1\right)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{\left(x^2+1\right)^2}$$

$$x^3 = Ax^3 + Ax + Bx^2 + B + Cx + D$$

$$\begin{cases} x^3 & \qquad \underline{A=1} \\ x^2 & \qquad \underline{B=0} \\ x & A+C=0 \to \underline{C=-1} \\ x^0 & B+D=0 \to \underline{D=0} \end{cases}$$

$$\int_0^\infty \frac{x^3}{\left(x^2+1\right)^2}dx = \int_0^\infty \frac{x}{x^2+1}dx - \int_0^\infty \frac{x}{\left(x^2+1\right)^2}dx$$

$$= \frac{1}{2}\int_0^\infty \frac{1}{x^2+1}d\left(x^2+1\right) - \frac{1}{2}\int_0^\infty \frac{1}{\left(x^2+1\right)^2}d\left(x^2+1\right)$$

$$= \left[\frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{2}\frac{1}{x^2+1}\right]_0^\infty$$

$\underline{=\infty}$ *diverges*

Exercise

Evaluate $\displaystyle\int_0^\infty \frac{1}{e^x+e^{-x}}dx$

Solution

$$\int_0^\infty \frac{1}{e^x+e^{-x}}dx = \int_0^\infty \frac{1}{e^x+e^{-x}}\frac{e^x}{e^x}dx$$

$$= \int_0^\infty \frac{e^x}{e^{2x}+1}dx$$

$$= \int_0^\infty \frac{1}{\left(e^x\right)^2+1}d\left(e^x\right)$$

$$= \arctan e^x \Big|_0^\infty$$

$$= \arctan(\infty) - \arctan(1)$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Exercise

Evaluate $\displaystyle\int_0^\infty \frac{e^x}{1+e^x}\, dx$

Solution

$$\int_0^\infty \frac{e^x}{1+e^x}\, dx = \int_0^\infty \frac{1}{1+e^x}\, d\left(e^x\right)$$

$$= \ln\left(1+e^x\right)\Big|_0^\infty$$

$$= \infty$$ *diverges*

Exercise

Evaluate $\displaystyle\int_0^\infty \cos \pi x\, dx$

Solution

$$\int_0^\infty \cos \pi x\, dx = \frac{1}{\pi}\sin \pi x\Big|_0^\infty$$

$$= \infty$$ *diverges*

Exercise

Evaluate $\displaystyle\int_0^\infty \sin\frac{x}{2}\, dx$

Solution

$$\int_0^\infty \sin\frac{x}{2}\, dx = -2\cos\frac{x}{2}\Big|_0^\infty$$

$$= \infty$$ *diverges*

Exercise

Evaluate the integral $\int_1^\infty \frac{dx}{(x+1)^9}$

Solution

$$\int_1^\infty \frac{dx}{(x+1)^9} = \int_1^\infty (x+1)^{-9}\, d(x+1)$$

$$= -\frac{1}{8}(x+1)^{-8} \Big|_1^\infty$$

$$= -\frac{1}{8}\left(0 - \frac{1}{2^8}\right)$$

$$\boxed{= \frac{1}{2,048}}$$

Exercise

Evaluate the integral $\int_1^\infty \frac{3x-1}{4x^3 - x^2}\, dx$

Solution

$$\frac{3x-1}{4x^3 - x^2} = \frac{3x-1}{x^2(4x-1)}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{4x-1}$$

$$3x - 1 = 4Ax^2 - Ax + 4Bx - B + Cx^2$$

$x^2 \quad 4A + C = 0 \quad \rightarrow \underline{C = -4}$

$x^1 \quad -A + 4B = 3 \quad \rightarrow \underline{A = 1}$

$x^0 \quad -B = -1 \quad \rightarrow \underline{B = 1}$

$$\int_1^\infty \frac{3x-1}{4x^3 - x^2}\, dx = \int_1^\infty \left(\frac{1}{x} + \frac{1}{x^2} - \frac{4}{4x-1}\right) dx$$

$$= \ln x - \frac{1}{x} - \int_1^\infty \frac{1}{4x-1}\, d(4x-1)$$

$$= \ln x - \frac{1}{x} - \ln(4x-1) \Big|_1^\infty$$

$$= \ln \frac{x}{4x-1} - \frac{1}{x} \Big|_1^{\infty}$$

$$= \ln \frac{1}{4} - 0 - \ln \frac{1}{3} - 1$$

$$= -\ln 4 + \ln 3 - 1$$

$$\boxed{= \ln \frac{3}{4} - 1}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{4}{x^2+16}\, dx$

Solution

$$\int_{-\infty}^{\infty} \frac{4}{x^2+16}\, dx = 2\int_0^{\infty} \frac{4}{x^2+4^2}\, dx$$

$$= 2 \arctan \frac{x}{4} \Big|_0^{\infty}$$

$$= 2 \arctan \infty - 2 \arctan 0$$

$$= 2\left(\frac{\pi}{2} - 0\right)$$

$$\boxed{= \pi}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{-1} \frac{dx}{(x-1)^4}$

Solution

$$\int_{-\infty}^{-1} \frac{dx}{(x-1)^4} = \int_{-\infty}^{-1} (x-1)^{-4}\, d(x-1)$$

$$= -\frac{1}{3} \frac{1}{(x-1)^3} \Big|_{-\infty}^{-1}$$

$$= -\frac{1}{3}\left(-\frac{1}{8} - 0\right)$$

$$\boxed{= \frac{1}{24}}$$

Exercise

Evaluate the integral $\int_0^{\infty} xe^{-x}\, dx$

Solution

$$\int_0^{\infty} xe^{-x}\, dx = -e^{-x}(x+1)\Big|_0^{\infty}$$

$$= -0 + 1$$

$$\underline{=1}$$

Exercise

Evaluate the integral $\int_0^{\infty} \frac{6x}{1+x^6}\, dx$

Solution

Let $u = x^2$

$du = 2xdx$

$$\int_0^{\infty} \frac{6x}{1+x^6}\, dx = \int_0^{\infty} \frac{3\, du}{1+u^3}$$

$$\frac{3}{1+u^3} = \frac{A}{1+u} + \frac{Bu+C}{1-u+u^2}$$

$$3 = A - Au + Au^2 + Bu + Bu^2 + C + Cu$$

$$\begin{cases} u^2 & A+B=0 \\ u^1 & -A+B+C=0 \\ u^0 & A+C=3 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 3 \qquad \Delta_A = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix} = 3$$

$A = \frac{3}{3} \underline{=1}$

$B = -A \underline{=-1}$

$C = 3 - A \underline{=2}$

$$\int_0^\infty \frac{6x}{1+x^6}\,dx = \int_0^\infty \frac{1}{1+u}\,du + \int_0^\infty \frac{-u+2}{1-u+u^2}\,du$$

$$= \int_0^\infty \frac{1}{1+u}\,d(u+1) - \frac{1}{2}\int_0^\infty \frac{2u-1+3}{1-u+u^2}\,du$$

$$= \ln(u+1)\Big|_0^\infty - \frac{1}{2}\int_0^\infty \frac{2u-1}{1-u+u^2}\,du - \frac{3}{2}\int_0^\infty \frac{1}{\left(u-\frac{1}{2}\right)^2+\frac{3}{4}}\,du$$

$$= \ln\left(x^2+1\right)\Big|_0^\infty - \frac{1}{2}\int_0^\infty \frac{1}{1-u+u^2}\,d\left(1-u+u^2\right) - \frac{3}{2}\int_0^\infty \frac{1}{\left(u-\frac{1}{2}\right)^2+\frac{3}{4}}\,d\left(u-\frac{1}{2}\right)$$

$$= \ln\left(x^2+1\right) - \frac{1}{2}\ln\left|1-u+u^2\right| - \frac{3}{2}\frac{2}{\sqrt{3}}\tan\left(\left(u-\frac{1}{2}\right)\frac{2}{\sqrt{3}}\right)\Big|_0^\infty$$

$$= \ln\left(x^2+1\right) - \frac{1}{2}\ln\left|1-x^2+x^4\right| - \sqrt{3}\tan\left(\frac{2x^2-1}{\sqrt{3}}\right)\Big|_0^\infty$$

$$= \ln\left(x^2+1\right) - \ln\sqrt{1-x^2+x^4} - \sqrt{3}\tan\left(\frac{2x^2-1}{\sqrt{3}}\right)\Big|_0^\infty$$

$$= \ln\frac{x^2+1}{\sqrt{1-x^2+x^4}} - \sqrt{3}\tan\left(\frac{2x^2-1}{\sqrt{3}}\right)\Big|_0^\infty$$

$$= \ln 1 - \sqrt{3}\tan(\infty) - \ln 1 + \sqrt{3}\tan\left(-\frac{1}{\sqrt{3}}\right)$$

$$= -\sqrt{3}\,\frac{\pi}{2} - \sqrt{3}\,\frac{\pi}{6}$$

$$\boxed{= -\frac{2\pi}{3}\sqrt{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^2 \frac{dx}{\sqrt[3]{|x-1|}}$

Solution

$$\int_0^2 \frac{dx}{\sqrt[3]{|x-1|}} = \int_0^1 \frac{dx}{\sqrt[3]{|x-1|}} + \int_1^2 \frac{dx}{\sqrt[3]{|x-1|}}$$

$$= -\int_0^1 (1-x)^{-1/3}\, d(1-x) + \int_1^2 (x-1)^{-1/3}\, d(x-1)$$

$$= -\frac{3}{2}(1-x)^{2/3} \Big|_0^1 + \frac{3}{2}(x-1)^{2/3} \Big|_1^2$$

$$= -\frac{3}{2}(0-1) + \frac{3}{2}(1-0)$$

$$= \frac{3}{2} + \frac{3}{2}$$

$$\underline{= 3}$$

$$\int_0^2 \frac{dx}{\sqrt[3]{|x-1|}} = \int_0^2 (x-1)^{-1/3}\, d(x-1)$$

$$= \frac{3}{2}(x-1)^{2/3} \Big|_0^2$$

$$= \frac{3}{2}(1+1)$$

$$\underline{= 3}$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^{1} \frac{dx}{x^2+2x+5}$

