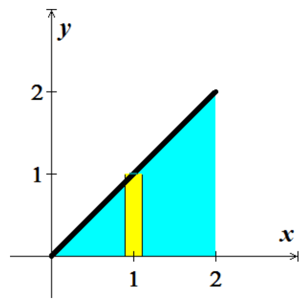
Solution   *Section* 1.4 – Volumes by Shells

Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the y-*axis*: $y = x$

Solution

$$V = 2\pi \int_0^2 x(x)\,dx \qquad V = 2\pi \int_a^b x\,f(x)\,dx$$



$$= 2\pi \int_0^2 x^2\,dx$$

$$= \frac{2\pi}{3} x^3 \Big|_0^2$$

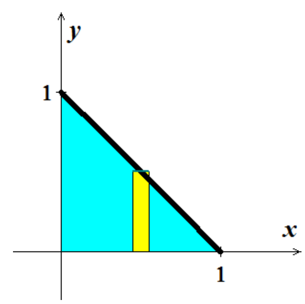
$$= \frac{16\pi}{3}$$

Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the y-*axis* $y = 1 - x$

Solution

$$V = 2\pi \int_0^1 x(1-x)\,dx \qquad V = 2\pi \int_a^b x\,f(x)\,dx$$



$$= 2\pi \int_0^1 \left(x - x^2\right)\,dx$$

$$= 2\pi \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1$$

$$= 2\pi \left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{\pi}{3}$$

Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the y-*axis* $y = \sqrt{x}$

Solution

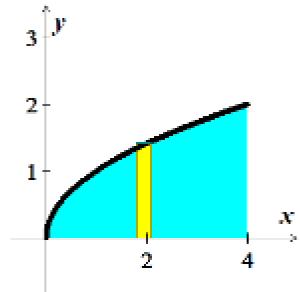
$$V = 2\pi \int_0^4 x\sqrt{x}\,dx \qquad V = 2\pi \int_a^b x\,f(x)\,dx$$

$$= 2\pi \int_0^4 x^{3/2}\ dx$$

$$= 2\pi \left(\frac{2}{5} x^{5/2} \right) \Big|_0^4$$

$$= \frac{4\pi}{5} \left(2^2 \right)^{5/2}$$

$$= \frac{128\pi}{5}$$



Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis* $\quad y = \frac{1}{2}x^2 + 1$

Solution

$$f(x) = 3 - \left(\frac{1}{2}x^2 + 1 \right) = 2 - \frac{1}{2}x^2$$

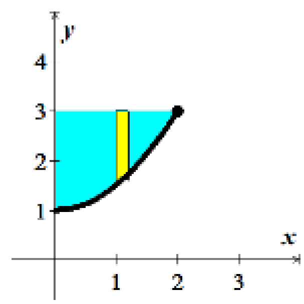
$$V = 2\pi \int_0^2 x\left(2 - \frac{1}{2}x^2 \right)\ dx \qquad V = 2\pi \int_a^b x\, f(x)\, dx$$

$$= 2\pi \int_0^2 \left(2x - \frac{1}{2}x^3 \right)\ dx$$

$$= 2\pi \left(x^2 - \frac{1}{8}x^4 \right) \Big|_0^2$$

$$= 2\pi (4 - 2)$$

$$= 4\pi$$



Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

$$y = \frac{1}{4}x^2, \quad y = 0, \quad x = 4$$

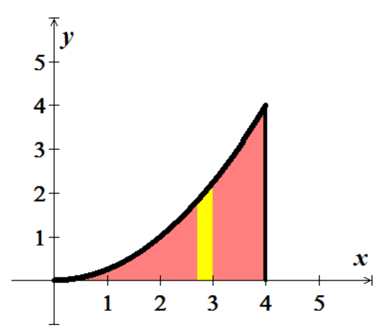
Solution

$$V = 2\pi \int_0^4 x\left(\frac{1}{4}x^2 \right) dx \qquad V = 2\pi \int_a^b x\big(f(x) - g(x) \big) dx$$

$$= \frac{\pi}{2} \int_0^4 x^3 dx$$

$$= \frac{\pi}{8} x^4 \Big|_0^4$$

$$= 32\pi$$



Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

$$y = \frac{1}{2}x^3, \quad y = 0, \quad x = 3$$

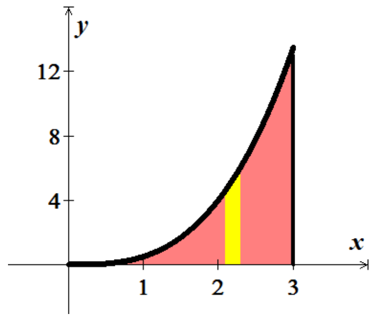
Solution

$$V = 2\pi \int_0^3 x\left(\frac{1}{2}x^3\right)dx \qquad V = 2\pi \int_a^b x\big(f(x) - g(x)\big)dx$$

$$= \pi \int_0^3 x^4 dx$$

$$= \frac{\pi}{5}x^5 \Big|_0^3$$

$$= \frac{243\pi}{5}$$



Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

$$y = x^2, \quad y = 4x - x^2$$

Solution

$$y = 4x - x^2 = x^2 \Rightarrow 2x^2 - 4x = 0 \rightarrow x = 0, \ 2$$

$$f(x) = 4x - x^2, \quad g(x) = x^2$$

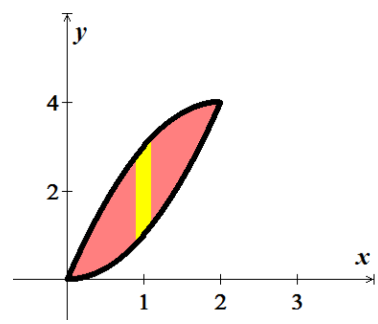
$$V = 2\pi \int_0^2 x\left(4x - x^2 - x^2\right)dx \qquad V = 2\pi \int_a^b x\big(f(x) - g(x)\big)dx$$

$$= 4\pi \int_0^2 \left(2x^2 - x^3\right)dx$$

$$= 4\pi\left(\frac{2}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^2$$

$$= 4\pi\left(\frac{16}{3} - 4\right)$$

$$= \frac{16\pi}{3}$$



Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

$$y = 9 - x^2, \quad y = 0$$

Solution

$y = 9 - x^2 = 0 \quad \to \quad \underline{x = \pm 3}$

$f(x) = 9 - x^2, \quad g(x) = 0$

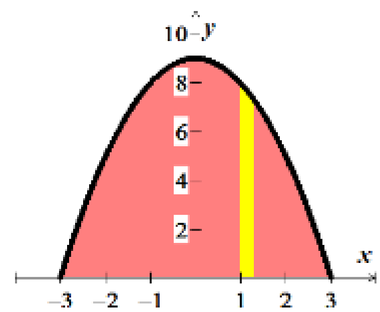
$$V = 2\pi \int_0^3 x\left(9 - x^2\right) dx \qquad\qquad V = 2\pi \int_a^b x\left(f(x) - g(x)\right) dx$$

$$= 2\pi \int_0^3 \left(9x - x^3\right) dx$$

$$= 2\pi \left(\frac{9}{2}x^2 - \frac{1}{4}x^4\right)\Big|_0^3$$

$$= 2\pi \left(\frac{81}{2} - \frac{81}{4}\right)$$

$$= \frac{81\pi}{2}$$



Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

$$y = 4x - x^2, \quad x = 0, \quad y = 4$$

Solution

$y = 4x - x^2 = 4 \Rightarrow x^2 - 4x + 4 \quad \to \quad \underline{x = 2}$

$f(x) = 4, \quad g(x) = 4x - x^2$

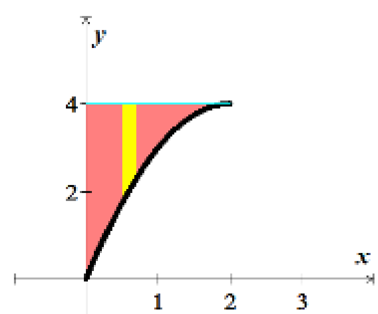
$$V = 2\pi \int_0^2 x\left(4 - 4x + x^2\right) dx \qquad\qquad V = 2\pi \int_a^b x\left(f(x) - g(x)\right) dx$$

$$= 2\pi \int_0^2 \left(4x - 4x^2 + x^3\right) dx$$

$$= 2\pi \left(2x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4\right)\Big|_0^2$$

$$= 2\pi \left(8 - \frac{32}{3} + 4\right)$$

$$= \frac{8\pi}{3}$$



Exercise

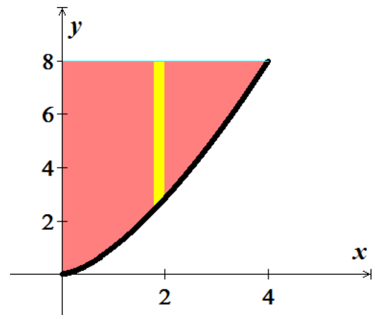
Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

$$y = x^{3/2}, \quad y = 8, \quad x = 0$$

Solution

$$y = x^{3/2} = 8 \Rightarrow x = \left(2^3\right)^{2/3} \quad \to \underline{x = 4}$$

$$f(x) = 8, \quad g(x) = x^{3/2}$$



$$V = 2\pi \int_0^4 x\left(8 - x^{3/2}\right)dx \qquad V = 2\pi \int_a^b x\left(f(x) - g(x)\right)dx$$

$$= 2\pi \int_0^4 \left(8x - x^{5/2}\right)dx$$

$$= 2\pi \left(4x^2 - \frac{2}{7}x^{7/2}\right)\Big|_0^4$$

$$= 2\pi \left(64 - \frac{256}{7}\right)$$

$$= \frac{384\pi}{7}$$

Exercise

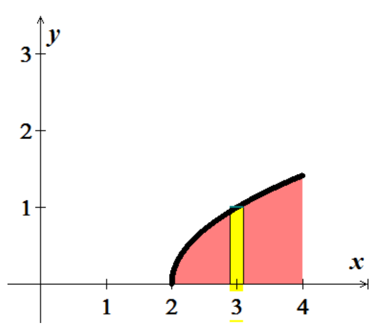
Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

$$y = \sqrt{x-2}, \quad y = 0, \quad x = 4$$

Solution

$$y = \sqrt{x-2} = 0 \quad \to \underline{x = 2}$$

$$f(x) = \sqrt{x-2}, \quad g(x) = 0$$



$$V = 2\pi \int_2^4 x\left(\sqrt{x-2}\right)dx \qquad V = 2\pi \int_a^b x\left(f(x) - g(x)\right)dx$$

$$= 2\pi \int_2^4 (u+2)u^{1/2}\, du \qquad u = x - 2 \quad x = u + 2 \quad du = dx$$

$$= 2\pi \int_2^4 \left(u^{3/2} + 2u^{1/2}\right) du$$

$$= 2\pi \left(\frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2}\right)\Big|_2^4$$

$$= 2\pi \left(\frac{2}{5}(x-2)^{5/2} + \frac{4}{3}(x-2)^{3/2}\right)\Big|_2^4$$

$$= 2\pi\left(\frac{8\sqrt{2}}{5} + \frac{8\sqrt{2}}{3}\right)$$

$$= 16\pi\sqrt{2}\left(\frac{1}{5} + \frac{1}{3}\right)$$

$$= \boxed{\frac{128\pi\sqrt{2}}{15}}$$

Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

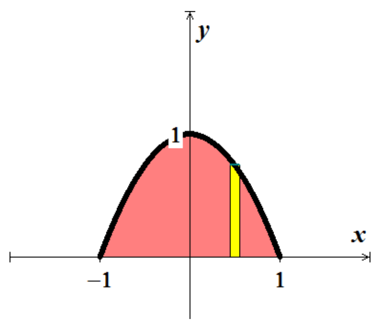
$$y = -x^2 + 1, \quad y = 0$$

Solution

$$y = -x^2 + 1 = 0 \quad \to \quad \underline{x = \pm 1}$$

$$f(x) = -x^2 + 1, \quad g(x) = 0$$

$$V = 2\pi\int_0^1 x\left(-x^2 + 1\right)dx \qquad V = 2\pi\int_a^b x\left(f(x) - g(x)\right)dx$$



$$= 2\pi\int_0^1 \left(-x^3 + x\right)\,dx$$

$$= 2\pi\left(-\frac{1}{4}x^4 + \frac{1}{2}x^2\right)\Big|_0^1$$

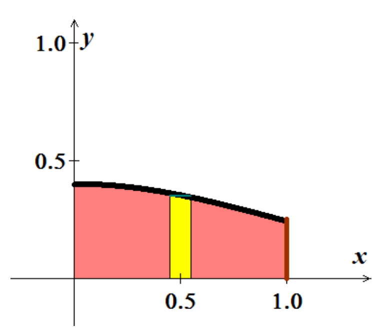
$$= 2\pi\left(-\frac{1}{4} + \frac{1}{2}\right)$$

$$= \boxed{\frac{\pi}{2}}$$

Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

$$y = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \quad y = 0, \quad x = 0, \quad x = 1$$



Solution

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \quad g(x) = 0$$

$$V = 2\pi\int_0^1 x\left(\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right)dx \qquad V = 2\pi\int_a^b x\left(f(x) - g(x)\right)dx$$

$$= -\sqrt{2\pi}\int_0^1 e^{-x^2/2}\,d\left(-\frac{x^2}{2}\right)$$

$$= -\sqrt{2\pi}\left(e^{-x^2/2}\right)\Big|_0^1$$

$$= -\sqrt{2\pi}\left(e^{-1/2} - 1\right)$$

$$= \sqrt{2\pi}\left(1 - \frac{1}{\sqrt{e}}\right)$$

Exercise

Use the shell method to find the volume of the solid generated by revolving the shaded region about the indicated axis

Solution

$$V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx$$

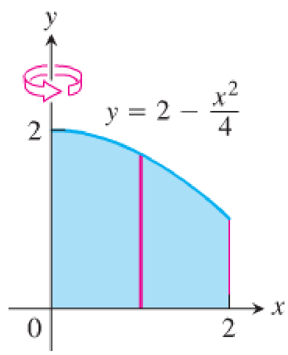
$$V = \int_0^2 2\pi(x)\left(2 - \frac{x^2}{4}\right)dx$$

$$= 2\pi \int_0^2 \left(2x - \frac{x^3}{4}\right)dx$$

$$= 2\pi \left(x^2 - \frac{x^4}{16}\right)\Big|_0^2$$

$$= 2\pi \left[\left(2^2 - \frac{2^4}{16}\right) - 0\right]$$

$$= 6\pi \ \ unit^3$$



Exercise

Use the shell method to find the volume of the solid generated by revolving the shaded region about the indicated axis

