Section 4.5 – Partial Orderings

Definition

A relation R on set S is called a partial ordering or partial order if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a partially ordered set, or poset, and is denoted by (S, R). Members of S are called elements of the poset.

Example

Show that the "greater than or equal" relation $(\geq)$ is a partial ordering on the set of integers

Solution

Because $a \geq a$ for every integer a, $\geq$ is reflexive.

If $a \geq b$ and $b \geq a$, then $a = b$. Hence, $\geq$ is symmetric.

If $a \geq b$ and $b \geq c$ imply that $a \geq c$. Hence, $\geq$ is transitive.

It follows that $(\geq)$ is a partial ordering on the set of integers and $(\mathbb{Z}, \ \geq)$ is a poset.

Example

Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set S.

Solution

Because $A \subseteq A$ whenever is a subset of S. $\subseteq$ is reflexive.

It is antisymmetric because $A \subseteq B$ and $B \subseteq A$ imply that $A = B$

$A \subseteq B$ and $B \subseteq C$ imply that $A \subseteq C$, Hence $\subseteq$ is transitive.

Hence, $\subseteq$ is a partial ordering on $P(S)$ and $\left(P(S), \ \subseteq\right)$ is a poset.

Example

Let R be the relation on the set of people such that xRy if x and y are people and x is older than y. Show that is not a partial ordering,

Solution

R is not reflexive, because no person is older than herself or himself $x \not{R} x$

R is antisymmetric because if a person x is older than y, then y is not older than x. That is xRy, then $y \not{R} x$.

The relation is transitive because a person x is older than y, then y is older than z, then x is older than z.

R is not a partial ordering.

Definition

The elements *a* and *b* of poset $(S,\ \preceq)$ are called comparable if either $a \preceq b$ or $b \preceq a$. When *a* and *b* are elements of *S* such that neither $a \preceq b$ nor $b \preceq a$, *a* and *b* are called incomparable.

Example

In the poset $(\mathbb{Z},\ |)$ are the integers 3 and 9 comparable? Are 5 and 7 comparable?

Solution

The integers 3 and 9 are comparable, because 3 | 9.

The integers 5 and 7 are ***incomparable***, because $5 \nmid 7$ and $7 \nmid 5$

Definition

If $(S,\ \preceq)$ is a poset and every two elements of *S* are comparable, *S* is called a ***totally ordered*** or ***linearly ordered*** set, and $\rightleftarrows$ is called a ***total order*** or a ***linear order***. A totally ordered set is also called a ***chain***.

Example

The poset $(\mathbb{Z},\ \leq)$ is totally ordered, because $a \leq b$ or $b \leq a$ whenever *a* and *b* are integers.

Example

The poset $\left(\mathbb{Z}^{+},\ |\right)$ is not totally ordered, because it contains elements that are incomparable, such as 5 and 7.

Definition

If $(S,\ \preceq)$ is well-ordered set if it is a poset such that $\rightleftarrows$ is a total ordering and every nonempty subset of *S* has a least element.

Example

The set of ordered pairs of positive integers, $\mathbb{Z}^{+} \times \mathbb{Z}^{+}$, with $\left(a_1,\ a_2\right) \preceq \left(b_1,\ b_2\right)$ if $a_1 < b_1$, or if $a_1 = b_1$ and $a_2 < b_2$ (Lexicographic ordering), is a well-ordered set.

The set ***Z***, with the usual ≤ ordering, is not well-ordered because the set of negative integers, which is a subset of ***Z***, has no least element.

Theorem – The Principle of Well-Ordered Induction

Suppose that S is a well-ordered set. Then $P(x)$ is true for all $x \in S$, if

Inductive Step: For every $y \in S$, if $P(x)$ is true for all $x \in S$ with $x \prec y$, then $P(y)$ is true.

Proof

Suppose it is not the case that $P(x)$ is true for all $x \in S$. Then there is an element $y \in S$ such that, $P(y)$ is false.

Consequently, the set $A = \{x \in S \mid P(x) \text{ } \textit{is false}\}$ is nonempty. Because S is well ordered, A has a least element a. By the choice of a as a least element of A, we know that $P(x)$ is true for all with $x \prec a$. This implies by the inductive step $P(a)$ is true. This contradiction shows that $P(x)$ must be true for all $x \in S$.

Example

Determine whether $(3,\ 5) \prec (4,\ 8)$, whether $(3,\ 8) \prec (4,\ 5)$, and whether $(4,\ 9) \prec (4,\ 11)$ in the poset $(\mathbb{Z} \times \mathbb{Z},\ \preceq)$, where $\preceq$ is the lexicographic ordering constructed from the usual $\le$ relation on $\mathbb{Z}$.