Solution

$$\int_{-1}^{1} \frac{dx}{x^2+2x+5} = \int_{-1}^{1} \frac{dx}{(x+1)^2+4}$$

$$= \frac{1}{2}\arctan\frac{x+1}{2} \Big|_{-1}^{1}$$

$$= \frac{1}{2}(\arctan 1 - \arctan 0)$$

$$\underline{= \frac{\pi}{8}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+5}$

Solution

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+5} = \int_{-\infty}^{\infty} \frac{dx}{(x+1)^2+4}$$

$$= \frac{1}{2}\arctan\frac{x+1}{2}\Big|_{-\infty}^{\infty}$$

$$= \frac{1}{2}\left(\arctan\infty - \arctan(-\infty)\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$\boxed{= \frac{\pi}{2}}$$

Exercise

Evaluate the integral $\displaystyle\int_{0}^{\infty} \cos x\, dx$

Solution

$$\int_{0}^{\infty} \cos x\, dx = \sin x\Big|_{0}^{\infty}$$

$$= \sin\infty - \sin 0$$

$$= \nexists$$

$= \text{doesn't exist} \quad \rightarrow$ ***diverges***

Exercise

Evaluate the integral $\displaystyle\int_{2}^{\infty} \frac{\cos\left(\frac{\pi}{x}\right)}{x^2}\, dx$

Solution

$$\int_{2}^{\infty} \frac{\cos\left(\frac{\pi}{x}\right)}{x^2}\, dx = -\frac{1}{\pi}\int_{2}^{\infty} \cos\left(\frac{\pi}{x}\right) d\left(\frac{\pi}{x}\right) \qquad d\left(\frac{\pi}{x}\right) = -\frac{\pi}{x^2}dx$$

$$= -\frac{1}{\pi}\sin\left(\frac{\pi}{x}\right)\Big|_{2}^{\infty}$$

$$= -\frac{1}{\pi}\left(\sin 0 - \sin\frac{\pi}{2}\right)$$

$$= -\frac{1}{\pi}(0-1)$$

$$\boxed{= \frac{1}{\pi}}$$

Exercise

Evaluate the integral $\int_{-\infty}^{a} \sqrt{e^x}\, dx$

Solution

$$\int_{-\infty}^{a} \sqrt{e^x}\, dx = \int_{-\infty}^{a} e^{x/2}\, dx$$

$$= 2e^{x/2} \Big|_{-\infty}^{a}$$

$$= 2\left(e^{a/2} - e^{-\infty}\right)$$

$$\boxed{= 2e^{a/2}}$$

Exercise

Evaluate the integral $\int_{0}^{\infty} \frac{e^x}{e^{2x}+1}\, dx$

Solution

$$\int_{0}^{\infty} \frac{e^x}{e^{2x}+1}\, dx = \int_{0}^{\infty} \frac{d\left(e^x\right)}{\left(e^x\right)^2+1}$$

$$= \tan^{-1} e^x \Big|_{0}^{\infty}$$

$$= \tan^{-1}\infty - \tan^{-1} 1$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$\boxed{= \frac{\pi}{4}}$$

Exercise

Evaluate the integral $\displaystyle\int_1^{\infty} \frac{dx}{x(x+1)}$

Solution

$$\int_1^{\infty} \frac{dx}{x(x+1)} = \int_1^{\infty} \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

$$= \ln|x| - \ln|x+1| \Big|_1^{\infty}$$

$$= \ln\left(\frac{x}{x+1}\right) \Big|_1^{\infty}$$

$$= \ln 1 - \ln\frac{1}{2}$$

$$= \ln 2$$

Exercise

Evaluate the integral $\displaystyle\int_1^{\infty} \frac{dx}{x^2(x+1)}$

Solution

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$1 = Ax^2 + Ax + Bx + B + Cx^2$$

$$\begin{cases} x^2 & A + C = 0 & \rightarrow \underline{C = 1} \\ x^1 & A + B = 0 & \rightarrow \underline{A = -1} \\ x^0 & B = 1 & \end{cases}$$

$$\int_1^{\infty} \frac{dx}{x^2(x+1)} = \int_1^{\infty} \left(-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right) dx$$

$$= -\ln x - \frac{1}{x} + \ln(x+1) \Big|_1^{\infty}$$

$$= \ln\frac{x+1}{x} - \frac{1}{x} \Big|_1^{\infty}$$

$$= \ln 1 - 0 - (\ln 2 - 1)$$

$$= 1 - \ln 2$$

Exercise

Evaluate the integral $\int_1^{\infty} \frac{3x^2+1}{x^3+x}\, dx$

Solution

$$\int_1^{\infty} \frac{3x^2+1}{x^3+x}\, dx = \int_1^{\infty} \frac{1}{x^3+x}\, d\left(x^3+x\right)$$

$$= \ln\left(x^3+x\right) \Big|_1^{\infty}$$

$$= \ln\infty - \ln 2$$

$$\underline{= \infty} \quad \textit{diverges}$$

Exercise

Evaluate the integral $\int_1^{\infty} \frac{1}{x^2}\sin\frac{\pi}{x}\, dx$

Solution

$$\int_1^{\infty} \frac{1}{x^2}\sin\frac{\pi}{x}\, dx = -\frac{1}{\pi}\int_1^{\infty} \sin\frac{\pi}{x}\, d\left(\frac{\pi}{x}\right) \qquad d\left(\frac{\pi}{x}\right) = -\frac{\pi}{x^2}dx$$

$$= \frac{1}{\pi}\cos\frac{\pi}{x} \Big|_1^{\infty}$$

$$= \frac{1}{\pi}(\cos 0 - \cos\pi)$$

$$= \frac{1}{\pi}(1+1)$$

$$\underline{= \frac{2}{\pi}}$$

Exercise

Evaluate the integral $\int_2^{\infty} \frac{dx}{(x+2)^2}$

Solution

$$\int_2^{\infty} \frac{dx}{(x+2)^2} = \int_2^{\infty} \frac{d(x+2)}{(x+2)^2}$$

$$= -\frac{1}{x+2} \Big|_2^{\infty}$$

$$= -\left(0 - \frac{1}{4}\right)$$

$$\boxed{= \frac{1}{4}}$$

Exercise

Evaluate the integral $\displaystyle\int_1^{\infty} \frac{\tan^{-1} x}{x^2+1}\, dx$

Solution

$$\int_1^{\infty} \frac{\tan^{-1} x}{x^2+1}\, dx = \int_1^{\infty} \tan^{-1} x\, d\left(\tan^{-1} x\right)$$

$$= \frac{1}{2}\left(\tan^{-1} x\right)^2 \Big|_1^{\infty}$$

$$= \frac{1}{2}\left(\left(\tan^{-1} \infty\right)^2 - \left(\tan^{-1} 1\right)^2\right)$$

$$= \frac{1}{2}\left(\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right)$$

$$= \frac{\pi^2}{2}\left(\frac{1}{4} - \frac{1}{16}\right)$$

$$= \frac{\pi^2}{2}\left(\frac{3}{16}\right)$$

$$\boxed{= \frac{3\pi^2}{32}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-3}^{1} \frac{dx}{(2x+6)^{2/3}}$

Solution

$$\int_{-3}^{1} \frac{dx}{(2x+6)^{2/3}} = \frac{1}{2}\int_{-3}^{1} (2x+6)^{-2/3}\, d(2x+6)$$

$$= \frac{3}{2}(2x+6)^{1/3} \Big|_{-3}^{1}$$

$$= \frac{3}{2}\left(\sqrt[3]{8} - 0\right)$$

$$= 3$$

Exercise

Evaluate the integral $\displaystyle\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

Solution

$$\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx = 2\int_0^1 e^{\sqrt{x}}\, d\left(\sqrt{x}\right)$$

$$= 2e^{\sqrt{x}} \Big|_0^1$$

$$= 2(e-1)$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\ln 3} \frac{e^x}{\left(e^x - 1\right)^{2/3}}\, dx$

Solution

$$\int_0^{\ln 3} \frac{e^x}{\left(e^x - 1\right)^{2/3}}\, dx = \int_0^{\ln 3} \left(e^x - 1\right)^{-2/3}\, d\left(e^x - 1\right)$$

$$= 3\left(e^x - 1\right)^{1/3} \Big|_0^{\ln 3}$$

$$= 3\left(\left(e^{\ln 3} - 1\right)^{1/3} - 0\right)$$

$$= 3(3-1)^{1/3}$$

$$= 3\sqrt[3]{2}$$

Exercise

Evaluate the integral $\displaystyle\int_1^2 \frac{dx}{\sqrt{x-1}}$

Solution

$$\int_1^2 \frac{dx}{\sqrt{x-1}} = \int_1^2 (x-1)^{-1/2}\, d(x-1)$$

$$= 2(x-1)^{1/2} \Big|_1^2$$

$$= 2(1-0)$$

$$= 2$$

Exercise

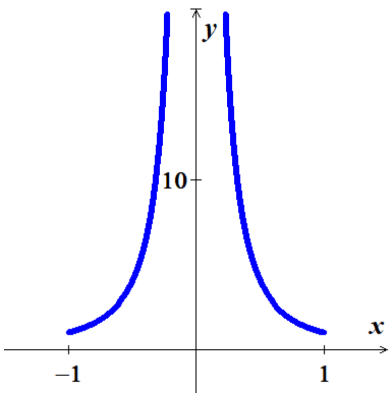
Evaluate the integral $\displaystyle\int_{-1}^1 \frac{dx}{x^2}$