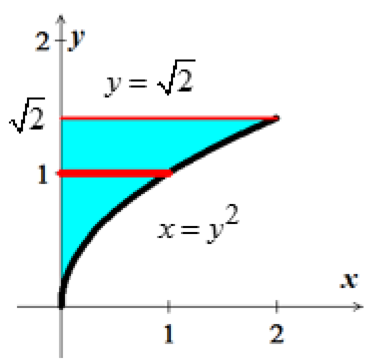
Solution

$$V = \int_0^{\sqrt{2}} 2\pi(y)\left(y^2\right)dy$$

$$= 2\pi \int_0^{\sqrt{2}} y^3 dy$$

$$= 2\pi \left(\frac{y^4}{4}\right)\Big|_0^{\sqrt{2}}$$

$$= 2\pi \ \ unit^3$$



Exercise

Use the shell method to find the volume of the solid generated by revolving the shaded region about the y-axis

Solution

$$V = \int_0^3 2\pi(x)\left(\frac{9x}{\sqrt{x^3+9}}\right)dx \qquad V = \int_a^b 2\pi\binom{shell}{radius}\binom{shell}{height}dx$$

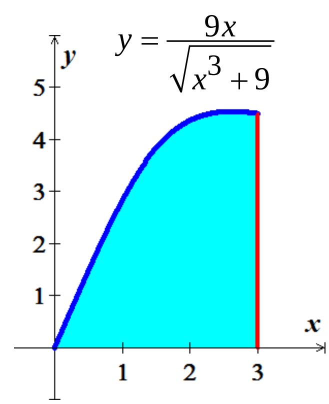
$$= 2\pi\int_0^3 \left(\frac{9x^2}{\sqrt{x^3+9}}\right)dx$$

$$= 2\pi\int_0^3 3\left(x^3+9\right)^{1/2} d\left(x^3+9\right) \qquad d\left(x^3+9\right) = 3x^2dx$$

$$= 6\pi\left[2\left(x^3+9\right)^{1/2}\right]_0^3 \qquad = 12\pi\left[\left(3^3+9\right)^{1/2} - (0+9)^{1/2}\right]$$

$$= 12\pi[6-3]$$

$$= 36\pi \ \ unit^3$$



Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = x^2, \quad y = 2 - x, \quad x = 0, \quad for\ x \ge 0$ about the y-axis.

Solution

$$V = \int_0^1 2\pi(x)\left((2-x)-x^2\right)dx \qquad V = \int_a^b 2\pi\binom{shell}{radius}\binom{shell}{height}dx$$

$$= 2\pi\int_0^1 x\left(2-x-x^2\right)dx$$

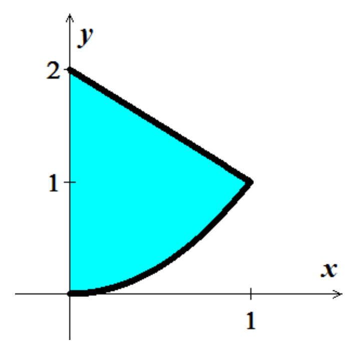
$$= 2\pi\int_0^1 \left(2x-x^2-x^3\right)dx$$

$$= 2\pi\left[x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right]_0^1$$

$$= 2\pi\left(1 - \tfrac{1}{3} - \tfrac{1}{4}\right)$$

$$= 12\pi\left(\frac{5}{12}\right)$$

$$= \frac{5\pi}{6} \ \ unit^3$$



Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y=2-x^2, \quad y=x^2, \quad x=0$ about the *y*-axis.

Solution

$$y=2-x^2=x^2 \quad \rightarrow \quad 2x^2=2 \Rightarrow x^2=1 \rightarrow \boxed{x=\pm 1}$$

Since *about y – axis,* $\quad a=x=0 \quad b=1$

$$V=\int_0^1 2\pi(x)\left(\left(2-x^2\right)-x^2\right)dx$$

$$=2\pi\int_0^1 x\left(2-2x^2\right)dx$$

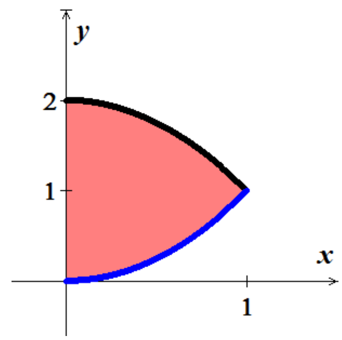
$$=4\pi\int_0^1\left(x-x^3\right)dx$$

$$=4\pi\left[\frac{1}{2}x^2-\frac{1}{4}x^4\right]_0^1$$

$$=4\pi\left[\frac{1}{2}-\frac{1}{4}\right]_0^1$$

$$=4\pi\left(\frac{1}{4}\right)$$

$$=\pi \ \ unit^3$$



Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y=\frac{3}{2\sqrt{x}}, \quad y=0, \quad x=1, \quad x=4$ about the *y*-axis.

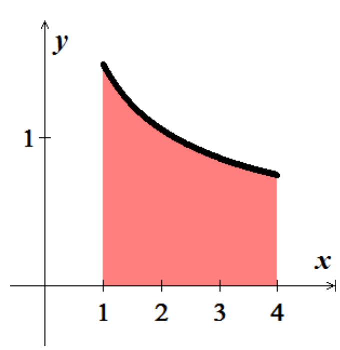
Solution

$$V=\int_1^4 2\pi(x)\left(\frac{3}{2\sqrt{x}}-0\right)dx$$

$$=\pi\int_1^4 x\left(3x^{-1/2}\right)dx$$

$$=3\pi\int_1^4 x^{1/2}dx$$

$$=3\pi\left[\frac{2}{3}x^{3/2}\right]_1^4$$



$$= 2\pi\left[4^{3/2} - 1^{3/2}\right]$$

$$= 2\pi(7)$$

$$\boxed{= 14\pi \ \ unit^3}$$

Exercise

Let $g(x) = \begin{cases} \dfrac{(\tan x)^2}{x} & 0 < x \le \frac{\pi}{4} \\ 0 & x = 0 \end{cases}$

a) Show that $x \cdot g(x) = (\tan x)^2, \quad 0 \le x \le \frac{\pi}{4}$

b) Find the volume of the solid generated by revolving the shaded region about the *y*-axis.

Solution

a) $x \cdot g(x) = \begin{cases} x \cdot \dfrac{(\tan x)^2}{x} & 0 < x \le \frac{\pi}{4} \\ x \cdot 0 & x = 0 \end{cases} \quad \Rightarrow x \cdot g(x) = \begin{cases} \tan^2 x & 0 < x \le \frac{\pi}{4} \\ 0 & x = 0 \end{cases}$

Since $x = 0 \quad \to \quad \tan x = 0 \qquad \Rightarrow x \cdot g(x) = \begin{cases} \tan^2 x & 0 < x \le \frac{\pi}{4} \\ \tan^2 x & x = 0 \end{cases}$

$$\Rightarrow \boxed{x \cdot g(x) = \tan^2 x \qquad 0 \le x \le \frac{\pi}{4}}$$

b) $V = 2\pi \displaystyle\int_0^{\pi/4} x \cdot g(x)\,dx \qquad\qquad V = \int_a^b 2\pi \begin{pmatrix} shell \\ radius \end{pmatrix} \begin{pmatrix} shell \\ height \end{pmatrix} dx$

$$= 2\pi \int_0^{\pi/4} \tan^2 x \, dx$$

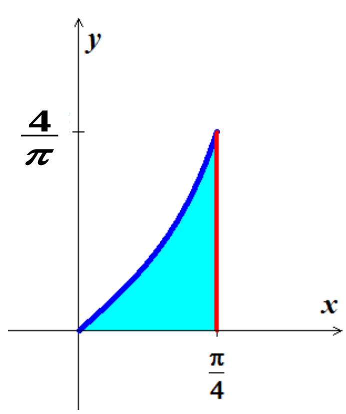
$$= 2\pi\left[\tan x - x\right]_0^{\pi/4}$$

$$= 2\pi\left[\left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) - (\tan 0 - 0)\right]$$

$$= 2\pi\left(1 - \frac{\pi}{4}\right)$$

$$= 2\pi\left(\frac{4-\pi}{4}\right)$$

$$\boxed{= \frac{4\pi - \pi^2}{2} \ \ unit^3}$$

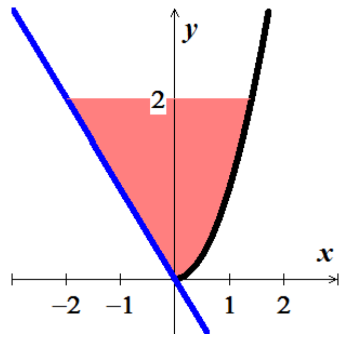


Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $x = \sqrt{y}, \quad x = -y, \quad y = 2$ about the x-axis.

Solution

$$x = \sqrt{y} = -y \quad \rightarrow y = 0 = c$$



$$V = \int_0^2 2\pi(y)\left(\sqrt{y} - (-y)\right)dy$$

$$= 2\pi \int_0^2 \left(y^{3/2} + y^2\right)dy$$

$$= 2\pi \left[\frac{2}{5}y^{5/2} + \frac{1}{3}y^3\right]_0^2$$

$$= 2\pi \left[\frac{2}{5}(2)^{5/2} + \frac{1}{3}(2)^3\right]$$

$$= 2\pi \left[\frac{8\sqrt{2}}{5} + \frac{8}{3}\right]$$

$$= 16\pi \left(\frac{3\sqrt{2} + 5}{15}\right)$$

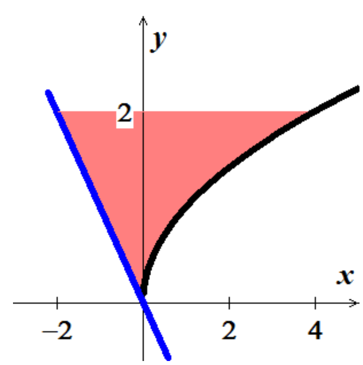
$$= \frac{16}{15}\pi\left(3\sqrt{2} + 5\right) \quad unit^3$$

Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $x = y^2, \quad x = -y, \quad y = 2, \quad y \ge 0$ about the x-axis.

Solution

$$x = y^2 = -y \quad \rightarrow y = 0 = c \quad d = 2$$



$$V = \int_0^2 2\pi(y)\left(y^2 - (-y)\right)dy$$

$$= 2\pi \int_0^2 \left(y^3 + y^2\right)dy$$

$$= 2\pi \left[\frac{1}{4}y^4 + \frac{1}{3}y^3\right]_0^2$$

$$= 2\pi \left(\frac{1}{4}(2)^4 + \frac{1}{3}(2)^3\right)$$

$$= 2\pi \left(4 + \frac{8}{3}\right)$$

$$= 2\pi\left(\frac{20}{3}\right)$$

$$= \frac{40\pi}{3} \ \ unit^3$$

Exercise

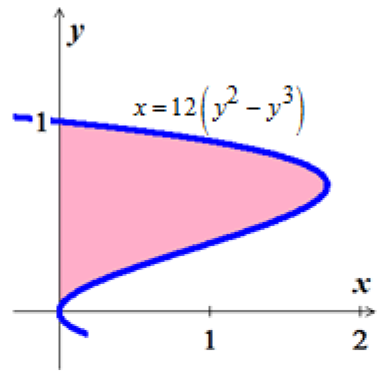
Use the shell method to find the volumes of the solids generated by revolving the shaded regions about the *indicated* axes.

a) The x-axis

b) The line $y = 1$

c) The line $y = \frac{8}{5}$

d) The line $y = -\frac{2}{5}$



Solution

a)
$$V = \int_0^1 2\pi(y)\cdot\left[12\left(y^2 - y^3\right)\right]dy$$

$$= 24\pi\int_0^1 \left(y^3 - y^4\right)dy$$

$$= 24\pi\left[\frac{y^4}{4} - \frac{y^5}{5}\right]_0^1$$

$$= 24\pi\left(\frac{1}{4} - \frac{1}{5}\right)$$

$$= 24\pi\left(\frac{1}{20}\right)$$

$$= \frac{6\pi}{5} \ \ unit^3$$

b)
$$V = \int_0^1 2\pi(1-y)\cdot\left[12\left(y^2 - y^3\right)\right]dy$$

$$= 24\pi\int_0^1 \left(y^2 - y^3 - y^3 + y^4\right)dy$$

$$= 24\pi\int_0^1 \left(y^2 - 2y^3 + y^4\right)dy$$

$$= 24\pi\left[\frac{y^3}{3} - \frac{y^4}{2} + \frac{y^5}{5}\right]_0^1$$

$$= 24\pi\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right)$$

$$= 24\pi\left(\frac{1}{30}\right)$$

$$\boxed{= \frac{4\pi}{5}\ \ unit^3}$$

c) $$V = \int_c^d 2\pi\begin{pmatrix} shell \\ radius \end{pmatrix}\begin{pmatrix} shell \\ height \end{pmatrix}dy$$

$$= 2\pi\int_0^1 \left(\frac{8}{5}-y\right)\cdot\left[12\left(y^2-y^3\right)\right]dy$$

$$= 24\pi\int_0^1 \left(\frac{8}{5}y^2-\frac{8}{5}y^3-y^3+y^4\right)dy$$

$$= 24\pi\int_0^1 \left(\frac{8}{5}y^2-\frac{13}{5}y^3+y^4\right)dy$$

$$= 24\pi\left[\frac{8}{15}y^3-\frac{13}{20}y^4+\frac{y^5}{5}\right]_0^1$$

$$= 24\pi\left(\frac{8}{15}-\frac{13}{20}+\frac{1}{5}\right)$$

$$= 24\pi\left(\frac{5}{60}\right)$$

$$\boxed{= 2\pi\ \ unit^3}$$

d) $$V = \int_0^1 2\pi\left(y+\frac{2}{5}\right)\cdot\left[12\left(y^2-y^3\right)\right]dy$$

