Solution   Section 2.10 – Related Rates

Exercise

If $y = x^2$ and $\frac{dx}{dt} = 3$, then what is $\frac{dy}{dt}$ *when* $x = -1$

Solution

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$
$$= 2x(3)$$
$$= 6x$$

$$\left.\frac{dy}{dt}\right|_{x=-1} = 6(-1)$$
$$= -6$$

Exercise

If $x = y^3 - y$ and $\frac{dy}{dt} = 5$, then what is $\frac{dx}{dt}$ *when* $y = 2$

Solution

$$\frac{dx}{dt} = \frac{dx}{dy}\frac{dy}{dt}$$
$$= \left(3y^2 - 1\right)(5)$$
$$= 5\left(3y^2 - 1\right)$$

$$\left.\frac{dx}{dt}\right|_{y=2} = 5\left(3(2)^2 - 1\right)$$
$$= 55$$

Exercise

A cube's surface area increases at the rate of 72 $in^2 / \sec$. At what rate is the cube's volume changing when the edge length is $x = 3$ *in*?

Solution

Cube's surface: $S = 6x^2$

$$\frac{dS}{dt} = 12x\frac{dx}{dt}$$
$$72 = 12x(3)$$
$$x = \frac{72}{26}$$

$\underline{= 2}$

$V = x^3$

$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$

$\left.\frac{dV}{dt}\right|_{x=3} = 3(3)^2(2)$

$\boxed{= 54 \ \ in^2 / sec}$

Exercise

The radius r and height h of a right circular cone are related to the cone's volume V by the equation $V = \frac{1}{3}\pi r^2 h$.

a) How is $\frac{dV}{dt}$ related to $\frac{dh}{dt}$ if $\boldsymbol{r}$ is constant?

b) How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ if $\boldsymbol{h}$ is constant?

c) How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ and $\frac{dh}{dt}$ if neither $\boldsymbol{r}$ nor $\boldsymbol{h}$ is constant?

Solution

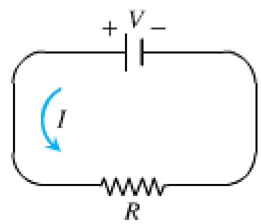
a) $\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt}$

b) $\frac{dV}{dt} = \frac{2}{3}\pi rh \frac{dr}{dt}$

c) $\frac{dV}{dt} = \frac{2}{3}\pi rh \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}$

Exercise

The voltage $\boldsymbol{V}$ (volts), current $\boldsymbol{I}$ (amperes), and resistance $\boldsymbol{R}$ (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 volt/sec while I is decreasing at the rate of $\frac{1}{3} amp / \sec$. Let t denote time in seconds.



a) What is the value of $\frac{dV}{dt}$?

b) What is the value of $\frac{dI}{dt}$?

c) What equation relates $\frac{dR}{dt}$ to $\frac{dV}{dt}$ and $\frac{dI}{dt}$?

d) Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amp. Is R increasing or decreasing?

Solution

a) $\boxed{\frac{dV}{dt} = 1 \ \ volt / sec}$

b) $\boxed{\dfrac{dI}{dt} = \dfrac{1}{3} \ \ amp / sec}$

c) $\dfrac{dV}{dt} = R\dfrac{dI}{dt} + I\dfrac{dR}{dt}$

$I\dfrac{dR}{dt} = \dfrac{dV}{dt} - R\dfrac{dI}{dt}$ $\qquad V = IR \ \Rightarrow \ R = \dfrac{V}{I}$

$\dfrac{dR}{dt} = \dfrac{1}{I}\left(\dfrac{dV}{dt} - \dfrac{V}{I}\dfrac{dI}{dt}\right)$

d) $\dfrac{dR}{dt} = \dfrac{1}{2}\left((1) - \dfrac{12}{2}\left(-\dfrac{1}{3}\right)\right)$

$\boxed{= \dfrac{3}{2} \ \ ohms / sec}$

R is increasing

Exercise

Let x and y be differentiable functions of t and let $s = \sqrt{x^2 + y^2}$ be the distance between the points $(x, 0)$ and $(0, y)$ in the xy–plane.

a) How is $\dfrac{ds}{dt}$ related to $\dfrac{dx}{dt}$ if y is constant?

b) How is $\dfrac{ds}{dt}$ related to $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ if neither x nor y is constant?

c) How is $\dfrac{dx}{dt}$ related to $\dfrac{dy}{dt}$ if s is constant?

Solution

$$s = \sqrt{x^2 + y^2} = \left(x^2 + y^2\right)^{1/2}$$

a) $\dfrac{ds}{dt} = \dfrac{1}{2}\left(x^2 + y^2\right)^{-1/2}\left(2x\dfrac{dx}{dt}\right)$

$\boxed{= \dfrac{x}{\sqrt{x^2+y^2}}\dfrac{dx}{dt}}$

b) $\dfrac{ds}{dt} = \dfrac{1}{2}\left(x^2 + y^2\right)^{-1/2}\left(2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt}\right)$

$= \dfrac{1}{\sqrt{x^2+y^2}}\left(x\dfrac{dx}{dt} + y\dfrac{dy}{dt}\right)$

$\boxed{= \dfrac{x}{\sqrt{x^2+y^2}}\dfrac{dx}{dt} + \dfrac{y}{\sqrt{x^2+y^2}}\dfrac{dy}{dt}}$

c) $s = \sqrt{x^2 + y^2}$

$$s^2 = x^2 + y^2$$

$$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$2x\frac{dx}{dt} = -2y\frac{dy}{dt}$$

$$\boxed{\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}}$$

Exercise

A 13-*foot* ladder is leaning against a house when its base starts to slide away. By the time the base is 12 *feet* from the house, the base is moving at the rate of 5 *ft/sec*.

a) How fast is the top of the ladder sliding down the wall then?

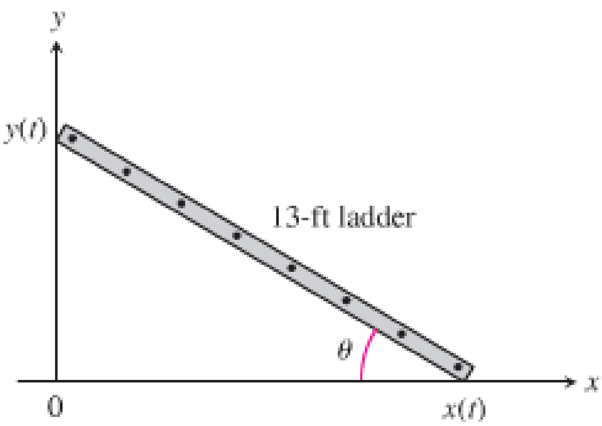
b) At what rate is the area of the triangle formed by the ladder, wall, and the ground changing then?

c) At what rate is the angle θ between the ladder and the ground changing then?

Solution

Given: $L = 13\ ft \quad x = 12 \quad \frac{dx}{dt} = 5\ ft / \sec$

$$y = \sqrt{13^2 - 12^2} = 5$$



a) $x^2 + y^2 = 13^2$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$y\frac{dy}{dt} = -x\frac{dx}{dt}$$

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

$$= -\frac{12}{5}(5)$$

$$\boxed{= -12\ \ ft / sec}$$

The ladder is sliding down the wall

b) Area of the triangle formed by the ladder and the walls is: $A = \frac{1}{2}xy$

$$\frac{dA}{dt} = \frac{1}{2}\left(y\frac{dx}{dt} + x\frac{dy}{dt}\right)$$

$$= \frac{1}{2}\left((5)(5) + (12)(-12)\right)$$

$$\boxed{= -19.5\ \ ft^2 / sec}$$

c) $\cos\theta = \frac{x}{13}$

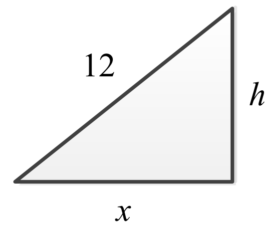
$$-\sin\theta \frac{d\theta}{dt} = \frac{1}{13}\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = -\frac{1}{13\sin\theta}\frac{dx}{dt}$$

$$= -\frac{1}{13\sin\theta}(5) \qquad \sin\theta = \frac{5}{13}$$

$$= -\frac{1}{13\left(\frac{5}{13}\right)}(5)$$

$$\boxed{= -1 \ \ rad / sec}$$



Exercise

A 13-*feet* ladder is leaning against a vertical wall when he begins pulling the foot of the ladder away from the wall at a rate of 0.5 *ft/s*. How fast is the top of the ladder sliding down the wall when the foot of the ladder is 5 *feet* from the wall?

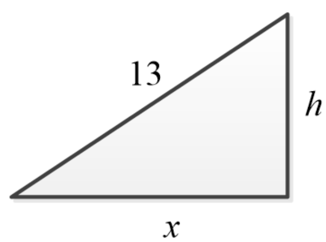
Solution

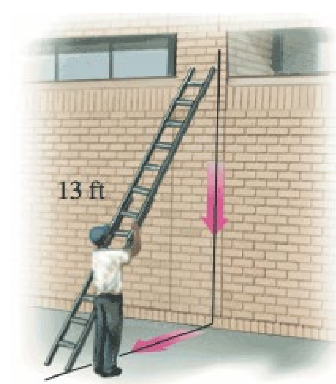
$$x^2 + h^2 = 13^2$$

$$x^2 + h^2 = 169$$

$$h = \sqrt{169 - 25}$$

$$\boxed{= 12}$$





$$2x\frac{dx}{dt} + 2h\frac{dh}{dt} = 0$$

$$\frac{dh}{dt} = -\frac{x}{h}\frac{dx}{dt}$$

$$= -\frac{5}{12}(0.5)$$

$$\boxed{= -\frac{5}{24} \ \ ft / sec}$$

So, the top of the ladder slides down the wall at $\frac{5}{24}$ *ft* / sec

Exercise

A 12-*feet* ladder is leaning against a vertical wall when he begins pulling the foot of the ladder away from the wall at a rate of 0.2 *ft/s*. What is the configuration of the ladder at the instant that the vertical speed of the top of the ladder equals the horizontal speed of the foot of the ladder?

Solution

$$x^2 + h^2 = 144$$

$$2x\frac{dx}{dt} + 2h\frac{dh}{dt} = 0$$

$$x\frac{dx}{dt} + h\frac{dh}{dt} = 0$$

The vertical speed of the top of the ladder equals the horizontal speed of the foot of the ladder.

$$\frac{dx}{dt} = 0.2$$

$$\frac{dh}{dt} = -0.2$$

$$0.2x - 0.2h = 0$$

$$x = h$$

Since $x = h$, the triangle is forming a $(45° - 45° - 90°)$ with

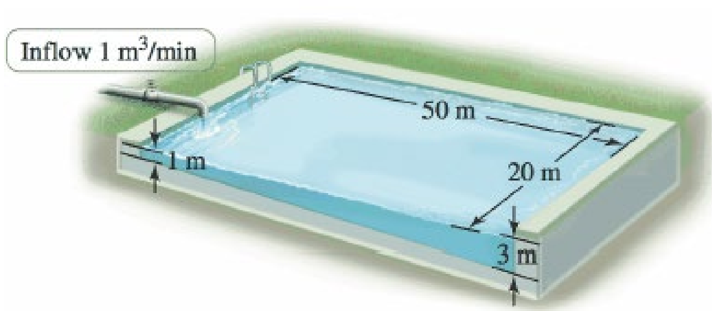
$$\begin{aligned} x = h &= 12\cos 45° \\ &= 6\sqrt{2} \end{aligned}$$

Exercise

A swimming pool is 50 *m* long and 20 *m* wide. Its length decreases linearly along the length from 3 *m* to 1 *m*. It is initially empty and is filed at a rate of 1 $m^3/\min$.

a) How fast is the water level rising 250 min after the filling begins?

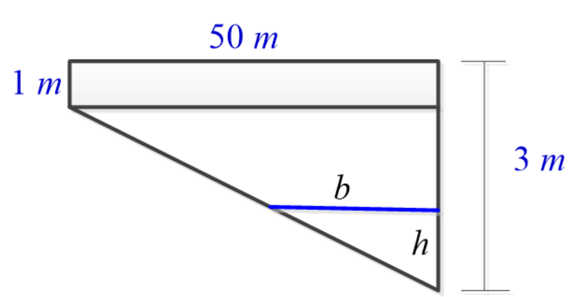
b) How long will it take to fill the pool?



Solution

$$\frac{h}{2} = \frac{b}{50}$$

$$b = 25h$$



The area of the side:

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{25}{2}h^2 \end{aligned}$$

For $0 \le h \le 2$

$$\begin{aligned} V(h) &= 12.5h^2(20) \\ &= 250h^2 \end{aligned}$$

For $2 < h \le 3$

$$V(h) = 250\times 2^2 + 50\times 20\times (h-2)$$

$$\boxed{= 1000h - 1000}$$

a) When $t = 250$ *min*

$$V = 250\ min \times 1\frac{m^3}{min}$$

$$\boxed{= 250\ m^3}$$

So $V(h) = 250h^2 = 250$

$\rightarrow \boxed{h = 1}$

$$\frac{dV}{dt} = 500h\frac{dh}{dt}$$

$$\boxed{= 1\ \frac{m^3}{min}}$$

$$\frac{dh}{dt} = \frac{1}{500}\ \frac{m}{\min}$$

$$\boxed{= .002\ \frac{m}{\min}}$$

b) $V(h) = 1000(3) - 1000$

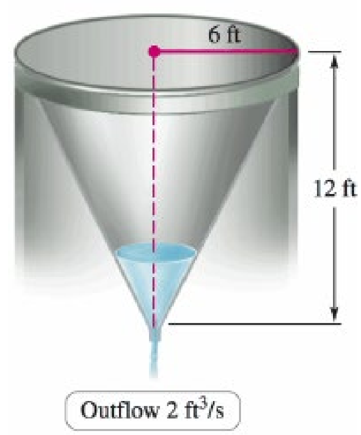
$$= 2000\ m^3$$

Since $\frac{dV}{dt} = 1\frac{m^3}{min}$

Then it will take 2,000 *minutes*.

Exercise

An inverted conical water tank with a height of 12 *feet* and a radius of 6 *feet* is drained through a hole in the vertex at a rate of 2 $ft^3/\sec$. What is the rate of change of the water depth when the water depth is 3 *feet*?



Solution

Given: $\frac{dV}{dt} = -2\frac{ft^3}{min}$

The water forms a cone with volume:

$$V = \frac{1}{3}\pi r^2 h$$

From the triangles:

$$\frac{r}{6} = \frac{h}{12} \quad \Rightarrow \quad r = \frac{1}{2}h$$

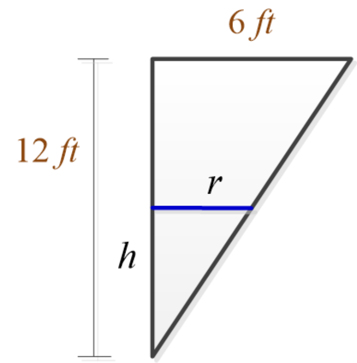
$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

$$= \boxed{\frac{1}{12}\pi h^3}$$

$$\frac{dV}{dt} = \frac{\pi h^2}{4} \cdot \frac{dh}{dt}$$

$$-2 = \frac{\pi 3^2}{4} \cdot \frac{dh}{dt}$$

$$\boxed{\frac{dh}{dt} = -\frac{8}{9\pi} \ ft/s}$$



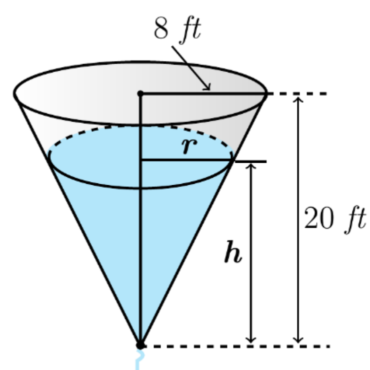
So, the depth of the water is decreasing at a rate of $\frac{8}{9\pi} \ ft/s$

Exercise

Water runs into a conical tank at the rate of $6\ ft^3/min$. The tank stands point down and has a height of 20 *feet* and a base radius of 8 *feet*. How fast is the water level rising when the water is 6 *feet* deep?

Solution

Given: $\frac{dV}{dt} = 6\ \frac{ft^3}{min}, \quad y = 20\ ft, \quad x = 8\ ft, \quad h = 6\ ft$



The water forms a cone with volume:

$$V = \frac{1}{3}\pi r^2 h$$

From the triangles:

$$\frac{r}{h} = \frac{8}{20} \quad \Rightarrow \quad r = \frac{2}{5}h$$

$$V = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h$$

$$= \boxed{\frac{4\pi}{75}h^3}$$

$$\frac{dV}{dt} = \frac{4\pi}{75}\left(3h^2\frac{dh}{dt}\right)$$

$$\frac{dV}{dt} = \frac{4\pi}{25}h^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{25}{4\pi h^2}\frac{dV}{dt}$$

$$= \frac{25}{4\pi(36)}(6)$$

$$= \boxed{\frac{25}{24\pi} \ ft/min}$$

The water level is rising at about 0.33157 ***ft/min***.

Exercise

Water runs into a conical tank at the rate of $5\ ft^3/min$. The tank stands point down and has a height of 10 *feet* and a base radius of 4 *feet*. How fast is the water level rising when the water is 6 *feet* deep?

Solution

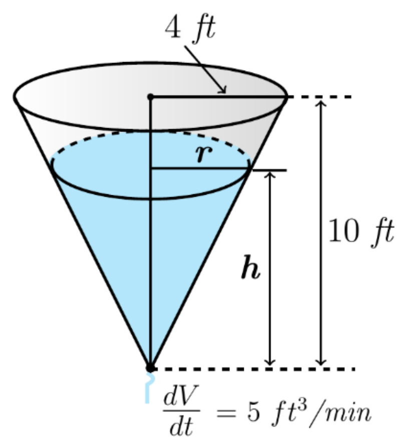
Given: $\frac{dV}{dt} = 5\ \frac{ft^3}{min}, \quad y = 10\ ft, \quad x = 4\ ft, \quad h = 6\ ft$

The water forms a cone with volume:

$$V = \frac{1}{3}\pi r^2 h$$

From the triangles:

$$\frac{r}{h} = \frac{4}{10} \quad \Rightarrow \quad r = \frac{2}{5}h$$



$$V = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h$$

$$= \frac{4\pi}{75}h^3$$

$$\frac{dV}{dt} = \frac{4\pi}{75}\left(3h^2\frac{dh}{dt}\right)$$

$$\frac{dV}{dt} = \frac{4\pi}{25}h^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{25}{4\pi h^2}\frac{dV}{dt}$$

$$= \frac{25}{4\pi(36)}(5)$$

$$= \frac{125}{144\pi}\ ft/min$$

The water level is rising at about 0.2763 ***ft/min***.

Exercise

Water runs into a conical tank at the rate of $5\ ft^3/min$. The tank stands point down and has a height of 20 *feet* and a base radius of 5 *feet*. How fast is the water level rising when the water is 4 *feet* deep?

Solution

Given: $\frac{dV}{dt} = 5\ \frac{ft^3}{min}, \quad y = 20\ ft, \quad x = 5\ ft, \quad h = 4\ ft$

The water forms a cone with volume:

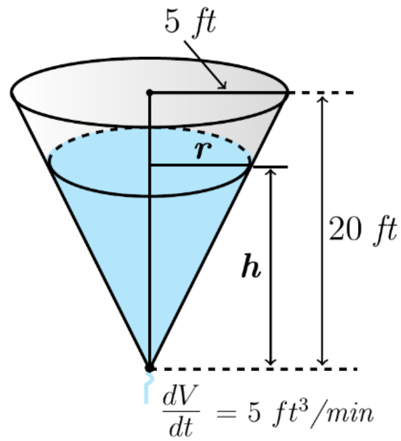
$$V = \frac{1}{3}\pi r^2 h$$

From the triangles:

$$\frac{r}{h} = \frac{5}{20} \quad \Rightarrow \quad r = \frac{1}{4}h$$

$$V = \frac{1}{3}\pi\left(\frac{1}{4}h\right)^2 h$$

$$\boxed{= \frac{\pi}{48}h^3}$$



$$\frac{dV}{dt} = \frac{\pi}{48}\left(3h^2\frac{dh}{dt}\right)$$

$$\frac{dV}{dt} = \frac{\pi}{16}h^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{16}{\pi h^2}\frac{dV}{dt}$$

$$= \frac{16}{\pi(16)}(5)$$

$$\boxed{= \frac{5}{\pi}\ ft/min}$$

The water level is rising at about 1.59155 ***ft/min***.

Exercise

Water runs into a conical tank at the rate of $4\ ft^3/min$. The tank stands point down and has a height of 20 *feet* and a base radius of 5 *feet*. How fast is the water level rising when the water is 5 *feet* deep?