Solution

$$\int_{-1}^1 \frac{dx}{x^2} = \int_{-1}^0 \frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2}$$

$$= -\frac{1}{x}\Big|_{-1}^0 - \frac{1}{x}\Big|_0^1$$

$$= -\infty \quad \textit{diverges}$$



$$\int_{-1}^1 \frac{dx}{x^2} = -\frac{1}{x}\Big|_{-1}^1$$

$$= -1-1$$

$$= -2$$

Exercise

Evaluate the integral $\displaystyle\int_0^2 \frac{dx}{(x-1)^2}$

Solution

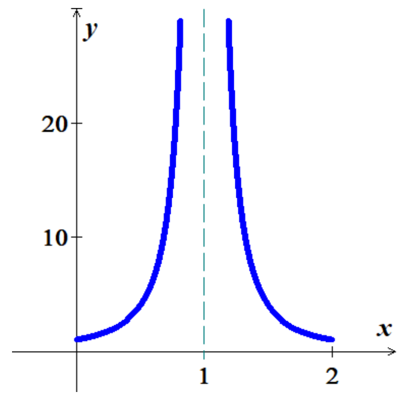
$$\int_0^2 \frac{dx}{(x-1)^2} = \int_0^1 \frac{dx}{(x-1)^2} + \int_1^2 \frac{dx}{(x-1)^2}$$

$$= \int_0^1 \frac{d(x-1)}{(x-1)^2} + \int_1^2 \frac{d(x-1)}{(x-1)^2}$$

$$= -\frac{1}{x-1}\Big|_0^1 - \frac{1}{x-1}\Big|_1^2$$

$$= -\infty - 1 - 1 + \infty$$

$$\underline{= \infty} \quad \textit{diverges}$$



$$\int_0^2 \frac{dx}{(x-1)^2} = -\frac{1}{x-1}\Big|_0^2$$

$$= -(1+1)$$

$$\underline{= -2}$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^{2} \frac{dx}{(x-1)^2}$

Solution

$$\int_{-1}^2 \frac{dx}{(x-1)^2} = \int_{-1}^1 \frac{dx}{(x-1)^2} + \int_1^2 \frac{dx}{(x-1)^2}$$

$$= \int_{-1}^1 \frac{d(x-1)}{(x-1)^2} + \int_1^2 \frac{d(x-1)}{(x-1)^2}$$

$$= -\frac{1}{x-1}\Big|_{-1}^1 - \frac{1}{x-1}\Big|_1^2$$

$$= -\infty - \frac{1}{2} - 1 + \infty$$

$$\underline{= \infty} \quad \textit{diverges}$$

Exercise

Evaluate the integral $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x^2-1}}$

Solution

$$\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x \Big|_1^\infty$$

$$= \sec^{-1}\infty - \sec^{-1}1$$

$$= \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2}$$

Exercise

Evaluate the integral $\displaystyle\int_0^\infty xe^{-x^2}\,dx$

Solution

$$\int_0^\infty xe^{-x^2}\,dx = -\frac{1}{2}\int_0^\infty e^{-x^2}\,d\left(-x^2\right)$$

$$= -\frac{1}{2}e^{-x^2}\Big|_0^\infty$$

$$= -\frac{1}{2}(0-1)$$

$$= \frac{1}{2}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^\infty xe^{-x^2}\,dx$

Solution

$$\int_{-\infty}^\infty xe^{-x^2}\,dx = -\frac{1}{2}\int_{-\infty}^0 e^{-x^2}\,d\left(-x^2\right) - \frac{1}{2}\int_0^\infty e^{-x^2}\,d\left(-x^2\right)$$

$$= -\frac{1}{2}e^{-x^2}\Big|_{-\infty}^0 - \frac{1}{2}e^{-x^2}\Big|_0^\infty$$

$$= -\frac{1}{2}(1-0) - \frac{1}{2}(0-1)$$

$$= -\frac{1}{2} + \frac{1}{2}$$

$$\underline{= 0}$$

$$\int_{-\infty}^{\infty} xe^{-x^2}dx = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2} d\left(-x^2\right)$$

$$= -\frac{1}{2}e^{-x^2} \Big|_{-\infty}^{\infty}$$

$$= -\frac{1}{2}(0-0)$$

$$\underline{= 0}$$

Exercise

Evaluate the integral $\int_{-\infty}^{\infty} \frac{x}{x^2+1}\, dx$

Solution

$$\int_{-\infty}^{\infty} \frac{x}{x^2+1}\, dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{x^2+1}\, d\left(x^2+1\right)$$

$$= \frac{1}{2}\ln\left(x^2+1\right) \Big|_{-\infty}^{\infty}$$

$$= \frac{1}{2}(\infty - \infty)$$

$\underline{= \infty}$ ***diverges***

Exercise

Evaluate the integral $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$

Solution

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \int_{-\infty}^{\infty} \frac{dx}{(x+1)^2+1}$$

$$= \int_{-\infty}^{\infty} \frac{d(x+1)}{(x+1)^2+1}$$

$$= \tan^{-1}(x+1)\Big|_{-\infty}^{\infty}$$

$$= \tan^{-1}\infty - \tan^{-1}(-\infty)$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$\boxed{= \pi}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+6x+12}$

Solution

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+6x+12} = \int_{-\infty}^{\infty} \frac{dx}{(x+3)^2+3}$$

$$= \int_{-\infty}^{\infty} \frac{d(x+3)}{(x+3)^2+3}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\frac{x+3}{\sqrt{3}}\Big|_{-\infty}^{\infty}$$

$$= \frac{1}{\sqrt{3}}\left(\tan^{-1}\infty - \tan^{-1}(-\infty)\right)$$

$$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$\boxed{= \frac{\pi}{\sqrt{3}}}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{2-\sqrt{3x}}$

Solution

Let $\quad u = 2 - \sqrt{3x} \quad \to \quad \sqrt{3x} = 2 - u$

$$du = -\frac{3}{2}(3x)^{-1/2}\, dx$$

$$dx = -\frac{2}{3}\sqrt{3x}\, du$$

$$= -\frac{2}{3}(2-u)\, du$$

$$\int \frac{dx}{2-\sqrt{3x}} = -\frac{2}{3}\int \frac{1}{u}(2-u)\ du$$

$$= -\frac{2}{3}\int \left(\frac{2}{u}-1\right)\ du$$

$$= -\frac{2}{3}\left(2\ln|u|-u\right)+C$$

$$= -\frac{2}{3}\left(2\ln\left|2-\sqrt{3x}\right|-2-\sqrt{3x}\right)+C$$

$$\underline{= -\frac{2}{3}\left(\ln\left(2-\sqrt{3x}\right)^2-2-\sqrt{3x}\right)+C}$$

Exercise

Evaluate the integral $\int \theta\ \cos(2\theta+1)\ d\theta$

Solution

		$\int \cos(2\theta+1)$
+	θ	$\frac{1}{2}\sin(2\theta+1)$
–	1	$-\frac{1}{4}\cos(2\theta+1)$

$$\int \theta\ \cos(2\theta+1)\ d\theta = \underline{\frac{1}{2}\theta\sin(2\theta+1)+\frac{1}{4}\cos(2\theta+1)+C}$$

Exercise

Evaluate the integral $\int \sqrt{x}\ \sqrt{1+\sqrt{x}}\ dx$

Solution

Let $u=\sqrt{x} \quad\Rightarrow\quad u^2=x$

$2udu=dx$

$$\int \sqrt{x}\ \sqrt{1+\sqrt{x}}\ dx = \int u\ \sqrt{1+u}\ \ (2udu)$$

$$= \int 2u^2\ \ (1+u)^{1/2}\ \ du$$

		$\int (1+u)^{1/2}$
+	$2u^2$	$\frac{2}{3}(1+u)^{3/2}$
−	$4u$	$\frac{4}{15}(1+u)^{5/2}$
+	4	$\frac{8}{105}(1+u)^{7/2}$

$$\int \sqrt{x}\,\sqrt{1+\sqrt{x}}\,dx = \frac{4}{3}u^2(1+u)^{3/2} - \frac{16}{15}u(1+u)^{5/2} + \frac{32}{105}(1+u)^{7/2} + C$$

$$= \frac{4}{3}x\left(1+\sqrt{x}\right)^{3/2} - \frac{16}{15}\sqrt{x}\left(1+\sqrt{x}\right)^{5/2} + \frac{32}{105}\left(1+\sqrt{x}\right)^{7/2} + C$$

Exercise

Find the area of the unbounded shaded region $\quad y = e^x, \quad -\infty < x \le 1$