<u>Solution</u>

Because 3 < 4, it follows that $(3,\ 5) \prec (4,\ 8)$ and that $(3,\ 8) \prec (4,\ 5)$. We have $(4,\ 9) \prec (4,\ 11)$, because the first entries of $(4,\ 9)$ and $(4,\ 11)$ are the same but 9 < 11.

Maximal and Minimal Elements

An element of a poset is called maximal if it is not less than any element of the poset. That is, a is ***maximal*** in the poset $(S,\ \preceq)$ if there is no element $b \in S$ such that $a \prec b$.

Similarly, an element of a poset is called minimal if it is not greater than any element of the poset. That is, a is ***minimal*** in the poset $(S,\ \preceq)$ if there is no element $b \in S$ such that $b \prec a$.

Maximal and minimal elements are easy to spot using a ***Hasse*** diagram. They are the "top" and "bottom" elements in the diagram.

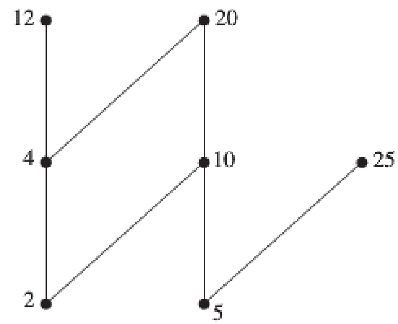
Sometimes there is an element in a poset that is greater than every other element. Such that an element is called the greatest element. That is, a s the ***greatest element*** of the poset $(S,\ \preceq)$

Example

Which elements of the poset ({2, 4, 5, 10, 12, 20, 25},, |) are maximal, and which are minimal?

Solution

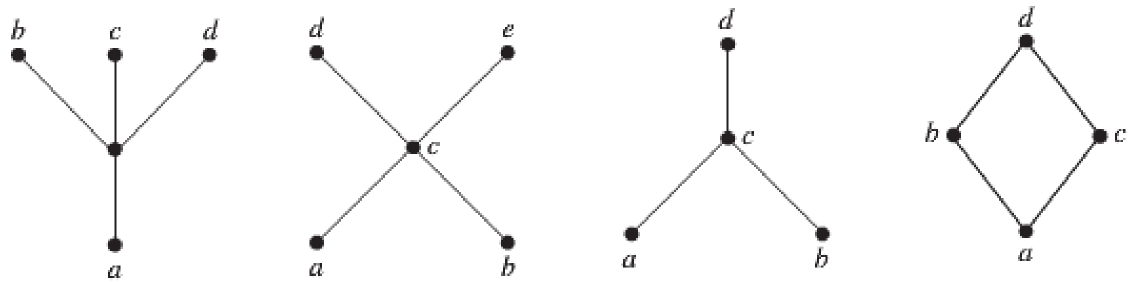
From the Hasse diagram, the poset shows that the maximal elements are 12, 20, and 25
The minimal elements are 2 and 5.



Hasse Diagram

Example

Determine whether the posets represented by each of the Hasse diagrams in figure below have greatest element and a least element.



(*a*) (*b*) (*c*) (*d*)

Solution

The least element of the poset with Hasse diagram (*a*) is a. This poset has no greatest element.
The poset with Hasse diagram (*b*) has neither a least nor a greatest element.
The poset with Hasse diagram (*c*) has no least element. Its greatest element is d.
The poset with Hasse diagram (*d*) has least element a and greatest element d.

Example

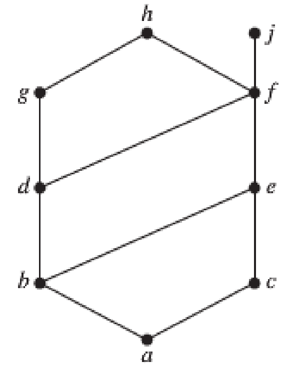
Let S be a set. Determine whether there is a greatest element and a least element in the poset $\left(P(S),\subseteq\right)$

Solution

The least element is the empty set, because $\varnothing \subseteq T$ for any subset T of S.
The greatest element in this poset, because $T \subseteq S$ whenever T is a subset of S.

Example

Find the lower and upper bounds of the subsets $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$ in the poset with the Hasse diagram shown in the figure.



Solution

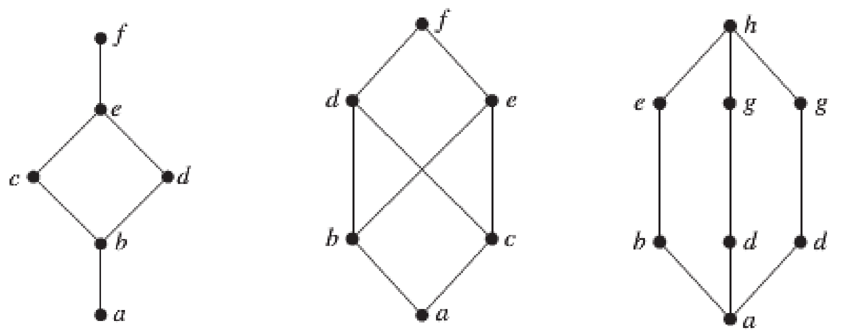
The upper bounds of $\{a, b, c\}$ are e, f, j and h and its only lower bound is a.
There is no upper bounds of $\{j, h\}$, and its lower bounds are $a, b, c, d, e,$ and f.
The upper bounds of $\{a, c, d, f\}$ are $f, h,$ and j, and its lower bound is a.