Solution

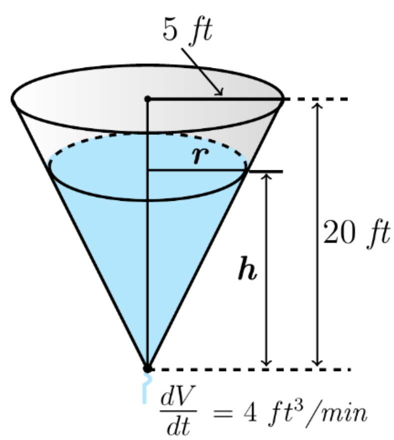
Given: $\frac{dV}{dt} = 4\ \frac{ft^3}{min}, \quad y = 20\ ft, \quad x = 5\ ft, \quad h = 5\ ft$

The water forms a cone with volume:

$$V = \frac{1}{3}\pi r^2 h$$

From the triangles:

$$\frac{r}{h} = \frac{5}{20} \quad \Rightarrow \quad r = \frac{1}{4}h$$



$$V = \frac{1}{3}\pi\left(\frac{1}{4}h\right)^2 h$$

$$\boxed{= \frac{\pi}{48}h^3}$$

$$\frac{dV}{dt} = \frac{\pi}{48}\left(3h^2\frac{dh}{dt}\right)$$

$$\frac{dV}{dt} = \frac{\pi}{16}h^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{16}{\pi h^2}\frac{dV}{dt}$$

$$= \frac{16}{\pi(25)}(4)$$

$$= \boxed{\frac{64}{25\pi}\ ft/min}$$

The water level is rising at about 0.814873 ***ft/min***.

Exercise

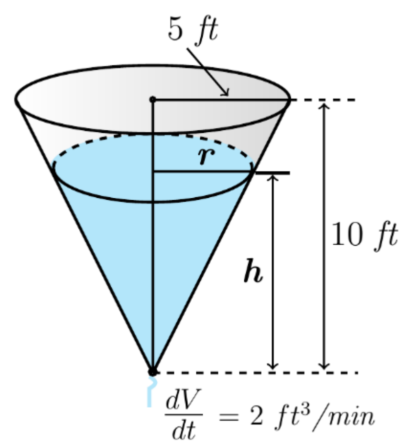
Water runs into a conical tank at the rate of $2\ ft^3/min$. The tank stands point down and has a height of 10 *feet* and a base radius of 5 *feet*. How fast is the water level rising when the water is 4 *feet* deep?

Solution

Given: $\frac{dV}{dt} = 2\ \frac{ft^3}{min}, \quad y = 10\ ft, \quad x = 5\ ft, \quad h = 4\ ft$

The water forms a cone with volume:

$$V = \frac{1}{3}\pi r^2 h$$



From the triangles:

$$\frac{r}{h} = \frac{5}{10} \quad \Rightarrow \quad r = \frac{1}{2}h$$

$$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$$

$$= \boxed{\frac{\pi}{12}h^3}$$

$$\frac{dV}{dt} = \frac{\pi}{12}\left(3h^2\frac{dh}{dt}\right)$$

$$= \frac{\pi}{4}h^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

$$= \frac{4}{\pi(16)}(2)$$

$$= \boxed{\frac{1}{2\pi}\ ft/min}$$

The water level is rising at about 0.159155 ***ft/min***.

Exercise

Water runs into a conical tank at the rate of $3\ ft^3/min$. The tank stands point down and has a height of 18 *feet* and a base radius of 6 *feet*. How fast is the water level rising when the water is 6 *feet* deep?

Solution

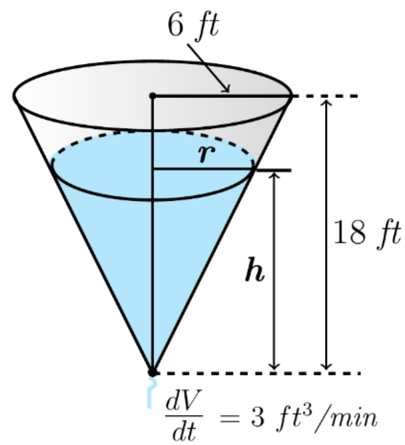
Given: $\frac{dV}{dt} = 3\ \frac{ft^3}{min}, \quad y = 18\ ft, \quad x = 6\ ft, \quad h = 6\ ft$

The water forms a cone with volume:

$$V = \frac{1}{3}\pi r^2 h$$

From the triangles:

$$\frac{r}{h} = \frac{6}{18} \Rightarrow r = \frac{1}{3}h$$



$$V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h$$

$$= \frac{\pi}{27}h^3$$

$$\frac{dV}{dt} = \frac{\pi}{27}\left(3h^2\frac{dh}{dt}\right)$$

$$= \frac{\pi}{9}h^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{9}{\pi h^2}\frac{dV}{dt}$$

$$= \frac{9}{\pi(36)}(3)$$

$$= \frac{3}{4\pi}\ ft/min$$

The water level is rising at about 0.23873 ***ft/min***.

Exercise

Water runs into a conical tank at the rate of $6\ ft^3/min$. The tank stands point down and has a height of 18 *feet* and a base radius of 6 *feet*. How fast is the water level rising when the water is 12 *feet* deep?

Solution

Given: $\frac{dV}{dt} = 6\ \frac{ft^3}{min}, \quad y = 18\ ft, \quad x = 6\ ft, \quad h = 12\ ft$

The water forms a cone with volume:

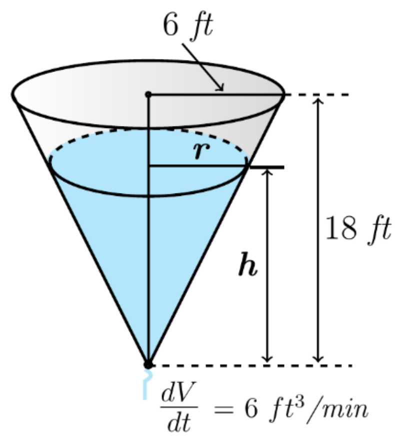
$$V = \frac{1}{3}\pi r^2 h$$

From the triangles:

$$\frac{r}{h} = \frac{6}{18} \Rightarrow r = \frac{1}{3}h$$

$$V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h$$

$$= \boxed{\frac{\pi}{27}h^3}$$



$$\frac{dV}{dt} = \frac{\pi}{27}\left(3h^2\frac{dh}{dt}\right)$$

$$= \frac{\pi}{9}h^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{9}{\pi h^2}\frac{dV}{dt}$$

$$= \frac{9}{\pi(144)}(6)$$

$$= \boxed{\frac{3}{8\pi}\ ft/min}$$

The water level is rising at about 0.119366 ***ft/min***.

Exercise

A hemispherical tank with a radius of 10 *m* is filled from an inflow pipe at a rate of 3 m^3/min. (Hint: The volume of a cap of thickness *h* sliced from a sphere of radius *r* is $\frac{\pi h^2(3r-h)}{3}$).

a) How fast is the water level rising when the water level is 5 *m* from the bottom of the tank?

b) What is the rate of change of the surface area of the water when the water is 5 *m* deep?

Solution

Given: $\frac{dV}{dt} = 3\frac{m^3}{min}$, $r = 10\ m$

a) $V(h) = \frac{1}{3}\pi h^2(3r-h)$

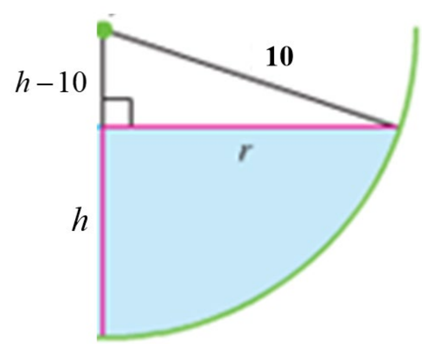
$$= 10\pi h^2 - \frac{1}{3}\pi h^3$$

$$\frac{dV}{dt} = \left(20\pi h - \pi h^2\right)\frac{dh}{dt}$$

When $h = 5\ m$

$$3 = (100-25)\pi\frac{dh}{dt}$$

$$\boxed{\frac{dh}{dt} = \frac{3}{75\pi}\ m/min}$$



b) $S = \pi r^2 \quad\Rightarrow\quad \frac{dS}{dt} = 2\pi r\frac{dr}{dt}$

From the right triangle:

$$10^2 = r^2 + (10-h)^2$$

$$100 = r^2 + 100 - 20h + h^2$$

$$20h = h^2 + r^2$$

$$h = 5, \quad r = \sqrt{100 - 25} = 5\sqrt{3}$$

$$20\frac{dh}{dt} = 2h\frac{dh}{dt} + 2r\frac{dr}{dt}$$

$$10\frac{dh}{dt} = h\frac{dh}{dt} + r\frac{dr}{dt}$$

$$5\sqrt{3}\frac{dr}{dt} = 10\frac{3}{75\pi} - 5\frac{3}{75\pi}$$

$$\frac{dr}{dt} = \frac{1}{5\sqrt{3}}\frac{15}{75\pi} = \frac{\sqrt{3}}{75\pi}$$

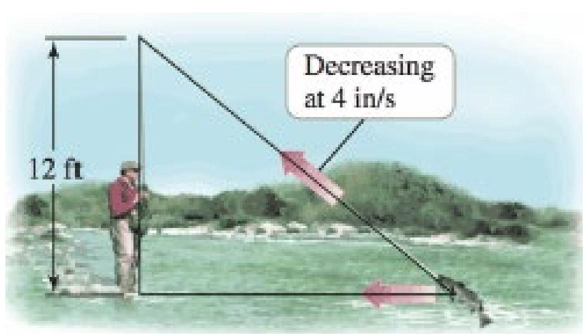
$$\frac{dS}{dt} = 2\pi r\frac{dr}{dt}$$

$$= 2\pi 5\sqrt{3}\frac{\sqrt{3}}{75\pi}$$

$$\boxed{= \frac{2}{5}\ \frac{m^2}{min}}$$

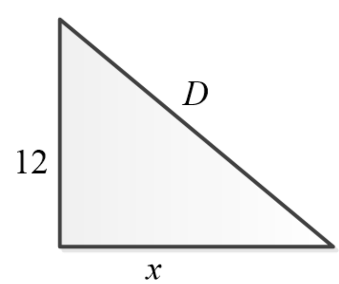
Exercise

A fisherman hooks a trout and reels in his line at 4 *in/sec*. Assume the trip of the fishing rod is 12 *feet* above the water directly above the fisherman and the fish is pulled horizontally directly towards the fisherman. Find the horizontal speed of the fish when it is 20 *feet* from the fisherman.



Solution

Let x be the distance between the fisherman's feet & fish.
Let D be the distance between the fisherman's head & the fish.



Given: $\frac{dD}{dt} = -4\ in/s, \quad x = 20\ ft$

$$D^2 = x^2 + 144$$

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{D}{x}\frac{dD}{dt}$$

$$= \frac{\sqrt{20^2 + 144}}{20}(-4)$$

$$\approx -4.66 \ \frac{in}{s}$$

The fish is moving toward the fisherman at about 4.66 *inches* per *second*.

Exercise

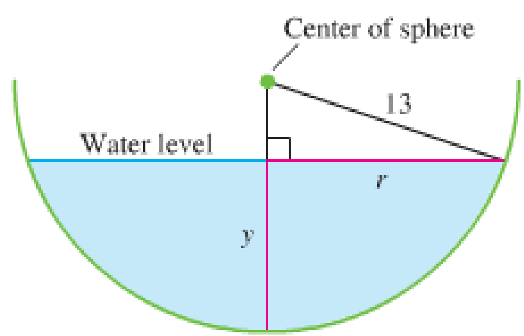
Water is flowing at the rate of 6 from a reservoir shaped like a hemispherical bowl of radius 13 ***m***. Answer the following questions, given that the volume of water in a hemispherical bowl of radius R is $V = \frac{\pi}{3}y^2(3R - y)$ when the water is y meters deep.

a) At what rate the water level changing when the water is 8 ***m*** deep?

b) What is the radius r of the water's surface when the water is y ***m*** deep?

c) At what rate is the radius r changing when the water is 8 ***m*** deep?

Solution



Given: $\frac{dV}{dt} = 6\ m^3/\min \quad R = 13\ m$

a) $V = \frac{\pi}{3}y^2(3R - y)$

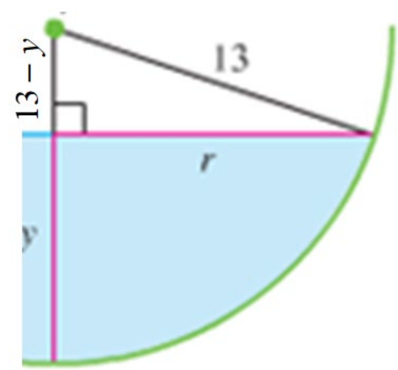
$$= \pi R y^2 - \frac{\pi}{3}y^3$$

$$\frac{dV}{dt} = \left(2\pi R y - \pi y^2\right)\frac{dy}{dt} \qquad \textit{Factor } \pi y$$

$$\frac{dV}{dt} = \pi y(2R - y)\frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{1}{\pi(8)(2(13) - (8))}(-6)$$

$$= -\frac{1}{24\pi}\ m/\min$$



b) The hemispherical is on the circle:

$$r^2 + (13 - y)^2 = 13^2$$

$$r^2 = 169 - \left(169 - 26y + y^2\right)$$

$$= 169 - 169 + 26y - y^2$$

$$= 26y - y^2$$

$$\boxed{r=\sqrt{26y-y^2}}$$

c) $r=\left(26y-y^2\right)^{1/2}$

$$\frac{dr}{dt}=\frac{1}{2}\left(26y-y^2\right)^{-1/2}(26-2y)\frac{dy}{dt}$$

$$=\frac{1}{2}\frac{26-2y}{\sqrt{26y-y^2}}\frac{dy}{dt}$$

$$\left.\frac{dr}{dt}\right|_{y=8}=\frac{1}{2}\frac{26-2(8)}{\sqrt{26(8)-(8)^2}}\left(-\frac{1}{24\pi}\right)$$

$$=-\frac{5}{288\pi}$$

$$\boxed{\approx 0.005526\ \ m/s}\quad or\quad \boxed{5.526\times10^{-3}\ \ m/s}$$

Exercise

A spherical balloon is inflated with helium at the rate of $100\pi\ ft^3/\min$. How fast is the balloon's radius increasing at the instant the radius is 5 *feet*? How fast the surface area increasing?

Solution

Given: $\quad \frac{dV}{dt}=100\pi\ \ ft^3/\min \quad r=5\,ft$

If $V=\frac{4}{3}\pi r^3$

$$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$$

$$\frac{dr}{dt}=\frac{1}{4\pi r^2}\frac{dV}{dt}$$

$$=\frac{1}{4\pi(5)^2}(100\pi)$$

$$\boxed{=1\ \ ft/min}$$

$$S=4\pi r^2$$

$$\frac{dS}{dt}=8\pi r\frac{dr}{dt}$$

$$=8\pi(5)(1)$$

$$\boxed{=40\pi\ \ ft^2/\min}$$

The rate of the surface area is increasing.

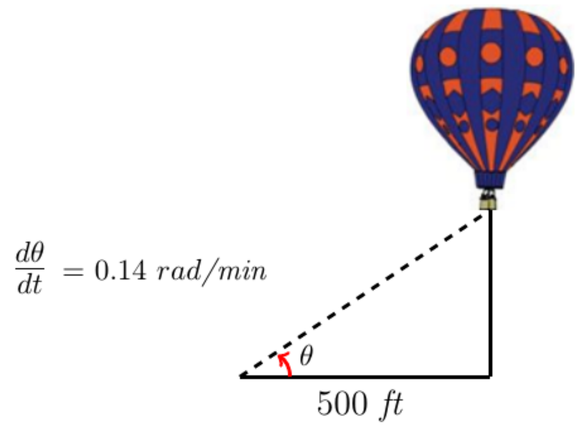
Exercise

A hot air balloon rising straight up from a level filed is tracked by a range finder 500 *feet* from the liftoff point. At the moment, the range finder's elevation angle is $\frac{\pi}{3}$, the angle is increasing at the rate of 0.14 *rad/min*. How fast is the balloon rising at that moment?

Solution

Given: $\frac{d\theta}{dt} = 0.14\ rad\ /\ \min \quad when \quad \theta = \frac{\pi}{3}$

$distance = 500\, ft$



$$\tan\theta = \frac{y}{500}$$

$$y = 500\tan\theta$$

$$\frac{dy}{dt} = 500\left(\sec^2\theta\right)\frac{d\theta}{dt}$$

$$= 500\left(\sec^2\frac{\pi}{3}\right)(0.14)$$

$$\boxed{= 280\ \ ft/min}$$

The balloon is rising at the rate of 280 *ft/min*.

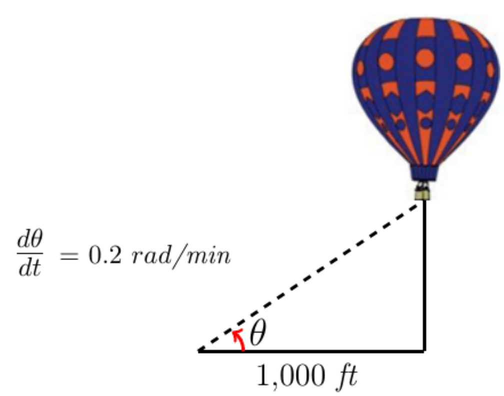
Exercise

A hot air balloon rising straight up from a level filed is tracked by a range finder 1,000 *feet* from the liftoff point. At the moment, the range finder's elevation angle is $\frac{\pi}{3}$, the angle is increasing at the rate of $0.2\ \ rad/min$. How fast is the balloon rising at that moment?

Solution

Given: $\frac{d\theta}{dt} = 0.2\ rad\ /\ \min \quad when \quad \theta = \frac{\pi}{3}$

$distance = 1{,}000\, ft$



$$\tan\theta = \frac{y}{1{,}000}$$

$$y = 1{,}000\tan\theta$$

$$\frac{dy}{dt} = 1{,}000\left(\sec^2\theta\right)\frac{d\theta}{dt}$$

$$= 1{,}000\left(\sec^2\frac{\pi}{3}\right)(0.2)$$

$$\boxed{= 800\ \ ft/min}$$

The balloon is rising at the rate of 800 *ft/min*.

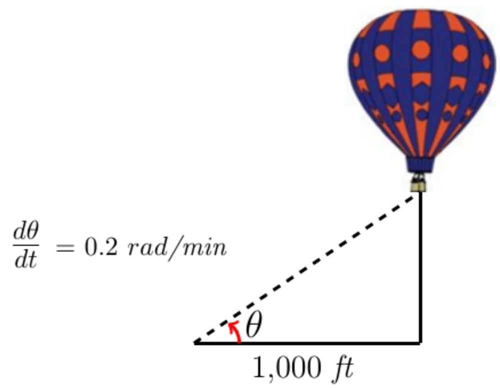
Exercise

A hot air balloon rising straight up from a level filed is tracked by a range finder 1,000 *feet* from the liftoff point. At the moment, the range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of $0.2\ \ rad/min$. How fast is the balloon rising at that moment?

Solution

Given: $\frac{d\theta}{dt}=0.2\ rad\ /\ \min \quad when \quad \theta=\frac{\pi}{4}$

$distance = 1{,}000\ ft$



$\tan\theta=\frac{y}{1{,}000}$

$y=1{,}000\tan\theta$

$$\frac{dy}{dt}=1{,}000\left(\sec^2\theta\right)\frac{d\theta}{dt}$$

$$=1{,}000\left(\sec^2\frac{\pi}{4}\right)(0.2)$$

$$=400\ \ ft/min$$

The balloon is rising at the rate of 400 *ft/min*.

Exercise

A balloon rising vertically above a level, straight road at a constant rate of 1 *ft/sec*. Just when the balloon is 65 *feet* above the ground, a bicycle moving at a constant rate of 17 *ft/sec* passes under it. How fast is the distance $s(t)$ between the bicycle and the balloon increasing 3 *sec* later?

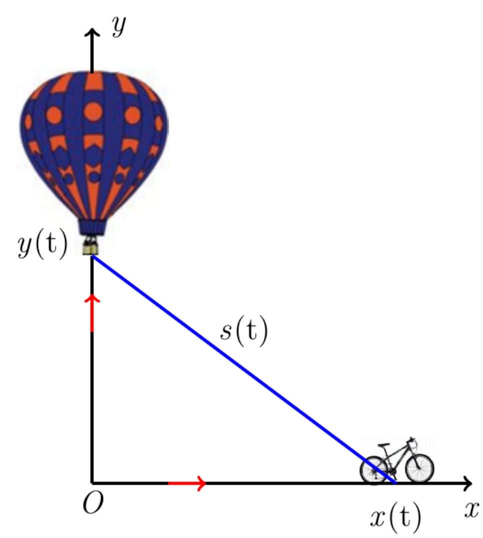
Solution

Given: $\frac{dy}{dt}=1\ \ ft\ /\ \sec \qquad y=65\ ft \qquad \frac{dx}{dt}=17\ \ ft\ /\ \sec$

Bicycle increasing 3 *sec*:

$x=vt=17(3)$

$=51\ \ ft$



$s^2=x^2+y^2$

$s=\sqrt{51^2+65^2}$

$\approx 83\ \ ft$

$$2s\frac{ds}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt}$$

$$\frac{ds}{dt}=\frac{1}{s}\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)$$

$$= \frac{1}{83}\left(51(17)+65(1)\right)$$

$$\boxed{= \frac{932}{83}\ ft/\sec}$$

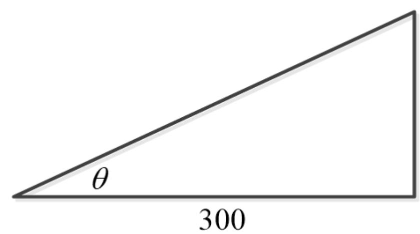
$$\boxed{\approx 11\ ft/\sec}$$

Exercise

An observer stands 300 *feet* from the launch site of a hot-air balloon. The balloon is launched vertically and maintains a constant upward velocity of 20 *ft/sec*. what is the rate of change of the angle of elevation of the balloon when it is 400 *feet* from the ground? The angle of elevation is the angle θ between the observer's line of sight to the balloon and the ground.