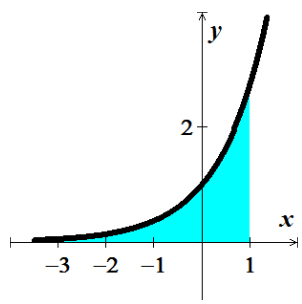
Solution

$$A = \int_{-\infty}^{1} e^x\,dx$$

$$= e^x \Big|_{-\infty}^{1}$$

$$= e - 0$$

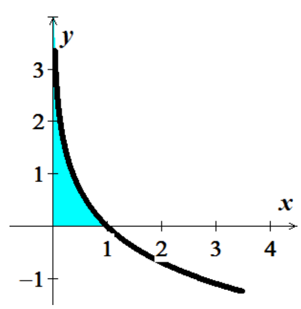
$$= e$$



Exercise

Find the area of the unbounded shaded region $\quad y = -\ln x$

Solution



$$A = -\int_{0}^{1} \ln x\,dx$$

$$= -(x\ln x - x)\Big|_{0}^{1}$$

$$= 1$$

Exercise

Find the area of the unbounded shaded region $y = \dfrac{1}{x^2+1}$

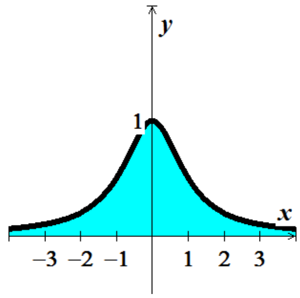
Solution

$$A = \int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$$

$$= \arctan x \Big|_{-\infty}^{\infty}$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$$

$$= \pi$$



Exercise

Find the area of the unbounded shaded region $y = \dfrac{8}{x^2+4}$

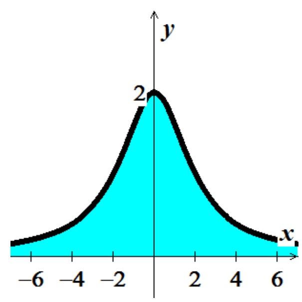
Solution

$$A = \int_{-\infty}^{\infty} \frac{8}{x^2+4}\,dx$$

$$= 4\arctan \frac{x}{2} \Big|_{-\infty}^{\infty}$$

$$= 4\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$= 4\pi$$



Exercise

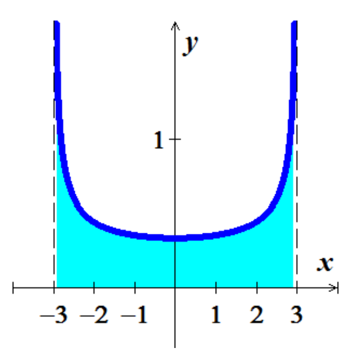
Find the area of the region *R* between the graph of $f(x) = \dfrac{1}{\sqrt{9-x^2}}$ and the *x-axis* on the interval $(-3,\ 3)$

(if it exists)

Solution

$$A = \int_{-3}^{3} \frac{dx}{\sqrt{9-x^2}}$$

$$= 2\int_{0}^{3} \frac{dx}{\sqrt{9-x^2}}$$



$$= 2\sin^{-1}\frac{x}{3}\Big|_0^3$$

$$= 2\left(\sin^{-1}1 - \sin^{-1}0\right)$$

$$\underline{= \pi \ \ unit^2}$$

Exercise

Find the volume of the region bounded by $f(x) = \left(x^2+1\right)^{-1/2}$ and the x-*axis* on the interval $[2, \infty)$ is revolved about the x-*axis*.

Solution

$$V = \pi \int_2^\infty \frac{1}{x^2+1}\,dx \qquad\qquad V = \pi \int_a^b \left(f(x)\right)^2 dx$$

$$= \pi \tan^{-1} x\Big|_2^\infty$$

$$= \pi\left(\tan^{-1}\infty - \tan^{-1}2\right)$$

$$\underline{= \pi\left(\frac{\pi}{2} - \tan^{-1}2\right) \ \ unit^3}$$

Exercise

Find the volume of the region bounded by $f(x) = \sqrt{\frac{x+1}{x^3}}$ and the x-*axis* on the interval $[1, \infty)$ is revolved about the x-*axis*.

Solution

$$V = \pi \int_1^\infty \frac{x+1}{x^3}\,dx \qquad\qquad V = \pi \int_a^b \left(f(x)\right)^2 dx$$

$$= \pi \int_1^\infty \left(\frac{1}{x^2} + x^{-3}\right) dx$$

$$= \pi\left(-\frac{1}{x} - \frac{1}{2}\frac{1}{x^2}\right)\Big|_1^\infty$$

$$= \pi\left(1 + \frac{1}{2}\right)$$

$$\underline{= \frac{3\pi}{2} \ \ unit^3}$$

Exercise

Find the volume of the region bounded by $f(x) = (x+1)^{-3}$ and the *x-axis* on the interval $[0, \infty)$ is revolved about the *y-axis*.

Solution

$$V = 2\pi \int_0^\infty x \frac{1}{(x+1)^3} dx \qquad\qquad V = 2\pi \int_a^b x \cdot f(x)\, dx \quad (Shell\ method)$$

$$= 2\pi \int_0^\infty \left(\frac{1}{(x+1)^2} - \frac{1}{(x+1)^3} \right) d(x+1)$$

$$\frac{x}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$x = Ax^2 + 2Ax + A + Bx + B + C$$

$$\begin{cases} \underline{A = 0} \\ 2A + B = 1 \ \rightarrow \underline{B = 1} \quad \underline{C = -1} \\ B + C = 0 \end{cases}$$

$$= 2\pi \left(\frac{-1}{x+1} + \frac{1}{2}\frac{1}{(x+1)^2} \right) \Big|_0^\infty$$

$$= 2\pi \left(1 - \frac{1}{2}\right)$$

$$\underline{= \pi \ \ unit^3}$$

Exercise

Find the volume of the region bounded by $f(x) = \frac{1}{\sqrt{x}\ln x}$ and the *x-axis* on the interval $[2, \infty)$ is revolved about the *x-axis*.

Solution

$$V = \pi \int_2^\infty \frac{1}{x \ln^2 x}\, dx \qquad\qquad V = \pi \int_a^b (f(x))^2\, dx$$

$$= \pi \int_2^\infty \frac{1}{\ln^2 x}\, d(\ln x)$$

$$= \pi \left(-\frac{1}{\ln x} \right) \Big|_2^\infty$$

$$= \pi \left(-0 + \frac{1}{\ln 2} \right)$$

$$= \boxed{\dfrac{\pi}{\ln 2}\ \ unit^3}$$

Exercise

Find the volume of the region bounded by $f(x) = \dfrac{\sqrt{x}}{\sqrt[3]{x^2+1}}$ and the *x-axis* on the interval $[0, \infty)$ is revolved about the *x-axis*.

Solution

$$V = \pi \int_0^\infty \frac{x}{\left(x^2+1\right)^{2/3}}\, dx \qquad V = \pi \int_a^b \left(f(x)\right)^2 dx$$

$$= \frac{\pi}{2} \int_0^\infty \left(x^2+1\right)^{-2/3} d\left(x^2+1\right)$$

$$= \frac{3\pi}{2} \left(x^2+1\right)^{1/3} \Big|_0^\infty$$

$$= \frac{3\pi}{2} (\infty - 1)$$

$$= \boxed{\infty \quad diverges}$$ So the volume doesn't exist

Exercise

Find the volume of the region bounded by $f(x) = \left(x^2-1\right)^{-1/4}$ and the *x-axis* on the interval $(1, 2]$ is revolved about the *y-axis*.

Solution

$$V = 2\pi \int_1^2 x\left(x^2-1\right)^{-1/4} dx \qquad V = 2\pi \int_a^b x \cdot f(x)\, dx \quad (Shell\ method)$$

$$= \pi \int_1^2 \left(x^2-1\right)^{-1/4} d\left(x^2-1\right)$$

$$= \frac{4\pi}{3} \left(x^2-1\right)^{3/4} \Big|_1^2$$

$$= \frac{4\pi}{3} (3)^{3/4}$$

$$= \boxed{\frac{4\pi}{3^{1/4}}\ \ unit^3}$$

Exercise

Find the volume of the region bounded by $f(x) = \tan x$ and the x-axis on the interval $\left[0, \frac{\pi}{2}\right)$ is revolved about the x-axis.

Solution

$$V = \pi \int_0^{\pi/2} \tan^2 x \, dx \qquad V = \pi \int_a^b (f(x))^2 \, dx$$

$$= \pi \int_0^{\pi/2} \left(\sec^2 x - 1\right) dx$$

$$= \pi (\tan x - x) \Big|_0^{\pi/2} \qquad \left(\tan \frac{\pi}{2} = \infty\right)$$

$$= \infty \quad \textit{diverges}$$ So the volume doesn't exist

Exercise

Find the volume of the region bounded by $f(x) = -\ln x$ and the x-axis on the interval $(0, 1]$ is revolved about the x-axis.

Solution

$$V = \pi \int_0^1 \ln^2 x \, dx \qquad V = \pi \int_a^b (f(x))^2 \, dx$$

$$u = \ln x \qquad dv = \ln x \, dx$$

$$du = \frac{dx}{x} \qquad v = x \ln x - x$$

$$u = \ln x \quad dv = dx \qquad du = \frac{dx}{x} \quad v = x \quad \rightarrow \int \ln x dx = x \ln x - \int dx = x \ln x - x$$

$$\int \ln^2 x \, dx = \ln x (x \ln x - x) - \int (\ln x - 1) dx$$

$$= x \ln^2 x - x \ln x - (x \ln x - x - x)$$

$$= x \ln^2 x - 2x \ln x + 2x$$

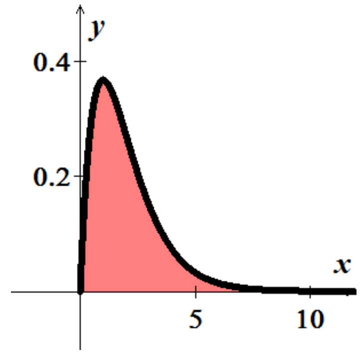
$$V = \pi \left(x \ln^2 x - 2x \ln x + 2x\right) \Big|_0^1$$

$$= 2\pi \ \textit{unit}^3$$

Exercise

Find the volume of the solid generated by revolving the region bounded by the graphs of $y = xe^{-x}$, $y = 0$, and $x = 0$ about the *x-axis*.