$$= 24\pi\int_0^1 \left(y^3-y^4+\frac{2}{5}y^2-\frac{2}{5}y^3\right)dy$$

$$= 24\pi\int_0^1 \left(\frac{3}{5}y^3-y^4+\frac{2}{5}y^2\right)dy$$

$$= 24\pi\left[\frac{3}{20}y^4-\frac{1}{5}y^4+\frac{2}{15}y^3\right]_0^1$$

$$= 24\pi\left(\frac{3}{20}-\frac{1}{5}+\frac{2}{15}\right)$$

$$= 24\pi\left(\frac{5}{60}\right)$$

$$\boxed{= 2\pi\ \ unit^3}$$

Exercise

Compute the volume of the solid generated by revolving the region bounded by the lines $y = x \quad and \quad y = x^2$ about each coordinate axis using

a) The *shell* method

b) The *washer* method

Solution

$$y = x = x^2 \Rightarrow x^2 - x = 0 \rightarrow \boxed{x = 0,\ 1}$$

a) x-axis

$$V = \int_0^1 2\pi(y)\cdot\left[\sqrt{y} - y\right]dy \qquad V = \int_c^d 2\pi\binom{shell}{radius}\binom{shell}{height}dy$$

$$= 2\pi\int_0^1 \left(y^{3/2} - y^2\right)dy$$

$$= 2\pi\left[\frac{2}{5}y^{5/2} - \frac{1}{3}y^3\right]_0^1$$

$$= 2\pi\left(\frac{2}{5} - \frac{1}{3}\right)$$

$$= \frac{2\pi}{15}\ \ unit^3$$

y-axis

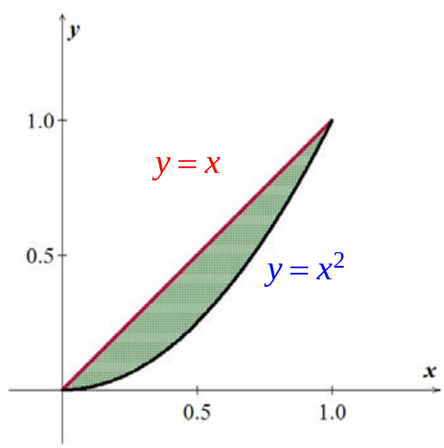
$$V = 2\pi\int_0^1 (x)\left(x - x^2\right)dx \qquad V = \int_a^b 2\pi\binom{shell}{radius}\binom{shell}{height}dx$$

$$= 2\pi\int_0^1 \left(x^2 - x^3\right)dx$$

$$= 2\pi\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1$$

$$= 2\pi\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= \frac{\pi}{6}\ \ unit^3$$



b) x-axis $\quad R(x) = x \quad and \quad r(x) = x^2$

$$V = \int_a^b \pi\left[R(x)^2 - r(x)^2\right]dx$$

$$= \pi\int_0^1 \left(x^2 - x^4\right)dx$$

$$= \pi\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1$$

$$= \pi\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$= \frac{2\pi}{15} \ \ unit^3$$

y-axis $\quad R(y) = \sqrt{y} \quad and \quad r(y) = y$

$$V = \int_c^d \pi\left[R(y)^2 - r(y)^2\right]dy$$

$$= \pi\int_0^1 \left(y - y^2\right)dy$$

$$= \pi\left[\frac{1}{2}y^2 - \frac{1}{3}y^3\right]_0^1$$

$$= \pi\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{\pi}{6} \ \ unit^3$$

Exercise

Use the *washer* method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = \sqrt{x}, \quad y = 2, \quad x = 0$ about

a) the x-axis

b) the y-axis

c) the line $x = 4$

d) the line $y = 1$

Solution

a) x-axis

$$V = \int_0^2 2\pi(y)\cdot\left(y^2 - 0\right)dy \qquad\qquad V = \int_c^d 2\pi\left(\begin{array}{c}shell\\radius\end{array}\right)\left(\begin{array}{c}shell\\height\end{array}\right)dy$$

$$= 2\pi\int_0^2 y^3 dy$$

$$= \frac{1}{2}\pi y^4\Big|_0^2$$

$$= \frac{1}{2}\pi(2)^4$$

$$= 8\pi \ \ unit^3$$

b) ***y-axis***

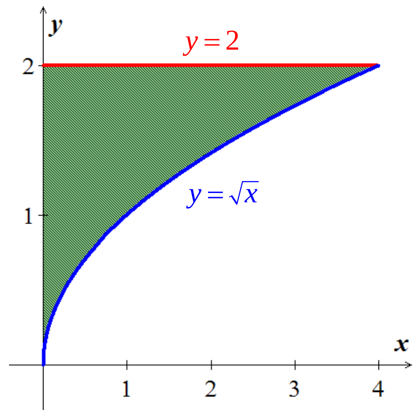
$$V = 2\pi\int_0^4 (x)\cdot\left(2-\sqrt{x}\right)dx \qquad V = \int_a^b 2\pi\begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix}dx$$

$$= 2\pi\int_0^4 \left(2x - x^{3/2}\right)dx$$

$$= 2\pi\left[x^2 - \tfrac{2}{5}x^{5/2}\right]_0^4$$

$$= 2\pi\left(16-\tfrac{64}{5}\right)$$

$$= \frac{32\pi}{5}\ \ \text{unit}^3$$



c) the line $x = 4$

$$V = \int_0^4 2\pi(4-x)\left(2-\sqrt{x}\right)dx \qquad V = \int_a^b 2\pi\begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix}dx$$

$$= 2\pi\int_0^4 \left(8-4x^{1/2}-2x-x^{3/2}\right)dx$$

$$= 2\pi\left[8x-\tfrac{8}{3}x^{3/2}-x^2-\tfrac{2}{5}x^{5/2}\right]_0^4$$

$$= 2\pi\left(32-\tfrac{64}{3}-16+\tfrac{64}{5}\right)$$

$$= \frac{224\pi}{15}\ \ \text{unit}^3$$

d) the line $y = 1$

$$V = 2\pi\int_0^2 (2-y)\left(y^2\right)dy \qquad V = \int_c^d 2\pi\begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix}dy$$

$$= 2\pi\int_0^2 \left(2y^2-y^3\right)dy$$

$$= 2\pi\left[\tfrac{2}{3}y^3-\tfrac{1}{4}y^4\right]_0^2$$

$$= 2\pi\left(\tfrac{16}{3}-\tfrac{16}{4}\right)$$

$$= \frac{32\pi}{12}$$

$$= \frac{8\pi}{3}\ \ \text{unit}^3$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $y=\frac{4}{x^3}$ and the lines $x=1$, *and* $y=\frac{1}{2}$ about

a) the x-axis;

b) the y-axis;

c) the line $x = 2$;

d) the line $y = 4$.

Solution

$$y=\frac{4}{x^3}=\frac{1}{2}$$

$$x^3=8 \quad\to\quad \underline{x=2|}$$

a) $x-axis \quad\to\quad$ *Washer Method*

$$V=\pi\int_1^2\left(\left(\frac{4}{x^3}\right)^2-\left(\frac{1}{2}\right)^2\right)dx$$

$$=\pi\int_1^2\left(16x^{-6}-\frac{1}{4}\right)dx$$

$$=\pi\left(-\frac{16}{5}x^{-5}-\frac{1}{4}x\right|_1^2$$

$$=\pi\left(-\frac{1}{10}-\frac{1}{2}+\frac{16}{5}+\frac{1}{4}\right)$$

$$=\pi\left(\frac{-2-10+64+5}{20}\right)$$

$$=\frac{57\pi}{20}\ \ unit^3$$

b) $y-axis \quad\to\quad$ *Shell Method*

$$V=2\pi\int_1^2 x\left(\frac{4}{x^3}-\frac{1}{2}\right)dx$$

$$=2\pi\int_1^2\left(\frac{4}{x^2}-\frac{1}{2}x\right)dx$$

$$=2\pi\left(-\frac{4}{x}-\frac{1}{4}x^2\right|_1^2$$

$$=2\pi\left(-2-1+4+\frac{1}{4}\right)$$

$$=2\pi\left(1+\frac{1}{4}\right)$$

$$=\frac{5\pi}{2}\ \ unit^3$$

c) $x = 2 \quad \to \quad \textit{Shell Method}$

$$V = 2\pi \int_1^2 (2 - x)\left(\frac{4}{x^3} - \frac{1}{2}\right) dx$$

$$= 2\pi \int_1^2 \left(8x^{-3} - 1 - \frac{4}{x^2} + \frac{1}{2}x\right) dx$$

$$= 2\pi \left(-\frac{4}{x^2} - x + \frac{4}{x} + \frac{1}{4}x^2\right)\Big|_1^2$$

$$= 2\pi\left(-1 - 2 + 2 + 1 + 4 + 1 - 4 - \frac{1}{4}\right)$$

$$= 2\pi\left(1 - \frac{1}{4}\right)$$

$$= \frac{3\pi}{2} \quad unit^3$$

d) $y = 4 \quad \to \quad \textit{Washer Method}$

$$R(x) = 4 - \frac{1}{2} = \frac{7}{2}$$

$$r(x) = 4 - \frac{4}{x^2}$$

$$V = \pi \int_1^2 \left(\left(\frac{7}{2}\right)^2 - \left(4 - \frac{4}{x^2}\right)^2\right) dx$$

$$= \pi \int_1^2 \left(\frac{49}{4} - 16 + \frac{32}{x^2} - 16x^{-4}\right) dx$$

$$= \pi \int_1^2 \left(-\frac{15}{4} + \frac{32}{x^2} - 16x^{-4}\right) dx$$

$$= \pi\left(-\frac{15}{4}x - \frac{32}{x} + \frac{16}{3x^3}\right)\Big|_1^2$$

$$= \pi\left(-\frac{15}{2} - 16 + \frac{2}{3} + \frac{15}{4} + 32 - \frac{16}{3}\right)$$

$$= \pi\left(16 - \frac{15}{4} - \frac{14}{3}\right)$$

$$= \pi\left(\frac{192 - 45 - 56}{12}\right)$$

$$= \frac{91\pi}{12} \quad unit^3$$

Exercise

The region in the first quadrant that is bounded by the curve $y = \frac{1}{\sqrt{x}}$, on the left by the line $x = \frac{1}{4}$, and below by the line $y = 1$ is revolved about the y-axis to generate a solid. Find the volume of the solid by

a) The *shell* method *b)* The *washer* method

Solution

a) The *shell* method

$$V = 2\pi \int_{1/4}^{1} x\left(\frac{1}{\sqrt{x}} - 1\right) dx$$

$$= 2\pi \int_{1/4}^{1} \left(x^{1/2} - x\right) dx$$

$$= 2\pi \left(\frac{2}{3}x^{1/2} - \frac{1}{2}x^2\right\vert_{1/4}^{1}$$

$$= 2\pi \left(\frac{1}{3} - \frac{1}{2} - \frac{1}{3} + \frac{1}{32}\right)$$

$$= \frac{11\pi}{48} \ unit^3$$

b) The *shell* method

$$y = \frac{1}{\sqrt{x}} \quad \to \quad x = \frac{1}{y^2}$$

$$x = \frac{1}{4} \quad \to \quad y = 2$$

$$V = \pi \int_{1}^{2} \left(\frac{1}{y^4} - \frac{1}{16}\right) dy$$

$$= \pi \left(-\frac{1}{3y^3} - \frac{1}{16}y\right\vert_{1}^{2}$$

$$= \pi \left(-\frac{1}{24} - \frac{1}{8} + \frac{1}{3} + \frac{1}{16}\right)$$

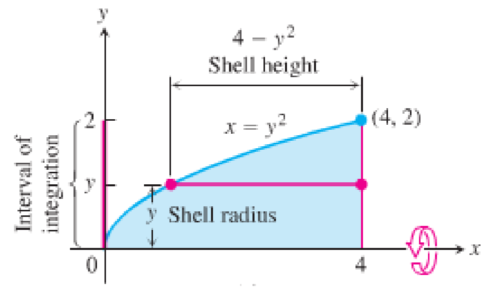
$$= \pi \left(\frac{7}{24} - \frac{1}{16}\right)$$

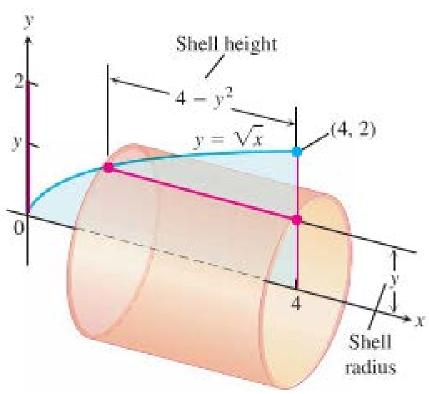
$$= \frac{11\pi}{48} \ unit^3$$

Exercise

The region bounded by the curve $y = \sqrt{x}$, the *x*-axis, and the line *x* = 4 is revolved about the *x*-axis to generate a solid. Find the volume of the solid.

Solution





$$V = 2\pi \int_c^d \begin{pmatrix} shell \\ radius \end{pmatrix} \begin{pmatrix} shell \\ height \end{pmatrix} dy$$

$$= 2\pi \int_0^2 (y)\left(4 - y^2\right) dy$$

$$= 2\pi \int_0^2 \left(4y - y^3\right) dy$$

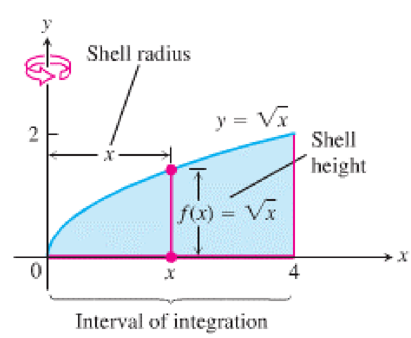
$$= 2\pi \left[2y^2 - \frac{y^4}{4} \right]_0^2$$

$$= 2\pi \left(2(2)^2 - \frac{(2)^4}{4} \right)$$

$$= 8\pi \ \ unit^3$$

Exercise

The region bounded by the curve $y = \sqrt{x}$, the *x*-axis, and the line *x* = 4 is revolved about the *y*-axis to generate a solid. Find the volume of the solid.