Solution

Given: $\frac{dh}{dt} = 20\ ft/s, \quad h = 400\ ft$



$$\tan\theta = \frac{h}{300}$$

$$\theta = \tan^{-1}\left(\frac{h}{300}\right)$$

$$\frac{d\theta}{dt} = \frac{1}{300\left(1+\left(\frac{h}{300}\right)^2\right)}\frac{dh}{dt}$$

$$= \frac{1}{300\left(1+\left(\frac{400}{300}\right)^2\right)}(20)$$

$$\boxed{\approx .024\ rad/sec}$$

Exercise

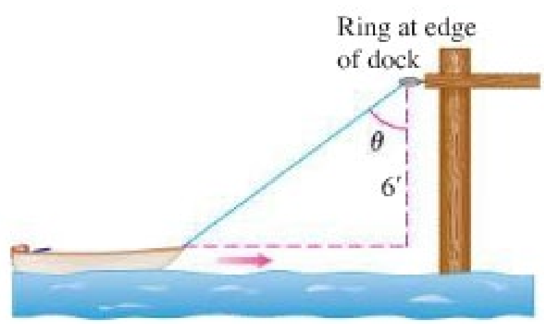
A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 *ft* above the bow. The rope is hauled in at rate of 2 *ft/sec*.

a) How fast is the boat approaching the dock when 10 *ft* of rope are out?

b) At what rate is the angle θ changing at this instant?

Solution

Given: $h = 6\ ft \quad \frac{ds}{dt} = -2\ ft/\sec$



a) $s = 10\,ft$

$$s^2 = x^2 + 6^2 \quad \Rightarrow \quad x = \sqrt{s^2 - 36}$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{s}{\sqrt{s^2 - 36}} \frac{ds}{dt}$$

$$\left.\frac{dx}{dt}\right|_{s=10} = \frac{10}{\sqrt{10^2 - 36}}(-2)$$

$$\underline{= -2.5 \ ft/\sec}$$

b) $\cos\theta = \frac{6}{s}$

$$-\sin\theta \frac{d\theta}{dt} = -\frac{6}{s^2}\frac{ds}{dt}$$

$$\frac{d\theta}{dt} = \frac{6}{\sin\theta s^2}\frac{ds}{dt} \qquad \sin\theta = \frac{x}{s} = \frac{\sqrt{10^2 - 36}}{10} = \frac{8}{10}$$

$$\frac{d\theta}{dt} = \frac{6}{(.8)10^2}(-2)$$

$$\underline{= -0.15 \ rad/\sec}$$

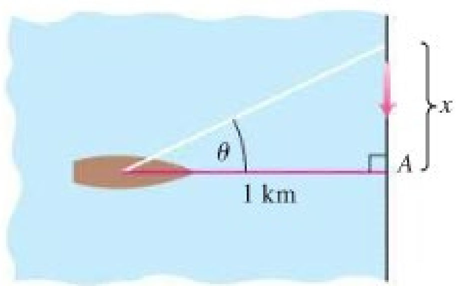
Exercise

The figure shows a boat 1 *km* offshore, sweeping the shore with a searchlight. The light turns at a constant rate $\frac{d\theta}{dt} = -0.6 \ rad / \sec$.

a) How fast is the light moving along the shore when it reaches point *A*?

b) How many revolutions per minute is 0.6 *rad/sec*?

Solution



Given: $\frac{d\theta}{dt} = -0.6 \ rad / \sec$

$$\tan\theta = \frac{x}{1}$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{dx}{dt}$$

a) At $x = 0 \Rightarrow \theta = 0$

$$\frac{dx}{dt} = \sec^2(0)(-0.6)$$

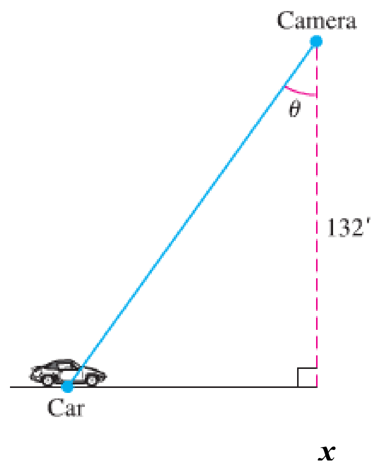
$$\underline{= -0.6}$$

$\therefore$ The speed of the light is 0.6 *km/sec* when it reaches point *A*.

b) $0.6\frac{rad}{sec}\frac{1\ rev}{2\pi\ rad}\frac{60\ sec}{1\ min} \underline{= \frac{18}{\pi} \ \frac{rev}{min}}$

Exercise

You are videotaping a race from a stand 132 *feet* from the track, following a car that is moving at 180 *mi/h* (264 *ft/sec*). How fast will your camera angle θ be changing when the car is right in front of you? A half second later?



Solution

$$\tan\theta = \frac{x}{132}$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{132}\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{132\sec^2\theta}\frac{dx}{dt}$$

$$= \frac{1}{132\sec^2(0)}(-264)$$

$$\underline{= -2 \ \ rad/\sec}$$

At half second later the car has traveled 132 *feet* right to the perpendicular

$$|\theta| = \frac{\pi}{4} \ \rightarrow \ \cos^2\theta = \frac{1}{2} \text{ , and } \frac{dx}{dt} = 264 \quad (\text{since } x \text{ increases})$$

$$\frac{d\theta}{dt} = \frac{1}{132(2)}(264)$$

$$\underline{= 1 \ \ rad/\sec}$$

Exercise

The coordinates of a particle in the metric *xy*–plane are differentiable functions of time *t* with $\frac{dx}{dt} = -1 \ m/\sec$ and $\frac{dy}{dt} = -5 \ m/\sec$. How fast is the particle's distance from the origin changing as it passes through the point (5, 12)?

Solution

Given: $\quad \frac{dx}{dt} = -1 \ \ m/\sec \quad \frac{dy}{dt} = -5 \ \ m/\sec$

$$s^2 = x^2 + y^2$$

$$s = \sqrt{x^2 + y^2} = \sqrt{5^2 + 12^2} \ \underline{= 13}$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$\left.\frac{ds}{dt}\right|_{(5,\ 12)} = \frac{1}{13}\big(5(-1) + 12(-5)\big)$$

$$\underline{= -5 \ \ m/\sec}$$

Exercise

Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 $in^3 / \min$.

a) How fast is the level in the pot rising when the coffee in the cone is 5 *in*. deep?

b) How fast is the level in the cone falling then?

Solution

$$r_{pot} = 3 \quad \frac{dV}{dt} = 10 \ in^3 / \min$$

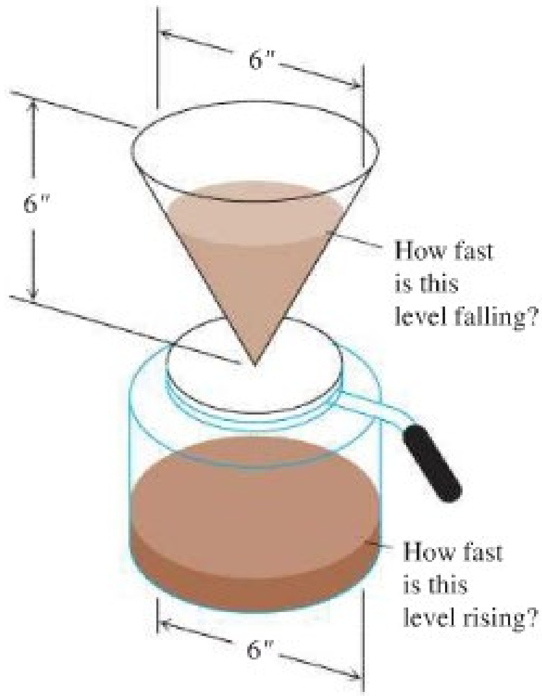
a) Let h be the height of the coffee in the pot.

Volume of the coffee:

$$V = \pi r^2 h$$
$$= 9\pi h$$

$$\frac{dV}{dt} = 9\pi \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{9\pi}\frac{dV}{dt}$$
$$= \frac{1}{9\pi}(10)$$
$$= \frac{10}{9\pi} \ in/min$$



b) Radius of the filter: $r = \frac{h}{2}$

Volume of the filter:

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$
$$= \frac{\pi h^3}{12}$$

$$\frac{dV}{dt} = \frac{\pi}{4}h^2 \frac{dh}{dt}$$

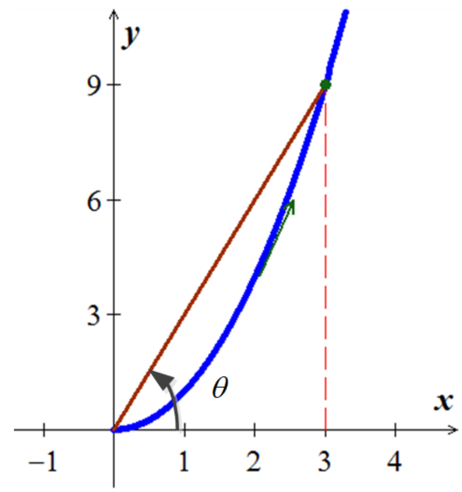
$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$
$$= \frac{4}{\pi (5)^2}(-10)$$
$$= -\frac{8}{5\pi} \ in/min$$

Exercise

A particle moves along the parabola $y = x^2$ *in* the first quadrant in such a way that its x-coordinate (measure in meters) increases at a steady 10 *m/sec*. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = 3$ *m*?

Solution

Given: $y = x^2 \quad v = \frac{dx}{dt} = 10 \ m/\sec \quad x = 3 \ m$



$$\tan\theta = \frac{y}{x} = \frac{x^2}{x} = x$$

$$\frac{d}{dt}\tan\theta = \frac{d}{dt}x$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{\sec^2\theta}\frac{dx}{dt} = \cos^2\theta \frac{dx}{dt}$$

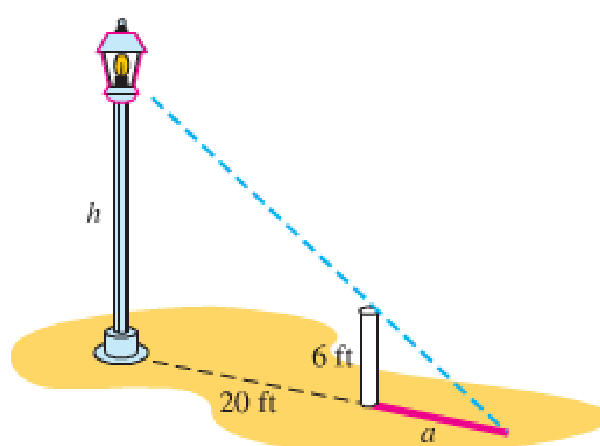
$$= \left(\frac{3}{\sqrt{9^2+3^2}}\right)^2 (10)$$

$$\underline{= 1 \ rad/\sec}$$

Exercise

To find the height of a lamppost, you stand a 6 *feet* pole 20 *feet* from the lamp and measure the length a of its shadow, finding it to be 15 *feet*, give or take an inch. Calculate the height of the lamppost using the value of $a = 15$ and estimate the possible error in the result.

Solution



$$\frac{h}{6} = \frac{20+a}{a} = \frac{35}{15}$$

$$= \frac{7}{3}$$

$$\underline{h = 14 \ ft}$$

$$\frac{h}{6} = \frac{20+a}{a}$$

$$ah = 120 + 6a$$

$$h = 6 + 120a^{-1}$$

$$\frac{dh}{dt} = -120a^{-2}\frac{da}{dt}$$

$$dh = -\frac{120}{a^2}da = -\frac{120}{15^2}\left(\pm\frac{1}{12}\right)$$

$$dh = -\frac{120}{a^2}da$$
$$= -\frac{120}{15^2}\left(\pm\frac{1}{12}\right)$$
$$= -\frac{120}{15^2}\left(\pm\frac{1}{12}\right)$$
$$\boxed{= \pm\frac{2}{45}\ ft} \qquad \boxed{\approx \pm 0.044\ \ ft}$$

Exercise

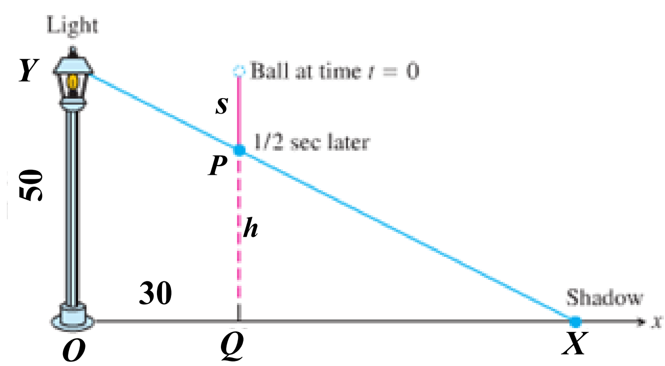
A light shines from the top of a pole 50 *feet* high. A ball is dropped from the same height from a point 30 *feet* away from the light. How fast is the shadow of the ball moving along the ground $\frac{1}{2}$ *sec* later?

(Assume the ball falls a distance $s = 16t^2$ *ft* in *t sec*.)

Solution

$$s = 16t^2$$
$$s + h = 50$$

Triangles *XOY* and *XQP* are similar:



$$\therefore \frac{XQ}{h} = \frac{OX}{50}$$
$$= \frac{30 + XQ}{50}$$

$$50|XQ| = 30h + h|XQ|$$

$$(50 - h)|XQ| = 30h$$

$$|XQ| = \frac{30h}{50-h}$$
$$= \frac{30(50-s)}{50-(50-s)}$$
$$= \frac{30\left(50-16t^2\right)}{50-50+16t^2}$$
$$= \frac{1500-480t^2}{16t^2}$$
$$= \frac{1500}{16t^2} - \frac{480t^2}{16t^2}$$
$$= \frac{1500}{16t^2} - 30$$

$$\frac{d}{dt}|XQ| = 1500\frac{-32t}{\left(16t^2\right)^2}$$
$$= 1500\frac{-32t}{256t^4}$$
$$= -\frac{375}{2t^3}$$

$$\left.\frac{d}{dt}|XQ|\right|_{t=\frac{1}{2}} = -\frac{375}{2\left(\frac{1}{2}\right)^3}$$
$$\boxed{= -1500\ ft/\sec}$$

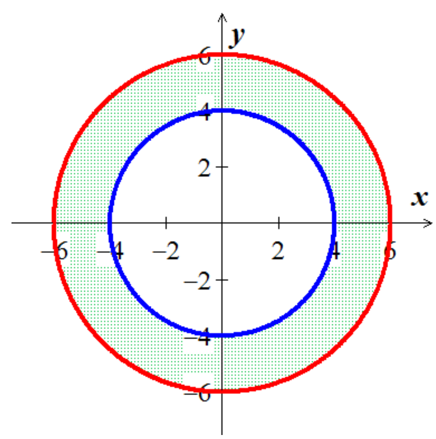
Exercise

A spherical iron ball 8 *in*. in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the rate of 10 $in^3/\min$, how fast is the thickness of the ice decreasing when it is 2 *in*. thick? How fast is the outer surface area of ice decreasing?

Solution

Given: $D = 8\ in \to r_1 = 4\ in \quad \frac{dV}{dt} = -10\ \text{in}^3/\min \quad think = 2\ \text{in}$

$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$
$$\left.\frac{dr}{dt}\right|_{r=6} = \frac{1}{4\pi(6)^2}(-10)$$
$$\boxed{= -\frac{5}{72\pi}\ in/min}$$



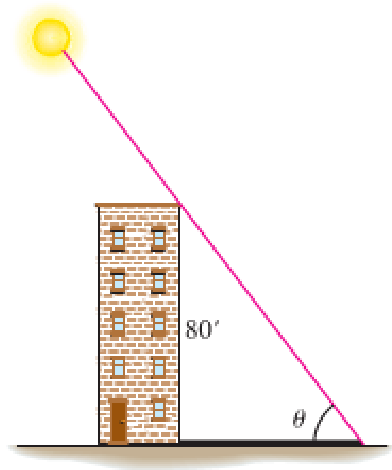
$$S = 4\pi r^2$$
$$\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$$
$$\left.\frac{dS}{dt}\right|_{r=6} = 8\pi(6)\left(-\frac{5}{72\pi}\right)$$
$$\boxed{= -\frac{10}{3}\ in^2/min}$$

The outer surface of the ice is decreasing at $-\frac{10}{3}\ in^2/\min$

Exercise

On a morning of a day when the sun will pass directly overhead, the shadow of an 80–*feet* building on level ground is 60 *feet* long. At the moment in question, the angle θ the sun makes with the ground is increasing at the rate of 0.27 °/ *min*. At what rate is the shadow decreasing?

Solution



Given: $x = 60\,ft \quad h = 80\,ft$

$$\frac{d\theta}{dt} = 0.27^\circ \min$$

$$= 0.27^\circ \frac{\pi rad}{180^\circ} \frac{1}{\min}$$

$$= \frac{3\pi}{2000} \quad rad / \min$$

$$\tan\theta = \frac{80}{x}$$

$$\frac{d}{dt}\tan\theta = \frac{d}{dt}\frac{80}{x}$$

$$\sec^2\theta \frac{d\theta}{dt} = -\frac{80}{x^2}\frac{dx}{dt}$$

$$\frac{dx}{dt} = \left| -\frac{x^2 \sec^2\theta}{80}\frac{d\theta}{dt} \right| \qquad \cos\theta = \frac{60}{\sqrt{60^2+80^2}} = \frac{60}{100} = \frac{3}{5}$$

$$= \frac{60^2\left(\frac{5}{3}\right)^2}{80}\left(\frac{3\pi}{2000}\right)$$

$$= 0.589 \;\; ft/\min$$

Exercise

A baseball diamond is a square 90 *feet* on a side. A player runs from first base to second at a rate of 16 *ft/sec*.

a) At what rate is the player's distance from third base changing when the player is 30 *feet* from first base?

b) At what rates are angles θ_1 *and* θ_2 changing at that time?

c) The player slides into second base at the rate of 15 *ft/sec*. At what rates are angles θ_1 *and* θ_2 changing as the player touches base?