Solution



$$V = \pi \int_0^\infty \left(xe^{-x}\right)^2 dx$$

$$= \pi \int_0^\infty x^2 e^{-2x}\, dx$$

$$= \pi e^{-2x}\left(-\frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{4}\right)\Big|_0^\infty$$

$$= \pi\left(0 + \frac{1}{4}\right)$$

$$\boxed{= \frac{\pi}{4}}$$

Exercise

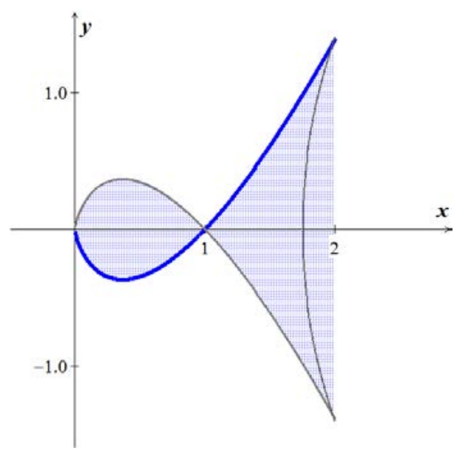
The region between the *x*-axis and the curve

$$f(x) = \begin{cases} 0, & x = 0 \\ x \ln x, & 0 < x \le 2 \end{cases}$$

is revolved about the *x*-axis to generate the solid.

Find the volume of the solid.

Solution



$$V = \pi \int_0^2 y^2\, dx$$

$$= \pi \int_0^2 (x \ln x)^2\, dx$$

$$u = (\ln x)^2 \qquad dv = x^2 dx$$

$$du = 2\frac{\ln x}{x}dx \qquad v = \frac{1}{3}x^3$$

$$= \frac{\pi}{3}x^3(\ln x)^2\Big|_0^2 - \frac{2\pi}{3}\int_0^2 x^2 \ln x\, dx$$

$$u = \ln x \qquad dv = x^2 dx$$

$$du = \frac{1}{x} dx \qquad v = \frac{1}{3} x^3$$

$$= \frac{\pi}{3}\left(8(\ln 2)^2 - 0\right) - \frac{2\pi}{9} x^3 \ln x \,\Big|_0^2 + \frac{2\pi}{9}\int_0^2 x^2\, dx$$

$$= \frac{8\pi}{3}(\ln 2)^2 - \frac{2\pi}{9}(8\ln 2 - 0) + \frac{2\pi}{27} x^3 \,\Big|_0^2$$

$$= \frac{8\pi}{3}(\ln 2)^2 - \frac{16\pi}{9}\ln 2 + \frac{16\pi}{27}$$

$$\boxed{= \frac{8\pi}{3}\left((\ln 2)^2 - \frac{2}{3}\ln 2 + \frac{2}{9}\right)}$$

Exercise

Consider the region satisfying the inequalities $y \le e^{-x}, \quad y \ge 0, \quad x \ge 0$

a) Find the area of the region

b) Find the volume of the solid generated by revolving the region about the *x-axis*.

c) Find the volume of the solid generated by revolving the region about the *y-axis*.

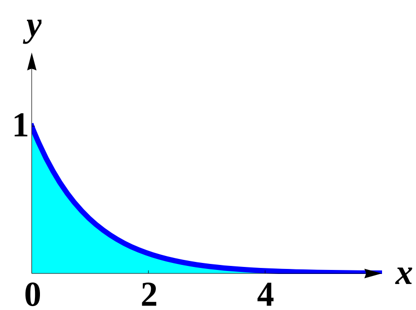
Solution

a) $A = \int_0^\infty e^{-x}\, dx$

$$= -e^{-x}\Big|_0^\infty$$

$$= -(0-1)$$

$$\boxed{=1}$$



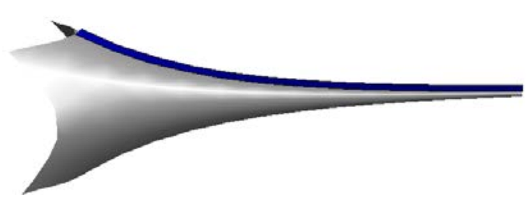
b) $V = \pi \int_0^\infty \left(e^{-x}\right)^2 dx$ $\qquad V = \pi\int_a^b \left(R(x)^2 - r(x)^2\right)dx$ (*Disk Method*)

$$= \pi \int_0^\infty e^{-2x}\, dx$$

$$= -\frac{\pi}{2} e^{-2x}\Big|_0^\infty$$

$$= -\frac{\pi}{2}(0-1)$$

$$\boxed{= \frac{\pi}{2}}$$

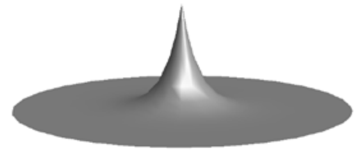


c) $V = 2\pi \int_0^{\infty} xe^{-x}\, dx \qquad V = 2\pi \int_a^b xf(x)\,dx \quad (Shell\ Method)$

$$= -2\pi e^{-x}(x+1)\Big|_0^{\infty}$$

$$= -2\pi(0-1)$$

$$\underline{= 2\pi}$$



Exercise

Consider the region satisfying the inequalities $\quad y \le \frac{1}{x^2}, \quad y \ge 0, \quad x \ge 1$

a) Find the area of the region

b) Find the volume of the solid generated by revolving the region about the *x-axis* .

c) Find the volume of the solid generated by revolving the region about the *y-axis* .

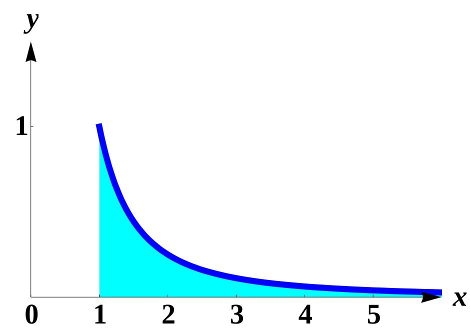
Solution

a) $A = \int_1^{\infty} \frac{1}{x^2}\, dx$

$$= -\frac{1}{x}\Big|_1^{\infty}$$

$$= -(0-1)$$

$$\underline{= 1}$$



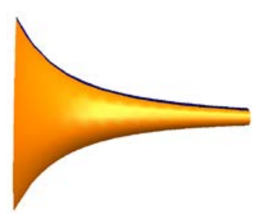
b) $V = \pi \int_0^{\infty} \left(\frac{1}{x^2}\right)^2 dx \qquad V = \pi \int_a^b \left(R(x)^2 - r(x)^2\right)dx \quad (Disk\ Method)$

$$= \pi \int_0^{\infty} x^{-4}\, dx$$

$$= -\frac{\pi}{3x^3}\Big|_1^{\infty}$$

$$= -\frac{\pi}{3}(0-1)$$

$$\underline{= \frac{\pi}{3}}$$



c) $V = 2\pi \int_0^{\infty} x\left(\frac{1}{x^2}\right) dx \qquad V = 2\pi \int_a^b xf(x)\,dx \quad (Shell\ Method)$

$$= 2\pi \int_0^{\infty} \frac{1}{x}\, dx$$

$$= 2\pi \ln x \Big|_1^{\infty}$$

$$\underline{= \infty} \quad \textit{Diverges}$$

Exercise

Find the perimeter of the hypocycloid of four cusps $x^{2/3} + y^{2/3} = 4$

Solution

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{y^{2/3}}{x^{2/3}}}$$

$$= \frac{\sqrt{x^{2/3}+y^{2/3}}}{x^{1/3}}$$

$$= \frac{\sqrt{4}}{x^{1/3}}$$

$$= 2x^{-1/3}$$

$$S = 4\int_0^8 2x^{-1/3}\, dx$$

$$= 12x^{2/3}\Big|_0^8$$

$$= 12(4-0)$$

$$\underline{= 48}$$

Exercise

Find the arc length of the graph $y = \sqrt{16-x^2}$ over the interval $[0,\ 4]$

Solution

$$y' = -\frac{x}{\sqrt{16-x^2}}$$

$$\sqrt{1+(y')^2}=\sqrt{1+\frac{x^2}{16-x^2}}$$

$$=\frac{4}{\sqrt{16-x^2}}$$

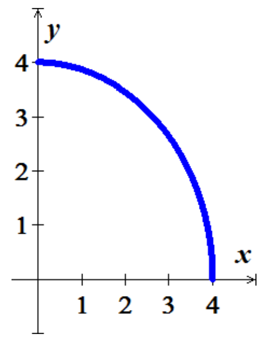
$$L=\int_0^4 \frac{4}{\sqrt{16-x^2}}\,dx$$

$$=4\arcsin\frac{x}{4}\Big|_0^4$$

$$=4(\arcsin 1-\arcsin 0)$$

$$=4\left(\frac{\pi}{2}\right)$$

$$\boxed{=2\pi}$$



Exercise

The region bounded by $(x-2)^2+y^2=1$ is revolved about the *y-axis* to form a torus. Find the surface area of the torus.