Lattices

A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a ***lattice***. Lattices have many special properties.

Example

Determine whether the posets represented by each of the Hasse diagrams are lattices



(a) (b) (c)

Solution

The posets represented by the Hasse diagrams in (***a***) and (***c***) are both lattices because in each poset every pair of elements has both a least upper bound and a greatest lower bound.
On the other hand, the poset with the Hasse diagram shown in (***b***) is not a lattice, because the elements b and c have no least upper bound. Each of the elements $d, e,$ and f is an upper bound, but none of these 3 elements precedes the other two with respect to the ordering of this poset.

Example

Is the poset $\left(\mathbb{Z}^+, \mid\right)$ a lattice?

Solution

Let a and b be two positive integers, The least upper bound and greatest lower bound of these 2 integers are the least common multiple and the greatest common divisor of these integers, respectively, as the reader should verify. It follows that this is a lattice.

Example

Determine whether the posets $(\{1,\ 2,\ 3,\ 4,\ 5\},\ |)$ and $(\{1,\ 2,\ 4,\ 8,\ 16\},\ |)$ are lattices

Solution

Because 2 and 3 have no upper bound in $(\{1,\ 2,\ 3,\ 4,\ 5\},\ |)$, they are certainly do not have a least upper bound. Hence, the first poset is not a lattice.

Every elements of the second poset have both a least upper bound and a greatest lower bound. The least upper bound of 2 elements in this poset is the larger of the elements and the greatest lower bound of 2 elements is the smaller of the elements. Hence, the second poset is a lattice.

Example

Determine whether $(P(S),\ \subseteq)$ is a lattice where *S* is a set.

Solution

Let *A* and *B* be 2 subsets of *S*. The least upper bound and the greatest lower bound of *A* and *B* are $A \cup B$ and $A \cap B$, respectively.

Hence, $(P(S),\ \subseteq)$ is a lattice.

Exercises *Section* **4.5 – Partial Orderings**

1. Which of these relations on {0, 1, 2, 3} are partial orderings? Determine the properties of a partial ordering that the others lack.

a) {(0, 0), (1, 1), (2, 2), (3, 3)}
b) {(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}
c) {(0, 0), (1, 1), (1, 2), (2, 2), (3, 3)}
d) {(0, 0), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)}
e) {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}
f) {(0, 0), (2, 2), (3, 3)}
g) {(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 3)}
h) {(0, 0), (1, 1), (1, 2), (2, 2), (3, 1), (3, 3)}
i) {(0, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (2, 3), (3, 0), (3, 3)}
j) {(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 3)}

2. Is (S, R) a poset If S is the set of all people in the world and $(a, b) \in R$, where a and b are people, if

a) a is a taller than b?
b) a is not taller than b?
c) $a = b$ or a is an ancestor of b?
d) a and b have a common friend?
e) a is a shorter than b?
f) a weighs more than b?
g) $a = b$ or a is a descendant of b?
h) a and b do not have a common friend?

3. Which of these are posets?

a) $(\mathbb{Z}, =)$	*b)* $(Z, \neq)$	*c)* $(Z, \geq)$	*d)* $(Z, \nmid)$
e) $(R, =)$	*f)* $(R, <)$	*g)* $(R, \leq)$	*h)* $(R, \neq)$

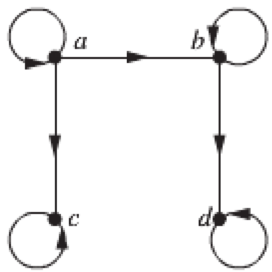
4. Determine whether the relations represented by these zero-one matrices are partial orders

a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *b)* $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *c)* $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *d)* $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

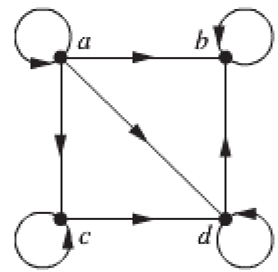
e) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ *f)* $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

5. Determine whether the relation with the directed graph shown is a partial order.

a)



b)



c)

6. Let $(S,\ R)$ be a poset. Show that $\left(S,\ R^{-1}\right)$ is also a poset, where R^{-1} is the inverse of R. The poset $\left(S,\ R^{-1}\right)$ is called the dual of $(S,\ R)$.

7. Draw the Hasse diagram for the "greater than or equal to" relation on {0, 1, 2, 3, 4, 5}