Solution

Given: $d_1 = 90\,ft \quad d_2 = 30\,ft \quad \frac{dx}{dt} = -16 \;\; ft / \sec$

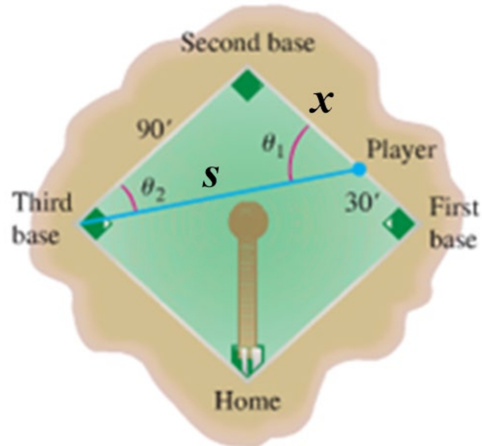
x: Distance between player and 2nd base

s: Distance between player and 3rd base

a) $x = 90 - 30 = 60\ ft$

$$s^2 = x^2 + 90^2$$

$$s = \sqrt{60^2 + 90^2}$$
$$= \sqrt{11700}$$
$$= 30\sqrt{13}$$



$$2s\frac{ds}{dt} = 2x\frac{dx}{dt}$$

$$\frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt}$$
$$= \frac{60}{30\sqrt{13}}(-16)$$
$$= -\frac{32}{\sqrt{13}}\ ft/sec$$
$$\approx -8.875\ ft/sec$$

b) $\sin\theta_1 = \frac{90}{s}$

$$\cos\theta_1 \frac{d\theta_1}{dt} = -\frac{90}{s^2}\frac{ds}{dt}$$

$$\frac{d\theta_1}{dt} = -\frac{90}{s^2\cos\theta_1}\frac{ds}{dt} \qquad \cos\theta_1 = \frac{x}{s}$$

$$\frac{d\theta_1}{dt} = -\frac{90}{s^2\frac{x}{s}}\frac{ds}{dt}$$
$$= -\frac{90}{s\cdot x}\frac{ds}{dt}$$
$$= -\frac{90}{30\sqrt{13}(60)}(-8.875)$$
$$\approx 0.123\ rad/\sec$$

$$\cos\theta_2 = \frac{90}{s}$$

$$-\sin\theta_2 \frac{d\theta_2}{dt} = -\frac{90}{s^2}\frac{ds}{dt}$$

$$\frac{d\theta_2}{dt} = \frac{90}{s^2\sin\theta_2}\frac{ds}{dt} \qquad \sin\theta_2 = \frac{x}{s}$$
$$= \frac{90}{s\cdot x}\frac{ds}{dt}$$
$$= \frac{90}{30\sqrt{13}(60)}(-8.875)$$

$$\boxed{\approx -0.123 \ \ rad/\sec}$$

c) $\dfrac{d\theta_1}{dt} = -\dfrac{90}{s^2\cos\theta_1}\dfrac{ds}{dt}$ $\qquad\qquad \dfrac{ds}{dt} = \dfrac{x}{s}\dfrac{dx}{dt}$

$$= -\frac{90}{s^2 \frac{x}{s}}\frac{x}{s}\frac{dx}{dt}$$

$$= -\frac{90}{s^2}\frac{dx}{dt}$$

$$= -\frac{90}{x^2+8100}\frac{dx}{dt}$$

Player slides into second base $\Rightarrow x = 0$

$$\left.\frac{d\theta_1}{dt}\right|_{x=0} = -\frac{90}{0^2+8100}(-15)$$

$$\boxed{=\frac{1}{6} \ \ rad/\sec}$$

$$\frac{d\theta_2}{dt} = \frac{90}{s^2\sin\theta_2}\frac{ds}{dt}$$

$$= \frac{90}{s^2 \frac{x}{s}}\frac{x}{s}\frac{dx}{dt}$$

$$= \frac{90}{s^2}\frac{dx}{dt}$$

$$= \frac{90}{x^2+8100}\frac{dx}{dt}$$

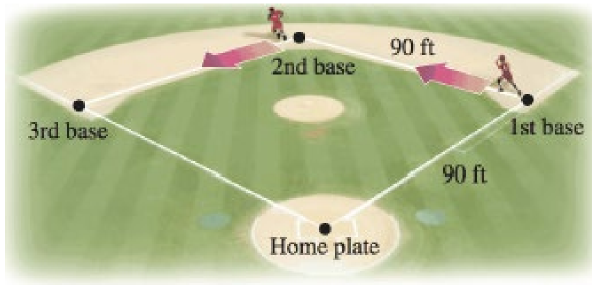
Player slides into second base $\Rightarrow x = 0$

$$\left.\frac{d\theta_2}{dt}\right|_{x=0} = \frac{90}{0^2+8100}(-15)$$

$$\boxed{=-\frac{1}{6} \ \ rad/\sec}$$

Exercise

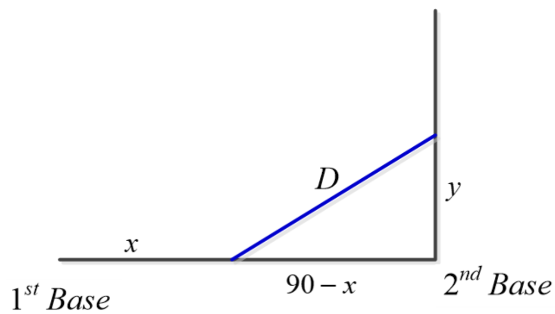
Runners stand at first and second base in a baseball game. At the moment, a ball is hit the runner at first base runs to second base at 18 *ft/s*; simultaneously the runner on second runs to third base at 20 *ft/s*. How fast is the distance between the runners changing 1 *sec* after the ball is hit?
(*Hint*: The distance between consecutive bases I 90 *feet* and the bases lie at the corners of a square.)



Solution

Given: $\frac{dx}{dt}=18\ ft/s,\quad \frac{dy}{dt}=20\ ft/s$

After 1 *sec*, $x=18 \quad and \quad y=20$



$$D^2=(90-x)^2+y^2$$

$$D=\sqrt{(90-18)^2+20^2}$$

$$=\sqrt{5584}$$

$$\approx 74.726$$

$$2D\frac{dD}{dt}=-2(90-x)\frac{dx}{dt}+2y\frac{dy}{dt}$$

$$D\frac{dD}{dt}=-(90-x)\frac{dx}{dt}+y\frac{dy}{dt}$$

$$\frac{dD}{dt}=\frac{-(90-18)(18)+20(20)}{74.726}$$

$$\approx -11.99 \quad ft/\sec$$

So, the distance between the runners is decreasing at a rate about 11.99 *feet* per *second*.

Exercise

The variables *x* and *y* are both differentiable functions of *t* and are related by the equation $y=x^2+3$.

Find $\frac{dy}{dt}$ when $x=1$, given $\frac{dx}{dt}=2$ when $x=1$.

Solution

$$\frac{dy}{dt}=\frac{d}{dt}\left(x^2+3\right)$$

$$=2x\frac{dx}{dt}$$

$= 2(1)(2)$

$\boxed{= 4}$

Exercise

A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius *r* of the outer ripple is increasing at a constant rate of 1 *foot* per *second*.

When the radius is 4 *feet*, at what rate is the total area ***A*** of the disturbed water changing?

Solution

Given: $\frac{dr}{dt} = 1$

Find: $\frac{dA}{dt}$ *when* $r = 4$

$$\frac{dA}{dt} = \frac{d}{dt}\left(\pi r^2\right)$$

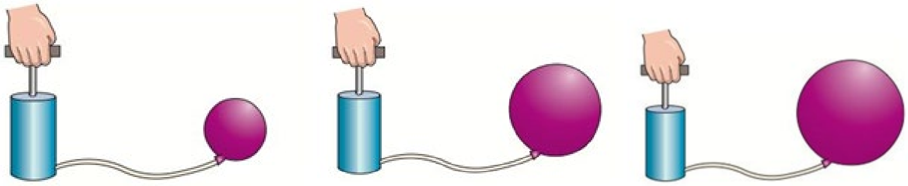
$$= 2\pi r \frac{dr}{dt}$$

$$= 2\pi(4)(1)$$

$$\boxed{= 8\pi \ ft^2/\sec}$$

Exercise

Air is being pumped into a spherical balloon at a rate of $4.5 \ ft^3/\min$.



Find the rate of change of the radius when the radius is 2 *feet*.

Solution

Given: $\frac{dV}{dt} = \frac{9}{2}$ ***Find***: $\frac{dr}{dt}$ *when* $r = 2$

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$= 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

$$= \frac{1}{16\pi}\frac{9}{2}$$

$$\boxed{= \frac{9}{32\pi} \ ft/\min}$$

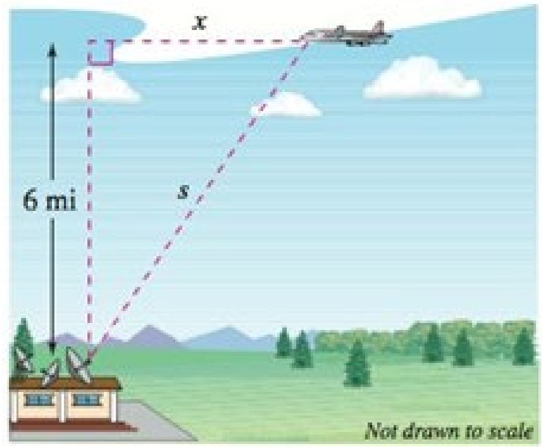
Exercise

An Airplane is flying on a flight path that will take it directly over a radar tracking station. The distance s is decreasing at a rate of 400 *mph* when $s = 10\ mi$. what is the speed of the plane?

Solution

Given: $\frac{ds}{dt} = -400 \quad when \quad s = 10$

Find: $\frac{dx}{dt} \quad when \quad s = 10\ \ \&\ \ x = 8$



$$s^2 = x^2 + 6^2$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{s}{x}\frac{ds}{dt}$$

$$= \frac{10}{8}(-400)$$

$$= -500\ \ mph$$

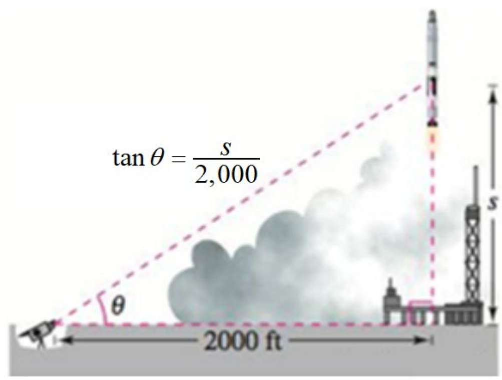
Exercise

Find the rate of change in the angle of elevation of the camera at 10 *seconds* after lift-off.

Solution

Given: $\frac{ds}{dt} = 100t$

Find: $\frac{d\theta}{dt} \quad when \quad t = 10\ \ \&\ \ s = 5{,}000$



$$\tan\theta = \frac{s}{2{,}000}$$

$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{2{,}000}\frac{ds}{dt}$$

$$hyp = \sqrt{s^2 + 2{,}000^2} \qquad \cos\theta = \frac{2{,}000}{\sqrt{s^2 + 2{,}000^2}}$$

$$\frac{d\theta}{dt} = \frac{\cos^2\theta}{2{,}000}\frac{ds}{dt}$$

$$= \frac{1}{2{,}000}\left(\frac{2{,}000}{\sqrt{5{,}000^2 + 2{,}000^2}}\right)^2 (100(10))$$

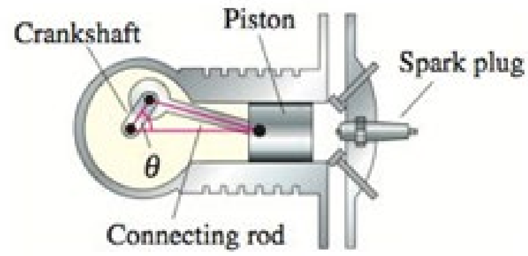
$$= \frac{1}{2}\left(\frac{2{,}000}{\sqrt{25\times10^6 + 4\times10^6}}\right)^2$$

$$= \frac{2}{29}\ \ rad/\sec$$

Exercise

In the engine, a 7–*inch* connecting rod is fastened to a crank of radius 3 *inches*, the crankshaft rotates counterclockwise at a constant rate of 200 *revolutions per minute*.

Find the velocity of the piston when $\theta = \frac{\pi}{3}$.



Solution

Given: $\frac{d\theta}{dt} = 400\pi$

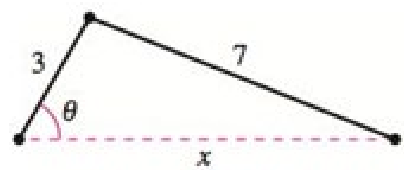
Find: $\frac{dx}{dt}$ *when* $\theta = \frac{\pi}{3}$

$7^2 = x^2 + 3^2 - 6x\cos\theta \qquad a^2 = b^2 + c^2 - 2bc\cos A \quad (\textit{Law of cosine})$

$x^2 - 6x\cos\theta = 40$

$\Rightarrow \left(\theta = \frac{\pi}{3}\right)$

$x^2 - 3x - 40 = 0 \rightarrow \underline{|x = 8}, \ \cancel{-5}$



$$2x\frac{dx}{dt} - 6\cos\theta\frac{dx}{dt} + 6x\sin\theta\frac{d\theta}{dt} = 0$$

$$(6\cos\theta - 2x)\frac{dx}{dt} = 6x\sin\theta\frac{d\theta}{dt}$$

$$\frac{dx}{dt} = \frac{6x\sin\theta}{6\cos\theta - 2x}\frac{d\theta}{dt}.$$

$$= \frac{6(8)\sin\frac{\pi}{3}}{6\cos\frac{\pi}{3} - 2(8)}(400\pi)$$

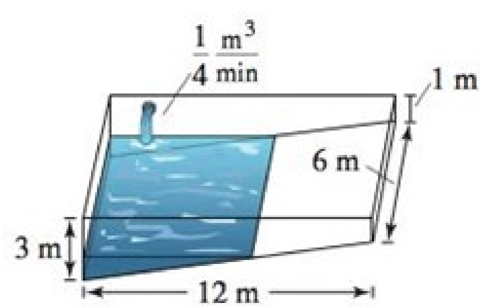
$$\underline{\approx -4018 \ \ in/\min \ |}$$

Exercise

A swimming pool is 12 *meters* long, 6 *meters* wide, 1 *meter* deep at the shallow end, and 3 *meters* deep at the deep end. Water is being pumped into the pool at $\frac{1}{4}\ m^3/\text{min}$, and there is 1 *meter* of water at the deep end.

a) What percent of the pool is filled?

b) At what rate is the water level rising?



Solution

Given: $\frac{dV}{dt} = \frac{1}{4}\ m^3/\text{min}$

a) Total Volume:

$$V = \frac{1}{2}(2)(12)(6) + (1)(12)(6)$$

$$\underline{= 144 \ \ m^3 \ |}$$

Volume of 1 m of water:

$$V = \frac{1}{2}(1)(6)(6)$$

$$= 18\ m^3$$

$$\%\text{ pool filled } = \frac{18}{144}(100\%)$$

$$= 12.5\%$$

b) Since there is 1 m of water in the pool, then $0 \le h \le 2$

$$\frac{b}{6} = \frac{h}{1}$$

$$b = 6h$$

$$V = \frac{1}{2}bh(6)$$

$$= 3(6h)h$$

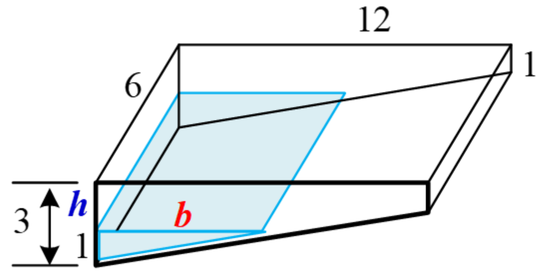
$$= 18h^2$$

$$\frac{dV}{dt} = 36h\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{36h}\frac{dV}{dt}$$

$$= \frac{1}{36(1)}\frac{1}{4}$$

$$= \frac{1}{144}\ m/min$$

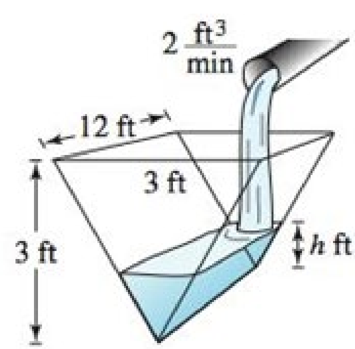


Exercise

A trough is 12 *feet* long and 3 *feet* across the top. Its ends are isosceles triangles with altitudes of 3 *feet*.

a) Water is being pumped into the trough at $2\ ft^3/\min$. How fast is the water level rising when the depth h is 1 *foot*?

b) The water is rising at a rate of $\frac{3}{8}\ in/\min$ when $h = 2$. Determine the rate at which water is being pumped into the trough.



Solution

Given: $\frac{dV}{dt} = 2\ ft^3/min$

a) $V = \frac{1}{2}bh(12) = 6bh$

$V = 6h^2 \quad since \quad b = h$

$$\frac{dV}{dt} = 12h\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{12h}\frac{dV}{dt} \qquad h = 1$$

$$= \frac{1}{12(1)}(2)$$

$$\boxed{= \frac{1}{6}\ ft/min}$$

b) ***Given***: $\frac{dh}{dt} = \frac{3}{8}\ in / \min,\ h = 2$

$$\frac{dV}{dt} = 12(2)\left(\frac{3}{8}\frac{in}{\min}\ \frac{1\ ft}{12\ in}\right)$$

$$\boxed{= \frac{3}{4}\ ft^3 / \min}$$

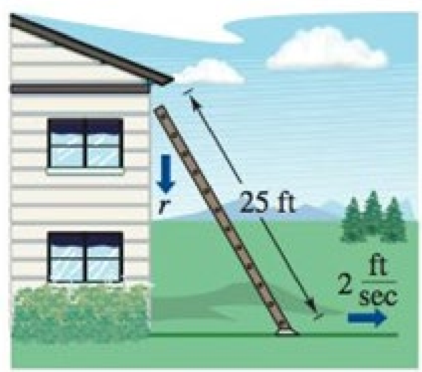
Exercise

A ladder 25 *feet* long is leaning against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 2 *feet* per *second*.

a) How fast is the top of the ladder moving down the wall when its base is 7 *feet*, 15 *feet*, and 24 *feet* from the wall?

b) Consider the triangle formed by the side of the house, the ladder, and the ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 7 *feet* from the wall.

c) Find the rate at which the angle between the ladder and the wall of the house is changing when the base of the ladder is 7 *feet* from the wall.



Solution

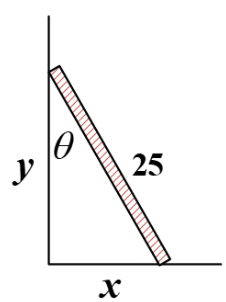
Given: $\frac{dx}{dt} = 2\ ft / \sec$

$$x^2 + y^2 = 25^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

$$\boxed{= -\frac{2x}{y}}$$



a) $x = 7$

$$y = \sqrt{25^2 - 7^2}$$

$$= \sqrt{576}$$

$$\boxed{= 24}$$

$$\frac{dy}{dt} = -\frac{2(7)}{24}$$

$$\boxed{= -\frac{7}{12}\ ft / \sec}$$

$x = 15$

$$y=\sqrt{25^2-15^2}$$
$$\underline{=20}$$

$$\frac{dy}{dt}=-\frac{2(15)}{20}$$
$$\boxed{=-\frac{3}{2}\ ft/\sec}$$

$x=24$

$$y=\sqrt{25^2-24^2}$$
$$\underline{=7}$$

$$\frac{dy}{dt}=-\frac{2(24)}{7}$$
$$\boxed{=-\frac{48}{7}\ ft/\sec}$$

b) $A=\frac{1}{2}xy$

$$\frac{dA}{dt}=\frac{1}{2}\left(y\frac{dx}{dt}+x\frac{dy}{dt}\right)$$

We have: $x=7;\quad y=24;\quad \frac{dy}{dt}=-\frac{7}{12};\quad \frac{dx}{dt}=2$

$$\frac{dA}{dt}=\frac{1}{2}\left(24(2)+7\left(-\frac{7}{12}\right)\right)$$
$$\boxed{=\frac{527}{24}\ ft^2/\sec}$$

c) $\tan\theta=\frac{x}{y}$

$$\sec^2\theta\ d\theta=\frac{1}{y}\frac{dx}{dt}-\frac{x}{y^2}\frac{dy}{dt}$$

$$d\theta=\cos^2\theta\left(\frac{1}{y}\frac{dx}{dt}-\frac{x}{y^2}\frac{dy}{dt}\right)$$

We have: $x=7;\quad y=24;\quad \frac{dy}{dt}=-\frac{7}{12};\quad \frac{dx}{dt}=2;\quad \cos\theta=\frac{y}{25}=\frac{24}{25}$

$$d\theta=\left(\frac{24}{25}\right)^2\left(\frac{1}{24}(2)-\frac{7}{24^2}\left(-\frac{7}{12}\right)\right)$$
$$=\left(\frac{24}{25}\right)^2\left(\frac{48\times12+49}{12\times24^2}\right)$$
$$=\frac{625}{12\times25^2}$$
$$\boxed{=\frac{1}{12}\ rad/\sec}$$

Exercise

A construction worker pulls a five-*meter* plank up the side of a building under construction by means of a rope tied to one end of the plank. Assume the opposite end of the plank follows a path perpendicular to the wall of the building and the worker pulls the rope at a rate of $0.15\ m/\sec$. How fast is the end of the plank sliding along the ground when it is 2.5 *meters* from the wall of the building?