Solution

$$2(x-2)+2yy'=0$$

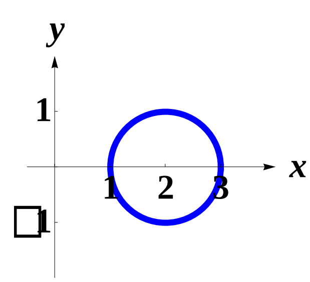
$$y'=-\frac{x-2}{y}$$

$$\sqrt{1+(y')^2}=\sqrt{1+\frac{(x-2)^2}{y^2}}$$

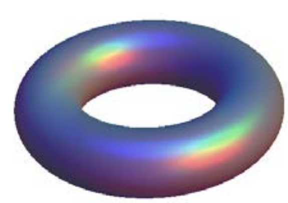
$$=\sqrt{\frac{y^2+(x-2)^2}{y^2}} \qquad (x-2)^2+y^2=1$$

$$=\frac{1}{y}$$

$$=\frac{1}{\sqrt{1-(x-2)^2}}$$



$$S=4\pi\int_1^3 \frac{x}{\sqrt{1-(x-2)^2}}\,dx$$



$$=4\pi\int_1^3 \frac{x-2+2}{\sqrt{1-(x-2)^2}}\,dx$$

$$= 4\pi \int_1^3 \frac{x-2}{\sqrt{1-(x-2)^2}}\, dx + 4\pi \int_1^3 \frac{2}{\sqrt{1-(x-2)^2}}\, dx$$

$$= -2\pi \int_1^3 \left(1-(x-2)^2\right)^{-1/2}\, d\left(1-(x-2)^2\right) + 8\pi \arctan(x-2)\Big|_1^3$$

$$= -4\pi\sqrt{1-(x-2)^2}\Big|_1^3 + 8\pi\left(\arctan(1) - \arctan(-1)\right)$$

$$= -4\pi(0-0) + 8\pi\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$\underline{= 8\pi^2}$$

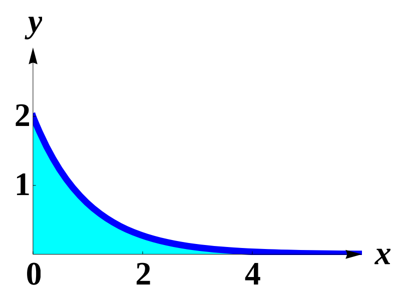
Exercise

Find the surface area formed by revolving the graph $y = 2e^{-x}$ on the interval $[0, \infty)$ about the *x-axis*

Solution

$$y' = -2e^{-x}$$

$$\sqrt{1+(y')^2} = \sqrt{1+4e^{-2x}}$$



$$S = 2\pi \int_0^\infty 2e^{-x}\sqrt{1+4e^{-2x}}\, dx$$

$$= -4\pi \int_0^\infty \sqrt{1+4\left(e^{-x}\right)^2}\, d\left(e^{-x}\right)$$

$$\int \sqrt{1+4u^2}\, du = \frac{1}{2}\int \sec^3\theta\, d\theta$$

$2u = \tan\theta \qquad \sqrt{4u^2+1} = \sec\theta$

$du = \frac{1}{2}\sec^2\theta\, d\theta$

$$u = \sec\theta \qquad dv = \sec^2\theta d\theta$$

$$du = \sec\theta\tan\theta d\theta \qquad v = \tan\theta$$

$$\int \sec^3\theta d\theta = \sec\theta\tan\theta - \int \tan\theta\left(\sec\theta\tan\theta d\theta\right)$$

$$= \sec\theta\tan\theta - \int \tan^2\theta\sec\theta\, d\theta$$

$$= \sec\theta\tan\theta - \int \left(\sec^2\theta - 1\right)\sec\theta\, d\theta$$

$$2\int \sec^3\theta d\theta == \sec\theta\tan\theta + \int \sec\theta\, d\theta$$

$$\int \sqrt{1+4u^2}\ du = \frac{1}{4}\left(\sec\theta\tan\theta + \ln\left|\sec\theta+\tan\theta\right|\right)$$

$$S = -\pi\left(2e^{-x}\sqrt{1+4e^{-2x}} + \ln\left|2e^{-x}+\sqrt{1+4e^{-2x}}\right|\right)\Big|_0^{\infty}$$

$$= -\pi\left(-2\sqrt{5} + \ln\left(2+\sqrt{5}\right)\right)$$

$$= \pi\left(2\sqrt{5} - \ln\left(2+\sqrt{5}\right)\right)$$

Exercise

The magnetic potential P at a point on the axis of a circular coil is given by

$$P = \frac{2\pi NIr}{k}\int_c^{\infty} \frac{1}{\left(r^2+x^2\right)^{3/2}}\ dx$$

Where $N, I, r, k,$ and c are constants. Find P.

Solution

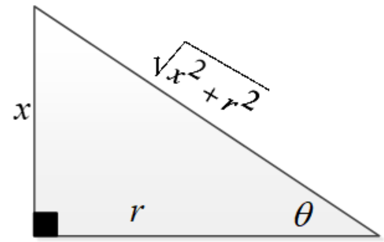
Let $K = \frac{2\pi NIr}{k}$

$$P = K\int_c^{\infty} \frac{1}{\left(r^2+x^2\right)^{3/2}}\ dx$$

$x = r\tan\theta \qquad \sqrt{x^2+r^2} = r\sec\theta$

$dx = r\sec^2\theta\ d\theta$

$$= K\int_c^{\infty} \frac{r\sec^2\theta}{r^3\sec^3\theta}\ d\theta$$



$$= \frac{K}{r^2}\int_c^{\infty} \cos\theta\ d\theta$$

$$= \frac{K}{r^2}\sin\theta\Big|_c^{\infty}$$

$$= \frac{K}{r^2}\frac{x}{\sqrt{r^2+x^2}}\Big|_c^{\infty}$$

$$= \frac{K}{r^2}\left(1-\frac{c}{\sqrt{r^2+c^2}}\right)$$

$$= \frac{2\pi NI\left(\sqrt{r^2+c^2}-c\right)}{kr\sqrt{r^2+c^2}}$$

Exercise

A "semi-infinite" uniform rod occupies the nonnegative x-*axis*. The rod has a linear density δ, which means that a segment of length dx has a mass of δdx. A particle of mass M is located at the point $(-a,\ 0)$. The gravitational force F that the rod exerts on the mass is given by

$$F = \int_0^\infty \frac{GM\delta}{(a+x)^2}\,dx$$

Where G is the gravitational constant. Find F.

Solution

$$F = \int_0^\infty \frac{GM\delta}{(a+x)^2}\,dx$$

$$= -\frac{GM\delta}{a+x}\Big|_0^\infty$$

$$\boxed{= \frac{GM\delta}{a}}$$

Exercise

Let R be the region bounded by the graph of $f(x) = x^{-p}$ and the x-*axis*

a) Let S be the solid generated when R is revolved about the x-*axis*. For what values of p is the volume of S finite for $0 < x \le 1$?

b) Let S be the solid generated when R is revolved about the y-*axis*. For what values of p is the volume of S finite for $0 < x \le 1$?

c) Let S be the solid generated when R is revolved about the x-*axis*. For what values of p is the volume of S finite for $x \ge 1$?

d) Let S be the solid generated when R is revolved about the y-*axis*. For what values of p is the volume of S finite for $x \ge 1$?

Solution

a) $V = \pi \int_0^1 \left(x^{-p}\right)^2 dx$ $\qquad V = \pi \int_a^b f(x)^2\,dx$

$$= \pi \int_0^1 x^{-2p}\,dx$$

$$= \pi \frac{x^{-2p+1}}{1-2p}\Big|_0^1$$

$$= \frac{\pi}{1-2p}\left(1 - 0^{-2p+1}\right)$$

The volume of S finite when $1-2p>0 \ \Rightarrow\ \boxed{p<\frac{1}{2}}$

b) $V = 2\pi \int_0^1 x \cdot x^{-p} dx \qquad\qquad V = 2\pi \int_a^b xf(x)dx$

$$= 2\pi \int_0^1 x^{1-p} dx$$

$$= \frac{2\pi}{2-p} x^{2-p} \Big|_0^1$$

$$= \frac{2\pi}{2-p}\left(1-0^{2-p}\right)$$

The volume of S finite when $2-p>0 \ \Rightarrow\ \boxed{p<2}$

c) $V = \pi \int_1^\infty \left(x^{-p}\right)^2 dx \qquad\qquad V = \pi \int_a^b f(x)^2 dx$

$$= \pi \int_1^\infty x^{-2p} dx$$

$$= \pi \frac{x^{-2p+1}}{1-2p} \Big|_1^\infty$$

$$= \frac{\pi}{1-2p}\left(\infty^{1-2p} - 1\right)$$

The volume of S finite when $1-2p<0 \ \Rightarrow\ \boxed{p>\frac{1}{2}}$ $\left(\frac{1}{\infty}=0\right)$

d) $V = 2\pi \int_0^1 x \cdot x^{-p} dx \qquad\qquad V = 2\pi \int_a^b xf(x)dx$

$$= 2\pi \int_0^1 x^{1-p} dx$$

$$= \frac{2\pi}{2-p} x^{2-p} \Big|_0^1$$

$$= \frac{2\pi}{2-p}\left(1-0^{2-p}\right)$$

The volume of S finite when $2-p>0 \ \Rightarrow\ \boxed{p<2}$

Exercise

The solid formed by revolving (about the x-*axis*) the unbounded region lying between the graph of $f(x)=\frac{1}{x}$ and the x-*axis* $(x\geq 1)$ is called ***Gabriel's Horn***.

Show that this solid has a finite volume and an infinite surface area

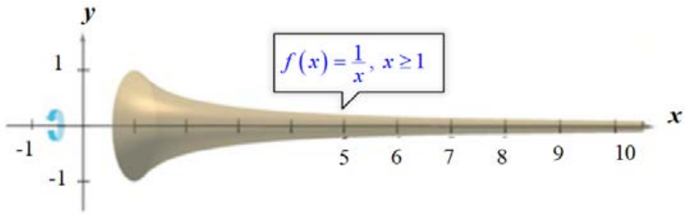
Solution

$$V=\pi\int_1^\infty \frac{1}{x^2}dx \qquad V=\pi\int_x^b (f(x))^2\,dx\ \ (\textit{disk method})$$

$$=-\pi\frac{1}{x}\Big|_1^\infty$$

$$=-\pi(0-1)$$

$$=\pi\ \ unit^3$$



$$f'(x)=-\frac{1}{x^2}$$

$$S=2\pi\int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx \qquad S=2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx$$

Since $\quad 1+\frac{1}{x^4}>1$ and $\int_1^\infty \frac{1}{x}\,dx$ diverges

Therefore the surface area in infinite.