Solution

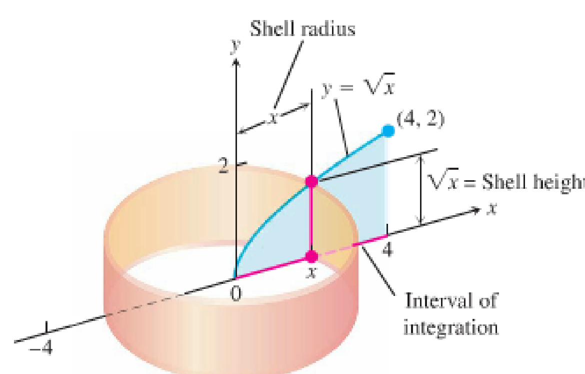
$$V = 2\pi \int_0^4 (x)\left(\sqrt{x}\right) dx \qquad V = \int_a^b 2\pi \begin{pmatrix} shell \\ radius \end{pmatrix} \begin{pmatrix} shell \\ height \end{pmatrix} dx$$

$$= 2\pi \int_0^4 x^{3/2} dx$$

$$= 2\pi \left[\frac{2}{5} x^{5/2} \right]_0^4$$

$$= \frac{4}{5}\pi \left[4^{5/2} \right]$$

$$= \frac{128\pi}{5} \ \ unit^3$$



Exercise

Find the volume of the solid generated by revolving the region bounded by $y = \sin x$ and the lines $x = 0, \quad x = \pi, \quad and \quad y = 2$ about the line $y = 2$.

Solution

About line $y = 2$

$$V = \pi \int_0^{\pi} (2 - \sin x)^2 \, dx$$

$$= \pi \int_0^{\pi} \left(4 - 4\sin x + \sin^2 x\right) dx$$

$$= \pi \int_0^{\pi} \left(4 - 4\sin x + \frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$$

$$= \pi \int_0^{\pi} \left(\frac{9}{2} - 4\sin x - \frac{1}{2}\cos 2x\right) dx$$

$$= \pi \left(\frac{9}{2}x + 4\cos x - \frac{1}{4}\sin 2x \right|_0^{\pi}$$

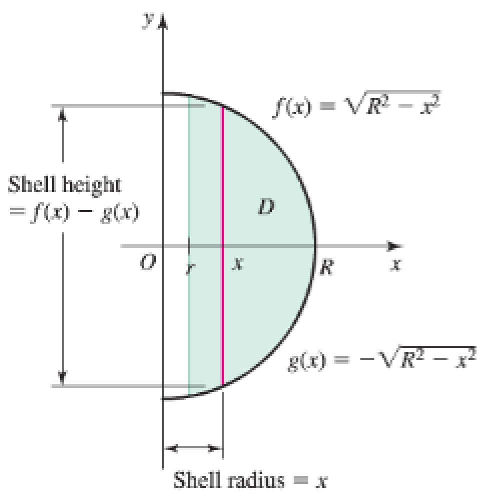
$$= \pi \left(\frac{9}{2}\pi - 4 - 4\right)$$

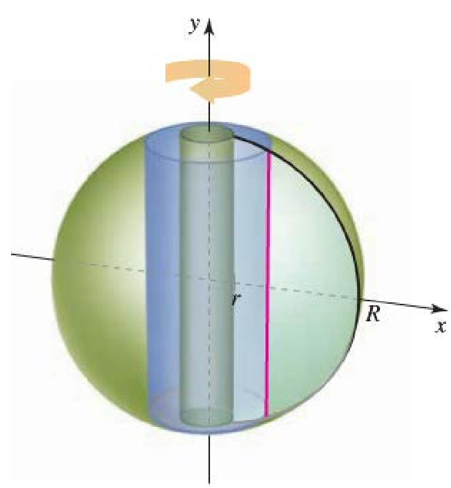
$$= \frac{9}{2}\pi^2 - 8\pi \ \ unit^3$$

Exercise

A cylinder hole with radius r is drilled symmetrically through the center of a sphere with radius R, where $r \le R$. What is the volume of the remaining material?

Solution





Let D be the region in the xy-plane bounded above by $f(x)=\sqrt{R^2-x^2}$, the upper half of the circle of radius R, and bounded below by $g(x)=-\sqrt{R^2-x^2}$, the lower half of the circle of radius R, for $r \le x \le R$.

The radius of a typical shell is x. Height is $f(x)-g(x)=2\sqrt{R^2-x^2}$

$$V=2\pi\int_r^R x\left(2\sqrt{R^2-x^2}\right)dx$$

$$=-2\pi\int_r^R \left(R^2-x^2\right)^{1/2} d\left(R^2-x^2\right)$$

$$=-\frac{4}{3}\pi\left(R^2-x^2\right)^{3/2}\Big|_r^R$$

$$=\frac{4}{3}\pi\left(R^2-r^2\right)^{3/2} \quad unit^3$$

Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y=\sqrt{x}, \quad y=2-x, \quad y=0$ about the x-axis.

Solution

$x=y^2$

$y=2-x^2=2-y^2 \Rightarrow y^2+y-2=0 \to y=\cancel{-2},\ 1$

Given: $y=0$

$$V=2\pi\int_0^1 y\left(2-y-y^2\right)dy$$

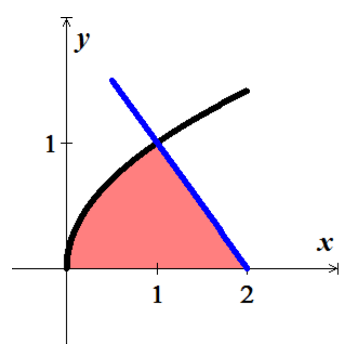
$$=2\pi\int_0^1\left(2y-y^2-y^3\right)dy$$

$$=2\pi\left(y^2-\frac{1}{3}y^3-\frac{1}{4}y^4\right)\Big|_0^1$$

$$=2\pi\left(1-\frac{1}{3}-\frac{1}{4}\right)$$

$$=2\pi\left(\frac{5}{12}\right)$$

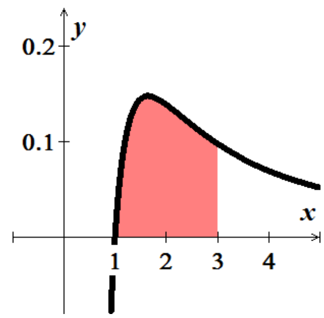
$$=\frac{5\pi}{6} \quad unit^3$$



Exercise

Find the volume of the region bounded by $y = \frac{\ln x}{x^2}$, $y = 0$, $x = 1$, *and* $x = 3$ revolved about the *y-axis*

Solution



$$V = 2\pi \int_1^3 x \frac{\ln x}{x^2} dx$$

$$= 2\pi \int_1^3 \ln x \, d(\ln x)$$

$$= \pi (\ln x)^2 \Big|_1^3$$

$$= \pi (\ln 3)^2 \ \ unit^3$$

Exercise

Find the volume of the region bounded by $y = \frac{e^x}{x}$, $y = 0$, $x = 1$, *and* $x = 2$ revolved about the *y-axis*

Solution

$$V = 2\pi \int_1^2 x \frac{e^x}{x} dx$$

$$= 2\pi \int_1^2 e^x dx$$

$$= 2\pi e^x \Big|_1^2$$

$$= 2\pi \left(e^2 - e\right) \ \ unit^3$$

Exercise

Find the volume of the region bounded by $y^2 = \ln x$, $y^2 = \ln x^3$, *and* $y = 2$ revolved about the *x-axis*

Solution

$$\begin{cases} y^2 = \ln x & \rightarrow \quad x = e^{y^2} \\ y^2 = \ln x^3 & \rightarrow \quad x = e^{y^2/3} \end{cases}$$

$$V = 2\pi \int_0^2 y \left(e^{y^2} - e^{y^2/3} \right) dy$$

$$= \pi \int_0^2 \left(e^{y^2} - e^{y^2/3} \right) d\left(y^2\right)$$

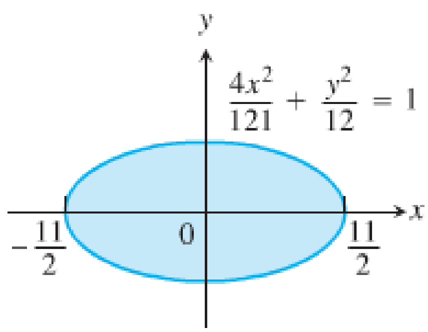
$$= \pi \left(e^{y^2} - 3e^{y^2/3} \right) \Big|_0^2$$

$$= \pi \left(e^4 - 3e^{4/3} - 1 + 3 \right)$$

$$= \pi \left(2 + e^4 - 3e^{4/3} \right) \ unit^3$$

Exercise

The profile of a football resembles the ellipse. Find the football's volume to the nearest *cubic inch*.



Solution

$$\frac{4x^2}{121} + \frac{y^2}{12} = 1$$

$$\frac{1}{12} y^2 = 1 - \frac{4}{121} x^2$$

$$y^2 = \frac{12}{121}\left(121 - 4x^2\right)$$

$$y = \sqrt{\frac{12}{121}\left(121 - 4x^2\right)}$$

$$V = \pi \int_{-11/2}^{11/2} \left(\sqrt{\frac{12}{121}\left(121 - 4x^2\right)} \right)^2 dx$$

$$= \frac{12\pi}{121} \int_{-11/2}^{11/2} \left(121 - 4x^2\right) dx$$

$$= \frac{12\pi}{121} \left(121x - \frac{4}{3} x^3 \right|_{-11/2}^{11/2}$$

$$= 2 \frac{12\pi}{121} \left(121x - \frac{4}{3} x^3 \right|_0^{11/2}$$

$$= \frac{24\pi}{11^2} \left(11^2 \left(\frac{11}{2}\right) - \frac{4}{3} \left(\frac{11}{2}\right)^3 \right)$$

$$= \frac{24\pi}{11^2}\left(11^3\right)\left(\frac{1}{2}-\frac{1}{6}\right)$$

$$= 264\pi\left(\frac{1}{3}\right)$$

$$= 88\pi \ \ unit^3$$

Exercise

Find the volume using both the disk/washer and shell methods of

$$y=(x-2)^3-2, \quad x=0, \quad y=25; \text{ revolved about the } y\text{-}axis$$

Solution

Using ***washers***:

$$(x-2)^3 = y+2 \quad \rightarrow \quad x = 2+\sqrt[3]{y+2}$$

$$x=0 \ \Rightarrow \ y=(-2)^3-2=-10$$

$$V = \pi\int_{-10}^{25}\left(2+\sqrt[3]{y+2}\right)^2 dy \qquad\qquad V=\pi\int_c^d f(y)^2\,dy$$

$$= \pi\int_{-10}^{25}\left(4+4(y+2)^{1/3}+(y+2)^{2/3}\right)d(y+2)$$

$$= \pi\left(4(y+2)+3(y+2)^{4/3}+\frac{3}{5}(y+2)^{5/3}\right)\Big|_{-10}^{25}$$

$$= \pi\left(108+3(27)^{4/3}+\frac{3}{5}(27)^{5/3}-\left(-32+3(-8)^{4/3}+\frac{3}{5}(-8)^{5/3}\right)\right)$$

$$= \pi\left(108+243+\frac{729}{5}+32-48+\frac{96}{5}\right)$$

$$= \pi(335+165)$$

$$= 500\pi \ \ unit^3$$

Using ***Shells***:

$$y=25 \quad \rightarrow x=2+\sqrt[3]{27}=5$$

$$V = 2\pi\int_0^5 x\left(25-(x-2)^3+2\right)dx \qquad\qquad V=2\pi\int_a^b x\left(f(x)-g(x)\right)dx$$

$$= 2\pi\int_0^5 x\left(27-x^3+6x^2-12x+8\right)dx$$

$$= 2\pi\int_0^5 \left(-x^4+6x^3-12x^2+35x\right)dx$$

$$= 2\pi\left(-\frac{1}{5}x^5 + \frac{3}{2}x^4 - 4x^3 + \frac{35}{2}x^2\right)\Big|_0^5$$

$$= 2\pi\left(-5^4 + \frac{3}{2}5^4 - 4(5)^3 + \frac{35}{2}(5)^2\right)$$

$$= 2\pi\left(-625 + \frac{1875}{2} - 500 + \frac{875}{2}\right)$$

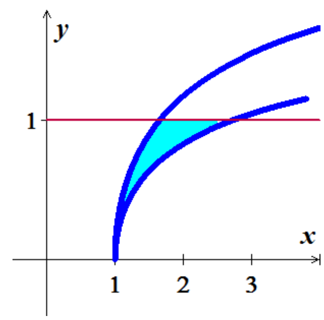
$$= 2\pi(250)$$

$$\underline{= 500\pi \ \ unit^3}$$

Exercise

Find the volume using both the disk/washer and shell methods of $y = \sqrt{\ln x}$, $y = \sqrt{\ln x^2}$, $y = 1$; revolved about the *x-axis*

Solution



Using ***washers***:

$$y = \sqrt{\ln x} = \sqrt{\ln x^2} \quad \to \quad \ln x = \ln x^2$$

$$x = x^2 \ \Rightarrow \ \underline{x = 0,\ 1}$$

$$y = 1 = \sqrt{\ln x} \quad \Rightarrow \quad \underline{x = e}$$

$$y = 1 = \sqrt{\ln x^2} \quad \Rightarrow \quad x^2 = e \ \to \ \underline{x = \sqrt{e}}$$

$$V = \pi\int_1^{\sqrt{e}}\left(\ln x^2 - \ln x\right)dx + \pi\int_{\sqrt{e}}^{e}(1 - \ln x)\,dx \qquad V = \pi\int_a^b\left(f(x)^2 - g(x)^2\right)dx$$

$$= \pi\int_1^{\sqrt{e}}(2\ln x - \ln x)\,dx + \pi\int_{\sqrt{e}}^{e}(1 - \ln x)\,dx$$

$$= \pi\int_1^{\sqrt{e}}(\ln x)\,dx + \pi\int_{\sqrt{e}}^{e}(1 - \ln x)\,dx \qquad \int \ln x\,dx = x\ln x - x$$

$$= \pi(x\ln x - x)\Big|_1^{\sqrt{e}} + \pi(2x - x\ln x)\Big|_{\sqrt{e}}^{e}$$

$$= \pi\left(\frac{1}{2}\sqrt{e} - \sqrt{e} + 1\right) + \pi\left(2e - e - 2\sqrt{e} + \frac{1}{2}\sqrt{e}\right)$$

$$= \pi\left(-\frac{1}{2}\sqrt{e} + 1 + e - 2\sqrt{e} + \frac{1}{2}\sqrt{e}\right)$$

$$= \pi\left(e - 2\sqrt{e} + 1\right)$$

$$\underline{= \pi\left(\sqrt{e} - 1\right)^2 \ \ unit^3}$$

Using ***Shells***:

$y = \sqrt{\ln x} \quad \Rightarrow \quad \underline{x = e^{y^2}|}$

$y = \sqrt{\ln x^2} \quad \Rightarrow \quad 2\ln x = y^2 \;\to\; \underline{x = e^{y^2/2}|}$

$$V = 2\pi \int_0^1 y\left(e^{y^2} - e^{y^2/2}\right)dy \qquad\qquad V = 2\pi\int_c^d y\big(p(y) - q(y)\big)dy$$

$$= \pi\int_0^1 e^{y^2} d\left(y^2\right) - 2\pi\int_0^1 e^{y^2/2} d\left(\tfrac{1}{2}y^2\right)$$

$$= \pi\left(e^{y^2} - 2e^{y^2/2}\right)\Big|_0^1$$

$$= \pi\left(e - 2e^{1/2} - 1 + 2\right)$$

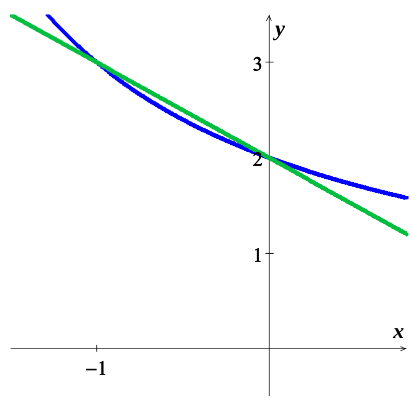
$$= \pi\left(e - 2\sqrt{e} + 1\right)$$

$$\boxed{= \pi\left(\sqrt{e} - 1\right)^2 \;\; unit^3}$$

Exercise

Find the volume using both the disk/washer and shell methods of $y = \dfrac{6}{x+3}$, $\quad y = 2 - x$; revolved about the x-*axis*

Solution



Using ***washers***:

$y = \dfrac{6}{x+3} = 2 - x$

$-x^2 - x + 6 = 6$

$x(x+1) = 0 \;\Rightarrow\; \underline{x = -1,\ 0|}$

$$V = \pi\int_{-1}^{0}\left((2-x)^2 - \frac{36}{(x+3)^2}\right)dx$$

$$= \pi\int_{-1}^{0} -(2-x)^2\, d(2-x) - \pi\int_{-1}^{0}\frac{36}{(x+3)^2}\, d(x+3)$$

$$= \pi\left(-\tfrac{1}{3}(2-x)^3 + \frac{36}{x+3}\right)\Big|_{-1}^{0}$$

$$= \pi\left(-\tfrac{8}{3} + 12 + 9 - 18\right)$$

$$\boxed{= \frac{\pi}{3} \;\; unit^3}$$

Using ***Shells***:

$$y=\frac{6}{x+3} \;\to\; x=\frac{6}{y}-3$$

$$y=2-x \;\to\; x=2-y$$

$$V=2\pi\int_2^3 y\left(2-y-\frac{6}{y}+3\right)dy \qquad\qquad V=2\pi\int_c^d y\left(p(y)-q(y)\right)dy$$

$$=2\pi\int_2^3\left(5y-y^2-6\right)dy$$

$$=2\pi\left(\frac{5}{2}y^2-\frac{1}{3}y^3-6y\right)_2^3$$

$$=2\pi\left(\frac{45}{2}-9-18-10+\frac{8}{3}+12\right)$$

$$=2\pi\left(\frac{151}{6}-25\right)$$

$$\boxed{=\frac{\pi}{3}\ \ unit^3}$$

Exercise

Use the shell method to find the volume of the solid generated by the revolving the plane region about the given line

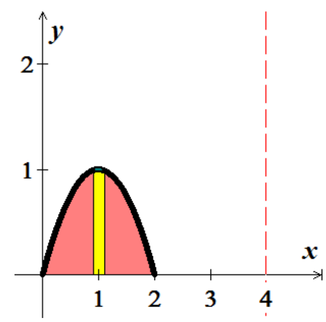
$$y=2x-x^2,\quad y=0,\quad \text{about the line}\quad x=4$$

Solution

$$y=2x-x^2=0 \quad \boxed{x=0,\ 2}$$

$$p(x)=4-x,\quad f(x)=2x-x^2$$

$$V=2\pi\int_0^2(4-x)\left(2x-x^2\right)dx \qquad\qquad V=2\pi\int_a^b p(x)\,f(x)\,dx$$



$$=2\pi\int_0^2\left(8x-6x^2+x^3\right)dx$$

$$=2\pi\left(4x^2-2x^3+\frac{1}{4}x^4\right)\Big|_0^2$$

$$=2\pi(16-16+4)$$

$$\boxed{=8\pi\ \ unit^3}$$

Exercise

Use the shell method to find the volume of the solid generated by the revolving the plane region about the given line

$$y=\sqrt{x}, \quad y=0, \quad x=4, \quad \textit{about the line} \quad x=6$$

Solution

$y=\sqrt{x}=0 \quad \underline{x=0}|$

$p(x)=6-x, \quad f(x)=\sqrt{x}$

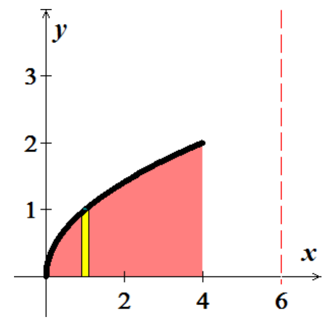
$$V=2\pi\int_0^4 (6-x)\left(\sqrt{x}\right)dx \qquad\qquad V=2\pi\int_a^b p(x)\,f(x)\;dx$$

$$=2\pi\int_0^4\left(6x^{1/2}-x^{3/2}\right)dx$$

$$=2\pi\left(4x^{3/2}-\frac{2}{5}x^{5/2}\right)\Big|_0^4$$

$$=2\pi\left(32-\frac{64}{5}\right)$$

$$=\frac{192\pi}{5}\ \ unit^3$$



Exercise

Use the shell method to find the volume of the solid generated by the revolving the plane region about the given line

$$y=x^2, \quad y=4x-x^2, \quad \textit{about the line} \quad x=4$$

Solution

$y=x^2=4x-x^2 \ \Rightarrow\ 2x^2-4x=0 \quad \underline{x=0,\ 2}|$

$p(x)=4-x, \quad f(x)=4x-x^2, \quad g(x)=x^2$

$$V=2\pi\int_0^2 (4-x)\left(4x-x^2-x^2\right)dx \qquad\qquad V=2\pi\int_a^b p(x)\big(f(x)-g(x)\big)\;dx$$

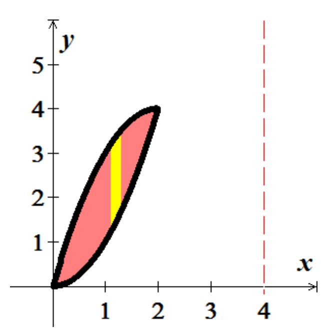
$$=2\pi\int_0^2 (4-x)\left(4x-2x^2\right)dx$$

$$=2\pi\int_0^2\left(16x-12x^2+2x^3\right)dx$$

$$=2\pi\left(8x^2-4x^3+\frac{1}{2}x^4\right)\Big|_0^2$$

$$=2\pi(32-32+8)$$

$$=16\pi\ \ unit^3$$



Exercise

Use the shell method to find the volume of the solid generated by the revolving the plane region about the given line

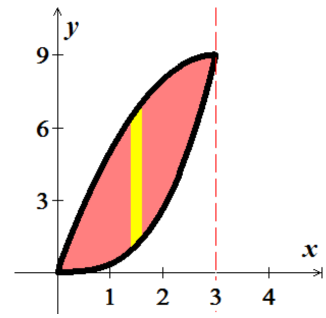
$$y = \frac{1}{3}x^3, \quad y = 6x - x^2, \quad \textit{about the line} \quad x = 3$$

Solution

$$y = \frac{1}{3}x^3 = 6x - x^2 \;\Rightarrow\; x\left(x^2 - 3x + 18\right) = 0 \quad \underline{x = 0,\ 3,\ \cancel{-6}}$$

$$p(x) = 3 - x, \quad f(x) = 6x - x^2, \quad g(x) = \frac{1}{3}x^3$$

$$V = 2\pi \int_0^3 (3 - x)\left(3x - x^2 - \frac{1}{3}x^3\right)dx \qquad V = 2\pi \int_a^b p(x)\big(f(x) - g(x)\big)\,dx$$



$$= 2\pi \int_0^3 \left(18x - 9x^2 + \frac{1}{3}x^4\right)dx$$

$$= 2\pi\left(9x^2 - 3x^3 + \frac{1}{15}x^5\right)\Big|_0^3$$

$$= 2\pi\left(81 - 81 + \frac{81}{5}\right)$$

$$= \frac{162\pi}{5} \ \ unit^3$$

Exercise

Use the disk method or shell method to find the volume of the solid generated vy revolving the region bounded by the graph of the equations about the given lines.

$y = x^3, \quad y = 0, \quad x = 2$

a) *the x-axis*   *b)* *the y-axis*   *c)* *the line* $x = 4$

Solution

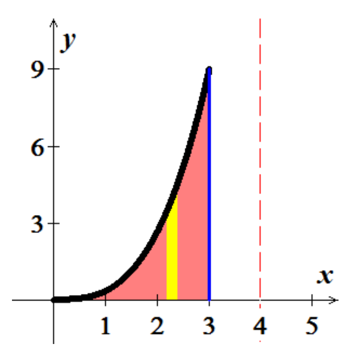
a) Using ***Disk method***:

$$f(x) = x^3, \quad g(x) = 0$$

$$V = \pi \int_0^2 x^6 dx \qquad V = \pi \int_a^b \left(\left(f(x)\right)^2 - \left(g(x)\right)^2\right)dx$$

$$= \frac{\pi}{7}x^7\Big|_0^2$$

$$= \frac{128\pi}{7} \ \ unit^3$$



b) Using ***Shell method***:

$$p(x) = x, \quad f(x) = x^3, \quad g(x) = 0$$

$$V = 2\pi \int_0^2 x\left(x^3\right)dx \qquad\qquad V = 2\pi \int_a^b p(x)\left(f(x)-g(x)\right)\,dx$$

$$= 2\pi \int_0^2 x^4\,dx$$

$$= \frac{2\pi}{5}x^5 \Big|_0^2$$

$$\boxed{= \frac{164\pi}{5}\ \ unit^3}$$

c) Using ***Shell method***:

$$p(x) = 4 - x, \quad f(x) = x^3, \quad g(x) = 0$$

$$V = 2\pi \int_0^2 (4-x)\left(x^3\right)dx \qquad\qquad V = 2\pi \int_a^b p(x)\left(f(x)-g(x)\right)\,dx$$

$$= 2\pi \int_0^2 \left(4x^3 - x^4\right)\,dx$$

$$= 2\pi\left(x^4 - \frac{1}{5}x^5\right)\Big|_0^2$$

$$= 2\pi\left(16 - \frac{32}{5}\right)$$

$$\boxed{= \frac{96\pi}{5}\ \ unit^3}$$

Exercise

Use the disk method or shell method to find the volume of the solid generated vy revolving the region bounded by the graph of the equations about the given lines.

$$y = \frac{10}{x^2}, \quad y = 0, \quad x = 1, \quad x = 5$$

a) *the x-axis*   *b)* *the y-axis*   *c)* *the line* $y = 10$

Solution

a) Using ***Disk method***:

$$R(x) = \frac{10}{x^2}, \quad r(x) = 0$$

$$V = \pi \int_1^5 100x^{-4}\,dx \qquad\qquad V = \pi \int_a^b \left(\left(R(x)\right)^2 - \left(r(x)\right)^2\right)dx$$

$$= -\frac{100}{3}\pi x^{-3}\Big|_1^5$$

$$= -\frac{100}{3}\pi\left(\frac{1}{125} - 1\right)$$

$= \dfrac{496\pi}{15}\ unit^3$

b) Using ***Shell method***:

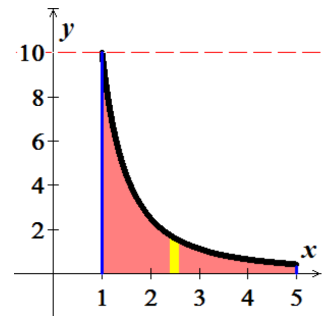
$p(x) = x, \quad f(x) = \dfrac{10}{x^2}, \quad g(x) = 0$

$$V = 2\pi \int_1^5 x\left(\frac{10}{x^2}\right)dx \qquad\qquad V = 2\pi\int_a^b p(x)\big(f(x) - g(x)\big)\ dx$$

$$= 20\pi \int_1^5 \frac{1}{x}\ dx$$

$$= 20\pi \ln x \Big|_1^5$$

$$= 20\pi \ln 5\ \ unit^3$$



c) Using ***Disk method***:

$R(x) = 10, \quad r(x) = 10 - \dfrac{10}{x^2}$

$$V = \pi \int_1^5 \left(100 - \left(10 - 10x^{-2}\right)^2\right)dx \qquad V = \pi\int_a^b \left(\left(R(x)\right)^2 - \left(r(x)\right)^2\right)dx$$

$$= \pi \int_1^5 \left(200x^{-2} - 100x^{-4}\right)dx$$

$$= 100\pi\left(-\frac{2}{x} + \frac{1}{3x^3}\right)\Bigg|_1^5$$

$$= 100\pi\left(-\frac{2}{5} + \frac{1}{375} + 2 - \frac{1}{3}\right)$$

$$= 100\pi\left(2 - \frac{274}{375}\right)$$

$$= 100\pi\left(\frac{476}{375}\right)$$

$$= \frac{1904\pi}{15}\ \ unit^3$$

Exercise

Let V_1 and V_2 be the volumes of the solids that result when the plane region bounded by $y = \dfrac{1}{x}$, $y = 0$, $x = \dfrac{1}{4}$, and $x = c$ $\left(where\ \ c > \dfrac{1}{4}\right)$ is revolved about the x-*axis* and the y-*axis*, respectively. Find the value of c for which $V_1 = V_2$

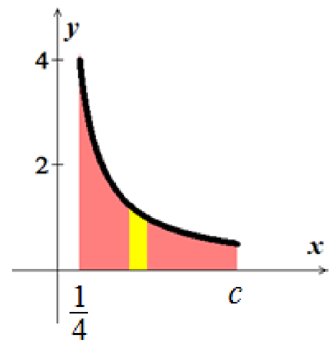
Solution

$$V_1 = \pi \int_{1/4}^{c} \frac{1}{x^2}\,dx \qquad\qquad V = \pi \int_a^b \left(R(x)^2 - r(x)^2 \right) dx$$

$$= -\pi \frac{1}{x} \Big|_{1/4}^{c}$$

$$= -\pi \left(\frac{1}{c} - 4 \right)$$

$$= \boxed{\frac{4c-1}{c}\pi}$$



$$V_2 = 2\pi \int_{1/4}^{c} x \frac{1}{x}\,dx$$

$$= 2\pi x \Big|_{1/4}^{c}$$

$$= \boxed{2\pi \left(c - \frac{1}{4} \right)}$$

Since $V_1 = V_2$

$$\frac{4c-1}{c}\pi = 2\pi \left(c - \frac{1}{4} \right)$$

$$4c - 1 = 2c^2 - \frac{1}{2}c$$

$$2c^2 - \frac{9}{2}c + 1 = 0$$

$$4c^2 - 9c + 2 = 0 \quad \rightarrow \quad \boxed{c = 2}, \ \cancel{\frac{1}{4}} \qquad \left(\frac{1}{4} \text{ has no volume} \right)$$

Exercise

The region bounded by $y = r^2 - x^2$, $y = 0$, and $x = 0$ is revolved about the *y-axis* to form a paraboloid. A hole, centered along the axis of revolution, is drilled through this solid. The hole has a radius k, $0 < k < r$. Find the volume of the resulting ring

a) By integrating with respect to x

b) By integrating with respect to y.