Solution

$$x^2 + y^2 = 25$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

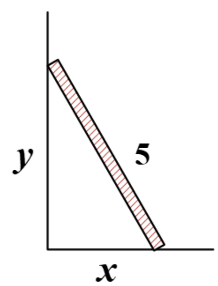
$$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$$

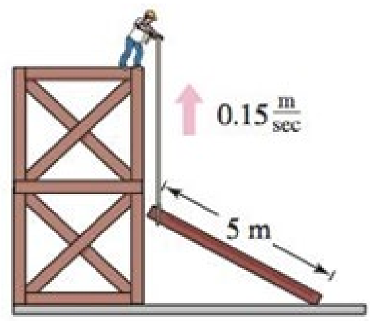
Given: $\frac{dy}{dt} = 0.15\ m/\sec \quad x = 2.5$

$$y = \sqrt{25 - 2.5^2} = \sqrt{18.75}$$

$$\frac{dx}{dt} = -\frac{\sqrt{18.75}}{2.5}(0.15)$$

$$\approx -0.26\ m/sec$$





Exercise

A winch at the top of a 12-*meter* building pulls a pipe of the same length to a vertical position. The winch pulls in rope at a rate of $-0.2\ m/\sec$. Find the rate of vertical change and the rate of horizontal change at the end of the pipe when $y = 6$

Solution

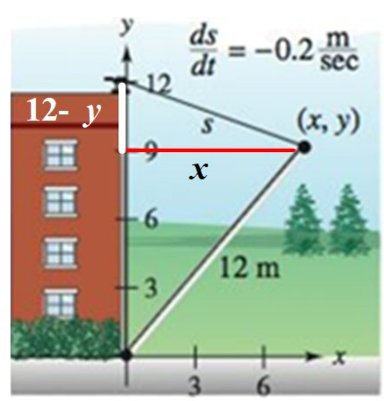
When $y = 6 \;\Rightarrow\; x = \sqrt{12^2 - 6^2} = \sqrt{108} = 6\sqrt{3}$

$$s = \sqrt{x^2 + (12 - y)^2}$$

$$= \sqrt{108^2 + (12 - 6)^2}$$

$$= \sqrt{144}$$

$$= 12$$



$$x^2 + (12 - y)^2 = s^2$$

$$2x\frac{dx}{dt} + 2(12 - y)(-1)\frac{dy}{dt} = 2s\frac{ds}{dt}$$

$$x\frac{dx}{dt} - (12 - y)\frac{dy}{dt} = s\frac{ds}{dt}$$

$$x^2 + y^2 = 12^2$$

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

$$\frac{dx}{dt}=-\frac{y}{x}\frac{dy}{dt}$$

$$x\left(-\frac{y}{x}\frac{dy}{dt}\right)-(12-y)\frac{dy}{dt}=s\frac{ds}{dt}$$

$$-y\frac{dy}{dt}-(12-y)\frac{dy}{dt}=s\frac{ds}{dt}$$

$$-12\frac{dy}{dt}=s\frac{ds}{dt}$$

$$\frac{dy}{dt}=-\frac{s}{12}\frac{ds}{dt} \qquad\qquad \frac{ds}{dt}=-0.2; \quad y=6; \quad x=6\sqrt{3}$$

$$=-\frac{12}{12}(-0.2)$$

$$\boxed{=\frac{1}{5}\ \ m/\sec}$$

$$\frac{dx}{dt}=-\frac{y}{x}\frac{dy}{dt}$$

$$=-\frac{6}{6\sqrt{3}}\frac{1}{5}$$

$$\boxed{=-\frac{\sqrt{3}}{15}\ \ m/\sec}$$

Exercise

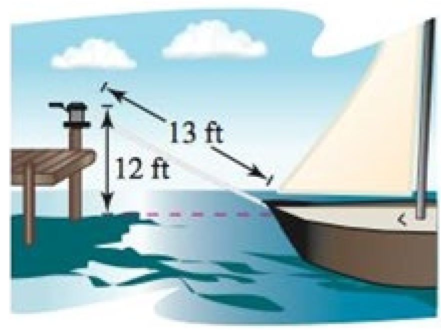
A boat is pulled into a dock by means of a winch 12 *feet* above the deck of the boat.

a) The winch pulls in rope at a rate of 4 *feet* per *second*. Determine the speed of the boat when there is 13 *feet* of rope out. What happens to the speed of the boat as it gets closer to the dock?

b) Suppose the boat is moving at a constant rate of 4 *feet* per *second*. Determine the speed at which the winch pulls in rope when there is a total of 13 *feet* of rope out. What happens to the speed at which the winch pulls in rope as the boat gets closer to the dock?

Solution

Let L be the length of the rope.



a) $L^2=12^2+x^2$

$$2L\frac{dL}{dt}=2x\frac{dx}{dt}$$

$$\frac{dx}{dt}=\frac{L}{x}\frac{dL}{dt}$$

Given: $\frac{dL}{dt}=-4\ \ ft/\sec$

When $L=13$

$$x=\sqrt{129-144}$$

$$= 5$$

$$\frac{dx}{dt} = \frac{13}{5}(-4)$$

$$= -\frac{52}{5} \ ft/\sec$$

b) $\frac{dx}{dt} = -4 \ ft/\sec; \quad L = 13$

$$\frac{dL}{dt} = \frac{x}{L}\frac{dx}{dt}$$

$$= \frac{5}{13}(-4)$$

$$= -\frac{20}{13} \ ft/\sec$$

$$\frac{dL}{dt} = \frac{\sqrt{L^2 - 144}}{L}(-4)$$

$$\lim_{L \to 12^+} \frac{dL}{dt} = -4 \lim_{L \to 12^+} \frac{\sqrt{L^2 - 144}}{L}$$

$$= 0$$

Exercise

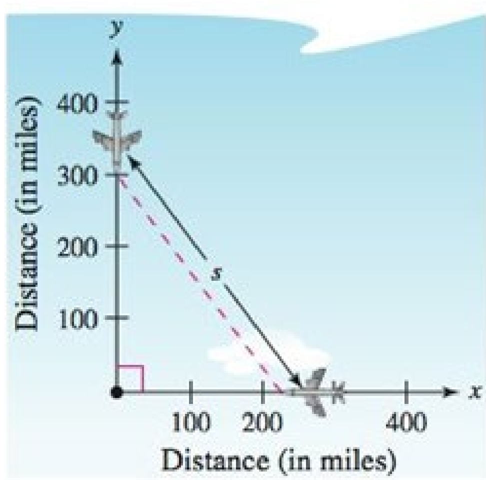
An air traffic controller spots two planes at the same altitude converging on a point as they fly at right angles to each other. One plane is 225 *miles* from the point moving at 450 *mph*. The other plane is 300 *miles* from the point moving at 600 *mph*.

a) At what rate is the distance between the planes decreasing?

b) How much time does the air traffic controller have to get one of the planes on a different flight path?

Solution

Given: $x = 225, \quad \frac{dx}{dt} = -450, \quad y = 300, \quad \frac{dy}{dt} = -600$



a) $s^2 = x^2 + y^2$

$$s = \sqrt{225^2 + 300^2}$$

$$= 375$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$= \frac{1}{375}\left(225(-450) + 300(-600)\right)$$

$$= -\frac{281,250}{375}$$

$$= -750 \ \ mph$$

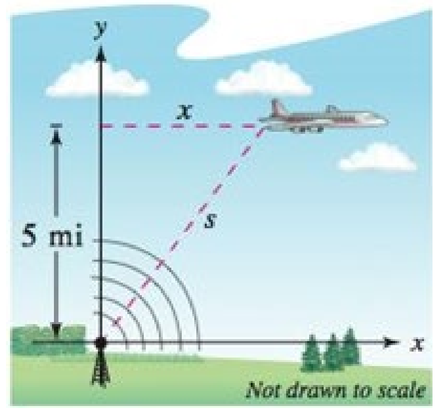
b) $t = \dfrac{375}{750} = \dfrac{1}{2} \ hr \quad = 30 \ \min$

Exercise

An airplane is flying at an altitude of 5 *miles* and passes directly over a radar antenna. When the plane is 10 *miles* away $(s = 10)$, the radar detects that the distance s is changing at a rate of 240 *mph*. What is the speed of the plane?

Solution

Given: $s = 10, \quad \dfrac{ds}{dt} = 240 \quad \dfrac{dy}{dt} = 0$



c) $s^2 = x^2 + y^2$

$$x = \sqrt{10^2 - 5^2}$$

$$= 5\sqrt{3}$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dx}{dt} = \frac{s}{x}\frac{ds}{dt}$$

$$= \frac{10}{5\sqrt{3}}(240)$$

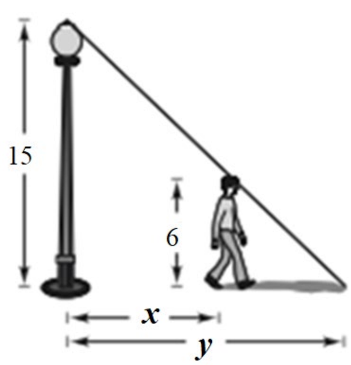
$$= 160\sqrt{3} \ \ mph$$

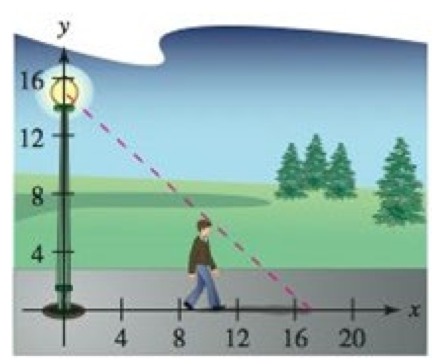
Exercise

A man 6 *feet* tall walks at a rate of 5 *feet* per *second* away from a light that is 15 *feet* above the ground.

a) When he is 10 *feet* from the base of the light, at what rate is the tip of his shadow moving?
b) When he is 10 *feet* from the base of the light, at what rate is the length of his shadow changing?

Solution





a) $\dfrac{y}{y - x} = \dfrac{15}{6}$

$$6y = 15y - 15x$$

$$9y = 15x$$

$$y = \frac{5}{3}x$$

$$\frac{dy}{dt} = \frac{5}{3}\frac{dx}{dt}$$

$$= \tfrac{5}{3}(5)$$

$$= \boxed{\frac{25}{3}\ ft/\sec}$$

b) $\dfrac{d(y-x)}{dt} = \dfrac{dy}{dt} - \dfrac{dx}{dt}$

$$= \frac{25}{3} - 5$$

$$= \boxed{\frac{10}{3}\ ft/\sec}$$

Exercise

A man 6 *feet* tall walks at a rate of 5 *feet* per *second toward* a light that is 20 *feet* above the ground.

a) When he is 10 *feet* from the base of the light, at what rate is the tip of his shadow moving?

b) When he is 10 *feet* from the base of the light, at what rate is the length of his shadow changing?

Solution

a) $\dfrac{y}{y-x} = \dfrac{20}{6}$

$6y = 20y - 20x$

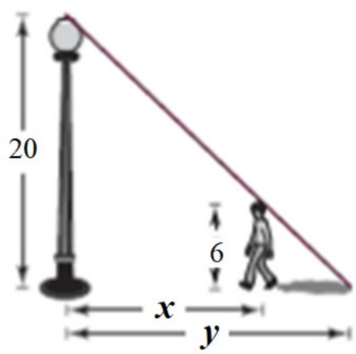
$14y = 20x$

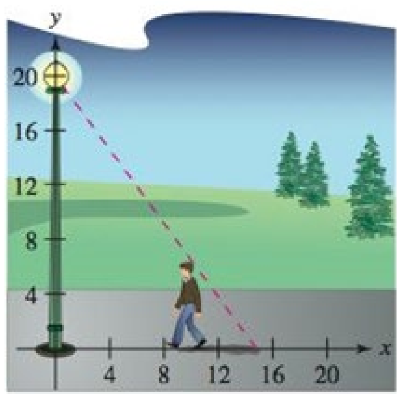
$y = \dfrac{10}{7}x$

$\dfrac{dy}{dt} = \dfrac{10}{7}\dfrac{dx}{dt}$

$= \dfrac{10}{7}(-5)$

$= \boxed{-\dfrac{50}{7}\ ft/\sec}$





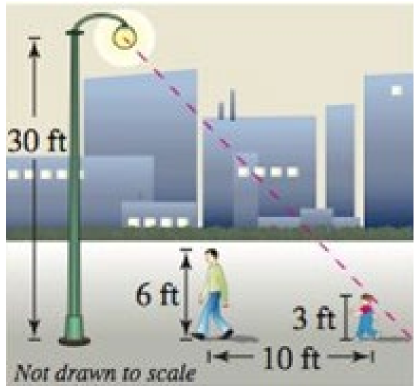
b) $\dfrac{d(y-x)}{dt} = \dfrac{dy}{dt} - \dfrac{dx}{dt}$

$$= -\frac{50}{7} - (-5)$$

$$= \boxed{-\frac{15}{7}\ ft/\sec}$$

Exercise

A man 6 *feet* tall walks at a rate of 5 $ft/\sec$ toward a streetlight that is 30 *feet* high. The man's 3-*foot*-tall child follows at the same speed, but 10 *feet* behind the man. At times, the shadow behind the child is caused by the man, and at other times, by the child.



a) Suppose the man is 90 *feet* from the streetlight. Show that the man's shadow extends beyond the child's shadow.

b) Suppose the man is 60 *feet* from the streetlight. Show that the child's shadow extends beyond the man's shadow.

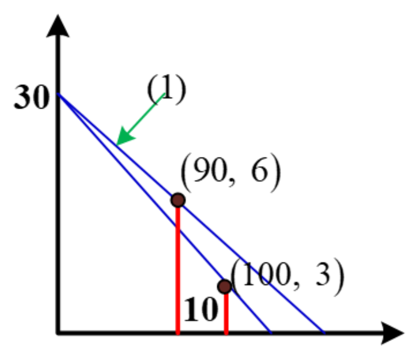
c) Determine the distance *d* from the man to the streetlight at which the tips of the two shadows are exactly the same distance from the streetlight.

d) Determine how fast the tip of the man's shadow is moving as a function of *x*, the distance between the man and the streetlight. Discuss the continuity of this shadow speed function.

Solution

a) Line (1): $y = \frac{6-30}{90-0}(x-0)+30$ $\qquad y = m\left(x - x_0\right) + y_0$

$$= -\frac{4}{15}x + 30$$



When $x = 100$

$$y = -\frac{400}{15} + 30$$

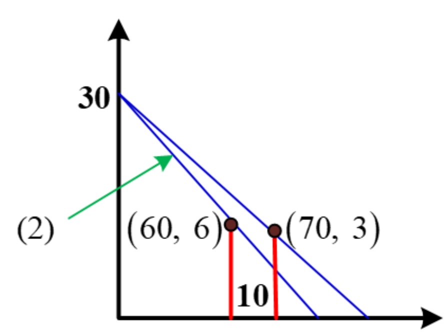
$$= \frac{10}{3} > 3$$

The shadow determined by the man extends beyond the shadow by the child.

b) Line (2): $y = \frac{6-30}{60-0}(x-0)+30$

$$= -\frac{24}{60}x + 30$$

$$= -\frac{2}{5}x + 30$$



When $x = 70$

$$y = -\frac{140}{5} + 30$$

$$= 2 < 3$$

The shadow determined by the child extends beyond the shadow by the man.

c) The points: $(0,\ 30)$, $(d,\ 6)$, and $(d+10,\ 3)$

$$\frac{30-6}{0-d}=\frac{30-3}{0-d-10}$$
$$-\frac{24}{d}=-\frac{27}{d+10}$$
$$24d+240=27d$$
$$3d=240$$
$$\boxed{d=80\ ft}$$

d) ***Given***: $\frac{dx}{dt}=-5$

The shadow is determined by the man

$$\frac{y}{30}=\frac{y-x}{6}$$
$$24y=30x$$
$$y=\frac{5}{4}x$$
$$\frac{dy}{dt}=\frac{5}{4}\frac{dx}{dt}$$
$$\boxed{=-\frac{25}{4}}$$

The shadow is determined by the child

$$\frac{y}{30}=\frac{y-x-10}{3}$$
$$27y=30x+300$$
$$y=\frac{10}{9}x+\frac{100}{9}$$
$$\frac{dy}{dt}=\frac{10}{9}\frac{dx}{dt}$$
$$\boxed{=-\frac{50}{9}}$$

$$\therefore\ \frac{dy}{dt}=\begin{cases}-\frac{50}{9} & 0<x<80\\ -\frac{25}{4} & x>80\end{cases}$$

$\frac{dy}{dt}$ is not continuous at $x=80$.

Exercise

A ball is dropped from a height of 20 *m*, 12 *m* away from the top of a 20-*meter* lamppost. The ball's shadow, caused by the light at the top of the lamppost, is moving along the level ground. How fast is the shadow moving 1 *second* after the ball is released?

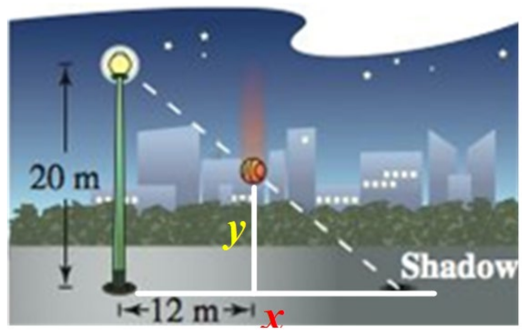
Solution

$y(t) = -4.9t^2 + 20 \qquad\qquad y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0 \quad g = 9.8\ m/s^2$

$\frac{dy}{dt} = -9.8t$

At $t = 1 \rightarrow \quad y(1) = 15.1\ m \quad y'(1) = -9.8\ \ m/s$

$\frac{y}{20} = \frac{x-12}{x}$



$xy = 20x - 240$

$y(1) = 15.1$

$20x - 15.1x = 240$

$\boxed{x = \frac{240}{4.9}}$

$x\frac{dy}{dt} + y\frac{dx}{dt} = 20\frac{dx}{dt}$

$(20-y)\frac{dx}{dt} = x\frac{dy}{dt}$

$$\begin{aligned}\frac{dx}{dt} &= \frac{x}{20-y}\frac{dy}{dt}\\ &= \frac{240}{4.9}\frac{1}{20-15.1}(-9.8)\\ &= -\frac{480}{4.9}\\ &= \boxed{-\frac{4800}{49}\ \ m/s}\end{aligned}$$

Exercise

The endpoints of a movable rod of length 1 *meter* have coordinates $(x,\ 0)$ and $(0,\ y)$. The position of the end on the *x-axis* is

$$x(t) = \frac{1}{2}\sin\frac{\pi t}{6} \qquad \text{where } t \text{ is the time in } seconds.$$

a) Find the time of one complete cycle of the rod.

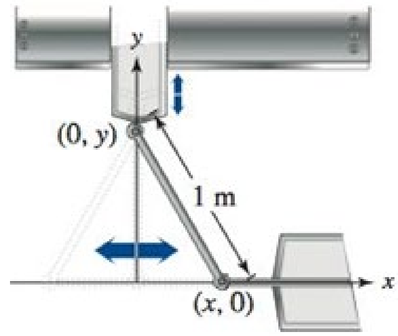
b) What is the lowest point reached by the end of the rod on the *y-axis*?

c) Find the speed of the *y-axis* endpoint when the *x-axis* endpoints is $\left(\frac{1}{4},\ 0\right)$

Solution

Given: $x(t)=\frac{1}{2}\sin\frac{\pi t}{6}\qquad x^2+y^2=1$

a) Period: $\frac{2\pi}{\pi/6}=12\ \text{sec}$



b) When $x=\frac{1}{2}$

$$y=\sqrt{1-\frac{1}{4}}$$

$$=\frac{\sqrt{3}}{2}$$

$\therefore$ The lowest point $\left(0,\ \frac{\sqrt{3}}{2}\right)$

c) When $x=\frac{1}{4}$

$$y=\sqrt{1-\frac{1}{16}}$$

$$=\frac{\sqrt{15}}{4}$$

$$x(t)=\frac{1}{2}\sin\frac{\pi t}{6}$$

$$\frac{dx}{dt}=\frac{\pi}{12}\cos\frac{\pi t}{6}$$

$$x^2+y^2=1$$

$$x\frac{dx}{dt}+y\frac{dy}{dt}=0$$

$$\frac{dy}{dt}=-\frac{x}{y}\frac{dx}{dt}$$

$$=-\left(\frac{1}{4}\cdot\frac{4}{\sqrt{15}}\right)\frac{\pi}{12}\cos\frac{\pi t}{6}\Big|_{t=1}$$

$$=-\frac{\pi}{12\sqrt{15}}\cos\frac{\pi}{6}$$

$$=-\frac{\pi}{12\sqrt{15}}\frac{\sqrt{3}}{2}$$

$$=-\frac{\pi}{24\sqrt{5}}$$

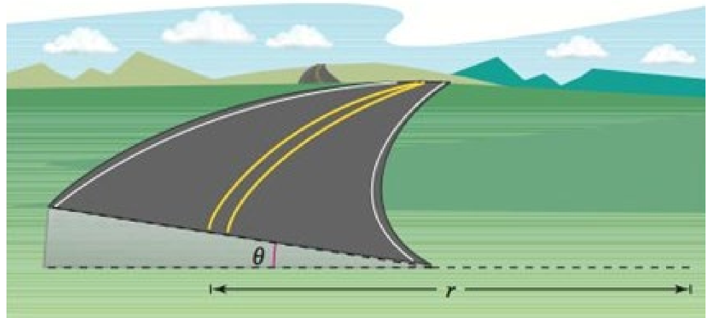
$$\textit{Speed}=\left|-\frac{\pi\sqrt{5}}{120}\right|$$

$$=\frac{\pi\sqrt{5}}{120}\ m/\sec$$

Exercise

Cars on a certain roadway travel on a circular arc of radius r. in order not to rely on friction alone to overcome to centrifugal force, the road is banked at an angle of magnitude θ from the horizontal. The banking angle must satisfy the equation $rg\tan\theta = v^2$, where v is the velocity of the cars and $g = 32\ ft/\sec^2$ is the acceleration due to gravity. Find the relationship between the related rates $\frac{dv}{dt}$ and $\frac{d\theta}{dt}$.

Solution



$$rg\tan\theta = v^2$$

$$32r\tan\theta = v^2$$

$$32r\sec^2\theta\frac{d\theta}{dt} = 2v\frac{dv}{dt}$$

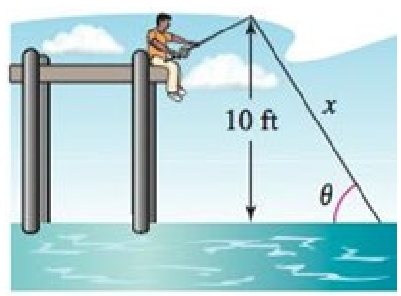
$$\boxed{\frac{dv}{dt} = \frac{16r}{v}\sec^2\theta\ \frac{d\theta}{dt}}$$

$$\boxed{\frac{d\theta}{dt} = \frac{v}{16r}\cos^2\theta\ \frac{dv}{dt}}$$

Exercise

A fish is reeled in at a rate of 1 $ft/\sec$ from a point 10 *feet* above the water. At what rate is the angle θ between the line and the water changing when there is a total of 25 *feet* of line from the end of the rod to the water?