Exercise

Water is drained from a 3000-*gal* tank at a rate that starts at 100 *gal/hr*. and decreases continuously by 5% /*hr*. If the drain left open indefinitely, how much water drains from the tank? Can a full tank be emptied at this rate?

Solution

Rate of the drain water: $r(t)=100(1-.05)^t$

$$=100(0.95)^t$$

$$=100e^{(\ln 0.95)t}$$

Total water amount drained:

$$D=\int_0^\infty 100e^{(\ln 0.95)t}dt$$

$$= \frac{100}{\ln 0.95} e^{(\ln 0.95)t} \Big|_0^{\infty}$$

$$= \frac{100}{\ln 0.95}(0-1)$$

$$\ln 0.95 < 0 \xrightarrow[t\to\infty]{} e^{(\ln 0.95)t} = e^{-\infty} = 0$$

$$\boxed{= -\frac{100}{\ln 0.95} \approx 1950 \ \ gal}$$

Since 1950 gal $<$ 3000 gal which it takes infinite time.

Therefore; the full 3,000–gallon tank cannot be emptied at this rate.

Exercise

Let $I(a) = \int_0^{\infty} \frac{dx}{(1+x^a)(1+x^2)}$, where a is a real number.

a) Evaluate $I(a)$ and show that its value is independent of a.

(***Hint***: split the integral into two integrals over $[0,\ 1]$ and $[1,\ \infty)$; then use a change of variables to convert the second integral into an integral over $[0,\ 1]$.)

b) Let f be any positive continuous function on $\left[0,\ \frac{\pi}{2}\right]$

Evaluate $\int_0^{\pi/2} \frac{f(\cos x)}{f(\cos x)+f(\sin x)} dx$

(***Hint***: Use the identity $\cos\left(\frac{\pi}{2}-x\right) = \sin x$)

Solution

a) $I(a) = \int_0^{\infty} \frac{dx}{(1+x^a)(1+x^2)}$

$$= \int_0^1 \frac{dx}{(1+x^a)(1+x^2)} + \int_1^{\infty} \frac{dx}{(1+x^a)(1+x^2)}$$

Let $u = \frac{1}{x} \Rightarrow x = \frac{1}{u}$

$$dx = -\frac{1}{u^2} du$$

$x = 1 \rightarrow u = 1$

$x = \infty \rightarrow u = 0$

$$I(a)=\int_0^1 \frac{dx}{\left(1+x^a\right)\left(1+x^2\right)}-\int_1^0 \frac{du}{u^2\left(1+u^{-a}\right)\left(1+u^{-2}\right)}$$

$$=\int_0^1 \frac{dx}{\left(1+x^a\right)\left(1+x^2\right)}+\int_0^1 \frac{du}{u^2\left(1+\frac{1}{u^a}\right)\left(1+\frac{1}{u^2}\right)}$$

$$=\int_0^1 \frac{dx}{\left(1+x^a\right)\left(1+x^2\right)}+\int_0^1 \frac{u^a du}{\left(1+u^a\right)\left(1+u^2\right)} \qquad (x=u)$$

$$=\int_0^1 \frac{dx}{\left(1+x^a\right)\left(1+x^2\right)}+\int_0^1 \frac{x^a du}{\left(1+x^a\right)\left(1+x^2\right)}$$

$$=\int_0^1 \frac{1+x^a}{\left(1+x^a\right)\left(1+x^2\right)}dx$$

$$=\int_0^1 \frac{dx}{1+x^2}$$

$$=\tan^{-1}x\Big|_0^1$$

$$=\tan^{-1}1-\tan^{-1}0$$

$$=\frac{\pi}{4}-0$$

$$\boxed{=\frac{\pi}{4}}$$

b) $$I=\int_0^{\pi/2}\frac{f(\cos x)}{f(\cos x)+f(\sin x)}dx$$

Let $u=\frac{\pi}{2}-x \quad \Rightarrow \quad x=\frac{\pi}{2}-u$

$dx=-du$

$$\cos\left(\frac{\pi}{2}-x\right)=\sin x$$

$x=1 \quad \to \quad u=1$

$x=\infty \quad \to \quad u=0$

$$I=-\int_{\pi/2}^{0}\frac{f(\sin u)}{f(\sin u)+f(\cos u)}du$$

$$= \int_0^{\pi/2} \frac{f(\sin u)}{f(\sin u) + f(\cos u)}\, du$$

$$2I = \int_0^{\pi/2} \frac{f(\cos x)}{f(\cos x) + f(\sin x)}\, dx + \int_0^{\pi/2} \frac{f(\sin x)}{f(\sin x) + f(\cos x)}\, dx$$

$$= \int_0^{\pi/2} dx$$

$$= x \Big|_0^{\pi/2}$$

$$= \frac{\pi}{2}$$

$$\boxed{I = \frac{\pi}{4}}$$

Exercise

Let R be the region bounded by $y = \ln x$, the x-*axis*, and the line $x = a$, where $a > 1$.

a) Find the volume $V_1(a)$ of the solid generated when R is revolved about the x-*axis* (as a function of a).

b) Find the volume $V_2(a)$ of the solid generated when R is revolved about the y-*axis* (as a function of a).

c) Graph V_1 and V_2. For what values of $a > 1$ is $V_1(a) > V_2(a)$?

Solution

a) $V_1(a) = \pi \int_1^a (\ln x)^2\, dx$

Let $z = \ln x \Rightarrow x = e^z$

$dx = e^z dz$

$$V_1(a) = \pi \int_1^a z^2 e^z\, dz$$

		$\int e^z dz$
+	z^2	e^z
−	$2z$	e^z
+	2	e^z

$$V_1(a) = \pi e^z\left(z^2 - 2z + 2\right)\Big|_1^a$$

$$= \pi x\left((\ln x)^2 - 2\ln x + 2\right)\Big|_1^a$$

$$= \pi\left[a\left((\ln a)^2 - 2\ln a + 2\right) - (\ln 1 - 2\ln 1 + 2)\right]$$

$$\boxed{= \pi\left(a\ln^2 a - 2a\ln a + 2a + 2\right)}$$

b) About *y-axis*

$$V_2(a) = 2\pi\int_1^a x\ln x\, dx$$

		$\int x dx$
+	$\ln x$	$\frac{1}{2}x^2$
−	$\frac{1}{x}$	$\int \frac{1}{2}x^2 dx$

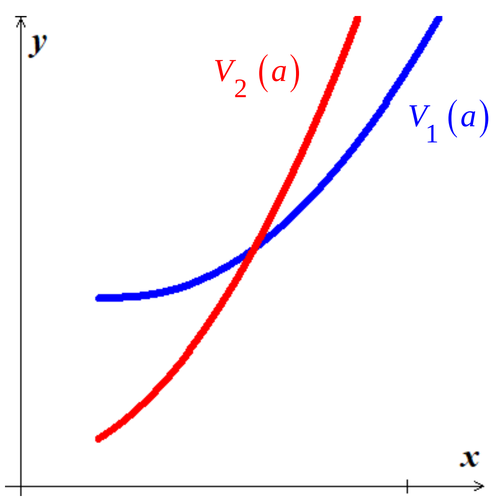
$$V_2(a) = 2\pi\left(\frac{1}{2}x^2\ln x\Big|_1^a - \frac{1}{2}\int_1^a x\, dx\right)$$

$$= 2\pi\left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right)\Big|_1^a$$

$$= \pi\left(a^2\ln a - \frac{1}{2}a^2 - \ln 1 + \frac{1}{2}\right)$$

$$\boxed{= \frac{\pi}{2}\left(2a^2\ln a - a^2 + 1\right)}$$

c)



$V_2(a) > V_1(a)$ when $a > 1$

Exercise

Let R be the region bounded by the graph of $f(x)=x^{-p}$ and the x-$axis$, for $x \ge 1$. Let V_1 and V_2 be the volumes of the solids generated when R is revolved about the x-$axis$ and the y-$axis$, respectively, if they exist.

a) For what values of p (if any) is $V_1 = V_2$?

b) Repeat part (a) on the interval $(0,\ 1]$.

Solution

a) $p = ?\ \ if\ \ V_1 = V_2$

$$V_1 = \pi \int_1^\infty \left(x^{-p}\right)^2 dx$$

$$= \pi \int_1^\infty x^{-2p}\, dx$$

$$= \frac{\pi}{1-2p} x^{-2p+1} \Big|_1^\infty$$

If $-2p+1 \ge 0 \quad \rightarrow \quad p \le \frac{1}{2}$

$\boxed{V_1 = \infty}$

If $p > \frac{1}{2}$

$$V_1 = \frac{\pi}{1-2p}(0-1)$$

$$\boxed{= \frac{\pi}{2p-1}}$$

$$V_2 = 2\pi \int_1^\infty x^{1-p}\, dx$$

$$= \frac{2\pi}{2-p} x^{2-p} \Big|_1^\infty$$

If $2 - p \ge 0 \quad \rightarrow \quad p \le 2$

$\boxed{V_2 = \infty}$

If $p > 2$

$$V_2 = \frac{2\pi}{2-p}(0-1)$$

$$\boxed{= \frac{2\pi}{p-2}}$$

$V_1 = V_2$

$\frac{\pi}{2p-1} = \frac{2\pi}{2-p}$

$\frac{1}{2p-1} = \frac{2}{2-p}$

$2-p = 4p-2$

$5p = 4$

$p = \frac{4}{5} < 2$

$\therefore V_1 = V_2$ only when both volumes are infinite.

b) $V_1 = \pi \int_0^1 x^{-2p}\, dx$

$= \frac{\pi}{1-2p} x^{-2p+1} \Big|_0^1$

$= \frac{\pi}{1-2p}(1-0)$

$\boxed{= \frac{\pi}{1-2p}}$

$V_2 = 2\pi \int_0^1 x^{1-p}\, dx$

$= \frac{2\pi}{2-p} x^{2-p} \Big|_0^1$

$= \frac{2\pi}{2-p}(1-0)$

$\boxed{= \frac{2\pi}{2-p}}$

$\frac{\pi}{1-2p} = \frac{2\pi}{2-p}$

$\frac{1}{1-2p} = \frac{2}{2-p}$

$2-p = 2-4p$

$\boxed{p = 0}$

$\therefore V_1 \neq V_2$ for any p.