Solution

a) $f(x) = r^2 - x^2, \quad g(x) = 0$

$$V = 2\pi \int_k^r x\left(r^2 - x^2 \right) dx \qquad\qquad V = 2\pi \int_a^b x\left(f(x) - g(x) \right) dx \quad (\textbf{\textit{Shell Method}})$$

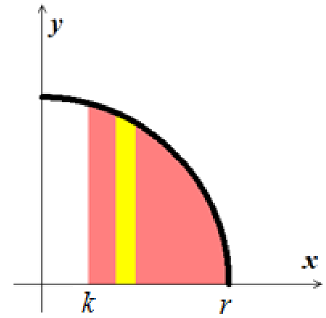
$$= 2\pi \int_k^r \left(r^2 x - x^3\right) dx$$

$$= 2\pi \left(\frac{1}{2} r^2 x^2 - \frac{1}{4} x^4\right)\Big|_k^r$$

$$= \frac{1}{2}\pi \left(2r^4 - r^4 - 2r^2 k^2 + k^4\right)$$

$$= \frac{1}{2}\pi \left(r^4 - 2r^2 k^2 + k^4\right)$$

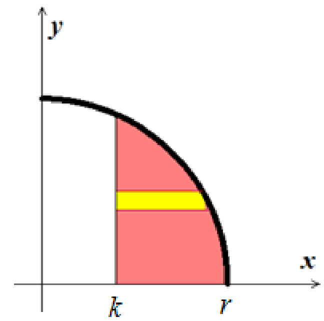
$$= \frac{1}{2}\pi \left(r^2 - k^2\right)^2 \quad unit^3$$



b) $y = r^2 - x^2 \quad \rightarrow \quad x = \sqrt{r^2 - y}$

$R(y) = \sqrt{r^2 - y}, \quad r(y) = k$

$$V = \pi \int_0^{r^2 - k} \left(r^2 - y - k^2\right) dy \qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2\right) dy$$



$$= \pi \left(\left(r^2 - k^2\right) y - \frac{1}{2} y^2\right)\Big|_0^{r^2 - k}$$

$$= \pi \left(\left(r^2 - k^2\right)^2 - \frac{1}{2}\left(r^2 - k^2\right)^2\right)$$

$$= \frac{1}{2}\pi \left(r^2 - k^2\right)^2 \quad unit^3$$

Exercise

The region R in the first quadrant bounded by the parabola $y = 4 - x^2$ and the coordinate axes is revolved about the y-axis to produce a dome-shaped solid. Find the volume of the solid in the following ways.

a) Apply the disk method and integrate with respect to y.

b) Apply the shell method and integrate with respect to x.

Solution

a) $y = 4 - x^2 \quad \rightarrow \quad x = \sqrt{4 - y}$

$$V = \pi \int_0^4 \left(\sqrt{4 - y}\right)^2 dy$$

$$= \pi \int_0^4 (4 - y)\, dy$$

$$= \pi \left(4y - \frac{1}{2} y^2\right)\Big|_0^4$$

$$= \pi(16-8)$$

$$\boxed{= 8\pi \ \ unit^3}$$

b) $$V = 2\pi \int_0^2 x\left(4-x^2\right)dx$$

$$= 2\pi \int_0^2 \left(4x - x^3\right)dx$$

$$= 2\pi \left(2x^2 - \frac{1}{4}x^4\right)\Big|_0^2$$

$$= 2\pi(8-4)$$

$$\boxed{= 8\pi \ \ unit^3}$$

Exercise

The region bounded by the curves $y = 1+\sqrt{x}$, $y = 1-\sqrt{x}$, and the line $x = 1$ is revolved about the *y*-axis. Find the volume of the resulting solid by

a) Integrating with respect to *x* and

b) Integrating with respect to *y*.

Solution

a) $$V = 2\pi \int_0^1 x\left(1+\sqrt{x}-1+\sqrt{x}\right)dx$$

$$= 2\pi \int_0^2 2x^{3/2}dx$$

$$= \frac{8}{5}\pi x^{5/2}\Big|_0^1$$

$$\boxed{= \frac{8\pi}{5} \ \ unit^3}$$

b) $y = 1+\sqrt{x} \quad \to \quad x = (y-1)^2$

$y = 1-\sqrt{x} \quad \to \quad x = (1-y)^2$

$x = 1 \quad \to \begin{cases} y = 0 \\ y = 2 \end{cases}$

$$V = \pi \int_0^2 \left(1^2 - \left((1-y)^2\right)^2\right)dy$$

$$= \pi \int_0^2 \left(1 - \left(1 - 2y + y^2\right)^2\right) dy$$

$$= \pi \int_0^2 \left(-y^4 + 4y^3 - 6y^2 - 4y\right) dy$$

$$= \pi \left(-\frac{1}{5}y^5 + y^4 - 2y^3 + 2y^2\right)\Big|_0^2$$

$$= \pi \left(-\frac{32}{5} + 16 - 16 + 8\right)$$

$$= \pi \left(8 - \frac{32}{5}\right)$$

$$= \frac{8\pi}{5} \ \ unit^3$$

Exercise

The region bounded by the graphs of $x = 0$, $x = \sqrt{\ln y}$, and $x = \sqrt{2 - \ln y}$ in the first quadrant is revolved about the y-axis. What is the volume of the resulting solid?

Solution

$$x = \sqrt{\ln y} \quad \to x^2 = \ln y \quad y = e^{x^2}$$

$$x = \sqrt{2 - \ln y} \quad \to x^2 = 2 - \ln y \quad y = e^{2 - x^2}$$

$$y = e^{2 - x^2} = e^{x^2}$$

$$2 - x^2 = x^2$$

$$2x^2 = 2 \quad \to \quad \underline{x = \pm 1}$$

$$V = 2\pi \int_0^1 x\left(e^{2 - x^2} - e^{x^2}\right) dx$$

$$= 2\pi \int_0^1 xe^{2 - x^2} dx - 2\pi \int_0^1 xe^{x^2} dx$$

$$= -\pi \int_0^1 e^{2 - x^2} d\left(2 - x^2\right) - \pi \int_0^1 e^{x^2} d\left(x^2\right)$$

$$= -\pi \left(e^{2 - x^2} + e^{x^2}\right)\Big|_0^1$$

$$= -\pi \left(e + e - e^2 - 1\right)$$

$$= \pi\left(e^2 - 2e + 1\right)$$

$$\underline{= \pi(e-1)^2 \ \ unit^3}$$

Exercise

The region bounded by $y = \left(1 - x^2\right)^{-1/2}$ and the x-axis over the interval $\left[0, \ \frac{\sqrt{3}}{2}\right]$ is revolved about the y-axis. What is the volume of the solid that is generated?

Solution

$$V = 2\pi \int_0^{\frac{\sqrt{3}}{2}} x\left(1 - x^2\right)^{-1/2} dx$$

$$= -\pi \int_0^{\frac{\sqrt{3}}{2}} \left(1 - x^2\right)^{-1/2} d\left(1 - x^2\right)$$

$$= -2\pi\left(1 - x^2\right)^{1/2} \Bigg|_0^{\frac{\sqrt{3}}{2}}$$

$$= -2\pi\left(\sqrt{1 - \frac{3}{4}} - 1\right)$$

$$= -2\pi\left(\frac{1}{2} - 1\right)$$

$$\underline{= \pi \ \ unit^3}$$

Exercise

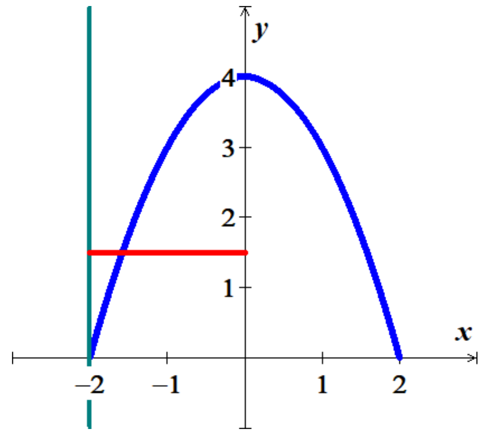
The region bounded by the graph $y = 4 - x^2$ and the x-axis over the interval $[-2, \ 2]$ is revolved about the line $x = -2$. What is the volume of the solid that is generated?

Solution

Using *Shell* Method radius: $x + 2$

$$V = 2\pi \int_{-2}^{2} (x + 2)\left(4 - x^2\right) dx$$

$$= 2\pi \int_{-2}^{2} \left(4x - x^3 + 8 - 2x^2\right) dx$$



$$= 2\pi \left(2x^2 - \frac{1}{4}x^4 + 8x - \frac{2}{3}x^3 \right|_{-2}^{2}$$

$$= 2\pi \left(8 - 4 + 16 - \frac{16}{3} - 8 + 4 + 16 - \frac{16}{3}\right)$$

$$= 2\pi \left(32 - \frac{32}{3}\right)$$

$$= \frac{128\pi}{3} \quad unit^3$$

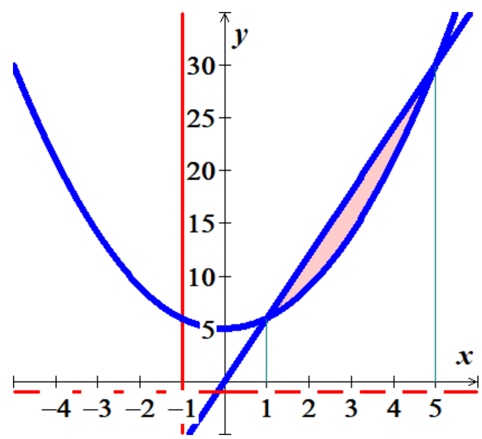
Exercise

The region bounded by the graph $y = 6x$ and $y = x^2 + 5$ is revolved about the line $y = -1$ and the line $x = -1$. Find the volumes of the resulting solids. Which one is greater?

Solution

$y = x^2 + 5 = 6x$

$x^2 - 6x + 5 = 0 \quad \rightarrow \quad \underline{x = 1,\ 5}$



About $y = -1$

Using *Washer* Method:

$R: \quad 6x - (-1) = 6x + 1$

$r: \quad x^2 + 5 + 1 = x^2 + 6$

$$V = \pi \int_1^5 \left[(6x+1)^2 - \left(x^2+6\right)^2\right] dx$$

$$= \pi \int_1^5 \left(36x^2 + 12x + 1 - x^4 - 12x^2 - 36\right) dx$$

$$= \pi \int_1^5 \left(-x^4 + 24x^2 + 12x - 35\right) dx$$

$$= \pi \left(-\frac{1}{5}x^5 + 8x^3 + 6x^2 - 35x \right|_1^5$$

$$= \pi \left(-625 + 1000 + 150 - 175 + \frac{1}{5} - 8 - 6 + 35\right)$$

$$= \pi \left(371 + \frac{1}{5}\right)$$

$$= \frac{1,856\pi}{5} \quad unit^3$$

About $x=-1$

Using *Shell* Method:

$height: \quad 6x-x^2-5$

$radius: \quad x-(-1)=x+1$

$$V=2\pi\int_1^5 (x+1)\left(6x-x^2-5\right)dx$$

$$=2\pi\int_1^5 \left(5x^2-x^3+x-5\right)dx$$

$$=2\pi\left(\frac{5}{3}x^3-\frac{1}{4}x^4+\frac{1}{2}x^2-5x\right|_1^5$$

$$=2\pi\left(\frac{625}{3}-\frac{625}{4}+\frac{25}{2}-25-\frac{5}{3}+\frac{1}{4}-\frac{1}{2}+5\right)$$

$$=2\pi\left(\frac{625}{3}-156+12-20\right)$$

$$=2\pi\left(\frac{625}{3}-164\right)$$

$$=\frac{256\pi}{3}\ \ unit^3$$

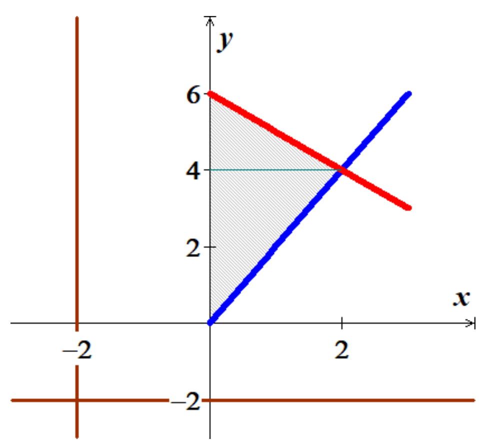
Exercise

The region bounded by the graph $y=2x$, $y=6-x$ and $y=0$ is revolved about the line $y=-2$ and the line $x=-2$. Find the volumes of the resulting solids. Which one is greater?