Solution



Given: $\frac{dx}{dt} = -1\ ft/\sec$

$$\sin\theta = \frac{10}{x}$$

$$\cos\theta\frac{d\theta}{dt} = -\frac{10}{x^2}\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = -\frac{10\sec\theta}{x^2}\frac{dx}{dt}$$

$$\sec\theta = \frac{x}{\sqrt{x^2-10^2}} = \frac{25}{\sqrt{25^2-10^2}} = \frac{25}{\sqrt{525}} = \frac{5}{\sqrt{21}}$$

$$= -\frac{10}{25^2}\frac{5}{\sqrt{21}}(-1)$$

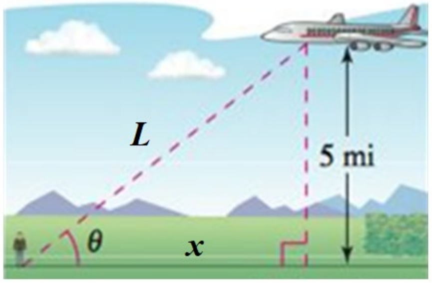
$$\boxed{= \frac{2}{25\sqrt{21}}\ rad/\sec}$$

Exercise

An airplane flies at an altitude of 5 *miles* toward a point directly over an observer. The speed of the plane is 600 *mph*. Find the rates at which the angle of elevation θ is changing when the angle is

$a)\ \ \theta = 30° \quad b)\ \ \theta = 60° \quad c)\ \ \theta = 75°$

Solution



Given: $y = 5; \quad \frac{dx}{dt} = -600\ mph$

$$\tan\theta = \frac{y}{x} = \frac{5}{x}$$

$$\sec^2\theta \frac{d\theta}{dt} = -\frac{5}{x^2}\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = -\frac{5\cos^2\theta}{x^2}\frac{dx}{dt}$$

Le L be the distance from observer to the plane. $\cos\theta = \frac{x}{L}$

$$\frac{d\theta}{dt} = -\frac{5}{x^2}\frac{x^2}{L^2}\frac{dx}{dt}$$

$$= -\frac{5}{L^2}\frac{dx}{dt} \qquad\qquad \sin\theta = \frac{5}{L} \ \rightarrow\ L = \frac{5}{\sin\theta}$$

$$= -5\frac{\sin^2\theta}{5^2}(-600)$$

$$= 120\sin^2\theta$$

a) $\theta = 30°$

$$\frac{d\theta}{dt} = 120\sin^2 30°$$

$$= \frac{120}{4}$$

$$= 30\ \ rad/hr$$

b) $\theta = 60°$

$$\frac{d\theta}{dt} = 120\sin^2 60°$$

$$= 120\left(\frac{3}{4}\right)$$

$$= 90\ \ rad/hr$$

c) $\theta = 75°$

$$\frac{d\theta}{dt} = 120\sin^2 75°$$

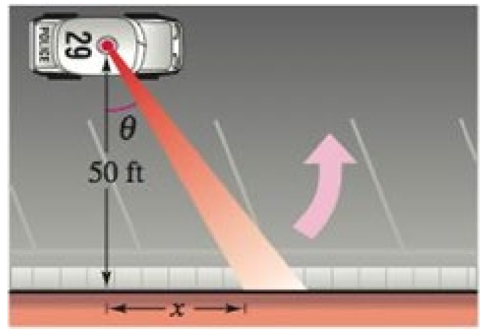
$$\approx 111.96\ \ rad/hr$$

Exercise

A patrol car is parked 50 *feet* from a long warehouse. The revolving light on top of the car turns at a rate of 30 *revolutions per minute*. How fast is the light beam moving along the wall when the beam makes angles of

$a)\ \theta = 30° \quad b)\ \theta = 60° \quad c)\ \theta = 70°$

With the perpendicular line from the light to the wall?



Solution

Given: $y = 50;\quad \frac{d\theta}{dt} = 30\ \frac{rev}{min}\frac{2\pi\ rad}{1\ rev}\frac{1\ \min}{60\ \sec}$

$$= \pi\ rad/\sec$$

$$\tan\theta = \frac{x}{50}$$

$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{50}\frac{dx}{dt}$$

$$\frac{dx}{dt} = 50\sec^2\theta\ \frac{d\theta}{dt}$$

$$\frac{dx}{dt} = 50\pi\sec^2\theta$$

a) $\theta = 30°$

$$\frac{dx}{dt} = 50\pi\sec^2 30°$$

$$= 50\pi\left(\frac{4}{3}\right)$$

$$= \frac{200\pi}{3}\ ft/\sec$$

b) $\theta = 60°$

$$\frac{dx}{dt} = 50\pi\sec^2 60°$$

$$= 50\pi(4)$$

$$= 200\pi\ \ ft/\sec$$

c) $\theta = 70°$

$$\frac{dx}{dt} = 50\pi\sec^2 70°$$

$$= 427.43\pi\ \ ft/\sec$$

Exercise

A wheel of radius 30 *cm* revolves at a rate of 10 *revolutions* per *second*. A dot is painted at a point *P* on the rim of the wheel.

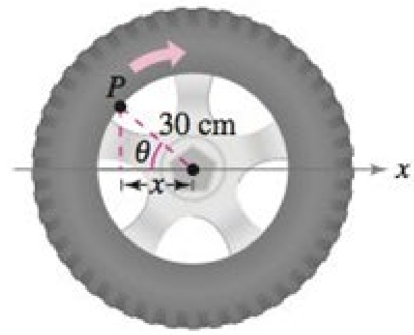
a) Find $\frac{dx}{dt}$ as a function of θ.

b) Graph the function.

c) When is the absolute value of the rate of change of *x* greatest?

d) When is it least?

e) Find $\frac{dx}{dt}$ when $\theta = 30°$ and $\theta = 60°$



Solution

Given: $r = 30$; $\frac{d\theta}{dt} = 10\ \frac{rev}{sec}\frac{2\pi\ rad}{1\ rev}$

$$= 20\pi\ rad/\sec$$

a) $\cos\theta = \frac{x}{30}$

$$-\sin\theta\frac{d\theta}{dt} = \frac{1}{30}\frac{dx}{dt}$$

$$\frac{dx}{dt} = -30\sin\theta\ \frac{d\theta}{dt}$$

$$= -600\pi\sin\theta$$

b)

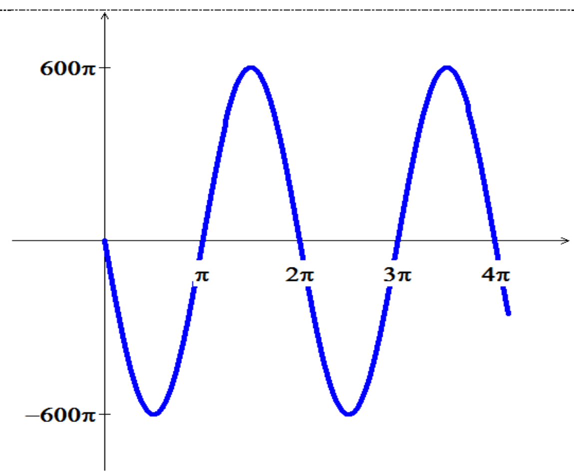
Amplitude: $|A| = 600\pi$

Period: $P = 2\pi$

Phase Shift: $\phi = 0$

VT: $y = 0$

x	***y***
0	0
$\frac{\pi}{2}$	-600π
π	0
$\frac{3\pi}{2}$	600π
2π	0



c) $|\sin\theta| = 1$ is the greatest value, therefore $\theta = \frac{\pi}{2} + n\pi$

$\therefore$ The greatest: $\frac{dx}{dt} = |-600\pi\sin\theta|$

$$= 600\pi$$

d) The least when $|\sin\theta| = 0 \ \rightarrow\ \theta = n\pi$:

$$\frac{dx}{dt} = 0$$

e) $\theta = 30°$

$$\frac{dx}{dt} = -600\pi \sin 30°$$

$$= -300\pi \ \ cm / \sec$$

$\theta = 60°$

$$\frac{dx}{dt} = -600\pi \sin 60°$$

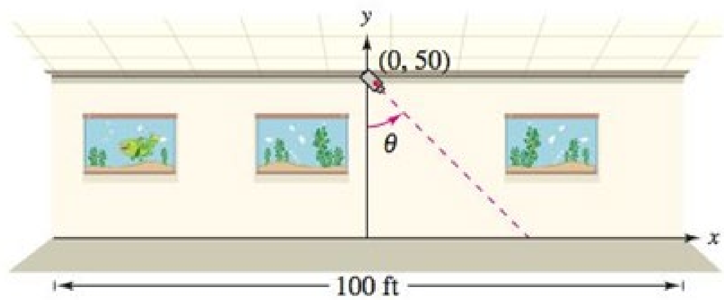
$$= -300\pi\sqrt{3} \ \ cm / \sec$$

Exercise

A security camera is centered 50 *feet* above a 100-*foot* hallway. It is easiest to design the camera with a constant angular rate of rotation, but this results in recording the images of the surveillance area at a variable rate. So, it is desirable to design a system with a variable rate of rotation and a constant rate of movement of the scanning beam along the hallway. Find a model for the variable rate of rotation when $\left|\frac{dx}{dt}\right| = 2 \ \ ft / \sec$

Solution

Given: $y = 50;\quad \left|\frac{dx}{dt}\right| = 2 \ \ ft / \sec$



$$\tan\theta = \frac{x}{50}$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{50}\frac{dx}{dt}$$

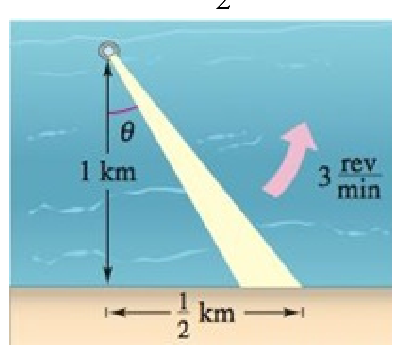
$$\frac{d\theta}{dt} = \frac{\cos^2\theta}{50}\frac{dx}{dt}$$

$$= \frac{\cos^2\theta}{50}(2)$$

$$= \frac{1}{25}\cos^2\theta \qquad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$$

Exercise

A rotating beacon is located 1 *km* off a straight shoreline. The beacon rotates at a rate of $3 \ \ rev / \min$. How fast (in km / hr) does the beam of light appear to be moving to a viewer who is $\frac{1}{2}\ km$ down the shoreline?



Solution

Given: $y = 1;\quad \frac{d\theta}{dt} = 3\ \frac{rev}{min}\,\frac{2\pi\ rad}{1\ rev}$

$$= 6\pi \ \ rad / \min$$

$$\tan\theta = x$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{dx}{dt}$$

$$\frac{dx}{dt} = \left(1 + \tan^2\theta\right)\frac{d\theta}{dt}$$

$$= \left(1 + \frac{1}{4}\right)\left(6\pi \frac{rad}{\min}\frac{60\ \min}{1\ hr}\right)$$

$$= 450\pi \quad km / hr$$

Exercise

A sandbag is dropped from a balloon at a height of 60 *m* when the angle of elevation to the sun is 30°. The position of the sandbag is

$$s(t) = 60 - 4.9t^2$$

Find the rate at which the shadow of the sandbag is traveling along the ground when the sandbag is at height of 35 *m*.

Solution

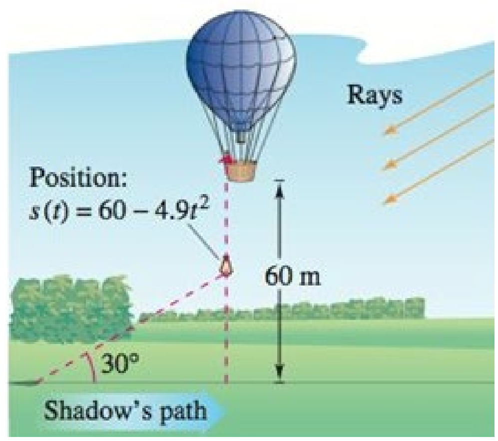
$$s(t) = 60 - 4.9t^2$$

$$\frac{d}{dt}s(t) = -9.8$$

$$s(t) = 60 - 4.9t^2 = 35$$

$$t^2 = \frac{25}{4.9}$$

$$t = \frac{5}{\sqrt{4.9}}$$



$$\tan 30° = \frac{s(t)}{x(t)} = \frac{1}{\sqrt{3}}$$

$$x(t) = \sqrt{3}\ s(t)$$

$$= 60\sqrt{3} - 4.9\sqrt{3}\ t^2$$

$$\frac{dx}{dt} = -9.8t\sqrt{3}$$

$$= -\frac{5(9.8)\sqrt{3}}{\sqrt{4.9}}$$

$$= -38.37 \quad m / \sec$$

Exercise

The distance between the head of a piston and the end of a cylindrical chamber is given by $x(t)=\frac{8t}{t+1}$ *cm*, for $t \ge 0$ (measured in seconds). The radius of the cylinder is 4 *cm*.

a) Find the volume of the chamber, for $t \ge 0$.

b) Find the rate of change of the volume $V'(t)$ for $t \ge 0$.

c) Graph the derivative of the volume function. On what intervals is the volume increasing? Decreasing?

Solution

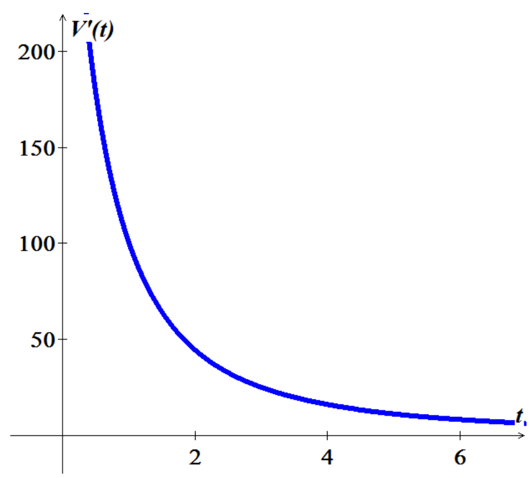
a)
$$V(t)=\pi r^2 h$$
$$=\pi(4)^2\frac{8t}{t+1}$$
$$=\frac{128\pi t}{t+1}\ cm^3$$

b)
$$V'(t)=\frac{128\pi}{(t+1)^2}$$

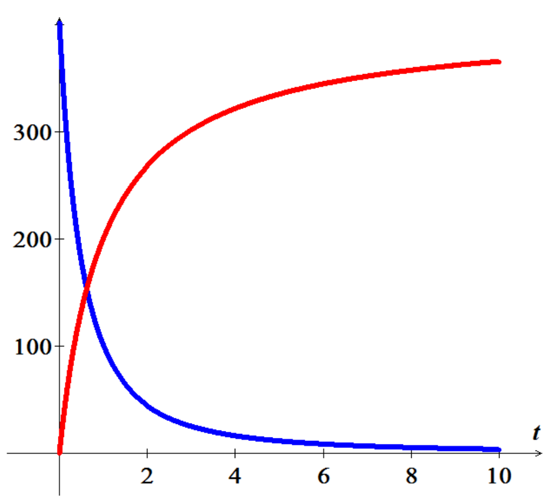
$$\left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$$

c)

t	$V'(t)$
0	128π
1	32π



Since the rate of change of the volume is strictly positive, the volume function must be increasing for $t>0$.



Exercise

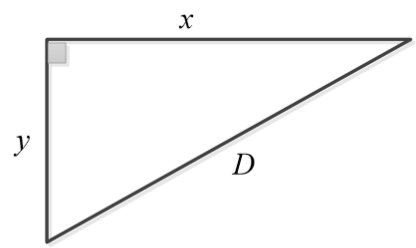
Two boats leave a dock at the same time. One boat travels south at 30 *mi/hr* and the other travels east at 40 *mi/hr*. after half an hour, how fast is the distance between the boats increasing?

Solution

Given: $\frac{dx}{dt} = 40 \ \ mi/hr, \quad \frac{dy}{dt} = 30 \ \ mi/hr, \quad t = 0.5 \ \ hr$

$t = 0.5 \quad \to x = 20 \quad y = 15$

$D^2 = x^2 + y^2$



$D = \sqrt{20^2 + 15^2}$

$\quad = 25$

$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$\frac{dD}{dt} = \frac{1}{D}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)\Big|_{\frac{dx}{dt}=40, \ \ \frac{dy}{dt}=30, \ \ D=25}$

$\quad = \frac{20(40)+15(30)}{25}$

$\quad = 50 \ \ mph$

Exercise

A spherical balloon is inflated at a rate of 10 $cm^3/\min$. At what rate is the diameter of the balloon increasing when the balloon has a diameter of 5 *cm*.

Solution

Given: $\frac{dV}{dt} = 10 \ \ cm^3/\min, \quad D = 5 \ cm$

$V = \frac{4}{3}\pi r^3$

$\quad = \frac{4}{3}\pi\left(\frac{D}{2}\right)^3$

$\quad = \frac{\pi}{6}D^3$

$\frac{dV}{dt} = \frac{\pi}{2}D^2\frac{dD}{dt}$

$\frac{dD}{dt} = \frac{2}{\pi D^2}\frac{dV}{dt}\Big|_{\frac{dV}{dt}=10 \quad D=5}$

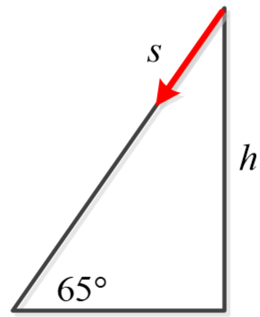
$\quad = \frac{2}{\pi 5^2}(10)$

$\quad = \frac{4}{5\pi} \ \ cm/min$

Exercise

A rope is attached to the bottom of a hot-air balloon that is floating above the flat field. If the angle of the rope to the ground remains 65° and the rope is pulled in at 5 *ft/s*, how quickly is the elevation of the balloon changing?

Solution



Given: $\frac{ds}{dt} = 5 \ ft/\sec, \quad \theta = 65°$

$$\sin 65° = \frac{h}{s}$$

$$h = s \ \sin 65°$$

$$\frac{dh}{dt} = \sin 65° \frac{ds}{dt}$$

$$= -5 \sin 65°$$

$$\boxed{\approx -4.53 \ ft/\sec}$$

Exercise

Water flows into a conical tank at a rate of $2 \ ft^3 / \min$. If the radius of the top of the tank is 4 *feet* and the height is 6 *feet*, determine how quickly the water level is rising when the water is 2 *feet* deep in the tank.

Solution

Given: $\frac{dV}{dt} = 2 \frac{ft^3}{min}, \quad y = 6\, ft, \quad x = 4\, ft, \quad h = 2\, ft$

The water forms a cone with volume: $V = \frac{1}{3}\pi r^2 h$

From the triangles: $\frac{r}{h} = \frac{4}{6}$

$$r = \frac{2}{3}h$$

$$V = \frac{1}{3}\pi \left(\frac{2}{3}h\right)^2 h$$

$$\boxed{= \frac{4\pi}{27}h^3}$$

$$\frac{dV}{dt} = \frac{4\pi}{27}\left(3h^2 \frac{dh}{dt}\right)$$

$$= \frac{4\pi}{9}h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{9}{4\pi h^2}\frac{dV}{dt}$$

$$= \frac{9}{4\pi(4)}(2)$$

$$\boxed{= \frac{9}{8\pi} \ ft/min}$$

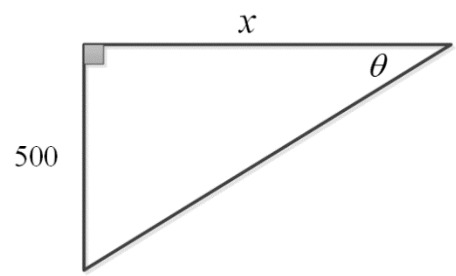
Exercise

A jet flies horizontally 500 *feet* directly above a spectator at an air show at 450 *mi/hr*. How quickly is the angle of elevation (between the ground and the line from the spectator to the jet) changing 2 seconds later?

Solution

Given: $\frac{dx}{dt} = 450\frac{mi}{hr}\frac{1\ hr}{3600\ sec}\frac{5280\ ft}{1\ mi}$

$= 660\ \ ft/\sec$



$\cot\theta = \frac{x}{500}$

$\theta = \cot^{-1}\frac{x}{500}$

$$\theta' = -\frac{\frac{d}{dt}\left(\frac{x}{500}\right)}{1+\left(\frac{x}{500}\right)^2}$$

$$= -\frac{\frac{1}{500}}{\frac{500^2+x^2}{500^2}}\frac{dx}{dt} \qquad t=2 \ \rightarrow\ x = 2(660) = 1320$$

$$= -\frac{500}{500^2+x^2}\frac{dx}{dt}\Bigg|_{\frac{dx}{dt}=660 \quad x=1320}$$

$$= -\frac{500}{500^2+1320^2}(660)$$

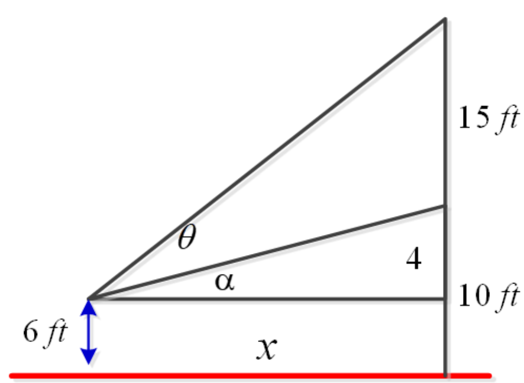
$\approx -0.166\ \ rad/sec$

Exercise

A man whose eyelevel is 6 *feet* above the ground walks toward a billboard at a rate of 2 *ft/s*. The bottom of the billboard is 10 *feet* above the ground, and it is 15 *feet* high. The man's viewing angle is the angle formed by the lines between the man's eyes and the top and bottom of the billboard. At what rate is the viewing angle changing when the man is 30 *feet* from the billboard?