Exercise

Let R_1 be the region bounded by the graph of $y=e^{-ax}$ and the *x-axis* on the interval $[0,\ b]$ where $a>0$ and $b>0$. Let R_2 be the region bounded by the graph of $y=e^{-ax}$ and the *x-axis* on the interval $[b,\ \infty)$. Let V_1 and V_2 be the volumes of the solids generated when R_1 and R_2 are revolved about the *x-axis*. Find and graph the relationship between a and b for which $V_1=V_2$.

Solution

Given: $R_1:\ \ y=e^{-ax}\quad [0,\ b]$

$R_2:\ \ y=e^{-ax}\quad [b,\ \infty]$

$$V_1=\pi\int_0^b\left(e^{-ax}\right)^2dx$$

$$=\pi\int_0^b e^{-2ax}\,dx$$

$$=-\frac{\pi}{2a}e^{-2ax}\Big|_0^b$$

$$\boxed{=-\frac{\pi}{2a}\left(e^{-2ab}-1\right)}$$

$$V_2=\pi\int_b^\infty\left(e^{-ax}\right)^2dx$$

$$=\pi\int_b^\infty e^{-2ax}\,dx$$

$$=-\frac{\pi}{2a}e^{-2ax}\Big|_b^\infty$$

$$\boxed{=\frac{\pi}{2a}e^{-2ab}}$$

$$V_1=V_2$$

$$-\frac{\pi}{2a}\left(e^{-2ab}-1\right)=\frac{\pi}{2a}e^{-2ab}$$

$$1-e^{-2ab}=e^{-2ab}$$

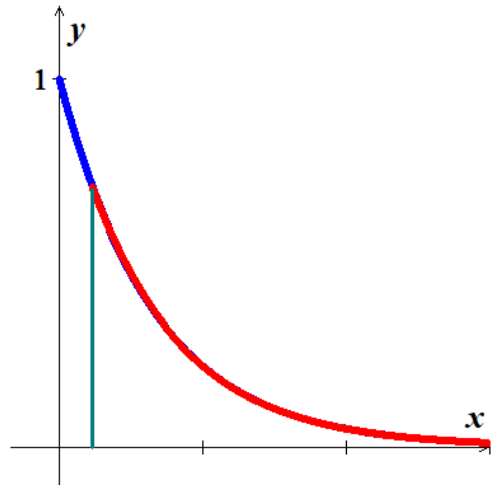
$$2e^{-2ab}=1$$

$$\frac{1}{e^{2ab}}=\frac{1}{2}$$

$$e^{2ab} = 2$$

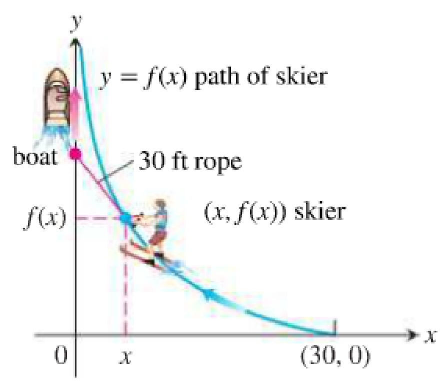
$$2ab = \ln 2$$

$$\boxed{ab = \frac{1}{2}\ln 2}$$



Exercise

Suppose that a boat is positioned at the origin with a water skier tethered to the boat at the point (30, 0) on a rope 30 *ft*. long. As the boat travels along the positive y-axis, the skier is pulled behind the boat along an unknown path $y = f(x)$, as shown



a) Show that $f'(x) = \dfrac{-\sqrt{900 - x^2}}{x}$

(*Hint*: Assume that the skier is always pointed directly at the boat and the rope is on line is on a line tangent to the path $y = f(x)$.)

b) Solve the equation in part (a) for $f(x)$, using $f(30) = 0$

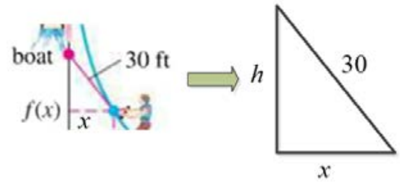
Solution

a) From the triangle: $h^2 + x^2 = 30^2 \quad \Rightarrow \quad h = \sqrt{900 - x^2}$

The slope of the tangent line (line is going down) is: $m = \dfrac{-\sqrt{900 - x^2}}{x}$

Thus, $f'(x)=\dfrac{-\sqrt{900-x^2}}{x}$

b) $f(x)=\displaystyle\int \frac{-\sqrt{900-x^2}}{x}\,dx$



$$x=30\sin\theta \quad \to dx=30\cos\theta\, d\theta, \quad 0<\theta<\frac{\pi}{2}$$

$$\sqrt{900-x^2}=\sqrt{900-900\sin^2\theta}=30\cos\theta$$

$$f(x)=-\int \frac{30\cos\theta}{30\sin\theta}(30\cos\theta)\,d\theta$$

$$=-30\int \frac{\cos^2\theta}{\sin\theta}\,d\theta$$

$$=-30\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta$$

$$=-30\int (\csc\theta-\sin\theta)\,d\theta$$

$$=-\ln\left|\csc\theta+\cot\theta\right|-30\cos\theta+C$$

$$=-\ln\left|\frac{30}{x}+\frac{\sqrt{900-x^2}}{x}\right|-\sqrt{900-x^2}+C$$

Given (30, 0), then

$$0=-\ln\left|\frac{30}{30}+\frac{\sqrt{900-30^2}}{30}\right|-\sqrt{900-30^2}+C$$

$$0=-\ln|1|+C \quad \Rightarrow \quad \underline{C=0}$$

$$\boxed{f(x)=-\ln\left|\frac{30}{x}+\frac{\sqrt{900-x^2}}{x}\right|-\sqrt{900-x^2}}$$

Exercise

Many chemical reactions are the result of the interaction of 2 molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If a is the amount of substance A and b is the substance B at time $t = 0$, and if x is the amount of product at time t, then the rate of formation of x may be given by the differential equation

$$\frac{dx}{dt} = k(a-x)(b-x) \quad \text{or} \quad \frac{1}{(a-x)(b-x)}\frac{dx}{dt} = k$$

Where k is a constant for the reaction. Integrate both sides of this equation to obtain a relation between x and t.

a) If $a = b$

b) If $a \neq b$

Assume in each case that $x = 0$ when $t = 0$

Solution

$$\frac{1}{(a-x)(b-x)}dx = kdt$$

a) $a = b \quad \Rightarrow \quad \frac{1}{(a-x)^2}dx = kdt$

$$\int \frac{1}{(a-x)^2}dx = \int kdt$$

$$\frac{1}{a-x} = kt + C$$

$$x(t=0) = 0 \quad \Rightarrow \quad \frac{1}{a} = C$$

$$\frac{1}{a-x} = kt + \frac{1}{a} = \frac{kat+1}{a}$$

$$a - x = \frac{a}{kat+1}$$

$$x = a - \frac{a}{kat+1}$$

$$= \frac{a^2kt}{kat+1}$$

b) $a \neq b \quad \Rightarrow \quad \frac{1}{(a-x)(b-x)}dx = kdt$

$$\int \frac{1}{(a-x)(b-x)}dx = \int kdt$$

$$\frac{1}{(a-x)(b-x)} = \frac{A}{a-x} + \frac{B}{b-x}$$

$$\begin{cases} -A-B=0 \\ bA+aB=1 \end{cases} \rightarrow \begin{cases} B=\dfrac{1}{a-b} \\ A=-\dfrac{1}{a-b} \end{cases}$$

$$\frac{-1}{a-b}\int \frac{1}{a-x}dx+\frac{1}{a-b}\int \frac{1}{b-x}dx=\int kdt$$

$$\frac{1}{a-b}\ln|a-x|-\frac{1}{a-b}\ln|b-x|=kt+C$$

$$\frac{1}{a-b}\ln\left|\frac{a-x}{b-x}\right|=kt+C$$

$$x(0)=0 \quad \Rightarrow \quad \boxed{\frac{1}{a-b}\ln\left(\frac{a}{b}\right)=C}$$

$$\frac{1}{a-b}\ln\left|\frac{a-x}{b-x}\right|=kt+\frac{1}{a-b}\ln\left(\frac{a}{b}\right)$$

$$\ln\left|\frac{a-x}{b-x}\right|=(a-b)kt+\ln\left(\frac{a}{b}\right)$$

$$\frac{a-x}{b-x}=e^{(a-b)kt+\ln\left(\frac{a}{b}\right)}$$

$$\frac{a-x}{b-x}=\frac{a}{b}e^{(a-b)kt}$$

$$a-x=b\frac{a}{b}e^{(a-b)kt}-x\frac{a}{b}e^{(a-b)kt}$$

$$x\left(\frac{a}{b}e^{(a-b)kt}-1\right)=ae^{(a-b)kt}-a$$

$$\boxed{x=\frac{abe^{(a-b)kt}-ab}{ae^{(a-b)kt}-b}}$$