Solution

$y=2x=6-x$

$3x=6 \quad \to \quad \underline{x=2} \quad (2,\ 4)$



About $y=-2$

Using *Shell* Method:

$height: \quad 6-y-\frac{1}{2}y=6-\frac{3}{2}y$

$radius: \quad y+2$

$$V=2\pi\int_0^4 (y+2)\left(6-\frac{3}{2}y\right)dy$$

$$=2\pi\int_0^4 \left(-\frac{3}{2}y^2+3y+12\right)dx$$

$$=2\pi\left(-\frac{1}{2}y^3+\frac{3}{2}y^2+12y\right|_0^4$$

$$= 2\pi(-32+24+48)$$

$$\underline{= 80\pi \ \ unit^3}$$

About $x = -2$

Using *Washer* Method:

$R: \quad 6 - y - (-2) = 8 - y$

$r: \quad x + 2 = \frac{1}{2}y + 2$

$$V = \pi \int_0^4 \left((8-y)^2 - \left(\frac{1}{2}y + 2\right)^2 \right) dy$$

$$= \pi \int_0^4 \left(64 - 16y - y^2 - \frac{1}{4}y^2 - 2y - 4 \right) dy$$

$$= \pi \int_0^4 \left(60 - 18y + \frac{3}{4}y^2 \right) dy$$

$$= \pi(16 - 144 + 240)$$

$$\underline{= 112\pi \ \ unit^3}$$

Exercise

The region R is bounded by the curves $x = y^2 + 2, \quad y = x - 4, \quad and \quad y = 0$

a) Write a single integral that gives the area of R.

b) Write a single integral that gives the volume of the solid generated when R is revolved about the x-axis.

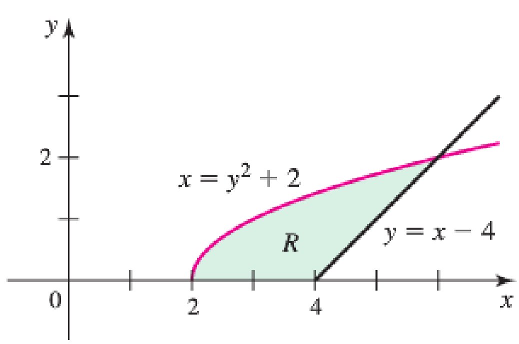
c) Write a single integral that gives the volume of the solid generated when R is revolved about the y-axis.

d) Suppose S is a solid whose base is R and whose cross sections perpendicular to R and parallel to the x-axis are semicircles. Write a single integral that gives the volume of S.

Solution

$y = x - 4 = y^2 - 2$

$y^2 - y - 2 = 0 \quad \to \quad \underline{y = \cancel{-1}, \ 2}$



a) $A = \displaystyle\int_0^2 \left[(y+4) - \left(y^2 + 2\right) \right] dy$

b) $V = 2\pi \displaystyle\int_0^2 y\left[(y+4) - \left(y^2 + 2\right) \right] dy$

c) About y-axis:

$Outer: \quad y+4$

$inner: \quad y^2+2$

$$V = \pi \int_0^2 \left[(y+4)^2 - \left(y^2+2\right)^2 \right] dy$$

d) $\perp R, \quad // \ x-axis \quad semicircle$

$$V = \int_0^2 A(y)\,dy$$

$$= \int_0^2 \frac{1}{2}\pi r^2 \ dy$$

$$= \frac{\pi}{2}\int_0^2 \left(\frac{y+4-y^2-2}{2}\right)^2 \ dy$$

$$= \frac{\pi}{8}\int_0^2 \left(y+2-y^2\right)^2 \ dy$$

Exercise

The region R is bounded by $y = \frac{1}{x^p}$ and the x-axis on the interval $[1,\ a]$, where $p>0$ and $a>1$.

Let V_x and V_y be the volumes of the solids generated when R is revolved about the x- and y-axes, respectively.

a) With $a=2$ and $p=1$, which is greater, V_x or V_y ?

b) With $a=4$ and $p=3$, which is greater, V_x or V_y ?

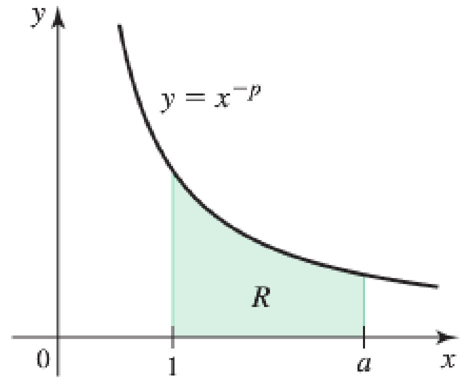
c) Find a general expression for V_x in terms of a and p. Note that $p=\frac{1}{2}$ is a special case, what is V_x when $p=\frac{1}{2}$?

d) Find a general expression for V_y in terms of a and p. Note that $p=2$ is a special case, what is V_y when $p=2$?

e) Explain how parts (*c*) and (*d*) demonstrate that $\lim\limits_{h\to 0} \frac{a^h-1}{h} = \ln a$

f) Find any values of a and p for which $V_x > V_y$

Solution



a) $p = 1 \quad a = 2$

$$V_x = \pi \int_1^2 \left(\frac{1}{x}\right)^2 dx$$

$$= -\pi \frac{1}{x} \Big|_1^2$$

$$= -\pi \left(\frac{1}{2} - 1\right)$$

$$= \frac{\pi}{2} \ \ unit^3$$

$$V_y = 2\pi \int_1^2 x\left(\frac{1}{x}\right) dx$$

$$= 2\pi x \Big|_1^2$$

$$= 2\pi \ \ unit^3$$

$$\therefore \ V_y > V_x$$

b) $p = 3 \quad a = 4$

$$V_x = \pi \int_1^4 \left(\frac{1}{x^3}\right)^2 dx$$

$$= \pi \int_1^4 x^{-6} dx$$

$$= -\frac{\pi}{5} x^{-5} \Big|_1^4$$

$$= -\frac{\pi}{5}\left(\frac{1}{104} - 1\right)$$

$$= \frac{1023\pi}{5120} \ \ unit^3$$

$$V_y = 2\pi \int_1^4 x\left(\frac{1}{x^3}\right) dx$$

$$= -2\pi \frac{1}{x} \Big|_1^4$$

$$= -2\pi \left(\frac{1}{4} - 1\right)$$

$$= \frac{3\pi}{2} \ \ unit^3$$

$\therefore \underline{V_y > V_x}$

c) $V_x = \pi \int_1^a \frac{1}{x^{2p}} dx$

For $p = \frac{1}{2}$

$$V_x = \pi \int_1^a \frac{1}{x} dx$$

$$= \pi \ln x \Big|_1^a$$

$$\underline{= \pi \ln a}$$

For $p \neq \frac{1}{2}$

$$V_x = \pi \int_1^a x^{-2p} dx$$

$$= \frac{\pi}{1-2p} x^{1-2p} \Big|_1^a$$

$$\underline{= \frac{\pi}{1-2p}\left(a^{1-2p} - 1\right)}$$

$$\therefore V_x = \begin{cases} \pi \ln a & \text{if } p = \frac{1}{2} \\ \frac{\pi}{1-2p}\left(a^{1-2p} - 1\right) & \text{if } p \neq \frac{1}{2} \end{cases}$$

d) $V_y = 2\pi \int_1^a x \frac{1}{x^p} dx$

$$= 2\pi \int_1^a x^{1-p}\, dx$$

For $p = 2$

$$V_y = 2\pi \int_1^a \frac{1}{x}\, dx$$

$$= 2\pi \ln x \Big|_1^a$$

$$\underline{= 2\pi \ln a}$$

For $p \neq 2$

$$V_y = 2\pi \int_1^a x^{1-p}\,dx$$

$$= \frac{2\pi}{2-p} x^{2-p} \Big|_1^a$$

$$\boxed{= \frac{2\pi}{2-p}\left(a^{2-p} - 1\right)}$$

$$\therefore V_y = \begin{cases} 2\pi \ln a & \text{if } \; p = 2 \\ \frac{2\pi}{2-p}\left(a^{2-p} - 1\right) & \text{if } \; p \neq 2 \end{cases}$$

e) From part (***c***):

$$V_x = \frac{\pi}{1-2p}\left(a^{1-2p} - 1\right)$$

Let $h = 1-2p$

$$h \to 0 \quad \Rightarrow \quad p \to \frac{1}{2}$$

$$\lim_{h\to 0} V_x = \lim_{p\to\frac{1}{2}} \frac{\pi}{1-2p}\left(a^{1-2p} - 1\right)$$

$$= \frac{\pi}{0}(1-1)$$

$$= \frac{0}{0}$$

$$= \lim_{p\to\frac{1}{2}} \frac{\pi(-2)a^{1-2p}\ln a}{-2}$$

$$\boxed{= \pi \ln a}$$

$$\lim_{h\to 0} V_x = \lim_{h\to 0} \frac{\pi}{h}\left(a^h - 1\right) = \pi \ln a$$

$$\boxed{\lim_{h\to 0} \frac{a^h - 1}{h} = \ln a}$$

From part (***d***):

$$V_y = \frac{2\pi}{2-p}\left(a^{2-p} - 1\right)$$

Let $h = 2-p$

$$h \to 0 \quad \Rightarrow \quad p \to 2$$

$$\lim_{h\to 0} V_y = \lim_{p\to 2} \frac{2\pi}{2-p}\left(a^{2-p} - 1\right)$$

$$= \frac{0}{0}$$

$$= \lim_{p\to 2} \frac{-2\pi a^{2-p}\ln a}{-1}$$

$$= 2\pi \ln a$$

$$\lim_{h\to 0} V_y = \lim_{h\to 0} \frac{2\pi}{h}\left(a^h - 1\right) = 2\pi \ln a$$

$$\boxed{\lim_{h\to 0} \frac{a^h - 1}{h} = \ln a}$$

f) No, V_y always is greater than V_x

$$\frac{1}{x^{2p}} < \frac{1}{x^{p-1}}$$

For $x > 1$ & $p > 0$

Exercise

Let R be the region bounded by the graph of $f(x) = cx(1-x)$ and the x-*axis* on $[0,\ 1]$. Find the positive value of c such that the volume of the solid generated by revolving R about the x-*axis* equals the volume of the solid generated by revolving R about the y-*axis*.

Solution

About x-*axis* : (Using Disks)

$$V = \pi \int_0^1 c^2 x^2 (1-x)^2\, dx$$

$$= \pi c^2 \int_0^1 x^2\left(1 - 2x + x^2\right) dx$$

$$= \pi c^2 \int_0^1 \left(x^2 - 2x^3 + x^4\right) dx$$

$$= \pi c^2 \left(\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5 \right|_0^1$$

$$= \pi c^2 \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right)$$

$$\boxed{= \frac{\pi c^2}{30}\ \ unit^3}$$

About y-*axis* : (Using Shell)

$$V = 2\pi \int_0^1 x \cdot cx(1-x)\, dx$$

$$= 2\pi c \int_0^1 \left(x^2 - x^3\right) dx$$

$$= 2c\pi\left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right|_0^1$$

$$= 2c\pi\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= \frac{\pi c}{6} \ \ unit^3$$

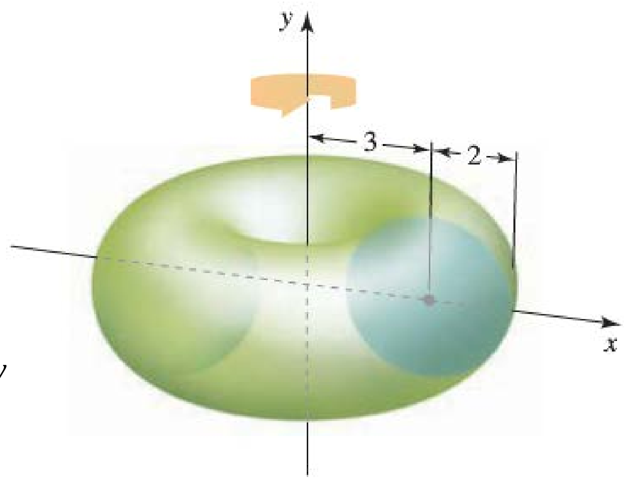
Exercise

Find the volume of the torus (doughnut formed when the circle of radius 2 centered at (3, 0) is revolved about the *y*-axis.

a) Use geometry to evaluate the integral

b) Use Shell method (use integral table)

Solution



$$(x-3)^2 + y^2 = 4 \quad \rightarrow \quad x = 3 \pm \sqrt{4-y^2}$$

$$V = \pi\int_{-2}^{2}\left(\left[f(y)\right]^2 - \left[g(y)\right]^2\right)dy$$

$$= \pi\int_{-2}^{2}\left(\left(3+\sqrt{4-y^2}\right)^2 - \left(3-\sqrt{4-y^2}\right)^2\right)dy$$

$$= 2\pi\int_{0}^{2} 2\left(6\sqrt{4-y^2}\right)dy$$

$$= 24\pi\int_{0}^{2}\sqrt{4-y^2}\ dy$$

a) $\sqrt{4-y^2}$ is a semi-circle with center (0, 0) and radius = 2, and since $0 \le y \ \& \ x \le 1$

Area = $\frac{1}{4}$(Area of this circle) $= \frac{1}{4}\pi(2)^2 = \pi$

$V = 24\pi^2 \ \ unit^3$

b) $V = 24\pi\left[\frac{1}{2}y\sqrt{4-y^2} + \frac{4}{2}\sin^{-1}\frac{y}{2}\right]_0^2$

$$= 24\pi\left(\frac{4}{2}\frac{\pi}{2}\right)$$

$$= 24\pi^2 \ \ unit^3$$

Exercise

The nose of a rocket is a solid of revolution of base radius r and height h that must join smoothly to the cylindrical body of the rocket. Taking the origin at the tip of the nose and the x-axis along the central axis of the rocket, various nose shapes can be obtained by revolving the cubic curve

$$y = f(x) = ax + bx^2 + cx^3$$

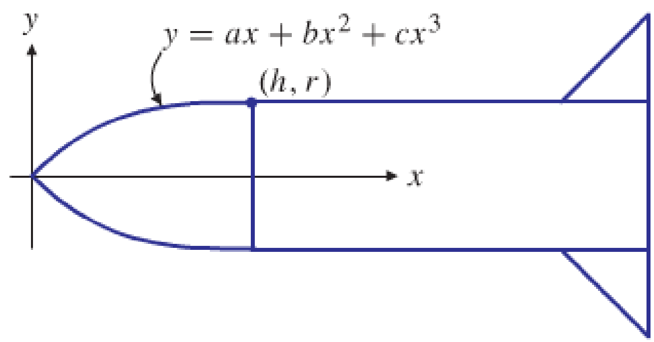
about x-axis. The cubic curve must have slope 0 at $x = h$, and its slope must be positive for $0 < x < h$. Find the particular cubic curve that maximizes the volume of the nose. Also show that his choice of the cubic makes the slope $\frac{dy}{dx}$ at the origin as large as possible and, hence, corresponds to the bluntest nose.