Solution

Given: $\frac{dx}{dt} = -2\ \ ft/s \quad x_0 = 30\ \ ft$

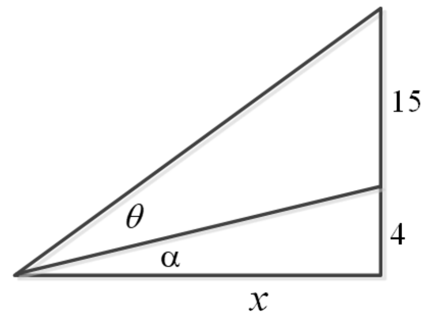


$\cot\alpha = \frac{x}{4} \ \rightarrow\ \alpha = \cot^{-1}\left(\frac{x}{4}\right)$

$\cot(\alpha+\theta) = \frac{x}{19}$

$\theta = \cot^{-1}\left(\frac{x}{19}\right) - \alpha$

$$= \cot^{-1}\left(\frac{x}{19}\right) - \cot^{-1}\left(\frac{x}{4}\right)$$



$$\theta' = -\frac{\frac{d}{dt}\left(\frac{x}{19}\right)}{1+\left(\frac{x}{19}\right)^2} + \frac{\frac{d}{dt}\left(\frac{x}{4}\right)}{1+\left(\frac{x}{4}\right)^2}$$

$$= -\frac{\frac{1}{19}\frac{dx}{dt}}{1+\frac{x^2}{361}} + \frac{\frac{1}{4}\frac{dx}{dt}}{1+\frac{x^2}{16}}$$

$$= \left(\frac{4}{x^2+16} - \frac{19}{x^2+361}\right)\frac{dx}{dt}\Bigg|_{\frac{dx}{dt}=-2 \quad x=30}$$

$$= (-2)\left(\frac{4}{900+16} - \frac{19}{900+361}\right)$$

$$= (-2)\left(\frac{1}{229} - \frac{19}{1,261}\right)$$

$$= (-2)\frac{-3090}{288,769}$$

$$\boxed{= \frac{6,180}{288,769}\ rad/sec} \qquad \boxed{\approx 0.0214\ rad/sec}$$

Exercise

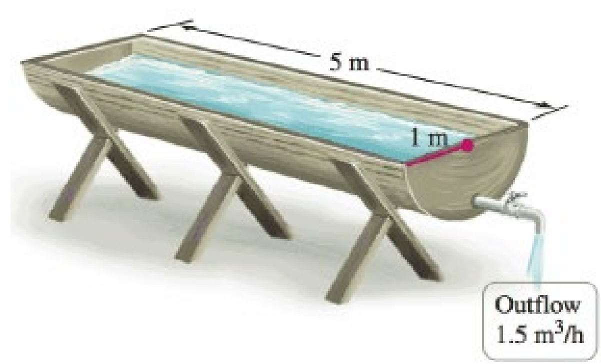
A trough is shaped like a half cylinder with length 5 *m* and radius 1 *m*. The trough is full of water when a valve is opened and water flows out of the bottom of the trough at a rate of 1.5 m^3 / hr.

(***Hint***: Area of the sector = $\frac{1}{2}r^2\theta$, *r* is the radius of a sector of the circle subtended by an angle of *θ*)

a) How fast is the water level changing when the water level is 0.5 *m* from the bottom of the through?

b) What is the rate of change of the surface area of the water when the water is 0.5 *m* deep?

Solution



$$\cos\frac{\theta}{2} = \frac{r-h}{h} \quad \Rightarrow \quad \frac{\theta}{2} = \cos^{-1}\frac{r-h}{h}$$

$$\boxed{\theta = 2\cos^{-1}\left(\frac{r-h}{h}\right)}$$

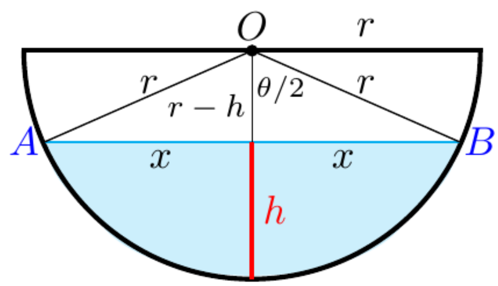
$$x^2 = r^2 - (r-h)^2$$

$$x = \sqrt{r^2 - r^2 + 2rh - h^2}$$

$$= \sqrt{2rh - h^2}$$

ΔAOB area is A_1

$$A_1 = 2\frac{1}{2}(r-h)\sqrt{2rh-h^2}$$
$$= (r-h)\sqrt{2rh-h^2}$$



Area of sector AOB is A_2

$$A_2 = \frac{1}{2}r^2\theta$$
$$= \frac{1}{2}r^2 2\cos^{-1}\left(\frac{r-h}{r}\right)$$
$$= r^2\cos^{-1}\left(\frac{r-h}{r}\right)$$

Area of cross-sectional:

$$\boxed{A = r^2\cos^{-1}\left(\frac{r-h}{r}\right) - (r-h)\sqrt{2rh-h^2}}$$

a) ***Given***: $r = 1 \quad \frac{dV}{dt} = -1.5 \quad = 0.5$

$$A = \cos^{-1}(1-h) - (1-h)\sqrt{2h-h^2}$$
$$V = 5A$$
$$= 5\cos^{-1}(1-h) - 5(1-h)\sqrt{2h-h^2}$$

$$\frac{dV}{dt} = 5\frac{1}{\sqrt{1-(1-h)^2}}\frac{dh}{dt} - 5\frac{1}{\sqrt{2h-h^2}}\left(-2h+h^2+(1-h)^2\right)\frac{dh}{dt}$$
$$= 5\frac{1}{\sqrt{1-(1-h)^2}}\frac{dh}{dt} - 5\frac{1}{\sqrt{2h-h^2}}\left(1-4h+2h^2\right)\frac{dh}{dt}$$

$$-\frac{3}{2} = \left(5\frac{1}{\sqrt{1-\frac{1}{4}}} - 5\frac{1}{\sqrt{1-\frac{1}{4}}}\left(1-2+\frac{1}{2}\right)\right)\frac{dh}{dt}$$
$$-\frac{3}{10} = \left(\frac{2}{\sqrt{3}} - \frac{2}{\sqrt{3}}\left(-\frac{1}{2}\right)\right)\frac{dh}{dt}$$
$$-\frac{3}{10} = \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right)\frac{dh}{dt}$$
$$-\frac{3}{10} = \sqrt{3}\frac{dh}{dt}$$
$$\boxed{\frac{dh}{dt} = -\frac{\sqrt{3}}{10}\ m/hr}$$

b) Surface: $S = 5(2x)$

$$S = 10\sqrt{2h-h^2}$$

$$\frac{dS}{dt} = 10\frac{1-h}{\sqrt{2h-h^2}}\frac{dh}{dt}$$

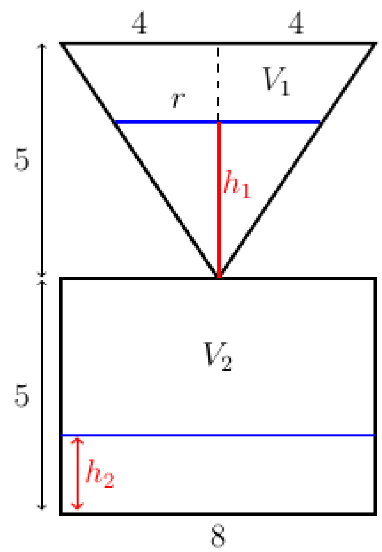
$$= 10\frac{\frac{1}{2}}{\sqrt{1-\frac{1}{4}}}\left(-\frac{\sqrt{3}}{10}\right)$$

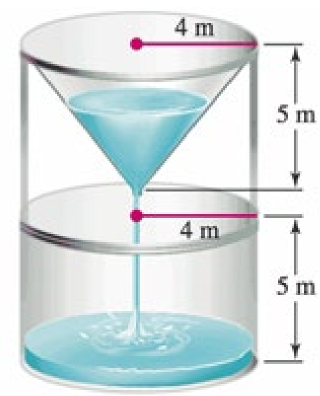
$$= -1 \ \ m^2/hr$$

Exercise

A conical tank with an upper radius of 4 *m* and a height of 5 *m* drains into a cylindrical tank with a radius of 4 *m* and a height of 5 *m*. If the water level in the conical tank drops at a rate of 0.5 *m/min*, at what rate does the water in the cylindrical tank rise when the water level in the conical tank is 3 *m*? 1 *m*?

Solution





$$V_1 = \frac{\pi}{3}r^2 h_1$$

$$\frac{r}{4} = \frac{h_1}{5} \quad \to \quad r = \frac{4h_1}{5}$$

$$V_1 = \frac{\pi}{3}\left(\frac{4h_1}{5}\right)^2 h_1$$

$$= \frac{16\pi}{75}h_1^3$$

$$\frac{dV_1}{dt} = \frac{16\pi}{25}h_1^2\frac{dh_1}{dt}$$

a) ***For*** $h_1 = 3$

Given: $\frac{dh_1}{dt} = -.5$

$$\frac{dV_1}{dt} = \frac{16\pi}{25}h_1^2\frac{dh_1}{dt}$$

$$= \frac{16\pi}{25}9\left(-\frac{1}{2}\right)$$

$$= -\frac{72\pi}{25} \ \ m^3/min$$

Volume of the lower tank:

$$V_2 = \pi r^2 h_2$$

$$= 16\pi h_2$$

$$\frac{dV_2}{dt} = 16\pi\frac{dh_2}{dt}$$

$$\frac{72\pi}{25} = 16\pi\frac{dh_2}{dt}$$

$$\boxed{\frac{dh_2}{dt} = \frac{9}{50}\ \ m/min}$$

b) **For** $h_1 = 1$

$$\frac{dV_1}{dt} = \frac{16\pi}{25}h_1^2\frac{dh_1}{dt}$$

$$= \frac{16\pi}{25}\left(-\frac{1}{2}\right)$$

$$\boxed{= -\frac{8\pi}{25}\ \ m^3/min}$$

$$\frac{dV_2}{dt} = 16\pi\frac{dh_2}{dt}$$

$$\frac{8\pi}{25} = 16\pi\frac{dh_2}{dt}$$

$$\boxed{\frac{dh_2}{dt} = \frac{1}{50}\ \ m/min}$$

Exercise

Two cylindrical swimming pools are being filled simultaneously at the same rate (in m^3/min). The smaller pool has a radius of 5 *m*, and the water level rises at a rate of 0.5 *m/min*. The larger pool has a radius of 8 *m*. How fast is the water level rising in the larger pool?

Solution

Small Pool:

$$V_s = \pi r^2 h_s$$

$$= 25\pi h_s$$

$$\frac{dV_s}{dt} = 25\pi\frac{dh_s}{dt}$$

$$= 25\pi\left(\frac{1}{2}\right)$$

$$\boxed{= 12.5\pi\ \ m^3/min}$$

Large Pool:

$$V_L = \pi r^2 h_L$$

$$= 64\pi h_L$$

$$\frac{dV_L}{dt} = 64\pi\frac{dh_L}{dt}$$

Since $\dfrac{dV_L}{dt} = \dfrac{dV_s}{dt}$

$$\frac{25\pi}{2} = 64\pi\frac{dh_L}{dt}$$

$$\boxed{\frac{dh_L}{dt} = \frac{25}{128}\ m/min}$$

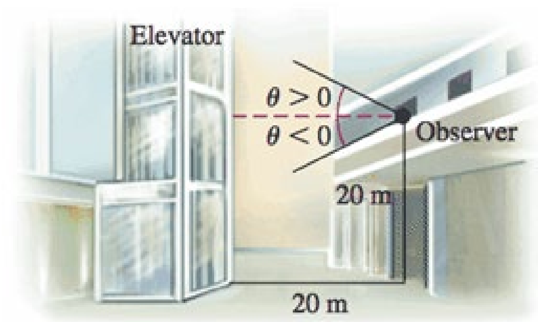
Exercise

An observer is 20 *m* above the ground floor of a large hotel atrium looking at a glass enclosed elevator shaft that is 20 *m* horizontally from the observer. The angle of elevation of the elevator is the angle that the observer's line of sight makes with the horizontal (it may be positive or negative).

a) Assuming that the elevator rises at a rate of 5 *m/s*, what is the rate of change of the angle of the angle of elevation when the elevator is 10 *m* above the ground?

b) When the elevator is 40 *m* above the ground?

Solution



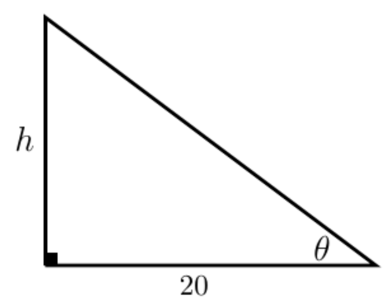
$$\tan\theta = \frac{h}{20} \quad \rightarrow \quad h = 20\tan\theta$$

$$\frac{dh}{dt} = 20\sec^2\theta\frac{d\theta}{dt} \quad (1)$$

a) ***Given***: $\dfrac{dh}{dt} = 5$

At $h = -10 \quad \rightarrow \quad \tan\theta = -\dfrac{10}{20} = -\dfrac{1}{2}$

$$\sec^2\theta = 1 + \frac{1}{4} = \boxed{\frac{5}{4}} \qquad \sec^2\theta = 1 + \tan^2\theta$$



$$(1) \rightarrow \quad \frac{d\theta}{dt} = \frac{1}{20\sec^2\theta}\frac{dh}{dt}$$

$$= \frac{1}{20\frac{5}{4}}(5)$$

$$\boxed{= \frac{1}{5}\ rad\ /\ sec}$$

b) 40 *m* above the ground $\Rightarrow$ $h = 20$

$$\tan\theta = \frac{20}{20} = 1$$

$$\sec^2\theta = 1 + 1 = \boxed{2} \qquad \sec^2\theta = 1 + \tan^2\theta$$

$$\frac{d\theta}{dt} = \frac{1}{20(2)}(5)$$

$$\boxed{= \frac{1}{8}\ rad\ /\ sec}$$

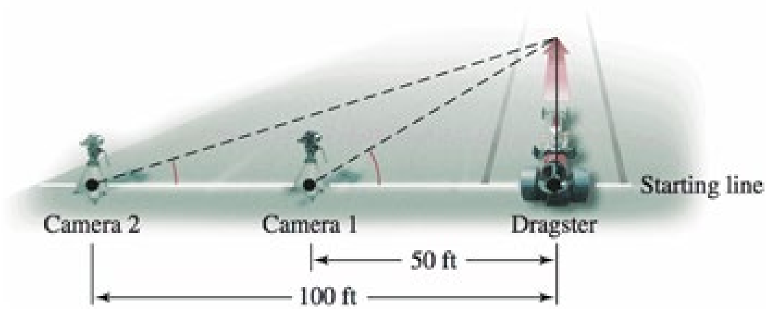
Exercise

A camera is set up at the starting line of a drag race 50 *ft*. from a dragster at the starting line (camera 1). Two seconds after the start race, the dragster has traveled 100 *ft*. and the camera is turning at 0.75 *rad/s* while filming the dragster.

a) What is the speed of the dragster at this point?

b) A second camera (camera 2) filming the dragster is located on the starting line 100 *ft*. away from the dragster at the start of the race. How fast is this camera turning 2 *sec* after the start of the race?

Solution



a) $\tan\theta = \dfrac{y}{50}$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{50}\frac{dy}{dt} \quad (1)$$

Given: $\dfrac{d\theta}{dt} = .75 = \dfrac{3}{4}$

$t = 2 \;\rightarrow\; y = 100$

$$\tan\theta = \frac{100}{50} = 2$$

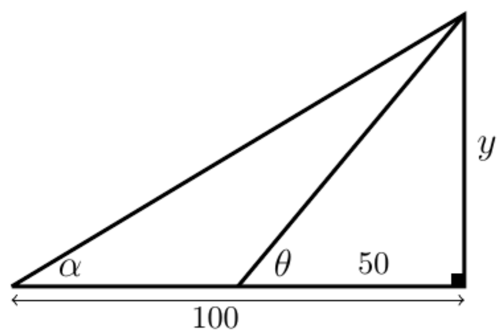
$$\sec^2\theta = 1 + 4 \qquad\qquad \sec^2\theta = 1 + \tan^2\theta$$

$$= 5$$

$$(1) \rightarrow \quad 5\left(\frac{3}{4}\right) = \frac{1}{50}\frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{375}{2} \;\; ft/\sec$$

$\therefore$ Speed at $t = 2$ *sec* is 187.5 $ft/\sec$



b) $\tan\alpha = \dfrac{y}{100}$

$$\sec^2\alpha \frac{d\alpha}{dt} = \frac{1}{100}\frac{dy}{dt}$$

$$\frac{d\alpha}{dt} = \frac{1}{100\sec^2\alpha}\frac{dy}{dt}$$

$$\tan\alpha = \frac{100}{100} = 1$$

$$\sec^2\theta = 1 + 1 \qquad\qquad \sec^2\theta = 1 + \tan^2\theta$$

$$= 2$$

$$\frac{d\alpha}{dt} = \frac{1}{200}\frac{375}{2}$$

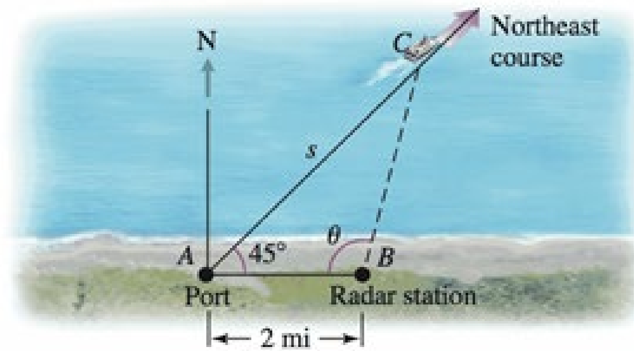
$$= \frac{15}{16} \;\; rad/\sec$$ Speed of the camera 2.

Exercise

A port and a radar station are 2 *mi* apart on a straight shore running east and west. A ship leaves the port at noon traveling northeast at a rate of 15 *mi/hr*. If the ship maintains its speed and course, what is the rate of change of the tracking angle θ between the shore and the line between the radar station and the ship at 12:30 PM?

Solution

$$C = 180° - 45° - \theta$$
$$= 135° - \theta$$
$$= \frac{3\pi}{4} - \theta$$



$$\frac{\sin\theta}{s} = \frac{\sin\left(\frac{3\pi}{4} - \theta\right)}{2}$$

$$2\sin\theta = s\left(\sin\frac{3\pi}{4}\cos\theta - \cos\frac{3\pi}{4}\sin\theta\right)$$
$$= s\left(\frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta\right)$$
$$= \frac{\sqrt{2}}{2}s\left(\cos\theta + \sin\theta\right)$$

$$4\sin\theta = \sqrt{2}s\left(\cos\theta + \sin\theta\right) \quad \times\frac{1}{\cos\theta}$$

$$4\tan\theta = \sqrt{2}s + \sqrt{2}s\tan\theta$$

$$\left(4 - \sqrt{2}s\right)\tan\theta = \sqrt{2}s$$

$$\tan\theta = \frac{\sqrt{2}s}{4 - \sqrt{2}s} \quad \to \quad \theta = \tan^{-1}\frac{\sqrt{2}s}{4 - \sqrt{2}s}$$

$$\frac{d\theta}{dt} = \frac{4\sqrt{2}}{\left(4 - \sqrt{2}s\right)^2} \cdot \frac{1}{1 + \left(\frac{\sqrt{2}s}{4 - \sqrt{2}s}\right)^2}\frac{ds}{dt} \qquad \left(\frac{ax+b}{cx+d}\right)' = \frac{ad - bc}{\left(cx+d\right)^2}$$

$$= \frac{4\sqrt{2}}{16 - 8\sqrt{2}s + 4s^2}\frac{ds}{dt}$$

$$= \frac{\sqrt{2}}{4 - 2\sqrt{2}s + s^2}\frac{ds}{dt} \qquad s = 7.5 = \frac{15}{2} \quad \frac{ds}{dt} = 15$$

$$= \frac{\sqrt{2}}{4 - 15\sqrt{2} + \frac{225}{4}}(15)$$

$$= \boxed{\frac{60\sqrt{2}}{241 - 60\sqrt{2}}\ rad/hr} \quad \approx 0.54\ \ rad/hr$$

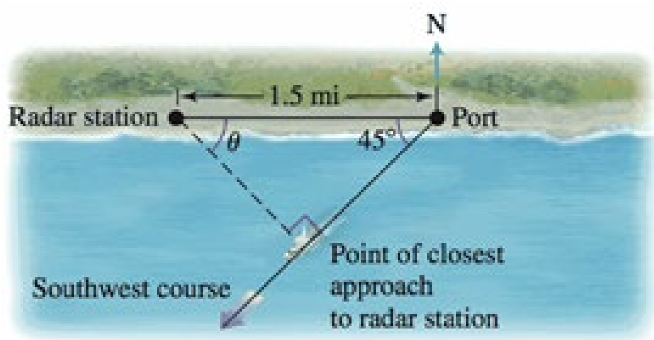
Exercise

A ship leaves port traveling southwest at a rate of 12 *mi/hr*. At noon, the ship reaches its closest approach to a radar station, which is on the shore 1.5 *mi* from the port. If the ship maintains its speed and course, what is the rate of change of the tracking angle θ between the radar station and the ship at 1:30 PM?

Solution

Let x be the distance the ship has traveled.

$$P = 180° - 45° - \theta$$
$$= 135° - \theta$$
$$= \frac{3\pi}{4} - \theta$$



$$\frac{\sin\theta}{x} = \frac{\sin\left(\frac{3\pi}{4} - \theta\right)}{\frac{3}{2}}$$

$$3\sin\theta = 2x\left(\sin\frac{3\pi}{4}\cos\theta - \cos\frac{3\pi}{4}\sin\theta\right)$$
$$= 2x\left(\frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta\right)$$
$$= \sqrt{2}x(\cos\theta + \sin\theta)$$

$$3\sin\theta = \sqrt{2}x(\cos\theta + \sin\theta) \quad \times\frac{1}{\cos\theta}$$

$$3\tan\theta = \sqrt{2}x + \sqrt{2}x\tan\theta$$

$$\left(3 - \sqrt{2}x\right)\tan\theta = \sqrt{2}x$$

$$\tan\theta = \frac{\sqrt{2}x}{3 - \sqrt{2}x} \quad \rightarrow \quad \theta = \tan^{-1}\frac{\sqrt{2}x}{3 - \sqrt{2}x}$$

$$\frac{d\theta}{dt} = \frac{3\sqrt{2}}{\left(3 - \sqrt{2}x\right)^2} \cdot \frac{1}{1 + \left(\frac{\sqrt{2}x}{3 - \sqrt{2}x}\right)^2}\frac{dx}{dt} \qquad \left(\frac{ax+b}{cx+d}\right)' = \frac{ad - bc}{(cx+d)^2}$$

$$= \frac{3\sqrt{2}}{9 - 6\sqrt{2}x + 4x^2}\frac{dx}{dt}$$

At 12:00: $\quad 2x^2 = 1.5^2 = \frac{9}{4} \quad \rightarrow \quad x = \frac{3}{2\sqrt{2}}$

At 1:30: $\quad x = \left(12 + \frac{12}{2}\right) + \frac{3}{2\sqrt{2}}$

$$= 18 + \frac{3}{2\sqrt{2}} \quad mi$$
$$\approx 19.06 \quad mi$$

$$\frac{d\theta}{dt} = \frac{3\sqrt{2}}{9 - 6\sqrt{2}(19.06) + 4(19.06)^2}(12) \qquad \frac{dx}{dt} = 12$$

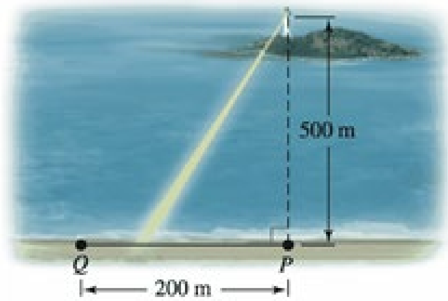
$$\approx 0.04 \quad rad/hr$$

Exercise

A lighthouse stands 500 *m* off of a straight shore, the focused beam of its light revolving four times each minute. *P* is the point on shore closest to the lighthouse and *Q* is a point on the shore 200 *m* from *P*.

a) What is the speed of the beam along the shore when it strikes the point *Q*?

b) Describe how the speed of the beam along the shore varies with the distance between *P* and *Q*. (neglect the height of the lighthouse)



Solution

a) $\tan\alpha = \dfrac{x}{500}$

$$\sec^2\alpha \frac{d\alpha}{dt} = \frac{1}{500}\frac{dx}{dt} \quad (1)$$

At Point *Q*:

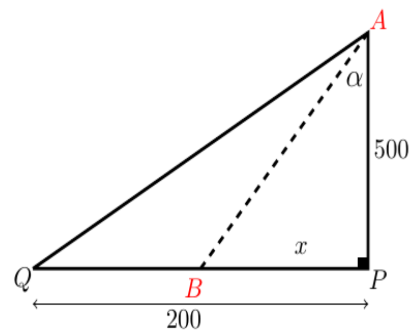
$$\tan\alpha = \frac{200}{500} = \frac{2}{5}$$

$$\sec^2\theta = 1 + \frac{4}{25} = \frac{29}{25} \qquad \sec^2\theta = 1 + \tan^2\theta$$

4 times each minute:

$$\frac{d\alpha}{dt} = 4\frac{2\pi}{1\ min}\frac{1\ min}{60\ sec}$$

$$= \frac{2\pi}{15}\ rad/\sec$$



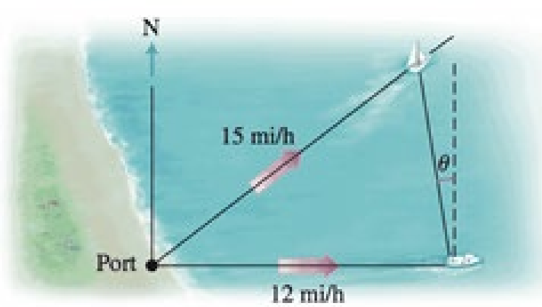
$$(1) \to \quad \frac{29}{25}\frac{2\pi}{15} = \frac{1}{500}\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{232\pi}{3}\ m/\sec$$

b) The beam moves slower when *B* near *P*, and faster as it approaches *Q* (or further away from *P*)

Exercise

A boat leaves a port traveling due east at 12 *mi/hr*. At the same time, another boat leaves the same port traveling northeast at 15 *mi/hr*. The angle $\boldsymbol{\theta}$ of the line between the boats is measured relative to due north.



What is the rate of change of this angle 30 *min*. after the boats leave the port? 2 *hr*. after the boats leave the port?

Solution

Given: $\frac{dx}{dt}=12 \quad \frac{dy}{dt}=15$

$$\frac{y}{\sin\left(\frac{\pi}{2}-\theta\right)}=\frac{x}{\sin\left(\frac{\pi}{4}+\theta\right)} \qquad (\textit{Law of Sines})$$

$$y\sin\left(\frac{\pi}{4}+\theta\right)=x\sin\left(\frac{\pi}{2}-\theta\right)$$

$$y\left(\sin\frac{\pi}{4}\cos\theta+\cos\frac{\pi}{4}\sin\theta\right)=x\left(\sin\frac{\pi}{2}\cos\theta-\cos\frac{\pi}{2}\sin\theta\right)$$

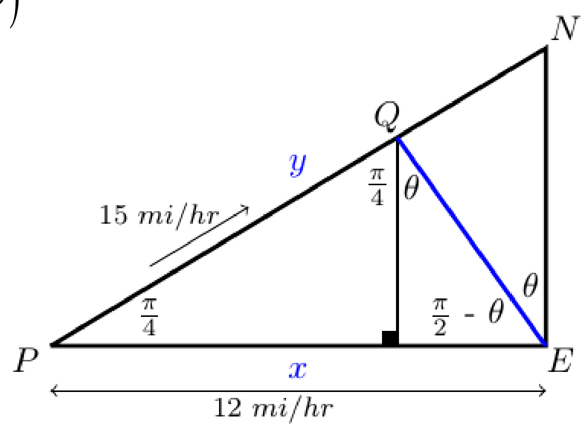
$$\frac{1}{\sqrt{2}}y(\cos\theta+\sin\theta)=x\cos\theta$$

$$\cos\theta+\sin\theta=\frac{x}{y}\sqrt{2}\cos\theta$$

$$1+\tan\theta=\frac{x}{y}\sqrt{2}$$

$$\tan\theta=\frac{x\sqrt{2}-y}{y}$$

$$\theta=\tan^{-1}\frac{x\sqrt{2}-y}{y}$$



$$\frac{d\theta}{dt}=\frac{\sqrt{2}y\frac{dx}{dt}-y\frac{dy}{dt}-\sqrt{2}x\frac{dy}{dt}+y\frac{dy}{dt}}{y^2}\cdot\frac{1}{1+\left(\frac{x\sqrt{2}-y}{y}\right)^2}$$

$$\left(\tan^{-1}u\right)'=\frac{u'}{1+u^2}$$

$$=\frac{\sqrt{2}\left(y\frac{dx}{dt}-x\frac{dy}{dt}\right)}{y^2+\left(x\sqrt{2}-y\right)^2}$$

After **30** *min* $\left(=\frac{1}{2}hr\right)$

$$\rightarrow \quad x=\frac{12}{2}=6\ mi \quad y=\frac{15}{2}=7.5\ mi$$

$$\frac{d\theta}{dt}=\frac{\sqrt{2}\left(\frac{15}{2}(12)-6(15)\right)}{\frac{225}{4}+\left(6\sqrt{2}-\frac{15}{2}\right)^2}$$

$$\underline{=0\,|}$$

After **2** *hrs.*

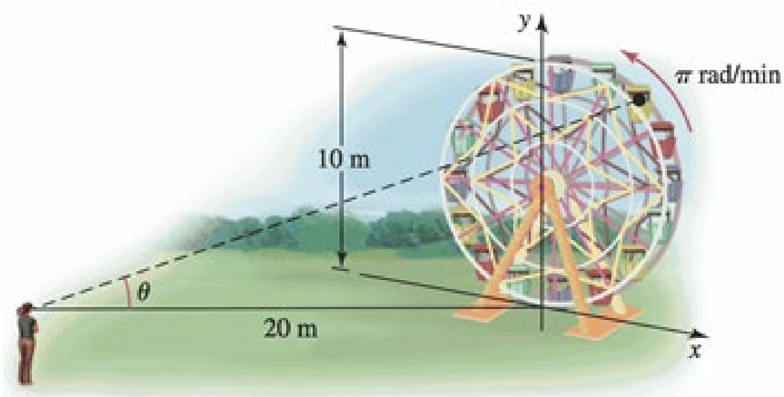
$$\rightarrow \quad x=12(2)=24\ mi \quad y=15(2)=30\ mi$$

$$\frac{d\theta}{dt}=\frac{\sqrt{2}\left(30(12)-24(15)\right)}{900+\left(24\sqrt{2}-30\right)^2}$$

$$\underline{=0\,|}$$

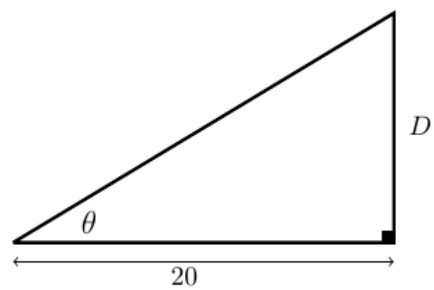
Exercise

An observer stands 20 *m* from the bottom of a 10-*m* tall Ferris wheel on a line that is perpendicular to the face of the Ferris wheel. The wheel revolves at a rate of π *rad/min* and the observer's line of sight with a specific seat on the wheel makes an angle $\boldsymbol{\theta}$ with the ground. 40 *seconds* after that seat leaves the lowest point on the wheel, what is the rate of change of $\boldsymbol{\theta}$?



Assume the observer's eyes are level with the bottom of the wheel.

Solution



$$\tan\theta = \frac{D}{20}$$

$$\theta = \tan^{-1}\frac{D}{20}$$

$$\frac{d\theta}{dt} = \frac{1}{20}\frac{1}{1+\frac{D^2}{400}}\frac{dD}{dt}$$

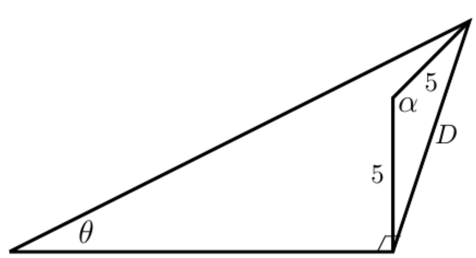
$$= \frac{20}{400+D^2}\frac{dD}{dt} \quad (1)$$

$$D^2 = 5^2+5^2-2(5)(5)\cos\alpha \qquad (\textit{Law of Cosines})$$

$$D^2 = 50-50\cos\alpha$$

$$2D\frac{dD}{dt} = 50\sin\alpha\frac{d\alpha}{dt}$$

$$\frac{dD}{dt} = \frac{25}{D}\sin\alpha\frac{d\alpha}{dt} \quad (2)$$



At $t = 40\ sec\frac{1\ min}{60\ sec} = \frac{2}{3}\ min$

Given: $\frac{d\alpha}{dt} = \pi\ \frac{rad}{min} \quad \to \quad \alpha = \frac{2\pi}{3}\ rad$

$$D = \sqrt{50-50\cos\frac{2\pi}{3}}$$

$$= \sqrt{50+25}$$

$$= 5\sqrt{3}$$

$$(2) \to \quad \frac{dD}{dt} = \frac{25}{5\sqrt{3}}\sin\left(\frac{2\pi}{3}\right)(\pi)$$

$$\frac{dD}{dt} = \frac{5\pi}{\sqrt{3}} \frac{\sqrt{3}}{2}$$

$$= \boxed{\frac{5\pi}{2}\ rad/min}$$

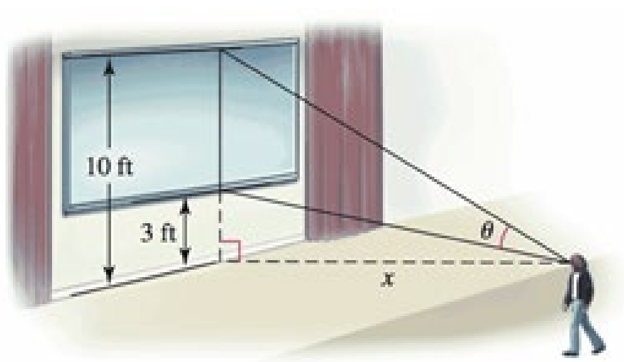
$$(1) \quad \frac{d\theta}{dt} = \frac{20}{400+75}\left(\frac{5\pi}{2}\right)$$

$$\boxed{\frac{d\theta}{dt} = \frac{2\pi}{19}\ rad/min}$$

Exercise

The bottom of a large theater screen is 3 *ft*. above your eye level and the top of the screen is 10 *ft*. above your eye level. Assume you walk away from the screen (perpendicular to the screen) at a rate of 3 *ft/s* while looking at the screen. What is the rate of change of the viewing angle $\boldsymbol{\theta}$ when you are 30 *ft*. from the wall on which the screen hangs, assuming the floor is flat?

Solution



$$\tan\alpha = \frac{3}{x}$$

$$\alpha = \tan^{-1}\frac{3}{x}$$

$$\tan(\alpha+\theta) = \frac{10}{x}$$

$$\alpha+\theta = \tan^{-1}\frac{10}{x}$$

$$\theta = \tan^{-1}\frac{10}{x} - \alpha$$

$$= \tan^{-1}\frac{10}{x} - \tan^{-1}\frac{3}{x}$$

$$\frac{d\theta}{dt} = \left(-\frac{10}{x^2}\right)\frac{1}{1+\frac{100}{x^2}}\frac{dx}{dt} - \left(-\frac{3}{x^2}\right)\frac{1}{1+\frac{9}{x^2}}\frac{dx}{dt}$$

$$= \left(\frac{3}{x^2+9} - \frac{10}{x^2+100}\right)\frac{dx}{dt} \qquad \frac{dx}{dt} = 3, \quad x = 30$$

$$= \left(\frac{3}{900+9} - \frac{10}{900+100}\right)(3)$$

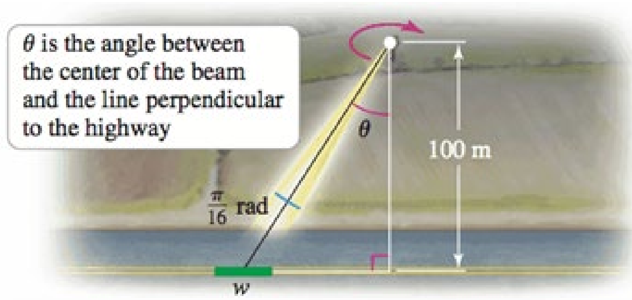
$$= 3\left(\frac{3}{909} - \frac{1}{100}\right)$$

$$= 3\frac{-609}{90900}$$

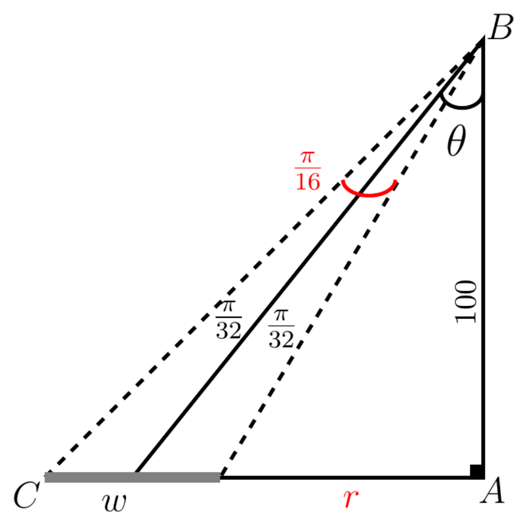
$$\boxed{= -\frac{203}{10100}\ rad/\sec}$$

Exercise

A revolving searchlight, 100 *m* from the nearest point on the center line of a straight highway, casts a horizontal beam along a highway. The beam leaves the spotlight at an angle of $\frac{\pi}{16}$ *rad* and revolves at a rate $\frac{\pi}{16}$ *rad/s*. Let $\boldsymbol{w}$ be the width of the beam as it sweeps along the highway and $\boldsymbol{\theta}$ be the angle that the center of the beam makes with the perpendicular to the highway. What is the rate of change of $\boldsymbol{w}$ when $\theta = \frac{\pi}{3}$? *Neglect the height of the lighthouse.*



Solution



$$\tan\left(\theta - \frac{\pi}{32}\right) = \frac{r}{100} \quad \Rightarrow \quad r = 100\tan\left(\theta - \frac{\pi}{32}\right)$$

$$\ell = AC$$

$$\tan\left(\theta + \frac{\pi}{32}\right) = \frac{\ell}{100} \quad \Rightarrow \quad \ell = 100\tan\left(\theta + \frac{\pi}{32}\right)$$

$$w = \ell - r$$

$$\frac{dw}{dt} = \frac{d\ell}{dt} - \frac{dr}{dt}$$

$$= \frac{d}{dt}\left(100\tan\left(\theta + \frac{\pi}{32}\right)\right) - \frac{d}{dt}\left(100\tan\left(\theta - \frac{\pi}{32}\right)\right)$$

$$= 100\left(\frac{1}{1+\left(\theta + \frac{\pi}{32}\right)^2} - \frac{1}{1+\left(\theta - \frac{\pi}{32}\right)^2}\right)\frac{d\theta}{dt}\Bigg|_{\frac{d\theta}{dt}=\frac{\pi}{16},\ \theta=\frac{\pi}{3}}$$

$$= 100\left(\frac{1}{1+\left(\frac{\pi}{3} + \frac{\pi}{32}\right)^2} - \frac{1}{1+\left(\frac{\pi}{3} - \frac{\pi}{32}\right)^2}\right)\frac{\pi}{16}$$

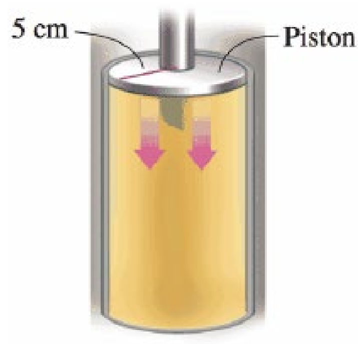
$$= \frac{25\pi}{4}\left(\frac{96^2}{96^2 + (35\pi)^2} - \frac{96^2}{96^2 + (29\pi)^2}\right)$$

$$= 57{,}600\pi\left(\frac{1}{96^2 + (35\pi)^2} - \frac{1}{96^2 + (29\pi)^2}\right)$$

$$\approx -4.9 \ \ m/\sec$$

Exercise

A piston is seated at the top of a cylindrical chamber with radius 5 *cm* when it starts moving into the chamber at a constant speed of 3 *cm/sec*. What is the rate of change of the volume of the cylinder when the piston is 2 *cm* from the base of the chamber?



Solution

$V = 25\pi h \qquad\qquad V = \pi r^2 h$

$$\frac{dV}{dt} = 25\pi \frac{dh}{dt}$$

$$= 25\pi(-3)$$

$$\underline{= -75\pi \ \ cm/s}$$