Solution

$$f'(x) = a + 2bx + 3cx^2$$

$$f'(h) = a + 2bh + 3ch^2 = 0$$

$$f(h) = ah + bh^2 + ch^3 = r$$



$$\begin{cases} a + 2bh + 3ch^2 = 0 \\ ah + bh^2 + ch^3 = r \end{cases}$$

$$\begin{cases} 2hb + 3h^2c = -a \\ h^2b + h^3c = r - ah \end{cases}$$

$$b = \frac{\begin{vmatrix} -a & 3h^2 \\ r - ah & h^3 \end{vmatrix}}{\begin{vmatrix} 2h & 3h^2 \\ h^2 & h^3 \end{vmatrix}} = \frac{2ah^3 - 3rh^2}{-h^4} = \frac{3r - 2ah}{h^2}$$

$$c = \frac{\begin{vmatrix} 2h & -a \\ h^2 & r - ah \end{vmatrix}}{\begin{vmatrix} 2h & 3h^2 \\ h^2 & h^3 \end{vmatrix}} = \frac{2rh - ah^2}{-h^4} = \frac{ah - 2r}{h^3}$$

$$f(x) = ax + \frac{3r - 2ah}{h^2}x^2 + \frac{ah - 2r}{h^3}x^3$$

$$(f(x))^2 = a^2x^2 + 2\frac{3r - 2ah}{h^2}ax^3 + 2\frac{ah - 2r}{h^3}ax^4 + \frac{(3r - 2ah)^2}{h^4}x^4$$
$$+ 2\frac{(3r - 2ah)(ah - 2r)}{h^5}x^5 + \frac{(ah - 2r)^2}{h^6}x^6$$

$$= a^2x^2 + \frac{6r-4ah}{h^2}ax^3 + \left(\frac{2a^2h-4ar}{h^3} + \frac{9r^2-12ahr+4a^2h^2}{h^4}\right)x^4$$
$$+ \frac{14ahr-12r^2-4a^2h^2}{h^5}x^5 + \frac{a^2h^2-4ahr+4r^2}{h^6}x^6$$

The volume of the nose cone is then

$$V(a) = \pi\int_0^h \left(f(x)\right)^2 dx$$

$$= \pi\int_0^h \left(\begin{array}{l} a^2x^2 + \frac{6r-4ah}{h^2}ax^3 + \left(\frac{2a^2h-4ar}{h^3} + \frac{9r^2-12ahr+4a^2h^2}{h^4}\right)x^4 \\ \quad + \frac{14ahr-12r^2-4a^2h^2}{h^5}x^5 + \frac{a^2h^2-4ahr+4r^2}{h^6}x^6 \end{array}\right)dx$$

$$= \pi\left|\frac{1}{3}a^2x^3 + \frac{1}{4}\frac{6r-4ah}{h^2}ax^4 + \frac{1}{5}\left(\frac{2a^2h-4ar}{h^3} + \frac{9r^2-12ahr+4a^2h^2}{h^4}\right)x^5\right.$$
$$\left. + \frac{1}{6}\frac{14ahr-12r^2-4a^2h^2}{h^5}x^6 + \frac{1}{7}\frac{a^2h^2-4ahr+4r^2}{h^6}x^7\right|_0^h$$

$$= \pi\left[\frac{1}{3}a^2h^3 + \frac{1}{4}\frac{6r-4ah}{h^2}ah^4 + \frac{1}{5}\left(\frac{2a^2h-4ar}{h^3} + \frac{9r^2-12ahr+4a^2h^2}{h^4}\right)h^5\right.$$
$$\left. + \frac{1}{6}\frac{14ahr-12r^2-4a^2h^2}{h^5}h^6 + \frac{1}{7}\frac{a^2h^2-4ahr+4r^2}{h^6}h^7\right]$$

$$= \pi\left[\frac{1}{3}a^2h^3 + \frac{3}{2}ah^2r - a^2h^3 + \frac{1}{5}\left(2a^2h^3 - 4ah^2r + 9hr^2 - 12ah^2r + 4a^2h^3\right)\right.$$
$$\left. + \frac{1}{6}\left(14ah^2r - 12hr^2 - 4a^2h^3\right) + \frac{1}{7}\left(a^2h^3 - 4ah^2r + 4hr^2\right)\right]$$

$$= \pi\left[-\frac{2}{3}a^2h^3 + \frac{3}{2}ah^2r + \frac{6}{5}a^2h^3 - \frac{16}{5}ah^2r + \frac{9}{5}hr^2\right.$$
$$\left. + \frac{7}{3}ah^2r - 2hr^2 - \frac{2}{3}a^2h^3 + \frac{1}{7}a^2h^3 - \frac{4}{7}ah^2r + \frac{4}{7}hr^2\right]$$

$$= \pi\left(\frac{1}{105}a^2h^3 + \frac{13}{210}ah^2r + \frac{13}{35}hr^2\right)$$

$$\boxed{= \frac{\pi h}{210}\left(2a^2h^2 + 13ahr + 78r^2\right)}$$

$$\frac{dV}{da} = \frac{d}{da}\left[\frac{\pi h}{210}\left(2a^2h^2 + 13ahr + 78r^2\right)\right]$$
$$= \frac{\pi h}{210}\left(4ah^2 + 13hr\right) = 0$$

$$4ah^2 + 13hr = 0 \quad \Rightarrow \quad a = -\frac{13r}{4h} \quad (CP)$$

Which is unacceptable since $a \ge 0$, and because $f'(x) > 0$ on $(0,\ h)$.

$$f'(x) = \frac{3ah-6r}{h^3}x^2 + \frac{6r-4ah}{h^2}x + a$$

$$x = \frac{-\frac{6r-4ah}{h^2} \pm \sqrt{\frac{36r^2-48ahr+16a^2h^2}{h^4} - \frac{12a^2h-24ar}{h^3}}}{\frac{6ah-12r}{h^3}}$$

$$= \frac{-6r+4ah \pm \sqrt{36r^2-48ahr+16a^2h^2-12a^2h^2+24ahr}}{6ah-12r} \cdot h$$

$$= \frac{-6r+4ah \pm \sqrt{36r^2-24ahr+4a^2h^2}}{6ah-12r} \cdot h$$

$$= \frac{-6r+4ah \pm \sqrt{(6r-2ah)^2}}{6ah-12r} \cdot h$$

$$= \frac{-6r+4ah \pm (6r-2ah)}{6ah-12r} \cdot h$$

$$\begin{cases} x_1 = \frac{-6r+4ah-6r+2ah}{6ah-12r} \cdot h = \frac{6ah-12r}{6ah-12r} \cdot h \underline{= h|} \\ x_2 = \frac{-6r+4ah+6r-2ah}{6ah-12r} \cdot h = \frac{2ah^2}{6ah-12r} \underline{= \frac{ah^2}{3ah-6r}|} \end{cases}$$

If $0 < x_2 < h \quad \Rightarrow \quad 0 < \frac{ah^2}{3ah-6r} < h \quad \to \quad 0 < ah^2 < 3ah^2 - 6rh$

$$0 < a < 3a - 6\frac{r}{h} \qquad \Rightarrow \ a > 0$$

$$-2a < -6\frac{r}{h} \quad \Rightarrow \quad a > \frac{3r}{h}$$

Hence, $0 \le a \le \frac{3r}{h}$.

We have

$$V(0) = \frac{\pi h}{210}\left(78r^2\right)$$

$$\underline{= \frac{13}{35}\pi r^2 h|}$$

$$V\left(\frac{3r}{h}\right) = \frac{\pi h}{210}\left(2\frac{9r^2}{h^2}h^2 + 13\frac{3r}{h}hr + 78r^2\right)$$

$$= \frac{\pi h}{210}\left(18r^2 + 39r^2 + 78r^2\right)$$

$$= \frac{\pi h}{210}\left(135r^2\right)$$

$$\underline{= \frac{9}{14}\pi r^2 h|}$$

The largest volume corresponds to $a = \frac{3r}{h}$, which is the largest allows value for a and so corresponds to the bluntest possible nose. The corresponding cubic $f(x)$ is

$$\begin{aligned} f(x) &= ax + \frac{3r - 2ah}{h^2}x^2 + \frac{ah - 2r}{h^3}x^3 \\ &= \frac{3r}{h}x - 3\frac{r}{h^2}x^2 + \frac{r}{h^3}x^3 \\ &= \boxed{\frac{r}{h^3}\left(3h^2x - 3hx^2 + x^3\right)} \end{aligned}$$

Exercise

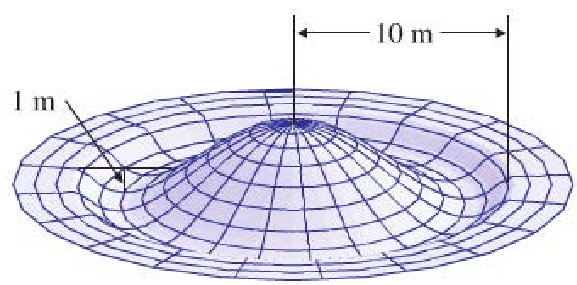
A landscaper wants to create on level ground a ring-shaped pool having an outside radius of 10 *m* and a maximum depth of 1 *m* surrounding a hill that will be built up using all the earth excavated from the pool. She decided to use a fourth-degree polynomial to determine the cross-sectional shape of the hill and pool bottom: at distance ***r*** *m* from the center of the development the height above or below normal ground level will be

$$h(r) = a\left(r^2 - 100\right)\left(r^2 - k^2\right) \quad m$$

For some $a > 0$, where k is the inner radius of the pool.
Find k and a so that the requirements given above are all satisfied.
How much earth must be moved from the pool to build the hill?

Solution

$$\begin{aligned} h'(r) &= a(2r)\left(r^2 - k^2\right) + a\left(r^2 - 100\right)(2r) \\ &= 2ar\left(r^2 - k^2 + r^2 - 100\right) = 0 \end{aligned}$$



$$\begin{aligned} 2r^2 - k^2 - 100 &= 0 \\ 2r^2 &= k^2 + 100 \\ r^2 &= \tfrac{1}{2}k^2 + 50 \end{aligned}$$

Since the depth must be 1 *m*, then

$$\begin{aligned} h(r) &= a\left(r^2 - 100\right)\left(r^2 - k^2\right) \\ -1 &= a\left(\tfrac{1}{2}k^2 + 50 - 100\right)\left(\tfrac{1}{2}k^2 + 50 - k^2\right) \\ -1 &= a\left(\tfrac{1}{2}k^2 - 50\right)\left(-\tfrac{1}{2}k^2 + 50\right) \\ -1 &= -a\left(50 + \tfrac{1}{2}k^2\right)\left(50 + \tfrac{1}{2}k^2\right) \\ 1 &= a\left(50 - \tfrac{1}{2}k^2\right)^2 \end{aligned}$$

$$a\left(100-k^2\right)^2=4$$

Volume of the pool:

$$V_P=2\pi a\int_k^{10} r\left(100-r^2\right)\left(r^2-k^2\right)dr$$

$$=2\pi a\int_k^{10} -\left(r^5-k^2r^3-100r^3+100k^2r\right)dr$$

$$=2\pi a\left(-\tfrac{1}{6}r^6+\tfrac{1}{4}k^2r^4+25r^4-50k^2r^2\right)\Big|_k^{10}$$

$$=2\pi a\left(-\tfrac{1}{6}10^6+\tfrac{1}{4}k^2 10^4+25\times 10^4-50k^2 10^2+\tfrac{1}{6}k^6-\tfrac{1}{4}k^6-25k^4+50k^4\right)$$

$$\underline{=2\pi a\left(\frac{250{,}000}{3}-2500k^2-\tfrac{1}{12}k^6+25k^4\right)}$$

Volume of the hill:

$$V_H=2\pi a\int_0^{k} r\left(r^2-100\right)\left(r^2-k^2\right)dr$$

$$=2\pi a\left(\tfrac{1}{6}r^6-\tfrac{1}{4}k^2r^4-25r^4+50k^2r^2\right)_0^k$$

$$=2\pi a\left(\tfrac{1}{6}k^6-\tfrac{1}{4}k^6-25k^4+50k^4\right)$$

$$\underline{=2\pi a\left(25k^4-\tfrac{1}{12}k^6\right)}$$

Volume of the pool = Volume of the hill

$$2\pi a\left(\frac{250{,}000}{3}-2500k^2-\tfrac{1}{12}k^6+25k^4\right)=2\pi a\left(25k^4-\tfrac{1}{12}k^6\right)$$

$$\frac{250{,}000}{3}-2500k^2-\tfrac{1}{12}k^6+25k^4=25k^4-\tfrac{1}{12}k^6$$

$$\frac{250{,}000}{3}-2500k^2=0$$

$$2500k^2=\frac{250{,}000}{3}$$

$$k^2=\frac{100}{3}\quad\Rightarrow\quad \boxed{k\approx 5.77}$$

$$a=\frac{4}{\left(100-k^2\right)^2}$$

$$= \frac{4}{\left(100-5.77^2\right)^2}$$

$$\approx 0.0009$$

$$V_H = 2\pi a\left(25k^4 - \frac{1}{12}k^6\right)$$

$$\approx 2\pi(0.0009)\left(25(5.77)^4 - \frac{1}{12}(5.77)^6\right)$$

$$\approx 140\ \ m^3$$